On the Capacity of Joint Time and Concentration Modulation for Molecular Communications

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Abstract—Most diffusion based molecular channels suffer from low information capacity due to the structure of the diffusion environment. To address this issue, this paper studies the capacity of the diffusion based molecular communication by exploiting both time and concentration level of the released molecules for information transfer. While the release time can, in general, be any real number, for the sake of tractability, we consider a discrete value for it, by dividing the transmission time interval into some sub-intervals. The transmitter releases molecules in one of the sub-intervals with a level of concentration both determined by input data, hereby applying joint time and concentration (JTAC) modulation. We derive the lower and upper bounds on the JTAC channel capacity. The observation time, at the receiver, which is equal to symbol period, is divided to some sub-intervals, not necessarily equal to the number of sub-intervals in the transmitter, and the number of received molecules in each sub-interval is counted. We propose three lower bounds, depending on how the receiver uses the number of molecules counted in the sub-intervals. In the first scheme, the receiver examines the sub-interval which has the maximum mutual information with channel inputs. Second scheme exploits the sum of received molecules in all sub-intervals to determine the concentration level. Then conditioned on the concentration level, for each sub-interval, the mutual information between the release time and the number of molecules received at the sub-interval computed, then the maximum of these mutual information terms determines the release time. In the last scheme, the concentration is detected using a similar approach to the second scheme, while the difference of received molecules in adjacent sub-intervals is utilized to detect the release time. A closed form lower bound expression has been derived for each case. Moreover, the symmetric Kullback-Liebler (KL) divergence metric is used to obtain a computable upper bound on the channel capacity. For this scenario, an optimum distribution for binary case has been found in [7]. In all of these CB models, released concentration is chosen (according to a specific distribution) as the channel input, and the maximum mutual information between the released concentration and the number of received molecules is computed for the capacity evaluation.

The timing channels has been proposed in [8] as an alternative way of communications in nano-networks and its capacity has been studied. To this end, the first hitting time probability has been considered to detect the release time of molecules. Then, accordingly, lower bounds on the channel capacity have been derived. In addition, Monte Carlo methods have been employed for calculating the achievable rates. [9] has studied timing channels with energy constraints. Timing molecular channels with drifts, which turns to an additive inverse Gaussian channel, has been introduced in [10], where capacity lower and upper bounds are presented for the memoryless case. In [11], the capacity of a channel with additive memoryless inverse Gaussian noise has been obtained in asymptotic regime. Finally in [12], assuming a limited life time for molecules, the authors have considered different detectors (first hitting time and average time) and have computed lower and upper bounds in each case (which have different noise characteristics).

Both CB and timing based modulations (TB), individually, have been proved to be useful in MC. The question here is that if it is beneficial to use joint TB and CB modulations. This question is challenging because their effects are correlated. Diffusion equation shows a strong correlation between effects of the concentration released and passage time of the...
molecules in the channel. In this work, our aim is to quantify these effects and describe corresponding channel properties. In fact, we want to answer the question that how using joint concentration and timing can improve the process of data transmission. In order to compute bounds on the channel capacity analytically, we assume that the transmitter chooses the release time among \( m \) discrete times instead of choosing from a continuous range. Our model uses different release times and different levels of molecules in transmitter to capture their effects on the transmission rate. In the receiver, the observation time (symbol period) is divided into some sub-intervals and based on the number of molecules received in different sub-intervals, the decision is made on the time and the level of molecules released by the transmitter at the related symbol transmission. We assume a memoryless channel, where residue molecules from previous time slots are removed from the environment using for example a specific enzymes and imposing a guard interval between successive transmissions [2]. We propose three lower bounds on the channel capacity of the JTAC modulation depending on how the counted number of molecules in different observation sub-intervals in the receiver is used as follows:

1) For each sub-interval, we compute the mutual information between the number of molecules received in the sub-interval with the channel inputs (the release time and level of concentration). The maximum of these mutual information provides a lower bound on the capacity.

2) Using the received molecules in total observation sub-intervals, the level of concentration is determined. Then, given the released concentration, the mutual information between the number of molecules received in each sub-interval and the release time is computed. The maximum of these mutual information terms is used to derive the second lower bound on the channel capacity.

3) Again like second bound, using the total received molecules, the level of concentration is determined. Then, given the concentration, the mutual information between the difference of the number of molecules received in adjacent sub-intervals and the release time, for each adjacent sub-interval, is computed, and the maximum of these mutual information terms is used to derive the third lower bound.

For each of above lower bounds, we derive a closed form expression. We also compute an upper bound on the channel capacity of the JTAC modulation. To this end, for each observation sub-interval, we compute an upper bound on the mutual information between the number of received molecules and the channel input (release time and concentration level) using the symmetric KL divergence introduced in [6]. To see how tight the derived bounds are, we also numerically compute the capacity of JTAC channel using Blahut-Arimoto algorithm.

Our numerical results show that lower bounds on channel capacity are tight specifically for the environments with large diffusion coefficient. Also our results demonstrate that using JTAC modulation significantly increases achievable rates compared to conventional concentration based (CB) and timing based (TB) modulations as it discussed in numerical section. More specifically, for the number of observation sub-intervals \( 20 \) and the number of distinct release time of \( 10 \) and in an environment with diffusion noise parameter equal to \( 2 \), we achieve up to 1.5 bits (\%50) higher rates than the CB. Also we see that in environments with lower \( \nu \) or noise parameter, \( c \), timing based modulation has a larger part in total achievable rates in channel. Also From the example considered, JTAC provides more robust transmission compared to the conventional CB. That is, in our schemes capacity falls more slowly when environmental noise parameter increases compared to CB in higher values of \( c \).

The rest of paper is organized as follows: in Section II we describe the system model. In Section III we provide lower bounds on the capacity of the channel (achievable rates) presenting the three schemes stated above. Section IV provides an upper bound on the channel capacity. In Section V numerical results are provided for the derived bounds. And finally in Section VI some concluding remarks are presented.

Notation: We denote random variables with upper case letters and their realizations with corresponding lower case letters. We use \( f(.) \) to represent probability density function (PDF) of continuous and mixed Random Variables. the probability mass function (PMF) of a discrete random variable \( X \) is denoted by \( P_X(x) \), and the probability mass function (PMF) of a discrete random variable \( X \) is donated by \( P_X(x) \). \( \text{erf}(x) \) is used to show error function given by \( \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \). For simplicity, throughout paper, \( h(.) \) represents differential entropy of a continuous random variable or entropy of a discrete random variable based on its argument. \( \log(.) \) is used to denote logarithm in natural basis. Basis for all other cases is specified in the related context. \( D_{KL} \) denotes Kullback-Leibler divergence metric. And finally \( \lambda_0 \) is used to denote noise molecules in the environment.

II. SYSTEM MODEL

Now, we describe the system model and its constraints. We consider a point to point MC system as follows:

- **Transmitter:** Transmitter is an exact concentration transmitter that completely controls the intensity and the time of released molecules. We have both average and maximum constraints on the released concentration (denoted by \( X \)),

\[
E\{X\} \leq E_m, \quad 0 \leq X \leq M.
\]

- **Transmission:** Transmission is time slotted with duration \( T_s \). To avoid ISI, the release time is restricted to be in interval \([IT_s, IT_s + \tau_x]\) in \( l \)-th time slot, where \( \tau_x \leq T_s \) is a design parameter based on the level of ISI [2]. We denote the release time by \( T_x \) which is a discrete random variable with \( m \) possible levels. In fact we have \( T_x = lT_s + j\sigma_x \), \( 0 \leq j \leq m - 1 \). The release time is assumed to have finite number of levels for the sake of tractability of capacity analysis. As \( m \) increases, we get closer to the continuous timing channels, in which the release time can be any time in the transmission interval. In each channel use, transmitter selects a time \( T_x = lT_s + j\sigma_x \), and concentration \( X = x \) consistent with (1). As said before, \( T_x \) is selected in a time interval with duration \( \tau_x \) which \( \tau_x \leq T_s \).

\[ \text{(2)} \] (see Fig. 1). Also note that since our channel model is a memoryless channel and our analysis are independent of a 1 This effect can be due to excessive molecules of the same type as the transmitted molecules that exist in diffusion environment from other sources. 

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1 This effect can be due to excessive molecules of the same type as the transmitted molecules that exist in diffusion environment from other sources.
specific time slot, in the following, without loss of generality, we assume \( t = 0 \), i.e., \( T_x = j \sigma_x \).

Receiver: We consider an ideal receiver that absorbs any molecule that hits its surface. Each observation time slot at the receiver is divided in \( n \) sub-intervals each with duration \( t_b \). Receiver counts the number of molecules that hit it in every \( t_b \) seconds. That is, we have \( n \) observations in each time slot where \( n = \frac{T_x}{t_b} \). The \( i \)-th observation (i.e., related to the number of molecules counted in the \( i \)-th sub-interval) is shown by \( Y_i \) (see Fig. 1). We assume that the molecules not arriving at the receiver in the duration of corresponding time slot, disappear, which is possible for example by injecting some specific enzymes in the environment [2]. Also note that the time taken by a molecule to hit the receiver is distributed according to a Lévy distribution [1]. The probability that each released molecule falls in the specific sub-interval \( i \) \((1 \leq i \leq n)\) with the assumption that the transmitter has released at time \( j \sigma_x \) is computed by integrating PDF of Lévy distribution in the corresponding sub-interval

\[
p_{ij} = \int_{(i-1) t_b}^{i t_b} L(j \sigma_x, c) dt, \quad (2)
\]

where

\[
L(j \sigma_x, c) = \frac{c}{\sqrt{2 \pi (t - j \sigma_x)^3}} \exp\left(\frac{-c}{2(t - j \sigma_x)}\right), \quad t - j \sigma_x \geq 0.
\]

\( c \) is Lévy distribution parameter, which is defined as \( \frac{\sigma^2}{2 \pi} \) where \( D \) is the diffusion coefficient of the environment and \( d \) is the distance between transmitter and receiver. As \( c \) increases (due to increase in \( d \) or decrease in \( D \)) the environment becomes more noisy and transmission gets more unreliable. From [4] and (3), we have:

\[
p_{ij} = \text{erf}(\sqrt{\frac{c}{2(t - t_b)}}) - \text{erf}(\sqrt{\frac{c}{2(t_b - j \sigma_x)}}).
\]

Based on the generalization of Bernoulli trials, if we define \( A = Pr(Y_1 = y_1, \ldots, Y_n = y_n; T_x = t_x, X = x) \) as the probability that \( y_i \) molecules fall in the \( i \)-th sub-interval \( i = 1, \ldots, n \), given \( T_x = t_x \) and \( X = x \) we have: \( A = \frac{y_1!y_2!\ldots y_n!(x-M_1)!}{y_1!y_2!\ldots y_n!} \prod_{i=1}^{n} p_{ij}^{y_i}(1 - \sum_j p_{ij})^{x-M_1} \), where \( M_1 = y_1 + \ldots + y_n \). For large values of \( x \), assuming that \( \frac{x}{y_i} \) equals to a constant value, \( \lambda \), we can use Poisson approximation as

\[
A = e^{-\lambda t} \frac{\lambda^{y_1}}{y_1!} \times e^{-\lambda t} \frac{(\lambda^{y_2})}{y_2!} \ldots e^{-\lambda t} \frac{(\lambda^{y_n})}{y_n!}, \quad (5)
\]

where we have \( \lambda_{ij} = xp_{ij}, 0 \leq j \leq m - 1 \), and \( p_{ij} \) is defined in [2]. So, the channel transition probability will be

\[
Pr(Y_i = y_i | y_1, \ldots, y_n; X = x, T_x = t_x) = \prod_i Pr(Y_i = y_i | x, T_x = t_x),
\]

where \( t_x = j \sigma_x, 0 \leq j \leq m - 1 \). Then from (5) we have:

\[
Pr(Y_i = y_i | x, T_x = t_x) = e^{-xp_{ij}} \frac{(xp_{ij})^{y_i}}{y_i!}.
\]

**Remark 1.** We neglect the impact of noise molecules in the environment throughout the paper (in contrast to Lévy noise which is inherent to diffusion environment) except in part B of section II and section III. When considering the environmental noise molecules, the channel transition probability is the same as (5), except that we have \( \lambda_{ij} = M_1 p_{ij} + \lambda_0 \).

**III. LOWER BOUNDS ON CHANNEL CAPACITY**

In this section, we provide three lower bounds on the JTAC channel capacity.

**A. First lower bound**

As stated before, in the first scheme, the receiver uses only the observation of a single sub-interval which has maximum mutual information with the input. This lower bound simply uses the information of one sub-interval and throws away information of other sub-intervals for the sake of the simplicity.

**Theorem 1.** A lower bound on the JTAC channel capacity based on maximum of mutual information between input and the number of received molecules in each different sub-intervals is

\[
C \geq \max_i I(X; T_x; Y_i) \geq \max_i R_i, \quad (7)
\]

where

\[
R_i = \log M + \frac{1}{2} \log \frac{M}{\mu} + \log \sqrt{\pi} \text{erf} \sqrt{\mu} + \alpha \mu
\]

\[- \log e^{\tau} \log \frac{1}{2} - \log \sqrt{\pi} \text{erf} \sqrt{\mu},\]

\[-1 \left(4 \sqrt{\frac{\mu}{12M^2}} \tan^{-1} \left(\sqrt{\frac{12M^2}{\mu}}\right) + 2 \sqrt{\pi} \log (1 + \frac{1}{12\mu^2M^2})\right) \] in (7), the receiver determines the \( i \)-th sub-interval that maximizes \( I(X; T_x; Y_i) \) for \( 1 \leq i \leq n \). To this

\[
p_{ij}^\ast = \max_j p_{ij}, \quad (8)
\]

**Proof.** To prove (7), the receiver determines the \( i \)-th sub-interval that maximizes \( I(X; T_x; Y_i) \) for \( 1 \leq i \leq n \). To this
end, we derive the bounds on this mutual information using Lemmas 1 and 2 given below. One strategy for computing lower bound on $I(X,T_x;Y_i)$ is to lower bound $h(Y_i)$ and upper bound $h(Y_i|X,T_x)$ in terms of input distributions and channel parameters. For computing a lower bound on $h(Y_i)$, we borrow a technique from [12], and then we upper bound $h(Y_i|X,T_x)$ based on the results on the entropy of Poisson random variable. Final result of the lower bound is in terms of $h(X,T_x)$, $E\{X\}$ and $p_{ij}$, which can be maximized under constraints (1) and (2). This concludes explicit lower bounds on $I(X,T_x;Y_i)$. To compute a lower bound on $h(Y_i)$ we need a variant of data processing theorem from [13].

Lemma 1. [13, Lemma 3.11] For any distributions $P$ and $Q$ on $X$ and any stochastic matrix $w = w(y|x): x \in X, y \in Y$ we have $D_{KL}(Pw||Qw) \leq D_{KL}(P||Q)$, i.e., every processing on distributions $P$ and $Q$ decreases their KL divergence.

Now, using above lemma we find a lower bound on $h(Y_i)$ in terms of input distribution of channel which could be maximized by choosing appropriate joint distribution on $(X,T_x)$ which we see in further discussions.

Lemma 2. A lower bound on entropy of output of JTAC Channel is as follows:
\[ h(Y_i) \geq h(X,T_x) - \log(\eta p^*_i) - m - \log(k). \]  
(9)

Proof. We use Lemma 1 by choosing an arbitrary input distribution $Q(.)$ and a specific input distribution with pdf
\[ P = f(x,T_x = j\sigma_x) = \frac{k}{\eta p_{ij}} e^{-\frac{1}{\eta} x}, \]  
(10)
where $k$ is selected such that $f(x,T_x = j\sigma_x)$ sums to one. In fact, we have $k = (\sum_{j=1}^{m} \frac{1}{\eta p_{ij}})^{-1}$. From (10), we have
\[ D_{KL}(Q(T_x,X)||P(T_x,X)) = \int_0^\infty \sum_{i=0}^{m} Q(T_x,X) \log(\frac{Q(T_x,X)}{P(X)} e^{-\frac{1}{\eta} x})dX \]  
(11)
\[ = -h(Q(X,T_x)) + 1 + \sum_{i=1}^{m} Q(T_x = j\sigma_x) \log(\eta p_{ij}). \]

Now let the PMF of the output of Poisson channel with input distributions $Q$ and $P$ be shown by $P_d^Q(.)$ and $P_d^G(.)$, respectively. From (10), we have:
\[ P_d^G = P_{Y_i}(y_i) = \sum_{t_x} \int_0^\infty \frac{k}{\eta p_{ij}} e^{-\frac{1}{\eta} x} p(y_i|x,t_x)dx, \]  
(12)
Using (10), it can be shown that:
\[ P_d^G = \sum_{j=1}^{m} \frac{k}{\eta p_{ij}} \left( \frac{1}{\eta p_{ij} + 1} \right)^i \left( \frac{\eta p_{ij}}{\eta p_{ij} + 1} \right)^y. \]  
(13)

Thus using the definition of KL divergence for $P_d^Q$ and $P_d^G$:
\[ D_{KL}(P_d^Q(.)||P_d^G(.)) = \sum_{y=0}^{\infty} P_d^Q(y_i) \log(\frac{P_d^G(y_i)}{\sum_{i=1}^{m} \frac{k}{\eta p_{ij}} \left( \frac{1}{\eta p_{ij} + 1} \right)^i \left( \frac{\eta p_{ij}}{\eta p_{ij} + 1} \right)^y}) \]  
\[ = -h(Y_i) - \sum_{y=0}^{\infty} P_d^Q(y_i) \log(\sum_{j=1}^{m} \frac{k}{\eta p_{ij} \eta p_{ij} + 1} \left( \frac{\eta p_{ij}}{\eta p_{ij} + 1} \right)^y). \]  
(14)

Applying Lemma 1 to (11) and (14) results in:
\[ h(Y_i) \geq h(X,T_x) - \sum_{j=1}^{m} Q(T_x = j\sigma_x) \log(\eta p_{ij}) - 1 \]  
(15)
\[ - \sum_{y=0}^{\infty} P_d^Q(y_i) \log(\sum_{j=1}^{m} \frac{k}{\eta p_{ij} \eta p_{ij} + 1} \left( \frac{\eta p_{ij}}{\eta p_{ij} + 1} \right)^y). \]

Now, we upper bound two sums in the above equation to conclude a simple form for the lower bound on $h(Y_i)$. By using $\log(x) \leq (x - 1)$, we obtain
\[ \sum_{y=0}^{\infty} P_d^Q(y_i) \log(\sum_{j=1}^{m} \frac{1}{\eta p_{ij} \eta p_{ij} + 1} \left( \frac{\eta p_{ij}}{\eta p_{ij} + 1} \right)^y) \]  
\[ \leq \sum_{y=0}^{\infty} P_d^Q(y_i) \log(1 + \left( \frac{\eta p_{ij}}{\eta p_{ij} + 1} \right)^y - 1) \]  
(16)
\[ = \sum_{y=0}^{\infty} \sum_{j=0}^{m-1} P_d^Q(y_i) \frac{1}{\eta p_{ij} \eta p_{ij} + 1} \left( \frac{\eta p_{ij}}{\eta p_{ij} + 1} \right)^y - 1 \]  
(17)
\[ \leq \sum_{y=0}^{\infty} \sum_{j=0}^{m-1} \frac{1}{\eta p_{ij}} \sum_{t_x} y \left( \frac{\eta p_{ij}}{\eta p_{ij} + 1} \right)^y - 1 \]  
(18)
\[ \sum_{y=0}^{\infty} \sum_{j=0}^{m-1} \frac{1}{\eta p_{ij}} - \eta p_{ij} - 1 = m - 1, \]

where (a) follows from the fact that $Y_i$ is an integer valued random variable and (b) is due to using the average of Geometric distribution.

Next, we upper bound $\sum_{j=1}^{m} Q(T_x = j\sigma_x) \log(\eta p_{ij})$ in (15). We know that $\log(p_{ij})$ is a monotonic function and above sum has a form of average of a random variable with probability distribution $Q(T_x)$. This sum can be upper bounded as:
\[ \sum_{j=1}^{m} Q(T_x = j\sigma_x) \log(\eta p_{ij}) \leq \log(\eta p^*_i), \]
(18)
where $p^*_i$ is defined in (8) in Combining (16) and (18) results in (9).

Lemma 3. An upper bound on the conditional entropy of the number of received molecules in a specific sub-interval $i$ is
\[ h(Y_i|X,T_x) \leq \frac{1}{2} \log(2\pi e) + \frac{1}{2} E\{\log(p^*_i X)\} + \frac{1}{2} E\{\log(1 + \frac{1}{12p^*_i X})\}. \]  
(19)

Proof. We have:
\[ h(Y_i|X,T_x) = \sum_{j=0}^{m} h(Y_i|X,T_x = j\sigma_x) P_t(x_j\sigma_x), \]  
(20)
where each entropy term can be upper bounded as follows (using [13] Theorem 8.6.5): since summing an independent uniform random variable with an arbitrarily random variable increases its variance, and by using the entropy of a Gaussian random variable, we have the following upper bound
\[ h(Y_i|X,T_x = j\sigma_x) \leq \frac{1}{2} \log(2\pi e) + \frac{1}{2} \log(E(p_{ij} X + \frac{1}{12})). \]
Using the Jensen’s inequality, we have:
\[
\begin{align*}
    h(Y_i|X,T_x = j\sigma_x) &\leq \frac{1}{2} \log(2\pi e) \\
    &+ \frac{1}{2} E\{\log(p_{ij}^X)\} + \frac{1}{2} E\{\log(1 + \frac{1}{12p_{ij}^X})\}.
\end{align*}
\]
(21)

Now, consider \(P_{T_x}^*\) as the distribution that maximizes \(\varphi(p)\). So,
\[
\begin{align*}
    h(Y_i|X,T_x) &\leq \sum_{j=0}^m \left( \frac{1}{2} \log(2\pi e) + \frac{1}{2} E\{\log(p_{ij}^X)\} + \frac{1}{2} E\{\log(1 + \frac{1}{12p_{ij}^X})\} \right) \hat{p}_{T_x}^*(j\sigma_x).
\end{align*}
\]

Noting that the right side of above equation is an increasing function of \(p_{ij}\), we achieve \(\varphi(p)\).\]

Using (9) and (19) and \(\partial J / \partial f(x) = -\log(f(x)) - 1 - \frac{1}{2} \log(x) + \Gamma_0 f(x) + \Gamma_1 x = 0\), solving for \(f(x)\), we have \(f(x) = \frac{k^2 x^2 e^{-k^2 x}}{\sqrt{\pi} \sqrt{x}}\). Applying constraints (1), we conclude
\[
f(x) = \frac{\sqrt{\mu}}{\sqrt{M \pi x} \erf(\sqrt{\mu})}. \tag{23}
\]

By substituting (23) and (24) in (22), we obtain
\[
\begin{align*}
    I(X, T_x; Y_i) &\geq \max \{ \log m \frac{1}{2} \log \frac{M}{\mu} + \log \sqrt{\pi} \erf(\sqrt{\mu}) + \alpha \mu \} \\
    &- \log E_m p^*_{ij} - m \log k - \frac{1}{2} \log 2\pi e - \frac{1}{2} \log p^*_i
\end{align*}
\]
(24)

where \(\alpha \) and \(\mu \) are defined in (7). This concludes the lower bound on the channel capacity given in (7).\]

B. Second lower bound

In this scheme, the receiver uses sum of received molecules in all sub-intervals to detect the transmitted concentration, while for the release time detection, the sub-interval with maximum mutual information with the released time is used.

**Theorem 2.** Using the sum of molecules counted in different sub-intervals, the JTAC channel capacity is lower bounded as
\[
C \geq \max(R_1, R_2), \tag{25}
\]

where
\[
R_1 = \log_2(m) - \log(c^*) - E_m \phi - \log(E_m p^*) - \log(m - 1) \\
- \log(2\pi e) - \log(k^*) + \log(12) + \max h(\text{Poisson}(M \tilde{p})) \tag{26}
\]

\[
R_2 = \log_2(m) - \log(c^*) - E_m \phi - \log(E_m p^*) - m \\
- \log(2\pi e) - \log(k^*) + \log(12) + \max h(\text{Poisson}(M \tilde{p})) \tag{27}
\]

\(\phi\) is computed by solving (62) given in Appendix A. Also note that \(c^*\) is computed using \(\phi\) derived in (62) and (60), \(Ei(\cdot)\) is a special function called exponential integral function [13] defined in (61) which is used to evaluate \(\phi\), and
\[
k^* = \sum_j \frac{1}{p^*_j}, \tag{28}
\]

\[
p^*_j = \max_{i=0}^m p_{ij}, \tag{29}
\]

\[
p^* = \max_j p^*_j. \tag{30}
\]

\[
\tilde{p}_i = \min_j p_{ij}. \tag{31}
\]

**Proof.** The proof is provided in Appendix A.\]

**Remark 2.** It will be shown in Section V that this scheme outperforms the one in Theorem 1 in some cases. In fact for large \(m\), we have \(R_2\) as tighter lower bound, while for small \(m\), \(R_1\) achieves higher rates.

C. Third lower bound

The third scheme uses the difference of received molecules in adjacent sub-intervals to detect the release time, while the detection of concentration is similar to the second scheme (i.e., using the sum of received molecules). That is, the receiver
counts the increase or decrease of received molecules in adjacent sub-intervals in order to detect transmitted time \( T_x \) conditioned on knowing the transmitted concentration level. Since this scheme could be of practical interest because of its simplicity, this bound could also give some insight on achievable rates by practical receivers.

The distribution of the difference of two independent Poisson random variables contains modified Bessel function \([16]\), and as a result computing the entropy becomes intractable in this case. Thus, we consider Gaussian approximation for Multinomial distribution as follows:

\[
Pr(Y_1 = y_1, ..., Y_n = y_n \mid X = x, T_x = t_x) = \frac{x!}{y_1!y_2!...y_n!}p_{ij}^{y_1}p_{2j}^{y_2}...p_{nj}^{y_n} = \prod_i Pr(y_i \mid x, t_x),
\]

where \( t_x = j \sigma_x \) and

\[
Pr(Y_i = y_i \mid X = x, T_x = t_x) = \frac{e^{-x}x^{y_i}}{\sqrt{2\pi x}p_{ij}}.
\]

Now, we have

\[
I(X, T_x; Y_1, ..., Y_n) = I(X; Y_1, ..., Y_n) + I(T_x; Y_1, ..., Y_n \mid X).
\]

To detect the concentration (the first term in (34)), we use the lower bound in [19], which is obtained by using received molecules in all observation sub-intervals. For the release time (the second term in (34)), based on the Markov chain,

\[
(X, T_x) \rightarrow (Y_1, ..., Y_n) \rightarrow f(Y_1, ..., Y_n) = Y_i - Y_{i-1},
\]

we have

\[
I(T_x; Y_1, ..., Y_n \mid X) \geq I(T_x; Y_i - Y_{i-1} \mid X).
\]

So we can find the following lower bound (i.e., achievable rate) on the JTAC channel capacity:

\[
C \geq I(X; Y_1, ..., Y_n) + \max I(T_x; Y_i - Y_{i-1} \mid X),
\]

which is computed in the following theorem.

**Theorem 3.** The JTAC channel capacity using the difference of received molecules in adjacent sub-intervals can be lower bounded as:

\[
C \geq \frac{1}{2m} \left( \log(M) \text{ erf} \left( \frac{M}{\sqrt{u}} \right) \right) - \frac{1}{2m \sqrt{u}} M^2 F_2 \left( \frac{1}{2}, \frac{3}{2}; \frac{3}{2}; \frac{M^2}{u} \right) + \sum_{k=0}^{r} \sum_{j} a_{kj} 2k m_{kij} + \frac{1}{2m} \sum_{i} \left( \frac{1}{2} \log(2^{\pi e} q_{ij}) \right) + \frac{1}{2} \left( \sum_{i} \log(p_{ij}) \right) - \log(y^*) - m \log(k') - \frac{1}{2m} \sum_{j} \log(p_{ij}).
\]

where \( F_2(\frac{1}{2}, \frac{3}{2}; \frac{3}{2}; x) \) is the Generalized Hypergeometric function \([15]\), \( m_{kij} \) and \( a_{ij} \) will be clarified in \([68]\) and \([69]\), and \( u \) is specified according to the distribution used in \([70]\).

**Proof.** A proof is given in Appendix B. \( \square \)

In next section we consider problem of finding Upper bounds on channel capacity.

**IV. Upper bounds On Channel Capacity**

To derive an upper bound on the channel capacity, first we obtain an upper bound on mutual information between inputs and number of received molecules in a specific sub-interval.

**Lemma 4.** For \( i \)-th sub-interval at the receiver we have

\[
I(X, T_x; Y_i) \leq \frac{E_m p_i^*}{M} (M - E_m) \log \left( \frac{M p_i^*}{\lambda_0} + 1 \right),
\]

where \( Y_i \) is the number of molecules received in \( i \)-th sub-interval and \( p_i^* \) is defined in [5].

**Proof.** We use the symmetric KL divergence, which has been defined in [6] as:

\[
D_{sym}(P \| Q) = D_{KL}(P \| Q) + D_{KL}(Q \| P).
\]

Now, we set

\[
P = p(x, t_x, y_i), \quad Q = p(x, t_x) p(y_i),
\]

and obtain an upper bound on \( I(X, T_x; Y_i) \). It is easy to show

\[
D_{sym}(P \| Q) = E_{p(x, t_x, y_i)} \log(p(y_i \mid x, t_x)) - E_{p(x, t_x)} \log(p(y_i \mid x, t_x)).
\]

Using the fact that \( E\{E[Y \mid X, T_x]\} = E\{E[Y \mid X]\} \), and from [6] we obtain

\[
E_{p(x, t_x, y_i)} \log(p(y_i \mid x, T_x = t_x)) = E\{(p_{ij} X + \lambda_0) \log(p_{ij} X + \lambda_0)\},
\]

\[
E_{p(x, t_x)} \log(p(y_i \mid X, T_x = t_x)) = E\{p_{ij} X + \lambda_0\} E\{\log(p_{ij} X + \lambda_0)\}.
\]

Combining above expressions with (38) results in,

\[
D_{sym} = \sum_{j=1}^{m} Pr(T_x = t_x) E\{(p_{ij} X + \lambda_0) \log(p_{ij} X + \lambda_0) \mid T_x = t_x\} - E\{p_{ij} X + \lambda_0\} E\{\log(p_{ij} X + \lambda_0)\}.
\]

Then, noting \( I(X, T_x; Y_i) \leq D_{sym} \), total upper bound is

\[
I(X, T_x; Y_i) \leq \sum_{j=1}^{m} Pr(T_x = t_x) E\{(p_{ij} X + \lambda_0) \log(p_{ij} X + \lambda_0) \mid T_x = t_x\} - E\{p_{ij} X + \lambda_0\} E\{\log(p_{ij} X + \lambda_0)\}.
\]

Since a binary random variable maximizes expression in (39), using similar steps as in \([6]\) for \( E_m \leq \frac{M}{2} \), we get

\[
\max(D_{sym}) = \frac{E_m p_i^*}{M} (M - E_m) \log \left( \frac{M p_i^*}{\lambda_0} + 1 \right).
\]

So, we have:

\[
I(X, T_x; Y_i) \leq \sum_{j=1}^{m} Pr(T_x = t_x) E\frac{E_m p_i^*}{M} (M - E_m) \log \left( \frac{M p_i^*}{\lambda_0} + 1 \right) \leq \frac{E_m p_i^*}{M} (M - E_m) \log \left( \frac{M p_i^*}{\lambda_0} + 1 \right),
\]

where \( E_m p_i^* = \frac{M p_i^*}{\lambda_0} + 1 \), (41)
TABLE I: Setup for numerical simulations for some typical values of Diffusion coefficient from [17] and \( d(\mu m) = 21.91 \).

| \( c \) | \( D(\mu m^2/s) \) | 0.1 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| \( c \) | \( D(\mu m^2/s) \) | 4800 | 480 | 240 | 160 | 120 | 96 |

where (a) follows from (8) and the fact that (40) is an increasing function of \( p_{ij} \).

**Theorem 4.** The capacity of the JTAC channel is upper bounded as:

\[
C \leq \sum_i \frac{E_m p_i^*}{M} (M - E_m) \log\left( \frac{M p_i^*}{\lambda_0} + 1 \right). \tag{42}
\]

**Proof.** Using memoryless property of channel, we have:

\[
I(X, T_x; Y_1, \ldots, Y_n) \leq \sum_i I(X, T_x; Y_i).
\]

Substituting (41) in the above equation we obtain (42). \( \square \)

### V. Numerical Results

In this section, we provide numerical evaluations of the proposed lower and upper bounds on the capacity of JTAC channel. We also numerically compute JTAC capacity using Blahut-Arimoto algorithm to examine how tight the bounds are. Moreover, we discuss the improvements compared with the concentration based (CB) modulation and timing based (TB) modulation. For CB, the information is only coded in the concentration based (CB) modulation and timing based (TB) modulation. For TB, the transmitter releases a fixed concentration of molecules in a sub-interval chosen according to the input message. Again by denoting the number of molecules received in \( i-th \) sub-interval by \( Y_i \), we consider \( \max_x I(T_x; Y_i | X = x) \), as a lower bound on the capacity of TB\(^3\). For CB channel, we use Blahut-Arimoto algorithm to compute its capacity numerically.

Fig. 3 depicts the proposed lower bounds and the upper bound versus the maximum concentration \( (M) \) and compares them with the numerically evaluated capacity of the JTAC and the CB channels. It is evident that the capacity and the bounds increase with \( M \), as expected. Also it is seen that the lower bound which is based on the difference of received molecules in adjacent sub-intervals (third scheme, given in (35)) provides higher achievable rates than the others. Since realizing receivers that detect the concentration of molecules in a limited number of time-slots seems to be more feasible compared to the receivers that have to detect the arrival time of molecules continuously, this suggests that third scheme could be considered for designing practical receivers to detect the release time. An important conclusion from Fig. 3a is that there is a relative large gap between the capacities of CB channel and JTAC. This indicates that using both time and concentration, to encode the message, results in significant improvement on achievable rates up to %50 (1.5 bits) compared to using only concentration. Fig. 3b depicts all bounds versus \( M \) with a lower diffusion noise parameter, i.e., \( c \). From Fig. 3a and Fig. 3b it can be seen that the proposed bounds become tighter at lower \( c \). Also it can be observed that in lower \( c \), the capacity increases more rapidly with \( M \) and obviously achieves higher values than in environments with larger \( c \). To study the effect of \( c \) on our bounds and the channel capacity, we consider a setup according to Table I where different values of diffusion coefficient are considered for a fixed distance between the transmitter and the receiver to simulate environments with different Lévy diffusion noise parameters, i.e., \( c \). Using this setup, Fig. 4 considers the effect of \( c \) on the channel capacity. We expect that with increasing \( c \), the capacity decreases due to the decrease in the diffusion coefficient of the environment (see Section II and Table I), which is the case for all achievable rates and the upper bound depicted in Fig. 2. Also Fig. 4 shows that with increasing \( c \), the capacity of CB channel for values of \( c \geq 2 \) falls more rapidly compared with the capacity (and even achievable rates) of JTAC channel. This suggests that using JTAC modulation provides a more robust strategy compared to conventional CB modulations in larger values of

---

\(^3\)Since \( I(T_x; Y_1, ..., Y_n | X = x) \) is of the form of a side information we could not use directly use Blahut-Arimoto to compute TB capacity numerically [18].
c (in this case 2) and JTAC is less sensitive to diffusion noise. Fig. 5 depicts the achievable rates and the upper bounds on the channel capacity versus the number of sub-intervals at the transmitter (m). As we increase m, we get closer to the continuous timing channel and thus we could achieve higher rates. But an important observation from Fig. 5 is that increasing the number of sub-intervals beyond a point which for this parameter setup is 50, has a little impact on the channel capacity. We observe similar results in other simulations with different environmental noise parameters; however, the point of saturation increases in less noisy channel (smaller value of c). This further motivates us considering the more practical discrete time-slotted timing channel (compared with the continuous timing channel). Fig. 6 shows the effect of the number of observation times at the receiver on the capacity. It is seen that by increasing the number of observation times in receiver, we can achieve higher rates as we expected. Finally Fig. 7a to 7c compares timing based rates (TB) with other bounds and channel capacity versus m. It is seen by comparing Fig. 7b and 7c that increase in n increases timing rates (thanks to the more accurate detection in receiver). And Fig. 7a and 7b compares TB rates with other bounds in environments with different c. As it seen from Fig. 7b the gap between lower bound on TB and CB has been reduced compare to Fig. 7a which indicates that in environments with small c, timing rate has larger portion in total rate of the channel and CB is more resistant to noise c compared to TB.

VI. CONCLUSION

In this paper, we introduced JTAC modulation for molecular diffusion channel, in which the information is modulated in both the release time and concentration of transmitted molecules. In order to analyze the JTAC performance in comparison with prior modulation schemes, more specifically concentration based (CB) modulation, we considered its capacity and derived three lower bounds and one upper bound on the capacity. We numerically evaluated the capacity of JTAC and CB modulation using Blahut-Arimoto algorithm and obtained a lower bound on the capacity of TB modulation. Our results indicate that the lower bound for the JTAC channel based on detecting the difference of the numbers of molecules in adjacent sub-intervals at the receiver provides tighter lower bound compared to the two other lower bounds. It is also observed that the capacity and achievable rates of the JTAC channel increase with the increase of the number of discrete release times at the transmitter or the number of observation sub-intervals at the receiver, up to a saturation point. The value of the saturation point increases, with increase in the number of sub-interval at the transmitter or by decrease in diffusion noise parameter (c). Our results further indicate that the JTAC modulation significantly improves the achievable rates compared to CB and TB modulations. For example, as shown in Fig. 5 we could achieve up to 1.5 bits per symbol higher rates compared to CB. And finally our numerical results indicated that in higher values of c JTAC falls less rapidly with increasing c, which suggest that using JTAC could provide more robust strategy for transmission in the environment with large c compared to CB. In our analysis, we neglected the ISI assuming a time gap between successive symbol intervals and left the analysis in the presence of ISI for future work. Another interesting problem is considering effect of flow on our results and its joint effect on both CB and TB. Also, Considering more realistic conditions, for example non-ideality of the transmitter and examining its effects are another area of future work.

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Fig. 6: Effect of $n$ (number of sub-intervals in receiver) on transmission rates with $c = 1$, $M = 15$, $m = 10$.

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APPENDIX A

PROOF OF (25)

Having access to the observations $Y_1, Y_2, ..., Y_n$, the receiver uses their sum, to decode the transmitted concentration $X$ and then conditioned on the transmitted $X$ the receiver uses the sub-interval which its observations has the maximum mutual information with the transmitted time $T_x$. Thus, we have:

$$I(X; T_x; Y_1, ..., Y_n) = I(X; Y_1, ..., Y_n) + I(T_x; Y_1, ..., Y_n|X),$$

(45)

where (a) follows since $(X, T_x) \rightarrow (Y_1, ..., Y_n) \rightarrow f(Y_1, ..., Y_n) = Y_1 + ... + Y_n$ forms a Markov chain and mutual information is non negative. We present proof in three steps.

Step 1) Lower bound on $I(X; Y_1 + Y_2 + ... + Y_n)$: Using the fact that conditioned on $T_x$ and $X$ the receiver observations $Y_i$ are independent Poisson random variable, we conclude that:

$$P(Y = y_1 + ... + y_n|X = x, T_x = j \sigma_x) = e^{-x \pi e} \frac{(x \pi e)^y}{y!},$$

(46)

Now consider:

$$I(X; Y_1 + ... + Y_n) = h(Y_1 + ... + Y_n) - h(Y_1 + ... + Y_n|X),$$

(47)

We lower bound the first term and upper bound the second term in (45) to conclude a lower bound on $I(X; Y_1 + ... + Y_n)$. By using similar steps as taken in proof of Lemma 2 we have:

$$h(Y_1 + ... + Y_n) \geq h(X, T_x) - \log(\eta p^*) - m - \log(k' \pi e).$$

(48)

For computing the upper bound on second term in (45), note that from (44) we have:

$$Pr(Y = y_1 + ... + y_n|X = x) = \sum_j e^{-x \pi e} \frac{(x \pi e)^y}{y!} Pr(T_x = j \sigma_x).$$

To compute the second term in (45), we should compute the entropy of above distribution. To do this, we define an auxiliary random variable $\theta$ as

$$\theta = \begin{cases} 
L_1 \sim \text{Poisson}(x p_1') & \text{with probability Pr}(T_x = \sigma_x) \\
L_m \sim \text{Poisson}(x p_m') & \text{with probability Pr}(T_x = m \sigma_x)
\end{cases}$$

Using the results from entropy of mixtures [19] we have:

$$h(Y_1 + Y_2 + ... + Y_n|X) = h(T_x) + \sum_j h(L_j) Pr(T_x = j \sigma_x).$$

(49)

Based on upper bound on Poisson distribution in [19], we upper bound (47) to conclude:

$$h(Y_1 + ... + Y_n|X) \leq h(T_x) + \sum_j Pr(T_x = j \sigma_x) \left( \frac{1}{2} \log(2\pi e) + \frac{1}{2} E\{\log((p_j' X))\} + \frac{1}{2} E\{\log(1 + \frac{1}{12p_j' X})\} \right).$$

(50)

Substituting (46) and (48) in (45), we have:

$$I(Y_1 + ... + Y_n; X) \geq h(X, T_x) - h(T_x) - \log(\eta p^*) - m - \log(k') - \sum_j Pr(T_x = j \sigma_x) \left( \frac{1}{2} \log(2\pi e) + \frac{1}{2} E\{\log((p_j' X))\} \right).$$

(51)
Step 2) Lower Bound on $I(T_x; Y_1, \ldots, Y_n | X)$: It is easy to show that:

$$\max_i I(T_x, Y_i | X) \leq I(T_x; Y_1, \ldots, Y_n | X),$$

So, we turn to compute (lower) bounds on $\max_i I(T_x, Y_i | X)$

We have:

$$I(T_x, Y_i | X) = h(Y_i | X) - h(Y_i | X, T_x).$$  \hfill (50)

In (19) we computed an upper bound on the second term of (50). So, it suffices to lower bound $h(Y_i | X)$ to obtain a lower bound on $I(T_x, Y_i | X)$. Using the distribution of $Y$ conditioned on $X$ is computed as:

$$P_r(Y_i = y_i | X = x) = \sum_j Pr(T_x = j \sigma_x) e^{(-z_{p_j} y_i)} / y_i!,$$

To lower bound entropy of $h(Y_i | X) = \sum_j Pr(T_x = j \sigma_x) (1 / 2 \log(2 \pi e)) + \frac{1}{2} E \{ \log(p_{ij} X) \}$

So total lower bound on $I(T_x; Y_i | X)$ is:

$$I(T_x; Y_i | X) \geq h(T_x) + h(\text{Poisson}(M \tilde{p}_i))$$

Remark 3: Using (19) and (52), $I(T_x; Y_i | X = x)$ can be lower bounded. From (19) for a specific event $\{ X = x \}$, we have,

$$h(Y_i | X = x) \geq \sum_j Pr(T_x = j \sigma_x) h(L_{ij}) - \sum_j Pr(T_x = j \sigma_x) (1 / 2 \log(p_{ij} x + 1 / 12)).$$

It is easy to see that this expression can be lower bounded as:

$$I(T_x; Y_i | X = x) \geq \log_2(m) + \frac{1}{m} \sum h(L_{ij}) - \log(p_{ij} x + 1 / 12).$$

This lower bound gives specific how using timing in JTAC gives us additional information (rates) compared to detecting only concentration.

Step 3) Deriving lower bounds on the channel capacity: By using bounds in (43), (49), and (53), we have the following lower bound on mutual information of inputs and outputs:

$$I(T_x, X_1, \ldots, Y_n) \geq h(X, T_x) + \max_i h(\text{Poisson}(M \tilde{p}_i))$$

$$- \log((\eta p^*) - m - \log(k'))$$

$$- \sum_j Pr(T_x = j \sigma_x) \frac{1}{2} \log(2 \pi e) + \frac{1}{2} E \{ \log(p_{ij} X) \}$$

$$+ \frac{1}{2} E \{ \log(1 + \frac{1}{12 p_{ij} X}) \} - \sum_j Pr(T_x = j \sigma_x) \frac{1}{2} \log(2 \pi e) + \frac{1}{2} E \{ \log(p_{ij} X) \} + \frac{1}{2} E \{ \log(1 + \frac{1}{12 p_{ij} X}) \}.$$
which simplifies to
\[
I(T_x, X; Y_1, ..., Y_n) \geq h(X, T_x) - \log(n\eta^*) - m \tag{56}
\]
\[
- \log(k') - \log(2\pi e) - \sum_j Pr(T_x = j\sigma_x)
\]
\[
\left(\frac{1}{2} \log(p_{j_1}) - \frac{1}{2} \log(12p_{i_1}^') + \frac{1}{2} E\{\log(12p_{j_1} X + 1)\}\right)
\]
\[
- \max_i \sum_j p(T_x = j\sigma_x) \left(\frac{1}{2} \log(p_{ij}) - \frac{1}{2} \log(12p_{ij})
\right)
\]
\[
+ \frac{1}{2} E\{\log(12p_{ij} X + 1)\}\right) + h(\text{Poisson}(M\tilde{p}_{i1}))
\]

Using \(p^* \geq \max_i p_{i1}^*\), we could lower bound the above expression to conclude the following simple form:
\[
I(T_x, X; Y_1, ..., Y_n) \geq h(X, T_x) - E\{\log(12p^* X + 1)\}
\]
\[
- \log(\eta^*) - m - \log(k') - \log(2\pi e) + \log(12) + \max_i h(\text{Poisson}(M\tilde{p}_{i1}))
\tag{57}
\]

It is easy to see that if we consider the noise molecules in \(\text{Poisson}(M\tilde{p}_{i1})\), we obtain
\[
I(T_x, X; Y_1, ..., Y_n) \geq h(X, T_x) - E\{\log(12p^* X + 1)\}
\]

By choosing appropriate distribution for \((X, T_x)\), we can maximize this lower bound. Noting that \(h(X, T_x) \leq h(X) + h(T_x)\), we use a uniform distribution for \(T_x\) which maximizes its entropy in a finite range, and choose \(X\) independent of \(T_x\) and according to the distribution that maximizes \(h(X) - E\{\log(12p^* X + 1)\}\). Using Lagrange multipliers, \(X\) has the following form:
\[
f_X(x) = \frac{c}{bc + 1} e^{\cdot x}, \quad b = 12p^*.
\tag{58}
\]

where \(\phi\) is computed such that constraint (1) is satisfied and \(c\) is such that this distribution integrates to one. Thus, we have:
\[
\frac{e^{\cdot Ei(\phi/2)} - Ei(\phi/2)}{\beta^2} = \frac{E_m}{c^{\cdot}},
\tag{59}
\]
\[
\frac{e^{\cdot Ei(\phi M/2) - Ei(\phi/2)}}{b^\beta} = \frac{1}{c^{\cdot}},
\tag{60}
\]

where \(Ei(.)\) is special function\(^6\) and is defined as:
\[
Ei(x) = - \int_{-x}^{\infty} e^{-t} \frac{dt}{t}.
\tag{61}
\]

Using (59) and (60), \(\phi\) is computed by solving the following equation:
\[
\frac{e^{\cdot Ei(\phi M/2) - Ei(\phi/2)}}{b^\beta} = \frac{E_m - 1}{b\phi(E_m - \frac{1}{\beta})}.
\tag{62}
\]

Also it is straightforward to see that:
\[
h(X) = - \log(c') + E\{\log(12p^* X + 1)\} - \eta\phi.
\tag{63}
\]

\(^5\)Here we use the fact that the expectation of a random variable is less than or equal to its maximum realization.

\(^6\)Exponential integral function

Substituting (58), (60), and (63) in (57) results in:
\[
I(T_x, X; Y_1, ..., Y_n) \geq \log_2(m) - \log(c) - \eta\phi - \log(\eta^*) - m - \log(2\pi e) - \log(k') + \log(12) + \max_i h(\text{Poisson}(M\tilde{p}_{i1}))
\]

where term \(h(\text{Poisson}(M\tilde{p}_{i1}))\) is computed with results of (20) as stated in (26).

Now, we consider the case of large \(m\). Noting \(\log(\frac{e}{m}) \leq \frac{e}{m} - 1\), a lower bound on \(h(Y_1)\) is
\[
\log_2(m) - \log(\eta^*) - (m - 1 - \log(k')).
\tag{64}
\]

Thus, we have
\[
I(T_x, X; Y_1, ..., Y_n) \geq \log_2(m) - \log(c) - \eta\phi - \log(\eta^*) - \log(m) - 1 - \log(2\pi e) - \log(k') + \log(12) + \max_i h(\text{Poisson}(M\tilde{p}_{i1}))
\]

which reduces to (27) for large \(m\), if the average constraint in (1) holds with equality.

**APPENDIX B**

**PROOF OF (35)**

To lower bound \(I(T_x; Y_i - Y_{i-1})\), we write
\[
I(T_x, Y_i - Y_{i-1}|X) = h(Y_i - Y_{i-1}|X) - h(Y_i - Y_{i-1}|X, T_x).
\tag{65}
\]

Using (32), we have:
\[
Pr(Y = y_i - y_{i-1}|X = x, T_x = t_x) = \frac{e^{\cdot e^{\cdot t}}}{\sqrt{2\pi x}(q_i^{\cdot})^2},
\tag{66}
\]

where \(q_{ij} = p_{ij} - p_{i-1j}\) and \(q_{ij}' = p_{ij} + p_{i-1j}\). Thus,
\[
Pr(Y_i - Y_{i-1} = y|X = x) = \sum_{\eta=1}^{n} Pr(T_x = j\sigma_x) \frac{e^{\cdot e^{\cdot t}}}{\sqrt{2\pi x}(q_i^{'})^2}.
\tag{67}
\]

**Lemma 5.** The entropy of mixture of Gaussian in (67) is lower bounded using Moments of its Gaussian components.

Proof. Proof is based on using Taylor series representation of logarithm of Gaussian Mixture and also using differential entropy. We use a similar approach of (21) for scalar Gaussian random variables. Without loss of generality, we assume \(E\{X\} = \eta\). For simplicity, we define the following notation for (67).
\[
g(y) \triangleq Pr(Y_i - Y_{i-1} = y|X = x) = \sum_{\eta=1}^{n} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2}
\]

where \(a_{ij}, b_{ij}\) and \(c_{ij}\) are evident from (67). Thus,
\[
\log(g(y)) = \sum_{\eta=0}^{n} z_{k} (y - y_{0})^{k} + R_{n},
\]

where \(y_{0} = xq_{11}\) and \(z_{k}\) is the coefficients of Taylor series for \(\log(g(y))\), and \(y_{0} = xq_{11}\). Therefore:
\[
h(Y_i - Y_{i-1}|X) = \int_{y} \sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2} \log(\sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2})dy,
\]

\[\int_{y} \sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2} \log(\sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2})dy,
\]

\[\int_{y} \sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2} \log(\sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2})dy,
\]

\[\int_{y} \sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2} \log(\sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2})dy,
\]

\[\int_{y} \sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2} \log(\sum_{\eta} a_{ij}e^{\cdot b_{ij}(y-c_{ij})^2})dy,
\]
\[(a) \geq \int_y \sum_j a_{ij} e^{b_{ij}(y-c_{ij})^2} \left( \sum_k z_k(y-y_0)^k \right) dy \]
\[(b) \geq \sum_k \sum_j a_{ij} z_k m_{kij}, \quad (68)\]

where (a) follows by approximating the logarithm with first \(r\) terms in Taylor series and (b) follows by using convexity of \(e^x\), Jensen’s inequality, and noting that \(m_{kij}\) is the non-central moments of a Gaussian random variable with mean \(\eta_{q_{il}}\) and variance \(\eta_{q_{il}}'\), computed using its central moments \(\mu_{il}\) as

\[m_{kij} = \sum_{l=0}^{k} \binom{k}{l} \mu_{il}^l (c_{ij} - y_0)^{n-l}. \quad (69)\]

which completes our proof. \(\square\)

Now, we consider the second term in (65).

**Lemma 6.** The entropy of probability distribution in (66) is:

\[ h(Y_i - Y_{i-1} | X, T) = \sum_j \Pr(T_x = j\sigma_x) \frac{1}{2} \log(2\pi e q'_{ij} + \frac{1}{2} E\{\log X\}). \]

**Proof.** We have:

\[ h(Y_i - Y_{i-1} | X, T_x) = \sum_j \Pr(T_x = t_x) h(Y_i - Y_{i-1} | X, T_x = j\sigma_x) \]
\[= \sum_j \Pr(T_x = t_x) \left( \frac{1}{2} \log(2\pi e q'_{ij} + \frac{1}{2} E\{\log X\}) \right). \]

where (a) follows from the Gaussian entropy. \(\square\)

By using Lemmas 5 and 6, we conclude that:

\[ I(T_x; Y_i - Y_{i-1} | X) \geq \sum_{k=0}^{r} \sum_j a_{ij} z_k m_{kij} \]
\[+ \sum_j \Pr(T_x = j\sigma_x) \left( \frac{1}{2} \log(2\pi e q'_{ij} + \frac{1}{2} E\{\log X\}) \right). \quad (70)\]

We propose using distribution \( f_{X,T_x}(x,j\sigma_x) = \frac{1}{\sqrt{\pi n} m} e^{-\frac{x^2}{u}} \) for computing the above lower bound (where \(u\) is chosen such that the distribution integrates to one), which results in:

\[ I(T_x; Y_i - Y_{i-1} | X) \geq \frac{1}{4m} (\log(M) \text{ erf} \frac{M}{\sqrt{u}}) \]
\[- \frac{1}{2m\sqrt{\pi u}} M_2 F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{M^2}{u} \right) \]
\[+ \sum_{k=0}^{r} \sum_j a_{ij} z_k m_{kij} + \frac{1}{2m} \sum_j \frac{1}{2} \log(2\pi e q'_{ij}). \]

Combining this bound with the lower bound in (49) (with the above proposed distribution \( f_{X,T_x}(x,j\sigma_x) \)) and considering constraint (1) hold with equality, we obtain (35).