NORMAL COMPLEX SURFACE SINGULARITIES WITH RATIONAL HOMOLOGY DISK SMOOTHINGS

HEESANG PARK, DONGSOO SHIN, AND ANDRÁS I. STIPSICZ

Abstract. We show that if the minimal good resolution graph of a normal surface singularity contains at least two nodes (i.e. vertex with valency at least 3) then the singularity does not admit a smoothing with Milnor fiber having rational homology equal to the rational homology of the 4-disk $D^4$ (called a rational homology disk smoothing). Combining with earlier results, this theorem then provides a complete classification of resolution graphs of normal surface singularities with a rational homology disk smoothing, verifying a conjecture of J. Wahl regarding such singularities. Indeed, together with a recent result of J. Fowler we get the complete list of normal surface singularities which admit rational homology disk smoothings.

1. Introduction

Let $(X,0)$ be a (germ of a) normal complex surface singularity. A smoothing of $(X,0)$ is a flat surjective map $\pi: (X,0) \to (\Delta,0)$, where $(X,0)$ has an isolated 3-dimensional singularity and $\Delta = \{ t \in \mathbb{C} \mid |t| < \epsilon \}$, such that $(\pi^{-1}(0),0)$ is smooth for every $t \in \Delta \setminus \{0\}$. Assume that $(X,0)$ is embedded in $(\mathbb{C}^N,0)$. Then there exists an embedding of $(X,0)$ in $(\mathbb{C}^N \times \Delta,0)$ such that the map $\pi$ is induced by the projection $\mathbb{C}^N \times \Delta \to \Delta$ to the second factor. The Milnor fiber $M$ of a smoothing $\pi$ of $(X,0)$ is defined by the intersection of a fiber $\pi^{-1}(t)$ ($t \neq 0$) near the origin with a small ball about the origin, that is, $M = \pi^{-1}(t) \cap B_\delta(0)$ ($0 < |t| \ll \delta \ll \epsilon$).

According to the general theory of Milnor fibrations (see Looijenga [16]), $M$ is a compact 4-manifold, with the link $L$ of the singularity $(X,0)$ as its boundary. In particular, the diffeomorphism type of $M$ depends only on the smoothing $\pi$; hence, the topological invariants of $M$ are invariants of the smoothing $\pi$. The 4-manifold $M$ has the homotopy type of a two-dimensional CW complex, thus we have $H_i(M,\mathbb{Z}) = 0$ for $i > 2$. Furthermore, by Greuel–Steenbrink [5, Theorem 2], the first Betti number $b_1(M)$ is zero. Therefore, an important invariant of $M$ (hence, of the smoothing $\pi$) is $H_2(M,\mathbb{Z})$, which is a finitely generated free abelian group. The Milnor number $\mu$ of the smoothing $\pi$ is given by the second Betti number $\mu = \dim H_2(M,\mathbb{Q})$.

If $\mu = 0$, that is, $H_i(M,\mathbb{Q}) = H_i(D^4,\mathbb{Q}) = 0$ for $i > 0$, we say that $M$ is a rational homology disk (QHD for short). Correspondingly, a smoothing $\pi$ with $\mu = 0$ is called a rational homology disk smoothing (‘QHD smoothing’ for short). For example, any cyclic quotient singularity of type $\frac{1}{pq}(1,pq-1)$ with two relatively prime integers $p > q$ admits a QHD smoothing. Indeed, according to Looijenga–Wahl [17 (5.10)] and Wahl [32 (5.9.1)], among cyclic quotient singularities these are the only ones having a QHD smoothing. Kollár–Shepherd-Barron [11] made substantial use of the fact that the QHD smoothing of a singularity of type $\frac{1}{pq}(1,pq-
(1) is a quotient of a smoothing of its index one cover; they invented the term “\(\mathbb{Q}\)-Gorenstein smoothing”. Singularities of type \(\frac{1}{pq}(1, pq - 1)\) play an important role in the Kollár–Shepherd-Barron–Alexeev (KSBA) compactification of moduli spaces of complex surfaces of general type. For instance, Y. Lee–J. Park [15] constructed a singular surface with singularities of type \(\frac{1}{pq}(1, pq - 1)\). Since the Milnor fibers of these singularities are topologically very simple, it is easy to control (topological) invariants of the smoothing of the singular surface. Hence, by smoothing the singular surface, they constructed examples of complex surfaces of general type with prescribed topological invariants. In particular, they constructed a point lying on the boundary of the KSBA compactification of a moduli space of complex surfaces of general type. Using similar ideas, many important examples of complex surfaces of general type have been constructed; see, for example, [10, 18, 20, 21, 22]. These constructions were motivated by the rational blow-down construction of Fintushel–Stern [4], and its generalization by J. Park [23]; in this smooth construction one substitutes the tubular neighbourhood of a configuration of surfaces in a 4-manifold intersecting each other according to the resolution graph of a singularity with a \(\mathbb{Q}\)HD smoothing of the same singularity. These constructions played a crucial role in constructing exotic differentiable structures on many 4-manifolds, cf. for example [4, 24, 25, 29].

Therefore it is an interesting problem to classify all normal surface singularities admitting a \(\mathbb{Q}\)HD smoothing. Such a singularity \((X, 0)\) must be rational; in particular the resolution dual graph is a tree and the vertices correspond to rational curves. Besides the cyclic quotient ones, further such examples were described by Wahl [32], and a list of such singularities (compiled by Wahl) was known to the experts, cf. the remark on the bottom of page 505 of de Jong–van Straten [3].

Using smooth topological ideas, in Stipsicz–Szabó–Wahl [30] strong necessary combinatorial conditions for the resolution graphs of singularities with a \(\mathbb{Q}\)HD smoothing has been derived. Besides the linear graphs (that is, the resolution graphs of cyclic quotient singularities of type \(\frac{1}{pq}(1, pq - 1)\)) the potential graphs were classified into six classes \(\mathcal{W}, \mathcal{M}, \mathcal{N}\) and \(\mathcal{A}, \mathcal{B}, \mathcal{C}\). In the first three classes the resolution graphs are all star-shaped (i.e. each admits a unique node), with the node having valency 3, and all these graphs are taut in the sense of Laufer [14]. (Recall that a singularity is called taut if it is determined analytically by its resolution graph.) The singularities corresponding to the graphs in \(\mathcal{W}, \mathcal{M}\) and \(\mathcal{N}\) all admit \(\mathbb{Q}\)HD smoothing.

The further three types \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{C}\) are defined by the following construction. Let \(\Gamma_A, \Gamma_B, \Gamma_C\) be the graphs given as follows.

\[
\begin{align*}
\Gamma_A: & \quad -3 \quad \bullet \quad -3 \\
\Gamma_B: & \quad -4 \quad \bullet \quad -2 \\
\Gamma_C: & \quad -6 \quad \bullet \quad -2
\end{align*}
\]

A non-minimal graph of type \(\mathcal{A}\) (or \(\mathcal{B}\) or \(\mathcal{C}\)) is a graph obtained as follows: Starting with the graph \(\Gamma_A\) (respectively, \(\Gamma_B\) or \(\Gamma_C\)), apply the following two blowing up operations:

(B-1) blow up the \((-1)\)-vertex

\[
\begin{align*}
& \quad \cdots \quad \bullet \quad \cdots \\
& \quad -1
\end{align*}
\]

\[
\begin{align*}
& \quad \cdots \quad \cdots \quad \cdots \\
& \quad -2
\end{align*}
\]
(B-2) or blow up any edge emanating from the $(-1)$-vertex

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \]

and repeat these procedures of blowing up (either the new $(-1)$-vertex or an edge emanating from it) finitely many times. The result is a non-minimal graph $\Gamma$.

A minimal graph $\Gamma$ of type $A$ (or $B$ or $C$) corresponding to a non-minimal graph $\Gamma$ of type $A$ (respectively, $B$ or $C$) is a graph obtained by

(M) modifying the unique $(-1)$-decoration of a non-minimal graph $\Gamma$ of type $A$ (respectively, $B$ or $C$) to $(-4)$ (respectively, $(-3)$ or $(-2)$).

The classes $A$, $B$ and $C$ are the collections of minimal graphs of the respective types. It is not hard to see that a graph in $A \cup B \cup C$ has at most one node of valency 4 (corresponding to the node of $\Gamma_A$, $\Gamma_B$ or $\Gamma_C$) and all the others are of valency 3.

Using methods of symplectic topology, in Bhupal–Stipsicz [11] star-shaped graphs admitting a QHD smoothing have been completely classified. In particular, it has been shown that if a minimal good resolution graph $\Gamma$ is star-shaped and corresponds to a singularity with a QHD smoothing, then $\Gamma$ is one of the graphs given by Figures 1 or 2.

Figure 1. Star-shaped graphs with one node of degree 3 corresponding to singularities with a QHD smoothing. We assume that $p, q, r \geq 0$.

In fact, in [33] Wahl conjectured that the only complex surface singularities admitting a QHD smoothing are the formerly known examples, which are all weighted homogeneous (hence, in particular, have resolution graphs with at most one node). In supporting this conjecture, Wahl showed that many graphs with exactly two nodes do not correspond to singularities with a QHD smoothing, cf. [33 Theorem 8.6]. The aim of this paper is to prove Wahl’s conjecture:
Figure 2. Star-shaped graphs with one node of degree 4 corresponding to singularities with a QHD smoothing. The quadruple $(a, b, c; d)$ is one of $\{(3, 3, 3; 4), (2, 4, 4; 3), (2, 3, 6; 2)\}$; furthermore $p \geq 0$.

**Main Theorem 1.1.** Suppose that $\Gamma$ is a minimal negative definite graph with at least two nodes. Then there is no complex surface singularity with resolution graph $\Gamma$ which admits a QHD smoothing.

This result, with the aforementioned result of Bhupal–Stipsicz [1], provides the following classification result:

**Corollary 1.2.** Suppose that $\Gamma$ is a minimal negative definite graph with the property that there is a singularity which admits a QHD smoothing and has $\Gamma$ as a resolution graph. Then $\Gamma$ is either the linear graph corresponding to one of the cyclic quotient singularities of type $\frac{1}{p}(1, pq - 1)$ (with $p > q > 0$ relatively prime) or $\Gamma$ is one of the graphs of Figures 1 or 2.

Indeed, the above result leads to the complete classification of complex normal surface singularities with QHD smoothing. Since the resolution graphs of cyclic quotient singularities and the graphs of Figure 1 are all taut by Laufer [14], for these cases the singularities themselves are determined by the resolution graph. A graph of Figure 2 does not determine a unique singularity — the analytic type depends on a complex number, the cross ratio of the four intersection points on the rational curve corresponding to the node of valency 4 with its four neighbours. According to a recent result of J. Fowler [6, Theorem 5(a)], for any graph in Figure 2 exactly one cross ratio determines a singularity admitting a QHD smoothing. This value of the cross ratio is also determined by Fowler [6]: it is anharmonic for $(a, b, c; d) = (3, 3, 3; 4)$, harmonic for $(a, b, c; d) = (2, 4, 4; 3)$, and 9 for $(a, b, c; d) = (2, 3, 6; 2)$.

Therefore, as a combination of Corollary 1.2 and the result of Fowler [6] we get the classification of singularities admitting a QHD smoothing:

**Corollary 1.3.** The set of complex normal surface singularities admitting a QHD smoothing is equal to the set of singularities we get by considering

- the cyclic quotient singularities of type $\frac{1}{p}(1, pq - 1)$ (with $p > q > 0$ relatively prime),
- the weighted homogeneous singularities corresponding to the taut graphs of Figure 1 and
- the weighted homogeneous singularities with resolution graphs of Figure 2 together with the cross ratios: anharmonic for $(a, b, c; d) = (3, 3, 3; 4)$, harmonic for $(a, b, c; d) = (2, 4, 4; 3)$, and 9 for $(a, b, c; d) = (2, 3, 6; 2)$.

**Remark 1.4.** It is known that a QHD smoothing component of a cyclic quotient singularity of type $\frac{1}{p}(1, pq - 1)$ (with $p > q > 0$ relatively prime) has dimension one, and the QHD smoothing can always be chosen to be a Q-Gorenstein smoothing. In [33, 34] Wahl verified the same properties for any weighted homogeneous surface singularity admitting a QHD smoothing. Hence, combined with Main Theorem 1.1 we conclude that any QHD smoothing of a normal surface singularity is Q-Gorenstein occurring over a one-dimensional smoothing component.
One of the main ideas of the proof of Theorem 1.1 is an extension of the result of Wahl in [33, §8] about graphs of two nodes. Let \((X, 0)\) be a germ of a rational surface singularity, and let \(\pi : V \rightarrow X\) be the minimal good resolution of \(X\) near \(0\) with \(E = \pi^{-1}(0)\) the exceptional set. Let \(E = \sum_{i=1}^{n} E_i\) be the decomposition of the exceptional divisor \(E\) into irreducible components \(E_i\) with \(E_i^2 = -d_i\). An irreducible component of the base space of the semi-universal deformation of \((X, 0)\) is called a smoothing component if a generic fiber over such a component is smooth. Every component of the base space of the semi-universal deformation of a rational surface singularity is a smoothing component, but their dimensions may vary. By Wahl [33, Theorem 8.1] a QHD smoothing component (i.e. a component containing a QHD smoothing, if any) of \((X, 0)\) has dimension

\[
h^1(V, \Theta_V(-\log E)) + \sum_{i=1}^{n} (d_i - 3),
\]

where \(\Theta_V(-\log E)\) is the sheaf of logarithmic vector fields (i.e. the dual of the sheaf \(\Omega_V(\log E)\) of logarithmic differentials); that is, it is the kernel of the natural surjection \(\Theta_V \rightarrow \bigoplus N_{E_i,V}^E\). In particular, if the above expression is nonpositive for a particular singularity, then it admits no QHD smoothing. The proof of our Main Theorem 1.1 will rest on the following technical result.

**Theorem 1.5.** Suppose that \((X, 0)\) is a rational surface singularity with resolution graph \(\Gamma\). Assume furthermore that \(\Gamma\) is of type \(A, B,\) or \(C\) and has at least two nodes. Then

\[
h^1(V, \Theta_V(-\log E)) + \sum_{i=1}^{n} (d_i - 3) \leq 0.
\]

This result immediately implies the main result of the paper:

**Proof of Main Theorem 1.1.** Suppose that \(\Gamma\) is a minimal negative definite graph with at least two nodes. Suppose furthermore that the singularity \((X, 0)\) has \(\Gamma\) as the resolution graph, and \((X, 0)\) admits a QHD smoothing. By Stipsicz–Szabó–Wahl [30] then \(\Gamma\) is of type \(A, B,\) or \(C\). By Wahl [33, Theorem 8.1] a QHD smoothing component has dimension given by Equation (1.1), which expression, by Theorem 1.5 is nonpositive. Consequently the smoothing component does not exist, concluding the proof.

The difficulty in proving Theorem 1.5 is that singularities with resolution graphs having at least two nodes are usually non-taut. Indeed, there may exist many analytically different singularities with the same resolution graph, and the dimension \(h^1(V, \Theta_V(-\log E))\) in Formula (1.1) depends on the analytic structure of the singularity \((X, 0)\).

In dealing with this difficulty, in Section 2 we prove that there exists a ‘natural’ singularity \((X_0, 0)\) with minimal good resolution \((V_0, E_0) \rightarrow (X_0, 0)\) that has the same weighted resolution graph (and the same cross ratio if any) as \((X, 0)\) such that

\[
h^1(V, \Theta_V(-\log E)) \leq h^1(V_0, \Theta_{V_0}(-\log E_0)).
\]

That is, the singularity \((X_0, 0)\) has maximal dimension \(h^1(V, \Theta_V(-\log E))\) among singularities having the same weighted resolution graph. So we may call \((X_0, 0)\) a ‘maximal’ singularity. By controlling how the expression of Formula (1.1) changes under the construction of the graphs in \(A, B,\) and \(C\), we verify Inequality (1.2) for the maximal singularities, eventually providing the proof of Theorem 1.5.

The singularity with the maximal dimension property has been already introduced by Laufer [13, Theorem 3.9] using the plumbing construction. In the last paragraph of Laufer [13, p. 93], he observed that \(h^1(V, \Theta_V(-\log E))\) is usually
maximal for the maximal singularity among singularities with the same resolution graph. Indeed, in [14] Theorem 3.1 Laufer proved the maximality property given by Inequality (1.3) for (pseudo) taut singularities, and used this fact to obtain a complete list of resolution graphs of such singularities [14]. In Theorem 2.6 we generalize Laufer’s observation for any rational surface singularity for which the resolution graph has nodes of valency 3 with at most one exception which is of valency 4.

Let us set up some more notation. Suppose that $\Gamma$ is a non-minimal graph of type $A$, $B$, or $C$, and define the augmented graph $\Gamma^\#$ as the graph obtained from $\Gamma$ by blowing up once the $(-1)$-vertex and (if needed) by (successively) blowing up an edge emanating from the $(-1)$-vertex according to its type as below:

- **type $A$**
  
  $\Gamma = -1 \quad \Gamma^\# = \quad -4 \quad -1 \quad -2 \quad -2$

- **type $B$**
  
  $\Gamma = -1 \quad \Gamma^\# = \quad -3 \quad -1 \quad -2$

- **type $C$**
  
  $\Gamma = -1 \quad \Gamma^\# = \quad -2 \quad -1$

Note that the minimal graph $\Gamma$ corresponding to $\Gamma$ can be obtained by deleting the redundant vertices and edges from $\Gamma^\#$ (three vertices and edges for type $A$, two for type $B$ and one for type $C$).

The paper is organized as follows. In Section 2 we prove that for a given resolution graph $\Gamma$ there is a natural singularity with $\Gamma$ as its resolution graph such that the $h^1$ appearing in the dimension formula of Equation (1.1) is maximal (among singularities with the same resolution graph $\Gamma$). For this we review the construction of some specific surfaces (called the plumbing surface). In Section 3 we verify some cohomological properties of these specific surfaces. Then, in Section 4 we provide formulae for the change of the dimension of Equation (1.1) under blow-ups and provide the proof of Theorem 1.5 which ultimately implies the main result of the paper.

Throughout this paper we work over the field of complex numbers.

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2. The plumbing schemes

In this section we prove that for a given negative definite weighted graph $\Gamma$ with certain properties, there is a normal surface singularity $(X_0, 0)$ (with minimal good resolution $(V_0, E_0) \rightarrow (X_0, 0)$) that has $\Gamma$ as its resolution graph and that $h^1(V_0, \mathcal{O}_{E_0}(-\log E_0))$ is maximal among singularities having the same weighted resolution graph (Corollary 2.8). For this we recall the definitions of plumbing surfaces and plumbing curves associated to a weighted graph, and we investigate their properties. (We refer to Laufer [13, Theorem 3.9] and Schüll [28] for constructions of these schemes.)

Let $\Gamma$ be a weighted graph which is a tree consisting of $(-d_i)$-vertices $E_i$ ($i = 1, \ldots, n$) with $d_i \geq 1$. Assume furthermore that the valencies of the nodes of $\Gamma$ are all equal to 3 possibly except exactly one node with valency 4. It is known that the analytic type of a singularity whose resolution graph has a node of valency 4 depends on the cross ratio of the node of valency 4. Throughout this paper a graph with a unique node of valency 4 (and all other nodes of valency 3) is always assumed to be given with a complex number $c \in \mathbb{C}$ ($c \neq 0, 1$), called the cross ratio of the graph.

2.1. Plumbing surfaces. For $i = 1, \ldots, n$, let $U_{i1} = \mathbb{C}^2$ ($k = 1, 2$) with coordinates $(x_{ik}, y_{ik})$. We glue $U_{i1}$ and $U_{i2}$ via the isomorphism

$$\phi_i : U_{i2} \setminus \{x_{i2} = 0\} \rightarrow U_{i1} \setminus \{x_{i1} = 0\}, \quad (x_{i2}, y_{i2}) \mapsto (1/x_{i2}, x_{i2}^d y_{i2}),$$

and obtain $V_i = U_{i1} \cup_{x_{i1}=0} U_{i2}$. The $(-d_i)$-vertex $E_i$ is realized as the zero section

$$E_i = \{y_{i1} = 0\} \cup \{y_{i2} = 0\}(\cong \mathbb{CP}^1) \subset V_i$$

of the $\mathbb{C}$-bundle $V_i$ over $\mathbb{CP}^1$ with $y_{ik}$ ($k = 1, 2$) as fiber coordinates.

We first define a two-dimensional (complex) analytic space $V_\Gamma$ associated to $\Gamma$, by gluing neighborhoods of the zero sections $E_i$'s of $V_i$'s together as explained below. If $E_i \cap E_j \neq \emptyset$ for $i \neq j$ (that is, if the two vertices $E_i$ and $E_j$ are connected by an edge in $\Gamma$), we glue a neighborhood of $E_i \subset V_i$ and that of $E_j \subset V_j$ as follows: For a fixed $i$, we place the (at most four) points $\{E_j \cap E_i \mid j \neq i\} \subset E_i$ at $x_{i1} = 0$, $x_{i2} = 0, x_{i1} = 1, x_{i1} = c (c \neq 0, 1)$, where $c$ is the cross ratio of the graph (if given). Choose $(x_{i1}, y_{i1}), (x_{i2}, y_{i2}), (x_{i1} - 1, y_{i1}), (x_{i1} - c, y_{i1})$ as local base coordinates of $V_i$ near $x_{i1} = 0$. Let $x_{i2} = 0, x_{i1} = 1$ and $x_{i1} = c$, respectively. Near a point of $E_i \cap E_j$ we glue a neighborhood of $E_i \subset V_i$ and that of $E_j \subset V_j$ by interchanging the above chosen base coordinates and fiber coordinates for $V_i$ and $V_j$.

Definition 2.1. The plumbing surface $S_\Gamma$ associated to $\Gamma$ is a germ of the two-dimensional analytic space $V_\Gamma$ along the one-dimensional curves $E = \cup E_i$.

We will show that some relevant cohomological properties of plumbing surfaces are independent of the choice of the cross ratio $c$. So, by slight abuse of notation, we denote the plumbing surface associated to $\Gamma$ by $S_\Gamma$ for simplicity, instead of recording also $c$ in the notation.

Remark 2.2. Let $\Gamma$ be a weighted graph and let $\Gamma'$ be a graph obtained by blowing up a vertex or an edge of $\Gamma$ in a way that $\Gamma'$ has the same number of valency 4 nodes as $\Gamma$. It is not hard to show that the plumbing surface $S_{\Gamma'}$ associated to $\Gamma'$ is equal to the surface $S_\Gamma$ blown up at the appropriate point on $S_\Gamma$.

Remark 2.3. For a non-minimal weighted graph $\Gamma$ of type $A$, $B$, or $C$, a model for the plumbing surface $S_{\Gamma}$ can be obtained as follows: Let $C_\infty$ be the negative section of the Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{F}^1} \oplus \mathcal{O}_{\mathbb{F}^1}(-1))$, i.e. $C_\infty$ is a section with $C_\infty \cdot C_\infty = -1$. Choose three distinct fibers $F_1, F_2, F_3$ of $\mathbb{F}_1$ intersecting $C_\infty$ at 0, 1, $\infty$, respectively. Then a concrete model for the plumbing surface $S_\Gamma$ can be obtained...
by appropriately blowing up a small neighborhood of the negative section $C_\infty$ and three distinct fibers $F_i$. Indeed, let $\Gamma_0$ be one of the weighted graphs $\Gamma_A, \Gamma_B, \Gamma_C$. After the appropriate sequence of blow-ups we can identify a configuration of curves (in the proper transform of the section $C_\infty$ and the three fibers $F_i$) which intersect each other according to $\Gamma_0$ in the resulting rational surface. The plumbing surface $S_{\Gamma_0}$ is a germ of the resulting rational surface along the curves. By further blowing up the curves at appropriate points, we can find a configuration of curves in the proper transform intersecting each other according to the given graph $\Gamma$. The germ of the surface along these curves then provides the plumbing surface $S_{\Gamma}$.  

2.2. Plumbing curves. Let $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$ and let $Z_i$ be the ideal sheaf of $E_i$ in $S_{\Gamma}$. We define the plumbing curve $Z_{\Gamma}(s)$ associated to $\Gamma$ and $s$ as a non-reduced one-dimensional scheme defined by the ideal sheaf $\prod_{i=1}^{n} Z_i^{s_i}$, which is the same as the plumbing construction of Laufer [13, Theorem 3.9]. For brevity, in case of $s = (1, \ldots, 1)$, we denote $Z_{\Gamma}(s)$ by $Z_{\Gamma}$. Here we briefly recall a more detailed construction of the plumbing curve $Z_{\Gamma}(s)$ given in Schüller [23, §3] and [27, §4]. Let $t_i = 2\{j \mid E_j \cap E_i \neq \emptyset\}$, and let $E_{t_i}$ $(1 \leq t \leq t_i)$ be the $t_i$ curves with $E_i \cap E_{t_i} \neq \emptyset$. We first define a 1-dimensional scheme $W_i$. Each $W_i$ consists of the following three affine open subschemes of $Z_{\Gamma}(s)$. (In what follows, if there is no node with valency 4, then one may remove the terms $y_{11} = c, y_{12} = c$ (c is the cross ratio) in the formulae): If $t_i = 1$ then

$$W_{1,12} = \text{Spec} \left( \mathbb{C}[x_{11}, y_{11}, y_{21}] / \left( x_{11}^{s_{11}} y_{11}^{s_{11}} \right) \right) \setminus \{ y_{11} = 1, y_{12} = c \}$$

If $t_i = 2$ then

$$W_{1,12} = \text{Spec} \left( \mathbb{C}[x_{11}, y_{11}, x_{22}, y_{21}] / \left( x_{11} x_{22} - 1, y_{11} - x_{22}^{s_{21}} y_{21}^{s_{21}} \right) \right) \setminus \{ y_{11} = 1, y_{12} = c \}$$

If $t_i = 3$ then

$$W_{1,12} = \text{Spec} \left( \mathbb{C}[x_{11}, y_{11}, x_{22}, y_{21}] / \left( x_{11} x_{22} - 1, y_{11} - x_{22}^{s_{21}} y_{21}^{s_{21}} \right) \right) \setminus \{ y_{11} = 1, y_{12} = c, y_{22} = 1, y_{21} = c \}$$

If $t_i = 4$ then

$$W_{1,12} = \text{Spec} \left( \mathbb{C}[x_{11}, y_{11}, x_{22}, y_{21}] / \left( x_{11} x_{22} - 1, y_{11} - x_{22}^{s_{21}} y_{21}^{s_{21}} \right) \right) \setminus \{ y_{11} = 1, y_{22} = 1 \}$$
The plumbing curve $Z_T(s)$ is given by gluing $W_i$ and $W_j$ in case $E_i \cap E_j \neq \emptyset$ by interchanging the base coordinates and the fiber coordinates for $W_i$ and $W_j$. That is, if $W_i \cap W_j = W_{i m_i} \cap W_{j m_j}$ for $1 \leq m_i, m_j \leq 2$, then we glue $W_i$ and $W_j$ by the relation

$$\tilde{x}_{im_i} = y_{jm_j},$$

$$y_{im_i} = \tilde{x}_{jm_j},$$

(2.1)

with $\tilde{x}_{im_i} = x_{im_i} - c$ if $W_j = W_{i q}$ with respect to $W_i$, or $\tilde{x}_{im_i} = x_{im_i} - 1$ if $W_j = W_{i s}$ with respect to $W_i$, or $\tilde{x}_{im_i} = x_{im_i}$ else, and analogously for $\tilde{x}_{jm_j}$.

2.3. Plumbing schemes and effective exceptional cycles. Let $(X, 0)$ be a germ of a rational surface singularity. Let $\pi : V \to X$ be the minimal good resolution of $X$ with $E = \pi^{-1}(0)$ the exceptional set. Let $E = \sum_{i=1}^n E_i$ be the decomposition of the exceptional set $E$ into irreducible components. Then the $E_i$’s have only normal crossings and $E_i \cong \mathbb{P}^1$. For $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$ let $Z(s) = \sum_{i=1}^n s_i E_i$ ($s_i \geq 1$) be an effective exceptional cycle supported on $E$. Let $\Gamma$ be the weighted graph corresponding to $E$.

In what follows we assume that the valencies of the vertices of $\Gamma$ are $\leq 3$ possibly except one node with valency 4 (as it is satisfied by graphs of type $A$, $B$, or $C$), although the same method would give the results for more general graphs. Furthermore, if there is a node of valency 4, say $E_n$, then we assume that the cross ratio $c$ of the graph $\Gamma$ is given as that of the four intersection points in $E_n$ by its four neighbours.

**Proposition 2.4** ([Laufer [13] Theorem 3.9], Schüller [28] Lemma 3.2]). The scheme $Z(s)$ can be obtained by gluing the open subsets $W_i$ of the plumbing curve $Z_T(s)$ with $s = (s_1, \ldots, s_n)$ by using various gluing maps.

**Proof.** The proof is given in the proof of Laufer [13] Theorem 3.9] or in that of Schüller [28] Lemma 3.2]. Here we briefly recall how to glue $W_i$ (for details see Schüller [28] Lemma 3.2)). There are open neighborhoods of $E_i$ in $Z$ isomorphic to $W_i$ for every $E_i$. For $E_i \cap E_j \neq \emptyset$, letting $m_i$, $m_j$, $\tilde{x}_{im_i}$, $\tilde{x}_{jm_j}$ as before, we glue $W_i$ and $W_j$ by the relations

$$\tilde{x}_{jm_j} = y_{im_i}(a_{y, ij} + \tilde{x}_{im_i}y_{im_j}p_{y, ij}),$$

$$y_{im_i} = \tilde{x}_{jm_j}(a_{x, ij} + \tilde{x}_{im_i}y_{im_j}p_{x, ij}),$$

(2.2)

for some $a_{x, ij}, a_{y, ij} \in \mathbb{C} \setminus \{0\}$ and $p_{x, ij}, p_{y, ij} \in \mathbb{C}[x_{im_i}, y_{im_j}]$. \hfill \Box

**Proposition 2.5** (Schüller [28] Proposition 3.14]). Let $Z(s) = \sum_{i=1}^n s_i E_i$ ($s_i \geq 1$) be an effective exceptional cycle supported on $E$. Then there exist an integral affine scheme $T$ and a locally trivial flat surjective map $f : X \to T$ such that $Z_T(s) = f^{-1}(t_0)$ for some closed point $t_0 \in T$ and $Z(s) \cong f^{-1}(t_1)$ for some $t_1$.

**Proof.** We briefly sketch the proof of Schüller [28] Proposition 3.14] for the convenience of the reader. Suppose that $Z(s)$ is defined by the relations in (2.2). Let

$$A = \mathbb{C}[u_{x, ij}, u_{y, ij}, u_{x, ij}^{-1}, u_{y, ij}^{-1}, u_x, u_y]$$

(2.3)

with $ij$ running over all $ij$ such that $W_i \cap W_j \neq \emptyset$. Here we put $u_{x, ij}^{-1}$ and $u_{y, ij}^{-1}$ in $A$ because $a_{x, ij}, a_{y, ij} \neq 0$ in the gluing map (2.2). Let $T = \text{Spec } A$. Then $X$ is defined as follows: $W_i \times T$ and $W_j \times T$ can be glued along $(W_i \cap W_j) \times T$ via

$$x_{ik_j} = y_{ik}(u_{y, ij} + x_{ik}y_{ik}p_{y, ij}u_y),$$

$$y_{ik_j} = x_{ik}(u_{x, ij} + x_{ik}y_{ik}p_{x, ij}u_x).$$

Then it is not difficult to show that the second projection $f : X \to T$ is flat, $Z(s) = f^{-1}(a_{x, 12}, a_{y, 12}, \ldots, a_{x, in}, a_{y, in}, 1, 1)$, and $Z_T(s) = f^{-1}(1, 1, \ldots, 1, 1, 0, 0)$. \hfill \Box
Next we compare \( h^1(Z(s), \Theta_{Z(s)}) \) and \( h^1(Z_{\Gamma}(s), \Theta_{Z_{\Gamma}(s)}) \):

**Theorem 2.6.** Let \((V, E) \to (X, 0)\) be the minimal good resolution of a rational surface singularity. Let \( E = \sum_{i=1}^{n} E_i \) be the decomposition of the exceptional set \( E \) into irreducible components. For \( s = (s_1, \ldots, s_n) \in \mathbb{N}^n \) let \( Z(s) = \sum_{i=1}^{n} s_i E_i \) (\( s_i \geq 1 \)) be an effective exceptional cycle supported on \( E \). Let \( \Gamma \) be the weighted dual graph corresponding to \( E \) (given with the same cross ratio of the node of valency 4 of \( E \), if any). Then we have

\[
h^1(Z(s), \Theta_{Z(s)}) \leq h^1(Z_{\Gamma}(s), \Theta_{Z_{\Gamma}(s)}).
\]

**Proof.** By the Mayer-Vietoris sequence (cf. Laufer [13, (3.10)] or Schüller [28, Lemma 3.4]), we have

\[
H^1(Z(s), \Theta_{Z(s)}) = \left( \bigoplus_{i \neq j} \Gamma(W_i \cap W_j, \Theta_{Z(s)}) \right) / \rho_{Z(s)} \left( \bigoplus_i \Gamma(W_i, \Theta_{Z(s)}) \right)
\]

\[
H^1(Z_{\Gamma}(s), \Theta_{Z_{\Gamma}(s)}) = \left( \bigoplus_{i \neq j} \Gamma(W_i \cap W_j, \Theta_{Z_{\Gamma}(s)}) \right) / \rho_{Z_{\Gamma}(s)} \left( \bigoplus_i \Gamma(W_i, \Theta_{Z_{\Gamma}(s)}) \right).
\]

(2.4)

where \( \rho_{Z(s)} \) and \( \rho_{Z_{\Gamma}(s)} \) are restriction maps. Furthermore, in computing them, by Laufer [13, (3.11)] or Schüller [28, (4.16)] it is enough to consider only elements of \( \bigoplus_{i \neq j} \Gamma(W_i \cap W_j, \Theta_{Z(s)}) \) of the form

\[
\sum_{a=1}^{s_{j-1}-s_i-1} \sum_{b=0}^{s_i-1} \alpha_{a b} x_i^a y_i^b \frac{\partial}{\partial x_i} + \sum_{c=0}^{s_j-1} \sum_{d=1}^{s_i-1} \beta_{c d} x_i^c y_i^d \frac{\partial}{\partial y_i}.
\]

(2.5)

We now consider the elements in \( \Gamma(W_i, \Theta_{Z(s)}) \) and \( \Gamma(W_i, \Theta_{Z_{\Gamma}(s)}) \). At first, note that \( \Gamma(W_i, \Theta_{Z(s)}) = \Gamma(W_i, \Theta_{Z_{\Gamma}(s)}) \). Let \( t_i = \sum_j (E_j \cap E_i) \) as before. Depending on \( t_i \), the elements of \( \Gamma(W_i, \Theta_{Z(s)}) = \Gamma(W_i, \Theta_{Z_{\Gamma}(s)}) \) are given as follows (cf. Laufer [13, pp. 86–87] and Laufer [14, (4.4)]; or Schüller [28, p. 68]): For any \( t_i \),

\[
x_{i1}^a y_{i1}^b \frac{\partial}{\partial y_{i1}}
\]

(2.6)

with \( 0 \leq a \leq v_i(b - 1) \), \( b > 0 \).

For \( t_i = 1, 2 \) we have

\[
x_{i1}^a y_{i1}^b \frac{\partial}{\partial x_{i1}}
\]

(2.7)

with \( 0 < a \leq v_i b + 1 \), \( b \geq 0 \). Additionally, for \( t_i = 1 \), we have

\[
y_{i2}^b \frac{\partial}{\partial x_{i1}}
\]

(2.8)

with \( b \geq 0 \). For \( t_i = 3 \) we have

\[
x_{i1}^a y_{i1}^b (x_{i1} - 1) \frac{\partial}{\partial x_{i1}}
\]

(2.9)

with \( 0 < a \leq v_i b \), \( b > 0 \). Finally for \( t_i = 4 \) we have

\[
x_{i1}^a y_{i1}^b (x_{i1} - 1)(x_{i1} - c) \frac{\partial}{\partial x_{i1}}
\]

(2.10)

with \( 0 < a \leq v_i b - 1 \), \( b > 0 \), where \( c \) is the cross ratio.

According to Schüller [28, Corollary 3.9], in order to compute \( h^1(Z(s), \Theta_{Z(s)}) \) and \( h^1(Z_{\Gamma}(s), \Theta_{Z_{\Gamma}(s)}) \), we first construct matrices \( M_{Z(s)} \) and \( M_{Z_{\Gamma}(s)} \) in the following way: For every intersection point \( x_{ij} \) of \( E_i \cap E_j \) and every element of Equation (2.5) we add one row to \( M_{Z(s)} \) and \( M_{Z_{\Gamma}(s)} \), respectively. Then for every \( W_i \) and for
every element of (2.7), (2.11), (2.15), or (2.16), we add one column to $M_{Z(s)}$ and $M_{\mathcal{Z}_{\mathcal{M}}(s)}$, respectively. The entries of the matrices $M_{Z(s)}$ and $M_{\mathcal{Z}_{\mathcal{M}}(s)}$ are the coefficients of the element associated to the column as an expression in the element associated to the row. Note that the two matrices $M_{Z(s)}$ and $M_{\mathcal{Z}_{\mathcal{M}}(s)}$ have the same number of rows, say $r$. The entries of $M_{\mathcal{Z}_{\mathcal{M}}(s)}$ are complex numbers determined by $\Gamma$ (and the cross ratio $c$, if given) and $s = (s_1, \ldots, s_n)$. On the other hand, the entries of $M_{Z(s)}$ are polynomials in $A$ of Equation (2.3). The difference between $M_{\mathcal{Z}_{\mathcal{M}}(s)}$ and $M_{Z(s)}$ is coming from the gluing data of Equations (2.1) and (2.2). Then it follows by (2.4) that

$$h^1(Z(s), \Theta_{Z(s)}(s)) = r - \text{rank} M_{Z(s)}$$

$$h^1(\mathcal{Z}_{\mathcal{M}}(s), \Theta_{\mathcal{Z}_{\mathcal{M}}(s)}(s)) = r - \text{rank} M_{\mathcal{Z}_{\mathcal{M}}(s)}.$$ 

Therefore, if $Z'$ is a nearby fiber of the deformation $f$ in Proposition 2.7, then we have $\text{rank} M_{Z(s)} = \text{rank} M_{Z'}$ because the rank is locally constant on the base space $T$. Therefore $h^1(Z(s), \Theta_{Z(s)})$ remains constant for the general fiber $Z'$ of the deformation $f$. The assertion then follows by upper semicontinuity. \hfill $\square$

Lemma 2.7. For $s \gg 0$, we have

$$h^1(V, \Theta_V(Z(s))) = h^1(Z(s), \Theta_{Z(s)})$$

Proof. This is a well-known fact; here we give a proof for the convenience of the reader. According to Burns–Wahl [2, Subsection (1.6)], there is an exact sequence

$$0 \to \Theta_{Z(s)} \to \Theta_V \to \bigoplus_{i=1}^n N_{E_i/V} \to 0. \quad (2.11)$$

Then we have the following commutative diagram:

\begin{align*}
0 & \to \Theta_{Z(s)} \to \Theta_V \to \bigoplus_{i=1}^n N_{E_i/V} \to 0 \\
0 & \to \Theta_V(-Z(s)) \xrightarrow{id} \Theta_V(-Z(s)) \to 0 \\
0 & \to \Theta_V(-\log E) \to \Theta_V \to \bigoplus_{i=1}^n N_{E_i/V} \to 0 \\
0 & \to \Theta_{Z(s)} \to \Theta_V \otimes \mathcal{O}_{Z(s)} \to \bigoplus_{i=1}^n N_{E_i/V} \to 0 \\
& \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 & \to \Theta_{Z(s)} \to \Theta_V \to \bigoplus_{i=1}^n N_{E_i/V} \to 0 \\
& \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 & \to \Theta_{Z(s)} \to \Theta_V \to \bigoplus_{i=1}^n N_{E_i/V} \to 0
\end{align*}

By the snake lemma, we get an exact sequence

$$0 \to \Theta_V(-Z(s)) \to \Theta_V(-\log E) \to \Theta_{Z(s)} \to 0. \quad (2.12)$$

Since $\Gamma$ is negative definite, one may choose $Z_0$ so that $Z_0 \cdot E_i < 0$ for all $i$ (that is, $-Z_0$ is ample). We have $H^i(V, \Theta_V(-Z_0)) = 0$ ($i = 1, 2$) by Kodaira vanishing, hence there is an isomorphism

$$H^1(V, \Theta_V(-\log E)) \to H^1(Z_0, \Theta_{Z_0}).$$

On the other hand, for any $Z(s) \geq Z_0$, the above isomorphism $H^1(V, \Theta_V(-\log E)) \to H^1(Z_0, \Theta_{Z_0})$ factors through

$$H^1(V, \Theta_V(-\log E)) \to H^1(Z(s), \Theta_{Z(s)}) \to H^1(Z_0, \Theta_{Z_0}).$$
Note that the first map is surjective; therefore it is an isomorphism. Hence we have
\[ h^1(V, \Theta_V(- \log E)) = h^1(Z(s), \Theta_{Z(s)}). \]

The combination of Theorem 2.6 and Lemma 2.7 immediately implies:

**Corollary 2.8.** With the notation as in Theorem 2.6,
\[ h^1(V, \Theta_V(- \log E)) \leq h^1(S, \Theta_{S}(- \log Z)). \]

\[ \square \]

### 3. Cohomological properties of plumbing schemes

Let \( \Gamma \) be a non-minimal graph of type \( \mathcal{A}, \mathcal{B}, \) or \( \mathcal{C} \). Let \( \Gamma^\sharp \) be the corresponding minimal graph, and let \( \Gamma^\sharp \) be the augmented graph corresponding to \( \Gamma \). The goal of this section is to compare \( h^1(S, \Theta_S(- \log Z)) \) to \( h^1(S, \Theta_{S}(- \log Z)) \) (see Theorem 3.6).

Suppose that \( Z = \sum_{i=1}^n E_i \) is the decompostion of the exceptional divisor \( Z_\Gamma \) in \( S_\Gamma \). Since \( \Gamma \) is negative definite, there is \( s_0 = (s_1, \ldots, s_n) \in \mathbb{N}^n \) such that
\[ Z_\Gamma(s_0) : E_i < 0 \]
for all \( i = 1, \ldots, n \), that is, \( -Z_\Gamma(s_0) \) is ample in \( S_\Gamma \). Set \( s = ms_0 = (ms_1, \ldots, ms_n) \).

**Lemma 3.1.** For \( m \gg 0 \), we have
\[ H^1_{\sharp}(\Theta_{S_\Gamma}(-Z_\Gamma(s))) = 0, \]
where \( H^1_{\sharp} \) means the cohomology with support on \( Z_\Gamma \).

**Proof.** Let \( F = \Theta_{S_\Gamma}(-Z_\Gamma(s)) \). In the following, for simplicity, we will denote \( S_\Gamma, Z_\Gamma, \Theta_{S_\Gamma}, \Theta_{Z_\Gamma}, Z_\Gamma \) by \( S, \overline{S}, \Theta, \overline{\Theta}, Z, \overline{Z} \), respectively.

Let \( \pi : \overline{S} \to \overline{X} \) be the map contracting \( Z \) to a point, say \( P \). Since \( H^1(\overline{S}, F) = 0 \) by Kodaira vanishing, we have the following commutative diagram with exact rows:
\[
\begin{array}{c}
0 \to \Gamma(\overline{X}, \pi_* F) \to \Gamma(\overline{X} - P, \pi_* F) \to H^1_{\sharp}(\pi_* F) \\
\end{array}
\]

By Laufer [12] Lemma 5.2, \( \pi_* F \) is coherent. Since \( V \) is Cohen-Macaualy at \( P \) (being two-dimensional and normal), depth\( P(\pi_* F) = 2 \) by Schlessinger [26] Lemma 1; hence \( H^1_{\sharp}(\pi_* F) = 0 \). Therefore \( \Gamma(\overline{X} - P, \pi_* F) \cong \Gamma(\overline{X} - P, \pi_* F) \); thus,
\[ \Gamma(\overline{S} - Z, F) \cong \Gamma(\overline{S} - Z, F), \]  
(3.1)

hence the assertion follows. \[ \square \]

**Remark 3.2.** The above lemma may be proved by a general result, the easy vanishing theorem of Wahl [35].

**Lemma 3.3.** For \( m \gg 0 \), we have \( H^0(Z, \Theta_{Z(s)}) = 0 \).

**Proof.** We use the same notations as in the proof of Lemma 3.1 for simplicity. From the short exact sequence (2.12), we have
\[ 0 \to H^0(\overline{S}, \overline{\Theta}(Z(s))) \to H^0(\overline{S}, \overline{\Theta}(\log Z)) \to H^0(\Theta_{Z(s)}) \to H^1(\overline{S}, \overline{\Theta}(Z(s))) \]
Since \( H^1(\overline{S}, \overline{\Theta}(Z(s))) = 0 \) by Kodaira vanishing, it is enough to show that
\[ H^0(\overline{S}, \overline{\Theta}(Z(s))) \to H^0(\overline{S}, \overline{\Theta}(\log Z)) \]
is an isomorphism.
By the definition of $\Theta(- \log Z)$, we have the short exact sequence

$$0 \to \Theta(- \log Z) \to \Theta \to \oplus \mathbb{N}_{Z_i} \to 0$$

where $Z = \sum_i Z_i$. So we have

$$H^0(\mathcal{S}, \Theta(- \log Z)) = H^0(\mathcal{S}, \Theta).$$

On the other hand, by Equation (3.1), we have

$$H^0(\mathcal{S}, \Theta(- Z(s))) = H^0(\mathcal{S}, \Theta(- Z(s))) = H^0(\mathcal{S}, Z, \Theta).$$

Using the depth argument as in the proof of Lemma 3.1, we have

$$H^0(\mathcal{S} \setminus Z, \Theta) = H^0(\mathcal{S}, \Theta);$$

hence the assertion follows. □

**Proposition 3.4.** For $m \gg 0$, the cohomology group $H^1(S_{TZ}, \Theta_{S_{TV}}(- Z_T(s)))$ depends only on the type $(A, B, \text{ or } C)$ of the graph $\Gamma$.

**Proof.** With the same notations as in the proof of Lemma 3.1, we have the following exact sequence:

$$0 = H^2(F) \to H^1(S^4, F) \to H^1(S^4 \setminus Z, F) \to H^2(S^4, F) = 0,$$  \hspace{1cm} (3.2)

where $H^2(F) = 0$ by Lemma 3.1 and $H^2(S^4, F) = 0$ because $F$ is a locally free sheaf on the germ $S^4$ of an analytic space along a one-dimensional curve (cf. Grauert [7, Satz 1, p. 355]). We first prove that $H^1(S^4 \setminus Z, F)$ and $H^2(F)$ in the above sequence depend only on the type of the graph $\Gamma$.

At first, we will show that $S^4 \setminus Z$ depends only on the type of the graph $\Gamma$; then, it is clear that $H^1(S^4 \setminus Z, F)$ depends only on the type of the graph $\Gamma$. This follows from the observation that

$$S^4 \setminus Z = (S^4 \setminus Z^2) \cup C_0,$$

where $C_0 = \text{Supp}(Z^2) \setminus \text{Supp}(Z)$. Since $S^4$ is obtained by blowing up (several times) the plumbing surface corresponding to the graph $\Gamma_A$, $\Gamma_B$, or $\Gamma_C$ according to its type, the complement $S^4 \setminus Z^2$ depends only on the type of the graph $\Gamma$. Furthermore $C_0$ depends only on the type of the graph $\Gamma$. Therefore $S^4 \setminus Z$ depends only on the type of the graph $\Gamma$ as asserted.

Next we prove that $H^2(F)$ depends only on the type of the graph $\Gamma$. In the following exact sequence

$$H^1(S, F) \longrightarrow H^1(S \setminus Z, F) \longrightarrow H^2(F) \longrightarrow H^2(S, F),$$

we have $H^1(S, F) = H^2(S, F) = 0$ by Kodaira vanishing; hence

$$H^2(F) \cong H^1(S \setminus Z, F).$$  \hspace{1cm} (3.3)

Therefore it is enough to show that $S \setminus Z$ depends only on the type of the graph. For $C_1 = S^4 \setminus S$, which depends only on the type of the graph, we have

$$S^4 \setminus Z = (S \setminus Z) \cup (C_1 \setminus Z).$$

Since $S^4 \setminus Z$ depends only on the type of the graph as we seen above, so does $S \setminus Z$. Hence $H^2(F)$ depends only on the type of the graph $\Gamma$.

Finally, the cohomology group $H^1(S^4, F)$ is the kernel of the connecting homomorphism

$$\phi : H^1(S^4 \setminus Z, F) \to H^2(F)$$

in the exact sequence Equation (3.2), which is just a restriction map because $H^2(F) \cong H^1(S \setminus Z, F)$ by Equation (3.3). Since the two spaces $S^4 \setminus Z$ and $S \setminus Z$
depend only on the type of the graph, so does $φ$. Therefore $H^1(S^t, F)$ in the exact sequence of Equation (3.2) also depends only on the type of the graph $Γ$ as asserted. □

The following result of Flenner-Zaidenberg shows how the cohomologies of logarithmic tangent sheaves change under blow-ups:

**Proposition 3.5** (Flenner–Zaidenberg [5, Lemma 1.5]). Let $S$ be a nonsingular surface, and let $D$ be a simple normal crossing divisor on $S$. Let $π: S' → S$ be the blow-up of $S$ at a point $p$ of $D$. Let $D' = f^∗(D)_{red}$.

(a) If $p$ is a smooth point of $D$, then there is an exact sequence

$$0 → π_*Θ_S(−log D') → Θ_{S}(−log D) → C → 0$$

where the constant sheaf $C$ is supported on $p$. Hence we have

$$C → H^1(S', Θ_{S'}(−log D')) → H^1(S, Θ_{S}(-log D)) → 0$$

and

$$H^2(S', Θ_{S'}(−log D')) ≅ H^2(S, Θ_{S}(-log D)).$$

(b) If $p$ is on two components of $D$, then

$$H^i(S', Θ_{S'}(−log D')) ≅ H^i(S, Θ_{S}(-log D))$$

for $i = 1, 2$. □

After these preparations, we are ready to turn to the proof of the main result of this section.

**Theorem 3.6.** Let $Γ$ be a non-minimal graph of type $A$, $B$, or $C$. Let $Γ$ be the corresponding minimal graph, and let $Γ^♯$ be the augmented graph corresponding to $Γ$. There is a constant $α$ which depends only on the type of $Γ$ (not on the graph $Γ$ itself and the cross ratio, if any) such that

$$h^1(S, Θ_{S}(−log ZΓ)) = h^1(S, Θ_{S}(−log Z_{Γ^♯})) = α - 1.

We then have

$$h^i(S, Θ_{S}(−log Z_{Γ^♯})) ≤ h^i(S, Θ_{S}(−log Z_{Γ})) - α + 1.$$

**Proof.** As in Equation (2.12), we have a short exact sequence

$$0 → Θ^♯(−Z(s)) → Θ^♯(−log Z) → Θ^♯_{Z(s)} → 0.$$<br>
By Lemma 2.7, $H^1(Θ^♯_{Z(s)}) ≅ H^1(S, Θ^♯(−log Z))$, and by Lemma 3.3 we have that $H^0(Σ, Θ^♯(−log Z)) = 0$. Therefore we have an exact sequence

$$0 → H^1(S, Θ^♯(−log Z)) → H^1(S, Θ^♯(−log Z)) → H^1(S, Θ^♯(−log Z)) → 0$$

with $H^2(S, Θ^♯(−log Z)) = 0$ by Grauert [7, Satz 1, p. 355] (as in the proof of Proposition 3.4). By the above Equation, we have

$$h^1(S, Θ^♯(−log Z)) = h^1(S, Θ^♯(−log Z)) - h^1(S, Θ^♯(−log Z)).$$

Here $H^1(S, Θ^♯(−log Z))$ and $H^1(S, Θ^♯(−log Z))$ are finite dimensional because $S^♯$ is a germ of an analytic space along one-dimensional curves (cf. Grauert [7, Satz 1, p. 355]).

On the other hand, consider the short exact sequence

$$0 → Θ^♯(−log Z) → Θ^♯(−log Z) → N_F → 0,$$

where $F$ is the redundant divisor, i.e. $F = Z^♯ − Z$ and $N_F = Z^♯ − Z$ for $F = Z$. Since $H^0(N_F) = 0$, we have

$$0 → H^1(S, Θ^♯(−log Z)) → H^1(S, Θ^♯(−log Z)) → H^1(N_F) → 0$$

Equation (3.6)
where $H^2(S^\sharp, \Theta^\sharp(- \log Z^\sharp)) = 0$ by Grauert [7, Satz 1, p. 355] as before. Then
$$h^1(S^\sharp, \Theta^\sharp(- \log Z^\sharp)) = h^1(S^\sharp, \Theta^\sharp(- \log Z^\sharp)) + h^1(N_F).$$ (3.7)

From Equations (3.5) and (3.7) we have
$$h_1(S^\sharp, \Theta^\sharp(- \log Z^\sharp)) = h_1(S^\sharp, \Theta^\sharp(- \log Z^\sharp)) + h_1(N_F) - h_1(S^\sharp, \Theta^\sharp(- Z(s))).$$ (3.8)

Set
$$\alpha = -h_1(N_F) + h_1(S^\sharp, \Theta^\sharp(- Z(s))).$$

The redundant divisor $F$ depends only on the type of the graph. By Proposition 3.4, the quantity $h^1(S^\sharp, \Theta^\sharp(- Z(s)))$ also depends only on the type of the graph $\Gamma$. Therefore the constant $\alpha$ depends only on the type of the graph $\Gamma$, and so the first assertion of the proposition follows.

For the second assertion, since $S^\sharp$ is obtained from $S_\Gamma$ by blowing up once the $(-1)$-vertex of $\Gamma$ and (if needed) blowing up edges emanating from the $(-1)$-vertex, it follows by Proposition 3.5 that
$$h^1(S^\sharp, \Theta^\sharp(- \log Z^\sharp)) \leq h^1(S_\Gamma, \Theta_{S_\Gamma}(- \log Z_\Gamma)) + 1.$$ Therefore the second inequality follows. $\square$

4. Singularities with no rational homology disk smoothings

In this section we prove the main technical result, Theorem 1.5 of the paper. The proof will rest on the following two lemmas, where we treat the case of zero or one node of valency 4. Recall that a graph $\Gamma$ is called $H$-shaped if it admits two nodes, both with valency 3, and we say that a graph $\Gamma$ is key-shaped if it admits two nodes with valencies 3 and 4, respectively. Recall that if a graph with a node $E$ of valency 4 is a resolution graph of a rational surface singularity, then $E^2 \leq -3$.

**Lemma 4.1.** Let $\Gamma_1$ be one of the following non-minimal $H$-shaped graphs of type $A$, $B$, or $C$:

\(\Gamma_1 = \)

(a) $\Gamma_1 = \)

(b) $\Gamma_1 = \)

where $a$ and $b$ are two of the integers in one of the triples $(3, 3, 3), (4, 4, 2), (6, 3, 2)$. Then the corresponding minimal graph $\Gamma_1$ is taut and we have
$$h^1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(- \log Z_{\Gamma_1})) = 0.$$

**Proof.** If $\Gamma_1$ is taut, it follows by Laufer [13, Theorem 3.10] that
$$h^1(Z_{\Gamma_1}(s), \Theta_{Z_{\Gamma_1}(s)}) = 0$$
for $s \gg 0$. Hence, by Lemma 2.7 we have
$$h^1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(- \log Z_{\Gamma_1})) = h^1(Z_{\Gamma_1}(s), \Theta_{Z_{\Gamma_1}(s)}) = 0.$$
Therefore it remains to prove the tautness of $\Gamma_1$.

For Case (a), the graph $\Gamma_1$ is of type $L_1 - J_1 - R_1$ for $d \geq 3$, and of type $L_2 - J_1 - R_1$ for $d = 2$ in the list of taut graphs of Laufer [14, Table IV, p. 139].

For Case (b), let $\Gamma_0$ be the blown-down graph of $\Gamma_1$:

$$\Gamma_0 = \begin{array}{c}
\bullet & -b \\
\bullet & -a & -d & -1
\end{array}$$

By the definition of the classes $A$, $B$, and $C$, one of the neighbours of the $(-1)$-vertex is a $(-2)$-vertex, while the other one is a $(-e)$-vertex with $e \geq 3$. We distinguish two cases according to whether the $(-2)$-vertex is between the $(-1)$-vertex and the node (motivated by the diagram above, we say that the $(-2)$-vertex is to the left of the $(-1)$-vertex), or the $(-2)$-vertex is on the other side of the $(-1)$-vertex (it is to the right of the $(-1)$-vertex).

Case 1: Suppose that the $(-2)$-vertex of $\Gamma_0$ is to the right of the $(-1)$-vertex, that is, $\Gamma_0$ is given as follows:

$$\Gamma_0 = \begin{array}{c}
\bullet & -b \\
\bullet & -a & -d & -1 & -2
\end{array}$$

where $e \geq 3$. Note that the $(-e)$-vertex $E$ and the $(-d)$-vertex $D$ (the node) may coincide.

If we blow down the $(-1)$-vertex of $\Gamma_0$ then the $(-2)$-vertex $A$ of $\Gamma_0$ becomes the $(-1)$-vertex of the new graph. Since a $(-1)$-vertex in a star-shaped graph of $A$, $B$, or $C$ with valencies $\leq 3$ is not a leaf, there must be a vertex attached to the right side of the $(-2)$-vertex $A$ of $\Gamma_0$. In summary, the non-minimal graph $\Gamma_1$ is of the form:

$$\Gamma_1 = \begin{array}{c}
\bullet & -b \\
\bullet & -a & -d & -1 & -2
\end{array}$$

where $e \geq 3$. If $E = D$, then the minimal graph $\Gamma_1$ is of type $L_1 - J_1 - R_8$ using the contraction $C_4$. If $E \neq D$, the graph $\Gamma_1$ is of type $L_1 - J_1 - R_8$ for $d \geq 3$ or $L_2 - J_1 - R_8$ for $d = 2$, concluding the argument in this subcase.

Case 2: Suppose now that the $(-2)$-vertex of $\Gamma_0$ is to the left of the $(-1)$-vertex. Then the non-minimal graph $\Gamma_1$ is given as follows:

$$\Gamma_1 = \begin{array}{c}
\bullet & -b \\
\bullet & -a & -d & -2 & -2
\end{array}$$

where $e \geq 3$. If $D = A$, then the minimal graph $\Gamma_1$ is of type $L_2 - J_1 - R_2$. Suppose that $D \neq A$. If there is a $(-d)$-vertex with $d \geq 3$ between $D$ and $A$, then $\Gamma_1$ is of type $L_1 - J_2 - R_3$ for $d \geq 3$, and, of type $L_2 - J_2 - R_3$ for $d = 2$. Assume that there is no $(-d)$-vertex with $d \geq 3$ between $D$ and $A$. If $d \geq 3$, then $\Gamma_1$ is of type $L_1 - J_2 - R_3$ using the contraction $C_3$. If $d = 2$, then $E$ is a leaf of the graph and $\Gamma_1$ is of type $L_2 - J_1 - R_2$. □
Remark 4.2. J. Wahl informed us that the example in [33, Remark 8.9] was erroneously claimed to be non-taut. The graph can be presented as $L_1 - J_1 - R_8$ using a contraction $C_4$ as we have seen above.

We have a similar result for key-shaped graphs.

**Lemma 4.3.** Let $\Gamma_1$ be a non-minimal key-shaped graph of type $A$, $B$, or $C$ given by:

\begin{align*}
&\text{(a)} \\
\Gamma_1 = & \begin{array}{c}
- b \\
- a & - d \\
- c \\
\end{array} & \begin{array}{c}
- 1 \\
\end{array}
\end{align*}

for $e \geq 3$; or

\begin{align*}
&\text{(b)} \\
\Gamma_1 = & \begin{array}{c}
- b \\
- a & - d \\
- c \\
\end{array} & \begin{array}{c}
- 2 \\
\end{array}
\end{align*}

where $(a, b, c)$ is one of the triples $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ and $d \geq 3$. Let $\Gamma_1$ be the corresponding minimal graph. Then we have

$$h_1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(-\log Z_{\Gamma_1})) = 1.$$ 

**Proof.** Suppose that the analytic structure of the singularity $X_{\Gamma_1}$ (obtained by contracting $Z_{\Gamma_1}$ in $S_{\Gamma_1}$) is determined by the graph $\Gamma_1$ and the analytic structure on the reduced exceptional set $Z_{\Gamma_1}$. By Laufer [14, Theorem 4.1] and Laufer [14, (4.1)], a necessary and sufficient condition for $\Gamma_1$ and $Z_{\Gamma_1}$ to determine the singularity $X_{\Gamma_1}$ is that the restriction map

$$H^1(Z_{\Gamma_1} (s), \Theta_{Z_{\Gamma_1} (s)}) \rightarrow H^1(Z_{\Gamma_1}, \Theta_{Z_{\Gamma_1}})$$

is an isomorphism. Since the analytic structure on $Z_{\Gamma_1}$ is uniquely determined by the cross ratio of the 4 intersection points on the $(-d)$-curve with valency 4, we have

$$h_1(Z_{\Gamma_1}, \Theta_{Z_{\Gamma_1}}) = 1.$$ 

Hence it follows by Lemma 2.7 that

$$h_1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(-\log Z_{\Gamma_1})) = h_1(Z_{\Gamma_1} (s), \Theta_{Z_{\Gamma_1} (s)}) = h_1(Z_{\Gamma_1}, \Theta_{Z_{\Gamma_1}}) = 1.$$ 

Therefore it is enough to show that the analytic structure of the singularity $X_{\Gamma_1}$ is determined by $\Gamma_1$ and $Z_{\Gamma_1}$. For this we will use the list in Laufer [14, Theorem 4.1] of all dual graphs for singularities which are determined by the graph and the analytic structure on the reduced exceptional set. The proof is similar to that of Lemma 4.1.

In Case (a) the graph $\Gamma_1$ is of type $L_1' - J_1 - R_1$ for $d \geq 5$, of type $L_1'' - J_1 - R_1$ for $d = 4$, and of type $L_2' - J_1 - R_1$ for $d = 3$ in the list of Laufer [14, Theorem 4.1].

For Case (b), as in the proof of Lemma 4.1 we have two cases:

**Case 1:** $\Gamma_1$ is given as follows:
\[ \Gamma_1 = \begin{array}{c}
\bullet \quad -b \\
\bullet \quad -a \\
\bullet \quad -1 \\
\bullet \quad E \\
\bullet \quad -2 \\

\end{array} \]

where \( e \geq 3 \). Notice that since \( d \geq 3 \) (for \( \Gamma_1 \) being a resolution graph of a rational surface singularity), the \((-1)\)-framed vertex in a star-shaped graph with valency 4 node and of type \( A, B, \) or \( C \) cannot be a leaf. For \( d \geq 5 \), \( \Gamma_1 \) is of type \( L'_1 - J_1 - R_8 \) possibly using the contraction \( C_3 \). For \( d = 4 \), \( \Gamma_1 \) is of type \( L''_1 - J_1 - R_8 \) possibly using the contraction \( C_3 \). For \( d = 3 \), if \( D = E \) then \( \Gamma_1 \) is of type \( L'_2 - J_1 - R_2, \) or if \( D \neq E \) then \( \Gamma_1 \) is of type \( L''_2 - J_1 - R_8 \).

**Case 2:** \( \Gamma_1 \) is given as follows:

\[ \Gamma_1 = \begin{array}{c}
\bullet \quad -b \\
\bullet \quad -a \\
\bullet \quad -d \\
\bullet \quad -1 \\
\bullet \quad -2 \\
\bullet \quad -2 \\
\bullet \quad -e \\

\end{array} \]

where \( e \geq 3 \). For \( d \geq 5 \), \( \Gamma \) is of the form \( L'_1 - J_2 - R_3 \) using the contraction \( C_3 \). For \( d = 4 \), \( \Gamma \) is of the form \( L''_1 - J_2 - R_3 \) using the contraction \( C_3 \). For \( d = 3 \), \( \Gamma \) is of the form \( L'_2 - J_2 - R_3 \) if there are only \((-2)\)-vertices between two nodes, or \( \Gamma \) is of the form \( L''_2 - J_2 - R_3 \) otherwise. \( \square \)

**Lemma 4.4.** Let \( \Gamma_1 \) be a non-minimal graph in Lemma 4.1 or Lemma 4.3 and let \( \Gamma_2 \) be the non-minimal graph obtained by blowing up once the \((-1)\)-vertex of \( \Gamma_1 \).

Then

\[ h^1(S_{\Gamma_2}, \Theta_{S_{\Gamma_2}}(-\log Z_{\Gamma_2})) = \alpha + \epsilon, \]

where \( \epsilon = 0 \) if \( \Gamma_1 \) is a graph in Lemma 4.1, \( \epsilon = 1 \) if \( \Gamma_1 \) is a graph in Lemma 4.3, and \( \alpha \) is given by Theorem 3.6 (and it depends only on the type of \( \Gamma_1 \)).

**Proof.** It follows from Theorem 3.6 that

\[ h^1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(-\log Z_{\Gamma_1})) = \alpha + \epsilon. \]

Since \( S_{\Gamma_1} \) is obtained from \( S_{\Gamma} \) by blowing up edges emanating from the \((-1)\)-curve (if needed), it follows by Proposition 3.5 that

\[ h^1(S_{\Gamma_2}, \Theta_{S_{\Gamma_2}}(-\log Z_{\Gamma_2})) = h^1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(-\log Z_{\Gamma_1})) = \alpha + \epsilon. \]

\( \square \)

**Theorem 4.5.** Let \( \Gamma_1 \) be a non-minimal graph in Lemma 4.1 or Lemma 4.3 and let \( \Gamma_2 \) be the non-minimal graph obtained by blowing up once the \((-1)\)-vertex of \( \Gamma_1 \).

Let \( \Gamma \) be a non-minimal graph obtained from \( \Gamma_2 \) by applying the blow-ups (B-1) and (B-2) (described in Section 1) finitely many times. Suppose that \( (X, 0) \) is a rational surface singularity with a resolution \((V, E)\) which admits the minimal graph \( \Gamma \) as the resolution graph. Let \( E = \sum_{i=1}^n E_i \) be the decomposition of the exceptional divisor into irreducible components \( E_i \) with \( E_i^2 = -d_i \). Then

\[ h^1(V, \Theta_V(-\log E)) + \sum_{i=1}^n (d_i - 3) \leq 0. \]
Proof. By Corollary 2.8 we have

\[ h^1(V, \Theta_V(-\log E)) \leq h^1(S_T, \Theta_{S_T}(-\log Z_T)). \]

Hence it is enough to show that

\[ h^1(S_{\Gamma}, \Theta_{S_{\Gamma}}(-\log Z_{\Gamma})) + \sum_{i=1}^n (d_i - 3) \leq 0. \]

Let \( m \) be the number of blow-ups of the \((-1)\)-vertices to obtain the non-minimal graph \( \Gamma \) from \( \Gamma_2 \). Since \( h^1(S_{\Gamma_2}, \Theta_{S_{\Gamma_2}}(-\log Z_{\Gamma_2})) = \alpha + \epsilon \) by Lemma 4.4, using Proposition 3.5 it follows that

\[ h^1(S_{\Gamma}, \Theta_{S_{\Gamma}}(-\log Z_{\Gamma})) \leq \alpha + \epsilon + m. \]

Then Theorem 3.6 implies that

\[ h^1(S_{\Gamma}, \Theta_{S_{\Gamma}}(-\log Z_{\Gamma})) \leq \epsilon + m + 1. \quad (4.1) \]

Note that for any minimal graph consisting of \((-e_i)\)-vertices, the blow-up procedure (B-1) with the modification (M) lowers the sum \( \sum_{i=1}^n (d_i - 3) \) by 1, while the procedure (B-2) with (M) leaves it unchanged; cf. the proof of Wahl [33, Theorem 8.6]. Since these sums for the starting graphs \( \Gamma_A, \Gamma_B, \Gamma_C \) are equal to 1, this sum for \( \Gamma_2 \) is \(-\epsilon - 1\). Since we blow up the \((-1)\)-vertices \( m \) times to obtain \( \Gamma \) from \( \Gamma_2 \), we have

\[ \sum_{i=1}^n (d_i - 3) = -\epsilon - 1 - m. \quad (4.2) \]

From Equation (4.1) and Equation (4.2) it follows that

\[ h^1(S_{\Gamma}, \Theta_{S_{\Gamma}}(-\log Z_{\Gamma})) + \sum_{i=1}^n (d_i - 3) \leq 0, \]

concluding the proof. \( \square \)

Now we are in the position of giving the proof of Theorem 1.5 which then implies the Main Theorem of the paper.

Proof of Theorem 1.5. Suppose first that \( \Gamma \) is of the form \( \Gamma_1 \) where \( \Gamma_1 \) is a non-minimal graph in Lemma 4.1 or Lemma 4.3. If \( \Gamma_1 \) is a non-minimal graph in Lemma 4.1 then we have \( h^1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(-\log Z_{\Gamma_1})) = 0 \) by Lemma 4.4 and it is easy to see that \( \sum_{i=1}^n (d_i - 3) = 0 \). Therefore, as in the proof of Wahl [33, Theorem 8.6], we have

\[ h^1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(-\log Z_{\Gamma_1})) + \sum_{i=1}^n (d_i - 3) = 0. \]

On the other hand, if \( \Gamma_1 \) is a non-minimal graph in Lemma 4.3 then we have \( h^1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(-\log Z_{\Gamma_1})) = 1 \) by Lemma 4.3. But, since \( \sum_{i=1}^n (d_i - 3) = -1 \), we have

\[ h^1(S_{\Gamma_1}, \Theta_{S_{\Gamma_1}}(-\log Z_{\Gamma_1})) + \sum_{i=1}^n (d_i - 3) = 0. \]

Suppose now that \( \Gamma \) is not of the form \( \Gamma_1 \) of Lemma 4.1 or Lemma 4.3. Let \( \Gamma_2 \) be the non-minimal graph obtained by blowing up the \((-1)\)-vertex of \( \Gamma_1 \) once (as in Theorem 4.5). Then \( \Gamma \) is obtained by applying the blow-ups (B-1) and (B-2) to \( \Gamma_2 \) finitely many times, and finally the modification (M). The assertion of the theorem now follows from Theorem 1.5. \( \square \)
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[35] J. Wahl. *A personal communication.*

Department of Mathematics, Konkuk University, Seoul 143-701, Korea
E-mail address: HeesangPark@konkuk.ac.kr

Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea
E-mail address: dsshin@cnu.ac.kr

Rényi Institute of Mathematics, Reáltanoda utca 13-15., Budapest 1053, Hungary
E-mail address: stipsicz@renyi.hu