The equations of motion of a secularly precessing elliptical orbit

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ABSTRACT
The equations of motion of a secularly precessing ellipse are developed using time as the independent variable. The equations are useful when integrating numerically the perturbations about a reference trajectory which is subject to secular perturbations in the node, the argument of pericenter and the mean motion. Usually this is done in connection with Encke’s method to ensure minimal rectification frequency. Similar equations are already available in the literature, but they are either given based on the true anomaly as the independent variable (Kyner & Bennett 1966), or in mixed mode with respect to the time through the use of a supporting equation to track the anomaly (Escobal 1965). The equations developed here form a complete and independent set of six equations in the time. Reformulations both of Escobal’s and Kyner and Bennett’s equations are also provided which lead to a more concise form.

Key words: method: analytical – celestial mechanics

1 INTRODUCTION
In many cases one is confronted with the evaluation of the spectral characteristics of specific perturbation accelerations on the orbital motion of planetary satellites. In several instances analytical evaluations can be provided. However, this is not always possible, and occasionally one may desire verification of the analytical results. Since the analytical development of perturbations is always based on some reference orbit, it becomes of interest in numerical verification work to be able to reproduce the same reference orbit. The simplest such orbit is an invariable ellipse, and Encke’s method can then be adopted for the numerical integration of perturbations about this nominal orbit. Encke’s method can be generalized to include non-Keplerian reference trajectories. The method can then be formulated in terms of the differential equation of the reference orbit $r^*(t)$ and the equation of the perturbation of the relative position $\delta r = r(t) - r^*(t)$ of the trajectory $r(t)$. These equations are

\begin{align*}
\frac{d^2 r^*}{dt^2} &= -\frac{\mu}{r^3} r^* + s(r^*, r^*, t), \\
\frac{d^2 \delta r}{dt^2} &= -\frac{\mu}{r^3} \{f(q) r^* + [1 + f(q)] \delta r\} + g(r, \dot{r}, r^*, \dot{r}^*, t),
\end{align*}

where $\mu = G(m_1 + m_2)$ is the gravitational parameter of two masses $m_1$ and $m_2$, $s(r, \dot{r}, t)$ is the disturbing acceleration which generates the non-Keplerian nominal trajectory, while $g$ is the complementary disturbing acceleration. This is defined as the difference between the actual disturbing acceleration $p(r, \dot{r}, t)$ acting on the body whose orbit is being propagated and the disturbing acceleration of the nominal trajectory,

$$g(r, \dot{r}, r^*, \dot{r}^*, t) = p(r, \dot{r}, t) - s(r^*, \dot{r}^*, t).$$

Encke’s parameter $q$ is defined (Battin 1987) as

$$q = -\frac{(2r^* + \delta r) \cdot \delta r}{(r^* + \delta r) \cdot (r^* + \delta r)},$$

and the auxiliary function $f(q)$ has the form

\[ f(q) = \frac{1}{1 + q} - q. \]
Note that in the unperturbed case \( s = p = 0 \) Encke’s equations (1) and (2) are exactly equivalent to the Two-Body equations of motion of the two trajectories. Of course, in this case, their use is limited to non-oscillating initial conditions. In general, however, the initial conditions for the perturbed case are such that \( \delta r (t_0) = \delta \hat{r} (t_0) = 0 \). Also note that Encke’s equations are completely equivalent to the original perturbation problem stated for trajectories \( r^* (t) \) and \( r (t) \). In analytical developments the approximation is usually made, however, of evaluating the disturbing acceleration \( p \) on the nominal trajectory, that is, of replacing \( g (r, \dot{r}, r^*, \dot{r}^*, t) \) with \( g (r^*, \dot{r}^*, t) = p (r^*, \dot{r}^*, t) - s (r^*, \dot{r}^*, t) \).

Kynan & Bennett (1966) (hereafter K&B) proposed a modification to Encke’s method which replaces the invariable ellipse with a precessing ellipse that includes the first-order secular effects of the Earth’s oblateness. This has the desirable effect of limiting the frequency of rectifications necessary to contain within specified bounds the unavoidable divergence of the perturbed orbit from the reference orbit. The K&B formulation is given using the true anomaly as the independent variable. For comparison with analytical theories based on time as the independent variable, like Kaula’s linear satellite theory (Kaula 1966), the K&B approach needs to be reformulated. This has been done by Escobal (1965). More recently Lundberg et al. (2000), on the basis of previous work by Lundberg et al. (1991), have introduced an extension of Encke’s method for application to long-arc orbit determination which uses a precessing and librating ellipse of variable shape based on the Escobal model. The behaviour of the K&B and the Escobal reference orbits are similar, but the K&B approach contains some periodic terms in addition to the purely secular terms of the time-wise approach of Escobal. Comparison with Kaula’s theory requires a time-wise approach, but implementation of Escobal’s original equations is cumbersome since they require the use of a supporting equation for the true anomaly, also a feature of the K&B approach. It is then desirable to investigate the possibility of improving on the available representations of the secular precessing ellipse. In the following we will review these classical methods and provide for each an alternative formulation. Finally we will develop a novel formulation with the time as independent variable, but distinctly different from Escobal’s, which leads to a vector differential equation of the second order, whose coefficients are functions of the dynamical state variables only.

## 2 THE SECULARLY PRECESSING ELLIPSE

The secularly precessing ellipse (SPE) can be defined through the secular rates of the angular elements either as a function of the true anomaly or as a function of the mean anomaly. Here we define the SPE for both cases providing analytical expressions for the secular rates of the angular elements due to the first zonal harmonic of the gravity field of the central body.

### 2.1 True anomaly \( f \) as independent variable

Given the gravitational parameter \( \mu_0 = G (m_1 + m_2) \), let the orbital elements at epoch be \( (a_0, e_0, i_0, \Omega_0, \omega_0, M_0) \) where the associated Keplerian mean motion is \( n_0 = \left( \mu_0 / a_0^3 \right)^{1/2} \). In the presence of a second degree zonal harmonic potential \( J_2 = -C_{20} \), the secular variations of the elements can be given as a function of the true anomaly \( f \) as

\[
\begin{align*}
\alpha &= a_0, \\
\Omega (f) &= \tau (f - f_0) + \Omega_0, \\
e &= e_0, \\
\omega (f) &= \eta (f - f_0) + \omega_0, \\
i &= i_0, \\
M (t) &= n (t - t_0) + M_0,
\end{align*}
\]

where the perturbed secular mean motion is given by

\[
n = n_0 (1 - \gamma),
\]

and the constant secular rates \( \tau \) and \( \eta \) due to the first zonal harmonic are given by Sterne (1960), Kynan & Bennett (1966)

\[
\tau = \frac{d\Omega}{df} = \frac{3}{2} \frac{J_2}{(1 - e^2)^2} \left( \frac{a_e}{a} \right)^2 \cos i,
\]

\[
\eta = \frac{d\omega}{df} = \frac{3}{4} \frac{J_2}{(1 - e^2)^2} \left( \frac{a_e}{a} \right)^2 \left( 4 - 5 \sin^2 i \right),
\]

with

\[
\gamma = -\frac{3}{2} J_2 \left( \frac{a_e}{a} \right)^2 \left( \frac{a}{r_0} \right)^3 \left[ 1 - 3 \sin^2 i \sin^2 (\omega_0 + f_0) \right],
\]

where \( r_0 = a (1 - e^2) / (1 + e \cos f_0) \). Note that the phoronomic elements \( \alpha, e \) and \( i \) do not show secular behavior.

### 2.2 Mean anomaly \( M \) as independent variable

If we choose time \( t \) as the independent variable, then the secular rates of the orbital elements assume the expressions

\[
\alpha = a_0, \\
\Omega (t) = \dot{\Omega} (t - t_0) + \Omega_0,
\]

Note that in the unperturbed case \( s = p = 0 \) Encke’s equations (1) and (2) are exactly equivalent to the Two-Body equations of motion of the two trajectories. Of course, in this case, their use is limited to non-oscillating initial conditions. In general, however, the initial conditions for the perturbed case are such that \( \delta r (t_0) = \delta \hat{r} (t_0) = 0 \). Also note that Encke’s equations are completely equivalent to the original perturbation problem stated for trajectories \( r^* (t) \) and \( r (t) \). In analytical developments the approximation is usually made, however, of evaluating the disturbing acceleration \( p \) on the nominal trajectory, that is, of replacing \( g (r, \dot{r}, r^*, \dot{r}^*, t) \) with \( g (r^*, \dot{r}^*, t) = p (r^*, \dot{r}^*, t) - s (r^*, \dot{r}^*, t) \).

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Equations (6) and equations (11) provide at any time \( t \) the secular evolution of the orbit elements of the reference orbit. As noted above, the two reference orbits are not exactly the same since the true anomaly is a periodic function of time. Given the initial Keplerian state vector \((a_0, e_0, \Omega_0, \omega_0, M_0)\) at epoch \( t_0 \), and the secular rates (8) and (9) and the constant (10) in the case of true anomaly as independent variable, or the secular rates (12) and (13) and the constant (15) if the mean anomaly is adopted as the independent variable, the nominal or reference trajectory \( \bar{r}(t) \)—hereafter simply indicated with \( r(t) \) for conciseness—is completely specified at any time \( t \). The corresponding Cartesian state \( \bar{r}(t), \bar{v}(t) \) can be computed by the following procedure

(i) Compute the nominal true anomaly \( f \) by solving first the modified Kepler’s equation

\[
\tilde{n}(t-t_0) + M_0 = E - e_0 \sin E,
\]

for the eccentric anomaly \( E \) and then by transforming to \( f \) via the usual Gauss formula

\[
\tan \frac{f}{2} = \sqrt{\frac{1+e_0}{1-e_0}} \tan \frac{E}{2}.
\]

The modified mean motion \( \tilde{n} \) is either \( n \) from equation (7) or \( \bar{n} \) from equation (14), depending respectively on whether the true anomaly or the time is assumed as the independent variable.

(ii) Compute the radius vector \( r \) and the radial velocity \( v_r \) by means of

\[
\begin{align*}
 r &= \frac{a (1-e^2)}{1+e \cos f}, & v_r &= \tilde{n}ae \sin f \sqrt{1-e^2}.
\end{align*}
\]

(iii) Finally position \( r(t) \) and velocity \( v(t) \) are given by

\[
\begin{align*}
 r(t) &= R(\Omega, i, \omega + f) r_{rtn}, \\
 v(t) &= \dot{R}(\Omega, i, \omega + f) r_{rtn} + R(\Omega, i, \omega + f) \dot{r}_{rtn},
\end{align*}
\]

where

\[
 r_{rtn} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}, \quad \dot{r}_{rtn} = \begin{pmatrix} \dot{r} \\ 0 \\ 0 \end{pmatrix}, \quad \ddot{r}_{rtn} = \begin{pmatrix} \ddot{r} \\ 0 \\ 0 \end{pmatrix},
\]

the vector \( \dot{r}_{rtn} \) having been defined for later use, and the rotation matrix \( R(\Omega, i, \omega + f) \) from the orbital reference frame (the RTN frame, defined by the local radial, transverse, and normal directions) to the inertial reference frame is given by

\[
 R(\Omega, i, \omega + f) = D(\Omega) C(i) B(\omega + f),
\]

with the elementary rotations defined as

\[
 D(\Omega) = \begin{pmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
 C(i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix},
\]
where the node \( \Omega \) and the argument of pericenter \( \omega \) have been updated using equations (6) or (11).

### 2.4 Comparison of the reference orbits

The two reference trajectories defined above are clearly different. The main difference between the two formulations is due to the equation of the center, that is, to the periodic nature of the difference between the true and the mean anomaly. Fundamentally this implies that the SPE in true anomaly oscillates periodically with respect to the SPE in mean anomaly.

### 3 THE SPE IN THE TRUE ANOMALY

This Section provides the formulation of the secularly precessing ellipse with the true anomaly as the independent variable. We first review the equations of motion developed by Kyner & Bennett (1966) and then provide a more concise reformulation.

#### 3.1 The Kyner and Bennett formulation

The equations of motion can be derived easily by differentiating twice equation (19) with respect to \( t \) and taking into account equation (6). We omit the derivation and give only the equations of motion which describe the motion on SPE. The formulation is essentially the same as the original formulation of Kyner and Bennett, that is

\[
\ddot{r} + \frac{\mu_0 (1 - \gamma)}{r^3} r = \mu_0 (1 - \gamma) \frac{1 + e \cos f}{r^3} \left\{ x^2 \frac{d^2 \mathbf{D}}{dt^2} \mathbf{CB} + 2 \tau (1 + \eta) \frac{d \mathbf{D}}{dt} \mathbf{C} \frac{d \mathbf{B}}{du} - (\eta^2 + 2 \eta) \mathbf{R} \right\} r_{\text{true}},
\]

(26)

the only differences being that we have factored \((1 - \gamma)^2\) on the right hand side and summed the Newtonian term with the last term on the right hand side of the original K&B equation (12). Note that here we have introduced the argument of latitude \( u = \omega + f \). The true anomaly appears explicitly in the equations of motion (26) through the rotation matrices \( \mathbf{B} \) and \( \mathbf{D} \) (equations 25 and 23 respectively). It is thus necessary to compute \( f \) either from the analytic procedure outlined previously, or by numerically integrating the auxiliary equation

\[
\dot{f} = \frac{n_0 (1 - \gamma)}{(1 - e^2)^{3/2}} \frac{1 + e \cos f}{a^2}.
\]

(27)

simultaneously with (26). Note that although the perturbed mean motion \( n \) appears, this equation is independent of the radius vector \( r \). The term \( 1 + e \cos f \) in equations (26) cannot be replaced with its Keplerian equivalent \( a / r \) unless the radius vector appearing at the denominator be computed by its Keplerian definition—which means going back to using the true anomaly again, or using the result from the integration of (27).

#### 3.2 A reformulation of the Kyner and Bennett approach

In this section we provide the derivation of an alternative form of the equations of motion (26). Starting from the kinematic representation (19) of the SPE we take the second derivative with respect to the time to obtain

\[
\frac{d^2 r}{dt^2} = \left( \frac{df}{dt} \right)^2 \frac{d^2 R}{dt^2} r_{\text{true}} + \frac{dR}{df} \left[ \frac{d^2 f}{dt^2} r_{\text{true}} + 2 \frac{df}{dt} \frac{dR}{df} r_{\text{true}} \right] + \dot{R} r_{\text{true}}.
\]

(28)

In keeping with the Kyner & Bennett approach (equations 6 and 8 - 10), every term is now reshaped into a form based on the true anomaly.

For the scalar components \( \dot{r} \) and \( \ddot{r} \) we need the first and second time derivatives of both the radius vector and the true anomaly. From (27) we immediately find

\[
\dot{f} = n \sqrt{1 - e^2} \frac{a^2}{r^2}.
\]

(29)

with \( n \) given by (7), and subsequently

\[
\ddot{f} = -2 \mu \frac{e \sin f}{r^3},
\]

(30)

where the perturbed gravitational parameter \( \mu \) is given by

\[
\mu = n_0^2 (1 - \gamma)^2 a^3.
\]

(31)

From the second of equations (18) follows that
\[
\frac{dr}{dt} = \frac{na}{\sqrt{1 - e^2}} e \sin f,
\]  
(32)

which can be differentiated to yield
\[
\frac{d^2r}{dt^2} = \frac{\mu}{r^2} e \cos f.
\]  
(33)

For the matrix terms we need the derivatives of the rotation matrix \( R \) with respect to true anomaly \( f \). These are easily computed from (22) and with the help of (6) they can be written as

\[
\frac{dR}{df} = \left[ \tau V + (1 + \eta) H \right] R,
\]
(38)

\[
\frac{d^2R}{df^2} = \left[ \tau^2 K + 2\tau (1 + \eta) N - (1 + \eta)^2 I \right] R,
\]
(39)

and

\[
(D_f R)_{r_{\text{in}}} = \left[ \tau V + (1 + \eta) H \right] r,
\]
(40)

\[
(D_2^f R)_{r_{\text{in}}} = \left[ \tau^2 K + 2\tau (1 + \eta) N - (1 + \eta)^2 \right] r.
\]
(41)

Substituting equations (34), (35), (38), (39), (40) and (41) into (28) we find
\[
\ddot{r} = n^2 (1 - e^2) \frac{a^2}{r^2} \left[ \tau^2 K + 2\tau (1 + \eta) N - (1 + \eta)^2 \right] r + \frac{\mu}{r^2} e \cos f r.
\]
(42)

Notice that in performing the substitutions all traces of the first derivative \( D_f R \) have been lost. Adding and subtracting the term \((\mu/r^2) r\), with \(\mu\) from (31), on the right hand side yields
\[
\ddot{r} + \frac{\mu}{r^2} r = n^2 (1 - e^2) \frac{a^2}{r^2} \left[ 1 - (1 + \eta)^2 \right] I + \tau^2 K + 2\tau (1 + \eta) N \right] r.
\]
(43)

This equation can now be put in the compact form
\[
\ddot{r} + \frac{\mu}{r^2} r = \frac{\mu p}{r^2} S r,
\]
(44)

where \( p = a (1 - e^2) \) is the orbital semi-latus rectum, or parameter of the ellipse, and the matrix \( S \) has been defined as
\[
S = \begin{pmatrix} A & 0 & B \sin \Omega \\ 0 & A & B \cos \Omega \\ 0 & 0 & C \end{pmatrix},
\]
(45)

with the constants
\[
A = 1 - (1 + \eta)^2 - \tau \left\{ \tau + 2 (1 + \eta) \cos i \right\},
\]
(46)

\[
B = 2\tau (1 + \eta) \sin i,
\]
(47)

\[
C = 1 - (1 + \eta)^2.
\]
(48)

In component form, with \( r = (x, y, z)^T \), the reformulated equations of motion of Kyner & Bennett read
\[
\ddot{x} + \frac{\mu}{r^3} x = \frac{\mu p}{r^2} \left\{ 1 - (1 + \eta)^2 - \tau \left\{ \tau + 2 (1 + \eta) \cos i \right\} \right\} x + 2\tau (1 + \eta) \sin i \sin \Omega \sin \Omega \quad z,
\]
\[
\ddot{y} + \frac{\mu}{r^3} y = \frac{\mu p}{r^2} \left\{ 1 - (1 + \eta)^2 - \tau \left\{ \tau + 2 (1 + \eta) \cos i \right\} \right\} y + 2\tau (1 + \eta) \sin i \cos \Omega \sin \Omega \quad z,
\]
\[
\ddot{z} + \frac{\mu}{r^3} z = \frac{\mu p}{r^2} \left\{ 1 - (1 + \eta)^2 \right\} z.
\]
(49)

Equations (41), although highly simplified with respect to equations (26) —note the absence of matrix multiplications— still need to be supported by equation (47), since true anomaly appears implicitly through the node \( \Omega \) (see equation (6)).
4 THE SPE IN THE MEAN ANOMALY

This Section provides the formulation of the secularly precessing ellipse with the time, or the mean anomaly, as the independent variable. We review the model due to Escobal (1965) and develop alternative formulations following essentially the same original line of development, but which are more concise for implementation.

4.1 The Escobal formulation

The procedure to obtain the equations of motion is well described in Escobal (1965). Here we give only the final result and the necessary auxiliary equation in case it is desired to fully integrate the equations of motion as a quick alternative to following the analytical update of the eccentric anomaly as envisioned in Escobal (1965). The equations of motion are derived from the decomposition of the radius vector

\[ \mathbf{r} = XP + YQ, \]  

in the Laplace reference frame, where \( P \) is the Hermann-Jacobi-Laplace unit vector and \( Q \) is a unit vector normal to \( P \) in the direction of increasing anomaly. The unit vectors \( P \) and \( Q \) are themselves defined as

\[
\begin{align*}
P_x &= \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i, \\
Q_x &= -\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i, \\
P_y &= \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i, \\
Q_y &= -\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i, \\
P_z &= \sin \omega \sin i, \\
Q_z &= \cos \omega \sin i.
\end{align*}
\]

The acceleration is then

\[
\ddot{r} + \frac{\ddot{\bar{r}}}{\bar{r}} \bar{r} = 2X\dot{P} + 2Y\dot{Q} + X\ddot{P} + Y\ddot{Q},
\]

where the modified gravitational parameter \( \bar{\mu} \) is given by

\[ \bar{\mu} = \bar{\bar{\mu}} a^3, \]

and the first and second derivatives of \( P = (P_x, P_y, P_z)^T \) and \( Q = (Q_x, Q_y, Q_z)^T \) are given by

\[
\begin{align*}
\dot{P}_x &= -\dot{\Omega}P_y + \omega Q_z, \\
\dot{Q}_x &= -\dot{\bar{\Omega}}Q_y - \bar{\omega}P_z, \\
\dot{P}_y &= \dot{\Omega}P_x + \omega Q_y, \\
\dot{Q}_y &= \bar{\Omega}Q_x - \dot{\bar{\omega}}P_y, \\
\dot{P}_z &= \omega Q_z, \\
\dot{Q}_z &= -\dot{\bar{\omega}}P_z,
\end{align*}
\]

and

\[
\begin{align*}
\ddot{P}_x &= -\left(\dot{\Omega}^2 + \dot{\omega}^2\right)P_x - 2\dot{\Omega}\dot{\omega}Q_y, \\
\ddot{Q}_x &= -\left(\dot{\bar{\Omega}}^2 + \dot{\bar{\omega}}^2\right)Q_x + 2\dot{\bar{\omega}}P_y, \\
\ddot{P}_y &= -\left(\dot{\Omega}^2 + \dot{\omega}^2\right)P_y + 2\dot{\Omega}\dot{\omega}Q_x, \\
\ddot{Q}_y &= -\left(\dot{\bar{\Omega}}^2 + \dot{\bar{\omega}}^2\right)Q_y - 2\dot{\bar{\omega}}P_z, \\
\ddot{P}_z &= -\dot{\omega}^2P_z, \\
\ddot{Q}_z &= -\dot{\bar{\omega}}^2Q_z.
\end{align*}
\]

Finally, \( X, \dot{X}, Y, \dot{Y} \) and \( \dot{Y} \) are defined either in terms of the eccentric anomaly \( E \) or in terms of the true anomaly \( f \) as

\[
\begin{align*}
X &= a (\cos E - e) = r \cos f, \\
\dot{X} &= \frac{\bar{\bar{n}}a^2}{r} \sin E = -\frac{\bar{\bar{n}}a}{\sqrt{1 - e^2}} \sin f, \\
Y &= a \sqrt{1 - e^2} \sin E = r \sin f, \\
\dot{Y} &= \frac{\bar{\bar{n}}a^2}{r} \sqrt{1 - e^2} \cos E = \frac{\bar{\bar{n}}a (e + \cos f)}{\sqrt{1 - e^2}},
\end{align*}
\]

where the perturbed mean motion \( \bar{n} \) is given by equation (14). As in the true anomaly-based Kyner and Bennett formulation the number of first order differential equation to integrate is effectively seven, that is, six equations for the state vector and one equation (uncoupled with the state) for eccentric anomaly

\[ \frac{dE}{dt} = \frac{\bar{n}}{1 - e \cos E}, \]

or the true anomaly

\[ \frac{df}{dt} = \frac{\bar{n}}{(1 - e^2)^{3/2}} \left(1 + e \cos f\right)^2. \]

Note that the equation for the anomaly is not redundant, but provides necessary information for the integration of the differential equations (54).

4.2 A reformulation of the Escobal approach

A simpler form of Escobal's equations of motion can be obtained by resolving Eq. (52) in inertial axes and substituting in it Eqs. (51), (55), (51) and (56) for the true anomaly to obtain

\[ \ddot{r} + \frac{\ddot{\bar{r}}}{\bar{r}} \bar{r} = -eA - \{B \dot{r} + A \cos f - \{C \dot{r} + D \} \sin f, \]

\( \ddot{\bar{r}} \)
where the components of the vectors $A$, $B$, $C$ and $D$ are

$$
A_x = \frac{2n a}{\sqrt{1 - e^2}} (\dot{\omega} P_x + \dot{\Omega} Q_y),
B_x = (\dot{\Omega}^2 + \dot{\omega}^2) P_x + 2\dot{\omega} \Omega Q_y,
A_y = \frac{2n a}{\sqrt{1 - e^2}} (\dot{\omega} P_y - \dot{\Omega} Q_x),
B_y = (\dot{\Omega}^2 + \dot{\omega}^2) P_y - 2\dot{\omega} \Omega Q_x,
A_z = \frac{2n a}{\sqrt{1 - e^2}} \dot{\omega} P_z,
B_z = \dot{\omega}^2 P_z,
C_x = -2\dot{\Omega} \dot{\omega} P_x + (\dot{\Omega}^2 + \dot{\omega}^2) Q_x,
D_x = \frac{2n a}{\sqrt{1 - e^2}} [-\dot{\Omega} P_y + \dot{\omega} Q_x],
C_y = 2\dot{\Omega} \dot{\omega} P_y + (\dot{\Omega}^2 + \dot{\omega}^2) Q_y,
D_y = \frac{2n a}{\sqrt{1 - e^2}} [\dot{\Omega} P_x + \dot{\omega} Q_y],
C_z = \dot{\omega}^2 Q_z,
D_z = \frac{2n a}{\sqrt{1 - e^2}} \dot{\Omega} \dot{\omega} Q_z.
$$

Equations (60) are written explicitly in the true anomaly, and this requires that for their numerical integration they be supplemented with equation (55).

Alternatively, one can choose to work with the eccentric anomaly, in which case the equations of motion read

$$
\ddot{f} + \frac{\ddot{\mu}}{r} r = e B' - \left\{ \frac{A'}{r} + B' \right\} \cos E - \left\{ C' + D' \frac{1}{r} \right\} \sin E,
$$

and the components of the several vector coefficient are

$$
A' = a (1 - e^2) A, \\
B' = a B, \\
C' = a \sqrt{1 - e^2} C, \\
D' = a \sqrt{1 - e^2} D.
$$

5 A NEW FORMULATION OF THE SPE IN THE MEAN ANOMALY $M$

In this section we give the detailed procedure we have followed to obtain a new form of the equations of motion representing the SPE in the mean anomaly. Starting from the kinematic representation of SPE (19) and taking the second derivative with respect to the time we have

$$
\frac{d^2 r}{dt^2} = R \frac{d^2 r_{\text{ret}}}{dt^2} + 2 \frac{dR}{dt} \frac{dr_{\text{ret}}}{dt} + \frac{d^2 R}{dt^2} r_{\text{ret}},
$$

which are to be used in connection with equations (12), (18), (19) and (11) that give the secular rates, the constant $\tilde{\gamma}$ and the evolution of the orbital elements with respect to the mean anomaly.

The time derivatives of the rotation matrix $R$ can be easily computed from (22) recalling (11). As in the case of the K&B reformulation of Section 3.2, the strategy is to define appropriate operators $D_k = d/dt$ and $D_k^2 = d^2/dt^2$ acting on the rotation matrix $R$. In Appendix B it is shown that these operators can be expressed as

$$
D_t = (\dot{\omega} + \dot{f}) \mathbf{H} + \dot{\Omega} \mathbf{V},
D_t^2 = (\dot{\omega} + \dot{f})^2 \mathbf{H}^2 + 2\dot{\Omega} (\dot{\omega} + \dot{f}) \mathbf{N} + \dot{\Omega}^2 \mathbf{K} + \dot{f} \mathbf{H}.
$$

If we now recall that the use of the time as independent variable implies from the second of (15) that

$$
\dot{e} = \frac{\ddot{\mu}}{r^2} e \sin f,
\dot{\mu} = \frac{\ddot{e}}{r^2} e \cos f,
$$

with $\ddot{\mu}$ given by (53), we can write

$$
\frac{dr_{\text{ret}}}{dt} = \ddot{\mu} e \cos f, \\
\frac{d^2 r_{\text{ret}}}{dt^2} = \ddot{\mu} e \cos f,
$$

and substitute (64), (65), (68) and (69) into (63), we get the equations of motion in the form

$$
\ddot{f} = \ddot{\mu} \frac{e \cos f}{r^3} r + \frac{\ddot{\mu}}{r} e \sin f - \frac{\ddot{\mu}}{r} e \sin f + e \left( \ddot{\omega} + \ddot{f} \right) \mathbf{H} + \dot{\Omega} \mathbf{V} + (\dot{\omega} + \dot{f}) \mathbf{H}^2 + 2\dot{\Omega} (\dot{\omega} + \dot{f}) \mathbf{N} + \dot{\Omega}^2 \mathbf{K} + \dot{f} \mathbf{H}.
$$

We now need expressions for the first and the second time derivatives of the true anomaly. These can be obtained from (58) and put in the form

$$
\dot{f} = \bar{n} \sqrt{1 - e^2} \frac{\ddot{\mu}}{r^2},
\ddot{f} = 2 \frac{n e}{r} \ddot{\mu} \sin f.
$$
We thus easily see that in the term with \( \dot{f}H \) as a factor cancels with the term \( f\dot{H} \). If at the same time we add the modified Keplerian term \( \mu \dot{r}r^3 \) to both the left and right hand sides, the equations of motion become

\[
\ddot{r} + \frac{\mu}{r^3}r = \frac{1 + e \cos f}{r^3}r + 2\frac{\hat{n}ae \sin f}{r\sqrt{1 - e^2}}(\dot{\omega}H + \dot{\Omega}V) r + \left[ (\dot{\omega} + \dot{f})^2 \dot{H}^2 + 2(\dot{\omega} + \dot{f})\dot{H}N + \dot{\Omega}^2 K \right] r. \tag{73}
\]

Considering that, on the basis of the relationship \( \text{[B17]} \), the action of \( H^2 \) on \( r \) is simply to reverse its sign,

\[
H^2 r = \mathbf{H}^2 \mathbf{r}_{\text{rtn}} = \mathbf{R} \mathbf{K} \mathbf{r}_{\text{rtn}} = -\mathbf{R}_r = -r, \tag{74}
\]

we can write

\[
(\dot{f})^2 H^2 r = -\frac{\mu}{r^3}r = -\frac{1 + e \cos f}{r^3}r, \tag{75}
\]

and thus eliminate the first term on the right hand side of \( \text{[B3]} \) with the term proportional to \((\dot{f})^2\). We thus obtain

\[
\ddot{r} + \frac{\mu}{r^3}r = 2\frac{\hat{n}ae \sin f}{r\sqrt{1 - e^2}}(\dot{\omega}H + \dot{\Omega}V) r + \left[ 2\dot{\Omega}(\dot{\omega} + \dot{f})N + \dot{\Omega}^2 K \right] r - (\dot{\omega}^2 + 2\dot{\omega})r. \tag{76}
\]

Collecting the terms depending on \( \dot{f} \) and substituting \( \dot{f} \) from \( \text{[B1]} \) and eliminating \( \hat{n} \) in favor of the modified area integral \( \dot{\hat{h}} = \hat{n}a^2 \sqrt{1 - e^2} \),

\[
\ddot{r} + \frac{\mu}{r^3}r = \left( E + \frac{\hat{e}}{pr} \sin f F + 2\frac{\dot{\hat{h}}}{r^2} G \right) r. \tag{77}
\]

If we now define

\[
E = \dot{\Omega}^2 K + 2\dot{\Omega}\omega N - \dot{\omega}^2, \tag{78}
\]

\[
F = \dot{\omega}H + \dot{\Omega}V, \tag{79}
\]

\[
G = \dot{\Omega}N - \dot{\omega}A, \tag{80}
\]

the equations of motion assume the more compact form

\[
\ddot{r} + \frac{\mu}{r^3}r = \left( E + \frac{\hat{e}}{pr} \sin f F + 2\frac{\dot{\hat{h}}}{r^2} G \right) r. \tag{81}
\]

The matrices \( E \) and \( G \) are upper triangular, while matrix \( F \) is antisymmetric. These three matrices are time-dependent through the longitude of the node, and are given explicitly as

\[
E(t) = \begin{pmatrix}
- (\dot{\Omega}^2 + 2\dot{\Omega}\omega \cos i + \dot{\omega}^2) & 0 & 2\dot{\Omega}\omega \sin i \sin \Omega(t) \\
0 & - (\dot{\Omega}^2 + 2\dot{\Omega}\omega \cos i + \dot{\omega}^2) & -2\dot{\Omega}\omega \sin i \cos \Omega(t) \\
0 & 0 & -\dot{\omega}^2
\end{pmatrix}, \tag{82}
\]

\[
F(t) = \begin{pmatrix}
\dot{\Omega} + \dot{\omega} \cos i & -\dot{\omega} \sin i \cos \Omega(t) \\
\dot{\omega} \sin i \cos \Omega(t) & -\dot{\omega} \sin i \sin \Omega(t) \\
\dot{\omega} \sin i \sin \Omega(t) & 0
\end{pmatrix} = -F^T(t), \tag{83}
\]

\[
G(t) = \begin{pmatrix}
- (\dot{\Omega} \cos i + \dot{\omega}) & 0 & \dot{\Omega} \sin i \cos \Omega(t) \\
0 & \dot{\Omega} \cos i + \dot{\omega} & -\dot{\Omega} \sin i \sin \Omega(t) \\
0 & 0 & -\dot{\omega}
\end{pmatrix}. \tag{84}
\]

It appears odd that the longitude of the pericenter does not appear explicitly in the formulation. However, we need to recall that the explicit dependence on time is restricted to the position of the orbital plane, which depends on the inclination \( i \) and the node \( \Omega \), only the latter of which is variable in our present model.

The final and crucial step is to recognize from \( \text{[B4]} \) that

\[
\frac{dr}{dt} = \frac{\hat{e}}{p} \sin f = \dot{r} \cdot \hat{r}, \tag{85}
\]

so that Eq. \( \text{[B1]} \) can be reduced to the form

\[
\ddot{r} + \frac{\mu}{r^3}r = \left[ E + \frac{2}{r} (\dot{\hat{r}} \cdot \hat{r}) F + 2\frac{\dot{\hat{h}}}{r^2} G \right] r. \tag{86}
\]

The right-hand side of this equation is the explicit form of the disturbing acceleration \( s(r, \hat{r}, t) \) appearing in equation \( \text{[1]} \). In component form, with \( r = (x, y, z)^T \), equation \( \text{[B4]} \) reads

\[
\dot{x} = - (\dot{\Omega}^2 + 2\dot{\Omega}\omega \cos i + \dot{\omega}^2) x + (2\dot{\Omega}\omega \sin i \sin \Omega) z + 2 \frac{x\dot{x} + y\dot{y} + z\dot{z}}{r^2} - (\dot{\Omega} \cos i + \dot{\omega}) y - (\dot{\omega} \sin i \cos \Omega) z \]

\[\tag{87}\]

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The Kyner and Bennett and the Escobal equations of motion defining the intermediary orbit as an ellipse subject to secular motion of its angular elements have been reviewed. In both cases a more compact formulation has been developed and presented, which is better suited for implementation. Escobal’s equations have also been reformulated and extended to allow the use of either the true or the eccentric anomaly, in addition to the mean anomaly. In fact, both these classical formulations are ‘redundant,’ in that they include one more differential equation than the minimum of six associated with the three degrees of freedom of the problem. The reason for this extra equation is due to the particular choice of variables, which makes it necessary to propagate the perturbed anomaly, be it the eccentric or the true anomaly, along with the dynamical variables.

The main contribution of the present work is the development of a novel formulation of the equations of motion of the secularly precessing ellipse that uses the time as the independent variable and requires no additional equation to account for the evolution of the anomaly. The final equation has the very compact form of a Two-Body equation of motion perturbed by a time-dependent acceleration containing three terms of degree 0 and ±1 in the radius vector. The explicit dependence on the time is only due to the presence of the longitude of the node in the forcing terms. It should be noted that the supporting equation for the anomaly cannot be simplified away for true anomaly-based theories like Kyner and Bennett’s.

The time-wise approach developed here can be used when it is desired to numerically verify the analytical propagation of perturbations based on Kaula-type linear theories, where the nominal trajectory is a secularly precessing orbit. In particular, it can be used to verify the perturbation spectrum. It can also be used when analyzing first-order perturbation effects on orbital arcs, or ephemerides, estimated from observational data. In that case the secular rates of the angular elements to be used in the present, novel formulation can be estimated very accurately by numerically fitting the estimated orbital ephemeris. Clearly, since they are based on observational data, these rates include the effects of all acting secular perturbations. When applying the present formulation to evaluate the effects, for instance, of neglected sources of perturbations along the given orbit, it is therefore necessary to exclude all terms generating secular perturbations from the forcing acceleration \( \mathbf{g} (\mathbf{r}, \dot{\mathbf{r}}, t) \) appearing on the right-hand side of equation (87). In this particular application, if Encke’s approach is adopted, no rectification is then needed, since the formulation is guaranteed not to generate any secular drift between the perturbed and the reference orbits. Applications of this novel formulation to the case of tidal perturbations will be the subject of a future contribution.

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APPENDIX A: THE OPERATORS $\mathcal{D}_\theta$ AND $\mathcal{D}_{\theta}^2$

The operators $\mathcal{D}_\theta$ and $\mathcal{D}_{\theta}^2$ were defined in Section 3.2 in terms of the matrices $V$, $H$, $K$ and $N$. Comparison of equations (34) and (35) with equations (36) and (37) makes it clear that, since $R$ is orthogonal, these matrices are defined as

\begin{align*}
V &= \frac{dD}{d\Omega}CBR^T, \\
H &= DC\frac{dB}{du}R^T, \\
K &= \frac{d^2D}{d\Omega^2}CBR^T, \\
N &= \frac{dD}{d\Omega}C\frac{dB}{du}R^T.
\end{align*}

(A1)

Now since $R^T = B^T C^T D^T$, it is straightforward to show that

\begin{align*}
V &= \frac{dD}{d\Omega}D^T = \begin{pmatrix} 0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}, \\
H &= DC\frac{dB}{du}B^T C^T D^T = \begin{pmatrix} 0 & -\cos i & -\sin i \cos \Omega \\
\cos i & 0 & -\sin i \sin \Omega \\
\sin i \cos \Omega & \sin i \sin \Omega & 0 \end{pmatrix}, \\
K &= \frac{d^2D}{d\Omega^2}D^T = \begin{pmatrix} -1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0 \end{pmatrix}, \\
N &= \frac{dD}{d\Omega}C\frac{dB}{du}B^T R^T = \begin{pmatrix} -\cos i & -\sin i \cos \Omega \\
0 & -\cos i & -\sin i \sin \Omega \\
0 & 0 & 0 \end{pmatrix},
\end{align*}

(A5)

(A6)

(A7)

(A8)

where we have repeatedly used the fact that

\[\frac{d\mathcal{B}}{du} = \begin{pmatrix} 0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}.
\]

(A9)

This equation, like (A5), shows a familiar property of rotation matrices (cf. Pars (1981)).

It remains to be shown that the action of the last operator $DCB_{uu}$ on the right hand side of equation (35) on $r_{\text{rtn}}$ is equivalent to multiplication by $-R$. This follows immediately once it is realized that

\[\frac{d^2B}{du^2} = S - B,
\]

(A10)

with

\[S = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \end{pmatrix}.
\]

(A11)

In fact, since $Sr_{\text{rtn}} = 0$, we have

\[DC\frac{d^2B}{du^2}r_{\text{rtn}} = DCr_{\text{rtn}} - Rr_{\text{rtn}} = -Rr_{\text{rtn}} = -r.
\]

(A12)

The full operator, analogous to $V$, $H$, etc., of course is defined as

\[DC\frac{d^2B}{du^2}R^T = DCSR^T - I,
\]

(A13)

but for the present purposes it is expedient to use its restriction to vectors having a null third component, which justifies our definition of the operator in equation (37).

APPENDIX B: THE OPERATORS $\mathcal{D}_\tau$ AND $\mathcal{D}_{\tau}^2$

For the first derivatives of rotation matrix $R$ (22) with respect to time we have

\[\frac{dR}{dt} = \frac{dD}{dt}CB + DC\frac{dB}{dt}.
\]

(B1)

The first term to the left hand side can be rewritten as
\[
\frac{d}{dt} \mathbf{C} \mathbf{B} = \hat{\Omega} \frac{d}{dt} \mathbf{C}^T \mathbf{D} \mathbf{C} \mathbf{B} = \hat{\Omega} \mathbf{V} \mathbf{R},
\]

where \( \mathbf{V} \) is the matrix already introduced in Eq. (A3).

The second term of (B1) can be reformulated as

\[
\mathbf{D} \frac{d}{dt} \mathbf{B} = \mathbf{u} \mathbf{D} \frac{d}{du} \mathbf{B} = (\dot{\omega} + \dot{f}) \mathbf{H} \mathbf{R},
\]

where \( \mathbf{H} \) is given by Eq. (A6). Collecting terms we have that the first derivatives of \( R \) is

\[
\frac{d}{dt} \mathbf{R} = \left[ (\dot{\omega} + \dot{f}) \mathbf{H} + \hat{\Omega} \mathbf{V} \right] \mathbf{R}.
\]

The second derivatives of \( \mathbf{R} \) can be computed from previous equation that is

\[
\frac{d^2}{dt^2} \mathbf{R} = \frac{d}{dt} \left[ (\dot{\omega} + \dot{f}) \mathbf{H} + \hat{\Omega} \mathbf{V} \right] \mathbf{R} + \left[ (\dot{\omega} + \dot{f}) \mathbf{H} + \hat{\Omega} \mathbf{V} \right]^2 \mathbf{R}.
\]

The first term of right hand side can be computed as

\[
\frac{d}{dt} \left[ (\dot{\omega} + \dot{f}) \mathbf{H} + \hat{\Omega} \mathbf{V} \right] \mathbf{R} = \ddot{f} \mathbf{H} + \hat{\Omega} (\dot{\omega} + \dot{f}) \mathbf{T},
\]

where we have substituted for

\[
\dot{\mathbf{H}} = \hat{\Omega} \mathbf{T},
\]

with

\[
\mathbf{T} = \begin{pmatrix} 0 & 0 & \sin \theta \sin \phi \\ 0 & 0 & -\sin \theta \cos \phi \\ -\sin \theta \sin \phi & \sin \theta \cos \phi & 0 \end{pmatrix}.
\]

Substituting and collecting terms in equation (B5) yields

\[
\frac{d^2}{dt^2} \mathbf{R} = \left[ (\dot{\omega} + \dot{f})^2 \mathbf{H}^2 + 2\hat{\Omega}(\dot{\omega} + \dot{f}) \mathbf{N} + \hat{\Omega}^2 \mathbf{V}^2 + \ddot{f} \mathbf{H} \right] \mathbf{R},
\]

where we have used

\[
\mathbf{Q} = \mathbf{H} \mathbf{V} + \mathbf{V} \mathbf{H} = \begin{pmatrix} -2 \cos \theta & 0 & \sin \theta \sin \phi \\ 0 & -2 \cos \phi & -\sin \phi \cos \phi \\ \sin \theta \sin \phi & \sin \phi \cos \phi & 0 \end{pmatrix},
\]

and

\[
\mathbf{Q} + \mathbf{T} = 2 \mathbf{N},
\]

with

\[
\mathbf{N} = \begin{pmatrix} -\cos \theta & 0 & \sin \theta \sin \phi \\ 0 & -\cos \phi & -\sin \phi \cos \phi \\ 0 & 0 & 0 \end{pmatrix}.
\]

It is now possible to define the operators \( \mathcal{D}_t \) and \( \mathcal{D}_t^2 \) as

\[
\mathcal{D}_t = (\dot{\omega} + \dot{f}) \mathbf{H} + \hat{\Omega} \mathbf{V},
\]

\[
\mathcal{D}_t^2 = (\dot{\omega} + \dot{f})^2 \mathbf{H}^2 + 2\hat{\Omega}(\dot{\omega} + \dot{f}) \mathbf{N} + \hat{\Omega}^2 \mathbf{V}^2 + \ddot{f} \mathbf{H},
\]

where we have used the identity

\[
\mathbf{V}^2 = \mathbf{K}.
\]

Note that these operators are intended for application to a position vector only. A further relationship is useful in connection with the operator \( \mathbf{H}^2 \). If we insert the identity \( \mathbf{B} \mathbf{B}^T \) after the factor \( \mathbf{C} \) in equation (A2) and note that \( \mathbf{B}^T (d\mathbf{B}/du) = \mathbf{V} \), it follows that

\[
\mathbf{H} = \mathbf{R} \mathbf{V} \mathbf{R}^T.
\]

Then, taking the square of both sides and multiplying by \( \mathbf{R} \) on the right we find

\[
\mathbf{H}^2 \mathbf{R} = \mathbf{R} \mathbf{V}^2 \mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{K}.
\]

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