WZW branes and gerbes

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Abstract

We reconsider the role that bundle gerbes play in the formulation of the WZW model on closed and open surfaces. In particular, we show how an analysis of bundle gerbes on groups covered by $SU(N)$ permits to determine the spectrum of symmetric branes in the boundary version of the WZW model with such groups as the target. We also describe a simple relation between the open string amplitudes in the WZW models based on simply connected groups and in their simple-current orbifolds.

1 Introduction

The WZW (Wess-Zumino-Witten) model [43], a version of a two-dimension sigma model with a group manifold $G$ as the target, constitutes an important laboratory for conformal field theory (CFT). It is a source of numerous rational models of CFT [34] and a building block of certain string vacua [8]. It is also closely connected to the topological 3-dimensional Chern-Simons gauge theory [14] [15]. It has been clear from the very start that the model involves topological effects of a new type which are due to the presence of the topological Wess-Zumino term in its action functional. The topological intricacies of the model appear already at the classical level as global obstructions in the definition of the action functional. Those obstructions lead to the quantization of the coupling constant (the level) of the model. The phenomenon is similar to the Dirac quantization of the magnetic monopole charge but in the loop space rather than in the physical space. In the physical space, it involves closed 3-forms instead of magnetic field 2-forms. Quite simple for simply connected groups $G$, this effect becomes more

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As is well known, a convenient mathematical framework for the Dirac monopoles, their quantization and the Bohm-Aharonov effect is provided by the theory of line bundles with hermitian connections. Up to isomorphism, such bundles may be characterized by certain sheaf cohomology classes. More exactly, they correspond to the elements of the real version of the degree 2 Deligne cohomology \[15,22\]. It was realized in \[23\] that the Deligne cohomology in degree 3 provides a mathematical language to treat the topological intricacies in the WZW model. The theory is somewhat analogous to the degree 2 case when the original space is replaced by its loop space. Indeed, a third degree real Deligne class determines a (unique up to isomorphism) hermitian line bundle with connection on the loop space \[23\]. The degree 3 theory appears, however, to be much richer. In particular, one of the basic constructions in degree 2, that of the parallel transport along curves, becomes that of the “parallel transport” around two-dimensional surfaces which may have different topology. For closed surfaces one obtains the \(U(1)\)-valued “holonomies” that enter the Feynman amplitudes of classical field configurations in the WZW model. For surfaces with boundary, the amplitudes take instead values in the product of lines associated to the boundary loops. New phenomena appear when the boundary components or their pieces are restricted to special submanifolds (D-branes) over which the Deligne cohomology class trivializes. The discussion in \[23\] extends easily to that case as was briefly evoked in \[24\]. This is precisely the situation that one confronts when studying boundary conditions in the WZW models that preserve (half of the) symmetries of the bulk theory. One of the main points of this paper is to show how the order 3 Deligne classes enter the classification of such boundary conditions, i.e. of the WZW branes.

Although the whole discussion may be made using the cohomological language, it is convenient to have at ones disposal geometric objects whose isomorphism classes are characterized by the degree 3 real Deligne cohomology classes. This was recognized in ref. \[6\] which proposed to use the theory of “gerbes” \[27\] to provide for such objects. It seems, that the most appropriate geometric notions are those of (hermitian) bundle gerbes with connection defined in \[5\] and of their stable isomorphisms introduced in \[30\]. The bundle gerbes with connection are simple geometric objects whose stable isomorphism classes are exactly described by the order 3 real Deligne cohomology. Their use allows to translate the cohomological discussions of \[23\] to a more geometric language which is indeed useful when discussing the issues related to branes. See also \[10,4,28\] for the discussions of gerbes in different, although related, contexts.

The paper is organized as follows. In Sect. 2 we recall the essential points of \[23\], with some of the details relegated to Sect. 10.1, and discuss their translation to the bundle-gerbe language, In Sect. 3 we present an explicit construction of gerbes over the \(SU(N)\) groups. Sect. 4 is devoted to the case of non-simply connected groups covered by \(SU(N)\). In parenthetical Sect. 5, we explain how to define gerbes on discrete quotient spaces, an issue which was previously discussed in the context of discrete torsion in \[39\]. How the construction from Sect. 4 fits into this general scheme is shown in Appendix B. In Sect. 6, we describe the line bundles with connection on the loop spaces induced by gerbes and relevant for the geometric description of closed string amplitudes. Sect. 7

subtle for non-simply connected ones leading to more involved selection rules for the level and, possibly, multiple (theta-)vacua of the quantum theory \[17\].
shows how those line bundles may be trivialized when restricted to loop spaces of branes and how to describe the brane structure in terms of gerbes. We also discuss the line bundles induced by gerbes on the space of paths with ends on branes, the open string counterpart of the loop space construction. In Sect. 8.1, we examine branes in the $SU(N)$ groups and in Sect. 8.2 the ones in the groups covered by $SU(N)$. In the latter case, we obtain a completely explicit description of the (symmetric) branes confirming the results based on studying consistency of quantum amplitudes. In Sect. 9, we evoke the bearing that the geometric constructions discussed in this paper have on the spectrum and the boundary partition functions of the WZW models based on the groups covered by $SU(N)$. We identify a general relation, that seems at least partially new in the context of the WZW theory, between the spaces of states for the boundary WZW models with non-simply connected groups and the ones for the models based on the covering groups. Sect. 10 briefly indicates how to extend this relation to general open string quantum amplitudes. In Section 11, we present a local description of the line bundles over the loop spaces and open path spaces induced by gerbes, discussed before in more abstract terms. Conclusions give a brief summary of what was achieved in the paper and list some open problems. More technical calculations referred to in the main text have been collected in Appendices.

2 Topological action functionals and gerbes

Let us start by recalling the some basic points of [23], changing the notations to more up-to-date ones.

2.1 Dirac monopoles and line bundles

Suppose that $B$ is a (magnetic field) closed 2-form on a manifold $M$. To describe a particle of unit charge moving in such a field along a trajectory $\varphi(t)$, one has to add to the action functional the coupling term $\int \varphi^* A$ where $A = d^{-1} B$ is the vector potential of $B$, i.e. a 1-form such that $dA = B$. The problem arises when $B$ is not exact so that there is no global $A$ (like for the magnetic field of a monopole). Dirac’s solution of the problem, when translated to a geometric language, is to define the Feynman amplitudes $e^{i \int \varphi^* d^{-1} B}$ of closed particle paths $\varphi$ as holonomies in a hermitian line bundle $L$ with (hermitian) connection $\nabla$ of curvature $\text{curv}(\nabla) = B$, provided such bundle exists. This is the case if the closed 2-form $\frac{1}{2\pi} B$ is integral in the sense that its periods over closed 2-cycles in $M$ are integers.

Let $(O_i)$ be a sufficiently fine open covering of $M$. We shall use the standard notation $O_{ij}, O_{ijk}$ etc. for the multiple intersections of the sets $O_i$. A choice of local sections $s_i : O_i \rightarrow L$ of length 1 gives rise to the local data $(g_{ij}, A_i)$ for $L$ such that $s_j = g_{ij} s_i$ and $\nabla s_i = \frac{1}{i} A_i s_i$. They have the following properties:

1. $g_{ij} = g_{ji}^{-1} : O_{ij} \rightarrow U(1)$ and on $O_{ijk}$

$$
\frac{g_{jk}}{g_{ik}} g_{ij}^{-1} g_{ij} = 1, \quad (2.1)
$$
2. $A_i$ are real 1-forms on $O_i$ such that
\[ dA_i = B, \tag{2.2} \]

3. on $O_{ij}$
\[ A_j - A_i = ig_{ij}^{-1}dg_{ij}. \tag{2.3} \]

If $s'_i$ correspond to a different choice of local sections so that $s'_i = f_is_i$ then
\[ g'_{ij} = g_{ij}f_jf_i^{-1} \quad \text{and} \quad A'_i = A_i + if_i^{-1}df_i. \tag{2.4} \]

The local data also naturally restrict to finer coverings. The two collections of local data are considered equivalent if they are related by (2.2) when restricted to a sufficiently fine common covering. The equivalence classes $w = [g_{ij}, A_i]$ may be viewed as (real, degree 2) Deligne (hyper-)cohomology classes \[ [23]. \]

The class of local data depends only on the bundle $L$ with connection and not on the choice of its local sections. Besides, isomorphic bundles give rise to the same Deligne class.

Denote by $W(M, B)$ the set of equivalence classes $w$ of local data corresponding to fixed $B$ and by $w(L)$ the class of local data of the bundle $L$. In fact, the line bundle $L$ together with its hermitian structure and connection may be reconstructed from the local data $(g_{ij}, A_i)$ up to isomorphism. One just takes the disjoint union $\bigcup_i(O_i \times \mathbb{C}) \equiv \bigcup_i(O_i \times \{i\} \times \mathbb{C})$ of trivial bundles and one divides it by the equivalence relation
\[ (x, i, g_{ij}z) \sim (x, j, z). \tag{2.5} \]

The covariant derivative given by $\nabla = d + \frac{1}{2}A_i$ on $O_i$ defines a connection on the quotient bundle. It follows that the elements of $W(M, B)$ are in one-to-one correspondence with the isomorphism classes of hermitian line bundles with connections. In particular, $W(M, B)$ in non-empty if and only if the 2-form $\frac{1}{2}B$ is integral. In the latter case, the cohomology group $H^1(M, U(1))$ acts on $W(M, B)$ in a free, transitive way, i.e. $W(M, B)$ is a $H^1(M, U(1))$-torsor. The action sends $w = [g_{ij}, A_i]$ to $uw = [u_{ij}g_{ij}, A_i]$, where $(u_{ij})$ is the Čech cocycle representing $u \in H^1(M, U(1))$. The latter group may be also viewed as that of characters of the fundamental group of $M$. The line bundles corresponding to $w$ and to $uw$ have holonomies differing by the corresponding character. In particular, the set $W(M, 0)$ of isomorphism classes of flat hermitian line bundles may be identified with the group $H^1(M, U(1))$. The multiplication in $H^1(M, U(1))$ corresponds to the tensor product of flat bundles.

The holonomy $\mathcal{H}(\varphi)$ in $L$ along a closed loop $\varphi : \ell \to M$ may be expressed using the local data of $L$. One splits $\ell$ into small closed intervals $b$ with common vertices $v$ in such a way that $b \subset O_{i_b}$ for some $i_b$, choosing also for each vertex $v$ and index $i_v$ so that $v \in O_{i_v}$. Then
\[ \mathcal{H}(\varphi) = \exp \left[ i \sum_b \int_b \varphi^*A_{i_b} \right] \prod_{v \in b} g_{i_v i_b}(\varphi(v)), \tag{2.6} \]
where the product $\prod_{v \in b}$ is taken with the convention that the entry following it is inverted if $v$ is the beginning of $b$. Since the holonomy depends only on the isomorphism class of $L$, the right hand side depends only on the class $w$ of the local data $(g_{ij}, A_i)$. More generally, for arbitrary curves $\varphi: \ell \to M$, the parallel transport in $L$ defines an element in $L_{\varphi(v_+)}^{-1} \otimes L_{\varphi(v_-)}$, where $v_{\pm}$ are the ends of $\ell$, $L_m$ denotes the fiber of $L$ over $m \in M$, and $L^{-1}$ is the bundle dual to $L$. Using local sections $s_{i\pm}$ such that $\varphi(v_{\pm}) \in O_{i\pm}$, this element may be represented by a number that is still given by the right hand side of (2.6). The value of (2.6) changes now upon changing the indices $i_{\pm}$ assigned to the endpoints of $\ell$ according to the identifications (2.5). It also changes when ones changes the local data $(g_{ij}, A_i)$ within the class $w$, but in the way consistent with isomorphisms of the line bundles reconstructed from such data.

2.2 Topological actions in two-dimensional field theories

In the two dimensional field theory, for example in the WZW model (see [23] for another example), one needs to make sense of action functionals written formally as $\int \phi^* d^{-1} H$ where $H$ is a closed but (possibly) not exact real 3-form $H$ on the target manifold $M$ in which the two-dimensional field $\phi$ takes values. This may be done in analogy to the one-dimensional prescription (2.6).

Suppose that, for a sufficiently fine covering $(O_i)$, one may choose the local data $(g_{ijk}, A_{ij}, B_i)$ with the following properties:

1. $g_{ijk} = g_{\sigma(i)\sigma(j)\sigma(k)}^{\text{sign}(\sigma)}: O_{ijk} \to U(1)$ and on $O_{ijkl}$
   \[ g_{jkl} g_{kli} g_{lij} g_{ijl}^{-1} = 1 \]
   (2.7)

2. $A_{ij} = -A_{ji}$ are real 1-forms on $O_{ij}$ and on $O_{ijk}$
   \[ A_{jk} - A_{ik} + A_{ij} = ig_{ijk}^{-1} dg_{ijk} \]
   (2.8)

3. $B_i$ are real 2-forms on $O_i$ such that
   \[ dB_i = H \]
   (2.9)

4. on $O_{ij}$,
   \[ B_j - B_i = dA_{ij} \]
   (2.10)

Such local data naturally restrict to finer coverings. Following [23], we shall consider two collections of local data equivalent if, upon restriction to a common sufficiently fine covering,

\[ g'_{ijk} = g_{ijk} \chi_{jk} \chi_{ik} \chi_{ij}^{-1} \]
(2.11,a)

\[ A'_{ij} = A_{ij} + \Pi_j - \Pi_i - i \chi_{ij}^{-1} d\chi_{ij} \]
(2.11,b)

\[ B'_i = B_i + d\Pi_i \]
(2.11,c)

for $\chi_{ij} = \chi_{ji}^{-1} : O_{ij} \to U(1)$ and real 1-forms $\Pi_i$ on $O_i$. The equivalence classes $w = [g_{ijk}, A_{ij}, B_i]$ may be viewed as Deligne (hyper-)cohomology classes in the degree
three $\mathbb{R}$. The set $W(M, H)$ of the classes corresponding to a given closed 3-form $H$ is non empty if and only if $\frac{1}{2\pi} H$ is integral in the sense that all its periods over closed 3-cycles in $M$ are integers. If this is the case then $W(M, H)$ is a $H^2(M, U(1))$-torsor, with the cohomology group $H^2(M, U(1))$ acting on $W(M, H)$ by

$$
(g_{ijk}, A_{ij}, B_i) \mapsto (u_{ijk}g_{ijk}, A_{ij}, B_i).
$$

(2.12)

If $H^3(M, \mathbb{Z})$ (or $H_2(M, \mathbb{Z})$) is without torsion, the above action is equivalent to

$$
(g_{ijk}, A_{ij}, B_i) \mapsto (g_{ijk}, A_{ij}, B_i + F),
$$

(2.13)

where $F$ is a closed 2-form on $M$. In the latter case, the class in $W(M, H)$ does not change if and only if $\frac{1}{2\pi} F$ is an integral 2-form. The equivalence of the two actions follows from the isomorphism $H^2(M, U(1)) \cong H^2(M, \mathbb{R})/H^2(M, 2\pi\mathbb{Z})$.

Let $\phi$ be a map from a compact oriented surface $\Sigma$ to $M$. One may triangulate $\Sigma$ in such a way that for each triangle $c$ there is an index $i_c$ such that $c \subset O_{i_c}$. We shall also choose indices $i_b$ for edges $b$ and $i_v$ for vertices $v$ so that $\phi(b) \subset O_{i_b}$ and $\phi(v) \in O_{i_v}$. The formal amplitudes $e^{i\int \phi^* d^{-1}H}$ may now be defined, as was first proposed in [2], by

$$
A(\phi) = \exp \left[ i \sum_c \int_c \phi^* B_{i_c} + i \sum_{b \subset c} \int_b \phi^* A_{i_c i_b} \right] \prod_{v \in b \subset c} g_{i_c i_b i_v} (\phi(v)),
$$

(2.14)

with the similar orientation conventions as in (2.6). It is straightforward to check that if $\partial \Sigma = \emptyset$ then $A(\phi)$ is independent of the choices of the triangulation and of the assignment of the covering indices and does not change under restrictions of the local data to finer coverings and under the equivalences (2.11).

Assume now that $\Sigma$ has a boundary $\partial \Sigma = \bigsqcup_s \ell_s$ with the boundary components $\ell_s$ that may be parametrized by the standard circle $S^1$. In this case the expression (2.14) still does not change if one modifies the triangulation and the index assignment in the interior of $\Sigma$, but it does change if the changes concern the boundary data. One may abstract from those changes a definition of a hermitian line bundle $L$ over the space $LM$ of loops in $M$ (or over the quotient of the latter by orientation-preserving reparametrizations) in such a way that

$$
A(\phi) \in \otimes_s L_{\phi|\ell_s}.
$$

(2.15)

The transition functions of the line bundle $L$ have been constructed in [23] where it was also shown that $L$ carries a natural connection whose curvature 2-form $\Omega$ is given by

$$
\langle \delta_1 \varphi, \delta_2 \varphi | \Omega(\varphi) \rangle = \int_\ell \varphi^*(\delta_2 \varphi) \iota(\delta_1 \varphi) H.
$$

(2.16)

For completeness, we include the explicit expressions from [23] in Sect. 10.1. For an equivalent choice of the local data $(g_{ijk}, A_{ij}, B_i)$, the bundle $L$ changes to an isomorphic one so that one obtains a natural map from $W(M, H)$ to $W(LM, \Omega)$. 
2.3 Bundle gerbes with connections

As we already mentioned, there are simple geometric objects, the (hermitian line) bundle gerbes with connection, whose appropriate isomorphism classes are described by elements of $W(M,H)$. Let us briefly recall this concept \[^{[35]}[^{[36]}].\]

Suppose that we are given a manifold map $\pi : Y \to M$ which admits local sections $\sigma_i : O_i \to Y$ over the sets of a sufficiently fine covering of $M$. Let $Y^{[n]} = Y \times_M Y \ldots \times_M Y$ denote the $n$-fold fiber product of $Y$. $Y^{[n]} = \{(y_1, \ldots, y_n) \in Y^n | \pi(y_1) = \ldots = \pi(y_n)\}$. We shall denote by $\pi^{[n]}$ the obvious map from $Y^{[n]}$ to $M$ and by $p_{n_1 \ldots n_k}$ the projection of $(y_1, \ldots, y_n)$ to $(y_{n_1}, \ldots, y_{n_k})$. A hermitian line bundle gerbe $G$ over $M$ with connection of curvature $H$ (shortly, a gerbe) is a quadruple $(Y,B,L,\mu)$ where

1. $B$ is a 2-form on $Y$ such that
   \[ dB = \pi^*H, \]  
   \hspace{1cm} (2.17)

2. $L$ is a hermitian line bundle with a connection $\nabla$ over $Y^{[2]}$ with curvature
   \[ \text{curv}(\nabla) = p_2^*B - p_1^*B, \]  
   \hspace{1cm} (2.18)

3. $\mu$ is an isomorphism of hermitian line bundles with connection over $Y^{[3]}$
   \[ \mu : p_{12}^*L \otimes p_{23}^*L \to p_{13}^*L, \]  
   \hspace{1cm} (2.19)

4. as isomorphisms of line bundles $p_{12}^*L \otimes p_{23}^*L \otimes p_{34}^*L$ and $p_{14}^*L$
   over $Y^{[4]}$
   \[ \mu \circ (\mu \otimes id) = \mu \circ (id \otimes \mu). \]  
   \hspace{1cm} (2.20)

The 2-form $B$ is called the curving of the gerbe. The isomorphism $\mu$ defines a structure of a groupoid on $L$ with the bilinear product $\mu : L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \to L_{(y_1,y_3)}$. The associativity of the product is guaranteed by (2.20). The bundle $L$ restricted to the diagonal composed of the elements $(y,y)$ may be naturally trivialized by the choice of the units of the groupoid multiplication and $\mu$ determines a natural isomorphism between $\kappa^*L$ and $L^{-1}$, where $\kappa(y_1,y_2) = (y_2,y_1)$. In order to elucidate the abstract definition copied from [\[^{[34]}\] (except for fixing the curving $B$ of the gerbe), let us immediately provide examples.

First, for an exact 3-form $H = dB$, the quadruple $(M,B, M^{[2]} \times \mathbb{C}, \cdot)$ with $Y = M$, with the trivial bundle $L$ over $M^{[2]} \cong M$, and with $\mu$ determined by the product of complex numbers, is a gerbe with curvature $H$.

Given a map $\pi : Y \to M$ admitting local sections and a hermitian line bundle $N$ over $Y$ with connection of curvature $F$, a simple example of a gerbe is provided by $G_N = (Y,F, p_1^*N^{-1} \otimes p_2^*N, \mu)$ with $\mu$ given by the obvious identification between $(N_y^{-1} \otimes N_{y_2}) \otimes (N_{y_2}^{-1} \otimes N_{y_3})$ and $N_y^{-1} \otimes N_{y_3}$. This is a gerbe with the vanishing curvature. Following \[^{[34]}[^{[35]}],\] we shall call gerbes $G_N$ trivial. Trivial gerbes are useful to recognize when a bundle $N$ is isomorphic to a pullback $\pi^*P$ of a hermitian bundle with connection
on $M$. This is the case if and only if there exists a unit length flat section $D$ (called a descent data) of the trivial gerbe $G_N$ bundle $p_1^*N^{-1} \otimes p_2^*N$ over $Y^{[2]}$ such that

$$
\mu(D \circ p_{12} \otimes D \circ p_{23}) = D \circ p_{13}.
$$

(2.21)

$D(y_1, y_2) : N_{y_1} \to N_{y_2}$ defines then an equivalence relation on $\sqcup_{y \in \pi^{-1}(m)} N_y$. Taking $P_m$ as the set of the equivalence classes, one obtains canonically a bundle $P$ and an isomorphism of $N$ with $\pi^*P$. We shall say that $P$ is obtained from $N$ and $D$ by the descent principle.

The next example will be central to our application of gerbes. Let $(g_{ijk}, A_{ij}, B_i)$ be local data on $M$ as described in the previous subsection. Take for $Y$ the disjoint union $\sqcup_i O_i$ with $\pi(x, i) = x$. Then $Y^{[n]} = \sqcup_{i_1,...,i_n} O_{i_1,...,i_n}$ and the projections $p_{i_1,...,i_n}$ are the inclusions of $O_{i_1,...,i_n}$ into $O_{i_1...i_n}$. We take as $L$ the trivial hermitian line bundle $Y^{[2]} \times \mathbb{C}$. The connection on $L$ will be given by $\nabla = d + \frac{1}{2} A_{ij}$ on $O_{ij}$ and the isomorphism $\mu$ by the multiplication by $g_{ijk}$ on $O_{ijk}$. The relation (2.17) is then assured by (2.21) and the equality (2.18) by (2.10). That $\mu$ preserves the connections follows from (2.8) and its associativity (2.20) is a consequence of (2.7).

Conversely, given a gerbe, one may define local data $(g_{ijk}, A_{ij}, B_i)$ the following way. One first chooses local sections $\sigma_i : O_i \to Y$ that induce local sections $\sigma_{i_1,...,i_n} \equiv (\sigma_{i_1}, ..., \sigma_{i_n})$ of $Y^{[n]}$ defined on intersections $O_{i_1,...,i_n}$. If the covering of $M$ is sufficiently fine, one may also choose unit length sections $s_{ij} : \sigma_{ij}(O_{ij}) \to L$ so that $s_{ji} = s_{ij}^{-1} \circ \kappa$. One defines the local data $(g_{ijk}, A_{ij}, B_i)$ by the relations

$$
B_i = \sigma_i^* B,
$$

$$
\sigma_{ij}^*(\nabla s_{ij}) = \frac{1}{2} A_{ij} s_{ij} \circ \sigma_{ij},
$$

(2.22a)

$$
\mu \circ (s_{ij} \circ \sigma_{ij} \otimes s_{jk} \circ \sigma_{jk}) = g_{ijk} s_{ik} \circ \sigma_{ik}.
$$

(2.22b)

(2.22c)

Properties (2.9) and (2.10) follow from (2.17) and (2.18). Equation (2.8) arises by covariantly differentiating (2.22c) along directions tangent to $\sigma_{ij}(O_{ij})$ with the use of relations $p_{12} \circ \sigma_{ij} = \sigma_{ij}$ etc. and of the fact that $\mu$ preserves the connections. Finally, the cocycle condition (2.7) follows from the associativity (2.20). For a trivial gerbe $G_N$, we may take $s_{ij} = (\chi_{ij}^{-1} \circ \pi^{[2]})(s_i^{-1} \otimes s_j)$, where $s_i$ are unit length sections $s_i : \sigma_i(O_i) \to N$ and $\chi_{ij} = \chi_{ji}^{-1} : O_{ij} \to U(1)$. One obtains then the local data

$$
(\chi_{ij}^{-1} \chi_{ik} \chi_{ij}^{-1}, \Pi_j - \Pi_i - i \chi_{ij}^{-1} d\chi_{ij}, d\Pi_i),
$$

(2.23)

where $\Pi_i$ is defined by the relation $\sigma_i^*(\nabla s_i) = \frac{1}{2} \Pi_i s_i \circ \sigma_i$.

Let us show that the class $w \in W(M, H)$ of the local data $(g_{ijk}, A_{ij}, B_i)$ depends only on the gerbe $G$ and not of the choices of local sections used in the construction of the data. First, restricting the sections $\sigma_i$ and, accordingly, $s_{ij}$ to a finer covering produces the restriction of the local data to that covering which, by definition, does not change the class in $W(M, H)$. For two choices of local sections, one may assume that they have been already restricted to a common covering with the sets $O_i$ sufficiently small. One has then to compare the local data induced by the two families of sections
where on the left hand side the sections of $L$ are multiplied using $\mu$, define $U(1)$-valued functions $\chi_{ij} = \chi_{ji}^{-1}$ on $O_{ij}$. Let $\Pi_i$ be 1-forms on $O_i$ given by

$$\tilde{\sigma}_i^* \nabla s_i = \frac{1}{i} \Pi_i s_i \circ \tilde{\sigma}_i.$$  

Relation (2.11,c) follows then from (2.13) and (2.24,a). Similarly, identity (2.11,b) is a consequence of (2.22,b), (2.24) and the fact that $\mu$ commutes with the covariant derivation. Finally, the associativity of the product defined by $\mu$ together with (2.22,c) and (2.24) implies (2.11,a). This shows that the local data $(g_{ijk}, A_{ij}, B_i)$ and $(g'_{ijk}, A'_{ij}, B'_i)$ define the same class $w \in W(M, H)$ which, consequently, depends only on the gerbe $G$. We shall denote this class by $w(G)$. Clearly, the class of the gerbe constructed from the local data $(g_{ijk}, A_{ij}, B_i)$ is the class of those data.

It is natural to inquire when two gerbes $G = (Y, B, L, \mu)$ and $G' = (Y', B', L', \mu')$ on $M$ with curvature $H$ define the same class $w \in W(M, H)$. A sufficient condition is that $Y = Y'$, $B = B'$ and that there exists a bundle isomorphism $\iota : L \to L'$ preserving the remaining structures. We shall call such gerbes isomorphic. It is clear, however, that this is not a necessary condition. For example, two gerbes constructed from equivalent local data on different open coverings of $M$ define the same class in $W(M, H)$ but may have spaces $Y$ and $Y'$ with different numbers of components. The appropriate geometric notion of a stable isomorphism of gerbes was introduced in [36]. It provides a necessary and sufficient condition for the equality $w(G_1) = w(G_2)$. We shall describe it now.

Let $G = (Y, B, L, \mu)$ be a gerbe and let $\omega : Z \to M$ be another map with local sections. Given also a map $\sigma : Z \to Y$ commuting with the projections on $M$, the pullback gerbe $\sigma^*G$ will be defined as $(Z, \sigma^*B, \sigma^*[2]*L, \sigma^*[3]*\mu)$. It has the same curvature as $G$. For two gerbes with the same $Y$, one may define their tensor product by taking the tensor product of the hermitian line bundles with connections over $Y^{[2]}$ and the tensor product $\mu \otimes \mu'$ as the groupoid multiplication. The curvings and the curvatures add under such operation. Let $G = (Y, B, L, \mu)$ and $G' = (Y', B', L', \mu')$ be two arbitrary gerbes over $M$. Take $Z = Y \times_M Y'$ and let $\sigma : Z \to Y$ and $\sigma' : Z \to Y'$ be the projections on the components in $Y \times_M Y'$. By definition, gerbes $G$ and $G'$ are stably isomorphic if there exists a line bundle $N$ over $Z$ and a line bundle isomorphism

$$\sigma^{[2]*}L \otimes p_1^*N^{-1} \otimes p_2^*N \overset{\iota}{\to} \sigma'^{[2]*}L'$$

defining an isomorphism between the gerbes $\sigma^*G \otimes G_N$ and $\sigma'^*G'$. In particular, this requires that the curvature $F$ of $N$ be equal to $\sigma^*B' - \sigma^*B$. The stable isomorphism of gerbes is an equivalence relation.

The line bundle isomorphism $\iota$ will be called the stable isomorphism between $G$ and $G'$. In general, it is not unique. If $\iota'$ is another stable isomorphisms between $G$ and $G'$ corresponding to line bundles $N'$ over $Z$ then it necessarily differs from $\iota$.  

\[ \sigma, s_{ij} \text{ and } \sigma'_{ij}, s'_{ij}. \]  

Let $\tilde{\sigma}_i = (\sigma_i, \sigma'_i) : O_i \to Y^{[2]}$ and let $s_i : \tilde{\sigma}_i(O_i) \to L$ be unit length sections of $L$. The relations

$$(s_i \circ \tilde{\sigma}_i)^{-1}(s_{ij} \circ \sigma_{ij})(s_j \circ \tilde{\sigma}_j) = \chi_{ij} s_{ij}' \circ \sigma_{ij}' ,$$

where on the left hand side the sections of $L$ are multiplied using $\mu$, define $U(1)$-valued functions $\chi_{ij} = \chi_{ji}^{-1}$ on $O_{ij}$. Let $\Pi_i$ be 1-forms on $O_i$ given by

$$\tilde{\sigma}_i^* \nabla s_i = \frac{1}{i} \Pi_i s_i \circ \tilde{\sigma}_i.$$
by an isomorphism between the trivial gerbes $G_N$ and $G_{N'}$. Such an isomorphism defines descent data for the bundle $N^{-1} \otimes N'$ so that, canonically, $N' \cong N \otimes \omega^*P$ for a bundle $P$ on $M$. Since $N$ and of $N'$ have the same curvatures, $P$ has to be a flat bundle. Conversely, the gerbes $G_N$ and $G_{N'}$ for $N' = N \otimes \omega^*P$ are canonically isomorphic if $P$ is a flat bundle on $M$.

**Remark.** It is easy to see that any pullback gerbe $\sigma^*G$ is stably isomorphic to $G$. Indeed, taking as the bundle $N$ over $Z \otimes_M Y$ the pullback of $L$ by the map $\sigma \times Id$ from $Z \otimes_M Y$ to $Y^{[2]}$, we observe that the groupoid multiplication $\mu$ defines a stable isomorphism

$$L(\sigma(z_1),\sigma(z_2)) \otimes N_{\sigma(z_1),y_1}^{-1} \otimes N_{\sigma(z_2),y_2} \rightarrow L_{(y_1,y_2)}$$

between $\sigma^*G$ and $G$. In fact, two gerbes $G$ and $G'$ are stable isomorphic if and only if they become isomorphic after the pullback to a common $Z$ (not necessarily equal to $Y \times_M Y'$) and the tensor multiplication by a trivial gerbe.

Clearly, the stably isomorphic gerbes have the same curvature $H$. Moreover, as it is easy to see, they give rise to the same class $w \in W(M,H)$. Indeed, under pullbacks of gerbes the local data do not change if we use the local sections $\sigma \circ \sigma_i : O_i \rightarrow Y$ and $s_{ij}$ for the gerbe before the pullback and $\sigma_i : O_i \rightarrow Z$ and $s_{ij} \circ \sigma^{[2]}$ for the pullback gerbe. Similarly, under tensor multiplication by a trivial gerbe, the local data change by (2.23), hence again stay in the same class. Converse is also true: if $w(G) = w(G')$ then $G$ and $G'$ are stably isomorphic. To prove this, it is enough to show that any gerbe $G = (Y, B, L, \mu)$ is stably isomorphic to the one constructed from its local data associated to the sections $\sigma_i$ and $s_{ij}$. This follows from the fact that the pullback of $G$ by the map $\sigma : \sqcup O_i \rightarrow Y$ equal to $\sigma_i$ on each $O_i$ is isomorphic to the local data gerbe, with the corresponding line bundle isomorphism given by the sections $s_{ij}$ (just recall how the local data and the corresponding gerbe are defined).

**Summarizing:** there is a one-to-one correspondence between the cohomology classes in $W(M,H)$ and the stable isomorphism classes of gerbes with curvature $H$.

### 3 Gerbes on groups $SU(N)$

Let $G$ be a connected, simply connected, simple compact group and let $g$ be its Lie algebra. We shall denote by $\text{tr}$ the non-degenerate bilinear invariant form on $g$ which allows to identify $g$ with its dual, by $t$ the Cartan subalgebra of $g$, by $r$ the rank of $g$, by $\Delta$ the set of the roots $\alpha$, by $\phi$ the highest root, by $\alpha^\vee$ and $\phi^\vee$ the coroots, by $e_\alpha$ the step generators corresponding to roots $\alpha$, by $\alpha_i, \alpha_i^\vee, \lambda_i$, and $\lambda_i^\vee$ for $i = 1, \ldots, r$, the simple roots, coroots, weights and coweights and by $Q, Q^\vee, P$ and $P^\vee$ the corresponding lattices. In particular,

$$g^C = t^C \oplus \left( \oplus_{\alpha \in \Delta} \mathbb{C} e_\alpha \right)$$

is the root decomposition of the complexification of $g$. The standard normalization of $\text{tr}$ requires that the long roots have length square 2 so that $\alpha^\vee = 2 \alpha / \text{tr} \alpha^2$. The
highest root $\phi = \phi^\vee = \sum k_i^\vee \alpha_i^\vee$, where $k_i^\vee$ are the dual Kac labels. The dual Coxeter number $h^\vee = 1 + \sum k_i^\vee$. The positive Weyl chamber $C_W \subset \mathfrak{t}$ is composed $\tau \in \mathfrak{t}$ such that $\text{tr} \tau \alpha_i \geq 0$ for each $i$ and the positive Weyl alcove $A_W$ is its subset restricted by the additional equality $\text{tr} \tau \phi \leq 1$. It is the $r$-dimensional simplex in $\mathfrak{t}$ with vertices $0$ and $\frac{1}{\alpha_i} \lambda_i$.

On $G$ we shall consider the unique up to normalization left- and right-invariant real closed 3-form

$$H = \frac{1}{12\pi} \text{tr} (g^{-1} dg)^3.$$ (3.2)

It will be convenient to parametrize the Lie algebra and the Lie group elements using the adjoint action of $G$. Elements in $\mathfrak{g}$ and in $G$ may be written, respectively, as

$$\gamma \tau \gamma^{-1} \quad \text{and} \quad \gamma e^{2\pi i \gamma^{-1}}$$ (3.3)

for some $\gamma \in G$ and $\tau \in \mathfrak{t}$. The group elements $\gamma$ are determined up to the right multiplication by $\gamma_0$ in the isotropy subgroups $G_0^\tau$ and $G_\tau$ composed of elements of $G$ commuting with $\tau$ and with $e^{2\pi i \gamma}$, respectively. Clearly, $G_0^\tau \subset G_\tau$. The subgroups $G_0^\tau$ and $G_\tau$ are connected. They correspond to the Lie subalgebras $\mathfrak{g}_\tau^0$ and $\mathfrak{g}_\tau$ of $\mathfrak{g}$ with complexifications

$$\mathfrak{g}_\tau^0 \subset \mathfrak{g}_\tau^C = \mathfrak{t}^C \oplus \left( \oplus_{\alpha \in \Delta_\tau^0} \mathbb{C} e_\alpha \right), \quad \mathfrak{g}_\tau^C = \mathfrak{t}^C \oplus \left( \oplus_{\alpha \in \Delta_\tau} \mathbb{C} e_\alpha \right),$$ (3.4)

where

$$\Delta_\tau^0 = \{ \alpha \in \Delta \mid \text{tr} \tau \alpha = 0 \}, \quad \Delta_\tau = \{ \alpha \in \Delta \mid \text{tr} \tau \alpha \in \mathbb{Z} \}. \quad (3.5)$$

The sets of Lie algebra and group elements $\{3.3\}$ with fixed $\tau$ form, respectively, the (co)adjoint orbit $O_\tau \subset \mathfrak{g}$ and the conjugacy class $C_\tau \subset G$. We have

$$O_\tau \cong G/G_\tau^0 \quad \text{and} \quad C_\tau \cong G/G_\tau$$ (3.6)

so that $O_\tau$ and $C_\tau$ are connected and simply connected.

The choice of $\tau$ in the parametrizations $\{3.3\}$ may be fixed if we demand that $\tau \in C_W$ or $\tau \in A_W$, respectively. Consider the open subsets $U_0 \subset \mathfrak{g}$ and $O_0 \subset G$ composed of elements of the form $\{3.3\}$ for $\tau \in A_W$ such that $\text{tr} \tau \phi < 1$. They are related by the exponential map $\mathfrak{g} \ni X \mapsto e^{2\pi i X} \in G$. Using the parametrization $\{3.3\}$ it is easy to see that the exponential map is injective on $U_0$ because $G_\tau^0 = G_\tau$ if $\text{tr} \tau \phi < 1$. Indeed, the last inequality implies that $\text{tr} \alpha \phi < 1$ for all positive roots. Similarly one shows that the derivative of the exponential map is invertible on $U_0$. It follows that the exponential map is a diffeomorphism between $U_0$ and $O_0$. Composing the latter with the homotopy $(t, X) \mapsto tX$ of $U_0$ and using the Poincare Lemma, one may obtain a 2-form $B_0$ on $O_0$ such that $dB_0 = H$. Explicitly, in the parametrization $\{3.3\}$,

$$B_0(\gamma e^{2\pi i \gamma^{-1}}) = Q(\gamma e^{2\pi i \gamma^{-1}}) + i \text{tr} \tau (\gamma^{-1} d\gamma)^2,$$ (3.7)

where

$$Q(\gamma e^{2\pi i \gamma^{-1}}) = \frac{1}{2\pi} \text{tr} \tau (\gamma^{-1} d\gamma) e^{2\pi i \gamma^{-1} d\gamma} e^{-2\pi i \gamma}.$$ (3.8)
These 2-forms will be the building blocks for the local data of a gerbe on $G$ with curvature $H$ for $G = SU(N)$.

The group $SU(N)$ has rank $r = N - 1$. It is simply laced so that $\alpha_i = \alpha_i^\vee$ and $\lambda_i = \lambda_i^\vee$. For the Cartan subalgebra composed of the diagonal $su(N)$ matrices, we may take

$$\alpha_i = \text{diag}(0, \ldots, 1, -1, \ldots, 0),$$

$$\lambda_i = \text{diag}(\frac{N-i}{N}, \ldots, \frac{N-i}{N}, \ldots, \frac{-i}{N})$$

with 1 and the last $\frac{N-i}{N}$ at the $(i-1)^{\text{th}}$ place counting from zero. The highest root is $\phi = \text{diag}(1, 0, \ldots, 0, -1)$, the Kac labels are $k_i^\vee = 1$ and the dual Coxeter number is equal to $N$. The center of $SU(N)$ is composed of the elements $z_i = e^{2\pi i \lambda_i}$ for $i = 0, 1, \ldots, r$, where we set $\lambda_0 = 0$. Let us consider the sets $O_i = z_iO_0 \subset SU(N)$. We may define 2-forms $B_i$ on $O_i$ by the pullback of $B_0$ from $O_0$:

$$B_i(g) = B_0(z_i^{-1} g).$$

Clearly, $dB_i = H$. If $g = \gamma e^{2\pi i \tau} \gamma^{-1}$ then $z_i^{-1} g = \gamma e^{2\pi i (\tau - \lambda_i)} \gamma^{-1}$. For each $i$ there is an element $w_i$ in the normalizer $N(T) \subset G$ of the Cartan subgroup $T \subset G$ such that if $\tau \in A_W$ then also $w_i(\tau - \lambda_i)w_i^{-1} \equiv \sigma_i(\tau)$ is in $A_W$. Explicitly, the element $w_i$ induces the Weyl group transformations

$$\text{diag}(a_0, \ldots, a_r) \mapsto w_i \text{diag}(a_0, \ldots, a_r) w_i^{-1} = \text{diag}(a_i, \ldots, a_r, a_0, \ldots, a_{i-1})$$

and $\sigma_i(\lambda_j) = \lambda_{[j-i]}$, where $[j-i] = (j-i) \mod N$. It follows that

$$z_i^{-1} \gamma e^{2\pi i r} \gamma^{-1} = \gamma e^{2\pi i (\tau - \lambda_i)} \gamma^{-1} = \gamma w_i^{-1} e^{2\pi i \sigma_i(\tau)} w_i \gamma^{-1}.$$  

Substituting into (3.7) and (3.8), we obtain

$$B_i(\gamma e^{2\pi i r} \gamma^{-1}) = Q(\gamma e^{2\pi i r} \gamma^{-1}) + i \text{tr} (\tau - \lambda_i)(\gamma^{-1} d\gamma)^2.$$  

The sets $O_i$ are composed of group elements $g$ such that $\tau \in A_W$ and $\text{tr} \alpha_i > 0$ in the parametrization (3.3). For such $\tau$, $G_{\tau} = G_{\tau - \lambda_i}$. In terms of the eigenvalues of the unitary matrices $g$ given by the entries of $e^{2\pi i \tau} = \text{diag}(e^{2\pi i a_0}, \ldots, e^{2\pi i a_r})$ such that $a_0 \geq \cdots \geq a_r$ and $a_0 - a_r \leq 1$, the sets $O_i$ are defined by the inequality $a_i > a_{i+1}$ and $O_0$ by $a_0 - a_r < 1$. Clearly, $G = \bigcup_{i=0,1,\ldots,r} O_i$.

In the first step in the construction of the gerbe on $SU(N)$ with curvature $H$ we set

$$Y = \bigcup_{i=0,1,\ldots,r} O_i \quad \text{and} \quad B|_{O_i} = B_i.$$  

As discussed before, the fiber products $Y[1] = \bigcup O_{i_1 \ldots i_n}$. To continue the construction of the gerbe, note that on the intersections $O_{ij}$ that form $Y[2]$ (with $i, j = 0, 1, \ldots$),

$$(B_j - B_i)(\gamma e^{2\pi i r} \gamma^{-1}) = -i \text{tr} \lambda_{ij}(\gamma^{-1} d\gamma)^2.$$  

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where $\lambda_{ij} \equiv \lambda_j - \lambda_i$. The expression on the right hand side coincides with the one for the Kirillov-Kostant symplectic form $F_{\lambda_{ij}}$ on the (co)adjoint orbit $O_{\lambda_{ij}}$ passing through $\lambda_{ij}$. More exactly, there is a map $O_{\lambda_{ij}} \ni g \to \rho_{ij}(g) \in O_{\lambda_{ij}}$ such that

$$B_j - B_i = \rho^*_j F_{\lambda_{ij}}.$$  \hfill (3.16)

This map is defined as follows. For $g \in O_{\lambda_{ij}}$, there exist two Lie algebra elements $X_i, X_j \in U_0$ such that $z^{-1}_i g = e^{2\pi i X_i}$ and $z^{-1}_j g = e^{2\pi i X_j}$. Then $\rho_{ij}(g) = X_i - X_j$. Explicitly, if $g = \gamma e^{2\pi i \tau \gamma^{-1}}$ then, as may be seen from (3.12), $X_i = \gamma w_i^{-1} \sigma_i(\tau) w_i \gamma^{-1} = \gamma(\tau - \lambda_i) \gamma^{-1}$ and similarly for $X_j$. Hence

$$\rho_{ij}(\gamma e^{2\pi i \tau \gamma^{-1}}) = \gamma \lambda_{ij} \gamma^{-1}. \tag{3.17}$$

Another way to see that the map (3.17) is well defined is to check that if $tr \tau \alpha_i > 0$ and $tr \tau \alpha_j > 0$ for $\tau \in A_W$ then the isotropy subgroup $G_\tau$ necessarily is contained in the isotropy subgroup $G^0_{\lambda_{ij}}$. Since for $i < j$

$$\lambda_{ij} = \text{diag}(\frac{i+j}{N}, \ldots, \frac{i-j}{N}, \ldots, \frac{N+i-j}{N}, \ldots, \frac{N-j}{N}, \ldots, \frac{i-j}{N}), \tag{3.18}$$

the isotropy subgroup $G^0_{\lambda_{ij}}$ is composed of block matrices $\gamma_0$ that preserve the subspace $V_{ij} \subset \mathbb{C}^N$ of vectors with vanishing first $i$ and last $N - j$ coordinates and its orthogonal complement. The coadjoint orbit $O_{\lambda_{ij}} \cong G/G^0_{\lambda_{ij}}$ may be identified with the Grassmannian $Gr_{ij}$ of $(j-i)$-dimensional subspaces $\gamma(V_{ij})$ in $\mathbb{C}^N$ with $\gamma \in SU(N)$.

The Kirillov-Kostant theory [30][31] provides an explicit construction of a hermitian line bundle $L_\lambda$ over the coadjoint orbit $O_{\lambda}$ with connection of curvature $F_{\lambda}$, provided that $\lambda$ is a weight which holds for $\lambda = \lambda_{ij}$. The bundle is obtained by dividing the trivial line bundle over $G$ by the equivalence relation

$$L_\lambda = (G \times \mathbb{C})/\sim_\lambda, \tag{3.19}$$

where

$$(\gamma, \zeta) \sim (\gamma \gamma_0, \chi_\lambda(\gamma_0^{-1} \zeta)), \tag{3.20}$$

for $\gamma_0 \in G^0_{\lambda}$. We shall denote the corresponding equivalence classes by $[\gamma, \zeta]_\lambda$. Above, $\chi_\lambda : G^0_{\lambda} \to U(1)$ stands for the group homomorphism (character) such that

$$\partial_i |_{t=0} \chi_\lambda(e^{itX_0}) = i \text{ tr } \lambda X_0 \tag{3.21}$$

for $X_0 \in g^0_{\lambda}$. Existence of $\chi_\lambda$ is guaranteed if $\lambda$ is a weight. The formula

$$\nabla = d + \text{ tr } \lambda(\gamma^{-1} d\gamma) \tag{3.22}$$

defines a connection in the trivial bundle over $G$ that descends to the bundle $L_\lambda$. The curvature of that connection is equal to $F_\lambda = -i \text{ tr } \lambda(\gamma^{-1} d\gamma)^2$. For $i < j,$

$$\chi_{\lambda_{ij}}(\gamma_0) = \det(\gamma_0 |_{V_{ij}}). \tag{3.23}$$
We shall also use an alternative description of the bundle $L_{\lambda ij}$. Let, for $i < j$, $E_{ij}$ be the tautological vector bundle over the Grassmannian $Gr_{ij}$ whose fiber at $\gamma(V_{ij})$ is this very subspace in $\mathbb{C}^N$. Clearly, $E_{ij}$ is a $(j-i)$-dimensional subbundle of the trivial bundle $Gr_{ij} \times \mathbb{C}^N$ from which it inherits the hermitian structure. It may be equipped with the connection

$$\nabla = P_{\gamma(V_{ij})}d$$

(3.24)

where $P_{\gamma(V_{ij})} = \gamma P_{V_{ij}} \gamma^{-1}$ denotes the orthogonal projection in $\mathbb{C}^N$ on $\gamma(V_{ij})$. The line bundle $L_{\lambda ij}$ may be identified with the top exterior power $\wedge^{j-i}E_{ij}$ of the bundle $E_{ij}$ by the mapping

$$[\gamma, \zeta] \lambda \mapsto \zeta \gamma e_i \wedge \ldots \wedge \gamma e_{j-1},$$

(3.25)

where $e_l$, $l = 0, 1, \ldots, r$, are vectors of the canonical basis of $\mathbb{C}^N$. It is easy to see that this mapping is compatible with the equivalence relation (3.20) and that it preserves the connection if we equip $\wedge^{j-i}E_{ij}$ with the one inherited from $E_{ij}$.

We may perform now the next step in the construction of the gerbe $G = (Y, B, L, \mu)$ on $SU(N)$ with $Y$ and $B$ given by (3.14). We shall define the line bundle $L$ with connection over $Y[2] = \sqcup O_{ij}$ by

$$L|_{O_{ij}} := \rho^*_{ij}L_{\lambda ij}.$$ 

(3.26)

Equation (3.16) guarantees that the curvature of $L$ satisfies requirement (2.18). We still have to construct the isomorphism $\mu$ providing $L$ with the groupoid structure. It will be given by the isomorphisms between the bundles on the triple intersections $O_{ijk}$

$$\mu_{ijk} : \rho^*_{ij}L_{\lambda ij} \otimes \rho^*_{jk}L_{\lambda jk} \longrightarrow \rho^*_{ik}L_{\lambda ik}. $$

(3.27)

We may assume that $i < j < k$. Then the isomorphism $\mu_{ijk}$ is determined by the natural map

$$(\gamma e_i \wedge \ldots \wedge \gamma e_{j-1}) \otimes (\gamma e_j \wedge \ldots \wedge \gamma e_{k-1})$$

$$\mapsto \gamma e_i \wedge \ldots \wedge \gamma e_{j-1} \wedge \gamma e_j \wedge \ldots \wedge \gamma e_{k-1}$$

(3.28)

and the associativity (2.20) becomes obvious.

This ends the construction of the gerbe $G = (Y, B, L, \mu)$ on the special unitary group $SU(N)$. Since $H^2(G, U(1)) = \{1\}$ for simply connected groups, $G$ is, up to stable isomorphism, a unique gerbe on $SU(N)$ with curvature $H$ given by (3.2). The tensor powers $G^k = (Y, kB, L^k, \mu^k)$ of $G$ for $k \in \mathbb{Z}$ give the gerbes on $SU(N)$ with curvature $kH$, again unique up to stable isomorphism. In particular, $G^{-1}$ is the gerbe dual to $G$ ($\mu^{-1}$ is the inverse of the transpose of $\mu$).

4 Gerbes on groups covered by $SU(N)$

Let us consider now the case of non-simply connected groups $G'$, quotients of simply connected groups $G$ by a subgroup $Z$ of their center. The closed 3-form $H$ of (3.2)
descends from $G$ to $G'$ to a 3-form $H'$ and we shall be interested in the gerbes on $G'$ with curvature proportional to $H'$. We shall restrict ourselves to the case when $G = SU(N)$ and $G' = SU(N)/\mathcal{Z}$, where $\mathcal{Z}$ is a cyclic group of order $N'$ such that $N = N'N''$. More explicitly, $\mathcal{Z} = \{z_a \mid N''$ divides $a\}$. As was shown in [17], the forms $\frac{1}{2\pi}kH'$ on $SU(N)/\mathcal{Z}$ are integral for even $k$ if $N'$ is even and $N''$ is odd and for integer $k$ in the other cases. In the present Section, we shall construct the corresponding gerbes $G'_k$ on $G'$. Not surprisingly in view of the discussion in Appendix 1 of [17], the construction reduces to solving a simple cohomological problem in the group cohomology of $\mathcal{Z} \cong \mathbb{Z}_{N'}$, see Appendix A for a brief summary on discrete group cohomology. The resulting gerbe will still be unique up to stable isomorphism since $H^2(G', U(1)) = \{1\}$ in the case at hand.

We shall take $Y' = \sqcup O_i$ with $O_i$ the open subsets of $SU(N)$ constructed before. $\pi'$ will be the natural projection from $Y'$ on the quotient group $SU(N)/\mathcal{Z}$. For the curving of the gerbe $G'_k$, we shall take the 2-form $B'$ equal to $kB$ on $O_i$, see (3.10). We have

$$Y'[n] = \{(g, i), (z_a^{-1}g, i'1), \ldots, (z_{a_{n-1}}^{-1}g, i'_{n-1}) \mid z_{am} \in \mathcal{Z}\},$$

for $g \in O_{i_1i_2\ldots i_{n-1}}$, where $i_m = [i'_m + a_m]$. We may then identify

$$Y'[n] \cong \sqcup_{a_1\ldots a_{n-1}} \sqcup_{i_1i_2\ldots i_{n-1}} O_{i_1i_2\ldots i_{n-1}}.$$

The hermitian line bundle $L'$ with connection over $O_{i_1} \subset Y'[2]$ should have the curvature

$$p_2^*(kB') = p_1^*(kB) = kB_j - kB_i = -ik \text{tr} \lambda_{ij}(\gamma^{-1}d\gamma)^2$$

in the parametrization $g = \gamma e^{2\pi i\tau} \gamma^{-1}$. We set

$$L'|_{O_{ij}} = \rho_{ij}^* L_{\lambda_{ij}}^k$$

where $L_{\lambda_{ij}}$ is the line bundle over the coadjoint orbit $O_{\lambda_{ij}}$ described in the previous section and $\rho_{ij}(\gamma e^{2\pi i\tau} \gamma^{-1}) = \gamma \lambda_{ij} \gamma^{-1}$. Recall that the elements in $L_{\lambda_{ij}}^k$ may be viewed as equivalence classes $[\gamma, \zeta]_{k\lambda_{ij}}$, see (3.2).

In the next step we should construct the isomorphism $\mu'$ of line bundles over $Y'[3]$ defining the groupoid multiplication in $L'$, see (2.19). The elements in $p_{12}^* L'$ over $g \in O_{ijl} \subset Y'[3]$ with $j = [j' + a]$ and $l = [l' + b]$ are given by the classes $[\gamma, \zeta]_{k\lambda_{ijl}}$. Those in $p_{13}^* L'$ by the classes $[\gamma, \zeta]_{k\lambda_{j'}}$. As for the elements of $p_{23}^* L'$, they correspond to the classes $[\gamma w_a^{-1}, \zeta]_{k\lambda_{j'l'a}}$, since $z_a^{-1}g = \gamma w_a^{-1}e^{2\pi i\sigma_a(r)}w_a \gamma^{-1}$, see (3.12), and $z_bz_a^{-1} = z_{[b-a]}$. The isomorphism $\mu'$ has then to be given by

$$\mu'([\gamma, \zeta]_{k\lambda_{ijl}} \otimes [\gamma w_a^{-1}, \zeta']_{k\lambda_{j'l'a}}) = [\gamma, u_{ijl} \zeta \zeta']_{k\lambda_{ijl}}$$

for $U(1)$-valued functions $u_{ijl}$ on $O_{ijl}$ whose dependence on $a$ and $b$ has been suppressed in the notation. These functions must be constant for $\mu'$ to preserve the
connections. Note that \( \mu' \) depends on the choice of matrices \( w_a \) defined up to the multiplication by elements of the Cartan subgroup \( T \). The latter dependence may, however, be absorbed in the choice of \( u_{ij} \).

As we show in Appendix B by a direct verification, the associativity of the product defined by \( \mu' \) imposes the condition

\[
  u_{j'[l-a][n-a]} u_{iln}^{-1} u_{ijn}^{-1} = x_{k, \lambda, \nu([a-b])} (w_{b} w_{a}^{-1} w_{b}^{1})^{-1}.
\]  

(4.6)

Upon taking \( i = j' = l' = n' = 0 \) and setting \( u_{ab} \equiv u_{a[b-a]} \), relation (4.6) reduces (upon the shift \( b \rightarrow [a + b], \ c \rightarrow [a + b + c] \)) to the condition

\[
  u_{bc} u_{[a+b]+1}^{-1} u_{[a+b]+c} w_{ab}^{-1} = x_{k, \lambda, c} (w_{[a+b]} w_{a}^{-1} w_{b}^{1}) \equiv U_{abc}
\]  

(4.7)

which may be interpreted in terms of the discrete group cohomology \( H^* (\mathcal{Z}, U(1)) \) with coefficients in \( U(1) \), see Appendix A. The \( U(1) \)-valued 3-cocycle \( (U_{abc}) \) on the cyclic group \( \mathcal{Z} \) satisfies the cocycle condition

\[
U_{bcd} U_{[a+b]+c}^{-1} U_{a+b+1} = 1
\]  

(4.8)

easy to verify with the use of the relation

\[
x_{k, \lambda, d} (w_{c} t w_{c}^{1}) = x_{k, \lambda, c} (t) x_{k, \lambda, d} (t)
\]  

(4.9)

holding for \( t \in T \). The condition (4.7) requires that \( (U_{abc}) \) be a coboundary. This does not have to be always the case since \( H^3 (\mathcal{Z}, U(1)) \cong \mathbb{Z}/N \), see Appendix A. Given a solution \((u_{ab})\) of (4.7),

\[
u_{ijl} = u_{a[b-a]} x_{k, \lambda, l} (w_{b} w_{a}^{-1} w_{b}^{1})
\]  

(4.10)

solves (4.6), as a straightforward check with the use of (4.9) shows.

We still have to study when (4.7) may be satisfied. In the action on the vectors of the canonical bases of \( \mathbb{C}^N \), the matrices \( w_a \) take the form \( w_a e_t = u_a e_{[t-a]} \), where \( u_a \) are diagonal matrices such that \( \det (u_a) = (-1)^{a(N-a)} \) assuring that \( \det (w_a) = 1 \). In particular, we may take \( u_a \) proportional to the unit matrix:

\[
u_a = \begin{cases} 
1, & \text{for } N' \text{ odd or } N'' \text{ even,} \\
(-1)^{a} & \text{for } N' \text{ even and } N'' \text{ odd,}
\end{cases}
\]  

(4.11)

where \( a = a'N'' \) (here and below, \( (1)^x = e^{i\alpha} \)). For that choice,

\[
w_{[a+b]} w_{a}^{-1} w_{b}^{1} = \begin{cases} 
1, & \text{for } N' \text{ odd or } N'' \text{ even,} \\
(-1)^{a} & \text{for } N' \text{ even and } N'' \text{ odd,}
\end{cases}
\] (4.12)

respectively, for \( m_{a'c'} = (a' + b')(N' - a' - b') - a'(N' - a') - b'(N' - b') \) and the associativity condition (4.7) becomes

\[
u_{bc} u_{[a+b]+c}^{-1} u_{[a+b]+c} u_{a}^{-1} u_{a}^{-1} u_{a}^{-1} = \begin{cases} 
1, & \text{for } N' \text{ odd or } N'' \text{ even,} \\
(-1)^{a} & \text{for } N' \text{ even and } N'' \text{ odd,}
\end{cases}
\] (4.13)
and may be solved by taking

\[ u_{ab} = \begin{cases} 
1 & \text{for } N' \text{ odd or } N'' \text{ even,} \\
(-1)^{-k(N''/N)\alpha'(N'-a)b'} & \text{for } N' \text{ even, } N'' \text{ odd and } k \text{ even.}
\end{cases} \]  

There is no solution for \( N' \) even and \( N'' \) and \( k \) odd.

This ends the construction of the gerbes \( \mathcal{G}'_k \) on \( SU(N)/Z \) with curvature \( kH' \) for all the values of \( k \) where the latter is an integral 3-form. Clearly, \( \mathcal{G}'_k \cong \mathcal{G}'_1^{k/2} \) for \( N' \) even and \( N'' \) odd.

### 5 Gerbes on discrete quotients

The above construction provides an illustration of a more general one of gerbes on spaces of orbits of a discrete group. The general case, that we shall briefly discuss in the present section which is somewhat parenthetical with respect to the main course of the exposition, sheds more light on the appearance of discrete group cohomology (the so-called "discrete torsion" [42]), as was noticed first in [39].

Suppose that \( \mathcal{G} \) is a gerbe on \( M \) with curvature \( H \) and that a finite group \( \Gamma \) acts on \( M \) preserving \( H \). We may ask the question if the action preserves \( \mathcal{G} \) in the sense that for each \( \gamma \in \Gamma \) the gerbe \( \gamma \mathcal{G} = (Y_\gamma, B, L, \mu) \), where \( Y_\gamma = Y \) as the space but has the projection on \( M \) replaced by \( \gamma \circ \pi \), is stably isomorphic to \( \mathcal{G} \). Recall that this means that there exists a hermitian line bundle \( N_\gamma \) over \( Z_\gamma = Y \times_M Y_\gamma \) with connection of curvature \( F_\gamma \) such that

\[ \sigma_*B = \sigma^*B + F_\gamma \]  

with \( \sigma, \sigma_\gamma \) denoting the projections from \( Z_\gamma \) to \( Y \) and \( Y_\gamma \), and that there exists an isomorphism

\[ \iota_\gamma : \sigma^*L \otimes p_1^*(N_\gamma)^{-1} \otimes p_2^*N_\gamma \rightarrow \sigma_\gamma^*L \]  

of hermitian line bundles with connection that preserves the groupoid multiplication. In particular,

\[ L_{(x_1, y_2)} \otimes (N_\gamma)^{-1}_{(x_1, y_2')} \otimes N_\gamma^{\gamma_{(x_2, y_2')}} \xrightarrow{\iota_\gamma} L_{(x_1', y_2')}. \]  

if \( \pi(y_1) = \pi(y_2) = \gamma \pi(y_1') = \gamma \pi(y_2') \). For \( \gamma = 1 \) one may take \( N_1 = L \) and \( \iota_1 \) defined by \( \mu \).

Suppose now that \( \Gamma \) acts without fixed points so that \( M/\Gamma \) is non-singular. We would like to construct a gerbe \( \mathcal{G}_\Gamma = (Y_\Gamma, B_\Gamma, L_\Gamma, \mu_\Gamma) \) on \( M/\Gamma \) with the curvature equal to the projection \( H' \) of \( H \). We shall set \( Y_\Gamma = Y \) with the projection \( \pi_\Gamma \) on \( M/\Gamma \) given by the composition of \( \pi : Y \rightarrow M \) with the canonical projection on the quotient space. The curving \( B_\Gamma \) will be taken equal to \( B \). Note that

\[ Y_\Gamma^{\Gamma} = \bigsqcup_{\gamma \in \Gamma} Z_\gamma. \]
Let us take
\[ L^\gamma_{\gamma} \big|_{Z_{\gamma}} = N^\gamma \otimes \pi^*_1 P^\gamma, \]  
(5.5)  
where \( P^\gamma \) is a flat line bundle on \( M \) and \( \pi_1(y, y') = \pi(y) \). Relation (5.1) assures then that the curvature of \( L^\gamma_{\gamma} \) is related to the curving by (2.18).

We still have to define the groupoid multiplication \( \mu_{\gamma} \) in \( L^\gamma_{\gamma} \) that is an isomorphism of line bundles over
\[ Y^{[3]}_{\gamma} = \bigcup_{\gamma_1, \gamma_2 \in \Gamma} \{(y, y', y'') \in Y^3 \mid \pi(y) = \gamma_1 \pi(y'), \pi(y') = \gamma_2 \pi(y'')\}. \]  
(5.6)  
On the \((\gamma_1, \gamma_2)\) component of \( Y^{[3]}_{\gamma} \),
\[ \mu_{\gamma} : p^*_2 N^{\gamma_1} \otimes \pi^*_1 P^{\gamma_1} \otimes p^*_3 N^{\gamma_2} \otimes \pi^*_2 P^{\gamma_2} \longrightarrow p^*_4 N^{\gamma_1 \gamma_2} \otimes \pi^*_3 P^{\gamma_1 \gamma_2}, \]  
(5.7)  
where \( \pi_n = \pi \circ p_n \). A necessary condition for existence of \( \mu_{\gamma} \) is that the bundle
\[ p^*_2 (N^{\gamma_1})^{-1} \otimes p^*_3 (N^{\gamma_2})^{-1} \otimes p^*_4 N^{\gamma_1 \gamma_2} \equiv \bar{R}^{\gamma_1 \gamma_2} \]  
(5.8)  
be isomorphic to a pullback \( \pi^*_3 \bar{R}^{\gamma_1 \gamma_2} \) of a flat bundle over \( M \). That this condition is fulfilled may be seen the following way. First note that the map \( \nu_{\gamma} \) of (5.3) defines isomorphisms
\[ (N^\gamma)_{(y_1, y'_1)} \longrightarrow N^\gamma_{(y_2, y'_2)} \otimes L_{(y_1, y'_2)} \otimes L^{-1}_{(y'_1, y'_2)} \]. \]  
(5.9)  
Combining the latter with the groupoid multiplication in \( L \) one obtains canonical isomorphisms between the fibers of the bundle \( \bar{R}^{\gamma_1 \gamma_2} \) over the triples \((y_1, y'_1, y''_1)\) and \((y_2, y'_2, y''_2)\) in \( Y^{[3]}_{\gamma} \) with the same projections on \( M \). Such isomorphisms define the descent data for the bundle \( \bar{R}^{\gamma_1 \gamma_2} \), see the discussion around (2.21). The existence of canonical bundle \( \bar{R}^{\gamma_1 \gamma_2} \) and of the canonical isomorphism \( \bar{R}^{\gamma_1 \gamma_2} \cong \pi^*_3 \bar{R}^{\gamma_1 \gamma_2} \) follows then by the descent principle. Besides, there exists a canonical isomorphism
\[ \bar{R}^{\gamma_1 \gamma_2} \otimes \bar{R}^{\gamma_1 \gamma_2 \gamma_3} \cong \bar{R}^{\gamma_1 \gamma_2 \gamma_3} \otimes (\gamma_1^{-1})^* \bar{R}^{\gamma_2 \gamma_3}, \]  
(5.10)  
as may be easily seen on the level of \( \bar{R} \) bundles.

To construct isomorphisms \( \mu_{\gamma} \) of (5.7) becomes then equivalent to specifying a family of isomorphisms \( \nu_{\gamma_1 \gamma_2} \) of flat bundles over \( M \)
\[ \nu_{\gamma_1 \gamma_2} : P^{\gamma_1} \otimes (\gamma_1^{-1})^* P^{\gamma_2} \rightarrow P^{\gamma_1 \gamma_2} \otimes R^{\gamma_1 \gamma_2}, \]  
(5.11)  
The associativity of \( \mu_{\gamma} \) becomes the condition
\[ \nu_{\gamma_1 \gamma_2 \gamma_3} \nu_{\gamma_1 \gamma_2} = \nu_{\gamma_1 \gamma_2 \gamma_3} (\gamma_1^{-1})^* \nu_{\gamma_2 \gamma_3}, \]  
(5.12)  
for isomorphisms between the bundles \( P^{\gamma_1} \otimes (\gamma_1^{-1})^* P^{\gamma_2} \otimes (\gamma_1^{-1})^* (\gamma_2^{-1})^* P^{\gamma_3} \) and the target bundles
\[ P^{\gamma_1 \gamma_2 \gamma_3} \otimes R^{\gamma_1 \gamma_2} \otimes R^{\gamma_1 \gamma_2 \gamma_3} \]  
and \[ P^{\gamma_1 \gamma_2 \gamma_3} \otimes R^{\gamma_1 \gamma_2 \gamma_3} \otimes (\gamma_1^{-1})^* R^{\gamma_2 \gamma_3}. \]
Given a family \( \iota_{\gamma_1, \gamma_2} \) of isomorphisms \([5.11]\) such that \([5.12]\) holds, we may obtain another family by multiplying \( \iota_{\gamma_1, \gamma_2} \) by an \( U(1) \)-valued 2-cocycle \( u_{\gamma_1, \gamma_2} \) satisfying

\[
\begin{align*}
\begin{array}{c}
u_{\gamma_1, \gamma_2} u_{\gamma_1, \gamma_2}^{-1} u_{\gamma_1, \gamma_2}^{-1} = 1.
\end{array}
\end{align*}
\]

The new family gives another solution for associative \( \mu_\gamma \). The coboundary choice \( u_{\gamma_1, \gamma_2} = \nu_{\gamma_1, \gamma_2} \nu_{\gamma_1}^{-1} \nu_{\gamma_2}^{-1} \) with \( U(1) \)-valued \( \nu \) leads to an isomorphic gerbe on \( M/\Gamma \) whereas the choices leading to non-trivial elements of \( H^2(\Gamma, U(1)) \) may give stably non-isomorphic gerbes.

The construction of gerbes \( G'_k \) on \( SU(N)/Z \) in the preceding section is an illustration of the general procedure described here, as we explain in detail in Appendix C.

Although above we have assumed that \( \Gamma \) acts without fixed points on \( M \), the above construction still goes through for general orbifolds provided that we redefine the spaces \( Y^{[n]}_\Gamma \) as

\[
Y^{[n]}_\Gamma := \{ (y, y_1, \gamma_1, \ldots, y_{n-1}, \gamma_{n-1}) \mid \pi(y) = \gamma_m \pi(y_m) \},
\]

i.e. keeping track of \( \gamma_m \in \Gamma \) (which could be recovered from \( y_m \)'s for free action of \( \Gamma \)). The resulting “orbifold gerbes” provide a natural tool for the treatment of strings on orbifolds in the background of closed 3-forms, see also \([40]\).

### 6 Bundle gerbes and loop-space line bundles

The construction of the amplitudes \( A(\phi) \) described in Sect. 2.2 with the use of the local data may be easily translated to the language of gerbes. In particular, the construction of an isomorphism class of hermitian line bundles with connection over the loop space \( LM \) from a class \( w \in W(M, H) \) may be lifted to a canonical assignment of a hermitian line bundle with connection to a gerbe on \( M \). In the present subsection, we shall describe those constructions that gain in simplicity when formulated with use of gerbes.

Let \( \mathcal{G} = (Y, B, L, \mu) \) be a gerbe on \( M \) with curvature \( H \) and, as before, \( \sigma_i : O_i \to Y \) be local sections. Let \( \phi \) be a map from a compact surface \( \Sigma \) to \( M \). For a sufficiently fine triangulation of \( \Sigma \) and a label assignment \( (c, b) \mapsto (i_c, i_b) \) such that \( \phi(c) \subset O_{i_c} \) and \( \phi(b) \subset O_{i_b} \), let us set

\[
\phi_c = \sigma_{i_c} \circ \phi|_c, \quad \phi_b = \sigma_{i_b} \circ \phi|_b, \quad \phi_{cb} = \sigma_{i_{c_b}} \circ \phi|_b,
\]

the latter for \( b \subset c \). These are lifts to \( Y \) or to \( Y^{[2]} \) of restrictions of \( \phi \) to the elementary cells. Denoting by \( \mathcal{H}(\cdot) \) the parallel transport in \( L \), we define:

\[
A(\phi) = \exp \left[ i \sum_c \int_c \phi^*_c B \right] \otimes_{b \subset c} \mathcal{H}(\phi_{cb}).
\]

Since \( \mathcal{H}(\phi_{cb}) \in \otimes_{v \in \partial \Sigma} L_{(y_c, y_b)} \), where \( y_c = \phi_c(v) \) and \( y_b = \phi_b(v) \) (with the convention that the dual fiber is taken if \( v \) is the beginning of \( b \)),

\[
A(\phi) \in \otimes_{v \in \partial c \subset c} L_{(y_c, y_b)}.
\]
The point is that if $\partial \Sigma = \emptyset$ then there is a canonical isomorphism, defined by the gerbe multiplication $\mu$, between the line in (6.3) and the complex line $\mathbb{C}$ so that the amplitude $A(\phi)$ may be naturally interpreted as a number. Indeed, fixing a vertex $v$ and going around as in Fig. 1,

\[
\begin{align*}
\text{Fig. 1}
\end{align*}
\]

we gather the contribution

\[
L(y_{b_1}, y_{c_1}) \otimes L(y_{c_1}, y_{b_2}) \otimes \cdots \otimes L(y_{c_n}, y_{b_1})
\]

(6.4)

to the line in (6.3) which is trivialized by subsequent application of $\mu$. That the result does not depend on where we start numbering the cells may be seen by choosing $y_v \in Y$ with $\pi(y_v) = \phi(v)$ and inserting $L(y_{b_r}, y_v) \otimes L(y_v, y_{b_{r+1}}) \cong \mathbb{C}$ at every second place in the chain (6.4). We may now use $\mu$ to trivialize the blocks

\[
L(y_{b_r}, y_v) \otimes L(y_v, y_{b_{r+1}}) \otimes L(y_{b_{r+1}}, y_{c_{r+1}})
\]

(6.5)

It is easy to check that the number obtained for $A(\phi)$ coincides with the one defined by the expression (2.14) with the use of the local data obtained from the sections $\sigma_i$ and $s_{ij}$. We give the proof in Appendix D. From the results of [23], it follows now that $A(\phi)$ does not depend on the choices of the local sections $\sigma_i$, of the lifts $\phi_c$, $\phi_b$ and $\phi_{cb}$, nor of the triangulation of $\Sigma$. Additionally, $A(\phi)$ is invariant under the composition of $\phi$ with orientation-preserving diffeomorphisms of $\Sigma$ and it goes to its inverse for diffeomorphisms reversing the orientation.

Suppose now that $\partial \Sigma = \sqcup \ell_s$. Consider the same expression (6.2). Proceeding as before, we may canonically reduce the line in (6.3) to

\[
\otimes_{s : v \in b \subseteq \ell_s} L(y_v, y_b)
\]

(6.6)

see Fig. 2 which replaces Fig. 1 in the boundary situation.
Let, for a closed loop $\varphi : \ell \to M$ and a sufficiently fine split of $\ell$,

$$\mathcal{L}_\varphi = \bigotimes_{v \in b \subseteq \ell} L_{(y_v, y_b)}$$  \hspace{1cm} (6.7)

with $y_v \in Y$ such that $\pi(y_v) = v$, and $y_b = \varphi_b(v)$, where $\varphi_b$ lift $\varphi|_b$ to $Y$. Let us show that the lines (6.7) are canonically isomorphic for different choices of $y_v$ and $\varphi_b$.

Let $\varphi'_b$ and $y'_v, y'_b = \varphi'_b(v)$ be another choice. Note that the parallel transport in $L$ along $(\varphi_b, \varphi'_b) : b \to Y^{[2]}$ defines a canonical trivialization of the line $\bigotimes_{v \in b \subseteq \ell} L_{(y_b, y'_b)}$. Similarly, the line $\bigotimes_{v \in b \subseteq \ell} L_{(y'_v, y_b)}$ is canonically trivial since each factor is accompanied by its dual.

Using also the product $\mu$, we obtain a chain of canonical isomorphisms

$$\bigotimes_{v \in b \subseteq \ell} L_{(y_v, y_b)} \cong \bigotimes_{v \in b \subseteq \ell} \left( L_{(y'_v, y_b)} \otimes L_{(y_v, y_b)} \otimes L_{(y_b, y'_b)} \right) \cong \bigotimes_{v \in b \subseteq \ell} L_{(y'_v, y'_b)} .$$ \hspace{1cm} (6.8)

Associativity of $\mu$ assures that the resulting isomorphisms are transitive so that we may free ourselves from the choice of local lifts in the definition of $\mathcal{L}_\varphi$ by passing to equivalence classes of elements related by the isomorphisms (6.8). Similarly, if we pass to a finer split of $\ell$ and use the restrictions of maps $\varphi_b$ to the new intervals, setting also $y_v = \varphi_b(v)$ for new vertices in the interior of the old intervals, then the net result on $\mathcal{L}_\varphi$ is to add trivial factors on the right hand side of (6.7). Dropping them is compatible with isomorphisms (6.8). In order to loose memory of the split used in (6.7), one may then define the projective limit $\mathcal{L}(G)_\varphi$ over trivializations of the lines obtained for fixed trivializations. All in all, we obtain this way a canonical hermitian line bundle $\mathcal{L}(G)$ over the loop space $LM$. Note that, by construction, $\mathcal{L}(G)_\varphi$ is invariant under orientation-preserving reparametrizations of $\ell$ and that the change of orientation gives rise to the dual line.

Comparing the lines (6.6) and (6.7) we infer that if $\partial \Sigma = \sqcup \ell_s$ then the amplitude (2.14) may be canonically defined as an element of the product of lines of $\mathcal{L}(G)$:

$$\mathcal{A}(\phi) \in \bigotimes_s \mathcal{L}(G)_{\phi|_{\ell_s}} .$$ \hspace{1cm} (6.9)

The hermitian line bundle $\mathcal{L}(G)$ may be equipped with a (hermitian) connection such that the parallel transport along the curve in the loop space $LM$ defined by $\phi : [0, 1] \times$
\[ \ell \to M \] is given by \( A(\phi) \). The curvature of this connection is equal to the 2-form \( \Omega \) on \( LM \) defined in (2.10). The amplitudes \( A(\phi) \) for arbitrary surfaces \( \Sigma \) provide a generalization of the parallel transport in the loop space.

How does the line bundle \( L(\mathcal{G}) \) and the amplitude \( A(\phi) \) depend on the gerbe? First, the line bundles \( L(\mathcal{G}) \) and \( L(\sigma^*\mathcal{G}) \), where \( \sigma^*\mathcal{G} \) is a pullback gerbe, are canonically isomorphic. Second, an isomorphism between gerbes \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) induces an isomorphism of the bundles \( L(\mathcal{G}_1) \) and \( L(\mathcal{G}_2) \). Third, the line bundle \( L(\mathcal{G}_1 \otimes \mathcal{G}_2) \) is canonically isomorphic to \( L(\mathcal{G}_1) \otimes L(\mathcal{G}_2) \). Finally, for a trivial gerbe \( \mathcal{G}_N \),

\[
L_\phi = \bigotimes_{v \in b C\ell} \left( N_{y_v}^{-1} \otimes N_{y_b} \right) \cong \bigotimes_{v \in b C\ell} N_{y_b} \cong \mathbb{C} \quad (6.10)
\]

where the last isomorphism is given by the parallel transport in \( N \) along \( \varphi_b \). It follows that a stable isomorphism between gerbes induces an isomorphism of the corresponding line bundles over \( LM \). If the gerbe \( \mathcal{G} \) is constructed from the local data then \( L(\mathcal{G}) \) is canonically isomorphic to the line bundle \( L \) over \( LM \) constructed from the local data described in Section 10.1.

In the language of the trivialization \( [3.10] \), the isomorphism between the lines \( L_\varphi \) corresponding to the isomorphic trivial gerbes \( \mathcal{G}_N \) and \( \mathcal{G}_N' \) for \( N' \cong N \otimes \pi^*P \) with \( P \) a flat bundle on \( M \) is given by the multiplication by the holonomy of \( P \) along \( \varphi \). It follows that the change of stable isomorphism between two gerbes obtained by composition with the isomorphism between the trivial gerbes \( \mathcal{G}_N \) and \( \mathcal{G}_N' \) multiplies the isomorphism of the line bundles over \( LM \) by the holonomy of \( P \).

### 7 Gerbes and branes

We have shown in the previous section that, given a gerbe \( \mathcal{G} = (Y, B, L, \mu) \) on \( M \) of curvature \( \mathcal{H} \), the formal amplitudes \( e^{i \int \phi d^{-1} \mathcal{H}} \) of a classical fields \( \phi : \Sigma \to M \) defined on the worldsheet \( \Sigma \) with boundary may be given sense as elements in the tensor product of lines of the line bundle \( L(\mathcal{G}) \) canonically associated to \( \mathcal{G} \), see (6.9). In general, \( L(\mathcal{G}) \) is a non-trivial bundle so that the amplitude cannot be naturally defined as numbers. Suppose, however, that the field \( \phi \) is restricted by the boundary conditions

\[
\phi(\ell_s) \in D_s \subset M, \quad (7.1)
\]

forcing its values on the boundary loops \( \ell_s \) of \( \Sigma \) to belong to submanifolds \( D_s \) of \( M \). Suppose moreover that \( D_s \) are chosen so that the line bundle \( L(\mathcal{G}) \) restricted to the space \( LD_s \) of loops in \( D_s \) becomes trivial. Upon a choice of trivializations of \( L(\mathcal{G})|_{LD_s} \), the amplitude \( A(\phi) \) may then be assigned a numerical value. Note that it is not necessary to require that the trivializations of \( L(\mathcal{G})|_{LD_s} \) flatten the connection.

Let us assume that a submanifold \( D \subset M \) is such that the restriction of the 3-form \( \mathcal{H} \) to \( D \) is exact: \( \mathcal{H}|_D = dQ \). To the 2-form \( Q \), we may associate a gerbe \( \mathcal{K} = (D, Q, D \times \mathbb{C}, \cdot) \) over \( D \) with curvature \( \mathcal{H}|_D \). Note that the corresponding hermitian line bundle \( L(\mathcal{K}) \) over \( LD \) is trivial but that its connection has a non-trivial curvature if \( \mathcal{H}|_D \neq 0 \). A natural way to assure the triviality of the restricted bundle \( L(\mathcal{G})|_{LD} \) and to provide for its trivializations is to assume that the restriction
\[ \mathcal{G}_D = (Y_D, B_D, L_D, \mu_D) \] of the gerbe \( \mathcal{G} \) to \( D \), where \( Y_D = \pi^{-1}(D) \), \( B_D = B|_{Y_D} \) etc., is stably isomorphic to the gerbe \( \mathcal{K} \). Explicitly, this means that there exist: a line bundle \( N \) over \( Y_D \) with connection of curvature \( F \) such that
\[ B_D + F = \pi_D^* Q \] (7.2)
and an isomorphism
\[ \iota : L_{|\mathcal{G}}{\otimes} p_1^* N^{-1} \otimes p_2^* N \rightarrow Y_D^2 \times \mathbb{C} \] (7.3)
of line bundles with connection over \( Y_D^2 \), compatible with the groupoid multiplication. By definition, a brane \( \mathcal{D} \) of \( \mathcal{G} \) with support \( D \) and curving \( Q \) is the quadruple \((D, Q, N, \iota)\). We shall consider two branes represented by collections \((D, Q, N, \iota)\) and \((D, Q, N', \iota')\) equivalent if the line bundles \( N \) and \( N' \) are isomorphic and \( \iota \) and \( \iota' \) are intertwined by the induced isomorphism of the trivial gerbes (note that such isomorphism is not unique). Non-equivalent branes with fixed support and curving correspond to \( N' \equiv N \otimes \pi_D^* P \), where \( P \) is a non-trivial flat bundle over \( D \). A choice of \( N \) and \( \iota \) induces canonically an isomorphism between \( \mathcal{L}(\mathcal{G})|_{\mathcal{D}} \) and the trivial hermitian line bundle \( \mathcal{L}(\mathcal{K}) \), see (6.10), with equivalent choices leading to the same isomorphism. Non-equivalent choices give rise to isomorphisms differing by multiplication by holonomy in a flat line bundle \( P \) over \( D \).

Given a gerbe \( \mathcal{G} \) with curvature \( H \), we may ask which submanifolds \( D \) with \( H|_D = dQ \) support branes with curving \( Q \). The obstructions to stable isomorphism of the gerbes \( \mathcal{G}_D \) and \( \mathcal{K} \) lie in the cohomology group \( H^2(D, U(1)) \) that acts freely and transitively on the set \( W(D, H|_D) \) of stable isomorphism classes of gerbes on \( D \) with curvature \( H|_D \). If the obstruction vanishes, then the cohomology group \( H^1(D, U(1)) \) (the group of isomorphism classes of flat hermitian bundles on \( D \)) acts freely and transitively on the set of equivalence classes (moduli) of branes \( \mathcal{D} \) with curving \( Q \) supported by \( D \). We shall see this in work in the next two sections.

Let \( IM \) be the space of open curves (strings) \( \varphi : [0, \pi] \rightarrow M \) and \( \mathcal{G} = (Y, B, L, \mu) \) a gerbe on \( M \) with curvature \( H \). The same construction that associated to \( \mathcal{G} \) a line bundle with connection over the loop space \( LM \), when applied to open curves, induces a hermitian line bundle with connection \( \mathcal{N} \) over the space
\[ \mathcal{Y} = \{ (\varphi, y_0, y_1) \in IM \times Y^2 \mid \pi(y_0) = \varphi(0), \pi(y_1) = \varphi(\pi) \} . \] (7.4)
The line bundle \( \mathcal{N} \) is composed from the fibers \( \mathcal{L}_\varphi \) of (6.7), with all the identifications as before except that one has to keep the memory of \( y_v = y_0 \) and \( y_e = y_1 \) for the end point vertices. The parallel transport in \( \mathcal{N} \) along a curve in \( IM \) is still determined by the amplitude \( \mathcal{A}(\phi) \) defined by (6.2) for \( \phi : [0, 1] \times [0, \pi] \rightarrow M \). The curvature of \( \mathcal{N} \) is given by the closed 2-form
\[ \Omega_{IM}(\varphi, y_0, y_1) = \Omega(\varphi) + B(y_0) - B(y_1) \] (7.5)
on \( \mathcal{Y} \), where \( \Omega \) defined by (2.10) with \( \ell = [0, \pi] \).

Given two branes \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \) with supports \( D_0 \) and \( D_1 \) of gerbe \( \mathcal{G} \), we may consider in the space \( IM \) of open strings the subspace
\[ I_{D_0 D_1} M = \{ \varphi : [0, \pi] \rightarrow M \mid \varphi(0) \in D_0, \varphi(\pi) \in D_1 \} . \] (7.6)
A slight modification of the construction described above permits now to define over $I_{D_0D_1}M$ a hermitian line bundle $L_{D_0D_1}(\mathcal{G}) \equiv L_{D_0D_1}$ with connection by setting

$$ (L_{D_0D_1})_{\phi} = (N_0)_{y_0} \otimes L_{\phi} \otimes (N_1)_{y_1}^{-1}, \quad (7.7) $$

where $L_{\phi}$ is given by (6.7). Due to the isomorphism (7.3), the lines obtained this way are canonically isomorphic also for different choices of $y_0$ and $y_1$, giving rise upon their identification to the fibers of $L_{D_0D_1}$. A choice of equivalent branes leads to (non-canonically) isomorphic bundles. The parallel transport in $L_{D_0D_1}$ is determined by

$$ A_{D_0D_1}(\phi) = A(\phi) \otimes \left( \otimes_{b \subset \ell_s} \mathcal{H}_s(\phi_b) \right) \quad (7.8) $$

for $\phi: [0,1] \times [0,\pi] \to M$, where $\ell_s$ denotes the piece of the boundary of $[0,1] \times [0,\pi]$ mapped into $D_s$ for $s = 0, 1$, $\mathcal{H}_s(\phi_b)$ stands for the parallel transport in $N_s$ along a lift $\phi_b$ of $\phi|_b$ to $Y$ and $A(\phi)$ is given by (6.2). The curvature of $L_{D_0D_1}$ is given by the 2-form on $I_{D_0D_1}M$

$$ \Omega_{D_0D_1} = \Omega + e_1^s Q_1 - e_0^s Q_0, \quad (7.9) $$

where $\Omega$ is as in (7.3) and $e_s$ are the evaluation maps,

$$ \varphi \xrightarrow{e_0} \varphi(0), \quad \varphi \xrightarrow{e_1} \varphi(\pi). \quad (7.10) $$

Fig. 3

More generally, suppose that $\phi: \Sigma \to M$ satisfies the boundary conditions (7.1) for $\ell_s$ being closed disjoint subintervals of the boundary loops of $\Sigma$, see Fig. 3. Then the amplitude defined by (7.8) satisfies

$$ A_{(D_s)}(\phi) \in \left( \otimes_{(s,s')} (L_{D_sD_s'})_{\phi\mid_{(s,s')}} \right) \otimes \left( \otimes_m L_{\phi\mid_{\ell_m}} \right), \quad (7.11) $$

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where $\ell_{(s,s')}$ are the boundary intervals bordering $\ell_s$ and $\ell_{s'}$ and $\ell_m$ are the boundary loops that do not contain intervals $\ell_s$. The curves $\phi_{(s,s')}$ stretch between the submanifolds $D_s$ and $D_{s'}$ and the line bundles $\mathcal{L}_{D_sD_{s'}}$ correspond to that geometry. The expression (7.8) generalizes the definition of the amplitude of a field to the case when local boundary conditions are imposed on pieces of the boundary of $\Sigma$.

In the quantum field theory, the amplitudes $A_{\mathcal{P}_0\mathcal{P}_1}(\phi)$ are “summed” (with additional scalar weights) over all fields satisfying the boundary conditions (7.4) resulting, at least formally, in a vector in the tensor product of Hilbert spaces of states, see Sect. 10 below. To each interval $\ell_{(s,s')}$ there corresponds a factor $\mathbb{H}_{\mathcal{P}_s\mathcal{P}_{s'}}$, the Hilbert space of states of the string stretching between the branes $\mathcal{D}_s$ and $\mathcal{D}_{s'}$ and to each $\ell_m$ a factor $\mathbb{H}$, the closed string space of states. Geometrically, spaces $\mathbb{H}_{\mathcal{P}_s\mathcal{P}_{s'}}$ are formed of sections of the corresponding line bundles $\mathcal{L}_{\mathcal{P}_0\mathcal{P}_1}$ and the space $\mathbb{H}$ of sections of $\mathcal{L}$ (more precisely, they are Hilbert space completions of spaces of sections). Even without going into the detailed construction of such spaces of states, the geometric classification of branes discussed above allows to obtain the spectrum of branes. We shall illustrate that in the next section on the example of the $SU(N)$ WZW theory and of its versions with groups covered by $SU(N)$.

8 Branes in the WZW model

In the WZW model, the candidate for the simplest form of the boundary condition that guarantees the conservation of half of the current algebra symmetries is to require that the values of the field $g : \Sigma \to G$ on the boundary loops $\ell_s \subset \partial \Sigma$ belong to the conjugacy classes $\mathcal{C}_\tau \subset G$. The closed 3-form $H$ of (3.2) becomes exact when restricted to a conjugacy class: $H|_{\mathcal{C}_\tau} = dQ_\tau$, where $Q_\tau$ is given by the expression (3.3) with constant $\tau$. The preservation of the current algebra symmetries requires that one sticks to that choice (or to its multiplicities) for the curving of branes supported by $\mathcal{C}_\tau$, see [1] or [25]. Now it is easy to check for which conjugacy classes the restriction of the gerbe on $G$ with curvature $kH$ is stably isomorphic to the gerbe $\mathcal{K} = (\mathcal{C}_\tau, kQ_\tau, \mathcal{C}_\tau \times \mathbb{C}, \cdot)$.

8.1 $SU(N)$ groups

Recall that for $G = SU(N)$, the gerbe $\mathcal{G}^k = (Y; kB, \mathcal{L}^k, \mu^k)$ with curvature $kH$ (unique, up to stable isomorphism) has $Y = \bigsqcup_{i=0}^{L} O_i$. We shall choose a coadjoint orbit $\mathcal{C}_\tau$ and denote $Z = \pi^{-1}(\mathcal{C}_\tau) = Y_{\mathcal{C}_\tau}$, where $\pi$ is the projection from $Y$ to $SU(N)$. Thus $Z = \sqcup Z_i$ where $Z_i = \mathcal{C}_\tau \cap O_i$. Since the sets $O_i \subset SU(N)$ are invariant under conjugations, $Z_i$ are either empty or equal to $\mathcal{C}_\tau$. Let $\omega = \pi|_Z$ and $\sigma : Z \to Y$ be the natural inclusion. We have to compare the restriction of the gerbe $\mathcal{G}^k$ to $\mathcal{C}_\tau$ with the pullback gerbe $\sigma^*\mathcal{K}$. On $Z_i$ the difference of the two curvings $kQ_\tau - kB_i|_{\mathcal{C}_\tau} \equiv F_{\tau i}$ has the form

$$F_{\tau i}(\gamma) e^{2\pi i \tau \gamma^{-1}} = -k i \text{tr} (\tau - \lambda_i)(\gamma^{-1} d\gamma)^2,$$

(8.1)

see (8.13). If the two gerbes $\mathcal{G}^k|_{\mathcal{C}_\tau}$ and $\mathcal{K}$ are stably isomorphic then the closed forms $F_{\tau i}$ must be curvature forms of hermitian line bundles $N|_{Z_i}$ and hence $\frac{1}{2\pi i} F_{\tau i}$ must be
integral. This condition is equivalent to the requirement that $k\tau$ be a weight. Indeed, in our case, $Z_i = C_\tau$ may be identified with the coadjoint orbits $\mathcal{O}_{k(\tau - \lambda_i)}$ by the maps $\gamma e^{2\pi i\tau} \gamma^{-1} \mapsto \gamma k(\tau - \lambda_i) \gamma^{-1}$ since the isotropy groups satisfy $G_\tau = G^0_{\tau - \lambda_i}$. Upon this identification, $F_{\tau i}$ becomes the Kirillov-Kostant symplectic form and integrality of $\frac{1}{2\pi} F_{\tau i}$ requires that $k(\tau - \lambda_i)$ be a weight.

In the latter case, the line bundles $N|_{Z_i}$ may be taken to be the pullbacks of the Kirillov-Kostant bundles $L_{k(\tau - \lambda_i)}$ by the identification of $Z_i$ with $\mathcal{O}_{k(\tau - \lambda_i)}$. Since the conjugacy classes are simply connected, the resulting line bundle $N$ over $Z$ is unique up to isomorphism. The mapping

$$[\gamma, \zeta]_{k\lambda_i} \otimes [\gamma, \zeta']_{k(\tau - \lambda_i)} \otimes [\gamma, \zeta'']_{k(\tau - \lambda_i)} \mapsto (y_1, y_2, \zeta \zeta' \zeta'')$$

for $y_1 = (g, i)$, $y_2 = (g, j)$ and $g = \gamma e^{2\pi i\tau} \gamma^{-1} \in C_\tau$, which is well defined because $\chi_{k\lambda_i}(\gamma_0) \chi_{k(\tau - \lambda_i)}(\gamma_0) = 1$ for $\gamma_0 \in G_\tau$, determines then the unique isomorphism (2.3) that commutes with the groupoid multiplication. It provides a stable isomorphism between the gerbes $G^\rho|_{C_\tau}$ and $\mathcal{K}$. Other choices of $N$ and $\iota$ lead to equivalent branes in the present case. We thus obtain for the $SU(N)$ WZW model a family of branes labeled by the weights $\lambda \in kA_W$, supported by the conjugacy classes $C_\tau$ with $\lambda = k\tau$. The weights in the dilated Weyl alcove $kA_W$ are called “integrable at level $k$” (2.3) and they also label the irreducible highest-weight representations of the level $k$ current algebra and the bulk primary fields of the model with conformal weights $h(\lambda) = \frac{tr(\lambda + 2\rho)}{2(k + h^2)}$.

### 8.2 Groups covered by $SU(N)$

We shall consider branes supported by the conjugacy classes in $G' = SU(N)/Z$ for $Z \cong Z_N$. Each conjugacy class in $G'$ is an image under the canonical projection from $G = SU(N)$ to $G'$ of a conjugacy class $C_\tau \subset G$ with $\tau \in A_W$. Recall that the elements $z_i$ of the center of $SU(N)$ act on $A_W$ by $\tau \mapsto \sigma_i(\tau)$. The conjugacy classes $C_{\sigma_0(\tau)}$ in $G$ for different $z_0 \in Z$ project to the same class in $G'$. This way the conjugacy classes in $G'$ may be labeled by the $Z$-orbits $[\tau]$ of elements in $A_W$. We shall denote the class in $G'$ corresponding to $[\tau]$ by $C'_{[\tau]}$. For any $\tau \in [\tau]$, $C'_{[\tau]}$ may be canonically identified with $C_\tau / Z_\tau$, where $Z_\tau$ is the subgroup of $Z$ leaving $\tau$ unchanged (it depends only on the orbit $[\tau]$). The 3-form $H'$ restricted to $C'_{[\tau]}$ still satisfies $H'|_{C'_{[\tau]}} = dQ'_\tau$, with $Q'_\tau$ denoting the the projection of $Q_\tau$ to the quotient space $C_\tau / Z_\tau$ and defining a 2-form on $C'_{[\tau]}$ that does not depend on $\tau \in [\tau]$. We obtain then the gerbe $\mathcal{K}' = (C'_{[\tau]}, kQ'_\tau, C'_{[\tau]} \times \mathbb{C}, \cdot)$ on $C'_{[\tau]}$.

Let us consider the space $Z' = \pi'^{-1}(C'_{[\tau]}) = \bigcup_{\tau \in [\tau]} Z_{\tau i}$, with $Z_{\tau i} = C_\tau \cap O_i$. Let $\sigma' : Z' \to Y'$ be the inclusion map and $\omega' = \pi'|_{Z'}$. We have to compare the restriction to $C'_{[\tau]}$ of the gerbe $G'_\lambda = (Y', B', L', \mu')$ on $G'$ constructed in Section 4 to the pullback gerbe $\sigma'^* \mathcal{K}'$. Over $Z_{\tau i}$ the difference of the curvings is $F_{\tau i}$, see (8.1), and the existence of the stable isomorphism between the two gerbes requires again that $k\tau$ be a weight, similarly as in the simply connected case. Let $N'$ denote the line bundle over $Z'$ that over $Z_{\tau i}$ coincides with the pullback of the Kirillov-Kostant bundle $L_{k(\tau - \lambda_i)}$ by the
be a cyclic subgroup of \(\mathbb{Z}_n\). In order to construct the primed version of the bundle isomorphism (7.3), let us consider the pairs \((y_1, y_2) \in \mathbb{Z}^2\) with \(y_1 = (g, i), \; y_2 = (z_a^{-1}g, j')\), \(g = \gamma e^{2\pi i \gamma^{-1}}\), \(z_a^{-1}g = \gamma w_a^{-1} e^{2\pi i \sigma_a(\gamma)} w_a \gamma^{-1}\) and the mapping

\[
[\gamma, \zeta]_{k \lambda ij} \otimes [\gamma, \zeta']_{k (\tau - \lambda i j)} \otimes [\gamma w_a^{-1}, \zeta'']_{k (\sigma_a(\gamma) - \lambda j')},
\]

\[\mapsto (y_1, y_2, v_{\tau,a} \zeta' \zeta''), \quad (8.3)
\]

where \(j = [j' + a]\) and \(v_{\tau,a} \in U(1)\). Note that \(v'\) is well defined since the isotropy subgroups satisfy

\[G_{\tau} = G^0_{\tau - \lambda i} = G^0_{\tau - \lambda j} = w_a^{-1} G^0_{\sigma_a(\gamma) - \lambda j'} w_a \subset G^0_{\lambda ij}\]

and for \(\gamma_0 \in G_{\tau}\), the product \(\chi_{k \lambda ij} (\gamma_0) \chi_{k (\tau - \lambda i j)} (\gamma_0) \chi_{k (\sigma_a(\gamma) - \lambda j')} (w_a \gamma_0 w_a^{-1}) = \chi_{k (\tau - \lambda i j)} (\gamma_0)\).

We have to choose \(v_{\tau,a}\) so that the isomorphism \(v'\) of hermitian line bundles with connections preserves also the groupoid multiplication. Let

\[V_{\tau,ab} = \chi_{k \sigma_{a+b}^b (\gamma)} (w_a w_b^{-1}) u_{ab}, \quad (8.4)\]

where \((u_{ab})\), a solution of (4.7), enters via (4.10) the definition of the groupoid multiplication \(\mu'\), see (4.3). As we prove in Appendix E, the requirement to preserve the groupoid multiplication imposes the cohomological relation

\[V_{\tau,ab} = v_{\sigma(a,b)} u_{\tau,ab}^{-1} v_{\tau,a}. \quad (8.5)\]

In Appendix F, we show that the 2-cochain \((V_{\tau,ab})\) on \(\mathbb{Z} \cong \mathbb{Z}^N\) with values in the group \(U(1)^{[\tau]}\) of \(U(1)\)-valued functions on the \(\mathbb{Z}\)-orbit \([\tau]\) is a 2-cocycle, i.e. that

\[V_{\sigma(a,b)} V_{\tau,ab}^{-1} V_{\tau,a[b+c]} V_{\tau,ab}^{-1} = 1. \quad (8.6)\]

Equation (8.5) requires that it be a coboundary, i.e. that it defines a trivial element in the cohomology group \(H^2(\mathbb{Z}, U(1)^{[\tau]}).\) This always holds since \(H^2(\mathbb{Z}, U(1)^{[\tau]} = \{1\},\) see Appendix A.

The multiplication of a solution of (8.5) by 1-cocycles \((v'_{\tau,a})\) satisfying the relation

\[v'_{\sigma(a,b)} v'_{\tau,ab}^{-1} v'_{\tau,a} = 1\]

gives all other solutions. Solutions differing by 1-boundaries \(v''_{\sigma(a)} v''_{\tau,a}\) lead to equivalent branes and the set of equivalence classes of branes supported by \(C^{[\tau]}\) forms a \(H^1(\mathbb{Z}, U(1)^{[\tau]}).\) Since \(H^1(\mathbb{Z}, U(1)^{[\tau]} = \mathbb{Z}_{\tau},\) see Appendix A, and \(\mathbb{Z}_{\tau} \cong H^1(C^{[\tau]}, U(1))\) and describes the moduli of flat line bundles on \(C^{[\tau]},\) this agrees with the general result about the classification of branes, see Sect. 7. Let \(Z_{\tau}\) be a cyclic subgroup of \(\mathbb{Z}\) of order \(n'\) and let \(n''\) and \(m''\) be such that \(n'n'' = N,\)

\[n'm'' = N'\] and \(n'' = m'' N''.\) Explicitly,

\[Z_{\tau} = \{ z_a \mid a = a'' n'' \quad \text{for} \quad a'' = 0, 1, \ldots, n' - 1 \}. \quad (8.7)\]
In order to generate all classes in $H^1(\mathcal{Z}, U(1)^{|\tau|})$ it is enough to take
\[ v'_{\tau,a} = (-1)^{2m} \]  
(8.8)
i.e. $\tau$-independent and equal to the characters of $\mathcal{Z}$. Besides $r$ above may be restricted
to integers between 0 and $n' - 1$ since there exists $(v''_\tau)$ such that $(-1)^{2m'} = (-1)^{2m''} = v''_{\sigma_\tau} v''_{\tau}^{-1}$.

It remains to describe explicitly a single solution of (8.5). Let $\tau_0 \in [\tau]$. Note that
\[ k\tau_0 = \sum_{i'=0}^{n''-1} n_{i'} \sum_{a''=0}^{n'-1} \lambda_{i'+a''n''} \]  
(8.9)
with $\sum_{i'=0}^{n''-1} n_{i'} = \frac{k}{n''}$ so that $n'$ has to divide $k$. The complicated case is when $N'$ is
even and $N''$ is odd and we shall deal with it first. Here, for $\frac{k}{n''}$ odd, $n'$ must be even
since $k$ is necessarily even. Let us choose the elements $w_a \in SU(N)$ inducing the
Weyl group transformations as at the end of Sect. 4. Let for $u \in U(1),$
\[ \chi_{\lambda}(u) = \sum_{i=0}^{N-1} u^{n_i} \]  
(8.10,a)
\[ \psi(a,b) = \chi_{k\lambda_a}(u_b) \cdot \begin{cases} 1 & \text{for } \frac{k}{n''} \text{ even,} \\ (-1)^{\frac{ab}{n''}} & \text{for } \frac{k}{n''} \text{ odd,} \end{cases} \]  
(8.10,b)
where $u_a \in U(1)$ are given by (4.11). The first formula may be viewed as extending
characters $\chi_{\lambda}$ to constant diagonal $U(N)$-matrices. The following properties of $\psi(a,b)$
are straightforward to verify:
\[ \psi(a + n'', b) = \psi(a, b + N) = \psi(a, b), \]  
(8.11,a)
\[ \psi(0, b) = 1, \quad \psi([a + b], c) = \psi(a, c) \psi(b, c), \]  
(8.11,b)
\[ \psi(a, [b + c])^{-1} \psi(a, b) \psi(a, c) = \chi_{k\lambda_a} (u_{[b+c]} u_b^{-1} u_c^{-1}). \]  
(8.11,c)
To each fixed weight $\lambda_0 = k\tau_0$ with $\tau_0 \in [\tau]$, we may assign a solution $(v^{\lambda_0}_{\tau,a})$ of (8.5)
given by
\[ v^{\lambda_0}_{\tau_0,a} = \chi_{k\tau_0}(u_a) \chi_{k\lambda_a}(u_a) \cdot \begin{cases} 1 & \text{for } \frac{k}{n''} \text{ even,} \\ (-1)^{\frac{2m}{n''}} & \text{for } \frac{k}{n''} \text{ odd,} \end{cases} \]  
(8.12,a)
\[ v^{\lambda_0}_{\sigma_\tau(\tau_0),a} = \psi(c,a)^{-1} v^{\lambda_0}_{\tau_0,a}. \]  
(8.12,b)
We shown in Appendix G that $(v^{\lambda_0}_{\tau,a})$ solves indeed (8.5) for $N'$ even and $N''$ odd. For
$N'$ odd or $N''$ even, since $V_{\tau,ab} = 1$, we may take the trivial solution of (8.5) $v^{\lambda_0}_{\tau,a} = 1$
(with a superfluous dependence on $\lambda_0$).

As we mentioned above, a general solution of (8.5) is obtained by multiplying the
particular solution $(v^{\lambda_0}_{\tau,a})$ by $(v'_{\tau,a})$ of (8.8). The bundle isomorphisms $i'$ obtained this
way determine stable isomorphisms between the restriction of gerbe $C'_k$ to $C'_{[\tau]}$ and gerbe $\mathcal{K}'$. Consequently, they determine the branes $(C'_{[\tau]}, kQ_{\tau}', N', \psi')$ supported by the conjugacy class $C'_{[\tau]}$ in $SU(N)/\mathbb{Z}$. As discussed in Sect 7, using the above structure, one may define the Wess-Zumino amplitudes (7.11) for the fields $\phi : \Sigma \to SU(N)/\mathbb{Z}$ satisfying boundary conditions (7.1) on subintervals of the boundary.

Note that the solutions $(\psi'_{\tau,a})$ of (8.3) differing by 1-coboundaries $(\psi''_{\sigma_a(\tau)}v_{\tau}'')^{-1}$ and giving rise to equivalent branes coincide for $a \in \mathcal{Z}_\tau$. Conversely, two solutions coinciding on $\mathcal{Z}_\tau$ necessarily differ by a 1-coboundary $(\psi''_{\sigma_a(\tau)}v_{\tau}'')^{-1}$. Indeed, they differ by a 1-cocycle $(\psi'_{\tau,a})$ such that $v_{\tau,a} = 1$ for $a \in \mathcal{Z}_\tau$. Setting $v_{\tau,0}' = 1$ for fixed $\tau_0 \in [\tau]$ and $v_{\sigma_a(\tau_0)} = v_{\tau_0,a}$ assures then that $v_{\tau,a}' = v_{\sigma_a(\tau)}v_{\tau}'$. Hence there is a one-to-one correspondence between the moduli of branes supported by $C'_{[\tau]}$ and the restrictions of the solutions $(\psi'_{\tau,a})$ of (8.3) to $a \in \mathcal{Z}_\tau$. The latter may be taken as products of the restrictions to $\mathcal{Z}_\tau$ of the special solutions $(\psi_{\tau,0}')$ assigned to $\lambda_0 = k\tau_0$ with $\tau_0 \in [\tau]$ by characters $\psi_{\lambda_0}$ of $\mathcal{Z}_\tau$ given by the right hand side of (8.8) with $a \in \mathcal{Z}_\tau$. As follows from (8.12), two pairs $(\lambda_0, \psi_{\lambda_0})$ and $(\lambda_0', \psi_{\lambda_0'}')$ give rise to the same restricted solution if

$$\lambda_0' = b\lambda_0 \quad \text{and} \quad \psi_{\lambda_0'}(a) = \phi_{\lambda_0}(b,a) \psi_{\lambda_0}(a), \quad (8.13)$$

with $b\lambda_0 = k\sigma_b^{-1}(\tau_0)$ and

$$\phi_{\lambda_0}(b,a) = \psi(b,a) v_{\tau_0,a}' / v_{\tau_0,a}'.$$  

(8.14)

Note that for any $b \in \mathcal{Z}$, $\phi_{\lambda_0}(b,a)$ must be a character of $\mathcal{Z}_\tau$ in its dependence on $a$ and that it satisfies a cocycle condition $\phi_{\lambda_0}(b,a) \phi_{\lambda_0}(c,a) = \phi_{\lambda_0}([b + c],a)$.

The upshot of the above discussion is that the set of moduli of symmetric branes in the $SU(N)/\mathbb{Z}$ WZW theory may be identified with the set of equivalence classes $[\lambda_0, \psi_{\lambda_0}]$ where $\lambda_0$ runs through the integrable weights and $\psi_{\lambda_0}$ through the characters of $\mathcal{Z}_{\tau_0}$ for $\lambda_0 = k\tau_0$, with the equivalence relation given by (8.13). This description of branes, obtained here from the Lagrangian considerations, agrees with the description of symmetric branes in simple current extension conformal field theories conjectured in [11] [11]. The general classification of the branes proposed there, basing on consistency considerations, involves equivalence classes of primary fields and characters of their “central stabilizers” that, for the $SU(N)$ WZW theory, reduce to the ordinary stabilizers $\mathcal{Z}_\tau$ in the simple current group $\mathcal{Z}$. The cocycle $\phi_{\lambda_0}(b,a)$ is not unique. If we multiply the special solution $(\psi'_{\tau,a})$ by a $\lambda_0$-dependent character $\rho_{\lambda_0}(a)$ of $\mathcal{Z}$, then

$$\phi_{\lambda_0}(b,a) \mapsto \phi_{\lambda_0}(b,a) \rho_{\lambda_0}(a)/\rho_{\lambda_0}(a).$$  

(8.15)

As we show in Appendix H, upon an appropriate choice of $\rho_{\lambda_0}(a)$, $\phi_{\lambda_0}(b,a)$ may be reduced to 1. In other words, it is possible to choose the solution $(\psi'_{\tau,a})$ so that, when restricted to $\mathcal{Z}_\tau$, it does not depend on $\lambda_0$.  

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9 Partition functions

Among the elementary quantum amplitudes of the WZW model are the partition functions. We shall describe them here in the simplest geometries: those of a torus for the bulk theory and of annulus for the boundary one, relating in the latter case the Lagrangian description with the use of gerbes to what was known from previous work. Although we shall concentrate on the example of the WZW model based on groups covered by $SU(N)$, the general picture should be similar for other WZW models.

9.1 Bulk case

The toroidal level $k$ partition functions are formally given by the functional integral over the toroidal amplitudes

$$Z(\tau) = \int e^{-S_\sigma(\phi)} A(\phi) \, D\phi,$$

(9.1)

where fields $\phi$ map the torus $T_\tau = \mathbb{C}/(2\pi\mathbb{Z} + \tau\mathbb{Z})$ with the modular parameter $\tau$ from the upper half plane to the group $G$, the sigma model action functional

$$S_\sigma(\phi) = \frac{k}{4\pi} \int \text{tr} (\phi^{-1} \partial \phi)(\phi^{-1} \bar{\partial} \phi),$$

(9.2)

and the amplitude $A(\phi)$ is obtained with the use of gerbe on $G$ with curvature $kH$. In the Hamiltonian language,

$$Z(\tau) = \text{tr}_H e^{2\pi i \tau (L_0 - \frac{c}{24}) - 2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})},$$

(9.3)

where $H$ is the closed-string Hilbert space composed of sections of the bundle $L$ over the loop group $LG$, operators $L_0, \bar{L}_0$ are the Virasoro generators and $c = \frac{k \dim(G)}{k + h^\vee}$ is the Virasoro central charge of the theory. For the connected, simply-connected simple compact groups,

$$H \cong \bigoplus \hat{V}_\lambda \otimes \bar{V}_\lambda,$$

(9.4)

where the sum is over the integrable weights, i.e. such that $\lambda \in kA_W$, and $\hat{V}_\lambda$ carries the unitary level $k$ irreducible representation of the current algebra $\hat{g}$ associated with group $G$ and the related action of the Virasoro algebra given by the Sugawara construction. Integrable weights $\lambda$ label the primary fields of the model with the fusion rule

$$\lambda_0 \ast \lambda_1 = \sum_\lambda N_{\lambda_0 \lambda_1}^\lambda \lambda.$$

(9.5)

The decomposition (9.4) implies that

$$Z(\tau) = \sum_\lambda |\hat{\chi}_\lambda(\tau)|^2,$$

(9.6)

1The modular parameter, for which we use the traditional notation, should not be confused with the Weyl alcove elements also denoted by $\tau$ in the present paper.
where \( \hat{\chi}_\lambda(\tau) = \text{tr}_{\hat{V}_\lambda} e^{2\pi i (L_0 - \frac{c}{24})} \) are the (restricted) level \( k \) affine characters of \( \hat{g} \) satisfying

\[
\hat{\chi}_\lambda(\tau) = e^{-2\pi i h(\lambda)} \hat{\chi}_\lambda(\tau + 1) = \sum_{\lambda'} S_{\lambda \lambda'}^{\lambda'}(\tau) = \sum_{\lambda'} \hat{S}_{\lambda \lambda'}^{\lambda'}(\tau),
\]

where \( h(\lambda) \) is the conformal weight of the primary field corresponding to \( \lambda \) and \( S_{\lambda \lambda'}^{\lambda'} = S_{\lambda \lambda'}^{\lambda'} = S_{\lambda \lambda'}^{\lambda'} \) are the elements of a unitary modular matrix \( S \) which enter the Verlinde formula for the fusion coefficients:

\[
N_{\lambda_0 \lambda_1}^\lambda = \sum_{\lambda'} S_{\lambda_0 \lambda}^{\lambda'} S_{\lambda \lambda_1}^{\lambda'} S_{\lambda_1 \lambda_0}^{\lambda'}. \tag{9.8}
\]

The toroidal partition functions for all connected non-simply connected simple compact groups were first obtained in [17]. For \( G' = SU(N)/Z \) with \( Z \cong \mathbb{Z} N' \), the integer level \( k \geq 0 \) is restricted by the condition of integrality of the 3-form \( \frac{k N'}{2} \text{tr} \lambda^2 = \frac{1}{2} k N' (N' - 1) \in \mathbb{Z} \).

The closed string Hilbert space

\[
\hat{H}' \cong \bigoplus_{z_a \in \mathbb{Z}} \bigoplus_{\lambda \in C_a} \hat{V}_\lambda \otimes \overline{V}_\lambda, \tag{9.10}
\]

where, as before, for integrable weights \( \lambda = k \tau = \sum_{i=0}^{N-1} n_i \lambda_i \) with \( \sum n_i = k \), the transformed weight

\[
a_\lambda = k \sigma^{-1}_a(\tau) = \sum n_i \lambda_{[i+a]}, \tag{9.11}
\]

and where

\[
C_a = \{ \lambda \mid \text{tr} \lambda \lambda_{N''} + \frac{ka}{2N''} \text{tr} \lambda_{N''}^2 \in \mathbb{Z} \}
\]

\[
= \{ \lambda \mid -\sum_{N''}^{n_{in}} + \frac{ka(N'' - 1)}{2N''} \in \mathbb{Z} \}. \tag{9.12}
\]

Consequently, the partition function

\[
Z'(\tau) = \sum_{z_a \in \mathbb{Z}} \sum_{\lambda \in C_a} \hat{\chi}_\lambda(\tau) \overline{\hat{\chi}_\lambda(\tau)}
\]

\[
= Z'(\tau + 1) = Z'(\tau). \tag{9.13}
\]

One obtains this a way a family of modular invariant sesquilinear combinations of characters \( \hat{\chi}_\lambda(\tau) \), for example for the case of the \( SU(2) \) group, the \( A \) and \( D \) series in the \( ADE \) classification [8] of modular invariants.

The above expressions for the toroidal partition functions coincide with the ones for the “simple current extensions” [32] of the \( SU(N) \) WZW theory by the simple current
group $Z$ generated by the simple current $J$ corresponding to the integrable weight $k\lambda_{N''}$ with the fusion rule

$$J^{a'*} \lambda = {}^{a'}\lambda. \quad (9.14)$$

for $a = a'N''$. The restriction on the level $k$ is expressed by demanding that the conformal weight $h(J) = \frac{k}{2} \text{tr} \lambda_{N''}^2$ multiplied by the order $N'$ of $J$ be an integer, which coincides with condition (9.9). The requirement $\lambda \in C_a$ is expressed by the monodromy charge with respect to the simple current $J$ and its modulo 2 refinement. The monodromy charge

$$Q_J(\lambda) = h(\lambda) + h(J) - h(J*\lambda) \mod 1$$

$$= -\text{tr} \lambda \lambda_{N''} \mod 1 = \sum_{N'} \text{inj}_i \mod 1 \quad (9.15)$$

is an important quantity of the conformal field theory. It is conserved in fusion and it relates the matrix elements of the modular matrix $S_{\lambda}^\lambda$ along the orbits of $Z$:

$$S_{\lambda'}^\lambda = e^{2\pi i a'Q_J(\lambda)} S_{\lambda}^\lambda. \quad (9.16)$$

The condition $\lambda \in C_a$ is equivalent to demanding that

$$Q_J(\lambda) + a'X \in \mathbb{Z}, \quad (9.17)$$

where

$$X = -\frac{k}{2} \text{tr} \lambda_{N''}^2 \mod 1 = -\frac{kN''(N'-1)}{2N'} \mod 1 \quad (9.18)$$

so that $2X = Q_J(J) \mod 1$.

When $h(J) = \frac{k}{2} \text{tr} \lambda_{N''}^2$ is an integer, the sets $C_a$ coincide for different $a$ and are preserved by the action of $Z$ on the set of integrable weights. In this case, the “pure simple current extension” in the terminology of [32], the partition function (9.13) may be rewritten as

$$Z'(\tau) = \frac{1}{N'} \sum_{\lambda \in C_0} \left| \sum_{z_\lambda \in Z} \tilde{\chi}_{x_\lambda}(\tau) \right|^2 = \sum_{\lambda \in \mathbb{Z}} |Z_{\lambda}| \sum_{\lambda \in [\lambda]} \tilde{\chi}_{\lambda}(\tau), \quad (9.19)$$

where $[\lambda]$ runs through the set of the $Z$-orbits in the set of integrable weights and $Z_{\lambda}$ denote the corresponding isotropy subgroups of $Z$ (we shall use this notation also below).

### 9.2 Boundary case

For the WZW model based on the simply-connected group $G = SU(N)$, the open string annular partition function corresponding to branes $D_s$ supported by the conjugacy classes $C_{r_s}$ with $s = 0, 1$ is formally given by the functional integral expression

$$Z_{C_{r_0}C_1}(T) = \int e^{-S_\phi(\phi)} A_{r_0C_1}(\phi) D\phi, \quad (9.20)$$
where $\phi : [0, T] \times [0, \pi] \to G$ with
\[
\phi(t, 0) \in C_{\tau_0}, \quad \phi(t, \pi) \in C_{\tau_1}, \quad \phi(T, x) = \phi(0, x),
\]
i.e. $\phi$ is periodic in the time direction. In the Hamiltonian language,
\[
Z_{D_0 D_1}(T) = \text{tr}_{\mathbb{H}^{D_0 D_1}} e^{-T(L_0 - \frac{c}{24})},
\]
where the open string Hilbert space $\mathbb{H}^{D_0 D_1}$ is composed of sections of the bundle $L_{D_0 D_1}$ over the space of open paths in $G$. Space $\mathbb{H}^{D_0 D_1}$ carries a unitary representation of the current algebra $\hat{g}$ that decomposes into the irreducible representations according to
\[
\mathbb{H}^{D_0 D_1} \cong \bigoplus_{\lambda} W_{\lambda_0 \lambda} \otimes \hat{V}_{\lambda},
\]
where $\lambda_s = k\tau_s$ and the multiplicity spaces $W_{\lambda_0 \lambda}$ may be naturally identified with the spaces of 3-point conformal blocks of the bulk group $SU(N)$ level $k$ WZW theory on the sphere with insertions of the primary fields corresponding to the integrable weights $\lambda_0$, $\lambda_1$ and $\lambda$. In particular, the dimension of the multiplicity spaces is given by the fusion coefficients:
\[
\dim(W_{\lambda_0 \lambda}) = N_{\lambda_0 \lambda}^{\lambda_1}.
\]
As the result, the annular partition functions of the $SU(N)$ WZW theory take the form
\[
Z_{D_0 D_1}(T) = \sum_{\lambda} N_{\lambda_0 \lambda}^{\lambda_1} \chi_{\lambda}(\frac{T}{2\pi}).
\]
In the dual Hamiltonian description, the annular partition functions may be described as closed string matrix elements
\[
Z_{D_0 D_1}(T) = \langle D_1 | e^{-\frac{2\pi i}{2} (L_0 + \bar{L}_0 - \frac{c}{24})} | D_0 \rangle.
\]
The states $|D_s\rangle$ corresponding to branes $D_s$ belong to (a completion of) the closed string Hilbert $\mathbb{H}$. They are combinations of the so called Ishibashi states $|\lambda\rangle$ representing the identity operators of the representation spaces $\hat{V}_{\lambda}$:
\[
|\lambda\rangle = \sum_i e_i^\lambda \otimes \bar{e}_i^\lambda
\]
for any orthonormal basis $(e_i^\lambda)$ of $\hat{V}_{\lambda}$. Explicitly, for the branes $D_s$ supported by the conjugacy classes $C_{\tau_s}$ with $\lambda_s = k\tau_s$,
\[
|D_s\rangle = \sum_{\lambda} \frac{S_{\lambda}}{\sqrt{S_{\lambda}}} |\lambda\rangle.
\]
Since
\[
\langle \lambda | e^{-\frac{2\pi i}{2} (L_0 + \bar{L}_0 - \frac{c}{24})} | \lambda' \rangle = \delta_{\lambda \lambda'} \hat{\chi}_{\lambda'}(\frac{2\pi i}{T}),
\]
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the right hand side of (9.26) is then equal to
\[
\sum_{X'} \frac{S'_{0'}^X}{S'_{0}} \lambda_X(\frac{2\pi i}{\tau}) ,
\] (9.30)
which indeed coincides with the right hand side of (9.23) in virtue of the modular property (9.7) of the affine characters and the Verlinde formula (9.8).

For the non-simply-connected group \( G' = SU(N)/Z \), the annular partition function corresponding to branes \( D'_s \) supported by the conjugacy classes \( C'_{[a_s]} \subset G' \) is given by the primed versions of (9.20) and (9.22):
\[
Z'_{D'_0,D'_1}(T) = \int e^{-k s(s) A'_{D'_0,D'_1}(\phi')} D\phi' = \text{tr}_{D'_0,D'_1} e^{-T(L_0-\frac{c}{12})}. \] (9.31)
The functional integral is now over fields \( \phi' : [0, T] \times [0, \pi] \to G' \) such that
\[
\phi'(t,0) \in C'_{[a_0]} , \quad \phi'(t,\pi) \in C'_{[a_1]} , \quad \phi'(T,x) = \phi'(0,x) .
\] (9.32)
Each \( \phi' \) may be lifted in \( N' \) different ways to a twisted periodic map \( \phi : [0, T] \times [0, \pi] \to G \) such that
\[
\phi(t,0) \in C_{a_0} , \quad \phi(t,\pi) \in C_{a_1} , \quad \phi(T,x) = z_a \phi(0,x)
\] (9.33)
for \( \tau_s \in [\tau_s] \) and \( z_a \in Z_{[\tau_a]} \cap Z_{[\tau_1]} \subset Z \). Expressing the functional integral over fields \( \phi' \) in terms of the one over their lifts leads to a natural representation for the boundary partition functions of \( G' \) WZW theory which does not seem to have appeared in the literature, although it is related to the well studied string theory construction of the branes in orbifold theories, see e.g. [12] [13]. Unlike the amplitudes \( A'_{D'_0,D'_1}(\phi) \), which are complex numbers, the ones of field \( \phi \) are line-bundle valued:
\[
A_{D_0,D_1}(\phi) = (L_{[\tau_0]}^{-1})_{\phi} \otimes (L_{D_0,D_1})^{-1}_\phi ,
\] (9.34)
where \( \varphi(x) = \phi(T,x) , \quad \tilde{\varphi} = \phi(0,x) = z_a^{-1} \varphi(x) \) and \( D_s \) are the branes of the group \( G \) theory supported by the conjugacy classes \( C_s \).

How do the amplitudes \( A_{D_0,D_1}(\phi) \) relate to \( A'_{D'_0,D'_1}(\phi') \)? The point is that the bundle gerbe \( G'_k \) on \( G' \) together with the branes \( D'_s \), determine canonically for any \( \varphi \in L_{[\tau_0]} \cap \tau_1 , \quad z_a \in Z \), and \( \tilde{\varphi} = z_a^{-1} \varphi \) a non-zero element
\[
\Phi(\tilde{\varphi},\varphi) = (L_{[\tau_0]}^{-1})_{\tilde{\varphi}} \otimes (L_{D_0,D_1})_\varphi ,
\] (9.35)
where the branes \( D_s = (C_s, kQ_{\tau_s}, N_s, \tau_s) \) are obtained by the restriction of the branes \( D'_s = (C'_{[a_s]}, kQ_{\tau_s}, N'_{[a_s]}, \tau_s) \) to the conjugacy classes \( C_{[a_s]} \) for \( \tau_s \in [\tau_s] \). We shall call such group \( G \) theory branes \( D_s \) compatible with \( D'_s \). Now, for \( \varphi = \phi(T, \cdot) \) and \( \tilde{\varphi} = \phi(0, \cdot) \), with \( z_a \in Z_{[\tau_a]} \cap Z_{[\tau_s]} \),
\[
A'_{D'_0,D'_1}(\phi') = \langle A_{D_0,D_1}(\phi) , \Phi(\tilde{\varphi},\varphi) \rangle
\] (9.36)
in the natural pairing. We describe a construction of the elements $\Phi(\tilde{\varphi}, \varphi)$, possessing the multiplicative property
\[
\Phi(\tilde{\varphi}, \varphi) \otimes \Phi(\tilde{\varphi}, \varphi) = \Phi(\tilde{\varphi}, \varphi)
\]  
(9.37)
for $\tilde{\varphi} = z_a^{-1} \varphi$ and $\tilde{\varphi} = z_0^{-1} \varphi$, in Appendix I. The construction relies on the concept of branes as formulated in the preceding sections.

The multiplication by $\Phi$ determines an isomorphism between the line bundles $L_{D_0 D_1}$ and $L_{D_0 D_1}$ covering the map $z_a^{-1} \varphi \mapsto \varphi$ between the spaces $I_{c_{\sigma_a(r_0)}, c_{\sigma_a(r_1)}}$ and $I_{c_{r_0} c_{r_1}}$ of open paths in $G$. Altogether, one obtains the action of $Z$ on the bundle
\[
\tilde{L}'_{D_0 D_1} = \bigcup_{(D_s)} L_{D_0 D_1}
\]  
(9.38)
where the union is taken over the branes $D_s$ compatible with $D_s$, $s = 0, 1$. The multiplicativity of this action follows from (9.37). The line bundle $L'_{D_0 D_1}$ over the space $I'_{D_0 D_1}$ of open paths in $G$ is canonically isomorphic with the quotient bundle $\tilde{L}'_{D_0 D_1}/Z$. By the formula
\[
(\sigma \Psi)(\varphi) = \Psi(z_a^{-1} \varphi) \otimes \Phi(z_a^{-1} \varphi, \varphi),
\]  
(9.39)
the action of $Z$ may be carried to sections of the line bundle $\tilde{L}'_{D_0 D_1}$ in the way that maps sections of $L_{D_0 D_1}$ to those of $L_{D_0 D_1}$. Finally, sections of the line bundle $L'_{D_0 D_1}$ may be identified with sections of $\tilde{L}'_{D_0 D_1}$ invariant under the action of $Z$.

The above gives rise to the following simple picture of the open string space of states $\mathbb{H}'_{D_0 D_1}$ of the group $G'$ WZW theory. The maps $\Psi \mapsto \sigma \Psi$ induce the (unitary) transformations
\[
U(a) : \mathbb{H}_{D_0 D_1} \longrightarrow \mathbb{H}_{D_0 D_1}
\]  
(9.40)
between the open string spaces of states of the group $G$ WZW theory. It may be shown that those transformations commute with the current algebra and hence also Virasoro algebra actions. Put together, they define a representation $U$ of the group $Z$ in the Hilbert space
\[
\mathbb{H}'_{D_0 D_1} = \bigoplus_{r_0 \in [r_0]} \mathbb{H}_{D_0 D_1}.
\]  
(9.41)
When restricted to $Z_{r_0} \cap Z_{r_1} \subset Z$, this representation acts diagonally, i.e. within the group $G$ open string spaces $\mathbb{H}_{D_0 D_1}$ with fixed $D_s$. The open string space states for the group $G'$ theory may be naturally identified with the $Z$-invariant families of states of the group $G$ theory:
\[
\mathbb{H}'_{D_0 D_1} \simeq P \mathbb{H}'_{D_0 D_1},
\]  
(9.42)
denoting for the projector on the \( Z \)-invariant subspace. The scalar product in the space \( \mathbb{H}_{\mathcal{D}'_0 \mathcal{D}'_1} \) should, however, be divided by \( N' \) with respect to the one inherited from \( \tilde{\mathbb{H}}_{\mathcal{D}'_0 \mathcal{D}'_1} \) to avoid the overcount.

For the annular partition function of the group \( G' \) theory one obtains this way the Hamiltonian expression:

\[
  Z'_{\mathcal{D}'_0 \mathcal{D}'_1} (T) = \frac{1}{N'} \sum_{\tau_0 \in \{ \tau_0 \}} \sum_{\tau_1 \in \{ \tau_1 \}} \sum_{z_0 \in \mathbb{Z} \cap \mathbb{Z} \tau_1} \sum_{\lambda} \left( \text{tr}_{W^\lambda_{\mathcal{D}'_0 \mathcal{D}'_1}} U(a) \right) \hat{\chi}_\lambda \left( \frac{T_1}{2\pi} \right),
\]

This indeed is compatible with the functional integral formula if rewrite the functional integral over fields \( \phi' \) in \((9.33)\) in terms of the one over fields \( \phi \) using relation \((9.36)\) and the equality of the sigma model actions \( S_\sigma (\phi') = S_\sigma (\phi) \). The factor \( \frac{1}{N'} \) takes care of the \( N' \)-fold overcount due to the fact that there are \( N' \) fields \( \phi \) corresponding to each \( \phi' \).

The commutation of the maps \( U(a) \) of \((9.40)\) with the the current algebra action implies that they descend to the multiplicity spaces in the decomposition \((9.23)\):

\[
  U(a) : W^\lambda_{\mathcal{D}'_0 \mathcal{D}'_1} \rightarrow W^a\lambda_{\mathcal{D}'_0 \mathcal{D}'_1}
\]

We may then rewrite expression \((9.44)\) for the open string partition function as

\[
  Z'_{\mathcal{D}'_0 \mathcal{D}'_1} (T) = \frac{1}{N'} \sum_{\gamma_0 \in \{ \gamma_0 \}} \sum_{\gamma_1 \in \{ \gamma_1 \}} \sum_{z_0 \in \mathbb{Z} \cap \mathbb{Z} \gamma_1} \sum_{\lambda} \left( \text{tr}_{W^\lambda_{\mathcal{D}'_0 \mathcal{D}'_1}} U(a) \right) \tilde{\chi}_\lambda \left( \frac{T_1}{2\pi} \right),
\]

i.e. in terms of the traces of the action of simple currents on the spaces of 3-point conformal blocks. The actions of simple currents on spaces of genus zero conformal blocs have been defined, up to phases, in \([21]\). In the action \((9.43)\), the phase freedom is fixed by the choice of brane structures \( \mathcal{D}'_s \) on the conjugacy classes \( \mathcal{C}'_{\{ \gamma_s \}} \) in \( G' \). Under the change of the brane structures \( \mathcal{D}'_s \) twisting the isomorphisms \( \iota' \) of \((8.3)\) by the multiplication of \( v_{\gamma_{s, a}} \) by \( v'_{\gamma_{s, a}} \) of \((8.8)\) with \( r = r_s \),

\[
  \Phi(\bar{\varphi}, \varphi) \mapsto (-1)^{\frac{2a_{m}}{N}} (-1)^{\frac{2a_{r}}{N}} \Phi(\bar{\varphi}, \varphi)
\]

inducing the transformation

\[
  U(a) \mapsto (-1)^{\frac{2a_{m}}{N}} (-1)^{\frac{2a_{r}}{N}} \frac{a_{r}}{U(a)},
\]

i.e. multiplying the representation \( U \) by the ratio of characters of \( Z \).

The annular partition functions for the simple current extension conformal field theories have been described in \([20, 41]\), see also \([37]\). They fit into the general scheme.
identified in [3]. In the dual Hamiltonian description they are given by the primed version of the matrix elements (9.26). The states $|D_s'> in (a completion of) the closed string boundary space $H'$ may now be expressed as combinations of the Ishibashi states $|\lambda,z_a>, in the diagonal components of $H'$, see (9.10). Let us denote by $E$ the corresponding set of labels, i.e.

$$E = \{ (\lambda,a) \mid \lambda \in C_{a}, z_a \in Z_{\lambda} \} .$$

Explicitly, for the branes $D_s'$ supported by the conjugacy classes $C'_{[\tau_s]}$ corresponding to the equivalence classes $[\lambda_s,\psi_{\lambda_s}]$, where $\lambda_s = k\tau_s$ and $\psi_{\lambda_s}$ are characters of $Z_{\tau_s}$, see the end of Sect. 8.2,

$$|D'_s> = \sum_{(\lambda,a) \in E} \frac{\Psi_{D'_s}^{(\lambda,a)}}{\sqrt{S_0^{\lambda}}} |\lambda,a>$$

with

$$\Psi_{D'_s}^{(\lambda,a)} = \frac{\sqrt{N'}}{|Z_{\lambda}|} S_\lambda^\lambda(a) \psi_{\lambda_s}(a) .$$

Here $S_\lambda^\lambda(a)$ are the matrix elements of modified unitary modular matrices non-zero only if $z_a \in Z_{\lambda} \cap Z_{\lambda'}$. They satisfy the identity

$$S_\lambda^\lambda(a) = \phi_{\lambda'}(b,a)^{-1} e^{2\pi i (Q(b) + a'X) v_{b,a}} S_\lambda^\lambda(a) ,$$

see [19][20]. The last relation, together with (9.17), assures that the right hand side of (9.51) is independent of the choice of $\lambda_s \in [\lambda_s]$ if $\phi_{\lambda}(b,a)$ is the same as the one used in the definition (8.13) of the equivalence classes $[\lambda_s,\psi_{\lambda_s}]$. Recall that the latter was fixed up to the transformations (8.15) with the help of which, as shown in Appendix H, it could be reduced to 1. As described in [19], up to a phase that does not depend on $\lambda$ and $\lambda'$, the matrix elements $S_{\lambda'}^\lambda(a)$ are equal to the entries $\tilde{S}_\lambda^\lambda$ of the modular matrix of the WZW theory based on the so called "orbit Lie algebra". In our case the latter theory is the level $\frac{k}{n'}$ one with group $SU(n'')$, where $n'$ is the order of the subgroup $Z_a$ of $Z$ generated by $z_a$ and $N = n'n''$. Writing

$$\lambda = \sum_{i=0}^{n''-1} n_i \sum_{\hat{a}=0}^{\hat{n}'-1} \lambda_{\hat{a}+\hat{n}''}$$

with $\sum_{i=0}^{n''-1} n_i = \frac{k}{n'}$, the corresponding weight of the orbit Lie algebra is

$$\tilde{\lambda} = \sum_{i=0}^{n-1} n_i \tilde{\lambda}_i ,$$

where $\tilde{\lambda}_i$ are the fundamental weights of $SU(n'')$. One has

$$S_{\lambda'}^\lambda(a) = (-1)^{\frac{k((n')^2-1)n''}{4n'}} \tilde{S}_\lambda^\lambda .$$

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In particular, \( S^\lambda_0(a) \) depends on \( a \) only through \( Z_a \). For fixed \( a \), the action of \( Z \) on the integral weights \( \lambda \) such that \( z_a \in Z_\lambda \) descends to the one of the quotient group \( \tilde{Z} \equiv Z / Z_a \) on the weights \( \tilde{\lambda} \). The induced action is generated by the fusion with the simple current \( J \) of the \( SU(n') \) theory with the weight \( \tilde{\lambda}_1 \). The identity \( (9.16) \) for the orbit group implies now that for \( z_b \in Z \),

\[
\tilde{S}^\tilde{\lambda}_0(\lambda') = e^{2\pi i b \tilde{Q}_J(\tilde{\lambda})} \tilde{S}^\lambda_{\lambda'}.
\]

(9.56)

In Appendix H we show that

\[
e^{2\pi i b \tilde{Q}_J(\tilde{\lambda})} = e^{2\pi i \nu'(Q_J(\lambda)+a'X)},
\]

(9.57)

so that \( (9.52) \) holds with \( \phi_{\lambda'}(b,a) \equiv 1 \), which is compatible with the results of Sect. 8.2.

Relations \( (9.50) \) and \( (9.51) \), with the use of \( (9.29) \) and \( (9.7) \), lead to the following expression for the annular partition function:

\[
Z'_{D_0' D_1'}(\tau) = \sum_{\lambda} N_{D_0' D_1'}^{D_0' D_1'}(\lambda) \tilde{\chi}_\lambda(\frac{T_i}{2\pi}),
\]

(9.58)

where

\[
N_{D_0' D_1'}^{D_0' D_1'}(\lambda) = \sum_{(\lambda',a) \in E} \frac{\psi_{D_0'}^{(\lambda,a)} \psi_{D_1'}^{(\lambda',a') \lambda'}}{S_0'} S_\lambda \delta_{\lambda\lambda'} \delta_{aa'}
\]

\[
= \frac{\nu'}{|Z_{\lambda_0}| |Z_{\lambda_1}|} \sum_{z_a \in Z_{\lambda_0} \cap Z_{\lambda_1}} \sum_{\lambda' \in C_a} \frac{S_{\lambda}'(a)}{S_0'} \frac{S_{\lambda_1}'(a)}{S_0'} \psi_{\lambda_0}(a) \psi_{\lambda_1}(a),
\]

(9.59)

where, again, the right hand side does not depend on the choice of \( \lambda_s \in [\lambda_s] \). The orthogonality relations \( 20 \)

\[
\sum_{D_0'} \psi_{D_0'}^{(\lambda,a)} \psi_{D_1'}^{(\lambda',a')} = \delta_{\lambda\lambda'} \delta_{aa'}
\]

(9.60)

guarantee that the matrices \( N_\lambda = (N_{D_0' D_1'}^{D_0' D_1'}) \), whose entries have to be nonnegative integers, represent the fusion algebra \( 3 \):

\[
\sum_{D_1'} N_{D_0' D_1'}^{D_0' D_1'}(\lambda) N_{D_0' D_1'}^{D_0' D_1'}(\lambda') = \sum_{\lambda''} N_{\lambda\lambda''} \nu' N_{D_0' D_1'}^{D_0' D_1'}(\lambda'').
\]

(9.61)

Search for the representations of the fusion algebra by matrices with entries that are nonnegative integers (the so called “NIM’s”) has been the basis of the approach to classification of boundary conformal field theories developed in \( 3 \), see also \( 38 \).

Expressions \( (9.58) \) with \( (9.59) \) are compatible with relation \( (9.46) \) if we assume the following formula for the traces of the action of \( Z_{\lambda_0} \cap Z_{\lambda_1} \) on the spaces of 3-point conformal blocks conjectured (up to multiplication by characters) in \( 19 \):

\[
\text{tr}_{W_{\lambda_0 \lambda_1}} U(a) = \sum_{\lambda'} \frac{S_{\lambda_0}'(a) S_{\lambda_1}'(a) S_{\lambda'}' \psi_{\lambda_0}(a)}{S_0'} \psi_{\lambda_1}(a)
\]

(9.62)
Indeed, the summation of (162) over the weights in fixed orbits gives

$$
\sum_{\lambda_0, \lambda_1, \lambda_2 \in [\lambda_s]} \tr_{W_{\lambda_0 \lambda_1 \lambda_2}} U(a) = \frac{1}{|z_{\lambda_0}| |z_{\lambda_1}|} \sum_{\lambda' \in C_a} \sum_{\lambda_0, \lambda_1, \lambda_2 \in [\lambda_s]} \frac{S^\lambda_{\lambda_1}(a) S^\lambda_{\lambda_2}(a) S^{\lambda'}_{\lambda_0}(a)}{S^0_{\lambda_0}(a)} \psi_{\lambda_0}(a) \psi_{\lambda_1}(a). \tag{9.63}
$$

Note that the sum over $\lambda'$ is effectively restricted to integrable weights fixed by $z_0$. With the use of transformation property (952), the sums over $z_b, z_c$ may be factored out as $\left| \sum_{z_b \in Z} e^{2\pi i b \langle Q_J(\lambda') + c'X \rangle} \right|^2$. Since the sum inside that factor divided by $N'$ represents the characteristic function of $C_a$, the identity (9.63) reduces to the relation

$$
\sum_{\lambda_0, \lambda_1, \lambda_2 \in [\lambda_s]} \tr_{W_{\lambda_0 \lambda_1 \lambda_2}} U(a) = \frac{(N')^2}{|z_{\lambda_0}| |z_{\lambda_1}|} \sum_{\lambda' \in C_a} \sum_{\lambda_0, \lambda_1, \lambda_2 \in [\lambda_s]} \frac{S^\lambda_{\lambda_1}(a) S^\lambda_{\lambda_2}(a) S^{\lambda'}_{\lambda_0}(a)}{S^0_{\lambda_0}(a)} \psi_{\lambda_0}(a) \psi_{\lambda_1}(a), \tag{9.64}
$$

where on the right hand side the weights $\lambda_s \in [\lambda_s]$ are fixed and the sum over $\lambda'$ is additionally constraint by the requirement $\lambda' \in C_a$. Equation (9.64), when inserted into (9.46), reproduces (9.58).

One may also consider partition functions that do not resolve different branes with the same support. Summing $Z_{c'_0, c'_1}^\prime(T)$ over the different brane structures $D_{c'_0}$ and $D_{c'_1}$ supported by the conjugacy classes $C_{[b_0]}$ and $C_{[b_1]}$, respectively, freezes $z_a$ in (9.46) to 1 and leads to the unresolved partition functions

$$
Z_{c'_0, c'_1}^\prime(T) = \sum_{\lambda_0, \lambda_1 \in [\lambda_s]} \sum_{\lambda_0, \lambda_1 \in [\lambda_s]} |z_{\lambda_0}| |z_{\lambda_1}| \tr_{W_{\lambda_0 \lambda_1}} e^{-T(L_{\lambda_0} - \pi i)}
= \sum_{\lambda_0, \lambda_1 \in [\lambda_s]} \sum_{\lambda_0, \lambda_1 \in [\lambda_s]} |z_{\lambda_0}| |z_{\lambda_1}| \sum_{\lambda} N_{\lambda_0 \lambda_1 \lambda} \lambda (T_{\lambda} - \pi i / 2), \tag{9.65}
$$

where, as usually, $\lambda_s = k\tau_s$. Rewriting the sums over the orbits $[\lambda_s]$ as sums over the group $Z$ and using the symmetry of the fusion coefficients $N_{\lambda_0 \lambda_1 \lambda} = N_{\lambda_1 \lambda_0 \lambda}$, we finally obtain

$$
Z_{c'_0, c'_1}^\prime(T) = \sum_{z_a \in Z} \sum_{\lambda} N_{\lambda_0 \lambda_1 \lambda} \lambda (T_{\lambda} - \pi i / 2), \tag{9.66}
$$

where on the right hand side, $\lambda_s$ are arbitrary elements in the $Z$-orbits $[\lambda_s]$. The relation between the annular partition functions with resolved and unresolved branes was discussed in (33) for the case of pure simple current extensions. The geometric approach based on gerbes should allow to recover within the Lagrangian framework similar relations to the ones discussed above for general orbifold conformal field theories.

### 10 Quantum amplitudes

The general quantum amplitudes of the WZW theory based on group $G$ are formally given by the functional integrals

$$
A_{(D_s)}(\Sigma) = \int e^{-S_\phi(\phi)} A_{(D_s)}(\phi) D\phi \tag{10.1}
$$
over fields $\phi : \Sigma \to G$ satisfying boundary conditions \((7.1)\) on closed disjoint subintervals $\ell_s$ of the boundary loops of $\Sigma$, with the amplitude $A_{(D_s)}(\phi)$ as in \((7.11)\).

Accordingly, we should have

$$A_{(D_s)}(\Sigma) \in \bigotimes_{(s,s')} \mathbb{H}_{D_s D_{s'}} \otimes \bigotimes_m \mathbb{H}.$$ \hfill (10.2)

The (purely) open string amplitudes have no external closed string factors $\mathbb{H}$ (although they may have closed strings states propagating in loops). In particular, if $\Sigma$ is a disc $O$ and fields $\phi$ are constrained to map three disjoint subintervals of the boundary into the supports of three branes $D_s$, $s = 0, 1, 2$, see Fig. 4,

![Fig. 4](image)

one obtains the quantum open string amplitude

$$A_{D_0 D_1 D_2}(O) \in \mathbb{H}_{D_0 D_1} \otimes \mathbb{H}_{D_1 D_2} \otimes \mathbb{H}_{D_2 D_0}$$ \hfill (10.3)

that encodes the operator product expansion of the boundary operators. The functional integral representation, together with the geometric interpretation of the relation \((9.42)\) between the spaces of open string states for group $G$ and group $G'$ WZW models leads to the following relation between the amplitudes \((10.3)\) for the two cases:

$$A'_{D'_0 D'_1 D'_2}(O) = (N')^2 \, P \otimes P \otimes P \, \tilde{A}'_{D'_0 D'_1 D'_2}(O)$$ \hfill (10.4)

with $P$ is given by \((9.43)\) and

$$\tilde{A}'_{D'_0 D'_1 D'_2}(O) = \bigoplus_{(D_s)} A_{D_0 D_1 D_2}(O) \in \bigoplus_{(D_s)} \mathbb{H}_{D_0 D_1} \otimes \mathbb{H}_{D_1 D_2} \otimes \mathbb{H}_{D_2 D_0}$$

$$\subset \mathbb{H}'_{D'_0 D'_1} \otimes \mathbb{H}'_{D'_1 D'_2} \otimes \mathbb{H}'_{D'_2 D'_0},$$ \hfill (10.5)

see \((9.41)\). The direct sums above are over branes $D_s$ compatible with $D'_s$, $s = 0, 1, 2$. An analogous relation, with $(N')^2$ replaced by $(N')^{M-1}$, holds for the disc amplitudes.
with $M$ subintervals mapped into brane supports. The latter amplitudes permit to
reconstruct the general open string amplitudes of the WZW theory by gluing. This
way one obtains a simple relation between the quantum open string amplitudes for the
theory based on the simply connected group $G$ and for its simple current orbifolds.
We postpone a more detailed discussion of that point to a future publication.

\section{Local constructions}

Let us describe at the end the local versions of the constructions inducing from a bundle
gerbe on manifold $M$ various geometric structures on the spaces of closed or open curves
(strings) in $M$.

\subsection{Closed strings}

We start by recalling from \cite{23} the construction of local data for a line bundle on $LM$
from local data $(g_{ijk}, A_{ij}, B_i)$ of a gerbe on an open covering $(O_i)$ of $M$.

Given a split of the circle $S^1$ into closed intervals $b$ with common vertices $v$ and an
assignment $b \mapsto i_b$ and $v \mapsto i_v$, that we collectively abbreviate as $I$, consider the open subset

\[ O_I = \{ \varphi \in LM \mid \varphi(b) \subset O_{i_b}, \varphi(v) \in O_{i_v} \} \quad (11.1) \]

of the loop space $LM$. Sets $O_I$ for different choices of $I$ cover $LM$ (we may discard $I$
for which $O_I = \emptyset$). Let us define 1-forms $A_I$ on $O_I \subset LM$ by

\[ \langle \delta \varphi, A_I(\varphi) \rangle = \sum_b \int_b \varphi^* i(\delta \varphi) B_{i_b} + \sum_{v \subset b} \langle \delta \varphi(v), A_{i_v,i_b} \rangle \quad (11.2) \]

with the usual sign convention in $\sum_v$.

If $I$ is the collection of $(b, v, i_b, i_v)$ and $J$ the one of $(b', v', j_b, j_v')$, consider the split
of $S^1$ into intersections $\overline{b}$ of the intervals $b$ and $b'$ and denote by $\overline{v}$ its vertices. The new split inherits two label assignments from the original ones. We set

\[
\begin{align*}
  i_b = i_b & \quad \text{if} \quad \overline{b} \subset b, \\
  j_b = j_b' & \quad \text{if} \quad \overline{b} \subset b', \\
  i_v = \begin{cases} i_v & \text{if} \quad \overline{v} = v, \\
  i_b & \text{if} \quad \overline{v} \subset \text{int}(b), \end{cases} &
  j_v = \begin{cases} j_v' & \text{if} \quad v = v', \\
  j_b' & \text{if} \quad \overline{v} \subset \text{int}(b'), \end{cases}
\end{align*}
\quad (11.3)\]

where $\text{int}(b)$ denotes the interior of $b$. Let us define functions $g_{IJ} : O_{IJ} \to U(1),$

\[ g_{IJ}(\varphi) = \exp \left[ i \sum_b \left( \varphi^* A_{ij_b} \right) \prod_{v \subset b} \left( g_{iv,jv_b}(\varphi(v)) / g_{iv,i_b}(\varphi(\overline{v})) \right) \right]. \quad (11.4) \]

The collection $(g_{IJ}, A_I)$ provides local data for a hermitian line bundle $L$ with connection
over $LM$. The curvature of $L$ is given by the 2-form $\Omega$ on $LM$, see (2.16). If
$(g'_{ijk}, A'_{ij}, B'_i)$ are equivalent local data related to the original ones by (2.11), then

\[ g'_{IJ} = g_{IJ} f_J f_I^{-1}, \quad A'_I = A_I + i f_I^{-1} df_I \quad (11.5) \]
for

\[ f_I^{-1}(\varphi) = \exp \left[ i \sum_b \int_b \varphi^* \Pi_{ib} \right] \prod_{v \in b} \chi_{ivb}(\varphi(v)) , \]

(11.6)
i.e. the local data \((g_{IJ}, A_I)\) change to equivalent ones, see (2.4).

### 11.2 Open strings

We may apply the previous constructions to the case of open curves. Using the same formulae (11.1), (11.2) and (11.4) to define an open covering \((O_I)\) of \(IM\), 1-forms \(A_I\) on \(O_I\) and \(U(1)\)-valued functions on \(O_{IJ}\), we obtain, however, a different structure. Now on \(O_{IJK}\), \(O_I\) and \(O_{IJ}\), respectively,

\[(g_{IJ} g^{-1}_{IK} g_{IJ}) = g_{i1j2k1} \circ e_1 / g_{i0j0k0} \circ e_0 ,\]

\[dA_I = \Omega + e_0^* B_{i0} - e_1^* B_{i1} ,\]

\[A_J - A_I - ig^{-1}_{IJ} dg_{IJ} = e_{0}^{*} A_{i0j0} - e_{1}^{*} A_{i1j1} ,\]

(11.7)

with \(\Omega\) given by (2.16) and \(e_s\) being the evaluation maps of (7.10).

Let \(D\) be a submanifold of \(M\) and let \(Q\) be a 2-form on \(D\) such that \(dQ = H|_D\). Suppose, moreover, that \(\Pi_i\) are one-forms on \(\tilde{O}_i = O_i \cap D\) and that \(\chi_{ij} = \chi^{-1}_{ji}\) are \(U(1)\)-valued functions on \(\tilde{O}_{ij}\) such that

1. On \(\tilde{O}_i\),

\[Q = B_i + d\Pi_i ,\]

(11.8)

2. On \(\tilde{O}_{ij}\)

\[0 = A_{ij} + \Pi_j - \Pi_i - i \chi^{-1}_{ij} d\chi_{ij} ,\]

(11.9)

3. On \(\tilde{O}_{ijk}\),

\[1 = g_{ijk} \chi^{-1}_{jk} \chi_{ik} \chi^{-1}_{ij} .\]

(11.10)

Provided that the covering \((O_I)\) is sufficiently fine, the existence of \((\chi_{ij}, \Pi_i)\) with the above properties is equivalent to the existence of a stable isomorphism between the restriction to \(D\) of the gerbe \(G\) constructed from the local data \((g_{ijk}, A_{ij}, B_i)\) and the gerbe on \(D\) obtained from the local data \((Q|_\tilde{O}_i , 0, 1)\). The choice of \((\chi_{ij}, \Pi_i)\) providing the stable isomorphism is determined up to the local data \((h_{ij}, R_i)\) of a flat hermitian line bundle over \(D\). The \(U(1)\)-valued functions \(f_I\) defined on the open subsets \(\tilde{O}_I \subset LD\) by (11.6) satisfy now

\[g_{IJ}|_{\tilde{O}_{IJ}} = f_J^{-1} f_I ,\]

(11.11)

and define a trivialization of the line bundle \(L|_{LD}\) defined on \(LD\) from the local data \((g_{IJ}|_{\tilde{O}_{IJ}})\). They permit to assign a numerical value

\[A_\nu(\phi) = A(\phi) f_I^{-1}(\phi|_\ell)\]

(11.12)
to the field $\phi : \Sigma \to M$ if $\partial \Sigma$ is composed of a single loop $\ell$ mapped by $\phi$ into $D$. Here $\mathcal{A}(\phi)$ is given by the expression (2.14) and the collection $I$ is obtained by restricting the triangulation of $\Sigma$ and corresponding label assignment to the boundary loop $\ell$. The result does not depend on the choices of the triangulation of $\Sigma$ neither on the label assignments. It does not change under the passage to equivalent local data $(g_{ij,k},A_{ij},B_i)$ on $M$ if we absorb the transformations (11.11) in the choice of $(\chi_{ij},\Pi_s^s)$ in (11.8) to (11.10). We may, however, always modify that choice by the local data $(h_{ij},R_i)$ of a flat hermitian bundle $P$ on $D$. Such a modification multiplies the amplitude $A_D^s(\phi)$ by the holonomy of $P$ along $\phi|\ell$. It gives the local description of the change of the stable isomorphism between the restricted gerbe and the gerbe constructed from the 2-form $Q$, as discussed in Sect. 7.

The generalization of the above discussion to the case when $\Sigma$ has multiple boundary components with $\phi$ mapping (some of) them into branes is straightforward.

Let $D_0$ and $D_1$ be two submanifolds of $M$ and $H|_{D_s} = dQ_s$ with the choices of the data $(\chi^s_{ij},\Pi^s_{ij})$ as above for each $D_s$. Consider the subspace $I_{D_0D_1} \subset IM$ of curves ending on the branes, see (7.6). We may adapt the definitions (11.2) and (11.4) to the present case by defining

$$A^D_{ij} = A_{ij} + e^s_0\Pi^s_{ij} - e^1_1\Pi^s_{ij},$$
$$g^D_{ij} = g_{ij} \left( \chi^0_{ij} \circ e_0 / \chi^1_{ij} \circ e_1 \right) .$$

One obtains this way local data $(g^D_{ij},A^D_P)$ of a hermitian line bundle $\mathcal{L}_{P_0P_1}$ with connection over $I_{D_0D_1} M$. The terms added in (11.2) and (11.4) are sensitive to the modification of $(\chi^s_{ij},\Pi^s_{ij})$ by local data of flat bundles $P_s$ on $D_s$. The net result is the multiplication of $\mathcal{L}_{P_0P_1}$ by $e^s_0P_0 \otimes e^s_1P_1^{-1}$. This does not effect the curvature $\Omega_D$ of $\mathcal{L}_{P_0P_1}$ given by (7.3). Upon a change to the equivalent local data $(g'_{ij,k},A'_{ij},B'_i)$, the relations (11.5) still hold for $f_I$ given by (11.6), provided we absorb the changes in the choice of $(\chi^s_{ij},\Pi^s_{ij})$. The line bundle $\mathcal{L}_{D_0D_1}$ constructed from the local data $(g^D_{ij},A^D_P)$ is canonically isomorphic to the line bundle $\mathcal{L}_{P_0P_1}(G)$ for the gerbe $G$ obtained from the local data $(g_{ij,k},A_{ij},B_i)$, provided that one uses the data $(\chi^s_{ij},\Pi^s_{ij})$ to construct the stable isomorphism $\iota_s$ of (7.3) between the restrictions of $G$ and $K_s$.

### 12 Conclusions

We have shown how the concept of a bundle gerbes with connection may be applied to resolve Lagrangian ambiguities in defining sigma models in the presence of the anti-symmetric tensor field $B$ determined locally up to closed form contributions. This was done both in closed string geometry and for open strings stretching between branes. Application of that approach to the WZW models based on groups covered by $SU(N)$ has permitted to recover within the Lagrangian approach the classification of symmetric branes. It has also allowed to make precise a straightforward relation between the quantum open string amplitudes for the WZW models based on simply connected groups and for their simple current orbifolds. Those relations are simpler than the ones for closed string amplitudes where the appearance of twisted sectors complicates the analysis. They should extend to more general orbifold theories.
There are further problems to which one may try to apply the geometric methods based on gerbes within the Lagrangian approach to conformal field theory. The case of non-orientable worldsheets has not been discussed in the present paper. The analysis of gerbes entering the WZW models with other groups, with applications to the classification of symmetric and symmetry-breaking branes in general non-simply connected groups, including the $SO(2N)/\mathbb{Z}_2$ case with discrete torsion, is an open problem. Neither have the coset models been treated within this framework. Super-symmetric extensions require a modification of the approach presented here to take care of the fermionic anomalies [13]. Finally, the bundle gerbes should be useful in analyzing open string models coupled to non-abelian Chan-Paton degrees of freedom, including the fractional branes in general orbifolds [12], and in the description of the Ramond-Ramond brane charges [4]. We plan to return to some of those issues in the future.

Appendix A

For reader’s convenience, we gather here the basic facts about discrete group cohomology, see [14] [11] [5]. Let $\Gamma$ be a discrete group with elements $\gamma_0, \gamma_1, \ldots$ and $\mathcal{U}$ an abelian group on which $\Gamma$ acts (possibly trivially). We shall use the multiplicative notation for the product both in $\Gamma$ and $\mathcal{U}$ and for the action of $\Gamma$ on $\mathcal{U}$. In our applications, $\Gamma$ will be a subgroup $\mathbb{Z}$ of the center of $SU(N)$ and $\mathcal{U}$ will be equal to $\mathcal{U}(1)$ or to the group of $\mathcal{U}(1)$-valued functions on the orbit of $\mathbb{Z}$ in the Weyl alcove of $su(N)$.

In general, the abelian group $C^n(\Gamma, \mathcal{U})$ of $n$-cochains on $\Gamma$ with values in $\mathcal{U}$ is composed of maps

$$\Gamma^n \ni (\gamma_1, \ldots, \gamma_n) \mapsto u_{\gamma_1,\ldots,\gamma_n} \in \mathcal{U}.$$  \hspace{1cm} (A.1)

Consider the group homomorphisms $d : C^n(\Gamma, \mathcal{U}) \to C^{n+1}(\Gamma, \mathcal{U})$ defined by the formula

$$(du)_{\gamma_1,\ldots,\gamma_{n+1}} = (\gamma_1 u_{\gamma_2,\ldots,\gamma_{n+1}}) \left( \prod_{m=1}^{n} u_{\gamma_1,\ldots,\gamma_m \gamma_{m+1},\ldots,\gamma_n+1}^{(-1)^m} \right) u_{\gamma_1,\ldots,\gamma_n}^{(-1)^{n+1}}.$$  \hspace{1cm} (A.2)

For $n = 0, 1, 2, 3$, the cases relevant for this paper, this gives

$$(du)_{\gamma} = (\gamma u)^{-1}$$

$$(du)_{\gamma_1,\gamma_2} = (\gamma_1 u_{\gamma_2}) u_{\gamma_1,\gamma_2}^{-1} u_{\gamma_1}$$

$$(du)_{\gamma_1,\gamma_2,\gamma_3} = (\gamma_1 u_{\gamma_2,\gamma_3}) u_{\gamma_1,\gamma_2,\gamma_3}^{-1} u_{\gamma_1,\gamma_2} u_{\gamma_1,\gamma_3}^{-1}$$

$$(du)_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} = (\gamma_1 u_{\gamma_2,\gamma_3,\gamma_4}) u_{\gamma_1,\gamma_2,\gamma_3,\gamma_4}^{-1} u_{\gamma_1,\gamma_2,\gamma_3} u_{\gamma_1,\gamma_2,\gamma_4} u_{\gamma_1,\gamma_3,\gamma_4}^{-1} u_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} u_{\gamma_1,\gamma_2,\gamma_3} u_{\gamma_1,\gamma_3,\gamma_4}^{-1} u_{\gamma_1,\gamma_2,\gamma_3} u_{\gamma_1,\gamma_2} u_{\gamma_1,\gamma_3}^{-1}.$$ \hspace{1cm} (A.3)

The square of $d$ vanishes and the cohomology groups of $\Gamma$ with values in $\mathcal{U}$ are defined as

$$H^n(\Gamma, \mathcal{U}) = \left\{ u \in C^n(\Gamma, \mathcal{U}) \mid du = 1 \right\} / dC^{n-1}(\Gamma, \mathcal{U}).$$ \hspace{1cm} (A.4)
For the special case of $\Gamma \cong \mathbb{Z}_p$ with generator $\gamma_0$ and $n = 1, 2, \ldots$,

$$H^{2n}(\Gamma, U) \cong \left\{ u \in U \mid \gamma u = u \text{ for } \gamma \in \Gamma \right\},$$

$$H^{2n-1}(\Gamma, U) \cong \left\{ u \in U \mid \prod_{r=0}^{n-1} (\gamma_0^ru) = 1 \right\},$$

(A.5)

see [5] [11]. In particular, for the trivial action of $\Gamma$ on $U(1)$,

$$H^{2n}(\Gamma, U(1)) \cong \{1\}, \quad H^{2n-1}(\Gamma, U(1)) \cong \mathbb{Z}_p$$

(A.6)

and for $U(1)^T$ being the group of $U(1)$-valued function on the set $T$ with the action of $\Gamma$ induced from that on $T$,

$$H^{2n}(\Gamma, U(1)^T) \cong \{1\}, \quad H^{2n-1}(\Gamma, U(1)^T) \cong \prod_{[\tau] \in \mathcal{T}/\mathcal{T}} \mathbb{Z}_{\tau},$$

(A.7)

where $\mathbb{Z}_{\tau}$ denotes the stabilizer subgroup of $\tau \in \mathcal{T}$ (which depends only on the $\Gamma$-orbit $[\tau]$ of $\tau$).

The discrete group cohomology should be distinguished from the one for Lie groups appearing also in the paper. The latter is defined as for general topological spaces, e.g. by the Čech construction.

Appendix B

To check the associativity of the product $\mu'$ of (4.3) over $O_{ij}^{l+i} \subset Y^{l+i}$ with $j = [j' + a]$, $l = [l' + b]$ and $n = [n' + c]$, we first calculate

$$\mu' \left( [\gamma, \zeta] \kappa_{\lambda ij} \otimes [\gamma w^{-1}_a, \zeta'] \kappa_{\lambda' [l-a]} \right) \otimes [\gamma w^{-1}_b, \zeta''] \kappa_{\lambda'' [n-b]}$$

$$\mu' \left( [\gamma, \zeta] \kappa_{\lambda ij} \otimes [\gamma w^{-1}_a, \zeta'] \kappa_{\lambda' [l-a]} \right) = [\gamma, \zeta] \kappa_{\lambda ij} \otimes [\gamma w^{-1}_a, \zeta'] \kappa_{\lambda' [l-a]} \otimes [\gamma w^{-1}_b, \zeta''] \kappa_{\lambda'' [n-b]}.$$

(B.1)

On the other hand, using the identity

$$[\gamma w^{-1}_b, \zeta''] \kappa_{\lambda'' [n-b]} = [\gamma w^{-1}_a w^{-1}_{b-a}, \zeta''] \kappa_{\lambda'' [n-b]}$$

that follows from the equivalence relation (3.20) since $w_{b-a}w_{a}w_{b}^{-1}$ lies in the Cartan subgroup $T$, we infer that

$$\mu' \left( [\gamma w^{-1}_a, \zeta'] \kappa_{\lambda' [l-a]} \otimes [\gamma w^{-1}_b, \zeta''] \kappa_{\lambda'' [n-b]} \right)$$

$$\mu' \left( [\gamma w^{-1}_a, \zeta'] \kappa_{\lambda' [l-a]} \otimes [\gamma w^{-1}_b, \zeta''] \kappa_{\lambda'' [n-b]} \right) = [\gamma w^{-1}_a, \zeta'] \kappa_{\lambda' [l-a]} \otimes [\gamma w^{-1}_b, \zeta''] \kappa_{\lambda'' [n-b]} \otimes [\gamma w^{-1}_b, \zeta''] \kappa_{\lambda'' [n-b]}.$$

(B.2)

Another application of $\mu'$ gives then:

$$\mu' \left( [\gamma, \zeta] \kappa_{\lambda ij} \otimes \mu' \left( [\gamma w^{-1}_a, \zeta'] \kappa_{\lambda' [l-a]} \otimes [\gamma w^{-1}_b, \zeta''] \kappa_{\lambda'' [n-b]} \right) \right)$$

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over $\gamma$ defines for the map $Y$ with $O\iota$ where $Z$ we shall take as $\Gamma$ the subgroup $N$ divisible by $Y$ the $(left\ or \ right)$ multiplication. Recall that in Section 4. We describe here the construction of the “quotient gerbe” $G_k'$ on $G' = SU(N)/Z$ along the lines of Section 5. The resulting gerbe coincides with the gerbe on $G'$ constructed in Section 4.

With $M = SU(N)$ and the gerbe $G^k = (Y, kB, L^k, \mu_k)$ constructed in Section 3 we shall take as $\Gamma$ the subgroup $Z \cong Z_{N'}$ of the center of $SU(N)$ acting on $M$ by the (left or right) multiplication. Recall that $Y = \bigcup_{i=0}^n O_i$. For $\gamma = z_a \in \Gamma$, where $a$ is divisible by $N'' = N/N'$,

$$Z_\gamma = Y \times_M Y_\gamma = \{(g, i), (z_a^{-1} g, j') \mid g \in O_i, \ z_a^{-1} g \in O_{j'}\}$$

$$\cong \bigcup_{i,j} O_{ij},$$

(C.1)

where $j = [j' + a]$. We may take the bundle $N^\gamma$ over $Z_\gamma$ to be equal to $\rho_{ij}^* L_{\lambda_{ij}}$ over the $O_{ij}$ component. Since

$$kB_j(z_a^{-1} g) - kB_i(g) = kB_j(g) - kB_i(g) = k \rho_{ij}^* F_{\lambda_{ij}}(g),$$

(C.2)

the relation (C.1) is satisfied. The isomorphism $\iota_\gamma$ of (5.4), upon taking

$$y_1 = (g, i_1), \ y_2 = (g, i_2), \ y'_1 = (z_a^{-1} g, j'_1), \ y'_2 = (z_a^{-1} g, j'_2)$$

(C.3)

with $g = \gamma e^{2\pi i \gamma^{-1} g} \in O_{i_1 j_1 j_2,}$, may be defined by

$$\iota_\gamma \left( [\gamma, \zeta]_{k_{\lambda_{ij}}} \right) = \left( [\gamma w^{-1}_a, \zeta \zeta'^{-1}]_{k_{\lambda_{ij}}} \right).$$

(C.4)

For $Y_\gamma = Y$ with the projection on $G'$ and $(y, y', y'') \in Y^{[3]}_\gamma$,

$$y = (g, i), \ y' = (z_a^{-1} g, j'), \ j = [j' + a], \ y'' = (z_a^{-1} g, l'), \ l = [l' + b],$$

(C.5)

the map

$$[\gamma, \zeta]_{k_{\lambda_{ij}}} \otimes [\gamma w^{-1}_a, \zeta']_{k_{\lambda_{j_1 j_2}}} \otimes [\gamma, \zeta'']_{k_{\lambda_{ij}}} \mapsto [g, \zeta^{-1} \zeta'^{-1} \zeta'', i.j.l]$$

(C.6)

defines for $\gamma_1 = z_a$ and $\gamma_2 = z_a^{-1} z_b$ an isomorphism between the bundle $R^{\gamma_1, \gamma_2}$, see (C.8), and the pullback of the flat bundle

$$R^{\gamma_1, \gamma_2} = \left( \bigcup_{i,j,l} O_{ijl} \times \mathbb{C} \right)/\sim$$

(C.7)

over $SU(N)$, where the equivalence relation $\sim$ is defined by

$$(g, \zeta_1, i_1 j_1 l_1) \sim (g, \zeta_2, i_2 j_2 l_2) \ \text{if} \ \zeta_1 = \zeta_2 \chi_{k_{\lambda_{ij}}} (w_b w^{-1}_a).$$

(C.8)
For \((y, y', y'', y''') \in Y^{[4]}\) with \(y''' = (z^{-1}_c g, n')\), \(n = [n' + c]\) and \(\gamma_3 = z^{-1}_b z_c\), the isomorphism \((5.10)\) identifies

\[
[g, \zeta, ijl] \otimes [g, \zeta', iln] \equiv \chi_{kk'}v_{[a-b]}(w_b w_a^{-1} w_{[b-a]}) \cdot [g, \zeta, ijn] \otimes [z^{-1}_a g, \zeta', j'[l-a][n-a]].
\] (C.9)

The flat bundle \(P^\gamma\) will be taken trivial and the isomorphisms \(\iota_{\gamma_1, \gamma_2}\) of \((5.11)\) will be defined by

\[
(g, \zeta) \otimes (z^{-1}_a g, \zeta') \rightarrow (g, \zeta \zeta') \otimes [g, u_{ijl}, ijl]
\] (C.10)

for \(g \in O_{ijl}\) and \(u_{ijl} \in U(1)\). According to the definition of the classes \([g, u_{ijl}, ijl]\), see \((C.10)\), we must have

\[
u_{i_2 j_2 l_2} = \chi_{kk'}v_{i_2}^l (w_b w_a^{-1} w_{[b-a]}) us_{i_1 j_1 l_1}
\] (C.11)

for the same \(a\) and \(b\) that we suppressed in the notation for \(u_{ijl}\). Equation \((4.10)\) is a special case of the above relation. Property \((5.12)\) reduces to \((4.6)\) which is consistent with the transformation properties \((C.11)\). It is then enough to consider \(u_{0ab} \equiv u_{a[b-a]}\) in which case \((4.6)\) reduces to \((4.7)\).

**Appendix D**

We shall prove here that the amplitude \(\mathcal{A}(\phi)\) of equation \((6.2)\), when interpreted as a number following the procedure described in Sect. 6, coincides with the expression \((2.14)\). First, note that \(\int_c \phi_c^* B = \int_c \phi^* B_\text{c}\). Next, observe that the holonomies in \(L\) are

\[
\mathcal{H}(\phi_{cb}) = \exp \left[ \int_b \phi_c^* A_{i\text{c}i}\right] \otimes s_{i\text{c}i}(yc, yb).
\]

We have then to compute the numbers assigned for every vertex \(v\) to

\[
s_{i_{1v}, i_{1v}}(yb_1, yc_1) \otimes s_{i_{2v}, i_{2v}}(yc_1, yb_2) \otimes \cdots \otimes s_{i_{nv}, i_{nv}}(yc_n, yb_n),
\]

see Fig. 1. For \(i_v\) such that \(v \in O_{i_v}\) and \(y_v = \sigma_{i_v}(\phi(v))\), we shall insert at every second place in the last chain the tensor \(s_{i_{br}, i_v}(yb_r, yc) \otimes s_{i_{c}, i_{br}}(yc, yb_r)\) mapped by \(\mu\) to \(1 \in L(y_{br}, y_{br})\). This permits to split the chain to the blocks

\[
s_{i_{v}, i_{br}}(y_v, y_{br}) \otimes s_{i_{1v}, i_{c}, i_{v}}(yb_r, yc) \otimes s_{i_{c}, i_{br+1}}(yc, y_{br+1}) \otimes s_{i_{br+1}, i_{c}}(yb_{br+1}, y_v).
\]

The latter give rise under \(\mu\) to the factors \(g_{i_{v}, i_{br+1}}(\phi(v)) g_{i_{br+1}, i_{c}}^{-1}(\phi(v))\) which build up the product appearing in \((2.14)\).
Appendix E

Let us show that the isomorphism $\iota'$ of (8.3) between $L' \otimes p_1^* N'^{-1} \otimes p_2^* N'$ and the trivial bundle $Z^{[2]} \times \mathbb{C}$ intertwines the groupoid multiplication if and only if the relation (8.3) holds. Let $y_1 = (g, i)$, $y_2 = (z^{-1}_a g, j')$, and $y_3 = (z^{-1}_b g, l')$ for $g = \gamma \exp i \pi \gamma^{-1}$. Consider the elements

$$f_1 = [\gamma, 1] k \lambda_{ij} \otimes [\gamma, 1]^{-1} \chi_{k(\sigma_a(\tau) - \lambda_j')} \otimes [\gamma w_a^{-1}, 1]_{k(\sigma_b(\tau) - \lambda_j')}$$

in the fiber $(L' \otimes p_1^* N'^{-1} \otimes p_2^* N')_{(y_1, y_2)}$ and

$$f_2 = [\gamma w_a^{-1}, 1] k \lambda_{ij} \otimes [\gamma w_a^{-1}, 1]^{-1} \chi_{k(\sigma_a(\tau) - \lambda_j')} \otimes [\gamma w_a^{-1} w_{[b-a]}, 1]_{k(\sigma_b(\tau) - \lambda_{j'})}$$

in the fiber $(L' \otimes p_1^* N'^{-1} \otimes p_2^* N')_{(y_2, y_3)}$. The product of those two elements,

$$\mu'(f_1 \otimes f_2) = [\gamma, u_{ij} k \lambda_{il} \otimes [\gamma, 1]^{-1} \chi_{k(\sigma_a(\tau) - \lambda_j')} \otimes [\gamma w_a^{-1} w_{[b-a]}, 1]_{k(\sigma_b(\tau) - \lambda_{j'})},$$

see (E.3), where the last tensor factor may be rewritten as

$$[\gamma w_b^{-1}, \chi_{k(\sigma_a(\tau) - \lambda_{j'})}(w_b w_a^{-1} w_{[b-a]}),]_{k(\sigma_b(\tau) - \lambda_{j'})}.$$ (E.4)

From the definition of the isomorphism $\iota'$ we have

$$\iota'(f_1) = (y_1, y_2, v_{\tau,a}),$$

$$\iota'(f_2) = (y_2, y_3, v_{\sigma_a(\tau), [b-a]}).$$

$$\iota'(\mu'(f_1 \otimes f_2)) = (y_1, y_3, v_{\tau,a} u_{iji} \lambda_{k(\sigma_b(\tau) - \lambda_{j'})}(w_b w_a^{-1} w_{[b-a]})).$$ (E.5,a, E.5,b, E.5,c)

The product of the first two elements of the trivial bundle over $Z^{[2]}$ is equal to the third one if and only if

$$v_{\tau,a} v_{\sigma_a(\tau), [b-a]} = v_{\tau,b} u_{iji} \chi_{k(\sigma_b(\tau) - \lambda_{j'})}(w_b w_a^{-1} w_{[b-a]})$$

$$= v_{\tau,b} \chi_{k(\sigma_b(\tau))}(w_b w_a^{-1} w_{[b-a]}) u_a[b-a],$$ (E.6)

where the last equality follows from (4.10). Upon the shift $b \mapsto [a + b]$ this reduces to (8.3).

Appendix F

Here we prove the cocycle identity (8.6). The left hand side is equal to

$$\chi_{k(\sigma_{[a+b+c]}(\tau))}(w_{[a+b]} w_{[b+c]}^{-1} w_{[a+c]}^{-1}) \chi_{k(\sigma_{[a+b+c]}(\tau))}(w_{[a+b]} w_{[a+b]}^{-1} w_{[a+b]} w_{[a+c]}^{-1})$$

$$\cdot \chi_{k(\sigma_{[a+b+c]}(\tau))}(w_{[a+b+c]} w_{[b+c]}^{-1} w_{[a+c]}^{-1}) \chi_{k(\sigma_{[a+b]})(\tau)}(w_{[a+b]} w_{[a+b]}^{-1} w_{[a+b]}^{-1})$$

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the identity (8.5) will follow if we show that

\[ \psi \mathrm{v} \sigma \ a, \ b \]

and it indeed coincides with the right hand side, as may be seen from the relation

\[ V_{\tau,ab} = \chi_{k\tau} (w_a^{-1} w_b^{-1} w_{a+b}) \chi_{k\lambda\sigma} (w_a^{-1} w_b^{-1} w_{a+b}) \ u_{ab} \]

following from the definition (8.4) and (F.2).
Appendix H

Here we show that the modification (8.15) with an appropriately chosen character \( \rho_{\lambda_0}(a) \) of \( \mathcal{Z} \) depending on \( \lambda_0 = k\tau_0 \) with \( \tau_0 \in [\tau] \) trivializes the cocycle \( \phi_{\lambda_0}(b,a) \) of (8.14). We only have to consider the case of \( N' \) even and \( N'' \) odd since for the other cases \( \phi_{\lambda_0}(b,a) = 1 \). First, for \( z_a \in \mathbb{Z}_\tau \), i.e. for \( a = a''n'' \),

\[
\psi(b,a) = (-1)^{a''(a' - a''b)n''} \cdot \begin{cases} 
1 & \text{for } \frac{k}{N} \text{ even,} \\
(-1)^{-a''b} & \text{for } \frac{k}{N} \text{ odd,}
\end{cases}
\]

see (8.10b). Let

\[
\rho^0_{\lambda_0}(a) = (-1)^a \sum_{i'=0}^{n''} i'n_{i'}, \quad \rho^1_{\lambda_0}(a) = (-1) \sum_{i'=0}^{n''} i'n_{i'},
\]

with \( \rho^1 \) defined for \( \frac{k}{N} \) odd. A direct check shows that for \( a = a''n'' \),

\[
\chi_{\lambda_0}^{-1}(u_a) \rho^0_{\lambda_0}(a)
\]

does not depend on the choice of \( \tau_0 \in [\tau] \). It follows that, for \( \frac{k}{N} \) even, \( \phi_{\lambda_0} \) is trivialized by (8.15) if we take \( \rho = \rho^0 \). Finally, for \( \frac{k}{N} \) odd, \( \sum_{i'=0}^{n''} i'n_{i'} \) preserves or changes its parity under the shift \( \lambda_0 \mapsto b\lambda_0 \) for \( b \) even or odd, respectively, so that

\[
\rho^0_{\lambda_0}(a) \rho^1_{\lambda_0}(a) = (-1)^{a''b(n''-1)}
\]

and, as the result, \( \phi_{\lambda_0}(b,a) \) is trivialized by (8.15) with \( \rho = \rho^1 \).

On the other hand, relations (9.52) and (9.56), together with the proportionality of the matrix elements \( S^\lambda_\lambda \) and \( S^\lambda_\lambda \) imply that

\[
\phi_{\lambda_0}(b,a) = e^{2\pi i b'(Q_J(\lambda)+a'X) - 2\pi i b Q_J(\lambda)}
\]

with \( \lambda \) another weight such that \( a\lambda = \lambda \). In particular, \( \phi_{\lambda_0}(b,a) \) appearing in (9.52) is \( \lambda_0 \)-independent. With the use of expressions (9.13), together with (9.53) and (9.54), one checks that

\[
e^{2\pi i b'Q_J(\lambda) - 2\pi i b Q_J(\lambda)}
\]

\[
= (-1)^{\frac{bk(n'-1)}{n''}} = \begin{cases} 
(-1)^b & \text{for } N' \text{ even, } N'' \text{ odd, } \frac{k}{N'} \text{ odd}, \\
1 & \text{otherwise.}
\end{cases}
\]

On the right hand side, the condition that \( \frac{k}{N} \) be odd may be replaced by the requirement that \( \frac{k}{N'} \) be odd if we at the same time we replace \( (-1)^b \) by \( (-1)^{a''b} \) for \( a = a''n'' \). On the other hand,

\[
e^{2\pi i a'b'X} = (-1)^{-\frac{a''bk(n'-1)}{n''}} = \begin{cases} 
(-1)^{-a''b} & \text{for } N' \text{ even, } N'' \text{ odd, } \frac{k}{N'} \text{ odd}, \\
1 & \text{otherwise.}
\end{cases}
\]
It follows that $\phi_{\lambda_0}(b,a)$, as given by (1.5), is equal to 1.

Appendix I

We construct here the canonical element $\Phi(\bar{\varphi},\varphi)$ of (9.35), where $\varphi:[0,\pi]\to G$ with $\varphi(0)\in C_{\omega}$, $\varphi(\pi)\in C_{\tau}$ and $\bar{\varphi} = z_0^{-1}\varphi$ for $z_0 \in \mathbb{Z}$. Let us choose, for a sufficiently fine split (partition) of $[0,\pi]$ into subintervals $b$, arbitrary lifts $\phi_b$ and $\bar{\phi}_b$ of $\varphi|_b$ and $\bar{\varphi}|_b$ to $Y = \mathbb{O}_i$. In other words,

$$\phi_b = (\varphi|_b,i_b), \quad \bar{\phi}_b = (\bar{\varphi}|_b,i_b)$$

(1.1)

for some choice of indices such that $\varphi(b)\subset O_{i_b}$ and $\bar{\varphi}(b)\subset O_{i_b}$. Let $\psi_b$ denote the mapping from $b \subset [0,\pi]$ to $Y_i[2]$ defined by

$$\psi_b(x) = (\bar{\phi}_b(x),\phi_b(x)).$$

(1.2)

We also choose lifts $y_v$ and $\bar{y}_v$ to $Y$ of $\varphi(v)$ and $\bar{\varphi}(v)$, respectively, for vertices $v$ of the partition of $[0,\pi]$. Let us set

$$\Phi(\bar{\varphi},\varphi) = \iota_0^{-1}(\bar{y}_0,\varphi(y_0),1) \otimes \iota_1^{-1}(\varphi(y_1),\bar{y}_1,\varphi(y_1),1) \otimes \left( \bigotimes_{b\subset [0,\pi]} \mathcal{H}_L(\psi_b) \right),$$

(1.3)

where $\iota_s'$ are the bundle isomorphisms given by (8.3). We shall show how $\Phi(\bar{\varphi},\varphi)$ may be considered in a canonical way as an element of the line $(L_{D_0D_1})_{\varphi}^{-1} \otimes (L_{D_0\bar{D}_1})_{\varphi}$.

As it stands,

$$\Phi(\bar{\varphi},\varphi) \in L'_{(y_0)} \otimes (N_0')_{y_0}^{-1} \otimes (N_0')_{y_0} \otimes L'_{(y_1)} \otimes (N_1')_{y_1} \otimes \left( \bigotimes_{b\subset [0,\pi]} L'_b(y_b,y_b) \right).$$

(1.4)

where $y_b = \phi_b(v)$ and $\bar{y}_b = \bar{\phi}_b(v)$. With the use of the groupoid multiplication, the last factor is canonically isomorphic to

$$\bigotimes_{v\in b\subset [0,\pi]} \left( L'_{(y_v,y_v)} \otimes L'_{(y_v,y_v)} \right).$$

(1.5)

But the line

$$L'_{(y_0)} \otimes \left( \bigotimes_{v\in b\subset [0,\pi]} L'_{(y_v,y_v)} \right) \otimes L'_{(y_1)}$$

(1.6)

is canonically trivial since the factors appear in dual pairs. We infer that, in a canonical way,

$$\Phi(\bar{\varphi},\varphi) \in (N_0')_{y_0}^{-1} \otimes \left( \bigotimes_{v\in b\subset [0,\pi]} L'_{(y_v,y_v)} \right) \otimes (N_1')_{y_1}$$

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\begin{equation}
\otimes (N_0')_{y(\epsilon)} \otimes \left( \otimes_{v \in b \subset [0, \pi]} L'_{y(v, y_b)} \right) \otimes (N_1')_{y(\pi)}^{-1}.
\end{equation}

(1.7)

Recalling that, by construction, the lines bundles $N_s'$ over the subsets $\pi^{-1}(C_{\tau_s}) \subset Y$ coincide with the bundles $N_s$ and using the definitions (6.7) and (7.7), we infer that the last line is canonically isomorphic with the line $(L_{\tilde{D}_0 \tilde{D}_1})_{\phi} \otimes (L_{D_0 D_1})_{\phi}$.

The fact that the isomorphisms $\iota_s'$ preserve the groupoid multiplication and the associativity of the groupoid multiplication in $L'$ result in the canonical identification (9.37). We leave the details to the reader. Finally, formula (9.36) is a consequence of the fact that, from the point of view of group $G'$, multiplication by $\Phi$ gives the canonical isomorphism used to identify two different realizations of the same fiber of the line bundle $L'_{D_0 D_1}$.

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