Semi-linear cooperative elliptic systems
involving Schrödinger operators:
Groundstate positivity or negativity.

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Abstract

We study here the behavior of the solutions to a 2 × 2 semi-linear cooperative
system involving Schrödinger operators (considered in its variational form):

\[ LU := (-\Delta + q(x))U = AU + \mu U + F(x, U) \quad \text{in} \quad \mathbb{R}^N \]

\[ U(x)|_{|x|\to\infty} \to 0 \]

where \( q \) is a continuous positive potential tending to \(+\infty\) at infinity; \( \mu \) is
a real parameter varying near the principal eigenvalue of the system; \( U \) is
a column vector with components \( u_1 \) and \( u_2 \) and \( A \) is a square cooperative
matrix with constant coefficients; \( F \) is a column vector with components \( f_1 \) and \( f_2 \) depending eventually on \( U \).

1 Introduction

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where \( q \) is a continuous positive potential tending to \(+\infty\) at infinity; \( U \) is
a column vector with components \( u_1 \) and \( u_2 \) and \( A \) is a square matrix with
constant coefficients; moreover \( A \) is a cooperative matrix (which means that
its coefficients outside the diagonal are non negative). \( F \) is a column vector
with components \( f_1 \) and \( f_2 \) depending eventually on \( U \). The real parameter
\( \mu \) varies near the principal eigenvalue of the system and plays a key role.
According to its position it determines not only the sign of the solutions but
also their position w.r.t. the groundstate.

Such systems have been intensively studied (very often for \( \mu = 0 \)) and mainly
for Dirichlet problems defined on bounded domains ( [16], [17], [18], [21], [20],
When the whole $\mathbb{R}^N$ is considered, as here, 2 cases are generally studied: either "Schrödinger systems" ([1], [2], [3], [7]), that is system involving Schrödinger operators, as here, or systems with a weight tending to 0 ([23], [6]). It is also possible to consider a combination of these 2 problems with a potential $q$ and a weight $g$:

$$LU := (-\Delta + q(x))U = g(x)AU + \mu g(x)U + F(x,U) \text{ in } \mathbb{R}^N$$

as far as $\frac{g}{q}$ tends to 0 at infinity which is the condition for having some compactness and therefore a discrete spectrum.

The first results on Schrödinger systems, when $F$ does not depend on $U$ (linear systems) deal with cooperative systems and with the Maximum Principle (MP) that is:

"If the data $F$ is non negative, $\neq 0$, then, any solution $U$ is non negative".

As for the case of one equation, this Maximum Principle holds for a parameter $\mu < \Lambda^*$, where $\Lambda^*$ is the principal eigenvalue of the system, which means that $LU - AU - \Lambda^* U = 0$ has a non zero solution which does not change sign.

For the classical case of an equation defined on a bounded domain with zero boundary conditions, $-\Delta u = \mu u + f(x), f > 0$, Clément and Peletier [14] have shown that the solution $u$ changes sign as soon as $\mu$ goes over $\lambda_1$, the first eigenvalue of the Dirichlet Laplacian defined on $\Omega$. More precisely there exists a small positive $\delta$, depending on $f$, such that for all $\mu \in (\lambda_1, \lambda_1 + \delta)$, $u < 0$. This phenomenon is known as "Anti-maximum Principle" ($\text{AMP}$).

In our present case, where we have no boundary, we have improved these results giving not only the sign of the solutions but also comparing the solutions with the groundstate (principal eigenfunction); it is what we call "groundstate positivity" ($\text{GSP}$) (resp. negativity) (resp. $\text{GSN}$). We extend in particular previous results established in [5] for linear systems to some semi-linear cooperative systems. For being not excessively technical, we limit our study to radial potentials and cooperative systems. Extensions to more general cases will appear somewhere else.

Our paper is organized as follows:
We recall first some previous results of the linear case that we use. Then we study a semi-linear equation. Finally we study a cooperative semi-linear system.
2 Linear Case: one equation

We shortly recall the case of a linear equation with a parameter \( \mu \) varying near the principal eigenvalue of the operator.

\[
(E) \quad Lu := (-\Delta + q(x))u = \mu u + f(x) \text{ in } \mathbb{R}^N, \\
\lim_{|x| \to +\infty} u(x) = 0.
\]

\((H_0)\) \( q \) is a positive continuous potential tending to \( +\infty \) at infinity.

We seek \( u \) in \( V \) where

\[
V := \{u \in L^2(\mathbb{R}^N) \text{ s.t. } \|u\|_V = (\int |
abla u|^2 + q(x)u^2)^{1/2} < \infty\}. 
\]

If \((H_0)\) is satisfied, the embedding of \( V \) into \( L^2(\mathbb{R}^N) \) is compact (see e.g. [19], [15]). Hence \( L \) possesses an infinity of eigenvalues tending to \( +\infty \):

\[
0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots, \lambda_k \to +\infty \text{ as } k \to \infty.
\]

**Notation:** \((\Lambda, \phi)\) We set from now on \( \Lambda := \lambda_1 \) the smallest one (which is positive and simple) and \( \phi \) the associated eigenfunction, positive and with \( L^2 \)-norm \( \|\phi\| = 1 \).

It is classical (see e.g. [24]) that if \( f \geq 0, \not\equiv 0 \), and \( \mu < \Lambda \), there exists exactly one solution which is positive: the positivity is ”improved”, or in other words, the (strong) maximum principle \((\text{MP})\) is satisfied:

\[
(\text{MP}) \quad f \geq 0, \not\equiv 0 \Rightarrow u > 0.
\]

Lately, as said above, another notion has been defined ([8], [10], [22]) the ”groundstate positivity” \((\text{GSP})\) (resp. ”negativity” \((\text{GSN})\)) which means that, there exists \( k > 0 \) such that the solution \( u > k\phi \) \((\text{GSP})\) (resp. \( u < -k\phi \) \((\text{GSN})\)).

We also say shortly ”fundamental positivity” or ”negativity”, or also ”\( \phi \)-positivity” or ”negativity”. Indeed these properties are more precise than MP or AMP. But for proving them, it is necessary to have a potential growing fast enough, a potential with a super quadratic growth.

In [10] a class \( \mathcal{P} \) of radial potentials is defined:

\[
\mathcal{P} := \{Q \in C(\mathbb{R}_+, \mathbb{R}_+^*)/\exists R_0 > 0, Q' > 0 \text{ a.e. on } [R_0, \infty), \int_{R_0}^\infty Q(r)^{-1/2} < \infty\}. 
\]
The last inequality holds precisely if \( Q \) is growing sufficiently fast, indeed faster than \( r^2 \) (the harmonic oscillator). In this paper we consider only a radial potential \( q \in \mathcal{P} \). Note that our proof is valid for more general potentials, in particular for perturbations of radial potential \([9]\) or \([10]\). We assume here

\[
(H_q') \quad q \text{ is radial and is in } \mathcal{P}
\]

**Remark 1**: Note that since \( q \) is in \( \mathcal{P} \) it satisfies \((H_q)\).

On \( f \) we assume

\[
(H_f^*) \quad f \in L^2(\mathbb{R}^N), \quad f^1 = \int f \phi > 0.
\]

For having more precise estimates on \( u \), in particular the "groundstate negativity" (GSN), we have to define another set \( X \) in which \( f \) varies, the set of "groundstate bounded functions":

\[
X := \{ h \in L^2(\mathbb{R}^N) : |h|/\phi \in L^\infty(\mathbb{R}^N) \},
\]

equipped with the norm \( \|h\|_X = \text{ess sup}_{\mathbb{R}^n}(|h|/\phi) \).

**Theorem 1**: Assume \((H_q')\) and \((H_f^*)\), \( f \in X \). For \( \mu < \Lambda \) or \( \Lambda < \mu < \lambda_2 \) there exists \( \delta > 0 \) (defined below) depending on \( f \) and a positive constant \( C \), depending on \( f \) such that if \( 0 < |\Lambda - \mu| < \delta \),

\[
\Lambda - \delta < \mu < \Lambda \Rightarrow u \geq \frac{C}{\Lambda - \mu} \phi > 0,
\]

\[
\Lambda < \mu < \Lambda + \delta \Rightarrow u \leq \frac{C}{\Lambda - \mu} \phi < 0.
\]

**Proof of Theorem 1**: Decompose now \( u \) and \( f \) in \((E)\) on \( \phi \) and its orthogonal:

\[
u = u^1 \phi + u^\perp; \quad f = f^1 \phi + f^\perp; \quad u^1 = \int u \phi, \quad \int u^\perp \phi = \int f^\perp \phi = 0;
\]

we derive from Equation \((E)\)

\[
(L - \mu)u^1 \phi = (\Lambda - \mu)u^1 \phi = f^1 \phi, \quad Lu^\perp = \mu u^\perp + f^\perp.
\]

Choose \( \mu < \Lambda \) or \( \Lambda < \mu < \lambda_2 \). From the first equation we derive

\[
u^1 = \frac{f^1}{(\Lambda - \mu)} \rightarrow \pm \infty \text{ as } (\Lambda - \mu) \rightarrow 0.
\]
By use of Theorem 3.2 (c) in [9] or [10], we know that the restriction of the resolvent \((L - \mu)^{-1}\) to \(X\) is bounded from \(X\) into itself. The following lemma is a direct consequence of this result as it is shown in the proof of the Theorem 3.4 in [9].

**Lemma 1**: There exists \(\delta_0\) small enough and there exists a constant \(c_0\) (depending on \(\delta_0\)) such that for all \(\mu\) with \(\Lambda - \delta_0 < \mu < \Lambda\) or \(\Lambda < \mu < \Lambda + \delta_0 < \lambda_2\),

\[-c_0\|f^\perp\|_X \leq \|u^\perp\|_X \leq c_0\|f^\perp\|_X.\]

Finally we take in account Lemma 1 and (3):

\[\|u^\perp\|_X \leq c_0\|f^\perp\|_X \text{ and } u = \frac{f^1}{\Lambda - \mu} \phi + u^\perp;\]

for \(|\Lambda - \mu| \to 0\), \(\frac{f^1}{\Lambda - \mu} \phi \to \pm \infty\) when \(u^\perp\) stays bounded. Hence, for \(|\Lambda - \mu|\) small enough, more precisely for \(|\Lambda - \mu| < \delta_1(f) := \frac{f^1}{c_0\|f^\perp\|_X}\), we have

\[\frac{f^1}{|\Lambda - \mu|} > c_0\|f^\perp\|_X.\]

We deduce that Theorem 1 is valid for \(\delta := \min\{\delta_0, \delta_1(f)\}\).

### 3 Semi-linear Schrödinger equation

We study now the case of a semi-linear equation. We first obtain bounds for the solutions, if they exist and then we show their existence via the method of "sub-super solutions". Finally, with additional assumptions, we prove the uniqueness of them.

Consider the semi-linear Schrödinger equation (SLSE)

\[(SLSE)\quad Lu := (-\Delta + q(x))u = \mu u + f(x, u) \text{ in } \mathbb{R}^N,\]

\[\lim_{|x| \to +\infty} u(x) = 0.\]

We assume that the potential \(q\) satisfies \((H'_q)\) and we denote as above by \((\Lambda, \phi)\) the principal eigenpair with \(\phi > 0\).

We work in \(L^2(\mathbb{R}^N)\) and we consider the problem in its variational formulation. We seek \(u\) in \(V\) for a suitable \(f\).

We assume that \(f\) satisfies:
\((H_f)\) \(f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}\) is a Caratheodory function \(i.e\). the function \(f(\cdot, u)\) is Lebesgue measurable in \(\mathbb{R}^N\), for every \(u(x) \in \mathbb{R}\) and the function \(f(x, \cdot)\) is continuous in \(\mathbb{R}\) for almost every \(x \in \mathbb{R}^N\). Moreover, \(f\) is such that

1. \(\forall u \in L^2(\mathbb{R}^N), \; f(\cdot, u) \in L^2(\mathbb{R}^N)\),
2. \(\exists \kappa > 0 \; s.t. \; \forall u \in V, \; f(x, u) \geq \kappa \phi(x) > 0\)
3. \(\exists K > \kappa > 0 \; s.t. \; \forall u \in V, \; f(x, u) \leq K \phi(x)\).

Later we also suppose
\(\forall x \in \mathbb{R}^N, \; u \to \frac{f(x, u)}{|u|}\) is strictly decreasing

**Remark 2**: Note that, by (ii) and (iii), for any \(u \in V, \; f(\cdot, u) \in X\) and hence the solutions, if they exist, are in \(X\).

Let a parameter \(\mu\) be given, with \(|\mu - \Lambda|\) “small enough”. In this section we prove groundstate positivity and negativity for the semi-linear Schrödinger equation.

**Theorem 2**: If \((H'_f)\) and \((H_f)\) are satisfied, then there exists \(\delta(f) > 0\) \((\delta = \delta(f) := \min\{\delta_0, \delta'_1(f) := \frac{\kappa}{c_0 K}\} \) where \(\delta_0\) and \(c_0\) are given in Lemma 1) such that, for \(0 < |\mu - \Lambda| < \delta\) there exists a solution \(u\) to \((SLES)\) such that

\[\|u\|_X \leq \frac{K}{|\Lambda - \mu|} + 2c_0 K.\]

Also
- for \(\Lambda - \delta < \mu < \Lambda, \; u \geq \frac{\kappa}{\Lambda - \mu} \phi > 0,\)
- for \(\Lambda < \mu < \Lambda + \delta < \lambda_2, \; u < \frac{K}{\Lambda - \mu} \phi < 0.\)

Moreover if \((H'_f)\) is satisfied, the solution to \((SLSE)\) is unique.

**Remark 3** If (ii) does not hold, for \(\mu < \Lambda,\) there exists a solution \(u\) such that

\[\|u\|_X \leq \frac{K}{|\Lambda - \mu|} + 2c_0 K.\]

The existence is classical (e.g. [2]) and the estimate follows from the proof below.
Proof of Theorem \[2\]
We do the proof in 3 steps: first maximum and anti-maximum principles, secondly existence of the solution such that \( u > \frac{\kappa}{\Lambda - \mu} \phi > 0 \) for \( \Lambda - \delta < \mu < \Lambda \) and such that \( u < \frac{K}{\Lambda - \mu} \phi < 0 \), for \( \Lambda < \mu < \Lambda + \delta \), and thirdly the uniqueness.

Step 1. Maximum and anti-maximum principles
We prove the positivity or negativity of the solutions exactly as for the linear case, but, since \( f \) depends on \( u \), we have to show that \( \delta \) (which depends on \( f \) in the linear case) is now uniform. This follows from hypotheses (\( \text{ii} \)) and (\( \text{iii} \)).

Let \( u \) be a solution to \( Lu = \mu u + f(x, u) \). For this \( u \), set
\[
f^1(u) = \int f(x, u)\phi(x)dx, \quad f^\perp(x, u) = f(x, u) - f^1(u)\phi(x).
\]
Also \( u^1 = \int u\phi(x)dx \) and \( u^\perp = u - u^1\phi \).

Note that, always by (\( \text{ii} \)) and (\( \text{iii} \)), \( 0 < \kappa \leq f^1(u) \leq K \).

With this decomposition, reporting in (SLSE), we obtain 2 equations:
\[
(L - \mu)u^1\phi = (\Lambda - \mu)u^1\phi = f^1\phi, \quad Lu^\perp = \mu u^\perp + f^\perp.
\]

Choose \( \mu < \Lambda \) or \( \Lambda < \mu < \lambda_2 \). From the first equation we derive
\[
u^1 = \frac{f^1}{\Lambda - \mu} \to \pm \infty \text{ as } (\Lambda - \mu) \to 0.
\]

Now we proceed exactly as for the linear case. By use of Theorem 3.2 (c) in \([9]\) or \([10]\), we know that the restriction of the resolvent \( (L - \mu)^{-1} \) to \( X \) is bounded from \( X \) into itself. So by (\( \text{iii} \)) and by Lemma \([11]\) there exists a \( \delta_0 \) small enough and there exists a constant \( c_0 \) (depending on \( \delta_0 \)) such that for all \( \mu \) with \( |\Lambda - \mu| < \delta_0 \),
\[
\|u^\perp\|_X \leq c_0\|f^\perp(x, u)\|_X \leq c_0\|f(x, u) - f^1(u)\phi(x)\|_X \leq 2c_0K.
\]

Write now
\[
u = \frac{f^1(u)}{\Lambda - \mu} \phi + u^\perp
\]

Hence \( \|u\|_X \leq \frac{f^1(u)}{|\Lambda - \mu|} + \|u^\perp\|_X \leq \frac{K}{|\Lambda - \mu|} + 2c_0K \). For \( |\Lambda - \mu| \to 0 \), \( \frac{f^1}{\Lambda - \mu} \phi \to \pm \infty \) when \( u^\perp \) stays bounded. For \( |\Lambda - \mu| \) small enough, that is here \( |\Lambda - \mu| < \delta_0' \) (\( f := \frac{\kappa}{2c_0K} \)), we get (since \( f^1 > 0 \))
\[
\frac{f^1}{|\Lambda - \mu|} \geq \frac{\kappa}{|\Lambda - \mu|} > 2c_0K \geq c_0\|f^\perp\|_X.
\]
Finally Maximum and anti-maximum principles are valid for 
\[ \delta(f) := \min\{\delta_0, \delta'(f)\}. \]

**Step 2. Existence of solutions**
We prove the existence of solutions by Schauder fixed point theory; for this purpose we need some classical elements: a set \( K^\pm \) constructed with the help of sub-super solutions and a compact operator \( T \) acting in \( K^\pm \) such that \( K^\pm \) stays invariant by \( T \): \( T(K^\pm) \subset K^\pm \).

1: "Sub-super solution":
- Case \( \Lambda - \delta < \mu < \Lambda \).

Obviously, by \((ii)\), \( u_0 = \frac{K}{\Lambda - \mu} \phi > 0 \) is a subsolution:

\[ L(u - u_0) = \mu(u - u_0) + f - (\Lambda - \mu)u_0 = \mu(u - u_0) + f - \kappa \phi \]

and by \((ii)\) and GSP, \( u - u_0 \geq 0 \).

Analogously \( v_0^0 = \frac{K}{\Lambda - \mu} \phi > 0 \) (\( K \) given in \((iii)\)) is a supersolution:

\[ Lu^0 = \frac{\Lambda}{\Lambda - \mu} K \phi = \Lambda u^0 = \mu u^0 + (\Lambda - \mu)u^0. \]

**Remark 4**: The sub- and supersolutions tend to \(+\infty\) as \( \mu \uparrow \Lambda \).

- Case \( \Lambda < \mu < \Lambda + \delta < \lambda_2 \). \( v^0 = \frac{K}{\Lambda - \mu} \phi < 0 \) is a supersolution. Indeed

\[ L(v^0 - u) = \mu(v^0 - u) + \kappa \phi - f \]

and by \((H_f)\) and the anti-maximum \( 0 > v^0 \geq u \).

Analogously, \( v_0 = \frac{K}{\Lambda - \mu} \phi < 0 \) is a subsolution.

**Remark 5**: The sub- and supersolutions tend to \(-\infty\) as \( \mu \downarrow \Lambda \).

**Remark 6**: Obviously, \( u_0 < u^0 \) for \( \Lambda - \delta < \mu < \Lambda \) (resp. \( v_0 < v^0 \) for \( \Lambda < \mu < \Lambda + \delta \)).

2: The operator \( T \)
We define \( T : u \in L^2 \rightarrow w = Tu \in V \), where \( w \in X \) is the unique solution to \( Lw = \mu w + f(x, u) \).

3: The invariant set \( K^+ := [u_0, u^0] \) for \( \Lambda - \delta < \mu < \Lambda \) (resp. \( K^- := [v_0, v^0] \) for \( \Lambda < \mu < \Lambda + \delta \)).
If $\mu < \Lambda$, by the maximum principle and the hypothesis (iii), $u \leq u^0$ implies $w \leq w^0$. Indeed,

$$L(u^0 - w) = \mu(u^0 - w) + (\Lambda - \mu)u^0 - f(x, u) = \mu(u^0 - w) + K\phi - f(x, u);$$

since, by (iii), $K\phi - f(x, u) \geq 0$, we apply the maximum principle and hence $w \leq u^0$. The 3 other cases lead to analogous calculation.

4: $T$ is compact in $X$.
First note that $K^+ \subset X$ (resp. $K^- \subset X$). $Lw - \mu w = f(x, u)$ can also be written

$$w = \frac{L - \mu I}{L - \mu I} f(x, u) = T(u).$$

Since by [10], [9], the resolvent $R(\mu) := (L - \mu I)^{-1}$ is compact in $X$ for $\mu \in (\Lambda - \delta, \Lambda)$ or $(\Lambda, \Lambda + \delta)$, and since $F : u \rightarrow f(x, u)$ is continuous, $T = R(\mu)F$ is compact.

We deduce from Schauder fixed point theory that there exists a solution to (SLSE) in $K^+$, (resp. in $K^-$).

**Step 3. Uniqueness**
For proving uniqueness we follow [13], p.57. First we assume not only $(H_f)$ but also $(H'_f)$. Assume that $u$ and $v$ are two solutions:

$$Lu = \mu u + f(x, u), \quad Lv = \mu v + f(x, v)$$

The solutions are in $X$ and we have shown that $u, v > u^0 > 0$ for $\Lambda - \delta < \mu < \Lambda$ (resp. $u, v < v^0 < 0$ for $\Lambda < \mu < \Lambda + \delta$). Hence we can write

$$\frac{Lu}{u} = \mu + \frac{f(x, u)}{u}; \quad \frac{Lv}{v} = \mu + \frac{f(x, v)}{v}.$$

By subtraction $q(x)$ and $\mu$ disappear. Multiply by $u^2 - v^2$ and integrate.

$$\int \left(-\Delta \frac{u}{u} + \Delta \frac{v}{v}\right)[u^2 - v^2] = \int \left[\frac{f(x, u)}{u} - \frac{f(x, v)}{v}\right][u^2 - v^2];$$

the last term is non positive by $(H'_f)$.

We transform exactly as in [13] the first term.

$$\int \left(-\Delta \frac{u}{u} + \Delta \frac{v}{v}\right)[u^2 - v^2] = \int |\nabla u - \frac{u}{v} \nabla v|^2 + |\nabla v - \frac{v}{u} \nabla u|^2 =$$

$$\int |v \nabla \frac{u}{v}|^2 + |u \nabla \frac{v}{u}|^2 \geq 0; \quad (4)$$

therefore both terms are equal to 0 and

$$u^2 - v^2 = 0 \Rightarrow u = v a.e.;$$

by regularity, $u = v$. 

9
4 Semi-linear cooperative system

We extend here to a class of semi-linear systems previous results shown in [5] where linear systems of the form $LU = \mu U + AU + F(x)$ are studied. We study for $a > 0$, $b > 0$, $c > 0$

$$(S) \quad \begin{cases} 
   Lu_1 = (\mu + a)u_1 + bu_2 + f_1(x, u_1) \\
   Lu_2 = cu_1 + (\mu + d)u_2 + f_2(x, u_2)
\end{cases} \quad \text{in } \mathbb{R}^N,$$

$u_1(x), u_2(x)|_{x \to \infty} \to 0.$

We write shortly $LU = \mu U + AU + F(x, U)$, where $A$ is the cooperative matrix with components $a, b, c, d$:

$$A = \begin{pmatrix} 
   a & b \\
   c & d
\end{pmatrix}.$$ 

**Notation** $(\xi_1, Y)$: Denote $\xi_1$ the largest eigenvalue of $A$ (the other one being denoted by $\xi_2$); $Y$ is the eigenvector associated with $\xi_1$:

$$AY = \xi_1 Y,$$

$$\xi_1 = \frac{a + d + \sqrt{(a - d)^2 + 4bc}}{2}.$$ 

An easy calculation shows that $(L - A)(Y \phi) = (\Lambda - \xi_1)Y \phi$; moreover here $Y \phi$ is with components which do not change sign: we choose both components of $Y$ positive:

$$y_1 = b > 0, \quad y_2 = \frac{d - a + \sqrt{(a - d)^2 + 4bc}}{2} > 0.$$ 

**Notation** $\Lambda^*$: $\Lambda^* := \Lambda - \xi_1$ is the principal eigenvalue of System $(S)$ with associated eigenvector $Y \phi$:

$$(L - A)(Y \phi) = (\Lambda - \xi_1)Y \phi = \Lambda^* Y \phi.$$ 

**Hypotheses:** We assume

$(H_A)$ $A$ is a $2 \times 2$ cooperative matrix with positive coefficients outside the diagonal.

$(H_F)$ $f_1, f_2 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are Caratheodory function i.e. the functions $f_1(\bullet, u_1)$ or $f_2(\bullet, u_2)$ are Lebesgue measurable in $\mathbb{R}^N$, for every $u_1(x)$ or $u_2(x)$ in $\mathbb{R}$ and the functions $f_1(x, \bullet)$, $f_2(x, \bullet)$ are continuous in $\mathbb{R}$ for almost every $x \in \mathbb{R}^N$. Moreover, $f_1, f_2$ are such that

$(i)$ $\forall u_1, u_2 \in L^2(\mathbb{R}^N), f_1(x, u_1), f_2(x, u_2) \in L^2(\mathbb{R}^N),$
\[ \exists \kappa > 0 \text{ s.t. } f_1(x, u_1), f_2(x, u_2) \geq \kappa \phi(x) \forall u_1, u_2 \in L^2(\mathbb{R}^N), \]

\[ \exists K > \kappa > 0 \text{ s.t. } f_1(x, u_1), f_2(x, u_2) \leq K \phi(x) \forall u_1, u_2 \in L^2(\mathbb{R}^N). \]

\((H_F') : \frac{f_1(x, u_1)}{|u_1|} \text{ and } \frac{f_2(x, u_2)}{|u_2|} \text{ are decreasing w.r.t. } u_1 \text{ and } u_2.\]

We introduce 2 sets:

\[ K^+ := \{(u_1, u_2) \in X^2 / u_1 \in \left(\frac{\kappa y_1 \phi}{\max(y_1, y_2)(\Lambda^* - \mu)}, \frac{K y_1 \phi}{\min(y_1, y_2)(\Lambda^* - \mu)}\right), \quad u_2 \in \left(\frac{\kappa y_2 \phi}{\max(y_1, y_2)(\Lambda^* - \mu)}, \frac{K y_2 \phi}{\min(y_1, y_2)(\Lambda^* - \mu)}\right)\} \]

for \( \mu < \Lambda^* \), and

\[ K^- := \{(u_1, u_2) \in X^2 / u_1 \in \left(\frac{K y_1 \phi}{\min(y_1, y_2)(\Lambda^* - \mu)}, \frac{\kappa y_1 \phi}{\max(y_1, y_2)(\Lambda^* - \mu)}\right), \quad u_2 \in \left(\frac{K y_2 \phi}{\min(y_1, y_2)(\Lambda^* - \mu)}, \frac{\kappa y_2 \phi}{\max(y_1, y_2)(\Lambda^* - \mu)}\right)\} \]

for \( \Lambda^* < \mu \).

**Theorem 3** If \((H_A)\) and \((H_F)\) are satisfied there exists \( \delta > 0 \), depending on \( f_1 \) and \( f_2 \) such that if \( \Lambda^* - \delta < \mu < \Lambda^* \) (resp. \( \Lambda^* < \mu < \Lambda^* + \delta \), (with \( \delta < \min\{\frac{\xi_2 - \xi_1}{2}, \lambda_2 - \Lambda\}\)) System \((S)\) has a solution which is in \( K^+ \), (resp. in \( K^- \)). Moreover, if \((H_F')\) is satisfied, the solution is unique.

**Proof of Theorem 3** We use of course the results above as well as previous results for linear systems obtained in [5] where Theorem 3 is shown for suitable assumptions on \( f_1 \) and \( f_2 \) (independent on \( u \)).

1. Maximum and anti-maximum principles

We diagonalize System \((S)\) thanks to the change of basis matrix \( P \), and we get a system of 2 equations. Here

\[ P = \begin{pmatrix} b & b \\ \xi_1 - a & \xi_2 - a \end{pmatrix}, \quad P^{-1} = \frac{1}{b(\xi_1 - \xi_2)} \begin{pmatrix} a - \xi_2 & b \\ \xi_1 - a & -b \end{pmatrix}, \]

Set

\[ D := P^{-1}AP = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}; \quad U = PV; \quad G := P^{-1}F. \] (5)

We obtain

\[ LV = DV + \mu V + G \] (6)
which is a system of 2 equations (with obvious notation):

\[ L v_1 = (\xi_1 + \mu) v_1 + g_1(u_1, u_2); \]
\[ L v_2 = (\xi_2 + \mu) v_2 + g_2(u_1, u_2). \]

Note that \( g_1 \) and \( g_2 \) are in \( X \).

The second equation, where the parameter \( \xi_2 + \mu \) stays away (below) from \( \Lambda \), has a \( \phi \) bounded solution \( v_2 \). Concerning the first equation, we apply Theorem 2 above. We compute \( g_1, g_2 \) and get

\[ (ii') \quad \exists \kappa' > 0 \text{ s.t. } g_1(x, u_1, u_2) \geq \kappa' \phi(x) \forall u_1, u_2 \in L^2(\mathbb{R}^N), \]

\[ (iii') \quad \exists K' > \kappa' > 0 \text{ s.t. } |g_2(x, u_1, u_2)| \leq K' \phi(x) \forall u_1, u_2 \in L^2(\mathbb{R}^N), \]

where \( \kappa' \) and \( K' \) are 2 positive constants depending on \( \kappa, K \) and on the coefficients of \( A \). This follows from \( \xi_1 - \xi_2 > 0 \) and \( (a - \xi_2) = \frac{a-d}{2} + \sqrt{(a-d)^2 + 4bc} \) with \( (a-d)^2 + 4bc > (a-d)^2 \), so that

\[ g_1 = \frac{1}{\xi_1 - \xi_2} [(a - \xi_2) f_1 + bf_2] > \kappa' \phi > 0. \]

Analogously we have \( g_1 < K' \phi \). Therefore Theorem 2 holds here with \( \delta = \min(\delta_0, \frac{\kappa'}{c_0K}, \frac{\xi_1 - \xi_2}{2}) \). Finally we deduce from the maximum principle for \( \Lambda^* - \delta < \mu < \Lambda^* \) that \( v_1 > \frac{\kappa'}{\Lambda^* - \mu} \phi > 0 \).

If \( \Lambda^* < \mu < \Lambda^* + \delta \), reasoning similarly, we deduce \( v_1 < \frac{K'}{\Lambda^* - \mu} \phi < 0 \). As \( \mu \to \Lambda^* \), \( v_1 \) tends to \( \infty \) when \( v_2 \) stays bounded. Indeed, by Remark 3

\[ \|v_2\|_X \leq \frac{K'}{|\Lambda - \xi_2 - \mu|} + 2c_0K' < \frac{2K'}{\xi_1 - \xi_2} + 2c_0K'; \]

the last inequality follows from \( \delta < \frac{\xi_1 - \xi_2}{2} \).

Now we go back to \( U = PV \).

\[ u_1 = av_1 + bv_2, \ u_2 = (\xi_1 - a)v_1 + (\xi_2 - a)v_2. \]

Combining the estimates above on \( v_1 \) and \( v_2 \), we conclude that, as \( |\Lambda^* - \mu| \to 0 \), there exists \( \delta^* \), depending only on \( L, A, \kappa, K \) such that as \( \mu \searrow \Lambda^* \), \( u_1 \) has the sign of \( a > 0 \) and \( u_2 > 0 \). If \( \mu \nearrow \Lambda^* \), \( u_1 \) has the sign of \( -a < 0 \) and \( u_2 < 0 \).

2. Existence of the solution in \( K^\omega_\delta \), (resp. in \( K^\omega_\delta \))
Sub-supersolutions:

1. Case $\Lambda^* - \delta^* < \mu < \Lambda^*$. Recall that $Y$ has positive components $y_1$ and $y_2$, and the principal eigenvector $\Phi = Y\phi$ satisfies

$$L\Phi - \mu\Phi - A\Phi = (\Lambda^* - \mu)\Phi.$$ 

Inspired by the case of one equation, we seek a subsolution $U_0$ of the form $cY\Phi$.

$$L(U - U_0) = A(U - U_0) + \mu(U - U_0) + (F(x, U) - (\Lambda^* - \mu)c\Phi).$$

For $c$ such that $F(x, U) - (\Lambda^* - \mu)cY\phi(x) > 0$, for $\mu < \Lambda^*$, we get $U - U_0 > 0$ by the maximum principle. Finally, since $F(x, U) - \kappa\max(y_1, y_2)Y\phi > 0$, a subsolution is

$$U_0 = \frac{\kappa}{\max(y_1, y_2)(\Lambda^* - \mu)}Y\phi.$$

Analogously $U^0 = \frac{K}{\min(y_1, y_2)(\Lambda^* - \mu)}Y\phi$ is a supersolution.

2. Case $\Lambda^* < \mu < \Lambda^* + \delta^*$. We have similar results with change of sign and replacing $K$ by $\kappa$.

$$V_0 = \frac{K}{\min(y_1, y_2)(\Lambda^* - \mu)}Y\phi$$

$$V^0 = \frac{\kappa}{\max(y_1, y_2)(\Lambda^* - \mu)}Y\phi$$

The operator $T$: We define $T : (u_1, u_2) \mapsto (w_1, w_2)$ where $(w_1, w_2)$ is the solution to the linear system

$$(S') \begin{cases} 
Lw_1 &= (a + \mu)w_1 + bw_2 + f_1(x, u_1) \\
Lw_2 &= cw_1 + (d + \mu)w_2 + f_2(x, u_2) 
\end{cases} \text{ in } \mathbb{R}^N. \quad \text{in } \mathbb{R}^N,$$

$$w_1(x), w_2(x)|_{x\to\infty} \to 0.$$ 

The rectangle: If $(u_1, u_2) \in \mathcal{K}_S^+$ for $\Lambda^* - \delta^* < \mu < \Lambda^*$ (resp. $(u_1, u_2) \in \mathcal{K}_S^-$ for $\Lambda^* < \mu < \Lambda^* + \delta^*$) then $(w_1, w_2) \in \mathcal{K}_S^+$ (resp $\mathcal{K}_S^-$). Indeed, for $\Lambda^* - \delta^* \mu < \Lambda^*$, this can be written with obvious notations

$$L(W - U_0) = (\mu + A)(W - U_0) + F;$$

for $\mu < \Lambda^*$, since $F$ has non negative components, $F \not\equiv 0$, then $W - U_0 > 0$. Analogously, we obtain the supersolution $U^0 - W > 0$. 

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We argue exactly as for one equation: $K^+_S$ or $K^-_S$ is invariant by $T$ and $LW = (A + \mu)W + F(x, U)$ can be written $W = (L - A - \mu I)^{-1} \hat{F}(x, U) = T(U)$. Since by [10] [9], the resolvent $R(\mu) := (L - \mu I)^{-1}$ is compact in $X$ for $\mu \in (\Lambda^* - \delta^*, \Lambda^*)$ or $(\Lambda^*, \Lambda^* + \delta^*)$, and since $\hat{F} : u \to F(x, u)$ is continuous, $T = R(\mu) \hat{F}$ is compact.

We apply the fixed point theorem. There exists a solution $U$.

3. Uniqueness

We assume now $(H'_p)$. assume there are 2 positive solutions $(u_1, u_2)$ and $(v_1, v_2)$ to (S); for the first equation we have $Lu_1 = (\mu + a)u_1 + bu_2 + f_1(x, u_1)$ and $Lv_1 = (\mu + a)v_1 + bv_2 + f_1(x, v_2)$. Since we are in $K^+$ (resp. $K^-$), divide by $bu_1$ the first equation and by $bv_1$ the second one and subtract:

$$
-\frac{\Delta u_1}{bu_1} + \frac{\Delta v_1}{bv_1} = \frac{u_2}{u_1} - \frac{v_2}{v_1} + \frac{f_1(x, u_1)}{bu_1} - \frac{f_1(x, v_1)}{bv_1}.
$$

Exactly as in [13] multiply by $(u_1^2 - v_1^2)$ and integrate; hence

$$
\int \left( -\frac{\Delta u_1}{bu_1} + \frac{\Delta v_1}{bv_1} \right) (u_1^2 - v_1^2) = \int \left( \frac{u_2}{u_1} - \frac{v_2}{v_1} + \frac{f_1(x, u_1)}{bu_1} - \frac{f_1(x, v_1)}{bv_1} \right) (u_1^2 - v_1^2).
$$

The first term is non-negative by (4):

$$
\int \left( -\frac{\Delta u_1}{bu_1} + \frac{\Delta v_1}{bv_1} \right) (u_1^2 - v_1^2) > 0.
$$

Then do exactly the same calculus with the second equation in (S) and add these two lines: we derive from (7) that $T_1 = T_2$ with

$$
T_1 = \int \left( -\frac{\Delta u_1}{bu_1} + \frac{\Delta v_1}{bv_1} \right) (u_1^2 - v_1^2) + \int \left( -\frac{\Delta u_2}{cu_2} + \frac{\Delta v_2}{cv_2} \right) (u_2^2 - v_2^2),
$$

$$
T_2 = \int \left( \frac{u_2}{u_1} - \frac{v_2}{v_1} + \frac{f_1(x, u_1)}{bu_1} - \frac{f_1(x, v_1)}{bv_1} \right) (u_1^2 - v_1^2) +
\int \left( \frac{u_1}{u_2} - \frac{v_1}{v_2} + \frac{f_2(x, u_2)}{cu_2} - \frac{f_2(x, v_2)}{cv_2} \right) (u_2^2 - v_2^2).
$$

Of course the 1st term $T_1$ is non-negative by (4). By $(H'_p)$,

$$
\int \left( \frac{f_1(x, u_1)}{bu_1} - \frac{f_1(x, v_1)}{bv_1} \right) (u_1^2 - v_1^2) + \int \left( \frac{f_2(x, u_2)}{cu_2} - \frac{f_2(x, v_2)}{cv_2} \right) (u_2^2 - v_2^2) < 0.
$$

We develop what is left and get

$$
\int \left( \frac{u_2}{u_1} - \frac{v_2}{v_1} \right) (u_1^2 - v_1^2) + \int \left( \frac{u_1}{u_2} - \frac{v_1}{v_2} \right) (u_2^2 - v_2^2) =
- \int \left( \sqrt{\frac{u_2v_1^2}{u_1}} - \sqrt{\frac{u_1v_2^2}{u_2}} \right)^2 - \int \left( \sqrt{\frac{u_2v_1^2}{v_1}} - \sqrt{\frac{u_1v_2^2}{v_2}} \right)^2 < 0
$$
Hence $T_1 = T_2 = 0$ and $u_1 = v_1, u_2 = v_2$. The solution is unique.

References

[1] A. Abakhti-Machachi, J. Fleckinger *Existence of positive solutions for non cooperative semilinear elliptic systems defined on an unbounded domain.* Pitman Research Notes in Math, 255, (1992) p.92-106.

[2] B. Alziary, N. Besbas *Anti-Maximum Principle for a Schrödinger Equation in $\mathbb{R}^N$, with a non radial potential.* RoMaKo, 59 (2005) pp 51-62.

[3] B. Alziary, L. Cardoulis J. Fleckinger, *Maximum principle and existence of solutions for elliptic systems involving Schrödinger operators.* Rev. R. Acad. Cienc. Exact. Fis. Nat. 91 (1) (1997), pp 47-52.

[4] B. Alziary, J. Fleckinger *Sign of the solutions to a non cooperative system.* Rostok Math. Kolloq., 71, (2016), p.3-13.

[5] B. Alziary, J. Fleckinger, *Blow up of the solutions to a linear elliptic system involving Schrödinger operators.* to appear.

[6] B. Alziary, J. Fleckinger, M. H. Lecureux, N. Wei *Positivity and negativity of solutions to $n \times n$ weighted systems involving the Laplace operator defined on $\mathbb{R}^N$, $N \geq 3$.* Electron. J. Diff. Eqns, 101, 2012, p.1-14.

[7] B. Alziary, J. Fleckinger, P. Takac, *Maximum and anti-maximum principles for some systems involving Schrödinger operator.* Operator Theory: Advances and applications, 110, 1999, p.13-21.

[8] B. Alziary, J. Fleckinger, P. Takac, *An extension of maximum and anti-maximum principles to a Schrödinger equation in $\mathbb{R}^N$.* Positivity, 5, (4), 2001, pp. 359-382

[9] B. Alziary, J. Fleckinger, P. Takac, *Groundstate positivity, negativity, and compactness for a Schrödinger operator in $\mathbb{R}^N$.* J. Funct. Anal., 245 (2007), 213–248. *Online: doi: 10.1016/j.jfa.2006.12.007.*

[10] B. Alziary, P. Takac *Compactness for a Schrödinger operator in the groundstate space over $\mathbb{R}^N$.* Electr. J Diff. Eq., Conf. 16, (2007) p.35-58.

[11] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces.* SIAM Re. 18, 4, 1976, p.620-709.
[12] H. Amann, *Maximum Principles and Principal Eigenvalues*. Ten Mathematical Essays on Approximation in Analysis and Topology, J. Ferrera, J. López-Gómez, F.R. Ruz del Portal ed., Elsevier, 2005, 1 - 60.

[13] H. Brezis, L. Oswald, *Remarks on sublinear elliptic equations*. Nonlinear Anal., T.M.A., 10, 1986, p.55-64.

[14] P. Clément, L. Peletier, *An anti-maximum principle for second order elliptic operators*. J. Diff. Equ. 34, 1979, p.218-229.

[15] D.E. Edmunds, W.D. Evans. Spectral Theory and Differential Operators. Oxford Science Publications, 1987.

[16] D.G. de Figueiredo, E. Mitidieri. *A maximum principle for an elliptic system and applications to semilinear problems*. S.I.A.M., J.Math.Anal., 17, 1986, p.836-849.

[17] D.G. de Figueiredo, E. Mitidieri. *Maximum principle for linear elliptic systems*. Quaterno Matematico 177, Dip.Sc. Mat., Univ. Trieste, 1988.

[18] D.G. de Figueiredo, E. Mitidieri. *Maximum principle for cooperative elliptic systems*. Comptes Rendus Acad. Sc. Paris, 310, 1990, p.49-52.

[19] J. Fleckinger. *Estimate of the number of eigenvalues for an operator of Schrödinger type*. Proceedings of the Royal Society of Edinburgh, 89A, 355-361, 1981.

[20] J. Fleckinger, J. Hernandez, F de Thelin. *On maximum principles and existence of positive solutions for some cooperative elliptic systems*. Diff and Int Eq., V.8, N.1, p.69-85, 1995.

[21] J. Hernandez. *Maximum Principles and Decoupling for Positive Solutions of Reaction Diffusion Systems*, In K.J. Brown, A.A. Lacey, eds. Reaction Diffusion Equations, Oxford, Clarendon Press, pp.199-224, 1990.

[22] M.H. Lécureux, *Comparison with groundstate for solutions of non cooperative systems for Schrödinger operators in $\mathbb{R}^N$*. Rostok Math. Kolloq. 65, 2010, p51-69.

[23] J. Fleckinger, H. Serag, *Semilinear cooperative elliptic systems on $\mathbb{R}^n$*. Rendiconti Mat. Ser. VII, V.15, (1995) p.89-108.

[24] M. Reed, B. Simon *Methods of modern mathematical physics IV. Analysis of operators*. Acad. Press, New York, 1978.
[25] G.Sweers *A Strong Maximum Principle for a Non-Cooperative Elliptic Systems*, S.I.A.M. J.Math.Anal., vol.20, pp.367-371, 1989.

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