Hamiltonian Approach to QCD in Coulomb Gauge: Perturbative Treatment of the Quark Sector

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Abstract We study the static gluon and quark propagator of the Hamiltonian approach to Quantum Chromodynamics in Coulomb gauge in one-loop Rayleigh–Schrödinger perturbation theory. We show that the results agree with the equal-time limit of the four-dimensional propagators evaluated in the functional integral (Lagrangian) approach.

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1 Introduction

Over the years there has been increased activity in Quantum Chromodynamics (QCD) in Coulomb gauge, starting with the seminal works of Gribov [1] and Zwanziger [2]. The use of Coulomb gauge is motivated by the fact that it is a so-called “physical gauge”. In fact, in QED Coulomb gauge fixing yields immediately the gauge invariant (transverse) part of the gauge field, i.e. physical degrees of freedom. Although this is not the case in QCD, one still expects that the transverse components of the gauge field contain the dominant part of the physical degrees of freedom.\textsuperscript{1}

Coulomb gauge has been mainly used in two approaches to QCD: i) in the Dyson–Schwinger equations (DSEs) based on the functional integral formulation of QCD [2, 4–10] and ii) in a variational approach based on the Hamiltonian formulation [11–20]. The general formulation of Yang–Mills theory within the Dyson–Schwinger approach in Coulomb gauge was set up in Ref. [5] and treated in one-loop perturbation theory in Refs. [6]. Thereby the results of covariant gauges were reproduced.

Since QCD is an asymptotically free theory, one expects its high-energy behaviour to be dominated by the perturbative results. The understanding of perturbation theory is therefore necessary for the regularization and renormalization of non-perturbative approaches. The perturbative treatment of the Yang–Mills sector of QCD within the Hamiltonian formulation was given in Refs. [21, 22]. In the present paper we extend the perturbative analysis to the quark sector of QCD. Some of the results presented below have been already obtained in Ref. [23]. Using the familiar Rayleigh–Schrödinger perturbation theory we calculate the static quark and gluon propagators to one-loop order within the Hamiltonian approach and show that they agree with the equal-time limit of the four-dimensional propagators evaluated in the more traditional functional integral approach [8]. The perturbative results obtained in the present paper for the static propagators are essential ingredients for the renormalization of the (non-perturbative) variational approach to the Hamiltonian formulation of QCD in Coulomb gauge to be presented elsewhere [24].

2 Perturbative Expansion of the QCD Hamilton Operator

The Hamilton operator of QCD in Coulomb gauge [25] reads in $d$ space dimensions

\[ H_{QCD} = -\frac{1}{2} \int d^dx \frac{\delta}{\delta A^a_i(\mathbf{x})} \frac{\delta}{\delta A^a_i(\mathbf{x})} + \frac{1}{4} \int d^dx B^a_i(\mathbf{x}) B^a_i(\mathbf{x}) + \int d^dx \psi^\dagger(\mathbf{x}) \left[ -i \alpha_i \partial_i + \beta m \right] \psi(\mathbf{x}) - g \int d^dx \psi^\dagger(\mathbf{x}) \alpha_i A^a_i(\mathbf{x}) \epsilon^a_{mn} \psi(\mathbf{x}) \]
\[ + \frac{g^2}{2} \int d^4x \, d^4y \, \mathcal{J}_A^{-1} \, n^a(x) \mathcal{A}_A \mathcal{F}^{ab}(x, y) \rho^b(y) \]  

(1)

where \( \alpha \) and \( \beta \) are the Dirac matrices, \( n^a \) (with \( a = 1, \ldots, N_c^2 - 1 \)) are the generators of the \( su(N_c) \) algebra in the fundamental representation, and

\[ \mathcal{J}_A = \text{Det} G_A^{-1}, \]

\[ \{G_A^{ab}(x, y)\}^{-1} = (-\delta^{ab} \partial^2 - g f^{acb} A^c(z) \partial_z) \delta(x-y) \]  

(2)

is the Faddeev–Popov determinant, with \( g \) being the bare coupling and \( f^{abc} \) the structure constants of \( su(N_c) \). Furthermore, \( B^a_t \) is the non-Abelian magnetic field, and

\[ F_A^{ab}(x, y) = \int d^4z \, G_A^{ab}(x, z) (-\partial_z^2) G_A^{cb}(z, y) \]  

(3)

is the so-called Coulomb kernel, which arises from the resolution of Gauss’s law in Coulomb gauge: It describes the Coulomb-like interaction between colour charges, whose density is given by

\[ \rho^a(x) = \rho^a_0(x) + \rho^a_1(x) \]

\[ = \psi^a(x) \gamma^a(\mathbf{p}) + f^{abc} A_c \gamma^b \delta(x) \]

(4)

to which both the quarks and the gluons contribute.

The fermion field operator \( \psi \) can be expanded in terms of the eigenspinors \( u(p, s), v(p, s) \) of the free Dirac Hamiltonian

\[ h_0(p) = \alpha \cdot p + \beta m \]

in the standard way

\[ \psi^a(x) = \int dp \, e^{i p \cdot x} \psi^a(p), \]

\[ \psi^a(p) = \frac{1}{\sqrt{2E_p}} [u(p, s) b^a(p, s) + v(-p, s) d^{a\dagger}(-p, s)], \]

where the index \( s = \pm 1 \) accounts for the two spin degrees of freedom. Furthermore, we have introduced the abbreviation

\[ \int dp \equiv \int \frac{d^4p}{(2\pi)^4}. \]

The spinors \( u(p, s), v(p, s) \) satisfy the eigenvalue equation

\[ h_0(p) u(p, s) = E_p u(p, s), \]

\[ h_0(p) v(-p, s) = -E_p v(-p, s), \]

(5)

with \( E_p = \sqrt{p^2 + m^2} \), and are normalized to

\[ u^\dagger(p, s) u(p', s') = 2E_p \delta_{ss'} = \psi^\dagger(p, s) \psi(p', s'), \]

\[ u^\dagger(p, s) \beta u(p', s') = 2m \delta_{ss'} = -\psi^\dagger(p, s) \beta \psi(p', s'), \]

\[ u^\dagger(p, s) v(-p', s') = 0. \]

(6)

The expansion coefficients \( b^m(p, s), d^{m\dagger}(p, s) \) are annihilation and creation operators satisfying the usual anti-commutation relations

\[ \{b^m(p, s), b^{m\dagger}(q, t)\} = \delta^{mm} \delta_{tt} (2\pi)^3 \delta(p-q), \]

\[ \{d^m(p, s), d^{m\dagger}(q, t)\} = \delta^{mm} \delta_{tt} (2\pi)^3 \delta(p-q), \]

which, with the normalization Eq. (6), ensure that the Fermi field in coordinate space has the required anticommutation relation

\[ \{\psi^m(x), \psi^{m\dagger}(y)\} = \delta^{mm} \delta(x-y). \]

For later convenience it is useful to introduce the following orthogonal projectors

\[ \Lambda_\pm (p) := \frac{1}{2} \pm \frac{\mathcal{H}_0(p)}{2E_p}, \]

(7)

which are (colour diagonal) Dirac matrices satisfying

\[ \Lambda_+(p) + \Lambda_-(p) = 1, \quad \Lambda_+(p) \Lambda_-(p) = 0, \]

\[ [\Lambda_\pm(p)]^2 = \Lambda_\pm(p). \]

Furthermore, from Eqs. (5) and (7) follows

\[ \sum_s \frac{u(p, s) \otimes u^\dagger(p, s)}{2E_p} = \Lambda_+(p), \]

\[ \sum_s \frac{v(p, s) \otimes v^\dagger(p, s)}{2E_p} = \Lambda_-(p). \]

(8)

The Hamiltonian Eq. (1) can be perturbatively expanded in powers of the coupling constant \( g \),

\[ H_{\text{QCD}} = H_0 + gH_1 + g^2 H_2 + \ldots \]

Since the perturbative treatment of the gluon sector within the Hamiltonian approach in Coulomb gauge was already given in Ref. [21] we will focus here on the perturbative treatment of the quark sector. The “unperturbed” Hamiltonian for the quarks is the free Dirac Hamiltonian, i.e. the third term on the r.h.s. of Eq. (1). Using the decomposition Eq. (4) of the quark field and the orthogonality relations (6), it acquires the standard form

\[ H_0 = H_0^Q + \int d^4p \, E_p^Q \left[ b^m(p, s) b^{m\dagger}(p, s) + d^m(p, s) d^{m\dagger}(p, s) \right], \]

where \( E_p^Q \) is the (negative divergent) zero-point energy. The vacuum state of the free Dirac theory is annihilated by the operators \( b \) and \( d \),

\[ b^m(p, s)|0\rangle_Q = 0, \quad d^m(p, s)|0\rangle_Q = 0, \]

(9)
and their Hermitian conjugate operators \( b^\dagger \) and \( d^\dagger \) generate the eigenstates of \( H_0 \), e.g.

\[
H_0 Q \cdot b^\dagger (p, s)|0\rangle_Q = (E^Q_0 + E_p b^\dagger (p, s))|0\rangle_Q, \\
H_0 Q \cdot d^\dagger (p, s)|0\rangle_Q = (E^Q_0 + E_p d^\dagger (p, s))|0\rangle_Q.
\]

The gauge field operator can be expanded as

\[
A_i^\dagger (x) = \int dp \, e^{ipx} A_i^\dagger (p),
\]

\[
A_i^\dagger (p) = \frac{1}{\sqrt{2p_i}} [a_i^\dagger (p) + a_i^\dagger (-p)]
\]

where \( a \) and \( a^\dagger \) are bosonic ladder operators satisfying

\[
[a_i^\dagger (p), a_j^\dagger (q)] = \delta^{ij} t_{ij}(p)(2\pi)^d \delta (p - q),
\]

with

\[
t_{ij}(p) = \delta_{ij} - \frac{p_ip_j}{p^2}
\]

being the transverse projector in momentum space. The unperturbed gluon Hamiltonian becomes

\[
H_0^{YM} = E_0^{YM} + \int dp \, |p| a_i^\dagger (p) a_i^\dagger (p)
\]

where \( E_0^{YM} \) is the irrelevant diverging zero-point energy of the gluons. The vacuum state of the free Yang–Mills sector is annihilated by the operators \( a \)

\[
a_i^\dagger (p)|0\rangle_{YM} = 0
\]

and the eigenstates of \( H_0^{YM} \) are generated by \( a^\dagger \).

The unperturbed QCD vacuum state is given by the tensor product

\[
|0\rangle = |0\rangle_{YM} \otimes |0\rangle_Q.
\]

with the quark and gluonic vacuum defined, respectively, by Eqs. (9) and (11). Expectation values of products of field operators obviously factorize in products of fermionic and gluonic expectation values, e.g.

\[
\langle 0 | A_1^\dagger A_2^\dagger A_3^\dagger | 0 \rangle = \langle 0 | A_1 | 0 \rangle \langle 0 | A_2 | 0 \rangle \langle 0 | A_3 | 0 \rangle, 
\]

for which Wick’s theorem holds: therefore, in perturbation theory all matrix elements can be expressed by the free static gluon

\[
\langle 0 | A_i^\dagger (p) A_j^\dagger (q) | 0 \rangle = \delta^{ij} t_{ij}(p)(2\pi)^d \delta (p + q),
\]

and quark Green functions

\[
\langle 0 | \psi_a^\dagger (p) \psi_a^\dagger (q) | 0 \rangle = \delta^{ab} n_{ij}(p)(2\pi)^d \delta (p - q) [A_+ (p)]_{\alpha \beta}, \\
\langle 0 | \psi_a^\dagger (p) \psi_b^\dagger (q) | 0 \rangle = \delta^{ab} n_{ij}(p)(2\pi)^d \delta (p - q) [A_- (p)]_{\beta \alpha},
\]

where the spinor indices have been written out explicitly.

The first-order perturbation is given by the minimal coupling term \( \psi^\dagger A \psi \), the fourth term on the r.h.s. of Eq. (1), and reads in momentum space

\[
H_1 = \int dp_1 dp_2 \psi_a^\dagger (p_1) [-i g_\alpha \delta^\gamma (p_1 - p_2)] \psi_1^\dagger (p_2).
\]

The second-order perturbation arises from the non-Abelian Coulomb interaction, last term in Eq. (1). Since this operator comes with a factor of \( g^2 \), we can replace the Coulomb kernel by its lowest-order expression, which is given by the negative inverse Laplacian [see Eqs. (2) and (3) with \( g = 0 \)], yielding

\[
H_2 = \frac{1}{2} \int dp_1 dp_2 dp_3 \psi_a^\dagger (p_1) \psi_a^\dagger (p_2) \times \frac{1}{(p_1 - p_2)^2} \psi_1^\dagger (p_3) \psi_1^\dagger (p_1 - p_2 + p_3).
\]

We have included here only the Coulomb interaction between fermionic charges: the coupling between the fermionic and gluonic colour charge through the Coulomb kernel does not contribute to the propagators to one-loop order and will henceforth be discarded.

3 Perturbative Corrections to the Vacuum State

In Rayleigh–Schrödinger perturbation theory the vacuum state is expanded in a power series

\[
|0\rangle_{QCD} \sim |0\rangle + g^2 |0\rangle_{QCD} + O(g^4),
\]

and the perturbative corrections to the wave function are chosen to be orthogonal to the unperturbed state

\[
|0\rangle^{(n)} = 0, \quad n \geq 1.
\]

The first- and second-order corrections to the vacuum state are

\[
|0\rangle^{(1)} = -\frac{\langle N | H_1 | 0 \rangle}{E_N - E_0} |N\rangle, \\
|0\rangle^{(2)} = -\frac{\langle N | H_2 | 0 \rangle + \langle N | H_1 | 0 \rangle^{(1)}}{E_N - E_0} |N\rangle
\]

where \( |N\rangle \) stands for a generic \( N \)-particle state with energy \( E_N \). Furthermore, \( E_0 = E_0^{YM} + E_0^Q \) is the energy of the perturbative QCD vacuum, which cancels, however, in the energy denominators. The series given in Eq. (16) is not normalized. The normalized state reads to the desired order \( O(g^2) \)

\[
|0\rangle_{QCD} = \left(1 - \frac{g^2}{2} |0\rangle^{(1)} |0\rangle + g^2 |0\rangle^{(2)} + O(g^3) \right).
\]

\footnote{For bosonic \( K \)-particle states a factor \( 1/K! \) has to be included to avoid multiple counting.}
It will not be necessary to evaluate \( |0(0)|^0 \) explicitly: this term merely cancels disconnected diagrams occurring in the evaluation of the propagators.

With the help of the projectors \( \Lambda_\pm \) [Eq. (7)] and of the sum rules (8) we obtain
\[
p^m(p,s)|0\rangle = \langle \Lambda_+(p) | a_\beta \psi^m_\alpha(p)|0\rangle \psi^0_\beta(p),
\]
\[
a^m(p,s)|0\rangle = \langle \Lambda_-(p) | a_\beta \psi^m_\alpha(p)|0\rangle \psi^0_\beta(p).
\]
Analogously, in view of Eq. (10) we have
\[
a^0(p)|0\rangle = \frac{1}{2} [pA^-_i(-p)|0\rangle \langle 0|A^+_i(p),
\]
where we have used \( A^-_i(p) = A_-(p) \). These relations allow us to express the matrix elements occurring in Eq. (17) in terms of field operators only, for which we can then use Wick’s theorem together with Eqs. (12) and (13).

The evaluation of the matrix elements in Eq. (17) with the operators \( H_1 \) and \( H_2 \) given by Eqs. (14) and (15) is now straightforward. As an example, we sketch here the evaluation of the first-order correction. From the form of the first-order perturbation [Eq. (14)] follows immediately that we need to consider in the sum in Eq. (17) the states with one gluon, one quark, and one antiquark. With Eqs. (19) and (20) we have therefore
\[
|0\rangle_{\text{first order}} = \int df \frac{\langle 0|\psi^0_\alpha(f_3) \psi^m_\alpha(f_2) \psi^0_\beta(f_1) H_1|0\rangle}{E_{f_1} + E_{f_2} + |f_3|} \times \langle \Lambda_+(f_3) | a_\beta | \Lambda_-(f_2) | a_\gamma \psi^0_\delta(f_1) \rangle |0\rangle.
\]
If we now insert the explicit form Eq. (14) of the first-order perturbation into the above expression, we are led to an expression containing the matrix element
\[
\langle 0|\Lambda^+_i(f_3) \psi^m_\alpha(f_2) \psi^0_\beta(f_1)|0\rangle \Lambda^-_i(p_1 - p_2) \psi^m_\alpha(p_2)|0\rangle.
\]
The fermionic and gluonic expectation values factorize and can be evaluated with the help of Eqs. (13) and (12). For the first-order correction to the QCD vacuum state given above it is straightforward to carry out the perturbative expansion of the quark and gluon propagators analogously to the perturbative treatment of the Yang-Mills sector given in Ref. [21].

### 4 One-Loop Perturbative Propagators

#### 4.1 Gluon Propagator

The perturbative correction to the gluon propagator
\[
q_{\text{QCD}}(0|A^+_i(p)A^+_j(q)|0)_{\text{QCD}} = \delta^{ab} (2\pi)^d \delta(p+q) t_{ij} D(p)
\]
can be evaluated by inserting Eq. (18) with the corrections given by Eqs. (21)–(23) into the above expression. The evaluation of the resulting matrix elements is straightforward. To one-loop level the gluon propagator is obtained in the form
\[
D(p) = \frac{1}{2|p|} \left[ 1 + g^2 D^{(1)}_{\Lambda^+}(p) + g^2 D^{(1)}_{\Lambda^-}(p) + \mathcal{O}(g^4) \right],
\]
where
\[
D^{(1)}_{\Lambda^+}(p) = \frac{g^2 N_c}{(d-1)p^2} \int d |q| \frac{1 - (\hat{p} \cdot \hat{q})^2}{|q||p+q|} \left[ \frac{|p+q|}{|p+q|+|p+q|^2} \right] \times \left[ (d-1)|q|^2 + (d-2)|p|^2 + (p \cdot q)^2 \right],
\]
and
\[
D^{(1)}_{\Lambda^-}(p) = \frac{g^2 N_c}{4(d-1)p^2} \int d |q| \frac{d - 2 + (\hat{p} \cdot \hat{q})^2}{|q|^2 (p+q)^2} q^2 - p^2.
\]
represents the one-loop correction arising from the Yang–Mills sector which was calculated in Refs. [21, 22], while
\[
D_Q^{(1)}(p) = \frac{t_{ij}(p)}{2(d-1)p^2} \int d_q \frac{2|p|E_q + E_{p+q}}{(|p| + E_q + E_{p+q})^2} \times \text{tr} \left[ \Lambda_+ (p+q) \alpha_i \Lambda_- (q) \alpha_j \right]
\]
is the contribution of the dynamical quarks. The Dirac trace can be explicitly taken and results in
\[
D_Q^{(1)}(p) = \frac{1}{(d-1)p^2} \int d_q \frac{|p| + E_q}{E_qE_{p+q}} \times \frac{(d-1)[E_qE_{p+q} + q \cdot (p+q) + m^2] - 2q_i t_{ij}(p) q_j}{(|p| + E_q + E_{p+q})^2},
\]  
(24)
As shown in Refs. [21, 22], the result of the Rayleigh–Schrödinger perturbation theory can be compared with the result [6, 8] of the more conventional perturbation theory in the Lagrangian (functional integral) approach in Coulomb gauge. The quark-loop contribution to the gluon form factor evaluated in Ref. [8] reads
\[
W_Q^{(1)}(p) = \frac{2}{(d-1)p^2} \int d^{d+1}q \times \frac{(d-1)[q^2 + q_i p_i + q \cdot (q + p) + m^2] - 2q_i t_{ij}(p) q_j}{(q^2 + m^2)((p+q)^2 + m^2)}
\]
(25)
where \(p^2 = p^2 + p^2\) is the squared Euclidean four-momentum. The dressing function of the equal-time propagator is related to the dressing function \(\beta\) of the more conventional perturbation theory in the Lagrangian (functional integral) approach in Coulomb gauge. The quark-loop contribution to the gluon form factor evaluated in Ref. [8] reads
\[
W_Q^{(1)}(p) = \frac{2}{(d-1)p^2} \int dp_1 W_Q^{(1)}(p_1) + \cdots \tag{26}
\]
Inserting Eq. (25) into Eq. (26) and performing the integrals over \(p_1\) and \(q_1\), Eq. (24) is indeed recovered. Furthermore, from the non-renormalization of the ghost-gluon vertex the well-known first coefficient of the QCD \(\beta\) function
\[
\beta(g) = \mu^2 \frac{\partial}{\partial \mu} g^2(\mu) = \frac{\beta_0}{(4\pi)^2} g^4 + \ldots,
\]
(27)
in presence of quarks is recovered in the present Hamiltonian approach.

4.2 Quark Propagator

The quark propagator is defined in the Hamiltonian approach by
\[
\frac{1}{2} \frac{1}{2} QCD [0] \left[ \psi_{\alpha}^\dagger (p) \psi_{\alpha} (q) \right] [0]_{QCD} = S_Q^{(\alpha)}(p) (2\pi)^d \delta(p - q).
\]
(28)
\[
S_Q^{(\alpha)}(p) = \delta^{\alpha \alpha} S_0(p) \delta(p - q) + \sum_{k+\ell > 0} g^{k+\ell} \psi^\dagger (p) \psi^\alpha (q) |0\rangle \langle 0 |. \tag{29}
\]
The evaluation of the matrix elements arising from the insertion of Eqs. (21) and (22) into Eq. (28) is somewhat lengthy but straightforward, and results in
\[
S(p) = S_0(p) \left[ 1 - g^2 \frac{C_F}{2} \int \frac{d_d q}{(p+q)^2} \delta(p+q) \frac{d_d-1}{2} \right] + \frac{g^2 C_F}{2} \int \frac{d_d q}{(p+q)^2} \delta(p+q) \frac{d_d-1}{2} \right] + \frac{g^2 C_F}{2} \int \frac{d_d q}{(p+q)^2} \delta(p+q) \frac{d_d-1}{2} \right] \tag{30}
\]
If the perturbed propagator is parameterized by
\[
S(p) = \frac{\mathcal{A}(p) \cdot p + \beta \mathcal{B}(p)}{2E_p}
\]
from Eq. (29) we can extract the one-loop expressions for the form factors \(\mathcal{A}\) and \(\mathcal{B}\) by taking the appropriate traces. This results in
\[
\mathcal{A}(p) = 1 - g^2 \frac{m^2 C_F}{2p^2 E_p^2} \int \frac{d_d q}{(2\pi)^d} E_q (p+q) \]
(31a)
(31b)
for the form factor of the kinetic term, and

$$\mathcal{B}(p) = m + m g^2 \frac{C_F}{2E_p} \int dq \frac{p \cdot (p + q)}{E_q(p + q)^2}$$

$$+ mg^2 \frac{C_F}{2E_p} \int dq \frac{p \cdot (p + q)}{E_q(p + q)(E_p + E_q + |p + q|)^2}$$

$$\times \left\{ \left(2E_p + E_q + |p + q| \right) \left[ (d - 3)p \cdot q + 2 \frac{[p \cdot (p + q)][q \cdot (p + q)]}{(p + q)^2} \right] \right.$$

$$+ (d - 1)[(p^2 - m^2)E_p + p^2|p + q| - m^2E_q] \right\}$$

(31b)

for the mass term.

As for the gluon propagator, these form factors can be compared with the results of the Lagrangian approach. In Ref. [8], the quark propagator (in Euclidean space) was parameterized in the form

$$S(p, p_4) = \frac{p_4 F_2(p) + \alpha \cdot p F_3(p) + \beta M(p)}{p_4^2 + p^2 + m^2}$$

(32)

where the dressing functions $F_2$, $F_3$, and $M$ are functions of $p^2$ and $p_4^2$. In this parameterization the $p_4 p_4$ component has been discarded, since it does not arise at one-loop level. The static propagator is obtained from the energy-dependent one [Eq. (32)] by integrating out the temporal component $p_4$ of the four-momentum

$$S(p) = \int \frac{dp_4}{2\pi} S(p, p_4).$$

(33)

Since the dressing function $F_2(p)$ is an even function of $p_4$, this component does not contribute to the equal-time propagator. Inserting Eq. (32) into Eq. (33) we find from Eq. (30) the identification

$$\int \frac{dp_4}{2\pi} \frac{F_2(p)}{p_4^2 + p^2 + m^2} = \mathcal{A}(p),$$

$$\int \frac{dp_4}{2\pi} \frac{M(p)}{p_4^2 + p^2 + m^2} = \mathcal{B}(p).$$

In fact, inserting here for $F_2(p)$ and $M(p)$ the results found in Ref. [8] and performing the integration over $p_4$ we recover for $\mathcal{A}(p)$ and $\mathcal{B}(p)$ the expressions given in Eqs. (31).

The renormalization of the quark propagator in the Rayleigh–Schrödinger perturbation theory can be worked out in the usual way. For this purpose we write the form factors (31) as

$$\mathcal{A}(p) = 1 + g^2 a_1, \quad \mathcal{B}(p) = m(1 + g^2 b_1),$$

where some regularization scheme has been assumed for the integrals defined by Eqs. (31). Inserting these expressions into Eq. (30) yields

$$S(p) = \frac{\alpha \cdot p(1 + g^2 a_1) + \beta m(1 + g^2 b_1)}{2\sqrt{p^2 + m^2}}.$$  (34)

Introducing the renormalized mass $m_R$ by

$$m = Z_m m_R, \quad Z_m = 1 + g^2 c_1,$$

the propagator Eq. (34) can be rewritten as

$$S(p) = \left(1 - g^2 \frac{c_1 m_R^2}{p^2 + m_R^2} \right) \times \frac{\alpha \cdot p(1 + g^2 a_1) + \beta m_R[1 + g^2(b_1 + c_1)]}{2\sqrt{p^2 + m_R^2}}.$$  (35)

Here the term in the parentheses arises from replacing the bare mass $m$ in the denominator of Eq. (34) by the renormalized one $m_R$. Expressing the bare propagator Eq. (27) in terms of renormalized quantities by

$$S(p) = Z_2 \frac{\alpha \cdot p + \beta m_R}{2\sqrt{p^2 + m_R}},$$

the quark mass and wave function renormalization constants, $Z_m$ and $Z_2$, must be chosen as

$$Z_m = 1 + g^2(a_1 - b_1), \quad Z_2 = 1 + g^2 \frac{p^2 a_1 + m_R^2 b_1}{p^2 + m_R^2}.$$  (36)

To obtain the explicit expressions for $a_1$, $b_1$ we use a momentum cut-off in Eq. (31). Alternatively, we can find these quantities by integrating the corresponding results of the Lagrangian functional integral approach [8] over the temporal component $p_4$ of the four-momentum. In this case, the divergent parts are for $d = 3 - 2\epsilon$

$$a_1^{\text{div}} = - \frac{C_F}{(4\pi)^2} \frac{p^2 + 4m_R^2}{p^2 + m_R^2} \frac{1}{\epsilon},$$

$$b_1^{\text{div}} = - \frac{C_F}{(4\pi)^2} \frac{2p^2 - m_R^2}{p^2 + m_R^2} \frac{1}{\epsilon}.$$  (36)

The coefficients in front of the $\epsilon$ pole in Eq. (36) are also found as factors multiplying $\ln \Lambda^2$ when the integrals in Eq. (31) are evaluated with a momentum cut-off $\Lambda$.

The two form factors $\mathcal{A}$, $\mathcal{B}$ cannot be separately renormalized, since the coefficients in Eq. (36) are momentum dependent. However, the quark mass and wave function renormalization constants [see Eq. (35)] result in the momentum-independent quantities

$$Z_m = 1 - \frac{g^2 C_F}{(4\pi)^2} \frac{3}{\epsilon} + \cdots, \quad Z_2 = 1 - \frac{g^2 C_F}{(4\pi)^2} \frac{1}{\epsilon} + \cdots.$$  (37a)

$$Z_2 = 1 - \frac{g^2 C_F}{(4\pi)^2} \frac{1}{\epsilon} + \cdots, \quad (37b)$$

In the Hamiltonian approach we work with $\psi^\dagger$ rather than $\bar{\psi} = \psi^\dagger \beta$. The formulae presented here differ from the ones in Ref. [8] by an overall matrix $\beta$. Lattice calculations [26] indicate that this component vanishes.
where the renormalization-scheme dependent terms have been discarded. In particular, Eq. (37a) is the (at this order perturbatively) gauge-invariant result for the mass renormalization constant, which agrees with the results for the pole mass from covariant gauges (see e.g. Refs. [27]). Let us also mention that the renormalization procedure carried out above necessitates the use of a fermion propagator [Eq. (27)] defined with the commutator, otherwise the fermion propagator is not multiplicatively renormalizable.

5 Conclusions

In this work we have extended the perturbative analysis of Yang–Mills theory within the Hamiltonian approach in Coulomb gauge performed in Ref. [21] to full QCD. The one-loop quark propagator as well as the quark-loop contribution to the gluon propagator have been calculated. Thereby the equal-time limit of the time-dependent propagators in the conventional functional integral (Lagrangian) formalism has been reproduced. Also the one-loop beta function has been recovered in the present Hamiltonian approach. The perturbative results obtained in this work are necessary ingredients for the renormalization of the non-perturbative propagators, which are currently under investigation.

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References
1. V. Gribov, Nucl. Phys. B139, 1 (1978).
2. D. Zwanziger, Nucl. Phys. B518, 237 (1998).
3. J. Heffner, H. Reinhardt, D.R. Campagnari, Phys. Rev. D85, 125029 (2012).
4. L. Baulieu, D. Zwanziger, Nucl. Phys. B548, 527 (1999).
5. P. Watson, H. Reinhardt, Phys. Rev. D75, 045021 (2007).
6. P. Watson, H. Reinhardt, Phys. Rev. D76, 125016 (2007); Phys. Rev. D77, 025030 (2008).
7. P. Watson, H. Reinhardt, Eur. Phys. J. C65, 567 (2010); Phys. Rev. D82, 125010 (2010).
8. C. Popovici, P. Watson, H. Reinhardt, Phys. Rev. D79, 045006 (2009).
9. P. Watson, H. Reinhardt, Phys. Rev. D85, 025014 (2012); Phys. Rev. D86, 125030 (2012).
10. C. Popovici, P. Watson, H. Reinhardt, Phys. Rev. D81, 105011 (2010); Phys. Rev. D83, 025013 (2011).
11. D. Schutte, Phys. Rev. D31, 810 (1985).
12. A.P. Szczepaniak, E.S. Swanson, Phys. Rev. D65, 025012 (2001).
13. C. Feuchter, H. Reinhardt, Phys. Rev. D70, 105021 (2004).
14. H. Reinhardt, C. Feuchter, Phys. Rev. D71, 105002 (2005).
15. D. Eppe, H. Reinhardt, W. Schleifenbaum, Phys. Rev. D75, 045011 (2007).
16. D. Eppe, H. Reinhardt, W. Schleifenbaum, A. Szczepaniak, Phys. Rev. D77, 085007 (2008).
17. D. Campagnari, H. Reinhardt, Phys. Rev. D82, 105021 (2010).
18. M. Pak, H. Reinhardt, Phys. Lett. B707, 566 (2012); Phys. Rev. D88, 125021 (2013).
19. H. Reinhardt, D. Campagnari, A. Szczepaniak, Phys. Rev. D84, 045006 (2011).
20. H. Reinhardt, J. Heffner, Phys. Lett. B718, 672 (2012); Phys. Rev. D88, 045024 (2013).
21. D. R. Campagnari, H. Reinhardt, A. Weber, Phys. Rev. D80, 025005 (2009).
22. D. Campagnari, A. Weber, H. Reinhardt, F. Astorga, W. Schleifenbaum, Nucl. Phys. B842, 501 (2011).
23. D. R. Campagnari, Ph.D. thesis, Universität Tübingen, Tübingen (2010).
24. D. R. Campagnari, H. Reinhardt, to be published.
25. N.H. Christ, T.D. Lee, Phys. Rev. D22, 939 (1980).
26. G. Burgio, M. Schroock, H. Reinhardt, M. Quandt, Phys. Rev. D86, 014506 (2012).
27. R. Tarrach, Nucl. Phys. B183, 384 (1981). N. Gray, D.J. Broadhurst, W. Grafe, K. Schilcher, Z. Phys. C48, 673 (1990). A.S. Kronfeld, Phys. Rev. D58, 051501 (1998).