ON THE REGULARITY OF ORIENTABLE MATROIDS

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ABSTRACT. We present two characterizations of regular matroids among orientable matroids and use them to give a measure of “how far” an orientable matroid is from being regular.

1. INTRODUCTION

A regular matroid is a matroid which is representable over any field; these objects have been heavily studied in the literature. The following are both characteristics of a regular matroid $M$:

1. Every basis of $M$ generates the same lattice of fundamental circuits.
2. The rank of the circuit lattice of $M$ is equal to corank$(M)$.

A proof of these statements is provided in Section 3.

Our main contribution in this paper is to prove that the first property above is actually an equivalence, and to provide a new proof that the second property is an equivalence:

Theorem 1. For $M$ an orientable matroid, the following are equivalent:

1. $M$ is regular.
2. Every basis of $M$ generates the same lattice of fundamental circuits.
3. The rank of the circuit lattice of $M$ is equal to corank$(M)$.

The equivalence of (1) and (3) is originally proven by Nickel in [7]. His proof relies on a result of Minty in [8], which gives that an orientable matroid is regular if and only if every circuit of the matroid is orthogonal to every cocircuit. Here, we give an alternate proof of the equivalence of (1) and (3), which relies on the equivalence of (1) and (2).

Although there are several other characterizations of regularity among orientable matroids—see for example [3], [4], and [5]—the ones given here can be used to provide a notion of how badly the matroid fails to be regular. This will be made precise in Section 4.

2. BACKGROUND

In this section we recall the definitions of oriented and regular matroids and set up notation. For a more comprehensive overview, we refer the reader to the book on matroids by Oxley [1] and the book on oriented matroids by Bjorner et al [2]. An oriented matroid $M$ is, informally speaking, a matroid together with a set of sign data which behaves well under circuit
elimination. More precisely, when \( E \) is the ground set of \( M \), a signed subset of \( E \) is a map \( X : E \to \{+,-,0\} \). The support of \( X \) is denoted \( \text{supp}(X) \) and defined to be \( \{e \in E : X(e) \neq 0\} \). We define \( X^+ = \{e \in E : X(e) = +\} \) and \( X^- = \{e \in E : X(e) = -\} \).

**Definition 2.** An oriented matroid \( M = (E,C) \) is a nonempty finite set \( E \) together with a collection \( C \) of signed subsets that satisfy the following axioms:

1. \( C \neq \emptyset \).
2. If \( C \in C \), then \( -C \in C \).
3. For all \( C_1, C_2 \in C \), if \( C_1 \subseteq C_2 \), then either \( C_1 = C_2 \) or \( -C_1 = C_2 \).
4. For all \( C_1, C_2 \in C \), \( e \in C_1^+ \cap C_2^+ \), and \( f \in (C_1^+ \setminus C_2^+) \cup (C_1^- \setminus C_2^-) \), there is a \( C_3 \in C \) such that \( C_3^+ \subseteq (C_1^+ \cup C_2^+) \setminus \{e\} \), \( C_3^- \subseteq (C_1^- \cup C_2^-) \setminus \{e\} \), and \( f \in C_3 \).

The elements of \( C \) are called *signed circuits*.

**Theorem 3.** [6, Theorem 3.1.1]

Let \( M \) be a matroid. The following are equivalent:

1. \( M \) is representable over \( \mathbb{Q} \) by a totally unimodular matrix.
2. \( M \) is representable over \( \mathbb{Q} \) by a unimodular matrix.
3. \( M \) is representable over any field \( \mathbb{F} \).
4. \( M \) is orientable and representable over \( \mathbb{F}_2 \).

If \( M \) satisfies any of the above conditions, we say \( M \) is regular.

Since matroids representable over ordered fields are orientable, it follows that all regular matroids are orientable. In fact, the oriented structure on a regular matroid is unique up to reorientation [2, Corollary 7.9.4]. It is well known that a matroid is regular if and only if its dual is.

Given a basis \( B \) of \( M \) and \( e \in E \setminus B \), denote as \( C(B,e) \) the oriented fundamental circuit of \( B \) and \( e \); that is, the unique oriented circuit whose support is contained in \( B \cup e \).

Throughout the rest of the paper, \( M \) will denote an orientable matroid of rank \( r \) with ground set \( E \). Let \( \mathcal{B} \) denote the set of bases of \( M \) and let \( C(B_i) \) and \( C^*(B_i) \) denote the set of oriented fundamental circuits and cocircuits, respectively, of a basis \( B_i \in \mathcal{B} \). Let \( \Lambda_i \) and \( \Lambda_i^* \) denote the lattices in \( \mathbb{Z}^E \) generated by \( C(B_i) \) and \( C^*(B_i) \), respectively. In defining these lattices, vectors with entries in \( \{+,-,0\} \) are treated as vectors in \( \mathbb{Z}^E \) with entries in \( \{1,-1,0\} \) in the obvious way.

### 3. Characterizations of regular matroids

We begin by proving the folklore result mentioned in Section [1].

**Proposition 4.** When \( M \) is a regular matroid, the following hold.

1. Every basis of \( M \) generates the same lattice of fundamental circuits.

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1. This condition is generally known as the strong circuit elimination axiom.
(2) The rank of the circuit lattice of $M$ is equal to $\text{corank}(M)$. That is, $\text{rank}(\Lambda) = \text{corank}(M)$.

**Proof.** Fix a totally unimodular matrix representation $A$ for $M$. The set of circuits of $M$ is defined to be the support-minimal elements of $\ker(A) \cap \mathbb{Z}^E$. Clearly $\ker(A)$ has dimension equal to $\text{corank}(M)$ over $\mathbb{Q}$. Any set of fundamental circuits of a single basis are independent over $\mathbb{Q}$, so they generate all of $\ker(A)$.

Totally unimodular matrices have totally unimodular kernels, by which we mean that there exists a matrix whose rows are a basis for the kernel which is itself totally unimodular. We consider the matrix whose rows are the fundamental circuits of a single basis; since the fundamental circuits form a basis for $\ker(A)$, this matrix is equivalent under a change of basis to one which is totally unimodular. Since these fundamental circuits form a unimodular matrix, they must also form an integral basis for $\ker(A) \cap \mathbb{Z}^E$. Since the fundamental circuits of each basis generate $\ker(A) \cap \mathbb{Z}^E$, it is clear that the first statement holds.

This implies that the maximum size of an integrally independent set of circuits is $\text{corank}(M)$, since this is the rank of the lattice of circuits of $M$. $\square$

Now we turn to proving that these properties of regular matroids are in fact characterizations.

**Proof of Theorem 4.** We begin by proving the first property in Theorem 4. Proposition 4 gives one direction, so it remains here to show that if each basis generates the same lattice of fundamental circuits, then $M$ is regular. Suppose that all the $\Lambda_i$ are equal, and fix an initial basis $B_0$. (Recall that $\Lambda_i$ denotes the lattice generated by the fundamental circuits of a basis $B_i$.) Let $A$ be the matrix defined by

$$A = [I_r | D]$$

where $r$ is the rank of $M$, the columns in $I_r$ (which denotes the $r \times r$ identity matrix) correspond to the elements of $B_0$, and the columns in $D$ correspond to $e \in E \setminus B_0$. The columns of $D$ are constructed according to their oriented fundamental circuit in $B_0$. That is, the linear dependence among the columns of $I_r$ and a column $e \in D$ produces the oriented fundamental circuit $C(B_0, e)$, and the column $e$ has entries in $\{0, \pm 1\}$.

It is clear that the entries of $A$ are all in $\{0, \pm 1\}$. The matrix $A$ defines some matroid, which we call $M'$. The circuits of $M'$ are the signed support of support-minimal elements of $\ker(A) \cap \mathbb{Z}^E$. By assumption, $C(B_0)$ generates the circuit lattice of $M'$; and by construction, $C(B_0) \subseteq C(M')$. Therefore, any circuit of $M$ must be a linear dependence in $A$, since any circuit $C$ of $M$ is a $\mathbb{Z}$-linear combination of elements of $C(B_0)$ and is therefore in $\ker(A)$. It remains to show that this linear dependence in $A$ is support-minimal in $M'$, i.e. support-minimal in $\ker(A) \cap \mathbb{Z}^E$. Suppose this linear
dependence is not support-minimal, so there exists a circuit $C$ of $M$ and a circuit $C'$ of $M'$ such that $\text{supp}(C') \subset \text{supp}(C)$. Then $C'$ is a support-minimal element of $\ker(A) \cap \mathbb{Z}^E$. Since it is an element of $\ker(A)$, it is some $\mathbb{Q}$-linear combination of elements of $C(B_0)$, since $C(B_0)$ forms a basis for $\ker(A)$, and can be written as $C' = q_1c_1 + \cdots + q_{c_\ell}c_\ell$ where the $q_i \in \mathbb{Q}$ and the $c_i$ are fundamental circuits of $B_0$. We will use the following lemma:

**Lemma 5.** The support of any $\mathbb{Q}$-linear combination of elements of $C(B_0)$ is a dependent set in $M$, provided that every basis of $M$ generates the same lattice of fundamental circuits.

**Proof.** Let $C' = q_1c_1 + \cdots + q_{c_\ell}c_\ell$ be a $\mathbb{Q}$-linear combination of elements of $C(B_0)$ as above. There exists an integer $t$ such that $tc'$ is in the circuit lattice of $M$ (we can take $t$ to be the least common denominator of the $q_i$ so that $tc'$ is an integer linear combination of the $c_i$). We will show that $C'$ is a dependent set in $M$. To the contrary, suppose that $C'$ is independent. Then it is a subset of some basis of $M$, which we call $B'$. In this case, $tc'$ cannot be in the lattice of circuits generated by $C(B')$. This holds since each element of $C(B')$ has exactly one element $e \in E \setminus B'$ in its support, and no two distinct elements of $C(B')$ contain the same such $e$ in their support. Therefore, it is impossible for all the $e \notin B'$ to cancel to give a linear combination of elements of $C(B')$ whose support is contained in $B'$. Therefore, $tc'$ is not in the lattice generated by $C(B')$, but it is in $\Lambda_0$, contradicting the hypothesis that all the $\Lambda_i$ are equal. This implies that $C'$ cannot be contained in any basis, so it must be dependent in $M$.

We apply Lemma 5 to conclude that $C' = q_1c_1 + \cdots + q_{c_\ell}c_\ell$ is a dependent set in $M$ whose support is properly contained in that of $C$, which contradicts the support-minimality of $C$ in $M$. Therefore no such $C'$ can exist, which means that $C$ is also a circuit in $M'$.

Therefore $C(M) \subseteq C(M')$. It remains to show that $C(M') \subseteq C(M)$ and that $A$ is unimodular. The second statement is proved below:

**Lemma 6.** $A$ is unimodular.

**Proof.** First, we will prove that it suffices to show that every support-minimal linear dependence of columns of $A$ can be written with coefficients of absolute value 1. If this is the case, then we can apply Cramer’s rule to prove the unimodularity of $A$.

Let $N$ be a square submatrix of $A$ which has rank $r$. We will say that the columns of such an $N$ form a basis for $A$. If in addition $\det(N) = \pm 1$, we will say that $N$ is a unimodular basis for $A$. Apply Cramer’s rule to a submatrix $N$ of $A$ whose columns form a basis for $A$ and a column $b$ of $A$ which is not in $N$. Then there exists a unique linear dependence among the columns of $N$ and $b$, which is the fundamental circuit $C(N,b)$. Assume for the moment that the coefficients of this dependence are in $\{1,0,-1\}$. If $\det(N) = \pm 1$, then $\det(N^i_b) \in \{1,0,-1\}$ for each $i$, where $N^i_b$ denotes the matrix obtained
from $N$ by replacing the $i^{th}$ column of $N$ with $b$. This proves that any basis which can be obtained from a unimodular basis via a single basis exchange move is itself unimodular. Any basis can be obtained from any other via a sequence of basis exchange moves, so it is sufficient that there exist some unimodular basis of $A$. Since $A$ contains a full-rank identity matrix as a submatrix, this holds. Therefore it remains to prove that every support-minimal linear dependence of columns of $A$ can be written with coefficients of absolute value 1.

Suppose there is some support-minimal linear dependence in $A$ which cannot be expressed in this way, so we have $a_1e_1 + \cdots + a_ke_k = 0$ for the $a_i$ nonzero integers not all having the same absolute value and the $e_i$ columns of $A$. (By abuse of notation, we let $e_1, \ldots, e_k$ denote any set of $k$ distinct columns of $A$; these need not be the first $k$ columns of $A$.) This linear dependence is in the kernel of $A$, so it can be written as some $Q$-linear combination of the fundamental circuits associated to $B_0$, since these form a basis for ker$(A)$. Therefore $a_1e_1 + \cdots + a_ke_k$ is a $Q$-linear combination of circuits in $C(B_0)$, so by Lemma 5 the signed support of this linear dependence forms a dependent set in $M$, which we will denote $C_k$. We divide into two cases. First, we assume that $C_k$ is actually a circuit of $M$, in which case the linear dependence in $M$ is actually support-minimal.

$C_k$ is a circuit of $M'$ by definition. By hypothesis, all $\Lambda_i$ are equal, so $C_k$ is in the $Z$-span of the fundamental circuits of $B_0$, that is, in $\Lambda_0$. Then we have that $C_k - (a_1e_1 + \cdots + a_ke_k)$ is also in the $Z$-span of elements of $C(B_0)$. By choosing an appropriate integer multiple $t$ of $C_k$, at least one term $e_i$ in $tC_k - (a_1e_1 + \cdots + a_ke_k)$ can be made to cancel. But this gives some new circuit of $M'$ whose support is properly contained in the support of $C_k$, contradicting the support-minimality of $C_k$.

Next, suppose $C_k$ is a dependent set in $M$ but not a circuit. Then there is some circuit of $M$ whose support is properly contained in that of $C_k$; call this circuit $C_j$. Since $C(M) \subseteq C(M')$, we would have $C_j$ also a circuit of $M'$, contradicting the support-minimality of $C_k$ in $M'$.

This gives that every support-minimal linear dependence of columns of $A$ can be written such that the coefficients in its support have absolute value 1. Combined with Cramer’s rule, this proves the unimodularity of $A$.

Since $A$ is unimodular, by Theorem 3 $M'$ must be regular. It remains to show that $M = M'$; equivalently, we can show that the circuits of $M$ are exactly the circuits of $M'$.

All $\Lambda_i$ are equal by hypothesis, and all $\Lambda'_j$ are equal by the fact that $M'$ is regular. (Here, we denote by $\Lambda'_j$ the lattice generated by the fundamental circuits of a basis $B'_j$ of $M'$.) Moreover, $\Lambda_0 = \Lambda'_0$ by construction of $A$. Therefore, $M$ and $M'$ have the same circuit lattice and a common basis, which gives that $C(M) = C(M')$. Therefore, since $C(M')$ is the set of
support-minimal elements of \( \ker(A) \cap \mathbb{Z}^E \), so is \( C(M) \). This implies that \( A \)
is a totally unimodular representation for \( M \), so \( M \) is regular.

Note that the dual statement for lattices generated by fundamental co-circuits of bases is also true. That is, \( M \) is regular if and only if all the \( \Lambda_i^* \) are the same.

We now turn to proving (3) of Theorem 1: an orientable matroid \( M \) is regular if and only if \( \text{rank}(\Lambda) = \text{corank}(M) \).

**Proof.** Proposition 4 gives that if \( M \) is regular, then \( \text{rank}(\Lambda) = \text{corank}(M) \).

So, suppose \( M \) is nonregular. By (2) of Theorem 1, there are two bases, call them \( B_1 \) and \( B_2 \), which do not generate the same circuit lattice. Then there is an oriented fundamental circuit \( c \) of \( B_2 \) which is not in the integral span of the circuits of \( B_1 \). In order to produce an independent set of circuits of size \( \text{corank}(M) + 1 \), we will show that there is no circuit in \( c \in C(B_1) \) which is in the integral span of \( \{c \cup C(B_1)\} \setminus \{c_1\} \). This will prove that minimum size of a generating set for the lattice generated by \( C(B_1) \cup c \) is equal to \( \text{corank}(M) + 1 \). Suppose to the contrary that

\[
c_1 = z_1c + z_2c_2 + \cdots + zkc_k
\]

for \( z, z_2, \ldots, z_k \in \mathbb{Z}, \ c_1 \in C(B_1), \) and \( c_2, \ldots, c_k \in C(B_1) \setminus \{c_1\} \). Since by assumption \( c \) is not in the \( \mathbb{Z} \)-span of \( C(B_1) \), we must have \( |z| > 1 \).

Since \( c_1 \) is a fundamental circuit of \( B_1 \), there exists a unique \( e_1 \in E \setminus B_1 \) which is in the support of \( c_1 \). This \( e_1 \) is not in the support of any of \( c_2, \ldots, c_k \), since these are fundamental circuits of \( B_1 \) distinct from \( c_1 \). Therefore, in order for the integral dependence to hold, \( e_1 \) must be in the support of \( c \). Since all nonzero entries in \( c_1 \) have absolute value 1, this implies that \( |z| = 1 \), contradicting the initial assumption that \( c \) is not in the \( \mathbb{Z} \)-span of \( C(B_1) \). Therefore, \( C(B_1) \cup c \) forms an integrally independent set of circuits of size \( \text{corank}(M) + 1 \), proving the first statement.

The second statement follows from the fact that \( M \) is regular if and only if its dual is.

Note that the maximum size of a set of integrally independent circuits of \( M \) is equal to the rank of the circuit lattice of \( M \). Since the set of fundamental circuits of a single basis is always linearly independent, it will always be the case that \( \text{rank}(\Lambda) \geq \text{corank}(M) \), and (3) of Theorem 1 is equivalent to the statement that this inequality is actually an equality if and only if \( M \) is regular.

4. Distance from regularity

The characterizations of regularity given in Section 3 are certainly not an exhaustive list; others can be found in [3], [4], and [5]. Most such characterizations come in two flavors. The first is a condition that a matroid is...
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regular if and only if it is representable over some finite set of fields. For example, an orientable matroid is regular if and only if is is representable over $\mathbb{F}_2$ (see (4) of Theorem 6). The other flavor is that of excluded minor characterizations. In [5], Tutte proves that a matroid is regular if and only if it does not contain a minor isomorphic to either the Fano matroid or its dual. It is also true that an orientable matroid is regular if and only if it contains no minor isomorphic to the uniform matroid $U_{2,4}$. (This follows from the facts that an oriented matroid is regular if and only if it is binary, and a matroid is binary if and only if it contains no $U_{2,4}$ minor.) Yuen uses this last characterization in [3] to prove that an oriented matroid $M$ is regular if and only if the number of circuit-cocircuit equivalence classes of orientations is equal to the number of bases. The characterizations in Theorem 1 can be used to provide a measure of “how badly” $M$ fails to be regular. We use (3) of this theorem to provide an irregularity parameter for an orientable matroid, and prove that it is monotone under taking minors.

We define this irregularity parameter to be the quantity $\rho(M) = \text{rank}(\Lambda) - \text{corank}(M)$. We have that $\rho(M) = 0$ if and only if $M$ is regular, so the value $\rho(M)$ may be thought of as a measure of how far $M$ is from being regular.

**Proposition 7.** $\rho(M)$ is additive on direct sums and monotone under taking minors.

**Proof.** Suppose $M \cong M_1 \oplus M_2$. Both the bases and the circuits of $M$ can be separated into the summands $M_1$ and $M_2$. The maximum number of integrally independent circuits of $M$ is equal to rank($\Lambda(M_1)$) + rank($\Lambda(M_2)$), and corank($M$) = corank($M_1$) + corank($M_2$). This immediately implies the first statement.

For the second statement, consider first the behavior of $\rho(M)$ on the contraction of some $e \in E$, denoted $M/e$. Contracting $M$ decreases the rank of the matroid by 1 but leaves the corank unchanged; that is, rank($M$) = rank($M/e$) + 1 and corank($M$) = corank($M/e$). It is obvious that the number of integrally independent circuits cannot increase under contraction, since any maximal independent set of circuits of $M$ will descend to a maximal independent set of circuits of $M/e$. Therefore $\rho(M/e) \leq \rho(M)$.

Since deletion is dual to contraction, $\rho(M)$ is also monotone under deletion.

Since $\rho$ is monotone under taking minors, the class of matroids such that $\rho(M) \leq n$ for any nonnegative integer $n$ is a minor-closed family. Clearly the only forbidden minor for $\rho(M) = 0$ is $U_{2,4}$, since any orientable matroid with $\rho(M) > 0$ is nonregular and therefore contains a $U_{2,4}$ minor. An easy computation shows that $\rho(U_{2,4}^2) = 2$, since the four circuits of $U_{2,4}^2$ form an integrally independent set and corank($U_{2,4}^2$) = 2. Therefore every nonregular matroid $M$ has $\rho(M) \geq 2$. It could be interesting to investigate whether
there is a finite list of forbidden minors characterizing the property $\rho \leq n$ for an integer $n$.

5. Acknowledgements

The author would like to thank Matt Baker for many helpful discussions, and Chi Ho Yuen for helpful feedback on an earlier version of this paper. She would also like to acknowledge the contribution of the many cups of coffee that went into producing these results.

References

[1] Oxley, Matroid Theory, volume 3., Oxford University Press, USA, 2006
[2] Anders Bjorner, Michael Las Vergnas, Bernd Sturmfels, Neil White, and Gunter Zeigler, Oriented Matroids, volume 46 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1999
[3] C.H. Yuen, On the number of circuit-cocircuit reversal classes of an oriented matroid, Preprint. Available at https://arxiv.org/pdf/1707.00342.pdf
[4] R.A. Pendavingh and S.M.H. van Zwam, Lifts of matroids over partial fields. J. Comb. Theory Series B, vol. 100, issue 1, pp. 26-67. 2010.
[5] W.T. Tutte, Lectures on Matroids. J. Res. Nat. Bur. Standards, Sect. B, vol. 69B, pages 1-47. 1965.
[6] N. White. Combinatorial geometries, volume 29 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1987.
[7] R. Nickel. Flows and Colorings in Oriented Matroids, 2012 (Ph.D. thesis).
[8] George J. Minty, On the axiomatic foundations of the theories of directed linear graphs, electrical networks and network programming. Journal of Mathematics and Mechanics, 15:485-520, 1966.