Nonminimal Einstein–Maxwell–Vlasov-axion model

Alexander B Balakin, Ruslan K Muharlyamov and Alexei E Zayats

Department of General Relativity and Gravitation, Institute of Physics, Kazan Federal University, Kremlevskaya str 18, Kazan 420008, Russia

E-mail: Alexander.Balakin@kpfu.ru, Ruslan.Muharlyamov@kpfu.ru and Alexei.Zayats@kpfu.ru

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Abstract
We establish a new self-consistent system of equations accounting for a nonminimal coupling of the cooperative gravitational, electromagnetic and pseudoscalar (axion) fields in a multi-component relativistic plasma. The axionic extension of the nonminimal Einstein–Maxwell–Vlasov theory is based on two consistent procedures. First, we use the Lagrange formalism to obtain nonminimal equations for the gravitational, electromagnetic and pseudoscalar fields with the additional sources generated in plasma. Second, we use the Vlasov version of the relativistic kinetic theory of the plasma, guided by the cooperative macroscopic electromagnetic, gravitational and axionic fields, to describe adequately the response of the plasma on the variations of these fields. In order to show the self-consistency of this approach we check directly the compatibility conditions for the master equations for the cooperative fields. Using these compatibility conditions we reconstruct the ponderomotive force, which acts on the plasma particles, and discuss the necessary conditions for existence of the distribution function of the equilibrium type.

Keywords: axion field, relativistic plasma, Vlasov model, nonminimal coupling
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1. Introduction

1.1. What does the Einstein–Maxwell–Vlasov-axion model describe?

The Einstein–Maxwell–Vlasov-axion model deals with the self-consistent theory of interaction between gravitational, electromagnetic, pseudoscalar (axion) fields and a relativistic multi-component plasma. The specific concept introduced by Vlasov into the plasma theory, namely, the concept of cooperative macroscopic field [1, 2] (or, equivalently, self-consistent, collective, average, mean field) had become the highly sought one in modern physics. Let us remind, that
the well-known Maxwell–Vlasov model (the basic element of modern plasma theory [3, 4])
deals with the cooperative electromagnetic field, which is created by the electrically charged
particles in plasma and which regulates the plasma state variations. From the mathematical
point of view, the Lorentz force, which enters the kinetic equation, contains the tensor of
electromagnetic field (the Maxwell tensor, \( F_{ik} \)), which is the solution of the Maxwell equations
with the electric current four-vector, averaged over the ensemble of these charged particles.
Thus, according to the Vlasov concept, the ensemble of plasma particles controls itself by
means of cooperative electromagnetic field. In this context, the Einstein–Vlasov model (see,
e.g., [5, 6]) deals with the cooperative gravitational field, which is created by some statistical
system (e.g., gas or plasma) and guides this system in a self-consistent manner. In the Einstein–
Vlasov models with scalar field (see, e.g., [7]) new (scalar) sources contribute to the creation
of gravitational field, in which the kinetic system evolves. The axionic extension of the Vlasov
idea is based on the introduction of cooperative pseudoscalar (axion) field; this model is
not yet elaborated, here we start to discuss the Vlasov-type models of the axionically active
plasma. When we combine the Einstein–Vlasov, Maxwell–Vlasov and axion-Vlasov models,
we deal with the cooperative gravitational-electromagnetic-pseudoscalar field, which regulates
the behavior of relativistic plasma and displays specific cross-interactions. To illustrate
the appearance of cross-interactions let us remind, for instance, the details of the theory
of relativistic gravitationally coupled plasma systems, studied in various contexts, e.g., in
[8–10]. In fact, we deal there with the application of the Einstein–Maxwell–Vlasov model,
and the coupling of cooperative gravitational and electromagnetic fields of the plasma is
the example of such cross-interaction. Clearly, the presented here Einstein–Maxwell–Vlasov–
axion model includes the description of the gravitational–electromagnetic, gravitational–axion,
axion–photon cross-couplings, and all these interactions can be unified on the base of the
Vlasov concept of the cooperative field of the plasma system.

1.2. What does the nonminimal coupling scheme add to the model?

The description of the nonminimal coupling of the gravitational field with scalar, pseudoscalar
(axion), electromagnetic, massive vector and gauge fields is based on the introduction of
specific cross-terms into the Lagrangian, which contain the Riemann tensor \( R_{ikmn} \), the Ricci
tensor, \( R_{ik} \), and Ricci scalar, \( R \), on the one hand, and the corresponding fields and their
derivatives, on the other hand. The theory of nonminimal coupling is elaborated in detail for
the scalar field (real and complex fields \( \Phi_1 \), as well as Higgs multiplets \( \Phi_{(a)} \)) (see, e.g., [11–13]
for review and references). Special attention in these investigations is focused on two models,
the first of them has the \( \xi R \Phi^2 \) coupling, and the second has the so-called nonminimal derivative
coupling [14, 15]. The study of a nonminimal coupling of gravity with electromagnetic field
started in [16] and this theory has been developed by many authors (see, e.g., [17–22]). Exact
solutions in the framework of nonminimal Einstein–Yang–Mills and Einstein–Yang–Mills–
Higgs theories are discussed in [23–25]. Nonminimal models of the axion–photon coupling
were considered in [26].

When we deal with the nonminimal Einstein–Maxwell–Vlasov-axion model, we should,
first, recover all the cross-terms appeared in the nonminimal Einstein–Maxwell-axion field
theory [26]. The second step in this way is connected with the description of the nonminimal
coupling of plasma to gravity. In [27, 28] one can find the models of the nonminimal coupling
of gravity to matter. We follow here the Vlasov concept and use another way: we introduce into
the kinetic equation the tidal (curvature induced) force linear in the Riemann tensor related to
the cooperative gravitational field in analogy with the Stokes, Langevin, antifriction, etc forces
considered in [29–31]. Thus, in our model we describe the nonminimal coupling of gravity
with fields (electromagnetic and pseudoscalar) and with matter (plasma particles).
1.3. Why this model could be interesting for applications to cosmology and astrophysics?

Our Universe expands with acceleration. This phenomenon, discovered in the observations of the Supernovae Ia [32–34], can be explained in two different ways. The first explanation is based on the hypothesis that there exists a dark energy [35–37], a cosmic substrate with negative pressure. Together with the dark matter [38–40], the existence of which is usually associated with the observations of the flat velocity curves of the spiral galaxies rotation [41, 42], the dark energy form the so-called cosmic dark fluid. The total contribution of the dark energy and dark matter into the energy balance of the Universe is estimated to be about 95%, thus the dark fluid guides the Universe late-time evolution and predetermines its fate. The second way of the explanation of the Universe accelerated expansion is connected with the generalization of the Einstein gravitational theory (see [13] for review and references), the nonminimal field theory being one of the versions of such generalization. Axions, hypothetical light massive pseudo-bosons, which appeared in the cosmological lexicon in the context of the strong CP-violation problem and spontaneous breaking of symmetry in the early Universe (see, e.g., [43–48]), are considered as one of the dark matter candidates (see, e.g., [49–51] and references therein). Keeping in mind that the contribution of the dark matter into the total Universe energy–density is about 23%, we should consider in detail the role of the pseudoscalar (axion) field in the Universe evolution, when we use the nonminimal field theory approach to its description.

Another motif to consider the Vlasov version of the Einstein–Maxwell–axion–plasma model is connected with the idea of self-regulation (self-guiding) of the cosmic dark fluid based on the Vlasov concept of a cooperative field. Such a self-regulation might, for instance, avoid the Big Rip scenario [52] of the late-time Universe evolution in analogy with the dynamic singularity avoidance in the \( F(R), f(T) \)-gravity models and their modifications, in the models with Chaplygin gas, in the model for the dark energy with various effective time-dependent equations of state (see, e.g., [53, 54]).

In addition, we keep in mind that plasma is an important constituent of many objects and media in our Universe, and that the photons emitted, scattered or deflected by the plasma particles, propagate in the environment of axionic dark matter. Thus, the minimal and nonminimal models of the axion–photon coupling in a relativistic plasma can clarify some properties of the electromagnetic waves emitted by astrophysical sources and detected by astronomers. In particular, one can expect detection of fingerprints of the axionic dark matter in the spectra of plasma frequencies and in the polarization rotation effects [55].

1.4. Our goal and structure of this paper

We are interested in establishing and detailed analysis of a new self-consistent Einstein–Maxwell–Vlasov-axion model, which describes evolutionary processes in a relativistic multi-component plasma guided by cooperative gravitational-electromagnetic-pseudoscalar (axion) fields. In this work we formulate both minimal and nonminimal Einstein–Maxwell–Vlasov-axion–plasma models using the combined Lagrange formalism and the kinetic approach. In order to show self-consistency of this combined approach we check directly the compatibility conditions for the coupled master equations describing the cooperative gravitational, electromagnetic and pseudoscalar (axion) fields in plasma.

This work is organized as follows. In section 2 we formulate the Vlasov approach to the description of the minimal Einstein–Maxwell-axion–plasma model. In subsection 2.1 we consider the Lagrangian composed of classical terms for the gravitational, electromagnetic, pseudoscalar fields, as well as the term describing the minimal axion–photon coupling, and a
general additive term for matter. In subsection 2.2 we obtain minimal master equations for the gravitational, electromagnetic, pseudoscalar (axion) fields with sources induced by plasma (in general form), and the corresponding compatibility conditions. In subsection 2.3 we formulate the kinetic equation of the Vlasov type (2.3.1), consider the moments of the distribution function (2.3.2), reconstruct the effective force using the compatibility conditions (2.3.3), discuss the problem of entropy balance (2.3.4), and discuss the structure of the distribution function of the equilibrium type as a solution to the kinetic equation with cooperative gravitational, electromagnetic and axion fields (2.3.5). Section 3 contains the nonminimal generalizations of the master equations for the gravitational, electromagnetic and axion fields, as well as the analysis of the generalized compatibility conditions. In section 4 we discuss the obtained results. Appendices A and B contain auxiliary and preparatory formulas, which are necessary for the self-consistent formulation of the model.

2. Minimal Einstein–Maxwell–Vlasov-axion model

2.1. Action functional

We start with the action functional

$$S(M) = \int d^4x \sqrt{-g} \left\{ \frac{R + 2\Lambda}{2\kappa} + \frac{1}{4} F_{mn} (F_{mn} + \phi F_{mn}^*) + L(m) \right\} - \frac{1}{2} \Psi_1^2 \left[ g_{nm} \phi \nabla_m \phi - m_{(A)}^2 \phi^2 - V(\phi^2) \right],$$

where $g_{ik}$ is the spacetime metric, $g$ is its determinant, $\nabla_k$ denotes the covariant derivative, $R$ is the Ricci scalar, $\Lambda$ is the cosmological constant, $\kappa$ is the Einstein constant. The Maxwell tensor $F_{mn}$ is expressed in terms of four-vector potential of the electromagnetic field $A_i$:

$$F_{mn} \equiv \nabla_m A_n - \nabla_n A_m.$$

The term $F_{mn}^* \equiv \frac{1}{2} \epsilon^{mnpq} F_{pq}$ describes the tensor dual to the Maxwell tensor; $\epsilon^{mnpq}$ is the Levi-Civita tensor, $E^{mnpq}$ is the absolutely antisymmetric Levi-Civita symbol with $E^{0123} = 1$. The dual Maxwell tensor satisfies the condition

$$\nabla_k F^{*}_{ik} = 0.$$

The symbol $\phi$ is used for the dimensionless pseudoscalar field; the axion field itself, $\Phi$, is considered to be proportional to this quantity $\Phi = \Psi_0 \phi$ with a coupling constant $\Psi_0$. The term $m_{(A)}$ is proportional to a (hypothetical) mass of an axion, $m_{(A)} = \frac{m_{(axion)}}{h}$; $h$ is the Planck constant; $V(\phi^2)$ is a potential of the pseudoscalar field. We use the signature $+,−,−,−$, and the units with $c = 1$. The term $L(m)$ stands for the Lagrangian of a matter interacting in general case with electromagnetic, pseudoscalar and gravitational fields; this term is considered to be the function of $A_i$, $\phi$ and $g_{ik}$.

2.2. Master equations for the cooperative fields

2.2.1. Electrodynamical equations. The variation of the action functional (1) with respect to the four-vector potential $A_i$ gives the electrodynamical equations

$$\nabla_i (F_{ik} + \phi F_{ik}^*) = - \frac{\delta L(m)}{\delta A_i},$$

which can be transformed into

$$\nabla_i F_{ik} = - F_{ik} \nabla_i \phi - t_i,$$
using the equations (3) and the standard definition of the electric current four-vector $I^i$, induced in plasma:

$$\frac{\delta L_{\text{(m)}}}{\delta A_i} = I^i. \tag{6}$$

The term $\delta L_{\text{(m)}} \approx \nabla_i \phi$ in (5) can be indicated as axionically induced effective current.

2.2.2. Master equation for the pseudoscalar (axion) field. The variation procedure with respect to the pseudoscalar field $\phi$ yields

$$\left[ \nabla^i \nabla_i + m^2_\phi \right] \phi = -\frac{1}{\Psi_0^2} \left[ \frac{1}{4} F^{mn} F_{mn} + J \right], \tag{7}$$

where the following definition is used for the scalar $J$:

$$J \equiv \frac{\delta L_{\text{(m)}}}{\delta \phi}. \tag{8}$$

Two source-like terms in the right-hand side of this equation appear: the first relates to the standard electromagnetic source for the axion field, the second one can be interpreted as a scalar current provided by the interaction of plasma particles with the axion field.

2.2.3. Master equations for the gravitational field. The variation of the action functional (1) with respect to the metric $g_{ik}$ gives the gravity field equations

$$R_{ik} - \frac{1}{2} R g_{ik} = \Lambda g_{ik} + \kappa \left[ T_{ik}^{\text{(EM)}} + T_{ik}^{(A)} + T_{ik}^{(m)} \right]. \tag{9}$$

Here the stress–energy tensor of the electromagnetic field is of the standard form

$$T_{ik}^{\text{(EM)}} \equiv \frac{1}{4} g_{ik} F_{mn} F_{mn} - \frac{1}{4} g_{ik} F_{im} F_{m}^{k}. \tag{10}$$

the stress–energy tensor of the pseudoscalar field is

$$T_{ik}^{(A)} \equiv \Psi_0^2 \left\{ \nabla_i \phi \nabla_k \phi - \frac{1}{2} g_{ik} \left[ \nabla^m \phi \nabla_m \phi - m^2_\phi \phi^2 - V(\phi^2) \right] \right\}, \tag{11}$$

and the stress–energy tensor of the plasma particles is given by the following expression:

$$T_{ik}^{(m)} = -\frac{2}{\sqrt{-g}} \frac{\delta \left[ \sqrt{-g} L_{\text{(m)}} \right]}{\delta g^{ik}}. \tag{12}$$

Let us stress that the variation of the term

$$\frac{1}{4} \sqrt{-g} \phi F^{mn} F_{mn} = \frac{1}{4} \phi L^{kmm} F_{ik} F_{mm} \tag{13}$$

vanishes, since this term does not depend on metric.

2.2.4. Compatibility conditions. Two compatibility conditions are well-known; the first one

$$\nabla_i I^i = 0, \tag{14}$$

is the differential consequence of the equation (5); the second condition follows from the Bianchi identities and yields

$$\nabla^k \left( R_{ik} - \frac{1}{2} R g_{ik} \right) = 0 = \kappa \nabla^k \left[ T_{ik}^{\text{(EM)}} + T_{ik}^{(A)} + T_{ik}^{(m)} \right]. \tag{15}$$

Using the equations (5) and (7) one can rewrite the last condition in the form

$$\nabla^k T_{ik}^{(m)} = F_{ik} I^k + J \nabla \phi. \tag{16}$$

Here we used the formula

$$F_{ik} F^{kmm} = -\frac{1}{2} g^{ik} F_{mm} F_{mm}. \tag{17}$$
which allows us to reduce the pseudotensor $F^i_m e^m_{4\text{im}}$ (see appendix A for details). Now we have to divide the stress–energy tensor of the matter $T^m_{ik}$ into two parts

$$T^m_{ik} = T^m_{ik}^{(m)} + T^m_{ik}^{(D)}.$$  \hspace{1cm} (18)

The first part, $T^m_{ik}^{(m)}$, relates to the (massive) plasma particles, and this stress–energy tensor will be represented below in terms of kinetic theory as a second-order moment of the distribution function. The term $T^m_{ik}^{(D)}$ corresponds to nonkinetic substrates, for instance, in the cosmological context it describes a dark fluid composed of a dark energy and dark matter. We suppose that $T^m_{ik}^{(D)}$ can be written in the standard form

$$T^m_{ik}^{(D)} = \rho^{(D)} U_i U_k + I^m_{(D)} U_k + I^m_{k(\text{D})} U_i + P^m_{ik},$$  \hspace{1cm} (19)

where $\rho^{(D)}$ is the energy–density (e.g., of the dark fluid), $I^m_i$ is the heat-flux four-vector, $U^k$ is the velocity four-vector of the substrate, and $P^m_{ik}$ is the pressure tensor.

2.3. Kinetic representation of the material sources in the equations for the cooperative fields

2.3.1. Axionic generalization of the Vlasov kinetic equation. The relativistic collisionless kinetic equation based on the Vlasov approach is of the form

$$\frac{p^i}{m_{(a)}} \left[ \frac{\partial}{\partial x^i} - \Gamma^i_{jl} \frac{\partial}{\partial p^j} \right] f_{(a)} + \frac{\partial}{\partial p^i} \left[ T^k_{ik} f_{(a)} \right] = 0,$$  \hspace{1cm} (20)

where $f_{(a)}$ is the distribution function of the particles of the sort $(a)$, $p^i$ is the momentum four-vector of the particle with the mass $m_{(a)}$. The four-vector $T^k_{ik}$ plays a role of effective force, which acts on the charged particle in the axion-active plasma.

2.3.2. Macroscopic moments. In the Vlasov model the electric current four-vector $I^i$ is considered to consist in a linear combination of first moments of the distribution functions

$$I^i = \sum_{(a)} e_{(a)} \int dP f_{(a)} p^i,$$  \hspace{1cm} (21)

where $e_{(a)}$ denotes the electric charge of the particle of the sort $(a)$; $dP = \sqrt{g} \, d^4p$ is the invariant integration volume in the momentum four-dimensional space. The stress–energy tensor of the particles is given by the second moment of the distribution function

$$T^{(m):ik} = \sum_{(a)} \int dP f_{(a)} p^i p^k.$$  \hspace{1cm} (22)

Since the particle momentum four-vector is normalized ($g_{ik} p^i p^k = m_{(a)}^2$), the trace of the stress–energy tensor is

$$T^{(m)} = \sum_{(a)} T_{(a)} = g_{ik} T^{(m):ik} = \sum_{(a)} m_{(a)}^2 \int dP f_{(a)}.$$  \hspace{1cm} (23)

i.e., it can be represented by the zero-order moment. The pseudoscalar source $\mathcal{J}$ can also be considered as a zero-order moment of the distribution function

$$\mathcal{J} = \sum_{(a)} \int dP f_{(a)} \mathcal{G}_{(a)}$$  \hspace{1cm} (24)

with the pseudoscalar quantity $\mathcal{G}_{(a)}$, which can, in principle, be decomposed with respect to the particle momentum four-vector $p^k$ as follows

$$\mathcal{G}_{(a)} = \alpha_{(a)} \phi + \beta_{(a)} p^k \nabla_k \phi + \gamma_{(a)} p^k F^m_{km} \nabla_m \phi + \cdots.$$  \hspace{1cm} (25)

The coefficients in this decomposition are introduced phenomenologically, but below we consider the constraints for them coming from compatibility conditions.
2.3.3. **Reconstruction of the effective force using the compatibility conditions.** The first compatibility condition (14) with (21) and (20) yields

$$\nabla_i F^i = - \sum_{(a)} e_{(a)} m_{(a)} \int dP \frac{\partial}{\partial p^i} \left[ F^i_{(a)} f_{(a)} \right] = 0, \quad (26)$$

i.e., this condition is satisfied for arbitrary force-like term $F^i_{(a)}$. The divergence of the stress–energy tensor yields

$$\nabla_k T^{(m)ik} = \nabla_k \sum_{(a)} dP f_{(a)} p^j p^k = - \sum_{(a)} m_{(a)} \int dP p^j \frac{\partial}{\partial p^k} \left[ F^k_{(a)} f_{(a)} \right]$$

$$= \sum_{(a)} m_{(a)} \int dP f_{(a)} F^k_{(a)} , \quad (27)$$

thus, according to (16) and (21), the force-like term $F^i_{(a)}$ has to satisfy the condition

$$\sum_{(a)} \int dP f_{(a)} \left[ m_{(a)} F^i_{(a)} - e_{(a)} F^i_{(a)} e_{(a)} p^m \right] = \mathcal{J} \nabla^i \phi - \nabla_i F^{(D)ik} . \quad (28)$$

For the sake of simplicity we suppose below that the stress–energy tensor $T^{(D)ik}$ is divergence-free, i.e., $\nabla_i T^{(D)ik} = 0$, but of course, it is easy to enlarge the formalism discussed below for the case, when the energy-momentum of the dark fluid is not a conserved quantity. Then according to (28) the term $F^i_{(a)}$ splits into the standard Lorentz force linear in the particle momentum four-vector, and the force $\mathcal{N}^i_{(a)}$ induced by the axion field

$$m_{(a)} F^i_{(a)} = e_{(a)} F^i_{(a)} p^i + \mathcal{N}^i_{(a)} . \quad (29)$$

For the quantity $\mathcal{N}^i_{(a)}$ we obtain the integral equation

$$\sum_{(a)} \int dP f_{(a)} \left[ \mathcal{N}^i_{(a)} - G_{(a)} \nabla^i \phi \right] = 0. \quad (30)$$

There are three principal possibilities to resolve this integral equation. Let us consider them in more detail.

(i) **There are no contact plasma–axion interactions.** The simplest version assumes that axions interact with plasma through the electromagnetic field only, so that the Lagrangian $L_{(m)}$ does not contain the pseudoscalar field $\phi$. This means that the source term $\mathcal{J}$ vanishes and the force $\mathcal{N}^i_{(a)}$ is absent. Thus the compatibility conditions (30) are assumed to be satisfied identically.

(ii) **Microscopic force and particle rest-mass variation.** The second version relates to the reconstruction of a microscopic force, which can include the particle momentum four-vector $p^i$, the axion field $\phi$ and its derivatives. Keeping in mind (30) such a force can be represented as $\mathcal{N}^i_{(a)} = G_{(a)} \nabla^i \phi$, and clearly this force is not orthogonal to the particle momentum four-vector $p^i$. This means that the particle rest mass is not conserved. In order to illustrate such a model let us assume that only the first term in (25) is nonvanishing, i.e., $G_{(a)} = \alpha_{(a)} \phi$, where $\alpha_{(a)}$ is constant. Then

$$\frac{1}{2} \frac{d}{dr} \left[ p_i p^i \right] = \frac{1}{m_{(a)}} \mathcal{N}^i_{(a)} p_i = \frac{1}{2} \frac{d}{dr} \left[ \alpha_{(a)} \phi^2 \right] , \quad (31)$$

and the square of the particle momentum four-vector

$$p_i p^i = m^2_{(a)} + \alpha_{(a)} \phi^2 \quad (32)$$

depends on coordinates through the axion field $\phi$.  

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(iii) Ponderomotive force in plasma induced by axions. The third variant to solve the integral equation (30) is to reconstruct a macroscopic force, which can contain some macroscopic moments of the distribution function; we indicate the force of this type as the ponderomotive force in analogy with the force appearing in the electrodynamics of continuous media. In our case the macroscopic force can be found using the following ansatz. Let the force be orthogonal to the particle four-momentum (thus providing the mass conservation), be quadratic in \( p^a \) and have the form

\[
\mathcal{R}^i_{(a)} = \left[ \delta^i_k \left( p_i p'^k - p'^i p_k \right) \right] v_{(a)} B^k, \quad \mathcal{R}^i_{(a)} p_i = 0, \tag{33}
\]

where \( v_{(a)} \) are constants introduced phenomenologically, and \( B^k \) are unknown functions. Then one obtains

\[
\sum_{(a)} \int \! dP f_{(a)} \mathcal{R}^i_{(a)} = S^i_k B^k, \tag{34}
\]

where

\[
S^i_k = \left[ \delta^i_k S^m - S^i_k \right], \quad S^i_k = \sum_{(a)} v_{(a)} T^i_{k(\alpha)}. \tag{35}
\]

Now, the condition (28) yields that the term \( B^k \) can be found using matrix equation

\[
S^i_k B^k = J \nabla^i \phi, \quad B^k = \tilde{S}^i_k J \nabla^i \phi, \tag{36}
\]

where \( \tilde{S}^i_k \) is the reciprocal matrix to the matrix \( S^i_k \), i.e., \( \tilde{S}^i_k S^k_l = \delta^i_l \). Of course, we assume that the matrix \( S^i_k \) is nondegenerated, i.e., \( \det S^i_k \neq 0 \). Finally, the force, which we search for, has the form

\[
\mathcal{R}^i_{(a)} = \left[ \delta^i_k \left( p_i p'^k - p'^i p_k \right) \right] v_{(a)} \tilde{\delta}^i_k J \nabla^i \phi. \tag{37}
\]

In order to illustrate the described procedure let us consider the following example. Let \( v_{(a)} = v \) (i.e., these parameters do not depend on sort of particles \( (a) \)), and let the stress–energy tensor of the plasma particles be described by the tensor for perfect fluid, i.e.,

\[
T^i_k = (W + P) U^i U^k - \delta^i_k P. \tag{38}
\]

Then the tensors \( \tilde{S}^i_k \) and \( \tilde{\delta}^i_k \) are, respectively,

\[
\tilde{S}^i_k = v \left[ \delta^i_k (W - 2P) - (W + P) U^i U_k \right], \tag{39}
\]

\[
\tilde{\delta}^i_k = \frac{1}{3vP(W - 2P)} \left[ 3P \delta^i_k - (W + P) U^k U_i \right]. \tag{40}
\]

Here \( W \) is the energy–density scalar, \( P \) is the Pascal pressure, \( U^i \) is the macroscopic velocity four-vector of the plasma system as a whole. Clearly, the reciprocal tensor exists, when \( P \neq 0 \) and \( W \neq 2P \). There is one interesting subcase: when \( W = 5P \) one obtains from (39)

\[
S^i_k = \frac{3v}{5} W L^i_k, \quad \tilde{S}^i_k = \frac{5}{3vW} L^i_k, \quad L^i_k = \delta^i_k - 2U^i U_k, \quad L^i_k L^j_l = \delta^i_j. \tag{41}
\]

Keeping in mind the standard equation of state \( P = (\gamma - 1)W \), we see that this case relates to the value \( \gamma = 6/5 \), i.e., the adiabatic index \( \gamma \) coincides with the upper critical value obtained in the Lane–Emden theory of Newtonian stars (see, e.g., [56]).
2.3.4. Entropy balance equation. The entropy production of the kinetic system, the evolution of which is guided by the kinetic equation (20) is connected with the action of the force $\mathcal{F}_{(a)}$ ($k_{(B)}$ is the Boltzmann constant, $\hbar = 2\pi\hbar$):

$$\sigma = \nabla_{\phi} S' = -k_{(B)} \sum_{(a)} \int dP f_{(a)} p^i \left[ \log \left( \frac{f_{(a)}}{f_{(a)}^0} \right) - 1 \right]$$

$$= k_{(B)} \sum_{(a)} m_{(a)} \int dP \log \left( \frac{f_{(a)}}{f_{(a)}^0} \right) \frac{\partial}{\partial p} \left[ F_{(a)}^{i} f_{(a)} \right]$$

$$= k_{(B)} \sum_{(a)} \int dP f_{(a)} \frac{\partial}{\partial p} \mathcal{F}_{(a)}^i.$$  \hspace{1cm} (42)

Clearly, the entropy production is absent when $\mathcal{F}_{(a)}^i$ vanishes, or does not depend on $p^i$, as in the case of microscopic force reconstruction. When the ponderomotive force exists, we find using (37) that the entropy production scalar is nonvanishing and is proportional to the pseudoscalar source-term $\mathcal{J}$:

$$\sigma = -3k_{(B)} \mathcal{J} \nabla_{\phi} \sum_{(a)} v_{(a)} \int dP f_{(a)} p^i.$$  \hspace{1cm} (43)

When $\mathcal{J} = 0$, we see that $\sigma = 0$, and the plasma is in the state of local equilibrium.

2.3.5. On distribution functions of the equilibrium type. Equilibrium-type distribution function

$$f_{(eq)}^{(a)} = h^{-3} \exp \left\{ \mu_{(a)} - \xi^{(a)} p^i \right\}$$  \hspace{1cm} (44)

satisfies the kinetic equation (20) with the force (29), (33), when

$$p^i \left[ \frac{\partial}{\partial x^i} \mu_{(a)} + e_{(a)} F_{ak} \xi^{(a)} + 3v_{(a)} \mathcal{B}_k \right] - p^i \left[ \nabla_{\phi} \xi^{(a)} - \xi^{(a)} v_{(a)} \mathcal{B}_k + g_{ak} \xi^{(a)} v_{(a)} \mathcal{B}_l \right] = 0.$$  \hspace{1cm} (45)

The momentum four-vector in this context is considered to be a random variable, thus, one obtains two sets of conditions, which are necessary the equation (45) to be satisfied. The first set of conditions

$$\frac{\partial}{\partial x^i} \mu_{(a)} = -e_{(a)} F_{ak} \xi^{(a)} + 3v_{(a)} \mathcal{B}_k.$$  \hspace{1cm} (46)

contains four equations for one unknown function $\mu_{(a)}$ (for the fixed index of the sort (a)). The integrability conditions $\frac{\partial^2}{\partial x^i \partial x^j} \mu_{(a)} = \frac{\partial^2}{\partial x^j \partial x^i} \mu_{(a)}$ are satisfied, when

$$e_{(a)} \left[ \frac{\partial}{\partial x^i} \mathcal{L}_i A_j - \frac{\partial}{\partial x^j} \mathcal{L}_i A_i \right] = -3v_{(a)} \left[ \frac{\partial}{\partial x^i} \mathcal{B}_j - \frac{\partial}{\partial x^j} \mathcal{B}_i \right].$$  \hspace{1cm} (47)

Here the symbol $\mathcal{L}_i$ denotes the Lie derivative along the four-vector $\xi^{(a)}$

$$\mathcal{L}_i A_j \equiv \xi^{(a)} \frac{\partial}{\partial x^i} A_j + A_i \frac{\partial}{\partial x^i} \xi^{(a)}.$$  \hspace{1cm} (48)

The second term in (45), quadratic in the particle momentum four-vector, gives the condition

$$\mathcal{L}_i g_{ak} \equiv \nabla_{\phi} \xi^{(a)} + \nabla_{\phi} \xi^{(a)} = v_{(a)} \left[ \xi^{(a)} \mathcal{B}_k + \xi^{(a)} \mathcal{B}_l - 2g_{ak} \xi^{(a)} \mathcal{B}_l \right].$$  \hspace{1cm} (49)

When $\mathcal{B}_i = 0$ it is the well-known Killing equation. When the right-hand side of the equation (49) is not vanishing, we deal with a generalization of the Killing equation. In a particular case, when $\xi^{(a)} = \xi$, and $v_{(a)} = v$ i.e., they do not depend on the sort of particle, and $\mathcal{B}_i = \mathcal{B} \xi_i$, the equation (49) reduces to the equation

$$\mathcal{L}_i g_{ak} = -2v \mathcal{B} [g_{ak} \xi^{(a)} \xi_i - \xi_i \xi_k].$$  \hspace{1cm} (50)
discussed in the works [29, 30] in the context of the antifriction force and accelerated expansion of the Universe. The equilibrium state in plasma exists if the equation (49) admits the existence of the time-like four-vectors \( \xi^a_i \), and if they coincide for all sorts of particles, i.e., \( \xi^a_i = \xi_i \). The solution of (47) is

\[
e^{(a)}_A A_j = -3v^{(a)}_j B_j + \frac{\partial}{\partial x^j} \Psi^{(a)},
\]

where \( \Psi^{(a)} \) is arbitrary scalar. Let us consider two different cases.

(i) General case, \( A_i \neq 0 \). When \( A_i \neq 0 \), the scalar \( \Psi^{(a)} \) can be eliminated using the gauge transformation of the potential four-vector

\[
A_j \rightarrow \tilde{A}_j + \nabla_j \tilde{\Psi}, \quad \Psi^{(a)} = e^{(a)}_A \xi^a_i \frac{\partial}{\partial x^i} \tilde{\Psi},
\]

thus \( \Psi^{(a)} \) can be put equal to zero. When (51) is satisfied, the scalar function \( \mu^{(a)} \) can be found from the equation

\[
\frac{\partial}{\partial x^i} \mu^{(a)} = -e^{(a)} \left[ F_{ik} \xi^k_i + \xi^i_A A_k \right] = -e^{(a)} \frac{\partial}{\partial x^i} \left[ \xi^k_i A_k \right],
\]

yielding

\[
\mu^{(a)} = \tilde{\mu}^{(a)} - e^{(a)} \xi^a_i A^i,
\]

where \( \tilde{\mu}^{(a)} \) is a constant. Thus, when, first, the four-vector \( \xi^i = \xi^i_a \) is time-like and satisfies the equation (49), second, the equation (51) is satisfied, then, using the standard definitions of the chemical potentials \( \mu^{(a)} \) of the macroscopic velocity four-vector \( U^i \) and of the temperature \( T \)

\[
\tilde{\mu}^{(a)} = \frac{\mu^{(a)}}{k(B)T}, \quad \xi^i = \frac{U^i}{k(B)T},
\]

one can rewrite the equilibrium type function (44) in the well-known form

\[
h^{(a)}_{\xi^{(eq)}} = \exp \left\{ \frac{\mu^{(a)} - U^i \left[ p^i + e^{(a)} A^i \right]}{k(B)T} \right\} \delta \left[ \sqrt{p^i p_i} - m^{(a)} \right].
\]

Let us mention that in this case the force-like term (33), quadratic in the particle four-momentum

\[
\mathcal{W}^{(a)}_{\xi} = -\frac{1}{2} \left[ \xi^a_i \nabla_k \Psi^{(a)} + \xi^k_i \nabla_j \Psi^{(a)} - 2g^{\alpha\beta} \xi^a_i \nabla_j \Psi^{(a)} \right],
\]

is proportional to the Lie derivative of the electromagnetic potential four-vector.

(ii) Special case. \( A_i = 0 \). When the macroscopic electromagnetic field in the electroneutral plasma vanishes, and thus \( A_i = 0 \), the equation (51) is satisfied, if the four-vector \( B^i_j \) is the gradient four-vector

\[
B^i_j = \frac{1}{5v^{(a)}} \frac{\partial}{\partial x^j} \Psi^{(a)}, \quad \mu^{(a)} = \tilde{\mu}^{(a)} - \Psi^{(a)}.
\]

As for the equation (49), it now is of the form

\[
\xi^a_k \xi^a_k = \frac{1}{2} \left[ \xi^a_i \nabla_k \Psi^{(a)} + \xi^k_i \nabla_j \Psi^{(a)} - 2g^{\alpha\beta} \xi^a_i \nabla_j \Psi^{(a)} \right].
\]

In other words, the contact interactions between axions and plasma particles can support an equilibrium state in the system, when the coupling parameters satisfy some specific conditions.
3. Nonminimal extension of the Einstein–Maxwell–Vlasov-axion–plasma model

3.1. Nonminimal extension of the Lagrangian

We consider now the total action functional as a sum of minimal and nonminimal contributions

\[ S = S_{(M)} + S_{(NM)}, \]

where \( S_{(M)} \) is given by (1) and

\[ S_{(NM)} = \int d^4x \sqrt{-g} \left\{ \frac{1}{4} R g^{ikmn} F_{ik} F_{mn} + \frac{1}{4} \chi_{(A)}^{ikmn} \phi F_{ik}^{*} F_{mn} \right\} \].

(61)

The quantity \( R g^{ikmn} \) is a nonminimal three-parameter susceptibility tensor \([21]\), which has a form

\[ R g^{ikmn} = q_1 R g^{ikmn} + q_2 R^{ikmn} + q_3 R^{ikmn}, \]

(62)

where the following auxiliary tensors are introduced

\[ g^{ikmn} \equiv \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{jm}), \]

(63)

\[ \eta^{ikmn} \equiv \frac{1}{2} (R^{im} g^{kn} - R^{in} g^{jm} + R^{jn} g^{im} - R^{jm} g^{in}). \]

(64)

The constants \( q_1, q_2 \) and \( q_3 \) are nonminimal parameters describing the linear coupling of the Maxwell tensor \( F_{mn} \) with curvature \([21]\). The quantity \( \chi_{(A)}^{ikmn} \), the nonminimal susceptibility tensor describing the linear coupling of the dual tensor \( F_{mn}^{*} \) with curvature, can be initially modeled as

\[ \chi_{(A)}^{ikmn} = Q_1 R g^{ikmn} + Q_2 \eta^{ikmn} + Q_3 R^{ikmn}, \]

(65)

using the direct analogy with the decomposition (62) and introducing three phenomenological parameters \( Q_1, Q_2 \) and \( Q_3 \). As in the previous case, the combination \( \phi F_{mn}^{*} \) gives the tensor quantity, thus \( \chi_{(A)}^{ikmn} \) is also a pure tensor. However, taking into account the relation (17) we conclude that the term \( \eta^{ikmn} \phi F_{mn}^{*} \) is proportional to \( R g^{ikmn} \phi F_{mn}^{*} \), and we have only two independent coupling constants among three parameters \( Q_1, Q_2 \) and \( Q_3 \). In the next subsection we discuss this problem in detail and motivate our ansatz that \( Q_2 = -Q_3 \).

The tensors \( R^{ikmn} \) and \( \chi_{(A)}^{ikmn} \), defined by (62) and (65), are skew-symmetric with respect to transposition of the indices \( i \) and \( k \), as well as \( m \) and \( n \). In addition the following relations take place

\[ R^{ikmn} = R^{mnik}, \quad \chi_{(A)}^{ikmn} = \chi_{(A)}^{mnik}, \]

(66)

which guarantee that the model under consideration does not contain the solutions of the skewon type \([57]\). The symmetric tensor

\[ \eta^{mn}_{(A)} \equiv \frac{1}{2} \eta (F^{mn} R_{ij} + F^{jm} R_{in}) + \eta_2 R g^{mn} + \eta_3 R^{mn}, \]

(67)

describes a nonminimal susceptibility for the pseudoscalar field in analogy with the Higgs fields \([25]\), but in this case the tensor \( \eta^{mn}_{(A)} \) contains an additional term linear in the Maxwell tensor. This term describes effects analogous to the so-called derivative coupling in the nonminimal scalar field theory \([14, 15]\). As for the tenth coupling constant \( \eta_{(A)} \), it is a direct analogue of the well-known coupling constant \( \xi \) in the nonminimal scalar field theory (see, e.g., \([11]\) for review and references).
3.2. One-parameter and two-parameters families of the models for nonminimal susceptibility tensor

The pseudoscalar $\chi^{ikmn}_{(A)} F^*_ik F^*_mn$ with the nonminimal susceptibility tensor (65) can be reduced to the pseudoscalar

$$\chi^{ikmn}_{(A)} F^*_ik F^*_mn = \left[ (Q_1 + \frac{1}{2} Q_2) Rg^{ikmn} + Q_3 Rg^{ikmn} \right] F^*_ik F^*_mn$$

(68)
due to the relation (17) proved in the appendix A. This means that, when we make the replacements

$$Q_1 = \overline{Q}_1 + Q, \quad Q_2 = \overline{Q}_2 - 2Q, \quad Q_3 = \overline{Q}_3$$

(69)

with arbitrary constant $Q$, the invariant $\frac{1}{4} \chi^{ikmn}_{(A)} \phi F^*_ik F^*_mn$ keeps the form and the master equations obtained by the variation of the corresponding term in the action functional (61) remain unchanged. In this sense we can consider the relations (69) as an analogue of gauge transformations. This fact gives us an argument to focus on the problem of an appropriate choice of the constants $Q_1, Q_2, Q_3$, which allows us to simplify the model.

3.2.1. Symmetry with respect to the left and right dualizations. Let us assume that the nonminimal susceptibility tensor satisfies the following symmetry condition

$$\chi^{ikmn}_{(A)} = \chi^{*ikmn}_{(A)}$$

(70)

which is equivalent to

$$\chi^{*ikmn}_{(A)} = -\chi^{ikmn}_{(A)}.$$  

(71)

Taking into account the following relations for the double-dual quantities

$$g^{*ikmn} = -g^{ikmn},$$

(72)

$$R^{*ikmn} = R^{ikmn} - Rg^{ikmn},$$

(73)

$$R^{*ikmn} = -R^{ikmn} + 2R^{ikmn} - Rg^{ikmn},$$

(74)

we can conclude that the symmetry condition (70) takes place for arbitrary curvature tensor, when $Q_2 + Q_1 = 0$. This condition has been implicitly used in the work [26]. The pseudoscalar (68) takes now the form

$$\chi^{ikmn}_{(A)} F^*_ik F^*_mn = \left[ (Q_1 - \frac{1}{3} Q_3) Rg^{ikmn} + Q_3 Rg^{ikmn} \right] F^*_ik F^*_mn.$$  

(75)

Using the Weyl tensor $C^{ikmn}$ and the standard decomposition of the Riemann tensor, we can also represent the susceptibility tensor as follows

$$\chi^{ikmn}_{(A)} = Q_1 C^{ikmn} + (Q_2 + Q_3) Rg^{ikmn} + (Q_1 - \frac{1}{3} Q_3) Rg^{ikmn}.$$  

(76)

The condition $Q_2 + Q_3 = 0$ excludes the term $Rg^{ikmn}$ from this decomposition, since only the Weyl tensor $C^{ikmn}$ and the tensor $g^{ikmn}$ possess symmetry with respect to left and right dualizations.

Since the constraint $\overline{Q}_2 + \overline{Q}_3 = 0$ will always be satisfied by the replacements (69) with the suitable choice of the fitting parameter $Q = -\frac{1}{2} (Q_2 + Q_3)$, without loss of generality, we can assume that the nonminimal susceptibility tensor $\chi^{ikmn}_{(A)}$ obeys the symmetry condition (70). Thus our model is effectively two-parameters and one parameter, say, $Q_2$, is the hidden one. Besides, we can impose an additional constraint on the constants $Q_1, Q_2, Q_3$, and therefore reduce our general model to a certain one-parameter submodel. Below we consider several examples of such submodels.
3.2.2. One-parameter submodels. Let us remind that the tensor \( R^{im} \)

\[
R^{im} \equiv g_{is} R^{s kmn} = \frac{1}{2} R q^{im}(3q_1 + q_2) + R^{im}(q_2 + q_3),
\]

vanishes in a generic (curved) spacetime, when the nonminimal couplings are linked by two relations \( 3q_1 + q_2 = 0 \) and \( q_2 + q_3 = 0 \). Analogously, the scalar \( R \)

\[
R \equiv g_{im} g_{in} R^{ikmn} = R(6q_1 + 3q_2 + q_3),
\]

takes zero value in the curved spacetime, when \( 3 \) vanishes, when \( 3 \) takes zero value, when \( 6q_1 + 3q_2 + q_3 = 0 \). Similarly, we obtain that \( \chi^{im}_{(A)} \equiv g_{km} \chi^{k mn}_{(A)} = \frac{1}{2} R q^{im}(3Q_1 + Q_2) + R^{im}(Q_2 + Q_3) \),

\[
\chi^{im}_{(A)} \equiv g_{km} g_{kn} \chi^{k mn}_{(A)} = R(6Q_1 + 3Q_2 + Q_3),
\]

takes zero value, when \( 6Q_1 + 3Q_2 + Q_3 = 0 \). Clearly, the condition \( 6Q_1 + 3Q_2 + Q_3 = 6\overline{Q}_1 + 3\overline{Q}_2 + \overline{Q}_3 = 0 \) is invariant with respect to the transformation (69), other conditions require special choices of the parameter \( Q \).

(a) The model with vanishing scalar \( \chi_{(A)} \). When \( 6Q_1 + 3Q_2 + Q_3 = 0 \) and thus \( \chi_{(A)} = 0 \), one obtains that the pseudoscalar (68) turns into

\[
\chi^{k mn}_{(A)} F_{ik} F_{mn} = Q_3 \left[ R^{k mn} - \frac{1}{2} R q^{k mn} \right] F_{ik} F_{mn}.
\]

Since the term \( 6Q_1 + 3Q_2 + Q_3 \) is invariant with respect to the transformation (69), this model contains only two arbitrary parameters, say, \( Q_2 \) and \( Q_3 \), but now \( Q_2 \) does not enter the invariant (81), i.e., it becomes effectively hidden.

(b) Weyl-type relation. A model, in which \( 3Q_1 + Q_2 = 0 \) and \( Q_2 + Q_3 = 0 \), appears when we suggest that \( \chi^{im}_{(A)} = 0 \), and thus the susceptibility tensor is proportional to the Weyl tensor

\[
\chi^{k mn}_{(A)} = Q_3 \varepsilon^{k mn}.
\]

These conditions are equivalent to \( 6Q_1 + 3Q_2 + Q_3 = 0 \) and \( Q_2 + Q_3 = 0 \), where the first one is invariant with respect to (69), while the second constraint just fixes the fitting parameter (see above). Therefore we can conclude that the previous condition with vanishing scalar \( \chi_{(A)} \) comes to this one by the suitable choice of \( Q \), and the pseudoscalar (68) takes the same form (81).

(c) Gauss–Bonnet-type relation. The model, for which the susceptibility tensor is proportional to the double-dual Riemann tensor

\[
\chi^{k mn}_{(A)} = -Q_3 \varepsilon^{k mn} R^{ik mn},
\]

is indicated in [21] as the Gauss–Bonnet-type model; such susceptibility tensor is divergence-free. In our context, this proportionality yields that the nonminimal parameters are coupled by two relations \( Q_1 - Q_3 = 0 \) and \( 2Q_1 + Q_2 = 0 \), and the pseudoscalar (68) is of the form

\[
\chi^{k mn}_{(A)} F_{ik} F_{mn} = Q_3 R^{k mn} F_{ik} F_{mn}.
\]

The condition \( \overline{Q}_1 - \overline{Q}_3 = Q_1 - Q_3 + Q = 0 \) can be satisfied by the choice \( Q = Q_1 - Q_1 \), i.e., the value of the fitting parameter \( Q \) is fixed. The condition \( 2Q_1 + Q_2 = 2\overline{Q}_1 + \overline{Q}_2 = 0 \) is invariant with respect to the transformations (69), thus, this requirement makes the model one-parameter. Let us mention, that the choice \( Q = Q_1 \), provides the relation \( \overline{Q}_1 = \overline{Q}_2 = 0 \) to be valid, thus we deal with the model related to \( \chi^{k mn}_{(A)} = Q_3 R^{k mn} \).
3.3. Nonminimal electrodynamic equations

Electrodynamic equations, which correspond to the action functional (60) with (1) and (61), are linear and have the standard form
\[ \nabla_i H^{ik} = -I^i. \]  
(85)

The excitation tensor \( H^{ik} \) and the Maxwell tensor \( F_{mn} \) are linked by the linear constitutive law (here we assume that \( Q_2 = -Q_3 \))
\[ H^{ik} = \mathcal{T}^{ik} + F^{ik} + R^{ikmn}F_{mn} + \left[ \phi \left( \mathcal{R}^{ik} + \chi^{ikmn} F_{mn} \right) \right]. \]  
(86)

The first term \( \mathcal{T}^{ik} \) given by
\[ \mathcal{T}^{ik} = -\frac{1}{2} \eta_1 \Psi_0^2 \left[ \left( R^m_{\phi n} \right)^i_0 \phi - R^m_{\phi n} \phi^i_0 \right] \nabla_m \phi, \]  
(87)
does not contain the Maxwell tensor and thus presents the so-called spontaneous polarization–magnetization tensor; in our case this quantity relates to the nonminimal polarization–magnetization of the axionically active medium, since it is linear in the Ricci tensor on the one hand, and in the four-gradient of the pseudoscalar (axion) field, on the other hand. The third term on the right-hand side, which is linear in the Maxwell tensor, is the curvature induced polarization–magnetization, appeared in the nonminimally extended pure Einstein–Maxwell model [21]. The contribution detailed in square brackets describes the axion–photon coupling, the terms \( \phi \) and \( F^{ik} \) enter the equations in the multiplicative form only. The nonminimal axion contribution \( \phi \chi^{ikmn} F_{mn} \) is a new term in comparison with the minimal model. The four-vector of the electric current has the standard Vlasov form (21) as in the minimal model; again, it satisfies the conservation law \( \nabla_i J^i = 0 \).

3.4. Nonminimal equation for the pseudoscalar (axion) field

Nonminimally extended master equation for the pseudoscalar \( \phi \) takes the form
\[ \nabla_m \left[ g^{mn} + \eta^{mn}_\alpha \right] \nabla_n \phi + \left[ m^2_{\phi \alpha} + V'(\phi^2) + \eta(R) \right] \phi = -\frac{1}{\Psi_0^2} \sum \left( \int dP f_\alpha \mathcal{G}_\alpha \right) + \frac{1}{4} F_{mn} \left( F^{mn}_\phi + \chi^{ikmn} F_{ik} \right), \]  
(88)
where \( \eta^{mn}_\alpha \) and \( \chi^{ikmn}_\alpha \) are given by (67) and (65), respectively. This equation is a nonminimal generalization of the master equation (7).

3.5. Nonminimal generalization of the equations for the gravitational field

Variation of the action functional (60) with (1) and (61) with respect to \( g^{ik} \) gives the nonminimally extended equations for the gravitational field
\[ R_{ik} - \frac{1}{2} R g_{ik} - \Lambda g_{ik} = \kappa \left[ T^{(m)}_{ik} + T^{(EM)}_{ik} + T^{(A)}_{ik} + T^{(NMEM)}_{ik} + T^{(NMA)}_{ik} \right], \]  
(89)
with the tensors \( T^{(EM)}_{ik} \) and \( T^{(A)}_{ik} \) given by (10) and (11), respectively. The nonminimal extension of the stress–energy tensor contains two contributions: first, \( T^{(NMEM)}_{ik} \) describing pure nonminimal electromagnetic part (see, e.g., [21] for details), second, the nonminimal axion part \( T^{(NMA)}_{ik} \). These tensors can be specified as follows:
\[ T^{(NMEM)}_{ik} = q_1 T^{(1)}_{ik} + q_2 T^{(2)}_{ik} + q_3 T^{(3)}_{ik}, \]  
(90)
\[ T^{(NMA)}_{ik} = (Q_1 - \frac{1}{2} Q_2) T^{(1)}_{ik} + Q_1 T^{(3)}_{ik} + \frac{1}{2} \Psi_0^2 \left[ \eta_1 T^{(4)}_{ik} + \eta_2 T^{(5)}_{ik} + \eta_3 T^{(6)}_{ik} + \eta(A) T^{(7)}_{ik} \right]. \]  
(91)
where we put $Q_2 = -Q_1$. Nine nonminimal contributions to the stress–energy tensor can be divided into two groups. The first group

$$T_{ik}^{(1)} = \frac{1}{2} \left[ \nabla_i \nabla_k - g_{ik} \nabla^l \nabla_l \right] [F_{mn} F^{mn}] - R_{km} F_{ik} - \frac{1}{2} F_{mn} F^{mn} \left( R_{ik} - \frac{1}{2} R g_{ik} \right),$$  \hspace{1cm} (92)$$

$$T_{ik}^{(2)} = -\frac{1}{2} g_{ik} \left[ \nabla_m \nabla_l (F_{mn} F^{kl}) - R_{lm} F^{mn} F_{ik} - F^{ln} (R_{il} F_{kn} + R_{ik} F_{ln}) - \frac{1}{2} \nabla^l \nabla_m (F_{im} F_{kn}) 
+ \frac{1}{2} \nabla_i (\nabla_l F_{mn}^n) + \nabla_k (F_{lm} F^{nm}) \right] - R_{im} F_{kn},$$  \hspace{1cm} (93)$$

$$T_{ik}^{(3)} = \frac{1}{2} g_{ik} R^{mn} F_{mn} F_{ls} - \frac{1}{2} F_{ls} (F_{im} R_{kn} + F_{ik} R_{ln}) - \frac{1}{2} \nabla_i \nabla_n \left[ F_{bn} F_{ik} + F_{ik} F_{bn} \right],$$  \hspace{1cm} (94)$$

does not contain pseudoscalar field. Other two terms

$$T_{ik}^{(4)} = \frac{1}{2} [\nabla_i \nabla_k - g_{ik} \nabla^l \nabla_l] [\phi F_{mn} F^{mn}] - \frac{1}{2} R_{ik} \phi F_{mn} F^{mn},$$  \hspace{1cm} (95)$$

$$T_{ik}^{(5)} = -\frac{1}{2} \nabla_m \nabla_n \left[ \phi (F_{bn} F_{ik} + F_{ik} F_{bn}) \right] + \frac{1}{2} \phi F^{mn} (F_{im} R_{kn} + F_{ik} R_{ln}),$$  \hspace{1cm} (96)$$

are linear in the pseudoscalar field $\phi$, and last four terms

$$T_{ik}^{(6)} = \nabla_m \phi R^{mn} F_{nk} \nabla_k + \frac{1}{2} g_{ik} \left[ \nabla_m \nabla_n - R_{mn} \right] [\nabla_{mn} \phi F^{mn}] + \nabla_{mn} \phi F^{mn} \left( R_{ik} - \frac{1}{2} R g_{ik} \right),$$  \hspace{1cm} (97)$$

$$T_{ik}^{(7)} = \left( \nabla_i \nabla_k - g_{ik} \nabla^l \nabla_l \right) \phi^2 - \left( R_{ik} - \frac{1}{2} R g_{ik} \right) \phi^2,$$  \hspace{1cm} (98)$$

are quadratic either in the four-gradient $\nabla_i \phi$, or in the axion field itself. In order to write these nine nonminimal contributions in an appropriate way we have used the Bianchi identities

$$\nabla_i R_{klmn} + \nabla_l R_{kmn} + \nabla_k R_{lmn} = 0,$$  \hspace{1cm} (101)$$

the properties of the Riemann tensor

$$R_{klmn} + R_{mkln} + R_{lkmn} = 0,$$  \hspace{1cm} (102)$$

as well as the rules for the commutation of covariant derivatives

$$\left( \nabla_i \nabla_k - \nabla_k \nabla_i \right) A^l = A^m R^l_{mlk}.$$  \hspace{1cm} (103)$$

3.6. Compatibility conditions for the nonminimal master equations

The Bianchi identity requires, as usual, that the total stress–energy tensor is divergence-free

$$\nabla^k \left[ T_{ik}^{(m)} + T_{ik}^{(EM)} + T_{ik}^{(A)} + T_{ik}^{(NMEM)} + T_{ik}^{(NMA)} \right] = 0,$$  \hspace{1cm} (104)$$

In the appendix B we show explicitly that this relation with nonminimal tensors $T_{ik}^{(NMEM)}$ and $T_{ik}^{(NMA)}$, given by (90)–(100), yields formally the same equation, as in the minimal model (see (16)). Thus, the requirement (30) for the force four-vector should be valid. Only one new detail appears: the pseudoscalar $G_{(a)}$, which is given by (25) in the minimal case, can be extended as follows:

$$G_{(a)} \to G_{(a)} + \lambda_{(a)} \phi R + \mu_{(a)} R^l \phi^l \nabla_i \phi + \nu_{(a)} R^{mn} \phi^m \nabla_i \phi + \omega_{(a)} R_{klnm}^i \phi^i F^{mn} \nabla_i \phi + \cdots,$$  \hspace{1cm} (105)$$

where the nonminimal part contains various convolutions of the Riemann tensor with the particle momentum four-vector, the Maxwell tensor, four-gradient of the axion field, etc.
4. Conclusions

We formulated the nonminimal Einstein–Maxwell–Vlasov-axion model, i.e., obtained the self-consistent system of master equations, which describes the evolution of nonminimally coupled gravitational (see subsection 3.5), electromagnetic (see subsection 3.3) and pseudoscalar (see subsection 3.4) fields in the multi-component relativistic plasma. We followed the combined approach: we used the Lagrange formalism to derive the nonminimally extended field equations, and the formalism of relativistic kinetic theory to link the pseudoscalar, vectorial and tensorial sources in the right-hand sides of the field equations with the macroscopic moments of the plasma distribution function. Then we checked directly the compatibility conditions to verify the self-consistency of this combined approach. We prepared the model for the next step: for cosmological and astrophysical applications.

In this work we follow the Vlasov concept of the cooperative fields. First of all, the gravitational field is considered to be the cooperative one: on the one hand, it governs the evolution of the electromagnetic and pseudoscalar (axion) fields and the plasma particle dynamics; on the other hand, these fields and plasma particles form the corresponding sources for the gravity field evolution. Second, we consider the cooperative electromagnetic field generated in plasma as a macroscopic field averaged over the statistical ensemble; the Lorentz force guiding the plasma particle contains this macroscopic electromagnetic field, and the electric current in the electrodynamic equations includes the first-order macroscopic moment of the distribution function. Third, evolution of the cooperative pseudoscalar (axion) field, on the one hand, is regulated by the cooperative gravitational and electromagnetic fields and by the pseudoscalar source induced in plasma; on the other hand, this axion field contributes to the total stress–energy tensor, the source for the gravitational field, forms specific current-like source in the electrodynamic equations, and acts on the plasma particles via the force appeared in the relativistic kinetic equation.

Only one element of the model is not yet fixed explicitly: the density $G(a)$ of the pseudoscalar source $J$ appeared in the master equation for the axion field (88). Its structure is assumed to be of the form (105) with (25). The coefficients $\alpha(a), ...$ etc, are introduced there phenomenologically. When this quantity is fixed, one can reconstruct the force $\mathcal{R}(a)$ (acting on the plasma particle) using the integral equation (30). We considered the forces of three types. First, when $G(a) = 0$, one obtains that $\mathcal{R}(a) = 0$, and we deal with plasma particles influenced by pure Lorentz force; this case relates to the vanishing entropy production. Second, when $\mathcal{R}(a) p_i \neq 0$, we deal with the model, in which the particle mass is not constant and depends on the square of the pseudoscalar field (32). We indicated the third version of the force, for which $\mathcal{R}(a) p_i = 0$, as the ponderomotive force; we illustrated the procedure of reconstruction of this force with the example of $\mathcal{R}(a)$ quadratic in the particle four-momentum (see (33)). We expect that further development of this theory and (probably) new experimental data will clarify the structure of this force.

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Appendix A

Let us transform the pseudotensor

$$F^{\mu}_{\nu\rho\sigma} F_{\rho\sigma} = \frac{1}{2} \epsilon^{\mu
u\kappa\lambda} F_{\kappa\lambda} F_{\rho\sigma}$$  \hspace{1cm} (A.1)
using the standard decomposition of the Maxwell tensor

$$F_{kl} = E_k U_l - E_l U_k - \epsilon_{kl} B^i U^i,$$  \hspace{1cm} (A.2)

where $E_i$ is the electric field four-vector, $B^i$ is the magnetic excitation four-vector, $U_i$ is the time-like velocity four-vector of the observer. The four-vectors $E_i$ and $B^i$ are orthogonal to the velocity four-vector, i.e., $E_i U^i = 0 = B^i U_i$. Taking into account the identity

$$\epsilon_{kl} U^k \epsilon_{ij} = -\delta_{ij},$$  \hspace{1cm} (A.3)

where $\delta_{ik}$ is the six-indices Kronecker tensor defined as

$$\delta_{ik} = \delta_{i}^{l} \delta_{k}^{m} + \delta_{i}^{m} \delta_{k}^{l} + \delta_{i}^{l} \delta_{k}^{m}, \quad \delta_{ik} = \delta_{i}^{l} \delta_{k}^{m} - \delta_{i}^{m} \delta_{k}^{l},$$  \hspace{1cm} (A.4)

we obtain by direct calculations that

$$\star F_{mn} F_{mq} = \epsilon_{nq} E_m B_m,$$  \hspace{1cm} (A.5)

The convolution with respect to $n$ and $q$ yields

$$\star F_{mn} F_{mn} = 4 E_m B_m,$$  \hspace{1cm} (A.6)

thus the formula

$$\frac{1}{4} \epsilon_{nq} \star F_{mn} F_{mn} - \star F_{mn} F_{mq} = 0$$  \hspace{1cm} (A.7)

is valid. This relation can be also interpreted as follows: the pseudotensorial analogue of the stress–energy tensor of the electromagnetic field is equal to zero identically.

**Appendix B**

In order to check the compatibility conditions in case of the nonminimally extended model (see (104)), we calculate sequentially the divergences of all elements of the total stress–energy tensor of the plasma, electromagnetic and pseudoscalar fields nonminimally coupled to gravity. First of all, we represent the divergence of the $T_{ik}^{(EM)}$ tensor as follows:

$$\nabla_i T_{ik}^{(EM)} = F_k n \nabla_i F_{mn} = F_k \{ F_m n + \star F_{mn} \nabla_m \phi + \nabla_m [ \tau_{mn} + \mathcal{R}^{mpq} F_{pq} + \phi \chi_{mpq} \star F_{pq} ] \},$$  \hspace{1cm} (B.1)

using the extended Maxwell equations (85) with (86). Then we transform the divergence of the tensor $T_{ik}^{(A)}$

$$\nabla_i T_{ik}^{(A)} = \Psi_0 \left[ \nabla_i \nabla_i \phi + m^2 \phi + V(\phi^2) \phi \right] \nabla_i \phi$$

$$= -\nabla_i \phi \left[ \Psi_0 \left[ \nabla_m ( \eta_{(A)} \nabla_m \phi ) + \eta_{(A)} R \phi \right] + \sum_{(a)} \int dP (S_{(A)} F_{ik}^{(a)}) \right],$$

using the nonminimal master equation for the pseudoscalar field (88). From the sum of these two divergencies we extract the terms linear in the parameter $q_1$ and compare it with the divergence of the tensor $T_{ik}^{(1)}$:

$$\nabla_i T_{ik}^{(1)} = -F_k n \nabla_i (RF_{mn}).$$  \hspace{1cm} (B.2)

Then we continue this procedure, using the following formulas for the terms, which include $q_2$ and $q_3$:

$$\nabla_i T_{ik}^{(2)} = -F_k n \nabla_i (R_{il} F_{mn} + F_{il} R_{mn}),$$  \hspace{1cm} (B.4)
When we deal with the term

$$\nabla^i T^{(1)}_{ik} = -\nabla^i (R^i {}^a_\phi F^a_{ik}) + \frac{1}{2} RF_{pq} F^{pq} \nabla_k \phi,$$

we collect the expressions in front of the parameter \(Q_1 - \frac{1}{2} Q_3\), since we assume here that \(Q_2 = -Q_3\) and take into account that \(T^{(2)}_{ik}\) can be transformed by using \((A.7)\) and be included into the term \(\frac{1}{2} T^{(1)}_{ik}\). Similarly, we compare the terms, which contain \(Q_3\), using

$$\nabla^i T^{(3)}_{ik} = -\frac{1}{2} F^i {}^a_\phi \nabla^i \left[ (R_{mpq} - a R_{mpq}) \phi F^{pq} \right] + \frac{1}{2} R_{mpq} F^{mn} F^{pq} \nabla_k \phi,$$

then compare the terms linear in \(\eta_1, \eta_2, \eta_3\), keeping in mind that

$$\nabla^i T^{(4)}_{ik} = -\frac{1}{2} F^i {}^a_\phi \nabla^i \left[ (R^i {}^a_\phi - R^i {}^a_{jk}) \nabla_k \phi \right] + \frac{1}{2} \nabla_k \phi \nabla^i \left[ (R_{ik} F^i + F_{ik} R^{ik}) \nabla_k \phi \right],$$

$$\nabla^i T^{(5)}_{ik} = \nabla_k \phi \nabla^i (R \nabla_l \phi),$$

and finally, the terms linear in \(\eta_A\), using the formula

$$\nabla^i T^{(7)}_{ik} = R \phi \nabla_k \phi.$$ 

Direct calculations show that all the terms linear in the mentioned coupling constants disappear, and the compatibility conditions \((104)\) written in the nonminimal case, reduce to the form \((16)\) obtained for the minimal case. In other words, the requirement \((30)\) for the force-like four-vector \(\xi\) in the plasma nonminimally coupled to gravity has the same form as in the minimal case, but now the pseudoscalar \(Q_{(a)}\) can contain the appropriate nonminimal terms in addition to the terms written in \((25)\).

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