Long-lived matter wave Bloch oscillations and dynamical localization by time-dependent nonlinearity management

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Abstract

We introduce a new method to achieve long-lived Bloch oscillations and dynamical localization of matter wave gap solitons in optical lattices. The method is based on the time-dependent modulations of the nonlinearity which can be experimentally implemented by means of the Feshbach resonance technique. In particular, we show that the width of the wavepacket is preserved if the time modulations of the nonlinearity are taken proportional to the curvature of the linear band spectrum which for most typical experimental settings are well approximated by harmonic time modulations of proper frequencies.

(Some figures in this article are in colour only in the electronic version)

The existence of nonspreading localized waves (or solitons) is a property of nonlinear systems which stems from the balance of nonlinearity and dispersion. In the case of Bose–Einstein condensates (BECs) tightly bounded in two spatial dimensions, bright matter solitons are known to exist when attractive two-body interactions compensate the vacuum dispersion. These waves were experimentally reported in [1]. An external periodic potential generated by two counter-propagating laser beams, i.e. an optical lattice (OL), applied to a BEC gives rise to a band-gap structure in the spectrum, introducing in this way artificial dispersion. In the effective mass approximation [2], one finds that signs of the atomic effective masses are different at opposite edges of each finite gap. Under this condition, the properties of matter waves change significantly: a BEC with repulsive inter-atomic interactions admits the existence of stationary gap solitons with chemical potentials inside spectrum band gaps. The experimental observation of these matter wave gap solitons was reported in [3]. Due to broken translational symmetry these solutions are intrinsically localized, especially for high strength OLs, so that any attempt to move them with the aid of an external force (such as, for example, the acceleration of the OL) usually leads to their destruction.

In order to control gap solitons and allow them to move in real and in reciprocal spaces, one has to properly design the nonlinearity in the system. Soliton existence indeed requires that the effective nonlinearity and the effective mass have opposite signs for all points in the Brillouin zone [2]. In recent papers [4, 5] it has been shown that this condition can be satisfied with the help of a nonlinear optical lattice, i.e. with a spatial modulation of the scattering length, and that it leads to the existence of Bloch oscillations and dynamical localization of matter waves which are long lived in the presence of nonlinearity. A nonlinear optical lattice implies a control of the scattering length in space, a task which is more difficult to implement than the modulation of the scattering length in time (the latter can be realized with external time-dependent magnetic fields via the usual Feshbach resonance technique [6]). It is then natural to ask if a time management of the nonlinearity can lead to the above long-lived Bloch and dynamical oscillatory behaviour in the nonlinear regime.

The aim of this paper is just devoted to this. In particular, we show for the first time that long-lived Bloch oscillations and dynamical localization of a matter wave gap soliton in an accelerated linear OL become possible in the...
presence of a properly designed time management of the nonlinearity. We remark that simultaneous effects of linear OLs and nonlinear time-dependent nonlinearities have been considered in the literature in various contexts. In the case of BECs this technique was used mainly for the stabilization of higher dimensional solitons [7], investigation of stable Feshbach resonance-managed discrete matter-wave solitons [8] and generation of bright and dark solitons [9]. No applications of this technique to nonlinear Bloch oscillations and dynamical localization phenomena have been suggested so far. We remark that the implementation of properly designed time-dependent nonlinearities for nondestructive matter wave transport in OLs can be of interest not only for BECs but also for nonlinear optics.

Let us start by considering a BEC described in the mean-field approximation by the one-dimensional Gross–Pitaevskii equation (GPE) with a time-varying s-wave scattering length $a_s(T)$ and cos-like OL accelerated in space according to the law $S(T)$:

$$i\hbar \frac{\partial \Psi}{\partial T} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial X^2} - V \cos \left[ 2\pi \left( X - S(T) \right) \right] \Psi + \frac{k^2}{a_s^2m^2} \left| \Psi \right|^2 \Psi. \quad (1)$$

Here $m$ and $\Psi$ denote the boson mass and the macroscopic wavefunction, $V$ is the OL amplitude, $T$ and $X$ are the time and the longitudinal coordinate, respectively, $d$ is the OL period and $a_\perp$ is the transverse trap size. Equation (1) is obtained from the full three-dimensional GPE in the case when the OL period $d$ is much bigger than the transversal trap size $a_\perp \ll d$ (for details of transition, see e.g. [9]). Introducing the dimensionless moving spatial coordinate $x = \pi (X - S(T))/d$, time $t = TE_R/h$ ($E_R = \hbar^2\pi^2/(2md^2)$ is the recoil energy) and wavefunction

$$\psi(x, t) = \frac{2d}{a_\perp} \sqrt{a_s(0)} \Psi(X, T) \exp \left[ \frac{i}{\hbar} \int_0^t \left( \frac{dS}{dt} \right)^2 dx \right] .$$

we obtain the normalized GPE

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} - v \cos(2x) \psi + \gamma(t) \psi + g(t)|\psi|^2\psi. \quad (2)$$

dimensionless OL amplitude $v = V/E_R$, time-dependent nonlinearity $g(t) = a_s(T)/a_s(0)$ and external force $\gamma(t) = (md/\pi E_R) d^2S/dT^2$, proportional to the acceleration of the OL. In the absence of the nonlinearity $g(t) \equiv 0$ and external force $\gamma(t) \equiv 0$, periodicity of the OL gives rise to a band-gap structure of both spectrum $\epsilon_n(q)$ and Bloch functions $\phi_n(q)$ of the linear problem $\epsilon_n(q)$, where $n$ is the band number (below, we consider only the first band $n = 1$ and therefore the index $n$ is omitted) and $q$ is the Bloch wavenumber. The spectrum is periodic in the reciprocal space with the period $2$ (in the chosen units): $\epsilon(q) = \epsilon(q + 2).

Under the influence of the external force, soliton motion in the real and reciprocal spaces obeys semiclassical laws for its centre in real and reciprocal spaces $X$ and $Q$, correspondingly [4, 5]:

$$\dot{X} = \frac{d\epsilon}{dq}|_{q=Q}, \quad \dot{Q} = -\gamma(t), \quad (3)$$

where the overdot stands for the time derivative. When the external force causes soliton motion, as follows from (3), the position of soliton centre $Q$ in the reciprocal space will be changed. At the same time, the existence of a gap soliton solution of equation (2) for a given $Q(t) = Q_0 - \int_0^t \gamma(T) dT$ ($Q_0$ is the initial coordinate of the soliton in the reciprocal space) is determined by the condition $M(Q(t))g(t) < 0$ for all times $t$ (here, $M(q) = [d^2\epsilon(q)/dq^2]^{-1}$ is the soliton effective mass). Thus, to prevent soliton destruction one should vary the nonlinearity $g(t)$ in such a manner as to keep the opposite sign with respect the effective mass (this assures that the above existence condition is satisfied for all $t$ and all $Q(t)$). Moreover, to facilitate the soliton motion, one should minimize the redistribution of particles, keeping the soliton width to be constant (this can be achieved by keeping the product of the effective mass and effective nonlinearity $g(t)\int_{X_1}^{X_2} |\psi_{Q(t)}(x)|^2 dx$ to be constant [4]). For this, it is sufficient to take $g(t)$ of the form

$$g(t) = -\frac{\int_{X_1}^{X_2} |\psi_{Q(t)}(x)|^2 dx}{M(Q(t))\int_{X_1}^{X_2} |\psi_{Q(t)}(x)|^2 dx} \gamma(t). \quad (4)$$

In the following, we apply the above nonlinear management scheme to induce long-lived nonlinear Bloch oscillations and dynamical localization of a gap soliton in an accelerated OL.

The Bloch oscillations of matter waves occur when the BEC is subject to a constant (dc) external force, i.e. for $\gamma(t) \equiv \Gamma \equiv 0$. For the particular choice of the lattice amplitude and the linear force, the above algorithm of designing the temporal dependence (4) of the nonlinearity results in curves depicted in figures 1(a) and (d) for $Q_0 = 1$ and $Q_0 = 0$, respectively. The gap-soliton dynamics under this $g(t)$ management is depicted in figures 1(b) and (e). Note that due to a proper choice of $g(t)$, the Bloch oscillations become very regular with the period and the spatial amplitude which coincide with the values $T_R = 2/|\Gamma| = 2 \times 10^3$ and $X_R = |\epsilon(1) - \epsilon(0)|/(2|\Gamma|) \approx 32.5\pi$ predicted from equations (3). This behaviour is in deep contrast with that depicted in figures 1(c) and (f) for the case of a constant nonlinearity, where the Bloch oscillations are accompanied by the decay of the wavepacket.

Similar results are obtained for the case of the dynamical localization. In this case, the motion of a soliton is turned on under the action of a time-periodic external force $\gamma(t) = \Gamma \cos(2\Omega t)$. When the frequency $\Omega$ is much smaller than the soliton energy $E$ (approximated by the chemical potentials of the initial stationary state for the initial $Q_0 = 0, 1$) at the same time fulfills the condition $\Omega = \Gamma \pi \beta_n$ with $\beta_n$ being the $n$th zero of the Bessel function $J_0(\beta_n) = 0 [10]$, one expects dynamical localization of the soliton, i.e. a regular oscillation in space induced by the ac force, provided its existence is guaranteed for all the energies belonging to the chosen band. In [5], this was achieved with the help of spatially non-uniform
The application of the external force, which starts to oscillate at time \( t = t_0 \)

\[
y'(t) = \begin{cases} 
\Gamma, & t < t_0, \\
\Gamma \cos \Omega(t - t_0), & t \geq t_0,
\end{cases}
\]  

and time-dependent nonlinearity \( g(t) \) (as depicted in figures 2(a) and (c)), results in the dynamical localization of the soliton, when its mean drift velocity becomes zero. These results are obtained for both the oscillation frequency, which corresponds to the first zero of the Bessel function (figure 2(b)), and the oscillation frequency, determined from the second zero of the Bessel function (figure 2(d)).

The numerical simulations, reported above, are performed for an exact numerical design of \( g(t) \). However, for typical values of \( v \geq 1 \) they are remarkably well reproduced using the dependence \( g(t) \) which stems from the tight-binding approximation. Since in typical experimental settings \( v \) is of the order of several recoil energies, expanding \( \varepsilon(q) \) in the Fourier series, we can consider only the two leading terms: \( \varepsilon(q) = \omega_0 - \omega_1 \cos(\pi q) \). This approximation already works well at \( v \geq 1 \), as one can see from table I of [11]. For the particular case \( v = 3 \) (for which our above-mentioned numerical simulations are performed), \( \omega_0 \approx -0.84, \omega_1 \approx 0.102 \). On the other hand, in this approximation the integral in equation (4) becomes practically a constant \(^4\), independent of \( q \) and, hence, of \( t \) and the temporal law (4) for nonlinearity \( g(t) \) simplifies to

\[
g(t) \approx -\cos(\pi \Omega(t)) \cos(\pi \Gamma t),
\]  
i.e. it is uniquely fixed by the dynamics of the wavepacket in the reciprocal space.

In this case of Bloch oscillations, the semiclassical equations of motion give \( X(t) = X_0 - \frac{\pi}{\Gamma} \cos(\pi Q_0)[1 - \cos(\pi \Gamma t)] \) and \( Q(t) = Q_0 - \Gamma t \) with \( X_0 \) being the initial coordinate of the soliton. The temporal dependence of the nonlinearity function then follows from equation (6) as

\[
g(t) \approx -\cos(\pi Q_0) \cos(\pi \Gamma t) \cos(\pi \Gamma t)
\]

At the same time for the dynamical localization case, the time-dependent nonlinearity can be approximated from equation (6) as

\[
g(t) \approx -\cos(\pi Q_0) \cos(\pi \Gamma t)
\]

To conclude, we have introduced a new method based on the time-dependent modulations of the nonlinearity to achieve long-lived Bloch oscillation and dynamical localization of gap solitons in OLs. The desired modulation is obtained directly from the curvature of the underlying band structure and can be experimentally implemented by means of time-dependent external magnetic fields via the Feshbach resonance technique. For strengths of the OL greater than one recoil energy (e.g. for most typical experimental settings), the shape of the time modulation can be fixed in a very simple manner using fields of proper frequencies.

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References

[1] Khaykovich L et al 2002 Science 296 1290
Strecker K E, Partridge G B, Truscott A G and Hulet R G 2002 Nature 417 150
[2] Konotop V V and Salerno M 2002 Phys. Rev. A 65 021602
Pu H et al 2003 Phys. Rev. A 67 043605

\(^4\) The deviation from a constant of the integral in (4), due to its dependence on \( Q_0 \), can be estimated in the Wannier basis as the overlapping integral of the Wannier functions of two adjacent sites, this giving for \( v = 3 \) an error of about 0.2%.

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[3] Eiermann B et al 2004 Phys. Rev. Lett. 92 230401
[4] Salerno M, Konotop V V and Bludov Yu V 2008 Phys. Rev. Lett. 101 030405
[5] Bludov Yu V, Konotop V V and Salerno M 2009 Eur. Phys. Lett. submitted
[6] Stwalley W C 1976 Phys. Rev. Lett. 37 1628
  Tiesinga E, Moerdijk A J, Verhaar B J and Stoof H T C 1992 Phys. Rev. A 46 R1167
  Inouye S, Andrews M R, Stenger J, Miesner H-J, Stamper-Kurn D M and Ketterle W 1998 Nature 392 151
  Stenger J, Inouye S, Andrews M R, Miesner H-J, Stamper-Kurn D M and Ketterle W 1999 Phys. Rev. Lett. 82 2422
[7] Porter M A, Chugunova M and Pelinovsky D E 2006 Phys. Rev. E 74 036610
[8] Abdullaev F Kh, Tsoy E N, Malomed B A and Kraenkel R A 2003 Phys. Rev. A 68 053606
[9] Brazhny V A and Konotop V V 2005 Phys. Rev. A 72 033615
[10] Dunlap D H and Kenkre V M 1986 Phys. Rev. B 34 3625
  Konotop V V, Chubykalo O A and Vázquez L 1993 Phys. Rev. E 48 563
  Cai D, Bishop A R, Grønbech-Jensen N and Salerno M 1995 Phys. Rev. Lett. 74 1186
[11] Alfimov G L, Kevrekidis P G, Konotop V V and Salerno M 2002 Phys. Rev. E 66 046608