Tunneling of persistent currents in coupled ring-shaped Bose–Einstein condensates

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Abstract

Considerable progress in experimental studies of atomic gases in a toroidal geometry opens up novel prospects for the investigation of fundamental properties of superfluid states and creation of new configurations for atomtronic circuits. In particular, a setting with Bose–Einstein condensates loaded in a dual-ring trap suggests a possibility to consider the dynamics of tunneling between condensates with different angular momenta. Accordingly, we address the tunneling in a pair of coaxial three-dimensional (3D) ring-shaped condensates separated by a horizontal potential barrier. A truncated (finite-mode) Galerkin model and direct simulations of the underlying 3D Gross–Pitaevskii equation are used for the analysis of tunneling superflows driven by an initial imbalance in atomic populations of the rings. The superflows through the corresponding Bose–Josephson junction are strongly affected by persistent currents in the parallel rings. Josephson oscillations of the population imbalance and angular momenta in the rings are obtained for co-rotating states and non-rotating ones. On the other hand, the azimuthal structure of the tunneling flow demonstrates formation of Josephson vortices (fluxons) with zero net current through the junction for hybrid states, built of counter-rotating persistent currents in the coupled rings.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The Josephson effect in alternating- and direct-current (ac and dc) forms was first discovered in a pair of two superconductors, the macroscopic wave functions of which are weakly coupled across the tunneling barrier [1, 2]. The ac Josephson effect in atomic Bose–Einstein condensates (BEC) was experimentally observed [3] in an atomic cloud loaded in the double-well potential, where a constant chemical-potential difference between the coupled condensates drives an oscillating atomic flow through the barrier, see also recent experimental work [4]. Josephson junctions are also known in excitonic condensates in bilayered materials [5].

On the other hand, persistent currents in toroidal atomic BECs were extensively investigated, both experimentally and theoretically, as a hallmark of superfluidity. The ring-shaped trap produces a large central hole around the axis of the condensate cloud. Thus, the core of persistent currents, alias vortex states, in toroidal traps is bounded by the potential structure, which makes even multicharged vortices robust. Further, the stability of the persistent current in a single ring suggests one to consider the impact of quantized angular momenta in two parallel-coupled superfluid rings on the Josephson effect in the dual-ring setting. In superconductors, geometrically similar ring-shaped long Josephson junctions are well-known objects [6–16], including their discrete version [17, 18].
Previous theoretical studies [19–29] have drawn considerable interest to systems of coupled circular BECs. Two identical parallel coaxial BEC rings, separated in the axial direction by a potential barrier, were considered in the context of the spontaneous generation of vortex lines by means of the Kibble–Zurek mechanism [31]. However, Josephson dynamics in such a symmetric double-ring system, to the best of our knowledge, has not been previously investigated. In the present work, we address tunneling of weakly coupled quantized superflows in a pair of stacked ring-shaped condensates. The schematic of the double-ring geometry is displayed in figure 1. We perform the analysis, in parallel, on the basis of the full three-dimensional (3D) Gross–Pitaevskii equation (GPE) for this setting and a simple finite-mode truncation, i.e. the Galerkin approximation (GA). In the general form, the GA is introduced and investigated in section 2. The GA predicts ac Josephson oscillations. Other remarkable dynamical states, which are produced by the numerical simulations, are natural modes in spin-orbit-coupled systems [32], are normal modes in spin-orbit-coupled systems [33]). Section 3 reports results of systematic simulations of the full GPE, which well corroborate the GA predictions. Thus, the six-mode GA identifies the minimal set of modes which adequately captures basic features of the two-coupled-rings system. Beyond the framework of the six-mode GA, the 3D simulations demonstrate that the vanishing of the net tunneling superflow in the hybrid configurations is associated with generation of Josephson vortices (fluxons) trapped in the Bose–Josephson junction (see [23]), which is also shown in section 3. The paper is concluded by section 4.

2. The GA

The basic set of modes which determines tunneling in the system of two weakly parallel-coupled rings can be identified by means of a finite-mode (truncated) approximation, i.e. GA, which replaces the underlying 3D GPE by a dynamical system with several degrees of freedom. In this section, we introduce the GPE for the present model, which is followed by the derivation of the GA and analysis of major scenarios for the dynamics of imbalance of the populations and angular momenta in the coupled annular-shaped BECs in the framework of this approximation.

2.1. Basic equations

The underlying GPE is [34]:

$$i\hbar \partial_t \Psi = -\frac{\hbar^2}{2M} \nabla^2 \Psi + V_{ext}(\mathbf{r})\Psi + g|\Psi|^2\Psi,$$

where $\tau$ is the temporal variable, $\nabla^2$ acts on the 3D wave function, $g = 4\pi \hbar^2 a_s / M$ is the nonlinearity strength, while $M = 3.819 \times 10^{-26}$ kg is the atomic mass and $a_s = 2.75$ nm the s-wave scattering length for the condensate composed of $^{23}$Na atoms. In cylindrical coordinates $(\rho \equiv \sqrt{x^2 + y^2}, \theta, z)$, the trapping potential includes terms which provide radial confinement centered at $\rho = \rho_0$ and a symmetric double-well potential in the vertical direction, with minima at $z = \pm z_0$:

$$V_{ext}(\rho, z) = \frac{1}{2} M \omega_z^2 (\rho - \rho_0)^2 + \frac{1}{2} M \omega_z^2 z^2 + U_b e^{-\frac{z^2}{2\sigma^2}},$$

where $z_0 = a_s \sqrt{2 \ln \left\{U_b / (M a_s^2 \omega_z^2)\right\}}$.

The 3D wave function of the double-ring system can be approximately written in terms of effective 1D wave functions of the two rings, $\phi(\theta)$ and $\psi(\theta)$, as

$$\Psi(\mathbf{r}) = \Psi_p(\rho) \left[ \Phi_+(z) \frac{\phi(\theta)}{\sqrt{\gamma}} + \Phi_-(z) \frac{\psi(\theta)}{\sqrt{\gamma}} \right],$$

with appropriate wave functions $\Psi(\rho)$ and $\Phi_\pm(z) = \Phi(z \mp z_0)$, centered, respectively, at $\rho = \rho_0$ and $z = \pm z_0$, and coefficients

$$\gamma = 8 \pi \alpha \sigma R^2 \int_0^{+\infty} |\Psi_p|^2 \rho d\rho \int_{-\infty}^{+\infty} |\Phi(z)|^2 dz,$$

$$R^{-2} = \int_0^{+\infty} |\Psi_p|^4 \rho^{-2} d\rho.$$
of ansatz (3) in equation (1) and averaging in the directions of \(\rho\) and \(z\), takes the form of [19, 20]

\[
i\partial_t \psi = -\frac{1}{2} \phi_{\theta\theta} + |\phi|^2 \psi - \kappa \psi,
\]

\[
i\partial_t \phi = -\frac{1}{2} \psi_{\theta\theta} + |\psi|^2 \phi - \kappa \phi,
\]

where \(t = (\hbar/2MR^2)^2\) is dimensionless time, and

\[
\kappa = \left| \int_{-\infty}^{\infty} \Phi^*_\rho(z) (\partial^2_{\rho}) \Phi_\rho(z) dz \right|.
\]

In equation (6), we keep coupling constant \(\kappa\) unscaled, to consider weak- and strong-coupling regimes separately. The sign of the nonlinear terms in equation (6) is self-repulsive, as we aim to focus on the case when the dynamics of persistent currents in the ring-shaped BEC is not subject to the modulations instability in the azimuthal direction, which occurs in the case of self-attraction. In the conservative system the total number of particles and angular momentum are conserved: \(N = N_1 + N_2 = \text{const},\ L = L_1 + L_2 = \text{const}\), where these quantities for the upper and lower rings are

\[
N_1 = \int_0^{2\pi} |\phi(\theta)|^2 d\theta, \quad N_2 = \int_0^{2\pi} |\psi(\theta)|^2 d\theta,
\]

\[
L_1 = -i \int_0^{2\pi} \phi^* \partial_{\theta\theta} \phi d\theta, \quad L_2 = -i \int_0^{2\pi} \psi^* \partial_{\theta\theta} \psi d\theta.
\]

It is relevant to compare equations (6) with similar equations which do not include the linear coupling (\(\kappa = 0\)), but feature nonlinear interaction between \(\phi\) and \(\psi\), representing a two-component wave function in the 1D ring, or approximate a 2D annular region [35]. In that case, a relevant problem is the miscibility–immiscibility transition in the binary condensate.

The configuration admitting counter-circulating flows, characterized by vorticities (topological charges) \(m_{1,2} = \pm m\), see figure 1, may be approximated by the following finite-mode ansatz, which takes into account the possible presence of the non-rotating component too:

\[
\phi(\theta, t) = a_0(t) + a_\pm(t) e^{i\mu \theta} + a_\mp(t) e^{-i\mu \theta},
\]

\[
\psi(\theta, t) = b_0(t) + b_\pm(t) e^{i\mu \theta} + b_\mp(t) e^{-i\mu \theta}.
\]

Effective evolution equations for amplitudes \(a_\pm\) and \(b_\pm\) can be derived from the Lagrangian of equation (6):

\[
\Lambda = \int_0^{2\pi} \left\{ \left(\frac{i}{2} (\phi^* \partial_{\theta} \phi + |\phi|^2 \phi) + \kappa |\phi|^2 \phi \right) + \text{c.c.} \right\} d\theta.
\]

The substitution of ansatz (9) in this expression and integration yields

\[
\frac{\Lambda}{2\pi} = \int_0^{2\pi} \left( a_0 \frac{da_0}{dt} + b_0 \frac{db_0}{dt} + a_\pm \frac{da_\pm}{dt} + b_\pm \frac{db_\pm}{dt} \right) d\theta + \text{c.c.} - H,
\]

where c.c. stands for the complex conjugate expression, and the Hamiltonian is

\[
H = \frac{m^2}{2} (|a_0|^2 + |a_\pm|^2 + |b_0|^2 + |b_\pm|^2) + \text{c.c.}.
\]

This system of six evolution equations represents the GA, i.e. the finite-mode truncation replacing the full GPE system. It admits an invariant reduction to four equations, by setting \(a_0 = b_0 = 0\). The GA provides an adequate simplification for diverse nonlinear systems [36–38], including GPE-based models of trapped BEC [39–41].

Further, the substitution of ansatz (9) in the definitions of the total number of particles and angular momentum, given by equation (8), yields the following expressions for the GA versions of these dynamical invariants:

\[
N/(2\pi) = |a_0|^2 + |b_0|^2 + \sum_{\pm} (|a_\pm|^2 + |b_\pm|^2),
\]

\[
L/(2\pi) = \pm m \sum_{\pm} (|a_\pm|^2 + |b_\pm|^2).
\]

In the general case, system (13) is equivalent to the Hamiltonian one with six degrees of freedom. The presence of only three dynamical invariants, represented by Hamiltonian (12) and the norm and angular momentum, given by equation (14), suggests that the GA system is not integrable, in agreement with the
well-known fact that the system of coupled GPEs (6) is a non-integrable one. The invariant reduction of equation (13) to four degrees of freedom, produced by setting \(a_0 = b_0 = 0\), is not integrable either.

### 2.2. Analysis of the GA

Equations (13), produced by the GA, admit simple invariant reductions for states with vorticities \((m_1, m_2) = (0, 0)\) and \((1, 1)\) or \((-1, -1)\), which correspond, severally, to ansatz (9) with \(a_\pm = b_\pm = 0\), and \(a_0 = b_0 = a_- = b_- = 0\) or \(a_0 = b_0 = a_+ = b_+ = 0\). In particular, for \((m_1, m_2) = (0, 0)\) system (13) reduces to a set of two equations:

\[
\begin{align*}
\frac{da_0}{dt} &= |a_0|^2 a_0 - \kappa b_0, \\
\frac{db_0}{dt} &= |b_0|^2 b_0 - \kappa a_0,
\end{align*}
\]

with \(\kappa > 0\). This system with two degrees of freedom is integrable, as it conserves Hamiltonian (20), written below, and quantities (14) (in this particular case, they amount to a single dynamical invariant).

On the other hand, the set of dynamical variables which include vorticities \((1, -1)\) or \((-1, 1)\) does not correspond to any invariant subsystem of equation (13) (except for the trivial case of \(\kappa = 0\)) with fewer than four degrees of freedom, hence its evolution is governed by the full system (13). We use this system in all cases—in particular, with the objective to test stability of the invariant reductions with respect to small perturbations which break the invariance. It is relevant to mention too that fixed points of equation (13) with \(|a_+| = |a_-|\) and \(|b_+| = |b_-|\) may describe standing patterns of the wave functions \(\sim \cos (mt\theta)\) and/or \(\sin (mt\theta)\), but we do not aim to address states of such types in the present work.

Thus, we simulated equation (13) with different inputs, corresponding to initial vorticity sets \((m_1, m_2) = (0, 0), (1, 1), (1, -1)\) and \((0, 1)\), by means of the standard Runge–Kutta algorithm. It was checked that the simulations indeed conserve the dynamical invariants given by equations (12) and (14).

Figures 2 and 5 display the dynamics initiated by inputs \((0, 0), (1, 1), (1, -1),\) and \((0, 1)\) with a small initial imbalance in the number of particles between the coupled rings, in the strong-coupling regime (\(\kappa = 50\), which is responsible for small oscillation period in this figure and similar ones). For the comparison’s sake, counterparts of these results, produced by full simulations of the 3D GPE (1), are displayed close to them in figures 3 and 6. Detailed discussion of the latter figures is given in the next section.

In figures 2(a) and (b) the numbers of particles oscillate similar to what is produced by the ac Josephson effect in the double-well setting (as one can see in figures 3(a) and (b)), we have obtained similar results from GPE. To support this conclusion, figure 4 demonstrates that the phase difference between the top and bottom rings, \(\delta \phi = \arg(\psi) - \arg(\psi)\), for the input with vorticities \((m_1, m_2) = (0, 0)\) (for one of the \((1, 1)\) type the situation is essentially the same) varies in time nearly linearly, which is a characteristic feature of the ac Josephson effect [3]. On the other hand, figures 5(a), (b) demonstrate that the Josephson oscillations, as predicted by the GA, vanish for the counter-rotating input, with \((m_1, m_2) = (-1, 1)\) as well as for one of type \((0, 1)\) (as one can see in figures 6(a) and (b) for \((-1, 1), (1, 0)\) respectively, we have obtained similar results from GPE).

It is also instructive to compare angular distributions of the inter-ring tunneling flow. To this end, figure 7 displays the density variation

\[
\delta n_\psi(\theta, t) \equiv |\psi(\theta, t)|^2 - |\psi(\theta, t = 0)|^2,
\]

as a function of angular coordinate \(\theta\). In particular, the state with \((m_1, m_2) = (1, 1)\) does not develop the spatial variation, while those of the \((1, -1)\) and \((0, 1)\) types naturally build...
spatially periodic patterns, with periods \(T_\theta = \pi\) and \(T_\theta = 2\pi\), respectively.

The analysis of experimental data for the ac Josephson effect, which was implemented in [3], has demonstrated that the relative difference in the number of particles

\[
\eta = (N_1 - N_2) / (N_1 + N_2),
\]

oscillates in time with a small amplitude \(\sim \delta \eta \approx 10^{-2}\).

It is relevant to consider restrictions on the amplitude of the oscillating superflow in more detail. In the framework of the GA, the amplitude of oscillations

\[
\delta \eta = \eta_{\text{max}} - \eta_{\text{min}},
\]

is determined by the initial value of the number-of-particle difference (17) and coupling constant \(\kappa\). Figure 8 displays values of \(\delta \eta\) averaged over ten oscillation periods for state \((m_1, m_2) = (0, 0)\) with real initial conditions. As expected, \(\delta \eta\) grows with \(\kappa\), and \(\delta \eta \to 0\) at \(\eta(t = 0) \to 0\). Note that the oscillation amplitude \(\eta\) has a sharp maximum at relatively small \(\eta(t = 0)\) and decays when the initial asymmetry grows, due to the repulsive nonlinear interactions, which resembles experimentally observed features of the macroscopic self-organized oscillations of BEC trapped in a double-well potential [42].

To gain further insight into the behavior of oscillation amplitude \(\delta \eta\), it is instructive to consider the simplest particular case, based on the system of two ordinary differential equations (15) for complex amplitudes \(a_0(t)\) and \(b_0(t)\). The complex amplitudes can be represented as

\[
a_0 = |a_0|e^{i\alpha(t)}, \quad b_0 = |b_0|e^{i\beta(t)},
\]

where \(|a_0| = \sqrt{A} \cos(\theta(t)), \quad |b_0| = \sqrt{A} \sin(\theta(t)), \quad A \equiv |a_0|^2 + |b_0|^2\) is the conserved total norm, and the distribution angle, \(\theta\), varies in interval \(0 \leq \theta \leq \pi/2\), so that \(\sin \theta, \cos \theta\) and norm \(A\) are always positive (or zero).

The present system with two degrees of freedom is integrable (as mentioned above), because it conserves \(A\) and the Hamiltonian

\[
H = \frac{1}{2}(|a_0|^4 + |b_0|^4) - \kappa(a_0^*b_0^* + a_0b_0),
\]

\[
\equiv \frac{1}{2} A^2 \left[ 1 - \frac{1}{2} \sin^2(2\theta) \right] - \kappa A \sin(2\theta) \cos(\alpha - \beta).
\]

In this case, asymmetry (17) amounts to

\[
\eta \equiv \frac{|a_0|^2 - |b_0|^2}{|a_0|^2 + |b_0|^2} \equiv \cos(2\theta),
\]

hence the largest asymmetry corresponds to largest \(|\cos(2\theta)|\), i.e. smallest \(\sin(2\theta)\).

If the input corresponds to real initial values of \(a_0(0)\) and \(b_0(0)\) of the same sign (i.e. initially one has \(\alpha_0 = \beta_0 = 0\) in equation (19)), the value of \(\sin(2\theta)\) is determined by the

Figure 4. The evolution of the phase difference between the wave functions in the upper and lower rings for the state generated by equation (13) with input \((m_1, m_2) = (0, 0)\) for different values of imbalance \(\eta\) and fixed \(\kappa = 50\).

Figure 3. The evolution of numbers of particles and angular momenta in the top and bottom rings for the states generated by simulations of equation (26) with inputs \((m_1, m_2) = (0, 0)\) (a); \((1, 1)\) (b). The scale of temporal variable in this figure is different from that in its GA counterpart in figure 2.
conservation of the Hamiltonian:
\[
\left[ \sin^2(\vartheta) \right]^2 + \frac{4\kappa}{A} \cos(\alpha - \beta) \sin(2\vartheta) \\
- \left[ \sin^2(2\vartheta_0) + \frac{4\kappa}{A} \sin(2\vartheta_0) \right] = 0,
\]
(22)

where \( \vartheta_0 \) is the initial value of \( \vartheta \) (so that constraint \( \sin(2\vartheta_0) \leq 1 \) holds). It is easy to see that a local minimum (or maximum) of \( \sin(2\vartheta_0) \), considered as a function of \( \cos(\alpha - \beta) \), while other parameters are fixed, does not exist. Indeed, differentiating equation (22) with respect to \( \cos(\alpha - \beta) \) and setting \( d(\sin(2\vartheta))/d(\cos(\alpha - \beta)) = 0 \) shows that this condition may hold solely at \( \sin(2\vartheta) = 0 \), but equation (22) does have root \( \sin(2\vartheta_0) = 0 \). Therefore, smallest and largest values of \( \sin(2\vartheta) \) may only be attained at extreme values of \( \cos(\alpha - \beta) \), i.e. \( \cos(\alpha - \beta) = \pm 1 \). The respective roots of equation (22) are
\[
\sin(2\vartheta) = \sin(2\vartheta_0),
\]
(23)
\[
\sin(2\vartheta) = 4\frac{\kappa}{A} + \sin(2\vartheta_0).
\]
(24)

This means that \( \sin(2\vartheta) \) never takes values smaller than \( \sin(2\vartheta_0) \), hence the asymmetry cannot be larger than its initial value. Note that, if \( \kappa \) is a small parameter, it follows from equations (23) and (24) that the amplitude of the oscillations of the asymmetry may be approximated by
\[
\delta(\cos(2\vartheta)) \approx 4A^{-1}|\tan(2\vartheta_0)| \kappa.
\]
(25)

This expression demonstrates that the amplitude of the asymmetry oscillations is proportional to \( \kappa \), and it has a sharp maximum at small values of \( \cos(2\vartheta_0) \), i.e. when the input has small asymmetry. These features are in good agreement with numerical simulations of equation (15) for the state with vorticities \( (m_1, m_2) = (0, 0) \), see figure 8.
3. 3D simulations

In the framework of the underlying GPE (1) with the external potential taken as per equation (2), the tunneling dynamics has been simulated, assuming values of the physical parameters \( \omega_1 = 2\pi \times 123 \text{ Hz}, \omega_2 = 2\pi \times 600 \text{ Hz}, U_0 = 80/\hbar \omega, \)
\( a = 0.3l, \quad l_c = \sqrt{\hbar/(M \omega)}, \)
\( 1.84 \mu \text{m}, \quad \rho_0 = 19.23 \mu \text{m}, \)
which are appropriate for the trapping pancake-shaped potential shown in Figure 1. The number of atoms is fixed as \( N = 6 \times 10^3 \). Further, we define dimensionless time, \( t = \tau \omega_j \), and coordinates, \( r \rightarrow r/l, \)
casting equation (1) in the scaled form:
\[
i \frac{\partial \Psi}{\partial t} = -\frac{1}{2} \nabla^2 \Psi + \Psi^{\text{ext}} + \tilde{g}|\Psi|^2 \Psi, \tag{26}\]
with \( \tilde{g} \equiv 4\pi a_0/\hbar = 0.0188 \). Numerical solutions of the GPE were used to calculate the superfluid-flow density as
\[
j(r) = \frac{i}{2}[\Psi (r) \nabla \Psi^{*}(r) - \Psi^{*}(r) \nabla \Psi(r)]. \tag{27}\]

Stationary states of the form \( \Psi(r,t) = \Psi(r)e^{-i\mu t} \), with chemical potential \( \mu \), can be found numerically by means of the imaginary-time-propagation method [43]. To obtain a stationary state with different vortex phase profiles in the upper and lower rings, we initiate the evolution in imaginary time with following input:
\[
\Psi(r) = |\Psi_0(x, y, z)|e^{iS(z)\theta}, \tag{28}
\]
where \( \Psi_0(x, y, z) \) is a numerically found solution with zero vorticity in both rings, \( \theta \) is, as above, the polar angle in the cylindrical coordinates, and integer topological charges \( m_1 \) and \( m_2 \) are imprinted with angular momenta carried by the top and bottom rings: \( S(z) = m_1 \) for \( z \geq 0 \) and \( S(z) = m_2 \) for \( z < 0 \), see a similar procedure used with a ‘peanut-shaped’ trapping potential developed in [44].

The dynamics of BEC in real time was simulated by means of the usual split-step fast-Fourier-transform method. In the simulations, a difference between the top and bottom condensates was seeded by a population imbalance in the initial state of the double-ring system. First, we find a stationary state in an asymmetric setting, with the potential of the bottom ring, at \( z = 0 \), biased by a constant term \( U_z > 0 \). Then, at \( t = 0 \) the biased state is quenched by suddenly turning the asymmetry off, \( U_z \rightarrow 0 \), which was followed by real-time simulations of equation (26) at \( t > 0 \). Each value of \( U_z \) introduces a specific value of the chemical-potential difference between the top and bottom condensates. We use \( U_z = 1.1 \hbar \omega \) for the simulations reported below, which is a physically relevant value of the bias. We have also simulated the evolution of asymmetrically quenched inputs in other forms (for example, using a 3D analog of a linearly-tilted double-well potential). It was found that the particular form of the initial asymmetry does not essentially affect the ensuing dynamics of the tunneling flows.

In the experiment, the double-ring system with different angular momenta in its top and bottom parts may appear spontaneously as a result of cooling, with different momenta, \( m_1 \) and \( m_2 \), being frozen into the two rings after the transition into the BEC state. Such hybrid states can also be created in a more controllable way. Indeed, the asymmetry of the density distribution in the top and bottom rings makes it possible to excite the vorticity by applying a stirring laser beam to one
Through the barrier between the rings, $j_f(x, y, z = 0)$, which is defined by equation (27). A set of such snapshots, taken at different moments of time, which are produced by simulations of equation (26), are displayed in figures 9–11. The angular distributions of $j_f(x, y, z = 0)$ in the Josephson junctions patterns of all the types considered here are in excellent agreement with the predictions of the GA, see figures 7 and 9–11.

Thus, the numerical simulations of the GA and full GPE predict the same effects in the superflow dynamics in the double ring: in the non-rotating $(0, 0)$ and co-rotating $(1, 1)$ states one observes generic Josephson oscillations of the total tunneling flow, while for the hybrid counter-rotating states of the $(1, −1)$ and $(-1, 1)$ types, as well as for the semi-vortex ones, with topological charges $(0, 1)$ and $(0, 1)$, the total tunneling flow vanishes, which resembles an effect known for cylindrical superconductive Josephson junctions [14, 15].

To gain a deeper insight into the hybrid dynamical state, produced by the input with different vorticities in the two rings, one may again use the superfluid current (27). To analyze its structure, we substitute ansatz (3) for the 3D wave function, with $\Phi_e = |\Phi_e|e^{-i\mu_1 l}, \Phi_b = |\Phi_b|e^{-i\mu_2 l}$ (wave functions of the stationary states), and $\phi = e^{im_1 \theta}, \bar{\psi} = e^{im_2 \theta}$. The substitution yields

$$J_\phi \sim \sin[(\mu_1 - \mu_2) l + (m_2 - m_1) \theta].$$

This simple relation agrees well with full 3D simulations, see figures 10 and 11, as well as with the GA predictions, see figures 7(b), (c). In particular, it follows from equation (29) that the total inter-ring currents for the states of the $(1, −1)$ and $(0, 1)$ types indeed vanish:

$$J_{\text{total}} = \int_0^{2\pi} j_\phi \, d\theta = 0,$$

as found above from the 3D simulations (figures 6(a), (b)) for $(−1, 1)$ and $(1, 0)$ and GA (figures 5(a), (b)) for $(1, −1)$ and $(0, 1)$.

On the other hand, for the states of the $(m_1, m_2) = (1, 1)$ and $(0, 0)$ types ($\phi = e^{i\theta}, \bar{\psi} = e^{i\theta}$, or $\phi = 1, \bar{\psi} = 1$, respectively in equation (3)), equations (27) and (30) produce a nonvanishing ac Josephson effect: $J_{\text{total}} \sim \sin[(\mu_1 - \mu_2) m]$. This conclusion is again in agreement with the results of both the 3D simulations (figures 3(a), (b) and 9) and the GA prediction (figures 2(a), (b) and 7(a)). Thus we conclude that, in the general case, the states of the type $(m_1, m_2)$ with $m_1 = m_2$ feature the ac Josephson effect, while ones with $m_1 \neq m_2$ produce zero total current.

Figure 9. Snapshots at different moments of time of the tunneling flow $j_f(x, y, z = 0)$ for the state with $(m_1, m_2) = (1, 1)$, as produced by simulations of equation (26).

Figure 10. Snapshots at different moments of time of the tunneling flow $j_f(x, y, z = 0)$ for the state with $(m_1, m_2) = (1, -1)$, as produced by simulations of equation (26).

Figure 11. Snapshots at different moments of time of the tunneling flow $j_f(x, y, z = 0)$ for the state with $(m_1, m_2) = (0, 1)$, as produced by simulations of equation (26).
Finally, angular distributions of the tunneling superflow $j_z$ for hybrid states of the ($-1, 1$) and ($1, 0$) types are displayed in figures 10 and 11. The structure of the tunneling flows suggests formation of a Josephson vortex (fluxon) in the ($1, 0$) state, and of a vortex–antivortex pair in the ($-1, 1$) configuration, see figure 12. For the symmetric unbiased initial state, one indeed observes stationary Josephson vortices. This result is similar to findings reported in [24], in which stationary vortices were obtained in an array of linearly-coupled one-dimensional BEC. However, following the application of the quench, in the biased two-ring system we observe, in figure 13, that fluxon cores rotate and bend. It is easy to see from equation (29) that the fluxon’s azimuthal cycling frequency is determined by the chemical-potential difference. These predictions of equation (29), produced by the GA, are found to be in excellent agreement with an extensive series of numerical simulations for different values of the initial population unbalance.

4. Conclusion and discussion

We have considered the bosonic Josephson junction between two atomic condensates loaded in parallel-coupled ring-shaped traps. The analysis is performed using the truncated model produced by the GA, and through direct systematic simulations of the underlying 3D GPE. The approximation reducing the full GPE to a system of linearly-coupled 1D equations is employed too. These approaches demonstrate that the ac Josephson effect with a uniform angular distribution of the superflow tunneling between the rings can be observed in states with equal vorticities in the parallel-coupled rings, $m_1 = m_2$. In hybrid states with different vorticities ($m_1 \neq m_2$), the total flow of the inter-ring tunneling vanishes, the angular distribution of the tunneling superflow being a periodic function of the angular coordinate with zero average. It is remarkable that the vanishing of the net tunneling superflow is accompanied by the formation of $N_f = |m_1 - m_2|$ Josephson vortices (fluxons) trapped in the Bose–Josephson junction. Detailed consideration of interactions between the fluxons in a regime of strong ring-ring coupling may be a relevant extension of the present work. Another relevant direction for the continuation of the analysis may be consideration of two-layer settings for spinor (two-component) condensates [48, 49].

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