Spin-valley locked topological edge states in a staggered chiral photonic crystal

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Abstract
Engineering pseudo-spin and valley degrees of freedom using quantum spin Hall and valley Hall effects has opened up remarkable possibilities for highly efficient and robust signal transport in time-reversal invariant photonic systems. Here we present a spin-valley locked photonic crystal that has distinct signs of chirality of sublattices in a honeycomb unit cell. We show that the photonic crystal has an insulating bulk dispersion and sublattice-dependent spin-valley coupled gapless edge states by exploiting a coupled dipole method and demonstrate valley-selective propagation by controlling spin state of an external dipole source. The interplay between spin, valley and sublattice shows a judicious way for one-way photon transport by using multiple degrees of freedom.

1. Introduction
Quantized Hall conductance of two-dimensional electron gas under a magnetic field has revealed the presence of a new phase of matter that cannot be explained by our previous understanding [1, 2]. Breaking of time-reversal symmetry (T) underlies the quantum Hall phase [1, 3], which is endowed with topologically protected chiral edge states [4]. The topological phases of matter also appear in T-invariant systems by means of spin [5] and valley [6, 7] degrees of freedom. The quantum spin Hall and valley Hall phases accompany helical edge states, group velocity of which are locked by spin and valley respectively. These topological phases of matter have originally been investigated in condensed matter physics and then found their analogies in other wave systems such as photonics [5, 8–11].

Photons have a spin-like quantity called pseudo-spin, which is referred to as spin hereafter for simplicity. The spin of photons can be defined by relative phase difference between electric and magnetic fields [12] or phase difference of electric field components [13, 14]. Emulation of photon spin by combining electromagnetic duality and bianisotropy paves a way to construct a quantum spin Hall system in photonics [15]. The quantum spin Hall phase manifests itself as spin-locked edge states and nonzero spin Chern number [5]. These spin-polarized edge states are robust under perturbations as long as the spin is not flipped. Meanwhile, a quantum valley Hall phase exhibits valley-locked edge states, which propagate unidirectionally in the absence of valley-flipping [8]. A necessary condition of the quantum valley Hall phase is breaking of inversion symmetry (P). Thus, the quantum valley Hall effect has been observed in a variety of P-broken photonic systems such as graphene [7, 16], valley photonic crystal [17–20], and transition metal dichalcogenide [21, 22].

Recently, topological phases which involve both spin and valley have been reported. Kane–Mele model that resembles a topological circuit network [23], alternatively stacked bilayer silicene [24], Bi or Sb-related structures [25, 26], and transition metal dichalcogenide mono/multi-layer [27–31] have been demonstrated to show a locking of spin and valley degrees of freedom. In photonics, the photonic crystals which have
broken $P$ by using polarization multiplexing [19] and bianisotropic response [32–34] also have shown the possibility of spin and valley dependent light transportation. Here, we present a $T$-invariant staggered chiral photonic crystal in a two-dimensional honeycomb lattice. The chiral property differs from the bianisotropy [35], which couples mutually orthogonal components of electric and magnetic fields, in that parallel components of the electric and magnetic fields are coupled [36]. Different signs of chirality of two sublattices in a unit cell break $P$ and result in a band gap whereas the honeycomb system with the same sign of chirality has Dirac points. We derive photonic Hamiltonian based on a coupled dipole method to calculate the photonic bulk and edge dispersion, in which three degrees of freedom, sublattice, spin and valley, are locked. Lastly, we confirm a unidirectional propagation of edge states in a zigzag-shaped domain wall of two different staggered chiral photonic crystals. The propagation direction of edge states can be controlled by electric and magnetic dipole moment of a point source.

2. Theoretical model and bulk band diagram

We consider a photonic crystal composed of hexagonally arranged particles with a lattice constant $a$ (figure 1(a)). A unit cell contains two particles that have the opposite signs of chirality ($\kappa$ and $-\kappa$, figure 1(b)). For simplicity, we call the sublattice denoted as yellow the first sublattice and the other the second sublattice. We assume that the photonic crystal is in quasi-static regime, namely, the distance between the nearest-neighbor sites is much smaller than wavelength ($a/\sqrt{3} \ll \lambda$). To obtain bulk dispersion of the photonic crystal, we exploit a coupled dipole method [37, 38], which has been used to examine bulk and boundary dispersion of one-dimensional [39, 40] and two-dimensional [41–44] topological lattices. Assuming that the radius of the particle is smaller than the separation [45], each particle can be approximated as a point dipole which has a dipole moment $\mathbf{P} = \left( \mathbf{p} \, \mathbf{m} \right)$ where $\mathbf{p}$ and $\mathbf{m}$ are electric and magnetic dipole moments respectively [37]. Dipole moment of each particle can be expressed in terms of polarizability tensor $\hat{\alpha}$ and electric and magnetic fields as [46]

$$\mathbf{P} = \hat{\alpha} \left( \frac{\mathbf{E}}{\mathbf{H}} \right),$$  

where $\hat{\alpha}$ can be written as

$$\hat{\alpha} = \begin{pmatrix} \hat{\alpha}_{ee} & \hat{\alpha}_{em} \\ \hat{\alpha}_{me} & \hat{\alpha}_{mm} \end{pmatrix}. \tag{2}$$

$\hat{\alpha}_{ee}(\hat{\alpha}_{mm})$ is an electric (magnetic) polarizability tensor of a form

$$\hat{\alpha}_{ee} = \hat{\alpha}_{mm} = \alpha \hat{I}_{3 \times 3} \tag{3}$$

where $\alpha$ is a scalar electric (magnetic) polarizability, and $\hat{I}_{3 \times 3}$ is $3 \times 3$ unity matrix. $\hat{\alpha}_{em}$ and $\hat{\alpha}_{me}$ associate coupling between electric and magnetic components that are parallel to each other as

$$\hat{\alpha}_{em} = -\hat{\alpha}_{me} = i\beta \hat{I}_{3 \times 3} \tag{4}$$

where $\beta$ is a scalar magneto-electric coupling polarizability, and is proportional to $\kappa$. Using the polarizability tensor, the dipole equation can be written self-consistently as

$$\hat{\alpha}^{-1}\mathbf{P}_n = \sum_{m \neq n} G_n^0(\mathbf{R}_n - \mathbf{R}_m)\mathbf{P}_m. \tag{5}$$

Here, the summation runs over the nearest-neighbors, $\mathbf{P}_n$ is a dipole moment of a particle at $\mathbf{R}_n$, and Green function is given as [46]

$$G_n^0(\mathbf{R}) = \frac{1}{ik} \begin{pmatrix} 0 & \nabla \nabla \nabla \\ -ik \nabla \nabla & 0 \end{pmatrix} \frac{\exp(ikR)}{R} \tag{6}$$

where $k$ is a wave vector, and $R$ is the magnitude of $\mathbf{R}$. By choosing $\mathbf{R}_0 = 0$ and using the Bloch theorem, $\mathbf{P}_n = \exp(i\mathbf{k}_l \cdot \mathbf{R}_n)\mathbf{P}$, where $\mathbf{k}_l$ is the in-plane momentum, the eigenvalue problem becomes

$$\hat{\alpha}^{-1}\mathbf{P} = \sum_n G_n^0(\mathbf{R}_n)\exp(-ik_l \cdot \mathbf{R}_n)\mathbf{P}. \tag{7}$$

For simplification, we define two scalars

$$\mu = \frac{\alpha}{\alpha^2 + \beta^2}, \tag{8}$$
Different signs of chirality of two sublattices break degeneracy in the whole BZ and symmetric about and introduces the band gap as confirmed by zero DoS in the band gap. In both cases, all bands are doubly and zero photonic density of states (DoS) at the frequency of the Dirac point (Figure 2(a)). When \( \omega = 0 \), the different signs of chirality of two sublattices break \( P \), which results in Dirac points at \( K \) and \( K' \) and zero photonic density of states (DoS) at the frequency of the Dirac point (Figure 2(a)). When \( \omega \neq 0 \), the different signs of chirality of two sublattices break \( P \) (Figure 2(b)). The breaking of \( P \) lifts the Dirac points and introduces the band gap as confirmed by zero DoS in the band gap. In both cases, all bands are doubly degenerate in the whole BZ and symmetric about \( \omega = 0 \) for reasons explained in Appendix A. Because the lattice is in \( xy \)-plane only, the eigenmodes are fully decoupled into eight in-plane \( (p_x = m_z = 0) \), four out-of-plane \( (p_y = p_z = m_x = m_y = 0) \), and zero markers modes. We only discuss in-plane modes unless otherwise stated. Bulk dispersion in the whole first BZ is shown in Figures 2(c) and (d) for better visualization. The composite Berry curvatures of those bands can be found in Appendix B.

Different signs of chirality of sublattices not only break \( P \), but also provide spin-dependent characteristics of eigenmodes. To understand the bulk eigenmodes, we define \( \Phi_E \) as a phase difference between \( x \) and \( y \) components of electric fields \( \Phi_E = \arg E_x - \arg E_y \) and examine the electric field.

![Figure 1](image) Design of the photonic crystals. (a) Schematic of the staggered chiral photonic crystal and (b) a unit cell. Yellow and green spheres represent a particle with positive \((+\kappa)\) and negative \((-\kappa)\) chirality, respectively. (c) The first BZ and high symmetry lines.

\[
v = \frac{\beta}{\alpha^2 - \beta^2}.
\]

Then \( \hat{\alpha}^{-1} = \hat{u}b_{x\times6} + iv\hat{M} \) for

\[
\hat{M} = \begin{pmatrix}
0_{3\times3} & -I_{3\times3} \\
I_{3\times3} & 0_{3\times3}
\end{pmatrix},
\]

where the opposite signs of the off-diagonal blocks are attributed to the opposite signs of chirality of two sublattices. Now equation (7) can be written in a form of an eigenvalue equation as

\[
u \hat{P} = \left[ \sum_n G^0(\hat{R}_n)e^{-ik_\parallel \hat{R}_n} - iv\hat{M} \right] \hat{P}.
\]

Components of the Green tensor that are responsible to the purely electric or magnetic dipole act on \( \hat{P} \) as

\[
(k^2\hat{I}_{3\times3} + \nabla \nabla) \frac{\epsilon_{ikR}}{R^n} \hat{P} = \frac{\epsilon_{ikR}}{R^n} \left[ (k^2R^2 + ikR - 1)\hat{I}_{3\times3} - (k^2R^2 + 3ikR - 3)\hat{R}_T\hat{R} \right] \hat{P},
\]

where \( \hat{R}_T = \hat{R}/R \) is a unit vector of \( \hat{R} \). Components of the Green tensor associated with the magneto-electric coupling act as

\[-ik\nabla \times \frac{\epsilon_{ikR}}{R^n} \hat{P} = \frac{\epsilon_{ikR}}{R^n} (k^2R^2 + ikR)\hat{R} \times \hat{P}.
\]

Because the wavelength is much smaller than the spacing, we apply the quasi-static approximation \( (kR \ll 1) \). Then the Green function can be simplified as

\[
G^0(\hat{R}) = \frac{1}{R^3} \begin{pmatrix}
-\hat{I}_{3\times3} + 3\hat{R}_T\hat{R} \\
\hat{0}_{3\times3} & -\hat{I}_{3\times3} + 3\hat{R}_T\hat{R}
\end{pmatrix}.
\]

Substitution of equation (14) to equation (11) gives the eigenvalue problem of the photonic crystal. The eigenvalue \( \nu \) is directly associated with the polarizabilities of the particle [equation (8)], and the eigenvector \( \hat{P} \) represents the dipole moment of eigenmode. Material dispersion is taken into account as \( u = (\omega - \omega_0)/A \) by using the first-order Taylor expansion where \( \omega \) is an angular frequency, \( \omega_0 \) is a resonant angular frequency, and \( A \) is a constant. For calculations, we used parameters given as

\[a = 1, A = 0.5 \times 10^7, \omega_0 = 0.5c/a \]

where \( c \) is speed of light in free-space, and \( v = 3 \) for nonzero chirality.

To obtain the bulk dispersion of the photonic crystal, we build an eigenvalue problem by estimating nearest-neighbor interactions and solve it along the high symmetry lines of the first Brillouin zone (BZ) shown in Figure 1(c). We consider three spatial coordinates, two field components (electric and magnetic) and two sublattices, and build 12-dimensional bulk Hamiltonian. Details of the bulk Hamiltonian are presented in Appendix A. Photonic crystal with \( \nu = 0 \) preserves \( P \), which results in Dirac points at \( K \) and \( K' \) and zero photonic density of states (DoS) at the frequency of the Dirac point (Figure 2(a)). When \( \nu \neq 0 \), the different signs of chirality of two sublattices break \( P \) (Figure 2(b)). The breaking of \( P \) lifts the Dirac points and introduces the band gap as confirmed by zero DoS in the band gap. In both cases, all bands are doubly degenerate in the whole BZ and symmetric about \( \omega = 0 \) for reasons explained in Appendix A. Because the lattice is in \( xy \)-plane only, the eigenmodes are fully decoupled into eight in-plane \( (p_x = m_z = 0) \), four out-of-plane \( (p_y = p_z = m_x = m_y = 0) \), and zero markers modes. We only discuss in-plane modes unless otherwise stated. Bulk dispersion in the whole first BZ is shown in Figures 2(c) and (d) for better visualization. The composite Berry curvatures of those bands can be found in Appendix B.
amplitude and $\Phi_E$ of eigenmodes at $K$ for the third ($n = 3$) and fourth ($n = 4$) bands (figure 3). When the particles have no chirality ($\nu = 0$), both sublattices are excited and have different spin states [figure 3(a)]. In contrast, when $\nu \neq 0$, the doublet exhibit radically different field distributions [figure 3(b) and (c)]. For $n = 3$, only the first sublattice is excited and has the negative phase [figure 3(b)]. Eigenmode of $n = 4$ shows the opposite results; field amplitude is localized at the second sublattice and has the positive phase [figure 3(c)]. These field distributions of the eigenmodes indicate that the doubly degenerated bands have spin-locked excitation of sublattices. At the other valley $K'$, the field amplitude appears the same while the sign of $\Phi_E$ is opposite.

3. Spin-valley locking of edge states

The interplay between spin, valley and sublattice appears not only in the bulk modes but also in edge dispersion. We solve an eigenvalue problem of a slab of photonic crystal to obtain the edge dispersion (figure 4(a)). The half of the slab consists of 10 unit cells aligned along $(x/2 + y\sqrt{3}/2)$-direction while the other half has two sublattices positioned reversely. Boundary conditions are set as open along the $x$-axis and periodic along the remaining boundaries. In the band gap, four edge states appear, two of which are
Figure 4. Spin-valley locking in the edge dispersion. (a) Schematic of a domain wall. (b) Edge dispersion of the in-plane modes. Black indicates the projected bulk states, and magenta and blue denote edge states that are localized at a mirror-symmetric interface and open boundaries, respectively. (c) Dipole moments of an edge state localized at the interface when $k_x = 0.7 \pi/a$ and $\omega = 0.48 c/a$. (d) Edge dispersion with spin-dependent color representation. Edge states at $K$ and $K'$ have valley-dependent spin states (up for $K$ and down for $K'$). (e) Edge dispersion of out-of-plane modes, in which the spin-valley locking does not appear.

localized at the open boundaries and have flat dispersions [blue, figure 4(b)] and the other two at the mirror-symmetric interface (magenta). Dipole moment distributions of the latter shows that they are localized at the interface [figure 4(c)]. These edge states originate from the breaking of $P$, which yields the quantum valley Hall phase.

To examine the spin of the edge states, we define spin state as $\Phi_p = \arg(p_x) - \arg(p_y)$. Note that $\Phi_p$ can be defined for two sublattices. $\Phi_p$ of the second sublattice shown in figure 4(d) proves that bulk dispersion is spin-split. Across the band gap, the edge states at the mirror-symmetric interface are also spin-valley locked, showing spin-up at projection of $K$ and spin-down at projection of $K'$. $\Phi_p$ of the other sublattice has the opposite spin-valley locking, spin-down at projection of $K$ and spin-up at projection of $K'$. Thus, the three degrees of freedom, which are spin, valley and sublattice, are all coupled to each other. This spin-valley locking originates from the magneto-electric coupling and is distinct to the photonic analogy of the quantum valley Hall phase observed in pure $P$-broken systems. However, such spin-valley locked characteristics only appear in in-plane modes and are not found in out-of-plane mode [figure 4(e)].

4. Helical edge states in real space

To confirm the spin-dependent unidirectional propagation in real space, we consider a bent domain wall between two staggered chiral photonic crystals [figure 5(a)]. The blue area consists of unit cells in which the first (second) sublattice has positive (negative) $\kappa$ as shown in figure 1(b). Unit cells of the yellow area have the opposite signs of chirality, that is, positive (negative) $\kappa$ for the second (first) sublattice. Gray areas are absorbing media that have an artificial loss given as $v \rightarrow v(1 + i)$. To calculate the dipole moment distributions, we add a source field term $F_{\text{ext}} = [E_{\text{ext}} H_{\text{ext}}]^T$ to the dipole equation [equation (5)]:

$$\hat{\alpha}^{-1}P_n = F_{\text{ext}} + \sum_{m \neq n} G^0(R_n - R_m)P_m.$$  

(15)

By using the Green tensor defined by equations (6), (12) and (13), each component of $F_{\text{ext}}$ can be obtained as

$$E_{\text{ext}} = \frac{e i k R}{R^3} \left[ (k^2 R^2 + ikR - 1)\hat{I}_{3 \times 3} - (k^2 R^2 + 3ikR - 3)\hat{R}^T \hat{R} \right] p_{\text{ext}},$$  

$$H_{\text{ext}} = \frac{e i k R}{R^3} \left[ (k^2 R^2 + ikR - 1)\hat{I}_{3 \times 3} - (k^2 R^2 + 3ikR - 3)\hat{R}^T \hat{R} \right] m_{\text{ext}}.$$  

(16)
Figure 5. Helical edge states in real space. (a) Schematic of a bent domain wall. Blue represents chiral photonic crystals shown in figure 1 and yellow corresponds to its inversion counterpart. Absorbing media (gray) are included to prevent wrapping of helical edge states. 36 × 24 unit cells are used. (b)–(d) Spatial distributions of dipole moments when the dipole moment of a source is \( p^{\text{ext}} = [10] \), \( m^{\text{ext}} = -i p^{\text{ext}} \), (c) \( p^{\text{ext}} = [1 - i]/\sqrt{2} \), \( m^{\text{ext}} = i p^{\text{ext}} \) and (d) \( p^{\text{ext}} = [1i]/\sqrt{2} \), \( m^{\text{ext}} = -i p^{\text{ext}} \).

Note that the off-diagonal components of the Green tensor [equation (13)] have no contribution because \( R \) and \( p^{\text{ext}} (m^{\text{ext}}) \) are in \( xy \)-plane. The dipole field distributions can be calculated as

\[
P = \left( i \hat{l}_{3} + i \varepsilon \hat{M} - \sum_{n} G^{\theta}(R_{n}) \right) F^{\text{ext}}. \tag{17}
\]

The dipole source is placed near the first sublattice in yellow area (denoted as red mark in figure 5). Frequency of the dipole source is set within the band gap as \( \omega = 0.5c/a \). Because the band gap has no allowed mode in bulk dispersion, only edge modes are excited. When the source is linearly polarized \( (p^{\text{ext}} = [10]^{\dagger}, m^{\text{ext}} = -i p^{\text{ext}}) \), which is equivalent to superposition of two opposite circular polarizations, the edge mode propagates bothways along the interface (figure 5(b)). On the other hand, when the source is spin-up \( (\Phi_{p} = \pi) \), the edge states that have spin-up only exist near \( K \) valley and thus, the edge wave propagates only to the right-hand side as shown in figure 5(c). In contrast, the edge mode propagates along the opposite side when the source is spin-down \( (\Phi_{p} = -\pi) \). If the source is positioned near the second sublattice, the propagation direction is reversed because of the spin–flipped dispersion. This real space simulation is consistent with the edge dispersion and proves that the edge modes are spin-valley locked.

While our work is purely numerical and theoretical, we briefly discuss the possibilities of an experimental realization of such a chiral photonic crystal. Colloidal based metamaterial particles that exhibit magneto-electric coupling [47, 48] can be a potential candidate for a sublattice of our system. Especially, the dielectric colloidal metamaterial consisting of a chiral cluster of silicon nanospheres [47] can be a probable choice to satisfy both the chirality and electromagnetic duality, although a more rigorous verification would require extraction of polarizabilities [49]. Arranging those dielectric particles in a honeycomb pattern in a staggered manner may lead to a realization of the spin-valley locked chiral photonic crystal.

5. Conclusion

In conclusion, we present a staggered chiral photonic crystal, in which spin, valley and sublattice degrees of freedom are locked. By assigning the opposite signs of chirality to two sublattices, we show that this \( P \)-broken system has a band gap, which is spanned by sublattice dependent spin-valley locked edge states.

Thus, the staggered chiral photonic crystal exhibits spin-locked excitation of edge states at a mirror-symmetric interface. In addition, we demonstrate direction controllable propagation of edge states in real space by using a dipole source. These spin-valley locking behaviors can be applied in diverse areas such as optical communication and topological spintronics.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.
Acknowledgments

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Appendix A. Bulk Hamiltonian from the coupled dipole method

The bulk Hamiltonian of the photonic crystal given as the expression in the square bracket in equation (11) can be written in a matrix form as

$$H = \begin{pmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{pmatrix} - ivM$$  \hspace{1cm} (A1)

where $\hat{G}_{ij} = G_0 (R_i - R_j) e^{-i k \cdot (R_i - R_j)}$. Considering that only nearest-neighbor couplings are included, $\hat{G}_{11}$ and $\hat{G}_{22}$ are $6 \times 6$ zero matrices, $\hat{G}_{12}$ can be written as

$$\hat{G}_{12} = \begin{pmatrix} G \\ \hat{0}_{3\times3} \\ \hat{G} \end{pmatrix}$$  \hspace{1cm} (A2)

and $\hat{G}_{21} = \hat{G}_{12}^\dagger$ is a Hermitian conjugate of $\hat{G}_{12}$. As can be seen from equation (14), $\hat{G}$ is defined as

$$\hat{G} = \sum_{i=1}^{3} \frac{1}{R_i^6} \left[ -\hat{I}_{3\times3} + 3 \hat{R}_i \hat{R}_i^\dagger \right] e^{-i k \cdot R_i}$$  \hspace{1cm} (A3)

where

$$R_1 = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} a, \quad R_2 = \begin{pmatrix} 0 \\ -1/\sqrt{3} \\ 0 \end{pmatrix} a, \quad R_3 = \begin{pmatrix} 1/2 \\ 1/2 \\ \sqrt{3}/2 \end{pmatrix} a$$  \hspace{1cm} (A4)

are vectors of the three nearest-neighbors in the Cartesian coordinate. Substitution of equation (A4) to equation (A3) gives

$$\hat{G} = \left( \frac{\sqrt{3}}{a} \right)^3 \left[ 2 e^{i k_x a} \begin{pmatrix} 5/4 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \frac{3\sqrt{3}}{4} \sin \frac{k_x a}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]$$

$$+ e^{i k_y a} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$  \hspace{1cm} (A5)

which can be also written as

$$\hat{G} \approx \hat{G}(K) + \frac{\partial \hat{G}}{\partial k} dk$$

$$= \begin{pmatrix} \hat{G}_{xy} & 0 \\ 0 & \hat{G}_{zz} \end{pmatrix}$$  \hspace{1cm} (A6)

where

$$\hat{G}_{xy} = \left( \frac{\sqrt{3}}{a} \right)^3 (A_0 \hat{\sigma}_0 + A_1 \hat{\sigma}_1 + A_3 \hat{\sigma}_3),$$

$$\hat{G}_{zz} = \left( \frac{\sqrt{3}}{a} \right)^3 (\frac{\sqrt{3}}{2} \delta k_x a - i \frac{2}{\sqrt{3}} \delta k_y a - i \frac{1}{2} \delta k_x \delta k_y a^2)$$  \hspace{1cm} (A7)
using the first-order Taylor expansion near $K$. Here $\hat{\sigma}_i$ are Pauli matrices, and $A_i$ are defined as
\[
A_0 = -\frac{\sqrt{3}}{4} \delta k_x a + \frac{i}{\sqrt{3}} \delta k_y a + \frac{i}{4} \delta k_z \delta k_y a^2,
\]
\[
A_1 = -\frac{9}{4} + \frac{3\sqrt{3}}{2} \delta k_y a - \frac{3\sqrt{3}}{4} \delta k_y a - i\frac{3}{8} \delta k_z \delta k_y a^2,
\]
\[
A_3 = -\frac{9}{4} - \frac{3\sqrt{3}}{8} \delta k_y a - \frac{\sqrt{3}}{4} \delta k_y a + i\frac{3\sqrt{3}}{8} \delta k_z \delta k_y a^2.
\]

Bulk Hamiltonian $H$ in equation (A1) can be written in terms of $\hat{G}$ and $\nu$ as
\[
H = \begin{pmatrix}
iv\hat{I}_{3\times3} & -i\nu\hat{I}_{3\times3} \\
\hat{G} & \hat{G} \\
\hat{G} & i\nu\hat{I}_{3\times3}
\end{pmatrix},
\]
and can be block-diagonalized to
\[
\tilde{H} = \begin{pmatrix}
+\hat{Q} & +\hat{Q} \\
-\hat{Q} & -\hat{Q}
\end{pmatrix},
\]
where $\hat{Q}$ is a $3 \times 3$ matrix satisfying $\hat{Q}^2 = \nu^2 \hat{I}_{3\times3} + \hat{G}\hat{G}$. Equation (A10) explains why all bands are doubly degenerate and symmetric to $\omega_0$. Furthermore, $\hat{Q}$ is also block-diagonal as
\[
\hat{Q} = \begin{pmatrix}
\hat{Q}_{11} & \hat{Q}_{12} \\
\hat{Q}_{21} & \hat{Q}_{22}
\end{pmatrix}.
\]

Because $x$ and $y$ components and $z$ component of $\hat{Q}$ are decoupled, in-plane and out-of-plane modes are decoupled. Thus, an in-plane (out-of-plane) polarized input source can excite only the in-plane (out-of-plane) modes selectively while leaving the other modes unaffected.

**Appendix B. Berry curvature and valley Chern number**

In this section, we present the composite Berry curvature and valley Chern number of the staggered chiral photonic crystal. Because all the bands are doubly degenerate, the Berry phase of mutually degenerate $m$ bands $N = n \oplus (n + 1) \oplus \cdots \oplus (n + m - 1)$ over a small loop enclosing four points $k_{ij}, k_{i+1j}, k_{i+1j+1}$ and $k_{ij+1}$ is calculated by using non-Abelian Wilson loop as [50, 51]
\[
\gamma^N_{\theta} = \text{Im} \ln \left[ \det (U^N_{k_{ij} \rightarrow k_{i+1j}} U^N_{k_{i+1j} \rightarrow k_{i+1j+1}} U^N_{k_{i+1j+1} \rightarrow k_{ij+1}} U^N_{k_{ij+1} \rightarrow k_{ij}}) \right],
\]
where $U^N_{k \rightarrow k'}$ is an $N$-rank matrix defined as
\[
(U^N_{k \rightarrow k'})_{ij} = \langle E^n_k | E^n_{k'} \rangle
= \int E^n_k(\mathbf{r}) \cdot \varepsilon(\mathbf{r}) E^n_{k'}(\mathbf{r}) d\mathbf{r}.
\]
$\mathbf{E}^n_k$ is an electric field of the $n$th band at $\mathbf{k}$. The composite Berry curvature $F^N_{\theta}$ is associated with the Berry phase as $F^N_{\theta} = \gamma^N_{\theta} / S$ where $S$ is the area of the closed loop. The composite valley Chern number $C^N_{K}$ can be calculated by integrating $F^N_{\theta}$ over the half BZ around the valley $K$ and dividing by $2\pi$:
\[
C^N_{K} = \frac{1}{2\pi} \int_{BZK} F^N d^2k.
\]
The half BZ around $K$ and $K'$ are indicated as purple and green in figure 6(a).
Figure 6. Berry curvature. (a) Purple (green) area indicates the half BZ centered at K (K'). (b) A map of Berry curvature for (left) \( n = \{1,2,7,8\} \) and (right) \( n = \{3,4,5,6\} \).

Figure 7. Spin-valley locking under resonant material properties. (a) Bulk and (b) surface dispersion calculated by assuming quadratic material dispersion: \( u = (\omega - \omega_0)^2/A \) where \( A = 0.5 \times 10^{14} \) and \( \omega_0 = 0.5c/a \).

Figure 8. Spin-valley locking beyond the quasi-static limit. (a) Bulk dispersion calculated under (black) and beyond (blue) the quasi-static limit when \( \lambda_c = 4\pi a \). (b-d) Surface dispersion calculated beyond the quasi-static limit when (b) \( \lambda_c = 4\pi a \), (c) \( \lambda_c = 2\pi a \), and (d) \( \lambda_c = 4a \).

When \( v \neq 0 \), the staggered chiral photonic crystal has two composite bands, \( 1 \oplus 2 \oplus 3 \oplus 4 \) and \( 5 \oplus 6 \oplus 7 \oplus 8 \) as shown in figure 2(b). Particularly, \( n \oplus (n + 1) \) bands for odd \( n \) are degenerate in the whole BZ while \( 1 \oplus 2 \) and \( 3 \oplus 4 \) (and equivalently \( 5 \oplus 6 \) and \( 7 \oplus 8 \)) are degenerate only at \( \Gamma \). The valley Chern number of our photonic crystal can only be obtained as a composite for bands \( 1 \oplus 2 \oplus 3 \oplus 4 \) and \( 5 \oplus 6 \oplus 7 \oplus 8 \), and are zero for both composites. Nonetheless, we can still examine the local Berry curvature of the \( n \oplus (n + 1) \) bands for odd \( n \) while keeping in mind that Berry curvature at the degeneracy is inaccurate. Although this Berry curvature are not correct at \( \Gamma \), it gives us useful information on how the eigenmodes wind locally in the remaining region.

The Berry curvatures of the \( n \oplus (n + 1) \) bands for odd \( n \) exhibit valley-dependent characteristics [figure 6(b)]; it is positive around \( K \) and negative around \( K' \) for \( 1 \oplus 2 \) and \( 7 \oplus 8 \) bands and opposite for \( 3 \oplus 4 \) and \( 5 \oplus 6 \). Whereas Berry curvatures generally have hot spots at \( K \) and \( K' \) in the pure quantum valley Hall phase, those of the staggered chiral photonic crystal show slightly different features. The Berry curvatures at \( K \) and \( K' \) are negligibly small while they increase near \( \Gamma \) as shown in figure 6(b). This distinct distributions of Berry curvature can be understood as the result of the coupling between spin, valley and...
sublattice degree of freedom; the staggered photonic crystal is not in either pure quantum spin Hall or valley Hall phase, but in a mixed state of those. Lastly, we calculate the Berry curvature of $n \oplus (n + 1)$ for odd $n$ around $K$ by replacing BZ in equation (B3) to BZ$_0 \setminus \{ \Gamma \}$. $C_k^{(n+1)} = 1/2$ for $n = 1$ and $n = 7$ while $C_k^{(n+1)} = -1/2$ for $n = 3$ and $n = 5$.

Appendix C. Spin-valley locking under resonant material properties and beyond the quasi-static limit

The material properties such as electric and magnetic polarizabilities may exhibit resonant behaviors. Thus, to confirm that the spin-valley locking also appears in the resonant regime, we assume quadratic material dispersion characterized by $u = (\omega - \omega_0)^2 / A$ instead of the linear dispersion $u = (\omega - \omega_0) / A$. The quadratic dispersion alters the bulk and surface dispersions but keeps the spin-valley locking (figure 7).

Lastly, we also examine the spin-valley locking beyond the quasi-static limit. In general, particles that strongly interact with light, namely particles in a resonant regime, have dimensions comparable to the wavelength. The center wavelength $\lambda_c = 2\pi c / \omega_0$ is $4\pi a$, which is much larger than the lattice constant. We investigate the bulk and surface dispersion without assuming the quasi-static approximation by using equations (6), (12) and (13) instead of equation (14). The spin-valley locking characteristics remains for several different center wavelengths, $\lambda_c = 4\pi a$, $\lambda_c = 2\pi a$, and $\lambda_c = 4a$ as shown in figure 8.

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