Rigorous derivation of the full primitive equations by scaled Boussinesq equations

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Abstract

The primitive equations of large-scale ocean dynamics form the fundamental model in geophysical flows. It is well-known that the primitive equations can be formally derived by hydrostatic approximation. On the other hand, the mathematically rigorous derivation of the primitive equations without coupling with the temperature is also known. In this paper, we generalize the above result from the mathematical point of view. More precisely, we prove that the scaled Boussinesq equations strongly converge to the full primitive equations as the aspect ratio parameter goes to zero, and the rate of convergence is of the same order as the aspect ratio parameter. The convergence result implies a rigorous justification of the hydrostatic approximation.

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Key Words: Boussinesq equations; Primitive equations; Hydrostatic approximation; Strong convergence

1 Introduction

The primitive equations are considered as the fundamental model in geophysical flows (see, e.g., [32, 33, 36, 39, 40]). For large-scale ocean dynamics, an important feature is that the vertical scale of ocean is much smaller than the horizontal scale, which means that we have to use the hydrostatic approximation to simulate the motion of ocean in the vertical direction. Based on this fact and the high accuracy of hydrostatic approximation, the primitive equations of ocean dynamics can be formally derived from the Boussinesq equations (see [10, 26]).

The rigorous mathematical derivation from the Navier-Stokes equations to the primitive equations was studied first by Azérad-Guillén [11] in a weak sense, then by Li-Titi [28] in a strong sense with error estimates, and finally by Furukawa et al. [14] in a strong sense but under relaxing the regularity on the initial condition. Subsequently, the strong convergence of solutions of the scaled Navier-Stokes equations to the corresponding ones of the primitive equations with only horizontal viscosity was obtained by Li-Titi-Yuan [30]. However, none of the above work has derived the full primitive equations, i.e., the primitive equations with full viscosity and diffusivity, from the mathematical point of view. Therefore, the aim of this paper is to derive the full primitive equations mathematically.

Let \( \Omega_\varepsilon = M \times (-\varepsilon, \varepsilon) \) be a \( \varepsilon \)-dependent domain, where \( M = (0, 1) \times (0, 1) \). Here, \( \varepsilon = H/L \) is called the aspect ratio, measuring the ratio of the vertical and horizontal scales of the motion, which is usually very small. Say, for large-scale ocean circulation, the ratio \( \varepsilon \sim 10^{-3} \ll 1 \).

Denote by \( \nabla_h = (\partial_x, \partial_y) \) the horizontal gradient operator. Then the horizontal Laplacian operator \( \Delta_h \) is given by

\[ \Delta_h = \nabla_h \cdot \nabla_h = \partial_{xx} + \partial_{yy}. \]
Let us consider the anisotropic Boussinesq equations defined on $\Omega_\varepsilon$

$$\begin{cases}
\partial_t u + (u \cdot \nabla)u + \nabla p - \theta k = \mu_h \Delta_h u + \mu_z \partial_z u, \\
\partial_t \theta + u \cdot \nabla \theta = \kappa_h \Delta_h \theta + \kappa_z \partial_z \theta, \\
\nabla \cdot u = 0,
\end{cases} \quad (1.1)$$

where the three dimensional velocity field $u = (v, w) = (v_1, v_2, w)$, the pressure $p$ and temperature $\theta$ are the unknowns. The unit vector $k = (0, 0, 1)$ points to the $z$-direction. $\mu_h$ and $\mu_z$ represent the horizontal and vertical viscosity coefficients respectively, while $\kappa_h$ and $\kappa_z$ represent the horizontal and vertical heat diffusion coefficients respectively.

Firstly, we transform the anisotropic Boussinesq equations (1.1), defined on the $\varepsilon$-dependent domain $\Omega_\varepsilon$, into the scaled Boussinesq equations defined on a fixed domain. To this end, we introduce the following new unknowns with subscript $\varepsilon$

$$u_{\varepsilon} = (v_{\varepsilon}, w_{\varepsilon}), \quad v_{\varepsilon}(x, y, z, t) = v(x, y, \varepsilon z, t),$$

$$w_{\varepsilon}(x, y, z, t) = \frac{1}{\varepsilon} w(x, y, \varepsilon z, t), \quad p_{\varepsilon}(x, y, z, t) = p(x, y, \varepsilon z, t),$$

$$\theta_{\varepsilon}(x, y, z, t) = \varepsilon \theta(x, y, \varepsilon z, t),$$

for any $(x, y, z) \in \Omega =: M \times (-1, 1)$ and for any $t \in (0, \infty)$.

Suppose that $\mu_\varepsilon = \kappa_\varepsilon = 1$ and $\mu_\varepsilon = \kappa_\varepsilon = \varepsilon^2$. Under these scalings, the anisotropic Boussinesq equations (1.1) defined on $\Omega_\varepsilon$ can be written as the following scaled Boussinesq equations

$$\begin{cases}
\partial_t v_{\varepsilon} - \Delta v_{\varepsilon} + (v_{\varepsilon} \cdot \nabla) v_{\varepsilon} + w_{\varepsilon} \partial_z v_{\varepsilon} + \nabla_h p_{\varepsilon} = 0, \\
\varepsilon^2 \left( \partial_t w_{\varepsilon} - \Delta w_{\varepsilon} + v_{\varepsilon} \cdot \nabla_h w_{\varepsilon} + w_{\varepsilon} \partial_z w_{\varepsilon} \right) + \partial_z p_{\varepsilon} - \theta_{\varepsilon} = 0, \\
\partial_t \theta_{\varepsilon} - \Delta \theta_{\varepsilon} + v_{\varepsilon} \cdot \nabla_h \theta_{\varepsilon} + w_{\varepsilon} \partial_z \theta_{\varepsilon} = 0, \\
\nabla_h \cdot v_{\varepsilon} + \partial_z w_{\varepsilon} = 0,
\end{cases} \quad (1.2)$$

defined on the fixed domain $\Omega$.

Next, we supply the scaled Boussinesq equations (1.2) with the following boundary and initial conditions

$$v_{\varepsilon}, w_{\varepsilon}, p_{\varepsilon} \text{ and } \theta_{\varepsilon} \text{ are periodic in } x, y, z, \quad (1.3)$$

$$(v_{\varepsilon}, w_{\varepsilon}, \theta_{\varepsilon})|_{z=0} = (v_0, w_0, \theta_0), \quad (1.4)$$

where $(v_0, w_0, \theta_0)$ is given. In addition, we also equip the system (1.2) with the following symmetry condition

$$v_{\varepsilon}, w_{\varepsilon}, p_{\varepsilon} \text{ and } \theta_{\varepsilon} \text{ are even, odd, even and odd with respect to } z, \text{ respectively.} \quad (1.5)$$

Noting that this symmetry condition is preserved by the scaled Boussinesq equations (1.2), i.e., it holds provided that the initial data satisfies this symmetry condition. Due to this fact, throughout this paper, we always suppose that the initial data satisfies

$$v_0, w_0 \text{ and } \theta_0 \text{ are periodic in } x, y, z, \text{ and are even, odd and odd in } z, \text{ respectively.} \quad (1.6)$$

In this paper, we will not distinguish in notation between spaces of scalar and vector-valued functions, in other words, we will use the same notation to denote both a space itself and its finite product spaces. For simplicity, we denote by notation $\| \cdot \|_p$ the $L^p(\Omega)$ norm.

For the proof of the global existence of weak solutions to the scaled Boussinesq equations (1.2), we refer to the work of Lions-Temam-Wang [26] Part IV. Specifically, for any initial data $(u_0, \theta_0) = (v_0, w_0, \theta_0) \in L^2(\Omega)$, with $\nabla \cdot u_0 = 0$, we can prove that there exists a global weak solution $(v_{\varepsilon}, w_{\varepsilon}, \theta_{\varepsilon})$ of the scaled Boussinesq equations (1.2), subject to boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5). Moreover, by the similar argument as Lions-Temam-Wang [26] Part IV, we can also show that it has a unique local strong solution $(v_{\varepsilon}, w_{\varepsilon}, \theta_{\varepsilon})$ for initial data $(u_0, \theta_0) = (v_0, w_0, \theta_0) \in H^1(\Omega)$, with $\nabla \cdot u_0 = 0$. The weak solutions of the scaled Boussinesq equations (1.2) are defined as follows.
Definition 1.1. Given \((u_0, \theta_0) = (v_0, w_0, \theta_0) \in L^2(\Omega)\), with \(\nabla \cdot u_0 = 0\). We say that a space periodic function \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) is a weak solution of the system \((1.2)\) corresponding to boundary and initial conditions \((1.3) - (1.4)\) and symmetry condition \((1.5)\), if

(i) \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon) \in C([0, \infty); L^2(\Omega)) \cap L^2_{\text{loc}}([0, \infty); H^1(\Omega))\) and

(ii) \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) satisfies the following integral equality

\[
\int_0^\infty \int_\Omega \left\{ (-v_\varepsilon \cdot \partial_t \psi_h - \varepsilon^2 w_\varepsilon \partial_t \varphi_3 - \theta_\varepsilon \partial_t \psi - \theta_\varepsilon \varphi_3) \right. \\
+ \left[ (u_\varepsilon \cdot \nabla) v_\varepsilon \nabla \varphi_h + \varepsilon^2 \nabla w_\varepsilon \cdot \nabla \varphi_3 + \nabla \theta_\varepsilon \cdot \nabla \psi \right] \\
+ \left[ (u_\varepsilon \cdot \nabla) v_\varepsilon \cdot \varphi_h + \varepsilon^2 (u_\varepsilon \cdot \nabla w_\varepsilon) \varphi_3 + (u_\varepsilon \cdot \nabla \theta_\varepsilon) \psi \right] \right\} dx dy dz dt \\
= \int_\Omega \left( v_\varepsilon \varphi_h(0) + \varepsilon^2 w_0 \varphi_3(0) + \theta_0 \varphi(0) \right) dx dy dz,
\]

for any spatially periodic function \((\varphi, \psi) = (\varphi_h, \varphi_3, \psi)\), with \(\varphi_0 = (\varphi_1, \varphi_2)\), such that \(\nabla \cdot \varphi = 0\) and \((\varphi, \psi) \in C^\infty(\Omega \times [0, \infty))\).

Remark 1.1. The initial value \(w_0\) for vertical velocity \(w_\varepsilon\) is uniquely determined by the divergence-free condition and symmetry condition \((1.5)\), which can be represented as

\[
w_0(x, y, z) = -\int_0^z \nabla_h \cdot v_0(x, y, \xi) \, d\xi,
\]

for any \((x, y) \in M\) and \(z \in (-1, 1)\).

Remark 1.2. Similar to the theory of three-dimensional Navier-Stokes equations, e.g., see Temam [38, Ch.III, Remark 4.1] and Robinson et al. [35, Theorem 4.6], we can prove that \((v_\varepsilon, w_\varepsilon)\) and \(\theta_\varepsilon\) satisfy the following energy inequalities

\[
\frac{1}{2} \left( \| v_\varepsilon(t) \|^2 + \varepsilon^2 \| w_\varepsilon(t) \|^2 \right) + \int_0^t \left( \| \nabla v_\varepsilon \|^2 + \varepsilon^2 \| \nabla w_\varepsilon \|^2 \right) ds \\
\leq \frac{1}{2} \left( \| v_0 \|^2 + \varepsilon^2 \| w_0 \|^2 \right) + \int_0^t \int_\Omega \theta_\varepsilon w_\varepsilon \, dx dy dz ds,
\]

and

\[
\frac{1}{2} \| \theta_\varepsilon(t) \|^2 + \int_0^t \| \nabla \theta_\varepsilon \|^2 ds \leq \frac{1}{2} \| \theta_0 \|^2,
\]

respectively, for a.e. \(t \in [0, \infty)\), as long as the weak solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) is obtained by Galerkin method.

When the aspect ratio \(\varepsilon \to 0\), the scaled Boussinesq equations \((1.2)\) formally converges to the full primitive equations

\[
\begin{aligned}
\partial_t v - \Delta v + (v \cdot \nabla_h) v + w \partial_z v + \nabla_h p &= 0, \\
\partial_z p - \theta &= 0, \\
\partial_t \theta - \Delta \theta + v \cdot \nabla_h \theta + w \partial_z \theta &= 0, \\
\nabla_h \cdot v + \partial_z w &= 0,
\end{aligned}
\]

We equip the limiting system \((1.10)\) with the same boundary and initial conditions \((1.3)-(1.4)\) and symmetry condition \((1.5)\) as the system \((1.2)\). In studying the well-posedness of system \((1.10)\), we observe that it is not necessary to give the initial condition for vertical velocity \(w\), since there is no dynamic equation for vertical velocity in the system. So we say that the system \((1.10)\) satisfies the initial condition \((1.4)\) just for the convenience of expression. Base on this fact and Remark \((1.1)\) only the initial conditions of horizontal velocity and temperature are given throughout the paper.
In consequence, the aim of this paper is to prove that the scaled Boussinesq equations (1.2) strongly converge to the full primitive equations (1.10) as the aspectation parameter tends to zero. It is crucial to point out that the global well-posedness of strong solutions to the full primitive equations with Neumann boundary conditions was established by Cao-Titi [10]. This well-posedness result will play an important role in proving that the system (1.2) strongly converge to the system (1.10).

Next we want to recall some results concerning the primitive equations. The global existence of weak solutions of the primitive equations with full viscosity and diffusivity was first given by Lions-Temam-Wang [25–27], but the question of uniqueness to this mathematical model is still unknown except for some special cases [5, 22, 29, 37]. Furthermore, the existence and uniqueness of strong solutions to this mathematical model in different setting are due to Cao-Titi [10], Kobelkov [41], Kukavica-Ziane [26,24], Hieber-Kashiwabara [15], Hieber et al. [17], as well as Giga et al. [15]. Subsequently, the study of the global strong solutions to the primitive equations is naturally carried out in the cases of partial dissipation. More details on these cases can be found in the work of Cao-Titi [11], Fang-Han [13], Li-Yuan [31], and Cao-Li-Titi [9]. However, the inviscid primitive equations with or without rotation is known to be ill-posed in Sobolev spaces, and its smooth solutions may develop singularity in finite time, see Renardy [34], Han-Kwan and Nguyen [16], Ibrahim-Lin-Titi [19], Wong [41], and Cao et al. [4].

Now we are to state the main results of this paper. Firstly, we remark that the global existence of weak solutions to the scaled Boussinesq equations (1.2) basically follows the proof in Lions-Temam-Wang [26, Part IV]. For initial data \((v_0, \theta_0) \in H^1(\Omega)\), it deduces from (1.7) that \((v_0, w_0, \theta_0) \in L^2(\Omega)\), which implies that the system (1.2) has a global weak solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\). For this case, we have the following strong convergence theorem.

**Theorem 1.1.** Given a periodic function pair \((v_0, \theta_0) \in H^1(\Omega)\) such that
\[
\int_{-1}^{1} \nabla_h \cdot v_0(x, y, z) dz = 0, \quad \int_{\Omega} v_0(x, y, z) dxdydz = 0, \quad \text{and} \quad \int_{\Omega} \theta_0(x, y, z) dxdydz = 0.
\]
Suppose that \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) is a global weak solution of the system (1.2), satisfying the energy inequality (1.8) and (1.9), and that \((v, \theta)\) is the unique global strong solution of the system (1.10), with the same boundary and initial conditions (1.5)-(1.6) and symmetry condition (1.7). Let
\[
(V_\varepsilon, W_\varepsilon, \Phi_\varepsilon) = (v_\varepsilon - v, w_\varepsilon - w, \theta_\varepsilon - \theta).
\]
Then, for any \(T > 0\), the following estimate holds
\[
\sup_{0 \leq t \leq T} \left( \frac{\| (V_\varepsilon, W_\varepsilon, \Phi_\varepsilon) \|_2^2}{t} \right) + \int_{0}^{T} \| \nabla (V_\varepsilon, W_\varepsilon, \Phi_\varepsilon) \|_2^2 dt \leq \varepsilon^2 \overline{J}_1(T),
\]
where \(\overline{J}_1(t)\) is a nonnegative continuously increasing function that does not depend on \(\varepsilon\). As a result, we have the following strong convergences
\[
(v_\varepsilon, \varepsilon w_\varepsilon, \theta_\varepsilon) \rightarrow (v, 0, \theta), \quad \text{in} \quad L^\infty([0, T]; L^2(\Omega)),
\]
\[
(\nabla v_\varepsilon, \varepsilon \nabla w_\varepsilon, \nabla \theta_\varepsilon, w_\varepsilon) \rightarrow (\nabla v, 0, \nabla \theta, w), \quad \text{in} \quad L^2([0, T]; L^2(\Omega)),
\]

and the rate of convergence is of the order \(O(\varepsilon)\).

Assume that initial data \((v_0, \theta_0)\) belongs to \(H^2(\Omega)\). Then from (1.7) it follows that \((v_0, w_0, \theta_0)\) belongs to \(H^1(\Omega)\). By the similar argument as Lions-Temam-Wang [26 Part IV], there exists a unique local strong solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) to the system (1.2) with the boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5). In this case, we also have the following strong convergence theorem.
Theorem 1.2. Given a periodic function pair \((v_0, \theta_0) \in H^2(\Omega)\) such that
\[
\int_{-1}^{1} \nabla_h \cdot v_0(x, y, z)dz = 0, \quad \int_{\Omega} v_0(x, y, z)dxdydz = 0, \quad \text{and} \quad \int_{\Omega} \theta_0(x, y, z)dxdydz = 0.
\]
Suppose that \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) is the unique local strong solution of the system \((1.3)\), and that \((v, \theta)\) is the unique global strong solution of the system \((1.10)\), with the same boundary and initial conditions \((1.3)-(1.4)\) and symmetry condition \((1.5)\). Let
\[
(V_\varepsilon, W_\varepsilon, \Phi_\varepsilon) = (v_\varepsilon - v, w_\varepsilon - w, \theta_\varepsilon - \theta).
\]
Then, for any \(T > 0\), there is a small positive constant \(\varepsilon(T) = \frac{2\sqrt{\varepsilon}}{3\sqrt{J_3(T)}}\) such that the system \((4.4)\) (see Section 4, below) has the following estimate
\[
\sup_{0 \leq t \leq T} \left( \| (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|^2_{H^2} \right)(t) + \int_{0}^{T} \| \nabla (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|^2_{H^1} dt \leq \varepsilon^2 J_3(T),
\]
provided that \(\varepsilon \in (0, \varepsilon(T))\), where \(J_3(t)\) is a nonnegative continuously increasing function that does not depend on \(\varepsilon\). As a result, we have the following strong convergences
\[
(v_\varepsilon, w_\varepsilon, \theta_\varepsilon) \rightarrow (v, w, \theta), \quad \text{in} \ L^\infty ([0, T]; H^1(\Omega)),
\]
\[
(\nabla v_\varepsilon, \varepsilon \nabla w_\varepsilon, \nabla \theta_\varepsilon, w_\varepsilon) \rightarrow (\nabla v, 0, \nabla \theta, w), \quad \text{in} \ L^2 ([0, T]; H^1(\Omega)),
\]
\[
w_\varepsilon \rightarrow w, \quad \text{in} \ L^\infty ([0, T]; L^2(\Omega)),
\]
and the rate of convergence is of the order \(O(\varepsilon)\).

Remark 1.3. (i) The convergence result in Theorem 1.1 or 1.2 implies a rigorous justification of hydrostatic approximation. In other words, the full primitive equations \((1.1)\) can be obtained by replacing the vertical momentum equation in the anisotropic Boussinesq equations \((1.1)\) with the hydrostatic equation
\[
\partial_z p - \theta = 0.
\]
(ii) The assumption \(\int_{-1}^{1} \nabla_h \cdot v_0 dz = 0\) is preserved by the hypothesis \((1.6)\) and the divergence-free condition \(\nabla \cdot v_0 = 0\). Moreover, the assumptions \(\int_{\Omega} v_0 dxdydz = 0\) and \(\int_{\Omega} \theta_0 dxdydz = 0\) are to ensure that \(u_\varepsilon, \theta_\varepsilon, u\) and \(\theta\) have integral average zero, so the Poincaré inequality can be conveniently used in the proofs of Theorem 1.1 and 1.2.

The rest of this paper is organized as follows. Some preliminary lemmas that will be used in subsequent sections are collected in Section 2. The proofs of Theorem 1.1 and 1.2 are presented in Section 3 and Section 4, respectively.

2 Preliminaries

For convenience, in this section we present some Ladyzhenskaya-type inequalities in three dimensions for a class of integrals without proving them, which will be frequently used throughout the paper.

Lemma 2.1. \((1.2)\) The following inequalities hold
\[
\int_{M} \left( \int_{-1}^{1} \varphi(x, y, z)dz \right) \left( \int_{-1}^{1} \psi(x, y, z)\phi(x, y, z)dz \right) dxdy \leq C \| \varphi \|^1_2 \left( \| \varphi \|^1_2 + \| \nabla_h \varphi \|^1_2 \right) \| \psi \|^1_2 \left( \| \psi \|^1_2 + \| \nabla_h \psi \|^1_2 \right) \| \phi \|_2,
\]
where \(M\) is the domain of integration.
\[
\int_M \left( \int_{-1}^1 \varphi(x,y,z)dz \right) \left( \int_{-1}^1 \psi(x,y,z)\phi(x,y,z)dz \right) dxdy \\
\leq C \|\psi\|_2^{1/2} \left( \|\phi\|_2^{1/2} + \|\nabla_h \psi\|_2 \right) \|\phi\|_2^{1/2} \left( \|\phi\|_2^{1/2} + \|\nabla_h \phi\|_2 \right) \|\varphi\|_2,
\]
for every \( \varphi, \psi, \phi \) such that the right-hand sides make sense and are finite, where \( C \) is a positive constant.

**Lemma 2.2.** (12) Let \( \varphi = (\varphi_1, \varphi_2, \varphi_3), \psi \) and \( \psi \) be periodic functions in \( \Omega \). Denote by \( \varphi_h = (\varphi_1, \varphi_2) \) the horizontal components of the function \( \varphi \). There exists a positive constant \( C \) such that the following estimate holds
\[
\left| \int_{\Omega} (\varphi \cdot \nabla \psi) \phi dxdydz \right| \leq C \|\nabla \varphi_h\|_2^{1/2} \|\Delta \varphi_h\|_2^{1/2} \|\nabla \psi\|_2^{1/2} \|\Delta \psi\|_2^{1/2} \|\phi\|_2,
\]
provided that \( \varphi \in H^1(\Omega) \), with \( \nabla \cdot \varphi = 0 \) in \( \Omega \), \( \int\varphi dxdydz = 0 \), and \( \varphi_3|_{z=0} = 0 \), \( \nabla \psi \in H^1(\Omega) \) and \( \phi \in L^2(\Omega) \).

## 3 Strong convergence for \( H^1 \) initial data

In this section, assume that initial data \((v_0, \theta_0) \in H^1(\Omega)\), where initial velocity \(v_0\) satisfies
\[
\int_{-1}^1 \nabla_h \cdot v_0(x,y,z)dz = 0, \text{ for all } (x,y) \in M,
\]
we prove that the scaled Boussinesq equations (1.2) strongly converge to the full primitive equations (1.1) as the aspect ration parameter \( \varepsilon \) goes to zero.

Under the assumption of initial data \((v_0, \theta_0) \in H^1(\Omega)\), the global well-posedness of strong solutions to the full primitive equations (1.1) with Neumann boundary conditions was established by Cao-Titi [11]. Consequently, we have the following well-posedness result for the case of periodic boundary conditions.

**Proposition 3.1.** Let \( v_0, \theta_0 \in H^1(\Omega) \) be two periodic functions with \( \int_{-1}^1 \nabla_h \cdot v_0(x,y,z)dz = 0 \) for all \((x,y) \in M\). Then the following assertions hold true:

(i) For any \( T > 0 \), there exists a unique global strong solution \((v, \theta)\) to the full primitive equations (1.1) subject to (1.3)-(1.5) such that
\[
(v, \theta) \in C([0, T]; H^1(\Omega)) \cap L^2([0, T]; H^2(\Omega)), \\
(\partial_t v, \partial_t \theta) \in L^2([0, T]; L^2(\Omega));
\]

(ii) The global strong solution \((v, \theta)\) to the system (1.1) satisfies the following energy estimate
\[
\sup_{0 \leq s \leq t} \left( \|v(s, \theta(s))\|_{H^1(\Omega)}^2 \right) \leq \int_0^t \left( \|v(\nabla \theta)\|_{H^1(\Omega)}^2 + \|\partial_t v, \partial_t \theta\|_2^2 \right) ds \leq \alpha_1(t), \tag{3.1}
\]
for any \( t \in [0, \infty) \), where \( \alpha_1(t) \) is a nonnegative continuously increasing function.

**Proposition 3.2.** Given a periodic function pair \((v_0, \theta_0) \in H^1(\Omega)\) with
\[
\int_{-1}^1 \nabla_h \cdot v_0 dz = 0 \text{ and } w_0(x, y, z) = -\int_0^z \nabla_h \cdot v_0(x, y, \xi)d\xi.
\]
Suppose that \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) is a global weak solution of the system (1.4), satisfying the energy inequality (1.8) and (1.9), and that \((v, \theta)\) is the unique global strong solution of the system (1.1). Then the following integral equality holds
\[
\left( \int_{\Omega} (v \cdot v + \varepsilon^2 w_\varepsilon w + \theta_\varepsilon \theta) dxdydz \right) (r) = \frac{\varepsilon^2}{2} \|w(r)\|_2^2
\]
for any \( r \in [0, \infty) \).

The Proposition 3.2 is formally obtained by testing the system (1.12) with \((v, w, \theta)\), while the rigorous proof for this proposition is due to the similar argument in Li-Titi [28] and Bardos et al. [2]. With the help of this proposition, we can estimate the difference function \((V, W, \Phi)\).

**Proposition 3.3.** Let \((V, W, \Phi) = (v - v, w - w, \theta - \theta)\). Under the same assumptions as in Proposition 3.2, the following estimate holds

\[
\sup_{0 \leq s \leq t} \left( \left\| (V, \varepsilon W, \Phi) \right\|_2^2 (s) + \int_0^t \| \nabla (V, \varepsilon W, \Phi) \|_2^2 ds \right) \leq \varepsilon^2 J_1(t),
\]

for any \( t \in [0, \infty) \), where

\[
J_1(t) = C e^{C(t + \alpha_1^2(t))} \left[ \alpha_1(t) + \alpha_2(t) + \left( \| v_0 \|_2^2 + \varepsilon^2 \| w_0 \|_2^2 + t \| \theta_0 \|_2^2 \right)^2 \right].
\]

Here \( C \) is a positive constant that does not depend on \( \varepsilon \).

**Proof.** Firstly, we multiply the first three equation in system (1.10) by \( v, w, \theta \), respectively, integrate over \( \Omega \times (0, r) \), and then integrate by parts to obtain

\[
\int_0^r \int_\Omega (v \cdot \partial_t v + \theta \cdot \partial_t \theta + \nabla v : \nabla v + \nabla \theta \cdot \nabla \theta) dxdydzdt
= \int_0^r \int_\Omega \left[ \partial_t v - (u \cdot \nabla) v \cdot v_e - (u \cdot \nabla \theta) \theta \right] dxdydzdt,
\]

(3.3)

note that the resultants have been added up. Next, replacing \((v, w, \theta)\) with \((v, w, \theta)\), a similar argument yields

\[
\frac{1}{2} \left( \| v(r) \|_2^2 + \| \theta(r) \|_2^2 \right) + \int_0^r \left( \| \nabla v \|_2^2 + \| \nabla \theta \|_2^2 \right) dt
= \frac{1}{2} \left( \| v_0 \|_2^2 + \| \theta_0 \|_2^2 \right) + \int_0^r \theta w dxdydzdt.
\]

(3.4)

According to Remark 1.2, the following energy inequalities hold

\[
\frac{1}{2} \left( \| v_e(r) \|_2^2 + \varepsilon^2 \| w_e(r) \|_2^2 \right) + \int_0^r \left( \| \nabla v_e \|_2^2 + \varepsilon^2 \| \nabla w_e \|_2^2 \right) dt
\leq \frac{1}{2} \left( \| v_0 \|_2^2 + \varepsilon^2 \| w_0 \|_2^2 \right) + \int_0^r \theta w_e dxdydzdt.
\]

(3.5)

and

\[
\frac{1}{2} \| \theta_e(r) \|_2^2 + \int_0^r \| \nabla \theta_e \|_2^2 dt \leq \frac{1}{2} \| \theta_0 \|_2^2.
\]

(3.6)
By the Hölder inequality and Young inequality, it deduces from (3.10) that
\[
\|v_{\varepsilon}(r)\|_2^2 + \varepsilon^2 \|w_{\varepsilon}(r)\|_2^2 + \int_0^r \left( \|\nabla v_{\varepsilon}\|_2^2 + \varepsilon^2 \|\nabla w_{\varepsilon}\|_2^2 \right) dt \\
\leq C \left( \|v_0\|_2^2 + \varepsilon^2 \|w_0\|_2^2 + r \|\theta_0\|_2^2 \right),
\]
where the divergence-free condition is used. Adding (3.5) and (3.6) we obtain
\[
\frac{1}{2} \left( \|v_{\varepsilon}(r)\|_2^2 + \varepsilon^2 \|w_{\varepsilon}(r)\|_2^2 + \|\theta_{\varepsilon}(r)\|_2^2 \right) \\
+ \int_0^r \left( \|\nabla v_{\varepsilon}\|_2^2 + \varepsilon^2 \|\nabla w_{\varepsilon}\|_2^2 + \|\nabla \theta_{\varepsilon}\|_2^2 \right) dt \\
\leq \frac{1}{2} \left( \|v_0\|_2^2 + \varepsilon^2 \|w_0\|_2^2 + \|\theta_0\|_2^2 \right) + \int_0^r \int_\Omega \varepsilon w_{\varepsilon} dxdydzdt.
\]

Now, we subtract the sum of (3.2) and (3.3) from the sum of (3.4) and (3.8) to write
\[
\frac{1}{2} \left( \|V_{\varepsilon}(r)\|_2^2 + \varepsilon^2 \|W_{\varepsilon}(r)\|_2^2 + \|\Phi_{\varepsilon}(r)\|_2^2 \right) \\
+ \int_0^r \left( \|\nabla V_{\varepsilon}\|_2^2 + \varepsilon^2 \|\nabla W_{\varepsilon}\|_2^2 + \|\nabla \Phi_{\varepsilon}\|_2^2 \right) dt \\
\leq \int_0^r \int_\Omega \left[ (u_{\varepsilon} \cdot \nabla \theta_{\varepsilon}) \theta + (u \cdot \nabla) \theta_{\varepsilon} + \Phi_{\varepsilon} W_{\varepsilon} \right] dxdydzdt \\
+ \int_0^r \int_\Omega \left[ [(u_{\varepsilon} \cdot \nabla)v_{\varepsilon} \cdot v + (u \cdot \nabla)v \cdot v_{\varepsilon}] dxdydzdt \\
+ \varepsilon^2 \int_0^r \int_\Omega \left[ (v_{\varepsilon} \cdot \nabla_h w_{\varepsilon}) w + w_{\varepsilon}(\partial_{\varepsilon} w_{\varepsilon}) w \right] dxdydzdt \\
+ \varepsilon^2 \int_0^r \int_\Omega \left[ - \left( \int_0^r \partial_t v \xi \right) \cdot \nabla_h W_{\varepsilon} - \nabla w \cdot \nabla W_{\varepsilon} \right] dxdydzdt \\
=: R_1 + R_2 + R_3 + R_4.
\]

In order to estimate the first integral term $R_1$ on the right-hand side of (3.9), we use the divergence-free condition and integration by parts to obtain
\[
R_1 := \int_0^r \int_\Omega \left[ (u_{\varepsilon} \cdot \nabla \theta_{\varepsilon}) \theta + (u \cdot \nabla) \theta_{\varepsilon} + \Phi_{\varepsilon} W_{\varepsilon} \right] dxdydzdt \\
= \int_0^r \int_\Omega \left[ (u_{\varepsilon} \cdot \nabla \theta_{\varepsilon}) \theta - (u \cdot \nabla) \theta_{\varepsilon} + \Phi_{\varepsilon} W_{\varepsilon} \right] dxdydzdt \\
= \int_0^r \int_\Omega \left[ (\nabla u_{\varepsilon} - u) \cdot \nabla \Phi_{\varepsilon} \right] \theta + \Phi_{\varepsilon} W_{\varepsilon} \right] dxdydzdt \\
= \int_0^r \int_\Omega \left[ (V_{\varepsilon} \cdot \nabla_h \Phi_{\varepsilon}) \theta + W_{\varepsilon} (\partial_{\varepsilon} \Phi_{\varepsilon}) \theta + \Phi_{\varepsilon} W_{\varepsilon} \right] dxdydzdt \\
=: R_{11} + R_{12} + R_{13}.
\]

Now we need to estimate the integral terms $R_{11}$, $R_{12}$ and $R_{13}$ on the right-hand side of (3.10). For the first integral term $R_{11}$, using the Hölder inequality, Sobolev embedding and Young inequality yields
\[
R_{11} := \int_0^r \int_\Omega \left( V_{\varepsilon} \cdot \nabla_h \Phi_{\varepsilon} \right) \theta dxdydzdt \\
\leq \int_0^r \|V_{\varepsilon}\|_3 \|\nabla_h \Phi_{\varepsilon}\|_2 \|\theta\|_6 dt.
\]
\[ \leq C \int_0^r \|V_\varepsilon\|_{L^2}^{1/2} \|\nabla V_\varepsilon\|_{L^2}^{1/2} \|\nabla h \cdot \Phi_\varepsilon\|_{L^2} \|\nabla \theta\|_{L^2} \, dt \]
\[ \leq C \int_0^r \|V_\varepsilon\|_{L^2}^2 \|\nabla \theta\|_{L^2}^2 \, dt + \frac{1}{24} \int_0^r \left( \|\nabla V_\varepsilon\|_{L^2}^2 + \|\nabla \Phi_\varepsilon\|_{L^2}^2 \right) \, dt. \]

For the second integral term \(R_{12}\), by the Lemma 2.1 from the Hölder inequality, Sobolev embedding, Poincaré inequality and Young inequality, it follows that

\[
R_{12} := \int_0^r \int_\Omega W_\varepsilon (\partial_x \Phi_\varepsilon \cdot \theta) \, dx dy dz dt
= \int_0^r \int_\Omega \left[ - (\partial_x W_\varepsilon) \Phi_\varepsilon \cdot \theta - W_\varepsilon \Phi_\varepsilon \partial_x \theta \right] \, dx dy dz dt
\leq C \int_0^r \|\nabla W_\varepsilon\|_{L^2} \|\Phi_\varepsilon\|_{L^2}^{1/2} \|\nabla \Phi_\varepsilon\|_{L^2}^{1/2} \|\nabla \theta\|_{L^2} \, dt
+ C \int_0^r \|\nabla W_\varepsilon\|_{L^2} \|\Phi_\varepsilon\|_{L^2}^{1/2} \|\nabla \Phi_\varepsilon\|_{L^2}^{1/2} \|\nabla \theta\|_{L^2} \, dt
\leq C \int_0^r \|\Phi_\varepsilon\|_{L^2}^2 \|\nabla \theta\|_{L^2}^2 \|\Delta \theta\|_{L^2}^2 \, dt + \frac{1}{24} \int_0^r \left( \|\nabla V_\varepsilon\|_{L^2}^2 + \|\nabla \Phi_\varepsilon\|_{L^2}^2 \right) \, dt,
\]

where the divergence-free condition and integration by parts are used. Finally, it remains to deal with the last integral term \(R_{13}\). Thanks to the Hölder inequality and Young inequality we reach

\[
R_{13} := \int_0^r \int_\Omega \Phi_\varepsilon W_\varepsilon \, dx dy dz dt
= \int_0^r \int_\Omega \Phi_\varepsilon \left( - \int_0^z \nabla h \cdot V_\varepsilon (x, y, \xi, t) \, d\xi \right) \, dx dy dz dt
\leq \int_0^r \int_M \left( \int_{-1}^1 |\Phi_\varepsilon| \, dz \right) \left( \int_{-1}^1 |\nabla h V_\varepsilon| \, dz \right) \, dx dy dz dt
\leq C \int_0^r \|\Phi_\varepsilon\|_{L^2}^2 \, dt + \frac{1}{24} \int_0^r \|\nabla V_\varepsilon\|_{L^2}^2 \, dt.
\]

Combining the estimates for \(R_{11}, R_{12}\) and \(R_{13}\) we have

\[
R_1 \leq C \int_0^r \left[ \|\Phi_\varepsilon\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \|\Delta \theta\|_{L^2}^2 \left( \|V_\varepsilon\|_{L^2}^2 + \|\Phi_\varepsilon\|_{L^2}^2 \right) \right] \, dt
+ \frac{1}{8} \int_0^r \left( \|\nabla V_\varepsilon\|_{L^2}^2 + \|\nabla \Phi_\varepsilon\|_{L^2}^2 \right) \, dt. \tag{3.11}
\]

Using the similar method as the first integral term \(R_1\) on the right-hand side of (3.9), the integral terms \(R_2\) and \(R_3\) can be bounded as

\[
R_2 := \int_0^r \int_\Omega \left[ (u_\varepsilon \cdot \nabla) v_\varepsilon \cdot v + (u \cdot \nabla) v \cdot v_\varepsilon \right] \, dx dy dz dt
\leq C \int_0^r \|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 \|V_\varepsilon\|_{L^2}^2 \, dt + \frac{1}{8} \int_0^r \|\nabla V_\varepsilon\|_{L^2}^2 \, dt \tag{3.12}
\]

and

\[
R_3 := \varepsilon^2 \int_0^r \int_\Omega \left[ (v_\varepsilon \cdot \nabla h w_\varepsilon) w + w_\varepsilon (\partial_x w_\varepsilon) w \right] \, dx dy dz dt
= \varepsilon^2 \int_0^r \int_\Omega \left[ w_\varepsilon \nabla h \cdot V_\varepsilon \int_0^z \nabla h \cdot v d\xi - v_\varepsilon \cdot \nabla h W_\varepsilon \int_0^z \nabla h \cdot v d\xi \right] \, dx dy dz dt
\leq C \varepsilon^2 \int_0^r \left( \|v_\varepsilon\|_{L^2}^2 \|\nabla v_\varepsilon\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + \varepsilon^4 \|w_\varepsilon\|_{L^2}^2 \|\nabla w_\varepsilon\|_{L^2}^2 \right) \, dt.
\]
Proof of Theorem 1.1.\footnote{Based on Proposition 3.3, we give the proof of Theorem 1.1.}

follows from (3.1) and (3.7) that

\[ G \leq \frac{1}{8} \int_{0}^{\tau} \left( \| \nabla V_{\varepsilon} \|_{2}^{2} + \varepsilon^{2} \| \nabla W_{\varepsilon} \|_{2}^{2} \right) dt, \quad (3.13) \]

respectively. By virtue of the Hölder inequality and Young inequality, we obtain

\[ R_{4} = \varepsilon^{2} \int_{0}^{\tau} \int_{Q} \left[ - \left( \int_{0}^{\tau} \nabla \cdot \left( v \partial_{t} \varepsilon w \right) \right) \cdot \nabla w_{\varepsilon} - \nabla \cdot \varepsilon w \cdot \nabla W_{\varepsilon} \right] dx dy dz dt \]

\[ \leq C \varepsilon^{2} \int_{0}^{\tau} \left( \| \partial_{t} v \|_{2}^{2} + \| \nabla w_{\varepsilon} \|_{2}^{2} + \| \partial_{x} w \|_{2}^{2} \right) dt + \frac{1}{8} \int_{0}^{\tau} \varepsilon^{2} \| \nabla W_{\varepsilon} \|_{2}^{2} dt \]

\[ \leq C \varepsilon^{2} \int_{0}^{\tau} \left( \| \partial_{t} v \|_{2}^{2} + \| \Delta v \|_{2}^{2} \right) dt + \frac{1}{8} \int_{0}^{\tau} \varepsilon^{2} \| \nabla W_{\varepsilon} \|_{2}^{2} dt, \quad (3.14) \]

Substituting (3.11) - (3.14) into (3.9) yields

\[ g(t) = \left( \| (V_{\varepsilon}, \Phi_{\varepsilon}) \|_{2}^{2} + \varepsilon^{2} \| W_{\varepsilon} \|_{2}^{2} \right) (t) + \int_{0}^{t} \left( \| \nabla (V_{\varepsilon}, \Phi_{\varepsilon}) \|_{2}^{2} + \varepsilon^{2} \| \nabla W_{\varepsilon} \|_{2}^{2} \right) ds \]

\[ \leq C \int_{0}^{t} \left[ \| \Phi_{\varepsilon} \|_{2}^{2} + \| \nabla \theta \|_{2}^{2} \| \Delta \theta \|_{2}^{2} \left( \| V_{\varepsilon} \|_{2}^{2} + \| \Phi_{\varepsilon} \|_{2}^{2} \right) \right] ds \]

\[ + C \int_{0}^{t} \| \nabla v \|_{2}^{2} \| \Delta \varepsilon \|_{2}^{2} \| V_{\varepsilon} \|_{2}^{2} ds + C \varepsilon^{2} \left( \| \partial_{t} v \|_{2}^{2} + \| \Delta v \|_{2}^{2} \right) \]

\[ + C \varepsilon^{2} \left( \| v_{\varepsilon} \|_{2}^{2} \| \nabla v_{\varepsilon} \|_{2}^{2} + \| \nabla \varepsilon \|_{2}^{2} \| \Delta v_{\varepsilon} \|_{2}^{2} + \varepsilon^{4} \| w_{\varepsilon} \|_{2}^{2} \| \nabla w_{\varepsilon} \|_{2}^{2} \right) \]

\[ \leq C \left( \| \nabla v \|_{2}^{2} + \| \Delta v \|_{2}^{2} + \| \nabla \theta \|_{2}^{2} + \| \Delta \theta \|_{2}^{2} \right) G(t) + C \varepsilon^{2} \left( \| \partial_{t} v \|_{2}^{2} + \| \Delta v \|_{2}^{2} \right) \]

\[ + C \varepsilon^{2} \left( \| v_{\varepsilon} \|_{2}^{2} \| \nabla v_{\varepsilon} \|_{2}^{2} + \| \nabla \varepsilon \|_{2}^{2} \| \Delta v_{\varepsilon} \|_{2}^{2} + \varepsilon^{4} \| w_{\varepsilon} \|_{2}^{2} \| \nabla w_{\varepsilon} \|_{2}^{2} \right). \]

Noting that the fact that \( G(0) = 0 \), and applying the Gronwall inequality to the above inequality, it follows from (3.11) and (3.14) that

\[ g(t) \leq C \varepsilon^{2} \exp \left\{ C \int_{0}^{t} \left( 1 + \| \nabla v \|_{2}^{2} + \| \Delta v \|_{2}^{2} + \| \nabla \theta \|_{2}^{2} + \| \Delta \theta \|_{2}^{2} \right) \right\} \]

\[ \times \left[ \int_{0}^{t} \left( \| \partial_{t} v \|_{2}^{2} + \| \Delta v \|_{2}^{2} + \| v_{\varepsilon} \|_{2}^{2} \| \nabla v_{\varepsilon} \|_{2}^{2} \right) ds \]

\[ + \int_{0}^{t} \left( \| \nabla v \|_{2}^{2} \| \Delta v \|_{2}^{2} + \varepsilon^{4} \| w_{\varepsilon} \|_{2}^{2} \| \nabla w_{\varepsilon} \|_{2}^{2} \right) ds \right] \]

\[ \leq C \varepsilon^{2} e^{C(t+\alpha_{1}(t))} \left[ \alpha_{1}(t) + \alpha_{2}(t) + \left( \| \nabla v \|_{2}^{2} + \| \Delta v \|_{2}^{2} + \| \nabla \theta \|_{2}^{2} + \| \Delta \theta \|_{2}^{2} \right)^{2} \right]. \]

This completes the proof. \( \square \)

Based on Proposition 3.3, we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** For any \( T > 0 \), thanks to Proposition 3.3, we have the following estimate

\[ \sup_{0 \leq t \leq T} \left( \| (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon}) \|_{2}^{2} \right)(t) + \int_{0}^{T} \| \nabla (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon}) \|_{2}^{2} dt \leq \varepsilon^{2} \mathcal{J}_{1}(T), \]

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Here $C$ is a positive constant that does not depend on $\varepsilon$. The above estimate implies that

$$
\begin{align*}
(v_\varepsilon, \varepsilon w_\varepsilon, \theta_\varepsilon) &\rightarrow (v, 0, \theta), \text{ in } L^\infty ([0, T]; L^2(\Omega)), \\
(\nabla v_\varepsilon, \varepsilon \nabla w_\varepsilon, \nabla \theta_\varepsilon) &\rightarrow (\nabla v, 0, \nabla \theta), \text{ in } L^2 ([0, T]; L^2(\Omega)).
\end{align*}
$$

By virtue of the divergence-free condition, $\nabla v_\varepsilon \rightarrow \nabla v$ in $L^2 ([0, T]; L^2(\Omega))$ yields

$$w_\varepsilon \rightarrow w \text{ in } L^2 ([0, T]; L^2(\Omega)).$$

Obviously, the rate of convergence is of the order $O(\varepsilon)$. The theorem is thus proved.

\[ \square \]

4 \textbf{Strong convergence for }$H^2$\textbf{ initial data}

In this section, assume that initial data $(v_0, \theta_0) \in H^2(\Omega)$ with

$$
\int_{-1}^1 \nabla_h \cdot v_0(x, y, z)dz = 0, \text{ for all } (x, y) \in M,
$$

we show that the scaled Boussinesq equations (1.2) strongly converge to the full primitive equations (1.10) as the aspect ratio parameter $\varepsilon$ goes to zero.

For the case of initial data $(v_0, \theta_0) \in H^2(\Omega)$, there is a unique local strong solution $(v_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ to the system (1.2), corresponding to the boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5). Denote by $T^*_\varepsilon$ the maximal existence time of this local strong solution.

Let

$$(U_\varepsilon, \Phi_\varepsilon, P_\varepsilon) = (V_\varepsilon, W_\varepsilon, \Phi_\varepsilon, P_\varepsilon),$$

$$(V_\varepsilon, W_\varepsilon, \Phi_\varepsilon, P_\varepsilon) = (v_\varepsilon - v, w_\varepsilon - w, \theta_\varepsilon - \theta, p_\varepsilon - p).$$

We subtract the system (1.10) from the system (1.2) and then lead to the following system

$$
\begin{align*}
\partial_t V_\varepsilon - \Delta V_\varepsilon + (U_\varepsilon \cdot \nabla) V_\varepsilon + (u \cdot \nabla) V_\varepsilon + (U_\varepsilon \cdot \nabla) v + \nabla_h P_\varepsilon &= 0, \\
\varepsilon^2 (\partial_t W_\varepsilon - \Delta W_\varepsilon + U_\varepsilon \cdot \nabla W_\varepsilon + U_\varepsilon \cdot \nabla w + u \cdot \nabla W_\varepsilon) + \partial_z P_\varepsilon &= 0, \\
- \Phi_\varepsilon + \varepsilon^2 (\partial_t w - \Delta w + u \cdot \nabla u) &= 0, \\
\partial_t \Phi_\varepsilon - \Delta \Phi_\varepsilon + U_\varepsilon \cdot \nabla \Phi_\varepsilon + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Phi_\varepsilon &= 0, \\
\nabla_h \cdot V_\varepsilon + \partial_z W_\varepsilon &= 0,
\end{align*}
$$

defined on $\Omega \times (0, T^*_\varepsilon)$.

The following proposition is a direct consequence of Proposition 3.3. Moreover, the basic energy estimate on the system (1.1)-(4.4) can also be obtained by the energy method.

**Proposition 4.1.** Suppose that $(v_0, \theta_0) \in H^2(\Omega)$, with $\int_{-1}^1 \nabla_h \cdot v_0 dz = 0$. Then the system (1.1)-(4.4) has the following basic energy estimate

$$
\sup_{0 \leq s \leq t} \left( \frac{\| (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|_2^2}{s} \right) + \int_0^t \frac{\| \nabla (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|_2^2}{ds} \leq C J_1(t),
$$

for any $t \in [0, T^*_\varepsilon)$, where

$$
J_1(t) = C e^{C(t + \alpha_1(t))} \left[ \alpha_1(t) + \alpha_2^2(t) + \left( ||v_0||_2^2 + \varepsilon^2 ||w_0||_2^2 + t ||\theta_0||_2^2 \right)^2 \right].
$$

Here $C$ is a positive constant that does not depend on $\varepsilon$. 

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Proof. Integrating the second equation in system (1.10) with respect to \( \mathbf{z} \) for any \( t \) and symmetry condition subject to boundary and initial conditions divergence-free condition, we can reformulate the system (1.10) as note that we have used the following fact that \( \int_0^1 \nabla \cdot v_0 dz = 0 \). Based on the above relation and divergence-free condition, we can reformulate the system (1.10) as

\[
\sup_{0 \leq s \leq t} \left( \| \Delta v \|_2^2 + \| \Delta \theta \|_2^2 \right) (s) + \int_0^t \left( \| \nabla \partial_t v \|_2^2 + \| \nabla \Delta v \|_2^2 + \| \nabla \partial_t \theta \|_2^2 + \| \nabla \Delta \theta \|_2^2 \right) ds \leq \alpha_2(t),
\]

for any \( t \in [0, \infty) \), where

\[
\alpha_2(t) = e^{C(t)} \left( \| v_0 \|_{H^2}^2 + \| \theta_0 \|_{H^2}^2 \right).
\]

Proposition 4.2. Suppose that \((v_0, \theta_0) \in H^2(\Omega)\), with \( \int_0^1 \nabla \cdot v_0 dz = 0 \). Then the system (1.10) has the following second order energy estimate

\[
\sup_{0 \leq s \leq t} \left( \| \Delta v \|_2^2 + \| \Delta \theta \|_2^2 \right) (s) + \int_0^t \left( \| \nabla \partial_t v \|_2^2 + \| \nabla \Delta v \|_2^2 + \| \nabla \partial_t \theta \|_2^2 + \| \nabla \Delta \theta \|_2^2 \right) ds \leq \alpha_2(t),
\]

Taking the \( L^2(\Omega) \) inner product of the equation (4.5) and (4.6) with \( \Delta (\Delta v - \partial_t v) \) and \( \Delta (\Delta \theta - \partial_t \theta) \) respectively, and integrating by parts, we have

\[
\begin{align*}
\frac{d}{dt} \left( \| \Delta v \|_2^2 + \| \Delta \theta \|_2^2 \right) + \left( \| \nabla \Delta v \|_2^2 + \| \nabla \Delta \theta \|_2^2 + \| \nabla \partial_t v \|_2^2 + \| \nabla \partial_t \theta \|_2^2 \right) & = \int_{\Omega} \nabla \left[ (v \cdot \nabla_h \theta - (\int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi) \partial_z \theta) \right] : \nabla (\Delta \theta - \partial_t \theta) dxdydz \\
& + \int_{\Omega} \nabla \left[ (v \cdot \nabla_h) v - (\int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi) \partial_z v \right] : \nabla (\Delta v - \partial_t v) dxdydz \\
& + \int_{\Omega} \nabla \left[ \int_0^z \nabla_h \theta(x, y, \xi, t) d\xi \right] : \nabla (\Delta v - \partial_t v) dxdydz \\
& =: R_1 + R_2 + R_3,
\end{align*}
\]

note that we have used the following fact that

\[
\int_{\Omega} \nabla h_p \cdot (v - \partial_t v) dxdydz = 0.
\]
We firstly estimate the integral term $R_1$ on the right-hand side of (4.7). Using the Lemma 2.2 and Young inequality gives

$$R_1 := \int_\Omega \nabla \left[ v \cdot \nabla_h \theta - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \partial_z \theta \right] \cdot \nabla (\Delta \theta - \partial_t \theta) \, dx \, dy \, dz$$

$$= \int_\Omega \nabla (u \cdot \nabla \theta) \cdot \nabla (\Delta \theta - \partial_t \theta) \, dx \, dy \, dz$$

$$= \int_\Omega (\partial_t u \cdot \nabla \theta + u \cdot \partial_t \nabla \theta) (\partial_t \Delta \theta - \partial_t \partial_t \theta) \, dx \, dy \, dz$$

$$\leq C \| \partial_t \nabla v \|_2^{1/2} \| \partial_t \Delta v \|_2^{1/2} \| \nabla \theta \|_2^{1/2} \| \Delta \theta \|_2^{1/2} (\| \partial_t \partial_t \theta \|_2 + \| \partial_t \Delta \theta \|_2)$$

$$+ C \| \nabla v \|_2^{1/2} \| \Delta v \|_2^{1/2} \| \nabla \theta \|_2^{1/2} \| \Delta \theta \|_2^{1/2} (\| \partial_t \partial_t \theta \|_2 + \| \partial_t \Delta \theta \|_2)$$

$$\leq C \left( \| \nabla \theta \|_2^2 \| \Delta \theta \|_2^2 + \| \nabla v \|_2^2 \| \Delta v \|_2^2 \right) \left( \| \Delta v \|_2^2 + \| \Delta \theta \|_2^2 \right)$$

$$+ \frac{1}{6} \left( \| \nabla \Delta v \|_2^2 + \| \nabla \partial_t \theta \|_2^2 + \| \nabla \Delta \theta \|_2^2 \right).$$

By the similar method as the integral term $R_1$ on the right-hand side of (4.7), the integral term $R_2$ can be bounded as

$$R_2 := \int_\Omega \nabla \left[ (v \cdot \nabla_h) v - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \partial_z v \right] : \nabla (\Delta v - \partial_t v) \, dx \, dy \, dz$$

$$= \int_\Omega (\partial_t u \cdot \nabla v + u \cdot \nabla \partial_t v) (\partial_t \Delta v - \partial_t \partial_t v) \, dx \, dy \, dz$$

$$\leq C \| \nabla v \|_2^2 \| \Delta v \|_2^2 \| \Delta \theta \|_2^2 + \frac{1}{6} \left( \| \nabla \partial_t v \|_2^2 + \| \nabla \Delta v \|_2^2 \right).$$

In order to estimate the third integral term $R_3$ on the right-hand side of (4.7), we split the gradient operator into two parts, $\nabla\theta_h$ and $\partial_z$. Then the Hölder inequality and Young inequality yield

$$R_3 := \int_\Omega \nabla \left[ \int_0^z \nabla_h \theta(x, y, \xi, t) d\xi \right] : \nabla (\Delta v - \partial_t v) \, dx \, dy \, dz$$

$$= \int_\Omega \nabla \theta_h \left[ \int_0^z \nabla_h \theta(x, y, \xi, t) d\xi \right] : \nabla_h (\Delta v - \partial_t v) \, dx \, dy \, dz$$

$$+ \int_\Omega \nabla \theta_h \cdot (\partial_t \Delta v - \partial_t \partial_t v) \, dx \, dy \, dz$$

$$= \int_\Omega \left[ \int_0^z \partial_t \nabla \theta_h(x, y, \xi, t) d\xi \right] : (\partial_t \Delta v - \partial_t \partial_t v) \, dx \, dy \, dz$$

$$+ \int_\Omega \nabla \theta_h \cdot (\partial_t \Delta v - \partial_t \partial_t v) \, dx \, dy \, dz$$

$$\leq C \left( \| \nabla \theta_h \|_2^2 + \| \Delta \theta_h \|_2^2 \right) + \frac{1}{6} \left( \| \nabla \partial_t v \|_2^2 + \| \nabla \Delta v \|_2^2 \right).$$

Combining the estimates for $R_1$, $R_2$ and $R_3$, we obtain

$$\frac{d}{dt} \left( \| \Delta v \|_2^2 + \| \Delta \theta \|_2^2 \right) + \frac{1}{2} \left( \| \nabla \Delta v \|_2^2 + \| \nabla \Delta \theta \|_2^2 + \| \nabla \partial_t v \|_2^2 + \| \nabla \partial_t \theta \|_2^2 \right)$$

$$\leq C \left( \| \nabla \theta \|_2^2 \| \Delta \theta \|_2^2 + \| \nabla v \|_2^2 \| \Delta v \|_2^2 + 1 \right) \left( \| \Delta v \|_2^2 + \| \Delta \theta \|_2^2 \right).$$
By virtue of the Gronwall inequality, it follows from (4.1) that

\[
\left(\|\Delta v\|^2 + \|\Delta \theta\|^2\right)(t) + \int_0^t \left(\|\nabla \Delta v\|^2 + \|\nabla \Delta \theta\|^2 + \|\nabla \partial_t v\|^2 + \|\nabla \partial_t \theta\|^2\right) ds \\
\leq \exp\left\{C \int_0^t \left(\|\nabla \partial_t \theta\|^2 + \|\nabla \partial_t v\|^2 + \|\Delta \theta\|^2 + \|\Delta v\|^2 + 1\right) ds\right\} \left(\|v_0\|^2 + \|\theta_0\|^2\right).
\]

The proof is completed.

With the help of Proposition 4.2, we can perform the first order energy estimate for the system (4.1)-(4.4) under some smallness condition.

**Proposition 4.3.** Suppose that \((v_0, \theta_0) \in H^2(\Omega)\), with \(\int_\Omega \nabla \cdot v_0 dz = 0\). Then there exists a small positive constant \(\lambda_0\) such that the system (4.1)-(4.4) has the following first order energy estimate

\[
\sup_{0 \leq s \leq t} \left(\|\nabla (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon)\|^2\right)(s) + \int_0^t \|\Delta (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon)\|^2 ds \leq C \mathcal{J}_2(t),
\]

for any \(t \in [0, T_\varepsilon^*)\), provided that

\[
\sup_{0 \leq s \leq t} \left(\|\nabla (V_\varepsilon, \Phi_\varepsilon)\|^2 + \varepsilon^2 \|\nabla W_\varepsilon\|^2\right)(s) \leq \lambda_0^2,
\]

where

\[
\mathcal{J}_2(t) = C e^{C [t + (1 + t^\alpha) \alpha_2^2(t)]} \left[\alpha_2(t) + \alpha_2^2(t)\right].
\]

Here \(C\) is a positive constant that does not depend on \(\varepsilon\).

**Proof.** Taking the \(L^2(\Omega)\) inner product of the first three equation in system (4.1)-(4.4) with \(-\Delta V_\varepsilon, -\Delta W_\varepsilon\) and \(-\Delta \Phi_\varepsilon\), respectively, then it follows from integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \left(\|\nabla (V_\varepsilon, \Phi_\varepsilon)\|^2 + \varepsilon^2 \|\nabla W_\varepsilon\|^2\right) + \|\Delta (V_\varepsilon, \Phi_\varepsilon)\|^2 + \varepsilon^2 \|\Delta W_\varepsilon\|^2 \\
= \int_\Omega (U_\varepsilon \cdot \nabla \Phi_\varepsilon + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Phi_\varepsilon) \Delta \Phi_\varepsilon dxdydz + \int_\Omega \Phi_\varepsilon \Delta W_\varepsilon dxdydz \\
+ \varepsilon^2 \int_\Omega (U_\varepsilon \cdot \nabla w + U_\varepsilon \cdot \nabla \psi + u \cdot \nabla \Phi_\varepsilon) \Delta W_\varepsilon dxdydz \\
+ \int_\Omega \|U_\varepsilon \cdot \nabla v + (u \cdot \nabla) V_\varepsilon + (U_\varepsilon \cdot \nabla) v\| \cdot \Delta \Phi_\varepsilon dxdydz \\
+ \varepsilon^2 \int_\Omega (\partial_t w - \Delta w + u \cdot \nabla w) \Delta W_\varepsilon dxdydz \\
=: R_1 + R_2 + R_3 + R_4,
\]

(4.8)

note that the resultants have been added up.

We now estimate the first integral term \(R_1\) on the right-hand side of (4.8), which will be split into two parts, \(R_{11}\) and \(R_{12}\). For the integral term \(R_{11}\), it deduces from the Lemma 2.2, Poincaré and Young inequality that

\[
R_{11} = \int_\Omega (U_\varepsilon \cdot \nabla \Phi_\varepsilon + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Phi_\varepsilon) \Delta \Phi_\varepsilon dxdydz \\
\leq C \|\nabla V_\varepsilon\|^{1/2} \|\Delta V_\varepsilon\|^{1/2} \|\nabla \Phi_\varepsilon\|^{1/2} \|\Delta \Phi_\varepsilon\|^{1/2} \|\Delta \Phi_\varepsilon\|_2
+ C \|\nabla V_\varepsilon\|^{1/2} \|\Delta \varepsilon\|^{1/2} \|\nabla \theta\|^{1/2} \|\Delta \theta\|^{1/2} \|\Delta \Phi_\varepsilon\|_2
\]
respectively. Due to the Hölder inequality, Lemma 2.2 and Young inequality, we reach

\[ R_{12} = \int_{\Omega} \Psi_2 \Delta W_{\varepsilon} \, dx \, dy \, dz \]

For another integral term \( R_{12} \), we use the Hölder inequality, Poincaré inequality and Young inequality to obtain

\[ R_{12} := \int_{\Omega} \Phi_2 \Delta W_{\varepsilon} \, dx \, dy \, dz = - \int_{\Omega} \nabla \Phi_2 \cdot \nabla W_{\varepsilon} \, dx \, dy \, dz \]

where we have used the fact that \( U_{\varepsilon} \) is divergence free. Combining the two estimates yields

\[ R_1 := \int_{\Omega} (U_\varepsilon \cdot \nabla \Phi_2 + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Phi_2) \Delta W_{\varepsilon} \, dx \, dy \, dz \]

By the similar method as the first integral term \( R_1 \) on the right-hand side of (4.8), the integral term \( R_2 \) and \( R_3 \) can be bounded as

\[ R_2 := \varepsilon^2 \int_{\Omega} (U_\varepsilon \cdot \nabla W_{\varepsilon} + U_\varepsilon \cdot \nabla w + u \cdot \nabla W_{\varepsilon}) \Delta W_{\varepsilon} \, dx \, dy \, dz \]

and

\[ R_3 := \int_{\Omega} (U_\varepsilon \cdot \nabla) V_\varepsilon + (u \cdot \nabla) V_\varepsilon + (U_\varepsilon \cdot \nabla)v \cdot \Delta V_\varepsilon \, dx \, dy \, dz \]

respectively. Due to the Hölder inequality, Lemma 2.2 and Young inequality, we reach

\[ R_4 := \varepsilon^2 \int_{\Omega} \left( \partial_t w - \Delta w + u \cdot \nabla w \right) \Delta W_{\varepsilon} \, dx \, dy \, dz \]

By substituting (4.9)-(4.12) into (4.8), we obtain
\[ \leq C_\sigma \epsilon^2 \left( \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 + \| \nabla \partial_v \|_2^2 + \| \nabla \Delta v \|_2^2 \right) \\
+ C_\sigma \left( \| \nabla (V, \Phi_v) \|_2^2 + \epsilon^2 \| \nabla W_\varepsilon \|_2^2 \right) \left( \| \Delta (V, \Phi_v) \|_2^2 + \epsilon^2 \| \Delta W_\varepsilon \|_2^2 \right) \\
+ 1 + \| \Delta \theta \|_2^2 \| \nabla \Delta \theta \|_2^2 + (1 + \epsilon^4) \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \right]. \\
}

Using the assumption given by the proposition

\[ \sup_{0 \leq s \leq t} \left( \| \nabla (V, \Phi_v) \|_2^2 + \epsilon^2 \| \nabla W_\varepsilon \|_2^2 \right) (s) \leq \lambda_0^2, \]

and choosing \( \lambda_0 = \sqrt{\frac{4}{2^{4\sigma}}}, \) it deduces from the above inequality that

\[ \frac{d}{dt} \left( \| \nabla (V, \Phi_v) \|_2^2 + \epsilon^2 \| \nabla W_\varepsilon \|_2^2 \right) + \left( \| \Delta (V, \Phi_v) \|_2^2 + \epsilon^2 \| \Delta W_\varepsilon \|_2^2 \right) \\
\leq C_\sigma \epsilon^2 \left( \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 + \| \nabla \partial_v \|_2^2 + \| \nabla \Delta v \|_2^2 \right) \\
+ C_\sigma \left[ 1 + \| \Delta \theta \|_2^2 \| \nabla \Delta \theta \|_2^2 + (1 + \epsilon^4) \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \right] \\
\times \left( \| \nabla (V, \Phi_v) \|_2^2 + \epsilon^2 \| \nabla W_\varepsilon \|_2^2 \right). \]

Noting that the fact that \((V, W, \Phi_v)\) is \(0\), and applying the Gronwall inequality to the above inequality, it follows from Proposition 4.3 that

\[ \left( \| \nabla (V, \Phi_v) \|_2^2 + \epsilon^2 \| \nabla W_\varepsilon \|_2^2 \right) (t) + \int_0^t \left( \| \Delta (V, \Phi_v) \|_2^2 + \epsilon^2 \| \Delta W_\varepsilon \|_2^2 \right) ds \\
\leq C_\sigma \epsilon^2 \exp \left\{ C_\sigma \int_0^t \left[ 1 + \| \Delta \theta \|_2^2 \| \nabla \Delta \theta \|_2^2 + (1 + \epsilon^4) \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \right] ds \right\} \\
\times \int_0^t \left( \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 + \| \nabla \partial_v \|_2^2 + \| \nabla \Delta v \|_2^2 \right) ds \\
\leq C_\sigma \epsilon^2 e^{C_\sigma \left[ (1 + \epsilon^4) \alpha_2 (t) \right]} \left[ \alpha_2 (t) + \alpha_2^2 (t) \right]. \]

The proof is thus completed. \( \square \)

For purpose of eliminating the effect of the smallness condition in Proposition 4.3, we have the following proposition.

**Proposition 4.4.** Let \( T_*^\varepsilon \) be the maximal existence time of the strong solution \((v_\varepsilon, w_\varepsilon, \phi_\varepsilon)\) to the system (1.12) corresponding to boundary and initial conditions (1.10), (1.11) and symmetry condition (1.5). Then, for any \( T > 0 \), there exists a small positive constant \( \varepsilon (T) = \frac{1}{3 \sqrt{J_2 (T)}} \) such that \( T_*^\varepsilon > T \) provided that \( \varepsilon \in (0, \varepsilon (T)) \). Furthermore, the system (4.1)–(4.4) has the following energy estimate

\[ \sup_{0 \leq t \leq T} \left( \| (V, \varepsilon W, \Phi_v) \|_H^2 \right) + \int_0^T \| \nabla (V, \varepsilon W, \Phi_v) \|_H^2 dt \leq \epsilon^2 \left( \overline{J}_1 (T) + \overline{J}_2 (T) \right), \]

where both \( \overline{J}_1 (t) \) and \( \overline{J}_2 (t) \) are nonnegative continuously increasing functions that do not depend on \( \varepsilon \).

**Proof.** For any \( T > 0 \), setting \( T_*^\varepsilon = \min \{ T_*^\varepsilon, T \} \), then it follows from Proposition 4.3 that

\[ \sup_{0 \leq t \leq T_*^\varepsilon} \left( \| (V, \varepsilon W, \Phi_v) \|_H^2 \right) + \int_0^{T_*^\varepsilon} \| \nabla (V, \varepsilon W, \Phi_v) \|_H^2 dt \leq \epsilon^2 \overline{J}_1 (T), \quad (4.13) \]
Proposition 4.4. This completes the proof.

For any \( t \), owing to Proposition 4.4, there exists a small positive constant that does not depend on \( \varepsilon \).

Let \( \lambda_0 \) be the constant from Proposition 4.4. Define

\[
\begin{align*}
\tilde{T} := \sup \left\{ t \in (0, T') \middle| \sup_{0 \leq s \leq t} \left( \| \nabla (V_{t}, \varepsilon W_{t}, \Phi_{t}) \|^2_{2} \right) (s) \leq \lambda_0^2 \right\}.
\end{align*}
\]

Thanks to Proposition 4.4, we have the following estimate

\[
\begin{align*}
\sup_{0 \leq s \leq t} \left( \| \nabla (V_{t}, \varepsilon W_{t}, \Phi_{t}) \|^2_{2} \right) (s) + \int_{0}^{t} \| \Delta (V_{t}, \varepsilon W_{t}, \Phi_{t}) \|^2_{2} ds \leq \varepsilon^2 \tilde{J}_2(T),
\end{align*}
\]

for any \( t \in [0, t'] \), where

\[
\begin{align*}
\tilde{J}_2(T) = C' e^{C[T + \alpha^2_2(T)]} \left[ \alpha_2(T) + \alpha_2^2(T) \right].
\end{align*}
\]

Here \( C' \) is a positive constant that does not depend on \( \varepsilon \). Choosing \( \varepsilon(T) = \frac{2\lambda_0}{3\sqrt{2}T(T)} \), it deduces from (4.14) that

\[
\begin{align*}
\sup_{0 \leq s \leq t} \left( \| \nabla (V_{t}, \varepsilon W_{t}, \Phi_{t}) \|^2_{2} \right) (s) + \int_{0}^{t} \| \Delta (V_{t}, \varepsilon W_{t}, \Phi_{t}) \|^2_{2} ds \leq \frac{4\lambda_0^2}{9},
\end{align*}
\]

for any \( t \in [0, t'] \) and for any \( \varepsilon \in (0, \varepsilon(T)) \), which leads to

\[
\begin{align*}
\sup_{0 \leq t < T'} \left( \| \nabla (V_{t}, \varepsilon W_{t}, \Phi_{t}) \|^2_{2} \right) (t) \leq \frac{4\lambda_0^2}{9} < \lambda_0^2.
\end{align*}
\]

From the definition of \( t' \), (4.14) implies \( t' = T' \). Therefore, the estimate (4.14) holds for \( t \in [0, T'] \) and for any \( \varepsilon \in (0, \varepsilon(T)) \).

We claim that \( T_{*} > T \) for any \( \varepsilon \in (0, \varepsilon(T)) \). If \( T_{*} \leq T \), then it is clear that

\[
\lim_{t \to (T_{*})^{-}} \sup_{0 \leq t < T} \left( \| \nabla (V_{t}, \varepsilon W_{t}, \Phi_{t}) \|^2_{2} \right) = \infty.
\]

Otherwise, the strong solution \((v_{t}, w_{t}, \theta_{t})\) to the system (1.2) can be extended beyond the maximal existence time \( T_{*} \). However, the above result contradicts to (4.14). This contradiction leads to \( T_{*} > T \), and hence \( T' = T \). Moreover, combining (4.13) with (4.14) yields the energy estimate in Proposition 4.4. This completes the proof. \( \square \)

Proof of Theorem 1.2 is shown as follows.

Proof of Theorem 1.2. For any \( T > 0 \), owing to Proposition 4.4, there exists a small positive constant \( \varepsilon(T) = \frac{2\lambda_0}{3\sqrt{2}T(T)} \) such that \( T_{*} > T \) provided that \( \varepsilon \in (0, \varepsilon(T)) \), which implies that the system (1.2) corresponding to boundary and initial conditions (1.3)–(1.4) and symmetry condition (1.5) has a unique strong solution \((v_{t}, w_{t}, \theta_{t})\) on the time interval \([0, T]\) for all \( \varepsilon \in (0, \varepsilon(T)) \). Furthermore, the following estimate holds

\[
\begin{align*}
\sup_{0 \leq t \leq T} \left( \| (V_{t}, \varepsilon W_{t}, \Phi_{t}) \|^2_{H^1} \right) (t) + \int_{0}^{T} \| \nabla (V_{t}, \varepsilon W_{t}, \Phi_{t}) \|^2_{H^1} dt \\
\leq \varepsilon^2 \left( \tilde{J}_1(T) + \tilde{J}_2(T) \right) =: \varepsilon^2 \tilde{J}_3(T),
\end{align*}
\]
where $\overline{f}_3(t)$ is a nonnegative continuously increasing function that does not depend on $\varepsilon$. From the above estimate it follows that

\[
(v_\varepsilon, v_\varepsilon, \theta_\varepsilon) \to (v, 0, \theta), \text{ in } L^\infty([0,T]; H^1(\Omega)),
\]
\[
(\nabla v_\varepsilon, \varepsilon \nabla w_\varepsilon, \nabla \theta_\varepsilon) \to (\nabla v, 0, \nabla \theta), \text{ in } L^2([0,T]; H^1(\Omega)).
\]

Due to the divergence-free condition, it deduces from $\nabla v_\varepsilon \to \nabla v$ in $L^2([0,T]; H^1(\Omega))$ and $v_\varepsilon \to v$ in $L^\infty([0,T]; H^1(\Omega))$ that

\[
w_\varepsilon \to w \text{ in } L^2([0,T]; H^1(\Omega)),
\]
\[
w_\varepsilon \to w \text{ in } L^\infty([0,T]; L^2(\Omega)),
\]

respectively. The theorem is thus proved.

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