Involution gradings in matrix algebras

Yuri Bahturin

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John’s, NL, A1C5S7, Canada

and

Department of Algebra, Faculty of Mathematics and Mechanics
Moscow State University
Moscow, 119992, Russia

Mikhail Zaicev

Department of Algebra, Faculty of Mathematics and Mechanics
Moscow State University
Moscow, 119992, Russia

1 Introduction

This paper is devoted to the correction of an error in the paper [5] in which
the classification of involution gradings on matrix algebras was derived from
the fact that in the decomposition of a graded matrix algebra as the tensor
product of an elementary and a fine component, these components remain
invariant under the involution.

2 Some notation and simple facts

Let $F$ be an arbitrary field, $A$ a not necessarily associative algebra over an $F$
and $G$ a group. We say that $A$ is a $G$-graded algebra, if there is a vector space

---

Email addresses: yuri@math.mun.ca (Yuri Bahturin),
zaicev@mech.math.msu.su (Mikhail Zaicev).

1 Work is partially supported by NSERC grant # 227060-04 and URP grant, Memorial
University of Newfoundland

2 Work is partially supported by RFBR, grant 06-01-00485a, and SSH-5666.2006.1

Preprint submitted to Elsevier Science 18 August 2021
sum decomposition

\[ A = \bigoplus_{g \in G} A_g, \]  

(1)
such that

\[ A_g A_h \subset A_{gh} \text{ for all } g, h \in G. \]  

(2)

A subspace \( V \subset A \) is called graded (or homogeneous) if \( V = \oplus_{g \in G} (V \cap A_g) \).

An element \( a \in R \) is called homogeneous of degree \( g \) if \( a \in A_g \). We also write \( \deg a = g \). The support of the \( G \)-grading is a subset

\[ \text{Supp } A = \{ g \in G \mid A_g \neq 0 \}. \]

Suppose now that \( F \) is of characteristic different from 2. If \( A \) is an associative algebra with involution \( \ast \) and, in addition to (2), one has

\[ (A_g)^* = A_g \text{ for all } g \in G. \]  

(3)

then we say that (1) is an involution preserving grading or simply an involution grading. In this case, given a graded subspace \( B \subset A \) we set

\[ H(B, \ast) = \{ b \in B \mid b^* = b \}, \text{ the set of symmetric elements of } B \]  

(4)

and

\[ K(B, \ast) = \{ b \in B \mid b^* = -b \}, \text{ the set of skew-symmetric elements of } B. \]  

(5)

If \( B \) is an associative subalgebra of \( A \) then \( B^{(-)} \) is a Lie subalgebra of \( A \), that is, with respect to \([x, y] = xy - yx\) while \( B^{(+)} \) is a Jordan subalgebra of \( A \), that is, with respect to \( x \circ y = xy + yx \). We always have \( B = B^{(-)} \oplus B^{(+)} \).

3 Reminder: Group gradings on matrix algebras

Below we briefly recall the results of [4], where the full description of abelian group gradings on the full matrix algebra has been given.

A grading \( R = \oplus_{g \in G} R_g \) on the matrix algebra \( R = M_n(F) \) is called elementary if there exists an \( n \)-tuple \((g_1, \ldots, g_n) \in G^n\) such that the matrix units \( E_{ij}, 1 \leq i, j \leq n \) are homogeneous and \( E_{ij} \in R_g \iff g = g_i^{-1}g_j \).
A grading is called fine if \( \dim R_g = 1 \) for any \( g \in \text{Supp } R \). A particular case of fine gradings is the so-called \( \varepsilon \)-grading where \( \varepsilon \) is \( n \)th primitive root of 1. Let \( G = \langle a \rangle_n \times \langle b \rangle_n \) be the direct product of two cyclic groups of order \( n \) and

\[
X_a = \begin{pmatrix}
\varepsilon^{n-1} & 0 & \ldots & 0 \\
0 & \varepsilon^{n-2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}, \quad X_b = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 \\
1 & 0 & \ldots & 0
\end{pmatrix}.
\]

Then

\[ X_aX_bX_a^{-1} = \varepsilon X_b, \quad X_a^n = X_b^n = I \tag{6} \]

and all \( X_a^iX_b^j, 1 \leq i, j \leq n \), are linearly independent. Clearly, the elements \( X_a^iX_b^j, i, j = 1, \ldots, n \), form a basis of \( R \) and all the products of these basis elements are uniquely defined by (7).

Now for any \( g \in G, g = a^ib^j \), we set \( X_g = X_a^iX_b^j \) and denote by \( R_g \) a one-dimensional subspace

\[ R_g = \langle X_a^iX_b^j \rangle. \tag{8} \]

Then from (7) it follows that \( R = \oplus_{g \in G} R_g \) is a \( G \)-grading on \( M_n(F) \) which is called an \( \varepsilon \)-grading.

Now let \( R = M_n(F) \) be the full matrix algebra over \( F \) graded by an abelian group \( G \). The following result has been proved in [4, Section 4, Theorems 5, 6] and [2, Subsection 2.2, Theorem 6, Subsection 2.3, Theorem 8].

**Theorem 1** Let \( F \) be an algebraically closed field of characteristic zero. Then as a \( G \)-graded algebra \( R \) is isomorphic to the tensor product

\[ R^{(0)} \otimes R^{(1)} \otimes \cdots \otimes R^{(k)} \]

where \( R^{(0)} = M_{n_0}(F) \) has an elementary \( G \)-grading, \( \text{Supp } R^{(0)} = S \) is a finite subset of \( G \), \( R^{(i)} = M_{n_i}(F) \) has the \( \varepsilon_i \) grading, \( \varepsilon_i \) being a primitive \( n_i \)th root of 1, \( \text{Supp } R^{(i)} = H_i \cong \mathbb{Z}_{n_i} \times \mathbb{Z}_{r_i} \), \( i = 1, \ldots, k \). Also \( H = H_1 \cdots H_k \cong H_1 \times \cdots \times H_k \) and \( S \cap H = \{ e \} \) in \( G \).

**Remark 1** It follows from a very general lemma in [4] that the support \( T \) of a fine grading on \( R = M_n \) is a subgroup of the grading group \( G \). Thus we have \( R = \bigoplus_{t \in T} R_t \) and \( R_t = \langle X_t \rangle \), for a nondegenerate matrix \( X_t \). Let us also recall that the product in \( R = M_n(F) \) with fine grading as above is defined by a bicharacter \( \alpha : T \times T \to F^* \) as follows: \( X_tX_u = \alpha(t, u)X_{tu} \), for any \( t, u \in T \).
The commutation relations in $R$ take the form $X_t X_u = \beta(t, u) X_u X_t$ where $\beta(t, u) = \alpha(t, u)/\alpha(u, t)$ is a skew-symmetric bicharacter on $T$ (see [1]).

Let us recall that any involution $\ast$ of $R = M_n$ can always be written as

$$
X^\ast = \Phi^{-1}(\tr X)\Phi
$$

where $\Phi$ is a nondegenerate matrix which is either symmetric or skew-symmetric and $X \mapsto \tr X$ is the ordinary transpose map. In the case where $\Phi$ is symmetric we call $\ast$ a transpose involution. If $\Phi$ is skew-symmetric $\ast$ is called a symplectic involution. Before we formulate the theorem describing involution gradings on $M_n$ in the case where the elementary and fine components are invariant under the involution, we need three (slightly modified) lemmas from [5]. The general restriction in [5] zero characteristic was not used in the proof of these particular lemmas. The first two deal with elementary involution gradings while the last with certain fine involution gradings. If $R$ has an involution $\ast$ then by $R^{(\pm)}$ we denote the space of symmetric (respectively skew-symmetric) matrices in $R$ under $\ast$.

The next lemma handles the case of an elementary grading compatible with an involution defined by a symmetric non-degenerate bilinear form.

**Lemma 1** Let $R = M_n(F)$, $n$ a natural number, be a matrix algebra with involution $\ast$ defined by a symmetric non-degenerate bilinear form. Let $G$ be an abelian group and let $R$ be equipped with an elementary involution $G$-grading defined by an $n$-tuple $(g_1, \ldots, g_n)$. Then $g_1^2 = \ldots = g_m^2 = g_{m+1}g_{m+l+1} = \ldots = g_{m+l}g_{m+2l}$ for some $0 \leq l \leq \frac{n}{2}$ and $m + 2l = n$. The involution $\ast$ acts as $X^\ast = (\Phi^{-1})^\tr X\Phi$ where

$$
\Phi = \begin{pmatrix}
I_m & 0 & 0 \\
0 & 0 & I_l \\
0 & I_l & 0
\end{pmatrix},
$$

where $I_s$ is the $s \times s$ identity matrix. Moreover, $R^{(-)}$ consists of all matrices of the type

$$
\begin{pmatrix}
P & S & T \\
-\tr T & A & B \\
-\tr S & C & -\tr A
\end{pmatrix},
$$

where $\tr P = -P$, $\tr B = -B$, $\tr C = -C$ and

$P \in M_m(F)$, $A, B, C, D \in M_l(F)$, $S, T \in M_{m \times l}(F)$

4
while $R^{(-)}$ consists of all matrices of the type
\[
\begin{pmatrix}
P & S & T \\
t^T & A & B \\
t^S & C & tA
\end{pmatrix},
\]
(11)
where $t^P = P$, $t^B = B$, $t^C = C$ and
\[
P \in M_m(F), \ A, B, C, D \in M_l(F), \ S, T \in M_{m \times l}(F).
\]
The next lemma deals with the case of an elementary grading compatible with an involution defined by a skew-symmetric non-degenerate bilinear form.

**Lemma 2** Let $R = M_n(F)$, $n = 2k$, be the matrix algebra with involution $\ast$ defined by a skew-symmetric non-degenerate bilinear form. Let $G$ be an abelian group and let $R$ be equipped with an elementary involution $G$-grading defined by an $n$-tuple $(g_1, \ldots, g_n)$. Then $g_1 g_{k+1} = \ldots = g_k g_{2k}$, the involution $\ast$ acts as $X^\ast = (\Phi^{-1})^T X \Phi$ where
\[
\Phi = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix},
\]
$I$ is the $k \times k$ identity matrix, $R^{(-)}$ consists of all matrices of the type
\[
\begin{pmatrix}
A & B \\
C & -tA
\end{pmatrix}, \ A, B, C \in M_k(F), \ t^B = B, t^C = C
\]
(12)
while $R^{(+)}$ consists of all matrices of the type
\[
\begin{pmatrix}
A & B \\
C & tA
\end{pmatrix}, \ A, B, C \in M_k(F), \ t^B = -B, t^C = -C.
\]
(13)

**Lemma 3** Let $R = M_2(F)$ be a $2 \times 2$ matrix algebra endowed with an involution $\ast : R \to R$ corresponding to a symmetric or skew-symmetric non-degenerate bilinear form with the matrix $\Phi$. The $(-1)$-grading of $M_2$ by $G = \langle a \rangle_2 \times \langle b \rangle_2$ is an involution grading if and only if one of the following holds:
(1) $\Phi$ is skew-symmetric,

$$\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K(R, \ast) = \text{Span } \{X_a, X_b, X_{ab}\}, \quad H(R, \ast) = \text{Span } \{X_e\};$$

(2) $\Phi$ is symmetric,

$$\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K(R, \ast) = \text{Span } \{X_a\}, \quad H(R, \ast) = \text{Span } \{X_e, X_b, X_{ab}\};$$

(3) $\Phi$ is symmetric,

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K(R, \ast) = \text{Span } \{X_{ab}\}, \quad H(R, \ast) = \text{Span } \{X_e, X_a, X_b\};$$

(4) $\Phi$ is symmetric,

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K(R, \ast) = \text{Span } \{X_b\}, \quad H(R, \ast) = \text{Span } \{X_e, X_a, X_{ab}\};$$

Notice that the involution in each case is already defined, say, in Case (1) one has

$$(\alpha X_e + \beta X_a + \gamma X_b + \delta X_{ab})^\ast = \alpha X_e - \beta X_a - \gamma X_b - \delta X_{ab}.$$

**Remark 2** If $R = M_n$ has a fine grading by a group $G$ with support an elementary abelian 2-subgroup $T$ then it is immediate from the previous lemma and a Remark 1 after Theorem 1 that $R$ has a basis $\{X_t \mid t \in T\}$ such that $X_tX_u = \alpha(t,u)X_{tu}$ where $\alpha(t,u) = \pm 1$ and for each $u \in T$ we have $X_u^{-1} = X_u = \alpha(u,u)X_u$.

We can now formulate the most general result available earlier, which describes gradings on a matrix algebra with involution (a weaker form of [5, Theorem 2], which is not true). With our additional assumption that the involution respects the fine and the elementary components of the grading, the proof of [5] works without changes. We remark here that this condition is always satisfied provided that the Sylow 2-subgroup of $G$ is cyclic.

**Theorem 2** Let $R = M_n(F) = \bigoplus_{g \in G} R_g$ be a matrix algebra over an algebraically closed field of characteristic zero graded by the group $G$ and $\text{Supp } R$
generates $G$. Suppose that $*: R \to R$ is a graded involution. Then $G$ is
abelian, and $R$ as a $G$-graded algebra is isomorphic to the tensor product
$R^{(0)} \otimes R^{(1)} \otimes \cdots \otimes R^{(k)}$ of a matrix subalgebra $R^{(0)}$ with elementary grading
and $R^{(1)} \otimes \cdots \otimes R^{(k)}$ a matrix subalgebra with fine grading. Suppose further
that both these subalgebras are invariant under the involution. Then $n = 2^km$ and

(1) $R^{(0)} = M_m(F)$ is as in Lemma 2 if $*$ is symplectic on $R^{(0)}$ or as in Lemma
1 if $*$ is transpose on $R_0$;
(2) $R^{(1)} \otimes \cdots \otimes R^{(k)}$ is a $T = T_1 \times \cdots \times T_k$-graded algebra and any $R^{(i)}, 1 \leq i \leq k$,
is $T_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra as in Lemma 3.
(3) A graded basis of $R$ is formed by the elements $Y \otimes X_{t_1} \otimes \cdots \otimes X_{t_k}$, where
$Y$ is an element of a graded basis of $R^{(0)}$ and the elements $X_{t_i}$ are of the
type (8), with $n = 2$, $t_i \in T_i$. The involution on these elements is given
canonically by

$$(Y \otimes X_{t_1} \otimes \cdots \otimes X_{t_k})^* = Y^* \otimes X_{t_1}^* \otimes \cdots \otimes X_{t_k}^* = \text{sgn}(t)(Y^* \otimes X_{t_1} \otimes \cdots \otimes X_{t_k}),$$

where $Y \in R^{(0)}$, $X_{t_i}$ are the elements of the basis of the canonical $(-1)$-
grading of $M_2$, $i = 1, \ldots, k$, $t = t_1 \cdots t_k \in T$, $\text{sgn}(t) = \pm 1$, depending on
the cases in Lemma 3.

In the next two sections we describe the antiautomorphisms of graded matrix
algebras in the general case, including that now we only assume that the base
field is algebraically closed of characteristic different from 2.

4 Antiautomorphisms of graded matrix algebras

We start this section with a result about the structure of fine gradings of
$R = M_n$ compatible with an antiautomorphism. This result is a far going
generalization of [5, Lemma 2]. Any antiautomorphism $\varphi$ of $R = M_n$ can
always be written as

$$\varphi * X = \Phi^{-1}({}^t X)\Phi$$

where $\Phi$ is a nondegenerate matrix and $X \mapsto {}^t X$ is the ordinary transpose
map. It is well-known that $\varphi$ is an involution if and only if $\Phi$ is either symmetric
or skew-symmetric. Recall that in the case where $\Phi$ is symmetric $\varphi$ is called a
transpose involution and if $\Phi$ is skew-symmetric then $\varphi$ is called a symplectic
involution.
Lemma 4 Let $R = M_n(F) = \bigoplus_{t \in T} R_t$ be the $n \times n$-matrix algebra with an $\varepsilon$-grading, $T = \langle a \rangle_n \times \langle b \rangle_n$. Let also $\varphi : R \to R$ be an antiautomorphism of $R$ defined by $\varphi * X = \Phi^{-1} tX\Phi$. If $\varphi * R_t = R_t$ for all $t \in T$ then $n = 2$, $\Phi$ coincides with the scalar multiple one of the matrices $I$, $X_a$, $X_b$ or $X_{ab}$ (see (6)).

Proof. First we consider the $\varphi$-action on $X_a$. Since $R_a$ is stable under $\varphi$,

$$\Phi^{-1} tX_a \Phi = \Phi^{-1} X_a \Phi = \alpha X_a$$

for some scalar $\alpha \neq 0$. Then

$$X_a \Phi X_a^{-1} = \alpha \Phi.$$  \hfill{(15)}

Since $X_a^n = I$, we obtain $\alpha^n = 1$, so that $\alpha = \varepsilon^j$ for some $0 \leq j \leq n - 1$.

Denote by $P$ the linear span of $I, X_a, \ldots, X_a^{n-1}$. Then $R = P \oplus X_b P \oplus \cdots \oplus X_b^{n-1} P$ as a vector space and the conjugation by $X_a$ acts on $X_b P$ as the multiplication by $\varepsilon^i$. In particular, all eigenvectors with eigenvalue $\varepsilon^i$ are in $X_b^j P$. It follows that $\Phi \in X_b^j P$, that is, $\Phi = X_b^j Q$ for some $Q \in P$.

Now we consider the action of $\varphi$ on $X_b$:

$$\varphi * X_b = \Phi^{-1} tX_b \Phi = \Phi^{-1} X_b^{-1} \Phi = \gamma X_b,$$

that is, $X_b \Phi X_b = \mu \Phi$ with $\mu = \gamma^{-1} \neq 0$. If we write $Q = \sum \alpha_i X_a^i$ then

$$X_b \Phi X_b = X_b^j \sum_i \alpha_i X_b X_a^i X_b = X_b^j \sum_i \alpha'_i X_a^i X_b^2 = \mu \Phi = \mu X_b^j \sum_i \alpha_i X_a^i.$$ \hfill{(16)}

In this case $X_b^j \sum_i \alpha'_i X_a^i X_b^2 = \mu X_b^j \sum_i \alpha_i X_a^i$ where the scalars $\alpha'_i$ can be explicitly computed using (15). Since the degrees in $X_a, X_b$ define the degrees in the $T$-grading, we can see that (16) immediately implies $X_b^2 = I$, i.e. $n = 2$.

As we have shown before, (15) implies $\Phi = X_b^j Q$ with $Q = \alpha_0 I + \alpha_1 X_a$. Since $n = 2$, the argument following (15) applies if we change $a$ and $b$ places so that $\Phi = X_a^k (\beta_0 I + \beta_1 X_b)$. Comparing these two expressions we obtain that $\Phi$ must be one of $I$, $X_a$, $X_b$, or $X_{ab}$, up to a scalar multiple. \hfill{\Box}

Now we make few remarks about the structure of elementary gradings on $M_n(F)$. Recall that a grading $M_n = R = \bigoplus_{g \in G} R_g$ is elementary if there exists an $n$-tuple $\tau = (g_1, \ldots, g_n) \in G^n$ such that the matrix units $E_{ij}, 1 \leq i, j \leq n$ are homogeneous and $E_{ij} \in R_{g_i} \iff g = g_i^{-1} g_j$. Elementary gradings arise from the gradings on vector spaces. Let $V = \text{Span} \{v_1, \ldots, v_n\}$ be a graded
vector space and \( \{ v_1, \ldots, v_n \} \) is a graded basis such that \( \deg v_i = g_i^{-1} \). Then any \( E_{ij} \) is a homogeneous linear transformation of \( V \) and \( \deg E_{ij} = g_i^{-1} g_j \).

Any permutation \( v_i \mapsto v_{\sigma(i)} \) of basis elements induces a graded automorphism of \( M_n = \text{End} V \) and the corresponding permutation on the \( n \)-tuple \( \tau = (g_1, \ldots, g_n) \). Hence we may permute the components of \( \tau \). Now suppose \( \tau \) has the form

\[
\tau = \left( t_1, \ldots, t_1, \ldots, t_m, \ldots, t_m \right)
\]

with \( t_1, \ldots, t_m \) pairwise distinct. In this case the identity component \( R_e \) is isomorphic to \( A_1 \oplus \cdots \oplus A_m \) where \( A_i \cong M_{p_i} \), for any \( i = 1, \ldots, m \) and consists of all block-diagonal matrices

\[
X = \text{diag}\{X_1, X_2, \ldots, X_m\}
\]

where \( X_j \) is a \( p_j \times p_j \)-matrix. Moreover, for any \( i \neq j \) the subspace \( A_i RA_j \) is graded and all \( X \in A_i RA_j \) are of degree \( t_i^{-1} t_j \) in the \( G \)-grading. As an easy consequence of this realization we obtain

**Lemma 5** Let \( R = M_n = \bigoplus_{g \in G} R_g \) be a matrix algebra with an elementary \( G \)-grading. If \( R_e \) is simple then the grading is trivial. If \( R_e \) is the sum of two simple components, \( R_e = A_1 \oplus A_2 \), then there exists \( g \in G, g \neq e \), such that \( A_1 RA_2 \subseteq R_g \).  

Now we consider a matrix algebra \( R = M_n \) with an involution \( * : R \to R \) preserving \( R_e \). Permuting \( t_1, \ldots, t_m \) in \( \tau \) we may assume that for any \( 1 \leq j \leq m - 1 \) either \( A_j^* = A_j \) or \( A_j^* = A_{j+1}, A_{j+1}^* = A_j \). In the first case \( A_j \) is simple and \( A_j RA_j = A_j \). In the second case \( B = A_j \oplus A_{j+1} \) is not simple but \( * \)-simple and \( A_j \simeq A_{j+1} \), i.e. \( A_j \) and \( A_{j+1} \) are matrix algebras of the same size \( s \). It is convenient to consider the subalgebra \( BRB \) as a subset of all matrices

\[
\text{diag}\{0, \ldots, 0, X, 0, \ldots, 0\}
\]

where \( X \) is \( 2s \times 2s \)-matrix on the respective position.

Next we consider a general \( G \)-graded matrix algebra \( R = M_n \). According to Theorem 1, \( R = C \otimes D \) where \( C \otimes I \) is a matrix algebra with elementary grading while \( I \otimes D \) is an algebra with fine grading.

**Lemma 6** Let \( R = C \otimes D = \bigoplus_{g \in G} R_g \) be a \( G \)-graded matrix algebra with an elementary grading on \( C \) and a fine grading on \( D \). Let \( \varphi : R \to R \) be an antiautomorphism on \( R \) preserving \( G \)-grading. Let also \( \varphi \) acts as an involution on the identity component \( R_e \) i.e \( \varphi^2 |_{R_e} = \text{Id} \). Then
1) $C_e \otimes I$ is $\varphi$-stable where $I$ is the unit of $D$ and hence $\varphi$ induces an involution $*$ on $C_e$;
2) there are subalgebras $B_1, \ldots, B_k \subseteq C_e$ such that $C_e = B_1 \oplus \cdots \oplus B_k$, $B_1 \otimes I, \ldots, B_k \otimes I$ are $\varphi$-stable and all $B_1, \ldots, B_k$ are $*$-simple algebras;
3) $\varphi$ acts on $R_e = C_e \otimes I$ as $\varphi \cdot X = S^{-1} tXS$ where $S = S_1 \otimes I + \cdots + S_k \otimes I$, $S_i \in B_i CB_i$ and $S_i = I_{p_i}$ if $B_i$ is $p_i \times p_i$-matrix algebra with transpose involution, $S_i = \begin{pmatrix} 0 & I_{p_i} \\ -I_{p_i} & 0 \end{pmatrix}$ if $B_i$ is $2p_i \times 2p_i$-matrix algebra with symplectic involution or $S_i = \begin{pmatrix} 0 & I_{p_i} \\ I_{p_i} & 0 \end{pmatrix}$ if $B_i \simeq M_{p_i} \oplus M_{p_i}$.
4) the centralizer of $R_e = C_e \otimes I$ in $R$ can be decomposed as $Z_1 D_1 \oplus \cdots \oplus Z_k D_k$ where $D_1, \ldots, D_k$ are $\varphi$-stable graded subalgebras of $R$ isomorphic to $D$ and $Z_i = Z_i' \otimes I$ where $Z_i'$ is the center of $B_i$;
5) $D$ as a graded algebra is isomorphic to $M_2 \oplus \cdots \oplus M_2$ where any factor $M_2$ has the fine $(-1)$-grading.

Proof. From Theorem 1 it follows that the identity component $R_e$ equals to $C_e \otimes I$. Since $R_e$ is $\varphi$-stable and $\varphi^2 = \text{Id}$ on $R_e$, the $\varphi$-action induces an involution $*$ on $C_e$. Since $C_e$ is semisimple it is a direct sum of $*$-simple algebras,

$$C_e = B_1 \oplus \cdots \oplus B_k. \quad (17)$$

Now 1), 2) and 3) follows from the classification of involution simple algebras [6].

Denote by $e_1, \ldots, e_k$ the units of $B_1, \ldots, B_k$, respectively. Clearly, the centralizer $Z$ of $C_e$ in $C$ is equal to $Z_1' \oplus \cdots \oplus Z_k'$ where $Z_i'$ is the center of $B_i$ and the centralizer of $R_e$ in $R$ coincides with $Z \otimes D = Z_1 D_1 \oplus \cdots \oplus Z_k D_k$ where $Z_i = Z_i' \otimes I$ and $D_i = e_i \otimes D$. Obviously the map $e_i \otimes d \mapsto d \in D$ is an isomorphism of graded algebras. Hence for proving 4) we only need to check that all $D_1, \ldots, D_k$ are $\varphi$-stable.

We fix $1 \leq i \leq k$ and consider $R' = (e_i \otimes I)R(e_i \otimes I) = C' \otimes D$ where $C' = e_i Ce_i$ and $D_i = e_i \otimes D$ is a graded subalgebra of $R'$. Then $R'$ is $\varphi$-stable since $\varphi \cdot (e_i \otimes I) = e_i \otimes I$. Also $R'_e = C'_e \otimes I$ with $C'_e = B_i$. If $B_i$ is simple then $C' = C'_e$ by Lemma 5. In this case $D_i$ is $\varphi$-stable since $\varphi$ preserves $R'_e$ and $D_i$ is the centralizer of $R'_e$ in $R'$.

Now suppose $B_i = A_1 \oplus A_2$ is the sum of two matrix algebras. First we will show that $C' \otimes I$ is a $\varphi$-stable graded subalgebra of $R$. Denote by $f_1, f_2$ the units of $A_1$ and $A_2$ respectively. Then $f_1, f_2 \in R_e$ and $\varphi$ permutes $f_1, f_2$. 

10
Moreover, \( f_1 C' f_2 \otimes I \) and \( f_2 C' f_1 \otimes I \) are graded subspaces. Since \( \varphi \ast f_1 \otimes I = f_2 \otimes I, \varphi \ast f_2 \otimes I = f_1 \otimes I \) we have

\[
\varphi \ast (f_1 C' f_2 \otimes I) \subseteq (f_1 \otimes I) R (f_2 \otimes I) = f_1 C' f_2 \otimes D.
\]

On the other hand, since by Lemma 5 there is \( g \in G \) such that \( f_1 C' f_2 \subseteq C' g \) for some \( g \in G \) it follows that

\[
\varphi \ast (f_1 C' f_2 \otimes I) \subseteq R g.
\]

Suppose now that \( x \in f_1 C' f_2, y \in D, x \) and \( y \) are homogeneous, \( \deg x = g, \deg y = h \) Then \( \deg (x \otimes y) = q \) if and only if \( h = e \) that is \( y = \lambda I \), for some scalar \( \lambda \). It follows that \( f_1 C' f_2 \otimes I \) is a \( \varphi \)-stable subspace. Similarly, \( \varphi \ast f_2 C' f_1 \otimes I \subseteq f_2 C' f_1 \otimes I, \) hence \( C' \otimes I \) is \( \varphi \)-stable.

Now from the decomposition \( R' = C' \otimes D \) it follows that \( D_i = e_i \otimes D \) is a \( \varphi \)-stable graded subalgebra.

For proving 5) we remark that \( D \) is isomorphic to, say, \( D_1 \) as a \( G \)-graded algebra and \( D_1 \) is \( \varphi \)-stable. So, it is enough to prove that \( D_1 \) is the tensor product of several copies of \( M_2 \). We decompose \( D_1 \) as the tensor product

\[
D_1 \simeq R_1 \otimes \cdots \otimes R_m
\]

where each \( R_i \) is a matrix algebra \( M_{n_i} \) with a fine \( \varepsilon_i \)-grading. Recall that \( H = \text{Supp} D_1 = H_1 \times \cdots \times H_m \) where \( H_i \simeq Z_{n_i} \times Z_{n_i} = \text{Supp} R_i, 1 \leq i \leq m \). Now since the \( G \)-grading on \( D_1 \) is \( \varphi \)-stable and

\[
R_i = \bigoplus_{h \in H_i} (D_1)_h
\]

it follows that \( \varphi \ast R_i = R_i \). Since any antiautomorphism \( \varphi \) on a matrix algebra acts as \( \varphi \ast X = \Phi^{-1} X \Phi \), we can apply Lemma 4. Now the proof of our lemma is complete.

In what follows we discuss the canonical form of the involution \( \varphi \) on the whole of \( R \). As mentioned, the \( \varphi \)-action on \( R \) is defined by

\[
\varphi \ast A = \Phi^{-1} tA \Phi
\]

for some matrix \( \Phi \). First let \( A \in R_e \). Consider the decomposition \( C_e = B_1 \oplus \cdots \oplus B_k \) found in Lemma 6. Then \( A = A_1 \otimes I + \cdots + A_k \otimes I \) with \( A_i \in B_i, 1 \leq i \leq k \). By Lemma 6 \( \varphi \) acts on \( A \) as

\[
\varphi \ast A = S^{-1} t A S.
\]
Hence the matrix $\Phi S^{-1}$ commutes with $A$ for any $A \in R_e$, that is $\Phi S^{-1}$ is an element of the centralizer of $R_e$ in $R$. Applying Claim 4) of Lemma 6 we obtain

$$
\Phi = S_1 Y_1 \otimes Q_1 + \cdots + S_k Y_k \otimes Q_k
$$

(18)

where $Q_i \in D$, $Y_i \in \mathbb{Z}'_i$, $1 \leq i \leq k$. Compute now the action of $\varphi^2$ on an arbitrary $A \in R$:

$$
\varphi^2 \ast A = \varphi \ast (\Phi^{-1} \cdot A \Phi) = \Phi^{-1} (\Phi^{-1} \cdot A \Phi) \Phi = (\Phi^{-1} \cdot \Phi)^{-1} A (\Phi^{-1} \cdot \Phi)
$$

Set $P = \Phi^{-1} \Phi$. Note that for any $T_i, T'_i \in B_i CB_i$ and $Q_i, Q'_i \in D$, $i = 1, \ldots, k$, the relation

$$
\left( \sum_i T_i \otimes Q_i \right) \left( \sum_i T'_i \otimes Q'_i \right) = \sum_i T_i T'_i \otimes Q_i Q'_i
$$

holds.

We compute the value of $P$:

$$
P = \Phi^{-1} \Phi = \sum_{i=1}^k (S_i Y_i)^{-1} S_i Y_i \otimes \cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i = \sum_{i=1}^k (S_i Y_i)^{-1} S_i Y_i \otimes \cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i
$$

(19)

**Lemma 7** All $Q_i$ in (19) satisfy $\cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i = \pm I$.

*Proof.* Obviously it is sufficient to prove the relation

$$
e_i \otimes \cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i = \pm e_i \otimes I
$$

in $D_i = e_i \otimes D$. Recall that $D_i$ is $\varphi$-stable (see Lemma 6) and $\varphi$ acts on $e_i \otimes X, X \in D$ as

$$
\varphi \ast (e_i \otimes X) = (S_i Y_i)^{-1} e_i \otimes (S_i Y_i) \otimes Q_i^{-1} X Q_i = e_i \otimes Q_i^{-1} X Q_i
$$

and by Lemma 4 $T_j = I, X_a, X_b$ or $X_{ab}$. In particular, $\cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i = \otimes \cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i$. By 5) Lemma 6 $D_i$ is the tensor product $M_2^{(1)} \otimes \cdots \otimes M_2^{(r)}$ of $2 \times 2$-matrix algebras with fine grading. As in the proof of 5) Lemma 6 we remark that all factors are $\varphi$-stable. Fix a factor $M_2^{(j)}$ and consider the action of $\varphi$ on $M_2^{(j)}$. Then

$$
\varphi \ast Y = T_j^{-1} \cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i
$$

and by Lemma 4 $T_j = I, X_a, X_b$ or $X_{ab}$. In particular, $\cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i = \pm I_2$ where $I_2$ is $2 \times 2$ identity matrix. Since $e_i \otimes Q_i^{-1} \cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i = T_j^{-1} (e_i \otimes \cdot \cdot \cdot \otimes (S_i Y_i)^{-1} S_i Y_i) T_j$ for all $X \in D$
where $T = T_1 \otimes \cdots \otimes T_r$, it follows that $e_i \otimes Q_i = \lambda T$ for some non-zero scalar $\lambda$. Hence $e_i \otimes Q_i$ satisfies a similar relation $e_i \otimes tQ_i^{-1}Q_i = \pm I$. \hfill \qed

We summarize what was done in this section as follows.

**Proposition 1** Suppose $R = M_n(F)$ is the full matrix algebra over an algebraically closed field of characteristic different from 2, graded by a finite abelian group $G$. Let $\varphi$ be a $G$-graded antiautomorphism of $R$ whose restriction to the identity component $R_e$ is of order two. Then $\varphi$ can be given as 

$$ \varphi = S_1 Y_1 \otimes Q_1 + \cdots + S_k Y_k \otimes Q_k $$

(20)

where $S_i$ and $Y_i$ are described in Lemma 6 and each $Q_i \in e_i \otimes D$ is such that $tQ_i^{-1}Q_i = \pm I$.

5 **Involutions on group graded matrix algebras**

In this section we preserve the notation introduced earlier except that we write $\varphi \ast X = X^\ast$. Our aim is to describe involutions on group graded matrix algebras. We will start with Equation (18), in which we additionally know from Lemma 7 that $tQ_i^{-1}Q_i = \pm I$. Let $g^{(p)}$ mean $g, \ldots, g$. Our aim is to prove the following.

**Theorem 3** Let $\varphi : X \to \Phi^{-1}X\Phi$ be an involution compatible with a grading of a matrix algebra $R$ by a finite abelian group $G$. Then, after a $G$-graded conjugation, we can reduce $\Phi$ to the form

$$ \Phi = S_1 \otimes X_{t_1} + \cdots + S_k \otimes X_{t_k} $$

(21)

where $S_i$ is one of the matrices $I$, $\left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right)$, or $\left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right)$ and each $X_{t_i}$ is a matrix spanning $D_{t_i}$, $t_i \in T$. The defining tuple of the elementary grading on $C$ should satisfy the following condition. We assume that the first $l$ of summands in (21) correspond to those $B_i$ in (17) which are simple and the remaining $k - l$ to $B_j$ which are not simple. Let the dimension of a simple $B_i$ be equal to $p_i^2$ and that of a non-simple $B_j$ to $2p_j^2$. Then the defining tuple has
the form

\[
\left( g_1^{(p_1)}, \ldots, g_l^{(p_l)}, (g_{l+1})^{(p_{l+1})}, (g_{l+1}^\prime)^{(p_{l+1})}, \ldots, (g_k^{(p_k)}, (g_k^\prime)^{(p_k)}) \right)
\]  

(22)

\[
g_1^2 t_1 = \cdots = g_l^2 t_l = g_{l+1}^2 t_{l+1} = \cdots = g_k^2 t_k.
\]  

(23)

Additionally, if \( \varphi \) is a transpose involution then each \( S_i \) is symmetric (skew-symmetric) at the same time as \( X_{t_i} \), for any \( i = 1, \ldots, k \). If \( \varphi \) is a symplectic involution, then each \( S_i \) is symmetric (skew-symmetric) if and only if the respective \( X_{t_i} \) is skew-symmetric (symmetric), \( i = 1, \ldots, k \).

Conversely, if we have a grading by a group \( G \) on a matrix algebra \( R \) defined by a tuple as in (22), for the component \( C \) with elementary grading, and by an elementary abelian 2-subgroup \( T \) as the support of the component \( D \) with fine grading and all of the above conditions are satisfied then (21) defines a graded involution on \( R \).

**Proof.** Choose \( X_u \in D_u, u \in T \), and consider \( (I \otimes X_u)^* \). Since \( \text{Span} \{ e_1, \ldots, e_k \} \otimes D \) is invariant with respect to \( \varphi \) we must have \( (I \otimes X_u)^* = \alpha_1 e_1 \otimes X_u + \cdots + \alpha_k e_k \otimes X_u \), for some scalars \( \alpha_1, \ldots, \alpha_k \). Now by Proposition 1 we have

\[
(I \otimes X_u)^* = \Phi^{-1}(I \otimes X_u)\Phi
\]

\[
= (Y_1^{-1}S_1^{-1} \otimes Q_1^{-1} + \cdots + Y_k^{-1}S_k^{-1} \otimes Q_k^{-1})
\]

\[
\times (e_1 \otimes iX_u + \cdots + e_k \otimes iX_u)(S_1Y_1 \otimes Q_1 + \cdots + S_kY_k \otimes Q_k)
\]

\[
= e_1 \otimes Q_1^{-1}X_u Q_1 + \cdots + e_k \otimes Q_k^{-1}X_u Q_k
\]

\[
= e_1 \otimes \alpha_1 X_u + \cdots + e_k \otimes \alpha_k X_u.
\]

It follows then that for each \( i = 1, \ldots, k \) we must have \( Q_i^{-1}X_u Q_i = \alpha_i X_u \) for all \( X_u \in e_i \otimes D_u \). As a result, the mapping \( X \to Q_i^{-1} X Q_i \) is a graded involution of an algebra with fine grading \( e_i \otimes D \). By Lemma 4 this mapping must have the form \( X_u \mapsto X_{t_i}^{-1} X_u X_{t_i} \), for some \( t_i \in T \). This allows us to conclude that our matrix \( \Phi \) can be chosen in the form

\[
\Phi = S_1Y_1 \otimes X_{t_1} + \cdots + S_kY_k \otimes X_{t_k}
\]  

(24)

where each \( Y_i \) is in the center of \( B_i \). We have that \( Y_i = \lambda_i e_i \) in every case where \( B_i \) is simple and \( Y_i = \xi_i e_i' + \zeta_i e_i'' \) in every case where \( B_i \) is not simple. Here \( e_i', e_i'' \) are the identities of simple components of \( B_i \) and \( e_i = e_i' + e_i'' \). Also any \( X_{t_i} \) is either symmetric or skew-symmetric.

Now let us check the conditions (23). This is done on case-by-case basis. If \( U = e_i U e_j \in C, 1 \leq i, j \leq l \) then \( \deg U = g_i^{-1}g_j \). Also, using (18) we obtain
that $U^* = e_jY_j^{-1}S_j^{-1}US_iYe_i \otimes X_{ij}^{-1}X_{ti}$ which is of degree $g_j^{-1}g_it_jt_i$ (we recall that by Lemma 4 all elements in $T$ are of order 1 or 2). Therefore, we have an equality $g_j^2t_i = g_j^2t_j$. If $U = e_iUe_j$, $1 \leq i \leq l$, $l + 1 \leq j \leq k$, then $\deg U = g_j^{-1}g_i$. We also have that $U^* = e_jY_j^{-1}S_j^{-1}US_iYe_i \otimes X_{ij}^{-1}X_{ti}$, which is of degree $(g_j^n)^{-1}g_it_jt_i$. It follows then that $g_j^ng_j^n t_j = g_i^2t_i$, also in accordance with (23). Finally, if $U = e_iUe_j$, $1 \leq i \leq j \leq k$, then $\deg U = (g_j^n)^{-1}g_j^n$ while $U^* = e_jY_j^{-1}S_j^{-1}US_iYe_i \otimes X_{ij}^{-1}X_{ti}$. Therefore, $\deg U^* = (g_j^n)^{-1}g_j^n t_jt_i$. It then follows that $g_j^ng_j^n t_j = g_i^ng_i^n t_i$, as required. The remaining three cases are in symmetry with the previous ones and produce the same results. By the way, these calculations also show that if a mapping is given by (14) where $\Phi$ is as in (21) satisfying (23) that this mapping is $G$-graded.

Now we need to eliminate $Y_1, \ldots, Y_k$ from the formula for $\Phi$. Recall the decomposition $R_e = B_1 \oplus \cdots \oplus B_k$ from Lemma 6. Each summand in (24) correspond to one of subalgebras $B_i$. Notice that if we apply an inner automorphism to $R$ then $\Phi$ is changed as a matrix of a bilinear form. If this automorphism is a conjugation by a matrix $P$ with identity bilinear grading then it is an isomorphism of graded algebras. In this isomorphic copy of $R = M_\omega$ the matrix of the involution $\varphi$ will take the form of $\Phi' = \Phi^\omega \Phi$. We build $P$ as $P = P_1 \otimes I + \cdots + P_k \otimes I$ where $P_i \in B_iCB_i$, for each $i = 1, \ldots, k$. If $B_i$ is simple then $Y_i = \xi I$. If $B_i$ is not simple then $Y_i = \zeta I + \xi I$. Here $e_i, \xi$ are the identities of simple components of $B_i$ and $e_i = e_i' + e_i''$. In the matrix form, $Y_i = \begin{pmatrix} \zeta I_i & 0 \\ 0 & \xi I_i \end{pmatrix}$.

Also, $S_i = \begin{pmatrix} 0 & I_i \\ I_i & 0 \end{pmatrix}$.

Notice that since $\varphi$ is an involution, $\Phi^{-1} \Phi = \omega I$ where $\omega = \pm 1$. In other words, $\Phi = \omega \Phi$. Now

$$
\Phi^\omega = Y_1 \xi S_1 \otimes \xi X_{t_1} + \cdots + Y_k \xi S_k \otimes \xi X_{t_k} = Y_1 \xi S_1 \otimes \alpha(t_1, t_1)X_{t_1} + \cdots + Y_k \xi S_k \otimes \alpha(t_k, t_k)X_{t_k}.
$$

Let us set $P_i = \frac{1}{\sqrt{\xi}} e_i$. If $B_i$ is simple then $S_i = \Phi^{-1} P_i S_i Y_i P_i = P_i S_i Y_i P_i = S_i$. 

15
If $B_i$ is not simple then it follows from $^t\Phi = \omega \Phi$ that $\zeta_i = \xi_i \omega \alpha(t_i, t_i)$. Then

$$S'_i = ^tP_i S_i Y_i P_i = P_i S_i Y_i P_i = \frac{1}{\xi_i} \begin{pmatrix} 0 & \xi_i I_{p_i} \\ \xi_i \omega \alpha(t_i, t_i) I_{p_i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{p_i} \\ \omega \alpha(t_i, t_i) I_{p_i} & 0 \end{pmatrix}.$$

For example, if $\varphi$ is a transpose involution, that is, $\Phi$ is symmetric, then $\omega = 1$ and the conjugation by $P$ as above reduces $\Phi$ to the form

$$\Phi = S'_1 \otimes X_{t_1} + \cdots + S'_k \otimes X_{t_k}$$

with

$$^tS' = ^tS'_1 \otimes \alpha(t_1, t_1) X_{t_1} + \cdots + ^tS'_k \otimes \alpha(t_k, t_k) X_{t_k}$$

so that, according to Remark 2, each $S'_i$ is symmetric if and only if $X_{t_i}$ is symmetric, as claimed. If $\varphi$ is a symplectic involution then $\omega = -1$ and using the same equations implies that $S'_i$ is symmetric if and only if $X_{t_i}$ is skew-symmetric.

The converse in the above theorem is immediate. $\square$

Now, for the determination of the gradings on simple matrix Jordan and Lie algebras, it is important to be able to compute the sets of symmetric and skew-symmetric elements of $R = M_n(F)$ under the involution just computed. A very simple remark is as follows:

$$H(R, *) = \text{Span} \{ A + A^* \mid A \text{ from a spanning set of } R \},$$

$$K(R, *) = \text{Span} \{ A - A^* \mid A \text{ from a spanning set of } R \}.$$

If $A = e_i U e_j \otimes X_u$ then $A^* = e_j S_j^{-1} U S_i e_i \otimes ^tX_{t_j} X_u X_{t_i}$. If we perform obvious calculations we obtain the sets of symmetric and skew-symmetric elements of $\varphi$ in the following form

$$H(R, *) = \text{Span} \{ e_i U e_j \otimes X_u + e_j S_j^{-1} U S_i e_i \otimes X_{t_j} X_u X_{t_i} \} \quad (25)$$

where $1 \leq i, j \leq k$, $u \in T$, and $U = e_i U e_i \in C$.

Quite similarly,

$$K(R, *) = \text{Span} \{ e_i U e_j \otimes X_u - e_j S_j^{-1} U S_i e_i \otimes X_{t_j} X_u X_{t_i} \} \quad (26)$$

where $1 \leq i, j \leq k$, $u \in T$, and $U = e_i U e_i \in C$. Here we simultaneously replaced $S_j^{-1}$ and $X_{t_j}^{-1}$ by $S_j$ and $X_{t_j}$.
Incidentally, this gives a canonical form for the simple graded Jordan algebras of the types $H(M_n, \ast)$ where $\ast$ is either transpose or symplectic involution (formula (25)), or a simple Lie algebra of the type $B_l, l \geq 2, C_l, l \geq 3, \text{ or } D_l, l \geq 5$(formula (26)), of which the forms suggested in [5] are a particular case.

References

[1] Bahturin, Yu.A., Montgomery, S., and M. V. Zaicev, *Generalized Lie solvability of associative algebras*, Proceedings of the International Workshop on Groups, Rings, Lie and Hopf Algebras (St. John’s), 1–23, Kluwer, Dordrecht, 2003.

[2] Bahturin, Y., and M. Zaicev, *Graded algebras and graded identities*, Polynomial identities and combinatorial methods (Pantelleria, 2001), 101–139, Lecture Notes in Pure and Appl. Math., 235, Dekker, New York, 2003.

[3] Bahturin, Y., and M. Zaicev, *Group gradings on matrix algebras*, Canad. Math. Bulletin, 45(2002), 499–508.

[4] Bahturin, Y., Sehgal, S, and M. Zaicev, *Group Gradings on Associative Algebras*, J. Algebra 241 (2001), 677–698.

[5] Bahturin, Y., Shestakov, I., and M. Zaicev, *Gradings on Simple Jordan and Lie Algebras*, J. Algebra 283 (2005), 849–868.

[6] Rowen, L. H., “Polynomial Identities in Ring Theory”, Academic Press, New York, 1980.