Exact Dirac Quantization of All 2-D Dilaton Gravity Theories

by

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ABSTRACT: The most general dilaton gravity theory in 2 spacetime dimensions is considered. A Hamiltonian analysis is performed and the reduced phase space, which is two dimensional, is explicitly constructed in a suitable parametrization of the fields. The theory is then quantized via the Dirac method in a functional Schrodinger representation. The quantum constraints are solved exactly to yield the (spatial) diffeomorphism invariant physical wave functionals for all theories considered. These wave functionals depend explicitly on the single configuration space coordinate as well as on the imbedding of space into spacetime (i.e. on the choice of time).

WIN-93-06
September, 1993
1 Introduction

Two dimensional dilaton gravity has recently been the subject of much work, in part because of its relation to string theory in non-critical dimensions\cite{1}, and in part because of its potential usefulness as a theoretical laboratory for examining questions concerning the end-point of Hawking radiation\cite{2}. Quantization of a variety of such models has been explicitly carried out using a variety of techniques\cite{3, 4, 5, 6, 7}. The purpose of this paper is to carry out the Dirac quantization of the most general 2-D dilaton gravity theory in the functional Schrodinger representation. By choosing a suitable parametrization for the fields we solve the quantum constraints exactly to find the unique quantum wave functional for all such theories. The particular technique that we use was first applied by Henneaux\cite{3} to Jackiw-Teitelboim gravity\cite{8}, and more recently in Ref.\cite{9} to spherically symmetric gravity. Similar methods have also been used in Ref.\cite{10} to quantize the CGHS model coupled to conformally invariant matter fields.

The most general action functional depending on the metric tensor $g_{\mu\nu}$ and a scalar field $\phi$ in two spacetime dimensions, such that it contains at most second derivatives of the fields can be written\cite{4}:

$$S[g, \phi] = \int d^2x \sqrt{-g} \left( \frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi - V(\phi) + D(\phi) R \right)$$

where $R$ is the Ricci scalar associated with $g_{\mu\nu}$ and $V(\phi)$ is an arbitrary function of $\phi$. $D(\phi)$ must be a differentiable function of $\phi$, such that $D(\phi) \neq 0$, and $dD(\phi)/d\phi \neq 0$ for any admissible value of $\phi$. Note that if $D(\phi) = constant$, the curvature term decouples and becomes a total divergence, leaving a single, propagating scalar degree of freedom.

As discussed in Ref.\cite{4}, the action in Eq.(1) contains only one independent function of $\phi$. This can be readily seen as follows: The kinetic term for the scalar field can be eliminated by means of the following field redefinition of the metric:

$$\bar{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu}$$

where $\Omega(\phi)$ is the solution to the differential equation:

$$\frac{1}{2} - \frac{1}{2} \frac{dD}{d\phi} \frac{d\ln \Omega}{d\phi} = 0$$

\footnote{We use the conventions for the curvature of Misner Thorne and Wheeler\cite{11}.}
In terms of $\overline{g}$ and a new scalar field $\overline{\phi} := D(\phi)$ the action takes the simple form:

$$S = \int d^2x \sqrt{-\overline{g}} (\overline{\phi} R - \overline{V}(\overline{\phi}))$$

(4)

where the new potential $\overline{V}$ is defined by:

$$\overline{V}(\overline{\phi}) := \frac{V(\phi(\overline{\phi}))}{\Omega^2(\phi(\overline{\phi}))}$$

(5)

Note that the conditions on $D(\phi)$ given above are sufficient, formally to guarantee the existence of $\overline{V}$. For example, in Jackiw-Teitelboim gravity\[^8\], $\overline{V} \propto \overline{\phi}$, while for the CGHS\[^2\] model $\overline{V} = constant$. In the case of spherically symmetric gravity\[^12, 9\] $\overline{V} \propto 1/\sqrt{\overline{\phi}}$.

We first outline the Hamiltonian and reduced phase space analysis of the Lagrangian in Eq.(4). The 2-D spacetime is assumed to be locally a direct product $R \times \Sigma$, where the spatial manifold, $\Sigma$, can at this stage be either open or closed. The metric can be parametrized as follows\[^13\]:

$$ds^2 = e^{2\rho} \left[ -\sigma^2 dt^2 + (dx + M dt)^2 \right]$$

(6)

where $x$ is a local coordinate for the spatial manifold $\Sigma$. In terms of this parametrization the action is (up to surface terms):

$$S = \int dt dx \left[ \frac{\dot{\phi}}{\sigma} (2 M \rho' + 2 M' - 2 \dot{\rho}) + \frac{\dot{\phi}'}{\sigma} (2 \sigma \sigma' - 2 M M' + 2 M \dot{\rho}
+ 2 \sigma^2 \rho' - 2 M^2 \rho') - \sigma^2 e^{2\rho} V(\phi) \right]$$

(7)

In the above, dots and primes denote differentiation with respect to time and space, respectively. The canonical momenta associated with the fields $\{\phi, \rho, \sigma, M\}$ are therefore:

$$\Pi_\phi = \frac{2}{\sigma} (M \rho' + M' - \dot{\rho})$$

(8)

$$\Pi_\rho = \frac{2}{\sigma} (-\dot{\phi} + M \dot{\phi}')$$

(9)

$$\Pi_\sigma = 0$$

(10)

$$\Pi_M = 0$$

(11)

\[^2\]We henceforth drop the bars over the fields.
Clearly (10) and (11) are primary constraints and $\sigma$ and $M$ play the role of Lagrange multipliers. The final canonical Hamiltonian, up to surface terms, is:

$$H_c = \int dx \left[ M\mathcal{F} + \sigma\mathcal{G} \right]$$  \hspace{1cm} (12)

where

$$\mathcal{F} := \rho'\Pi_\rho + \phi'\Pi_\phi - \Pi_\rho' \approx 0$$ \hspace{1cm} (13)

$$\mathcal{G} := 2\phi'' - 2\phi'\rho' - \frac{1}{2}\Pi_\phi\Pi_\rho + e^{2\rho}V(\phi) \approx 0$$ \hspace{1cm} (14)

are secondary constraints. (The notation $\approx 0$ denotes “weakly vanishing in the Dirac sense[14, 15].) It is straightforward to verify that the above constraints are first class, and that no further constraints appear in the Dirac algorithm. $\mathcal{F}$ is the generator of spatial diffeomorphisms on $\Sigma$, whereas $\mathcal{G}$ is the analogue of the Hamiltonian constraint in general relativity and generates time translations. Moreover, $\mathcal{F}$ and $\mathcal{G}$ satisfy the standard constraint algebra associated with diffeomorphism invariant theories in two spacetime dimensions[13]. Note that while $\mathcal{G}$ is quadratic in the momenta, as expected, it is off-diagonal in field space: neither $\Pi^2_\phi$, nor $\Pi^2_\rho$ appears. This is due to the fact that, in this field parametrization, there is no kinetic term for the scalar field. (cf. Eq.(11)). As we shall see later, this is the crucial feature of our chosen parametrization that allows the quantum theory to be solved exactly using these techniques.

In the non-compact case, a surface term must be added to the canonical Hamiltonian to make Hamilton’s equations agree with the Euler Lagrange equations[15]. The specific form of this surface term is determined in part by the boundary conditions on the fields, which in principle vary from one specific model to another.

For completeness we write down Hamilton’s equations for the momenta

$$\dot{\Pi}_\phi = \{\Pi_\phi, H\} = -2\sigma'' - 2(\sigma\phi)' - \sigma e^{2\rho}dV/d\phi + (M\Pi_\phi)'$$ \hspace{1cm} (15)

$$\dot{\Pi}_\rho = \{\Pi_\rho, H\} = (M\Pi_\rho)' - 2(\sigma\phi)' - 2\sigma e^{2\rho}V(\phi)$$ \hspace{1cm} (16)

Apart from the Lagrange multipliers $\sigma$ and $M$ there are two fields, $(\{\phi, \rho\})$ and their conjugate momenta in the canonical Hamiltonian. Given the existence of two first class constraints, there are no propagating modes in the theory. The reduced
phase space is finite dimensional. We will now show that there are two phase space
degrees of freedom. It is useful to define the following linear combination of the
constraints:
\[ \tilde{G} := -e^{-2\rho}(\phi'G + \frac{1}{2}\Pi_{\rho}F) \]
\[ = (C[\rho, \Pi_{\rho}, \phi, \Pi_{\phi}])' \]  \hspace{1cm} (17)
where we have defined the functional \( C \) by
\[ C[\rho, \Pi_{\rho}, \phi, \Pi_{\phi}] := \left( e^{-2\rho} \left( \frac{\Pi_{\rho}^2}{4} - (\phi')^2 \right) - j(\phi) \right) \]  \hspace{1cm} (18)
with \( j(\phi) \) a solution to the equation:
\[ \frac{dj(\phi)}{d\phi} = V(\phi) \]  \hspace{1cm} (19)
As long as \( e^{-2\rho}\phi' \neq 0 \), the set of constraints \( \{ \tilde{G}, F \} \) are equivalent to the original
set \( \{ G, F \} \). \( C \) commutes with both \( \tilde{G} \) and \( F \) and Eq. (17) implies that the constant
mode of the functional \( C \) is unconstrained. Thus \( C \) is a physical observable. The
momentum canonically conjugate to \( C \) must also be physical in the Dirac sense, and
is found to be:
\[ P := -\int dx \frac{2e^{2\rho}\Pi_{\rho}}{(\Pi_{\rho}^2 - 4(\phi')^2)} \]  \hspace{1cm} (20)
It also commutes with the constraints, and has a Poisson bracket with \( C \) of:
\[ \{ C, P \} = 1 \]  \hspace{1cm} (21)
As expected, both \( C \) and \( P \) are global variables: \( C \) is constant on \( \Sigma \) while its conjugate
is defined as a spatial integral.

The reduced phase space for spherically symmetric gravity has been discussed in
some detail by Thiemann and Kastrup\cite{7}. In that case one can show\cite{9} that for static
configurations, the variable \( C \) is equal to the ADM energy of the system. The momen-
tum variable \( P \) on the other hand is associated with global spacetime diffeomorphisms
that transform static solutions into stationary solutions. As discussed in detail for
4-D cosmological models with two Killing vectors by Ashtekar and Samuel\cite{16}, this
explains an apparent discrepancy between the size of the covariant solution space, and
the size of the reduced phase space. For example, in spherically symmetric gravity Birkhoff’s theorem implies that there is only a one parameter family of distinct solutions, while as argued above and in [7], the reduced phase space is two dimensional: the extra phase space variable corresponds to spacetime diffeomorphisms that cannot be implemented canonically. In the present context, this can be verified by noting that an infinitesimal change in $P$ is generated by the constraint $\tilde{G}$ as follows:

$$\delta P = \{P, \int dx \xi(x) \tilde{G}\} = \int dx \xi'(x)$$

(22)

This is zero in the compact case, and in the non-compact case for test functions $\xi$ that vanish at the boundaries. Since gauge transformations are defined precisely in terms of such test functions, $P$ is gauge invariant in this sense and commutes with the constraints as claimed above. However, the observable $P$ can nonetheless be changed by a “non-canonical” diffeomorphism; one for which $\xi$ does not obey the usual boundary conditions. From the spacetime viewpoint, one must consider the following coordinate transformation:

$$t \rightarrow t + f(r)$$

(23)

For constant $f(r)$, this is merely a temporal translation. However, if $f'(r) \neq 0$ such a transformation changes a static solution to one that is stationary. In particular, under the infinitesimal form of Eq.(23), the lapse function goes from zero to:

$$\delta M = \sigma^2 f'(r)$$

(24)

while,

$$\delta P = \int dx \frac{e^{2\rho} \sigma}{\sigma'} f'$$

(25)

For static solutions, it can be shown that $\frac{e^{2\rho} \sigma}{\sigma'} = \text{constant}$. A comparison of Eq.(22) and Eq.(24) therefore shows that, when starting from static solutions, $\xi(r) \propto f(r)$. Thus, as claimed in Refs.[16] and [7] the observable $P$ is related to the existence of stationary but non-static solutions to the field equations. Of course, the above analysis is only relevant in the non-compact case: when $\Sigma$ is compact, changes in $P$ cannot be implemented by the transformations Eq.(22) or Eq.(25).
We now proceed with the Dirac quantization of the theory in the functional Schrodinger representation, in which states are given by functionals of the fields, namely
\[ \Psi = \Psi[\phi, \rho] \] (26)

As usual in the Schrodinger representation, we define the momentum operators as (functional) derivatives:
\[ \hat{\Pi}_\phi = -i\hbar \delta \delta \phi(x) \] (27)
\[ \hat{\Pi}_\rho = -i\hbar \delta \delta \rho(x) \] (28)

The next step\[3, 9\], is to note that the constraints \( F \) and \( G \) (or equivalently \( F \) and \( \tilde{G} \) can be solved for the momenta as follows:
\[ \Pi_\phi \approx g[\rho, \phi]Q[C; \phi, \rho] \] (29)
\[ \Pi_\rho \approx Q[C; \phi, \rho] \] (30)

where we have defined:
\[ Q[C; \phi, \rho] := 2\sqrt{\left(\phi'\right)^2 + (C + j(\phi))e^{2\rho}} \] (31)
\[ g[\phi, \rho] := 4\phi'' - 4\phi' \rho' + 2e^{2\rho}V(\phi) \] (32)

In Eq.(31), \( C \) is a constant of integration that corresponds precisely to the observable defined in Eq.(18).

We now define physical states \( \Psi_{phys}[\phi, \rho] \), as those that are annihilated by the constraints:
\[ \left( \hat{\Pi}_\phi - \frac{g[\phi, \rho]}{Q[C; \phi, \rho]} \right) \Psi_{phys} = 0 \]
\[ \left( \hat{\Pi}_\rho - Q[C; \phi, \rho] \right) \Psi_{phys} = 0 \] (33)

The solution to these constraints take the form:
\[ \Psi_{phys}[C; \phi, \rho] = \exp \left( \frac{i}{\hbar} S[C; \phi, \rho] \right) \] (34)

\[^3\text{The constraints } \Pi_M \approx 0 \text{ and } \Pi_\sigma \approx 0 \text{ will require the wave functional to be independent of the arbitrary Lagrange multipliers } M \text{ and } \sigma. \text{ We therefore drop them from the following discussion.}\]
where $S$ satisfies the linear, coupled functional differential equations:

\[
\frac{\delta}{\delta \phi} S[C; \phi, \rho] = \frac{g[\phi, \rho]}{Q[C; \phi, \rho]}, \quad \frac{\delta}{\delta \rho} S[C; \phi, \rho] = Q[C; \phi, \rho]
\] (35)

The second of these equations can be directly integrated, since it does not involve spatial derivatives, to yield:

\[
S[C; \phi, \rho] = \int dx \left[ Q + \phi' \ln \left( \frac{2\phi' - Q}{2\phi' + Q} \right) \right] + F[C; \phi]
\] (36)

where $F[C; \phi]$ as an arbitrary functional independent of $\rho$. Remarkably, the remaining operator constraint, applied to Eq.(36) yields the simple result that $F[C; \phi] = F[C] = \text{constant}$ (independent of $\phi$). Thus physical states are described by wave functionals of the form:

\[
\Psi_{\text{phys}}[\phi, \rho] = \chi[C] \exp \frac{i}{\hbar} \int dx \left[ Q + \phi' \ln \left( \frac{2\phi' - Q}{2\phi' + Q} \right) \right],
\] (37)

where $\chi[C] = \exp i/\hbar F[C]$ is a completely arbitrary function of the configuration space coordinate $C$. This arbitrariness in the wave functional is a consequence of the fact that the reduced Hamiltonian for the system (in terms of $C$ and $P$) vanishes identically.

The wave functional in Eq.(37) is invariant under spatial diffeomorphisms generated by the quantum constraint:

\[
\hat{F} = \rho' \hat{\Pi}_\rho + \phi' \hat{\Pi}_\phi - \frac{\partial}{\partial x} \hat{\Pi}_\rho
\] (38)

It is also annihilated by the Hamiltonian constraint, with factor ordering:

\[
\hat{G} = \frac{1}{2} g[\phi, \rho] - \frac{1}{2} Q \hat{\Pi}_\rho Q^{-1} \hat{\Pi}_\rho
\] (39)

We also note that the phase of the wave functional becomes complex when $Q$ is imaginary. This is consistent with traditional quantum mechanics: $Q^2 < 0$ corresponds to classically forbidden regions in which the classical momenta are imaginary (cf.
The phase can also pick up an imaginary part when the argument of the logarithm is negative, i.e. when

\[ 4(\phi')^2 - Q^2 < 0 \]  

(40)

The physical significance of this contribution is less clear at this stage. However, it is interesting to note that in the case of spherically symmetric gravity, Eq.(40) is satisfied for static solutions expressed in Kruskal coordinates when \( r < 2m \).

Finally, we remark that the wave functional \( \Psi_{phys} \) yields, as expected, a consistent representation for the physical phase space in the Schrodinger representation. That is, one can explicitly verify the relation:

\[-i\hbar \frac{\partial}{\partial C} \Psi_{phys} = P \Psi_{phys}\]  

(41)

where \( P \) is equal to the expression in Eq.(20) evaluated on the constraint surface.

The solution Eq.(37) has very interesting features that highlight several important issues in quantum gravity. As the notation indicates, the wave functional is an explicit function of the physical configuration space coordinate \( C \) as well a functional of the imbedding variables \( \rho \) and \( \phi \). Invariance of the wave functional under spatial diffeomorphisms guarantees that one of the two functions is redundant: it can be trivially eliminated by choosing an appropriate spatial coordinate. The remaining function is essentially the time variable in the problem. Different choices correspond to different time slicings. The fact that the wave functional depends in a non-trivial way on this choice of time slice (as opposed to the choice of spatial coordinate) is sometimes referred to as the many-fingered time problem in quantum gravity [17]. However, in the present context it can also be interpreted as a consequence of the fact that the solution Eq.(37) which solves the Hamiltonian constraint is analogous to a time dependent Schrodinger state.

We have so far avoided the question of the correct Hilbert space measure. One can perhaps ask whether there exists a functional measure on the space of \( \phi \) and \( \rho \) that makes this state normalizable. Alternatively, one can first choose a spatial coordinate and time slicing and then define the measure only on the (one-dimensional) space of physical observables. This question can perhaps best be studied in the context of particular models, such as spherically symmetric gravity or the CGHS model. In any case, it is hoped that the existence of the above exact solution in a the most general theory of 2-D dilaton gravity provides a useful laboratory for the study of this, and other fundamental questions in quantum gravity.
Finally we remark that in Ref.[9], the solution Eq.(37) was applied to spherically symmetric gravity in order to find the quantum wave functional for an isolated black hole. This wave functional was shown to have interesting properties consistent with the presence of a quantum mechanical instability (i.e. Hawking radiation.) There is in fact a large class dilaton gravity theories that possess Schwarzschild type black hole solutions. A complete analysis of the exact quantum wave functional for these theories, along the lines of Ref.[9] is currently in progress.

Acknowledgements We are grateful to Steve Carlip for helpful conversations and a critical reading of the manuscript. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

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