1. Introduction and main results

1.1. Pair correlation. Let \((x_n)_{n=1}^\infty\) be a sequence on the one-dimensional Torus \(\mathbb{T} \cong [0,1]\). A natural object of interest is the behavior of gaps between the first \(N\) elements on a local scale. If the sequence is comprised of i.i.d. uniformly distributed random variables, then, for all \(s > 0\),

\[
\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{s}{N} \right\} = 2s \quad \text{almost surely.}
\]

We show that being close to Poissonian pair correlation for few values of \(s\) is enough to deduce global regularity statements: if, for some \(0 < \delta < 1/2\), a set of points \(\{x_1, \ldots, x_N\}\) satisfies

\[
\frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{s}{N} \right\} \leq (1 + \delta)2s \quad \text{for all } 1 \leq s \leq (8/\delta) \sqrt{\log N},
\]

then the discrepancy \(D_N\) of the set satisfies \(D_N \leq 8^{1/3} + N^{-1/3} \delta^{-1/2}\). We also show that distribution properties are reflected in the global deviation from the Poissonian pair correlation

\[
N^2 D_N^2 \leq \frac{2}{N} \int_0^{N/2} \frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{s}{N} \right\} - 2s^2 ds \lesssim N^2 D_N^2,
\]

where the lower is bound is conditioned on \(D_N \gtrsim N^{-1/3}\). The proofs use a connection between exponential sums, the heat kernel on \(\mathbb{T}\) and spatial localization. Exponential sum estimates are obtained as a byproduct. We also describe a connection to diaphony and several open problems.

1.2. A local result. The first result shows that being close to Poissonian pair correlation for a small range of values of \(s\) can be enough to conclude global regularity results. We quantify regularity using the discrepancy \(D_N\) of a finite point set, defined in the usual manner as the maximal deviation between empirical and uniform distribution

\[
D_N = \sup_{I \subset \mathbb{T}} \left| \frac{\# \{x_1, x_2, \ldots, x_N\} \cap I}{N} - |I| \right|,
\]

where the supremum ranges over all intervals \(I \subset \mathbb{T}\).
Theorem 1. Let \( \{x_1, \ldots, x_N\} \subset [0, 1] \) and \( 0 < \delta \ll 1 \) such that

\[
\frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{s}{N} \right\} \leq (1 + \delta)2s \quad \text{for all} \quad 1 \leq s \leq (8/\delta)\sqrt{\log N},
\]

then the discrepancy of the set satisfies \( D_N \lesssim \delta^{1/3} + \delta^{-1/2}N^{-1/3} \).

This should be compared to a result of Grepstad & Larcher [3] that being \( \delta \)--close to Poissonian pair correlation for \( s \in \{1, \ldots, \delta^{-3}\} \) implies \( D_N \lesssim \delta \). This result and Theorem 1 are clearly of the same flavor but cover somewhat different scaling regimes – we have no reason to assume that these results are optimal. There should be many other interesting results along these lines.

Open Problem (Global regularity via local pair correlation statistics). What is the smallest range of values of \( s \) for which one needs to require approximate Poissonian pair correlation statistics to ensure some regularity of the distribution?

The proof of Theorem 1 is based on the use of Fourier analysis to obtain an exponential sum estimate: we show that for \( 0 < \delta \ll 1 \) the assumptions of Theorem 1 imply

\[
\sum_{k \neq 0} \left| \sum_{n=1}^{N} e^{2\pi ikx_n} \right|^2 \lesssim \delta N^2.
\]

It is instructive to study the case of randomly chosen points: then each of the \( \sim \delta^{3/2}N \) squared exponential sums are of expected size \( \sim N \) and the expression would be \( \sim \delta^{3/2}N^2 \). It is not clear to us whether such a bound holds or whether the assumptions in Theorem 1 allow for point sets that are substantially different from randomly chosen points.

1.3. A global result. We show that well-distributed sequences have pair correlation globally close to Poissonian. For somewhat irregular sets, \( D_N \gtrsim N^{-1/3} \), the converse direction also holds.

Theorem 2. Let \( \{x_1, \ldots, x_N\} \subset [0, 1] \). Then

\[
\frac{2}{N} \int_0^{N/2} \left| \frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{s}{N} \right\} - 2s \right|^2 \, ds \lesssim N^2 D_N^2.
\]

Moreover, if \( D_N \gtrsim N^{-1/3} \), then

\[
\frac{2}{N} \int_0^{N/2} \left| \frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{s}{N} \right\} - 2s \right|^2 \, ds \gtrsim N^2 D_N^2.
\]

The statement is sharp in the regime \( D_N \sim 1 \) since upper and lower bound match but it is very clearly not sharp anywhere else. In particular, it would be quite nice to see whether one could possibly obtain results of such a flavor for a more restricted range of values of \( s \).

Open problem. Can Theorem 2 be improved/sharpened/localized?

1.4. Concluding Remarks. The proof of Theorem 2 makes use of LeVeque’s upper bound [8] derived from the Erdős-Turan inequality; the arising exponential sum is sometimes called the diaphony [23]

\[
F_N := \left( 2 \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \sum_{n=1}^{N} e^{2\pi ikx_n} \right|^2 \right)^{1/2} = \left( \frac{\pi^2}{2N^2} \sum_{m,n=1}^{N} (1 - 2 \{x_m - x_n\}^2 - \frac{1}{3}) \right)^{1/2}.
\]

A byproduct of our proof of Theorem 2 is the following Corollary.

Corollary. Let \( \{x_1, \ldots, x_N\} \subset [0, 1] \). Then

\[
A = \frac{2}{N} \int_0^{N/2} \left| \frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{s}{N} \right\} - 2s \right|^2 \, ds
\]

is bounded from above by

\[
A \leq \pi^{-2} F_N^2 (\{x_n - x_m : 1 \leq n, m \leq N\})^2 + 1.
\]
Since there are many results dealing with diaphony of deterministic sequence (see for example [11, 12, 13]), this Corollary could suggest that there might be some hope of getting refined results for the pair correlation of deterministic sequences. There is another series of results that seem connected in spirit: given a set \( \{x_1, \ldots, x_N\} \subset \mathbb{T} \), we may consider the difference set \( \{x_i - x_j : 1 \leq i, j \leq N\} \) and ask how the discrepancy \( D_N \) of the set relates to the discrepancy of the difference set \( D_{N^2} \). Improving earlier results by Vinogradov [21] and van der Corput & Pisot [20], Cassels [3] showed

\[
D_N \lesssim \sqrt{D_{N^2}(1 + \log D_{N^2})}.
\]

Motivated by this result, we quickly note another approach that follows rather quickly from the Erdős-Turan inequality but may prove useful for such problems or even be of independent interest.

**Proposition.** Let \( \{x_1, \ldots, x_N\} \subset \mathbb{T} \). There is a discrepancy bound

\[
D_N \leq \frac{\sqrt{\log N}}{N} \sum_{m,n=1}^{N^\prime} \min \left\{ \log N, \log \left( \frac{1}{4 \sin (\pi (x_m - x_n))^2} \right) \right\}.
\]

We observe that

\[
\int_0^{1/2} \log \left( \frac{1}{4 \sin^2 \pi x} \right) dx = 0,
\]

which indicates that there should be cancellation in the sum if the set of points has a pair correlation close to Poissonian. While this approach might not yield localized estimates it could conceivably lead to results along the lines of Theorem 2. Finally, unconnected to these other results, we note two other curious by-product of the proof of Theorem 2. The first is another proof which indicates that there should be cancellation in the sum if the set of points has a pair correlation (\( \pi^2 / 6 \) given after the proof). The second implication is an exponential sum estimate.

**Corollary.** For all \( \{x_1, \ldots, x_N\} \subset \mathbb{T} \)

\[
N \sum_{k=1}^{N} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \leq \sum_{k=1}^{\infty} \frac{2}{k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 + \pi^2 N^2.
\]

One interpretation of that inequality is that any finite set of points cannot only be irregular with respect to odd frequencies. It could be interesting to see whether this inequality is part of a larger family of inequalities, at least visually it seems to have a certain interpolatory flavor. We conclude by remarking that a weaker notion was already introduced in [18], where it was shown that if \( (x_n)_{n=1}^{\infty} \) is a sequence on \( \mathbb{T} \), \( 0 < \alpha < 1 \) and for all \( s > 0 \)

\[
\lim_{N \to \infty} \frac{1}{N^{2-\alpha}} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{s}{N^\alpha} \right\} = 2s \quad \text{a.s.,}
\]

then the sequence \( (x_n) \) is uniformly distributed. We note that this interpolates between Poissonian pair correlation (\( \alpha = 1 \)) and a classical notion of uniform distribution (\( \alpha = 0 \)).

**Open problem.** Do 'most' sequences satisfy this property for some \( 0 < \alpha < 1 \)?

It seems conceivable that \( D_N \lesssim N^{-\beta} \) would imply the property for all \( \alpha < \beta \). However, there are other natural notions that could be of interest, we specifically mention conditions like

\[
\int_{s-\frac{t}{N}}^{s+\frac{t}{N}} \frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{t}{N} \right\} dt \to \infty \quad \text{for } s \geq 1/2
\]

or, for \( s \gg 1 \) and \( u = o(s) \),

\[
\frac{1}{2u} \int_{s-u}^{s+u} \frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : |x_m - x_n| \leq \frac{t}{N} \right\} dt \to \infty \quad 2s + o(u).
\]
2. Proof of Theorem 1

2.1. Preliminaries. We will use the Jacobi $\theta$–function given by

$$\theta_t(x) = \sum_{k \in \mathbb{Z}} e^{-4\pi^2 k^2 t} e^{2\pi i k x} = 1 + 2 \sum_{k=1}^{\infty} e^{-4\pi^2 k^2 t} \cos 2\pi k t.$$ 

Basic properties are $\theta_t(x) \geq 0$ and

$$\int_\mathbb{T} \theta_t(x) dx = 1.$$ 

We will use it as a tool that allows us to localize functions: convolution with $\theta_t$ is easy to compute since its Fourier series is explicit. Simultaneously, convolution has little effect on the function since $\theta_t(x)$ is highly localized: $\theta_t(x)$ acts as the heat kernel on $\mathbb{T}$ and thus, for $t$ small, is well-approximated by the Euclidean heat kernel

$$\theta_t(x) \sim \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}.$$ 

There are various ways of making this notion precise, one of them being that the heat kernel $k_t$ on $\mathbb{R}$ satisfies

$$k_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} \quad \text{and} \quad \theta_t(x) = \sum_{k \in \mathbb{Z}} k_t(x + k).$$

The second ingredient that we need is a fairly basic rearrangement statement: its underlying idea is far from novel but this particular case may not have been stated before (though it can be proved in the usual completely standard manner).

**Lemma.** Let $f : [0, \infty] \to \mathbb{R}_{\geq 0}$ be a strictly monotonically decreasing function and suppose that the finite measure $\mu$ on $[0, \infty]$ satisfies $\mu([0, x]) \leq \phi(x)$ for all $\alpha < x < \beta$ for some $\phi \in C^1$. Then

$$\int_0^\infty f d\mu \leq f(0)\phi(\alpha) + \int_\alpha^\beta f(x)\phi'(x)dx + f(\beta)\mu(\mathbb{R}_{\geq 0} \setminus [0, \beta]).$$

The proof is an elementary rearrangement argument, see e.g. Lieb & Loss [9], and is left to the reader. Indeed, it is not difficult to see that the right-hand side is sharp and the extremal measure $\mu$ can be characterized: it has point mass $\phi(\alpha)$ in 0, the absolutely continuous density $\phi'(x)dx$ on $[\alpha, \beta]$ and another point mass at $\beta$. We will not use the characterization and, when applying the result, replace the last term by the larger quantity $f(\beta)\mu(\mathbb{R}_{\geq 0})$.

2.2. Proof of the Theorem.

**Proof.** For $t$ small, one summand dominates the remaining expression. We start the argument by using an idea from [11 19]: for arbitrary $X > 0$

$$\sum_{|k| \leq \delta^{3/2} N} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \leq e^{4\pi^2 \delta} \sum_{k \in \mathbb{Z}} e^{-4\pi^2 k^2 / (\delta N)^2} \sum_{n=1}^{N} e^{2\pi i k (x_m - x_n)}$$

$$= e^{4\pi^2 \delta} \sum_{k \in \mathbb{Z}} e^{-4\pi^2 k^2 / (\delta N)^2} \sum_{m,n=1}^{N} e^{2\pi i k (x_m - x_n)}$$

$$= e^{4\pi^2 \delta} \sum_{m,n=1}^{N} \sum_{k \in \mathbb{Z}} \theta(\delta N)^{-2} (x_m - x_n).$$

We introduce the measure (given as the finite sum of Dirac measures)

$$\mu = \sum_{m,n=1}^{N} \delta_{x_m - x_n}.$$
The function \( \theta(\delta N)^{-2} \) is monotonically decaying away from the origin and the measure \( \mu \) satisfies
\[
\mu([-s, s]) \leq (1 + \delta) 2s N^2 \quad \text{for all} \quad \frac{1}{N} \leq s \leq \frac{8 \sqrt{\log N}}{\delta}.
\]
This implies, using the symmetry of \( \mu \) and the previous Lemma,
\[
\int_T \theta(\delta N)^{-2}(x) d\mu \leq \theta(\delta N)^{-2}(0)(1 + \delta) 2N + (1 + \delta) 2N^2 \int_{1/N}^{(8/\delta) \sqrt{\log N}/N} \theta(\delta N)^{-2}(x) dx
\]
\[
+ \theta(\delta N)^{-2} \left( \frac{8 \sqrt{\log N}}{\delta N} \right) N^2.
\]
We observe that
\[
\int_{1/N}^{(8/\delta) \sqrt{\log N}/N} \theta(\delta N)^{-2}(x) dx \leq \frac{1}{2} \int_T \theta(\delta N)^{-2}(x) dx = \frac{1}{2}.
\]
We also observe that
\[
\theta(\delta N)^{-2} \left( \frac{8 \sqrt{\log N}}{\delta N} \right) N^2 \leq (1 + o(1)) \frac{\delta N}{\sqrt{4\pi}} \exp \left( -\frac{\delta^2 N^2}{4} \frac{64 \log N}{N^2} \frac{\delta^2}{2} \right) N^2 \ll 1.
\]
Furthermore
\[
\theta(\delta N)^{-2}(0) \sim (1 + o(1)) \frac{\delta N}{\sqrt{4\pi}}
\]
where \( o(1) \to 0 \) as \( N \to \infty \). Summing up, we obtain
\[
\sum_{|k| \leq \delta^{1/2} N} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \lesssim e^{4\pi^2 \delta} \left( \delta N^2 + (1 + \delta) N^2 \right).
\]
and thus, subtracting the value \( N^2 \) coming from \( k = 0 \),
\[
\sum_{k \neq 0} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \lesssim \delta N^2.
\]
We can now employ LeVeque’s inequality \([5]\) to conclude that
\[
ND(N) \lesssim \left( \sum_{k=1}^{\infty} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \right)^{1/3} \lesssim N^{1/3} \left( \delta N^2 + \sum_{k=\delta^{1/2} N}^{\infty} \frac{1}{k^2} \right)^{1/3} \lesssim N \left( \delta + \sum_{k=\delta^{1/2} N}^{\infty} \frac{1}{k^2} \right)^{1/3} \lesssim N \left( \delta + \frac{1}{\delta^{3/2} N} \right)^{1/3}.
\]
We use the trivial estimate \( \lesssim N^2 \) on the remaining exponential sum
\[
N^{1/3} \left( \delta N^2 + \sum_{k=\delta^{1/2} N}^{\infty} \frac{1}{k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \right)^{1/3} \lesssim N \left( \delta + \sum_{k=\delta^{1/2} N}^{\infty} \frac{1}{k^2} \right)^{1/3} \lesssim N \left( \delta + \frac{1}{\delta^{3/2} N} \right)^{1/3}.
\]
It is easily seen that
\[
\left( \delta + \frac{1}{\delta^{3/2} N} \right)^{1/3} \lesssim \begin{cases} 
\delta^{1/3} \quad & \text{if} \ \delta \geq N^{-2/5} \\
N^{-1/3} \delta^{-1/2} \quad & \text{if} \ \delta \lesssim N^{-2/5}.
\end{cases}
\]
It is easy to pinpoint where the argument is lossy: in the absence of more information, we assume that the measure \( \mu \) is clustered immediately outside of \( s = (8/\delta) \sqrt{\log N} \). If this could be excluded, then further improvements could be obtained from the same argument.
3. Proof of Theorem 2.

Proof. We use $\chi$ to denote, as usual, the characteristic function of a set and start by rewriting the problem as (note the change of scale $s/N \to s$)

$$\# \{1 \leq m \neq n \leq N : |x_m - x_n| \leq s\} = \left< \left( \sum_{i=1}^{N} \delta_{x_i} \right) \ast \chi_{[-s,s]} \right> \sum_{i=1}^{N} \delta_{x_i} - N.$$

Plancherel's theorem implies

$$\left< \left( \sum_{i=1}^{N} \delta_{x_i} \right) \ast \chi_{[-s,s]} \right> \sum_{i=1}^{N} \delta_{x_i} - N = \sum_{k \in \mathbb{Z}} \hat{\chi}_{[-s,s]}(k) \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 - N$$

and removing the frequency $k=0$ allows us to rewrite the expression as

$$\sum_{k \in \mathbb{Z}} \hat{\chi}_{[-s,s]}(k) \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 - N = \sum_{k \in \mathbb{Z}, k \neq 0} \hat{\chi}_{[-s,s]}(k) \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 + 2sN^2 - N$$

This implies that the quantity

$$\frac{A}{2} = \int_0^{1/2} \left< \frac{1}{N} \# \{1 \leq m \neq n \leq N : |x_m - x_n| \leq s\} - 2Ns \right>^2 ds$$

can be written as

$$\frac{B}{2} = \int_0^{1/2} \left< \frac{1}{N} \sum_{k \in \mathbb{Z}, k \neq 0} \hat{\chi}_{[-s,s]}(k) \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 - 1 \right>^2 ds.$$

We square the expression and deal with the three terms separately. The first term is

$$\frac{1}{N^2} \int_0^{1/2} \sum_{k, m \in \mathbb{Z}, k \neq 0, m} \hat{\chi}_{[-s,s]}(k) \hat{\chi}_{[-s,s]}(m) \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \sum_{n=1}^{N} e^{2\pi i m x_n} \right|^2 \int_0^{1/2} \hat{\chi}_{[-s,s]}(k) \hat{\chi}_{[-s,s]}(m) ds,$$

which can be rearranged as

$$\frac{1}{N^2} \sum_{k, m \in \mathbb{Z}, k \neq 0, m} \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \sum_{n=1}^{N} e^{2\pi i m x_n} \right|^2 \int_0^{1/2} \hat{\chi}_{[-s,s]}(k) \hat{\chi}_{[-s,s]}(m) ds.$$

We quickly compute all arising integrals: for $k \in \mathbb{Z}$ and $k \neq 0$,

$$\hat{\chi}_{[-s,s]}(k) = \int_{-1/2}^{1/2} \chi_{|y| \leq s} e^{-2\pi i k y} dy ds = \frac{\sin (2k\pi s)}{k\pi}.$$

It is easy to see that the expression vanishes when integrated over $[0, 1/2]$ if $k$ is even (and $k \neq 0$). If $k$ is odd, then

$$\int_0^{1/2} \frac{\sin (2k\pi s)}{k\pi} ds = \int_{k/2}^{1/2} \frac{\sin (2k\pi s)}{k\pi} ds = \frac{1}{k^2 \pi^2}.$$

Moreover, for $k, m \in \mathbb{Z}$, $|k| \neq |m|$,

$$\int_0^{1/2} \hat{\chi}_{[-s,s]}(k) \hat{\chi}_{[-s,s]}(m) ds = \int_0^{1/2} \frac{\sin (2k\pi s) \sin (2m\pi s)}{m\pi} ds = 0.$$

Finally, we remark that

$$\int_0^{1/2} \hat{\chi}_{[-s,s]}(k)^2 ds = \int_0^{1/2} \hat{\chi}_{[-s,s]}(k)^2 ds$$

$$= \int_0^{1/2} \left( \frac{\sin (2k\pi s)}{k\pi} \right)^2 ds = \frac{1}{k^2 \pi^2} \int_0^{1/2} \sin (2k\pi s)^2 ds = \frac{1}{4k^2 \pi^2}.$$
Therefore, the expression simplifies to
\[
\frac{1}{N^2} \sum_{k \in \mathbb{Z}} \frac{1}{2k^2 \pi^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 = \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4.
\]

The second term simplifies to
\[
-\frac{2}{N} \sum_{k \in \mathbb{Z}, k \neq 0} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \int_{0}^{1/2} \left| \hat{\chi}_{[-s,s]}(k) \right|^2 ds = -\frac{2}{N} \sum_{k \in \mathbb{Z}, k \neq 0} \frac{1}{k^2 \pi^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2
\]
\[
= -\frac{4}{N} \sum_{k \in \mathbb{N} \setminus (2\mathbb{N})} \frac{1}{k^2 \pi^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2
\]

and the third term is trivially 1/2. Altogether,
\[
A = \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{2}{\pi^2 k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 - \frac{1}{N} \sum_{k \in \mathbb{N} \setminus (2\mathbb{N})} \frac{8}{k^2 \pi^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 + 1.
\]

The argument shows that we have an essentially explicit expression for the squared deviation; the remaining difficulty is to estimate the two exponential sums. The inequality of LeVeque [8] bounds the second term in size from above by
\[
\frac{1}{N} \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \lesssim \frac{1}{N^2} \left( \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \right) \lesssim ND_N^2.
\]

We can use the same inequality to also conclude that
\[
\frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{4\pi^2 k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 \lesssim \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 \lesssim N^2 D_N^2.
\]

We will now compute a lower bound for this term as well: recall the Erdős-Turan inequality
\[
ND_N \leq \inf_{K \in \mathbb{N}} \frac{N}{K+1} + 3 \sum_{k=1}^{K} \frac{1}{k} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|.
\]

Denoting the right-hand side by \(E_N\), we summarize the Erdős-Turan inequality as \(E_N \gtrsim ND_N\) and conclude that
\[
ND_N \lesssim \sum_{k=1}^{N/E_N} \frac{1}{k} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|.
\]

Using Hölder’s inequality \(L^{4/3} \times L^4 \rightarrow L^1\), we obtain
\[
ND_N \lesssim \sum_{k=1}^{N/E_N} \frac{1}{k} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right| \leq \left( \sum_{k=1}^{N/E_N} \frac{1}{k^{2/3}} \right)^{3/4} \left( \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 \right)^{1/4}
\]
\[
\lesssim \left( \frac{N}{E_N} \right)^{1/4} \left( \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 \right)^{1/4}
\]

and thus
\[
\frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 \gtrsim ND_N E_N \gtrsim N^2 D_N^5.
\]
Finally, whenever $N^2D_N^5 \gtrsim ND_N^2$ (which occurs for $D_N \gtrsim N^{-1/3}$), the positive terms dominates the negative term and

$$A \gtrsim \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{4\pi^2 k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 \gtrsim N^2 D_N^5.$$ 

\[ \square \]

**Remark.** The proof has an amusing consequence if we plug in the set \{0, 1/2\}. We observe that

$$A = 2 \int_{0}^{1/2} \left( \frac{1}{N} \# \{1 \leq m \neq n \leq N : |x_m - x_n| \leq s\} - 2Ns \right)^2 ds = 2 \int_{0}^{1/2} (4s)^2 ds = \frac{4}{3}.$$ 

At the same time, the exponential sum is very easy

$$\sum_{n=1}^{2} e^{2\pi i k x_n} = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}.$$ 

The formula

$$A = \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{2}{\pi^2 k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 - \frac{1}{N} \sum_{k \in \mathbb{N}\setminus\{2N\}} \frac{8}{k^2\pi^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 + 1$$

obtained in the proof of Theorem 2 thus simplifies to

$$4 = \frac{1}{3} \sum_{k=1}^{\infty} \frac{2}{\pi^2 k^2} (16 \cdot 1, \text{k is even}) + 1 = \frac{1}{4} \sum_{k=1}^{\infty} \frac{32}{\pi^2 (2k)^2} + 1 = 2 \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2} + 1$$

and thus $\zeta(2) = \pi^2/6$.

### 3.1. Proof of the Corollaries

The second Corollary is easy to establish. We have

$$0 \leq A = \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{2}{\pi^2 k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 - \frac{1}{N} \sum_{k \in \mathbb{N}\setminus\{2N\}} \frac{8}{k^2\pi^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 + 1$$

and the desired inequality follows from a re-formulation. The first Corollary can be see as follows

$$A \leq \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{2}{\pi^2 k^2} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^4 + 1 = \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{2}{\pi^2 k^2} \left| \sum_{m,n=1}^{N} e^{2\pi i k (x_n - x_m)} \right|^2 + 1.$$ 

The desired inequality then follows from the definition of diaphony $F_N$.

### 4. Proof of the Proposition

**Sketch of Proof.** We use the Erdős-Turan inequality to conclude that

$$ND_N \lesssim \frac{N}{k} \left| \sum_{n=1}^{N} e^{2\pi i k x_n} \right| \lesssim \left( \sum_{k=1}^{N} \frac{1}{k} \right)^{1/2} \left( \sum_{k=1}^{N} \frac{1}{k} \sum_{n=1}^{N} e^{2\pi i k x_n} \right)^{1/2} \lesssim \sqrt{\log N} \left( \sum_{m,n=1}^{N} \sum_{k=1}^{N} \frac{\cos (2\pi k (x_m - x_n))}{k} \right)^{1/2}$$

The main inside is that the inner sum resembles a well-known Fourier series

$$\sum_{k=1}^{\infty} \frac{\cos (2\pi k x)}{k} = \log \left( \frac{1}{4\sin^2 \pi x} \right)$$

and there is fast convergence away from the integers. Close to the origin, we may use

$$\left| \sum_{k=1}^{N} \frac{\cos (2\pi k x)}{k} \right| \lesssim \log N$$
and the transition region can be dealt with by standard methods.

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