On the $N_f$ and $a$ dependence of $B_K$

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We present results of a study of $B_K$ using tadpole improved gauge-invariant staggered operators. Using three ensembles of $16^3 \times 32$ configurations with varying numbers of dynamical flavors, we observe a small dependence on $N_f$. Using 7 quenched ensembles at different values of $\beta$, we extrapolate to $a = 0$.

1. Introduction

The $B_K$ parameter serves to parameterize the weak hadronic matrix element responsible for $K^0 - \bar{K}^0$ mixing. Since this mixing gives us the only CP violation observed to date, $B_K$ is a crucial link between the measured quantity $\epsilon$ and the parameters of the Standard Model. Lattice calculations are well suited for the study of $B_K$ parameter, and it has by now received much attention. After an early round of calculations\cite{1–3}, the statistics have now been raised to a level which allows one to examine some of the fine points of the calculation, such as checks on the reliability of one-loop lattice perturbation theory\cite{4}, the chiral behavior and nondegenerate quark masses\cite{5,7}, the dependence of $B_K$ on the lattice spacing\cite{3,6,7} and the number of dynamical flavors\cite{8}. In this note we offer more information on these latter two points.

2. Calculational Setup

Table 1

| $N_f$ | $\beta$ | $N_{\text{config}}$ | $m_qa$ |
|-------|---------|---------------------|--------|
| 0     | 6.05    | 306                 | 0.384(5) |
| 2     | 5.7     | 83                  | 0.384(4) |
| 4     | 5.4     | 69                  | 0.391(7) |

For the dynamical fermion comparison we use lattices of geometry $16^3 \times 32$, with parameters as given in table 1. The quenched configurations were generated on the Ohio Supercomputer Center T3D, while the dynamical configurations with two and four flavors of $m_qa = 0.01$ staggered fermions were generated on the 256-node Columbia machine. The parameters were chosen so as to make the scales of the lattices exceedingly close (and equal to approximately $(2 \text{ GeV})^{-1}$), as determined from the $\rho$-meson mass in chiral limit (see Fig. 1 and Ref. \cite{9}). We employ 9 values of (degenerate) valence $d$ and $s$ quark masses from $m_q = .01$ to $m_q = .05$.

![Figure 1. Data and linear fit for $m_qa$ vs. quark mass, for three sets of configurations with $N_f=0,2$ and 4. (See the talk by D. Chen [9].)](image)

For the continuum limit study we generated 7 ensembles of quenched configurations as listed in table 2 and used 7 to 9 values of $m_q$.
Table 2
Quenched Ensembles for Continuum Extrapolation

| $\beta$ | Geometry | $N_{\text{config}}$ | $m_g$ |
|---------|----------|----------------------|-------|
| 5.70    | $16^4 \times 32$ | 259                 | .01 to .08 |
| 5.80    | $16^3 \times 32$ | 200                 | .01 to .04 |
| 5.90    | $16^4 \times 32$ | 200                 | .01 to .04 |
| 6.00    | $16^3 \times 32$ | 221                 | .01 to .04 |
| 6.05    | $16^3 \times 32$ | 306                 | .01 to .04 |
| 6.10    | $24^3 \times 32$ | 60                  | .01 to .04 |
| 6.20    | $24^3 \times 48$ | 121                 | .005 to .035 |

For creating kaons (at rest) we use a wall of $U(1)$ noise on timeslice $t = 0$, i.e. complex random numbers $\xi_\vec{x}$ at each space point such that $\langle \xi_\vec{x} \xi_\vec{y}^\dagger \rangle = \delta_\vec{x},\vec{y}$. This is statistically equivalent to computing a collection of delta-function sources. In particular, our wall creates only pseudoscalars. We use a lattice duplicated in the time direction, with periodic boundary conditions in space and time (see Fig. 2). Computing propagators on the doubled lattice, we obtain forward-and backward-going propagators which we use for computing $B_K$. That is, if $G_\pi(t)$ is the $\pi$ propagator on the doubled lattice, then our operator correlation functions are schematically of the form $G_\pi(t)G_\pi(t + N_i)$, where $N_i = 32$ or 48.

We employ three kinds of operators: Landau gauge, gauge invariant, and tadpole improved. Landau gauge operators are defined by fixing the gauge and omitting explicit links in non-local operators. For gauge-invariant operators we supply the links, averaging over all shortest paths. Tadpole-improved operators are gauge-invariant operators, but with all links rescaled by $u_0/\Delta$, where $u_0 = P^{1/4}$, $P$ is the average plaquette, and $\Delta$ is the number of links needed to connect fermion fields. We opted for tadpole-improved operators on all configurations, using the others on a subset of configurations for checks.

The matching between continuum and lattice operators is of the form

$$O_i^{\text{cont}} = (\delta_{ij} + \frac{g^2}{16\pi^2}(\gamma_{ij}\log(\frac{\pi}{\mu a}) + C_{ij}))O^{\text{lat}}_j,$$

where $\gamma_{ij}$ is the one-loop anomalous dimension matrix, and $C_{ij}$ are finite coefficients, which can be sizable. We take these from the calculations of Refs. [10,11]. For the continuum scheme, we choose NDR, quoting results either at scale $\mu = \pi/a$ or at $\mu = 2$ GeV. We use the $\overline{\text{MS}}$ coupling constant $g^2_{\overline{\text{MS}}} = 1/g^2_{\text{bare}} + 0.02461 - 0.00704 N_f$. To check how well

Figure 2. We use periodic boundary conditions in space and time, and the lattice is duplicated in time direction.

Figure 3. $B_K$ with (lower points) and without (upper points) one-loop perturbative matching. The points are artificially displaced horizontally for clarity.
the perturbation theory works, we computed all three operators on a subset of the \( N_f = 2 \) ensemble, finding that after one-loop corrections are put in, the matrix elements agree within our statistical error. For the bulk of the calculation we used tadpole-improved operators exclusively.

![Figure 4](image)

Figure 4. Data and fit for \( B_K \) vs. \( m_K^2 \) on the quenched ensemble. The vertical line marks the physical kaon mass.

3. Results for \( N_f \) Dependence

Figs. 4 and 5 show the results for \( B_K \) on three ensembles of configurations. Values at 9 quark mass points are fitted to the form expected from chiral perturbation theory, 

\[
B_K = a + bm_K^2 + cm_K^2 \ln m_K^2.
\]

The \( N_f = 4 \) and \( N_f = 2 \) curves are similar in shape, while the quenched curve crosses between the other two. While this is perfectly allowed, we should also inject a small note of caution—our ensembles have the same \( \rho \)-masses, but these masses are presumably affected to some degree by the finite volume. If this effect is sizable and depends significantly on \( N_f \), our curves could shift a little.

Taking the results at face value, we note that the \( N_f = 2 \) and \( N_f = 0 \) results lie nearly on top of each other at the kaon mass, consistent with our earlier results [8]. Also, most of the \( N_f = 2 \) data lie below \( N_f = 0 \), consistent with the observation by other groups that quenching seems to increase \( B_K \) slightly (see, e.g. ref. [14]). However, the \( N_f = 4 \) data turn this picture upside down. Fig. 5 shows our final values for \( B_K \), obtained at the physical kaon mass and by extrapolation to the chiral limit. We see that the interpolated \( N_f = 3 \) result is a few percent higher than quenched.

4. Continuum Extrapolation

Performing the same analysis on the quenched ensembles, we obtain the result shown in figure 7, where we plot \( B_K(NDR, \mu = 2 \text{ GeV}) \) versus the scale as determined from \( m_\rho \). The data are well fit by the quadratic form 

\[
B_K(a) = B_K(a = 0) + (a\Lambda_2)^2 + (a\Lambda_4)^4,
\]

where the scale of the power correction parameters turns out to be typical of QCD: \( \Lambda_2 \approx 650 \text{ MeV}, \Lambda_4 \approx 650 \text{ MeV} \). Alternatively, we note that we can avoid making reference to the possibly problematic \( m_\rho \) by using the scaling form

\[
a(\beta) = a_0 \left( \frac{16\pi^2}{11g^2} \right)^{-\beta} \exp\left( \frac{-8\pi^2}{11g^2} \right)
\]

where we take \( g \) here to be the \( \overline{MS} \) coupling. This amounts to shuffling around the \( a^4 \) corrections,
and in practice tends to straighten the data out. That is to say, much of the curvature in figure 7 might be ascribed to scaling violations in $m_{\rho}$ itself. To quote a final value we make the conservative choice of a linear fit to the four points with $\beta \geq 6.0$, and obtain

$$B_K|_{a=0,N_f=0} = 0.573 \pm 0.015.$$ 

5. Conclusions

From the dynamical comparison, we find that $B_K(N_f=3)$ is $(5 \pm 2)\%$ larger than $B_K(N_f=0)$. Combining with the $a = 0$ extrapolation we we quote our current central value $B_K$ in the real world:

$$B_K(N_{DR}, \mu = 2 \text{ GeV}, N_f = 3, a = 0) = 0.60 \pm 0.02$$

Remaining uncertainties include possible finite-size effects in the dynamical ensemble, higher order perturbative corrections in the matching, and higher order chiral ($m_s - m_d$) effects. A study of hadronic weak matrix elements relevant for $\epsilon'/\epsilon$ using the same techniques and ensembles is currently underway.

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