On Riemannian manifolds admitting $W_2$-curvature tensor

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ON RIEMANNIAN MANIFOLDS ADMITTING $W_2$-CURVATURE TENSOR

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Abstract. The object of this paper is to study the properties of flat spacetimes under some conditions regarding the $W_2$-curvature tensor. In the first section, several results are obtained on the geometrical symmetries of this curvature tensor. It is shown that in a spacetime with $W_2$-curvature tensor filled with a perfect fluid, the energy momentum tensor satisfying the Einstein’s equations with a cosmological constant is a quadratic conformal Killing tensor. It is also proved that a necessary and sufficient condition for the energy momentum tensor to be a quadratic Killing tensor is that the scalar curvature of this space must be constant. In a radiative perfect fluid, it is shown that the sectional curvature is constant.

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1. INTRODUCTION

In 1970, Pokhariyal and Mishra, [9] introduced a new curvature tensor in an n-dimensional manifold $(M, g)$ denoted by $W_2$ and defined by

$$W_2(X, Y, Z, U) = R(X, Y, Z, U) + \frac{1}{n-1}(g(X, Z)S(Y, U) - g(Y, Z)S(X, U)).$$ (1.1)

Some authors have studied the $W_2$-curvature tensor on some special manifolds before, [3, 5, 8, 16].

A non-flat n-dimensional Riemannian manifold $(M, g)$ is called generalized recurrent Riemannian manifold [4] if its curvature tensor satisfies the condition

$$\nabla R = A \otimes R + B \otimes G$$ (1.2)

where $A$ and $B$ non-zero 1-forms. $\otimes$ is the tensor product, $\nabla$ denotes the Levi-Civita connection, and $G$ is a tensor type (0,4) given by

$$G(X, Y, Z, U) = g(X, U)g(Y, Z) - g(X, Z)g(Y, U)$$ (1.3)
for all $X, Y, Z, U \in \chi(M)$, where $\chi(M)$ is the Lie algebra of smooth vector fields on $M$. This manifold is denoted by $(GK_n)$. These manifolds have been studied by some authors before, [1, 2, 10, 11], etc.

In this paper, the properties of a $W_2$-flat spacetime are studied. Some theorems on the $W_2$ curvature tensor are proved.

2. SOME PROPERTIES OF $W_2$-CURVATURE TENSOR

**Definition 1.** Let $(M, g)$ be a manifold with Levi-Civita connection $\nabla$. A quadratic Killing tensor is a generalization of a Killing vector and is defined as a second order symmetric tensor $A$ satisfying the condition [13, 15]

$$(\nabla_X A)(Y, Z) + (\nabla_Y A)(Z, X) + (\nabla_Z A)(X, Y) = 0$$

(2.1)

**Definition 2.** Let $(M, g)$ be a manifold with Levi-Civita connection $\nabla$. A quadratic conformal Killing tensor is an analogous generalization of a conformal Killing vector and is defined as a second order symmetric tensor $A$ satisfying the condition [13, 15]

$$(\nabla_X A)(Y, Z) + (\nabla_Y A)(Z, X) + (\nabla_Z A)(X, Y) = k(X)g(Y, Z) + k(Y)g(Z, X) + k(Z)g(X, Y).$$

(2.2)

Now, we have the following theorems:

**Theorem 1.** If the Ricci tensor of $M$ admitting $W_2$-curvature tensor is a quadratic conformal Killing tensor then $\bar{W}_2(Y, Z)$ (which is type of (0, 2)) is also quadratic conformal Killing tensor.

**Proof.** By (1.1) we have

$$\bar{W}_2(Y, Z) = \frac{n}{n-1}(S(Y, Z) - \frac{r}{n}g(Y, Z))$$

(2.3)

where $S(Y, Z)$ and $r$ are the Ricci tensor and the scalar curvature of $M$, respectively. If we take the covariant derivative of (2.3), we find

$$(\nabla_X \bar{W}_2)(Y, Z) = \frac{n}{n-1}(\nabla_X S)(Y, Z) - \frac{1}{n-1}(\nabla_X g)(Y, Z)$$

(2.4)

Permutating the indices $X, Y, Z$ cyclically in (2.4) and adding the three equations, we get

$$(\nabla_X \bar{W}_2)(Y, Z) + (\nabla_Y \bar{W}_2)(Z, X) + (\nabla_Z \bar{W}_2)(X, Y)$$

$$= \frac{n}{n-1}((\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y))$$

$$- \frac{1}{n-1}((\nabla_X g)(Y, Z) + (\nabla_Y g)(Z, X) + (\nabla_Z g)(X, Y)).$$

(2.5)
If we assume that the Ricci tensor of $M$ is a quadratic conformal Killing tensor, from (2.2) and (2.5), we obtain that
\[
\begin{align*}
(\nabla_X \tilde{W}_2)(Y, Z) + (\nabla_Y \tilde{W}_2)(Z, X) + (\nabla_Z \tilde{W}_2)(X, Y) \\
= & \left(\frac{n}{n-1} a(X) - \frac{1}{n-1} \nabla_X r\right) g(Y, Z) \\
& + \left(\frac{n}{n-1} a(Y) - \frac{1}{n-1} \nabla_Y r\right) g(Z, X) \\
& + \left(\frac{n}{n-1} a(Z) - \frac{1}{n-1} \nabla_Z r\right) g(X, Y). \quad (2.6)
\end{align*}
\]
By taking $\frac{n}{n-1} a(X) - \frac{1}{n-1} \nabla_X r = a(X)$, it can be seen that
\[
(\nabla_X \tilde{W}_2)(Y, Z) + (\nabla_Y \tilde{W}_2)(Z, X) + (\nabla_Z \tilde{W}_2)(X, Y) \\
= a(X) g(Y, Z) + a(Y) g(Z, X) + a(Z) g(X, Y).
\]
This relation completes the proof. □

**Theorem 2.** Let the Ricci tensor of $M$ admitting a $W_2$-curvature tensor be a quadratic Killing tensor. A necessary and sufficient condition for $\tilde{W}_2(X, Y)$ to be a quadratic Killing tensor is that the scalar curvature of $M$ be constant.

**Proof.** Let us consider that the Ricci tensor of $M$ is a quadratic Killing tensor. From equations (2.1) and (2.5), we have
\[
(\nabla_X \tilde{W}_2)(Y, Z) + (\nabla_Y \tilde{W}_2)(Z, X) + (\nabla_Z \tilde{W}_2)(X, Y) \\
= a(X) g(Y, Z) + a(Y) g(Z, X) + a(Z) g(X, Y). \quad (2.7)
\]
If $\tilde{W}_2(X, Y)$ is a quadratic Killing tensor then we get
\[
(\nabla_X r) g(Y, Z) + (\nabla_Y r) g(Z, X) + (\nabla_Z r) g(X, Y) = 0 \quad (2.8)
\]
Walker’s Lemma,[14] states the following. If $a(X, Y)$ and $b(X)$ are numbers that satisfy $a(X, Y) = a(Y, X)$ and
\[
a(X, Y) b(Z) + a(Y, Z) b(X) + a(Z, X) b(Y) = 0 \quad (2.9)
\]
for all $X, Y, Z$, then either all the $a(X, Y)$ are zero or all the $b(X)$ are zero. Hence, by the above Lemma, we get from (2.8) and (2.9) that either $g(X, Y) = 0$ or $\nabla_X r = 0$. As $g(X, Y) \neq 0$, we get $\nabla_X r = 0$, i.e., the scalar curvature of $M$ is constant. Conversely, if the scalar curvature is constant, then, from (2.7), $\tilde{W}_2(X, Y)$ is a quadratic Killing tensor. Thus, the proof is completed. □

The geometrical symmetries of a Riemannian manifold are expressed through the equation
\[
L_\xi A - 2\Omega A = 0 \quad (2.10)
\]
where \( A \) represents a geometrical/physical quantity \( L_\xi \), denotes the Lie derivative with respect to the vector field \( \xi \) and \( \Omega \) is a scalar, [7].

One of the most simple and widely used example is the metric inheritance symmetry for which \( A = g \) in (2.10); in this case, \( \xi \) is the Killing vector field if \( \Omega \) is zero, i.e.,

\[
(L_\xi g)(X, Y) = 2\Omega g(X, Y)
\]  

(2.11)

A Riemannian manifold \( M \) is said to admit a symmetry called a curvature collineation (CC) provided there exists a vector field \( \xi \) such that

\[
(L_\xi R)(X, Y)Z = 0
\]  

(2.12)

where \( R(X, Y)Z \) is the Riemannian curvature tensor, [6].

Now, we shall investigate the role of such symmetry inheritance for the \( W_2 \)-curvature tensor of a Riemannian manifold.

**Theorem 3.** If a Riemannian manifold \( M \) admitting \( W_2 \)-curvature tensor with a Killing vector \( \xi \) is a (CC) then the Lie derivative of the \( W_2 \)-curvature tensor is zero.

**Proof.** If \( \xi \) is a Killing vector of \( M \) then we have

\[
(L_\xi g)(X, Y) = 0
\]  

(2.13)

Since \( M \) admits a (CC) then we have also from (2.12)

\[
(L_\xi S)(X, Y) = 0
\]  

(2.14)

By taking the Lie derivative of (1.1) and using the equations (2.10) and (2.11), we obtain

\[
(L_\xi W)(X, Y)Z = 0
\]

The proof is completed.

\[ \square \]

**Theorem 4.** If a Riemannian manifold \( M \) admitting \( W_2 \)-curvature tensor with a conformal Killing vector \( \xi \) has a symmetry inheritance then the \( W_2 \)-curvature tensor has also symmetry inheritance property.

**Proof.** Let us assume that \( \xi \) of \( M \) is a conformal Killing vector and \( M \) has a symmetry inheritance. By taking the Lie derivative of (1.1) and using the equations (2.10) and (2.11), we find

\[
(L_\xi W)(X, Y, Z, U) = 2\Omega R(X, Y, Z, U)
\]

\[
+ \frac{2\Omega}{n-1}(g(X, Z)S(Y, U) - g(Y, Z)S(X, U))
\]

\[
= 2\Omega W(X, Y, Z, U).
\]

The proof is completed.

\[ \square \]
3. $W_2$-FLAT SPACETIMES

**Theorem.** [12] A $W_2$-flat space is an Einstein space, i.e,

$$S(X,Y) = \frac{r}{n} g(X,Y)$$

(3.1)

We denote the $W_2$ flat spacetime as $(W_2 FS)_n$ and we consider that our space is a perfect fluid. In local coordinates, a perfect fluid is a spacetime $(M, g)$ satisfying the Einstein equations

$$S_{ij} - \frac{r}{2} g_{ij} + \lambda g_{ij} = k T_{ij}$$

(3.2)

where $S_{ij}$ and $r$ denote the Ricci tensor and the scalar curvature, respectively. $\lambda$ is the cosmological constant and $T_{ij}$ is the energy-momentum tensor. We can state the following theorem:

**Theorem 5.** For a $(W_2 FS)_4$, the energy-momentum tensor satisfying the Einstein’s equations with a cosmological constant is locally symmetric.

**Proof.** In a $(W_2 FS)_4$, by (3.1), we have

$$S_{ij} = \frac{r}{4} g_{ij}$$

(3.3)

and then we get from (3.2) and (3.3)

$$T_{ij} = \frac{1}{k} (\lambda - \frac{r}{4}) g_{ij}$$

(3.4)

By taking the covariant derivative of (3.4), we get

$$\nabla_k T_{ij} = -\frac{1}{4k} (\nabla_k r) g_{ij}.$$  

(3.5)

Since $r$ is constant in $(W_2 FS)_4$,

$$\nabla_k r = 0$$

(3.6)

then (3.5) reduces to

$$\nabla_k T_{ij} = 0.$$  

(3.7)

This completes the proof. □

**Theorem 6.** For a $(W_2 S)_4$, whose the Ricci tensor is quadratic Killing, a necessary and sufficient condition the energy-momentum tensor satisfying the Einstein’s equations with a cosmological constant be a quadratic Killing tensor is that the scalar curvature of this space must be constant.

**Proof.** If $T_{ij}$ and $R_{ij}$ are quadratic Killing tensors, (3.5) reduces to

$$(\nabla_k r) g_{ij} + (\nabla_i r) g_{jk} + (\nabla_j r) g_{ki} = 0$$

(3.8)
By Walker’s Lemma,[14] from (2.9) we get $\nabla_k r = 0$. Thus we can say that the scalar curvature of this space is constant. Conversely, if $r$ is constant, from (3.5), it can be obtained that

$$\nabla_k T_{ij} + \nabla_i T_{jk} + \nabla_j T_{ki} = 0$$

Thus, the proof is completed.

Now, we consider that a radiative perfect fluid in $(W_2 S)_4$. Thus, we have

$$T_{ij} = \sigma a_i a_j, \quad a_i a^i = -1 \quad (3.9)$$

In this case, by taking the covariant derivative of (3.9), we find

$$\nabla_k T_{ij} = (\nabla_k \sigma) a_i a_j + \sigma ((\nabla_k a_i) a_j + (\nabla_k a_j) a_i) \quad (3.10)$$

Multiplying (3.9) by $g^{ij}$, we get

$$T = -\sigma \quad (3.11)$$

By taking the covariant derivative of (3.11), it can be found that

$$\nabla_k T = -\nabla_k \sigma \quad (3.12)$$

Now, we can state the following theorem:

**Theorem 7.** For a radiative perfect fluid in $(W_2 S)_4$, if the energy-momentum tensor satisfying the Einstein’s equations with a cosmological constant is generalized recurrent then the integral curves of the vector field $a_i$ are geodesics.

**Proof.** For a radiative perfect fluid in $(W_2 S)_4$, if $T_{ij}$ is generalized recurrent then from (1.2)

$$\nabla_k T_{ij} = \lambda_k T_{ij} + \beta_k g_{ij} \quad (3.13)$$

and

$$\nabla_k T = \lambda_k T + 4\beta_k \quad (3.14)$$

Moreover, by (3.12) and (3.14), we get

$$\nabla_k \sigma = \lambda_k \sigma - 4\beta_k \quad (3.15)$$

Putting equations (3.13) and (3.15) in (3.10), we have the following relation

$$\lambda_k T_{ij} + \beta_k g_{ij} = (\lambda_k \sigma - 4\beta_k) a_i a_j + \sigma ((\nabla_k a_i) a_j + (\nabla_k a_j) a_i) \quad (3.16)$$

By (3.9), (3.16) takes the form

$$\lambda_k \sigma a_i a_j + \beta_k g_{ij} = \lambda_k \sigma a_i a_j - 4\beta_k a_i a_j + \sigma ((\nabla_k a_i) a_j + (\nabla_k a_j) a_i) \quad (3.17)$$

Multiplying (3.17) by $g^{ij}$ and using (3.9), we obtain

$$2\sigma a^i (\nabla_k a_i) = 0 \quad (3.18)$$

Since, in a radiative perfect fluid, $\sigma \neq 0$ then from (3.18), we finally get

$$a^i (\nabla_k a_i) = 0 \quad (3.19)$$

From (3.19), we can say that the integral curves of the vector field $a_i$ are geodesics. In this case, the proof is completed.
Now, by using the above theorem, we can state the following:

**Theorem 8.** For a radiative perfect fluid in \((W_2FS)_4\), the sectional curvature is constant.

**Proof.** For a radiative perfect fluid in \((W_2FS)_4\), the scalar curvature of this space is constant. Thus, we can say that for a radiative perfect fluid in \((W_2FS)_4\), the scalar curvature of this space is constant.

The sectional curvature of a Riemannian manifold \(M\) is in the form

\[
K(\pi) = \frac{R_{hijk}X^hY^iX^jY^k}{(g_{ij}g_{hk} - g_{ik}g_{jh})X^hY^iX^jY^k}.
\]

(3.20)

By (1.1), if \(M\) is a \(W_2\) flat manifold then we have

\[
R_{ijkh} = \frac{1}{n-1}(g_{jk}S_{ih} - g_{ik}S_{jh}).
\]

(3.21)

Thus, multiplying (3.21) by \(g^{ih}\) and using (3.21) again, we get

\[
R_{ijkh} = \frac{r}{n(n-1)}(g_{jk}g^{ih} - g_{ik}g^{jh}).
\]

(3.22)

Finally, from (3.20) and (3.22), it can be seen that

\[
K(\pi) = \frac{r}{n(n-1)}.
\]

□

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