Self-similarity in an Exchangeable Site-Dynamics Model

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Abstract
A system of sites indexed by \( \mathbb{N} \) are each assigned an initial fitness value in \([0, 1]\). Each unit of time an environment is sampled for the system and a proposed fitness, uniform in \([0, 1]\), is sampled for each site, all independent of each other and the past. The environment is good with probability \( p \) and is bad otherwise. The fitness of each site is then updated to the maximum or the minimum of its current fitness and the proposed fitness according to whether the environment is good or bad. We study the empirical fitness distribution, a probability-valued Markov process, and prove that it converges in distribution to a unique stationary distribution. Under the stationary distribution the system exhibits a self-similar (typically) non-smooth site-exchangeable behavior whereas the fitness distribution for each site is smooth. Our analysis is done by identifying a class of iterated function systems for which we prove ergodicity and provide a probabilistic representation of the stationary distribution. This also yields a close connection to the fractals literature.

Keywords Markov chain · Exchangeable stochastic process · Self-similar measure · Population biology

1 Introduction
We consider a discrete-time model on state space \([0, 1]^\mathbb{N}\). At each time unit \( t \in \mathbb{Z}_+ \), each site \( n \in \mathbb{N} \) has “fitness” \( \eta_t(n) \in [0, 1] \). The system evolves in time as follows. Let \( p \) be a fixed number in \((0, 1)\). At any time \( t \geq 0 \) we generate a Bernoulli random variable \( B_{t+1} \) with parameter \( p \), and independently a sequence \((U_{t+1}(n) : n \in \mathbb{N})\) of IID (that is, independent and identically distributed) uniform random variables on \([0, 1]\). We update the model according to the following rules.

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• If $B_{t+1} = 1$ then for every $n \in \mathbb{N}$, $\eta_{t+1}(n) = \max(\eta_t(n), U_{t+1}(n))$.
• If $B_{t+1} = 0$ then for every $n \in \mathbb{N}$, $\eta_{t+1}(n) = \min(\eta_t(n), U_{t+1}(n))$.

We think of the $(B_t : t \in \mathbb{Z}_+)$ as a time-evolving environment. At a given time the environment can be “good” or “bad”, it is the same for all sites. The impact of the environment at a given site, however, depends on the current site fitness as well as its own “luck”, independently of what happens to the other sites.

The resulting process is a Markov chain. Additionally, the evolution of each individual site is also a Markov chain. Thus, the entire system can be viewed as a system of infinitely many coupled Markov chains. Of particular interest is the resulting joint structure.

The site-exchangeability of the model allows, through de Finetti’s theorem to define an auxiliary CDF-valued process, describing the empirical fitness distribution across the system where CDF stands for cumulative distribution function. We show that this process converges to a stationary distribution on CDFs which is non-degenerate and self-similar across all fitness levels at the same time. More precisely, the proportions of sites with fitness level below any given threshold are self-similar with full support on $[0, 1]$ and with a.s. local exponents varying continuously over $(0, 1)$. These measures are part of a class that has appeared in the fractals literature [7]. The model we present here exhibits a continuum of them jointly. Notably, the self-similarity only appears at the system-wide scale as the fitness at individual sites under the stationary distribution is smooth.

This model is related to the so-called “catastrophe” models, see for instance [4] and [3]. In particular a model introduced in [6] and studied in [2] is reminiscent of our model. We now describe it. At every discrete time the population increases by one unit with probability $p$ or is subjected to a catastrophe (i.e. it loses a random number of individuals) with probability $1 - p$. Hence, similarly to the present model the evolution is affected by external environmental changes. A major difference with the present model, however, is that catastrophe models only track the overall size of the population. We track every individual in the population.

The paper is organized as follows. In Sect. 2 we state our main convergence results and provide probabilistic descriptions of the limits. We first introduce and discuss an auxiliary process describing the evolution of the empirical fitness distribution in Sect. 2.1 and then turn to our model in Section 2.2. Some of the proofs are deferred to Sect. 4. In Sect. 3 we explore the self-similar nature of the limits obtained through an analytic representation. In Sect. 4 we present a framework generalizing the dynamics in our model and prove convergence results as well as some properties of the limiting distributions.

2 Convergence Results

The main tool in our analysis of the the model is an auxiliary process $\Theta$, representing the empirical fitness distribution across the system. We introduce and study this process in Sect. 2.1, and use the results to derive our main result, Theorem 3.

2.1 An Auxiliary Exchangeable Process

Assume that the initial distribution is exchangeable (e.g. example IID) then for each $t$, the family of random variables (RV in short) $(\eta_t(n) : n \in \mathbb{N})$ is an exchangeable sequence. Let

$$I_t(n, u) = 1_{[\eta_t(n) \leq u]}.$$
Then for every \( t \) and \( u \), the family \( (I_t(n, u) : n \in \mathbb{N}) \) is an exchangeable sequence. In particular, it follows from de Finetti’s theorem that there exists a random variable \( \Theta_t(u) \), measurable with respect to the exchangeable \( \sigma \)-algebra \( \mathcal{E} \) such that the distribution of \( (I_t(n, u) : n \in \mathbb{N}) \), conditioned on \( \mathcal{E} \) is IID with a Bernoulli distribution with parameter \( \Theta_t(u) \). Furthermore, \n\begin{equation}
\Theta_t(u) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} I_t(n, u), \text{ a.s.}
\end{equation}

Consider now the evolution of \( \Theta \) over time. If \( B_{t+1} = 1 \) then \n\begin{equation}
I_{t+1}(n, u) = I_t(n, u) I_{[U_{t+1} \leq u]},
\end{equation}

On the other hand if \( B_{t+1} = 0 \) then \n\begin{equation}
I_{t+1}(n, u) = I_t(n, u) + (1 - I_t(n, u)) I_{[U_{t+1} \leq u]}.
\end{equation}

From the Law of Large Numbers for exchangeable sequences and the independence of \( \eta_t \) and \( (U_{t+1}(n) : n \in \mathbb{N}) \) we obtain \n\begin{equation}
\Theta_{t+1}(u) = \begin{cases} 
\Theta_t(u)u & \text{if } B_{t+1} = 1 \\
\Theta_t(u) + (1 - \Theta_t(u))u & \text{if } B_{t+1} = 0.
\end{cases}
\end{equation}

Though our motivation came from studying \( \eta \), the dynamics in (2) define an auxiliary Markov chain on the set of cumulative distribution functions supported on \([0, 1]\). Our first result identifies its stationary distribution.

**Theorem 1** Let \( \Theta \) be the Markov chain on the set of CDFs supported on \([0, 1]\) with dynamics defined by (2).

Let \( G_0, G_1, \ldots \) be IID Geom(1 - \( p \))-distributed RVs, and for \( k \in \mathbb{Z}_+ \), let \( T_k = G_0 + \cdots + G_k \). Then the distribution of the random CDF \( \Theta_t(\cdot) \) converges as \( t \to \infty \) to that of the random CDF \( \Theta_\infty \) given by \n\begin{equation}
\Theta_\infty(u) = \sum_{k=0}^{\infty} u^T_k \left( \frac{1-u}{u} \right)^k, \quad u \in (0, 1).
\end{equation}

The convergence in the theorem implies that the process possesses a unique stationary distribution given by the expression in (3), though this can be also verified by a direct calculation. The proof of the Theorem is deferred to Sect. 4.2, as a special case of more general dynamics.

Returning to \( \eta \) with an initial distribution which is exchangeable and \( \Theta \) defined through (1), the distribution of \( \Theta_t(\cdot) \) determines the distribution of \( \eta_t \), conditioned on \( \mathcal{E} \).

Note that for every fixed \( u \), the process \( t \to \Theta_t(u) \) is a \([0, 1]\)-Markov chain, whose stationary distribution equal to the distribution of \( \Theta_\infty(u) \). Let \( \mu_u \) be the distribution of \( \Theta_\infty(u) \) and let \( G_u \) denote its CDF. Then \( \mu_u \) is self-similar with respect to the iterated function system \( \{S_0, S_1\}, S_j : [0, 1] \to [0, 1] \) with \( S_0(x) = u + (1-u)x \) and \( S_1(x) = ux \). Indeed, from the dynamics (2), \( \mu_u \) satisfies (10). In Fig. 1 we present an illustration of \( G_u \) for several values of \( u \). We study properties of \( G_u \) in Sect. 3.

The moments of \( \Theta_\infty(u) \) can be computed through iterations of the following formula:

**Proposition 2** For \( k = 1, 2, \ldots \)

\n\begin{equation}
E[\Theta_\infty^k(u)] = \frac{1-p}{1-pu^k - (1-p)(1-u)^k} \sum_{j=0}^{k-1} \binom{k}{j} u^{k-j} (1-u)^j E[\Theta_\infty^j(u)].
\end{equation}
Proof Use (10) to obtain that for any bounded Borel function \( f : [0, 1] \rightarrow \mathbb{R} \),
\[
E[f(\Theta_\infty(u))] = pE[f(u\Theta_\infty(u))] + (1 - p)E[f(u + (1 - u)\Theta_\infty(u))],
\]
(4)
Where the expectation is with respect to \( \mu_u \). Fixing \( \lambda \) and letting \( f(z) = e^{\lambda z} \) and let \( \varphi_u(\lambda) = E[f(\Theta_\infty(u))] \) be the moment generating function for \( \Theta_\infty(u) \). Then since \( \Theta_\infty(u) \) takes values in \([0, 1]\), \( \varphi_u \) is entire, and \( \varphi_u^{(k)}(0) = E[\Theta_\infty(u)^k] \). From (4), we have
\[
\varphi_u(\lambda) = p\varphi_u(u\lambda) + (1 - p)e^{u\lambda}\varphi_u((1 - u)\lambda).
\]
By taking derivatives with respect to \( \lambda \) we obtain,
\[
\varphi_u^{(k)}(\lambda) = p u^k\varphi_u^{(k)}(u\lambda) + (1 - p) \sum_{j=0}^{k} \binom{k}{j} u^{k-j} e^{\lambda u} (1 - u)^j \varphi_u^{(j)}((1 - u)\lambda).
\]
The result follows by taking \( \lambda = 0 \) and doing the algebra. \( \square \)

The results above are related to a long standing open problem. Consider the following iterated function system. Let \( \{T_0, T_1\}, T_j : [0, 1] \rightarrow [0, 1] \) by \( T_0(x) = 1 - u + ux \) and \( T_1(x) = ux \). In [5] examples for \( u \)'s in \((1/2, 1)\) are given for which the stationary distribution corresponding to the system \( \{T_0, T_1\} \) is continuous but singular with respect to the Lebesgue measure. It is also known that the stationary distribution is absolutely continuous for some values in \((1/2, 1)\) and singular for all values in \((0, 1/2)\). Unfortunately our method does not apply to the system \( \{T_0, T_1\} \). One main difference being that the intersection of the images of \([0, 1]\) under \( S_0 \) and \( S_1 \) is a singleton, while the intersection of the respective images under \( T_0 \) and \( T_1 \) is a non-trivial interval \((1 - u, u)\) when \( u \in (1/2, 1) \). As far as we know the question of determining for which \( u \)'s in \((1/2, 1)\) the stationary measure is singular is still open, see also the discussion in [1,p. 24]. However, we prove the following representation for
the unique stationary distribution of the system \( \{T_0, T_1\} \). For a fixed \( u \) in \((0, 1)\), the stationary distribution has the same distribution as

\[
\frac{1 - u}{u} \sum_{k \geq 0} u^T_k,
\]

where \( T_k = G_0 + \cdots + G_k \), and \( G_0, G_1, \ldots \) are IID Geom\((1 - p)\). This representation formula is an application of Corollary 9.

2.2 Convergence for Our Model

In light of the results of the previous section, if the initial distribution of \( \eta \) is exchangeable, then the random variables \( (\eta_t(n) : n \in \mathbb{N}) \) are exchangeable for every \( t \in \mathbb{Z}_+ \). Hence, for \( N \in \mathbb{N} \) and every \((u_1, u_2, \ldots, u_N)\) in \([0, 1]^N\),

\[
P\left( \bigcap_{n=1}^{N} \{\eta_t(n) \leq u_n\} \mid \mathcal{E} \right) = \prod_{n=1}^{N} P(\eta_t(n) \leq u_n | \mathcal{E}) = \prod_{n=1}^{N} \Theta_t(u_n). \quad (5)
\]

Move generally, we have the following:

**Theorem 3** Let \( \pi_p \) be a probability distribution on \([0, 1]^\mathbb{N}\) whose finite dimensional distributions are given by

\[
\pi_p \left( \bigcap_{n=1}^{N} [0, u_n] \times [0, 1]^\mathbb{N} \right) = E \left[ \prod_{n=1}^{N} \Theta_\infty(u_n) \right], \quad N \in \mathbb{N}, \; u_1, \ldots, u_N \geq 0.
\]

Then for every initial distribution \( \eta_0 \) (not necessarily exchangeable), the distribution of \( \eta_t \) converges weakly to \( \pi_p \). In particular, \( \pi_p \) is the unique stationary distribution for the dynamics of the process \( t \rightarrow \eta_t(\cdot) \).

**Proof of Theorem 3** Consider two copies of the process constructed from the same sequences of Bernoulli “environments” and Uniform “luck” variables. The first \( t \rightarrow \tilde{\eta}_t(\cdot) \) will have some arbitrary initial distribution. The second \( t \rightarrow \eta_t(\cdot) \) will have a deterministic and exchangeable (hence constant) initial distribution: \( \eta_0(n) = \frac{1}{2} \) for all \( n \in \mathbb{N} \). Each unit of time we update both sequences using the rules presented in the first paragraph of Sect. 1. This construction implies that if \( \tilde{\eta}_t(n) = \eta_t(n) \), then \( \tilde{\eta}_{t+s}(n) = \eta_{t+s}(n) \) for all \( s \geq 0 \). Moreover, initially, \( |\tilde{\eta}_0(n) - \eta_0(n)| \leq \frac{1}{2} \) for all \( n \in \mathbb{N} \). A straightforward induction leads to the following. For each \( n \in \mathbb{N} \)

1. the process \( t \rightarrow |\eta_t(n) - \tilde{\eta}_t(n)| \) is nonincreasing;
2. conditioned on the entire past of the two chains up to time \( t \), inclusive, the probability of \( \eta_{t+1}(n) = \tilde{\eta}_{t+1}(n) \) is bounded below by \( \frac{1}{2} \min(p, 1 - p) \). This is a lower bound for the probability of the union of the two disjoint events:

   (a) \( U_{t+1}(n) \) larger than \( \max(\eta_t(n), \tilde{\eta}_t(n)) \) and \( B_{t+1} = 1 \);
   (b) \( U_{t+1}(n) \) smaller than \( \min(\eta_t(n), \tilde{\eta}_t(n)) \) and \( B_{t+1} = 0 \).

We omit the simple details of the inductions. This implies that at some random time \( T_n \), dominated by Geom\(\left(\frac{1}{2} \min(p, 1 - p)\right)\) and such that for all \( s \geq T_n \),

\[ \eta_s(n) = \tilde{\eta}_s(n). \]
Fixing \( N \) and letting \( \tau_N = \max(T_1, \ldots, T_N) \) we have that for any continuous and bounded function \( F : \mathbb{R}^N \rightarrow \mathbb{R} \),

\[
|E[F(\eta_t(1), \ldots, \eta_t(N))] - E[F(\eta_t(1), \ldots, \eta_t(N))]| \leq \|F\|_{\infty} P(\tau_N > t). \tag{6}
\]

Note that the r.h.s. converges to 0 as \( t \) goes to infinity. From (5) and Theorem 1 we have that

\[
\lim_{t \to \infty} E\left[F(\eta_t(1), \ldots, \eta_t(N))\right] = E\left[\int \cdots \int F(u_1, \ldots, u_N) d\prod_{n=1}^{N} \Theta_{\infty}(u_n)\right].
\]

This along with (6) prove that the finite dimensional distributions of \( \tilde{\eta}_t(\cdot) \) converge to the respective distributions under \( \pi_p \). This in turn implies that the distribution of \( \tilde{\eta}_t \) converges to \( \pi_p \) weakly. Moreover, this shows that \( \pi_p \) is the unique stationary distribution for our dynamics. \( \square \)

As a consequence of Theorem 3 the CDF at a single site under \( \pi_p \) is

\[
F_p(u) = E[\Theta_{\infty}(u)].
\]

Setting \( k = 1 \) in Proposition 2 we obtain

\[
F_p(u) = \frac{(1 - p)u}{1 - pu - (1 - p)(1 - u)} = \frac{(1 - p)u}{(1 - p)u + p(1 - u)}, \tag{7}
\]

leading to the following:

**Corollary 4** The single site chain \((\eta_t(n) : t \in \mathbb{Z}_+)\) at any fixed site \( n \) has a unique stationary distribution with CDF \( F_p \) given by (7) and density \( f_p \),

\[
f_p(u) = \frac{p}{1 - p} \left(\frac{F_p(u)}{u}\right)^2 = \frac{p(1 - p)}{((1 - 2p)u^2 + p)^2}
\]

Note the following:

\[
F_p(u) = F_{1-u}(1 - p) \tag{8}
\]

as well as

\[
f_p(0+) = \frac{1 - p}{p} = \frac{1}{f_p(1-)}.
\]

## 3 Self-similarity

In this section we explore some of basic properties the distribution of \( \Theta_{\infty}(u) \) for each \( u \). We obtain a representation for its CDF (in the same spirit of the ternary representation for the Cantor function) and use it to derive some basic properties: continuity of the measure and local exponents (which yield singularity with respect to the Lebesgue measure).

### 3.1 Representation

Fix \( u \in (0, 1) \) and define the function system \( \{S_0, S_1\}, S_j : [0, 1] \rightarrow [0, 1] \) by \( S_0(x) = u + (1 - u)x \) and \( S_1(x) = ux \). Then (2) can be written as

\[
\Theta_{t+1}(u) = S_{B_{t+1}}(\Theta_t(u)) \tag{9}
\]
It therefore follows that a probability distribution \( \mu_u \) on \([0, 1] \) is stationary for \( \Theta(u) \) if and only if
\[
\mu_u = (1 - p)\mu_u \circ S_0^{-1} + p\mu_u \circ S_1^{-1}.
\]
(10)
That is, \( \mu_u \) is self-similar with respect to the function system \( \{S_0, S_1\} \) and the probability vector \((1 - p, p)\), see [7].

As the images of \( S_0 \) and \( S_1 \) are \([u, 1] \) and \([0, u] \), it follows that \( \mu_u \) satisfies (10) if and only if its CDF \( G_u \) satisfies
\[
G_u(uz) = pG_u(z)
\]
(11)
\[
G_u(u + (1 - u)z) = p + (1 - p)G_u(z)
\]
(12)
for all \( z \in [0, 1] \). Here is an explicit formula for the CDF of \( \Theta_\infty(u) \) on a dense set.

**Proposition 5** Let \( u \in (0, 1) \). Let \( D \) be the set of numbers of the form
\[
y = \sum_{l=1}^{m} u^n_l (1 - u)^{l-1}
\]
(13)
for some \( m \in \mathbb{N} \) and for \( n_1, n_2, \ldots, n_m \) as follows.

- If \( m = 1 \), then \( n_1 \in \mathbb{Z}_+ \cup \{\infty\} \).
- If \( m > 1 \), then
  - \( n_1, \ldots, n_{m-1} \in \mathbb{N} \cup \{\infty\} \) satisfying \( n_1 \leq \cdots \leq n_{m-1} \); and
  - \( n_m \in \mathbb{Z}_+ \cup \{\infty\} \) satisfying \( n_{m-1} \leq n_m + 1 \).

1. The set \( D \) is dense in \([0, 1]\).
2. Let \( G_u \) denote the CDF of \( \Theta_\infty(u) \). Then, \( G_u \) is continuous, and for \( y \) defined by (13),
\[
G_u(y) = \sum_{l=1}^{m} p^n_l (1 - p)^{l-1}.
\]
(14)

It follows from the proposition that for \( u = p \), we have \( G_p(y) = y \) for all \( y \) in \( D \). Since this is a dense set, it follows that \( \mu_p \) is uniform on \([0, 1]\).

From (11) \( G_u(u^n) = p^n \), for all \( n \in \mathbb{N} \). Then by right continuity, \( G_u(0) = 0 \). Note that (14) is a consequence of (11) and (12). The rest of the proposition will be proved below.

**Proof of Proposition 5** The continuity of \( G_u \) is proved below in Proposition 10. We show now that \( D \) is dense in \([0, 1]\).

Let \( J_{1,d} = \{1, u, u^2, \ldots, 0\} \), and let \( J_{1,u} = \{u + (1 - u)x : x \in J_{1,d}\} \). Then every element in \( y \in J_{1,u} \) is of the form \( u + (1 - u)u^k \) for some \( k \in \mathbb{Z}_+ \cup \{\infty\} \) and by (12) for such \( y \), \( G_u(y) = p + (1 - p)p^k \). Iterating the definition, for \( m \geq 1 \) let
\[
J_{m+1,d} = \left\{ u^k x : x \in J_{m,u}, k \in \mathbb{Z}_+ \cup \{\infty\} \right\}
\]
(15)
\[
J_{m+1,u} = \{u + (1 - u)x : x \in J_{m+1,d}\}
\]
(16)

We will now prove that for every \( m \in \mathbb{N} \), \( J_{m,d} \) is the set of numbers expressible as (13). The case \( m = 1 \) follows directly from the definition of \( J_{1,d} \). We continue to the general case and will apply induction, with the base case being \( m = 2 \). From (15), we have
\[
J_{2,d} = \left\{ u^{1+k_2} + (1 - u)u^{k_1+k_2} : k_1, k_2 \in \mathbb{Z}_+ \cup \{\infty\} \right\}.
\]
Therefore letting \( n_1 = 1 + k_2 \in \mathbb{N} \cup \{\infty\} \) and \( n_2 = k_1 + k_2 \in \mathbb{Z}_+ \), we also have \( n_1 \leq n_2 + 1 \), and (13) holds for \( m = 2 \).

We turn to the induction step. From the induction hypothesis on \( J_{m,d} \) and the definition of \( J_{m,u} \), we have that \( y \in J_{m,u} \) if and only if

\[
y = u + (1 - u) \sum_{l=1}^{m} u^{n_l} (1 - u)^{l-1}.
\]

and then from (15), \( y \in J_{m+1,d} \) if and only if there exists \( k_{m+1} \in \mathbb{Z}_+ \cup \{\infty\} \) such that

\[
y = u^{k_{m+1}+1} + \sum_{l=1}^{m} u^{n_l+k_{m+1}} (1 - u)^l
\]

\[
= u^{k_{m+1}+1} + u^{n_1+k_{m+1}} (1 - u) + u^{n_2+k_{m+1}} (1 - u)^2 + \ldots + u^{n_m+k_{m+1}} (1 - u)^m
\]

\[
= \sum_{l=1}^{m+1} u^{n'_l} (1 - u)^{l-1},
\]

where \( n'_1 = k_{m+1} + 1 \), and for \( l = 2, \ldots, m + 1, n'_l = n_{l-1} + k_{m+1} \). By the induction hypothesis,

- \( 1 \leq n_1 \leq \ldots \leq n_{m-1} \), and therefore it follows that \( 1 \leq n'_1 \leq \ldots \leq n'_m \).
- \( n_{m-1} \leq n_m + 1 \), and therefore

\[
n'_m = n_{m-1} + k_{m+1} \leq n_m + 1 + k_{m+1} = n'_{m+1} + 1.
\]

This completes the proof that for all \( m \geq 1 \), \( J_{m,d} \) is the set of numbers expressible as (13).

Finally, it remains to show that the union of \( J_{m,d}, m \in \mathbb{Z}_+ \), is dense in \([0, 1]\). Let

\[
R_m = \sum_{j=1}^{m} (1 - u)^{j-1}.
\]

Then \( R_\infty = \lim_{m \to \infty} R_m = \frac{1}{u} \).

Let \( x \in (0, 1) \), and let \( n_1 = \min\{n : u^n \leq x\} \). Then necessarily, \( n_1 \in \mathbb{N} \), and \( x \in [u^{n_1}, u^{n_1-1}] \). Since

\[
u^{n_1} = u^{n_1} R_1 < u^{n_1} R_2 < \ldots < u^{n_1} R_\infty = u^{n_1-1},
\]

there exists \( m_1 \in \mathbb{N} \) such that

\[
u^{n_1} R_{m_1} \leq x \leq u^{n_1} R_{m_1+1}.
\]

Set \( x_1 = u^{n_1} R_{m_1} \), and let \( x_2 = x - x_1 \). Then \( x_2 \in u^{n_1}[0, (1 - u)^{m_1}] \). Therefore we can write \( x_2 = u^{n_1}(1 - u)^{m_1} y_1 \) where \( y_1 \in [0, 1) \). In other words,

\[
x = x_1 + x_2 = \underbrace{u^{n_1} R_{m_1}}_{\in J_{m_1,d}} + u^{n_1}(1 - u)^{m_1} y_1,
\]

where \( y_1 \in [0, 1) \). If \( y_1 = 0 \), we stop. Otherwise, we iterate the process for \( y_1 \). That is, we find natural numbers \( n_2 \) and \( m_2 \) such that

\[
y_1 = u^{n_2} R_{m_2} + u^{n_2}(1 - u)^{m_2} y_2
\]

where \( y_2 \in [0, 1) \). Hence,

\[
x = u^{n_1} R_{m_1} + u^{n_1+n_2}(1 - u)^{m_1} R_{m_2} + u^{n_1+n_2}(1 - u)^{m_1+m_2} y_2.
\]
It is easy to check that \( u^{n_1} R_{m_1} + u^{n_1+n_2} (1-u)^m_1 R_{m_2} \) is in \( J_{m_1+m_2} \). If \( y_2 = 0 \) we stop. If not we continue this process to get arbitrarily close to \( x \). 

\[ \Box \]

### 3.2 Local Exponents and Singularity

From Proposition 5 we know that the probability measure \( \mu_u \) has no atoms. In this section we will explore the local behavior of this measure. Our analysis will show that with the exception of the case \( p = u \), \( \mu_u \) is singular with respect to the Lebesgue measure.

Recall that \( D \) denotes the set of all points in the form (13). Then \( D \) is countable and dense, and therefore \( \mu_u(D) = 0 \). We will show that the local exponent of \( \mu_u \) on \( D \) is quite different from the local exponent outside of \( D \), see Fig. 2.

#### 3.2.1 Local Exponent on \( D \)

We will first look at the behavior of \( G_u \) near points in \( D \). Let \( y \in D \)

\[
y = \sum_{l=1}^{m} u^{n_l} (1-u)^{l-1}
\]

be of the form \( m \geq 2 \) and \( n_1, \ldots, n_m \) all finite. We make this restriction only to simplify the argument. First, we consider \( \delta \) of the form \( \delta = u^{n_m+k} (1-u)^m \). Then, by (14),

\[
G_u(y + \delta) - G_u(y) = p^{n_m+k} (1-p)^m
\]

\[
= u^{\frac{\ln p}{\ln u} (n_m+k)} (1-p)^m
\]

\[
= (\delta / (1-u)^m)^{\frac{\ln p}{\ln u}} (1-p)^m
\]

\[
= C(y, u, p) \delta^{\frac{\ln p}{\ln u}}.
\]

Next, if \( \delta \) is positive and \( \delta < u^{n_m}(1-u)^m \) there exists a positive integer \( k \) and corresponding \( \delta_1 = u^{n_m+k} (1-u)^m \), \( \delta_2 = u^{n_m+k-1} (1-u)^m \) so that \( \delta \in [\delta_1, \delta_2] \). Then

\[
G_u(y + \delta_1) - G_u(y) \leq G_u(y + \delta) - G_u(y) \leq G_u(y + \delta_2) - G_u(y),
\]

and from (19) we obtain

\[
\frac{\ln p}{\ln u} \ln \delta_1 \leq \ln(G_u(y + \delta) - G_u(y)) - \ln C(y, u, p) \leq \frac{\ln p}{\ln u} \ln \delta_2.
\]

As \( \delta \to 0^+ \), \( k \to \infty \) and consequently, \( \ln \delta_2 \sim \ln \delta_1 \sim \ln \delta \to -\infty \). The Squeeze Theorem and (19) give

\[
\lim_{\delta \to 0^+} \frac{\ln(G_u(y + \delta) - G_u(y))}{\ln \delta} = \frac{\ln p}{\ln u}
\]

We will now approach the same \( y \) from the left.

\[
y = \sum_{l=1}^{m} u^{n_l} (1-u)^{l-1}
\]

\[
= \sum_{l=1}^{m-1} u^{n_l} (1-u)^{l-1} + u^{n_m} (1-u)^{m-1}
\]
\[
= \sum_{l=1}^{m-1} u^n(1-u)^{l-1} + \sum_{l'=m}^{\infty} u^{n+1}(1-u)^{l'-1}.
\]

For \( k \in \mathbb{N} \), let \( \delta \) be of the form
\[
\delta = \sum_{l'=m+k}^{\infty} u^n(1-u)^{l'-1} = u^n(1-u)^{m+k-1}.
\]

Note that as \( k \to \infty, \delta \to 0 \). With this choice,
\[
y - \delta = \sum_{l=1}^{m-1} u^n(1-u)^{l-1} + \sum_{l'=m}^{m+k-1} u^{n+1}(1-u)^{l'-1}
\]
and so the by the continuity of \( G_u \) we have
\[
G_u(y) - G_u(y-\delta) = \sum_{l'=m+k}^{\infty} p^n(1-p)^{l'-1} = p^n(1-p)^{m+k-1}
\]
\[
= p^n(1-u)\left(\frac{\ln(1-p)}{\ln(1-u)}\right)
= p^n \left(\frac{\delta}{u^n}\right) \left(\frac{\ln(1-p)}{\ln(1-u)}\right)
= C'(y,u,p)\delta \left(\frac{\ln(1-p)}{\ln(1-u)}\right).
\]

The analysis used to obtain the right-limit in (20) from limits along \( \delta \) of a specific type can be repeated here as well, and so we conclude from (21) that
\[
\lim_{\delta \to 0} \frac{\ln |G_u(y-\delta) - G_u(y)|}{\ln \delta} = \frac{\ln(1-p)}{\ln(1-u)}
\]

Note that when the right hand side in (20) or (22) is larger or equal to 1, then the respective one-sided derivative exists, and is equal to zero if the limit is strictly larger than 1. Furthermore, \( \ln p < \ln u \) if and only if \( \ln(1-p) > \ln(1-u) \), therefore exactly one of the one-sided limits is larger than 1 except for \( u = p \).

### 3.2.2 Local Exponent Outside of \( D \)

To ease notation we will freeze \( u \) and drop the dependence on it. For each \( t \in \mathbb{N} \), let \( \epsilon = (\epsilon_1, \ldots, \epsilon_t) \in \{0,1\}^t \). Write \( |\epsilon| = \sum \epsilon_i \). Recall that \( S_0(x) = u + (1-u)x, S_1(x) = ux \) and \((p_0, p_1) = (1-p, p)\).

Let
\[
I(\epsilon) = S_{\epsilon_t} \circ \cdots \circ S_{\epsilon_1}([0,1]).
\]

Note that we compose from the right rather than the left. Let \( \mathcal{F}_t = \{I(\epsilon) : \epsilon \in \{0,1\}^t\} \). Since \( [0,1] = S_0([0,1]) \cup S_1([0,1]), \) it follows from induction that the union of all elements in \( \mathcal{F}_t \) is \([0,1]\). In what follows, if \( I \) is an interval, write \( |I| \) for its length. If \( \epsilon \in \{0,1\}^t \), then \( |I(\epsilon)| = (1-u)^{|\epsilon|}|u^{|\epsilon|} \).

\[
\sum_{\epsilon \in \{0,1\}^t} |I(\epsilon)| = \sum_{k=0}^{t} \binom{t}{k} (1-u)^{t-k} u^k = 1,
\]
Therefore the intervals in $F_t$ are non overlapping. We also observe that the points which belong to exactly two intervals are elements in the countable set $D$.

Suppose $x \in [0, 1] \cap D^c$. Then for every $t$ there exists a unique element $I_t(x) \in F_t$ such that $x \in I_t(x)$. Also since every element in $F_{t+1}$ is a subset of a unique element in $F_t$, it follows that $I_{t+1}(x) \subset I_t(x)$. As a result,

$$
\bigcap_{t=1}^{\infty} I_t(x) = \{x\},
$$

and there exists a unique sequence $\epsilon \in \{0, 1\}^N$ such that $x \in I(\epsilon_1, \ldots, \epsilon_t)$ for all $t$. Note that $\epsilon_1, \epsilon_2, \ldots$ are all functions of $x$, and that for every $I \in F_t$, $x \in I$ if and only if $I = I(\epsilon_1(x), \epsilon_2(x), \ldots, \epsilon_t(x))$.

Let’s sample $X$ according to $\mu$ and fix $(\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_t) \in \{0, 1\}^t$. Then

$$
P(X \in I(\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_t)) = p_0 \mu \left(x : S_0 x \in S_{\epsilon'_2} \cdots S_{\epsilon'_t}(0, 1)\right)
$$

$$
+ p_1 \mu \left(x : S_1 x \in S_{\epsilon'_2} \cdots S_{\epsilon'_t}(0, 1)\right)
$$

$$
= p_{\epsilon'_1} \mu(x : x \in S_{\epsilon'_2} \circ \cdots \circ S_{\epsilon'_t}(0, 1)).
$$

Iterating,

$$
P(X \in I(\epsilon'_1, \ldots, \epsilon'_t)) = \prod_{i=1}^t p_{\epsilon'_i}.
$$

Equivalently,

$$
P(\epsilon_1(X) = \epsilon'_1, \epsilon_2(X) = \epsilon'_2, \ldots, \epsilon_t(X) = \epsilon'_t) = \prod_{i=1}^t p_{\epsilon'_i}.
$$

That is, when $X$ is sampled according to $\mu$, the RVs $(\epsilon_1(X), \epsilon_2(X), \ldots)$ are IID Bern($p_1$). Let $B_t(x) = \sum_{i=1}^t \epsilon_i(x)$ and recall that $I_t(x)$ is the unique interval in $F_t$ containing $x$. Then by construction $\mu(I_t(x)) = (1 - p)^{t - B_t(x)} p^{B_t(x)}$, while $|I_t(x)| = (1 - u)^{t - B_t(x)} u^{B_t(x)}$. Therefore by the law of large numbers

$$
\lim_{t \to \infty} \frac{\ln \mu(I_t(x))}{\ln |I_t(x)|} = \frac{p \ln p + (1 - p) \ln(1 - p)}{p \ln u + (1 - p) \ln(1 - u)}, \quad \mu \text{ a.s.} \quad (23)
$$

**Proposition 6** For every $u \neq p$ the measure $\mu_u$ is singular with respect to the Lebesgue measure.

**Proof** The r.h.s of (23) is in $(0, 1]$ and is equal to 1 if and only if $u = p$, in which case $\mu_u$ is uniform. Assume now $u \neq p$. Let $N$ be the differentiability set of $G_u$ in $(0, 1)$. Since $G_u$ is nondecreasing, $N$ has Lebesgue measure one. However, (23) shows that $\mu_u(N) = 0$. Therefore $\mu_u$ is singular with respect to the Lebesgue measure.

\[\square\]

## 4 Self-affine Markov Chains

In this section we present a generalization of the dynamics in given in (2) and obtain the corresponding limit results, which we then use to prove Theorem 1 as a special case. We begin with the general setup and a limit result,
Fig. 2 The local exponents for \(\mu_u\) as a function of \(u\), with \(p = 0.4\). The red graph represents the minimum of the left and right limits from (22) and (20). The blue graph represents the a.s. limit of (23) (Color figure online)

4.1 Setup and Convergence of Marginals

Fix \(K \in \mathbb{N}\). Let \((p_0, \ldots, p_K)\) be a probability vector with strictly positive entries. That is

\[
\min_{0 \leq j \leq K} p_j > 0, \quad \sum_{j=0}^{K} p_j = 1 \tag{24}
\]

Also, let

\[
\begin{align*}
0 &\leq a_j(u) \leq a_j(u) + b_j(u) \leq 1 & j = 0, \ldots, K, u \in [0, 1) \\
\{ j \to (a_j(u), b_j(u)) \} &\text{ is one-to-one for every } u \in [0, 1). \tag{25}
\end{align*}
\]

For each \(u\) and \(j = 0, \ldots, K\), let \(S_j(u)x = a_j(u) + b_j(u)x\) be an affine map from \([0, 1)\) into \([0, 1]\).

Next, let \((B_t : t \in \mathbb{N})\) be IID with \(P(B_1 = j) = p_j, \ j = 0, \ldots, K\). \tag{26}

We now define a family of Markov chains indexed by \(u \in [0, 1)\) according to the rule:

\[
\begin{align*}
\Theta_0(\cdot) &\in [0, 1] \\
\Theta_t(u) &\equiv S_{B_t}(u)\Theta_{t-1}(u) = a_{B_t}(u) + b_{B_t}(u)\Theta_{t-1}(u). \tag{27}
\end{align*}
\]

Assumption 1 Let \(K \in \mathbb{N}\). Assume that (24), (25), (26) hold, and let \((\Theta_t(\cdot) : t \in \mathbb{Z}_+)\) be as in (27).

Finally, define

\[
N_{t,j} = |\{s \leq t : B_s = j\}|, \ j = 0, \ldots, K.
\]

We have the following

**Theorem 7** Let Assumption 1 hold. Then for every \(u \in [0, 1)\)

1. the process \((\tilde{\Theta}_t(u) : t \in \mathbb{Z}_+)\) defined as

\[
\tilde{\Theta}_t(u) = \sum_{s=1}^{t} a_{B_s}(u) \prod_{i=0}^{K} b_{i}^{N_{s-1,i}(u)} + \Theta_0(u) \prod_{i=0}^{K} b_{i}^{N_{0,i}(u)}.
\]

is identically distributed as \((\Theta_t(u) : t \in \mathbb{Z}_+)\).
2. 
\[
\lim_{t \to \infty} \tilde{\Theta}_t(u) = \tilde{\Theta}_\infty(u) = \sum_{s=1}^{\infty} a_{B_s}(u) \prod_{i=0}^{K} b_i^{N_t-1,i}(u), \text{ a.s.}
\] (28)

3. 
\[
E|\tilde{\Theta}_t(u) - \tilde{\Theta}_\infty(u)| \leq \rho'(u) \frac{2 - \rho(u)}{1 - \rho(u)},
\]
where \( \rho(u) = \sum_{i=0}^{K} p_i b_i(u) \in (0, 1) \).

**Proof** From (27) we see that \( \Theta_t(u) \) is a deterministic function of \( \Theta_0, B_0, \ldots, B_t \) and of \( a_0(u), \ldots, a_K(u), b_0(u), \ldots, b_K(u) \). In order to keep the notation simple, in what follows we fix \( u \), and suppress the dependence on it.

By (27) we get
\[
\Theta_t = a_{B_t} + b_{B_t}a_{B_{t-1}} + b_{B_t}b_{B_{t-1}}a_{B_{t-2}} + \cdots + b_{B_t}b_{B_{t-1}} \cdots b_{B_2}a_{B_1} + b_{B_t}b_{B_{t-1}} \cdots b_{B_1} \Theta_0.
\]

Now fix \( t \in \mathbb{N} \). For \( r = 1, \ldots, t \), let \( \tilde{B}_r = B_{t-r+1}, \) so \( \tilde{B}_1 = B_t, \tilde{B}_2 = B_{t-1}, \ldots \). Also, let \( \tilde{N}_{r,i} = |\{1 \leq r \leq t : \tilde{B}_r = i\}| \). Then,
\[
N_{t,i} - N_{s,i} = 1_{\{B_{r+1}=i\}} + \cdots + 1_{\{B_r=i\}} = 1_{\{\tilde{B}_{t+1}=i\}} + \cdots + 1_{\{\tilde{B}_{t-s}=i\}} = \tilde{N}_{t-s,i},
\]
and \( B_s = \tilde{B}_{t-s+1} \). With this, we can write
\[
a_{B_s} \prod_{s<k \leq t} b_{B_k} = a_{B_s} \prod_{i=0}^{K} b_i^{N_{t,i}-N_{s,i}} = a_{\tilde{B}_{t-s+1}} \prod_{i=0}^{K} b_i^{\tilde{N}_{t-s,i}},
\]
and so changing the summation from \( s \) to \( r = t-s+1 \), we obtain
\[
\Theta_t = \sum_{r=1}^{t} a_{\tilde{B}_r} \prod_{i=0}^{K} b_i^{\tilde{N}_{t-i,i}} + \Theta_0 \prod_{i=0}^{K} b_i^{\tilde{N}_{t,i}}.
\]
Since the joint distribution of \( \tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_t \) coincides with that of \( B_1, \ldots, B_t \), it follows that
\[
\Theta_t \overset{\text{dist}}{=} \tilde{\Theta}_t := \sum_{s=1}^{t} a_{B_s} \prod_{i=0}^{K} b_i^{N_{t,i}-N_{s,i}} + \Theta_0 \prod_{i=0}^{K} b_i^{N_{t,i}}.
\] (29)
Note that this equality in distribution holds also for the function \( \Theta_t(u) \). Note that the expression on the right hand side is a partial sum of a series. By (24) \( \max_j p_j < 1 \), and by (25), \( \max_j b_j \leq 1 \). In particular, \( \rho = \rho(u) = \sum_{i=0}^{K} p_i b_i \leq 1 \) with equality if and only if \( b_j = 1 \) for all \( j \), which violates our assumptions. Using the formula for the probability generating function of a multinomial distribution,
\[
E \left[ \prod_{i=0}^{K} b_i^{N_{t-1,i}} \right] = \left( \sum_{i=0}^{K} b_i p_i \right)^{s-1} = \rho^{s-1},
\]
monotone convergence guarantees that the partial sum in (29) converges a.s. to $\tilde{\Theta}_\infty$ which is defined by (28). We also have

$$\tilde{\Theta}_\infty - \tilde{\Theta}_t = \sum_{s=t+1}^{\infty} a_{B_s} \prod_{i=0}^{K} b_i^{N_{s-1,i}} - \Theta_0 \prod_{i=0}^{K} b_i^{N_{t,i}}.$$ 

Therefore,

$$E|\tilde{\Theta}_\infty - \tilde{\Theta}_t| \leq \sum_{s=t+1}^{\infty} (\rho^{s-1}) + \rho^t$$

$$= \rho^t \left( \frac{1}{1-\rho} + 1 \right)$$

$$= \rho^t \frac{2-\rho}{1-\rho}.$$ 

$$\square$$

### 4.2 Convergence of CDFs and Proof of Theorem 1

We will make the following assumptions

**Assumption 2** Assumption 1 holds, and

1. For every $z \in [0, 1]$ and $i \in \{0, \ldots, K\}$, the function $u \rightarrow S_i(u)z$ is right-continuous and nondecreasing.
2. The function $u \rightarrow \Theta_0(u)$ right-continuous and non-decreasing on $[0, 1)$ and has range contained in $[0, 1]$.

Observe that under Assumption 2, for every $t \in \mathbb{Z}_+$, the function $u \rightarrow \Theta_t(u)$ can be extended to a CDF by letting

$$\Theta_t(u) = \begin{cases} 
0 & u < 0 \\
1 & u \geq 1
\end{cases}$$

(30)

**Proposition 8** Let Assumption 2 hold. Then the distribution of the random CDFs $\Theta_t(\cdot)$ converges as $t \to \infty$ to the distribution of the random CDF $\tilde{\Theta}_\infty$:

$$\tilde{\Theta}_\infty(u) = \begin{cases} 
0 & u < 0; \\
\sum_{s=1}^{\infty} a_{B_s}(u) \prod_{i=0}^{K} b_i^{N_{s-1,i}}(u) & u \in [0, 1); \ a.s. \\
1 & u \geq 1.
\end{cases}$$

**Proof** We extend $\tilde{\Theta}_t(\cdot)$ to $\mathbb{R}$ analogously to (30). As a non-decreasing and right-continuous function is determined by the values it attains on the rationals, and all finite-dimensional distributions of $\Theta_t(\cdot)$ and of $\tilde{\Theta}_t(\cdot)$ coincide, it follows the two function-valued processes $(\Theta_t(\cdot) : t \in \mathbb{Z}_+)$ and $(\tilde{\Theta}_t(\cdot) : t \in \mathbb{Z}_+)$ are identically distributed. Since a.s. convergence implies convergence in distribution, it is enough to show that $\tilde{\Theta}_t(\cdot)$ converges a.s. to the prescribed limit.

Let $(u_n : n \in \mathbb{N})$ be a sequence in $[0, 1)$ increasing to 1, and let $A_n$ be the event that the right hand side of (29) converges for $u = u_n$. By (27), $\Theta_t(u)$ is a (random) composition of
the functions $S_0 \ldots, S_K$, all of which are increasing in $u$, so we have $A_{n+1} \subseteq A_n$. Now let $A = \cap_{n=1}^{\infty} A_n$. Then,

$$P(A) = \lim_{n \to \infty} P(A_n) = 1.$$ 

Therefore, on $A$, $\lim_{t \to \infty} \Psi_t(u)$ exists for all $u \in [0,1)$, and is equal to $\Theta_\infty(u)$, the expression on the right hand side of (28). By monotone convergence, this latter expression is right-

continuous on $[0,1)$. We have therefore shown that $\Psi_t(\cdot)$ converges pointwise to $\Theta_\infty(\cdot)$ on $[0,1)$ a.s. Finally, extend $\Theta_\infty$ to $\mathbb{R}$ according to (30) and the result follows.

\[\square\]

**Corollary 9** Let $K = 1$, $a_0(u) = 0$, and $(p_0, p_1) = (p, 1-p)$ for some $p \in (0,1)$. Then, for $u \in [0,1)$,

$$\lim_{t \to \infty} \Theta_t(u) = \frac{a_1(u)}{b_0(u)} \sum_{k=0}^{\infty} b_0^T_k(u) \left( \frac{b_1(u)}{b_0(u)} \right)^k \quad \text{a.s.,} \quad (31)$$

where $T_k = G_0 + \cdots + G_k$, and $G_0, G_1, \ldots$ are IID Geom$(1-p)$.

**Proof** Since $a_0(u) = 0$, the summation in the expression for $\Theta_\infty(u)$ in Theorem 7 is only over those $s$ such that $B_s = 1$. Let $T_{s-1} = 0$ and continue inductively, letting $T_k = \inf\{t > T_{k-1} : B_t = 1\}$, $k \in \mathbb{Z}$. Then $(T_k - T_{k-1} : k \in \mathbb{N})$ is an IID sequence with distribution Geom$(1-p)$, or, equivalently, $T_k$ is the partial sum of exactly $k+1$ IID Geom$(1-p)$. Note that $N_{T_k-0,1} = k$ and $N_{T_k-1,0} = T_k - 1 - k$. Therefore,

$$\Theta_\infty(u) = \sum_{k=0}^{\infty} a_1(u) b_0^{T_k} (u) b_1^{T_k} (u)$$

$$= \frac{a_1(u)}{b_0(u)} \sum_{k=0}^{\infty} b_0^T_k(u) \left( \frac{b_1(u)}{b_0(u)} \right)^k .$$

\[\square\]

We now prove Theorem 1.

**Proof of Theorem 1** The theorem corresponds to $K = 1$, $p_0 = p$, $p_1 = 1-p$, $S_0(x) = ux$ and $S_0(x) = u + (1-u)x$. That is,

$(a_0(u), b_0(u)) = (0, u), \quad (a_1(u), b_1(u)) = (u, 1-u)$.

Applying Corollary 9 gives (3). This proves (3).

Since the function on the right hand side of (31) is continuous a.s. it follows that the limit holds for all $u \in (0,1)$, a.s. This implies that the distribution of the random function $\Psi_t(\cdot)$ converges as $t \to \infty$ to the distribution of the function on the right hand side of (31). \[\square\]

### 4.3 Continuity of the Self-similar Measure

**Proposition 10** Let Assumption 1 hold and assume now that $b_i(u) > 0$ for all $i$ and that $S_i(u)$ is $1-1$ for all $i$, and also that for $i \neq j$, the intersection of the images of $S_i(u)$ and $S_j(u)$ is either empty or contains exactly one element. Then the distribution of $\Theta_\infty(u)$ is continuous and its CDF satisfies

$$G_u(z) = \frac{G_u(S_i'(u)z) - \sum_{i < i'} p_i}{p_{i'}}$$

for all $i' \in \{0, \ldots, K\}$, $u \in [0,1)$ and $z \in \mathbb{R}$. 

\[\square\] Springer
Proof Since each of the Markov chains \((\Theta_t(u) : t \in \mathbb{Z}_+)\), \(u \in [0, 1]\) converges as \(t \to \infty\) to a unique distribution indexed by \(u\), it follows that the limiting distribution is the unique stationary distribution for the given dynamics. On the other hand, if \(\mu_u\) is the stationary distribution for that chain, then
\[
\mu_u(\cdot) = \sum_{i=0}^{K} p_i \mu_u \circ S_i^{-1}(u)(\cdot).
\]

Hence, for every \(x\),
\[
\mu_u([x]) = \sum_{i=0}^{K} p_i \mu_u \circ S_i(u)^{-1}([x]),
\]
(32)
and there can be at most two distinct \(i\)'s such that \(S_i(u)^{-1}([x])\) is not empty.

We now prove that \(\mu_u\) has no atoms. By contradiction, assume that the set of atoms is not empty. Since the cumulative distribution function \(G_u\) is increasing and every atom for \(\mu_u\) is a discontinuity point for \(G_u\) there are at most countably many atoms for \(\mu_u\). Since the sum over all atoms \(\sum_{z} \mu_u([z]) \leq 1\) it is easy to see that \(\mu_u([z])\) attains a maximum for some \(z = x\).

Then either
1. \(x\) is in the image of \(S_i(u)\) for exactly one \(i\). Let \([z] = S_i(u)^{-1}([x])\). By (32)
\[
\begin{align*}
\mu_u([x]) &= p_i \mu_u \circ S_i(u)^{-1}([x]) \\
&= p_i \mu_u([z]) \\
&< \mu_u([x]),
\end{align*}
\]
where we used the assumed maximality of \(\mu_u([x])\). This is a contradiction;
2. or \(x\) is the image of \(S_i(u)\) and \(S_j(u)\) for \(i \neq j\). Then, there is a unique \(i \in \{0, \ldots, K - 1\}\) such that \(S_i(u)1 = x = S_{i+1}(u)0\). By (32) \(\mu_u([x])\) is equal to \(p_i \mu_u([1]) + p_{i+1} \mu_u([0])\).

By the maximality of \(\mu_u([x])\), it follows that \(\mu_u([0]) = \mu_u([1]) = \mu_u([x])\) and \(p_i + p_{i+1} = 1\). Therefore, we necessarily have \(K = 1\) and \(i = 0\). We are now back to case 1 for \(x = 0\) and \(x = 1\). That is, \(\mu_u\) reaches a maximum at 0 and at 1 but 0 and 1 are each the image of a single \(S_i(u)\). As in case 1 above this leads to a contradiction.

Therefore \(\mu_u\) has no atoms and the corresponding distribution function is continuous.

We now turn to the proof of the second statement in Proposition 10.

If \(I_i\) is the image of \([0, 1]\) under \(S_i(u)\), then \(\mu_u(I_i) = p_i\), and if \(G_u\) is the CDF of \(\mu_u\),
\[
G_u(x) = \sum_{i=0}^{K} p_i G_u(S_i^{-1}(u)(x)).
\]

If \(x \in I_i\), the sum above becomes
\[
G_u(x) = \sum_{i < i'} p_i + p_{i'} G_u(S_{i'}^{-1}(u)(x)) + 0.
\]

We can rewrite this by letting \(z = S_{i'}^{-1}(u)(x) = (x - a_{i'}(u))/b_{i'}(u)\) to obtain the following,
\[
G_u(z) = \frac{G_u(S_{i'}(u)z) - \sum_{i < i'} p_i}{p_{i'}}.
\]

\(\square\)
Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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