Operational characterization of general quantum correlation via complex weak value measurement

Agung Budiyono and Hermawan K. Dipojono

Department of Engineering Physics,
Bandung Institute of Technology, Bandung, 40132, Indonesia and
Research Center for Nanoscience and Nanotechnology,
Bandung Institute of Technology, Bandung, 40132, Indonesia

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Abstract

The last two decades have witnessed significant progress on the understanding the quantum correlation more general than entanglement, wherein even a separable state may yield correlation that cannot be emulated by any classical object. Such a general nonclassical correlation is not only intriguing from the fundamental point of view, but it has also been recognized as a resource in a variety of quantum information processing tasks and quantum technology. Here, we propose a characterization of the general quantum correlation in bipartite system in terms of direct laboratory operations using weak measurement with postselection. We define a quantity based on the imaginary part of weak values obtained via weak measurement of a local basis followed by a postselection of another local basis, and an optimization procedure over all possible choices of the two bases. We show that it satisfies certain desirable requirements for a quantifier of general quantum correlations. It may be statistically interpreted as the minimum genuine quantum share of uncertainty. It is a faithful witness of entanglement for general pure states, giving an observable lower bound to a scaled square root of the linear entropy of entanglement. We then suggest an information theoretic interpretation of the general quantum correlation in a multipartite state as the minimum mean absolute error in an optimal estimation of any local measurement basis, based on the outcomes of local projective measurement on the state, in the worst case scenario.

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*Electronic address: agungbymlati@gmail.com
I. INTRODUCTION

Nonclassical correlation in multipartite quantum systems was originally associated with entangled states, namely states that cannot be prepared by any set of local operation and classical communication (LOCC) [1]. Studies in the last couple of decades however show that most unentangled (i.e., separable) states are somewhat nonclassically correlated, manifested in the various forms of discord-like quantum correlations [2–26]. See Refs. [27–29] for recent reviews. Conceptually, these general quantum correlations in which entanglement is a subset, arise from the noncommutativity between the multipartite quantum state and some set of local quantum observables, rather than from the nonseparability of the state. The nonclassical correlations beyond entanglement have received a growing attention both from the fundamental viewpoint and also practically to better understand the physical origin of the quantum advantages in certain information processing tasks and schemes of quantum technology which consume very little or no entanglement [30–37]. Different approaches have been proposed to characterize and quantify the general quantum correlation, using quantum information theoretical concepts which take on an input of the state of the multipartite system. They thus indirectly need a full quantum state tomography for its evaluation. It remains intriguing whether it is possible to characterize and quantify general quantum correlation directly in terms of transparent laboratory operations.

On the other hand, there is a measurement scheme in which the noncommutativity between the state and the observable appears directly at the pointer of the measuring device. Consider the weak measurement of a Hermitian observable $O$ with the preselected state $\rho$ followed with the postselection on the state $|\phi\rangle$ via a normal, i.e., strong projective measurement. One then obtains the following complex weak value [38–40]:

$$O^w(\phi|\rho) = \frac{\langle\phi|O|\phi\rangle}{\langle\phi|\phi\rangle},$$

with its real and imaginary parts that can be directly read-off respectively from the average shift of the position and momentum of the pointer of the measuring device [41, 42]. The complex nature of the weak value has been exploited recently to devise a method for direct measurement of a quantum state or its quasiprobability representation [43–45]. Notice in particular that the imaginary part of the weak value, i.e.,

$$\text{Im}\{O^w(\phi|\rho)\} = \frac{1}{2i} \frac{\langle\phi|[O,\rho]|\phi\rangle}{\langle\phi|\phi\rangle},$$

(2)
directly captures the noncommutativity between the state and the observable which is the
origin of various nonclassical properties of quantum systems encoded in the quantum state.
It is thus natural to ask if one can use the operational scheme of weak measurement with
postselection to directly characterize and quantify the quantum correlation beyond entan-
glement arising from the noncommutativity between the multipartite state and some set of
local observables.

In this paper, we argue for an affirmative answer to the above question. First, given a
preselected multipartite state, we operationally define a quantity from the imaginary part
of the weak values obtained by making weak measurement over a complete set of local
orthonormal basis of the subsystems, followed by postselection over the product of the local
bases, and optimized over all possible choices of these two bases. We demonstrate that it
satisfies certain plausible properties expected for a quantifier of quantum correlation beyond
entanglement. It sets a lower bound to the negativity of quantumness [16, 46]. We argue that
it may be interpreted as the minimum genuine quantum share out of the total uncertainty
of the local measurement basis in the multipartite state. For general pure states, it gives
an observable lower bound, hence a witness, to the scaled square root of the linear entropy
of entanglement. Our approach thus offers a definition of general quantum correlation in
terms of direct laboratory operations based on weak measurement with postselection, which
might suggest a fresh insight into its role as a resource in quantum information processing.
Moreover, it leads to the following information theoretical interpretation of the general
quantum correlation: a multipartite state is quantumly correlated if and only if the optimal
estimation of any local basis, based on the outcomes of the local projective measurement
over the state, must yield a nonvanishing mean absolute error in the worst case scenario,
with the minimum value that is given by the quantum correlation. On the other hand, a
multipartite state is classical devoid of quantum correlation if and only if, local measurement
over the state can be used to make a sharp optimal estimation of a certain local basis.

II. NONCLASSICAL STATES, NONCOMMUTATIVITY, AND NONREAL
WEAK VALUE

Consider a composite of two subsystems $AB$ with a quantum state that is represented
by a density matrix $\rho_{AB}$ acting on the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_{A(B)}$ is the
Hilbert space of the subsystem $A(B)$. In general multipartite setting, a separable state is intuitively defined as a state that can be prepared by LOCC \[1\]. For the above bipartite system, a separable state can thus in general be expressed as a classical statistical mixture of the product states, i.e.,

$$\rho_{AB}^S = \sum_k p_k \rho_A^k \otimes \rho_B^k,$$

where $\rho_A^k$ is the density matrix of the subsystem $A$ on $\mathcal{H}_A$, and $\{p_k\}$ are mixing probabilities: $p_k \geq 0$, $\sum_k p_k = 1$. Any other states, i.e., nonseparable states, are entangled. For pure states, separability is equivalent to the absence of nonclassical correlation. Remarkably, allowing impurity to the bipartite state may lead to nonclassical correlations even when it is separable. That is, various different schemes have been revealed in the last couple of decades which show that mixed separable states may contain correlation that cannot be accessed by any classical object \[2–29\]. For instance, such a separable state may yield a global modification as a response to any local measurement \[2,3,15\].

By contrast, classical intuition suggests that for a genuine classical multipartite state, there is at least a local measurement which does not lead to a global modification to the state. In the bipartite setting, there are two classes of states which conform to the above intuition \[9,33\]. One is a class of so-called classical-quantum (or, $A$-classically correlated) states, which are those that can be expressed as

$$\rho_{AB}^{CQ} \doteq \sum_k p_k |k\rangle \langle k|_A \otimes \rho_B^k,$$

where $\{|k\rangle_A\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_A$ of subsystem $A$, $\{\rho_B^k\} \in \mathcal{H}_B$ are states of subsystem $B$, and $\{p_k\}$ are probabilities, i.e., $p_k \geq 0$, $\sum_k p_k = 1$. The quantum-classical (or, $B$-classically correlated) state is defined analogously by exchanging the role of the parties $A$ and $B$. The other class consists of bipartite states with density matrix which take the following form:

$$\rho_{AB}^{CC} \doteq \sum_{k,l} p_{kl} |k\rangle \langle k|_A \otimes |l\rangle \langle l|_B,$$

where $\{|k\rangle_A\}$ and $\{|l\rangle_B\}$ are respectively orthonormal basis of subsystem $A$ and $B$, and $\{p_{kl}\}$ are joint probabilities so that $p_{kl} \geq 0$, $\sum_{k,l} p_{kl} = 1$. Such states are called classical-classical (or, totally classically correlated) states. One can see that applying local projective (von Neumann) measurement $\{\Pi_A^k \otimes \mathbb{I}_B\}$ and $\{\Pi_A^k \otimes \Pi_B^{l}\}$ (where $\Pi^x \doteq |x\rangle \langle x|$ is one-dimensional projector onto the subspace spanned by $|x\rangle$), respectively, to the classical-quantum states and classical-classical states, without learning or recording the outcomes, leave them unmodified,
i.e., $\sum_k (\Pi^k_A \otimes I_B) g^{CQ}_{AB} (\Pi^k_A \otimes I_B) = g^{CQ}_{AB}$ and $\sum_{k,l} (\Pi^k_A \otimes \Pi^l_B) g^{CC}_{AB} (\Pi^k_A \otimes \Pi^l_B) = g^{CC}_{AB}$. This can be seen as due to the fact that the local measurement basis and the bipartite state commute, i.e., $[ (\Pi^k_A \otimes I_B), g^{CQ}_{AB} ] = 0$ and $[ (\Pi^k_A \otimes \Pi^l_B), g^{CC}_{AB} ] = 0$, for all $k, l$. Any bipartite state that cannot be represented either in the forms of Eqs. (3) or (4) thus contain nonclassical correlation arising from the noncommutativity between the state and the local measurement basis, even if it is separable.

Various mathematical characterizations of the classical bipartite states of the types of Eqs. (3) and (4) have led researchers to develop different quantifiers and measures of quantum correlation beyond entanglement, by applying quantum information theoretical concepts: quantum mutual information, von-Neumann entropy, distance and infidelity between two density matrices, Wigner-Yanase skew information or quantum Fisher information [28]. All these quantifiers essentially quantify the difference between the bipartite state under scrutiny and the classical bipartite states of Eqs. (3) and (4). Moreover, they indirectly or directly reflects the noncommutativity between the bipartite states and some set of local observables.

To evaluate the nonclassical correlation in a bipartite quantum state using these quantifiers, we typically need an input of the associated density matrix, thus requires a tomography of the full quantum state. In order to better understand the operational meaning of the quantum correlation beyond entanglement, it is desirable to have a quantifier of general quantum correlation whose definition directly and transparently translates into laboratory operations.

On the other hand, as noted in the previous section, the imaginary part of the weak value can be used to directly probe the noncommutativity between a preselected bipartite quantum state and the local measurement basis as follows. Suppose we wish to probe the noncommutativity between a bipartite state $\varrho_{AB}$ and a local measurement basis $\{\Pi^k_A\}$ on subsystem $A$. To do this, one first makes a weak measurement of $\Pi^k_A \otimes I_B$ with the preselected state $\varrho_{AB}$ followed by a postselection on $|\phi_{AB}\rangle$, reads the imaginary part of the weak value from the average shift of the pointer momentum, and multiply it with the probability of the successful postselection, to get

$$C_A[\Pi^k_A \otimes I_B; \varrho_{AB}, |\phi_{AB}\rangle] \doteq \text{Im} \left\{ \frac{\langle \phi_{AB}|(\Pi^k_A \otimes I_B)\varrho_{AB}|\phi_{AB}\rangle}{\langle \phi_{AB}|\varrho_{AB}|\phi_{AB}\rangle} \right\} \langle \phi_{AB}|\varrho_{AB}|\phi_{AB}\rangle$$

$$= \frac{1}{2\iota} \langle \phi_{AB}|[(\Pi^k_A \otimes I_B), \varrho_{AB}]|\phi_{AB}\rangle. \tag{5}$$
One can then see that for the classical-quantum state of Eq. (3), there is a local measurement basis \{\Pi_A^k\} such that \([\Pi_A^k \otimes \mathbb{I}_B, \varrho_{AB}^{CQ}] = 0\), and we have \(C_A[\Pi_A^k \otimes \mathbb{I}_B; \varrho_{AB}^{CQ}, |\phi_{AB}\rangle] = 0\), for all \(k\). Moreover, choosing a postselection state \(|\phi_{AB}\rangle\) such that its associated density matrix does not commute with the preselected state \(\varrho_{AB}\) and the local measurement \(\Pi_A^k \otimes \mathbb{I}_B\), then \(C_A[\Pi_A^k \otimes \mathbb{I}_B; \varrho_{AB}, |\phi_{AB}\rangle] = 0\) implies \([\Pi_A^k \otimes \mathbb{I}_B, \varrho_{AB}] = 0\). Hence, in this setting, a nonvanishing \(C_A[\Pi_A^k \otimes \mathbb{I}_B; \varrho_{AB}, |\phi_{AB}\rangle]\) captures the noncommutativity between \(\Pi_A^k \otimes \mathbb{I}_B\) and \(\varrho_{AB}\). A similar observation applies to the set of classical-classical states. Namely, for each classical-classical state of Eq. (4), there must be a pair of local bases \{\Pi_A^k\} on \(A\) and \{\Pi_B^l\} on \(B\) so that \([\Pi_A^k \otimes \Pi_B^l, \varrho_{AB}^{CC}] = 0\), and we have \(C_A[\Pi_A^k \otimes \Pi_B^l; \varrho_{AB}, |\phi_{AB}\rangle] = 0\). Moreover, choosing \(|\phi_{AB}\rangle\) whose density matrix does not commute with \(\varrho_{AB}\) and \(\Pi_A^k \otimes \Pi_B^l\), then \(C_A[\Pi_A^k \otimes \Pi_B^l; \varrho_{AB}, |\phi_{AB}\rangle] = 0\) implies \([\Pi_A^k \otimes \Pi_B^l, \varrho_{AB}] = 0\). We note that the noncommutativity between the state and a Hermitian observable can also be directly checked via quantum variance capturing the uncertainty of the observable in the state. However, quantum variance also takes the contribution of the uncertainty due to the statistical, i.e., classical mixing of states, so that it is not genuinely quantum.

III. GENERAL NONCLASSICAL CORRELATION FROM LOCAL WEAK VALUE MEASUREMENT

We want to devise a quantifier of general quantum correlation in a bipartite state from \(C_A[\Pi_A^k \otimes \mathbb{I}_B; \varrho_{AB}, |\phi_{AB}\rangle]\) defined in Eq. (5) so that it can be observed directly using weak measurement with postselection. To this end, given an arbitrary bipartite quantum state \(\varrho_{AB}\), let us define the following quantity which maps the state to a nonnegative real number:

\[
C_A^Q[\varrho_{AB}] = \min_{\Pi_A^k} \max_{\{\phi_A, \phi_B\}} \sum_{a,\phi_A,\phi_B} |C_A[\Pi_A^k \otimes \mathbb{I}_B; \varrho_{AB}, |\phi_A, \phi_B\rangle]| \\
= \min_{\Pi_A^k} \max_{\{\phi_A, \phi_B\}} \sum_{a,\phi_A,\phi_B} |\text{Im}\left\{\frac{\langle \phi_A, \phi_B | (\Pi_A^k \otimes \mathbb{I}_B) \varrho_{AB} | \phi_A, \phi_B \rangle}{\langle \phi_A, \phi_B | \varrho_{AB} | \phi_A, \phi_B \rangle}\right\}| \\
\times \langle \phi_A, \phi_B | \varrho_{AB} | \phi_A, \phi_B \rangle \\
= \min_{\Pi_A^k} \max_{\{\phi_A, \phi_B\}} \sum_{a,\phi_A,\phi_B} |\text{Im}\{\langle \phi_A, \phi_B | (\Pi_A^k \otimes \mathbb{I}_B) \varrho_{AB} | \phi_A, \phi_B \rangle\}| \\
= \min_{\Pi_A^k} \max_{\{\phi_A, \phi_B\}} \sum_{a,\phi_A,\phi_B} \frac{1}{2} |\langle \phi_A, \phi_B | [(\Pi_A^k \otimes \mathbb{I}_B), \varrho_{AB}] | \phi_A, \phi_B \rangle|, \tag{6}
\]
where \( \{ |\phi_A, \phi_B\rangle \} \) is the set of postselection basis of the Hilbert space \( \mathcal{H}_{AB} \), assumed to be given by the tensor product \( \{ |\phi_A, \phi_B\rangle \} = \{ |\phi_A\rangle \otimes |\phi_B\rangle \} \), with \( \{ |\phi_{A(B)}\rangle \} \) is the basis of the Hilbert space \( \mathcal{H}_{A(B)} \). From the second line, \( C_A^Q[\varrho_{AB}] \) can thus be directly translated operationally as follows. We first make weak measurement of the local measurement basis \( \{ \Pi_a^\alpha \} \) on the subsystem \( A \) with a preselected bipartite state \( \varrho_{AB} \) and a postselection given by the product basis \( \{ |\phi_A, \phi_B\rangle \} \), read the absolute imaginary part from the average shift of the pointer momentum, average over the probability of successful postselection, maximize over all product of postselection bases \( \{ |\phi_A, \phi_B\rangle \} \in \mathcal{B}_{\{ |\phi_A\rangle \otimes |\phi_B\rangle \}} \), and finally minimize over all possible local measurement bases \( \{ \Pi_a^\alpha \} \in \mathcal{B}_{\{ \Pi_a^\alpha \}} \). Here, \( \mathcal{B}_{\{ |\phi_A\rangle \otimes |\phi_B\rangle \}} \) and \( \mathcal{B}_{\{ \Pi_a^\alpha \}} \) denote respectively the set of all local postselection bases \( \{ |\phi_A, \phi_B\rangle \} \) of \( \mathcal{H}_{AB} \) and local measurement bases \( \{ \Pi_a^\alpha \} \) of \( \mathcal{H}_A \). \( C_A^Q[\varrho_{AB}] \) is intended to quantify the nonclassical correlation in the bipartite state \( \varrho_{AB} \) arising from the noncommutativity between \( \varrho_{AB} \) and any local measurement basis \( \{ \Pi_a^\alpha \} \) of the subsystem \( A \). Similarly, to quantify the nonclassical correlation in the bipartite state \( \varrho_{AB} \) which arises from the noncommutativity between \( \varrho_{AB} \) and any local measurement basis \( \{ \Pi_a^\alpha \otimes \Pi_b^\beta \} \), we define the following quantity which maps the density matrix to a nonnegative real number:

\[
C_{AB}^Q[\varrho_{AB}] = \min_{\{ \Pi_a^\alpha \otimes \Pi_b^\beta \} \in \mathcal{B}_{\{ \Pi_a^\alpha \otimes \Pi_b^\beta \}}} \max_{\{ |\phi_A, \phi_B\rangle \}} \sum_{a,b,\phi_A,\phi_B} \left| \text{Im} \left\{ \frac{\langle \phi_A, \phi_B \rangle (\Pi_a^\alpha \otimes \Pi_b^\beta) \varrho_{AB} | \phi_A, \phi_B \rangle}{\langle \phi_A, \phi_B \rangle | \varrho_{AB} | \phi_A, \phi_B \rangle} \right\} \right| \\
\times \langle \phi_A, \phi_B \rangle | \varrho_{AB} | \phi_A, \phi_B \rangle \\
= \min_{\{ \Pi_a^\alpha \otimes \Pi_b^\beta \} \in \mathcal{B}_{\{ \Pi_a^\alpha \otimes \Pi_b^\beta \}}} \max_{\{ |\phi_A, \phi_B\rangle \}} \sum_{a,b,\phi_A,\phi_B} \left| \text{Im} \left\{ \langle \phi_A, \phi_B \rangle (\Pi_a^\alpha \otimes \Pi_b^\beta) \varrho_{AB} | \phi_A, \phi_B \rangle \right\} \right| \\
= \min_{\{ \Pi_a^\alpha \otimes \Pi_b^\beta \} \in \mathcal{B}_{\{ \Pi_a^\alpha \otimes \Pi_b^\beta \}}} \max_{\{ |\phi_A, \phi_B\rangle \}} \sum_{a,b,\phi_A,\phi_B} \left| \frac{1}{2} \langle \phi_A, \phi_B \rangle ([\Pi_a^\alpha \otimes \Pi_b^\beta], \varrho_{AB}) | \phi_A, \phi_B \rangle \right|,
\]

where, in this case, the minimization is carried out over all possible local measurement bases \( \{ \Pi_a^\alpha \otimes \Pi_b^\beta \} \in \mathcal{B}_{\{ \Pi_a^\alpha \otimes \Pi_b^\beta \}} \) of \( \mathcal{H}_{AB} \).

Mathematically, the maximization over the postselection basis \( \{ |\phi_A, \phi_B\rangle \} \) is necessary to avoid trivial result that for any local measurement basis \( \{ \Pi_a^\alpha \} \), one can choose the postselection basis as \( \{ |\phi_A\rangle \} = \{ |a\rangle \} \), so that \( C_A[\Pi_a^\alpha \otimes \mathbb{I}_B, \varrho_{AB}, |\phi_A, \phi_B\rangle] = 0 \). For a similar reason, we emphasize that the search for the minimum over the local measurement bases is performed after the maximization over the postselection bases is done. Otherwise, for each postselection basis \( \{ |\phi_A, \phi_B\rangle \} \), we can always choose the local measurement basis \( \{ \Pi_a^\alpha \} = \{ |\phi_A\rangle \langle \phi_A| \} \).
to get \( C_A[\Pi^a_A \otimes I_B; \varrho_{AB}, |\phi_A, \phi_B\rangle] = 0 \). Conceptually, for a given local measurement basis \( \{\Pi^a_A\} \) (or, \( \{(\Pi^a_A \otimes \Pi^b_B)\} \)), maximizing over all possible postselection bases \( \{|\phi_A, \phi_B\rangle\} \) means that we are seeking for the largest incompatibility between \( \{(\Pi^a_A \otimes I_B)\} \) (or, \( \{(\Pi^a_A \otimes \Pi^b_B)\} \)) and \( \varrho_{AB} \). On the other hand, the minimization over the local measurement bases \( \{\Pi^a_A\} \) for \( C_A^Q[\varrho_{AB}] \) (respectively, over \( \{\Pi^a_A \otimes \Pi^b_B\} \) for \( C_{AB}^Q[\varrho_{AB}] \)) means that we search for those classical-quantum state of Eq. (3) (respectively, classical-classical state of Eq. (4)) that is least incompatible with \( \varrho_{AB} \).

We further emphasize that the projective postselection measurement is over product of local bases, i.e., \( \{|\phi_A, \phi_B\rangle\} = \{|\phi_A\rangle \otimes |\phi_B\rangle\} \), so as not to generate nonclassical correlation. Hence, both the weak measurement and the postselection are local operations, so that any nonclassical correlation captured by \( C_A^Q[\varrho_{AB}] \) or \( C_{AB}^Q[\varrho_{AB}] \) must originate inherently from the bipartite state under scrutiny. We shall here on refer to \( C_A^Q[\varrho_{AB}] \) of Eq. (6) and \( C_{AB}^Q[\varrho_{AB}] \) of Eq. (7), respectively, as one-sided and two-sided w-nonclassical correlation. Finally, the above definition can be extended to more than two parties straightforwardly, while retaining clear operational meaning. For example, the one-sided w-nonclassical correlation in the state \( \varrho_{ABC} \) of tripartite system \( ABC \) is defined as \( C_A^Q[\varrho_{ABC}] = \min_{\{\Pi^a_A\}} \max_{\{|\phi_A, \phi_B, \phi_C\rangle\}} \sum_{\phi_A, \phi_B, \phi_C} \text{Im}\{\langle \phi_A, \phi_B, \phi_C| (\Pi^a_A \otimes I_B \otimes I_C) \varrho_{ABC} |\phi_A, \phi_B, \phi_C\rangle\} , \) where the postselection on the subsystem \( C \) is given by the orthonormal basis of subsystem \( C \). Let us mention that completely different approaches to characterize general quantum correlation based on weak measurement but without postselection, and thus independent of the concept of weak value, were reported in Refs. [47, 48].

Consider first the one-sided w-nonclassical correlation \( C_A^Q[\varrho_{AB}] \) defined in Eq. (6). We argue that it satisfies the following desirable properties for a quantifier of quantum correlation [28]:

(i) Faithful, i.e., \( C_A^Q[\varrho_{AB}] \) is vanishing if and only if the bipartite state \( \varrho_{AB} \) belongs to the class of classical-quantum state of Eq. (3).

(ii) Invariant under local unitary transformation, i.e., \( C_A^Q[(U_A \otimes U_B) \varrho_{AB} (U_A^\dagger \otimes U_B^\dagger)] = C_A^Q[\varrho_{AB}] \), where \( U_{A(B)} \) is any unitary operator applying locally on subsystem \( A(B) \).

(iii) Nonincreasing under any local completely positive trace preserving (CPTP) operation \( \Lambda_B \) on subsystem \( B \), namely on the party whose nonclassical correlation is not being quantified, i.e., \( C_A^Q[(I_A \otimes \Lambda_B) \varrho_{AB}] \leq C_A^Q[\varrho_{AB}] \).
Let us sketch the proofs of the above properties and discuss a couple of others.

To establish property (i), first, let us suppose that the bipartite state \( \varrho_{AB} \) belongs to the class of classical-quantum state, namely there exists an orthonormal basis \( \{ \Pi_{\alpha}^{A} = |k\rangle \langle k|_{A} \} \) on the subsystem \( A \) so that \( \varrho_{AB} \) can be expressed as in Eq. (3). Then, we can choose \( \{ \Pi_{\alpha}^{A} \} \) as the local measurement basis in the definition of \( C_{\alpha}^{A}[\varrho_{AB}] \) of Eq. (6). In this case, since \( [(\Pi_{\alpha}^{A} \otimes I_{B}), \varrho_{AB}] = 0 \) for all \( k \), we have \( C_{\alpha}^{A}[\varrho_{AB}] = 0 \). Conversely, suppose \( C_{\alpha}^{A}[\varrho_{AB}] = 0 \). Then, from the definition of \( C_{\alpha}^{A}[\varrho_{AB}] \) in Eq. (6), there must be an orthonormal measurement basis \( \{ \Pi_{\alpha}^{A} \} \) of the subsystem \( A \) so that \( \langle \phi_{A}, \phi_{B} | [(\Pi_{\alpha}^{A} \otimes I_{B}), \varrho_{AB}]| \phi_{A}, \phi_{B} \rangle = 0 \) for all \( k \) and for all possible choices of the postselection bases \( \{ |\phi_{A}, \phi_{B} \rangle \} \). This can only be true if \( [(\Pi_{\alpha}^{A} \otimes I_{B}), \varrho_{AB}] = 0 \) for all \( k \). It first implies that \( \varrho_{AB} \) is separable. Moreover, it also implies that \( [\Pi_{\alpha}^{A}, \varrho_{A}] = 0 \) for all \( k \), where \( \varrho_{A} = \text{Tr}_{B} \varrho_{AB} \), which means that \( \{ \Pi_{\alpha}^{A} \} \) is the eigenprojectors of \( \varrho_{A} \). Hence, we have \( \varrho_{A} = \sum_{k} p_{k} \Pi_{\alpha}^{A} \) for some \( \{ p_{k} \} \), \( p_{k} \geq 0 \), \( \sum_{k} p_{k} = 1 \).

Finally, any separable state \( \varrho_{AB} \) with the reduced density matrix \( \varrho_{A} = \sum_{k} p_{k} \Pi_{\alpha}^{A} \) must take the form of classical-quantum state of Eq. (3).

The property (ii) of invariant under local unitary transformation can be shown as follows:

\[
C_{\alpha}^{A}[(U_{A} \otimes U_{B}) \varrho_{AB}(U_{A}^{\dagger} \otimes U_{B}^{\dagger})]
= \min_{\{ \Pi_{\alpha}^{A} \}} \max_{\{ |\phi_{A}, \phi_{B} \rangle \}} \sum_{a,\phi_{A},\phi_{B}} |\text{Im}\{ \langle \phi_{A}, \phi_{B} | (U_{A} \otimes U_{B})(U_{A}^{\dagger} \otimes U_{B}^{\dagger}) \cdot (\Pi_{\alpha}^{A} \otimes I_{B}) (U_{A} \otimes U_{B}) \varrho_{AB}(U_{A}^{\dagger} \otimes U_{B}^{\dagger}) | \phi_{A}, \phi_{B} \rangle \}| \\
= \min_{\{ \Pi_{\alpha}^{A} \}} \max_{\{ |\phi_{A}, \phi_{B} \rangle \}} \sum_{a,\phi_{A}',\phi_{B}'} |\text{Im}\{ \langle \phi_{A}', \phi_{B}' | (U_{A} \Pi_{\alpha}^{A} U_{A}^{\dagger} \otimes I_{B}) \varrho_{AB} | \phi_{A}', \phi_{B}' \rangle \}| \\
= \min_{\{ \Pi_{\alpha}^{A} \}} \max_{\{ |\phi_{A}, \phi_{B} \rangle \}} \sum_{a,\phi_{A}',\phi_{B}'} |\text{Im}\{ \langle \phi_{A}', \phi_{B}' | (\Pi_{\alpha}^{A} \otimes I_{B}) \varrho_{AB} | \phi_{A}', \phi_{B}' \rangle \}| \\
= C_{\alpha}^{A}[\varrho_{AB}].
\]

Here, we have inserted an identity \( (U_{A} \otimes U_{B})(U_{A}^{\dagger} \otimes U_{B}^{\dagger}) = I \) in the second line. To get the third line, we have defined a new postselection basis: \( \{ |\phi_{A}', \phi_{B}' \rangle \} = (U_{A}^{\dagger} \otimes U_{B}^{\dagger}) | \phi_{A} \rangle \otimes | \phi_{B} \rangle \}, \) where we have made use of the assumption that the original postselection basis is a tensor product of the local bases of subsystems \( A \) and \( B \). Moreover, the fourth line is obtained by identifying a new set of local measurement basis as \( \{ \Pi_{\alpha}^{A} \} = \{ U_{A} \Pi_{\alpha}^{A} U_{A}^{\dagger} \} \). Noting that the above transformation of bases do not change the sets \( B_{\{ |\phi_{A} \rangle \otimes | \phi_{B} \rangle \}} \) and \( B_{\{ \Pi_{\alpha}^{A} \}} \) over which we respectively perform the maximization and the minimization in Eq. (6), we thus have \( \max_{\{ |\phi_{A}, \phi_{B} \rangle \}}(\cdot) = \max_{\{ |\phi_{A}', \phi_{B}' \rangle \}}(\cdot) \) and \( \min_{\{ \Pi_{\alpha}^{A} \}}(\cdot) = \min_{\{ \Pi_{\alpha}^{A} \}}(\cdot) \). This observation gives the
last equality in Eq. (8).

To establish the property (iii), first, we recall that according to the Stinespring’s theorem, any CPTP operation or quantum channel can be implemented by a dilation on a larger Hilbert space, wherein the system is made contact with an ancilla in a state $\varrho_E$ on the Hilbert space $\mathcal{H}_E$, let them interact via some global unitary, and then followed by partial tracing over the ancilla as: $(I_A \otimes \Lambda_B)[\varrho_{AB}] = \text{Tr}_E\{(I_A \otimes U_{BE})(\varrho_{AB} \otimes \varrho_E)(I_A \otimes U_{BE})^\dagger\}$, where $U_{BE}$ is the unitary interaction applying on the subsystem $B$ and the ancilla $E$. Using this, we then have

$$C^\mathcal{O}_A[(I_A \otimes \Lambda_B)[\varrho_{AB}]] = \min_{\{\Pi^A_a\}} \max_{\{\phi_A, \phi_B\}} \sum_{a,\phi_A,\phi_B} |\text{Im}\{\langle \phi_A, \phi_B | (\Pi^A_a \otimes I_B) \cdot \text{Tr}_E\{(I_A \otimes U_{BE})(\varrho_{AB} \otimes \varrho_E)(I_A \otimes U_{BE})^\dagger\}|\phi_A, \phi_B\}| | \leq \min_{\{\Pi^A_a\}} \max_{\{\phi_A, \phi_B, \phi_E\}} \sum_{a,\phi_A,\phi_B,\phi_E} |\text{Im}\{\langle \phi_A, \phi_B, \phi_E | (\Pi^A_a \otimes I_B \otimes I_E) \cdot (I_A \otimes U_{BE})(\varrho_{AB} \otimes \varrho_E)(I_A \otimes U_{BE})^\dagger|\phi_A, \phi_B, \phi_E\}| | \leq \min_{\{\Pi^A_a\}} \max_{\{\phi_A, \phi_B, \phi_E\}} \sum_{a,\phi_A,\phi_B,\phi_E} |\text{Im}\{\langle \phi_A, \phi'_B, \phi_E | (\Pi^A_a \otimes I_B \otimes I_E) \cdot (\varrho_{AB} \otimes \varrho_E)|\phi_A, \phi'_B, \phi_E\}| |,$$

where the basis $\{|\phi_E\rangle\}$ of the Hilbert space $\mathcal{H}_E$ of the ancilla inserted into the second line is arbitrary, and we have chosen $\{|\phi_E\rangle\}$ so that the unitarily transformed postselection basis of the whole system-ancilla is again factorizable, i.e., $\{(I_A \otimes U_{BE})^\dagger|\phi_A\rangle \otimes |\phi_B\rangle \otimes |\phi_E\rangle\} = |\phi_A\rangle \otimes |\phi'_B\rangle \otimes |\phi_E\rangle\}$. Such a basis can always be chosen given the interaction unitary $U_{BE}$. To see this, we note that the interaction unitary must take the form $U_{BE} = e^{-ig(O_B,O_E)}$, where $g(O_B, O_E)$ is the Hermitian generator, and $O_B$ and $O_E$ are Hermitian operators acting respectively on the subsystem $B$ and the ancilla $E$. We then choose $\{|\phi_E\rangle\}$ as the eigenvectors of $O_E$ with the eigenvalues $\{\phi_E\}$ so that $U_{BE}^\dagger|\phi_B\rangle \otimes |\phi_E\rangle = e^{ig(O_B,\phi_E)}|\phi_B\rangle \otimes |\phi_E\rangle = |\phi'_B\rangle \otimes |\phi_E\rangle$, where $|\phi'_B\rangle \doteq e^{ig(O_B,\phi_E)}|\phi_B\rangle$, as claimed. Note that $|\phi'_B\rangle$ now depends on the index $\phi_E$. Finally, from Eq. (9), noting that $\langle \phi_E|\varrho_E|\phi_E\rangle$ is nonnegative and real, and $\sum_{\phi_E} \langle \phi_E|\varrho_E|\phi_E\rangle = $
1, we obtain:

\[
C_q^Q[\mathbb{I}_A \otimes \Lambda_B] \hat{q}_{AB} \\
\leq \min_{\{\Pi^a\}} \max_{\{\phi_A, \phi_B\}} \sum_{a, \phi_A, \phi_B} |\text{Im}\{\langle \phi_A, \phi_B | (\Pi^a_A \otimes \Pi^a_B) \hat{q}_{AB} | \phi_A, \phi_B \rangle\}| \langle \hat{q}_E | \hat{q}_E \rangle \\
= \sum_{\phi_E} C_q^Q[\hat{q}_{AB}] \langle \hat{q}_E | \hat{q}_E \rangle = C_q^Q[\hat{q}_{AB}],
\]

(10)

where, in the last line, we have made use of the fact that, for each \(\hat{q}_E\), the set \(\mathcal{B}_{\{\phi_A, \phi_B\}}\) of the postselection bases \(\{|\phi_A, \phi_B\}\) is the same as the set \(\mathcal{B}_{\{\phi_A, \phi_B\}}\) of the postselection bases \(\{|\phi_A, \phi_B\}\) so that \(\max_{\{\phi_A, \phi_B\}}(\cdot) = \max_{\{\phi_A, \phi_B\}}(\cdot)\) which is independent of \(\hat{q}_E\).

Besides satisfying the above three plausible constraints, the one-sided w-nonclassical correlation \(C_q^Q[\hat{q}_{AB}]\) also has the following properties. First, it is convex, i.e., \(C_q^Q[\sum_k p_k \hat{q}_E^k] \leq \sum_k p_k C_q^Q[\hat{q}_E^k]\), where \(\{p_k\}\) are probabilities, due to triangle inequality and the fact that \(p_k \geq 0\). Moreover, the one-sided w-nonclassical correlation is nonincreasing when one or more of the parties are discarded (ignored). To show this, without losing generality, consider for example a tripartite state \(\hat{q}_{ABC}\). One then has

\[
C_q^Q[\hat{q}_{ABC}] \\
= \min_{\{\Pi^a\}} \max_{\{\phi_A, \phi_B, \phi_C\}} \sum_{a, \phi_A, \phi_B, \phi_C} |\text{Im}\{\langle \phi_A, \phi_B, \phi_C | (\Pi^a_A \otimes \Pi^a_B \otimes \Pi^a_C) \hat{q}_{ABC} | \phi_A, \phi_B, \phi_C \rangle\}| \\
= \sum_{\phi_C} \langle \hat{q}_C | \hat{q}_{ABC} \rangle = \text{Tr}_C \{\hat{q}_{ABC}\}. \\
\]

(11)

The equality is reached when the subsystem \(C\) is totally uncorrelated with the rest, i.e., \(C_q^Q[\hat{q}_{AB} \otimes \hat{q}_C] = C_q^Q[\hat{q}_{AB}]\), by virtue of the fact that \(\langle \hat{q}_C | \hat{q}_C \rangle = 1\). Hence, adding or removing an (independent) ancilla does not change the w-nonclassical correlation as intuitively expected. The above property straightforwardly generalizes to any number of parties as \(C_q^Q[\hat{q}_{12...N}] \geq C_q^Q[\hat{q}_{12...N-1}] \geq \cdots \geq C_q^Q[\hat{q}_{123}] \geq C_q^Q[\hat{q}_{12}]\), where \(\hat{q}_{1...k} = \text{Tr}_{k+1...N} \{\hat{q}_{1...N}\}\).

One can check that the two-sided w-nonclassical correlation \(C_{AB}^Q[\hat{q}_{AB}]\) defined in Eq. (7) also satisfies property (i) of faithfulness, i.e., it is vanishing if and only if the bipartite state \(\hat{q}_{AB}\) takes the form of classical-classical state of Eq. (4). Moreover, it is also invariant under local unitary transformation satisfying property (iii). Property (iii), however, cannot be defined for the two-sided quantum correlation \(C_{AB}^Q[\hat{q}_{AB}]\) in the bipartite setting. Instead,
one can prove for a tripartite state that $C^Q_{AB}[(I_A \otimes I_B \otimes \Lambda_C)\varrho_{ABC}] \leq C^Q_{AB}[\varrho_{ABC}]$, where $\Lambda_C$ is a completely positive trace preserving operation on $C$. One can also easily see that $C^Q_{AB}[\varrho_{AB}] \leq C^Q_{AB}[\varrho_{ABC}]$, with equality when $\varrho_{ABC} = \varrho_{AB} \otimes \varrho_C$. Beside these, the two-sided w-nonclassical correlation satisfies the following inequality:

$$C^Q_{ABC}[\varrho_{ABC}] \leq C^Q_{AB}[\varrho_{AB}] \leq C^Q_{AB}[\varrho_{ABC}] = \varrho_{AB} = \text{Tr}_C(\varrho_{ABC}) = \varrho_{AB}. \quad (12)$$

by noting that $\sum_{c,\phi_C} \langle \phi_C | (I_A \otimes I_B \otimes \Pi_C) \varrho_{ABC} | \phi_C \rangle = \text{Tr}_C \varrho_{ABC} = \varrho_{AB}$. It is the state of $AB$ after the strong projective measurement of the local basis $\{\Pi_C\}$ on $\varrho_{ABC}$ without reading the outcomes.

Finally, another important requirement for a quantifier to be a bonafide measure of quantum correlation is that for pure states, it should reduce to a corresponding measure of entanglement [28]. This captures the intuition that for pure states, separability is equivalent to no nonclassical correlation. We have not yet been able to clarify this important issue. However, in the next section, we prove that for general pure states of arbitrary finite dimension, the one-sided w-nonclassical correlation defined in Eq. (6) gives an observable lower bound to a scaled linear entropy of entanglement, which for two qubits pure state reduces to the entanglement concurrence [49]. Hence, for pure states, the w-coherence can be seen as a faithful witness of entanglement.

Let us further show that, for any generic mixed state, the two-sided w-nonclassical correlation sets an observable lower bound to the negativity of quantumness, a measure of general quantum correlation which quantifies the amount of entanglement that can be activated via a local measurement of the subsystem [16, 46]. First, starting from the definition of the
two-sided w-nonclassical correlation in Eq. (7), we have

\[ C_{AB}^{Q}[q_{AB}] \]

\[ = \min_{\{\Pi_{A}^{a} \otimes \Pi_{B}^{b}\}} \max_{\{\phi_{A}, \phi_{B}\}} \sum_{a,b,\phi_{A},\phi_{B}} | \text{Im}\{\langle \phi_{A}, \phi_{B} | (\Pi_{A}^{a} \otimes \Pi_{B}^{b}) q_{AB} \phi_{A}, \phi_{B}\} | \]

\[ \leq \min_{\{\Pi_{A}^{a} \otimes \Pi_{B}^{b}\}} \max_{\{\phi_{A}, \phi_{B}\}} \sum_{a,b,\phi_{A},\phi_{B} d' \neq a,b \neq b} \sum_{a',b'} | \langle a, b | q_{AB} | a', b' \rangle | \]

\[ \times | \langle \phi_{A} | a \rangle \langle \phi_{B} | b \rangle \langle a' \phi_{A} \rangle \langle b' \phi_{B} \rangle | \]

\[ \leq \min_{\{\Pi_{A}^{a} \otimes \Pi_{B}^{b}\}} \sum_{a' \neq a,b \neq b} | \langle a, b | q_{AB} | a', b' \rangle | \]

\[ \times \max_{\{\phi_{A}\}} \left| \sum_{a'} \langle \phi_{A} | a \rangle \langle a' \phi_{A} \rangle \right| \max_{\{\phi_{B}\}} \left| \sum_{b'} \langle \phi_{B} | b \rangle \langle b' \phi_{B} \rangle \right|, \quad (13) \]

where we have inserted the completeness relations \( \sum_{a'} | a' \rangle \langle a' | = \mathbb{I}_{A} \) and \( \sum_{b'} | b' \rangle \langle b' | = \mathbb{I}_{B} \).

One the other hand, using Cauchy-Schwartz relations \( \sum_{a'} | a' \rangle \langle a' | = \mathbb{I}_{A} \) and \( \sum_{b'} | b' \rangle \langle b' | = \mathbb{I}_{B} \). Given the measurement basis \( \{|a(b)\}\), the equalities in Eq. (14) are attained when the postselection basis \( \{|\phi_{A(B)}\}\) is mutually unbiased with the measurement basis so that \( |\langle \phi_{A(B)} | a(b) \rangle | = 1/\sqrt{d_{A(B)}} \) for all \( a(b) \) and \( \phi_{A(B)} \), where \( d_{A(B)} \) is the dimension of the Hilbert space of subsystem \( A(B) \). Upon inserting Eq. (14) into Eq. (13), we finally obtain

\[ C_{AB}^{Q}[q_{AB}] \leq \min_{\{\{a(b)\}\}} \sum_{a' \neq a,b \neq b} | \langle a, b | q_{AB} | a', b' \rangle | \]

\[ \leq C_{AB}^{Q,l_{1}}[q_{AB}]. \quad (15) \]

The quantity on the right-hand side is just the measure of general quantum correlation based on \( l_{1}-\text{norm} \) measure of (local) coherence [28] which is equal to twice of the total (two-sided) negativity of quantumness [16, 46]. We emphasize that unlike the measure of general quantum correlation based on \( l_{1}-\text{norm} \) (local) coherence and negativity of quantumness, the w-nonclassical correlation is in principle directly observable via weak measurement and postselection.
We proceed to give a concrete computation of the w-nonclassical correlation in a simple bipartite state. We first note that, as is also the case for the quantifiers of general quantum correlation proposed in the literature, the calculation of the w-nonclassical correlation defined in Eqs. (6) and (7) is in general analytically intractable, involving optimization over all possible relevant bases, which in general takes the form of an optimization of multivariable nonlinear function. For example, consider the following maximally entangled two-qubit state

\[ |\Psi_{AB}^+\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle, \tag{16} \]

where \{0, 1\} is the computational basis, i.e., the eigenstates of Pauli \( \sigma_z \) matrix. For the purpose of computation, we express the postselection bases for the Hilbert spaces of subsystem A and B, i.e., \{\( |\phi_{A(B)}\rangle \)\} = \{\( |\phi_{A(B)}^+\rangle, |\phi_{A(B)}^-\rangle \)\}, using the following parameterization:

\[
\begin{align*}
|\phi_{A(B)}^+\rangle & = \cos \frac{\alpha_{A(B)}}{2} |0\rangle_{A(B)} + \sin \frac{\alpha_{A(B)}}{2} e^{i\beta_{A(B)}} |1\rangle_{A(B)}, \\
|\phi_{A(B)}^-\rangle & = \sin \frac{\alpha_{A(B)}}{2} |0\rangle_{A(B)} - \cos \frac{\alpha_{A(B)}}{2} e^{i\beta_{A(B)}} |1\rangle_{A(B)},
\end{align*}
\]

\[ \tag{17} \]

where \( 0 \leq \alpha_{A(B)} \leq \pi \), and \( 0 \leq \beta_{A(B)} \leq 2\pi \). One can scan over all the tensor product of the postselection bases for the qubit A and B, i.e., all the bases \{\( |\phi_A, \phi_B\rangle \)\} \( \in \mathcal{B}_{(|\phi_A\rangle \otimes |\phi_B\rangle)} \), by varying the angles \( (\alpha_A, \beta_A) \) and \( (\alpha_B, \beta_B) \) over their ranges of values. Furthermore, let us parameterize all the possible local measurement bases on subsystem A as \( \{\Pi_A^\mu\} = \{\Pi_A^{\mu+}, \Pi_A^{\mu-}\} \), where \( \Pi_A^{\mu\pm} \equiv |\mu\rangle \langle \mu|_A \), with

\[
\begin{align*}
|\mu+\rangle_A & = \cos \frac{\theta_A}{2} |0\rangle_A + \sin \frac{\theta_A}{2} e^{i\eta_A} |1\rangle_A, \\
|\mu-\rangle_A & = \sin \frac{\theta_A}{2} |0\rangle_A - \cos \frac{\theta_A}{2} e^{i\eta_A} |1\rangle_A,
\end{align*}
\]

\[ \tag{18} \]

\( 0 \leq \theta_A \leq \pi \), \( 0 \leq \eta_A \leq 2\pi \). Inserting all the above ingredients into the one-sided nonclassical correlation \( C_A^{\Omega}[g_{AB}] \) of Eq. (6), we obtain

\[
\begin{align*}
C_A^{\Omega}[|\Psi_{AB}^+\rangle \langle \Psi_{AB}^+|] &= \min_{\{\mu_A\}} \max_{\{\phi_A, \phi_B\}} \sum_{\mu_A, \phi_A, \phi_B} |\text{Im}\{\langle \phi_A, \phi_B| (\Pi_A^\mu \otimes \mathbb{I}_B)|\Psi_{AB}^+\rangle \langle \Psi_{AB}^+| \phi_A, \phi_B\rangle\}| \\
&= \min_{\{\theta_A, \eta_A\}} \max_{\{\alpha_A, \alpha_B, \beta_A, \beta_B\}} |\sin \alpha_A \sin \alpha_B \sin (\beta_A + \beta_B) \cos \theta_A - \cos \alpha_A \sin (\beta_B + \eta_A) \sin \theta_A \\
&\quad - \sin \alpha_A \cos \alpha_B \sin (\beta_A - \eta_A) \sin \theta_A| \\
&= 0.99999 \ldots. \tag{19}
\end{align*}
\]
See Appendix A for the detailed derivation of the above trigonometric function that has to be optimized, i.e., maximized over the parameters \((\alpha_A, \alpha_B, \beta_A, \beta_B)\) and then followed by choosing the minimum over \((\theta_A, \eta_A)\).

Next, as an example of a separable state with a nonvanishing one-sided w-nonclassical correlation, we consider the Werner state for \(2 \times 2\) dimension:

\[
\rho_{AB}^W = \frac{1 - p}{4} (I_A \otimes I_B) + p |\Psi_{AB}^+\rangle \langle \Psi_{AB}^+|.
\] (20)

It is known that the above Werner state is separable for \(p < 1/3\). The one-sided w-nonclassical correlation can be straightforwardly computed to get, noting Eq. (19),

\[
C_A^Q[\rho_{AB}^W] = pC_A^Q[|\Psi_{AB}^+\rangle \langle \Psi_{AB}^+|] \approx p \geq 0,
\] (21)

where the first equality is obtained directly from the definition in Eq. (6). Hence, it is nonvanishing even in the regime when the Werner state is separable. We note that the above value of the w-nonclassical correlation for the Werner state is equal to the square root of the geometric discord based on Hilbert-Schmidt distance [34].

IV. STATISTICAL AND INFORMATION THEORETIC MEANING

A. W-nonclassical correlation as the witness for genuine local quantum uncertainty and pure state entanglement

Recall that the one-sided and two-sided w-nonclassical correlations are defined via the noncommutativity between the local measurement bases and the multipartite state. Such noncommutativity is one of the sources of uncertainty in quantum measurement. It is therefore instructive to discuss the relation between the w-nonclassical correlations and the local quantum uncertainty. Let us first consider the one-sided w-nonclassical correlation. Then, as shown in the Appendix we obtain the following inequality

\[
C_A^Q[\rho_{AB}] \leq \min_{(I_{A}^{a})} \sum_{a} \left( \text{Tr}\{(I^{a}_{A} \otimes I_{B})^{2} \rho_{AB}\} - \text{Tr}\{(I^{a}_{A} \otimes I_{B}) \rho_{AB}\}^{2}\right)^{1/2} \\
= \min_{(I_{A}^{a})} \sum_{a} \Delta(I_{A}^{a} \otimes I_{B})[\rho_{AB}],
\] (22)
where $\Delta^2_{\hat{O}}[\rho] \doteq \text{Tr}\{\hat{O}^2\rho\} - (\text{Tr}\{\hat{O}\rho\})^2$ is the quantum variance of $\hat{O}$ in the state $\rho$ which takes into account both the genuine quantum uncertainty arising from the noncommutativity between $O$ and $\rho$, and the classical uncertainty associated with classical mixing in the state $\rho$. Similarly, for the two-sided w-nonclassical correlation, we obtain

$$C^Q_{AB}[\varrho_{AB}] \leq \min_{\{\Pi_a^A \otimes \Pi_b^B\}} \sum_{a,b} \Delta_{\{\Pi_a^A \otimes \Pi_b^B\}}[\varrho_{AB}].$$

Hence, the strength of the w-nonclassical correlation is limited by the minimum of the total sum of the uncertainty of each element of the local measurement basis quantified by the quantum standard deviation. Conversely, the nonclassical correlation captured by $C^Q_{A}[\varrho_{AB}]$ and $C^Q_{AB}[\varrho_{AB}]$ give the lower bounds to the quantum uncertainty of any local basis, i.e., from Eq. (22), for any local basis $\{\Pi_k^A\}$, we have

$$\sum_k \Delta_{\{\Pi_k^A \otimes I^B\}}[\varrho_{AB}] \geq \min_{\{\Pi_k^A\}} \sum_a \Delta_{\{\Pi_a^A \otimes I^B\}}[\varrho_{AB}] \geq C^Q_{A}[\varrho_{AB}],$$

and similarly, for any local measurement bases $\{\Pi_k^A \otimes \Pi_l^B\}$, we have $\sum_{k,l} \Delta_{\{\Pi_k^A \otimes \Pi_l^B\}}[\varrho_{AB}] \geq \min_{\{\Pi_k^A \otimes \Pi_l^B\}} \sum_{a,b} \Delta_{\{\Pi_a^A \otimes \Pi_b^B\}}[\varrho_{AB}] \geq C^Q_{AB}[\varrho_{AB}]$. The w-nonclassical correlation can therefore be seen as one of the sources of the quantum uncertainty in the measurement of the local basis which arises from the noncommutativity between the local basis and the multipartite state.

The above observation suggests to interpret $C^Q_{A}[\varrho_{AB}]$ and $C^Q_{AB}[\varrho_{AB}]$ as the minimum genuine quantum share of the local uncertainty, out of the total local quantum uncertainty quantified by the quantum standard deviation. Recall that the latter also includes the uncertainty arising from the classical mixing of states, and possibly from other sources. To this end, it is intriguing to compare the w-nonclassical correlation with the Wigner-Yanase skew information [50], which is also based on the noncommutativity between the state and a Hermitian observable and has been argued to quantify the genuine quantum share of uncertainty arising from quantum noncommutativity [51]. Let us mention that the minimum value of the Wigner-Yanase skew information over certain set of local nondegenerate observables is proposed in Ref. [25] as a quantifier of nonclassical correlation beyond entanglement. We note that unlike the Wigner-Yanase skew information, w-nonclassical correlation $C^Q_{A}[\varrho_{AB}]$ and $C^Q_{AB}[\varrho_{AB}]$ are operationally appealing, in that they have transparent interpretation in terms of direct laboratory operations based on weak measurement and postselection.

Next, noting that $\text{Tr}\{(\Pi_a^A \otimes I^B)^2\varrho_{AB}\} = \text{Tr}\{(\Pi_a^A \otimes I^B)\varrho_{AB}\} = p(a|\varrho_A;\Pi_a^A)$ is just the probability to get the outcome $a$ in the local projective measurement $\{\Pi_a^A\}$ over the state
$\varrho_A = \text{Tr}_B \{\varrho_{AB}\}$ of the subsystem $A$, and inserting into Eq. (22), we obtain

$$C_A^Q[\varrho_{AB}] \leq \min_{\{\Pi^a_A\}} \sum_a \left[ p(a|\varrho_A; \Pi^a_A) - p(a|\varrho_A; \Pi^a_A)^2 \right]^{1/2}$$

$$\leq \min_{\{\Pi^a_A\}} d_A^{1/2} \left[ 1 - \sum_a p(a|\varrho_A; \Pi^a_A)^2 \right]^{1/2}$$

$$= \min_{\{\Pi^a_A\}} d_A^{1/2} S_2^{1/2} \{\{p(a|\varrho_A; \Pi^a_A)\}\},$$  \hspace{1cm} (24)

where we have again used the Cauchy-Schwartz inequality and the normalization $\sum_a p(a|\varrho_A; \Pi^a_A) = 1$ to get the second line, and

$$S_2[\{p(a|\varrho_A; \Pi^a_A)\}] = 1 - \sum_a p(a|\varrho_A; \Pi^a_A)^2$$  \hspace{1cm} (25)

is the Tsallis entropy with the entropy index 2.

Let us further show that the minimum of the Tsallis entropy on the right-hand side of Eq. (24) is obtained when the local measurement basis $\{\Pi^a_A\}$ is just given by the eigenprojectors of the reduced density matrix $\varrho_A$. Suppose that the reduced density matrix has the following spectral decomposition: $\varrho_A = \sum_j \pi_j(\varrho_A) \Pi^B_A$, where $\{\Pi^B_A = |\eta_j\rangle \langle \eta_j|_A\}$ are the eigenprojectors, and $\{\pi_j(\varrho_A)\}$, $\pi_j(\varrho_A) \geq 0$, $\sum_j \pi_j(\varrho_A) = 1$, are the associated eigenvalues. First, we have $p(a|\varrho_A; \Pi^a_A) = \text{Tr}\{\Pi^a_A \varrho_A\} = \sum_j \pi_j(\varrho_A) \text{Tr}\{\Pi^a_A \Pi^B_A\}$. Inserting this into the definition of the Tsallis entropy in Eq. (25), and noting the convexity of the quadratic function, we get, using the Jensen inequality,

$$S_2[\{p(a|\varrho_A; \Pi^a_A)\}] = 1 - \sum_a \left( \sum_j \pi_j(\varrho_A) \text{Tr}\{\Pi^a_A \Pi^B_A\} \right)^2$$

$$\geq 1 - \sum_a \sum_j \pi_j(\varrho_A)^2 \text{Tr}\{\Pi^a_A \Pi^B_A\}$$

$$= 1 - \sum_j \pi_j(\varrho_A)^2 = S_2[\{\pi_j(\varrho_A)\}],$$  \hspace{1cm} (26)

where we have used the normalization $\sum_j \text{Tr}\{\Pi^B_A \Pi^a_A\} = \langle \eta_j|\eta_j \rangle = 1$. By comparing Eq. (25) to Eq. (26), one finds that the equality in Eq. (26), namely, the minimum of $S_2[\{p(a|\varrho_A; \Pi^a_A)\}]$, is reached when $p(a|\varrho_A; \Pi^a_A) = \pi_a(\varrho_A)$, i.e., when $\{\Pi^a_A\} = \{\Pi^\eta_a\}$, as claimed. From Eqs. (24) and (26), we thus have

$$C_A^Q[\varrho_{AB}] \leq d_A^{1/2} S_2^{1/2} \{\{\pi_j(\varrho_A)\}\}. $$  \hspace{1cm} (27)

Finally, notice that the Tsallis entropy $S_2[\{\pi_j(\varrho_A)\}]$ on the right-hand side of Eq. (27) is
just the linear entropy associated with the reduced density matrix $\rho_A = \sum_j \pi_j (\rho_A) \Pi_A^j$, i.e.,

$$S_2[\{\pi_a(\rho_A)\}] = 1 - \sum_a \pi_a(\rho_A)^2 = 1 - \text{Tr}[\rho_A^2].$$  \hspace{1cm} (28)

It quantifies the mixedness or impurity of the reduced density matrix $\rho_A = \text{Tr}_B \rho_{AB}$. For a pure bipartite state $|\psi_{AB}\rangle$, the linear entropy can be used as a measure of entanglement, i.e.,

$$E_l[|\psi_{AB}\rangle \langle \psi_{AB}|] = 1 - \text{Tr}[\rho_A^2] = 1 - \text{Tr}[\rho_B^2] = S_2[\{\pi_a(\rho_A)\}], \quad \rho_{A(B)} = \text{Tr}_{B(A)}\{|\psi_{AB}\rangle \langle \psi_{AB}|\}. \hspace{1cm} \text{Eqs. (27) and (28) therefore show that for general pure bipartite states } |\psi_{AB}\rangle, \text{ the one-sided w-nonclassical correlation } C_Q^A[|\psi_{AB}\rangle \langle \psi_{AB}|] \text{ can be seen as a witness for the quantum entanglement in } |\psi_{AB}\rangle, \text{ namely, it gives a lower bound to the scaled square root of the linear entropy of entanglement as}$$

$$C_Q^A[|\psi_{AB}\rangle \langle \psi_{AB}|] \leq d_A^{1/2}(1 - \text{Tr}\{\rho_A^2\})^{1/2}. \hspace{1cm} (29)$$

Moreover, since $C_Q^A[|\psi_{AB}\rangle \langle \psi_{AB}|]$ is vanishing if and only if $|\psi_{AB}\rangle$ is factorizable, it is a faithful witness for the pure state entanglement. We note that for two-qubit pure state, denoting the Schmidt coefficients as $\lambda_+$ and $\lambda_-$, the right-hand side of Eq. (29) is just given by the entanglement concurrence [49], i.e.,

$$d_A^{1/2}(1 - \text{Tr}\{\rho_A^2\})^{1/2} = \sqrt{2} \sqrt{1 - (\lambda_+^2 + \lambda_-^2)} = 2\lambda_+\lambda_-,$$

where we have used $\lambda_+^2 + \lambda_-^2 = 1$. We emphasize that unlike the linear entropy of entanglement on the right-hand side of Eq. (29), the w-nonclassical correlation on the left-hand side has a transparent operational interpretation and, at least in principle, can be observed directly via weak measurement with postselection. Finally, note that the maximum value of the right-hand side of Eqs. (27) or (29) over all bipartite quantum states is reached when $\pi_j(\rho_A) = 1/d_A$ for all $j$, to give $\max_{\rho_A} d_A^{1/2}(1 - \text{Tr}\{\rho_A^2\})^{1/2} = \sqrt{d_A - 1}$. For pure bipartite state, this is the case when it is maximally entangled, i.e., $|\psi_{AB}\rangle = \sum_i \frac{1}{\sqrt{d_A}} |a_i\rangle |b_i\rangle$.

It is however not known if for the maximally entangled state, the w-nonclassical correlation $C_Q^A[|\psi_{AB}\rangle \langle \psi_{AB}|]$ on the left-hand side of Eq. (27) is also maximized.

**B. W-nonclassical correlation as minimum mean absolute error in optimal estimation of local basis based on local measurement on the multipartite state**

The expression of the w-nonclassical correlation of Eq. (6) and (7) in terms of the imaginary part of the weak value naturally suggests an information theoretic interpretation based on the statistical interpretation of weak values. First, we note that as argued in Refs.
the real part of the weak value $O^w(\phi|\varrho)$ of Eq. (11) can be seen as the optimal estimate of the Hermitian observable $O$ given prior information about preparation represented by the quantum state $\varrho$, based on the outcomes of a projective measurement $\{\Pi^a\}$, $\sum_{\phi} \Pi^\phi = \mathbb{I}$, on $\varrho$. Here, optimal means that the estimate minimizes the associated mean-squared estimation error. Moreover and importantly, in this case, the minimum mean-squared estimation error is just given by the variance of the imaginary part of the weak value. Noting this fact, it is argued in Ref. [55] that the imaginary part of the weak value can be seen as the strength of the error in a single shot estimation via the introduction of an unbiased global random variable. See also Refs. [56, 57] for the case of continuous quadrature variables.

With this observation in mind, $C^Q_A[\varrho_{AB}]$ defined in Eq. (6) can then be interpreted as follows. Suppose we wish to estimate the outcome of a local projector on subsystem $A$, i.e., an element of the local measurement basis $\Pi^k_A \in \{\Pi^a_A\}$, based on the outcomes $(\phi_A, \phi_B)$ of a complete set of projective measurement of $\{\Pi^a_A \otimes \Pi^a_B\}$, $\Pi^\phi_{A(B)} = |\phi_{A(B)}\rangle \langle \phi_{A(B)}|$, on the prepared bipartite state represented by $\varrho_{AB}$. Then, according to the information theoretical interpretation of weak value alluded to above, the mean absolute estimation error is given by the absolute imaginary part of the weak value of $\Pi^k_A \otimes \mathbb{I}_B$ with the preselected state $\varrho_{AB}$ and the postselected state $|\phi_A, \phi_B\rangle$, averaged over the probability of the successful postselection, i.e.,

$$\langle |\mathcal{E}_{\Pi^k_A}[\varrho_{AB}; |\phi_A, \phi_B\rangle] \rangle = \sum_{\phi_A, \phi_B} |\text{Im}\left\{\frac{\langle \phi_A, \phi_B | (\Pi^k_A \otimes \mathbb{I}_B) \varrho_{AB} | \phi_A, \phi_B \rangle}{\langle \phi_A, \phi_B | \varrho_{AB} | \phi_A, \phi_B \rangle}\right\}| \langle \phi_A, \phi_B | \varrho_{AB} | \phi_A, \phi_B \rangle \rangle. \quad (30)$$

We then consider the worst case scenario wherein the total sum of mean absolute error of the estimation of $\{\Pi^a_A\}$ is maximized over all possible measurement $\{\Pi^\phi_A \otimes \Pi^\phi_B\} \in \mathcal{B}(\{\phi_A\otimes|\phi_B\rangle\}$, i.e., $\max_{\{|\phi_A, \phi_B\rangle\}} \sum_a \langle |\mathcal{E}_{\Pi^a_A}[\varrho_{AB}; |\phi_A, \phi_B\rangle] \rangle$. Finally, among these worst case scenarios, we choose the local basis $\{\Pi^a_A\}$ so that the total mean absolute estimation error is the least (i.e., the best of the worst scenario). This gives us the one-sided w-nonclassical correlation of Eq. (6), i.e.,

$$\min_{\{\Pi^a_A\}} \max_{\{|\phi_A, \phi_B\rangle\}} \sum_a \langle |\mathcal{E}_{\Pi^a_A}[\varrho_{AB}; |\phi_A, \phi_B\rangle] \rangle = \min_{\{\Pi^a_A\}} \max_{\{|\phi_A, \phi_B\rangle\}} \sum_a \text{Im}\{\langle \phi_A, \phi_B | (\Pi^a_A \otimes \mathbb{I}_B) \varrho_{AB} | \phi_A, \phi_B \rangle \} = C^Q_A[\varrho_{AB}]. \quad (31)$$

This is just a minimax estimation problem. An analogous interpretation can be made for
the two-sided w-nonclassical correlation $C_{AB}^Q[\rho_{AB}]$ of Eq. (7).

The above information theoretic interpretation in terms of optimal estimation of local basis based on local measurement on the multipartite state, suggests a new characterization of the quantum versus classically correlated multipartite states of Eqs. (3) or (4) as follows. A multipartite state is quantumly correlated if and only if the optimal estimation of any local basis based on the local projective measurement on the state must yield a nonvanishing mean absolute error in the worst case scenario, with the minimum value that is given by the w-nonclassical correlation. By contrast, a multipartite state is classical devoid of quantum correlation, if and only if the outcomes of local projective measurement on it can be used to make a sharp (i.e., with vanishing estimation error) optimal estimation of a certain local basis. Such a negative characterization of general quantum correlation is similar to the characterization of quantum correlation expressed by no-go theorem for quantum local broadcasting [33].

V. SUMMARY AND REMARKS

We have proposed a characterization of nonclassical correlation beyond entanglement in terms of transparent and direct laboratory operations. We operationally defined a quantity based on the imaginary part of weak values obtained via the weak measurement and postselection, together with an optimization procedure, which captures the minimum incompatibility between the multipartite state under scrutiny and any local measurement basis. We showed that it satisfies certain desirable requirements for a quantifier of general nonclassical correlation in the multipartite state. We argued that it may be statistically interpreted as the minimum genuine quantum part of local uncertainty arising from the incompatibility between the local measurement basis and the multipartite state. It is a faithful witness for pure state entanglement giving an observable lower bound to a scaled square root of the linear entropy of entanglement. Moreover, we have offered an information theoretic interpretation of the nonclassical correlation as the minimum mean absolute error of the optimal estimation of a local basis, based on the outcomes of local projective measurement on the multipartite state, in the worst case scenario.

Our results suggest intriguing fundamental links between the quantum-classical division and contrast captured by general nonclassical correlation, and the concept of weak value.
in the multipartite setting. Note that the ability of the scheme of weak measurement with postselection to directly characterize general nonclassical correlation is related to the fact that a similar scheme can be used to directly observe the quantum state or its equivalent quasiprobability representation \[41, 42\], in which, the nonclassical features of quantum systems, including general nonclassical correlation, are encoded. In both schemes for the direct measurement of quantum states and direct characterization of general quantum correlation, the complex nature of the weak value, especially the imaginary part which captures the noncommutativity between the quantum state and observable, are crucial. It is hoped that the weak measurement with postselection may offer a fresh operational framework to characterize and quantify other nonclassical features of quantum mechanics.

Acknowledgments

Appendix A: Computation of Eq. (19)

Let us first compute each term in the sum in Eq. (19) for different elements of the local measurement and postselection bases. From Eq. (18), it is convenient to express the local measurement basis as

\[
|\mu^+\rangle\langle \mu^+| = \cos^2 \frac{\theta}{2} |0\rangle \langle 0| + \frac{1}{2} \sin \theta e^{-i\eta} |0\rangle \langle 1| + \frac{1}{2} \sin \theta e^{i\eta} |1\rangle \langle 0|.
\]

One then obtains, for the weak measurement of $|\mu^+\rangle\langle \mu^+|$ with the postselection state $|\phi_A^+, \phi_B^+\rangle$, using the expression for $|\mu^+\rangle\langle \mu^+|$ of Eq. (A1) and noting the expression for the postselection states in Eq. (17),

\[
C_{\phi_A^+, \phi_B^+}^{\mu^+} \langle \Psi_{AB}^+ | \Psi_{AB}^+ \rangle \]

\[
\doteq |\text{Im} \{ \langle \phi_A^+, \phi_B^+ | (|\mu^+\rangle \langle \mu^+|_A \otimes I_B) |\Psi_{AB}^+ \rangle \langle \Psi_{AB}^+ | \phi_A^+, \phi_B^+ \} | \]

\[
= |I_{++}^{00} \cos^2 \frac{\theta}{2} + I_{++}^{01} \sin \theta + I_{++}^{10} \sin^2 \frac{\theta}{2}|. \quad (A2)
\]
where each term on the right-hand side takes the form:

\[
\begin{align*}
I_{++}^{00} &= \frac{1}{8} \sin \alpha_A \sin \alpha_B \sin(\beta_A + \beta_B); \\
I_{++}^{01} &= \frac{1}{8} \cos^2 \frac{\alpha_A}{2} \sin \alpha_B (\sin(-\beta_B - \eta_A) + \frac{1}{8} \sin \alpha_A \sin^2 \frac{\alpha_B}{2} \sin(\beta_A - \eta_A)); \\
I_{++}^{10} &= \frac{1}{8} \sin \alpha_A \cos^2 \frac{\alpha_B}{2} \sin(-\beta_A + \eta_A) + \frac{1}{8} \sin^2 \frac{\alpha_A}{2} \sin \alpha_B \sin(\beta_B + \eta_A); \\
I_{++}^{11} &= -\frac{1}{8} \sin \alpha_A \sin \alpha_B \sin(\beta_A + \beta_B). 
\end{align*}
\]

On the other hand, for the weak measurement of \(|\mu-\rangle \langle \mu-|\) with the postselection state \(|\phi_A+, \phi_B+\rangle\), using the expression for \(|\mu-\rangle \langle \mu-|\) of Eq. (A1), we get

\[
C_{\phi_A+, \phi_B+}^{\mu-} \langle \Psi_{AB}^{+}\rangle \langle \Psi_{AB}^{+}\rangle \\
\frac{1}{8} \sum \{\phi_{A}, \phi_{B}\} |(\mu-\rangle \langle \mu-| \otimes \mathbb{I}_B) \langle \Psi_{AB}^{+}\rangle \langle \Psi_{AB}^{+}\rangle |\phi_{A}, \phi_{B}+\rangle\rangle \\
= |I_{++}^{00} \sin^2 \frac{\theta_A}{2} - I_{++}^{10} \sin \theta_A - I_{++}^{11} \cos^2 \frac{\theta_A}{2}|. 
\]

Similarly, one can compute the cases for the weak measurement of \(|\mu_{A}\pm\rangle \langle \mu_{A}\pm|\) with the remaining postselection states \(|\phi_{A}+, \phi_{B}-\rangle, |\phi_{A}-, \phi_{B}+\rangle, \) and \(|\phi_{A}-, \phi_{B}-\rangle\). One can further show by inspection that \(|C_{\phi_A+, \phi_B+}^{\mu_{A}+} \langle \Psi_{AB}^{+}\rangle \langle \Psi_{AB}^{+}\rangle| = |C_{\phi_A+, \phi_B+}^{\mu_{A}-} \langle \Psi_{AB}^{+}\rangle \langle \Psi_{AB}^{+}\rangle| = |C_{\phi_A-, \phi_B+}^{\mu_{A}+} \langle \Psi_{AB}^{+}\rangle \langle \Psi_{AB}^{+}\rangle| = |C_{\phi_A-, \phi_B+}^{\mu_{A}-} \langle \Psi_{AB}^{+}\rangle \langle \Psi_{AB}^{+}\rangle| = |C_{\phi_A-, \phi_B-}^{\mu_{A}+} \langle \Psi_{AB}^{+}\rangle \langle \Psi_{AB}^{+}\rangle| = |C_{\phi_A-, \phi_B-}^{\mu_{A}-} \langle \Psi_{AB}^{+}\rangle \langle \Psi_{AB}^{+}\rangle|,\)

where, upon inserting Eq. (A3) into Eq. (A2), we have

\[
|C_{\phi_A+, \phi_B+}^{\mu_{A}+} \langle \Psi_{AB}^{+}\rangle \langle \Psi_{AB}^{+}\rangle| = \frac{1}{8} |\sin \alpha_A \sin \alpha_B \sin(\beta_A + \beta_B) \cos \theta_A - \cos \alpha_A \sin \alpha_B \sin(\beta_B + \eta_A) \sin \theta_A - \sin \alpha_A \cos \alpha_B \sin(\beta_A - \eta_A) \sin \theta_A|. 
\]

Inserting these terms into the definition of the one-sided w-correlation of Eq. (6), we obtain Eq. (19). Hence, we need to perform the maximization of nonlinear function of Eq. (19) over \((\alpha_A, \alpha_B, \beta_A, \beta_B)\), and followed by choosing the minimum over \((\theta_A, \eta_A)\). We do this by using the subroutine fminsearch in a simple matlab code to get the maximum over \((\alpha_A, \alpha_B, \beta_A, \beta_B)\), followed with the application of subroutine min of matlab to search for the minimum over \((\theta_A, \eta_A)\).
Appendix B: Proof of Eq. (22)

First, we have, from Eq. (6),

\[
C^Q_A[\varrho_{AB}] = \min_{\{\Pi^a_A\}} \max_{\{\phi\}} \sum_a \sum_\phi \left| \text{Im} \left\{ \frac{\text{Tr} \{ \Pi^\phi_{AB} (\Pi^a_A \otimes I_B) \varrho_{AB} \}}{\text{Tr} \{ \Pi^\phi_{AB} \varrho_{AB} \}} \right\} \right| \text{Tr} \{ \Pi^\phi_{AB} \varrho_{AB} \} \leq \sum_a \left[ \sum_\phi \left( \left\{ \frac{\text{Tr} \{ \Pi^{\phi^*}_{AB} (\Pi^a_A \otimes I_B) \varrho_{AB} \}}{\text{Tr} \{ \Pi^{\phi^*}_{AB} \varrho_{AB} \}} \right\}^2 - \left( \sum_\phi \text{Re} \{ \text{Tr} \{ \Pi^{\phi^*}_{AB} (\Pi^a_A \otimes I_B) \varrho_{AB} \} \} \right)^2 \right]^{1/2}. \tag{B1} \]

Here \(\{|\phi^*\rangle\} = \{|\phi^*_A, \phi^*_B\}\) is the postselection basis which achieves the maximum, \(\Pi^{\phi^*}_{AB} = \Pi^{\phi^*}_A \otimes \Pi^{\phi^*}_B = |\phi^*_A\rangle \langle \phi^*_A| \otimes |\phi^*_B\rangle \langle \phi^*_B|\), and \(\{\Pi^a_A\}\) is the local measurement basis which achieves the minimum. Moreover, we have applied the Cauchy-Schwartz inequality and the completeness relation for the postselection basis, i.e., \(\sum_\phi \text{Tr} \{ \Pi^{\phi^*}_{AB} \varrho_{AB} \} = 1\), to get the third and fourth lines. Next, applying again the Cauchy-Schwartz uncertainty relation for bounded operators, \(|\text{Tr} \{X Y^\dagger\}|^2 \leq \text{Tr} \{X X^\dagger\} \text{Tr} \{Y Y^\dagger\}\), to the numerator in the first term on the right-hand side of Eq. (B1), i.e.,

\[
\left( \text{Tr} \{ \Pi^{\phi^*}_{AB} (\Pi^a_A \otimes I_B) \varrho_{AB} \} \right)^2 = \left( \text{Tr} \{ (\Pi^{\phi^*}_A)_{1/2} (\Pi^a_A \otimes I_B) (\Pi^{\phi^*}_A)_{1/2} (\varrho_{AB}) (\Pi^{\phi^*}_A)_{1/2} \} \right)^2 \leq \text{Tr} \{ \Pi^{\phi^*}_{AB} (\Pi^a_A \otimes I_B) \varrho_{AB} (\Pi^a_A \otimes I_B) \} \text{Tr} \{ \varrho_{AB} \Pi^{\phi^*}_{AB} \}. \tag{B2} \]

and using the completeness relation \(\sum_\phi \Pi^{\phi^*}_A = I\), we finally obtain Eq. (22), i.e.,

\[
C^Q_A[\varrho_{AB}] \leq \sum_a \left( \text{Tr} \{ (\Pi^a_A \otimes I_B) \varrho_{AB} \} - \text{Tr} \{ (\Pi^a_A \otimes I_B) \varrho_{AB} \} \right)^2)^{1/2}. \tag{B3} \]

By following similar steps as above, we also obtain, for the two-sided w-nonclassical correlation,

\[
C^Q_{AB}[\varrho_{AB}] \leq \sum_{a^*} \left( \text{Tr} \{ (\Pi^a_A \otimes \Pi_{B}^{b^*}) \varrho_{AB} \} - \text{Tr} \{ (\Pi^a_A \otimes \Pi_{B}^{b^*}) \varrho_{AB} \} \right)^2)^{1/2}, \tag{B4} \]

where \(\{\Pi^a_A \otimes \Pi_{B}^{b^*}\}\) is the product of the local measurement bases for subsystems A and B which achieves the minimum.

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