Inverse obstacle scattering problems with a single incident wave and the logarithmic differential of the indicator function in the enclosure method

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Received 4 April 2011, in final form 17 June 2011
Published 13 July 2011
Online at stacks.iop.org/IP/27/085006

Abstract
This paper gives a remark on the enclosure method by considering inverse obstacle scattering problems with a single incident wave whose governing equation is given by the Helmholtz equation in two dimensions. It is concerned with the indicator function in the enclosure method. The previous indicator function is essentially real-valued since only its absolute value is used. In this paper, another method for the use of the indicator function is introduced. The method employs the logarithmic differential with respect to the independent variable of the indicator function and yields directly the coordinates of the vertices of the convex hull of unknown polygonal sound-hard obstacles or thin ones. The convergence rate of the obtained formulae is better than that of the previous indicator function. Some other applications of this method are also given.

1. Introduction and statements of the main results

The aim of this paper is to add further new knowledge on the enclosure method. The enclosure method was originally introduced in [5, 6] for inverse boundary value problems for elliptic equations. The method aims at extracting information about the location and shape of unknown discontinuity embedded in a known reference medium that gives an effect on the propagation of the signal, such as an obstacle, inclusion, crack, etc. The method can be divided into two versions. One is a version that employs infinitely many pairs of input and output data, that is, the Dirichlet-to-Neumann map (or Neumann-to-Dirichlet map). Another is a version that employs a single set of input and output data. We call this second version the single measurement version of the enclosure method. We refer the reader to [12–14] for recent applications of the single measurement version of the enclosure method.
This paper is concerned with the indicator function in the single measurement version of the enclosure method. In [9], having the single measurement version of the enclosure method, the author considered the reconstruction issue of inverse obstacle scattering problems of acoustic wave in two dimensions. The problem is to reconstruct a two-dimensional obstacle from the Cauchy data on a circle surrounding the obstacle of the total wave field generated by a single incident plane wave with a fixed wave number \( k > 0 \). The author established an extraction formula of the value of the support function at a generic direction which yields information about the convex hull of polygonal sound-hard obstacles. However, the indicator function used in [9] is essentially real-valued since only its absolute value is used. In this paper, another method for the use of the indicator function is introduced. It is shown that the function used in [9] is essentially real-valued since only its absolute value is used. In this paper, another method for the use of the indicator function is introduced. It is shown that the logarithmic differential with respect to the independent variable of the indicator function yields directly the coordinates of the vertices of the convex hull of unknown polygonal sound-hard obstacles or thin ones.

Let us describe our main results. First consider a polygonal obstacle denoted by \( D \), that is: \( D \subseteq \mathbb{R}^2 \) takes the form \( D_1 \cup \cdots \cup D_m \) with \( 1 \leq m < \infty \) where each \( D_j \) is open and a polygon, \( \overline{D}_j \cap \overline{D}_{j'} = \emptyset \) if \( j \neq j' \).

The total wave field \( u \) outside the obstacle \( D \) takes the form \( u(x; d, k) = e^{ikx \cdot d} + w(x) \) with \( k > 0, d \in S^1 \) and satisfies
\[
\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D},
\]
\[
\frac{\partial u}{\partial v} = 0 \quad \text{on} \quad \partial D,
\]
\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial w}{\partial r} - ikw \right) = 0,
\]
where \( r = |x| \) and \( v \) denotes the unit outward normal relative to \( \mathbb{R}^2 \setminus \overline{D} \). The last condition above is called the Sommerfeld radiation condition. Some further information about \( u \) is in order. \( u \) belongs to \( C^\infty(\mathbb{R}^2 \setminus \overline{D}) \) and satisfies \( u|_{\mathbb{R}^2 \setminus \overline{D} \cap B} \in H^1((\mathbb{R}^2 \setminus \overline{D}) \cap B) \) for a large open disc \( B \) containing \( \overline{D} \). This restricts the singularity of \( u \) at the corners of \( D \). The boundary condition for \( \partial u/\partial v \) on \( \partial D \) means that \( D \) is a sound-hard obstacle and should be considered as a weak sense.

Note that, in this paper, \( k \) is just positive and there is no other assumption on \( k \).

Let \( B_R \) be an open disc with radius \( R \) centered at a fixed point satisfying \( \overline{D} \subseteq B_R \). We assume that \( B_R \) is known. Our data are \( u = u(\cdot; d, k) \) and \( \partial u/\partial v \) on \( \partial B_R \) for a fixed \( d \) and \( k \), where \( v \) is the unit outward normal relative to \( B_R \). Let \( \omega \) and \( \omega^\perp \) be two unit vectors perpendicular to each other. We always assume that the orientation of \( \omega^\perp \) and \( \omega \) coincides with \( e_1 \) and \( e_2 \), and thus \( \omega^\perp \) is unique.

Recall the support function of \( D \): \( h_D(\omega) = \sup_{x \in D} x \cdot \omega \). The values of the support function give the signed distances from the origin of coordinates to the support lines of \( D \) and the convex hull \([D]\) of \( D \) by the formula \([D] = \cap_{\omega \in S^1} \{ x \in \mathbb{R}^2 \mid x \cdot \omega < h_D(\omega) \}\).

We say that \( \omega \) is regular with respect to \( D \) if the set \( \partial D \cap \{ x \in \mathbb{R}^2 \mid x \cdot \omega = h_D(\omega) \} \) consists of only one point. In other words, when \( t \) moves from \( t = \infty \) to \( -\infty \), the line \( x \cdot \omega = t \) which is perpendicular to direction \( \omega \) descends from infinity and firstly touches a single point on \( \partial D \) at \( t = h_D(\omega) \). Since \( D \) is assumed to be polygonal, the point should be a vertex of \([D]\).

Set \( c_\tau(\omega) = \tau \omega + i\sqrt{\tau^2 + k^2} \omega^\perp \) with \( \tau > 0 \). Let \( v_\tau(x) = e^{i\cdot c_\tau(\omega)} \). This \( v \) satisfies the Helmholtz equation in the whole plane.
Define
\[ I(\tau; \omega, d, k) = \int_{\partial BR} \left( \frac{\partial u}{\partial v} v_\tau - \frac{\partial v}{\partial v} u \right) dS. \] (1.1)

This complex-valued function of \( \tau \) is called the indicator function in the single measurement version of the enclosure method. In [9], Ikehata has established the formula
\[ \lim_{\tau \to \infty} \frac{1}{\tau} \log |I(\tau; \omega, d, k)| = h_D(\omega), \] (1.2)
provided \( \omega \) is regular with respect to \( D \). In this formula, one makes use of only the absolute value of the indicator function \( I(\tau; \omega, d, k) \). Thus, one needs two regular directions \( \omega \) for determining a single vertex of the convex hull of \( D \) since formula (1.2) gives only a single line on which the vertex lies. Here, we present a method for the use of the complex values of the indicator function which directly yields the coordinates of a vertex of the convex hull of \( D \) with indicator functions for a single regular direction \( \omega \).

Since \( (\omega \perp)^\perp = -\omega \), we have \( \sqrt{\tau^2 + k^2} \omega + i\tau \omega^\perp = ic\tau (\omega \perp) \). This gives
\[ \partial_\tau v_\tau = \frac{i}{\sqrt{\tau^2 + k^2}} x \cdot c\tau (\omega \perp) v_\tau \]
and thus we have
\[ \frac{\partial}{\partial v} (\partial_\tau v_\tau) = \frac{i}{\sqrt{\tau^2 + k^2}} [c\tau (\omega \perp) \cdot v + (x \cdot c\tau (\omega \perp))(c\tau (\omega) \cdot v)] v_\tau \] (1.3)
and
\[ I'(\tau; \omega, d, k) = \int_{\partial BR} \left( \frac{\partial u}{\partial v} \cdot \partial_\tau v_\tau - \frac{\partial}{\partial v} (\partial_\tau v_\tau) u \right) dS. \] (1.4)

Our first result is the following theorem.

**Theorem 1.1.** Let \( \omega \) be regular with respect to \( D \). Let \( x_0 \in \partial D \) be the point with \( x_0 \cdot \omega = h_D(\omega) \). There exists a \( \tau_0 > 0 \) such that, for all \( \tau \geq \tau_0 \), \( |I(\tau; \omega, d, k)| > 0 \) and the formula
\[ \lim_{\tau \to \infty} \frac{I'(\tau; \omega, d, k)}{I(\tau; \omega, d, k)} = h_D(\omega) + ix_0 \cdot \omega^\perp \] (1.5)
is valid.

Theorem 1.1 means that, as \( \tau \to \infty \) the logarithmic differential of \( I(\tau; \omega, d, k) \) with respect to \( \tau \) converges to \( x_0 \cdot \omega + ix_0 \cdot \omega^\perp \). The convergence rate of (1.5) is better than that of (1.2). For this see remark 2.1 in section 2.

Here, we explain why (1.5) gives further information about \( D \). Let \( \omega \) be regular with respect to \( D \). We denote by \( x(\omega) = (x(\omega)_1, x(\omega)_2) \) the single point in \( \partial D \cap \{ x \in \mathbb{R}^2 | x \cdot \omega = h_D(\omega) \} \). Since it holds that
\[ x \cdot (\omega + i\omega^\perp) = (x_1 - ix_2)(\omega_1 + i\omega_2), \quad x \in \mathbb{R}^2, \] (1.6)
from (1.5) we have
\[ x(\omega)_1 = \text{Re} \left\{ (\omega_1 + i\omega_2) \lim_{\tau \to \infty} \left( \frac{I'(\tau; \omega, d, k)}{I(\tau; \omega, d, k)} \right) \right\} \] (1.7)
and
\[ x(\omega)_2 = \text{Im} \left\{ (\omega_1 + i\omega_2) \lim_{\tau \to \infty} \left( \frac{I'(\tau; \omega, d, k)}{I(\tau; \omega, d, k)} \right) \right\}. \] (1.8)
The set $I(D)$ of all directions which are not regular with respect to $D$ is finite. Let $I(D) = \{ (\cos \theta_j, \sin \theta_j) \mid 0 \leq \theta_1 < \cdots < \theta_N < 2\pi \}$ and $k = 1, 2$. Define

$$x(\theta)_k = \begin{cases} x(\cos \theta, \sin \theta)_k, & \text{if } \theta \in [0, 2\pi[ \setminus \{ \theta_j \mid j = 1, \ldots, N \}, \\ \infty, & \text{if } \theta = \theta_j, j = 1, \ldots, N, \end{cases}$$

and extend it as the $2\pi$-periodic function of $\theta \in \mathbb{R}$. Since $D$ is polygonal, both $x(\theta)_1$ and $x(\theta)_2$ are piece-wise constant and one of which has discontinuity at $\theta = \theta_j$ in the following sense: for each $j$ it holds that $x(\theta_j + 0)_1 \neq x(\theta_j - 0)_1$ or $x(\theta_j + 0)_2 \neq x(\theta_j - 0)_2$. Therefore, one can expect that computing both $x(\omega)_1$ and $x(\omega)_2$ for sufficiently many $\omega$ via formulae (1.7) and (1.8), one can estimate $I(D)$. This is new information extracted from the indicator function in the enclosure method.

Another implication of theorem 1.1 is the following idea. Given $y \in \mathbb{R}^2$, define

$$I(\tau; y, \omega, d, k) = e^{-y \cdot (\tau \omega + i \sqrt{\tau^2 + k^2} \omega^\perp)} I(\tau; \omega, d, k).$$

This corresponds to substitute $x \mapsto e^{(x - y) \cdot (\tau \omega + i \sqrt{\tau^2 + k^2} \omega^\perp)}$ instead of $v_\tau$ into (1.1). Since

$$\lim_{\tau \to \infty} I'(\tau; y, \omega, d, k) I(\tau; y, \omega, d, k) = (x_0 - y) \cdot (\omega + i \omega^\perp),$$

it follows from (1.5) that

$$\lim_{\tau \to \infty} \frac{I'(\tau; y, \omega, d, k)}{I(\tau; y, \omega, d, k)} = |x_0 - y|.$$

This together with (1.6) yields that

$$\lim_{\tau \to \infty} \frac{|I'(\tau; y, \omega, d, k)|}{|I(\tau; y, \omega, d, k)|} = |x_0 - y|.$$

Since $x_0$ is the unique point which minimizes the function $y \mapsto |x_0 - y|$, one possible alternative idea to find $x_0$ is to consider the minimization problem of the following function for a suitable $\tau$:

$$y \mapsto \frac{|I'(\tau; y, \omega, d, k)|}{I(\tau; y, \omega, d, k)}.$$

Since this paper concentrates on only the theoretical issue of the enclosure method, we leave the numerical implementation of this idea for future research.

The result can be extended to a thin obstacle case. Let $\Sigma$ be the union of finitely many disjoint closed piece-wise linear segments $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$. Assume that there exists a simply connected open set $D$ such that $D$ is a polygon and each $\Sigma_j$ consists of sides of $D$. We assume that $\overline{D} \subset B_R$ with a $R > 0$. We denote by $\nu$ the unit outward normal on $\partial D$ relative to $B_R \setminus \overline{D}$ and set $v^+ = v$ and $v^- = -v$ on $\Sigma$. Given $k > 0$ and $d \in S^1$, let $u = u(x), x \in \mathbb{R}^2 \setminus \Sigma$, be the solution of the scattering problem

$$(\Delta + k^2)u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Sigma,$$

$$\frac{\partial u^\pm}{\partial v^\pm} = 0 \quad \text{on} \quad \Sigma,$$

$$\lim_{r \to \infty} r^{\frac{1}{2}} \left( \frac{\partial u}{\partial r} - ik w \right) = 0.$$
Figure 1. The case when every end point of $\Sigma_1$ and $\Sigma_2$ satisfies $x \cdot \omega < h_\Sigma(\omega)$.

Figure 2. The case when there is an end point $x_0$ of some $\Sigma_j$ such that $x_0 \cdot \omega = h_\Sigma(\omega)$ and $d$ is not perpendicular to $\nu$ on $\Sigma_j$ near the point.

where $w = u - e^{i k \cdot d}$, $u^+ = u|_{\mathbb{R}^2 \setminus D}$ and $u^- = u|_{D}$. Note that this is a brief description of the problem and for the exact one see [9]. Define

$$ I_\Sigma(\tau; \omega, d, k) = \int_{\partial B_R} \left( \frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u \right) dS. $$

The following is our second result.

**Theorem 1.2.** Let $\omega$ be regular with respect to $\Sigma$. If every end point of $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ satisfies $x \cdot \omega < h_\Sigma(\omega)$, then there exists a $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$, $|I_\Sigma(\tau; \omega, d, k)| > 0$ and the formula

$$ \lim_{\tau \to \infty} \frac{I_\Sigma'(\tau; \omega, d, k)}{I_\Sigma(\tau; \omega, d, k)} = h_\Sigma(\omega) + ix_0 \cdot \omega \quad (1.9) $$

is valid.

If there is an end point $x_0$ of some $\Sigma_j$ such that $x_0 \cdot \omega = h_\Sigma(\omega)$, then, for $d$ that is not perpendicular to $\nu$ on $\Sigma_j$ near the point, the same conclusions as above are valid.

Let $R > 0$. In this paper, we denote by $B_R(x_0)$ the open disc with radius $R$ centered at $x_0$. Note that $\nu$ on $\Sigma_j \cap B_\eta(x_0)$ for sufficiently small $\eta > 0$ becomes a constant vector if $x_0$ is an end point of $\Sigma_j$. See also figures 1 and 2 for illustrations of the conditions in theorem 1.2.

In [9], under the same assumption as in theorem 1.2, it is shown that

$$ \lim_{\tau \to \infty} \frac{1}{\tau} \log |I_\Sigma(\tau; \omega, d, k)| = h_\Sigma(\omega). \quad (1.10) $$
Thus, (1.9) also adds further knowledge on the use of the indicator function in the thin obstacle case.

A brief outline of this paper is as follows. Theorems 1.1 and 1.2 are proved in sections 2 and 3, respectively. Both proofs employ some previous computation results performed in [9] for the proof of (1.2) and (1.10); however, some nontrivial modifications of the computation are also required. The idea of using the logarithmic differential of the original indicator function developed in this paper can be applied to several other previous applications of the enclosure method published in [5, 7, 8, 10, 11, 13, 14]. In section 4, two applications of the argument for the proof of theorem 1.1 are given. In the appendix, first for the reader’s convenience we give the proof of proposition 2.1, which ensures an expansion of the solution of the Helmholtz equation at a corner. The proof is focused on some technical part that is different from the case when \( k = 0 \). Second, a proof of a formula which is important for the computation of the expansion of \( I'(\tau; \omega, d, k) \) as \( \tau \to \infty \) is given. Third, a proof of some estimates that are needed for the proof of an application presented in section 4 is given.

Note also that in sections 2 and 3 we simply write \( v_\tau = v \).

Finally, it should be pointed out that the uniqueness issue of the inverse scattering problems with a single incident plane wave for polygonal or polyhedral obstacles has been extensively studied. See [3, 15] and references therein.

2. Proof of theorem 1.1

Let \( x_0 \) denote the single point of the set \( \{ x \mid x \cdot \omega = h_D(\omega) \} \cap \partial D \). \( x_0 \) has to be a vertex of \( D_j \) for some \( j \). The internal angle of \( D_j \) at \( x_0 \) is less than \( \pi \) and thus \( 2\pi - \text{internal angle} \), which we denote by \( \Theta_1 \), satisfies \( \pi < \Theta_1 < 2\pi \).

If one chooses a sufficiently small \( \eta > 0 \), then one can write

\[
B_\eta(x_0) \cap (B_R \setminus \overline{D}) = \{ x_0 + r \cos \theta a + \sin \theta a^\perp \mid 0 < r < \eta, 0 < \theta < \Theta_1 \},
\]

where \( a = \cos p \omega^\perp + \sin p \omega, a^\perp = -\sin p \omega^\perp + \cos p \omega, -\pi < p < 0; \Gamma_p = \{ x_0 + ra \mid 0 < r < \eta \}, \Gamma_q = \{ x_0 + r(\cos \Theta a + \sin \Theta a^\perp) \mid 0 < r < \eta \} \) Note that the orientation of \( a, a^\perp \) coincides with that of \( e_1, e_2 \). See also figure 1 of [5].

The quantity \( -p \) means the angle between two vectors \( \omega^\perp \) and \( a \). \( p \) satisfies \( \Theta > \pi + (-p) \).

Set \( q = \Theta - 2\pi + p \). Then, we have \( -\pi < q < p < 0 \) and the expression

\[
\Gamma_p = \{ x_0 + r(\cos p \omega^\perp + \sin p \omega) \mid 0 < r < \eta \},
\]

\[
\Gamma_q = \{ x_0 + r(\cos q \omega^\perp + \sin q \omega) \mid 0 < r < \eta \}.
\]

This is the meaning of \( p \) and \( q \).

We set

\[
u(r, \theta) = u(x), \quad x = x_0 + r(\cos \theta a + \sin \theta a^\perp).
\]

The following proposition describes the behavior of \( u(r, \theta) \) as \( r \to 0 \).

Proposition 2.1 (Proposition 4.2 in [9]). Let \( \eta \) satisfy \( \eta \ll 1/2k \). Then, there exists a sequence \( \alpha_1, \alpha_2, \ldots, \alpha_m, \ldots \) such that

(1) for each \( s \in [0, 2] \)

\[
u(r, \theta) = \sum_{m=1}^\infty \alpha_m J_{\mu_m}(kr) \cos \mu m \theta, \quad \text{in } H^1(B_{\eta}(x_0) \cap (B_R \setminus \overline{D}))
\]
where
\[ \mu_m = \frac{(m - 1)\pi}{\Theta} \]
and \( J_{\mu_m} \) denotes the Bessel function of order \( \mu_m \) given by the formula
\[
J_{\mu_m}(z) = \left( \frac{z}{2} \right)^{\mu_m} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (n + 1 + \mu_m)} \left( \frac{z}{2} \right)^{2n};
\]
(2) as \( m \rightarrow \infty \)
\[
|\alpha_m| = O \left( \frac{\Gamma(1 + \mu_m)}{\sqrt{\mu_m}} \frac{1}{\eta_k} \right); \]
(3) for each \( l = 1, \ldots, \) there exists a positive number \( C_l \) such that, for all \( r \in ]0, \eta[ \),
\[
|u(r, 0) - \sum_{m=1}^{l} \alpha_m J_{\mu_m}(kr)| \leq C_l r^{\mu_l+1},
\]
\[
|u(r, \Theta) - \sum_{m=1}^{l} \alpha_m (-1)^{m-1} J_{\mu_m}(kr)| \leq C_l r^{\mu_l+1}, \quad 0 < r < \eta.
\]

In [9], the proof is omitted since it can be performed along the same line as the proof in the case when \( k = 0 \), as given in [4]. However, there is a technical difference from the case when \( k = 0 \) and so to make sure, and for the reader’s convenience, in the appendix we give the proof which focused on the difference.

Let \( s = \sqrt{\tau^2 + k^2 + \tau} \). We have
\[
\tau = \frac{1}{2} \left( s - \frac{k^2}{s} \right), \quad \sqrt{\tau^2 + k^2} = \frac{1}{2} \left( s + \frac{k^2}{s} \right).
\]

Recall that the proof of formula (1.2) is based on the following two facts in [9].

- As \( s \rightarrow \infty \), the complete asymptotic expansion
\[
e^{-i\sqrt{\tau^2 + k^2} x_0 \cdot \omega} e^{-\tau h_D(\omega)} I'(\tau; \omega, d, k) \sim -i \sum_{n=2}^{\infty} \frac{e^{i\frac{1}{2} \mu_n k \mu_n} \alpha_n (e^{i\beta_n \mu_n} + (-1)^n e^{i\gamma_n \mu_n})}{s^{\mu_n}},
\]
(2.2)
is valid.

- \( \exists n \geq 2 \alpha_n (e^{i\beta_n \mu_n} + (-1)^n e^{i\gamma_n \mu_n}) \neq 0 \). Thus, the quantity
\[
m^* = \min[m \geq 2 |\alpha_m (e^{i\beta_n \mu_n} + (-1)^n e^{i\gamma_n \mu_n})| \neq 0]
\]
is well defined. \( m^* \) depends on \( k, d, D \) and \( \omega \).

For the proof of theorem 1.1, we compute the asymptotic expansion of \( I'(\tau; \omega, d, k) \) as \( s \rightarrow \infty \). The result is
\[
\sqrt{\tau^2 + k^2} e^{-i\sqrt{\tau^2 + k^2} x_0 \cdot \omega} e^{-\tau h_D(\omega)} I'(\tau; \omega, d, k)
\]
\[
= -i \sum_{m=2}^{n} \alpha_m (e^{i\beta_n \mu_n} + (-1)^n e^{i\gamma_n \mu_n}) (e^{i\mu_n \mu_n} k \mu_n + i x_0 \cdot c_\tau(\omega^2)) \frac{e^{i\frac{1}{2} \mu_n k \mu_n}}{s^{\mu_n}} + O \left( \frac{1}{s^{\mu_n+1}} \right).
\]
(2.3)

The proof of (2.3) is given in subsection 2.1. Here, we show how to prove theorem 1.1 by assuming (2.3).
Since \( \alpha_m[e^{ip/h} + (-1)^m e^{iq/h}] = 0 \) for all \( m \) with \( m < m^* \) and \( \beta \equiv \alpha_m[e^{ip/h} + (-1)^m e^{iq/h}] \neq 0 \), from (2.2) we have

\[
\lim_{\tau \to \infty} e^{\mu \omega^*} e^{-i(\tau^2 + k^2)h_D(\omega)} I(\tau; \omega, d, k) = -i\beta e^{i\mu \omega^*} \left( k \right)^{\mu \omega^*}. \tag{2.4}
\]

On the other hand, since

\[
\lim_{\tau \to \infty} \frac{i x_0 \cdot c_s(\omega^2)}{\sqrt{\tau^2 + k^2}} = x_0 \cdot \omega + i x_0 \cdot \omega^\perp,
\]

it follows from (2.3) that

\[
\lim_{\tau \to \infty} \tau^{\mu \omega^*} e^{-i(\tau^2 + k^2)h_D(\omega)} I'(\tau; \omega, d, k) = -i\beta(x_0 \cdot \omega + i x_0 \cdot \omega^\perp) e^{i\mu \omega^*} \left( k \right)^{\mu \omega^*}. \tag{2.5}
\]

A combination of (2.4) and (2.5) ensures the validity of (1.5). This completes the proof of theorem 1.1.

**Remark 2.1.** Since \( c_s(\omega^2) \) depends on \( \tau \) and thus \( s, (2.3) \) is not the complete asymptotic expansion. However, using (2.2), (2.3) and the expression

\[
-\frac{s}{2}(\omega + i \omega^\perp) + \frac{k^2}{2s}(\omega - i \omega^\perp),
\]

one can easily obtain the following expansion:

\[
e^{-i(\tau^2 + k^2)h_D(\omega)} \left\{ \sqrt{\tau^2 + k^2} I' - \left( \frac{s}{2} x_0 \cdot (\omega + i \omega^\perp) + \frac{k^2}{2s} x_0 \cdot (\omega - i \omega^\perp) \right) I \right\} = i \sum_{n=2}^{\infty} \alpha_m[e^{ip/h} + (-1)^m e^{iq/h}] \frac{\mu_n e^{i\mu_n} k^{\mu_n}}{s^{\mu_n}} + O \left( \frac{1}{s^{\mu_n+1}} \right),
\]

where \( I = I(\tau; \omega, d, k) \) and \( I' = I'(\tau; \omega, d, k) \). Note that we have used \( \pi < \Theta \leq 2\pi \).

The above formula yields the second term of the expansion of the logarithmic differential of the indicator function as \( \tau \to \infty \):

\[
\frac{I'(\tau; \omega, d, k)}{I(\tau; \omega, d, k)} = h_D(\omega) + i x_0 \cdot \omega^\perp + \frac{\mu_n}{\tau} + O \left( \frac{1}{\tau^2} \right).
\]

Thus, the convergence rate of (1.5) is better than that of (1.2) since (2.4) yields

\[
\frac{1}{\tau} \log |I(\tau; \omega, d, k)| = h_D(\omega) - \frac{\mu_n}{\tau} + O \left( \frac{1}{\tau} \right).
\]

From this viewpoint one can say that (1.5) is an improvement of (1.2). Note also that both formulae show that the accuracy of the approximation depends on the size of \( \mu_n \equiv (m^* - 1)\pi/\Theta \).

**Remark 2.2.** By (1.4), \( I'(\tau; \omega, d, k) \) can be considered as another indicator function; however, it is easy to see that from (1.2) and (1.5) when \( x_0 \neq 0 \) and (2.3) when \( x_0 = 0 \) we have

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log |I'(\tau; \omega, d, k)| = h_D(\omega).
\]

Thus, from this formula we cannot obtain any new information about \( D \).
Integration by parts gives

$$I'(\tau; \omega, d, k) = \int_D u \frac{\partial}{\partial \nu} v \, dS.$$ 

Localizing this integral at $x_0$, we have, modulo exponentially decaying terms as $\tau \to \infty$,

$$e^{-i\sqrt{\tau^2 + k^2} x_0 \omega} e^{-i\theta_0(\omega)} I'(\tau; \omega, d, k) \sim e^{-i\sqrt{\tau^2 + k^2} x_0 \omega} e^{-i\theta_0(\omega)} \int_{\Gamma_p} u \frac{\partial}{\partial \nu} v \, dS + e^{-i\sqrt{\tau^2 + k^2} x_0 \omega} e^{-i\theta_0(\omega)} \int_{\Gamma_q} u \frac{\partial}{\partial \nu} v \, dS \equiv I_p(\tau) + I_q(\tau). \quad (2.6)$$

We have $v = \sin p \omega^\perp - \cos p \omega$ and $x = x_0 + r(\cos p \omega^\perp + \sin p \omega)$ on $\Gamma_p$; $v = -\sin q \omega^\perp + \cos q \omega$ and $x = x_0 + r(\cos q \omega^\perp + \sin q \omega)$ on $\Gamma_q$.

From those we have:

- on $\Gamma_p$
  
  $$c_r(\omega^\perp) \cdot v = \tau \sin p + i\sqrt{\tau^2 + k^2} \cos p,$$

  $$c_r(\omega) \cdot v = -(\tau \cos p - i\sqrt{\tau^2 + k^2} \sin p),$$

  $$x \cdot c_r(\omega^\perp) = x_0 \cdot c_r(\omega^\perp) + r(\tau \cos p - i\sqrt{\tau^2 + k^2} \sin p),$$

  $$v = e^{i\theta_0(\omega)} e^{i\sqrt{\tau^2 + k^2} x_0 \omega} e^{i\tau \sin p} e^{i\sqrt{\tau^2 + k^2} \tau \cos p},$$

- on $\Gamma_q$

  $$c_r(\omega^\perp) \cdot v = -(\tau \sin q + i\sqrt{\tau^2 + k^2} \cos q),$$

  $$c_r(\omega) \cdot v = \tau \cos q - i\sqrt{\tau^2 + k^2} \sin q,$$

  $$x \cdot c_r(\omega^\perp) = x_0 \cdot c_r(\omega^\perp) + r(\tau \cos q - i\sqrt{\tau^2 + k^2} \sin q),$$

  $$v = e^{i\theta_0(\omega)} e^{i\sqrt{\tau^2 + k^2} x_0 \omega} e^{i\tau \sin q} e^{i\sqrt{\tau^2 + k^2} \tau \cos q}. \quad (2.7)$$

It follows from (1.3) and (2.7) that

$$\sqrt{\tau^2 + k^2} I_p(\tau) = \{i(\tau \sin p + i\sqrt{\tau^2 + k^2} \cos p) - x_0 \cdot c_r(\omega^\perp)(\tau \cos p - i\sqrt{\tau^2 + k^2} \sin p)\} \times \int_0^\eta u(r, 0) e^{i\tau \sin p} e^{i\sqrt{\tau^2 + k^2} \tau \cos p} \, dr$$

$$- i(\tau \cos p - i\sqrt{\tau^2 + k^2} \sin p)^2 \int_0^\eta r u(r, 0) e^{i\tau \sin p} e^{i\sqrt{\tau^2 + k^2} \tau \cos p} \, dr. \quad (2.9)$$

It follows also from (1.3) and (2.8) that

$$\sqrt{\tau^2 + k^2} I_q(\tau) = -i\{i(\tau \sin q + i\sqrt{\tau^2 + k^2} \cos q) - x_0 \cdot c_r(\omega^\perp)(\tau \cos q - i\sqrt{\tau^2 + k^2} \sin q)\} \times \int_0^\eta u(r, \Theta) e^{i\tau \sin q} e^{i\sqrt{\tau^2 + k^2} \tau \cos q} \, dr$$

$$+ i(\tau \cos q - i\sqrt{\tau^2 + k^2} \sin q)^2 \int_0^\eta r u(r, \Theta) e^{i\tau \sin q} e^{i\sqrt{\tau^2 + k^2} \tau \cos q} \, dr. \quad (2.10)$$
Here, we make use of (2.1). Since $\alpha_1 J_0(k|x-x_0|)$ satisfies the Helmholtz equation in the whole plane, the indicator function for $u$ coincides with that for $u - \alpha_1 J_0(k|x-x_0|)$. Thus one may assume, in advance, that $\alpha_1 = 0$ in the computation of the integrals in (2.9) and (2.10).

Since $p$ and $q$ satisfies $-\pi < q < p < 0$, we have $\sin p < 0$ and $\sin q < 0$. This gives, for $\theta = p, q$ and $\mu > 0$,

$$\int_0^\eta r^\mu e^{ir\sin\theta} dr = O(\tau^{-((\mu+1)})). \quad (2.11)$$

Set

$$I_\mu(\tau, \theta) = \int_0^\eta J_\mu(kr) e^{ir\sin\theta} e^{i\sqrt{r^2+k^2}r\cos\theta} dr,$$

$$K_\mu(\tau, \theta) = \int_0^\eta r J_\mu(kr) e^{ir\sin\theta} e^{i\sqrt{r^2+k^2}r\cos\theta} dr. \quad (2.12)$$

It follows from (2.1) and (2.11) that

$$\int_0^\eta u(r, 0) e^{ir\sin\theta} e^{i\sqrt{r^2+k^2}r\cos\theta} \cos p dr = \sum_{m=2}^n \alpha_m I_{\mu_m}(\tau, p) + O(\tau^{-(\mu_m+1)}),$$

$$\int_0^\eta r u(r, 0) e^{ir\sin\theta} e^{i\sqrt{r^2+k^2}r\cos\theta} \cos p dr = \sum_{m=2}^n \alpha_m K_{\mu_m}(\tau, p) + O(\tau^{-(\mu_m+2)}),$$

$$\int_0^\eta u(r, \theta) e^{ir\sin\theta} e^{i\sqrt{r^2+k^2}r\cos\theta} \cos q dr = \sum_{m=2}^n (\alpha_m(-1)^{m-1} I_{\mu_m}(\tau, q) + O(\tau^{-(\mu_m+1)}),$$

$$\int_0^\eta r u(r, \theta) e^{ir\sin\theta} e^{i\sqrt{r^2+k^2}r\cos\theta} \cos q dr = \sum_{m=2}^n (\alpha_m(-1)^{m-1} K_{\mu_m}(\tau, q) + O(\tau^{-(\mu_m+2)}).$$

Substituting these into (2.9) and (2.10), we obtain

$$\sqrt{\tau^2+k^2} I_p(\tau) = i[\tau \sin p + i\sqrt{\tau^2+k^2} \cos p] - x_0 \cdot c_1(\omega^+)(\tau \cos p - i\sqrt{\tau^2+k^2} \sin p)]$$

$$\times \sum_{m=2}^n \alpha_m I_{\mu_m}(\tau, p)$$

$$- i(\tau \cos p - i\sqrt{\tau^2+k^2} \sin p)^2 \sum_{m=2}^n \alpha_m K_{\mu_m}(\tau, p) + O(\tau^{-(\mu_m+1)}) \quad (2.13)$$

and

$$\sqrt{\tau^2+k^2} I_q(\tau) = -i[\tau \sin q + i\sqrt{\tau^2+k^2} \cos q] - x_0 \cdot c_1(\omega^+)(\tau \cos q - i\sqrt{\tau^2+k^2} \sin q)]$$

$$\times \sum_{m=2}^n \alpha_m(\tau)^{m-1} I_{\mu_m}(\tau, q)$$

$$+ i(\tau \cos q - i\sqrt{\tau^2+k^2} \sin q)^2 \sum_{m=2}^n \alpha_m(\tau)^{m-1} K_{\mu_m}(\tau, q) + O(\tau^{-(\mu_m+1)}). \quad (2.14)$$

Let $\mu = \mu_m$. We have already established that

**Proposition 2.2** ([9]). As $s \longrightarrow \infty$, we have

$$(\tau \cos \theta - i\sqrt{\tau^2+k^2} \sin \theta) I_\mu(\tau, \theta) = \frac{i e^{i(\theta+\frac{\pi}{2})k^\mu}}{s^\mu} + O(s^{-\infty}). \quad (2.15)$$
The main problem is to compute the asymptotic expansion of the quantity \((\tau \cos \theta - i\sqrt{\tau^2 + k^2 \sin \theta})^2 K_\mu(\tau, \theta)\).

We prove

**Proposition 2.3.** As \(s \to \infty\), we have

\[
(\tau \cos \theta - i\sqrt{\tau^2 + k^2 \sin \theta})^2 K_\mu(\tau, \theta) = i(1 - \xi)^2(1 + \xi + \mu(1 - \xi))(1 - \xi)^{-\frac{1}{2} \Gamma(\frac{1}{2} + \mu)k^2 s^{\mu}} + O(s^{-\infty}),
\]

where \(\xi = (k/s)^2 e^{2i\theta}\).

**Proof.** The following formula can be proven along the same line as (3.3) of lemma 3.1 in [9]: for each \(l' = 0, 1, \ldots\)

\[
K_\mu(\tau, \theta) = \sum_{j=0}^{l'} \frac{(-1)^j}{j! \Gamma(1 + j + \mu)} \left(\frac{k}{2}\right)^{2j+\mu} \int_0^\infty r^{2j+\mu+1} e^{ir \sin \theta} e^{\sqrt{\tau^2 + k^2 r \cos \theta}} dr + O\left(\frac{1}{s^{2(l'+1)+\mu}}\right).
\]

(2.16)

Here, we recall (3.4) of lemma 3.2 in [9]: as \(s \to \infty\),

\[
\int_0^\infty r^\sigma e^{ir \sin \theta} e^{\sqrt{\tau^2 + k^2 r \cos \theta}} dr = \sum_{n=0}^l \frac{L_{\sigma,n}(\theta)}{\sigma^n e^{2n+1}} + O\left(\frac{1}{s^{\sigma+2(l+1)+1}}\right),
\]

where \(\sigma \geq 0, l = 0, 1, \ldots\) and

\[
L_{\sigma,n}(\theta) = i e^{i\theta} e^{i(\theta + \frac{1}{2})} \Gamma(\sigma + n + 1) \frac{\Gamma(\sigma + 1)}{n!}.
\]

Applying this with \(\sigma = 2j + \mu + 1\) to (2.17), we have

\[
K_\mu(\tau, \theta) = \sum_{j=0}^{l'} \frac{(-1)^j}{j! \Gamma(1 + j + \mu)} \left\{ \left(\frac{k}{2}\right)^{2j+\mu} \sum_{n=0}^l \frac{L_{2j+\mu+1,n}(\theta)}{s^{2j+\mu+2n+2}} + O\left(\frac{1}{s^{2j+\mu+2l+2}}\right) \right\}
\]

\[
+ O\left(\frac{1}{s^{2l+1}+\mu}\right) + O\left(\frac{1}{s^{2l+1}+\mu}\right).
\]

(2.17)

(2.18)

Now let \(l = l'\). Then, (2.18) becomes

\[
K_\mu(\tau, \theta) = \sum_{n=0}^l \left( \sum_{n_1+n_2=n} \frac{(-1)^n}{n_2! \Gamma(1 + n_2 + \mu)} \left(\frac{k}{2}\right)^{2n+\mu} \frac{L_{2n+\mu+1,n_1}(\theta)}{s^{2n+\mu+2}} \right) \times \frac{1}{s^{2n+\mu+2}} + O\left(\frac{1}{s^{2(l+1)+\mu}}\right).
\]

(2.19)
Write
\[
\sum_{n_1+n_2=\pi} \frac{(-1)^{n_2}}{n_2! \Gamma(1+n_2+\mu)} \left( \frac{k}{2} \right)^{2n_2+\mu} L_{2n_2+\mu} \left( \frac{\theta}{2} \right) \sum_{n_1+n_2=\pi} \frac{(-1)^{n_2}}{n_2! \Gamma(1+n_2+\mu)} \left( \frac{k}{2} \right)^{2n_2+\mu} \\
\times i e^{i\theta} e^{(i\theta+\frac{\pi}{2})(2n_2+\mu)+2} (-k^2 e^{2i\theta}) n_1! \Gamma(2n_2+\mu+n_1+2) n_1 \Gamma(1+n_2+\mu) \\
= 2^2 i e^{i\theta} e^{(i\theta+\frac{\pi}{2})(\mu+1)} k^\mu (-k^2 e^{2i\theta}) n_1! \Gamma(1+n_2+\mu) \\
= 2^{2i} e^{i\theta} e^{(i\theta+\frac{\pi}{2})(\mu+1)} k^\mu (-k^2 e^{2i\theta}) n_1! \Gamma(1+n_2+\mu). \tag{2.20}
\]

Here, we make use of the following formula:
\[
\sum_{n_1+n_2=\pi} \frac{(-1)^{n_2}}{n_1! n_2! \Gamma(1+n_2+\mu)} = (-1)^n (n+1)(n+1+\mu). \tag{2.21}
\]

This corresponds to (3.5) of lemma 3.3 in [9]; however, the proof needs a careful modification of that of lemma 3.3. See the appendix for the proof.

From (2.19), (2.20) and (2.21) one obtains
\[
K_\mu(\tau, \theta) = 2^2 i e^{i\theta} e^{(i\theta+\frac{\pi}{2})(\mu+1)} k^\mu \sum_{n=0}^\infty \frac{(k^2 e^{2i\theta}) n(n+1)(n+1+\mu)}{\tau^{2n+\mu+2}} + O \left( \frac{1}{\tau^{2(\mu+1)+\mu+2}} \right)
\]
and thus
\[
(\cos \theta - i \sqrt{\tau^2 + k^2 \sin^2 \theta})^2 K_\mu(\tau, \theta) = \frac{(s e^{-i\theta})^2}{2^2}(1 - \xi)^2 \times \left\{ 2^2 i e^{i\theta} e^{(i\theta+\frac{\pi}{2})(\mu+1)} k^\mu \sum_{n=0}^\infty \frac{(k^2 e^{2i\theta}) n(n+1)(n+1+\mu)}{\tau^{2n+\mu+2}} \right\} + O \left( \frac{1}{\tau^{2(\mu+1)+\mu+2}} \right)
\]
\[
= e^{-2i\theta}(1 - \xi)^2 \times \left\{ i e^{i\theta} e^{(i\theta+\frac{\pi}{2})(\mu+1)} k^\mu \sum_{n=0}^\infty \frac{(k^2 e^{2i\theta}) n(n+1)(n+1+\mu)}{\tau^{2n+\mu}} \right\} + O \left( \frac{1}{\tau^{2(\mu+1)+\mu}} \right)
\]
\[
= i(1 - \xi)^2 \sum_{n=0}^\infty \frac{(k^2 e^{2i\theta}) n(n+1)(n+1+\mu)}{\tau^{2n+\mu}} + O \left( \frac{1}{\tau^{2(\mu+1)+\mu}} \right) \tag{2.22}
\]

Let $|\xi| < 1$. Since
\[
\sum_{n=0}^\infty \xi^n (n+1) = (1 - \xi)^{-2}, \quad \sum_{n=0}^\infty \xi^n (n+1)^2 = (1 + \xi)(1 - \xi)^{-3},
\]
we have
\[
\sum_{n=0}^\infty \xi^n (n+1)(n+1+\mu) = (1 + \xi + \mu(1 - \xi))(1 - \xi)^{-3}
\]
and for each fixed $l$
\[
\sum_{n=0}^\infty \xi^n (n+1)(n+1+\mu) = (1 + \xi + \mu(1 - \xi))(1 - \xi)^{-3} + O(|\xi|^{l+1}).
\]
Substituting this with \( \zeta = (k/s)^2 e^{2i\theta} \) into (2.22), we have
\[
(\tau \cos \theta - i\sqrt{\tau^2 + k^2} \sin \theta)^2 K_\mu(\tau, \theta)
\]
\[
= i(1 - \zeta)^2(1 + \zeta + \mu(1 - \zeta))(1 - \zeta)^{-3} \frac{i e^{i(\theta + \zeta)\mu k^2}}{s^{\mu}} + O \left( \frac{1}{s^{2(\mu+1)\tau^2}} \right).
\]
Since \( l \) can be arbitrary large we obtain (2.16).

We continue the proof of (2.3). One can write
\[
\tau \sin \theta + i\sqrt{\tau^2 + k^2} \cos \theta = \frac{ie^{-i\theta}}{2} \left[ 1 + \left( \frac{k}{s} \right)^2 e^{i2\theta} \right]
\]
and
\[
\tau \cos \theta - i\sqrt{\tau^2 + k^2} \sin \theta = \frac{s e^{-i\theta}}{2} \left[ 1 - \left( \frac{k}{s} \right)^2 e^{i2\theta} \right].
\]
From these and (2.15) one obtains
\[
\left\{ (\tau \sin \theta + i\sqrt{\tau^2 + k^2} \cos \theta) - x_0 \cdot c_\tau(\omega^1)(\tau \cos \theta - i\sqrt{\tau^2 + k^2} \sin \theta) \right\} I_\mu(\tau, \theta)
\]
\[
= i(1 + \zeta)(1 - \zeta)^{-1} - x_0 \cdot c_\tau(\omega^1) \frac{ie^{i(\theta + \zeta)\mu k^2}}{s^{\mu}} + O(s^{-\infty}).
\]
Since
\[
i(1 + \zeta)(1 - \zeta)^{-1} - i(1 + \zeta)^2(1 + \zeta + \mu(1 - \zeta))(1 - \zeta)^{-3} = -i\mu,
\]
it follows from (2.13), (2.14), (2.16) and (2.23) that
\[
\sqrt{\tau^2 + k^2} I_p(\tau) \equiv \sum_{m=2}^n n \alpha_m (i\mu m + x_0 \cdot c_\tau(\omega^1)) e^{i(\theta + \zeta)\mu k^2} s^{\mu m} + O \left( \frac{1}{s^{\mu+1}} \right)
\]
and
\[
\sqrt{\tau^2 + k^2} I_q(\tau) \equiv \sum_{m=2}^n n \alpha_m (-1)^m (i\mu m + x_0 \cdot c_\tau(\omega^1)) e^{i(\theta + \zeta)\mu k^2} s^{\mu m} + O \left( \frac{1}{s^{\mu+1}} \right).
\]
Now from these and (2.6) we obtain (2.3).

3. Proof of theorem 1.2

First consider the case when every end point of \( \Sigma_1, \Sigma_2, \ldots, \Sigma_m \) satisfies \( x \cdot \omega < h_\Sigma(\omega) \). Then, \( x_0 \in \Sigma \) with \( x_0 \cdot \omega = h_\Sigma(\omega) \) should be a vertex of \( D \) and a point where two segments in some \( \Sigma_j \) meet. We take the same polar coordinates as those in section 2.

Integration by parts gives
\[
I_p(\tau; \omega, d, k) = I_p(\tau; \omega, d, k) = \int_{\Sigma} [u] \frac{\partial}{\partial \nu} \frac{\partial}{\partial t} v \ dS,
\]
where \( [u] = u^+|_{\partial D} - u^-|_{\partial D} \). Localizing this integral at \( x_0 \), we have, modulo exponentially decaying as \( \tau \to \infty \),
\[
e^{-i\sqrt{\tau^2 + k^2} x_0 \cdot \omega^1} e^{-\tau h_\Sigma(\omega)} I_p(\tau; \omega, d, k)
\]
\[
\sim e^{-i\sqrt{\tau^2 + k^2} x_0 \cdot \omega^1} e^{-\tau h_\Sigma(\omega)} \int_{\Gamma_p} [u] \frac{\partial}{\partial \nu} \frac{\partial}{\partial t} v \ dS
\]
\[
+ e^{-i\sqrt{\tau^2 + k^2} x_0 \cdot \omega^1} e^{-\tau h_\Sigma(\omega)} \int_{\Gamma_q} [u] \frac{\partial}{\partial \nu} \frac{\partial}{\partial t} v \ dS
\]
\[
= I_p(\tau) + I_q(\tau). \tag{3.1}
\]
It follows from (1.3) and (2.7) that
\[
\sqrt{\tau^2 + k^2} I_p(\tau) = i \left[ (r \sin \varphi + i \sqrt{\tau^2 + k^2} \cos \varphi) - x_0 \cdot c_t (\omega^2) (\tau \cos \varphi - i \sqrt{\tau^2 + k^2} \sin \varphi) \right]
\times \int_0^\eta (u^+(r, 0) - u^-(r, 2\pi)) e^{tr \sin \varphi} e^{j\sqrt{\tau^2 + k^2} \tau} dr
\]
\[\quad - i(\tau \cos \varphi - i \sqrt{\tau^2 + k^2} \sin \varphi)^2 \times \int_0^\eta r (u^+(r, 0) - u^-(r, 2\pi)) e^{tr \sin \varphi} e^{j\sqrt{\tau^2 + k^2} \tau} dr. \tag{3.2}\]

It follows also from (1.3) and (2.8) that
\[
\sqrt{\tau^2 + k^2} I_q(\tau) = -i \left\{ (r \sin \varphi + i \sqrt{\tau^2 + k^2} \cos \varphi) - x_0 \cdot c_t (\omega^2) (\tau \cos \varphi - i \sqrt{\tau^2 + k^2} \sin \varphi) \right\}
\times \int_0^\eta (u^+(r, \varphi) - u^-(r, \varphi)) e^{tr \sin \varphi} e^{j\sqrt{\tau^2 + k^2} \tau} dr
\]
\[\quad + i(\tau \cos \varphi - i \sqrt{\tau^2 + k^2} \sin \varphi)^2 \times \int_0^\eta r (u^+(r, \varphi) - u^-(r, \varphi)) e^{tr \sin \varphi} e^{j\sqrt{\tau^2 + k^2} \tau} dr. \tag{3.3}\]

By proposition 4.4 in [9], we have
\[
u^+ (r, \varphi) = \sum_{m=1}^\infty \alpha^+_m I_{\mu^+_m} (kr) \cos \mu^+_m \varphi, \quad 0 < r < \eta, \quad 0 \leq \varphi \leq \Theta, \]
\[
u^- (r, \varphi) = \sum_{m=1}^\infty \alpha^-_m I_{\mu^-_m} (kr) \cos \mu^-_m (\varphi - \Theta), \quad 0 < r < \eta, \quad \Theta \leq \varphi \leq 2\pi, \]
where \(\mu^+_m = (m - 1)\pi / \Theta\) and \(\mu^-_m = (m - 1)\pi / (2\pi - \Theta)\). For the precise meaning of this expansion see [9]. From these and \(\mu^+_m < \mu^-_m\) we have
\[
[u]_p = [u^+(r, 0) - u^-(r, 2\pi)] = \sum_{m=1}^l \left( J_{\mu^+_m} (kr) \alpha^+_m + J_{\mu^-_m} (kr) \alpha^-_m (1)^m \right) + O(r^{|\mu^+_m|}), \tag{3.4}\]
\[
[u]_q = [u^+(r, \varphi) - u^-(r, \varphi)] = -\sum_{m=1}^l \left( J_{\mu^+_m} (kr) \alpha^+_m (1)^m + J_{\mu^-_m} (kr) \alpha^-_m \right) + O(r^{|\mu^-_m|}). \]

It follows from (3.4) and (2.11) that
\[
\int_0^\eta (u^+(r, 0) - u^-(r, 2\pi)) e^{tr \sin \varphi} e^{j\sqrt{\tau^2 + k^2} \tau} dr
\]
\[\quad = \sum_{m=1}^n \left( \alpha^+_m I_{\mu^+_m} (\varphi, p) + \alpha^-_m (1)^m I_{\mu^-_m} (\varphi, p) \right) + O(\tau^{-|\mu^-_m|+1}), \]
\[
\int_0^\eta r (u^+(r, 0) - u^-(r, 2\pi)) e^{tr \sin \varphi} e^{j\sqrt{\tau^2 + k^2} \tau} dr
\]
\[\quad = \sum_{m=1}^n \left( \alpha^+_m K_{\mu^+_m} (\varphi, p) + \alpha^-_m (1)^m K_{\mu^-_m} (\varphi, p) \right) + O(\tau^{-|\mu^-_m|+2}). \]
\[
\int_0^\eta (u^+(r, \Theta) - u^-(r, \Theta)) e^{ir \sin \varphi} e^{i\sqrt{\tau^2 + k^2} r \cos \varphi} \, dr \\
= - \sum_{m=1}^n [\alpha_m^+ (-1)^m J_{\mu_m^+}(\tau, q) + \alpha_m^- i K_{\mu_m^-}(\tau, q)] + O(\tau^{-m\mu_m^+}),
\]

\[
\int_0^\eta r (u^+(r, \Theta) - u^-(r, \Theta)) e^{ir \sin \varphi} e^{i\sqrt{\tau^2 + k^2} r \cos \varphi} \, dr \\
= - \sum_{m=1}^n [\alpha_m^+ (-1)^m K_{\mu_m^+}(\tau, q) + \alpha_m^- i K_{\mu_m^-}(\tau, q)] + O(\tau^{-m\mu_m^+}).
\]

Substituting these into (3.2) and (3.3), we obtain
\[
\sqrt{\tau^2 + k^2} I_p(\tau) = \{[(\sin p + i\sqrt{\tau^2 + k^2} \cos p) - x_0 \cdot c_r(\omega^\perp)(\tau \cos p - i\sqrt{\tau^2 + k^2} \sin p)] \\
\times \sum_{m=1}^n [\alpha_m^+ i J_{\mu_m^+}(\tau, p) + \alpha_m^- (-1)^m J_{\mu_m^-}(\tau, p)] \\
- i(\tau \cos p - i\sqrt{\tau^2 + k^2} \sin p)^2 \\
\times \sum_{m=1}^n [\alpha_m^+ K_{\mu_m^+}(\tau, p) + \alpha_m^- (-1)^m K_{\mu_m^-}(\tau, p)] + O(\tau^{-m\mu_m^+})
\] (3.5)

and
\[
\sqrt{\tau^2 + k^2} I_q(\tau) = -i[(\sin q + i\sqrt{\tau^2 + k^2} \cos q) - x_0 \cdot c_r(\omega^\perp)(\tau \cos q - i\sqrt{\tau^2 + k^2} \sin q)] \\
\times \sum_{m=1}^n [\alpha_m^+ (-1)^m J_{\mu_m^+}(\tau, q) + \alpha_m^- i J_{\mu_m^-}(\tau, q)] \\
+ i(\tau \cos q - i\sqrt{\tau^2 + k^2} \sin q)^2 \\
\times \sum_{m=1}^n [\alpha_m^+ (-1)^m K_{\mu_m^+}(\tau, q) + \alpha_m^- K_{\mu_m^-}(\tau, q)] + O(\tau^{-m\mu_m^+}).
\] (3.6)

It follows from (3.5), (3.6), (2.16) and (2.23) that
\[
\sqrt{\tau^2 + k^2} I_p(\tau) = \sum_{m=1}^n \alpha_m^+ (i\mu_m^+ + x_0 \cdot c_r(\omega^\perp)) \frac{e^{i(q^\perp \cdot \omega^\perp)\mu_m^+} k^{\mu_m^+}}{\gamma_{\mu_m^+}} \\
+ \sum_{m=1}^n \alpha_m^- (-1)^m (i\mu_m^- + x_0 \cdot c_r(\omega^\perp)) \frac{e^{i(q^\perp \cdot \omega^\perp)\mu_m^-} k^{\mu_m^-}}{\gamma_{\mu_m^-}} + O\left(\frac{1}{\gamma^{m\mu_m^+}}\right)
\]

and
\[
\sqrt{\tau^2 + k^2} I_q(\tau) = \sum_{m=1}^n \alpha_m^+ (-1)^m (i\mu_m^+ + x_0 \cdot c_r(\omega^\perp)) \frac{e^{i(q^\perp \cdot \omega^\perp)\mu_m^+} k^{\mu_m^+}}{\gamma_{\mu_m^+}} \\
+ \sum_{m=1}^n \alpha_m^- (i\mu_m^- + x_0 \cdot c_r(\omega^\perp)) \frac{e^{i(q^\perp \cdot \omega^\perp)\mu_m^-} k^{\mu_m^-}}{\gamma_{\mu_m^-}} + O\left(\frac{1}{\gamma^{m\mu_m^+}}\right).
\]
Now it follows from this and (3.1) that
\[ \sqrt{\tau^2 + k^2} e^{-i\sqrt{\tau^2+k^2} \lambda_0} w^* e^{-i \sqrt{h_\Sigma(\alpha)} I_\Sigma'(\tau; \alpha, d, k) = \left( -1 \right)^m e^{i k \mu_\alpha} \left( -\mu_\alpha^+ + i x_0 \cdot c_e \phi_{\Sigma}(\alpha) \right) \frac{e^{i q k_\alpha^+ \mu_\alpha^+}}{\sqrt{\mu_\alpha^+}} + O \left( \frac{1}{\sqrt{\mu_\alpha^+}} \right). \] (3.7)

Since we have the following cancellation [9]:
\[ (-1)^m e^{i \mu_\alpha} + e^{i \mu_\alpha^+} = 0, \] (3.8)
(3.7) becomes
\[ \sqrt{\tau^2 + k^2} e^{-i\sqrt{\tau^2+k^2} \lambda_0} w^* e^{-i \sqrt{h_\Sigma(\alpha)} I_\Sigma'(\tau; \alpha, d, k) = \left( -1 \right)^m e^{i k \mu_\alpha} \left( -\mu_\alpha^+ + i x_0 \cdot c_e \phi_{\Sigma}(\alpha) \right) \frac{e^{i q k_\alpha^+ \mu_\alpha^+}}{\sqrt{\mu_\alpha^+}} + O \left( \frac{1}{\sqrt{\mu_\alpha^+}} \right). \] (3.9)

By the way, we have already known in [9] that
\[ e^{-i\sqrt{\tau^2+k^2} \lambda_0} w^* e^{-i \sqrt{h_\Sigma(\alpha)} I_\Sigma'(\tau; \alpha, d, k) = \left( -1 \right)^m e^{i k \mu_\alpha} \left( -\mu_\alpha^+ + i x_0 \cdot c_e \phi_{\Sigma}(\alpha) \right) \frac{e^{i q k_\alpha^+ \mu_\alpha^+}}{\sqrt{\mu_\alpha^+}} + O \left( \frac{1}{\sqrt{\mu_\alpha^+}} \right). \] (3.10)
and there exists \( m \geq 2 \) such that \( \alpha^*_m \left( e^{i \mu_\alpha} + (-1)^m e^{i \mu_\alpha^+} \right) \neq 0 \). Having these together with (3.9), hereafter we take the same course as the obstacle case.

The case when there is an end point of some \( \Sigma_j \) such that \( x \cdot \alpha = h_\Sigma(\alpha) \) corresponds to the case when \( p = q \). We omit its description.

**Remark 3.1.** From (3.9) and (3.10) we see that the field \( u^{-}(r, \theta) \) never affects the asymptotic behavior of \( I_\Sigma'(\tau; \alpha, d, k) \) and \( I_\Sigma'(\tau; \alpha, d, k) \) as \( \tau \rightarrow \infty \) modulo rapidly decreasing. The key point is cancellation (3.8).

4. Some other applications

In this section, we present some implications of the argument performed for the proof of theorem 1.1.

4.1. From the far-field pattern of the scattered wave for a single incident plane wave

Let \( u = e^{ikx} d + w \) be the same as that of theorem 1.1. It is well known that \( w \) has the asymptotic expansion as \( r \rightarrow \infty \) uniformly with respect to \( \phi \in S^1 \):
\[ u(r\phi) = \frac{e^{ikr}}{\sqrt{r}} F(\phi; d, k) + O \left( \frac{1}{r^{3/2}} \right). \]
The coefficient \( F(\phi; d, k) \) is called the far-field pattern of the scattered wave \( w \) at direction \( \phi \).

In this subsection, we present a direct formula that extracts the coordinates of the vertices of the convex hull of unknown polygonal sound hard obstacles \( D = D_1 \cup \cdots \cup D_m \) from the far-field pattern for fixed \( d \) and \( k \).
We identify \( \varphi = (\varphi_1, \varphi_2) \) with the complex number given by \( \varphi_1 + i\varphi_2 \) and denote it by the same symbol \( \varphi \). Given \( N = 1, \ldots, \tau > 0, \omega \in S^1 \) and \( k > 0 \), define the function \( g_N(\cdot; \tau, k, \omega) \) on \( S^1 \) by the formula

\[
g_N(\varphi; \tau, k, \omega) = \frac{1}{2\pi} \sum_{|m| \leq N} \left( \frac{ik\varphi}{(\tau + \sqrt{\tau^2 + k^2})\omega} \right)^m.
\]

Then, we have

\[
\partial_\tau g_N(\varphi; \tau, k, \omega) = -\frac{1}{2\pi} \frac{1}{\sqrt{\tau^2 + k^2}} \sum_{1 \leq |m| \leq N} m \left( \frac{ik\varphi}{(\tau + \sqrt{\tau^2 + k^2})\omega} \right)^m.
\]

In this subsection, \( B_R \) denotes the open disc centered at the origin of the coordinates with radius \( R \) and we assume that \( \overline{D} \subset B_R \).

**Theorem 4.1.** Let \( \omega \) be regular with respect to \( D \). Let \( \beta_0 \) be the unique positive solution of the equation

\[
2e^s + \log s = 0.
\]

Let \( \beta \) satisfy \( 0 < \beta < \beta_0 \). Let \( \{\tau(N)\}_{N=1}^{\infty} \) be an arbitrary sequence of positive numbers satisfying, as \( N \to \infty \),

\[
\tau(N) = \beta N e^R + O(1).
\]

Then, the formula

\[
\lim_{N \to \infty} \frac{\int_{S^1} F(-\varphi; d, k) \partial_\tau g_N(\varphi; \tau(N), k, \omega) \, dS(\varphi)}{\int_{S^1} F(-\varphi; d, k) g_N(\varphi; \tau(N), k, \omega) \, dS(\varphi)} = h_D(\omega) + i x_0 \cdot \omega \perp \quad (4.1)
\]

is valid.

In [11], we have shown that, under the same choice of \( \tau(N) \) and \( \omega \) being regular with respect to \( D \),

\[
\lim_{N \to \infty} \frac{1}{\tau(N)} \log \left| \int_{S^1} F(-\varphi; d, k) g_N(\varphi; \tau(N), k, \omega) \, dS(\varphi) \right| = h_D(\omega).
\]

Thus, theorem 4.1 corresponds to theorem 1.1.

Let us describe the proof of theorem 4.1. The starting point is the following identity which is a consequence of formula (2.9) in [2]:

\[
-\sqrt{8\pi k} e^{\pi/4} \int_{S^1} F(-\varphi; d, k) g_N(\varphi; \tau, k, \omega) \, dS(\varphi)
\]

\[
= I(\tau; \omega, d, k) + \int_{\partial B_R} \left( \frac{\partial u}{\partial v} (v_{s_\tau} - v_\tau) - \frac{\partial v}{\partial v} (v_{s_\tau} - v_\tau) u \right) \, dS,
\]

where

\[
v_{s_\tau}(y) = \int_{S^1} e^{iy} g_N(\varphi; \tau, k, \omega) \, dS(\varphi).
\]

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By theorem 2.1 in [11] we know that the second term on the right-hand side of (4.2) has the bound $e^{-\tau(N)\delta} O(N^{-\infty})$ as $N \to \infty$. Since $e^{-\tau(N)\delta} = O(e^{\tau(N)\delta})$, it follows from (2.4) and (4.2) that

$$\lim_{N \to \infty} \tau(N)^{\mu_*} e^{-i\sqrt{\tau(N)\delta}k^2 \zeta_0 \omega^2} e^{-\tau(N)\delta} \frac{8\pi k}{e^{\pi/4}} \int_{S^1} F(-\psi; d, k) g_N(\phi; \tau(N), k, \omega) \, dS(\phi)$$

$$= i\beta e^{\frac{\pi i}{2}} \frac{k}{2} \mu_*$$ (4.3)

Similar to (4.2), we have

$$- \frac{\sqrt{8\pi k}}{e^{\pi/4}} \int_{S^1} F(-\psi; d, k) \partial_t g_N(\phi; \tau, k, \omega) \, dS(\phi)$$

$$= I'(\tau; \omega, d, k) + \int_{\partial B_R} \left( \frac{\partial u}{\partial v} (v_{h, g_N} - \partial_t v_t) - \frac{\partial}{\partial v} (v_{h, g_N} - \partial_t v_t) u \right) \, dS,$$ (4.4)

where

$$v_{h, g_N}(y) = \int_{S^1} e^{ik \psi} \partial_t g_N(\phi; \tau, k, \omega) \, dS(\phi).$$

The following lemma corresponds to theorem 2.1 in [11] and see the appendix for the proof.

**Lemma 4.1.** We have, as $N \to \infty$,

$$e^{\eta \tau(N)} \sup_{|y| \leq R}\left| \int_{S^1} e^{ik \psi} \partial_t g_N(\phi; \tau(N), k, \omega) \, dS(\phi) - \partial_t v_t(y)|_{\tau=\tau(N)} \right|$$

$$+ e^{\theta \tau(N)} \sup_{|y| \leq R} \left| \nabla \int_{S^1} e^{ik \psi} \partial_t g_N(\phi; \tau(N), k, \omega) \, dS(\phi) - \partial_t v_t(y)|_{\tau=\tau(N)} \right|$$

$$= O(N^{-\infty}).$$

Now from lemma 4.1, (4.3) and (2.5) we obtain

$$\lim_{N \to \infty} \tau(N)^{\mu_*} e^{-i\sqrt{\tau(N)\delta}k^2 \zeta_0 \omega^2} e^{-\tau(N)\delta} \frac{8\pi k}{e^{\pi/4}} \int_{S^1} F(-\psi; d, k) g_N(\phi; \tau(N), k, \omega) \, dS(\phi)$$

$$= i\beta (x_0 \cdot \omega + ix_0 \cdot \omega^\perp) e^{\frac{\pi i}{2}} \mu_* \frac{k}{2}.$$

From this together with (4.4) yields (4.1).

### 4.2. From the Cauchy data of the scattered wave for a single point source

Let $y \in \mathbb{R}^2 \setminus \overline{D}$. Let $E = E_D(x, y)$ be the unique solution of the scattering problem:

$$(\triangle + k^2)E = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D},$$

$$\frac{\partial}{\partial v} E = - \frac{\partial}{\partial v} \Phi_0(\cdot, y) \quad \text{on} \quad \partial D,$$

$$\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial E}{\partial r} - i k E \right) = 0,$$

where

$$\Phi_0(x, y) = \frac{1}{4} H_0^{(1)}(k|x - y|)$$

and $H_0^{(1)}$ denotes the Hankel function of the first kind [16].
The total wave outside $D$ exerted by the point source located at $y$ is given by the formula

$$\Phi_D(x, y) = \Phi_0(x, y) + E_D(x, y), \quad x \in \mathbb{R}^2 \setminus D.$$  

In this subsection we consider the following problem.

**Inverse problem.** Let $R_1 > R$. We denote by $B_R$ and $B_{R_1}$ the open discs centered at a common point with radius $R$ and $R_1$, respectively. Assume that $\overline{D} \subset B_R$. Fix $k > 0$ and $y \in \partial B_{R_1}$.

Extract information about the location and shape of $D$ from $\Phi_D(x, y)$ given at all $x \in \partial B_R$.

Define

$$J(\tau; \omega, y, k) = \int_{\partial B_R} \left( \frac{\partial}{\partial v} \Phi_D(x, y) \cdot v_r(x; \omega) - \frac{\partial}{\partial v} v_r(x; \omega) \cdot \Phi_D(x, y) \right) dS(x).$$

Then, we have

$$J'(\tau; \omega, y, k) = \int_{\partial B_R} \left( \frac{\partial}{\partial v} \Phi_D(x, y) \cdot \partial_{\tau} v_r - \frac{\partial}{\partial v} (\partial_{\tau} v_r) \cdot \Phi_D(x, y) \right) dS(x).$$

Note that $(\partial/\partial v)\Phi_D(x, y)$ for $x \in \partial B_R$ can be computed from $\Phi_D(x, y)$ given at all $x \in \partial B_R$ by solving an exterior Dirichlet problem for the Helmholtz equation. See [13] for this point. This remark applies also to $(\partial/\partial v)u$ on $\partial B_R$ in theorems 1.1 and 1.2.

A combination of the proof of theorem 1.2 in [13] and the same argument as performed in the proof of theorem 1.1 yields the following formula.

**Theorem 4.2.** Assume that

$$\text{diam } D < \text{dist}(D, \partial B_{R_1}).$$

Let $\omega$ be regular with respect to $D$. Let $x_0 \in \partial D$ be the point with $x_0 \cdot \omega = h_D(\omega)$. Then, there exists a $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$, $|J(\tau; \omega, d, k)| > 0$ and the formula

$$\lim_{\tau \to \infty} \frac{J'(\tau; \omega, y, k)}{J(\tau; \omega, y, k)} = h_D(\omega) + i x_0 \cdot \omega$$

is valid.

Note that it is an open problem whether one can drop condition (4.5). For more information about this see [13].

**Acknowledgments**

This research was partially supported by Grant-in-Aid for Scientific Research (C) (no 21540162) of Japan Society for the Promotion of Science. The author thanks Takashi Ohe for useful discussion.

**Appendix**

**Proof of proposition 2.1**

**Proof.** For each $m$, define

$$u_m(r) = \int_0^\Theta u(r, \theta) \cos \mu_m \theta \, d\theta, \quad r \in ]0, 2\eta[.$$

Then, we see that $u_m$ satisfies the equation

$$- (ry')' + \left( \frac{\mu_m^2}{r} - k^2 r \right) y = 0 \text{ in } ]0, 2\eta[.$$
and thus this yields that there exist numbers \( \alpha_m, \beta_m \) such that

\[
u_m(r) = \alpha_m J_{\mu_m}(kr) + \beta_m Y_{\mu_m}(kr),
\]

where \( Y_{\mu_m}(kr) \) denotes the Bessel function of the second kind ([16]). Then, a similar argument performed in [4] and the behavior of \( J_{\mu_m}(kr) \) and \( Y_{\mu_m}(kr) \) as \( r \to 0 \) ([16]), one concludes that \( \beta_m = 0 \) for all \( m \geq 1 \). Substituting \( \nu_m(r) = \alpha_m J_{\mu_m}(kr) \) into the inequality

\[
\int_0^{2\eta} r|\nu'_m(r)|^2 \, dr \leq \|\nabla \nu\|_{L^2(B_2(\eta) \cap (B_R \setminus \Omega))}^2 < \infty,
\]

we obtain

\[
\alpha_m^2 \int_0^{2\eta} r|[J_{\mu_m}(kr)]'|^2 \, dr \leq C,
\] (A.1)

where \( C \) is a positive constant independent of \( m \).

Thus, the problem is to estimate the integral on the left-hand side of (A.1) from below. For this purpose, we make use of the following formula which can be checked directly:

\[
[J_{\mu_m}(kr)]' = \mu_m r J_{\mu_m}(kr) - k J_{\mu_m}(kr)_{-1} + k J_{\mu_m}(kr)_{+1}.
\]

From this we have

\[
r|[J_{\mu_m}(kr)]'|^2 = \frac{\mu_m^2}{r} |J_{\mu_m}(kr)|^2 - 2\mu_m k J_{\mu_m}(kr) J_{\mu_m}(kr)_{+1} + k^2 |J_{\mu_m}(kr)_{+1}|^2
\]

and thus

\[
\int_0^{2\eta} r|[J_{\mu_m}(kr)]'|^2 \, dr \geq \mu_m \int_0^{2\eta} \frac{1}{r} |J_{\mu_m}(kr)|^2 \, dr - 2\mu_m k \int_0^{2\eta} J_{\mu_m}(kr) J_{\mu_m}(kr)_{+1} \, dr.
\] (A.2)

By formula (37) on page 338 in [1] and a change of independent variable we have

\[
k \int_0^{2\eta} J_{\mu_m}(kr) J_{\mu_m}(kr)_{+1} \, dr = \int_0^{2\eta} J_{\mu_m}(r) J_{\mu_m}(r)_{+1} \, dr
= \sum_{n=0}^{\infty} |J_{\mu_m+1}(2\eta k)|^2.
\] (A.3)

Furthermore, from (A.3) and the recurrence relation of the Bessel functions

\[
\mu_m J_{\mu_m}(kr) = k r (J_{\mu_m-1}(kr) + J_{\mu_m+1}(kr))
\]

which can be checked directly we have

\[
\int_0^{2\eta} \frac{1}{r} |J_{\mu_m}(kr)|^2 \, dr = \frac{k}{\mu_m} \int_0^{2\eta} J_{\mu_m-1}(kr) J_{\mu_m}(kr) \, dr + \frac{k}{\mu_m} \int_0^{2\eta} J_{\mu_m}(kr) J_{\mu_m}(kr)_{+1} \, dr
= \frac{1}{\mu_m} \int_0^{2\eta} J_{\mu_m-1}(r) J_{\mu_m}(r) \, dr + \frac{1}{\mu_m} \int_0^{2\eta} J_{\mu_m}(r) J_{\mu_m}(r)_{+1} \, dr
= \frac{1}{\mu_m} \sum_{n=0}^{\infty} |J_{\mu_m+n}(2\eta k)|^2 + \frac{1}{\mu_m} \sum_{n=0}^{\infty} |J_{\mu_m+n+1}(2\eta k)|^2
= \frac{1}{\mu_m} |J_{\mu_m}(2\eta k)|^2 + \frac{2}{\mu_m} \sum_{n=0}^{\infty} |J_{\mu_m+n+1}(2\eta k)|^2.
\]

This together with (A.2) and (A.3) yields

\[
\int_0^{2\eta} r|[J_{\mu_m}(kr)]'|^2 \, dr \geq \mu_m |J_{\mu_m}(2\eta k)|^2.
\] (A.4)
Since
\[ J_{\mu_n}(2\eta k) = \frac{(\eta k)^{\mu_n}}{\Gamma(1 + \mu_m)} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(1 + \mu_m)}{\Gamma(1 + \mu_m + n)} (\eta k)^{2n} \right\} \]
and
\[ \frac{\Gamma(1 + \mu_m)}{\Gamma(1 + \mu_m + n)} = \frac{1}{\prod_{j=1}^{n}(j + \mu_m)} \leq \frac{1}{n!}, \]
it holds that
\[ |J_{\mu_n}(2\eta k)| \geq \frac{(\eta k)^{\mu_n}}{\Gamma(1 + \mu_m)} (1 - (e^{\eta k} - 1)). \]
Thus, choosing \( \eta k \) in such a way that \( e^{\eta k} - 1 < 1/2 \), that is, \( \eta k < \log(3/2) \), we obtain
\[ |J_{\mu_n}(2\eta k)| \geq \frac{1}{2} \frac{(\eta k)^{\mu_n}}{\Gamma(1 + \mu_m)}. \]
This together with (A.4) gives the following estimate:
\[ \int_0^{2\eta} r |\{J_{\mu_n}(kr)\}'|^2 \, dr \geq \frac{1}{4} \frac{\mu_m}{\Gamma(1 + \mu_m)^2} (\eta k)^{2\mu_n}. \] (A.5)
Then, (A.1) and (A.5) give the estimate in (2). The remaining parts of the statements are consequences of (2) and the completeness of \((\cos \mu_m \theta)^{\infty}_{m=1}\) in \(L^2([0, \Theta])\).

Proof of (2.21)

Since
\[ \Gamma(n + 2 + l + \mu) = \left\{ \prod_{j=1}^{n+1}(j + l + \mu) \right\} \Gamma(1 + l + \mu), \]
one can rewrite
\[ \sum_{n_1+n_2=n} \frac{(-1)^{n_2}}{n_1!n_2!} \Gamma(1 + n_2 + \mu) \]
\[ = \sum_{n_1+n_2=n} \frac{(-1)^{n_2}}{n_1!n_2!} \prod_{j=1}^{n_2+1}(j + n_2 + \mu) \]
\[ = \frac{1}{n!} \sum_{n_1+n_2=n} \frac{(-1)^{n_2}}{n_1!n_2!} \left( \frac{d}{dx} \right)^{n+1} \left\{ x^{n_1+n_2+\mu} \right\}_{x=1} \]
\[ = \frac{1}{n!} \left( \frac{d}{dx} \right)^{n+1} \left\{ (1-x)^n x^{n_1+\mu} \right\}_{x=1} \]
\[ = \frac{n+1}{n!} \left( \frac{d}{dx} \right)^{n} \left\{ (1-x)^n \right\}_{x=1} \cdot \frac{d}{dx} x^{n_1+\mu} \]
\[ = (-1)^{n+1} (n+1)(n+1+\mu). \]

Proof of lemma 4.1

In [11], we have already known that
\[ v_{\mu_n}(y) - v_{\tau}(y) = - \sum_{m>N} \left\{ \left( \tau - \sqrt{\tau^2 + k^2} \omega \right)^m \right\} J_m(kr) e^{im\theta} \]
\[ - \sum_{m>N} \left\{ \left( \tau + \sqrt{\tau^2 + k^2} \omega \right)^m \right\} J_m(kr) e^{-im\theta}, \] (A.6)
where $y = (r \cos \theta, r \sin \theta)$. Since

$$
\frac{(r \mp \sqrt{r^2 + k^2}) \bar{z}}{k^m} = m \left( \frac{(r \pm \sqrt{r^2 + k^2}) \bar{z}}{k^m} \right) - 1 \left( \frac{(r \mp \sqrt{r^2 + k^2}) \bar{z}}{k^{m-1}} \right) \frac{m}{k^{m-1}} \sqrt{r^2 + k^2},
$$

from (A.6) we have

$$
\sqrt{r^2 + k^2} (v_{h_{0z} g_0}(y) - \partial_\tau v_z(y)) = \sum_{m<N} m \left( \frac{(r \mp \sqrt{r^2 + k^2}) \bar{z}}{k^m} \right) J_m(kr) e^{im\theta}
$$

$$
- \sum_{m>N} m \left( \frac{(r + \sqrt{r^2 + k^2}) \bar{z}}{k^m} \right) J_m(kr) e^{-im\theta}.
$$

Since

$$
|J_m(kr)| \leq \left( \frac{kr}{2} \right)^m \frac{1}{m!},
$$

it follows that

$$
\sqrt{r^2 + k^2} |v_{h_{0z} g_0}(y) - \partial_\tau v_z(y)| \leq C (N + 1) E(\tau; N + 1), \tag{A.7}
$$

where $C > 0$ is independent of $N$ and $\tau$ and

$$
E(\tau; N) = \frac{1}{N!} \left( \frac{R(\tau + \sqrt{\tau^2 + k^2})}{2} \right)^N e^{R(\tau + \sqrt{\tau^2 + k^2})/2}.
$$

By virtue of the choice of $\tau(N), \beta$ and the Stirling formula (cf [16]), we have

$$
e^{R(\tau(N))} E(\tau(N); N + 1) = O(N^{-\infty}).
$$

This is the key point, see [11] for details of the derivation. This together with (A.7) yields half of the desired estimates. The remaining estimate can also be given similarly.

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