A new kind of index theorem

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Dedicated to Krzysztof Wojciechowski on his 50th Birthday

Abstract

Index theory has had profound impact on many branches of mathematics. In this note we discuss the context for a new kind of index theorem. We begin, however, with some operator-theoretic results. In [11] Berger and Shaw established that finitely cyclic hyponormal operators have trace-class self-commutators. In [9], [31] Berger and Voiculescu extended this result to operators whose self-commutators can be expressed as the sum of a positive and a trace-class operator. In this note we show this result can’t be extended to operators whose self-commutators can be expressed as the sum of a positive and a $S_p$-class operator. Then we discuss a conjecture of Arveson [4] on homogeneous submodules of the $m$-shift Hilbert space $H^2_m$ and propose some refinements of it.

Further, we show how a positive solution would enable one to define $K$-homology elements for subvarieties in a strongly pseudo-convex domain with smooth boundary using submodules of the corresponding Bergman module. Finally, we discuss how the Chern character of these classes in cyclic cohomology could be defined and indicate what we believe the index to be.

0 Introduction

The complex Hilbert space $H$ is said to be a Hilbert module over the algebra $A$ if $H$ is a unital module over $A$. This is equivalent to a representation of $A$ on $H$. In the last two decades, there has been considerable interest in the study of Hilbert modules for various classes of algebras, in part as an approach to multivariate operator theory. In [20] $A$ was assumed to be a function algebra and module multiplication to be bounded. Other authors (e.g. [29], [23])

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have considered other kinds of algebras. More recently, there has been an interest in modules for which $A$ is the algebra of polynomials $\mathbb{C}[z]$ with various assumptions such as (1) coordinate functions act contractively or (2) they act as a spherical contraction. In [2] Arveson, considered the latter case and identified the $m$-shift space $H^2_m$ as having particularly nice properties. In the course of his studies [4], he raised a question about the almost reductivity of the submodules of $H^2_m \otimes \mathbb{C}^k$, for $1 \leq k < \infty$, generated by homogeneous polynomials; that is, modules for which the coordinate multipliers and their adjoints have compact or $p$-summable cross-commutators. In [5] he established this result for submodules generated by monomials. (Also, see [9], [27], for some subsequent work on this topic.) In [16], the author extended this result to a class of commuting weighted shifts which includes the $m$-shift and Bergman and Hardy modules for the ball.

In this note we discuss Arveson’s conjecture in full generality and more. We suggest, in particular, that submodules of Bergman modules over strongly pseudo-convex domains of $\mathbb{C}^m$ with smooth boundary determined by subvarieties are $p$-reductive for $p > m$. Moreover, in such a case they determine odd $K$-homology classes [13] for the space equal to the intersection of the subvariety with the boundary of the domain. Further, one could define a Chern character using the cyclic cohomology of Connes [14]. We conjecture that this class is the one determined by the almost complex structure on the intersection of the subvariety and the boundary. Such a result would be a new kind of index theorem.

We begin by considering some results of Berger [9], which extended his earlier theorem with Shaw [11] in operator theory. The latter result established that self-commutators of hyponormal operators are trace-class in the presence of finite cyclicity.

My interest in the question of almost reductivity was spurred by Arveson and resulted from an ongoing dialogue with him on his work on this topic. This rather unusual note was the subject of conference talks given in 2005 at IUPUI, Penn State and Roskilde University and is presented here to bring to the attention of other researchers, what we believe to be most promising and interesting topic.

1 Results in Operator Theory

Recall that the bounded linear operator $T$ on the Hilbert space $\mathcal{H}$ is said to be hyponormal if the self-commutator $[T^*, T] = T^*T - TT^*$ is a positive operator. In [9] Berger and Shaw demonstrated the surprising result that a finitely cyclic hyponormal operator has a trace-class self-commutator. There is also an estimate of the trace involving the degree of cyclicity and the area of the spectrum of $T$ but that inequality will not concern us at this time.

Recall that an operator $X$ on $\mathcal{H}$ is said to belong to $\mathcal{S}_p$, the Schatten–von Neumann class,
for $1 \leq p < \infty$, if $X$ is compact and the eigenvalues of $(X^*X)^{1/2}$ belong to $\ell^p$ (cf. [25]). Now $S_1$ consists of the trace-class operators. One knows that $S_p$ is a Banach space with dual space $S_q$ with $\frac{1}{p} + \frac{1}{q} = 1$, for $1 \leq p < \infty$, if we identify $S_\infty$ with $L(H)$, the space of all bounded operators on $H$. Further, $S_p$ is a two-sided ideal in $S_\infty = L(H)$.

In subsequent years, Berger [9] extended the Berger–Shaw Theorem to cover a larger class of operators which is the class we shall consider. (There was also related work by Voiculescu [31] and Carey–Pincus on this class.) For $1 \leq p < \infty$, we’ll say that an operator $T$ on $H$ belongs to $A_p$ if $[T^*, T] = P + C$, where $P \geq 0$ and $C$ is in $S_p$. Observe that all hyponormal operators are in $A_p$ as are all operators $T$ for which $[T^*, T]$ is in $S_p$. Observe also for $p = 1$, that there is a well-defined trace on the self-commutators of operators in $A_1$ taking values in $(-\infty, \infty]$ and that for $T$ in $A_1$ we have $[T^*, T]$ trace-class iff this trace is finite.

Finally, we will let $A_0$ denote the operators $T$ for which $[T^*, T] = P + C$ with $P$ positive and $C$ compact.

The class of hyponormal operators is closed under restriction to invariant subspaces. That is, if $T$ is hyponormal and $V$ is an invariant subspace for $T$, then $T|_V$ is hyponormal. The following lemma shows the same is true for class $A_p$.

**Lemma 1 ([9], [31]).** If $T$ belongs to $A_p$ for $1 \leq p \leq \infty$ or $p = 0$, and $V$ is an invariant subspace for $T$, then $T|_V$ is in $A_p$.

**Proof.** If one writes $T = (\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix})$ relative to the decomposition $H = V \oplus V^\perp$, and $Q$ is the orthogonal projection on to $V$, then

$$[(T|_V)^*, (T|_V)] = Q[T^*, T]Q + QTQ^\perp T^*Q,$$

where $Q^\perp = I - Q$. Since $[T^*, T] = P + C$, with $P \geq 0$ and $C$ in $S_p$, we have

$$[(T|_V)^*, (T|_V)] = (QPQ + QTQ^\perp T^*Q) + QCQ$$

and the first sum on the right-hand side is positive while $QCQ$ is in $S_p$. \qed

The following result is a special case of a result due to Berger [9]. We reproduce the proof since it is short and we believe deserves to be better known.

**Proposition 2.** If $T$ is in $A_1$ and $V$ and $\{V_n\}$ are invariant subspaces for $T$ such that each $V_n$ is finite dimensional, $V_n \subset V_{n+1}$ for all $n$ and $\bigcup_{n=1}^{\infty} V_n$ is dense in $V$, then $[(T|_V)^*, (T|_V)]$ is trace-class.

**Proof.** Let $Q$ and $\{Q_n\}$ be the orthogonal projections onto $V$ and $\{V_n\}$, respectively. Then $\{Q_n\}$ converges in the strong operator topology to $Q$. Adopting the same notation as in the preceding proof for the representation of the self-commutators, we have $[(T|_{V_n})^*, (T|_{V_n})] = P_n + C_n$ for
each \( n \) and \([ (T|_V)^* , (T|_V) ] = P + C\). Moreover, the sequence \( \{ P_n \} \) converges strongly to \( P \) while the sequence \( \{ C_n \} \) converges to \( C \) in the norm on \( S_1 \).

Since \( T|_{\mathcal{V}_n} \) is finite rank, we have \( \text{Tr}[ (T|_{\mathcal{V}_n})^* , (T|_{\mathcal{V}_n}) ] = 0 \) and hence \( 0 \leq \text{Tr} P_n \leq \| C_n \|_1 \) for all \( n \). Further, we have \( \| C_n \|_1 \to \| C \|_1 \) which implies that \( \lim \text{Tr} P_n \leq M < \infty \) and hence \( \text{Tr} P \leq M \) using a variant of Fatou’s Lemma. Therefore, \( P \) is trace-class from which the result follows. \( \square \)

Actually Berger proved a stronger result. Suppose we have another invariant subspace \( \mathcal{V}_0 \) contained in all the \( \mathcal{V}_n \) so that the dimension of \( \mathcal{V}_n / \mathcal{V}_0 \) is finite for all \( n \) and \( T|_{\mathcal{V}_0} \) is in \( A_1 \). Then the preceding argument yields the same conclusion, namely, that the self-commutator of \( T|_V \) is in \( S_1 \).

We would like to obtain the analogous result for operators in \( A_p \). In an earlier version of this note we thought we had proved it.\(^1\) Unfortunately, the following example shows it to be false.

**Example 3.** Consider the weighted unilateral shift \( S_n \) defined on \( \ell^2 \) with the standard basis \( \{ e_k \}_{k=1}^\infty \) so that

\[
S_n e_k = \begin{cases} \sqrt{\frac{k}{n}} e_{k+1}, & 1 \leq k \leq n. \\ e_{k+1}, & n < k. \end{cases}
\]

An easy calculation shows that \( \|[S_n^*, S_n]\|_p = n^{1-p} \) for \( 1 \leq p < \infty \). If \( \mathcal{V}_n \) is the subspace of \( \ell^2 \) spanned by \( \{ e_k \}_{k=n}^\infty \), then \( S_n \mathcal{V}_n \subset \mathcal{V}_n \) and \( \|[S|_{\mathcal{V}_n}^*, (S|_{\mathcal{V}_n})]\|_p = 1 \) for all \( n \). Moreover, if we set \( S = \bigoplus_{n=1}^\infty S_n \) acting on \( \bigoplus_{n=1}^\infty \ell^2 \) and \( \mathcal{V} = \bigoplus_{n=1}^\infty \mathcal{V}_n \), then \( \|[S^*, S]\|_p < \infty \) but \( \|[S|_{\mathcal{V}}^*, (S|_{\mathcal{V}})]\|_p = \infty \) for all \( p, 1 < p < \infty \).

We conclude that \( S^* \) is in \( A_p \), \( \mathcal{V}_n^\perp \) is spanned by generalized eigenvectors for \( S^* \) (hence one can construct the desired sequence of finite dimensional approximates for it) but \( S^*|_{\mathcal{V}^\perp} \) is not \( p \) almost reductive. Observe that \( \mathcal{V}^\perp \) and \( \mathcal{V} \) are not finitely cyclic for \( S^* \) and \( S \), respectively.

We now reframe Proposition 2 in a setting which makes the hypotheses more transparent.

**Theorem 0.** Let \( T \) be an operator in \( A_1 \) and \( \mathcal{V} \) be an invariant subspace for \( T \) spanned by generalized eigenvectors for \( T|_\mathcal{V} \). Then \( ([T|_\mathcal{V}]^*, (T|_\mathcal{V})] \) is in \( S_1 \).

**Proof.** Let \( \{ f_k \} \) be a sequence of generalized eigenvalues for \( T|_\mathcal{V} \) which spans \( \mathcal{V} \). Further, let \( \mathcal{V}_n \) be the invariant subspace for \( T \) generated by \( \{ f_k \}_{k=1}^n \). Then the \( \{ \mathcal{V}_n \} \) are nested, finite dimensional and their union is dense in \( \mathcal{V} \). The result now follows from Proposition 2. \( \square \)

As we indicated above, our original goal was to extend Proposition 2 to \( A_p \) and thereby extend Theorem 0 to this class. Unfortunately, Example 3 shows this is impossible.

\(^{1}\)We want to thank W.B. Arveson for pointing out the mistake in the earlier version of this note.
Since a finite dimensional invariant subspace for $T$ is spanned by the generalized eigenvectors for it, the hypotheses of the foregoing theorem is the only way to fulfill the condition in Proposition 2. Berger introduced the notion of an invariant subspace $\mathcal{V}_0$ being effectually full in $\mathcal{V}$ by requiring the denseness in $\mathcal{V}$ of the set of vectors in $\mathcal{V}$ that some nonzero polynomial in $T$ takes into $\mathcal{V}_0$. This hypothesis enabled him to satisfy the weaker hypotheses we have mentioned earlier.

A question which presents itself at this point is whether Theorem 0 might hold for all invariant subspaces of $T$ without any restriction. The following example shows that this is not the case.

**Example 4.** Consider the Bergman space $B^2(\mathbb{D})$ for the unit disk $\mathbb{D}$ which can be defined as the closure of the analytic polynomials in $L^2(\mathbb{D})$ relative to planar Lebesgue measure. Further, consider the operator $T = M_z^*$, where $M_z$ is multiplication by $z$ on $B^2(\mathbb{D})$. Note that $T$ lies in $A_1$.

Suppose we have invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for $T$ such that $0 \subset \mathcal{M} \subset \mathcal{N} \subset B^2(\mathbb{D})$. If both $T|_{\mathcal{M}}$ and $T|_{\mathcal{N}}$ are in $A_1$, then the same is true for the compression of $T$ to the semi-invariant subspace $\mathcal{N}/\mathcal{M}$. (This is essentially Theorem 1 in [16].) However, one knows (cf. [1]) that there is a semi-invariant subspace for the Bergman shift so that the restriction of the Bergman shift to it realizes each proper contraction operator on a separable Hilbert space. Hence we can obtain a semi-invariant subspace for which the self-commutator of the restriction is not even compact.

Here would be a good place to record a result which is a refinement of Theorem 1 of [16].

**Theorem 1.** If $\mathcal{M}_1$ and $\mathcal{M}_2$ are essentially reductive modules for the algebra $A$ and $X : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a module map having closed range, then both $\ker X$ and $\text{ran } X$ are essentially reductive.

**Proof.** If we write $\mathcal{M}_1 = (\ker X)^\perp \oplus \ker X$ and $\mathcal{M}_2 = \text{ran } X \oplus (\text{ran } X)^\perp$, then $X$ has the form $(X_0 \ 0)$, where $X_0$ is an invertible operator from $(\ker X)^\perp$ onto $\text{ran } X$.

For $T$ any operator between Hilbert spaces, let $\hat{T}$ denote its image in the corresponding Calkin algebra, that is, modulo the compact operators. Since both $\mathcal{M}_1$ and $\mathcal{M}_2$ are essentially reductive, for $\varphi$ in $A$ the elements $A_\varphi$ and $B_\varphi$ are normal, where $A_\varphi$ and $B_\varphi$ denote the operators defined by module multiplication by $\varphi$.

Let $A_\varphi = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$ and $B_\varphi = \begin{pmatrix} B_{11} & B_{12}^* \\ 0 & B_{22}^* \end{pmatrix}$ be the representations of $A_\varphi$ and $B_\varphi$ relative to the decompositions of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. If we consider the images of all these operators in their respective Calkin algebras, we can apply the Fuglede–Putnam Theorem to conclude that the relationship $\hat{\hat{X}} A_\varphi = B_\varphi \hat{X}$ implies that $\hat{A}_\varphi \hat{X}^* = \hat{X}^* B_\varphi$ and therefore, we have $\hat{X}^* \hat{X} A_\varphi = A_\varphi \hat{X}^* \hat{X}$. This equation in turn implies that $\hat{A}_\varphi^{21} \hat{X}_0^* \hat{X}_0 = 0$ and thus $\hat{A}_\varphi^{21} = 0$ since $\hat{X}_0^* \hat{X}_0$ is invertible. However, $\hat{A}_\varphi^{21} = 0$ for all $\varphi$ in $A$ means that $\mathcal{M}_1$ is essentially reductive.
Working with $\mathcal{M}_2$ we conclude that $\hat{X}^*\hat{B}_\varphi = \hat{B}_\varphi \hat{X}^*$ and hence $\hat{X}_0\hat{X}_0^*\hat{B}_\varphi^2 = 0$. Again this implies that $\mathcal{M}_2$ is essentially reductive which completes the proof.

Unfortunately, since no appropriate analogue of the Fuglede–Putnam Theorem is known for the $p$-summable case, such a proof won’t allow us to conclude $p$-reductivity of the kernel and range if $\mathcal{M}_1$ and $\mathcal{M}_2$ are. However, in [4] Arveson gives such a result for a specific class of Hilbert modules and module maps.

## 2 Almost Reductive Hilbert Modules

We now show that Theorem 4 enables one to settle a question about cross-commutators for some submodules in the multi-variable setting so long as they have trace-class cross-commutators. Since the conjecture of Arveson [4] motivated this study, let us begin by considering it in some detail.

Recall that $H^2_m$, the $m$-shift Hilbert space for $1 \leq m < \infty$, is defined using the symmetric Fock space and is a module over $\mathbb{C}[z]$. Moreover, Arveson showed that multiplication by each coordinate function $Z_i$ acts contractively on $H^2_m$ and all cross-commutators $[Z_i^*, Z_j]$ lie in $\mathcal{S}_p$ for $p > m$ and $1 \leq i, j \leq m$. Arveson conjectured that the restriction operators $Y_i = Z_i|_S$ and their adjoints also have $\mathcal{S}_p$ cross-commutators for any submodule $S$ of $H^2_m \otimes \mathbb{C}^k$ for $1 \leq k < \infty$ generated by homogeneous polynomials. Moreover, he established the result for $S$ generated by monomials. He has also developed [4] a theory of “standard Hilbert modules” in an effort to establish his conjecture. Another proof of the result for monomial submodules was given in [16] and it also covered certain commuting weighted shifts. Also, Arveson showed that the general case for homogeneous submodules of $H^2_m$ for $m = 2$ followed from a result of Guo [26]. Finally, a generalization to the case of certain pairs of commuting weighted shifts was recently obtained in [27].

The simple matrix calculation used in [16] and the proof of Theorem 4 shows that if $T_1$ and $T_2$ are two operators on a Hilbert space $\mathcal{H}$ with $\mathcal{G}$ an invariant subspace for both such that $[T_1, T_2]$ lies in $\mathcal{S}_p$ for $1 \leq p < \infty$ or $p = 0$, then the compressions $S_i = T_i|_G$ have $[S_i^*, S_j]$ in $\mathcal{S}_p$ for $1 \leq k < \infty$ generated by homogeneous polynomials. Moreover, he established the result for $S$ generated by monomials. He has also developed [4] a theory of “standard Hilbert modules” in an effort to establish his conjecture. Another proof of the result for monomial submodules was given in [16] and it also covered certain commuting weighted shifts. Also, Arveson showed that the general case for homogeneous submodules of $H^2_m$ for $m = 2$ followed from a result of Guo [26]. Finally, a generalization to the case of certain pairs of commuting weighted shifts was recently obtained in [27].

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We consider the case of commuting weighted shifts using the notation of [16]. We have a weight set $\Lambda$ for the index set $A_m$, $1 \leq m < \infty$, with the Hilbert space $\mathcal{M}_\Lambda$ and the weighted shifts defined by the coordinate functions $Z_i$, $1 \leq i \leq m$. The weight set $\Lambda$ satisfies $(\ast)$ if the shifts are contractive, $(\ast\ast)$ if all cross-commutators of the coordinate multipliers and their adjoints are compact, and $(\ast\ast)_p$ if the latter operators lie in $\mathcal{S}_p$. Actually, in the following result one can replace $(\ast)$ by assuming only $(\ast)'$ that the $Z_i$ are only bounded.
Theorem 2. If $\Lambda$ is a weight set satisfying (") and (**)$_1$, $\mathcal{S}$ is a submodule of $\mathcal{M}_\Lambda \otimes \mathbb{C}_k$, $1 \leq k < \infty$, so that $S^\perp$ is generated by polynomials, and $Y_i = Z_i|_S$, then the cross-commutators $[Y_i^*, Y_j]$ are in $S_1$ for $1 \leq i, j \leq m$.

Proof. If we set $T = Z_i^*$ for some fixed $i$, then $S^\perp$ is invariant for $T$. Moreover, $S^\perp$ is spanned by polynomials. Therefore, $T$ and $S^\perp$ satisfy the hypotheses of Theorem 0 which implies that $[Y_i, Y_j^*]$ lies in $S_1$ for all $1 \leq i \leq m$. Here we are using the fact that the self-commutator of the restriction of $Z_i$ to $S$ lies in $S_1$ iff the same is true for the restriction of $Z_i^*$ to $S^\perp$.

Now if we take $T = Z_j^* + Z_k^*$ for $1 \leq j \neq k \leq m$, then $T$ and $S^\perp$ again satisfy the hypotheses of Theorem 0. Therefore, we have $[Y_j + Y_k, Y_j^* + Y_k^*]$ lies in $S_1$. Since $[Y_j, Y_j^*]$ and $[Y_k, Y_k^*]$ lie in $S_1$, we conclude that the real part of $[Y_j, Y_k^*]$ is in $S_1$. Repeating the argument for $T = Z_j^* + iZ_k^*$, we see that the imaginary part of $[Y_j, Y_k^*]$ is in $S_1$ which completes the proof.

Let $P_n$ denote the subspace of $\mathbb{C}[z]$ consisting of homogeneous polynomials of degree $n$. If $S$ is generated by homogeneous polynomials, then $S = \oplus(S \cap P_n)$. This in turn implies that $S^\perp = \oplus(S^\perp \cap P_n)$ and hence $S^\perp$ is generated by polynomials. Thus Theorem 0 applies to homogeneous submodules. Instead of assuming that $S^\perp$ is generated by polynomials, which are joint generalized eigenvectors for the adjoint of coordinate multipliers, we could assume more generally that $S^\perp$ is spanned by such vectors.

Observe that we can’t consider $\mathcal{M}_\Lambda \otimes \ell^2$, where $\ell^2$ is the infinite-dimensional Hilbert space since the cross-commutators on it would no longer be in $S_1$. However, if we consider a finite direct sum of block weighted shifts satisfying the analogues of (")", (**)$_1$, then the result does carry over and the blocks could be infinite, so long as the cross-commutators are still in $S_1$.

While most natural examples of multi-variate Hilbert modules are not 1-reductive, one can obtain a family of nontrivial examples in the context of commuting weighted shifts.

Example 5. For $m > 1$, if the weight set is taken to be: $\lambda_\alpha = \{(1 + \alpha_1 + \alpha_2 + \cdots + \alpha_m)!!\}^{-\delta}$, then $\mathcal{M}_\Lambda$ is 1-reductive if $\delta > \frac{m-1}{2}$ and the $Z_i$ are in $S_2$ if $\delta > \frac{m}{2}$. Thus $\mathcal{M}_\Lambda$ is a nontrivial example of a 1-reductive Hilbert module for $\delta$ satisfying $\frac{m}{2} \geq \delta > \frac{m-1}{2}$ and Theorem 2 applies.

As we have indicated, originally we had hoped that Theorem 0 would extend to $S_p$, $p > 1$, but as Example 5 indicates, this is not the case. Another approach would be to represent either the submodule or the corresponding quotient module as the kernel or cokernel of a closed module map to which Theorem 1 applies. The difficulty here is that the module map must have closed range and we know few conditions that guarantee that.

Since most natural examples of multivariate Hilbert modules are $p$-reductive only for $p > 1$, this approach reveals little about the validity of Arveson’s conjecture in general either for $H^2_m$ or other natural examples. Even though that is the case, let us describe what we believe is a natural setting for the conjecture.
Let \( \Omega \) be a bounded, strongly pseudo-convex domain in \( \mathbb{C}^m \) with smooth boundary and \( B^2(\Omega) \) be the Bergman space, that is, the subset of functions \( f \) in \( L^2(\Omega) \) relative to volume measure for which \( \bar{\partial}f = 0 \) taken in the sense of distributions. One knows [30] that the module action on \( B^2(\Omega) \) by functions holomorphic on a neighborhood of the closure of \( \Omega \) is \( p \)-reductive for \( p > m \). That is, cross-commutators of these multiplication operators and their adjoints lie in \( S_p \). For \( Z \) a variety of \( \Omega \), let \( B^2_Z(\Omega) \) be the functions in \( B^2(\Omega) \) that vanish on \( Z \) and let \( Q_Z \) be the quotient module \( B^2(\Omega)/B^2_Z(\Omega) \) (cf. [18]). One can show that \( Q_Z \) is a contractive Hilbert module over \( A(\Omega) \) with support in the closure of \( Z \). Moreover, since \( B^2(\Omega) \) is a kernel Hilbert space and evaluation at \( z \) in \( \Omega \) is continuous, there is a vector \( k_z \) in \( B^2(\Omega) \) for which \( f(z) = \langle f, k_z \rangle_{B^2(\Omega)} \) for \( f \) in \( B^2(\Omega) \). The vectors \( \{k_z\} \) are joint eigenvectors for the adjoint of the operators defined by the module action. Moreover, one has that \( Q_Z \) is the closed span of \( \{k_z | z \in Z\} \). Therefore, this example satisfies the same kind of hypotheses as in Theorem 2.

More generally, one can see that one could consider any submodule of \( B^2(\Omega) \) defined as the orthogonal complement of a collection of eigenvectors \( \{k_z\} \) and their partial derivatives, which are also generalized eigenvectors for the adjoint of module action. These submodules include the closures of ideals in the algebra of functions holomorphic on some neighborhood of the closure of \( \Omega \). In particular, one can consider not just the functions that vanish on a subvariety but those that vanish to higher order. Moreover, using the result in [16] we see that if these submodules are \( p \)-reductive for \( p > m \), then the quotient module obtained from them are also \( p \)-reductive for \( p > m \).

Although the evidence for such a result is perhaps scant we are optimistic enough to formulate:

**Conjecture 6.** If \( S \) is a submodule of \( B^2(\Omega) \) such that \( S^\perp \) is spanned by joint generalized eigenvectors for the adjoint of the operators defined by the module action, then both \( S \) and \( S^\perp \) are \( p \)-reductive for \( p > m \).

This result, even in the multiplicity one case, would be of considerable interest. For a submodule obtained as the closure of a principal ideal \( I \) in \( \mathbb{C}[z] \), the result is equivalent to the weighted Bergman space defined for the measure \( |p|^2 d \text{Vol} \) on \( \Omega \) being \( p \)-reductive for \( p > m \), where \( p(z) \) is a generator for \( I \). However, one might expect, if Conjecture 6 holds, for the generalization to finite multiplicity to also be valid.

**Conjecture 7.** The same conclusion as in Conjecture 6 for submodules of \( B^2(\Omega) \otimes \mathbb{C}^k \).

There is an even stronger result possible which would be very useful in our considerations of the following section. (See [17] and [19] for the necessary definitions.)

**Conjecture 8.** If \( \mathcal{M} \) is a finite rank, quasi-free, \( p \)-reductive Hilbert module over \( A(\Omega) \) and \( S \) is a submodule for which \( S^\perp \) is spanned by generalized eigenvectors for the adjoint of the
operators defined by the module action, then $S$ and $S^\perp$ are $p$-reductive.

It is quite likely that some additional “regularity” hypotheses on $\mathcal{M}$ are necessary for the last conjecture to hold.

There is another way to frame the final conjecture using a notion introduced in [17]. Recall that a Hilbert module $\mathcal{M}$ is said to belong to class $(PS)$ if it is spanned by the generalized eigenvectors for the adjoint of the operators defined by the module action.

**Conjecture 8′.** Let $\mathcal{H}$ be a finite rank quasi-free, $p$-reductive Hilbert module over the algebra $A(\Omega)$. If $\mathcal{M}$ is a submodule of $\mathcal{H}$ such that $\mathcal{H}/\mathcal{M}$ belongs to the class $(PS)$, then $\mathcal{M}$ is $p$-reductive.

### 3 $K$-Homology Classes

Let $\mathcal{H}$ be a $p$-reductive Hilbert module over the algebra $A$ and $\mathcal{J}(\mathcal{H})$ be the $C^*$-algebra generated by the operators defined by module multiplication on $\mathcal{H}$ and let $C(\mathcal{H})$ be the commutator ideal in $\mathcal{J}(\mathcal{H})$. Then $C(\mathcal{H})$ consists of compact operators and hence $(\mathcal{J}(\mathcal{H}) + K(\mathcal{H}))/K(\mathcal{H})$ is a commutative $C^*$-algebra. Therefore this quotient $C^*$-algebra is isometrically isomorphic to $C(X_\mathcal{H})$ for some compact Hausdorff space $X_\mathcal{H}$. In [15], it is shown for $A$ a commutative Banach algebra that $X_\mathcal{H}$ can be identified with a closed subset of the maximal ideal space $M_A$. Similarly, if $A = \mathbb{C}[z]$ and the module action of the coordinate functions are all contractive operators, then one can identify $X_\mathcal{H}$ as a closed subset of the unit polydisk $\partial \mathbb{D}^m$.

In any case, since we have the short exact sequence $0 \to K(\mathcal{H}) \to \mathcal{J}(\mathcal{H}) + K(\mathcal{H}) \to C(X_\mathcal{H}) \to 0$, one always obtains an odd $K$-homology element, denoted $[\mathcal{H}]$, in $K_1(X_\mathcal{H})$. While we hope to investigate these classes more thoroughly after additional cases of the conjecture have been established, we want to draw attention here to a few natural questions and raise a few more conjectures. Our aim is to show why these are interesting questions. We focus on the case of Bergman spaces over strongly pseudo-convex domains with smooth boundary.

**Theorem 3.** Let $\Omega$ be a bounded strongly pseudo-convex domain in $\mathbb{C}^m$ with smooth boundary, $B^2(\Omega)$ be the Bergman module, $\mathcal{Z}$ be a subvariety of $\Omega$, $B^2_\mathcal{Z}(\Omega)$ be the submodule of functions in $B^2(\Omega)$ that vanish on $\mathcal{Z}$ and $Q_\mathcal{Z}$ be the quotient module $B^2(\Omega)/B^2_\mathcal{Z}(\Omega)$. If $Q_\mathcal{Z}$ is a $p$-reductive module for the algebra of functions holomorphic on some neighborhood of $\partial \Omega$, then $[Q_\mathcal{Z}]$ is in $K_1(\mathcal{Z} \cap \partial \Omega)$.

**Proof.** The only thing requiring proof is the fact that $X_{Q_\mathcal{Z}} \subseteq \mathcal{Z} \cap \partial \Omega$. This follows from the fact that $X_{B^2(\Omega)} = \partial \Omega$ and that $B^2_\mathcal{Z}(\Omega)$ is a Hilbert module over $A(\Omega)/A_\mathcal{Z}(\Omega)$. $\square$

The question arises as to which element of $K_1(\mathcal{Z} \cap \partial \Omega)$ is obtained. One can show in some cases such as $\Omega = \mathbb{B}^m$ that it is the fundamental class, taking multiplicity into account,
determined by the complex structure on $\Omega$ or the spin$^c$-structure on $\partial \Omega$ (or the negative of these classes) and I conjecture that this is true in general.\textsuperscript{2} One problem which arises is that $\partial \Omega \cap Z$ need not be a manifold.

One can show by various means that the $K_1$-classes determined by $B^2(\mathbb{B}^m)$ and $H^2_m$ are equal. In fact, the same seems to be true for any kernel Hilbert module over $\mathbb{B}^m$ that is essentially reductive. (An argument showing this fact would follow from Conjecture 8.) I suspect the same thing is true for the $K_1$-classes obtained for a subvariety $Z$, that is, the $K_1$-class doesn’t depend on the kernel Hilbert module over $\Omega$ with which one begins.

Finally, there is one other issue I would like to raise before concluding. We will again frame it in the context of submodules of Bergman modules. Although one can show that $B^2_2(\Omega)$ is $p$-reductive for $p > m$, it is not $p$-reductive for any smaller $p$. That is, it has the same degree of “smoothness” (cf. \textsuperscript{4}) as does $B^2(\Omega)$. However, I don’t believe that is the case for $Q_Z$. In particular, in \textsuperscript{10}, I showed that its smoothness depends on the dimension of $Z$ or the degree of the Hilbert polynomial \textsuperscript{22} for $Q_Z$. I will formulate one final conjecture, that an analogous result holds in general. We state it only for the case of the unit ball.

\textbf{Conjecture 9.} Let $I$ be an ideal in $\mathbb{C}[z]$ and $S$ be the submodule obtained from its closure in $B^2(\mathbb{B}^m)$. Then the quotient module $B^2(\mathbb{B}^m)/S$ is $q$-reductive for $q > \dim(Z \cap \mathbb{B}^m)$, where $Z$ is the zero variety of $I$.\textsuperscript{4}

There is another line of investigation possible here if this conjecture holds. If $Q_Z$ is $p$-reductive for $q > \dim(Z \cap \mathbb{B}^m)$, then it should be possible to define a cyclic cohomology class following Connes \textsuperscript{14} which will be the Chern character of $[Q_Z]$. One interesting question is how this class varies when the subvariety $Z$ changes. For example, suppose one considers $Z_c = \{z \in \Omega | \rho(z) = c\}$ for $c$ in $\mathbb{C}$. As one knows, for some $c$, $Z_c \cap \partial \Omega$ will be a manifold while for others, it is not. Moreover, there is also the issue of $\partial \Omega \cap Z_c$ being a manifold while $Z_c$ has singularities in $\Omega$.

One fascinating example to consider would be the presentation of the exotic spheres found by Brieskorn \textsuperscript{12}. Recall that he exhibited analytic polynomials for which an exotic sphere is obtained from the intersection of the zero variety of the polynomial in $\mathbb{C}^n$ with spheres of small diameter. Although the precise polynomials he used are not homogeneous, this example indicates that one is likely to obtain interesting varieties in our context.

I believe different techniques will be needed to establish such a conjecture. The result in \textsuperscript{21} provides a lower bound on $p$ if $[Q_Z]$ is indeed a fundamental class for $\partial \Omega \cap Z$.

\textsuperscript{2}I thank Paul Baum for discussions on how to define such a $K$-homology class which is related to our earlier work \textsuperscript{17}.

\textsuperscript{3}In \textsuperscript{27} the $K$-homology class obtained for homogeneous modules in $B^2(\mathbb{H}^2)$ is consistent with this conjecture.

\textsuperscript{4}In \textsuperscript{27} this conjecture is verified in case $\dim(Z \cap \mathbb{B}^m) \leq 1$. 

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