Introductory remarks

Understanding the nature of confinement has been a challenging task for many years. Several scenarios with different candidates for the relevant gluonic excitations were proposed, but no closed picture has emerged yet (it is even still debated whether confinement is predominantly an infrared or an ultraviolet phenomenon). Also a connection of confinement to chiral symmetry and its breaking has been conjectured, but not been shown either.

In recent work [1, 2, 3] we have explored the idea of connecting quantities sensitive to confinement to spectral sums for Dirac and covariant Laplace operators. These ideas were developed further in [4, 5, 6] where it was shown that also other quantities such as quark propagators and heat kernels may be turned into order parameters for the breaking of center symmetry. Spectral sums provide a natural decomposition into infrared (IR) and ultraviolet (UV) parts and allow one to analyze their respective role in confinement, as studied numerically using quenched [2, 3, 4, 5, 6] and dynamical [7] lattice configurations.

In this letter we build on those results and develop a new order parameter for center symmetry. In particular we Fourier transform the quark condensate (that turns into the chiral condensate in the massless limit) with respect to a U(1)-valued temporal boundary condition. This procedure corresponds to an equivalence class of Polyakov loops, thereby being an order parameter for the center symmetry. We explore the duality relation between the quark condensate and these dressed Polyakov loops numerically, using quenched lattice QCD configurations below and above the QCD phase transition. It is demonstrated that the Dirac spectrum responds differently to changing the boundary condition, in a manner that reproduces the expected Polyakov loop pattern. We find the dressed Polyakov loops to be dominated by the lowest Dirac modes, in contrast to thin Polyakov loops investigated earlier.

PACS numbers: 12.38.Aw, 11.15.Ha, 11.10.Wx

Dual quark condensate

We construct a new order parameter for finite temperature QCD by considering the quark condensate for U(1)-valued temporal boundary conditions for the fermions. Fourier transformation with respect to the boundary condition defines the dual condensate. This quantity corresponds to an equivalence class of Polyakov loops, thereby being an order parameter for the center symmetry. We explore the duality relation between the quark condensate and these dressed Polyakov loops numerically, using quenched lattice QCD configurations below and above the QCD phase transition. It is demonstrated that the Dirac spectrum responds differently to changing the boundary condition, in a manner that reproduces the expected Polyakov loop pattern. We find the dressed Polyakov loops to be dominated by the lowest Dirac modes, in contrast to thin Polyakov loops investigated earlier.

Dual quark condensate

We begin our discussion with recalling the definition of the chiral condensate \( \Sigma \). The starting point is the scalar dependence of the IR modes on the boundary condition. The dressed Polyakov loop can be written as a spectral sum for Dirac eigenvalues. Using quenched SU(3) gauge configurations below and above the critical temperature, we show that dressed loops are IR dominated (in contrast to conventional thin Polyakov loops [2]) and that the deconfinement transition is signaled by a changing dependence of the IR modes on the boundary condition.

Center transformation property of the dressed Polyakov loop is independent of the quark mass parameter \( m \). It is particularly interesting that the mass parameter \( m \) may be used to relate the chiral condensate and the conventional thin Polyakov loop. In the limit where the quark mass parameter \( m \) is sent to infinity, the dressed Polyakov loop reduces to the thin loop that winds once. Conversely, for the more interesting limit \( m \to 0 \) we recover the dual of the conventional chiral condensate. We stress that this dual has a vanishing value below \( T_c \), while it acquires a non-zero value above the transition.

In the second step the fermions were integrated out and the remaining expectation value \( \langle \Sigma(m, V) \rangle \) of the fermion bilinear \( \bar{\psi}\psi \) evaluated at finite volume \( V \) and mass \( m \),

\[
\Sigma(m, V) = -\int \frac{d^4x}{V} \left\langle \bar{\psi}(x)\psi(x) \right\rangle = \frac{1}{V} \left\langle \text{Tr} \left[ (m + D)^{-1} \right] \right\rangle_G.
\]

In the second step the fermions were integrated out and the remaining expectation value \( \langle \ldots \rangle_G \) is the path integral over the gauge fields with gauge action and fermion determinant included in the weight factor. \( D \) denotes the Dirac operator at vanishing quark mass. We refer to \( \Pi \) as the ”quark condensate”. The chiral condensate, i.e., the proper order parameter for chiral symmetry breaking is obtained through a double limit, where first the 4-volume \( V \) is sent to infinity and then the quark mass \( m \) to zero: \( \Sigma = \lim_{m \to 0} \lim_{V \to \infty} \Sigma(m, V) \).

An important result, that we will return to later, is the Banks-Casher formula [9] which relates the chiral condensate to the density \( \rho(0) \) of eigenvalues at the origin:
\( \Sigma = \pi \rho(0) \). Below the critical temperature \( T_c \), where chiral symmetry is broken (\( \Sigma \neq 0 \)), the eigenvalue density \( \rho(0) \) at the origin is non-vanishing, while above \( T_c \) the Dirac spectrum develops a gap, \( \rho(0) \) vanishes and chiral symmetry is restored (\( \Sigma = 0 \)). Chiral symmetry breaking is a feature of full QCD, i.e., massless fermions are taken into account. However, the above discussed mechanism of a non-vanishing spectral density \( \rho(0) \) below \( T_c \) and a spectral gap above \( T_c \) is known to hold also for the quenched case, i.e., pure gauge theory (see, e.g., [10] for a lattice study of this property).

We now define the dual quark condensate which we will later identify as an order parameter for center symmetry. We work in a finite Euclidean volume with temporal extent \( \beta = 1/k_BT \). For the fermion fields \( \psi \) we use the generalized temporal boundary conditions \( \psi(\vec{x}, \beta) = e^{i\varphi} \psi(\vec{x}, 0) \). The canonical choice is antiperiodic, i.e., \( \varphi = \pi \), while here we allow for arbitrary values \( \varphi \in [0, 2\pi] \). For the spatial directions the fermions obey periodic boundary conditions, and the gauge fields are periodic in all four directions.

Now the quark condensate is considered for an arbitrary boundary angle \( \varphi \) indicated by a subscript for the Dirac operator. The "dual quark condensate" \( \Sigma_n \) is defined as the Fourier transform with respect to \( \varphi \),

\[
\overline{\Sigma}_n(m, V) = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\varphi n} \left\langle \text{Tr} \left[ (m + D_\varphi)^{-1} \right] \right\rangle_G ,
\]

(2)

where the index \( n \) is an integer. This gauge invariant quantity, in particular the case \( n = 1 \), will be investigated analytically and numerically now.

**Relation to dressed Polyakov loops**

Using the lattice regularization, we now discuss the relation of the dual condensate to equivalence classes of Polyakov loops (dressed Polyakov loops). To be specific, we use the staggered lattice Dirac operator (other lattice Dirac operators give equivalent results)

\[
D_{xy} = \sum_{\mu=1}^{4} \frac{\eta_\mu(x)}{2a} \left[ U_\mu(x) \delta_{x+\mu, y} - U_\mu(x-\mu) \right] \delta_{x-\mu, y} ,
\]

(3)

with the staggered sign function \( \eta_\mu(x) = (-1)^{x_1 + \ldots + x_{\mu-1}} \). The coordinates \( x, y \) run over all sites of a 4-dimensional \( L^3 \times N_t \) lattice with lattice spacing \( a \). The gauge link variables \( U_\mu(x) \) are elements of the gauge group \( SU(N) \).

The \( U(1) \)-valued temporal fermionic boundary conditions are most conveniently introduced by attaching the boundary phase to the temporal link on the last time-slice, \( U_4(\vec{x}, N_t) \rightarrow e^{i\varphi} U_4(\vec{x}, N_t) \) (at the moment \( \varphi \) is held fixed and the Fourier transformation of [4] is performed only later in Eq. (3)). Inserting this into the Dirac operator, we evaluate the propagator in [4] for sufficiently large \( m \) as a geometric series

\[
\text{Tr} \left[ (m + D_\varphi)^{-1} \right] = \frac{1}{m} \sum_{k=0}^{\infty} \frac{(-1)^k}{m^k} \text{Tr} \left[ (D_\varphi)^k \right] .
\]

(4)

The Dirac operator [3] contains only terms that connect nearest neighbors. The power \( (D_\varphi)^k \) corresponds to a chain of \( k \) hops. The trace in [4] is over color- and spacetime indices. The latter trace implies that the chains of loops have to form closed loops \( l \) (thus on a lattice with even numbers of sites in all directions \( k \) must be even).

Consequently, the sum in [4] can be reexpressed as a sum over the set \( \mathcal{L} \) of all possible closed loops on the lattice,

\[
\text{Tr} \left[ (m + D_\varphi)^{-1} \right] = \frac{1}{m} \sum_{l \in \mathcal{L}} \frac{s(l)}{(2am)^{|l|}} \text{Tr} \left[ \prod_{(x, \mu) \in \mathcal{L}} U_\mu(x) \right] .
\]

(5)

The remaining trace \( \text{Tr}_c \) is over the color indices of the ordered product of all link variables \( U_\mu(x) \) in a loop \( l \), and we use \( U_{-\mu}(x) = U_\mu(x-\mu) \). By \( s(l) \) we denote the sign of a particular loop \( l \) which is obtained as product of the staggered sign factors. Each step in the loop comes with a factor of \( 1/2am \) from the discretization in [3] and the normalization in [4]. The number of steps, i.e., the length of the loop is denoted by \( |l| \). The loops may close around the boundary. When they close around the temporal boundary, they pick up a factor of \( \exp(i\varphi) \) if they run forward in time, and a factor of \( \exp(-i\varphi) \) for backward running. Denoting the number of times a loop \( l \) winds around the compact time direction by its winding number \( q(l) \in \mathbb{Z} \), we obtain the factor \( \exp(iq(l)) \) in [5].

This is how the boundary angle \( \varphi \) can be used to distinguish between closed loops of different winding number [1]. When expression [5] is inserted into the formula [4] for the dual condensate, the \( \varphi \)-integration with the additional Fourier factor \( \exp(-iqn) \) projects to loops of a particular winding number \( n \). We finally obtain

\[
\overline{\Sigma}_n(m, V) = \frac{1}{V} \sum_{l \in \mathcal{L}(n)} \frac{s(l)}{(2am)^{|l|}} \left\langle \text{Tr} \left[ \prod_{(x, \mu) \in \mathcal{L}} U_\mu(x) \right] \right\rangle_G ,
\]

(6)

where the sum now runs over the set \( \mathcal{L}(n) \) of loops that wind \( n \)-times around the compact time direction. The case of \( n = 1 \), i.e., the dual condensate \( \overline{\Sigma}_1(m, V) \) which corresponds to loops that wind exactly once, is what we refer to as the "dressed Polyakov loop". From [6] it is obvious that in the large-\( m \) limit the dominant contribution is the conventional thin Polyakov loop (as this is the shortest loop winding once).

Let us finally discuss the behavior of the dual condensates under a center transformation, \( U_4(\vec{x}, t_0) \rightarrow z U_4(\vec{x}, t_0) \), where all temporal links on a time-slice, i.e., at some fixed \( t_0 \), are multiplied with an element \( z \) of the center of the gauge group. The gauge action and the measure are invariant under this transformation, but the center symmetry is broken spontaneously at the critical temperature \( T_c \), signaling the transition to the deconfined phase [8]. Both the thin Polyakov loop and the dressed Polyakov loop are order parameters for center symmetry. More generally a loop that winds \( n \)-times around the compact time direction picks up a net factor of \( z^n \) and thus our dual condensates transform under a center transformation as \( \overline{\Sigma}_n \rightarrow z^n \overline{\Sigma}_n \). In particular the dressed
Polyakov loop, i.e., the dual condensate for $n = 1$, transforms as $\Sigma_1 \to z \Sigma_1$, which is the same transformation law as for the thin Polyakov loop.

Spectral sums and numerical analysis

An elegant way to express the dual condensate (2) is as a spectral sum over all Dirac eigenvalues $\lambda_\varphi(i)$ (again evaluated for boundary angle $\varphi$),

$$\Sigma_n(m, V) = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\varphi n} \sum_i \left( (m + \lambda_\varphi(i))^{-1} \right). \quad (7)$$

First of all this spectral representation consists of a finite sum over the eigenvalues (due to the regularization on a finite lattice), whereas the representation (6) contains an infinite sum over loops (even on a finite lattice).

Secondly, the sum on the rhs. of (7) can easily be computed numerically and then allows for the appealing possibility to study the role of individual parts of the spectrum. Since the (purely imaginary) eigenvalues appear in the denominator, we expect the dual condensate to be dominated by IR modes (for small mass $m$) and thus to have a well-defined continuum limit (see also the discussion of the convergence of the spectral sums in [8]). We stress that the above discussed properties under center transformations are independent of the mass parameter $m$. Naturally one is interested in small values of $m$, where the spectral sums are IR dominated. For the Figures 1 and 2 below we use masses near the strange quark mass and only in Fig. 3, where we illustrate the transition to the large mass behavior, we increase the mass to $m = 1$ GeV.

For our numerical analysis of the spectral representation [11] we compute complete spectra of the staggered Dirac operator [3] at different boundary angles $\varphi$ with parallel LAPACK routines. We use quenched SU(3) gauge configurations generated from the Lüscher-Weisz gauge action [12]. The scale was determined [13] from the Sommer parameter setting $r_0 = 0.5$ fm. We use $L^3 \times N_4$ lattices with $L$ ranging from 8 to 14 and $N_4$ from 4 to 8, and adjust the couplings such that we have ensembles below and above the critical temperature $T_c$ $\sim$ 300 MeV. For the gauge configurations above $T_c$ the thin Polyakov loop has a non-vanishing expectation value and for our numerical study we use configurations where the thin Polyakov loop is real. Below we discuss how the spectral sum for the dressed Polyakov loop may generate also the two possible complex phases. All errors shown are statistical errors from the Jackknife method.

We begin with analyzing the $\varphi$-dependence of the integrand in (7). In Fig. 1 we plot this integrand versus $\varphi$ for two values of $am$ comparing an ensemble below $T_c$ to one above $T_c$. It is obvious that below $T_c$ the integrand is essentially constant, while above $T_c$ it shows a pronounced cosine type of behavior. Integrating over $\varphi$ with the weight $\exp(-i\varphi)$, i.e., $n = 1$, gives a vanishing dressed Polyakov below $T_c$, while above $T_c$ a non-vanishing value is observed. We conclude that the transition from confinement to deconfinement leads to a different response of the spectral sums to the changing temporal fermion boundary conditions. Below we will demonstrate that the IR modes play the dominant role in this process.

Above $T_c$, when using instead of an ensemble with real thin Polyakov loop, configurations where the thin Polyakov loop has one of the two complex phases, the integrand is shifted by $\pm 2\pi/3$, but otherwise has the same form as depicted in Fig. 1. Integrating this shifted integrand shows that the dressed Polyakov loop produces the same $Z_3$ phase pattern as the thin Polyakov loop.

Now we demonstrate that the dressed Polyakov loop does indeed signal the phase transition. In Fig. 2 we show the results for the dressed Polyakov loop at $m = 100$ MeV as a function of $T$. The necessary $\varphi$-integration was implemented with the extended Simpson rule using typically 8 or 16 values of $\varphi$. It is obvious that below $T_c$ the dressed Polyakov loop vanishes, while above $T_c$ it assumes a non-vanishing value, signaling that the center
symmetry is broken in the deconfining phase. Another interesting connection between quark condensate and dressed Polyakov loop is obtained by using the Banks-Casher type of representation already addressed above. After performing the consecutive limits of infinite volume and vanishing mass, the chiral condensate can be written as the density of eigenvalues at the origin also for arbitrary boundary angle $\varphi$. The dual condensate is then obtained by integrating the $\varphi$-dependent spectral density $\rho(\varphi)$, and for the case of $n = 1$ we find

$$\tilde{\Sigma}_1 = \int_0^{2\pi} \frac{d\varphi}{2} e^{-i\varphi} \rho(\varphi).$$

Below $T_c$ the spectral density at the origin is constant as a function of $\varphi$ and a vanishing dressed Polyakov loop emerges. More interesting is the situation above $T_c$, where a non-trivial $\varphi$ dependence is necessary for a non-vanishing dressed Polyakov loop. Naively one would think that above $T_c$ the spectral density at the origin must be zero, such that the chiral condensate may vanish. However, in [11] (for a different phase convention) it was shown that the spectral gap depends on the relative phase between the boundary angle $\varphi$ and the phase $\theta$ of the Polyakov loop. If $\varphi$ equals the negative Polyakov loop phase the gap closes completely, giving rise to a non-zero spectral density. Inserting $\rho(\varphi) \propto \delta(\varphi + \theta)$ in (8) one obtains a non-vanishing dressed Polyakov loop above $T_c$ with the correct phase $\theta$.

Let us now consider the individual contributions $C(\lambda) = (2\pi V)^{-1} \int d\varphi \exp(-i\varphi)(m + \lambda \varphi)^{-1}/\varphi$ to the spectral sum (7). The upper plots of Fig. 3 show $|C(\lambda)/\tilde{\Sigma}_1|$ as a function of $|\lambda|$. Since the case of vanishing quark mass $m$ can be obtained only through a limiting procedure, we compare two different values of $m$. In both cases the largest contributions come from the IR end of the spectrum. However, for the smaller mass the contributions beyond the deep IR have died out completely.

When considering the relative role of IR and UV contributions, one must take into account that the density of eigenvalues increases strongly with increasing $|\lambda|$. Thus in the lower plots of Fig. 3 we show the accumulated contribution $A(|\lambda|) = \sum_{|\lambda| \leq |\lambda|} C(\lambda)$ to the final value of $1$ becomes monotonic, and only the IR modes give sizable contributions to the spectral sum for the dressed Polyakov loop. The scale up to which eigenvalues are relevant grows with $T$.

Concluding remarks

In this letter we have shown that a duality transformation of the quark condensate with respect to the fermionic temporal boundary condition gives rise to an order parameter for center symmetry. This order parameter can be viewed as a set of closed loops with the same winding number around compact time. For the case of single winding, which we refer to as the dressed Polyakov loop, our observable interpolates between the quark condensate (via a Fourier transform) and the thin Polyakov loop, in the limits of vanishing and infinite mass, respectively.

We studied the corresponding spectral sums of Dirac operator eigenvalues numerically for quenched gauge configurations. It was shown that the transition from the confined to the deconfined phase is seen as a different dependence of the spectral sums on the fermionic boundary condition. Decomposing the spectral sum shows that the main signal comes from the IR part of the spectrum.

As an outlook we stress that our theoretical considerations also hold for other gauge groups (where, e.g., for SU(2) the phase transition can be of different order) and for the case of dynamical quarks, the consequences of which would be interesting to study numerically.

Acknowledgments

We thank C.B. Lang, K. Langfeld, W. Söldner, T. Kovacs, P. van Baal, J. Verbaarschot and A. Wipf for valuable discussions. The numerical analysis was done at the ZID, University of Graz. E.B. and C.G. are supported by NAWI-GASS and the FWF (DK WI2033-N08 and P 20330-N16) and F.B. by DFG (BR 2872/4-1).
[1] C. Gattringer, Phys. Rev. Lett. 97, 032003 (2006).
[2] F. Bruckmann, C. Gattringer, C. Hagen, Phys. Lett. B 647, 56 (2007).
[3] C. Hagen, F. Bruckmann, E. Bilgici, C. Gattringer, PoS(LATTICE 2007)289 [arXiv:0710.0294 [hep-lat]].
[4] F. Synatschke, A. Wipf, C. Wozar, Phys. Rev. D 75, 114003 (2007).
[5] F. Synatschke, A. Wipf and K. Langfeld, [arXiv:0803.0271 [hep-lat]].
[6] E. Bilgici, C. Gattringer, [arXiv:0803.1127] [hep-lat].
[7] W. Söldner, PoS(LATTICE2007) 222.
[8] L. McLerran, B. Svetitsky, Phys. Rev. D 24, 450 (1981).
[9] T. Banks, A. Casher, Nucl. Phys. B 169, 103 (1980).
[10] C. Gattringer, P. E. L. Rakow, A. Schäfer, W. Söldner, Phys. Rev. D 66, 054502 (2002).
[11] C. Gattringer, S. Schaefer, Nucl. Phys. B 654, 30 (2003).
[12] M. Lüscher, P. Weisz, Commun. Math. Phys. 97, 59 (1985); E: 98, 333 (1985); G. Curci, P. Menotti, G. Paffuti, Phys. Lett. B 130, 205 (1983); E: B 135, 516 (1984).
[13] C. Gattringer, R. Hoffmann, S. Schaefer, Phys. Rev. D 65, 094503 (2002).