Green functions and twist correlators for $N$ branes at angles.

Igor Pesando

$^1$Dipartimento di Fisica, Università di Torino and I.N.F.N. - sezione di Torino
Via P. Giuria 1, I-10125 Torino, Italy

ipesando@to.infn.it

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Abstract

We compute the Green functions and correlator functions for $N$ twist fields for branes at angles on $T^2$ and we show that there are $N - 2$ different configurations labeled by an integer $M$ which is roughly associated with the number of reflex angles of the configuration. In order to perform this computation we use a $SL(2,\mathbb{R})$ invariant formulation and geometric constraints instead of Pochammer contours. In particular the $M = 1$ or $M = N - 1$ amplitude can be expressed without using transcendental functions. We determine the amplitudes normalization from $N \rightarrow N - 1$ reduction without using the factorization into the untwisted sector. Both the amplitudes normalization and the OPE of two twist fields are unique (up to one constant) when the $\epsilon \leftrightarrow 1 - \epsilon$ symmetry is imposed. For consistency we find also an infinite number of relations among Lauricella hypergeometric functions.

keywords: D-branes, Conformal Field Theory

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1 Introduction and conclusions

Since the beginning, D-branes have been very important in the formal development of string theory as well as in attempts to apply string theory to particle phenomenology and cosmology. However, the requirement of chirality in any physically realistic model leads to a somewhat restricted number of possible D-brane set-ups. An important class are intersecting brane models where chiral fermions can arise at the intersection of two branes at angles. An important issue for these models is the computation of Yukawa couplings and flavour changing neutral currents.

Besides the previous computations many other computations often involve correlators of twist fields and excited twist fields. It is therefore important and interesting in its own to be able to compute these correlators. As known in the literature [1] and explicitly shown in [2] for the case of magnetized branes these computations boil down to the knowledge of the Green function in presence of twist fields and of the correlators of the plain twist fields.

In this technical paper we have analyzed the $N$ twist fields amplitudes at tree level for open strings localized at $D$-branes intersections on $T^2$ using the classical path integral approach [1]. The subject has been explored in many papers and in both the branes at angles setup and the magnetic branes setup see for example ([12], [3], [4], [5], [6], [7], [8], [9]).

We have shown that there are different sectors with different amplitudes. Sectors are labeled by an integer $M_{cw}$ ($1 \leq M_{cw} \leq N - 2$) and that the number of sectors is equal to the number of reflex angles formed by the brane configuration. This means that for example all the configurations in fig. (1) have different amplitudes. In particular the quantum amplitudes with $M_{cw} = 1$ can be expressed using elementary functions only. This result generalizes the result previously obtained for both four point amplitudes ([6], [7]) and for the $N$ point amplitudes [5] where only the special case $M = N - 2$ were considered. Since the $N = 4 M = 2$ amplitude has also been obtained by a different approach in ([10], [11]). it would be interesting to understand how this can come about in this different setup.

We have also obtained the normalizations (up to one constant) of both two twist fields OPE and amplitudes. This result has been achieved using three ingredients: the consistency of the $N$ twist fields correlator factorization into $N - 1$ twist fields one, the canonical normalization of the 2 twist correlator $\langle \sigma_{\epsilon}(x)\sigma_{1-\epsilon}(y) \rangle = 1/(x-y)^{(1-\epsilon)}$ and the assumption of the symmetry of under $\sigma_{\epsilon} \leftrightarrow \sigma_{1-\epsilon}$.

Finally we have computed the Green functions in presence of $N$ twist fields and we have shown that in order to do so there needs three different kinds of derivatives instead of the usual two which are needed in the closed string case.

This paper is organized as follows. In section 2 we review the geometrical framework of branes at angles and we fix our conventions. In this section we discuss carefully how to make use of the doubling trick in presence of multiple cuts and the existence of local and global constraints. In section 3 we show the
existence of $N - 2$ different sectors and compute the corresponding classical solutions. We show also explicitly the results for the $N = 3$ and $N = 4$ cases. Moreover using the known relation between closed string and open string amplitudes [13] we express the classical action as a sum of products of holomorphic and antiholomorphic parts. Details on this computation are given in appendix A. In section 4 we compute the Green functions for the different sectors and give explicit expressions for $N = 3$ and $N = 4$ cases. In particular we discuss the existence of infinite relations among polynomial of Lauricella hypergeometric functions which must follow from the consistency of the procedure. Finally in section 5 we compute the quantum correlators of $N$ twists and their normalization factors. In particular we show that the $M_{cw} = 1$ sector amplitudes can be expressed as a product of elementary functions. Moreover we discuss how $N - 1$ twist fields amplitudes can be obtained from $N$ twist fields ones. A mathematically curious consequence is that certain determinant of order $N - 2$ involving Lauricella hypergeometric functions of order $N - 3$ are expressible as product of powers.

2 Review of branes at angles

The Euclidean action for the string configuration is given by

$$S = \frac{1}{4\pi\alpha'} \int d\tau_E \int_0^\pi d\sigma \left( \partial_\alpha X^I \right)^2 = \frac{1}{4\pi\alpha'} \int_H d^2u \left( \partial X \bar{\partial} \bar{X} + \bar{\partial} X \partial \bar{X} \right)$$

Figure 1: The four different cases with $N = 6$. a) $M_{ccw} = 2$ and $M_{cw} = 4$ where $M_{ccw}$ is measured counterclockwise and $M_{cw}$ clockwise. b) $M_{ccw} = 3$ and $M_{cw} = 3$. c) $M_{ccw} = 4$ and $M_{cw} = 2$. d) $M_{ccw} = 5$ and $M_{cw} = 1$. 

where \( u \in H \), the upper half plane, \( d^2 u = e^{2\pi \tau} d\tau d\sigma = \frac{du}{2\pi i} d\bar{u} \) and \( I = 1, 2 \) so that \( X = \frac{1}{\sqrt{2}}(X^1 + iX^2) \), \( \bar{X} = X^* \). The complex string coordinate is a map from the upper half plane to a closed polygon \( \Sigma \) in \( \mathbb{C} \), i.e. \( X : H \rightarrow \Sigma \subset \mathbb{C} \). For example in fig. 2 we have pictured the interaction of \( N = 4 \) branes at angles \( D_i \) with \( i = 1, \ldots N \). The interaction between brane \( D_i \) and \( D_{i+1} \) is at \( f_i \in \mathbb{C} \) where we use the rule that index \( i \) is defined modulo \( N \).

\[
\begin{align*}
\sigma &= \pi \\
\sigma &= 0 \\
D_1 &\quad D_2 &\quad D_3 &\quad D_4
\end{align*}
\]

Figure 2: Map from the Minkowskian worldsheet to the target polygon \( \Sigma \).

### 2.1 The local description

Locally at the interaction point \( f_i \) the boundary conditions for the brane \( D_i \) are given by

\[
Re(e^{-i\pi \alpha_i} X'_L|_{\sigma=0}) = Im(e^{-i\pi \alpha_i} X_{loc}|_{\sigma=0}) - g_i = 0
\]

while those for the brane \( D_{i+1} \) by

\[
Re(e^{-i\pi \alpha_{i+1}} X'_L|_{\sigma=\pi}) = Im(e^{-i\pi \alpha_{i+1}} X_{loc}|_{\sigma=\pi}) - g_{i+1} = 0
\]

with

\[
f_i = \frac{e^{i\pi \alpha_{i+1}} g_i - e^{i\pi \alpha_i} g_{i+1}}{\sin \pi (\alpha_{i+1} - \alpha_i)}
\]

When we write the Minkowskian string expansion as \( X(\sigma, \tau) = X_L(\tau + \sigma) + X_R(\tau - \sigma) \) the previous boundary conditions imply (and not become since they are not completely equivalent because of zero modes)

\[
X'_L \text{ loc}(\xi) = e^{i2\pi \alpha_i} X'_R \text{ loc}(\xi), \quad X'_L \text{ loc}(\xi + \pi) = e^{i2\pi \alpha_{i+1}} X'_R \text{ loc}(\xi - \pi)
\]

or in a more useful way in order to explicitly compute the mode expansion

\[
X'_L \text{ loc}(\xi + 2\pi) = e^{i2\pi \epsilon_i} X'_L \text{ loc}(\xi), \quad X'_R \text{ loc}(\xi + 2\pi) = e^{-i2\pi \epsilon_i} X'_R \text{ loc}(\xi)
\]
where we have defined

\[ \epsilon_i = \begin{cases} 
(\alpha_{i+1} - \alpha_i) & \alpha_{i+1} > \alpha_i \\
1 + (\alpha_{i+1} - \alpha_i) & \alpha_{i+1} < \alpha_i
\end{cases} \tag{7} \]

so that \(0 < \epsilon_i < 1\) and there is no ambiguity in the phase \(e^{2\pi \epsilon_i}\) entering the boundary conditions. The quantity \(\pi \epsilon_i\) is the angle between the two branes \(D_i\) and \(D_{i+1}\) measured counterclockwise as shown in fig. 3. A consequence of this definition is that \(\epsilon\) becomes \(1 - \epsilon\) when we flip the order of two branes. For example the angles in fig. 4 become those in fig. 5 when we reverse the order we count the branes, i.e. when we follow the boundary clockwise instead of counterclockwise the physics must obviously not change.

We introduce as usual the Euclidean fields \(X_{\text{loc}}(u, \bar{u}), \bar{X}_{\text{loc}}(u, \bar{u})\) by a worldsheet Wick rotation in such a way they are defined on the upper half plane by \(u = e^{\tau E + i\sigma} \in \mathbb{H}\). The previous choice of having brane \(D_i\) at \(\sigma = 0\) (2) and brane \(D_{i+1}\) at \(\sigma = \pi\) (3) implies that in the local description where

Figure 3: The connection between \(\epsilon\) and the geometrical angles \(\alpha\)s defining the branes.

Figure 4: A polygon \(\Sigma\) with an reflex angle and branes counted counterclockwise with \(N = 4\) and \(M_{\text{ccw}} = 3\).

Figure 5: A polygon \(\Sigma\) with an reflex angle and branes counted clockwise with \(N = 4\) and \(M_{\text{cw}} = 1\).
the interaction point is at \( x = 0 \) \( D_i \) is mapped into \( x > 0 \) and \( D_{i+1} \) into \( x < 0 \). The boundary conditions \( \{5\} \) can then immediately be written as

\[
\partial X_{\text{loc}}(x + i0^+) = e^{i2\pi \alpha_i} \partial X_{\text{loc}}(x - i0^+) \quad 0 < x, \\
\partial X_{\text{loc}}(x + i0^+) = e^{i2\pi \alpha_{i+1}} \partial X_{\text{loc}}(x - i0^+) \quad x < 0
\]

(8)

and similarly relations for \( \tilde{X} \) which can be obtained by complex conjugation. When we add to the previous conditions the further constraints

\[
X(0, 0) = f_i, \quad \tilde{X}(0, 0) = f_i^* \quad (9)
\]

we obtain a system of conditions which are equivalent to the original ones \( \{2\} \quad \{3\} \).

In order to express the boundary conditions \( \{6\} \) in the Euclidean formulation it is better to introduce the local fields defined on the whole complex plane by the doubling trick as

\[
\partial \chi_{\text{loc}}(z) = \begin{cases} \\
\partial X_{\text{loc}}(u) & z = u \text{ with } \text{Im} \ z > 0 \text{ or } z \in \mathbb{R}^+ \\
e^{i2\pi \alpha_i} \partial \tilde{X}_{\text{loc}}(\bar{u}) & z = \bar{u} \text{ with } \text{Im} \ z < 0 \text{ or } z \in \mathbb{R}^+ 
\end{cases}
\]

\[
\partial \tilde{\chi}_{\text{loc}}(z) = \begin{cases} \\
\partial \tilde{X}_{\text{loc}}(u) & z = u \text{ with } \text{Im} \ z > 0 \text{ or } z \in \mathbb{R}^+ \\
e^{-i2\pi \alpha_i} \partial X_{\text{loc}}(\bar{u}) & z = \bar{u} \text{ with } \text{Im} \ z < 0 \text{ or } z \in \mathbb{R}^+ 
\end{cases} \quad (10)
\]

In this way we can write eqs \( \{6\} \) as

\[
\partial \chi_{\text{loc}}(e^{i2\pi \delta}) = e^{i2\pi \epsilon_i} \partial \chi_{\text{loc}}(\delta), \quad \partial \tilde{\chi}_{\text{loc}}(e^{i2\pi \delta}) = e^{-i2\pi \epsilon_i} \partial \tilde{\chi}_{\text{loc}}(\delta) \quad (11)
\]

Notice that while the two Minkowskian boundary conditions \( \{6\} \) are one the complex conjugate of the other the previous Euclidean ones are independent and each is mapped into itself by complex conjugation therefore the Euclidean classical solutions for \( \chi \) and \( \tilde{\chi} \) are independent.

The quantization of the string with given boundary conditions yields

\[
X_{\text{loc}}(u, \bar{u}) = f_i + \frac{1}{2} \sqrt{2 \alpha'} e^{i\pi \alpha_i} \sum_{n=0}^{\infty} \left[ \frac{\tilde{\alpha}(i)_n}{n + 1 - \epsilon_i} u^{-(n + 1) + \epsilon_i} - \frac{\alpha_{i}^\dagger}{n + \epsilon_i} u^{n + \epsilon_i} \right] \\
+ \frac{1}{2} \sqrt{2 \alpha'} e^{i\pi \alpha_i} \sum_{n=0}^{\infty} \left[ -\frac{\tilde{\alpha}(i)^\dagger}{n + 1 - \epsilon_i} \bar{u}^{n + 1 - \epsilon_i} + \frac{\alpha_{i}^\dagger}{n + \epsilon_i} \bar{u}^{-(n + \epsilon_i)} \right] \\
\tilde{X}_{\text{loc}}(u, \bar{u}) = f_i^* + \frac{1}{2} \sqrt{2 \alpha'} e^{-i\pi \alpha_i} \sum_{n=0}^{\infty} \left[ -\frac{\tilde{\alpha}(i)_n}{n + 1 - \epsilon_i} u^{n + 1 - \epsilon_i} + \frac{\alpha_{i}^\dagger}{n + \epsilon_i} u^{-(n + \epsilon_i)} \right] \\
+ \frac{1}{2} \sqrt{2 \alpha'} e^{-i\pi \alpha_i} \sum_{n=0}^{\infty} \left[ \frac{\tilde{\alpha}(i)_n}{n + 1 - \epsilon_i} \bar{u}^{-(n + 1 - \epsilon_i)} - \frac{\alpha_{i}^\dagger}{n + \epsilon_i} \bar{u}^{n + \epsilon_i} \right] \quad (12)
\]

with non trivial commutation relations \( (n, m \geq 0) \)

\[
[\alpha_{i}^\dagger_n, \alpha_{i}^\dagger_m] = (n + \epsilon_i) \delta_{m,n}, \quad [\tilde{\alpha}_{i}^\dagger_n, \tilde{\alpha}_{i}^\dagger_m] = (n + 1 - \epsilon_i) \delta_{m,n} \quad (13)
\]
and vacuum defined in the usual way by
\[ \alpha_{(i)n}[T_i] = \bar{\alpha}_{(i)n}[T_i] = 0 \quad n \geq 0 \]  
(14)
The vacuum is then generated from the twist operator \( \sigma_{\epsilon_i,f_i} \) which depends both on the twist \( \epsilon_i \) and on the position \( f_i \in \mathbb{C} \). The dependence on the twist \( \epsilon_i \) can be read f.x. from the OPEs
\[ \partial X(u)\sigma_{\epsilon_i,f_i}(x) \sim (u-x)^{\epsilon_i-1}(\partial X\sigma_{\epsilon_i,f_i})(x) \]
\[ \partial X(u)\sigma_{\epsilon_i,f_i}(x) \sim (u-x)^{-\epsilon_i}(\partial X\sigma_{\epsilon_i,f_i})(x) \]  
(15)
which can be deduced from the local computations
\[ \partial X_{\text{loc}}(u)[T_i] \sim u^{\epsilon_i-1} \left(-\frac{1}{2}i\sqrt{2\alpha'}e^{i\pi \alpha_{i+1}}\alpha_{(i)0}^+|T_i]\right), \quad \partial \bar{X}_{\text{loc}}(u)[T_i] \sim u^{-\epsilon_i} \left(-\frac{1}{2}i\sqrt{2\alpha'}e^{-i\pi \alpha_{i-1}}\bar{\alpha}_{(i)0}^+|T_i\right) \]  
(16)

On the other side the dependence on \( f_i \) can be read from the OPE
\[ e^{ik\cdot X(z,\bar{z})}\sigma_{\epsilon_i,f_i}(x) \sim |z|^{-\alpha'k^2}e^{-\frac{1}{2}R^2(\epsilon_i)\alpha'k^2}e^{ik\cdot f_i}\sigma_{\epsilon_i,f_i}(x) \]  
(17)
which can be deduced from the local computation
\[ |z|^{-\alpha'k^2}e^{-\frac{1}{2}R^2(\epsilon_i)\alpha'k^2}:e^{ik\cdot X_{\text{loc}}(z,\bar{z})}:|T_i\rangle \sim |z|^{-\alpha'k^2}e^{ik\cdot f_i}e^{-\frac{1}{2}R^2(\epsilon_i)\alpha'k^2}|T_i\rangle \]  
(18)
upon the identification \[ \left[15\right] e^{ik\cdot X(z,\bar{z})} \leftrightarrow |z|^{-\alpha'k^2}e^{-\frac{1}{2}R^2(\epsilon_i)\alpha'k^2}:e^{ik\cdot X_{\text{loc}}(z,\bar{z})}: \] with \( R^2(\epsilon_i) = 2\psi(1) - \psi(\epsilon_i) - \psi(1 - \epsilon_i), \psi(z) = \frac{d\ln \Gamma(z)}{dz} \) being the digamma function. Notice that there is no obvious way of computing the angles \( \alpha_i \) and \( \alpha_{i+1} \) from OPEs.

### 2.2 Global description

In the local description, where the interaction point is at \( x = 0 \), \( D_i \) is mapped into \( x > 0 \) and \( D_{i+1} \) into \( x < 0 \) this means that in the global description the world sheet interaction points are mapped on the boundary of the upper half plane so that \( x_{i+1} < x_i \). The global equivalent of the local boundary conditions eq.s \[ \left[8\right] \] become
\[ \partial X_L(x+i0^+) = e^{i2\pi \alpha_i}i\partial \bar{X}_R(x-i0^+) \quad x_i < x < x_{i-1} \]
\[ \partial \bar{X}_L(x+i0^+) = e^{-i2\pi \alpha_i}i\partial X_R(x-i0^+) \quad x_i < x < x_{i-1} \]  
(19)
To the previous constraints one must also add
\[ X_{\text{loc}}(x_i, \bar{x}_i) = f_i, \quad \bar{X}_{\text{loc}}(x_i, \bar{x}_i) = f_i \]  
(20)
in order to get a system of boundary conditions equivalent to the original ones \[ \left[2,3\right] \]. When we introduce the global fields defined on the whole complex
plane by the doubling trick as

\[
\partial \tilde{X}(z) = \left\{ \begin{array}{ll}
\partial \tilde{X}(u) & z = u \text{ with } Im\ z > 0 \text{ or } z \in \mathbb{R} - [x_N, x_2] - [x_1, \infty] \\
e^{-i2\pi \alpha_2} \partial \tilde{X}(\tilde{u}) & z = \tilde{u} \text{ with } Im\ z < 0 \text{ or } z \in \mathbb{R} - [x_N, x_2] - [x_1, \infty]
\end{array} \right.
\]

(21)

the local boundary conditions \([6]\) can be written in the global formulation as

\[
\partial \mathcal{X}(x_i + e^{i2\pi} \delta) = e^{i2\pi \epsilon_i} \partial \mathcal{X}(x_i + \delta)
\]

\[
\partial \tilde{\mathcal{X}}(x_i + e^{i2\pi} \delta) = e^{-i2\pi \epsilon_i} \partial \tilde{\mathcal{X}}(x_i + \delta).
\]

For the proper definition of the global constraints which follow from eq.s \([20]\), for example when dealing with the derivatives of the Green functions as in section \([11]\) it is worth noticing the behavior of the previously introduced fields under complex conjugation when \(z\) is restricted to \(z \in \mathbb{C} - (-\infty, x_2] - [x_1, \infty]\)

\[
[\partial \mathcal{X}(z)]^* = e^{-i2\pi \alpha_2} \partial \mathcal{X}(z \rightarrow \tilde{z}) = \partial \tilde{\mathcal{X}}(\tilde{z}) = \left\{ \begin{array}{ll}
\tilde{\partial} \tilde{X}(\tilde{u}) & \tilde{z} = \tilde{u} \\
e^{-i2\pi \alpha_2} \partial \tilde{X}(u) & \tilde{z} = u
\end{array} \right.
\]

(23)

where \(\partial \mathcal{X}(z \rightarrow \tilde{z})\) means that the holomorphic \(\partial \mathcal{X}(z)\) is evaluated at \(\tilde{z}\).

The previous expressions also show that it is not necessary to introduce the antiholomorphic fields \(\partial \mathcal{X}(\tilde{z})\) and \(\partial \tilde{\mathcal{X}}(\tilde{z})\) which it is possible to construct applying the doubling trick on \(\partial \tilde{X}(\tilde{u})\) and \(\partial \tilde{\mathcal{X}}(\tilde{u})\) respectively.

3 The path integral approach

Following the by now classic method \([11]\) we compute twists correlators by the path integral

\[
\langle \sigma_{\epsilon_1, f_1}(x_1) \ldots \sigma_{\epsilon_N, f_N}(x_N) \rangle = \int_{\mathcal{M}((x_i, \epsilon_i, f_i))} D\mathcal{X} e^{-S_E} \tag{24}
\]

\footnote{It is also possible to perform the doubling trick by defining

\[
\partial \mathcal{X}(z) = \left\{ \begin{array}{ll}
\partial \mathcal{X}(u) & z = u \text{ with } Im\ z > 0 \text{ or } z \in \mathbb{R} - [x_N, x_1] \\
e^{i2\pi \alpha_1} \partial \mathcal{X}(\tilde{u}) & z = \tilde{u} \text{ with } Im\ z < 0 \text{ or } z \in \mathbb{R} - [x_N, x_1]
\end{array} \right.
\]

and similarly for \(\partial \tilde{\mathcal{X}}(z)\) but then all the formulae require a cyclic permutation of the indexes as \(2 \rightarrow 1 \rightarrow N \rightarrow 1\) so that the anharmonic ratio \([28]\) becomes \(\omega_z = \frac{(z-x_2)(x_1-x_N)}{(z-x_N)(x_1-x_2)}\). This is not a cyclic permutation for all indexes, i.e it is not \(i \rightarrow i-1\) and hence all the \(x_{j \neq 1,2,N}\) are mapped to \(\omega_j < 0\), nevertheless \(\sum_{j=2}^{N} \equiv \sum_{j \neq 1} \rightarrow \sum_{j \neq N} \equiv \sum_{j=1}^{N-1}\) where in order to perform the change of indexes we have rewritten \(\sum_{j=2}^{N}\) as \(\sum_{j \neq 1}\) and similarly for the product \(\prod_{j=2}^{N} \rightarrow \prod_{j=1}^{N-1}\). There is also a third possibility and amounts to a cyclic permutation \(2 \rightarrow 1 \rightarrow N \rightarrow N-1 \rightarrow \ldots \rightarrow 2\).}

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where $\mathcal{M}\left(\{x_i, \epsilon_i, f_i\}\right)$ is the space of string configurations satisfying the boundary conditions \([19]\) and \([20]\). Since the integral is quadratic we can then efficiently separate the classical fields from the quantum fluctuations as

$$X(u, \bar{u}) = X_{cl}(u, \bar{u}) + X_q(u, \bar{u})$$

(25)

where $X_{cl}$ satisfies the previous boundary conditions while $X_q$ satisfies the same boundary conditions but with all $f_i = 0$. After this splitting we obtain

$$\langle \sigma_{\epsilon_1, f_1}(x_1) \ldots \sigma_{\epsilon_N, f_N}(x_N) \rangle = N(x_i, \epsilon_i) e^{-S_{E, cl}(x_i, \epsilon_i, f_i)}$$

(26)

The explicit expressions for $X_{cl}$ and $\bar{X}_{cl}$ given in eq.s \([32, 33]\) show that they vanish when $f_i = 0$ hence also the classical action evaluated for $f_i = 0$ $S_{E, cl}(x_i, \epsilon_i, f_i = 0)$ is zero. Actually because of translational invariance what said before works even when all $f_i$ are equal, i.e. when $f_i = f$ and therefore we can identify

$$\langle \sigma_{\epsilon_1, f}(x_1) \ldots \sigma_{\epsilon_N, f}(x_N) \rangle$$

(27)

Our strategy is therefore first to compute the classical contribution in the rest of this section and then compute the quantum contribution in section \([5]\).

### 3.1 The classical solution

We want now to write the general solution for $\partial \mathcal{X}$ and $\partial \bar{\mathcal{X}}$ in a way that the $SL(2, \mathbb{R})$ symmetry is manifest. To this purpose we introduce the anharmonic ratio

$$\omega_z = \frac{(z - x_N)(x_2 - x_1)}{(z - x_1)(x_2 - x_N)}$$

(28)

and the corresponding ones $\omega_j$ where $z$ has been replaced with $x_j$. In particular we get $\omega_N = 0$, $\omega_2 = 1$ and $\omega_1 = -\infty$. The choice of $\omega_1 = -\infty$ is dictated by the request that powers are defined as $(\omega - \omega_i)^\epsilon = |\omega - \omega_i|^\epsilon e^{i\epsilon \phi}$ where $\phi = \arg(\omega - \omega_i)$ is counted from the real axis with range $(-\pi, \pi)$ so that all cuts must be towards $-\infty$, as it is shown in fig. \([6]\).

We can now write the general solutions as

$$\partial \mathcal{X}(z) = \frac{\partial \omega_z}{\partial z} \sum_{n=0}^{N-M-2} a_n(\omega_j) \partial_\omega \mathcal{X}^{(n)}(\omega_z)$$

$$\partial \bar{\mathcal{X}}(z) = e^{-i2\pi \alpha_2} \frac{\partial \omega_z}{\partial z} \sum_{r=0}^{M-2} b_r(\omega_j) \partial_\omega \bar{\mathcal{X}}^{(r)}(\omega_z)$$

(29)

where we have defined the basis

$$\partial_\omega \mathcal{X}^{(n)}(\omega_z) = \prod_{j=2}^{N} (\omega_z - \omega_j)^{-(1-\epsilon_j)} \omega_z^n, \quad 0 \leq n \leq N - M - 2$$

$$\partial_\omega \bar{\mathcal{X}}^{(r)}(\omega_z) = \prod_{j=2}^{N} (\omega_z - \omega_j)^{-\epsilon_j} \omega_z^r, \quad 0 \leq r \leq M - 2$$

(30)
and we have also defined the integer

$$M = \sum_{i=1}^{N} \epsilon_i$$

(31)

When the target polygon $\Sigma$ is followed counterclockwise this integer $M$ is equal to the number of reflex angles plus 2 since every acute angle internal the target polygon is $\pi - \pi \epsilon$ while every reflex one is $2\pi - \pi \epsilon$ as shown in fig. 7. In a similar way when the target polygon is followed clockwise $M$ is the number of acute angles minus 2. Nevertheless it is important to notice how polygons having $M$ and $M' = N - M$ both measured counterclockwise or clockwise are not the same polygons as it shown in fig. 1 in the case $N = 6$. To distinguish between these two cases it is necessary to compare expression 7 with the phases $\alpha_i$ as derived from the geometrical relations $f_{i+1} - f_i = \pm e^{i\pi \alpha_{i+1}} |f_{i+1} - f_i|$ (where the sign depends on the case) as shown in fig. 8.
Figure 8: The connection between $f_{i+1} - f_i$ and the geometrical angle $\alpha_{i+1}$ defining the brane.

Also when changing the $f$s while keeping fixed the $\epsilon$s the shape may change as shown in fig. [9] for $N = 4$ and $M_{ccw} = 2$ and in fig. [10] for $N = 4$ and $M_{ccw} = 3$. From now on we measure $M$ clockwise in not otherwise stated. Since the number of reflex angles must be less or equal than $N - 3$ we deduce that $2 \leq M_{ccw} \leq N - 1$ or $1 \leq M_{cw} \leq N - 2$ and hence there are $N - 2$ different sectors.

Figure 9: The four different cases with $N = 4$ and $M_{ccw} = 2$ and $M_{cw} = 2$ which can be obtained moving the brane whose intersection points are the empty circles.

Figure 10: The four different cases with $N = 4$ and $M_{ccw} = 3$ and $M_{cw} = 1$ which can be obtained moving whose intersection points are the empty circles.

In the previous expressions (29) $\frac{\partial \omega_{z}}{\partial z}$ ensures the proper transformation under $SL(2, \mathbb{R})$ and the product $\prod_{j=2}^{N} (\omega_{z} - \omega_{j})^{-(1-\epsilon_{j})}$ and the corresponding

\[ [X_{cl}(u, \bar{u}; \{1 - \epsilon\}, \{f^{*}\})]^{*} = X_{cl}(u, \bar{u}; \{\epsilon\}, \{f\}) \] maps $M_{ccw}$ into $M_{cw} = N - M_{ccw}$ because it is like the map $X \rightarrow X^{*}$ which reverses the order in which a circuit is followed. Hence it does not map $M_{ccw}$ into $M'_{ccw} = N - M_{ccw}$.

\[ 2 \]
one for \( \partial \bar{X} \) yields the proper monodromies around all the point, \( x_1 \) included. The extrema of the summations, i.e. the maximum allowed values of \( n \) and \( r \) are chosen in order to have a finite action and in particular their values are determined by the analysis of the behavior of the solutions around \( z = x_1 \) and not around \( z = \infty \) as one would naively expect. This happens because the solutions \( (29) \) behave as \( O(1/z^2) \) at \( z = \infty \) because of the factor \( \partial \omega \partial z \).

The powers of the products \( \prod_{j=2}^{N} (\omega_z - \omega_j)^{-1-\epsilon_j} \) and \( \prod_{j=2}^{N} (\omega_z - \omega_j)^{-\epsilon_j} \) are chosen in order to get a finite \( X_{cl}(u, \bar{u}) \) at the singular points, explicitly using the definitions \( (21) \) and the expressions \( (29) \) we can write\(^{3} \)

\[
X_{cl}(u, \bar{u}) = f_1 + \sum_{n=0}^{N-M-2} a_n(\omega_j) \int_{x_1; z \in H} u dz \frac{\partial \omega_z}{\partial z} \prod_{j=2}^{N} (\omega_z - \omega_j)^{-(1+\epsilon_j)} \omega_z^n
\]

\[
= f_1 + \sum_{n=0}^{N-M-2} a_n(\omega_j) \int_{-\infty; \omega \in H} d\omega \prod_{j=2}^{N} (\omega - \omega_j)^{-(1+\epsilon_j)} \omega^n
\]

\[
\prod_{r=0}^{M-2} b_r(\omega_j) \left[ \int_{-\infty; \omega \in H} \frac{\partial \omega_z}{\partial z} \prod_{j=2}^{N} (\omega_z - \omega_j)^{-\epsilon_j} \omega^r \right]^{*}
\]

(32)

where in the last step we have used the explicit definition of the power to connect the integral performed in lower half plane with that performed in the upper half plane. In a similar way we can write

\[
\bar{X}_{cl}(u, \bar{u}) = f_1^* + e^{-i2\pi \alpha_2} \sum_{n=0}^{N-M-2} a_n(\omega_j) \left[ \int_{-\infty; \omega \in H} d\omega \prod_{j=2}^{N} (\omega - \omega_j)^{-(1+\epsilon_j)} \omega^n \right]^* \]

\[
+ e^{-i2\pi \alpha_2} \sum_{r=0}^{M-2} b_r(\omega_j) \int_{-\infty; \omega \in H} d\omega \prod_{j=2}^{N} (\omega - \omega_j)^{-\epsilon_j} \omega^r
\]

(33)

where the coefficients are again \( a \) and \( b \) and not \( a^* \) and \( b^* \) as naively expected because we computed both \( X_{cl} \) and \( \bar{X}_{cl} \) using the definitions of \( \partial X \) and \( \partial \bar{X} \) \( (21) \) which mix both \( \partial X \) and \( \partial X \). On the other side \( \bar{X}_{cl} = (X_{cl})^* \) hence there are constraints on the coefficients \( a \) and \( b \), i.e. \( a^* = e^{-i2\pi \alpha_2} a \) and similarly for \( b \) but these constraints are precisely the ones needed to solve the equations \( (34) \) when one takes into account the geometrical requirements that \( \bar{f}_{i+1} - \bar{f}_i = e^{i\pi \alpha_1} |f_{i+1} - f_i| \) as shown in fig. \( (8) \).

In order to determine the \( N-M-1 \) functions \( a(\omega_j) \) \( (j \neq 1, 2, N) \) and the \( M-1 \) \( b(\omega_j) \) we need simply to impose the \( N-2 \) geometrical constraints

\[
X_{cl}(x_{i+1}, \bar{x}_{i+1}) - X_{cl}(x_i, \bar{x}_i) = f_{i+1} - f_i \quad i = 2, \ldots N-1
\]

(34)

\(^{3}\text{Because of way we have chosen the cuts we have } (\omega_z - \omega_j)^{\alpha}|^{*} = (\omega_z - \omega_j)^{\alpha} \text{ when } \omega_j \text{ is real.}\)
There is actually one more equation one can obviously impose, the one with $i = 1$ but it turns out to be linearly dependent on the previous ones when the geometrical constraints on $f$ and $\epsilon$ are imposed. It is worth noticing that the previous constraints have an obvious geometrical meaning differently from the use of Pochammer path used in the literature. The explicit solution of the previous constraints is given by solving the linear system of $N - 2$ equations

$$
\sum_{n=0}^{N-M-2} (-1)^{i-1} I_{i,n}(1 - \epsilon_j) a_n + \sum_{r=0}^{M-2} I_{i,n}^{(N)}(\epsilon_j) b_r = e^{-i\pi \sum_{j=2}^{i} \epsilon_j} (f_i - f_{i+1}) \quad i = 2, \ldots N - 1
$$

(35)

where we have introduced the real valued integrals

$$
I_{i,n}^{(N)}(\alpha_j) = \int_{\omega_{i+1}}^{\omega_i} d\omega \prod_{j=2}^{N} |\omega - \omega_j|^{-\alpha_j} \omega^{n}
$$

(36)

which are connected to type D Lauricella generalized hypergeometric function when $i, i + 1 \neq 1$ by

$$
I_{i,0}^{(N)}(\alpha_j) = \prod_{j \neq 1,i,i+1} \frac{\omega_j - \omega_{i+1}^{-\alpha_j} |\omega_i - \omega_{i+1}|^{1-\alpha_{i+1}}}{\omega_i - \omega_{i+1}}
$$

\[
\cdot \frac{1}{B(1 - \alpha_i, 1 - \alpha_{i+1})} \cdot F_D^{(N-3)}(1 - \alpha_i; \{\alpha_{j \neq 1,i,i+1}; 2 - \alpha_i; \alpha_{i+1}; \xi_a\})
\]

(37)

where the parameters $\xi_a (a = 1, \ldots N - 3)$ are given by

$$
\xi_a = \begin{cases} 
\frac{\omega_i - \omega_{i+1}}{\omega_{a+1} - \omega_{i+1}} & 1 \leq a \leq i - 2 \\
\frac{\omega_{i+1} - \omega_{i+3}}{\omega_{a+3} - \omega_{i+1}} & i - 1 \leq a \leq N - 3
\end{cases}
$$

(38)

In particular for $N = 3 F_D^{(0)}$ is Euler Beta function $B$ and for $N = 4 F_D^{(1)}$ is

---

4 The net effect of using the real valued integrals $I_{i,n}^{(N)}(\epsilon)$ is simply the phase $e^{+i\pi \sum_{j=2}^{i} \epsilon_j}$ which multiply the real integral. In particular for $I_{i,n}^{(N)}(1 - \epsilon)$ we get $(-1)^{i-1} e^{-i\pi \sum_{j=2}^{i} \epsilon_j}$ and this explains the alternating sign which appears weird at first sight.

5 All these integrals can be expressed using $I_{i,0}^{(N)}$ since $\omega_N = 0$, explicitly $I_{i,n}^{(N)}(\alpha_j) = I_{i,0}^{(N)}(\alpha_j - \delta_{j,N})$ but we have introduced this redundancy for notational simplicity.

Moreover we have used $\omega_{N+1} = \omega_1 = -\infty$ because indexes are defined mod $N$. 
a plain hypergeometric $\, _2 F_1$, explicitly the previous expressions become

\[
I_{2,n}^{(3)}(1 - \epsilon_j) = B(\epsilon_2, \epsilon_3 + n) \\
I_{2,n}^{(4)}(1 - \epsilon_j) = \omega_3^{\epsilon_3-1+n} (1 - \omega_3)^{\epsilon_2+\epsilon_3-1} \frac{1}{B(\epsilon_3, \epsilon_2)} \, _2 F_1(\epsilon_3; 1 - n - \epsilon_4; \epsilon_2 + \epsilon_3; \frac{\omega_3 - 1}{\omega_3}) \\
I_{3,n}^{(4)}(1 - \epsilon_j) = \omega_3^{\epsilon_3+\epsilon_4+n-1} \frac{1}{B(\epsilon_4 + n, \epsilon_3)} \, _2 F_1(1 - \epsilon_2; n + \epsilon_4; \epsilon_3 + \epsilon_4 + n; \omega_3)
\]  

(39)

We are now ready to compute the classical action for our solution. Using the explicit expression for $\partial \lambda \times \partial \bar{\lambda}$ we can write

\[
S_{cl} = \frac{1}{8\pi \alpha'} \left[ \sum_{n,m=0}^{N-M-2} (e^{-i\pi \alpha_2 a_n}) (e^{-i\pi \alpha_2 a_m}) \int_{\mathbb{C}} d^2 \omega \, \prod_{j=2}^{N} |\omega - \omega_j|^{-2(1-\epsilon_j)} \omega^n \bar{\omega}^m \\
+ \sum_{r,s=0}^{M-2} (e^{-i\pi \alpha_2 b_r})(e^{-i\pi \alpha_2 b_s}) \int_{\mathbb{C}} d^2 \omega \, \prod_{j=2}^{N} |\omega - \omega_j|^{-2\epsilon_j} \omega^r \bar{\omega}^s \right]
\]

(40)

where an overall factor $\frac{1}{2}$ appears because we have extended the integration domain from the upper half plane to the whole complex plane. Notice that $e^{-i\pi \alpha_2 a}, e^{-i\pi \alpha_2 b} \in \mathbb{R}$ which is however not enough to use their moduli $|a|$ and $|b|$ in the previous expression. As explained in appendix $A$ using the technique developed in $[13]$ the previous integrals can be expressed as a product of holomorphic and antiholomorphic contour integrals as

\[
\int_{\mathbb{C}} d^2 \omega \, \prod_{j=2}^{N} |\omega - \omega_j|^{-2\epsilon_j} \omega^n \bar{\omega}^m = \\
= \sum_{i=2}^{N-1} \sum_{l=i+1}^{N} \sin \left( \pi \sum_{j=i+1}^{l} \epsilon_j \right) I_{i,n}^{(N)}(\epsilon) f_{l,m}^{(N)}(\epsilon)
\]

(41)

3.2 The explicit $N = 3, \, M_{cw} = 1 \, (M_{ccw} = 2)$ case

Let us start examining the $M_{cw} = 1$ computation. In this case we see immediately that $\partial \bar{\lambda}$ is identically zero and that the only unknown is $a_0$ which is not a function but simply a constant. The eq.s (35) reduce simply to

\[
- a_0 e^{i\pi \epsilon_2} B(\epsilon_2, \epsilon_3) = f_2 - f_3
\]

(42)

because $I_{2,n}^{(3)}(\alpha_j) = B(1 - \alpha_2, 1 + n - \alpha_3)$ where $B(\epsilon_2, \epsilon_3)$ is Euler beta function. The complete solution is then

\[
X_{cl}^{(3,1,cw)}(u, \bar{u}) = f_3 + e^{i\pi (1 - \epsilon_2)}(f_2 - f_3) \frac{B(1, \epsilon_3)}{B(\epsilon_2, \epsilon_3)} \, _2 F_1(\epsilon_3; 1 - \epsilon_2; \epsilon_3 + 1; \omega_u) \omega_u^{\epsilon_3}
\]

(43)

\[e^{-i\pi \alpha_2 a_n}\] is real as discussed before but it is by no means assured that it is positive.
Consider now the $M_{ccw} = 2$ case where only $b_0$ is different from zero and therefore $\partial x = 0$. Proceeding as before we get

$$b_0 e^{i\pi \epsilon_2} B(1 - \epsilon_2, 1 - \epsilon_3) = f_2 - f_3$$

from which follows

$$X^{(3,2,ccw)}_{cl}(u, \bar{u}) = f_3 + \frac{e^{-i\pi \epsilon_2}(f_2 - f_3)}{B(1, 1 - \epsilon_3)B(1 - \epsilon_2, 1 - \epsilon_3)} \left[ 2F1\left(1 - \epsilon_3, \epsilon_2; 2 - \epsilon_3; \omega_u\right) \omega_u^{1-\epsilon_3}\right]^*$$

(45)

which explicitly shows the equivalence $[X^{(N,(N-M)_{ccw})}_{cl}(u, \bar{u}; \{1 - \epsilon\}, \{f^*\})]^* = X^{(N,M_{ccw})}_{cl}(u, \bar{u}; \{\epsilon\}, \{f\})$.

### 3.3 The explicit $N = 4$, $M_{cw} = 1$ ($M_{ccw} = 3$) case

In this case we again immediately realize that $\partial x$ is identically zero and that the only unknowns are the two functions $a_0(\omega_3)$ and $a_1(\omega_3)$. The eqs (35) reduce simply to

$$\sum_{n=0}^{1} (-)^{i-1} I^{(4)}_{t,n}(1 - \epsilon_3) a_n(\omega_3) = e^{-i\pi \sum_{j=2}^{i} \epsilon_j} (f_i - f_{i+1}) \quad i = 2, 3$$

(46)

with $I^{(4)}$s given explicitly in eqs (39). The classical solution then reads

$$X^{(4,1,ccw)}_{cl}(u, \bar{u}) = f_1 + \sum_{n=0}^{1} a_n(\omega_3) \int_{-\infty}^{\omega_u} d\omega \ (\omega - 1)^{\epsilon_2 - 1}(\omega - \omega_3)^{\epsilon_3} \omega^{n+\epsilon_4 - 1}$$

(47)

The $X^{(4,3,ccw)}_{cl}(u, \bar{u})$ solution can then be obtained as $X^{(4,3)}_{cl}(u, \bar{u}; \{\epsilon\}, \{f\}) = [X^{(4,1)}_{cl}(u, \bar{u}; \{1 - \epsilon\}, \{f^*\})]^*$.

### 3.4 The explicit $N = 4$, $M = 2$ case

In this case $M$ can be understood either as $M_{ccw}$ or as $M_{cw}$ which of the two can be only decided looking at the phases $\{\alpha_i\}$. The unknowns are the two functions $a_0(\omega_3)$ and $b_0(\omega_3)$ and eqs (35) reduce simply to

$$(-)^{i-1} I^{(4)}_{t,0}(1 - \epsilon_3) a_0(\omega_3) + I^{(0)}_{t,0}(\epsilon_3) b_0(\omega_3) = e^{-i\pi \sum_{j=2}^{i} \epsilon_j} (f_i - f_{i+1}) \quad i = 2, 3$$

(48)

hence the classical solution reads

$$X^{(4,2)}_{cl}(u, \bar{u}) = f_1 + a_0(\omega_3) \int_{-\infty}^{\omega_u} d\omega \ (\omega - 1)^{\epsilon_2 - 1}(\omega - \omega_3)^{\epsilon_3} \omega^{\epsilon_4 - 1}$$

$$+ b_0(\omega_3) \left[ \int_{-\infty}^{\omega_u} d\omega \ (\omega - 1)^{-\epsilon_2}(\omega - \omega_3)^{-\epsilon_3} \omega^{-\epsilon_4}\right]^*$$

(49)
3.5 Wrapping contributions

The wrapping contributions have been studied in [16] for the N=3 case and in [5] for the case $M = N - 2$ and there is not any difference among the different $M$ values therefore the results obtained there are valid. Let us anyhow quickly review them. Given a minimal $N$-polygon in $T^2$ with vertexes $\{f_i\}$, i.e. with all vertexes in the fundamental cell, we can consider non minimal polygons which wrap the $T^2$. These can be easily described as polygons which have vertexes $\{\tilde{f}_i\}$ in the covering $R^2$ where $T^2 \equiv R^2/\Lambda$ with the lattice defined as $\Lambda = \{(n_1e_1 + n_2e_2|n_1, n_2 \in \mathbb{Z}\}$. These configurations give an additive contribution to the classical path integral as

$$\langle \sigma_{\epsilon_1,f_1}(x_1) \ldots \sigma_{\epsilon_N,f_N}(x_N) \rangle_{T^2} = \mathcal{N}(x_i, \epsilon_i) \sum_{\tilde{f}_i} e^{-S_{E,cl}(x_i, \epsilon_i, \tilde{f}_i)}$$  \hspace{1cm} (50)

In order to determine the possible vertexes $\{\tilde{f}_i\}$ without redundancy it is necessary to keep a vertex fixed and then expand the polygon. For definiteness we keep fixed the vertex $\tilde{f}_1 = f_1$ which lies at the intersection between $D_N$ and $D_1$. We then move the next vertex $f_2$ along the $D_1$ brane. Explicitly we write $\tilde{f}_2 = \tilde{f}_1 + (f_2 - f_1) + n_1t_1 = f_2 + n_1t_1$ with $n_1 \in \mathbb{Z}$ and $t_1$ the shortest tangent vector to $D_1$ which is in $\Lambda$. We can now continue for all the other vertexes for which we have $\tilde{f}_i = \tilde{f}_{i-1} + (f_i - f_{i-1}) + n_{i-1}t_{i-1} = f_i + \sum_{k=1}^{i-1} n_k t_k$. For consistency we need requiring $\tilde{f}_{N+1} \equiv \tilde{f}_1 = f_1$, therefore the possible wrapped polygons are obtained from the solution of the Diophantine equation

$$\sum_{i=1}^{N} n_i t_i = 0$$  \hspace{1cm} (51)

which cannot be solved in general terms but only on a case by case basis as discussed in [5].

4 Green functions for $N \geq 3$

Having determined the classical solution we now compute the Green functions in presence of twist fields both as an intermediate step toward the computation of the quantum part of correlators and as a key ingredient to the computation of excited twist fields correlators.

Following partially the literature we define the following quantities for the quantum fluctuations which are connected with the derivatives of the Green functions as

$$g_{(N,M)}(z, w; \{x_i\}) = \frac{\langle \partial \mathcal{X}_q(z) \partial \mathcal{X}_q(w) \sigma_{\epsilon_1,f}(x_1) \ldots \sigma_{\epsilon_N,f}(x_N) \rangle}{\langle \sigma_{\epsilon_1,f}(x_1) \ldots \sigma_{\epsilon_N,f}(x_N) \rangle}$$

$$h_{(N,M)}(z, w; \{x_i\}) = \frac{\langle \partial \mathcal{X}_q(z) \partial \mathcal{X}_q(w) \sigma_{\epsilon_1,f}(x_1) \ldots \sigma_{\epsilon_N,f}(x_N) \rangle}{\langle \sigma_{\epsilon_1,f}(x_1) \ldots \sigma_{\epsilon_N,f}(x_N) \rangle}$$

$$l_{(N,M)}(z, w; \{x_i\}) = \frac{\langle \partial \mathcal{X}_q(z) \partial \mathcal{X}_q(w) \sigma_{\epsilon_1,f}(x_1) \ldots \sigma_{\epsilon_N,f}(x_N) \rangle}{\langle \sigma_{\epsilon_1,f}(x_1) \ldots \sigma_{\epsilon_N,f}(x_N) \rangle}$$  \hspace{1cm} (52)
We do not need to consider functions involving antiholomorphic quantities because $\partial X$ and $\overline{\partial X}$ are related to $\partial X$ and $\overline{\partial X}$ as in eq.s (23).

Quantum fluctuations are required to satisfy the boundary conditions

$$Re(e^{-i\pi\alpha_1}\partial_y X_q|_{y=0}) = Im(e^{-i\pi\alpha_1}X_q|_{y=0}) = 0 \quad x_{i+1} < x < x_i.$$  (53)

These conditions that can reformulated as a set of local constraints

$$\partial X_q(x_i + e^{2i\pi}\delta) = e^{2i\pi\epsilon_1}\partial X_q(x_i + \delta), \quad \partial\tilde{X}_q(x_i + e^{2i\pi}\delta) = e^{2i\pi\epsilon_1}\partial\tilde{X}_q(x_i + \delta)$$  (54)

and as a set of global constraints

$$X_q(x_i, \bar{x}_i) = X_q(x_{i+1}, \bar{x}_{i+1}), \quad \tilde{X}_q(x_i, \bar{x}_i) = \tilde{X}_q(x_{i+1}, \bar{x}_{i+1}).$$  (55)

In the spirit of what done in the previous section we use a $SL(2,\mathbb{R})$ invariant formulation and we write

$$g_{(N,M)}(z, w; \{x_i\}) = \frac{1}{(z-w)^2} \prod_{j\neq 1} \frac{(\omega_z - \omega_j)^{s_j}}{(\omega_w - \omega_j)^{s_j}} \sum_{n=0}^{N-1} \sum_{s=0}^{M} a_{ns}(\omega_j) \omega_z^nx_w^s,$$

$$h_{(N,M)}(z, w; \{x_i\}) = e^{-2i\pi\alpha_2} \frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w} \sum_{\tau,\rho=0}^{M-2} \sum_{b_{\tau}\rho}(\omega_j) \partial_{\omega}X^{(\tau)}(\omega_z) \partial_{\omega}X^{(\rho)}(\omega_w),$$

$$l_{(N,M)}(z, w; \{x_i\}) = e^{2i\pi\alpha_2} \frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w} \sum_{c_{nm}(\omega_j)} \partial_{\omega}X^{(n)}(\omega_z) \partial_{\omega}X^{(m)}(\omega_w)$$  (56)

where $a_{ns}(\omega_j), b_{\tau}\rho(\omega_j)$ and $c_{nm}(\omega_j)$ are unknown functions of the anharmonic ratios $\omega_j\neq 1,2,N$.

Let us rapidly review the ingredients of the previous construction. The factors $\frac{1}{(z-w)^2}, \frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w}$ and $\frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w}$ are there to ensure the proper $SL(2,\mathbb{R})$ transformations. The powers of the singular parts have been chosen in order to reproduce the singularities of OPEs

$$\partial X(z) \partial \tilde{X}(w) \sim \frac{1}{(z-w)^2} + O(1)$$  (57)

$$\partial X(z) \sigma_{\epsilon,f}(x) \sim (z-x)^{\epsilon-1}(\partial X \sigma_{\epsilon,f})(x)$$

$$\partial \tilde{X}(z) \sigma_{\epsilon,f}(x) \sim (z-x)^{-\epsilon}(\partial \tilde{X} \sigma_{\epsilon,f})(x)$$  (58)

where $(\partial X \sigma_{\epsilon,f})(x)$ and $(\partial \tilde{X} \sigma_{\epsilon,f})(x)$ are excited twists. In particular eq.s (54) are the same of eq.s (58) as it should be since twist operators have been introduced to generate (54) constraints.

The upper bounds of the summation ranges are fixed by request that singularities $z \to x_1$ and $w \to x_1$ are not worse than those in eq.s (58) while the lower bound is fixed from the $z \to x_N$ and $w \to x_N$ limits.

There is another consistency condition: when $x_i \to x_j$ we must obtain the corresponding Green function with $N \to N-1$. It is worth stressing that at
first sight there is a further constraint. Usually the OPE between two twists is written as

\[ \sigma_{\epsilon_i,f}(x_i) \sigma_{\epsilon_j,f}(x_j) \sim \begin{cases} (x_i - x_j)^{-\epsilon_i\epsilon_j} \mathcal{M}(\epsilon_i, \epsilon_j) \sigma_{\epsilon_i+\epsilon_j,f}(x_j) & \epsilon_i + \epsilon_j < 1 \\ (x_i - x_j)^{-\epsilon_i\epsilon_j(1-\epsilon_i)} \mathcal{N}(\epsilon_i, \epsilon_j) \sigma_{\epsilon_i+\epsilon_j-1,f} & \epsilon_i + \epsilon_j > 1 \end{cases} \]  

(59)

with \( \mathcal{M}(\epsilon_i, \epsilon_j) = \mathcal{N}(\epsilon_i, \epsilon_j) = 1 \). We will discuss that it is not possible to set both \( \mathcal{M} \) and \( \mathcal{N} \) to one in section 5.3. Now we would however comment on the fact that the previous expression is written without higher terms leading to the wrong impression that all the omitted terms are descendants. If it were true that the OPE (59) has no other primaries in the rhs this would imply that the derivatives of Green functions are analytic functions of the variables \( x_i \) too since the overall singularity in \( x_i - x_j \) due to the power factor would cancel between the numerator and the denominator. This is not true as the explicit computations show but in the \( M_{cw} = 1 \), \( M_{ccw} = N - 1 \) case and the reason is that the previous OPE involves actually an infinite number of primary fields with powers of OPE coefficients which do not differ by integers, explicitly for \( \epsilon_i + \epsilon_j < 1 \)

\[
\sigma_{\epsilon_i,f}(x_i) \sigma_{\epsilon_j,f}(x_j) \sim (x_i - x_j)^{\epsilon_i\epsilon_j} \mathcal{M}(\epsilon_i, \epsilon_j) \sigma_{\epsilon_i+\epsilon_j,f}(x_j) \\
+ \sum_{k=1} c_k (x_i - x_j)^{\epsilon_i\epsilon_j + k(\epsilon_i + \epsilon_j)} [(\partial X)^k \sigma_{\epsilon_i+\epsilon_j,f}](x_j) + \ldots
\]

(60)

where \( c_k \) are certain numbers and \( \ldots \) stands for other primaries and descendants. These primary fields have a simple interpretation as the states associated to the Hilbert space of twisted string since all of them have conformal dimensions which differ by multiples of \( \pm \epsilon \).

To continue and write in a more compact way the following expressions we define

\[
P(\omega_z, \omega_w) = \prod_{j=2}^{N} \frac{(\omega_z - \omega_j)^{\epsilon_j-1}}{(\omega_w - \omega_j)^{\epsilon_j}}
\]

\[
S(\omega_z, \omega_w) = \sum_{n=0}^{N-M} \sum_{s=0}^{M} a_{ns}(\omega_j)\omega_z^n \omega_w^s
\]

(61)

When we impose the constraint from the \( z \to w \) limit given in eq. (57) we get

\[
S(\omega_w, \omega_w) = \sum_{n=0}^{N-M} \sum_{s=0}^{M} a_{ns}(\omega_j)\omega_w^{n+s} = \prod_{j=2}^{N} (\omega_w - \omega_j)
\]

\[
\frac{\partial S}{\partial \omega_z} \bigg|_{\omega_z=\omega_w} = \sum_{n=0}^{N-M} \sum_{s=0}^{M} na_{ns}(\omega_j)\omega_w^{n+s-1} = \sum_{j=2}^{N} \frac{1 - \epsilon_j}{\omega_w - \omega_j} \cdot \prod_{l=2}^{N} (\omega_w - \omega_l)
\]

(62)
or the equivalent equations with \( w \to z \) and \( \epsilon \to 1 - \epsilon \). These are \((N + 1) + N\) equations for \( a_{ns} \) but only \(2N\) are independent since both imply that \( a_{N-M, M} = 0 \). Generically, i.e. for \( M_{cw} \neq 1 \) these equations are not sufficient to fix the \((N - M + 1)(M + 1)\) unknowns \( a_{ns} \) and must be supplemented by the constraints which follow from eqs. (55). These further constraints allow also to fix the remaining \((M - 1)^2 + (N - M - 1)^2\) unknowns functions \( b_{rs} \) and \( c_{nm} \). For example from the first equation in (55) we get

\[
X_q(x_i, \bar{x}_i) - X_q(x_{i+1}, \bar{x}_{i+1}) = \int_{x_{i+1}}^{x_i} dX_q = \int_{x_{i+1}}^{x_i} dx [\partial X_q(x + i0^+) + \bar{\partial} X_q(x - i0^+)] = \int_{x_{i+1}}^{x_i} dx [\partial X_q(x + i0^+) + e^{i2\pi \alpha_2} \bar{\partial} X_q(x - i0^+)] = 0
\]

which implies the constraints\(^7\)

\[
\int_{x_{i+1}}^{x_i} dx \ g_{(N,M)}(x + i0^+, w) + e^{i2\pi \alpha_2} h_{(N,M)}(x - i0^+, w) = 0
\]

\[
\int_{x_{i+1}}^{x_i} dx \ l_{(N,M)}(z, x - i0^+) + e^{i2\pi \alpha_2} g_{(N,M)}(z, x + i0^+) = 0
\]

These can be explicitly written as

\[
\sum_{s=0}^{M} \omega_w^s \sum_{n=0}^{N-M} a_{ns}(\omega_j) \int_{\omega_{i+1} \omega \in H} \frac{d\omega}{(\omega - \omega_w)^2} \prod_{j=2}^{N} (\omega - \omega_j)^{\epsilon_j-1} \omega^n (\omega - \omega_{i+1} \omega \in H) \prod_{j=2}^{N} (\omega - \omega_j)^{-\epsilon_j} \omega^r = 0
\]

\[
\sum_{s=0}^{M} \omega_w^s \sum_{r=0}^{M-2} b_{rs}(\omega_j) \int_{\omega_{i+1} \omega \in H} \frac{d\omega}{(\omega - \omega_w)^2} \prod_{j=2}^{N} (\omega - \omega_j)^{-\epsilon_j} \omega^s
\]

As in the case for the classical solution only \(N - 2\) of the previous intervals give independent constraints, let us say \( i = 2, \ldots N - 1 \). All these constraints are then sufficient to fix completely and uniquely all the coefficients. These constraints are actually much more than needed since they equate a polynomial in \( \omega_w \) or in \( \omega_z \) to an analytic function. If we expand around \( \omega_w = \omega_z = \infty \) and consider only the polynomial part we have enough equations to fix all

\(^7\) It is worth noticing that the segment \([x_1, x_i]\) is followed for one addend above and for the other below the cut (it works also the other way round w.r.t. the main text). This ensures that both addends have the same phase modulus \( \pi \). Consistency among possible formulations of the constraints is due to \([g(z, w)]^* = g(\bar{z}, \bar{w}), [h(z, w)]^* = e^{i4\pi \alpha_2} h(\bar{z}, \bar{w}) \) and \([l(z, w)]^* = e^{-i4\pi \alpha_2} l(\bar{z}, \bar{w}) \).
the unknowns since to the previous $2N$ constraints in eqs. (62) we add $M - 1$
eq 1$equations in $\omega_w$ times $N - 2$ intervals and $N - M - 1$ equations for $\omega_z$ times $N - 2$ intervals. Actually the previous equations are already overdetermined in $a$ since the $2N$ eq.s (62) and the $(N - 2)(M - 1)$ ones in $\omega_w$ are sufficient for fix both $a$ and $b$ and similarly for the ones in $\omega_z$ therefore for consistency we suppose that this overdetermined system is consistent as well as all the remaining equations obtained from the polar part in $\omega_w$ and $\omega_z$. As far as the consistency of the functions $a$ determined in the two ways we have checked it in particular limits in the explicit cases treated afterward. Moreover all constraints derived from the polar part are polynomials in the integrals $I^{(N)}$ (36) since the functions $a$ and $b$ are solutions of a linear system whose coefficients are precisely the $I^{(N)}$s. Nevertheless it is easy to show that all constraints must be equivalent to a relation with polynomial coefficients in $\omega_j \neq 1, 2, N$ and $I^{(N)}$ and at most linear in $\hat{I}^{(N+1)}$ (see eq. (78)) with one of the parameters equal to 2. This can be seen as follows. It is possible to split $g_{(N,M)}(z, w; \{x_i\})$ in a singular part and a regular one as

$$g_{(N,M)}(z, w; \{x_i\}) = g_{s(N,M)}(z, w; \{x_i\}) + g_{r(N,M)}(z, w; \{x_i\})$$

$$g_{s(N,M)}(z, w; \{x_i\}) = \frac{1}{(z - w)^2} \prod_{j \neq 1} (\omega_z - \omega_j)^{r_j - 1} \sum_{n=0}^{N-M-2} \sum_{s=0}^{M} a_{(0)ns}(\omega_j) \omega_z^n \omega_w^s$$

$$g_{r(N,M)}(z, w; \{x_i\}) = \frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w} \sum_{n=0}^{N-M-2} \sum_{s=0}^{M} \tilde{a}_{ns}(\omega_j) \partial_{\omega_z} \tilde{X}^{(n)}(\omega_z) \partial_{\omega_w} \tilde{X}^{(r)}(\omega_w)$$

(67)

This splitting is completely arbitrary and therefore it is not uniquely defined but it can be made unique imposing further conditions such for example the request of setting to zero all the $a_{n,s} = k - n$ but the two with lowest $n$ as for example in eq. (85) for the $(N = 4, M = 2)$ case or the request that the singular part $g_{s(N,M)}$ goes into $g_{s(N-1,M')}$ when $x_i \rightarrow x_j$ as shown in appendix B and explicitly in eq. (153) for $(N = 4, M = 2)$ case. Once fixed by a “gauge choice” the singular part the regular one is fixed by the global boundary conditions.

Actually if we choose to split $g$ into a regular part and a singular part (which is fixed not uniquely by the OPEs) as in eqs. (67) the equation (65)

---

Eq.s (62) divide naturally the set of unknowns $\{a_{ns}\}$ into subsets $\{a_{ns}\}_{s+n=k}$ and for each of these subsets there are two linear equations.
can be written as

\[
\sum_{n=0}^{N-M} \sum_{s=0}^{M} \omega_s^N a_n(\omega_j) \int_{\omega_{j+1};\omega \in H}^\omega \frac{d\omega}{(\omega - \omega_w)^2} \prod_{j=2}^{N} (\omega - \omega_j)^{r_j} \omega^n
\]

\[
+ \sum_{n=0}^{N-M} \sum_{s=0}^{M} \omega_s^N \tilde{a}_{ns}(\omega_j) \int_{\omega_{j+1};\omega \in H}^\omega d\omega \partial_\omega \chi^{(n)}(\omega)
\]

\[
+ \sum_{n=0}^{N-M} \sum_{s=0}^{M} \omega_s^N \sum_{r=0}^{M-2} b_{nr}(\omega_j) \int_{\omega_{j+1};\omega \in H}^\omega d\omega \partial_\omega \chi^{(r)}(\omega) = 0 \quad (68)
\]

which reveals that the singular part \(g_{s(N,M)}\) contributes with a term linear in \(i^{(N+1)}\) (with one parameter equal to 2 because of the term \((\omega - \omega_w)^{-2}\)) while the other terms have rational coefficient in \(\omega_j \neq 1,2,N\) and \(I^{(N)}\) once we plug the solution for the coefficients back. In a similar way we can write the equation corresponding to (66) as

\[
\sum_{n=0}^{N-M} \sum_{s=0}^{M} \omega_s^N a_n(\omega_j) \int_{\omega_{j+1};\omega \in H}^\omega \frac{d\omega}{(\omega - \omega_w)^2} \prod_{j=2}^{N} (\omega - \omega_j)^{r_j} \omega^n
\]

\[
+ \sum_{n=0}^{N-M} \sum_{s=0}^{M} \omega_s^N \tilde{a}_{ns}(\omega_j) \int_{\omega_{j+1};\omega \in H}^\omega d\omega \partial_\omega \chi^{(n)}(\omega)
\]

\[
+ \sum_{n=0}^{N-M} \sum_{s=0}^{M} \omega_s^N \sum_{r=0}^{M-2} b_{nr}(\omega_j) \int_{\omega_{j+1};\omega \in H}^\omega d\omega \partial_\omega \chi^{(r)}(\omega) = 0 \quad (69)
\]

Having determined the derivatives of the Green functions we can reconstruct the actual Green functions as

\[
G^{X\tilde{X}}_{(N,M)}(u, \bar{u}; v, \bar{v}; \{x_i\}) = \int_{x_i;u' \in H}^{u} du' \int_{x_j;v' \in H}^{v} dv' g_{(N,M)}(u', v'; \{x_i\})
\]

\[
+ e^{-i2\pi \alpha_2} \int_{x_i;u' \in H}^{u} du' \int_{x_j;v' \in H}^{v} dv' l_{(N,M)}(u', v'; \{x_i\})
\]

\[
+ e^{i2\pi \alpha_2} \int_{x_i;\bar{u}' \in H^-}^{\bar{u}} d\bar{u}' \int_{x_j;\bar{v}' \in H^-}^{\bar{v}} d\bar{v}' h_{(N,M)}(\bar{u}', \bar{v}'; \{x_i\})
\]

\[
+ \int_{x_i;\bar{u}' \in H^-}^{\bar{u}} d\bar{u}' \int_{x_j;\bar{v}' \in H^-}^{\bar{v}} d\bar{v}' g_{(N,M)}(\bar{u}', \bar{v}'; \{x_i\})
\]
and

\[ G^{XX}_{(N,M)}(u, \bar{u}; v, \bar{v}; \{x_i\}) = \int_{x_i; u' \in H}^{u} du' \int_{x_j; v' \in H}^{v} dv' \, l_{(N,M)}(u', v'; \{x_i\}) \]
\[ + e^{i2\pi \alpha_2} \int_{x_i; u' \in H}^{u} du' \int_{x_j; v' \in H}^{v} dv' \, g_{(N,M)}(u', v'; \{x_i\}) \]
\[ + e^{i2\pi \alpha_2} \int_{x_i; \bar{u}' \in H}^{\bar{u}} du' \int_{x_j; \bar{v}' \in H}^{\bar{v}} dv' \, g_{(N,M)}(\bar{u}', \bar{v}'; \{x_i\}) \]
\[ + e^{i4\pi \alpha_2} \int_{x_i; \bar{u}' \in H}^{\bar{u}} du' \int_{x_j; \bar{v}' \in H}^{\bar{v}} dv' \, h_{(N,M)}(\bar{u}', \bar{v}'; \{x_i\}) \]

(71)

and

\[ G^{XX}_{(N,M)}(u, \bar{u}; v, \bar{v}; \{x_i\}) = \int_{x_i; u' \in H}^{u} du' \int_{x_j; v' \in H}^{v} dv' \, h_{(N,M)}(u', v'; \{x_i\}) \]
\[ + e^{-i2\pi \alpha_2} \int_{x_i; u' \in H}^{u} du' \int_{x_j; v' \in H}^{v} dv' \, g_{(N,M)}(v', u'; \{x_i\}) \]
\[ + e^{-i2\pi \alpha_2} \int_{x_i; \bar{u}' \in H}^{\bar{u}} du' \int_{x_j; \bar{v}' \in H}^{\bar{v}} dv' \, g_{(N,M)}(\bar{u}', \bar{v}'; \{x_i\}) \]
\[ + e^{-i4\pi \alpha_2} \int_{x_i; \bar{u}' \in H}^{\bar{u}} du' \int_{x_j; \bar{v}' \in H}^{\bar{v}} dv' \, l_{(N,M)}(\bar{u}', \bar{v}'; \{x_i\}) \]

(72)

where the arbitrariness of the lower integration limit is due to the constraints \[64\] which allow to change \(x_i \to x_k\).

### 4.1 The explicit \(N = 3, \, M_{cw} = 1\) case

In this case \(g_{(3,1)}\) is completely fixed by the local constraints to be

\[ g_{(3,1)}(z, w; \{x_i\}) = \frac{1}{(z-w)^2} \frac{1}{(\omega_{z} - 1)\omega_{w}^{c_{1}-1} \omega_{z}^{\epsilon_{2}-1}} \left[ (1-\epsilon_{2}-\epsilon_{3})\omega_{z}^{2} + (\epsilon_{2}+\epsilon_{3})\omega_{z}\omega_{w} - (1-\epsilon_{2})\omega_{z} - \epsilon_{3}\omega_{w} \right] \]

(73)

while \(h_{(3,1)} = 0\) since \(M = 1\). We get therefore a constraint from eq. \[65\] or the equivalent form \[68\] which read

\[ \omega_{w} \int_{0}^{1} d\omega \frac{1}{(\omega-\omega_{w})^{2}} (\omega-1)^{c_{1}-1} \omega^{\epsilon_{3}-1} [\epsilon_{2} + \epsilon_{3}] \omega - \epsilon_{3} \]
\[ + \int_{0}^{1} d\omega \frac{1}{(\omega-\omega_{w})^{2}} (\omega-1)^{c_{2}-1} \omega^{\epsilon_{3}-1} [(1-\epsilon_{2})\omega^{2} - (1-\epsilon_{3})\omega] = 0 \]

(74)
which can be read either as a constraint on the hypergeometric functions

\[
(1 - \epsilon_2 - \epsilon_3) \frac{B(2 + \epsilon_3, \epsilon_2)}{\omega_w} 2F_1(2, 2 + \epsilon_3; 2 + \epsilon_2 + \epsilon_3; \frac{1}{\omega_w}) \\
- (1 - \epsilon_3) \frac{B(1 + \epsilon_3, \epsilon_2)}{\omega_w} 2F_1(2, 1 + \epsilon_3; 1 + \epsilon_2 + \epsilon_3; \frac{1}{\omega_w}) \\
+ \left[ (\epsilon_2 + \epsilon_3) \frac{B(1 + \epsilon_3, \epsilon_2)}{\omega_w} 2F_1(2, 1 + \epsilon_3; 1 + \epsilon_2 + \epsilon_3; \frac{1}{\omega_w}) - \epsilon_3 \frac{B(\epsilon_3, \epsilon_2)}{\omega_w} 2F_1(2, 1 + \epsilon_3; \epsilon_2 + \epsilon_3; \frac{1}{\omega_w}) \right] \omega_w = 0
\]

or as infinite constraints on the coefficients of the \(\omega_w\) expansion which relate different Beta functions. We are now left to determine \(l_{(3,1)}\) from eq. (66), explicitly

\[
\omega_z^2 \int_0^1 d\omega \frac{1}{(\omega_z - \omega)^2 (\omega - 1)^{-\epsilon_2} \omega^{-\epsilon_3} (1 - \epsilon_2 - \epsilon_3)} \\
+ \omega_z \int_0^1 d\omega \frac{1}{(\omega_z - \omega)^2 (\omega - 1)^{-\epsilon_2} \omega^{-\epsilon_3} [(1 - \epsilon_2 - \epsilon_3) \omega^2 - (1 - \epsilon_3) \omega]} \\
+ \int_0^1 d\omega \frac{1}{(\omega_z - \omega)^2 (\omega - 1)^{-\epsilon_2} \omega^{-\epsilon_3} (-\epsilon_3 \omega)} \\
+ c_{00} \int_0^1 d\omega (\omega - 1)^{\epsilon_2 - 1} \omega^{\epsilon_3 - 1} = 0
\]

or

\[
\omega_z^2 (1 - \epsilon_2 - \epsilon_3) \tilde{I}_{2,0}^{(4)}(\epsilon_j; 2) \\
+ \omega_z [(1 - \epsilon_2 - \epsilon_3) \tilde{I}_{2,2}^{(4)}(\epsilon_j; 2) - (1 - \epsilon_3) \tilde{I}_{2,1}^{(4)}(\epsilon_j; 2)] \\
- \epsilon_3 \tilde{I}_{2,1}^{(4)}(\epsilon_j; 2) + c_{00} \tilde{I}_{2,0}^{(3)}(\epsilon_j) = 0
\]

where we have introduced the function

\[
\tilde{I}_{i,n}^{(N)}(\alpha_j; \beta) = \int_{\omega_{i+1}}^\omega d\omega \prod_{j=2}^N (\omega - \omega_j)^{-\alpha_j} (\omega - \omega_w)^{-\beta} \omega^n
\]

which is a slight modification of our previous definition and is still connected to the Lauricella functions \(F_D^{(n)}\). In the \(\omega_z \rightarrow \infty\) limit we can determine the unique unknown coefficient and hence the \(l_{(3,1)}\) normalization to be

\[
c_{00} = -(1 - \epsilon_2 - \epsilon_3) \frac{B(1 - \epsilon_2, 1 - \epsilon_3)}{B(\epsilon_2, \epsilon_3)}
\]

We get also infinite constraints from the subleading orders in \(\omega_z\) or plugging the previous value for \(c_{00}\) back into eq. (76) an equation of the form \(\sum B \ 2F_1 = 0\) as eq. (75).
4.2 The explicit $N = 4$, $M = 1$ case

Again as the case before $g_{(4,1)}$ is completely fixed by the local constraints only to be

$$g_{(4,1)}(z, w; \{ x_i \}) = \frac{1}{(z - w)^2} \frac{(\omega_z - 1)^{\epsilon_2 - 1}}{(\omega_z - \omega_3)^{\epsilon_2}} \frac{(\omega_z^2 - \omega_3)^{\epsilon_3 - 1}}{(\omega_z^2 - \omega_3)^{\epsilon_3}} \frac{\omega_z^{\epsilon_4 - 1}}{\omega_z^{\epsilon_4}} \left[ \epsilon_1 \omega_z^3 + (1 - \epsilon_1) \omega_z^2 \omega_3 - [(1 - \epsilon_2 - \epsilon_4) + (1 - \epsilon_2 - \epsilon_4)] \omega_z^2 
- [(\epsilon_3 + \epsilon_4) + (\epsilon_2 + \epsilon_4) \omega_3] \omega_z + (1 - \epsilon_4) \omega_3 \omega_z + \epsilon_4 \omega_3 \omega_4 \right]$$

(80)

and $h_{(4,1)} = 0$ since $M = 1$. As in the $(3,1)$ case from eq. (65) or the equivalent form (68) we get the constraints

$$-\omega_w^2 [(1 - \epsilon_3 - \epsilon_4) + (1 - \epsilon_2 - \epsilon_4) \omega_3] \tilde{I}^{(5)}_{i,0} (1 - \epsilon_j; 2)$$
$$+ \omega_w [(1 - \epsilon_1) \tilde{I}^{(5)}_{i,0} (1 - \epsilon_j; 2) - (\epsilon_3 + \epsilon_4) + (\epsilon_2 + \epsilon_4) \omega_3] \tilde{I}^{(5)}_{i,1} (1 - \epsilon_j; 2) + \epsilon_4 \omega_3 \tilde{I}^{(5)}_{i,0} (1 - \epsilon_j; 2)$$
$$+ \epsilon_1 \tilde{I}^{(4)}_{i,0} (1 - \epsilon_j) + (1 - \epsilon_4) \omega_3 \tilde{I}^{(4)}_{i,1} (1 - \epsilon_j) = 0$$

(81)

In particular notice that $\tilde{I}^{(5)}_{i,n} \sim F_D^{(2)}$ is the Appell function.

We can proceed to determine the $l_{(4,1)}$ function. This amounts to fixing the four functions $c_0, c_1, c_{10}, c_{11}$ from eq. (66) which reads

$$(-1)^{i+1} \left\{ \omega_z^3 \epsilon_1 \tilde{I}^{(5)}_{i,0} (\epsilon_j; 2) + \omega_z^2 (1 - \epsilon_1) \tilde{I}^{(5)}_{i,1} (\epsilon_j; 2) - \omega_z [(1 - \epsilon_3 - \epsilon_4) + (1 - \epsilon_2 - \epsilon_4) \omega_3] \tilde{I}^{(5)}_{i,1} (1 - \epsilon_j; 2)$$
$$- \omega_z [(\epsilon_3 + \epsilon_4) + (\epsilon_2 + \epsilon_4) \omega_3] \tilde{I}^{(5)}_{i,0} (\epsilon_j; 2) + \omega_z (1 - \epsilon_4) \omega_3 \tilde{I}^{(5)}_{i,0} (\epsilon_j; 2)$$
$$+ c_0 \tilde{I}^{(4)}_{i,0} (1 - \epsilon_j) + c_1 \tilde{I}^{(4)}_{i,1} (1 - \epsilon_j) + \omega_z c_{10} \tilde{I}^{(4)}_{i,0} (1 - \epsilon_j) + \omega_z c_{11} \tilde{I}^{(4)}_{i,1} (1 - \epsilon_j) = 0$$

(82)

for $i = 2, 3$. When we consider the $\omega_z \rightarrow \infty$ limit we get two sets of equations, the one from the coefficient of $\omega_z$

$$(-1)^{i+1} \epsilon_1 \tilde{I}^{(4)}_{i,0} (\epsilon_j) + c_10 \tilde{I}^{(4)}_{i,0} (1 - \epsilon_j) + c_{11} \tilde{I}^{(4)}_{i,1} (1 - \epsilon_j) = 0$$

(83)

and the other from the coefficient of $\omega_z^0$

$$(-1)^{i+1} (1 + \epsilon_1) \tilde{I}^{(4)}_{i,1} (\epsilon_j) + c_{00} \tilde{I}^{(4)}_{i,0} (1 - \epsilon_j) + c_{01} \tilde{I}^{(4)}_{i,1} (1 - \epsilon_j) = 0$$

(84)

plus an infinite set of constraints from the coefficients of the polar expansion in $\omega_z$ or, equivalently plugging the previous value back into eq. (82) and equation of the form $(2F_1)^2 F_D^{(2)} + \sum (2F_1)^3 = 0$ analogously to eq. (75).

4.3 The explicit $N = 4$, $M = 2$ case

This is the first case where there are more unknown coefficients than equations from the local constraints and therefore we must use the global constraints to
fix completely \( g(4,2) \) and determine both \( h(4,2) \) and \( l(4,2) \) which are now both not vanishing. We can nevertheless fix the singular part \( g_{s(4,2)} \) by choosing \( a_{20} = 0 \) so we can get

\[
g_{s(4,2)} = \frac{1}{(z - w)^2} \frac{(\omega_z - 1)\epsilon_2^{-1} (\omega_z - \omega_3)^{\epsilon_3-1} \omega_z^{\epsilon_1-1}}{(\omega_w - \omega_3)\epsilon_3 \omega_w^{\epsilon_3}}
\]

\[
\{ \epsilon_1 \omega_z^2 \omega_w + (1 - \epsilon_1) \omega_z \omega_w^2 - [(2 - \epsilon_3 - \epsilon_4) + (2 - \epsilon_2 - \epsilon_4) \omega_3] \omega_z \omega_w \\
+ [(1 - \epsilon_3 - \epsilon_4) + (1 - \epsilon_2 - \epsilon_4) \omega_3] \omega_w^2 + (1 - \epsilon_4) \omega_3 \omega_2 + \epsilon_4 \omega_3 \omega_w \}
\]

Using the global constraints for \( g \) and \( h \) as given in eq. (68) for \( i = 2, 3 \) it is then possible to determine \( a_{00} \) (which corresponds to \( a_{20} \) after the split of \( g(4,2) \) into a regular and singular part) and \( b_{00} \), in particular taking the \( \omega_w \to \infty \) limit we get

\[
\begin{align*}
\bar{a}_{00} & I_{2,0}^4 (1 - \epsilon) - b_{00} I_{2,0}^4 (\epsilon) = -(1 - \epsilon_1) I_{2,1}^4 (1 - \epsilon) - [(1 - \epsilon_3 - \epsilon_4) + (1 - \epsilon_2 - \epsilon_4) \omega_3] I_{2,0}^4 (1 - \epsilon) \\
\bar{a}_{00} & I_{3,0}^4 (1 - \epsilon) + b_{00} I_{3,0}^4 (\epsilon) = -(1 - \epsilon_1) I_{3,1}^4 (1 - \epsilon) - [(1 - \epsilon_3 - \epsilon_4) + (1 - \epsilon_2 - \epsilon_4) \omega_3] I_{3,0}^4 (1 - \epsilon)
\end{align*}
\]

(86)

where the minus sign in the lhs of the first line is due a careful treatment of phases. In the limit \( \omega_w \to \infty \) eq. (69) allows to fix \( c_{00} \) and again \( \bar{a}_{00} \) as

\[
\begin{align*}
\bar{a}_{00} & I_{2,0}^4 (\epsilon) - c_{00} I_{2,0}^4 (1 - \epsilon) = -\epsilon_1 I_{2,1}^4 (\epsilon) \\
\bar{a}_{00} & I_{3,0}^4 (\epsilon) + c_{00} I_{3,0}^4 (1 - \epsilon) = -\epsilon_1 I_{3,1}^4 (\epsilon)
\end{align*}
\]

(87)

The two previous ways of fixing \( \bar{a}_{00} \) must be compatible and this can be easily verified at least in the \( \omega_3 \to 1^- \) limit.

5 The quantum twists correlators

In this section we want to compute the \( N \) twists correlators in the \( N - 2 \) different sectors determined by \( M \). We can generically write the \( N \) twists correlators in the \( M \) sector as

\[
\prod_{i=1}^{N} \sigma_{i,f_i}(x_i) = A_{(N,M)}(\omega_j \neq 1,2,N) e^{-S_{E,M}(\sigma_{i,f_i})} \prod_{1 \leq i < j \leq N}(x_i - x_j)^{\Delta_{ij}}
\]

(88)

The powers \( \Delta_{ij} \) can be completely fixed as follows. From the proper behavior for \( x_i \to \infty \) we get the constraints \( \sum_{i \neq i} \Delta_{ij} = 2\Delta(\sigma_{i,f_i}) = \epsilon_i (1 - \epsilon_i) \) where we have defined \( \Delta_{ji} = \Delta_{ij} \) for \( j > i \). Now redefining \( A \to A \prod_{3 \leq i \leq N-1} (\omega_i - \omega_j) \Delta_{ij} \prod_{3 \leq i \leq N-1} (1 - \omega_i) \Delta_{2i} \prod_{3 \leq i \leq N-1} \omega_i^{\Delta_{iN}} \) and remembering that \( \omega_2 = 1, \omega_N = 0 \) and \( (\omega_i - \omega_j) \propto (x_i - x_j) \) we can set all \( \Delta \)s to zero but \( \Delta_{1i}, \Delta_{12}, \Delta_{1N} \) (\( 3 \leq i \leq N-1 \)) and \( \Delta_{2N} \) which can now be fixed by the first set of constraints. Therefore we can choose a “gauge” where the previous correlator can be
\[
\prod_{i=1}^{N} \sigma_{\epsilon_i, f_i}(x_i) = \frac{1}{\prod_{3 \leq i \leq N-1} (x_1 - x_i)^{\epsilon_i(1-\epsilon_i)}} \\
\cdot \frac{1}{(x_1 - x_2)^{\frac{1}{2}[\epsilon_1(1-\epsilon_1)-\sum_{i=3}^{N-1} \epsilon_i(1-\epsilon_i)+\epsilon_2(1-\epsilon_2)-\epsilon_N(1-\epsilon_N)]}} \\
\cdot \frac{1}{(x_1 - x_N)^{\frac{1}{2}[-\epsilon_1(1-\epsilon_1)+\sum_{i=3}^{N-1} \epsilon_i(1-\epsilon_i)+\epsilon_N(1-\epsilon_N)-\epsilon_2(1-\epsilon_2)]}} \\
\cdot \frac{1}{(x_2 - x_N)^{\frac{1}{2}[2\epsilon_2(1-\epsilon_2)+\epsilon_N(1-\epsilon_N)-\epsilon_1(1-\epsilon_1)+\sum_{i=3}^{N-1} \epsilon_i(1-\epsilon_i)]}} \\
\cdot A_{(N,M)}(\omega_j \neq 1, 2, N) \ e^{-S_{E,cl}(x_i, \epsilon_i, f_i)}
\]

We can now proceed in the usual way. We first compute the expectation value of the energy-momentum tensor as

\[
\langle\langle T(z) \rangle\rangle = \frac{\langle \partial \chi_q(z) \partial \bar{\chi}_q(z) \sigma_{\epsilon_1, f}(x_1) \cdots \sigma_{\epsilon_N, f}(x_N) \rangle}{\langle \sigma_{\epsilon_1, f}(x_1) \cdots \sigma_{\epsilon_N, f}(x_N) \rangle}
\]

then using the OPE

\[
T(z) \sigma_{\epsilon_i, f_i}(x_i) \sim \frac{\epsilon_i(1-\epsilon_i)}{(z-x_i)^2} + \frac{\partial_x \sigma_{\epsilon_i, f_i}(x_i)}{z-x_i} + O(1)
\]

we compute

\[
\partial_x \ln \prod_{i=1}^{N} \sigma_{\epsilon_i, f}(x_i) = \lim_{z \to x_i} \left( z - x_i \right) \left[ \langle\langle T(z) \rangle\rangle - \epsilon_i(1-\epsilon_i) \right]
\]

The function \( A_{(N,M)} \) in the quantum case where \( f_i = f \) can be determined from eq. (88) \((j \neq 1, 2, N)\) as

\[
\partial x_i \ln \prod_{i=1}^{N} \sigma_{\epsilon_i, f}(x_i) = - \sum_{j \neq i} \frac{\Delta_{ij}}{x_j - x_i} + \frac{\partial \omega_j}{\partial x_j} \frac{\partial \ln A_{(N,M)}}{\partial \omega_j}
\]

5.1 The \( M_{ccw} = N - 1, M_{cw} = 1 \) cases

Using the expansion (61) for \( S \) and the constraints (62) we can easily deduce that

\[
\langle\langle T(z) \rangle\rangle = \frac{1}{2} \left( \frac{\partial \omega_j}{\partial z} \right)^2 \left[ \sum_{j=2}^{N} \frac{\epsilon_j}{(\omega_j - \omega_j)^2} - \left( \sum_{j=2}^{N} \frac{\epsilon_j}{\omega_j - \omega_j} \right)^2 \right] + \prod_{j=2}^{N} \frac{1}{\omega_j} \frac{\partial^2 S}{\partial \omega_j^2} \bigg|_{\omega_j = \omega_j}
\]
It then follows that \((j \neq 1, 2, N)\)

\[
\frac{\partial}{\partial x_j} \log \left( \prod_{i=1}^{N} \sigma_{\epsilon_i, f_i = f(x_i)} \right) = -\frac{\epsilon_j}{x_j - x_1} \\
+ \epsilon_j \left[ -\sum_{l \neq 1, j} \frac{\epsilon_l}{x_j - x_l} + \frac{M - \epsilon_1}{x_j - x_1} \right] \\
+ \frac{1}{2} \prod_{l \neq 1, j} \frac{1}{\omega_j - \omega_l} \frac{\partial^2 S}{\partial \omega_j \partial \omega_l} \bigg|_{\omega_w = \omega_z = \omega_j}
\]

(95)

from which we can obtain \(A_{(N,M)}\) using eq. (93) to get

\[
\frac{\partial \ln A_{(N,M)}}{\partial \omega_j} = \frac{1}{2} \prod_{l=2; l \neq j}^{N} \frac{1}{\omega_j - \omega_l} \frac{\partial^2 S}{\partial \omega_j \partial \omega_l} \bigg|_{\omega_w = \omega_z = \omega_j} + \sum_{l=2; l \neq j}^{N} \frac{\Delta_{jl} - \epsilon_j \epsilon_l}{\omega_j - \omega_l}
\]

(96)

The main issue is then to compute \(\frac{\partial^2 S}{\partial \omega_w \partial \omega_z}\). This can be done immediately in two cases, i.e. \(M_{cw} = 1\) for which \(\frac{\partial^2 S}{\partial \omega_w \partial \omega_z} = 0\) since the maximum \(\omega_w\) power is 1 and \(M_{ccw} = N - 1\) for which \(\frac{\partial^2 S}{\partial \omega_z \partial \omega_z} = 0\) since the maximum \(\omega_z\) power is 1 as it is obvious from eq. (61). In the former case we get

\[
\langle \prod_{i=1}^{N} \sigma_{\epsilon_i, f_i = f(x_i)} \rangle = C_{(N,M=1)}(\epsilon) \prod_{3 \leq j < l \leq N-1} (\omega_j - \omega_l)^{-\epsilon_j \epsilon_l} \\
\times \prod_{3 \leq l \leq N-1} (1 - \omega_l)^{-\epsilon_2 \epsilon_l} \\
\times \prod_{3 \leq j \leq N-1} \omega_j^{-\epsilon_j \epsilon_N} \\
\times \frac{1}{(x_1 - x_2)^{\frac{1}{2} \epsilon_1 (1 - \epsilon_1) - \sum_{i=3}^{N-1} \epsilon_i (1 - \epsilon_i) + \epsilon_2 (1 - \epsilon_2) - \epsilon_N (1 - \epsilon_N)}} \\
\times \frac{1}{(x_1 - x_N)^{\frac{1}{2} \epsilon_1 (1 - \epsilon_1) - \sum_{i=3}^{N-1} \epsilon_i (1 - \epsilon_i) + \epsilon_N (1 - \epsilon_N) - \epsilon_2 (1 - \epsilon_2)}} \\
\times \frac{1}{(x_2 - x_N)^{\frac{1}{2} \epsilon_2 (1 - \epsilon_2) + \epsilon_N (1 - \epsilon_N) - \epsilon_1 (1 - \epsilon_1) + \sum_{i=3}^{N-1} \epsilon_i (1 - \epsilon_i)}}
\]

(97)

while in the latter we get the same result but with the substitution \(\epsilon \rightarrow 1 - \epsilon\) by expanding \(\omega_z\) around \(\omega_w\). The coefficients \(C_{N,1}\) and \(C_{N,N-1}\) will be fixed in section 5.4 and are given in eqs (137).

### 5.2 The \(N \geq 4\) and \(N - 2 \geq M \geq 2\) cases

For all the other cases it is enough to use a slight modification of the technique used in (13) (see also [5]).

The main idea of this approach is to define a new basis for the classical solutions (see eqs (29)) and consequently for the non singular part of the derivative of the Green function \(g(z, w)\) (see eq. (67)) which are closed under certain operations needed to compute the correlators.
We start therefore by defining a new basis for the classical solutions

\[
\partial_\omega \mathcal{X}^{(I)}(\omega) = \prod_{j=2}^{N} (\omega - \omega_j)^{-1} \prod_{l \in S_I} (\omega - \omega_l) \quad I \in S
\]

\[
\partial_\omega \bar{\mathcal{X}}^{(I)}(\omega) = \prod_{j=2}^{N} (\omega - \omega_j)^{-c_j} \prod_{l \in \bar{S}_I} (\omega - \omega_l) \quad \bar{I} \in \bar{S}
\]

where we have defined two ordered sets

\[
S = \{N - M - 1 \text{ arbitrary different indexes chosen among } 3, \ldots, N-1\} \quad \text{(99)}
\]

and

\[
\bar{S} = \{M - 1 \text{ arbitrary different indexes chosen among } 3, \ldots, N-1\} \quad \text{(100)}
\]

and the subsets \(S_I = S - \{I\}\) for any \(I \in S\) and similarly for \(\bar{S}_I\). In order to be able to define the previous basis as a linear combination of the original one \((30)\) we need that both \(n \geq 0\) and \(r \geq 0\), i.e. \(N - 2 \geq M \geq 2\). In particular what follows works even if either \(S_I = \emptyset\) or \(\bar{S}_I = \emptyset\), i.e. \(M = N - 2\) or \(M = 2\) for example when \(N = 4\) and \(M = 2\).

We can now expand the regular part of \(g\) and \(h, l\) as

\[
g_{r(N,M)}(z, w; \{x_i\}) = \frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w} \sum_{I \in S} \sum_{l \in S} \bar{a}_{II}(\omega_j) \partial_\omega \mathcal{X}^{(I)}(\omega_z) \partial_\omega \bar{\mathcal{X}}^{(I)}(\omega_w)
\]

\[
h_{r(N,M)}(z, w; \{x_i\}) = \frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w} \sum_{I \in S} \sum_{J \in S} \bar{b}_{IJ}(\omega_j) \partial_\omega \mathcal{X}^{(J)}(\omega_z) \partial_\omega \bar{\mathcal{X}}^{(J)}(\omega_w)
\]

\[
l_{r(N,M)}(z, w; \{x_i\}) = \frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w} \sum_{I \in S} \sum_{J \in S} \bar{c}_{IJ}(\omega_j) \partial_\omega \mathcal{X}^{(J)}(\omega_z) \partial_\omega \bar{\mathcal{X}}^{(J)}(\omega_w)
\]

\[(101)\]

Then we can find a solution of the first of the constraints in eq. \((64)\) as

\[
g_{s(N,M)}(z, w; \{x_i\}) = g_{s(N,M)}(z, w; \{x_i\}) - \frac{\partial \omega_z}{\partial z} \sum_{i=1}^{N-2} (W^{-1})^i \partial_\omega \mathcal{X}^{(I)}(\omega_z) \int_{I_i} \frac{d\omega}{(\omega - \omega_w)^2} P \Sigma_0(\omega, \omega_w)
\]

\[
= g_{s(N,M)}(z, w; \{x_i\}) - \frac{\partial \omega_w}{\partial w} \sum_{i=1}^{N-2} (W^{-1})^i \partial_\omega \bar{\mathcal{X}}^{(I)}(\omega_w) \int_{I_i} \frac{d\omega}{(\omega - \omega_z)^2} P \Sigma_0(\omega, \omega)
\]

\[(102)\]

where \(S_0(\omega_z, \omega_w)\) is the same as in the second equation in \((61)\) but for the singular part of \(g\), i.e. with coefficients \(a_{(0)}\) as in eq. \((67)\). Moreover we have
These integrals correspond to the intervals for which coefficient.

\[ W_i^I = \int_{\omega_{i+2}}^{\omega_{i+1}} d\omega \partial_{\omega} \mathcal{A}^{(I)}(\omega + i0^+) \quad i = 1, \ldots N - 2, \quad I \in S \]

\[ W_i^I = \int_{\omega_{i+2}}^{\omega_{i+1}} d\omega \partial_{\omega} \tilde{\mathcal{A}}^{(I)}(\omega - i0^+) \quad i = 1, \ldots N - 2, \quad I \in \tilde{S} \]

(103)

From this expression is immediate to compute the energy-momentum tensor expectation value which can be split into a singular part as in eq. (94) but with the substitution \( S \) and eq. (106) would contain the sum over all possible \( J \)

\[
\langle \langle T_r(w) \rangle \rangle = -\frac{\partial \omega_J}{\partial w} \sum_{i=1}^{N-2} (W^{-1})^i \int_{I_i} \frac{d\omega}{(\omega - \omega_J)^{3-\epsilon_J}} \prod_{j=2; j \neq J}^{N} \frac{1}{\prod_{j \in S} (\omega_J - \omega_j)} S_0(\omega, \omega_J)
\]

(104)

where \( j \notin S_J \) means \( j \in \{2, \ldots N\} \setminus S_J \). If we consider \( J \in S \) we can then evaluate

\[
\lim_{w \rightarrow x_J} \langle \langle T_r(w) \rangle \rangle = -\frac{\partial \omega_J}{\partial x_J} \sum_{i=1}^{N-2} (W^{-1})^i \int_{I_i} \frac{d\omega}{(\omega - \omega_J)^{3-\epsilon_J}} \prod_{j=2; j \neq J}^{N} \frac{1}{\prod_{j \in S} (\omega_J - \omega_j)} S_0(\omega, \omega_J)
\]

(105)

Now following [14] we rewrite the integrand as

\[
\frac{1}{(\omega - \omega_J)^{3-\epsilon_J}} \prod_{j=2; j \neq J}^{N} \frac{1}{\prod_{j \in S} (\omega_J - \omega_j)} S_0(\omega, \omega_J) = \partial_{\omega_J} \partial_{\omega} \mathcal{A}^{(J)}(\omega) + \sum_{L \in S} T_L^J \partial_{\omega} \mathcal{A}^{(L)}(\omega)
\]

(106)

. The leading singularity is \( O((\omega - \omega_J)^{-2+\epsilon_J}) \) because \( S_0(\omega_J, \omega_J) = 0 \) as follows from the first equation in (62) when evaluated for \( \omega_w = \omega_J \). Moreover when the left hand side is subtracted the leading singularity and multiplied for \( \prod_{j=2}^{N}(\omega - \omega_J)^{1-\epsilon_J} / \prod_{j \in S} (\omega - \omega_j) \) we are left with a rational function with poles at \( \omega_J \) \( (I \in S) \) which vanish at \( \omega = \infty \) as the right hand side. Because of the sum over \( \omega \) in eq. (105) the only \( T^J_L \) needed is

\[
T^J_L = -(1 - \epsilon_J) \sum_{l \in S_J} \frac{1}{\omega_J - \omega_l} + \frac{1}{2} \prod_{l \in S_J} \frac{1}{\omega_J - \omega_l} \prod_{l \notin S} \frac{1}{\omega_J - \omega_l} \partial_{\omega_J}^2 S_0(\omega_J, \omega_J)
\]

(107)

---

9 Again as in eqs (64) it is important that the integration is once above and once below the cut as this ensures that both integrals have the same phase modulus \( \pi \).

10 There are two integrals which are actually divergent but their sum is however convergent. These integrals correspond to the intervals for which \( \omega_J \) is a boundary point.

11 We restrict to the case \( J \in S \) because otherwise eq. (105) would contain the sum over all \( I \in S \) and eq. (106) would contain the sum over all possible \( \partial_{\omega_J} \partial_{\omega} \mathcal{A}^{(J)}(\omega) \) each with a non trivial coefficient.
When we insert this value into eq. (105) and add the contribution from the singular part which has the same expression as eq. (95) with $S \rightarrow S_0$ and $\epsilon \rightarrow 1 - \epsilon$ since we have here $\partial^2_{\omega_i} S_0$ we get

$$
\partial_{x_J} \log \left( \prod_{i=1}^{N} \sigma_{\epsilon_i, f_i = f}(x_i) \right) = -\frac{(1 - \epsilon_J)(2 - \epsilon_J)}{x_J - x_1}
- 2(1 - \epsilon_J) \left[ \sum_{l \neq 1, J} \frac{1 - \epsilon_l}{x_J - x_l} + \frac{M - N + 1 - \epsilon_1}{x_J - x_1} \right]
- \frac{\partial \omega_J}{\partial x_J} \left[ \sum_{i=1}^{N-2} (W^{-1})^i_j \partial_{\omega_j} W_i^J - (1 - \epsilon_J) \sum_{l \in S_J} \frac{1}{\omega_j - \omega_l} \right]
$$

(108)

from which the dependence on $S_0$ has disappeared but we are left with a dependence on $\partial \omega_J W_i^J$. Differently from what done in [14] we cannot rely on the fact that twists have both an holomorphic and antiholomorphic dependence in order to end the computation using

$$
\det W = \sum_{i=1}^{N-2} (W^{-1})^i_j \partial_{\omega_j} W_i^J + \sum_{i=1}^{N-2} (W^{-1})^i_j \partial_{\omega_j} W_i^J
$$

(109)

and

$$
\partial_{\omega_j} W_i^J \neq J = \frac{\epsilon_I}{\omega_J - \omega_I} \left( W_i^J - W_i^I \right)
$$

(110)

Instead we have to rely on the second of the (64) constraints (or better its complex conjugate which is has the same expression with the substitution $\pm i0^+ \rightarrow \mp i0^+$). Analogously as before we require $J \in S$ and we get

$$
\partial_{x_J} \log \left( \prod_{i=1}^{N} \sigma_{\epsilon_i, f_i = f}(x_i) \right) = -\frac{\epsilon_J(1 + \epsilon_J)}{x_J - x_1}
- 2\epsilon_J \left[ \sum_{l \neq 1, J} \frac{\epsilon_l}{x_J - x_l} + \frac{M + \epsilon_1}{x_J - x_1} \right]
- \frac{\partial \omega_J}{\partial x_J} \left[ \sum_{i=1}^{N-2} (W^{-1})^i_j \partial_{\omega_j} W_i^J - \epsilon_J \sum_{l \in S_J} \frac{1}{\omega_j - \omega_l} \right]
$$

(111)

then only if $J = J \in S \cap \bar{S}$ we can average the previous expressions (108) and (111) Into this average we can use the analogous expression of eq. (110)

$$
\partial_{\omega_j} W_i^{J \neq j} = \frac{1 - \epsilon_J}{\omega_j - \omega_J} \left( W_i^J - W_i^J \right)
$$

(112)
\[ \begin{align*}
\sum_{i=1}^{N-2} (W^{-1})_i^j \partial_{\omega_j} W_i^j + \sum_{i=1}^{N-2} (W^{-1})_i^j \partial_{\omega_j} W_i^j &= \partial_{\omega_j} \det W - \sum_{l \in S_j} (W^{-1})_i^j \partial_{\omega_j} W_i^j - \sum_{l \in S_j} (W^{-1})_i^j \partial_{\omega_j} W_i^j \\
&= \partial_{\omega_j} \det W + \sum_{l \in S_j} \frac{\epsilon_l}{\omega_j - \omega_l} + \sum_{l \in S_j} \frac{1 - \epsilon_l}{\omega_j - \omega_l}
\end{align*} \]

(113)

to get finally
\[ \partial_{\omega_j} \log A_{(N,M)}(\omega_j) = \partial_{\omega_j} \log \left[ \left( \det W \right)^{-\frac{1}{2}} \prod_{l \in S_j} (\omega_J - \omega_l)^{\frac{1}{2}} \left( \prod_{l \in S_j} (\omega_J - \omega_l)^{\Delta_{Jl} - \frac{1}{2}[(1-\epsilon_J)(1-\epsilon_l)+\epsilon_J\epsilon_l]} \right) \right] \]

(114)

The previous equation is valid only for \( J \in S \cap \bar{S} \) but if, in either \( S \) or in \( \bar{S} \) there is at least one further element than those contained in \( S \cap \bar{S} \) or if \( S \cap \bar{S} \) contains all the independent \( \omega_j \) as in the \( N = 4 \) case, we can deduce that
\[ A_{(N,M)}(\omega_j) = \text{const} \left( \det W \right)^{-\frac{1}{2}} \prod_{\text{ord}(I) < \text{ord}(J); I,J \in S} (\omega_I - \omega_J)^{\frac{1}{2}} \prod_{\text{ord}(I) < \text{ord}(J); I,J \in S} (\omega_I - \omega_J)^{\frac{1}{2}} \]

\[ \prod_{2 \leq j < l \leq N} (\omega_J - \omega_l)^{\Delta_{Jl} - \frac{1}{2}[(1-\epsilon_J)(1-\epsilon_l)+\epsilon_J\epsilon_l]} \]

(115)
as a consequence of the independence of the result under a change of the elements of \( S \) and/or \( \bar{S} \). Under the change \( S \to S' = (S \setminus \{I_0\}) \cup \{I_1\} \) the integrals \( W_{(S) i}^{(L)} \) in eqs (113) transform as
\[ W_{(S') i}^{(L)} = \frac{\omega_{I_1} - \omega_L}{\omega_{I_0} - \omega_L} W_{(S) i}^{(L)} + \frac{\omega_{I_1} - \omega_{I_0}}{\omega_L - \omega_{I_0}} W_{(S) i}^{(I_0)} \quad L \neq I_0 \]

\[ W_{(S') i}^{(I_1)} = W_{(S) i}^{(I_0)} \]

(116)

so that the transformation of the determinant
\[ \det W_{S',S} = \det W_{S,S} \prod_{L \in S_{I_0}} \frac{\omega_{I_1} - \omega_L}{\omega_{I_0} - \omega_L} \]

(117)
is what is needed to compensate the change \( \prod_{\text{ord}(I) < \text{ord}(J); I,J \in S'} (\omega_I - \omega_J)^{\frac{1}{2}} \rightarrow \prod_{\text{ord}(I) < \text{ord}(J); I,J \in S} (\omega_I - \omega_J)^{\frac{1}{2}} \).

\(^{12}\) In the expression we have used \( \text{ord}(I) \) to indicate the order of \( I \) in the ordered set \( S \).
The final expression for the $N$ twists correlator in the $M$ sector is then

$$
\langle \prod_{i=1}^{N} \sigma_{i,j}=f(x_i) \rangle = C_{(N,M)}(\epsilon) \frac{\prod_{3 \leq j < l \leq N-1}(\omega_j - \omega_l)^{-\frac{1}{2}[(1-\epsilon_j)(1-\epsilon_l)+\epsilon_j \epsilon_l]}}{\prod_{3 \leq i \leq l \leq N-1}(x_i - x_j) \epsilon_i (1-\epsilon_i)}
$$

\begin{align*}
&\cdot \frac{\prod_{3 \leq j \leq N-1}(1 - \omega_j)^{-\frac{1}{2}[(1-\epsilon_j)(1-\epsilon_j)+\epsilon_j \epsilon_j]}}{(x_1 - x_2)^{\frac{1}{2}[(1-\epsilon_j)(1-\epsilon_j)+\epsilon_j \epsilon_j]}}
&\cdot \prod_{3 \leq j \leq N-1} \omega_j^{-\frac{1}{2}[(1-\epsilon_j)(1-\epsilon_N)+\epsilon_j \epsilon_N]}
&\cdot \frac{1}{(x_1 - x_N)^{\frac{1}{2}[(1-\epsilon_j)(1-\epsilon_N)+\epsilon_j (1-\epsilon_N)]}}
&\cdot (\det W_{S,S})^{-\frac{1}{2}} \prod_{\text{ord}(I) < \text{ord}(J) ; I,J \in S} (\omega_I - \omega_J)^{\frac{1}{2}} \prod_{\text{ord}(I) < \text{ord}(J) ; I,J \in S} (\omega_I - \omega_J)^{\frac{1}{2}} \tag{118}
\end{align*}

Notice that the previous expression is true even if there is only one element in $S$ in which case the product $\prod (\omega_I - \omega_J)^{\frac{1}{2}}$ is simply 1. Similarly for the $S$ case. In subsection 5.3 we will fix the constant $C_{(N,M)}(\epsilon)$.

### 5.3 $N-1$ amplitudes from $N$ amplitudes

We want to check the consistency of the results of the previous section. We do this by making $x_{j+1}$ coalesce with $x_j$ and so deducing the $N-1$ twists correlators from $N$ twists ones.

We start noticing that from the $(N, M)$ sector we can generically compute both $(\tilde{N}, \tilde{M}) = (N-1, M)$ and $(\tilde{N}, \tilde{M}) = (N-1, M-1)$ sectors depending whether $\epsilon_j + \epsilon_{j+1} < 1$ or $\epsilon_j + \epsilon_{j+1} > 1$. Exceptions are the $M = 1$ case where only $\tilde{M} = 1$ is possible and $M = N-1$ where only $\tilde{M} = \tilde{N} - 1 = N - 2$ is possible.

#### 5.3.1 $(N, 1)$ into $(N-1, 1)$ case

Starting from eq. (97) we can very easily take the limit $x_{j+1} \rightarrow x_j$. When we use

$$
\epsilon_j (1 - \epsilon_j) + \epsilon_{j+1} (1 - \epsilon_{j+1}) = \tilde{\epsilon}_j (1 - \tilde{\epsilon}_j) + 2 \epsilon_j \epsilon_{j+1} \quad \tilde{\epsilon}_j = \epsilon_j + \epsilon_{j+1} \tag{119}
$$

and

$$
\omega_j - \omega_{j+1} = \frac{(x_{j+1} - x_j)(x_N - x_1) x_2 - x_1}{(x_j - x_1)(x_{j+1} - x_1) x_2 - x_N} \quad \frac{1}{2} \tag{120}
$$

we find

$$
\langle \prod_{i=1}^{N} \sigma_{i,j}=f(x_i) \rangle \sim_{x_{j+1} \rightarrow x_j} (x_j - x_{j+1})^{\epsilon_j \epsilon_{j+1}} M(\epsilon_j, \epsilon_{j+1}) \prod_{i=1}^{N-1} \sigma_{i,j}=f(x_i) \tag{121}
$$

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and the consistency relation for the normalizations

\[ C_{(N,1)}(\epsilon) = C_{(N-1,1)}(\tilde{\epsilon}) M(\epsilon_j, \epsilon_{J+1}) \]  

(122)

where \( \tilde{\epsilon} \) are the twists of the \((N-1,1)\) theory defined by \( \tilde{\epsilon}_j = \epsilon_j \) for \( j < J \), \( \tilde{\epsilon}_j = \epsilon_j + \epsilon_{J+1} \) for \( j = J \) and \( \tilde{\epsilon}_j = \epsilon_{j+1} \) for \( j > J \). Actually all the previous equations work even when we consider the \( \omega_j \to \infty \) \((x_j \to x_1)\) limit.

5.3.2 \((N, N-1)\) into \((N-1, N-2)\) case

In a way completely analogous to that done in the previous subsection we get

\[ \langle \prod_{i=1}^{N} \sigma_{\epsilon_i,J}(x_i) \rangle_{x_{J+1} \to x_J} (x_J - x_{J+1})^{-(1-\epsilon_{J})(1-\epsilon_{J+1})} N(\epsilon_j, \epsilon_{J+1}) \langle \prod_{i=1}^{N-1} \sigma_{\epsilon_i,J}(x_i) \rangle \]  

(123)

and the consistency relation the consistency relation for the normalization coefficients

\[ C_{(N,1)}(\epsilon) = C_{(N-1,1)}(\tilde{\epsilon}) N(\epsilon_j, \epsilon_{J+1}) \]  

(124)

where \( \tilde{\epsilon} \) are the twists of the \((N-1,1)\) theory defined by \( \tilde{\epsilon}_j = \epsilon_j \) for \( j < J \), \( \tilde{\epsilon}_j = \epsilon_j + \epsilon_{J+1} - 1 \) for \( j = J \) and \( \tilde{\epsilon}_j = \epsilon_{j+1} \) for \( j > J \). Again all works in the \( \omega_j \to \infty \) \((x_j \to x_1)\) limit.

5.3.3 \((N, M)\) into \((N-1, M)\) with \(2 \leq M \leq N-2\) case

In this case we start from the general expression \[118\] and choose the sets \( S \) and \( \tilde{S} \) so that \( J \in S, J \notin \tilde{S} \) and \( J+1 \notin S \) then we now show that the new sets \( \tilde{S} \) and \( \tilde{\tilde{S}} \) are given by \( \tilde{S} = S_J = S \setminus \{J\} \) and \( \tilde{\tilde{S}} = \tilde{S} \).

The previous choices are dictated by the need of having a simple and clean way of computing the limit of \( \det W_{S,\tilde{S}} \). In particular while the interval \([\omega_{J+1}, \omega_J]\) vanishes \( \partial \chi_S^{(J)} \) and \( \partial \tilde{\chi}_S^{(J)} \) become the new \( \partial \chi_S^{(J)} \) and \( \partial \tilde{\chi}_S^{(J)} \) and \( \partial \chi_S^{(J)} \) develops a not integrable singularity at \( \omega_J = \omega_{J+1} \) and gives the leading singularity of \( \det W_{S,\tilde{S}} \), explicitly we find \[13\]

\[ \det W_{S,\tilde{S}} \sim W_{S,\tilde{S}}^{(J)} \]  

(125)

with

\[ W_{S,\tilde{S}}^{(J)} = (\omega_J - \omega_{J+1})^{1-\epsilon_j-\epsilon_{J+1} - i\sigma_{\epsilon_j,J}} B(\epsilon_j, \epsilon_{J+1}) \prod_{l \neq J,L} (\omega_J - \omega_l)^{-\epsilon_l} \prod_{l \in S_J} (\omega_J - \omega_L) \]  

(126)

where \( B(\cdot, \cdot) \) is Euler Beta function. Using these results into \[118\] with a not so short computation we find the expected result

\[ \langle \prod_{i=1}^{N} \sigma_{\epsilon_i,J}(x_i) \rangle_{x_{J+1} \to x_J} (x_J - x_{J+1})^{-\epsilon_j \epsilon_{J+1}} M(\epsilon_j, \epsilon_{J+1}) \langle \prod_{i=1}^{N} \sigma_{\epsilon_i,J}(x_i) \rangle \]  

(127)

\[ \text{See appendix C for an example of the computations involved in the special case } N = 4, M = 2. \]
and a relation among the amplitude normalizations and the OPE normalization in eq. (59) which up to a phase reads

$$C_{(N,M)}(\epsilon) \left[B(\epsilon_J, \epsilon_{J+1})\right]^{-\frac{1}{2}} = C_{(N-1,M)}(\tilde{\epsilon}) \ M(\epsilon_J, \epsilon_{J+1})$$  \hspace{1cm} (128)$$

where $\tilde{\epsilon}$ are the twists of the $(N - 1, M)$ theory, i.e. $\tilde{\epsilon}_j = \epsilon_j$ for $j < J, \tilde{\epsilon}_J = \epsilon_J + \epsilon_{J+1}$ for $j = J$ and $\tilde{\epsilon}_j = \epsilon_{j+1}$ for $j > J$.

It is worth noticing that the previous result (127) shows that eq. (118) is valid even when $S$ has only one element. If we perform the reduction from this case, i.e. with $(N + 1, N - 1)$ and we compare with the expression for the $(N, N - 1)$ amplitudes we deduce that

$$\det W_{(\theta, \bar{S})} \prod_{\text{ord}(I) < \text{ord}(J); I \in \bar{S}} (\omega_I - \omega_J)^{-1} \propto \prod_{2 \leq j < I \leq N} (\omega_I - \omega_I)^{\epsilon_I + \epsilon_I - 1}$$  \hspace{1cm} (129)$$

where $W_{(\theta, \bar{S})}$ is simply the matrix $\| W_{I,J} \|$. In other words certain determinants of order $N - 2$ (card($\bar{S}$) = $M - 1 = N - 2$) of Lauricella hypergeometric functions of order $N - 3$ (since all $W_{I,J}$ can be expressed using $I^{(N)}$) are a product of powers. This could point to that also the general $\det W_{S, \bar{S}}$ may be expressed as an elementary function.

For the special case where both $S$ and $\bar{S}$ have just one element, i.e. for $N = 4, M = 2$ a direct and little different computation is needed but the result is the same.

For checking the consistency of the approach and of the normalization coefficients we determine in the next section it is worth considering the $\omega_J \rightarrow \infty$ limit. The result for the normalization coefficients in this case is based on the relation

$$\det W_{S, \bar{S}} \sim W_{(S, \bar{S})}^{(J)} \omega_J^{\epsilon_J(N-2M-1)} \det W_{S, \bar{S}} \sim B(\epsilon_J, 1 - \epsilon_1 - \epsilon_J) \omega_J^{\epsilon_J(N-2M-1)-\epsilon_1} \det W_{S, \bar{S}}$$ \hspace{1cm} (130)$$

and reads

$$C_{(N,M)}(\epsilon) \left[B(\epsilon_J, 1 - \epsilon_1 - \epsilon_J)\right]^{-\frac{1}{2}} = C_{(N-1,M)}(\tilde{\epsilon}) \ M(\epsilon_J, \epsilon_1)$$  \hspace{1cm} (131)$$

with the new twists given by $\tilde{\epsilon}_1 = \epsilon_1 + \epsilon_J, \tilde{\epsilon}_j = \epsilon_j$ for $1 < j < J$ and $\tilde{\epsilon}_j = \epsilon_{j+1}$ for $j > J$.

**5.3.4 (N, M) into (N - 1, M - 1) case**

In this case we can choose the sets $S$ and $\bar{S}$ so that $\bar{J} \in S, \bar{J} \notin S$ and $\bar{J}+1 \notin S$ then it is possible to show as in the previous case that the new sets $\bar{S}$ and $\bar{S}$ are given by $\bar{S} = S$ and $\bar{S} = \bar{S}_J = S \setminus \{J\}$.

In particular it is possible to find analogously as before that the determinant behaves in the $x_J \rightarrow x_{J+1}$ limit as

$$\det W_{S, \bar{S}} \sim W_{(S, \bar{S})}^{(J)} \det W_{S, \bar{S}}$$ \hspace{1cm} (132)$$
and the amplitude reduction gives

$$\langle \prod_{i=1}^{N} \sigma_{\epsilon, f_i = f(x_i)} \rangle \sim (x_J - x_{J+1})^{-(1-\epsilon_J)(1-\epsilon_{J+1})} \langle \prod_{i=1, i \neq J}^{N} \sigma_{\epsilon, f_i = f(x_i)} \rangle \quad (133)$$

It follows a relation among the amplitude normalizations and the OPE normalization in eq. (59) which up to a phase reads

$$C_{(N,M)}(\epsilon) \left[ B(1 - \epsilon_J, 1 - \epsilon_{J+1}) \right]^{-\frac{1}{2}} = C_{(N-1,M-1)}(\tilde{\epsilon}) \mathcal{N}(\epsilon_J, \epsilon_{J+1}) \quad (134)$$

where \( \tilde{\epsilon} \) are the twists of the \((N-1,M-1)\) theory, i.e. \( \tilde{\epsilon}_j = \epsilon_j \) for \( j < J \), \( \tilde{\epsilon}_J = \epsilon_J + \epsilon_{J+1} - 1 \) for \( j = J \) and \( \tilde{\epsilon}_j = \epsilon_{j+1} \) for \( j > J \).

As in the previous subsection starting from the \((N+1,2)\) amplitude and reducing it to \((N,1)\) we deduce that

$$\det W_{(S, \emptyset)} \prod_{\text{ord}(I) < \text{ord}(J); I, J \in S} (\omega_I - \omega_J)^{-1} \propto \prod_{2 \leq j < l \leq N} (\omega_j - \omega_l)^{1 - \epsilon_j - \epsilon_l} \quad (135)$$

where \( W_{(S, \emptyset)} \) is simply the matrix \( \| W_i \| \).

### 5.4 Amplitudes and OPEs normalization

We normalize the 2-point amplitude as

$$\langle \sigma_\epsilon(x)\sigma_{1-\epsilon}(y) \rangle = \frac{1}{(x-y)^{\epsilon(1-\epsilon)}} \quad (136)$$

This normalization is not unique since any redefinition as \( \sigma_\epsilon \to R(\epsilon)\sigma_\epsilon \) with \( R(\epsilon) R(1 - \epsilon) = 1 \) would work. In particular this kind of redefinition can only be seen in amplitudes with at least three twist fields since it leaves unchanged amplitudes involving two twist fields and an arbitrary number of untwisted fields therefore it cannot be fixed factorizing a 4 twists into an untwisted channel. If we require the normalizations to be invariant under the symmetry \( \epsilon \leftrightarrow 1 - \epsilon \) then all the normalizations are completely fixed (up one constant \( k \) and phases) to be

$$C_{(N,1)} = k^{N-2} \left[ \prod_{j=1}^{N} \frac{\Gamma(1 - \epsilon_j)}{\Gamma(\epsilon_j)} \right]^{1/4}$$

$$C_{(N,M)} = k^{N-2} \left[ \prod_{j=2}^{N} \frac{\Gamma(\epsilon_j)\Gamma(1 - \epsilon_j)}{\Gamma(\epsilon_1)\Gamma(1 - \epsilon_1)} \right]^{1/4} \quad 2 \leq M \leq N - 2$$

$$C_{(N,N-1)} = k^{N-2} \left[ \prod_{j=1}^{N} \frac{\Gamma(\epsilon_j)}{\Gamma(1 - \epsilon_j)} \right]^{1/4} \quad (137)$$
along with the OPE normalizations

\[
\mathcal{M}(\alpha, \beta) = k \left[ \frac{\Gamma(1-\alpha) \Gamma(1-\beta) \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(1-\alpha - \beta)} \right]^{1/4}
\]

\[
= k \left[ \frac{\Gamma(1-\alpha) \Gamma(1-\beta) \Gamma(1-\gamma)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma)} \right]^{1/4} \quad \alpha + \beta + \gamma = 1
\]

\[
\mathcal{N}(\alpha, \beta) = k \left[ \frac{\Gamma(\alpha) \Gamma(\beta) \Gamma(2-\alpha-\beta)}{\Gamma(1-\alpha) \Gamma(1-\beta) \Gamma(\alpha + \beta - 1)} \right]^{1/4}
\]

\[
= k \left[ \frac{\Gamma(\alpha) \Gamma(\beta) \Gamma(\delta)}{\Gamma(1-\alpha) \Gamma(1-\beta) \Gamma(1-\delta)} \right]^{1/4} \quad \alpha + \beta + \delta = 2
\]

which also respect the symmetry \( \epsilon \leftrightarrow 1-\epsilon \) as \( \mathcal{N}(\alpha, \beta) = \mathcal{M}(1-\alpha, 1-\beta) \).

It is at first sight surprising that there is not symmetry among the twist operators in the \( M \neq 1, N = 1 \) case but this is due to two reasons. The first is our choice of using a \( SL(2, \mathbb{R}) \) invariant formalism which singles out some points and the second is that not all twist operators are on the same footing since some couples of twists sum to a quantity less than one while others to one bigger than one. These normalization are the “square root” of the ones found in [17] for the \( N = 4 \) closed string case and matches those obtained for \( N = 3 \) in the magnetic brane case in [9] and for \( N = 4 \) case in [10].

Let us see how we can get the previous results by exploiting the consequences of equations of the previous subsections such as eqs (128) and (134). First we notice that we can always normalize the 2-points correlator as chosen because the generic normalization factor \( C_{(2,1)}(\epsilon, 1-\epsilon) \) is symmetric in the exchange \( \epsilon \leftrightarrow 1-\epsilon \) hence we can redefine the twist operators as \( \sigma_\epsilon = \tilde{\sigma}_\epsilon / \sqrt{C_{(2,1)}(\epsilon, 1-\epsilon)} \).

From the reduction \( (N = 3, M = 1) \) to \( (\tilde{N} = 2, \tilde{M} = 1) \) with the help of eq. (122) we find that \( \mathcal{M}(\alpha, \beta) = C_{(3,1)}(\alpha, \beta, \gamma) \) with \( \alpha + \beta + \gamma = 1 \) has the following basic symmetries

\[
\mathcal{M}(\alpha, \beta) = \mathcal{M}(\beta, \alpha) = \mathcal{M}(\alpha, 1-\alpha - \beta)
\]

(139)

and all the others which follow from them.

In a similar way from the \( (N = 3, M = 2) \) to \( (\tilde{N} = 2, \tilde{M} = 1) \) reduction and from eq. (124) we find

\[
\mathcal{N}(\alpha, \beta) = \mathcal{N}(\beta, \alpha) = \mathcal{N}(\alpha, 2-\alpha - \beta)
\]

(140)

Now we can consider the \( (N = 4, M = 2) \) to \( (\tilde{N} = 3, \tilde{M} = 1) \) reduction in two different ways. Either with \( (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \rightarrow (\epsilon_1, \epsilon_2, \epsilon_3 + \epsilon_4 - 1) \) which implies

\[
C_{(4,2)}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)[B(1-\epsilon_3, 1-\epsilon_4)B(\epsilon_2, \epsilon_3 + \epsilon_4 - 1)]^{1/2} = \mathcal{N}(\epsilon_3, \epsilon_4)C_{(3,1)}(\epsilon_1, \epsilon_2, \epsilon_3 + \epsilon_4 - 1)
\]

(141)
or with \((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \rightarrow (\epsilon_1, \epsilon_2 + \epsilon_3, \epsilon_4 - 1)\) which implies

\[
C_{(4,2)}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)[B(1 - \epsilon_3, 1 - \epsilon_2)B(\epsilon_4, \epsilon_3 + \epsilon_2 - 1)]^{\frac{1}{2}} = \mathcal{N}(\epsilon_3, \epsilon_2)C_{(3,1)}(\epsilon_1, \epsilon_2 + \epsilon_3 - 1, \epsilon_4)
\]

Now taking the ratio of the two previous equations and using the symmetries of \(\mathcal{M}\) and \(\mathcal{N}\) we are led to the minimal ansatz

\[
\mathcal{M}(\alpha, \beta) = k[\Gamma(\alpha)\Gamma(\beta)\Gamma(1 - \alpha - \beta)]^a[\Gamma(1 - \alpha)\Gamma(1 - \beta)\Gamma(\alpha + \beta)]^b
\]

\[
\mathcal{N}(\alpha, \beta) = k[\Gamma(\alpha)\Gamma(\beta)\Gamma(2 - \alpha - \beta)]^c[\Gamma(1 - \alpha)\Gamma(1 - \beta)\Gamma(\alpha + \beta - 1)]^d
\]

which gives an overconstrained system when plugged back into the ratio constraint whose solution is \(a = -b\) and \(c = -d = \frac{1}{2} + a\). This solution immediately yields both \(C_{(3,2)}\) and \(C_{(4,2)}\). Imposing the symmetry \(\epsilon \leftrightarrow 1 - \epsilon\) then selects \(a = -\frac{1}{4}\). It is then easy to generalize to the full expressions. These can be checked in different limits also when we consider \(\omega_j \rightarrow \infty\) using to eq. \([131]\).

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### A Details on rewriting the classical action.

We want to give some details on the use of KLT technique for reducing the integral

\[
J^{(N)}(\alpha + n, \bar{\alpha} + \bar{n}) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \prod_{j=2}^{N} (x + iy - \omega_j)^{\alpha_j + n_j}(x - iy - \omega_j)^{\bar{\alpha}_j + \bar{n}_j}
\]

(144)

with \(n_j, \bar{n}_j \in \mathbb{Z}\) to the sum of products of an holomorphic and antiholomorphic integral. First we interpret the previous integral in \(y\) as a line integral in the complex plane \(Y = t + iy\). In the variable \(Y\) the integrand has cuts in \(\pm(\omega_j - x)\), the main issue is then to properly define the phase of

\[
(x + Y - \omega)^\alpha(x - Y - \omega)^\bar{\alpha} = |x + Y - \omega|^{\alpha}|x - Y - \omega|^{\bar{\alpha}}e^{i(\phi + \bar{\phi})}.
\]

(145)

The proper choice is shown in fig. (11) and is constrained by the request that when \(Y = iy\) and \(\alpha = \bar{\alpha}\) then \(\phi + \bar{\phi} = 0\).

We can then rotate clockwise the path in \(Y\) plane, change variables as \(\xi = x + t, \eta = x - t\) and then we can rewrite the \(J^{(N)}\) integral as

\[
J^{(N)}(\alpha + n, \bar{\alpha} + \bar{n}) = -\frac{i}{2} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta \prod_{j=2}^{N} |\xi - \omega_j|^{\alpha_j} |\eta - \omega_j|^{\bar{\alpha}_j}
\]

\[
\times (\xi - \omega_j)^{n_j} (\eta - \omega_j)^{\bar{n}_j}
\]

\[
\times e^{-i\pi \alpha_j \theta(\omega_j - \xi) \theta(\eta - \omega_j)} e^{-i\pi \bar{\alpha}_j \theta(\xi - \omega_j) \theta(\omega_j - \eta)}
\]

(146)
Figure 11: Proper definition of angles $\phi$ and $\bar{\phi}$ and therefore of phases when $x - \omega > 0$. When $x - \omega < 0$ we substitute $(x - \omega) \to -(x - \omega)$.

If $\bar{\alpha} = \alpha$ then we can proceed as in KLT. We fix $\xi$ and we exam the $\eta$ integral. Each factor of the integrand can then be rewritten as

$$|\eta - \omega_j|^\alpha_j e^{-i\pi \alpha_j} \theta((\omega_j - \xi)(\eta - \omega_j)) = \theta(\omega_j - \xi) \left[ \omega_j - (\eta + i0^+) \right]^\alpha_j$$

$$+ \theta(-\omega_j + \xi) e^{+i\pi \alpha_j} \left[ \omega_j - (\eta - i0^+) \right]^\alpha_j \quad (147)$$

when we choose the phase in the complex $\eta$ plane as in fig. (12), obviously other choices would do the job as well. In words this means that when $\xi < \omega_j$ we run above the cut from $-\infty$ to $\omega_j$ in the complex $\eta$ plane while we run below the cut when $\omega_j < \xi$. Hence the original integral can be written as

$$J^{(N)}(\alpha + n, \alpha + \bar{n}) = -\frac{i}{2} \sum_{i=N-1}^{2} \int_{-\omega_{i+1}}^{\omega_i} d\xi \prod_{j=2}^{N} |\xi - \omega_j|^\alpha_j (\xi - \omega_j)^{\bar{\alpha}_j}$$

$$\times e^{i \sum_{l=i}^{N} \alpha_l} \int_{C_i} d\eta \prod_{j=2}^{N} (\omega_j - \eta)^{\alpha_j} (\eta - \omega_j)^{\bar{\alpha}_j} \quad (148)$$

where the path $C_i$ is given in fig. (13). In particular the integrals $\int_{\omega_2}^{\infty} d\xi$ and $\int_{-\infty}^{\omega_N} d\xi$ do not contribute since the integrals over $d\eta$ runs either above or
below the cuts and are zero because of Jordan lemma. We can then rewrite the $C_i$ integral as an integral above (or below depending the cases) the cuts plus a remainder. The final result is then

$$J^{(N)}(\alpha + n, \alpha + \tilde{n}) = -\sum_{i=2}^{N-1} \sum_{l=i+1}^{N} \sin \left( \pi \sum_{j=i+1}^{l} \alpha_j \right)$$

$$\times \int_{\omega_i}^{\omega_{i+1}} d\xi \prod_{j=2}^{N} |\xi - \omega_j|^\alpha_j (\xi - \omega_j)^{\tilde{n}_j}$$

$$\times \int_{\omega_{i+1}}^{\omega} d\eta \prod_{j=2}^{N} |\omega_j - \eta|^\alpha_j (\eta - \omega_j)^{\tilde{n}_j}$$

(149)

B  Fixing the singular part of $g(z, w)$ in a consistent way with $N \rightarrow N - 1$ reduction

Let us suppose that all coefficients $c_{n,s}(\omega_j)$ depend on $\omega_j$ ($3 \leq j \leq N - 1$) in an analytic way. We want to show that it is then possible to fix then in a recursive way starting from those of the $N = 3, M = 1$ case. This can be done considering two limits $x_j \rightarrow x_N$, i.e. $\omega_j \rightarrow 0$ and $x_j \rightarrow x_1$, i.e. $\omega_j \rightarrow \infty$.

Combining the two cases when $\epsilon_1 + \epsilon_j < 1$ and $\epsilon_j + \epsilon_N < 1$ we get

$$\tilde{c}_{n,s}^{(N,M)}(\omega, \epsilon) = c_{n-1,s}^{(N-1,M)}(\tilde{\omega}, \tilde{\epsilon}) - c_{n,s}^{(N-1,M)}(\tilde{\omega}, \tilde{\epsilon})\omega_j$$

$$\tilde{c}_{0,s}^{(N,M)}(\omega, \epsilon) = -c_{n,s}^{(N-1,M)}(\tilde{\omega}, \tilde{\epsilon})\omega_j$$

$$c_{N-M,s}^{(N,M)}(\omega, \epsilon) = c_{N-M-1,s}^{(N-1,M)}(\tilde{\omega}, \tilde{\epsilon})$$

(150)

when $1 \leq n \leq N - M - 1$, $0 \leq s \leq M$ and where we have defined

$$\left\{ \begin{array}{ll}
\tilde{\epsilon}_i = \epsilon_i & i = 1, \ldots j - 1 \\
\tilde{\epsilon}_i = \epsilon_{i+1} & i = j, \ldots N - 2 \\
\tilde{\epsilon}_{N-1} = \epsilon_N + \epsilon_j - \theta(\epsilon_N + \epsilon_j > 1)
\end{array} \right.$$
and
\[
\begin{align*}
\hat{e}_1 &= \epsilon_1 + \epsilon_j - \theta(\epsilon_1 + \epsilon_j > 1) \\
\hat{e}_i &= \epsilon_i & i = 2, \ldots, j - 1 \\
\hat{e}_i &= \epsilon_{i+1} & i = j, \ldots, N - 1
\end{align*}
\] (152)

and similar relations between \( \hat{\omega} \) with \( \omega \) and \( \hat{\dot{\omega}} \) with \( \dot{\omega} \). For example applying the previous formula to the \( N = 4, M = 2 \) case we get
\[
g_s^{(4,2)}(z, w) = \frac{1}{(z - w)^2} \prod_{2 \leq j \leq 4} \left( \frac{\omega_j - \omega_j^j}{\omega_j - \omega_j^1} \right) \left\{ \epsilon_1 \omega_j^2 \omega_j w + (1 - \epsilon_1) \omega_j \omega_j^2 \right. \\
+ (1 - \epsilon_1 - \epsilon_2) \omega_j^2 - [\epsilon_3 + \epsilon_4 + (\epsilon_1 + \epsilon_3) \omega_3] \omega_j \omega_j + (1 - \epsilon_2 - \epsilon_4) \omega_3 \omega_j^2 \right. \\
+ (1 - \epsilon_4) \omega_3 \omega_j + \epsilon_4 \omega_3 \omega_j^2 \right\} 
\] (153)

\section{C Some details on the \( N = 4 \) reduction}

As an example of the way we performed the \( N \to N - 1 \) we give now some details on the \( N = 4, M = 2 \) case, in particular we consider \( \theta \to 0 \) when \( \epsilon_3 + \epsilon_4 > 1 \). Under this conditions we want to compute the behavior of
\[
\det W = \begin{vmatrix} W_1^3 & W_1^3 \\ W_2^3 & W_3^3 \end{vmatrix} 
\] (154)

where we have chosen \( S = \bar{S} = \{3\} \). The different entries of the determinant have the following limits
\[
W_1^3 = \int_{\omega_3}^1 d\omega (\omega - 1)^{\epsilon_2 - 1} (\omega - \omega_3)^{\epsilon_3 - 1} \omega^{\epsilon_4 - 1} \sim \int_{0}^1 d\omega (\omega - 1)^{\epsilon_2 - 1} \omega^{(\epsilon_3 + \epsilon_4 - 1) - 1} \\
= e^{i\pi(\epsilon_2 - 1)} B(\epsilon_2, \epsilon_3 + \epsilon_4 - 1), 
\] (155)

\[
W_2^3 = \int_0^{\omega_3} d\omega (\omega - 1)^{\epsilon_2 - 1} (\omega - \omega_3)^{\epsilon_3 - 1} \omega^{\epsilon_4 - 1} \\
= \omega_3^{\epsilon_3 + \epsilon_4 - 1} \int_{0}^{1} dt (\omega_3 t - 1)^{\epsilon_2 - 1} (t - 1)^{\epsilon_3 - 1} t^{\epsilon_4 - 1} \\
\sim \omega_3^{\epsilon_3 + \epsilon_4 - 1} e^{i\pi(\epsilon_2 + \epsilon_3)} B(\epsilon_3, \epsilon_4), 
\] (156)

and
\[
W_1^3 = \int_{\omega_3, \omega \in H^1} d\omega (\omega - 1)^{-\epsilon_2} (\omega - \omega_3)^{-\epsilon_3} \omega^{-\epsilon_4} \\
= \omega_3^{1 - \epsilon_3 - \epsilon_4} \int_{1}^{1/\omega_3} dt (\omega_3 t - 1)^{-\epsilon_2} (t - 1)^{-\epsilon_3} t^{-\epsilon_4} \\
\sim e^{-i\pi\epsilon_2 \omega_3^{1 - \epsilon_3 - \epsilon_4}} \int_{1}^{+\infty} dt (t - 1)^{-\epsilon_3} t^{-\epsilon_4} = e^{-i\pi\epsilon_2 \omega_3^{1 - \epsilon_3 - \epsilon_4}} B(\epsilon_3 + \epsilon_4 - 1, 1 - \epsilon_3) 
\] (157)
finally the limit of the last entry can be obtained using again the substitution
\[ \omega = \omega_3 t \] to be
\[
W_2^3 = \int_{0, \omega \in H^-} \omega (\omega - 1)^{-\epsilon_2} (\omega - \omega_3)^{-\epsilon_3} \omega^{-\epsilon_4} \sim e^{+i\pi(\epsilon_2 + \epsilon_3)\omega_3^{1-\epsilon_3-\epsilon_4}} B(1 - \epsilon_3, 1 - \epsilon_4)
\]
(159)

Inserting all the previous asymptotic behaviors into the determinant we get its \( \omega_3 \to 0, \epsilon_3 + \epsilon_4 > 1 \) limit to be
\[
\det W = \begin{vmatrix}
 e^{i\pi(\epsilon_2-1)} B(\epsilon_2, \epsilon_3 + \epsilon_4 - 1) & e^{i\pi\epsilon_2 \omega_3^{1-\epsilon_3-\epsilon_4}} B(\epsilon_3 + \epsilon_4 - 1, 1 - \epsilon_3) \\
 e^{i\pi(\epsilon_3+\epsilon_4)\omega_3^{1+\epsilon_4-1}} B(\epsilon_3, \epsilon_4) & e^{i\pi(\epsilon_2+\epsilon_3)\omega_3^{1-\epsilon_3-\epsilon_4}} B(1 - \epsilon_3, 1 - \epsilon_4)
\end{vmatrix}
\sim \omega_3^{(1-\epsilon_3)+(1-\epsilon_4)-1} B(1 - \epsilon_3, 1 - \epsilon_4) B(\epsilon_2, 1 - (1 - \epsilon_3) - (1 - \epsilon_4))
\]
(160)

where it is worth noticing that we can drop the relative phases since only one product is the leading one. This happens luckily also for all the other computations which are needed to compute all the \( N \to N - 1 \) reduction.

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