Gaussian and hermite Ornstein–Uhlenbeck processes

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ABSTRACT
In the present paper we study the asymptotic behavior of the auto-covariance function for Ornstein–Uhlenbeck (OU) processes driven by Gaussian noises with stationary and non-stationary increments and for Hermite OU processes. Our results are generalizations of the corresponding results of Cheridito et al. and Kaarakka and Salminen.

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1. Introduction
Let $B^H := \{B^H_t, t \in \mathbb{R}\}$ denote a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$, that is, $B^H$ is a centered Gaussian process with covariance function

$$E(B^H_t B^H_s) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s \in \mathbb{R}.$$ 

Consider the fractional Ornstein–Uhlenbeck (fOU) process $X^H := \{X^H_t, t \in \mathbb{R}\}$ defined as the solution to the Langevin equation

$$dX^H_t = -\theta X^H_t dt + dB^H_t,$$  \quad (1.1)

where $\theta \in \mathbb{R}$.

Notice that if $\theta > 0$ and the initial condition $X^H_0 = \int_{-\infty}^0 e^{\theta s} dB^H_s$, then the unique strong solution $X^H$ of (1.1) is a stationary Gaussian sequence and since $E(X^H_t X^H_0) \rightarrow 0$ as $t \rightarrow \infty$ (according to [4, Theorem 2.3]), a criterion for stationary Gaussian processes (see [18, Theorem 6.6]) gives that the process is ergodic.

The fOU process is one of the most studied and widely applied stochastic process. It represents interesting model for stochastic dynamics with memory, with applications to e.g. finance, telecommunication networks and physics. In the finance context, several researchers in recent years have been interested in studying statistical estimation problems for fOU processes. The statistical analysis of equations driven by fBm is obviously more recent. The
development of stochastic calculus with respect to the fBm allowed to study such models. On the other hand, the long-range dependence property makes the fBm important driving noise in modeling several phenomena arising, for instance, from volatility modeling in finance. Let us mention some important results in this field where the volatility exhibits long-memory, which means that the volatility today is correlated to past volatility values with a dependence that decays very slowly. The authors of [5–8] considered the problem of option pricing under a stochastic volatility model that exhibits long-range dependence. More precisely they assumed that the dynamics of the volatility are described by the Equation (1.1), where the Hurst parameter $H$ is greater than 1/2. On the other hand, the paper [14] on rough volatility contends that the short-time behavior indicates that the Hurst parameter $H$ in the volatility is less than 1/2. Furthermore, the drift parameter estimation for fractional-noise-driven Ornstein–Uhlenbeck processes, in particular fOU, has also attracted the interest of many researchers recently. We refer the interested readers to [1, 2, 9, 10, 12, 13, 15, 17, 24] and references therein.

There are two possible definitions for the fOU process. One can define this stochastic process as the solution of (1.1). Alternatively, one can define the fOU process as the Lamperti transform of the fBm $B^H$:

$$Z_t^{0,H} = e^{-\theta t} B_t^H, \quad t \geq 0, \quad (1.2)$$

which is a stationary process. In the case when $H = \frac{1}{2}$, $\theta > 0$, the process (1.1), with $X_0 = \int_0^\infty e^{\theta t} dB_t^H$, and the process (1.2) have the same finite dimensional distributions, thanks to the Lévy characterization theorem. On the other hand, when $H \neq \frac{1}{2}$, the probability distributions of (1.1) and (1.2) are different. The process given by (1.2) can be also be expressed as the solution to the Langevin-type stochastic equation,

$$dX_t = -\theta X_t dt + dY_t^{(0)}, \quad t \geq 0, \quad (1.3)$$

with initial condition $X_0 = B_t^H$, where the noise $\{Y_t^{(0)}, t \geq 0\}$ is given by the formula $Y_t^{(0)} := \int_0^t e^{-\theta s} dB_s^H$ with $a_s^{(0)} := \frac{H}{H} e^{-\theta s}$. Motivated by this, [16] introduced a fractional Ornstein–Uhlenbeck process of the second kind as the solution to a Langevin-type stochastic equation, namely

$$dX_t = -\theta X_t dt + dY_t^{(1)}, \quad t \geq 0, \quad (1.4)$$

with initial condition $X_0 = 0$, where the noise $Y_t^{(1)} = \int_0^t e^{-\theta s} dB_s^H$ with $a_t := a_t^{(1)} = He^{-\theta t}$. Whereas, the process given by (1.1) is usually called the fractional Ornstein–Uhlenbeck process of the first kind.

Let us describe what is proved in [4] and [16], about the processes (1.1) and (1.4), respectively. In [4], Cheridito et al. showed that the solution to Langevin Equation (1.1), with $X_0 = \int_0^\infty e^{\theta t} dB_t^H$, is stationary and the decay of its auto-covariance function behaves as a power function. However, in [16], Kaarakka and Salminen proved when $H > \frac{1}{2}$ that the solution of the Langevin Equation (1.4), with $X_0 = \int_0^\infty e^{\theta t} dY_t^{(1)}$, is a stationary process and its auto-covariance function decays exponentially.

The purpose of this paper is to provide a general approach to study these properties for more general Gaussian and Hermite processes that are of the form
\[dX_t = -2X_t dt + dG_t.\]

We will study stationarity and ergodicity properties, and the decay of the auto-covariance function of such processes. These facts play an important role in stochastic analysis and in different applications, and for these reasons the topic has been extensively studied in the literature. For instance, they can be used to study different parameters describing such Gaussian or Hermite processes. Of particular interest for turbulence theory is the large and small lags limit behavior of the auto-covariance function of the fOU process, which has been proposed as a representation of homogeneous Eulerian turbulent velocity, see [23].

The paper is organized as follows. In Section 2 we provide some keys lemmas needed in order to state the main results of the present paper. In Section 3 we study the ergodicity and stationarity properties and the decay of the auto-covariance function for Hermite Ornstein–Uhlenbeck processes of the first kind and for Gaussian Ornstein–Uhlenbeck processes of the second kind. In Section 4 we end the paper by studying the auto-covariance function for Ornstein–Uhlenbeck processes driven by a Gaussian noise with non-stationary increments.

Throughout the paper, the symbol \(C\) stands for a generic constant, whose value can change from one line to another. Moreover, for \(t \to \infty\), we will write \(f(t) \sim g(t)\), provided that \(f(t)/g(t) \to 1\).

2. Key lemmas

Let \(G := \{G_t, t \in \mathbb{R}\}\) be a measurable process defined on some probability space \((\Omega, \mathcal{F}, P)\) (here, and throughout the text, we assume that \(\mathcal{F}\) is the sigma-field generated by \(G\)). The following assumption is required.

\((A)\) \(G_0 = 0\) a.s., and there exists a constant \(\gamma \in (0, 1)\) such that for every \(q \geq 2\) there is a constant \(c_q > 0\) satisfying

\[E\left[|G_t - G_s|^q\right] \leq c_q|t - s|^{q\gamma} \quad \text{for all } s, t \in \mathbb{R}.\]  

(2.1)

Note that, if \((A)\) holds, then by the Kolmogorov-Centsov theorem, we can conclude that for all \(\varepsilon \in (0, \gamma)\), the process \(G\) admits a modification with \((\gamma - \varepsilon)\)-Hölder continuous paths, still denoted \(G\) in the sequel.

Remark 2.1. Corollary 2.8.14 in [20] tells us that, inside a fixed Wiener chaos, all the \(L^q\)-norms are equivalent. Hence, if \(G\) is a Gaussian or Hermite process, the assumption \((A)\) is equivalent to \(G_0 = 0\) a.s., and

\[E\left[|G_t - G_s|^2\right] \leq C|t - s|^{2\gamma} \quad \text{for all } s, t \in \mathbb{R}.\]

Let us recall some basic elements of pathwise Riemann-Stieltjes integral, which are helpful for some of the arguments we use. For any \(\alpha \in (0, 1]\), a function \(f : [a, b] \to \mathbb{R}\) is said to be \(\alpha\)-Hölder continuous function if

\[\sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{|t - s|^{\alpha}} < \infty.\]
If \( f, g : [a, b] \to \mathbb{R} \) are Hölder continuous functions of orders \( \alpha \) and \( \beta \), respectively, with \( \alpha + \beta > 1 \), Young [27] proved that the Riemann–Stieltjes integral \( \int_{a}^{b} f_i d g_i \) exists. Hence, if \( G \) is a process satisfying \((A)\), then the stochastic integral \( \int_{a}^{b} u_i dG_i \) is well defined as a pathwise Riemann–Stieltjes integral provided that the trajectories of the process \( \{u_i, t \in \mathbb{R}\} \) are \( \alpha \)-Hölder continuous functions on any finite interval for some \( \alpha > 1 - \gamma \).

**Lemma 2.2.** Let \( \{G_t, t \in \mathbb{R}\} \) be a measurable process satisfying the assumption \((A)\) and let \( \xi \in L^0(\Omega) \). Then, for every \( \theta \in \mathbb{R} \) and \(-\infty < s < t < \infty\),

\[
\int_{s}^{t} e^{\theta r} dG_r = e^{\theta t} G_t - e^{\theta s} G_s - \theta \int_{s}^{t} e^{\theta r} G_r dr. \tag{2.2}
\]

In addition, if we assume \( \theta > 0 \), we have for every \( t \in \mathbb{R} \), \( \int_{-\infty}^{t} e^{\theta r} dG_r \) is well defined as a Riemann–Stieltjes integral and

\[
\int_{-\infty}^{t} e^{\theta r} dG_r = e^{\theta t} G_t - \theta \int_{-\infty}^{t} e^{\theta r} G_r dr, \tag{2.3}
\]

and the unique continuous solution to the equation

\[
X_t = \xi - \theta \int_{0}^{t} X_s ds + G_t, \quad t \geq 0, \tag{2.4}
\]

is given by

\[
X_t = e^{-\theta t} \left( \xi + \int_{0}^{t} e^{\theta s} dG_s \right), \quad t \geq 0. \tag{2.5}
\]

**Proof.** Since \( G \) satisfies \((A)\), the integral \( \int_{s}^{t} e^{\theta r} dG_r \) as stated above is well defined as a pathwise Riemann–Stieltjes integral. So, the claim (2.2) can be immediately obtained by integrating by parts (see, e.g. [26, Theorem 2.21]).

Now, let us prove (2.3). Suppose that \( \theta > 0 \). Using the same argument as in the proof of (2.7) in [12], we have, for any \( \gamma < \delta \), \( \lim_{|t| \to \infty} \frac{G_t}{|t|^{\delta}} = 0 \) almost surely. This implies that for all \( t \in \mathbb{R} \), \( \int_{-\infty}^{t} e^{\theta r} G_r dr \) exists as a Riemann integral, which, by [26, Theorem 2.21], implies that the Riemann–Stieltjes integral \( \int_{-\infty}^{t} e^{\theta r} dG_r \) exists too and (2.3) holds.

Finally, a continuous function \( X \) is solution to the Equation (2.4) if and only if the function \( u(t) = \int_{0}^{t} X_s ds, t \geq 0, \) is solution to the linear differential equation

\[
u' (t) = -\theta u(t) + \xi + G_t, \quad u(0) = 0,
\]

which has the unique solution

\[
u(t) = e^{-\theta t} \int_{0}^{t} e^{\theta s} (\xi + G_s) ds, \quad t \geq 0.
\]

As a consequence, the unique continuous solution \( X \) that solves (2.4) is given by

\[
X_t = -\theta e^{-\theta t} \int_{0}^{t} e^{\theta s} (\xi + G_s) ds + \xi + G_t
\]

\[
e^{-\theta t} \left( \xi + \int_{0}^{t} e^{\theta s} dG_s \right), \quad t \geq 0,
\]

where the latter equality comes from (2.2). This completes the proof.

\( \square \)
In what follows $R_G(s, t) := E(G_s G_t)$, $s, t \in \mathbb{R}$, denotes the covariance function of the process $G$.

**Lemma 2.3.** Let $\theta \in \mathbb{R}$ and $-\infty < s < t \leq u < v < \infty$. Assume that (A) holds and $\frac{\partial^2 R_G}{\partial y \partial x}$ is a continuous function on $\mathbb{R}^2 \setminus \{(x, y) | x = y\}$. Then

$$E \left( \int_s^t e^{\theta x} dG_x \int_u^v e^{\theta y} dG_y \right) = \int_u^v \int_s^t e^{\theta x} e^{\theta y} \frac{\partial^2 R_G}{\partial y \partial x} (x, y) dxdy, \tag{2.6}$$

provided that the integral on the right-hand side converges.

In addition, if we assume $\theta > 0$, the identity (2.6) is also valid for $s = -\infty$. In other words, for every $\theta > 0$ and $-\infty < t < u < v < \infty$,

$$E \left( \int_{-\infty}^t e^{\theta x} dG_x \int_u^v e^{\theta y} dG_y \right) = \int_u^v \int_{-\infty}^t e^{\theta x} e^{\theta y} \frac{\partial^2 R_G}{\partial y \partial x} (x, y) dxdy, \tag{2.7}$$

provided that the integral on the right-hand side converges.

**Proof.** Let us first suppose $t = u$. Using (2.2), we have

$$E \left( \int_s^t e^{\theta x} dG_x \int_t^v e^{\theta y} dG_y \right) = \left. e^{\theta x} e^{\theta y} R_G(t, v) \right|_{x = t}^{x = s} e^{\theta x} e^{\theta y} R_G(s, v) + e^{\theta x} e^{\theta y} R_G(s, t)$$

$$- \theta e^{\theta x} \int_s^t e^{\theta y} R_G(t, y) dy + \theta e^{\theta x} \int_t^v e^{\theta y} R_G(s, y) dy - \theta e^{\theta x} \int_s^t e^{\theta y} R_G(x, v) dx$$

$$+ \theta e^{\theta x} \int_s^t e^{\theta y} R_G(x, t) dx + \theta^2 \int_t^v \int_s^t e^{\theta x} e^{\theta y} R_G(x, y) dxdy.$$

On the other hand, by integrating by parts, we have

$$- \theta e^{\theta y} \int_s^t e^{\theta x} R_G(x, t) dx = -e^{\theta y} \left. e^{\theta x} R_G(t, v) \right|_{x = t}^{x = s} e^{\theta y} R_G(s, v) + e^{\theta y} \int_s^t e^{\theta x} \frac{\partial R_G}{\partial x} (x, v) dx,$$

$$\theta e^{\theta x} \int_s^t e^{\theta y} R_G(x, t) dx = e^{2\theta t} R_G(t, t) - \theta e^{\theta x} e^{\theta y} R_G(s, t) - \theta e^{\theta x} \int_s^t e^{\theta y} \frac{\partial R_G}{\partial x} (x, t) dx,$$

and

$$\theta^2 \int_t^v \int_s^t e^{\theta x} e^{\theta y} R_G(x, y) dxdy = \theta \int_t^v \int_s^t e^{\theta y} e^{\theta t} R_G(t, y) dy - \theta \int_t^v e^{\theta y} e^{\theta t} R_G(s, y) dy$$

$$- \theta \int_t^v \int_s^t e^{\theta x} e^{\theta y} \frac{\partial R_G}{\partial x} (x, y) dxdy.$$

These equalities imply

$$E \left( \int_s^t e^{\theta x} dG_x \int_t^v e^{\theta y} dG_y \right) = e^{\theta y} \int_s^t e^{\theta x} \frac{\partial R_G}{\partial x} (x, v) dx - e^{\theta t} \int_s^t e^{\theta x} \frac{\partial R_G}{\partial x} (x, t) dx$$

$$- \theta \int_t^v \int_s^t e^{\theta x} e^{\theta y} \frac{\partial R_G}{\partial x} (x, y) dxdy.$$
Further,
\[-\theta \int_t^\nu \int_s^t e^{\rho x} e^{\rho y} \frac{\partial R_G}{\partial x} (x, y) dx dy = -e^{\rho v} \int_s^t e^{\rho x} \frac{\partial R_G}{\partial x} (x, v) dx + e^{\rho \alpha} \int_s^t e^{\rho x} \frac{\partial R_G}{\partial x} (x, t) dx + \int_t^v \int_s^t e^{\rho x} e^{\rho y} \frac{\partial^2 R_G}{\partial y \partial x} (x, y) dx dy,\]
which proves (2.6) for \( t = u. \)

Let us now suppose \( t < u. \) From above we deduce that
\[
E \left( \int_s^t e^{\rho x} dG_x \int_u^v e^{\rho y} dG_y \right) = E \left( \int_s^t e^{\rho x} dG_x \int_s^v e^{\rho y} dG_y \right) - E \left( \int_s^t e^{\rho x} dG_x \int_t^u e^{\rho y} dG_y \right)
\]
\[
= \int_t^v \int_s^t e^{\rho x} e^{\rho y} \frac{\partial^2 R_G}{\partial y \partial x} (x, y) dx dy - \int_u^v \int_s^t e^{\rho x} e^{\rho y} \frac{\partial^2 R_G}{\partial y \partial x} (x, y) dx dy
\]
which completes the proof of (2.6).

Finally, using (2.3) and following the same arguments as above, the claim (2.7) follows.

\[\square\]

Remark 2.4. Lemma 2.3 was proved in [4, Lemma 2.1] in the case when \( G = B^H \) is a fBm with Hurst parameter \( H \in (0, \frac{1}{2}) \cup \left( \frac{1}{2}, 1 \right]. \)

Lemma 2.5. Let \( \rho \) be a positive measurable function on \( \mathbb{R}. \) Then, for all \( 0 < s < t, \)
\[e^{-\theta t} e^{-\theta s} \int_s^t e^{\rho x} e^{\rho y} \rho (y - x) dx dy \leq e^{-\theta (t-s)} \int_0^t e^{\rho x} e^{\rho y} \rho (y - x) dx dy,\]
provided that the integral on the right-hand side converges.

Proof. By change of variables \( u = x - s \) and \( v = y - s, \) we obtain
\[
e^{-\theta t} e^{-\theta s} \int_s^t e^{\rho x} e^{\rho y} \rho (y - x) dx dy = e^{-\theta (t-s)} \int_0^t e^{\rho u} e^{\rho v} \rho (v - u) du dv
\leq e^{-\theta (t-s)} \int_0^t e^{\rho u} e^{\rho v} \rho (v - u) du dv,
\]
which finishes the proof. \[\square\]

Lemma 2.6. Let \( \gamma \in \left( 0, \frac{1}{2} \right) \cup \left( \frac{1}{2}, 1 \right) \) and \( \theta > 0. \) Then, as \( t \to \infty, \)
\[
e^{-\theta t} \int_0^t e^{\rho x} e^{\rho y} (y - x)^{2\gamma-2} dx dy \sim \frac{t^{2\gamma-2}}{\theta^2}.
\]
As a consequence, there exists a constant \( C > 0 \) that depends only on \( \theta \) and \( \gamma \) such that for every \( t - s > 2, \)
\[ e^{-\theta t} e^{-\theta s} \int_s^t \int_0^s e^{\theta x} e^{\theta y} (y - x)^{2\gamma - 2} \, dx \, dy \leq C(t - s)^{2\gamma - 2}. \]  

\[ (2.9) \]

**Proof.** First we prove (2.8). Let \( t > 2 \). Making the change of variables \( u = y - x \), we get

\[
e^{-\theta t} \int_0^t \int_{-\infty}^0 e^{\theta x} e^{\theta y} (y - x)^{2\gamma - 2} \, dx \, dy = e^{-\theta t} \int_0^t \int_y^\infty e^{-\theta u} e^{2\gamma - 2} \, dudy
\]

\[
e^{-\theta t} \int_0^t du e^{-\theta u} u^{2\gamma - 2} \int_0^{u+t} dy e^{2\gamma y}
\]

\[
e^{-\theta t} \int_0^t e^{-\theta u} u^{2\gamma - 2} (e^{2\theta(u+t)} - 1) \, du
\]

\[ = \frac{e^{-\theta t}}{2\theta} \int_0^t e^{-\theta u} u^{2\gamma - 2} (e^{2\theta u} - 1) \, du + \frac{e^{-\theta t}}{2\theta} \int_t^\infty e^{-\theta u} u^{2\gamma - 2} (e^{2\theta u} - 1) \, du. \]  

(2.10)

Furthermore,

\[
A_{1,t} = \int_0^t e^{-\theta u} u^{2\gamma - 2} (e^{2\theta u} - 1) \, du
\]

\[
A_{1,t} = \int_0^t e^{-\theta u} u^{2\gamma - 2} (e^{2\theta u} - 1) \, du + \int_t^\infty e^{-\theta u} u^{2\gamma - 2} (e^{2\theta u} - 1) \, du.
\]

(2.11)

Since \( e^{2\theta u} - 1 \to 2\theta \) as \( u \to 0 \), then the function \( e^{\frac{2\theta u}{u} - 1} \) is bounded on \((0,1]\), that is, \( \sup_{u \in (0,1]} e^{\frac{2\theta u}{u} - 1} < C < \infty \). This implies that

\[
\int_0^1 e^{-\theta u} u^{2\gamma - 2} (e^{2\theta u} - 1) \, du \leq C \int_0^1 e^{-\theta u} u^{2\gamma - 1} \, du
\]

\[
\leq C \int_0^1 u^{2\gamma - 1} \, du
\]

\[
= \frac{C}{2\gamma}.
\]

(2.12)

On the other hand, as \( t \to \infty \),

\[
\frac{e^{-\theta t}}{2\theta} \int_1^t e^{-\theta u} u^{2\gamma - 2} (e^{2\theta u} - 1) \, du = \frac{e^{-\theta t}}{2\theta} \int_1^t e^{\theta u} u^{2\gamma - 2} \, du - \frac{e^{-\theta t}}{2\theta} \int_1^t e^{-\theta u} u^{2\gamma - 2} \, du
\]

\[
\sim \frac{t^{2\gamma - 2}}{2\theta^2},
\]

where we used the fact that \( e^{-\theta t} \int_1^t e^{-\theta u} u^{2\gamma - 2} \, du \leq Ce^{-\theta t} \), and by L'Hôpital's Rule, we have

\[ (2.13) \]
\[
\lim_{t \to \infty} \int_0^t e^{\theta u} u^{2\gamma - 2} du = \lim_{t \to \infty} \frac{e^{\theta t} t^{2\gamma - 2}}{2 \theta t^{2\gamma - 2} e^{\theta t} (\theta + (2\gamma - 2)t^{-1})} = \lim_{t \to \infty} \frac{1}{2 \theta (\theta + (2\gamma - 2)t^{-1})} = \frac{1}{20^2}.
\]

Thus, combining (2.11), (2.12), and (2.13), we deduce that, as \( t \to \infty \),
\[
A_{1,t} \sim \frac{t^{2\gamma - 2}}{20^2}.
\]

For \( A_{2,t} \), we have, as \( t \to \infty \),
\[
A_{2,t} = \frac{e^{-\theta t}}{2\theta} \int_t^\infty e^{-\theta u} u^{2\gamma - 2} (e^{\theta t} - 1) du
= \frac{e^{\theta t}}{2\theta} \int_t^\infty e^{-\theta u} u^{2\gamma - 2} du - \frac{e^{-\theta t}}{2\theta} \int_t^\infty e^{-\theta u} u^{2\gamma - 2} du
\sim \frac{t^{2\gamma - 2}}{20^2},
\]
where we used the fact that \( \frac{e^{\theta t}}{2\theta} \int_t^\infty e^{-\theta u} u^{2\gamma - 2} du \leq C t^{2\gamma - 2} e^{-\theta t} \), and by using L'Hôpital's Rule, we have
\[
\lim_{t \to \infty} \int_t^\infty e^{-\theta u} u^{2\gamma - 2} du = \lim_{t \to \infty} \frac{-e^{\theta t} t^{2\gamma - 2}}{2 \theta t^{2\gamma - 2} e^{-\theta t} (-\theta + (2\gamma - 2)t^{-1})}
= \lim_{t \to \infty} \frac{-1}{2 \theta (-\theta + (2\gamma - 2)t^{-1})} = \frac{1}{20^2}.
\]

Therefore, (2.10), (2.14), and (2.15) prove (2.8).

The estimate (2.9) is a direct consequence of (2.8) and Lemma 2.5.

\[\square\]

**Lemma 2.7.** Assume that \( \gamma \in (0, 1) \) and \( \theta > 0 \). Then, as \( t \to \infty \),
\[
e^{-\theta t} \int_0^t \int_{-\infty}^\infty e^{\theta x} e^{\theta y} \left( e^{\frac{x}{\gamma}} \pm e^{\frac{x}{\gamma}} \right) dy dx
\sim e^{-\min(\theta,\gamma^{-1})t} \times \begin{cases} 
\int_0^\infty (e^{\theta u} - e^{-\theta u}) \left( e^{\frac{u}{\gamma}} \pm e^{\frac{u}{\gamma}} \right) du & \text{if } \theta < \frac{1}{\gamma} - 1, \\
\frac{t}{2\theta} & \text{if } \theta = \frac{1}{\gamma} - 1, \\
\frac{1}{\theta^2 - (\frac{1}{\gamma} - 1)^2} & \text{if } \theta > \frac{1}{\gamma} - 1,
\end{cases}
\]

and
\[
e^{-\theta t} \int_0^t \int_{-\infty}^\infty e^{\theta x} e^{\theta y} \left[ \left( e^{\frac{x}{\gamma}} \pm e^{\frac{x}{\gamma}} \right)^{2\gamma - 2} - \left( e^{\frac{x}{\gamma}} - e^{-\frac{x}{\gamma}} \right)^{2\gamma - 2} \right] dy dx
\sim e^{-\min(\theta,\gamma^{-1})t} \times \begin{cases} 
\int_0^\infty (e^{\theta u} - e^{-\theta u}) \left[ \left( e^{\frac{u}{\gamma}} \pm e^{\frac{u}{\gamma}} \right)^{2\gamma - 2} - \left( e^{\frac{u}{\gamma}} - e^{-\frac{u}{\gamma}} \right)^{2\gamma - 2} \right] du & \text{if } \theta < \frac{2}{\gamma} - 1, \\
\frac{(2\gamma - 2)e^{\frac{\theta}{\gamma}}}{\theta} & \text{if } \theta = \frac{2}{\gamma} - 1, \\
\frac{4\gamma^2 - 4}{\theta^2 - (\frac{2}{\gamma} - 1)^2} & \text{if } \theta > \frac{2}{\gamma} - 1.
\end{cases}
\]
Consequently, there exists a constant $C > 0$ that depends only on $\theta$ and $\gamma$ such that for every $t - s > 2$,

$$
e^{-\theta t}e^{-ht} \int_s^t \int_0^1 e^{0x}e^{0y} (e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2} dxdy \leq C \begin{cases} e^{-\min(\theta, \frac{1}{\gamma} - 1)|t-s|} & \text{if } \theta \neq \frac{1}{\gamma} - 1, \\ te^{-\min(\theta, \frac{1}{\gamma} - 1)|t-s|} & \text{if } \theta = \frac{1}{\gamma} - 1, \end{cases}$$

and

$$
e^{-\theta t}e^{-ht} \int_s^t \int_0^1 e^{0x}e^{0y} \left[(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2} - (e^{x \gamma} - e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}\right] dxdy \leq C \begin{cases} e^{-\min(\theta, \frac{1}{\gamma} - 1)|t-s|} & \text{if } \theta \neq \frac{2}{\gamma} - 1, \\ te^{-\min(\theta, \frac{1}{\gamma} - 1)|t-s|} & \text{if } \theta = \frac{2}{\gamma} - 1. \end{cases} \tag{2.19}$$

**Proof.** Let us prove (2.16). Let $t > 2$. Making the change of variables $u = y - x$, we get

$$e^{-\theta t} \int_0^t \int_{-\infty}^0 e^{0x}e^{0y} (e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2} dxdy = e^{-\theta t} \int_0^t \int_{-\infty}^\infty e^{-\theta u}e^{\theta y} (e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2} dudy$$

$$= e^{-\theta t} \int_0^t \int_0^\infty e^{-\theta u}e^{\theta y} (e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2} dudy$$

$$= \frac{e^{-\theta t}}{2\theta} \int_0^t \left(e^{-\theta u}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) du + \frac{e^{-\theta t}}{2\theta} \int_t^\infty e^{-\theta u}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) du \right).$$

Further,

$$B_{1,t} = \frac{e^{-\theta t}}{2\theta} \int_0^t \left(e^{-\theta u}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) du \right)$$

$$= \frac{e^{-\theta t}}{2\theta} \int_0^1 \left(e^{-\theta u}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) du + \frac{e^{-\theta t}}{2\theta} \int_1^t \left(e^{-\theta u}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) du \right) \right).$$

Since $\frac{2\theta u - 1}{u} \to 2\theta$, $\frac{e^{x \gamma} - e^{-y \frac{\gamma}{\theta}}}{u} \to \frac{1}{\gamma}$ and $e^{x \gamma} + e^{-y \frac{\gamma}{\theta}} \to 2$ as $u \to 0$, then

$$\int_0^1 e^{-\theta u}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) du \leq C \int_0^1 (u + u^{2\gamma - 1}) du < \infty.$$

On the other hand,

$$e^{-\theta t}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) \sim e^{-(\frac{1}{\gamma} - 1)\theta} \text{ as } t \to \infty. \tag{2.21}$$

Thus, for $\theta < \frac{1}{\gamma} - 1$, we obtain, using (2.21), $\int_0^\infty e^{-\theta u}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) du < \infty$. In this case, we have as $t \to \infty$,

$$\frac{e^{-\theta t}}{2\theta} \int_1^t e^{-\theta u}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) du \sim \frac{e^{-\theta t}}{2\theta} \int_1^\infty e^{-\theta u}(e^{x \gamma} + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2}(e^{2\theta u} - 1) du.$$

For $\theta > \frac{1}{\gamma} - 1$, since $0 \leq (1 + e^{-y \frac{\gamma}{\theta}})^{2\gamma - 2} \to 1$ as $u \to \infty$, there exists $C > 0$ such that
\[
\int_1^t e^{-\theta u}(e^{\frac{\theta}{2}} \pm e^{-\frac{\theta}{2}})^{2\gamma-2}(e^{2\theta u} - 1)\, du \geq C \int_1^t e^{(\frac{\theta}{1+\gamma})u}(1 - e^{-2\theta})\, du \to \infty
\]
as \( t \to \infty \). Combining this together with (2.21) and L'Hôpital's Rule, we get

\[
\lim_{t \to \infty} \frac{\int_1^t e^{-\theta u}(e^{\frac{\theta}{2}} \pm e^{-\frac{\theta}{2}})^{2\gamma-2}(e^{2\theta u} - 1)\, du}{20e^{\theta t}e^{-(\frac{1}{1+\gamma})t}} = \lim_{t \to \infty} \frac{2\theta e^{\theta t}e^{-(\frac{1}{1+\gamma})t}}{20e^{\theta t}} \left( \frac{1}{\gamma} + 1 \right)
\]

In this situation, we have as \( t \to \infty \),

\[
\frac{e^{-\theta t}}{20} \int_1^t e^{-\theta u}(e^{\frac{\theta}{2}} \pm e^{-\frac{\theta}{2}})^{2\gamma-2}(e^{2\theta u} - 1)\, du \sim \frac{e^{-(\frac{1}{1+\gamma})t}}{20\left( \theta - \frac{1}{\gamma} + 1 \right)}.
\]

For \( \theta = \frac{1}{\gamma} - 1 \), since \( 0 < (1 \pm e^{-\frac{\theta}{2}})^{2\gamma-2} \to 1 \) as \( u \to \infty \), there exists \( C > 0 \) such that, as \( t \to \infty \),

\[
\int_1^t e^{-\theta u}(e^{\frac{\theta}{2}} \pm e^{-\frac{\theta}{2}})^{2\gamma-2}(e^{2\theta u} - 1)\, du \geq C \int_1^t (1 - e^{-2\theta})\, du \to \infty.
\]

Combining this together with (2.21) and L'Hôpital's Rule leads to

\[
\lim_{t \to \infty} \frac{\int_1^t e^{-\theta u}(e^{\frac{\theta}{2}} \pm e^{-\frac{\theta}{2}})^{2\gamma-2}(e^{2\theta u} - 1)\, du}{t} = \lim_{t \to \infty} \frac{(1 \pm e^{-\frac{\theta}{2}})^{2\gamma-2}(1 - e^{-2\theta})}{t} = 1.
\]

In this case, we get as \( t \to \infty \),

\[
\frac{e^{-\theta t}}{20} \int_1^t e^{-\theta u}(e^{\frac{\theta}{2}} \pm e^{-\frac{\theta}{2}})^{2\gamma-2}(e^{2\theta u} - 1)\, du \sim \frac{te^{-\theta t}}{20}.
\]

Thus, as \( t \to \infty \),

\[
B_{1,t} \sim e^{-\min(0,\frac{1}{\gamma} - 1)t} \times \begin{cases} 
\int_0^\infty (e^{\theta u} - e^{-\theta u})(e^{\frac{\theta}{2}} \pm e^{-\frac{\theta}{2}})^{2\gamma-2} \, du & \text{if } \theta < \frac{1}{\gamma} - 1, \\
t \frac{1}{20} & \text{if } \theta = \frac{1}{\gamma} - 1, \\
\frac{1}{20\left( \theta - \frac{1}{\gamma} + 1 \right)} & \text{if } \theta > \frac{1}{\gamma} - 1.
\end{cases}
\]

For \( B_{2,t} \), we have

\[
\begin{cases} 
\int_0^\infty (e^{\theta u} - e^{-\theta u})(e^{\frac{\theta}{2}} \pm e^{-\frac{\theta}{2}})^{2\gamma-2} \, du & \text{if } \theta < \frac{1}{\gamma} - 1, \\
t \frac{1}{20} & \text{if } \theta = \frac{1}{\gamma} - 1, \\
\frac{1}{20\left( \theta - \frac{1}{\gamma} + 1 \right)} & \text{if } \theta > \frac{1}{\gamma} - 1.
\end{cases}
\]
\[ B_{2,t} = \frac{e^{-\theta t}}{2\theta} \int_t^\infty e^{-\theta u} (e^{\frac{u}{\gamma}} + e^{-\frac{u}{\gamma}})^{2\gamma-2} (e^{2\theta t} - 1) \, du \]

\[ = \frac{e^{\theta t}(1 - e^{-2\theta t})}{2\theta} \int_t^\infty e^{-\theta u} e^{-(\frac{1}{\gamma} - 1)u} (1 \pm e^{-\frac{u}{\gamma}})^{2\gamma-2} \, du, \]

where, as \( t \to \infty \),

\[ e^{\theta t} \int_t^\infty e^{-\theta u} e^{-(\frac{1}{\gamma} - 1)u} (1 \pm e^{-\frac{u}{\gamma}})^{2\gamma-2} \, du \sim \frac{e^{-(\frac{1}{\gamma} - 1)u}}{\theta + \frac{1}{\gamma} - 1} \]

since, by L'Hôpital's Rule, we have

\[ \lim_{t \to \infty} \frac{e^{\theta t} \int_t^\infty e^{-\theta u} e^{-(\frac{1}{\gamma} - 1)u} (1 \pm e^{-\frac{u}{\gamma}})^{2\gamma-2} \, du}{e^{\theta t} e^{-(\frac{1}{\gamma} - 1)u}} = \lim_{t \to \infty} \frac{1}{\theta + \frac{1}{\gamma} - 1} (1 \pm e^{-\frac{u}{\gamma}})^{2\gamma-2} \]

\[ = \frac{1}{\theta + \frac{1}{\gamma} - 1}. \]

Hence, as \( t \to \infty \),

\[ B_{2,t} \sim \frac{e^{-(\frac{1}{\gamma} - 1)t}}{2\theta(\theta + \frac{1}{\gamma} - 1)}. \quad (2.23) \]

Then, using (2.20), (2.22), and (2.23), we obtain (2.16).

By similar arguments as above and the fact

\[ (e^{\theta u} - e^{-\theta u}) \left[ (e^{\frac{u}{\gamma}} + e^{-\frac{u}{\gamma}})^{2\gamma-2} - (e^{\frac{u}{\gamma}} - e^{-\frac{u}{\gamma}})^{2\gamma-2} \right] \sim (4\gamma - 4)e^{-(\frac{1}{\gamma} - 1)t} \quad \text{as} \quad t \to \infty, \]

the claim (2.17) follows.

The estimate (2.18) and (2.19) are direct consequences of (2.16) and (2.17), respectively, combined with Lemma 2.5.

Consider the Ornstein–Uhlenbeck process \( X := \{X_t, t \geq 0\} \) defined by the following linear stochastic differential equation

\[ X_0 = 0, \quad dX_t = -\theta X_t \, dt + dG_t, \quad t \geq 0, \quad (2.24) \]

where the process \( G \) satisfies (A) and \( \theta \in \mathbb{R} \). According to Lemma 2.2, the solution of the Equation (2.24) can be expressed explicitly as

\[ X_t = e^{-\theta t} \int_0^t e^{\theta r} \, dG_r = G_t - \theta e^{-\theta t} \int_0^t e^{\theta r} \, G_r \, dr. \quad (2.25) \]

This implies that

\[ E(X_t^2) = R_G(t, t) - 2\theta e^{-\theta t} \int_0^t e^{\theta r} R_G(r, t) \, dr + \theta^2 e^{-2\theta t} \int_0^t \int_0^t e^{\theta r} e^{\theta s} R_G(r, s) \, dr \, ds. \quad (2.26) \]

If \( \theta > 0 \), then, according to (2.3),

\[ Z_t = \int_{-\infty}^t e^{-\theta(t-s)} \, dG_s. \quad (2.27) \]
is well defined as a Riemann–Stieltjes integral, so we can write
\[ X_t = Z_t - e^{-\theta t}Z_0, \quad t \geq 0. \] (2.28)

We will also make use of the following lemmas.

**Lemma 2.8** ([11]). Let \( g : [0, \infty) \times [0, \infty) \to \mathbb{R} \) be a symmetric function such that \( \frac{\partial g}{\partial s}(s, r) \) and \( \frac{\partial g}{\partial \theta r}(s, r) \) are integrable on \((0, t) \times [0, t)\) for all \( t > 0 \). Then, for every \( t > 0 \),
\[
\Delta g(t) := g(t, t) - 2\theta e^{-\theta t} \int_0^t g(s, t)e^{\theta s}ds + \theta^2 e^{-2\theta t} \int_0^t \int_0^t g(s, r)e^{\theta(s+r)}drds
\]
\[
= 2e^{-2\theta t} \int_0^t e^{\theta s} \frac{\partial g}{\partial s}(s, 0)ds + 2e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr \frac{\partial^2 g}{\partial s \partial r}(s, r)e^{\theta r}.
\] (2.29)

The following lemma is an immediate consequence of (2.25) and (2.6).

**Lemma 2.9** ([10]). Let \( G \) be a measurable process satisfying (A) and \( X \) is the solution of the Equation (2.24). Assume that \( \frac{\partial R_G}{\partial \theta x} \) is continuous on \( \mathbb{R}^2 \setminus \{(x, y) | x = y\} \). Then, for every \( 0 < s < t \), we have
\[
E(X_sX_t) = e^{-\theta(t-s)}E(X_s^2) + e^{-\theta s}e^{-\theta s} \int_s^t e^{\theta v} \int_0^s e^{\theta u} \frac{\partial^2 R_G}{\partial u \partial v}(u, v)dudv, \quad s, t \geq 0,
\] (3.1)
provided that the integral on the right-hand side converges.

### 3. Langevin equations driven by noises with stationary increments

Recall that a process \( G = \{G_t, t \in \mathbb{R}\} \) has stationary increments if, for all \( s \in \mathbb{R}, \{G_t - G_0, t \in \mathbb{R}\} \) has the same finite distributions as \( \{G_{t+s}, G_s, t \in \mathbb{R}\} \).

The existence and uniqueness of stationary solutions to Langevin equations driven by noise processes with stationary increments are discussed in the following theorem.

**Theorem 3.1.** Let \( \{G_t, t \in \mathbb{R}\} \) be a measurable process with stationary increments satisfying (A). Assume \( \theta > 0 \). Then,

a. The solution \( Z_t = \int_{-\infty}^t e^{-\theta(t-s)}dG_s, t \geq 0, \) of the equation
\[
dZ_t = -\theta Z_t dt + dG_t, Z_0 = \int_{-\infty}^0 e^{\theta s}dG_s, \quad t \geq 0,
\] (3.1)
is a stationary process.

b. In addition, if we assume that the function \( \rho_G(t) := E(G_t^2), t \in \mathbb{R}, \) is twice continuously differentiable on \( \mathbb{R} \setminus \{0\}, \) then
\[
E(Z_tZ_0) = e^{-\theta t}E(Z_0^2) + \frac{e^{-\theta t}}{2} \int_0^t \int_{-\infty}^0 e^{\theta u}e^{\theta v} \rho_G''(v - u)dudv, \quad t \geq 0,
\] (3.2)
and the process \( X, \) given by (2.25), satisfies
\[
E(X_sX_t) = e^{-\theta(t-s)}E(X_s^2) + e^{-\theta s}e^{-\theta s} \int_s^t e^{\theta v} \int_0^s e^{\theta u} \rho_G''(v - u)dudv, \quad s, t \geq 0,
\] (3.3)
\[
|E(X_t^2) - E(Z_0^2)| \leq Ce^{-\theta t}, \quad t \geq 0.
\]  
(3.4)

Also,
\[
E(Z_0^2) = \frac{\theta}{2} \int_0^\infty e^{-\theta t} \rho_G(t) dt,
\]  
(3.5)

provided that the integrals above converge.

**Proof.** Applying Lemma 2.2 for \( \zeta = \int_0^t e^{\theta s} dG_s \), the unique solution to the Equation (3.1) can be expressed as
\[
Z_t = \int_{-\infty}^t e^{-\theta(t-s)} dG_s = G_t - \theta e^{-\theta t} \int_{-\infty}^t e^{\theta s} G_s ds, \quad t \geq 0.
\]

On the other hand, it follows from [3, Theorem 2.1] that the process
\[
G_t = \theta e^{-\theta t} \int_{-\infty}^t e^{\theta s} G_s ds, \quad t \geq 0,
\]
is a unique-in-law stationary solution to the Langevin Equation (3.1). Thus the part (a) is proved.

Let us prove the part (b). Since \( G_0 = 0 \) a.s. and \( G \) has stationary increments, we have
\[
R_G(u,v) = \frac{1}{2} [\rho_G(u) + \rho_G(v) - \rho_G(v-u)] \quad \text{for all } u, v \in \mathbb{R}. \tag{3.6}
\]
Combining this with
\[
Z_t = Z_0 + e^{-\theta} \int_0^t e^{-\theta s} dG_s,
\]  
(2.7) and \( \frac{\partial^2 R_G}{\partial u \partial v} (u,v) = \rho''_G(v-u) \) for all \( u < v \), we deduce that
\[
E(Z_t Z_0) = e^{-\theta} E(Z_0^2) + e^{-\theta} \int_0^t \int_{-\infty}^0 e^{\theta u} e^{\theta v} \frac{\partial^2 R_G}{\partial v \partial u} (u,v) du dv
+ e^{-\theta} E(Z_0^2) + \frac{e^{-\theta t}}{2} \int_{-\infty}^t \int_{-\infty}^0 e^{\theta u} e^{\theta v} \rho''_G(v-u) du dv,
\]
which proves (3.2). The claim (3.3) is a direct consequence of (2.30) and \( \frac{\partial^2 R_G}{\partial v \partial u} (u,v) = \rho''_G(v-u) \) for all \( u \neq v \). Furthermore, the inequality (3.4) follows immediately from (2.28) and the stationarity of \( Z \).
Now, it remains to prove (3.5). According to (2.3) and (3.6), we can write

\[ E(Z_0^2) = \theta^2 \int_{-\infty}^{0} \int_{-\infty}^{0} e^{\theta u} e^{\theta v} R_G(u, v) \, du \, dv \]

\[ = \theta^2 \int_{-\infty}^{0} \int_{-\infty}^{0} e^{\theta u} e^{\theta \rho_G(u)} \, du \, dv - \theta^2 \int_{-\infty}^{0} \int_{-\infty}^{v} e^{\theta u} e^{\theta \rho_G(v - u)} \, du \, dv \]

\[ = \theta \int_{-\infty}^{0} e^{\theta u} \rho_G(u) \, du - \theta^2 \int_{-\infty}^{0} \int_{0}^{\infty} e^{-\theta x} e^{2\theta \rho_G(x)} \, dx \, dv \]

\[ = \theta \int_{0}^{\infty} e^{-\theta x} \rho_G(x) \, dx. \]

Therefore the proof is complete. \( \square \)

As examples we consider Hermite Ornstein–Uhlenbeck processes of the first kind and Gaussian Ornstein–Uhlenbeck processes of the second kind and study the decay of their auto-covariance functions.

### 3.1. Fractional Ornstein–Uhlenbeck processes

Here we consider the fractional Ornstein–Uhlenbeck process

\[ X^H_t := e^{-\theta t} \int_{0}^{t} e^{\theta y} dB^H_y, \quad (3.7) \]

that is, the solution to the Langevin Equation (2.24) in the case when \( G = B^H \) is a fractional Brownian motion with Hurst parameter \( H \in (0, 1) \).

Since \( B^H \) is Gaussian and

\[ E(B^H_t - B^H_s)^2 = |t - s|^{2H}; \quad s, t \geq 0, \]

we deduce that the assumption (\( A \)) holds for \( G = B^H \), according to Remark 2.1.

So, if \( \theta > 0 \), the integral

\[ Z^H_t := \int_{-\infty}^{t} e^{-\theta(t-s)} dB^H_s \quad (3.8) \]

is well defined as a pathwise Riemann–Stieltjes integral and we have

\[ X^H_t = Z^H_t - e^{-\theta t} Z^H_0. \]

Let us now state properties of the processes \( X^H \) and \( Z^H \), defined by (3.7) and (3.8), respectively.

**Theorem 3.2.** Assume that \( H \in (0, 1) \) and \( \theta > 0 \). Let \( X^H \) and \( Z^H \) be the processes defined by (3.7) and (3.8), respectively. Then

i. \( Z^H \) is an ergodic stationary Gaussian process.

ii. For any integer \( p \geq 1 \), there exists \( C > 0 \) depending only on \( \theta, H \) and \( p \) such that
• if $p$ is even, $|E[(X_t^H)^p] - E[(Z_0^H)^p]| \leq Ce^{-\theta t}$ for all $t \geq 0$, with
  $$E[(Z_0^H)^p] = \frac{p!}{2^p \left(\frac{p}{2}\right)!} \left(\frac{H!^2 (2H)}{\theta^{2H}}\right)^{\frac{p}{2}},$$

• if $p$ is odd, $E[(X_t^H)^p] = E[(Z_0^H)^p] = 0$.

iii. If $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $E(Z_t^HZ_0^H) \sim \frac{t^{2H-2}}{\theta^2}$ as $t \to \infty$. If $H = \frac{1}{2}$, $E(Z_t^2Z_0^2) = \frac{e^{-\theta t}}{2\pi}$.

iv. If $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, there exists $C > 0$ depending only on $\theta$ and $H$ such that
  $$E(X_t^H X_s^H) \leq C|t-s|^{2H-2}$$
  for all $|t-s| > 2$,

and if $H = \frac{1}{2}$, $E(X_t^\frac{1}{2}X_t^\frac{1}{2}) \leq Ce^{-\theta|t-s|}$ for all $|t-s| > 2$.

**Proof.** These claims can easily be obtained using Theorem 3.1 and (2.8). The claims (i) and (iii) have been obtained previously by [4] and [3]. For the point (ii), it follows from [13] that
  $$E[(Z_0^H)^p] = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ \frac{p!}{2^p \left(\frac{p}{2}\right)!} \left(\frac{H!^2 (2H)}{\theta^{2H}}\right)^{\frac{p}{2}} & \text{if } p \text{ is even.} \end{cases}$$

Combining this with (i), Gaussianity of $X^H$ and $Z^H$ and (2.28), the claim (ii) is obtained. The fourth part of Theorem 3.2 is an obvious consequence of the third part and the decomposition (2.28).

**Remark 3.3.** When $\theta < 0$, the properties of the processes $X^H$ and $Z^H$ given by (3.7) and (3.8), respectively, are very different from those corresponding to the case $\theta > 0$ given in Theorem 3.2. For instance, if $\theta < 0$, $e^{-\theta t}X_t^H \to 0$ as $t \to \infty$ and in $L^2(\Omega)$ as $t \to \infty$. We refer only to [11] and [12] for information about this case and additional references.

### 3.2. Hermite Ornstein–Uhlenbeck processes

The Hermite process $G^{(q, H)} := \left\{G_t^{(q, H)}, t \in \mathbb{R}\right\}$ of order $q \geq 1$ and Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ is defined as a multiple Wiener-Itô integral of the form

$$G_t^{(q, H)} = d(q, H)\int_{\mathbb{R}} dW(y_1) \ldots \int_{\mathbb{R}} dW(y_q) \left(\int_0^t (s - y_1)^{-\frac{1}{2}+\frac{qH}{2}} \ldots (s - y_q)^{-\frac{1}{2}+\frac{qH}{2}} \frac{ds}{s}\right)$$

(3.9)

for every $t \in \mathbb{R}$, where $X_t^\alpha = X^\alpha 1_{(0, \infty)}(x)$, $\int_0^t := -\int_{-t}^0$ if $t \leq 0$, and $\{W(y), y \in \mathbb{R}\}$ is a Wiener process, whereas $d(q, H)$ is a normalizing positive constant chosen to ensure that $E\left[(G_t^{(q, H)})^2\right] = 1$.

Except for Gaussianity, Hermite processes of order $q \geq 2$ share many properties with the fBm (corresponding to $q = 1$). First note that, according to the fact that $G^{(q, H)}$ is Hermite,
and Remark 2.1, we deduce that the assumption (A) holds for $G = G^{(q,H)}$. Moreover, the Hermite process (3.9) is $H$-self-similar and it has stationary increments. Its covariance coincides with the covariance of the fBm for all $q \geq 1$, that is, for every $q \geq 1$,

$$E\left(G^{(q,H)}_t - G^{(q,H)}_s\right)^2 = |s - t|^{2H}; s, t \in \mathbb{R},$$

(3.10)

The class of Hermite processes also includes the Rosenblatt process which is obtained for $q = 2$. The Hermite process is non-Gaussian if $q \geq 2$. These processes have attracted a lot of interest in the recent past (see the monographs [21, 25] and the references therein).

The Wiener integral of a deterministic function $f$ with respect to a Hermite process $G^{(q,H)}$ which we denote by $\int_{\mathbb{R}} f(u) dG^{(q,H)}_u$, has been constructed by [19].

We recall that the stochastic integral $\int_{\mathbb{R}} f(u) dG^{(q,H)}_u$ is well-defined for any $f$ belonging to the space $|H|$ of functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)f(v)||u - v|^{2H-2}dudv < \infty.$$ Further, for any $f, g \in |H|$, that

$$E\left[\int_{\mathbb{R}} f(u) dG^{(q,H)}_u \int_{\mathbb{R}} g(v) dG^{(q,H)}_v\right] = H(2H - 1)\int_{\mathbb{R}} \int_{\mathbb{R}} f(u) g(v) |u - v|^{2H-2}dudv.$$ (3.11)

Now, let us consider the Hermite Ornstein–Uhlenbeck process

$$X_t^{(q,H)} := e^{-\theta t} \int_0^t e^{\theta s} dG^{(q,H)}_s, \quad t \geq 0,$$ (3.12)

that is, the solution to the Langevin Equation (2.24) in the case when $G = G^{(q,H)}$ is the Hermite process of order $q \geq 1$ and Hurst parameter $H \in (\frac{1}{2}, 1)$, according to Lemma 2.2. Further, if $\theta > 0$, the process

$$Z_t^{(q,H)} := \int_{-\infty}^t e^{-\theta(t-s)} dG^{(q,H)}_s, \quad t \geq 0,$$ (3.13)

is well defined in the Riemann–Stieltjes sense, and we have

$$X_t^{(q,H)} = Z_t^{(q,H)} - e^{-\theta t} Z_0^{(q,H)}, \quad t \geq 0.$$ By Theorem 3.1, $Z^{(q,H)}$ is stationary. Furthermore, using (3.11), stationarity of $Z^H$, and $H > \frac{1}{2}$, we have, for every $s, t \geq 0$,

$$E\left(Z_t^{(q,H)} Z_0^{(q,H)}\right) = E\left(Z_t^H Z_0^H\right) \quad \text{for all } \theta > 0,$$

and

$$E\left(X_t^{(q,H)} X_0^{(q,H)}\right) = E\left(X_t^H X_0^H\right) \quad \text{for all } \theta \in \mathbb{R},$$

where $X^H$ and $Z^H$ are the processes given by (3.7) and (3.8), respectively. Combining the results above with Theorem 3.1 and Theorem 3.2 leads to the following theorem.
Theorem 3.4. Assume that $H > \frac{1}{2}$ and $\theta > 0$. Let $X^{(q,H)}$ and $Z^{(q,H)}$ be the processes defined by (3.12) and (3.13), respectively. Then

- $Z^{(q,H)}$ is a stationary process, and $E \left[ \left( Z^{(q,H)}_0 \right)^2 \right] = \frac{H \Gamma(2H)}{\theta^{2H}}$.
- There exists a constant $C > 0$ depending only on $\theta$ and $H$ such that, for all $t \geq 0$,
  \[
  \left| E \left[ \left( X^{(q,H)}_t \right)^2 \right] - \frac{H \Gamma(2H)}{\theta^{2H}} \right| \leq Ce^{-\theta t}.
  \]
- $E(Z^{(q,H)}_t Z^{(q,H)}_0) \sim \frac{t^{2H-2}}{\theta^2}$ as $t \to \infty$.
- There exists a constant $C > 0$ depending only on $\theta$ and $H$ such that, for all $|t - s| > 2$, $E(X^{(q,H)}_t X^{(q,H)}_s) \leq C|t - s|^{2H-2}$.

3.3. Gaussian Ornstein–Uhlenbeck processes of the second kind

Let $U := \{U_t, t \geq 0\}$ be a Gaussian process satisfies the assumption (A). In addition, we assume that the process $U$ is $\gamma$-self-similar, that is, $\{U_{bt}, t \geq 0\} \stackrel{\text{law}}{=} \{b^\gamma U_t, t \geq 0\}$ for all $b > 0$. Hence, the integrals $\int_0^t e^{-s}dU_a$ if $t \geq 0$ and $\int_0^t e^{-s}dU_a$ if $t < 0$, with $a_t := \theta e^t$, are well defined as Riemann–Stieltjes integrals, and let $Y^{(1)}_{t, U} := \left\{ Y^{(1)}_{t, U}, t \in \mathbb{R} \right\}$ denote the process defined by $Y^{(1)}_{t, U} := \int_0^t e^{-s}dU_a$ if $t \geq 0$ and $Y^{(1)}_{t, U} := -\int_0^t e^{-s}dU_a$ if $t < 0$.

Let us introduce the following processes,

\[
L_t := e^{-t}U_{a_t} - U_{a_0}, \quad t \in \mathbb{R}, \quad \eta_t := \int_0^t e^{-s}U_{a_s}ds \text{ for } t \geq 0, \quad \eta_t := -\int_0^t e^{-s}U_{a_s}ds \text{ for } t < 0.
\] (3.14)

Integrating by parts, we get

\[
Y^{(1)}_{t, U} = L_t + \eta_t \quad \text{for all } t \in \mathbb{R}.
\] (3.15)

Define

\[
f_U(x) := \gamma^{-\gamma}R_U\left( e^{\frac{x}{\gamma}}, e^{-\frac{x}{\gamma}} \right) = \gamma^{-\gamma}E\left( U_{e^{\frac{x}{\gamma}}} U_{e^{-\frac{x}{\gamma}}} \right), \quad x \in \mathbb{R}.
\] (3.16)

We will make use of the following lemmas.

Lemma 3.5. Let $\{L_t, t \in \mathbb{R}\}$ and $\{\eta_t, t \in \mathbb{R}\}$ be the processes given by (3.14), and let $f_U$ be the even function defined by (3.16). Then, for every $s, t \in \mathbb{R}$,

\[
E(\eta_t \eta_s) = h_U(s) + h_U(t) - h_U(|t - s|),
\] (3.17)

\[
E(L_t L_s) = f_U(|t - s|) - f_U(t) - f_U(s) + f_U(0),
\] (3.18)

and

\[
E(L_t \eta_s) + E(L_s \eta_t) = 0,
\] (3.19)

where $h_U(t) := \int_0^t (|t| - x)f_U(x)dx$ for all $t \in \mathbb{R}$.
Hence, for every \(s,t \in \mathbb{R}\),
\[
R^{(i)}_{Y_{U}}(s,t) = E\left( Y_{s,U}^{(i)} Y_{t,U}^{(i)} \right) = E(L_{s}L_{t}) + E(\eta_{s}\eta_{t}),
\]
(3.20)
and, if we suppose that \(f_{U}\) is twice continuously differentiable on \(\mathbb{R} \setminus \{0\}\), we have for every \(s,t \in \mathbb{R}\) with \(s \neq t\),
\[
\frac{\partial^{2}R^{(i)}_{Y_{U}}}{\partial t \partial s}(s,t) = f''_{U}(|t-s|) - f''_{U}(|t-s|).
\]
(3.21)
Moreover, for every \(s,t \in \mathbb{R}\),
\[
E\left[ (Y_{s,U}^{(i)} - Y_{t,U}^{(i)})^2 \right] = 2f_{U}(0) - 2f_{U}(|t - s|) + 2h_{U}(|t - s|),
\]
(3.22)
which implies that the Gaussian process \(Y_{U}^{(i)}\) has stationary increments.

In addition, if we suppose that \(\int_{0}^{\infty} |f_{U}(x)|dx < \infty\), then the process \(Y_{U}^{(1)}\) satisfies the assumption \((A)\).

**Proof.** Using similar arguments as in [1], the statements (3.17)–(3.20) can be immediately proved. For (3.21), it follows from (3.17) to (3.20) that
\[
\frac{\partial^{2}R^{(i)}_{Y_{U}}}{\partial t \partial s}(s,t) = h''_{U}(|t-s|) - f''_{U}(|t-s|)
\]
for every \(s,t \in \mathbb{R}\) such that \(s \neq t\). Furthermore, it is clear that \(h''_{U}(x) = f''_{U}(x)\). Thus (3.21) is obtained. The estimate (3.22) follows directly from (3.17)–(3.20).

Let us now prove that \(Y_{U}^{(i)}\) satisfies the assumption \((A)\). Since \(U\) is Gaussian, then, using (3.15), the process \(Y_{U}^{(1)}\) is Gaussian. Combining this result with Remark 2.1, we see that, in order to show that \(Y_{U}^{(1)}\) satisfies \((A)\), it suffices to prove that for some \(C > 0\),
\[
E\left[ (Y_{s,U}^{(1)} - Y_{t,U}^{(1)})^2 \right] \leq C|t - s|^\gamma \quad \text{for all } s,t \in \mathbb{R}.
\]
From (3.22) we have, for every \(s,t \in \mathbb{R}\),
\[
E\left[ (Y_{s,U}^{(1)} - Y_{t,U}^{(1)})^2 \right] \leq 2|f_{U}(0) - f_{U}(|t - s|)| + 2|h_{U}(|t - s|)|.
\]
Since \(|h'_{U}(x)| = \int_{0}^{\infty} f_{U}(x)dx| \leq \int_{0}^{\infty} |f_{U}(x)|dx =: C_{f} < \infty\) for all \(x \geq 0\), we deduce that \(|h_{U}(x)| \leq C_{f}|x|\) for all \(x \geq 0\).

On the other hand, since \(U\) is \(\gamma\)-self-similar, we get for all \(x \geq 1\),
\[
|f_{U}(0) - f_{U}(x)| \leq |f_{U}(0)| + |f_{U}(x)| \leq 2|f_{U}(0)| \leq 2|f_{U}(0)||x|^\gamma.
\]
Moreover, for all \(0 < x < 1\),
\[
|f_{U}(0) - f_{U}(x)| = \gamma^{\frac{\gamma}{2}} |e^{-x}E(U_{1}U_{\hat{x}}) - E(U_{1}^{2})|
\leq \gamma^{\frac{\gamma}{2}} \left| (e^{-x} - 1)E(U_{1}U_{\hat{x}}) \right| + \left| E(U_{1}(U_{\hat{x}} - U_{1}) \right|
\leq C(|x| + |x|^\gamma) \leq C|x|^\gamma,
\]
where we used \(e^{x} - 1 \rightarrow -1\) as \(x \to 0\) and for all \(0 < x < 1\),
\[ |E(U_1(U_\xi - U_1))| \leq (E(U_1^2))^{\frac{1}{2}} (E(U_\xi^2 - U_1^2))^{\frac{1}{2}} \leq C \left| \frac{e^x - 1}{x} \right|^2 |x|^\gamma \leq C|x|^{\gamma}, \]

according to the assumption \((A)\). Thus the desired result is obtained. \(\square\)

**Remark 3.6.** Since \(\{Y^{(1)}_{t, U}, t \in \mathbb{R}\}\) has stationary increments, note that
\[
\rho''_{Y^{(1)}_U}(t - s) = \frac{\partial^2 R_{Y^{(1)}_U}(s, t)}{\partial t \partial s} \neq f_U(|t - s|) - f''_U(|t - s|),
\]
according to (3.6) and (3.21).

**Lemma 3.7.** Define for every \(\gamma > 0\),
\[ m_\gamma(x) := (e^{\frac{x}{\sqrt{\gamma}}} - e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma}, \quad n_\gamma(x) := (e^{\frac{x}{\sqrt{\gamma}}} + e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma}. \]
Then, for every \(\gamma \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)\),
\[ m''_\gamma(x) - m_\gamma(x) = \frac{2(2\gamma - 1)}{\gamma} (e^{\frac{x}{\sqrt{\gamma}}} - e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma - 2}, \quad (3.23) \]

and
\[ n''_\gamma(x) - n_\gamma(x) = \frac{2(1 - 2\gamma)}{\gamma} (e^{\frac{x}{\sqrt{\gamma}}} + e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma - 2}. \quad (3.24) \]

**Proof.** Let us prove (3.23). We have
\[ m''_\gamma(x) = \left(1 - \frac{1}{2\gamma}\right) (e^{\frac{x}{\sqrt{\gamma}}} - e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma - 1} (e^{\frac{x}{\sqrt{\gamma}}} + e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma + 1} + \frac{1}{2\gamma} (e^{\frac{x}{\sqrt{\gamma}}} - e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma}. \]
This leads to
\[ m''_\gamma(x) - m_\gamma(x) = \left(1 - \frac{1}{2\gamma}\right) (e^{\frac{x}{\sqrt{\gamma}}} - e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma - 1} \left( (e^{\frac{x}{\sqrt{\gamma}}} + e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma} - (e^{\frac{x}{\sqrt{\gamma}}} - e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma} \right) \]
\[ = 4 \left(1 - \frac{1}{2\gamma}\right) (e^{\frac{x}{\sqrt{\gamma}}} - e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma - 2} \]
\[ = \frac{2(2\gamma - 1)}{\gamma} (e^{\frac{x}{\sqrt{\gamma}}} - e^{-\frac{x}{\sqrt{\gamma}}})^{2\gamma - 2}, \]
which proves (3.23). Similar reasoning gives (3.24). \(\square\)

Now, let us consider the Ornstein–Uhlenbeck process of the second kind \(X_U := \{X_{t, U}, t \geq 0\}\), defined as the unique (pathwise) solution to
\[ X_{0, U} = 0, \quad dX_{t, U} = -\theta X_{t, U}dt + dY^{(1)}_{t, U}, \quad t \geq 0. \quad (3.25) \]

According to Lemma 3.5, the process \(Y^{(1)}_{t, U}\) satisfies the assumption \((A)\). So, using Lemma 2.2, the unique solution of (3.25) can be written as
\[ X_{t,U} = e^{-\theta t} \int_0^t e^{\theta s} dY_{s,U}^{(1)}, \quad t \geq 0. \] (3.26)

Moreover, for any \( \theta > 0 \), the process
\[ Z_{t,U} := \int_{-\infty}^t e^{-\theta(t-s)} dY_{s,U}^{(1)}, \quad t \geq 0 \] (3.27)
is well defined as a Riemann–Stieltjes integral.

Hence we can also write
\[ X_{t,U} = Z_{t,U} - e^{-\theta t} Z_{0,U}, \quad t \geq 0. \]

Furthermore, since \( Y_{U}^{(1)} \) has stationary increments, it follows from (3.5) and (3.22) that
\[
E\left(Z_{0,U}^2\right) = 0 \int_0^\infty \left(f_U(0) - f_U(t) + h_U(t)\right)e^{-\theta t} dt \\
= f_U(0) - \int_0^\infty f_U(t)e^{-\theta t} dt + \theta \int_0^\infty e^{-\theta t} \int_0^t (t-x)f_U(x) dx dt \\
= f_U(0) - \int_0^\infty f_U(t)e^{-\theta t} dt + \frac{1}{\theta} \int_0^\infty f_U(x)e^{-\theta x} dx \\
= f_U(0) + \left(\frac{1}{\theta} - \theta\right) \int_0^\infty f_U(t)e^{-\theta t} dt. \] (3.28)

Now we will apply the results above to fractional, subfractional and bifractional Ornstein–Uhlenbeck processes of the second kind.

### 3.3.1. Fractional Ornstein–Uhlenbeck processes of the second kind

Here we consider the fractional Ornstein–Uhlenbeck process of the second kind \( X_{B^H} := \{X_{t,B^H}, t \geq 0\} \), defined as the unique solution to (3.25) when \( U = B^H \) is a fBm with Hurst parameter \( H \in (0,1) \). More precisely,
\[
X_{0,B^H} = 0, \quad dX_{t,B^H} = -\theta X_{t,B^H} dt + dY_{t,B^H}^{(1)}, \quad t \geq 0. \] (3.29)

In this case we have, according to [1, Section 4], \( \int_0^\infty |f_{B^H}(x)| dx < \infty \). Moreover, for all \( x \in \mathbb{R} \),
\[
f_{B^H}(x) = H^{2H} R_{B^H}(e^{\tilde{m}} - e^{\tilde{m}H}) \\
= \frac{H^{2H}}{2} \left[ e^x + e^{-x} - (e^{\tilde{m}H} - e^{\tilde{m}})^{2H} \right] \\
= \frac{H^{2H}}{2} \left[ e^x + e^{-x} - m_H(x) \right],
\]
where the function \( m_H(x) \) is defined in Lemma 3.7. Moreover, using the latter equation, Remark 3.6 and Lemma 3.7, we get
\[
\rho_{B^H}''(x) = f_{B^H}(x) - f_{B^H}''(x) = (2H - 1)H^{2H-1}(e^{\tilde{m}H} - e^{-\tilde{m}})^{2H-2}. \] (3.30)

**Theorem 3.8.** Assume that \( H \in (0,1) \) and \( \theta > 0 \). Let \( \{X_{t,B^H}, t \geq 0\} \) and \( \{Z_{t,B^H}, t \geq 0\} \) be the processes defined by (3.26) and (3.27) for \( U = B^H \), respectively. Then
• \( \{Z_{t,BH}, t \geq 0\} \) is an ergodic stationary Gaussian process, and
\[
E\left(Z_{0,BH}^2\right) = f_{BH}(0) + \left(\frac{1}{\theta} - \theta\right) \int_0^{\infty} f_{BH}(t)e^{-\theta t} dt.
\]

• There exists a constant \( C > 0 \) depending only on \( \theta \) and \( H \) such that, for all \( t \geq 0 \),
\[
\left| E\left(X_{t,BH}^2\right) - E\left(Z_{0,BH}^2\right) \right| \leq Ce^{-\theta t}.
\]

• If \( H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \), then, as \( t \to \infty \),
\[
E(Z_{t,BH}Z_{0,BH}) \sim (2H - 1)H^{2H-1}e^{-\min(\theta, \frac{1}{H} - 1)t}
\]
\[
\times \begin{cases} 
\int_0^{\infty} (e^{\theta u} - e^{-\theta u})(e^{\frac{u}{H}} - e^{-\frac{u}{H}})^{2H-2} du & \text{if } \theta < \frac{1}{H} - 1, \\
\frac{t}{2\theta} & \text{if } \theta = \frac{1}{H} - 1, \\
\frac{1}{\theta^2 - \left(\frac{1}{H} - 1\right)^2} & \text{if } \theta > \frac{1}{H} - 1.
\end{cases}
\]

• If \( H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \), there exists a constant \( C > 0 \) depending only on \( \theta \) and \( H \) such that, for all \( |t - s| > 2 \),
\[
E(X_{t,BH}X_{s,BH}) \leq C \begin{cases} 
eq \frac{1}{H} - 1, \\
eq \frac{1}{H} - 1. \\
\end{cases}
\]

Proof. Except the ergodicity of \( Z_{BH} \), the results of Theorem 3.8 can be immediately obtained by using Theorem 3.1, (2.16), (2.18), (3.28), and (3.30). The ergodicity of \( Z_{BH} \) is an immediate conclusion of the fact that \( Z_{BH} \) is a stationary Gaussian process and its auto-covariance function asymptotically vanishes (see, for instance, [22, Example 2.2.8]).

Remark 3.9. Note that the third and fourth parts of Theorem 3.8 are valid for all \( H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \). However, these two estimates have been proved in [16] only when \( H \in \left(\frac{1}{2}, 1\right) \) with \( \theta \neq \frac{1}{H} - 1 \).

3.3.2. Subfractional Ornstein–Uhlenbeck processes of the second kind

The subfractional Brownian motion (subfBm) \( S^H := \{S^H_t, t \geq 0\} \) with parameter \( H \in (0,1) \) is a centered Gaussian process with covariance function
\[
E(S^H_t S^H_s) = t^{2H} + s^{2H} - \frac{1}{2} \left( (t + s)^{2H} + |t - s|^{2H} \right).
\]

Note that, when \( H = \frac{1}{2} \), \( S^H \) is a standard Brownian motion. The subfBm \( S^H \) is self-similar with exponent of self-similarity \( H \in (0,1) \) and its increments are non-stationary. Moreover, using
\[
E(S^H_t - S^H_s)^2 \leq (2 - 2^{2H-1})|s - t|^{2H}; s, t \geq 0,
\]
the fact that \( S^H \) is Gaussian and Remark 2.1, we deduce that the assumption (A) holds for \( G = S^H \). On the other hand, according to [1, Section 4], \( \int_0^\infty |f_{S^H}(x)|dx < \infty \). Moreover, for all \( x \in \mathbb{R} \),

\[
 f_{S^H}(x) = R_{S^H}(e^{\tilde{m}}, e^{\tilde{n}})
\]

\[
 = e^x + e^{-x} - \frac{1}{2} \left[ (e^{\tilde{m}} + e^{\tilde{n}})^{2H} + (e^{\tilde{m}} - e^{\tilde{n}})^{2H} \right]
\]

\[
 = e^x + e^{-x} - \frac{1}{2} [n_H(x) + m_H(x)],
\]

where the functions \( n_H(x) \) and \( m_H(x) \) are defined in Lemma 3.7. Further, using the latter equation, Remark 3.6 and Lemma 3.7, we get

\[
 \rho''_{Y_{S^H}}(x) = f_{S^H}(x) - f''_{S^H}(x) = (2H - 1)H^{2H - 1} \left[ (e^{\tilde{m}} + e^{\tilde{n}})^{2H-2} - (e^{\tilde{m}} - e^{\tilde{n}})^{2H-2} \right].
\]

Now let us consider the subfractional Ornstein–Uhlenbeck process of the second kind \( X_{S^H} := \{X_{t,S^H}, t \geq 0\} \), defined as the unique solution to (3.25) when \( U = S^H \). In other words, \( X_{S^H} \) is the solution to the equation

\[
 X_{0,S^H} = 0, \quad dX_{t,S^H} = - \theta X_t dt + dY_{1,S^H}, \quad t \geq 0,
\]

Using similar arguments as in Section 3.3.1, we deduce the following result.

**Theorem 3.10.** Assume that \( H \in (0,1) \) and \( \theta > 0 \). Let \( \{X_{t,S^H}, t \geq 0\} \) and \( \{Z_{t,S^H}, t \geq 0\} \) be the processes defined by (3.26) and (3.27) for \( U = S^H \), respectively. Then

- \( \{Z_{t,S^H}, t \geq 0\} \) is an ergodic stationary Gaussian process, and

\[
 E\left(Z_{0,S^H}^2\right) = f_{S^H}(0) + \left( \frac{1}{\theta} - \theta \right) \int_0^\infty f_{S^H}(t)e^{-\theta t}dt.
\]

- There exists a constant \( C > 0 \) depending only on \( \theta \) and \( H \) such that, for all \( t \geq 0 \),

\[
 \left| E\left(X_{t,S^H}^2\right) - E\left(Z_{0,S^H}^2\right) \right| \leq Ce^{-\theta t}.
\]

- If \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \), then, as \( t \to \infty \),

\[
 E(Z_{t,S^H}Z_{0,S^H}) \sim (2H - 1)H^{2H - 1}e^{-\min(0,1-H)}t
\]

\[
 \times \left\{ \begin{array}{ll}
 \int_0^\infty (e^{\theta u} - e^{-\theta u}) \left[ (e^{\tilde{m}} + e^{\tilde{n}})^{2H-2} - (e^{\tilde{m}} - e^{\tilde{n}})^{2H-2} \right] du & \text{if } \theta < \frac{2}{H-1}, \\
 \frac{(2H - 2)\theta e^\theta}{\theta} & \text{if } \theta = \frac{2}{H-1}, \\
 \frac{4H - 4}{\theta^2 - (2/H-1)^2} & \text{if } \theta > \frac{2}{H-1}.
\end{array} \right.
\]

- If \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \), there exists a constant \( C > 0 \) depending only on \( \theta \) and \( H \) such that, for all \( |t - s| > 2 \),
\[ E(X_{t,s}, X_{s,t}) \leq C \begin{cases} e^{-\min(\theta, \frac{1}{H}-1)|t-s|} & \text{if } \theta \neq \frac{2}{H} - 1, \\ te^{-\min(\theta, \frac{1}{H}-1)|t-s|} & \text{if } \theta = \frac{2}{H} - 1. \end{cases} \]

### 3.3.3. Bifractional Ornstein–Uhlenbeck processes of the second kind

Let \( B^{H,K} := \{B^t_{H,K}, t \geq 0\} \) be a bifractional Brownian motion (bifBm) with parameters \( H \in (0, 1) \) and \( K \in (0, 1] \). This means that \( B^{H,K} \) is a centered Gaussian process with the covariance function

\[ E(B_{s}^{H,K} B_{t}^{H,K}) = \frac{1}{2^{K}} \left( (t^{2H} + s^{2H})^{K} - |t - s|^{2HK} \right). \]

The case \( K = 1 \) corresponds to the fBm with Hurst parameter \( H \). The process \( B^{H,K} \) verifies

\[ E\left( |B_{t}^{H,K} - B_{s}^{H,K}|^{2} \right) \leq 2^{1-K} |t - s|^{2HK}. \]

Combining this with the fact that \( B^{H,K} \) is Gaussian and Remark 2.1, we deduce that the assumption (A) holds for \( G = B^{H,K} \).

Furthermore, according to [1, Section 4], \( \int_{0}^{\infty} |f_{B^{H,K}}(x)| \, dx < \infty \). We can also write, for all \( x \in \mathbb{R} \),

\[ f_{B^{H,K}}(x) = R_{B^{H,K}}(e^{i\pi x}, e^{i\pi x}) = \frac{1}{2^{K}} \left[ (e^{i\pi x} + e^{-i\pi x})^{K} - (e^{i\pi x} - e^{-i\pi x})^{2HK} \right] = \frac{1}{2^{K}} \left[ n_{B}^{(x)}(x) + m_{HK}(x) \right], \]

where the functions \( n_{B}^{(x)}(x) \) and \( m_{HK} \) are defined in Lemma 3.7. Moreover, using the latter equation, Remark 3.6 and Lemma 3.7, we get

\[ \rho_{Y_{1}^{(1)}}^{(1)}(x) = f_{B^{H,K}}(x) - f_{1}^{(1)}(x) = \frac{(HK)^{2HK}(K - 1)}{2^{K-2}K} (e^{i\pi x} + e^{-i\pi x})^{K-2} + \frac{(HK)^{2HK-1}(2HK - 1)}{2^{K-1}} (e^{i\pi x} - e^{-i\pi x})^{2HK-2}. \]

Now let us consider the bifractional Ornstein–Uhlenbeck process of the second kind \( X_{B^{H,K}} := \{X_{t, B^{H,K}}, t \geq 0\} \), defined as the unique solution to (3.25) when \( U = B^{H,K} \). In other words, \( X_{B^{H,K}} \) is the solution to the equation

\[ X_{0, B^{H,K}} = 0, \quad dX_{t, B^{H,K}} = -\theta X_{t} dt + dY_{1, B^{H,K}}^{(1)}, \quad t \geq 0. \]

Similar arguments as in Section 3.3.1 lead the following result.

**Theorem 3.11.** Assume that \( H, K \in (0, 1) \) and \( \theta > 0 \). Let \( \{X_{t, B^{H,K}}, t \geq 0\} \) and \( \{Z_{t, B^{H,K}}, t \geq 0\} \) be the processes defined by (3.26) and (3.27) for \( U = B^{H,K} \), respectively. Then
• \{Z_{t,B^H,K}, t \geq 0\} is an ergodic stationary Gaussian process, and
\[
E\left(Z_{0,B^H,K}^2\right) = f_{B^H,K}(0) + \left(\frac{1}{\theta} - \theta\right) \int_0^\infty f_{B^H,K}(t)e^{-\frac{\theta}{t}} dt.
\]

• There exists a constant \(C > 0\) depending only on \(\theta\) and \(H\) such that, for all \(t \geq 0\),
\[
\left| E\left(X_{t,B^H,K}^2\right) - E\left(Z_{0,B^H,K}^2\right) \right| \leq Ce^{-\theta t}.
\]

• If \(H \in (0, \frac{1}{2})\) with \(HK \neq \frac{1}{2}\), then, as \(t \to \infty\),
\[
E(Z_{t,B^H,K}Z_{0,B^H,K}) \sim \frac{(HK)^{2HK (K-1)}}{2K-2}e^{-\min\left(\theta, \frac{1}{HK}-1\right)t}
\times \begin{cases} 
\int_0^\infty \left(e^{\theta u} - e^{-\theta u}\right)(e^\frac{u}{HK} + e^{-\frac{u}{HK}})^{K-2} du & \text{if } \theta < \frac{2}{K} - 1, \\
\frac{t}{2\theta} & \text{if } \theta = \frac{2}{K} - 1, \\
\frac{1}{\theta^2 - \left(\frac{2}{K} - 1\right)^2} & \text{if } \theta > \frac{2}{K} - 1.
\end{cases}
\]

• If \(H \in (\frac{1}{2}, 1)\) with \(HK \neq \frac{1}{2}\), then, as \(t \to \infty\),
\[
E(Z_{t,B^H,K}Z_{0,B^H,K}) \sim \frac{(HK)^{2HK-1}(2HK-1)}{2K-1}e^{-\min\left(\theta, \frac{1}{HK}-1\right)t}
\times \begin{cases} 
\int_0^\infty \left(e^{\theta u} - e^{-\theta u}\right)(e^\frac{u}{HK} - e^{-\frac{u}{HK}})^{2HK-2} du & \text{if } \theta < \frac{1}{HK} - 1, \\
\frac{t}{2\theta} & \text{if } \theta = \frac{1}{HK} - 1, \\
\frac{1}{\theta^2 - \left(\frac{1}{HK} - 1\right)^2} & \text{if } \theta > \frac{1}{HK} - 1.
\end{cases}
\]

• If \(H \in (0, \frac{1}{2})\) with \(HK \neq \frac{1}{2}\), there exists a constant \(C > 0\) depending only on \(\theta\) and \(H\) such that, for all \(|t-s| > 2\),
\[
E(X_{t,B^H,K}X_{s,B^H,K}) \leq C \begin{cases} 
\text{e}^{-\min\left(\theta, \frac{1}{HK}-1\right)|t-s|} & \text{if } \theta \neq \frac{2}{K} - 1, \\
t\text{e}^{-\min\left(\theta, \frac{1}{HK}-1\right)|t-s|} & \text{if } \theta = \frac{2}{K} - 1.
\end{cases}
\]

• If \(H \in (\frac{1}{2}, 1)\) with \(HK \neq \frac{1}{2}\), there exists a constant \(C > 0\) depending only on \(\theta\) and \(H\) such that, for all \(|t-s| > 2\),
\[
E(X_{t,B^H,K}X_{s,B^H,K}) \leq C \begin{cases} 
\text{e}^{-\min\left(\theta, \frac{1}{HK}-1\right)|t-s|} & \text{if } \theta \neq \frac{1}{HK} - 1, \\
t\text{e}^{-\min\left(\theta, \frac{1}{HK}-1\right)|t-s|} & \text{if } \theta = \frac{1}{HK} - 1.
\end{cases}
\]
4. Langevin equations driven by Gaussian processes with non-stationary increments

This section deals with non-stationary Gaussian Ornstein–Uhlenbeck processes. More precisely, we consider two examples of Gaussian Ornstein–Uhlenbeck process of the form (2.24), where the driving process $G$ is Gaussian but it does not have stationary increments.

We will make use of the following technical lemma.

**Lemma 4.1.** Let $\gamma \in (0,1)$ and $\lambda > 0$. Then there exists a constant $C>0$ depending only on $\lambda$ and $\gamma$ such that, for all $t \geq 1$,

$$e^{-\lambda t} \int_0^t s^{\gamma-1} e^{\lambda s} \, ds \leq Ct^{\gamma-1}.$$ 

**Proof.** We have, for every $t \geq 1$,

$$e^{-\lambda t} \int_0^t s^{\gamma-1} e^{\lambda s} \, ds = e^{-\lambda t} \int_0^{\gamma/2} s^{\gamma-1} e^{\lambda s} \, ds + e^{-\lambda t} \int_{\gamma/2}^t s^{\gamma-1} e^{\lambda s} \, ds$$

$$\leq e^{-\lambda t/2} \int_0^t s^{\gamma-1} \, ds + \left(\frac{t}{2}\right)^{\gamma-1} e^{-\lambda t} \int_{\gamma/2}^{t} e^{\lambda s} \, ds$$

$$\leq C \left(t^{\gamma-1} e^{-\lambda t/2} + t^{\gamma-1}\right)$$

$$\leq Ct^{\gamma-1},$$

which completes the proof. \qed

4.1. Subfractional Ornstein–Uhlenbeck process

Here we consider the Ornstein–Uhlenbeck process $X^S_H := \{X^S_H(t), t \geq 0\}$ defined by the following linear stochastic differential equation

$$X^S_H(0) = 0, \quad dX^S_H(t) = -\theta X^S_H(t) \, dt + dS^H_t, \quad (4.1)$$

where $S^H$ is a subfBm with Hurst parameter $H \in (0,1)$, defined in Section 3.3.2. In this case, we can write that, for every $s, t \geq 0$,

$$R^S_H(s, t) = R^B_H(s, t) + \frac{1}{2} \left(t^{2H} + s^{2H} - (t+s)^{2H}\right)$$

$$=: R^B_H(s, t) + g^S_H(s, t), \quad (4.2)$$

where $B^H$ is a fBm with Hurst parameter $H \in (0,1)$.

Note that the process $S^H$ does not have stationary increments and so the Gaussian process $Z^S_H := \int_{-\infty}^t e^{-\theta(t-s)} \, dS^H_s$, with $\theta > 0$, is non-stationary. Thus, in this section, we will only discuss properties of the process $X^S_H$.

**Theorem 4.2.** Assume that $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ and $\theta > 0$. Let $X^S_H$ be the process defined by (4.1). Then, there exists a constant $C > 0$ depending only on $\theta$ and $H$ such that, for all $t > 2$,
\[
\left| \mathbb{E} \left[ \left( X_t^{sH} \right)^2 \right] \right| - \frac{H \Gamma (2H)}{\partial (2H)} \leq Ct^{2H-2}, \quad (4.3)
\]

and for all \( |t - s| > 2 \),
\[
\mathbb{E} \left[ X_t^{sH} X_t^{sH} \right] \leq C |t - s|^{2H-2}. \quad (4.4)
\]

**Proof.** Using (2.26), (4.2), and (2.29), we deduce that, for every \( t > 2 \),
\[
\mathbb{E} \left[ \left( X_t^{sH} \right)^2 \right] = \mathbb{E} \left[ \left( X_t^H \right)^2 \right] + \Delta_{g_{sH}} (t)
\]
\[
= 2e^{-2\theta t} \int_0^t e^{\theta s} \frac{\partial g_{sH}}{\partial s} (s, 0) ds + 2e^{-2\theta t} \int_0^t dse^{\theta s} \int_0^s dr \frac{\partial^2 g_{sH}}{\partial s \partial r} (s, r)e^{\theta r},
\]
where \( X^H \) is the process given in Theorem 3.2.

It is easy to check that for every \( s, r > 0 \),
\[
\frac{\partial g_{sH}}{\partial s} (s, 0) = 0, \quad \text{and} \quad \frac{\partial^2 g_{sH}}{\partial s \partial r} (s, r) = -H(2H - 1)(r + s)^{2H-2}.
\]

According to the claim (ii) in Theorem 3.2, we have
\[
\left| \mathbb{E} \left[ \left( X_t^H \right)^2 \right] \right| - \frac{H \Gamma (2H)}{\partial (2H)} \leq Ce^{-\theta t}, \quad t \geq 0.
\]

Therefore, in order to prove (4.3), it is enough to check that
\[
e^{-2\theta t} \int_0^t dse^{\theta s} \int_0^s dr (r + s)^{2H-2} e^{\theta r} \leq Ct^{2H-2}, \quad t > 2.
\]

On the other hand, for all \( t > 2 \),
\[
e^{-2\theta t} \int_0^t dse^{\theta s} \int_0^s dr (r + s)^{2H-2} e^{\theta r} \leq e^{-2\theta t} \int_0^t dse^{\theta s} \int_0^{2\sqrt{rs}} dr (2\sqrt{rs})^{2H-2} e^{\theta r}
\]
\[
= 2^{2H-2} \left( e^{-\theta t} \int_0^t s^{H-1} e^{\theta s} ds \right)^2 
\]
\[
\leq Ct^{2H-2},
\]
where the latter equality comes from Lemma 4.1. Thus, (4.3) is obtained.

Now we prove (4.4). From Lemma 2.9, it follows immediately that, for all \( |t - s| > 2 \),
\[
\mathbb{E} \left[ X_t^{sH} X_t^{sH} \right] = e^{-\theta (t-s)} \mathbb{E} \left[ \left( X_s^{sH} \right)^2 \right] + e^{-\theta t} e^{-\theta s} \int_s^t e^{\theta v} \int_0^v e^{\theta u} \frac{\partial^2 R_{sH}}{\partial u \partial v} (u, v) dudv
\]
\[
\leq C |t - s|^{2H-2},
\]
where we used (4.3), (2.9) and the fact that for every \( u, v > 0 \) with \( u \neq v \),
\[
\left| \frac{\partial^2 R_{sH}}{\partial u \partial v} (u, v) \right| = H|2H - 1|(v + u)^{2H-2} - |v - u|^{2H-2}
\]
\[
\leq 2H|2H - 1|v - u|^{2H-2}.
\]
\[\square\]
4.2. Bifractional Ornstein–Uhlenbeck process

Consider the Ornstein-Uhlenbeck process \( X^{B^H,K} := \{X_t^{B^H,K}, t \geq 0\} \) defined by the following linear stochastic differential equation

\[
X_0^{B^H,K} = 0, \quad dX_t^{B^H,K} = -\theta X_t^{B^H,K} dt + dB_t^{H,K}, \tag{4.5}
\]

where \( B^{H,K} \) is a bifBm with Hurst parameters \( H, K \in (0,1) \), defined in Section 3.3.3. In this case, we can write that, for every \( s, t \geq 0 \),

\[
R^{B^H,K}(s,t) = \frac{1}{2^K-1} R^{B^H,K}(s,t) + \frac{1}{2^K} \left[ (t^{2H} + s^{2H})^K - t^{2HK} - s^{2HK} \right] \tag{4.6}
\]

where \( B^{H,K} \) is a fBm with Hurst parameter \( HK \in (0,1) \).

Note that the process \( S^H \) does not have stationary increments and so the Gaussian process \( Z^{B^H,K}_t := \int_0^t e^{-\theta(t-s)} dB^{H,K}_s \), with \( \theta > 0 \), is non-stationary. Here we will only discuss properties of the process \( X^{B^H,K} \).

**Theorem 4.3.** Assume that \( 0 > 0 \) and \( H, K \in (0,1) \) with \( HK \neq \frac{1}{2} \). Let \( X^{B^H,K} \) be the process defined by (4.5). Then, there exists a constant \( C > 0 \) depending only on \( \theta \) and \( H \) such that, for all \( t > 2 \),

\[
\left| E \left[ \left( X_t^{B^H,K} \right)^2 \right] - \frac{HK\Gamma(2HK)}{2^{K-1} \theta^{2HK}} \right| \leq Ct^{2HK-2}, \tag{4.7}
\]

and for all \( |t-s| > 2 \),

\[
E\left( X_t^{B^H,K} X_s^{B^H,K} \right) \leq C \begin{cases} |t-s|^{2HK-2H-1} & \text{if } 0 < H < \frac{1}{2} \\ |t-s|^{2HK-2} & \text{if } \frac{1}{2} \leq H < 1. \end{cases} \tag{4.8}
\]

**Proof.** Using (2.26), (4.6), and (2.29), we deduce that, for every \( t > 2 \),

\[
E\left( X_t^{B^H,K} \right)^2 = \frac{1}{2^{K-1}} E\left( X_t^{HK} \right)^2 + A_{g_{B^H,K}}(t)
= 2e^{-2\theta t} \int_0^t e^{\theta s} \frac{\partial g_{B^H,K}}{\partial s}(s,0) ds + 2e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr \frac{\partial^2 g_{B^H,K}}{\partial s \partial r}(s,r) e^{\theta r},
\]

where the process \( X^{HK} \) is defined in Theorem 3.2.

It is easy to check that for every \( s, r > 0 \),

\[
\frac{\partial g_{B^H,K}}{\partial s}(s,0) = 0, \quad \text{and} \quad \frac{\partial^2 g_{B^H,K}}{\partial s \partial r}(s,r) = \frac{(2H)K(K-1)}{2^K} (r^{2H} + s^{2H})^{K-2} (rs)^{2H-1}.
\]

According to the claim (ii) in Theorem 3.2, we have

\[
\left| E\left[ \left( X_t^{HK} \right)^2 \right] - \frac{HK\Gamma(2HK)}{\theta^{2HK}} \right| = O(e^{-\theta t}) \quad \text{as } t \to \infty.
\]
Thus, in order to prove (4.7), it is enough to check that for every \( t > 2 \),
\[
e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr (r^{2H} + s^{2H})^{K-2} (rs)^{2H-1} e^{\theta r} \leq Ct^{2H-2}.
\]

On the other hand, for all \( t > 2 \),
\[
e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr (r^{2H} + s^{2H})^{K-2} (rs)^{2H-1} e^{\theta r}
\leq e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr (2r^{H} s^{H})^{K-2} (rs)^{2H-1} e^{\theta r}
= 2^{K-3} \left( e^{-\theta t} \int_0^t s^{HK-1} e^{\theta s} ds \right)^2
\leq Ct^{2H-2},
\]

where the latter equality follows immediately from Lemma 4.1. Therefore, (4.7) is obtained. Let us prove (4.8). According to Lemma 2.9, we have
\[
E \left( X_s^{B^{h,K}} X_t^{B^{h,K}} \right) = e^{-\theta (t-s)} E \left[ \left( X_s^{B^{h,K}} \right)^2 \right] + e^{-\theta t} e^{-\theta s} \int_s^t e^{\theta r} \int_0^r e^{\theta u} \frac{\partial^2 R_{B^{h,K}}}{\partial u \partial v} (u, v) dudv. \tag{4.9}
\]

If \( H < \frac{1}{2} \), and using Lemma 4.1, we have, for every \( s < t \),
\[
e^{-\theta t} e^{-\theta s} \int_s^t e^{\theta r} \int_0^r e^{\theta u} \frac{\partial^2 R_{B^{h,K}}}{\partial u \partial v} (u, v) dudv
\leq C e^{-\theta t} e^{-\theta s} \int_s^t e^{\theta r} \int_0^r e^{\theta u} (u^{2H} + v^{2H})^{K-2} (uv)^{2H-1} dudv
\leq C e^{-\theta t} e^{-\theta s} \int_s^t e^{\theta r} \int_0^r e^{\theta u} (v^{2H})^{K-2} (uv)^{2H-1} dudv
= C \left[ e^{-\theta s} \int_s^t e^{\theta u} u^{2H-1} du \right] \left[ e^{-\theta t} \int_s^t e^{\theta r} v^{2HK-2H-1} dv \right]
\leq C e^{-\theta t} \int_s^t e^{\theta r} v^{2HK-2H-1} dv
= C e^{-\theta (t-s)} \int_0^{t-s} e^{\theta y} (s + y)^{2HK-2H-1} dy
= C \left[ e^{-\theta (t-s)} \int_0^{t-s} e^{\theta y} (s + y)^{2HK-2H-1} dy + e^{-\theta (t-s)} \int_{t-s}^{t} e^{\theta y} (s + y)^{2HK-2H-1} dy \right]
\leq C \left[ e^{-\theta (t-s)} + (t-s)^{2HK-2H-1} \right].
\]

Combining this with (4.9) and (4.7), we obtain (4.8) for \( H < \frac{1}{2} \).
If $H \geq \frac{1}{2}$, then (4.8) holds, due to (4.9), (4.7), (2.9) and the fact that, for every $0 < u < v$,

$$\frac{\partial^2 R_{B^{H,K}}(u, v)}{\partial u \partial v} (u, v) = \frac{(2H)^2 K(1 - K)}{2^K} (u^{2H} + v^{2H})^{K-2} (uv)^{2H-1}$$

$$\leq \frac{(2H)^2 K(1 - K)}{2^K} (2u^H v^H)^{K-2} (uv)^{2H-1}$$

$$\leq \frac{(2H)^2 K(1 - K)}{4} v^{2HK-2}$$

$$\leq \frac{(2H)^2 K(1 - K)}{4} |v - u|^{2HK-2}.$$ 

\[\square\]

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