The regularization continuation method with an adaptive time step control for linearly equality-constrained optimization problems

Xin-long Luo · Hang Xiao

Abstract This paper considers the regularization continuation method and the trust-region updating strategy for the linearly equality-constrained optimization problem. The proposed method utilizes the linear conservation law of the regularization method such that it does not need to compute the correction step for preserving the feasibility other than the previous continuation methods and the quasi-Newton updating formulas for the linearly equality-constrained optimization problem. Moreover, the new method uses the L-BFGS method as the preconditioning technique to improve its computational efficiency in the well-posed phase, and it uses the inverse of the regularization two-sided projected Hessian matrix as the pre-conditioner to improve its robustness. Numerical results also show that the new method is more robust and faster than the traditional optimization method and the recent continuation method. Finally, the global convergence analysis of the new method is also given.

Keywords continuation method · preconditioned technique · trust-region method · linear conservation law · regularization method

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1 Introduction

In this article, we consider the following linearly equality-constrained optimization problem

\[
\min_{x \in \mathbb{R}^n} f(x)
\]

subject to \( Ax = b \), \( \text{(1)} \)

where \( A \in \mathbb{R}^{m \times n} \) is a matrix and \( b \in \mathbb{R}^m \) is a vector. This problem has many applications in engineering fields such as the visual-inertial navigation of an unmanned aerial vehicle maintaining the horizontal flight \([10,35]\), and there are many practical methods to solve it such as the sequential quadratic programming (SQP) method \([27,45]\), the penalty function method \([14]\).

For the constrained optimization problem (1), the continuation method \([2,11,17,25,47,57]\) is another method other than the traditional optimization method such as SQP or the penalty function method. The advantage of the continuation method over the SQP method is that the continuation method is capable of finding many local optimal points of the non-convex optimization problem by tracking its trajectory, and it is even possible to find the global optimal solution \([5,49,61]\). However, the computational efficiency of the classical continuation methods is lower than that of the traditional optimization method such as SQP. Recently, the reference \([35]\) gives a continuation method with the trusty time-stepping scheme for the problem (1), which is faster than SQP and the penalty method. In order to improve the computational efficiency and the robustness of the continuation method for the large-scale optimization problem further, we consider a special limited-memory BFGS (L-BFGS) as the preconditioned technique in the well-posed phase and use the inverse of the regularization two-sided projected Hessian matrix as the preconditioner in the ill-posed phase. Moreover, the new method utilizes the linear conservation law of the regularization method and it does not need to compute the correction step for preserving the feasibility other than the previous continuation method \([35]\) and the quasi-Newton method \([45,56]\).

The rest of the paper is organized as follows. In section 2, we give the regularization continuation method with the switching preconditioned technique and the trust-region updating strategy for the linearly equality-constrained optimization problem (1). In section 3, we analyze the global convergence of this new method. In section 4, we report some promising numerical results of the new method, in comparison to the traditional optimization method such as SQP (the built-in subroutine fmincon.m of the MATLAB2020a \([42]\) and the alternating direction method of multipliers (ADMM \([8]\), only for convex problems)), and the recent continuation method such as Ptctr \([35]\) for some large-scale problems. Finally, we give some discussions and conclusions in section 5.
2 The adaptive regularization continuation method

In this section, we give the regularization continuation method with the switching preconditioned technique and an adaptive time-step control based on the trust-region updating strategy [12] for the linearly equality-constrained optimization problem (1). Firstly, we consider the regularization projected Newton flow based on the KKT conditions of linearly constrained optimization problem. Then, we give the regularization continuation method with the trust-region updating strategy to follow this special ordinary differential equations (ODEs). The new method uses the L-BFGS updating method as the preconditioned technique to improve its computational efficiency in the well-posed phase, and it uses the inverse of the regularization two-sided projected Hessian matrix as the pre-conditioner to improve its robustness in the ill-posed phase. Finally, we give a preprocessing method for the infeasible initial point.

2.1 The regularization projected Newton flow

For the linearly constrained optimization problem (1), it is well known that its optimal solution $x^*$ needs to satisfy the Karush-Kuhn-Tucker conditions (p. 328, [45]) as follows:

$$\nabla_x L(x, \lambda) = \nabla f(x) + A^T \lambda = 0, \tag{2}$$

$$Ax - b = 0. \tag{3}$$

where the Lagrangian function $L(x, \lambda)$ is defined by

$$L(x, \lambda) = f(x) + \lambda^T (Ax - b). \tag{4}$$

Similarly to the method of the negative gradient flow for the unconstrained optimization problem [22], from the first-order necessary conditions (2)-(3), we can construct a dynamical system of differential-algebraic equations for problem (1) [13, 32, 33, 34, 50] as follows:

$$\frac{dx}{dt} = -\nabla L_x(x, \lambda) = -\left(\nabla f(x) + A^T \lambda\right), \tag{5}$$

$$Ax - b = 0. \tag{6}$$

By differentiating the algebraic constraint (6) with respect to $t$ and substituting it into the differential equation (5), we obtain

$$A \frac{dx}{dt} = -A \left(\nabla f(x) + A^T \lambda\right) = -A \nabla f(x) - AA^T \lambda = 0. \tag{7}$$

If we assume that matrix $A$ has full row rank further, from equation (7), we obtain

$$\lambda = -\left( AA^T \right)^{-1} A \nabla f(x). \tag{8}$$
By substituting \( A \) of equation (8) into equation (5), we obtain the projected gradient flow [57] for the constrained optimization problem (1) as follows:

\[
\frac{dx}{dt} = -\left(I - A^T (AA^T)^{-1} A\right) \nabla f(x) = -P g(x), \tag{9}
\]

where \( g(x) = \nabla f(x) \) and the projection matrix \( P \) is defined by

\[
P = I - A^T (AA^T)^{-1} A. \tag{10}
\]

It is not difficult to verify \( P^2 = P \). That is to say, the projection matrix \( P \) is symmetric and its eigenvalues are zero or one. From Theorem 2.3.1 in p. 73 of [19], we know that its matrix 2-norm is

\[
\|P\| = 1. \tag{11}
\]

We denote \( P^+ \) as the Moore-Penrose generalized inverse of the projection matrix \( P \) (p. 11, [56]). Since the projection matrix \( P \) is symmetric and \( P^2 = P \), it is not difficult to verify

\[
P^+ = P. \tag{12}
\]

Actually, from equation (12), we have \( PP^+ P = P(P)P = P = P^+ \), \( P^+ PP^+ = P^3 = P \), \( (P^+ P)^T = P^+ P = P \) and \( (PP^+)^T = PP^+ = P \).

Furthermore, from equation (10), we have \( AP = 0 \). We denote \( \mathcal{N}(A) \) as the null space of \( A \). Since the rank of \( A \) is \( m \), we know that the rank of \( \mathcal{N}(A) \) equals \( n - m \) and there are \( n - m \) linearly independent vectors \( x_i (i = 1, \ldots, n - m) \) to satisfy \( Ax_i = 0 (i = 1, \ldots, n - m) \). From equation (10), we know that those \( n - m \) linearly independent vectors \( x_i (i = 1, \ldots, n - m) \) satisfy \( Px_i = x_i (i = 1, \ldots, n - m) \). That is to say, the projection matrix \( P \) has \( n - m \) linearly independent eigenvectors associated with eigenvalue 1. Consequently, the rank of \( P \) is at least \( n - m \). By combining it with \( AP = 0 \), we know that \( P \) spans the null space of \( A \).

\textbf{Remark 1} \ If \( x(t) \) is the solution of the ODE (9), it is not difficult to verify that \( x(t) \) satisfies \( A(dx/dt) = 0 \). That is to say, if the initial point \( x_0 \) satisfies \( Ax_0 = b \), the solution \( x(t) \) of the generalized projected gradient flow (9) also satisfies the feasibility \( Ax(t) = b, \forall t \geq 0 \). This linear conservation property is very useful when we construct a structure-preserving algorithm [20,51,52] to follow the trajectory of the ODE (9) and obtain its steady-state solution \( x^\star \).

If we assume that \( x(t) \) is the solution of the ODEs (9), by using the property \( P^2 = P \), we obtain

\[
\frac{df(x)}{dt} = (\nabla f(x))^T \frac{dx}{dt} = - (\nabla f(x))^T P \nabla f(x) = -g(x)^T P^2 g(x) = -\|P g(x)\|^2 \leq 0.
\]

That is to say, \( f(x) \) is monotonically decreasing along the solution curve \( x(t) \) of the dynamical system (9). Furthermore, the solution \( x(t) \) converges to \( x^\star \) when \( f(x) \) is lower bounded and \( t \) tends to infinity [22,49,57], where \( x^\star \) satisfies the first-order
Karush-Kuhn-Tucker conditions (2)-(3). Thus, we can follow the trajectory $x(t)$ of the ODE (9) to obtain its steady-state solution $x^*$, which is also one stationary point of the original optimization problem (1).

However, since the Jacobian matrix of $Pg(\cdot)$ is rank-deficient, we will confront the numerical difficulties when we use the explicit ODE method to follow the projected gradient flow (9) [3, 6, 7]. In order to mitigate the stiffness of the ODE (9), we can use the generalized inverse $(PV^2 f(x)P)^+$ of the two-sided projected Hessian matrix $PV^2 f(x)P$ as the preconditioner for the ODE (9), which is used similarly to the system of nonlinear equations [36], the unconstrained optimization problem [22, 38], the linear programming problem [37] and the underdetermined system of nonlinear equations [39].

Firstly, we integrate the ODE (9) from zero to $t$, then we obtain
\[
x(t) = x(t_0) - \int_0^t Pg(x(\tau))d\tau = x(t_0) - P \int_0^t g(x(\tau))d\tau.
\] (13)

Thus, if we denote $z(t) = -\int_0^t g(x(\tau))d\tau$, from equation (13), we have
\[
x(t) = x(t_0) + Pz(t).
\] (14)

By substituting it into the ODE (9), we obtain
\[
P \frac{dz(t)}{dt} = -Pg(x(t_0) + Pz(t)).
\] (15)

Then, by using the generalized inverse $(PV^2 f(x(t_0) + Pz(t))P)^+$ of the Jacobian matrix $PV^2 f(x(t_0) + Pz(t))P$ as the preconditioner for the ODE (15), we have
\[
P \frac{dz(t)}{dt} = - (PV^2 f(x(t_0) + Pz(t))P)^+ Pg(x(t_0) + Pz(t)).
\] (16)

We reformulate equation (16) as
\[
(PV^2 f(x(t_0) + Pz(t))P) \frac{dPz(t)}{dt} = -Pg(x(t_0) + Pz(t)).
\] (17)

By substituting $Pz(t) = x(t) - x(t_0)$ into equation (17), we obtain the projected Newton flow for problem (1) as follows:
\[
(PV^2 f(x)P) \frac{dx(t)}{dt} = -Pg(x).
\] (18)

Although the projected Newton flow (18) mitigates the stiffness of the ODE such that we can adopt the explicit ODE method to integrate it on the infinite interval, there are two disadvantages yet. One is that the two-side projected Hessian matrix $PV^2 f(x)P$ may be not positive. Consequently, it can not ensure the objective function $f(x)$ is monotonically decreasing along the solution $x(t)$ of the ODE (18). The other is that the solution $x(t)$ of the ODE (18) can not ensure to satisfy the linear conservation law, i.e., $Adx(t)/dt = 0$. In order to overcome these two disadvantages,
we use the similar regularization technique of solving the ill-posed problem [21, 58] for the projected Newton flow (18) as follows:

$$\left(\sigma(x)I + PV^2 f(x)P\right) \frac{dx(t)}{dt} = -Pg(x), \quad (19)$$

where the scalar regularization function $\sigma(x)$ satisfies $\sigma(x) + \mu_{\text{min}}(PV^2 f(x)P) \geq \sigma_{\text{min}} > 0$ and $\mu_{\text{min}}(PV^2 f(x)P)$ represents the smallest eigenvalue of $PV^2 f(x)P$.

**Remark 2** If we assume that $x(t)$ is the solution of the ODE (19), from the property $AP = 0$, we have

$$A \left(\sigma(x)I + PV^2 f(x)P\right) \frac{dx(t)}{dt} = -APg(x) = 0.$$ 

Consequently, we obtain $A\sigma(x)dx(t)/dt = 0$. By integrating it, we obtain $Ax(t) = Ax(t_0) = b$. That is to say, the solution $x(t)$ of the ODE (19) satisfies the linear conservation law $Ax = b$.

**Remark 3** From Remark 2, we know that the solution $x(t)$ of the ODE (19) satisfies $Ax(t) = b$. Furthermore, $P$ is the null space of $A$. By combining these two properties, we obtain $x(t) = x_0 + Pz(t)$. Thus, we have $dx(t)/dt = Pd\zeta(t)/dt$. By substituting the property $P^2 = P$ into it, we obtain $Pdx(t)/dt = P^2d\zeta(t)/dt = P\zeta(t)/dt = dx(t)/dt$. Consequently, from equation (19) and the assumption $\sigma(x) + \lambda_{\text{min}}(PV^2 f(x)P) \geq \sigma_{\text{min}} > 0$, we obtain

$$\frac{df(x(t))}{dt} = (\nabla f(x))^T \frac{dx(t)}{dt} = (\nabla f(x))^T P \frac{dx(t)}{dt} = (Pg(x))^T \frac{dx(t)}{dt} = -\lambda_{\text{min}}(P^2 f(x)P)^{-1} (Pg(x)) \leq 0.$$ 

That is to say, the objective function $f(\cdot)$ is monotonically decreasing along the solution $x(t)$ of the ODE (19). Furthermore, the solution $x(t)$ converges to $x^\ast$ when $f(x)$ is lower bounded and $\|PV^2 f(x)P\| \leq M$ [22, 29, 49, 57], where $M$ is a positive constant and $x^\ast$ is the stationary point of the regularization projected Newton flow (19). Thus, we can follow the trajectory $x(t)$ of the ODE (19) to obtain its stationary point $x^\ast$, which is also one stationary point of the original optimization problem (1).

2.2 The regularization continuation method

The solution curve $x(t)$ of the ODE (19) can not be efficiently followed on an infinite interval by the general ODE method such as backward differentiation formulas (BDFs, the subroutine ode15s.m of the MATLAB R2020a environment) [3, 6, 7, 24]. Thus, we need to construct the particular method for this problem. We apply the first-order explicit Euler method [53] to the ODE (19), then we obtain the regularization projected Newton method:

$$\left(\sigma_k I + PV^2 f(x_k)P\right) d_k = -Pg(x_k), \quad (20)$$

$$x_{k+1} = x_k + \alpha_k d_k, \quad (21)$$
where $\alpha_k$ is the time step. If we let $\alpha_k = 1$, the regularization projected Newton method is the Levenberg-Marquardt method \cite{28, 31, 43}.

Since the time step $\alpha_k$ of the regularization projected Newton method (20)-(21) is restricted by the numerical stability \cite{53}. That is to say, for the linear test equation $dx/dt = -\lambda x$, its time step size $\alpha_k$ is restricted by the stable region $|1 - \lambda \alpha_k| \leq 1$. Therefore, the large time step can not be adopted in the steady-state phase. In order to avoid this disadvantage, similarly to the processing technique of the nonlinear equations \cite{36, 39, 37} and the unconstrained optimization problem \cite{38}, we replace $\alpha_k$ with $\Delta t_k / (1 + \Delta t_k)$ in equation (21) and let $\sigma_k = \sigma_0 / \Delta t_k$ in equation (20). Then, we obtain the regularization continuation method:

$$B_k d_k = -P g(x_k), \quad d_k s_k = \frac{\Delta t_k}{1 + \Delta t_k} d_k, \quad (22)$$

$$x_{k+1} = x_k + s_k, \quad (23)$$

where $\Delta t_k$ is the time step and $B_k = (\sigma_0 / \Delta t_k)I + P \nabla^2 f(x_k)P$ or its quasi-Newton approximation.

**Remark 4** The time step $\Delta t_k$ of the regularization continuation method (22)-(23) is not restricted by the numerical stability. Therefore, the large time step $\Delta t_k$ can be adopted in the steady-state phase such that the regularization continuation method (22)-(23) mimics the projected Newton method near the stationary point $x^*$ and it has the fast local convergence rate. The most of all, the new step size $\alpha_k = \Delta t_k / (\Delta t_k + 1)$ is favourable to adopt the trust-region updating strategy to adjust the time step $\Delta t_k$ such that the regularization continuation method (22)-(23) accurately follows the trajectory of the regularization flow (19) in the transient-state phase and achieves the fast convergence rate near the stationary point $x^*$.

When $B_k$ is updated by the BFGS quasi-Newton formula \cite{4, 15, 18, 54}:

$$B_{k+1} = B_k + \frac{y_k y_k^T}{y_k^T s_k} - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k}, \quad B_0 = I, \quad (24)$$

where $y_k = P g(x_{k+1}) - P g(x_k)$, $s_k = x_{k+1} - x_k$, there is an invariable property for the transformation matrix $P$ and we state it as the following lemma 1.

**Lemma 1** Assume that $B_k$ is updated by the BFGS quasi-Newton formula (24) and $s_k$ is solved by equation (22), then we have $P(B_k - I) = B_k - I$ and $P s_k = s_k$ for $k = 0, 1, 2, \ldots$.

**Proof.** We prove this property by induction. When $k = 0$, from $P^2 = P$, we have $P(B_0 - I) = 0 = B_0 - I$ and $P s_0 = s_0$. We assume that $P(B_l - I) = B_l - I$ and $P s_l = s_l$ when $k = l$. Then, when $k = l + 1$, from $P^2 = P$, $P y_l = P g(x_{l+1}) - P g(x_l) = y_l$ and
equation (24), we have
\[
P B_{l+1} = P B_l + \frac{P y_l y_l^T}{y_l^T s_l} - \frac{P B_l s_l y_l^T}{s_l^T B_l s_l} = P + B_l - I + \frac{y_l y_l^T}{y_l^T s_l} - \frac{(P + B_l - I) s_l y_l^T}{s_l^T B_l s_l} = P - I + B_l,
\]
\[
= P - I + B_l - I = P - B_l - I.
\]
(25)

Consequently, we obtain \(P(B_{l+1} - I) = B_{l+1} - I\).

From equation (22) and equation (25), we have
\[
P B_{l+1} d_{l+1} = (P B_{l+1}) d_{l+1} = (P + B_{l+1} - I) d_{l+1}
\]
\[
= -P^2 g(x_{l+1}) = -P g(x_{l+1}) = B_{l+1} d_{l+1},
\]
(26)

Consequently, from equation (26), we have \(P d_{l+1} = d_{l+1}\). By combining it with equation (22), we obtain \(P s_{l+1} = s_{l+1}\). Therefore, according to the principle of induction, the conclusion is true.

**Remark 5** From equations (22)-(23), Lemma 1 and the property \(AP = 0\) of the projection matrix \(P\), it is not difficult to verify \(A x_k = 0\). Thus, if the initial point \(x_0\) is feasible, i.e. \(Ax_0 = b\), \(x_k\) also satisfies the linear constraint \(Ax_k = b\). That is to say, the regularization continuation method (22)-(23) satisfies the linear conservation law such that it does not need to compute the correction step for preserving the linear feasibility other than the previous continuation method and the quasi-Newton formula [35] for the linearly equality-constrained optimization problem.

2.3 The adaptive time step control

Another issue is how to adaptively adjust the time step size \(\Delta t_k\) at every iteration. We borrow the adjustment technique of the trust-region radius from the trust-region method due to its robustness and its fast convergence rate [12, 62]. According to the linear conservation law of the regularization continuation method (22)-(23), \(x_{k+1}\) will preserve the feasibility if \(Ax_k = b\). That is to say, \(x_{k+1}\) satisfies \(Ax_{k+1} = b\). Therefore, we use the objective function \(f(x)\) instead of the nonsmooth penalty function \(f(x) + \sigma \|Ax - b\|\) as the merit function. Similarly to the stepping-time scheme of the ODE method for the unconstrained optimization [23, 31, 35], we also need to construct a local approximation model of \(f(x)\) around \(x_k\). Here, we adopt the following quadratic function as its approximation model:
\[
g_k(x_k + s) = f(x_k) + s^T g_k + \frac{1}{2} s^T B_k s, 
\]
(27)
where \(g_k = \nabla f(x_k)\) and \(B_k = (\sigma_0/\Delta t_k) I + P Q^2 f(x_k) P\) or its quasi-Newton approximation.
In order to save the computational time, from the regularization continuation method (22)-(23), we simplify the quadratic model \( q_k(x_k + s_k) - q(x_k) \) as follows:

\[
m_k(s_k) = g_k^T s_k - \frac{0.5 \Delta t_k}{1 + \Delta t_k} g_k^T s_k = 1 + 0.5 \Delta t_k g_k^T s_k \approx q_k(x_k + s_k) - q_k(x_k). \quad (28)
\]

We enlarge or reduce the time step \( \Delta t_k \) at every iteration according to the following ratio:

\[
\rho_k = \frac{f(x_k) - f(x_k + s_k)}{m_k(0) - m_k(s_k)}. \quad (29)
\]

A particular adjustment strategy is given as follows:

\[
\Delta t_{k+1} = \begin{cases} 
\gamma_1 \Delta t_k, & \text{if } 0 \leq |1 - \rho_k| \leq \eta_1, \\
\Delta t_k, & \text{else if } \eta_1 < |1 - \rho_k| < \eta_2, \\
\gamma_2 \Delta t_k, & \text{others},
\end{cases} \quad (30)
\]

where the constants are selected as \( \eta_1 = 0.25, \gamma_1 = 2, \eta_2 = 0.75, \gamma_2 = 0.5 \) according to our numerical experiments. We accept the trial step \( s_k \) and let \( x_{k+1} = x_k + s_k \), when \( \rho_k \geq \eta_a \) and the approximation model \( m_k(0) - m_k(s_k) \) satisfies the Armijo sufficient descent condition:

\[
m_k(0) - m_k(s_k) \geq \eta_m \| s_k \| \| p_{s_k} \|, \quad (31)
\]

where \( \eta_a \) and \( \eta_m \) are the small positive constants such as \( \eta_a = \eta_m = 1.0 \times 10^{-6} \). Otherwise, we discard it and let \( x_{k+1} = x_k \).

Remark 6 This new time-stepping scheme based on the trust-region updating strategy has some advantages compared to the traditional line search strategy [30]. If we use the line search strategy and the damped projected Newton method (20)-(21) to follow the trajectory \( x(t) \) of the projected Newton flow (19), in order to achieve the fast convergence rate in the steady-state phase, the time step size \( \alpha_k \) of the damped projected Newton method is tried from 1 and reduced by half with many times at every iteration. Since the linear model \( f(x_k) + g_k^T s_k \) may not approximate \( f(x_k + s_k) \) well in the transient-state phase, the time step \( \alpha_k \) will be small. Consequently, the line search strategy consumes the unnecessary trial steps in the transient-state phase. However, the selection scheme of the time step based on the trust-region strategy (29)-(30) can overcome this shortcoming.

2.4 The switching preconditioned technique

For the large-scale problem, the numerical evaluation of the projected Hessian matrix \( P \nabla^2 f(x_k) P \) consumes much time. In order to overcome this shortcoming, in the well-posed phase, we use the limited memory BFGS quasi-Newton formula (see [4, 15, 18, 40, 54] or pp. 222-230, [45]) to approximate the regularized projected Hessian matrix \( \frac{\partial}{\partial x_k} \left( \alpha_k I + \frac{\alpha_k}{2} \nabla^2 f(x_k) P \right) \) of the regularization continuation method (22)-(23).
Recently, Ullah, Sabi’u and Shah [59] give an efficient L-BFGS updating formula for the system of monotone nonlinear equations. Furthermore, the reference [38] also tests its efficiency for some unconstrained optimization problems. Therefore, we adopt the L-BFGS updating formula to approximate $I/\Delta t_k + P \nabla^2 f(x_k) P$ in the well-posed phase via slightly revising it as

\[
B_{k+1} = \begin{cases} 
I - \frac{s_k y^T_k}{s_k^T y_k} + \frac{y y^T_k}{y^T_k y_k}, & \text{if } |s^T_k y_k| > \theta \|s_k\|^2, \\
I, & \text{otherwise},
\end{cases}
\]  

(32)

where $s_k = x_{k+1} - x_k$, $y_k = P \nabla f(x_{k+1}) - P \nabla f(x_k)$ and $\theta$ is a small positive constant such as $\theta = 10^{-6}$.

By using the Sherman-Morrison-Woodburg formula (P. 17, [56]), from equation (32), when $|y^T_k s_k| \neq \theta \|s_k\|^2$, we obtain the inverse of $B_{k+1}$ as follows:

\[
B_{k+1}^{-1} = I - \frac{y y^T_k s_k + s_k y y^T_k}{y^T_k s_k} + \frac{2 \theta y^T_k y_k}{(y^T_k s_k)^2} y y^T_k.
\]  

(33)

The initial matrix $B_0$ can be simply selected as an identity matrix. From equation (33), it is not difficult to verify

\[
B_{k+1} s_k = \frac{y^T_k s_k y_k}{y^T_k y_k}.
\]

That is to say, $B_{k+1}$ satisfies the scaling quasi-Newton property.

The L-BFGS updating formula (32) has some nice properties such as the symmetric positive definite property and the positive lower bound of its eigenvalues.

**Lemma 2** When $|s^T_k y_k| > \theta \|s_k\|^2$, $B_{k+1}$ is symmetric positive definite and its eigenvalues are greater than $(\theta^2 \|s_k\|^2) / (2 \|y_k\|^2)$ and less than 2. Consequently, when $|s^T_k y_k| > \theta \|s_k\|^2$, the eigenvalues of $B_{k+1}^{-1}$ are greater than 1/2 and less than $\frac{2 \|y_k\|^2}{\theta^2 \|y_k\|^2}$.

**Proof.** (i) For any nonzero vector $z \in \mathbb{R}^n$, from equation (32), we have

\[
z^T B_{k+1} z = \|z\|^2 - \left(\frac{z^T s_k}{\|s_k\|^2}\right)^2 + \left(\frac{z^T y_k}{\|y_k\|^2}\right)^2 \geq \left(\frac{z^T y_k}{\|y_k\|^2}\right)^2 \geq 0.
\]  

(34)

In the first inequality of (34), we use the Cauchy-Schwartz inequality $\|z^T s_k\| \leq \|z\| \|s_k\|$ and its equality holds if only if $z = ts_k$. Therefore, $B_{k+1}$ is symmetric semi-positive definite. When $z = ts_k$, since $s^T_k y_k \neq 0$, from equation (34), we have $z^T B_{k+1} z = t^2 \left(s^T_k y_k\right)^2 / \|y_k\|^2 > 0$. Consequently, $B_{k+1}$ is symmetric positive definite when $s^T_k y_k \neq 0$.

(ii) It is not difficult to know that there exist at least $n - 2$ linearly independent vectors $z_1, z_2, \ldots, z_{n-2}$ to satisfy $z^T_k s_k = 0, z^T_k y_k = 0 (i = 1, \ldots, n - 2)$. That is to say, matrix $B_{k+1}$ defined by equation (32) has at least $(n - 2)$ linearly independent eigenvectors associated with eigenvalues of 1. We denote the other two eigenvalues
of $B_{k+1}$ as $\mu_{i}^{k+1} (i = 1:2)$. We denote $\text{tr}(C) = \sum_{i=1}^{n} c_{ii}, C \in \mathbb{R}^{n \times n}$. Then, we have $\text{tr}(B_{k+1}) = \sum_{i=1}^{n} (\mu_{i}^{k+1}) = \mu_{1}^{k+1} + \mu_{2}^{k+1} + (n - 2)$. By substituting it into equation (32), we obtain

$$
\begin{align*}
\mu_{1}^{k+1} + \mu_{2}^{k+1} &= \text{tr}(B_{k+1}) - (n - 2) \\
&= \text{tr}(I) - \text{tr}\left( \frac{s_{k}^{T} y_{k}}{s_{k}^{T} s_{k}} \right) + \text{tr}\left( \frac{y_{k} y_{k}^{T}}{y_{k}^{T} y_{k}} \right) - (n - 2) = 2,
\end{align*}
$$

(35)

where we use the property $\text{tr}(AB^{T}) = \text{tr}(B^{T}A)$ of matrices $A, B \in \mathbb{R}^{m \times n}$. Since matrix $B_{k+1}$ is symmetric semi-positive definite, we know that its eigenvalues are greater than or equal to 0, namely $\mu_{i}^{k+1} \geq 0 (i = 1, 2)$. By substituting it into equation (35), we obtain

$$
\mu_{i}^{k+1} \leq \mu_{1}^{k+1} + \mu_{2}^{k+1} = 2, \quad i = 1, 2.
$$

(36)

Furthermore, the symmetric matrix $B_{k+1}$ has other $(n - 2)$ eigenvalues of 1. Therefore, by combining it with equation (36), we know that the eigenvalues of matrix $B_{k+1}$ are less than or equal to 2.

We denote $\mu_{i}^{k+1} (i = 1:n)$ as the eigenvalues of $B_{k+1}$. Then, we have $\mu_{i}^{k+1} = 1 (i = 3:n)$. By using the property $\det(B_{k+1}) = \prod_{i=1}^{n} (\mu_{i}^{k+1}) = \mu_{1}^{k+1} \mu_{2}^{k+1}$, from equation (32), we obtain

$$
\begin{align*}
\mu_{1}^{k+1} \mu_{2}^{k+1} &= \det(B_{k+1}) = \det\left( \left( 1 + \frac{y_{k} y_{k}^{T}}{y_{k}^{T} y_{k}} \right) \right) \left( I - \left( 1 + \frac{y_{k} y_{k}^{T}}{y_{k}^{T} y_{k}} \right)^{-1} \right) \left( \frac{s_{k}^{T}}{s_{k}^{T} s_{k}} \right) \\
&= \det\left( I + \frac{y_{k} y_{k}^{T}}{y_{k}^{T} y_{k}} \right) \det\left( I - \left( I + \frac{y_{k} y_{k}^{T}}{y_{k}^{T} y_{k}} \right)^{-1} \right) \left( \frac{s_{k}^{T}}{s_{k}^{T} s_{k}} \right) \\
&= 2 \left( 1 - \frac{1}{\|s_{k}\|^{2}} \right) \left( I + \frac{y_{k} y_{k}^{T}}{y_{k}^{T} y_{k}} \right)^{-1} \left( s_{k}^{T} \right) \\
&= 2 \left( 1 - \frac{1}{\|s_{k}\|^{2}} \right) \left( I - \frac{y_{k} y_{k}^{T}}{2 y_{k}^{T} y_{k}} \right) \left( s_{k}^{T} \right) = \frac{(s_{k}^{T} y_{k})^{2}}{y_{k}^{T} y_{k} \|s_{k}\|^{2}}.
\end{align*}
$$

(37)

From equation (36), we know $\mu_{i}^{k} \leq 2 (i = 1, 2)$. By substituting it into equation (37), we obtain

$$
\mu_{i}^{k+1} = \frac{1}{2} \frac{(s_{k}^{T} y_{k})^{2}}{\|s_{k}\|^{2} \|y_{k}\|^{2}} , \quad i = 1, 2.
$$

(38)

By combining it with $\mu_{i}^{k+1} = 1 (i = 3:n)$, we have

$$
\mu_{i}^{k+1} \geq \min \left\{ 1, \frac{1}{2} \frac{(s_{k}^{T} y_{k})^{2}}{\|s_{k}\|^{2} \|y_{k}\|^{2}} \right\} = \frac{1}{2} \frac{(s_{k}^{T} y_{k})^{2}}{\|s_{k}\|^{2} \|y_{k}\|^{2}} \geq \frac{1}{2} \|s_{k}\|^{2} \|y_{k}\|^{2},
$$

(39)

where we use the Cauchy-Schwarz inequality $|s_{k}^{T} y_{k}| \leq \|s_{k}\| \|y_{k}\|$. 

Since the matrix $B_{k+1}$ is symmetric positive definite when $|x_k^T y_k| > \theta \|s_k\|^2$, the inverse of $B_{k+1}$ exists. Furthermore, the eigenvalues of $B_{k+1}^{-1}$ equal $1/\mu_i(B_{k+1})$ ($i = 1:n$), where $\mu_i(C)$ represents the $i$-th eigenvalue of matrix $C \in \mathbb{R}^{n \times n}$. Therefore, by combining it with equations (36) and (39), we know that the eigenvalues of $B_{k+1}^{-1}$ are greater than $1/2$ and less than $(2\|y_k\|^2)/(\theta^2\|s_k\|^2)$ when $|x_k^T y_k| > \theta \|s_k\|^2$. □

According to our numerical experiments[38], the L-BFGS updating formula (32) works well for most problems and the objective function decreases very fast in the well-posed phase. However, for the ill-posed problems, the L-BFGS updating formula (32) will approach the stationary solution $x^*$ very slow in the ill-posed phase. Furthermore, it fails to get close to the stationary solution $x^*$ sometimes.

In order to improve the robustness of the regularization continuation method (22)-(23), we adopt the inverse $B_{k+1}^{-1}$ of the regularization two-side projected Hessian matrix as the preconditioner in the ill-posed phase, where $B_{k+1}$ is defined by

$$B_{k+1} = \frac{\sigma_0}{\Delta t_{k+1}} I + PV^2 f(x_{k+1})P.$$  \hspace{1cm} (40)

Now, the problem is how to automatically identify the ill-posed phase and change to the inverse of the regularization two-side projected Hessian matrix from the L-BFGS updating formula (32). Here, we adopt the simple criterion. That is to say, we regard that the regularization continuation method (22)-(23) is in the ill-posed phase once there exists the time step $\Delta t_k \leq 10^{-3}$.

In the ill-posed phase, the computational time of the projected Hessian matrix is heavy if we update the projected Hessian matrix $PV^2 f(x_k)P$ at every iteration. In order to save the computational time of the projected Hessian evaluation, we set $B_{k+1} = B_k$ when $m_k(0) - m_k(x_k)$ approximates $f(x_k) - f(x_k + s_k)$ well, where the approximation model $m_k(s_k)$ is defined by equation (28). Otherwise, we update $B_{k+1} = \left(\frac{\sigma_0}{\Delta t_{k+1}} I + PV^2 f(x_{k+1})P\right)$ in the ill-posed phase. In the ill-posed phase, a practice updating strategy is give by

$$B_{k+1} = \begin{cases} B_k, & \text{if } |1 - \rho_k| \leq \eta_1, \\ \frac{\sigma_0}{\Delta t_{k+1}} I + PV^2 f(x_{k+1})P, & \text{otherwise}, \end{cases}$$  \hspace{1cm} (41)

where $\rho_k$ is defined by equations (28)-(29) and $\eta_1 = 0.25$.

For a real-world problem, the analytical Hessian matrix $V^2 f(x_k)$ may not be offered. Thus, in practice, we replace the projected Hessian matrix $PV^2 f(x_k)P$ with its difference approximation as follows:

$$PV^2 f(x_k)P \approx \left[ \frac{P g(x_k + \epsilon P e_1) - P g(x_k)}{\epsilon}, \ldots, \frac{P g(x_k + \epsilon P e_n) - P g(x_k)}{\epsilon} \right],$$  \hspace{1cm} (42)

where $e_i$ represents the unit vector whose elements equal zeros except for the $i$-th element which equals 1, and the parameter $\epsilon$ can be selected as $10^{-6}$ according to our numerical experiments.
The regularization continuation method

2.5 The treatment of rank-deficient problems and infeasible initial points

For a real-world problem, matrix $A$ may be deficient-rank. We assume that the rank of $A$ is $r$ and we use the QR decomposition (pp.276-278, [19]) to factor $A^T$ into a product of an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ and an upper triangular matrix $R \in \mathbb{R}^{n \times m}$ as follows:

$$A^T E = QR = \begin{bmatrix} Q_1 | Q_2 \\ R_1 | 0 \end{bmatrix},$$

(43)

where $E \in \mathbb{R}^{m \times m}$ is a permutation matrix, $R_1 = R(1 : r, 1 : m)$ is upper triangular matrix and its diagonal elements are non-zero, and $Q_1 = Q(1 : n, 1 : r)$, $Q_2 = Q(1 : n, (r + 1) : n)$ satisfy $Q_1^T Q_1 = I$, $Q_2^T Q_2 = I$ and $Q_1^T Q_2 = 0$. Then, we reduce the linear constraint $Ax = b$ as

$$Q_1^T x = b_r,$$

(44)

where $b_r = (R_1 R_1^T)^{-1} (R_1 (E^T b))$.

From equations (10) and (44), we simplify the projection matrix $P$ as

$$P = I - Q_1 Q_1^T = Q_2 Q_2^T.$$

(45)

In practical computation, we adopt the different formulas of the projection matrix $P$ according to $r \leq n/2$ or $r > n/2$. Thus, we give the computational formula of the projected gradient $P g_k$ as follows:

$$P g_k = \begin{cases} g_k - Q_1 (Q_1^T g_k), & \text{if } r \leq \frac{1}{2} n, \\ Q_2 (Q_2^T g_k), & \text{otherwise}. \end{cases}$$

(46)

where $r$ is the number of columns of $Q_1$, i.e. the rank of $A$.

For a real-world optimization problem (1), we probably meet the infeasible initial point $x_0$. In other words, the initial point can not satisfy the constraint $Ax = b$. We handle this problem by solving the following projection problem:

$$\min_{x \in \mathbb{R}^n} \|x - x_0\|^2 \quad \text{subject to} \quad Q_1^T x = b_r,$$

(47)

where $b_r = (R_1 R_1^T)^{-1} (R_1 (E^T b))$. By using the Lagrangian multiplier method to solve problem (47), we obtain the initial feasible point $x_0^F$ of problem (1) as follows:

$$x_0^F = x_0 - Q_1 (Q_1^T x_0 - b_r).$$

(48)

For convenience, we set $x_0 = x_0^F$ in line 4 of Algorithm 1.

According to the above discussions, we give the detailed implementation of the regularization continuation method and the trust-region updating strategy for the linearly equality-constrained optimization problem (1) in Algorithm 1.
Algorithm 1 The regularization continuation method and the trust-region updating strategy for linearly constrained optimization problems (the Rcmtr method)

**Input:** the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the linear constraint $Ax = b$, $A, b \in \mathbb{R}^{m \times n}$, the initial point $x_0$ (optional), the terminated parameter $\varepsilon$ (optional).

**Output:** the optimal approximation solution $x^\ast$.

1. If $x_0$ or $\varepsilon$ is not provided, then we set $x_0 = \text{ones}(\theta, 1) \text{ or } \varepsilon = 10^{-6}$.
2. Initialize the parameters: $\eta_1 = 10^{-4}$, $\eta_2 = 10^{-10}$, $\gamma_1 = 0.25$, $\gamma_2 = 0.75$, $\gamma_3 = 0.5$, $\theta = 10^{-6}$, $\max\,\text{itc} = 300$. Set $\theta_0 = 10^{-1}$, $\Delta \theta_0 = 10^{-2}$, $\text{flag仿真_phase} = 0$, $\text{flag仿真_trialstep} = 1$, $\Delta x_{-1} = 0$, $\Delta y_{-1} = 0$, $\rho_{-1} = 0$, $B_{k-1} = I$, $H_{k-1} = I$, $R_{k-1} = I$, $\text{itc} = 0$.
3. Factorize matrix $A^T$ such that $A^T E = Q_k R_k$ with the QR decomposition (43). By solving $(R_k R_k^T)^{-1} b_r = R_k (E^T b_r)$, we obtain $b_r$.
4. Compute $x_0 \leftarrow x_0 - Q_k (Q_k^T x_0 - b_r)$, such that $x_0$ satisfies the linear constraint $Ax_0 = b$.
5. Set $k = 0$. Evaluate $f_0 = f(x_0)$ and $g_0 = \nabla f(x_0)$.
6. Compute the projected gradient $p_{k0} = P_{R_k} g_0$ according to the formula (46).
7. while $([(\|p_{k0}\| > \varepsilon) \& (\text{itc} < \max\,\text{itc})])$ do
8. \hspace{1em} $\text{itc} = \text{itc} + 1$;
9. \hspace{1em} if $\Delta \theta_k < 10^{-3}$ then
10. \hspace{2em} Set $\text{flag仿真_phase} = 1$
11. \hspace{1em} end if
12. \hspace{1em} if ($\text{flag仿真_phase} = 0$) then
13. \hspace{2em} if ($\text{flag仿真_trialstep} = 1$) then
14. \hspace{3em} if $[\|\Delta x_{k-1}\| > \theta \|\Delta y_{k-1}\|^2]$ then
15. \hspace{4em} $d_k = - \left( p_{k1} - \frac{\gamma_1 (g_{k-1}^T p_{k1}) + \gamma_2 \Delta y_{k-1}^T p_{k1}}{\gamma_1 \Delta x_{k-1}^T + \gamma_2 \Delta y_{k-1}^T} \Delta x_{k-1} \right)$.
16. \hspace{3em} else
17. \hspace{4em} $d_k = -p_{k1}$.
18. \hspace{3em} end if
19. \hspace{2em} end if
20. \hspace{1em} end if
21. \hspace{1em} if ($\text{flag仿真_trialstep} = 0$) then
22. \hspace{2em} Set $B_k = (\gamma_0 / \Delta \theta_k) I + H_k$ and decompose $B_k = Q_k R_k$ with the QR decomposition.
23. \hspace{2em} else if $|(\|p_{k-1}\| - 1| > 0.25)$ then
24. \hspace{3em} Evaluate $H_k = P_{R_k} f(x_k) P$ from equation (42).
25. \hspace{3em} Set $B_k = (\gamma_0 / \Delta \theta_k) I + H_k$ and decompose $B_k = Q_k R_k$ with the QR decomposition.
26. \hspace{2em} else
27. \hspace{3em} $Q_k = Q_{k-1}$, $R_k = R_{k-1}$
28. \hspace{2em} end if
29. \hspace{1em} By solving $R_k d_k = -Q_k^T p_{k1}$, we obtain $d_k$.
30. \hspace{1em} end if
31. \hspace{1em} Set $x_k = x_{k-1} + d_k$ and $x_{k+1} = x_k + d_k$.
32. Evaluate $f_{k+1} = f(x_{k+1})$ and compute the ratio $p_k$ from equations (28)-(29).
33. \hspace{1em} if $p_k \geq \eta_1$ and $x_k$ satisfies the sufficient descent condition (31)) then
34. \hspace{2em} Set flag仿真_trialstep = 1 and evaluate $r_{k+1} = \nabla f(x_{k+1})$.
35. \hspace{2em} Compute $p_{k+1} = p_{R_k} r_{k+1}$ according to the formula (46). Set $y_k = p_{k+1} - p_k$.
36. \hspace{2em} else
37. \hspace{3em} Set flag仿真_trialstep = 0, $x_{k+1} = x_k$, $f_{k+1} = f_k$, $p_{k+1} = p_k$, $y_{k+1} = y_k$, $d_{k+1} = d_k$.
38. \hspace{2em} end if
39. \hspace{1em} Adjust the time step size $\Delta \theta_{k+1}$ based on the trust-region updating strategy (30).
40. \hspace{1em} Set $k \leftarrow k + 1$.
41. end while
3 Algorithm Analysis

In this section, we analyze the global convergence of the regularization continuation method (22)-(23) with the trust-region updating strategy and the switching preconditioned technique for the linearly equality-constrained optimization problem (i.e. Algorithm 1). Firstly, we give a lower-bounded estimate of \( m_k(0) - m_k(s_k) \) \( (k = 1, 2, \ldots) \). This result is similar to that of the trust-region method for the unconstrained optimization problem [48]. For simplicity, we assume that the rank of matrix \( A \) is full.

We denote the feasible set \( S_f \) as
\[
S_f = \{ x : Ax = b \}. \tag{49}
\]

In the following analysis of Algorithm 1, the function \( f \) is assumed to satisfy Assumption 1.

**Assumption 1** Assume that the function \( f(\cdot) \) is twice continuously differential and there exists a positive constant \( M \) such that
\[
\| \nabla^2 f(x) \| \leq M, \tag{50}
\]
holds for all \( x \in S_f \).

By combining the property \( \| P \| = 1 \) of the projection matrix \( P \), from the assumption (50), we obtain
\[
\| P \nabla^2 f(x) P \| \leq \| P \| \| \nabla^2 f(x) \| \| P \| = \| \nabla^2 f(x) \| \leq M. \tag{51}
\]

According to the property of the matrix norm, we know that the absolute eigenvalue of \( P \nabla^2 f(x) P \) is less than \( M \). If we denote \( \mu (P \nabla^2 f(x) P) \) as the eigenvalue of \( P \nabla^2 f(x) P \), we know that the eigenvalue of \( \left( \frac{\sigma_0}{\Delta t} I + P \nabla^2 f(x) P \right) \) is \( \frac{\sigma_0}{\Delta t} + \mu (P \nabla^2 f(x) P) \). Consequently, from equation (51), we known that
\[
\frac{\sigma_0}{\Delta t} I + P \nabla^2 f(x) P \succ 0, \quad x \in S_f, \quad \text{when} \quad \Delta t < \frac{\sigma_0}{M}. \tag{52}
\]

**Lemma 3** Assume that the quadratic model \( q_k(x) \) is defined by equation (28) and \( s_k \) is computed by the regularization continuation method (22)-(23), where matrices \( B_k \) \( (k = 1, 2, \ldots) \) are updated by the L-BFGS formula (32) in the well-posed phase. Then, we have
\[
m_k(0) - m_k(s_k) \geq \frac{\Delta t_k}{4(1 + \Delta t_k)} \| P g_k \|^2 \geq c_m \| P g_k \| ||s_k||, \tag{53}
\]
where \( c_m \) is a positive constant, \( P g_k = P \nabla f(x_k) \) and the projection matrix \( P \) is defined by equation (10).
Proof. From \( y_{k-1} = Pg(x_k) - Pg(x_{k-1}) \) and \( AP = 0 \), we have \( Ay_{k-1} = 0 \). By combining it with \( As_{k-1} = 0 \), from the L-BFGS formula (32) and the regularization continuation method (22)-(23), we obtain \( As_k = 0 \), namely \( x_{k+1} \in S_f \) and \( s_k = Pz_k \) for a vector \( z_k \in \mathbb{R}^n \). Consequently \( Ps_k = P^2 z_k = Pz_k = s_k \). By induction, we know that \( Ps_k = s_k \) \((k = 0, 1, 2, \ldots)\). Furthermore, from the L-BFGS formula and Lemma (2), we know that the eigenvalues of \( B_k^{-1} \) are greater than \( 1/2 \). By combining them into equation (28) and using the symmetric Shur decomposition (p. 440, [19]) of \( B_k^{-1} \), we obtain

\[
m_k(0) - m_k(s_k) = -\frac{1 + 0.5\Delta t_k}{1 + \Delta t_k} s_k^T s_k = -\frac{1 + 0.5\Delta t_k}{1 + \Delta t_k} B_k^{-1} s_k \geq \frac{1 + 0.5\Delta t_k}{1 + \Delta t_k} \frac{\Delta t_k}{2(1 + \Delta t_k)} \|Ps_k\|^2. \tag{54}
\]

By substituting the property \((1 + 0.5\Delta t_k)/(1 + \Delta t_k) \geq (0.5 + 0.5\Delta t_k)/(1 + \Delta t_k) = 0.5\) into equation (54), we have

\[
m_k(0) - m_k(s_k) \geq \frac{\Delta t_k}{4(1 + \Delta t_k)} \|Ps_k\|^2. \tag{55}
\]

From equation (51), we have

\[
\|y_{k-1}\| = \|Pg(x_{k-1}) - Pg(x_{k-2})\| = \left\| \int_0^1 PV^2 f(x_{k-2} + ts_{k-1})s_{k-1}dt \right\|
\]

\[
= \left\| \int_0^1 PV^2 f(x_{k-2} + ts_{k-1})Ps_{k-1}dt \right\| \leq \int_0^1 \|PV^2 f(x_{k-2} + ts_{k-1})P\| \|s_{k-1}\|dt
\]

\[
\leq M\|s_{k-1}\|. \tag{56}
\]

From Lemma 2, we know that the eigenvalues of \( B_k \) are greater than \( \frac{\theta^2\|s_{k-1}\|^2}{2\|y_{k-1}\|^2} \). By substituting equation (56) into it, we know that the eigenvalues of \( B_k \) are greater than \( \theta^2/(2M^2) \). Furthermore, from the symmetric Shur decomposition (p. 440, [19]), we know that there exists an orthogonal matrix \( U_k \) such that \( B_k = U_k^T \text{diag} \left( \mu_1^k, \ldots, \mu_n^k \right) U_k \), where \( \mu_1^k \geq \mu_2^k \geq \cdots \geq \mu_n^k \) are the eigenvalues of the symmetric matrix \( B_k \). Thus, we obtain

\[
\|B_k s_k\|^2 = \left\| (U_k B_k U_k^T) U_k s_k \right\|^2 = (U_k s_k)^T \text{diag} \left( (\mu_1^k)^2, \ldots, (\mu_n^k)^2 \right) (U_k s_k)
\]

\[
\geq \left( \frac{\theta^2}{2M^2} \right)^2 s_k^T U_k^T U_k s_k = \left( \frac{\theta^2}{2M^2} \right)^2 \|s_k\|^2, \text{ i.e. } \|B_k s_k\| \geq \frac{\theta^2}{2M^2} \|s_k\|. \tag{57}
\]

By combining it with equations (22) and (55), we obtain

\[
m_k(0) - m_k(s_k) \geq \frac{\Delta t_k}{4(1 + \Delta t_k)} \|Pz_k\|^2 = \frac{1}{4} \|Pz_k\|\|B_k s_k\| \geq \frac{\theta^2}{8M^2} \|Pz_k\|\|s_k\|. \tag{58}
\]

We denote \( c_w = \theta^2/(8M^2) \). Then, from equation (58), we obtain the result (53). \( \square \)
Similarly to the estimation of equation (57), from equation (22) and the symmetric property of $M$, where we use the property that the absolute eigenvalues of $\lambda P \lambda$ where $\lambda$ is computed by the regularization continuation method (22)-(23), we obtain

$$m_k(0) - m_k(s_k) \geq \frac{\Delta t_k}{4(1 + \Delta t_k)} \| P s_k \|^2 \geq c_b \| P s_k \| \| s_k \|, \quad (59)$$

where $c_b$ is a positive constant, $P s_k = P \lambda^2 f(x_k)$ and the projection matrix $P$ is defined by equation (10).

**Proof.** From equations (22)-(23) and $B_k = \left( \frac{\sigma_0}{\Delta t_k} I + PV^2 f(x_k) P \right)$, we have

$$P \left( \frac{\sigma_0}{\Delta t_k} I + PV^2 f(x_k) P \right) s_k = -\frac{\Delta t_k}{1 + \Delta t_k} p^2 g_k. \quad (60)$$

By substituting $P^2 = P$ into the above equation (60), we obtain $P s_k = s_k$. Consequently, by combining it with the property $AP = 0$, we obtain $\lambda s_k = 0$, i.e. $x_k \in S_f$ if $x_k \in S_f$. By induction, we obtain $x_k \in S_f (k = 1, 2, \ldots)$ when $x_0 \in S_f$. Therefore, according to the assumption $\Delta t_k \leq 2M$, from equation (52), we know

$$\left( \frac{\sigma_0}{\Delta t_k} I + PV^2 f(x_k) P \right) \succ 0. \quad (61)$$

From equations (22), (61) and $P s_k = s_k$, by using the symmetric Shur decomposition (p. 440, [19]), we have

$$-s^T_k g_k = -\langle P s_k \rangle^T g_k = -s^T_k (P s_k) = \frac{\Delta t_k}{1 + \Delta t_k} P s_k \left( \frac{\sigma_0}{\Delta t_k} I + PV^2 f(x_k) P \right)^{-1} P s_k \geq \frac{\Delta t_k}{1 + \Delta t_k} \frac{1}{\sigma_0 / \Delta t_k + \| PV^2 f(x_k) P \|} \| P s_k \|^2 \geq \frac{\Delta t_k}{1 + \Delta t_k} \frac{1}{\sigma_0 / \Delta t_k + M} \| P s_k \|^2. \quad (62)$$

Similarly to the estimation of equation (57), from equation (22) and the symmetric Shur decomposition (p. 440, [19]), we have

$$\| B_k s_k \| = \left\| \left( \frac{\sigma_0}{\Delta t_k} I + PV^2 f(x_k) P \right) s_k \right\| \geq \left( \frac{\sigma_0}{\Delta t_k} - M \right) \| s_k \|, \quad (63)$$

where we use the property that the absolute eigenvalues of $PV^2 f(x_k) P$ are less than $M$. From equation (22) and by substituting equation (63) into equation (62), we obtain

$$-s^T_k g_k \geq \frac{\sigma_0 / \Delta t_k - M}{\sigma_0 / \Delta t_k + M} \| P s_k \| \| s_k \| \geq \frac{2M - M}{2M + M} \| P s_k \| \| s_k \| = \frac{1}{3} \| P s_k \| \| s_k \|, \quad (64)$$

where we use the assumption $\Delta t_k \leq \sigma_0 / (2M)$ and the monotonically increasing property of $\alpha(t) = (t - M) / (t + M) when t > M$. 

**Lemma 4** Assume that the quadratic model $q_k(x)$ is defined by equation (28) and $s_k$ is computed by the regularization continuation method (22)-(23), where $B_k = \left( \frac{\sigma_0}{\Delta t_k} I + PV^2 f(x_k) P \right)$ and $\Delta t_k \leq \frac{\sigma_0}{2M}$ in the ill-posed phase. Then, we have
From the approximation model (28) and the estimation (64), we have
\[ m_k(0) - m_k(s_k) = -\frac{1 + 0.5\Delta_t}{1 + \Delta_t} \quad \text{st} \quad 1 \quad \frac{1}{3}\frac{1 + 0.5\Delta_t}{1 + \Delta_t} \geq \frac{1}{3}\frac{1 + 0.5\Delta_t}{1 + \Delta_t} \| p_{x_k} \| \| s_k \| \]
\[ = \frac{1}{3}\frac{1 + 0.5(1 + \Delta_t)}{1 + \Delta_t} \| p_{x_k} \| \| s_k \| \geq \frac{1}{6}\| p_{x_k} \| \| s_k \|. \quad (65) \]
where we use the property \( 0.5 + 0.5(1 + \Delta_t) \geq 0.5(1 + \Delta_t) \). We denote \( c_b = 1/6 \).

Then, from equation (65), we obtain the estimation (59).

In order to prove that \( p_{x_k} \) converges to zero when \( k \) tends to infinity, we need to estimate the lower bound of time step sizes \( \Delta_t \) (\( k = 1, 2, \ldots \)).

**Lemma 5** Assume that \( f \) satisfies Assumption 1 and the sequence \( \{x_k\} \) is generated by Algorithm 1. Then, there exists a positive constant \( \delta_0 \) such that
\[ \Delta_t \geq \gamma \Delta_t \]
holds for all \( k = 1, 2, \ldots \), where \( \Delta_t \) is adaptively adjusted by the trust-region updating strategy (28)-(30).

**Proof.** From the first-order Taylor expansion, we have
\[ f(x_k + s_k) = f(x_k) + \int_0^1 s_k^T g(x_k + ts_k) dt. \quad (67) \]
Thus, from equations (28)-(29), (67), the Armijo sufficient descent condition (31) and the assumption (50), we have
\[ |p_k - 1| = \left| \frac{f(x_k) - f(x_k + s_k) - (m_k(0) - m_k(s_k))}{m_k(0) - m_k(s_k)} \right| \]
\[ \leq \left| \int_0^1 s_k^T g(x_k + ts_k) dt \right| + \frac{0.5\Delta_t}{1 + 0.5\Delta_t} \quad \text{st} \quad \frac{0.5\Delta_t}{1 + 0.5\Delta_t} \]
\[ \leq \frac{0.5M\|s_k\|^2}{m_k(0) - m_k(s_k)} \| p_{x_k} \| \| s_k \| \leq \frac{0.5\Delta_t}{1 + 0.5\Delta_t} \quad \text{st} \quad \delta_0 \Delta_t \]
\[ \leq \frac{0.5\Delta_t}{1 + 0.5\Delta_t}. \quad (68) \]

From Lemma 3 and Lemma 4, we know that there exists a constant \( \eta_m \) such as \( \eta_m = \min\{c_w, c_h\} \) such that the approximation model \( m_k(0) - m_k(s_k) \) satisfies the Armijo sufficient descent condition (31) when \( \Delta_t \leq 1/(2M) \) and the function \( f \) satisfies Assumption 1. By substituting the sufficient descent condition (31) into equation (68), we obtain
\[ |p_k - 1| \leq \frac{0.5M}{\eta_m} \| p_{x_k} \| \| s_k \| \leq \frac{0.5\Delta_t}{1 + 0.5\Delta_t}. \quad (69) \]
When $B_k$ is updated by the L-BFGS formula (32) in the well-posed phase, from Lemma 2, we know that the eigenvalues of $B_k^{-1}$ are less than max $\left\{ 1, \frac{2\|y_k-1\|^2}{\theta^2\|s_k-1\|^2} \right\}$. By combining it with equations (32) and (51), we obtain

$$
\|s_k\| = \frac{\Delta t_k}{1 + \Delta t_k} \left\| B_k^{-1} p_{s_k} \right\| \leq \frac{\Delta t_k}{1 + \Delta t_k} \max \left\{ 1, \frac{2\|y_k-1\|^2}{\theta^2\|s_k-1\|^2} \right\} \|p_{s_k}\| \leq \frac{\Delta t_k}{1 + \Delta t_k} \max \left\{ 1, \frac{2\|P_g(x_k-1 + s_k-1) - P_g(x_k-1)\|^2}{\theta^2\|s_k-1\|^2} \right\} \|p_{s_k}\| \leq \frac{\Delta t_k}{1 + \Delta t_k} \max \left\{ 1, \frac{2\|M\|^2}{\theta^2} \right\} \|p_{s_k}\| = \frac{\Delta t_k}{1 + \Delta t_k} L_u \|p_{s_k}\|, \hspace{1cm} (70)
$$

where we denote $L_u = \max \left\{ 1, \frac{2\|M\|^2}{\theta^2} \right\}$.

When $B_k = \left( \frac{\sigma_0}{\Delta t_k} I + P\nabla^2 f(x_k) P \right)$ and $\Delta t_k \leq \sigma_0/(2M)$, from equations (22) and (51), we have

$$
\|s_k\| = \frac{\Delta t_k}{1 + \Delta t_k} \left\| B_k^{-1} p_{s_k} \right\| = \frac{\Delta t_k}{1 + \Delta t_k} \left\| \left( \frac{\sigma_0}{\Delta t_k} I + P\nabla^2 f(x_k) P \right)^{-1} p_{s_k} \right\| \leq \frac{\Delta t_k}{1 + \Delta t_k} \frac{1}{\sigma_0/\Delta t_k - M} \|p_{s_k}\| \leq \frac{\Delta t_k}{1 + \Delta t_k} \frac{1}{\sigma_0/\Delta t_k - M} \|p_{s_k}\|. \hspace{1cm} (71)
$$

Thus, when $B_k$ are updated by the formula (41) and $\Delta t_k \leq \sigma_0/(2M)$ in the ill-posed phase, from equation (71), we have

$$
\|s_k\| \leq \frac{\Delta t_k}{1 + \Delta t_k} \frac{1}{\sigma_0/\Delta t_k - M} \|p_{s_k}\|. \hspace{1cm} (72)
$$

We denote $L_u = \max\{L_u, 1/M\}$. By substituting equations (70) and (72) into equation (69), when $\Delta t_k \leq \sigma_0/(2M)$, we obtain

$$
|\rho_k - 1| \leq \frac{0.5ML_u}{\eta_m} \frac{\Delta t_k}{1 + \Delta t_k} + \frac{0.5\Delta t_k}{1 + 0.5\Delta t_k} \leq \frac{0.5ML_u}{\eta_m} \frac{\Delta t_k}{1 + \Delta t_k} \frac{1}{0.5 + 0.5\Delta t_k} \leq \frac{ML_u + \eta_m}{2\eta_m} \frac{\Delta t_k}{1 + \Delta t_k}. \hspace{1cm} (73)
$$

We denote

$$
\delta_M \triangleq \min \left\{ \frac{2\eta_1\eta_m}{ML_u + \eta_m}, \frac{\sigma_0}{2M}, \Delta t_0 \right\}. \hspace{1cm} (74)
$$

Then, from equations (73)-(74), when $\Delta t_k \leq \delta_M$, it is not difficult to verify

$$
|\rho_k - 1| \leq \eta_1. \hspace{1cm} (75)
$$

We assume that $K$ is the first index such that $\Delta t_K \leq \delta_M$ where $\delta_M$ is defined by equation (74). Then, from equations (74)-(75), we know that $|\rho_k - 1| \leq \eta_1$. According to the time step adjustment formula (30), $x_K + s_K$ will be accepted and the time step $\Delta t_{K+1}$ will be enlarged. Consequently, $\Delta t_k \geq \gamma_2 \delta_M$ holds for all $k = 1, 2, \ldots$. \hfill \Box
By using the result of Lemma 5, we prove the global convergence of Algorithm 1 for the linear equality-constrained optimization problem (1) in Theorem 1.

**Theorem 1** Assume that $f$ satisfies Assumption 1 and $f(x)$ is lower bounded when $x \in S_f$, where $S_f$ is defined by equation (49). The sequence $\{x_k\}$ is generated by Algorithm 1. Then, we have

$$\lim_{k \to \infty} \inf \|Pg_k\| = 0,$$

where $g_k = \nabla f(x_k)$ and the projection matrix $P$ is defined by equation (10).

**Proof.** We prove the result (76) by contradiction. Assume that there exists a positive constant $\varepsilon$ such that

$$\|Pg_k\| > \varepsilon$$

holds for all $k = 0, 1, 2, \ldots$

According to Lemma 5 and Algorithm 1, we know that there exists an infinite subsequence $\{x_{k_i}\}$ such that the trial steps $s_{k_i}$ ($i = 1, 2, \ldots$) are accepted. Otherwise, all steps are rejected after a given iteration index, then the time step size will keep decreasing, which contradicts (66). Therefore, from equations (29), (31) and (77), we have

$$f(x_0) - \lim_{k \to \infty} f(x_k) = \sum_{k=0}^{\infty} (f(x_k) - f(x_{k+1})) \geq \sum_{i=0}^{\infty} (f(s_{k_i}) - f(s_{k_i} + s_{k_i})) \geq \eta_\alpha \sum_{i=0}^{\infty} \|Pg_{k_i}\| \|s_{k_i}\| \geq \eta_\alpha \varepsilon \sum_{i=0}^{\infty} \|s_{k_i}\|.$$  

(78)

Since $f(x)$ is lower bounded when $x \in S_f$ and the sequence $\{f(x_k)\}$ is monotonically decreasing, we have $\lim_{k \to \infty} f(x_k) = f^*$. By substituting it into equation (78), we obtain

$$\lim_{i \to \infty} \|s_{k_i}\| = 0.$$  

(79)

When $B_k$ is updated by the L-BFGS formula (2) in the well-posed phase, from Lemma 2, we know $\|B_k\| \leq 2$. When $B_k$ is updated by the formula (41) in the ill-posed phase, from equations (51) and (66), we know that $\|B_k\| \leq \left(\frac{\sigma_0}{\gamma_2 \sigma_M} + M\right)$. We denote

$$L_B = \max \left\{2, \left(\frac{\sigma_0}{\gamma_2 \sigma_M} + M\right)\right\}.$$  

(80)

By substituting equations (66) and (80) into equation (22), we obtain

$$\|Pg_{k_i}\| = \frac{1 + \Delta k_i}{\Delta k_i} \|B_{k_i} s_{k_i}\| = \left(1 + \frac{1}{\Delta k_i}\right) \|B_{k_i} s_{k_i}\| \leq \left(1 + \frac{1}{\gamma_2 \sigma_M}\right) L_B \|s_{k_i}\|.$$  

(81)

By substituting equation (81) into equation (79), we obtain

$$\lim_{i \to \infty} \|Pg_{k_i}\| = 0,$$

which contradicts the assumption (76). Consequently, the result (76) is true. □
4 Numerical Experiments

In this section, we conduct some numerical experiments to test the performance of Algorithm 1 (the Rcmtr method). The codes are executed by a HP notebook with the Intel quad-core CPU and 8Gb memory in the MATLAB R2020a environment [42]. The two-sided projected Hessian matrix \( P\nabla^2 f(x)P \) of Algorithm 1 is approximated by the difference formula (42).

SQP [16,18,45,60] is the traditional-representative method for the constrained optimization problems. Ptctr is the recent continuation method and significantly better than SQP for linearly equality-constrained optimization problems according to the numerical results in [35]. Therefore, we select these two typical methods as the basis for comparison. The implementation code of SQP is the built-in subroutine fmincon.m of the MATLAB2020a environment [42].

We select 57 linearly equality-constrained optimization problems from [1,35,44,55] as the test problems, some of which are the unconstrained optimization problems [1,44,55] and we add the same linear constraint \( Ax = b \), where \( b = 2 \ast \text{ones}(n, 1) \) and \( A \) is defined as follows:

\[
A_1 = \begin{bmatrix}
2 & 1 & 0 & \cdots & 0 & 0 \\
1 & 2 & 1 & \cdots & 0 & 0 \\
. & . & . & \ddots & . & . \\
0 & 0 & 0 & \cdots & 2 & 1 \\
0 & 0 & 0 & \cdots & 1 & 2 \\
0 & 0 & 0 & \cdots & 1 & 2
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
2 & 2 & 2 & \cdots & 2 & 2 \\
1 & 1 & 1 & \cdots & 1 & 1 \\
. & . & . & \ddots & . & . \\
2 & 2 & 2 & \cdots & 2 & 2 \\
1 & 1 & 1 & \cdots & 1 & 1
\end{bmatrix}, \quad A = [A_1, A_2].
\] (82)

The alternating direction method of multipliers (ADMM [8]) is an efficient method for some convex optimization problems and studied by many researchers in recent years. Therefore, we also compare Rcmtr with ADMM for 17 linearly equality-constrained convex optimization problems of those 57 test problems. The compared ADMM subroutine [8] is downloaded from the web site https://web.stanford.edu/~boyd/papers/admm/.

The termination conditions of the four compared methods are all set by

\[
\|\nabla_x L(x_k, \lambda_k)\|_\infty \leq 1.0 \times 10^{-6},
\]  \( k = 1, 2, \ldots \), (83)

\[
\|Ax_k - b\|_\infty \leq 1.0 \times 10^{-6}, \quad k = 1, 2, \ldots ,
\] (84)

where the Lagrange function \( L(x, \lambda) \) is defined by equation (4) and \( \lambda \) is defined by equation (8).

We test those 57 problems with \( n = 2 \) to \( n \approx 1000 \). The numerical results are arranged in Tables 1-2 for the convex problems, and Tables 3-4 for the non-convex problems. The computational time and the number of iterations of Rcmtr, Ptctr and SQP are illustrated in Figure 1 and Figure 2, respectively. From Table 1 and Table 2, we find that Rcmtr can solve those convex linearly constrained-equality optimization problems.
problems. However, there are 3 convex problems of 17 convex problems can not be solved well by Ptctr and SQP, respectively. ADMM can not work well for those 17 test convex problems.

From Table 3 and Table 4, we find that Rcmtr can solve those 40 non-convex linearly equality-constrained optimization problems well except for a particularly difficult problem (Stretched V Function [55]). For this problem, Ptctr and SQP can not solve it, too. There are two non-convex problems and five non-convex problems of 40 non-convex problems can not be solved by Ptctr and SQP, respectively. Furthermore, from Tables 2-4 and Figure 1, we find that the computational time of Rcmtr is significantly less than those of Ptctr and SQP for most of test problems, respectively.

From those numerical results, we find that Rcmtr works significantly better than the other three methods. One of the reasons is that Rcmtr uses the L-BFGS method (33) as the preconditioned technique to follow their trajectories in the well-posed phase. Consequently, Rcmtr only involves three pairs of the inner product of two vectors and one matrix-vector product ($p_{k+1} = P_g s_k$) to obtain the trial step $s_k$ and involves about $(n - m)n$ flops at every iteration in the well-posed phase. However, Ptctr needs to solve a linear system of equations with an $n \times n$ symmetric definite coefficient matrix and involves about $\frac{1}{2}n^3$ flops (p. 169, [19]) at every iteration. SQP needs to solve a linear system of equations with dimension $(m + n)$ when it solves a quadratic programming subproblem at every iteration (pp. 531-532, [45]) and involves about $\frac{1}{2}(m + n)^3$ flops (p. 116, [19]).

![Fig. 1](image-url)  
*Fig. 1: The computational time (s) of Ptctr, Rcmtr and SQP for test problems.*
Table 1: Numerical results of Rcmtr and ADMM for convex problems.

| Problems                        | Rcmtr         | ADMM         |
|---------------------------------|---------------|--------------|
|                                | Steps (time)  | $f(x^\ast)$ (KKT) | Steps (time) | $f(x^\ast)$ (KKT) |
| Exam. 1 Kim Problem 1 [26,35]   | 13 (0.24)     | 7.27e+03 (3.44e-07) | 3 (0.04)     | 2.20e+04 (40.00) (failed) |
| (n = 1000, m = n/2)             |               |              |              |                          |
| Exam. 2 LLS Problem 1 [35]     | 17 (0.42)     | 1.44e+03 (9.37e-07) | 21 (0.07)    | 2.76e+03 (6.00) (failed) |
| (n = 1200, m = n/3)             |               |              |              |                          |
| Exam. 3 Osborne Problem 1 [35,46]| 1 (0.55)      | 7.15e+02 (1.27e-15) | 60 (0.18)    | 8.48e+02 (2.80) (failed) |
| (n = 1200, m = 2/3n)            |               |              |              |                          |
| Exam. 4 Mak Problem [35,41]     | 11 (0.47)     | 97.96 (7.74e-07)  | 4 (0.05)     | 1.32e+02 (1.00) (failed) |
| (n = 1000, m = n/2)             |               |              |              |                          |
| Exam. 5 LLS Problem 2 [35]     | 14 (0.66)     | 82.43 (7.54e-08)  | 12 (0.04)    | 8.90e+03 (32.00) (failed) |
| (n = 1000, m = n/2)             |               |              |              |                          |
| Exam. 6 Osborne Problem 2 [35,46]| 14 (0.97)     | 5.14e+02 (8.73e-07) | 60 (0.29)    | 7.66e+02 (2.80) (failed) |
| (n = 1200, m = n/2)             |               |              |              |                          |
| Exam. 7 Carlberg Problem [9,35] | 15 (0.74)     | 1.19e+04 (1.66e-06) | 3 (0.04)     | 1.40e+04 (32.00) (failed) |
| (n = 1000, m = n/2)             |               |              |              |                          |
| Exam. 8 Kim Problem 2 [26,35]   | 21 (1.59)     | 4.22e+04 (1.43e-06) | 3 (0.33)     | 3.28e+05 (1.92e+03) (failed) |
| (n = 1000, m = n/2)             |               |              |              |                          |
| Exam. 9 Yamashita Problem [35,61]| 25 (2.62)     | 0.50 (3.67e-07)  | 16 (0.06)    | 45.01 (0.50) (failed)   |
| (n = 1200, m = n/3)             |               |              |              |                          |
| Exam. 10 Quartic With Noise     | 7 (0.08)      | 1.01e+02 (2.96e-07) | 400 (0.40)   | 1.91e+02 (3.96) (failed) |
| Function [1] (n = 1000, m = n/2)|               |              |              |                          |
| Exam. 11 Rotated Hyper Ellipsoid| 6 (2.50)      | 1.25e+05 (8.30e-06) | 400 (1.04)   | 1.26e+05 (2.00e+05) (failed) |
| Function [55] (n = 1000, m = n/2)|             |              |              |                          |
| Exam. 12 Sphere Function [55]   | 1 (0.08)      | 1.67e+02 (3.13e-15) | 400 (0.27)   | 1.96e+02 (2.00) (failed) |
| (n = 1000, m = n/2)             |               |              |              |                          |
| Exam. 13 Sum Squares Function   | 28 (4.08)     | 4.08e+04 (1.58e-06) | 400 (0.32)   | 4.16e+04 (9.98e+02) (failed) |
| [55] (n = 1000, m = n/2)        |               |              |              |                          |
| Exam. 14 Trid Function [55]     | 38 (2.61)     | 5.82e+02 (5.36e-07) | 400 (0.36)   | 5.88e+02 (3.99) (failed) |
| (n = 1000, m = n/2)             |               |              |              |                          |
| Exam. 15 Booth Function [55]    | 13 (1.00e-03) | 9.00 (1.98e-07)  | 18 (1.00e-03) | 45.00 (30.00) (failed)   |
| (n = 2, m = n/2)                |               |              |              |                          |
| Exam. 16 Matyas Function [55]   | 17 (1.00e-04) | 0.18 (4.44e-07)  | 18 (2.00e-03) | 2.50 (5.20) (failed)    |
| (n = 2, m = n/2)                |               |              |              |                          |
| Exam. 17 Zakharov Function [55] | 21 (8.00e-03) | 7.31 (1.65e-07)  | 21 (1.00e-03) | 4.33e+02 (1.87e+03) (failed) |
| (n = 10, m = n/2)               |               |              |              |                          |

5 Conclusions

In this paper, we give the regularization continuation method with the trust-region updating strategy (Rcmtr) for linearly equality-constrained optimization problems. Moreover, we utilizes the linear conservation law of the regularization method and the
Table 2: Numerical results of Ptctr, Rcmtr and SQP for convex problems.

| Problems                          | Ptctr (steps (time)) | Rcmtr (steps (time)) | SQP (steps (time)) |
|-----------------------------------|----------------------|----------------------|--------------------|
|                                   | J(x) (KKT)           | J(x) (KKT)           | J(x) (KKT)         |
| Exam. 1 Kim Problem 1 [26,35]     | 11 (0.56)            | 13 (0.24)            | 2 (0.36)           |
| (n = 1000, m = n/2)               | 7.27e+03 (5.79e-08)  | 7.27e+03 (3.44e-07)  | 7.27e+03 (8.30e-13) |
| Exam. 2 LLS Problem 1 [35,46]     | 17 (1.01)            | 17 (0.42)            | 13 (2.59)          |
| (n = 1200, m = n/3)               | 1.44e+03 (7.36e-07)  | 1.44e+03 (9.37e-07)  | 1.44e+03 (3.42e-07) |
| Exam. 3 Osborne Problem 1 [35,46] | 12 (1.01)            | 1 (0.55)             | 3 (1.48)           |
| (n = 1200, m = 2/3n)             | 7.15e+02 (2.30e-07)  | 7.15e+02 (1.27e-15)  | 7.14e+02 (2.22e-15) |
| Exam. 4 Mak Problem [35,41]       | 11 (0.59)            | 11 (0.47)            | 8 (1.18)           |
| (n = 1000, m = n/2)              | 97.96 (3.50e-07)     | 97.96 (7.74e-07)     | 97.96 (1.34e-10)   |
| Exam. 5 LLS Problem 2 [35]       | 14 (0.69)            | 14 (0.66)            | 11 (1.65)          |
| (n = 1000, m = n/2)              | 82.43 (8.79e-08)     | 82.43 (7.54e-08)     | 82.43 (1.76e-09)   |
| Exam. 6 Osborne Problem 2 [35,46] | 13 (1.04)            | 14 (0.97)            | 15 (5.86)          |
| (n = 1200, m = n/2)              | 5.14e+02 (1.79e-07)  | 5.14e+02 (8.75e-07)  | 5.14e+02 (1.75e-06) |
| Exam. 7 Carlberg Problem [9,35]   | 10 (0.54)            | 15 (0.74)            | 14 (1.96)          |
| (n = 1000, m = 2/3n)             | 1.19e+04 (1.23e-07)  | 1.19e+04 (1.66e-06)  | 1.19e+04 (1.13e-05) |
| Exam. 8 Kim Problem 2 [26,35]     | 12 (0.73)            | 21 (1.59)            | 29 (3.27)          |
| (n = 1000, m = n/2)              | 4.22e+04 (6.14e-06)  | 4.22e+04 (1.43e-06)  | 4.22e+04 (3.05e-06) |
| Exam. 9 Tamakshita Problem [35,61]| 16 (0.89)            | 25 (2.62)            | 14 (2.64)          |
| (n = 1200, m = n/3)              | 0.50 (4.39e-07)      | 0.50 (3.67e-07)      | 0.50 (1.01e-07)    |
| Exam. 10 Quartic With Noise Function [1] | 9 (0.42) | 7 (0.08) | 4 (0.64) | 1.01e+02 (1.25e-09) |
| (n = 1000, m = n/2)              | 1.01e+02 (3.14e-07)  | 1.01e+02 (2.69e-07)  | 1.01e+02 (1.25e-09) |
| Exam. 11 Rotated Hyper Ellipsoid Function [55] | 8 (0.72) | 6 (2.50) | 400 | 1.46e+05 (3.22e+02) (failed) |
| (n = 1000, m = n/2)              | 1.25e+05 (2.08e-04)  | 1.25e+05 (8.30e-06)  | 1.46e+05 (3.22e+02) (failed) |
| Exam. 12 Sphere Function [35]     | 10 (0.43)            | 1 (7.50e-02)         | 3 (0.44)           |
| (n = 1000, m = n/2)              | 1.67e+02 (1.11e-07)  | 1.67e+02 (3.13e-15)  | 1.67e+02 (7.67e-10) |
| Exam. 13 Sum Squares Function [35] | 17 (9.77)            | 28 (4.08)            | 400 (44.36)        |
| (n = 1000, m = n/2)              | 1.89e+04 (1.85e-04)  | 4.08e+04 (1.58e-06)  | 4.08e+04 (1.01e-02) (failed) |
| Exam. 14 Traj Function [55]       | 304 (9.18)           | 38 (2.61)            | 400 (44.05)        |
| (n = 1000, m = n/2)              | 5.82e+02 (8.34e-04)  | 5.82e+02 (5.36e-07)  | 5.82e+02 (1.56e-04) (failed) |
| Exam. 15 Booth Function [55]      | 12 (1.00e-04)        | 13 (1.00e-03)        | 17 (6.00e-03)      |
| (n = 2, m = n/2)                 | 9.00 (1.74e-07)      | 9.00 (1.98e-07)      | 9.00 (3.55e-15)    |
| Exam. 16 Matyas Function [55]     | 11 (4.00e-03)        | 17 (1.00e-04)        | 3 (5.00e-03)       |
| (n = 2, m = n/2)                 | 0.18 (1.87e-08)      | 0.18 (4.44e-07)      | 0.18 (1.67e-16)    |
| Exam. 17 Zakharov Function [55]   | 15 (6.00e-03)        | 21 (8.00e-03)        | 21 (7.00e-03)      |
| (n = 10, m = n/2)                | 7.31 (2.93e-08)      | 7.31 (1.65e-07)      | 7.31 (8.50e-06)    |

The quasi-Newton method such that it does not need to compute the correction step other than the previous continuation method. The new continuation method uses the inverse of the regularization two-sided projected Hessian matrix as the pre-conditioner to improve its robustness, which is other than the previous quasi-Newton methods. Numerical results show that Rcmtr is more robust and faster than the traditional optimization method such as SQP (the built-in subroutine fmincon.m of the MATLAB2020a envi-
Table 3: Numerical results of Ptctr, Rcmtr, SQP for large-scale nonconvex problems.

| Problems                  | Ptctr steps (time) | Ptctr f(x) (KKT) | Rcmtr steps (time) | Rcmtr f(x) (KKT) | SQP steps (time) | SQP f(x) (KKT) |
|---------------------------|-------------------|------------------|-------------------|------------------|-----------------|----------------|
| Exam. 1 LLS Problem 3 [35] (n = 1000, m = n/2) | 38 (2.45) | 1.96e+02 (1.17e-05) | 25 (10.27) | -3.03e+03 (4.86e-07) | 42 (7.70) | 1.88e+02 (7.97e-06) |
| Exam. 2 Ackly Function [55] (n = 1000, m = n/2) | 1 (0.11) | 2.64 (1.87e-07) | 1 (7.10e-02) | 2.64 (7.50e-07) | 2 (0.37) | 2.42 (1.94e-07) |
| Exam. 3 Rosenbrock Function [55] (n = 1000, m = n/2) | 9 (0.64) | 9.26e+03 (9.03e-06) | 20 (0.78) | 9.26e+03 (2.15e-06) | 400 (44.68) | 9.26e+03 (5.00e-03) (failed) |
| Exam. 4 Dixon-Price Function [55] (n = 1000, m = n/2) | 400 (15.54) | 8.97e+04 (2.42e-02) (failed) | 25 (2.35) | 9.00e+04 (1.74e-09) | 400 (46.97) | 9.26e+04 (1.28e+05) (failed) |
| Exam. 5 Griewank Function [55] (n = 1000, m = n/2) | 20 (0.73) | 0.86 (4.81e-07) | 12 (0.35) | 0.86 (4.40e-08) | 9 (1.12) | 0.86 (1.07e-10) |
| Exam. 6 Levy Function [55] (n = 1000, m = n/2) | 70 (1.83) | 71.06 (2.36e-08) | 56 (0.12) | 71.06 (8.25e-07) | 31 (3.82) | 71.06 (1.11e-07) |
| Exam. 7 Molecular Energy Function [41] (n = 1000, m = n/2) | 30 (0.94) | 4.69e+02 (4.38e-07) | 55 (0.75) | 4.69e+02 (8.66e-07) | 16 (2.04) | 4.69e+02 (1.71e-06) |
| Exam. 8 Powell Function [55] (n = 1000, m = n/2) | 11 (0.67) | 4.26e+03 (1.77e-06) | 17 (9.50e-02) | 4.26e+03 (4.52e-07) | 364 (41.34) | 4.26e+03 (1.38e-04) (failed) |
| Exam. 9 Kastegrin Function [55] (n = 1000, m = n/2) | 20 (0.63) | 2.93e+03 (6.42e-07) | 24 (0.11) | 2.93e+03 (1.56e-06) | 7 (1.00) | 4.44e+03 (1.13e-06) |
| Exam. 10 Schwefel Function [55] (n = 1000, m = n/2) | 109 (3.17) | 4.19e+05 (3.74e-07) | 71 (1.16) | 4.19e+05 (4.16e-06) | 51 (5.49) | 4.02e+05 (2.31e-06) |
| Exam. 11 Styblinski Tang Function [55] (n = 1000, m = n/2) | 76 (2.05) | -9.61e+03 (1.12e-05) | 89 (8.02) | -9.61e+03 (4.57e-06) | 172 (21.46) | -2.56e+04 (7.03e-04) (failed) |
| Exam. 12 Sperber Function [55] (n = 1000, m = n/2) | 6 (0.53) | 2.65e+03 (1.43e-06) | 8 (8.40e-02) | 2.65e+03 (8.92e-07) | 3 (0.47) | 2.65e+03 (1.42e-05) |
| Exam. 13 Streched V Function [55] (n = 1000, m = n/2) | 1 (0.39) | 3.10e-03 (1.08e+05) (failed) | 16 (18.86) | 2.89e+02 (2.41e-03) (failed) | 6 (0.95) | 1.25 (34.15) (failed) |

The regularization continuation method [42], the recent continuation method such as Ptctr [35] and the alternating direction method of multipliers (ADMM [8]). Therefore, Rcmtr is worth exploring further, and we will extend it to the nonlinearly constrained optimization problem in the future.

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Table 4: Numerical results of Ptctr, Rcmtr, SQP for small-scale nonconvex problems.

| Problems | Ptctr (time) | Rcmtr (time) | SQP (time) |
|----------|--------------|--------------|------------|
| Exam. 1 Beale Function [55] (n = 2, m = n/2) | 11 (2.00e-03) | 23 (2.00e-03) | 10 (2.00e-03) |
| Exam. 2 Branin Function [55] (n = 2, m = n/2) | 26 (4.00e-04) | 10 (1.00e-04) | 9 (8.00e-04) |
| Exam. 3 Eason Function [55] (n = 2, m = n/2) | 10 (1.00e-04) | 25 (1.00e-04) | 8 (6.00e-04) |
| Exam. 4 Mostak Function [1] (n = 2, m = n/2) | 12 (1.00e-02) | 11 (1.00e-02) | 6 (5.00e-03) |
| Exam. 5 Levy Function N. 13 [55] (n = 2, m = n/2) | 8 (4.00e-03) | 10 (1.00e-04) | 9 (8.00e-04) |
| Exam. 6 McCormick Function [55] (n = 12, m = n/2) | 12 (3.00e-03) | 13 (1.00e-04) | 5 (5.00e-03) |
| Exam. 7 Pern Function d, b [55] (n = 4, m = n/2) | 25 (4.00e-03) | 40 (1.00e-02) | 28 (8.00e-03) |
| Exam. 8 Power Sum Function [55] (n = 4, m = n/2) | 1 (6.00e-03) | 1 (2.00e-03) | 2 (6.00e-03) |
| Exam. 9 Price Function [1] (n = 2, m = n/2) | 8 (6.00e-03) | 9 (1.10e-02) | 11 (6.00e-03) |
| Exam. 10 Bobachevsky Function [55] (n = 2, m = n/2) | 9 (4.00e-03) | 11 (4.00e-03) | 9 (6.00e-03) |
| Exam. 11 Coonville Function [55] (n = 4, m = n/2) | 13 (2.00e-03) | 26 (1.00e-02) | 11 (5.00e-03) |
| Exam. 12 Drop Wave Function [55] (n = 2, m = n/2) | 10 (2.00e-03) | 9 (2.00e-03) | 6 (5.00e-03) |
| Exam. 13 Schaffer Function [55] (n = 2, m = n/2) | 14 (5.00e-03) | 13 (3.00e-03) | 8 (5.00e-03) |
| Exam. 14 3rd-Hump Camel Function [55] (n = 2, m = n/2) | 10 (4.00e-03) | 18 (2.00e-03) | 11 (5.00e-03) |
| Exam. 15 Three-Hump Camel Function [55] (n = 2, m = n/2) | 15 (4.00e-03) | 24 (1.00e-04) | 7 (4.00e-03) |
| Exam. 16 Treganoni Function [1] (n = 2, m = n/2) | 11 (4.00e-03) | 13 (1.00e-04) | 9 (4.00e-03) |
| Exam. 17 Box Bettes Exponential Quadratic Function [1] (n = 3, m = 2) | 20 (2.00e-02) | 35 (8.00e-03) | 13 (6.00e-03) |
| Exam. 18 Chachnau Function [1] (n = 2, m = n/2) | 8 (5.00e-03) | 9 (5.00e-03) | 6 (3.00e-03) |
| Exam. 19 Eggholder Function [55] (n = 2, m = n/2) | 17 (3.00e-03) | 22 (2.00e-02) | 9 (6.00e-03) |
| Exam. 20 Exp2 Function [1] (n = 2, m = n/2) | 11 (2.00e-03) | 15 (3.00e-03) | 6 (5.00e-03) |
| Exam. 21 Hansen Function [1] (n = 2, m = n/2) | 9 (6.00e-03) | 8 (2.00e-03) | 6 (5.00e-03) |
| Exam. 22 Himmelblau 3-D Function [55] (n = 3, m = 2) | 13 (2.00e-03) | 22 (2.00e-03) | 3 (4.00e-03) |
| Exam. 23 Holder Table Function [55] (n = 2, m = n/2) | 13 (2.00e-03) | 16 (2.00e-03) | 5 (5.00e-03) |
| Exam. 24 Michalewicz Function [55] (n = 2, m = n/2) | 15 (4.00e-03) | 16 (3.00e-03) | 2 (4.00e-03) |
| Exam. 25 Schaffer Function N. 4 [55] (n = 4, m = n/2) | 8 (5.00e-03) | 11 (1.00e-04) | 6 (6.00e-03) |
| Exam. 26 Trefethen 4 Function [1] (n = 2, m = n/2) | 9 (5.00e-03) | 24 (3.00e-04) | 8 (6.00e-03) |
| Exam. 27 Zettl Function [1] (n = 2, m = n/2) | 11 (4.00e-03) | 17 (1.00e-04) | 11 (4.00e-03) |
Fig. 2: The number of iterations of Ptctr, Rcmtr and SQP for test problems.

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