A Predictive SO(10) Scheme for Fermion Masses and Mixings*

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Abstract

We present a predictive scheme for fermion masses and mixings inspired by supersymmetric SO(10) in which the gauge hierarchy problem is resolved without fine tuning the parameters. There are six predictions in the flavor sector, all consistent with the present data. The scheme reproduces the familiar asymptotic relations $m_b = m_t$ and $m_d m_s m_b = m_e m_\mu m_\tau$. A new expression for $V_{cb}$ is obtained in terms of the quark masses. The remaining predictions involve the quark mixing angles $V_{us}$ and $V_{ub}$, as well as the parameter $\tan\beta$ which turns out to be close to $m_t/m_b$.

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1. Introduction

In this paper we present a scheme for fermion masses and mixings motivated by supersymmetric $SO(10)$ grand unification (GUT) in which the gauge hierarchy problem is overcome without fine tuning. There are six predictions in the flavor sector, all of which are phenomenologically consistent and compatible with the existence of a heavy top quark. A simple derivation in $SO(10)$ of the quark and lepton mass matrices that lead to these predictions will be presented.

The familiar asymptotic relations $m_b^0 = m_\tau^0$ and $m_d^0m_s^0m_b^0 = m_e^0m_\mu^0m_\tau^0$ (see [1]) are reproduced in our scheme, while a new sum rule for $m_b^0$ in terms of the quark mixing angle $V_{cb}$ is obtained. Two of the remaining three predictions are for $V_{us}$ and $V_{ub}$, resembling those from the Fritzsch scheme [2], while the third prediction is for the ratio of the two Higgs vacuum expectation values, $\tan\beta \approx m_t/m_b$ [3].

The model presented here provides a unified framework for some of the well–known and successful relations for the quark masses and mixing angles which are compatible with a heavy top quark. Since the up–quark, down–quark and the charged–lepton mass matrices are assumed to have similar forms, their derivation from an underlying $SO(10)$ GUT is relatively straightforward.

The $SO(10)$ model is the simplest grand unified example that assembles all fermions of a given family into a single irreducible representation, the spinorial $16$. As such, it has proven useful to work with $SO(10)$ in attempting to model fermion masses and mixings. It also has the potential to relate the neutrino masses and mixings to those of the quarks and charged leptons [4]. The doublet–triplet mass–splitting problem that all GUT models must address has an elegant resolution in SUSY $SO(10)$ without any fine tuning of the parameters [5,6,7]. We are also encouraged by recent attempts to derive realistic supersymmetric $SO(10)$ GUTs from superstrings in the free

\[1\] We use the superscript $^0$ to denote GUT scale quantities.
fermionic formulation [8,9].

In the $SO(10)$ model of fermion masses that we consider, we shall require that the doublet–triplet mass splitting problem be resolved without fine tuning. Furthermore, motivated by our desire to preserve the successful prediction of $\sin^2 \theta_W$, we assume that $SO(10)$ breaks directly to the minimal supersymmetric standard model (MSSM). These two requirements typically call for a non–minimal Higgs system at the GUT scale which we exploit to arrive at a predictive scheme for quark and lepton masses and mixings.

2. Model

We shall adopt the mechanism developed in Ref. [6,7] to solve the doublet–triplet splitting problem naturally in SUSY $SO(10)$, making use of an old suggestion of Dimopoulos and Wilczek [5]. This involves the coupling of Higgs $10$’s with an adjoint $45: 10_1, 45.10_2$. If the $45$ acquires a vacuum expectation value (VEV) along the $(B – L)$ direction, $\langle 45 \rangle = \text{diag.}(a, a, a, 0, 0) \times i\tau_2$, this coupling gives GUT scale masses to the color–triplets in the $10$’s, while leaving massless the $SU(2)_L$ doublets to be identified with $H_u$ and $H_d$ of MSSM. The superpotential term $(M_{\text{GUT}}^{10_2})$ will make the doublets in $10_2$ superheavy, leaving MSSM as the effective low energy theory.

The minimal Higgs sector that can break the $SO(10)$ gauge symmetry down to MSSM involves one $45$ and a spinorial $16 + \overline{16}$ [7]. However, the low energy theory below the GUT scale in this case is not quite MSSM. In particular, some of the Higgs(ino) superfields turn out to have intermediate scale masses. Here, since we require that the theory below the GUT scale be MSSM, it will be necessary to have a slightly more extended Higgs system at the GUT scale.

The Higgs system that we employ for the symmetry breaking involves two sectors. One sector breaks $SO(10)$ down to $SU(5)$. This can be achieved for example, via the following superpotential involving a $45$ ($A$) and a $16 + \overline{16}$ ($C + \overline{C}$):

$$W_1 = m_1 C C + m_2 A^2 + \lambda_1 C C A \ .$$ (1)
This superpotential induces a VEV for $C$ and $\bar{C}$ along the $SU(5)$ singlet direction, and for $A$ along $SU(5) \times U(1)_X$: $\langle A \rangle = \text{diag.}(a, a, a, a, a) \times i\tau_2$.

The second sector breaks $SO(10)$ down to $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. The superpotential involves a $54$ ($S$) and a second adjoint $A'$ [10]:

$$W_2 = m_3 S^2 + m_4 A'^2 + \lambda_2 S^3 + \lambda_3 A'^2 S.$$  \hspace{1cm} (2)

The resulting VEVs of $S$ and $A'$ are $\langle S \rangle = \text{diag.}(s, s, s, -\frac{3}{2}s, -\frac{3}{2}s) \times 1$, $\langle A' \rangle = \text{diag.}(a', a', a', 0, 0) \times i\tau_2$. The $A'$ field in this sector is also responsible for the doublet–triplet splitting via the superpotential terms involving Higgs $10$–plets (denoted by $H$ and $H'$):

$$W_3 = HA'H' + m_5 H'^2.$$  \hspace{1cm} (3)

In order to keep the masses of the electroweak Higgs doublets at the correct scale, the VEV of $A'$ should be along $(B - L)$ to a high degree of accuracy. This requires that the $A'$ coupling to the $(A, C)$ sector be very weak, since any such coupling will induce an $A'$ VEV along the $SU(5) \times U(1)_X$ singlet direction as well. Now the $(S, A')$ sector should be linked with the $(A, C)$ sector, otherwise pseudo-Goldstone bosons will result. The simplest way to link the two sectors without upsetting the VEV pattern of $A'$ is [6] by a coupling involving another adjoint: $\text{Tr}(AA'A'')$. This term, due to its complete antisymmetry, vanishes at the minimum, and thus does not affect the VEV of $A'$. Yet, it gives GUT scale masses to all the would-be pseudo-Goldstone bosons. The $A''$ superfield can have its own superpotential and its VEV can in general be written as $\langle A'' \rangle = \text{diag.}a''(1 + z, 1 + z, 1 + z, 1 - \frac{3}{2}z, 1 - \frac{3}{2}z) \times i\tau_2$. Here $z = 0$ would correspond to the VEV being along $U(1)_X$, while $z = -1 \ (2/3)$ would preserve $I_{3R} \ (B - L)$.

Since our focus here is the fermion Yukawa sector, we shall not get into the details of symmetry breaking, except to note that one can find discrete symmetries that would lead to the superpotential given in Eqs. (1)-(3) along with the $\text{Tr}(AA'A'')$ term, while preventing other terms which potentially can upset the gauge hierarchy. Note that one has the option of using gauge
singlet superfields with GUT scale VEVs to induce some of the mass terms in Eqs. (1)-(3).

We now proceed to show that the above Higgs structure can lead to a predictive set of mass matrices for the quarks and charged leptons, provided one introduces a flavor–dependent discrete symmetry.

3. Fermion Mass Matrices

In our scheme, only the third family receives mass from a renormalizable operator in the superpotential. The second family masses as well as the mixing angle arise from dimension 4 operators in the superpotential, suppressed by one inverse power of $M$, where $M$ is a scale much larger than the GUT scale. The first family masses and mixings will come from dimension 5 operators. It is entirely conceivable that these non–renormalizable operators arise from integrating out some vector–like families with masses of order $M$. The inter–generational mass hierarchy is thus related to the small ratio $(M_{\text{GUT}}/M)$, an idea discussed by numerous authors in the past [11].

In our discussions we shall not assume any special structure for these ‘effective’ operators. They will only be constrained by the $SO(10)$ gauge symmetry and some discrete flavor symmetry. In particular, if one contraction of the group indices is allowed in a given nonrenormalizable operator, we shall allow for all possible group contractions. This should make the derivation of such ‘effective’ operators from an underlying theory somewhat easier, since no special care is needed in the way the heavy fields are integrated out.

As indicated above, the third family acquires its mass from a direct coupling to the Higgs $10$–plet $H$. All the other masses will involve the adjoint fields $A$ and $A''$ with VEVs specified earlier. In order to generate the first family masses and mixings, a singlet field $X$ with a GUT scale VEV is also employed. The Yukawa superpotential is given by

$$W_{\text{Yuk}} = h_{33}\psi_3^T C\gamma_a\psi_3 H_a + h_{23}\psi_2^T C\gamma_{(abc)} \psi_3 A''_{ab} H_c + h'_{23}\psi_2^T C\gamma_a \psi_3 A''_{ab} H_b + h_{12}\psi_1^T C\gamma_{(abc)} \psi_2 A_{ab} H_c X + h'_{12}\psi_1^T C\gamma_a \psi_2 A_{ab} H_b X .$$

(4)
Here $\psi_i, (i = 1 - 3)$ stand for the three fermion families belonging to the 16 of $SO(10)$, $C$ is the $SO(10)$ charge conjugation matrix, and $\gamma_{\{abc\}}$ is the totally antisymmetric combination of the $SO(10)$ gamma matrices:

$$\gamma_{\{abc\}} = \gamma_a \gamma_b \gamma_c - \gamma_a \gamma_c \gamma_b - \gamma_b \gamma_a \gamma_c + \gamma_c \gamma_a \gamma_b + \gamma_b \gamma_c \gamma_a - \gamma_c \gamma_b \gamma_a.$$  

In Eq. (4), appropriate powers of $M$ in the denominators are to be understood.

As an example of a discrete symmetry that would lead to the Yukawa terms of Eq. (4), consider the transformations of the relevant fields under $z_5$: $(\psi_1, \psi_2, \psi_3) \sim (\omega^2, \omega, 1), A'' \sim \omega^4, X \sim \omega^2$, where $\omega^5 = 1$. This guarantees the radiative stability of the Yukawa couplings. Note that in the higher dimensional operators in Eq. (4), both the 10 and 120 ‘effective’ operators contribute, i.e., all possible contraction of the $SO(10)$ group indices have been included.

The couplings $h_{33}, h_{23}, h_{12}$ can be made real by field redefinitions, while $h'_{12}, h'_{23}$ and $z$ are in general complex parameters.

To arrive at the charged fermion mass matrices from Eq. (4), it is necessary to determine the couplings of all the components of 16, with the light doublets $(H_u, H_d)$ in $H$. This is carried out by adopting a definite set of $SO(10)$ gamma matrices [12]. The fields $H_u$ and $H_d$ in the notation of Ref. [12] are $H_u = (H_9 + iH_{10})/\sqrt{2}$ and $H_d = (H_9 - iH_{10})/\sqrt{2}$. Consequently, the $h_{23}$ term in Eq. (4) can be expanded as (after absorbing $a''/M$ into $h_{23}$, and similarly $\langle AX \rangle/M^2$ into $h_{12}$)

$$\sqrt{2}ih_{23}H_u \left[ u_2 u_3 \left( -Q_u + 2Q_w - (1 - \frac{3}{2}z) \right) + u_3 u_2 \left( -Q_u + 2Q_u - (1 - \frac{3}{2}z) \right) \right]$$

$$+ \sqrt{2}ih_{23}H_u \left[ \nu_2 \nu_3 \left( -Q_\nu + 2Q_\nu - (1 - \frac{3}{2}z) \right) + \nu_3 \nu_2 \left( -Q_\nu + 2Q_\nu - (1 - \frac{3}{2}z) \right) \right]$$

$$+ \sqrt{2}ih_{23}H_d \left[ d_2 d_3 \left( -Q_d + 2Q_d - (1 - \frac{3}{2}z) \right) + d_3 d_2 \left( -Q_d + 2Q_d - (1 - \frac{3}{2}z) \right) \right]$$

$$+ \sqrt{2}ih_{23}H_d \left[ e_2 e_3 \left( -Q_e + 2Q_e - (1 - \frac{3}{2}z) \right) + e_3 e_2 \left( -Q_e + 2Q_e - (1 - \frac{3}{2}z) \right) \right].$$  

(6)
Similarly, the $h'_{23}$ term gives rise to
\[
\sqrt{2}h'_{23}(1-\frac{3}{2}z) \left[ -H_u(u_2u'_3 + u_3u'_2 + \nu_2\nu'_3 + \nu_3\nu'_2) + H_d(d_2d'_3 + d_3d'_2 + e_2e'_3 + e_3e'_2) \right].
\] (7)

Here we have defined the “charges” of quarks and leptons to be $Q = X + 6 \left(\frac{Y}{2}\right)$, where $X$ is the $U(1)_X$ charge normalized such that the $16$ of $SO(10)$ decomposes into $SU(5) \times U(1)_X$ as $16 \rightarrow 1_5 + 5_{-3} + 10_1$. These charges are given by
\[
Q_u = 1 + z, \quad Q_{u^c} = 1 - 4z, \quad Q_d = 1 + z, \quad Q_{d^c} = -3 + 2z, \\
Q_e = -3 - 3z, \quad Q_{e^c} = 1 + 6z, \quad Q_\nu = -3 - 3z, \quad Q_{\nu^c} = 5.
\] (8)

We can now write down the mass matrices for the up–quark, down quark and charged leptons. Let us define $a_{23} \equiv \sqrt{2}ih_{23}, \ r_{23}a_{23} \equiv \sqrt{2}ih'_{23}(1 - \frac{3}{2}z)$, $a_{12} \equiv \sqrt{2}ih_{12}, r_{12} \equiv \sqrt{2}ih'_{12}$. The mass matrices are then given by
\[
M_u = v_u \begin{pmatrix} 0 & -a_{12}r_{12} & 0 \\ -a_{12}r_{12} & 0 & -a_{23}(\frac{15}{2}z + r_{23}) \\ 0 & -a_{23}(\frac{15}{2}z + r_{23}) & a_{33} \end{pmatrix}
\]
\[
M_d = v_d \begin{pmatrix} 0 & a_{12}(-6 + r_{12}) & 0 \\ a_{12}(6 + r_{12}) & 0 & a_{23}(-6 + \frac{3}{2}z + r_{23}) \\ 0 & a_{23}(6 + \frac{3}{2}z + r_{23}) & a_{33} \end{pmatrix}
\]
\[
M_t = v_d \begin{pmatrix} 0 & a_{12}(-6 + r_{12}) & 0 \\ a_{12}(-6 + r_{12}) & 0 & a_{23}(6 + \frac{7}{2}z + r_{23}) \\ 0 & a_{23}(6 + \frac{7}{2}z + r_{23}) & a_{33} \end{pmatrix}
\] (9)

A few remarks about (9) are in order. By construction, they have a Fritzsch–like texture. However, there is an important difference in that the matrices are not symmetric. This is because both the symmetric $10$ and the antisymmetric $120$ ‘effective’ operators contribute to the $(12)$ and $(23)$ entries in (9). The net contribution is then neither symmetric nor antisymmetric. This implies in particular that the top–quark mass in our present scheme can
be larger than the 150 GeV limit [13] set by the Fritzsch ansatz (the limit is \( \leq 90 \text{ GeV} \) without the renormalization group considerations [14]). The matrices in (9) admit a top quark as heavy as about 200 GeV.

Another point to note is that the VEV of \( A'' \), being proportional to the parameter \( z \), breaks the \( SU(5) \) symmetry. Thus the bad predictions of minimal \( SU(5) \), namely \( m_0^s = m_0^\mu \) and \( m_0^d = m_0^e \) will be corrected.

As noted earlier, the parameters \((z, r_{12}, r_{23})\) are in general complex. In the analysis that follows, we shall assume that CP is a good symmetry of the Lagrangian. It can be spontaneously broken by the VEV’s of \( A, A'' \) or \( X \). It is easy to see that if CP is a good symmetry of the superpotential in (1)-(3), it admits a solution where \( \langle A \rangle \) and \( \langle A'' \rangle \) are real. If this solution is chosen, the only possible source of CP violation in the mass matrix is in the phase of the singlet field \( X \). If two such fields are present, there is, in general, a phase in \( r_{12} \) which cannot be rotated away. We shall assume in the analysis that follows that \( r_{12} \) is complex, while \((z, r_{23})\) are real.

The assumption of spontaneous CP violation is of course motivated by the desire to reduce the number of arbitrary parameters. Within the context of supersymmetry, there is another reason. It is a well known problem that the new phases in the soft SUSY breaking sector of the MSSM, unless somehow suppressed, will lead to unacceptably large values of the neutron and electron electric dipole moments. One way to suppress these potentially dangerous contributions is to assume that CP violation has a spontaneous origin [15].

With only \( r_{12} \) complex, there are 8 parameters \((a_{33}, a_{23}, a_{12}, r_{23}, r_{12}, \alpha, z, v_u/v_d)\) in Eq. (9) \((\alpha \text{ is the phase of } r_{12})\) to fit 14 observables (6 quark masses, 3 lepton masses, 3 quark mixing angles, one CP phase and \( \tan \beta \)). This results in 6 predictions in the flavor sector as advocated.

Two of the predictions of the model are

\[
m_b^0 \simeq m_\tau^0; \quad \tan \beta \simeq \frac{m_t}{m_b}\]

which can be seen by considering the \((33)\) elements of \( M_{u,d,l} \). The resulting \( b \)-quark mass at low energies is known to be consistent with data [3].
Another prediction of the model is

\[ m_d^0 m_s^0 m_b^0 = m_e^0 m_\mu^0 m_\tau^0 \]  

(11)

which follows from the determinants of \( M_d \) and \( M_l \). This is one of the Georgi–Jarlskog relations [1], which also fits the low energy data quite well.

The remaining predictions have to do with the three quark mixing angles. The first one relates \( V_{cb} \) to ratios of quark and lepton masses. This relation is new, and so we shall explain its derivation in some detail. To arrive at it one can safely ignore the masses and mixings involving the first family. Define

\[ P \equiv \eta_P \left| \frac{m_d^0 m_b^0}{m_t^0 m_s^0} \right| ; \quad Q \equiv \eta_Q \left| \frac{m_s^0 m_\tau^0}{m_\mu^0 m_\mu^0} \right| \]  

(12)

where \( \eta_{P,Q} = \pm 1 \), depending on the signs of the fermion masses. One can see that

\[ z = \frac{2(1 - Q)}{3 + 7Q - 10PQ} , \]  

(13)

\[ r_{23}^2 = \frac{225 [(1 + Q - 4PQ)^2 - 4Q]}{(3 + 7Q - 10PQ)^2} . \]  

(14)

Numerically, the parameters \(|P|\) and \(|Q|\) lie in the range

\[ |P| = \left( \frac{1}{5.4} \text{ to } \frac{1}{7.0} \right) ; \quad |Q| = \left( \frac{1}{2.9} \text{ to } \frac{1}{3.7} \right) \]  

(15)

where we have used the relation \( m_d^0 m_d^0 = m_\mu^0 m_\mu^0 \), extrapolated the light quark masses from low energies to the top mass scale (\( \sim 160 \text{ GeV} - 190 \text{ GeV} \)) using \( \alpha_s(M_Z) = 0.12 \), and imposed the phenomenological constraint \( m_s/m_d = (15 \text{ to } 25) \).

Since \( r_{23}^2 > 0 \),

\[ |1 + Q - 4PQ| \geq 2\sqrt{|Q|} . \]  

(16)

Depending on the sign factors \( \eta_P \) and \( \eta_Q \) there are then four possibilities,
(i) For $\eta_P = +, \eta_Q = +$, Eq. (16) translates into an inequality\footnote{For brevity, we denote $m_t^0$ by $t^0$ and so on.}

\[
\frac{|t^0|}{c^0} \geq \frac{4|\varphi^0|}{\left(1 - \left|\frac{e_0}{\mu_0}d_0^0\right|^{1/4}\right)^2}.
\]  

(17)

This leads to a lower limit on $m_t^{\text{phys}}$ of about 165 GeV.

(ii) If $\eta_P = -, \eta_Q = -$, then

\[
|P| \leq \frac{(1 - |Q| - 2\sqrt{|Q|})}{4|Q|}
\]

(18)

which cannot be satisfied since the right hand side is negative.

(iii) If $\eta_P = +, \eta_Q = -$,

\[
|1 - |Q| + 4|PQ|| \geq 2\sqrt{|Q|},
\]

leading to the constraint

\[
\frac{|t^0|}{c^0} \leq \frac{4|\varphi^0|}{\left(2\left|\frac{e_0}{\mu_0}d_0^0\right|^{1/4} + \left|\frac{e_0}{\mu_0}d_0^0\right|^{1/2} - 1\right)}.
\]  

(20)

This results in an upper limit on $m_t^{\text{phys}}$ of about 125 GeV which is inconsistent with the data.

(iv) If $\eta_P = -, \eta_Q = +$, Eq. (16) leads to the inequality $1 + |Q| + 4|PQ| \geq 2\sqrt{|Q|}$, which is automatically satisfied.

The allowed solutions therefore are $\eta_P = \pm, \eta_Q = +$. Using these sign factors, the asymptotic expression for $V_{cb}$ can be readily obtained. It is given by the relation $|V_{cb}^0| = |a_{23}/a_{33}||6 - 9z - 2r_{23}|$, with $|a_{23}/a_{33}| = |c^0/t^0|^{1/2}/|r_{23} - (15z/2)^2|^{1/2}$. Using (13),(14), this yields ($x \equiv |c^0\mu^0/d^0|^{1/2}$, $y \equiv |c^0\tau^0/t^0\mu^0|$)

\[
|V_{cb}^0| = \frac{1}{\sqrt{2}} \left|\frac{\mu^0}{\tau^0}\right|^{1/2} \frac{2x - 2\eta_py \pm ((1 + x - 4\eta_py)^2 - 4x)^{1/2}}{(1 + x - 2\eta_py)^{1/2}}.
\]

(21)
Here we have used the relation $|m_\nu^0/m_\mu^0| = |m_\nu^0/m_\mu^0|^1/2$ which is a consequence of the identity $|m_d^0 m_s^0| = |m_s^0 m_\mu^0|$. The $\pm$ sign in (21) corresponds to choosing $r_{23}$ to be $\mp$. Only the positive sign for $r_{23}$ will lead to an acceptable $V_{cb}$.

It is instructive to approximate (21) in the limit of an infinite top–quark, even though the finite top mass effects will turn out to be significant. In this limit, $|V_{cb}^0|$ can be written as (for $r_{23}$ positive)

$$|V_{cb}^0| \simeq \frac{1}{\sqrt{2}} |\mu_0^0|^{1/2} \frac{1}{\sqrt{1 + x}} \left[ (3x - 1) + \eta_F \frac{c_0^0 \tau_0^0}{t_0^0 \mu_0^0} \left( \frac{1 + 12x + 3x^2}{1 - x^2} \right) \right].$$  \hspace{1cm} (22)

Note that in the strict Georgi–Jarlskog limit, viz., $|\mu_0^0| = 3|s_0^0|$, $|d_0^0| = 3|e_0^0|$, $x = 1/3$ and the dominant $(3x - 1)$ term in (22) vanishes. As a consequence, the model admits very low values of $|V_{cb}|$.

Another useful expression for $|V_{cb}^0|$ which resembles the Fritzsch relation is

$$|V_{cb}^0| = \sqrt{\frac{s_0^0}{b_0^0} \left( \frac{6 - \frac{3}{2} z - r_{23}}{6 - \frac{3}{2} z + r_{23}} \right)^{1/2} - \sqrt{\frac{c_0^0}{t_0^0} \frac{15z + r_{23}}{t_0^0}} \left( \frac{1}{2} - \frac{1}{2} \right)^{1/2}}. \hspace{1cm} (23)$$

It is evident from (23) that for $r_{23} \geq 0$, the $\sqrt{s_0^0/b_0^0}$ term has a suppression factor while the $\sqrt{c_0^0/t_0^0}$ term is enhanced (for $z \simeq 1/4$), thereby yielding values of $|V_{cb}^0|$ that are significantly smaller than those from the Fritzsch ansatz.

Relation (21) for $|V_{cb}^0|$ can be extrapolated to low energies by using the renormalization group equations corresponding to $\tan \beta \simeq m_t/m_b$. The running factors for the relevant quantities to go from the weak to the GUT scale can be expressed analytically as (for $\tan \beta \simeq m_t/m_b$)

$$\eta_{KM} = \left( 1 - \frac{Y_t}{Y_f} \right)^{1/7}; \eta_{s/b} = \eta_{c/t} = \left( 1 - \frac{Y_t}{Y_f} \right)^{2/7};$$

$$\eta_{\mu/\tau} = \left( \frac{\alpha_1}{\alpha_G} \right)^{1879/37224} \left( \frac{\alpha_3}{\alpha_G} \right)^{2/3} \left( \frac{Y_t}{Y_\tau} \right)^{-3/8} \left( 1 - \frac{Y_t}{Y_\tau} \right)^{3/14}. \hspace{1cm} (24)$$

Here $Y_t = h_t^2$ is the square of the top-quark Yukawa coupling, $Y_f (\simeq 1.1)$ is the weak scale value of $Y_t$ corresponding to ‘infinite’ value at the GUT scale, and $Y_\tau$ is the weak scale value of the $\tau$ Yukawa coupling–squared.
The predicted low energy values for $|V_{cb}|$ are plotted in Figures 1 and 2 (corresponding to $\eta_P = \pm 1$) as a function of the mass ratio $|m_s/m_d|$. We have used the three-loop QCD and 1 loop QED beta and gamma functions to evolve the light quark masses from low energies to the top-quark threshold (which is also assumed to be the SUSY threshold). These running factors, corresponding to $\alpha_s(M_Z) = 0.12$ are $(\eta_u, \eta_{d,s}, \eta_c, \eta_b, \eta_{e,\mu,\tau}) = (0.401, 0.404, 0.460, 0.646, 0.982, 0.984)$, where the low energy scale is taken to be $1$ GeV for $(u,d,s)$, $1.27$ GeV for $c$, and $4.25$ GeV for $b$. In the square root in Eq. (21), only the relative negative sign (corresponding to $r_{23} \geq 0$) leads to an acceptable value of $|V_{cb}|$. This relative minus sign has been used in Figures 1 and 2. Clearly, both signs of $\eta_P$ result in a consistent prediction for $|V_{cb}|$. A 'central' value of $V_{cb} = 0.04$ corresponds to the prediction $m_s/m_d = (18, 19.5, 24)$ for $\eta_P = +1$ and $m_s/m_d = (22, 21, 17.5)$ for $\eta_P = -1$, with $m_t^{\text{phys}} = (165, 174, 190)$ GeV respectively.

Finally, the remaining two mixing angle predictions of the model are

$$|V^0_{us}| \approx \sqrt{\frac{d^0}{s^0} \left(1 - \frac{|V^0_{cb}|}{|V^0_{ub}|} \frac{c^0_s}{s^0} \frac{1}{1/2} e^{i\alpha} \right) - \eta_P e^{i\alpha} \sqrt{\frac{u^0}{c^0}}}$$

$$|V^0_{ub}| = \sqrt{\frac{u^0}{c^0} V^0_{cb} - e^{i\alpha} \frac{s^0}{b^0} \frac{d^0}{b^0} \frac{1}{6 - r_{12}} \frac{1}{6 + r_{12}} \frac{6 - \frac{3}{2} Z + r_{23}}{6 - \frac{3}{2} Z - r_{23}}} \right)^{1/2} \right) \right). \quad (25)
$$

In the expression for $|V^0_{ub}|$, the second term is numerically smaller than the first term, although not negligible. Taking the phase $\alpha \simeq \pi/2$, we find that $|V_{ab}| \simeq (0.002 \sim 0.003)$. This and the $|V_{us}|$ prediction resemble those from the Fritzsch ansatz, and are quite consistent with the present data. The CP parameter turns out to be $J \approx (2 - 3) \times 10^{-5}$.

4. Conclusions

We have presented a scheme for fermion masses inspired by supersymmetric $SO(10)$ in which the gauge hierarchy is implemented without fine tuning. This typically calls for a non–minimal Higgs sector which we exploit in de-
riving expression (9) for the mass matrices. Assuming spontaneous CP vio-
lution, we are led to 6 predictions in the flavor sector which work very well,
especially with a heavy top quark.

There exist several ansatzes for the fermion mass matrices in the literature
[16-21]. The Fritzsch ansatz is one of the simplest and therefore attractive.
However, it would appear [13] that this ansatz is excluded by the recent
Fermilab data on the top quark mass. The Georgi–Jarlskog mass matrices
have generated renewed attention [16-17]. They lead to 6 predictions in
the flavor sector. Owing to the up–down asymmetry in this scheme, the
derivation of these mass matrices from an underlying theory is somewhat
nontrivial [17]. In comparison, the scheme presented here can be obtained
from SUSY GUTs without too much effort. There are some approaches with
more than six flavor predictions [19], and it would be interesting to see the
realization of these matrices from an underlying GUT or related symmetries.

As for the neutral sector, small neutrino masses can be easily accom-
modated in our scheme. The Dirac neutrino mass matrix is completely de-
termined in the model (see Eq. (6)). As for the right handed ($\nu_R$) Ma-
ajorana mass matrix, we find that there are three choices which lead to a
predictive neutrino spectrum [4,21]. They correspond to the non–zero en-
tries in the Majorana matrix being \{$(11), (23), (32)$\}, or \{(22), (13), (31)$\} or
\{$(33), (12), (21)$\}, all of which result in a non–singular $\nu_R$ matrix. In each
case there is a one–parameter family of solutions for the neutrino mass ra-
tios and the mixing angles. We plan to discuss the detailed phenomenology
of such spectra and their implications for neutrino oscillations in a separate
paper.

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Figure Captions

Fig. 1. A plot of $|V_{cb}|$ versus $|m_s/m_d|$ corresponding to $\eta_P = +1$. $\alpha_s(M_Z) = 0.12$ has been used along with $m_c(m_c) = 1.27 \text{ GeV}$. The three curves correspond to three different values of $m_t^{\text{phys}}$.

Fig. 2. Same plot as in Fig. 1, but for $\eta_P = -1$. 