COLEMAN-GURTIN TYPE EQUATIONS WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. We present a new formulation and generalization of the classical theory of heat conduction with or without fading memory which includes the usual heat equation subject to a dynamic boundary condition as a special case. We investigate the well-posedness of systems which consist of Coleman-Gurtin type equations subject to dynamic boundary conditions, also with memory. Nonlinear terms are defined on the interior of the domain and on the boundary and subject to either classical dissipation assumptions, or to a nonlinear balance condition in the sense of [11]. Additionally, we do not assume that the interior and the boundary share the same memory kernel.

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1. Introduction

In recent years there has been an explosive growth in theoretical results concerning dissipative infinite-dimensional systems with memory including models arising in the theory of heat conduction in special materials and the theory of phase-transitions. The mathematical and physical literature, concerned primarily with qualitative/quantitative properties of solutions to these models, is quite extensive and much of the work before 2002 is largely referenced in the survey paper by Grasselli and Pata [19]. More recent results and updates can be found in [6] [8] [9] [10] (cf. also [16] [17]). A basic evolution equation considered in these references is that for an homogeneous and isotropic heat conductor occupying a d-dimensional (bounded) domain $\Omega$ with sufficiently smooth boundary $\Gamma = \partial \Omega$ and reads

$$\partial_t u - \omega \Delta u - (1 - \omega) \int_0^\infty k_\Omega (s) \Delta u (x, t - s) ds + f (u) = 0, \quad (1.1)$$

in $\Omega \times (0, \infty)$. Here $u = u (t)$ is the (absolute) temperature distribution, $\omega > 0$, $r = -f (u (t))$ is a temperature dependent heat supply, and $k_\Omega : [0, \infty) \to \mathbb{R}$ is a continuous nonnegative function, smooth on $(0, \infty)$ and vanishing at infinity, and summable. As usual, (1.1) is derived by assuming the following energy balance equation

$$\partial_t e + \text{div} (q) = r$$

Date: October 30, 2014.

2000 Mathematics Subject Classification. 35B25, 35B40, 35B41, 35K57, 37L30, 45K05.

Key words and phrases. Coleman-Gurtin equation, dynamic boundary conditions with memory, heat conduction, heat equations.
by considering the following relationships:
\[ e = e_\infty + c_0 u, \quad q = -\omega \nabla u - (1 - \omega) \int_0^\infty k_\Omega (s) \nabla u(x, t - s) \, ds, \]

for some constants \( e_\infty, c_0 > 0 \). Equation (1.1) is always subject to either homogeneous Dirichlet (\( u = 0 \)) or Neumann boundary conditions (\( \partial_n u = 0 \)) on \( \Gamma \times (0, \infty) \). The first one asserts that the temperature is kept constant and close to a given reference temperature at \( \Gamma \) for all time \( t > 0 \), while the second “roughly” states that the system is thermally isolated from outside interference. This equation is also usually supplemented by the “initial” condition \( u : (-\infty, 0) \to \mathbb{R} \) such that
\[ u|_{t \in (-\infty, 0]} = \bar{u} \text{ in } \Omega. \]

These choices of boundary conditions, although help simplify substantially the mathematical analysis of (1.1)-(1.3), are actually debatable in practice since in many such systems it is usually difficult, if not impossible, to keep the temperature constant at \( \Gamma \) for all positive times without exerting some additional kind of control at \( \Gamma \) for \( t > 0 \). A matter of principle also arises for thermally isolated systems in which, in fact, the correct physical boundary condition for (1.1) turns out to be the following
\[ q \cdot n = \omega \partial_n u + (1 - \omega) \int_0^\infty k_\Omega (s) \partial_n u(x, t - s) \, ds = 0 \text{ on } \Gamma \times (0, \infty), \]

see, for instance, [3, Section 6]. Indeed, the condition \( \partial_n u = 0 \) on \( \Gamma \times (0, \infty) \) implies (1.2), say when \( u \) is a sufficiently smooth solution of (1.1)-(1.2), but clearly the converse cannot hold in general.

In the classical theory of heat conduction, it is common to model a wide range of diffusive phenomena including heat propagation in homogeneous isotropic conductors, but generally it is assumed, as above, that surface (i.e., boundary) conditions are completely static or stationary. In some important cases this perspective neglects the contribution of boundary sources to the total heat content of the conductor. A first step to remedy this situation was done in Goldstein [18] for heat equations. The approach presented there introduces dynamic boundary conditions into an ad hoc fashion and lacks some rigor in the case of reaction-diffusion equations. In the next section of the paper we will make use of the usual physical principles and present a new formulation and generalization of the classical theory. Our general approach follows that of Coleman and Mizel [5] which regards the second law of thermodynamics as included among the laws of physics and which is compatible with the principle of equipresence in the sense of Truesdell and Toupin (see Section 2). Thus, this new formulation is expected to give a solid foundation to the arguments employed in derivations of the heat equation with “dynamic” boundary conditions developed in Goldstein [18], or in models for phase transitions developed in Gal and Grasselli [13,14]. Accounting for the presence of boundary sources, the new formulation naturally leads to dynamic boundary conditions for the temperature function \( u \) and that contain the above static conditions (especially, (1.4)) as special cases (see Section 2). In particular, we derive on \( \Gamma \times (0, \infty) \), the following boundary condition for (1.1):
\[ \partial_t u - \nu \Delta_\Gamma u + \omega \partial_n u + g(u) \]
\[ + (1 - \omega) \int_0^\infty k_\Omega (s) \partial_n u(x, t - s) \, ds + (1 - \nu) \int_0^\infty k_\Gamma (s) (-\Delta_\Gamma + \beta) u(x, t - s) \, ds \]
\[ = 0, \]

for some \( \nu \in (0, 1) \) and \( \beta > 0 \). Here \( k_\Gamma : [0, \infty) \to \mathbb{R} \) is also a smooth nonnegative, summable function over \( (0, \infty) \) such that \( k_\Gamma \) is vanishing at infinity. The last two boundary terms on the left-hand side of equation (1.5) are due to contributions coming from a (linear) heat exchange rate between the bulk \( \Omega \) and the boundary \( \Gamma \), and boundary fluxes, respectively (cf. Section 2).

Our goal in this paper is to extend the previous well-posedness results of [7,8,9,10,12,15,17] and [11,12,15] in the following directions:

- by allowing general boundary processes take place also on \( \Gamma \), equation (1.1) is now subject to boundary conditions of the form (1.5);
- we consider more general functions \( f, g \in C^1(\mathbb{R}) \) satisfying either classical dissipation assumptions, or more generally, nonlinear balance conditions allowing for bad behavior of \( f, g \) at infinity;
- we develop a general framework allowing for both weak and smooth initial data for (1.1), (1.5), and possibly different memory functions \( k_\Omega, k_\Gamma \).
we extend a Galerkin approximation scheme whose explicit construction is crucial for the existence of strong solutions.

The paper is organized as follows. In Section 3 we provide the functional setup. In Section 4 we prove theorems concerning the well-posedness of the system, based on (1.4), (1.5), generated by the new formulation. In the subsequent section, we present a rigorous formulation and examples in which (1.5) naturally occurs for (1.4).

2. Derivation of the model equations

To begin let us consider a bounded domain \( \Omega \subset \mathbb{R}^d \) which is occupied by a rigid body. The region \( \Omega \) is assumed to be bounded by a smooth boundary \( \Gamma := \partial \Omega \) which is assumed to be at least Lipschitz continuous. As usual, a thermodynamic process taking place in \( \Omega \) is defined by five basic functions, that is, the specific internal energy \( e \), the absolute temperature \( u = u(x,t) \), the heat flux \( q = q(x,t) \), the heat supply \( h(x,t) \), absorbed by the material at \( x \in \Omega \), and possibly furnished by the external world (i.e., thermodynamic processes that occur outside of \( \Omega \)). All these quantities, defined per unit volume and unit time, are scalars except for \( q \in \mathbb{R}^d \) which is a vector.

The classical theory \([4, 5]\) of heat conduction in the body \( \Omega \) ignores any heat contribution which may be supplied from processes taking place on \( \Gamma \) and, hence, this situation is never modelled by the theory. This is the case in many applications, in calorimetry, which go back to problems that occur as early as the mid 1950’s, see [3, Chapter I, Section 1.9, pg. 22-24]. A typical example arises when a given body \( \Omega \) is in perfect thermal contact with a thin metal sheet, possibly of different material \( \Gamma = \partial \Omega \) completely insulating the body \( \Omega \) from contact with, say, a well-stirred hot or cold fluid. The assumption made is that the metal sheet \( \Gamma \) is sufficiently thin such that the temperature \( v(t) \) at any point on \( \Gamma \) is constant across its thickness. Since the sheet \( \Gamma \) is in contact with a fluid it will either heat or cool the body \( \Omega \) in which case the heat supplied to \( \Omega \) is due to both \( \Gamma \) and the body of fluid, not to mention the fact that the temperature distribution in the sheet is also affected by heat transfer between \( \Gamma \) and the interior \( \Omega \). Since the outershell \( \Gamma \) is in perfect contact with the body \( \Omega \), it is reasonable to assume by continuity that the temperature distribution \( u(t) \) in \( \Omega \), in an infinitesimal layer near \( \Gamma \) is equal to \( v(t) \), for all times \( t > \delta \), that is, \( u(t)|_\Gamma = v(t) \) for all \( t > \delta \); they need not, of course, be equal at \( t = \delta \), where \( \delta \) is the (initial) starting time. When \( \rho_1, \rho_2 \) correspond to the densities of \( \Omega \) and \( \Gamma \), respectively, and \( c_1, c_2 \) denote the heat capacities of \( \Omega \) and \( \Gamma \), respectively, this example can be modelled by the balance equation

\[
\rho_1 c_1 \partial_t u = -\operatorname{div}(q) + h_\Omega \quad \text{in} \quad \Omega \times (\delta, \infty),
\]

suitably coupled with an equation for \( \Gamma \), by considering the heat balance of an element of area of the sheet \( \Gamma \), which is

\[
\rho_2 c_2 \partial_t u = q \cdot n - \operatorname{div}_\Gamma(q_\Gamma) + l_\Gamma \quad \text{in} \quad \Gamma \times (\delta, \infty).
\]

Here \( n \in \mathbb{R}^d \) denotes the exterior unit normal vector to \( \Gamma \), \( l_\Gamma(x,t) \) is an external heat supply and \( q_\Gamma \) is a tangential heat flux on \( \Gamma \) while \( \operatorname{div}_\Gamma \) is the surface divergence whose definition is given below. Note that the correct term to couple the balance equations for \( \Omega \) and \( \Gamma \) is given by \( q \cdot n \), since this is used to quantify a (linear) heat exchange rate across \( \Gamma \) from \( \Omega \) in all directions normal to the boundary \( \Gamma \). The system (2.1)–(2.2) is also important in control problems for the heat equation, say when a specific temperature distribution at the boundary \( \Gamma \) is desired (see [21]).

As mentioned earlier, in the classical theory on heat conduction one usually ignores boundary contributions by either prescribing the temperature on \( \Gamma \) or assuming that the flux across the surface \( \Gamma \) from \( \Omega \) is null, or simply, by invoking Newton’s law of cooling which states that the flux across the surface is directly proportional to temperature differences between the surface and the surrounding medium. In the sequel, it is our goal to include general boundary processes into the classical theory of heat conduction. To this end, in order to define a complete thermodynamic process in \( \Omega = \Omega \cup \Gamma \), as in the previous example, we need to add four more response functions, that is, the specific surface energy \( e_\Gamma(x,t) \), the specific surface entropy density \( \eta_\Gamma(x,t) \), the tangential heat flux \( q_\Gamma(x,t) \in \mathbb{R}^{d-1} \), and the external heat supply \( h_\Gamma(x,t) \), all defined for \( x \in \Gamma \), per unit area and unit time. It is assumed that the absolute (local) temperature \( u(t) \) is sufficiently smooth up to \( \Omega \) as a function of the spatial coordinate. We now introduce the following definition.
• We say that the set of nine time-dependent variables constitutes a complete thermodynamic process in \( \overline{\Omega} \) if the following conservation law holds, not only for \( \Omega \), but also for any subdomain \( \Omega_0 \subset \Omega \) and any part \( \Gamma_0 \subset \Gamma \):

\[
\int_{\Omega} \dot{\varepsilon}_{\Omega} dx + \int_{\Gamma} \dot{\varepsilon}_{\Gamma} d\sigma = - \int_{\Omega} \text{div}(q) dx - \int_{\Gamma} \text{div}_{\Gamma}(q_{\Gamma}) d\sigma + \int_{\Omega} h_{\Omega} dx + \int_{\Gamma} h_{\Gamma} d\sigma. \tag{2.3}
\]

In (2.3), \( dx \) denotes the volume element, \( d\sigma \) is the element of surface area and the superimposed dot denotes the time-derivative. Note that in general, the external heat supply \( h_{\Gamma} \) on \( \Gamma \) must also depend, possibly in a nonlinear fashion, on the heat content exchanged across \( \Gamma \) from \( \Omega \), i.e., \( h_{\Gamma} = f(q \cdot n) + l_{\Gamma} \), for some function \( f \), where \( l_{\Gamma} \) accounts either for the heat supply coming solely from \( \Gamma \) or some other source outside of \( \Gamma \), see the above example (2.1)-(2.2). In order to give a rigorous definition to \( \text{div}_{\Gamma}(q_{\Gamma}) \), we regard \( \Gamma \) as a compact Riemannian manifold without boundary, endowed with the natural metric inherited from \( \mathbb{R}^d \), given in local coordinates by \( \tau \) and with fundamental form \((\tau_{ij})_{i,j=1,...,d-1} \). A scalar-valued function \( w \in C^\infty(\Gamma) \) induces an element of the dual space of \( T_{\Gamma} \) via the directional derivative of tangential vectors at \( x \in \Gamma \). Clearly, \( T_{\Gamma} \) is a Hilbert space when endowed with scalar product induced from \( \mathbb{R}^d \). For a tangential vector field \( q_{\Gamma} \in C^\infty(\Gamma) \), that is, \( q_{\Gamma}(x) \in T_{\Gamma} \), for \( x \in \Gamma \), the surface divergence, \( \text{div}_{\Gamma}(q_{\Gamma}) \), is in the local coordinates \( \tau \) for \( \Gamma \),

\[
\text{div}_{\Gamma}(q_{\Gamma})(\tau) = \frac{1}{\sqrt{|\tau|}} \sum_{i=1}^{d-1} \partial_i (\sqrt{|\tau|} q_i(\tau)),
\]

where \( q_i \) are the components of \( q_{\Gamma} \) with respect to the basis \( \{\partial_1 \tau,...,\partial_{d-1} \tau\} \) of \( T_{\Gamma} \) and \( |\tau| = \det(\tau_{ij}) \).

Moreover, we can define the surface gradient \( \nabla_{\Gamma} u \) as a unique element of \( T_{\Gamma} \) corresponding to this dual space element via a natural isomorphism, that is,

\[
\nabla_{\Gamma} u(\tau) = \sum_{i,j=1}^{d-1} \tau_{ij} \partial_i u(\tau) \partial_j \tau,
\]

with respect to the canonical basis \( \{\partial_1 \tau,...,\partial_{d-1} \tau\} \) of \( T_{\Gamma} \). For a multi-index \( \alpha \in N_0^d \), the operator \( \nabla_{\Gamma}^\alpha u \) is defined by taking iteratively the components of \( \nabla_{\Gamma} u \). It is worth emphasizing that our form of the first law (2.3) is equivalent to

\[
\dot{\varepsilon}_{\Omega} = -\text{div}(q) + h_{\Omega} \text{ in } \Omega, \text{ and } \dot{\varepsilon}_{\Gamma} = -\text{div}_{\Gamma}(q_{\Gamma}) + h_{\Gamma} \text{ on } \Gamma, \tag{2.4}
\]

under suitable smoothness assumptions on the response functions involved in (2.3). Equation (2.3) may be called the law of conservation of total energy or the extended First Law of Thermodynamics. For each such complete thermodynamic process, let us define the total rate of production of entropy in \( \bar{\Omega} = \Omega \cup \Gamma \) to be

\[
\Upsilon := \int_{\Omega} \dot{\varepsilon}_{\Omega} dx + \int_{\Gamma} \dot{\varepsilon}_{\Gamma} d\sigma - \int_{\Omega} h_{\Omega} dx + \int_{\Omega} \frac{q}{u} dx + \int_{\Gamma} \frac{q_{\Gamma}}{u} d\sigma - \int_{\Gamma} \frac{h_{\Gamma}}{u} d\sigma, \tag{2.5}
\]

where we regard \( q/u \) as a vectorial flux of entropy in \( \Omega \), \( h_{\Omega}/u \) as a scalar supply of entropy produced by radiation from inside the body \( \Omega \), \( h_{\Gamma}/u \) is viewed as a scalar supply of entropy produced by radiation from \( \Gamma \) and \( q_{\Gamma}/u \) is a tangential flux of entropy on \( \Gamma \). More precisely, we define \( \Upsilon \) to be the difference between the total rate of change in entropy of \( \bar{\Omega} \) and that rate of change which comes from the heat supplies in both \( \Omega \) and \( \Gamma \), and both the inward and tangential fluxes. We postulate the following extended version of the Second Law of Thermodynamics as follows.

• For every complete thermodynamic process in \( \bar{\Omega} \) the inequality

\[
\Upsilon \geq 0 \tag{2.6}
\]

must hold for all \( t \), not only in \( \bar{\Omega} \), but also on all subdomains \( \Omega_0 \subset \Omega \) and all parts \( \Gamma_0 \subset \Gamma \), respectively.\footnote{When (2.6) holds on all parts \( \Omega_0 \subset \Omega \), it is understood that all the boundary integrals in (2.5) drop out; in the same fashion, when (2.6) is satisfied for all parts \( \Gamma_0 \subset \Gamma \), the bulk integrals are also omitted from the definition of \( \Upsilon \).} For obvious reasons, we will refer to the inequality \( \Upsilon \geq 0 \) as the extended Clausius-Duhem inequality. Finally, a complete thermodynamic process is said to be admissible in \( \bar{\Omega} \) if it is compatible with a set of constitutive conditions given on the response functions introduced above, at each point of \( \bar{\Omega} \) and at all times \( t \).
Of course, for the postulate to hold, the various response functions must obey some restrictions, including the usual ones which are consequences of the classical Clausius-Duhem inequality. In particular, the entropy \( \eta_\Omega \) at each point \( x \in \Omega \) must be determined only by a function of the specific internal energy \( e_\Omega \), and the temperature \( u \) at \( x \in \Omega \) is determined only by a relation involving \( e_\Omega \) and \( \eta_\Omega \). More precisely, it turns out that for the postulate to hold on any \( \Omega_0 \subset \Omega \), both the internal energy \( e_\Omega \) and the entropy function \( \eta_\Omega \) must be constitutively independent of any higher-order stress tensors \( \nabla^\gamma u \) for any \( \gamma \geq 1 \), such that they are only functions of the local temperature, i.e., it follows that

\[
\begin{align*}
  e_\Omega = e_\Omega (u) \quad \text{and} \quad \eta_\Omega = \eta_\Omega (u),
\end{align*}
\]

respectively, cf. [5, Theorem 1, pg. 251]. Indeed, our postulate implies that the local form of the second law must hold also on any subdomain \( \Omega_0 \) of \( \Omega \); this implies that

\[
\begin{align*}
  \gamma_\Omega := \left( \eta_\Omega - \frac{h_\Omega}{u} + \text{div} \left( \frac{\eta_\Omega}{u} \right) \right) \geq 0 \quad \text{in} \quad \Omega
\end{align*}
\]

and

\[
\begin{align*}
  \gamma_\Gamma := \left( \eta_\Gamma - \frac{h_\Gamma}{u} + \text{div} \left( \frac{\eta_\Gamma}{u} \right) \right) \geq 0 \quad \text{on} \quad \Gamma.
\end{align*}
\]

From [5], we know that \( \gamma_\Omega \geq 0 \) in the body \( \Omega \) if and only if

\[
q \cdot \nabla u \leq 0,
\]

for all values \( u, \nabla u, \ldots, \nabla^\gamma u \), with \( q = q (u, \nabla u, \nabla^2 u, \ldots, \nabla^\gamma u) \). This inequality is called the heat conduction inequality in \( \Omega \). In fact, this inequality was established in [20] under more general constitutive assumptions on \( \eta_\Omega, q \) and \( e_\Omega \), excluding memory effects, as functionals of the entropy field over the entire body \( \Omega \) at the same time.

We now find necessary and sufficient set of restrictions on the remaining functions \( \eta_\Gamma, e_\Gamma, q_\Gamma \). As in [5], we assume a formulation of constitutive equations to be compatible with the Principle of Equipresence in the sense of Truesdell and Toupin [28, pg. 293], which basically states that “a variable present as an independent variable in one constitutive equation should be so present in all”. In the present formulation, the material at \( x \in \Gamma \) is characterized by the response functions \( \eta_\Gamma, e_\Gamma \) and \( q_\Gamma \), which give the functions \( \eta_\Gamma (x, t), e_\Gamma (x, t) \) and \( q_\Gamma (x, t) \), respectively, when the values \( \nabla^j \Gamma u (x, t) \) are known for \( j = 0, 1, 2, \ldots, \alpha \). Dropping the hats for the sake of convenience and by force of this principle, we assume that

\[
\begin{align*}
  e_\Gamma &= e_\Gamma \left( u, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right), \quad (2.11) \\
  \eta_\Gamma &= \eta_\Gamma \left( u, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right), \quad (2.12) \\
  q_\Gamma &= q_\Gamma \left( u, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right). \quad (2.13)
\end{align*}
\]

Furthermore, we assume that for any fixed values of \( \nabla^j \Gamma u \), the response function \( e_\Gamma \) is smooth in the first variable \( u \), i.e., we suppose \( \frac{\partial e_\Gamma}{\partial u} (u, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u) \neq 0 \). This implies that there exist new response functions, say \( \tilde{\eta}_\Gamma, \tilde{e}_\Gamma \) and \( \tilde{q}_\Gamma \), which can be used to write (2.11)-(2.13) in the following form:

\[
\begin{align*}
  u &= \tilde{u} \left( e_\Gamma, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right), \quad (2.14) \\
  \eta_\Gamma &= \tilde{\eta}_\Gamma \left( e_\Gamma, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right), \quad (2.15) \\
  q_\Gamma &= \tilde{q}_\Gamma \left( e_\Gamma, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right). \quad (2.16)
\end{align*}
\]

For each fixed values of the tensors \( \nabla^j \Gamma u, j = 0, 1, 2, \ldots, \alpha \), the variable \( \tilde{u} \left( \cdot, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right) \) is determined through the inverse function of \( e_\Gamma \), given by (2.11), such that \( \tilde{\eta}_\Gamma \) and \( \tilde{q}_\Gamma \) are defined by

\[
\begin{align*}
  \tilde{\eta}_\Gamma \left( e_\Gamma, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right) &= \eta_\Gamma \left( \tilde{u} \left( e_\Gamma, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right), \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right), \\
  \tilde{q}_\Gamma \left( e_\Gamma, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right) &= q_\Gamma \left( \tilde{u} \left( e_\Gamma, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right), \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \right).
\end{align*}
\]

Note that with \( u (x, t) \) specified for all \( x \) and \( t \), equations (2.11)-(2.13) give \( \eta_\Gamma (x, t), e_\Gamma (x, t) \) and \( q_\Gamma (x, t) \), for all \( x \) and \( t \), in which case the local form of the First Law (see also (2.1)) determines also \( h_\Gamma \). In particular, every temperature distribution \( u (x, t) > 0 \) with \( x \) varying over \( \Gamma \), determines a unique complete thermodynamic process in \( \Gamma \). By a standard argument in [5, pg. 249], in (2.11)-(2.13) we may regard not only \( e_\Gamma, \nabla \Gamma u, \nabla^2 \Gamma u, \ldots, \nabla^\alpha \Gamma u \), but also their time-derivatives \( \dot{e}_\Gamma, \nabla \dot{\Gamma} u, \nabla^2 \dot{\Gamma} u, \ldots, \nabla^\alpha \dot{\Gamma} u \), to form a set of quantities which can be chosen independently at one fixed point \( x \in \Gamma \) and time.
For each complete thermodynamic process in \( \Omega \), the second energy balance equation (2.4) allows us to write (2.9) as
\[
\gamma_T = \eta_T - \frac{q_T}{u} + \text{div} \left( \frac{\partial x}{u} \right) = \eta_T - \frac{e_T}{u} + q_T \cdot \nabla \left( \frac{1}{u} \right).
\]
(2.17)

Since \( q_T \) and \( \eta_T \) must be given by (2.16) and (2.15), at any point \((x, t)\), we have
\[
\dot{\eta}_T = \frac{\partial \eta_T}{\partial t} \dot{e}_T + \sum_{j=1}^{\alpha} \left( \frac{\partial \eta_T}{\partial u_{i_1,i_2,...,i_j}} \right) \dot{u}_{i_1,i_2,...,i_j},
\]
where the summation convention is used and where in local coordinates of \( \Gamma \), \( u_{i_1,i_2,...,i_j} = (\nabla^T u)_{i_1,i_2,...,i_j} \). It follows that
\[
\gamma_T = \left( \frac{\partial \eta_T}{\partial t} - \frac{1}{u} \right) \dot{e}_T + \sum_{j=1}^{\alpha} \left( \frac{\partial \eta_T}{\partial u_{i_1,i_2,...,i_j}} \right) \dot{u}_{i_1,i_2,...,i_j} - \frac{1}{u^2} \tilde{q}_T \cdot \nabla \gamma u.
\]
(2.18)

In order for \( \gamma_T \geq 0 \) to hold on \( \Gamma \) (but also on all parts \( \Gamma_0 \subset \Gamma \)), according to (2.9) and our postulate, it is necessary and sufficient that
\[
\frac{\partial \eta_T}{\partial t} = 0, \quad j = 1, 2, ..., \alpha.
\]
(2.19)

It follows from (2.19) that the functions \( \eta_T \) and \( \eta_T \) from (2.14) and (2.15) cannot depend on \( \nabla^T u, \nabla^T u, ..., \nabla^T u \), and they must reduce to functions of the scalar variable \( e_T \) only, i.e., \( \eta_T = \eta_T (e_T), u = \tilde{u}_T (e_T) \). These functions must also obey the first equation of (2.19); hence, the variables \( \nabla^T u, \nabla^T u, ..., \nabla^T u \) must also be dropped out of equations (2.14) and (2.15) to get
\[
e_T = e_T (u) \quad \text{and} \quad \eta_T = \eta_T (u).
\]
(2.20)

Consequently, with this reduction we observe that (2.18) becomes
\[
\gamma_T = \frac{1}{u^2} \tilde{q}_T \cdot \nabla \gamma u,
\]
for all temperature fields \( u > 0 \) and \( q_T \) given by (2.16). Thus, in order to have \( \gamma_T \geq 0 \) on \( \Gamma \), it is necessary and sufficient that \( \tilde{q}_T \cdot \nabla \gamma u \leq 0 \), or equivalently,
\[
q_T (u, \nabla^T u, \nabla^T u, ..., \nabla^T u) \cdot \nabla \gamma u \leq 0,
\]
(2.21)

for all values \( u, \nabla^T u, \nabla^T u, ..., \nabla^T u \). We call (2.21) the heat conduction inequality on \( \Gamma \). Therefore, we have established that a necessary and sufficient condition for the extended Clausius-Duhem inequality to hold for all complete thermodynamic processes on \( \Omega \) is that both the conduction inequalities (2.10) and (2.21) in \( \Omega \) and \( \Gamma \), respectively, hold. An interesting consequence is that the following choices of \( q = -k_T (u) \nabla u \) and \( q_T = -k_T (u) \nabla \gamma u \), where \( k_T, k_T > 0 \) are the thermal conductivity of \( \Omega \) and \( \Gamma \), respectively, are covered by this theory. Such choices were assumed by the theories developed in [13], [14], [18] for the system (2.1)-(2.2).

Motivated by the above result, we now wish to investigate more general constitutive conditions for the response functions involved in (2.5), by allowing them to depend also explicitly on histories up to time \( t \) of the temperature and/or the temperature gradients at \( x \). Following the approach of [4], using the abbreviations \( g_\Omega := \nabla u, g_\Gamma := \nabla \gamma u \), we consider a fixed point \( x \in \Omega \), and define the functions \( u^t, g_\Omega^t, g_\Gamma^t \) as the histories up to time \( t \) of the temperature and the temperature gradients at \( x \). More precisely, we let
\[
\begin{align*}
u^t (x, s) &= u (x, t - s), \\
g_\Omega^t (x, s) &= g_\Omega (x, t - s) \\
g_\Gamma^t (x, s) &= g_\Gamma (x, t - s),
\end{align*}
\]
for all \( s \in [0, \infty) \), on which these functions are well-defined. For a complete thermodynamic process in \( \Omega \), we define the following energy densities on \( \Omega \) and \( \Gamma \), respectively, by
\[
\psi_\Omega := e_\Omega - u^t g_\Omega, \quad \psi_\Gamma := e_\Gamma - u^t g_\Gamma.
\]
(2.22)

Of course, knowledge of \( e_\Omega, e_\Gamma \) and \( g_\Omega, g_\Gamma \) obviously determine \( \psi_\Omega \) and \( \psi_\Gamma \) by these relations. We now consider a new generalization of the constitutive equations for (2.7), (2.20) and the bulk and surface fluxes \( q, q_T \), respectively. We shall investigate the implications that the second law (2.6) has on these functions. We assume that the material at \( x \in \Omega \) is characterized by three constitutive functionals \( P_\Omega \), \( H_\Omega \) and \( g \), in the bulk, \( \Omega \), and three more constitutive functionals \( P_\Gamma \), \( H_\Gamma \) and \( q_T \), on the surface \( \Gamma \), which
give the present values of \( \psi_\Omega, \psi_\Gamma, \eta_\Omega, \eta_\Gamma, q \) and \( q_r \) at any \( x \), whenever the histories are specified at \( x \). Note that the restrictions of the functions \( u^t, g^t_\Omega, g^t_\Gamma \) on the open interval \((0, \infty)\), denoted here by \( u^t_t, g^t_{\Omega,t}, g^t_{\Gamma,t} \), are called past histories. Since a knowledge of the histories \((u^t, g^t_\Omega, g^t_\Gamma)\) is equivalent to a knowledge of the past histories \((u^t_t, g^t_{\Omega,t}, g^t_{\Gamma,t})\), and the present values \( u^t(0) = u, g^t_\Omega(0) = g_\Omega(t), g^t_\Gamma(0) = g_\Gamma(t) \), it suffices to consider

\[
\begin{align*}
\psi_\Omega &= P_\Omega(u^t_t, g^t_{\Omega,t}), & \psi_\Gamma &= P_\Gamma(u^t_t, g^t_{\Gamma,t}), \\
\eta_\Omega &= H_\Omega(u^t_t, g^t_{\Omega,t}), & \eta_\Gamma &= H_\Gamma(u^t_t, g^t_{\Gamma,t}), \\
q &= q(u^t_t, g^t_{\Omega,t}), & q_r &= q_r(u^t_t, g^t_{\Gamma,t}),
\end{align*}
\]  

(2.23)

where the Principle of Equipresence is assumed in (2.23). We further suppose that all the functionals in (2.23) obey the principle of fading memory as formulated in [6] (cf. also [20, Section 5]). In particular, this assumption means that “deformations and temperatures experienced in the distant past should have less effect on the present values of the entropies, energies, stresses, and heat fluxes than deformations and temperatures which occurred in the recent past”. Such assumptions can be made precise through the so-called “memory” functions \( m_\Omega, m_\Gamma \), which characterize the rate at which the memory fades both in the body \( \Omega \) and on the surface \( \Gamma \), respectively. In particular, we may assume that both functions \( m_S(\cdot) \), \( S \in \{\Omega, \Gamma\} \), are positive, continuous functions on \((0, \infty)\) decaying sufficiently fast to zero as \( s \to \infty \). In this case, we let \( D_S \) denote the common domain for the functionals \( P_S, H_S \) and \( q_S \) \( (q_\Omega = q) \), as the set of all pairs \((u^t_t, g^t_S)\) for which \( u^t_t > 0 \) and \( \|(u^t_t, g^t_S)\| < \infty \), where

\[
\|(u^t_t, g^t_S)\|^2 := |u^t_t(0)|^2 + |g^t_S(0)|^2 + \int_0^\infty |u^t_t(s)|^2 m_S(s) \, ds + \int_0^\infty (g^t_S(s) \cdot g^t_S(s)) m_S(s) \, ds,
\]  

(2.24)

and where \( S \in \{\Omega, \Gamma\} \). Furthermore, for each \( S \in \{\Omega, \Gamma\} \) we assume as in [4] that \( P_S, H_S \), and \( q_S \) \( (q_\Omega = q) \) are continuous over \( D_S \) with respect to the norm (2.24), but also that \( P_S \) is continuously differentiable over \( D_S \) in the sense of Fréchet, and that the corresponding functional derivatives are jointly continuous in their arguments.

In order to observe the set of restrictions that the postulate \((2.7)\) puts on the response functions, we recall \((2.4)\) and substitute \((2.22)\) into the local forms \((2.8), (2.9)\) to derive the following (local) forms of the extended Clusius-Duhem inequality on \( \Omega \):

\[
\begin{align*}
\dot{\psi}_\Omega + i\eta_\Omega + \frac{1}{u} q_\Omega \cdot \nabla u &\leq 0 \quad \text{in } \Omega, \\
\dot{\psi}_\Gamma + i\eta_\Gamma + \frac{1}{u} q_\Gamma \cdot \nabla_\Gamma u &\leq 0 \quad \text{on } \Gamma.
\end{align*}
\]  

(2.25)

We recall that a complete thermodynamic process is admissible in \( \Omega \) if it is compatible with the set of constitutive conditions given in \((2.22)\) at each point \( x \) and at all times \( t \). Since we believe that our postulate \((2.6)\) should hold for all time-dependent variables compatible with the extended law of balance of energy in \((2.5)\), it follows from [4, Theorem 6] (cf. also [6, Section 6, Theorem 1]) that the Clusius-Duhem inequalities \((2.22)\) imply for each \( S \in \{\Omega, \Gamma\} \) that

\begin{itemize}
  \item The instantaneous derivatives of \( P_S \) and \( H_S \) with respect to \( g_S \) are zero; more precisely,

\[
D_{g_S} P_S = D_{g_S} H_S = 0.
\]

  \item The functional \( H_S \) is determined by the functional \( P_S \) through the entropy relation:

\[
H_S = -D_u P_S.
\]

  \item The modified heat conduction inequalities

\[
\frac{1}{u^t} (q_S \cdot g_S) \leq \sigma_S, \quad S \in \{\Omega, \Gamma\},
\]

(with \( q_\Omega = q \)) hold for all smooth processes in \( \overline{\Omega} \) and for all \( t \).
\end{itemize}

Above, \( \sigma_S \) denotes the internal/boundary dissipation

\[
\sigma_S(t) := \frac{1}{u(t)} \left[ \delta_u P_S \left( u^t, g^t_S \mid \dot{u}^t_r \right) + \delta_{g_S} P_S \left( u^t, g^t_S \mid \dot{g}^t_{S,r} \right) \right],
\]

at time \( t \), corresponding to the histories \((u^t, g^t_S)\), where \( \dot{u} \) is the present rate of change of \( u \) at \( x \), \( \dot{u}^t \) is the past history of the rate of change of \( u \) at \( x \), and so on. Moreover, \( D_{g_S} P_S, \delta_u P_S \) and \( \delta_{g_S} P_S \) denote
the following linear differential operators

\[ D_{gs} P_S (u^t, g^t_S) \cdot l = \left( \frac{\partial}{\partial y} P_S \left( u^t, g^t_S, u, g_S + yl \right) \right)_{y=0}, \]

\[ \delta_u P_S \left( u^t, g^t_S \mid k \right) = \left( \frac{\partial}{\partial y} P_S \left( u^t + yk, g^t_S, u, g_S \right) \right)_{y=0}, \]

\[ \delta_{gS} P_S \left( u^t, g^t_S \mid \kappa \right) = \left( \frac{\partial}{\partial y} P_S \left( u^t, g^t_S + y\kappa, u, g_S \right) \right)_{y=0}, \]

with identities which hold clearly for \((u^t, g^t_S) \in D_S, S \in \{\Omega, \Gamma\}, l \in \mathbb{R} \times \mathbb{R} \) (\(\zeta_\Omega = d, \zeta_\Gamma = d - 1\)), and all \((k, \kappa)\) such that

\[ \int_0^\infty |k(s)|^2 m_S(s) \, ds < \infty, \int_0^\infty |\kappa(s)|^2 m_S(s) \, ds < \infty. \]

To derive a simple model which is sufficiently general (see (2.28), (2.29) below), we need to consider a set of constitutive equations for \(e_S, g_S, S \in \{\Omega, \Gamma\}\), which comply with the above implications that the second law has on the response functions associated with a given complete thermodynamic process in \(\Omega\). A fairly general assumption is to consider small variations in the absolute temperature and temperature gradients on both \(\Omega\) and \(\Gamma\), respectively, from equilibrium reference values (cf. (2.1)-(2.2)). We take

\[ e_\Omega (u) = e_{\Omega,\infty} + \rho_\Omega \zeta_\Omega u, \quad e_\Gamma (u) = e_{\Gamma,\infty} + \rho_\Gamma \zeta_\Gamma u, \]

where the involved positive constants \(e_{\Omega,\infty}, e_{\Gamma,\infty}, \rho_\Omega, \rho_\Gamma\) denote the internal energies at equilibrium, the specific heat capacities and material densities of \(S \in \{\Omega, \Gamma\}\), respectively. In addition, we assume that the internal and boundary fluxes satisfy the following constitutive equations:

\[ q(t) = -\omega \Delta u - (1 - \omega) \int_0^\infty m_\Omega(s) \nabla u^t(s) \, ds, \]

\[ q_\Gamma(t) = -\nu \Delta \Gamma u - (1 - \nu) \int_0^\infty m_\Gamma(s) \nabla \Gamma u^t(s) \, ds, \]

for some constants \(\omega, \nu \in (0, 1)\). Of course, when \(m_S = 0, S \in \{\Omega, \Gamma\}\), we recover in (2.26) the usual Fourier laws. Thus, in this context the constants \(\omega, \nu\) correspond to the instantaneous conductivities of \(\Omega\) and \(\Gamma\), respectively. Furthermore, we assume in (2.4) nonlinear temperature dependent heat sources \(h_S, S \in \{\Omega, \Gamma\}\), namely, we take

\[ h_\Omega(t) := -f(u(t)) - \alpha (1 - \omega) \int_0^\infty m_\Omega(s) u(x, t - s) \, ds, \]

\[ h_\Gamma(t) := -g(u(t)) - q \cdot n - \beta (1 - \nu) \int_0^\infty m_\Gamma(s) u(x, t - s) \, ds, \]

for some \(\beta > 0, \alpha > 0\), where the source on \(\Gamma\), \(h_\Gamma\) is also assumed to depend linearly on heat transport from inside of \(\Omega\) in directions normal to the boundary \(\Gamma\). With these assumptions, (2.4) yields the following system with memory

\[ \partial_t u - \omega \Delta u - (1 - \omega) \int_0^\infty m_\Omega(s) \Delta u(x, t - s) \, ds + f(u) \]

\[ = 0, \quad \text{in } \Omega \times (0, \infty), \quad \text{subject to the boundary condition} \]

\[ \partial_t u - \nu \Delta \Gamma u + \omega \partial_n u + (1 - \omega) \int_0^\infty m_\Omega(s) \partial_n u(x, t - s) \, ds \]

\[ + (1 - \nu) \int_0^\infty m_\Gamma(s) (-\Delta \Gamma + \beta) u(x, t - s) \, ds + g(u) \]

\[ = 0, \quad \text{on } \Gamma \times (0, \infty). \]

It is worth emphasizing that a different choice \(e_\Gamma(u) = e_{\Gamma,\infty} \) in (2.4) leads to a formulation in which the boundary condition (2.29) is not dynamic any longer in the sense that it does not contain the term \(\partial_t u\) anymore. This stationary boundary condition can be also reduced to (1.1) by a suitable choice of
the parameters $\beta, \nu$ and the history $m_{\Gamma}$ involved in (2.26) and (2.27). On the other hand, it is clear that if we (formally) choose $m_S = \delta_0$ (the Dirac mass at zero), for each $S \in \{\Omega, \Gamma\}$, equations (2.28)-(2.29) reduce into the following system

$$\begin{cases}
\partial_t u - \Delta u + \mathbf{f}(u) = 0, & \text{in } \Omega \times (0, \infty), \\
\partial_t u - \Delta_{\Gamma} u + \partial_n u + \mathbf{f}(u) = 0, & \text{on } \Gamma \times (0, \infty),
\end{cases}$$

(2.30)

where $\mathbf{f}(x) := g(x) + (1 - \nu) \beta x$, $\mathbf{f}(x) := f(x) + (1 - \omega) \alpha x, x \in \mathbb{R}$. The latter has been investigated quite extensively recently in many contexts (i.e., phase-field systems, heat conduction phenomena with both a dissipative and non-dissipative source $\mathbf{f}$, Stefan problems, and many more). We refer the reader to recent investigations pertaining the system (2.30) in [1, 11, 12, 14, 13, 15], and the references therein.

3. Past history formulation and functional setup

As in [8] (cf. also [19]), we can introduce the so-called integrated past history of $u$, i.e., the auxiliary variable

$$\eta^t_s(x, s) = \int_0^s u(x, t - y) dy,$$

for $s, t > 0$. Setting

$$\mu_{\Omega}(s) = -\omega^{-1} (1 - \omega) m_{\Omega}'(s), \quad \mu_{\Gamma}(s) = -\nu^{-1} (1 - \nu) m_{\Gamma}'(s),$$

(3.1)

assuming that $m_S, S \in \{\Omega, \Gamma\}$, is sufficiently smooth and vanishing at $\infty$, formal integration by parts into (2.28)-(2.29) yields

$$(1 - \omega) \int_0^\infty \mu_{\Omega}(s) \Delta u(x, t - s) ds = \omega \int_0^\infty \mu_{\Omega}(s) \Delta \eta^t_s(x, s) ds,$$

$$\left(1 - \omega\right) \int_0^\infty \mu_{\Omega}(s) \partial_n u(x, t - s) ds = \omega \int_0^\infty \mu_{\Omega}(s) \partial_n \eta^t_s(x, s) ds,$$

and

$$(1 - \nu) \int_0^\infty \mu_{\Gamma}(s) \left(-\Delta_{\Gamma} u(t - s) + \beta u(t - s)\right) ds = \nu \int_0^\infty \mu_{\Gamma}(s) \left(-\Delta_{\Gamma} \eta^t_s(s) + \beta \eta^t_s(s)\right) ds.$$  

(3.2)

Thus, we consider the following formulation.

**Problem P.** Find a function $(u, \eta^t)$ such that

$$\partial_t u - \omega \Delta u - \omega \int_0^\infty \mu_{\Omega}(s) \Delta \eta^t_s(s) ds + \alpha \omega \int_0^\infty \mu_{\Omega}(s) \eta^t_s(x, s) ds + f(u) = 0,$$

(3.3)

in $\Omega \times (0, \infty)$,

$$\partial_t u - \nu \Delta_{\Gamma} u + \omega \partial_n u + \omega \int_0^\infty \mu_{\Omega}(s) \partial_n \eta^t_s(s) ds$$

$$+ \nu \int_0^\infty \mu_{\Gamma}(s) \left(-\Delta_{\Gamma} \eta^t_s(s) + \beta \eta^t_s(s)\right) ds + g(u)$$

$$= 0,$$

(3.4)

on $\Gamma \times (0, \infty)$, and

$$\partial_t \eta^t_s(s) + \partial_s \eta^t(s) = u(t), \text{ in } \overline{\Omega} \times (0, \infty),$$

(3.5)

subject to the boundary conditions

$$\eta^t(0) = 0, \text{ in } \overline{\Omega} \times (0, \infty)$$

(3.6)

and initial conditions

$$u(0) = u_0 \text{ in } \Omega, \quad u(0) = v_0 \text{ on } \Gamma,$$

(3.7)

and

$$\eta^0(s) = \eta_0 \text{ in } \Omega, \quad \eta^0(s) = \xi_0 \text{ on } \Gamma.$$  

(3.8)

Note that we do not require that the boundary traces of $u_0$ and $\eta_0$ equal to $v_0$ and $\xi_0$, respectively. Thus, we are solving a much more general problem in which equation (3.3) is interpreted as an evolution
equation in the bulk $\Omega$ properly coupled with the equation \([3.3]\) on the boundary $\Gamma$. Finally, we note that $\eta_0, \xi_0$ are defined by

$$
\eta_0 = \int_0^s u_0(x, -y) \, dy, \text{ in } \Omega, \text{ for } s > 0, \\
\xi_0 = \int_0^s v_0(x, -y) \, dy, \text{ on } \Gamma, \text{ for } s > 0.
$$

However, from now on both $\eta_0$ and $\xi_0$ will be regarded as independent of the initial data $u_0, v_0$. Indeed, below we will consider a more general problem with respect to the original one. In order to give a more rigorous notion of solutions for problem \([3.3]-[3.8]\), we need to introduce some terminology and the functional setting associated with this system.

In the sequel, we denote by $\|\cdot\|_{L^2(\Gamma)}$ and $\|\cdot\|_{L^2(\Omega)}$ the norms on $L^2(\Gamma)$ and $L^2(\Omega)$, whereas the inner products in these spaces are denoted by $\langle \cdot, \cdot \rangle_{L^2(\Gamma)}$ and $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$, respectively. Furthermore, the norms on $H^s(\Omega)$ and $H^s(\Gamma)$, for $s > 0$, will be indicated by $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{H^s(\Gamma)}$, respectively. The symbol $\langle \cdot, \cdot \rangle$ stands for pairing between any generic Banach spaces $V$ and its dual $V^*$; $(u, v)^tr$ will also denote the vector-valued function $(u_n)$. Constants below may depend on various structural parameters such as $|\Omega|$, $|\Gamma|$, $\ell_1$, $\ell_2$, etc., and these constants may even change from line to line. Furthermore, we denote by $K(R)$ a generic monotonically increasing function of $R > 0$, whose specific dependance on other parameters will be made explicit on occurrence.

Let us now define the basic functional setup for \([3.3]-[3.8]\). From this point on, we assume that $\Omega$ is a bounded domain of $\mathbb{R}^3$ with boundary $\Gamma$ which is of class $C^2$. To this end, consider the space $\mathcal{X}^2 = L^2(\Omega, d\mu_0)$, where $d\mu_0 = dx|\Omega| \oplus d\sigma$, such that $dx$ denotes the Lebesgue measure on $\Omega$ and $d\sigma$ denotes the natural surface measure on $\Gamma$. It is easy to see that $\mathcal{X}^2 = L^2(\Omega, dx) \oplus L^2(\Gamma, d\sigma)$ may be identified under the natural norm

$$
\|u\|_{\mathcal{X}^2}^2 = \int_\Omega |u(x)|^2 \, dx + \int_\Gamma |u(x)|^2 \, d\sigma.
$$

Moreover, if we identify every $u \in C(\overline{\Omega})$ with $U = (u|\Omega, u|\Gamma) \in C(\Omega) \times C(\Gamma)$, we may also define $\mathcal{X}^2$ to be the completion of $C(\overline{\Omega})$ in the norm $\|\cdot\|_{\mathcal{X}^2}$. In general, any function $u \in \mathcal{X}^2$ will be of the form $u = (u_1, u_2)$ with $u_1 \in L^2(\Omega, dx)$ and $u_2 \in L^2(\Gamma, d\sigma)$, and there need not be any connection between $u_1$ and $u_2$. From now on, the inner product in the Hilbert space $\mathcal{X}^2$ will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X}^2}$. Next, recall that the Dirichlet trace map $\text{tr}_D : C^\infty(\overline{\Omega}) \to C^\infty(\Gamma)$, defined by $\text{tr}_D(u) = u|\Gamma$, extends to a linear continuous operator $\text{tr}_D : H^r(\Omega) \to H^{r-1/2}(\Gamma)$, for all $r > 1/2$, which is onto for $1/2 < r < 3/2$. This map also possesses a bounded right inverse $\text{tr}_D^{-1} : H^{r-1/2}(\Gamma) \to H^r(\Omega)$ such that $\text{tr}_D(\text{tr}_D^{-1}\psi) = \psi$, for any $\psi \in H^{r-1/2}(\Gamma)$. We can thus introduce the subspaces of $H^r(\Omega) \times H^{r-1/2}(\Gamma)$ and $H^r(\Omega) \times H^r(\Gamma)$, respectively, by

$$
\mathcal{V}_0^r := \{U = (u, \psi) \in H^r(\Omega) \times H^{r-1/2}(\Gamma) : \text{tr}_D(u) = \psi\},
$$

$$
\mathcal{V}^r := \{U = (u, \psi) \in \mathcal{V}_0^r : \text{tr}_D(u) = \psi \in H^r(\Gamma)\},
$$

for every $r > 1/2$, and note that $\mathcal{V}^r_0, \mathcal{V}^r$ are not product spaces. However, we have the following dense and compact embeddings $\mathcal{V}^{r_1}_0 \subset \mathcal{V}^{r_2}_0$, for any $r_1 > r_2 > 1/2$ (by definition, this also true for the sequence of spaces $\mathcal{V}^{r_1} \subset \mathcal{V}^{r_2}$). Naturally, the norm on the spaces $\mathcal{V}^r_0, \mathcal{V}^r$ are defined by

$$
\|U\|_{\mathcal{V}^r_0}^2 := \|u\|_{H^r(\Omega)}^2 + \|\psi\|_{H^{r-1/2}(\Gamma)}^2, \quad \|U\|_{\mathcal{V}^r}^2 := \|u\|_{H^r(\Omega)}^2 + \|\psi\|_{H^r(\Gamma)}^2.
$$

In particular, the norm in the spaces $\mathcal{V}^1_0, \mathcal{V}^1$ can be defined as in terms of the following equivalent norms:

$$
\|U\|_{\mathcal{V}^1_0} := \left(\omega \|\nabla u\|^2_{L^2(\Omega)} + \nu \|\nabla \psi\|^2_{L^2(\Gamma)} + \beta \nu \|\psi\|^2_{L^2(\Gamma)}\right)^{1/2}, \nu > 0,
$$

$$
\|U\|_{\mathcal{V}^1} := \left(\omega \|\nabla u\|^2_{L^2(\Omega)} + \alpha \omega \|u\|^2_{L^2(\Gamma)}\right)^{1/2}.
$$

Now we introduce the spaces for the memory vector-valued function $(\eta, \xi)$. For a given nonnegative, not identically equal to zero, and measurable function $\theta_S, S \in \{\Omega, \Gamma\}$, defined on $\mathbb{R}_+$, and a real Hilbert space
Consider the linear boundary value problem, Lemma 3.1.\[ \text{Moreover, for each } r > 1/2 \text{ we define } \]
\[
L^2_{\partial_0 \oplus \partial_r} (\mathbb{R}^+; \mathcal{V}') \simeq L^2_{\partial_0} (\mathbb{R}^+; \mathcal{V}_0') \oplus L^2_{\partial_r} (\mathbb{R}^+; H^r (\Gamma))
\]
as the Hilbert space of \( \mathcal{V}' \)-valued functions \( (\eta, \xi)^{tr} \) on \( \mathbb{R}_+ \) endowed with the inner product
\[
\left\langle \begin{pmatrix} \eta_1 \\ \xi_1 \end{pmatrix}, \begin{pmatrix} \eta_2 \\ \xi_2 \end{pmatrix} \right\rangle_{L^2_{\partial_0 \oplus \partial_r} (\mathbb{R}^+; \mathcal{V}')} = \int_0^\infty \left( \theta_\Omega(s) \langle \eta_1(s), \eta_2(s) \rangle_{H^r(\Gamma)} + \theta_\Gamma(s) \langle \xi_1(s), \xi_2(s) \rangle_{H^r(\Gamma)} \right) ds.
\]
Consequently, for \( r > 1/2 \) we set
\[
\mathcal{M}_\Omega^0 := L^2_{\mu_0} (\mathbb{R}^+; L^2(\Omega)), \quad \mathcal{M}_\Omega := L^2_{\mu_0} (\mathbb{R}^+; \mathcal{V}_0'), \quad \mathcal{M}_r := L^2_{\mu_r} (\mathbb{R}^+; H^r(\Gamma))
\]
and
\[
\mathcal{M}_{\Omega, r}^0 := L^2_{\mu_0 \oplus \mu_r} (\mathbb{R}^+; \mathcal{X}^2), \quad \mathcal{M}_{\Omega, r} := L^2_{\mu_0 \oplus \mu_r} (\mathbb{R}^+; \mathcal{V}').
\]
Clearly, because of the topological identification \( H^r(\Omega) \simeq \mathcal{V}_0' \), one has the inclusion \( \mathcal{M}_{\Omega, r} \subset \mathcal{M}_{\Omega}^0 \) for each \( r > 1/2 \). In the sequel, we will also consider Hilbert spaces of the form \( W^{k,2}_{\mu_0} (\mathbb{R}^+; \mathcal{V}_0') \) for \( k \in \mathbb{N} \).

When it is convenient, we will also use the notation
\[
\mathcal{H}^{s, r}_{\Omega, r} := \mathcal{X}^2 \times \mathcal{M}_{\Omega, r}, \quad \mathcal{H}^{s, r}_{\Omega, \Gamma} := \mathcal{V}^s \times \mathcal{M}_{\Omega, r} \text{ for } s, r \geq 1.
\]

For matter of convenience, we will also set the inner product in \( \mathcal{M}_{\Omega, r}^1 \), as follows:
\[
\left\langle \begin{pmatrix} \eta_1 \\ \xi_1 \end{pmatrix}, \begin{pmatrix} \eta_2 \\ \xi_2 \end{pmatrix} \right\rangle_{L^2_{\partial_0 \oplus \partial_r} (\mathbb{R}^+; \mathcal{V})} = \omega \int_0^\infty \theta_\Omega(s) \left\langle \begin{pmatrix} \nabla \eta_1(s) \\ \nabla \xi_1(s) \end{pmatrix}, \begin{pmatrix} \nabla \eta_2(s) \\ \nabla \xi_2(s) \end{pmatrix} \right\rangle_{L^2(\Omega)} + \alpha \langle \eta_1(s), \eta_2(s) \rangle_{L^2(\Omega)} ds
\]
\[
+ \nu \int_0^\infty \theta_\Gamma(s) \left\langle \begin{pmatrix} \nabla \eta_1(s) \\ \nabla \xi_1(s) \end{pmatrix}, \begin{pmatrix} \nabla \eta_2(s) \\ \nabla \xi_2(s) \end{pmatrix} \right\rangle_{L^2(\Gamma)} + \beta \langle \xi_1(s), \xi_2(s) \rangle_{L^2(\Gamma)} ds.
\]

The following basic elliptic estimate is taken from [13] Lemma 2.2.

Lemma 3.1. Consider the linear boundary value problem,
\[
\begin{cases}
-\Delta u = p_1 \text{ in } \Omega, \\
-\Delta u + \partial_\Omega u + \beta u = p_2 \text{ on } \Gamma.
\end{cases}
\]
If \( (p_1, p_2)^{tr} \in H^s(\Omega) \times H^s(\Gamma) \), for \( s \geq 0 \) and \( s + \frac{1}{2} \notin \mathbb{N} \), then the following estimate holds for some constant \( C > 0 \),
\[
\|u\|_{H^{s+2}} + \|u\|_{H^{s+2}(\Gamma)} \leq C \left( \|p_1\|_{H^s} + \|p_2\|_{H^s(\Gamma)} \right).
\]

We also recall the following basic inequality from [11] Lemma A.2.

Lemma 3.2. Let \( s > 1 \) and \( u \in H^1(\Omega) \). Then, for every \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \sim \varepsilon^{-1} \) such that,
\[
\|u\|_{L_{\gamma}^p(\Omega)} \leq \varepsilon \|\nabla u\|_{L^2(\Omega)} + C_\varepsilon \left( \|u\|_{L_\Omega^p(\Omega)}^p + 1 \right),
\]
where \( \gamma = \max\{s, 2(s-1)\} \).

Next, we consider the linear (self-adjoint, positive) operator \( C \psi := C_\beta \psi = -\Delta \psi + \beta \psi \) acting on \( D(C) = H^2(\Gamma) \). The basic (linear) operator, associated with problem (3.3)-(3.5), is the so-called “Wentzell” Laplace operator. Recall that \( \omega \in (0, 1) \). We let
\[
A_{\alpha, \beta, \nu, \omega}^W \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} -\omega \Delta + \alpha \omega I & 0 \\ \omega \partial_\Omega (\cdot) & \nu C \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]
\[
= A_{\alpha, \beta, \nu, \omega}^W \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \nu C u_2 \end{pmatrix},
\]
with

\[ D \left( A_{W}^{\alpha,\beta,\nu,\omega} \right) := \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{Y} : -\Delta u_1 \in L^2(\Omega), \ \omega \partial_n u_1 - \nu C u_2 \in L^2(\Gamma) \right\}, \tag{3.16} \]

where \( \mathbb{Y} := \mathbb{V}_0^1 \) if \( \nu = 0 \), and \( \mathbb{Y} := \mathbb{V}^1 \) if \( \nu > 0 \). It is well-known that \( (A_{W}^{\alpha,\beta,\nu,\omega}, D(A_{W}^{\alpha,\beta,\nu,\omega})) \) is self-adjoint and nonnegative operator on \( X^2 \) whenever \( \alpha, \beta, \nu \geq 0 \), and \( A_{W}^{\alpha,\beta,\nu,\omega} > 0 \) if either \( \alpha > 0 \) or \( \beta > 0 \). Moreover, the resolvent operator \( (I + A_{W}^{\alpha,\beta,\nu,\omega})^{-1} \in \mathcal{L}(X^2) \) is compact. Moreover, since \( \Gamma \) is of class \( C^2 \), then \( D(A_{W}^{\alpha,\beta,\nu,\omega}) = \mathbb{V}^2 \) if \( \nu > 0 \). Indeed, for any \( \alpha, \beta \geq 0 \) with \( (\alpha, \beta) \neq (0, 0) \), the map \( \Psi : U \rightarrow A_{W}^{\alpha,\beta,\nu,\omega}U \), when viewed as a map from \( \mathbb{V}_2 \) into \( X^2 = L^2(\Omega) \times L^2(\Gamma) \), is an isomorphism and there exists a positive constant \( C_* \), independent of \( U = (u, \psi)^\dagger \), such that

\[ C_*^{-1} \| U \|_{\mathbb{V}_2} \leq \| \Psi(U) \|_{X^2} \leq C_* \| U \|_{\mathbb{V}_2}, \tag{3.17} \]

for all \( U \in \mathbb{V}_2 \) (cf. Lemma 3.1). Whenever \( \nu = 0 \), by elliptic regularity theory and \( U \in D(A_{W}^{\alpha,\beta,0,\omega}) \) one has \( u \in H^{3/2}(\Omega) \) and \( \psi = \text{tr}_D(u) \in H^1(\Gamma) \), since the Dirichlet-to-Neumann map is bounded from \( H^1(\Gamma) \) to \( L^2(\Gamma) \); hence \( D(A_{W}^{\alpha,\beta,0,\omega}) = \mathbb{W} \), where \( \mathbb{W} \) is the Hilbert space equipped with the following (equivalent) norm

\[ \| U \|_\mathbb{W}^2 := \| U \|_{\mathbb{V}_2}^2 + \| \Delta u \|_{L^2(\Omega)}^2 + \| \partial_n u \|_{L^2(\Gamma)}^2. \]

We refer the reader to more details to e.g., [1], [15], [2] and the references therein. We now have all the necessary ingredients to introduce a rigorous formulation of problem \( P \) in the next section.

4. Variational formulation and well-posedness

We need the following hypotheses for problem \( P \). For the function \( \mu_S, S \in \{ \Omega, \Gamma \} \), given by (3.1), we consider the following assumptions (cf. e.g. [8], [10] and [17]). Assume

\[ \mu_S \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+), \tag{4.1} \]

\[ \mu_S(s) \geq 0 \text{ for all } s \geq 0, \tag{4.2} \]

\[ \mu_S'(s) \leq 0 \text{ for all } s \geq 0. \tag{4.3} \]

These assumptions are equivalent to assuming that \( m_S(s), S \in \{ \Omega, \Gamma \} \), is a bounded, positive, nonincreasing, convex function of class \( C^2 \). These conditions are commonly used in the literature (see, for example, [8], [10] and [17]) to establish existence and uniqueness of continuous global weak solutions for Coleman-Gurtin type equations subject to Dirichlet boundary conditions.

As far as natural conditions for the nonlinear terms are concerned, we assume \( f, g \in C^1(\mathbb{R}) \) satisfy the sign conditions

\[ f'(s) \geq -M_f, \ g'(s) \geq -M_g, \text{ for all } s \in \mathbb{R}, \tag{4.4} \]

for some \( M_f, M_g > 0 \) and the growth assumptions, for all \( s \in \mathbb{R} \),

\[ |f(s)| \leq \ell_1(1 + |s|^{r_1 - 1}), \ ||s|^{r_2 - 1}, \ |g(s)| \leq \ell_2(1 + |s|^{r_2 - 1}), \tag{4.5} \]

for some positive constants \( \ell_1 \) and \( \ell_2 \), and where \( r_1, r_2 \geq 2 \). Let now

\[ \bar{g}(s) := g(s) - \nu/\beta s, \text{ for } s \in \mathbb{R}. \tag{4.6} \]

In addition, we assume there exists \( \varepsilon \in (0, \omega) \) so that the following balance condition

\[ \liminf_{|s| \to \infty} \frac{f(s)s + \ell_1^{(1)} \bar{g}(s)}{|s|^{r_1}} > 0 \]

holds for \( r_1 \geq \max\{r_2, 2(r_2 - 1)\} \). The number \( C_\Omega > 0 \) is the best Sobolev constant in the following Sobolev-Poincaré inequality

\[ \| u - \langle u \rangle_\Gamma \|_{L^2(\Omega)} \leq C_\Omega \| \nabla u \|_{L^2(\Omega)}, \ \langle u \rangle_\Gamma := \frac{1}{|\Gamma|} \int_\Gamma \text{tr}_D(u) \, d\sigma, \tag{4.7} \]

for all \( u \in H^1(\Omega) \), see [25] Lemma 3.1].
The assumption (4.7) deserves some additional comments. Suppose that that for \( |y| \to \infty \), both the internal and boundary functions behave accordingly to the following laws:

\[
\lim_{|y| \to \infty} \frac{f'(y)}{|y|^{p-2}} = (r_1 - 1) c_f, \quad \lim_{|y| \to \infty} \frac{g'(y)}{|y|^{q-2}} = (r_2 - 1) c_g,
\]

for some \( c_f, c_g \in \mathbb{R} \setminus \{0\} \). In particular, it holds

\[
f(y) y \sim c_f |y|^{r_1}, \quad g(y) y \sim c_g |y|^{r_2} \quad \text{as} \quad |y| \to \infty.
\]

For the case of bulk dissipation (i.e., \( c_f > 0 \)) and anti-dissipative behavior at the boundary \( \Gamma \) (i.e., \( c_g < 0 \)), assumption (4.7) is automatically satisfied provided that \( r_1 > \max \{ r_2, 2(r_2 - 1) \} \). Furthermore, if \( 2 < r_2 < 2(r_2 - 1) = r_1 \) and

\[
c_f > \frac{1}{4 \varepsilon} \left( \frac{C_\Omega |\Gamma| c_g r_2}{|\Omega|} \right)^2,
\]

for some \( \varepsilon \in (0, \omega) \), then once again (4.7) is satisfied. In the case when \( f \) and \( g \) are sublinear (i.e., \( r_1 = r_2 = 2 \) in (4.5)), the condition (4.7) is also automatically satisfied provided that

\[
\left( c_f + \frac{|\Gamma|}{|\Omega|} c_g \right) > \frac{1}{\varepsilon} \left( \frac{C_\Omega |\Gamma| c_g}{|\Omega|} \right)^2
\]

for some \( \varepsilon \in (0, \omega) \). Of course, when both the bulk and boundary nonlinearities are dissipative, i.e., there exist two constants \( C_f > 0, C_g > 0 \) such that, additionally to (4.5),

\[
\begin{cases}
  f(s) s \geq C_f |s|^{r_1}, \\
  g(s) s \geq C_g |s|^{r_2},
\end{cases}
\]

for all \( |s| \geq s_0 \), for some sufficiently large \( s_0 > 0 \), condition (4.7) can be dropped and is no longer required (see [11]).

In order to introduce a rigorous formulation for problem \( P \), we define

\[
D(T) := \left\{ \Phi = \begin{pmatrix} \eta \varepsilon t \\ \xi \varepsilon t \end{pmatrix} \in \mathcal{M}_{\Omega, \Gamma} : \partial_s \Phi \in \mathcal{M}_{\Omega, \Gamma}, \; \Phi(0) = 0 \right\}
\]

and consider the linear (unbounded) operator \( T : D(T) \to \mathcal{M}_{\Omega, \Gamma} \) by

\[
T \Phi = -\left( \frac{d^2}{dt^2} + \frac{\partial}{\partial x} \right) \Phi = \begin{pmatrix} \eta^{t} \\ \xi^{t} \end{pmatrix} \in D(T).
\]

The follow result can be proven following [19, Theorem 3.1].

**Proposition 4.1.** The operator \( T \) with domain \( D(T) \) is an infinitesimal generator of a strongly continuous semigroup of contractions on \( \mathcal{M}_{\Omega, \Gamma} \), denoted \( e^{Tt} \).

As a consequence, we also have (cf., e.g. [23, Corollary IV.2.2]).

**Corollary 4.2.** Let \( T > 0 \) and assume \( U = \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix} \in L^1(0, T; \mathcal{V}^1) \). Then, for every \( \Phi_0 \in \mathcal{M}_{\Omega, \Gamma} \), the Cauchy problem for \( \Phi^t = \begin{pmatrix} \eta^t \\ \xi^t \end{pmatrix} \),

\[
\begin{cases}
  \partial_t \Phi^t = T \Phi^t + U(t), \quad \text{for} \; t > 0, \\
  \Phi^0 = \Phi_0,
\end{cases}
\]

has a unique (mild) solution \( \Phi \in C([0, T]; \mathcal{M}_{\Omega, \Gamma}) \) which can be explicitly given as

\[
\Phi^t(s) = \begin{cases}
  \int_{0}^{s} U(t - y) dy, \quad \text{for} \; 0 < s \leq t, \\
  \Phi_0(s - t) + \int_{s}^{t} U(t - y) dy, \quad \text{for} \; s > t,
\end{cases}
\]

cf. also [8, Section 3.2] and [19, Section 3].
Remark 4.3. (i) Note that, from assumption \(4.3\), the following inequality holds for all \(\Phi \in D(T)\).

\[
\langle T\Phi, \Phi \rangle_{\mathcal{M}_{1,G}} \leq 0
\]  

(4.16)

(ii) If \(\Phi_0 \in D(T)\) and \(\partial_t U \in L^1(0,T;\mathbb{V}^2)\), the function \(\Phi^t\) given by \(4.15\) satisfies \(4.13\) in the strong sense a.e. on \((0,T)\), for any \(T > 0\).

We are now ready to introduce the rigorous (variational) formulation of problem \(P\).

**Definition 4.4.** Let \(\alpha, \beta > 0\), \(\omega, \nu \in (0,1)\) and \(T > 0\). Given \((u_{v0}) \in \mathbb{V}^2\), \((\rho_{0}) \in \mathcal{M}_{1,G}^1\), we seek to find functions \(U(t) = (u_{v(t)}, \Phi^t)\) with the following properties:

\[
U \in L^\infty(0,T;\mathbb{V}^2) \cap L^2(0,T;\mathbb{V}^1), \quad \Phi \in L^\infty(0,T;\mathcal{M}_{1,G}^1),
\]  

(4.17)

\[
u \in L^\infty(\Omega \times (0,T)), \quad v \in L^2(\Gamma \times (0,T)),
\]  

(4.18)

\[
\partial_t U \in L^2(0,T;\mathbb{V}^1) \oplus \left(L^2(\Omega \times (0,T)) \times L^2(\Gamma \times (0,T))\right),
\]  

(4.19)

\[
\partial_t \Phi \in L^2\left(0,T;W_{\mu_0 \oplus \mu_T}^{-1,2}(\mathbb{R}_+;\mathbb{V}^1)\right).
\]  

(4.20)

\((U, \Phi^t)\) is said to be a weak solution to problem \(P\) if \(v(t) = \text{tr}_D(u(t))\) and \(\xi^t = \text{tr}_D(\eta^t)\) for almost all \(t \in (0,T]\), and \((U(t), \Phi^t)\) satisfies, for almost all \(t \in (0,T]\),

\[
\langle \partial_t U(t), \Xi \rangle_{\mathbb{V}^2} + \left(\mu_{v0}^0, \omega U(t), \Xi\right)_{\mathbb{X}^2} + \int_0^\infty \mu_\Omega(s) \left(\mu_{\mathcal{A}^0,0,0}^0, \nu(t), \Xi\right)_{\mathbb{X}^2} ds + \nu \int_0^\infty \mu_\Omega(s) \left(C\xi^t(\mathcal{S}, \Omega), \eta^t\right)_{L^2(\Gamma)} ds + \left(F(U(t)), \Xi\right)_{\mathbb{X}^2} = 0,
\]  

(4.21)

for all \(\Xi = (\nu, \omega) \in \mathbb{V}^1 \oplus (L^2(\Omega) \times L^2(\Gamma))\), all \(\Pi = (\rho_{v0}) \in \mathcal{M}_{1,G}^1\) and

\[
U(0) = U_0 = (u_{v0}, v_0)^{tr}, \quad \Phi^0 = \Phi_0 = (\eta_{0}, \xi_{0})^{tr}.
\]  

(4.22)

Above, we have set \(F : \mathbb{R}^2 \to \mathbb{R}^2\),

\[
F(U) := \left(\frac{f(u)}{\tilde{g}(v)}\right),
\]

with \(\tilde{g}\) defined as in \(4.7\). The function \([0,T] \ni t \mapsto (U(t), \Phi^t)\) is called a global weak solution if it is a weak solution for every \(T > 0\).

In the sequel, if the initial datum \((U_0, \Phi_0)\) is more smooth, the following notion of strong solution will also become important.

**Definition 4.5.** Let \(\alpha, \beta > 0\), \(\omega, \nu \in (0,1)\) and \(T > 0\). Given \((u_{v0}) \in \mathbb{V}^1\), \((\rho_{0}) \in \mathcal{M}_{1,G}^2\), the pair of functions \(U(t) = (u_{v(t)}, \Phi^t)\) satisfying

\[
U \in L^\infty(0,T;\mathbb{V}^2) \cap L^2(0,T;\mathbb{V}^2),
\]  

(4.23)

\[
\Phi \in L^\infty(0,T;\mathcal{M}_{1,G}^1),
\]

\[
\partial_t U \in L^\infty(0,T;\mathbb{V}^1) \cap L^2(0,T;\mathbb{X}^2),
\]  

(4.24)

\[
\partial_t \Phi \in L^2\left(0,T;L^{2,2}_{\rho_0 \oplus \mu_T}(\mathbb{R}_+;\mathbb{X}^2)\right),
\]

is called a strong solution to problem \(P\) if \(v(t) = \text{tr}_D(u(t))\) and \(\xi^t = \text{tr}_D(\eta^t)\) for almost all \(t \in (0,T]\), and additionally, \((U(t), \Phi^t)\) satisfies \(4.21\), a.e. for \(t \in (0,T]\), for all \(\Xi \in \mathbb{V}^1\), \(\Pi \in \mathcal{M}_{1,G}^1\), and

\[
U(0) = U_0 = (u_{v0}, v_0)^{tr}, \quad \Phi^0 = \Phi_0 = (\eta_{0}, \xi_{0})^{tr}.
\]  

(4.24)

The function \([0,T] \ni t \mapsto (U(t), \Phi^t)\) is called a global strong solution if it is a strong solution for every \(T > 0\).
Remark 4.6. Note that a strong solution is incidently more smooth than a weak solution in the sense of Definition 4.4. Moreover, on account of standard embedding theorems the regularity \( U \in L^\infty (0, T; \mathcal{V}^1) \cap L^2 (0, T; \mathcal{V}^2) \) implies that

\[
 u \in L^\infty (0, T; L^p (\Omega)) \cap L^q (0, T; L^p (\Omega))
\]

for any \( p \in (6, \infty) \), \( 1 \leq q \leq 4p/(p-6) \), and \( \text{tr}_\Gamma (u) \in L^\infty (0, T; L^4 (\Omega)) \), for any \( s \in (1, \infty) \).

Another notion of strong solution to problem \( P \), although weaker than the notion in Definition 4.5, can be introduced as follows.

Definition 4.7. The pair \( U = (u, \Phi) \) and \( \Phi = (\Phi_t, \Phi_{tt}) \) is called a quasi-strong solution of problem \( P \) on \([0, T)\) if \( (U(t), \Phi_t) \) satisfies the equations \( 4.21-4.22 \) for all \( \Xi \in \mathcal{V}' \), \( \Pi \in \mathcal{M}^1_{\Omega, T} \), almost everywhere on \([0, T)\) and if it has the regularity properties:

\[
\begin{align*}
 U & \in L^\infty (0, T; \mathcal{V}^1) \cap W^{1,2} (0, T; \mathcal{V}^1), \\
 \Phi & \in L^\infty (0, T; D(T)), \\
 \partial_t U & \in L^\infty (0, T; \mathcal{V}^2), \\
 \partial_t \Phi & \in L^\infty (0, T; \mathcal{M}^1_{\Omega, \Gamma}).
\end{align*}
\]

As before, the function \([0, T) \ni t \mapsto (U(t), \Phi_t) \) is called a global quasi-strong solution if it is a quasi-strong solution for every \( T > 0 \).

Our first result in this section is contained in the following theorem. It allows us to obtain generalized solutions in the sense of Definition 4.4.

Theorem 4.8. Assume \( 4.4 \)–\( 4.7 \) and \( 4.10 \)–\( 4.11 \) hold. For each \( \alpha, \beta > 0 \), \( \omega, \nu \in (0, 1) \) and \( T > 0 \), and for any \( U_0 = (u_0, v_0)^T \in \mathbb{R}^2 \), \( \Phi_0 = (\eta_0, \xi_0)^T \in \mathcal{M}^1_{\Omega, \Gamma} \), there exists at least one (global) weak solution \( (U, \Phi) \in C ([0, T]; \mathcal{M}^1_{\Omega, T}) \) to problem \( P \).

Proof. The proof is divided into several steps. Much of the motivation for the above theorem comes from [11]. Indeed, the dissipativity induced by the balance condition \( 4.7 \) will be exploited to obtain an \textit{apriori} bound. Of course, several modifications need to be made in order to incorporate the dynamic boundary conditions with memory into the framework.

Step 1. (An \textit{apriori} bound) To begin, we derive an \textit{apriori} energy estimate for any (sufficiently) smooth solution \((U, \Phi)\) of problem \( P \). Under the assumptions of the theorem, we claim that the following estimate holds:

\[
\begin{align*}
\|U(t)\|_{\mathcal{X}_2}^2 + \|\Phi(t)\|^2_{\mathcal{M}^1_{\Omega, \Gamma}} & - 2 \langle T \Phi, \Phi_t \rangle_{\mathcal{M}^1_{\Omega, \Gamma}} + 2 \int_0^t \left( \|U(\tau)\|_{\mathcal{V}_2}^2 + \|u(\tau)\|_{\mathcal{V}'_{\Omega, \Gamma}}^2 \right) d\tau \\
& \leq C_T \left( 1 + \|U(0)\|_{\mathcal{X}_2}^2 + \|\Phi(0)\|^2_{\mathcal{M}^1_{\Omega, \Gamma}} \right),
\end{align*}
\]

for all \( t \in [0, T] \), for some constant \( C > 0 \), independent of \((U, \Phi)\) and \( t \).

We now show \( 4.29 \). In Definition 4.4 we are allowed to take, for almost all \( t \in [0, T] \),

\[
\Xi = U(t) = (u(t), u(t)|_{\Gamma})^T \in \mathcal{V}_2 \cap (L^\gamma (\Omega) \times L^\omega (\Gamma))
\]

and

\[
\Pi = \Phi = (\eta^t, \xi^t) \in \mathcal{M}^1_{\Omega, \Gamma}.
\]

Then we obtain the differential identities

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|U\|_{\mathcal{X}_2}^2 + \langle A^0, \mu, \omega, U, U \rangle_{\mathcal{X}_2} + \langle \Phi, U \rangle_{\mathcal{M}^1_{\Omega, \Gamma}} + \langle F(U), U \rangle_{\mathcal{X}_2} &= 0, \tag{4.30}
\end{align*}
\]

where

\[
\begin{align*}
\langle \Phi, U \rangle_{\mathcal{M}^1_{\Omega, \Gamma}} &= \omega \int_0^\infty \mu_\Omega (s) \left( \nabla \eta^t (s), \nabla u \right)_{L^2 (\Omega)} + \alpha \langle \eta^t (s), u \rangle_{L^2 (\Omega)} \, ds \\
&\quad + \nu \int_0^\infty \mu_\Gamma (s) \left( \nabla \xi^t (s), \nabla u \right)_{L^2 (\Gamma)} + \beta \langle \xi^t (s), u \rangle_{L^2 (\Gamma)} \, ds \\
&= \int_0^\infty \mu_\Omega (s) \left( A^{0,0,0, \omega} \Phi^t (s), U \right)_{\mathcal{X}_2} \, ds + \nu \int_0^\infty \mu_\Gamma (s) \left( C \xi^t (s), u \right)_{L^2 (\Gamma)} \, ds,
\end{align*}
\]
Following (2.22) and (3.11), we estimate the product with $F$ as follows:

$$\frac{1}{2} \frac{d}{dt} \|\Phi^t\|_{\mathcal{M}^1_{\Omega, T}}^2 = \langle T\Phi^t, \Phi^t \rangle_{\mathcal{M}^1_{\Omega, T}} + \langle U, \Phi^t \rangle_{\mathcal{M}^1_{\Omega, T}},$$

which hold for almost all $t \in [0, T]$. Adding these identities together and recalling (4.11), we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \|U\|_{X^2}^2 + \|\Phi^t\|_{\mathcal{M}^1_{\Omega, T}}^2 \right) - \langle T\Phi^t, \Phi^t \rangle_{\mathcal{M}^1_{\Omega, T}} + \left( \omega \|\nabla u\|_{L^2(\Omega)}^2 + \nu \|\nabla u\|_{L^2(\Gamma)}^2 + \beta \|u\|_{L^2(\Gamma)}^2 \right)$$

$$\leq - \langle f(u), u \rangle_{L^2(\Omega)} - \langle \overline{g}(u), u \rangle_{L^2(\Gamma)}.$$}

Following (2.22) and (3.11), we estimate the product with $F$ on the right-hand side of (4.33), as follows:

$$\langle F(U), U \rangle_{X^2} = \langle f(u), u \rangle_{L^2(\Omega)} + \langle \overline{g}(u), u \rangle_{L^2(\Gamma)}$$

$$= \int_{\Omega} \left( f(u)u + \frac{\|\Gamma\|}{|\Omega|} \overline{g}(u)u \right) \, dx - \frac{|\Gamma|}{|\Omega|} \int_{\Gamma} \left( \overline{g}(u)u - \frac{1}{|\Gamma|} \int_{\Gamma} \overline{g}(u)u \, d\sigma \right) \, dx.$$

Exploiting Poincaré inequality (4.8) and Young’s inequality, we see that for all $\varepsilon \in (0, \omega)$,

$$\frac{|\Gamma|}{|\Omega|} \int_{\Omega} \left( \overline{g}(u)u - \frac{1}{|\Gamma|} \int_{\Gamma} \overline{g}(u)u \, d\sigma \right) \, dx \leq C_\Omega \frac{|\Gamma|}{|\Omega|} \int_{\Omega} |\nabla (\overline{g}(u)u)| \, dx$$

$$= C_\Omega \frac{|\Gamma|}{|\Omega|} \int_{\Omega} |\nabla u(\overline{g}(u)u + \overline{g}(u))| \, dx$$

$$\leq \varepsilon \|\nabla u\|_{L^2(\Omega)}^2 + \frac{C^2_\Omega |\Gamma|^2}{4\varepsilon |\Omega|^2} \int_{\Omega} |\overline{g}'(u)u + \overline{g}(u)|^2 \, dx.$$

Combining (4.34)-(4.35) and applying assumption (4.7) yields

$$\langle F(U), U \rangle_{X^2} \geq \delta \|u\|_{L^2(\Omega)}^2 - \varepsilon \|\nabla u\|_{L^2(\Omega)}^2 - C\delta,$$

for some positive constants $\delta$ and $C\delta$ that are independent of $U$, $t$ and $\varepsilon$. Plugging (4.36) into (4.33) gives, for almost all $t \in [0, T]$,

$$\frac{1}{2} \frac{d}{dt} \left( \|U\|_{X^2}^2 + \|\Phi^t\|_{\mathcal{M}^1_{\Omega, T}}^2 \right) - \langle T\Phi^t, \Phi^t \rangle_{\mathcal{M}^1_{\Omega, T}} + \left( \omega - \varepsilon \right) \|\nabla u\|_{L^2(\Omega)}^2$$

$$\leq \delta \|u\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla u\|_{L^2(\Gamma)}^2 + \beta \|u\|_{L^2(\Gamma)}^2.$$
Hence, this justifies our choice of test function for the first of (4.21). Concerning the second equation of (4.21), in view of (4.38) and the representation formula (4.15) we have

\[ T\Phi^i(s) = -\partial_s \Phi^i(s) = \begin{cases} -U(t-s) & \text{for } 0 < s \leq t, \\ -\partial_s \Phi_0(s-t) + U(t-s) & \text{for } s > t. \end{cases} \]

Then, with a given \( \Phi_0 \in \mathcal{M}_{\Omega, \Gamma}, \partial_s \Phi_0(\cdot) \in W^{-1,2}_{\mu_0 \oplus \mu_T} (\mathbb{R}_+; \mathbb{V}^1) \), we conclude

\[ \partial_t \Phi \in L^2 \left( 0, T; W^{-1,2}_{\mu_0 \oplus \mu_T} (\mathbb{R}_+; \mathbb{V}^1) \right). \]  

(4.44)

This concludes Step 1.

Step 2. (A Galerkin basis) First, for any \( \alpha, \beta \geq 0 \) we recall that \( \left( A_{W}^{\alpha, \beta, \nu, \omega} \right)^{-1} \in \mathcal{L}(\mathcal{X}^2) \) is compact provided that either \( \beta > 0 \) or \( \alpha > 0 \). This means that, for \( i \in \mathbb{N} \), there is a complete system of eigenfunctions \( \Psi_i^{\alpha, \beta, \nu, \omega} = (\vartheta_i^{\alpha, \beta, \nu, \omega}, \psi_i^{\alpha, \beta, \nu, \omega})^{\operatorname{tr}} \) of the eigenvalue problem

\[ A_{W}^{\alpha, \beta, \nu, \omega} \Psi_i^{\alpha, \beta, \nu, \omega} = \lambda_i \Psi_i^{\alpha, \beta, \nu, \omega} \text{ in } \mathcal{X}^2 \]

with

\[ \Psi_i^{\alpha, \beta, \nu, \omega} \in D \left( A_{W}^{\alpha, \beta, \nu, \omega} \right) \cap \left( C^2(\overline{\Omega}) \times C^2(\Gamma) \right), \]

see \[\text{Appendix}]. The eigenvalues \( \lambda_i = \lambda_i^{\alpha, \beta, \nu, \omega} \in (0, \infty) \) may be put into increasing order and counted according to their multiplicity to form a divergent sequence going to infinity. In addition, also due to standard spectral theory, the related eigenfunctions form an orthogonal basis in \( \mathbb{V}^1 \) that is orthonormal in \( \mathbb{X}^2 \). Note that for each \( i \in \mathbb{N} \), the pair \( (\lambda_i, \vartheta_i) \in \mathbb{R}_+ \times C^2(\overline{\Omega}) \), \( \vartheta_i = \vartheta_i^{\alpha, \beta, \nu, \omega} \), is a classical solution of the elliptic problem

\[ \begin{cases} -\omega \Delta \vartheta_i + \alpha \vartheta_i = \lambda_i \vartheta_i, \\ -\nu \Delta \Gamma (\vartheta_i|\Gamma) + \omega \vartheta_i + \beta \nu \vartheta_i|_{\Gamma} = \lambda_i \vartheta_i|_{\Gamma}, \end{cases} \]

in \( \Omega \), on \( \Gamma \).  

(4.45)

It remains to select an orthonormal basis \( \{ \zeta_i \}_{i=1}^{\infty} \) of \( \mathcal{M}_{\Omega, \Gamma}^{1} = L^2_{\mu_0 \oplus \mu_T} (\mathbb{R}_+; \mathbb{V}^1) \) that also belongs to \( D(\mathcal{T}) \cap W_{\mu_0 \oplus \mu_T}^{1,2} (\mathbb{R}_+; \mathbb{V}^1) \). We can choose vectors \( \zeta_i = \xi_i \Psi_i^{\alpha, \beta, \nu, \omega} \), with eigenvectors \( \Psi_i^{\alpha, \beta, \nu, \omega} \in D(A_{W}^{\alpha, \beta, \nu, \omega}) \) satisfying (4.45) above, such that \( \{ \xi_i \}_{i=1}^{\infty} \subset C_c^\infty (\mathbb{R}_+) \) is an orthonormal basis for \( L^2_{\mu_0 \oplus \mu_T} (\mathbb{R}_+) \). This choice will be crucial for the derivation of strong solutions in the section later.

Let \( T > 0 \) be fixed. For \( n \in \mathbb{N} \), set the spaces

\[ X_n = \text{span} \left\{ \Psi_1^{\alpha, \beta, \nu, \omega}, \ldots, \Psi_n^{\alpha, \beta, \nu, \omega} \right\} \subset \mathbb{X}^2, \quad X_\infty = \bigcup_{n=1}^{\infty} X_n, \]

and

\[ M_n = \text{span} \{ \zeta_1, \zeta_2, \ldots, \zeta_n \} \subset \mathcal{M}_{\Omega, \Gamma}, \quad M_\infty = \bigcup_{n=1}^{\infty} M_n. \]

Obviously, \( X_\infty \) is a dense subspace of \( \mathbb{V}^1 \). For each \( n \in \mathbb{N} \), let \( P_n : \mathbb{X}^2 \rightarrow X_n \) denote the orthogonal projection of \( \mathbb{X}^2 \) onto \( X_n \) and let \( Q_n : \mathcal{M}_{\Omega, \Gamma}^1 \rightarrow M_n \) denote the orthogonal projection of \( \mathcal{M}_{\Omega, \Gamma}^1 \) onto \( M_n \). Thus, we seek functions of the form

\[ U_n(t) = \sum_{i=1}^{n} a_i(t) \Psi_i^{\alpha, \beta, \nu, \omega} \quad \text{and} \quad \Phi_n^i(s) = \sum_{i=1}^{n} b_i(t) \zeta_i(s) \Psi_i^{\alpha, \beta, \nu, \omega} \]  

(4.46)

that will satisfy the associated discretized problem \( P_n \) described below. The functions \( a_i \) and \( b_i \) are assumed to be (at least) \( C^2(0, T) \) for \( i = 1, \ldots, n \). By definition, note that

\[ u_n(t) = \sum_{i=1}^{n} a_i(t) \vartheta_i^{\alpha, \beta, \nu, \omega} \quad \text{and} \quad u_n(t)|_{\Gamma} = \sum_{i=1}^{n} a_i(t) \vartheta_i^{\alpha, \beta, \nu, \omega}|_{\Gamma}, \]

(4.47)

also

\[ \eta_n^i(s) = \sum_{i=1}^{n} b_i(t) \zeta_i(s) \quad \text{and} \quad \ell_n^i(s) = \sum_{i=1}^{n} b_i(t) \zeta_i(s)|_{\Gamma}. \]  

(4.48)

As usual, to approximate the given initial data \( U_0 \in \mathbb{X}^2 \) and \( \Phi_0 \in \mathcal{M}_{\Omega, \Gamma}^1 \), we take \( U_{n0} \in \mathbb{V}^1 \) such that \( U_{n0} \rightarrow U_0 \) (in \( \mathbb{X}^2 \)), since \( \mathbb{V}^1 \) is dense in \( \mathbb{X}^2 \), and \( \Phi_{n0} \rightarrow \Phi_0 \) (in \( \mathcal{M}_{\Omega, \Gamma}^1 \)).
For $T > 0$ and for each integer $n \geq 1$, the weak formulation of the approximate problem $P_n$ is the following: find $(U_n, \Phi_n)$, given by (4.46) such that, for all $\bar{U} = (\bar{u}, \bar{v}) \in X_n$ and $\bar{\Phi} = (\bar{\eta}, \bar{\xi}) \in M_n$, the equations

$$\langle \partial_t U_n, \bar{U} \rangle_{X^2} + \left\langle A^{0,\beta,\nu,\omega} U_n, \bar{U} \right\rangle_{X^2} + \left\langle \Phi_n, \bar{U} \right\rangle_{M^{1,\beta}_{1,\Gamma}} + \left\langle P_n F(U_n), \bar{U} \right\rangle_{X^2} = 0$$

and

$$\langle \partial_t \Phi_n, \bar{\Phi} \rangle_{M^{1,\beta}_{1,\Gamma}} = \left\langle \partial \Phi_n, \Phi_n \right\rangle_{M^{1,\beta}_{1,\Gamma}} + \langle U_n, \bar{\Phi} \rangle_{M^{1,\beta}_{1,\Gamma}}$$

(4.49)

hold for almost all $t \in (0, T)$, subject to the initial conditions

$$\langle U_n(0), \bar{U} \rangle_{X^2} = \langle U_0, \bar{U} \rangle_{X^2} \quad \text{and} \quad \langle \Phi_n(0), \bar{\Phi} \rangle_{M^{1,\beta}_{1,\Gamma}} = \langle \Phi_0, \bar{\Phi} \rangle_{M^{1,\beta}_{1,\Gamma}}.$$  

(4.50)

To show the existence of at least one solution to (4.49)-(4.51), we now suppose that $n$ is fixed and we take $\bar{U} = \Psi_k$ and $\bar{\Phi} = \zeta_k$ for some $1 \leq k \leq n$. Then substituting the discretized functions (4.46) into (4.49)-(4.51), we easily arrive at a system of ordinary differential equations in the unknowns $a_k = a_k(t)$ and $b_k = b_k(t)$ on $X_n$ and $M_n$, respectively. We need to recall that

$$\langle P_n F(U_n), U_k \rangle = \langle F(U_n), P_n U_k \rangle = \langle F(U_n), U_k \rangle.$$  

Since $f, g \in C^1(\mathbb{R})$, we may apply Cauchy’s theorem for ODEs to find that there is $T_n \in (0, T)$ such that $a_k, b_k \in C^2(0, T_n)$, for $1 \leq k \leq n$ and both (4.49) and (4.50) hold in the classical sense for all $t \in [0, T_n]$. This argument shows the existence of at least one local solution to problem $P_n$ and ends Step 2.

Step 3. (Boundedness and continuation of approximate maximal solutions) Now we apply the (uniform) apriori estimate (4.52) which also holds for any approximate solution $(U_n, \Phi_n)$ of problem $P_n$ on the interval $[0, T_n]$, where $T_n < T$. Owing to the boundedness of the projectors $P_n$ and $Q_n$ on the corresponding spaces, we infer

$$\|U_n(t)\|_{X^2}^2 + \|\Phi_n\|_{M^{1,\beta}_{1,\Gamma}}^2 < 2 \left( \|\Phi_0\|_{M^{1,\beta}_{1,\Gamma}}^2 + \|\Phi_n(0)\|_{M^{1,\beta}_{1,\Gamma}}^2 \right)^{1/2} + \|\Phi_n(0)\|_{M^{1,\beta}_{1,\Gamma}}^2.$$  

(4.52)

for some constant $C_T > 0$ independent of $n$ and $t$. Hence, every approximate solution may be extended to the whole interval $[0, T]$, and because $T > 0$ is arbitrary, any approximate solution is a global one. As in Step 1, we also obtain the uniform bounds (4.38)-(4.44) for each approximate solution $(U_n, \Phi_n)$. Thus,

$$U_n \text{ is uniformly bounded in } L^\infty(0, T; X^2),$$  

(4.53)

$$U_n \text{ is uniformly bounded in } L^2(0, T; V^1),$$  

(4.54)

$$u_n \text{ is uniformly bounded in } L^r(\Omega \times (0, T)),$$  

(4.55)

$$u_n \text{ is uniformly bounded in } L^{r^1}(\Gamma \times (0, T)),$$  

(4.56)

$$\Phi_n \text{ is uniformly bounded in } L^\infty(0, T; M_{1,\Gamma}^1),$$  

(4.57)

$$F(U_n) \text{ is uniformly bounded in } L^r(\Omega \times (0, T)) \times L^{r^1}(\Gamma \times (0, T)),$$  

(4.58)

$$\partial_t U_n \text{ is uniformly bounded in } L^2(0, T; (V^1)^*) \oplus L^{r^1}(\Omega \times (0, T)) \times L^{r^2}(\Gamma \times (0, T)),$$  

(4.59)

$$\partial_t \Phi_n \text{ is uniformly bounded in } L^2(0, T; W_{\mu,\nu}^{-1,1,2}(\mathbb{R}_+; V^1)).$$  

(4.60)

This concludes Step 3.

Step 4. (Convergence of approximate solutions) By Alaoglu’s theorem (cf. e.g. [24, Theorem 6.64]) and the uniform bounds (4.38)-(4.55), there is a subsequence of $(U_n, \Phi_n)$, generally not relabelled, and functions $U$ and $\Phi$, obeying (4.38)-(4.44), such that as $n \to \infty$,

$$U_n \rightharpoonup U \text{ weakly-* in } L^\infty(0, T; X^2),$$  

(4.61)

$$U_n \rightarrow U \text{ weakly in } L^2(0, T; V^1),$$  

$$u_n \rightarrow u \text{ weakly in } L^r(\Omega \times (0, T)),$$  

$$u_n \rightarrow u \text{ weakly in } L^{r^1}(\Gamma \times (0, T)),$$  

$$\Phi_n \rightarrow \Phi \text{ weakly-* in } L^\infty(0, T; M_{1,\Gamma}^1).$$  

(4.61)
Moreover, setting \(k_S := (-\mu_S)^{1/2} \geq 0, S \in \{\Omega, \Gamma\}\) we have
\[
\partial_t U_n \to \partial_t U \text{ weakly in } L^2 \left(0, T; (\mathcal{V}^1)^* \right) \oplus \left(L'^i(\Omega \times (0, T)) \times L'^i(\Gamma \times (0, T)) \right),
\]
(4.62)
\[
\Phi_n \to \Phi \text{ weakly in } L^2 \left(0, T; \tilde{L}^2_{k_S^2 k_T^2} (\mathbb{R}_+; \mathcal{V}^1) \right),
\]
(4.63)
owing to the bound on \((T\Phi_n, \Phi_n)_{\mathcal{M}_{1,1,1}}\) from \((4.52)\) and
\[
\partial_t \Phi_n \to \partial_t \Phi \text{ weakly in } L^2 \left(0, T; W^{-1,2}_{\mu_0} (\mathbb{R}_+; \mathcal{V}^1) \right).
\]
(4.64)
Indeed, we observe that the last of \((4.61)\) and integration by parts yield, for any \(\zeta \in C_0^\infty (J; C_0^\infty (\mathbb{R}_+; \mathcal{V}^1))\),
\[
\int_0^T \langle \partial_t \Phi_n, \zeta \rangle_{\mathcal{M}_{1,1,1}} \, dt = - \int_0^T \langle \Phi_n, \partial_t \zeta \rangle_{\mathcal{M}_{1,1,1}} \, dt \to - \int_0^T \langle \Phi, \partial_t \zeta \rangle_{\mathcal{M}_{1,1,1}} \, dt,
\]
and that \(\Phi^t \in C(0, T; W^{-1,2}_{\mu_0} (\mathbb{R}_+; \mathcal{V}^1))\). We can exploit the second of \((4.61)\) and \((4.62)\) to deduce
\[
U_n \to U \text{ strongly in } L^2 \left(0, T; \mathcal{X}^2 \right),
\]
(4.65)
by application of the Agmon-Lions compactness criterion since \(\mathcal{V}^1\) is compactly embedded in \(\mathcal{X}^2\). This last strong convergence property is enough to pass to the limit in the nonlinear terms since \(f, g \in C^1\) (see, e.g., \([11, 15]\)). Indeed, on account of standard arguments (cf. also \([1]\)) we have
\[
P_n F(U_n) \to F(U) \text{ weakly in } L^2 \left(0, T; \mathcal{V}^1 \right).
\]
(4.66)
The convergence properties \((4.61)-(4.65)\) allow us to pass to the limit as \(n \to \infty\) in equation \((4.49)\) in order to recover \((4.21)\), using standard density arguments. Indeed, in order to pass to the limit in the equations for memory, we use \((4.63)\) and the following distributional equality
\[
\int_0^T \langle \partial_t \Phi^t, \zeta \rangle_{\mathcal{M}_{1,1,1}} \, dt = - \int_0^T \langle \Phi^t, \partial_t \zeta \rangle_{\mathcal{M}_{1,1,1}} \, dt - \int_0^T \langle \Phi^t, \partial_t \zeta \rangle_{\mathcal{M}_{1,1,1}} \, dt.
\]
Thus, we also get the last two equations of \((4.21)\) by virtue of the last of \((4.61)\).

Step 5. (Continuity of the solution) According to the description for problem \(P\), see \((4.21)\), we have
\[
\partial_t U \in L^2 \left(0, T; (\mathcal{V}^1)^* \right) \oplus \left(L'^i(\Omega \times (0, T)) \times L'^i(\Gamma \times (0, T)) \right),
\]
\[
\partial_t \Phi \in L^2 \left(0, T; W^{-1,2}_{\mu_0} (\mathbb{R}_+; \mathcal{V}^1) \right).
\]
(4.67)
Since the spaces \(L^2 \left(0, T; (\mathcal{V}^1)^* \right), \ L'^i(\Omega \times (0, T)) \times L'^i(\Gamma \times (0, T))\) are the dual of \(L^2 \left(0, T; \mathcal{V}^1 \right)\) and \(L'^i(\Omega \times (0, T)) \times L'^i(\Gamma \times (0, T))\), respectively, recalling \((4.61)\), we can argue exactly as in the proof of \([11]\) Proposition 2.5 to deduce that \(U \in C \left([0, T]; \mathcal{X}^2 \right)\). Finally, owing to \(U \in L^2(0, T; \mathcal{V}^1)\) and Corollary \((4.12)\) it follows that \(\Phi \in \mathcal{C} \left([0, T]; \mathcal{M}_{1,1,1} \right)\). Thus, both \(U(0)\) and \(\Phi(0)\) make sense and the equalities \(U(0) = U_0\) and \(\Phi^0 = \Phi_0\) hold in the usual sense due to the strong convergence of \(U_{0n} \to U_0\) in \(\mathcal{X}^2\), and \(\Phi_{0n} \to \Phi_0\) in \(\mathcal{M}_{1,1,1}\), respectively. The proof of the theorem is finished. \(\square\)

When both the bulk and boundary nonlinearities are dissipative (i.e., \((4.12)\) holds in place of the balance \((4.7)\)), we also have the following.

**Theorem 4.9.** Assume \((4.21), (4.53)\) and \((4.13), (4.12)\) hold. For each \(\alpha, \beta > 0, \omega, \nu \in (0, 1)\) and \(T > 0,\) and for any \(U_0 = (u_0, v_0)^t \in \mathcal{X}^2, \Phi_0 = (\eta_0, \xi_0)^t \in \mathcal{M}_{1,1,1}\), there exists at least one (global) weak solution \((U, \Phi) \in C \left([0, T]; \mathcal{M}_{1,1,1} \right)\) to problem \(P\) in the sense of Definition \((4.4)\)

**Proof.** The proof is essentially the same as the proof of Theorem \((4.8)\) with the exception that one employs the estimate
\[
f(u) u \geq C_f |u|^{\alpha} - C_1, \quad \bar{g}(u) u \geq C_g |u|^{\beta} - C_2, \quad \forall s \in \mathbb{R},
\]
in place of \((4.36)\), owing to \((4.12)\). This implies the same a priori estimate \((4.29)\). \(\square\)

Finally, we also have uniqueness of the weak solution in some cases.
Proposition 4.10. Let \((U_i, \Phi_i)\) be any two weak solutions of problem \(P\) in the sense of Definition 4.4. Then the following estimate holds:

\[ \|U_1(t) - U_2(t)\|_{X^2} + \|\Phi_1(t) - \Phi_2(t)\|_{M^1_{\Omega, \Gamma}} \leq \left(\|U_1(0) - U_2(0)\|_{X^2} + \|\Phi_1^0 - \Phi_2^0\|_{M^1_{\Omega, \Gamma}}\right) e^{Ct}, \tag{4.68} \]

for some constant \(C > 0\) independent of time, \(U_i\) and \(\Phi_i\).

Proof. Set \(\bar{U} = U_1 - U_2, \bar{\Phi} = \Phi_1 - \Phi_2\). The function \((\bar{U}, \bar{\Phi})\) satisfies the equations:

\[
\begin{align*}
\left\langle \partial_t \bar{U}(t), \bar{V} \right\rangle_{X^2} + & \left\langle A^{0, \beta, \nu, \omega}_{W} \bar{U}(t), \bar{V} \right\rangle_{X^2} + \left\langle F(U_1) - F(U_2), \bar{V} \right\rangle_{X^2} \\
+ & \int_0^\infty \mu_\Omega(s) \left\langle A^{0, 0, 0, 0}_{W} \bar{\Phi}^s, \bar{V} \right\rangle_{X^2} ds + \nu \int_0^\infty \mu_\Gamma(s) \left\langle C\bar{\xi}^s, \bar{v} \right\rangle_{L^2(\Gamma)} ds \\
= & 0,
\end{align*}
\]

and

\[ \left\langle \partial_t \bar{\Phi}^s, -T\bar{\Phi}^s - \bar{U}(t), \Pi \right\rangle_{M^1_{\Omega, \Gamma}} = 0, \tag{4.70} \]

for all \((V, \Pi) \in \left(V^1 \oplus (L^1(\Omega) \times L^\infty(\Gamma))\right) \times M^1_{\Omega, \Gamma}\), subject to the associated initial conditions

\[ \bar{U}(0) = U_1(0) - U_2(0) \quad \text{and} \quad \bar{\Phi}^0 = \Phi_1^0 - \Phi_2^0. \]

Multiplication of (4.69) by \(V = \bar{U}(t)\) in \(X^2\) and multiplication of (4.70) by \(\Pi = \bar{\Phi}^t\) in \(M^1_{\Omega, \Gamma}\), followed by summing the resulting identities, leads us to the differential inequality

\[ \frac{d}{dt} \left(\|U_1 - U_2\|^2_{X^2} + \|\Phi_1 - \Phi_2\|^2_{M^1_{\Omega, \Gamma}}\right) \leq -2 \left\langle F(U_1) - F(U_2), \bar{U} \right\rangle_{X^2} \]

\[ = -2 \left\langle f(u_1) - f(u_2), u_1 - u_2 \right\rangle_{L^2(\Omega)} - 2 \left\langle g(v_1) - g(v_2), v_1 - v_2 \right\rangle_{L^2(\Gamma)}, \tag{4.71} \]

Employing assumption (4.4) on the nonlinear terms, we easily find that

\[ \frac{d}{dt} \left(\|U_1 - U_2\|^2_{X^2} + \|\Phi_1 - \Phi_2\|^2_{M^1_{\Omega, \Gamma}}\right) \leq C \|U_1 - U_2\|^2_{X^2}, \tag{4.72} \]

for some \(C = C(M_f, M_g, \beta) > 0\). Application of the standard Gronwall lemma to (4.72) yields the desired claim (4.68). \(\square\)

In the final part of this section, we turn our attention to the existence of global strong solutions for problem \(P\). First, assuming that the interior and boundary share the same memory kernel, we can derive the existence of strong solutions in the case when the bulk and boundary nonlinearities have supercritical polynomial growth of order at most \(7/2\). Let \(\overline{f}, \overline{g}\) denote the primitives of \(f\) and \(g\), respectively, such that \(\overline{f}(0) = \overline{g}(0) = 0\).

Theorem 4.11. Let (4.4)-(4.7) be satisfied for \(\mu_\Omega \equiv \mu_\Gamma\), and assume that \(f, g \in C^1(\mathbb{R})\) satisfy the following assumptions:

(i) \(|f'(s)| \leq \ell_1 (1 + |s|^r_1)\), for all \(s \in \mathbb{R}\), for some (arbitrary) \(1 \leq r_1 < \frac{3}{2}\).

(ii) \(|g'(s)| \leq \ell_2 (1 + |s|^r_2)\), for all \(s \in \mathbb{R}\), for some (arbitrary) \(1 \leq r_2 < \frac{5}{2}\).

(iii) (4.4) holds and there exist constants \(C_i > 0\), \(i = 1, ..., 4\), such that

\[ f(s) s \geq -C_1 |s|^2 - C_2, \quad g(s) s \geq -C_3 |s|^2 - C_4, \quad \forall s \in \mathbb{R}. \tag{4.73} \]

Given \(\alpha, \beta > 0\), \(\omega, \nu \in (0, 1)\), \((U_0, \Phi_0) \in H^0_{\Omega, \Gamma}\), there exists a unique global strong solution \((U, \Phi)\) to problem \(P\) in the sense of Definition 4.7.

Proof. Step 1 (The existence argument). By Remark 4.6 it suffices to deduce additional regularity for \((U, \Phi)\). In order to get the crucial estimate we rely once again on various dissipative estimates. First, we notice that using the condition of (4.73), we obtain

\[ \left\langle F(U_n), U_n \right\rangle_{X^2} \geq -C_F \left(\|U_n\|^2_{X^2} + 1\right), \]
for some $C_F > 0$. Thus, arguing in the same fashion as in getting (4.39), in view of Gronwall’s lemma we obtain
\[
\|U_n(t)\|_{L^2}^2 + \|\Phi_n(t)\|^2_{M^2_{\Omega \Gamma}} \leq C_T \left( 1 + \|U(0)\|^2_{L^2} + \|\Phi^0(t)\|^2_{M^2_{\Omega \Gamma}} \right),
\]
where $C_T \sim e^{CT}$, for some $C > 0$ which is independent of $T$, $n$, $t$.

Next, we derive an estimate for $U_n \in L^\infty(0, T; \Gamma^1)$ and $\Phi_n \in L^\infty(0, T; M^2_{\Omega \Gamma})$. We use again the scheme (4.49)−(4.51) in which we test equation (4.49) with the function
\[
\overline{U} = Z_n := \left( z_n \right)_{z_n \in \Gamma}, \quad z_n := \sum_{i=1}^n a_i(t) \lambda_i \theta^{\alpha, \beta, \nu, \omega}_i \in C^2 \left( [0, T) \times \overline{\Omega} \right).
\]
We get
\[
\langle \partial_t U_n, Z_n \rangle_{\Gamma^2} + \left\langle A_n^{0, \beta, \nu, \omega} U_n, Z_n \right\rangle_{\Gamma^2} + \left\langle \Phi_n^t(s), Z_n \right\rangle_{M^2_{\Omega \Gamma}} + \langle F(U_n), Z_n \rangle_{\Gamma^2} = 0.
\]
Moreover, testing (4.50) with \(\overline{\Phi} = \Xi_n^t := \left( \varphi_n^t \right)_{\varphi_n^t \in \Gamma}, \quad \varphi_n^t := \sum_{i=1}^n b_i(t) \varphi_i(s) \lambda_i \theta^{\alpha, \beta, \nu, \omega}_i = \sum_{i=1}^n b_i(t) \lambda_i \zeta_i(s) \)
we find
\[
\langle \partial_t \Phi_n^t, \Xi_n^t \rangle_{M^2_{\Omega \Gamma}} = \langle T \Phi_n^t, \Xi_n^t \rangle_{M^2_{\Omega \Gamma}} + \left\langle U_n, \Xi_n^t \right\rangle_{M^2_{\Omega \Gamma}}.
\]
Indeed, \((Z_n, \Xi_n^t) \in X_n \times M_n\) is admissible as a test function in (4.49)−(4.50). Recalling (4.40), we further notice that \(Z_n = A_n^{\alpha, \beta, \nu, \omega} U_n\) and \(\Xi_n^t = A_n^{\alpha, \beta, \nu, \omega} \Phi_n^t\), respectively, due to the fact that the eigenpair \((\lambda_i, \theta^{\alpha, \beta, \nu, \omega}_i)\) solves (4.39). Owing to these identities and (4.31), we have
\[
\langle \Phi_n^t(s), Z_n \rangle_{M^2_{\Omega \Gamma}} = \mu_\Omega(s) \left| \left. A_n^{0, \beta, \nu, \omega} \Phi_n^t(s) \right|_{\Omega^1} \right|^2_{L^2(\Gamma)} ds + \nu \int_0^\infty \mu_\Gamma(s) \left| \left. C_{\lambda_n^t}(s) \right|_{\Omega^1} \right|^2_{L^2(\Gamma)} ds
\]
\[
= \langle U_n, \Xi_n^t \rangle_{M^2_{\Omega \Gamma}}.
\]
Adding relations (4.75)−(4.76) together, and using (4.77) we further deduce
\[
\frac{1}{2} \frac{d}{dt} \left( \|U_n\|^2_{L^2(\Omega)} + \|\Xi_n^t\|^2_{L^2(\Omega)} \right) + \|U_n\|^2_{L^2(\Omega)} + \|\Xi_n^t\|^2_{\Gamma^2} = \alpha \omega \langle u_n, z_n \rangle_{L^2(\Omega)} - \langle F(U_n), Z_n \rangle_{\Gamma^2},
\]
and
\[
\langle T \Phi_n^t, \Xi_n^t \rangle_{M^2_{\Omega \Gamma}} = \mu_\Omega(s) \left| \left. A_n^{\alpha, \beta, \nu, \omega} T \Phi_n^t \Xi_n^t \right|_{\Omega^1} \right|^2_{L^2(\Omega)} ds = \frac{1}{2} \int_0^\infty \mu_\Omega(s) \left| \left. \Xi_n^t(s) \right|_{\Omega^1} \right|^2_{\Gamma^2} ds,
\]
thanks to the fact that $\mu_\Omega \equiv \mu_\Gamma$. We begin estimating both terms on the right-hand side of (4.78). The first one is easy,
\[
\alpha \omega \langle u_n, z_n \rangle_{L^2(\Omega)} \leq \delta \|z_n\|^2_{L^2(\Omega)} + C_\delta \|u_n\|^2_{L^2(\Omega)},
\]
for any $\delta \in (0, 1]$. To bound the last term we integrate by parts in the following way:
\[
\langle F(U_n), Z_n \rangle_{\Gamma^2} = \int_\Omega f(u_n) \left( -\omega \Delta u_n + \alpha \omega u_n \right) dx + \int_\Gamma \tilde{g}(u_n) \left( -\nu \Delta \Gamma u_n + \omega \partial_n u_n + \nu \beta u_n \right) d\sigma
\]
\[
= \omega \int_\Omega f'(u_n) |\nabla u_n|^2 dx + \nu \int_\Gamma \tilde{g}'(u_n) |\nabla \Gamma u_n|^2 d\sigma
\]
\[
+ \alpha \omega \int_\Omega f(u_n) u_n dx + \nu \beta \int_\Gamma \tilde{g}(u_n) u_n d\sigma
\]
\[
+ \omega \int_\Gamma \left( \tilde{g}(u_n) - f(u_n) \right) \partial_n u_n d\sigma.
\]
By assumptions (4.4) and (4.73), we can easily find a positive constant $C_{\delta}$ independent of $t, T$ and $n$ such that

$$\omega \int_{\Omega} f^\ast(u_n) |\nabla u_n|^2 \, dx + \nu \int_{\Gamma} \bar{g}^\ast(u_n) |\nabla T u_n|^2 \, d\sigma \geq -M_f \omega \|\nabla u_n\|^2_{L^2(\Omega)} - M_g \nu \|\nabla T u_n\|^2_{L^2(\Gamma)}$$

(4.82)

and

$$\alpha \omega \int_{\Omega} f(u_n) u_n \, dx + \nu \beta \int_{\Gamma} \bar{g}(u_n) u_n \, d\sigma \geq -C \left(\|u_n\|^2_{L^2} + 1\right).$$

(4.83)

In order to estimate the last boundary integral on the right-hand side of (4.81), we observe that due to assumptions (i)-(ii) it suffices to estimate boundary integrals of the form

$$I := \int_{\Gamma} u_n^{r+1} \partial_n u_n \, d\sigma, \quad \text{for some } r < 5/2.$$

Indeed, due to classical trace regularity and embedding results, for every $\delta \in (0, 1]$ we have

$$I \leq \|\partial_n u_n\|_{H^{1/2}(\Gamma)} \|u_n^{r+1}\|_{H^{-1/2}(\Gamma)} \leq \delta \|u_n\|^2_{H^2(\Omega)} + C_{\delta} \|u_n^{r+1}\|^2_{H^{-1/2}(\Gamma)}.$$  

(4.84)

It remains to estimate the last term in (4.84). To this end, we employ the basic Sobolev embeddings $H^{1/2}(\Gamma) \subset L^4(\Gamma)$ and $H^1(\Gamma) \subset L^s(\Gamma)$, for any $s \in \left(\frac{6}{5}, \infty\right)$, respectively. Owing to elementary Holder inequalities, we deduce that

$$\|u_n^{r+1}\|^2_{H^{-1/2}(\Gamma)} = \sup_{\psi \in H^{1/2}(\Gamma): \|\psi\|_{H^{1/2}(\Gamma)} = 1} |\langle u_n^{r+1}, \psi \rangle|^2$$

(4.85)

for some positive constant $C$ independent of $u, n, t, T$, for sufficiently large $s \in \left(\frac{6}{5}, \infty\right)$, where $\bar{s} := 4s / (3s - 4) > 4/3$. Exploiting now the interpolation inequality

$$\|u\|_{L^{s}(\Gamma)} \leq C \|u\|^{1/(2r)}_{H^2(\Gamma)} \|u\|^{1-1/(2r)}_{L^{2s}(\Gamma)},$$

provided that $r = 1 + 2/\bar{s} < 5/2$, we further infer from (4.85) that

$$\|u_n^{r+1}\|^2_{H^{-1/2}(\Gamma)} \leq C \|u_n\|^2_{H^1(\Gamma)} \|u_n\|^2_{H^2(\Gamma)} \|u_n\|^{2r-1}_{L^2(\Gamma)}$$

(4.86)

$$\leq \eta \|u_n\|^2_{H^1(\Gamma)} + C \|u_n\|^2_{H^1(\Gamma)} \left(\|u_n\|^2_{H^2(\Gamma)} \|u_n\|^{2(2r-1)}_{L^2(\Gamma)}\right),$$

for any $\eta \in (0, 1)$. Inserting (4.86) into (4.84) and choosing a sufficiently small $\eta = \delta/C_{\delta}$, by virtue of (4.17), we easily deduce

$$I \leq \delta \|Z_n\|^2_{L^2} + C_{\delta} \|u_n\|^2_{H^1(\Gamma)} \left(\|u_n\|^2_{H^2(\Gamma)} \|u_n\|^{2(2r-1)}_{L^2(\Gamma)}\right).$$

(4.87)

Thus, setting

$$\Xi(t) := \|U_n(t)\|^2_{V_1} + \|\Xi_n\|^2_{L^2_{\nu,1}(R^+; X^2)},$$

$$\Lambda(t) := C_{\delta} \left(1 + \|u_n\|^2_{H^1(\Gamma)} \|u_n\|^{2(2r-1)}_{L^2(\Gamma)}\right),$$

it follows from (4.78), (4.80), (4.83) and (4.87) that

$$\frac{d}{dt} \Xi(t) - 2 \langle T \Phi_n^t, \Xi_n^t \rangle_M + (2 - \delta) \|Z_n\|^2_{L^2} \leq \Xi(t) \Lambda(t),$$

(4.88)

for a sufficiently small $\delta \in (0, 1]$. Gronwall’s inequality together with (4.73) yields

$$\|U_n(t)\|^2_{V_1} + \|\Xi_n\|^2_{L^2_{\nu,1}(R^+; X^2)} + \int_0^t \left(\|Z_n(\tau)\|^2_{L^2} - 2 \langle T \Phi_n^\tau, \Xi_n^\tau \rangle_M\right) \, d\tau$$

(4.89)

$$\leq C_T \left(\|U(0)\|^2_{V_1} + \|\Xi^0\|^2_{L^2_{\nu,1}(R^+; X^2)}\right),$$

owing to the boundedness of the (orthogonal) projectors $P_n : X^2 \to X_n$ and $Q_n : M^1_{\Omega, \Gamma} \to M_n$, and the fact that $\Lambda \in L^1(0, T)$, for any $T > 0$. 
From (4.89), recalling (4.17), we obtain the following uniform (in \( n \)) bounds for each approximate solution \((U_n, \Phi_n)\):

\[
\begin{align*}
U_n &\text{ is uniformly bounded in } L^\infty \left(0, T; \mathcal{V}^1 \right), \\
U_n &\text{ is uniformly bounded in } L^2 \left(0, T; \mathcal{V}^2 \right), \\
\Phi_n &\text{ is uniformly bounded in } L^\infty \left(0, T; \mathcal{M}_{{\Omega_1, \Gamma}}^2 \right), \\
\Phi_n &\text{ is uniformly bounded in } L^2 \left(0, T; L_{k_0}^2 \left(\mathbb{R}_+; \mathcal{V}^2 \right) \right).
\end{align*}
\]

(4.90) (4.91) (4.92) (4.93)

Observe now that by (4.49)-(4.50), we also have

\[
\langle \partial_t U_n, \overline{U} \rangle_{\mathcal{X}^2} = \langle \partial_t U_n, P_n \overline{U} \rangle_{\mathcal{X}^2}
\]

\[
= - \langle \Lambda^0_{W, \beta, \nu, \omega} U_n, P_n \overline{U} \rangle_{\mathcal{X}^2} - \langle \Phi_n, P_n \overline{U} \rangle_{\mathcal{M}_{{\Omega_1, \Gamma}}} - \langle F(U_n), P_n \overline{U} \rangle_{\mathcal{X}^2}
\]

(4.94)

and

\[
\langle \partial_t \Phi_n, \overline{\Phi} \rangle_{\mathcal{M}_{{\Omega_1, \Gamma}}} = \langle \partial_t \Phi_n, Q_n \overline{\Phi} \rangle_{\mathcal{M}_{{\Omega_1, \Gamma}}}
\]

\[
= \langle T \Phi_n, Q_n \overline{\Phi} \rangle_{\mathcal{M}_{{\Omega_1, \Gamma}}} + \langle U_n, Q_n \overline{\Phi} \rangle_{\mathcal{M}_{{\Omega_1, \Gamma}}},
\]

respectively. Thus, from the uniform bounds (4.90)-(4.93), we deduce by comparison in equations (4.94)-(4.95) that

\[
\begin{align*}
\partial_t U_n &\text{ is uniformly bounded in } L^\infty \left(0, T; (\mathcal{V}^1)^* \right) \cap L^2 \left(0, T; \mathcal{X}^2 \right), \\
\partial_t \Phi_n &\text{ is uniformly bounded in } L^2 \left(0, T; L_{\mu_1}^2 \left(\mathbb{R}_+; (\mathcal{V}^1)^* \right) \right) \cap L^\infty \left(0, T; L_{\mu_1}^2 \left(\mathbb{R}_+; \mathcal{V}^2 \right) \right).
\end{align*}
\]

(4.96) (4.97)

We are now ready to pass to the limit as \( n \) goes to infinity. On account of the above uniform inequalities, we can find \( U \) and \( \Phi \) such that, up to subsequences,

\[
U_n \rightarrow U \text{ weakly * in } C([0, T]; \mathcal{X}^2).
\]

(4.98) (4.99) (4.100) (4.101) (4.102) (4.103) (4.104)

Due to (4.98) and (4.102) and the classical Agmon-Lions compactness theorem, we also have

\[
U_n \rightarrow U \text{ strongly in } C([0, T]; \mathcal{X}^2).
\]

(4.104)

Thanks to (4.98)-(4.103) and (4.104), we can easily control the nonlinear terms in (4.49)-(4.50). By means of the above convergence properties, we can pass to the limit in these equations and show that \((U, \Phi)\) solves (4.21) in the sense of Definition 4.5.

Finally, uniqueness follows from Proposition 4.10 owing to assumption (4.4). The proof of the theorem is finished. \( \square \)

**Remark 4.12.** Observe that the assumption \( \mu_2 \equiv \mu_\Gamma \) in Theorem 4.11 is crucial for the identity (4.77) to hold. Without it, cancellation in (4.78) does not generally occur and (4.79) does not hold.

We now let

\[
h_f(s) = \int_0^s f'(\tau) \tau d\tau \text{ and } h_g(s) = \int_0^s g'(\tau) \tau d\tau.
\]

The next result states that there exist strong solutions, albeit in a much weaker sense than in Theorem 2.11, even when the interior and boundary memory kernels \( \mu_S(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) do not coincide but both decay exponentially fast as \( s \) goes to infinity.
Theorem 4.13. Let \( f \), \( g \in C^1(\mathbb{R}) \) satisfy the following conditions:

(i) \(|f'(s)| \leq \ell_1 \left(1 + |s|^2 \right)\), for all \( s \in \mathbb{R}\).

(ii) \(|g'(s)| \leq \ell_2(1 + |s|^{r_2})\), for all \( s \in \mathbb{R}\), for some (arbitrary) \( r_2 > 2\).

(iii) \( (4.4)\) holds and there exist \( C_i > 0, i = 1, \ldots, 8\), such that

\[
\begin{align*}
&f(s) \geq -C_1 |s|^2 - C_2, \quad g(s) \geq -C_3 |s|^2 - C_4, \quad \forall s \in \mathbb{R} \\
h_f(s) \geq -C_5 |s|^2 - C_6, \quad h_g(s) \geq -C_7 |s|^2 - C_8, \quad \forall s \in \mathbb{R}.
\end{align*}
\]

(4.105)

In addition, assume there exist constants \( \delta_S > 0 \) such that

\[
\mu_S(s) + \delta_S \mu_S(s) \leq 0, \quad \text{for all} \quad s \in \mathbb{R}^+, \quad S \in \{\Omega, \Gamma\}.
\]

(4.106)

Given \( \alpha, \beta > 0, \omega, \nu \in (0, 1), \) \( (U_0, \Phi_0) \in \mathbb{V}^2 \times (\mathcal{M}^{2}_{\Omega, \Gamma} \cap D(T)) \), there exists a unique global quasi-strong solution \((U, \Phi)\) to problem \( P \) in the sense of Definition 4.7.

Proof. It suffices to provide bounds for \((U, \Phi)\) in the (more regular) spaces in \((4.25)-(4.28)\). With reference to problem \( P_n \), we consider the approximate problem of finding \((U_n, \Phi_n)\) of the form \((4.10)\) such that, \((U_n, \Phi_n)\) already satisfies \((4.49)-(4.50)\), and

\[
\begin{align*}
\langle \partial_t U_n, \nabla \rangle_{\mathcal{X}_2} + \langle A^{0, \beta, \nu, \omega}_W U_n, \nabla \rangle_{\mathcal{X}_2} + \langle \partial_t \Phi^t_n, \Phi \rangle_{\mathcal{M}^{1}_{\Omega, r}} \quad (4.107) \\
= -\left( f(u) \partial_t u_n, \tilde{u} \right)_{L^2(\Omega)} - \left( g(\tilde{u}) \partial_t \tilde{u}_n, \tilde{v} \right)_{L^2(\Gamma)}
\end{align*}
\]

and

\[
\begin{align*}
\langle \partial_t \Phi^t_n, \Phi \rangle_{\mathcal{M}^{1}_{\Omega, r}} = \langle T \partial_t \Phi^t_n, \Phi \rangle_{\mathcal{M}^{1}_{\Omega, r}} + \langle \partial_t U_n, \Phi \rangle_{\mathcal{M}^{1}_{\Omega, r}} \quad (4.108)
\end{align*}
\]

hold for almost all \( t \in (0, T) \), for all \( \nabla U = (\tilde{u}, \tilde{v})^{\text{T}} \in X_n \) and \( \Phi = (\tilde{\eta}, \tilde{\xi})^{\text{T}} \in M_n \); moreover, the function \((U_n, \Phi_n)\) fulfils the conditions \( U_n(0) = P_n U_0, \Phi^0_n = Q_n \Phi^0 \) and

\[
\partial_t U_n(0) = P_n \hat{U}_0, \quad \partial_t \Phi^t_n = Q_n \hat{\Phi}^t_0,
\]

(4.109)

where we have set

\[
\begin{align*}
\hat{U}_0 := -A^{0, \beta, \nu, \omega}_W U_0 - \int_0^{\infty} \mu_\Omega(s) A^{0, 0, 0, \omega}_W \Phi_0(s) ds - \nu \int_0^{\infty} \mu_\Gamma(s) \left( C_\xi_0(s) \right) ds - F(U_0), \\
\hat{\Phi}^0 := T \Phi_0(s) + U_0.
\end{align*}
\]

Note that, if \( U_0 \in \mathbb{V}^2 \) and \( \Phi^0 \in D(T) \cap \mathcal{M}^2_{\Omega, \Gamma} \), then \((\hat{U}_0, \hat{\Phi}^0) \in \mathbb{X}^2 \times \mathcal{M}^{0,1}_{\Omega, \Gamma} = \mathcal{H}^{0,1}_{\Omega, \Gamma} \), owing to the continuous embeddings \( H^2(\Omega) \subset L^\infty(\Omega), \) \( H^2(\Gamma) \subset L^\infty(\Gamma) \). In particular, owing to the boundedness of the projectors \( P_n \) and \( Q_n \) on the corresponding subspaces, we have

\[
\| (\partial_t U_n(0), \partial_t \Phi^t_n) \|_{\mathcal{H}^{0,1}_{\Omega, \Gamma}} \leq K(R),
\]

(4.110)

for all \((U_0, \Phi^0) \in \mathbb{V}^2 \times (D(T) \cap \mathcal{M}^2_{\Omega, \Gamma})\) such that \( \| (U_0, \Phi^0) \|_{\mathcal{H}^{2,2}_{\Omega, \Gamma}} \leq R \), for some positive monotone nondecreasing function \( K \). Indeed, according to assumptions \((4.41)-(4.43)\), we can infer that

\[
0 \leq \int_0^{\infty} \mu_S(s) ds = \mu_S^0 < \infty, \quad \text{for each} \quad S \in \{\Omega, \Gamma\},
\]

(4.111)

such that repeated application of Jensen’s inequality yields

\[
\begin{align*}
\left\| \int_0^{\infty} \mu_\Omega(s) A^{0, \beta, \nu, \omega}_W \Phi_0(s) ds \right\|_{\mathcal{X}_2}^2 \leq \mu_\Omega^0 \int_0^{\infty} \mu_\Omega(s) \left\| A^{0, 0, 0, \omega}_W \Phi_0(s) \right\|_{\mathcal{X}_2}^2 ds \\
\leq C \mu_\Omega^0 \int_0^{\infty} \mu_\Omega(s) \left\| \Phi_0(s) \right\|_{H^2_2}^2 ds
\end{align*}
\]

and

\[
\begin{align*}
\left\| \int_0^{\infty} \mu_\Gamma(s) C_\xi_0(s) ds \right\|_{L^2(\Gamma)}^2 \leq \mu_\Gamma^0 \int_0^{\infty} \mu_\Gamma(s) \left\| C_\xi_0(s) \right\|_{L^2(\Gamma)}^2 ds \\
\leq C \mu_\Gamma^0 \int_0^{\infty} \mu_\Gamma(s) \left\| \Phi_0(s) \right\|_{H^2_2(\Gamma)}^2 ds
\end{align*}
\]
Our starting point is the validity of the energy estimate (4.74) which holds on account of the first assumption of (4.105). Next we proceed to take \( \overline{U} = \partial_t U_n(t) \) in (4.107) and \( \overline{\Phi} = \partial_t \Phi^\prime_n(s) \) in (4.108), respectively, by noting that this choice \((\overline{U}, \overline{\Phi})\) is an admissible test function. Summing the resulting identities and using (4.3), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \|\partial_t U_n\|^2_{H^1} + \|\partial_t \Phi^\prime_n\|^2_{H^1} \right) - \langle T \partial_t \Phi^\prime_n, \partial_t \Phi^\prime_n \rangle_{\mathcal{M}^1_{1,1}} + \left( \omega \|\nabla \partial_t u_n\|^2_{L^2(\Omega)} + \nu \|\nabla \partial_t u_n\|^2_{L^2(\Gamma)} + \beta \|\partial_t u_n\|^2_{L^2(\Gamma)} \right) = - \left( f'(u_n) \partial_t u_n, \partial_t u_n \right)_{L^2(\Omega)} - \left( g'(u_n) \partial_t u_n, \partial_t u_n \right)_{L^2(\Gamma)} \leq \max(M_f, M_g) \|\partial_t u_n\|^2_{L^2}.
\]

Thus, integrating (4.112) with respect to \( \tau \in (0, t) \), by application of Graw’s inequality, we have the estimate

\[
\left\| \left( \partial_t U_n(t), \partial_t \Phi^\prime_n \right)_{\mathcal{H}^2} \right\|^2 + \int_0^t \left( 2 \left\| \partial_t U_n(\tau) \right\|^2_{H^1} + \left\| \partial_t \Phi^\prime_n \right\|^2_{L^2(\Omega \oplus \mathbb{R}^+; V^1)} \right) d\tau \leq K_T(R),
\]

for all \( t \geq 0 \) and all \( R > 0 \) such that \( \left\| \left( U_0, \Phi^0 \right) \right\|_{\mathcal{H}^2} \leq R \). Thanks to (4.113), we deduce the uniform bounds

\[
\partial_t U_n \in L^\infty(0, T; H^1) \cap L^2(0, T; V^1),
\]

\[
\partial_t \Phi^\prime_n \in L^\infty(0, T; H^1) \cap L^2(0, T; L^2(\Omega \oplus \mathbb{R}^+; V^1)),
\]

which establishes (4.114)-(4.115) for the approximate solution \((U_n, \Phi^\prime_n)\).

We now establish a bound for \( U_n \in L^\infty(0, T; V^1) \) in a different way from the proof of Theorem 4.11. For this estimate, the uniform regularity in (4.114)-(4.115) is crucial. To this end, we proceed to take \( \overline{U} = U_n(t) \) in (4.107) in order to derive

\[
\frac{d}{dt} \left( \|U_n\|^2_{V^1} + \|\partial_t U_n, U_n\|_{H^1}^2 \right) + 2 \int_\Omega h_f(u_n) \, dx + 2 \int_\Gamma h_g(u_n) \, d\sigma
\]

\[
= 2 \left\| \partial_t U_n \right\|^2_{L^2} - 2 \langle \partial_t \Phi^\prime_n, U_n \rangle_{\mathcal{M}^1_{1,1}}.
\]

Moreover, using (4.114) and owing to the Cauchy-Schwarz and Young inequalities and the second of (4.105), the following basic inequality holds:

\[
C_* \|U_n\|^2_{V^1} - K_T(R)
\]

\[
\leq \|U_n\|^2_{V^1} + \langle \partial_t U_n, U_n \rangle_{H^1}^2 + 2 \int_\Omega h_f(u_n) \, dx + 2 \int_\Gamma h_g(u_n) \, d\sigma
\]

\[
\leq C \|U_n\|^2_{V^1} - K_T(R),
\]

for some constants \( C_*, C > 0 \) and some function \( K_T > 0 \), all independent of \( n \) and \( t \). Finally, for any \( \eta > 0 \) we estimate

\[
- \langle \partial_t \Phi^\prime_n, U_n \rangle_{\mathcal{M}^1_{1,1}} \leq \eta \|U_n\|^2_{V^1} + C_\eta \int_0^\infty \mu_\Omega(s) \left\| \partial_t \eta_n(s) \right\|^2_{H^1} \, ds + C_\eta \int_0^\infty \mu_\Gamma(s) \left\| \partial_t \xi_n(s) \right\|^2_{H^1(\Gamma)} \, ds
\]

\[
\leq \eta \|U_n\|^2_{V^1} - C_\eta \delta^\prime \int_0^\infty \mu_\Omega(s) \left\| \partial_t \eta_n(s) \right\|^2_{H^1} \, ds - C_\eta \delta^\prime \int_0^\infty \mu_\Gamma(s) \left\| \partial_t \xi_n(s) \right\|^2_{H^1(\Gamma)} \, ds,
\]

where in the last line we have employed assumption (4.106). Thus, from (4.116) we obtain the inequality

\[
\frac{d}{dt} \left( \|U_n\|^2_{V^1} + \|\partial_t U_n, U_n\|_{H^1}^2 \right) + 2 \int_\Omega h_f(u_n) \, dx + 2 \int_\Gamma h_g(u_n) \, d\sigma
\]

\[
\leq C_\eta \|U_n(t)\|^2_{V^1} - \Lambda_2(t),
\]

for a.e. \( t \in (0, T) \), where we have set

\[
\Lambda_2(t) := 2 \|\partial_t U_n\|^2_{H^1} - 2 \langle \partial_t \Phi^\prime_n, U_n \rangle_{\mathcal{M}^1_{1,1}}.
\]
We now observe that $\Lambda_2 \in L^1(0, T)$ on account of (4.72), (4.114)-(4.115) and (4.117)-(4.118), because $\partial_t U_n(0) \in \mathcal{X}^2$ by (4.109). Thus, observing (4.117), the application of Gronwall’s inequality to (4.119) yields the desired uniform bound
\[
U_n \in L^\infty \left(0, T; \mathcal{V}^1\right).
\] (4.120)
Finally, by comparison in equation (4.95), by virtue of the uniform bounds (4.120) and (4.115) we also deduce
\[
\Phi_n(t) \in L^\infty \left(0, T; \mathcal{M}^2_{1, R}\right)
\] (4.121)
uniformly with respect to all $n \geq 1$. In particular, it holds $\Phi_n \in L^\infty (0, T; \mathcal{D}(r))$ uniformly. Finally, by (4.120) and assumptions (i)-(ii), we also have
\[
F(U_n) \in L^\infty \left(0, T; \mathcal{V}^2\right).
\] We can pass to the limit as $n \to \infty$ in (4.114)-(4.115), (4.120) and (4.121) to find a limit point $(U, \Phi)$ with the properties stated in (4.25)-(4.28). Passage to the limit in equations (4.49)-(4.50) and in particular, thanks to (4.113)-(4.115) and the fact that
\[
\Phi \in H^2(0, T; \mathcal{V}^2),
\]
we deduce
\[
\Phi(t) \in H^2 \left(0, T; \mathcal{V}^2\right),
\]
which is enough to deduce
\[
U \in L^\infty \left(0, T; \mathcal{V}^2\right),
\]
from (4.109), we have $\Phi(t) \in H^2 \left(0, T; \mathcal{V}^2\right)$, and therefore,
\[
\|\partial_t u\|_{L^\infty(0, T; L^2(\Omega))} + \|\partial_t t\|_{L^\infty(0, T; L^2(\Gamma))} \leq C_T,
\]
thanks to (4.113)- (4.118) and the fact that $U \in \mathcal{V}^2$. It follows that $(U, \Phi)$ also satisfies the following elliptic system
\[
\begin{align*}
-\omega \Delta u + \omega \omega u + f(u) &= H_\Omega, \quad \text{a.e. in } \Omega \times (0, T), \\
-\nu \Delta \Gamma u + \omega \partial_\nu u + \nu \partial_\nu \Phi + \tilde{g}(u) &= H_\Gamma, \quad \text{a.e. in } \Gamma \times (0, T),
\end{align*}
\] (4.123)
where we have set
\[
\begin{align*}

\left(H_\Omega \atop H_\Gamma\right) := \left(\begin{array}{c}
\mu \int_0^t \rho \alpha(s) \left(-\Delta \eta^\prime(s) + \alpha \eta^\prime(s) ds - \partial_t u\right) \\
\nu \int_0^t \mu \eta^\prime(s) C \xi \eta^\prime(s) ds - \partial_t u
\end{array}\right).
\end{align*}
\]
We now observe that in view of (4.110), we have $\Phi(t) = (\eta^\prime, \xi)^T \in L^\infty \left(0, T; \mathcal{L}^2_0(\mathbb{R}^+; \mathcal{V}^2)\right)$, and therefore,
\[
\|H_\Omega\|_{L^\infty(0, T; L^2(\Omega))} + \|H_\Gamma\|_{L^\infty(0, T; L^2(\Gamma))} \leq C_T,
\] (4.124)
owing to (4.114) (see the scheme developed in Theorem 4.13 cf. (4.107)-(4.109) and (4.112)-(4.113)). Thus, we can apply a regularity result for the system (4.123) from [22, Lemma A.2] to further deduce
\[
u(t) \in H^2(0, T; \mathcal{V}^2),
\]
and therefore,
\[
\|\partial_t u\|_{L^\infty(0, T; L^2(\Omega))} \leq C_T
\]
(4.125)
On the other hand, to prove the first bound on $U$ from (4.122), we need to apply Lemma 3.1 with the obvious choices:
\[
p_1 := H_\Omega - f(u), \quad p_2 := -\tilde{g}(u) + H_\Gamma.
\]
Owing to (4.123) and once again to (4.124), it is not difficult to realize that $p_1 \in L^\infty \left(0, T; L^2(\Omega)\right)$, $p_2 \in L^\infty \left(0, T; L^2(\Gamma)\right)$, which is enough to deduce
\[
u(t) \in H^2(0, T; \mathcal{V}^2),
\]
and therefore,
\[
\|\partial_t u\|_{L^\infty(0, T; L^2(\Omega))} \leq C_T
\]
(4.126)
The final bound in (4.127) is an immediate consequence of (4.115). The proof is finished. \qed
Remark 4.15. In Theorem 4.13, since the initial datum $(U_0, \Phi_0)$ belongs to $\mathcal{V}^2 \times (\mathcal{M}_{\Omega,T}^2 \cap D(T))$ it would be desirable to prove that

$$U \in L^\infty(0,T;\mathcal{V}^2), \quad \Phi \in L^\infty(0,T;\mathcal{M}_{\Omega,T}^2),$$

as well. Unfortunately, we cannot deduce (4.127) as in the proof of Theorem 4.11 because generally, $\mu_\Omega \neq \mu_T$; see Remark 4.12.

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