One step beyond: The excursion set approach with correlated steps

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ABSTRACT
We provide a simple formula that accurately approximates the first crossing distribution of barriers having a wide variety of shapes, by random walks with a wide range of correlations between steps. Special cases of it are useful for estimating halo abundances, evolution, and bias, as well as the nonlinear counts in cells distribution. We discuss how it can be extended to allow for the dependence of the barrier on quantities other than overdensity, to construct an excursion set model for peaks, and to show why assembly and scale dependent bias are generic even at the linear level.

Key words: large-scale structure of Universe

1 INTRODUCTION
In hierarchical clustering models, to estimate cluster abundances at any given time one must estimate the abundance of sufficiently overdense regions in the initial conditions (Press & Schechter 1974). The problem is to find those regions in the initial conditions that are sufficiently overdense on a given smoothing scale, but not on a larger scale. The framework for not double-counting smaller overdense regions that are embedded in larger ones is known as the Excursion Set approach (Epstein 1983; Bond et al. 1991; Lacey & Cole 1993; Sheth 1998).

In this approach, one looks at the overdensity around any given random point in space as a function of smoothing scale. The resulting curve resembles a random walk, whose height tends to zero on very large smoothing scales. In this overdensity versus scale plane, the critical density for collapse defines another curve, which we will call the barrier. The double-counting problem is solved by asking for the largest smoothing scale on which the walk first crosses the barrier. It is fairly straightforward to solve this problem numerically, by direct Monte-Carlo simulation of the path integrals (Bond et al. 1991). This is particularly simple if the steps in the walk are independent, but one can also include correlations between the steps, whose nature depends on the underlying fluctuation field (i.e., for a Gaussian field, on the power spectrum) and the form of the smoothing filter.

When the steps are independent, exact solutions for constant (Bond et al. 1991) or linear (Sheth 1998) barriers are known. Exact solutions for more general barrier shapes are not known, but good analytic approximations are available (Sheth & Tormen 2002; Lam & Sheth 2009). In the appropriate units, these solutions are self-similar, i.e. independent of the form of the power spectrum (of course, they depend strongly on the barrier shape). However, an exact solution of the first crossing problem for correlated steps is still unknown. The main goal of the present work is to provide a simple formula which works for a wide variety of barrier shapes, smoothing filters, and power spectra. This is done in Section 2 – our main result is equation (5), and it is explicitly not self-similar. Section 3 describes a number of extensions of this calculation – having to do with walks conditioned to pass through a certain point in a certain way, or with barriers whose height depends on hidden variables. A final section summarizes our results, indicating how we expect our work to be used when fitting to the halo abundances which, recent simulations indicate, are not quite self-similar.

2 FIRST CROSSING DISTRIBUTION WITH CORRELATED STEPS
In what follows, we will assume that the underlying fluctuation field is Gaussian. We comment on non-Gaussian field in the Discussion section. In hierarchical models, the variance of the fluctuation field is a monotonic function of the smoothing scale (the exact relation depending on the shape of the power spectrum and the smoothing window.) Therefore we can use the terms smoothing scale and variance interchangeably, and we will use $s$ to denote the variance.

Let $B(s)$ denote the height of the barrier on ‘scale’ $s$, and $p(\delta, s)$ the probability that the walk has height $\delta$ on this scale. We will assume that $\langle \delta \rangle \equiv 0$, so that $s \equiv \langle \delta^2 \rangle$. We
would like to write down the probability \( f(s) \) that \( \delta < B(S) \) for all \( S < s \) and \( \delta > B(s) \) at \( s \). When the steps in the walk are uncorrelated the two conditions separate, simplifying the analysis. For correlated steps, this simplicity is lost.

### 2.1 The completely correlated limit

Recently Paranjape et al. (2012) have argued that there is considerable virtue in thinking of the limiting case in which the steps in the walk are completely correlated. In this case, if the walk had height \( \delta_0 \) on scale \( S_0 \), then it has height \( \delta = \delta_0 \sqrt{s/S_0} \) on scale \( s \), so \( \delta \) is a smooth monotonic function of \( s \). Therefore, if \( \delta \) first exceeds \( B(s) \) on scale \( s \), it was certainly below \( B(s) \) on all \( S < s \), and one need not account for this requirement explicitly. Hence, the first crossing distribution is just

\[
  f_{cc}(s) = \frac{\partial}{\partial s} \int_{B(s)}^{\infty} d\delta \, p(\delta, s) = -\frac{\exp(-B^2(s)/2s)}{2\pi s} \frac{dB(s)/\sqrt{s}}{ds}\, \text{for } s < S_{\text{min}}, \tag{1}
\]

where \( S_{\text{min}} \) is the scale on which \( B(s)/\sqrt{s} \) is minimum (in many cases of interest \( S_{\text{min}} = \infty \)). Note that this relates the shape of \( f(s) \) to that of \( p(\delta, s) \) on the same scale.

Paranjape et al. (2012) show that, despite the strong assumption about the deterministic smoothness of the walks, this expression provides a very good description of the first crossing distribution (at small \( s \)) even when the steps are not completely correlated. Physically, this is because one thinks of real walks as stochastic zig-zags superimposed on smooth completely correlated trajectories at small \( s \) a walk has not fluctuated enough to depart significantly from its deterministic counterpart. One might thus expect that there is danger of double counting trajectories with two or more crossings only when \( s \) becomes large.

In cosmology the large \( s \) regime is not nearly as interesting as the small. The discussion above suggests that it will be useful to construct an expansion in terms of the number of times walks cross the barrier. The first step in this program is to assume that no walks double-cross, but the actual correlation structure scatters their crossing scale around the completely correlated prediction. Accounting for this fact alone should allow to estimate \( f(s) \) with a greater regime of accuracy than equation (1). Accounting for one earlier crossing should be even more accurate, and so on.

### 2.2 A bivariate approximation for the strongly correlated regime

To proceed, we assume that when steps are strongly correlated one can replace the requirement that \( \delta(S) < B(S) \) for all \( S < s \), which is a condition on all the steps in the walk prior to \( s \), with the milder requirement that \( \delta(S - \Delta s) < B(S - \Delta s) \) for \( \Delta s \to 0 \) — a condition on the one preceding step. Because now the walk values on the two scales are independent, the analysis is more involved than when the steps were completely correlated, but the increase in complexity is relatively minor because we only require a bivariate distribution.

For small \( \Delta s \) we can expand both \( \delta \) and \( B \) in a Taylor series. The condition on the walk height at the previous step means that \( \delta(s) - \Delta s \partial^2/\partial s^2 < B(s) - \Delta s \partial B/\partial s \). If we use primes to denote derivatives with respect to \( s \), then the first crossing distribution of interest is given by the fraction of walks which have

\[
  B(s) \leq \delta \leq B(s) + \Delta s (\delta' - B') \quad \text{and} \quad \delta' \geq B'(s) \tag{2}
\]

That is to say,

\[
  f(s) \, ds = \lim_{\Delta s \to 0} \int_{B'}^{B(s)+\Delta s(\delta'-B')} \int_{B(s)}^{\infty} d\delta' \, d\delta \, p(\delta, \delta') = \Delta s \, p(B(s)) \int_{B'}^{\infty} d\delta' \, p(\delta'(B')(\delta' - B')). \tag{2}
\]

For a Gaussian field \( p(B(s), s) \) is Gaussian, and the conditional distribution in the integral is also Gaussian, with shifted mean and reduced variance. If we define

\[
  \gamma^2 \equiv \frac{\langle \delta \delta' \rangle^2}{\langle \delta^2 \rangle \langle \delta'^2 \rangle} \quad \text{and} \quad \Gamma^2 = \frac{\gamma^2}{1-\gamma^2}, \tag{3}
\]

the mean of \( p(\delta'(B)|B) \) is \( \langle \delta'(B) \rangle = \gamma B \langle \delta'^2 \rangle^{1/2}/\langle \delta^2 \rangle^{1/2} \) and the variance is \( \langle \delta'^2 \rangle (1-\gamma^2) \). (Our notation was set by the fact that, for Gaussian smoothing filters, \( \gamma \) equals the quantity spectral quantity \( \sigma_1/\sigma_2 \) which Bardeen et al. 1986 called \( \gamma \). For tophat filters, the integrals over the power spectrum which define our \( \gamma \) are given by Paranjape et al. 2011.) Note that \( \langle \delta \delta' \rangle = 1/2 \), and thus \( \langle \delta'(B) \rangle = B/2s \).

Before we evaluate the integral, note that the completely correlated approximation corresponds to the limit in which \( \gamma = 1 \) and \( \delta'(B) \) becomes a delta function centered on \( B/2s \). The integral then yields \( B/2s - B' = -\sqrt{\delta} \partial(B/\sqrt{s})/\partial s \), and the resulting expression is consistent with equation (1). Notice that our analysis has indeed extended the completely correlated solution by replacing the delta function with a Gaussian whose width depends explicitly on the underlying power spectrum.

In the generic case, one still has an integral over a single Gaussian distribution. If we define

\[
  x = \frac{\delta' - B'}{\langle \delta'^2 \rangle^{1/2}}, \quad \beta(s) = \frac{B(s)}{\sqrt{s}} \quad \text{and} \quad \beta_1(s) = -\beta(s) \frac{\partial \ln \beta(s)}{\partial \ln \sqrt{s}} \tag{4}
\]

(the reason for our choice of sign for \( \beta_1 \) will become clear later), then equation (2) gives

\[
  f(s) = \frac{e^{-\beta^2(s)/2}}{2\gamma \sqrt{2\pi}} \int_0^\infty dx \, e^{-\frac{(x-x')^2}{2\sqrt{2\pi}}} \left( 1 + \frac{\text{erf}(\Gamma \beta_1 \sqrt{2})}{2} + \frac{e^{-\beta^2(s)/2}}{2\gamma \sqrt{2\pi}} \beta_1 \right). \tag{4}
\]

Evaluating the integral yields

\[
  f(s) = \frac{e^{-\beta^2(s)/2}}{2\sqrt{2\pi}} \beta_1 \left[ 1 + \frac{\text{erf}(\Gamma \beta_1 \sqrt{2})}{2} + \frac{e^{-\beta^2(s)/2}}{2\gamma \sqrt{2\pi}} \beta_1 \right]. \tag{5}
\]

(recall \( \beta_1 \) depends on \( s \)); this is our main result.

Comparison with equation (1) shows that the term in square brackets above represents the correction to the completely correlated solution. There are two points to be made here. First, this correction term depends on \( \Gamma \), indicating that the shape of the first crossing distribution depends explicitly on the form of the underlying power spectrum. In this respect, walks with correlated steps are fundamentally different from walks with uncorrelated steps. Second, when \( \beta_1 \gg 1 \) then this term tends to unity so the first crossing distribution reduces to that for completely correlated walks. To see what large \( \beta_1 \) implies, it is useful to consider some special cases.
Figure 1. Distribution of the scale $y = \delta_0^2/s$ on which walks first cross the barrier $\delta_c(1 + \alpha s/\delta_0^2)$. Histograms show numerical Monte Carlo results for Gaussian smoothing of $P(k) \propto k^{-1.2}$, for which $\Gamma^2 = 9/10$; top to bottom are for barriers with $\alpha = -1$, 0 and 0.5. For comparison, the two dotted curves show the corresponding distributions for $\alpha = 0$ and $-1$ when steps are uncorrelated. Smooth curves show Eq. (5) with $\Gamma$ and the appropriate values of $\alpha$; the agreement indicates that our formula works well for a wide variety of barrier shapes. Symbols with error bars show results for a constant barrier ($\alpha = 0$) and tophat smoothing of a ΛCDM $P(k)$. This shows that, in contrast to when steps are uncorrelated, the first crossing distribution does indeed depend (weakly) on $P(k)$. The solid curve shows Eq. (5) with $\Gamma^2 = 9/10$ and the appropriate values of $\alpha$; the agreement shows that our formula works well for a wide range of $\Gamma$.

### 2.3 When the barrier has constant height

For a constant barrier, $B(s) = \delta_c$ and $\beta_c(s) = \delta_c/\sqrt{s}$. The latter is conventionally denoted as $\nu$, so that $\beta_c \gg 1$ corresponds to large $\nu$. In terms of $\nu$, equation (2) is

$$v_f(\nu) = \nu e^{-\nu^2/2} \left[ \frac{1 + \text{erf}(\Gamma \nu/\sqrt{2})}{2} + e^{-\nu^2/2} \right].$$

In this form, it is clear that $\Gamma^{-1}$ acts as the $\nu$-scale below which the correction to the completely correlated distribution becomes important. (Note however that our $\Gamma$ is not the same parameter defined by Peacock & Heavens 1990 and used in Paranjape et al. 2012.)

Figure 1 compares this formula to distributions generated via Monte-Carlo simulation of the walks (following Bond et al. 1991). We present results for two rather different power spectra and smoothing filters, with two different values of $\Gamma$. Our first choice is Gaussian smoothing of $P(k) \propto k^{\gamma}$ with $\gamma = -1.2$, for which $\Gamma^2 = (n+3)/2 = 9/10$. Our second is tophat smoothing of a ΛCDM power spectrum; in this case $\gamma$ itself depends on scale, with $\gamma \approx 1/2$ (and hence $\Gamma \approx 1/3$) on the scales where $\nu \approx 1$. The figure shows that equation (5), with $\Gamma^2 = 9/10$ and 1/3 respectively, provides an excellent approximation to the Monte-Carlo distributions.

Note in particular that, for ΛCDM, ignoring the scale dependence of $\Gamma$ (by simply using 1/3 on all scales) works very well. This demonstrates that Eq. (5) is a simple and accurate approximation for the CDM family of models.

### 2.4 Linear barriers

For linear barriers $B(s) = \delta_0(1 + \alpha s/\delta_0^2)$, making $\beta(s) = \nu + \alpha/\nu$ and $\beta_c = \nu - \alpha/\nu$. Figure 1 shows results for $\alpha = -1$ and 1/2, both for Gaussian smoothing of $P(k) \propto k^{-1.2}$; Equation (5) with $\Gamma^2 = 9/10$ works very well. Note in particular that our formula is quite different from the Inverse Gaussian distribution associated with uncorrelated steps.

### 3 EXTENSIONS

The analysis above has been so simple, and its results so accurate, that it is interesting and natural to extend it to a variety of other problems, some of which we outline below.

#### 3.1 Halo bias

The excursion set approach is often used to quantify the correlation between halo abundances and their environment. This is done by computing the ratio of $f(s|\delta_0, S_0)$, the first crossing distribution subject to the additional constraint that the walks passed through some $\delta_0$ on some large scale $S_0$ before first crossing $\delta_c$ on scale $s \gg S_0$, to $f(s)$.

Motivated by the previous Section, we now set

$$f(s|\delta_0, S_0) = \int_{B'} \int_{B(s)} d\delta p(\delta, \delta'|\delta_0, S_0)$$

in the limit where $\Delta s \to 0$. This will give

$$f(s|\delta_0, S_0) = p(\delta|\delta_0, S_0) \int_{B'} d\delta' p(\delta'|B, \delta_0) (\delta' - B'),$$

which is very similar to equation (2), only with modified mean values and variances. The Gaussian outside the integral has mean $\langle \delta|\delta_0 \rangle = \delta_0 (\delta_0/B_0)/S_0$ and variance $s(1 - \xi^2)$, where $\xi \equiv \langle \delta \delta_0 \rangle/\sqrt{s}S_0$. The one inside has mean $\langle \delta'|B, \delta_0 \rangle = (\mu_B + \mu_0 \delta_0)/(1 - \xi^2)$, where $\mu_B = \langle \delta' \delta_0 \rangle/(\delta_0/B_0)$ and $\mu_0 = \langle \delta' \delta_0 \rangle/\langle \delta \delta_0 \rangle/\langle \delta \delta_0 \rangle/s$. The first Gaussian obeys the scaling assumed by Paranjape et al. (2012), who showed it was a good approximation to their conditional Monte Carlo distributions. Therefore, it is interesting to see if this scaling also holds for the integral.

If $\langle \delta \delta' \rangle \approx 0$ (or, more carefully stated, if the term containing this quantity is smaller than the other) then $\mu_B = \langle \delta' \delta_0 \rangle/s$ and $\mu_0 = -\langle \delta \delta_0 \rangle/\langle \delta \delta_0 \rangle/s$, so $\langle \delta'|B, \delta_0 \rangle = \mu_B + \mu_0 B$. This is the same rescaling of $B$ as for the first Gaussian, in this approximation $f(s|\delta_0, S_0)$ follows the scaling with $\delta_0$ that was assumed by Paranjape et al. (2012). Therefore, the conclusions of Paranjape & Sheth (2012) about the difference between real and Fourier space bias factors also hold for our calculation. In particular, the Fourier space bias $b_1$ will be simply given by differentiating $f(s)$ with respect to $B(0)$, whereas the real space cross correlation will carry an additional factor of $\langle \delta \delta_0 \rangle/s$. For a constant barrier $b_1 = \partial \ln f(\nu)/\partial \delta_0$ yields

$$b_1 = \frac{\nu^2 - 1}{\nu^2} + \frac{1}{\nu^2} \left[ 1 + \frac{\text{erf}(\Gamma \nu/\sqrt{2})}{2} \sqrt{2\pi \nu} \right]^{-1}. \quad (9)$$
The first term on the right hand side is the bias associated with completely correlated walks. The correction term is negligible for \(\nu \gg 1\), and it tends to \(1/\delta\), when \(\nu \ll 1\).

The expression above assumes that \((\delta ')\) is negligible. This is reasonable when \(S_0 \ll S\); e.g., for Gaussian smoothing filters, \((\delta ') \approx \sigma_b^2 / 2\sigma_f^2\) falls to zero faster than \((\delta \delta_b)\). But at intermediate scales it is not, and \(f(s|\delta_b, S_0)\) will have additional dependence on \(\delta_b\). In Musso, Paranjape & Sheth (2012) we show that this will generically introduce scale dependence into the Fourier space bias factors, even at the linear level.

3.2 Conditional mass functions

The excursion set theory can also be used to compute halo progenitor mass functions and merger rates (Lacey & Cole 1993). For this, we need the joint distribution \(f(s, b, S, B)\) of walks that first cross the barrier \(B\) on scale \(S\), and first cross the barrier \(b > B\) on scale \(s > S\). Dividing by \(f(B, S)\), which we already have, will give the conditional distribution \(f(s|b, S, B)\). The most natural way to estimate it to set

\[
f(s|b, S, B) \approx \frac{1}{\Delta S} \int_{B}^{\infty} \int_{\Delta S}^{\Delta S} \int_{\Delta S}^{\Delta S} \int_{\Delta S}^{\Delta S} \frac{p(\delta, \delta', \Delta, \Delta')}{f(S) dS} \]

(Strictly speaking, we would adjust the limits on the integrals over \(\Delta\) to ensure that \(\Delta < b\).) We have written the final expression in a suggestive form, to show that the left hand side should be thought of as a weighted average over first crossing distributions, each with its own value of \(\Delta\).

In the limit where the scales are very different, \(s \gg S\), it should be a good approximation to ignore the \((\delta ')\) and \((\delta \Delta')\) correlations. This makes \(b\) \(\sim B\), \(\Delta^* \sim B\) and \((\delta \Delta')\) \(\sim B\), where \(G\) is the same quantity as \(\gamma\), but on the scale \(S\). This means there is some dependence on \(\Delta^*\). Note that if there were no correlation with \(\Delta^*\) (in effect \(G \approx 0\)) then the conditional distribution would have the same form as the unconditional one, except that \(b \rightarrow b - BS \approx S\), and \(s \rightarrow S(S/S)^2\), where we have defined \(S = \Delta \approx \Delta\)). This is similar to the rescaling for sharp-k filtering, for which \(S = S\). However, for Gaussian smoothing of a power law \(P(k)\), \(G \approx \gamma\), so one may not set \(G \rightarrow 0\) if one does not also set \(\gamma \rightarrow 0\). Therefore, the case with \(\gamma \approx 0\) may be more relevant. In this case \((\delta \Delta) \approx \gamma(1-G^2)(b-\delta B, \Delta')\).

3.3 Dependence of \(B\) on parameters other than \(\delta\)

In triaxial collapse models, the barrier is a function of the values of the initial deformation tensor (rather than just its trace). This makes

\[
f(s|b, S, B) = \int_{-e}^{e} dp \int_{-e}^{e} dp' \int_{-\infty}^{\infty} d\delta' \int_{B'}^{\infty} d\delta \]

where \(B' = e'(\partial B/\partial e) + p'(\partial B/\partial p)\). If the scale dependence of \(e\) and \(p\) can be neglected, then this will simplify to

\[
f(s) = \int_{-\infty}^{\infty} dp \int_{e}^{e'} dp p(e, p, \delta(e, p)) \int_{0}^{\infty} d\delta' \delta' p(\delta'|e, p, \delta(e, p)).
\]

Further, if the correlation between \(\delta'\) and \(e\) and \(p\) can be neglected, then

\[
f(s) = \int_{-\infty}^{\infty} dp g(e, p|\delta(e, p)) f(s|\delta(e, p)),
\]

and the first crossing distribution is that for \(\delta(e, p)\), weighted by the probability of having \(e\) and \(p\). If we use equation (A3) of Sheth et al. (2001) for the distribution of \(e\) and \(p\) given \(\delta\), then our analysis should be thought of as generalizing their equation (9). In particular, because the result is a weighted sum of first crossing distributions, it exhibits exactly the sort of stochasticity discussed in Appendix C of Paranjape et al. (2011). Performing this calculation more carefully is a subject of ongoing work.

3.4 Excursion set model of peaks

One of the great virtues of our approach is that it provides a simple way to compare the excursion set description with that for peaks. The relation is particularly simple for Gaussian smoothing filters, since then \(\gamma\) of our equation (3) is the same parameter which plays an important role in peaks theory. This correspondence means that our parameter \(x\) is essentially the same as the peak curvature parameter; the only difference is that we define the derivative with respect to the variance \(s\), whereas peaks theory derivatives are with respect to smoothing scale \(R\). This means that our integrals over \(\delta'\) are really just integrals over curvature, which makes intuitive sense. Hence, to implement our prescription for peaks, we only need to account for the fact that the distribution of curvatures around a peak position differs from that around random positions. If we write our equation (3)
as \((\beta_s/2)e^{-\beta_s^2/2}/\sqrt{2\pi}\) times \(\langle x \rangle/\beta_s\), then we need only replace \(x\) \(\rightarrow \langle x f(x) \rangle\), where \(f(x)\) is given by equation (A15) of Bardeen et al. (1986).

For a constant barrier, our \(\gamma/\beta_s\) equals their \(x_s\), so our expression is simply their equation (A14) weighted by \(x/x_s\) and integrated over \(x\). (Their additional factor of \((2\pi)^{3/2} R_0^3\) is just the usual \(m/\bar{\rho}\) factor in excursion set theory, which converts from mass fractions to halo abundances.) Omitting the \(x/x_s\) factor when performing the integral over \(x\) yields the usual expression for peaks (their equation A18).

Therefore, one might think of their (A18) as representing the ‘completely correlated limit’ for peaks, whereas our analysis yields what should be thought of as the moving barrier excursion set model for peaks.

4 DISCUSSION

We presented a formula, equation (5), which provides an excellent description (see Fig. 1) of the first crossing distribution of a large variety of barriers, by walks exhibiting a large range of correlations (i.e., it is valid for a wide class of power spectra and smoothing filters). As we discuss below, we expect a special case of it – equation (6) – to be a good physically motivated fitting formula for halo abundances.

We then showed that our approach provides a simple expression for the first crossing distribution associated with walks conditioned to pass through a certain point, and hence a simple expression for how halo bias factors are modified because of the correction term (equation (9)). We sketched why a generic feature of this approach is that even the linear bias factor should be scale dependent. We also showed how to approximate the first crossing distributions associated with two non-intersecting barriers: the probability that a walk first crosses barrier \(B\) on some scale \(S\) and then \(b > B\) on scale \(s > S\) (equation (10)). This exhibit ‘assembly bias’ so they may provide useful approximations for halo progenitor mass functions and merger rates.

Finally, we argued that our approach makes it particularly easy to see how the first crossing distribution is modified if the barrier depends on hidden parameters, such as those associated with the triaxial collapse model (Section 5.3) or peaks (Section 5.4). In our approach, peaks differ from random positions only in that the integral over \(x\) in equation (4) is modified. A similar integral in the expression for peak abundances leads to scale dependent bias even at the linear level (Desjacques et al. 2010), and is why we now expect \(k\)-dependent bias even for random positions.

We are not the first to have considered the correlated steps problem. Peacock & Heavens (1990) identified \(\Gamma\) (our equation 4) as the key parameter. Their approximation for \(f(s)\) is more accurate than more recent approximations (Maggiore & Riotto 2010; Achitouv & Crocce 2011) which are, in any case, restricted to special combinations of power spectra and smoothing filter (Paranjape et al. 2012). However, our equation 4 is simpler and more accurate for a wider range of barrier shapes, power spectra and smoothing filters than any of these previous studies, and it can be easily extended. Besides the problems outlined above, an obvious direction would be to include non-Gaussianity, along the lines of Musso & Paranjape (2012). This extension is conceptually straightforward – equations (2) will hold also for a non-Gaussian field – and is currently being investigated.

We argued that this accuracy stems from the small \(s\) behavior of the walks. In this regime, which is of most interest in cosmology, \(f(s)\) is well-described by equation (1) for completely correlated walks (Paranjape et al. 2012). These walks are deterministic: the distribution of their heights is a delta function. For real walks, instead, it has a width that increases with \(s\). Equation (4) was obtained by allowing for this broader distribution of heights in the one step prior to the crossing. However, it does not explicitly account for walks which may have criss-crossed the barrier more than once. Musso et al. (2012) discuss how accounting for more zigs and zags may yield even greater accuracy at larger \(s\).

Equation (5) is explicitly the completely correlated first crossing rate times a correction factor. It depends on power spectrum only because this factor depends on \(\Gamma\). Since this factor is small at small \(s\), in this regime (which is the usual one in cosmology) one should expect to see departures from self-similarity, but they should be small. E.g., for a barrier of constant height, the first crossing distribution becomes equation (9), and the correction factor becomes important at \(\nu \leq \Gamma^{-1}\), with \(\Gamma^{-1} \sim 3\) for the \(\Lambda\)CDM family of \(P(k)\).

In cosmology the first crossing distribution is often used as a fitting formula. For this purpose, one might treat either \(\delta_c\) or \(\Gamma\) or both as free parameters (even though equation (4) describes the first crossing distribution very well with no free parameters). In this case, the value of \(\delta_c\) will depend on a variety of factors (Sheth et al. 2001; Maggiore & Riotto 2010b; Paranjape et al. 2012), and we expect \(\Gamma\) to depend on the effective slope of the power spectrum. In this sense, our formula provides a simple way to understand, interpret and quantify the departures from self-similarity which simulations are just beginning to show.

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