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Extreme amenability of abelian $L_0$ groups. (English) [Zbl 1311.28002]
J. Funct. Anal. 263, No. 10, 2978-2992 (2012).

The definition of amenability (introduced by J. von Neumann [Fundam. Math. 13, 73–116 (1929; JFM 55.0151.01) ; ibid. 13, 333 (1929; JFM 55.0151.02)]) was in terms of fixed point properties on compacta (f.p.c.). In particular, a topological group $G$ is defined to be extremely amenable in that every continuous action of $G$ on a compact (Hausdorff) space $X$ has a fixed point in $X$. Discrete and locally compact groups do not generally have this property as they can be free-acting. Previously, the notion of extreme amenability theory had been well-defined only for semigroups, cf. T. Mitchell [Trans. Am. Math. Soc. 122, 195–202 (1966; Zbl 0146.12101)]. Classical amenability of groups concerns fixed points for continuous ‘affine’ actions of a group on a compact convex locally convex vector space.

V. Pestov had originally assumed in his definition of extreme amenability that $G$ was already an amenable group. This restriction was subsequently avoided by introducing submeasures; these are are subadditive measures i.e., not having the property $\mu(U)\cup\mu(V) = \mu(U)+\phi(V)$ for disjoint $U$ and $V$. Let $\phi$ a submeasure on $(X,B)$ where $\phi : B \to [0, +\infty)$ and $B$ a Boolean algebra of subsets of $X$. The submeasure is called diffuse if for every $\varepsilon > 0$ there is a finite cover of $X$ by sets $X_i$ each of which has submeasure less than $\varepsilon$. D. Maharam [Ann. Math. (2) 48, 154–167 (1947; Zbl 0029.20401)] dealing with a ‘control problem’ characterised measure algebras (in the class of complete Boolean algebras), based on von Neumann [Problem 163, The Scottish Book (1937)]. She showed that a complete Boolean algebra is metrisable if and only if it supports a strictly positive submeasure with a ‘continuity’ property involving vanishing of infinite intersections.

The first example of an extreme amenable group was by W. Herer and J. P. R. Christensen [Math. Ann. 213, 203–210 (1975; Zbl 0311.28002)] using spectral theory, for the additive group $\mathbb{R}$ with Lebesgue measure. A measure was called pathological (in that there is no finitely additive measure dominated by the submeasure cf., Maharam [loc. cit.]). They called a metrisable group exotic if it does not have any non trivial strongly continuous unitary representations in $\mathcal{B}(H)$, $H$ a Hilbert space and in this case proved the existence of a fixed point. Later other methods of constructing extremely amenable groups became available.

$L^0(G, \phi)$ ($X$ arbitrary compact Hausdorff and hidden) is defined to be the completion, for a topology defined by convergence in submeasure, of the set of all submeasurable functions from $X$ to $G$ with finite range and it is given a group structure by pointwise multiplication. In effect it is made up by submeasurable sections of a fibre-bundle with base $X$, the fibres being copies of $G$. $L^0(X, \mu)$, where $\mu$ is the Lebesgue counting measure, is not extremely amenable since $X$ acts freely on the sections.

Extreme amenability theory is an extended version of the ‘H. Furstenberg program’ which concerned large Polish topological groups; cf. M. Gromov and V. D. Milman [Ann. J. Math. 105, 843–854 (1983; Zbl 0522.53039)] who were concerned with sequences of compact subgroups with dense union. Existence of a concentration of measures was established leading to what are called Lévy groups (after P. Lévy’s [Leçons d’analyse fonctionnelle. Paris: Gauthier-Villars (1922; JFM 48.0453.01)] on concentration of measures on the surface of spheres in high-dimensional Euclidean spaces).

Theorems on extreme amenability of groups can sometimes be understood as Ramsey-type theorems cf. F. P. Ramsey [On a Problem of Formal Logic, Proceedings London Mathematical Society (1930)]. These can be something like ‘for every partition of a large structured object into classes one of the classes contains a large structured sub-object’. However, Ramsey gives no information about which class this would be. Ramsey theory here, for example, concerns finding monochromatic sets, cf. I. Farah and S. Solecki [J. Funct. Anal. 255, No. 2, 471–493 (2008; Zbl 1172.22002)] used a simplicial complex to to determine monochromatic sets for the Ramsey theory.

To construct simplicial complices, denoted $K$, one considers a simplex as set comprising $k$ points, taken to be vertices; a simplex having $k+1$ vertices it is said to have dimension $k$. It has a purely combinatorial description. It could also be closed under the operation of taking subsets. Denote the largest dimension of the component simplices by $d$. Every $d$-dimensional simplicial complex can be realised as polyhedra in a
A (vertex) colouring of a graph $\Gamma$ is or of a simplicial complex a function defined on the set of its vertices such that no two vertices connected with an edge are assigned the same value. The chromatic number $\chi(\Gamma)$ is the minimal number of colors required to colour the vertices.

J. Lovász [J. Combinatorial Theory Ser A 25 (1978)] proved the conjecture [M. Kneser, Jahresber. Deutsch. Math.-Verein 58 (1955)] that the graph – with all $k$-element subsets of $[n] = (1, 2, \ldots, n)$ as vertices and all pairs of disjoint sets as edges – has chromatic number $n$. His proof used the Borsuk-Ulam theorem and has been adapted to obtain lower bounds for the chromatic number of finite graphs [K. Borsuk, Fundam. Math. 21, 35–38 (1933; JFM 59.1254.01)], states that every continuous transformation of the sphere $S^n$ into $\mathbb{R}^n$ collapses a pair of antipodal points onto each other. The theory was initiated by L. A. Lyusternik [Monatsh. Math. Phys. 37, 125–130 (1930; JFM 56.1133.04)] and by L. G. Schnirelmann and Borsuk in the 30’s. Schnirel'mann’s proof was in terms of a three-colouring of $S^2$.

A barycentric subdivision $sd(F)$ of a (partially ordered) simplicial complex $F$ is effected by declaring its vertices to be its set of nonempty simplices (while remaining ordered). Repeating the process, given a sufficient amount of subdivisions one may approximate any continuous function between polyhedra by simplicial maps.

Generalisations of Borsuk/Ulam to mappings $S^n \to \mathbb{R}^m$ collapse a collection of points under the action a continuous function to a single point. Farah/Solecki [loc. cit.] use a version of Borsuk/Ulam due to A. Yu. Volovikov [Mat. Sb. (1979)] (see also M. Nakaoka [Osaka J. Math. 7, 443–449 (1970; Zbl 0218.57010]) and H. J. Munkholm [Math. Scand. 24, 167–185 (1969; Zbl 0186.57501)]); this relies on $X$ being connected paracompact Hausdorff and a fixed-point-free mapping $f : X \to X$ such that the cohomology indices $H^i(X, \mathbb{Z}/p)$ all vanish mod $p$. Farah/Solecki adapted Kneser graphs to a simplicial complex $K$ with a grid of vertices embedded in the product of copies of $\mathbb{Z}$ to get, for every continuous map $f : \|K\| \to \mathbb{R}^{2d+1}$, a $\mathbb{Z}/p$-orbit that collapses to a single point in $\|K\|$. They do this with a single prime $p$ then go by induction to prove the general case using several primes $p_k$ to get a simplicial complex $\mathbb{Z}/p^\infty$ which is the join of $n$ copies. The vertices are denoted by $(i, r) : i < n, r \in \mathbb{Z}/p$. The simplices are formed from subsets which form chains with respect to inclusion. The p-complex has $s \in \mathbb{Z}/p$ acting as $(i, r) \mapsto (i, s + r)$.

The author’s main theorem deals with supposedly general abelian groups $G$; the reviewer asserts that for the article under review these have to be restricted to algebraically finitely generated $G$. He tries to construct a comprehensive infinite system of undirected measurable graphs embedded in $X$. He subdivides $X$ into partitions $\mathcal{P}$ of $X$ into disjoint sets. The inductive process from $G$ to $L^0(G)$ extends actions $\mathbb{Z} \to G$ to mappings $\mathbb{Z}^\infty \to L^0(G)$. His Definition 4 connects by edges the vertices of undirected graphs $(k_A : A \in \mathcal{P})$ and $(\ell_A : A \in \mathcal{P})$ if $k_A = \ell_A + 1$ on all of the space except a set of $\phi$-size less than $\epsilon$. He uses $\epsilon \in (0, 1)$ as a parameter for fineness of the systems of graphs.

However his definition is improper as such; he needs to consider a symmetrisation of the graphs which then also provides edge-transitivity. The reviewer notes that the system connects vertices by edges if and only they lie within the same partition element for a given $\epsilon$. It seems that his theorem is proved only $\phi$-almost everywhere on $X$ and that he uses ultralilter limits to select the graph he desires; much is dependent on the universality of his grid. His the basic argument seems to be valid but with an overcomplicated description.

The author’s main theorem is that $L^0(G, \phi)$ is extremely amenable for ‘every’ abelian $G$ and diffuse $\phi$.

A subset $A$ of an (additive) group $H$ is called left syndetic if there is a compact $K \subset H$ such that $H = K + A$. In order to prove his main theorem, the author, in his Lemma 5, uses a coloured version of Theorem 3.4.9 of [V. Pestov, Dynamics of infinite-dimensional groups. University Lecture Series 40. Providence, RI: American Mathematical Society (AMS) (2006; Zbl 1123.37003)]. Pestov gave necessary and sufficient conditions for extreme amenability of a topological group $H$ (acting on $X$) in terms of left syndetic sets of $H$, for example, if and only if for every left syndetic set $S \subset H$, the set $S - S$ is everywhere dense in $H$.

To verify the second part of Lemma 5, viz. assuming that $L^0(G, \phi)$ is not extremely amenable to deduce...
that the chromatic number of his graphs must be finite (except for partitions of small φ-size). If G is an ‘arbitrary’ (i.e., finitely generated) abelian group then the L^0 group with diffuse submeasure φ has no finite upper bound on the chromatic numbers as P runs over all the measurable partitions with ϵ ∈ (0, 1).

To prove his assertion one would test whether a symmetric neighbourhood of S does not intersect gS when g ≠ e. This would show that the result is true φ-almost everywhere in X (something like almost surely true).

The author uses the hypothesis that G is an abelian topological group with diffuse submeasure such that L^0(G) is not extremely amenable, viz., there exists a syndectic set S and an element of L^0 which is not in S−S. He assumes, without giving any justification, that there are a finite number of left translates by g ∈ G needed to cover G and concludes by induction that there are at most a finite number colours for his partition scheme for L^0. The reviewer perceives the result is correct though his notation is extremely complicated. The reviewer prefers using the fibre structure of L^0(G) so that the elements of L^0(G) are measurable sections of the bundle.

To complete the proof his main theorem the author reverts to a Farah/Solecki simplicial complex. He proves that chromatic numbers of his graphs diverge when ϕ = Γ proves that chromatic numbers of his graphs diverge when ϵ > 0 the author constructs a function F : N → N which diverges slowly as n → ∞ and the chromatic numbers of the graphs are greater than F(n^2). This F(n) is the analogue of inverse of the M_n as described in Farah/Solecki Lemma 4.3 (used when they showed that for a certain submeasure L^0 is not Lévy).

It is not clear in the statement first part of Lemma 5, whether the author means ‘every’ or ‘any’ abelian group. He is trying to prove that if G is abelian and admits no finitely bounded colourings then it is not extremely amenable. He makes two different assumptions for his argument by contradiction, viz., that the graphs admit a finite colouring and also assumes without proof that L^0(Z, φ) is not extremely amenable (which should be the desired conclusion). This non-extreme amenability has not even been proved by Farah/Solecki even for compact groups (see their Proposition 4.5). Corollary 2 is suspect as stated.

**Reviewer:** Aubrey Wulfsohn (Coventry)

**MSC:**

- 28A20 Measurable and nonmeasurable functions, sequences of measurable functions, modes of convergence
- 46A16 Not locally convex spaces (metrizable topological linear spaces, locally bounded spaces, quasi-Banach spaces, etc.)
- 28A60 Measures on Boolean rings, measure algebras
- 05C55 Generalized Ramsey theory
- 43A07 Means on groups, semigroups, etc.; amenable groups
- 54D80 Special constructions of topological spaces (spaces of ultrafilters, etc.)

**Keywords:**

topological dynamics; extremely amenable group; abelian group; simplicial complex; Kneser graph; measurable graph; Borsuk-Ulam theorem; compact Hausdorff space; submeasure; ultrafilter

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