**K-finite matrix elements of irreducible Harish–Chandra modules are hypergeometric**

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We show that each K-finite matrix element of an irreducible Harish-Chandra module can be obtained from spherical functions by a finite collection of operations.

1. **Notation.** Let \( G \) be a linear semisimple Lie group, let \( K \) be the maximal compact subgroup. Let \( \mathfrak{g} \) be the Lie algebra of \( G \), let \( \mathfrak{u}(\mathfrak{g}) \) be its universal enveloping algebra. Let \( L_X, R_X \), where \( X \in \mathfrak{g} \), be the left and right Lie derivatives on \( G \). Denote by \( \mathfrak{u}_l(\mathfrak{g}) \) (resp., \( \mathfrak{u}_r(\mathfrak{g}) \)) the algebra of differential operators on \( G \) generated by left (resp., right) derivatives.

By \( \Psi_s(g) \) we denote the spherical functions on \( G \), \( s \) is the standard parameter of a spherical function (see [5]).

For a finite-dimensional representation \( \xi \) of \( G \), denote by \( \mathfrak{M}(\xi) \) the space of finite linear combinations of matrix elements of \( \xi \).

2. **Formulation of the result.** Let \( V \) be an irreducible Harish-Chandra module (see, for instance, [3]) over \( G \), denote by \( \pi(g) \) operators of representation of \( G \) in some completion of \( V \). Let \( \sigma \) ranges in the set \( \hat{K} \) of all irreducible representations of \( K \). Let \( V = \bigoplus_{\sigma} V_\sigma \) be the decomposition of \( V \) into a direct sum of \( K \)-isotipical components.

**Proposition.** a) Let \( V \) be an irreducible Harish-Chandra module in a general position. Let \( V^\circ \) be the dual module. Let \( v \in V_\sigma, w \in V_\tau^\circ \). There exists an irreducible finite dimensional representation \( \xi \) of \( G \) and \( s \) such that the matrix element \( \{\pi(g)v, w\} \) is a finite sum

\[
\{\pi(g)v, w\} = \sum_j h_j(g) \cdot p_jq_j \Psi_s(g)
\]

where \( h_j \in \mathfrak{M}(\xi) \), \( p_j \in \mathfrak{u}_l(\mathfrak{g}) \), \( q_j \in \mathfrak{u}_r(\mathfrak{g}) \).

b) For an arbitrary Harish-Chandra module, each \( K \)-finite matrix element admits a representation

\[
\{\pi(g)v, w\} = \lim_{s \to s_0} \sum_j h_j(s; g) \cdot p_j(s)q_j(s) \Psi_s(g),
\]

where \( h_j(s; g) \) is an element \( \mathfrak{M}(\xi) \) depending in a parameter \( s \), and \( p_j(s) \in \mathfrak{u}_l(\mathfrak{g}) \), \( q_j(s) \in \mathfrak{u}_r(\mathfrak{g}) \). Moreover, the degrees of \( p_j(s), q_j(s) \), and the number of summands are uniformly bounded in \( s \) (for fixed \( V, \sigma, \tau \)).

3. **Proof.** First, we introduce additional notation.

Denote by \( P \) the minimal parabolic subgroup in \( G \), consider the decomposition \( P = MAN \), where \( N \) is the nilpotent radical, \( MA \) is the Levi factor of \( P \), \( M \) is the compact subgroup, \( A \simeq (\mathbb{R}^+)^k \) is the vector subgroup.

Consider the flag space \( G/P \), consider the corresponding Grassmannians, i.e., factor-spaces \( G/Q_\alpha \), where \( Q_\alpha \supset P \) are maximal parabolics in \( G \). Equip all the spaces \( G/P, G/Q_\alpha \) with \( K \)-invariant measures (this allows to define Jacobians below). For \( \omega \in G/P \), denote by \( \omega_\alpha \) its image under the map \( G/P \rightarrow G/Q_\alpha \).

For \( g \in G \), denote by \( J_\alpha(g, \omega) \) the Jacobian of the transformation \( g : G/Q_\alpha \rightarrow G/Q_\alpha \) at the point \( \omega_\alpha \).

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Let $\mu$ be a character $A \to \mathbb{C}^*$, let $\tau$ be an irreducible representation of $M$. We denote by $\mu \otimes \tau$ the representation of $P = MAN$, that is $\mu$ on $A$, $\tau$ on $M$ and is trivial on $N$. By $\text{Ind}_P^G(\mu \otimes \tau)$ we denote the representation of $G$ induced from $\mu \otimes \tau$, i.e., a representation of principal (nonunitary) nondegenerate series.

By the Subquotient Theorem, each Harish-Chandra module can be realized as a subquotient (and even as a subrepresentation) in some representation $\text{Ind}_P^G(\mu \otimes \tau)$, see, for instance, [6]. Hence, it is sufficient to prove the statement on matrix elements for the representations $\text{Ind}_G^P(\mu \otimes \tau)$ (they can be reducible).

Let $\rho$ be a spherical representation with parameter $s$. Let $h \in V$ be a spherical vector, let $h^\circ$ be the spherical vector in $V^\circ$. Vectors $v \in V_\sigma$, $w \in (V^\circ)_\tau$ can be represented in the form

$$v = \left( \sum a_\alpha \prod X_{\alpha_j} \right) \cdot h, \quad w = \left( \sum b_\beta \prod Y_{\beta_i} \right) \cdot h^\circ$$

for some $X_{\alpha_j}, Y_{\beta_i} \in \mathfrak{g}$. Thus

$$\{ \pi(g)v, w \} = \{ \pi(g) \left( \sum a_\alpha \prod X_{\alpha_j} \right) \cdot h, \left( \sum b_\beta \prod Y_{\beta_i} \right) \cdot h^\circ \} = \{ \left( \sum b_\beta \prod (-Y_{\beta_i}) \right) \pi(g) \left( \sum a_\alpha \prod X_{\alpha_j} \right) \cdot h, h^\circ \} = \{ \left( \sum b_\beta \prod (-L Y_{\beta_i}) \right) \left( \sum a_\alpha \prod (R X_{\alpha_j}) \right) \Psi_s(g) \cdot h, h^\circ \}$$

$\text{C}^\star$. Consider an induced representation $\pi = \text{Ind}_P^G(\chi \otimes 1)$, where $1$ denotes one-dimensional representation of $M$. If $\chi = \chi_s$ is in a general position (in fact $s \in \mathbb{C}^k$ is outside a locally finite family of complex hyperplanes), then $\pi$ is an irreducible spherical representation. This situation was considered in $\text{A}^\star$. Now examine the case of reducible $\pi$. For this, we must follow continuity of matrix elements as functions of parameters $s$.

For $t_\alpha \in \mathbb{C}$ define the representation

$$\rho_t(g)f(\omega) := f(g\omega) \prod J_\alpha(g, \omega)^{t_\alpha}$$

of $G$ in the space of functions on $G/P$. It can be readily checked that this family of representations coincides with the family $\text{Ind}_P^G(\chi \otimes 1)$, where $\chi = \chi_s$ ranges in all the characters of $A$ (and the dependence $s = s(t)$ is some linear transformation).

Thus, we obtain a realization of the family $\text{Ind}_P^G(\chi \otimes 1)$ such that the action of $K$ is independent in $\chi$ and operators of representation are continuous functions in $s$.

$\text{C}^\star$. Let $\xi$ be an irreducible finite-dimensional representation of $G$ in the space $H$. Following [7], we consider the tensor product

$$\pi \otimes \xi = \text{Ind}_P^G(\chi \otimes 1) \otimes \xi = \text{Ind}_P^G(\chi \otimes \xi \big|_P)$$

The representation $\xi \big|_P$ is reducible and it admits a finite filtration with irreducible subquotients

$$H_1 \supset H_2 \supset H_3 \supset \ldots$$
The nilpotent subgroup $N \subset P$ acts in the subquotients $H_j/H_{j+1}$ in a trivial way.

The representations of subgroup $MA \subset P$ in $H_j/H_{j+1}$ have the form $\mu_j \otimes \tau_j$ for some characters $\mu_j$ of $A$ and some irreducible representations $\tau_j$ of $M$.

Thus, the representation $\pi \otimes \rho$ has a filtration, whose subquotients are representations of principal series having the form $\text{Ind}_G^H(\chi \cdot \mu \otimes \tau)$.

$D^\star$. Fix a representation $\hat{\pi}$ of $M$ and a character $\hat{\chi}$ of $A$. We intend to realize $\text{Ind}_G^H(\hat{\mu} \otimes \hat{\pi})$ as subquotient in an appropriate tensor product (1).

We can choose a representation $\xi$ of $G$ such that the restriction of $\xi$ to $M$ contains $\hat{\tau}$. Then restriction of $\xi$ to $P = MAN$ contains a subquotient of form $\hat{\mu} \otimes \hat{\pi}$ with a certain character $\mu$.

Next, we choose a character $\chi$ of $A$ such that $\chi \cdot \hat{\mu} = \hat{\chi}$. Thus we obtain that $\text{Ind}_G^H(\chi \otimes 1) \otimes \xi$ contains a given representation $\text{Ind}_G^H(\hat{\mu} \otimes \hat{\pi})$ as a subquotient.

$E^\star$. $K$-finite matrix elements of $\text{Ind}_G^H(\hat{\mu} \otimes \hat{\pi})$ are contained in $K$-finite matrix elements of $\text{Ind}_G^H(\chi \otimes \xi)$. The latter matrix elements are finite linear combinations of products of $K$-finite matrix elements of $\text{Ind}_G^H(\chi)$ and matrix elements of $\xi$. This finishes proof of a) and b).

**4. An application. Domains of holomorphy of matrix coefficients.**

**Corollary.** Each domain of holomorphy $\Omega \subset G_\mathbb{C}$ of all the spherical functions is a domain of holomorphy of all the $K$-finite matrix elements of all the irreducible Harish-Chandra modules over $G$.

In particular, all such matrix elements are holomorphic in the Akhiezer–Gindikin domain $\mathcal{H}$ (this is obtained in §).

**Corollary.** There is a submanifold $Y \subset G_\mathbb{C}$, such that for each irreducible Harish-Chandra module over $G$, the matrix-valued function $g \mapsto \rho(g)$ can be extended to a holomorphic function on the universal covering space of $G_\mathbb{C} \setminus Y$.

**Proof.** Spherical functions on $G$ are multivalued holomorphic functions on $G_\mathbb{C}$ having singularities (branching) on a prescribed manifold $Y \subset G_\mathbb{C}$, see §. Hence, for Harish-Chandra modules in a general position, there is nothing to prove.

To prove the statement for exceptional values of $s$, we must follow details of $B^\star$.

Denote by $V$ the space of $K$-finite functions on the flag space $G/P$ (see $B^\star$).

Denote by $1 \in V$ the function $f(\omega) = 1$. Fix $\sigma, \tau \in \widetilde{K}$. Fix $w \in V_\sigma$, $w^* \in V_\tau$. Let consider the representations $\text{Ind}_P^G(\chi_s \otimes 1)$ and follow a behavior of the corresponding matrix element as a function of $s$.

Denote by $\mathcal{U}^N(g) \subset \mathcal{U}(g)$ the subspace consisting of all elements of degree $\leq N$. Let $N$ be sufficiently large, such that $\mathcal{U}^N(g) \cdot 1$ contains the whole subspace $V_\sigma$ for all generic characters $\chi$. Consider a collection $r_1, r_2, \ldots \in \mathcal{U}^N$ such that for some generic $\chi$

1. $r_j \cdot 1$ are linear independent in $\text{Ind}_P^G(\chi_s)
2. their linear combinations contains $V_\tau$.

These properties remain valid for all $s$ outside a certain algebraic submanifold $M$ in the space of parameters. Thus we express our vector $w$ as a linear combination of $r_j$, $w = \sum c_j(s) r_j(s)$, where $c_j$ are certain rational functions.

Applying $A^\star$, we obtain, that our matrix element has a form $\Xi(s) \Psi_s(g)$, where $\Xi(s)$ is an element of $\mathcal{U}_l \otimes \mathcal{U}_e$ depending rationally in $s$.

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2 A proof. Denote by $G_c$ the compact form of $G$. Consider the induced representation $\text{Ind}_P^G_c(\hat{\pi})$. Let $\xi$ be its irreducible subrepresentation. We consider $\xi$ as a representation of $G_c$.
Now let \( s_0 \) be an exceptional value of \( s \). Let \( w \in V_\sigma, w^\circ \in V_\tau \). Consider a holomorphic curve \( \gamma(\varepsilon) \) (with \( \gamma(0) = s_0 \)) avoiding singularities of \( \Xi \) and singular values of the parameter \( s \). A priory, the function

\[
F(\varepsilon, g) = \Xi(\gamma(\varepsilon))\Psi_{\gamma(\varepsilon)}(g)
\]

is holomorphic in the domain \( 0 < |\varepsilon| < \delta, g \in G_C \setminus Y \) and has a pole on the submanifold \( \varepsilon = 0 \). But we know, that that \( F(\varepsilon, g) \) has a finite limit as \( g \in G \) and \( \varepsilon \to 0 \). Hence it has no pole, and hence it is holomorphic at \( \varepsilon = 0 \). In particular it is holomorphic on the submanifold \( \varepsilon = 0 \), and this is the desired statement. □

A product of such matrices \( \rho(g_1) \cdot \rho(g_2) \) generally is divergent but sometimes it is well-defined (see [10], [8]). For each \( X \in g_C, g \in G_C \setminus Y \), we have

\[
\frac{d}{d\varepsilon}\rho(\exp(\varepsilon X)g) = \rho(X)\rho(g)
\]

Remark. There are exceptional situations, when a unitary representation admits a continuation to the whole complex group or its subsemigroup, apparently these cases are well-understood, see [13], [11], [12]. (Sections 1.1, 4.4, 5.4, 7.4-7.6, 9.7), [9], [2]. May be there are other (non-semigroup) cases of unexpectedly large (non-semigroup) domain of holomorphy. As far as I know, this problem never was considered.

5. Nonlinear semisimple Lie groups. For universal coverings of the groups \( SU(p, q), Sp(2n, \mathbb{R}), SO^*(2n) \) our construction survives, we only must replace spherical functions by appropriate Heckman–Opdam hypergeometric functions, see [4], Chapter 1.

I do not know, is it possible to express matrix elements of universal covering groups of \( SO(p, q) \) and \( SL(n, \mathbb{R}) \) in the terms of Heckman–Opdam hypergeometric functions.

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