Solutions of the Bogomolny Equation on $\mathbb{R}^3$
with Certain Type of Knot Singularity

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1 Introduction

This paper studies the moduli space of solutions to the Bogomolny equation on $\mathbb{R}^3$ with a knot singularity.

On $\mathbb{R}^3$, suppose $A$ is an $su(2)$ valued 1-form, $\Phi$ is an $su(2)$ valued function. Let $d_A = d + [A \wedge \cdot]$, $F^A = dA + \frac{1}{2}[A \wedge A]$. Then the Bogomolny equation can be written as

$$V(A, \Phi) = \star F^A - d_A \Phi = 0,$$

where $\star$ is the 3-dimensional Hodge star operator.

Moduli space of the Bogomolny equation on $\mathbb{R}^3$ with certain boundary condition at $\infty$ has been well studied for a long time. See for example [1], [2] and [3]. This paper studies the moduli space of solutions to the Bogomolny equation on $\mathbb{R}^3$ with a knot singularity.

Fix a constant $\gamma \in (0, \frac{1}{2})$ and a smooth knot $K$ in $\mathbb{R}^3$ with length $l$, parametrized by $t \in [0, l]$ with $|K'(t)| = 1$ (so that $K(0) = K(l)$). Let $\rho$ be the distance to the knot $K$. The interesting solutions to the Bogomolny equation in this paper have a $\gamma$ monodromy along the the meridian of the knot when $\rho \to 0$.

Section 2 defines what it means to have a “knot singularity with monodromy $\gamma$”. This definition requires a fiducial configuration (i.e., an $(A, \Phi)$ pair) that has a knot singularity with monodromy $\gamma$. Any configuration that is “close enough” to the fiducial one near the knot is defined to have the same type of singularity and monodromy.
Section 3 and Section 4 assume that \( \gamma \in (0, \frac{1}{8}) \cup \left( \frac{3}{8}, \frac{1}{2} \right) \) and extend the fiducial solution to the Bogomolny equation with knot singularity to the entire \( \mathbb{R}^3 \setminus K \). Having the fiducial solution defined globally, I glue it with certain standard solutions on \( \mathbb{R}^3 \) with mass \( M \) and magnetic number \( k \) to get more solutions. The glued solutions have the same singularity as the fiducial one, i.e., a \( \gamma \) monodromy. One of the most important results of this part is, for a generic \( M \) (which is a real number representing the “mass” of the monopole), a neighborhood of the glued solution in the moduli space is a finite dimensional manifold. The reason for the assumption of \( \gamma \in (0, \frac{1}{8}) \cup \left( \frac{3}{8}, \frac{1}{2} \right) \) is due to some technical issues, which I have no method to get rid of.

Section 5 still assumes \( \gamma \in (0, \frac{1}{8}) \cup \left( \frac{3}{8}, \frac{1}{2} \right) \) but deals with the moduli space near more general solutions to the Bogomolny equation with a knot singularity. The best result I can get is, any solution of this kind always has a neighborhood with a real analytic structure (this comes from the fact that the linearization of the Bogomolny equation at the solution is a Fredholm operator in certain Hilbert spaces).

Section 6 proves a local regularity theorem which says that, for a solution to the Bogomolny equation, having a curvature with finite \( L^2 \) norm (which is defined in Section 2) near the knot is enough to guarantee that it has at most a knot singularity with \( \gamma \) monodromy for some \( \gamma \). This regularity theorem is inspired by [4]. Having this local regularity theorem and re-examining the definition of knot singularities in Section 2, the fiducial configuration is essentially unnecessary.

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2 The Bogomolny equations near the knot

2.1 Parametrization and metric near a knot

Let \( N \) be the normal bundle of \( K \). Choose an orthonormal frame of \( N \), namely \( \{ e_1(s), e_2(s) \} \). Then \( (s, z_1, z_2) \) gives a parametrization in a \( 2\epsilon \)-tubular neighborhood of \( K \) for some small \( \epsilon > 0 \) by

\[
(s, z_1, z_2) \rightarrow K(s) + z_1 e_1(s) + z_2 e_2(s) \in \mathbb{R}^3 \quad (z_1^2 + z_2^2 \leq 4\epsilon^2).
\]
This tubular neighborhood can be identified with a small disc bundle of $K$ contained in $N$ whose radius is $2\epsilon$. Let $N_{2\epsilon}$ denote both the tubular neighborhood and this small disc bundle. Let $\rho$ be the distance function to the knot. When using the notation $N_{\epsilon}$ (or $N_{2\epsilon}, N_{100\epsilon}, N_{\delta}$ and ect.), it is often assumed that $\rho$ is smooth and has no critical points in $N_{\epsilon}\setminus K$.

Besides the standard flat metric induced from $\mathbb{R}^3$ (that is $g_0 = dx_1^2 + dx_2^2 + dx_3^2$), there is another flat metric of $N_{2\epsilon}$ given by $ds^2 + dz_1^2 + dz_2^2$.

Let $\chi_{\epsilon}$ be a cut-off function which is supported in $N_{2\epsilon}$ and 1 on $N_{\epsilon}$. Consider a metric adapted to the knot as the following:

$$g_{\epsilon} = (1 - \chi_{\epsilon})(dx_1^2 + dx_2^2 + dx_3^2) + \chi_{\epsilon}(ds^2 + dz_1^2 + dz_2^2).$$

Let $z = z_1 + iz_2 = re^{i\theta}$. Then $ds^2 + dz_1^2 + dz_2^2 = ds^2 + d\rho^2 + \rho^2 d\theta^2$. It can be proved (see lemma 7.6 in the appendix) that $|g_{\epsilon} - g_0| = O(\rho)$, and $|\nabla g_{\epsilon} - \nabla g_0| = O(1)$.

Throughout this section, the metric on $N_{\epsilon}$ is always assumed to be $g_{\delta}$, and will not be reflected on subscripts.

### 2.2 Differential “0+1” forms

A differential “0+1” form is a direct sum of a 1-form and a 0-form.

**Definition 2.1.** An $su(2)$-valued “0+1” form is also called a **configuration**. In particular, let $a$ be a $su(2)$-valued 1-form, $\varphi$ be a $su(2)$-valued function, then $\psi = a + \varphi$ is a $su(2)$-valued “0+1” form, or a configuration.

Typically, under any given metric, I use $\{1, \tau_1, \tau_2, \tau_3\}$ to denote an orthonormal basis of “0+1” forms, where $\tau_1, \tau_2, \tau_3$ form a basis of 1-forms. For example, under the metric $g_0$, one possible choice is $\tau_1 = dx_1, \tau_2 = dx_2, \tau_3 = dx_3$.

On $N_{\delta}\setminus K$, however, it is convenient to introduce the following notations:

$$\bar{\tau}_1 = dz_1, \bar{\tau}_2 = dz_2, \tau_\rho = d\rho, \tau_\theta = \frac{1}{\rho}d\theta, \bar{\tau}_3 = \tau_s = ds.$$  

Then under the metric $g_{\delta}$, both $\{1, \bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3\}$ and $\{1, \tau_\rho, \tau_\theta, \tau_s\}$ are orthonormal basis for “0+1” forms.

Each metric induces a Clifford structure on the spaces of “0 + 1” forms. To be more precise, suppose $\tau$ is any 1-form, and let $\mathcal{g}(\tau)$ represent the Clifford
multiplication represented by $\tau$. Then $\varrho(\tau)$ sends any “0 + 1” form $\psi = a + \varphi$ to $<\tau, a> + * (\tau \wedge a) + \varphi \tau$, where $< , >$ is the inner product of 1-forms induced by the metric, $*$ is the Hodge star operator (which also depends on the metric), $\varphi \tau$ is just the pointwise multiplication. If $\psi$ is $su(2)$-valued, then $\varrho(\tau)(\psi)$ is also $su(2)$-valued.

For convenience, I also use $\tau_i, \tau_\rho$ etc. themselves to represent the Clifford multiplication that they represent (that is $\varrho(\tau_i), \varrho(\tau_\rho)$ etc.). According to the context, this won’t cause any ambiguity.

2.3 The Bogomolny equations and its linearization

Suppose $A$ is an $su(2)$ valued 1-form, $\Phi$ is an $su(2)$ valued function. Let $d_A = d + [A \wedge \cdot], F = dA + \frac{1}{2}[A \wedge A]$. Then the Bogomolny equations is written as $V(A, \Phi) = *F_A - d_A \Phi = 0$, where $*$ is the Hodge star operator. Let $\Psi$ denote a configuration $(A, \Phi)$. It is convenient to write $\Psi = A + \Phi$ as an $su(2)$-valued “0 + 1” form.

Let $V(\Psi) = *F_A - d_A \Phi$. Suppose $\psi = a + \varphi$ is a variation of $\Psi$, then

$$V(\Psi + \psi) = V(\Psi) + *d_A a - d_A \varphi - a \wedge \Phi + \frac{1}{2} * [a \wedge a] - [a \wedge \varphi]$$

$$= V(\Psi) + L_\varphi(\psi) + Q(\psi, \psi),$$

where $L_\varphi, Q$ are the linear part and the quadratic part respectively.

Using an extra gauge fixing condition $*d_A *a + [\Phi, \varphi] = 0$, one can define the extended linearization operator $\tilde{L}_\varphi$ on $\psi = a + \varphi$ as the following:

$$\tilde{L}_\varphi(\psi) = *d_A a - d_A \varphi - a \wedge \Phi + *d_A a + [\Phi, \varphi] = D_A + [\Phi, \cdot],$$

where $D_A \psi = D_A (a + \varphi) = *d_A a - d_A \varphi + *d_A a$. The formal adjoint of $\tilde{L}_\varphi$ is

$$\tilde{L}_\varphi^\dagger = D_A - [\Phi, \cdot].$$

In terms of Clifford multiplication, they can also be written as:

$$\tilde{L}_\varphi = \sum_j \varrho(\tau_j) \nabla_j^A + [\Phi, \cdot], \quad \tilde{L}_\varphi^\dagger = \sum_j \varrho(\tau_j) \nabla_j^A - [\Phi, \cdot],$$

where $\{\tau_j\}$ is any orthonormal basis of 1-forms and $\nabla_j^A$ is the covariant derivative under the connection $A$ along the $\tau_j$ direction (where $\tau_j$ is identified with a unit vector via the metric).
2.4 The behavior of $\tilde{L}$ on $N_\epsilon \setminus K$

Suppose $\psi$ is a smooth configuration (or equivalently, a smooth $su(2)$-valued “0+1” form) on $N_\epsilon \setminus K$. Suppose $\psi = \varphi - a_1 \bar{\tau}_1 - a_2 \bar{\tau}_2 - a_3 \bar{\tau}_3$. Then $\alpha = a_1 - ia_2$, $\beta = a_3 + i\varphi$ are $sl(2, \mathbb{C}) = su(2) \otimes \mathbb{C}$-valued functions. It is convenient to write a configuration $\psi$ on $N_\epsilon \setminus K$ in terms of two $sl(2, \mathbb{C})$-valued functions as $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ sometimes (instead of a “0+1” form).

Suppose $\Psi = A + \Phi$ is the fiducial configuration, let $A = A_z dz_1 + A_{z_2} dz_2 + A_\rho ds = A_\rho d\rho + \rho A_\theta d\theta + A_\sigma ds$.

Let $\nabla_{z_1} = \partial_{z_1} + [A_{z_1}, \cdot], \nabla_{z_2}^A = \partial_{z_2} + [A_{z_2}, \cdot], \nabla_s^A = \partial_s + [A_s, \cdot], \nabla_{\rho} = \partial_{\rho} + [A_{\rho}, \cdot], \nabla_{\theta} = \partial_{\theta} + \rho [A_{\theta}, \cdot]$, then

$$\nabla_{z_1}^A i \nabla_{z_2} = e^{i\theta} (\nabla_{\rho} - \frac{i}{\rho} \nabla_{\theta}), \quad \nabla_{z_1}^A = e^{-i\theta} (\nabla_{\rho} + \frac{i}{\rho} \nabla_{\theta}).$$

$$\tilde{L}_\psi \psi = \tilde{L}_\psi \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} (\nabla_{z_1}^A + i \nabla_{z_2}) \alpha + \nabla_s^A \beta - i[i\Phi, \beta] \\ (\nabla_{z_1}^A - i \nabla_{z_2}) \beta - \nabla_s^A \alpha - i[i\Psi, \alpha] \end{pmatrix} = \begin{pmatrix} e^{i\theta} (\nabla_{\rho} + \frac{i}{\rho} \nabla_{\theta}) \alpha + \nabla_s^A \beta - i[i\Phi, \beta] \\ e^{-i\theta} (\nabla_{\rho} - \frac{i}{\rho} \nabla_{\theta}) \beta - \nabla_s^A \alpha - i[i\Psi, \alpha] \end{pmatrix}.$$  

$$\tilde{L}_\psi^* \psi = \tilde{L}_\psi^* \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} (\nabla_{z_1}^A + i \nabla_{z_2}) \alpha + \nabla_s^A \beta + i[i\Phi, \beta] \\ (\nabla_{z_1}^A - i \nabla_{z_2}) \beta - \nabla_s^A \alpha + i[i\Psi, \alpha] \end{pmatrix} = \begin{pmatrix} e^{i\theta} (\nabla_{\rho} + \frac{i}{\rho} \nabla_{\theta}) \alpha + \nabla_s^A \beta + i[i\Phi, \beta] \\ e^{-i\theta} (\nabla_{\rho} - \frac{i}{\rho} \nabla_{\theta}) \beta - \nabla_s^A \alpha + i[i\Psi, \alpha] \end{pmatrix}.$$  

When $\Psi = \check{\Psi} = \gamma \sigma d\theta + M \sigma$, $\tilde{L}_\Phi$ is abbreviated as $\tilde{L}$.

$$\tilde{L}_\psi = \tilde{L} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} e^{i\theta} (\partial_{\rho} + \frac{i}{\rho} \partial_{\theta}) \alpha + \partial_t \beta + \frac{\gamma e^{i\theta}}{\rho} [i\sigma, \alpha] - M[i\sigma, \beta] \\ e^{-i\theta} (\partial_{\rho} - \frac{i}{\rho} \partial_{\theta}) \beta - \partial_t \alpha - \frac{\gamma e^{-i\theta}}{\rho} [i\sigma, \beta] - M[i\sigma, \alpha] \end{pmatrix},$$

$$\tilde{L}_\psi^* = \tilde{L}^* \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} e^{i\theta} (\partial_{\rho} + \frac{i}{\rho} \partial_{\theta}) \alpha + \partial_t \beta + \frac{\gamma e^{i\theta}}{\rho} [i\sigma, \alpha] + M[i\sigma, \beta] \\ e^{-i\theta} (\partial_{\rho} - \frac{i}{\rho} \partial_{\theta}) \beta - \partial_t \alpha - \frac{\gamma e^{-i\theta}}{\rho} [i\sigma, \beta] + M[i\sigma, \alpha] \end{pmatrix}.$$  

2.5 Fourier decomposition on $N_\epsilon \setminus K$

Consider the following three operators: $\partial_{\theta}, \partial_s, [i\sigma, \cdot]$ acting on $sl(2, \mathbb{C})$-valued functions. They are pairwise commutative and they have eigenvalues $(im, ik, 2j)$ on the common “eigenvector” $e^{-im\theta - iks} h_j$, where $m, \frac{kl}{2\pi} \in \mathbb{Z}$ and $j \in \{\pm 1, 0\}$. 
Note in particularly when $\Psi = \hat{\Psi} = \gamma \sigma d\theta + M\sigma$ is chosen to be fiducial, $j = 0$ corresponds to the $\varphi^\parallel$ component and $j = \pm 1$ corresponds to the $\varphi^\perp$ component. This is assumed throughout this section.

**Definition 2.2.** Any smooth $sl(2, \mathbb{C})$-valued function $\alpha$ on $N_\epsilon \setminus K$ can be decomposed into infinite sums of those common “eigenvectors” with coefficients that depend only on $\rho$. Each such common eigenvector with its coefficients is called a **Fourier component** of $\alpha$.

**Corollary 2.3.** Suppose $\psi$ is any smooth configuration $N_\epsilon \setminus K$. Fix $s$ and $\rho$, then

$$\min\{4\gamma^2, \frac{1 - 2\gamma}{2}\} \int_0^{2\pi} (|\partial_\theta \psi|^2 + |\psi^\perp|^2) d\theta \leq \int_0^{2\pi} |\nabla^\hat{\Psi}_\theta \psi|^2 d\theta \leq (1 + 4\gamma^2) \int_0^{2\pi} (|\partial_\theta \psi|^2 + |\psi^\perp|^2) d\theta,$$

$$\min\{4\gamma^2, 1 - 2\gamma\} \int_0^{2\pi} |\psi^\perp|^2 d\theta \leq \int_0^{2\pi} |\nabla^\hat{\Psi}_\theta \psi|^2 d\theta.$$

**Proof.** Each Fourier component $e^{-im\theta - iksh_j}$ is also an eigenvector of the operator $\partial_\theta + \gamma \sigma \cdot \cdot$ with eigenvalue $-i(m + 2\gamma j)$.

Suppose $\psi_0 = w(\rho)e^{-im\theta - iksh_j}$ is one Fourier component of $\psi$. Then

$$\int_0^{2\pi} |\nabla^\hat{\Psi}_\theta \psi_0|^2 d\theta = \int_0^{2\pi} |\partial_\theta \psi_0 + \gamma \sigma \cdot \psi_0|^2 d\theta = (m + 2\gamma j)^2 \int_0^{2\pi} |\psi_0|^2 d\theta,$$

$$\int_0^{2\pi} |\partial_\theta \psi_0|^2 d\theta = m^2 \int_0^{2\pi} |\psi_0|^2 d\theta, \quad \int_0^{2\pi} |\psi^\perp_0|^2 d\theta = j^2 \int_0^{2\pi} |\psi_0|^2 d\theta.$$

Recall that $m$ is an integer and $j = 0$ or $\pm 1$, it is easy to see

$$\min\{4\gamma^2, \frac{1 - 2\gamma}{2}\}(m^2 + j^2) \leq (m + 2\gamma j)^2 \leq (1 + 4\gamma^2)(m^2 + j^2),$$

$$\min\{4\gamma^2, 1 - 2\gamma\} j^2 \leq (m + 2\gamma j)^2.$$

Hence the corollary is true. \qed

**2.6 Some (semi-)Hilbert norms on** $N_\epsilon \setminus K$

Throughout this section, fix a small enough $\delta > 0$, and assume $\epsilon \leq \delta$.

Let $\sigma = \left( \begin{array}{cc} -i & 0 \\ 0 & i \end{array} \right) \in su(2)$. Fix a real constant $M \neq 0$ and a fiducial constant $su(2)$-valued function $\hat{\Phi} \equiv M\sigma$. Under the inner product of the Killing form of $su(2)$, suppose $\psi = \psi^\parallel + \psi^\perp$, where $\psi^\parallel, \psi^\perp$ are parallel and perpendicular to $\hat{\Phi}$ respectively.
Definition 2.4. Define the following Hilbert semi-norms on $N_{\epsilon}\backslash K$:

$$\|\psi\|_{L^2_{N,\epsilon}} = \left(\int_{N_{\epsilon}\backslash K} (\rho |\psi|^2_{g_\delta}) d^3x\right)^{\frac{1}{2}}, \quad \|\psi\|_{\tilde{H}^1_{N,\epsilon}} = \left(\int_{N_{\epsilon}\backslash K} (\rho |\nabla \psi|^2_{g_\delta} + \frac{1}{\rho} |\psi|^2_{g_\delta}) d^3x\right)^{\frac{1}{2}}.$$ 

Here $d^3x$ and $g_\delta$ mean that the metric $g_\delta$ is used in the expression, with $\delta$ being a small enough constant such that $\delta > \epsilon$. Since $g_\delta = g_\epsilon$ on $N_\epsilon$ for any $\delta > \epsilon$, the norms don’t depend on $\delta$. The subscript “$N,\epsilon$” indicate the norms are defined on $N_{\epsilon}\backslash K$.

Let $L^2_{\epsilon}$ and $\tilde{H}^1_{N,\epsilon}$ be the completion of smooth configurations (or smooth $su(2)$-valued function, smooth $su(2)$-valued differential forms, smooth $su(2)$-valued tensors, etc. according to the context) on $N_\epsilon \backslash K$ with respect to the corresponded semi-norms.

2.7 The distinguished fiducial configuration on $N_\epsilon$

Recall $\gamma \in (0, \frac{1}{2})$ is a fixed constant.

Definition 2.5. Let $\hat{A}^\gamma = \gamma \sigma d\theta$. Recall that $\hat{\Phi}^\gamma = M\sigma$. Then $\hat{\Psi}^\gamma = \hat{A}^\gamma + \hat{\Phi}^\gamma = \gamma \sigma d\theta + M\sigma$ is called the distinguished fiducial connection on $N_\epsilon$. The superscript $\gamma$ may be omitted if it causes no confusion.

Remark 2.6. If $\gamma \notin (0, \frac{1}{2})$, since any $\Psi^\gamma$ is $SO(3)$ gauge equivalent to $\Psi^{\gamma \pm \frac{1}{2}}$, $\Psi^\gamma$ is eventually gauge equivalent to some $\gamma' \in [0, \frac{1}{2})$ version of $\Psi^{\gamma'}$. The case $\gamma' = 0$ simply means there is no knot singularity. So only $\gamma \in (0, \frac{1}{2})$ suits the interests of this paper.

Consider the following connection:

$$\nabla \hat{A} = \partial_\rho \otimes d\rho + (\partial_\theta + \gamma \sigma) \otimes d\theta + \partial_s \otimes ds.$$ 

Definition 2.7. Let

$$\|\psi\|^2_{\tilde{H}^1_{N,\epsilon}} = \int_{N_{\epsilon}\backslash K} \rho (|\nabla \hat{A}\psi|^2 + |\hat{\Phi}, \psi|^2) d^3x.$$ 

(The metric) $g_\delta$ is used to define the above integral.

Proposition 2.8. The semi-norms $H^1_{N,\epsilon}$ and $\tilde{H}^1_{N,\epsilon}$ are equivalent semi-norms.
Proof.

$$\|\psi\|_{H_{N,\varepsilon}}^2 = \int_0^t \int_0^{2\pi} \int_0^\varepsilon \rho^2 (|\partial_\rho \psi|^2 + \frac{1}{\rho^2} |\partial_\theta \psi + \gamma [\sigma, \psi]|^2 + |\partial_s \psi|^2 + M^2 |\sigma, \psi|^2) d\rho d\theta ds.$$ 

$$\|\psi\|_{\tilde{H}_{N,\varepsilon}}^2 = \int_0^t \int_0^{2\pi} \int_0^\varepsilon \rho^2 (|\partial_\rho \psi|^2 + \frac{1}{\rho^2} (|\partial_\theta \psi|^2 + |\psi|^2) + |\partial_s \psi|^2) d\rho d\theta ds.$$ 

Since (see corollary 2.3) \( \min \{4\gamma^2, \frac{(1 - 2\gamma)^2}{2}\} \int_0^{2\pi} (|\partial_\theta \psi|^2 + |\psi|^2) d\theta \leq \int_0^{2\pi} |\partial_\theta \psi + \gamma [\sigma, \psi]|^2 d\theta \)

\[ \leq (1 + 4\gamma^2) \int_0^{2\pi} (|\partial_\theta \psi|^2 + |\psi|^2) d\theta, \quad \text{and} \quad 0 \leq \rho^2 M^2 |\sigma, \psi|^2 \leq 4(M\epsilon)^2 |\psi|^2, \]

so \( \min \{4\gamma^2, \frac{(1 - 2\gamma)^2}{2}\} \cdot \|\psi\|_{H_{N,\varepsilon}}^2 \leq \|\psi\|_{\tilde{H}_{N,\varepsilon}}^2 \leq (1 + 4\gamma^2 + 4(M\epsilon)^2) \cdot \|\psi\|_{H_{N,\varepsilon}}^2, \)

and the lemma is proved. \( \square \)

**Remark 2.9.** Suppose \( \gamma, M \) and \( \delta \) are fixed. Then the norms \( H_{N,\varepsilon} \) and \( \tilde{H}_{N,\varepsilon} \) bound each other uniformly for any \( \epsilon \leq \delta \).

**Proof.** For any \( \epsilon \leq \delta \),

\[ \min \{4\gamma^2, \frac{(1 - 2\gamma)^2}{2}\} \cdot \|\psi\|_{H_{N,\varepsilon}}^2 \leq \|\psi\|_{\tilde{H}_{N,\varepsilon}}^2 \leq (1 + 4\gamma^2 + 4(M\delta)^2) \cdot \|\psi\|_{H_{N,\varepsilon}}^2. \]

\( \square \)

**Remark 2.10.** The norm defined by \( (\int_{N_\varepsilon \setminus K} |\nabla^4 \psi|^2 d^3x)^{\frac{1}{4}} \) is also equivalent to \( H_{N,\varepsilon} \) and \( \tilde{H}_{N,\varepsilon} \).

### 2.8 Some preliminary analysis on the semi-norm \( \tilde{H}_{N,\varepsilon} \)

This subsection derives some technical inequalities related to \( \tilde{H}_{N,\varepsilon} \) (some variations of Hardy types, Sobolev types etc.) that I’m going to use.

**Lemma 2.11.** Suppose \( \psi \) is smooth on \( N_\varepsilon \setminus K \) with finite \( \tilde{H}_{N,\varepsilon} \) norm, then

\[ \lim_{\rho \to 0} \int_{\partial N_\rho} |\psi|^2 \rho d\Omega = \lim_{\rho \to 0} \int_0^t \int_0^{2\pi} \rho |\psi|^2 d\theta ds = 0, \]

where \( \rho d\Omega = \rho d\theta ds \) is the volume form of \( \partial N_\rho \) under the metric \( g_\varepsilon \).
Proof. Fix $\theta, s$, then

$$|\psi(2\rho) - \psi(\rho)| \leq \int_{\rho}^{2\rho} |\partial_t \psi(t)| dt$$

$$\leq (\rho)^{\frac{1}{4}} \left( \int_{\rho}^{2\rho} |\partial_t \psi(t)|^2 dt \right)^{\frac{1}{2}}$$

$$\leq \rho^{-\frac{1}{4}} \left( \int_{\rho}^{2\rho} t^2 |\partial_t \psi(t)|^2 dt \right)^{\frac{1}{2}}.$$ 

So

$$\rho^\frac{1}{4} |\psi(\rho)| \leq \frac{(2\rho)^{\frac{1}{2}} |\psi(2\rho)|}{\sqrt{2}} + \left( \int_{\rho}^{2\rho} t^2 |\partial_t \psi(t)|^2 dt \right)^{\frac{1}{2}}.$$ 

Using the following inequality: $(a + b)^2 \leq \frac{3}{2} a^2 + 9b^2$,

$$\rho |\psi(\rho)|^2 \leq \frac{3}{2} \left( \frac{(2\rho)^{\frac{1}{2}} |\psi(2\rho)|}{\sqrt{2}} \right)^2 + 9 \left( \int_{\rho}^{2\rho} t^2 |\partial_t \psi(t)|^2 dt \right)^{\frac{1}{2}}$$

$$= \frac{3}{4} (2\rho |\psi(2\rho)|^2) + 9 \left( \int_{\rho}^{2\rho} t^2 |\partial_t \psi(t)|^2 dt \right).$$

Hence

$$\int_0^l \int_0^{2\pi} \rho |\psi(\rho)|^2 d\theta ds \leq \frac{3}{4} \int_0^l \int_0^{2\pi} (2\rho |\psi(2\rho)|^2 d\theta ds + 9 \int_0^l \int_0^{2\pi} t^2 |\partial_t \psi(t)|^2 dt d\theta ds.$$

$$\int_{\partial N_{\rho}} |\psi|^2 \rho d\Omega \leq \frac{3}{4} \int_0^l \int_0^{2\pi} |\psi|^2 \rho d\Omega + 9 \int_0^l \int_0^{2\pi} t^2 |\partial_t \psi(t)|^2 dt d\theta ds.$$

From $\psi \in \mathbb{H}_{N, \epsilon}$,

$$\int_0^l \int_0^2 \int_0^{2\pi} t^2 |\partial_t \psi(t)|^2 dt d\theta ds \leq \| \psi \|^2_{\mathbb{H}_{N, \epsilon}} < \infty,$$

then

$$\lim_{\rho \to 0} 9 \int_0^2 \int_0^{2\pi} t^2 |\partial_t \psi(t)|^2 dt d\theta ds = 0,$$

which implies

$$\lim_{\rho \to 0} \int_0^l \int_0^{2\pi} \rho |\psi(\rho)|^2 d\theta ds = 0.$$

\[\square\]

Lemma 2.12. For any $\eta \in (0, 1)$,

$$\eta (1-\eta) \int_0^l \int_0^{2\pi} \int_0^\epsilon \rho^2 |\partial_\rho \psi| d\rho d\theta ds \leq \int_0^l \int_0^{2\pi} \int_0^\epsilon \rho^2 |\partial_\rho \psi| d\rho d\theta ds + \eta \int_0^l \int_0^{2\pi} \epsilon \psi(\epsilon)|^2 d\theta ds,$$

$$\eta \int_0^l \int_0^{2\pi} \epsilon |\psi(\epsilon)|^2 d\theta ds \leq \int_0^l \int_0^{2\pi} \int_0^\epsilon \rho^2 |\partial_\rho \psi|^2 d\rho d\theta ds + \eta (1+\eta) \int_0^l \int_0^{2\pi} \epsilon |\psi|^2 d\rho d\theta ds.$$
Assume Proposition 2.13.

Proof. Using the previous lemma

\[
\int_0^l \int_0^{2\pi} \epsilon |\psi(\epsilon)|^2 d\theta ds = \int_0^l \int_0^{2\pi} \int_0^\epsilon \partial_\rho (\rho |\psi(\rho)|^2) d\rho d\theta ds \\
= \int_0^l \int_0^{2\pi} \int_0^\epsilon \rho \partial_\rho (|\psi(\rho)|^2) + |\psi(\rho)|^2 d\rho d\theta ds \\
\geq \int_0^l \int_0^{2\pi} \int_0^\epsilon ((1 - \eta)|\psi(\rho)|^2 - \frac{1}{\eta} \rho^2 |\partial_\rho |\psi|^2| d\rho d\theta ds,
\]

and

\[
\int_0^l \int_0^{2\pi} \epsilon |\psi(\epsilon)|^2 d\theta ds = \int_0^l \int_0^{2\pi} \int_0^\epsilon \rho \partial_\rho (|\psi(\rho)|^2) + |\psi(\rho)|^2 d\rho d\theta ds \\
\leq \int_0^l \int_0^{2\pi} \int_0^\epsilon \left( \frac{1}{\eta} \rho^2 |\partial_\rho |\psi|^2 + (1 + \eta)|\psi(\rho)|^2 \right) d\rho d\theta ds.
\]

Suppose \(Q(\psi_1, \psi_2)\) is a bi-linear form whose inputs are two configurations and whose outputs can be either a function or a configuration or any other object that has a norm. The bi-linear form defined in subsection 2.3 is an example, but this proposition can be suited for more general bi-linear forms, although I’m using the same notation \(Q\).

Proposition 2.13. Assume

\[|Q(\psi_1, \psi_2)| \leq C |\psi_1||\psi_2| \quad \text{point-wise for some constant } C > 0.\]

Then there exists a constant \(c_1 > 0\) which depends only on the knot, for any \(\psi_1, \psi_2 \in \overline{\mathbb{H}}_{N,\epsilon}\),

\[\|Q(\psi_1, \psi_2)\|_{L^2}^2 \leq C c_1 \|\psi_1\|_{\overline{\mathbb{H}}_{N,\epsilon}}^2 + \epsilon \int_0^l \int_0^{2\pi} |\psi^{(\epsilon)}(\epsilon)|^2 d\theta ds (\|\psi_2\|_{\overline{\mathbb{H}}_{N,\epsilon}}^2 + \epsilon \int_0^l \int_0^{2\pi} |\psi^{(\epsilon)}(\epsilon)|^2 d\theta ds).\]

Proof. From the previous lemma,

\[
\int_{N_\epsilon} (\rho |\nabla \psi|^2 + \frac{1}{\rho} |\psi|^2) d^3 x = \|\psi\|_{\overline{\mathbb{H}}_{N,\epsilon}}^2 + \int_{N_\epsilon} \frac{1}{\rho} |\psi^{(\epsilon)}|^2 d^3 x \\
= \|\psi\|_{\overline{\mathbb{H}}_{N,\epsilon}}^2 + \int_0^l \int_0^{2\pi} \int_0^\epsilon |\psi^{(\epsilon)}|^2 d\rho d\theta ds \\
\leq \|\psi\|_{\overline{\mathbb{H}}_{N,\epsilon}}^2 + 4 \int_0^l \int_0^{2\pi} \int_0^\epsilon \rho |\partial_\rho |\psi^{(\epsilon)}|^2 d\rho d\theta ds + 2\epsilon \int_0^l \int_0^{2\pi} |\psi^{(\epsilon)}(\epsilon)|^2 d\theta ds, \\
\leq 5 \|\psi\|_{\overline{\mathbb{H}}_{N,\epsilon}}^2 + 2\epsilon \int_0^l \int_0^{2\pi} |\psi^{(\epsilon)}(\epsilon)|^2 d\theta ds.
\]

On the other hand, suppose \(\epsilon \leq \delta < 1\), by theorem 7.3 in the appendix,

\[
\int_{N_\epsilon} (\rho |\nabla \psi|^2 + \frac{1}{\rho} |\psi|^2) d^3 x \geq \frac{1}{C_0} \left( \int_{N_\epsilon} \rho^3 |\psi|^6 d^3 x \right) \frac{1}{2},
\]

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Proposition 2.14. Suppose \(|\psi|^{4}d^{3}x\rangle \leq (\int_{N_{\varepsilon}} \frac{1}{\rho}|\psi|^{2}d^{3}x\rangle \frac{1}{\rho} (\int_{N_{\varepsilon}} \rho^{\alpha}|\psi|^{\alpha}d^{3}x\rangle \frac{1}{\rho} \leq (c_{0})^\frac{\gamma}{2} \int_{N_{\varepsilon}} (\rho|\nabla \psi|^{2} + \frac{1}{\rho}|\psi|^{2})d^{3}x.

Hence

\[\|Q(\psi_{1}, \psi_{2})\|^{2}_{L^{2}} \leq C \int_{N_{\varepsilon}} \rho_{1}^{\alpha}|\psi|^{2} |\psi_{2}|^{2}d\Omega \leq C \int_{N_{\varepsilon}} \rho_{1}^{\alpha}|\psi|^{2}d\Omega \leq \int_{N_{\varepsilon}} \rho_{1}^{\alpha}|\psi|^{2}d\Omega \leq C \|\psi\|^{2}_{L^{2}} + \epsilon \int_{0}^{2\pi} \int_{0}^{2\pi} |\psi_{1}(\epsilon)|^{2}d\theta d\sigma + \epsilon \int_{0}^{2\pi} \int_{0}^{2\pi} |\psi_{2}(\epsilon)|^{2}d\theta d\sigma.\]

\[\text{Letting } c_{1} = 25(c_{0})^\frac{\gamma}{2}, \text{ the proof is finished.}\]

2.9 Features of \(\tilde{L}\) on \(N_{\varepsilon}\)

To study the operator \(\tilde{L}\) and \(\tilde{L}^{\dagger}\) on each Fourier component, suppose \(\alpha = ue^{-i(m+1)\theta - iks}, \beta = ve^{-im\theta - iks}\) s.t. \([i\sigma, u] = 2ju, [i\sigma, v] = 2jv,\ egregious\). then

\[\tilde{L}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} (\partial_{\rho} + \frac{(m + 1 + 2\gamma j)}{\rho})u + (-ik - 2Mj)v e^{-im\theta - iks} \\ (\partial_{\rho} - \frac{(m + 2\gamma j)}{\rho})v + (ik - 2Mj)u e^{-i(m+1)\theta - iks} \end{pmatrix},\]

\[\tilde{L}^{\dagger}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} (\partial_{\rho} + \frac{(m + 1 + 2\gamma j)}{\rho})u + (-ik + 2Mj)v e^{-im\theta - iks} \\ (\partial_{\rho} - \frac{(m + 2\gamma j)}{\rho})v + (ik + 2Mj)u e^{-i(m+1)\theta - iks} \end{pmatrix}.\]

Consider \(\tilde{L}\) as an operator from \(\mathbb{H}_{N_{\varepsilon}}\) to \(L^{2}_{N_{\varepsilon}}\) (and it is obviously bounded).

Proposition 2.14. Suppose \(\psi \in \mathbb{H}_{N_{\varepsilon}}\) is smooth, and suppose \(\gamma \in (0, \frac{1}{8})\), then

\[\|\tilde{L}(\psi)\|^{2}_{L^{2}} + (2\gamma - 24\gamma^{3}) \int_{\partial N_{\varepsilon}} |\psi^{\perp}|^{2}\rho d\Omega \geq 2\gamma^{3}\|\psi\|^{2}_{\mathbb{H}_{N_{\varepsilon}}} + \frac{\epsilon}{2} \int_{\partial N_{\varepsilon}} (\tau_{\rho} \psi, \tau_{s}\nabla_{s} A^{4} \psi > + \frac{1}{\rho} \psi, \tau_{\rho}\nabla_{\rho} A^{4} \psi > + \psi, [M_{\sigma}, \psi]) \rho d\Omega,\]

where \(\rho d\Omega\) is the volume form on \(\partial N_{\varepsilon}\).

The same inequality is also true for \(\tilde{L}^{\dagger}\), except changing \(M\) to \(-M\).

Proof. I prove the inequality for \(\tilde{L}\) here, and the case of \(\tilde{L}^{\dagger}\) is true for the same reason.
With out loss of generality, assume $\psi$ is smooth and has only one Fourier component: \( \psi = \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = \left( \begin{array}{c} u e^{-i(m+1)\theta - ikt} \\ v e^{-im\theta - ikt} \end{array} \right) \), with \([i\sigma, u] = 2ju, [i\sigma, v] = 2jv\).

(Here, $u, v$ depends only on $\rho$.) Let $\lambda = m + 1 + 2j\gamma$.

In this setting, $\tau_{\varphi} = \frac{z_1 \bar{\tau}_1 + z_2 \bar{\tau}_2}{\rho}$ sends \( \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \) to \( \left( \begin{array}{c} i e^{-i\theta} \\ i e^{i\theta} \alpha \end{array} \right) \); $\tau_{\theta}$ sends \( \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \) to \( \left( \begin{array}{c} e^{-i\theta} \\ -e^{i\theta} \alpha \end{array} \right) \); $\tau_{s}$ sends \( \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \) to \( \left( \begin{array}{c} -i\alpha \\ i\beta \end{array} \right) \).

So I have the following identities:

1. 
\[
\frac{\epsilon}{2} \int_{\partial N_\epsilon} (\langle \tau_{\rho} \psi, \tau_{s} \nabla^A \psi \rangle + \frac{1}{\rho} \langle \tau_{\rho} \psi, \tau_{\theta} \nabla^A \psi \rangle + \langle \tau_{\rho} \psi, [M\sigma, \psi] \rangle) \rho d\Omega \\
= \epsilon^2(\langle u(\epsilon), (-ik + 2Mj)v(\epsilon) \rangle) + \frac{1}{2} \epsilon(\lambda |u(\epsilon)|^2 + (1 - \lambda)|v(\epsilon)|^2);
\]

2. 
\[
\int_{\partial N_\epsilon} |\psi|^2 \rho d\Omega = \epsilon(|u(\epsilon)|^2 + |v(\epsilon)|^2);
\]

3. 
\[
\|\psi\|^2_{\mathbb{H}_{N,\epsilon}} = \int_0^\epsilon (\rho^2(|\partial_{\rho}u|^2 + |\partial_{\rho}v|^2) + (k^2 + 4M^2 j^2)\rho^2(|u|^2 + |v|^2) + (\lambda^2|u|^2 + (1 - \lambda)^2|v|^2)) d\rho.
\]

(Since $\varphi \in \mathbb{H}_{N,\epsilon}$, and by lemma 2.11 $\lim_{\rho \to 0} \rho(|u|^2 + |v|^2) = 0.$)

4. 
\[
\left\| \tilde{L}\psi \right\|^2_{L^2} = \int_{N_\epsilon} \rho|\tilde{L}\psi|^2 d^3 x \\
= \int_0^\epsilon \rho^2((\partial_{\rho} + \frac{\lambda}{\rho}) u + (-ik + 2Mj)v)^2 + |(\partial_{\rho} - \frac{\lambda - 1}{\rho}) v + (ik + 2Mj)u|^2) d\rho \\
= \int_0^\epsilon \rho^2(|\partial_{\rho}u|^2 + |\partial_{\rho}v|^2) d\rho + (k^2 + 4M^2 j^2) \int_0^\epsilon \rho^2(|u|^2 + |v|^2) d\rho \\
- \lambda(1 - \lambda) \int_0^\epsilon (|u|^2 + |v|^2) d\rho + \int_0^\epsilon \rho^2 \partial_{\rho}(\langle u, (-ik + 2Mj)v \rangle) d\rho \\
+ \epsilon^2(\langle u(\epsilon), (-ik + 2Mj)v(\epsilon) \rangle) + \epsilon(\lambda |u(\epsilon)|^2 + (1 - \lambda)|v(\epsilon)|^2).
\]

(The last identity above uses integration by part and the fact that $\lim_{\rho \to 0} \rho(|u|^2 + |v|^2) = 0.$)
Since \((\frac{1}{4} + 4\gamma^2)(1 - 4\gamma^2) > \frac{1}{4}\), by Cauchy-Schwarz inequality,

\[
\begin{align*}
(\frac{1}{4} + 4\gamma^2) \int_0^\epsilon \rho^2(|\partial_\rho u|^2 + |\partial_\rho v|^2) d\rho + (1 - 4\gamma^2)(k^2 + 4M^2j^2) \int_0^\epsilon \rho^2(|u|^2 + |v|^2) d\rho \\
+ \int_0^\epsilon \rho^2(\partial_\rho u < u, (-ik + 2Mj)v >) d\rho &\geq 0.
\end{align*}
\]

So

\[
\|\tilde{L}\psi\|_{L^2}^2 \geq \left(\frac{3}{4} - 4\gamma^2\right) \int_0^\epsilon \rho^2(|\partial_\rho u|^2 + |\partial_\rho v|^2) d\rho + 4\gamma^2(k^2 + 4M^2j^2) \int_0^\epsilon \rho^2(|u|^2 + |v|^2) d\rho \\
- \lambda(1 - \lambda) \int_0^\epsilon (|u|^2 + |v|^2) d\rho + \epsilon^2(< u(\epsilon), (-ik + 2Mj)v(\epsilon) >) + \epsilon(\lambda|u(\epsilon)|^2 + (1 - \lambda)|v(\epsilon)|^2).
\]

There are three different cases:

1. \(\lambda \in (0, 1)\), so \(\lambda = 2\gamma\) or \(\lambda = 1 - 2\gamma\);

2. \(\lambda \notin [0, 1]\) and \(\lambda\) is not an integer;

3. \(\gamma\) is an integer.

**Case 1**: \(\lambda \in (0, 1)\), so that \(\lambda = 2\gamma\) or \(\lambda = 1 - 2\gamma\). Using lemma 2.12 and the fact that \(\gamma \in (0, \frac{1}{8})\),

\[
\lambda(1 - \lambda) \int_0^\epsilon (|u|^2 + |v|^2) d\rho \leq \left(\frac{3}{4} - 8\gamma^2\right) \int_0^\epsilon \rho^2(|\partial_\rho u|^2 + |\partial_\rho v|^2) d\rho + (3 - 32\gamma^2)\gamma\epsilon(|u(\epsilon)|^2 + |v(\epsilon)|^2),
\]

and \(2\gamma^3 \int_0^\epsilon (\lambda^2 |u|^2 + (1 - \lambda)^2 |v|^2) d\rho \leq 2\gamma^2 \int_0^\epsilon \rho^2(|\partial_\rho u|^2 + |\partial_\rho v|^2) d\rho + 2\gamma^2(\lambda|u(\epsilon)|^2 + (1 - \lambda)|v(\epsilon)|^2),\)

so \(\|\tilde{L}\psi\|_{L^2}^2 \geq \left(\frac{3}{4} - 4\gamma^2\right) \int_0^\epsilon \rho^2(|\partial_\rho u|^2 + |\partial_\rho v|^2) d\rho + 4\gamma^2(k^2 + 4M^2j^2) \int_0^\epsilon \rho^2(|u|^2 + |v|^2) d\rho \\
- \lambda(1 - \lambda) \int_0^\epsilon (|u|^2 + |v|^2) d\rho + \epsilon^2(< u(\epsilon), (-ik + 2Mj)v(\epsilon) >) + \epsilon(\lambda|u(\epsilon)|^2 + (1 - \lambda)|v(\epsilon)|^2)
\]

\[
= 2\gamma^3 \left(\rho^2(|\partial_\rho u|^2 + |\partial_\rho v|^2) + (k^2 + 4M^2j^2)\rho^2(|u|^2 + |v|^2) + (\lambda^2|u|^2 + (1 - \lambda)^2|v|^2)\right) d\rho \\
+ \epsilon^2(< u(\epsilon), (-ik + 2Mj)v(\epsilon) >) + \frac{1}{2}\epsilon(\lambda|u(\epsilon)|^2 + (1 - \lambda)|v(\epsilon)|^2)
\]

\[
= 2\gamma^3\|\psi\|_{H^{N,\epsilon}}^2 + \frac{\epsilon}{2} \int_{\partial N_\epsilon} \langle < \tau_\rho \psi, \tau_\rho \nabla_\theta^A \psi > + \frac{1}{\epsilon} < \tau_\rho \psi, \tau_\rho \nabla_\theta^A \psi > + < \tau_\rho \psi, [M_\sigma, \psi] > \rangle d\Omega \\
- (2\gamma - 24\gamma^3)\|\psi\|_{L^2}^2.
\]
So the lemma is proved in this case.

**Case 2:** \( \lambda \notin [0,1] \) and \( \lambda \) is not an integer. Without loss of generality, assume \( \lambda > 1 \). Form lemma 2.12

\[
(\lambda - 1)\epsilon |v(\epsilon)|^2 \leq \int_0^\epsilon \rho^2 |\partial_\rho v|^2 d\rho + \lambda(\lambda - 1) \int_0^\epsilon |v|^2 d\rho.
\]

\[
\|\hat{L}\psi\|_{L^2}^2 \geq \left(\frac{3}{4} - 4\gamma^2\right) \int_0^\epsilon \rho^2 |\partial_\rho u|^2 + |\partial_\rho v|^2 d\rho + 4\gamma^2 (k^2 + 4M^2j^2) \int_0^\epsilon \rho^2 |u|^2 + |v|^2 d\rho \\
+ \lambda(\lambda - 1) \int_0^\epsilon (|u|^2 + |v|^2) d\rho + \epsilon^2 < u(\epsilon), (-ik + 2Mj)\epsilon u(\epsilon) > + \epsilon(\lambda |u(\epsilon)|^2 - (\lambda - 1)|v(\epsilon)|^2)
\]

\[
\geq 4\gamma^2 \int_0^\epsilon (\rho^2 (|\partial_\rho u|^2 + |\partial_\rho v|^2) + (k^2 + 4M^2j^2)\rho^2 (|u|^2 + |v|^2)) d\rho + \frac{1}{2}\lambda(\lambda - 1) \int_0^\epsilon (|u|^2 + |v|^2) d\rho \\
+ \epsilon^2 < u(\epsilon), (-ik + 2Mj)\epsilon u(\epsilon) > + \frac{1}{2}\epsilon(\lambda |u(\epsilon)|^2 - (\lambda - 1)|v(\epsilon)|^2)
\]

\[
\geq 4\gamma^2 \int_0^\epsilon (\rho^2 (|\partial_\rho u|^2 + |\partial_\rho v|^2) + (k^2 + 4M^2j^2)\rho^2 (|u|^2 + |v|^2) + (\lambda^2 |u|^2 + (1 - \lambda)^2 |v|^2)) d\rho \\
+ \epsilon^2 < u(\epsilon), (-ik + 2Mj)\epsilon u(\epsilon) > + \frac{1}{2}\epsilon(\lambda |u(\epsilon)|^2 + (1 - \lambda) |v(\epsilon)|^2)
\]

\[
= 4\gamma^2 \|\psi\|_{H_{N,\epsilon}}^2 + \frac{\epsilon}{2} \int_{\partial N_\epsilon} (\tau_\rho \nabla_s A \psi + \frac{1}{\epsilon} \tau_\rho, \tau_\theta \nabla_\theta A \psi > + \tau_\rho, [M\sigma, \psi] >) d\Omega.
\]

So in this case the lemma is also proved.

**Case 3:** \( \lambda \) is an integer, and \( j = 0 \). If \( \lambda \notin [0,1] \), then the argument for Case 2 is still valid. So I only need to prove it when \( \lambda = 0 \) or 1. Without loss of generality, assume \( \lambda = 0 \), in which case \( \psi^\perp = 0 \).

\[
\|\hat{L}\psi\|_{L^2}^2 \geq \left(\frac{3}{4} - 4\gamma^2\right) \int_0^\epsilon \rho^2 |\partial_\rho u|^2 + |\partial_\rho v|^2 d\rho + 4\gamma^2 k^2 \int_0^\epsilon \rho^2 (|u|^2 + |v|^2) d\rho \\
+ \epsilon^2 < u(\epsilon), -i\kappa v(\epsilon) > + \epsilon|v(\epsilon)|^2
\]

\[
\geq 4\gamma^2 \int_0^\epsilon \rho^2 (|\partial_\rho u|^2 + |\partial_\rho v|^2) d\rho + 4\gamma^2 k^2 \int_0^\epsilon \rho^2 (|u|^2 + |v|^2) d\rho + \epsilon^2 < u(\epsilon), -i\kappa v(\epsilon) > + \frac{1}{2}\epsilon|v(\epsilon)|^2
\]

\[
\geq 4\gamma^2 \int_0^\epsilon (\rho^2 (|\partial_\rho u|^2 + |\partial_\rho v|^2) + |v|^2 + k^2\rho^2 (|u|^2 + |v|^2)) d\rho + \epsilon^2 < u(\epsilon), -i\kappa v(\epsilon) > + \frac{1}{2}\epsilon|v(\epsilon)|^2
\]

\[
= 4\gamma^2 \|\psi\|_{H_{N,\epsilon}}^2 + \frac{\epsilon}{2} \int_{\partial N_\epsilon} (\tau_\rho, \tau_\sigma \nabla_s A \psi > + \frac{1}{\epsilon} \tau_\rho, \tau_\theta \nabla_\theta A \psi > + \tau_\rho, [M\sigma, \psi] >) d\Omega.
\]

The proof is finished.
The following two remarks follow directly from the above proof.

**Remark 2.15.** The condition $\psi \in H_{N,\epsilon}$ can be replaced by a weaker condition:

$$\liminf_{\rho \to 0} \int_{\partial N_\rho} |\psi|^2 \rho d\Omega = \liminf_{\rho \to 0} \int_0^I \int_0^{2\pi} \rho |\psi(\rho)|^2 d\theta ds = 0,$$

and the proof is still valid.

**Remark 2.16.** If $\gamma \in \left(\frac{3}{8}, \frac{1}{2}\right)$, then the same argument leads to (using $(1-2\gamma)$ to substitute $2\gamma$)

$$\left\| L(\psi) \right\|_{L^2}^2 + (1 - 2\gamma - 3(1 - 2\gamma)^3) \int_{\partial N_\epsilon} |\psi|^2 \rho d\Omega \geq \frac{1}{4}(1-2\gamma)^3 \left\| \Psi \right\|_{H_{N,\epsilon}}^2 + \frac{\epsilon}{2} \int_{\partial N_\epsilon} \left( \langle \tau_\rho \psi, \tau_\sigma \nabla A \psi \rangle + \frac{1}{\epsilon} < \tau_\rho \psi, \tau_\theta \nabla A \psi \rangle + \langle \tau_\rho \psi, [M_\sigma, \psi] \rangle \right) \rho d\Omega.$$

### 3 The Hilbert spaces on $\mathbb{R}^3$

#### 3.1 The distinguished fiducial configuration on $\mathbb{R}^3\setminus K$

Recall $\delta \gg \epsilon$, and recall $\chi_\delta$ was a function supported in $N_{2\delta}$ and is 1 in $N_\delta$.

**Definition 3.1.** Extend the distinguished fiducial configuration on $N_\epsilon \setminus K$ to $\mathbb{R}^3\setminus K$ as the following:

$$\hat{A} = \gamma \sigma \omega, \hat{\Phi} = M \sigma,$$

where $\omega$ is a closed 1-form on $\mathbb{R}^3\setminus K$ such that $\omega$ has bounded support and $\omega = d\theta$ on $N_{2\delta}$. Let $\hat{\Psi} = \hat{A} + \hat{\Phi}$ be the distinguished fiducial configuration on $\mathbb{R}^3\setminus K$.

Clearly the distinguished fiducial configuration is a solution to the Bogomolny equation with $\hat{A} = 0$ far away from the knot. The notation $L$ with subscript “$\Psi$” missed is the extended linearization operator w.r.t. the distinguished fiducial configuration.

Throughout this section, the metric on $\mathbb{R}^3\setminus K$ is $g_\delta$ by default.

Let $\rho_\epsilon = \min \{ \rho, \epsilon \}$. Let $\nabla \hat{A} = \nabla + [\hat{A}, \cdot]$. Consider the following norms:

$$\left\| \psi \right\|_{L^2}^2 = \int_{\mathbb{R}^3\setminus K} (\rho_\epsilon |\psi|^2) d^3 x, \quad \left\| \psi \right\|_{H_\epsilon}^2 = \int_{\mathbb{R}^3\setminus K} \rho_\epsilon (|\nabla \hat{A} \psi|^2 + |\hat{\Phi} \psi|^2) d^3 x.$$
Remember the metric $g_\delta$ on $\mathbb{R}^3$ is used here for some fixed $\delta \geq \epsilon$ to define norm and volume form in the integration, even it is not reflected in the notations.

The spaces $L^2_\epsilon$ and $\mathbb{H}_\epsilon$ denote the completion of smooth configurations (or smooth $su(2)$-valued function, smooth $su(2)$-valued differential forms, smooth $su(2)$-valued tensors, etc. according to the context) with bounded support on $\mathbb{R}^3 \setminus K$ with finite corresponded norms. It is obvious that different $\epsilon$ give equivalent $\mathbb{H}_\epsilon$ norms (same for $L^2_\epsilon$).

**Proposition 3.2.** Suppose $\psi \in \mathbb{H}_\epsilon$, then the restriction of $\psi$ on $N_\epsilon \setminus K$ is in $\mathbb{H}_{N,\epsilon}$, and in particular

$$\lim_{\rho \to 0} \int_{\partial N_\rho} |\psi|^2 \rho d\Omega = 0.$$  

*Proof.* This is because of proposition 2.8 and lemma 2.11. \hfill \Box

**Proposition 3.3.** Suppose $\psi \in \mathbb{H}_\epsilon$, then

$$\int_{\mathbb{R}^3 \setminus K} |\psi|^2 d^3 x \leq \max\left\{ \frac{\epsilon}{4\gamma^2}, \frac{2\epsilon}{(1-2\gamma)^2}, \frac{1}{4M^2 \epsilon} \right\} \|\psi\|_{\mathbb{H}_\epsilon}^2.$$  

*Proof.* This is because

$$\min\{4\gamma^2, \frac{(1-2\gamma)^2}{2}\} \int_{N_\epsilon \setminus K} |\psi|^2 d^3 x \leq \epsilon \|\psi\|_{\mathbb{H}_{N,\epsilon}}^2$$  

and

$$\int_{\mathbb{R}^3 \setminus N_\epsilon} |\psi|^2 d^3 x \leq \frac{\epsilon}{4M^2 \epsilon} \int_{\mathbb{R}^3 \setminus N_\epsilon} M^2 ||[\sigma, \psi]]||^2 d^3 x.$$  

*Proof.* This is because

$$\min\{8\gamma - 4\gamma^3, 4(1-2\gamma) - 4(1-2\gamma)^3\} \int_{\partial N_\epsilon} |\psi|^2 \rho d\Omega \leq \epsilon \int_{\mathbb{R}^3 \setminus N_\epsilon} |\nabla \tilde{\Phi}_\epsilon|^2 d^3 x.$$  

*Proof.* The proof defers to Subsection 3.5. \hfill \Box

**Proposition 3.5.** Supposing $|M|$ is large enough (which depends on the knot), then there exists an $\epsilon_2 \leq \delta$, such that if $\psi \in \mathbb{H}_{\epsilon_2}$ is smooth, then

$$\min\{8\gamma - 4\gamma^3, 4(1-2\gamma) - 4(1-2\gamma)^3\} \int_{\partial N_{\epsilon_2}} |\psi|^2 \rho d\Omega \leq \epsilon \int_{\mathbb{R}^3 \setminus N_{\epsilon_2}} (|\nabla \tilde{\Phi}_\epsilon|^2 + ||\tilde{\Phi}_\epsilon||^2) d^3 x.$$  

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Proof. The proof defers to Subsection 3.5.

Proposition 3.6. Suppose $Q(\psi_1, \psi_2)$ is a bi-linear form such that

$$|Q(\psi_1, \psi_2)| \leq C|\psi_1||\psi_2| \quad \text{point-wise.}$$

Supposing moreover $Q(\psi_1, \psi_2)$ satisfies

$$Q(\psi^\perp_1, \psi^\perp_2) = 0,$$

then for any $\psi_1, \psi_2 \in H_\epsilon$, and for some possibly larger $C_1$ that depends on $C$, $\epsilon$ and the knot,

$$\|Q(\psi_1, \psi_2)\|_{L^2} \leq C_1\|\psi_1\|_{H_\epsilon}\|\psi_2\|_{H_\epsilon}.$$

Proof. Without loss of generality, it may be assumed that $\psi_1, \psi_2$ are smooth and have bounded support, and $\psi_1 = \psi^\perp_1$. Then

$$\|Q(\psi_1, \psi_2)\|_{L^2}^2 = \int_{\mathbb{R}^3 \setminus K} \rho |Q(\psi_1, \psi_2)|^2 d^3x = \|Q(\psi_1, \psi_2)\|_{L^2_{N, \epsilon}}^2 + \epsilon \int_{\mathbb{R}^3 \setminus N_\epsilon} |Q(\psi_1, \psi_2)|^2 d^3x.$$

On $N_\epsilon$,

$$\|Q(\psi_1, \psi_2)\|_{L^2_{N, \epsilon}}^2 \leq C\|\psi_1\|_{H_{N, \epsilon}}^2 \|\psi_2\|_{H_{N, \epsilon}}^2 + \epsilon \int_0^1 \int_0^{2\pi} |\psi_2^\epsilon(\epsilon)|^2 d\theta ds,$$

where $\epsilon \int_0^1 \int_0^{2\pi} |\psi_2^\epsilon(\epsilon)|^2 d\theta ds$ can be bounded by $\|\psi_2\|_{H_\epsilon}$ from theorem 7.1 and theorem 7.2 in the appendix. So there exists a constant $\tilde{C} > 0$ such that

$$\|Q(\psi_1, \psi_2)\|_{L^2_{N, \epsilon}} \leq \tilde{C}\|\psi_1\|_{H_\epsilon}\|\psi_2\|_{H_\epsilon}.$$

On the other hand,

$$\left( \int_{\mathbb{R}^3 \setminus N_\epsilon} |Q(\psi_1, \psi_2)|^2 d^3x \right)^{\frac{1}{2}} \leq C \left( \int_{\mathbb{R}^3 \setminus N_\epsilon} |\psi_1|^2 |\psi_2|^2 d^3x \right)^{\frac{1}{2}}$$

$$\leq C \left( \int_{\mathbb{R}^3 \setminus N_\epsilon} |\psi_1|^6 d^3x \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3 \setminus N_\epsilon} |\psi_1|^6 d^3x \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3 \setminus N_\epsilon} |\psi_2|^6 d^3x \right)^{\frac{1}{3}}.$$

Using the assumption $\psi_1 = \psi^\perp_1$ and theorem 7.6 in the appendix, the above expression can also be bounded (up to a constant) by $\|\psi_1\|_{H_\epsilon}^2 \|\psi_2\|_{H_\epsilon}^2$. Hence the proposition is true. \qed
3.3 Fredholm theory of $\tilde{L}$

Lemma 3.7. Suppose $\gamma \in (0, \frac{1}{8})$, $\tilde{L}$ as a bounded linear operator from $\mathbb{H}_\epsilon$ to $\mathbb{L}_\epsilon^2$ has closed range and finite kernel.

The same statement is also true when $\gamma \in \left(\frac{3}{8}, \frac{1}{2}\right)$.

Proof. I prove the case $\gamma \in (0, \frac{1}{8})$ here, and the case of $\gamma \in \left(\frac{3}{8}, \frac{1}{2}\right)$ follows by the same argument.

Suppose $\{\tau_j\}$ is an othonormal basis of 1-form, then

$$\int_{\mathbb{R}^3 \setminus \mathbb{N}_{\epsilon_2}} |\tilde{L}\psi|^2 d^3x = \int_{\mathbb{R}^3 \setminus \mathbb{N}_{\epsilon_2}} |D_A\psi + [M\sigma, \psi]|^2 d^3x$$

$$= \int_{\mathbb{R}^3 \setminus \mathbb{N}_{\epsilon_2}} (|\sum_j \tau_j \nabla_j^A \psi|^2 + 2 \sum_j <\tau_j \nabla_j^A \psi, [M\sigma, \psi]> + |[M\sigma, \psi]|^2) d^3x$$

$$= \int_{\mathbb{R}^3 \setminus \mathbb{N}_{\epsilon_2}} (|\nabla_j^A \psi|^2 + 4M^2|\psi|^2 + \sum_j \nabla_j (<\tau_j \psi, \tau_j \nabla_j^A \psi>) - \sum_j \nabla_j (<\psi, \tau_j [M\sigma, \psi]>)d^3x$$

$$= \int_{\mathbb{R}^3 \setminus \mathbb{N}_{\epsilon_2}} (|\nabla_j^A \psi|^2 + 4M^2|\psi|^2 + \sum_j \nabla_j (<\tau_j \psi, \sum_i \tau_i \nabla_i^A \psi + [M\sigma, \psi]>) - \frac{1}{2} \sum_j \nabla_j^2 (|\psi|^2)) d^3x$$

$$= \int_{\mathbb{R}^3 \setminus \mathbb{N}_{\epsilon_2}} (|\nabla_j^A \psi|^2 + 4M^2|\psi|^2) d^3x$$

$$+ \int_{\partial \mathbb{N}_{\epsilon_2}} \left(-\frac{1}{\epsilon_2} <\tau_\rho \psi, \tau_\theta \nabla_\theta^A \psi> - <\tau_\rho \psi, \tau_\sigma \nabla_\sigma^A \psi> - <\tau_\rho \psi, [M\sigma, \psi]> \right) d\omega.$$

In the above expression, $\nabla_j$ should be understood as covariant derivatives w.r.t the metric $g_\delta$ in the dual of $\tau_j$ direction, so that the expression doesn’t depend on the othonormal basis that is chosen. The last step is the Stokes theorem.

On the other hand, from proposition 2.14

$$\left\| \tilde{L}(\psi) \right\|_{\mathbb{L}_2^2}^2 + (2\gamma - 24\gamma^3) \int_{\partial \mathbb{N}_{\epsilon}} |\psi|^2 \rho d\omega$$

$$\geq 2\gamma^3 \|\Psi\|_{\mathbb{H}_\epsilon}^2 + \frac{\epsilon}{2} \int_{\partial \mathbb{N}_{\epsilon}} (<\tau_\rho \psi, \tau_\sigma \nabla_\sigma^A \psi> + \frac{1}{\epsilon} <\tau_\rho \psi, \tau_\theta \nabla_\theta^A \psi> + <\tau_\rho \psi, [M\sigma, \psi]> \right) \rho d\omega.$$

So

$$2 \int_{\mathbb{N}_{\epsilon} \setminus K} \rho |\tilde{L}\psi|^2 d^3x + \epsilon \int_{\mathbb{R}^3 \setminus \mathbb{N}_{\epsilon}} |\tilde{L}\psi|^2 d^3x + (4\gamma - 48\gamma^3) \int_{\partial \mathbb{N}_{\epsilon}} |\psi|^2 \rho d\omega$$

$$\geq 4\gamma^3 \|\Psi\|_{\mathbb{H}_\epsilon}^2 + \epsilon \int_{\mathbb{R}^3 \setminus \mathbb{N}_{\epsilon}} (|\nabla_j^A \psi|^2 + 4M^2|\psi|^2) d^3x.$$
So
\[ 2\|\tilde{L}\psi\|_{L^2_{\rho}}^2 + (4\gamma - 48\gamma^3) \int_{\partial \mathcal{N}_e} |\psi\|^2 \rho d\Omega \geq 4\gamma^3 \|\psi\|_{H_{\epsilon}}^2. \]

By theorem \[7.4\] \((\int_{\partial \mathcal{N}_e} |\psi\|^2 \rho d\Omega)^{\frac{1}{2}}\) is compact relative to \(H_{\epsilon}\), hence \(\tilde{L}\) has closed range and finite kernel.

**Remark 3.8.** To prove the inequality
\[ 2\|\tilde{L}\psi\|_{L^2_{\rho}}^2 + (4\gamma - 48\gamma^3) \int_{\partial \mathcal{N}_e} |\psi\|^2 \rho d\Omega \geq 4\gamma^3 \|\psi\|_{H_{\epsilon}}^2, \]
the condition \(\psi \in \mathbb{H}_{\epsilon_2}\) at the beginning can be substituted by a weaker condition, which is
\[ \lim \inf_{\rho \to 0} \int_{\partial \mathcal{N}_o} |\psi|^2 \rho d\Omega = 0, \]
and the proof is still valid.

**Proposition 3.9.** Same condition as the above lemma, \(\tilde{L}\) also has finite cokernel. Hence \(\tilde{L}\) is a Fredholm map from \(\mathbb{H}_{\epsilon}\) to \(L^2_{\rho}\).

**Proof.** Relative to inner product from the space \(L^2_{\rho}\), the adapted formal adjoint of \(\tilde{L}\) satisfies \(\tilde{L}^\dagger_{\epsilon} : \tilde{L}^\dagger_{\epsilon} \left( \frac{1}{\rho_{\epsilon}} \psi \right) = \frac{1}{\rho_{\epsilon}} \tilde{L}^\dagger(\psi)\).

Suppose \(\frac{1}{\rho_{\epsilon}} \psi \in L^2_{\epsilon}\) and \(\tilde{L}^\dagger_{K,\epsilon} \left( \frac{1}{\rho_{\epsilon}} \psi \right) = 0\), so that \(\tilde{L}^\dagger \psi = 0\). Then
\[ \int_{\mathbb{R}^3 \setminus K} \rho_{\epsilon} \left| \frac{1}{\rho_{\epsilon}} \psi \right|^2 d^3x = \int_{\mathbb{R}^3 \setminus K} \left| \frac{1}{\rho_{\epsilon}} \psi \right|^2 d^3x < +\infty. \]

Since
\[ \int_0^\epsilon \int_{\partial \mathcal{N}_o} \frac{1}{\rho} |\psi|^2 \rho d\Omega d\rho = \int_{\mathbb{R}^3 \setminus K} \frac{1}{\rho_{\epsilon}} |\psi|^2 d^3x \leq \int_{\mathbb{R}^3 \setminus K} \frac{1}{\rho_{\epsilon}} |\psi|^2 d^3x < +\infty, \]
so
\[ \lim \inf_{\rho \to 0} \int_{\partial \mathcal{N}_o} |\psi|^2 \rho d\Omega = 0. \]

Having the above limit instead of the condition \(\psi \in \mathbb{H}_{\epsilon}\), then the same argument as lemma \[3.7\] can be used to show that
\[ 2\|\tilde{L}^\dagger \psi\|_{L^2_{\rho}}^2 + (4\gamma - 48\gamma^3) \int_{\partial \mathcal{N}_e} |\psi\|^2 \rho d\Omega \geq 4\gamma^3 \|\psi\|_{H_{\epsilon}}^2. \]

By a local elliptic regularity theorem (see \[7.9\]), \(\psi\) is smooth. Hence \((4\gamma - 48\gamma^3) \int_{\partial \mathcal{N}_e} |\psi\|^2 \rho d\rho\) is finite. Together with \(\tilde{L}^\dagger \psi = 0\), it implies that \(\psi\) is in \(\mathbb{H}_{\epsilon}\).

However, by the same argument as lemma \[3.7\] \(\tilde{L}\) as a map from \(\mathbb{H}_{\epsilon}\) to \(L^2_{\rho}\) has finite kernel. So the cokernel of \(\tilde{L}\) in \(L^2_{\rho}\) has only finite dimension. 

\[ \square\]
Proposition 3.10. Supposing $|M|$ is large enough (which depends on $K$ and $\gamma$), then both the kernel and cokernel of $\tilde{L}$ are 0.

Proof. I still prove the case $\gamma \in (0, \frac{1}{8})$, and the other case that $\gamma \in (\frac{3}{8}, \frac{1}{2})$ follows by the same reason. By lemma 3.4

$$(4\gamma - 48\gamma^3) \int_{\partial N_\varepsilon} |\psi^+|^2 \rho d\Omega \leq \frac{\varepsilon_2}{2} \int_{\mathbb{R}^3 \setminus N_{2\varepsilon}} (|\nabla^A \psi^+|^2 + 4M^2|\psi^+|^2) d^3x.$$ 

So from the proof of lemma 3.7

$$2 \int_{N_{2\varepsilon} \setminus \gamma} \rho |\tilde{L}\psi|^2 d^3x + \varepsilon_2 \int_{\mathbb{R}^3 \setminus N_{2\varepsilon}} |\tilde{L}\psi|^2 d^3x \geq 4\gamma^3 \|\psi\|^2_{\mathbb{H}_{N, 2\varepsilon}} + \frac{\varepsilon_2}{2} \int_{\mathbb{R}^3 \setminus N_{2\varepsilon}} (|\nabla^A \psi|^2 + 4M^2|\psi^+|^2) d^3x.$$ 

Hence there exists a constant $C > 0$ which depents only on $\gamma$, such that

$$\|\tilde{L}\psi\|^2_{L^2_\varepsilon} \geq 2\gamma^3 \|\psi\|^2_{\mathbb{H}_{2\varepsilon}}.$$ 

So $\tilde{L}$ has 0 kernel in $\mathbb{H}_{2\varepsilon}$.

The same argument also shows that $\tilde{L}^\dagger$ has 0 kernel in $\mathbb{H}_{2\varepsilon}$, which implies that $\tilde{L}$ has 0 cokernel in $L^2_\varepsilon$ (by the same reason as proposition 3.9). This finishes the proof.

Note that although the proof above used a special $\varepsilon_2$, since different $\varepsilon$ give equivalent $\mathbb{H}_\varepsilon$ and equivalent $L^2_\varepsilon$ norms, the result is true for any $\varepsilon$. To be more precise, if $|M|$ is large enough,

$$\|\tilde{L}\psi\|^2_{L^2_\varepsilon} \text{ or } \|\tilde{L}^\dagger\psi\|^2_{L^2_\varepsilon} \geq \frac{2\gamma^3 \min\{\varepsilon, \varepsilon_2\}^2}{\max\{\varepsilon, \varepsilon_2\}^2} \|\psi\|^2_{\mathbb{H}_\varepsilon}.$$ 

Corollary 3.11. For any $M \neq 0$, the index of $\tilde{L}$ is always 0.

Proof. This is because a path connected set of Fredholm operators have the same index. 

Proposition 3.12. Suppose $\gamma \in (0, \frac{1}{8}) \cup (\frac{3}{8}, \frac{1}{2})$. Let $C$ be the set of all $M \in \mathbb{R} \setminus \{0\}$, such that $\tilde{L}$ has 0 kernel and cokernel. Then
• *C* is a bounded set;

• *C* has no accumulation points except 0 (which may or may not be an accumulation point).

**Proof.** The compactness of *C* follows from proposition 3.10 Only the second statement needs to be proved.

Suppose on the contrary *M*₀ ≠ 0 is an accumulation point of *C*, then there exists a sequence *ψₖ ∈ ℍₖ* and *Mₖ ∈ ℝ \ {0}* such that,

\[
\lim_{k \to +\infty} M_k = M_0, \quad Dψ_k + M_k[σ, ψ_k] = 0.
\]

The definition of ℍₖ depends on *M*. However, it may be assumed that all *Mₖ* have the same sign with *M₀*, so they give equivalent norms and cause no ambiguity. In particular, without loss of generality, *M₀* is used to define ℍₖ in this proof. It may also be assumed that ‖*ψ_k‖ₖ,ψₖ = 1 for all *k*.

\[
(Dψ_k + M₀[σ, ψ_k]) = (M₀ − M_k)[σ, ψ_k] \to 0 \quad \text{in} \quad ℍₖ \quad \text{as} \quad k \to +∞.
\]

Since *D + M₀[σ, ·]* has only finite dimensional kernel and ‖*ψ_k‖ₖ,ψₖ = 1, there exists a subsequence of *ψ_k* that converges to an element in ker(*D + M₀[σ, ·]*) without loss of generality, it may be assumed that *ψ_k* itself converges to *ψ₀ ∈ ker(*D + M₀[σ, ·]*)

\[
0 = \int_{ℝ³ \setminus K} <Dψ_k + M_k[σ, ψ_k], [σ, ψ₀]> \, d³x
\]

\[
\geq |M_k + M₀| \int_{ℝ³ \setminus K} <[σ, ψ_k], [σ, ψ₀]> \, d³x - |D[σ, ψ_k], ψ₀| > + <[σ, ψ_k], Dψ₀ > \, d³x
\]

\[
\geq |M_k + M₀| \int_{ℝ³ \setminus K} |ψₖ|² \, d³x - \liminf_{ρ \to 0} \int_{∂Nρ} |ψₖ| ||ψ₀|ρdΩ - |M_k + M₀| \int_{ℝ³ \setminus K} |ψₖ|² − |ψ₀|² \, d³x.
\]

Since \(\lim_{k \to +\infty} ψ_k = ψ₀\) in ℍₖ,

\[
\lim_{k \to +\infty} |M_k + M₀| \int_{ℝ³ \setminus K} |ψₖ|² − |ψ₀|² \, d³x ≤ 2|M₀|(\int_{ℝ³ \setminus K} |ψ₀|² \, d³x)^{½} \lim_{k \to +\infty} (\int_{ℝ³ \setminus K} |ψₖ|² − |ψ₀|² \, d³x)^{½} = 0
\]

\[
\liminf_{ρ \to 0} \int_{∂Nρ} |ψₖ|² |ψ₀|²ρdΩ ≤ \lim_{ρ \to 0}(\int_{∂Nρ} |ψₖ|²ρdΩ)^{½}(\int_{∂Nρ} |ψ₀|²ρdΩ)^{½} = 0
\]

So \(0 ≥ \limsup_{k \to +\infty} |M_k + M₀| \int_{ℝ³ \setminus K} |ψ₀|² \, d³x = 2|M₀| \int_{ℝ³ \setminus K} |ψ₀|² \, d³x\).

Since *ψ₀* is smooth by theorem 7.39 it implies that *ψ₀* = 0. So *ψ₀* = ψ₀. Suppose *ψ₀ = fσ* where *f* is a smooth function on ℝ³ \ *K* with bounded support. Then *df = 0* and *f = 0* by Hodge theory. Hence *ψ₀ = 0* which contradicts the fact that ‖*ψ₀‖ₖ,ψₖ = 1. □
3.4 Adjusting the metric

**Proposition 3.13.** If I use the metric $g_0$ instead of $g_δ$ to define the norm $H_δ$ and $L^2_δ$, then I still get an equivalent norm.

**Proof.** This is because the two metrics $g_0$ and $g_δ$ can bound each other uniformly on $\mathbb{R}\setminus\gamma$.

Note that not only the norms $H_δ$ and $L^2_δ$, but also the extended linear operator $\tilde{L}$ depends on the metric.

**Proposition 3.14.** Assume $δ$ is small enough. If I use $g_0$ instead of $g_δ$ to define $\tilde{L}$ as a map from $H_δ$ to $L^2_δ$, then it is still a Fredholm operator with the same index and 0 kernel.

**Proof.** In this proof I use temporary notations $\tilde{L}_0$ and $\tilde{L}_δ$ to denote the extended linearization operator defined using $g_0$ and $g_δ$ respectively. To be precise, let $ψ = a + \varphi$, any version of $\tilde{L}$ can be written as

$$\tilde{L} = *d A a - d \hat{A} \varphi + *d \hat{A} * a.$$

The only difference between the two versions of $\tilde{L}$ is the star operator $*$. Two metrics differ only on $N_{2δ}$, so their difference $\tilde{L}_0 - \tilde{L}_δ$ is an operator supported on $N_{2δ}$. By theorem 7.6 in the appendix,

$$|g_0 - g_δ| = O(ρ)$$

and

$$|∇g_0 - ∇g_δ| = O(1),$$

so there exists a $c > 0$, such that

$$|(\tilde{L}_δ ψ - \tilde{L}_0 ψ)| \leq c(|ψ| + ρ|∇\hat{A} ψ|).$$

Without loss of generality, it may be assumed that $ε = ε_2$. Using lemma 2.12,

$$\|\tilde{L}_0 ψ - \tilde{L}_δ ψ\|_{L^2_δ}^2 \leq c \int_{N_{2δ}} (ρ_ερ^2|∇\hat{A} ψ|^2 + ρ_ε|ψ|^2)d^3x ≤ cδ^2 \int_{N_{2δ}} ρ_ε|∇ψ|^2d^3x + cε^2 \int_{N_{2δ}} \frac{1}{ρ_ε}|ψ|^2d^3x \leq 5cδ^2\|ψ\|^2_{H_{N,ε}} + 2cε^2 \int_{∂N_ε} |ψ|^2dΩ.$$

From corollary 7.3 in the appendix with $η = 1$, and fix some $δ_1 ≥ δ$, there exists a constant

$$ε^2 \int_{∂N_ε} |ψ|^2dΩ ≤ (C + δ_1)ε^2 \int_{R^3\setminus N_ε} |∇A ψ|^2d^3x ≤ δ(C + δ_1)ε \int_{R^3\setminus N_ε} |∇A ψ|^2d^3x.$$

From the proof of lemma 3.7,

$$\|\tilde{L}_δ ψ\|_{L^2_δ}^2 ≥ (\frac{1}{C(γ)})\|ψ\|^2_{H_{N,ε}} + \frac{ε}{4} \int_{R^3\setminus N_ε} (|∇A ψ|^2 + \frac{(2Mε)^2}{ε^2}|ψ^⊥|^2)d^3x.$$
So
\[ \|L_0 \psi\|_{L^2}^2 \geq \left( \frac{1}{C(\gamma)} - 5C\delta^2 \right) \|\psi\|_{H^1}^2 + \frac{\epsilon}{4} \int_{\mathbb{R}^3 \setminus N_{\epsilon}} ((1-4\delta(C+\delta_1))|\nabla A\psi|^2 + \frac{(2M\epsilon)^2}{\epsilon^2} |\psi|^2) d^3 x. \]

Therefore if \( \delta \) is small enough, \( L_\delta - L_0 \) is compact relative to \( L_\delta \) as an operator from \( H_\epsilon \) to \( L^2_\epsilon \).

### 3.5 Some proofs

**Proof of proposition 3.4.** There exists an \( \epsilon_1 \leq \delta \), for any \( \epsilon \leq \epsilon_1 \), if \( \psi \in H_\epsilon \) is smooth, then if \( \psi \in H_\epsilon \) is smooth, then

\[ \min \{2\gamma - \gamma^3, (1 - 2\gamma) - (1 - 2\gamma)^3\} \int_{\partial N_{\epsilon}} |\psi|^2 \rho d\Omega \leq \epsilon \int_{\mathbb{R}^3 \setminus N_{\epsilon}} |\nabla \tilde{A}_\psi|^2 d^3 x. \]

**Proof.** It may be assumed that \( \psi = \psi^\perp \). I only prove the case \( \gamma \leq \frac{1}{4} \), in which \( \min \{2\gamma - \gamma^3, (1 - 2\gamma) - (1 - 2\gamma)^3\} = 2\gamma - \gamma^3 \) for convenience. The other case \( \left( \frac{1}{4} < \gamma < \frac{1}{2} \right) \) follows by the same reason.

The proof has two steps:

**Step 1** proves the following statement: there exists an \( \epsilon_1 \leq \delta \) such that, if \( \psi \in H_\epsilon \) is smooth, then

\[ (2\gamma - \gamma^3) \int_{\partial N_{\epsilon}} |\psi|^2 \rho d\Omega \leq \epsilon_1 \int_{\mathbb{R}^3 \setminus N_{\epsilon}} |\nabla \tilde{A}_\psi|^2 d^3 x. \]

Choose any \( 0 < 2\epsilon_1 \leq \delta \), then from theorem 7.1 and theorem 7.2 there exists some constant \( C(2\epsilon_1) \), such that

\[ 2\epsilon_1 \int_{\mathbb{R}^3 \setminus N_{2\epsilon_1}} |\nabla \tilde{A}_\psi|^2 d^3 x \geq C(2\epsilon_1) \int_{\partial N_{2\epsilon_1}} |\psi|^2 \rho d\Omega. \]

If \( C(2\epsilon_1) \geq 2\gamma - \gamma^3 \), step 1 is done. Otherwise, \( C(2\epsilon_1) < 2\gamma - \gamma^3 \).

By corollary 2.3

\[ 4\gamma^2 \int_0^{2\pi} |\psi|^2 d\theta \leq \int_0^{2\pi} |\partial_\theta \psi + \gamma [\sigma, \psi]|^2 d\theta, \]

so

\[ \int_0^{2\pi} |\nabla \tilde{A}_\psi|^2 d\theta \geq \int_0^{2\pi} (|\partial_\rho \psi|^2 + \frac{4\gamma^2}{\rho^2} |\psi|^2) d\theta. \]
Choose $\mu > 0$ small enough such that

$$4\gamma^2 - 2\mu C(2\epsilon_1)\mu - 2^{2\mu}C(2\epsilon_1)^2 \geq 0,$$

$$\int_{\mathbb{R}^3 \setminus N_\epsilon} |\nabla \hat{A}\psi|^2 d^3 x \geq \int_{N_{2\epsilon_1} \setminus N_\epsilon} |\nabla \hat{A}\psi|^2 d^3 x + \frac{C(2\epsilon_1)}{2\epsilon_1} \int_{\partial N_{2\epsilon_1}} |\psi|^2 \rho d\Omega$$

$$\geq \int_0^l \int_0^{2\pi} \int_{\epsilon_1}^{2\epsilon_1} (\rho |\partial_\rho \psi|^2 + \frac{4\gamma^2}{\rho} |\psi|^2) d\rho d\theta ds + C(2\epsilon_1) \int_0^l \int_0^{2\pi} |\psi(2\epsilon_1)|^2 d\theta ds$$

$$= \left[ \left( \frac{4\gamma^2}{\rho} \right) \int_0^l \int_0^{2\pi} \int_{\epsilon_1}^{2\epsilon_1} |\nabla \hat{A}\psi|^2 d^3 x \right] \frac{C(2\epsilon_1)}{2\epsilon_1} \int_{\partial N_{2\epsilon_1}} |\psi|^2 \rho d\Omega$$

$$\geq \int_0^l \int_0^{2\pi} \int_{\epsilon_1}^{2\epsilon_1} (\rho |\partial_\rho \psi|^2 + \frac{4\gamma^2}{\rho} |\psi|^2) d\rho d\theta ds + \frac{2^{2\mu}C(2\epsilon_1)}{\epsilon_1} \int_{\partial N_{2\epsilon_1}} |\psi|^2 |\hat{A}_\psi|^2 d^2 x$$

Let $C(\epsilon_1) = 2^{2\mu}C(2\epsilon_1)$. Again if $C(\epsilon_1) \geq 2\gamma - \gamma^3$, then step 1 is done. Otherwise, substitute $2\epsilon_1$ with $\epsilon_1$, repeat the above process until the constant be greater or equal than $2\gamma - \gamma^3$, which finishes the first step.

**Step 2** proves that, for any $\epsilon \leq \epsilon_1$, if $\psi \in \mathbb{H}_\epsilon$ is smooth, then

$$(2\gamma - \gamma^3) \int_{\partial N_{\epsilon}} |\hat{A}\psi|^2 \rho d\Omega \leq \epsilon \int_{\mathbb{R}^3 \setminus N_\epsilon} |\nabla \hat{A}\psi|^2 d^3 x.$$
so the proposition is proved. □

Proof of proposition 3.5: Suppose $|M|$ is large enough (which depends on the knot), then there exists an $\epsilon_2 \leq \delta$, if $\psi \in H_{\epsilon_2}$ is smooth, then if $\psi \in H_{\epsilon_2}$ is smooth, then

$$\min \{8\gamma - 4\gamma^3, 4(1 - 2\gamma) - 4(1 - 2\gamma)^3\} \int_{\partial N_{\epsilon_2}} |\psi|^2 |\rho d\Omega \leq \epsilon \int_{\mathbb{R}^3 \setminus N_{\epsilon_2}} (|\nabla \tilde{A} \psi|^2 + |[\tilde{\Phi}, \psi]|^2) d^3x.$$

Proof. The proof is similar with the proof of proposition 3.4. From the start, there exists a constant $C(2\epsilon_1)$, such that

$$2\epsilon_1 \int_{\mathbb{R}^3 \setminus N_{2\epsilon_1}} (|\nabla \tilde{A} \psi|^2 + 4M^2|\psi|^2) d^3x \geq C(2\epsilon_1) \int_{\partial N_{2\epsilon_1}} |\psi|^2 |\rho d\Omega.$$

If $C(2\epsilon_1) \geq 8\gamma - 4\gamma^3$, then the proof is done. Otherwise, $C(2\epsilon_1) < 8\gamma - 4\gamma^3$.

Since $|M|$ is large enough, it can be assumed that

$$4(M\epsilon_1)^2 - 2C(2\epsilon_1) - 4C(2\epsilon_1)^2 \geq 0.$$
Let $C(\epsilon_1) = 2C(2\epsilon_1)$. Again if $C(\epsilon_1) \geq 8\gamma - 4\gamma^3$, then the proof is finished. Otherwise, substitute $2\epsilon_1$ with $\epsilon_1$, repeat the above process until the constant be greater or equal than $8\gamma - 4\gamma^3$, which is valid since $M$ is assumed to be large enough. This finished the proof.

\section{Solutions from the gluing}

\subsection{Admissible approximate solutions}

In general, fixing any configuration (or $su(2)$-valued “0+1” form) $\Psi = A + \Phi$ to be set as the fiducial configuration, one can define an adjusted version of Hilbert norms as the following:

$$\|\psi\|_{H_{A,\epsilon}}^2 = \int_{\mathbb{R}^3 \setminus K} \rho_\epsilon(|\nabla A\psi|^2 + ||[\Phi, \psi]|^2) d^3x.$$  

The extended linearization operator is also defined w.r.t. the fiducial configuration:

$$\tilde{L}_\Psi = *d_A a - d_A \varphi - a \wedge \Phi + *d_A * a + [\Phi, \varphi].$$

Clearly $\tilde{L}_\Psi$ is a bounded linear map from $H_{\psi,\epsilon}$ to $L^2_{\psi,\epsilon}$. However, it may not be Fredholm, unless some extra constraints are satisfied.

Choose an origin $O$ for $\mathbb{R}^3$ and let $r$ be the distance to the origin. Let $\tilde{\Psi} = \tilde{A} + \tilde{\Phi}$ be the distinguished fiducial configuration.

**Definition 4.1.** If a configuration $\Psi = A + \Phi$ on $\mathbb{R}^3 \setminus K$ satisfies the following constraints:

- there exists a constant $c > 0$ such that $\lim_{r \to \infty} |\Phi| = c$;
- $\limsup_{r \to +\infty} r(|\nabla A\Phi| + |F^A|) < +\infty$;
- $\Psi = \tilde{\Psi}$ on $N_{100\delta}$, and in particular $V(\Psi) = F^A = \nabla A\Psi = 0$ on $N_{100\delta}$;
- for some small enough $\mu > 0$ to be determined, $|V(\Psi)| \leq \mu$ pointwise.

Then $\Psi$ is called an **admissible approximate solution**.

**Proposition 4.2.** Suppose $Q(\psi_1, \psi_2)$ is a bi-linear form such that

$$|Q(\psi_1, \psi_2)| \leq C|\psi_1||\psi_2| \text{ point-wise.}$$

Suppose moreover $Q(\psi_1, \psi_2)$ satisfies

$$Q(\psi_1^\sigma, \psi_2^\sigma) = 0,$$
then for any \( \psi_1, \psi_2 \in \mathbb{H}_{\psi, \epsilon} \), and for some possibly larger \( C_1 \) that depends on \( C, \epsilon \) and the knot,

\[
\|Q(\psi_1, \psi_2)\|_{L^2} \leq C_1 \|\psi_1\|_{\mathbb{H}_{\psi, \epsilon}} \|\psi_2\|_{\mathbb{H}_{\psi, \epsilon}}.
\]

**Proof.** The proof is similar with the proof of proposition 3.6 and omitted. \( \Box \)

### 4.2 \( \tilde{L}_\psi \) is Fredholm with 0 cokernel

Consider a large enough number \( R > 0 \), such that \( N_{100\delta} \) is contained entirely in the ball centered at \( O \) of radius \( R \), namely \( B_R(0) \), and suppose \( |\Phi| > \frac{\epsilon}{2} \) outside \( B_R(0) \).

**Proposition 4.3.** Assume \( \Psi \) is an admissible approximate solution, then \( \tilde{L}_\psi \) as a bounded linear operator from \( \mathbb{H}_{\psi, \epsilon} \) to \( L^2_\epsilon \) has finite dimensional kernel and closed range.

**Proof.**

\[
\int_{\mathbb{R}^3 \setminus N_{\epsilon}} |\tilde{L}_\psi \psi|^2 \, d^3x = \int_{\mathbb{R}^3 \setminus N_{\epsilon}} |d_\psi + [\Phi, \psi]|^2 \, d^3x
\]

\[
= \int_{\mathbb{R}^3 \setminus N_{\epsilon}} (|\sum_j \tau_j \nabla_j^\lambda \psi|^2 + 2 \sum_j \langle \tau_j \nabla_j^\lambda \psi, [\Phi, \psi] \rangle + \|\Phi, \psi\|^2) \, d^3x
\]

\[
= \int_{\mathbb{R}^3 \setminus N_{\epsilon}} \left( \sum_j |\nabla_j^\lambda \psi|^2 + \|\Phi, \psi\|^2 + \sum_j \langle \psi, \tau_j (\sum_l \tau_l \nabla_l^\lambda \phi) \rangle + \sum_j \langle \psi, \tau_j \tau_l F_{jl} \psi \rangle \right)
\]

\[
+ \sum_{j \neq l} \nabla_j \langle \psi, \tau_j \nabla_l^\lambda \phi \rangle + \sum_j \nabla_j \langle \psi, \tau_j \nabla_l^\lambda \phi \rangle) \, d^3x
\]

\[
= \int_{\mathbb{R}^3 \setminus N_{\epsilon}} \left( \sum_j |\nabla_j^\lambda \psi|^2 + \|\Phi, \psi\|^2 + \sum_j \langle \psi, \tau_j (\sum_l \tau_l \nabla_l^\lambda \phi) \rangle + \sum_j \langle \psi, \tau_j \tau_l F_{jl} \psi \rangle \right)
\]

\[
+ \int_{\partial N_{\epsilon}} (-\frac{1}{\epsilon} < \tau_\rho \psi, \tau_\theta \nabla_\theta^\lambda \psi > - \tau_\rho \psi, \tau_\sigma \nabla_\sigma^\lambda \psi > - \tau_\rho \psi, [\sum_l \tau_l \sigma \nabla_l^\lambda \phi] >) \, d\Omega
\]

The only term that is new compared to the proof of lemma 3.7 is the following one:

\[
\int_{\mathbb{R}^3 \setminus N_{\epsilon}} \sum_j \langle \psi, \tau_j ((\sum_l \tau_l \sigma \nabla_l^\lambda \phi) - V(\psi)) \rangle \, d^3x.
\]

Let

\[
Q_\psi(\psi_1, \psi_2) = \sum_j \langle \psi_1, \tau_j ((\sum_l \tau_l \sigma \nabla_l^\lambda \phi) \rangle, \psi_2 \rangle, \quad P_\psi(\psi_1, \psi_2) = \sum_j \langle \psi_1, \tau_j (V(\psi) \rangle, \psi_2 \rangle.
\]

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Applying the argument from the proof of lemma 3.7,
\[
\int_{\mathbb{R}^3 \setminus K} \rho \abs{\tilde{L}_\psi \psi}^2 d^3 x \geq \frac{1}{C} \|\psi\|_{H^{\rho,\epsilon}}^2 - 2 \int_{\mathbb{R}^3 \setminus N_\epsilon} |Q_\Psi(\psi, \psi)| d^3 x - \int_{\mathbb{R}^3 \setminus N_\epsilon} |P_\Psi(\psi, \psi)| d^3 x.
\]

A similar computation on \(\tilde{L}_\psi^\dagger\) shows that
\[
\int_{\mathbb{R}^3 \setminus K} \rho |\tilde{L}_\psi^\dagger \psi|^2 d^3 x \geq \frac{1}{C} \|\psi\|_{H^{\rho,\epsilon}}^2 - \int_{\mathbb{R}^3 \setminus N_\epsilon} P_\Psi(\psi, \psi) d^3 x.
\]

By assuming \(|V(\Psi)| \leq \mu\) to be small enough, together with proposition 4.2, then
\[
\int_{\mathbb{R}^3 \setminus N_\epsilon} P_\Psi(\psi, \psi) d^3 x \leq \frac{1}{2C} \|\psi\|_{H^{\rho,\epsilon}}^2.
\]

Note that since \(\lim_{r \to +\infty} \rho r |F^A| = 0\), by proposition 4.2, there exists a large enough ball \(B \subset \mathbb{R}^3\) such that
\[
\int_{\mathbb{R}^3 \setminus B} |Q_\Psi(\psi, \psi)| d^3 x \leq \frac{1}{4C} \|\psi\|_{H^{\rho,\epsilon}}^2.
\]

So
\[
\int_{\mathbb{R}^3 \setminus K} \rho \abs{\tilde{L}_\psi \psi}^2 d^3 x \geq \frac{1}{4C} \|\psi\|_{H^{\rho,\epsilon}}^2 - 2 \int_{B \setminus N_\epsilon} |Q_\Psi(\psi, \psi)| d^3 x, \quad \int_{\mathbb{R}^3 \setminus K} \rho \abs{\tilde{L}_\psi \psi}^2 d^3 x \geq \frac{1}{2C} \|\psi\|_{H^{\rho,\epsilon}}^2,
\]

where \(\int_{B \setminus N_\epsilon} |Q_\Psi(\psi, \psi)| d^3 x\) is compact relative to \(\|\psi\|_{H^{\rho,\epsilon}}^2\) by theorem 7.4, which is enough to finish the proof by the same argument with Subsection 3.3.

\[\square\]

### 4.3 Moduli space of real solutions near an admissible approximate solution

Suppose \(\Psi = A + \Phi\) is an admissible approximate solution. Hence \(\tilde{L}_\psi\) is Fredholm with 0 cokernel as a map from \(\mathbb{H}_{\Psi,\epsilon}\) to \(L^2_{\epsilon}\).

**Definition 4.4.** Suppose \(\psi \in \mathbb{H}_{\Psi,\epsilon}\) satisfies the following equations:

\[
\begin{align*}
V(\Psi + \psi) &= V(\Psi) + L_\Psi(\psi) + Q(\psi, \psi) = 0; \\
\tilde{L}_\psi \psi - L_\Psi \psi &= *dA *a + [\Phi, \varphi] = 0.
\end{align*}
\]

Then \(\psi\) is called a **solution to the Bogomolny equation with gauge fixing condition w.r.t. \(\Psi\)**.
Theorem 4.5. There exists a small open neighbourhood of 0 in $\mathbb{H}_{\Psi,\epsilon}$, namely $U$, such that solutions to the Bogomolny equation with gauge fixing condition w.r.t. $\Psi$ in $U$ are 1-1 correspondent with $(\ker \hat{L}_\Psi) \cap U$.

Proof. Since $\hat{L}_\Psi$ is Fredholm with 0 cokernel, there exists an $C > 0$, such that for any $W \in \mathbb{L}^2$, there exists a unique $\psi \in (\ker \hat{L}_\Psi)\perp \subset \mathbb{H}_{\Psi,\epsilon}$, such that $\hat{L}_\Psi(\psi) = W$. Moreover, $\|\psi\|_{\mathbb{H}_{\Psi,\epsilon}} \leq C\|W\|_{\mathbb{L}^2}$.

Choose any $\psi_0 \in (\ker \hat{L}_\Psi) \cap U$. Consider $\psi_1, \psi_2, \ldots, \in (\ker \hat{L}_\Psi)\perp$ such that

$$\hat{L}_\Psi(\psi_1) = -V(\Psi + \psi_0) = -V(\Psi) - Q(\psi_0, \psi_0);$$
$$\hat{L}_\Psi(\psi_2) = -V(\Phi + \psi_0 + \psi_1) = -2Q(\psi_0, \psi_1) - Q(\psi_1, \psi_1);$$
$$\vdots$$
$$\hat{L}_\Psi(\psi_n) = -V(\Phi + \psi_0 + \cdots + \psi_{n-1}) = -2Q(\psi_0 + \cdots + \psi_{n-2}, \psi_{n-1}) - Q(\psi_{n-1}, \psi_{n-1}).$$

Since $U$ is a small neighborhood and $\mu$ is also small enough, it may be assumed that $C^2C_\Psi(\mu + C_\Psi\|\psi_0\|_{\mathbb{H}_{\Psi,\epsilon}}^2 + 2\|\psi_0\|_{\mathbb{H}_{\Psi,\epsilon}}) \leq \frac{1}{24}$. So

$$\|\psi_1\|_{\mathbb{H}_{\Psi,\epsilon}} \leq C\|V(\Psi)\|_{\mathbb{L}^2} + \|Q(\psi_0, \psi_0)\|_{\mathbb{L}^2}) \leq C(\mu + C_\Psi\|\psi_0\|_{\mathbb{H}_{\Psi,\epsilon}}^2) \leq \frac{1}{2}\frac{1}{24}CC_\Psi;$$
$$\|\psi_2\|_{\mathbb{H}_{\Psi,\epsilon}} \leq CC_\Psi\|\psi_1\|_{\mathbb{H}_{\Psi,\epsilon}}(\|\psi_1\|_{\mathbb{H}_{\Psi,\epsilon}} + 2\|\psi_0\|_{\mathbb{H}_{\Psi,\epsilon}}) \leq \frac{1}{2}\|\psi_1\|_{\mathbb{H}_{\Psi,\epsilon}} \leq \frac{1}{2}\frac{1}{24}CC_\Psi;$$
$$\|\psi_3\|_{\mathbb{H}_{\Psi,\epsilon}} \leq CC_\Psi\|\psi_2\|_{\mathbb{H}_{\Psi,\epsilon}}(\|\psi_2\|_{\mathbb{H}_{\Psi,\epsilon}} + 2\|\psi_1\|_{\mathbb{H}_{\Psi,\epsilon}} + 2\|\psi_0\|_{\mathbb{H}_{\Psi,\epsilon}}) \leq \frac{1}{2}\|\psi_2\|_{\mathbb{H}_{\Psi,\epsilon}} \leq \frac{1}{2}\frac{1}{24}CC_\Psi;$$
$$\vdots$$
$$\|\psi_n\|_{\mathbb{H}_{\Psi,\epsilon}} \leq CC_\Psi\|\psi_{n-1}\|_{\mathbb{H}_{\Psi,\epsilon}}(\|\psi_{n-1}\|_{\mathbb{H}_{\Psi,\epsilon}} + 2\|\psi_{n-2}\|_{\mathbb{H}_{\Psi,\epsilon}} + \cdots + 2\|\psi_0\|_{\mathbb{H}_{\Psi,\epsilon}}) \leq \frac{1}{2}\|\psi_{n-1}\|_{\mathbb{H}_{\Psi,\epsilon}} \leq \frac{1}{2}\frac{1}{24}CC_\Psi.$$

Hence $\psi = \sum_{n=0}^{+\infty} \psi_n$ is in $\mathbb{H}_{\Psi,\epsilon}$ which is clearly a solution to the Bogomolny equation with gauge fixing condition w.r.t. $\Psi$. In addition, $\psi - \psi_0 \in \ker(\hat{L}_\Psi)\perp$, so $\psi_0$ is the projection of $\psi$ on $\ker(\hat{L}_\Psi)$.

On the other hand, suppose $\hat{\psi} \in U$ is any solution to the Bogomolny equation with gauge fixing condition w.r.t. $\Psi$ whose projection on $\ker(\hat{L}_\Psi)$ is also $\psi_0$. So $\hat{\psi} - \psi \in (\ker \hat{L}_\Psi)\perp$. In particular,

$$\|\hat{\psi} - \psi\|_{\mathbb{H}_{\Psi,\epsilon}} \leq C\|\hat{L}_\Psi(\hat{\psi} + \psi)\|_{\mathbb{L}^2} \leq C\|Q(\hat{\psi} + \psi, \hat{\psi} - \psi)\|_{\mathbb{L}^2} \leq CC_\Psi\|\hat{\psi} + \psi\|_{\mathbb{H}_{\Psi,\epsilon}}\|\hat{\psi} - \psi\|_{\mathbb{H}_{\Psi,\epsilon}}.$$

Since $U$ is small enough, it can be assumed that $CC_\Psi\|\hat{\psi} + \psi\|_{\mathbb{H}_{\Psi,\epsilon}} < 1$, which implies that $\hat{\psi} = \psi$. Hence the theorem is proved. \qed
4.4 The 1-monopole solution

Definition 4.6. Choose a distinguished point $P$ in $\mathbb{R}^3$. Choose a Cartesian coordinates of $\mathbb{R}^3$ centered at $P$, namely $\{w_1, w_2, w_3\}$. Let $w = M(w_1 + iw_2), t = Mw_3, r = \sqrt{|w|^2 + t^2}$. The following solution $\Psi = \Phi + A = \Phi + A_1dw_1 + A_2dw_2 + A_3dw_3$ is called the 1-monopole:

\[
\begin{align*}
\Phi &= \frac{M(r \coth r - 1)}{2r}((t \cosh r - r \sinh r)(-i 0) + (0 w) + (0 -\bar{w} 0)), \\
A_3 &= \frac{iM}{2r}(\frac{r \coth r - 1}{r \cosh r - r \sinh r})(0 w) \\
A_1 + iA_2 &= \frac{M}{2r((r \cosh r - r \sinh r)(-\cosh r r + \frac{1}{\sinh r})0 w) + (t + r^2 - r \coth r)0 0).
\end{align*}
\]

Suppose $U = \frac{(r + t)e^{-r}}{|w|}$, so $r \cosh r - t \sinh r = \frac{|w|}{2}U + U^{-1}$, and $t \cosh r - r \sinh r = \frac{|w|}{2}U - U^{-1}$.

\[
\begin{align*}
\Phi &= (\coth r - 1)\frac{M}{U + U^{-1}}(\frac{U - U^{-1}}{2})(-i 0) + \frac{1}{|w|}(0 w) \\
A_3 &= \frac{iM}{|w|}(\coth r - 1)(\frac{1}{U + U^{-1}})(0 w) \\
A_1 + iA_2 &= \frac{M}{|w|(U + U^{-1})}((-\cosh r r + \frac{1}{\sinh r})0 w) - \frac{1}{r}(t - r^2 + r \coth r)0 0.
\end{align*}
\]

Some calculations:

\[
\begin{align*}
F_{12} &= \nabla_3 \Phi = \partial_3 \Phi + [A_3, \Phi] \\
&= \frac{M^2}{(r^3 - \frac{t}{r \sinh^2 r})(U + U^{-1})}(\frac{U - U^{-1}}{2})(-i 0) + \frac{1}{|w|}(0 w) \\
&\quad - \frac{M^2}{(r \coth r - 1)(r \cosh r - r \sinh r)(U + U^{-1})}(\frac{U - U^{-1}}{2})(0 w) - \frac{1}{r}(t - r^2 + r \coth r)0 0; \\
F_{23} + iF_{31} &= (\nabla_1 + i\nabla_2)\Phi = (\partial_1 + i\partial_2)\Phi + [A_1 + iA_2, \Phi] \\
&= \frac{M^2}{(r^3 - \frac{t}{r \sinh^2 r})(U + U^{-1})}(\frac{U - U^{-1}}{2})(-i 0) + \frac{1}{|w|}(0 w) \\
&\quad - \frac{M^2w}{(r \coth r - 1)(r \cosh r - r \sinh r)(U + U^{-1})}(t(-i 0) + \frac{1}{r}(t - r^2 + r \coth r)0 0.
\end{align*}
\]

Let $v = \frac{1}{2(U + U^{-1})}(U - U^{-1})(-i 0) + \frac{1}{|w|}(0 z)\bar{z} 0)$, which is a unit vector. Let $v^\perp$ denote any unit vector that is perpendicular to $v$, finally
\[
\begin{cases}
  \Phi = M(\coth r - \frac{1}{r^2})v, \\
  F_{12} = M^2\left(\frac{t}{r^3} - \frac{t}{\sinh^2 r}\right)v - M^2(\coth r - \frac{1}{r^2})\frac{\rho}{r} v^\perp, \\
  F_{23} + i F_{31} = M^2\left(\frac{z}{r^3} - \frac{z}{r \sinh^2 r}\right)v - M^2(\coth r - \frac{1}{r^2})\sqrt{\rho^2 + 2\ell^2} v^\perp.
\end{cases}
\]

The half line \(|w| = 0, t \geq 0\) is called the “Dirac ray”. It is easy to see that

- \(|\nabla A \Psi| = |F^A| = O\left(\frac{1}{r^2}\right), \ \Phi = Mv + O\left(\frac{1}{r}\right);

- at a point whose distance to the Dirac ray \(d \geq \frac{2}{M}\) so that \((r-t)^2 e^r \geq \rho,

\[
\frac{(u - u^{-1})}{(u + u^{-1})} = -1 + \frac{2\rho^2}{\rho^2 + (r-t)^2 e^{2r}} = -1 + O(e^{-r}),
\]

then

\[
v = -\frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + O(e^{-r}).
\]

**Remark 4.7.** The 1-monopole defined here may be in an unfamiliar format to experts. In fact, under a gauge transformation, the 1-monopole is gauge equivalent to the following solution to the Bogomolny equation, which is better known (see for example Page 104 in [3]):

\[
\begin{cases}
  \Phi = \left(\frac{1}{r \tanh r} - \frac{1}{r^2}\right)(w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3), \\
  A = \left(\frac{1}{r \sinh r} - \frac{1}{r^2}\right)((w_2 \sigma_3 - w_3 \sigma_2)dw_1 + (w_3 \sigma_1 - w_1 \sigma_3)dw_2 + (w_1 \sigma_2 - w_2 \sigma_1)dw_3).
\end{cases}
\]

where \(\sigma_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.

**4.5 The admissible approximate solution from gluing**

Suppose \(\psi_1, \psi_2, \psi_3, \ldots, \psi_n\) are 1-monopole solutions centered at \(P_1, P_2, \ldots, P_n\) and have Dirac rays \(L_1, L_2, \ldots, L_n\) respectively. Suppose \(d > 0\) is a large enough constant and \(R \gg d\) is even larger. Let \(V_i(R)\) be the open set of points whose distance to the Dirac ray \(L_i\) are less than \(R\). Suppose moreover any two of the corresponded open neighbourhoods \(V_1(R), V_2(R), \ldots, V_n(R)\) together with \(B_R(O)\) (the ball of radius \(R\) centered at the origin) do not intersect. Let \(V_0(R)\) denote \(B_R(O)\). Let \(r\) be the distance function to the origin \(O\). Then
Proposition 4.8. Let $\Psi = \sum_{j=0}^{n} \tilde{\chi}_j \Psi_j$, where each $\tilde{\chi}_j$ is a suitable cut-off function that is 1 on $V_j(d)$ and $|\nabla \tilde{\chi}_j| \leq \frac{1}{d^r}$. Moreover, it may be assumed that $\sum_{j=0}^{n} \tilde{\chi}_j = 1$. Given that $d$ is large enough and $R \gg d$, then $\Psi$ is an admissible approximate solution.

Proof. All the requirements in definition 4.1 can be examined straightforwardly. \qed

5 Other solutions in $\mathbb{H}_\epsilon$

5.1 Some basic analysis

Suppose $\hat{A} - A \in \mathbb{H}_{N,\epsilon}$ is a $su(2)$-valued 1-form on $N_\epsilon$, such that

$$A = A_\rho d\rho + \rho A_\theta d\theta + A_s ds,$$

and consider the following connection

$$\nabla \hat{A} = \nabla + A = (\partial_\rho + A_\rho) \otimes d\rho + (\partial_\theta + \rho A_\theta) \otimes d\theta + (\partial_s + A_s) \otimes ds,$$

then

Lemma 5.1. $\nabla A$ and $\nabla \hat{A}$ has the same monodromy near $K$ in the following sense

$$\lim_{\rho \to 0} \int_0^l |\int_0^{2\pi} (\rho A_\theta - \gamma \sigma) d\theta| ds = 0.$$

Proof.

$$\lim_{\rho \to 0} \int_0^l |\int_0^{2\pi} (\rho A_\theta - \gamma \sigma) d\theta| ds \leq \lim_{\rho \to 0} (2\pi l)^{\frac{1}{2}} (\int_0^l \int_0^{2\pi} |\rho A_\theta - \gamma \sigma|^2 d\theta ds)^{\frac{1}{2}} = 0.$$

I need a technical lemma that will be used later:

Lemma 5.2. Assuming $\gamma \in (0, \frac{1}{8})$, then for $\epsilon > 0$ small enough,

$$\gamma^2 \|\psi\|^2_{\mathbb{H}_{\epsilon}, \epsilon} + \int_{N_\epsilon \setminus K} \frac{1}{\rho} ((\frac{3}{16} - 2\gamma^2) |\psi|^2 + (1 - 2\gamma^2) |\nabla A^4 \psi|^2 + \langle \psi, \tau_s \nabla A^4 \psi \rangle) d^3 x \geq 0.$$
Proof.

\[
\begin{aligned}
&\int_{N \setminus K} \frac{1}{\rho} \left( \frac{3}{16} - 2\gamma^2 \right) |\psi|^2 + (1 - 2\gamma^2) |\nabla^A_\theta \psi|^2 + \langle \psi, \tau_s \nabla^A_\theta \psi \rangle \right) d^3 x \\
\geq &\int_{N \setminus K} \frac{1}{\rho} \left( \frac{3}{16} - 2\gamma^2 \right) |\psi|^2 + (1 - 2\gamma^2) |\nabla^\hat{A}_\theta \psi|^2 + \langle \psi, \tau_s \nabla^\hat{A}_\theta \psi \rangle \right) d^3 x \\
&- \int_{N \setminus K} (2 < (\nabla^\hat{A}_\theta \psi + \psi), [(\hat{A}_\theta - A_\theta), \psi] > + \rho [((\hat{A}_\theta - A_\theta), \psi] |^2) d^3 x \\
\geq &\int_{N \setminus K} (2(\rho|\nabla^\hat{A}_\theta \psi|^2 + |\psi|^2)|\hat{A} - A||\psi| + \rho|\hat{A} - A|^2 |\psi|^2) d^3 x,
\end{aligned}
\]

where

\[
2 \int_{N \setminus K} \rho|\nabla^\hat{A}_\theta \psi| |\hat{A} - A||\psi|^2 d^3 x \leq 2 \left( \int_{N \setminus K} \rho|\nabla^\hat{A}_\theta \psi|^2 d^3 x \right)^{\frac{1}{2}} \left( \int_{N \setminus K} \rho|\hat{A} - A|^4 d^3 x \right)^{\frac{1}{2}} \left( \int_{N \setminus K} \rho|\psi|^4 d^3 x \right)^{\frac{1}{2}},
\]

\[
2 \int_{N \setminus K} |\hat{A} - A||\psi|^2 d^3 x \leq 2 \left( \int_{N \setminus K} \rho |\hat{A} - A|^2 d^3 x \right)^{\frac{1}{2}} \left( \int_{N \setminus K} \rho |\psi|^4 d^3 x \right)^{\frac{1}{2}},
\]

\[
\int_{N \setminus K} \rho|\hat{A} - A|^2 |\psi|^2 d^3 x \leq \left( \int_{N \setminus K} \rho |\hat{A} - A|^4 d^3 x \right)^{\frac{1}{2}} \left( \int_{N \setminus K} \rho |\psi|^4 d^3 x \right)^{\frac{1}{2}}.
\]

Because \( \hat{A} - A \in H^\epsilon \), when \( \epsilon \) is small, both \( \int_{N \setminus K} \rho |\hat{A} - A|^4 d^3 x \) and \( \int_{N \setminus K} \rho |\nabla^\hat{A}_\theta \psi|^2 d^3 x \) can be arbitrarily small. And both \( \int_{N \setminus K} \rho |\hat{A} - A|^4 d^3 x \) and \( \left( \int_{N \setminus K} \rho |\psi|^4 d^3 x \right)^{\frac{1}{2}} \) can be bounded (up to a constant which doesn’t depend on \( \epsilon \)) by \( \|\psi\|^2_{H^\epsilon} \). Hence the lemma is true.

\[
5.2 \text{ Fredholmness of } L_\Psi
\]

Let \( V = *F^A - d_A \Phi \), and \( V = \sum_j V_j \tau_j = \sum_j (*F - d_A \Phi)_j \). In particular, if \( \Psi = A + \Phi \) is a solutions to the Bogomolny equation, then \( V = 0 \). This
subsection proves that for such $\Psi$, $\tilde{L}_\psi$ is a Fredholm operator from $\mathbb{H}_{\psi,\epsilon}$ to $\mathbb{L}^2_\epsilon$. 

\[
\int_{N \setminus K} \rho |\tilde{L}_\psi \psi|^2 d^3x = \int_{N \setminus K} \rho |D_A \psi + [\Phi, \psi]|^2 d^3x \\
= \int_{N \setminus K} \rho (\sum_j \tau_j \nabla_j \psi|^2 + 2 \sum_j <\tau_j \nabla_j \psi, [\Phi, \psi]> + |[\Phi, \psi]|^2)d^3x \\
= \int_{\mathbb{R}^3 \setminus N} \rho (\sum_j |\nabla_j \psi|^2 + |[\Phi, \psi]|^2 + \sum_j (<\psi, \tau_j ([*F^A + d_A \Phi]_j, \psi)> \\
+ \sum_j \nabla_j(<\tau_j \psi, (\sum_l \tau_l \nabla_l \psi) + [\Phi, \psi]>) - \frac{1}{2} \sum_j \nabla_j^2(\psi^2))d^3x \\
= \int_{N \setminus K} \rho (\sum_j |\nabla_j \psi|^2 + |[\Phi, \psi]|^2 + \sum_j (<\psi, \tau_j ([*F^A + d_A \Phi]_j, \psi)>d^3x \\
+ \int_{N \setminus K} <\psi, \frac{\tau_s \nabla_A \psi - \tau_\phi \nabla_A \psi + \tau_\rho [\Phi, \psi] > d^3x \\
+ \int_{\partial N} <\psi, -\tau_s \nabla_A \psi + \epsilon (\tau_\phi \nabla_A \psi - \tau_\rho [\Phi, \psi]) > \rho d\Omega \\
+ \liminf_{\rho \to 0} \rho \int_{\partial N_\rho} <\psi, \tau_s \nabla_A \psi - \tau_\phi \nabla_A \psi - [\Phi, \psi] > \rho d\Omega \\
= \int_{N \setminus K} \rho (\sum_j |\nabla_j \psi|^2 + |[\Phi, \psi]|^2 + \sum_j (<\psi, \tau_j ([*F^A + d_A \Phi]_j, \psi)>d^3x \\
+ \int_0^1 \int_0^{2\pi} \rho <\psi, \frac{\tau_s \nabla_A \psi - \tau_\phi \nabla_A \psi + \tau_\rho [\Phi, \psi] > d\rho d\theta ds \\
- \epsilon^2 \int_0^1 \int_0^{2\pi} (<\psi, \frac{\tau_s \nabla_A \psi - \tau_\phi \nabla_A \psi + \tau_\rho [\Phi, \psi] > |_{\rho=\epsilon})d\theta ds \\
= \int_{N \setminus K} \rho (\sum_j |\nabla_j \psi|^2 + |[\Phi, \psi]|^2 + \sum_j (<\psi, \tau_j ([*F^A + d_A \Phi]_j, \psi)>d^3x \\
+ \frac{1}{2} \int_0^1 \int_0^{2\pi} \rho^2 \partial_\rho (<\psi, \tau_\phi \nabla_A \psi - \tau_\rho [\Phi, \psi] > d\rho d\theta ds + \int_{N \setminus K} \frac{1}{\rho} <\psi, \tau_s \nabla_A \psi > d^3x \\
+ \frac{1}{2} \epsilon^2 \int_0^1 \int_0^{2\pi} (<\psi, \tau_\phi \nabla_A \psi - \tau_\rho [\Phi, \psi] > |_{\rho=\epsilon})d\theta ds \\
= \int_{N \setminus K} \rho (\sum_j |\nabla_j \psi|^2 + |[\Phi, \psi]|^2 + \sum_j (<\psi, \tau_j ([*F^A + d_A \Phi]_j, \psi)>d^3x \\
+ \frac{1}{2} \int_{N \setminus K} \rho \partial_\rho (<\psi, \tau_\phi \nabla_A \psi - \tau_\rho [\Phi, \psi] > d^3x + \int_{N \setminus K} \frac{1}{\rho} <\psi, \tau_s \nabla_A \psi > d^3x \\
+ \frac{1}{2} \epsilon \int_{\partial N} <\psi, \tau_\phi \nabla_A \psi - \tau_\rho [\Phi, \psi] > \rho d\Omega, 
\]
where
\[
\frac{1}{2} \int_{N_c \setminus K} \rho \partial_{\rho} (\langle \psi, -\tau_{\theta} \nabla^A_s \psi + \tau_{\rho} [\Phi, \psi] \rangle) d^3 x
\]
\[
= \int_{N_c \setminus K} \rho < \nabla^A_{\rho} (\psi), -\tau_{\theta} \nabla^A_s \psi + \tau_{\rho} [\Phi, \psi] > d^3 x + \frac{1}{2} \int_{N_c \setminus K} \rho < \psi, \tau_{\theta} [F_{s\rho}, \psi] + \tau_{\rho} [\nabla^A \Phi, \psi] > d^3 x
\]
\[
\leq \left( \frac{1}{4} + 4\gamma^2 \right) \int_{N_c \setminus K} \rho |\nabla^A_{\rho} (\psi)|^2 d^3 x + (1 - 4\gamma^2) \int_{N_c \setminus K} \rho (| - \tau_{\theta} \nabla^A_s \psi + \tau_{\rho} [\Phi, \psi]|^2) d^3 x
\]
\[
+ \frac{1}{2} \int_{N_c \setminus K} \rho < \psi, \tau_{\theta} [F_{s\rho}, \psi] + \tau_{\rho} [\nabla^A \Phi, \psi] > d^3 x
\]
\[
= \left( \frac{1}{4} + 4\gamma^2 \right) \int_{N_c \setminus K} \rho |\nabla^A_{\rho} (\psi)|^2 d^3 x + (1 - 4\gamma^2) \int_{N_c \setminus K} \rho (|\nabla^A_s \psi|^2 + |[\Phi, \psi]|^2) d^3 x
\]
\[
+ \int_{N_c \setminus K} \rho < \psi, \frac{1}{2}(\tau_{\theta} [F_{s\rho}, \psi] + \tau_{\rho} [\nabla^A \Phi, \psi]) + (1 - 4\gamma^2)(\tau_{\rho}[\nabla^A \Phi, \psi]) > d^3 x.
\]

Then
\[
\int_{N_c \setminus K} \rho |\tilde{L}_{\psi} \psi|^2 d^3 x
\]
\[
\geq \left( \frac{3}{4} - 4\gamma^2 \right) \int_{N_c \setminus K} \rho |\nabla^A_{\rho} (\psi)|^2 d^3 x + 4\gamma^2 \int_{N_c \setminus K} \rho (|\nabla^A_s \psi|^2 + |[\Phi, \psi]|^2) d^3 x
\]
\[
+ \int_{N_c \setminus K} \frac{1}{2}(|\nabla^A_{\rho} \psi|^2 + < \psi, \tau_{\rho} \nabla^A_s \psi >) d^3 x + \int_{N_c \setminus K} \rho (\sum_j (\langle \psi, \tau_{\rho} [F_j + dA j, \psi] \rangle)) d^3 x
\]
\[
+ \frac{1}{2} \epsilon \int_{\partial N_c} \psi, \tau_{\theta} \nabla^A_s \psi - \tau_{\rho} [\Phi, \psi] > \rho d\Omega
\]
\[
- \int_{N_c \setminus K} \rho < \psi, \tau_{\theta} [F_{s\rho}, \psi] + \tau_{\rho} [\nabla^A \Phi, \psi]) + (1 - 4\gamma^2)(\tau_{\rho}[\nabla^A \Phi, \psi]) > d^3 x
\]
\[
\geq \left( \frac{3}{4} - 4\gamma^2 \right) \int_{N_c \setminus K} \rho |\nabla^A_{\rho} (\psi)|^2 d^3 x + 4\gamma^2 \int_{N_c \setminus K} \rho (|\nabla^A_s \psi|^2 + |[\Phi, \psi]|^2) d^3 x
\]
\[
+ \int_{N_c \setminus K} \frac{1}{2}(|\nabla^A_{\rho} \psi|^2 + < \psi, \tau_{\rho} \nabla^A_s \psi >) d^3 x - 2 \int_{N_c \setminus K} \rho ([F^A] + |dA \Phi|) |\psi \land \psi| d^3 x
\]
\[
+ \frac{1}{2} \epsilon \int_{\partial N_c} \psi, \tau_{\theta} \nabla^A_s \psi - \tau_{\rho} [\Phi, \psi] > \rho d\Omega,
\]

where $[\psi \land \psi]$ is a quadratic term that involve wedge product and the Lie bracket in $su(2)$.
On the other hand
\[
\epsilon \int_{\mathbb{R}^3 \setminus N_e} |\bar{L}_\psi \psi|^2 d^3 x = \int_{\mathbb{R}^3 \setminus N_e} \left| \sum_j \tau_j \nabla^A_j \psi + [\Phi, \psi] \right|^2 d^3 x
\]
\[
= \epsilon \int_{\mathbb{R}^3 \setminus N_e} \left( \sum_j \tau_j \nabla^A_j \psi \right)^2 + \sum_j \left( \tau_j \nabla^A_j \psi, [\Phi, \psi] \right) + \left| [\Phi, \psi] \right|^2 d^3 x
\]
\[
= \epsilon \int_{\mathbb{R}^3 \setminus N_e} \left( \sum_j \tau_j \nabla^A_j \psi \right)^2 + \sum_j \left( <\psi, \tau_j [(\ast F^A + d_A \Phi)_j, \psi] > + \left| [\Phi, \psi] \right|^2 \right) d^3 x
\]
\[
+ \epsilon \int_{\partial N_e} \left( \frac{1}{\epsilon} <\tau_{j, \psi}, \tau_{s} \nabla^A_{\psi} \psi \right) > - \tau_{j, \psi}, \tau_{s} \nabla^A_{\psi} \psi > - \tau_{j, \psi}, [\Phi, \psi] > \right) \rho d\Omega.
\]
So
\[
2 \int_{N_e \setminus K} \rho |\bar{L}_\psi \psi|^2 d^3 x + \epsilon \int_{\mathbb{R}^3 \setminus N_e} |\bar{L}_\psi \psi|^2 d^3 x
\]
\[
\geq \left( \frac{3}{2} - 8\gamma^2 \right) \int_{N_e \setminus K} \rho \left( \nabla^A_{\rho} (\psi)^2 d^3 x + 8\gamma^2 \right) \int_{N_e \setminus K} \rho \left( |\nabla^A_{\rho} (\psi)|^2 + \left| [\Phi, \psi] \right|^2 \right) d^3 x
\]
\[
+ 2 \int_{\mathbb{R}^3 \setminus N_e} \int_0^{2\pi} \int_0^\pi \left( |\nabla^A_{\rho} (\psi)|^2 + <\psi, \tau_{s} \nabla^A_{\rho} \psi > \right) \rho d\theta d\sigma + \int_{\partial N_e} <\psi, \tau_{s} \nabla^A_{\rho} \psi > \rho d\Omega
\]
\[
+ \epsilon \int_{\mathbb{R}^3 \setminus N_e} \left( \sum_j \tau_j \nabla^A_j \psi \right)^2 + \sum_j \left( <\psi, \tau_j [V_j, \psi] > \right) + \left| [\Phi, \psi] \right|^2 d^3 x
\]
\[
- \int_{\mathbb{R}^3 \setminus N_e} \left( | \ast F^A + d_A \Phi || \psi \wedge \psi || \right) d^3 x - 4 \int_{N_e \setminus K} \rho \left( |F^A| + |d_A \Phi| || \psi \wedge \psi || \right) d^3 x
\]
\[
\geq \frac{3}{2} - 8\gamma^2 \int_{N_e \setminus K} \rho \left( |\nabla^A_{\rho} (\psi)|^2 d^3 x - \int_{\mathbb{R}^3 \setminus N_e} \left( | \ast F^A + d_A \Phi || \psi \wedge \psi || \right) d^3 x
\]
\[
+ 2 \int_{N_e \setminus K} \frac{1}{\rho} \left( (1 - 4\gamma^2)|\nabla^A_{\rho} (\psi)|^2 + <\psi, \tau_{s} \nabla^A_{\rho} \psi > \right) d^3 x + \int_{\partial N_e} <\psi, \tau_{s} \nabla^A_{\rho} \psi > \rho d\Omega
\]
\[
- 4 \int_{N_e \setminus K} \rho \left( |F^A| + |d_A \Phi| || \psi \wedge \psi || \right) d^3 x
\]
\[
\geq \frac{3}{2} - 8\gamma^2 \int_{N_e \setminus K} \rho |\nabla^A_{\rho} (\psi)|^2 d^3 x + \int_{\partial N_e} <\psi, \tau_{s} \nabla^A_{\rho} \psi > \rho d\Omega
\]
\[
+ \int_{N_e \setminus K} \frac{1}{\rho} \left( (3 - 4\gamma^2)|\psi|^2 + (2 - 4\gamma^2)|\nabla^A_{\rho} (\psi)|^2 + 2 <\psi, \tau_{s} \nabla^A_{\rho} \psi > \right) d^3 x
\]
\[
- \int_{\mathbb{R}^3 \setminus N_e} \left( | \ast F^A + d_A \Phi || \psi \wedge \psi || \right) d^3 x - 4 \int_{N_e \setminus K} \rho \left( |F^A| + |d_A \Phi| || \psi \wedge \psi || \right) d^3 x.
\]
The last step above follows from lemma 2.12
So
\[
\| \bar{L}_\psi \psi \|^2 \geq 4\gamma^2 \| \psi \|_{H^2_{\rho, \psi}}^2 - \frac{3}{8} \int_{\partial N_e} |\psi|^2 \rho d\Omega - \frac{1}{2} \int_{\partial N_e} |\nabla^A_{\rho} (\psi)| \rho d\Omega
\]
\[
+ \int_{N_e \setminus K} \frac{1}{\rho} \left( (3 - 16 - 2\gamma^2)|\psi|^2 + (1 - 2\gamma^2)|\nabla^A_{\rho} (\psi)|^2 + <\psi, \tau_{s} \nabla^A_{\rho} \psi > \right) d^3 x
\]
\[
- \frac{1}{2} \int_{\mathbb{R}^3 \setminus N_e} \left( | \ast F^A + d_A \Phi || \psi \wedge \psi || \right) d^3 x - 2 \int_{N_e \setminus K} \rho \left( |F^A| + |d_A \Phi| || \psi \wedge \psi || \right) d^3 x.
\]
\[ \gamma^2 \| \psi \|_{H_{\Phi, \epsilon}}^2 + \int_{N_{\epsilon} \setminus K} \frac{1}{\rho} \left( \left( \frac{3}{16} - 2 \gamma^2 \right) |\psi|^2 + (1 - 2 \gamma^2) |\nabla_{\theta}^A \psi|^2 + \langle \psi, \tau_{\theta} \nabla_{\theta}^A \psi \rangle \right) d^3x \geq 0. \]

Hence
\[
\left\| \tilde{L}_{\Phi} \psi \right\|_{L^2}^2 \geq 3 \gamma^2 \| \psi \|_{H_{\Phi, \epsilon}}^2 - \frac{3}{8} \int_{\partial N_{\epsilon}} \| \psi \|^2 \rho d\Omega - \frac{1}{2} \int_{\partial N_{\epsilon}} \| \psi \| |\nabla_{\theta}^A \psi| \rho d\Omega - \frac{1}{2} \int_{\mathbb{R}^3 \setminus N_{\epsilon}} \left( |\ast F^A + d_A \Phi| |\psi \wedge \psi| \right) d^3x - 2 \int_{N_{\epsilon} \setminus K} \rho (|F^A| + |d_A \Phi|) |\psi \wedge \psi| d^3x,
\]
where
\[
2 \int_{N_{\epsilon} \setminus K} \rho (|F^A| + |d_A \Phi|) |\psi|^2 d^3x \leq 2 \left( \int_{N_{\epsilon} \setminus K} \rho (|F^A| + |d_A \Phi|)^2 d^3x \right) \frac{1}{2} \left( \int_{N_{\epsilon} \setminus K} \rho |\psi|^4 d^3x \right) \frac{1}{2}.
\]

Since \( \epsilon \) is small enough, \( (\int_{N_{\epsilon} \setminus K} \rho (|F^A| + |d_A \Phi|)^2 d^3x)^{\frac{1}{2}} \) can be arbitrarily small and \( (\int_{N_{\epsilon} \setminus K} \rho |\psi|^4 d^3x)^{\frac{1}{2}} \) can be bounded above (up to a constant which is doesn’t depend on \( \epsilon \)) by \( \| \psi \|_{H_{\Phi, \epsilon}}^2 \). In particular, it may be assumed that
\[
2 \int_{N_{\epsilon} \setminus K} \rho (|F^A| + |d_A \Phi|) |\psi|^2 d^3x \leq \gamma^2 \| \psi \|_{H_{\Phi, \epsilon}}^2.
\]

So
\[
\left\| \tilde{L}_{\Phi} \psi \right\|_{L^2}^2 \geq 2 \gamma^2 \| \psi \|_{H_{\Phi, \epsilon}}^2 - \frac{3}{8} \int_{\partial N_{\epsilon}} \| \psi \|^2 \rho d\Omega - \frac{1}{2} \int_{\partial N_{\epsilon}} \left( |\psi| |\nabla_{\theta}^A \psi| \right) \rho d\Omega - \frac{1}{2} \int_{\mathbb{R}^3 \setminus N_{\epsilon}} \left( |\ast F^A + d_A \Phi| |\psi \wedge \psi| \right) d^3x.
\]

By the same reason
\[
\left\| \tilde{L}_{\Phi} \psi \right\|_{L^2}^2 \geq 2 \gamma^2 \| \psi \|_{H_{\Phi, \epsilon}}^2 - \frac{3}{8} \int_{\partial N_{\epsilon}} \| \psi \|^2 \rho d\Omega - \frac{1}{2} \int_{\partial N_{\epsilon}} \left( |\psi| |\nabla_{\theta}^A \psi| \right) \rho d\Omega - \frac{1}{2} \int_{\mathbb{R}^3 \setminus N_{\epsilon}} \left( |V| |\psi \wedge \psi| \right) d^3x.
\]

Note that in particular, supposing \( V = \tilde{L}_{\Phi} \psi = 0 \) and \( \frac{1}{\rho} \psi \in L^2_{\epsilon} \), then
\[
0 = \left\| \tilde{L}_{\Phi} \psi \right\|_{L^2}^2 \geq \gamma^2 \| \psi \|_{H_{\Phi, \epsilon}}^2 - \frac{1}{8} \int_{\partial N_{\epsilon}} \| \psi \|^2 \rho d\Omega - \int_{\partial N_{\epsilon}} \left( |\psi| |\nabla_{\theta}^A \psi| \right) \rho d\Omega,
\]
which implies that \( \psi \in H_{\Phi, \epsilon} \).
On the other hand

\[ \left\| \hat{L}_\psi \psi \right\|_{L^2_x} + \int_{\mathbb{R}^3 \setminus N_{\varepsilon}} \left( |\ast F^A + d_A \Phi| |(\psi \wedge \psi)| \right) d^3 x \geq \frac{1}{2} \min_{\{ \varepsilon \leq \varepsilon_1 \leq 2\varepsilon \}} \left\| \hat{L}_\psi \psi \right\|_{L^2_x}^2 \]

\[ \geq \frac{1}{2} \min_{\{ \varepsilon \leq \varepsilon_1 \leq 2\varepsilon \}} \left( \gamma^2 \left\| \psi \right\|_{H^{\psi, \varepsilon}}^2 - \frac{1}{8} \int_{\partial N_{\varepsilon_1}} |\psi|^2 \rho d\Omega - \int_{\partial N_{\varepsilon_1}} (|\psi|^2 |\nabla^A \psi|) \rho d\Omega \right) \]

\[ \geq \frac{1}{2} \gamma^2 \left\| \psi \right\|_{H^{\psi, \varepsilon}}^2 - \frac{1}{8} \varepsilon \int_{N_{2\varepsilon_0} \setminus N_{\varepsilon_0}} |\psi|^2 d^3 x - \left( \frac{4}{\varepsilon} \int_{N_{2\varepsilon_0} \setminus N_{\varepsilon_0}} |\psi|^2 d^3 x \right)^{\frac{3}{2}} \left\| \psi \right\|_{H^{\psi, \varepsilon}}. \]

The same inequality is true for \( \tilde{L}_\psi \psi \). So both \( \hat{L}_\psi \) and \( \tilde{L}_\psi \) as maps from \( H^{\psi, \varepsilon} \) to \( L^2 \) have finite kernel and closed range. Hence \( \tilde{L}_\psi \) as a map from \( H^{\psi, \varepsilon} \) to \( L^2 \) is Fredholm.

### 5.3 Real analytical structure

There is a direct corollary of the Fredholmness of \( \tilde{L}_\psi \):

**Theorem 5.3.** There exists a small open neighbourhood of 0 in \( H^{\psi, \varepsilon} \), namely \( U \), and a real analytical map \( f \) from \( (\ker \hat{L}_\psi) \cap U \) to \( \text{coker} \tilde{L}_\psi \subset L^2 \), such that solutions to the Bogomolny equation with gauge fixing condition w.r.t. \( \Psi \) in \( U \) are 1-1 correspondent with \( f^{-1}(0) \).

**Proof.** Since \( \tilde{L}_\psi \) is Fredholm, there exists a \( C > 0 \), such that for any \( W \in L^2 \). Letting \( W^\perp \) be the projection of \( W \) onto \( (\ker \tilde{L}_\psi)^\perp \subset L^2 \), then there exists a unique \( \psi \in (\ker \tilde{L}_\psi)^\perp \subset H^{\psi, \varepsilon} \), such that \( \tilde{L}_\psi(\psi) = W^\perp \). Moreover, \( \left\| \psi \right\|_{H^{\psi, \varepsilon}} \leq C \left\| W^\perp \right\|_{L^2} \).

It may also be assumed that \( \left\| Q(\psi, \psi) \right\|_{L^2} \leq C \left\| \psi \right\|_{H^{\psi, \varepsilon}}^2 \) by proposition 4.2

The proof has two steps:

**Step 1:** Assume \( U \) is a small enough open neighbourhood of 0 in \( H^{\psi, \varepsilon} \). For any \( \psi_0 \in (\ker \tilde{L}_\psi) \cap U \), there exists a unique \( w(\psi_0) \in (\ker \tilde{L}_\psi)^\perp \), such that

\[ \tilde{L}_\psi(w(\psi_0)) + Q(\psi_0 + w(\psi_0), \psi_0 + w(\psi_0)) \in (\coker \tilde{L}_\psi) \subset L^2. \]

The proof of the above statement is similar with the proof of theorem 4.2 and here it is: choose any \( \psi_0 \in (\ker \tilde{L}_\psi) \cap U \). Consider \( \psi_1, \psi_2, \ldots, \in (\ker \tilde{L}_\psi)^\perp \) such that

\[ \tilde{L}_\psi(\psi_1) = -Q(\psi_0, \psi_0)^\perp; \]

\[ \tilde{L}_\psi(\psi_2) = -2Q(\psi_0, \psi_1)^\perp - Q(\psi_1, \psi_1)^\perp; \]

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\[ \tilde{L}_\psi (\psi_n) = -2Q(\psi_0 + \cdots + \psi_{n-2}, \psi_{n-1})^\perp - Q(\psi_{n-1}, \psi_{n-1})^\perp. \]

Here \((\cdot)^\perp\) is the projection to \((\ker L_\psi)^\perp\) in \(L^2_x\).

Since \(U\) is a small neighborhood, it may be assumed that
\[ C^2 \|\psi_0\|_{H_{\varphi, \epsilon}}^2 + \|\psi_0\|_{H_{\varphi, \epsilon}} \leq \frac{1}{24C^2}. \]

So
\[ \|\psi_1\|_{H_{\varphi, \epsilon}} \leq C\|Q(\psi_0, \psi_0)\|_{L^2} \leq C^2 \|\psi_0\|_{H_{\varphi, \epsilon}}^2 \leq \frac{1}{24C^2}; \]
\[ \|\psi_2\|_{H_{\varphi, \epsilon}} \leq C^2 \|\psi_1\|_{H_{\varphi, \epsilon}} (\|\psi_1\|_{H_{\varphi, \epsilon}} + 2\|\psi_0\|_{E_{\varphi, \epsilon}}) \leq \frac{1}{2} \|\psi_1\|_{H_{\varphi, \epsilon}} \leq \frac{1}{25C^2}; \]
\[ \vdots \]
\[ \|\psi_n\|_{E_{\varphi, \epsilon}} \leq C^2 \|\psi_{n-1}\|_{H_{\varphi, \epsilon}} (\|\psi_{n-1}\|_{H_{\varphi, \epsilon}} + 2\|\psi_{n-2}\|_{H_{\varphi, \epsilon}} + \cdots + 2\|\psi_0\|_{E_{\varphi, \epsilon}}) \leq \frac{1}{2} \|\psi_{n-1}\|_{H_{\varphi, \epsilon}} \leq \frac{1}{2n+3C^2}. \]

Therefore letting \(w(\psi_0) = \sum_{n=1}^{+\infty} \psi_n\) in \(H_{\varphi, \epsilon}\) satisfies the requirement (except the uniqueness).

To prove the uniqueness, supposing that \(\hat{w}(\psi_0)\) also satisfies the requirement, then
\[ \|\hat{w}(\psi_0) - w(\psi_0)\|_{H_{\varphi, \epsilon}} \leq C\|Q(\hat{w}(\psi_0), \psi_0, \hat{w}(\psi_0) - w(\psi_0))\|_{L^2} \]
\[ \leq C^2 \|\hat{w}(\psi_0) + w(\psi_0)\|_{H_{\varphi, \epsilon}} \|\hat{w}(\psi_0) - w(\psi_0)\|_{H_{\varphi, \epsilon}}. \]

Since \(U\) is small enough, it can be assumed that \(C^2 \|\hat{w}(\psi_0) + w(\psi_0)\|_{H_{\varphi, \epsilon}} < 1\), which implies that \(\hat{w}(\psi_0) = w(\psi_0)\).

**Step 2:** Having \(w(\psi_0)\) defined, it is obvious that, for any \(\psi_0 \in (\ker L_\psi) \cap U\), there exists an element \(\psi_1\) such that \(L_\psi(\psi_0 + \psi_1) + Q(\psi_0 + \psi_1, \psi_0 + \psi_1) = 0\) if and only if \(\psi_1 = w(\psi_0)\) and \(L_\psi(\psi_0 + w(\psi_0)) + Q(\psi_0 + w(\psi_0), \psi_0 + w(\psi_0)) = 0\), and if and only if the projection of \(Q(\psi_0 + w(\psi_0), \psi_0 + w(\psi_0))\) onto \((\ker L_\psi)^\perp\) is 0.

Let \(f(\psi_0)\) be the projection of \(Q(\psi_0 + w(\psi_0), \psi_0 + w(\psi_0))\) onto \((\ker L_\psi)^\perp\). Then \(f\) is a map from \((\ker L_\psi) \cap U\) to \((\ker L_\psi)^\perp\) which satisfies the requirements. \(\square\)
6 Regularity theorem near the knot

The goal of this section is to prove the following theorem:

**Theorem 6.1.** Suppose \( A \) is an \( su(2) \)-valued 1-form on \( N, K \), representing a connection on the trivial \( SU(2) \)-bundle. Suppose the curvature \( F^A \) has finite \( L^2 \) norm. By shrinking \( \epsilon \), it may be assumed that the \( L^2 \) norm of \( F^A \) is small enough. Then there exists a constant \( C \) which doesn’t depend on \( A \) and \( \epsilon \), and a flat fiducial connection namely \( A^f = \gamma \sigma d\theta + \tilde{\gamma} \sigma ds \) for some \( \gamma \) and \( \tilde{\gamma} \), and a gauge transformation \( u \), such that after the gauge transformation (so that \( A \) becomes \( u(A) \)),

\[
\int_{N, K} \rho(|\nabla (u(A) - A^f)|^2 + |\sigma, (u(A) - A^f)|^2) d^3x \leq C \| F^A \|_{L^2}^2.
\]

In particular, letting \( \hat{A} = \gamma \sigma d\theta \), then \( u(A) - \hat{A} \) has finite \( H^N, \epsilon \) norm.

6.1 Some preliminary Uhlenbeck type analysis

**Lemma 6.2.** (Uhlenbeck lemma) There exists a large enough constant \( C \) such that, supposing \( B_r \) is a ball of radius \( r \), and supposing

\[
\int_{B_r} |F^A|^2 d^3x \leq \frac{1}{Cr},
\]

then after some gauge transformation \( u \), letting \( u(A) = d^*(uAu^{-1} - (du)u^{-1}) \),

\[
\left\{ \begin{array}{l}
d^*(u(A)) = 0, \quad \ast u(A)|_{\partial B_r} = 0; \\
\int_{B_r} (|\nabla A|^2 + \frac{1}{r^2} |A|^2) d^3x \leq C \int_{B_r} |F^A|^2 d^3x.
\end{array} \right.
\]

Moreover, the above gauge transformation \( u \) is unique up to a constant gauge transformation.

**Proof.**

\[
\int_{B_r} |F^A|^2 d^3x \leq \left( \int_{B_r} |F^A|^2 d^3x \right)^{\frac{3}{2}} \left( \int_{B_r} 1 d^3x \right)^{\frac{1}{2}} \leq \left( \frac{4\pi}{3C^3} \right)^{\frac{1}{2}}.
\]

Since \( C > 0 \) is large enough, if may be assumed that \( \left( \frac{4\pi}{3C^3} \right)^{\frac{1}{2}} \) is smaller than the Uhlenbeck’s constant. Hence the lemma can be proved by a standard Uhlenbeck’s argument. \( \square \)
Lemma 6.3. Suppose $B_1, B_2$ are two open balls in $\mathbb{R}^3$ that overlap. Let $V$ be an open subset such that $V \subset \bar{V} \subset B_1 \cap B_2$. Suppose $A$ is an su(2)-connection on $B_1 \cup B_2$ such that $\|F^A\|_{L^2(B_1 \cup B_2)}$ is small enough, where $F^A$ is the curvature. Suppose $a, b$ are the su(2)-valued 1-forms that represent $A$ in the Uhlenbeck gauges on $B_1, B_2$ respectively. On $B_1 \cap B_2$, they are related by a gauge transformation $g$, i.e., $a = gb^{-1} - (dg)g^{-1}$ on $B_1 \cap B_2$. Then

1) there exists a constant $C > 0$ which doesn’t depend on $A$, such that any two points $S_1, S_2 \in V$,

$$|g(S_1) - g(S_2)| \leq C\|F_A\|_{L^2(B_1 \cup B_2)}^2;$$

2) for any point $S \in V$,

$$\int_{B_1 \cap B_2} (|g - g(V)|^2 + |\nabla g|^2 + |\nabla \nabla g|^2) d^3x \leq C\|F_A\|_{L^2(B_1 \cup B_2)}^2.$$

Proof. Consider the identity $d^* dg = a \cdot ag - 2ag \cdot b + gb \cdot b$.

Let $L = a \cdot ag - 2ag \cdot b + gb \cdot b$, then for some constant $C_0 > 0$,

$$\|L\|_{L^2(B_1 \cup B_2)} \leq C_0(\|a\|_{L^4(B_1 \cup B_2)} + \|b\|_{L^4(B_1 \cup B_2)})^2 \leq C_0^2\|F_A\|_{L^2(B_1 \cup B_2)}^2.$$

The second inequality above is from the Uhlenbeck gauge condition and Sobolev embedding.

On $B_1 \cap B_2$,

$$d^* dg = L, \quad \text{and} \quad \frac{\partial g}{\partial \bar{n}} = 0,$$

where $\bar{n}$ is the outside normal vector of $\partial(B_1 \cap B_2)$.

So (2) is a direct corollary of the standard elliptic regularity theorem.

To prove (1), supposing $G(x, y)$ is the Green’s function on $B_1 \cap B_2$ with Neumann boundary condition, then for any point $x$ on $V$,

$$g(x) = \int_{B_1 \cap B_2} L(y)G(x, y)dy + c,$$ where $c$ is a constant that doesn’t depend on $x$.

$$|g(S_1) - g(S_2)| \leq \int_{B_1 \cap B_2} |L(y)||G(S_1, y) - G(S_2, y)|d^3y.$$

Note that for any fixed $x$, the function $G(x, y)$ has only one singularity at $y = x$ which $\sim \frac{1}{\text{dist}(x, \cdot)}$, where $\text{dist}(x, \cdot)$ is the distance to the point $x$, and
Lemma 6.5. (gluing Uhlenbeck gauges) Let $U$ and $R$

Remark 6.4. When rescaling $B_1 \cup B_2$ by a factor $R$, the above inequalities should be

\begin{align}
(1) \quad |g(S_1) - g(S_2)| & \leq CR\|F_A\|_{L^2(B_1 \cup B_2)}^2; \\
(2) \quad \int_{B_1 \cap B_2} \left( \frac{1}{R^3} |g - g(V)|^2 + \frac{1}{R} |\nabla g|^2 + R |\nabla \nabla g|^2 \right) d^3 x \leq CR\|F_A\|_{L^2(B_1 \cup B_2)}^2;
\end{align}

where $C$ is independent with $R$.

Lemma 6.5. (gluing Uhlenbeck gauges) Let $U_1$ and $U_2$ be two bounded open sets in $\mathbb{R}^3$ with smooth boundary that have a nonempty connected overlap. Suppose $\chi$ is a smooth cut-off function which is 0 on $U_1 \setminus U_2$ and 1 on $U_2 \setminus U_1$ and $|\nabla \chi|, |\nabla \nabla \chi|$ are bounded (note that the existence of $\chi$ has some requirements on $U_1$ and $U_2$). Suppose $A$ is an $su(2)$-valued 1-form on $U_1 \cup U_2$, viewed as a connection on the trivial $SU(2)$-bundle. Suppose $U$ is an open set that covers $U_1 \cup U_2$ and suppose the curvature $F^A$ has small enough $L^2$ norm on $U$. Let $u_1$ and $u_2$ be $SU(2)$-valued functions on $U_1$ and $U_2$ respectively, viewed as gauge transformations. Suppose there exists a constant $C > 0$ such that

$$|u_1^{-1}u_2 - 1| \leq C \int_U |F^A|^2 d^3 x \quad \text{on } U_1 \cap U_2 \text{ pointwise.}$$

$$\int_{U_1 \cap U_2} (|u_1^{-1}u_2 - 1|^2 + |\nabla (u_1^{-1}u_2)|^2 + |\nabla \nabla (u_1^{-1}u_2)|^2) d^3 x \leq C \int_U |F^A|^2 d^3 x.$$

Let $a_1 = u_1^{-1}Au_1 + u_1^{-1}du_1, a_2 = u_2^{-1}Au_2 + u_2^{-1}du_2$. Assume

$$\int_{U_i} (|\nabla a_i|^2 + |a_i|^2) d^3 x \leq C \int_U |F^A|^2 d^3 x, \quad \text{for } i = 1, 2.$$

Then there exists an $SU(2)$-valued function $u$ on $U_1 \cup U_2$ (still viewed as a gauge transformation) such that letting $a = u^{-1}Au + u^{-1}du$, then

(1) $u = u_1$ on $U_1 \setminus U_2$ and $u = u_2$ on $U_2 \setminus U_1$.
\( (2) \) \( |u^{-1}u_i - 1| \leq C \int_U |F^A|^2 d^3x \) on \( U_1 \cap U_2 \) for \( i = 1, 2 \);

\( (3) \) and there exists a constant \( C' \) such that
\[ \int_{U_1 \cap U_2} (|\nabla a|^2 + |a|^2) d^3x \leq C' \int_U |F^A|^2 d^3x. \]

**Proof.** Suppose \( e^v = u_2^{-1}u_2 \) on \( U_1 \cap U_2 \). Letting \( u = u_1 e^{xv} \) on \( U_1 \) and \( u = u_2 \) on \( U_2 \setminus U_1 \), then obviously \( u \) satisfies (1) (2). To prove (3), it is enough to show that
\[ \int_{U_1 \cap U_2} (|\nabla a|^2 + |a|^2) d^3x \leq C' \int_U |F^A|^2 d^3x. \]

\[ = \int_{U_1 \cap U_2} (|\nabla (e^{-xv}a_1 e^{xv} + e^{-xv}d(e^{xv}))|^2 + |e^{-xv}a_1 e^{xv} + e^{-xv}d(e^{xv})|^2) d^3x \]
\[ \leq 5 \int_{U_1 \cap U_2} (|\nabla (e^{-xv})|^2 |a_1|^2 + |\nabla a_1|^2 + |a_1|^2 |\nabla (e^{xv})|^2 + |\nabla \nabla (e^{xv})|^2 + |\nabla (e^{-xv})|^2 |\nabla (e^{xv})|^2) d^3x \]
\[ + 2 \int_{U_1 \cap U_2} (|a_1|^2 + |\nabla (e^{xv})|^2) d^3x \]
\[ \leq C_1 \int_{U_1 \cap U_2} (|\nabla a_1|^2 + (|a_1|^2 + |\nabla \chi|^2 |v|^2 + |\nabla v|^2) (|\nabla \chi|^2 |v|^2 + |\nabla v|^2) + |\nabla \nabla \chi|^2 |v|^2 + |\nabla v|^2 + |\nabla v|^2 |a_1|^2 + |\nabla v|^2) d^3x \]
\[ \leq C_2 \int_{U_1 \cap U_2} (|\nabla a_1|^2 + (|a_1|^2 + |v|^2 + |\nabla v|^2) (|v|^2 + |\nabla v|^2) + |v|^2 + |\nabla v|^2 + |a_1|^2 + |\nabla v|^2) d^3x \]
\[ \leq C' \int_U |F^A|^2 d^3x, \]

where \( C_1, C_2, C' \) are constants. \( \Box \)

**Remark 6.6.** When re-scaling \( U_1 \cup U_2 \) and \( U \) by a factor \( R \), the above theorem should be stated as following: Assume
\[ |u^{-1}u - 1| \leq CR \int_U |F^A|^2 d^3x \] on \( U_1 \cap U_2 \) pointwise.

\[ \int_{U_1 \cup U_2} \left( \frac{1}{R^3} |u^{-1}u_i - 1|^2 + \frac{1}{R} |\nabla (u^{-1}u_2)|^2 + R |\nabla \nabla (u^{-1}u_2)|^2 \right) d^3x \leq CR \int_U |F^A|^2 d^3x. \]

Let \( a_1 = u_1^{-1}Au_1 + u_1^{-1}du_1, a_2 = u_2^{-1}Au_2 + u_2^{-1}du_2 \). Assume
\[ \int_{U_i} (R |\nabla a_i|^2 + \frac{1}{R} |a_i|^2) d^3x \leq CR \int_U |F^A|^2 d^3x, \] for \( i = 1, 2 \).

Then there exists an \( SU(2) \)-valued function \( u \) on \( U_1 \cup U_2 \) (still viewed as a gauge transformation) such that letting \( a = u^{-1}Au + u^{-1}du \), then
(1) \( u = u_1 \) on \( U_1 \setminus U_2 \) and \( u = u_2 \) on \( U_2 \setminus U_1 \);

(2) \( |u^{-1} u_i - 1| \leq CR \int_u |F^A|^2 d^3 x \) on \( U_1 \cap U_2 \) for \( i = 1, 2 \);
then
\[
\int_{U_1 \cup U_2} (R|\nabla a|^2 + \frac{1}{R}|a|^2) d^3 x \leq C'R \int_U |F^A|^2 d^3 x.
\]

**Proof.**
\[
\int_{U_1 \cap U_2} (R|\nabla a|^2 + \frac{1}{R}|a|^2) d^3 x = \int_{U_1 \cap U_2} (R|\nabla (e^{-\chi v} a_1 e^{\chi v} + e^{-\chi v} d(e^{\chi v}))|^2 + \frac{1}{R}|e^{-\chi v} a_1 e^{\chi v} + e^{-\chi v} d(e^{\chi v})|^2) d^3 x
\leq 100 \int_{U_1 \cap U_2} (R|\nabla a_1|^2 + R|a_1|^2 |\nabla (\chi v)|^2 + R|\nabla \nabla (\chi v)|^2 + R|\nabla (\chi v)|^4) d^3 x
\]
\[
\quad + 100 \int_{U_1 \cap U_2} (\frac{1}{R}|a_1|^2 + \frac{1}{R}|\nabla (\chi v)|^2) d^3 x
\leq C_1 \int_{U_1 \cap U_2} (R|\nabla a_1|^2 + R(|a_1|^2 + \frac{1}{R^2} |v|^2 + |\nabla v|^2)(\frac{1}{R^2} |v|^2 + |\nabla v|^2) + \frac{1}{R^3} |v|^2 + R|\nabla \nabla v|^2) d^3 x
\]
\[
\quad + C_1 \int_{U_1 \cap U_2} (\frac{1}{R}|a_1|^2 + \frac{1}{R^3} |v|^2 + \frac{1}{R} |\nabla v|^2) d^3 x
\leq C'R \int_U |F^A|^2 d^3 x,
\]}

where \( C_1, C' \) are constants. □

### 6.2 Open covering

Suppose \( F^A \in L^2_\varepsilon \),
\[
\int_{\mathbb{R}^3 \setminus K} \rho_\varepsilon |F^A|^2 d^3 x < +\infty.
\]

Consider an open covering \( \{U_\alpha\} \) of \( N_\varepsilon \setminus K \), where each \( \alpha \) represents a triple \((n_\rho, n_\theta, n_s)\). To be more precise, let \( t \) be a large enough integer such that \( 2t \varepsilon \gg l \) and \( t \gg 1000 \), then \( n_\rho, n_\theta, n_s \) are nonnegative integers such that \( n_\rho \in [2, +\infty), n_\theta \in [0, t - 1], n_s \in [0, 2^{t+n_\rho} - 1] \). Define \( U_\alpha = U_{n_\rho, n_\theta, n_s} \) as the following:

\[
\rho \in \left( \frac{\varepsilon}{2(3n_\rho + 1)}, \frac{\varepsilon}{2(3n_\rho - 4)} \right), \theta \in \left( \frac{2\pi(3n_\theta - 4)}{3t + 2}, \frac{2\pi(3n_\theta + 1)}{3t + 2} \right), s \in \left( \frac{(3n_s - 4)l}{3 \times 2^{t+n_\rho} + 2}, \frac{(3n_s + 1)l}{3 \times 2^{t+n_\rho} + 2} \right).
\]

where the value of \( \theta \) is defined modulo \( 2\pi \) and the value of \( s \) modulo \( l \), the subscripts \( n_\theta \) and \( n_s \) are also defined in the sense of modulo \( 3t + 2 \) and modulo \( 3 \times 2^{t+n_\rho} + 2 \) respectively.
Let the point $P_\alpha$ be
\[ P_\alpha = P_{n_\rho, n_\eta, n_s} = \{ r = \frac{\epsilon}{2^{3n_\rho}}, \theta = \frac{6\pi n_\theta}{3t+2}, s = \frac{3n_s l}{3 \times 2(t+n_\rho) + 2} \}. \]

Two open sets or points are called “neighbors” if exactly one of the three subscripts differ by 1.

The open cover $U_\alpha$ is a sub open cover of open balls $B_\alpha = B_{n_\rho, n_\eta, n_s}$, such that the ball $B_{n_\rho, n_\eta, n_s}$ has radius $2^{-n_\rho} \eta$, where $\eta$ is a constant. Each ball $B_{n_\rho, n_\eta, n_s}$ is a subset of $N_{2^{-(n_\rho-10)}} \setminus N_{2^{-(n_\rho+10)}}$, but covers $U_{n_\rho, n_\eta, n_s}$.

From lemma 6.2, assuming $\epsilon$ is small enough, on each ball $B_\alpha$, there exists an $SU(2)$-valued function $u_\alpha$ such that letting $A_\alpha = u_\alpha(A)$ on $B_\alpha$, then
\[
\begin{cases}
  d^*(A_\alpha) = 0, & *A_\alpha|_{\partial B_\alpha} = 0; \\
  \int_{B_\alpha} |\nabla A_\alpha|^2 + \frac{1}{\rho^2} |A_\alpha|^2 d^3x \leq C \int_{B_\alpha} |F| A|^2 d^3x.
\end{cases}
\]

Using the freedom of a constant gauge transformation, it may be assumed that
- $u_{n_\rho, n_\eta, n_s}(P_{n_\rho, n_\eta, n_s}) = u_{n_\rho, (n_\eta+1), n_s}(P_{n_\rho, n_\eta, n_s})$ for all $n_\eta = 1, 2, \ldots, t - 1$ (note that 0 is excluded);
- $u_{n_\rho, 1, n_s}(P_{n_\rho, 0, n_s}) = u_{n_\rho, 1, (n_s+1)}(P_{n_\rho, 0, n_s})$ for all $n_s = 1, 2, 3, \ldots, 2^{t+n_\rho} - 1$ (note that 0 is excluded again);
- $u_{n_\rho, 1, 1}(P_{n_\rho, 0, 0}) = u_{(n_\rho+1), 1, 1}(P_{n_\rho, 0, 0})$ for all $n_\rho = 1, 2, 3, \ldots$;
- Letting $g_{n_\rho, n_s} = u_{n_\rho, 0, n_s}^{-1} u_{n_\rho, 1, n_s}$, then $g_{n_\rho, n_s}(P_{n_\rho, 1, n_s})$ is called the $\theta$ monodromy of $A$ at the level $(n_\rho, n_s)$, denoted as $\Gamma^\theta_{n_\rho, n_s}$;
- Letting $g_{n_\rho} = u_{n_\rho, 1, 0}^{-1} u_{n_\rho, 1, 1}$, then $g_{n_\rho}(P_{n_\rho, 1, 1})$ is called the $\phi$ monodromy of $A$ at the level $n_\rho$, denoted as $\Gamma^\phi_{n_\rho}$.

Note that even all the above conditions are satisfied, there is still one more freedom to choose the Uhlenbeck gauges (change all the above $u$ by a same constant gauge transformation), which will be used later.

**Corollary 6.7.** Suppose $U_{n_\rho, n_s} = \bigcup_{n_\eta} U_{n_\rho, n_\eta, n_s}$, $U_{n_\rho} = \bigcup_{n_s} U_{n_\rho, n_s}$. There exists a constant $C > 0$.
- Suppose $\alpha = (n_\rho, n_\eta, n_s)$, $\alpha'$ is either $(n_\rho + 1, n_\eta, n_s)$, $(n_\rho, n_\eta + 1, n_s)$ (in which case $n_\eta \neq 0$) or $(n_\rho, n_\eta, n_s)$ (in which case $n_s \neq 0$). Suppose $P \in U_{\alpha} \cap U_{\alpha'}$ is any point in the overlap. Then
\[
|u_\alpha(P) - u_{\alpha'}(P)| \leq C \int_{U_{n_\rho} \setminus U_{n_\rho+1}} |F|^2 d^3x.
\]
• Suppose \( \alpha = (n_{\rho}, 0, n_s) \), \( \alpha' = (n_{\rho}, 1, n_s) \). Suppose \( P \in U_{\alpha} \cap U_{\alpha'} \) is any point in the overlap. Then
\[
|u_\alpha(P) \Gamma_{n_{\rho},n_s}^\theta - u_{\alpha'}(P)| \leq C \int_{U_{n_{\rho+1}}} \rho |F^A|^2 d^3 x.
\]

• Suppose \( \alpha = (n_{\rho}, n_\theta, 0) \), \( \alpha' = (n_{\rho}, n_\theta, 1) \). Suppose \( P \in U_{\alpha} \cap U_{\alpha'} \) is any point in the overlap. Then
\[
|u_\alpha(P) \Gamma_{n_{\rho}}^s - u_{\alpha'}(P)| \leq C \int_{U_{n_{\rho+1}}} \rho |F^A|^2 d^3 x.
\]

**Proof.** This is a direct corollary of lemma 6.3. ∎

**Corollary 6.8.** Assuming \( \int_{N_n} \rho |F^A|^2 d^3 x \leq +\infty \), then there exist a \( \Gamma^\theta \in SU(2) \) and a \( \Gamma^s \in SU(2) \), such that
\[
\lim_{n_{\rho} \to +\infty} \Gamma_{n_{\rho},n_s}^\theta = \Gamma^\theta \text{ uniformly in } n_s, \quad \lim_{n_{\rho} \to +\infty} \Gamma_{n_{\rho}}^s = \Gamma^s.
\]

Moreover, \( \Gamma^\theta \) and \( \Gamma^s \) commute with each other.

**Proof.** Taking \( \Gamma_{n_{\rho},n_s}^\theta \) as an example, choosing any point \( P_1 \in U_{(n_{\rho+1},0,n_s)} \cap U_{n_{(n_{\rho+1}),1,n_s}} \cap U_{n_{\rho},0,n_s} \cap U_{n_{\rho},1,n_s} \) (note that this set is not empty). Then
\[
|\Gamma_{n_{\rho},n_s}^\theta - u^{-1}_{(n_{\rho+1}),0,n_s}(P_1)u_{(n_{\rho+1}),1,n_s}(P_1)| \leq C \int_{U_{n_{\rho+1}}} \rho |F^A|^2 d^3 x.
\]
\[
|\Gamma_{n_{\rho+1},n_s}^\theta - u^{-1}_{(n_{\rho+1}),0,n_s}(P_1)u_{(n_{\rho+1}),1,n_s}(P_1)| \leq C \int_{U_{n_{\rho+1}}} \rho |F^A|^2 d^3 x.
\]
\[
|u_{(n_{\rho+1}),0,n_s}(P_1)u_{(n_{\rho+1}),1,n_s}(P_1) - u^{-1}_{(n_{\rho+1}),0,n_s}(P_1)u_{(n_{\rho+1}),1,n_s}(P_1)| \\
\leq |u_{(n_{\rho+1}),0,n_s}(P_1)u_{(n_{\rho+1}),0,n_s}(P_1) - u_{(n_{\rho+1}),0,n_s}(P_1)u_{(n_{\rho+1}),1,n_s}(P_1)| \\
\leq |u_{(n_{\rho+1}),0,n_s}(P_1)u_{(n_{\rho+1}),0,n_s}(P_1) - 1| + |u_{(n_{\rho+1}),1,n_s}(P_1)u_{(n_{\rho+1}),1,n_s}(P_1) - 1| \leq 2C \int_{U_{n_{\rho+1}}} \rho |F^A|^2 d^3 x.
\]

Hence
\[
|\Gamma_{n_{\rho},n_s}^\theta - \Gamma_{n_{\rho+1},n_s}^\theta| \leq 4C \int_{U_{n_{\rho+1}}} \rho |F^A|^2 d^3 x.
\]

Since \( \int_{N_n \setminus K} \rho |F^A|^2 d^3 x < +\infty \), for any \( n_s \) the sequence \( \Gamma_{n_{\rho},n_s}^\theta \) is a Cauchy sequence, hence there exists an \( \Gamma_{n_s}^\theta = \lim_{n_{\rho} \to +\infty} \Gamma_{n_{\rho},n_s}^\theta \). (The convergence is uniform in \( u_s \)). On the other hand,
\[
|\Gamma_{n_{\rho},n_s}^\theta - u^{-1}_{(n_{\rho+1}),0,n_s}(P_1)u_{(n_{\rho+1}),1,n_s}(P_1)| \leq C \int_{U_{n_{\rho+1}}} \rho |F^A|^2 d^3 x.
\]
\[ |\Gamma_{n,1}^{\theta} - u_{n,1}^{-1}(P_1)u_{n,1,1}(P_1)| \leq C \int_{U_{n,1} \cup U_{n+1}} \rho |F^A|^2 d^3x. \]

\[ |u_{n,1}^{-1}(P_1)u_{n,1,1}(P_1) - u_{n,0,1}^{-1}(P_1)u_{n,1,1}(P_1)| \leq |u_{n,0,1}^{-1}(P_1)u_{n,0,1,1}(P_1)| \]

\[ \leq |u_{n,0,1}(P_1)u_{n,0,1,1}(P_1) - 1| + |u_{n,1,1}(P_1)u_{n,1,1,1}(P_1) - 1| \leq 2C \int_{U_{n,1} \cup U_{n+1}} \rho |F^A|^2 d^3x. \]

So

\[ |\Gamma_{n,s}^{\theta} - \Gamma_{n,s}^{\theta, +1}| \leq 4C \int_{U_{n,1} \cup U_{n+1}} \rho |F^A|^2 d^3x. \]

Hence \( \Gamma_{n,s}^{\theta} \) are the same for different \( n_s \), i.e., there exists an \( \Gamma_{n,s}^{\theta} = \lim_{n_s \to +\infty} \Gamma_{n,s}^{\theta} \) which convergent uniformly in \( u_s \).

The existence of \( \Gamma^s \) can be proved by the similar reason and skipped.

Finally, to prove \( \Gamma^\theta \) and \( \Gamma^s \) commute, note that for any \( P_1 \in U_{n,0} \cup U_{n,1} \cup U_{n+1} \),

\[ |\Gamma_{n,1}^{\theta} - u_{n,1}^{-1}(P_1)u_{n,1,1}(P_1)| \leq C \int_{U_{n,1} \cup U_{n+1}} \rho |F^A|^2 d^3x. \]

\[ |\Gamma_{n,1}^{s} - u_{n,1,1,0}(P_1)u_{n,1,1}(P_1)| \leq C \int_{U_{n,1} \cup U_{n+1}} \rho |F^A|^2 d^3x. \]

\[ |\Gamma_{n,0}^{\theta} - u_{n,0,1,0}^{-1}(P_1)u_{n,1,0,1}(P_1)| \leq C \int_{U_{n,1} \cup U_{n+1}} \rho |F^A|^2 d^3x. \]

\[ |\Gamma_{n,0}^{s} - u_{n,0,1,0}^{-1}(P_1)u_{n,0,1,1}(P_1)| \leq C \int_{U_{n,1} \cup U_{n+1}} \rho |F^A|^2 d^3x. \]

So

\[ |\Gamma^\theta \Gamma^s - \Gamma^s \Gamma^\theta| = \lim_{n_s \to +\infty} |\Gamma_{n,s}^{\theta} \Gamma_{n,s}^{s} - \Gamma_{n,s}^{s} \Gamma_{n,s}^{\theta}| \]

\[ = \lim_{n_s \to +\infty} |u_{n,0,0}^{-1}(P_1)u_{n,1,1}(P_1) - u_{n,1,0,1}(P_1)| = 0. \]

Recall that there is an extra freedom of a constant gauge transformation not be used in the definitions of local Uhlenbeck gauges. This freedom can be used to change \( \Gamma^\theta \) and \( \Gamma^s \) up to conjugacy. In particular, it can be assumed that \( \Gamma^\theta = e^{2\pi \gamma \sigma} \) and \( \Gamma^s = e^{2\pi \tilde{\gamma} \sigma} \) for some \( \gamma, \tilde{\gamma} \in (0,1) \).
6.3 Regularity of a solution near the knot

**Theorem 6.9.** One can adapt all the $u_\alpha$ to $\tilde{u}_\alpha$ and let $\tilde{A}_\alpha = \tilde{u}_\alpha(A)$, such that for some possibly larger $\tilde{C}$,

\[ \int_{U_\alpha} |\nabla A_\alpha|^2 + \frac{1}{\rho^2} |A_\alpha|^2 \, d^3x \leq \tilde{C} \sum_{\alpha' \text{ is a neighbor of } \alpha} \int_{U_{\alpha'}} |F^{\alpha'}|^2 \, d^3x. \]

- Suppose $\alpha = (n_\rho, n_\theta, n_s)$, $\alpha'$ is either $(n_\rho + 1, n_\theta, n_s)$, $(n_\rho, n_\theta + 1, n_s)$ (in which case $n_\theta \neq 0$) or $(n_\rho, n_\theta, n_s)$ (in which case $n_s \neq 0$), then $\tilde{u}_\alpha = \tilde{u}_{\alpha'}$ on $U_\alpha \cap U_{\alpha'}$.

- Suppose $\alpha = (n_\rho, 0, n_s)$, $\alpha' = (n_\rho, 1, n_s)$, then $\tilde{u}_\alpha \Gamma^\theta = \tilde{u}_{\alpha'}$ on $U_\alpha \cap U_{\alpha'}$.

- Suppose $\alpha = (n_\rho, n_\theta, 0)$, $\alpha' = (n_\rho, n_\theta, 1)$, then $\tilde{u}_\alpha \Gamma^s = \tilde{u}_{\alpha'}$ on $U_\alpha \cap U_{\alpha'}$.

**Proof.** Suppose $\chi(t)$ is a smooth cut-off function which is 0 when $t \leq 0$ and 1 when $t \geq 1$. It may also be assumed that $\chi(t) = \chi(1 - t)$. Let $\chi(a, b, t) = \chi\left(\frac{t - a}{b - a}\right)$.

Let $\tilde{U}_\alpha = \bigcup_{\alpha' \text{ is a neighbor of } \alpha} U_{\alpha'}$. Define the following functions on $\tilde{U}_{n_\rho, n_\theta, n_s}$ which depend only one of the parameters $\rho, \theta$ and $s$ respectively:

1. Let $\chi^\rho_{\alpha} = \chi^\rho_{n_\rho, n_\theta, n_s}(\rho) = \chi(\frac{\epsilon}{2^{3n_\rho - 1} \cdot 2^{3n_\theta - 1}}, \rho)$ be a function which depends only on $\rho$. For convenience, let $\chi^{\rho+1}_{\alpha}$ denote $\chi^{\rho}_{(n_\rho + 1), n_\theta, n_s}$.

2. Let $\chi^\theta_{\alpha} = \chi^\theta_{n_\rho, n_\theta, n_s}(\theta) = \chi(\frac{2\pi(3n_\theta - 4)}{3t + 2}, \frac{2\pi(3n_\theta - 2)}{3t + 2}, \theta)$ be a function which depends only on $\theta$. For convenience, let $\chi^{\theta+1}_{\alpha}$ denote $\chi^{\theta}_{n_\rho, (n_\theta + 1), n_s}$.

3. Let $\chi^s_{\alpha} = \chi^s_{n_\rho, n_\theta, n_s}(s) = \chi(\frac{3n_\theta - 4}{3}, \frac{(3n_\theta - 2)}{3}, s)$ be a function which depends only on $s$. For convenience, let $\chi^{s+1}_{\alpha}$ denote $\chi^s_{n_\rho, n_\theta, (n_s + 1)}$.

There are three steps to construct $\tilde{u}_\alpha$:

**Step 1:** Let $v^\rho_\alpha = v^\rho_{n_\rho, n_\theta, n_s}$ be the $su(2)$-valued function on $U_{n_\rho, (n_\theta - 1), n_s} \cap U_{n_\rho, n_\theta, n_s}$ such that $v^\rho_\alpha = \begin{cases} u^{n_\rho, (n_\theta - 1), n_s}_{n_\rho, n_\theta, n_s} & \text{when} \ n_\theta \neq 1 \\ (u^{n_\rho, (n_\theta - 1), n_s}_{n_\rho, n_\theta, n_s} \Gamma^\theta)^{-1} u^{n_\rho, n_\theta, n_s}_{n_\rho, n_\theta, n_s} & \text{when} \ n_\theta = 1 \end{cases}$

and let $v^{\rho+1}_{n_\rho, n_\theta, n_s}$ be the $su(2)$-valued function on $U_{n_\rho, n_\theta, n_s} \cap U_{n_\rho, (n_\theta + 1), n_s}$ such that $v^{\rho+1}_{\alpha} = \begin{cases} u^{n_\rho, n_\theta, n_s}_{n_\rho, n_\theta, n_s} & \text{when} \ n_\theta \neq 0 \\ (u^{n_\rho, n_\theta, n_s}_{n_\rho, n_\theta, n_s} \Gamma^\theta)^{-1} u^{n_\rho, (n_\theta + 1), n_s}_{n_\rho, n_\theta, n_s} & \text{when} \ n_\theta = 0 \end{cases}$.
Let
\[ \hat{u}_\alpha = \begin{cases} u_\alpha e^{-\chi^0_\alpha v^0_\alpha} & \text{on } U_{n_p, (n_\theta - 1), n_s} \cap U_{n_p, n_\theta, n_s}; \\ e^{\chi^0_\alpha v^0_\alpha} u_\alpha & \text{on } U_{n_p, n_\theta, n_s} \cap U_{n_p, (n_\theta + 1), n_s}; \\ u_\alpha & \text{elsewhere on } U_{n_p, n_\theta, n_s}. \end{cases} \]

2. Let \( v^\rho_\alpha = v^\rho_{n_p, n_\theta, n_s} \) be the \( su(2) \)-valued function on \( U_{n_p, n_\theta, (n_\theta - 1)} \cap U_{n_p, n_\theta, n_s} \) such that
\[ e^{v^\rho_\alpha} = \begin{cases} u_1^{-1} u_{n_p, n_\theta, (n_\theta - 1)} & \text{when } n_\theta \neq 1 \\ (u_{n_p, n_\theta, (n_\theta - 1)} \Gamma^s)^{-1} u_{n_p, n_\theta, n_s} & \text{when } n_\theta = 1 \end{cases} \]
and let \( v^{\rho + 1}_{n_p, n_\theta, n_s} \) be the \( su(2) \)-valued function on \( U_{n_p, n_\theta, n_s} \cap U_{n_p, n_\theta, (n_\theta + 1)} \) such that
\[ e^{v^{\rho + 1}_{\alpha}} = \begin{cases} u_1^{-1} & \text{when } n_\theta \neq 0 \\ (u_{n_p, n_\theta, (n_\theta + 1)} \Gamma^s)^{-1} u_{n_p, n_\theta, (n_\theta + 1)} & \text{when } n_\theta = 0. \end{cases} \]

Let
\[ \hat{\tilde{u}}_\alpha = \begin{cases} \hat{u}_\alpha e^{-\chi^0_\alpha v^0_\alpha} & \text{on } U_{n_p, n_\theta, (n_\theta - 1)} \cap U_{n_p, n_\theta, n_s}; \\ e^{\chi^0_\alpha v^0_\alpha} \hat{u}_\alpha & \text{on } U_{n_p, n_\theta, n_s} \cap U_{n_p, n_\theta, (n_\theta + 1)}; \\ \hat{u}_\alpha & \text{elsewhere on } U_{n_p, n_\theta, n_s}. \end{cases} \]

3. Let \( v^\rho_\alpha = v^\rho_{n_p, n_\theta, n_s} \) be the \( su(2) \)-valued function on \( U_{n_p, (n_\theta - 1), n_\theta, n_s} \) such that \( e^{v^\rho_\alpha} = u_1^{-1} u_{n_p, n_\theta, n_s} \) and let \( v^{\rho + 1}_{n_p, n_\theta, n_s} \) be the \( su(2) \)-valued function on \( U_{n_p, n_\theta, n_s} \cap U_{n_p, (n_\theta + 1), n_\theta, n_s} \) such that
\[ e^{v^{\rho + 1}_{\alpha}} = u_1^{-1} u_{n_p, n_\theta, n_s} u_{n_p, (n_\theta + 1), n_\theta, n_s}. \]

Let \( \tilde{u}_\alpha = \begin{cases} \hat{u}_\alpha e^{-\chi^0_\alpha v^0_\alpha} & \text{on } U_{(n_p - 1), n_\theta, n_s} \cap U_{n_p, n_\theta, n_s}; \\ e^{\chi^0_\alpha v^0_\alpha} \hat{u}_\alpha & \text{on } U_{n_p, n_\theta, n_s} \cap U_{(n_p + 1), n_\theta, n_s}; \\ \hat{u}_\alpha & \text{elsewhere on } U_{n_p, n_\theta, n_s}. \end{cases} \]

Applying the argument of lemma 6.5 twice on each of the above steps, it is then proved that \( \tilde{u}_\alpha \) constructed above satisfies all the requirements.

Let \( A^f = \gamma \sigma d\theta + \frac{\gamma}{2\pi l} \sigma ds \). It is flat and trivial on each open ball \( B_{n_p, n_\theta, n_s} \). However \( A^f \) is not trivial globally. One way to describe \( A^f \) is to set \( A^f \) to be the trivial connection on each ball \( B_{n_p, n_\theta, n_s} \) but with a nontrivial constant gauge transformation on \( B_{n_p, 0, n_s} \cap B_{n_p, 1, n_s} \), namely \( \tilde{g}_{n_p, n_s} = e^{2\pi \gamma \sigma} \) and a nontrivial constant gauge transformation on \( B_{n_p, 1, 0} \cap B_{n_p, 1, 1} \).

Clearly in the above gauge, on each \( U_{n_p, n_\theta, n_s} \), letting \( A^f_\alpha = 0 \), then
\[ \int_{U_\alpha} |\nabla (\tilde{A}_\alpha - A^f_\alpha)|^2 + \frac{1}{\rho^2} |\tilde{A}_\alpha - A^f_\alpha|^2 d^3 x \leq \tilde{C} \sum_{\alpha'} \int_{U_{\alpha'}} |F^A|^2 d^3 x. \]

Or equivalently, for some possibly larger \( \tilde{C} \),
\[ \int_{U_\alpha} (\rho |\nabla A^f_\alpha (\tilde{A}_\alpha - A^f_\alpha)|^2 + \frac{1}{\rho} |\tilde{A}_\alpha - A^f_\alpha|^2) d^3 x \leq \tilde{C} \sum_{\alpha'} \int_{U_{\alpha'}} \rho |F^A|^2 d^3 x. \]

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The above expression is gauge invariant. In particular, on each \( U_\alpha \), there exists a gauge transformation \( v_\alpha \) that transforms \( A_\alpha^I \) back to \( A^I = \gamma \sigma d\theta + \frac{\dot{\gamma}}{2\pi l} \sigma ds \) and also make \( v_\alpha \circ \tilde{u}_\alpha \) a global gauge transformation (i.e., they agree on all intersections of different \( U_\alpha \)). Letting \( \tilde{A} = v_\alpha(\tilde{A}_\alpha) \) (it is defined globally, hence the subscript is not needed), then on each \( U_\alpha \),

\[
\int_{U_\alpha} (\rho |\nabla A^I(\tilde{A} - A^I)|^2 + \frac{1}{\rho} |\tilde{A} - A^I|^2) d^3x \leq \tilde{C} \sum_{\alpha'} is either \( \alpha \) or a neighbor of \( \alpha \) \( \int_{U_{\alpha'}} \rho |F^A|^2 d^3x \).

Summing over all the \( U_\alpha \), for some possibly larger \( \tilde{C} \),

\[
\int_{N_\varepsilon} (\rho |\nabla A^I(\tilde{A} - A^I)|^2 + \frac{1}{\rho} |\tilde{A} - A^I|^2) d^3x \leq \tilde{C} \int_{N_\varepsilon} \rho |F^A|^2 d^3x.
\]

**Corollary 6.10.** Suppose \( \Psi = A + \Phi \) is a solution to the Bogomolny equation on \( N_\varepsilon \setminus K \) and

\[
\int_{N_\varepsilon \setminus K} \rho |F^A|^2 < +\infty,
\]

then \( \Psi \) is gauge equivalent to a \( u(\Psi) \) such that, either \( u(\Psi) \) has no knot singularity or for some \( \gamma \), using \( \tilde{\Psi} = \tilde{A} + \tilde{\Phi} = \gamma \sigma d\theta \) to define \( \mathbb{H}_{N_\varepsilon} \), \( u(\Psi) - \tilde{\Psi} \in \mathbb{H}_{N_\varepsilon} \).

**Proof.** There exists a gauge transformation \( u \) and some \( A^I = \gamma \sigma d\theta + \frac{\dot{\gamma}}{2\pi l} \sigma ds \) such that

\[
\int_{N_\varepsilon} (\rho |\nabla A^I (u(A) - A^I)|^2 + \frac{1}{\rho} |u(A) - A^I|^2) d^3x \leq C \int_{N_\varepsilon} \rho |F^A|^2 d^3x.
\]

If \( \gamma = 0 \) then \( u(A) \) has no knot singularity. Otherwise, the above inequality implies that \( u(A) - A^I \in \mathbb{H}_{N_\varepsilon} \). Note that \( A^I - \tilde{A} \in \mathbb{H}_{N_\varepsilon} \), hence \( u(A) - \tilde{A} \in \mathbb{H}_{N_\varepsilon} \). On the other hand, by the Bogomolny equation, \( \nabla u(A) \Phi = *F^u(A) \), hence \( \Phi - \tilde{\Phi} \in \mathbb{H}_{N_\varepsilon} \).

### 7 Appendix: some inequalities

**Lemma 7.1.** Let \( \Psi = A + \Phi \) be a fixed fiducial configuration. Suppose \( \psi \) is a smooth configuration on \( \mathbb{R}^3 \setminus K \) with bounded support, \( B_R(0) \) is a ball of radius \( R > 0 \) in \( \mathbb{R}^3 \) centered at origin such that \( N_{100\delta} \subset B_R(0) \) (hence there \( g_{\delta} \) and \( g_0 \) are the same on \( \mathbb{R}^3 \setminus B_R(0) \)). Suppose \( d^2x = r^2 dr d\Theta \), where \( d\Theta \) is the normalized spherical form. Then

\[
2 \int_{\mathbb{R}^3 \setminus B_R(0)} |\nabla A \psi|^2 d^3x \geq \frac{1}{R} \int_{\partial B_R(0)} |\psi(R)|^2 R^2 d\Theta + \frac{1}{2} \int_{\mathbb{R}^3 \setminus B_R(0)} \frac{1}{r^2} |\psi|^2 d^3x.
\]
Proof.

\[
2 \int_{R^3 \setminus B_R(0)} |\nabla^A \psi|^2 \, d^3 x + \frac{1}{2} \int_{R^3 \setminus B_R(0)} \frac{1}{r^2} |\psi|^2 \, d^3 x \\
\geq 2 \int_{R^3} \int_{\partial B_r(0)} r^2 |\nabla^A \psi|^2 \, d\Omega d\theta + \frac{1}{2} \int_{R^3} \int_{\partial B_r(0)} |\psi|^2 \, d\Omega d\theta \\
\geq - \int_{R^3} \int_{\partial B_r(0)} 2r < \nabla^A \psi, \psi > \, d\Omega d\theta = - \int_{R^3} \int_{\partial B_r(0)} r \partial_r(|\psi|^2) \, d\Omega d\theta \\
= \frac{1}{R} \int_{\partial B_R(0)} |\psi(R)|^2 R^2 \, d\Omega + \int_{R^3} \int_{\partial B_r(0)} |\psi|^2 \, d\Omega d\theta \\
= \frac{1}{R} \int_{\partial B_R(0)} |\psi(R)|^2 R^2 \, d\Omega + \int_{R^3} \int_{\partial B_r(0)} r^2 |\psi|^2 \, d^3 x.
\]

\[\square\]

**Theorem 7.2.** Same condition as the lemma [7.1]. Suppose \( \epsilon \leq \delta \). There exists a constant \( C \) which depends on \( \epsilon \) and the knot \( K \), but not on \( A \), such that

\[
\int_{\partial N_\epsilon} |\psi|^2 \rho d\Omega + \int_{B_R(0) \setminus N_\epsilon} |\psi|^2 \, d^3 x \leq C \int_{R^3 \setminus N_\epsilon} |\nabla^A \psi|^2 \, d^3 x,
\]

where \( \rho d\Omega \) is the volume form on \( \partial N_\epsilon \). Since \( g_\delta \) and \( g_0 \) can bound each other uniformly, it doesn’t matter which metric to use in the expression.

**Proof.** From lemma [7.1] \( \int_{\partial B_R(0)} |\psi|^2 \, R^2 \, d\Omega \) can be bounded (up to a constant) by \( \int_{R^3 \setminus N_\epsilon} |\nabla^A \psi|^2 \, d^3 x \). By a trace embedding theorem, \( \int_{\partial N_\epsilon} |\psi|^2 \rho d\Omega + \int_{\partial B_R(0)} |\psi|^2 \, R^2 \, d\Omega \) can be bounded (up to a constant) by \( \int_{B_R(0) \setminus N_\epsilon} (|\nabla(|\psi|^2)| + |\psi|^2) \, d^3 x \leq \int_{R^3 \setminus N_\epsilon} |\nabla^A \psi|^2 \, d^3 x + 5 \int_{B_R(0) \setminus N_\epsilon} |\psi|^2 \, d^3 x \). So it remains to show \( \int_{\partial B_R(0)} |\psi|^2 \, R^2 \, d\Omega \) can be bounded (up to a constant) by \( \int_{R^3 \setminus N_\epsilon} |\nabla^A \psi|^2 \, d^3 x + \int_{\partial B_R(0)} |\psi|^2 \, R^2 \, d\Omega \).

Suppose on the contrary, there exists a sequence \( \{\psi_n\} \) such that

\[
\lim_{n \to +\infty} \left( \int_{R^3 \setminus N_\epsilon} |\nabla^A \psi_n|^2 \, d^3 x + \int_{\partial B_R(0)} |\psi_n|^2 \, R^2 \, d\Omega \right) = 0, \text{ but } \int_{B_R(0) \setminus N_\epsilon} |\psi_n|^2 \, d^3 x = \text{Vol}(B_R(0) \setminus N_\epsilon) > 0.
\]

Then

\[
\limsup_{n \to +\infty} \int_{B_R(0) \setminus N_\epsilon} |\nabla(|\psi_n|^2)| \, d^3 x \leq 2 \limsup_{n \to +\infty} \left( \left( \int_{B_R(0) \setminus N_\epsilon} |\nabla^A \psi_n|^2 \, d^3 x \right)^{\frac{1}{2}} \left( \int_{B_R(0) \setminus N_\epsilon} |\psi_n|^2 \, d^3 x \right)^{\frac{1}{2}} \right) = 0.
\]

So \( |\psi_n|^2 \) have bounded \( L_1^1(B_R(0) \setminus N_\epsilon) \) norms. By a Rellich lemma, there exists a subsequence of \( \{|\psi_n|^2\} \), namely \( \{|\psi_{n_k}|^2\} \), that converges to some
$f \in L^1_1(B_R(0) \setminus N_\epsilon)$.

Hence $\int_{B_R(0) \setminus N_\epsilon} |\nabla f| d^3x = 0$, and $\int_{B_R(0) \setminus N_\epsilon} f d^3x = \text{Vol}(B_R(0) \setminus N_\epsilon)$, so $f$ is the constant function 1.

By a trace embedding thereom,

$$\lim_{k \to +\infty} \int_{\partial B_R(0)} ||\psi_n^k||^2 - 1| R^2 d\Theta = 0,$$

which contradicts the assumption

$$\lim_{n \to +\infty} \int_{\partial B_R(0)} |\psi_n|^2 R^2 d\Theta = 0.$$

$\square$

**Corollary 7.3.** *Same condition as lemma 7.1.* There exists a constant $C$ that doesn’t depend on $\epsilon$, such that, supposing $\epsilon \leq \delta$, and supposing $\eta > 0$, then

$$\epsilon^{\eta-1} \int_{\partial N_\epsilon} |\psi|^2 \rho d\Omega \leq \left( C\delta^{\eta-1} + \frac{\delta^\eta}{\eta} \right) \int_{\mathbb{R}^3 \setminus N_\epsilon} |\nabla^A \psi|^2 d^3x.$$

**Proof.** Suppose $C$ is the constant from theorem 7.2 such that

$$\int_{\partial N_\delta} |\psi|^2 \rho d\Omega \leq C \int_{\mathbb{R}^3 \setminus N_\delta} |\nabla^A \psi|^2 d^3x.$$

Then for any $\epsilon \leq \delta$,

$$\epsilon^{\eta-1} \int_{\partial N_\epsilon} |\psi|^2 \rho d\Omega = \delta^{\eta-1} \int_{\partial N_\delta} |\psi|^2 \rho d\Omega - \int_\epsilon^{\delta} \int_{\partial N_\rho} \partial_\rho (\rho^\eta |\psi|^2) d\rho d\Omega$$

$$\leq C\delta^{\eta-1} \int_{\mathbb{R}^3 \setminus N_\delta} |\nabla^A \psi|^2 d^3x - (\eta + 1) \int_\epsilon^{\delta} \int_{\partial N_\rho} \rho^{\eta-1} |\psi|^2 \rho d\rho d\Omega + 2 \int_\epsilon^{\delta} \int_{\partial N_\rho} \rho^\eta |\nabla^A \psi| |\psi| \rho d\rho d\Omega$$

$$\leq C\delta^{\eta-1} \int_{\mathbb{R}^3 \setminus N_\delta} |\nabla^A \psi|^2 d^3x + \frac{1}{(\eta + 1)} \int_\epsilon^{\delta} \int_{\partial N_\rho} \rho^{\eta-1} |\nabla^A \psi|^2 \rho d\rho d\Omega$$

$$\leq (C\delta^{\eta-1} + \frac{\delta^\eta}{\eta}) \int_{\mathbb{R}^3 \setminus N_\delta} |\nabla^A \psi|^2 d^3x.$$

$\square$

**Theorem 7.4.** *(A Rellich theorem)* Suppose $S$ is a compact subset of $\mathbb{R}^3 \setminus K$, and suppose $\psi \in H_{\Psi, \epsilon}$ for some fixed smooth configuration $\Psi$.

(1) The following integral is compact relative to $||\psi||^2_{H_{\Psi, \epsilon}}$: $\int_S |\psi|^2 d^3x$, which means, any sequence in $H_{\Psi, \epsilon}$ with bounded $H_{\Psi, \epsilon}$ norms has a sub-sequence
that the above integral converges.

(2) For any $\epsilon > 0$, the following integral is also compact relative to $\|\psi\|_{H^{0,\epsilon}}$:

$$
\int_{\partial N_\epsilon} |\psi|^2 \rho d\Omega.
$$

\textbf{Proof.} (1) Using theorem 7.2 by assuming $S \subset B_R(0) \setminus N_\epsilon$ for some $R$ and $\epsilon$, it is clear that the Sobolev norm $W^{1,2}(S)$ of $|\psi|$ on $S$ is bounded above by $\|\psi\|_{H^{0,\epsilon}}$. Hence (1) follows by a Rellich theorem on $S$.

(2) Letting $S$ be the completion of $N_\epsilon \setminus N_4$. Then the Sobolev norm $W^{1,2}(S)$ of $\psi$ on $S$ is bounded above by $\|\psi\|_{H^{0,\epsilon}}$. Since $\partial N_\epsilon$ is one component of $\partial S$, (2) follows from a compact trace embedding theorem. \qed

\textbf{Theorem 7.5.} Suppose $\Psi = A + \Phi$ is the background fiducial configuration and $\epsilon \leq \delta < 1$. There exists a constant $c_1 > 0$ which doesn’t depend on $\epsilon$ or $\Psi$, such that supposing $\psi$ is a smooth configuration on $N_\epsilon \setminus K$ with finite $H^{0,\epsilon}$ norm, then

$$
\left( \int_{N_\epsilon} \rho^3 |\psi|^6 d\tilde{x} \right)^{\frac{1}{3}} \leq c_0 \int_{N_\epsilon} (\rho |\nabla \psi|^2_{g_0} + \frac{1}{\rho} |\psi|^2_{g_0}) d\tilde{x}.
$$

Again, since $g_0$ and $g_0$ can bound each other uniformly, the inequality is also true if $g_0$ is used instead of $g_5$. Here $g_5$ is used just for convenience.

\textbf{Proof.} Throughout this proof, the metric $g_0$ is used by default, and the subscripts are omitted for convenience.

$$
|\psi(\rho, \theta, s)|^4 \leq \int_0^l (|\partial_s (|\psi(\rho, \theta, s_0)|^4)| + \frac{1}{l} |\psi(\rho, \theta, s_0)|^4) ds_0
\leq \int_0^l (4|\partial_s (|\psi(\rho, \theta, s_0)|)| |\psi(\rho, \theta, s_0)|^3 + \frac{1}{l} |\psi(\rho, \theta, s_0)||\psi(\rho, \theta, s_0)|^3) ds_0
\leq (16 + \frac{1}{l^2})^\frac{1}{2} \left( \int_0^l (|\partial_s (|\psi(\rho, \theta, s_0)|)| + |\psi(\rho, \theta, s_0)|^2) ds_0 \right)^\frac{1}{2} \left( \int_0^l |\psi(\rho, \theta, s_0)|^6 ds_0 \right)^\frac{1}{2}.
$$

Similarly,

$$
|\psi(\rho, \theta, s)|^4 \leq (16 + \frac{1}{(2\pi)^2})^\frac{1}{2} \left( \int_0^{2\pi} (|\partial_\theta (|\psi(\rho, \theta_0, s)|)|^2 + |\psi(\rho, \theta_0, s)|^2) d\theta_0 \right)^\frac{1}{2} \left( \int_0^{2\pi} |\psi(\rho, \theta_0, s)|^6 d\theta_0 \right)^\frac{1}{2},
$$

$$
\rho^3 |\psi(\rho, \theta, s)|^4 \leq \int_0^\epsilon (|\partial_\rho (\rho_0^3|\psi(\rho_0, \theta, s)|)|^4) + \frac{1}{\epsilon} \rho_0^3 |\psi(\rho_0, \theta, s)|^4) d\rho_0
\leq \int_0^\epsilon (4\rho_0^3|\partial_\rho (|\psi(\rho_0, \theta, s)|)| |\psi(\rho_0, \theta, s)|^3 + 4\rho_0^3|\psi(\rho_0, \theta, s)||\psi(\rho_0, \theta, s)|^3) d\rho_0
\leq 4(\int_0^\epsilon \rho_0^4 |\partial_\rho (|\psi(\rho_0, \theta, s)|)|^2 + |\psi(\rho_0, \theta, s)|^2) d\rho_0)^\frac{1}{2} \left( \int_0^\epsilon \rho_0^4 |\psi(\rho_0, \theta, s)|^6 d\rho_0 \right)^\frac{1}{2}.
$$

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where \( A = 2(16 + \frac{1}{t^2})^{\frac{1}{4}}(16 + \frac{1}{(2\pi)^2})^{\frac{1}{4}} \) and doesn’t depend on \( \epsilon \), which proves the theorem. \( \square \)

**Lemma 7.6.** There exists a constant \( C \) such that, for any vectors \( \mathbf{v}_1, \mathbf{v}_2 \) with norm 1 (it doesn’t matter which metric be use to normalize \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \), because the two metrics can bound each other uniformly),

\[
|g_0(\mathbf{v}_1, \mathbf{v}_2) - g_0(\mathbf{v}_1, \mathbf{v}_2)| \leq C \rho, \quad |\nabla g_0(\mathbf{v}_1, \mathbf{v}_2) - \nabla g_0(\mathbf{v}_1, \mathbf{v}_2)| \leq C.
\]
Proof. By definition,

\[ g_\delta - g_0 = \chi_\delta((ds^2 + dz_1^2 + dz_2^2) - (dx_1^2 + dx_2^2 + dx_3^2)). \]

Let \( g = ds^2 + dz_1^2 + dz_2^2 \) in \( N_{2\delta} \). Consider its orthonormal basis \( \{\partial_s, \partial_{z_1}, \partial_{z_2}\} \). Recall that under the local parametrization, \((s, z_1, z_2)\) represents the point \( K(s) + z_1e_1(s) + z_2e_2(s) \). So

\[ \partial_s = K'(s) + z_1e'_1(s) + z_2e'_2(s), \quad \partial_{z_1} = e_1(s), \quad \partial_{z_2} = e_2(s). \]

Suppose \( e'_1(s) = \kappa_1(s)K'(s) + \omega(s)e_2(s), \quad e'_2(s) = \kappa_2(s)K'(s) - \omega(s)e_1(s) \), \( \kappa''(s) = -\kappa_1(s)e_1(s) - \kappa_2(s)e_2(s) \). Under the metric \( g_0 \),

\[
|\partial_s|_{g_0}^2 &= 1 + \rho^2 + 2z_1\kappa_1(s) + 2z_2\kappa_2(s),
|\partial_{z_1}|_{g_0}^2 = |\partial_{z_2}|_{g_0}^2 = 1,
\]

\[
<\partial_s, \partial_{z_1} >_{g_0} = -z_2\omega(s), \quad <\partial_s, \partial_{z_2} >_{g_0} = z_1\omega(s).
\]

Thus

\[ |g_0 - g| = O(\rho), \quad \text{so} \quad |g_\delta - g_0| \leq C\rho. \]

Moreover

\[
\nabla(|\partial_s|_{g_0}^2) = 2\rho\nabla\rho + 2\kappa_1(s)\nabla(z_1) + 2\kappa_2(s)\nabla(z_2) + (2z_1\kappa'_1(s) + 2z_2\kappa'_2(s))(\nabla s) = O(1),
\]

\[
\nabla|\partial_{z_1}|_{g_0}^2 = \nabla|\partial_{z_2}|_{g_0}^2 = 0, \quad \nabla(<\partial_s, \partial_{z_1} >_{g_0}) = -\nabla(z_2)\omega(s) - z_2\omega'(s)(\nabla s) = O(1),
\]

\[
<\partial_s, \partial_{z_2} >_{g_0} = \nabla(z_1)\omega(s) + z_1\omega'(s)(\nabla s) = O(1).
\]

So \( |\nabla g_0 - \nabla g| = O(1), \) and \( |\nabla g_0 - \nabla g_\delta| = O(1). \)

\( \square \)

**Theorem 7.7.** *(Sobolev embedding on \( \mathbb{R}^3 \setminus N_c \)) Suppose \( \psi \) is a smooth configuration on \( \mathbb{R}^3 \setminus N_c \) with bounded support. Let \( A \) be a fiducial connection. Then there exists a constant \( C > 0 \) which doesn’t depend on \( A \) and \( \psi \) such that

\[
\int_{\mathbb{R}^3 \setminus N_c} |\psi|^6 d^3x \leq C \int_{\mathbb{R}^3 \setminus N_c} |\nabla^A \psi|^2 d^3x.
\]

**Proof.** Note that \( \nabla(|\psi|^2) \leq |\nabla^A \psi|^2 \), then the theorem is just a standard 3-dimensional Sobolev inequality.

\( \square \)

**Theorem 7.8.** *(A Hardy’s inequality) Suppose \( \eta \in (0, 1) \). Then there exists a constant \( C_\eta \). For any ball \( B_R \subset \mathbb{R}^3 \) of radius \( R \) for some \( R > 0 \), let \( B_{\eta R} \) be the ball with the same center as \( B_R \). Suppose \( f \) is a smooth function on \( B_R \). Let \( r \) be the distance to the center of the ball. Then

\[
\int_{B_{\eta R}} \frac{1}{r} |f|^2 d^3x \leq C_\eta (\frac{1}{R} \int_{B_R} |f|^2 d^3x + R \int_{B_R} |\nabla f|^2 d^3x).
\]

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Remark: A direct corollary of the above theorem is \( \int_{B_R} \frac{1}{r} |f|^2 \, d^3x \) can be bounded above (up to a constant) by \( C_\gamma \left( \int_{B_R} |f|^2 \, d^3x + R \int_{B_R} |\nabla f|^2 \, d^3x \right) \).

Note that since the Green’s function to the Laplacian operator in 3-dimensional space has a point singularity of type \( \sim \frac{1}{r} \), the above inequality can be used to give a point-wise estimate of a function on a bounded open set in \( \mathbb{R}^3 \) based on its Laplacian and certain boundary conditions.

Proof. Let \( d^3x = r^2 \, d\Omega \), where \( d\Omega \) be the spherical volume form. Suppose \( \chi(r) \) is a cut-off function which equals 1 when \( 0 \leq r \leq \eta R \) and 0 when \( r = R \). It can be assumed that \( |\chi'(r)| \sim \frac{1}{R} \) up to a constant which depends on \( \eta \) but independent with \( R \). Then

\[
\int_{B_{\eta R}} \frac{1}{r} |f|^2 \, d^3x = \int_{\partial B_r} \int_0^\eta \int_0^R |f|^2 r \, dr \, d\Omega \leq \int_{\partial B_r} \int_0^R \chi(r)^2 |f(r)|^2 r \, dr \, d\Omega,
\]

and

\[
\int_{\partial B_r} \int_0^R \chi(r)^2 |f(r)|^2 r \, dr \, d\Omega
= -\int_{\partial B_r} \int_0^R \frac{d}{dr} (\chi(r)|f(r)|) \chi(r)|f(r)|^2 r \, dr \, d\Omega
\leq \frac{1}{2} \int_{\partial B_r} \int_0^R \chi(r)^2 |f(r)|^2 r \, dr \, d\Omega + \frac{1}{2} \int_{\partial B_r} \int_0^R \left( |\chi'(r)f(r)|^2 + |\chi(r)f'(r)|^2 \right) r^3 \, dr \, d\Omega
\leq \frac{1}{2} \int_{\partial B_r} \int_0^R \chi(r)^2 |f(r)|^2 r \, dr \, d\Omega + R \int_{\partial B_r} \int_0^R \left( |\chi'(r)f(r)|^2 + |\chi(r)f'(r)|^2 \right) r^2 \, dr \, d\Omega
\leq \frac{1}{2} \int_{\partial B_r} \int_0^R \chi(r)^2 |f(r)|^2 r \, dr \, d\Omega + \frac{C_\eta}{2} \left( \frac{1}{R} \int_{B_R} |f|^2 \, d^3x + R \int_{B_R} |\nabla f|^2 \, d^3x \right).
\]

Theorem 7.9. (A local regularity theorem)

Supposing \( B_R(p) \) is an open ball, and supposing \( \Psi \) is a smooth configuration on \( B_R(P) \), and \( \psi \) is a configuration in \( L^2(B_R(p)) \) such that \( \tilde{L}_\Psi \psi = 0 \), then \( \psi \) is smooth on \( B_{\frac{3}{2}}(p) \). More generally, if \( \psi \) is locally in \( L^2 \) (for example, \( \psi \in H_{\Psi,\epsilon} \)) and \( \tilde{L}_\Psi \psi = 0 \) (or \( \tilde{L}_{\Psi}^\dagger \psi = 0 \)), then \( \psi \) is smooth.

Proof. This can be proved by a standard elliptic regularity argument. \( \square \)
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