SUSPENSION HOMOTOPY OF 4-MANIFOLDS AND THE SECOND 2-LOCAL COHOMOTOPY SETS

PENGCHENG LI

Abstract. In this paper we determine the homotopy type of the (double) suspension of a closed, smooth, connected, orientable 4-manifold $M$, whose integral homology groups can have 2-torsion. Moreover, the decomposition results are applied to give a characterization of the second 2-local cohomotopy set $\pi^2(M; \mathbb{Z}_2)$, which is the set of homotopy classes of based maps from $M$ into the 2-local sphere $S^2_{(2)}$.

1. Introduction

Recently, research on the homotopy properties of manifolds has emerged in two directions. The first direction is the loop homotopy of manifolds, which can be traced back to Beben and Wu’s work [BW15] in 2011. After them, many people made efforts to promote the development of this project, such as Beben, Theriault and Huang [BT14, BT22, HT22]. On the other hand, in 2019 So and Theriault [ST] exhibited rich applications of the suspension homotopy of 4-manifold in gauge groups and current groups, which are important study objects in geometry and physics. Hereafter, this topic has also been widely studied, such as [Hua21a, Hua22, Hua21b, CS22, HL].

This paper contributes to a further research on the suspension homotopy type of manifolds. In the above related literature, due to some intractable obstructions, the authors usually avoid to handle the 2-torsions in the integral homology groups of the manifolds. For example, So and Theriault [ST] required the 4-manifolds are 2-torsion-free in integral homology, Huang [Hua21b] restricts to 6-manifolds with integral homology groups containing no 2 or 3-torsion, while Cutler and So [CS22], Huang and Li [HL] respectively studied the suspension homotopy of simply-connected 6-manifolds and 7-manifolds after localization away from 2.

In this paper we developed new technique and tools in homotopy theory to obtain relatively complete (see comments below Theorem 1.1) characterizations of the suspension homotopy of a given 4-manifold $M$ which can have 2-torsion in homology. For instance, we successfully apply certain homotopy properties of some $(n - 1)$-connected finite CW-complexes of dimension at most $n + 3$ to obtain the homotopy decompositions of $\Sigma^2 M$. Moreover, the Ponstnikov squaring operation (1.1) and the Pontryagin squaring operation (1.2) appear to be powerful in the characterizations of the first two homology sections of the homology decomposition of $\Sigma M$, see Section 4.2.

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To make sense of the Introduction, we need the following notions and notations. Let $G$ be an abelian group and let $n$ be a non-negative integer, denote by $H_n(X; G)$ (resp., $H^n(X; G)$) the $n$-th (singular) homology (resp., cohomology) group of $X$ with coefficients in $G$, denote by $P^n(G)$ the $n$-dimensional Peterson space (cf. [Nei10]) which admits a unique non-trivial reduced integral cohomology group $G$ in dimension $n$. In particular, for integers $n, k \geq 2$, we denote by $Z/k$ the group of integers modulo $k$. Recall the Peterson space have the cell structure

$$P^n(k) = P^n(Z/k) = S^{n-1} \cup_k e^n,$$

which admits the obvious inclusion $i_{n-1}$ of the bottom sphere $S^{n-1}$ into $P^n(k)$ and the pinch map $q_n$ onto $S^n$. For each $n \geq 3$, there is a generator $\eta_n \in \pi_{n+1}(P^n(2^r))$ satisfying the formula

$$q_n \eta_n \simeq \eta_n,$$

see Lemma 2.1, where $\eta_n : S^{n+1} \to S^n$ is the iterated suspensions of the Hopf map $\eta : S^3 \to S^2$. For a homomorphism $\phi : G \to G'$ of groups, ker$(\phi)$ and im$(\phi)$ denote the kernel and the image subgroups of $\phi$, respectively.

To deal with 2-torsions in $H^*(M; Z)$, we shall employ the following cohomology operations. Let $X$ be a connected CW-complex. For each $r \geq 1$, there are unstable cohomology operations: the Postnikov square

$$(1.1) \quad \nabla_0 : H^1(X; Z/2^r) \to H^3(X; Z/2^{r+1})$$

and the Pontryagin square

$$(1.2) \quad \nabla_1 : H^2(X; Z/2^r) \to H^4(X; Z/2^{r+1}).$$

These two operations were carefully studied by Whitehead [Whi50, Whi51]. In particular, the Postnikov square $\nabla_0$ is a homomorphism, while the Pontryagin square $\nabla_1$ is a quadratic function with respect to the cup product $\smile$:

$$(1.3) \quad \nabla_1(-x) = \nabla_1(x), \quad \nabla_1(nx) = n^2 \nabla_1(x),$$

$$\nabla_1(x + y) = \nabla_1(x) + \nabla_1(y) + j_r(x \smile y),$$

where $j : Z/2^r \to Z/2^{r+1}$ is the canonical inclusion homomorphism. Moreover, $\nabla_0$ is the suspension operation of $\nabla_1$:

$$(1.4) \quad \sigma \nabla_0 = \nabla_1 \sigma,$$

where $\sigma : H^*(X; G) \to H^{*+1}(\Sigma X; G)$ is the suspension isomorphism. The Adem relations

$$Sq^3 = Sq^1 Sq^2, \quad Sq^3 Sq^1 + Sq^2 Sq^2 = 0$$

yields the secondary operation $\Theta_n$ based on the relation $\varphi_n \theta_n = 0$ with

$$(1.5) \quad \theta_n = \left(\frac{Sq^2 Sq^1}{Sq^2}\right) : K_n \to K_{n+3} \times K_{n+2},$$

$$\varphi_n = (Sq^1, Sq^2) : K_{n+3} \times K_{n+2} \to K_{n+4},$$

where $n \geq 1, K_n = K_m(Z/2)$ denotes the Eilenberg-MacLane space of type $(Z/2, m)$. For each $r \geq 1$, the higher order Bockstein operations

$$(1.6) \quad \beta_r : H^*(X; Z/2) \to H^{*+1}(X; Z/2)$$
are inductively defined by setting $\beta_1$ as the usual Bockstein homomorphism associated to the short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0;$$

for $r \geq 2$, $\beta_r$ is defined on the intersection of $\ker(\beta_i)$, $i < r$, and taking values in the quotient by the $\text{im}(\beta_i)$, $i < r$. This is also indicated by the dashed arrow in (1.6). See [Har02, Section 5.2] for more details. Note that the higher Bocksteins $\beta_r$ and the sequence $\Theta = \{\Theta_n\}_{n \geq 1}$ are both stable (cf. [Har02, 4.2.2]):

$$\Omega \beta_r = \beta_r, \quad \Omega \Theta_{n+1} = \Theta_n.$$

Let $M$ be a closed, smooth, connected, orientable 4-manifold. By Poincaré duality and the universal coefficient theorem for cohomology, the homology groups $H_*(M; \mathbb{Z})$ are given by the following table:

| $i$  | 0, 4 | 1  | 2  | 3  | $\geq 5$ |
|------|------|----|----|----|---------|
| $H_i(M; \mathbb{Z})$ | $\mathbb{Z}$ | $\mathbb{Z}^m \oplus T$ | $\mathbb{Z}^d \oplus T$ | $\mathbb{Z}^m$ | 0       |

Table 1. $H_*(M; \mathbb{Z})$

where $m, d$ are non-negative integers, and $T$ is a finitely generated torsion abelian group. Let $T_2 = \bigoplus_{j=1}^n \mathbb{Z}/2^{r_j}$ be the 2-primary component of $T$.

Now it is prepared to state our first main theorem.

**Theorem 1.1.** Let $M$ be a closed, smooth, connected, orientable 4-manifold with integral homology $H_*(M; \mathbb{Z})$ given by Table 1.

1. Suppose that $M$ is spin, then $\Sigma^2 M$ has two possible homotopy types:

   (a) If $\Theta(H^1(M; \mathbb{Z}/2)) = 0$, then there is a homotopy equivalence

   $$\Sigma^2 M \simeq (\bigvee_{i=1}^m (S^3 \vee S^5)) \vee (\bigvee_{i=1}^d S^4) \vee P^4(T) \vee P^5(T) \vee S^6.$$ 

   (b) If $\Theta(H^1(M; \mathbb{Z}/2)) \neq 0$, then

   $$\Sigma^2 M \simeq (\bigvee_{i=1}^m (S^3 \vee S^5)) \vee (\bigvee_{i=1}^d S^4) \vee P^4\left(\frac{T}{\mathbb{Z}/2^{r_{j_0}}}\right) \vee P^5(T) \vee A^6(2^{r_{j_0}} \eta^2),$$

   where $j_0$ is the maximum of the indices $j \leq n$ such that

   $\Theta(x) \neq 0, \beta_{r_j}(x) \neq 0, \ x \in H^1(M; \mathbb{Z}/2).$

2. Suppose that $M$ is non-spin and $\Theta(H^1(M; \mathbb{Z}/2)) = 0$, then $\Sigma^2 M$ has three possible homotopy types:

   (a) If for any $u \in H^4(\Sigma^2 M; \mathbb{Z}/2)$ with $\text{Sq}^2(u) \neq 0$ and any $v \in \ker(\text{Sq}^2)$, there hold

   $$\beta_r(u + v) = 0, \quad u + v \notin \text{im}(\beta_s), \quad \forall \ r, s \geq 1,$$

   then there is a homotopy equivalence

   $$\Sigma^2 M \simeq (\bigvee_{i=1}^m (S^3 \vee S^5)) \vee (\bigvee_{i=1}^{d-1} S^4) \vee P^4(T) \vee P^5(T) \vee C^6_\eta,$$
(b) Suppose that for any \( u \in H^2(M; \mathbb{Z}/2) \) with \( \text{Sq}^2(u) \neq 0 \) and any \( v \in \ker(\text{Sq}^2) \), there hold
\[
u + v \notin \text{im}(\beta_s), \ \forall \ s \geq 1,
\]
while there exist \( u' \in H^2(M; \mathbb{Z}/2) \) with \( \text{Sq}^2(u') \neq 0 \) and \( v' \in \ker(\text{Sq}^2) \) such that
\[
\beta_r(u' + v') \neq 0 \quad \text{for some } r \geq 1.
\]
Then there is a homotopy equivalence
\[
\Sigma^2 M \simeq \bigvee_{i=1}^{m} (S^3 \vee S^5) \vee \bigvee_{i=1}^{d} S^4 \vee P^4(T) \vee P^5 \left( \frac{T}{\mathbb{Z}/2^j} \right) \vee C^6_{r,1},
\]
where \( j_1 \) is the maximum of the indices \( j \leq n \) such that
\[
\text{Sq}^2(u') \neq 0, \quad \beta_r(u' + v') \neq 0.
\]
(c) Suppose that there exist \( u \in H^2(M; \mathbb{Z}/2) \) with \( \text{Sq}^2(u) \neq 0 \) and \( v \in \ker(\text{Sq}^2) \) such that
\[
u + v \in \text{im}(\beta_r) \quad \text{for some } r,
\]
then there is a homotopy equivalence
\[
\Sigma^2 M \simeq \left( \bigvee_{i=1}^{m} (S^3 \vee S^5) \right) \vee \left( \bigvee_{i=1}^{d} S^4 \right) \vee P^4 \left( \frac{T}{\mathbb{Z}/2^j} \right) \vee P^5(T) \vee A^6(\tilde{\eta}_{r,2}),
\]
where \( j_2 \) is the minimum of the indices \( j \leq n \) such that \( u + v \in \text{im}(\beta_{r_j}) \).

We omit the discussion of the case where \( M \) is non-spin and \( \Theta \) acts non-trivially on \( H^1(M; \mathbb{Z}/2) \). In this case, the double suspension of \( M \) also has three possible homotopy types, which are similar to the second part (2) of Theorem 1.1. The main difference is that there exists some index \( j_0 \), which is determined by \( \Theta \) as in (1) and is usually different from \( j_2 \), such that \( A^6(2^{r_0} \eta^2) \) is a wedge summand of \( \Sigma^2 M \). Moreover, there maybe exists non-trivial Whitehead products, one may need to consider \( \Sigma^3 M \) to obtain a complete characterizations of the suspension homotopy of \( M \).

We also study the homotopy type of the suspension \( \Sigma M \) in terms of the Postnikov square \( \mathcal{P}_0 \) (or equivalently the Pontryagin square \( \mathcal{P}_1 \)).

**Theorem 1.2.** Let \( M \) be a closed, smooth, connected, orientable 4-manifold with \( H_*(M; \mathbb{Z}) \) given by Table 1 and \( r_1 \leq r_2 \leq \cdots \leq r_n \). If the Postnikov square
\[
\mathcal{P}_0: H^1(M; \mathbb{Z}/2^{r_0}) \to H^3(M; \mathbb{Z}/2^{r_0+1})
\]
is trivial, then the desuspensions of the homotopy decompositions of \( \Sigma^2 M \) in Theorem 1.1 give the homotopy decompositions of \( \Sigma M \).

If \( H_*(M; \mathbb{Z}) \) contains no 2-torsion (i.e., \( T_2 = 0 \)), then the homotopy decomposition \( \Sigma M \simeq \bigvee_{i=1}^{m} S^2 \vee \Sigma W \) (4.1) implies that the Pontryagin square
\[
\mathcal{P}_1: H^1(\Sigma M; \mathbb{Z}/2^{r_0}) \to H^3(\Sigma M; \mathbb{Z}/2^{r_0+1})
\]
is trivial, hence so is \( \mathcal{P}_0 \) by (1.4). Hence Theorem 1.2 extends So and Theriault’s results [ST, Theorem 1.1]. However, the author didn’t find any
other 4-manifolds $M$ satisfying $\mathfrak{P}_0(H^1(M;\mathbb{Z}/2\mathbb{Z})) = 0$. This is also the reason that the above theorem was arranged after Theorem 1.1.

In addition to applications in geometry and physics, see [ST, Hua21a], the suspension homotopy of manifolds also contributes to characterize the 2-local cohomotopy sets $\pi^2(M;\mathbb{Z}(2)) = [M, S^5_2]$, where $X(2)$ is the 2-localization of the space $X$. The homotopy set $\pi^2(X;G) = [X, M_\mathbb{Z}(G)]$ with $G$ cyclic over $p$-local integer $\mathbb{Z}(p)$ are called the modular cohomotopy sets, see [LPW] for more study or reference on this topic. When localized at 2, the EHP fibre sequence (cf. [Jam55])

\[(1.7) \quad \Omega^2 S^5 \xrightarrow{P} S^2 \xrightarrow{E} \Omega S^3 \xrightarrow{H} \Omega S^5 \]

yields an exact sequence

\[
\pi^3(\Sigma^2 M;\mathbb{Z}(2)) \xrightarrow{H_2} \pi^5(\Sigma^2 M;\mathbb{Z}(2)) \xrightarrow{P} \pi^2(\Sigma M;\mathbb{Z}(2)) \xrightarrow{E} \pi^3(\Sigma M;\mathbb{Z}(2)) \xrightarrow{H_1} \pi^5(\Sigma M;\mathbb{Z}(2)),
\]

where $E$ is the suspension and $H_1, H_2$ are induced by the second James-Hopf invariant $H$. Here we use the identification $[M, \Omega^2 Y] = [\Sigma M, Y]$. Let $\alpha \in \pi^3(\Sigma M;\mathbb{Z}(2))$ be given. Then it follows that

1. $E^{-1}(\alpha)$ is non-empty if and only if $H_1(\alpha) = 0$; i.e., $\alpha \in \ker(H_1)$.
2. When $E^{-1}(\alpha)$ is non-empty, there is a bijection $E^{-1}(\alpha) \approx \ker(H_2)$.

As an application to 2-local cohomotopy set $\pi^2(M;\mathbb{Z}(2))$, we compute $\ker(H_1)$ and $\ker(H_2)$ in terms of homotopy decompositions in Theorem 1.1 and 1.2.

**Theorem 1.3.** Let $M$ be closed, smooth, connected, orientable 4-manifold with $H_4(M;\mathbb{Z})$ given by Table 1. Suppose that $M$ is spin or that $M$ is non-spin with trivial action of $\Theta$ on $H^1(M;\mathbb{Z}/2)$. If $C_{r_1}$ is a wedge summand of $\Sigma^2 M$, then there is an isomorphism

\[\ker(H_2) \cong \bigoplus_{i=1}^m \mathbb{Z}(2) \oplus \bigoplus_{j=1}^n \mathbb{Z}/2^{r_j-1} \oplus \mathbb{Z}/2^{r_{j1}};\]

otherwise we have

\[\ker(H_2) \cong \bigoplus_{i=1}^m \mathbb{Z}(2) \oplus \bigoplus_{j=1}^n \mathbb{Z}/2^{r_j-1}.\]

If, in addition, the Postnikov square $\mathfrak{P}_0: H^1(M;\mathbb{Z}/2\mathbb{Z}) \to H^3(M;\mathbb{Z}/2\mathbb{Z})$ is trivial with $r_n \geq 1 \geq r_2 \geq r_1$, then the suspension $E: \pi^2(M;\mathbb{Z}(2)) \to \pi^3(\Sigma M;\mathbb{Z}(2))$ is surjective.

The paper is organized as follows. In Section 2 we review some homotopy theory of partial $(n-1)$-connected $(n+3)$-dimensional CW-complexes and certain cohomology operations. Section 3 mainly introduces an useful criteria to determine the homotopy type of suspensions and the matrix method to analyse the homotopy type of homotopy cofibres of maps from a single space.
into a wedge sum. Section 4 determines the homotopy type of the double
suspension of the 4-manifold $M$ and Section 4.2 studies the homotopy type
of $\Sigma M$ in terms of the Pontryagin squaring operations. The proofs of The-
orem 1.1 and 1.2 appear at the end of Section 4 and 4.2, respectively. The
basic method to determine the homotopy type of the suspensions are the
homology decomposition. In Section 5 we apply Theorem 1.2 and the EHP
fibre sequence (1.7) to prove Theorem 1.3.

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2. Preliminaries

Throughout, all spaces $X, Y, \cdots$ are based connected CW-complexes, and
$[X, Y]$ is the set of based homotopy classes of based maps from $X$ to $Y$.
We identify a map $f$ with its homotopy class in notation. For composable
maps $g$ and $f$, denote by $gf$ or $g \circ f$ the composition of $g$ with $f$. Unless
otherwise specified, $CX$ denotes the reduced mapping cone of a space $X$,
and $C_f$ denotes the homotopy cofibre of a given map $f: X \to Y$. For a
cyclic group $G$, $G(x)$ means $x$ is a generator of $G$.

2.1. Some homotopy theory of mod $2^r$ Moore spaces. Let $n, k \geq 2$
and let $P^n(k) = S^{n-1} \cup_k e^n$ be the $n$-dimensional mod $k$ Moore space. There
is a homotopy cofibration for $P^n(k)$:

$$S^{n-1} \xrightarrow{k} S^{n-1} \xrightarrow{i_{n-1}} P^n(k) \xrightarrow{q_n} S^n,$$

where $i_{n-1}$ and $q_n$ are the canonical inclusion and projection, respectively.
Recall that if 2 doesn’t divide $k$, then

$$\pi_n(P^n(k)) = \pi_{n+1}(P^n(k)) = 0, \quad \forall \ n \geq 3.$$

For each $r, s \geq 1$ and $n \geq 3$, there exists a map (with $n$ omitted in notation)

$$\chi^r_s: P^{n+1}(2^r) \to P^{n+1}(2^s)$$

satisfying the following relation formulas (cf. [BH91]):

$$\chi^r_s i_n = \begin{cases} i_n, & r \geq s; \\ 2^{s-r} i_n, & r < s. \end{cases} \quad q_n + 1 \chi^r_s = \begin{cases} 2^{r-s} q_{n+1}, & r \geq s; \\ q_{n+1}, & r < s. \end{cases}$$

Note that a multiple $t \alpha$ (or written as $t \cdot \alpha$) of an element $\alpha \in \pi_k(X)$ coincides
with the composite $\alpha \circ t$.

Lemma 2.1. Let $r \geq 1$ and $n \geq 3$ be integers.

1. $\pi_{n-1}(P^n(2^r)) \cong \mathbb{Z}/2^r \langle i_{n-1} \rangle$.
2. $\pi_3(P^3(2^r)) \cong \mathbb{Z}/2^{r+1} \langle i_3 \eta \rangle$, $\pi_{n+1}(P^{n+1}(2^r)) \cong \mathbb{Z}/2 \langle i_3 \eta \rangle$.
3. There are isomorphisms

$$\pi_{n+1}(P^n(2^r)) \cong \pi_{n+2}(P^{n+1}(2^r)) \cong \begin{cases} \mathbb{Z}/4 \langle \tilde{\eta} \rangle, & r = 1; \\ \mathbb{Z}/2 \langle \tilde{\eta} \rangle \oplus \mathbb{Z}/2 \langle i_2 \eta \rangle, & r \geq 2, \end{cases}$$

where $\tilde{\eta}$ satisfies the formulas

$$\tilde{\eta} = \chi^1_r \tilde{\eta} = \eta, \quad q_n \tilde{\eta} = \eta, \quad \tilde{\eta}_1 = 2 \eta^2 q_{n+2}.$$
Dually, there are isomorphisms
\[
\pi^n(P^{n+2}(2^t)) \cong \begin{cases} 
\mathbb{Z}/4(\bar{\eta}_1), & r = 1; \\
\mathbb{Z}/2(\bar{\eta}_r) \oplus \mathbb{Z}/2(\eta^2q_{n+2}), & r \geq 2,
\end{cases}
\]
where \(\bar{\eta}_r\) satisfies the formula
\[
\bar{\eta}_r^2 = 2\bar{\eta}_r = \eta^2q_{n+2}.
\]
(5) The composite \(\nu' = \bar{\eta}_1\bar{\eta}_1\) is a generator of \(\pi_6(S^3) \cong \mathbb{Z}/4\).

Proof. (1) The isomorphism holds by the Hurewicz theorem.
(2) By [Bau96, bottom of page 19, top of page 20], there hold
\[
\pi_n(P^n(2^r)) \cong \begin{cases} 
\Gamma(\mathbb{Z}/2^t) \cong \mathbb{Z}/2^{t+1}, & n = 3; \\
\mathbb{Z}/2 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2, & n \geq 4.
\end{cases}
\]
Here \(\Gamma(\mathbb{Z}/2^t)\) is the Whitehead’s quadratic group, see [Bau96] or [Whi50].

(3) By [Bau96, Proposition 11.1.12], \(\pi_4(P^3(2^t))\) is isomorphic to the stable homotopy group \(\pi^4_*(P^3(2^t))\), whose generators and the relations \((2.2)\) refer to [BH91].
(4) The isomorphisms and the relation formulas follow by \((3)\) under the Spanier-Whitehead duality:
\[
\pi^n(P^{n+2}(2^t)) \cong \pi_{n+2}(P^{n+1}(2^t)).
\]
(5) See [Gra14, Section 2]. \(\square\)

For simplicity we still denote \(\bar{\eta}_r: S^{n+1} \rightarrow P^n(2^t)\) the iterated suspensions of the generator \(\bar{\eta}_r\) of \(\pi_4(P^3(2^t))\). Combining \((2.1)\) and \((2.2)\), we have

**Corollary 2.2.** Let \(r, s \geq 1\). There hold relations
\[
\chi_s\bar{\eta}_r = \begin{cases} 
\bar{\eta}_r, & s \geq r; \\
2^{-s}\bar{\eta}_s, & s \leq r.
\end{cases}
\]

### 2.2. Some elementary \(A^2_n\)-complexes.

We call a finite CW-complex \(X\) an \(A^2_n\)-complex if \(X\) is \((n-1)\) connected and \(\dim(X) \leq n + k\). It is well-known that elementary (or called indecomposable) \(A^1_n\)-complexes consists of spheres \(S^n, S^{n+1}\) and Moore spaces \(P^{n+1}(p^r)\) with \(p\) odd primes and \(r \geq 1\). One may consult [Cha50, ZLP19, ZP21, BH91] for more homotopy theory of such complexes. We need the following elementary \(A^2_n\)-complexes, \(n \geq 3\):

\[
\begin{align*}
C^n_{\eta} &= S^n \cup \eta \mathcal{C}S^{n+1} = \Sigma^{n-2}CP^2, \\
C^n_r &= P^{n+1}(2^r) \cup i_n\eta \mathcal{C}S^{n+1}, \quad C^{n+2, l} = S^n \cup \eta q_{n+1} \mathcal{C}P^{n+1}(2^l), \\
C^{n+2, l} &= P^{n+1}(2^r) \cup i_{n+1}\eta q_{n+1} \mathcal{C}P^{n+1}(2^l), \\
A^{n+3}(\eta^2) &= S^n \cup \eta^2 \mathcal{C}S^{n+2}, \\
A^{n+3}(\bar{\eta}_r) &= P^{n+1}(2^r) \cup i_{n+1}\eta q_{n+1} \mathcal{C}S^{n+2}, \\
A^{n+3}(2^r\eta^2) &= P^{n+1}(2^r) \cup i_{n+1}\eta q_{n+1} \mathcal{C}S^{n+2}.
\end{align*}
\]

Here the first four \(A^2_n\)-complexes are the *elementary Chang-complexes* (due to Chang [Cha50]), and the last two spaces are the only two \(A^2_n\)-complexes with the homology groups:

\[
H_n \cong \mathbb{Z}/2^r, \quad H_{n+3} = H_0 \cong \mathbb{Z}, \quad H_i = 0 \text{ for } i \neq 0, n, n+3.
\]
Compare [Bau96, Theorem 10.3.1]. One can also refer to [ZP17, Li] for more homotopy theory of Chang-complexes. Note that all of the above $A_n^3$-complexes desuspend: they can be defined for $n \geq 2$.

2.3. Cohomology operations. We give some lemmas about the cohomology operations defined in Section 1.

2.3.1. Squaring operations. Recall that the Steenrod square

$$\text{Sq}^2 : H^n(-; \mathbb{Z}/2) \to H^{n+2}(-; \mathbb{Z}/2)$$

is a stable cohomology operation such that $\text{Sq}^2(x) = x^2$ for any cohomology class $x$ of dimension 2, cf. [Hat02, Section 4.L].

Lemma 2.3 (cf. [ZP17]). For any $n \geq 3$, the Steenrod square

$$\text{Sq}^2 : H^n(C; \mathbb{Z}/2) \to H^{n+2}(C; \mathbb{Z}/2)$$

is an isomorphism for each (elementary) Chang-complex $C$.

Lemma 2.4. For each $n \geq 2, r \geq 1$, the Steenrod square

$$\text{Sq}^2 : H^{n+1}(A^{n+3}(\tilde{\eta}_r); \mathbb{Z}/2) \to H^{n+3}(A^{n+3}(\tilde{\eta}_r); \mathbb{Z}/2)$$

is an isomorphism.

Proof. By (2.2) there is a homotopy commutative diagram of homotopy cofibrations (in which rows and columns are all homotopy cofibrations):

$$
\begin{array}{cccc}
* & \longrightarrow & S^n & \longrightarrow & S^n \\
\downarrow & & \downarrow \iota_n & & \downarrow \\
S^{n+2} & \xrightarrow{\tilde{\eta}_r} & P^{n+1}(2^r) & \longrightarrow & A^{n+3}(\tilde{\eta}_r) \\
\downarrow \eta & & \downarrow \eta_{n+1} & & \downarrow d \\
S^{n+2} & \xrightarrow{\eta} & S^{n+1} & \longrightarrow & C^{n+3}
\end{array}
$$

It follows that $d^* : H^k(C^{n+3}_{\eta}; \mathbb{Z}/2) \to H^k(A^{n+3}(\tilde{\eta}_r); \mathbb{Z}/2)$ is an isomorphism for $k = n+1, n+3$. The isomorphism in the lemma then follows by Lemma 2.3 and the commutative square

$$
\begin{array}{ccc}
H^{n+1}(C^{n+3}_{\eta}; \mathbb{Z}/2) & \xrightarrow{\text{Sq}^2} & H^{n+3}(C^{n+3}_{\eta}; \mathbb{Z}/2) \\
\cong & \downarrow d & \cong \downarrow d \\
H^{n+1}(A^{n+3}(\tilde{\eta}_r); \mathbb{Z}/2) & \xrightarrow{\text{Sq}^2} & H^{n+3}(A^{n+3}(\tilde{\eta}_r); \mathbb{Z}/2)
\end{array}
$$

□

Lemma 2.5. Let $X$ be a simply-connected finite 4-dimensional CW-complex. Then the Pontryagin square

$$\mathfrak{P}_1 : H^2(X; \mathbb{Z}/2^r) \to H^4(X; \mathbb{Z}/2^{r+1})$$

is given by $\mathfrak{P}_1(x) = x^2$, the cup product modulo $2^{r+1}$.

Proof. See [Whi49, page 74, (10.1)].

□
Lemma 2.6. Let \( r \geq 1 \) and let

\[ S^3 \xrightarrow{t \cdot \eta^2} P^3(2^r) \xrightarrow{i} C(t), \]

be a homotopy cofibration with \( t \in \mathbb{Z}/2^{r+1} \). Then for any \( u \geq r \), the Pontryagin square

\[ \mathfrak{P}_1: H^2(C(t); \mathbb{Z}/2^u) \rightarrow H^4(C(t); \mathbb{Z}/2^{u+1}) \]

is given by \( \mathfrak{P}_1(x) = ty \), where \( x, y \) are the cohomology classes of dimension 2 and 4, respectively. It follows that \( \mathfrak{P}_1 \) is trivial if and only if \( t = 0 \).

Proof. It is well-known that the Hopf invariant map \( H: \pi_3(S^2) \rightarrow \mathbb{Z} \) is a homomorphism with \( H(\eta) = 1 \) (cf. [Hat02, Proposition 4B.1]). Hence \( H(t\eta) = t \). Let \( C_{t\eta} \) be the cofibre of the map \( t\eta: S^3 \rightarrow S^2 \). Then by Lemma 2.5 we have

\[ (2.3) \quad \mathfrak{P}_1(x) = x^2 = H(t\eta)y = ty, \]

where \( x \in H^2(C_{t\eta}; \mathbb{Z}/2^u) \) and \( y \in H^4(C_{t\eta}; \mathbb{Z}/2^{u+1}) \) are the respective generators.

There are homotopy cofibrations

\[ S^3 \xrightarrow{t\eta} S^2 \xrightarrow{i_2} C_{t\eta}, \]
\[ S^2 \xrightarrow{2^r i_2} C_{t\eta} \xrightarrow{j} C(t), \]

where \( j \) is the canonical inclusion map. The following induced commutative diagrams

\[ H^2(C_{t\eta}; \mathbb{Z}/2^u) \xrightarrow{j^*} H^2(C(t); \mathbb{Z}/2^u) \]
\[ \xrightarrow{\mathfrak{P}_1} H^4(C_{t\eta}; \mathbb{Z}/2^{u+1}) \]
\[ \xrightarrow{j^*} H^4(C(t); \mathbb{Z}/2^{u+1}) \]

together with (2.3) then implies that the Pontryagin square on the bottom row is given by the formula

\[ \mathfrak{P}_1(x) = y^2 = ty, \]

where \( x, y \) are the cohomology classes of dimension 2, 4, respectively. Since the cohomology group \( H^4(C(t); \mathbb{Z}/2^{u+1}) \cong \mathbb{Z}/2^{l+1} \), we complete the proof of the statements in the lemma. \( \square \)

2.3.2. Higher order cohomology operations. Recall the secondary cohomology operations

\[ (2.4) \quad \Theta_n: S_n(X) \rightarrow T_n(X), \]

based on the relation \( \psi_n^\theta_n = 0 \) (1.5), where

\[ S_n(X) = \ker(\theta_n) = \ker(\mathrm{Sq}^2) \cap \ker(\mathrm{Sq}^2 \mathrm{Sq}^1) \]
\[ T_n(X) = \coker(\Omega \varphi_n) = H^{n+3}(X; \mathbb{Z}/2)/\im(\mathrm{Sq}^1 + \mathrm{Sq}^2). \]

Lemma 2.7. Let \( n \geq 2, r \geq 1 \). For \( X = A^{n+3}(\eta^2) \) or \( A^{n+3}(2^r \eta^2) \), the secondary operation \( \Theta_n \) acts non-trivially on \( H^n(X; \mathbb{Z}/2) \); that is,

\[ 0 \neq \Theta_n: H^n(X; \mathbb{Z}/2) \rightarrow H^{n+3}(X; \mathbb{Z}/2). \]
Proof. For $X = A^{n+3}(\eta^2)$ or $A^{n+3}(2^r\eta)$, we compute that
\[
S_n(X) = H^n(X; \mathbb{Z}/2) \cong \mathbb{Z}/2, \\
T_n(X) = H^{n+3}(X; \mathbb{Z}/2) \cong \mathbb{Z}/2.
\]
The proof of $\Theta_n \neq 0$ for $X = A^{n+3}(\eta^2)$ refers to [Har02, page 96]. There is a homotopy cofibration
\[
S^n \xrightarrow{\Theta_n^{2r}} A^{n+3}(\eta^2) \xrightarrow{j} A^{n+3}(2^r\eta),
\]
which induces the following commutative square
\[
\begin{array}{c}
H^n(A^{n+3}(2^r\eta); \mathbb{Z}/2) \\
\downarrow j^* \\
H^n(A^{n+3}(\eta^2); \mathbb{Z}/2)
\end{array}
\]
\[
\begin{array}{c}
\Theta_n \downarrow \\
\Theta_{n \neq 0} \downarrow j^*
\end{array}
\]
Thus $\Theta \neq 0$ for $X = A^{n+3}(2^r\eta)$.

The higher order Bocksteins (1.6)
\[
\beta_r : H^n(X; \mathbb{Z}/2) \longrightarrow H^{n+1}(X; \mathbb{Z}/2)
\]
are helpful to detect torsion elements of $H_*(X; \mathbb{Z})$ or $H^*(X; \mathbb{Z})$.

Lemma 2.8 (cf. [MT68], page 173 and 61). The following statements hold:

1. The higher Bockstein $\beta_r$ detects the degree $2^r$ map on $S^n$; in other words, for each $r \geq 1$, there are exactly one non-trivial higher Bockstein
\[
\beta_r : H^{n-1}(P^n(2^r); \mathbb{Z}/2) \rightarrow H^n(P^n(2^r); \mathbb{Z}/2).
\]

2. For each $r \geq 1$, elements of $H^*(X; \mathbb{Z}/2)$ coming from free integral homology class lie in $\ker(\beta_r)$ and not in $\operatorname{im}(\beta_r)$.

3. If $z \in H^{n+1}(X; \mathbb{Z})$ generates a direct summand $\mathbb{Z}/2^r$ for some $r$, then there exist generators $z' \in H^n(X; \mathbb{Z}/2)$ and $z'' \in H^{n+1}(X; \mathbb{Z}/2)$ such that
\[
\beta_r(z') = z'', \quad \beta_i(z') = \beta_i(z'') = 0 \text{ for } i < r.
\]

3. Analysis methods

In this section we list some auxiliary lemmas that simply the proof arguments in the next section. These lemmas appear to be applicable to other similar problems as well, so we leave them in a separate section.

We say that a map $f : X \rightarrow Y$ is homologically trivial if it induced trivial homomorphism $f_* : H_*(X) \rightarrow H_*(Y)$ for each $i$.

Lemma 3.1 ([Hat02], Theorem 4H.3). Let $X$ be a simply-connected space of dimension $N$. Write $H_i = H_i(X)$. Then there is a sequence $X_2 \subseteq X_3 \subseteq \cdots \subseteq X_{n}$ of subcomplexes $X_j$ of $X$ such that

1. $i_* : H_j(X_0) \cong H_j(X)$ for $j \leq n$ and $H_j(X_0) = 0$ for $j > n$.

2. $X_2 = M_2(H_2), X_N = X$. 

Consider the following commutative diagrams:

\[
M_n(H_{n+1}) \xrightarrow{k_n} X_n \xrightarrow{i_n} X_{n+1} \rightarrow M_{n+1}(H_{n+1})
\]

with \(k_n\) homologically trivial.

Note that we have the canonical inclusions \(X^n \subseteq X_n \subseteq X^{n+1}\), where \(X^k\) denotes the \(k\)-skeleton of \(X\). The map \(k_n\) above is called the \(n\)-th \(k'\)-invariant, and plays a key role in the homology decomposition of \(X\). For instance, \(k_n\) is null-homotopic if and only if \(X_n \simeq X_{n-1} \vee M_n(H_nX)\).

**Lemma 3.2.** Let \(f: \bigvee_{i=1}^n A_i \rightarrow \bigvee_{j=1}^m B_j\) be a map which induces trivial homomorphism in cohomology groups with coefficients in abelian groups \(G\) and \(G'\). Let

\[
f_j = p_j \circ f, \quad f_{i,j} = f_j \circ i_i = p_j \circ f \circ i_i,
\]

where \(i_i: A_i \rightarrow \bigvee_{i=1}^n A_i\) and \(p_j: \bigvee_{j=1}^m B_j \rightarrow B_j\) are respectively the canonical inclusion and projection, \(1 \leq i \leq n, 1 \leq j \leq m\).

1. If \(H^*(C_f;G)\) contains no non-trivial cup products, then so do \(H^*(C_{f_i};G)\) and \(H^*(C_{f_{i,j}};G), \forall i, j\).
2. If the cohomology operation \(\mathcal{O}: H^k(C_f;G) \rightarrow H^l(C_f;G')\) is trivial, then so are the operations

\[
\mathcal{O}_j: H^k(C_{f_j};G) \rightarrow H^l(C_{f_j};G'),
\]

\[
\mathcal{O}_{i,j}: H^k(C_{f_{i,j}};G) \rightarrow H^l(C_{f_{i,j}};G').
\]

where \(\mathcal{O}_j\) and \(\mathcal{O}_{i,j}\) are the cohomology operation of the same type as \(\mathcal{O}\).

**Proof.** (1) The statement (1) is due to [ST, Lemma 4.2].

(2) By the proof of [ST, Lemma 4.2], for any integer \(k \geq 0\) and any coefficient group \(G\), there are monomorphisms

\[
d_j^*: H^k(C_{f_j};G) \rightarrow H^k(C_f;G),
\]

and epimorphisms

\[
d_{i,j}^*: H^k(C_{f_{i,j}};G) \rightarrow H^k(C_{f_j};G).
\]

Consider the following commutative diagrams:

\[
\begin{array}{ccc}
H^k(C_f;G) & \xleftarrow{d_j^*} & H^k(C_{f_j};G) & \xrightarrow{d_{i,j}^*} & H^k(C_{f_{i,j}};G) \\
\mathcal{O} & & \mathcal{O}_j & & \mathcal{O}_{i,j} \\
H^l(C_f;G') & \xleftarrow{d_j^*} & H^l(C_{f_j};G') & \xrightarrow{d_{i,j}^*} & H^l(C_{f_{i,j}};G').
\end{array}
\]

It follows that \(\mathcal{O}_j\) is the restriction of \(\mathcal{O}\), and \(\mathcal{O}_{i,j}\) is induced by \(\mathcal{O}_j\). Thus if \(\mathcal{O}\) is trivial, then so are \(\mathcal{O}_j, \mathcal{O}_{i,j}\). \(\square\)

The following lemma is useful to determine the homotopy type of a suspension, see [HL, Lemma 6.4] or [ST, Lemma 5.6].

**Lemma 3.3.** Let \(S \xrightarrow{f} \bigvee_{i=1}^n A_i\bigvee B \xrightarrow{g} \Sigma C\) be a homotopy fibration of simply-connected CW-complexes. Let \(p_j: \bigvee_i A_i \rightarrow A_j\) be the canonical
projection onto the wedge summand $A_j$, $j = 1, \cdots, n$. Suppose that each composition
$$f_j : S \overset{f}{\longrightarrow} \bigvee_i A_i \overset{p_j}{\longrightarrow} A_j$$
is null-homotopic, then there is a homotopy equivalence
$$\Sigma C \simeq \bigvee_{i=1}^n A_i \vee D,$$
where $D$ is the homotopy cofibre of the composition $S \overset{f}{\longrightarrow} (\bigvee_i A_i) \vee B \overset{q_B}{\longrightarrow} B$ with $q_B$ the obvious projection.

3.1. Matrix methods. Let $X = \Sigma X'$, $Y_i = \Sigma Y_i'$ be suspensions, $i = 1, 2, \cdots, n$. Let
$$i_l : Y_l \longrightarrow \bigvee_{j=i} Y_i, \quad p_k : \bigvee_{i=1}^n Y_i \longrightarrow Y_k$$
be respectively the canonical inclusions and projections, $1 \leq k, l \leq n$. By the Hilton-Milnor theorem, we may write a map $f : X \rightarrow \bigvee_{i=1}^n Y_i$ as
$$f = \sum_{k=1}^n i_k \circ f_k + \theta,$$
where $f_k = p_k \circ f : X \rightarrow Y_k$ and $\theta$ satisfies $\Sigma \theta = 0$. The first part $\sum_{k=1}^n i_k \circ f_k$ is usually represented by a vector:
$$u_f = (f_1, f_2, \cdots, f_n)^t.$$ We say that $f$ is completely determined by its components $f_k$ if $\theta = 0$; in this case, denote $f = u_f$. Let $h = \sum_{k,l} i_l h_{kl} p_k$ be a self-map of $\bigvee_{i=1}^n Y_i$ which is completely determined by its components $h_{kl} = p_k \circ h \circ i_l : Y_l \rightarrow Y_k$. Denote by
$$M_h := (h_{kl})_{n \times n} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{pmatrix}$$
Then the composition law $h(f + g) \simeq hf + hg$ implies that the product
$$M_h (f_1, f_2, \cdots, f_n)^t$$
given by the matrix multiplication represents the composite $h \circ f$. Two maps $f = u_f$ and $g = u_g$ are called equivalent, denoted by
$$(f_1, f_2, \cdots, f_n)^t \sim (g_1, g_2, \cdots, g_n)^t,$$
if there is a self-homotopy equivalence $h$ of $\bigvee_{i=1}^n Y_i$, which can be represented by the matrix $M_h$, such that
$$M_h (f_1, f_2, \cdots, f_n)^t \simeq (g_1, g_2, \cdots, g_n)^t.$$ Recall that the above matrix multiplication refers to elementary row operations in matrix theory; and note that the homotopy cofibres of the maps $f = u_f$ and $g = u_g$ are homotopy equivalent if $f$ and $g$ are equivalent.

The following lemma serves as an example of the above matrix method.
Lemma 3.4. Define $X$ by the homotopy cofibration
\[
S^4 \xrightarrow{(f_1,f_2,\ldots,f_n)^t} \bigvee_{j=1}^n V_j \longrightarrow X,
\]
where $f_j : S^4 \to V_j$, $j = 1, \ldots, n$.

(1) If $V_j = S^3$ for $j = 1, 2, \ldots, n$ and $f_{j_0} = \eta$ for some $j_0$, then there is a homotopy equivalence
\[
X \simeq C_\eta^5 \vee \bigvee_{j \neq j_0} S^3.
\]

(2) If $V_j = P^4(2^{r_j})$ for $j = 1, 2, \ldots, n$, and $f_j = i_3 \eta$ for some $j$, then there is a homotopy equivalence
\[
X \simeq C_{r_1}^5 \vee \bigvee_{j \neq j_1} P^4(2^{r_j}),
\]
where $j_1 = \max\{1 \leq j \leq n \mid f_j = i_3 \eta\}$.

Proof. (1) If there are a unique $f_{j_0} = \eta$, the statement clearly holds. We may assume that $f_1 = \eta$, $f_i = \varepsilon_i \cdot \eta$, $\varepsilon_i \in \{0, 1\}$. Then
\[
\left(\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
-\varepsilon_2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\varepsilon_n & 0 & \cdots & 1
\end{array}\right) \left(\begin{array}{c}
\eta \\
\varepsilon_2 \cdot \eta \\
\vdots \\
\varepsilon_n \cdot \eta
\end{array}\right) \simeq \left(\begin{array}{c}
\eta \\
0 \\
\vdots \\
0
\end{array}\right).
\]
It follows that there exists a self-homotopy equivalence $e_S$ of $\bigvee_{j=1}^n S^3$ such that
\[
e_S f \sim (\eta, 0, \ldots, 0)^t,
\]
and hence there is a homotopy equivalence
\[
X = C_f \simeq C_{e_S f} \simeq C_\eta^5 \vee \bigvee_{j=2}^m S^3.
\]

(2) The statement clearly holds if there is a unique $j$ such that $f_j = i_3 \eta$. Let $j_1$ be defined in the lemma. If there is an index $j_2$ such that
\[
f_{j_2} = i_3 \eta \in \pi_4(P^4(2^{r_{j_2}})),
\]
then the matrix multiplication
\[
\begin{pmatrix}
1_p & 0 \\
-\chi^r_s & 1_p
\end{pmatrix}
\begin{pmatrix}
i_3 \eta \\
i_3 \eta
\end{pmatrix} \simeq \begin{pmatrix}
i_3 \eta \\
0
\end{pmatrix},
\]
implies that $(f_{j_1}, f_{j_2})^t \sim (f_{j_1}, 0)^t$. By induction, it follows that there exists a self-homotopy equivalence $e_P$ of $\bigvee_{j=1}^m P^4(2^{r_j})$ such that
\[
e_P \circ (f_1, f_2, \ldots, f_n)^t \simeq (0, \ldots, 0, i_3 \eta, 0, \ldots, 0)^t,
\]
where $i_3 \eta$ in the latter vector lies in the $j_1$-th position. Thus we have a homotopy equivalence
\[
X = C_f \simeq C_{r_{j_1}}^5 \vee \bigvee_{j \neq j_1} P^4(2^{r_j}).
\]
\[\square\]
4. Suspension homotopy types of $M$

By (1) and [ST, Lemma 5.1], there is a homotopy equivalence

$$\Sigma M \simeq \bigvee_{i=1}^{m} S^2 \vee \Sigma W,$$

where $W$ is a CW-complex with integral homology given by the following table:

| $i$ | 0, 4 | 1 | 2 | 3 | $\geq 5$ |
|-----|------|---|---|---|--------|
| $H_i(W)$ | $\mathbb{Z}$ | $T$ | $\mathbb{Z}^d \oplus T$ | $\mathbb{Z}^m$ | 0 |

**Table 2.** $H_*(W; \mathbb{Z})$

By Lemma 3.1 and Table 2, there are homotopy cofibrations

$$\bigvee_{i=1}^{d} S^2 \vee P^3(T) \xrightarrow{\mathbb{Z}^k_3} P^3(T) \to W_3,$$

$$\bigvee_{i=1}^{m} S^3 \xrightarrow{k_4} W_3 \to W_4, \quad S^5 \xrightarrow{k_5} W_4 \to \Sigma W,$$

where $k_3, k_4, k_5$ are homologically trivial maps. Let $T_2 = \bigoplus_{j=1}^{n} \mathbb{Z}/2^{r_j}$ be the 2-primary component of $T$ and write $T = T_2 \oplus T_{\neq 2}$. For each $k \geq 3$, there are homotopy equivalences (cf. [Nei10])

$$P^k(T) \simeq P^k(T_2) \oplus P^k(T_{\neq 3}) \simeq \left( \bigvee_{j=1}^{n} P^k(2^{r_j}) \right) \vee P^k(T_{\neq 3}).$$

**Lemma 4.1.** There is a homotopy equivalence

$$W_3 \simeq \left( \bigvee_{i=1}^{d} S^3 \right) \vee P^3(T) \vee P^4(T).$$

**Proof.** By (4.2), there is a homotopy cofibration

$$\left( \bigvee_{i=1}^{d} S^2 \right) \vee P^3(T) \xrightarrow{\mathbb{Z}^k_3} P^3(T) \to W_3,$$

where $f$ is a homologically trivial map with its two components of the following types:

$$f^S_1 : \left( \bigvee_{i=1}^{d} S^2 \right) \hookrightarrow \left( \bigvee_{i=1}^{d} S^2 \right) \vee P^3(T) \xrightarrow{f} P^3(T);$$

$$f^T_2 : P^3(T) \hookrightarrow \left( \bigvee_{i=1}^{d} S^2 \right) \vee P^3(T) \xrightarrow{f} P^3(T).$$

Here the arrows "\hookrightarrow" denote the obvious inclusions. $f^S_1$ and $f^T_2$ are both homologically trivial. Then the Hurewicz isomorphism $\pi_2(P^3(T)) \cong$...
$H_2(P^3(T))$ implies that both $f^S_1$ and the composite

$$S_T \xrightarrow{j} P^3(T) \xrightarrow{f^T} P^3(T)$$

are null-homotopic, where $j$ is the canonical inclusion of the bottom wedge sum $S_T$ of 2-spheres into $P^3(T)$. Let $m_T: S_T \to S_T$ be the attaching map of $P^3(T)$. There is a diagram of homotopy cofibrations (in which rows are columns are homotopy cofibrations):

$$
\begin{array}{ccc}
S_T & \to & * \\
\downarrow{m_T} & & \downarrow{i_2\circ m_T} \\
S_T & \xrightarrow{0} & P^3(T) \\
\downarrow{i} & & \downarrow{} \\
P^3(T) & \xrightarrow{k_3^P} & P^3(T) \\
\end{array}
$$

It follows that

$$C_{k_3^P} \simeq P^3(T) \vee P^4(T),$$

and hence there is a homotopy equivalence

$$W_3 \simeq \bigvee_{i=1}^d S^3 \vee C_{k_3^P} \simeq \left( \bigvee_{i=1}^d S^3 \right) \vee P^3(T) \vee P^4(T).$$

\[\square\]

**Lemma 4.2.** There is a homotopy equivalence

$$W_4 \simeq \left( \bigvee_{i=1}^d S^3 \right) \vee P^4(T) \vee C_{g_2}$$

for some homologically trivial map $g_2: \bigvee_{i=1}^m S^3 \to P^3(T)$. Moreover, there is a homotopy equivalence

$$\Sigma W_4 \simeq \left( \bigvee_{i=1}^d S^4 \right) \vee P^4(T) \vee P^5(T) \vee \bigvee_{i=1}^m S^5.$$

**Proof.** By (4.2) and Lemma 4.1, there is a homotopy cofibration

$$\bigvee_{i=1}^m S^3 \xrightarrow{g} \left( \bigvee_{i=1}^d S^3 \right) \vee P^3(T) \vee P^4(T) \to W_4$$

with $g$ a homologically trivial map. $g$ is determined by the following components

$$g_1: S^3 \to \bigvee_{i=1}^m S^3 \xrightarrow{g} \left( \bigvee_{i=1}^d S^3 \right) \vee P^3(T) \vee P^4(T) \to \bigvee_{i=1}^d S^3 \to S^3,$$

$$g_2: S^3 \to \bigvee_{i=1}^m S^3 \xrightarrow{g} \left( \bigvee_{i=1}^d S^3 \right) \vee P^3(T) \vee P^4(T) \to P^3(T),$$

$$g_3: S^3 \to \bigvee_{i=1}^m S^3 \xrightarrow{g} \left( \bigvee_{i=1}^d S^3 \right) \vee P^3(T) \vee P^4(T) \to P^4(T).$$
Here the unlabelled maps are the obvious inclusions and projections. \( g_1, g_2, g_3 \) are all homologically trivial. The Hurewicz theorem implies that both \( g_1 \) and \( g_3 \) are null-homotopic. Then by Lemma 3.3 we get the first statement.

To prove the second homotopy equivalence, it suffices to show that if \( f: S^4 \to \Sigma^4(T) \) is homologically trivial, then \( f \) is null-homotopic. Consider the following homologically trivial components of \( f \)

\[
f_1: S^4 \xrightarrow{f} \Sigma^4(T) \to \Sigma^4(T_{\neq 2}),
\]

\[
f_2^j: S^4 \xrightarrow{f} \Sigma^4(T) \to \Sigma^4(T_2) \to \Sigma^4(T_{\neq 2}), \quad j = 1, 2, \ldots, n.
\]

\( f_1 \) is clearly null-homotopic, because \( \pi_4(\Sigma^4(p^r)) = 0 \) for odd primes \( p \). Observe that \( W_4 = \Sigma W^4 \) is a suspension, the Steenrod square \( Sq^2 \) acts trivially on \( H^2(W_4; \mathbb{Z}/2) \). By Lemma 3.2 (2), \( Sq^2 \) acts trivially on \( H^2(C_{f^2}; \mathbb{Z}/2) \).

Note that the homotopy cofibre of the generator \( i_3 \eta \) of \( \pi_4(\Sigma^4(2^r)) \) is \( C_{f^2} \),

it follows by Lemma 2.3 that \( f_2^j \) is null-homotopic for each \( j = 1, 2, \ldots, n \). Thus \( f \) is null-homotopic, by Lemma 3.3.

4.1. Homotopy types of \( \Sigma^2M \). By (4.2) and Lemma 4.2, there is a homotopy cofibration

\[
S^5 \xrightarrow{h} \bigvee_{i=1}^d S^4 \lor P^4(T) \lor P^5(T) \lor \bigvee_{i=1}^m S^5 \to \Sigma^2W
\]

with \( h \) a homologically trivial map. Since \( \pi_5(P^4(p^r)) = \pi_5(P^5(p^r)) = 0 \) for any odd primes \( p \), Lemma 3.3 indicates that there is a homotopy equivalence

\[
\Sigma^2W \simeq P^4(T_{\neq 2}) \lor P^5(T_{\neq 2}) \lor \bigvee_{i=1}^m S^5 \lor C_{\varphi},
\]

where \( \varphi: S^5 \to \bigvee_{i=1}^d S^4 \lor P^4(T_2) \lor P^5(T_2) \) is a homologically trivial map. \( \varphi \) has the following three types of components:

\[
\varphi_1: S^5 \xrightarrow{\varphi_1} \bigvee_{i=1}^d S^4 \lor P^4(T_2) \lor P^5(T_2) \to S^4,
\]

\[
\varphi_2^j: S^5 \xrightarrow{\varphi_2^j} \bigvee_{i=1}^d S^4 \lor P^4(T_2) \lor P^5(T_2) \to P^4(T_2) \to P^4(2^r),
\]

\[
\varphi_3^j: S^5 \xrightarrow{\varphi_3^j} \bigvee_{i=1}^d S^4 \lor P^4(T_2) \lor P^5(T_2) \to P^5(T_2) \to P^5(2^r),
\]

where \( j = 1, 2, \ldots, n \) and the unlabelled maps are the obvious projections.

**Proposition 4.3.** If \( Sq^2\left( H^3(\Sigma^2W; \mathbb{Z}/2) \right) = 0 \), then the homotopy type of \( \Sigma^2W \) is determined by the secondary operation \( \Theta \) (2.4) and the higher Bockstein \( \beta_r \). Explicitly, if \( \Theta(H^3(C_{\varphi}; \mathbb{Z}/2)) = 0 \), then there is a homotopy equivalence

\[
C_{\varphi} \simeq \bigvee_{i=1}^d S^4 \lor P^4(T_2) \lor P^5(T_2) \lor S^6;
\]
otherwise we have
\[ C_\varphi \simeq \left( \bigvee_{i=1}^{d} S^4 \right) \vee P^4\left( \frac{T_2}{\mathbb{Z}/2^{r_0}} \right) \vee P^5(T_2) \vee A_6^* \left( 2^{r_0} \eta^2 \right), \]
where \( j_0 \) is the maximum of the indices \( j \) satisfying
\[ \Theta(x) \neq 0, \beta_j(x) \neq 0, \ x \in H^3(C_\varphi; \mathbb{Z}/2). \]

**Proof.** By assumption and (4.3), \( Sq^2 \) acts trivially on \( H^4(C_\varphi; \mathbb{Z}/2) \), and hence so does \( Sq^2 \) on \( H^4(C_{\varphi_1}; \mathbb{Z}/2) \), \( H^4(C_{\varphi_j}; \mathbb{Z}/2) \) for each \( k = 2, 3 \) and \( j = 1, 2, \ldots, n \), by Lemma 3.2 (2). It follows by Lemma 2.3 and 2.4 that \( \varphi_1, \varphi_2, \ldots, \varphi_n \) are null-homotopic, and
\[ \varphi_j = y_j \cdot i_3 \eta^2, \quad y_j \in \mathbb{Z}/2, \quad j = 1, 2, \ldots, n. \]

By Lemma 2.7, the coefficients \( y_j \) can be detected by the secondary operation \( \Theta \). There are possibly many such indices \( j \), however, similar arguments to that in the proof of Lemma 3.4 show that there exists a homotopy equivalence \( e \) of \( P^4(T_2) \) such that
\[ e(\varphi_1, \varphi_2, \ldots, \varphi_n) \simeq (0, \ldots, 0, \varphi_j, 0, \ldots, 0) \]
with \( j_0 \) described in the proposition. The proof then completes by applying Lemma 3.3. \( \square \)

**Proposition 4.4.** If \( Sq^2 \left( H^4(\Sigma^2 W; \mathbb{Z}/2) \right) \neq 0 \) and \( \Theta \left( H^3(C_\varphi; \mathbb{Z}/2) \right) = 0 \), then the homotopy type of \( \Sigma^2 W \) can be characterized as follows.

1. Suppose that for any \( u \in H^4(\Sigma^2 M; \mathbb{Z}/2) \) with \( Sq^2(u) \neq 0 \) and any \( v \in \ker(Sq^2) \), there hold
\[ \beta_r(u + v) = 0, \quad u + v \notin \ker(Sq^2), \quad \forall \ r, s \geq 1, \]
then there is a homotopy equivalence
\[ C_\varphi \simeq \left( \bigvee_{i=1}^{d-1} S^4 \right) \vee P^5(T_2) \vee P^4(T_2) \vee C_\eta^6. \]

2. Suppose that for any \( u \in H^4(\Sigma^2 M; \mathbb{Z}/2) \) with \( Sq^2(u) \neq 0 \) and any \( v \in \ker(Sq^2) \), there hold
\[ u + v \notin \ker(Sq^2), \quad \forall \ s \geq 1, \]
while there exist \( u' \in H^4(\Sigma^2 M; \mathbb{Z}/2) \) with \( Sq^2(u') \neq 0 \) and \( v' \in \ker(Sq^2) \) such that
\[ \beta_r(u' + v') \neq 0 \text{ for some } r \geq 1. \]
Then there is a homotopy equivalence
\[ C_\varphi \simeq \left( \bigvee_{i=1}^{d} S^4 \right) \vee P^5\left( \frac{T_2}{\mathbb{Z}/2^{r_1}} \right) \vee P^4(T_2) \vee C_{\eta_1}^6 \]
with \( j_1 \) the maximum of indices \( j \) such that
\[ Sq^2(u') \neq 0, \beta_{j_1}(u' + v') \neq 0, u', v' \in H^4(\Sigma^2()). \]
3. Suppose that there exist $u \in H^4(\Sigma^2 M; \mathbb{Z}/2)$ with $Sq^2(u) \neq 0$ and $v \in \ker(Sq^2)$ such that

$$u + v \in \text{im}(\beta_r)$$

for some $r \geq 1$, then there is a homotopy equivalence

$$C_\phi \simeq \left( \bigvee_{i=1}^d S^4 \right) \vee P^5(T_2) \vee P^4(\frac{T_2}{\mathbb{Z}/2^r}) \vee A^6(\tilde{\eta}_{r_2})$$

with $j_2$.

**Proof.** By the Hilton-Milnor theorem, Lemma 2.1 and 2.7, and the assumption that $\Theta(H^3(C_\phi; \mathbb{Z}/2)) = 0$, we can put

(4.4)

$$\varphi = \sum_{j=1}^d x_i \cdot \eta + \sum_{j=1}^n y_j \cdot i_4 \eta + \sum_{k=1}^n z_k \cdot \tilde{\eta}_{rk} + \theta,$$

where $\theta$ is a linear combination of Whihthead products in $\pi_5(P^4(T_2))$.

By Lemma 2.3 and 2.4, we see that at least one of these coefficients $x_i, y_j, z_k$ is non-zero.

(1) Under the conditions in (1), we deduce from Lemma 2.8 that $u$ comes from a free integral homology class. It follows that

$$y_j = z_k = 0, \text{ and } x_i = 1 \text{ for some } i$$

in the expression (4.4). By Lemma 3.4 (1), we may assume that there is exactly one $j$ such that $x_j = 1$. Thus by Lemma 3.3, we get the homotopy equivalence in (1).

(2) The arguments are similar to (1), the conditions (2) implies that

$$x_i = z_k = 0, \text{ and } y_j = 1 \text{ for some } j,$$

while Lemma 3.4 (2) guarantees that we may assume that there is exactly one such $j$, which equals to $j_1$ described in the proposition. The homotopy equivalence in (2) then follows by Lemma 3.3.

(3) The conditions (3) imply that

$$z_k \equiv 1 \pmod{2} \text{ for some } k.$$ 

By Lemma 3.4, we may firstly assume that

$$x_1 = \varepsilon \in \mathbb{Z}/2, \quad x_i = 0 \text{ for } i > 1;$$

$$y_{j_0} = \varepsilon \in \mathbb{Z}/2, \quad y_j = 0 \text{ for } j \neq j_0.$$

Note that $(\varepsilon, \varepsilon) \neq (1, 1)$, because

$$\begin{pmatrix} \eta \\ i_4 \eta \end{pmatrix} \sim \begin{pmatrix} \eta \\ 0 \end{pmatrix} : S^5 \rightarrow S^4 \vee P^5(2r).$$

By the relation $q_4 \tilde{\eta}_{rk} = \eta$ in (2.2), we have

$$\begin{pmatrix} \eta \\ \tilde{\eta}_{rk} \end{pmatrix} \sim \begin{pmatrix} 0 \\ \tilde{\eta}_{rk} \end{pmatrix}, \quad \begin{pmatrix} i_4 \eta \\ \tilde{\eta}_{rk} \end{pmatrix} \sim \begin{pmatrix} 0 \\ \tilde{\eta}_{rk} \end{pmatrix}. $$
It follows that $z_k \equiv 1 \pmod{2}$ implies that $\epsilon = \varepsilon = 0$. On the other hand, Corollary 2.2 implies that
\[ \left( \tilde{\eta}_r \right) \sim \left( \tilde{\eta}_s \right) \text{ for } r \leq s. \]
Thus up to homotopy we may assume that $x_i = y_j = 0$ and there exists exactly one $z_{k_0} \equiv 1 \pmod{2}$ with $k_0$ described in the proposition. Then we get the homotopy equivalence in (3) by Lemma 3.3.

Proof of Theorem 1.1. It is well-known that a closed, smooth, connected, orientable 4-manifold $M$ is spin if and only if the Steenrod square $Sq^2$ acts trivially on $H^2(M; \mathbb{Z}/2)$. The homotopy types of $\Sigma^2 M$ in Theorem 1.1 then are obtained by (4.1, 4.3), Proposition 4.3 and 4.4.

4.2. Homotopy types of $\Sigma M$ and the Pontryagin squares. This section aims to give a proof of Theorem 1.2, which characterizes homotopy types of the suspension $\Sigma M$ under certain hypothesis. By (1.4), there hold equivalence relations
\[ P_1(H^2(M; \mathbb{Z}/2^r)) = 0 \iff P_1(H^2(\Sigma M; \mathbb{Z}/2^r)) = 0. \]
Recall the homotopy equivalence (4.1) and the homotopy cofibrations (4.2).

Lemma 4.5. If the Pontryagin square $P_1$ acts trivially on $H^2(\Sigma M; \mathbb{Z}/2^r)$, then so does $P_1$ on $H^2(W_4; \mathbb{Z}/2^r)$.

Proof. By Lemma 3.1 and the universal coefficient theorem for cohomology, the canonical inclusion $i: W_4 \to \Sigma W$ induces isomorphisms
\[ i^*: H^2(\Sigma W; \mathbb{Z}/2^r) \cong H^2(W_4; \mathbb{Z}/2^r), \]
\[ i^*: H^4(\Sigma W; \mathbb{Z}/2^{r+1}) \cong H^2(W_4; \mathbb{Z}/2^{r+1}). \]
If $P_1$ acts trivially on $H^2(\Sigma M; \mathbb{Z}/2^r)$, then so does $P_1$ on $H^2(\Sigma W; \mathbb{Z}/2^r)$, by (4.1). The following commutative diagram
\[ H^2(\Sigma W; \mathbb{Z}/2^r) \xrightarrow{P_1=0} H^4(\Sigma W; \mathbb{Z}/2^{r+1}) \]
\[ \cong i^* \]
\[ H^2(W_4; \mathbb{Z}/2^r) \xrightarrow{P_1} H^4(W_4; \mathbb{Z}/2^{r+1}) \]
then implies $P_1 = 0$ on the second row.

Lemma 4.6. If $r_1 \leq r_2 \leq \cdots \leq r_n$ and the Pontryagin square $P_1$ acts trivially on $H^2(\Sigma M; \mathbb{Z}/2^{r_n})$, then there is a homotopy equivalence
\[ W_4 \simeq \bigvee_{i=1}^{d} S^3 \vee \bigvee_{i=1}^{m} S^4 \vee P^3(T) \vee P^4(T). \]

Proof. By Lemma 4.1, it suffices to show the homologically trivial component
\[ g_2: S^3 \to P^3(T) \]
is null-homotopic. By Lemma 3.3 and arguments in the proof of [ST, Lemma 5.7], it suffices to show that the components
\[ g_j^2 : S^3 \xrightarrow{g_j^2} P^3(T_2) \xrightarrow{P^3} P^3(2^{r_j}) \]
are null-homotopic for each \( j = 1, 2, \ldots, n \).

The assumption and Lemma 4.5 imply that the Pontryagin square
\[ \mathcal{P}_1 : H^2(W_4; \mathbb{Z}/2^r) \to H^4(W_4; \mathbb{Z}/2^{r+1}) \]
is trivial. By the universal coefficient theorem for cohomology, \( g_j^1 \) induces trivial homomorphism in mod \( 2^r \) or mod \( 2^r + 1 \) cohomology groups, and hence by Lemma 3.2 (2), the Pontryagin square
\[ \mathcal{P}_1 : H^2(C_{g_j^2}; \mathbb{Z}/2^r) \to H^4(C_{g_j^2}; \mathbb{Z}/2^{r+1}) \]
is trivial. Then it follows by Lemma 2.1 and 2.6 that \( g_j^2 \) is null-homotopic, \( j = 1, 2, \ldots, n \). \( \square \)

**Proof of Theorem 1.2.** By Lemma 4.6 and (4.2), there is a homotopy cofibration
\[ S^4 \xrightarrow{k_5} W_4 \simeq \bigvee_{i=1}^d S^3 \cup \bigvee_{i=1}^m S^4 \cup P^3(T) \cup P^4(T) \to \Sigma W \]
with \( k_5 \) homologically trivial. Since \( \pi_4(P^3(2^{r_j})) = \pi_4(P^4(2^{r_j})) = 0 \), Lemma 3.3 implies that there is a homotopy equivalence
\[ \Sigma W \simeq \bigvee_{i=1}^m S^4 \cup P^3(T_{\neq 2}) \cup P^4(T_{\neq 2}) \cup C_{\phi}, \]
where \( \phi : S^4 \to (\bigvee_{i=1}^d S^3) \cup P^4(T_2) \cup P^3(T_2) \) is a homologically trivial map. Compare (4.3). The discussion on the homotopy type of \( \Sigma W \) is totally parallel to that of \( \Sigma^2 W \) in Subsection 4.1, see Proposition 4.3 and 4.4. The proof then completes by the homotopy equivalence (4.1). \( \square \)

5. Application to the 2-local cohomotopy set \( \pi^2(M; \mathbb{Z}(2)) \)

In this section we compute some EHP sequence and give a proof of Theorem 1.3. Firstly from Theorem 1.2 we immediately get the following two corollaries.

**Corollary 5.1.** Let \( M, j_0, j_1, j_2 \) be given by Theorem 1.1 and let \( \Theta \) be the secondary operation defined by (2.4). Denote
\[ G_c = \bigoplus_{i=1}^m \pi_5(S^5_{(2)}) \oplus \bigoplus_{j=1}^n \pi_5(P^5(2^{r_j})). \]

(1) If \( M \) is spin and \( \Theta(H^1(M; \mathbb{Z}/2)) = 0 \), then there are isomorphisms
\[ \pi^5(\Sigma^2 M; \mathbb{Z}(2)) \cong G_c \oplus \begin{cases} \pi_6(S^5), & \text{if } \Theta(H^1(M; \mathbb{Z}/2)) = 0; \\ \pi_5(A^6(2^{r_0} \eta^2); \mathbb{Z}(2)), & \text{otherwise}. \end{cases} \]
(2) Suppose that $M$ is non-spin and $\Theta(H^1(M; \mathbb{Z}/2)) = 0$, then there are isomorphisms

$$
\pi^5(\Sigma^2 M; \mathbb{Z}/2) \cong G \oplus \begin{cases} 
\pi^5(C_0^5; \mathbb{Z}/2), & \text{if } C_0^5 \subseteq \Sigma^2 M; \\
\pi^5(C_{r,j}^5; \mathbb{Z}/2), & \text{if } C_{r,j}^5 \subseteq \Sigma^2 M; \\
\pi^5(A^5(\eta_{r,j_2}); \mathbb{Z}/2), & \text{if } A^5(\eta_{r,j_2}) \subseteq \Sigma^2 M.
\end{cases}
$$

Here $X \subseteq \Sigma^2 M$ means $X$ is a wedge summand of $\Sigma^2 M$.

Corollary 5.2. Let $M, j_0, j_1, j_2$ be given by Theorem 1.1 and let $\Theta$ be the secondary operation defined by (2.4). Assume that $r_1 \leq r_2 \leq \cdots \leq r_n$ and the Postnikov square

$$
\Psi_0: H^1(M; \mathbb{Z}/2^n) \to H^3(M; \mathbb{Z}/2^{r_n+1})
$$

is trivial.

(1) If $M$ is spin, then there are isomorphisms

$$
\pi^5(\Sigma M; \mathbb{Z}/2) \cong \begin{cases} 
\pi^5(S_0^5), & \text{for } \Theta(H^1(M; \mathbb{Z}/2)) = 0; \\
\pi^5(\Sigma^2 M; \mathbb{Z}/2), & \text{otherwise}.
\end{cases}
$$

(2) If $M$ is non-spin and $\Theta(H^1(M; \mathbb{Z}/2)) = 0$, then there are isomorphisms

$$
\pi^5(\Sigma M; \mathbb{Z}/2) \cong \begin{cases} 
\pi^5(C_0^5; \mathbb{Z}/2), & \text{if } C_0^5 \subseteq \Sigma M; \\
\pi^5(C_{r,j}^5; \mathbb{Z}/2), & \text{if } C_{r,j}^5 \subseteq \Sigma M; \\
\pi^5(A^5(\eta_{r,j_2}); \mathbb{Z}/2), & \text{if } A^5(\eta_{r,j_2}) \subseteq \Sigma M.
\end{cases}
$$

Here $Y \subseteq \Sigma M$ means $Y$ is a wedge summand of $\Sigma M$.

Recall the EHP fibration sequence localized at 2:

$$
\Omega^2 S^5 \xrightarrow{P} S^2 \xrightarrow{E} \Omega S^3 \xrightarrow{H} \Omega S^5,
$$

where $E$ is the suspension, $H$ is the second James-Hopf invariant.

Lemma 5.3 (cf. [Gra14, Tod62]). Localized at 2, there is an EHP exact sequence

$$
\pi_{k+n+2}(S^{2n+1}) \xrightarrow{P} \pi_{k+n}(S^n) \xrightarrow{E} \pi_{k+n+1}(S^{n+1}) \xrightarrow{H} \pi_{k+n+1}(S^{2n+1}) \xrightarrow{P} \pi_{k+n-1}(S^n).
$$

(1) Let $\iota_m$ be the identity map on $S^m$. Then

$$
HP(\iota_{2n+1}) = (1 + (-1)^n)\iota_{2n-1},
$$

where $P(\iota_{2n+1}) \in \pi_{2n-1}(S^n)$. Especially, $H(\eta^2) = 0 \in \pi^5(S_0^5)$.

(2) $\pi_6(S^3) \cong \mathbb{Z}/4(\nu')$ embeds in the short exact sequence

$$
0 \to \pi_5(S^2) \xrightarrow{E} \pi_6(S^3) \xrightarrow{H} \pi_6(S^5) \to 0.
$$

That is, $2\nu' = \eta^3$ and $H(\nu') = \eta$.

Lemma 5.4. For maps $\alpha: \Sigma Y \to S^{n+1}$ and $\beta: X \to Y$, there hold

$$
H(\alpha \circ (\Sigma \beta)) = H(\alpha) \circ (\Sigma \beta).
$$
Proof. There is a commutative diagram
\[
\begin{array}{ccc}
[\Sigma X, S^{n+1}] & \xrightarrow{H} & [\Sigma X, S^{2n+1}] \\
\downarrow{\cong} & & \downarrow{\cong} \\
[\Sigma Y, S^{n+1}] & \xrightarrow{H} & [\Sigma Y, S^{2n+1}]
\end{array}
\]
Thus \( H(\alpha \circ (\Sigma \beta)) = H((\Sigma \beta)^*(\alpha)) = (\Sigma \beta)^*(H(\alpha)) = H(\alpha) \circ (\Sigma \beta). \)

\[\square\]

Lemma 5.5. \( H : [P^5(2^r), S^3] \to [P^5(2^r), S^5] \) is non-trivial: \( H(\eta_r) \neq 0 \). It follows that \( \text{coker}(H) \cong \mathbb{Z}/2^{r-1} \), which is trivial for \( r = 1 \).

**Proof.** The EHP sequence (5.1) induces the exact sequence
\[
\pi^3(P^5(2^r)) \xrightarrow{H} \pi^5(P^5(2^r)) \xrightarrow{\nu} \pi^3(P^3(2^r)) \xrightarrow{E} \pi^3(P^4(2^r)) \to 0
\]

\[
\pi^3(P^5(2^r)) \xrightarrow{\cong} \mathbb{Z}/2^r \xrightarrow{\cong} \mathbb{Z}/2^r \xrightarrow{\cong} \mathbb{Z}/2 \xrightarrow{\cong} 0
\]

Hence the homomorphism \( H \) is non-trivial.

\[\square\]

Lemma 5.6. \( H : [A^6(2^{r_0} \eta^2), S^3] \to [A^6(2^{r_0} \eta^2), S^5] \) is an isomorphism.

**Proof.** The generators of the domain and codomain groups, both isomorphic to \( \mathbb{Z}/2 \), are given by Lemma 5.9. Then by Lemma 5.4 and 5.3,

\( H(\nu'q_6) = H(\nu'q_6) = \eta q_6. \)

\[\square\]

Lemma 5.7. For \( X = C^{n+2}_r, C^n_{r+2} \), \( n \geq 2, r \geq 1 \), let \( q_k : X \to S^k \) be the possible canonical pinch maps, \( k \leq n+2 \).

(1) \([C^2_r, S^2] = 0, [C^4_r, S^2] \cong \mathbb{Z}/2^{r+1}(\eta q_3).\)

(2) \([C^5_r, S^3] \cong \mathbb{Z}(\zeta) \) with \( \zeta \) satisfying \( \zeta^4 = 2 \); \([C^5_r, S^3] \cong \mathbb{Z}(q_5).\)

(3) \([C^5_r, S^3] \cong \mathbb{Z}/2(q_{q_4}), [C^5_r, S^3] \cong \mathbb{Z}(q_5).\)

(4) \([C^6_r, S^3] = 0, [C^6_r, S^5] \cong \mathbb{Z}/2^{r+1}(q_5).\)

**Proof.** The group structures and generators can be easily computed by applying the exact functor \([- , S^k]\) to appropriate homotopy cofibrations for elementary Chang-complexes in [ZP17, Section 3.2]; the details are omitted here.

\[\square\]

Lemma 5.8. Let \( r \geq 1 \) be an integer.

(1) \( H : [C^5_r, S^3_{(2)}] \to [C^5_r, S^5_{(2)}] \) is injective, and

\( H : [C^6_r, S^3_{(2)}] \to [C^6_r, S^5_{(2)}] \)

is surjective.

(2) \( H : [C^5_r, S^3_{(2)}] \to [C^5_r, S^5_{(2)}] \) is trivial, and the cokernel of

\( H : [C^6_r, S^3_{(2)}] \to [C^6_r, S^5_{(2)}] \)

is isomorphic to \( \mathbb{Z}/2^r \).
Proof. (1) Localized at 2, the EHP fibration (5.1) induces an exact sequence
\[ [\mathbb{C}P^2, S^2] \xrightarrow{E} [\mathbb{C}^5, S^3] \xrightarrow{H} \mathbb{C}^5 \xrightarrow{S^5}. \]
By Lemma 5.7, \([\mathbb{C}P^2, S^2(p)] = 0\), hence \(H\) is injective. The second \(H\) is surjective for \([\mathbb{C}^5, S^5]\) = 0.

(2) By Lemma 5.7, the first \(H\) is a homomorphism \(\mathbb{Z}/2 \rightarrow \mathbb{Z}(2)\), hence it is trivial. For the second one, consider the induced EHP exact sequence:
\[
\begin{array}{c}
[\mathbb{C}^6, S^3] \xrightarrow{H} [\mathbb{C}^6, S^5] \xrightarrow{P} [\mathbb{C}^4, S^2] \xrightarrow{E} [\mathbb{C}^5, S^3] & \rightarrow 0 \\
[\mathbb{C}^6, S^3] \xrightarrow{H} \mathbb{Z}/2r+1 \rightarrow \mathbb{Z}/2r+1 \xrightarrow{E} \mathbb{Z}/2 & \rightarrow 0
\end{array}
\]
The exactness then implies that \(\text{coker}(H) \cong \text{ker}(E) \cong \mathbb{Z}/2^r\). □

Lemma 5.9. Let \(i_P: \mathbb{P}^3(2r) \rightarrow A^5(2r\eta^2)\) and \(q_k: A^5(2r\eta^2) \rightarrow S^k\) be the canonical inclusion and projections, \(k = 3, 5\). There are isomorphisms of groups:

1. \([A^5(2r\eta^2), S^3] \cong \mathbb{Z}/2r+1(q_3).\)
2. \([A^5(2r\eta^2), S^4] \cong \mathbb{Z}/2(q_4)\).
3. \([A^5(2r\eta^2), S^5] \cong \mathbb{Z}(q_5)\).
4. \([A^5(2r\eta^2), S^3] \cong \mathbb{Z}/2(q_4')\), where \(q_4' \in \pi_6(S^3) \cong \mathbb{Z}/4\) is a generator.

Proof. There are homotopy cofibrations for \(A^5 = A^5(2r\eta^2)\):
\[
\begin{aligned}
S^4 \xrightarrow{i_2\eta^2} \mathbb{P}^3(2r) & \xrightarrow{i_P} A^5 \xrightarrow{q_3} S^5, \\
S^2 \xrightarrow{i_22r} A^5(\eta^2) & \xrightarrow{i} A^5 \xrightarrow{q_3} S^3.
\end{aligned}
\]
Applying \([-j, S^k]\) to the above homotopy cofibrations, one can easily obtain the groups \([X^j, S^k]\) in the Lemma. We give the proof of \([A^5, S^3]\) here as an example and omit the proof of other groups.

Consider the induced exact sequence
\[
[A^5(\eta^2), S^3] \xrightarrow{i_22r} [S^3, S^3] \xrightarrow{q_3} [A^5, S^3] \xrightarrow{i_2} A^5(\eta^2), S^3] \rightarrow 0.
\]
It is easy to compute that \([A^5(\eta^2), S^3] = 0\) and \([A^5(\eta^2), S^3] \cong \mathbb{Z}/2 \oplus \mathbb{Z}\), where \(\mathbb{Z}/2\) is generated by \(\eta'q_6\), and \(\mathbb{Z}\) is generated by a map \(\xi\) satisfying \(\xi \circ i_3 = 2\), the map of degree 2 on \(S^3\). It follows that
\[
\text{coker}((i_22r)^*) \cong \mathbb{Z}/2r+1(1) \xrightarrow{q_3} [A^5, S^3].
\]
□

Recall the following EHP exact sequence:
\[
\pi^3(\Sigma^2M; \mathbb{Z}(2)) \xrightarrow{H_2} \pi^5(\Sigma^2M; \mathbb{Z}(2)) \xrightarrow{P} \pi^2(M; \mathbb{Z}(2)) \xrightarrow{E} \pi^3(\Sigma M; \mathbb{Z}(2)) \xrightarrow{H_1} \pi^5(\Sigma M; \mathbb{Z}(2)),
\]
where we use \(H_1, H_2\) to distinguish the different homomorphisms induced by the second James-Hopf invariant \(H\).
Proof of Theorem 1.3. By exactness, the suspension $E$ is surjective if and only if $H_1 = 0$.

(1) Firstly suppose that $M$ is spin.

Case (a): $\Theta(H^1(M; \mathbb{Z}/2)) = 0$. By Corollary 5.1 (1), $\text{coker}(H_2)$ is a direct sum of the cokernels of the following three types homomorphisms:

\[
\begin{align*}
H_{2,S^5}: [S^5, S^3] &\to [S^5, S^5], \\
H_{2,5^6}: [S^6, S^3] &\to [S^6, S^5], \\
H_{2,j}: [P^5(2^j), S^3] &\to [P^5(2^j), S^5], j = 1, 2, \ldots, n.
\end{align*}
\]

By Lemma 5.3, we see that $H_{2,S^5} = 0$ and $H_{2,5^6}$ is surjective. The cokernels $\text{coker}(H_{2,j})$ are given by Lemma 5.5. Thus

\[
\text{coker}(H_2) \cong \bigoplus_{i=1}^{m} \mathbb{Z}_{(2)} \oplus \bigoplus_{j=1}^{n} \mathbb{Z}/2^{j-1}.
\]

If $\mathfrak{P}_0(H^1(M; \mathbb{Z}/2^n)) = 0$, then Corollary 5.2 implies that homomorphism $H_1: \pi^3(\Sigma M; \mathbb{Z}(2)) \to \pi^5(\Sigma M; \mathbb{Z}(2))$ is determined by $H: \pi_5(S^3) \to \pi_5(S^5)$. Since $H(\eta^2) = 0$, we have $H_1 = 0$. Case (b): $\Theta(H^1(M; \mathbb{Z}/2)) \neq 0$. By Lemma 5.6,

\[
H_{2,5^0}: [A^6(2^{r_0} \eta^2), S^3] \to [A^6(2^{r_0} \eta^2), S^5]
\]

is an isomorphism, and hence by Corollary 5.1 (1), we get

\[
\text{coker}(H_2) = \bigoplus_{i=1}^{m} \mathbb{Z}_{(2)} \oplus \bigoplus_{j=1}^{n} \mathbb{Z}/2^{j-1}.
\]

If $\mathfrak{P}_0(H^1(M; \mathbb{Z}/2^n)) = 0$, then Corollary 5.2 and Lemma 5.9 show that $H_1$ is determined by the map

\[
H: [A^5(2^{r_0} \eta^2), S^3] \cong \mathbb{Z}/2^{r_0+1} \to [A^5(2^{r_0} \eta^2), S^5] \cong \mathbb{Z},
\]

which is clearly trivial. Thus $H_1 = 0$.

(2) Suppose that $M$ is non-spin and $\Theta(H^1(M; \mathbb{Z}/2)) = 0$.

Case (a): $C^6_\eta$ is a wedge summand of $\Sigma^2 M$. By Corollary 5.1 (2) and Lemma 5.7 and 5.5, we get

\[
\text{coker}(H_2) \cong \bigoplus_{i=1}^{m} \mathbb{Z}_{(2)} \oplus \bigoplus_{j=1}^{n} \mathbb{Z}/2^{j-1}.
\]

If $\mathfrak{P}_0(H^1(M; \mathbb{Z}/2^n)) = 0$, then Corollary 5.2 and Lemma 5.8 (1) show that $H_1 = 0$.

Case (b): $C^6_{r_j}$ is a wedge summand of $\Sigma^2 M$. By Corollary 5.1 (2) and Lemma 5.8 (2), we have

\[
\text{coker}(H_2) \cong \bigoplus_{i=1}^{m} \mathbb{Z}_{(2)} \oplus \bigoplus_{j=1}^{n} \mathbb{Z}/2^{r_j-1} \oplus \mathbb{Z}/2^{r_j1}.
\]

If $\mathfrak{P}_0(H^1(M; \mathbb{Z}/2^n)) = 0$, then Corollary 5.2 and Lemma 5.8 (2) imply $H_1 = 0$. 


Case (c): $A^6(\tilde{\eta}_r)$ is a wedge summand of $\Sigma^2 M$. The homotopy cofibration
\[ S^5 \xrightarrow{i_r} P^4(2^r) \xrightarrow{i} A^6(\tilde{\eta}_r) \xrightarrow{q_6} S^6, \]
yields the exact sequence
\[ [P^5(2^r), S^5] \xrightarrow{(\tilde{\eta}_r)^*} [S^6, S^5] \xrightarrow{q_6^*} [A^6(\tilde{\eta}_r), S^5] \to 0, \]
which, together with the relation $q_0\tilde{\eta}_r = \eta$ (2.2), implies that
\[ [A^6(\tilde{\eta}_r), S^5] = 0. \]
Thus by Corollary 5.1 (2) and Lemma 5.5, we get the isomorphism
\[ \text{coker}(H_2) \cong \bigoplus_{i=1}^m \mathbb{Z}_{(2)} \oplus \bigoplus_{j=1}^n \mathbb{Z}/2^r_j - 1. \]

If $\mathcal{Q}_0(H^1(M; \mathbb{Z}/2^{r_\ast})) = 0$, Corollary 5.1 (2) implies that $H_1$ is determined by
\[ H: [A^5(\tilde{\eta}_r), S^3] \to [A^5(\tilde{\eta}_r), S^3]. \]
From the homotopy cofibration $S^4 \xrightarrow{i_r} P^3(2^r) \xrightarrow{i} A^5(\tilde{\eta}_r)$ we compute that
\[ [A^5(\tilde{\eta}_r), S^3] \cong \mathbb{Z}/2^{r - 1}, \quad [A^5(\tilde{\eta}_r), S^3] \cong \mathbb{Z}. \]
Thus $H$, and therefore $H_1$ is trivial in this case. □

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Department of Mathematics, Southern University of Science and Technology, Shenzhen 518055, P. R. China

Email address: lipc@sustech.edu.cn