A Dynamical System Analysis of Three Fluid cosmological Model

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In Friedman-Robertson-Walker flat spacetime, we consider a three fluid cosmological model which contains dark matter, dark energy and baryonic matter in the form of perfect fluid with a barotropic equation of state. Dark matter is taken in form of dust and dark energy is described by a scalar field with a potential $V(\phi)$. Einstein’s field equations are reduced to an autonomous dynamical system by suitable redefinition of basic variables. Considering exponential potential for the scalar field, critical points are obtained for the autonomous system. Finally stability of the critical points and cosmological implications are analyzed.

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I. INTRODUCTION

A large number of recent observational data, including Type Ia supernovae [1, 2], Large Scale Structure [3, 4] and Cosmic Microwave Background anisotropies [5, 6], Baryon Acoustic Oscillations [7] have provided evidences for a spatially flat universe which suffered two accelerated phases - an early acceleration phase (inflation), which occurred prior to the radiation dominated era and a recently initiated present time accelerated expansion. This late time accelerated expansion is attributed to a mysterious exotic matter with large negative pressure called dark energy. The nature of dark energy is unknown but the feature of dark energy is that it remains unclustered at all scales whereas gravitational clustering of baryons and nonbaryonic cold dark matter can be seen. The combined analysis of the different cosmological observations suggests that the universe consists of about 70% dark energy, 25% dark matter and 5% baryons and radiation. As the origin of dark energy is unknown, so several candidates have been proposed to describe it. The simplest choice for dark energy is the cosmological constant [8–10]. This so-called ΛCDM model is able to explain most of the current observational data. But it suffers from two problems - coincidence problem and fine tuning problem. The lack of a reasonable explanation for the cosmological constant problems has led researchers to explore other routes to explain the observations. So dynamical dark energy models have been proposed as alternatives in the literature. A wide class of scalar field models have been introduced to model dark energy. Scalar field models including quintessence [11, 12], K-essence [13, 14], tachyon etc have attracted lots of attention [15–19].

In this paper, we consider a FRW model of the Universe which contains three types of matter - dark matter, dark energy and baryonic matter [20]. The dark matter is in form of dust. The dark energy is described by a scalar field and the baryons are described by a perfect fluid. As a scalar field can be considered to be a perfect fluid in effect, so the matter in the universe is considered in terms of two perfect fluids and a dust. All three fluids are assumed to be self-interacting and minimally coupled to gravity.

We will study the model qualitatively and will check for viable cosmological solution considering cosmological constraints and observational data. From Einstein’s field equations along with Klein Gordan equation we form an autonomous system. In order to study the dynamical character of the system, critical points are obtained and corresponding cosmological models are analyzed. Feasible cosmological solutions should depict our present universe as global attractor i.e all the possible initial conditions lead to the observed percentages of dark energy and dark matter, and once reached, they remain fixed forever. For this reason we will focus on the stability of critical points i.e cosmological models which are attractors. For theoretical details of dynamical system method we would suggest to
go through the books [21–23].

The paper is organized as follows: next section describes the basic equations and physical background of the model. Section III describes formation of autonomous system. Phase space analysis for both 3D and 2D autonomous system are discussed in detail in section IV. Summary is written in section V.

II. BASIC EQUATIONS

The standard cosmology suggests that our universe went through the radiation dominated era, followed by matter dominated phase and then according to recent observations the universe is going through an accelerated expansion. For consistence any theoretical model should coincide with the history of the universe. The homogeneous and isotropic flat FRW space time is chosen as the model of the universe. The universe is assumed to be filled up with non interacting dark matter, dark energy and baryon. Dark matter is considered in the form of dust having energy density $\rho_m$ and dark energy is driven by a scalar field $\phi$ with potential $V(\phi)$ whose energy density and pressure is given by

$$\rho_d = \frac{1}{2} \epsilon \dot{\phi}^2 + V(\phi) \quad \text{and} \quad p_d = \frac{1}{2} \epsilon \dot{\phi}^2 - V(\phi)$$

respectively. We will have quintessence ( real $\phi$) if $\epsilon = 1$ or phantom field (imaginary $\phi$) if $\epsilon = -1$. We have chosen $\epsilon = 1$ here.

The 'baryonic matter' is described by a perfect fluid with linear equation of state $p_b = (\nu - 1)\rho_b$ where $\rho_b, p_b$ are density and pressure of the fluid and satisfies $\frac{2}{3} < \nu \leq 2$ where $\nu$ is the adiabatic index of the fluid. In particular $\nu = 1$ and $\nu = \frac{4}{3}$ correspond to dust matter and radiation respectively. All three fluids interact minimally.

The Einstein field equations can be written as (here $k = 8\pi G$ is gravitational constant, $c = 1$ )

$$3H^2 = k(\rho_m + \rho_d + \rho_b) \quad (2)$$

$$2\dot{H} = -\frac{k}{2}(\rho_m + \rho_b + p_b + \dot{\phi}^2) \quad (3)$$

where $H$ is the Hubble parameter.

Klein-Gordon equation for the scalar field is

$$\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0 \quad (4)$$

The energy conservation relations take the form

$$\dot{\rho}_m + 3H \rho_m = 0 \quad (5)$$

$$\dot{\rho}_d + 3H (\rho_d + p_d) = 0 \quad (6)$$

$$\dot{\rho}_b + 3H (\rho_b + p_b) = 0 \quad (7)$$
From (5) we get, \( \rho_m = \frac{A}{a^3} \), \( A \) is an integration constant, 'a(t)' is scale factor of the universe. 

Equation (7) implies that \( \rho_b = C a^{-3\nu} \) and \( p_b = (\nu - 1) C a^{-3\nu} \), where \( \frac{2}{3} < \nu \leq 2, \nu \neq 1 \).

Now from (2), (3) and (4), we obtain the system of equations as

\[
3H^2 = k\left(\frac{A}{a^3} + \frac{1}{2} \dot{a}^2 + V(\phi) + C a^{-3\nu}\right) \tag{8}
\]

\[
2 \dot{H} = -\frac{k}{2} (A a^{-3} + \nu C a^{-3\nu} + \dot{\phi}^2) \tag{9}
\]

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0 \tag{10}
\]

### III. FORMATION OF AUTONOMOUS SYSTEM

The above evolution equations are highly non-linear, so we will study our cosmological model qualitatively. At first we will formulate an autonomous system from (8)- (10). We will discuss the properties and cosmological consequences based on the existence, stability of critical points and value of the cosmological parameters. We employ the dimensionless variables \[19, 24\],

\[
x = \sqrt[6]{\frac{k}{H}} \dot{\phi} \quad \text{and} \quad y = \sqrt[3]{\frac{k}{H \sqrt{V(\phi)}}} \tag{11}
\]

It is conventional to write fractional energy densities as

\[
\Omega_m = \frac{k A a^{-3}}{3H^2} \quad \text{and} \quad \Omega_b = \frac{k C a^{-3\nu}}{3H^2} \tag{12}
\]

Now the Friedmann equation (8) can be rewritten as

\[
\Omega_m + \Omega_b + x^2 + y^2 = 1 \tag{13}
\]

From (9), we have

\[
\dot{H} = -3H^2 (x^2 + \frac{\Omega_m}{2} + \frac{\nu \Omega_b}{2}) \tag{14}
\]

Now, \( 0 \leq \Omega_b \leq 1 \Rightarrow \)

\[
0 \leq \Omega_m + x^2 + y^2 \leq 1 \tag{15}
\]

The evolution of the dynamical system is described by the autonomous system

\[
\frac{dx}{dN} = 3x(x^2 - 1 + \frac{\Omega_m}{2} + \frac{\nu \Omega_b}{2}) - \sqrt{\frac{3}{2k}} \frac{V'(\phi)}{V(\phi)} y^2 \tag{16}
\]

\[
\frac{dy}{dN} = y[3(x^2 + \frac{\nu \Omega_b}{2} + \frac{\Omega_m}{2}) + \sqrt{\frac{3}{2k}} \frac{V'(\phi)}{V(\phi)} x] \]

\[
\frac{d\Omega_m}{dN} = -3\Omega_m [1 - \Omega_m - 2x^2 - \nu \Omega_b] \]
where $N \equiv \ln a$.

The effective equation of state for this three fluid model has the expression

$$\omega_{\text{eff}} = \frac{p_\phi + p_b}{\rho_m + \rho_d + \rho_b} = -1 + 2x^2 + \nu(1 - \Omega_m - x^2 - y^2) + \Omega_m \tag{17}$$

For cosmic acceleration, $\omega_{\text{eff}} < -\frac{1}{3}$ is required.

The equation state for dark energy is given by

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{x^2 - y^2}{x^2 + y^2} \tag{18}$$

and the density parameter for the scalar field is

$$\Omega_\phi = \frac{k \rho_\phi}{3H^2} = x^2 + y^2 \tag{19}$$

It should be noted that the physical region in the phase plane is constrained by the requirement that the energy density be non-negative i.e $\Omega_m \geq 0$ and $\Omega_b \geq 0$. So (15) restricts the dependent variables $x$ and $y$ to be on the circular cylinder $x^2 + y^2 \leq 1$. The equality sign indicates that there is no longer any dark matter.

IV. PHASE SPACE ANALYSIS AND CRITICAL POINTS

A. 3D autonomous system

We assume $\frac{V'(\phi)}{V(\phi)} = \text{constant}$ i.e potential of the scalar field is exponential, then we can write

$$\sqrt{\frac{3}{2k} V'(\phi)} = \text{constant} = \alpha \text{ (say)}.$$

We rewrite the autonomous system as

$$\frac{dx}{dN} = 3x[x^2 - 1 + \frac{\Omega_m}{2} + \frac{\nu}{2}(1 - \Omega_m - x^2 - y^2)] - \alpha y^2 \tag{20}$$

$$\frac{dy}{dN} = y[3x^2 + 3\frac{\nu}{2}(1 - \Omega_m - x^2 - y^2) + 3\frac{\Omega_m}{2} + \alpha x] \tag{21}$$

$$\frac{d\Omega_m}{dN} = -3\Omega_m[1 - \Omega_m - 2x^2 - \nu(1 - \Omega_m - x^2 - y^2)] \tag{22}$$

In order to obtain the critical points of the system (20)-(22), we set $\frac{dx}{dN} = 0$, $\frac{dy}{dN} = 0$ and $\frac{d\Omega_m}{dN} = 0$.

An attractor is one of the stable critical points of the system. Of course, the critical points need to satisfy (15) for validity. Clearly, $(0, 0, 0)$ is a critical point. Other critical points are $(0, 0, 1)$, $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 0, \Omega_m)$ if $\nu = 1$ and $(\frac{3}{2\alpha}(\nu - 1), \pm \frac{3}{2\alpha}(1 - \nu), 1 - \frac{9}{2}(1 - \nu)^2)$. Considering linear perturbations about the critical point $(x_c, y_c, \Omega_{mc})$ into the dynamical system equa-
TABLE I. Equilibrium points, related parameters and Eigen values

| Equilibrium point | x  | y  | $\Omega_m$ | $\Omega_\phi$ | $\Omega_b$ | $\omega_{eff}$ | Eigen Values |
|-------------------|----|----|-------------|--------------|------------|---------------|--------------|
| $C_1$             | 0  | 0  | 0           | 0            | 0          | 1             | $-1 + \nu - 3(1 - \nu), \frac{3\nu}{2}, 3(\frac{\nu}{2} - 1)$ |
| $C_2$             | 1  | 0  | 0           | 1            | 0          | 1             | $6(1 - \frac{\nu}{2}) , 3 + \alpha, 3(1 - \nu)$ |
| $C_3$             | -1 | 0  | 0           | 1            | 0          | 1             | $6(1 - \frac{\nu}{2}) , 3 - \alpha, 3(1 - \nu)$ |
| $C_4$             | 0  | 0  | 0           | 0            | 0          | 0             | $-\frac{\alpha}{2} , \frac{3\nu}{2}, 3(1 - \nu)$ |
| $C_5$             | $\frac{1}{2\nu}(\nu - 1) \pm \frac{\delta}{2\nu}(1 - \nu) < 1 - \frac{9(1 - \nu)^2}{2\alpha^2} + \frac{9(1 - \nu)^2}{2\alpha^2} > 0$ | 0 | 0 | 0 | 0 | 0 | $\text{in table II}$ |

The linearized perturbation matrix takes the form

$$A = \begin{bmatrix}
9x_c^2(1 - \frac{\nu}{2}) - 3(1 - \frac{\Omega_{mc}}{2}) & -3\nu y_c x_c - 2\alpha y_c & \frac{3\nu}{2}(1 - \nu) \\
+3(1 - \Omega_{mc})\frac{\nu}{2} & 3y_c x_c(2 - \nu) + \alpha y_c & 3x_c^2(1 - \frac{\nu}{2}) + \frac{3\nu}{2} + \frac{3\Omega_{mc}}{2}(1 - \nu) & \frac{3\nu}{2}(1 - \nu) \\
& +\alpha x_c - \frac{9\nu y_c^2}{2} & 6\Omega_{mc}(1 - \nu) + 3x_c^2(2 - \nu) & -3\nu y_c^2 - 3(1 - \nu) \\
-6x_c\Omega_{mc}(\nu - 2) & -6\nu y_c\Omega_{mc} & 6\Omega_{mc}(1 - \nu) + 3x_c^2(2 - \nu) & \end{bmatrix}$$

Hyperbolic critical points of the system, related cosmological parameters, eigen values are given in table I.

$C_1$ represents non accelerating universe completely dominated by baryonic matter. This equilibrium point is unstable because one eigen value is positive. $C_2$ and $C_3$ represents universe dominated by dark energy and these two points are unstable, one eigen value being positive. Late time acceleration is denied for these points. The stability of dark energy dominated phase will determine the fate of the universe. The critical point $C_4$ is dominated by dark matter describing a non accelerating universe. It is unstable in nature. From history of the universe, we expect that the matter dominated era should be unstable, otherwise it would not enter dark energy dominated phase. There should exist unstable matter dominated phase along with stable dark energy dominated phase. From the table I, we see that no critical point matches with present day observation. There is another critical point $(0,0,\Omega_m)$, if $\nu = 1$ which is non hyperbolic in nature as two eigen values are zero. Hence, we can not investigate the local stability of the system at this point and it does not have any stable manifold by center manifold theorem. This model represents completely dark matter dominated phase without late time acceleration.

For different values of $\nu$ and $\alpha$, $C_5$ will denote different critical points. These points are listed in table II with their eigen values. $C_{5a}$, $C_{5b}$ (in table II) denotes same critical point but for different $\nu$ and
### TABLE II. $C_5$ for different $\nu$ and $\alpha$, related parameters and Eigen values

| Equilibrium point | $\alpha$ | $\nu$ | $x$    | $y$    | $\Omega_m$ | $\Phi$ | $\Omega_b$ | $\omega_{eff}$ | Eigen Values           |
|-------------------|---------|-------|--------|--------|------------|--------|------------|-------------------|------------------------|
| $C_{5a}$          | -2.028  | 1.8   | -0.5916| 0.5916 | .3         | .699   | .001       | 0                 | $-0.0096 \pm 2.0543i$, $-0.2356$ |
| $C_{5b}$          | -1.52   | 1.6   | -0.5916| 0.5916 | .3         | .699   | .001       | 0                 | $-0.027 \pm 1.8655i$, $-0.0055$ |
| $C_{5c}$          | 2       | 1.3   | .225   | -0.225 | .898       | .101   | .001       | 0                 | $-0.9939 \pm .4031i$, 1.6365    |
| $C_{5d}$          | 1.3     | 1.2   | 0.375  | -0.375 | .718       | .281   | .001       | 0                 | $-0.7040 \pm 0.7392i$, 1.2323   |
| $C_{5e}$          | -0.5    | 0.8   | 0.57   | 0.57   | 0.35       | .649   | .001       | 0                 | $-0.527 \pm 0.5944i$, $-0.3735$ |
| $C_{5f}$          | -1.5    | 0.9   | 0.1    | -0.1   | .98        | .02    | 0          | 0                 | $-1.446, 1.304, -0.3055$        |
| $C_{5g}$          | -0.5    | 1.2   | -0.6   | 0.6    | 0.28       | .72    | 0          | 0                 | $0.1174 \pm 1.4189i$, 0.1121    |
| $C_{5h}$          | 1.5     | 0.75  | -0.25  | 0.25   | .87        | .125   | .005       | 0                 | $-0.828 \pm 0.18i$, $-1.21$     |

We can see that $C_{5a}$, $C_{5b}$, $C_{5e}$ and $C_{5h}$ have all the eigen values negative. So we can say that they may represent stable phase. Except $C_{5h}$, the corresponding stable universe is largely dominated by dark energy, and dark matter density is very close to presently available observed value, but for $C_{5h}$, the universe is dominated by dark matter. The points do not explain late time acceleration although. Rest of the critical points are unstable points. $C_{5c}$, $C_{5d}$ and $C_{5f}$ are largely dominated by dark matter whereas $C_{5g}$ is dominated by dark energy. $C_{5c}$, $C_{5d}$ and $C_{5f}$ represent contracting universe as $y < 0$ for these points. None of these points describes viable model.

#### B. 2D Autonomous system

The above 3D system i.e (20)-(22) can be reduced to a 2D autonomous system if we do not consider time evolution of dark matter. Recent observations predict that $\Omega_m$ is approximately 0.30, and considering some bounds we take the value of $\Omega_m$ within the range 0.24 - 0.34.

Then the system becomes

\[
\frac{dx}{dN} = 3x[x^2 - 1 + \frac{\Omega_m}{2} + \frac{\nu}{2}(1 - \Omega_m - x^2 - y^2)] - \alpha y^2 \tag{23}
\]

\[
\frac{dy}{dN} = y[3x^2 + 3\frac{\nu}{2}(1 - \Omega_m - x^2 - y^2) + 3\frac{\Omega_m}{2} + \alpha x] \tag{24}
\]

where $\Omega_m$ is constant lying within 0.24 < $\Omega_m$ < 0.34.

Evidently, (0, 0) is a critical point of this system. When $y = 0$ and $x \neq 0$, we get $x = \pm \sqrt{1 + \frac{\Omega_m(\nu-1)}{2(1-\frac{\nu}{2})}}$. and for $x = 0$ and $y \neq 0$, we obtain $y = \pm \sqrt{1 - \Omega_m + \frac{\Omega_m}{2}}$. We have taken the help of figures drawn for this system for non zero $x$ and $y$ using MATLAB. Different critical points are obtained for different values of $\alpha$, $\nu$. Figures are drawn for these points. These points are listed in the tables III to IX. From
TABLE III. Critical points for $\Omega_\phi = .32$, $\nu = 1.9$ and $\alpha = 1.7$, related parameters

| Equilibrium point | x   | y   | $\Omega_\phi$ | $\omega_{eff}$ | Nature       |
|-------------------|-----|-----|---------------|----------------|--------------|
| $C_{2a}$          | 0   | 0   | 0             | 0.58           | saddle       |
| $C_{2b}$          | -0.44 | 0.76 | 0.77         | -0.29          | stable node  |
| $C_{2c}$          | -0.44 | -0.76 | 0.77         | -0.29          | stable node  |
| $C_{2d}$          | 1.58 | 0   | 2.56         | 4.31           | unstable node|
| $C_{2e}$          | -1.48 | 0   | 2.19         | 4.31           | unstable     |

TABLE IV. Critical points for different $\Omega_m = .3$, $\nu = 1.8$ and $\alpha = -2$, related parameters

| Equilibrium point | x   | y   | $\Omega_\phi$ | $\omega_{eff}$ | Nature       |
|-------------------|-----|-----|---------------|----------------|--------------|
| $C_{2a}$          | 0   | 0   | 0             | 0.56           | saddle       |
| $C_{3b}$          | 0.52 | 0.70 | 0.76         | -0.16          | stable node  |
| $C_{3c}$          | 0.52 | -0.70 | 0.76       | -0.16          | stable node  |

TABLE V. Critical points for $\Omega_m = 0.2$, $\nu = 1.6$ and $\alpha = 1.1$, related parameters

| Equilibrium point | x   | y   | $\Omega_\phi$ | $\Omega_b$ | $\omega_{eff}$ | Nature       |
|-------------------|-----|-----|---------------|------------|----------------|--------------|
| $C_{2a}$          | 0   | 0   | 0             | 0.8        | 0.48           | saddle       |
| $C_{4a}$          | -0.33 | 0.89 | 0.90         | 0.001      | -0.58          | stable node  |
| $C_{4c}$          | -0.33 | 0.89 | 0.90         | 0.001      | -0.58          | stable node  |

TABLE VI. Critical points for $\Omega_m = 0.22$, $\nu = 1.2$ and $\alpha = 1.1$, related parameters

| Equilibrium point | x   | y   | $\Omega_\phi$ | $\Omega_b$ | $\omega_{eff}$ | Nature       |
|-------------------|-----|-----|---------------|------------|----------------|--------------|
| $C_{2a}$          | 0   | 0   | 0             | 0.78       | 0.156          | saddle       |
| $C_{5a}$          | -0.32 | 0.84 | 0.82         | 0.001      | -0.57          | stable node  |
| $C_{5c}$          | -0.32 | -0.84 | 0.82        | 0.001      | -0.57          | stable node  |

TABLE VII. Critical points for $\Omega_m = 0.24$, $\nu = 1.3$ and $\alpha = 3$, related parameters

| Equilibrium point | x   | y   | $\Omega_\phi$ | $\Omega_b$ | $\omega_{eff}$ | Nature       |
|-------------------|-----|-----|---------------|------------|----------------|--------------|
| $C_{2a}$          | 0   | 0   | 0             | 0.76       | 0.228          | saddle       |
| $C_{6a}$          | -0.58 | 0.47 | 0.56         | 0.20       | 0.17           | stable node  |
| $C_{6c}$          | -0.58 | 0.47 | 0.56         | 0.20       | 0.17           | stable node  |

TABLE VIII. Critical points for $\Omega_m = 0.24$, $\nu = 1.3$ and $\alpha = 1.3$, related parameters

| Equilibrium point | x   | y   | $\Omega_\phi$ | $\Omega_b$ | $\omega_{eff}$ | Nature       |
|-------------------|-----|-----|---------------|------------|----------------|--------------|
| $C_{2a}$          | 0   | 0   | 0             | 0.76       | 0.228          | saddle       |
| $C_{7b}$          | -0.40 | 0.87 | 0.91         | 0.001      | 0.54           | stable node  |
| $C_{7c}$          | -0.40 | -0.87 | 0.91        | 0.001      | 0.54           | stable node  |
Phase space of the autonomous system of three fluid cosmological model

**FIG. 1.** Phase portrait for the system, related table III.

**TABLE IX.** Critical points for \( \Omega_{m} = 0.2, \nu = 1.6 \) and \( \alpha = -1.1 \), related parameters

| Equilibrium point | x   | y   | \( \Omega_{\phi} \) | \( \Omega_{b} \) | \( \omega_{eff} \) | Nature        |
|------------------|-----|-----|---------------------|-----------------|----------------|----------------|
| \( C_{2a} \)    | 0   | 0   | 0.8                | 0.48            |                | saddle        |
| \( C_{4b} \)    | 0.33| 0.89| 0.90               | 0.001           | -0.58          | stable node   |
| \( C_{4c} \)    | 0.33| -0.89| 0.90              | 0.001           | -0.58          | stable node   |

**FIG. 2.** Phase portrait for the system, related table IV.

**FIG. 3.** Phase portrait for the system, related table V.

**FIG. 4.** Phase portrait for the system, related table VI.

**FIG. 5.** Phase portrait for the system, related table VII.

**FIG. 6.** Phase portrait for the system, related table VIII.
table III (see Fig 1) and IV (see Fig 2) we find that there are critical points, namely, $C_{2b}, C_{2c}, C_{3b}, C_{3c}$ which are stable node in nature, but they violate (13) marginally. $C_{2d}, C_{2e}$ are far outside of the range so are not considered. We did not list this type of critical points for other values of $\alpha, \nu$ in the tables or discussion. We notice that in table V and VI (Fig 3 and Fig 4) $C_{4b}, C_{4c}$ or $C_{5b}, C_{5c}$ are stable nodes and these points describe late time acceleration. Table VII, VIII (Fig 5 and Fig 6) give critical points $C_{6b}, C_{6c}$ or $C_{7b}, C_{7c}$ which are also stable node though late time acceleration is not possible for these points. Table IX (Fig 7) gives the same critical point as in table V. All the critical points listed here like $C_{2b}, C_{2c}, C_{4b}, C_{4c}$ represent dark energy dominated universe. We need to remember $y < 0$ represents contracting universe. This type of solution will remain invalid.

V. SUMMARY

We can see that in three dimension no viable cosmological solution exists which satisfy both dark energy domination and late time acceleration. But in two dimension, $C_{5b}$ is the most favorable solution representing dark energy dominated universe having late time acceleration. $C_{5c}$ is not considered as $y < 0$ represents contracting universe. $C_{4b}$ may be considered another solution though bounds from (13) should be remembered. Thus three fluid cosmological model may represent a cosmological solution.
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