Passing the Limits of Pure Local Search for the Maximum Weight Independent Set Problem in $d$-Claw Free Graphs

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Abstract
In this paper, we consider the task of computing an independent set of maximum weight in a given $d$-claw free graph $G = (V,E)$ equipped with a positive weight function $w : V \rightarrow \mathbb{R}_{>0}$. The local improvement algorithm $SquareImp$ proposed by Berman [2], that achieves a performance ratio of $\frac{2}{3} + \epsilon$ for any $\epsilon > 0$, has remained unimproved for the last twenty years. Recently, Neuwohner [14] has shown how to improve this to $\frac{4}{5} - \frac{1}{63,790,993}$ by taking into account a broader class of local improvements. By considering local improvements of logarithmic size, she further obtained approximation ratios of $\frac{1}{1+\epsilon}$ in quasi-polynomial time, where $0 \leq \epsilon_d \leq 1$ and $\lim_{d \to \infty} \epsilon_d = 0$ [15]. For the special case of the $d-1$-Set Packing Problem, she showed how to get down to a polynomial running time by means of color coding. On the other hand, she provided examples showing that no local improvement algorithm considering local improvements of size $O(\log(|V(G)|))$ with respect to some power $w^\alpha$ of the weight function, $\alpha \in \mathbb{R}$, can yield an approximation guarantee better than $\frac{2}{3}$. This result seems to be the end of the story of (pure) local search for the Maximum Weight Independent Set Problem in $d$-claw free graphs since the lower bound of $\frac{2}{3}$ is asymptotically tight.

However, it turns out that if one considers local improvements that arise by dropping vertex weights and running an algorithm devised for the unweighted setting on certain sub-instances of the given one, one can get beyond the $\frac{2}{3}$-threshold and obtain approximation guarantees of $\frac{1}{1+\epsilon}$ in quasi-polynomial time. Again, for the special case of $d-1$-Set Packing instances, a polynomial running time can be ensured. Starting from this result, we also conduct a more general investigation of the relation between approximation guarantees for the unweighted and weighted variants of both the Maximum Weight Independent Set Problem and the $d-1$-Set Packing problem. In doing so, we can show that for any constant $\sigma > 0$, there exists a constant $\tau > 0$ such that a (quasi-)polynomial time $1 + \sigma \cdot (d-2)$-approximation for the unweighted $d-1$-Set Packing Problem (the Maximum Cardinality Independent Set problem in $d$-claw free graphs) implies a (quasi-)polynomial time $1 + \tau \cdot (d-2)$-approximation for the weighted $d-1$-Set Packing Problem (the Maximum Weight Independent Set problem in $d$-claw free graphs).

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1 Introduction
For $d \geq 1$, a $d$-claw $C_d$ [2] is defined to be a star consisting of one center node and a set $T_C$ of $d$ additional vertices connected to it, which are called the talons of the claw (see Figure 1). Moreover, similar to [2], we define a 0-claw to be a graph consisting only of a single vertex $v$, which is regarded as the unique element of $T_C$ in this case. An undirected graph $G = (V,E)$ is said to be $d$-claw free if none of its induced subgraphs forms a $d$-claw. For example, 1-claw free graphs do not possess any edges, while 2-claw free graphs are disjoint unions of cliques. For natural numbers $k \geq 3$, the Maximum Weight Independent Set Problem (MWIS) in $k+1$-claw free graphs is often studied as a generalization of the weighted $k$-Set Packing Problem, which is defined as follows: Given a family $S$ of sets, each of size at most $k$, together with a positive weight function $w : S \rightarrow \mathbb{R}_{>0}$, the task is to find a disjoint sub-collection of $S$ of maximum weight. By considering the conflict graph $G_S$ associated with an instance of
the weighted \(k\)-Set Packing Problem, the vertices of which are given by the sets in \(S\) and the edges of which represent non-empty set intersections, one obtains a weight preserving one-to-one correspondence between feasible solutions to the \(k\)-Set Packing Problem and independent sets in \(G_S\), which can be shown to be \(k+1\)-claw free.

For the cardinality variant, considerable progress has been made during the last decade. The first improvement over the approximation guarantee of \(k\) achieved by a simple greedy approach was obtained by Hurkens and Schrijver in 1989 \[11\], who showed that for any \(\epsilon > 0\), there exists a constant \(p_\epsilon\) for which a local improvement algorithm that first computes a maximal collection of disjoint sets and then repeatedly applies local improvements of constant size at most \(p_\epsilon\), until no more exist, yields an approximation guarantee of \(\frac{k}{2} + \epsilon\). In this context, a disjoint collection \(X\) of sets contained in the complement of the current solution \(A\) is considered a local improvement of size \(|X|\) if the sets in \(X\) intersect at most \(|X| - 1\) sets from \(A\), which are then replaced by the sets in \(X\), increasing the cardinality of the found solution. Hurkens and Schrijver also proved that a performance guarantee of \(\frac{k}{2}\) is best possible for a local search algorithm only considering improvements of constant size, while Hazan, Safra and Schwartz \[10\] established in 2006 that no \(o\left(\frac{k}{\log k}\right)\)-approximation algorithm is possible in general unless \(P = NP\). At the cost of a quasi-polynomial runtime, Halldórsson \[9\] could prove an approximation factor of \(\frac{k+2}{3}\) by applying local improvements of size logarithmic in the total number of sets. Cygan, Grandoni and Mastrolilli \[7\] managed to get down to an approximation factor of \(\frac{k+1}{3} + \epsilon\), still with a quasi-polynomial runtime.

The first polynomial time algorithm improving on the result by Hurkens and Schrijver was obtained by Sviridenko and Ward \[17\] in 2013. By combining means of color coding with the algorithm presented in \[9\], they achieved an approximation ratio of \(\frac{k+2}{3}\). This result was further improved to \(\frac{k+1}{3} + \epsilon\) for any fixed \(\epsilon > 0\) by Cygan \[6\], obtaining a polynomial runtime doubly exponential in \(\frac{1}{\epsilon}\). The best approximation algorithm for the unweighted \(k\)-Set Packing Problem in terms of performance ratio and running time is due to Fürer and Yu from 2014 \[8\], who achieved the same approximation guarantee as Cygan, but a runtime that is only singly exponential in \(\frac{1}{\epsilon}\).

Concerning the unweighted version of the MWIS in \(d\)-claw free graphs, as remarked in \[17\], both the result of Hurkens and Schrijver as well as the quasi-polynomial time algorithms by Halldórsson and Cygan, Grandoni and Mastrolilli translate to this more general context, yielding approximation guarantees of \(\frac{d+1}{2} + \epsilon\), \(\frac{d+1}{3}\) and \(\frac{d}{3} + \epsilon\), respectively. However, it is not clear how to extend the color coding approach relying on coloring the underlying universe to the setting of \(d\)-claw free graphs \[17\]. When it comes to the weighted variant of the problem, even less is known. For \(d \leq 3\), it is solvable in polynomial time (see \[12\] and \[16\] for the unweighted, \[13\] for the weighted variant), while for \(d \geq 4\), again no \(o\left(\frac{d}{\log d}\right)\)-approximation algorithm is possible unless \(P = NP\) \[10\]. Moreover, in contrast to the unit weight case, considering local improvements the size of which is bounded by a constant can only slightly improve on the performance ratio of \(d - 1\) obtained by the greedy algorithm, since Arkin and Hassin have shown that such an approach
yields an approximation ratio no better than $d - 2$ in general [1]. Similar to the unweighted case, given an independent set $A$, an independent set $X$ is called a \textit{local improvement} of $A$ if it is disjoint from $A$ and the total weight of the neighbors of $X$ in $A$ is strictly smaller than the weight of $X$. Despite the negative result in [1], Chandra and Halldórsson [5] have found that if one does not perform the local improvements in an arbitrary order, but in each step augments the current solution $A$ by an improvement $X$ that maximizes the ratio between the total weight of the vertices added to and removed from $A$ (if exists), the resulting algorithm, which the authors call \textit{BestImp}, approximates the optimum solution within a factor of $\frac{2d}{d-1}$.

By scaling and truncating the weight function to ensure a polynomial number of iterations, they obtain a $2d + \epsilon$-approximation algorithm for the MWIS in $d$-claw free graphs.

As already mentioned, for the last twenty years, the algorithm \textit{SquareImp} suggested by Berman [2] has been unchallenged. It iteratively applies local improvements of the squared weight function that arise as sets of talons of claws in $C$, until no more exist. In doing so, \textit{SquareImp} achieves an approximation ratio of $\frac{2d}{d-1}$, leading to a polynomial time $2d + \epsilon$-approximation algorithm for any $\epsilon > 0$. Berman also provides an example for $w \equiv 1$ showing that his analysis is tight. It consists of a bipartite graph $G = (V,E)$ the vertex set of which splits into a maximal independent set $A = \{1, \ldots, d - 1\}$ such that no claw improves $|A|$ and an optimum solution $B = (\binom{a}{1} \cup \binom{a}{2})$, where the set of edges is given by $E = \{(a,b) : a \in A, b \in B, a \neq b\}$. As the example uses unit weights, he also concludes that applying the same type of local improvement algorithm for a different power of the weight function does not provide further improvements.

However, as also implied by the result in [11], while no small improvements forming the set of talons of a claw in the input graph exist in the tight example given by Berman, once this additional condition is dropped, improvements of small constant size can be found quite easily (see Figure 2). This in turn indicates that considering a less restricted class of local improvements may result in a better approximation guarantee. Based on this observation, Neuwohner [14] recently managed to slightly improve on the approximation guarantee provided by Berman by studying the case where his analysis of the algorithm \textit{SquareImp} is 'almost tight' and pointing out how to find a different type of local improvement in this case. By conducting a more in-depth analysis of the structure of instances that are 'close to tight', Neuwohner [15] further showed that considering (a certain type of) local improvement of logarithmic size can yield approximation guarantees arbitrarily close to $\frac{d-1}{2}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{(Part of) the tight instance provided in [2].}
\end{figure}
for $d \to \infty$. On the other hand, she also proved that an approximation guarantee of $\frac{d-1}{2}$ is actually best possible for this kind of approach. This fact appears to be particularly striking as on the other hand, her analysis shows that instances for which the analysis can be close to being tight are locally unweighted in that the weights of adjacent vertices from an optimum solution and from the chosen solution can only differ by a factor very close to 1. But for the unit weight case, the best known approximation guarantees achievable by considering local improvements of logarithmic size get arbitrarily close to $\frac{d}{4}$. As a consequence, we study an algorithm that first checks for local improvements of logarithmically bounded size like Neuwohner’s algorithm LogImp [15], and, as soon as no more exist, proceeds by constructing sub-instances in which the weights of neighboring vertices in $A$ and $V \setminus A$ are close, applying an algorithm for the unit weight variant and checking whether this results in a local improvement of $w^2$. As the best known algorithms for the unweighted variant yield an approximation guarantee of $\frac{d}{4}$, meaning that we gain a factor of $\frac{2}{3}$ compared to the approximation ratio of SquareImp, we have enough slack to make up for the inaccuracies resulting from the slight deviations between the weights of neighboring vertices to obtain an improved approximation guarantee of $\frac{d}{2} - \Omega(d)$.

As this brief sketch indicates, a similar argument can be conducted for any approximation algorithm for the unweighted problem the approximation guarantee of which is by a constant factor away from what we have for the unweighted case. This raises the question of how the best approximation ratio achievable for the weighted problem via this approach depends on the approximation guarantee we are given in the unweighted setting. While for the sake of readability, we do not want to spend too much effort in the (rather tedious) optimization of constants for the hypothetical scenario where we have a better approximation ratio for unweighted $k$-Set Packing than the guarantee of $\frac{k}{k+1} + \epsilon$ [6], [8], which has not been improved for 7 years, we settle for a qualitative statement the proof of which gives a rough guideline of how to proceed. More precisely, we show that for any constant $\sigma > 0$, there exists a constant $\tau > 0$ such that a (quasi-)polynomial time $1 + \tau \cdot (d-2)$-approximation for the Maximum Cardinality Independent Set problem in $d$-claw free graphs (the unweighted $d-1$-Set Packing Problem) implies a (quasi-)polynomial time $1 + \sigma \cdot (d-2)$-approximation for the Maximum Weight Independent Set problem in $d$-claw free graphs (the weighted $d-1$-Set Packing Problem). Given that unless $P = NP$, we have a lower bound of $\Omega(\frac{k}{k \log k})$ for even the unweighted $k$-Set Packing problem, this more or less covers the range of approximation guarantees we can expect. The rest of this paper is organized as follows:

In Section 2, we review Berman’s algorithm SquareImp and give a short overview of the analysis pointing out the results we reuse in the analysis of our algorithm. Moreover, we recall some of the definitions from [15] that we reemploy. In Section 3, we finally present the algorithm we study in this paper. We also prove our main result, stating that for $d \geq 5$, our algorithm yields a $d^2 - \Omega(d)$-approximation. Given that the Maximum Weight Independent Set problem in $d$-claw free graphs, is, however, $NP$-hard for $d \geq 4$, Section 4 elaborates on the case $d = 4$. In Section 5, we investigate the relation between approximation guarantees for the unweighted and the weighted problem variants. Finally, Section 6 concludes the paper.

2 Preliminaries

2.1 Results from [2]

In this section, we shortly recap the definitions and main results from [2] that we will employ in the analysis of our local improvement algorithm. We first introduce some basic notation.
Algorithm 1 SquareImp [2]

Input: an undirected $d$-claw free graph $G = (V,E)$ and a positive weight function $w : V \to \mathbb{R}_{>0}$

Output: an independent set $A \subseteq V$

1. $A \leftarrow \emptyset$
2. while there exists a claw $C$ in $G$ that improves $w^2(A)$ do
   3. $A \leftarrow A \setminus \left(N(T_C,A) \cup T_C\right)$
3. return $A$

that is needed for its formal description.

Definition 1 (neighborhood [2]). Given an undirected graph $G = (V,E)$ and subsets $U, W \subseteq V$ of vertices, we define the neighborhood $N(U, W)$ of $U$ in $W$ as

$$N(U, W) := \{w \in W : \exists u \in U : \{u, w\} \in E \vee u = w\}.$$ 

In order to simplify notation, for $u \in V$ and $W \subseteq V$, we write $N(u, W)$ instead of $N(\{u\}, W)$. 

Notation 2. Given a weight function $w : V \to \mathbb{R}$ and some $U \subseteq V$, we write $w^2(U) := \sum_{u \in U} w^2(u)$. Observe that in general, $w^2(U) \neq (w(U))^2$.

Definition 3 ([2]). Given an undirected graph $G = (V,E)$, a positive weight function $w : V \to \mathbb{R}_{>0}$ and an independent set $A \subseteq V$, we say that a vertex set $B \subseteq V$ improves $w^2(A)$ if $B$ is independent in $G$ and $w^2(A \setminus N(B,A) \cup B) > w^2(A)$ holds. For a claw $C$ in $G$, we say that $C$ improves $w^2(A)$ if its set of talons $T_C$ does.

Note that an independent set $B$ improves $A$ if and only if we have $w^2(B) > w^2(N(B,A))$ (see Proposition [14]).

Using the notation introduced above, Berman’s algorithm SquareImp [2] can now be formulated as in Algorithm 1. Observe that by positivity of the weight function, every $v \notin A$ such that $A \cup \{v\}$ is independent constitutes the talon of a 0-claw improving $w^2(A)$, so the algorithm returns a maximal independent set.

The main idea of the analysis of SquareImp presented in [2] is to charge the vertices in $A$ for preventing adjacent vertices in an optimum solution $A^*$ from being included into $A$. The latter is done by spreading the weight of the vertices in $A^*$ among their neighbors in the maximal independent set $A$ in such a way that no vertex in $A$ receives more than $\frac{2}{d}$ times its own weight. The suggested distribution of weights proceeds in two steps:

First, each vertex $u \in A^*$ invokes costs of $\frac{w(v)}{2}$ at each $v \in N(u, A)$, leaving a remaining weight of $w(u) - \frac{w(N(u,A))}{2}$ to be distributed. (Note that this term can be negative.)

In a second step, each vertex in $u$ sends an amount of $w(u) - \frac{w(N(u,A))}{2}$ to a heaviest neighbor it possesses in $A$, which is captured by the following definition of charges:

Definition 4 (charges [2]). Let $G = (V,E)$ be an undirected graph and let $w : V \to \mathbb{R}_{>0}$ be a positive weight function. Further assume that an independent set $A^* \subseteq V$ and a maximal independent set $A \subseteq V$ are given. We define a map charge : $A^* \times A \to \mathbb{R}$ as follows:

For each $u \in A^*$, pick a vertex $v \in N(u, A)$ of maximum weight and call it $n(u)$. Observe that this is possible, because $A$ is a maximal independent set in $G$, implying that $N(u, A) \neq \emptyset$, since either $u \in A$ or $u$ possesses a neighbor in $A$. 


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Next, for $u \in A^*$ and $v \in A$, define
\[
\text{charge}(u, v) := \begin{cases} 
    w(u) - \frac{1}{2} w(N(u, A)), & \text{if } v = n(u) \\
    0, & \text{otherwise}
\end{cases}
\]

The definition of charges directly implies the subsequent statement:

**Corollary 5** ([2]). In the situation of Definition 4, we have
\[
w(A^*) = \sum_{u \in A^*} \frac{w(N(u, A))}{2} + \sum_{u \in A^*} \text{charge}(u, n(u)) \\
\leq \sum_{u \in A^*} \frac{w(N(u, A))}{2} + \sum_{u \in A^*: \text{charge}(u, n(u)) > 0} \text{charge}(u, n(u)).
\]

The analysis proposed by Berman now proceeds by bounding the total weight sent to the vertices in $A$ during the two steps of the cost distribution separately. Lemma 6 bounds the weight received in the first step, while Lemma 7 and Lemma 8 take care of the total charges invoked. The following results appear in [2], but we have slightly modified the way they are formulated to suit our purposes. Matching proofs, which are partly easier than those presented in [2], can be found in Appendix A.

**Lemma 6** ([2]). In the situation of Definition 4, if the graph $G$ is $d$-claw free for some $d \geq 2$, then
\[
\sum_{u \in A^*} \frac{w(N(u, A))}{2} \leq \frac{d - 1}{2} \cdot w(A).
\]

**Lemma 7** ([2]). In the situation of Definition 4, for $u \in A^*$ and $v \in A$ with $\text{charge}(u, v) > 0$, we have
\[
w^2(u) - w^2(N(u, A) \setminus \{v\}) \geq 2 \cdot \text{charge}(u, v) \cdot w(v).
\]

**Lemma 8** ([2]). Let $G = (V, E)$ be $d$-claw free and $w : V \to \mathbb{R}_{>0}$. Let further $A^*$ be an independent set in $G$ of maximum weight and let $A$ be independent in $G$ with the property that no claw improves $w^2(A)$. Then for each $v \in A$, we have
\[
\sum_{u \in A^*: \text{charge}(u, v) > 0} \text{charge}(u, v) \leq \frac{w(v)}{2}.
\]

By combining Corollary 5 with the previous lemmata, one obtains Theorem 9 which states an approximation guarantee of $\frac{d}{2}$.

**Theorem 9** ([2]). Let $G = (V, E)$ be $d$-claw free and $w : V \to \mathbb{R}_{>0}$. Let further $A^*$ be an independent set in $G$ of maximum weight and let $A$ be independent in $G$ with the property that no claw improves $w^2(A)$. Then
\[
w(A^*) \leq \sum_{u \in A^*} \frac{w(N(u, A))}{2} + \sum_{u \in A^*: \text{charge}(u, n(u)) > 0} \text{charge}(u, n(u)) \leq \frac{d}{2} \cdot w(A).
\]
2.2 Results from [15]

We further reuse some of the definitions from [15]:

► **Definition 10** (Claw-shaped local improvement). Let \((G, w)\) be an instance of the MWIS in \(d\)-claw free graphs and let \(A \subseteq V(G)\) be independent. We call a local improvement \(X\) **claw-shaped** if \(|X| = 1\) and \(N(X, A) = \emptyset\) or if there is \(v \in A\) such that \(\{v\} \cup X\) induces a \(|X|\)-claw in \(G\) centered at \(v\), that is if \(X\) is the set of talons of some claw in \(G\) centered in \(A\), if the center is non-empty.

Note that if there is no claw-shaped improvement, we know that \(A\) is maximal and can define two fixed maps \(n : V\setminus A \to A\) mapping \(u\) to an element of \(N(u, A)\) of maximum weight and \(n_2 : \{u \in V\setminus A : |N(u, A)| \geq 2\} \to A\) mapping \(u\) to an element of \(N(u, A)\) of maximum weight.

► **Definition 11** (Circular improvement). Let \(\kappa \in (0, 1)\) with \(\frac{1}{\kappa} \in \mathbb{N}^+\). Let \((G, w)\) be an instance of the MWIS in \(d\)-claw free graphs, let \(A\) be a maximal independent set and let two fixed maps

- \(n : V\setminus A \to A\) mapping \(u\) to an element of \(N(u, A)\) of maximum weight and
- \(n_2 : \{u \in V\setminus A : |N(u, A)| \geq 2\} \to A\) mapping \(u\) to an element of \(N(u, A)\) of maximum weight

be given.

We call a local improvement \(X \subseteq V\setminus A\) **circular** if there exists \(U \subseteq X \cap \{u \in V\setminus A : |N(u, A)| \geq 2\}\) with \(|U| \leq \frac{2}{\kappa} \log(|V|)\) such that

(i) \(C := (\bigcup_{u \in U} \{n(u), n_2(u)\}, \{e_u = \{n(u), n_2(u)\}, u \in U\})\) is a cycle, where the edge set is considered as a multiset and two parallel edges are regarded as the edge set of a cycle of length 2.

(ii) If we define \(Y_v := \{x \in X \setminus U : n(x) = v\}\) for \(v \in A\), then \(|Y_v| \leq d - 1\) for all \(v \in A\) and moreover, \(X = U \cup \bigcup_{v \in V(C)} Y_v\).

(iii) For each \(u \in U:\)

\[
\begin{align*}
    w^2(u) + \frac{1}{2} \cdot w^2(Y_n(u) \cup Y_{n_2(u)}) &> \frac{w^2(n(u)) + w^2(n_2(u))}{2} \\
    &+ w^2(N(u, A) \setminus \{n(u), n_2(u)\}) \\
    &+ \frac{1}{2} \cdot \sum_{x \in Y_n(u)} w^2(N(x, A) \setminus \{n(u)\}) \\
    &+ \frac{1}{2} \cdot \sum_{x \in Y_{n_2}(u)} w^2(N(x, A) \setminus \{n_2(u)\})
\end{align*}
\]

Note that it has been shown in [15] how to search for circular improvements in quasi-polynomial, and, for the special case of \(k\)-Set Packing, in polynomial time.

After having recapitulated the results from [2] and [15] that we will reemploy in our analysis, we are now prepared to introduce and study our algorithm.

# 3 Improving the Approximation Factor

## 3.1 The Local Improvement Algorithm

► **Definition 12** (Local improvement). Given a \(d\)-claw free graph \(G = (V, E)\), a strictly positive weight function \(w : V \to \mathbb{R}_{>0}\), and an independent set \(A \subseteq V\), we call an independent set \(X \subseteq V\) a **local improvement** of \(w^2(A)\) of size \(|X|\) if \(w^2(A \setminus N(X, A) \cup X) > w^2(A)\).
Proposition 13. Let $G, w$ and $A$ be as in Definition 12. If $X$ is a local improvement of $w^2(A)$, then $A \setminus N(X, A) \cup X$ is independent in $G$.

Proof. As $A$ and $X$ are independent, no pair of vertices from $A \setminus N(X, A)$ or from $X$ can be adjacent. By definition of $N(X, A)$, there is no edge in $E$ connecting $A \setminus N(X, A)$ to $X$, which completes the proof.

Proposition 14. Let $G, w$ and $A$ be as in Definition 13. Then an independent set $X$ constitutes a local improvement of $A$ if and only if we have $w^2(N(X, A)) < w^2(X)$.

Proof. By Definition 1, we have $w^2(A, N(X, A) \cup X) = w^2(A) - w^2(N(X, A)) + w^2(X)$, implying the claim.

For each value of $d$, we fix two constants $\epsilon$ and $\xi$. They satisfy a bunch of inequalities that pop up during our analysis and are listed in Appendix B. The precise values we choose for these constants can be read from Table 1. We further need the following definition:

Definition 15 (helpful vertex). Let $v \in A$. We say that a vertex $u \in N(v, V \setminus A)$ is helpful for $v$ if

1. (i) $|N(u, A)| \geq 2$
   (ii) $v \in \{n(u), n_2(u)\}$
   (iii) $(1 + \epsilon)^{-1} \cdot \max\{w(u), w(n(u))\} \leq w(n_2(u)) \leq w(n(u)) \leq (1 + \epsilon) \cdot w(u)$
   (iv) $w(N(u, A) \setminus \{n(u), n_2(u)\}) \leq \epsilon \cdot w(u)$ or
   (v) $w(N(u, A) \setminus \{n(u)\}) \leq \epsilon \cdot w(u)$.

2. (i) $v = n(u)$ and
   (ii) $(1 + \epsilon)^{-1} \cdot w(u) \leq w(n(u)) \leq (1 + \epsilon) \cdot w(u)$ and
   (iii) $w(N(u, A) \setminus \{n(u)\}) \leq \epsilon \cdot w(u)$.

For $u \in V \setminus A$, we define $\text{help}(u) := \{v \in A : u$ is helpful for $v\}$.

Let MIS be a quasi-polynomial time $\rho$-approximation algorithm for the Maximum Cardinality Independent Set Problem in $d$-claw free graphs that can be implemented to run in polynomial time on conflict graphs of (known) $d - 1$-Set Packing instances. Note that the state-of-the-art algorithm by Fürer and Yu [8] satisfies this condition for $\rho = \frac{d + 2}{\log(k)}$ for some arbitrarily chosen, but fixed $\tilde{\epsilon} > 0$. In particular, we can assume $\rho \leq \frac{d + 2}{3}$. As it is NP-hard to approximate $k$-Set Packing even within $\Theta\left(\frac{k}{\log(k)}\right)$, it is reasonable to assume $\rho > 1$ in the following. All in all, we obtain

$$1 < \rho \leq \frac{d + \epsilon}{3}.$$  

(1)

The remainder of Section 3 is dedicated to the analysis of Algorithm 2 for the Maximum Weight Independent Set Problem in $d$-claw free graphs. The main result of this paper is given by the following theorem:

Theorem 16. For $d \geq 5$, Algorithm 2 yields an approximation guarantee of

$$\frac{d}{2} \cdot \xi \cdot \left(d - 2 - \frac{1}{d - 1} \cdot \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} - \frac{d - 2}{d - 1} \cdot \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - \epsilon^2}\right).$$

If one chooses $\kappa = \frac{1}{\sqrt{d - 1}}$, $\rho := \frac{d + \epsilon}{3}$, $\epsilon := 0.084$ and $\xi := 0.01$, this is bounded by $0.4986 \cdot d + 0.0202$. 


After having stated our main result, we want to start by convincing ourselves of the correctness of the latter bound. It yields the lowest asymptotic growth of the approximation guarantee at each point, we have \( w(A) \) equals zero initially and must increase by at least one in each iteration. On the other hand, we run Algorithm 2 with the integral weight function and rescale the weight function. Then, we delete vertices that we call it a polynomial number of times. To this end, observe that the values of \( w \) in lines 2, 5, 8, and 18 of Algorithm 3 are independent. For lines 2, 5, and 8, this is clear by the definition. As far as line 18 is concerned, note that \( A \) in lines 2, 5, 8 and 18 of Algorithm 3 is independent. This follows immediately from the fact that we maintain the property that \( A \) is independent throughout the algorithm because \( \emptyset \) is independent and Proposition \([13] \) tells us that none of our update steps can harm this invariant, provided that each of the sets \( X \) in lines 2, 5, 8, and 18 of Algorithm 3 is independent. For lines 2, 5, and 8, this is clear by the definition. As far as line 18 is concerned, note that \( X \subseteq (V(H)\setminus A) = V_{\text{help}} \) and that \( X \) is independent in \( H \) since it defines an independent set in an induced subgraph of \( H \). As \( H \) contains all edges in \( G \) between vertices from \( V_{\text{help}} \), it follows that \( X \) is independent in \( G \). Next, observe that Algorithm 2 is guaranteed to terminate because no set \( A \) can be attained twice, given that \( w^2(A) \) strictly increases in each iteration of the while-loop except the last one, and there are only finitely many possibilities. Furthermore, each iteration runs in quasi-polynomial (considering \( d \) a constant) time and, if \( G \) is the conflict graph of a given \( d - 1 \)-Set Packing instance, even in polynomial time. To see this, first note that we can check for a local improvement of size at most 3 and for a claw-shaped improvement in polynomial time \( O(|V|^{d-1} \cdot (|V| + |E|)) \), assuming \( d \geq 4 \). Moreover, we have seen in \([13] \) how to check for the existence of a circular improvement in quasi-polynomial, and, for the conflict graphs of set packing instances, in polynomial time. As also MIS itself runs in quasi-polynomial, and, for the conflict graphs of set packing instances, in polynomial time, it remains to see that we call it a polynomial number of times. To this end, observe that the values of \( A_{\geq L} \) and \( X \subseteq U \) only depend on \( \min \{ w(v) : w(v) \geq L, v \in V \} \) or \( \max \{ w(v) : w(v) \leq U, v \in V \} \), respectively, so it suffices to consider the \( |V| \) values \( \{ w(v), v \in V \} \) for \( L \) and \( U \).

In order to achieve a polynomial number of iterations, we scale and truncate the weight function as explained in \([5] \) and \([2] \). Given a constant \( N > 1 \), we first compute a greedy solution \( A' \) and rescale the weight function \( w \) such that \( w(A') = N \cdot |V| \) holds. Note that the greedy algorithm yields an approximation guarantee of \( d - 1 \), which is preserved when rescaling the weight function. Then, we delete vertices \( v \) of truncated weight \( |w(v)| = 0 \) and run Algorithm 2 with the integral weight function \( |w| \). In doing so, we know that \( |w|^2(A) \) equals zero initially and must increase by at least one in each iteration. On the other hand, at each point, we have

\[
|w|^2(A) \leq w^2(A) \leq (w(A))^2 \leq (d - 1)^2 \cdot w^2(A') = (d - 1)^2 \cdot N^2 \cdot |V|^2,
\]

```
Algorithm 2  Local improvement algorithm

Input: an undirected \( d \)-claw free graph \( G = (V, E) \), a positive weight function \( w : V \rightarrow \mathbb{R}_{>0} \)
Output: an independent set \( A \subseteq V \)
1  \( A \leftarrow \emptyset \)
2  \( continue \leftarrow \text{true} \)
3  \( \text{while} \ continue \ \text{do} \)
4    \( continue \leftarrow \text{RunIteration}(G, A, w) \)
5  \( \text{return} \ A \)
```

The latter bound yields the lowest asymptotic growth of the approximation guarantee achievable by our approach (up to the fact that the constants are rounded, of course). However, for small values of \( d \), this need not be optimum, and in particular, this value is not even smaller than \( d^2 \) for \( d \leq 14 \). For this reason, Table 7 displays (approximately) optimum choices of \( \epsilon \) and \( \xi \) and the resulting approximation guarantees for \( d \leq 14 \). After having stated our main result, we want to start by convincing ourselves of the correctness of Algorithm 2. First, note that Algorithm 2 is correct in the sense that it returns an independent set. This follows immediately from the fact that we maintain the property that \( A \) is independent throughout the algorithm because \( \emptyset \) is independent and Proposition \([13] \) tells us that none of our update steps can harm this invariant, provided that each of the sets \( X \) in lines 2, 5, 8, and 18 of Algorithm 3 is independent. For lines 2, 5, and 8, this is clear by the definition. As far as line 18 is concerned, note that \( X \subseteq (V(H)\setminus A) = V_{\text{help}} \) and that \( X \) is independent in \( H \) since it defines an independent set in an induced subgraph of \( H \). As \( H \) contains all edges in \( G \) between vertices from \( V_{\text{help}} \), it follows that \( X \) is independent in \( G \).
Algorithm 3 RunIteration($G, A, w$)

**Input:** an undirected $d$-claw free graph $G = (V, E)$, a positive weight function $w : V \rightarrow \mathbb{R}_{>0}$, $A \subseteq V(G)$ independent

**Output:** Whether a local improvement was found.

1. if $\exists$ local improvement $X$ such that $|X| \leq 3$ then
2. $A \leftarrow A \setminus N(X, A) \cup X$
3. return true

4. if $\exists$ claw-shaped improvement $X$ then
5. $A \leftarrow A \setminus N(X, A) \cup X$
6. return true

7. if $\exists$ circular improvement $X$ then
8. $A \leftarrow A \setminus N(X, A) \cup X$
9. return true

10. $w_{\text{min}} \leftarrow \min_{v \in V} w(v)$
11. $w_{\text{max}} \leftarrow \max_{v \in V} w(v)$
12. $V_{\text{help}} \leftarrow \{u \in V \setminus A : \text{help}(u) \neq \emptyset\}$
13. $H \leftarrow (A \cup V_{\text{help}}, \{(u, v) : u \in V_{\text{help}}, v \in A, v \in \text{help}(u)\} \cup E(G[V_{\text{help}}]))$
14. for $L \in [w_{\text{min}}, w_{\text{max}}]$ do
15. $A_{\geq L} \leftarrow \{v \in A : w(v) \geq L\}$
16. $V_{\geq L} \leftarrow A_{\geq L} \cup \{u \in V(H) \setminus A : w(u) \geq L \land N_H(u, A) \subseteq A_{\geq L}\}$
17. $X \leftarrow \text{MIS}(H[V_{\geq L}])$
18. $X \leftarrow A \setminus X$
19. for $U \in [w_{\text{min}}, w_{\text{max}}]$ do
20. $X \leq U \leftarrow \{u \in X : w(u) \leq U\}$
21. if $w^2(X \leq U) > w^2(N(X \leq U, A))$ then
22. $A \leftarrow A \setminus N(X \leq U, A) \cup X \leq U$
23. return true
24. return false

which bounds the total number of iterations by the latter term. Finally, if $r > 1$ specifies the approximation guarantee achieved by Algorithm 2, $A$ denotes the solution it returns and $A^*$ is an independent set of maximum weight with respect to the original respectively the scaled, but untruncated weight function $w$, we know that

$$r \cdot w(A) \geq r \cdot |w|(A) \geq |w|(A^*) \geq w(A^*) - |A^*| \geq w(A^*) - |V| \geq \frac{N - 1}{N} \cdot w(A^*),$$

so the approximation ratio increases by a factor of at most $\frac{N}{N-1}$.

### 3.2 Analysis of the Performance Ratio

We now move on to the analysis of the approximation guarantee. Denote some optimum solution by $A^*$ and denote the solution found by Algorithm 2 by $A$. Observe that by positivity of the weight function, $A$ must be a maximal independent set, as adding a vertex would certainly yield a claw-shaped local improvement of $w^2(A)$.

Our goal is to show Theorem 16 that is, we want to prove that for $d \geq 5$, Algorithm 2 yields
an approximation guarantee of
\[
\frac{d}{2} - \xi \cdot \left( d - 2 - \frac{1}{d-1} \cdot \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} - \frac{d - 2}{d - 1} \cdot \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - \epsilon^2} \right).
\]

We use some notation as well as most of the analysis of the algorithm SquareImp by Berman. In particular, we employ the same definition of neighborhoods and charges. Observe that this is well-defined since the solution \( A \) returned by our algorithm must constitute a maximal independent set in the given graph because when Algorithm 2 terminates, there is no more local improvement of size at most 3 present.

\begin{definition}[contribution] Define a contribution map
\[
\text{contr} : A^* \times A \to \mathbb{R}_{\geq 0}
\]
by setting
\[
\text{contr}(u, v) := \begin{cases} 
\max \left\{ 0, \frac{w^2(u) - w^2(N(u, A) \setminus \{v\})}{w(v)} \right\}, & \text{if } v \in N(u, A) \\
0, & \text{else} 
\end{cases}
\]
\end{definition}

\begin{proposition} For each \( v \in A \), we have \( \sum_{u \in A^*} \text{contr}(u, v) \leq w(v) \).
\end{proposition}

\begin{proof}
If \( v \in A^* \), this is true because \( N(v, A^*) = N(v, A) = \{v\} \) and \( \text{contr}(v, v) = w(v) \) in this case.

If \( v \notin A^* \), the set \( T \) of vertices sending positive contributions to \( v \) constitutes the set of talons of a claw centered at \( v \) and \( \sum_{u \in T} \text{contr}(u, v) > w(v) \) would imply that \( T \) constitutes a local improvement of \( w^2 \).
\end{proof}

\begin{proposition} For each \( u \in A^* \), we have
\[
\sum_{v \in A} \text{contr}(u, v) \geq \text{contr}(u, n(u)) \geq 2 \cdot \text{charge}(u, n(u)).
\]
\end{proposition}

\begin{proof}
The first inequality follows by non-negativity of the contribution, which also implies the second inequality in case \( \text{charge}(u, n(u)) \leq 0 \). If \( \text{charge}(u, n(u)) > 0 \), Lemma 7 provides the desired statement.
\end{proof}
Definition 20 (support). A support map is a map \( s : A^* \rightarrow 2^A \) with the property that for each \( u \in A^* \), we have \( s(u) \subseteq N(u, A) \).

To prove our main theorem, we consider the support map \( \text{supp}(u) \) that is given by

\[
\text{supp}(u) := \left\{ v \in N(u, A) \backslash \text{help}(u) : v \neq n(u) \lor \text{contr}(u, n(u)) \leq \frac{w(n(u))}{2} \right. \\
\left. \lor \left( 1 + \sqrt{\frac{\epsilon}{2}} \right) \cdot w(u) \leq w(n(u)) \right. \\
\left. \lor w(N(u, A) \backslash \{n(u)\}) > \epsilon \cdot w(u) \right\}.
\]

Definition 21 (missing neighbor). For \( v \in A \) with \( |N(v, A^*)| < d - 1 \), we say that \( v \) has \( d - 1 - |N(v, A^*)| \) missing neighbors.

We want to show that a large fraction (in terms of weight) of all vertices in \( A \) has only very few neighbors that are missing or not helpful for \( v \) (recall Definition 15). To this end, we need to see how a vertex \( v \) profits from neighbors of the aforementioned types.

Definition 22 (profit). Given a support map \( s \), define a profit map \( p : A^* \times A \rightarrow \mathbb{R}_{\geq 0} \) by

\[
p(u, x) := \begin{cases} 
\frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(s(u))}, & x \in s(u) \\
0, & \text{otherwise.}
\end{cases}
\]

Note that as all weights are positive, \( v \in s(u) \) implies \( 2 \cdot w(s(u)) > 0 \), so we do not divide by zero here.

Denote the profit map that arises from \( \text{supp} \) by profit.

Lemma 23. Let \( u \in A^* \) such that

= \( w(N(u, A) \backslash \{n(u)\}) \leq \epsilon \cdot w(u) \) and

= \( n(u) \in \text{supp}(u) \).

Then \( (1 + \epsilon) \cdot w(u) < w(n(u)) \).

Proof. First, note that we cannot have \( u \in A^* \cap A \): If this were the case, then \( u = n(u) \) and \( N(u, A) = \{n(u)\} \) and \( \text{contr}(u, n(u)) = w(u) = w(n(u)) \), implying that \( n(u) \notin \text{supp}(u) \), a contradiction. Hence, \( u \in V \backslash A \). As \( n(u) \in \text{supp}(u) \subseteq N(u, A) \backslash \text{help}(u) \), \( u \) is not helpful for \( n(u) \) and we must have \( w(n(u)) < (1 + \epsilon)^{-1} \cdot w(u) \) or \( (1 + \epsilon) \cdot w(u) < w(n(u)) \). Assume that we were in the first case. Then

\[
w^2(N(u, A)) = w^2(n(u)) + w^2(N(u, A) \backslash \{n(u)\}) \leq \frac{w^2(u)}{(1 + \epsilon)^2} + \epsilon^2 \cdot w^2(u)
\]

\[
= w^2(u) \cdot \frac{1 + \epsilon^2 + 2 \epsilon^3 + \epsilon^4}{1 + 2 \epsilon + \epsilon^2} \leq w^2(u).
\]

But this implies that \( \{u\} \) constitutes a claw-shaped improvement, a contradiction.

Lemma 24. Let \( u \in A^* \) such that

= \( w(N(u, A)) \leq (2 + \epsilon) \cdot w(u) \),

= \( w(N(u, A) \backslash \{n(u)\}) \leq \epsilon \cdot w(u) \), and

= \( (1 + \epsilon) \cdot w(u) < w(n(u)) \).

Then for \( z \in \text{supp}(u) \), we have \( \text{profit}(u, z) \geq \xi \).
Proof. If \( \text{supp}(u) = \emptyset \), we are done, so assume that this is not the case. As \( w(N(u, A) \setminus \{n(u)\}) \leq \epsilon \cdot w(u) \), we have

\[
\text{contr}(u, n(u)) \cdot w(n(u)) \geq w^2(u) - w^2(N(u, A) \setminus \{n(u)\}) \geq (1 - \epsilon^2) \cdot w^2(u).
\]

If further

\[
w(u) > \sqrt{\frac{1}{2 \cdot (1 - \epsilon^2)}} \cdot w(n(u)),
\]

then the previous estimate implies that

\[
\text{contr}(u, n(u)) \geq (1 - \epsilon^2) \cdot \frac{w^2(u)}{w(n(u))} \geq \frac{w(n(u))}{2}.
\]

By Definition 20, this yields \( n(u) \notin \text{supp}(u) \) or \( (1 + \sqrt{2}) \cdot w(u) \leq w(n(u)) \). We get

\[
\sum_{v \in N(u, A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u))
\]

\[
\geq \text{contr}(u, n(u)) \cdot w(n(u)) - (2 \cdot w(u) - w(N(u, A))) \cdot w(n(u))
\]

\[
\geq w^2(u) - w^2(N(u, A) \setminus \{n(u)\}) - (2 \cdot w(u) - w(N(u, A))) \cdot w(n(u))
\]

\[
\geq w^2(u) - w(N(u, A) \setminus \{n(u)\}) \cdot w(N(u, A) \setminus \{n(u)\}) - (2 \cdot w(u) - w(N(u, A))) \cdot w(n(u))
\]

\[
= (w(u) - w(n(u)))^2 + (w(n(u)) - w(N(u, A) \setminus \{n(u)\})) \cdot w(N(u, A) \setminus \{n(u)\}).
\]

In case \( (1 + \sqrt{2}) \cdot w(u) \leq w(n(u)) \), observing that

\[
w(N(u, A) \setminus \{n(u)\}) \leq \epsilon \cdot w(u) < \left(1 + \sqrt{2} \right) \cdot w(u) \leq w(n(u)),
\]

this leads to

\[
\sum_{v \in N(u, A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u))
\]

\[
\geq (w(u) - w(n(u)))^2 \geq \left( w(n(u)) - \frac{w(n(u))}{1 + \sqrt{2}} \right)^2
\]

\[
= \frac{\epsilon}{(1 + \sqrt{2})^2} \cdot w^2(n(u))
\]

\[
\geq \frac{\epsilon}{2 \cdot (1 + \sqrt{2})} \cdot w(n(u)) \cdot w(u)
\]

\[
\geq \frac{\epsilon}{2 \cdot (1 + \sqrt{2})} \cdot (2 + \epsilon) \cdot w(n(u)) \cdot w(N(u, A))
\]

\[\geq \xi \cdot w(n(u)) \cdot w(\text{supp}(u)).\]

On the other hand, if \( n(u) \notin \text{supp}(u) \), we obtain

\[
\sum_{v \in N(u, A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u))
\]

\[
\geq (w(n(u)) - \epsilon \cdot w(u)) \cdot w(N(u, A) \setminus \{n(u)\})
\]

\[
\geq (w(n(u)) - \epsilon \cdot w(n(u))) \cdot w(N(u, A) \setminus \{n(u)\}) \mid n(u) \notin \text{supp}(u)
\]

\[
\geq (1 - \epsilon) \cdot w(n(u)) \cdot w(\text{supp}(u)).
\]
This shows that for \( z \in \text{supp}(u) \), we get
\[
\text{profit}(u, z) = \sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \over 2 \cdot w(\text{supp}(u)) \\
\geq \left(1 - \epsilon \right) \cdot w(n(u)) \cdot w(\text{supp}(u)) \over 2 \cdot w(n(u)) = 1 - \epsilon \geq \frac{25}{2} \geq \xi.
\]
Finally, if we have
\[
w(u) \leq \sqrt{2 \cdot (1 - \epsilon^2) \cdot w(n(u))},
\]
then
\[
\sum_{v \in N(u, A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u)) \geq \text{profit}(u, z) \geq \left(1 - \sqrt{2 \cdot (1 - \epsilon^2)} \right) ^2 \cdot w^2(n(u)).
\]
Together with
\[
w(\text{supp}(u)) \leq w(N(u, A)) \leq w(n(u)) + \epsilon \cdot w(u) \leq \left(1 + \epsilon \cdot \sqrt{2 \cdot (1 - \epsilon^2)} \right) \cdot w(n(u)),
\]
this shows that for \( z \in \text{supp}(u) \), we get
\[
\text{profit}(u, z) = \sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \over 2 \cdot w(\text{supp}(u)) \\
\geq \left(1 - \sqrt{2 \cdot (1 - \epsilon^2)} \right) ^2 \cdot w^2(n(u)) \over 2 \cdot \left(1 + \epsilon \cdot \sqrt{2 \cdot (1 - \epsilon^2)} \right) \cdot w^2(n(u)) = \left(1 - \sqrt{2 \cdot (1 - \epsilon^2)} \right) ^2 \geq \frac{25}{2} \geq \xi.
\]

\begin{lemma}
For each \( u \in A^* \), we have
\[
\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \geq w(N(u, A)) - 2 \cdot w(u).
\]
Moreover, for each \( u \in A^* \) with \( |N(u, A)| \geq 2 \), we have the following:
\[
\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \\
\geq \frac{[w(u) - w(n(u))]^2 + (w(n(u)) - w(n_2(u))) \cdot (w(N(u, A)) - w(n(u))}{w(n(u))}
\]
\end{lemma}
\[
\sum_{v \in N(u,A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \\
\geq \frac{2 \cdot w(u) \cdot (w(u) - w(n(u))) + w(n_2(u)) \cdot (w(n(u)) - w(n_2(u)))}{w(n(u))} \\
+ \frac{w(N(u,A) \setminus \{n(u), n_2(u)\}) \cdot (w(n(u)) - 2 \cdot w(N(u,A) \setminus \{n(u), n_2(u)\}))}{w(n(u))}
\]

(4)

\[
\sum_{v \in N(u,A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \\
\geq \frac{(w(u) - w(n(u)))^2}{w(n(u))} \\
+ \frac{(w(n(u)) - \min\{w(n_2(u)), w(N(u,A) \setminus \{n(u), n_2(u)\})\}) \cdot w(N(u,A) \setminus \{n(u), n_2(u)\})}{w(n(u))}
\]

(5)

**Proof.** (4):

\[
\sum_{v \in N(u,A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \geq -2 \cdot \text{charge}(u, n(u)) = w(N(u,A)) - 2 \cdot w(u)
\]

by non-negativity of the contribution.

(5):

\[
\sum_{v \in N(u,A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u)) \\
\geq \text{contr}(u, n(u)) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u)) \\
\geq w^2(u) - w^2(N(u,A) \setminus \{n(u)\}) - (2 \cdot w(u) - w(N(u,A))) \cdot w(n(u)) \\
\geq w^2(u) - w(n_2(u)) \cdot (w(N(u,A)) - w(n(u))) - (2 \cdot w(u) - w(N(u,A))) \cdot w(n(u)) \\
= w^2(u) - w(n(u)) \cdot (w(N(u,A)) - w(n(u))) - (2 \cdot w(u) - w(N(u,A))) \cdot w(n(u)) \\
+ (w(n(u)) - w(n_2(u))) \cdot (w(N(u,A)) - w(n(u))) \\
= (w(u) - w(n(u)))^2 + (w(n(u)) - w(n_2(u))) \cdot (w(N(u,A)) - w(n(u))).
\]

Division by \(w(n(u)) > 0\) yields the claim.
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\[ \sum_{v \in N(u,A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u)) \]
\[ \geq \text{contr}(u, n(u)) \cdot w(n(u)) + \text{contr}(u, n_2(u)) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u)) \]
\[ \geq w^2(u) - w^2(N(u,A) \setminus \{n(u)\}) - w^2(n_2(u)) \]
\[ - (2 \cdot w(u) - w(N(u,A))) \cdot w(n(u)) \]
\[ = 2 \cdot w^2(u) - 2 \cdot w^2(N(u,A) \setminus \{n(u), n_2(u)\}) - w^2(n_2(u)) \]
\[ = 2 \cdot w^2(u) - 2 \cdot w(u) \cdot w(n(u)) - 2 \cdot w^2(N(u,A) \setminus \{n(u), n_2(u)\}) \]
\[ + w(n(u)) \cdot w(n_2(u)) \cdot w(N(u,A) \setminus \{n(u), n_2(u)\}) \cdot w(n(u)) \]
\[ \geq 2 \cdot w(u) \cdot (w(u) - w(n(u))) + w(n_2(u)) \cdot (w(n(u)) - w(n_2(u))) \]
\[ + w(N(u,A) \setminus \{n(u), n_2(u)\}) \cdot (w(n(u)) - 2 \cdot w(N(u,A) \setminus \{n(u), n_2(u)\})) \]

Again, division by $w(n(u)) > 0$ yields the claim.

\[ \sum_{v \in N(u,A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u)) \]
\[ \geq \text{contr}(u, n(u)) \cdot w(n(u)) + \text{contr}(u, n_2(u)) \cdot w(n(u)) \]
\[ \geq w^2(u) - w^2(N(u,A) \setminus \{n(u)\}) - w^2(n_2(u)) \]
\[ - (2 \cdot w(u) - w(N(u,A))) \cdot w(n(u)) \]
\[ = (w(u) - w(n(u)))^2 + (w(n(u)) - w(n_2(u))) \cdot w(n_2(u)) \]
\[ + w(n(u)) \cdot w(N(u,A) \setminus \{n(u), n_2(u)\}) - w^2(N(u,A) \setminus \{n(u), n_2(u)\}) \]
\[ \geq (w(u) - w(n(u)))^2 + w(n(u)) \cdot w(N(u,A) \setminus \{n(u), n_2(u)\}) \]
\[ - \min\{w(n_2(u)), w(N(u,A) \setminus \{n(u), n_2(u)\})\} \cdot w(N(u,A) \setminus \{n(u), n_2(u)\}) \]
\[ = (w(u) - w(n(u)))^2 \]
\[ + (w(n(u)) - \min\{w(n_2(u)), w(N(u,A) \setminus \{n(u), n_2(u)\})\}) \cdot w(N(u,A) \setminus \{n(u), n_2(u)\}) \]

Here, the inequality $\max_{x \in N(u,A) \setminus \{n(u), n_2(u)\}} w(x) \leq \min\{w(n_2(u)), w(N(u,A) \setminus \{n(u), n_2(u)\})\}$ follows since $n(u)$ and $n_2(u)$ are two vertices in $N(u,A)$ of maximum weight and by non-negativity of weights. Once more, division by $w(n(u)) > 0$ yields the claim.

\[ \textbf{Lemma 26.} \ Let u \in A^* \text{ and } z \in A \text{ such that } z \in \text{supp}(u). \text{ Then profit}(u,z) \geq \xi. \]

\[ \textbf{Proof.} \ Case 1: w(N(u,A)) \geq (2 + \epsilon) \cdot w(u). \]
Then, for \(z \in \text{supp}(u)\), we obtain
\[
\text{profit}(u, z) = \sum_{v \in N(u,A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \geq \frac{w(N(u, A)) - 2 \cdot w(u)}{2 \cdot w(\text{supp}(u))} \geq \frac{1}{2} - \frac{1}{2 + \epsilon} = \frac{\epsilon}{4 + 2\epsilon} \geq \xi.
\]
Hence, we can assume that \(w(N(u, A)) < (2 + \epsilon) \cdot w(u)\) in the following.
Note that if \(|N(u,A)| = 1\), then \(v \in N(u,A) = \{n(u)\}\), implying that \(n(u) = v \in \text{supp}(u)\).
Hence, we get \(w(N(u, A)) = w(0) = 0 \leq \epsilon \cdot w(u)\).

Hence, we can apply Lemma \[23\] and Lemma \[24\] to conclude the statement of the lemma. As \(A\) is maximal, we can, therefore, assume \(|N(u,A)| \geq 2\) in the following. Observe that this in particular implies \(u \in A'\setminus A\).

**Case 2:** \(w(n(u)) > (1 + \epsilon) \cdot w(u)\).

Then \(w(N(u, A)\setminus \{n(u)\}) = w(N(u, A)) - w(n(u)) < (2 + \epsilon) \cdot w(u) - (1 + \epsilon) \cdot w(u) = w(u)\).

If \(w(N(u, A)) - w(n(u)) \leq \epsilon \cdot w(u)\), then Lemma \[24\] yields the desired statement. So assume that \(w(N(u, A)) - w(n(u)) > \epsilon \cdot w(u)\).

Our goal is to apply \[3\]. We have
\[
(w(n(u)) - w(n_2(u)))(w(N(u, A)) - w(n(u))) \geq (w(n(u)) - w(n_2(u))) \cdot \max\{\epsilon \cdot w(u), w(n_2(u))\}.
\]
As a function of \(w(n_2(u))\), the right hand side attains its minimum on the interval \([0, \epsilon \cdot w(u)]\) at \(\epsilon \cdot w(u)\), and is concave on the interval \([\epsilon \cdot w(u), w(u)]\). Recall that
\[
w(n_2(u)) \leq w(N(u, A)\setminus \{n(u)\}) < w(u).
\]
Hence, we get
\[
w(n(u)) - w(n_2(u)) \cdot \max\{\epsilon \cdot w(u), w(n_2(u))\} \\
\geq \min\{w(n(u)) - \epsilon \cdot w(u), \epsilon \cdot w(u) \cdot (w(n(u)) - w_2(u)) \cdot w(u)\} \\
= w(u) \cdot w(n(u)) \cdot \min\left\{\left(1 - \epsilon \cdot \frac{w(u)}{w(n(u))}\right) \cdot \epsilon, 1 - \frac{w(u)}{w(n(u))}\right\} \\
\geq w(u) \cdot w(n(u)) \cdot \min\left\{\left(1 - \frac{\epsilon}{1 + \epsilon}\right) \cdot \epsilon, 1 - \frac{1}{1 + \epsilon}\right\} \\
= \frac{\epsilon}{1 + \epsilon} \cdot w(u) \cdot w(n(u))
\]
since \(w(n(u)) > (1 + \epsilon) \cdot w(u)\) by assumption. As a consequence, for \(z \in \text{supp}(u)\),
\[
\text{profit}(u, z) = \sum_{v \in N(u,A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \\
\leq \frac{w(N(u, A)) - w(n_2(u)) \cdot (w(N(u, A)) - w(n(u)))}{2 \cdot w(\text{supp}(u))} \\
\geq \frac{\epsilon \cdot w(u)}{2 \cdot (1 + \epsilon) \cdot w(N(u, A))} \\
> \frac{\epsilon}{2 \cdot (1 + \epsilon) \cdot (2 + \epsilon)} \geq \xi.
\]
Case 3: \( w(n_2(u)) < (1 + \epsilon)^{-1} \cdot w(u) \leq w(n(u)) \leq (1 + \epsilon) \cdot w(u) \).

Case 3.1: \( w(N(u, A)) - w(n(u)) \leq \epsilon \cdot w(u) \)

Then \( u \) is helpful for \( n(u) \) and \( \text{supp}(u) \subseteq N(u, A) \setminus \{n(u)\} \), implying that

\[
w(\text{supp}(u)) \leq w(N(u, A)) - w(n(u)).
\]

As a consequence, for \( z \in \text{supp}(u) \),

\[
\text{profit}(u, z) = \frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(\text{supp}(u))} \geq \frac{(w(n(u)) - w(n_2(u))) \cdot (w(N(u, A)) - w(n(u)))}{2 \cdot w(n(u))} \geq \frac{w(n(u)) - w(N(u, A) \setminus \{n(u)\})}{2 \cdot w(n(u))} \geq \frac{w(n(u)) - \epsilon \cdot w(u)}{2 \cdot w(n(u))} \geq \frac{w(n(u)) \cdot (1 - \epsilon (1 + \epsilon))}{2} = \frac{1 - \epsilon (1 + \epsilon)}{2} \geq \frac{1}{1 + \epsilon}.
\]

Case 3.2: \( w(N(u, A)) - w(n(u)) > \epsilon \cdot w(u) \) and \( w(n_2(u)) < \frac{1}{1 + \epsilon} \cdot w(n(u)) \)

We have

\[
w(N(u, A)) - w(n(u)) > \epsilon \cdot w(u) \geq \frac{\epsilon}{1 + \epsilon} \cdot w(n(u)).
\]

By using the same reasoning as in Case 2, we get

\[
(w(n(u)) - w(n_2(u))) \cdot (w(N(u, A)) - w(n(u))) \geq (w(n(u)) - w(n_2(u))) \cdot \max \left\{ \frac{\epsilon}{1 + \epsilon} \cdot w(n(u)), w(n_2(u)) \right\} \geq \min \left\{ \left( w(n(u)) - \frac{1}{1 + \epsilon} \cdot w(n(u)) \right) \cdot \frac{1}{1 + \epsilon} \cdot w(n(u)), \left( w(n(u)) - \frac{\epsilon}{1 + \epsilon} \cdot w(n(u)) \right) \cdot \frac{\epsilon}{1 + \epsilon} \cdot w(n(u)) \right\} = \frac{\epsilon}{(1 + \epsilon)^2} \cdot w^2(n(u)).
\]

As a consequence, for \( z \in \text{supp}(u) \),

\[
\text{profit}(u, z) = \frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(\text{supp}(u))} \geq \frac{(w(n(u)) - w(n_2(u))) \cdot (w(N(u, A)) - w(n(u)))}{2 \cdot w(n(u)) \cdot w(\text{supp}(u))} \geq \frac{\epsilon \cdot w^2(n(u))}{2 \cdot (1 + \epsilon)^2 \cdot w(n(u))} \geq \frac{\epsilon}{2 \cdot (1 + \epsilon)^3} \geq \frac{1}{1 + \epsilon} \cdot w(n(u)).
\]

since \( w(N(u, A)) < (2 + \epsilon) \cdot w(u) \leq (2 + \epsilon) \cdot (1 + \epsilon) \cdot w(n(u)) \).

Case 3.3: \( w(N(u, A)) - w(n(u)) > \epsilon \cdot w(u) \) and \( w(n_2(u)) \geq \frac{1}{1 + \epsilon} \cdot w(n(u)) \).

Then

\[
w(n_2(u)) \leq w(n(u)) \leq (1 + \epsilon) \cdot w(n_2(u)) < (1 + \epsilon) \cdot (1 + \epsilon)^{-1} \cdot w(u) = w(u). \quad (6)
\]
Using this, we obtain

\[
\sum_{v \in N(u,A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u))
\]

\[\geq 2 \cdot w(u) \cdot (w(u) - w(n(u))) + w(n_2(u)) \cdot (w(n(u)) - w(n_2(u)))
+ w(N(u,A) \setminus \{n(u), n_2(u)\}) \cdot (w(n(u)) - 2 \cdot w(N(u,A) \setminus \{n(u), n_2(u)\}))
\]

\[\geq 2 \cdot w(n_2(u)) \cdot (w(u) - w(n(u))) + w(n_2(u)) \cdot (w(n(u)) - w(n_2(u)))
+ w(N(u,A) \setminus \{n(u), n_2(u)\}) \cdot (w(n(u)) - 2 \cdot w(N(u,A) \setminus \{n(u), n_2(u)\}))
\]

\[= w(n_2(u)) \cdot (w(u) - w(n_2(u)))
+ w(N(u,A) \setminus \{n(u), n_2(u)\}) \cdot (w(n(u)) - 2 \cdot w(N(u,A)) + 2 \cdot w(n(u)) + 2 \cdot w(n_2(u)))
\]

\[= w(n_2(u)) \cdot (w(u) - w(n_2(u)))
+ w(N(u,A) \setminus \{n(u), n_2(u)\}) \cdot (w(n(u)) - 2 \cdot w(N(u,A)))
\]

\[= w(n_2(u)) \cdot (w(u) - w(n_2(u)))
+ w(N(u,A) \setminus \{n(u), n_2(u)\}) \cdot (3 + 2 \cdot (1 + \epsilon)^{-1}) \cdot w(n(u)) - 2 \cdot (2 + \epsilon) \cdot w(u)
\]

\[= w(n_2(u)) \cdot (w(u) - w(n_2(u)))
\]

\[\geq (1 + \epsilon)^{-1} \cdot w(n(u)) \cdot (w(u) - (1 + \epsilon)^{-1} \cdot w(u))
\]

\[= \frac{\epsilon}{(1 + \epsilon)^2} \cdot w(n(u)) \cdot w(u).
\]

This implies that for \(z \in \text{supp}(u),\)

\[
\text{profit}(u, z) = \frac{\sum_{v \in N(u,A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(\text{supp}(u))}
\]

\[\geq \frac{\epsilon \cdot w(u) \cdot w(n(u))}{2 \cdot (1 + \epsilon)^2 \cdot w(n(u)) \cdot w(N(u,A))}
\]

\[= \frac{\epsilon \cdot w(u)}{2 \cdot (1 + \epsilon)^2 \cdot (w(N(u,A)))} \geq 2 \cdot (1 + \epsilon)^2 \cdot (2 + \epsilon) \geq \zeta.
\]

**Case 4:** \((1 + \epsilon)^{-1} \cdot w(u) \leq w(n_2(u)) \leq w(n(u)) \leq (1 + \epsilon) \cdot w(u)
**Case 4.0:** \(w(n_2(u)) < (1 + \epsilon)^{-1} \cdot w(n(u)).

As \((1 + \epsilon)^{-1} \cdot w(u) \leq w(n_2(u)) < (1 + \epsilon)^{-1} \cdot w(n(u)),\) we must have \(w(u) < w(n(u)).\) By \([3]\), we obtain

\[
\sum_{v \in N(u,A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u))
\]

\[\geq (w(u) - w(n(u)))^2 + (w(n(u)) - w(n_2(u))) \cdot (w(N(u,A)) - w(n(u)))
\]

\[\geq (w(n(u)) - w(n_2(u))) \cdot w(n_2(u))
\]

\[\geq (1 - (1 + \epsilon)^{-1}) \cdot w(n(u)) \cdot (1 + \epsilon)^{-1} \cdot w(u)
\]

\[= (1 - (1 + \epsilon)^{-1}) \cdot (1 + \epsilon)^{-1} \cdot w(n(u)) \cdot w(u).
\]
For $z \in \text{supp}(u)$, this yields
\[
\text{profit}(u, z) = \frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(\text{supp}(u))} \geq \frac{(1 - (1 + \epsilon)^{-1}) \cdot (1 + \epsilon)^{-1} \cdot w(n(u)) \cdot w(u)}{2 \cdot w(n(u)) \cdot w(N(u, A))} \geq \frac{(1 - (1 + \epsilon)^{-1}) \cdot (1 + \epsilon)^{-1} \cdot w(u)}{2 \cdot (2 + \epsilon) \cdot w(u)} \geq \epsilon \cdot 2^{25} \geq \xi.
\]

**Case 4.1:** $(1 + \epsilon)^{-1} \cdot w(n(u)) \leq w(n_2(u))$ and $w(N(u, A) \setminus \{n(u), n_2(u)\}) \leq \epsilon \cdot w(u)$:

Then $u$ is helpful for both $n(u)$ and $n_2(u)$ and we get $\text{supp}(u) \subseteq N(u, A) \setminus \{n(u), n_2(u)\}$.

**Case 4.1.1**: $w(n(u)) \geq w(u)$. Now, if
\[
w(n(u)) - w(n_2(u)) \geq \frac{w(N(u, A) \setminus \{n(u), n_2(u)\})}{2},
\]
then for $z \in \text{supp}(u)$,
\[
\text{profit}(u, z) = \frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(\text{supp}(u))} \geq \frac{(w(n(u)) - w(n_2(u))) \cdot (w(N(u, A) - w(n(u)))}{2 \cdot w(n(u)) \cdot w(\text{supp}(u))} \geq \frac{(w(n(u)) - w(n_2(u))) \cdot w(N(u, A) \setminus \{n(u), n_2(u)\})}{2} \geq \frac{w(N(u, A) \setminus \{n(u), n_2(u)\})}{2} \cdot (1 + \epsilon)^{-1} \cdot w(n(u)) \Rightarrow \frac{1}{4 \cdot (1 + \epsilon)} \geq \xi.
\]

On the other hand, if $w(n(u)) - w(n_2(u)) < \frac{w(N(u, A) \setminus \{n(u), n_2(u)\})}{2}$, then
\[
w(N(u, A)) = w(n(u)) + w(n_2(u)) + w(N(u, A) \setminus \{n(u), n_2(u)\}) = 2 \cdot w(n(u)) - (w(n(u)) - w(n_2(u))) + w(N(u, A) \setminus \{n(u), n_2(u)\}) \geq 2 \cdot w(n(u)) - w(N(u, A) \setminus \{n(u), n_2(u)\}) + w(N(u, A) \setminus \{n(u), n_2(u)\}) \geq 2 \cdot w(u) + \frac{w(N(u, A) \setminus \{n(u), n_2(u)\})}{2}.
\]

This implies
\[
\text{profit}(u, z) = \frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(\text{supp}(u))} \geq \frac{w(N(u, A)) - 2 \cdot w(u)}{2 \cdot w(\text{supp}(u))} \geq \frac{w(N(u, A))}{2} - w(u) \geq \frac{w(N(u, A) \setminus \{n(u), n_2(u)\})}{w(N(u, A) \setminus \{n(u), n_2(u)\})} \geq \frac{1}{4} \geq \xi.
\]
Case 4.1.2: \( w(n(u)) < w(u) \).

Then
\[
\begin{align*}
w(n_2(u)) & \leq w(n(u)) < w(u).
\end{align*}
\]

Hence,
\[
\begin{align*}
\sum_{v \in N(u,A)} \text{contr}(u,v) \cdot w(n(u)) & - 2 \cdot \text{charge}(u,n(u)) \cdot w(n(u)) \\
& \geq 2 \cdot w(u) \cdot (w(n(u)) - w(n_2(u))) + w(N(u,A) \setminus \{n(u),n_2(u)\}) \cdot (w(n(u)) - w(n_2(u))) \\
& + w(N(u,A) \setminus \{n(u),n_2(u)\}) \cdot (w(n(u)) - 2 \cdot w(N(u,A) \setminus \{n(u),n_2(u)\})) \\
& \geq w(N(u,A) \setminus \{n(u),n_2(u)\}) \cdot w(n(u)) - 2 \cdot w(N(u,A) \setminus \{n(u),n_2(u)\}) \\
& \geq w(N(u,A) \setminus \{n(u),n_2(u)\}) \cdot w(n(u)) - 2 \cdot \epsilon \cdot w(u) \\
& = w(N(u,A) \setminus \{n(u),n_2(u)\}) \cdot w(n(u)) \cdot (1 - 2 \cdot \epsilon \cdot (1 + \epsilon)).
\end{align*}
\]

This yields
\[
\begin{align*}
\text{profit}(u,z) & = \frac{\sum_{v \in N(u,A)} \text{contr}(u,v) - 2 \cdot \text{charge}(u,n(u))}{2 \cdot w(\text{supp}(u))} \\
& = \frac{\left( \sum_{v \in N(u,A)} \text{contr}(u,v) - 2 \cdot \text{charge}(u,n(u)) \right) \cdot w(n(u))}{2 \cdot w(N(u,A)) \cdot w(\text{supp}(u))} \\
& \geq \frac{w(N(u,A) \setminus \{n(u),n_2(u)\}) \cdot w(n(u)) \cdot (1 - 2 \cdot \epsilon \cdot (1 + \epsilon))}{2 \cdot w(N(u,A)) \cdot w(N(u,A) \setminus \{n(u),n_2(u)\})} \\
& = \frac{1 - 2 \cdot \epsilon \cdot (1 + \epsilon)}{2} \geq \xi.
\end{align*}
\]

Case 4.2: \((1 + \epsilon)^{-1} \cdot w(n(u)) \leq w(n_2(u))\) and \( w(N(u,A) \setminus \{n(u),n_2(u)\}) > \epsilon \cdot w(u) \):

We want to apply (5). As \( w(N(u,A)) < (2 + \epsilon) \cdot w(u) \) and
\[
(1 + \epsilon)^{-1} \cdot w(u) \leq w(n_2(u)) \leq w(n(u)),
\]
we have
\[
\begin{align*}
w(n(u)) & - w(N(u,A) \setminus \{n(u),n_2(u)\}) \\
& \geq w(n(u)) - ((2 + \epsilon) \cdot w(u) - w(n(u)) - (1 + \epsilon)^{-1} \cdot w(u)) \\
& = 2 \cdot w(n(u)) - (2 + \epsilon - (1 + \epsilon)^{-1}) \cdot w(u) \\
& \geq (2 - ((2 + \epsilon) - (1 + \epsilon)^{-1}) \cdot (1 + \epsilon)) \cdot w(n(u)) \\
& = (3 - (2 + \epsilon) \cdot (1 + \epsilon)) \cdot w(n(u)).
\end{align*}
\]

Hence,
\[
\begin{align*}
\text{profit}(u,z) & = \frac{\sum_{v \in N(u,A)} \text{contr}(u,v) - 2 \cdot \text{charge}(u,n(u))}{2 \cdot w(\text{supp}(u))} \\
& \geq \frac{\left( w(n(u)) - w(N(u,A) \setminus \{n(u),n_2(u)\}) \right) \cdot w(N(u,A) \setminus \{n(u),n_2(u)\})}{2 \cdot w(N(u,A))} \\
& \geq \frac{(3 - (2 + \epsilon) \cdot (1 + \epsilon)) \cdot w(n(u)) \cdot \epsilon \cdot w(u)}{2 \cdot w(n(u)) \cdot (2 + \epsilon) \cdot w(u)} \\
& \geq \frac{(3 - (2 + \epsilon) \cdot (1 + \epsilon)) \cdot \epsilon}{4 + 2\epsilon} \geq \xi.
\end{align*}
\]
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Case 5: $w(n_2(u)) \leq w(n(u)) < (1 + \epsilon)^{-1} \cdot w(u)$:

Case 5.1: $2 \cdot w(N(u, A) \setminus \{n(u), n_2(u)\}) \leq w(n(u))$:

We get

$$\sum_{v \in N(u, A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u))$$

This leads to

$$\text{profit}(u, z) = \frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(\text{supp}(u))}$$

$$= \left(\frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(n(u)) \cdot w(\text{supp}(u))}\right) \cdot w(n(u))$$

$$\geq \frac{2 \cdot w(u) \cdot (w(n(u)) - w(n(u)))}{2 \cdot w(n(u)) \cdot w(N(u, A))}$$

$$= \frac{w(u) \cdot \left(\frac{w(u)}{w(n(u))} - 1\right)}{w(N(u, A))}$$

$$\geq \frac{w(u) \cdot (1 + \epsilon - 1)}{(2 + \epsilon) \cdot w(u)}$$

$$\geq \frac{\epsilon}{2 + \epsilon} \leq \xi.$$

Case 5.2: $2 \cdot w(N(u, A) \setminus \{n(u), n_2(u)\}) > w(n(u))$:

We get

$$\sum_{v \in N(u, A)} \text{contr}(u, v) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u))$$

$$\geq \text{contr}(u, n(u)) \cdot w(n(u)) - 2 \cdot \text{charge}(u, n(u)) \cdot w(n(u))$$

$$\geq (w(u) - w(n(u)))^2$$

$$+ (w(n(u)) - \min\{w(n_2(u)), w(N(u, A) \setminus \{n(u), n_2(u)\})\}) \cdot w(N(u, A) \setminus \{n(u), n_2(u)\})$$

Note that both summands are non-negative since all weights are positive and $w(n_2(u)) \leq w(n(u))$.

Case 5.2.1: $w(n(u)) \leq \epsilon \cdot w(u) + \min\{w(n_2(u)), w(N(u, A) \setminus \{n(u), n_2(u)\})\}$:

Then

$$3 \cdot w(n(u)) - 2 \cdot \epsilon \cdot w(u) \leq w(n(u)) + w(n_2(u)) + w(N(u, A) \setminus \{n(u), n_2(u)\})$$

$$= w(N(u, A)) \leq (2 + \epsilon) \cdot w(u)$$
and, hence, \( w(n(u)) \leq \left( \frac{2}{3} + \varepsilon \right) \cdot w(u) \). As a consequence, we obtain, for \( z \in \text{supp}(u) \),

\[
\text{profit}(u, z) = \frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(\text{supp}(u))} \geq \frac{w(u) - w(n(u))}{2 \cdot w(n(u)) \cdot w(N(u, A))} \geq \frac{(\frac{1}{2} - \varepsilon)^2 \cdot (w(u))^2}{2 \cdot (\frac{2}{3} + \varepsilon) \cdot w(u) \cdot (2 + \varepsilon) \cdot w(u)} = \frac{(\frac{1}{2} - \varepsilon)^2}{4 \cdot (2 + \varepsilon) \cdot (2 + \varepsilon)} \geq \xi
\]

Case 5.2.2: \( w(n(u)) > \varepsilon \cdot w(u) + \min\{w(n_2(u), w(N(u, A) \setminus \{n(u), n_2(u)\})\}) \):

Recall that by assumption on Case 5.2, we also have

\[
w(N(u, A) \setminus \{n(u), n_2(u)\}) > \frac{w(n(u))}{2}.
\]

Therefore, we obtain, for \( z \in \text{supp}(u) \),

\[
\text{profit}(u, z) = \frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{2 \cdot w(\text{supp}(u))} \geq \frac{w(n(u)) - \min\{w(n_2(u), w(N(u, A) \setminus \{n(u), n_2(u)\})\}) \cdot w(N(u, A) \setminus \{n(u), n_2(u)\})}{2 \cdot w(n(u)) \cdot w(N(u, A))} \geq \frac{\varepsilon \cdot w(n(u))}{2 \cdot w(n(u)) \cdot (2 + \varepsilon) \cdot w(u)} = \frac{\varepsilon}{4 \cdot (2 + \varepsilon)} \geq \xi
\]

Lemma 27. Let \( s \) be a support map and let \( p \) be the corresponding profit map. Then we have

\[
w(A^*) \leq \frac{d}{2} \cdot w(A) - \sum_{v \in A} \sum_{u \in A^*} p(u, v) \cdot w(v) - \sum_{v \in A} \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v).
\]

Proof. By definition of charges, we have

\[
w(A^*) = \sum_{u \in A^*} \frac{w(N(u, A))}{2} + w(u) - \frac{w(N(u, A))}{2} = \sum_{u \in A^*} \frac{w(N(u, A))}{2} + \text{charge}(u, n(u)) = \sum_{v \in A} \frac{|N(v, A^*)|}{2} \cdot w(v) + \sum_{u \in A^*} \text{charge}(u, n(u)) = \frac{d - 1}{2} \cdot w(A) - \sum_{v \in A} \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v) + \sum_{u \in A^*} \text{charge}(u, n(u)).
\]

As a consequence, we need to show that

\[
\sum_{u \in A^*} \text{charge}(u, n(u)) \leq \frac{w(A)}{2} - \sum_{v \in A} \sum_{u \in A^*} p(u, v) \cdot w(v)
\]
to establish the claim. This is equivalent to
\[ \sum_{u \in A^*} 2 \cdot \text{charge}(u, n(u)) + \sum_{u \in A^*} \sum_{v \in A} 2 \cdot p(u, v) \cdot w(v) \leq w(A). \]

We obtain
\[
\begin{align*}
\sum_{u \in A^*} 2 \cdot \text{charge}(u, n(u)) &+ \sum_{u \in A^*} \sum_{v \in E} 2 \cdot p(u, v) \cdot w(v) \\
&= \sum_{u \in A^*} 2 \cdot \text{charge}(u, n(u)) \\
&+ \sum_{u \in A^*} \sum_{v \in E: s(u) \neq \emptyset} \sum_{v \in N(u, A)} \frac{\text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u))}{w(s(u))} \cdot w(v) \\
&= \sum_{u \in A^*} 2 \cdot \text{charge}(u, n(u)) \\
&+ \sum_{u \in A^*} \sum_{v \in E: s(u) \neq \emptyset} \sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, n(u)) \leq w(A).
\end{align*}
\]

This finishes the proof.

\textbf{Corollary 28.}

\[ w(A^*) \leq \frac{d}{2} \cdot w(A) - \sum_{v \in A} \sum_{u \in A^*} \text{profit}(u, v) \cdot w(v) - \sum_{v \in A} \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v). \]

\textbf{Definition 29.} In account with Algorithm 3, define the graph \( H \) as follows:
- \( V(H) := A \cup \{ u \in V \setminus A : \text{help}(u) \neq \emptyset \} \)
- \( E(H) := \{(u, v) : u \in V \setminus A, v \in A, v \in \text{help}(u)\} \cup E(G[\{ u \in V \setminus A : \text{help}(u) \neq \emptyset \}]), \)
and let \( H^* := H[A \cup A^*] \). Observe that as \( A^* \) is independent, \( H^* \) does not contain any edge from \( E(G[\{ u \in V \setminus A : \text{help}(u) \neq \emptyset \}]). \)

\textbf{Lemma 30.} Let \( \delta \in [0, \frac{1}{2}] \), let \( s \) be a support map and let \( p \) be the corresponding profit map with the property that for each \( u \in A^* \) and each \( v \in s(u) \), we have \( p(u, v) \geq \delta \). Then
\[ w(A^*) \leq \frac{d}{2} \cdot w(A) - \frac{d - 2}{2} \cdot w(A \cap A^*) - \delta \cdot \sum_{v \in A \setminus A^*} (d - 1 - \{|u : N(v, A^*) : v \notin \text{help}(u) \cup s(u)| - |\delta_{H^*}(v)|\} \cdot w(v).\]
As we will reuse this statement at a later point with a different definition of help, we point out that the only property of help that we need for the argument below is the role it plays in the definition of $H$ and $H^*$.

Proof. By Lemma 27 we know that

$$w(A^*) \leq \frac{d}{2} \cdot w(A) - \sum_{u \in A} \sum_{v \in A^*} p(u, v) \cdot w(v) - \sum_{v \in A} \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v)$$

$$= \frac{d}{2} \cdot w(A) - \sum_{u \in A^*} \sum_{v \in A} p(u, v) \cdot w(v) - \sum_{v \in A} \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v)$$

$$\leq \frac{d}{2} \cdot w(A) - \sum_{u \in A^*} \sum_{v \in s(u)} \delta \cdot w(v) - \sum_{v \in A \setminus A^*} \delta \cdot \left( d - 1 - |N(v, A^*)| \right) \cdot w(v)$$

$$\leq \frac{d}{2} \cdot w(A) - \sum_{v \in A \setminus A^*} \sum_{u \in N(v, A^*): v \in s(u)} \delta \cdot w(v) - \sum_{v \in A \setminus A^*} \delta \cdot \left( d - 1 - |N(v, A^*)| \right) \cdot w(v)$$

$$\leq \frac{d}{2} \cdot w(A) - \sum_{v \in A \setminus A^*} \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v)$$

$$\leq \frac{d}{2} \cdot w(A) - \frac{d - 2}{2} \cdot w(A \cap A^*)$$

$$= \frac{d}{2} \cdot w(A) - \delta \cdot \sum_{v \in A \setminus A^*} (d - 1 - \left( |N(v, A^*)| - |\{u \in N(v, A^*) : v \in s(u)\}| \right)) \cdot w(v)$$

$$= \frac{d}{2} \cdot w(A) - \frac{d - 2}{2} \cdot w(A \cap A^*) - \delta \cdot \sum_{v \in A} (d - 1 - \left| \{u \in N(v, A^*) : v \notin s(u)\} \right|) \cdot w(v)$$
By Lemma 26, we know that we can apply Lemma 30 with \( n \) neighbors in the independent set \( v \) and for each \( u \in A^* \), \( v \in \text{help}(u) \) implies \( u \in N(v, A^*) \). Hence, for \( v \in A \setminus A^* \), we have
\[
|\{ u \in N(v, A^*) : v \in \text{help}(u) \}| = |\{ u \in V \setminus A : v \in \text{help}(u) \}| = |\delta_H(v)|.
\]
Plugging this into the above computations yields
\[
w(A^*) \leq \frac{d}{2} \cdot w(A) - \frac{d - 2}{2} \cdot w(A \cap A^*)
- \delta \cdot \sum_{v \in A \setminus A^*} (d - 1 - |\{ u : N(v, A^*) : v \notin \text{help}(u) \cup s(u) \}| - |\delta_H(v)|) \cdot w(v)
\]
as claimed.

By Proposition 18 and the definition of \( \text{supp} \), we know that each \( v \in A \) has at most one neighbor \( u \in N(v, A^*) \) such that \( v \notin \text{help}(u) \cup \text{supp}(u) \) because each such neighbor \( u \) satisfies \( n(u) = v \) and contributes more than \( \frac{w(v)}{2} \) to \( v \). Define
\[A' := \{ v \in A \setminus A^* : \exists u \in N(v, A^*) : v \notin \text{supp}(u) \cup \text{help}(u) \},\]
and for \( v \in A' \), denote the unique \( u \in N(v, A^*) \) with \( v \notin \text{supp}(u) \cup \text{help}(u) \) by \( t(v) \).

**Lemma 31.**
\[
w(A^*) \leq \frac{d}{2} \cdot w(A) - \xi \cdot \sum_{v \in A'} (d - 2 - |\delta_H(v)|) \cdot w(v)
- \xi \cdot \sum_{v \in A \setminus A^*} (d - 1 - |\delta_H(v)|) \cdot w(v)
\]

**Proof.** By Lemma 26, we know that we can apply Lemma 30 with \( \text{supp} \), \( \text{profit} \) and \( \delta = \xi \). This results in a bound of
\[
w(A^*) \leq \frac{d}{2} \cdot w(A) - \frac{d - 2}{2} \cdot w(A \cap A^*)
- \xi \cdot \sum_{v \in A \setminus A^*} (d - 1 - |\{ u : N(v, A^*) : v \notin \text{help}(u) \cup \text{supp}(u) \}| - |\delta_H(v)|) \cdot w(v)
= \frac{d}{2} \cdot w(A) - \frac{d - 2}{2} \cdot w(A \cap A^*)
- \xi \cdot \sum_{v \in A'} (d - 2 - |\delta_H(v)|) \cdot w(v)
- \xi \cdot \sum_{v \in A \setminus (A^* \cup A')} (d - 1 - |\delta_H(v)|) \cdot w(v)
\]
by definition of $A'$. For $v \in A \cap A^*$, we have $N(v, A^*) = \{v\}$ and $\delta_{H^*}(v) = \emptyset$, and hence,

$$
\frac{d - 2}{2} \cdot w(v) \geq \xi \cdot (d - 1) \cdot w(v) = \xi \cdot (d - 1 - |\delta_{H^*}(v)|) \cdot w(v).
$$

This leads to

$$
w(A^*) \leq \frac{d}{2} \cdot w(A) - \frac{d - 2}{2} \cdot w(A \cap A^*)
- \xi \cdot \sum_{v \in A'} (d - 2 - |\delta_{H^*}(v)|) \cdot w(v)
- \xi \cdot \sum_{v \in A \setminus (A^* \cup A')} (d - 1 - |\delta_{H^*}(v)|) \cdot w(v)
\leq \frac{d}{2} \cdot w(A) - \xi \cdot \sum_{v \in A \setminus A^*} (d - 1 - |\delta_{H^*}(v)|) \cdot w(v)
- \xi \cdot \sum_{v \in A \setminus (A^* \cup A')} (d - 1 - |\delta_{H^*}(v)|) \cdot w(v)
\leq \frac{d}{2} \cdot w(A)\tag{21}.
$$

as claimed. ▶

Lemma 32. Let $S$ be a finite set, $\varphi : S \to \mathbb{R}_{\geq 0}$, $\mu : S \to \mathbb{R}_{\geq 0}$ and $\eta > 0$ such that

$$
\sum_{s \in S} \varphi(s) \cdot \mu(s) > \eta \cdot \varphi(S).
$$

Let further $0 < \lambda < 1$. Then there exists $x > 0$ such that

$$
\sum_{s \in S : \varphi(s) \geq x} \mu(s) > \lambda \cdot \eta \cdot |\{s \in S : \varphi(s) \geq x\}|.
$$

Proof. Assume towards a contradiction that this were not the case. We get

$$
\sum_{s \in S} \varphi(s) \cdot \mu(s) = \sum_{s \in S} \lambda^{-1} \cdot \lambda \cdot \varphi(s) \cdot \mu(s)
= \sum_{s \in S} \lambda^{-1} \cdot \int_0^\infty \varphi(s) \mu(s)dx
= \sum_{s \in S} \lambda^{-1} \cdot \int_0^\infty \mu(s) \mathbb{1}_{\lambda \cdot \varphi(s) \geq x}dx
= \lambda^{-1} \cdot \int_0^\infty \sum_{s \in S} \mu(s) \mathbb{1}_{\lambda \cdot \varphi(s) \geq x}dx
= \lambda^{-1} \cdot \int_0^\infty \sum_{s \in S : \varphi(s) \geq x} \mu(s)dx.
$$
\[
\leq \lambda^{-1} \cdot \int_0^\infty \lambda \cdot \eta \cdot \left| \{ s \in S : \varphi(s) \geq x \} \right| dx \\
= \eta \cdot \int_0^\infty \left| \{ s \in S : \varphi(s) \geq x \} \right| dx \\
= \eta \cdot \sum_{s \in S} \int_0^\infty \mathbb{1}_{\varphi(s) \geq x} dx \\
= \eta \cdot \sum_{s \in S} \int_0^{\varphi(s)} 1 dx \\
= \eta \cdot \varphi(S) < \sum_{s \in S} \varphi(s) \cdot \mu(s),
\]
a contradiction. This finishes the proof. \(\blacklozenge\)

**Lemma 33.** Let \(S_1\) and \(S_2\) be finite sets, \(\varphi : S_1 \cup S_2 \to \mathbb{R}_{>0}\) and \(\eta > 0\) such that
\[
|S_1| > \eta \cdot |S_2|.
\]
Let further \(0 < \lambda\). Then, there exists \(x \in \mathbb{R}_{>0}\) such that
\[
\sum_{s \in S_1 : \varphi(s) \leq x} \varphi(s) > \lambda \cdot \eta \cdot \sum_{s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x} \varphi(s).
\]

**Proof.** As \(S_1\) and \(S_2\) are finite and \(S_1\) is not empty (since its cardinality is positive), let \(\Phi := \max_{s \in S_1 \cup S_2} \varphi(s) > 0\). Moreover, let
\[
x_0 := \inf \{ x \geq 0 : \left| \{ s \in S_1 : \varphi(s) \leq x \} \right| > \eta \cdot \left| \{ s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x \} \right| \}.
\]
Observe that \(x_0\) exists because for \(x = \max \{1, \lambda\} \cdot \Phi\), the sets we obtain are just \(S_1\) and \(S_2\), which satisfy the given condition. Hence, we take the infimum over a non-empty set of values.

We want to prove that
\[
\sum_{s \in S_1 : \varphi(s) \leq x_0} \varphi(s) > \lambda \cdot \eta \cdot \sum_{s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x_0} \varphi(s).
\]
First, note that by positivity of \(\varphi\), we must have \(x_0 \geq \min \{ \varphi(s) : s \in S_1 \} > 0\) because otherwise, by definition of the infimum, there has to be \(0 \leq \beta < \min \{ \varphi(s) : s \in S_1 \}\) for which
\[
0 = \left| \{ s \in S_1 : \varphi(s) \leq \beta \} \right| > \eta \cdot \left| \{ s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot \beta \} \right| \geq 0,
\]
a contradiction.

Next, we show that
\[
|\{ s \in S_1 : \varphi(s) \leq x_0 \}| > \eta \cdot |\{ s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x_0 \}|.
\]
Let
\[
x := \min \{ x > x_0 : x \in \{ \varphi(s) : s \in S_1 \} \cup \{ \lambda \cdot \varphi(s) : s \in S_2 \} \},
\]
where \( \min \emptyset := \infty \). Then \( \bar{x} > x_0 \) and by definition of the infimum, there is \( x_0 \leq \beta < \bar{x} \) such that

\[
\{ s \in S_1 : \varphi(s) \leq \beta \} > \eta \cdot \{ s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot \beta \}.
\]

But now, for each \( s \in S_1 \), we have

\[
\varphi(s) \leq x_0 \iff \varphi(s) \leq \beta
\]

and for each \( s \in S_2 \), we have

\[
\varphi(s) \leq \lambda^{-1} \cdot x_0 \iff \lambda \cdot \varphi(s) \leq \beta \iff \varphi(s) \leq \lambda^{-1} \cdot \beta.
\]

Hence,

\[
\{ s \in S_1 : \varphi(s) \leq x_0 \} > \eta \cdot \{ s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x_0 \}.
\]

To simplify notation, let \( S'_1 := \{ s \in S_1 : \varphi(s) \leq x_0 \} \) and \( S'_2 := \{ s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x_0 \} \). Note that by definition, \( |S'_1| > \eta \cdot |S'_2| \) and in particular, \( |S'_1| > 0 \). Observe that by minimality of \( x_0 \), for \( 0 < x < x_0 \), we have

\[
|\{ s \in S'_1 : \varphi(s) \leq x \}| = |\{ s \in S'_1 : \varphi(s) \leq x \}| \leq \eta \cdot |\{ s \in S'_2 : \varphi(s) \leq \lambda^{-1} \cdot x \}|
\]

\[
= \eta \cdot |\{ s \in S'_2 : \varphi(s) \leq \lambda^{-1} \cdot x \}|.
\]

We compute

\[
x_0 \cdot |S'_1| = \sum_{s \in S'_1} \varphi(s) + x_0 - \varphi(s)
\]

\[
= \varphi(S'_1) + \sum_{s \in S'_1} x_0 - \varphi(s)
\]

\[
= \varphi(S'_1) + \sum_{s \in S'_1} \int_{S'_1}^x 1 dx
\]

\[
= \varphi(S'_1) + \sum_{s \in S'_1} \int_0^{x_0} 1_{x \geq \varphi(s)} dx
\]

\[
= \varphi(S'_1) + \int_0^{x_0} \sum_{s \in S'_1} 1_{x \geq \varphi(s)} dx
\]

\[
= \varphi(S'_1) + \int_0^{x_0} |\{ s \in S'_1 : \varphi(s) \leq x \}| dx
\]

\[
\geq \varphi(S'_1) + \eta \int_0^{x_0} |\{ s \in S'_2 : \varphi(s) \leq \lambda^{-1} \cdot x \}| dx
\]

\[
= \varphi(S'_1) + \eta \int_0^{x_0} \sum_{s \in S'_2} 1_{\varphi(s) \leq \lambda^{-1} \cdot x} dx
\]

\[
= \varphi(S'_1) + \eta \sum_{s \in S'_2} \int_0^{x_0} 1_{\varphi(s) \leq \lambda^{-1} \cdot x} dx
\]

\[
= \varphi(S'_1) + \eta \sum_{s \in S'_2} \int_0^{x_0} 1_{\lambda \varphi(s) \leq x} dx
\]

\[
= \varphi(S'_1) + \eta \sum_{s \in S'_2} \int_{\lambda \varphi(s)}^x 1 dx
\]
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\[ = \varphi(S'_1) + \eta \sum_{s \in S'_3} x_0 - \lambda \cdot \varphi(s) \]
\[ = \varphi(S'_1) + x_0 \cdot \eta \cdot |S'_2| - \lambda \cdot \eta \cdot \varphi(S'_2). \]

This results in

\[ \varphi(S'_1) \geq \lambda \cdot \eta \cdot \varphi(S'_2) + x_0 \cdot (|S'_1| - \eta \cdot |S'_2|) > \lambda \cdot \eta \cdot \varphi(S'_2), \]

where the last inequality follows since $|S'_1| > \eta \cdot |S'_2|$ and $x_0 > 0$. This finishes the proof.

In the following, we distinguish the two cases where the weight of $A'$ is large, i.e. $w(A') > \gamma \cdot w(A)$ and the case where the weight of $A'$ is small, i.e. $w(A') \leq \gamma \cdot w(A)$ for a threshold $\gamma$ that we will choose later. In the first case, we will proceed by showing that we obtain the claimed approximation guarantee because if this were not the case, we could either find a circular improvement or a local improvement of size 3. In the second case, we prove that one of the unweighted instances we consider would yield a local improvement if the approximation guarantee were not as desired.

We start by considering the case where $w(A')$ is large. For the argument we aim at in this situation, we need the following two lemmata, which are, however, also true if $w(A')$ is small.

▶ Lemma 34. If there exists $u \in A^* \setminus A$ such that $\text{help}(u) = \{n(u)\}$ and $n(u) \in A'$, then $\{u, t(n(u))\}$ constitutes a local improvement.

Proof. First, note that $u$ and $t(n(u))$ are distinct because $u$ is helpful for $n(u)$, but $t(n(u))$ is not. By the definition of helpful vertices (Definition 15) and by our choice of $t(n(u))$, we know that

(i) $(1 + \epsilon)^{-1} \cdot w(u) \leq w(n(u)) \leq (1 + \epsilon) \cdot w(u),$
(ii) $w(N(u, A) \setminus \{n(u)\}) \leq \epsilon \cdot w(u),$
(iii) $w(n(u)) < (1 + \sqrt{2}) \cdot w(t(n(u)))$ and
(iv) $w(N(t(n(u)), A) \setminus \{n(u)\}) \leq \epsilon \cdot w(t(n(u))).$

Let $X := \{u, t(n(u))\}$. Then

\[ w^2(N(X, A)) \leq w^2(n(u)) + w^2(N(u, A) \setminus \{n(u)\}) + w^2(N(t(n(u)), A) \setminus \{n(u)\}) \]
\[ \leq w^2(n(u)) + w((N(u, A) \setminus \{n(u)\}))^2 + w((N(t(n(u)), A) \setminus \{n(u)\}))^2 \]
\[ \leq w^2(n(u)) + \epsilon^2 \cdot w^2(u) + \epsilon^2 \cdot w^2(t(n(u))) \]
\[ \leq \left(1 + \frac{\epsilon^2}{2}\right) \cdot w^2(u) + \left(1 + \frac{\sqrt{2} \epsilon^2}{2}\right) \cdot w^2(t(n(u))) + \epsilon^2 \cdot w^2(u) + \epsilon^2 \cdot w^2(t(n(u))) \]
\[ = \left(1 + \frac{\epsilon^2}{2} + \epsilon^2\right) \cdot w^2(u) + \left(1 + \frac{\sqrt{2} \epsilon^2}{2} + \epsilon^2\right) \cdot w^2(t(n(u))) \]
\[ \leq w^2(u) + w^2(t(n(u))) = w^2(X). \]

Hence, $X$ is a local improvement.

▶ Lemma 35. If there exists $u \in A^* \setminus A$ such that $n(u)$ and $n_2(u)$ are contained in $A' \cap \text{help}(u)$, then $\{t(n(u)), u, t(n_2(u))\}$ is a local improvement.

Proof. By the definition of help($u$), we know that

(i) $(1 + \epsilon)^{-1} \cdot \max\{w(u), w(n(u))\} \leq w(n_2(u)) \leq w(n(u)) \leq (1 + \epsilon) \cdot w(u)$ and
(ii) $w(N(u, A) \setminus \{n(u), n_2(u)\}) \leq \epsilon \cdot w(u).$
Moreover, the definition of the vertices \( t(n(u)) \) and \( t(n_2(u)) \) tells us that they are distinct because \( n(t(n(u)) = n(u) \) and \( n(t(n_2(u)) = n_2(u) \), and different from \( u \) since neither of them is helpful for \( n(u) \) or \( n_2(u) \), respectively. In addition to that, the fact that \( (n(u), n_2(u)) \) is not contained in the support of \( t(n(u)) \) \( t(n_2(u)) \) tells us that

(i) \( w(n(u)) < (1 + \sqrt{2}) \cdot w(t(n(u))), w(n_2(u)) < (1 + \sqrt{2}) \cdot w(t(n_2(u))), \)

(ii) \( w(N(t(n(u)), A) \{ n(u) \}) \leq \epsilon \cdot w(t(n(u))) \) and \( w(N(t(n_2(u)), A) \{ n_2(u) \}) \leq \epsilon \cdot w(t(n_2(u))) \).

Let \( X := \{ t(n(u)), u, t(n_2(u)) \} \). Then

\[
\begin{align*}
    w^2(N(X, A)) & \leq w^2(n(u)) + w^2(n_2(u)) + w^2(N(u, A) \{ n(u), n_2(u) \}) \\
                       & \quad + w^2(N(t(n(u)), A) \{ n(u) \}) + w^2(N(t(n_2(u)), A) \{ n_2(u) \}) \\
                       & \leq w^2(n(u)) + w^2(n_2(u)) + (w(N(u, A) \{ n(u), n_2(u) \}))^2 \\
                       & \quad + (w(N(t(n(u)), A) \{ n(u) \}))^2 + (w(N(t(n_2(u)), A) \{ n_2(u) \}))^2 \\
                       & \leq w^2(n(u)) + w^2(n_2(u)) + \epsilon^2 \cdot w^2(u) \\
                       & \quad + \epsilon^2 \cdot w^2(t(n(u))) + w^2(t(n_2(u))) \\
                       & = w^2(n(u)) - (1 - \epsilon^2) \cdot w^2(t(n(u))) + w^2(n_2(u)) - (1 - \epsilon^2) \cdot w^2(t(n_2(u))) \\
                       & \quad + \epsilon^2 \cdot w^2(u) + w^2(t(n(u))) + w^2(t(n_2(u))) \\
                       & \leq (w^2(n(u)) + w^2(n_2(u))) \cdot \left( 1 - (1 - \epsilon^2) \cdot \left( 1 + \frac{\epsilon}{\sqrt{2}} \right)^{-2} \right) \\
                       & \quad + \epsilon^2 \cdot w^2(u) + w^2(t(n(u))) + w^2(t(n_2(u))) \\
                       & \leq 2 \cdot (1 + \epsilon)^2 \cdot \left( 1 - (1 - \epsilon)^2 \cdot \left( 1 + \frac{\epsilon}{\sqrt{2}} \right)^{-2} \right) + \epsilon^2 \cdot w^2(u) \\
                       & \quad + w^2(t(n(u))) + w^2(t(n_2(u))) \\
                       & \leq w^2(u) + w^2(t(n(u))) + w^2(t(n_2(u))) = w^2(X).
\end{align*}
\]

Hence, \( X \) constitutes a local improvement as claimed.

\[ \blacksquare \]

\textbf{Lemma 36.} Let \( \gamma \in (0, 1) \) and consider a run of Algorithm 3. If \( w(A') > \gamma \cdot w(A) \) and

\[
\sum_{v \in A'} |\delta_H(v)| \cdot w(v) \geq \frac{(2 + \kappa) \cdot (1 + \epsilon^2)}{(2 + \epsilon) \cdot \gamma} \cdot w(A'),
\]

then we find a local improvement of size at most 3, a claw-shaped improvement, or a circular improvement in the next iteration.

\textbf{Proof.} If there exists a local improvement of size at most 3 or a claw-shaped improvement, we are done, so assume that this is not the case. Consider the multi-graph \( J \) given by

\( V(J) = A \) and

\( E(J) := \{ e_u := \{ n(u), n_2(u) \} : u \in A^* \setminus A : \text{help}(u) = \{ n(u), n_2(u) \} \} \).

Moreover, let \( J^* \) be the sub-graph of \( J \) with vertex set \( A \) and edge set

\( E(J^*) = \{ e \in E(J) : e \cap A' \neq \emptyset \} \).

We want to show that \( J^* \) contains a cycle of logarithmic size. To this end, first observe that by Lemma 34 we know that for each \( v \in A' \), we have \( |\delta_H(v)| = |\delta_H(v) \setminus (\delta_H(v) \setminus A) \} \) because \( v \) does not have any neighbors \( u \) in \( H^* \) such that \( \text{help}(u) = \{ n(u) \} \). Additionally, Lemma 35 tells us that \( A' \) is an independent set in \( J^* \). As by construction, \( A \setminus A' \) is independent in \( J^* \), too, we
can conclude that $J^*$ is bipartite with bipartitions $A'$ and $A \setminus A'$.
We want to show that $J^*$ has a sub-graph that is dense enough to contain a cycle of logarithmic size by making use of Lemma 32. To this end, we have to calculate the weighted sum of degrees in $J^*$. Note that by Definition 15, for each edge $\{n(u), n_2(u)\} \in E(J^*)$, we have $(1 + \epsilon)^{-1} \cdot w(n_2(u)) \leq w(n(u)) \leq w(n_2(u))$. Throughout the following calculation, we denote edges of $J^*$ in such a way that the first vertex is from $A'$ and the second one is from $A \setminus A'$. We compute

$$\sum_{v \in A} w(v) \cdot |\delta_{J^*}(v)| = \sum_{(x, y) \in E(J^*)} w(x) + w(y) \geq \sum_{(x, y) \in E(J^*)} w(x) + (1 + \epsilon)^{-1} \cdot w(x) = (1 + (1 + \epsilon)^{-1}) \cdot \sum_{(x, y) \in E(J^*)} w(x) = (1 + (1 + \epsilon)^{-1}) \cdot \sum_{v \in A'} w(v) \cdot |\delta_{J^*}(v)| = (1 + (1 + \epsilon)^{-1}) \cdot \sum_{v \in A'} w(v) \cdot |\delta_{J^*}(v)| \geq (1 + (1 + \epsilon)^{-1}) \cdot \frac{2 + \kappa}{2 + \epsilon} \cdot w(A') \geq (1 + (1 + \epsilon)^{-1}) \cdot \frac{2 + \kappa}{2 + \epsilon} \cdot w(A) = (2 + \epsilon) \cdot \frac{(2 + \kappa) \cdot (1 + \epsilon)}{2 + \epsilon} \cdot w(A) = (1 + \epsilon) \cdot (2 + \kappa) \cdot w(A).$$

Define
- $S := A,$
- $\mu(v) := |\delta_{J^*}(v)|,$
- $\varphi(v) := w(v)$
- $\eta := (1 + \epsilon) \cdot (2 + \kappa)$ and
- $\lambda := (1 + \epsilon)^{-1}.$

Then Lemma 32 tells us that there is $x > 0$ such that

$$\sum_{v \in A : w(v) \geq (1 + \epsilon) \cdot x} |\delta_{J^*}(v)| > (2 + \kappa) \cdot |\{v \in A : w(v) \geq x\}|.$$

As for $v \in A$ with $w(v) \geq (1 + \epsilon) \cdot x$, we know that every neighbor of $v$ in $J^*$ has weight at least $(1 + \epsilon)^{-1} \cdot (1 + \epsilon) \cdot x = x$, we can infer that

$$\sum_{v \in A : w(v) \geq x} |\delta_{J^*}(v)| \geq \sum_{v \in A : w(v) \geq (1 + \epsilon) \cdot x} |\delta_{J^*}(v)| > (2 + \kappa) \cdot |\{v \in A : w(v) \geq x\}|.$$

In particular, as $J^*$ does not contain any loops, the strict inequality tells us that

$$|\{v \in A : w(v) \geq x\}| \geq 2.$$

Moreover, we obtain

$$|E(J^*[\{v \in A : w(v) \geq x\}])| = \frac{1}{2} \cdot \sum_{v \in A : w(v) \geq x} |\delta_{J^*}(v)| > \frac{1}{2} \cdot |\{v \in A : w(v) \geq x\}| \geq \left(1 + \frac{\kappa}{2}\right) \cdot |\{v \in A : w(v) \geq x\}|.$$
Now, Lemma 3.2 from [3] and the fact that \( \frac{1}{k} \in \mathbb{N}^+ \) allow us to conclude that the graph \( J^*[\{v \in A : w(v) \geq x\}] \) contains a cycle of length at most \( \frac{2}{k} \cdot \log(|A|) \leq \frac{2}{k} \cdot \log(|V(G)|) \). Call this cycle \( C \) and let \( U \) be the set of vertices from \( A^c \) that induce the edges of \( C \). We show that \( X := U \cup \bigcup_{v \in V(C) \cap A^c} \{t(v)\} \) defines a local improvement (which is clearly circular). Note that for \( v \neq v' \) we have \( t(v) \neq t(v') \) since \( n(t(v)) = v \neq v' = n(t(v')) \), and that for \( u \in U \) and \( v \in V(C) \cap A^c \), we have \( t(v) \neq u \). To this end, observe that \( t(v) = u \) implies \( n(u) = n(t(v)) = v \). But by definition of \( J^* \), this yields \( v \in \text{help}(u) \), whereas \( v \not\in \text{help}(t(v)) \), a contradiction. Hence, we obtain a disjoint union as claimed.

To see that \( X \) defines a local improvement/satisfies the third condition from Definition 11, we need to show that for each edge \( \{n(u), n_2(u)\} = \{v, z\} \in E(C) \) such that \( v \in A^c \) and \( z \in A \setminus A^c \), we have

\[
2 \cdot w^2(u) + w^2(t(v)) > w^2(v) + w^2(z) + 2 \cdot w^2(N(u, A) \setminus \{v, z\}) + w^2(N(t(v), A) \setminus \{v\}).
\]

Note that summing up these inequalities then yields

\[
2 \cdot w^2(X) > 2 \cdot w^2(V(C)) + 2 \cdot w^2(N(X, A) \setminus V(C)) \geq 2 \cdot w^2(N(X, A)),
\]

implying that \( X \) is a local improvement. Recall that as \( \text{help}(u) = \{n(u), n_2(u)\} = \{v, z\} \), we have

\[
\max\{w(v), w(z)\} \leq (1 + \epsilon) \cdot w(u)
\]

and

\[
w(N(u, A) \setminus \{v, z\}) = w(N(u, A) \setminus \{n(u), n_2(u)\}) \leq \epsilon \cdot w(u).
\]

Besides, by definition of \( t(v) \), we have

\[
w(v) \leq \left( 1 + \sqrt{\frac{\epsilon}{2}} \right) \cdot w(t(v))
\]

and

\[
w(N(t(v), A) \setminus \{v\}) \leq \epsilon \cdot w(t(v)).
\]

Using this, we obtain

\[
w^2(v) + w^2(z) + 2 \cdot w^2(N(u, A) \setminus \{v, z\}) + w^2(N(t(v), A) \setminus \{v\})
\]

\[
\leq w^2(v) + w^2(z) + 2 \cdot (w(N(u, A) \setminus \{v, z\}))^2 + (w(N(t(v), A) \setminus \{v\}))^2
\]

\[
\overset{13}{\leq} w^2(v) + w^2(z) + 2 \cdot (w(N(u, A) \setminus \{v, z\}))^2 + \epsilon^2 \cdot w^2(t(v))
\]

\[
\overset{11}{\leq} w^2(v) + w^2(z) + 2 \cdot \epsilon^2 \cdot w^2(u) + \epsilon^2 \cdot w^2(t(v))
\]

\[
= w^2(v) - (1 - \epsilon^2) \cdot w^2(t(v)) + w^2(z) + 2 \cdot \epsilon^2 \cdot w^2(u) + w^2(t(v))
\]

\[
\overset{10}{\leq} \left( 1 - (1 - \epsilon^2) \cdot \left( 1 + \sqrt{\frac{\epsilon}{2}} \right)^{-2} \right) \cdot w^2(v) + w^2(z) + 2 \cdot \epsilon^2 \cdot w^2(u) + w^2(t(v))
\]

\[
\overset{9}{\leq} \left( 1 + \epsilon^2 \cdot \left( 1 - (1 - \epsilon^2) \cdot \left( 1 + \sqrt{\frac{\epsilon}{2}} \right)^{-2} \right) + 1 \right) + 2 \cdot \epsilon^2 \cdot w^2(u) + w^2(t(v))
\]

\[
\overset{8}{\leq} 2 \cdot w^2(u) + w^2(t(v)).
\]

This finishes the proof.
Corollary 37. Let $\gamma \in (0, 1)$ such that
\[
\gamma > \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{(2 + \epsilon) \cdot (d - 2) - \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon}}.
\] (14)

If $w(A') > \gamma \cdot w(A)$ and Algorithm 2 terminates, then we have
\[
w(A^*) \leq \frac{d}{2} \cdot w(A) - \xi \cdot \left( \gamma \cdot (d - 2) - \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} \right) \cdot w(A).
\]

Proof. If $w(A') > \gamma \cdot w(A)$ and the algorithm terminates, then
\[
\sum_{v \in A'} |\delta_{H^*}(v)| \cdot w(v) < \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{(2 + \epsilon) \cdot \gamma} \cdot w(A')
\]
and, hence,
\[
\sum_{v \in A'} (d - 2 - |\delta_{H^*}(v)|) \cdot w(v) > \left( d - 2 - \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} \right) \cdot w(A')
\]
\[
\geq \gamma \cdot \left( d - 2 - \frac{2 \cdot (2 + \kappa) \cdot (1 + \epsilon)^2}{(2 + \epsilon) \cdot \gamma} \right) \cdot w(A)
\]
\[
= \left( \gamma \cdot (d - 2) - \frac{2 \cdot (2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} \right) \cdot w(A).
\]
Applying Lemma 31 yields the desired statement.

Lemma 38. If $\sum_{v \in A} |\delta_{H^*}(v)| \cdot w(v) > \frac{2 \rho \cdot (1 + \epsilon)^3}{1 - \epsilon^2} \cdot w(A)$, then the algorithm finds a local improvement in the next iteration.

Proof. In case there is an improving claw, this is clear, so assume there is none. Furthermore, the assumption of the lemma implies that $A \neq \emptyset$, and, therefore, $w(A) > 0$ by positivity of weights. To simplify notation, we define $\theta := \frac{\sum_{v \in A} |\delta_{H^*}(v)| \cdot w(v)}{w(A)} > 0$. Then we have
\[
\sum_{v \in A} |\delta_{H^*}(v)| \cdot w(v) = \theta \cdot w(A)
\] (15)
and
\[
\theta > \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - \epsilon^2} \Leftrightarrow 1 > \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{\theta} + \epsilon^2.
\] (16)

Compliant with Algorithm 3, for $x \in R_{\geq 0}$, let
\[
A_{\geq x} = \{ v \in A : w(v) \geq x \}
\]
and
\[
V_{\geq x} := A_{\geq x} \cup \{ u \in V(H) \setminus A : w(u) \geq x \land N_H(u, A) \subseteq A_{\geq x} \}.
\]
Finally, let $H_{\geq x} := H[V_{\geq x}]$ and $H^*_{\geq x} = H^*[V_{\geq x} \cap (A \cup A^*)]$.

Claim 39. For $v \in A$ with $(1 + \epsilon) \cdot x \leq w(v)$, we have $v \in V(H^*_{\geq x})$ and $\delta_{H^*_{\geq x}}(v) = \delta_{H^*}(v)$.

Proof. First, by choice of $x$, we have $v \in A_{\geq x} \subseteq V(H^*_{\geq x})$. Next, by definition of $H^*$, we know that for $\{ u, v \} \in \delta_{H^*}(v) \subseteq \delta_{H}(v)$, $u$ is helpful for $v$. As a consequence, one of the following applies:
\( N_H(u, A) = \{n(u)\} = \{v\} \) and \((1 + \epsilon)^{-1} \cdot w(u) \leq w(v) \leq (1 + \epsilon) \cdot w(u)\).

\( N_H(u, A) = \{n(u), n_2(u)\} \) and

\[
(1 + \epsilon)^{-1} \cdot \max\{w(u), w(n(u)), w(n_2(u))\} = (1 + \epsilon)^{-1} \cdot \max\{w(u), w(n(u))\}
\leq w(n_2(u)) \leq w(n(u)) \leq (1 + \epsilon) \cdot w(u).
\]

In either case, \( u \) and all neighbors of \( u \) in \( H \) are of weight at least \( \frac{w(u)}{1+\epsilon} \geq x \). In particular, \( w(u) \geq x \) and \( N_H(u, A) \subseteq A_{\geq x} \). This implies that \( u \in V_{\geq x} \), and, therefore, \( \{u, v\} \in E(H^*_x) \).

As \( \{u, v\} \in \delta_{H^*_x}(v) \) was arbitrarily chosen, we can conclude that \( \delta_{H^*_x}(v) = \delta_H^*(v) \). \hfill \( \blacktriangleleft \)

Our next goal is to show the following statement:

\[ \sum_{v \in A_{\geq x}^*} |\delta_{H_{\geq x}^*}(v)| > \frac{\theta}{1+\epsilon} \cdot |A_{\geq x}^*|. \]

**Proof.** We want to apply Lemma 32. To this end, set

- \( S := A \),
- \( \mu(v) := |\delta_H(v)| \),
- \( \varphi(v) := w(v) \),
- \( \lambda := (1 + \epsilon)^{-1} \) and
- \( \eta := \theta \).

Then Lemma 32 tells us that there is \( x^* \) such that

\[ \sum_{v \in A_{(1+\epsilon) \cdot x^*}} |\delta_{H_{\geq x^*}}(v)| \geq \sum_{v \in A_{(1+\epsilon) \cdot x^*}} |\delta_{H^*}(v)| \geq \frac{\theta}{1+\epsilon} \cdot |A_{\geq x^*}|. \]

\( \blacktriangleleft \)

Note that in particular, the strict inequality implies that \( \sum_{v \in A_{\geq x^*}} |\delta_{H_{\geq x^*}}(v)| > 0 \). As all vertices in \( A \cap A^* \) are isolated in \( H^* \), \( H^*[V(H^*) \setminus (A \cap A^*)] \) is bipartite with bipartitions \( A \setminus A^* \) and \( V(H^*) \setminus A \) and all vertices from \( A^* \) have degree at most 2 in \( H^* \), we get

\[ 2 \cdot |A^* \cap V_{\geq x^*}| \geq \sum_{u \in A^* \cap V_{\geq x^*}} |\delta_{H_{\geq x^*}}^*(u)| = \sum_{v \in A_{\geq x^*}} |\delta_{H_{\geq x^*}}(v)| > \frac{\theta}{1+\epsilon} \cdot |A_{\geq x^*}|. \]

This implies

\[ |A^* \cap V_{\geq x^*}| > \frac{\theta}{2 \cdot (1+\epsilon)} \cdot |A_{\geq x^*}|. \]

Given that \( A^* \cap V_{\geq x^*} \) is independent in \( H \), we know that the algorithm MIS, applied to \( H_{\geq x^*} \), finds an independent set \( \bar{X} \) in \( H_{\geq x^*} \) of size at least

\[ |\bar{X}| > \frac{\theta}{2 \cdot (1+\epsilon)} \cdot |A_{\geq x^*}|. \]

Note that the strict inequality is inherited from the strict inequality on \( |A^* \cap V_{\geq x^*}| \) and \( |A_{\geq x^*}| \). Further observe that as \( H_{\geq x^*} \) is an induced sub-graph of \( H \), \( \bar{X} \) is independent in \( H \), too. Define \( X := \bar{X} \setminus A \). As \( \bar{X} \cap A \subseteq V_{\geq x^*} \cap A = A_{\geq x^*} \), we get

\[ |X| > \frac{\theta}{2 \cdot (1+\epsilon)} \cdot |A_{\geq x^*}| - |\bar{X} \cap A| \geq \frac{\theta}{2 \cdot (1+\epsilon)} \cdot |A_{\geq x^*} \setminus \bar{X}|. \]
where the last inequality follows from the fact that \( \frac{\theta}{2(1+\epsilon)\rho} > 1 \) by (16). Moreover, as \( \bar{X} \) is independent in \( H \), no vertex in \( X \) is adjacent (in \( H \)) to a vertex in \( A_{\geq x} \), implying that \( N_H(X, A_{\geq x^*}) \subseteq A_{\geq x} \setminus \bar{X} \). Hence, we obtain

\[
|X| > \frac{\theta}{2(1+\epsilon)\rho} \cdot |N_H(X, A_{\geq x^*})|.
\]

For \( x \geq 0 \), let \( X^\leq_x := \{ v \in X : w(v) \leq x \} \).

\( \triangleright \) **Claim 41.** There is \( x > 0 \) such that \( w^2(X^\leq_x) > \frac{\theta}{2(1+\epsilon)\rho} \cdot w^2(N_H(X^\leq_x, A_{\geq x^*})) \).

**Proof.** We want to apply Lemma 33. To this end, let

- \( S_1 := X \),
- \( S_2 := N_H(X, A_{\geq x^*}) \),
- \( \varphi(s) := w^2(s) > 0 \) for \( s \in S_1 \cup S_2 \),
- \( \eta := \frac{\theta}{2(1+\epsilon)\rho} \), and
- \( \lambda := (1+\epsilon)^{-2} \).

In this setting, Lemma 33 tells us that there is \( x > 0 \) such that

\[
w^2(X^\leq_x) > \frac{\theta}{2 \cdot (1+\epsilon)^3 \cdot \rho} \cdot w^2(N_H(X^\leq_x, A_{\geq x^*})).
\]

By construction of \( H \), for \( \{u, v\} \in E(H) \) with \( u \in V \setminus A \), \( v \in A \) and \( w^2(u) \leq x \), we have \( w(v) \leq (1+\epsilon) \cdot w(u) \) and, hence, \( w^2(v) \leq (1+\epsilon)^2 \cdot w^2(u) \leq (1+\epsilon)^2 \cdot x \). These facts imply that

\[
w^2(X^\leq_x) > \frac{\theta}{2 \cdot (1+\epsilon)^3 \cdot \rho} \cdot w^2(N_H(X^\leq_x, A_{\geq x})).
\]

\( \triangleright \)

It remains to see that \( X^\leq_x \) as implied by the claim constitutes a local improvement of the squared weight function. To this end, as \( X \subseteq V_{\geq x} \setminus A \), we know that for \( u \in X \), we have \( N_H(u, A) \subseteq A_{\geq x} \). Therefore,

\[
w^2(X^\leq_x) > \frac{\theta}{2 \cdot (1+\epsilon)^3 \cdot \rho} \cdot w^2(N_H(X^\leq_x, A)),
\]

and it suffices to bound \( w^2(N(X^\leq_x, A) \setminus N_H(X^\leq_x, A)) \). By construction of \( H \), we know that for each \( u \in X^\leq_x \subseteq V(H) \setminus A \), there exists \( v \in N(u, A) \) for which \( u \) is helpful. In particular, \( N_H(u, A) \) contains all vertices for which \( u \) is helpful and moreover,

\[
w(N(u, A) \setminus N_H(u, A)) \leq \epsilon \cdot w(u)
\]

and

\[
w^2(N(u, A) \setminus N_H(u, A)) \leq \epsilon^2 \cdot w^2(u).
\]

This yields

\[
w^2(N(X^\leq_x, A) \setminus N_H(X^\leq_x, A)) \leq \epsilon^2 \cdot w^2(X^\leq_x),
\]

and, hence,

\[
w^2(X^\leq_x) \leq \left( \frac{2 \cdot (1+\epsilon)^3 \cdot \rho}{\theta} + \epsilon^2 \right) \cdot w^2(X^\leq_x)
\]

\[
> w^2(N_H(X^\leq_x, A)) + w^2(N(X^\leq_x, A) \setminus N_H(X^\leq_x, A))
\]

\[
= w^2(X^\leq_x, A).
\]

This shows that \( X^\leq_x \) yields a local improvement.
\textbf{Corollary 42.} If \( w(A') \leq \gamma \cdot w(A) \) when the algorithm terminates, then we have

\[
w(A^*) \leq \left( \frac{d}{2} - \xi \cdot \left( d - 1 - \gamma - \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - \epsilon^2} \right) \right) \cdot w(A).
\]

\textbf{Proof.} By Lemma 38 we know that

\[
\sum_{v \in A} |\delta_H(v)| \cdot w(v) \leq \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - \epsilon^2} \cdot w(A).
\]

By Lemma 31 we obtain

\[
w(A^*) \leq \frac{d}{2} \cdot w(A) - \xi \cdot \sum_{v \in A'} (d - 2 - |\delta_H(v)|) \cdot w(v)
\]

\[
= \frac{d}{2} \cdot w(A) - \xi \cdot w(A) - \xi \cdot \sum_{v \in A} (d - 2 - |\delta_H(v)|) \cdot w(v)
\]

\[
\leq \frac{d}{2} \cdot w(A) - (1 - \gamma) \cdot \xi \cdot w(A) - \xi \cdot \sum_{v \in A} (d - 2 - |\delta_H(v)|) \cdot w(v)
\]

\[
= \frac{d}{2} \cdot w(A) - \xi \cdot \sum_{v \in A} (1 - \gamma) \cdot w(v) - \xi \cdot \sum_{v \in A} (d - 2 - |\delta_H(v)|) \cdot w(v)
\]

\[
= \frac{d}{2} \cdot w(A) - \xi \cdot \sum_{v \in A} (d - 2 + (1 - \gamma) - |\delta_H(v)|) \cdot w(v)
\]

\[
\leq \frac{d}{2} \cdot w(A) - \xi \cdot \left( d - 1 - \gamma - \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - \epsilon^2} \right) \cdot w(A).
\]

\textbf{By combining Corollary 37 and Corollary 42 for, we finally obtain Theorem 16.} For easier readability, we restate it once again:

\textbf{Theorem 16} For \( d \geq 5 \), Algorithm 2 yields an approximation guarantee of

\[
\frac{d}{2} - \xi \cdot \left( d - 2 - \frac{1}{d - 1} \cdot \frac{(2 + \kappa) \cdot (1 + \epsilon)}{2 + \epsilon} - \frac{d - 2}{d - 1} \cdot \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - \epsilon^2} \right).
\]

\textbf{Proof.} Let

\[
\gamma := 1 + \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{(2 + \epsilon) \cdot (d - 1)} - \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{(1 - \epsilon^2) \cdot (d - 1)}.
\]

Our choice of constants implies that \( \gamma \in (0, 1) \): We pick \( \kappa \leq \epsilon \), implying that

\[
\gamma = 1 + \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{(2 + \epsilon) \cdot (d - 1)} - \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{(1 - \epsilon^2) \cdot (d - 1)} \leq 1 + \frac{(1 + \epsilon)^2}{d - 1} - \frac{2 \cdot (1 + \epsilon)^3}{(1 - \epsilon^2) \cdot (d - 1)} < 1.
\]
On the other hand, 
\[
\frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{(1 - e^2) \cdot (d - 1)} \leq \frac{2 \cdot \frac{d + \epsilon}{3} \cdot (1 + \epsilon)^3}{(1 - e^2) \cdot (d - 1)} = \frac{2 \cdot d + \epsilon \cdot 1 + \epsilon^3}{3 \cdot d - 1 \cdot 1 - e^2} \leq \frac{2 \cdot 5 + \epsilon \cdot (1 + \epsilon^3)}{4 \cdot 1 - e^2} < 1.
\]

If \( w(A') > \gamma \cdot w(A) \), then Corollary 37, which is applicable by (24), tells us that 
\[
w(A^*) \leq \frac{d}{2} \cdot w(A) - \xi \cdot \left( \gamma \cdot (d - 2) - \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} \right) \cdot w(A).
\]

Plugging in our choice of \( \gamma \) yields
\[
\gamma \cdot (d - 2) - \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} = \left( 1 + \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{(2 + \epsilon) \cdot (d - 1)} \right) \cdot \left( \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{(1 - e^2) \cdot (d - 1)} \right) \cdot (d - 2) - \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon}
\]
\[
d = 2 - 2 \left( \frac{d - 2}{d - 1} - 1 \right) \cdot \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{(2 + \epsilon) \cdot (d - 1)} \cdot \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - e^2}.
\]

This finishes the proof for the case \( w(A') > \gamma \cdot w(A) \). On the other hand, if \( w(A') \leq \gamma \cdot w(A) \), then Corollary 42 yields
\[
w(A^*) \leq \left( \frac{d}{2} - \xi \cdot \left( d - 1 - \gamma - \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - e^2} \right) \right) \cdot w(A).
\]

By our choice of \( \gamma \), we get
\[
d - 1 - \gamma = \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - e^2}
\]
\[
d = d - 2 - \frac{1}{d - 1} \cdot \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} = \left( 1 - \frac{1}{d - 1} \right) \cdot \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - e^2}
\]
\[
d = d - 2 - \frac{1}{d - 1} \cdot \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} = \frac{d - 2}{d - 1} \cdot \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - e^2}.
\]

This concludes the proof.

Note that we can only find parameters \( \epsilon \) and \( \kappa \) for which
\[
d - 2 - \frac{1}{d - 1} \cdot \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{2 + \epsilon} = \frac{d - 2}{d - 1} \cdot \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{1 - e^2}
\]
is positive for \( d \geq 5 \).

**4 The case \( d = 4 \)**

In order to keep the paper at a reasonable length, we have also decided against dealing with case \( d = 4 \) respectively \( k = 3 \), which requires a slightly different approach, in full details. However, as it seems a little unsatisfying to only present an improved approach for \( d \geq 5 \).
Respectively $k \geq 4$, given that the MWIS in $d$-claw free graphs/ the $k$-Set Packing Problem is $NP$-hard for $d = 4/k = 3$ already, we would like to shortly point out the reason why a slightly modified proof is needed for $d = 4$, and give an idea of how to conduct it.

To illustrate the difficulty we are facing in the case $d = 4$, consider the following instance of the MWIS in 4-claw free graphs (see Figure 3): Let $m \in \mathbb{N}_{>0}$, take a cycle of length $8m$, and color its vertices with the colors red, blue, yellow and blue in an alternating manner. Now, as the half-perimeter of the cycle is divisible by 4, each red vertex is opposed by another red vertex. For each pair of opposite red vertices, add another blue vertex and connect it to both red ones. Moreover, for each yellow vertex, add a new blue vertex and connect it to the yellow vertex. In the resulting graph $G$, every vertex has degree at most 3, so it is 4-claw free and corresponds to a 3-Set Packing instance by associating each vertex with its set of incident edges.

Both the red and yellow as well as the blue vertices form an independent set. Assume that $A^*$ consists of all blue vertices, $A'$ is the set of yellow vertices, and $A \setminus A^*$ is the set of red vertices. Further assume that for each yellow vertex $v$, $t(v)$ is the blue vertex which is only connected to $v$. Then for each yellow vertex $v$, $t(v)$ is the only neighbor of $v$ in $A^*$ that is not connected to any other vertex in $A$. Additionally, no two yellow vertices share a common blue neighbor. Hence, none of the improvements of constant size considered in Lemma 34 and Lemma 35 exists. Moreover, the graph $J^*$ that we consider in the proof of Lemma 36 is the cycle of length $4m$ arising from the cycle of length $8m$ by shortcutting all blue vertices, and as the total number of vertices in our instance is $11m$, $J^*$ does not contain a cycle of logarithmic size for $m$ large enough. Finally, $H^*$ does not contain the vertices $\{t(v), v \in A'\}$, so in $H^*$, there are at most $5m$ blue vertices. Hence, the ratio between the number of blue and the number of yellow and red vertices in $H^*$ is at most $\frac{3}{2} < \frac{4}{3}$, implying that at least a straightforward application of the same arguments as in the proof of Lemma 35 is impossible. To overcome this issue, one has to get rid of those vertices $u$ such that $n(u) \notin \text{help}(u) \cup \text{supp}(u)$. Once this has been accomplished, in case every vertex in $A$ had $d - 1 = 3$ helpful neighbors, the ratio $\frac{|A^*|}{|A|}$ would be at least $\frac{3}{2} > \frac{4}{3}$, implying that there would be enough slack between the approximation guarantee of $\text{MIS}$ and the minimum ratio $\frac{|A^*|}{|A|}$ in case there are all neighbors are helpful to get an improved approximation guarantee in the same spirit as in Lemma 38.
Now, a simple way to ensure that there are no vertices \( u \) such that \( n(u) \notin \text{help}(u) \cup \text{supp}(u) \) is to just declare those vertices \( u \) as helpful for \( n(u) \), too, or to add \( n(u) \) to their support. By definition of the support, such a vertex \( u \) has to satisfy \( w(n(u)) \leq (1 + \sqrt{5}) \cdot w(u) \) and \( w(N(u, A) \setminus \{n(u)\}) \leq \epsilon \cdot w(u) \). If further \( w(u) \leq (1 + \sqrt{5}) \cdot w(n(u)) \), we can consider \( u \) as helpful and can still conduct essentially the same argument as before, we just have to replace some of the factors \((1 + \epsilon) \) by \((1 + \sqrt{5})\). On the other hand, if \( w(u) > (1 + \sqrt{5}) \cdot w(n(u)) \), \([3]\) tells us that we can afford to add \( n(u) \) to the support of \( u \). Consequently, although these considerations result in a much smaller value of \( \epsilon \), which is why we decided to present the proof for \( d \geq 5 \) instead, they give a better approximation guarantee than \( \frac{d}{2} \) for the case \( d = 4 \), too.

### 5 The relation between weighted and unweighted \( k \)-Set Packing

Although applicable to any \( \rho \)-approximation for the MWIS with \( \rho \leq \frac{d}{2} + \epsilon \), the previous analysis was tailored to the case where we actually use the approximation algorithm for the Maximum Cardinality Independent Set Problem/ unweighted \( k \)-Set Packing implied by \([8]\). In particular, when working with the small value of \( \epsilon \) enforced by the proximity of the approximation ratio of \( \frac{d}{2} \) we want to improve upon and the approximation guarantee of \( \frac{d}{2} + \tau \cdot (d - 2) \), it is not hard to see from the statement of Theorem \([16]\) that we get stuck above an approximation guarantee of \( \frac{d}{2} - \xi \cdot (d - 2) \), no matter how small \( \rho \) becomes.

While we do not want to put too much effort into optimizing the analysis for the hypothetical case that a significant improvement in the approximation guarantee for the unweighted case is achieved, in the following, we are still going to prove the following statement:

\[ \textbf{Theorem 43.} \text{ For any constant } \sigma \in (0, 1), \text{ there exists a constant } \tau \in (0, 1) \text{ with the following property: If there are } d_0 \in \mathbb{N}_{\geq 4} \text{ and an algorithm } A \text{, that, given } d \geq d_0 \text{ and an instance of the (unweighted) MIS in } d \text{-claw free graphs as input, computes a } 1 + \tau \cdot (d - 2) \text{-approximate solution, then there is an algorithm } A' \text{ that, given } d \geq d_0 \text{ and an instance of the (weighted) MWIS in } d \text{-claw free graphs as input, computes a } 1 + \tau \cdot (d - 2) \text{-approximate solution by calling } A \text{ a polynomial number of times and performing further operations running in quasi-polynomial time. For the special case of } k \text{-Set Packing, the additional operations can be performed in polynomial time.} \]

Note that we express the approximation guarantees as \( 1 + \chi \cdot (d - 2) \) with \( \chi \in (0, 1) \) instead of \( \chi \cdot (d - 2) \) to ensure they are not smaller than 1. (However, by essentially the same arguments used in the beginning of the proof to get from an approximation guarantee of \( 1 + \tau \cdot (d - 2) \) to an approximation guarantee of \( \tau \cdot d \), one can prove the same theorem with approximation guarantees of \( \tau \cdot d \) and \( \sigma \cdot d \) instead.) Moreover, observe that it suffices to deal with the case \( d \geq 4 \) since for \( d \leq 3 \), the problem at hand is solvable in polynomial time anyways. Also, note that as the best lower bound on the approximation guarantee achievable for \( k \)-Set Packing in case \( P \neq NP \) is \( \Omega \left( \frac{1}{\log d} \right) \), the statement of the theorem is not "trivially true" in the sense that the assumption would be known to directly imply \( P = NP \). Finally, observe that Theorem \([13]\) provides a way to translate statements about the approximation hardness of the weighted \( k \)-Set Packing Problem to the unweighted setting.

The remainder of this section is now dedicated to the proof of Theorem \([43]\) By \([1]\) and since the MWIS in \( d \)-claw free graphs for \( d \geq 4 \) generalizes 3-Set Packing, we know that there is a constant \( C > 1 \) with the property that unless \( P = NP \), there is no \( C \)-approximation for the MWIS in \( d \)-claw free graphs. Let \( \delta := \frac{\sigma}{2} \in (0, \frac{1}{2}) \) and let \( \alpha := \delta^{-2} \), \( \beta := \delta \) and \( m := \lceil \delta^{-3} \rceil \).
Moreover, choose
\[
\bar{\tau} := \frac{(1 - \beta) \cdot \sigma}{3 \cdot m \cdot \alpha^4} = \frac{(1 - \frac{\alpha}{2}) \cdot \sigma^g}{768 \cdot \frac{8}{\sigma^2}}
\]  
and let \( \tau := (1 + \frac{\alpha}{2})^{-1} \). Assume that there are \( d_0 \) and \( A \) as in the statement of the theorem. If \( 1 + \tau \cdot (d_0 - 2) < C \), then \( P = NP \) and there exists a polynomial time algorithm for the decision variant of the MWIS in \( d \)-claw free graphs. By performing binary search, we obtain a polynomial time algorithm that computes the optimum value of a solution and by querying how including/excluding a vertex changes the optimum, we can incrementally construct and optimum solution in polynomial time. Hence, we may assume that \( 1 + \tau \cdot (d_0 - 2) \geq C \). This yields \( \frac{\alpha}{2} \cdot \tau \cdot (d - 2) \geq 1 \) for every \( d \geq d_0 \), showing that
\[
\bar{\tau} \cdot d \geq \frac{\tau \cdot (d - 2)}{\tau \cdot (d - 2)} = \frac{1 + \frac{\alpha}{2}}{\tau \cdot (d - 2)} + \frac{1}{\tau \cdot (d - 2)} \\
\tau \cdot (d - 2) \geq \tau \cdot (d - 2) + 1
\]
for all \( d \geq d_0 \). Hence, the approximation guarantee of \( \bar{\tau} \cdot d \) for all \( d \geq d_0 \).

We modify our algorithm as follows: In each iteration, for each \( \text{Algorithm 3}. \) For its analysis, we define a new support map \( \text{Definition 22} \), where our renewed definition of helpful vertices is employed in the sub-routine \( \text{RunIteration} \) (Algorithm 3). For its analysis, we define a new support map

**Definition 44** (helpful vertex). Let \( u \in V \setminus A \) and let
\[
i_u := \min\{|N(u, A)|+1, m+1, \min\{i \in \{1, \ldots, |N(u, A)|\} : w(n_i(u)) \notin [\alpha^{-1} \cdot w(u), \alpha \cdot w(u)]\}\},
\]
where \( \min \emptyset := \infty \). We set
\[
\text{help}(u) := \begin{cases} (n_i(u) \leq i \leq n_{\text{end}}) \to A \\
\emptyset,w^2(n_i(u) \leq i \leq |N(u, A)|) \leq \beta \cdot w^2(u)
\end{cases}
\]
otherwise.

We consider Algorithm 4, where our renewed definition of helpful vertices is employed in the sub-routine \( \text{RunIteration} \) (Algorithm 3). For its analysis, we define a new support map

**Definition 45** (support). For \( u \in V \setminus A \), we define
\[
\text{supp}(u) := N(u, A) \setminus \text{help}(u).
\]
By applying Definition 42, we obtain a corresponding profit map.

**Lemma 46.** For \( u \in A^* \setminus A \) and \( z \in \text{supp}(u) \), we have \( \text{profit}(u, z) \geq \frac{1}{2} - \delta \).

**Proof.** If there exists \( z \in \text{supp}(u) \), then in particular \( N(u, A) \notin \emptyset \).

**Case 1:** \( w(n_i(u)) \geq \alpha \cdot w(u) \). Then we obtain
\[
\text{profit}(u, z) = \frac{\sum_{v \in N(u, A)} \text{contr}(u, v) - 2 \cdot \text{charge}(u, v)}{2 \cdot w(\text{supp}(u))} \geq \frac{2}{2} - \frac{w(N(u, A))}{2 \cdot w(N(u, A))} = \\
1 - \frac{w(u)}{w(N(u, A))} > 1 - \frac{1}{\alpha} \geq 1 - \frac{1}{2} - \delta^2 \geq 1 - \delta.
\]
Case 2: \(w(n_1(u)) < \alpha^{-0.5} \cdot w(u)\). The fact that there is no claw-shaped improvement implies that

\[
\alpha^{-0.5} \cdot w(u) \cdot w(N(u, A)) > w(n_1(u)) \cdot w(N(u, A)) \geq \alpha^{-2}(N(u, A)) \geq w^2(u),
\]

implying that \(w(N(u, A)) > \alpha^{-0.5} \cdot w(u)\). By a similar argument as in Case 1, we obtain

\[
\text{profit}(u, z) \geq \frac{1}{2} - \frac{1}{\alpha^{0.5}} = \frac{1}{2} - \delta.
\]

Case 3: \(\alpha^{-0.5} \cdot w(u) \leq w(n_1(u)) \leq \alpha \cdot w(u)\) and there is \(1 \leq j \leq \min\{\|N(u, A)\|, m\}\) such that \(w(n_j(u)) < \alpha^{-1} \cdot w(u)\).

If \(j = 1\), then the fact that there is no claw-shaped improvement implies that

\[
\alpha^{-1} \cdot w(u) \cdot w(N(u, A)) > w(n_1(u)) \cdot w(N(u, A)) \geq \alpha^{-2}(N(u, A)) \geq w^2(u),
\]

implying that \(w(N(u, A)) > \alpha \cdot w(u)\). Hence, we can perform the same argument as in Case 1 to conclude the desired statement.

Next, assume that \(j \geq 2\). If \(w^2(\{n_i(u), j \leq i \leq |N(u, A)|\}) \leq \beta \cdot w^2(u)\), then

\[
\text{help}(u) = \{n_i(u), 1 \leq i < j\},
\]

and we obtain

\[
\sum_{v \in N(u, A)} \text{contrib}(u, v) \cdot w(n_1(u)) - 2 \cdot \text{charge}(u, n_1(u)) \cdot w(n_1(u)) \geq w^2(u) - w^2(\{n_1(u)\}) - (2 \cdot w(u) - w(N(u, A))) \cdot w(n_1(u)) = (w(u) - w(n_1(u)))^2 + w(n_1(u)) \cdot w(N(u, A) \setminus \{n_1\}) - w^2(N(u, A) \setminus \{n_1\}) = (w(u) - w(n_1(u)))^2 + \sum_{i=2}^{|N(u, A)|} (w(n_1(u) - w(n_i(u))) \cdot w(n_i(u)) \geq \sum_{i=2}^{|N(u, A)|} (w(n_1(u) - w(n_i(u))) \cdot w(n_i(u)) \geq w(n_1(u)) \geq 1 - \alpha^{-0.5} \cdot w(u) \cdot \sum_{i=2}^{|N(u, A)|} (1 - \alpha^{-0.5}) \cdot w(n_1(u)) \cdot w(n_i(u)) = (1 - \alpha^{-0.5}) \cdot w(n_1(u)) \cdot w(\{n_i(u), j \leq i \leq |N(u, A)|\}) = (1 - \alpha^{-0.5}) \cdot w(n_1(u)) \cdot w(\text{supp}(u)).
\]

This yields

\[
\text{profit}(u, z) \geq \frac{1 - \alpha^{-0.5}}{2} = \frac{1}{2} - \frac{\delta}{2} > \frac{1}{2} - \delta.
\]

If \(w^2(\{n_i(u), j \leq i \leq |N(u, A)|\}) > \beta \cdot w^2(u)\), then

\[
\alpha^{-1} \cdot w(u) \cdot w(\{n_i(u), j \leq i \leq |N(u, A)|\}) > w^2(\{n_i(u), j \leq i \leq |N(u, A)|\}) > \beta \cdot w^2(u),
\]

leading to \(w(\{n_i(u), j \leq i \leq |N(u, A)|\}) \geq \alpha \beta \cdot w(u)\). Analogously to Case 1, this gives

\[
\text{profit}(u, z) \geq \frac{1}{2} - \frac{1}{\alpha \beta} = \frac{1}{2} - \frac{1}{\delta^{-2}} = \frac{1}{2} - \delta.
\]
To this end, first consider the case where

\[ \forall 1 \leq j \leq \min\{\lfloor N(u, A)\rfloor, m\} : w(n_j(u)) \geq \alpha^{-1} \cdot w(u). \]

If \( |N(u, A)| \leq m \), all vertices in \( N(u, A) \) are helpful for \( u \) (\( \nu^u = |N(u, A)| + 1 \) in this case and the vertex set the squared weight of which has to be bounded by \( \beta \cdot w^2(u) \) is empty), and \( \text{supp}(u) = \emptyset \), a contradiction to \( z \in \text{supp}(u) \). Hence, \( m < |N(u, A)| \) and \( w(N(u, A)) \geq \sum_{i=1}^{m} w(n_i(u)) \geq m \cdot \alpha^{-1} \cdot w(u) \). By the argument employed in Case 1, this results in

\[ \text{profit}(u, z) \geq \frac{w(N(u, A)) - 2 \cdot w(u)}{2 \cdot w(N(u, A))} \geq \frac{1}{2} - \frac{\alpha}{m} = \frac{1}{2} - \frac{\delta^{-2}}{\lfloor \delta^{-3} \rfloor} \geq \frac{1}{2} - \frac{\delta^{-2}}{\delta^{-3}} = \frac{1}{2} - \delta. \]

Fix an optimum solution \( A^* \). In account with Algorithm 3, define the graph \( H \) as follows:

\begin{itemize}
    \item \( V(H) := A \cup \{ u \in V \setminus A : \text{help}(u) \neq \emptyset \} \)
    \item \( E(H) := \{ \{u, v\} : u \in V \setminus A, v \in A, v \in \text{help}(u) \} \cup E(G[\{ u \in V \setminus A : \text{help}(u) \neq \emptyset \}]), \)
\end{itemize}

and let \( H^* := H[A \cup A^*] \). Observe that as \( A^* \) is independent, \( H^* \) does not contain any edge from \( E(G[\{ u \in V \setminus A : \text{help}(u) \neq \emptyset \}]) \).

**Lemma 47.** In each iteration of Algorithm 3 we have

\[ w(A^*) \leq \frac{d}{2} \cdot w(A) - \left( \frac{1}{2} - \delta \right) \cdot \sum_{v \in A} \max\{0, d - 2 - |\delta_{H^*}(v)|\} \cdot w(v). \]

**Proof.** By Lemma 27 we know that we have

\[ w(A^*) \leq \frac{d}{2} \cdot w(A) - \sum_{v \in A} \sum_{u \in \text{A}} \text{profit}(u, v) - \sum_{v \in A} \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v). \]

Hence, it suffices to show that for each \( v \in A \), we have

\[ \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v) + \sum_{u \in \text{A}} \text{profit}(u, v) \geq \left( \frac{1}{2} - \delta \right) \cdot \max\{0, d - 2 - |\delta_{H^*}(v)|\} \cdot w(v). \]

To this end, first consider the case where \( v \in A \cap A^* \). Then \( N(v, A^*) = \{ v \} \) and

\[ \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v) = \frac{d - 2}{2} \cdot w(v) \geq \left( \frac{1}{2} - \delta \right) \cdot \max\{0, d - 2 - |\delta_{H^*}(v)|\} \cdot w(v). \]

Next, let \( v \in A \setminus A^* \). Then \( N(v, A^*) \subseteq V \setminus A \) because if there were \( w \in N(v, A^*) \cap A \), then \( v \notin A^* \) would imply \( v \neq w \) and hence, the independent set \( A \) would contain two adjacent vertices, a contradiction. As a consequence, we know that

\[ |\delta_{H^*}(v)| = |\{ u \in V \setminus A : v \in \text{help}(u) \}| = |\{ u \in N(v, A^*) : v \in \text{help}(u) \}| \]

since we have seen that \( N(v, A^*) \subseteq V \setminus A \) and by Definition 44 we know that \( v \in \text{supp}(u) \) implies \( v \in N(u, A) \) respectively \( u \in N(v, A^*) \). Now, by Definition 45 we know that

\[ N(v, A^*) = \{ u \in N(v, A^*) : v \in \text{help}(u) \} \cup \{ u \in N(v, A^*) : v \in \text{supp}(u) \}. \]

This leads to

\[ |\{ u \in N(v, A^*) : v \in \text{supp}(v) \}| = |N(v, A^*)| - |\{ u \in N(v, A^*) : v \in \text{help}(v) \}| = |N(v, A^*)| - |\delta_{H^*}(v)| \geq 0. \]
By Lemma 46 and non-negativity of the profit, we finally obtain
\[
\frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v) + \sum_{u \in A^*} \text{profit}(u, v)
\geq \frac{d - 1 - |N(v, A^*)|}{2} \cdot w(v) + \sum_{u \in N(v, A^*) \cap \text{supp}(v)} \text{profit}(u, v)
\geq \left(\frac{1}{2} - \delta\right) \cdot (d - 1 - |N(v, A^*)|) \cdot w(v) + |\{u \in N(v, A^*) : v \in \text{supp}(v)\}| \cdot \left(\frac{1}{2} - \delta\right) \cdot w(v)
\geq \left(\frac{1}{2} - \delta\right) \cdot (d - 1 - |N(v, A^*)|) \cdot w(v) + (|N(v, A^*)| - |\delta_H \cdot (v)|) \cdot \left(\frac{1}{2} - \delta\right) \cdot w(v)
= \left(\frac{1}{2} - \delta\right) \cdot (d - 1 - |\delta_H \cdot (v)|) \cdot w(v)
\geq \left(\frac{1}{2} - \delta\right) \cdot \max\{0, d - 2 - |\delta_H \cdot (v)|\} \cdot w(v),
\]
where the last part follows from the fact that \(|\delta_H \cdot (v)| \leq |N(v, A^*)| \leq d - 1\) by d-claw freeness of \(G\).

\[\blacktriangleleft\]

Corollary 48. If \(\sum_{v \in A} |\delta_H \cdot (v)| \cdot w(v) \leq \sigma \cdot (d - 2) \cdot w(A)\), then \(w(A^*) \leq (1 + \sigma \cdot (d - 2)) \cdot w(A)\).

**Proof.** By Lemma 47, we get
\[
w(A^*) \leq \frac{d}{2} \cdot w(A) - \left(\frac{1}{2} - \delta\right) \cdot \sum_{v \in A} \max\{0, d - 2 - |\delta_H \cdot (v)|\} \cdot w(v)
\leq \frac{d}{2} \cdot w(A) - \left(\frac{1}{2} - \delta\right) \cdot \sum_{v \in A} (d - 2 - |\delta_H \cdot (v)|) \cdot w(v)
= \frac{d}{2} \cdot w(A) - \left(\frac{1}{2} - \delta\right) \cdot (d - 2) \cdot w(A) + \left(\frac{1}{2} - \delta\right) \cdot \sum_{v \in A} |\delta_H \cdot (v)| \cdot w(v)
\leq w(A) + \delta \cdot (d - 2) \cdot w(A) + \left(\frac{1}{2} - \delta\right) \cdot \sigma \cdot (d - 2) \cdot w(A)
\leq w(A) + \frac{\sigma}{2} \cdot (d - 2) \cdot w(A) + \frac{\sigma}{2} \cdot (d - 2) \cdot w(A)
= (1 + \sigma \cdot (d - 2)) \cdot w(A).
\]

As a consequence, we are done if we can show that the assumptions of Corollary 48 are always met when Algorithm 3 terminates. To show this, we just adapt the proof of Lemma 38 to fit into our new setting.

**Lemma 49.** \(\sum_{v \in A} |\delta_H \cdot (v)| \cdot w(v) > \sigma \cdot (d - 2) \cdot w(A)\), then Algorithm 3 finds a local improvement in the next iteration.

**Proof.** In case there is an improving claw, this is clear, so assume there is none. Furthermore, the assumption of the lemma implies that \(A \neq \emptyset\), and, therefore, \(w(A) > 0\) by positivity of weights. Compliant with Algorithm 3 for \(x \in \mathbb{R}_{\geq 0}\), let
\[
A_{\geq x} = \{v \in A : w(v) \geq x\}
\]
\[
V_{\geq x} := A_{\geq x} \cup \{u \in V(H) \setminus A : w(u) \geq x \land N_H(u, A) \subseteq A_{\geq x}\}.
\]
Finally, let \(H_{\geq x} := H[V_{\geq x}]\) and \(H^*_{\geq x} = H^*[V_{\geq x} \cap (A \cup A^*)]\).
Claim 50. For \( x > 0 \) and \( v \in A \) with \( \alpha^2 \cdot x \leq w(v) \), we have \( v \in V(H_{\geq x}) \) and \( \delta_{H_{\geq x}}(v) = \delta_H(v) \).

Proof. First, by choice of \( x \), we have \( v \in A_{\geq x} \subseteq V(H_{\geq x}) \). Next, by definition of \( H^* \), we know that for \( \{u, v\} \in \delta_H(v) \subseteq \delta_H(u) \), \( u \) is helpful for \( v \) and we have \( w(v) \leq \alpha \cdot w(u) \), implying that \( w(u) \geq \alpha^{-1} \cdot w(v) \geq x \) since \( \alpha \geq 1 \). Moreover, every neighbor \( z \) of \( u \) in \( H^* \) is helpful for \( u \) and satisfies \( w(z) \geq \alpha^{-1} \cdot w(u) \geq \alpha^{-2} \cdot w(v) \geq x \). Consequently, \( u \) and all neighbors of \( u \) in \( H \) are of weight at least \( x \). In particular, \( w(u) \geq x \) and \( N_H(u, A) \subseteq A_{\geq x} \). This implies that \( u \in V_{\geq x} \) and, therefore, \( \{u, v\} \in E(H^*_{\geq x}) \). As \( \{u, v\} \in \delta_H(v) \) was arbitrarily chosen, we can conclude that \( \delta_{H^*_{\geq x}}(v) = \delta_H(v) \).

Our next goal is to show the following statement:

Claim 51. There is \( x^* \in [w_{min}, w_{max}] \) such that

\[
\sum_{v \in A_{\geq x^*}} |\delta_{H^*_{\geq x^*}}(v)| > \frac{\sigma \cdot (d - 2)}{\alpha^2} \cdot |A_{\geq x^*}|.
\]

Proof. We want to apply Lemma 32 To this end, set

- \( S := A \),
- \( \mu(v) := |\delta_H(v)| \),
- \( \varphi(v) := w(v) \),
- \( \lambda := \alpha^{-2} \) and
- \( \eta := \sigma \cdot (d - 2) \).

Then Lemma 32 tells us that there is \( x^* \) such that

\[
\sum_{v \in A_{\geq x^*}} |\delta_{H^*_{\geq x^*}}(v)| \geq \sum_{v \in A_{\geq x^*}} |\delta_H(v)| > \frac{\sigma \cdot (d - 2)}{\alpha^2} \cdot |A_{\geq x^*}|.
\]

Note that in particular, the strict inequality implies that \( \sum_{v \in A_{\geq x^*}} |\delta_{H^*_{\geq x^*}}(v)| > 0 \). As all vertices in \( A \cap A^* \) are isolated in \( H^* \), \( H^*[V(H^*) \setminus (A \cap A^*)] \) is bipartite with bipartitions \( A \setminus A^* \) and \( V(H^*) \setminus A \) and all vertices from \( A^* \) have degree at most \( m \) in \( H^* \), we get

\[
m \cdot |A^* \cap V_{\geq x^*}| \geq \sum_{u \in A^* \cap V_{\geq x^*}} |\delta_{H^*_{\geq x^*}}(u)| = \sum_{v \in A_{\geq x^*}} |\delta_{H^*_{\geq x^*}}(v)| > \frac{\sigma \cdot (d - 2)}{\alpha^2} \cdot |A_{\geq x^*}|.
\]

This implies

\[
|A^* \cap V_{\geq x^*}| > \frac{\sigma \cdot (d - 2)}{m \cdot \alpha^2} \cdot |A_{\geq x^*}|.
\]

Given that \( A^* \cap V_{\geq x^*} \) is independent in \( H \), we know that the algorithm MIS, applied to \( H_{\geq x^*} \), finds an independent set \( \tilde{X} \) in \( H_{\geq x^*} \) of size at least

\[
|\tilde{X}| > \frac{1}{\tau \cdot d} \cdot \frac{\sigma \cdot (d - 2)}{m \cdot \alpha^2} \cdot |A_{\geq x^*}| \geq \frac{\sigma}{3 \cdot \tau \cdot m \cdot \alpha^2} \cdot |A_{\geq x^*}|.
\]

Note that the strict inequality is inherited from the strict inequality on \( |A^* \cap V_{\geq x^*}| \) and \( |A_{\geq x^*}| \). Further observe that as \( H_{\geq x^*} \) is an induced sub-graph of \( H \), \( \tilde{X} \) is independent in \( H \), too. Define \( X := \tilde{X} \setminus A \). As \( \tilde{X} \cap A \subseteq V_{\geq x^*} \cap A = A_{\geq x^*} \), we get

\[
|X| > \frac{\sigma}{3 \cdot \tau \cdot m \cdot \alpha^2} \cdot |A_{\geq x^*}| - |X \cap A| \geq \frac{\sigma}{3 \cdot \tau \cdot m \cdot \alpha^2} \cdot |A_{\geq x^*} \setminus \tilde{X}|,
\]
where the last inequality follows from the fact that \( \frac{\sigma}{3 \cdot \frac{\bar{\sigma}}{\bar{\sigma}}, m, \alpha^2} > 1 \) by (17) and since \( \beta \in (0, 1) \). Moreover, as \( \bar{X} \) is independent in \( H \), no vertex in \( \bar{X} \) is adjacent (in \( H \)) to a vertex in \( A_{\geq x} \cap \bar{X} \), implying that \( N_H(X, A_{\geq x}) \subseteq A_{\geq x} \setminus \bar{X} \). Hence, we obtain

\[
|X| > \frac{\sigma}{3 \cdot \frac{\bar{\sigma}}{\bar{\sigma}}, m, \alpha^2} \cdot |N_H(X, A_{\geq x})|.
\]

For \( x \geq 0 \), let \( X^{\leq x} := \{ v \in X : w(v) \leq x \} \).

\[ \triangleright \text{Claim 52. There is } x > 0 \text{ such that } w^2(X^{\leq x}) > \frac{\sigma}{3 \cdot \frac{\bar{\sigma}}{\bar{\sigma}}, m, \alpha^4} \cdot w^2(N_H(X^{\leq x}, A_{\geq x})). \]

\[ \text{Proof. We want to apply Lemma 33. To this end, let} \]

\[ S_1 := X, \]

\[ S_2 := N_H(X, A_{\geq x}), \]

\[ \varphi(s) := w^2(s) > 0 \text{ for } s \in S_1 \cup S_2, \]

\[ \eta := \frac{\sigma}{3 \cdot \frac{\bar{\sigma}}{\bar{\sigma}}, m, \alpha^4} \] and

\[ \lambda := \alpha^2. \]

In this setting, Lemma 33 tells us that there is \( x > 0 \) such that

\[
w^2(X^{\leq x}) > \frac{\sigma}{3 \cdot \frac{\bar{\sigma}}{\bar{\sigma}}, m, \alpha^4} \cdot w^2(\{ v \in N_H(X, A_{\geq x}) : w^2(v) \leq \alpha^2 \cdot x \}).
\]

By construction of \( H \), for \( \{u, v\} \in E(H) \) with \( u \in V \setminus A, v \in A \) and \( w^2(u) \leq x \), we have \( w(v) \leq \alpha \cdot w(u) \) and, hence, \( w^2(v) \leq \alpha^2 \cdot w^2(u) \leq \alpha^2 \cdot x \). These facts imply that

\[
w^2(X^{\leq x}) > \frac{\sigma}{3 \cdot \frac{\bar{\sigma}}{\bar{\sigma}}, m, \alpha^4} \cdot w^2(N_H(X^{\leq x}, A_{\geq x})).
\]

\[ \triangleright \]

It remains to see that \( X^{\leq x} \) as implied by the claim constitutes a local improvement of the squared weight function. To this end, as \( X \subseteq V_{\geq x} \setminus A \), we know that for \( u \in X \), we have \( N_H(u, A) \subseteq A_{\geq x} \). Therefore,

\[
w^2(X^{\leq x}) > \frac{\sigma}{3 \cdot \frac{\bar{\sigma}}{\bar{\sigma}}, m, \alpha^4} \cdot w^2(N_H(X^{\leq x}, A)),
\]

and it suffices to bound \( w^2(N(X^{\leq x}, A) \setminus N_H(X^{\leq x}, A)) \). By construction of \( H \), we know that for each \( u \in X^{\leq x} \subseteq V(H) \setminus A \), there exists \( v \in N(u, A) \) for which \( u \) is helpful. In particular, \( N_H(u, A) \) contains all vertices for which \( u \) is helpful and moreover,

\[
w^2(N(u, A) \setminus N_H(u, A)) \leq \beta \cdot w^2(u)
\]

by Definition 44. This yields

\[
w^2(N(X^{\leq x}, A) \setminus N_H(X^{\leq x}, A)) \leq \beta \cdot w^2(X^{\leq x}),
\]

and, hence,

\[
w^2(X^{\leq x}) \geq \frac{3 \cdot \frac{\bar{\sigma}}{\bar{\sigma}}, m, \alpha^4}{\sigma} \cdot w^2(X^{\leq x}) + \beta \cdot w^2(X^{\leq x})
\]

\[ > w^2(N_H(X^{\leq x}, A)) + w^2(N(X^{\leq x}, A) \setminus N_H(X^{\leq x}, A))
\]

\[ = w^2(X^{\leq x}, A).
\]

This shows that \( X^{\leq x} \) yields a local improvement.
6 Conclusion

In this paper, we have seen how to asymptotically beat the lower bound of $\frac{d-1}{2}$, which is best possible for pure local search [15], for the Maximum Weight Independent Set Problem in $d$-claw free graphs. While for the general MWIS, we can only do that at the cost of a quasi-polynomial running time, the algorithm we suggest can be implemented to run in polynomial time on instances of the weighted $k$-Set Packing Problem with $k \geq 4$. Although the absolute improvements we achieve are rather small, we still believe that this result is interesting, given that most previous approaches to weighted $k$-Set Packing Problem rely on pure local search, and are, hence, doomed to produce no better approximation guarantee than $\frac{k}{2}$. Given that these qualitative statements appear to be much more interesting than the precise value of the constants, these are only partly optimized, giving an idea of the order of magnitude of improvements that are achievable with this approach, whilst keeping the rather lengthy and tedious calculations at a minimum.

Finally, we manage to relate the approximation guarantees for the unweighted and the weighted versions of the problem(s) at hand. In doing so, we are the first ones to establish such a connection, which we believe to be a good starting point for future research.

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A Proofs of Lemmata from the Analysis of SquareImp

**Proof of Lemma 6.** As \( A^* \) is independent in \( G \), each \( v \in V \) satisfies \( |N(v, A^*)| \leq d - 1 \), because either \( v \in A^* \) and \( N(v, A^*) = \{v\} \), or \( v \not\in A^* \) and \( N(v, A^*) \) constitutes the set of talons of a claw centered at \( v \), provided it is non-empty.

**Proof of Lemma 7.** \( \text{charge}(u, v) > 0 \) implies \( v = n(u) \in N(u, A) \) and, therefore,

\[
\begin{align*}
2 \cdot \text{charge}(u, v) \cdot w(v) & = (2 \cdot w(u) - w(N(u, A))) \cdot w(v) \\
& = 2 \cdot w(u) \cdot w(v) - w(N(u, A)) \cdot w(v) \\
& \leq w^2(u) + w^2(v) - w(N(u, A)) \cdot w(v) \\
& = w^2(u) - w(N(u, A)) \cdot w(v) \\
& \leq w^2(u) - w^2(N(u, A) \setminus \{v\})
\end{align*}
\]

From this, we get

\[
2 \cdot \text{charge}(u, v) \cdot w(v) \leq w^2(u) - w^2(N(u, A) \setminus \{v\})
\]

as claimed.

**Proof of Lemma 8.** Assume for a contradiction that

\[
\sum_{u \in A^*: \text{charge}(u, v) > 0} \text{charge}(u, v) > \frac{w(v)}{2}
\]
for some $v \in A$. Then $v \not\in A^*$ since

$$\{ u \in A^* : \text{charge}(u, v) > 0 \} = \{ v \} = N(v, A) = N(v, A^*)$$

and

$$\sum_{u \in A^* : \text{charge}(u, v) > 0} \text{charge}(u, v) = \text{charge}(v, v) = \frac{w(v)}{2}$$

otherwise. Hence, $T := \{ u \in A^* : \text{charge}(u, v) > 0 \}$ forms the set of talons of a claw centered at $v$. By Lemma 7, it satisfies

$$w^2(T) = \sum_{u \in T} w^2(u) > \sum_{u \in T} w^2(N(u, A) \setminus \{ v \}) + w^2(v) \geq w^2(N(T, A)), $$

contradicting the fact that no claw improves $w^2(A)$. \hfill \blacktriangleleft

### B  Inequalities satisfied by our choice of constants

$$2 \epsilon^3 + \epsilon^4 \leq 2 \epsilon$$  

(18)

$$3 + 2 \cdot (1 + \epsilon)^{-1} - 2 \cdot (2 + \epsilon) \cdot (1 + \epsilon) \geq 0$$  

(19)

$$\frac{2}{3} \cdot \frac{5 + \epsilon}{4} \cdot \frac{(1 + \epsilon)^3}{1 - \epsilon^2} < 1$$  

(20)

$$2 \cdot (1 + \epsilon)^2 \cdot \left( 1 - (1 - \epsilon)^2 \cdot \left( 1 + \frac{\sqrt{\epsilon}}{2} \right)^{-2} \right) + \epsilon^2 < 1$$  

(21)

$$\max \left\{ \frac{(1 + \epsilon)^2}{2}, \frac{(1 + \sqrt{\epsilon})^2}{2} \right\} + \epsilon^2 < 1$$  

(22)

$$(1 + \epsilon)^2 \cdot \left( 1 - (1 - \epsilon^2) \cdot \left( 1 + \frac{\sqrt{\epsilon}}{2} \right)^{-2} \right) + 2 \cdot \epsilon^2 < 2$$  

(23)

$$1 + \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{(2 + \epsilon) \cdot (d - 1)} - \frac{2 \cdot \rho \cdot (1 + \epsilon)^3}{(1 - \epsilon^2) \cdot (d - 1)} \geq \frac{(2 + \kappa) \cdot (1 + \epsilon)^2}{(2 + \epsilon) \cdot (d - 2)}$$  

(24)

$$\frac{1 - \epsilon}{2} \geq \frac{1 - \epsilon \cdot (1 + \epsilon)}{2} \geq \frac{1 - 2 \cdot \epsilon \cdot (1 + \epsilon)}{2} \geq \xi$$  

(25)

$$\frac{(1 - \sqrt{\frac{1}{2(1 - \epsilon^2)}})^2}{2 \cdot \left( 1 + \epsilon \cdot \sqrt{\frac{1}{2(1 - \epsilon^2)}} \right)} \geq \xi$$  

(26)
Instead of plugging all of our choices of constants into the above inequalities and verifying them, we briefly explain how we came up with the values displayed in Table 1 so that the dedicated reader may reproduce them. First, we compute the maximum value of $\epsilon > 0$ that satisfies all of the constraints (18)-(23). This yields a value of $\epsilon > 0.084$. Next, for a given value of $\epsilon$, we need to compute the maximum value of $\xi$ subject to (25)-(31). For this purpose, we regard the left hand side of each of these constraints as a function in $\epsilon$. It turns out that for the range of values $\epsilon \in [0, 0.084]$ we are interested in, (27) is the most restrictive constraint (see Figure 4). Hence, for a given value of $\epsilon \in (0, 0.084]$, we can set $\xi(\epsilon) := \frac{\epsilon^2}{2(1 + \epsilon)}$. Now, we can plug this, together with $\kappa := \epsilon$ (actually, we want to choose $\kappa := \frac{1}{\epsilon^{\frac{1}{2}}}$, but this only gives a slightly better approximation guarantee) and $\rho := \frac{d + \epsilon}{4}$ into the approximation guarantee from Theorem 16 which then becomes a function in $d$ and $\epsilon$. For fixed $d$, we can, hence, (approximately) compute the optimum value of $\epsilon$, e.g. by using a computer algebra system. Moreover, the function we obtain is linear in $d$, so to achieve the best asymptotic behavior, we choose $\epsilon$ such that the coefficient of $d$ is minimized. This actually results in the maximum possible value of $\epsilon = 0.084$. Finally, we check that (24) is satisfied. To this end, note that due to our choice of $\kappa := \epsilon$ (a smaller choice of $\kappa$ only makes the constraint less tight) and $\rho := \frac{d + \epsilon}{4}$, the function $(d, \epsilon) \mapsto \text{LHS of (24)} - \text{RHS of (24)}$ is monotonically decreasing in $\epsilon \in (0, 1)$ and monotonically increasing in $d$. Hence, to see that $\epsilon \leq 0.084$ satisfies (24) for $d \geq 15$, it suffices to check the case $d = 15$.

We remark that since some of the estimates we employ in our analysis are rather crude, the approximation guarantees we obtain are most likely not the best ones achievable with our approach. However, they give an impression of the order of magnitudes of the improvements we can expect, while trying to keep the rather tedious and lengthy calculations to a minimum.
(a) The functions given by (25)-(31) (colored in red, orange, yellow, green, light blue, dark blue and fuchsia), meaning that the order of the rightmost function values, from top to bottom, is (25), (26), (30), (31), (28), (29), (27)).

(b) The functions given by (26)-(31) (colored in orange, yellow, green, light blue, dark blue and fuchsia), meaning that the order of the rightmost function values, from top to bottom, is (26), (30), (31), (28), (29), (27)).

Figure 4 The bounds on $\xi$ implied by (25)-(31).