Stability of motion and thermodynamics in charged black holes in $f(T)$ gravity

G.G.L. Nashed$^a$ and Emmanuel N. Saridakis$^{b,c,d}$

$^a$Centre for Theoretical Physics, The British University, P.O. Box 43, El Sherouk City, Cairo 11837, Egypt
$^b$National Observatory of Athens, Lofos Nymfon, 11852 Athens, Greece
$^c$CAS Key Laboratory for Researches in Galaxies and Cosmology, Department of Astronomy, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China
$^d$School of Astronomy, School of Physical Sciences, University of Science and Technology of China, Hefei 230026, P.R. China

E-mail: nashed@bue.edu.eg, msaridak@noa.gr

Received November 18, 2021
Revised March 4, 2022
Accepted April 15, 2022
Published May 11, 2022

Abstract. We investigate the stability of motion and the thermodynamics in the case of spherically symmetric solutions in $f(T)$ gravity using the perturbative approach. We consider small deviations from general relativity and we extract charged black hole solutions for two charge profiles, namely with or without a perturbative correction in the charge distribution. We examine their asymptotic behavior, we extract various torsional and curvature invariants, and we calculate the energy and the mass of the solutions. Furthermore, we study the stability of motion around the obtained solutions, by analyzing the geodesic deviation, and we extract the unstable regimes in the parameter space. We calculate the inner (Cauchy) and outer (event) horizons, showing that for larger deviations from general relativity or larger charges, the horizon disappears and the central singularity becomes a naked one. Additionally, we perform a detailed thermodynamic analysis examining the temperature, entropy, heat capacity and Gibb’s free energy. Concerning the heat capacity we find that for larger deviations from general relativity it is always positive, and this shows that $f(T)$ modifications improve the thermodynamic stability, which is not the case in other classes of modified gravity.

Keywords: modified gravity, Exact solutions, black holes and black hole thermodynamics in GR and beyond, dark energy theory

ArXiv ePrint: 2111.06359
1 Introduction

There are both theoretical and observational motivations for the construction of gravitational modifications, namely of extended theories of gravity that possess general relativity as a particular limit, but which in general exhibit a richer structure [1]. The first is based on the fact that since general relativity is non-renormalizable one could hope that more complicated extensions of it would improve the renormalizability properties [2]. The second motivation is related to the observed features of the Universe, and in particular the need to describe its two accelerated phases, namely one at early times (inflation) and one at late times (dark energy era). The usual approach in the construction of gravitational modifications is to start from the Einstein-Hilbert action and extend it in various ways [3]. Nevertheless, one can start from the equivalent torsional formulation of gravity, and in particular from the Teleparallel Equivalent of General Relativity (TEGR) [4–7], and modify it accordingly, obtaining $f(T)$ gravity [8–10], $f(T,G)$ gravity [11], $f(T,B)$ gravity [12–14], scalar-torsion theories [15–17], etc. Torsional gravity can lead to interesting cosmological phenomenology and hence it has attracted a large amount of research [8, 18–44].

Additionally, torsional and $f(T)$ gravity exhibit novel and interesting black hole and spherically symmetric solutions too [45–68]. In particular, spherically symmetric solutions with a constant torsion scalar $T$ have been studied in [69–71], while cylindrically charged black holes solutions using quadratic and cubic forms of $f(T)$ have been derived [72–75]. Moreover, by using the Noether’s symmetry approach, static spherically black hole solutions have been investigated in [76]. In similar lines, the research of static spherically symmetric solutions using $f(T)$ corrections on TEGR was the focus of interest in many studies using the perturbative approach [77–81], while vacuum regular BTZ black hole solutions in Born-Infeld gravity have been extracted in [82, 83].

Although spherically symmetric solutions in $f(T)$ gravity has been investigated in many works, the stability of motion around them has not been examined in detail. This issue is quite...
crucial, having in mind that modifications of general relativity are known to present various instabilities is various regimes of the parameter space. Hence, in this work we aim to derive charged spherically symmetric solution in $f(T)$ gravity using the perturbative approach, and then examine the stability of motion and thermodynamic properties.

The arrangement of the manuscript is as follows: in section 2 we extract the charged black-hole solutions for $f(T)$ gravity, using the perturbative approach, for two charge profiles. In section 3 we study the properties of the extracted perturbative solutions, and in particular their asymptotic forms, the invariants, and their energy. In section 4 we proceed to the investigation of the stability of motion around the solutions, by extracting and analyzing the geodesic deviation. Moreover, in section 5 we study in detail the thermodynamic properties, focusing on the temperature, entropy, heat capacity and Gibbs free energy. The final section 6 is reserved for conclusions and discussion.

2 Charged black hole solutions in $f(T)$ gravity

Let us extract charged black hole solutions following the perturbative approach. As usual, in torsional gravity as the dynamical field we use the orthonormal tetrad, whose components are $h_{a\mu}$, with Latin indices (from 0 to 3) denoting the tangent space and Greek indices (from 0 to 3) marking the coordinates on the manifold. The relation between the tetrad and the manifold metric is $g_{\alpha\beta} = \eta_{ij} h_{i\alpha} h_{j\beta}$, with $\eta_{ij}$ the Minkowski metric $\eta_{ij} = \text{diag.}(-1, +1, +1, +1)$. The torsion tensor is given as $T_{a\mu\nu} := \partial_{\mu} h_{a\nu} - \partial_{\nu} h_{a\mu} - \partial_{\lambda} h_{a\lambda\mu}$.

The action of $f(T)$ gravity, alongside a minimally coupled electromagnetic sector, is [46, 47]

$$ S_{f(T)} = \int d^4x \ |h| \left( \frac{1}{2\kappa^2} f(T) + \mathcal{F} \right), $$

(2.1)

with $\kappa = 8\pi G$ the gravitational constant, and $|h| = \text{det}(h_{a\mu}) = \sqrt{-g}$. The torsion scalar $T$ is written as

$$ T_{a\mu\nu} := \frac{1}{2}(K_{a\mu\nu} - h_{a\mu} T_{\lambda\nu} + h_{a\nu} T_{\lambda\mu} + h_{a\lambda\mu} T_{\nu} - T_{a\mu\nu}), $$

(2.2)

with the contortion tensor being $K_{a\mu\nu} := -\frac{1}{2} (T_{\mu\nu} - T_{\nu\mu} - T_{a}^{\mu\nu})$. Additionally, $\mathcal{F}$ is the gauge-invariant Lagrangian of electromagnetism given as $\mathcal{F} = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$ [84]. Variation of action (2.1) with respect to the tetrad yields the field equations [85]:

$$ \zeta_{a\mu} := \frac{1}{4} f(T) h_{a\mu} + f_{T} \left[ T_{\nu a}^{b\mu} S_{b\nu}^{\mu} + \frac{1}{8} \partial_{\nu}(h S_{a\mu}) \right] + f_{TT} S_{a\mu}^{\nu} \partial_{\nu} T - \frac{1}{2} \kappa^{2} \Theta_{a\mu} = 0, $$

(2.2)

with $f_{T} \equiv \partial f/\partial T$ and $f_{TT} \equiv \partial^2 f/\partial T^2$, and where the electromagnetic stress-energy tensor is

$$ \Theta_{a\mu} = F_{\alpha\alpha} F^{\mu\alpha} - \frac{1}{4} \delta_{a\mu} F_{\alpha\beta} F^{\alpha\beta}. $$

(2.3)

Moreover, variation of action (2.1) with respect to the Maxwell field gives

$$ \partial_{\nu} (\sqrt{-g} F^{\mu\nu}) = 0. $$

(2.4)

We can rewrite equation (2.2) purely in terms of spacetime indices by contracting with $g_{\mu\rho}$ and $h^{a}_{\alpha}$, resulting to

$$ H_{a\rho} = \frac{1}{2} \kappa^{2} \Theta_{a\rho}. $$

(2.5)
The symmetric part of (2.5) was sourced by the energy-momentum tensor (2.3), while their anti-symmetric part is a vacuum constraint for the considered matter models. The latter is equal to the variation of the action with respect to the flat spin-connection components [86, 87], namely

\[ H_{(\sigma \rho)} = \frac{1}{2} \kappa^2 \Theta_{(\sigma \rho)}, \quad H_{[\sigma \rho]} = 0. \]  

(2.6)

The explicit forms of these equations can be seen in eqs. (26) and (30) of [16] by setting the scalar field \( \phi \) to zero, however we do not display them here since we will derive the spherically symmetric field equations directly from (2.2).

We proceed by focusing on spherically symmetric solutions. Employing the spherical coordinates \( (t, r, \theta, \phi) \) we write the suitable spherically symmetric tetrad space as:

\[
b^\rho_\mu = \begin{pmatrix}
\sqrt{a} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{b}} \cos(\phi) \sin(\theta) & r \cos(\phi) \cos(\theta) & -r \sin(\phi) \sin(\theta) \\
0 & \frac{1}{\sqrt{b}} \sin(\phi) \sin(\theta) & r \sin(\phi) \cos(\theta) & r \cos(\phi) \sin(\theta) \\
0 & 0 & -r \sin(\theta) & 0
\end{pmatrix}, \tag{2.7}
\]

where \( a \equiv a(r) \) and \( b \equiv b(r) \) are two positive \( r \)-dependent functions. The above tetrad corresponds to the usual metric

\[ ds^2 = -a(r) dt^2 + \frac{dr^2}{b(r)} + r^2 d\Omega^2, \]  

(2.8)

with \( d\Omega^2 = (d\theta^2 + \sin^2 \theta d\phi^2) \). Using (2.7) the torsion scalar becomes

\[ T = \frac{2 \left[ 1 - \sqrt{b(r)} \right] \left[ r a'(r) - a(r) \sqrt{b(r)} + a(r) \right]}{r^2 a(r) b(r)}. \]  

(2.9)

Note that \( T \) becomes zero in the case \( a = b \rightarrow 1 \).

Inserting the above tetrad choice into the field equations (2.2) we acquire

\[
\zeta_t' = \frac{1}{4} f + \frac{\sqrt{b} (r a' + 2a) + r a b' - r b a' - 2 b a}{2 r^2 a b^2} f_T + \frac{(\sqrt{b} - 1)}{r b} T' f_T T - \frac{Q^2}{2 ab} = 0, \tag{2.10}
\]

\[
\zeta_r' = \frac{\sqrt{b} (r a' + 2a) - 2(r a' + a)}{2 r^2 a b} f_T f_T + \frac{f}{4} + \frac{Q^2}{2 ab} = 0, \tag{2.11}
\]

\[
\zeta_\theta' = \zeta_\phi' = \frac{2 a \sqrt{b} - (r a' + 2a)}{4 r a b} T' f_T T + \frac{f}{4} + \frac{Q^2}{2 ab} \left[ \frac{b(r^2 a'^2 - 6 r a a' - 2 r^2 a a'' - 4 a'^2) + a(2 a + r a')(r b' + 4 b^3/2) - 4 a b^2)}{8 r^2 a^2 b^2} \right] f_T = 0, \tag{2.12}
\]

where primes denote derivatives with respect to \( r \). In the above equations we have introduced the components of the electric field \( Q_\mu = [Q(r), 0, 0, 0] \), where \( F_{\mu \nu} = Q_{\mu,\nu} - Q_{\nu,\mu} \). Hence, the non-vanishing components of the Maxwell field are

\[
\frac{Q'[a(r b' - 4b) + r b a'] - 2 r b a Q''}{2 r a^2 b^2} = 0. \tag{2.13}
\]

Note that equations (2.10)–(2.13) coincide with those of [88] when \( Q = 0 \).
In the following we solve the above equations to first-order expansion around the Reissner-Nordström background, which allows us to extract analytical solutions (since in general the torsion scalar is not a constant, in which case one has the simple Reissner-Nordström solution). Hence, we assume the perturbative general vacuum charged solution as

\[ a(r) = 1 - \frac{2M}{r} + \frac{s^2}{r^2} + \epsilon a_1(r), \quad (2.14) \]
\[ b(r) = 1 - \frac{2M}{r} + \frac{s^2}{r^2} + \epsilon b_1(r), \quad (2.15) \]
\[ Q(r) = -\frac{s}{r} + \epsilon Q_1(r). \quad (2.16) \]

Finally, concerning the \( f(T) \) function we will consider the power-law form

\[ f(T) = T + \frac{1}{2} \alpha \epsilon T^2, \quad (2.17) \]

with \( \alpha \) the usual parameter of the \( T^2 \) term and \( \epsilon \ll 1 \) the small tracking parameter used to quantify the expansion in a consistent way [77, 88]. The above expression in the limit \( \epsilon \to 0 \) recovers Teleparallel Equivalent of General Relativity, and it is known to be a good approximation for every realistic \( f(T) \) gravity [89–91], since the extra term quantifies the deviation from General Relativity.

Substituting (2.14)–(2.17) into (2.10)–(2.12), keeping \( \epsilon \) terms up to first order, we obtain:

\[
\zeta^r = \frac{\epsilon}{r^8 g^4} \left\{ \alpha (\varrho - 1) \left[ (10 \varrho^2 + 5 \varrho + 1)(\varrho - 1)^2 r^4 + 2 s^2 (8 \varrho^2 + 4 \varrho + 1)(\varrho - 1) r^2 + s^4 (3 \varrho + 1) \right] - r^4 s^2 a_1 - r^7 \varrho b_1' + (\varrho^2 r^2 - 2 r^2 + s^2) r^4 \varrho b_1 + 2 \varrho^2 r^6 s Q_1' \right\} = 0,
\]
\[
\zeta^\varrho = \zeta^\varphi = \frac{\epsilon}{4 r^{10} \varrho^4} \left\{ 2 \varrho \varphi \left[ (5 \varrho^2 + 4 \varrho + 1)(\varrho - 1)^4 r^6 + s^2 (\varrho - 1)^2 (8 \varrho^3 - 11 \varrho^2 - 6 \varrho - 3) r^4 
+ s^4 r^4 (3 - 10 \varrho^2 + 7 \varrho^2 - s^2) + 2 \varrho^4 r^4 a_1'' - 8 \varrho^4 r^4 s Q_1' \right] 
+ \varrho^2 r^7 [(1 - 3 \varrho^2) r^2 + s^2] a_1' - \varrho^6 r^7 [(\varrho^2 + 1) r^2 - s^2] b_1' + a_1 [r^8 (1 - \varrho^4) - 2 r^6 s^2 + r^4 s^4] 
+ \varrho^4 b_1 [(\varrho^4 - 1) r^8 + 2 r^6 s^2 - r^4 s^4] \right\} = 0,
\quad (2.18) \]

while the Maxwell field equation (2.13) becomes

\[ 2 \varrho^4 \varphi Q''_\varphi + 4 \varrho^4 r^4 Q''_1 - s \{ \varrho^2 r^2 a_1' - \varrho^6 \varphi^2 b_1' - [(\varrho^2 - 1) r^2 + s^2] (a_1 - \varrho^2 b_1) \} = 0, \quad (2.19) \]

where \( \varrho = \sqrt{1 - \frac{2M}{r} + \frac{s^2}{r^2}} \). We solve the above equations separately in the cases where \( Q_1(r) = 0 \) and \( Q_1(r) \neq 0 \), namely the cases with or without a perturbative correction in the charge profile.
• Case I: $Q_1(r) = 0$:

In this case differential equations (2.18) and (2.19) admit the solution:

$$a_1(r) = \frac{1}{s^6 r^6 \bar{q} q_1} \left\{ 4 s^5 \alpha \bar{q} \left( 5 q_1^{3/2} q^2 \tan^{-1} \Theta ight) 
+ s \left( r M (5 s^2 - 6 M^2) + 2 M^4 + s^2 (M^2 - 2 s^2) \right) \tan^{-1} \Theta_1 \right\} 
+ q_1 \left\{ 4 \alpha s^2 r^5 \ln q \left( (s^2 - 3 M^2) r + M (M^2 + s^2) \right) 
+ q \left\{ 2 r^6 \alpha \ln q \left( 2 M^4 + M_2^2 s^2 - 2 s^4 \right) - s^2 \left\{ 8 r^5 \alpha \ln(r) \left( (s^2 - 3 M^2) r + M^3 + s^2 M \right) 
- s^2 \left[ s_2 c r^6 - (12 \alpha M + s^2 c_1) r^5 + \alpha (4 r^4 q_1^2 - 4 / 3 s^4 r^2 + 3 / 5 s^2) \right] \right\} 
- 20 r^2 s^2 \alpha \left[ r^3 (4 / 3 s^2 - 2 M^2) + 2 / 15 M s^4 + r^2 M (M^2 - 1 / 3 s^2) \right] 
+ r s^2 / 3 (M^2 - 1 / 5 s^2) \right\} \right\}, \quad (2.20)$$

$$b_1(r) = -\frac{4}{s^6 r^6 \bar{q}^2 q_1^2} \left\{ s r^4 \alpha \bar{q}^{3/2} q_1^{5/2} \left\{ 5 q_1^{3/2} [s^2 (2 M + r) - 3 r M^2] \tan^{-1} \Theta \n+ 2 s \left( 2 r + 5 M \right) s^4 - M^2 s^2 (6 M + 11 r) + 10 r M^4 \right\} \tan^{-1} \Theta_1 \right\} 
+ q_1^2 \left\{ 2 s^2 r^4 \alpha \bar{q}^{3/2} [3 M (M + r) s^2 - s^4 - 5 r M^3] \ln \bar{q} 
+ r^4 \alpha s^2 (2 M^4 + 2 M^2 s^2 - s^4) (2 M r - s^2) \alpha \ln q_1 
+ s^2 \left\{ 4 r^4 \alpha s^2 \ln(r) [5 M^3 r - 3 M s^2 (M + r) + s^4] - 8 / 3 \alpha r s^{10} \n+ 1 / 10 \alpha s^8 (23 \alpha \bar{q}^{3/2} + 40 M r^2 + 100 r^3) 
+ r s^6 [28 \alpha r^4 - r^3 / 2 (\bar{q}^{3/2} c_2 + 168 \alpha M) + 6 \alpha r^2 M^2 - 8 / 3 \alpha \bar{q}^{3/2} - 4 M \alpha \bar{q}^{3/2}] 
+ r^3 s^3 \left\{ 46 / 3 \alpha r^4 - 128 M r^3 \alpha + r^2 [(M c_2 - c_1 / 2) \bar{q}^{3/2} + 188 M^2 \alpha] 
+ 8 \alpha (65 M^3 / 3 + 4 \bar{q}^{3/2}) + 16 M \alpha \bar{q}^{3/2} \right\} 
- 2 r^4 M s^2 \alpha \left[ 2 M^2 r^2 - 11 M r^3 + M \bar{q}^{3/2} + 3 r^4 + 16 M^3 r \right] 
+ 6 \alpha \bar{q}^{3/2} \left\{ \right\} \right\} \right\}, \quad (2.21)$$

where $\Theta = \frac{M r - s^2}{s q}$, $\Theta_1 = \frac{M r - c}{q_1}$ and $q_1 = \sqrt{M^2 - s^2}$. Expressions (2.20), (2.21) are the solution of the field equations (2.2) and (2.4) up to $O(\epsilon)$.

• Case II: $Q_1(r) \neq 0$:

In this case the solution of (2.18) and (2.19) in the case $Q_1(r) \neq 0$ for the metric
functions is
\[
a_1(r) = c_4 + \frac{1}{r^2 a_{5/2}} \left\{ \int \frac{1}{r^6} \left[ \alpha \left[ a^{1/2} r^2 (s^2 r^4 - r^6 - s^6 + r^2 s^2) + 2r \varrho (3r^4 s^2 - 2r^6 - s^6) \\
+ r^2 \varrho^{3/2} (5s^4 - 3r^4 + 6r^2 s^2) + 2r \varrho^2 (10s^4 + 8r^4 - 22r^2 s^2) \\
+ \varrho^{5/2} (18s^4 + 15r^4 - 23r^2 s^2) + 2r \varrho^3 (10r^2 - 9s^2) + 3r^2 \varrho^{7/2} \right] \\
- 2r^7 \varrho^{5/2} \partial_2'} dr + c_3 \alpha^{5/2} \right\},
\]
(2.22)
while inserting these into (2.19) we finally acquire
\[
Q_1(r) = \frac{1}{15 r^5 \varrho \partial_1 s^6} \left\{ 30 \alpha r^4 \varrho s \epsilon [4M^3 r - 2Mr s^2 - 3M^2 s^2 + s^4 \ln(r^2 \varrho^2) \\
+ 30 \alpha r^4 \varrho \tanh^{-1} \left( \frac{M - r}{\varrho \partial_1} \right) [(8M^4 - 8M^2 s^2 + s^4) r + M s^2 (5s^2 - 6M^2)] \\
+ 75 \alpha r^4 \varrho^{5/2} \tanh^{-1} \left( \frac{Mr - s^2}{s \varrho \partial_1} \right) (5M^2 r - 4Ms^2 - s^2 r) \\
+ 3000 \alpha r^4 \partial_1 \ln 2 [(5M^2 - \epsilon^2) r - 4M \epsilon^2] - 15r^5 \alpha s \varrho \partial_1 \ln \varrho \partial_1^2 (6M^2 - 5s^2) \\
- 60r^4 \alpha s \varrho \partial_1 \ln r [4M^3 r - 2Mr s^2 - 3M^2 s^2 + s^4] \\
+ s \partial_1 \left[ \psi [15 (rc_4 - c_3) r^4 s^5 + 2s^2 \alpha (3s^6 + 15r^4 s^2 - 10s^4 r^2 + 30M s^2 r^3 - 60M^2 r^4)] \\
+ 5s^4 \alpha (25r^4 - 125Mr^3 - 2s^4 + 23s^2 r^2 - 15r^2 M^2 - 6s^2 r M) \\
+ 25 \alpha r^5 M^3 (29r - 30M) + 25M s^2 r^4 \alpha (29M^2 - 13s^2 - 19Mr) \right] \right\}.
\]
(2.24)
Hence, we have extracted the spherically symmetric solutions in the case of a quadratic deviation from Teleparallel Equivalent of General Relativity, which as we mentioned above is the first correction in every realistic $f(T)$ gravity. We stress here that all the above expressions in the limit $\epsilon \to 0$ recover the Schwarzschild results, hence our solutions provide the corrections on the latter brought about the $f(T)$ gravity.

3 Properties of the solutions

In this section we examine the properties of the extracted perturbative solutions, and in particular their asymptotic forms, the invariants, and their energy.
3.1 Asymptotic forms

In the case $Q_1(r) = 0$, the asymptotic form of the solutions (2.20), (2.21), up to $O(\epsilon)$ become

$$a(r) \approx 1 - \frac{2M}{r} + \frac{s^2}{r^2} + \epsilon \left\{ c_2 - \frac{c_1}{r} - \frac{\alpha}{r} \left[ \frac{68}{3M} + \frac{10M^2}{s^2q_1} - \frac{6}{q_1} - \frac{76M}{3s^2} + \frac{20\ln(2M - 2s)}{s} - \frac{40M^2\ln(2M - 2s)}{s^3} \right] \right\} + O\left(\frac{1}{r^3}\right), \quad (3.1)$$

$$b(r) \approx 1 - \frac{2M}{r} + \frac{s^2}{r^2} + \epsilon \left\{ c_1 - \frac{2Mc_2}{r} + \frac{\alpha}{r} \left[ \frac{68}{3M} + \frac{10M^2}{s^2q_1} - \frac{6}{q_1} - \frac{76M}{3s^2} + \frac{20\ln(2M - 2s)}{s} - \frac{40M^2\ln(2M - 2s)}{s^3} \right] \right\}, \quad (3.2)$$

and thus the metric (2.8) becomes Minkowski for $r \to \infty$.

On the other hand, in the case $Q_1(r) \neq 0$, the asymptotic forms of the solutions (2.22), (2.23) up to $O(\epsilon)$ become

$$a(r) = 1 - \frac{2M}{r} + \frac{s^2}{r^2} + \epsilon \left\{ c_3 \left[ 1 + \frac{2}{s} \right] + \frac{c_4}{r} - \frac{\alpha}{r} \left[ \frac{8M^3\ln(r)}{s^4} - \frac{8M\ln(r)}{s^2} - \frac{136M^3}{3s^2q_1^2} + \frac{30M^2}{3q_1^2} + 76M \right] + 2\tanh^{-1} \left( \frac{M}{s} \right) \left[ \frac{32M^2}{s^3} - \frac{20M^4}{s^5} - \frac{12}{s} \right] + \frac{2sc_3}{r^2} + O\left(\frac{1}{r^3}\right) \right\}, \quad (3.3)$$

$$b(r) = 1 - \frac{2M}{r} + \frac{s^2}{r^2} - 5\epsilon \left\{ 12Mc_3 + 3c_4s \right\} - \frac{\alpha}{r} \left[ \frac{60M^3}{s^4} + \frac{24M^3\ln(r)}{s^4} - \frac{76M}{s^2} - 4\tanh^{-1} \left( \frac{M}{s} \right) \left( \frac{24M^2}{s^3} + \frac{9}{s} + \frac{15M^4}{s^5} \right) \right] + \frac{\alpha}{3s^3r^2} \left[ 240sM^4 - 208M^2s^3 - 120M^2s^3\ln(r) + 36s^4 \right. \right.$$  

$$\left. - 48\tanh^{-1} \left( \frac{M}{s} \right) \left( 3M^3s^4 - 10M^3s^2 + 5M^5 \right) \right] + \frac{(16M^2 - s^3)c_3 + 4c_4sM}{sr^2} + O\left(\frac{1}{r^3}\right) \right\}, \quad (3.4)$$

which also become Minkowski in the limit $r \to \infty$. 
3.2 Invariants

Let us examine the behavior of various invariants in the obtained solutions. For the case \( Q_1(r) = 0 \), and inserting the asymptotic forms (3.1), (3.2) into the tetrad (2.7) and metric (2.8), and then into the various tensor definitions we respectively acquire the following expressions for the torsion tensor square, the torsion vector square, the torsion scalar, the Kretschmann scalar, the Ricci tensor square, and the Ricci scalar:

\[
T_{\mu\nu\lambda}T_{\mu\nu\lambda} = \frac{16M + 8\epsilon(c_1 - 2M c_2)}{r^3} - \frac{16s^2 - 4M^2 + \epsilon [68M^2 c_2^2 - 16s^2 c_2 - 4Mc_1 + 15c_1^2 \epsilon + 8M^2 c_2 - 64Mc_1 c_2 \epsilon]}{2r^4} + O\left(\frac{1}{r^5}\right), \quad (3.5)
\]

\[
T^\mu T_\mu = \frac{4(2M + \epsilon(c_1 - 2Mc_2 + \epsilon c_1 c_2 - 2Mc_2^2))}{r^3} + O\left(\frac{1}{r^4}\right), \quad (3.6)
\]

\[
T(r) = \frac{4c_2^2(2Mc_2 - c_1)}{r^3} + \frac{4M^2 - 8s^2 - \epsilon(8M^2 c_2 - 8s^2 c_2 - 4Mc_1) + \epsilon^2[9c_1^2 + 76M^2 c_2^2 - 16s^2 c_2^2 - 56Mc_1 c_2]}{2r^4} + O\left(\frac{1}{r^5}\right), \quad (3.7)
\]

\[
R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} = \frac{48M^2 + 48\epsilon M(c_1 - 2c_2 M) + 12\epsilon^2(c_1^2 - 6Mc_1 c_2 + 16M^2 c_2^2)}{r^6} + O\left(\frac{1}{r^7}\right), \quad (3.8)
\]

\[
R^{\mu\nu} R_{\mu\nu} = \frac{4c_2^2c_2^2[c_1 - 2Mc_2]}{r^6} + \frac{2s^2[2s^2 - 4s^2 c_2 - \epsilon^2(2c_1^2 - 13Mc_1 c_2 - 8s^2 c_2^2 + 18M^2 c_2^2)]}{r^8} + O\left(\frac{1}{r^9}\right), \quad (3.9)
\]

\[
R = -\frac{\epsilon^2(10M^2 c_2^2 - 2s^2 c_2^2 - 9Mc_1 c_2 + 2c_1^2)}{r^4} - \frac{\epsilon^2(48M^2 c_2^2 - 24s^2Mc_2^2 - 44M^2 c_1 c_2 + 10Mc_1^2 + 11s^2 c_1 c_2)}{r^5} - \frac{\epsilon^2(168M^4 c_2^2 - 128s^2Mc_2^2 + 10s^2 c_2^2 - 156M^3 c_1 c_2 + 81Ms^2 c_1 c_2 + 36M^2 c_1^2 - 10s^2 c_1^2)}{r^6} - \frac{\epsilon^2}{r^7}\left(112M^3 c_1^2 + 512M^5 c_2^2 + 372M^2 s^2 c_1 c_2 - 16M^3 \alpha c_2 + 60Ms^2 c_2^2 + 96Ms^4 c_2^2 - 31s^4 c_1 c_2 - 520M^3 q^2 c_2^2 - 480M^4 c_1 c_2\right) + \frac{16\alpha M^3}{r^7} + O\left(\frac{1}{r^8}\right). \quad (3.10)
\]

Similarly, for the case \( Q_1(r) \neq 0 \) we obtain the same expressions, and the only difference is in the torsion tensor square, which now becomes

\[
T^{\mu\nu\lambda}T_{\mu\nu\lambda} = \frac{32\alpha(3s^4 - 8M^2 s^2 + 5M^4)}{r^3 s^5} + \frac{24s^4 \epsilon(2Mc_1 + c_2) - 2M\right] + 32\epsilon M(3c_1 s^3 + 19\alpha s^2 - 15\alpha M^2)}{3r^3 s^4} + O\left(\frac{1}{r^4}\right). \quad (3.11)
\]
The above invariants reveal the presence of the singularity at \( r = 0 \) as expected, which is more mild than the case of simple TEGR, a known feature of higher-order torsional theories [8, 64, 68, 88].

3.3 Energy

One of the advantages of teleparallel formulation of gravity is the easy handling of the energy calculations, which is not the case in usual curvature formulation [8]. We start with the gravitational energy-momentum, \( P^a \), which in integral form in four dimensions is [92]

\[
P^a = - \int_V d^3x \partial_i \Pi^a_{\alpha},
\]

where \( V \) is the three-dimensional volume and \( \Pi^a_{\alpha} = -4\pi S_{a0i}^{\alpha} \) is expressed in terms of the superpotential components. In the TEGR limit, namely for \( f_T = 1 \), the above expression reduces to the form given in [93].

We start with the case \( \mathcal{Q}_1(r) = 0 \). Inserting the tetrad functions (3.1), (3.2) into the tetrad (2.7) we can calculate the involved superpotential component as

\[
S_{001}^{001} = \frac{6M^2s^3(\epsilon c_2 - 1) + 6sM(76M\alpha - 3c_1s^2 - 68\alpha s^2) + 60M\alpha \ln(2(M - s)(2M^2 - s^2))}{6Mr^2s^3(1 + c_2\epsilon)}
\]

which substituted into (3.12) leads to

\[
P^0 = E \approx M + \epsilon \left\{ \left( \frac{15}{3s^2q_1} \right) M - \frac{20}{s^3} \frac{(2M^2 - s^2) \ln[2(M - s)]}{s^3} \right\} - \frac{M^2 + s^2}{2r} - 5\epsilon \alpha \left\{ \left( \frac{15}{3s^2q_1} \right) M - \frac{20}{s^3} \frac{(2M^2 - s^2) \ln[2(M - s)]}{s^3} \right\} = M + O \left( \frac{1}{r} \right),
\]

where \( \mathcal{M} \approx M - \epsilon \alpha \frac{38M}{3s^2} \) is the Arnowitt-Deser-Misner (ADM) mass that contains \( M \) and \( \epsilon \) up to first order.

In the case \( \mathcal{Q}_1(r) \neq 0 \), the above procedure leads to

\[
S_{001}^{001} = \frac{-M}{r^2} + \epsilon \left[ 12\alpha(5M^4 - 8M^2s^2 + 3s^4) \tanh^{-1} \left( \frac{M}{s^2} \right) + 12Mc_4s^4(s + 2) + 4sM\alpha(17s^2 - 15M^2) + 3c_4s^5 \right],
\]

and then to

\[
P^0 = E \approx M + \epsilon \left[ \frac{12\alpha \tanh^{-1} \left( \frac{M}{s^2} \right)(5M^4 - 8M^2s^2 + 3s^4) + 3Mc_4s^4(s + 2) + 4sM\alpha(17s^2 - 15M^2) + 3c_4s^5}{3s^5} \right] - \frac{M^2 + s^2}{2r} - 5\alpha \left[ \frac{12\alpha \tanh^{-1} \left( \frac{M}{s^2} \right)(10M^4 - 16M^2s^2 + 6s^4) + 3c_4s^4(3M^2s + 6M^2 - 2s^2 + s^3) + 8sM^2\alpha(19s^2 - 15M^2) + 6c_4M^2s^5}{6rs^5} \right],
\]
\[ E \approx \mathcal{M}_1 + O \left( \frac{1}{r} \right), \]  

(3.17)

where \( \mathcal{M}_1 = M + \epsilon \left[ \frac{68\alpha M + 33\alpha(s+2)}{3s^2} \right] \). Note that for \( \epsilon = 0 \), i.e. in the TEGR limit, we recover the well-known energy expression of Reissner-Nordström spacetime [94].

4 Geodesic deviation and stability of motion

In this section we proceed to the examination of the stability of motion around the obtained black hole solutions, investigating the geodesic deviation. The geodesic equations of a test particle in the gravitational field are given by

\[
\frac{d^2 x^\alpha}{d\lambda^2} + \left\{ \frac{\alpha}{\beta\rho} \right\} \frac{dx^\beta}{d\lambda} \frac{dx^\rho}{d\lambda} = 0,
\]

(4.1)

where \( s \) denotes the affine connection parameter and \( \left\{ \frac{\alpha}{\beta\rho} \right\} \) the Levi-Civita connection. The geodesic deviation equations acquire the form [95]

\[
\frac{d^2 \psi^\sigma}{d\lambda^2} + 2 \left\{ \frac{\sigma}{\mu\nu} \right\} \frac{dx^\mu}{d\lambda} \frac{d\psi^\nu}{d\lambda} + \left\{ \frac{\sigma}{\mu\nu} \right\}, \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \psi^\rho = 0,
\]

(4.2)

where \( \psi^\rho \) is the 4-vector deviation.

In the case of the spherically symmetric ansatz (2.8) the above expressions give

\[
\frac{d^2 t}{d\lambda^2} = 0,
\]

\[
\frac{1}{2} a'(r) \left( \frac{dt}{d\lambda} \right)^2 - r \left( \frac{d\phi}{d\lambda} \right)^2 = 0,
\]

\[
\frac{d^2 \theta}{d\lambda^2} = 0,
\]

\[
\frac{d^2 \phi}{d\lambda^2} = 0,
\]

(4.3)

and therefore for the geodesic deviation we finally obtain

\[
\frac{d^2 \psi^1}{d\lambda^2} + b(r)a'(r) \frac{dt}{d\lambda} \frac{d\psi^0}{d\lambda} - 2rb(r) \frac{d\phi}{d\lambda} \frac{d\psi^3}{d\lambda} + \left\{ \frac{1}{2} [a'(r)b'(r) + b(r)a''(r)] \left( \frac{dt}{d\lambda} \right)^2 - [b(r) + rb'(r)] \left( \frac{d\phi}{d\lambda} \right)^2 \right\} \psi^1 = 0,
\]

\[
\frac{d^2 \psi^0}{d\lambda^2} + \frac{b'(r) dt}{b(r) d\lambda} \frac{d\psi^1}{d\lambda} = 0,
\]

\[
\frac{d^2 \psi^2}{d\lambda^2} + \left( \frac{d\phi}{d\lambda} \right)^2 \psi^2 = 0,
\]

\[
\frac{d^2 \psi^3}{d\lambda^2} + \frac{2d\phi}{r d\lambda} \frac{d\psi^1}{d\lambda} = 0.
\]

(4.4)
Using the circular orbit $\theta = \frac{\pi}{2}$, $\frac{d\theta}{d\lambda} = 0$, and $\frac{d\lambda}{d\lambda} = 0$, we acquire $\left(\frac{d\phi}{d\lambda}\right)^2 = a'(r) - r a'(r)$ and 
$\left(\frac{dt}{d\lambda}\right)^2 = \frac{a'(r)}{2a(r)} - r a'(r)$, and thus equations (4.4) can be rewritten as

\[
\frac{d^2\psi^1}{d\phi^2} + a(r)a'(r)\frac{dt}{d\phi} \frac{d\psi^0}{d\phi} - 2ra(r)\frac{d\psi^3}{d\phi} + \left\{ \frac{1}{2} \left[ a''(r) + a(r)a''(r) \right] \right\} \frac{d\psi^1}{d\phi} = 0,
\]
\[
\frac{d^2\psi^2}{d\phi^2} + \psi^2 = 0,
\]
\[
\frac{d^2\psi^0}{d\phi^2} + \frac{a'(r)}{a(r)} \frac{dt}{d\phi} \frac{d\psi^1}{d\phi} = 0,
\]
\[
\frac{d^2\psi^3}{d\phi^2} + \frac{2}{r} \frac{d\psi^1}{d\phi} = 0.
\]

(4.5)

The second equation of (4.5) corresponds to a simple harmonic motion, which indicates that the plane $\theta = \frac{\pi}{2}$ is stable. Moreover, the other equations of (4.5) have solutions of the form:

\[
\psi^0 = \zeta_1 e^{i\sigma \phi}, \quad \psi^1 = \zeta_2 e^{i\sigma \phi}, \quad \text{and} \quad \psi^3 = \zeta_3 e^{i\sigma \phi}, \quad (4.6)
\]

where $\zeta_1, \zeta_2$ and $\zeta_3$ are constants. Substituting (4.6) into (4.5), we extract the stability of motion condition as:

\[
3a b b' - \sigma^2 a b' - 2r b^{3/2} a'^{3/2} - r a b^3 + r a b' a' + r a b a'' > 0.
\]

(4.7)

Equation (4.7) has the following solution in terms of the metric potentials

\[
\sigma^2 = \frac{3a b b' - 2r b^{3/2} a'^{3/2} - r a b^3 + r a b' a' + r a b a''}{a^2 b^2} > 0.
\]

(4.8)

Hence, in order to conclude on the stability of motion around the obtained black-hole solutions, for the case $Q_1(r) = 0$ in the above expressions we insert $a(r)$ and $b(r)$ from (3.1), (3.2), while for the case $Q_1(r) \neq 0$ from (3.4).

In order to present the above results in a more transparent way, in figure 1 we depict the behavior of $\sigma^2$ for various choices of the model parameters, for the two cases $Q_1(r) = 0$ and $Q_1(r) \neq 0$ separately. Note that for $Q_1(r) = 0$ we always obtain stability of motion as expected, while for $Q_1(r) \neq 0$ we find potentially unstable regions.

5 Thermodynamics

In this section we perform an analysis of the thermodynamic properties of the obtained black-hole solutions. Since the nature of the solutions and especially their thermodynamic features change for $Q_1(r) = 0$ and $Q_1(r) \neq 0$, in the following we examine the two cases separately.

5.1 Thermodynamics of the black hole solution with $Q_1(r) = 0$

We start by investigating the black-hole solution of the case $Q_1(r) = 0$ given in (3.1), (3.2). In the left graph of figure 2 we display the metric potentials $a(r)$ and $b(r)$. As we can see,
Figure 1. The black-hole stability of motion parameter $\sigma^2$ versus $r$. Left graph: $Q_1(r) = 0$ with $M = 9$, $c_1 = c_2 = s = 1$ and various choices of the model parameters $\alpha$ and $\epsilon$ in Planck mass units. Right graph: $Q_1(r) \neq 0$ with $M = 0.01$, $c_1 = c_2 = 1$ and various choices of the model parameters $\alpha$, $\epsilon$ and $s$ in Planck mass units.

$a(r)$ may exhibit two horizons while $b(r)$ does not. In the right graph of figure 2 we focus on $a(r)$, in order to make more transparent the behavior of its possible two horizons, acquired by solving $a(r) = 0$, namely $r_-$ which denotes the inner Cauchy horizon of the black hole and $r_+$ which is the outer event horizon. In particular, for small $\alpha$ values, namely small deviations from general relativity, we obtain two horizons, however as $\alpha$ increases there is a specific value in which the two horizons become degenerate ($r_- = r_+ = r_d$), while for larger values the horizon disappears and the central singularity becomes a naked one. This is a known feature of torsional gravity, namely for some regions of the parameter space naked singularities appear [46, 47, 73]. Finally, let us calculate the total mass contained within the event horizon $r_+$. We find the mass-radius expression as

$$M_+ \equiv M(\nabla_+) \approx \frac{r_+^2}{2}, \quad (5.1)$$

where $M_+$ is given by eq. (3.14) for $r_+$ in place of $r$.

We proceed by examining the temperature. The Hawking black-hole temperature is defined as [96–99]

$$T_+ \equiv T(r_+) = \frac{a'(r_+)}{4\pi}, \quad (5.2)$$

with $r = r_+$ the event horizon, which satisfies $a'(r_+) \neq 0$. Additionally, in the framework of $f(T)$ gravity, the black-hole entropy is given by [100–102]

$$S_+ \equiv S(r_+) = \frac{A}{4f_T(r_+)}, \quad (5.3)$$

where $A$ is the area. Inserting the $f(T)$ form (2.17) and the solution (3.1), (3.2) into the above definitions we find

$$T_+ \approx \frac{3r_+^2s^3 + \epsilon \alpha [30s^2r_+ + 11s^3 + 30r_+^3 \ln(2/r_+) + 45s r_+^2] + 3s^3r_+^2c_2}{12\pi r_+^3}, \quad (5.4)$$
Figure 2. Left graph: the two metric potentials $g_{tt} \equiv a(r)$ and $g_{rr} \equiv b(r)$ given in (3.1), (3.2) versus $r$, for the black hole solution with $Q_1(r) = 0$, for $M = 9$, $c_1 = c_2 = 1$, $\alpha = 0.1$, $\epsilon = 0.1$, and $s = 4$, in Planck mass units. Right graph: the metric potential $a(r)$ versus $r$, for $M = 9$, $c_1 = 1$, $s = 4$ and various values of the model parameters $c_2$, $\alpha$ and $\epsilon$, in Planck mass units. $r_-$ and $r_+$ are the inner and outer horizons respectively, while $r_d$ is the degenerate horizon in which the above two coincide.

Figure 3. The temperature (left graph) and entropy (right graph) versus the horizon, for the black hole solution with $Q_1(r) = 0$, for $c_1 = 1$, $c_2 = 5$ and $s = 4$ and for various values of the model parameters $\alpha$ and $\epsilon$, in Planck mass units.

\[ S_+ \simeq \pi r_+^2 \left[ 1 + 2\epsilon \alpha \left( \frac{4}{r_+^2} + \frac{M_+ \epsilon - \epsilon f^\epsilon}{r_+^4} \right) \right]. \quad (5.5) \]

These expressions indicate that for $\epsilon = 0$ we recover the standard general-relativity temperature and entropy. In figure 3 we depict the temperature and entropy versus the horizon, for various values of the model parameters. As we can see the entropy is always positive and exhibits a quadratic behavior, while the temperature is always positive when $\epsilon > 0$ but for vanishing $\epsilon$ it is positive only for $r_d > r_+$.

We now focus on the heat capacity, which is a crucial quantity concerning the thermodynamic stability [103–105], since our perturbative approach to the black-hole solution
allows for an easy calculation. The heat capacity at the event horizon is defined as \[106–108\]:

\[
C_+ \equiv C(r_+) \approx \frac{\partial M_+}{\partial T_+} = \frac{\partial M_+}{\partial r_+} \left( \frac{\partial T_+}{\partial r_+} \right)^{-1},
\]

(5.6)

and positive heat capacity implies thermodynamic stability. Substituting (5.1) and (5.4) into (5.6) we obtain the heat capacity as

\[
C_+ \approx \frac{2\pi r_+^2}{3s^3} \left[ \epsilon \alpha(30r_+ + 7s - 60r \ln 2) - 3s^3 \right].
\]

(5.7)

Expression (5.7) implies that \(C_+\) does not diverge and thus we do not have a second-order phase transition. In the left graph of figure 4 we depict \(C_+\) as a function of the horizon. As we can see, in the \(\epsilon = 0\) case we have \(C_+ < 0\) due to the negative derivative of the temperature, as expected for the Reissner Nordström black hole. Nevertheless, for \(\epsilon > 0\) we obtain positive heat capacity. This is one of the main results of the present work, namely that \(f(T)\) modifications improve the thermodynamic stability. Note that this is not the case in other gravitational modifications, since for instance in \(f(R)\) gravity the heat capacity is positive only conditionally \([109–111]\).

We close this subsection by the examination of the Gibb’s free energy. In terms of the mass, temperature and entropy at the event horizon this is defined as \([102, 112]\):

\[
G(r_+) = M(r_+) - T(r_+)S(r_+).
\]

(5.8)

Inserting (5.1), (5.4) and (5.5) into (5.8), we obtain

\[
G_+ \equiv G(r_+)
\]

\[
= 3s^3(3s^2 + r_+^2) + \epsilon \alpha [30r_+^3 \ln(2r_+) + 31sr_+^2 + 3s^3r_+ - 30s^2r_+ - 231s^3] + 3\epsilon s^3r_+(r_+c_2 - c_1).
\]

(5.9)

In the right graph of figure 4 we depict the behavior of Gibb’s free energy. As we observe it is always positive, for both \(\epsilon = 0\) and \(\epsilon > 0\).
Figure 5. Left graph: The two metric potentials $g_{tt} \equiv a(r)$ and $g_{rr} \equiv b(r)$ given in (3.3), (3.4) versus $r$, for the black hole solution with $Q_1(r) \neq 0$, for $M = 9$, $c_1 = c_2 = 1$, $\alpha = 0.1$, $\epsilon = 0.1$, and $s = 4$, in Planck mass units. Right graph: the metric potential $a(r)$ versus $r$, for $M = 9$, $c_3 = 1$, $c_4 = 1$ and various values of the model parameters $\alpha$, $\epsilon$ and $s$, in Planck mass units. $r_-$ and $r_+$ are the inner and outer horizons respectively, while $r_d$ is the degenerate horizon in which the above two coincide.

5.2 Thermodynamics of the black hole solution with $Q_1(r) \neq 0$

In this subsection we repeat the above thermodynamic analysis in the case of the black hole solution for $Q_1(r) \neq 0$ given in (3.3), (3.4). In the left graph of figure 5 we depict the metric potentials $a(r)$ and $b(r)$, and as we observe $a(r)$ may exhibit two horizons while $b(r)$ does not. In the right graph of figure 5 we present $a(r)$. Similarly to the previous subsection, we see that for small $\alpha$ values, namely small deviations from general relativity, we obtain two horizons, however as $\alpha$ increases there is a specific value in which the two horizons become degenerate ($r_− = r_+ = r_d$), while for larger values the horizon disappears and the central singularity becomes a naked one. However, the interesting feature is that for the same $\alpha$ value, the parameter $s$ that quantifies the charge profile also affects the horizon structure, and in particular larger $s$ leads to the appearance of the naked singularity.

The mass-radius relation takes the form

$$M_+ = \frac{s^2 + r_+^2}{2r_+} + \epsilon \left[ \frac{3sc_3(2s^2 + 2r_+^2 + sr_+^2) + 3s^2c_4r_+ + 56\alpha(s^2 + r_+^2)}{s^2r_+} \right], \quad (5.10)$$

and it is plotted in figure 6, where we can verify that $M_+$ is always positive.

For the temperature (5.2) we obtain

$$T_+ \simeq \frac{3s^2(r_+^2 - s^2) + \epsilon[r_+^2(56\alpha + 3s^2c_3 + 6c_3s) + 6s^2(4\alpha - sc_3)]}{12\pi s^2r_+^3}. \quad (5.11)$$

Moreover, for the entropy (5.3) we find

$$S_+ \simeq \frac{\pi}{r_+^3} \left[ r_+^5 - 2\epsilon\alpha M(r_+M - 2s^2) \right], \quad (5.12)$$

which again for $\epsilon = 0$ recovers the general relativity result. In figure 7 we depict the temperature and entropy versus the horizon, for various values of the model parameters. We mention that in this case both temperature and entropy may acquire negative values, however the entropy, which is always quadratically increasing, is positive when $r_+ > r_d$. 

– 15 –
Figure 6. The mass-radius relation (5.10) for the black hole solution with $Q_1(r) \neq 0$, for $c_3 = 1$, $c_4 = 0$, and $s = 4$, and various values of the model parameters $\alpha$, $\epsilon$ in Planck mass units.

Figure 7. The temperature (left graph) and entropy (right graph) versus the horizon, for the black hole solution with $Q_1(r) \neq 0$, for $M = 9$, $c_3 = c_4 = 1$ and $s = 4$, and various values of the model parameters $\alpha$, $\epsilon$ in Planck mass units.

For the heat capacity $C_+ = \frac{\partial M_+}{\partial r_+} \left( \frac{\partial T}{\partial r_+} \right)^{-1}$, using (5.10) and (5.11), we acquire

$$C_+ \simeq -2\pi(2s^2 + r_+^2 - 2s^2\epsilon c_3 - 8 - \epsilon\alpha).$$  

(5.13)

Expression (5.13) implies that $C_+$ does not diverge and therefore we do not have a second-order phase transition. In the left graph of figure 8 we present $C_+$ as a function of the horizon. As we can see, in the $\epsilon = 0$ case we have $C_+ < 0$ due to the negative derivative of the temperature, as expected for the Reissner Nordström black hole. Nevertheless, for $\epsilon > 0$, namely in the case where the $f(T)$ correction is switched on, we may obtain positive heat capacity. Finally, for the Gibb’s free energy $G(r_+) = M(r_+) - T(r_+)S(r_+)$, using (5.10), (5.11) and (5.12), we find

$$G_+ \simeq \frac{3s^2(r_+^2 + s^2) + \epsilon[3c_3s(2r_+^2 + sr_+^2 + 6s^2) + 6s^2r_+c_4 + 8\alpha(7r_+^2 + 11s^2)]}{12s^2r_+^2}. \quad (5.14)$$
In the right graph of figure 8 we present Gibb’s free energy as a function of the horizon. As we can see for both $\epsilon = 0$ and $\epsilon > 0$ it is always positive.

6 Conclusions and discussion

We investigated the stability of motion and thermodynamics in the case of spherically symmetric solutions in $f(T)$ gravity using the perturbative approach. In particular, we considered small deviations from teleparallel equivalent of general relativity and we extracted charged black hole solutions for two charge profiles, namely with or without a perturbative correction in the charge distribution. Firstly, we examined their asymptotic behavior showing that for large distances they become Minkowski. Then we extracted various torsional and curvature invariants, which revealed the presence of the central singularity as expected. Moreover, we calculated the energy and the mass of the solutions. As we showed, all results recover the general relativity ones in the case where the $f(T)$ deviation goes to zero.

As a next step we investigated the stability of motion around the obtained black hole solutions, by extracting and studying the geodesic deviation of a test particle in their gravitational field. Assuming a secular orbit, we extracted the corresponding stability of motion condition in terms of the metric potentials. As we saw, in the case where the perturbative correction to the charge profile is absent the solution is always stable, however in the case where it is present we obtained unstable regimes in the parameter space.

Additionally, we performed a detailed analysis of the thermodynamic properties of the black hole solutions. In particular, we extracted the inner (Cauchy) and outer (event) horizons, the mass profile, the temperature, the entropy, the heat capacity and the Gibb’s free energy. As we showed, for small $\alpha$ values, namely small deviations from general relativity, we obtain the two horizons, however as $\alpha$ increases there is a specific value in which the two horizons become degenerate, and for larger values the horizon disappears and the central singularity becomes a naked one, a known feature of torsional gravity. Furthermore, we saw that for the same $\alpha$ value, the parameter $s$ that quantifies the charge profile also affects the horizon structure, and in particular larger $s$ leads to the appearance of the naked singularity.
Concerning the temperature and entropy, we showed that although there are regimes in which they become negative, for $r_+ > r_d$ they are always positive definite. Concerning the heat capacity we saw that it does not diverge and thus we do not have a second-order phase transition. However, the most interesting result is that it becomes positive for larger deviations from general relativity, which shows that $f(T)$ modifications improve the thermodynamic stability, which is not the case in other gravitational modifications. Finally, for the Gibb’s free energy, we showed that it is always positive, for all torsional additions and for both charge-profile cases.

In summary, the present work indicates that torsional modification of gravity may have an advantage comparing to other gravitational modification classes, when stability issues are raised, which may serve as an additional motivation for the corresponding investigations. One particular interesting issue is to investigate in detail whether torsional modified gravity leads to smoother (weaker) central singularities comparing to general relativity or curvature modified gravity. This issue will be the focus of interest of a separate project.

References

[1] CANTATA collaboration, Modified Gravity and Cosmology: An Update by the CANTATA Network, arXiv:2105.12582 [nSPIRE].
[2] K.S. Stelle, Renormalization of Higher Derivative Quantum Gravity, Phys. Rev. D 16 (1977) 953 [nSPIRE].
[3] S. Capozziello and M. De Laurentis, Extended Theories of Gravity, Phys. Rept. 509 (2011) 167 [arXiv:1108.6266] [nSPIRE].
[4] A. Unzicker and T. Case, Translation of Einstein’s attempt of a unified field theory with teleparallelism, physics/0503046 [nSPIRE].
[5] R. Aldrovandi and J.G. Pereira, Teleparallel Gravity: An Introduction, Springer, Dordrecht, The Netherlands (2013) [ISBN:978-94-007-5143-9].
[6] T. Shirafuji and G.G.L. Nashed, Energy and momentum in the tetrad theory of gravitation, Prog. Theor. Phys. 98 (1997) 1355 [gr-qc/9711010] [nSPIRE].
[7] J.W. Maluf, The teleparallel equivalent of general relativity, Annalen Phys. 525 (2013) 339 [arXiv:1303.3897] [nSPIRE].
[8] Y.-F. Cai, S. Capozziello, M. De Laurentis and E.N. Saridakis, $f(T)$ teleparallel gravity and cosmology, Rept. Prog. Phys. 79 (2016) 106901 [arXiv:1511.07586] [nSPIRE].
[9] G.R. Bengochea and R. Ferraro, Dark torsion as the cosmic speed-up, Phys. Rev. D 79 (2009) 124019 [arXiv:0812.1205] [nSPIRE].
[10] E.V. Linder, Einstein’s Other Gravity and the Acceleration of the Universe, Phys. Rev. D 81 (2010) 127301 [Erratum ibid. 82 (2010) 109902] [arXiv:1005.3039] [nSPIRE].
[11] G. Kofinas and E.N. Saridakis, Teleparallel equivalent of Gauss-Bonnet gravity and its modifications, Phys. Rev. D 90 (2014) 084044 [arXiv:1404.2249] [nSPIRE].
[12] S. Bahamonde, C.G. Böhmer and M. Wright, Modified teleparallel theories of gravity, Phys. Rev. D 92 (2015) 104042 [arXiv:1508.05120] [nSPIRE].
[13] L. Karpathopoulos, S. Basilakos, G. Leon, A. Paliathanasis and M. Tsamparlis, Cartan symmetries and global dynamical systems analysis in a higher-order modified teleparallel theory, Gen. Rel. Grav. 50 (2018) 79 [arXiv:1709.02197] [nSPIRE].
[14] C.G. Boehmer and E. Jensko, Modified gravity: A unified approach, Phys. Rev. D 104 (2021) 024010 [arXiv:2103.15906] [nSPIRE].
[15] C.-Q. Geng, C.-C. Lee, E.N. Saridakis and Y.-P. Wu, “Teleparallel” dark energy, Phys. Lett. B 704 (2011) 384 [arXiv:1109.1092] [SPIRE].

[16] M. Hohmann, L. Järvi and U. Ualikhanova, Covariant formulation of scalar-torsion gravity, Phys. Rev. D 97 (2018) 104011 [arXiv:1801.05786] [SPIRE].

[17] S. Bahamonde, K.F. Dialektopoulos and J. Levi Said, Can Horndeski Theory be recast using Teleparallel Gravity?, Phys. Rev. D 100 (2019) 064018 [arXiv:1904.10791] [SPIRE].

[18] R. Zheng and Q.-G. Huang, Growth factor in $f(T)$ gravity, JCAP 03 (2011) 002 [arXiv:1010.3512] [SPIRE].

[19] W. El Hanafy and G.G.L. Nashed, Exact Teleparallel Gravity of Binary Black Holes, Astrophys. Space Sci. 361 (2016) 68 [arXiv:1507.07377] [SPIRE].

[20] K. Bamba, C.-Q. Geng, C.-C. Lee and L.-W. Luo, Equation of state for dark energy in $f(T)$ gravity, JCAP 01 (2011) 021 [arXiv:1011.0508] [SPIRE].

[21] S. Capozziello, V.F. Cardone, H. Farajollahi and A. Ravanpak, Cosmography in $f(T)$-gravity, Phys. Rev. D 84 (2011) 043527 [arXiv:1108.2789] [SPIRE].

[22] H. Wei, X.-J. Guo and L.-F. Wang, Noether Symmetry in $f(T)$ Theory, Phys. Lett. B 707 (2012) 298 [arXiv:1112.2270] [SPIRE].

[23] J. Amorós, J. de Haro and S.D. Odintsov, Bouncing loop quantum cosmology from $F(T)$ gravity, Phys. Rev. D 87 (2013) 104037 [arXiv:1305.2344] [SPIRE].

[24] G. Otalora, Cosmological dynamics of tachyonic teleparallel dark energy, Phys. Rev. D 88 (2013) 063505 [arXiv:1305.5896] [SPIRE].

[25] K. Bamba, S.D. Odintsov and D. Sáez-Gómez, Conformal symmetry and accelerating cosmology in teleparallel gravity, Phys. Rev. D 88 (2013) 084042 [arXiv:1308.5789] [SPIRE].

[26] J.-T. Li, C.-C. Lee and C.-Q. Geng, Einstein Static Universe in Exponential $f(T)$ Gravity, Eur. Phys. J. C 73 (2013) 2315 [arXiv:1302.2688] [SPIRE].

[27] M. Malekjani, N. Haidari and S. Basilakos, Spherical collapse model and cluster number counts in power law $f(T)$ gravity, Mon. Not. Roy. Astron. Soc. 466 (2017) 3488 [arXiv:1609.01964] [SPIRE].

[28] G. Farrugia and J. Levi Said, Stability of the flat FLRW metric in $f(T)$ gravity, Phys. Rev. D 94 (2016) 124054 [arXiv:1701.00134] [SPIRE].

[29] G.G.L. Nashed, Schwarzschild solution in extended teleparallel gravity, EPL 105 (2014) 10001 [arXiv:1501.00974] [SPIRE].

[30] E.N. Saridakis, Introduction to teleparallel and $f(T)$ gravity and cosmology, in 14th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG14), Rome, Italy, (2015), World Scientific, New York, U.S.A. (2017), pg. 1135.

[31] J.-Z. Qi, S. Cao, M. Biesiada, X. Zheng and H. Zhu, New observational constraints on $f(T)$ cosmology from radio quasars, Eur. Phys. J. C 77 (2017) 502 [arXiv:1708.08603] [SPIRE].

[32] Y.-F. Cai, C. Li, E.N. Saridakis and L. Xue, $f(T)$ gravity after GW170817 and GRB170817A, Phys. Rev. D 97 (2018) 103513 [arXiv:1801.05827] [SPIRE].

[33] H. Abedi, S. Capozziello, R. D’Agostino and O. Luongo, Effective gravitational coupling in modified teleparallel theories, Phys. Rev. D 97 (2018) 084008 [arXiv:1803.07171] [SPIRE].

[34] A. El-Zant, W. El Hanafy and S. Elgammal, $H_0$ Tension and the Phantom Regime: A Case Study in Terms of an Infrared $f(T)$ Gravity, Astrophys. J. 871 (2019) 210 [arXiv:1809.09390] [SPIRE].
[35] F.K. Anagnostopoulos, S. Basilakos and E.N. Saridakis, *Bayesian analysis of f(T) gravity using fσ data*, Phys. Rev. D **100** (2019) 083517 [arXiv:1907.07533] [inSPIRE].

[36] Y.-F. Cai, M. Khurshudyan and E.N. Saridakis, *Model-independent reconstruction of f(T) gravity from Gaussian Processes*, Astrophys. J. **888** (2020) 62 [arXiv:1907.10813] [inSPIRE].

[37] S.-F. Yan, P. Zhang, J.-W. Chen, X.-Z. Zhang, Y.-F. Cai and E.N. Saridakis, *Interpreting cosmological tensions from the effective field theory of torsional gravity*, Phys. Rev. D **101** (2020) 121301 [arXiv:1909.06388] [inSPIRE].

[38] A. Awad, W. El Hanafy, G.G.L. Nashed, S.D. Odintsov and V.K. Oikonomou, *Constant-roll Inflation in f(T) Teleparallel Gravity*, JCAP **07** (2018) 026 [arXiv:1710.00682] [inSPIRE].

[39] W. El Hanafy and G.G.L. Nashed, *Phenomenological Reconstruction of f(T) Teleparallel Gravity*, Phys. Rev. D **100** (2019) 083535 [arXiv:1910.04160] [inSPIRE].

[40] D. Wang and D. Mota, *Can f(T) gravity resolve the H0 tension?*, Phys. Rev. D **102** (2020) 063530 [arXiv:2003.10095] [inSPIRE].

[41] W. El Hanafy and E.N. Saridakis, *f(T) cosmology: from Pseudo-Bang to Pseudo-Rip*, JCAP **09** (2021) 019 [arXiv:2011.15070] [inSPIRE].

[42] G.G.L. Nashed, *Brane World black holes in Teleparallel Theory Equivalent to General Relativity and their Killing vectors, Energy, Momentum and Angular-Momentum*, Chin. Phys. B **19** (2010) 020401 [arXiv:0910.5124] [inSPIRE].

[43] M. Hashim, W. El Hanafy, A. Golovnev and A.A. El-Zant, *Toward a concordance teleparallel cosmology. Part I. Background dynamics*, JCAP **07** (2021) 052 [arXiv:2010.14964] [inSPIRE].

[44] X. Ren, T.H.T. Wong, Y.-F. Cai and E.N. Saridakis, *Data-driven Reconstruction of the Late-time Cosmic Acceleration with f(T) Gravity*, Phys. Dark Univ. **32** (2021) 100812 [arXiv:2103.01260] [inSPIRE].

[45] C.G. Boehmer, A. Mussa and N. Tamanini, *Existence of relativistic stars in f(T) gravity*, Class. Quant. Grav. **28** (2011) 245020 [arXiv:1107.4455] [inSPIRE].

[46] P.A. Gonzalez, E.N. Saridakis and Y. Vasquez, *Circularly symmetric solutions in three-dimensional Teleparallel, f(T) gravity and Maxwell-f(T) gravity*, JHEP **07** (2012) 053 [arXiv:1110.4024] [inSPIRE].

[47] S. Capozziello, P.A. Gonzalez, E.N. Saridakis and Y. Vasquez, *Exact charged black-hole solutions in D-dimensional f(T) gravity: torsion vs curvature analysis*, JHEP **02** (2013) 039 [arXiv:1210.1098] [inSPIRE].

[48] R. Ferraro and F. Fiorini, *Spherically symmetric static spacetimes in vacuum f(T) gravity*, Phys. Rev. D **84** (2011) 083518 [arXiv:1109.4209] [inSPIRE].

[49] T. Wang, *Static Solutions with Spherical Symmetry in f(T) Theories*, Phys. Rev. D **84** (2011) 024042 [arXiv:1102.4410] [inSPIRE].

[50] K. Atazadeh and M. Mousavi, *Vacuum spherically symmetric solutions in f(T) gravity*, Eur. Phys. J. C **73** (2013) 2272 [arXiv:1212.3764] [inSPIRE].

[51] W. El Hanafy and G.G.L. Nashed, *Exact Teleparallel Gravity of Binary Black Holes*, Astrophys. Space Sci. **361** (2016) 68 [arXiv:1507.07377] [inSPIRE].

[52] M.E. Rodrigues, M.J.S. Houndjo, J. Tossa, D. Momeni and R. Myrzakulov, *Charged Black Holes in Generalized Teleparallel Gravity*, JCAP **11** (2013) 024 [arXiv:1306.2280] [inSPIRE].

[53] G.G.L. Nashed, *Spherically symmetric charged-dS solution in f(T) gravity theories*, Phys. Rev. D **88** (2013) 104034 [arXiv:1311.3131] [inSPIRE].

[54] G.G.L. Nashed, *Spherically Symmetric Solution in (1 + 4)-Dimensional f(T) Gravity Theories*, Adv. High Energy Phys. **2014** (2014) 830109 [arXiv].
[75] G.G.L. Nashed and W. El Hanafy, Analytic rotating black hole solutions in N-dimensional $f(T)$ gravity, *Eur. Phys. J. C* 77 (2017) 90 [arXiv:1612.05106] [INSPIRE].

[76] A. Paliathanasis et al., New Schwarzschild-like solutions in $f(T)$ gravity through Noether symmetries, *Phys. Rev. D* 89 (2014) 104042 [arXiv:1402.5935] [INSPIRE].

[77] A. DeBenedictis and S. Ilijic, Spherically symmetric vacuum in covariant $F(T) = T + \frac{\alpha}{2} T^2 + \mathcal{O}(T^3)$ gravity theory, *Phys. Rev. D* 94 (2016) 124025 [arXiv:1609.07465] [INSPIRE].

[78] S. Bahamonde, J. Levi Said and M. Zubair, Solar system tests in modified teleparallel gravity, *JCAP* 10 (2020) 024 [arXiv:2006.06750] [INSPIRE].

[79] S. Bahamonde, K. Flathmann and C. Pfeifer, Photon sphere and perihelion shift in weak $f(T)$ gravity, *Phys. Rev. D* 100 (2019) 084064 [arXiv:1907.10858] [INSPIRE].

[80] M.L. Ruggiero and N. Radicella, Weak-Field Spherically Symmetric Solutions in $f(T)$ gravity, *Phys. Rev. D* 91 (2015) 104014 [arXiv:1501.02198] [INSPIRE].

[81] G.G.L. Nashed, Quadratic and cubic spherically symmetric black holes in the modified teleparallel equivalent of general relativity: energy and thermodynamics, *Class. Quant. Grav.* 38 (2021) 125004 [arXiv:2105.05688] [INSPIRE].

[82] C.G. Böhmer and F. Fiorini, The regular black hole in four dimensional Born–Infeld gravity, *Class. Quant. Grav.* 36 (2019) 12LT01 [arXiv:1901.02966] [INSPIRE].

[83] C.G. Böhmer and F. Fiorini, BTZ gems inside regular Born-Infeld black holes, *Class. Quant. Grav.* 37 (2020) 185002 [arXiv:2005.11843] [INSPIRE].

[84] J. Plebański, Lectures on non-linear electrodynamics, NORDITA Publishing, Stockholm, Sweden (1970).

[85] M. Krššák and E.N. Saridakis, The covariant formulation of $f(T)$ gravity, *Class. Quant. Grav.* 33 (2016) 115009 [arXiv:1510.08432] [INSPIRE].

[86] A. Golovnev, T. Koivisto and M. Sandstad, On the covariance of teleparallel gravity theories, *Class. Quant. Grav.* 34 (2017) 145013 [arXiv:1701.08271] [INSPIRE].

[87] M. Hohmann, L. Järv, M. Krššák and C. Pfeifer, Teleparallel theories of gravity as analogue of nonlinear electrodynamics, *Phys. Rev. D* 97 (2018) 104042 [arXiv:1711.09930] [INSPIRE].

[88] S. Bahamonde, K. Flathmann and C. Pfeifer, Photon sphere and perihelion shift in weak $f(T)$ gravity, *Phys. Rev. D* 100 (2019) 084064 [arXiv:1907.10858] [INSPIRE].

[89] S. Nesseris, S. Basilakos, E.N. Saridakis and L. Perivolaropoulos, Viable $f(T)$ models are practically indistinguishable from ΛCDM, *Phys. Rev. D* 88 (2013) 103010 [arXiv:1308.6142] [INSPIRE].

[90] R.C. Nunes, S. Pan and E.N. Saridakis, New observational constraints on $f(T)$ gravity from cosmic chronometers, *JCAP* 08 (2016) 011 [arXiv:1606.04359] [INSPIRE].

[91] C. Li, Y. Cai, Y.-F. Cai and E.N. Saridakis, The effective field theory approach of teleparallel gravity, $f(T)$ gravity and beyond, *JCAP* 10 (2018) 001 [arXiv:1803.09818] [INSPIRE].

[92] S.C. Ulhoa and E.P. Spaniol, On the gravitational energy–momentum vector in $f(T)$ theories, *Int. J. Mod. Phys. D* 22 (2013) 1350069.

[93] J.W. Maluf, J.F. da Rocha-Neto, T.M.L. Toribio and K.H. Castello-Branco, Energy and angular momentum of the gravitational field in the teleparallel geometry, *Phys. Rev. D* 65 (2002) 124001 [gr-qc/0204035] [INSPIRE].

[94] G.G.L. Nashed and T. Shirafuji, Reissner-Nordstrom spacetime in the tetrad theory of gravitation, *Int. J. Mod. Phys. D* 16 (2007) 65 [arXiv:0704.3898] [INSPIRE].

[95] R.A. D’Inverno, *Introducing Einstein’s relativity*, Clarendon Press, Oxford, U.K. (1992).
[96] A. Sheykhi, *Higher-dimensional charged f(R) black holes*, Phys. Rev. D 86 (2012) 024013 [arXiv:1209.2960] [INSPIRE].

[97] A. Sheykhi, *Thermodynamics of apparent horizon and modified Friedmann equations*, Eur. Phys. J. C 69 (2010) 265 [arXiv:1012.0383] [INSPIRE].

[98] S.H. Hendi, A. Sheykhi and M.H. Dehghani, *Thermodynamics of higher dimensional topological charged AdS black branes in dilaton gravity*, Eur. Phys. J. C 70 (2010) 703 [arXiv:1002.0202] [INSPIRE].

[99] A. Sheykhi, M.H. Dehghani and S.H. Hendi, *Thermodynamic instability of charged dilaton black holes in AdS spaces*, Phys. Rev. D 81 (2010) 084040 [arXiv:0912.4199] [INSPIRE].

[100] R.-X. Miao, M. Li and Y.-G. Miao, *Violation of the first law of black hole thermodynamics in f(T) gravity*, JCAP 11 (2011) 033 [arXiv:1107.0515] [INSPIRE].

[101] G. Cognola, O. Gorbunova, L. Sebastiani and S. Zerbini, *On the Energy Issue for a Class of Modified Higher Order Gravity Black Hole Solutions*, Phys. Rev. D 84 (2011) 023515 [arXiv:1104.2814] [INSPIRE].

[102] Y. Zheng and R.-J. Yang, *Horizon thermodynamics in f(R) theory*, Eur. Phys. J. C 78 (2018) 682 [arXiv:1806.09858] [INSPIRE].

[103] G.G.L. Nashed, *Stability of the vacuum nonsingular black hole*, Chaos Solitons Fractals 15 (2003) 841 [gr-qc/0301008] [INSPIRE].

[104] Y.S. Myung, *Instability of rotating black hole in a limited form of f(R) gravity*, Phys. Rev. D 84 (2011) 024048 [arXiv:1104.3180] [INSPIRE].

[105] Y.S. Myung, *Instability of a Kerr black hole in f(R) gravity*, Phys. Rev. D 88 (2013) 104017 [arXiv:1309.3346] [INSPIRE].

[106] K. Nouicer, *Black holes thermodynamics to all order in the Planck length in extra dimensions*, Class. Quant. Grav. 24 (2007) 5917 [Erratum ibid. 24 (2007) 6435] [arXiv:0706.2749] [INSPIRE].

[107] I. Dymnikova and M. Korpusik, *Thermodynamics of Regular Cosmological Black Holes with the de Sitter Interior*, Entropy 13 (2011) 1967, http://www.mdpi.com/1099-4300/13/12/1967.

[108] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, Phys. Rev. D 60 (1999) 064018 [hep-th/9902170] [INSPIRE].

[109] E. Elizalde, G.G.L. Nashed, S. Nojiri and S.D. Odintsov, *Spherically symmetric black holes with electric and magnetic charge in extended gravity: physical properties, causal structure, and stability analysis in Einstein’s and Jordan’s frames*, Eur. Phys. J. C 80 (2020) 109 [arXiv:2001.11357] [INSPIRE].

[110] G.G.L. Nashed and E.N. Saridakis, *New rotating black holes in nonlinear Maxwell f(R) gravity*, Phys. Rev. D 102 (2020) 124072 [arXiv:2010.10422] [INSPIRE].

[111] G.G.L. Nashed and S. Nojiri, *Non-trivial black hole solutions in f(R) gravitational theory*, Phys. Rev. D 102 (2020) 124022 [arXiv:2012.05711] [INSPIRE].

[112] W. Kim and Y. Kim, *Phase transition of quantum corrected Schwarzshild black hole*, Phys. Lett. B 718 (2012) 687 [arXiv:1207.5318] [INSPIRE].