A Theory of Intermittency Renormalization of 1D Gaussian Multiplicative Chaos Measures

Dmitry Ostrovsky

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Abstract

A theory of intermittency differentiation is developed for a general class of 1D Gaussian Multiplicative Chaos measures including the measure of Bacry and Muzy on the interval and circle as special cases. An exact, non-local functional equation is derived for the derivative of a general functional of the total mass of the measure with respect to intermittency. The formal solution is given in the form of an intermittency expansion and proved to be a renormalized expansion in the centered moments of the total mass of the measure. The full intermittency expansion of the Mellin transform of the total mass is computed for a class of Gaussian Multiplicative Chaos measures. The theory is shown to extend to the dependence structure of the measure. For application, the intermittency expansion of the Bacry-Muzy measure on the circle is computed exactly, and the Morris integral probability distribution is shown to reproduce the moments of the total mass and the intermittency expansion, resulting in the conjecture that it is the distribution of the total mass. It is conjectured in general that the intermittency expansion is convergent for smooth test functions and captures the distribution of the total mass uniquely.

1 Introduction

The theory of Gaussian Multiplicative Chaos (GMC) measures has greatly advanced since its inception in 1972 by Mandelbrot [22], who introduced the key ingredients of what is now known as GMC under the name of the limit lognormal measure, cf. also his review [23]. The mathematical foundation of the subject was laid down by Kahane [19], who created a comprehensive, mathematically rigorous theory of multiplicative chaos measures based on his theory of convergence of a particular class of positive martingales. The theory was advanced further around 2000 with the introduction of the conical set construction by Barral and Mandelbrot [4] and Schmitt and Marsan [40] and assumed its modern form with the theory of infinitely divisible multiplicative chaos measures of Bacry and Muzy [3], [24] that is based on a spectral representation of infinitely divisible processes of Rajput and Rosinski [39]. The theory of Bacry and Muzy was limited to multiplicative chaos on a finite interval. It has since been extended in the gaussian case to multiple dimensions by Robert and Vargas [37], to other geometric shapes such as the circle by Astala et. al [1], as well as to critical multiplicative chaos by Duplantier et. al [10] and Barral et. al. [6], and most recently to super-critical multiplicative chaos by Madaule et. al. [21]. In the general infinitely divisible case the theory of Bacry and Muzy was further advanced by Barral and Jin [5] and we derived key invariance properties of the underlying infinitely divisible field in [28] and a formula for the moments of the total mass in [35].
The interest in multiplicative chaos derives from its remarkable property of multifractality, from complexity of mathematical problems that it poses such as understanding its stochastic dependence structure, and from the many applications in mathematics and theoretical and statistical physics, in which it naturally appears. Without aiming for comprehension, we can mention applications to conformal field theory and quantum gravity [7], [9], [36], statistical mechanics of disordered energy landscapes and extrema of the 2D gaussian free field [12], [14], [15], [18], [20], [34], a theory of conformal weldings [1], and even conjectured applications to the behavior of the Riemann zeta function on the critical line [16], [33].

A fundamental open problem in the theory of GMC is to calculate the distribution of the total mass of the chaos measure and, more generally, understand its stochastic dependence structure, i.e. the joint distribution of the measure of several subsets of the set, on which it is defined. The contribution of this paper is to advance this problem in dimension one (1D) for a general class of 1D GMC measures having the property that the positive integer moments of its total mass are known in the form of a multiple integral of Selberg type. For example, the moments of the total mass of the Bacry-Muzy GMC measure are given by the classical Selberg integral [2] on the interval and by the Morris integral on the circle [12], [34]. The primary challenge of recovering the distribution from the moments is that the moments become infinite at any level of intermittency (also referred to as the inverse temperature in the statistical physics literature). Hence, the problem of recovering the distribution from the moments is that of renormalization, i.e. of removing infinity from the moments and re-summing them so as to reconstruct the distribution. In the special case of the Bacry-Muzy GMC measure on the interval we developed in a series of papers [25]–[30] the theory of intermittency differentiation that allowed us to compute the full high-temperature (low intermittency) expansion of the Mellin transform of the total mass and effectively reconstruct the Mellin transform by summing the intermittency expansion. We then checked that the resulting expression is the Mellin transform of a valid probability distribution, known as the Selberg integral probability distribution, having the properties that its positive integer moments are given by the Selberg integral and that the asymptotic expansion of its Mellin transform coincides with the intermittency expansion. Thus, we constructed a good candidate for the distribution of the total mass in the sub-critical regime and then calculated the critical distribution as a simple limiting case, cf. [32], [34].

It is worth emphasizing that the main achievement of the intermittency differentiation approach is that it provides an exact mechanism for renormalization. It is possible in some cases to guess the Mellin transform, in the sense of a function of a complex variable whose restriction to the finite interval of positive integers, where the moments are finite, coincides with the moments, cf. [8], [12], [14], [13], [34]. While this method produces the same formulas for the Mellin transform of the total mass of the Bacry-Muzy GMC measure on the interval and circle as ours, it does not capture the distribution uniquely because it operates on the moments directly and the moment problem is not determinate.

The contribution of this paper is threefold. Our main contribution is to extend the theory of intermittency differentiation to a general 1D GMC measure on both the interval and circle. The GMC measure is defined as the exponential functional of a regularized gaussian process with logarithmic covariance in the limit of zero regularization. We show that the key intermittency invariance of the underlying gaussian process, which we originally derived in [25] in the Bacry-Muzy case on the interval, naturally extends to the general case. This observation allows us to carry over our original approach to the general 1D GMC measure. In particular, we prove that the intermittency expansion is an exactly renormalized expansion in the centered moments of the total mass. As an application,
we treat the case of the Bacry-Muzy GMC on the circle and compute the Mellin transform of the total mass exactly, recovering the results of [12] and [34] that were obtained heuristically. The second contribution is to give a comprehensive exposition of our approach. Originally, the derivations of the intermittency differentiation rule, of intermittency expansions, and of the Mellin transform were presented in different publications with varying degrees of generality. The goal of this paper is to collect them all in one place so as to emphasize their generality and make our approach accessible to a wider audience. Finally, the third contribution is to compile a list of conjectures that might lead to further advances in the future, the primary of which is that the intermittency expansion is convergent for a class of smooth test functions and therefore captures the distribution of the total mass of the 1D GMC measure uniquely.

Our paper is limited to the problem of the distribution of the total mass of 1D GMC measures and to the derivation of the intermittency expansion of its Mellin transform. We will not attempt to review the theory of the Selberg and Morris integral probability distributions, which are conjectured to be the distributions of the total mass of the Bacry-Muzy GMC measure on the interval and circle, respectively, as it would lead us to the subject of Barnes beta probability distributions [31], [32] that is outside the scope of this paper and that we recently reviewed in [34]. We will also not review the theory of multiplicative chaos measures per se and refer the interested reader to [38] for a general review in the gaussian case, to [30] and [33] for detailed reviews of the gaussian case on the interval and to [28] and [35] for the infinitely divisible case on the interval.

Our results are mathematically rigorous except for the derivation of the intermittency differentiation rule, which is exact in the sense of equality of formal power series but not mathematically rigorous.

The plan of the paper is as follows. In Section 2 we briefly review the general 1D GMC construction following the ideas of Bacry and Muzy and state the formula for the moments of the total mass. In Section 3 we state the rule of intermittency differentiation and explain its origin in the simplest case of integer moments. In Section 4 we derive the intermittency expansion and prove that it is an exactly renormalized expansion in the centered moments. In Section 5 we calculate the high temperature (low intermittency) expansion of the Mellin transform in terms of the expansion of log-moments in intermittency. In Section 6 we extend the intermittency differentiation approach to multiple subintervals, i.e. the dependence structure of the GMC measure. In Section 7 we give a formal derivation of the intermittency differentiation rule from the first principles. In Section 8 we calculate the distribution of the total mass of the Bacry-Muzy GMC measure on the circle and relate it to the Morris integral probability distribution. In Section 9 we list a number of key conjectures and open problems. Conclusions are given in Section 10. The appendix presents the general 1D GMC measure on the interval as a deformation of the Bacry-Muzy construction.

## 2 A Brief Review of 1D Gaussian Multiplicative Chaos

Consider a stationary gaussian process having the general logarithmic covariance of the form

\[
\text{Cov} \left[ \omega_{\mu, \varepsilon}(s), \omega_{\mu, \varepsilon}(t) \right] = \begin{cases} 
-\mu \log r(s-t), & \varepsilon \leq |s-t| < 1, \\
\mu \left( -\log r(\varepsilon) + (1 - \frac{|s-t|}{\varepsilon}) \frac{1}{\varepsilon} \log r(\varepsilon) \right), & |s-t| \leq \varepsilon.
\end{cases}
\]  

(1)

The parameter $\mu > 0$ is known as intermittency and is often written in the form $\mu = 2\beta^2$, in which case $\beta$ is referred to as the inverse temperature. The process $\omega_{\mu, \varepsilon}(t)$ is defined on $t \in (0,1)$. Its mean
is defined by the condition
\[ E[\omega_{\mu, \epsilon}(s)] = -\frac{1}{2} \text{Var}[\omega_{\mu, \epsilon}(s)]. \tag{2} \]

The function \( r(t) \) is assumed to have the following properties\(^1\):

1. \( r(t) \) is smooth and even on \((-1, 0) \cup (0, 1)\),
2. \( \lim_{t \to 0} t \frac{d}{dt} \log r(t) = 1 \),
3. \( (s, t) \to -\log r(s - t) \) is positive definite. \( \tag{5} \)

We will also impose one of the two boundary conditions,

1. \( r(1) = 1 \), or
2. \( r(t) = r(1 - t) \). \( \tag{6a} \) \( \tag{6b} \)

corresponding to the process being defined on the interval or circle, respectively. The two main examples are

1. \( r(t) = |t| \), \( \tag{7} \)
2. \( r(t) = |1 - e^{2\pi i t}| \). \( \tag{8} \)

The first is the Bacry-Muzy process \(^2\) and the second is its circular version first considered in \(^1\).

It is shown in the Appendix that the gaussian process defined in (1) can be constructed by properly generalizing the construction of Bacry and Muzy. Note that the condition in (2) implies

\[ E[e^{\omega_{\mu, \epsilon}(s)}] = 1. \tag{9} \]

Now, consider the associated random measure (also known as the partition function),

\[ M_{\mu, \epsilon}[\varphi](t) \buildrel \Delta \over = \int_0^t \varphi(s) e^{\omega_{\mu, \epsilon}(s)} ds. \tag{10} \]

We will assume for simplicity that \( \varphi(s) > 0 \). The measure is normalized so that

\[ E[M_{\mu, \epsilon}[\varphi](t)] = \int_0^t \varphi(s) ds. \tag{11} \]

The theory of Kahane \(^3\) implies that the limit is a non-trivial random measure \( M_\mu(ds) \) for a range of \( \mu \in [0, \mu_c) \) so that

\[ \lim_{\epsilon \to 0} M_{\mu, \epsilon}[\varphi](t) = \int_0^t \varphi(s) M_\mu(ds). \tag{12} \]

The positive integer moments of the total mass\(^4\) can be calculated up to some critical value. Denote

\[ S_n[\varphi](\mu) = E\left[ \left( \int_0^1 \varphi(s) M_\mu(ds) \right)^n \right]. \tag{13} \]

\(^1\) Positive definiteness in the interval case follows from the other conditions provided \( d^2/dt^2 \log r(t) < 0 \) and \( d/dt|_{t=0} \log r(t) > 0 \), cf. the Appendix. Also, in the circular case, the condition \( \epsilon \leq |s - t| < 1 \) is replaced with \( \epsilon \leq |s - t| \leq 1 - \epsilon \).

\(^2\) By a slight abuse of terminology, we refer to any integral of the form \( \int_0^1 \varphi(t) M_\mu(dt) \) as the total mass.
Then, the standard gaussian calculation shows that the $n$th moment is

$$
S_n[\varphi](\mu) = \int_{[0,1]^n} \prod_{i=1}^n \Phi(s_i) \prod_{i<j} r(s_i - s_j)^{-\mu} ds_1 \cdots ds_n, \ n < 2/\mu,
$$

(14)

and is infinite otherwise. In fact, it is easy to see that the contribution of the region where the integrand is large, i.e. where the points are within $\varepsilon$ apart, is of the order

$$
O(\varepsilon^{n-1} r(\varepsilon)^{-\mu n(n-1)/2}).
$$

(15)

As $r(\varepsilon) \sim \text{const} \varepsilon$ as $\varepsilon \to 0$ by (4), the condition for the existence of the integral is then

$$
n < 2/\mu,
$$

(16)

so that the exponent in (15) is positive. We also note that the moments scale quadratically,

$$
\mathbb{E} \left[ \left( \int_0^t M_\mu(ds) \right)^n \right] \sim \text{const} t^{n-\mu n(n-1)/2}, \ t \to 0.
$$

(17)

This means that the multifractal spectrum of the measure is

$$
\zeta(q) = q - \frac{1}{2} \mu q (q - 1).
$$

(18)

Relying on the theory of Bacry and Muzy [24], the measure is non-degenerate provided

$$
\zeta'(1) > 0,
$$

(19)

and the positive moments for $q > 1$ are finite if

$$
\zeta(q) > 1.
$$

(20)

The first condition gives us

$$
\mu_c = 2,
$$

(21)

and the second recovers (16). These conditions are well-known in the case of the Bacry-Muzy GMC measure on the interval and circle. Throughout this paper it is tacitly assumed that $\mu < 2$, i.e. we are in the sub-critical regime.

### 3 Statement of the Intermittency Differentiation Rule

In this section we will state the fundamental rule of intermittency differentiation and give it a simple derivation in a special case. The formal derivation in the general case is given in Section 7. We note that it is an open problem to produce a mathematically rigorous derivation.

Consider the general functional of the (generalized) total mass of the form

$$
v(\mu, f, F) \equiv \mathbb{E} \left[ F \left( \int_0^1 e^{\mu f(s)} \varphi(s) M_\mu(ds) \right) \right],
$$

(22)
where \( f(s) \) are \( F(x) \) are sufficiently smooth but otherwise arbitrary. It is understood that the integration with respect to \( M_\mu(ds) \) is in the sense of \( \varepsilon \to 0 \) limit so that \( v(\mu, f, F) = \lim_{\varepsilon \to 0} v_\varepsilon(\mu, f, F) \) and \( v_\varepsilon(\mu, f, F) \triangleq E\left[ F\left( \int_0^1 e^{\mu f(s)} \varphi(s) M_\mu,e(ds) \right) \right] \) with \( M_\mu,e(ds) \) as in (10). Also, let \( g(s_1, s_2) \) be defined by

\[
g(s_1, s_2) \triangleq -\log r(s_1 - s_2).
\]

Then, we have the following main result extending the corresponding result for the Bacry-Muzy GMC measure on the interval, cf. [25]–[30], in the form of a functional Feynman-Kac equation, in which the intermittency plays the role of time.

**Theorem 3.1 (Rule of Intermittency Differentiation)** The expectation \( v(\mu, f, F) \) is invariant under intermittency differentiation and satisfies

\[
\frac{\partial}{\partial \mu} v(\mu, f, F) = \int_{[0, 1]} v(\mu, f + g(\cdot, s), F(1)) e^{\mu f(s)} f(s) \varphi(s) ds +
\]

\[
\frac{1}{2} \int_{[0, 1]^2} v(\mu, f + g(\cdot, s_1) + g(\cdot, s_2), F(2)) e^{\mu f(s_1) + f(s_2) + g(s_1, s_2)} \times
\]

\[
\times g(s_1, s_2) \varphi(s_1) \varphi(s_2) ds(2).
\]

The mathematical content of (24) is that differentiation with respect to the intermittency parameter \( \mu \) is equivalent to a combination of two functional shifts induced by the \( g \) function. It is clear that both terms in (24) are of the same functional form as the original functional in (22) so that Theorem 3.1 allows us to compute derivatives of all orders. In fact,

\[
v(\mu, f + g(\cdot, s), F(1)) = E\left[ F'\left( \int_0^1 e^{\mu f(t)} \varphi(t) r(s - t)^{-\mu} M_\mu(dt) \right) \right],
\]

\[
v(\mu, f + g(\cdot, s_1) + g(\cdot, s_2), F(2)) = E\left[ F''\left( \int_0^1 e^{\mu f(t)} \varphi(t) r(s_1 - t)^{-\mu} r(s_2 - t)^{-\mu} M_\mu(dt) \right) \right].
\]

The origin of the differentiation rule can be most easily explained by examining the case of the positive integer moments,

\[
F(x) = x^n.
\]

In this case one can give an elementary derivation that is based on the formula for the moments in (14). We have by (14),

\[
v(\mu, f, F) = \int_{[0, 1]^n} \prod_{i=1}^n \varphi(s_i) e^{\mu f(s_i)} \prod_{i<j}^n r(s_i - s_j)^{-\mu} ds^{(n)}, n < 2/\mu.
\]

The integrand is manifestly symmetric in \( (s_1 \cdots s_n) \) so that

\[
\frac{\partial}{\partial \mu} v(\mu, f, F) = n \int_{[0, 1]} ds f(s) \varphi(s) e^{\mu f(s)} \left[ \int_{[0, 1]^{n-1}} \prod_{i=1}^{n-1} \varphi(s_i) e^{\mu f(s_i)} r(s - s_i)^{-\mu} \prod_{i<j}^n r(s_i - s_j)^{-\mu} ds^{(n-1)} \right] +
\]

\[
+ \frac{1}{2} n(n - 1) \int_{[0, 1]^2} ds_1 ds_2 g(s_1, s_2) \varphi(s_1) \varphi(s_2) e^{\mu f(s_1) + f(s_2) + g(s_1, s_2)} \times
\]

*\( \varphi(s) \) is fixed and dropped from the list of arguments for brevity.*
\[
\times \left[ \int_{[0,1]^{n-2}} \prod_{i=1}^{n-2} \phi(s_i) e^{\mu f(s_i)} r(s_1 - s_i)^{-\mu} r(s_2 - s_i)^{-\mu} \prod_{i<j} r(s_i - s_j)^{-\mu} ds^{(n-2)} \right].
\]

(29)

It is elementary to see from (14), (25), (26) that for \( F(x) = x^\mu \) we have the identities

\[
v(\mu, f + g(\cdot, s), F^{(1)}) = n \int_{[0,1]^{n-1}} \prod_{i=1}^{n-1} \phi(s_i) e^{\mu f(s_i)} r(s - s_i)^{-\mu} \prod_{i<j} r(s_i - s_j)^{-\mu} ds^{(n-1)},
\]

(30)

\[
v(\mu, f + g(\cdot, s_1) + g(\cdot, s_2), F^{(2)}) = n(n-1) \int_{[0,1]^{n-2}} \prod_{i=1}^{n-2} \phi(s_i) e^{\mu f(s_i)} r(s_1 - s_i)^{-\mu} r(s_2 - s_i)^{-\mu} \times
\]

\[
\times \prod_{i<j} r(s_i - s_j)^{-\mu} ds^{(n-2)}.
\]

(31)

Thus, the terms in the square brackets in (29) are the same first and second derivative terms as in (24).

### 4 The Intermittency Expansion and Renormalization

The structure of the intermittency differentiation rule implies that one can represent the solution in the form of an expansion in \( \mu \), at least formally. As an immediate corollary of Theorem 3.1, we obtain the following expansion (cf. (26) for the special case of the Baccry-Muzy GMC measure on the interval). Let

\[
x \triangleq \frac{1}{\int_0^1 \phi(s) \, ds}.
\]

(32)

**Theorem 4.1 (Intermittency Expansion)** The functional of the total mass \( \mathbb{E} \left[ F \left( \int_0^1 \phi(s) M_\mu(ds) \right) \right] \) has the formal intermittency expansion

\[
\mathbb{E} \left[ F \left( \int_0^1 \phi(s) M_\mu(ds) \right) \right] = F(x) + \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \left[ \sum_{k=2}^{2n} F^{(k)}(x) H_{n,k}(\phi) \right].
\]

(33)

The expansion coefficients \( H_{n,k}(\phi) \) are given by the binomial transform of the derivatives of the positive integer moments

\[
H_{n,k}(\phi) = \frac{(-1)^k}{k!} \sum_{l=2}^{k} (-1)^l \binom{k}{l} x^{k-l} \left. \frac{\partial^n \phi}{\partial \mu^n} \right|_{\mu=0},
\]

(34)

It is clear that the expansion coefficients \( H_{n,k}(\phi) \) are determined uniquely by the moments. The reason for this is that they are independent of \( F(x) \), so one can in particular take \( F(x) = x^\mu \) and then use binomial inversion to compute them.

**Corollary 4.2 (Intermittency Renormalization)** The intermittency expansion in (33) is an exactly renormalized expansion in the centered moments of \( \int_0^1 \phi(s) M_\mu(ds) \). Indeed, we have the identity

\[
\mathbb{E} \left[ F \left( \int_0^1 \phi(s) M_\mu(ds) \right) \right] = F(x) + \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \left[ \sum_{k=0}^{\infty} \frac{F^{(k)}(x)}{k!} \frac{\phi^n}{\partial \mu^n} \right]_{\mu=0} \mathbb{E} \left[ \left( \int_0^1 \phi(s) M_\mu(ds) - x^\mu \right)^2 \right].
\]

(35)
This result follows from a simple observation that
\[
\left. \frac{\partial^n}{\partial \mu^n} \right|_{\mu=0} E \left[ \left( \int_0^1 \varphi(s) M_\mu(ds) - x \right)^k \right] = k! H_{n,k}(\varphi) \tag{36}
\]
and the much less trivial fact that the expansion coefficients satisfy the key renormalizability identity
\[
H_{n,k}(\varphi) = 0 \quad \forall k > 2n \tag{37}
\]
so that the \(k\) sum in (35) is finite, and (35) is equivalent to (33). Of course, if the moments were all finite, then one could also write
\[
E \left[ F \left( \int_0^1 \varphi(s) M_\mu(ds) \right) \right] = F(x) + \sum_{n=1}^\infty \frac{\mu^n}{n!} E \left[ \left( \int_0^1 \varphi(s) M_\mu(ds) - x \right)^k \right] \tag{38}
\]
which is the naive expansion in the centered moments. This is not possible in our case. The formal equivalence of the expansion in (35) to that in (38) shows that we have removed infinity from the moments and so found an exactly renormalized solution.

The proof of (37) follows from the structure of the formula for the moments. Recalling \(g(s,t) = -\log r(s-t)\), we can write
\[
H_{n,k}(\varphi) = \frac{(-1)^k}{k!} \sum_{i=2}^k (-1)^i \binom{k}{i} x^{k-i} \int [\prod_{i=1}^l \varphi(s_i)] \left[ \sum_{i<j} \sum_{s_i,s_j} \right] ds^{(l)}. \tag{39}
\]
The claim is that (37) follows from this equation and the fact that \(g(s,t)\) is symmetric. Let \(\mathcal{S}_k^l\) denote the set of all subsets of \(\{1 \cdots k\}\) consisting of exactly \(l\) elements. Let
\[
h_{n,k}(s) \equiv \frac{(-1)^k}{k!} \prod_{i=1}^k \varphi(s_i) \sum_{i=2}^k (-1)^i \sum_{\sigma \in \mathcal{S}_k^l} \left[ \sum_{i<j} \sum_{s_i,s_j} \right] \tag{40}
\]
Clearly, \(h_{n,k}(s)\) is symmetric in \(s_1 \ldots s_k\). The size of \(\mathcal{S}_k^l\) is \(\binom{k}{l}\) so that we have the identity for any \(l \leq k\),
\[
\int_{[0,1]^k} \prod_{i=1}^k \varphi(s_i) \sum_{\sigma \in \mathcal{S}_k^l} \left[ \sum_{i<j} \sum_{s_i,s_j} \right] ds^{(k)} = \binom{k}{l} x^{k-l} \int_{[0,1]^l} \prod_{i=1}^l \varphi(s_i) \left[ \sum_{i<j} \sum_{s_i,s_j} \right] ds^{(l)}, \tag{41}
\]
and, therefore, \(H_{n,k}(\varphi) = \int_{[0,1]^k} h_{n,k}(s) ds^{(k)}\). The interest in the symmetrized representation is explained in the following proposition, cf. [26].

**Lemma 4.3** Up to the prefactor \(\prod_{i=1}^k \varphi(s_i)/k!\), the coefficients \(h_{n,k}(s)\) are the same as those terms in the multinomial expansion of
\[
\left( \sum_{i<j} g(s_i,s_j) \right)^n \tag{42}
\]
that involve all the indices \(s_1 \cdots s_k\).
This is best illustrated by an example. Let \( n = 3 \) and \( k = 4 \). Then, abbreviating \( g_{ij} \triangleq g(s_i, s_j) \), we have for \( h_{3,4} \)

\[
3g_{12}g_{34}^2 + 3g_{34}g_{12}^2 + 3g_{13}g_{24}^2 + 3g_{24}g_{13}^2 + 3g_{14}g_{23}^2 + 3g_{23}g_{14}^2 + 6g_{12}g_{23}g_{34} + 6g_{12}g_{13}g_{14} + 6g_{12}g_{23}g_{24} + 6g_{13}g_{23}g_{34} + 6g_{12}g_{14}g_{23} + 6g_{13}g_{14}g_{24} + 6g_{12}g_{14}g_{34} + 6g_{13}g_{14}g_{24} + 6g_{12}g_{24}g_{34} + 6g_{14}g_{23}g_{34} + 6g_{14}g_{23}g_{24} + 6g_{13}g_{23}g_{24} + 6g_{12}g_{13}g_{24}.
\]

(43)

It follows that the coefficients \( H_{n,k}(\varphi) \) must satisfy (37) as each term in the multinomial expansion in (42) contains the product of exactly \( n \) factors, and each individual factor involves two distinct indices. These \( n \) factors together must involve all the \( k \) indices, which is only possible if \( k \leq 2n \).

The fundamental open problem is to show that the intermittency expansion in Theorem 4.1 determines the distribution of the total mass uniquely, cf. Section 9 for details.

5 Calculation of the High Temperature Expansion of the Mellin Transform

In this section we will assume that the moments are known in closed form. Then, one can compute the expansion coefficients \( H_{n,k}(\varphi) \) recursively and effectively sum the series in the special case of \( F(x) = x^q, q \in \mathbb{C} \).

Recall the definition of the complete exponential Bell polynomials \( Y_n(x_1 \cdots x_n) \),

\[
\exp \left( \sum_{k=1}^{\infty} \frac{x_k}{k!} t^k \right) = \sum_{n=0}^{\infty} Y_n(x_1 \cdots x_n) \frac{t^n}{n!}.
\]

(44)

This is understood as the equality of formal power series. Let us assume that the expansion of the logarithm of the moments in \( \mu \) is known. In other words,

\[
\log \int \prod_{i=1}^{l} \phi(s_i) \prod_{i<j}^{l} r(s_i - s_j)^{-\mu} ds = l \log x + \sum_{p=1}^{\infty} c_p(l) \mu^p,
\]

(45)

where it is understood that the coefficients \( c_p(l) \) depend in \( \varphi \). Then, by the definition of Bell polynomials, we have

\[
\left. \frac{\partial^n S_l}{\partial \mu^n} \right|_{\mu = 0}^{\varphi} = x^l Y_n(1!c_1(l), \cdots, n!c_n(l)).
\]

(46)

The recurrence relation of the Bell polynomials,

\[
Y_{n+1} = \sum_{r=0}^{n} \binom{n}{r} Y_{n-r} x_{r+1}, \quad Y_0 = 1,
\]

(47)

implies the following recurrence for the expansion coefficients, cf. [27].
Lemma 5.1 The expansion coefficients are uniquely determined by

\[ H_{n+1,k}(\varphi) = x^k A_{n,k} + \sum_{r=0}^{n-1} \binom{n}{r} \sum_{t=2}^{k-r} x^{k-t} H_{n-r,t}(\varphi) B_{r,t,k}, \quad n \geq 0, k \geq 2, \]  

\[ A_{n,k} \triangleq (-1)^k \frac{(n+1)!}{k!} \sum_{l=2}^{k} (-1)^l \binom{k}{l} c_{n+1}(l), \]  

\[ B_{r,t,k} \triangleq (-1)^k \frac{(r+1)!}{k!} \sum_{l=2}^{k} (-1)^l \binom{k}{l} \binom{l}{t} c_{r+1}(l). \]

We will now consider the special case of the Mellin transform. The intermittency expansion in this case takes on the form

\[ E \left[ \left( \int_0^1 \varphi(s) M_\mu(ds) \right)^q \right] = x^q + \sum_{n=1}^{\infty} \frac{\mu^n}{n!} f_n(q), \]  

\[ f_n(q) = \sum_{k=2} (q)_k x^{q-k} H_{n,k}(\varphi), \quad n = 1, 2, 3, \ldots, \]

where the sum has been extended to infinity by \( (q)_k \triangleq q(q-1)(q-2)\cdots(q-k+1) \), and it is understood that the coefficients \( f_n(q) \) depend in \( \varphi \). The key result is that these coefficients can be computed explicitly in the special case when \( c_p(l) \) are polynomials in the moment order \( l \) by means of the recurrence in Lemma 5.1. This generalizes the calculation of the high temperature expansion of the Mellin transform in [27].

Theorem 5.2 (Intermittency Expansion of the Mellin Transform) Let \( c_p(l) \) be a polynomial in the moment order \( l \). Then, the expansion coefficients of the Mellin transform satisfy the recurrence\(^4\)

\[ f_{n+1}(q) = n! \sum_{r=0}^{n} \frac{f_{r}(q)}{(n-r)!} (r+1) c_{r+1}(q), \quad f_0(q) = x^q, \]

and the intermittency expansion of the Mellin transform is

\[ E \left[ \left( \int_0^1 \varphi(s) M_\mu(ds) \right)^q \right] = x^q \exp \left( \sum_{r=1}^{\infty} \mu^r c_r(q) \right). \]

Thus, the expansion for the complex moments is obtained by replacing \( l \) with \( q \) in the expansion of the positive integer moments. The significance of this result is that it allows one to reconstruct the distribution from the dependence of the moments on \( \mu \).

Proof The assumption of polynomial dependence of \( c_p(l) \) on \( l \) means that we can write

\[ c_p(l) = R_p \left( \frac{d}{dz} \right) e^{zl} \]

for a polynomial \( R_p(z) \) of some degree depending on \( p \). Given the recurrence in Lemma 5.1 and the definition of \( f_n(q) \) we can write

\[ f_{n+1}(q) = x^q \left[ \sum_{k=2}^{\infty} (q)_k A_{n,k} \right] + \sum_{r=0}^{n-1} \binom{n}{r} \sum_{t=2}^{\infty} x^{q-t} H_{n-r,t} \left[ \sum_{k=t}^{\infty} (q)_k B_{r,t,k} \right]. \]

\(^4\)In [27] and [30] this recurrence is written in terms of \( b_r(q) = (r+1) c_{r+1}(q) \).
We will now show that the representation of coefficients in (55) implies the identities
\[ \sum_{k=2}^{\infty} (q)_k A_{n,k} = (n+1)! c_{n+1}(q), \] (57)
\[ \sum_{k=t}^{\infty} (q)_k B_{r,t,k} = (q)_t (r+1)! c_{r+1}(q). \] (58)

In fact, substituting the expression for \( A_{n,k} \) in Lemma 5.1 and using the identity
\[ \sum_{l=2}^{k} (-1)^l \binom{k}{l} e^{zl} = (1 - e^z)^k + ke^z - 1, \] (59)
we get the expression
\[ \sum_{k=2}^{\infty} (q)_k A_{n,k} = (n+1)! R_{n+1} \left( \frac{d}{dz} \right) \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} (q)_k \left( (1 - e^z)^k + ke^z - 1 \right), \] (60)
and (57) follows from the identity
\[ \sum_{k=2}^{\infty} \frac{d^k}{k!} (q)_k = (1 + a)^q - qa - 1. \] (61)

Similarly, using the identity
\[ \sum_{l=t}^{k} (-1)^l \binom{k}{l} \binom{l}{t} e^{zl} = (-1)^t \binom{k}{t} e^{zt} (1 - e^z)^{k-t}, \] (62)
we obtain the expression
\[ \sum_{k=t}^{\infty} (q)_k B_{r,t,k} = (r+1)! R_{r+1} \left( \frac{d}{dz} \right) \sum_{k=t}^{\infty} \frac{(-1)^{k-t}}{(k-t)!} (1 - e^z)^{k-t}, \] (63)
and (58) follows from the identity
\[ \sum_{k=t}^{\infty} \frac{d^{k-t}}{(k-t)!} (q)_k = (q)_t (1 + a)^{q-t}. \] (64)

Having established (57) and (58), it remains to observe that (53) is now equivalent to (56) and (54) follows from the recurrence relation of Bell polynomials in (47). □

The restriction of the polynomial dependence of \( c_p(l) \) on the moment order \( l \) is not overly restrictive. For example, in the case of the Bacry-Muzy GMC measure on the interval and circle and \( \varphi(s) \) such that the moments are given by the Selberg or Morris integral, respectively, \( c_p(l) \) is known analytically and can be expressed as a function of \( l \) in terms of Bernoulli polynomials, cf. [27] and Section 8 below.

The expansion of the Mellin transform in \( \mu \) is not convergent in general and should be interpreted as the asymptotic expansion, however see Conjecture 9.1 in Section 9.

We end this section with an extension of Theorem 5.2 to a general transform of the total mass, extending the corresponding result for the Bacry-Muzy GMC measure on the interval, cf. [28].
Corollary 5.3 (Intermittency Expansion of the General Transform) Consider the normalized random variable
\[
\tilde{M} \triangleq \frac{1}{x} \int_0^1 \varphi(s) M_{\mu}(ds).
\]

Given constants \(a\) and \(s\) and a smooth function \(F(s)\), the intermittency expansion of the general transform of \(\log \tilde{M}\)
\[
E\left[F(s + a \log \tilde{M})\right] = \sum_{n=0}^{\infty} F_n(a, s) \frac{\mu^n}{n!}
\]
is determined by \(F_0(a, s) = F(s)\), and
\[
F_{n+1}(a, s) = \sum_{r=0}^{n} \frac{n!}{(n-r)!} (r+1)c_{r+1}\left(a \frac{d}{ds}\right) F_{n-r}(a, s).
\]

This result shows that the solution for the general transform is obtained by replacing \(q\) with \(ad/ds\) in the solution for the Mellin transform in Theorem 5.2.

6 Intermittency Renormalization and Dependence Structure

In this section we will extend the intermittency differentiation rule and theory of intermittency renormalization to the joint distribution of the GMC measure. We will state all results without proof as the proofs are very similar to those in the previous sections and can be found in the special case of the Bacry-Muzy GMC on the interval in [29]. Throughout this section, we let \(\varphi(s) = 1\) for simplicity.

Let \(I_j, j = 1 \cdots N\), denote \(N\) non-overlapping subintervals of \((0,1)\) and \(F_j(x), j = 1 \cdots N\), denote \(N\) smooth functions. Consider a general functional
\[
v(\mu, \tilde{f}, \tilde{F}, \tilde{I}) = E\left[\prod_{j=1}^{N} F_j\left(\int_{I_j} e^{\mu f_j(s)} M_{\mu}(ds)\right)\right].
\]

We will use \(|I|\) to denote the length of \(I\), \(I^k\) to denote the \(k\)-dimensional product \(I \times \cdots \times I\). Finally, we will write \(\tilde{f} + \tilde{g}(\cdot, s)\) to denote the vector function with components \(f_j(u) + g(u, s), j = 1 \cdots N\), where \(g(u, s)\) is as in [23]. Then, we have the following rule of intermittency differentiation for multiple subintervals.

Theorem 6.1 (Intermittency Differentiation for Multiple Subintervals)
\[
\frac{\partial}{\partial \mu} v(\mu, \tilde{f}, \tilde{F}, \tilde{I}) = \sum_{j=1}^{N} \int_{I_j} f_j(s) e^{\mu f_j(s)} v(\mu, \tilde{f} + \tilde{g}(\cdot, s), F_1 \cdots F_j^{(1)} \cdots F_N, \tilde{I}) ds +
\]
\[
+ \frac{1}{2} \sum_{j=1}^{N} \int_{I_j} \int_{I_j} \exp(\mu(f_j(s_1) + f_j(s_2) + g(s_1, s_2))) g(s_1, s_2)^*\]

\[
\times v(\mu, \tilde{f} + \tilde{g}(\cdot, s_1) + \tilde{g}(\cdot, s_2), F_1 \cdots F_j^{(2)} \cdots F_N, \tilde{I}) ds_1 ds_2 +
\]

\[
+ \sum_{j<k} \int_{I_j} \int_{I_k} \exp(\mu(f_j(s_1) + f_k(s_2) + g(s_1, s_2))) g(s_1, s_2)^*\]

\[
\times v(\mu, \tilde{f} + \tilde{g}(\cdot, s_1) + \tilde{g}(\cdot, s_2), F_1 \cdots F_j^{(1)} \cdots F_k^{(1)} \cdots F_N, \tilde{I}) ds_1 ds_2.
\]
The structure of the intermittency derivative thus amounts to certain functional shifts that are induced by the \( g \) function. In the special case of \( N = 1 \) and \( I_1 = (0, 1) \) the last term drops out and we recover the rule of intermittency differentiation in Theorem 3.1.

We are primarily interested in the joint distribution of \( \int_{I_1} M_\mu(ds), \cdots, \int_{I_N} M_\mu(ds) \). The intermittency expansion of the joint distribution follows immediately from the differentiation rule.

**Theorem 6.2** (Intermittency Expansion) There exist universal expansion coefficients \( H_{n;k_1 \cdots k_N} \) such that

\[
E \left[ \prod_{j=1}^{N} F_j \left( \int_{I_j} M_\mu(ds) \right) \right] = \prod_{j=1}^{N} F_j (|I_j|) + \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \sum_{2 \leq \sum k_j \leq 2n} H_{n;k_1 \cdots k_N} \prod_{j=1}^{N} F_j^{(k_j)} (|I_j|). \tag{70}
\]

The coefficients \( H_{n;k_1 \cdots k_N} \) are universal in the sense of being independent of \( F_1 \cdots F_N \) and can be determined from the joint positive integer moments of \( \int_{I_1} dM_\mu(ds), \cdots, \int_{I_N} dM_\mu(ds) \). Denote

\[
S_{q_1 \cdots q_N} (\mu, \vec{I}) \equiv E \left[ \prod_{j=1}^{N} \left( \int_{I_j} M_\mu(ds) \right)^{q_j} \right]. \tag{71}
\]

**Corollary 6.3** Given \( 2 \leq k_1 + \cdots + k_N \leq 2n \), the expansion coefficients satisfy

\[
H_{n;k_1 \cdots k_N} = \frac{(-1)^{k_1+\cdots+k_N}}{k_1! \cdots k_N!} \sum_{q_j \leq k_j} (-1)^{q_1+\cdots+q_N} \prod_{j=1}^{N} |I_j|^{k_j-q_j} \left( \frac{k_j}{q_j} \right) \frac{\partial^n}{\partial \mu^n} \bigg|_{\mu=0} S_{q_1 \cdots q_N} (\mu, \vec{I}). \tag{72}
\]

The key renormalizability property of (71) now takes on the following form.

**Theorem 6.4** The expansion coefficients as defined in (72) for all indices \( k_1, \cdots, k_N \) satisfy

\[
H_{n;k_1 \cdots k_N} = 0 \text{ if } \sum_{j=1}^{N} k_j > 2n. \tag{73}
\]

Hence, as before, the intermittency expansion is an exactly renormalized expansion in the joint centered moments. Finally, the formula for the joint integer moments is

\[
S_{k_1 \cdots k_N} (\mu, \vec{I}) = \int \cdots \int \prod_{i=1}^{N} \left[ \prod_{p=0}^{k_i} r(s_{ip} - s_{iq})^{-\mu} \right] \prod_{i<j} \left[ \prod_{p=0}^{k_i} \prod_{q=0}^{k_j} r(s_{ip} - s_{jq})^{-\mu} \right] d\vec{s}_1 \cdots d\vec{s}_N. \tag{74}
\]

where \( \vec{s}_j, j = 1 \cdots N \), denotes the vector \( \vec{s}_j \equiv (s_{j1}, \cdots, s_{jk_j}) \) such that \( s_{ij} \in I_j \forall i \). It is an open question how to compute these integrals, even for \( r(t) = |t| \) and \( N = 2 \), cf. [29] and [35].

## 7 Derivation of the Intermittency Differentiation Rule

The derivation of (24) from first principles is essentially based on the intermittency invariance of the underlying Gaussian field, see [25], [26], and [28] for an extension to the general infinitely divisible field. For our purposes in this section, it is convenient to define another Gaussian field by

\[
\omega_{\mu,L,N}(s) \equiv \omega_{\mu,N}(s) + \mathcal{N}(0, \mu \log L), \tag{75}
\]

where \( L \geq 1 \) and \( \mathcal{N}(0, \mu \log L) \) is an independent Gaussian random variable with variance \( \mu \log L \).
Lemma 7.1 (Intermittency Invariance) Let $B(t)$ be the standard Brownian motion independent of $\omega_{\mu, L, \epsilon}(s)$. Then, we have the equality of gaussian processes in law,

$$B(\delta) + \omega_{\mu, 1, \epsilon}(s) = \omega_{\mu - \delta, 1, \epsilon}(s) + \delta + \frac{\delta}{2},$$

(76)

which are viewed as random functions of $s$ on the interval $s \in (0, 1)$ at fixed $0 < \delta < \mu$ and $\epsilon$. $\tilde{\omega}_{\delta, \epsilon, \epsilon}(s)$ denotes an independent copy of the $\omega_{\mu, L, \epsilon}(s)$ process at the intermittency parameter $\delta$ and $L = 2.718 \ldots$.

Proof The gaussian processes on the left- and right-hand sides of the equation are stationary with continuous sample paths so it is sufficient to compare their means and covariances. We have in general

$$E[\omega_{\mu, L, \epsilon}(t)] = -\frac{\mu}{2} \left( \epsilon \frac{d}{d\epsilon} \log r(\epsilon) + \log \frac{L}{r(\epsilon)} \right),$$

(77a)

$$\text{Cov}[\omega_{\mu, L, \epsilon}(t), \omega_{\mu, L, \epsilon}(s)] = \mu \log \frac{1}{r(t-s)}, \epsilon \leq |t-s| \leq 1,$$

(77b)

$$|t-s| \leq \epsilon.$$

Then, the covariance of the left-hand side of (76) is

$$\text{Cov}[B(\delta) + \omega_{\mu, 1, \epsilon}(t), B(\delta) + \omega_{\mu, 1, \epsilon}(s)] = \delta + \mu \log \frac{1}{r(t-s)}, \epsilon \leq |t-s| \leq 1,$$

(78a)

$$\text{Cov}[B(\delta) + \omega_{\mu, 1, \epsilon}(t), \omega_{\mu, 1, \epsilon}(s)] = \delta + \mu \left( \log \frac{1}{r(\epsilon)} + \left( 1 - \frac{|s-t|}{\epsilon} \right) \epsilon \frac{d}{d\epsilon} \log r(\epsilon) \right), \epsilon \leq |t-s| \leq 1.$$  

(78b)

The covariance of the right-hand side of (76) is

$$\text{Cov}[\omega_{\mu - \delta, 1, \epsilon}(t) + \tilde{\omega}_{\delta, \epsilon, \epsilon}(t), \omega_{\mu - \delta, 1, \epsilon}(s) + \tilde{\omega}_{\delta, \epsilon, \epsilon}(s)] = (\mu - \delta) \log \frac{1}{r(t-s)} + \delta \log \frac{e^{r(t-s)}}{r(t-s)},$$

(79a)

$$= (\mu - \delta) \left( \log \frac{1}{r(\epsilon)} + \left( 1 - \frac{|s-t|}{\epsilon} \right) \epsilon \frac{d}{d\epsilon} \log r(\epsilon) \right) + \delta \left( \log \frac{e^{r(\epsilon)}}{r(\epsilon)} + \left( 1 - \frac{|s-t|}{\epsilon} \right) \epsilon \frac{d}{d\epsilon} \log r(\epsilon) \right),$$

(79b)

for $\epsilon \leq |t-s| \leq 1$ and $|t-s| < \epsilon$, respectively. Thus, the covariances are the same. As the mean is defined to be $-\frac{1}{2}$ variance, so are the means. \hfill \blacksquare

We need three additional lemmas. In what follows, we will occasionally write $\omega(s)$ without any subscript to mean $\omega_{\mu, L, \epsilon}(s)$ to simplify notation.

Lemma 7.2 (Girsanov) Let $s_1$ and $s_2$ be any two distinct times, $s_1, s_2 \in (0, 1)$, and let $C(s, t)$ denote the covariance function of $\omega(s)$ in (77b) and (77c). Let

$$u(\mu, f, F) \triangleq F \left( \int_0^1 e^{f(s)+\omega(s)} ds \right),$$

(80)
where \( f(x) \) is continuous and \( F(x) \) is smooth. Then,

\[
E \left[ u(\mu, f + C(\cdot, s_1), F) \right] = E \left[ u(\mu, f, F) e^{\omega(s_1)} \right],
\]
\[
E \left[ u(\mu, f + C(\cdot, s_1) + C(\cdot, s_2), F) \right] = e^{-C(s_1, s_2)} E \left[ u(\mu, f, F) e^{\omega(s_1) + \omega(s_2)} \right].
\]

**Proof** We have by construction

\[
E \left[ \exp(\omega(s)) \right] = 1
\]

for all \( s \). Introduce an equivalent probability measure

\[
d\mathcal{D} \equiv e^{\omega(s_1)} d\mathcal{P},
\]

where \( \mathcal{P} \) is the original probability measure corresponding to \( E \). Then, the law of the process \( s \to \omega(s) + C(s, s_1) \) with respect to \( \mathcal{D} \) equals the law of the original process \( s \to \omega(s) \) with respect to \( \mathcal{P} \). Indeed, it is easy to show that the two processes have the same finite-dimensional distributions by computing their characteristic functions. The computation is straightforward. The continuity of sample paths can then be used to conclude that the equality of all finite-dimensional distributions implies the equality in law.

To verify (82), it is sufficient to remark that the measure

\[
d\mathcal{D} \equiv e^{\omega(s_1) + \omega(s_2) - C(s_1, s_2)} d\mathcal{P}
\]

is a probability measure that is equivalent to the original probability measure \( \mathcal{P} \). It follows that the finite-dimensional distributions of the process \( s \to \omega(s) + C(s, s_1) + C(s, s_2) \) with respect to \( \mathcal{D} \) are the same as those of the original process \( s \to \omega(s) \) with respect to \( \mathcal{D} \). The rest of the argument goes through verbatim.

It is convenient to introduce the quantity \( g_{L,E}(s,t) \) by

\[
\mu_{L,E}(s,t) = \text{Cov} \left[ \omega_{\mu,L,E}(s), \omega_{\mu,L,E}(t) \right],
\]

where the covariance is defined in (77b) and (77c).

**Lemma 7.3** Let \( f(\delta, s) \) be an arbitrary continuous function that vanishes as \( \delta \to 0 \). Let \( \mathcal{D}(s) \equiv e^{f(\delta, s)} + \omega_{g_{L,E}(s)} - 1 \). Then, given any distinct \( s_1, \ldots, s_k \in (0, 1) \), as \( \delta \to 0 \),

\[
E[\mathcal{D}(s_1) \mathcal{D}(s_2)] = (e^{f(\delta, s_1)} - 1) (e^{f(\delta, s_2)} - 1) + \delta g_{L,E}(s_1, s_2) + o(\delta),
\]
\[
E[\mathcal{D}(s_1) \cdots \mathcal{D}(s_k)] = (e^{f(\delta, s_1)} - 1) \cdots (e^{f(\delta, s_k)} - 1) + o(\delta), \quad k \neq 2.
\]

**Proof** It is easy to show that for any subset \( \sigma \) of \( \{1, \cdots, k\} \)

\[
E \left[ \exp \left( \sum_{i \in \sigma} \omega_{\delta, L,E}(s_i) \right) \right] = \exp \left( \sum_{i < j, i, j \in \sigma} \delta g_{L,E}(s_i, s_j) \right).
\]

Let \( \mathcal{S}_p \) denote the set of all subsets of \( \{1, \cdots, k\} \) that consist of exactly \( p \) indices, \( p = 0 \cdots k \), with the convention that the only element of \( \mathcal{S}_0 \) is the empty set. Then, given \( k \) distinct numbers, we have the algebraic identity

\[
(a_1 - 1) \cdots (a_k - 1) = \sum_{p=0}^{k} (-1)^{k-p} \sum_{\sigma \in \mathcal{S}_p} \prod_{i \in \sigma} a_i,
\]
taking all empty sums to mean zero and empty products to mean one. It is easily verified by induction.

If we now expand the brackets on the left-hand side of (86) and make use of (87) and (88), we obtain

$$
\sum_{p=0}^{k} (-1)^{k-p} \sum_{\sigma \in \mathcal{P}_p} \exp \left( \sum_{i \in \sigma} f(\delta, s_i) + \delta \sum_{i<j} g_{L,\epsilon}(s_i, s_j) \right). 
$$

(89)

It remains to expand this expression in $\delta$ and recall that $f(\delta, \epsilon) \to 0$ as $\delta \to 0$ by assumption. There results

$$
\sum_{p=0}^{k} (-1)^{k-p} \sum_{\sigma \in \mathcal{P}_p} e^{\sum_{i \in \sigma} f(\delta, s_i)} + \delta \sum_{p=0}^{k} (-1)^{k-p} \sum_{\sigma \in \mathcal{P}_p} \sum_{i<j} g_{L,\epsilon}(s_i, s_j) + o(\delta).
$$

(90)

By (88), the first term in (90) is exactly $\prod_{i=1}^{k} (\exp(f(\delta, s_i)) - 1)$ that occurs on the right-hand side of (86). It is not difficult to see that the second term in (90) equals $\delta g_{L,\epsilon}(s_1, s_2)$ if $k = 2$ and is zero otherwise.

**Lemma 7.4** Let $f(s)$ be a continuous function and $F(x)$ be smooth. Let

$$
u_{\epsilon}(z, \mu, f, F) \triangleq F \left( z \int_0^1 e^{\mu f(s) + \omega_\mu L,\epsilon(s)} ds \right).$$

(91)

Then, there holds the following identity

$$
\frac{\partial}{\partial \mu} \nu_{\epsilon}(z, \mu, f, F) = z \nu_{\epsilon}(z, \mu, f, F^{(1)}) \int_0^1 e^{\mu f(s) + \omega_\mu L,\epsilon(s)} f(s) ds
$$

$$
- \lim_{\delta \to 0} \left[ \frac{1}{\delta} \sum_{k=1}^{\infty} \nu_{\epsilon}(z, \mu, f, F^{(k)}) \left( z \int_0^1 e^{\mu f(s) + \omega_\mu L,\epsilon(s)}(e^{\omega_\epsilon(s)} - 1) ds \right)^k \right],
$$

(92)

where

$$
\omega_\epsilon(s) \triangleq \omega_{\mu - \delta, L,\epsilon}(s) - \omega_{\mu, L,\epsilon}(s).
$$

(93)

**Proof** The result follows from representing

$$
\int_0^1 e^{(\mu - \delta) f(s) + \omega_{\mu - \delta, L,\epsilon}(s)} ds = \int_0^1 e^{\mu f(s) + \omega_{\mu, L,\epsilon}(s)} ds - \delta \int_0^1 e^{\mu f(s) + \omega_{\mu, L,\epsilon}(s)} f(s) ds +
$$

$$
+ \int_0^1 e^{\mu f(s) + \omega_{\mu, L,\epsilon}(s)}(e^{\omega_\epsilon(s)} - 1) ds + o(\delta),
$$

(94)

and Taylor expanding in the “small” parameter

$$
\int_0^1 e^{\mu f(s) + \omega_{\mu, L,\epsilon}(s)}(e^{\omega_\epsilon(s)} - 1) ds
$$

that vanishes as $\delta \to 0$.

We can now give a formal derivation of Theorem 3.1.
Proof} The main idea of the proof is to consider a stochastic flow and derive the corresponding Feynman-Kac equation regarding intermittency as time. We can assume $\phi(s) = 1$ without any loss of generality. Let $u_\varepsilon(z, \mu, f, F) \triangleq F\left(z \int_0^1 e^{\mu f(s)+\omega_{\mu,1}(s)} ds\right)$ and let $v_\varepsilon(z, \mu, f, F)$ be its expectation, $v_\varepsilon(z, \mu, f, F) \triangleq \mathbb{E}[u_\varepsilon(z, \mu, f, F)]$, so that $v(\mu, f, F) = \lim_{\varepsilon \to 0} v_\varepsilon(1, \mu, f, F)$. The starting point is the limit
\[ A \triangleq \frac{\partial}{\partial \varepsilon} \bigg|_{\delta=0} \mathbb{E}^* \left[ v_\varepsilon \left( z e^{B(\delta)}, \mu, f, F \right) \right], \] (95)
where $B(t)$ is the standard Brownian motion independent of $\omega_{\mu,1}(s)$, and the star is used to distinguish the expectation with respect to $B(t)$ from that with respect to $\omega_{\mu,1}(s)$. By the backward Kolmogorov equation, we have
\[ A = \frac{1}{2} \frac{\partial^2}{\partial z^2} \bigg|_{z=0} v_\varepsilon(z, \mu, f, F) = \frac{1}{2} \left[ z \frac{\partial}{\partial z} + z^2 \frac{\partial^2}{\partial z^2} \right] v_\varepsilon(z, \mu, f, F). \] (96)

On the other hand, this limit can be computed in a different way. By Lemma 7.1 there holds the following equality in law
\[ e^{B(\delta)} \int_0^1 e^{\mu f(s)+\omega_{\mu,1}(s)} ds = e^{\delta} \int_0^1 e^{\mu f(s)+\omega_{\mu-\delta,1}(s)+\omega_{\mu,\varepsilon}(s)} ds. \] (97)
Thus, to compute the limit in (95), we need to expand
\[ \mathbb{E}^* \left[ F \left( z e^{B(\delta)} \int_0^1 e^{\mu f(s)+\omega_{\mu-\delta,1}(s)+\omega_{\mu,\varepsilon}(s)} ds \right) \right] - v_\varepsilon(z, \mu, f, F) \] (98)
in $\delta$ up to $o(\delta)$ terms. The star now indicates the expectation with respect to $\omega_{\delta,\varepsilon}(s)$, which is independent of $\omega_{\mu-\delta,1}(s)$ by construction. Let $\omega_{\varepsilon}(s) \triangleq \omega_{\mu-\delta,1}(s) - \omega_{\mu,1}(s)$ as in (93) and
\[ \omega_{\varepsilon}(s) \triangleq \omega_{\delta,\varepsilon}(s). \] (99)
While we do not know how to expand either $\omega_{\varepsilon}(s)$ or $\omega_{\varepsilon}(s)$ in $\delta$, they both clearly vanish as $\delta \to 0$. It follows that the expression in (98) can be written as
\[ \mathbb{E}^* \left[ u_\varepsilon(z, \mu, f, F^{(1)}) \frac{\delta}{2} z \int_0^1 e^{\mu f(s)+\omega_{\mu,1}(s)} ds + \sum_{k=1}^\infty \frac{\delta^k}{k!} u_\varepsilon(z, \mu, f, F^{(k)}) g^{(k)} \right] + o(\delta), \] (100)
\[ g^{(k)} \triangleq \int_0^1 e^{\mu f(s)+\omega_{\mu,1}(s)} (e^{\omega_{\varepsilon}(s)+\omega_{\varepsilon}(s)} - 1) ds. \] (101)
The advantage of this representation is that the only $\omega_{\varepsilon}$ dependence is in $\omega_{\varepsilon}(s)$. This allows us to compute the $\mathbb{E}^*$ expectation in (98). Indeed, (98) entails two expectations: the $\mathbb{E}$ with respect to $\omega_{\varepsilon}$ process inherited from the definition of $v_\varepsilon(z, \mu, f, F)$ and the $\mathbb{E}^*$ expectation with respect to $\bar{\omega}_{\varepsilon}$ process. Interchanging their order, it follows from (100) that computing the $\mathbb{E}^*$ expectation is now reduced to computing $\mathbb{E}^* [g^{(k)}]$. As $\omega_{\varepsilon}(s)$ and $\omega_{\varepsilon}(s)$ are independent processes, it follows from Lemma 7.3 applied to $\mathcal{B}(s) = \exp(\omega_{\varepsilon}(s)+\omega_{\varepsilon}(s)) - 1$ with $f(\delta, s) \triangleq \omega_{\varepsilon}(s)$ that the $\mathbb{E}^*$ expectation equals
\[ \mathbb{E}^*[\mathcal{B}(s_1) \mathcal{B}(s_2)] = (e^{\omega_{\varepsilon}(s_1)} - 1) (e^{\omega_{\varepsilon}(s_2)} - 1) + \delta g_{\varepsilon,\varepsilon}(s_1, s_2) + o(\delta), \] (102)
\[ \mathbb{E}^*[\mathcal{B}(s_1) \cdots \mathcal{B}(s_k)] = (e^{\omega_{\varepsilon}(s_1)} - 1) \cdots (e^{\omega_{\varepsilon}(s_k)} - 1) + o(\delta), \] (103)
Collecting what we have shown so far, we obtain

\[ A = \frac{z}{2} \mathbb{E} \left[ u_e(z, \mu, f, F^{(1)}) \int_0^1 e^{\mu f(s) + \omega_{0,1,\varepsilon}(s)} ds \right] + \]

\[ \lim_{\delta \to 0} \frac{1}{\delta} \sum_{k=1}^{\infty} \frac{z^k}{k!} \mathbb{E} \left[ u_e(z, \mu, f, F^{(k)}) \left( \int_{[0,1]} e^{\mu f(s) + \omega_{0,1,\varepsilon}(s)} (e^{\omega_{0,1,\varepsilon}(s)} - 1) ds \right)^k \right] + \]

\[ \frac{z^2}{2} \mathbb{E} \left[ u_e(z, \mu, f, F^{(2)}) \int_{[0,1]^2} e^{\mu f(s_1) + \omega_{0,1,\varepsilon}(s_1) + \mu f(s) + \omega_{0,1,\varepsilon}(s_2) g_{e,e}(s_1, s_2) ds_2} ds_1 \right]. \tag{104} \]

By Lemma 7.4 the \( \delta \to 0 \) limit that is involved in (104) equals

\[ -\frac{\partial}{\partial \mu} v_e(z, \mu, f, F) + z \mathbb{E} \left[ u_e(z, \mu, f, F^{(1)}) \int_{[0,1]} e^{\mu f(s) + \omega_{0,1,\varepsilon}(s)} f(s) ds \right]. \tag{105} \]

Observing that \( g_{e,e}(s_1, s_2) = 1 + g_{1,1,e}(s_1, s_2) \) and substituting (105) into (104), we obtain

\[ A = \left( \frac{z}{2} \frac{\partial}{\partial z} + \frac{z^2}{2} \frac{\partial^2}{\partial z^2} \right) v_e(z, \mu, f, F) - \frac{\partial}{\partial \mu} v_e(z, \mu, f, F) + \]

\[ z \int_{[0,1]} \mathbb{E} \left[ u_e(z, \mu, f, F^{(1)}) e^{\omega_{0,1,\varepsilon}(s)} \right] e^{\mu f(s)} f(s) ds + \]

\[ \frac{z^2}{2} \int_{[0,1]^2} \left[ u_e(z, \mu, f, F^{(2)}) e^{\omega_{0,1,\varepsilon}(s_1) + \omega_{0,1,\varepsilon}(s_2)} \right] e^{\mu f(s_1) + \mu f(s_2) g_{1,1,e}(s_1, s_2) ds_2} ds_1. \tag{106} \]

Finally, upon evaluating the expectations in (106) by Lemma 7.2, comparing the resulting expression for \( A \) with that in (106), and then letting \( z = 1 \) and \( \varepsilon \to 0 \), we arrive at (24).

### 8 Bacry-Muzy GMC Measure on the Circle

In this section we will apply the general theory to the special case of the Bacry-Muzy GMC on the circle. For our purposes in this section it is convenient to introduce the quantity

\[ \tau \triangleq \frac{2}{\mu}. \tag{107} \]

Let \( r(t) \) be as in (8). Recall the Morris integral, cf. Chapters 3 and 4 of [11],

\[ \int_{[-\frac{\pi}{2}, \frac{\pi}{2}]} \prod_{l=1}^n e^{2\pi i \theta (\lambda_1 - \lambda_2) / 2} \left| 1 + e^{2\pi i \theta} \right|^{\lambda_1} \left| e^{2\pi i \theta} - e^{2\pi i \theta} \right|^{-2/\tau} d \theta = \]

\[ = \prod_{j=0}^{n-1} \frac{\Gamma(1 + \lambda_1 + \lambda_2 - j \tau) \Gamma(1 - (j+1) \tau)}{\Gamma(1 + \lambda_1 - j \tau) \Gamma(1 + \lambda_2 - j \tau) \Gamma(1 - (j+1) \tau)}. \tag{108} \]

To bring it to the form of (14), we let

\[ \varphi(s) = |1 - e^{2\pi i s}|^{2\lambda}, \tag{109} \]
for some $\lambda \geq 0$ so that the moments are

$$S_\mu[\varphi](\mu) = \int_{[0,1]^n} \prod_{i=1}^n |1 - e^{2\pi i \theta_i}|^{2\lambda} \prod_{k<l} |e^{2\pi i \theta_k} - e^{2\pi i \theta_l}|^{-\mu} d\theta, \quad (110)$$

$$= \prod_{j=0}^{n-1} \frac{\Gamma(1 + 2\lambda - \frac{j}{\tau}) \Gamma(1 - (j+1) \tau)}{\Gamma(1 + \lambda - \frac{j}{\tau})^2 \Gamma(1 - \frac{j}{\tau})}. \quad (111)$$

**Lemma 8.1 (Log Moment Expansion)** The expansion (115) of the log moments in $\mu$ near $\mu = 0$ is

$$\log S_\mu[\varphi](\mu) = l \left( \log \Gamma(1 + 2\lambda) - 2 \log \Gamma(1 + \lambda) \right) + \sum_{p=1}^{\infty} c_p(l) \mu^p, \quad (112)$$

$$c_p(l) = \frac{1}{p^{2p}} \left[ \left( \zeta(p, 1 + 2\lambda) - 2\zeta(p, 1 + \lambda) \right) \frac{B_{p+1}(l) - B_{p+1}}{p+1} + \zeta(p) \frac{B_{p+1}(l + 1) - B_{p+1}}{p+1} - l \zeta(p) \right]. \quad (113)$$

**Proof** This is a simple corollary of (111) and the following formulas involving the Hurwitz zeta function and Bernoulli polynomials,

$$\log \Gamma(a + z) = \log \Gamma(a) + \sum_{p=1}^{\infty} \frac{(-z)^p}{p} \zeta(p, a), \quad (114)$$

$$\sum_{j=x}^{y} j^p = \frac{B_{p+1}(y+1) - B_{p+1}(x)}{p+1}, \quad (115)$$

and the convention $\zeta(1, a) = -\psi(a)$, $\zeta(p, 1) = \zeta(p)$. 

By Theorem 5.2, we know the asymptotic expansion of the Mellin transform. We now wish to construct a positive probability distribution having the properties that its moments are given by (111) and the asymptotic expansion of its Mellin transform coincides with the series in Theorem 5.2.

**Theorem 8.2 (Morris Integral Probability Distribution)** The function

$$\mathcal{M}(q|\tau, \lambda_1, \lambda_2) = \frac{\tau^\frac{q}{2}}{\Gamma(1 - \frac{q}{\tau})} \frac{\Gamma_2(\tau(\lambda_1 + \lambda_2 + 1) + 1 - q | \tau)}{\Gamma_2(\tau(\lambda_1 + \lambda_2 + 1) + 1 | \tau)} \times$$

$$\times \frac{\Gamma_2(\tau(1 + \lambda_1) + 1 | \tau)}{\Gamma_2(\tau(1 + \lambda_1) + 1 - q | \tau)} \frac{\Gamma_2(\tau(1 + \lambda_2) + 1 | \tau)}{\Gamma_2(\tau(1 + \lambda_2) + 1 - q | \tau)} \quad (116)$$

reproduces the product in (108) when $q = n < \tau$ and is the Mellin transform of the distribution

$$M_{(\tau, \lambda_1, \lambda_2)}(q) = \frac{\tau^{1/\tau}}{\Gamma(1 - 1/\tau)} \beta_{22}^{-1}(\tau, b_0 = \tau, b_1 = 1 + \tau \lambda_1, b_2 = 1 + \tau \lambda_2) \times$$

$$\times \beta_{1,0}^{-1}(\tau, b_0 = \tau(\lambda_1 + \lambda_2 + 1) + 1), \quad (117)$$

where $\beta_{22}^{-1}(a,b)$ is the inverse Barnes beta of type $(2,2)$ and $\beta_{1,0}^{-1}(a,b)$ is the independent inverse Barnes beta of type $(1,0)$. In particular, $\log M_{(\tau, \lambda_1, \lambda_2)}$ is infinitely divisible.
We refer the reader to [32] for a review of the double gamma function \( \Gamma_2(z \mid \tau) \) and to [34] for a review of Barnes beta distributions. The proof is given in [34]. In the special case of \( \lambda_1 = \lambda_2 = 0 \) this result first appeared in [12].

**Theorem 8.3 (Asymptotic Expansion)** The asymptotic expansion of \( \log \mathcal{M}(q \mid \tau, \lambda_1, \lambda_2) \) in \( \tau \) in the limit \( \tau \to \infty \) is:

\[
\log \mathcal{M}(q \mid \tau, \lambda_1, \lambda_2) \sim q \left( \log \Gamma(1 + \lambda_1 + \lambda_2) - \log \Gamma(1 + \lambda_1) - \log \Gamma(1 + \lambda_2) \right) + \sum_{p=1}^{\infty} \frac{1}{p \tau^p} \left[ (\zeta(p, 1 + \lambda_1 + \lambda_2) - \zeta(p, 1 + \lambda_1) - \zeta(p, 1 + \lambda_2)) \frac{B_{p+1}(q) - B_{p+1}}{p+1} + \zeta(p) \frac{B_{p+1}(q+1) - B_{p+1}}{p+1} - q \zeta(p) \right].
\]

**Proof** The first step is to express \( \mathcal{M}(q \mid \tau, \lambda_1, \lambda_2) \) in terms of the Alexeiewsky-Barnes \( G \)-function. We have the identity:

\[
\mathcal{M}(q \mid \tau, \lambda_1, \lambda_2) = \frac{1}{\Gamma(q(1 - \frac{1}{\tau}))} \frac{G(\tau(\lambda_1 + \lambda_2 + 1) + 1 \mid \tau)}{G(\tau(1 + \lambda_1 + \lambda_2 + 1) + 1 - q \mid \tau)} \frac{G(q \mid \tau)}{G(1 + \lambda_1 + 1 \mid \tau)} \times \frac{G(\tau(1 + \lambda_1) + 1 - q \mid \tau)}{G(\tau(1 + \lambda_2) + 1 - q \mid \tau)}.
\]

cf. [30] for the relationship between \( \Gamma_2(z \mid \tau) \) and \( G(z \mid \tau) \).

**Lemma 8.4** Let

\[
I(q \mid a, \tau) \triangleq \int_0^\infty dx \frac{e^{-ax} - 1}{e^{x\tau} - 1} \left[ \frac{e^x - 1}{e^x - 1} + q \left( \frac{q^2 - q}{2} \right) x \right].
\]

Then, \( I(q \mid a \tau, \tau) \) has the asymptotic expansion:

\[
I(q \mid a \tau, \tau) \sim \sum_{r=1}^{\infty} \frac{\zeta(r+1, 1 + a)}{r+1} \left( \frac{B_{r+2}(q) - B_{r+2}}{r+2} \right) / \tau^{r+1}
\]

in the limit \( \tau \to +\infty \) and

\[
I(q \mid a, \tau) = \log \frac{G(1 + a + \tau \mid \tau)}{G(1 - q + a + \tau \mid \tau)} - q \log \left[ \Gamma(1 + \frac{a}{\tau}) \right] + \left( \frac{q^2 - q}{2\tau} \right) \psi(1 + \frac{a}{\tau}).
\]

The proof is given in [30]. It remains to apply this lemma to each of the four ratios of the \( G \)-functions in (119) and collect the terms.

The Morris integral probability distribution thus has the required properties and so is naturally conjectured to be the distribution of the total mass of the Bacry-Muzy GMC measure on the circle, cf. Conjecture [9.4] below.
9 Conjectures and Open Questions

In this section we will present a number of key conjectures and some open questions that are associated with our work.

Conjecture 9.1 (Uniqueness) Let \( x = \int_0^1 \phi(s) \, ds \). The intermittency expansion,
\[
E \left[ F \left( \int_0^1 \phi(s) \, M_\mu \, ds \right) \right] = F(x) + \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \left[ \sum_{k=2}^{2n} F^{(k)}(x) H_{n,k}(\varphi) \right],
\] (123)
is convergent for sufficiently smooth \( F(x) \) in a neighborhood of \( \mu = 0 \) and its sum coincides with the left-hand side. In other words, the coefficients of the expansion capture the distribution uniquely.

It is known that the intermittency expansion is convergent for finite ranges of positive and negative integer moments of the Bacry-Muzy measure on the interval, cf. Propositions 4.1 and 4.2 in [27].

Conjecture 9.2 (Infinite Divisibility) The distribution of
\[
\log \int_0^1 \phi(s) \, M_\mu \, ds
\] (124)
is infinitely divisible.

This is known for the Selberg and Morris integral probability distributions, cf. [30] and [34] and was first discovered in the special case of \( \lambda_1 = \lambda_2 = 0 \) in [27].

Conjecture 9.3 (Polynomial Behavior of Log-Moments) The coefficients \( c_p(l) \) of the expansion of log-moments in \( \mu \) in (45) are polynomials in the moment order \( l \).

This is known for the Selberg and Morris integral probability distributions, cf. [27] and Lemma 8.1 above.

Conjecture 9.4 (GMC on the Circle) Let \( M_\mu \, ds \) denote the GMC measure on the circle described in Section 8 and \( M_\tau,\lambda_1,\lambda_2 \) denote the special case of the Morris integral probability distribution as in Theorem [8.2] with \( \tau = 2/\mu \). Then,
\[
\int_0^1 |1 - e^{2\pi is} |^{2\lambda} M_\mu \, ds = M_\tau,\lambda_1,\lambda_2.
\] (125)
The reason for the restriction \( \lambda_1 = \lambda_2 \) is that \( M_\tau,\lambda_1,\lambda_2 \) is real-valued whereas the generalized total mass corresponding to the full Morris integral is not in general, unless \( \lambda_1 = \lambda_2 \). This conjecture first appeared in [12] for \( \lambda_1 = \lambda_2 = 0 \) and in [34] in general.

Conjecture 9.5 (GMC on the Interval) Let \( M_\mu \, ds \) denote the GMC measure on the interval with \( r(t) \) as in (7) and \( \tau = 2/\mu \). Then, the distribution of
\[
\int_0^1 s^{\lambda_1} (1-s)^{\lambda_2} M_\mu \, ds
\] (126)
has the Mellin transform
\[ \mathcal{M}(q \mid \tau) \triangleq \mathbb{E} \left[ \left( \int_0^1 M_\mu(ds) \right)^q \right], \quad (128) \]
is self-dual (involution invariant) under the transformation
\[ \tau \rightarrow \frac{1}{\tau}, \quad q \rightarrow \frac{q}{\tau}, \quad (129) \]
\[ \mathcal{M} \left( \frac{q}{\tau} \right) (2\pi)^{-\frac{q}{2}} \Gamma^\frac{q}{2}(1-\tau) \Gamma(1-\frac{q}{\tau}) = \mathcal{M}(q \mid \tau)(2\pi)^{-q} \Gamma^q(1-\tau) \Gamma(1-q). \quad (130) \]

This is known for both Morris and Selberg integral probability distributions and was first discovered in [14], cf. also [13], [32], [34] for extensions to nonzero \( \lambda_1, \lambda_2 \) and [8] for self-duality in the model of the 2D Gaussian Free Field (GFF) restricted to circles, cf. (134) below.

**Conjecture 9.8 (Centered GMC Measure)** Considered the centered version of the underlying Gaussian field,
\[ \tilde{\omega}_{\mu, \varepsilon}(u) \triangleq \omega_{\mu, \varepsilon}(u) - \omega_{\mu, \varepsilon}(0). \quad (131) \]
Let \(-1/\mu - 1/2 < \text{Re}(q) < 2/\mu\), then
\[ \lim_{\varepsilon \to 0} \mu^{(\log r(-1))(1-q)} \mathbb{E} \left[ \left( \int_0^1 \varphi(u) e^{\tilde{\omega}_{\mu, \varepsilon}(u)} du \right)^q \right] = \mathbb{E} \left[ \left( \int_0^1 r(u)^{\mu q} \varphi(u) M_\mu(du) \right)^q \right]. \quad (132) \]
\[ ^{5} \text{This holds for } \mathcal{M}(q, \lambda_1, \lambda_2) \text{ in } (160) \text{ provided it is multiplied by } (2\pi)^q \text{ and the } \lambda \text{ s transform } \lambda_i = \tau \lambda_i, \text{ cf. } [34] \text{ for details.} \]
This is motivated by conjectured mod-gaussian limit theorems and based on Girsanov’s theorem for
gaussian fields, cf. [33] for details. For example, combining this conjecture with (127), one obtains
for the Bacry-Muzy GMC on the interval
\[
\lim_{\varepsilon \to 0} e^{\mu(\log e - 1) \frac{2q+1}{q}} \mathbb{E} \left[ \left( \int_0^1 e^{\tilde{\omega}_{\mu,\varepsilon}(s)} \, ds \right)^q \right] = \tau^\varepsilon (2\pi)^q \Gamma^{-q}(1 - 1/\tau) \frac{\Gamma_2(1+q+\tau|\tau)}{\Gamma_2(1+2q+\tau|\tau)} \times \frac{\Gamma_2(1-q+\tau|\tau)}{\Gamma_2(1+\tau|\tau)} \frac{\Gamma_2(2+q+2\tau|\tau)}{\Gamma_2(2+2\tau|\tau)},
\]
(133)

We also want to mention a few open questions.

1. Are there any examples of \( r(t) \) different from (7) and (8) for which the moments in (14) and the
   full intermittency expansion can be computed in closed form and the intermittency expansion
   can be re-summed to give the Mellin transform of a valid probability distribution? This is
   particularly interesting for the model of the 2D Gaussian Free Field (GFF) restricted to circles
   that was recently considered in [8] and corresponds to
   \[
   r(t) = \left| \frac{1 - e^{2\pi i t}}{1 - q e^{2\pi i t}} \right|
   \]
   (134)
   for some \( q \in (-1, 1) \).

2. Is the distribution of the total mass always expressible in terms of Barnes beta distributions (as
   is the case of both Selberg and Morris integral distributions)?

3. How can one generalize the key renormalizability property in (37) and, more generally, the theory
   of intermittency expansions to the case of infinitely divisible multiplicative chaos measures
   (in the sense of Bacry and Muzy)? This is particularly interesting as both the key intermittency
   invariance in (76) and formula for the positive integer moments in (14) are known in
   the general infinitely divisible case, cf. [28] for the former and [35] for the latter. Hence, one
   should in principle be able to derive the corresponding intermittency differentiation rule and
   intermittency expansion.

4. Is it possible to make our derivation of the intermittency differentiation rule mathematically
   rigorous?

5. What is the dependence structure of GMC measures? We showed that the dependence structure
   of the GMC measure can be recovered, in the sense of intermittency expansions, from the
   joint integer moments. For example, for two subintervals \( I_1 \) and \( I_2 \) and \( r(t) = |t| \) one needs to
   calculate for all \( n \) and \( m \)
   \[
   \mathbb{E} \left[ \left( \int_{I_1} M_{\mu}(ds) \right)^n \left( \int_{I_2} M_{\mu}(ds) \right)^m \right] = \int_{P_1 \times P_2} \prod_{k<l} |x_k - x_l|^{-\mu} \, dx,
   \]
   (135)
   The calculation of such integrals presents a particular challenge, cf. [29] and [35] for details.

6. How to extend the theory to non-stationary GMC measures? The simplest non-trivial example
   is to replace \( \omega_{\mu,\varepsilon}(s) \) with the centered process \( \omega_{\mu,\varepsilon}(s) - \omega_{\mu,\varepsilon}(0) \) as in (131) above. This
is particularly interesting in the context of extrema of a regularized version of the fractional Brownian motion with zero Hurst index considered in [13] and [17] and mesoscopic statistics of Riemann zeroes considered in [33].

7. How to extend the theory of intermittency expansions to complex functionals of the total mass? For example, the Morris integral probability distribution is defined and is real-valued for $\lambda_1 \neq \lambda_2$ in Theorem 8.2. Yet, the corresponding functional of the total mass is complex-valued, unless $\lambda_1 = \lambda_2$, in spite of the fact the integer moments of the total mass are real-valued. Therefore, one needs to consider more general moments than just the integer moments of the total mass and develop the corresponding theory of intermittency renormalization.

8. What is the correct definition of Selberg’s integral for the general 1D GMC measure? A natural proposal, which is consistent with the Selberg and Morris integrals, is

$$S(\lambda_1, \lambda_2, \lambda) = \int_{[0,1]^n} \prod_{i=1}^n r(s_i)^{\lambda_1} r(1-s_i)^{\lambda_2} \prod_{i<j} r(s_i - s_j)^{2\lambda} ds_1 \cdots ds_n.$$  \hspace{1cm} (136)

Can this integral be computed or otherwise characterized analytically?

10  Conclusions

We have presented a theory of renormalization of 1D GMC measures. The theory is based on the rule of intermittency differentiation. The rule is an exact, non-local functional equation that prescribes how to differentiate a general class of functionals of the GMC measure with respect to intermittency. A repeated application of this rule leads to a perturbative expansion of the functional in intermittency known as the intermittency expansion (or the high temperature expansion). The intermittency expansion is a renormalized expansion in the centered moments of the total mass of the GMC measure. We have shown that the full intermittency expansion of the Mellin transform can be computed for a class of GMC measures having the property that the logarithm of integer moments of the total mass is polynomial in the moment order. We have illustrated the theory with the case of the periodized Bacry-Muzy GMC measure on the circle. We have explicitly computed the intermittency expansion, summed it in closed form, and showed that the resulting Mellin transform is the Mellin transform of the Morris integral probability distribution, which is then conjectured to be the distribution of the total mass on the circle.

Our derivation of the intermittency differentiation rule is exact but not mathematically rigorous. It is based on the intermittency invariance of the underlying gaussian field, which we formulated for a general 1D GMC measure in this paper. This invariance is a technical devise that substitutes for the non-existent Markov property of the underlying gaussian field and allows one to derive a Feynman-Kac equation for the distribution of the total mass by considering a stochastic flow in intermittency (as opposed to time in the classical framework of diffusions). The intermittency invariance gives two ways of evaluating the limit of the flow, which results in the differentiation rule. The first way is the backward Kolmogorov equation, the second way involves detailed analysis of certain infinite series expansions, combined with a combinatorial property of the measure, and applications of Girsanov’s theorem for gaussian fields. The rule is non-local, \textit{i.e.} involves the measure of a continuum of subintervals of the given interval due to the very strong stochastic dependence of the measure.
We have presented several key conjectures and formulated a number of open questions that we hope will help stimulate future research on GMC measures. Our main conjecture is that the intermittency expansion is convergent for smooth test functions of the total mass and sufficiently small intermittency and therefore captures the distribution of the total mass uniquely. Our approach is not limited to the problem of total mass or GMC measures. We have shown that the intermittency differentiation approach has a natural extension to the joint distribution of the mass of subintervals, i.e. the dependence structure of the GMC measure, and can reconstruct the joint distribution form the corresponding joint moments. The main challenge on this path is the actual computation of the joint moments. Similarly, both the intermittency invariance of the underlying gaussian field and the multiple integral representation of the integer moments of the total mass have known generalizations to the infinitely divisible multiplicative chaos measure of Bacry and Muzy. The main open questions in the infinitely divisible case are the computation of the moments and derivation of the intermittency differentiation rule.

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A A Review of the GMC on the Interval

The construction of $\omega_{\mu,\varepsilon}(s)$ is based on the idea of using conical sets as in Bacry and Muzy [24] and modifying the intensity measure to match the desired covariance. The conical sets live on the upper half-plane in the case of the boundary condition in (6a) and on the torus in the case of (6b).

The starting point is a gaussian independently scattered random measure $P$ on the time-scale plane $\mathbb{H}+ = \{(t, l), l > 0\}$, distributed uniformly with respect to some positive intensity measure $\rho$. This means that $P(A)$ is a gaussian random variable for measurable subsets $A \subset \mathbb{H}_+$. The property of being independently scattered means that $P(A)$ and $P(B)$ are independent if $A$ and $B$ do not intersect. Uniform distribution with respect to $\rho$ means that the characteristic function of $P(A)$ is given by

$$E\left[e^{iqP(A)}\right] = e^{\mu \phi(q) \rho(A)}, \quad q \in \mathbb{R},$$

(137)

where $\mu > 0$ is the intermittency parameter and $\phi(q)$ is the logarithm of the characteristic function of the underlying gaussian distribution and is given by

$$\phi(q) = -i\frac{q^2}{2} - \frac{q^2}{2}.$$  

(138)

The mean is chosen in such a way that

$$\phi(-i) = 0$$

so that $E[e^{P(A)}] = 1 \forall A \subset \mathbb{H}_+$,

(139)

which gives rise to the martingale property of the limit measure. The existence of such random measures is established in [39]. Next, following Bacry and Muzy [24], we introduce special conical
sets $\mathcal{A}_\varepsilon(u)$ in the time-scale plane defined by

$$
\mathcal{A}_\varepsilon(u) = \left\{ (t,l) \mid |t-u| \leq \frac{l}{2} \text{ for } \varepsilon \leq l \leq 1 \text{ and } |t-u| \leq \frac{1}{2} \text{ for } l \geq 1 \right\}.
$$

The last preparatory step is to define a family of gaussian processes with dependent increments $\omega_{\mu,\varepsilon}(u)$ by

$$
\omega_{\mu,\varepsilon}(u) = P\left(A_\varepsilon(u)\right).
$$

It is clear that $\omega_{\mu,\varepsilon}(u)$ and $\omega_{\mu,\varepsilon}(v)$ are dependent in general if $|u-v| < 1$ and are independent otherwise so this case corresponds to $r(1) = 1$.

We now wish to choose the intensity measure $\rho(\text{d}t \text{d}l)$ in such a way that the process $\omega_{\mu,\varepsilon}(u)$ has the covariance given in (1). We make the ansatz

$$
\rho(\text{d}t \text{d}l) = \frac{f(l)}{l^2} \text{d}t \text{d}l.
$$

**Lemma A.1** Define $f(l)$ by

$$
\frac{f(l)}{l^2} = -\frac{d^2}{dl^2} \log r(l), \ l \in (0,1).
$$

$$
f(l) = \frac{d}{dz}\bigg|_{z=1} \log r(z), \ l \geq 1.
$$

and assume that these quantities are positive. Then, the process $\omega_{\mu,\varepsilon}(u)$ in (141) with the intensity measure defined in (142) has the covariance specified in (1).

Note that for $r(l) = l$ we recover the original result of Bacry and Muzy, namely,

$$
f(l) = 1.
$$

**Proof** We need to compute the $\rho$ measure of the intersection of $\mathcal{A}_\varepsilon(u)$ and $\mathcal{A}_\varepsilon(v)$. Defining

$$
g(z) = \int_z^1 \frac{f(l)}{l^2} \text{d}l,
$$

it is easy to show that we have the identity

$$
\int_z^1 g(x) \text{d}x = \int_z^1 \frac{f(l)}{l^2} (l-z) \text{d}l.
$$

Using the definition of $f(l)$ in (143) and (144), we have the additional identities

$$
g(z) = -\frac{d}{dx}\bigg|_{x=1} \log r(x) + \frac{d}{dz} \log r(z),
$$

$$
\int_z^1 g(x) \text{d}x = -(1-z) \frac{d}{dl}\bigg|_{l=1} \log r(l) - \log r(z).
$$

Assume first that $\varepsilon \leq |u-v| < 1$. Let $z = v-u$, $u < v$. Then, for $z \in [\varepsilon,1]$,

$$
\rho\left(\mathcal{A}_\varepsilon(u) \cap \mathcal{A}_\varepsilon(v)\right) = (1-z) \int_1^\infty \frac{f(l)}{l^2} \text{d}l + \int_z^1 \frac{f(l)}{l^2} (l-z) \text{d}l.
$$
It follows from (144), (147), and (149), that the $\rho$ measure of the intersection equals $-\log r(z)$ as desired.

Now, we need to consider the case of $z \leq \epsilon$. The $\rho$ measure of the intersection is

$$\rho\left(\mathscr{A}_\epsilon(u) \cap \mathscr{A}_\epsilon(v)\right) = (1 - z) \int_1^\infty \frac{f(l)}{l^2} dl + \int_\epsilon^1 \frac{f(l)}{l^2} (1 - z) dl.$$

Using (144) and (147), we can write for the $\rho$ measure of the intersection,

$$\rho\left(\mathscr{A}_\epsilon(u) \cap \mathscr{A}_\epsilon(v)\right) = \int_\epsilon^1 g(x) dx + (\epsilon - z) g(\epsilon) + \left(1 - z\right) \frac{d}{dl}\bigg|_{l=1} \log r(l).$$

Finally, substituting (148) and (149) and noticing several cancellations, we get

$$\rho\left(\mathscr{A}_\epsilon(u) \cap \mathscr{A}_\epsilon(v)\right) = -\log r(\epsilon) + \left(1 - \frac{z}{\epsilon}\right) \frac{d}{dl}\bigg|_{l=\epsilon} \log r(l).$$

Hence, we arrive at the desired form of covariance in (1). □

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