THE RESTRICTED KIRILLOV–RESHETIKHIN MODULES FOR THE CURRENT AND TWISTED CURRENT ALGEBRAS

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INTRODUCTION

In this paper we define and study a family of $\mathbb{Z}_+$–graded modules for the polynomial valued current algebra $\mathfrak{g}[t]$ and the twisted current algebra $\mathfrak{g}[t]^\sigma$ associated to a finite–dimensional classical simple Lie algebra $\mathfrak{g}$ and a non–trivial diagram automorphism of $\mathfrak{g}$. The modules which we denote as $KR(m\omega_i)$ and $KR^\sigma(m\omega_i)$ respectively are indexed by pairs $(i, m)$, where $i$ is a node of the Dynkin diagram and $m$ is a non–negative integer, and are given by generators and relations. These modules are indecomposable, but usually reducible, and we describe their Jordan–Holder series by giving the corresponding graded decomposition as a direct sum of irreducible modules for the underlying finite–dimensional simple Lie algebra. Moreover, we prove that the modules are finite–dimensional and hence restricted, i.e., there exists an integer $n \in \mathbb{Z}_+$ depending only on $\mathfrak{g}$ and $\sigma$ such that $(\mathfrak{g} \otimes t^n)KR(m\omega_i) = 0$. It turns out that this graded decomposition is exactly the one predicted in [9, Appendix A], [10, Section 6] coming from the study of the Bethe Ansatz in solvable lattice models.

Our interest in these modules and the motivation for calling them the Kirillov–Reshetikhin modules arises from the fact that when we specialize the grading by setting $t = 1$, (or equivalently putting $q = 1$ in the formulae in [9], [10]) the character of the module is exactly the one predicted in [13], [14] for a family of irreducible finite–dimensional modules for the Yangian of $\mathfrak{g}$. Analogous modules for the untwisted quantum affine algebra associated to $\mathfrak{g}$ are also known to exist with this decomposition [2] and these have been studied from a combinatorial viewpoint in [9], [15], [10], [17]. One of the methods used in [2] involves passing to the $q = 1$ limit of the modules for the quantum affine algebra, although the resulting modules are not graded or restricted. The methods used in [2] require a number of complicated results from the representation theory of the untwisted quantum affine algebras which have not been proved in the twisted case.

In this paper we show that the Kirillov–Reshetikhin modules can be studied in the non–quantum case. Thus we define graded restricted analogues and compute their graded characters without resorting to the quantum situation. As a consequence we have a mathematical interpretation of the parameter $q$ which appears in the fermionic formulae in [9], [10]. Then we prove that the abstractly defined modules $KR(m\omega_i)$ have a concrete construction as follows: the $\mathfrak{g}[t]$–module structure of the “fundamental” Kirillov–Reshetikhin modules (in most cases these modules correspond to taking $m = 1$) is described explicitly and then the module $KR(m\omega_i)$ is realized as a canonical submodule of a tensor product of the fundamental modules.

VC was partially supported by the NSF grant DMS-0500751.
Another motivation for our interest is the connection with Demazure modules, \([4, 5, 6, 7]\). Thus we are able to prove that the Kirillov–Reshetikhin modules for the current algebras are isomorphic as representations of the current algebra to the Demazure modules in multiples of the basic representation of the current algebras. Our methods also work for some nodes of the other exceptional algebras, we explain this together with the reasons for the difficulties for the exceptional algebras in the concluding section of this paper.

1. Preliminaries

1.1. Notation. Let \(\mathbb{Z}_+\) (resp. \(\mathbb{N}\)) be the set of non-negative (resp. positive) integers. Given a Lie algebra \(\mathfrak{g}\), let \(\mathcal{U}(\mathfrak{g})\) denote the universal enveloping algebra of \(\mathfrak{g}\) and \(\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]\) the polynomial valued current Lie algebra of \(\mathfrak{g}\). The Lie algebra \(\mathfrak{g}[t]\) and its universal enveloping algebra are \(\mathbb{Z}_+\)-graded where the grading is given by the powers of \(t\). We shall identify \(\mathfrak{g}\) with the subalgebra \(\mathfrak{g} \otimes 1\) of \(\mathfrak{g}[t]\).

1.2. Classical simple Lie algebras. For the rest of the paper \(\mathfrak{g}\) denotes a complex finite-dimensional simple Lie algebra of type \(A_n, B_n, C_n\) or \(D_n, n \geq 1\), and \(\mathfrak{h}\) a Cartan subalgebra of \(\mathfrak{g}\). Let \(I = \{1, \cdots, n\}\) and \(\{\alpha_i : i \in I\}\) (resp. \(\{\omega_i : i \in I\}\)) be a set of simple roots (resp. fundamental weights) of \(\mathfrak{g}\) with respect to \(\mathfrak{h}\), \(R^+\) (resp. \(Q, P\)) be the corresponding set of positive roots (resp. root lattice, weight lattice) and let \(Q^+, P^+\) be the \(\mathbb{Z}_+\)-span of the simple roots and fundamental weights respectively. It is convenient to set \(\omega_0 = 0\). Let \(\theta \in R^+\) be the highest root. For \(i \in I\), let \(\varepsilon_i : Q \rightarrow \mathbb{Z}\) be defined by requiring \(\eta = \sum_{i \in I} \varepsilon_i(\eta)\alpha_i, \eta \in Q\). We assume throughout the paper that the simple roots are indexed as in \([1]\). Given \(\alpha \in R\), let \(\mathfrak{g}_\alpha\) be the corresponding root space. Fix non-zero elements \(x^\pm_\alpha, h_\alpha \in \mathfrak{h}\), such that

\[
[h_\alpha, x^\pm_\alpha] = \pm 2x^\pm_\alpha, \quad [x^+_\alpha, x^-_\alpha] = h_\alpha.
\]

Set \(\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm \alpha}\). Given any subset \(J \subset I\) let \(\mathfrak{g}(J)\) be the subalgebra of \(\mathfrak{g}\) generated by the elements \(x^\pm_{\alpha_j}, j \in J\). The sets \(R(J), P(J)\) etc. are defined in the obvious way.

Let \((\ , \ )\) be the form on \(\mathfrak{h}^*\) induced by the restriction of the Killing form of \(\mathfrak{g}\) to \(\mathfrak{h}\) normalized so that \((\theta, \theta) = 2\) and set \(d_j = 2/(\alpha_j, \alpha_j)\).

1.3. Finite-dimensional \(\mathfrak{g}\)-modules. Given \(\lambda \in P^+\), let \(V(\lambda)\) be the irreducible finite-dimensional \(\mathfrak{g}\)-module with highest weight vector \(v_\lambda\), i.e., the cyclic module generated by \(v_\lambda\) with defining relations:

\[
n^+ v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda, \quad (x^-_\alpha)^{\lambda(h_\alpha)+1} v_\lambda = 0.
\]

For \(\mu \in P\), let \(V(\lambda)_\mu = \{v \in V(\lambda) : hv = \mu(h)v, \ h \in \mathfrak{h}\}\). Any finite-dimensional \(\mathfrak{g}\)-module \(M\) is isomorphic to a direct sum \(\bigoplus_{\lambda \in P^+} V(\lambda)^{\oplus m_\lambda(M)}\), \(m_\lambda(M) \in \mathbb{Z}_+\).

Proposition. \([18]\) Let \(\lambda, \mu \in P^+\). Then \(V(\lambda) \otimes V(\mu)\) is generated as a \(\mathfrak{g}\)-module by the element \(v_\lambda \otimes v^*_\mu\) and relations:

\[
h(v_\lambda \otimes v^*_\mu) = (\lambda + \mu^*)(h)(v_\lambda \otimes v^*_\mu),
\]

and

\[
(x^+_\alpha)^{-\mu^*(h_\alpha)+1} (v_\lambda \otimes v^*_\mu) = (x^-_\alpha)^{-\lambda(h_\alpha)+1} (v_\lambda \otimes v^*_\mu) = 0,
\]

for all \(\alpha \in R^+\). Here \(\mu^*\) is the lowest weight of \(V(\mu)\) and \(0 \neq v^*_\mu \in V(\mu)_{\mu^*}\). \(\square\)
1.4. Graded Modules. Given a g–module $M$, we regard it as a $g[t]$–module by setting $(x \otimes t^r)m = 0$ for all $m \in M$, $r \in \mathbb{N}$, and denote the resulting graded $g[t]$–module by $ev_0(M)$. Let $V = \oplus_{s \in \mathbb{Z}_+} V[s]$ be a graded representation of $g[t]$ with $\dim(V[s]) < \infty$. Note that each $V[s]$ is a $g$–module. For any $s \in \mathbb{Z}_+$, let $V(s)$ be the $g[t]$–quotient of $V$ by the submodule $\oplus_{s' \leq s} V[s']$. Clearly

$$V(s)/V(s + 1) \cong ev_0(V[s]),$$

and the irreducible constituents of $V$ are just the irreducible constituents of $ev_0(V[s])$, $s \in \mathbb{Z}_+$.

2. The Kirillov-Reshetikhin modules for $g[t]$

2.1. The modules KR($m\omega_i$).

Definition. Given $i \in I$ and $m \in \mathbb{Z}_+$, let KR($m\omega_i$) be the $g[t]$–module generated by an element $v_{i,m}$ with relations,

$$n^+[t]v_{i,m} = 0, \quad hv_{i,m} = m\omega_i(h)v_{i,m}, \quad (h \otimes t^r)v_{i,m} = 0, \quad h \in h, r \in \mathbb{N},$$

and

$$(x_{\alpha_i}^\pm)^{m+1}v_{i,m} = (x_{\alpha_i}^- \otimes t)v_{i,m} = x_{\alpha_i}^+v_{i,m} = 0, \quad j \in I \{i\}. \quad \square$$

Note that the modules KR($m\omega_i$) are graded modules since the defining relations are graded. The following is trivially checked.

Lemma. For all $i \in I$, $m \in \mathbb{Z}_+$, the module $ev_0(V(m\omega_i))$ is a quotient of KR($m\omega_i$). \hfill \square

Remark. These modules were defined and studied initially in \cite{2} Section 1] (where they were denoted as $W(i,m)$) as quotients of the finite–dimensional Weyl modules defined in \cite{4}. In this paper however, we shall show directly that the modules KR($m\omega_i$) are finite–dimensional as a consequence of the analysis of their $g$–module structure.

2.2. The graded character of KR($m\omega_i$). We now state the main result of this section. We need some additional notation. For $i \in I$, $m \in \mathbb{Z}_+$, let $P^+(i,m) \subseteq P^+$ be defined by:

$$P^+(i,m) = P^+(i,d_i) + P^+(i,m - d_i), \quad m > d_i,$$

where $P^+(i,1) = \omega_i$ if $\varepsilon_i(\theta) = d_i$ and in the other cases,

$$P^+(i,1) = \{\omega_i, \omega_{i-2}, \cdots, \omega_{i-\ell}\}, \quad g = D_n,$$

$$P^+(i,2) = \{2\omega_i, 2\omega_{i-2}, \cdots, 2\omega_{i-\ell}\}, \quad g = B_n,$$

$$P^+(i,\ell) = \{2\omega_i, 2\omega_{i-2}, \cdots, 2\omega_{i-\ell}\}, \quad g = C_n,$$

where $\ell \in \{0,1\}$ and $i = \ell \mod 2$. Let $\mu_0, \cdots, \mu_k$ be the unique enumeration of the sets $P^+(i,d_i)$ chosen so that

$$\mu_j - \mu_{j+1} \in R^+, \quad \mu_j - \mu_{j+2} \in Q^+ \backslash R^+, \quad 0 \leq j \leq k - 1.$$ 

Given $m = d_i m_0 + m_1$ with $0 \leq m_1 < d_i$ and $m \in P^+(i,m)$, we can clearly write $m = m_1 \omega_i + \mu_j + \cdots + \mu_{j_\ell}$, where $j_r \in \{0,1, \cdots, k\}$ for $1 \leq r \leq m_0$. We say that the expression
is reduced if each $j_r$ is minimal with the property that $\mu - \mu_{j_1} - \cdots - \mu_{j_r} \in P^+(i, m - r)$. Such an expression is clearly unique and we set

$$|\mu| = \sum_{r=1}^{m_0} j_r.$$  

**Theorem.** 

(i) Let $i \in I, m, s \in \mathbb{Z}_+$. We have,

$$KR(m\omega_i)[s] \cong \bigoplus_{\{\mu \in P^+(i,m): |\mu| = s\}} V(\mu).$$

(ii) Write $m = \tilde{d}m_0 + m_1$, where $0 \leq m_1 < \tilde{d}$. The canonical homomorphism of $\mathfrak{g}[t]$–modules

$$KR(m\omega_i) \to KR(m_1\omega_i) \otimes KR(d_i\omega_i) \oplus m_0$$

mapping $v_{i,m} \mapsto v_{i,m_1} \otimes v_{i,d_i}^{m_0}$ is injective.

The rest of the section is devoted to the proof of this result.

2.3. Elementary properties of $KR(m\omega_i)$.

**Proposition.**

(i) We have

$$KR(m\omega_i) = \bigoplus_{\mu \in \mathfrak{h}^*} KR(m\omega_i)_\mu$$

and $KR(m\omega_i)_\mu \neq 0$ only if $\mu \in m\omega_i - Q^+$.

(ii) Regarded as a $\mathfrak{g}$–module, $KR(m\omega_i)$ and $KR(m\omega_i)[s], s \in \mathbb{Z}_+$, are isomorphic to a direct sum of irreducible finite–dimensional representations of $\mathfrak{g}$.

(iii) For all $0 \leq r \leq m$, there exists a canonical homomorphism $KR(m\omega_i) \to KR(r\omega_i) \otimes KR((m - r)\omega_i)$ of graded $\mathfrak{g}[t]$–modules such that $v_{i,m} \mapsto v_{i,r} \otimes v_{i,(m-r)}$.

**Proof.** Part (i) follows by a standard application of the PBW theorem. For (ii) it suffices, by standard results, to show that $KR(m\omega_i)$ is a sum of finite–dimensional $\mathfrak{g}$–modules. Note that the defining relations of $KR(m\omega_i)$ imply that $U(\mathfrak{g})v_{i,m} \cong V(m\omega_i)$ as $\mathfrak{g}$–modules. Hence

$$(2.3) \quad (x_\alpha^{m\omega_i(h_\alpha)} + 1)v_{i,m} = 0$$

for all $\alpha \in R^+$. For $v \in KR(m\omega_i)_\mu$ we have $U(\mathfrak{g})v = U(\mathfrak{n}^-)U(\mathfrak{n}^+)v$. Part (ii) implies that $U(\mathfrak{n}^+)v$ is a finite–dimensional vector space. Further since the action of $\mathfrak{n}^-$ on $\mathfrak{n}^-\mathfrak{z}[t]$ given by the Lie bracket is locally nilpotent and since $v \in U(\mathfrak{n}^-\mathfrak{z}[t])v_{i,m}$, it follows from (2.3) that $x_\alpha$ acts nilpotently on $v$. This proves that $U(\mathfrak{g})v$ is finite–dimensional and part (ii) is established. Part (iii) is clear from the defining relations of the modules. 

**Corollary.** For $i \in I, m \in \mathbb{Z}_+$, we have

$$KR(m\omega_i) = \bigoplus_{\mu \in P^+} V(\mu)^{\otimes m_{\mu}(i,m)}.$$  

In particular, $KR(0) \cong \mathbb{C}$. 

\[ \square \]
2.4. An upper bound for $m_\mu(i,m)$. The following result was proved in [2, Theorem 1] under the assumption that $KR(m\omega_i)$ is finite-dimensional. An inspection of the proof shows however, that the only place this is used is to write $KR(m\omega_i)$ as a direct sum of irreducible $\mathfrak{g}$–modules. But (as we shall see in the twisted case, where we do give a proof of the analogous proposition) this only requires the weaker result proved in Proposition 2.3.

Proposition. As a $\mathfrak{g}$–module we have

$$KR(m\omega_i) \cong \bigoplus_{\mu \in P^+(i,m)} V(\mu)^{\oplus m_\mu},$$

where $m_\mu \in \{0,1\}$.

Th next corollary is immediate since $P^+(i,m)$ is a finite set by definition.

Corollary. For all $i \in I$, the modules $KR(m\omega_i)$ are finite–dimensional. \hfill \Box

2.5. Proof of Theorem 2.2 the cases $\varepsilon_i(\theta) = 1$, $m \in \mathbb{Z}_+$ and $\varepsilon_i(\theta) = \tilde{d}_i$, $m = 1$. In these cases, it follows from the definition that $P^+(i,m) = \{m\omega_i\}$. Since $ev_0(V(m\omega_i))$ is a $\mathfrak{g}[t]$–module quotient of $KR(m\omega_i)$, part (i) is immediate from Proposition 2.3. Part (ii) of the theorem is now obvious since the canonical inclusion $V(m\omega_i) \to V(\omega_i)^{\oplus m}$ of $\mathfrak{g}$–modules is obviously also an inclusion of the $\mathfrak{g}[t]$–modules $ev_0(V(m\omega_i)) \to ev_0(V(\omega_i))^{\oplus m}$.

Since $\varepsilon_i(\theta) = 1$ for all $1 \leq i \leq n$ if $\mathfrak{g}$ is of type $A_n$, the theorem is proved in this case and we assume for the rest of this section that $\mathfrak{g}$ is not of type $A_n$.

2.6. An explicit construction in the case $\varepsilon_i(\theta) = 2$ and $m = \tilde{d}_i$.

Let $V_s$, $0 \leq s \leq k$, be $\mathfrak{g}$–modules such that

$$(x \otimes t)v = p_s(x \otimes v), \quad (x \otimes t^r)v = 0, \quad r \geq 2,$$

for all $x \in \mathfrak{g}$, $v \in V_s$, $1 \leq s \leq k$. Clearly,

$$V[s] \cong_{\mathfrak{g}} V_s, \quad 0 \leq s \leq k.$$

Moreover, if the maps $p_s$, $0 \leq s \leq k – 1$, are all surjective and if $V_0 = U(\mathfrak{g})v_0$ then $V = U(\mathfrak{g}[t])v_0$.

Proposition. Let $i \in I$ be such that $\varepsilon_i(\theta) = 2$ and let $\mu_s \in P^+(i,\tilde{d}_i)$, $0 \leq s \leq k$. The modules $V(\mu_s)$, $0 \leq s \leq k$, satisfy (2.4) and the resulting $\mathfrak{g}[t]$–module is isomorphic to $KR(\tilde{d}_i\omega_i)$. In particular,

$$KR(\tilde{d}_i\omega_i)[j] \cong_{\mathfrak{g}} V(\mu_j), \quad 0 \leq j \leq k.$$
Proof. Using Proposition 1.3 it is easy to see that
\[ \text{Hom}_\mathfrak{g}(\mathfrak{g} \otimes V(\mu_s), V(\mu_{s+1})) \cong \text{Hom}_\mathfrak{g}(\mathfrak{g}, V(\mu_s) \otimes V(\mu_{s+1})) \neq 0 \]
for all \( 0 \leq s \leq k - 1 \). It is not hard to check (see [8], [11]) that as \( \mathfrak{g} \)-modules,
\[ \wedge^2 \mathfrak{g} \cong \mathfrak{g} \oplus V(\nu), \]
where \( \nu = 2\omega_1 + \omega_2 \) if \( \mathfrak{g} \) is of type \( C_n \), \( \nu = \omega_1 + 2\omega_3 \) if \( \mathfrak{g} \) is of type \( B_3 \), \( \nu = \omega_1 + \omega_3 + \omega_4 \) if \( \mathfrak{g} \) is of type \( D_4 \), and \( \nu = \omega_1 + \omega_3 \) otherwise. Since \( \mu_s - \mu_{s+2} \notin R \), it follows that \( \text{Hom}_\mathfrak{g}(\mathfrak{g} \otimes V(\mu_s), V(\mu_{s+2})) = 0 \). To prove that \( \text{Hom}_\mathfrak{g}(V(\nu) \otimes V(\mu_s), V(\mu_{s+2})) = 0 \), it suffices to prove that
\[ \text{Hom}_\mathfrak{g}(V(\mu_s), V(\nu) \otimes V(\mu_{s+2})) = 0. \]

Suppose that \( 0 \neq p \in \text{Hom}_\mathfrak{g}(V(\mu_s), V(\nu) \otimes V(\mu_{s+2})) \). A simple computation using the explicit formulas for the fundamental weights in terms of the simple roots (see [11] for instance) shows that
\[ \mu_{s+2} + \nu - \mu_s \in Q^+_t, \quad J = \{1, \ldots, s - 1\}. \]
Hence
\[ 0 \neq p(v_{\mu_s}) \in U(\mathfrak{g})v_\nu \otimes U(\mathfrak{g})v_{\mu_{s+2}}, \]
which implies that
\[ p(U(\mathfrak{g})v_{\mu_s}) \subset U(\mathfrak{g})v_\nu \otimes U(\mathfrak{g})v_{\mu_{s+2}}. \]
The formulas given for \( \mu_s \) shows that \( \mu_s(h_J) = 0 \). This means that \( U(\mathfrak{g})v_{\mu_s} \cong \mathcal{C} \) which implies that
\[ p(\mathcal{C}) \cong \mathcal{C} \subset (U(\mathfrak{g})v_\nu \otimes U(\mathfrak{g})v_{\mu_{s+2}}). \]
But this is impossible since by standard results \( U(\mathfrak{g})v_{\mu_{s+2}} \) is not isomorphic to the \( \mathfrak{g} \)-dual of \( U(\mathfrak{g})v_\nu \). Hence \( p = 0 \) and we have proved that the modules \( V(\mu_s) \), \( 0 \leq s \leq k - 1 \), satisfy the conditions of [2,4]. Let \( KR(d_iw_i) \) be the resulting \( \mathfrak{g}[t] \)-module. Since the modules \( V(\mu_s) \) are all irreducible, it follows moreover that
\[ KR(d_iw_i) = U(\mathfrak{g}[t])v_{d_iw_i}. \]

To see that \( KR(d_iw_i) \cong KR(d_iw_i) \) it suffices, by Proposition 2.4, to prove that \( KR(d_iw_i) \) is a quotient of \( KR(d_iw_i) \). Since \( \mu_0 = m\omega_i \) and \( \mu_0 - \mu_1 \in R^+ \), we see that we must have,
\[ p_0 \left( (n^+ \oplus \mathfrak{h} \oplus Cx^-_{\alpha_i} \oplus v_{m\omega_i}) = 0. \right. \]
This proves that
\[ n^+[t]v_{m\omega_i} = 0, \quad hv_{m\omega_i} = (m\omega_i)(h)v_{m\omega_i}, \quad (h \otimes t^r)v_{m\omega_i} = (x^-_{\alpha_i} \otimes t)v_{m\omega_i} = 0, \quad r \geq 1. \]
Finally, since \( KR(m\omega_i) \) is obviously finite-dimensional, it follows that
\[ (x^-_{\alpha_i}m\omega_i)^{1+1}v_{m\omega_i} = x^-_{\alpha_i}v_{m\omega_i} = 0, \quad j \in I \setminus \{i\}, \]
and the proof of the proposition is complete. \( \square \)

Given \( \mu \in P^+(i, m) \), \( m = d_im_0 + m_1 \), \( 0 \leq m_1 < d_i \), with reduced expression
\[ \mu = m_1\omega_i + m_{j_1} + \cdots + m_{j_m}, \]
set \( s_j = \#\{r : j_r \geq j\}, 0 \leq j \leq k, \) and
\[ x_\mu = (x^-_{\mu_{k-1}} - \mu_k \otimes t)^{s_k} \cdots (x^-_{\mu_0 - \mu_1} \otimes t)^{s_1} \in U(n^-[t]). \]
It is easily seen that
\[ x_\mu v_{i,m} \in KR(m\omega_i)[[\mu]] \cap KR(m\omega_i)_{\mu}. \]

**Corollary.** Let \( i \in I \) be such that \( \varepsilon_i(\theta) = 2 \). For \( 0 \leq s \leq k \) we have
\[ x_{\mu_s} v_{i,d_i} \neq 0, \quad h(x_{\mu_s} v_{i,d_i}) = \mu_s(h)(x_{\mu_s} v_{i,d_i}), \quad n^+(x_{\mu_s} v_{i,d_i}) = 0, \quad h \in h, \]
or, equivalently,
\[ x_{\mu_s} v_{\mu_0} = v_{\mu_s}. \]

**Proof.** Suppose that \( \mu = \mu_s \) for some \( 1 \leq s \leq k \). We first prove that there exists \( x_s \in n^- \) such that
\begin{equation}
(2.5) \quad p_s(x_s \otimes v_{\mu_s}) = v_{\mu_{s+1}}.
\end{equation}
For this, note that by Proposition 1.3
\[ v = p_s(x_\theta \otimes v_{\mu_s}) \neq 0. \]
Since \( V(\mu_{s+1}) \) is irreducible there exists \( x'_s \in U(n^+) \) such that \( x'_s v = v_{\mu_{s+1}} \), which gives
\[ x'_s(p_s(x_\theta \otimes v_{\mu_s})) = p_s(\text{ad}(x'_s)x_\theta \otimes v_{\mu_s}) = v_{\mu_{s+1}}, \]
proving (2.5). Setting \( x_s = \text{ad}(x'_s)x_\theta \in n^-_{\mu_s - \mu_{s+1}} \) it follows that \( x_s \) is a non-zero scalar multiple of \( x_{\mu_s - \mu_{s+1}} \) for some \( 0 \leq s \leq k \) and we get
\[ (x_{\mu_s - \mu_{s+1}} \otimes t)v_{\mu_s} \neq 0. \]
An obvious induction on \( s \) now proves that
\[ x_{\mu_s} v_{i,d_i} \neq 0, \quad 1 \leq s \leq k - 1. \]
The other two statements of the corollary are now immediate. \( \square \)

Recall that
\[ KR(d_i \omega_i)(r) \cong KR(d_i \omega_i)/\{ \oplus_{r' \geq r} KR(m\omega_i)[r'] \}. \]
Let \( v_r \) be the image of \( v_{d_i \omega_i} \) in \( KR(d_i \omega_i)(r) \). By Corollary 2.6 we see that
\begin{equation}
(2.6) \quad x_{\mu_{r-1}} v_r \neq 0, \quad x_{\mu_r}, v_r = 0, \quad 0 \leq r \leq r' \leq k.
\end{equation}

2.7. **Proof of Theorem 2.2** the case \( \varepsilon_i(\theta) = 2, \ m \geq d_i \).

If \( m = d_i \), then part (i) is the statement of Proposition 2.6 and part (ii) is trivially true. Assume that \( m > d_i \) and write \( m = d_i m_0 + m_1, \ 0 \leq m_1 < d_i \). Note that since \( m_1 \in \{0,1\} \) we know by the earlier case that
\[ KR(m_1 \omega_i) \cong ev_0(V(m_1 \omega_i)). \]
Set
\[ \tilde{KR}(m\omega_i) = U(g[l])(v_{m_1 \omega_i} \otimes v_{d_i \omega_i} \otimes m_0) \subset KR(m_1 \omega_i) \otimes KR(d_i \omega_i)^{\otimes m_0}. \]
It is easily checked that $\tilde{KR}(m\omega_i)$ is a graded quotient of $KR(m\omega_i)$. Using Proposition 2.4 we see that part (i) of the theorem follows if we prove that for all $s \in \mathbb{Z}_+$,

$$
(2.7) \quad \tilde{KR}(m\omega_i)[s] \cong \bigoplus_{\{\mu \in P^+(i,m) : |\mu| = s\}} V(\mu).
$$

In particular this proves that $\tilde{KR}(m\omega_i) \cong KR(m\omega_i)$ and so proves part (ii) of the theorem as well.

Let $\mu = m_1\omega_i + \mu_{j_1} + \cdots + \mu_{j_{m_0}}$ be a reduced expression for $\mu$, set

$$
K = KR(m_1\omega_i) \otimes KR(\tilde{d}_i\omega_i) (j_1 + 1) \otimes \cdots \otimes KR(\tilde{d}_i\omega_i) (j_{m_0} + 1),
$$

and let $\tilde{K} = U(g[t]) (v_{m_1\omega_i} \otimes v_{j_1} \otimes \cdots \otimes v_{j_{m_0}})$. Clearly $K$ and $\tilde{K}$ are graded quotients of $KR(m_1\omega_i) \otimes KR(\tilde{d}_i\omega_i)^{\otimes m_0}$ and $KR(\omega_i)$ respectively. Let

$$
\pi : KR(m_1\omega_i) \otimes KR(\tilde{d}_i\omega_i)^{\otimes m_0} \to K, \quad \pi(\tilde{KR}(m\omega_i)) = \tilde{K},
$$

be the canonical surjective morphism of $g[t]$–modules.

Using (2.6) and the comultiplication of $U(g[t])$ one computes easily that

$$
(2.8) \quad x_{\mu} \left( \pi(v_{m_1\omega_i} \otimes v_{\tilde{d}_i\omega_i}^{\otimes m_0}) \right) = v_{m_1\omega_i} \otimes x_{\mu_{j_1}} v_{j_1} \otimes \cdots \otimes x_{\mu_{j_{m_0}}} v_{j_{m_0}} \neq 0.
$$

Corollary 2.6 implies that

$$
n^+ \left( v_{m_1\omega_i} \otimes x_{\mu_{j_1}} v_{j_1} \otimes \cdots \otimes x_{\mu_{j_{m_0}}} v_{j_{m_0}} \right) = 0,
$$

and

$$
v_{m_1\omega_i} \otimes x_{\mu_{j_1}} v_{j_1} \otimes \cdots \otimes x_{\mu_{j_{m_0}}} v_{j_{m_0}} \in K[[\mu]] \cap K_{\mu}.
$$

It follows immediately that

$$
\tilde{K}[\mu]\cong_{\tilde{g}} (V(\mu) \otimes N)
$$

for some $g$–submodule $N \subset \tilde{K}$ which proves (2.7).

3. The twisted algebras

We use the notation of the previous section freely.

3.1. Preliminaries. Throughout this section we let $\epsilon \in \{0,1\}$. From now on $g$ denotes a Lie algebra of type $A_n$ or $D_n$ and $\sigma : g \to g$ the non–trivial diagram automorphism of order two. The statements in this subsection can be found in [12]. Thus we have

$$
g = g_0 \oplus g_1, \quad h = h_0 \oplus h_1, \quad n^+ = n_0^+ \oplus n_1^+, \quad n^- = n_0^- \oplus n_1^-,
$$

where

$$
g_0 = \{ x \in g : \sigma(x) = x \}, \quad g_1 = \{ x \in g : \sigma(x) = -x \}.
$$

For any subalgebra $a$ of $g$ with $\sigma(a) \subset a$ we set $a_0 = a \cap g_0$, and we have $a = a_0 \oplus a_1$. The subalgebra $g_0$ is a simple Lie algebra with Cartan subalgebra $h_0$ and we let $I_0$ be the index set for the corresponding set of simple roots numbered as in [11].

Although in this and the following sections it is convenient to use the ambient Lie algebra $g$, we do not need any other data associated with it. Thus, all representations, roots, weights, the maps $\varepsilon_i$ and so on will always be those associated with the fixed point algebra $g_0$. 
We have $g_0$ is of type $C_n$ if $g$ is of type $A_{2n-1}$ and of type $B_n$ if $g$ is of type $A_{2n}$ or $D_{n+1}$. Let $(R_0)_s$ be the set of short roots of $g_0$ and $(R_0)_l$ the set of long roots. The adjoint action of $g_0$ on $g$ makes $g_1$ into an irreducible representation of $g_0$ and we have

$$g_1 = h_1 \bigoplus_{\mu \in h_0^*} (g_1)_\mu, \quad g_1 \cong V(\phi),$$

where $\phi \in Q_0^+$ is the highest short root of $g_0$ if $g$ is of type $A_{2n-1}$ or $D_{n+1}$ and twice the highest short root if $g$ is of type $A_{2n}$. Further, if we set

$$R_1 = \{ \mu \in h_0^*: (g_1)_\mu \neq 0 \} \setminus \{0\}, \quad R_1^+ = R_1 \cap Q_0^+$$

then $R_1 = (R_0)_s$ if $g$ is of type $A_{2n-1}$ or $D_{n+1}$ and

$$R_1 = R_0 \cup \{2\alpha: \alpha \in (R_0)_s\}$$

if $g$ is of type $A_{2n}$. In all cases, $\dim(g_1)_\alpha = 1$ for $\alpha \in R_1$.

**Lemma.**

(i) **The maps** $g_1 \otimes g_1 \to g_0$ and $g_0 \otimes g_1 \to g_1$ **defined by** $x \otimes y \to [x, y]$ **are surjective homomorphism of** $g_0$-**modules.**

(ii) **If** $\alpha \in R_0 \cap R_1$, **there exists** $h \in h_1$ **such that** $[h, y_\alpha^+] = x_\alpha^-$. \hfill \Box

**3.2. The twisted current algebra.** Extend $\sigma$ to an automorphism $\sigma_t: g[t] \to g[t]$ by extending linearly the assignment,

$$\sigma_t(x \otimes t^r) = \sigma(x) \otimes (-t)^r, \quad x \in g, \quad r \in \mathbb{Z}_+.$$ 

If $a \subset g$ is such that $\sigma(a) \subset a$, let $a[t]^\sigma$ be the set of fixed points of $\sigma_t$. Clearly

$$a[t]^\sigma = a_0 \otimes C[t^2] \oplus a_1 \otimes tC[t^2].$$

and

$$g[t]^\sigma = n_-^-[t]^\sigma \oplus h[t]^\sigma \oplus n^+[t]^\sigma.$$ 

**3.3. The modules** $KR^\sigma(m\omega_i)$.

**Definition.** For $i \in I_0$, $m \in \mathbb{Z}_+$, let $KR^\sigma(m\omega_i)$ be the $g[t]^\sigma$-module generated by an element $v_{i,m}^\sigma$ with relations,

$$n^+[t]^\sigma v_{i,m}^\sigma = 0, \quad h_0 v_{i,m}^\sigma = m\omega_i(h_0)v_{i,m}^\sigma, \quad (h_\epsilon \otimes t^{2r-\epsilon})v_{i,m}^\sigma = x_{\alpha_j}^- v_{i,m}^\sigma = 0,$$

for all $h_\epsilon \in h_\epsilon$, $r \in \mathbb{Z}_+, j \neq i$,

$$v_{i,m}^\sigma = 0,$$

and

$$v_{i,m}^\sigma v_{i,m}^\sigma = (y_{\alpha_i}^- \otimes t)v_{i,m}^\sigma = 0. \hfill \Box$$
Note that when \( g \) is of type \( A_{2n} \) or if \( \alpha_i \in (R_0)_s \) it can be seen, by using Lemma 3.3(ii), that the relation \((x_\alpha^\sigma \otimes t^2v^\sigma_{i,m}) = 0\) is actually a consequence of \((y_\alpha^\sigma \otimes t)v^\sigma_{i,m} = 0\). The modules \( KR^\sigma(m\omega_i) \) are clearly graded modules for the graded algebra \( g[t]^\sigma \). Any \( g_0 \)-module \( V \) can be regarded as a module for \( g[t]^\sigma \) by setting

\[
(x_\epsilon \otimes t^{2r^+})v = 0, \quad v \in V, \quad x_\epsilon \in g_\epsilon, \quad r \in \mathbb{Z}_+,
\]

and we denote the corresponding module by \( ev^\sigma_0(V) \). The next lemma is easily checked.

**Lemma.** For all \( i \in I_0 \) and \( m \in \mathbb{Z}_+ \) the module \( ev^\sigma_0(V(m\omega_i)) \) is a quotient of \( KR^\sigma(m\omega_i) \). \( \square \)

### 3.4. The graded character of \( KR^\sigma(m\omega_i) \)

For \( i \in I_0 \) and \( m \in \mathbb{N} \), let \( P_0^+(i,m)^\sigma \) be the subset of \( P_0^+ \) defined by

\[
P_0^+(i,m)^\sigma = P_0^+(i,d_i^\sigma)^\sigma + P_0^+(i,m-d_i^\sigma)^\sigma,
\]

where

\[
d_i^\sigma = 1, \quad g \neq A_{2n},
\]

\[
d_i^\sigma = 2, \quad i \neq n, \quad d_n^\sigma = 4, \quad g = A_{2n},
\]

and, for \( 1 \leq m \leq d_i^\sigma, \) \( P_0^+(i,m)^\sigma \) is given as follows. If \( g \) is of type \( A_{2n} \), then

- \( P_0^+(i,1)^\sigma = \{\omega_i\} \)
- \( P_0^+(i,2)^\sigma = \{2\omega_i, 2\omega_{i-1}, \cdots, 2\omega_1, 0\}, \quad i \neq n \)
- \( P_0^+(n,2)^\sigma = \{2\omega_n\} \)
- \( P_0^+(n,3)^\sigma = \{3\omega_n\} \)
- \( P_0^+(n,4)^\sigma = \{4\omega_n, 2\omega_{n-1}, \cdots, 2\omega_1, 0\} \)

If \( g \) is of type \( A_{2n-1} \), then

- \( P_0^+(i,1)^\sigma = \{\omega_i, \omega_{i-2}, \cdots, \omega_1\} \)

where \( i \in \{0,1\} \) and \( i = \overline{i} \mod 2 \). Finally if \( g \) is of type \( D_{n+1} \), then

- \( P_0^+(i,1)^\sigma = \{\omega_i, \omega_{i-1}, \cdots, 0\}, \quad i \neq n \)
- \( P_0^+(n,1)^\sigma = \{\omega_n\} \)

In particular \( P_0^+(i,m)^\sigma \) is finite. Fix an enumeration \( \mu_s, \quad 0 \leq s \leq k \), of the sets \( P_0^+(i,m)^\sigma \), \( 1 \leq m \leq d_i^\sigma \) by requiring

\[
\mu_s - \mu_{s+1} \in R_1^+, \quad \mu_s - \mu_{s+2} \notin (R_0^+ \cup R_1^+).
\]

For \( \mu \in P_0^+(i,m)^\sigma \) define a reduced expression and \( |\mu| \in \mathbb{Z}_+ \) analogously to the untwisted case.

The main result is the following.

**Theorem.** Let \( i \in I_0, \) \( m \in \mathbb{Z}_+ \).

(i) For all \( s \in \mathbb{Z}_+ \),

\[
KR^\sigma(m\omega_i)[s] \cong_{g_0} \bigoplus_{\mu \in P_0^+(i,m)^\sigma : |\mu|=s} V(\mu).
\]
(ii) Write \( m = d_i^2 m_0 + m_1 \), where \( 0 \leq m_1 < d_i^2 \). The canonical homomorphism of \( \mathfrak{g}^\sigma[t] \)-modules

\[
KR^\sigma(m_\omega_i) \rightarrow KR^\sigma(m_1_\omega_i) \otimes KR^\sigma(d_i^2 \omega_i) \otimes m_0
\]

is injective.

The proof of the theorem proceeds as in the untwisted case.

**3.5. Elementary Properties of \( KR^\sigma(m_\omega_i) \).**

The next proposition is the twisted version of Proposition 2.3 and is proved in the same way.

**Proposition.**

(i) We have

\[
KR^\sigma(m_\omega_i) = \bigoplus_{\mu \in \mathfrak{h}_0^+} KR^\sigma(m_\omega_i)_\mu
\]

and \( KR^\sigma(m_\omega_i)_\mu \neq 0 \) only if \( \mu \in m_\omega_i - Q_0^+ \).

(ii) Regarded as a \( \mathfrak{g}_0 \)-module, \( KR^\sigma(m_\omega_i) \) and \( KR^\sigma(m_\omega_i)[s] \), \( s \in \mathbb{Z}_+ \), are isomorphic to a direct sum of irreducible finite–dimensional representations of \( \mathfrak{g}_0 \). In particular if \( W_0 \) is the Weyl group of \( \mathfrak{g}_0 \), then \( KR^\sigma(m_\omega_i)_\mu \neq 0 \) iff \( KR^\sigma(m_\omega_i)_w \mu \neq 0 \) for all \( w \in W_0 \).

(iii) Given \( 0 \leq r \leq m \), there exists a canonical homomorphism \( KR^\sigma(m_\omega_i) \rightarrow KR^\sigma(r_\omega_i) \otimes KR((m-r)_\omega_i) \) of graded \( \mathfrak{g}[t] \)-modules such that \( v^\sigma_{i,m} \mapsto v^\sigma_{i,r} \otimes v^\sigma_{i,m-r} \).

\( \square \)

**Corollary.** We have

\[
KR^\sigma(m_\omega_i) \cong \bigoplus_{\mu \in \mathfrak{h}_0^+} V(\mu)^{m_\mu(i,m)}.
\]

In particular \( KR^\sigma(0) \cong \mathbb{C} \).

\( \square \)

**3.6. An upper bound for \( m_\mu(i,m) \).**

**Proposition.** For all \( i \in I_0 \), \( m \in \mathbb{Z}_+ \), we have

\[
KR^\sigma(m_\omega_i) \cong_{\mathfrak{g}_0} \bigoplus_{\mu \in \mathfrak{h}_0^+ (i,m)} V(\mu)^{m_\mu},
\]

where \( m_\mu \in \{0,1\} \).

**Corollary.** For all \( i \in I_0 \) and \( m \in \mathbb{Z}_+ \), the modules \( KR^\sigma(m_\omega_i) \) are finite–dimensional.

\( \square \)

We postpone the proof of this proposition to the next section.

**3.7. An explicit construction of the modules \( KR^\sigma(m_\omega_i) \), \( i \in I_0 \), \( 1 \leq m \leq d_i^2 \).**

The next lemma is standard and can be found in [8], [19]. Let \( \iota : \mathfrak{g}_0 \rightarrow \wedge^2(\mathfrak{g}_1) \) be the \( \mathfrak{g}_0 \)-module map such that \([ , ] \cdot \iota = \text{id}\).

**Lemma.** (a) If \( \mathfrak{g} \) is of type \( D_{n+1} \), \( n \geq 2 \), then \( \wedge^2(\mathfrak{g}_1) \cong_{\mathfrak{g}_0} \iota(\mathfrak{g}_0) \).

(b) If \( \mathfrak{g} \) is of type \( A_n \), \( n \neq 3 \), we have \( \wedge^2(\mathfrak{g}_1) \cong \iota(\mathfrak{g}) \oplus V(\nu) \), where

(i) \( \nu = \omega_1 + \omega_3 \) if \( \mathfrak{g} \) is of type \( A_{2n-1} \), \( n \geq 3 \),
for all $p$.

Moreover, if the maps resulting from the last $V$ are maps of type $A_2n$, $n \geq 3$,

(i) $\nu = 2\omega_1 + \omega_2$ if $g$ is of type $A_{2n}$, $n \geq 3$,

(ii) $\nu = 2\omega_1 + 2\omega_2$ if $g$ is of type $A_{4}$,

(iii) $\nu = 6\omega_1$ if $g$ is of type $A_2$.

Assume that $V_s$, $0 \leq s \leq k$, are $g_0$-modules, let $p_s : g_1 \otimes V_s \rightarrow V_{s+1}$, $0 \leq s \leq k - 1$, be $g_0$-module maps, and set $p_k = 0$. Set also $q_s = p_{s+1}(1 \otimes p_s) \in \text{Hom}_{g_0}(g_1 \otimes g_1 \otimes V_s, V_{s+2})$. Suppose that one of the following two conditions hold:

(a) $q_s(\wedge^2(g_1) \otimes V_s) = 0$, $\forall$ $0 \leq s \leq k - 2$,

(b) for all $0 \leq s \leq k - 2$, $y, z, w \in g_1$,

(i) $q_s(V(y) \otimes V_s) = 0$, $p_{s+2}(1 \otimes p_{s+1})(1 \otimes p_s)(((z \wedge w) \otimes y - y \otimes (z \wedge w)) \otimes v) = 0$.

Let $x \in g_0$, $y \in g_1$, $v \in V_s$. The following formulas define a graded $g[t]^\sigma$-module structure on $V = \oplus_{s=1}^k V_s$.

$$(x \otimes t^2)v = q_s(t(x) \otimes v), \quad (y \otimes t)v = p_s(y \otimes v),$$

and

$$(x \otimes t^{2r})v = (y \otimes t^{2r-1})v = 0,$$

for all $r \geq 2$ and $0 \leq s \leq k$. Furthermore,

$$V[s] \cong_{g_0} V_s.$$}

Moreover, if the maps $p_s$, $0 \leq s \leq k - 1$ are all surjective and $V_0 = U(g[t]^\sigma)v_0$, then the resulting $g[t]^\sigma$-module is cyclic on $v_0$.

**Proposition.** Let $i \in I_0$, $1 \leq m \leq d_i^\sigma$. The modules $V(\mu_s)$, $0 \leq s \leq k$, satisfy (3.4) (resp. (3.3)) if $g$ is of type $A_n$ (resp. $D_n$). The resulting $g[t]^\sigma$-module is isomorphic to $KR^\sigma(m\omega_i)$ and

$$KR^\sigma(m\omega_i)[s] \cong_{g_0} V(\mu_s).$$

**Proof.** If $k = 0$ there is nothing to prove since it follows from Proposition 3.6 and Lemma 3.3 that $KR^\sigma(m\omega_i) \cong \text{ev}_\nu^\sigma(V(m\omega_i))$. Assume that $k > 0$. Using Proposition 1.3 it is easy to see that $\text{Hom}_{g_0}(V(\mu_s) \otimes V(\mu_s+1), g_1) \neq 0$ and hence

$$\text{Hom}_{g_0}(g_1 \otimes V(\mu_s), V(\mu_{s+1})) \neq 0.$$

Suppose first that $g$ is of type $A_n$. Equation (3.4) holds if we prove that

$$\text{Hom}_{g_0}(\wedge^2(g_1) \otimes V_s, V_{s+2}) = 0, \quad \forall \quad 0 \leq s \leq k - 2.$$

Since $\mu - \mu_{s+2} \notin R_0$, it is immediate that $\text{Hom}_{g_0}(g_0 \otimes V(\mu_s), V(\mu_{s+2})) = 0$. To prove that $\text{Hom}_{g_0}(V(\nu) \otimes V(\mu_s), V(\mu_{s+2})) = 0$, it suffices to prove that

$$\text{Hom}_{g_0}(V(\mu_s), V(\nu) \otimes V(\mu_{s+2})) = 0.$$}

This is done exactly as in the untwisted case by noting that $\mu_{s+2} + \nu - \mu_s = \sum_{j=1}^{s-1} k_j \alpha_j$ for some $k_j \in \mathbb{Z}_+$. 

If \( g \) is of type \( D_{n+1}, n \geq 2 \), then Lemma 3.7 implies that the first condition in (3.5) is trivially satisfied. If \( n = 3 \) the second condition is also trivially true since \( k \leq 3 \). If \( n > 3 \), let \( N \) be the \( g_0 \)-submodule of \( T^3(g_1) \) spanned by elements of the form \((x \otimes y - y \otimes x) \otimes z - z \otimes (x \otimes y - y \otimes x)\), with \( x, y, z \in g_1 \). Note that

\[
N \cap \wedge^3(g_1) = 0, \quad N \subset (g_1 \otimes \wedge^2(g_1) + \wedge^2(g_1) \otimes g_1).
\]

Now, it is not hard to see that,\[\text{if } s = 0, \quad g_1 \otimes \wedge^2(g_1) \cong_{g_0} V(\omega_1 + \omega_2) \oplus V(\omega_3) \oplus V(\omega_1), \quad n > 3,\]
and that the \( g_0 \)-submodule \( V(\omega_3) \) occurs in \( T^3(g_1) \) with multiplicity one. It follows that

\[
N \cong V(\omega_1 + \omega_2)^{\oplus r_1} \oplus V(\omega_1)^{\oplus r_2}, \quad 0 \leq r_1, r_2 \leq 2.
\]

We now prove that

\[
\text{Hom}_{g_0}(N \otimes V(\mu_s), V(\mu_{s+3})) = 0, \quad 0 \leq s \leq k - 3,
\]
which establishes the second condition in (3.5). Since \((V(\mu_s) \otimes V(\mu_{s+3}))_\omega = 0\), it follows that \(\text{Hom}_{g_0}(V(\omega_1) \otimes V(\mu_s), V(\mu_{s+3})) = 0\) and we are left to show that

\[
\text{Hom}_{g_0}(V(\omega_1 + \omega_2) \otimes V(\mu_s), V(\mu_{s+3})) = 0.
\]

For this it suffices to prove that

\[
\text{Hom}_{g_0}(V(\mu_s), V(\omega_1 + \omega_2) \otimes V(\mu_{s+3})) = 0.
\]

This is done as usual by noting that \( \mu_{s+3} + \omega_i + \omega_2 - \mu_s \) is in the span of the elements \( \alpha \), \( 1 \leq r \leq s - 1 \) and by observing that the \((g_0)_r\)-module \( U((g_0)_r)v_{\omega_1 + \omega_2} \subset V(\omega_1 + \omega_2)\) is not dual to the \((g_0)_r\)-module \( U((g_0)_r)v_{\mu_{s+3}} \subset V(\mu_{s+3})\), where \( J = \{1, \cdots, s - 1\} \).

This proves that if we fix non–zero maps \( p_\alpha \in \text{Hom}_{g_0}(g_1 \otimes V(\mu_s), V(\mu_{s+1})) \), then we can construct a graded cyclic \( g[t]^{\sigma}\)-module

\[
V = U(g[t])v_{\mu_0} = \bigoplus_{s=0}^k V(\mu_s), \quad V[s] \cong V(\mu_s), \quad 0 \leq s \leq k.
\]

As in the untwisted case, to complete the proof it suffices, in view of Proposition 3.6, to prove that \( V \) is a quotient of \( KR^\sigma(\omega_\alpha) \). Since \( \mu_0 = m\omega_i \) and \( \mu_0 - \mu_1 \in R^\sigma_0 \setminus \{\alpha_i\} \) we get

\[
p_0((n_1^+ \oplus h_1 \oplus C \bar{y}_{\alpha_i}) \otimes v_{m\omega_i}) = 0,
\]

i.e.,

\[
(n_1^+ \otimes t^{2r-1})v_{m\omega_i} = (h_1 \otimes t^{2r-1})v_{m\omega_i} = (y_{\alpha_i}^{-1} \otimes t)v_{m\omega_i} = 0, \quad r \geq 1.
\]

For the same reasons,

\[
q_0((n_0^+ \oplus h_0 \oplus C x_{\alpha_i}^{-1}) \otimes v_{m\omega_i}) = 0.
\]

Hence we get

\[
(n_0^+ \otimes t^{2r})v_{m\omega_i} = (h_0 \otimes t^{2r})v_{m\omega_i} = (x_{\alpha_i}^{-1} \otimes t^{2r})v_{m\omega_i} = 0, \quad r \geq 1.
\]

The relations (3.2) follow since \( V \) is obviously finite–dimensional which completes the proof that \( V \) is a quotient of \( KR^\sigma(\omega_\alpha) \).
Given \( \mu \in P_0^+(i,m)^\sigma \), \( m = d_i^\sigma m_0 + m_1 \), \( 0 \leq m_1 < d_i^\sigma \), with reduced expression
\[
\mu = m_1 \omega_i + m_{j_1} + \cdots + m_{j_{m_0}},
\]
set \( s_j = \#\{ r : j_r \geq j \}, 0 \leq j \leq k \), and
\[
y_\mu = (y_{\mu_{k-1} - \mu_k} \otimes t)^{s_k} \cdots (y_{\mu_0 - \mu_1} \otimes t)^{s_1} \in U(n^-[t]^\sigma).
\]
It is easily seen that
\[
y_\mu v_\mu^\sigma \in KR(m_\omega_i)[[\mu]] \cap KR(m_\omega_i)_\mu.
\]
The next corollary is proved in the same way as Corollary 2.6 and we omit the details.

**Corollary.** For \( 0 \leq s \leq k \) we have
\[
y_{\mu_s} v_{\mu_s}^\sigma \neq 0, \quad h(y_{\mu_s} v_{\mu_s}^\sigma) = \mu_s(h)(y_{\mu_s} v_{\mu_s}^\sigma), \quad n^-(y_{\mu_s} v_{\mu_s}^\sigma) = 0, \quad h \in \mathfrak{h}_0,
\]
or, equivalently,
\[
y_{\mu_s} v_{\mu_0} = v_{\mu_s}.
\]

### 3.8. The modules \( \tilde{KR}^\sigma(m_\omega_i) \) and the completion of the proof of Theorem 3.4

For \( m = d_i^\sigma m_0 + m_1 \in \mathbb{Z}_+ \), \( i \in I_0 \), set
\[
\tilde{KR}^\sigma(m_\omega_i) = U(\mathfrak{g}[t]^\sigma)(v_{m_1 \omega_i}^\sigma \otimes (v_{d_i^\sigma \omega_i}^\sigma)^{\otimes m_0}) \subset KR^\sigma(m_1 \omega_i) \otimes KR^\sigma(d_i^\sigma \omega_i)^{\otimes m_0}.
\]
The next proposition is proved in exactly the same way as the corresponding result in Section 2.7 for the untwisted case, by using Proposition 3.7 and Corollary 3.7 and clearly completes the proof of Theorem 3.4.

**Proposition.** Let \( i \in I_0 \), \( m \in \mathbb{N} \). Then
\[
\tilde{KR}^\sigma(m_\omega_i)[s] \cong_{\mathfrak{g}_0} \bigoplus_{\mu \in P_0^+(i,m)^\sigma : |\mu| = s} V(\mu).
\]
In particular,
\[
\tilde{KR}^\sigma(m_\omega_i) \cong KR^\sigma(m_\omega_i)
\]
as \( \mathfrak{g}[t]^\sigma \)-modules. \[\square\]

### 4. Proof of Proposition 3.6

We will use the following remark repeatedly without further comment. Let \( x \in (\mathfrak{g}_e)_{\alpha} \), \( \alpha \in R_e \), and \( r \in \mathbb{Z}_+ \) be such that \( (x \otimes t^{2r + r}) v_{i,m}^\sigma = 0 \). Then \( (x \otimes t^{2r + 2s + \epsilon}) v_{i,m}^\sigma = 0 \) for all \( s \in \mathbb{Z}_+ \). To see this, observe that if \( h \in \mathfrak{h}_e \), then by definition we have \( (h \otimes t^{2s + \epsilon}) v_{i,m}^\sigma = 0 \). If \( x \in (\mathfrak{g}_e)_{\alpha} \) for some \( \alpha \in R_e \), choose \( h \in \mathfrak{h}_0 \) such that \([h, x] = x\). This gives
\[
0 = [h \otimes t^2, x \otimes t^r] v_{i,m}^\sigma = (x \otimes t^{r+2}) v_{i,m}^\sigma,
\]
thus proving the remark.
4.1. The case $\alpha_i \in R_1$ with $\varepsilon_i(\phi) = 1$. The following Lemma establishes the proposition in this case.

**Lemma.** Suppose that $i \in I_0$ is such that $\alpha_i \in R_1$ and $\varepsilon_i(\phi) = 1$ (in other words $i = n$ if $\mathfrak{g}$ is of type $D_{n+1}$ and $i = 1$ if $\mathfrak{g}$ is of type $A_{2n-1}$). Then

$$KR^\sigma(m\omega_i) \cong ev_0^h(V(m\omega_i)).$$

**Proof.** It is straightforward to check that we can write

$$y_\beta^{-} = [x_{\beta}, y_{\alpha}], \quad x_\gamma^{-} = [y_\beta, y_\gamma],$$

for some $\beta \in R_0^+ \cup R_1^+$ with $\varepsilon_i(\beta) = 0$, $\varepsilon_i(\gamma) = 1$. It follows from (3.3) and Proposition 3.5 that $(y_\alpha^- \otimes t)v_{i,m}^\sigma = 0$. Since for all $\alpha \in R_1^+$ we have $y_\alpha^- \otimes t \in U(n_0^+)(y_\alpha^- \otimes t)$ it follows from (3.4) that $(y_\alpha^- \otimes t)v_{i,m}^\sigma = 0$. It follows now from (4.1) that $(x_\alpha^- \otimes t^2)v_{i,m}^\sigma = 0$ and hence that $(x_\alpha^- \otimes t^2)v_{i,m}^\sigma = 0$ for all $\alpha \in R_1^+$. This proves that $KR^\sigma(m\omega_i) = U(n_0) v_{i,m}^\sigma$ and the Lemma is proved.

4.2. The subalgebras $n^-[t]_\max^\sigma(i)$.

From now on we assume that $i \in I_0$ is such that either $\alpha_i \notin R_1$ or $\varepsilon_i(\phi) = 2$. Let $\max(i) \in \{1, 2\}$ be the maximum value of the restriction of $\varepsilon_i$ to $R_1^+$, and let $n^-[t]_\max^\sigma(i)$ be the subalgebra of $n^-[t]_\max^\sigma(i)$ generated by

$$\{y_\alpha^- \otimes t : \alpha \in R_1^+, \varepsilon_i(\alpha) = \max(i)\}.$$ If $\mathfrak{g}$ is of type $A_n$, then it is easy to see that

$$n^-[t]_\max^\sigma(i) = \bigoplus_{\alpha \in R_1^+: \varepsilon_i(\alpha) = \max(i)} C(y_\alpha^- \otimes t),$$

while if $\mathfrak{g}$ is of type $D_{n+1}$, we have

$$n^-[t]_\max^\sigma(i) = \left( \bigoplus_{\alpha \in R_1^+: \varepsilon_i(\alpha) = 1} C(y_\alpha^- \otimes t) \right) \oplus \left( \bigoplus_{\alpha \in R_1^+: \varepsilon_i(\alpha) = 2} C(x_\alpha^- \otimes t^2) \right).$$

4.3. Further relations satisfied by $v_{i,m}^\sigma$.

**Proposition.**

(i) Let $\alpha \in R_1^+$. If $\varepsilon_i(\alpha) < \max(i)$ we have

$$(y_\alpha^- \otimes t^{2r+1})v_{i,m}^\sigma = 0 \quad \text{for all} \quad r \in \mathbb{Z}_+,$$

while if $\varepsilon_i(\alpha) = \max(i)$ we have

$$(y_\alpha^- \otimes t^{2r+3})v_{i,m}^\sigma = 0 \quad \text{for all} \quad r \in \mathbb{Z}_+.$$

(ii) If $\mathfrak{g}$ is of type $A_n$, then for all $\alpha \in R_0^+$, $r \in \mathbb{Z}_+$, we have

$$(x_\alpha^- \otimes t^{2r+2})v_{i,m}^\sigma = 0.$$
(iii) If $\mathfrak{g}$ is of type $D_{n+1}$, then for all $\alpha \in R_0^+$ and $r \in \mathbb{Z}_+$, we have

$$(x^-_\alpha \otimes t^{2r+2\varepsilon_i(\alpha)})v^\sigma_{i,m} = 0, \quad i < n.$$  

Proof. Let $\alpha \in R_1^+$. Suppose that $\varepsilon_i(\alpha) = 0$. Then $s_\alpha (m\omega_i - \alpha) = m\omega_i + \alpha$ and hence $(KR^\sigma (m\omega_i))_{m\omega_i - \alpha} = 0$. This shows that

$$x^-_\alpha v^\sigma_{i,m} = (y^-_\alpha \otimes t)v^\sigma_{i,m} = 0.$$  

If $\varepsilon_i(\alpha) = 1 < \max(i)$, then we can choose $j \in I_0$ such that $\beta = \alpha - \alpha_j \in R_1^+ \cap R_0^+$. If $j = i$, then $\varepsilon_i(\beta) = 0$ and we get by using (3.3) that

$$(y^-_\alpha \otimes t)v^\sigma_{i,m} = [x^-_\beta , y^-_\alpha \otimes t]v^\sigma_{i,m} = 0,$$

and if $j \neq i$, then $\varepsilon_i(\alpha_j) = 0$ and we get

$$(y^-_\alpha \otimes t)v^\sigma_{i,m} = [x^-_\beta , y^-_\alpha \otimes t]v^\sigma_{i,m} = x^-_\alpha (y^-_\beta \otimes t)v^\sigma_{i,m}.$$  

Repeating the argument with $\beta$ we eventually get $(y^-_\alpha \otimes t)v^\sigma_{i,m} = 0$, thus proving (i). The proofs of (ii) and (iii) are similar and we omit the details. □

Corollary.

(i) As vector spaces we have

$$(4.4) \quad KR^\sigma (m\omega_i) = U(n^-_0)U(n^-[t]_{\max(i)})v^\sigma_{i,m}.$$  

In particular $KR^\sigma (m\omega_i)$ is finite-dimensional.

(ii) If $\mathfrak{g}$ is of type $A_2$ and $1 \leq m \leq 3$, then we have

$$KR^\sigma (m\omega_i) \cong ev_0 (V(m\omega_i))$$

and Proposition 3.6 is proved in these cases.

Proof. A straightforward application of the Poincaré-Birkhoff-Witt theorem gives (4.4). Since $n^- \oplus n^-[t]_{\max(i)}$ is a finite-dimensional Lie algebra, it follows that for all $\mu \in P$

$$\dim(KR^\sigma (m\omega_i)_\mu) < \infty.$$  

Proposition 3.6 now implies that if $KR^\sigma (m\omega_i)$ is infinite-dimensional, there must be infinitely many elements $\nu_r \in P^+$ with $KR^\sigma (m\omega_i)_{\nu} \neq 0$ and hence $\nu_k \in P^+ \cap (m\omega_i - Q^+)$. But this is a contradiction since it is well-known that the set $P^+ \cap (m\omega_i - Q^+)$ is finite.

If $\mathfrak{g}$ is of type $A_2$ and $m \leq 3$, then $i = 1$ and $(m\omega_1 - \phi)(h_{\alpha_1}) < 0$ since $\varepsilon_1(\phi) = 2$. On the other hand, since $\varepsilon_1(\phi - \alpha_1) = 1$, we see that

$$x^+_{\alpha_1} (y^-_\phi \otimes t)v^\sigma_{i,m} = (y^-_{\phi - \alpha_1} \otimes t)v^\sigma_{i,m} = 0.$$  

Since $KR^\sigma (m\omega_i)$ is isomorphic to a direct sum of finite-dimensional $\mathfrak{g}_0$–modules this forces $(y^-_\phi \otimes t)v^\sigma_{i,m} = 0$ which proves the corollary. □
4.4. Proof of Proposition 3.6. Recall the enumeration $\mu_0, \cdots, \mu_k$ of the sets $P_0^+(i, d_i^\sigma)^\sigma$ and set $\phi_s = \mu_{k-s} - \mu_{k-s+1} \in R_1^+$ for $1 \leq s \leq k$. Given $r = (r_1, \cdots, r_k) \in \mathbb{Z}_+^k$, set $\eta_r = \sum_{s=1}^k r_s \phi_s \in Q^+$ and

$$y_r = (y_{\phi_1} \otimes t)^{r_1} \cdots (y_{\phi_k} \otimes t)^{r_k}.$$  

Note that $\eta_r + \eta_r' = \eta_{r+r'}$ for all $r, r' \in \mathbb{Z}_+^k$. For $1 \leq s \leq k$, let $e_s \in \mathbb{Z}_+^k$ be the element with one in the $s^{th}$ place and zero elsewhere. As usual $\leq$ is the partial order on $P_0$ defined by $\mu \leq \mu'$ iff $\mu' - \mu \in Q_0^+$. 

Proposition. We have,

$$n_0^+ y_r v_{i,m}^\sigma \in \sum_{\eta_r' < \eta_r} \mathbf{U}(n_0) y_r v_{i,m}^\sigma. \tag{4.5}$$

$$KR^\sigma (m \omega_i) = \sum_{r \in \mathbb{Z}_+^k} \mathbf{U}(g_0) y_r v_{i,m}^\sigma. \tag{4.6}$$

Assuming this proposition we complete the proof of Proposition 3.6 as follows. By Proposition 3.5 we can pick a $g_0$-module $W^0$ such that

$$KR^\sigma (m \omega_i) \cong_{g_0} \mathbf{U}(g_0) v_{i,m}^\sigma \oplus W^0.$$  

If $W^0 = 0$ there is nothing to prove. Otherwise, let $\text{pr}_{W^0}$ be the $g_0$-projection of $KR^\sigma (m \omega_i)$ onto $W^0$. Using (4.6) we see that there exists $r \in \mathbb{Z}_+^k$ such that $\text{pr}_{W^0}(y_r) \neq 0$. Choose $r_1 \in \mathbb{Z}_+^k$ such that $y_{r_1} = \text{pr}_{W^0}(y_{r_1} v_{i,m}^\sigma) \neq 0$ and such that $\eta_{r_1}$ is minimal: i.e., if $\text{pr}_{W^0}(y_{r_1} v_{i,m}^\sigma) \neq 0$ for some $r \in \mathbb{Z}_+^k$, then $\eta_{r_1} - \eta_r \notin Q_0^+$. Using (4.5) we get

$$n_0^+ \text{pr}_{W^0}(y_{r_1} v_{i,m}^\sigma) = \text{pr}_{W^0}(n_0^+ y_{r_1} v_{i,m}^\sigma) = 0,$$

i.e.,

$$n_0^+ (v_{r_1}^\sigma) = 0.$$  

Thus we can write

$$KR^\sigma (m \omega_i) \cong_{g_0} \mathbf{U}(g_0) v_{i,m}^\sigma \oplus \mathbf{U}(g_0) v_{r_1}^\sigma \oplus W^1,$$

for some $g_0$-submodule $W^1$. Repeating the argument we find that there exists $r_s, s \in \mathbb{Z}_+$, such that

$$KR^\sigma (m \omega_i) \cong \mathbf{U}(g_0) v_{i,m}^\sigma \oplus \bigoplus_{s \geq 1} \mathbf{U}(g_0) v_{r_s}^\sigma,$$

with $n_0^+ v_{r_s}^\sigma = 0$. Since

$$v_{r_s}^\sigma \in KR^\sigma (m \omega_i)_{m \omega_i - \sum_{j=1}^k s_j \phi_j},$$

for some $s_j \in \mathbb{Z}_+$ we get

$$m \omega_i - \sum_{j=1}^k s_j \phi_j = (m - s_k d_k^\sigma) \omega_i + (s_k - s_{k-1}) \mu_1 + \cdots + (s_2 - s_1) \mu_{k-1} + s_1 \mu_k \in P_0^+.$$  

An inspection of the sets $P_0^+(i, d_i^\sigma)^\sigma$ shows that this implies

$$m \geq d_i^\sigma s_k, \quad s_j \leq s_{j+1}, \quad 1 \leq j \leq k - 1,$$

which in turn implies that the weight of $v_{r_s}^\sigma$ is in $P_0^+(i, m)^\sigma$. Since $r_s \neq r_s'$ if $s \neq s'$, Proposition 3.6 is now proved. It remains to prove Proposition 4.4.
4.4.1. **Proof of (4.5).** It is useful to recall our convention that if \( \alpha \in R^+_1 \), then \( y^-_\alpha \) denotes an arbitrary non-zero element of the space \((g_1)_{-\alpha}\).

**The case when \( g \) is of type \( D_{n+1} \) or \( A_{2n} \).**

We proceed by induction on \( n \). To see that induction begins for the series \( D_{n+1} \) at \( n = 2 \) note that then, \( n^+[t]_{\text{max}(1)} = C(y_0^+ + 2 \otimes t) \) and hence the result follows from Corollary 4.3(i). For \( A_{2n} \) it begins at \( n = 1 \) by noting that \( n_0^+[t]_{\text{max}(1)} = C(y_0^+ \otimes t) \) and \( \varepsilon_1(\phi - \alpha) = 1 < \text{max}(1) \) and again using Corollary 4.3.

For the inductive step set \( J = \{2, \ldots, n \} \) and let \( j \in J \). We have
\[
x_j^+ y_{r'} v^\sigma_{i,m} = (y_{\phi_1} \otimes t)^{r_1} x_j^+ y_{r-r_1e_1} v^\sigma_{i,m} \in (y_{\phi_1} \otimes t)^{r_1} \sum_{r' \in S} U((n_0^-)_{J'}) y_{r'} \subset \sum_{r' \in S} U((n_0^-)_{J})(y_{\phi_1} \otimes t)^{r_1} y_{r'}
\]
where
\[
S = \{r' \in \mathbb{Z}^+_k : r'_1 = 0, \eta_{r'} < \eta_{r-r_1e_1}\}.
\]
The last inclusion is a consequence of the fact that \([y_{\phi_1}, n_0^-] = 0\). It is now simple to check that \( \eta_r > \eta_{r'+r_1e_1} \).

Suppose now that \( j = 1 \) and that \( g \) is of type \( A_{2n} \). We have
\[
x_{\alpha_1}^+ y_r v^\sigma_{i,m} = (y_{\alpha_1+2} \otimes t)y_{r-e_1} v^\sigma_{i,m} = \delta_{1,1} x_{\alpha_1}^+ y_{r-e_1+2} v^\sigma_{i,m},
\]
where \( \delta_{1,1} \) is one if \( i = 1 \) and zero otherwise. If \( g \) is of type \( D_{n+1} \), we have
\[
(4.7) \quad x_{\alpha_1}^+ y_r v^\sigma_{i,m} = y_{r-e_1+2} v^\sigma_{i,m} + (x_{\theta} \otimes t^2)y_{r_2-e_1} v^\sigma_{i,m},
\]
where we recall that \( \theta = \alpha_1 + 2 \sum_{j=2}^n \alpha_j \). Since \( \eta_{r-e_1+2} < \eta_r \) it remains to prove that
\[
(x_{\theta} \otimes t^2)y_{r_2-e_1} v^\sigma_{i,m} \in \sum_{\eta_{r'} < \eta_r} U(n_0^-) y_{r'} v^\sigma_{i,m}.
\]
Now
\[
(4.8) \quad x_{\alpha_1}^- y_{r_2-e_1+2} v^\sigma_{i,m} = (x_{\theta} \otimes t^2)y_{r_2-e_1} v^\sigma_{i,m} + y_{r-e_1+2} v^\sigma_{i,m}.
\]
Since \( \eta_r > \eta_{r-e_1+2} \) the inductive step is proved.

**The case when \( g \) is of type \( A_{2n-1} \).**

We note that induction begins at \( n = 2 \) by using Corollary 4.3 since in this case we have when \( i = 2 \) that \( n^+[t]_{\text{max}(2)} = C(y_0^+ + 2 \otimes t) \). Set \( J = \{2, \ldots, n \} \). For the inductive step, note that if \( i \) is odd, then \( \phi_i \in R^+_j \) and by the induction hypothesis it suffices to consider only the case \( x_{\alpha_1}^+ y_r v^\sigma_{i,m} \). Since
\[
[x_{\alpha_1}, y_r] = 0,
\]
the result is immediate.

Assume then that \( i \) is even and let \( j \neq 2 \). We have
\[
x_j^+ y_r v^\sigma_{i,m} = (y_{\phi_1} \otimes t)^{r_1} x_j^+ y_{r-r_1e_1} v^\sigma_{i,m} \in (y_{\phi_1} \otimes t)^{r_1} \sum_{r' \in S} U((n_0^-)_{J'}) y_{r'} \subset \sum_{r' \in S} U((n_0^-)_{J})(y_{\phi_1} \otimes t)^{r_1} y_{r'}
\]
where
\[
S = \{r' \in \mathbb{Z}^+_k : r'_1 = 0, \eta_{r'} < \eta_{r-r_1e_1}\}.
\]
If \( j = 2 \), we get
\[
x_{\alpha_2}^+ y_{r_{i,m}} = (y_{\alpha_1 + \alpha_2 + 2 \sum_{s=3}^{n} \alpha_s} \otimes t) y_{r_{i,m}} = \delta_{i,2} x_{\alpha_1 + \alpha_2 + \alpha_3} y_{r_{i,m}} - e_1 v_{i,m}.
\]
This completes the proof of the inductive step.

4.4.2. **Proof of** (4.6). Let \( \mathfrak{k}^+_i \subset n^+_0 \) be the subalgebra spanned by elements \( \{ x^+_\alpha : \varepsilon_i(\alpha) = 0 \} \). It is easily seen that the subalgebra \( n^{-}[t]_{\max(i)}^\sigma \) is a module for \( \mathfrak{k}_i \) via the adjoint action, i.e.,
\[
[\mathfrak{k}_i, n^{-}[t]_{\max(i)}^\sigma] \subset n^{-}[t]_{\max(i)}^\sigma.
\]
This action defines a \( \mathfrak{k}_i \)-module structure on \( U(n^{-}[t]_{\max(i)}^\sigma) \) and let \( \rho : \mathfrak{k}_i \to \text{End}(U(n^{-}[t]_{\max(i)}^\sigma)) \) denote the corresponding homomorphism of Lie algebras.

**Lemma.** We have
\[
U(n^{-}[t]_{\max(i)}^\sigma) = \sum_{r \in Z^+_i} \rho(U(\mathfrak{k}^+_i)) y_r.
\]

Assuming the Lemma we complete the proof of (4.6) as follows. Let \( g \in U(n^{-}[t]_{\max(i)}^\sigma) \) and write \( g = \sum_{r \in Z^+_i} \rho(x_r) y_r \) for some \( x_r \in U(\mathfrak{k}^+_i) \). Then,
\[
g r_{i,m} = \sum_{r \in Z^+_i} \rho(x_r) y_r v_{i,m} = \sum_{r \in Z^+_i} x_r y_r v_{i,m} \in \sum_{r \in Z^+_i} U(g_0) y_r v_{i,m},
\]
where the second equality uses the fact that \( x_r v_{i,m} = 0 \). An application of Corollary 4.3(i) now gives (4.6).

The proof of Lemma 4.4.2 is straightforward and we give a proof when \( g \) is of type \( D_{n+1} \). The other cases are similar. If \( i = 1 \), there is nothing to prove and we assume from now on that \( 1 < i < n \). We proceed by induction on \( n \), with induction beginning at \( n = 2 \) by the preceding comment. Let \( J = \{ 2, \cdots, n \} \), by the induction hypothesis we have
\[
U(n^{-}[t]_{\max(i) \cap n^+_j}[t]_\sigma) = \sum_{r \in Z^+_i : r_1 = 0} \rho(U(\mathfrak{k}^+_i \cap n^+_j)) y_r,
\]
and since \( [y_{\phi_1}, n^{-}[t]_\sigma] = [y_{\phi_1}, n^+_j] = 0 \) we get
\[
(4.9) \quad \sum_{r_1 \in Z^+_i} (y_{\phi_1} \otimes t)^{r_1} U(n^{-}[t]_{\max(i) \cap n^+_j}[t]_\sigma) = \sum_{r \in Z^+_i} \rho(U(\mathfrak{k}^+_i \cap n^+_j)) y_r.
\]
Thus the lemma is established if we prove that
\[
(4.10) \quad \rho(U(\mathfrak{k}^+_i)) \sum_{r_1 \in Z^+_i} (y_{\phi_1} \otimes t)^{r_1} U(n^{-}[t]_{\max(i) \cap n^+_j}[t]_\sigma) = U(n^{-}[t]_{\max(i)})\).
\]
For \( 1 \leq j \leq i - 1 \), set
\[
\theta_j = \alpha_1 + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_n),
\]
and
\[
x_s = (x_{\theta_1}^- \otimes t^2)^{s_1} \cdots (x_{\theta_{i-1}}^- \otimes t^2)^{s_{i-1}}, \quad s \in Z^+_{i-1}.
\]
By the PBW theorem we have,
\[ U(\mathfrak{n}^-[t]_{\text{max}(i)}) = \sum_{s \in \mathbb{Z}_+^i, r \in \mathbb{Z}_+} x_s(y_{\phi_1} \otimes t)^r U(\mathfrak{n}^-[t]_{\text{max}(i) \cap \mathfrak{a}_j}^r [t]_{\sigma}), \]
and hence (4.11) follows if we prove that for all \( s \in \mathbb{Z}_+^i, r \in \mathbb{Z}_+ \) we have
\[ x_s(y_{\phi_1} \otimes t)^r U(\mathfrak{n}^-[t]_{\text{max}(i) \cap \mathfrak{a}_j}^r [t]_{\sigma}) \in \rho(U(t_1^\pm)) \sum_{r_1 \in \mathbb{Z}_+} (y_{\phi_1} \otimes t)^{r_1} U(\mathfrak{n}^-[t]_{\text{max}(i) \cap \mathfrak{a}_j}^{r_1} [t]_{\sigma}). \]

To prove (4.11) let \( r_1 \in \mathbb{Z}_+, g \in U(\mathfrak{n}^-[t]_{\text{max}(i) \cap \mathfrak{a}_j}^{r_1} [t]_{\sigma}) \). For \( 2 \leq 2s_1 \leq r_1 \) we get
\[ \rho(x_{\alpha_1}^+)^{s_1}(y_{\phi_1} \otimes t)^{r_1} g = \sum_{p=0}^{s_1} (x_{\phi_1}^- \otimes t^2)^p (y_{\phi_1} \otimes t)^{r_1-s_1-p} (y_{\phi_2} \otimes t)^{s_1-p} g, \]
which proves (4.11) for all \( s \) with \( s_j = 0 \) for \( j > 1 \).

Assume that (4.11) is established for all \( s \) such that \( s_j = 0 \) for all \( j \geq r \). To prove that it holds for \( s \) with \( s_j = 0 \) for \( j > r > 1 \), suppose first that \( s_r = 1 \). Then,
\[ \rho(x_{\alpha_r}^+) x_{s-e_r+e_{r-1}}(y_{\phi_1} \otimes t)^{r_1} g = x_s(y_{\phi_1} \otimes t)^{r_1} g + x_{s-e_r+e_{r-1}}(y_{\phi_1} \otimes t)^{r_1} \rho(x_{\alpha_r}^+) g, \]
and hence we get by using (4.9) that (4.11) holds for \( x_s(y_{\phi_1} \otimes t)^{r_1} g \) when \( s_r = 1 \). An obvious induction on \( s_r \) gives the result in general.

5. Demazure modules and concluding remarks

5.1. The relation between the modules \( KR(m\omega_i) \) and Demazure modules.

5.1.1. The affine Kac–Moody algebras. Let \( \mathbb{C}[t, t^{-1}] \) be the ring of Laurent polynomials in an indeterminate \( t \). Let \( \widehat{\mathfrak{g}} \) be the affine Lie algebra defined by
\[ \widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d, \]
where \( c \) is central and
\[ [x \otimes t^k, y \otimes t^l] = [x, y] \otimes t^{s+k} + s\delta_{s+k,0} (x, y) c, \]
\[ [d, x \otimes t^s] = sx \otimes t^s, \]
for all \( x, y \in \mathfrak{g}, s, k \in \mathbb{Z} \) and where \( \langle , , \rangle \) is the Killing form of \( \mathfrak{g} \). Set
\[ \widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \]
and regard \( \mathfrak{h}^* \) as a subspace of \( \widehat{\mathfrak{h}}^* \) by setting \( \lambda(c) = \lambda(d) = 0 \) for \( \lambda \in \mathfrak{h}^* \). For \( 0 \leq i, j \leq n \) let \( \Lambda_i \in \widehat{\mathfrak{h}}^* \) be defined by \( \Lambda_i(h_j) = \delta_{i,j} \), and let \( \widehat{P}^+ \subset \widehat{\mathfrak{h}}^* \) be the non–negative integer linear span of \( \Lambda_i, 0 \leq i \leq n \). Define \( \delta \in \widehat{\mathfrak{h}}^* \) by
\[ \delta|_\mathfrak{h} = 0, \quad \delta(d) = 1, \quad \delta(c) = 0. \]
The elements \( \alpha_i \in \widehat{\mathfrak{h}}^*, 0 \leq i \leq n \), where \( \alpha_0 = \delta - \theta \) are a set of simple roots for \( \widehat{\mathfrak{g}} \) with respect to \( \widehat{\mathfrak{h}} \). Let \( \widehat{Q}^+ \) be the subset of \( \widehat{P}^+ \) spanned by the elements \( \alpha_i, 0 \leq i \leq n \). Let \( \widehat{W} \) be the extended affine Weyl group and \( \langle , , \rangle \) be the \( \widehat{W} \)–invariant form on \( \widehat{\mathfrak{h}}^* \) obtained by requiring \( (\Lambda_i, \alpha_j) = \delta_{ij} \) for all \( 0 \leq i, j \leq n \) and \( (\delta, \alpha_i) = 0 \) for all \( 0 \leq i \leq n \).
5.1.2. The highest weight integrable modules. Given \( \Lambda = \sum_{i=0}^{n} m_i \Lambda_i \in \hat{P}^+ \), let \( L(\Lambda) \) be the \( \mathfrak{g} \)-module generated by an element \( v_\Lambda \) with relations:

\[
\mathfrak{g} [t] v_\Lambda = 0, \quad (n^+ + 1) v_\Lambda = 0, \quad (h \otimes 1) v_\Lambda = \Lambda(h) v_\Lambda, \\
(x_{\alpha_i}^- \otimes 1)^{m_i + 1} v_\Lambda = 0, \quad (x_{\alpha_i}^+ \otimes t^{-1})^{m_{i+1}} v_\Lambda = 0,
\]

for \( 1 \leq i \leq n \). This module is known to be irreducible and integrable (see [12]). The next proposition also can be found in [12].

**Proposition.** Let \( \Lambda \in \hat{P}^+ \).

(i) We have

\[
L(\Lambda) = \bigoplus_{\Lambda' \in \hat{P}} L(\Lambda)', \quad \text{where} \quad L(\Lambda)' = \{ v \in L(\Lambda) : hv = \Lambda'(h)v, \ h \in \mathfrak{h} \}.
\]

Moreover, \( \dim(L(\Lambda)') < \infty, \dim(L(\Lambda)) = 1, \) and \( L(\Lambda) = 0 \) if \( \Lambda - \Lambda' \notin \hat{Q}^+ \).

(ii) The set

\[
\text{wt}(L(\Lambda)) = \{ \Lambda' \in \hat{P} : L(\Lambda)' \neq 0 \} \subset \Lambda - \hat{Q}^+.
\]

is preserved by \( \hat{W} \) and

\[
\dim(L(\Lambda)_{w\Lambda}) = \dim(L(\Lambda)') \forall \ w \in \hat{W}, \ \Lambda' \in \hat{P}.
\]

In particular, \( \dim(L(\Lambda)_{w\Lambda}) = 1 \) for all \( w \in \hat{W} \).

\( \square \)

5.1.3. The Demazure modules. From now on we fix \( \Lambda = m \Lambda_0 \) for some \( m \in \mathbb{Z}_+ \) and we also assume for simplicity that \( \mathfrak{g} \) is of type \( A_n \) or \( D_n \). Given \( w \in \hat{W} \), let \( v_{w\Lambda} \) be a non–zero element in \( L(\Lambda)_{w\Lambda} \) and let

\[
D(w\Lambda) = U(\mathfrak{g}[t]) v_{w\Lambda} \subset L(\Lambda).
\]

It is clear from Proposition 5.1.2 that \( D(w\Lambda) \) is finite–dimensional for all \( w \in \hat{W} \).

**Proposition.** Let \( \Lambda = m \Lambda_0 \in \hat{P}^+ \) for some \( m > 0 \) and \( w \in \hat{W} \) be such that \( w\Lambda|_\mathfrak{h} = m \omega_i \) for some \( 1 \leq i \leq n \). Then, \( D(w\Lambda) \) is isomorphic to \( KR(m \omega_i) \).

**Proof.** To prove the proposition we first show that \( v_{w\Lambda} \) satisfies the defining relations of \( KR(m \omega_i) \). It was shown in [3] that the element \( v_{w\Lambda} \) satisfied the relations

\[
n^+[t] v_{w\Lambda} = 0, \quad (h \otimes t^r) v_{w\Lambda} = 0, \quad h \in \mathfrak{h}, \ r \in \mathbb{N}.
\]

The proof given in [3] Proposition 1.4.3] was for the special case when \( \mathfrak{g} \) is of type \( A_n \), but works for any simple Lie algebra. Since \( D(w\Lambda) \) is finite–dimensional the only relation that remains to be checked is that

\[
(x_{\alpha_i}^- \otimes t) v_{w\Lambda} = 0.
\]

If not, then \( L(\Lambda)_{w\Lambda - \alpha_i + \delta} \neq 0 \) and hence we find that \( L(\Lambda)_{\Lambda - w^{-1} \alpha_i + \delta} \neq 0 \). Now,

\[
(w\Lambda, \alpha_i) = m = (m \Lambda_0, w^{-1} \alpha_i)
\]

implies that \( w^{-1} \alpha_i = \alpha + \delta \) for some \( \alpha \in R \) and hence \( L(\Lambda)_{\Lambda - \alpha} \neq 0 \). This forces

\[
\alpha \in R^+, \quad L(\Lambda)_{s_\alpha(\Lambda - \alpha)} \neq 0.
\]
Since $\Lambda = m\Lambda_0$, we find that $s_\alpha(\Lambda - \alpha) = \Lambda + \alpha$ and hence $L(\Lambda)_{\Lambda+\alpha} \neq 0$ which is a contradiction. This proves that $D(w\Lambda)$ is a quotient of $KR(m\omega_i)$. The fact that it as an isomorphism of $g[t]$–modules is immediate from [7, Theorem 1].

\[\square\]

5.2. The case of exceptional Lie algebras. For the exceptional Lie algebras the definition and elementary properties of the modules $KR(m\omega_i)$ and their twisted analogs are exactly the same as for the classical Lie algebras. Moreover, the $g$–module decompositions of the modules $KR(m\omega_i)$ (resp. $KR^\sigma(m\omega_i)$) can be shown to coincide with the conjectural decompositions given in [9], [10], [14] as long as $i$ is such that the maximum value of $\varepsilon_i(\alpha) \leq \hat{d}_i$ for all $\alpha \in R^+$. In the other cases, the main difficulty lies in proving Proposition 2.4 (resp. Proposition 3.6). One reason for this is that the multiplicity of the irreducible representations occurring in a given graded component can be bigger than one. The construction of the fundamental Kirillov–Reshetikhin modules is also much more complicated since the number of irreducible components is large, in the case of $E_8$ and the trivalent node for instance, the total number of components is conjectured to be three hundred and sixty eight with twenty four non–isomorphic isotypical components. In fact in this case, a general conjecture for the structure of the non–fundamental modules is not available and the combinatorics seems formidable.

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