CONTROL OF THE NON-GEOMETRICALLY INTEGRAL REDUCTIONS

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Abstract. — For a geometrically integral projective scheme, we give an upper bound of the product of the norms of its non-geometrically integral reductions over an arbitrary number field in this paper. We take the adelic viewpoint to absorb the former ideas on this subject.

Résumé (Côntrole des réductions non-géométriquement intégres)
Pour un schéma projectif géométriquement intégre, on donne une majoration du produit des normes de ses réductions non-géométriquement intére sur un corps de nombres arbitraire. On prend le point de vue adélique afin d’absorber les idées antérieures autour de ce sujet.

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1. Introduction

Let $X \hookrightarrow \mathbb{P}^n_K \to \text{Spec} K$ be a geometrically integral closed sub-scheme over the number field $K$, $\mathcal{X} \hookrightarrow \mathbb{P}^n_K \to \text{Spec} \mathcal{O}_K$ be its Zariski closure, and $\mathcal{X}_p = \mathcal{X} \times \text{Spec} \mathcal{O}_K \to \text{Spec} F_p$ for $p \in \text{Spm} \mathcal{O}_K$. By [7, Théorème 9.7.7], the set

\[(1) \quad \mathcal{Q}(\mathcal{X}) = \{ p \in \text{Spm} \mathcal{O}_K | \mathcal{X}_p \to \text{Spec} F_p \text{ is not geometrically integral} \}\]

is finite.

It is an important subject to give a numerical description of the set $\mathcal{Q}$. In fact, we are interested in the upper bound of $\sum_{p \in \mathcal{Q}} \log N(p)$ or a lower bound such
that for all maximal ideals whose norm is larger than this, the reductions are always geometrically integral.

1.1. History. — Traditionally, we only focus on the case of hypersurfaces when $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, and there are fruitful results reported on this subject. By [12, Exercise 2.4.1], we only need to study whether the polynomial defining this hypersurface is absolutely irreducible over the residue field. For simplicity, some works only focus on the case of plan curves.

To the author’s knowledge, this subject was considered by A. Ostrowski in [13] at first, but it is implicit. In [17], W. M. Schmidt gave an explicit estimate, which is refined by [9] (see also [10]).

In [14], W. M. Ruppert transferred the criterion of absolute irreducibility of polynomials into the existence of certain polynomial solutions of a kind of partial differential equations, where he applied the de Rham cohomology of a kind of complexes. By this result, he gave an upper bound of non-geometrically reductions for the case of arbitrary hypersurfaces, and a sharper upper bound for the case of plane curves. This result improved the former results, and was generalized by [18] and [1] to different directions.

In [21], U. Zannier gave an upper bound depending on the multi-degree of a polynomial $f(x, y)$ over $\mathbb{Z}$. This result is improved by W. M. Ruppert in [16] by refining his method in [14]. In [15], he considered a special kind of plane curves and gave a better upper bound.

In [5], Shuhong Gao and V. M. Rodrigues applied Newton polytopes to refine the estimate in [16], where they involved the number of integral points of Newton polytopes into the estimate.

1.2. Adelic viewpoint. — In this paper, we will give such an upper bound for the case of an arbitrary number field. Let $X \hookrightarrow \mathbb{P}^n_K$ be a hypersurface, and we consider the Zariski closure $\mathcal{X}$ of $X$ in $\mathbb{P}^n_{\mathcal{O}_K}$. In this case $\mathcal{X} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K}$ can be defined by a primitive $\mathcal{O}_K$-coefficient equation if and only if $\mathcal{O}_K$ is a principle ideal domain.

Similar to the method in [11] to study the non-reduced reductions over an arbitrary number field, we introduce the adelic viewpoint to overcome this obstruction. We consider the $K$-coefficient polynomial defining $X \hookrightarrow \mathbb{P}^n_K$ as coefficients in the adelic ring $\mathbb{A}_K$, and then we can obtain a primitive $\mathbb{A}_{\mathcal{O}_K}$-coefficient polynomial by multiplying an element in $\mathbb{A}_K$ which does not change the height of polynomial in the adelic sense. Then for each $p \in \text{Spm} \mathcal{O}_K$, the $p$-part of this primitive polynomial of $\mathbb{A}_{\mathcal{O}_K}$-coefficients is primitive over $\mathcal{O}_{K,p}$, which defines $\mathcal{X}_p \hookrightarrow \mathbb{P}^n_{\mathcal{O}_{K,p}}$ from $\mathcal{X} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K}$ via the base change $\text{Spec} \mathcal{O}_{K,p} \rightarrow \text{Spec} \mathcal{O}_K$. Then we can consider the reduction type of each $\mathcal{X}_p$ modulo $p$.

For the case of geometrically integral hypersurfaces, we use a numerical criteria of geometrically integral of Ruppert [14, Satz 3, Satz 4]. For the general case, we use the theory of Chow form and Cayley form to reduce it to the case of hypersurfaces, which is similar to that in [11, §7]. In fact, we have the following estimate.

**Theorem 1.1 (Theorem 4.5).** — Let $X$ be a geometrically integral closed subscheme of $\mathbb{P}^n_K$ of dimension $d$ and degree $\delta$, and $\mathcal{X}$ be the Zariski closure of $X$ in
Then we have
\[ \frac{1}{[K : Q]} \sum_{p \in \mathcal{O}(X)} \log N(p) \leq (\delta^2 - 1) h(X) + C(n, d, \delta), \]
where \( h(X) \) is a kind of heights of \( X \) and \( N(p) = \#(\mathcal{O}_K/p) \). We will give the above constant \( C(n, d, \delta) \) explicitly in Theorem 4.5, and we have \( C(n, d, \delta) \ll_n \delta^3 \).

If we consider the case of plane curves (\( d = 1 \) and \( n = 2 \)) and use the naive height (see Definition 2.2), we are able to obtain \( C(n, d, \delta) \ll_n \delta^2 \log \delta \) in the above estimate, see Proposition 4.3.

1.3. Structure of the article. — This paper is organized as follows. In §2, we introduce the useful notions on Diophantine geometry and Arakelov geometry. In §3, we introduce some results of Ruppert for a numerical criteria of the geometrically integral property. In §4, we give a such upper bound for the case of hypersurfaces by the above results of Ruppert under the adelic viewpoint, and an upper bound for the general case by applying the theory of Chow form and Cayley form.

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2. Height functions

The height of arithmetic varieties is a kind of invariants which evaluates the arithmetic complexity of this varieties. In order to define this, we introduce some knowledge of Arakelov geometry and Diophantine geometry.

2.1. Normed vector bundles. — The normed vector bundle is one of the main research objects in Arakelov geometry. Let \( K \) be a number field and \( \mathcal{O}_K \) be its ring of integers. A normed vector bundle on \( \text{Spec} \mathcal{O}_K \) is a pair \( E = \left( E, (\| \cdot \|_v)_{v \in M_K, \infty} \right) \), where:
- \( E \) is a projective \( \mathcal{O}_K \)-module of finite rank;
- \( (\| \cdot \|_v)_{v \in M_K, \infty} \) is a family of norms, where \( \| \cdot \|_v \) is a norm over \( E \otimes \mathcal{O}_{K,v} \mathbb{C} \) which is invariant under the action of \( \text{Gal}(\mathbb{C}/K_v) \).

If all the norms \( (\| \cdot \|_v)_{v \in M_K, \infty} \) are Hermitian, we call \( E \) a Hermitian vector bundle on \( \text{Spec} \mathcal{O}_K \). In particular, if \( \text{rk} \mathcal{O}_K(E) = 1 \), we say that \( E \) is a Hermitian line bundle since all Archimedean norms are Hermitian in this case.
2.2. Height of arithmetic varieties. — In this part, we will introduce a kind of height functions of arithmetic varieties defined by the arithmetic intersection theory developed by Gillet and Soulé in [6], which is first introduced by Faltings in [3, Definition 2.5], see also [19, III.6].

**Definition 2.1 (Arakelov height).** — Let $K$ be a number field, $\mathcal{O}_K$ be its ring of integers, $\mathcal{E}$ be a Hermitian vector bundle of rank $n+1$ over $\text{Spec} \mathcal{O}_K$, and $\mathcal{L}$ be a Hermitian line bundle on $\mathbb{P}(\mathcal{E})$. Let $X$ be a pure dimensional closed sub-scheme of $\mathbb{P}(\mathcal{E}_K)$ of dimension $d$, and $\mathcal{X}$ be the Zariski closure of $X$ in $\mathbb{P}(\mathcal{E})$. The Arakelov height of $X$ is defined as the arithmetic intersection number

$$\frac{1}{[K : \mathbb{Q}]} \deg \left( \hat{c}_1(\mathcal{L})^{d+1} \cdot [\mathcal{X}] \right),$$

where $\hat{c}_1(\mathcal{L})$ is the arithmetic first Chern class of $\mathcal{L}$ (cf. [19, Chap. III.4, Proposition 1] for its definition). This height is noted by $h_{\mathcal{L}}(X)$ or $h_{\mathcal{L}}(\mathcal{X})$.

2.3. Height of hypersurfaces. — Let $X$ be a hypersurface in $\mathbb{P}^n_K$. By [8, Proposition 7.6, Chap. I], $X$ is defined by a homogeneous polynomial. We define a height function of hypersurfaces by considering its polynomial of definition.

**Definition 2.2 (Naive height).** — Let $f(T_0, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} a_{i_0, \ldots, i_n} T_0^{i_0} \cdots T_n^{i_n}$ be a polynomial. We define the naive height of $f(T_0, \ldots, T_n)$ as

$$H_K(f) = \prod_{v \in M_K} \max_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} \left\{ |a_{i_0, \ldots, i_n}|_v^{[K_v : \mathbb{Q}_v]} \right\},$$

and $h(f) = \frac{1}{[K : \mathbb{Q}]} \log H_K(f)$. In addition, if $f(T_0, \ldots, T_n)$ is homogeneous and defines the hypersurface $X \hookrightarrow \mathbb{P}^n_K$, we define the naive height of $X$ as

$$h(X) = h(f).$$

2.4. Adelic height. — In order to work over arbitrary number field, we will introduce the so-called adelic height of a polynomial, which has been applied to study the non-reduced reductions in [11].

Let $K$ be a number field, $\mathcal{O}_K$ be its ring of integers. In addition, we denote by

$$\mathcal{A}_K = \left\{ (a_v)_{v \in M_K} \mid a_v \in \mathcal{O}_{K,v} \text{ except a finite number of } v \in M_K \right\},$$

the adelic ring of $K$, by

$$\mathcal{A}_{\mathcal{O}_K} = \{ (a_v)_{v \in M_K} \mid a_v \in \mathcal{O}_{K,v} \text{ for all } v \in M_K \}$$

the integral adelic ring of $K$, and by $\Delta : K \hookrightarrow \mathcal{A}_K$ the diagonal embedding. Let $a = (a_v)_{v \in M_K} \in \mathcal{A}_K$, we define

$$|a|_{\mathcal{A}_K} = \prod_{v \in M_K} |a_v|_{v}^{[K_v : \mathbb{Q}_v]}.$$
Definition 2.3 (Local part). — Let \( \{a_{i_0, \ldots, i_n}\} = \{(a^v_{i_0, \ldots, i_n})_{v \in M_K}\} \) be a finite family of elements in \( \mathbb{A}_K \) with the indices \((i_0, \ldots, i_n) \in \mathbb{N}^{n+1}\), and

\[
f(T_0, T_1, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} a_{i_0, i_1, \ldots, i_n} T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}
\]

be a non-zero polynomial in \( \mathbb{A}_K[T_0, \ldots, T_n] \). For each \( v \in M_K \), we denote by

\[
f^{(v)}(T_0, T_1, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} a^v_{i_0, i_1, \ldots, i_n} T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}
\]

the \( v \)-part of \( f(T_0, T_1, \ldots, T_n) \), or by \( f^{(p)}(T_0, T_1, \ldots, T_n) \) for \( p \in \text{Spm} \mathcal{O}_K \) corresponding to the place \( v \in M_{K, f} \), which is called the \( p \)-part.

Definition 2.4 (Adelic height). — Let

\[
f(T_0, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} a_{i_0, \ldots, i_n} T_0^{i_0} \cdots T_n^{i_n}
\]

be a polynomial with coefficients in \( \mathbb{A}_K \), where we denote \( a_{i_0, \ldots, i_n} = (a^v_{i_0, \ldots, i_n})_{v \in M_K} \in \mathbb{A}_K \) for every index \((i_0, \ldots, i_n)\) in the above polynomial. We define the \textit{adelic height} of \( f \) as

\[
H_{\mathbb{A}_K}(f) = \prod_{v \in M_K} \max_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} \{|a^v_{i_0, \ldots, i_n}|_v\}^{[K_v : Q_v]}.
\]

In addition, we denote \( h(f) = \frac{1}{[K : \mathbb{Q}]} \log H_{\mathbb{A}_K}(f) \).

Let

\[
f(T_0, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} a_{i_0, \ldots, i_n} T_0^{i_0} \cdots T_n^{i_n}
\]

be a polynomial with coefficients in \( \mathbb{K} \), and \( c \in \mathbb{A}_K \) with \( |c|_{\mathbb{A}_K} = 1 \). Let

\[
g(T_0, T_1, \ldots, T_n) = \sum_{(i_0, i_1, \ldots, i_n) \in \mathbb{N}^{n+1}} c \Delta(a_{i_0, i_1, \ldots, i_n}) T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}
\]

be the polynomial with coefficients in \( \mathbb{A}_K \). Then by definition, we have

(2) \( H_{\mathbb{A}_{K}}(g) = H_{\mathbb{K}}(f) \),

where \( H_{\mathbb{K}}(f) \) is defined at Definition 2.2.

Let

\[
f(T_0, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} a_{i_0, \ldots, i_n} T_0^{i_0} \cdots T_n^{i_n}
\]

be a polynomial with coefficients in \( \mathbb{K} \). By [11, Lemme 3.11], there exists \( c \in \mathbb{A}_K \) with \( |c|_{\mathbb{A}_K} = 1 \), such that for each \( v \in M_{K, f} \), we have

\[
\max_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} \{|c \Delta(a_{i_0, \ldots, i_n})|_v\} = 1.
\]

Let \( b_{i_0, \ldots, i_n} = c \Delta(a_{i_0, \ldots, i_n}) \), then

(3) \( F(T_0, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} b_{i_0, \ldots, i_n} T_0^{i_0} \cdots T_n^{i_n} \in \mathbb{A}_K[T_0, \ldots, T_n] \).
which is called the adelicly primitive polynomial of $f$.

3. Criterion of non-geometrically hypersurfaces

Let $X$ be a geometrically integral hypersurface of $\mathbb{P}^n_K$ defined by the homogeneous polynomial $f(T_0, \ldots, T_n)$, and $\mathcal{X}$ be the Zariski of $X$ in $\mathbb{P}^n_{O_K}$. For all $p \in \text{Spm} \ O_K$, in order to study the reduction of $\mathcal{X} \hookrightarrow \mathbb{P}^n_{O_K} \to \text{Spec} \ O_K$ at $p$, we factor the reduction through the localization at $p$. More precisely, we have the following Cartesian diagram:

$$
\begin{array}{cccc}
\mathcal{X}_{\mathbb{F}_p} & \longrightarrow & \mathcal{X}_{O_K,p} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^n_{\mathbb{F}_p} & \longrightarrow & \mathbb{P}^n_{O_K,p} & \longrightarrow & \mathbb{P}^n_{O_K} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec} \mathbb{F}_p & \longrightarrow & \text{Spec} \ O_{K,p} & \longrightarrow & \text{Spec} \ O_K.
\end{array}
$$

By definition, $\mathcal{X}_{O_K,p} \hookrightarrow \mathbb{P}^n_{O_K,p}$ is defined by the $p$-part $F(p)(T_0, \ldots, T_n)$ of $F(T_0, \ldots, T_n)$ (see Definition 2.3), which is primitive over $O_{K,p}$ by the construction of $F(T_0, \ldots, T_n)$ at (3).

By [12, Exercise 2.4.1] (see [11, Corollaire 6.2] for a projective version), for arbitrary $p \in \text{Spm} \ O_K$, the fact that the polynomial $F(p)(T_0, \ldots, T_n)$ modulo $p[T_0, \ldots, T_n]$ is not absolutely irreducible over $\mathbb{F}_p$ is verified if and only if $\mathcal{X}_{\mathbb{F}_p}$ is not geometrically integral over $\text{Spec} \mathbb{F}_p$. So in order to control the set $Q(\mathcal{X})$ introduced at (1), we need to study the absolute irreducibility of $F(p)(T_0, \ldots, T_n) \mod p[T_0, \ldots, T_n]$ for all $p \in \text{Spm} \ O_K$. For this target, we refer the following two results of W. M. Ruppert.

The first one is for the case of plane curves.

**Proposition 3.1** ([14], Satz 3). — Let

$$g(T_0, \ldots, T_2) = \sum_{(i_0, i_1, i_2) \in \mathbb{N}^3 \atop i_0 + i_1 + i_2 = \delta} b_{i_0, i_1, i_2} T_0^{i_0} T_1^{i_1} T_2^{i_2}$$

be a homogeneous polynomial of degree $\delta$ over an algebraically closed field $k$. Then there exists a family of homogeneous polynomial $\{\phi_j\}_{j \in J} \in \mathbb{Z}[b_{i_0, i_1, i_2}]$ with the index set $J$ and variables $\{b_{i_0, i_1, i_2}\} (i_0, i_1, i_2) \in \mathbb{N}^3, i_0 + i_1 + i_2 = \delta$, which are of degree $\delta^2 - 1$ and length smaller than $\delta^{3\delta^2 - 3}$, such that

1. If $F$ is reducible, then $\phi_j(b_{i_0, i_1, i_2}) = 0$ for every $j \in J$;
2. If $F$ is irreducible and $k$ is of characteristic $0$, then there exists at least one $j \in J$, such that $\phi_j(b_{i_0, i_1, i_2}) \neq 0$.

The second one is for the case of general hypersurfaces.
Proposition 3.2 ([14], Satz 4). — Let
\[ g(T_0, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} b_{i_0, \ldots, i_n} T_0^{i_0} \cdots T_n^{i_n} \]
be a homogeneous polynomial of degree \( \delta \) over an algebraically closed field \( k \). Then there exists a family of homogeneous polynomials \( \{ \phi_j \}_{j \in J} \in \mathbb{Z}[b_{i_0, \ldots, i_n}] \) with the index set \( J \) and variables \( \{ b_{i_0, \ldots, i_n} \mid (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}, \ i_0 + \cdots + i_n = \delta \} \), which are of degree \( \delta^2 - 1 \) and length smaller than \( \delta^2 - 1 \left[ \left( \frac{n+\delta}{\delta} \right)^3 \delta^2 - 1 \right] \), such that
1. If \( F \) is reducible, then \( \phi_j(b_{i_0, \ldots, i_n}) = 0 \) for every \( j \in J \);
2. If \( F \) is irreducible and \( k \) is of characteristic 0, then there exists at least one \( j \in J \), such that \( \phi_j(b_{i_0, \ldots, i_n}) \neq 0 \).

4. Control of the non-geometrically integral reductions

By Proposition 3.1 and 3.2, W. M. Ruppert gives a control of non-geometrically integral reductions of hypersurfaces in \( \mathbb{P}_k^n \) in [14, Korollar 1, Korollar 2], where the notion "geometrically integral" means that on \( \text{Spec} \, \mathbb{Q} \). This is an effective version of a theorem of Ostrowski [13]. In this part, we will give such a control for the case over an arbitrary number field \( K \) for general projective schemes.

4.1. Non-geometrically integral reductions of hypersurfaces. — For the case of hypersurfaces, by applying Proposition 3.1 and 3.2 to an adelically primitive polynomial (3), we have the following two results. Since their proofs are quite similar, we only provide the detailed proof for the case of general hypersurfaces.

Proposition 4.1. — Let \( X \hookrightarrow \mathbb{P}_K^n \) be a geometrically integral hypersurface of degree \( \delta \), \( \mathcal{X} \) be its Zariski closure in \( \mathbb{P}_{\mathcal{O}_K}^n \), \( \mathcal{X}_p = \mathcal{X} \times_{\text{Spec} \mathcal{O}_K} \text{Spec} \, \mathcal{F}_p \), and
\[ \mathcal{Q}(\mathcal{X}) = \{ p \in \text{Spm} \mathcal{O}_K \mid \mathcal{X}_p \to \text{Spec} \, \mathcal{F}_p \text{ is not geometrically integral} \} . \]
We have
\[ \frac{1}{[K : \mathbb{Q}]} \sum_{p \in \mathcal{Q}(\mathcal{X})} \log N(p) \leq (\delta^2 - 1) h(X) + C(n, \delta), \]
where \( N(p) = \#(\mathcal{O}_K / p) \), \( h(X) \) is the classic height of \( X \) in \( \mathbb{P}_K^n \), and the constant
\[ C(n, \delta) = (\delta^2 - 1) \left( 3 \delta^2 + \delta \log 3 + \log \left( \frac{n+\delta}{\delta} \right) \right) . \]

Proof. — Suppose \( X \) is defined by the homogeneous polynomial of \( K \)-coefficients
\[ f(T_0, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} a_{i_0, \ldots, i_n} T_0^{i_0} \cdots T_n^{i_n} \]
and

$$F(T_0, \ldots, T_n) = \sum_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} b_{i_0, \ldots, i_n} T_0^{i_0} \cdots T_n^{i_n}$$

be a adelically primitive polynomial of the above $f(T_0, \ldots, T_n)$ constructed as (3). We use the notations in Proposition 3.2, and we choose an index $j \in J$ of the polynomial $\phi_j(b_{i_0, \ldots, i_n})$ with variables $b_{i_0, \ldots, i_n}$, such that $\phi_j(a_{i_0, \ldots, i_n}) \neq 0$ for the coefficients of $f(T_0, \ldots, T_n)$.

For each $p \in \text{Spm} \mathcal{O}_K$, since $b_{i_0, \ldots, i_n}^{(p)} \in \mathcal{O}_{K,p}$, we have $|\phi_j(b_{i_0, \ldots, i_n}^{(p)})|_p \leq 1$ if $\phi_j(b_{i_0, \ldots, i_n}^{(p)}) \neq 0$. By definition, if the maximal ideal $p \in \mathcal{Q}(\mathcal{X})$, we have $|\phi_j(b_{i_0, \ldots, i_n}^{(p)})|_p < 1$. Then we obtain

$$\frac{1}{[K : \mathbb{Q}]} \sum_{p \in \mathcal{Q}(\mathcal{X})} \log N(p) \leq - \sum_{p \in \mathcal{Q}(\mathcal{X})} \frac{[K_p : \mathbb{Q}_p]}{[K : \mathbb{Q}]} \log \left( |\phi_j(b_{i_0, \ldots, i_n}^{(p)})|_p \right)$$

$$\leq - \sum_{p \in \text{Spm} \mathcal{O}_K} \frac{[K_p : \mathbb{Q}_p]}{[K : \mathbb{Q}]} \log \left( |\phi_j(b_{i_0, \ldots, i_n}^{(p)})|_p \right)$$

$$= \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_{K,\infty}} \log \left( |\phi_j(b_{i_0, \ldots, i_n}^{(v)})|_v \right).$$

In order to estimate $\log \left( |\phi_j(b_{i_0, \ldots, i_n}^{(v)})|_v \right)$ for a fixed $v \in M_{K,\infty}$, from the properties of $\phi_j$ given in Proposition 3.2, we have

$$\log \left( |\phi_j(b_{i_0, \ldots, i_n}^{(v)})|_v \right) \leq (\delta^2 - 1) \log \left( \max_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} \left\{ |b_{i_0, \ldots, i_n}^{(v)}|_v \right\} \right)$$

$$+ (\delta^2 - 1) \left( 3 \log \delta + \delta \log 3 + \log \left( \frac{n + \delta}{\delta} \right) \right).$$

Then we obtain

$$\frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_{K,\infty}} \log \left( |\phi_j(b_{i_0, \ldots, i_n}^{(v)})|_v \right)$$

$$\leq \frac{\delta^2 - 1}{[K : \mathbb{Q}]} \sum_{v \in M_{K,\infty}} \log \left( \max_{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}} \left\{ |b_{i_0, \ldots, i_n}^{(v)}|_v \right\} \right)$$

$$+(\delta^2 - 1) \left( 3 \log \delta + \delta \log 3 + \log \left( \frac{n + \delta}{\delta} \right) \right)$$

$$= (\delta^2 - 1) h(X) + (\delta^2 - 1) \left( 3 \log \delta + \delta \log 3 + \log \left( \frac{n + \delta}{\delta} \right) \right),$$

where the last equality is from (2) and (3). Then we have the assertion. \qed
Remark 4.2. — With all the notations in Proposition 4.1, we have $C(n, \delta) \ll_n \delta^3$.

By applying the method of the proof of Proposition 4.1 to Proposition 3.1, we have the following estimate for the case of plane curves.

**Proposition 4.3.** — Let $X \hookrightarrow \mathbb{P}^2_K$ be a geometrically integral plane curve of degree $\delta$, $\mathcal{X}$ be its Zariski closure in $\mathbb{P}^2_{\mathcal{O}_K}$, $\mathcal{X}_p = \mathcal{X} \times_{\text{Spec} \mathcal{O}_K} \text{Spec} \mathbb{F}_p$, and

$$\mathcal{Q}(\mathcal{X}) = \{ p \in \text{Spec} \mathcal{O}_K | \mathcal{X}_p \to \text{Spec} \mathbb{F}_p \text{ is not geometrically integral} \}.$$  

We have

$$\frac{1}{|K : Q|} \sum_{p \in \mathcal{Q}(\mathcal{X})} \log N(p) \leq (\delta^2 - 1)h(X) + C(n, \delta),$$

where $N(p) = \#(\mathcal{O}_K/p)$, $h(X)$ is the classic height of $X$ in $\mathbb{P}^n_K$ as Definition 2.2, and the constant $C(n, \delta) = (3\delta^2 - 3) \log \delta$.

**Remark 4.4.** — With all the notations in Proposition 4.3, we have $C(n, \delta) \ll_n \delta^2 \log \delta$, which has a better dependence on the degree than the case of general hypersurfaces provided in Proposition 4.1. If we only consider the dependence on the degree of plane curves, this estimate has the same as the later improvements.

4.2. Non-geometrically reductions of general projective schemes. — In order to study the non-geometrically reductions of general schemes, it is significant to understand the reductions over their Chow form or Cayley form. Then we will reduce the general case to that of hypersurfaces.

4.2.1. Chow form and Cayley form. — We resume the construction of Cayley forms briefly. For more details applied in the quantitative arithmetics, we refer the readers to [2, §3], see also [11, §2] for the application to the study of the non-reduced reductions.

Let $\mathcal{E}$ be an hermitian vector bundle of rank $n + 1$ over $\text{Spec} \mathcal{O}_K$, and $X \hookrightarrow \mathbb{P}(\mathcal{E}_K)$ be a geometrically integral closed sub-scheme of dimension $d$ and degree $\delta$. We denote

$$\theta : \mathcal{E}^\vee_K \otimes \mathcal{E}_K \to \bigwedge^d \mathcal{E}_K$$

the homomorphism which maps $\xi \otimes (x_0 \wedge \cdots \wedge x_n)$ to

$$\sum_{i=0}^d (-1)^i \xi(x_i) x_0 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_d.$$  

Let $\Gamma$ be the sub-variety of $\mathbb{P}(\mathcal{E}_K) \times_{\text{Spec} \mathcal{O}_K} \mathbb{P}(\bigwedge^{d+1} \mathcal{E}_K)$ which classifies the all the points $(\xi, \alpha)$ such that $\theta(\xi \otimes \alpha) = 0$. Let $p : \mathbb{P}(\mathcal{E}_K) \times_{\text{Spec} \mathcal{O}_K} \mathbb{P}(\bigwedge^{d+1} \mathcal{E}_K) \to \mathbb{P}(\mathcal{E}_K)$ and $q : \mathbb{P}(\mathcal{E}_K) \times_{\text{Spec} \mathcal{O}_K} \mathbb{P}(\bigwedge^{d+1} \mathcal{E}_K) \to \mathbb{P}(\bigwedge^{d+1} \mathcal{E}_K)$ be the two canonical projections.

By [2, Proposition 3.4] or [11, Proposition 2.6], the scheme $q(\Gamma \cap \rho^{-1}(X))$ is a geometrically integral hypersurface in $\mathbb{P}(\bigwedge^{d+1} \mathcal{E}_K)$ of degree $\delta$. We denote by $\psi_{X,K} \in \text{Sym}^\delta_{\mathcal{O}_K} (\bigwedge^{d+1} \mathcal{E}_K)$ the element which defines this hypersurface, and by $\Psi_{X,K}$ the $K$-linear sub-space of $\text{Sym}^\delta_{\mathcal{O}_K} (\bigwedge^{d+1} \mathcal{E}_K)$ generated by $\psi_{X,K}$.

Let $\Psi_{X}$ be the saturation of $\Psi_{X,K}$ in $\text{Sym}^\delta_{\mathcal{O}_K} (\bigwedge^{d+1} \mathcal{E}_K)$, which means $\Psi_{X}$ is the largest sub-$\mathcal{O}_K$-module of $\text{Sym}^\delta_{\mathcal{O}_K} (\bigwedge^{d+1} \mathcal{E}_K)$ such that $\Psi_{X} \otimes \mathcal{O}_K K = \Psi_{X,K}$ and $\text{Sym}^\delta_{\mathcal{O}_K} (\bigwedge^{d+1} \mathcal{E}_K)/\Psi_{X}$ is a torsion-free $\mathcal{O}_K$-module.
By [1, §4.3.2 (i), (iv)], the construction of Cayley form commutes with the extension from \( X \hookrightarrow \mathbb{P}(\mathcal{E}_K) \) to \( \mathcal{X} \hookrightarrow \mathbb{P}(\mathcal{E}) \), et commutes with the base change from \( \mathcal{O}_K \) to its residue field, see [1, §4.3.1] for more details for the above argument. Then in order to control the non-geometrically integral reductions of \( \mathcal{X} \rightarrow \text{Spec} \mathcal{O}_K \), we are able to consider the non-geometrically reductions of its Cayley form.

4.2.2. Control the non-geometrically reductions. — With the above constructions, we consider the non-geometrically reductions of general projective schemes below. We pick \( \mathcal{E} = (\mathcal{O}_K \oplus (n+1)K, (\parallel \cdot \parallel_\nu)_{\nu \in M_{K,\infty}}) \), where for each \( \nu \in M_{K,\infty} \), the norm \( \parallel \cdot \parallel_\nu \) maps \( (x_0, \ldots, x_n) \) to \( \sqrt{|x_0|^2 + \cdots + |x_n|^2} \). In this case, we denote \( \mathbb{P}(\mathcal{E}_K) \) and \( \mathbb{P}(\mathcal{E}) \) by \( \mathbb{P}^n_K \) and \( \mathbb{P}^n_{\mathcal{O}_K} \) respectively for simplicity.

Theorem 4.5. — With all the above notations and conditions. Let \( X \) be a geometrically integral closed sub-scheme of \( \mathbb{P}^n_K \) of dimension \( d \) and degree \( \delta \), \( \mathcal{X} \) be the Zariski closure of \( X \) in \( \mathbb{P}^n_{\mathcal{O}_K} \), \( \mathcal{X}_{\mathbb{F}_p} = \mathcal{X} \times_{\text{Spec} \mathcal{O}_K} \text{Spec} \mathbb{F}_p \), and

\[
\mathcal{Q}(\mathcal{X}) = \{ \mathfrak{p} \in \text{Spm} \mathcal{O}_K | \mathcal{X}_{\mathbb{F}_p} \to \text{Spec} \mathbb{F}_p \text{ is not geometrically integral} \}.
\]

We denote \( N(n, d) = \binom{n+1}{d+1} - 1, N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p}) \), and \( H_m = 1 + \cdots + \frac{1}{m} \). Then we have

\[
\frac{1}{[K : Q]} \sum_{\mathfrak{p} \in \mathcal{Q}(\mathcal{X})} \log N(\mathfrak{p}) \leq (\delta^2 - 1)h_{\mathcal{O}(1)}(X) + C'(n, d, \delta),
\]

where \( \mathcal{O}(1) \) is equipped with the corresponding Fubini-Study metrics, \( h_{\mathcal{O}(1)}(X) \) is the Arakelov height of \( X \) in \( \mathbb{P}^n_K \) as Definition 2.1, and the constant

\[
C'(n, d, \delta) = (\delta^2 - 1) \left( 3 \log \delta + \log \left( \frac{N(n, d) + \delta}{\delta} \right) + \left( (N(n, d) + 1) \log 2 + 4 \log(N(n, d) + 1) + \log 3 - \frac{1}{2} H_{N(n, d)} \right) \delta \right).
\]

Proof. — Let

\[
\mathcal{Q}(\Psi_X) = \{ \mathfrak{p} \in \text{Spm} \mathcal{O}_K | \Psi_X \otimes \mathcal{O}_K \mathbb{F}_p \\
\text{isn’t generated by an absolutely irreducible polynomial} \}.
\]

Then by [12, Exercise 2.4.1], the fact \( \mathfrak{p} \in \mathcal{Q}(\Psi_X) \) is verified if and only if the Cayley form of \( \mathcal{X}_{\mathbb{F}_p} \) is not geometrically integral over \( \text{Spec} \mathbb{F}_p \), which is verified if and only if \( \mathcal{X} \times_{\text{Spec} \mathcal{O}_K} \text{Spec} \mathbb{F}_p \to \text{Spec} \mathbb{F}_p \) is not geometrically integral. So we obtain

\[
\frac{1}{[K : Q]} \sum_{\mathfrak{p} \in \mathcal{Q}(\mathcal{X})} \log N(\mathfrak{p}) = \frac{1}{[K : Q]} \sum_{\mathfrak{p} \in \mathcal{Q}(\Psi_X)} \log N(\mathfrak{p}).
\]
By Proposition 4.1, we have
\[
\frac{1}{[K : \mathbb{Q}]} \sum_{p \in \mathcal{O}(\psi_X)} \log N(p)
\leq (\delta^2 - 1) h(\psi_{X,K}) + (\delta^2 - 1) \left( 3 \log \delta + \delta \log 3 + \log \left( \frac{N(n, d)}{\delta} + \delta \right) \right),
\]
where \(h(\psi_{X,K})\) is defined at Definition 2.2.

By (25), (26) and (27) in the proof of [11, Théorème 7.1], we have
\[
h(\psi_{X,K}) - h_{\text{U}(1)}(X) \leq (N(n, d) + 1) \delta \log 2 + 4 \delta \log(N(n, d) + 1) - \frac{1}{2} \delta H_{N(n,d)}.
\]
So we obtain the assertion by combining the above estimates.

**Remark 4.6.** — We consider the constant \(C'(n, d, \delta)\) defined in Theorem 4.5. Then we have \(C'(n, d, \delta) \ll_n \delta^3\). Due to the comparison of heights, we have the same estimate of this constant to the case of curves and of general dimensions if we choose the Arakelov height.

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