On the Mass of Two Dimensional Quantum Black Hole

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ABSTRACT

For the two dimensional dilaton-coupled quantum gravity model, we give the local black hole mass, which is an analogue of what was first introduced by Fischler, Morgan and Polchinski in the four dimensional gravitational systems. We analyze the original CGHS model with this local mass and find that the local mass is decreasing in the future direction on the matter shock-wave line, while it stays constant at past null infinity.

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Recently two dimensional black hole physics has been attracting much interest in studying the evaporation of black holes. It raised some hope of solving problems associated with the Hawking radiation \[1\]. Callan, Giddings, Harvey and Strominger \[2\] studied the string-inspired two dimensional toy model (CGHS model). They presented the general solutions at the classical level and showed that their model has a solution having a black hole which is formed by the matter shock wave. And also they discussed about the evaporation of the black hole at the quantum level. Subsequently many features of the CGHS model and its modified versions have been vigorously studied by many people \[3 – 8\].

In this paper we shall study the evaporation of the black hole concentrating on the mass of the black hole in the original CGHS model. First we will define a local function which gives the mass of a black hole at the classical level. And then we will analyze the behavior of the mass function at the quantum level.

The CGHS model without the conformal matter fields is given by

\[
S = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} (R + 4(\nabla \phi)^2 + 4\lambda^2),
\]

where \(\phi\) is a dilaton field and \(\lambda\) is a constant. Following the Arnowitt-Deser-Misner (ADM) formulation, the two dimensional metric \(g\) is written by

\[
g_{ab} = \begin{pmatrix}
-N_0^2 + \frac{N_0^2}{\gamma} & N_1 \\
N_1 & \gamma
\end{pmatrix},
\]

where \(N_0\) (\(N_1\)) is known as lapse (shift) and \(\gamma\) stands for a dynamical degree of freedom of gravitational sector. \(\gamma\) and the dilaton \(\phi\) are the dynamical degrees of freedom in the present model Eq.(1). In terms of the canonical variable, that is, \(\gamma, \phi\) and their conjugate momenta, \(\pi_\gamma, \pi_\phi\), the above action is rewritten as,

\[
S = \int \pi_\phi \dot{\phi} + \pi_\gamma \dot{\gamma} - N_0 \mathcal{H}_0 - N_1 \mathcal{H}_1.
\]

Here the dot stands for time derivative and \(\mathcal{H}_0\) (\(\mathcal{H}_1\)) is the generator of time (space)
reparametrization;

\[
\mathcal{H}_0 = 2 \sqrt{\gamma} e^{2\phi} \pi_\gamma \pi_\phi + 4 \gamma \sqrt{\gamma} e^{2\phi} \pi_\gamma^2 - \frac{e^{-2\phi} (\phi')^2}{\sqrt{\gamma}} - \sqrt{\gamma} e^{-2\phi} \lambda^2 ,
\]
\[
\mathcal{H}_1 = -\frac{\gamma'}{\gamma} \pi_\gamma + \frac{\phi'}{\gamma} \pi_\phi ,
\]

where the prime denotes the spatial derivative. The lapse and shift are Lagrange multipliers, which leads to the constraint equations,

\[
\mathcal{H}_0(\pi_\phi, \phi, \pi_\gamma, \gamma) = 0 ,
\]
\[
\mathcal{H}_1(\pi_\phi, \phi, \pi_\gamma, \gamma) = 0 .
\]

We find that the combination of the above constraints,

\[
\Phi \equiv \frac{2\phi'}{\lambda \sqrt{\gamma}} \times \mathcal{H}_0 - \frac{4\gamma \pi_\gamma e^{2\phi}}{\lambda} \times \mathcal{H}_1 ,
\]

becomes the total derivative of a local function,

\[
\Phi = \mathcal{M}' = 0 ,
\]
\[
\mathcal{M} \equiv \frac{4\gamma \pi_\gamma^2}{\lambda} e^{2\phi} - \frac{(\phi')^2}{\lambda \gamma} e^{-2\phi} + \lambda e^{-2\phi} .
\]

This quantity \( \mathcal{M} \) is the two dimensional version of a local mass which is first introduced by Fischler Morgan and Polchinski [9] in the study of a spherically symmetric four dimensional gravitational system. It also plays an important role in studying the evaporation of the four dimensional black hole in Ref.[10].

The local mass function (8) becomes

\[
\mathcal{M} = \frac{1}{\lambda} \left[ 4 e^{-2\rho} e^{-2\phi} \partial_+ \phi \partial_- \phi + \lambda^2 e^{-2\phi} \right]
\]

in the conformal gauge;

\[
g_{++} = -\frac{1}{2} e^{2\rho} , \quad g_{+0} = g_{-0} = 0 ,
\]

where the light-cone coordinates are \( x^\pm = x^0 \pm x^1 \). Plugging the classical static
black hole solution of mass $M$,

$$ds^2 = e^{2\rho} dx^+ dx^- = \frac{dx^+ dx^-}{\frac{M}{\lambda} - \lambda^2 x^+ x^-}, \quad \phi = \rho,$$

into the above mass function, we find

$$\mathcal{M}(x^+, x^-) = M.$$  \hspace{1cm} (12)

Furthermore, in Ref.[2] there presented an example of the formation of the black hole by the shock wave of a conformal matter field $f$ traveling in the $x^-$ direction at $x^+ = x_0^+$. The stress tensor of the matter is given by

$$\frac{1}{2} \partial_+ f \partial_+ f = \frac{M}{\lambda x_0^+} \delta(x^+ - x_0^+),$$

where $M$ is a parameter representing the magnitude of the shock-wave, which is shown to be the mass of a black hole. Then the classical solution is

$$ds^2 = e^{2\rho} dx^+ dx^- = \frac{dx^+ dx^-}{\frac{M}{\lambda x_0} (x^+ - x_0^+) \theta(x^+ - x_0^+) - \lambda^2 x^+ x^-},$$

$$\rho = \phi.$$  \hspace{1cm} (14)

Calculating the local mass function for this geometry, we get

$$\mathcal{M}(x^+, x^-) = M \theta(x^+ - x_0^+).$$  \hspace{1cm} (15)

The value becomes zero in the Linear Dilaton Vacuum (LDV) region while beyond the matter shock-wave line it becomes $M$ as expected.

So far we have seen that the mass function gives the mass of the black hole at the classical level. Eq.(15) means that the evaporation of the black hole does not occur classically since the mass of the black hole does not vanish towards the future null infinity.
Now we consider how the mass function behaves at the quantum level. At the one-loop level, the quantum corrections are the contributions of the conformal anomaly of the matter fields and that from the gravitational sector. We incorporate the quantum effect through including the following term which comes from the trace anomaly into the action,

$$\kappa \partial_+ \partial_- \rho,$$

where $\kappa$ depends on the number of the matter fields and here we assume that it is a large positive number. Then we have the following equations at this level,

$$\partial_+ \partial_- \phi = \left(1 - \frac{\kappa}{2} e^{2\phi}\right) \partial_+ \partial_- \rho,$$

$$2 \left(1 - \kappa e^{2\phi}\right) \partial_+ \partial_- \phi - 4 \left(1 - \frac{\kappa}{2} e^{2\phi}\right) \partial_+ \phi \partial_- \phi - \left(1 - \frac{\kappa}{2} e^{2\phi}\right) \lambda^2 e^{2\rho} = 0.\quad(18)$$

Once incorporating the quantum effect, one finds that the model is no longer exactly solvable. Many features in classical theory become different. For example, $\phi$ does not stay equal to $\rho$, and the analysis breaks down due to the singularity at some value of $\phi$, etc. On the other hand, Linear Dilaton Vacuum (LDV) is still a solution of quantum system, that is, LDV is stable in this quantum theory.

Then we will analyze on a narrow region above the matter shock-wave line $x^+ = x_0^+$ and also the past null infinity region. We assume that the fields $\phi$ and $\rho$ take the classical values on the line, which guarantees that the solution approaches the classical one asymptotically.

Since the mass function is local we might see the spacetime point where the evaporation completes. However the quantum effect will be large at that point and since we are restricted to the one-loop level, the point may be beyond our scope in the present paper.

Now we shall see the behavior of the mass function with the quantum correction. On the matter shock-wave line, the mass function(9) can be obtained if one
knows $\partial_+ \phi (x_0^+, x^-)$. It can be calculated from Eq.(18) [4],

$$
\partial_+ \phi (x_0^+, x^-) = -\frac{1}{2x_0^+} + \frac{M}{2\lambda x_0^+} \frac{1}{\sqrt{w} \sqrt{w - \kappa}},
\tag{19}
$$

where

$$
w \equiv -\lambda^2 x_0^+ x^-.
\tag{20}
$$

Then we have

$$
M \bigg|_{x^+ = x_0^+} = M \sqrt{\frac{w}{w - \kappa}}.
\tag{21}
$$

$\mathcal{M}$ diverges at $w = \kappa$ where the two dimensional curvature is singular [3, 4]. The quantum effect is very large at that singular point and our analysis has already been broken down there. On the other hand, at the apparent horizon on the matter shock-wave line [4],

$$
x^-_{AH} = -\sqrt{\left( \frac{M}{\lambda^3 x_0^+} \right)^2 + \left( \frac{\kappa}{2\lambda^2 x_0^+} \right)^2} - \frac{\kappa}{2\lambda^2 x_0^+},
\tag{22}
$$

or

$$
w_{AH} = \sqrt{\left( \frac{M}{\lambda} \right)^2 + \left( \frac{\kappa}{2} \right)^2} + \frac{\kappa}{2}.
\tag{23}
$$

$\mathcal{M}$ is finite and towards the past null infinity $x^- \to -\infty$ it decreases to $M$.

Similarly we can calculate $\partial_+ \phi (x_0^+ + \epsilon, x^-)$ with small $\epsilon$ and $\partial_+ \rho (x_0^+, x^-)$ from Eqs.(17) and (18). And hence we obtain the mass function at $x^+ = x_0^+ + \epsilon$,

$$
\mathcal{M} \bigg|_{x^+ = x_0^+ + \epsilon} = \frac{M \sqrt{w}}{\sqrt{w} - \kappa} + \frac{M \epsilon}{x_0^+} \left[ \frac{M w}{\lambda (w - \kappa)^2} - \frac{5 \kappa \sqrt{w}}{4 (w - \kappa)^{3/2}} - \frac{2 M w}{\lambda \kappa (w - \kappa)} \right. 
\tag{24}
$$

$$
\left. + \frac{2 M \sqrt{w}}{\lambda \kappa \sqrt{w - \kappa}} + \frac{\sqrt{w}}{4 \sqrt{w} - \kappa} \log \frac{w}{w - \kappa} \right].
$$

And if we assume that the quantum fluctuation becomes small towards the past null infinity and the fields become almost classical there, we can calculate the
mass function similarly as,
\[ \mathcal{M}|_{x^- \to -\infty} = M. \]  
(25)

Furthermore, the derivative of the mass function with respect to \( x^- \) is obtained by
\[ \partial_- \mathcal{M}|_{x^- \to -\infty} = 0. \]  
(26)

The mass function is shown in figure 1.

Fig. 1

The behavior of the mass function has the following features: i) it decreases along the negative \( x^- \) direction on the matter shock-wave line; ii) it decreases in the \( x^+ \) direction on the matter shock-wave line \( (x^+_0, -\infty < x^- < x^-_{AH}) \); iii) at \( x^- \to -\infty \) the mass function is constant and its derivative with respect to \( x^- \) is zero.

We have defined the local mass function (9) which gives the mass of the black hole in the two dimensional dilaton gravity system at the classical level. The evaporation of the black hole at the quantum level implies that the mass function decreases to zero. We found that the mass function decreases in the \( x^+ \) direction on the matter shock wave line, however it was not detected that the mass function becomes below the classical value due to the quantum effect in our analysis. Further analysis towards the future null infinity or along the apparent horizon is necessary.

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FIGURE CAPTIONS

1) The local mass function Eq.(8). The region painted black is over the singularity. The mass function in the blank region is beyond our present analysis.