CHARACTERIZATION OF ENTROPY FOR SPACING SHIFTS

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ABSTRACT. Suppose $P \subseteq \mathbb{N}$ and let $(\Sigma_P, \sigma_P)$ be the space of a spacing shift. We show that if entropy $h_{\sigma_P} = 0$ then $(\Sigma_P, \sigma_P)$ is proximal. Also $h_{\sigma_P} = 0$ if and only if $P = \mathbb{N} \setminus E$ where $E$ is an intersective set. Moreover, we show that $h_{\sigma_P} > 0$ implies that $P$ is a $\Delta^*$ set; and by giving a class of examples, we show that this is not a sufficient condition. Then there is enough results to solve question 5 given in [J. Banks et al., Dynamics of Spacing Shifts, Discrete Contin. Dyn. Syst., to appear].

INTRODUCTION AND DEFINITIONS

In this paper we give a characterization of entropy of a spacing shifts by the combinatorial property of the set $P \subseteq \mathbb{N}$ which defines a spacing shift. A detailed study for spacing shifts can be found in [1], so we here only consider the basic definitions and notions needed for our task.

A topological dynamical system (TDS) is a pair $(X, T)$ such that $X$ is a compact metric space and $T$ is a continuous surjective self map. The orbit closure of a point $x$ in $(X, T)$ is the set $\overline{O}(x) = \{T^n(x) : n \in \mathbb{N}\}$. A system $(X, T)$ is transitive if it has a point $x$ such that $\overline{O}(x) = X$. Also a point $x$ is recurrent if for every neighborhood $U$ of $x$ there exists $n \neq 0$ such that $T^n(x) \in U$. We let $N(x, U) = \{n \in \mathbb{N} : T^n(x) \in U\}$ and $N(U, V) = \{n \in \mathbb{N} : T^n(U) \cap V \neq \emptyset\}$ where $U$ and $V$ are open sets.

Let $x_1, x_2 \in X$. One says that $(x_1, x_2) \in X \times X$ is a proximal pair if

$$\liminf_{n \to \infty} d(T^n(x_1), T^n(x_2)) = 0;$$

and a TDS is called proximal if all $(x_1, x_2) \in X \times X$ are proximal pairs.

Let $A = \{a_i\}_{i \in \mathbb{N}}$ be an increasing sequence of natural numbers. Then $s = a_1 + a_2 + \ldots + a_i$, $i < j$ is called a partial finite sum of $A$. The finite sums of $A$ denoted by $FS(A)$ is the set of all partial finite sums. A set $F \subseteq \mathbb{N}$ is called IP-set if it contains the finite sums of some sequence of natural numbers. Let $\mathcal{IP}$ be the set of all IP-sets.

A set $D \subseteq \mathbb{N}$ is called $\Delta$-set if there exists an increasing sequence of natural numbers $S = (s_n)_{n \in \mathbb{N}}$ such that the difference set $\Delta(S) = \{s_i - s_j : i > j\} \subseteq D$. Denote by $\Delta$ the set of all $\Delta$-sets. Any IP-set is a $\Delta$-set; for let $S = (a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots)$. A collection $\mathcal{F}$ of non-empty subsets of $\mathbb{N}$ is called a family if it is hereditary upward: if $F \in \mathcal{F}$ and $F \subseteq F'$, then $F' \in \mathcal{F}$. The dual family $\mathcal{F}^*$, is defined to be all subsets of $\mathbb{N}$ that meets all sets in $\mathcal{F}$. That is

$$\mathcal{F}^* = \{G \subseteq \mathbb{N} : G \cap F \neq \emptyset, \forall F \in \mathcal{F}\}.$$

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Hence $IP^*$ and $\Delta^*$ are the dual family of $IP$ and $\Delta$ respectively.

The notions for a subset of natural numbers such as $\Delta$ or $IP$ are structural notions. For instance, an $IP$-set is more structured than a $\Delta$-set. Other structures are also defined [7], [3]. There are also notions for largeness which are defined by means of different densities on subsets of natural numbers. See [7], [2] for a rather complete treatment for both of these notions. Let $A \subseteq \mathbb{N}$. Then

$$d(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$$

is called the upper density of $A$. Also the lower density is defined as

$$d^*(A) = \liminf_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n}.$$ 

When $d(A) = d^*(A)$ then it is called the density of $A$ and is denoted by $d(A)$. The upper Banach density of $A$ is denoted by $d^*(A)$ and is defined as

$$d^*(A) = \limsup_{N_i, M_i \to \infty} \frac{|A \cap \{M_i, M_i + 1, \ldots, N_i\}|}{N_i - M_i + 1}.$$ 

When there is $k \in \mathbb{N}$ such that all the intervals in $\mathbb{N} \setminus A$ have length less than $k$, then $A$ is called syndetic. The length of the largest of such intervals will be called the gap of $A$. Clearly, $d(A) > 0$ for any syndetic set $A$. The dual of syndetic sets are thick sets; a set is thick if and only if $d^*(A) = 1$. We say $A$ is thickly syndetic if for every $N$ the positions where consecutive elements of length $N$ begins form a syndetic set.

Note that $\Delta^*$-sets are highly structured and are syndetic [3]. Another of such large and structured subsets of $\mathbb{N}$ are Bohr sets. We say that a subset $A \subset \mathbb{N}$ is a Bohr set if there exist $m \in \mathbb{N}, \alpha \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and open set $U \subset \mathbb{T}^m$ such that

$$\{n \in \mathbb{N} : n \alpha \in U\}$$

is in $A$. In particular, every $k\mathbb{N}$ is a Bohr set.

**Definition 0.1.** For any set $P \subset \mathbb{N}$ define a spacing shift to be the subshift

$$\Sigma_P = \{s \in \Sigma : s_i = s_j = 1 \Rightarrow |i - j| \in P \cup \{0\}\}.$$ 

For any $y \in \Sigma_P$ we associate a set $A_y = \{i : y_i = 1\}$. it is clear that $A_y - A_y \subset P$. Therefore, notions of largeness and structure for $A_y$ gives the same notions for incidence of 1’s for $y$. That is we set

$$d(y) := d(A_y) = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} y_i}{n} = \lim_{n \to \infty} \frac{|A_y \cap \{1, \ldots, n\}|}{n}.$$ 

Similarly, $d^*(y)$, $d(y)$ and $d^*(y)$ can be defined.

By Definition 0.1 it is clear that $A_y - A_y \subset P$.

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1. Zero Entropy Gives Proximality

The following questions arises in [1] Question 5.

"Is there \( P \) such that \( \mathbb{N} \setminus P \) does not contain \( IP \)-set but \( \Sigma_P \) is proximal? What about positive topological entropy? Are these two properties (i.e proximity and zero entropy) essentially different in the context of spacing shifts?"

We give positive answer to the first question but we will show that if \( \mathbb{N} \setminus P \) contains \( \Delta \)-set (and hence \( IP \)-set), then the entropy is zero. Also we will show that zero entropy in spacing shifts implies proximality.

For any \( x, y \in \Sigma_P \) let

\[
F_{xy}(t) = \liminf_{n \to \infty} \frac{1}{n}|\{0 \leq m \leq n - 1 : d(\sigma^m(x), \sigma^m(y)) < t\}|.
\]

**Remark 1.1.** In [1] the authors show that if there are \( x, y \in \Sigma_P, t > 0 \) such that \( F_{xy}(t) < 1 \) then \( h_{\sigma_P} > 0 \). If such \( x, y \) and \( t \) exist, then there is some \( y' \in \Sigma_P \) such that \( d(y') > 0 \). Because let \( t = 2^{-l} \) then there exists an increasing sequence \( \{q_i\}_{i=1}^\infty \) and \( \epsilon > 0 \) such that either \( |\{0 \leq j \leq q_i : x_j \neq y_j\}| > \frac{\gamma \epsilon}{t^2} \) or \( |\{0 \leq j \leq t_i : y_j \neq 0\}| > \frac{\gamma \epsilon}{t^2} \). Hence \( d(x) \) or \( d(y) \) is positive.

In [1] Lemma 3.5, it has been proved that if \( \mathbb{N} \setminus P \) contains an \( IP \)-set then \( d(y) = 0 \), for any \( y \in \Sigma_P \). We give a stronger result with a simpler proof.

**Theorem 1.2.** If \( \mathbb{N} \setminus P \) contains a \( \Delta \)-set then \( d^*(y) = 0 \) for all \( y \in \Sigma_P \).

**Proof.** If \( y \in \Sigma_P \), then \( A_y - A_y \subset P \). But if there is \( y \) such that \( d^*(y) > 0 \) then \( A_y - A_y \) is a \( \Delta^* \)-set [5] and \( \mathbb{N} \setminus P \) cannot have a \( \Delta \)-set. \( \square \)

The following result is a reformulation of two results in [1].

**Theorem 1.3.** If for all \( y \in \Sigma_P \), \( d(y) = 0 \), then

1. \( h_{\sigma_P} = 0 \).
2. \( \sigma_P \) is proximal.

**Proof.** (1) and (2) are proved in [1] Theorem 3.6 and [1] Theorem 3.11 respectively for the case when \( \mathbb{N} \setminus P \) contains an \( IP \)-set. The proof of these theorems are based on the fact that if \( \mathbb{N} \setminus P \) contains an \( IP \)-set then \( d(y) = 0 \), for any \( y \in \Sigma_P \). Then this last result will lead to the both conclusions. \( \square \)

Again the proof of this Theorem is a minor alteration of in the proof of [1] Theorem 3.18.

**Theorem 1.4.** There exists some \( y \in \Sigma_P \) with \( d^*(y) > 0 \) if and only if \( h_{\sigma_P} > 0 \).

**Proof.** First suppose there exists a point \( y \in \Sigma_P \) such that \( d^*(y) > 0 \), so for some \( l \) there exist two increasing sequences \( \{M_i\}_{i=1}^\infty \), \( \{N_i\}_{i=1}^\infty \) and \( \gamma > 0 \) such that

\[
|\{M_i \leq j \leq N_i : y_{[j, j+l]} \neq 0^{l+1}\}| \geq (N_i - M_i)\gamma.
\]

So

\[
|\{M_i \leq j \leq N_i : y_j \neq 0\}| \geq \frac{(N_i - M_i)\gamma}{l+1}.
\]

Then by definition we have

\[
h_{\sigma_P} \geq \lim_{N_i-M_i \to \infty} \frac{1}{N_i - M_i} \log(2^{\frac{(N_i - M_i)\gamma}{l+1}}) > 0.
\]

Conversely, if for any \( y \in \Sigma_P \), \( d^*(y) = 0 \) then \( d(y) = 0 \) and the proof follows from Theorem [13]. \( \square \)
An immediate consequence of the above theorem is that if $P$ is not $\Delta^*$, then $h_{\sigma P} = 0$. In particular, this sorts out the second question.

By Theorem 1.3 if $h_{\sigma P} > 0$, then there is a $y \in \Sigma P$ such that $d(y) > 0$. Combining this with the results of the above Theorem we have:

**Corollary 1.5.** There is a point $y \in \Sigma P$ with $d(y) > 0$ if and only if for some $y', d^*(y') > 0$.

The following gives an answer to the third question. Moreover, this result and the fact that when $P$ misses an IP-set then it is not $\Delta^*$ and so has zero entropy is an answer for the first question as well.

**Theorem 1.6.** If $h_{\sigma P} = 0$ then $\Sigma P$ is proximal.

*Proof.* Suppose $h_{\sigma P} = 0$. Then by Theorem 1.3 for any $y \in \Sigma P$ we have $d^*(y) = 0$ which implies that $d(\{i : y_i = 0\}) = 1$. Hence for any two points $x, y \in \Sigma P$, $d(\{i : x_i = 0\} \cap \{i : y_i = 0\}) = 1$ and this in turn implies that $\Sigma P$ is proximal. 

\[\square\]

### 1.1. A necessary condition for transitivity.

Still there is not a characterization for $P$ to have $\Sigma P$ transitive. This also has been put as a question in [1, Question 1]. A necessity is the following.

**Theorem 1.7.** Suppose $\Sigma P$ is transitive. Then $P$ is an IP–IP set.

*Proof.* For any TDS such as $(X, T)$, the return times of a recurrence point $x$ to any non-empty open set $U$, that is, $N(x, U) = \{n \in \mathbb{N} : T^n(x) \in U\}$ is an IP-set [6, Theorem 2.17]. Now let $y$ be a transitive point. Then $y$ is a recurrence point and $N(y, [1])$ is an IP-set. But $N(y, [1]) = \{y_i : y_i = 1\} = A_y$ and so $A_y - A_y \subset P$ and as a result $P$ is an IP–IP set. 

An application of the above theorem is that any thick subset of natural numbers is an IP–IP set. This is because $\Sigma P$ is weak mixing if and only if $P$ is thick and if a TDS is weak mixing, then it is transitive, in fact, totally transitive: $(\Sigma P, \sigma^n)$ is transitive for all $n = 0, 1, \ldots$

It is not hard to see that for any infinite subset of $\mathbb{N}$ such as $A$, $P = FS(A) - FS(A)$ is a transitive system. On the other hand, let $k \geq 3, p_2 > p_1$ and $p_2 - p_1 \neq kn$ for any $n \in \mathbb{N}$. Now if $P = k\mathbb{N} \cup \{p_1, p_2\}$, then $\Sigma P$ is not transitive, however it is clearly IP–IP set. Because it contains an IP–IP set such as $k\mathbb{N}$.

By now we understand that this is the structure in $P$ and not density which gives interesting dynamics to our spacing shifts systems. For instance, if $P$ is not a $\Delta$-set-set, then for all $y \in \Sigma P$, $\sum_{i=1}^{\infty} y_i < \infty$. This gives a very simple dynamics to $\Sigma P$. In fact, it is an equicontinuous system where any point will be attracted to $0^\infty$ eventually. We may choose $P$ to have high density. As an example, for any \( \varepsilon > 0 \) let $\frac{1}{k} < \varepsilon$ and set $P = \mathbb{N} \setminus k\mathbb{N}$. Then $d(P) = 1 - \varepsilon$ and since $k\mathbb{N}$ is a $\Delta^*$-set $P$ does not contain any $\Delta$-set.

2. **Combinatorial Characterization for Zero Entropy**

In section 1, we showed that $P$ must be at least $\Delta^*$ set, that is a highly structured and large set to have positive entropy. Here we show that even if $P$ is a $\Delta^*$ set, it is not guaranteed that $h_{\sigma P} > 0$.

One calls $E \subset \mathbb{N}$ a density intersective set if for any $A \subset \mathbb{N}$ with positive upper Banach density, $E \cap (A - A) \neq \emptyset$. For instance, any IP-set is a density intersective
set. In fact, if $R \subset \mathbb{N}$ is an IP-set and $p(\cdot)$ is a polynomial such that $p(\mathbb{N}) \subset \mathbb{N}$, then $E = \{p(n) : n \in R\}$ is a density intersective set [4].

**Theorem 2.1.** $h_{\sigma_P} = 0$ if and only if $P = \mathbb{N} \setminus E$ where $E$ is a density intersective set.

**Proof.** Suppose $h_{\sigma_P} = 0$. If $E = \mathbb{N} \setminus P$ is not density intersective, then there must be a set $A$ with positive upper Banach density such that $A - A \subseteq P$. Choose $y \in \Pi_{i=0}^{\infty} \{0, 1\}$ such that $y_i = 1$ if and only if $i \in A$. Then $y \in \Sigma_P$ and $A = A_y$. But this is absurd by Theorem 1.4.

For the other side, if $E$ is density intersective, then $P$ does not contain any $A - A$ where $A$ is as above. Therefore, for all $y \in \Sigma_P$, $d^*(y) = 0$ which implies $h_{\sigma_P} = 0$. □

It is an easy exercise to show that $\{n^2 : n \in \mathbb{N}\}$ does not contain any $\Delta$-set. So $P = \mathbb{N} \setminus E$ is a $\Delta^*$ set and by the above theorem, $h_{\sigma_P} = 0$.

**2.1. Positive entropy with no non-zero periodic points.** Any spacing shift has $0^\infty$ as its periodic point. But a spacing shift has a non-zero periodic point of period $k$ if and only if $P$ contains $k\mathbb{N}$ [1, Lemma 2.6]. This implies there is a point $y$ with $d(y) \geq \frac{1}{k}$ and so by Theorem 1.4 we have positive entropy.

**Theorem 2.2.** There is $P$ such that $\Sigma_P$ has positive entropy with no non-zero periodic points.

**Proof.** A theorem of Kříž [8] states that there is a set $A$ with positive upper Banach density whose difference set contains no Bohr set. So let $y = \{y_i\}_{i \in \mathbb{N}}$ be defined by $y_i = 1$ if $i \in A$ and zero otherwise. Set $P = A - A$. Then $y \in \Sigma_P$, $A_y = A$ and $\overline{d}(y) = \overline{d}(A) > 0$. Therefore, $h_{\sigma_P} > 0$ and since $P$ does not contain any Bohr set it does not contain any $k\mathbb{N}$ and the proof is complete. □

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