A CLASSICAL $N = 4$ SUPER W-ALGEBRA

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Abstract:

I construct classical superextensions of the Virasoro algebra by employing the Ward identities of a linearly realized subalgebra. For the $N = 4$ superconformal algebra, this subalgebra is generated by the $N = 2 \ U(1)$ supercurrent and a spin 0 $N = 2$ superfield. I show that this structure can be extended to an $N = 4$ super $W_3$ algebra, and give the complete form of this algebra.
1. Introduction

Extensions of the Virasoro algebra containing higher spin currents, so called W-algebras [1,2], play a central role in many areas of two dimensional physics. They originally appeared as Hamiltonian structures on integrable hierarchies [3], and since have been recognized as a symmetry in for example conformal field theories [4], Toda theories [5] and two dimensional gravity [6,7]. Recently, there has been progress in constructing W-algebraic analogues of string theory [8].

Unfortunately, the explicit form of W-algebras is very complicated, except in the simplest cases when only a few currents of low spin are present, and as a consequence, relatively few W-algebras have been constructed explicitly. For example, an infinite series of algebras $W_n$ are known to exist containing currents for all values of the spin from 2 to $n$ [2,5], but only $W_3$ [1] and $W_4$ [9] have been constructed explicitly. The $W_n$ all have $N = 2$ superextensions [10], but again only for $n = 3$ [11,12] and $n = 4$ [13] the explicit structure is known. Although the Virasoro algebra has superextensions with arbitrarily high $N$ [14], for the $W_3$ algebra no extensions with $N > 2$ have been constructed previously.

Direct constructions of W-algebras are facilitated considerably by recognizing the maximal linearly realized subalgebra $\mathcal{H}$ with respect to which all remaining currents transform as primaries, and imposing covariance with respect to this $\mathcal{H}$. In general, $\mathcal{H}$ is a proper extension of the Virasoro algebra. In the case of $N = 2$ supersymmetric W-algebras, $\mathcal{H}$ is the $N = 2$ superconformal algebra, and the construction can be simplified considerably by using $N = 2$ superfields. In this paper, I construct an $N = 4$ super $W_3$ algebra. In this case, $\mathcal{H}$ is a proper subalgebra of the linear $N = 4$ superconformal algebra [15], generated by the $N = 2$ supercurrent $J$ and a spin 0 $N = 2$ superfield $\Phi$. The structure of $\Phi$ is somewhat unusual: its lowest component is the inverse derivative of a spin 1 current. No nonlocalities occur, because only (super)derivatives of $\Phi$ enter the algebra.

I will use the dual formalism recently introduced in [13]. In this approach, one constructs a three dimensional action, analogous in some respects to a Chern-Simons action, from which the Ward identities of the algebra follow as equations of motion. This formulation contains, besides the generating currents, a set of dual one-forms. These one-forms satisfy Maurer-Cartan equations which also follow as equations of motion from the action. Combined with the Ward identities, the Maurer-Cartan equations form an integrable system which is equivalent to the Poisson bracket formulation of the algebra. The integrability of the system is equivalent to the Jacobi identities of the Poisson bracket formulation.

The construction of the action bears a close resemblance to the construction of correlators in conformal field theory, in particular in the way the Ward identities of the subalgebra $\mathcal{H}$ are utilized. The spins in this subalgebra are descendants of the identity, while the remaining spins are primaries with respect to $\mathcal{H}$. The nontrivial task is to construct a set of...
three point functions for the primaries and their composites, consistent with \( \mathcal{H} \) covariance. All correlators involving \( \mathcal{H} \) currents then follow from the couplings of the primaries by the \( \mathcal{H} \) Ward identities, and only a finite number of them are nonzero. The three dimensional action is the effective action of all correlators.

The explicit form of the \( \mathcal{H} \) dressings of the primary couplings is extremely complicated in all but the simplest cases, and they are responsible for the unwieldy form of the largest \( W \)-algebras constructed thus far. It is however very straightforward to determine these dressings. They are simply the \( \mathcal{H} \) covariantizations of the terms in the effective action determined by the couplings of the primaries, and can be constructed in the familiar Yang-Mills fashion by exploiting the anomaly in the \( \mathcal{H} \) Ward identities. This anomaly is analogous to the inhomogeneous terms in the transformation law of Yang-Mills gauge fields, and can be used to cancel terms in the \( \mathcal{H} \) variation of the effective action caused by derivatives.

The organization of the rest of this paper is as follows: In Section 2, I construct as an example the \( SO(N) \) symmetric \( N \)-extended superconformal algebras and discuss the relationship between nonlinear and linear versions of these algebras. As a preparation to the \( N = 4 \) super \( W_3 \) algebra, I discuss in Section 3 the \( N = 2 \) superfield structure of the \( N = 4 \) superconformal algebra. In Section 4, I derive the couplings of the primaries of the \( N = 4 \) super \( W_3 \) algebra, and Section 5 contains some conclusions. The full structure of the action is given in an appendix.

2. Linear and nonlinear superconformal algebras

In this paper I follow the dual approach to \( W \)-algebras recently introduced in [13]. To illustrate this formalism in a simple setting, I first consider the generic nonlinear \( SO(N) \) symmetric superconformal algebra [14]. The generators of this algebra are the stress tensor \( T \), of spin 2, \( N \) supersymmetry generators \( X_a \), of spin \( \frac{3}{2} \), and a set of \( SO(N) \) currents \( J_{ab} \), of spin 1, where \( a, b \) run from 1 to \( N \). I also introduce a set of gauge potentials \( \mu \), of spin \( -1 \), \( \chi_a \), of spin \( -\frac{1}{2} \), and \( A_{ab} \), of spin 0.

Let the subalgebra \( \mathcal{H} \) described in the Introduction be the semidirect sum of the Virasoro algebra and the \( SO(N) \) affine Lie algebra. The nontrivial primary fields are then the \( X_a \), and there are no couplings possible between these primaries. Thus, the only nontrivial couplings are the \( \mathcal{H} \) dressings of the propagators, and the three dimensional action takes the following simple form

\[
S = \int_{\Sigma} \int dx \left( 2TF - 2X_a \nabla \chi_a + J_{ab} F_{ab} + \mu \mu'' + \chi_a \chi_a'' |_{\text{cov}} + \frac{1}{2} A_{ab} A'_{ab} \right) .
\]  

(2.1)

Here, \( F = d\mu + \mu \mu' \) is the field strength of \( \text{diff}S^1 \), \( \nabla \chi_a = d\chi_a - \frac{1}{2} \mu' \chi_a + \mu \chi'_a + QA_{ab} \chi_b \) is the covariant derivative of \( \chi_a \), and \( F_{ab} = dA_{ab} + \mu A_{ab}' + QA_{ac} A_{cb} \) is the \( SO(N) \) field.
strength. Primes denote derivatives with respect to the $S^1$ coordinate $x$, and $\chi''|_{\text{cov}}$ denotes a covariantized second derivative of $\chi_a$ which is given explicitly below. All fields are forms on the two dimensional surface $\Sigma$ with values in the tensor algebra of $\text{diff}S^1$, and the lagrangian is a spin 1 two-form on $\Sigma$ (modulo an inhomogeneous term in its transformation law which is the conformal anomaly).

The action (2.1) is a generalization of the Chern-Simons action for a finite dimensional Lie group and contains in fact the $SO(N)$ Chern-Simons action. It consists of a kinetic piece pairing the generators of the algebra and the gauge fields, and a potential describing the various couplings between the fields. The quadratic piece of the potential is diagonal and corresponds to the central extension of the algebra, while all higher couplings are contained in the covariantizations of $\chi_a\chi_a''|_{\text{cov}}$.

Ignoring for the moment the matter fields $X_a$, $\chi_a$, the equations of motion corresponding to (2.1) are

$$F = 0,$$  \hspace{1cm} (2.2)

$$F_{ab} = 0,$$  \hspace{1cm} (2.3)

$$\nabla T + \mu''' + \frac{1}{2} J_{ab} A'_{ab} = 0,$$  \hspace{1cm} (2.4)

$$\nabla J_{ab} + A'_{ab} = 0.$$  \hspace{1cm} (2.5)

Eqs. (2.2,3) are the Maurer-Cartan equations of $\text{diff}S^1 \oplus \text{Map}(S^1, SO(N))$ and eqs. (2.4,5) are the Ward identities of the central extension $\mathcal{H} = \text{Vir} \oplus \hat{SO}(N)$. Using (2.2,3), one easily shows that the terms $\int dx \mu'''$ and $\frac{1}{2} \int dx A_{ab} A'_{ab}$ appearing in (2.1) are closed two forms. They correspond to the Gel’fand-Fuchs cocycle and the central extension of affine $SO(N)$ respectively.

The explicit form of $\chi_a''|_{\text{cov}}$ is determined by demanding that the remaining piece of the lagrangian $\int dx \chi_a\chi_a''|_{\text{cov}}$ is likewise closed, still using the decoupled Ward identities (2.4,5). This is done in much the same way as in Yang-Mills theory, exploiting the nonhomogeneity of (2.4,5) to covariantize the derivatives in $\chi_a\chi_a''$. First, one calculates the exterior derivate of $\chi_a\chi_a''$ and observes that it can be cancelled by adding terms linear in the $\mathcal{H}$ currents. One then calculates the exterior derivative of these additional terms and cancels it by terms quadratic in the $\mathcal{H}$ currents. At this point the process must terminate since all $\mathcal{H}$ currents have dimension $\geq 1$, and in general the process will terminate as long as all $\mathcal{H}$ currents have positive dimension.

In this way one obtains

$$\int dx \chi_a\chi_a''|_{\text{cov}} = \int dx \left( \chi_a\chi_a'' + \frac{1}{2} \chi_a\chi_a T - 2Q\chi_a\chi_b J_{ab} - \frac{1}{8} \chi^2 J_{ab} J_{ab} + Q^2 \chi_a\chi_b J_{ac} J_{cb} \right).$$  \hspace{1cm} (2.6)
The $SO(N)$ coupling is determined by imposing the integrability condition $d^2 = 0$ on the equations of motion of the full action (2.1). This gives $Q^2 = -\frac{1}{4}$.

Notice that for $N = 2$ the terms quadratic in $J$ cancel. In this case, the algebra is in fact linear, and the action can be rewritten concisely in terms of superfields.

Let $\theta = \theta_1 + \theta_2$ and $\bar{\theta} = \theta_1 - \theta_2$ denote the chiral combinations of two fermionic coordinates $\theta_1, \theta_2$, and let $D = \partial_{\bar{\theta}} + \theta \partial_x$, $\bar{D} = \partial_{\theta} + \bar{\theta} \partial_x$ denote the corresponding superderivatives. Then $\phi' = \frac{i}{2} (\bar{D}D + D\bar{D})\phi$ is the $x$-derivative, while I define $\dot{\phi} = \frac{i}{2} (\bar{D}D - D\bar{D})\phi$. Let $\mu$ and $J$ be $N = 2$ superfields of spin $-\frac{1}{2}$ and $1$ respectively, and consider the following action

$$S = \int_{\Sigma} \int dx d\bar{\theta} d\theta (2JF + \mu \dot{\mu}) ,$$

where $F = d\mu + \mu \mu' + \frac{1}{2} D\mu D\mu$. The equations of motion corresponding to (2.7)

$$\nabla J + \dot{\mu}' ,$$

are easily seen to be integrable. Here, $\nabla J$ is the $N = 2$ supercovariant derivative, which for a field $\phi$ of spin $s$ and charge $q$ reads

$$\nabla \phi = d\phi + s \mu' \phi + \mu \dot{\phi} + \frac{1}{2} \bar{D}D\phi + \frac{1}{2} D\mu \bar{D}\phi + q \dot{\mu}\phi .$$

In $x$-space, (2.7) reduces to the original action (2.1) for $N = 2$.

For $N = 3$, a linear algebra [15] can be constructed in the same way using $N = 3$ superfields. The action is

$$S = \int_{\Sigma} \int dx d^3\theta (2\Lambda F + \mu D^3\mu) ,$$

where $\Lambda$ is a spin $\frac{1}{2}$ supercurrent, $F = d\mu + \mu \mu' + \frac{1}{4} D_a \mu D_a \mu$ and $D^3 = D_1 D_2 D_3$. One easily verifies that the equations of motion are integrable so that this action indeed defines a linear $N = 3$ superconformal algebra. The $x$-space action is

$$S = \int_{\Sigma} \int dx (2TF - 2X_a \nabla \chi_a + 2J_a F_a - 2\Lambda \nabla \lambda$$

$$+ \mu \mu'' + \chi_a \chi_a'' + A_a A_a' + \lambda^2$$

$$+ \frac{1}{2} \chi^2 T + i \epsilon_{abc} \chi_a \chi_b J_c + \lambda \chi_a J_a - \Lambda \chi_a A_a') .$$

The difference with (2.1) is that the terms quadratic in $J$ have been replaced by nonminimal couplings involving the spin $\frac{1}{2}$ fields $\Lambda, \lambda$. The $\mathcal{H}$ subalgebra is now the Virasoro algebra, so that $X_a, J_a$ and $\Lambda$ are all to be regarded as matter primaries. The last two terms in (2.12)
represent the three-point coupling of the primaries. The coefficients of these couplings are obtained by imposing the integrability of the equations of motion.

For \( N \geq 4 \), no simple superfield formalism is available since in the present approach the Gel’fand-Fuchs cocycle does not have an extension to \( N \geq 4 \) superspace. For \( N = 4 \), a linear algebra does exist [15] the most general form of which has been given in [16]. The action for the (untwisted) singly extended algebra of [15] is

\[
S = \int_{\Sigma} \int dx \left( 2TF - 2X_a \nabla \chi_a + J_{ab} F_{ab} - 2\Lambda_a \nabla \lambda_a + 2K \nabla B \\
+ \mu \mu'' + \chi_a \chi''_a + \frac{1}{2} A_{ab} A'_{ab} + \lambda^2 + BB' \\
+ \frac{1}{2} \chi^2 T - 2Q \chi_a \chi'_b J_{ab} \\
+ \lambda_a \chi_b J_{ab} - \Lambda_a \chi_b A'_{ab} + \lambda_a \chi_a K - \Lambda_a \chi_a B' \right),
\]

where the tilde denotes dualization of the \( SO(4) \) indices. The \( J^2 \) terms of (2.1) have now been replaced by terms involving the \( \Lambda_a \) of spin \( \frac{1}{2} \) and the \( U(1) \) current \( K \), with dual one-forms \( \lambda_a \) and \( B \) respectively. In this case, the superconformal algebra already has three independent couplings. These couplings are in general much harder to determine than the \( H \) covariantizations, and it would be extremely hard to construct a \( W_3 \) extension in this \( x \)-space formulation. As I show in the next section, the action (2.13) can be written in terms of \( N = 2 \) superfields, and in this formulation the entire potential is again just a sum of covariantized central extensions, with no nontrivial couplings. This formulation will be the basis of my construction of \( N = 4 \) \( W_3 \).

3. \( N = 2 \) superfields for the \( N = 4 \) superconformal algebra

The spin content of the \( N = 4 \) superconformal algebra of the previous section is \( (2)^1, (3/2)^4, (1)^7, (1/2)^4 \), while the spins of the corresponding one-forms are \( (-1)^1, (-1/2)^4, (0)^7, (1/2)^4 \). These spins can be grouped into \( N = 2 \) superfields as follows. For the one-forms, one defines a spin \(-1\) superfield \( \mu \), two spin \(-\frac{1}{2}\) superfields \( \chi, \bar{\chi} \) and a spin 0 superfield \( A \), where the highest component of \( A \) is the derivative of the \( U(1) \) connection \( B \), i.e. \( A \mapsto A + \theta \lambda_+ + \bar{\theta} \lambda_+ + \theta \bar{\theta} B' \). For the generators, one can then define a spin 1 superfield \( J \) (the \( N = 2 \) supercurrent of the previous section), two spin \( \frac{1}{2} \) superfields \( \Lambda, \bar{\Lambda} \) and a spin 0 superfield \( \Phi \), which reads explicitly \( \Phi \mapsto \partial^{-1} K + \theta \Lambda_+ + \bar{\theta} \Lambda_+ + \theta \bar{\theta} J \), where \( \partial^{-1} K \) denotes the inverse \( x \)-derivative of the \( U(1) \) current \( K \), and \( J \) is one of the \( SO(4) \) currents. The action will be local as long as \( \Phi \) occurs only through its (super)derivatives or multiplied by a factor the highest component of which is an \( x \)-derivative, as in the kinetic term \( \Phi \nabla A \).

The fermionic superfields can have a \( U(1) \) charge \( q \), which turns out to be 0, and a super \( U(1) \) charge \( Q \), which is the coupling to the super \( U(1) \) connection \( A \). The \( N = 2 \) superconformal algebra has a charge conjugation symmetry, which I will refer to as
"parity". $J$ and $\mu$ have negative and positive parity respectively, and in general bosonic generators and dual one-forms have opposite parity. $\Lambda$ and $\bar{\Lambda}$ are parity conjugates, as are $\chi$ and $\bar{\chi}$, and it turns out that $\Phi$ and $A$ have positive and negative parity respectively. The entire lagrangian has negative parity, and spin 0.

To write a central term for the spin 0 field $A$, one again has to use the inverse derivative. It is easy to check that the term $A\partial^{-1}\dot{A}$ reduces in $x$-space to the usual central terms for the component fields.

The action for the $N = 4$ superconformal algebra can now be written simply as

$$S = \int_{\Sigma} \int dx d\bar{\theta} d\theta \left( 2JF + \bar{\Lambda}\nabla\chi - \Lambda\nabla\bar{\chi} + 2\Phi\nabla A + \mu\dot{\mu}' + \bar{\chi}\chi|_{\text{cov}} + A\partial^{-1}\dot{A} \right),$$

(3.1)

where $\chi|_{\text{cov}}$ denotes the derivative covariantized with respect to the $H$ subgroup generated by $J$ and $\Phi$. The $H$ Ward identities are

$$\nabla J + \dot{\mu}' + \frac{1}{2}\bar{D}AD\Phi + \frac{1}{2}DA\bar{D}\Phi = 0,$$

(3.2)

$$\nabla\Phi + \partial^{-1}\dot{A} = 0,$$

(3.3)

and one easily obtains

$$\int dx \bar{\chi}\chi|_{\text{cov}} = \int dx \left( \frac{1}{2}\bar{\chi}\chi T + Q(\bar{\chi}\bar{D}\chi D\Phi + \bar{\chi}D\chi D\Phi - \bar{\chi}\Phi') \right),$$

(3.4)

where the charge is determined to be $Q^2 = \frac{1}{4}$. Notice that in this expression $\Phi$ only enters through its (super)derivatives, so that indeed no nonlocalities occur. The fact that $\Phi$ enters the Ward identities (3.2,3) only through its derivatives guarantees that this will be true in general.

In $x$-space, the action (3.1) expands to the action (2.13) of the previous section.

For comparison with the $W_3$ algebra discussed in the next section, I now discuss what happens if one omits the superfields $\Phi$, $A$. In that case, the integrability condition $d^2 = 0$ fails on $\chi$, and instead one has

$$d^2\chi = -\frac{1}{8}(\bar{\chi}'\chi - \bar{\chi}\chi' + \bar{D}\bar{\chi}D\chi + D\bar{\chi}\bar{D}\chi)\chi.$$

(3.5)

The terms between brackets are exactly the appropriate coupling of a spin 0, parity-odd superfield, and this is in fact how one deduces the parity of $A$.

If one tries to construct an $N = 4$ super $W_3$ algebra with the spin content expected for linearly realized supersymmetry, i.e. $(3)^1$, $(5/2)^4$, $(2)^6$, $(3/2)^4$, $(1)^1$, one encounters an exact analogue of eq. (3.5): $d^2$ vanishes on the spin $\frac{3}{2}$ superfields up to a term which can be cancelled by coupling to a parity-odd spin 0 superconnection. Thus, the entire $H$ subalgebra of the $N = 4$ superconformal algebra, generated by $J$ and $\Phi$, carries over to the $W_3$ algebra.
4. The $N=4$ super $W_3$ algebra

In this section, I show that a consistent set of couplings can be constructed for the primaries of the $N=4$ super $W_3$ algebra. I start with the canonical spin content of the last paragraph, and introduce the spin 0 superfield $\Phi$ when such becomes necessary. Thus, consider the $N=2$ superfields $J$, of spin 1 and negative parity, parity conjugates $X, \bar{X}$, of spin $\frac{3}{2}$ and charge $\pm q$, and $S$, of spin 2 and positive parity. The corresponding dual one-forms are the superfields $\mu$, of spin $-1$ and positive parity, parity conjugates $\psi, \bar{\psi}$, of spin $-\frac{3}{2}$ and charge $\pm q$, and $\nu$, of spin $-2$ and negative parity. At this point, the $H$ subgroup is just the $N=2$ superconformal algebra.

The following is a systematic way to determine the couplings between the primaries and their composites.

i): Write down general ansätze $d\nu_i \mapsto \sum \omega_{ij}$ for the derivatives of all primary one-forms $\nu_i$ in terms of the primary one-forms and currents. Here, $j$ labels different sectors, the terms in $\omega_{ij}$ only differing in their derivative structure and not in their field content.

ii): First, consider only the coupling of each primary to the $H$ subalgebra. Thus, set the $H$ covariant derivative of each primary to zero, and calculate $d\omega_{ij}$ for each $i, j$. The terms containing derivatives of the anomaly in the $H$ Ward identities can be cancelled by higher order terms, but all other terms should vanish. (For example, in the present case where $H$ is the $N=2$ superconformal algebra, all terms which do not contain $\dot{\mu}'$ or a derivative thereof should vanish.) This condition determines most, if not all, relative coefficients in each sector.

iii): Next, turn on the anomalies. That is, for each current $S_i$ set $\nabla S_i + D\nu_i = 0$, where $\nu_i$ is the one-form dual to $S_i$ and the differential operator $D$ is the propagator for $\nu_i$ determined by the central extension. Each coupling $d\nu_i \mapsto \omega_{ij}$ in the ansatz comes from a term $S_i\omega_{ij}$ in the potential, and gives rise to a term $D\nu_i\omega_{ij}$ in the derivative of the potential. Requiring the sum of these contributions to vanish gives relations between the coefficients in different sectors. This condition follows from the associativity of the operator product algebra.

iv): The final step also requires the knowledge of the rules $d\nu_0 \mapsto \omega_{0j}$ where $\nu_0$ denote the one-forms belonging to $H$. These follow from the first order covariantizations of the central terms. Then, calculate $d^2\nu_i$ using the full set of rules $d\nu_i \mapsto \sum \omega_{ij}$, and require these expressions to vanish. This determines all remaining parameters in the ansatz.

In the present case, the action has the form

$$S = \int \Sigma \int dx d\theta d\bar{\theta} \left( 2J\dot{F} + 2X\nabla \dot{\psi} - 2X\nabla \bar{\psi} + 2S\nabla \nu + \mu \dot{\mu}' + \bar{\psi} \dddot{\psi}'' - \frac{a}{2} \bar{\psi} \dddot{\psi}'' + \nu \dddot{\nu}'' + \text{higher couplings} \right),$$

(4.1)
where \( q \) and \( s = -\frac{3}{2} \) are the (unknown) charge and spin of \( \psi \). From the first order covariantizations of the central terms one obtains

\[
d\mu \mapsto \frac{1}{3}(q^2 - \frac{q}{2})\bar{\psi}\psi'' - \frac{1}{3}(q^2 - 3)\bar{\psi}'\psi' + \frac{1}{3}(q^2 - \frac{q}{4})\bar{\psi}''\psi + \frac{1}{6}q\bar{\psi}'\psi - \frac{1}{6}q\bar{\psi}'\psi' - \frac{1}{6}\bar{\psi}'\psi
\]

\[+ \frac{1}{3}(q + \frac{3}{2})\bar{\psi}\psi D\psi' - \frac{1}{3}(q + \frac{3}{2})\bar{\psi}'\psi D\psi' - \frac{1}{3}(q - \frac{3}{2})\bar{\psi}D\psi' + \frac{1}{3}(q - \frac{3}{2})\bar{\psi}'D\psi
\]

\[- 2\nu'' + 3\nu'' - \dot{\nu}'' - \frac{3}{2}D\nu D\nu' + 2D\nu' D\nu' - \frac{3}{2}D\nu D\nu'\]  

\[(4.2)\]

After step \( ii\) the ansätze are the following

\[d\psi \mapsto \alpha_0\psi''\nu + \alpha_1\psi'\nu
\]

\[= \left(\frac{1}{3}(q\alpha_0 - (q^2 - \frac{1}{2})\alpha_1)\psi'\nu + (-\alpha_0 + \frac{1}{2}q\alpha_1)\psi''\nu + \frac{1}{6}(\alpha_0 - q\alpha_1)\bar{\psi}'\nu - \frac{1}{6}\bar{\psi}'\nu'\right)
\]

\[+ \left(\frac{1}{3}(q\alpha_0 - \frac{1}{3}q\alpha_1)\psi''\nu + \frac{1}{3}(-q\alpha_0 + (q^2 - \frac{3}{4})\alpha_1)\psi'\nu\right)
\]

\[+ \left(\frac{1}{2}(\alpha_0 - (q + \frac{1}{2})\alpha_1)D\psi'\bar{\psi}\nu + \frac{1}{2}(\alpha_0 - (q - \frac{1}{2})\alpha_1)\bar{\psi}'D\psi\nu\right)
\]

\[+ \frac{1}{2}(-\alpha_0 + (q + \frac{1}{2})\alpha_1)D\psi\bar{\psi}\nu' + \frac{1}{2}(-\alpha_0 + (q - \frac{1}{2})\alpha_1)\bar{\psi}D\psi\nu',
\]

\[(4.3)\]

\[d\nu \mapsto \beta_0(\frac{1}{5}\nu''\nu + \frac{1}{10}\nu'\nu' + \frac{1}{10}\bar{\psi}\nu D\nu' - \frac{1}{10}\bar{\psi}'D\nu
\]

\[+ \gamma_0(\bar{\psi}'\psi' - \bar{\psi}'\psi'') + \gamma_1(\bar{\psi}'\psi' - \bar{\psi}'\psi') + \gamma_2D\psi'\bar{\psi}D\psi + \gamma_3\bar{\psi}D\psi\bar{\psi}D\psi\]  

\[(4.4)\]

modulo terms involving the zero-forms. The latter follow from the terms in the potential of degree \( \geq 2 \) in the zero-forms, which can be parametrized as

\[V_{\geq 2} = \zeta_0\nu\nu'\bar{\psi}\nu + \zeta_1(\nu\bar{\psi}\nu\bar{\psi}'\nu D\nu - \text{p.c.}) + \zeta_2(\nu\bar{\psi}\nu\bar{\psi}'\nu D\nu' - \text{p.c.})
\]

\[+ \eta_0\bar{\psi}\nu\bar{\psi}'\nu X + \theta_0(\nu\bar{\psi}'\nu S\bar{\psi}'\nu X - \text{p.c.})\]  

\[(4.5)\]

where p.c. denotes the parity conjugate. The \( \gamma_i \) and \( \zeta_i \) are restricted by the relations

\[-\frac{1}{2}\gamma_0 + (1 + q)\gamma_1 - (\frac{3}{2} - q)\gamma_3 = 0, -\frac{1}{2}\gamma_0 - (1 - q)\gamma_1 - (\frac{3}{2} + q)\gamma_2 = 0, \frac{1}{2}\zeta_0 - \zeta_1 - 4\zeta_2 = 0.
\]

Step \( iii\) determines the \( \gamma_i \) in terms of the \( \alpha_i \) and \( q \)

\[\gamma_0 = -\frac{1}{9}(q^2 - \frac{q}{4})(\alpha_0 - q\alpha_1),
\]

\[\gamma_1 = \frac{1}{18}(q^2 - \frac{q}{4})\alpha_1,
\]

\[\gamma_2 = \frac{1}{18}(q - \frac{3}{2})(\alpha_0 - \alpha_1),
\]

\[\gamma_3 = -\frac{1}{18}(q + \frac{3}{2})(\alpha_0 + \alpha_1),
\]

\[(4.6)\]

\[(4.7)\]

\[(4.8)\]

\[(4.9)\]

consistently with the above conditions on the \( \gamma_i \). The charge is determined in step \( iv\) to be \( q = -\frac{5}{2} \), up to a sign which corresponds to interchanging \( \psi \) and \( \bar{\psi} \). Continuing with step \( iv\) one finds

\[\alpha_0 = \frac{3}{2}\sqrt{-1}, \quad \alpha_1 = -\frac{3}{2}\sqrt{-1},
\]

\[8\]
\[ \beta_0 = 5\sqrt{-1}, \]
\[ \gamma_0 = \sqrt{-1}, \quad \gamma_1 = -\frac{1}{2}\sqrt{-1}, \quad \gamma_2 = -\sqrt{-1}, \quad \gamma_3 = 0, \]
\[ \zeta_0 = -12, \quad \zeta_1 = -6, \quad \zeta_2 = 0, \]
\[ \eta_0 = 12, \quad \theta_0 = 12, \]
\[ \text{where the sign ambiguity of } \sqrt{-1} \text{ corresponds to the freedom of flipping the signs of } \nu \text{ and } S. \]

With the above choice for the parameters, \( d^2 \) vanishes on all one-forms except on \( \psi \), where instead one finds

\[
d^2 \psi = \left( -3(\bar{\psi}\psi''' - \bar{\psi}'\psi' + \bar{\psi}''\psi' - \bar{\psi}'''\psi + \bar{\psi}\psi' - \bar{\psi}'\psi) \\
+ 5(\bar{\psi}\psi'' - \bar{\psi}'\psi' + \bar{\psi}''\psi + \bar{\psi}'\psi' - \bar{\psi}'\psi' + \bar{\psi}''\psi) \\
- 2(D\bar{\psi}D\psi'' - D\bar{\psi}'D\psi' + D\bar{\psi}''D\psi) \\
+ 8(D\bar{\psi}D\psi'' - D\bar{\psi}'D\psi' + D\bar{\psi}''D\psi) \right) \psi. \tag{4.11}
\]

Notice that the expression multiplying \( \psi \) has odd parity. Also, its highest component is a total \( x \)-derivative, and therefore it can potentially be cancelled by coupling \( \psi \) to the spin 0 superfield \( A \) of the \( N = 4 \) superconformal algebra, so that its covariant derivative becomes

\[
\nabla \psi = d\psi - \frac{3}{2} \mu' \psi + \mu \psi' - \frac{5}{2} \dot{\mu} + QA \psi. \tag{4.12}
\]

One finds that indeed the right hand side of (4.11) is cancelled by the first order covariantizations of the central terms for \( \bar{\psi}, \psi \) if the coupling constant \( Q = \sqrt{-6} \).

This completes the analysis of the elementary couplings between the primaries. To construct the full action is now a matter of covariantizing all terms in the lagrangian implied by these couplings. Although the result, which is given in an appendix, is extremely lengthy, this is a straightforward computation, and could presumably have been done by a machine.

5. Conclusion

In this paper I gave some examples of classical \( W \)-algebras in a dual formulation, and in particular constructed a new \( N = 4 \) super extension of the \( W_3 \) algebra. I illustrated how the explicit structure of \( W \)-algebras can be split into a set of couplings between generators which are primaries of a subalgebra \( \mathcal{H} \), and covariantizations with respect to this subalgebra. In practice, the couplings between the primaries are usually found to be much simpler in structure than the \( \mathcal{H} \) covariantizations, which can be extremely lengthy. I showed that the couplings can be constructed independently of the bulk of the \( \mathcal{H} \) covariantizations, and
although I have not proven this, the existence of a consistent set of couplings presumably implies the existence of the whole algebra.

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Appendix
In this appendix I give the complete lagrangian for the $N = 4$ super $W_3$ algebra of Section 4 in $N = 2$ superspace. Below, $Q = \sqrt{-6}$ denotes the charge of the fermionic superfields.

$$
\mathcal{L} = JF + \bar{X} \nabla \psi - X \nabla \bar{\psi} + S \nabla \nu + \Phi \nabla A
+ \frac{1}{2} A \partial^{-1} \dot{A} + \frac{1}{2} \mu \dot{\mu}'
+ \frac{1}{2} \bar{\psi} \dot{\psi}' - \frac{5}{6} \bar{\psi} \dot{\psi}'''
- \frac{11}{2} \bar{\psi} \dot{\psi}''J + \frac{5}{2} \bar{\psi} \dot{\psi}'' \dot{J} + \frac{1}{2} \bar{\psi} D \psi' \bar{D} J - 2 \bar{\psi} \bar{D} \psi' D J
- \frac{19}{4} \bar{\psi} \dot{\psi}' J - \frac{5}{12} \bar{\psi} \dot{\psi}' \dot{J} + \frac{5}{4} \bar{\psi} \dot{\psi}' J + \frac{1}{4} \bar{\psi} \dot{\psi} \dot{J}
+ \frac{1}{3} \bar{\psi} D \psi \bar{D} J - \frac{4}{3} \bar{\psi} D \psi D J' - \frac{4}{3} \bar{\psi} \dot{J}''
+ Q \left( \frac{1}{2} \bar{\psi} \bar{D} \psi'' D \Phi + \frac{1}{2} \bar{\psi} D \psi'' \bar{D} P + \frac{5}{2} \bar{\psi} \dot{\psi}'' \dot{\Phi} - \frac{1}{2} \bar{\psi} \dot{\psi}'' \Phi' \right.
+ \bar{\psi} D \psi' D \Phi' + \bar{\psi} D \psi' D \Phi' + \frac{5}{2} \bar{\psi} \dot{\psi}' \Phi' - \frac{1}{2} \bar{\psi} \dot{\psi}'' \Phi'
+ \frac{1}{2} \bar{\psi} \bar{D} \psi D \Phi' + \frac{1}{2} \bar{\psi} D \psi D \Phi'' + \frac{5}{6} \bar{\psi} \dot{\psi}' \dot{\Phi}' - \frac{1}{2} \bar{\psi} \dot{\psi}'' \Phi''
\left. + \frac{25}{8} \bar{\psi} \dot{\psi} J \dot{J}^2 + 3 \bar{\psi} \dot{\psi} J \dot{J} - \frac{14}{3} \bar{\psi} D \psi J D J + \frac{4}{3} \bar{\psi} D \psi J \bar{D} J \right)
- \frac{1}{2} \bar{\psi} \dot{\psi} J \dot{J} - 10 \bar{\psi} \dot{\psi} J \dot{J} + \frac{4}{3} \bar{\psi} \bar{D} J D J
+ Q \left( \frac{1}{2} \bar{\psi} \bar{D} \psi' J D \Phi + \frac{5}{2} \bar{\psi} D \psi' J \bar{D} \Phi \right.
+ \frac{11}{4} \bar{\psi} \dot{\psi} J \dot{J} \Phi - \frac{7}{2} \bar{\psi} \psi' J \Phi' - \frac{1}{2} \bar{\psi} \dot{\psi}' \bar{D} J D \Phi - 2 \bar{\psi} \phi' \bar{D} J D \Phi - \frac{5}{2} \bar{\psi} \psi' J \Phi
\left. + \frac{5}{2} \bar{\psi} \bar{D} \psi J D \Phi' + \frac{1}{2} \bar{\psi} D \psi J D \Phi' + 2 \bar{\psi} \phi \bar{D} \psi D J \Phi - \frac{1}{2} \bar{\psi} D \psi \bar{D} J \dot{\Phi} \right.
+ \frac{5}{4} \bar{\psi} \bar{D} \psi J' D \Phi + \frac{5}{4} \bar{\psi} D \psi J' D \Phi + \frac{1}{4} \bar{\psi} D \psi J \bar{D} \Phi + \frac{1}{4} \bar{\psi} D \psi J \bar{D} \Phi
+ \frac{1}{4} \bar{\psi} \dot{\psi} J \dot{J} \Phi' - \frac{7}{2} \bar{\psi} \psi' J \Phi'' - \frac{1}{2} \bar{\psi} \psi \bar{D} J \bar{D} \Phi' - 2 \bar{\psi} \phi D J \bar{D} \Phi'
+ \frac{5}{4} \bar{\psi} \dot{\psi} J' \Phi + \frac{1}{2} \bar{\psi} \psi' J \Phi - \frac{1}{4} \bar{\psi} \psi' J \Phi' - \frac{1}{4} \bar{\psi} \psi J \Phi'
- \frac{1}{4} \bar{\psi} \psi J D J \Phi' - \frac{1}{4} \bar{\psi} \psi D J' D \Phi')
- \frac{1}{4} \bar{\psi} \dot{\psi}'' \bar{D} \Phi D \Phi + \frac{5}{2} \bar{\psi} \psi' \bar{D} \Phi D \Phi
- \bar{\psi} \bar{D} \psi' D \Phi D \Phi + \bar{\psi} \bar{D} \psi' D \Phi D \Phi' - \frac{1}{4} \bar{\psi} D \psi' \bar{D} \Phi D \Phi' - \frac{1}{4} \bar{\psi} D \psi' \bar{D} \Phi \Phi
- \frac{1}{8} \bar{\psi} \psi' \bar{D} \Phi D \Phi' - \frac{13}{8} \bar{\psi} \psi' \bar{D} \Phi D \Phi' - \frac{5}{2} \bar{\psi} \psi' \bar{D} \Phi \Phi'^{2} + \frac{5}{2} \bar{\psi} \psi' \bar{D} \Phi'^{2} \right)

\begin{align*}
&+ \frac{3}{4} \bar{\psi} \bar{\psi} \bar{D} \Phi' D \Phi + \frac{1}{2} \bar{\psi} \bar{\psi} \bar{D} \Phi' D \Phi' + \frac{1}{8} \bar{\psi} \bar{\psi} \Phi'^2 - \frac{1}{8} \bar{\psi} \bar{\psi} \Phi^2 \\
&- \frac{3}{4} \bar{\psi} \bar{D} \psi (D \Phi' D \Phi' + D \Phi' D \Phi'' - D \Phi' \bar{D} \Phi' - D \Phi \bar{D} \Phi') \\
&- \frac{1}{3} \bar{\psi} D \Phi (D \Phi' \Phi' + D \Phi' \Phi' + \bar{D} \Phi' \Phi + \bar{D} \Phi \bar{D} \Phi') \\
&- \frac{4}{3} \bar{\psi} \psi (\bar{D} \Phi'' D \Phi + 2 \bar{D} \Phi' D \Phi' + \bar{D} \Phi D \Phi'') \\
&+ Q^2 (- \frac{1}{2} \bar{\psi} \bar{\psi}' \bar{D} \Phi D \Phi - \bar{\psi} \bar{D} \psi' \Phi D \Phi - \bar{\psi} \bar{D} \psi' \Phi \bar{D} \Phi) \\
&- \bar{\psi} \bar{\psi}' \bar{D} \Phi' D \Phi - \bar{\psi} \bar{\psi}' \bar{D} \Phi' D \Phi' - \frac{5}{2} \bar{\psi} \bar{\psi}' \Phi^2 + \bar{\psi} \bar{\psi}' \Phi + \frac{1}{2} \bar{\psi} \bar{\psi} \Phi^2 \\
&- \frac{1}{2} \bar{\psi} D \psi \Phi' D \Phi' - \frac{1}{6} \bar{\psi} D \psi \Phi' D \Phi - \bar{\psi} D \psi \Phi D \Phi' - \bar{\psi} D \psi \Phi D \Phi' \\
&- \frac{1}{2} \bar{\psi} \bar{D} \psi' \bar{D} \Phi' D \Phi' - \bar{\psi} \bar{D} \psi' \bar{D} \Phi' D \Phi' - \frac{1}{2} \bar{\psi} \bar{D} \psi' \bar{D} \Phi' D \Phi' \\
&- \frac{5}{2} \bar{\psi} \bar{\psi} \Phi' \Phi' + \frac{1}{2} \bar{\psi} \bar{\psi} \Phi' \Phi' + \bar{\psi} \bar{\psi} \Phi'' \Phi) \\
&- 8 \bar{\psi} \bar{\psi}' J^3 \\
&+ Q (3 \bar{\psi} \bar{D} \psi J^2 D \Phi + 3 \bar{\psi} D \psi J^2 D \Phi) \\
&- 3 \bar{\psi} \psi J^2 \Phi' + \frac{35}{3} \bar{\psi} \bar{\psi} J^2 \Phi - \frac{4}{3} \bar{\psi} \bar{\psi} J \bar{D} J D \Phi - \frac{14}{3} \bar{\psi} \bar{\psi} J \bar{D} J \bar{D} \Phi) \\
&+ 3 \bar{\psi} \psi J \bar{D} \Phi D \Phi - \frac{35}{3} \bar{\psi} \bar{\psi}' J \bar{D} \Phi D \Phi \\
&- \frac{7}{3} \bar{\psi} \bar{D} \psi (D \bar{J} \bar{D} \Phi D \Phi - J \bar{\Phi} D \Phi + J \bar{\Phi}' D \Phi) \\
&+ \frac{2}{3} \bar{\psi} \bar{D} \psi (\bar{D} J \bar{D} \Phi D \Phi - J \bar{\Phi} D \Phi - J \bar{\Phi}' D \Phi) \\
&- \frac{1}{3} \bar{\psi} \bar{D} \psi J \bar{D} \Phi D \Phi - \frac{5}{3} \bar{\psi} \bar{\psi} J \bar{D} \Phi D \Phi + \frac{2}{3} \bar{\psi} \bar{\psi} \bar{D} \phi (\Phi' - \bar{\Phi}) D \Phi + \frac{2}{3} \bar{\psi} \bar{\psi} \bar{D} \phi (\Phi' + \bar{\Phi}) D \Phi \\
&- \frac{16}{5} \bar{\psi} \bar{\psi} J \bar{D} \Phi' D \Phi - \frac{14}{5} \bar{\psi} \bar{\psi} J \bar{D} \Phi' D \Phi' - \frac{1}{3} \bar{\psi} \psi J \Phi'^2 + \frac{1}{3} \bar{\psi} \psi J \Phi^2 \\
&+ Q^2 (- \frac{5}{2} \bar{\psi} \bar{\psi}' J \bar{D} \Phi D \Phi - \frac{5}{2} \bar{\psi} \bar{D} \psi J \bar{\Phi} D \Phi - \frac{5}{2} \bar{\psi} \bar{D} \psi \bar{D} \Phi \bar{D} \Phi \\
&- \frac{5}{2} \bar{\psi} \psi J \bar{D} \Phi' D \Phi - \frac{5}{2} \bar{\psi} \psi J \bar{D} \Phi' D \Phi' + \frac{5}{2} \bar{\psi} \psi J \bar{\Phi}' - \frac{11}{2} \bar{\psi} \psi J \bar{\Phi}' \\
&+ \frac{5}{2} \bar{\psi} \bar{D} \psi J \bar{D} \Phi D \Phi + 2 \bar{\psi} \bar{D} \psi J \bar{D} \Phi D \Phi - \frac{5}{4} \bar{\psi} \bar{\psi} J \bar{D} \Phi D \Phi - \frac{5}{4} \bar{\psi} \bar{\psi} J \bar{D} \Phi D \Phi) \\
&+ Q \left( \frac{19}{4} \bar{\psi} \bar{\psi}' \bar{D} \Phi D \Phi - \frac{5}{4} \bar{\psi} \bar{\psi}' \bar{D} \Phi D \Phi \\
&+ \frac{3}{4} \bar{\psi} \bar{D} \psi D \Phi' \bar{D} \Phi D \Phi + \frac{1}{2} \bar{\psi} \bar{D} \psi D \Phi' \bar{D} \Phi D \Phi - \frac{9}{8} \bar{\psi} \bar{D} \psi \Phi'^2 D \Phi + \frac{1}{8} \bar{\psi} \bar{D} \psi \Phi'^2 D \Phi \\
&+ \bar{\psi} \bar{D} \psi \Phi' D \Phi + \frac{1}{4} \bar{\psi} D \psi \Phi' \bar{D} \Phi + \frac{1}{8} \bar{\psi} D \psi \Phi'^2 D \Phi + \frac{1}{8} \bar{\psi} D \psi \Phi'^2 D \Phi \\
&+ \frac{9}{4} \bar{\psi} \psi \Phi' D \Phi' D \Phi' + \frac{21}{12} \bar{\psi} \psi \Phi' D \Phi' D \Phi + \frac{5}{12} \bar{\psi} \psi \Phi' \bar{D} \Phi' D \Phi + \frac{7}{4} \bar{\psi} \psi \Phi \bar{D} \Phi D \Phi \\
&- \frac{5}{24} \bar{\psi} \psi \Phi^3 + \frac{5}{24} \bar{\psi} \psi \Phi'^2 + \frac{1}{8} \bar{\psi} \psi \Phi' \Phi'^2 - \frac{1}{8} \bar{\psi} \psi \Phi'^3 \right) \\
&+ Q^2 \left( \bar{\psi} \psi \Phi \bar{D} \Phi D \Phi + \frac{1}{2} \bar{\psi} D \psi \Phi'^2 D \Phi + \frac{1}{2} \bar{\psi} D \psi \Phi'^2 D \Phi \\
&+ \frac{1}{2} \bar{\psi} \psi \Phi' D \Phi D \Phi + \bar{\psi} \psi \Phi' \bar{D} \Phi' D \Phi + \bar{\psi} \psi \Phi \bar{D} \Phi D \Phi' - \frac{1}{2} \bar{\psi} \psi \Phi'^2 D \Phi' + \frac{5}{6} \bar{\psi} \psi \Phi^3 \right) \\
&- (12 + 3Q^2) \bar{\psi} \psi J^2 \bar{D} \Phi D \Phi \\
&+ (10Q + \frac{5}{2} Q^3) \bar{\psi} \psi J \bar{\Phi} \bar{D} \Phi D \Phi \\
&+ \left( \frac{1}{2} - \frac{15}{8} Q^2 - \frac{1}{2} Q^4 \right) \bar{\psi} \psi \Phi'^2 \bar{D} \Phi D \Phi + \left( - \frac{1}{2} - \frac{1}{8} Q^2 \right) \bar{\psi} \psi \Phi'^2 \bar{D} \Phi D \Phi \\
&+ \frac{1}{2} \bar{\nu} \nu''
\end{align*}
\begin{align*}
+ \nu \nu'' J + 3 \nu \bar{D} \nu'' D J + \nu \bar{J} + 4 \nu \bar{D} \nu' D J' + \frac{1}{2} \nu \bar{J}' + \frac{3}{2} \nu \bar{D} \nu D J'' \\
- \frac{1}{2} \nu \nu' J^2 - \nu \bar{D} \nu J D J - \frac{1}{2} \nu \nu J J' + 2 \nu \nu' J J - \frac{15}{2} \nu \nu' \bar{D} J D J \\
- \frac{1}{2} \nu \bar{D} \nu J D J' + 9 \nu \bar{D} \nu \bar{J} D J - \frac{3}{2} \nu \bar{D} \nu J' D J \\
+ \frac{1}{2} \nu \nu'' \bar{D} \Phi D \Phi + \frac{3}{2} \nu \bar{D} \nu'' (P' - \Phi) D \Phi \\
+ \frac{1}{2} \nu \nu' \bar{D} \Phi' D \Phi - \frac{1}{2} \nu \nu' \bar{D} \Phi D \Phi' + \frac{1}{2} \nu \nu' \Phi'^2 - \frac{1}{2} \nu \nu' \Phi^2 \\
+ 2 \nu \bar{D} \nu' (\Phi'' - \Phi') D \Phi + 2 \nu \bar{D} \nu' (\Phi' - \Phi) D \Phi' \\
+ \frac{1}{4} \nu \nu' \bar{D} \Phi'' D \Phi - \frac{1}{4} \nu \nu' \bar{D} \Phi D \Phi'' + \frac{1}{2} \Phi' \Phi'' - \frac{1}{2} \nu \nu' \Phi \Phi' \\
+ \frac{3}{2} \nu \bar{D} \nu (\Phi'' D \Phi + 2 \Phi'' D \Phi' + \Phi' D \Phi'' - \Phi'' D \Phi - 2 \Phi' D \Phi' - \Phi \Phi'') \\
- \nu \bar{D} \nu J^2 D J - \nu \nu' J^3 \\
- \frac{1}{2} \nu \nu' J \bar{D} \Phi D \Phi - \frac{1}{2} \nu \bar{D} \nu J (\Phi' - \Phi) D \Phi - \frac{1}{2} \nu \bar{D} \nu J (\Phi' - \Phi) D \Phi \\
+ \frac{3}{2} \nu \bar{D} \nu \bar{D} \Phi \Phi D \Phi - \frac{3}{2} \nu \bar{D} \nu \Phi \Phi D \Phi + \frac{3}{2} \nu \bar{D} \nu \Phi'^2 - \frac{9}{2} \nu \bar{D} \nu \Phi^2 \\
- \frac{1}{4} \nu \bar{D} \nu J (\Phi'' D \Phi + \Phi' D \Phi' - \Phi \Phi' D \Phi'') \\
- \frac{3}{2} \nu \nu' J^2 \bar{D} \Phi D \Phi - \nu \bar{D} \nu J D J \bar{D} \Phi D \Phi - \frac{1}{2} \nu \bar{D} \nu J^2 (\Phi' - \Phi) D \Phi \\
- \frac{23}{16} \nu \nu' \Phi^2 \bar{D} \Phi D \Phi + \frac{23}{16} \nu \nu' \Phi'^2 \bar{D} \Phi D \Phi \\
+ \frac{5}{8} \nu \bar{D} \nu (\Phi' - \Phi) D \Phi' \Phi D \Phi - \frac{1}{10} \nu \bar{D} \nu \Phi' \Phi^2 D \Phi - \frac{1}{10} \nu \bar{D} \nu \Phi \Phi'^2 D \Phi \\
+ \frac{9}{16} \nu \bar{D} \nu \Phi'^3 D \Phi + \frac{1}{10} \nu \bar{D} \nu \Phi^3 D \Phi \\
- 2 \alpha \hat{X} (\psi'' \nu - \psi' \nu - \frac{1}{2} \bar{D} \psi' \bar{D} \nu - \bar{D} \psi' \bar{D} \nu \\
+ \frac{13}{2} \psi' \nu + \frac{1}{2} \psi' \nu - \frac{1}{2} \psi \psi + \frac{1}{2} \psi \psi' \\
+ \frac{1}{3} \bar{D} \psi \bar{D} \nu' + \frac{2}{3} \bar{D} \psi \bar{D} \nu' \psi'' - \psi'' \nu \\
+ \frac{17}{3} \psi' \nu J - 3 \psi \nu J - \frac{5}{3} \bar{D} \psi \bar{D} \nu J - \frac{5}{3} \bar{D} \psi \bar{D} \nu J \\
+ \frac{13}{3} \bar{D} \psi \bar{D} \nu D J - \frac{5}{3} \bar{D} \psi \bar{D} \nu D J - \frac{11}{3} \psi \bar{D} \nu D J - \frac{7}{3} \psi \bar{D} \nu D J \\
\frac{7}{3} \psi \nu J - \psi \nu J - \frac{5}{3} \psi \nu J - \psi \nu J \\
+ Q (D \psi' \nu D \Phi - D \psi' \nu D \Phi - 2 \psi' \nu \Phi + \psi' \nu \Phi' + \psi \nu \Phi \\
- D \psi \nu D \Phi' - D \psi \nu D \Phi' + \psi \nu \Phi'' - \psi \nu \Phi' \\
+ \frac{1}{2} \psi' \bar{D} \nu \bar{D} \Phi - \psi' \bar{D} \nu \bar{D} \Phi + \bar{D} \psi \bar{D} \nu \bar{D} \Phi + \frac{1}{2} \bar{D} \psi \bar{D} \nu \bar{D} \Phi \\
- \frac{1}{4} \bar{D} \psi \bar{D} \nu \bar{D} \Phi - \frac{1}{4} \bar{D} \psi \bar{D} \nu \bar{D} \Phi + \frac{1}{4} \bar{D} \psi \bar{D} \nu \bar{D} \Phi + \frac{1}{4} \bar{D} \psi \bar{D} \nu \bar{D} \Phi 
\end{align*}
\[-\frac{13}{12} \psi \dot{\phi} - \frac{1}{4} \psi \phi' + \frac{1}{4} \dot{\psi} \phi' - \frac{1}{4} \psi \phi' \]
\[+ \frac{7}{2} \psi \ddot{D} \nu D \phi' - \psi D \nu \ddot{D} \phi' - \frac{1}{3} \psi \ddot{D} \nu' D \phi + \frac{2}{3} \psi D \nu' \ddot{D} \phi \]
\[+ 8 \psi \nu J^2 \]
\[+ Q(-3 \dot{D} \psi \nu J D \phi - 3D \psi \nu J D \phi + \frac{5}{3} \dot{D} \psi \nu J D \phi - \frac{5}{3} \psi \nu D \nu J D \phi \]
\[- \frac{17}{2} \psi \nu J \phi + 3 \psi \nu J \phi' + \frac{13}{2} \psi \nu J D \phi + \frac{5}{3} \psi \nu \ddot{D} J D \phi \]
\[+ \frac{4}{3} \psi \nu D \phi' D \phi + \frac{1}{3} \psi \nu D \phi D \phi' - \frac{5}{6} \psi \nu \phi'^2 + \frac{5}{6} \psi \nu \phi^2 \]
\[+ Q^2(\psi' s D \phi D \phi + \frac{1}{3} \psi \nu \ddot{D} \phi D \phi - \frac{1}{4} \psi \nu' D \phi D \phi \]
\[+ \ddot{D} \psi \nu \ddot{D} \phi + D \psi \nu \ddot{D} \phi - \frac{1}{2} \psi \nu \ddot{D} \nu D \phi - \psi \nu \ddot{D} \phi \]
\[+ \psi \nu \ddot{D} \phi' D \phi + \psi \nu \ddot{D} \phi D \phi' + \psi \nu \phi'^2 - \psi \nu \phi \phi' \]
\[+ (8 + 3Q^2)\psi \nu J \ddot{D} \phi D \phi \]
\[- \frac{3}{2} Q \psi \nu (\dot{\phi} + \phi') \ddot{D} \phi D \phi - Q^3 \psi \nu \ddot{D} \phi D \phi \]
\[+ \beta_0(\frac{1}{2} \nu \nu' S + \frac{1}{2} \nu \ddot{D} \nu' D S + \frac{1}{4} \nu \nu' S' + \frac{5}{20} \nu \nu' \ddot{S} + \frac{3}{10} \nu \nu' D S' \]
\[+ \frac{7}{10} \nu \nu' J S + \frac{9}{10} \nu \ddot{D} \nu S D J - \frac{1}{10} \nu \ddot{D} \nu J D S \]
\[+ \frac{7}{20} \nu \nu' D \phi D \phi S + \frac{9}{20} \nu \ddot{D} \nu (\phi' - \dot{\phi}) D \phi S - \frac{1}{20} \nu \ddot{D} \nu D \phi D \phi D \phi D \phi S \]
\[+ \gamma_0(-\frac{4}{3} \ddot{\psi} \psi S + \frac{2}{3} \ddot{\psi} \ddot{D} \psi D S - \ddot{\psi} \psi S' + \frac{1}{3} \ddot{\psi} \ddot{S} - 4 \dot{\psi} \ddot{S} \]
\[+ Q(\frac{1}{3} \ddot{\psi} \ddot{\phi} S - \frac{2}{3} \ddot{\psi} \ddot{D} \phi D S) - 2 \ddot{\psi} \ddot{\phi} \phi D \phi S \]
\[+ \zeta_0(\frac{1}{2} \nu \nu' X X - \frac{1}{2} \nu \ddot{D} \nu X D X - \frac{1}{2} Q \nu \ddot{D} \nu D \phi X X) \]
\[+ \frac{1}{2} \eta_0 \ddot{\psi} \ddot{x} D X + \theta_0 \nu \ddot{S} X \]
\[- p.c. \]

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