Ising model on Cayley trees: a new class of Gibbs measures and their comparison with known ones

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Abstract. For the Ising model on Cayley trees we give a very wide class of new Gibbs measures. We show that these new measures are extreme under some conditions on the temperature. We give a review of all known Gibbs measures of the Ising model on trees and compare them with our new measures.

Keywords: classical phase transitions
1. Introduction

The well known nearest-neighbor (n.n.) Ising model on the Cayley tree still offers new interesting phenomena (see e.g. [4–6, 12] for recent results). Here we widely extend the set of known Gibbs measures of this model.

The Cayley tree \( \Gamma^k \) of order \( k \geq 1 \) is an infinite tree, i.e. a connected graph without cycles, such that exactly \( k+1 \) edges originate from each vertex. Let \( \Gamma^k = (V, L) \) where \( V \) is the set of vertices and \( L \) the set of edges. Two vertices \( x \) and \( y \) are called nearest neighbors if there exists an edge \( l \in L \) connecting them. We will use the notation \( l = \langle x, y \rangle \). A collection of distinct nearest neighbor pairs \( \langle x, x_1 \rangle, \langle x_1, x_2 \rangle, ..., \langle x_{d-1}, y \rangle \) is called a path from \( x \) to \( y \). The distance \( d(x, y) \) on the Cayley tree is the number of edges of the shortest path from \( x \) to \( y \).

For a fixed \( x^0 \in V \), called the root, we set
\[
W_n = \{ x \in V \mid d(x, x^0) = n \}, \quad V_n = \bigcup_{m=0}^{n} W_m
\]
and denote
\[
S(x) = \{ y \in W_{n+1} : d(x, y) = 1 \}, \quad x \in W_n,
\]
the set of direct successors of \( x \).

The n.n. Ising model is then defined by the formal Hamiltonian
\[
H(\sigma) = -J \sum_{\langle x, y \rangle \in V} \sigma(x)\sigma(y). \tag{1.1}
\]

Here the first sum runs over n.n. vertices \( \langle x, y \rangle \), the spins \( \sigma(x) \) take values \( \pm 1 \), and the real parameter \( J \) stands for the interaction energy.

The (finite-dimensional) Gibbs distributions over configurations at inverse temperature \( \beta = 1/T \) and general boundary fields are defined by
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\[ \mu_n(\sigma_n) = Z_n^{-1}(h) \exp \left\{ \beta J \sum_{(x,y) \subseteq V_n} \sigma(x)\sigma(y) + \sum_{x \in W_n} h_x \sigma(x) \right\} \]  

with partition functions given by

\[ Z_n(h) = \sum_{\sigma_n} \exp \left\{ \beta J \sum_{(x,y) \subseteq V_n} \sigma(x)\sigma(y) + \sum_{x \in W_n} h_x \sigma(x) \right\}. \]

Here the spin configurations \( \sigma_n \) belong to \( \{-1, +1\}^{V_n} \) and \( h = \{h_x \in \mathbb{R}, \ x \in V\} \) is a collection of real numbers that stands for a (generalized) boundary condition.

The probability distributions (1.2) are said to be compatible if for all \( \sigma_{n-1} \)

\[ \sum_{\omega_n} \mu_n(\sigma_{n-1}, \omega_n) = \mu_{n-1}(\sigma_{n-1}) \]

where the configurations \( \omega_n \) belong to \( \{-1, +1\}^{W_n} \).

It is well known (see chapter 2 of [12] for a detailed proof) that this compatibility condition is satisfied if and only if for any \( x \in V \) the following equation holds

\[ h_x = \sum_{y \in S(x)} f_\theta(h_y), \]

where

\[ \theta = \tanh(\beta J), \quad f_\theta(h) = \arctanh(\theta \tanh h). \]

Namely, for any boundary condition satisfying the functional equation (1.6) there exists a unique Gibbs measure which is called a splitting Gibbs measure (SGM). Moreover, according to [8, theorem 12.6], any extreme Gibbs measure is SGM; since each Gibbs measure can be represented by extreme ones, the full description problem of Gibbs measures of the Ising model is reduced to the description the class of SGMs. Therefore, in this paper we consider only SGMs and omit the word ‘splitting’.

A boundary condition satisfying (1.6) is called compatible.

The paper is organized as follows. The results are given in section 2. Section 3 contains a review of all known Gibbs measures of the Ising model on Cayley trees and their comparison with the new measures of this paper. Proofs are given in section 4.

2. Results

Here we consider the half-tree. Namely the root \( x^0 \) has \( k \) nearest neighbors.

We construct below new solutions of the functional equation (1.6). Consider the following matrix
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\begin{align*}
M &= \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 \\
a_2 & a_1 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4 \\
b_2 & b_1 & b_4 & b_3
\end{pmatrix},
\end{align*}

where \( a_i, b_j \) are non-negative integers and
\[ a_1 + a_2 + a_3 + a_4 = k, \quad b_1 + b_2 + b_3 + b_4 = k. \] (2.1)

This matrix defines the number of times the values \( \pm h, \pm l \) occur in the set \( S(x) \) for each \( h_x \in \{\pm h, \pm l\} \). More precisely, the boundary condition \( h = \{h_x, x \in V\} \) with fields taking values \( \pm h, \pm l \) defined by the following steps:

(i) if at vertex \( x \) we have \( h_x = h \), then the function \( h_y \), which gives real values to each vertex \( y \in S(x) \) by the following rule
\[
\begin{aligned}
&h \text{ on } a_1 \text{ vertices of } S(x); \\
&-h \text{ on } a_2 \text{ of the remaining vertices;} \\
&l \text{ on } a_3 \text{ of the remaining vertices;} \\
&-l \text{ on } a_4 \text{ of the remaining vertices.}
\end{aligned}
\]

(ii) if at vertex \( x \) we have \( h_x = l \), then the function has the values
\[
\begin{aligned}
&h \text{ on } b_1 \text{ vertices of } S(x); \\
&-h \text{ on } b_2 \text{ of the remaining vertices;} \\
&l \text{ on } b_3 \text{ of the remaining vertices;} \\
&-l \text{ on } b_4 \text{ of the remaining vertices.}
\end{aligned}
\]

If at vertex \( x \) we have \( h_x = -h \) (resp. \( -l \)) then we multiply the above formulas by \( -1 \).

(See figure 1 for an example of such a function.)

It is easy to see that the boundary conditions in the above construction are compatible iff \( h \) and \( l \) satisfy the following system of equations:
\[
\begin{aligned}
h &= (a_1 - a_2) f_\theta(h) + (a_3 - a_4) f_\theta(l) \\
l &= (b_1 - b_2) f_\theta(h) + (b_3 - b_4) f_\theta(l),
\end{aligned}
\] (2.2)

where \( a_i \) and \( b_i \) are given in matrix \( M \).

Denote
\[
a = a_1 - a_2, \quad b = a_3 - a_4, \quad c = b_1 - b_2, \quad d = b_3 - b_4. \] (2.3)

By condition (2.1) we have \( a, b, c, d \in \{-k, -k + 1, \ldots, k - 1, k\} \). Then the system (2.2) has the form
\[
\begin{aligned}
h &= af_\theta(h) + bf_\theta(l) \\
l &= cf_\theta(h) + df_\theta(l).
\end{aligned}
\] (2.4)

**Theorem 1.** Independently of the parameters the system of equation (2.4) has solution \((0, 0)\), and if \( |(bc - ad)\theta^2 + (a + d)\theta| > 1 \) then there are at least three distinct solutions \((0, 0), (\pm h_*, \pm l_*)\), where \( h_*, l_* > 0 \).
As was mentioned above, for any boundary condition satisfying the functional equation (1.6) there exists a unique Gibbs measure, thus by the solutions \((h, l)\) mentioned in theorem 1, we can construct new Gibbs measures, denoted by \(\mu_{h,l}\). These measures also depend on the choice of the value of the root, and differ in cases of non-uniqueness of theorem 1.

**Theorem 2.** Let \(\theta > 0\) (i.e. \(J > 0\), the ferromagnetic Ising model) then

1. If \(hl = 0\) then the corresponding measure \(\mu_{h,l}\) is extreme for \(\theta \in (\frac{1}{k}, \frac{1}{\sqrt{k}})\), where the measure \(\mu_{h,l}\) exists;
2. The measures \(\mu_{h,l}\), with \(h > 0, l > 0\), are extreme as soon as they exist.

Proofs are given in section 4.

### 3. Relation of the measures \(\mu_{h,l}\) to known ones

**Translation invariant measures.** (see e.g. [3, 8, 11]) Such measures correspond to \(h_x \equiv h\), i.e. constant functions. These measures are particular cases of our measures mentioned in theorem 2 which can be obtained for \(a = a_1 - a_2 = k\), i.e. \(a_1 = k, a_2 = a_3 = a_4 = 0\).

In this case the condition (1.6) reads

\[
h = kf_\theta(h).
\]

The equation (3.1) has a unique solution \(h = 0\), if \(\theta \leq \theta_c = 1/k\) and three distinct solutions \(h = 0, \pm h_* (h_* > 0)\), when \(\theta > \theta_c\).

Let us denote by \(\mu_0, \mu_\pm\) the corresponding Gibbs measures and recall the following known results for the ferromagnetic Ising model (\(\theta \geq 0\)):

1. If \(\theta \leq \theta_c\), \(\mu_0\) is unique and extreme.
2. If \(\theta > \theta_c\), \(\mu_-\) and \(\mu_+\), are extreme.
3. \(\mu_0\) is extreme if and only if \(\theta < 1/\sqrt{k}\).
Let $h$ be a boundary condition satisfying (1.6) on $\Gamma^{k_0}$. For $k \geq k_0 + 1$ define the following boundary condition on $\Gamma^k$:
\[
\tilde{h}_x = \begin{cases} 
    h_x, & \text{if } x \in V^{k_0} \\
    0, & \text{if } x \in V^k \setminus V^{k_0},
\end{cases} 
\]
(3.2)
where $V^k$ denote the set of vertex of $\Gamma^k$. Namely, to each vertices of $V^{k_0}$ one adds $k - k_0$ successors with vanishing value of the boundary condition. It is obvious the b.c. $\tilde{h}$ satisfies the compatibility condition (1.6). In this way one constructs a new set of Gibbs measures that are extreme in the range $1/k_0 < \theta < 1/\sqrt{k}$ (see [1] for details).

In case $h$ is translation invariant on $\Gamma^{k_0}$ then the corresponding measures of this construction can be obtained by theorem 1 for $a_1 = k_0$, $a_2 = b_1 = b_2 = 0$, $a_3 + a_4 = k - k_0$ and $l = 0$. (See figure 2 for an example.)

But in the case when $h$ is not translation invariant, the measures of ART do not coincide with measures of theorem 1.

**Bleher-Ganikhodjaev construction.** Consider an infinite path $\pi = \{x^0 = x_0 < x_1 < \ldots\}$ on the half Cayley tree (the notation $x < y$ meaning that paths from the root to $y$ go through $x$). Associate to this path a collection $h^\pi$ of numbers given by the condition
\[
h^\pi_x = \begin{cases} 
    -h_x, & \text{if } x \prec x_n, x \in W_n, \\
    h_x, & \text{if } x_n \prec x, x \in W_n, \\
    h_{x_n}, & \text{if } x = x_n.
\end{cases} 
\]
(3.3)
$n = 1, 2, \ldots$ where $x \prec x_n$ (resp. $x_n \prec x$) means that $x$ is on the left (resp. right) from the path $\pi$ and $h_{x_n} \in [-h_x, h_x]$ are arbitrary numbers. For any infinite path $\pi$, the collection of numbers $h^\pi$ satisfying relations (1.6) exists and is unique (see [2]).

A real number $t = t(\pi)$, $0 \leq t \leq 1$ can be assigned to the infinite path and the set $h^{\pi(t)}$ is uniquely defined. The set of numbers $h^{\pi(t)}$ being distinct for different $t \in [0, 1]$, it is also the case for the corresponding Gibbs measures. One thus obtains uncountably many Gibbs measures and they are extreme. For each fixed $t$ the ground state configuration of such measure contains a unique interface path $\pi(t)$. Using theorem 1
we can construct new classes of measures which have infinitely many interface paths. Let us give these measures precisely:

Let \( k \geq 2, a_1 \geq 2 \) such that \( k - a_1 \) an even positive integer. In theorem 1 take \( l = h \) and \( a_2 + a_4 = a_3, b_i = a_i, i = 1, 2, 3, 4 \) and \( l = h \). The bold paths are going to infinity and each is an interphase path (separating ‘+’ and ‘−’ values).

Periodic Gibbs measures. Let \( G_k \) be a free product of \( k + 1 \) cyclic groups of the second order with generators \( a_1, a_2, \ldots, a_{k+1} \), respectively.

It is known that there exists an one-to-one correspondence between the set of vertices \( V \) of the Cayley tree \( \Gamma_k \) and the group \( G_k \).

**Definition 1.** Let \( \tilde{G} \) be a normal subgroup of the group \( G_k \). The set \( h = \{h_x : x \in G_k\} \) is said to be \( \tilde{G} \)-periodic if \( h_y x = h_x \) for any \( x \in G_k \) and \( y \in \tilde{G} \).

Let \( G_k^{(2)} = \{x \in G_k : \text{the length of word } x \text{ is even}\} \).

Note that \( G_k^{(2)} \) is the set of even vertices (i.e. with even distance to the root). Consider the boundary conditions \( h^\pm \) and \( h^\mp \):

\[
h^\pm_x = \begin{cases} h_x, & \text{if } x \in G_k^{(2)} \\ -h_x, & \text{if } x \in G_k \setminus G_k^{(2)} \end{cases}
\]

and denote by \( \mu^{(\pm)} \) the corresponding Gibbs measures.

**Figure 3.** This is an example of the function \( h_x \) on the vertices of the Cayley tree of order 5. Here \( a_1 = 4, a_2 = 1, a_3 = a_4 = 0 \). One can take the same function for \( a_1 = 3, a_2 = 0, a_3 = a_4 = 1, b_i = a_i, i = 1, 2, 3, 4 \) and \( l = h \). The bold paths are going to infinity and each is an interphase path (separating ‘+’ and ‘−’ values).
The $\tilde{G}$- periodic solutions of equation (1.6) are either translation-invariant ($G_k$-periodic) or $G_k^{(2)}$-periodic (see [7]), they are solutions to
\begin{equation}
  u = kf_0(v), \quad v = kf_\theta(u). \tag{3.5}
\end{equation}

In the ferromagnetic case only translation invariant b.c. can be found. In the anti-ferromagnetic case ($\theta \leq 0$) the system (3.5) has a unique solution $h = 0$ if $\theta \geq -1/k$, and three distinct solutions $h = 0$, $h^\pm$ if $\theta < -1/k$.

Let us also recall that for the antiferromagnetic Ising model:

(1) If $\theta \geq -1/k$, $\mu_0$ is unique and extreme.
(2) If $\theta < -1/k$, $\mu^{(\pm)}$ and $\mu^{(\mp)}$, are extreme.

see [8, 12].

We note that these measures are particular cases of measures of theorem 1 which can be obtained for $a_1 = 0$, $a_2 = \varepsilon$, i.e. $a_3 = a_4 = 0$. (See figure 4, for $k = 3$).

Weakly periodic Gibbs measures. Following [13, 14], recall the notion of weakly periodic Gibbs measures.

Let $G_k/\hat{G}_k = \{H_1, ..., H_r\}$ be a factor group, where $\hat{G}_k$ is a normal subgroup of index $r \geq 1$.

**Definition 2.** A set $h = \{h_x, x \in G_k\}$ is called $\hat{G}_k$—weakly periodic, if $h_x = h_{x^\downarrow}$, for any $x \in H_i$, $x^\downarrow \in H_j$, where $x^\downarrow$ denotes the ancestor of $x$.

Weakly periodic b.c. $h$ coincide with periodic ones if $h_x$ is independent of $x^\downarrow$.

We recall results known for the cases of index two. Note that any such subgroup has the form
\begin{equation}
  H_A = \left\{ x \in G_k : \sum_{i \in A} \omega_x(a_i) \quad \text{is even} \right\}, \tag{3.6}
\end{equation}

where $\emptyset \neq A \subseteq N_k = \{1, 2, \ldots, k + 1\}$, and $\omega_x(a_i)$ is the number of $a_i$ in a word $x \in G_k$.

We consider $A \neq N_k$; when $A = N_k$ weak periodicity coincides with standard periodicity.

Let $G_k/H_A = \{H_0, H_1\}$ be the factor group, where $H_0 = H_A$, $H_1 = G_k \setminus H_A$. Then, in view of (1.6), the $H_A$-weakly periodic b.c. has the form
\begin{equation}
  h_x = \begin{cases} 
  h_1, & x \in H_0, \ \ x^\downarrow \in H_0, \\
  h_2, & x \in H_0, \ \ x^\downarrow \in H_1, \\
  h_3, & x \in H_1, \ \ x^\downarrow \in H_0, \\
  h_4, & x \in H_1, \ \ x^\downarrow \in H_1,
\end{cases} \tag{3.7}
\end{equation}

where the $h_i$ satisfy the following equations:
\begin{equation}
  \begin{cases} 
  h_1 = |A| f_\theta(h_3) + (k - |A|) f_\theta(h_1), \\
  h_2 = (|A| - 1) f_\theta(h_3) + (k + 1 - |A|) f_\theta(h_1), \\
  h_3 = (|A| - 1) f_\theta(h_2) + (k + 1 - |A|) f_\theta(h_4), \\
  h_4 = |A| f_\theta(h_2) + (k - |A|) f_\theta(h_4). \tag{3.8}
\end{cases}
\end{equation}

It is obvious that the following sets are invariant with respect to the operator $W : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defined by RHS of (3.8):

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\[ I_1 = \{h \in \mathbb{R}^4 : h_1 = h_2 = h_3 = h_4\}, \quad I_2 = \{h \in \mathbb{R}^4 : h_1 = h_4; h_2 = h_3\}, \]

\[ I_3 = \{h \in \mathbb{R}^4 : h_1 = -h_4; h_2 = -h_3\}. \]

It is obvious to see that

– measures corresponding to solutions on \( I_1 \) are translation invariant, i.e. particular cases of the measures given in theorem 1.

– measures corresponding to solutions on \( I_2 \) are weakly periodic, which coincide with the measures given in theorem 1 for \( a_1 = k - |A|, a_2 = 0, a_3 = |A|, a_4 = 0, b_1 = k + 1 - |A|, b_2 = 0, b_3 = |A| - 1, b_4 = 0. \)

– measures corresponding to solutions on \( I_3 \) are weakly periodic, which coincide with the measures given in theorem 1 for \( a_1 = k - |A|, a_2 = 0, a_3 = 0, a_4 = |A|, b_1 = k + 1 - |A|, b_2 = 0, b_3 = 0, b_4 = |A| - 1. \) (See figure 5.)

Moreover, the system (3.8) was solved only in cases \( |A| = 1 \) and \( |A| = k \) (see [13, 14]). Thus theorem 1 gives, in particular, new weakly periodic measures.

**Remark 1.** The remaining known Gibbs measures are called: Zachary measures (see e.g. part (b) of theorem 12.31 in [8]), Higuchi’s non-translation-invariant measures (see [9]), Alternating Gibbs measures (see [6]) and weakly periodic measures for subgroups of index 4 (see chapter 2 of [12] for details). All these measures correspond to functions \( h_x \) with more than 4 distinct values. Thus these measures are different from the measures mentioned in theorem 1.
4. Proofs

4.1. Proof of theorem 1

Proof. To prove the theorem we shall use the following (simply checked) properties of the function \( f_\theta(x) \).

**Lemma 1.** The function \( f_\theta \) has the following properties:

1. \( f_\theta(-x) = -f_\theta(x) \), i.e. it is odd function of \( x \);
2. \( f_{-\theta}(x) = -f_\theta(x) \), i.e. it is odd function of \( \theta \);
3. \( \lim_{x \to \infty} f_\theta(x) = \text{arctanh}(\theta) \);
4. \( \frac{d}{dx} f_\theta(0) = \theta, \quad 0 < \frac{d}{dx} f_\theta(x) \leq \theta, \quad \theta > 0 \);
5. \( \frac{d^2}{dx^2} f_\theta(x) < 0, \quad x > 0, \quad \theta > 0 \).
6. The equation \( h = m f_\theta(h) \) (where \( m \geq 1, \theta \in (-1, 1) \)) has unique solution \( h = 0 \), if \( -1 < \theta \leq \frac{1}{m} \), and three solutions \( h = 0, \pm h_*, h_* > 0 \), if \( \frac{1}{m} < \theta < 1 \).
If \( a = b = c = d = 0 \) then the system (2.4) has the unique solution \((h,l) = (0,0)\). Thus to have a non-zero solution it is necessary to have a non-zero parameter, i.e. \( a^2 + b^2 + c^2 + d^2 > 0 \). Since \( h \) and \( l \) play a symmetric role (up to a renaming of parameters), it suffices to consider the following cases:

(1) \( a = b = 0 \). In this case from system (2.4) we get \( h = 0 \) and \( l = df_\theta(l) \). Part 2 of lemma 1 allows us to assume \( d \geq 0 \) only (otherwise we can change \( \theta \) by \(-\theta\)). The case \( d = 0 \) gives \( l = 0 \). So it remains \( d \in \{1, \ldots, k\} \). By part 6 of lemma 1 we get (for \( a = b = 0 \)) the system (2.4) has

- unique solution \((0,0)\), if \( d = 0 \) or \(-1 < \theta \leq \frac{1}{2}\);
- three solutions \((0,0), (\pm l_\ast), l_\ast > 0\), if \( \frac{1}{d} < \theta < 1 \), \( d \neq 0 \).

(2) \( a = 0, b \neq 0 \). In this case from the first equation of the system (2.4) we get \( h = bf_\theta(l) \). Then from the second equation we obtain

\[
l = g(l) = cf_\theta(bf_\theta(l)) + df_\theta(l). \tag{4.1}
\]

Using lemma 1 one can see that \( g(0) = 0, g(-l) = -g(l), g'(0) = bcd^2 + d\theta \) and \( g(l) \) is a bounded function of \( l \). Moreover, if \( |g'(0)| > 1 \) (i.e. 0 is unstable fixed point of \( g \)) then there is a sufficiently small neighborhood of \( l = 0: (-\varepsilon, +\varepsilon) \) such that \( g(l) < l, \) for \( l \in (-\varepsilon, 0) \) and \( g(l) > l, \) for \( l \in (0, +\varepsilon) \). For \( l \in (0, \varepsilon) \) the iterates \( g^{(n)}(l) \) remain > 0, monotonically increase and hence converge to a limit, \( l_\ast \geq 0 \) which solves (4.1). However, \( l_\ast > 0 \) as 0 is unstable. Then since \( g \) is odd function of \( l, -l_\ast \) also solves (4.1). Thus

- If \( |g'(0)| = |bcd^2 + d\theta| > 1 \) then the system (2.4) has at least three solutions:

\[
(0,0), (\pm bf_\theta(l_\ast), \pm l_\ast).
\]

(3) \( a \neq 0, b = 0 \). In this case from the first equation of (2.4) we obtain \( h = af_\theta(h) \). As above without loss of generality here we assume that \( a > 0 \). Then part 6 of lemma 1 gives that the last equation has up to three solutions: \( 0, \pm h_\ast \). The case \( h = 0 \) reduces the second equation of (2.4) to \( l = df_\theta(l) \) which is also the equation of the form mentioned in the part 6 of lemma 1. The cases \( h = \pm h_\ast \) reduces the second equation to

\[
l = \pm cf_\theta(h_\ast) + df_\theta(l) = \pm \frac{c}{a} h_\ast + df_\theta(l). \tag{4.2}
\]

Analysis of solutions to this equation was done in lemma 12.27 of [8]: denote

\[
\tilde{h} = \max_{l \geq 0} |df_\theta(l) - l|.
\]

For \( \theta > 0 \) and \( d \geq 1 \) the equation (4.2) has

- (i) a unique solution \( l_\ast \) when \( \frac{c}{a} h_\ast > \tilde{h} \) or \( \frac{c}{a} h_\ast = \tilde{h} = 0 \),
- (ii) two distinct solutions \( l_- < l_+ \) when \( \frac{c}{a} h_\ast = \tilde{h} > 0 \), and
- (iii) three distinct solutions \( l_- < l_0 < l_+ \) when \( \frac{c}{a} h_\ast < \tilde{h} \).

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Summarizing we get following nine solutions:

- If $\theta > \frac{1}{d}$ then there are three solutions to (2.4):
  
  $$(0, 0), \ (0, l_*)\ , \ (0, -l_*).$$

  For $h_*$ satisfying above-mentioned condition (iii) we have six new solutions
  
  $$(\pm h_*, \pm l_-), \ (\pm h_*, \pm l_0), \ (\pm h_*, \pm l_+).$$

(4) $ab \neq 0$. In this case from the first equation of (2.4) we get

$$f_\theta(l) = \frac{1}{b} (h - af_\theta(h)).$$

Using this from the second equation we obtain

$$l = cf_\theta(h) + \frac{d}{b} (h - af_\theta(h)) = \varphi(h) \equiv \frac{1}{b} [(bc - ad) f_\theta(h) + dh].$$

Consequently, the first equation of (2.4) can be written as

$$h = \psi(h) \equiv af_\theta(h) + b f_\theta \left( \frac{1}{b} [(bc - ad) f_\theta(h) + dh] \right). \tag{4.3}$$

It is easy to see that $\psi(0) = 0$ and similarly as case 2) one can show that the equation (4.3) has at least three solutions if $|\psi'(0)| = |\theta||(bc - ad)\theta + a + d| > 1$. Thus the are at least three solutions to (2.4) of the form:

$$(h_i, \varphi(h_i)), \ i = 1, 2, 3. \quad \square$$

4.2. Proof of theorem 2

**Proof.** We use a result of [10] to establish a bound for reconstruction insolvability corresponding to the Gibbs measure $\mu_{h,l}$. Because, it is known that if $\mu$ is a Gibbs measure (not necessarily with free-boundary-condition, i.e. $h_x \neq 0$ is allowed) of an associated spin system, the fact that reconstruction is impossible for $\mu$ is equivalent to saying that $\mu$ is an extremal Gibbs measure of the spin system. Below following [10] we give more details of this equivalence and apply it for our measures.

Let us first give some necessary definitions from [10]. For $k \geq 2$, let $T_k$ denote a half tree, i.e. the infinite rooted $k$-ary tree (in which every vertex has $k$ children). Consider an initial finite complete subtree $T$, that is a tree of the following form: in the rooted tree $T_k$, take all vertices at distance $\leq d$ from the root, plus the edges joining them, where $d$ is a fixed constant. We identify subgraphs of $T$ with their vertex sets and write $E(A)$ for the edges within a subset $A$ and $\partial A$ for the boundary of $A$, i.e. the neighbors of $A$ in $(T \cup \partial T) \setminus A$.

In [10] the key ingredients are two quantities, $\kappa$ and $\gamma$, which bound the probabilities of percolation of disagreement down and up the tree, respectively. Both are properties
of the collection of Gibbs measures \( \{ \mu_\tau \} \), where the boundary condition \( \tau \) is fixed and \( \mathcal{T} \) ranges over all initial finite complete subtrees of \( \mathbb{T}^k \). For a given subtree \( \mathcal{T} \) of \( \mathbb{T}^k \) and a vertex \( x \in \mathcal{T} \), we write \( \mathcal{T}_x \) for the (maximal) subtree of \( \mathcal{T} \) rooted at \( x \) that is a tree given by \( \mathcal{T} \cap \mathbb{T}^k_x \), with \( \mathbb{T}^k_x \) the half tree with root \( x \). We draw the trees with the root at the top and the leaves at the bottom. When \( x \) is not the root of \( \mathcal{T} \), let \( \mu_{\mathcal{T}_x} \) denote the (finite-volume) Gibbs measure in which the parent of \( x \) has its spin fixed to \( s \) and the configuration on the bottom boundary of \( \mathcal{T}_x \) (i.e. on \( \partial \mathcal{T}_x \setminus \{ \text{parent of } x \} \)) is specified by \( \tau \).

For two measures \( \mu_1 \) and \( \mu_2 \) on \( \Omega \), \( \| \mu_1 - \mu_2 \|_x \) denotes the variation distance between the projections of \( \mu_1 \) and \( \mu_2 \) onto the spin at \( x \), i.e.

\[
\| \mu_1 - \mu_2 \|_x = \frac{1}{2} (| \mu_1(\sigma(x) = -1) - \mu_2(\sigma(x) = -1) | + | \mu_1(\sigma(x) = 1) - \mu_2(\sigma(x) = 1) |).
\]

Denote by \( \Omega_\mathcal{T} \) the set of configurations \( \sigma \) given on \( \mathcal{T} \cup \partial \mathcal{T} \) that agree with \( \tau \) on \( \partial \mathcal{T} \), i.e. \( \tau \) specifies a boundary condition on \( \mathcal{T} \). For any \( \eta \in \Omega_\mathcal{T} \) and any subset \( A \subseteq \mathcal{T} \), the Gibbs distribution on \( A \) conditional on the configuration outside \( A \) being \( \eta \) is denoted by \( \mu_\mathcal{T}^{\eta} \).

Let \( \eta_x^{x,s} \) be the configuration \( \eta \) with the spin at \( x \) set to \( s \).

Following [10, p 165] define

\[
\kappa \equiv \kappa(\mu) = \sup_{x \in \mathbb{T}^k} \max_{s,s'} \| \mu^{x}_{\mathcal{T}_x} - \mu^{x'}_{\mathcal{T}_x} \|_x;
\]

\[
\gamma \equiv \gamma(\mu) = \sup_{A \subseteq \mathbb{T}^k} \max_{A} \| \mu^{y,s}_{A} - \mu^{y,s'}_{A} \|_x,
\]

where the supremum is taken over all subsets \( A \subseteq \mathbb{T}^k \), the maximum is taken over all boundary conditions \( \eta \), all sites \( y \in \partial A \), all neighbors \( x \in A \) of \( y \), and all spins \( s, s' \in \{-1, 1\} \).

As the main ingredient we apply [10, theorem 4.3], from which it follows that the Gibbs measure \( \mu \) is extreme if \( k \kappa \gamma < 1 \).

To use the above-mentioned condition for the given choices of solutions to (1.6) we have to bound the corresponding \( \kappa \) and \( \gamma \) and show that \( k \kappa \gamma < 1 \).

For both \( \kappa \) and \( \gamma \), we need to bound a quantity of the form \( \| \mu^{\eta}_{A} - \mu^{\eta}_A \|_z \), where \( y \in \partial A \) and \( z \in A \) is a neighbor of \( y \). The key observation of [10] is that this quantity can be expressed very cleanly in terms of the ‘magnetization’ at \( z \), i.e. the ratio of probabilities of a \((-\) - spin and a \((+\) - spin at \( z \). It will actually be convenient to work with the magnetization without the influence of the neighbor \( y \); let \( \mu^{\eta}_{A} \) denote the Gibbs distribution with boundary condition \( \eta \), except that the spin at \( y \) is free (or equivalently, the edge connecting \( z \) to \( y \) is erased).

\[\Box\]

**Proposition 1.** Let \( \mu \) be one of measures \( \mu_{h,1} \). For any subset \( A \subseteq \mathcal{T} \), any boundary configuration \( \eta \), any site \( y \in \partial A \) and any neighbor \( z \in A \) of \( y \), we have

\[
\| \mu^{\eta}_{A} - \mu^{\eta}_A \|_z = K_\beta(R_z),
\]
where
\[ R_z = \frac{\mu_A^* \sigma(z) = -}{\mu_A^* \sigma(z) = +} = e^{-2h_z}, \]

here \( h_z \) is a compatible function constructed in section 2 using \( h \) and \( l \) by steps (i) and (ii). The function \( K_\beta \) is defined by
\[ K_\beta(a) = \frac{1}{e^{-2\beta J}a + 1} - \frac{1}{e^{2\beta J}a + 1}. \]

**Proof.** The prove is similar to the proof of [10, proposition 4.2].

Note that \( R_z \in [0, +\infty) \). It is easy to check that \( K_\beta(a) \) is an increasing function in the interval \([0, 1]\), decreasing in the interval \([1, +\infty]\), and is maximized at \( a = 1 \). Therefore, we can always bound \( \kappa \) and \( \gamma \) from above by \( K_\beta(1) = \theta = \tanh(\beta J) \). Thus the bounds of \( \kappa \) and \( \gamma \) can be controlled by the magnetization \( R_z \). The bound will be better than \( \theta \) when \( R_z \) differs from 1 for any \( z \).

To prove part 1 of theorem 2 we use estimates \( \gamma \leq \theta \) and \( \kappa \leq \theta \) because \( hl = 0 \) gives that \( R_z = 1 \) for some \( z \in V \). Thus condition \( k\gamma \kappa < 1 \) gives \( k\theta^2 < 1 \) and the part 1 follows.

Now we shall prove part 2. For the Gibbs measure \( \mu_{h,l} \) corresponding to a solution \((h, l)\) of (2.4) we denote
\[
\begin{align*}
\mathcal{H}^\pm &= \{x \in V : h_z = \pm h\}, \\
\mathcal{L}^\pm &= \{x \in V : h_z = \pm l\}, \\
\alpha &= e^{-\beta J}, \quad A = e^{2h}, \quad C = e^{2l}, \\
F(x) &= \frac{\alpha + x}{1 + \alpha x}.
\end{align*}
\]

Then \( R_z \) corresponding to \( \mu_{h,l} \) has the following form
\[
R_z = \begin{cases} 
A, & z \in \mathcal{H}^+ \\
1/A, & z \in \mathcal{H}^- \\
C, & z \in \mathcal{L}^+ \\
1/C, & z \in \mathcal{L}^-,
\end{cases}
\]
where \( A \neq 1, C \neq 1 \) (since \( h \neq 0 \) and \( l \neq 0 \)) and satisfy the following system of equations
\[
\begin{align*}
A &= [F(A)]^{a_1-a_2} [F(C)]^{a_3-a_4}, \\
C &= [F(A)]^{b_1-b_2} [F(C)]^{b_3-b_4}.
\end{align*}
\] (4.5)

To check the extremality condition \( k\kappa \gamma < 1 \) for \( \mu_{h,l} \) we use estimation \( \gamma < \theta \). To bound \( \kappa \) we use that \( R_z \) has values \( A, 1/A, C, 1/C \). Thus we have
\[ \kappa \leq \max\{K_\beta(s) : s \in \{A, 1/A, C, 1/C\}\}. \]

Without loss of generality we take \( K_\beta(A) = \max\{K_\beta(s) : s \in \{A, 1/A, C, 1/C\}\} \), because \( A \) and \( C \) play similar role. We shall use the following formula (see lemma 4.3 of [10]):
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\[ K_\beta(A) = \frac{1}{k} \cdot \frac{A}{J(A)} \cdot J'(A), \]

where \( J(x) = (F(x))^k \). Note that under condition \(|\theta||(bc - ad)\theta + a + d| > 1\) the solution \( h = \frac{1}{2} \ln A \) of (4.3) is an attracting (stable) fixed point for \( \psi \). Moreover, it is known that \( h < h^\ast \) (see proof of part 1) of theorem 12.31 in [8], where \( h^\ast > 0 \) is solution to \( h = k f\theta(h) \) (for \( \theta > \frac{1}{k} \)). Note that \( e^{2h^\ast} \) is an attractive fixed point of \( J(x) \), i.e. \( J'(e^{2h^\ast}) < 1 \). Since \( 0 < A = e^{2h} \leq e^{2h^\ast} \) we have \( J(A) \geq A \) and \( J'(A) \leq 1 \) for \( \theta > \frac{1}{k} \). Consequently, we get
\[
\kappa \leq K_\beta(A) = \frac{1}{k} \cdot \frac{A}{J(A)} \cdot J'(A) \leq \frac{1}{k}.
\]

Hence \( k^\gamma \kappa < \theta < 1 \). \( \Box \)

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