Dissipative backward stochastic differential equations with locally Lipschitz nonlinearity.

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Abstract

In this paper we study a class of backward stochastic differential equations (BSDEs) of the form

\[ dY_t = -AY_t dt - f_0(t, Y_t) dt - f_1(t, Y_t, Z_t) dt + Z_t dW_t, \quad 0 \leq t \leq T; \quad Y_T = \xi \]

in an infinite dimensional Hilbert space $H$, where the unbounded operator $A$ is sectorial and dissipative and the nonlinearity $f_0(t, y)$ is dissipative and defined for $y$ only taking values in a subspace of $H$. A typical example is provided by the so-called polynomial nonlinearities. Applications are given to stochastic partial differential equations and spin systems.

Key words Backward stochastic differential equations, stochastic evolution equations.

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1 Introduction

Let $H, K$ be real separable Hilbert spaces with norms $\| \cdot \|_H$ and $\| \cdot \|_K$. Let $W$ be a cylindrical Wiener process in $K$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{\mathcal{F}_t\}_{t \in [0,T]}$ denote its natural augmented filtration. Let $\mathcal{L}^2(K, H)$ be the Hilbert space of Hilbert-Schmidt operators from $K$ to $H$. 
We are interested in solving the following backward stochastic differential equation

\[ dY_t = -AY_t dt - f(t, Y_t, Z_t) dt + Z_t dW_t, \quad 0 \leq t \leq T, \quad Y_T = \xi \]  

(1)

where \( \xi \) is a random variable with values in \( H \), \( f(t, Y_t, Z_t) = f_0(t, Y_t) + f_1(t, Y_t, Z_t) \) and \( f_0, f_1 \) are given functions, and the operator \( A \) is an unbounded operator with domain \( D(A) \) contained in \( H \). The unknowns are the processes \( \{Y_t\}_{t \in [0,T]} \) and \( \{Z_t\}_{t \in [0,T]} \), which are required to be adapted with respect to the filtration of the Wiener process and take values in \( H \), \( L^2(K,H) \) respectively.

In finite dimensional framework such type of equations has been solved by Pardoux and Peng [12] in the nonlinear case. They proved an existence and uniqueness result for the solution of the equation (1) when \( A = 0 \), the coefficient \( f(t, y, z) \) is Lipschitz continuous in both variables \( y \) and \( z \), and the data \( \xi \) and the process \( \{f(t, 0, 0)\}_{t \in [0,T]} \) are square integrable. Since this first result, many papers were devoted to existence and uniqueness results under weaker assumptions. In finite dimension, when \( A = 0 \), the Lipschitz condition on the coefficient \( f \) with respect to the variable \( y \) is replaced by a monotonicity assumption; moreover, more general growth conditions in the variable \( y \) are formulated. Let us mention the contribution of Briand and Carmona [1], for a study of polynomial growth in \( L^p \) with \( p > 2 \), and the work of Pardoux [11] for an arbitrary growth. In [13] Pardoux and Rascanu deal with a BSDE involving the subdifferential of a convex function; in particular, one coefficient is not everywhere defined for \( y \in \mathbb{R}^k \).

In other works the existence of the solution is proved when the data, \( \xi \) and the process \( \{f(t, 0, 0)\}_{t \in [0,T]} \), are in \( L^p \) for \( p \in (1, 2) \). El Karoui, Peng and Quenez [4] treat the case when \( f \) is Lipschitz continuous; in [2] this result is generalized to the case of a monotone coefficient \( f \) (both for equations on a fixed and on a random time interval) and is studied even the case \( p = 1 \).

In the infinite-dimensional framework Hu and Peng [6], and Oksendal and Zhang [10] give an existence and uniqueness result for the equation with an operator \( A \), infinitesimal generator of a strongly continuous semigroup and the coefficient \( f \) Lipschitz in \( y \) and \( z \). Pardoux and Rascanu [14] replace the operator \( A \) with the subdifferential of a convex function and assume that \( f \) is dissipative, everywhere defined and continuous with respect to \( y \), Lipschitz with respect to \( z \) and with linear growth in \( y \) and \( z \).

Special results deal with stochastic backward partial differential equations (BSPDEs): we recall in particular the works of Ma and Yong [8] and
Earlier, Peng [16] studied a backward stochastic partial differential equation and regarded the classical Hamilton-Jacobi-Bellman equation of optimal stochastic control as special case of this problem.

Our work extends these results in a special direction. We consider an operator $A$ which is the generator of an analytic contraction semigroup on $H$ and a coefficient $f(t, y, z)$ of the form $f_0(t, y) + f_1(t, y, z)$. The coefficient $f_1(t, y, z)$ is assumed to be bounded and Lipschitz with respect to $y$ and $z$. The term $f_0(t, y)$ is defined for $y$ only taking values in a suitable subspace $H_\alpha$ of $H$ and it satisfies the following growth condition for some $1 < \gamma < 1/\alpha$, $S \geq 0$, $\mathbb{P}$-a.s.

$$|f_0(t, y)|_H \leq S(1 + \|y\|_{H_\alpha}^\gamma) \quad \forall t \in [0, T], \quad \forall y \in H_\alpha.$$  

Following [6], we understand the equation (1) in the following integral form

$$Y_t - \int_t^T e^{(s-t)A}[f_0(s, Y_s) + f_1(s, Y_s, Z_s)]ds + \int_t^T e^{(s-t)A}Z_s dW_s = e^{(T-t)A}\xi,$$

requiring, in particular, that $Y$ takes values in $H_\alpha$. This requires generally that the final condition also takes values in the smaller space $H_\alpha$. We take as $H_\alpha$ a real interpolation space which belongs to the class $J_\alpha$ between $H$ and the domain of an operator $A$ (see Section 2). Moreover $f_0(t, \cdot)$ is assumed to be locally Lipschitz from $H_\alpha$ into $H$ and dissipative in $H$. We prove (Theorem 5) that if $\xi$ takes its values in the closure of $D(A)$ in $H_\alpha$ and is such that $\|\xi\|_{H_\alpha}$ is essentially bounded, then equation (2) has a unique mild solution, i.e. there exists a unique pair of progressively measurable processes $Y : \Omega \times [0, T] \to H_\alpha$, $Z : \Omega \times [0, T] \to L^2(K; H)$, satisfying $\mathbb{P}$-a.s. equality (2) for every $t$ in $[0, T]$ and such that $\mathbb{E} \sup_{t \in [0, T]} \|Y_t\|_{H_\alpha}^2 + \mathbb{E} \int_0^T \|Z_t\|_{L^2(K; H)}^2 dt < \infty$.

This result extends former results concerning the deterministic case to the stochastic framework: see [7], where previous works of Fujita - Kato [5], Pazy [15] and others are collected. In these papers similar assumptions are made on the coefficients $f_0$, $f_1$ and on the operator $A$.

The plan of the paper is as follows. In Section 2 some notations and definitions are fixed. In Section 3 existence and uniqueness of the solution of a simplified equation are proved, where $f_1$ is a bounded progressively measurable process which does not depend on $y$ and $z$. In Section 4, applying the previous result, a fixed point argument is used in order to prove our main result on existence and uniqueness of a mild solution of (2). Section 5 is devoted to applications.
2 Notations and setting

The letters $K$ and $H$ will always denote two real separable Hilbert spaces. Scalar product is denoted by $\langle \cdot, \cdot \rangle$; $L^2(K; H)$ is the separable Hilbert space of Hilbert-Schmidt operators from $K$ to $H$ endowed with the Hilbert-Schmidt norm. $W = \{W_t\}_{t \in [0, T]}$ is a cylindrical Wiener process with values in $K$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\{F_t\}_{t \in [0, T]}$ is the natural filtration of $W$, augmented with the family of $\mathbb{P}$-null sets of $\mathcal{F}$.

Next we define several classes of stochastic processes with values in a Banach space $X$.

- $L^2(\Omega \times [0, T]; X)$ denotes the space of measurable $X$-valued processes $Y$ such that $\left[ \mathbb{E} \int_0^T |Y_\tau|^2 d\tau \right]^{1/2}$ is finite, identified up to modification.

- $L^2(\Omega; C([0, T]; X))$ denotes the space of continuous $X$-valued processes $Y$ such that $\left[ \mathbb{E} \sup_{\tau \in [0, T]} |Y_\tau|^2 \right]^{1/2}$ is finite, identified up to indistinguishability.

- $C^\alpha([0, T]; X)$ denotes the space of $\alpha$-Hölderian functions on $[0, T]$ with values in $X$ such that $[f]_\alpha = \sup_{0 \leq x < y \leq T} \frac{|f(x) - f(y)|}{(y - x)^\alpha} < \infty$.

Now we need to recall several preliminaries on semigroup and interpolation spaces. We refer the reader to [7] for the proofs and other related results.

A linear operator $A$ in a Banach space $X$, with domain $D(A) \subset X$, is called sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $M > 0$ such that

\[
\begin{align*}
(i) & \quad \rho(A) \supseteq S_{\theta, \omega} = \{ \lambda \in \mathbb{C} : \lambda \neq \omega, |\text{arg}(\lambda - \omega)| < \theta \}, \\
(ii) & \quad \|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|} \quad \forall \lambda \in S_{\theta, \omega} \quad (3)
\end{align*}
\]

where $\rho(A)$ is the resolvent set of $A$. For every $t > 0$, (3) allows us to define a linear bounded operator $e^{tA}$ in $X$, by means of the Dunford integral

\[
e^{tA} = \frac{1}{2\pi i} \int_{\omega + r \gamma_{r, \eta}} e^{\lambda} (\lambda I - A)^{-1} d\lambda, \quad t > 0,
\]

where, $r > 0, \eta \in (\pi/2, \pi)$ and $\gamma_{r, \eta}$ is the curve $\{ \lambda \in \mathbb{C} : |\text{arg}\lambda| = \eta, |\lambda| \geq r \} \cup \{ \lambda \in \mathbb{C} : |\text{arg}\lambda| \leq \eta, |\lambda| = r \}$, oriented counterclockwise. We also set
$e^{0A}x = x, \forall x \in X$. Since the function $\lambda \mapsto e^{t\lambda}R(\lambda, A)$ is holomorphic in $S_{\theta,\omega}$, the definition of $e^{tA}$ is independent of the choice of $r$ and $\eta$. If $A$ is sectorial, the function $[0, +\infty) \rightarrow L(X), t \mapsto e^{tA}$, with $e^{tA}$ defined by (4) is called analytic semigroup generated by $A$ in $X$. We note that for every $x \in X$ the function $t \mapsto e^{tA}x$ is analytic (and hence continuous) for $t > 0$. $e^{tA}$ is a strongly continuous semigroup if and only if $D(A)$ is dense in $X$; in particular this holds if $X$ is a reflexive space.

We need to introduce suitable classes of subspaces of $X$.

**Definition 1.** Let $(\alpha, p)$ be two numbers such that $0 < \alpha < 1$, $1 \leq p \leq \infty$ or $(\alpha, p) = (1, \infty)$. Then we denote with $D_A(\alpha, p)$ the space

$$D_A(\alpha, p) = \{ x \in X : t \mapsto v(t) = \| t^{1-\alpha-1/p}Ae^{tA}x \| \in L^p(0, 1) \}$$

where $\| x \|_{D_A(\alpha, p)} = \| x \|_X + \| [x]_\alpha = \| x \|_X + \| v \|_{L^p(0,1)}$.

(We set as usual $1/\infty = 0$).

We recall here some estimates for the function $t \mapsto e^{tA}$ when $t \rightarrow 0$, which we will use in the sequel. For convenience, in the next proposition we set $D_A(0, p) = X, p \in [1, \infty]$.

**Proposition 1.** Let $(\alpha, p), (\beta, p) \in (0, 1) \times [1, +\infty] \cup \{(1, \infty)\}, \alpha \leq \beta$. Then there exists $C = C(p; \alpha, \beta)$ such that

$$\| t^{-\alpha+\beta}e^{tA} \|_{L(D_A(\alpha, p), D_A(\beta, p))} \leq C, \quad 0 < t \leq 1.$$  

**Definition 2.** Let $0 \leq \alpha \leq 1$ and let $D, X$ be Banach spaces, $D \subset X$. A Banach space $Y$ such that $D \subset Y \subset X$ is said to belong to the class $J_\alpha$ between $X$ and $D$ if there is a constant $C$ such that $\| x \|_Y \leq C \| x \|_X^{1-\alpha}\| x \|_D^\alpha, \quad \forall x \in D$. In this case we write $Y \in J_\alpha(X, D)$.

Now we give the definition of solution to the BSDE:

$$Y_t - \int_t^T e^{(s-t)A}[f_0(s, Y_s) + f_1(s, Y_s, Z_s)]ds + \int_t^T e^{(s-t)A}Z_s dW_s = e^{(T-t)A}\xi,$$

(5)

**Definition 3.** A pair of progressively measurable processes $(Y, Z)$ is called mild solution of (5) if it belongs to the space $L^2(\Omega; C([0, T]; H_\alpha)) \times L^2(\Omega \times [0, T]; L^2(K, H))$ and $\mathbb{P}$-a.s.solves the integral equation (5) on the interval $[0, T]$.

We finally state a lemma needed in the sequel. It is a generalization of the well known Gronwall’s lemma. Its proof is given in the Appendix.
Lemma 1. Assume $a, b, \alpha, \beta$ are nonnegative constants, with $\alpha < 1$, $\beta > 0$ and $0 < T < \infty$. For any nonnegative process $U \in L^1(\Omega \times [0, T])$, satisfying $\mathbb{P}$-a.s. $U_t \leq a(T-t)^{-\alpha} + b \int_t^T (s-t)^{\beta-1} \mathbb{E} F U_s ds$ for almost every $t \in [0, T]$, it holds $\mathbb{P}$-a.s. $U_t \leq aM(T-t)^{-\alpha}$, for almost every $t \in [0, T]$. $M$ is a constant depending only on $b, \alpha, \beta, T$.

3 A simplified equation

As a preparation for the study of (2), in this section we consider the following simplified version of that equation:

$$ Y_t - \int_t^T e^{(s-t)A} [f_0(s, Y_s) ds + f_1(s)] ds + \int_t^T e^{(s-t)A} Z_s dW_s = e^{(T-t)A} \xi, \quad (6) $$

for all $t \in [0, T]$.

We suppose that the following assumptions hold.

Hypothesis 2.

1. $A : D(A) \subset H \to H$ is a sectorial operator. We also assume that $A$ is dissipative, i.e. it satisfies $\langle Ay, y \rangle \leq 0$, $\forall y \in D(A)$;

2. for some $0 < \alpha < 1$ there exists a Banach space $H_\alpha$ continuously embedded in $H$ and such that
   (i) $D_A(\alpha, 1) \subset H_\alpha \subset D_A(\alpha, \infty)$;
   (ii) the part of $A$ in $H_\alpha$ is sectorial in $H_\alpha$.

3. the final condition $\xi$ is an $\mathcal{F}_T$-measurable random variable defined on $\Omega$ with values in the closure of $D(A)$ with respect to $H_\alpha$-norm. We denote this set $\overline{D(A)}^{H_\alpha}$. Moreover $\xi$ belongs to $L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; H_\alpha)$;

4. $f_0 : \Omega \times [0, T] \times H_\alpha \to H$ satisfies:
   
   i) $\{f_0(t, y)\}_{t \in [0, T]}$ is progressively measurable $\forall y \in H_\alpha$;
   
   ii) there exist constants $S > 0$, $1 < \gamma < 1/\alpha$ such that $\mathbb{P}$-a.s.
   \[ |f_0(t, y)|_H \leq S(1 + ||y||_{H_\alpha}^\gamma) \quad t \in [0, T], y \in H_\alpha; \]
   
   iii) for every $R > 0$ there is $L_R > 0$ such that $\mathbb{P}$-a.s.
   \[ |f_0(t, y_1) - f_0(t, y_2)|_H \leq L_R ||y_1 - y_2||_{H_\alpha} \]
   for $t \in [0, T]$ and $y_i \in H_\alpha$ with $||y_i||_{H_\alpha} \leq R$;
iv) there exists a number $\mu \in \mathbb{R}$ such that $\mathbb{P}$-a.s., $\forall t \in [0, T]$, $y_1, y_2 \in H$,\[
< f_0(t, y_1) - f_0(t, y_2), y_1 - y_2 >_H \leq \mu |y_1 - y_2|_H^2; \quad (7)
\]

5. $f_1 : \Omega \times [0, T] \to H$ is progressively measurable and for some constant $C > 0$ it satisfies $\mathbb{P}$-a.s. $|f_1(t)|_H \leq C$, for $t \in [0, T]$.

**Remark 1.** We note that the pair $(Y, Z)$ solves the BSDE (6) with final condition $\xi$ and drift $f = f_0 + f_1$ if and only if the pair $(\bar{Y}, \bar{Z}) := (e^{\lambda T} Y_t, e^{\lambda T} Z_t)$ is a solution of the same equation with final condition $e^{\lambda T} \xi$ and drift $f'(t, y) := f'_0(t, y) + f'_1(t)$ where $f'_0(t, y) = e^{\lambda t}(f_0(t, e^{-\lambda t} y), \lambda y)$, $f'_1(t) = e^{\lambda t} f_1(t)$. If we choose $\mu = \lambda$, then $f'_0$ satisfies the same assumption as $f_0$, but with (7) replaced by $< f_0(t, y_1) - f_0(t, y_2), y_1 - y_2 >_H \leq 0$. If this last condition holds, then $f_0$ is called dissipative. Hence, without loss of generality, we shall assume until the end that $f_0$ is dissipative, or equivalently that $\mu = 0$ in (7).

### 3.1 A priori estimates

We prove a basic estimate for the solution in the norm of $H$.

**Proposition 2.** Suppose that Hypothesis 2 holds; if $(Y, Z)$ is a mild solution of (6) on the interval $[a, T]$, $0 \leq a \leq T$, then there exists a constant $C_1$, which depends only on $|\xi|_{L^\infty(\Omega, H)}$ and on the constants $S$ of 4.ii) and $C$ of 5. such that $\mathbb{P}$-a.s. $\sup_{a \leq t \leq T} ||Y_t||_H \leq C_1$. In particular the constant $C_1$ is independent of $a$.

**Proof.** Let the pair $(Y, Z) \in L^2(\Omega, C([a, T]; H_0) \times L^2(\Omega \times [a, T]; L^2(K; H))$ satisfy (6). Let us introduce the operators $J_n = n(nI - A)^{-1}$, $n > 0$. We note that the operators $AJ_n$ are the Yosida approximations of $A$ and they are bounded. Moreover $|J_n x - x| \to 0$ as $n \to \infty$, for every $x \in H$. We set $Y^n_t = J_n Y_t$, $Z^n_t = J_n Z_t$. It is readily verified that $Y^n$ admits the Itô differential
\[
dY^n_t = -AY^n_t dt - J_n f(t, Y_t) dt - J_n f_1(t) dt + Z^n_t dW_t, \quad Y^n_T = J_n \xi.
\]

Applying the Itô formula to $|Y^n_t|_H^2$, using the dissipativity of $A$, we obtain
\[
|Y^n_T|_H^2 + \int_T^t |Z^n_s|_{L^2(K; H)}^2 ds \leq |J_n \xi|_H^2 + 2 \int_t^T < J_n f_0(s, Y_s), Y^n_s >_H ds + 2 \int_t^T < J_n f_1(s), Y^n_s >_H ds - \int_t^T < J_n f_0(s, Y_s), Y^n_s >_H ds + 2 \int_t^T < Y^n_s, Z^n_s dW_s >_H.
\]
\[\tag{8}\]
We note that \( \int_t^T < J_n f_0(s, Y_s) + J_n f_1(s), Y^n_s >_H \, ds \to \int_t^T < f_0(s, Y_s) + f_1(s), Y_s >_H \, ds \) by dominated convergence, as \( n \to \infty \). Moreover by the dominated convergence theorem we have \( \int_t^T ||(Z^n_s)^* Y^n_s - Z^*_s Y_s||^2_H \, ds \to 0 \) \( \mathbb{P} \)-a.s. and it follows that \( \int_t^T < Y^n_s, Z^n_s dW_s >_H \to \int_t^T < Y_s, Z_s dW_s >_H \) in probability. If we let \( n \to \infty \) in (8) we obtain

\[
|Y_1|_H^2 + \int_t^T ||Z_s||^2_{L^2(K,H)} \, ds \leq |\xi|_H^2 + 2 \int_t^T < f_0(s, Y_s) + f_1(s), Y_s >_H \, ds \\
- 2 \int_t^T < Y_s, Z_s dW_s >_H .
\]

Recalling (7), that we assume to hold with \( \mu = 0 \), it follows that

\[
|Y_1|_H^2 + \int_t^T ||Z_s||^2_{L^2(K,H)} \leq \\
\leq |\xi|_H^2 + 2 \int_t^T < f_0(s,0), Y_s >_H + 2 \int_t^T < f_1(s), Y_s >_H \, ds + \\
- 2 \int_t^T < Y_s, Z_s dW_s >_H \\
\leq |\xi|_H^2 + \int_t^T |f(s,0)|^2_H \, ds + \int_t^T |f_1(s)|^2_H \, ds + 2 \int_t^T |Y_s|^2_H \, ds + \\
- 2 \int_t^T < Y_s, Z_s dW_s >_H .
\]

Now, since \( \sup_{0 \leq t \leq T} |f(t,0)|^2_H \leq S^2 \) and since the stochastic integral \( \int_a^t < Y_s, Z_s dW_s >_H , t \in [a,T] \) is a martingale, if we take the conditional expectation given \( \mathcal{F}_t \) we have

\[
|Y_t|_H^2 \leq \mathbb{E}^{\mathcal{F}_t} \{ |\xi|_H^2 + 2 \mathbb{E}^{\mathcal{F}_t} \int_t^T |Y_s|^2_H \, ds + \\
+ \mathbb{E}^{\mathcal{F}_t} \int_t^T |f(s,0)|^2_H \, ds + \mathbb{E}^{\mathcal{F}_t} \int_t^T |f_1(s)|^2_H \, ds \} \\
\leq |\xi|_{L^\infty(\Omega,H)}^2 + (S^2 + C^2) T + 2 \mathbb{E}^{\mathcal{F}_t} |Y_s|^2_H \, ds .
\]

Since \( Y \) belongs to \( L^2(\Omega; C([a,T]; H_\alpha)) \) and, consequently, \( ||Y||^2_{H_\alpha} \in L^1(\Omega \times [0,T]) \), we can apply Lemma 1 to \( |Y|^2_H \) and conclude that

\[
|Y_t|^2_H \leq (||\xi||_{L^\infty(\Omega,H)}^2 + [S^2 + C^2] T + 2T e^{2T}).
\]

Now we will show that the result of Proposition 2, together with the growth condition satisfied by \( f_0 \), yields an a priori estimate on the solution in the \( H_\alpha \)-norm.
Let $0 < \alpha < 1$ and let $\gamma > 1$ be given by 4.ii). We fix $\theta = \alpha \gamma$ and consider the Banach space $D_A(\theta, \infty)$ introduced in Definition 1. It is easy to check (see [7]) that, if we take $\theta \in (0, 1)$, then $H_\alpha$ contains $D_A(\theta, \infty)$ and belongs to the class $J_{\alpha/\theta}$ between $D_A(\theta, \infty)$ and $H$, hence the following inequality is satisfied:

$$|x|_{H_\alpha} \leq c|x|_{D_A(\theta, \infty)}^{1-\frac{\theta}{\theta_1}}, \quad x \in D_A(\theta, \infty).$$

(9)

Proposition 3. Suppose that Hypothesis 2 is satisfied. Let $(Y, Z)$ be a mild solution of (6) in $[a, T]$, $a \geq 0$ and assume that there exists two constants $R > 0$ and $K > 0$, possibly depending on $a$, such that, $\mathbb{P}$-a.s.,

$$\sup_{t \in [a, T]} \|Y_t\|_{H_\alpha} \leq R, \quad \sup_{t \in [a, T]} |Y_t|_H \leq K.$$  (10)

Then the following inequality holds $\mathbb{P}$-a.s.:

$$|Y_t|_{L^\infty(\Omega, D_A(\theta, \infty))} \leq C_2 \frac{1}{(T-t)^{\theta_1-\alpha}}, \quad a \leq t < T$$  (11)

with $C_2$ depending on the operator $A$, $\|\xi\|_{L^\infty(\Omega, H_\alpha)}$, $\theta$, $\alpha$, $K$, $C$ of 5. and $S$ of 4.ii) of Hypothesis 2.

Proof. Taking the conditional expectation given $\mathcal{F}_t$ in equation (6) we find

$$Y_t = \mathbb{E}^{\mathcal{F}_t} \left( e^{(T-t)A} \xi + \int_t^T e^{(s-t)A} [f_0(s, Y_s) + f_1(s)] ds \right), \quad a \leq t \leq T.$$  (12)

Consequently, we have

$$\|Y_t\|_{D_A(\theta, \infty)} \leq \mathbb{E}^{\mathcal{F}_t} \|e^{(T-t)A} \xi\|_{D_A(\theta, \infty)}$$

$$+ \mathbb{E}^{\mathcal{F}_t} \int_t^T \|e^{(s-t)A} [f_0(s, Y_s) + f_1(s)]\|_{D_A(\theta, \infty)} ds, \quad a \leq t \leq T.$$  (13)

Since $H_\alpha \subset D_A(\alpha, \infty)$, we have

$$\mathbb{E}^{\mathcal{F}_t} \|e^{(T-t)A} \xi\|_{D_A(\theta, \infty)} \leq$$

$$\leq \mathbb{E}^{\mathcal{F}_t} \|e^{(T-t)A} \|_{L(D_A(\alpha, \infty), D_A(\theta, \infty))} \|\xi\|_{L^\infty(\Omega, D_A(\alpha, \infty))}$$

$$\leq C_0 \frac{1}{(T-t)^{\theta_1-\alpha}} \|\xi\|_{L^\infty(\Omega, H_\alpha)},$$

with $C_0 = C_0(\alpha, \theta, \infty)$, where in the last inequality we use Proposition 1.
Moreover

\[ \mathbb{E}^F_t \int_t^T ||e^{(s-t)A}[f_0(s, Y_s) + f_1(s)]||_{D_A(\theta, \infty)} ds \leq \]
\[ \leq \mathbb{E}^F_t \int_t^T ||e^{(s-t)A}||_{L(H,D_A(\theta, \infty))} ||f_0(s, Y_s) + f_1(s)||_H ds \leq \]
\[ \leq \mathbb{E}^F_t \left( \int_t^T \frac{C_1}{(s-t)^\theta} ||f_0(s, Y_s)||_H + ||f_1(s)||_H \right) ds \leq \]
\[ \leq \mathbb{E}^F_t \int_t^T \frac{C_1}{(s-t)^\theta} \left[ S(1 + ||Y_s||_{H_0}) + C \right] ds. \]

In the inequality we used Hypotheses 4.ii) and 5. and Proposition 1. Recalling (9), we conclude that the last term is dominated by

\[ \mathbb{E}^F_t \int_t^T \frac{C_1}{(s-t)^\theta} \left[ S(1 + c||Y_s\gamma(1-\alpha)/\theta||_{Y_s} ||Y_s||_{D_A(\theta, \infty)}) + C \right] ds = \]
\[ = \mathbb{E}^F_t \int_t^T \frac{C_1}{(s-t)^\theta} \left[ S(1 + c||Y_s\gamma(1-\alpha)/\theta||_{Y_s} ||Y_s||_{D_A(\theta, \infty)}) + C \right] ds, \]

by choosing \( \theta = \alpha \gamma \). By the second inequality in (10) this can be estimated by

\[ \int_t^T \frac{C_1}{(s-t)^\theta} S(1 + cK\gamma(1-\alpha)/\theta \mathbb{E}^F_t ||Y_s||_{D_A(\theta, \infty)}) ds \leq \int_t^T \frac{C_1}{(s-t)^\theta} (C + S) ds + \int_t^T \frac{C_1}{(s-t)^\theta} ScK\gamma(1-\alpha)/\theta \mathbb{E}^F_t ||Y_s||_{D_A(\theta, \infty)} ds. \]

Hence by (13) and (14) it follows

\[ ||Y_t||_{D_A(\theta, \infty)} \leq \frac{C_0}{(T-t)^{\theta-\alpha}} ||\xi||_{L^\infty(\Omega,H_0)} + \int_t^T \frac{C_1}{(s-t)^\theta} (C + S) ds \]
\[ + \int_t^T \frac{C_1}{(s-t)^\theta} ScK\gamma(1-\alpha)/\theta \mathbb{E}^F_t ||Y_s||_{D_A(\theta, \infty)} ds, \]

and (11) follows from Lemma 1. In order to justify the application of Lemma 1, we need to prove that \( ||Y||_{D_A(\theta, \infty)} \) belongs to \( L^1(\Omega \times [a, T]) \). This also follows from (13) and (14) since, for some constant \( K_1 \),

\[ ||Y_t||_{D_A(\theta, \infty)} \leq \]
\[ \leq \frac{K_1}{(T-t)^{\theta-\alpha}} ||\xi||_{L^\infty(\Omega,H_0)} + \mathbb{E}^F_t \left[ \sup_{s \in [a,T]} (1 + ||Y_s||_{H_0}) \right] \int_t^T \frac{ds}{(s-t)^\theta} \]
\[ \leq \frac{K_1}{(T-t)^{\theta-\alpha}} ||\xi||_{L^\infty(\Omega,H_0)} + (1 + R) \int_t^T \frac{ds}{(s-t)^\theta}. \]
3.2 Local existence and uniqueness

We prove that, under Hypothesis 2, there exists a unique solution of (6) on an interval \([T - \delta, T]\) with \(\delta\) sufficiently small.

To treat the ordinary integral in the left hand side of (6), we need the following result, whose proof can be found in [7], Proposition 4.2.1 and Lemma 7.1.1.

**Lemma 3.** Let \(\phi \in L^\infty((a, T); H),\) \(0 < a < T\) and set

\[
v(t) = \int_t^T e^{(s-t)A} \phi(s) ds, \quad a \leq t \leq T.\]

If \(0 < \alpha < 1,\) then \(v \in C^{1-\alpha}([a, T]; D_A(\alpha, 1))\) and there is \(G_0 > 0,\) not depending on \(a,\) such that

\[
\|v\|_{C^{1-\alpha}([a, T]; D_A(\alpha, 1))} \leq G_0 \|\phi\|_{L^\infty((a, T); H)}.
\]

Since \(D_A(\alpha, 1) \subset H_\alpha,\) we also have \(v \in C^{1-\alpha}([a, T]; H_\alpha)\) and there is \(G > 0,\) not depending on \(a,\) such that

\[
\|v\|_{C^{1-\alpha}([a, T]; H_\alpha)} \leq G \|\phi\|_{L^\infty((a, T); H)}.
\]

**Theorem 4.** Let us assume that Hypothesis 2 holds, except possibly 4.iv). Then there exists \(\delta > 0\) such that the equation (6) has a unique local mild solution \((Y, Z) \in L^2(\Omega; C([T - \delta, T]; H_\alpha)) \times L^2(\Omega \times [T - \delta, T]; L^2(K; H)).\)

**Remark 2.** The dissipativity condition 4.iv) only plays a role in obtaining the a priori estimate in \(H\) (Proposition 2) and consequently global existence, as we will see later.

**Proof.** Let \(M_\alpha := \sup_{0 \leq t \leq T} \|e^{tA}\|_{L(H_\alpha)}\). We fix a positive number \(R\) such that \(R \geq 2M_\alpha \|\xi\|_{L^\infty(\Omega_\alpha; H_\alpha)}.\) This implies that \(\sup_{0 \leq t \leq T} \|e^{tA} \xi\|_{H_\alpha} \leq R/2\) \(\mathbb{P}\)-a.s. Moreover, let \(L_R\) be such that

\[
|f_0(t, y_1) - f_0(t, y_2)|_H \leq L_R|y_1 - y_2|_{H_\alpha}, \quad 0 \leq t \leq T, \quad \|y_i\|_{H_\alpha} \leq R.
\]

We recall that the space \(L^2(\Omega; C([T - \delta, T]; H_\alpha))\) is a Banach space endowed with the norm \(Y \rightarrow (\mathbb{E} \sup_{t \in [T - \delta, T]} \|Y_t\|_{H_\alpha}^2)^{1/2}.\) We define

\[
\mathbb{K} = \{Y \in L^2(\Omega; C([T - \delta, T], H_\alpha)) : \sup_{t \in [T - \delta, T]} \|Y_t\|_{H_\alpha} \leq R, \quad a.s.\}.
\]
It easy to check that $K$ is a closed subset of $L^2(\Omega; C([T - \delta, T]; H_\alpha))$, hence a complete metric space (with the inherited metrics). We look for a local mild solution $(Y, Z)$ in the space $K$. We define a nonlinear operator $\Gamma : K \to K$ as follows: given $U \in K$, $Y = \Gamma(U)$ is the first component of the mild solution $(Y, Z)$ of the equation

$$Y_t - \int_t^T e^{(s-t)A}[f_0(s, U_s)ds + f_1(s)]ds + \int_t^T e^{(s-t)A}Z_s dW_s = e^{(T-t)A}\xi$$

(15) for $t \in [T - \delta, T]$. Since $U \in K$ we have $\mathbb{P}$-a.s.

$$|f_0(t, U_t) + f_1(t)|_H \leq S(1 + ||U_t||_{H_\alpha}) + C \leq S(1 + R^\gamma) + C,$$

(16) for all $t$ in $[T - \delta, T]$. Hence $f_0(\cdot, U_t) + f_1(\cdot)$ belongs to $L^2(\Omega \times [T - \delta, T]; H)$ and, by a result of Hu and Peng [6], there exists a unique pair $(Y, Z) \in L^2(\Omega \times [T - \delta, T]; H) \times L^2(\Omega \times [T - \delta, T]; L^2(K; H))$ satisfying (15). Moreover, by taking the conditional expectation given $\mathcal{F}_t$, $Y$ has the following representation

$$Y_t = \mathbb{E}^{\mathcal{F}_t}(e^{(T-t)A}\xi + \int_t^T e^{(s-t)A}[f_0(s, U_s) + f_1(s)]ds).$$

We will show that $\Gamma$ is a contraction for the norm of $L^2(\Omega, C([T - \delta, T]; H_\alpha))$ and maps $K$ into itself, if $\delta$ is sufficiently small; clearly, its unique fixed point is the required solution of the BSDE.

We first check the contraction property. Let $U^1, U^2 \in K$. Then

$$\Gamma(U^1)_t - \Gamma(U^2)_t = Y^1_t - Y^2_t = \mathbb{E}^{\mathcal{F}_t}[\int_t^T e^{(s-t)A}(f_0(s, U^1_s) - f_0(s, U^2_s))ds].$$

Let $v(t) = \int_t^T e^{(s-t)A}(f_0(s, U^1_s) - f_0(s, U^2_s))ds$. Then, noting that $v(T) = 0$ and recalling Lemma 3, for $t \in [T - \delta, T]$

$$||Y^1_t - Y^2_t||_{H_\alpha} = ||\mathbb{E}^{\mathcal{F}_t}v(t)||_{H_\alpha} \leq \mathbb{E}^{\mathcal{F}_t}|v(t)||_{H_\alpha} \leq \delta^{1-\alpha}\mathbb{E}^{\mathcal{F}_t}|v||_{C^{(1-\alpha)}([T - \delta, T]; H_\alpha)}$$

$$\leq \frac{G\delta^{(1-\alpha)}L_R}{\alpha} \mathbb{E}^{\mathcal{F}_T}[|f_0(\cdot, U^1_s) - f_0(\cdot, U^2_s)||_{L^\infty([T - \delta, T], H)}]$$

$$\leq G\delta^{(1-\alpha)}L_R \mathbb{E}^{\mathcal{F}_T} \sup_{t \in [T - \delta, T]} ||U^1_t - U^2_t||_{H_\alpha} =: M_t,$$

where $\{M_t, t \in [T - \delta, T]\}$ is a martingale. Hence, by Doob’s inequality

$$\mathbb{E} \sup_{t \in [T - \delta, T]} ||Y^1_t - Y^2_t||_{H_\alpha} \leq \mathbb{E} \sup_{t \in [T - \delta, T]} |M_t|^2 \leq 2\mathbb{E}|M_T|^2 =$$

$$= 2G^2L^2_R\delta^{2(1-\alpha)} \mathbb{E} \sup_{t \in [T - \delta, T]} ||U^1_t - U^2_t||_{H_\alpha}^2.$$
If $\delta \leq \delta_0 = 2GLR^{(1-\alpha)/2}$, then $\Gamma$ is a contraction with constant $1/2$.

Next we check that $\Gamma$ maps $\mathbb{K}$ into itself. For each $U \in \mathbb{K}$ and $t \in [T-\delta, T]$ with $\delta \leq \delta_0$ we have

$$
\sup_{t \in [T-\delta, T]} ||\Gamma(U)_t||_{H_\alpha} \leq \sup_{t \in [T-\delta, T]} ||Y_t||_{H_\alpha} \leq \sup_{t \in [T-\delta, T]} \mathbb{E}^F_t ||e^{(T-t)A}\xi||_{H_\alpha} + \sup_{t \in [T-\delta, T]} \mathbb{E}^F_t \left|\int_t^T e^{(s-t)A}[f_0(s, U_s) + f_1(s)]ds\right|_{H_\alpha}
$$

$$
\leq R/2 + \sup_{t \in [T-\delta, T]} \mathbb{E}^F_t \left|\int_t^T e^{(s-t)A}[f_0(s, U_s) + f_1(s)]ds\right|_{H_\alpha}
$$

$$
\leq R/2 + \sup_{t \in [T-\delta, T]} \mathbb{E}^F_t \int_t^T \left|e^{(s-t)A}[f_0(s, U_s) + f_1(s)]\right|_{D_A(\alpha, 1)}ds,
$$

where in the last inequality we have used the fact that $D_A(\alpha, 1) \subset H_\alpha$. Now, by Proposition 1, and from 4.i) and 5., it follows that

$$
||e^{(s-t)A}[f_0(s, U_s) + f_1(s)]||_{D_A(\alpha, 1)} \leq ||e^{(s-t)A}||_{L(H, D_A(\alpha, 1))} |f_0(s, U_s) + f_1(s)|_{H_\alpha} \leq \frac{C_\alpha}{(s-t)^\alpha} [S(1 + ||U_s||_{H_\alpha}) + C].
$$

Then, since $U \in \mathbb{K}$, we arrive at

$$
\sup_{t \in [T-\delta, T]} ||\Gamma(U)_t||_{H_\alpha} \leq
$$

$$
\leq R/2 + \sup_{t \in [T-\delta, T]} \mathbb{E}^F_t \int_t^T \frac{C_\alpha}{(s-t)^\alpha} [S(1 + ||U_s||_{H_\alpha}) + C]ds
$$

$$
\leq R/2 + \sup_{t \in [T-\delta, T]} \mathbb{E}^F_t \int_t^T \frac{C_\alpha}{(s-t)^\alpha} [S(1 + R^\gamma) + C]ds
$$

$$
\leq R/2 + C_\alpha S \left[\frac{(1 + R^\gamma) + C}{1 - \alpha}\right] \delta^{1-\alpha},
$$

where $C_\alpha$ depends on $A, \alpha$. Hence, if $\delta \leq \delta_0$ is such that $C_\alpha S \left[\frac{(1 + R^\gamma) + C}{1 - \alpha}\right] \delta^{1-\alpha}$ is less or equal to $R/2$, then $\sup_{t \in [T-\delta, T]} ||\Gamma(U)_t||_{H_\alpha} \leq R$. Due to Lemma 3, $\mathbb{P}$-a.s. the function $t \mapsto Y_t - \mathbb{E}^F_t e^{(T-t)A}\xi$ belongs to $C[T-\delta, T]; H_\alpha)$; moreover, the map $t \mapsto \mathbb{E}^F_t e^{(T-t)A}\xi$ belongs to $C[T-\delta, T]; H_\alpha)$, since $\xi$ is a random variable taking values in $D(A)_{H_\alpha}$. Therefore, $\mathbb{P}$-a.s. $Y \in C([T-\delta, T]; H_\alpha)$ and $\Gamma$ maps $\mathbb{K}$ into itself and has a unique fixed point in $\mathbb{K}$.

**Remark 3.** By Lemma 3, using properties of analytic semigroups, it can be proved that for every fixed $\omega$ the range of the map $\Gamma$ is contained in $C^{1-\beta}([T-\delta, T-\epsilon]; D_A(\beta, 1))$ for every $\epsilon \in (0, \delta)$, $\beta \in [0, 1]$.  

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3.3 Global existence

Now we are able to prove a global existence theorem for the solution of the equation (6), using all the results presented above.

**Theorem 5.** If Hypothesis 2 is satisfied, the equation (6) has a unique mild solution \((Y, Z) \in L^2(\Omega; C([0, T], H_\alpha)) \times L^2(\Omega \times [0, T]; L^2(K; H))\).

**Proof.** By Theorem 4 equation (6) has a unique mild solution \((Y^1, Z^1) \in L^2(\Omega; C([T - \delta_1, T], H_\alpha)) \times L^2(\Omega \times [T - \delta_1, T]; L^2(K; H))\) on the interval \([T - \delta_1, T]\), for some \(\delta_1 > 0\). By Proposition 2 we know that there exists a constant \(C_1\) such that \(\mathbb{P}\text{-a.s.}\)

\[
|Y_{T-\delta_1}|_H \leq C_1. \tag{17}
\]

We recall that the constant \(C_1\) depends only on \(|\xi|_{L^\infty(\Omega,H)}\) and on the constants \(S\) of 4.ii) and \(C\) of 5. and is independent of \(\delta_1\). Moreover, by Proposition 3, there exists a constant \(C_2\) such that \(\mathbb{P}\text{-a.s.}\)

\[
||Y_{T-\delta_1}||_{L^\infty(\Omega,D_A(\theta,\infty))} \leq C_2 \frac{1}{\delta_1^{\theta-\alpha}}, \tag{18}
\]

with \(C_2\) depending on the operator \(A\), \(||\xi||_{L^\infty(\Omega,H_\alpha)}\), \(\theta\), \(\alpha\), \(C_1\). This implies that \(Y_{T-\delta_1}\) belongs to \(L^\infty(\Omega;H_\alpha)\) and it can be taken as final value for the problem

\[
Y_t - \int_t^{T-\delta_1} e^{(s-t)A} [f_0(s,Y_s)ds + f_1(s)]ds + \int_t^{T-\delta_1} e^{(s-t)A} Z_s dW_s = e^{(T-\delta_1-t)A} Y_{T-\delta_1} \tag{19}
\]

on an interval \([T - \delta_1 - \delta_2, T - \delta_1]\), for some \(\delta_2 > 0\). As in the proof of Theorem 4, we fix a positive number \(R_2\) such that

\[R_2 = 2M_\alpha \frac{C_2}{\delta_1^{\theta-\alpha}} \geq 2M_\alpha ||Y_{T-\delta_1}||_{L^\infty(\Omega,D_A(\theta,\infty))}.\]

By Theorem 4 there exists a pair of progressively measurable processes \((Y^2, Z^2)\) in \(L^2(\Omega; C([T - \delta_1 - \delta_2, T - \delta_1]; H_\alpha)) \times L^2(\Omega \times [T - \delta_1 - \delta_2, T - \delta_1]; L^2(K, H))\) which solves (19) on the interval \([T - \delta_1 - \delta_2, T - \delta_1]\) where \(\delta_2\) depends on the operator \(A\), \(\alpha\), \(R_2\). We note that the continuity in \(T - \delta_1\) of \(Y^2\) follows from the fact that \(Y_{T-\delta_1}\) takes values in \(D_A(\alpha,1)\) (see Remark 3), so that \(Y_{T-\delta_1}\) takes values in \(\overline{D(A)}^{H_\alpha}\). Now, the process \(Y_1\) defined by \(Y_1^1\) on the interval \([T - \delta_1, T]\) and by \(Y_1^2\) on \([T - \delta_1 - \delta_2, T - \delta_1]\) belongs
to $L^2(\Omega; C([T - \delta_1 - \delta_2, T]; H))$ and it easy to see that it satisfies (6) in the whole interval $[T - \delta_1 - \delta_2, T]$. Consequently, by Proposition 2, $\mathbb{P}$-a.s., $|Y_{T-\delta_1-\delta_2}|_H \leq C_1$ with $C_1$ the constant in (17), and by (18)

$$
\|Y_{T-\delta_1-\delta_2}\|_{L^\infty(\Omega, D_A(\theta, \infty))} \leq \frac{C_2}{(\delta_1 + \delta_2)^{\theta - \alpha}} \leq \frac{C_2}{\delta_1^{\theta - \alpha}},
$$

(20)

where $C_2$ is the same constant as in (18). Again, $Y_{T-\delta_1-\delta_2}$ can be taken as initial value for problem

$$
Y_t - \int_t^{T-\delta_1-\delta_2} e^{(s-t)A}[f_0(s, Y_s)ds + f_1(s)]ds + \int_t^{T-\delta_1-\delta_2} e^{(s-t)A}Z_s dW_s = e^{(T-\delta_1-\delta_2)A}Y_{T-\delta_1-\delta_2}
$$

(21)

on the interval $[T - \delta_1 - \delta_2 - \delta_3, T - \delta_1 - \delta_2]$, where $\delta_3$ will be fixed later. In this case, by (20), we can choose

$$
R_3 = R_2 = 2M_\alpha \frac{C_2}{\delta_1^{\theta - \alpha}} \geq 2M_\alpha \|Y_{T-\delta_1-\delta_2}\|_{L^\infty(\Omega, D_A(\theta, \infty))}
$$

and prove that there exists a unique mild solution $(Y^3, Z^3)$ of (21) on the interval $[T - \delta_1 - \delta_2 - \delta_3, T - \delta_1 - \delta_2]$, with $\delta_3 = \delta_2$. So we extend the solution to $[T - \delta_1 - 2\delta_2, T]$. Proceeding this way we prove the global existence to (6) on $[0, T]$. \hfill \Box

4 The general case

We can now study the equation:

$$
Y_t - \int_t^T e^{(s-t)A}[f_0(s, Y_s) + f_1(s, Y_s, Z_s)]ds + \int_t^T e^{(s-t)A}Z_s dW_s = e^{(T-t)A} \xi
$$

(22)

We require that the function $f_1$ satisfy the following assumptions:

Hypothesis 6.

1. there exists $K \geq 0$ such that $\mathbb{P}$-a.s.

$$
|f_1(t, y, z) - f_1(t, y', z')|_H \leq K|y - y'|_H + K||z - z'||_{L^2(K; H)},
$$

for every $t \in [0, T], y, y' \in H, z, z' \in L^2(K; H),}$
2. there exists $C \geq 0$ such that $\mathbb{P}$-a.s. $|f_1(t, y, z)|_H \leq C$, for every $t \in [0, T], y \in H, z \in \mathcal{L}^2(K; H)$.

**Theorem 7.** If Hypotheses 2 and 6 hold, then equation (22) has a unique solution in $L^2(\Omega; C([0, T]; H_a)) \times L^2(\Omega \times [0, T]; \mathcal{L}^2(K; H))$.

**Proof.** Let $\mathcal{M}$ be the space of progressive processes $(Y, Z)$ in the space $L^2(\Omega \times [0, T]; H) \times L^2(\Omega \times [0, T]; \mathcal{L}^2(K; H))$ endowed with the norm

$$|||(Y, Z)|||^2_F = \mathbb{E} \int_0^T e^{\beta s}(|Y_s|^2_H + |Z_s|^2_{\mathcal{L}^2(K; H)}) ds,$$

where $\beta$ will be fixed later. We define $\Phi : \mathcal{M} \to \mathcal{M}$ as follows: given $(U, V) \in \mathcal{M}$, $(Y, Z) = \Phi(U, V)$ is the unique solution on the interval $[0, T]$ of the equation

$$Y_t - \int_t^T e^{(s-t)A}[f_0(s, Y_s) ds + f_1(s, U_s, V_s)] ds + \int_t^T e^{(s-t)A} Z_s dW_s = e^{(T-t)A} \xi.$$

By Theorem 5 the above equation has a unique mild solution $(Y, Z)$ which belongs to $L^2(\Omega; C([0, T]; H_a)) \times L^2(\Omega \times [0, T]; \mathcal{L}^2(K; H))$. Therefore $\Phi(\mathcal{M}) \subset \mathcal{M}$. We will show that $\Phi$ is a contraction for a suitable choice of $\beta$; clearly, its unique fixed point is the required solution of (22). We take another pair $(U', V') \in \mathcal{M}$ and apply Proposition 3.1 in [3] to the difference of two equations. We obtain

$$\mathbb{E} \int_0^T e^{\beta s} [\beta |Y_t^1 - Y_t^2|^2_H + |Z_s^1 - Z_s^2|^2_{\mathcal{L}^2(K; H)}] ds \leq$$

$$\leq 2\mathbb{E} \int_0^T e^{\beta s} \left( f_0(s, Y_s^1) + f_1(s, U_s^1, V_s^1) - f_0(s, Y_s^2) - f_1(s, U_s^2, V_s^2), Y_s^1 - Y_s^2 \right)_H ds$$

$$\leq 2\mathbb{E} \int_0^T e^{\beta s} K(|U_s^1 - U_s^2|_H + |V_s^1 - V_s^2|_{\mathcal{L}^2(K; H)}) |Y_s^1 - Y_s^2|_H ds$$

$$\leq \mathbb{E} \int_0^T e^{\beta s} (|U_s^1 - U_s^2|^2_H + |V_s^1 - V_s^2|^2_{\mathcal{L}^2(K; H)}) / 2 + 4K^2 |Y_s^1 - Y_s^2|^2_H ds,$$

where we have used 4.iv) of Hypothesis 2 and 1. of Hypothesis 6. Choosing $\beta = 4K^2 + 1$, we obtain the required contraction property.

### 5 Applications

In this section we present some backward stochastic partial differential problems which can be solved with our techniques.
5.1 The reaction-diffusion equation

Let $D$ be an open and bounded subset of $\mathbb{R}^n$ with a smooth boundary $\partial D$. We choose $K = L^2(D)$. This choice implies that $dW_t/dt$ is the so-called "space-time white noise". Moreover, since Hilbert-Schmidt operators on $L^2(D)$ are represented by square integrable kernels, the space $L^2(L^2(D), L^2(D))$ can be identified with $L^2(D \times D)$. We are given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ generated by $W$ and augmented in the usual way. Let us consider a non symmetric bilinear, coercive continuous form $a : H^1_0(D) \times H^1_0(D) \to \mathbb{R}$ defined by

$$a(u, v) := - \int_D \sum_{i,j} a_{ij}(x) D_i u(x) D_j v(x) dx,$$

where the coefficients $a_{ij}$ are Lipschitz continuous and there exists $\alpha > 0$ such that $\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2$ for every $x \in \bar{D}, \xi \in \mathbb{R}^n$. Let $A$ be the operator associated with the bilinear form $a$ such that $< Au, v >_{L^2(D)} = a(u, v), v \in H^1_0(D)$ and $u \in D(A)$. It is known that, in this case, $D(A) = H^2(D) \cap H^1_0(D)$, where $H^2(D)$ and $H^1_0(D)$ are the usual Sobolev spaces.

We consider for $t \in [0,T]$ and $x \in D$ the backward stochastic problem written formally

\[
\begin{cases}
\frac{\partial Y(t,x)}{\partial t} = AY(t,x) + r(Y(t,x)) + g(t,Y(t,x),Z(t,x),x) + Z(t,x) \frac{\partial W(t,x)}{\partial t} & \text{on } \Omega \times [0,T] \times \bar{D} \\
Y(T,x) = \xi(x) & \text{on } \Omega \times D \\
Y(t,x) = 0 & \text{on } \Omega \times [0,T] \times \partial D
\end{cases}
\]

We suppose the following.

**Hypothesis 8.**

1. $r : \mathbb{R} \to \mathbb{R}$ is a continuous, increasing and locally Lipschitz function;
2. $r$ satisfies the following growth condition: $|r(x)| \leq S(1 + |x|^\gamma)$ $\forall x \in \mathbb{R}$ for some $\gamma > 1$;
3. $g$ is a measurable real function defined on $[0,T] \times \mathbb{R} \times L^2(D \times D) \times D$ and there exists a constant $K > 0$ such that

$$|g(t,y_1,z_1,x) - g(t,y_2,z_2,x)| \leq K(|y_1 - y_2| + ||z_1 - z_2||_{L^2(D \times D)})$$

for all $t \in [0,T], y_1, y_2 \in \mathbb{R}, z_1, z_2 \in L^2(D), x \in D;$
1. there exists a real function $h$ in $L^2(D \times D)$ such that $\mathbb{P}$-a.s. $|g(t, y, z, x)| \leq K_1 h(x)$ for all $t \in [0, T], y \in \mathbb{R}, z \in L^2(D), x \in D$;

5. $\xi$ belongs to $L^\infty(\Omega; H^2(D) \cap H^1_0(D))$.

We define the operator $A$ by $(Ay)(x) = Ay(x)$ with domain $D(A) = H^2(D) \cap H^1_0(D)$. We set $f_0(t, y)(x) = -r(y(t, x))$ for $t \in [0, T], x \in D$ and $y$ in a suitable subspace of $H$ which will be determined below. For $t \in [0, T], x \in D, y \in L^2(D), z \in L^2(D \times D)$ we define $f_1$ as the operator $f_1(t, y, z)(x) = -g(t, y(t, x), z(t, x), x)$. Then problem (23) can be written in abstract way as

$$dY_t = -AY_t dt - f_0(t, Y_t) dt - f_1(t, Y_t, Z_t) dt + Z_t dW_t, \quad Y_T = \xi.$$

Under the conditions in Hypothesis 8, the assumptions in Hypotheses 2, 6 are satisfied. The operator $A$ is a closed operator in $L^2(D)$ and it is the infinitesimal generator of an analytic semigroup in $L^2(D)$ satisfying $\|e^{tA}\|_{L(H)} \leq 1$ (see [17], Chapter 3). In particular, by Lumer-Philips theorem, $A$ is dissipative. The non linear function $f_0(t, \cdot) : L^{2\gamma}(D) \to L^2(D)$, $y \mapsto -r(y)$ is locally Lipschitz. We look for a space of class $J_\alpha$ between $H$ and $D(A)$ where $f_0$ is well defined and locally Lipschitz. It is well known (see [18]) that the fractional order Sobolev space $W^{\beta, 2}(D)$ is of class $J_{\beta/2}$ between $L^2(D)$ and $H^2(D)$ for every $\beta \in (0, 2)$. Hence the space $H_\alpha$ defined by $H_\alpha = W^{\beta, 2}(D)$ if $\beta < 1$, by $W^{\beta, 2}(D) \cap H^1_0(D)$ if $\beta = 1$ is of class $J_{\beta/2}$ between $H$ and $D(A)$. Moreover the restriction of $A$ on $H_\alpha$ is a sectorial operator ([18]). By the Sobolev embedding theorem, $W^{\beta, 2}$ is contained in $L^q(D)$ for all $q$ if $\beta \geq \frac{n}{2}$, and in $L^{2n/(n-2\beta)}(D)$ if $\beta < \frac{n}{2}$. If we choose $\beta \in (0, 2)$ we have $W^{\beta, 2}(D) \subset L^{2\gamma}(D)$ for $n < 4\frac{\gamma}{\gamma - 1}$. It is clear that $f_0$ is locally Lipschitz with respect to $y$ from $H_\alpha$ into $H$. It is easy to verify that $f_0$ satisfies 4.ii) of Hypothesis 2 with $\gamma = 2n + 1$ and that it is dissipative with constant $\mu = 0$. The function $f_1$ is Lipschitz uniformly with respect to $y$ and $z$ and it is bounded. The final condition $\xi$ takes values in $D(A)^{H_\alpha}$ and belongs to $L^\infty(\Omega; H_\alpha)$. Hence we can apply the global existence theorem and state that the above problem has a unique mild solution $(Y, Z) \in L^2(\Omega; C([0, T]; H_\alpha)) \times L^2(\Omega \times [0, T]; L^2(K, H))$.

5.2 A spin system

Let $Z$ be the one-dimensional lattice of integers. Its elements will be interpreted as atoms. A configuration is a real function $y$ defined on $Z$. The value $y(n)$ of the configuration at the point $n$ can be viewed as the state of the atom $n$. 

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We consider an infinite system of equations

\[ dY^n_t = -a_n Y^n_t \, dt + \sum_{|n-j| \leq 1} V(Y^n_t - Y^j_t) \, dt + Z^n_t \, dW_t \quad n \in \mathbb{Z}, \quad 0 \leq t \leq T \]  

(24)

\[ Y^n_T = \xi_n \quad n \in \mathbb{Z}, \]

where \( Y^n \) and \( Z^n \) are real processes, and \( V : \mathbb{R} \rightarrow \mathbb{R} \).

Let \( l^2(\mathbb{Z}) \) be the usual Hilbert space of square summable sequences. To study system (24) we apply results of previous sections. To fit our assumption in Hypotheses 2 and 6, we suppose the following

**Hypothesis 9.**

1. \( W^n, n \in \mathbb{Z} \) are independent standard real Wiener processes;
2. \( a = \{a_n\}_{n \in \mathbb{Z}} \) is a sequence of nonnegative real numbers;
3. \( \xi = \{\xi_n\}_{n \in \mathbb{Z}} \) is a random variable belonging to \( L^\infty(\Omega, l^2(\mathbb{Z})) \);
4. the function \( V : \mathbb{R} \rightarrow \mathbb{R} \) is defined by \( V(x) = x^{2k+1} \quad k \in \mathbb{N} \).

We will study system (24) regarded as a backward stochastic evolution equation for \( t \in [0, T] \)

\[ dY_t = (AY_t + f_0(t, Y_t)) \, dt + Z_t \, dW_t, \quad Y_T = \xi \]  

(25)

on a properly chosen Hilbert space \( H \) of functions on \( \mathbb{Z} \).

To reformulate problem (24) in the abstract form (25), we set \( K = H = l^2(\mathbb{Z}) \). We set \( W_t = \{W^n_t\}_{n \in \mathbb{Z}}, t \in [0, T] \). By 1. of Hypothesis 9, \( W \) is a cylindrical Wiener process in \( H \) defined on \( (\Omega, \mathcal{F}, P) \). We define the operator \( A \) by

\[ A(y) = (a_n y_n)_n, \quad D(A) = \{y \in l^2(\mathbb{Z}) \text{ such that } \sum_{n \in \mathbb{Z}} a_n^2 y_n^2 < \infty \}. \]

It is easy to prove that \( A \) is a self-adjoint operator in \( l^2(\mathbb{Z}) \), hence the infinitesimal generator of a sectorial semigroup. The coefficient \( f_0 \) is given by \( (f_0(t, y))_n = (V(y_{n+1} - y_n) + V(y_{n-1} - y_n)), \quad t \in [0, T], y \in D(f_0) \) where \( D(f_0) = \{y \in l^2(\mathbb{Z}) \text{ such that } \sum_{n \in \mathbb{Z}} |x_{n+1} - x_n|^{2(2k+1)} < +\infty \}. \) Under Hypothesis 9, \( A, f_0, \xi \) satisfy Hypotheses 2 and 6. We observe that in this case the domain of \( f_0 \) is the whole space \( H \): if \( y \in l^2(\mathbb{Z}) \) then

\[ \left\{ \sum_{n \in \mathbb{Z}} |y_{n+1} - y_n|^{2(2k+1)} \right\}^{1/(2k+1)} \leq \left\{ \sum_{n \in \mathbb{Z}} |y_{n+1} - y_n|^2 \right\}^{1/2} \leq 2||y||l^2(\mathbb{Z}). \]
Consequently, we can take $H_\alpha$ with $\alpha = 0$, i.e. $H_0 = H$. The function $f_0$ is dissipative. Namely

$$< f_0(t, y) - f_0(t, y'), y - y' >_{l^2(\mathbb{Z})} =$$

$$= \sum_{n \in \mathbb{Z}} \{(y_{n+1} - y_n)(2k+1) + (y_{n-1} - y_n)(2k+1)\} +$$

$$+ \{(y'_{n+1} - y'_n)(2k+1) + (y'_{n-1} - y'_n)(2k+1)\}[y_n - y'] =$$

$$= - \sum_{n \in \mathbb{Z}} \{(y_{n+1} - y_n)(2k+1) - (y_{n-1} - y_n)(2k+1)\}[y_n - y']$$

and the last term is negative. Moreover, $f_0$ satisfies 4.ii) of Hypothesis 2 with $\gamma = 2k + 1$. The map $f_0$ is also locally Lipschitz from $H$ in to $H$.

Then by Theorem 7, problem (25) has a unique mild solution $(Y, Z) \in L^2(\Omega, C([0, T]; H)) \times L^2(\Omega \times [0, T]; L^2(K, H))$.

6 Appendix

This section is devoted to the proof of Lemma 1. Assume first that $\beta = 1$. Using recursively the inequality $U_t \leq a(T - t)^{-\alpha} + b \int_t^T \mathbb{E}^{\mathcal{F}_s} U_s ds$ we can easily prove that

$$U_t \leq a(T - t)^{-\alpha} + \int_t^T a \sum_{k=1}^{n-1} b^k (r-t)^{k-1} \frac{1}{(k-1)!} (T-r)^\alpha dr +$$

$$+ b \mathbb{E}^{\mathcal{F}_T} \int_t^T \frac{(b(r-t))^{n-1}}{(n-1)!} U_r dr.$$

The last term in the above inequality tends to zero as $n$ tends to infinity for each $t$ in the interval $[0, T]$. Thus

$$U_t \leq a(T - t)^{-\alpha} + a \sum_{k=1}^{\infty} \int_t^T \frac{b^k (T-t)^{k-1}}{(k-1)!} (T-r)^\alpha dr \leq$$

$$\leq a(T - t)^{-\alpha} + ab(T-t) \int_t^T \frac{1}{(T-r)^\alpha} dr \leq$$

$$\leq a(T - t)^{-\alpha} + ab(T-t) \frac{1}{1-\alpha} (T-t)^{1-\alpha} \leq a(T - t)^{-\alpha} M$$

where $M = 1 + be^{b(T-t)^{1-\alpha}} T$.

In the case $\beta \neq 1$ a similar proof can be given, based on recursive use of the inequality $U_t \leq a(T - t)^{-\alpha} + b \int_t^T (s-t)^{\beta-1} \mathbb{E}^{\mathcal{F}_s} U_s ds$. 

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References

[1] Ph. Briand and R. Carmona. BSDEs with polynomial growth generators. *J. Appl. Math. Stochastic Anal.*, 13(3):207–238, 2000.

[2] Ph. Briand and B. Deylon and Y. Hu and E. Pardoux and L. Stoica. $L^p$ solutions of backward stochastic differential equations. *Stochastic Process. Appl.*, 108(1):109–129, 2003.

[3] F. Confortola. Dissipative backward stochastic differential equations in infinite dimensions. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 9(1):155–168, 2006.

[4] N. El Karoui and S. G. Peng and M. C. Quenez. Backward Stochastic Differential equations in Finance. *Math. Finance*, 7(1):1–71, 1997.

[5] H. Fujita and T. Kato. On the Navier-Stokes initial value problem I. *Arch. Rational Mech. Anal.* 16:269–315, 1964.

[6] Y. Hu and S. G. Peng. Adapted solution of a backward semilinear stochastic evolution equation. *Stochastic Anal. Appl.*, 9(4):445–459, 1991.

[7] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems* volume 16 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel 1995.

[8] J. Ma and J. Yong Adapted solution of a degenerate backward SPDE, with applications. *Stochastic Process. Appl.* 70:59–84, 1997.

[9] J. Ma and J. Yong On linear, degenerate backward stochastic partial differential equations. *Probab. theory Related Fields* 113:135–170 1999.

[10] B. Oksendal and T. Zhang. On backward stochastic partial differential equations, 2001. Preprint.
[11] E. Pardoux. BSDEs, weak convergence and homogenization of semilinear PDEs. *Nonlinear analysis, differential equations and control (Montreal, QC, 1998)*, 503–549, NATO Sci. Ser. C Math. Phys. Sci., 528, Kluwer Acad. Publ., Dordrecht, 1999.

[12] É. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems and Control Lett.* 14:55–61, 1990.

[13] E. Pardoux and A. Răşcanu. Backward stochastic differential equations with subdifferential operator and related variational inequalities. *Stochastic Process. Appl.*, 76(2):191–215, 1998.

[14] E. Pardoux and A. Răşcanu. Backward stochastic variational inequalities. *Stochastics Stochastics Rep.*, 67(3-4):159–167, 1999.

[15] A. Pazy *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, (1983).

[16] S. Peng Stochastic Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.*, 30:284–304, 1992.

[17] H. Tanabe *Equations of evolution*. Monographs and Studies in Mathematics, 6. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.

[18] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators vol. 18 of North-Holland Mathematical Library* North-Holland Publishing Co., Amsterdam-New York, 1978