Spontaneous Spin Textures in Dipolar Spinor Condensates: A Dirac String Gas Approach

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We study the spontaneous spin textures induced by magnetic dipole-dipole interaction in ferromagnetic spinor condensates under various trap geometries. At the mean-field level, we show the system is dual to a Dirac string gas with a negative string tension in which the ground state spin texture can be easily determined. We find that three-dimensional condensates prefer a meron-like vortex texture, quasi one-dimensional condensates favor the axially polarized flare texture, while condensates in quasi two dimensions exhibit either a meron texture or an in-plane polarized texture.

The magnetic dipole-dipole interaction (DDI) is known to play a significant role in determining the long range behaviors of ultracold gases of spinful atoms, especially those with a large spin \( \frac{1}{2} \). In the ferromagnetic spinor condensates of spinful bosons with nonzero spin expectation values \( \langle F \rangle \), the DDI usually leads to rich spin textures that strongly depend on the trap geometry and the initial state preparation [3][15]. Meanwhile, the existence of spontaneous spin textures in dipolar spinor condensates has been verified by numerical simulations based on the mean-field Gross-Pitaevskii equation [8][12]. How-ever, due to the complicated form of DDI, there still lacks an explicit and efficient theoretical way to understand the spin textures.

In this letter, we develop an analytical approach to the problem by introducing a duality at the mean-field level from a dipolar spinor condensate to a weakly interacting Dirac string gas with a negative string tension. We find that the ground state is reached by forming as many closed strings with small enough curvatures as possible. Based on this principle, we are able to determine the spin textures in various trap geometries used in cold atom experiments. We show the ground state spin texture of a large three-dimensional (3D) spherical condensate is a meron-like vortex texture, while the axially polarized flare texture [10] is shown to be favored in quasi one-dimensional (1D) condensates. In quasi two-dimensional (2D) pancake traps, a phase transition driven by the pancake radius is found between an in-plane polarized texture and a meron texture. In particular, multiple merons can be seen to arise in quasi 2D traps when the aspect ratio is comparatively large, as is found numerically in Ref. [3]. The Dirac string gas picture thus offers a greatly simplified way of understanding the spin textures in dipolar condensates.

We begin with the description of the ferromagnetic spinor condensates at low energies. In general, a spinor condensate of integral spin \( F \) atoms is characterized by a spinor order parameter of \( 2F + 1 \) components \( \Psi_m(r) \) \( (m = -F, -F + 1, \cdots, F) \), which represents the coherent amplitude of annihilating a boson in the spin state \( |F, m\rangle \) at position \( r \) [12][10]. At the mean field level, the number of atoms per unit volume in the condensate is

\[
n(r) = \sum_m \langle \Psi_m^*(r) \Psi_m(r) \rangle = \frac{1}{\sqrt{n(r)}} \Psi(r),
\]

while the expectation of the local spin is \( \langle F(r) \rangle = \varphi^2(r) \bar{F} \varphi(r) \), where \( \varphi(r) = \Psi(r)/\sqrt{n(r)} \) is the normalized spinor order parameter, and \( \bar{F} \) is the spin matrix of the spin \( F \) representation. In a slow enough trap potential, we can approximate the particle number density \( n(r) = n_0 \) as a constant. Up to a U(1) × SU(2) transformation, the normalized spinor order parameter \( \varphi(r) \) is determined by the local interactions, which are characterized by \( F + 1 \) scattering lengths \( a_{F,J} \) between two atoms with total spin \( 2J \) \( (0 \leq J \leq F) \) [10]. A spinor condensate is said to be ferromagnetic if \( \varphi(r) = (0, \cdots, 0, 1, 0, \cdots, 0) \) in a certain basis, where the only nonzero component is \( m = F_0 > 0 \).

At low energies, the long wave length fluctuation of the order parameter in the coset space of degeneracy gives rise to multiple gapless Goldstone modes. In a ferromagnetic spinor condensate, there is exactly one quadratic dispersion mode corresponding to the fluctuation of the local spin [13][16]. In the presence of the magnetic dipole-dipole interaction, the effective low energy spin Hamiltonian of a ferromagnetic spinor condensate can be written as

\[
H = H_0 + H_D,
\]

where

\[
H_0 = \frac{\alpha}{M} \int d^3 r (\nabla \mathbf{F}(r))^2,
\]

\[
H_D = \frac{\lambda}{2} \int d^3 r_1 d^3 r_2 \frac{\mathbf{F}_1 \cdot \mathbf{F}_2 - 3(\mathbf{F}_1 \cdot \mathbf{r}_{12})(\mathbf{F}_2 \cdot \mathbf{r}_{12})}{4\pi r_{12}^3},
\]

are the kinetic energy of the quadratic Goldstone mode and the DDI energy, respectively. We have used here the normalized local spin field \( \mathbf{F}(r) = (\mathbf{F}(r))/F_0 \) that satisfies \( |\mathbf{F}(r)| = 1 \) for later convenience. \( M \) denotes the particle mass, \( \alpha \) is defined by \( \alpha = n_0 \hbar^2 F(F + 1) - F_0^2/4 \), and the interaction parameter \( \lambda \) is given by \( \lambda = \mu_0 (g_F \mu_B)^2 n_0^2 F_0^2 \), where \( g_F \) is the Landé factor and \( \mu_B \) is the Bohr magneton. In writing the \( H_D \) term, we have used the abbreviations \( \mathbf{F}_i = \mathbf{F}(r_i) \) and \( \mathbf{r}_{12} = r_{12}\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2 \), where \( \mathbf{r}_{12} \) is the corresponding unit vector.

It is difficult to infer the low energy spin texture configurations directly from the perspective of local spins, since the spin-spin interaction in \( H_D \) is highly anisotropic.
However, it is possible to understand $H_D$ more naturally in terms of Dirac strings. Consider a chain of atoms that are arranged uniformly along a curve as is shown in Fig. (a), with their spins aligned head-to-tail and parallel to the tangent of the curve. Since each atom carries a magnetic dipole moment $\mathbf{m}_F = g_F \mu_B \langle \mathbf{F} \rangle$, the curve can be exactly viewed as a Dirac string, with a positive magnetic monopole at one end and a negative one at the other. If we parameterize the curve as $r(l)$ with an affine parameter $l$, and require $|dr/dl| = 1$, the local spin field on the curve is then $\mathbf{F} = dr/dl$. We can easily calculate the dipolar interaction between such a Dirac string and a spin magnetic moment $\mathbf{m}_F$ not belonging to the curve:

$$E_{\mathbf{m}_F} = \mu_0 Q_m \int dl \frac{r^2}{4\pi} \frac{dr}{dl} \cdot \mathbf{m}_F - 3 \mathbf{m}_F \cdot \mathbf{r}$$

$$E_{\mathbf{m}_F} = \mu_0 Q_m \left( \frac{r_i}{4\pi r_f} - \frac{r_f}{4\pi r_i} \right) \cdot \mathbf{m}_F = -\mathbf{m}_F \cdot \mathbf{B}_{str}$$

where we have defined $Q_m = g_F \mu_B F_0/b$ with the spacing between atoms on the string $b \sim n_0^{-1/3}$, while $r_i$ and $r_f$ are the positions of the ends of the string relative to $\mathbf{m}_F$. This means that effectively the magnetic moment $\mathbf{m}_F$ only feels the magnetic field of the monopoles at the ends of the string, which carry monopole charges $\pm Q_m$ respectively. Remarkably, a closed Dirac string has no interaction energy with other spins at all.

Further, it is not hard to prove that the interaction energy between two Dirac strings is solely given by the interaction of the four monopoles at the ends of the strings, as is shown in Fig. (b). The interaction energy between two monopoles $Q_{m_1}$ and $Q_{m_2}$ is simply given by the Coulomb law, $E_{Q}(1,2) = \mu_0 Q_{m_1} Q_{m_2}/4\pi r_{12}$. Two parallel(anti-parallel) Dirac strings will therefore repulse(attract) each other. Still, a closed string does not interact with any other strings. The Dirac strings are therefore almost free under the dipolar interaction except for their ends.

In the continuum limit, we can view the flow lines of the local spin field $\mathbf{F}(r)$ of the condensate as Dirac strings, each of which consists of a single column of atoms. We denote the cross sectional area of a string as $\sigma$, which is around $n_0^{-2/3}$. The spacing between atoms on a string is then $b = (\mu_0 \alpha \sigma)^{-1}$, and the monopole charge can be expressed as $Q_m = g_F \mu_B n_0 \alpha F_0 \sigma$. Though the bulk of a Dirac string has no interaction with any other strings, it has a negative self interaction energy. Consider a spin on a Dirac string of length $L \gg b$. Locally, we can view the string as a straight line, so the spin is attracted by the other spins on the line. The attraction energy the spin feels is then estimated to be $-E_{F_0} \times \mu_0 (g_F \mu_B F_0)^2 \lambda d^2 n_0 \sim \mu_0 (g_F \mu_B F_0)^2 n_0 = \lambda n_0$. To determine the coefficient of proportionality, we can consider a 3D spherical condensate with all spins polarized along $z$ direction (Fig. (b)), and compute the dipolar interaction energy $E_d$ sensed by the spin at the center. In the perspective of spins and using the spherical coordinates, this energy is

$$E_d = \frac{\lambda}{n_0} \int_0^R 2\pi r^2 dr \int_0^\pi d\theta \sin \theta \frac{1 - 3 \cos^2 \theta}{4\pi r^3} = 0 \quad (3)$$

where $R$ is the radius, and $\theta$ is the polar angle. In the perspective of Dirac strings, the spin feels the attraction energy $E_{F_0}$ from its own string and a repulsion from monopoles at the ends of all the other strings. In cylindrical coordinates $(\rho, \varphi, z)$ where the cross sectional area of a string is $d\sigma = d\rho d\varphi$, this energy is calculated as

$$E_d = E_{F_0} + \frac{2\lambda}{n_0} \int_0^R 2\pi \rho d\rho \sqrt{\rho^2 - \rho_0^2} \frac{\lambda}{4\pi R^3} = E_{F_0} + \frac{\lambda}{3n_0} \quad (4)$$

From Eq. (3) and Eq. (4), we see the attraction energy is $E_{F_0} = -\lambda/3n_0$. The self interaction energy of a Dirac string of length $L$ and cross sectional area $d\sigma$ is then

$$E_{str}^{(d)}(L) = \frac{E_d}{2} \times n_0 L d\sigma = -\frac{1}{6} \lambda L d\sigma$$

This means a Dirac string has a negative string tension $T_{str} = -\lambda d\sigma/6$, and thus the whole space will be filled with Dirac strings. The action for such a Dirac string can be written as $S_{str} = (T_{str}/\hbar) \int dtdl = T_{str} \Sigma/\hbar$, where $\Sigma$ is the worldsheet area of the string. The Dirac string therefore corresponds to a classical bosonic string theory with a $U(1)$ gauge group $[17]$.

Now we turn to the kinetic energy $H_0$, and interpret it in the language of Dirac strings. By dividing the gradient operator into $\nabla = \nabla_{||} + \nabla_{\perp}$, where $||$ and $\perp$ stands for parallel and perpendicular to the string respectively, we can rewrite $H_0$ into two parts: The parallel part contributes an additional energy to the string self energy, so
that it now reads

$$E_{\text{str}}(L) = d\sigma \int_0^L dl \left[ \frac{1}{6} \lambda + \frac{\alpha}{M} \left( \frac{d^2 r}{dl^2} \right)^2 \right]. \quad (6)$$

Note that $|d^2 r/dl^2|$ is the curvature of the string. The perpendicular part produces a nonnegative contact interaction between two Dirac strings $L_1$ and $L_2$:

$$E_I(L_1, L_2) = d\sigma_1 d\sigma_2 \frac{3\alpha}{4M} \int_0^{L_1} dl_1 \int_0^{L_2} dl_2 \delta^3(r_{12})$$

$$\times \left( \frac{\mathcal{F}_1 - \mathcal{F}_2}{r_{12}} \right)^2 \left[ 3 - 5 \left( \frac{\mathcal{F}_1 \cdot r_{12}}{r_{12}^2} \right) \frac{\mathcal{F}_2 \cdot r_{12}}{r_{12}^2} \right], \quad (7)$$

where we have used the notations $r_{12} = r_1(l_1) - r_2(l_2)$ and $\mathcal{F}_i = dr_i(l_i)/dl_i$. Though it looks awkward, this interaction simply means two Dirac strings close to each other prefer to be coplanar so that $(\mathcal{F}_1 - \mathcal{F}_2)/r_{12} \to 0$. Therefore, we see the ferromagnetic dipolar condensate at the mean-field level is dual to a Dirac string gas with a string self energy $E_{\text{str}}$, a monopole interaction energy $E_Q$ and a contact interaction energy $E_I$. With $E_{\text{str}}$ mostly coming from $H_D$ and $E_I$ from $H_0$, this duality in some sense interchanges the roles of the kinetic energy and the interaction energy, and is thus a strong-weak duality.

The ground state spin texture can be easily obtained in the Dirac string gas picture. First, since an open string can always lower its monopole interaction energy by attaching its two ends together and forming a closed string, a strong dipolar condensate will prefer as many closed strings as possible. Further, to reduce the kinetic energy $H_0$, the close strings will tend to have a smaller average curvature and be locally coplanar to each other. These constitute the guiding rules for identifying the spin textures in a condensate under various trap geometries.

We now consider a 3D spherical dipolar condensate with radius $R$. Based on the guiding rules above, the most natural spin texture one can expect is a meron-like vortex texture as shown in Fig. 2(b). Since circular closed strings possess the smallest average curvature, the region $\rho > R_0$ (in cylindrical coordinates) is fulfilled with parallel circular strings, where $R_0$ is a characteristic length scale to be determined. All the circular strings are symmetric about the $z$ axis, so that locally they are always coplanar. At $\rho < R_0$, the circular strings gradually give way to spiral strings proceeding along the $z$ direction that have a lower kinetic energy $H_0$, and finally at $\rho = 0$ the string becomes vertical and straight. Consequently, there will be monopoles distributed on the $\rho < R_0$ patches of the sphere’s surface, with a monopole area density $n_A \approx \pm Q_m/d\sigma = \pm g_F \mu_B n_0 R_0$. The total energy of the dipolar condensate can be estimated as

$$E_{AD} \approx \frac{\lambda}{6} V + \int_{\rho \geq R_0} R_0^3 \frac{d^3 r}{\rho^2} \frac{\alpha}{M} \frac{1}{\rho^2} + \frac{2 g_0 (\pi R_0^3 n_A)^2}{4 \pi R_0}.$$  \quad (8)

$$\approx -\frac{\lambda}{6} V + \frac{4\pi \alpha}{M} R_0 \log \frac{R}{R_0} + \frac{\pi}{2} R_0^3,$$

where $V = 4\pi R^3/3$ is the volume of the condensate, and $R_0/R$ is assumed small. The second term comes from curvatures of the strings, and the third term is the monopole interaction energy $E_Q$. By minimizing the total energy with respect to $R_0$, we get the characteristic length $R_0 = (10\alpha/3\lambda)^{1/3} R^{1/3}$. The condition for the moneron-like spin texture to arise is then roughly $R > R_0$, or $R > (10\alpha/3\lambda)^{1/3}$. As expected, the spin textures are more likely to arise for massive and large spin atoms.

It is easy to see the moneron-like spin texture is stable. When perturbed, the circular strings may be tilted, misaligned, or distorted. In any case, the total energy is raised higher. In the examples shown in Fig. 2(c) and Fig. 2(d), parallel open strings have to arise to keep the strings dense, so they cost a monopole interaction energy. In another case shown in Fig. 2(e), no monopole occurs, but the strings are no longer locally coplanar to each other, which increases the kinetic energy $H_0$.

This analysis can be readily applied to the dipolar condensates in other trap geometries. A practical geometry is the axially symmetric trap where the boundary is given by $x^2 + y^2 + (z/\alpha)^2 = R^2$. By setting $\alpha \gg 1$, we get a quasi-1D cigar trap of length $L_c = 2AR$. In this case, circular closed strings are no longer energetically favorable, since their curvatures are large. Instead, vertical strings along $z$ direction are preferred so that the amount of monopoles on the surface is minimal. In a realistic trap where the particle density becomes lower near the surface, the vertical strings will be bent outward, forming a flare texture as shown in Fig. 3(a). By an energy estimation similar to the above, one can show that circular strings begin to occur when $A \sim (\lambda M/\alpha)^{1/3} L_c^{2/3}$ or smaller, leading to a crossover from the flare texture to...
the 3D meron-like vortex texture.

In the opposite limit \( A \ll 1 \), the trap is a quasi-2D pancake of radius \( R \). Correspondingly, there are two candidate ground state spin textures. The first spin texture is shown in Fig. 3(b), where all strings are in-plane and polarized along the same direction. The total energy of this texture consists of the string self-energy and the surface monopole interaction energy, or explicitly

\[
E^{(I)}_{2D} \approx -\frac{\lambda}{6} V + w \lambda R^3 A^2 \log \frac{A}{\alpha},
\]

where \( V \) is the volume of the condensate, and \( w > 0 \) is a numerical factor. The second spin texture is the 2D meron texture shown in Fig. 3(c). Similar to the 3D case, there is a characteristic radius \( R_{2D} \). The region outside \( R_{2D} \) is fulfilled with in-plane circular strings, while inside \( R_{2D} \) strings become spiral and finally vertical at the center. In the case the half height of the pancake \( z_h = AR \ll R_{2D} \), the total energy of the configuration can be estimated as [19]

\[
E^{(II)}_{2D} \approx -\frac{\lambda}{6} V + \frac{4\pi\alpha}{M} z_h \log \frac{R_{2D}}{R} + \frac{\pi\lambda}{2} z_h R_{2D}^2.
\]

Minimizing the energy we get \( R_{2D} = (4\alpha/\lambda M)^{1/2} \), which is independent of \( z_h \) or \( R \). At a fixed half height \( z_h \), one can show that the in-plane polarized texture is favored when \( R < R_c \), where \( R_c \) is a critical radius determined by solving \( E^{(I)}_{2D} = E^{(II)}_{2D} \), while the meron texture arises when \( R > R_c \). We note the two textures are separated by a phase transition at \( R = R_c \) instead of a crossover.

When the pancake trap is not axially symmetric, multiple merons may occur. As an example, in a rectangular trap with aspect ratio \( q = 2 \), the single-meron texture in Fig. 3(a) costs a kinetic energy \( \propto z_h R/R_{2D} \) near the center. Instead, the in-plane polarized texture in Fig. 3(b) only costs a kinetic energy \( \propto z_h \log(R/R_{2D}) \), thus it will be favored at large \( R \). The two merons are aligned antiparallel to lower the DDI energy. Similarly, a \( q \)-meron texture may arise for a trap with aspect ratio \( q \), which agrees with the numerical results in Ref. [9].

In conclusion, we have shown that the duality to the Dirac string gas picture is an efficient way of determining the spontaneous spin texture in a dipolar spinor condensate. We expect this method to be further employed in future work to study the spin textures in external fields and the dynamics of spin textures. In addition, it will also be useful and intriguing to possibly construct a quantum version of this duality, instead of the one at the mean-field level presented here.

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