ASYMPTOTIC STABILITY OF VISCOUS CONTACT WAVE AND RAREFACTION WAVES FOR THE SYSTEM OF HEAT-CONDUCTIVE IDEAL GAS WITHOUT VISCOITY

LILI FAN, GUIQIONG GONG, AND SHAOJUN TANG*

This paper is concerned with the Cauchy problem of heat-conductive ideal gas without viscosity. We show that, for the non-viscous case, if the strengths of the wave patterns and the initial perturbation are suitably small, the unique global-in-time solution exists and asymptotically tends toward the corresponding the viscous contact wave or the composition of a viscous contact wave with rarefaction waves determined by the initial condition, which extended the results by Huang-Li-Matsumura [13], where they treated the viscous and heat-conductive ideal gas.

Keywords: Non-viscous; asymptotic behavior; viscous contact wave; rarefaction waves.

Mathematics Subject Classification 2010: 35Q35, 35B35, 35L65

Contents

1. Introduction 2  
2. Preliminaries and Main Results 4  
2.1. Viscous Contact Wave 4  
2.2. Composition Waves 6  
3. Stability Analysis 9  
3.1. Reformed the System 9  
3.2. A Priori Estimates 11  
References 21

*Corresponding author. Email address: shaojun.tang@whu.edu.cn (S. J. Tang).
1. Introduction

We consider the Cauchy problem for the equation of heat-conductive ideal gas without viscosity

\[
\begin{align*}
    &v_t - u_x = 0, \\
    &u_t + p_x = 0, \\
    &\left(e + \frac{u^2}{2}\right)_t + (pu)_x = \kappa \left(\frac{\theta}{v}\right)_x,
\end{align*}
\]

with the following prescribed initial data and the far field state

\[
\begin{align*}
    &\left(v, u, \theta\right)(x, 0) = \left(v_0, u_0, \theta_0\right)(x), \quad x \in \mathbb{R}, \\
    &\left(v, u, \theta\right)(\pm \infty, t) = \left(v_{\pm}, u_{\pm}, \theta_{\pm}\right), \quad t > 0.
\end{align*}
\]

Here \( v(x, t) > 0, u(x, t), \theta(x, t) > 0, e(x, t) > 0 \) and \( p(x, t) \) are the specific volume, fluid velocity, internal energy, absolute temperature and pressure of the gas, respectively, while \( \kappa > 0 \) denotes the heat conduction coefficient. \( v_{\pm}(> 0), \theta_{\pm}(> 0) \) and \( u_{\pm}(\in \mathbb{R}) \) are given constants and the initial data \( (v_0(x), u_0(x), \theta_0(x)) \) are assumed to satisfy \( \inf_{x \in \mathbb{R}} v_0(x) > 0, \inf_{x \in \mathbb{R}} \theta_0(x) > 0 \) and the compatibility conditions \( (v_0, u_0, \theta_0)(\pm \infty) = (v_{\pm}, u_{\pm}, \theta_{\pm}) \).

Throughout this paper, we are concerned with the ideal and polytropic fluids and in such a case, \( p \) and \( e \) are given by the following state equations

\[
p = \frac{R \theta}{v} = Av^{-\gamma}e^{s - \frac{\gamma - 1}{\gamma - 1}}, \quad e = \frac{R}{\gamma - 1} \theta + \text{const.},
\]

where \( s \) is the entropy, \( \gamma > 1 \) is the adiabatic exponent and both \( A \) and \( R \) are positive constants.

If \( \kappa = 0 \), we can rewrite the system (1.1) as

\[
\begin{align*}
    &v_t - u_x = 0, \\
    &u_t + p_x = 0, \\
    &\left(e + \frac{u^2}{2}\right)_t + (pu)_x = 0.
\end{align*}
\]

It is well-known that the above system has the three eigenvalues: \( \lambda_1 = -\sqrt{\gamma p/v} < 0, \lambda_2 = 0, \lambda_3 = -\lambda_1 > 0 \), where the second characteristic field is linear degenerate and the others are genuinely nonlinear. The solutions of this inviscous equations with the Riemann initial data

\[
\left(v, u, \theta\right)(x, 0) = \begin{cases} 
    (v_-, u_-, \theta_-), & x < 0, \\
    (v_+, u_+, \theta_+), & x > 0.
\end{cases}
\]

has the basic Riemann solutions which are dilation invariant: shock wave, rarefaction wave, contact discontinuity, and certain linear combinations of these basic wave patterns [39]. The inviscid system (1.4) is a typical example of hyperbolic conservation laws and is of great importance to study the large-time asymptotic behavior.
of solutions of the corresponding viscous system (1.1). In fact, if the unique global
entropy to its Riemann problem (1.4), (1.5) consists of shock wave, rarefaction wave,
contact discontinuity, and/or their linear superpositions, then the large time behav-
ior of the Cauchy problem (1.1), (1.2) of the corresponding viscous conservation
laws is expected to be precisely described by the corresponding viscous shock wave,
rarefaction wave, viscous contact discontinuity and/or their linear superpositions.

The rigorous mathematical justifications of the above expectation is one of the
hottest topics in the field of nonlinear partial differential equations, especially in the
field of nonlinear dissipative hyperbolic conservation laws. In fact, new phenomena
have been discovered and new techniques, such as the weighted characteristic energy
method and the approximate Green function method have been developed based on
the intrinsic properties of the underlying nonlinear waves, cf. [13, 22, 23, 25, 27, 40, 41]
and the references cited therein.

By combining these methods together with the fundamental energy method, some
interesting results have been obtained for the compressible Navier-Stokes equations:

- For the case when both viscosity coefficient $\mu$ and the heat conductivity
  coefficient $\kappa$ are positive constants, many excellent results have been obtained
  for the nonlinear stability of some basic wave patterns consisting of diffusion
  waves, viscous shock waves, rarefaction waves, viscous contact discontinuities
  and/or their certain linear superpositions with small perturbation, cf. [25] for
diffusion waves, [20,22,23,81] for viscous shock profiles, [24,31] for rarefaction
waves, [15,17,18,19] for viscous contact discontinuities, [14] for the composition
of viscous shock profiles of different family, [13] for the composition a viscous
contact wave and rarefaction waves. For the corresponding results with large
initial perturbation, see [5, 7, 16, 19, 32, 34, 36, 38] and the reference cited therein.
- For the case when $\mu > 0$, $\kappa = 0$, Liu and Zeng [26] studied the large-time
  behavior of solutions around a constant state for this case. For the three-
dimensional case, the global existence and the temporal convergence rate of
solutions was obtained by Duan and Ma [6].
- When $\mu = 0$, $\kappa > 0$, if the corresponding Riemann solution consists of two
  shock waves form different families, the nonlinear stability of the composition
  of viscous shock waves from different families was investigated by Fan and
  Matsumura in [8], while the nonlinear stability of viscous contact waves
together with the temporal decay rates was obtained by Ma and Wang in [29].
  In all these results, both the strengths of the underlying wave patterns and
  the initial perturbation are assumed to be small.

From the above results, a natural question is: Whether can we get the nonlinear
stability of combination of viscous contact wave with rarefaction waves for the case
$\mu = 0$ and $\kappa > 0$? This is the motivation of our work. As a continuation of [8],
this manuscript showed that, if the strengths of the viscous waves and the initial
perturbation are suitably small, the unique global-in-time solution exists and asymptotically tends toward the corresponding viscous contact wave or the composition of a viscous contact wave with rarefaction waves from different families. Our result generalizes the corresponding results obtained by Huang, Li and Matsumura in [13] for the case of \( \mu > 0, \kappa > 0 \) to the case of \( \mu > 0, \kappa = 0 \) and extends the result of Ma and Wang [29] for the nonlinear stability of the viscous contact waves to the nonlinear stability of the composition of a viscous contact wave with rarefaction waves from different families.

Now we outline the main difficulties of the problem and our strategy to overcome the difficulties. The first difficulty is due to the fact that the system (1.1) is of less dissipation caused by the fact that \( \mu = 0 \), thus we need more subtle estimates to recover the regularity and dissipation for the components of the hyperbolic part. We shall overcome this difficulty by manipulating several new energy estimates and also looking for the perturbed solution for the integrated system of (1.1) in \( C([0, \infty), H^2) \) to control the nonlinearity of the hyperbolic part, instead of the usual \( C([0, \infty), H^1) \) in [13]. Secondly, our stability results included the superposition of the rarefaction waves with the viscous contact discontinuity, which will lead to the degenerate characteristics. Thanks to the estimates on the heat kernel function obtained by Huang, Li and Matsumura established in [13], we can still close the energy type estimates for our non-viscous case and get the desired results.

Before concluding this section, it is worth pointing out that there are many results on the nonlinear stability of basic wave patterns to some hyperbolic conservation laws with dissipation, cf. [9, 27, 40] for hyperbolic conservation laws with artificial viscosity, [10–12, 21, 35, 37] for compressible Navier-Stokes equations with density and/or temperature dependent transportation coefficients, and [1–4] for compressible fluid models of Korteweg type, and so on.

The rest of the paper is arranged in the following way: in the next section, we will give some elementary properties of the viscous contact wave and rarefaction wave and state the main results. Main theorem will be proved in the section 3.

2. Preliminaries and Main Results

To show our main results, in this section, we will construct the two desired viscous contact wave and viscous rarefaction waves for (1.1) and state the main results. For each \( z_- := (v_-, u_-, \theta_-) \), we can see our situation takes place provided \( z_+ := (v_+, u_+, \theta_+) \) is located on a quarter of a curved surface in a small neighborhood of \( z_- \). In what follows, as [8], the neighborhood of \( z_- \) denoted by \( \Omega_- \) is given by

\[
\Omega_- = \{(v, u, \theta) \mid |(v - v_-, u - u_-, \theta - \theta_-)| \leq \delta \},
\]

where \( \delta \) is a positive constant depending only on \( z_- \).

2.1. Viscous Contact Wave. we firstly recall the viscous contact wave \((\tilde{v}, \tilde{u}, \tilde{\theta})\) for the compressible system (1.1) defined in [18]. For the Riemann problem (1.4), (1.5),
it is known that the contact discontinuity solution \( \tilde{Z}(x,t) := (\tilde{V}, \tilde{U}, \tilde{\Theta})(x,t) \) takes the form

\[
(\tilde{V}, \tilde{U}, \tilde{\Theta})(x,t) = \begin{cases} 
(\tilde{v}_-, \tilde{u}_-, \theta_-), & x < 0, \ t > 0, \\
(\tilde{v}_+, \tilde{u}_+, \theta_+), & x > 0, \ t > 0,
\end{cases}
\]

(2.1)

provided that

\[
u_-=u_+\text{,} \quad p_- = \frac{R\theta_-}{v_-} = \frac{R\theta_+}{v_+} = p_+.
\]

(2.2)

In the setting of the compressible Navier-Stokes system \( \text{(1.1)} \), the smooth approximate wave \((\tilde{v}, \tilde{u}, \tilde{\theta})\) to the contact wave behaves as a diffusion wave due to the dissipation effect and we call this wave "viscous contact wave". It can be constructed as follows. Since the pressure is constant under the condition \( \text{(2.7)} \), we set

\[
p_+ = \frac{R\tilde{\theta}}{v},
\]

which indicates the leading part of the energy equation \( \text{(1.1)}_3 \) is

\[
\frac{R}{\gamma-1}\tilde{\theta}_t + p_+\tilde{u}_x = \kappa \left( \frac{\tilde{\theta}_x}{v} \right)_x.
\]

(2.4)

Meanwhile the equation \( \tilde{v}_t = \tilde{u}_x \) leads to a nonlinear diffusion equation

\[
\left\{ \begin{array}{l}
\tilde{\theta}_t = a \left( \frac{\tilde{\theta}_x}{\tilde{\theta}} \right)_x, \\
\Theta(\pm, t) = \theta_\pm,
\end{array} \right.
\]

(2.5)

which has a unique self similarity solution \( \tilde{\theta}(x,t) = \tilde{\theta}(\xi), \ \xi = \frac{x}{\sqrt{1+t}} \). Furthermore, on the one hand, \( \tilde{\theta}(\xi) \) is a monotone function, increasing if \( \theta_+ > \theta_- \) and decreasing if \( \theta_+ < \theta_- \); on the other hand, there exists some positive constant \( \delta_0 \), such that for \( \delta = |\theta_+ - \theta_-| \leq \delta_0 (\leq \overline{\delta}) \), \( \tilde{\theta} \) satisfies

\[(1 + t)|\tilde{\theta}_x| + (1 + t)^{\frac{\gamma}{2}}|\tilde{\theta}_x| + |\tilde{\theta} - \theta_\pm| \lesssim \delta e^{-\frac{x^2}{4(1+t)}}, \quad \text{as} \quad |x| \to \infty.\]

(2.6)

Once \( \tilde{\theta} \) is defined, the viscous contact profile

\[
Z^c(x,t) := (V^c, U^c, \Theta^c)(x,t)
\]

is determined as follows:

\[
\Theta^c = \tilde{\theta}, \quad V^c = \frac{R\tilde{\theta}}{p_+}, \quad U^c = u_- + \frac{\kappa(\gamma - 1)}{\gamma R} \tilde{\theta}_x.
\]

(2.7)

It is straightforward to check that \( Z^c(x,t) \) satisfies

\[
\|(Z^c - \tilde{Z})(t)\|_{L^p} = O(\kappa(\frac{1}{p})^{\frac{1}{p'}})(1 + t)^{\frac{1}{2p}}, \quad p \geq 1,
\]

which means the nonlinear diffusion wave \( Z^c(x,t) \) approximates the contact discontinuity \( \tilde{Z}(x,t) \) to the Euler system \( \text{(1.1)} \) in \( L^p(p \geq 1) \) norm. Moreover, the viscous
contact wave \(Z^c(x, t)\) solves the compressible Navier-Stokes system without viscosity (1.1) as

\[
\begin{align*}
V_t^c - U_x^c &= 0, \\
U_t^c + \left( \frac{R \Theta^c}{V^c} \right)_x &= U_t^c, \\
\frac{R}{\gamma - 1} \Theta^c_t + p_x^c &= \kappa \left( \frac{\Theta^c}{V^c} \right)_x.
\end{align*}
\]  

Our first main result is as follow:

**Theorem 2.1.** For any given \(z_-\), assume that \(z_+ \in \Omega_-\) satisfies (2.2), let \(Z^c(x, t)\) is the viscous contact wave defined in (2.7) with strength \(\delta = |\theta_+ - \theta_-|\). There exist positive constants \(\epsilon_1\) and \(\delta_1\) (\(\leq \delta_0\)), such that if \(\delta < \delta_1\) and the initial data satisfies

\[
\| (v_0(\cdot) - V(\cdot, 0), u_0(\cdot) - U(\cdot, 0), \theta_0(\cdot) - \Theta(\cdot, 0)) \|_2 \leq \epsilon_1,
\]

then the Cauchy problem (1.1), (1.3) admits a unique global solution \((v, u, \theta)(t, x)\) satisfies

\[
(v - V^c, u - U^c, \theta - \Theta^c)(t, x) \in X[0, +\infty)
\]

and

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |(v - V^c, u - U^c, \theta - \Theta^c)(x, t)| = 0,
\]

where the solution space \(X(0, t)\) is defined in (3.5).

### 2.2. Composition Waves

When the relation (2.2) is fails, the basic theory of conservation laws [39] implies that for any given constant state \(z_-\) and \(z_+ \in \Omega_-\) (\(\delta\) is suitably small), the Riemann problem (1.4), (1.5) has a unique solution. Based on this, our second purpose is concerned with the stability of superposition of a viscous contact wave with rarefaction waves. In this situation, we assume that

\[
z_+ \in R_1CR_3(z_-) \subseteq \Omega_-,
\]

where

\[
R_1CR_3(z_-) := \left\{ (v, u, \theta) \in \Omega_- \middle| s \neq s_- \right\}
\]

\[
u \geq u_- - \int_{v_-}^{\frac{1}{e^{1/R}}(s-s_-)^{v}} \lambda_1(\eta, s_-) d\eta, \quad u \geq u_- - \int_{e^{-R}}^{\frac{1}{e^{1/R}}(s-s_-)^{v}} \lambda_3(\eta, s) d\eta
\]

with the entropy \(s\) in (1.3) is defined as follows:

\[
s = \frac{R}{\gamma - 1} \ln \frac{R \theta}{A} + R \ln v, \quad s_\pm = \frac{R}{\gamma - 1} \ln \frac{R \theta_\pm}{A} + R \ln v_\pm.
\]

It is known that there exists some suitably small \(\delta_1 > 0\) such that for

\[
|\theta_- - \theta_+| \leq \delta_1,
\]
there exists a unique pair of points $z^m_- := (v^m, u^m, \theta^m_-)$ and $z^m_+ := (v^m_+, u^m_+, \theta^m_+)$ in $\Omega_-$ satisfying
\[ \frac{R \theta^m_+}{v^m_-} = \frac{R \theta^m_-}{v^m_+} := p_m \] (2.14)
and
\[ |v^m_+ - v_-| + |u^m - u_+| + |\theta^m_+ - \theta_-| \lesssim |\theta_+ - \theta_-|. \] (2.15)
Moreover, the point $z^m_-$ belongs to the 1-rarefaction wave curve $z^r_1 := (v^r_1, u^r_1, \theta^r_1)(\xi)$ which connected with $z_-$, while $z^m_+$ belongs to the 3-rarefaction wave curve $z^r_3 := (v^r_3, u^r_3, \theta^r_3)(\xi)$ which connected with $z_+$, that is, the 1-rarefaction wave is the weak solution of Riemann problem of the Euler system [12] with the following Riemann data
\[ z^r_1(x, 0) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v^m_-, u^m_-, \theta^m_-), & x > 0 \end{cases} \]
and the 3-rarefaction wave with Riemann data as
\[ z^r_3(x, 0) = \begin{cases} (v^m_+, u^m_+, \theta^m_+), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0 \end{cases} \]
Since rarefaction waves $z^r_i(\xi)(i = 1, 3)$ are not smooth enough, it is convenient to construct smooth approximations of them. Motivated by [20], we start with the problem of the Burgers equation:
\[ \begin{cases} w^r_t + w^r w^r_x = 0, & x \in \mathbb{R}, \ t > 0, \\ w^r(0, x) = w^r_0(x) := \frac{1}{2}(w_r + w_l) + \frac{1}{2}(w_r - w_l) \tanh(x), \end{cases} \] (2.16)
where $w_l = \lambda_1(v_-, s_-)$, $w_r = \lambda_1(v_+, s_-)$. Let $w(x, t)$ be the unique global solution of (2.16), then $Z^r_1(x, t) := (V^r_1, U^r_1, \Theta^r_1)(x, t)$ are defined by
\[ \begin{cases} \lambda_1(V^r_1, s_-) = w(x, t), \\ U^r_1 = u_+ - \int_{v^-}^{V^r_1} \lambda_1(\eta, s_-) d\eta, \\ \Theta^r_1 = \Theta_- (v_-)^{1-\gamma}(V^r_1)^{1-\gamma} \end{cases} \] (2.17)
with
\[ w_l = \lambda_1(v_-, s_-), \quad w_r = \lambda_1(v^m_-, s_-) \]
and the smooth approximation of $(v^r_3, u^r_3, \theta^r_3)(\xi)$ is given by $Z^r_3(x, t) := (V^r_3, U^r_3, \Theta^r_3)(x, t)$ constructed by the same manner as (2.17)
\[ \begin{cases} \lambda_3(V^r_3, s_+) = w(x, t), \\ U^r_3 = u_+ - \int_{v^m_+}^{V^r_3} \lambda_3(\eta, s_+) d\eta, \\ \Theta^r_3 = \Theta_+(v_+)^{1-\gamma}(V^r_3)^{1-\gamma}, \end{cases} \] (2.18)
where \( w(x, t) \) is defined in (2.16) with
\[
    w_l = \lambda_3(v^m_+, s_+), \quad w_r = \lambda_3(v_+, s_+).
\]
And since the condition (2.14), (2.15), \( z^m_+ \) is connected by the viscous contact wave \( Z^c(x,t) \) constructed in (2.25), (2.21). Finally, we investigate some properties of \( Z^c_i(x,t)(i = 1,3) \) and \( Z^c(x,t) \) as [13], divided \( \mathbb{R} \times [0,t] \) into three parts \( \mathbb{R} \times [0,t] = \Omega_1 \cup \Omega_2 \cup \Omega_3 \) with
\[
    \Omega_1 = \{(x,t)|2x < \lambda_1(v^m_-, s_-)t\},
    \Omega_3 = \{(x,t)|2x > \lambda_3(v^m_+, s_+)_t\},
    \Omega_c = \{(x,t)|\lambda_1(v^m_-, s_-)t \leq 2x \leq \lambda_3(v^m_+ s_+)t\},
\]
and it holds that

**Lemma 2.1.** ([13]) For any given \( z_- \), assume that \( z_+ \in R_1CR_3(z_-) \subseteq \Omega_- \), then the smooth rarefaction wave \( Z^c_i(x,t)(i = 1,3) \) and \( Z^c \) satisfy:

(i) \( V^r_{it} = U^r_{ix} > 0(i = 1,3) \) for all \( x \in \mathbb{R}, \ t > 0 \).

(ii) For \( 1 \leq p \leq \infty \) and \( i = 1,3 \), it holds for
\[
    \| (V^r_{ix}, U^r_{ix}, \Theta^r_{ix})(t) \|_{L^p} \lesssim \min\{\delta, \delta^\beta (1 + t)^{-1 + \frac{1}{p}}\},
\]
\[
    \| (V^r_{ixx}, U^r_{ixx}, \Theta^r_{ixx})(t) \|_{L^p} \lesssim \min\{\delta, (1 + t)^{-1}\}.
\]

(iii) In \( \Omega_c \), we have for \( i = 1,3 \)
\[
    |Z^r_{ix}| + |Z^1_i - z^-_m| + |Z^3_i - z^m_+| \leq \delta e^{-c(|x|+t)}
\]
and in \( \Omega_i(i = 1,3) \)
\[
    |Z^c_i| + |Z^c_1 - z^m_+| \leq \delta e^{-c(|x|+t)},
    |Z^r_{ix}| + |Z^1_i - z^-_m| + |Z^3_i - z^m_+| \leq \delta e^{-c(|x|+t)}.
\]

(iv) For the rarefaction wave \( z^r_i(\frac{x}{t}) \) \((i = 1,3)\), it holds
\[
    \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |Z^r_i(x,t) - z^r_i(\frac{x}{t})| = 0.
\]

Setting \( Z(x,t) := (V,U,\Theta)(x,t) \) is
\[
\begin{cases}
    V(x,t) = V^r_i(x,t) + V^c(x,t) + V^3_i(x,t) - v^m_+ - v^m_-, \\
    U(x,t) = U^r_i(x,t) + U^c(x,t) + U^3_i(x,t) - 2u^m, \\
    \Theta(x,t) = \Theta^r_i(x,t) + \Theta^c(x,t) + \Theta^3_i(x,t) - \theta^m_+ - \theta^m_+.
\end{cases}
\]
Then our second main result is as follows:

**Theorem 2.2.** For any given \( z_- \), assume that \( z_+ \in R_1CR_3(z_-) \subseteq \Omega_- \) with \( |\theta_+ - \theta_-| \leq \delta_1 \). There exist positive constants \( \epsilon_2 \) and \( \delta_2(\leq \min\{\overline{\delta}, \delta_1\}) \), such that if \( \delta < \delta_2 \) and the initial data satisfies
\[
    \|(v_0(\cdot) - V(\cdot,0), u_0(\cdot) - U(\cdot,0), \theta_0(\cdot) - \Theta(\cdot,0))\|_2 \leq \epsilon_2,
\]
\[
\begin{aligned}
    \text{(2.21)}
\end{aligned}
\]
then the Cauchy problem \((1.1), (1.3)\) admits a unique global solution \((v, u, \theta)(x, t)\) satisfying

\[
(v - V, u - U, \theta - \Theta) \in X([0, +\infty))
\]

and

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left\{ |(v - v_1^c - V^c - v_3^m + v_4^m)(x, t)|, 
|u - u_1^c - U^c - u_3^m + 2u_4^m)(x, t)|, 
|\theta - \theta_1^c - \Theta^c - \theta_3^m + \theta_4^m)(x, t)| \right\} = 0
\]

(2.22)

where the solution space \(X(0, t)\) is defined in (3.5).

\textbf{Remark 2.1.} if \(z_m = z_+ = (v_m, u_m, \theta_m)\), we can also get the stability of the composition of two rarefaction waves.

3. Stability Analysis

In this section, we will prove the asymptotic behavior of the solution for nonviscous system \((1.1), (1.2)\). If \(z_m = z_+\), Theorem 2.2 will coincide with the result of Theorem 2.1, here we will show the stability of the composition wave only.

3.1. Reformed the System. \(Z(x, t) := (V, U, \Theta)(x, t)\) defined in (2.20) satisfies

\[
\begin{align*}
V_t - U_x &= 0, \\
U_t + P_x &= -R_1, \\
\frac{R}{\gamma - 1} \Theta_t + PU_x &= \kappa \left( \frac{\Theta}{V} \right)_x - R_2,
\end{align*}
\]

(3.1)

where

\[
P := \frac{R \Theta}{V}, \quad P_i = \frac{R \Theta V_i^r}{V^r_i} \quad (i = 1, 3),
\]

\[
R_1 := -(P - P_1 - P_3 - p^c)_x + U_1^c := R_1^1 + U_1^c,
\]

\[
R_2 := \left\{ (p_m - P)U_x^c + (P_1 - P)U_{1x}^r + (P_3 - P)U_{3x}^r \right\}
+ \kappa \left\{ \left( \frac{\Theta}{V} \right)_x - \left( \frac{\Theta^c}{V^c} \right)_x \right\} := R_2^1 + R_2^2.
\]

(3.2)

Let the perturbation function is

\[
(\phi, \psi, \xi) := (v, u, \theta) - (V, U, \Theta),
\]

then the reformed equation is

\[
\begin{align*}
\phi_t - \psi_x &= 0, \\
\psi_t + \left( \frac{R \xi}{v} - \frac{P \phi}{v} \right)_x &= R_1, \\
\frac{R}{\gamma - 1} \xi_t + p \psi_x + (p - P)U_x &= \kappa \left( \frac{\xi}{v} - \frac{\Theta \phi}{v^c} \right)_x + R_2
\end{align*}
\]

(3.3)
with the initial data
\[(\phi, \psi, \xi)(x, 0) = (\phi_0, \psi_0, \xi_0)(x) = (v_0(x) - V(x, 0), u_0(x) - U(x, 0), \theta_0(x) - \Theta(x, 0)).\]

And the solution space is defined as
\[X(0, t) := \{(\phi, \psi, \xi) | (\phi, \psi, \xi) \in C([0, t), H^2(\mathbb{R})), (\phi, \psi) \in L^2([0, t), H^1(\mathbb{R})), \xi \in L^2([0, t), H^2(\mathbb{R})) \}. \]

(3.5)

The local existence is known in [34].

**Proposition 3.1.** (Local existence) Under the assumptions stated in Theorem 2.1, the Cauchy problem (3.3), (3.4) admits a unique smooth solution \((\phi, \psi, \xi)(x, t) \in X(0, t_1)\) for some sufficient small \(t_1 > 0\), and \((\phi, \psi, \xi)(x, t)\) satisfies
\[
\sup_{0 \leq t \leq t_1} \| (\phi, \psi, \xi)(t) \|^2_2 \leq 2 \| (\phi_0, \psi_0, \xi_0) \|^2_2. \tag{3.6}
\]

Suppose that \((\phi, \psi, \xi)(x, t)\) has been extended to the time \(t > t_1\), we want to get the following a priori estimates to obtain a global solution.

**Proposition 3.2.** (A prior estimates) Under the assumptions stated in Theorem 2.1, there exist positive constants \(\epsilon_2 \leq 1, \delta_2 \leq \min\{\delta_1, \delta, 1\}\) and \(C\), such that \((\phi, \psi, \xi) \in X([0, t])\) for some \(t > 0\) satisfying
\[
N(t) = \sup_{0 \leq \tau \leq t} \| (\phi, \psi, \xi)(\tau) \|_2 \leq \epsilon_2, \tag{3.7}
\]
\[
\delta = |\theta_+ - \theta_-| < \delta_2,
\]
it follows the estimate
\[
\sup_{0 \leq \tau \leq t} \| (\phi, \psi, \xi)(\tau) \|_2^2 + \int_0^t \left( \| (\phi_x, \psi_x)(\tau) \|_1^2 + \| \xi_x(\tau) \|_2^2 \right) d\tau \lesssim \| (\phi_0, \psi_0, \xi_0) \|_2^2 + \delta_2. \tag{3.8}
\]

Once Proposition 3.2 is proved, we can extend the unique local solution \((\phi, \psi, \xi)(x, t)\) obtained in Proposition 3.1 to \(t = \infty\), moreover, estimate (3.8) implies that
\[
\int_0^{\infty} \left( \| (\phi_x, \psi_x, \xi_x)(t) \|_2^2 + \left| \frac{d}{dt} \| (\phi_x, \psi_x, \xi_x)(t) \|_2^2 \right| \right) dt \lesssim +\infty, \tag{3.9}
\]
which together with Sobolev inequality easily leads to the asymptotic behavior (2.22), this concludes the proof of Theorem 2.1. In the rest of this section, our main task is to show this a priori estimates.
3.2. A Priori Estimates. At first, we show the basic estimates.

Lemma 3.1. Under the assumptions in proposition 3.2, we have

\[ \|\langle \phi, \psi, \xi \rangle(t)\|^2 + \int_0^t \int_\mathbb{R} (|\alpha U_{1x}| + |U_{3x}|)(\phi^2 + \xi^2 + \xi_x^2)dxdt \]

\[ \lesssim \|\langle \phi_0, \psi_0, \xi_0 \rangle\|^2 + \delta + \delta^{\frac{1}{2}} \int_0^t \|\langle \phi_x, \psi_x \rangle(\tau)\|^2d\tau \]

(3.10)

+ \delta \int_0^t \frac{1}{1 + \tau} \int_\mathbb{R} (\phi^2 + \xi^2)e^{\frac{2}{1 + \tau}}dxdt.

Proof. The first step, by using lemma 2.1, we need get some estimates of \( R_1 \) and \( R_2 \). Since \( R\Theta^c = p_m V^c \), direct calculation yields that

\[ R_1^1 = R_1 \left( \frac{\Theta_1^1}{V_1^3} + \frac{\Theta_3^r}{V_{cd}^2} - \frac{\Theta}{V} \right) \]

\[ = R\Theta_{1x}^r ((V_1^r)^{-1} - V^{-1}) + R\Theta_{3x}^r ((V_3^r)^{-1} - V^{-1}) \]

\[ + R\Theta_{x}^{cd} ((V_{cd})^{-1} - V^{-1}) + RV_{1x}^r \left( \frac{\Theta}{V_2} - \frac{\Theta_1^r}{V_1^r} \right) \]

\[ + RV_{3x}^r \left( \frac{\Theta}{V_2} - \frac{\Theta_3^r}{V_3^r} \right) + RV_{x}^{cd} \left( \frac{\Theta}{V_2} - \frac{\Theta^c}{V_2} \right). \]

(3.11)

It is easy to compute

\[ |\Theta_{1x}^r ((V_1^r)^{-1} - V^{-1})| \lesssim |\Theta_{1x}^r| |V_3^r - V_m^r| + |V^c - v_m^r| \lesssim \delta e^{-c(|x|+t)} \]

and we can treat the other terms on the righthand side of (3.11) in the same way to obtain

\[ |R_1^1| \lesssim \delta e^{-c(|x|+t)} \]

(3.12)

and

\[ |R_1| \lesssim |R_1^1| + |U_{1x}^c| \lesssim \delta e^{-c(|x|+t)} + \frac{\delta}{(1+t)^{\frac{3}{2}}} e^{\frac{-\alpha x^2}{1+t}}. \]

(3.13)

Similarly we have

\[ |R_2^1| \lesssim \delta e^{-c(|x|+t)}. \]

(3.14)

Since

\[ R_2^2 = \kappa \left( \frac{\Theta_{1x}^r}{V} + \frac{\Theta_3^r}{V} \right)_x + \kappa \left( \frac{\Theta_{x}^{cd}}{V} - \frac{\Theta_x^{cd}}{V_{cd}} \right)_x := R_2^{21} + R_2^{22} \]

and

\[ R_2^{21} \lesssim \left( |\Theta_{1xx}^r| + |\Theta_3^{3xx}| + |\Theta_{1x}^r||V_{1x}^r| + |\Theta_{3x}^r||V_{3x}^r| \right) \]

\[ + |\Theta_{1x}^r||V_{3x}^r + |V_{x}^{cd}| + |\Theta_3^{3x}||V_{1x}^r| + |V_{x}^{cd}|), \]
it follows from (2.6) and Lemma 2.1 that
\[
|R_{21}| \lesssim \delta^{\frac{1}{8}} (1 + t)^{-\frac{7}{8}},
\]
\[
|R_{22}| \lesssim \left( |\Theta_{xx}| + |\Theta_{x}| |V^c_x| \right) \left( |V^r_3 - v^m_3| + |V^r_1 - v^-_m| \right)
+ |\Theta_{x}^c| (|V^r_{1x}| + |V^r_{3x}|) \lesssim \delta e^{-c|x|+t}.
\]
(3.15)

Thus we get
\[
|R_2| \lesssim \delta^{\frac{1}{8}} (1 + t)^{-\frac{7}{8}}.
\]
(3.16)

And by the direct calculation, we can also obtain
\[
|(R_{1x}, R_{1xx})(x, t)| \lesssim \delta e^{-c|x|+t} + \frac{\delta}{(1 + t)^{\frac{7}{8}}} e^{-\frac{c}{1+t}},
\]
(3.17)

Then multiplying (3.3)_1 by \(-R\Theta \left( \frac{1}{v} - \frac{1}{V} \right)\), (3.3)_2 by \(\psi\) and (3.3)_3 by \(\xi/\theta\), respectively, and adding the results together, we get
\[
\left\{ R\Theta \Phi \left( \frac{v}{V} \right) + \frac{1}{2} \psi^2 + \frac{R\Theta_t}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) \right\}_t + \frac{\kappa}{v^2} \xi_x^2 + H_1 + Q_1 + Q_2
\]
\[
= R_1 \psi + R_2 \frac{\xi}{\theta},
\]
(3.18)

where
\[
H_1 := (p - P) \psi - \frac{\kappa}{\theta} \left( \frac{\xi_x}{v} - \frac{\Theta_x \psi}{v V} \right),
\]
\[
Q_1 := -R\Theta_t \Phi \left( \frac{v}{V} \right) + \frac{P U_x}{v V} \psi^2 + \frac{R\Theta_t}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\xi}{\theta} (p - P) U_x,
\]
(3.19)
\[
Q_2 := -\frac{\kappa}{\theta^2 v^2} \xi_x - \frac{\kappa}{\theta^2 v V} \Theta_x + \frac{\kappa}{\theta^2 v V} \Theta_x,
\]

here
\[
\Phi(s) = s - 1 - \ln s.
\]

In the process of the calculation, we have used the following equalities
\[
-R\Theta \left( \frac{1}{v} - \frac{1}{V} \right) \phi_t = \left\{ R\Theta \Phi \left( \frac{v}{V} \right) \right\}_t + \frac{P U_x}{v V} \phi^2 - R\Theta_t \Phi \left( \frac{v}{V} \right),
\]
(3.20)
\[
\frac{R \xi_t}{\gamma - 1} \frac{\xi}{\theta} = \left\{ R\Theta_t \Phi \left( \frac{\theta}{\Theta} \right) \right\}_t + \frac{R\Theta_t}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right)
\]

and the equations in (3.3).
Since the composite waves affected each other, we will try to divide it the viscous contact wave $Z_c$ and viscous rarefaction wave $Z_r(i = 1, 3)$,

$$
-R\Theta_t = (\gamma - 1)(P_r U_{1x} + P_3 U_{3x}) + (\gamma - 1)\left(p_m U_x - \kappa \left(\frac{\Theta^c}{V_c}x\right)\right)
$$

$$
= (\gamma - 1)P(U_{1x} + U_{3x}) + (\gamma - 1)(P_1^r - P)U_{1x}^r
$$

$$
+ (\gamma - 1)(P_3^r - P)U_{3x}^r + (\gamma - 1)\left(p_m U_x - \kappa \left(\frac{\Theta^c}{V_c}x\right)\right),
$$

(3.21)

thus

\[Q_1 = -R\Theta_t \Phi\left(\frac{v}{V}\right) + \frac{R\Theta_t}{\gamma - 1} \Phi\left(\frac{\Theta}{\theta}\right) + \frac{P(U_{1x} + U_{3x} + U_x^c)}{vV} \phi^2\]

$$
+ \frac{\xi}{\theta}(p - P)(U_{1x} + U_{3x} + U_x^c)
$$

\[= : (|U_{1x}^r| + |U_{3x}^r|)Q_{11} + Q_{12},\]

where

\[Q_{11} := (\gamma - 1)P\Phi\left(\frac{v}{V}\right) - P\Phi\left(\frac{\Theta}{\theta}\right) + \frac{P}{vV} \phi^2 + \frac{\xi}{\theta}(p - P)
\]

$$
= P\left(\frac{\theta V}{\Theta v} - 1 + \gamma \left(\frac{\theta}{\Theta}\right) - \left(\log\frac{\theta}{\Theta} + (\gamma - 1) \log \frac{v}{V}\right)\right)
$$

(3.23)

\[\geq C(\phi^2 + \xi^2)
\]

and

\[Q_{12} := U_x^c\left(\frac{P\phi^2}{vV} + (\gamma - 1)p_m \Phi\left(\frac{v}{V}\right) + p_m \Phi\left(\frac{\Theta}{\theta}\right) + \frac{\xi}{\theta}(p - P)\right)
\]

$$
+ (P_1^r - P)U_{1x}^r\left[(\gamma - 1)\Phi\left(\frac{v}{V}\right) - \Phi\left(\frac{v}{V}\right)\right]
$$

$$
- \kappa \left(\frac{\Theta^c}{V_c}x\right)\left[(\gamma - 1)\Phi\left(\frac{v}{V}\right) - \Phi\left(\frac{v}{V}\right)\right]
$$

$$
+ (P_3^r - P)U_{3x}^r\left[(\gamma - 1)\Phi\left(\frac{v}{V}\right) - \Phi\left(\frac{v}{V}\right)\right]
$$

\[\lesssim |(U_x^c, \Theta^c_{xx}, \Theta^c_{xx} V_x^c)|(\phi^2 + \xi^2) + \delta(U_{1x}^r + U_{3x}^r)(\phi^2 + \xi^2).
\]

Meanwhile,

\[|Q_2| \lesssim (N(t) + \delta + \frac{1}{8})\xi^2 + \Theta_x^2(\phi^2 + \xi^2),
\]

(3.25)
where
\[
\Theta_x^2 \lesssim (\Theta_{1x}^2 + \Theta_{3x}^2) + (\Theta^c_x^2) \\
\lesssim \Theta_{1x}^2 + \Theta_{3x}^2 + \frac{\delta}{1 + t} e^{-\frac{c\tau^2}{1 + \tau}}.
\] (3.26)

Therefore, after integrating (3.19) on \([0, t] \times \mathbb{R}\) and using the above estimates, we obtain
\[
\| (\phi, \psi, \xi)(t) \|^2 + \int_0^t \int_{\mathbb{R}} \left( \left| U_{1x}^r \right| + \left| U_{3x}^r \right| \right) (\phi^2 + \xi^2) + \xi_x^2) dx \, d\tau \\
\lesssim \| (\phi_0, \psi_0, \xi_0) \|^2 + \int_0^t \frac{\delta}{1 + \tau} \int_{\mathbb{R}} (\phi^2 + \xi^2) e^{-\frac{c\tau^2}{1 + \tau}} \, dx \, d\tau \\
+ \int_0^t \int_{\mathbb{R}} (\Theta_{1x}^2 + \Theta_{3x}^2) (\phi^2 + \xi^2) \, dx \, d\tau + \int_0^t \int_{\mathbb{R}} (|\psi||R_1| + |\xi||R_2|) \, dx \, d\tau.
\] (3.27)

Noticing that \(\| (\Theta_{1x}^r, \Theta_{3x}^r) \|_\infty \lesssim \delta \frac{t}{(1 + t)^{\frac{7}{8}}},\) one can easily get
\[
\int_0^t \int_{\mathbb{R}} (\Theta_{1x}^r + \Theta_{3x}^r) (\phi^2 + \xi^2) \, dx \, d\tau \\
\lesssim \delta^\frac{7}{8} \int_0^t \| (\phi, \xi) \|_{1, 2}^2 (1 + \tau)^{-\frac{7}{8}} \, d\tau \\
\lesssim \delta^\frac{7}{8} \int_0^t \| (\phi, \xi) \| \| (\phi_x, \xi_x) \| (1 + \tau)^{-\frac{7}{8}} \, d\tau \\
\lesssim \delta^\frac{7}{8} \int_0^t \| (\phi, \xi) \| \| (\phi_x, \xi_x) \|^2 \, d\tau + \delta^\frac{7}{8} \int_0^t \| (\phi, \xi) \|^2 (1 + \tau)^{-\frac{7}{8}} \, d\tau \\
\lesssim \delta^\frac{7}{8} \int_0^t \| (\phi_x, \xi_x) \| \, d\tau + \delta^\frac{7}{8}.
\] (3.28)

For the last term in (3.27), we have the following estimate
\[
\int_0^t \int_{\mathbb{R}} (|\psi||R_1| + |\xi||R_2|) \, dx \, d\tau \\
\lesssim \delta \int_0^t \| \psi \|_\infty (\int_{\mathbb{R}} e^{-c(|x|+\tau)} \, dx + \int_{\mathbb{R}} \frac{\delta}{(1 + \tau)^{\frac{7}{8}}} e^{-\frac{c\tau^2}{1 + \tau}} \, dx) \, d\tau \\
+ \delta^\frac{7}{8} \int_0^t \| \xi \|_\infty (1 + \tau)^{-\frac{7}{8}} \, d\tau \\
\lesssim \delta \int_0^t \| \psi \|_\frac{1}{2} \| \psi_x \|_\frac{1}{2} (1 + \tau)^{-1} \, d\tau + \delta^\frac{7}{8} \int_0^t \| \xi \|_\frac{1}{2} \| \xi_x \|_\frac{1}{2} (1 + \tau)^{-\frac{7}{8}} \, d\tau
\] (3.29)
\[ \lesssim \delta \int_0^t \| (\psi_x, \xi_x) \|^2 d\tau + \delta \frac{2}{3} \int_0^t \| (\psi, \xi) \|^2 (1 + \tau)^{-\frac{2}{3}} d\tau \]

\[ \lesssim \delta \int_0^t \| (\psi_x, \xi_x) \|^2 d\tau + \delta. \]

Inserting (3.28) and (3.29) into (3.27), we can get (3.10) and this completes the proof of Lemma 3.1.

Then we claim that there exists some positive constant \( C \) such that

**Lemma 3.2.** \((13)\) Under the assumptions in proposition 3.2, we have

\[ \int_0^t \frac{1}{1 + \tau} \int_\mathbb{R} (\phi^2 + \psi^2 + \xi^2) e^{-\frac{x^2}{1 + \tau}} dx d\tau \]

\[ \lesssim 1 + \int_0^t \| (\phi_x, \psi_x, \xi_x) \|^2 d\tau + \int_0^t \int_\mathbb{R} (|U_{1x}^r| + |U_{3x}^r|)(\phi^2 + \psi^2 + \xi^2) dx d\tau. \]  

(3.30)

For \( \alpha > 0 \), we define the following heat kernel \( \omega(x, t) \) which will play an essential role in the later estimates. We define

\[ \omega(x, t) = (1 + t)^{-\frac{3}{2}} e^{-\frac{x^2}{1 + t}}, \quad g(x, t) = \int_{-\infty}^x \omega(y, t) dy. \]

It is easy to check that

\[ 4\alpha g_t = g_{xx}, \quad \| g(\cdot, t) \|_{L^\infty} = \sqrt{\pi} \alpha^{-\frac{1}{2}}. \]

The proof (3.30) is divided into two parts:

\[ \int_0^t \int_\mathbb{R} \omega^2 ((R\xi - P\phi)^2 + \psi^2) dx d\tau \]

\[ \lesssim 1 + \int_0^t \| (\phi_x, \psi_x, \xi_x) (\tau) \|^2 d\tau + \delta \int_0^t \int_\mathbb{R} \omega^2 (\phi^2 + \psi^2 + \xi^2) dx d\tau \]

\[ + \delta \int_0^t \int_\mathbb{R} (|U_{1x}^r| + |U_{3x}^r|)(\phi^2 + \psi^2 + \xi^2) dx d\tau \]  

(3.31)

and for any \( \eta > 0 \),

\[ \int_0^t \int_\mathbb{R} \omega^2 (R\xi + (\gamma - 1)P\phi)^2 dx d\tau \]

\[ \lesssim 1 + \int_0^t \| (\phi_x, \psi_x, \xi_x) (\tau) \|^2 d\tau + (\delta + \eta) \int_0^t \int_\mathbb{R} \omega^2 (\phi^2 + \psi^2 + \xi^2) dx d\tau, \]  

(3.32)

\[ + \delta \int_0^t \int_\mathbb{R} (|U_{1x}^r| + |U_{3x}^r|)(\phi^2 + \psi^2 + \xi^2) dx d\tau. \]
Here we used the same method as in [13], the only difference lied in the loss of diffusion term of fluid velocity, but that’s not the main term. We omit the detail for simplify.

We now turn to obtain the higher order estimates, further calculation yields the following Lemma

**Lemma 3.3.** Under the assumptions in proposition 3.2, we have

\[
\|(\phi_x, \psi_x, \xi_x)(t)\|^2 + \int_0^t \|\xi_{xx}(\tau)\|^2 d\tau \leq \|(\phi_0, \psi_0, \xi_0)\|^2 + \delta + (\delta + N(t)) \int_0^t \|(\phi_x, \psi_x)(\tau)\|^2 d\tau.
\]  

(3.33)

**Proof.** Multiplying (3.31) by \(\frac{P}{v} \phi_x\), (3.32) by \(\psi_x\) and (3.33) by \(\frac{\xi_x}{\theta}\), respectively, and adding the resulting equations together, then we have

\[
\left\{ \frac{P}{2} \phi_x^2 + \frac{\psi_x^2}{2} + \frac{R \xi_x^2}{2} \frac{1}{(\gamma - 1)\theta} \right\}_t + \frac{\kappa}{v\theta} \xi_{xx} + H_2 + J_2 = R_1 \phi_x + R_2 \frac{\xi_x}{\theta},
\]

(3.34)

where

\[
H_2 = (p-P) \psi_x + \frac{\xi_x}{\theta} \left( (p-P) U_x + \left( \frac{\kappa \xi_x}{v} - \frac{\kappa \Theta_x \phi}{v V} \right)_x \right),
\]

\[
J_2 = \left( \frac{\xi_x}{\theta} \right)_x \left( \left( \frac{\kappa \xi_x}{v} - \frac{\kappa \Theta_x \phi}{v V} \right)_x - (p-P) U_x \right) - \frac{\kappa}{v\theta} \xi_{xx} - \left( \frac{P}{2} \right)_t \phi^2
\]

\[
- \frac{R}{2(\gamma - 1)} \left( \frac{1}{\theta} \right)_t \xi^2 - \left( \frac{R}{v} \right)_x \xi_{xx} + \left( \frac{P}{v} \right)_x \phi_{xx} + p_x \psi_x \xi_x
\]

\[
= O(1) \left( N(t) + \delta + \eta \right) \|\phi_x, \psi_x, \xi_x\|^2 + \|V_x, U_x, \Theta_x, \Theta_{xx}\|^2 (\phi^2 + \xi^2).
\]

After integrating (3.34) on \([0, t] \times \mathbb{R}\), we get

\[
\|(\phi_x, \psi_x, \xi_x)(t)\|^2 + \int_0^t \|\xi_{xx}(\tau)\|^2 d\tau
\]

\[
\leq \|(\phi_0, \psi_0, \xi_0)\|^2 + (\delta + N(t) + \eta) \int_0^t \|(\phi_x, \psi_x, \xi_x)(\tau)\|^2_1 d\tau + \int_0^t \int_\mathbb{R} \left( |\Theta_{xx}| + |\Theta_x| \right)^2 (\phi^2 + \xi^2) dx d\tau + \int_0^t \int_\mathbb{R} \left( |R_1 \phi_x| + |R_2 \xi_x| \right) dx d\tau,
\]

(3.36)

where \(\eta > 0\) is a constant suitably small, the last two terms on the right hand side of (3.36) can be treated similarly as (3.28) and (3.29), respectively. Then, with the
help of the results of Lemma 3.1 and Lemma 3.2 we can deduce the estimate (3.33), and this completes the proof of Lemma 3.3.

**Lemma 3.4.** Under the assumption in proposition 3.2, we have

\[
\|((\phi_{xx}, \psi_{xx}, \xi_{xx})(t)\|^2 + \int_0^t \|\xi_{xxx}(\tau)\|^2 d\tau
\leq \|((\phi_{0xx}, \psi_{0xx}, \xi_{0xx})\|^2 + \delta + (\delta + N(t)) \int_0^t \|((\phi_x, \psi_x)(\tau)\|^2 d\tau.
\]

Proof. Multiplying (3.31) by \( \xi_{xxx} \), (3.32) by \( \psi_{xxx} \) and (3.33) by \( \xi_{xxx} \), respectively, and adding the results together, it is easily to get

\[
\left\{ \frac{P}{2v} \phi_{xx}^2 + \psi_{xx}^2 + \frac{R \xi_{xx}^2}{2(\gamma - 1)\theta} \right\}_t + \frac{\kappa}{v\theta} \xi_{xxx}^2 + H_3 + J_3
\]

\[
=R_{1xx} \psi_{xx} + R_{1xx} \frac{\xi_{xxx}}{v},
\]

where

\[
H_3 = (p - P)_{xx} \psi_{xx} + \frac{\xi_{xxx}}{\theta} \left( (p - P)U_x - \left( \frac{\kappa \xi_x}{v} - \frac{\kappa \Theta_x \phi}{vV} \right)_x \right),
\]

\[
J_3 = \left( \frac{\xi_{xx}}{\theta} \right)_x \left( \frac{\kappa \xi_x}{v} - \frac{\kappa \Theta_x \phi}{vV} \right)_x - (p - P)U_x - \frac{\kappa}{v\theta} \xi_{xxx}^2
\]

\[
- \left( \frac{P}{2v} \right)_t \phi_{xx}^2 + \left( \frac{R}{2(\gamma - 1)\theta} \right) \xi_{xx}^2 + (2p_x \psi_{xx} + p_{xx} \xi_{xx}) \frac{\xi_{xx}}{\theta}
\]

\[
+ 3 \left( \frac{R}{v} \right)_x \xi_{xx} \psi_{xx} - 3 \left( \frac{P}{v} \right)_x \phi_{xx} \psi_{xx} + 2J_3^1 + J_3^2.
\]

Here \( J_3^1, J_3^2 \) are the following equalities

\[
J_3^1 := \psi_{xxx} \left( \frac{P}{v} \right)_x \phi_x - \left( \frac{R}{v} \right)_x \xi_x,
\]

\[
J_3^2 := \psi_{xxx} \left( \frac{P}{v} \right)_{xx} \phi - \left( \frac{R}{v} \right)_{xx} \xi_x.
\]

Meanwhile, we can get

\[
J_3^1 = \left\{ \psi_{xx} \left( \frac{P}{v} \right)_x \phi_x - \left( \frac{R}{v} \right)_x \xi_x \right\} \psi_{xx} \left( \frac{P}{v} \right)_x \phi_x - \left( \frac{R}{v} \right)_x \xi_x
\]

\[= \left\{ \psi_{xx} \left( \frac{P}{v} \right)_x \phi_x - \left( \frac{R}{v} \right)_x \xi_x \right\} + O(1)(N(t) + \delta) |(\phi_x, \xi_x, \phi_{xx}, \psi_{xx}, \xi_{xx})|^2.\]
and
\[
\psi_{xxx}(\frac{P}{v})_{xx} \phi
= \psi_{xxx} \phi \left( \frac{P_{xx}}{v} + 2P_x \left( \frac{1}{v} \right)_x + P \left( \frac{2v_x^2 - V_{xx}}{v^2} \right) \right) - \frac{P\phi}{v^2} \phi_{xx} \psi_{xxx}
\]
\[
= - \frac{P\phi}{v^2} \phi_{xx} \psi_{xxx} + \left\{ \psi_{xx} \phi \left[ \frac{P_{xx}}{v} + 2P_x \left( \frac{1}{v} \right)_x + P \left( \frac{2v_x^2 - V_{xx}}{v^2} \right) \right] \right\}_x
\]
\[
- \left\{ \frac{P\phi^2}{2} \psi_{xx} \psi_{xxx} \right\}_t + \left\{ \frac{P\phi^2}{2} \right\}_t \phi \left( \frac{P_{xx}}{v} + 2P_x \left( \frac{1}{v} \right)_x + P \left( \frac{2v_x^2 - V_{xx}}{v^2} \right) \right)
\]
\[
+ O(1)(N(t) + \delta) |(\phi_x, \xi_x, \phi_{xx}, \psi_{xx}, \xi_{xx})|^2 + ||(\Theta_x, \Theta_{xx})|^2| \phi, \xi \|^2.
\]

Similar to the estimate of \( \psi_{xxx}(\frac{P}{v})_{xx} \phi \), we have
\[
\psi_{xxx} \left( \frac{R_x}{v} \right)_{xx} \xi = \left\{ R_x \frac{\phi_x^2}{2} \right\}_t + \left\{ \psi_{xx} \xi \left( \frac{V_{xx}}{v^2} - \frac{2v_x^2}{v^3} \right) \right\}_x + ||(\Theta_x, \Theta_{xx})|^2| \phi, \xi \|^2
\]
\[
+ O(1)(N(t) + \delta) |(\phi_x, \xi_x, \phi_{xx}, \psi_{xx}, \xi_{xx})|^2.
\]

Therefore, it holds
\[
J_3 = \left\{ \frac{R_x \phi_x^2}{2} \right\}_t + \left\{ \psi_{xx} \phi \left[ \frac{P_{xx}}{v} + 2P_x \left( \frac{1}{v} \right)_x + P \left( \frac{2v_x^2 - V_{xx}}{v^2} \right) \right] \right\}_x
\]
\[
+ \left\{ \psi_{xx} R_x \left( \frac{V_{xx}}{v^2} - \frac{2v_x^2}{v^3} \right) \right\}_x + \left\{ \psi_{xx} \left( \frac{P}{v} \right)_x - \left( \frac{R}{v} \right) \right\}_x \xi_x \right\}_x
\]
\[
+ O(1)(N(t) + \delta) |(\phi_x, \xi_x, \phi_{xx}, \psi_{xx}, \xi_{xx})|^2 + ||(\Theta_x, \Theta_{xx})|^2| \phi, \xi \|^2.
\]

After integrating (3.38) on \([0, t] \times \mathbb{R}\), we get
\[
||\phi_{xx} \psi_{xx}, \xi_{xx}\|_X(t) + \int_0^t \|\xi_{xxx}(\tau)\|^2 d\tau
\]
\[
\lesssim ||\phi_{00}, \psi_{00}, \xi_{00}||^2_2 + (\delta + N(t)) \int_0^t ||\phi_{xx} \psi_{xx}, \xi_{xx}\|_X^2 d\tau
\]
\[
+ \int_0^t \int_\mathbb{R} (||\Theta_{xx}|| + ||\Theta_x||^2)(\phi, \xi)^2 dx d\tau
\]
\[
+ \int_0^t \int_\mathbb{R} R_{1xx} \psi_{xx} + R_{2xx} \xi_{xx} dx d\tau.
\]
For the estimate of the last term in (3.43), we have

\[
\int_0^t \int_\mathbb{R} |R_{1xx} \psi_{xx}| dx \, d\tau \\
\leq \delta \int_0^t \left\| \psi_{xx}(\tau) \right\| \left\{ \left( \int_\mathbb{R} e^{-2|x|} e^{-2\tau} \, dx \right)^{\frac{1}{2}} + \frac{1}{(1 + \tau)^{\frac{3}{2}}} \right\} \, d\tau \tag{3.44}
\]

\[
\leq \delta \int_0^t (1 + t)^{-\frac{5}{4}} \left\| \psi_{xx}(\tau) \right\|^2 \, d\tau + \delta
\]

and

\[
\int_0^t \int_\mathbb{R} |R_{2xx} \xi_{xx}| dx \, d\tau \\
\leq \delta^{\frac{1}{8}} \int_0^t \| \xi_{xx} \|_\infty (1 + \tau)^{-\frac{7}{8}} \, d\tau \tag{3.45}
\]

\[
\leq \delta^{\frac{1}{8}} \int_0^t \| \xi_{xx} \|^{\frac{3}{2}} \| \xi_{xxx} \|^{\frac{1}{2}} (1 + \tau)^{-\frac{7}{8}} \, d\tau
\]

\[
\leq \delta^{\frac{1}{8}} \int_0^t \| \xi_{xx} \| \| \xi_{xxx} \| \, d\tau + \delta^{\frac{1}{8}} \int_0^t (1 + \tau)^{-\frac{7}{8}} \, d\tau
\]

\[
\leq \delta^{\frac{1}{8}} \int_0^t \| \xi_{xx} \|^2 \, d\tau + \delta^{\frac{1}{8}},
\]

then using the results of Lemma 3.1-Lemma 3.3, we can get (3.37) and this completes the proof of Lemma 3.4.

Combining the results of Lemma 3.1-Lemma 3.4, we known that

\[
\|(\phi, \psi, \xi)(t)\|_2^2 + \int_0^t \| \xi_{x}(\tau) \|^2 \, d\tau \\
\lesssim \|(\phi_0, \psi_0, \xi_0)\|_2^2 + \delta + (\delta + N(t) + \eta) \int_0^t \| (\phi_x, \psi_x)(\tau) \|^2 \, d\tau.
\]

Based on these analysis, we now deal with the last term on the right hand in (3.46).

**Lemma 3.5.** Under the assumption in proposition 3.2, we have

\[
\int_0^t \| (\phi_x, \psi_x)(\tau) \|^2 \, d\tau \lesssim \|(\phi_0, \psi_0, \xi_0)\|_2^2 + \delta.
\]

Proof. Multiplying (3.3)_2 by $-\frac{P}{2}\phi_x$, and (3.3)_3 by $\psi_x$, respectively, and adding all the resultant equations, we have

$$\frac{R}{\gamma - 1}\xi\psi_t - \frac{P}{2}\phi_t\psi + \frac{P^2}{2v}\phi_x^2 + \frac{P}{2}\psi_x^2$$

$$= \frac{P}{2}\psi_x^2 - \frac{P}{2}\phi_x\psi + \frac{P}{2}\phi_x \left( \left( \frac{R}{2v} \right)_x - \left( \frac{P}{v} \right)_x \phi + R_1 \right) - \frac{R}{\gamma - 1}\xi_x\psi_t$$

(3.48)

Integrating (3.48) on $[0,t] \times \mathbb{R}$ and using the inequality (3.46), it holds that

$$\int_0^t \int_\mathbb{R} \left( \phi_x^2 + \psi_x^2 \right) dx \, d\tau$$

$$\lesssim \| (\phi, \psi, \xi) \|_1^2 + \| (\phi_0, \psi_0, \xi_0) \|_1^2 + \int_0^t \| \xi_x(\tau) \|_1^2 d\tau$$

(3.49)

$$+ \left( \frac{1}{4} + \delta + N(t) \right) \int_0^t \int_\mathbb{R} \left( \phi_x^2 + \psi_x^2 \right) dx \, d\tau + \int_0^t \int_\mathbb{R} \left( |\Theta_{xx}| + |\Theta_x| \right)^2 (\phi^2 + \psi^2) dx \, d\tau$$

$$+ \int_0^t \int_\mathbb{R} |R_1\phi_x + R_2\psi_x| dx \, d\tau.$$

Similar as the estimates of (3.28) and (3.29), we can easily get

$$\int_0^t \int_\mathbb{R} \left( \phi_x^2 + \psi_x^2 \right) dx \, d\tau$$

(3.50)

$$\lesssim \| (\phi_0, \psi_0, \xi_0) \|_2^2 + \delta + (\delta + N(t)) \int_0^t \| (\phi_x, \psi_x)(\tau) \|_1^2 d\tau.$$

Similarly, multiplying (3.3)_2 by $-\frac{P}{2}\phi_{xx}$, (3.3)_3 by $\psi_{xx}$, respectively, and integrating the result on $[0,t] \times \mathbb{R}$, then by using (3.46), we can also get

$$\int_0^t \int_\mathbb{R} \left( \phi_{xx}^2 + \psi_{xx}^2 \right) dx \, d\tau$$

(3.51)

$$\lesssim \| (\phi_0, \psi_0, \xi_0) \|_2^2 + \delta + (\delta + N(t)) \int_0^t \| (\phi_x, \psi_x)(\tau) \|_1^2 d\tau.$$

Adding the results of (3.50) with (3.51), we can get (3.47) and this completes the proof of Lemma 3.5. \qed

Inserting (3.47) into (3.46), then it yields (3.8), that is the result of Proposition 3.2.
Acknowledgments: The authors are grateful to Professor A. Matsumura for his support and advice. This work was supported by the Fundamental Research Funds for the Central Universities and three grants from the National Natural Science Foundation of China under contracts 11301405, 11671309 and 11731008, respectively.

REFERENCES

[1] Z. Z. Chen, L. He and H. J. Zhao, Nonlinear stability of traveling wave solutions for the compressible fluid models of Korteweg type, *J. Math. Anal. Appl.* **422** (2015) 1213–1234.
[2] Z. Z. Chen and Q. H. Xiao, Nonlinear stability of viscous contact wave for the one-dimensional compressible fluid models of Korteweg type, *Math. Methods Appl. Sci.* **36** (2013) 2265–2279.
[3] Z. Z. Chen, L. J. Xiong and Y. J. Meng, Convergence to the superposition of rarefaction waves and contact discontinuity for the 1-D compressible Navier-Stokes-Korteweg system, *J. Math. Anal. Appl.* **412** (2014) 646–663.
[4] Z. Z. Chen and H. J. Zhao, Existence and nonlinear stability of stationary solutions to the full compressible Navier-Stokes-Korteweg system, *J. Math. Pures Appl.* **101** (2014) 330–371.
[5] R. J. Duan, H. X. Liu and H. J. Zhao, Nonlinear stability of rarefaction waves for the compressible Navier-Stokes equations with large initial perturbation, *Trans. Amer. Math. Soc.* **361**(1) (2009) 453–493.
[6] R. J. Duan and H. F. Ma, Global existence and convergence rates for the 3-D compressible Navier-Stokes equations without heat conductivity, *Indiana Univ. Math. J.* **57** (2008) 2299–2319.
[7] L. L. Fan, H. X. Liu, T. Wang and H. J. Zhao, Inflow problem for the one-dimensional compressible Navier-Stokes equations under large initial perturbation, *J. Differential Equations* **257** (2014) 3521–3553.
[8] L. L. Fan and A. Matsumura, Asymptotic stability of a composite wave of two viscous shock waves for a one-dimensional system of non-viscous and heat-conductive ideal gas, *J. Differential Equations* **258** (2015) 1129–1157.
[9] J. Goodman, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, *Arch. Rational Mech. Anal.* **95** (1986) 325–344.
[10] L. He, S. J. Tang and T. Wang, Stability of viscous shock waves for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity, *Acta Math. Sci. Ser. B Engil. Ed.* **36** (2016) 34–48.
[11] B. K. Huang and Y. K. Liao, Global stability of viscous contact wave with rarefaction waves for compressible Navier-Stokes equations with temperature-dependent viscosity, *Math. Models Methods Appl. Sci.* **27** (2017) 2321–2379.
[12] B. K. Huang, L. S. Wang and Q. H. Xiao, Global nonlinear stability of rarefaction waves for compressible Navier-Stokes equations with temperature and density dependent transport coefficients, *Kinet. Relat. Models* **9** (2016) 469–514.
[13] F. M. Huang, J. Li and A. Matsumura, Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier-Stokes system, *Arch. Rational Mech. Anal.* **197** (2010) 89–116.
[14] F. M. Huang and A. Matsumura, Stability of a composite wave of two viscous shock waves for the full compressible Navier-Stokes equation, *Comm. Math. Phys.* **289** (2009) 841–861.
[15] F. M. Huang, A. Matsumura and X. D. Shi, On the stability of contact discontinuity for compressible Navier-Stokes equations for free boundary, *Osaka J. Math.* **41** (2004) 193–210.
[16] F. M. Huang and T. Wang, Stability of superposition of viscous contact wave and rarefaction waves for compressible Navier-Stokes system, *Indiana Univ. Math. J.* **65**(6) (2016) 1833–1875.
[17] F. M. Huang, A. Matsumura and Z. P. Xin, Stability of contact discontinuities for the 1-D compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.* **179** (2005) 55–77.
[18] F. M. Huang, Z. P. Xin and T. Yang, Contact discontinuity with general perturbation for gas motions, *Adv. Math.* **219** (2008) 1246–1297.
[19] F. M. Huang and H. J. Zhao, On the global stability of contact discontinuity for compressible Navier-Stokes equations, *Rend. Semin. Mat. Univ. Padova* **109**(2003) 283–305.
[20] S. Kawashima and A. Matsumura, Asymptotic stability of travelling wave solutions of systems for one-dimensional gas motion, *Comm. Math. Phys.* **101** (1985) 97–127.
[21] H. X. Liu, T. Yang, H. J. Zhao and Q. Y. Zou, One-dimensional compressible Navier-Stokes equations with temperature dependent transport coefficients and large data, *SIAM J. Math. Anal.* **46** (2014) 2185–2228.
[22] T. P. Liu, Nonlinear stability of shock waves for viscous conservation laws, *Mem. Am. Math. Soc.* **56**(1985) 1–108.
[23] T. P. Liu, Shock wave for compressible Navier-Stokes equations are stable, *Comm. Pure Appl. Math.* **39**(1986) 565–594.
[24] T. P. Liu and Z. P. Xin, Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations, *Comm. Math. Phys.* **118**(1988) 451–465.
[25] T. P. Liu and Y. N. Zeng, Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws, *Mem. Amer. Math. Soc.* **125**(599) (1997) pp. viii+120.
[26] T. P. Liu and Y. N. Zeng, Compressible Navier-Stokes equations with zero heat conductivity, *J. Differential Equations* **153**(2) (1999) 225–291.
[27] T. P. Liu and Y. N. Zeng, Time-asymptotic behavior of wave propagation around a viscous shock profile, *Comm. Math. Phys.* **290**(1) (2009) 23–82.
[28] T. P. Liu and Y. N. Zeng, Shock waves in conservation laws with physical viscosity, *Mem. Amer. Math. Soc.* **234**(1105) (2015) pp. vi+168.
[29] S. X. Ma and J. Wang, Decay rates to viscous contact waves for the compressible Navier-Stokes equations, *J. Math. Phys.* **57** (2016) 1–14.
[30] A. Matsumura and K. Nishihara, Asymptotics toward the rarefaction waves of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.* **3** (1986) 1–13.
[31] A. Matsumura and K. Nishihara, On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.* **2**(1) (1985) 17–25.
[32] A. Matsumura and K. Nishihara, Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas, *Comm. Math. Phys.* **144** (1992) 325–335.
[33] A. Matsumura and K. Nishihara, Global asymptotics toward the rarefaction waves for solutions of viscous p-system with boundary effect, *Quart. Appl. Math.* **58** (2000) 69–83.
[34] K. Nishihara, T. Yang and H. J. Zhao, Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations, *SIAM J. Math. Anal.* **35** (2004) 1561–1597.
[35] L. Wan and T. Wang, Symmetric flows for compressible heat-conducting fluids with temperature dependent viscosity coefficients, *J. Differential Equations* **262** (2017) 5939–5977.
[36] L. Wan, T. Wang and H. J. Zhao, Asymptotic stability of wave patterns to compressible viscous and heat-conducting gases in the half-space, *J. Differential Equations* **261** (2016) 5949–5991.
[37] T. Wang and H. J. Zhao, One-dimensional compressible heat-conducting gas with temperature-dependent viscosity, *Math. Models Methods Appl. Sci.* **26** (2016) 2237–2275.
[38] T. Wang, H. J. Zhao and Q. Y. Zou, One-dimensional compressible Navier-Stokes equations with large density oscillation, *Kinet. Relat. Models* **6** (2013) 649–670.
[39] J. Smoller, *Shock Wave and Reaction-Diffusion Equations*, 2nd Edn. (Springer-Verlag, New York, 1994).
A. Szepessy and Z. P. Xin, Nonlinear stability of viscous shock waves, Arch. Ration. Mech. Anal. 122 (1993) 53–103.

Z. P. Xin, On nonlinear stability of contact discontinuities. In: Hyperbolic problems: theory, numerics, applications (World Sci. Publishing, River Edge, NJ, 1996), pp. 249–257.

Lili Fan: School of Mathematics and Computer Science, Wuhan Polytechnic University, Wuhan 430023, P. R. China
E-mail address: fll810@live.cn

Guiqiong Gong: School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P. R. China
E-mail address: gongguiqiong@yeah.net

Shaojun Tang: School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P. R. China
E-mail address: shaojun.tang@whu.edu.cn