Nonlocalized clustering in nuclei

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\section*{ABSTRACT}
Cluster structures in nuclei have been studied for half a century and one central problem is how to understand and describe the relative motion of clusters. In 2001, the THSR (Tohsaki-Horiuchi-Schuck-Röpke) wave function was proposed for exploring the possible $\alpha$ condensation in $n\alpha$ nuclei. This novel microscopic cluster wave function provides us with a new perspective for understanding general cluster correlations in nuclei. In this article, we will review a new understanding for cluster structures, that is the concept of nonlocalized clustering, which is inspired by the THSR wave function. The concept of nonlocalized clustering is characterized as the dimensional size parameter rather than the traditional inter-cluster distance parameter. Based on this concept, a container picture is introduced for describing and understanding the complicated motion of clusters in nuclei.

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\section*{1. Introduction}
Clustering, as one of the most interesting and crucial phenomena of many-body dynamics, exists abundantly in a variety of physical systems [1–3]. On a macro scale, the mutual gravity can draw galaxies together into a cluster that is usually several millions of light years across. The formed galaxy cluster [4] consists of hundreds or thousands of galaxies and makes very splendid collective motion in...
the universe. In Chemistry, in a micro-scale system, some atomic or molecular units can be aggregated to the atomic or molecular clusters, e.g. the typical 20-atoms metal Cu$_{20}$, Ag$_{20}$ and Au$_{20}$ clusters [5,6]. The nature for objects to congregate to clustering on different scales is quite amazing.

In nuclear systems, especially in light nuclei, the formation of clustering is enhanced due to the saturating two-nucleon force and Pauli exclusion principle. Unlike the shell-model picture [7], in the cluster model [8], the nucleus can be considered as the assembly of clusters or subunits that are composed of some nucleons. In $\alpha$-conjugate nuclei ($4n$ nuclei), the $\alpha$-clustering correlation is becoming very prominent due to the high stability of $\alpha$ cluster [9–12]. In this case, the relative motion of $\alpha$ clusters becomes a significant freedom of the nucleus, which can be considered as some kind of enhanced cluster correlations in the many-body system [13]. In heavier nuclei, the observed $\alpha$ and some clusters decay phenomena and some theoretical calculations [14–17] indicate that the cluster has a certain probability to be preformed on the surface of the nucleus before decay. In nuclear matter, the formation of light clusters in low density is also a very important subject [18–22].

Cluster structures in nuclei have been studied for quite a long time [8,25–27] and the history even begins before the discovery of neutron. At early stage, the higher binding energies of $n\alpha$ nuclei, like $^8$Be, $^{12}$C, $^{16}$O, and the observed $\alpha$-decay from heavy radioactive nuclei provide some clues about possible cluster structures in some nuclei [28,29]. For example, as for the ground state of $^8$Be, it can decay into two $\alpha$ particles with a large reduced width, so it is very likely that the ground state of $^8$Be has a molecule-like 2$\alpha$ cluster structure. In Figure 1, the Ikeda diagram shows with the increase in excitation energies, various clustering degrees of freedom activate and different cluster structures can be formed around threshold energies, which indicate that the cluster structures are the general feature of light nuclei. Indeed, for the low-excitation-energy region in light $\alpha$-conjugate nuclei, some rotational bands or states can be understood easily from the view of cluster structures while they are difficult to be explained from the shell-model picture, e.g. the inversion doublet bands of $^{20}$Ne and $^{16}$O [27,30].

Since 1960s, microscopic cluster models [31] have been developed remarkably for studying various clustering phenomena in nuclei. Three important and representative cluster models are resonating group method (RGM), generator coordinate method (GCM), and orthogonality condition model (OCM) (semi-microscopical cluster model), which take into account properly the effect of the antisymmetrization and Pauli exclusion principle. Many experimental data related with clustering in $4n$ light nuclei have been reproduced well using these models [27]. We here want to emphasize the GCM Brink wave function and the corresponding understanding for cluster correlations.

The localized clustering is a traditional understanding for cluster structures in nuclei. Early times, there were not too much knowledge for cluster structures and $4n$ nuclei were considered just as a collection of structureless rigid $\alpha$ particles
undergoing the localized motion [32]. Obviously, that was an oversimplified idea. However, this concept for rigid or geometric cluster structures was gradually accepted due to the development of the Brink cluster wave function [33]. The Brink wave function is parameterized by the geometry of positions of the cluster centers. Thus, the geometrical cluster structure was obtained by the energy variational method without any a priori assumption. It seems that, rigid cluster structures or concept of localized clustering can also be supported in the microscopic cluster model.

Recently, the proposed THSR wave function [34], based a concept of nonlocalized clustering, has brought a new perspective to the Hoyle state [34–36]. The Hoyle state is now considered to be an Σ condensate state, in which three α clusters occupy the same (0S) orbit and make a nonlocalized motion. On the other hand, the 3α RGM wave function for the ground state of 12C was found to have about 93% squared overlap with a single THSR wave function [37,38]. Furthermore, using a Hybrid-THSR-Brink wave function, the concept of nonlocalized clustering can be extended to the inversion doublet band in 20Ne[39,40]. It was concluded that, in general cluster systems, the dynamics prefers nonlocalized clustering rather than the traditional localized clustering. Based on this new concept, a container picture was proposed for the description of cluster structures [41]. The spirit of the THSR wave function is developing to study a much wider range of nuclear systems [42–44]. In this review paper, we will give an introduction for these important progress from the THSR wave function, especially the concept of nonlocalized clustering.

Figure 1. Ikeda Diagram.
Notes: This diagram shows possible subunit clusters can appear around the corresponding threshold energies. The threshold energies for the breakup of clusters in MeV are shown. This diagram is adopted from Ref. [23] and the original one is from Ref. [24].
This review paper is organized as follows. In Section 2, we will introduce briefly some microscopic cluster wave functions, especially the THSR wave function, which is a completely new idea for dealing with cluster structures in nuclei. In Section 3, the concept of nonlocalized clustering will be introduced, which will be explained by the calculated results from the inversion doublet band in $^{20}$Ne using a Hybrid-THSR-Brink wave function. In Section 4, based on the concept of nonlocalized clustering, the container picture will be discussed. In Section 5, we will give a summarized discussion and also discuss the possible future work.

2. Microscopic cluster wave functions

The cluster models, especially microscopic cluster models always play a central role for studying cluster structures in nuclei. RGM and GCM are two most important traditional microscopic cluster models, by which many cluster states have been extensively studied and reported. See the review Refs. [23,31,45].

The RGM [46,47] was the first microscopic cluster model for dealing with the relative motion of clusters in nuclei. In RGM, neutrons and protons are divided into different clusters, which are continually being broken up and reformed in various ways due to the antisymmetrization effect. As for the two-cluster system, the RGM wave function can be written as,

$$\Phi_R = \mathcal{A} \{ \chi(\xi) \phi_{c_1} \phi_{c_2} \}.$$  

Here, $\phi_{c_1}$ and $\phi_{c_2}$ are the shell-model wave functions of cluster $c_1$ and $c_2$, respectively. $\mathcal{A}$ is the antisymmetrization operator that exchanges the nucleons of two clusters. $\xi$ is the relative coordinate between clusters and $\chi(\xi)$ is the relative wave function of two clusters. The essential point of RGM is to obtain this explicit relative wave function $\chi(\xi)$ in Equation (1). The solution of RGM is equivalent to the full solution of a many-body Schrödinger’s equation in a cluster system.

Another traditional and more popular microscopic cluster model is the GCM Brink cluster model, which was widely used for the description of cluster structures in nuclei [48–50]. We begin with the general $n$-cluster Brink wave function [33],

$$\Phi^B(R_1, \ldots, R_n) = n_0 \mathcal{A} \{ \psi_{c_1}(R_1) \psi_{c_2}(R_2) \cdots \psi_{c_n}(R_n) \}$$  

$$= \frac{1}{\sqrt{A!}} \text{Det}[\phi_1 \phi_2 \cdots \phi_A].$$  

Here, $\psi_{c_i}(R_i)$ represents the $i$th cluster wave function with the generator coordinate $R_i$. $\phi_i$ represents the $i$th nucleon wave function, which is usually expressed by the harmonic oscillator wave function. $\mathcal{A}$ is the antisymmetrization operator that exchanges the nucleons of different clusters. It can be seen that the Brink wave function $\Phi^B(R_1, \ldots, R_n)$ can be written as a Slater determinant, which is
useful for the practical calculations using the developed matrix technique [51].
This is a very important reason why the Brink wave function can be so widely used in nuclear cluster physics.

At the same time, the Brink model also provides a very simple picture for understanding the motion of clusters. Each cluster in the system can move around some fixed points by the generator coordinates \( \{ R_1, \ldots, R_n \} \). By the unconstrained variational calculations, some geometry and rigid cluster structures can be obtained. Many \( 4n \) nuclei have been systematically explored using Brink cluster wave functions [49,50,52]. However, we should be very careful about the concept of localization revealed by the Brink wave function. In the following chapter, we will mainly focus on this problem.

The GCM [53] is a general method to describe the collective motion in nuclei. With a linear combination of Brink wave functions, the GCM-Brink wave function is becoming a very powerful tool for studying cluster structures in nuclei. The general GCM-Brink wave function can be written as [33],

\[
\Phi_{GCM}^B = \int dR f(R) \Phi^B(R). \tag{4}
\]

Here, \( R \) is the generator coordinate and \( f(R) \) is the weight function and it can be determined by discretization techniques by numerical evaluation [54]. By adopting enough mesh points for the generator coordinate \( R \) and solving the following Hill-Wheeler equation [55], the obtained \( \Phi_{GCM}^B \) can be considered as the full solution of the cluster systems [56].

\[
\sum_{R'} \langle \Phi^B(R) | \hat{H} - E | \Phi^B(R') \rangle f(R') = 0. \tag{5}
\]

In 2001, to study \( \alpha \)-cluster condensation in \( n\alpha \) cluster system, Tohsaki, Horiiuchi, Schuck, and Röpke proposed a new type cluster wave function, which is called THSR wave function [34],

\[
\Psi_{n\alpha}(\beta) = \int dR_1 \cdots dR_n \exp \left[ -\frac{R_1^2 + R_2^2 + \cdots + R_n^2}{\beta^2} \right] \Phi_{n\alpha}^B(R_1, \ldots, R_n) \tag{6}
\]

\[
= N_0 \exp \left[ -\frac{2nX_G^2}{B^2} \right] \phi \left\{ \prod_{i=1}^n \exp \left[ -\frac{2(X_i - X_G)^2}{B^2} \right] \phi(\alpha_i) \right\}. \tag{7}
\]

Here \( B^2 = b^2 + 2\beta^2 \) and \( b \) is the size parameter of the harmonic-oscillator wave function of \( \alpha \) cluster. \( X_G \) is the center-of-mass coordinate of \( n\alpha \) clusters. The \( n\alpha \) wave function in Equation (6) can be understood as an analogy to BCS wave function for pairing [57], which both describe the similar condensation phenomena in spite of different systems. The THSR wave function in Equation (6) connects the traditional \( n\alpha \) Brink wave function by a very simple Gaussian integral of the generator coordinates. After integral in Equation (6), we can clearly see the novel
THSR wave function has a quite different form for the relative wave function part. The clusters can almost move freely restricted by only one size parameter $\beta$ rather than the localized motion confined by many inter-cluster parameters in the Brink wave function. On the other hand, to describe the $n\alpha$ condensation problem, the Brink cluster model needs $n-1$ independent parameters with respect to the inter-cluster distance and calculations are becoming very complicated. In the above THSR wave function, only one size-parameter $\beta$ is sufficient [58].

After 15 years’ development for this microscopic cluster wave function, the THSR wave function has had a significant impact on the nuclear cluster physics [42,59]. The first striking achievement from the THSR wave function is the establishment of $\alpha$ condensation for the Hoyle state in $^{12}\text{C}$ at the excitation energy $E_x = 7.65$ MeV. Early on, Horiuchi proposed that the Hoyle state had a $^8\text{Be}(0^+_1)+\alpha$ structure with weakly correlated based on the OCM calculations [60]. OCM is a semi-microscopic cluster model that approximates treatments of the Pauli principle, in which relative wave functions of clusters are required to be orthogonal to the forbidden states. After that, the weakly coupled $3\alpha$ structure was confirmed by the full microscopic $3\alpha$ cluster calculations [61]. Now, it is found that the constructed simple $3\alpha$ THSR wave function has overlaps of more than 95% with the corresponding RGM wave function, which gives a very strong support to this $\alpha$ condensation picture [38,57]. The second achievement from the THSR wave function is, the spirit of THSR wave function is not only suitable for the special condensation or gas-like cluster states but also a general feature of cluster states in nuclei [40,41], which will be discussed in the next part.

3. Nonlocalized clustering

The original THSR wave function (See Equation (6)) [34] is designed for the study of $\alpha$ condensation states. Characterized by a new size parameter rather than the conventional inter-cluster distance parameter, the THSR wave functions have been quite successful for the $n\alpha$ gas-like cluster states, e.g. even the single THSR wave functions can give very exact description of the Hoyle state and $2\alpha$ state [62]. The so high-percentage description of the gas-like cluster states could not be an accident and the physics underlying THSR wave function should be further digged.

$^{20}\text{Ne}$ is one of the most important nuclei in nuclear cluster physics, which has a very typical $\alpha+^{16}\text{O}$ cluster structure and has been studied by many nuclear models [30,64–67]. At early times, even the cluster structures in nuclei have not been completely confirmed, Horiuchi and Ikeda proposed that [63] the observed ground-state and $K^\pi=0^-$ bands in $^{20}\text{Ne}$, which were difficult to be explained from the shell model at that time, should be regarded as being an inversion doublet band arising from the hetero-polar di-nucleus configuration of $\alpha+^{16}\text{O}$ structure. See the schematic diagram in Figure 2. This kind of inversion doublet band in $^{20}\text{Ne}$ can be considered as the direct manifestation of cluster structures.
in $^{20}\text{Ne}$. Furthermore, in asymmetry two-cluster systems, the inversion doublet band can be regarded as a clear indication of the existence of the localized cluster structure together with the observation of large cluster decay widths. In this case, it seems we have to regard states of the inversion doublet band of $^{20}\text{Ne}$ as having a $\alpha + ^{16}\text{O}$ localized clustering.

Now, we have known that the inversion doublet band of $^{20}\text{Ne}$ in Figure 2 origins from the $\alpha + ^{16}\text{O}$ cluster structure. Further studies indicate that the ground-state band of $^{20}\text{Ne}$ has a very compact shell-model-like cluster structure while the negative-parity band has a well-developed cluster structure. Therefore, to extend the THSR wave function to general cluster states, the $^{20}\text{Ne} (\alpha + ^{16}\text{O})$ is a very suitable cluster system. In the first step, the ground-state band of $^{20}\text{Ne}$ was described well using the generalized THSR wave function [39], which is a natural extended version of the original THSR wave function from $n\alpha$ cluster systems to general cluster systems. Next, to describe the $K^\pi = 0^+$ band in $^{20}\text{Ne}$ by the spirit of THSR wave function, a Hybrid-THSR-Brink wave function [40] was introduced,

$$\Psi_{\text{Ne}}(\beta, S) \propto A \left\{ \exp \left[ - \sum_{k=x,y,z} \frac{8(r-S)^2}{5(b^2 + 2\beta_k^2)} \right] \phi_{\alpha} \phi_{16\text{O}} \right\}.$$  (8)

Here $\beta \equiv (\beta_x, \beta_y, \beta_z)$, $r = X_2 - X_1$. $X_1$ and $X_2$ represent the center-of-mass coordinates of the $\alpha$ and $^{16}\text{O}$ clusters, respectively. All calculations here
Table 1. Squared overlaps [41] between the single normalized projected THSR-type wave functions $\Phi_{\text{THSR}}^{\text{Min}}$ corresponding to the minimum energies and the normalized Brink GCM wave functions of cluster states of $^{20}\text{Ne}$.

| States   | $0^+$ | $2^+$ | $4^+$ | $1^-$ | $3^-$ |
|----------|-------|-------|-------|-------|-------|
| $|\langle \Phi_{\text{THSR}}^{\text{Min}} | \Phi_{\text{Brink GCM}} \rangle |^2$ | 0.9929 | 0.9879 | 0.9775 | 0.9998 | 0.9987 |

are performed with restriction to axially symmetric deformation, that is, $S \equiv (0,0,S_z)$. Spin and parity eigen functions are obtained by angular momentum projection technique [39]. In this hybrid wave function, the Brink wave function and the THSR wave function can be obtained at the limit of $\beta \to 0$ and $S_z \to 0$, respectively. Next, we will show how the localized or nonlocalized clustering problem is clarified using this flexible hybrid wave function [40,41].

Firstly, we performed variational calculations for parameters $\beta$ and $S_z$ using the projected Hybrid-THSR-Brink wave function. Figure 3 shows the energy curves of $J^\pi = 0^+, 2^+, 1^-$, and $3^-$ states with two sets of $\beta$ values, which represent the width of Gaussian relative wave function in the Hybrid-THSR-Brink wave function. In the first set, the value of $\beta$ parameter is fixed at 0. Then, the Hybrid-THSR-Brink wave function becomes nothing but the Brink wave function. Figure 3 shows the calculated energy curves of $^{20}\text{Ne}$ and this kind of parabole-type energy curves from the Brink cluster model can be found in many literature for different cluster systems [25,68], which were usually considered to be the evidences to support the localized clustering in nuclei. The parameter $S_z$ for the inter-cluster distance was regarded as a dynamical parameter for describing the cluster systems. For instance, the minimum energy of the ground state of $^{20}\text{Ne}$ appears at $S_z = 3.0$ fm. This non-zero $S_z$ seems to indicate the $\alpha + ^{16}\text{O}$ cluster structure in $^{20}\text{Ne}$ favors the localized clustering, which is just the traditional understanding of the localized clustering. However, if we adopt the other set of $\beta$ values, which were obtained by the unconstrained variational calculations using the Hybrid-THSR-Brink wave function, it was found that minimum points for the low-lying states of the inversion doublet band in $^{20}\text{Ne}$ appear at $S_z = 0$. In this case, the Hybrid-THSR-Brink wave function becomes a nearly pure THSR wave function and the parameter $S_z$ does not play any physical role in describing the cluster structures. Therefore, the localized clustering from the Brink wave function is misleading. The appearance of non-zero $S_z$ is just because the relative wave function in Brink wave function is adopted a very narrow width.

Furthermore, Table 1 shows the squared overlaps between the single optimum THSR wave functions and the corresponding Brink GCM wave functions. The high-percentage squared overlaps indicate that the obtained single THSR wave functions for the inversion doublet band are almost 100% equivalent to the corresponding full solutions from GCM. This is a strong support for the concept of nonlocalized clustering. Therefore, it was concluded that the THSR wave function characterized as the nonlocalized clustering, is more appropriate for
Figure 3. Energy curves of $\alpha+^{16}\text{O}$ cluster in $^{20}\text{Ne}$ with different widths of Gaussian relative wave functions in the Hybrid-THSR-Brink wave function. Note: This figure is adopted from Ref. [41].

describing the cluster structure in $^{20}\text{Ne}$ and the new parametrization by $\beta$ can be considered as the correct dynamic parameter for the description of the cluster structures in nuclei.

The THSR wave function has been extended to describe ordinary cluster states and also the negative-parity states. Since the THSR wave function is constructed in a nonlocalized way, we conclude that the nonlocalized clustering should be the general feature of cluster states in nuclei [41]. However, it should be noted that, in a two-cluster system, clusters cannot overlap too much due to the Pauli exclusion principle, which causes the effectively localized spatial distribution in the two-cluster system. In this situation, the prolate THSR wave function has the parity-violating deformation of $\alpha+^{16}\text{O}$. Another more typical example is the description of $\alpha$-linear-chain states using the THSR wave function. In Ref. [69], it was reported that in the description of the $\alpha$-linear-chain states for $^{12}\text{C}$ and $^{16}\text{O}$, the single THSR wave functions are almost 100% equivalent to the corresponding superposed Brink wave functions on one dimension direction. Interestingly, the nonlocalized picture is very different from the traditional idea of the localized Brink wave function while the density distributions are shown to have localized $\alpha$ clusters due to the inter-$\alpha$ Pauli repulsion. Figure 4 shows the intrinsic densities of the $3\alpha$- and $4\alpha$-linear-chain states from the intrinsic THSR wave functions, which indicate that the clear spatial localization of clusters is attributed to the inter-cluster Pauli principle. Generally speaking, the dynamics prefers nonlocalized clustering but kinematics makes the cluster system look like localized clustering, especially it is more enhanced in the two-cluster system. In three-cluster or multi-cluster systems with low density, the clusters have more space and will have a more nonlocalized clustering feature.
4. Container picture for cluster structures

We have shown that the THSR wave function is not only suitable for describing α condensation states and gas-like cluster states but also ordinary cluster states with normal density. Therefore, various cluster states in nuclear systems can be described in a unified way due to the introduced new dimension size parameter. The container picture is used for expressing the new understanding of cluster dynamics, in which the clusters can almost move freely in a β-size container and only cannot overlap too much due to Pauli principle. The central quantity of cluster dynamics is the size parameter, which is quite different from the traditional inter-cluster distance parameter and reflects the character of nonlocalized clustering. In Ref. [41], details about the features of the container picture are explained.

The container picture of cluster dynamics consists of three important ingredients [41]. The first is to regard the motion of clusters as being almost independent and it can be described by the nonlocalized lowest orbit in some kind mean-field potential of clusters. The second is the collective excitation of the cluster system, which can be described by the Hill-Wheeler equation with respect to the size parameter(s) β. The third is the inter-cluster Pauli repulsion principle, which can be considered as the origin of molecular structures of clusters.

The container picture provides us with a very natural and simple picture for describing and understanding various cluster states in nuclei. As we know, the internal wave functions of clusters are usually adopted as the simple shell model wave functions. When the size parameter $\beta \to 0$, the corresponding container wave function will become the SU(3) shell model wave function [70,71], which is the reason why this cluster wave function also can describe very well the compact
cluster or shell-model-like cluster states. The container with larger size parameter has more spatial expansion, in which the clusters can move in a wider volume and the gas-like cluster states can be formed. In an infinity container, \( (\beta \to +\infty) \), the correlations between clusters will disappear and clusters make completely free motions. In addition, the container can also have some-type shape due to the introduced deformed \( \beta \) parameter. For example, if we restrain \( \beta_x \to 0, \beta_y \to 0 \), the linear chain structure can be obtained in the \( \beta_z \)-size container. This flexible and powerful container picture is very promising to be extended to more general cluster systems.

5. Future work

In this review paper, a new concept in nuclear cluster physics was reviewed, that is the concept of nonlocalized clustering. This concept is original from the THSR wave function and it is clarified further using a Hybrid-THSR-Brink wave function by calculations of the inversion doublet band in \(^{20}\)Ne. It was also found that the Pauli exclusion principle plays a very important role in the formation of molecular cluster structures in nuclei. Furthermore, based on this new concept, the container picture was proposed for understanding the cluster structures. In the container picture, the conventional inter-cluster distance parameter was not adopted again, instead of that, a new size parameter for the cluster system was introduced, which can be considered as the relevant dynamical parameter for the relative motion of clusters. The container picture provides us with a unified picture for understanding and describing the \( \alpha \) condensation states, gas-like cluster states, developed cluster states, and compact or shell-model-like cluster states in nuclei. In the future, it is very promising the container picture can be extended and developed in a more wide range.

The cluster structures in neutron-rich nuclei have been observed from experiments [72] and also studied intensively by AMD (Antisymmetrized Molecular Dynamics) [73]. The excess neutrons around the core clusters usually tend to form some kind of molecular-like structures. Recently, it was reported [74,75] that, the concept of nonlocalized clustering has been extended to the cluster structures of isotopes of Be. In the future, it is hoped that, by constructing more general THSR-type wave function, various molecular-like cluster structures, like \( \pi \)-orbit structures or \( \sigma \)-orbit structures in some neutron-rich nuclei can be obtained in a very natural way in the container picture.

Nowadays, more and more resonance states of nuclear cluster systems have been observed from experiments [76]. As we know, the microscopic cluster models play a central role for the description of cluster structures in nuclei. However, how to describe this kind of multi-cluster resonance states is still a challenge for the present microscopic cluster models. Therefore, to develop a microscopic cluster model for the description of this kind of exotic molecular structures in excited states is required. Recently, combined with the radius constraint method,
the THSR wave function has been extended to describe some resonance cluster states [77,78]. Radius constraint method [79] is a bound-state-approximation method for removing the continuum states by superposing eigenvectors of root-mean-square (RMS) radius with respect to smaller RMS radius eigenvalues. The complex scaling method [80] is a very powerful method for dealing with resonance states by the transformation of the Schrödinger equation. In the future, the THSR wave function is very promising to be combined with the complex scaling method for dealing with resonance states.

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References

[1] K. Wildermuth and Y.C. Tang, *A Unified Theory of the Nucleus*, Vieweg, Braunschweig, 1977.
[2] P. Jena, *Physics and Chemistry of Small Clusters*, Springer Science & Business Media, New York, 2013.
[3] R.N. Grimes, J. Chem. Educ. 81 (2004) p.657.
[4] H. Wu, S. Allen, R. Wechsler, et al., *Precision Cosmology with Galaxy Cluster Surveys*, Stanford University, Stanford, 2011.
[5] J. Wang, G. Wang and J. Zhao, Chem. Phys. Lett. 380 (2003) p.716.
[6] G. Wang, *Cluster Physics*, Shanghai Scientific & Technical, Shanghai, 2003.
[7] R.D. Lawson, *Theory of the Nuclear Shell Model*, Clarendon Press, Oxford, 1980.
[8] Y. Akaishi, S.A. Chin, H. Horiuchi and K. Ikeda, *Cluster Models and Other Topics*, World Scientific, Singapore, 1987.
[9] Z. Ren and G.O. Xu, Phys. Rev. C 36 (1987) p.456.
[10] Z. Ren and G.O. Xu, Phys. Rev. C 38 (1988) p.1078.
[11] Z. Ren and G.O. Xu, J. Phys. G: Nucl. Part. Phys. 15 (1989) p.465.
[12] W.B. He, Y.G. Ma, X.G. Cao, et al., Phys. Rev. Lett. 113 (2014) p.032506.
[13] S.J. Wang and W. Cassing, Ann. Phys. 159 (1985) p.328.
[14] D. Ni and Z. Ren, Phys. Rev. C 81 (2010) p.024315.
[15] D. Ni and Z. Ren, Phys. Rev. C 87 (2013) p.027602.
[16] C. Xu and Z. Ren, Phys. Rev. C 73 (2006) p.041301.
[17] H.B. Yang, Z.Y. Zhang, J.G. Wang, Z.G. Gan, et al., Eur. Phys. J. A 51 (2015) p.88.
[18] G. Röpke, L. Münchow and H. Schulz, Nucl. Phys. A 379 (1982) p.536.
[19] G. Röpke, A. Schnell, et al., Phys. Rev. Lett. 80 (1998) p.3177.
[20] T. Sogo, G. Röpke and P. Schuck, Phys. Rev. C 82 (2010) p.034322.
[21] T. Sogo, G. Röpke and P. Schuck, Phys. Rev. C 81 (2010) p.064310.
[22] J. Dong, W. Zuo and J. Gu, Phys. Rev. C 87 (2013) p.014303.
[23] W. von Oertzen, M. Freer and Y. Kanada-En’yo, Phys. Rep. 432 (2006) p.43.
[24] K. Ikeda, N. Takigawa and H. Horiuchi, Prog. Theor. Phys. Suppl. E68 (1968) p.464.
[25] K. Ikeda, T. Marumori, et al., Prog. Theor. Phys. Suppl. 52 (1972) p.1.
[26] D.M. Brink, J. Phys. Conf. Ser. 111 (2008) p.012001.
[27] Y. Fujisawa, H. Horiuchi, et al., Prog. Theor. Phys. Suppl. 68 (1980) p.29.
[28] W.A. Tyrrell, K.G. Carroll and H. Margenau, Phys. Rev. 55 (1939) p.790.
[29] P.C. Sood and P.C. Joshi, Prog. Theor. Phys. 45 (1971) p.1697.
[30] T. Matsuse, M. Kamimura and Y. Fukushima, Prog. Theor. Phys. 53 (1975) p.706.
[31] S. Saito, Prog. Theor. Phys. Suppl. 62 (1977) p.11.
[32] W. Wefelmeier, Z. Für Phys. Hadrons Nucl. 107 (1937) p.332.
[33] D. Brink, The Alpha-Particle Model of Light Nuclei, in Proceedings of the International School of Physics “Enrico Fermi”, Course 36, C. Bloch, ed., Academic Press, New York, 1966, p.247.
[34] A. Tohsaki, H. Horiuchi, P. Schuck and G. Röpke, Phys. Rev. Lett. 87 (2001) p.192501.
[35] T. Yamada and P. Schuck, Eur. Phys. J. Hadrons Nucl. 26 (2005) p.185.
[36] T. Yamada, Y. Funaki, H. Horiuchi, G. Röpke, et al., Clusters in Nuclei, Springer, Berlin, 2012.
[37] Y. Funaki, A. Tohsaki, H. Horiuchi, P. Schuck, et al., Eur. Phys. J. A 24 (2005) p.321.
[38] Y. Funaki, A. Tohsaki, H. Horiuchi, P. Schuck, et al., Phys. Rev. C 67 (2003) p.051306.
[39] B. Zhou, Z. Ren, C. Xu, Y. Funaki, et al., Phys. Rev. C 86 (2012) p.014301.
[40] B. Zhou, Y. Funaki, H. Horiuchi, Z. Ren, et al., Phys. Rev. Lett. 110 (2013) p.262501.
[41] B. Zhou, Y. Funaki, H. Horiuchi, Z. Ren, et al., Phys. Rev. C 89 (2014) p.034319.
[42] Y. Funaki, H. Horiuchi and A. Tohsaki, Prog. Part. Nucl. Phys. 82 (2015) p.78.
[43] C. Xu, Z. Ren, G. Röpke, P. Schuck, et al., Phys. Rev. C 93 (2016) p.011306.
[44] G. Röpke, P. Schuck, Y. Funaki, H. Horiuchi, et al., Phys. Rev. C 90 (2014) p.034304.
[45] M. Freer, Rep. Prog. Phys. 70 (2007) p.2149.
[46] J.A. Wheeler, Phys. Rev. 52 (1937) p.1083.
[47] J. Wheeler, Phys. Rev. 52 (1937) p.1107.
[48] W. Rae, A. Merchant and J. Zhang, Phys. Lett. B 321 (1994) p.1.
[49] J. Zhang and W. Rae, Nucl. Phys. A 564 (1993) p.252.
[50] J. Zhang, W. Rae and A. Merchant, Nucl. Phys. A 575 (1994) p.61.
[51] H. Horiuchi, Prog. Theor. Phys. Suppl. 62 (1977) p.90.
[52] D. Brink, H. Friedrich, A. Weiguny and C. Wong, Phys. Lett. B 33 (1970) p.143.
[53] J.J. Griffin and J.A. Wheeler, Phys. Rev. 108 (1957) p.311.
[54] P. Ring and P. Schuck, The Nuclear Many-Body Problem, Springer, Dordrecht, 2004.
[55] D. Galetti and A.F.R. de Toledo Piza, Phys. Rev. C 17 (1978) p.774.
[56] H. Horiuchi, Prog. Theor. Phys. 43 (1970) p.375.
[57] Y. Funaki, H. Horiuchi, W. von Oertzen, et al., Phys. Rev. C 80 (2009) p.064326.
[58] A. Tohsaki, Front. Phys. 6 (2011) p.320.
[59] P. Schuck, Y. Funaki, H. Horiuchi, G. Röpke, A. Tohsaki and T. Yamada, Phys. Scripta 91 (2016) p.123001.
[60] H. Horiuchi, Prog. Theor. Phys. 53 (1975) p.447.
[61] E. Uegaki, S. Okabe, Y. Abe and H. Tanaka, Prog. Theor. Phys. 57 (1977) p.1262.
[62] Y. Funaki, H. Horiuchi, A. Tohsaki, P. Schuck, et al., Prog. Theor. Phys. 108 (2002) p.297.
[63] H. Horiuchi and K. Ikeda, Prog. Theor. Phys. 40 (1968) p.277.
[64] J. Hiura, Y. Abe, S. Saitō and O. Endō, Prog. Theor. Phys. 42 (1969) p.555.
[65] A. Arima and S. Yoshida, Phys. Lett. B 40 (1972) p.15.
[66] M. LeMere, Y.C. Tang and D.R. Thompson, Phys. Rev. C 14 (1976) p.23.
[67] Y. Chiba, M. Kimura and Y. Taniguchi, Phys. Rev. C 93 (2016) p.034319.
[68] J. Hiura, F. Nemoto and H. Bandô, Prog. Theor. Phys. Suppl. 52 (1972) p.173.
[69] T. Suhara, Y. Funaki, B. Zhou, H. Horiuchi, et al., Phys. Rev. Lett. 112 (2014) p.062501.
[70] K.T. Hecht, Phys. Rev. C 16 (1977) p.2401.
[71] K. Hecht and W. Zahn, Nucl. Phys. A 318 (1979) p.1.
[72] Z. Yang, Y. Ye, Z. Li, J. Lou, et al., Phys. Rev. Lett. 112 (2014) p.162501.
[73] Y. Kanada-En’yo, Y. Taniguchi and M. Kimura, J. Phys. Conf. Ser. 111 (2008) p.012002.
[74] M. Lyu, Z. Ren, B. Zhou, Y. Funaki, et al., Phys. Rev. C 91 (2015) p.014313.
[75] M. Lyu, Z. Ren, B. Zhou, Y. Funaki, et al., Phys. Rev. C 93 (2016) p.054308.
[76] S. Aoyama, T. Myo, K. Katô and K. Ikeda, Prog. Theor. Phys. 116 (2006) p.1.
[77] Y. Funaki, Phys. Rev. C 92 (2015) p.021302.
[78] B. Zhou, A. Tohsaki, H. Horiuchi and Z. Ren, Phys. Rev. C 94 (2016) p.044319.
[79] Y. Funaki, H. Horiuchi and A. Tohsaki, Prog. Theor. Phys. 115 (2006) p.115.
[80] A.T. Kruppa and K. Katô, Prog. Theor. Phys. 84 (1990) p.1145.