Emergence of scalar matter from spinfoam model

Peng Xu\(^*\) and Yongge Ma\(^+\)

Department of Physics, Beijing Normal University, Beijing 100875, China

A spinfoam model of 3D gravity non-minimally coupled with a scalar field is studied. By discretization of the scalar field, the model is worked out precisely in a purely combinational way. It is shown that the quantum physics of the scalar matter are totally encoded into the modified dynamics of SU(2) spin-network states which describe the quantum geometry of space. It turns out that the physics of the scalar matter coupled with gravity manifested in the low energy scale can be viewed as the phenomena emerged from this microscopical construction. This gives rise to a radical observation on the issue of the unification of geometry and matter.

The search for the mergence of quantum mechanics and general relativity has a long history that can be traced back to the time of Einstein and Dirac. Over the past twenty years the so-called loop quantum gravity (LQG) has made considerable progress in quantizing general relativity background independently. While the kinematics of LQG is well established, the dynamics of the theory is still a thorny problem\(^{[1]}\). Spinfoam models were first introduced as candidates to solve this problem\(^{[2]}\). Remarkably, a large number of distinct approaches to the issue of quantum gravity converge to this formalism\(^{[7, 8]}\). Spinfoam models can be interpreted as the Feynman sum over histories of the evolutions of quantum geometries. Penrose’s idea, that quantum spacetime is still a thorny problem\(^{[1, 2, 3, 4]}\), was somehow re-emerged in LQG by the construction of spin-network states describing the quantum geometry\(^{[9]}\).

A specified spinfoam is a two complex \(\Gamma\) constructed from the dual of the triangulation \(\Delta\) of the spacetime manifold with its faces colored by spins \(j_f\) and edges colored by intertwinors \(\tau_e\), which encode the geometrical data of the simplicial manifold. The dynamics is then determined by the sum over amplitudes contributed by all possible spinfoams

\[
Z = \sum_{\Gamma} w(\Gamma) \sum_{j_f, \tau_e} \prod_{f \in \Gamma} A_f(j_f) \prod_{e \in \Gamma} A_e(\tau_e) \prod_{v \in \Gamma} A_v(j_f, \tau_e),
\]

where \(w(\Gamma)\) denotes the weight associated to each triangulation, \(A_f\), \(A_e\) and \(A_v\) are the amplitudes associated to each face, edge and vertex of \(\Gamma\) respectively. The choices of different functions \(w(\Gamma)\) and the amplitudes \(A_f\), \(A_e\) and \(A_v\) define different models. This picture can be viewed as a 2D generalization of Feynman diagrams, and the vertex amplitude plays the similar role as in standard QFT. In recent years the most active research fields of spinfoam models are the analysis of their low energy limits\(^{[11, 12]}\), their connection with canonical programs\(^{[13, 14]}\), and the matter couplings. The problem of matter couplings in a quantum gravity theory is of extremely importance, because it is necessary to understand how matter fields interact with gravity in a fully quantum mechanical way. Various methods have been proposed to solve this problem in the formalism of spinfoams. One of them is to incorporate the Feynman diagrams of matter interactions into the spinfoams\(^{[15, 16, 17, 18]}\). A surprising result is that the effective dynamics of a scalar field coupled to 3d spinfoams can be described by a non-commutative field theory, which encodes the information of quantum spacetime\(^{[19]}\).

Fermions and gauge fields coupled to 3D spinfoam are studied in\(^{[21, 22]}\). Another more ambitious approach to this issue was proposed in\(^{[23, 24]}\), where particles of standard model are suggested as the local excitations of the quantum states of spacetime. In this letter we propose a new approach to incorporate a scalar field \(\phi\) into 3d spinfoams. Here the dynamics is determined by a new vertex amplitude \(A_v(j_f, \tau_e, \Phi_e)\) which involves the degrees of freedom of both the geometry and the scalar matter. The attractive property of this new model is that it can be casted into a modified dynamics of pure quantum geometry. Thus the scalar matter manifested itself in the low energy scale can be viewed as being emerged from this microscopical construction.

We start with the physical system of a massless Klein-Gordon field \(\phi\) coupled to the gravitational field with Riemannian signature defined on a 3D manifold \(\mathcal{M}\). The standard action reads

\[
S[g, \phi] = \int_{\mathcal{M}} \sqrt{|g|} \left( R[g] - g^{ab} \nabla_a \phi \nabla_b \phi \right).
\]

It can be shown that the classical dynamics of this action is “conformally” equivalent to that of

\[
S[\tilde{g}, \tilde{\phi}] = \int_{\mathcal{M}} \sqrt{\tilde{|\phi|}} \sqrt{|\tilde{g}|} R[\tilde{g}],
\]

under the transformations \(g_{ab} = |\tilde{\phi}| \tilde{g}_{ab}\) and \(\phi = -\frac{1}{2} \ln |\tilde{\phi}|\)\(^{[25]}\). The action \(\mathcal{S}\) is also of physical interest because it can be obtained by the symmetric reduction from a 4D spacetime with a hypersurface orthogonal Killing vector field\(^{[23]}\). In the first order formalism this action is written in terms of \((\epsilon^{I}_a, \Omega^{J}_a)\) as

\[
S[\epsilon, \omega, \phi] = \int_{\mathcal{M}} \sqrt{\phi} |e_{IJK} \epsilon^I \wedge \Omega^J(\omega)|\]

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\(^*\)xupeng@mail.bnu.edu.cn

\(^+\)mayg@bnu.edu.cn
where \( e_a^I \) is the soldering one-form, \( \omega_{ab}^{IJ} \) is the spin connection and \( \Omega_{e_a}^{IJ} \) is the curvature of \( \omega_{ab}^{IJ} \). The quantum theory is determined by the partition functional

\[
Z = \int D[\phi] D[e] D[\omega] e^{-i \int_M \sqrt{|\phi|} \epsilon_{IJK} e^I \wedge \Omega^{JK}}. \tag{4}
\]

We introduce the triangulation \( \Delta \) of \( M \) and its dual \( \Delta^* \) as the regulator of the system. Point, segment and triangle of \( \Delta \) are denoted as \( p, s \) and \( t \), and vertex, edge and face of \( \Delta^* \) are denoted as \( v, e \) and \( f \). The triad field \( e_a^I \) is smeared along segments, and the term \( \sqrt{|\phi|} \epsilon_{ab}^{IJ} \), as a \( su(2) \) valued two form, is smeared along the faces of \( \Delta^* \) that dual to the segments,

\[
E_a^I = \int e^I, \quad \Omega_{e_a}^{IJ}(\phi) = \int \sqrt{|\phi|} \Omega_{e_a}^{IJ} \tag{5}
\]

The triangulation are supposed to be fine enough, so that on the faces the scalar \( \sqrt{|\phi|} x \) can be approximated by a constant \( |\phi| x \). Then Eq. (5) reads

\[
\Omega_{e_a}^{IJ}(\phi) = \int \sqrt{|\Phi_f|} \Omega_{e_a}^{IJ} = \sqrt{|\Phi_f|} \Omega_{e_a}^{IJ} \tag{6}
\]

where \( \Omega_{e_a}^{IJ} = \int \Omega_{ab}^{IJ} \) and \( \sqrt{|\Phi_f|} \) can be treated as the weight of the variable \( \Omega_{e_a}^{IJ} \). The smeared curvature \( \Omega_{e_a}^{IJ} \) can be related to the holonomy \( U_f \) along the boundary \( e_f^1 \circ \ldots \circ e_f^n \) of \( f \) as

\[
g_{e_f^1} \circ g_{e_f^2} \circ \ldots \circ g_{e_f^n} = U_f = e^{\Omega_f} = 1_g + \Omega_f + \ldots, \tag{7}
\]

where \( g_{e_f^i} = \mathcal{P} \exp(\int_{e_f^i} \omega^{IJ}) \) is the group element associated to each edge. By the regularization, the degrees of freedom left are \( (E_a^I, g_e, \Phi_f) \), and the partition functional takes the form

\[
Z = \sum_{\Delta} \prod_{s} \int dE_s^I \prod_{e} \int dg_e \prod_{f} \int d\Phi_f e^{-i \sum f \text{tr}(E_s^I \sqrt{|\Phi_f|})}, \tag{8}
\]

where the weight \( w(\Delta) \) associated to each triangulation are assumed to be the same and set to 1 for simplicity. This regularized partition functional will approach Eq. (4) when the triangulation becomes finer and finer. For a fixed triangulation, integrating out the triad \( E_a^I \), we have

\[
Z_{\Delta} = \int \sum_{f} \prod_{f} d\Phi_f \prod_{e} \int dg_e \prod_{f} \int \sqrt{|\Phi_f|} \epsilon_{IJK} \chi^{IJ}(G_{e_f^1} \circ \ldots \circ G_{e_f^n}), \tag{9}
\]

where \( G_{e_f^i} = \mathcal{P} \exp(\int_{e_f^i} \sqrt{|\Phi_f|} \omega^{IJ}) \) is a \( SU(2) \) element that depends on \( (\Phi_f, g_{e_f^i}) \), \( \chi^{IJ} \) is the dimension of the representation \( J_f \), and \( \chi^{IJ}(g) \) is the character of the group element \( g \) in the representation \( J_f \). Though the vertex amplitude \( A_v \) is not worked out explicitly, the dynamics does involve both the representation \( J \) and the scalar \( \Phi \). Now we consider the sector that the weight \( \sqrt{|\Phi_f|} \) is discrete, i.e., the discretization of the scalar field. We rewrite the action (3) as

\[
S[e, \omega, \phi] = \int_M C \sqrt{|\phi|} \epsilon_{IJK} e^I \wedge \Omega^{JK}(\omega) \tag{9}
\]

and let \( \sqrt{|\Phi_f|/C} \) to take values in \( \mathbb{Z}^+ \cup \{0\} \), which are denoted as \( N_f \). The constant multiplier \( C \) will not affect the physics, and the steps of \( C N_f \epsilon_{IJK} e^I \wedge \Omega^{JK}(\omega) \) in the action can be made as small as possible by choosing appropriate \( C \). This ensures that the regularized partition functional (8) can be approximated as good as one wants. The partition functional now reads

\[
Z^N_{\Delta} = \int \prod_s \int dE_s^I \prod_f \int dg \sum_{N_f} \rho(\bar{N}) e^{-i N_f \sum f \text{tr}(E_s^I \Omega_{e_a}^{IJ})}, \tag{10}
\]

where \( \bar{N} \) denotes \( \{N_{f_1}, N_{f_2}, \ldots, \} \), and \( \rho(\bar{N}) \) is the weight for each configuration \( \bar{N} \), which comes from the measure \( d\Phi_f \) in Eq. (8). Integrating out \( E_a^I \) we have

\[
Z^N_{\Delta} = \int \prod_f \int dg \sum_{N_f} \rho(\bar{N}) \prod_{f} \delta(U_f \circ \ldots \circ U_f) \tag{11}
\]

where \( \delta_{\alpha_1, \alpha_2, \ldots, \alpha_n}^{\alpha_1' \alpha_2' \ldots \alpha_n'}(g_{e_f^1}^{\alpha_1} g_{e_f^2}^{\alpha_2} \ldots g_{e_f^n}^{\alpha_n}) \) denotes the representation matrix of \( g \) which belongs to the representation \( J_f \). We have chosen a special but natural ordering of the \( N_f \) copies of the group elements \( g_{e_f^i} \) in the above derivation, which plays a key role in our model. The ambiguity caused by this ordering can be controlled by means of fine-enough triangulations. The result is interpreted as that for each face \( f \) of \( \Delta^* \) we associate \( N_f \) copies of “virtual” faces which are all colored by the same representation \( J_f \). The indexes of the representation matrices of edges which bound the \( N_f \) faces are contracted with each other following the order

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**FIG. 1:** (a) The \( N_f \) “virtual” faces. (b) Three “bunch” of faces that joint at each edge.
expressed in Eq. (11). See Fig. 1(a) for a graphic presentation, where the contraction wind around each face \( f \) for \( N_f \) times. There will be one group element, denoted as \( g_e \), for each face \( f \) whose contraction with the next one will cross the "virtual" faces. We will show that the choices of these crossing edges do not affect the physics. For each edge \( e \) of \( \Delta^* \), there are three "bundles" of faces which joint at it (see Fig. 1(b)). According to Eq. (11), there is one following integral for each edge, 

\[
\int dg_e \bigotimes_{l=1}^{N_N} R^{j_{l1}}(g_e)_{\alpha_{l1}}^{\beta_{l1}} \bigotimes_{j=1}^{N_f} R^{j_{lj}}(g_e)_{\gamma_{lj}}^{\beta_{lj}} \bigotimes_{k=1}^{N_f} R^{j_{lk}}(g_e)_{\gamma_{lk}}^{\beta_{lk}} = \sum_t \left( \bigotimes_{i=1}^{N_f} \bigotimes_{1}^{2} \bigotimes_{3}^{4} \bigotimes_{5}^{6} \bigotimes_{7}^{8} \right),
\]

where \( P_{inv}^{\otimes_{N_f} \otimes_{j_1} \otimes_{j_2} \otimes_{j_3} \otimes_{j_4} \otimes_{j_5} \otimes_{j_6} \otimes_{N_f}} \) is the projector into the invariant subspace \( Inv \), and \( \lambda_{11}^{...} \rho_{N_f}^{...} \) form an othnormal basis of the invariant subspace. These tensors are symmetric with respect to the indexes from the same "bunch" of faces. For the sake of readability, we will neglect these parentheses in the following context. This operator \( P_{inv} \) and the subspace \( Inv \) are not null only if the corresponding representations coupled together satisfy certain compatible conditions \( a \). The edge integral (12) can be illustrated as Fig. 2 by means of the graphical techniques in performing \( SU(2) \) tensor calculus \( \Box \), where the rectangle on the left denotes the integral \( \int dg_e \). According to the structure of \( \Delta^* \), there are four edges which joint at each vertex, and hence six "bundles" of faces joint at each vertex. Thus, for each vertex we have four integrals of the form (12) with the corresponding indexes contracted, i.e.,

\[
\int dg_{e_1} dg_{e_2} dg_{e_3} dg_{e_4}
\]

\[
\bigotimes_{i=1}^{N_f} R^{j_{l1}}(g_e)_{\alpha_{l1}}^{\beta_{l1}} \bigotimes_{j=1}^{N_f} R^{j_{lj}}(g_e)_{\gamma_{lj}}^{\beta_{lj}} \bigotimes_{k=1}^{N_f} R^{j_{lk}}(g_e)_{\gamma_{lk}}^{\beta_{lk}} = \sum_t \left( \bigotimes_{i=1}^{N_f} \bigotimes_{1}^{2} \bigotimes_{3}^{4} \bigotimes_{5}^{6} \bigotimes_{7}^{8} \right),
\]

where the free indexes will contract with that of the nearby vertices. The resulted amplitude reads

\[
A_e(t_1, t_2, t_3, t_4) = e^{i \sum_{\gamma}^{N_f} (t_1) \lambda_{N_f}^{(1)} \lambda_{N_f}^{(2)} \lambda_{N_f}^{(3)} \lambda_{N_f}^{(4)} \lambda_{N_f}^{(5)} \lambda_{N_f}^{(6)} \lambda_{N_f}^{(7)} \lambda_{N_f}^{(8)}},
\]

where the Einstein summation convention is adopted. The result is represented by means of the graphical techniques as Fig. 3. The presence of crossing edges only cause a permutation of the indexes of the same "bunch" of faces and hence does not affect the amplitudes \( A_e \). Thus we arrived at the final form of the partition functional of Eq. (11),

\[
\mathcal{Z}_N = \sum_{\Delta} \sum_{j_f, N_f, e} \rho(N) \prod_{f \in \Delta^*} \bigotimes_{j_f} \prod_{e \in \Delta^*} A_e(j_f, N_f, e).
\]

The boundary states that encode the physical information on a hypersurface \( \Sigma \) which intersects with the spin-foams can be described as the "generalized" \( SU(2) \) spin-network states \( T_{\gamma, N_f, e} \) consisting of network-like graphs \( \gamma \) with nodes labeled by the intertwinors \( \iota_e \) and links labeled by both the representations \( j_f \) and the integers \( N_f \). The remarkable result comes up as that these states \( T_{\gamma, N_f, e} \) can be casted into the pure \( SU(2) \) spin-network
states $T_{\gamma, N, j, \iota}$, which constitute the basis of the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ of 3D quantum gravity [1, 2, 3]. This is done by replacing the $(N_f, j_f)$-labeled links by $N_f$ copies of $j_f$-labeled links which joint at the same nodes as FIG. 4. With the different choices of $N_f$ and $j_f$, $T_{\gamma, N, j, \iota}$ will run over a special subset $T$ of $\mathcal{H}_{\text{kin}}$. Hence the information of the scalar field on the hypersurface are totally encoded into the standard $SU(2)$ spin-network states $T_{\gamma, N, j, \iota}$ or the states of quantum geometry. Thus the kinematics of this model has been casted into that of the 3D quantum geometry. The evolution of $T_{\gamma, N, j, \iota}$ is generated by the vertices as FIG. 5 and the transition amplitudes $A_{\iota}(j_f, N_f, \iota)$ can be calculated in a purely combinatorial way by means of the recouplings of angular momentums. The dynamics can be interpreted as the quantum evolutions from the initial boundary states $T_{\gamma, N, j, \iota}$ to the final boundary states $T_{\gamma, N, j, \iota}'$. This is equivalent to the choice of the physical inner product $< T^j, T^i >_{\text{phys}}$ between the kinematical states. By the formal relation [6]

$$< T^j, T^i >_{\text{phys}} = \int D[N] \langle T^j, e^{-i \int_S \mathbf{H}[N]} T^i \rangle_{\text{kin}},$$

one may go inversely to construct the Hamiltonian operator $\mathbf{H}$ defined on $\mathcal{H}_{\text{kin}}$ which generates this dynamics. The new dynamics will reduce to that of Ponzano-Regge model, as one would expect, when $\phi$ is assumed to be a constant field and $N_f$ is set to 1 for all faces. It is a natural extension of the dynamics of pure gravity by the new transition amplitudes [13] coming also from the couplings of $SU(2)$ representations. As a generalization of Penrose's idea [9], both the kinematics and the dynamics of the scalar matter and gravity are built into these couplings of the angular momentums. In the full quantum situation, the physics of this system can be casted into the dynamics of the pure quantum geometry, while, in the semiclassical situation, the physics manifests itself as the dynamics of the geometry coupled with the scalar matter. Thus the scalar field can be viewed as a phenomenon emerged from the microscopical system of quantum spinfoams. This gives rise to a remarkable and radical observations that the geometry and matter fields appearing in the low energy scale may originate from a single microscopical construction, which may be spinfoams in accordance with above viewpoint.

To summarize, the spinfoam model of the system [9] is worked out. This model shed some new lights on the issue of matter couplings in LQG and spinfoam formalism. The distinct property of this model gives rise to a radical observation on the issue of unification of geometry and matter. Being the convergent point of distinct approaches of quantum gravity, spinfoams may also be the convergent point of geometry and matter.

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[1] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. 21, R53 (2004).
[2] C. Rovelli, Quantum Gravity, (Cambridge University Press, Cambridge, 2004).
[3] T. Thiemann, Modern Canonical Quantum General Relativity, (Cambridge University Press, Cambridge, 2007).
[4] M. Han, Y. Ma and W. Huang, Int. J. Mod. Phys. D 16, 1397 (2007).
[5] M. Reisenberger and C. Rovelli, Phys. Rev. D 56, 3490-3508 (1997).
[6] C. Rovelli, Phys. Rev. D 59, 104015 (1999).
[7] A. Perez, Class. Quant. Grav. 20, R43 (2003).
[8] D. Oriti, arXiv:gr-qc/0311066v1
[9] R. Penrose, In Quantum Theory and Beyond ed. T. Bastin. (Cambridge University Press, Cambridge, 1971).
[10] C. Rovelli and L. Smolin, Phys. Rev. D 52, 5743 (1995).
[11] L. Modesto and C. Rovelli, Phys. Rev. Lett. 95, 191301 (2005).
[12] E. Alesci and C. Rovelli, Phys. Rev. D 76, 104012 (2007); 77, 044024 (2008).
[13] K. Noui and A. Perez Class. Quant. Grav. 22, 1739 (2005).
[14] E. Alesci and F. Sardelli, Phys. Rev. D 78, 104009 (2008).
[15] L. Freidel and D. Louapre, Class. Quant. Grav. 21, 5685 (2004); L. Freidel and E. Livine, Class. Quant. Grav. 23, 2021 (2006).
[16] K. Noui and A. Perez, Class. Quant. Grav. 22, 4489 (2005).
[17] L. Freidel, D. Oriti and J. Ryan, arXiv:gr-qc/0506067.
[18] D. Oriti and J. Ryan, Class. Quant. Grav. 23, 6543 (2006).
[19] L. Freidel and E. Livine, Phys. Rev. Lett. 96, 221301 (2006).
[20] D. Oriti, H. Pfeiffer, Phys. Rev. D 66, 124010, (2002).
[21] W. Fairbairn, Gen. Rel. Grav. 39, 427 (2007).
[22] S. Speziale, Class. Quant. Grav. 24, 5139 (2007).
[23] S. Bilson-Thompson, F. Markopoulou and L. Smolin, Class. Quant. Grav. 24 3975 (2007).
[24] L. Smolin and Y.D. Wan, Nucl. Phys. B 796, 331 (2008);
[25] Y.D. Wan, Nucl. Phys. B 814, 1 (2009).
[26] H. He, Y. Ma and X. Yang, Int. J. Mod. Phys. D 12 1961 (2003).
[26] D. Brink and R. Satchler, Angular Momentum (Claredon Press, Oxford, 1968).
