Abstract. In the context of a theorem of Richter, we establish a similarity between $C_0$-semigroups of analytic 2-isometries \( \{T(t)\}_{t \geq 0} \) acting on a Hilbert space \( \mathcal{H} \) and the multiplication operator semigroup \( \{M_{\phi_t}\}_{t \geq 0} \) induced by \( \phi_t(s) = \exp(-st) \) for \( s \) in the right-half plane \( \mathbb{C}_+ \) acting boundedly on weighted Dirichlet spaces on \( \mathbb{C}_+ \). As a consequence, we derive a connection with the right shift semigroup \( \{S_t\}_{t \geq 0} \)

\[
S_tf(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq t, \\
f(x-t) & \text{if } x > t,
\end{cases}
\]

acting on a weighted Lebesgue space on the half line \( \mathbb{R}_+ \) and address some applications regarding the study of the invariant subspaces of $C_0$-semigroups of analytic 2-isometries.

1. Introduction

The concept of a 2-isometry was introduced by Agler in the early eighties (cf. [1]); this is related to notions due to J. W. Helton (see [8] and [9]) and characterized in terms of their extension properties (see [2]). Recall that a bounded linear operator \( T \) on a separable, infinite dimensional complex Hilbert space \( \mathcal{H} \) is called a 2-isometry if it satisfies

\[
T^*T^2 - 2T^*T + I = 0,
\]

where \( I \) denotes the identity operator. In addition, such operators are called analytic if no nonzero vector is in the range of every power of \( T \). It turns out that \( M_z \), i.e. the multiplication operator by \( z \), acting on the classical Dirichlet space, is a cyclic analytic 2-isometry. But, moreover, in [15] (see also [14]) Richter proved that any cyclic analytic 2-isometry is unitarily equivalent to \( M_z \) acting on a generalized Dirichlet space \( D(\mu) \).

More precisely, let \( \mu \) be a finite non-negative Borel measure on the unit circle \( \mathbb{T} \) and \( D(\mu) \) the generalized Dirichlet space associated to \( \mu \), that is, the Hilbert space consisting of analytic functions on the unit disc \( \mathbb{D} \) such that

\[
\int_{\mathbb{D}} |f'(z)|^2 \left( \int_{|\xi|=1} \frac{1-|z|^2}{|\xi-z|^2} d\mu(\xi) \right) \frac{dm(z)}{\pi} < \infty,
\]
is finite (here \(dm(z)\) denotes the Lebesgue area measure in \(\mathbb{D}\)). Note that if \(\mu = 0\), the space \(D(\mu)\) is defined to be the classical Hardy space \(H^2\) and for non-zero, finite, non-negative Borel measures \(\mu\) on \(T\), the space \(D(\mu)\) is contained in the Hardy space (see [7, Chapter 7]). Then Richter’s Theorem reads as follows:

**Theorem (Richter).** Let \(T\) be a bounded linear operator on an infinite dimensional complex Hilbert space \(H\). Then the following conditions are equivalent:

(i) \(T\) is an analytic 2-isometry with \(\dim \ker T^* = 1\),

(ii) \(T\) is unitarily equivalent to \((M_z, D(\mu))\) for some finite non-negative Borel measure on \(T\), where

\[
\|f\|_{D(\mu)}^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 \left(\int_{|\xi| = 1} \frac{1 - |z|^2}{|\xi - z|^2} d\mu(\xi)\right) \frac{dm(z)}{\pi}.
\]

One of the main applications of Richter’s Theorem concerns the study of the invariant subspaces for the multiplication operator \(M_z\) in the spaces \(D(\mu)\) and its relationship with the classical Beurling Theorem for the Hardy space \(H^2\) (see [3]). For instance, regarding the Dirichlet space \(D = D\left(\frac{d\xi}{2\pi}\right)\), Richter and Sundberg [16] proved that any closed, invariant subspace \(\mathcal{M}\) under \(M_z\) satisfies that \(\dim \mathcal{M} \ominus z \mathcal{M} = 1\). Moreover, if \(\varphi \in \mathcal{M} \oplus z \mathcal{M}\) with \(\|\varphi\|_D = 1\), then \(|\varphi(z)| \leq 1\) for \(|z| \leq 1\) and \(\mathcal{M} = \varphi D(m_\varphi)\), where \(dm_\varphi\) is the measure on \(T\) given by \(dm_\varphi(\xi) = |\varphi(\xi)|^2 \frac{d\xi}{2\pi}\). For general \(D(\mu)\) spaces, an analogous result holds. We refer the reader to Chapters 7 and 8 in the recent monograph “A primer on the Dirichlet space” [7] for more on the subject.

Motivated by the Beurling-Lax Theorem and the work carried out by Richter, the aim of this work is taking further the study of the 2-isometries and considering \(C_0\)-semigroups of 2-isometric operators. In particular, we will establish a similarity between \(C_0\)-semigroups of analytic 2-isometries \(\{T(t)\}_{t \geq 0}\) acting on a Hilbert space \(H\) and the multiplication operator semigroup \(\{M_{\phi_t}\}_{t \geq 0}\) induced by \(\phi_t(s) = \exp(-st)\) for \(s\) in the right-half plane \(\mathbb{C}_+\) acting boundedly on weighted Dirichlet spaces \(D_{\mathbb{C}_+}(\nu)\) on \(\mathbb{C}_+\) (see Definition 2.3). As a consequence, by means of the Laplace transform, we derive a connection with the right shift semigroup \(\{S_t\}_{t \geq 0}\)

\[
S_tf(x) = \begin{cases} 
0 & \text{if } 0 \leq t \leq x, \\
 f(x-t) & \text{if } x > t,
\end{cases}
\]

acting on a weighted Lebesgue space on the half line \(\mathbb{R}_+\). Finally, some applications regarding the study of the invariant subspaces of \(C_0\)-semigroups of analytic 2-isometries are also discussed in Section 3.

## 2. \(C_0\)-Semigroups of Analytic 2-Isometries

First, we introduce some basic concepts and terminology regarding \(C_0\)-semigroups of bounded linear operators. For more on this topic, we refer the reader to the Engel–Nagel monograph [6].
A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ of operators on a Hilbert space $H$ is a family of bounded linear operators on $H$ satisfying the functional equation
\[
\begin{aligned}
T(t + s) &= T(t)T(s) \quad \text{for all } t, s \geq 0, \\
T(0) &= I,
\end{aligned}
\]
and such that $T(t) \to I$ in the strong operator topology as $t \to 0^+$. Given a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$, there exists a closed and densely defined linear operator $A$ that determines the semigroup uniquely, called the generator of $\{T(t)\}_{t \geq 0}$, defined by means of
\[
Ax := \lim_{t \to 0^+} \frac{T(t)x - x}{t},
\]
where the domain $D(A)$ of $A$ consists of all $x \in H$ for which this limit exists (see [6, Chapter II], for instance). Although the generator is, in general, an unbounded operator, it plays an important role in the study of a $C_0$-semigroup, reflecting many of its properties.

However, if $1$ is in the resolvent of $A$, that is, in the set
\[
\rho(A) = \{ \lambda \in \mathbb{C} : (A - \lambda I) : D(A) \subset H \to H \text{ is bijective} \},
\]
then $(A - I)^{-1}$ is a bounded operator on $H$ by the Closed Graph Theorem, and the Cayley transform of $A$ defined by
\[
V := (A + I)(A - I)^{-1}
\]
is a bounded operator on $H$, since $V - I = 2(A - I)^{-1}$. Therefore $V$ determines the semigroup uniquely, since $A$ does. This operator is called the cogenerator of the $C_0$-semigroup $\{T(t)\}_{t \geq 0}$. Observe that $1$ is not an eigenvalue of $V$.

Recall that if $A$ is a closed operator, then the spectral bound $s(A)$ of $A$ is defined by
\[
s(A) := \sup \{ \Re \lambda : \lambda \in \sigma(A) \},
\]
where $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is the spectrum of $A$, and in case that $A$ is the generator of a $C_0$-semigroup, then $s(A)$ is always dominated by the growth bound of the semigroup, that is,
\[
-\infty \leq s(A) \leq w_0 = \inf \left\{ w \in \mathbb{R} : \exists M_w \geq 1 \text{ such that } ||T(t)|| \leq M_w e^{wt} \text{ for all } t \geq 0 \right\}.
\]
Indeed, if $r(T(t))$ denotes the spectral radius of $T(t)$, it follows that $w_0 = \frac{1}{t} \log r(T(t))$ for each $t > 0$ (see [6, Section 2, Chapter IV], for instance). The following lemma will be useful in the context of our main result later.

**Lemma 2.1.** Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on a separable, infinite dimensional complex Hilbert space $H$ consisting of 2-isometries and $A$ its generator. Then $1 \in \rho(A)$ and therefore, the cogenerator $V$ of $\{T(t)\}_{t \geq 0}$ is well-defined.
For $t \geq 0$, $T(t)$ satisfies
\[T(t)^nT(t)^n - nT(t)^*T(t) + (n - 1)I = 0,\]
and so
\[\|T(t)^nx\|^2 = n\|T(t)x\|^2 - (n - 1)\|x\|^2\]
for $x \in \mathcal{H}$. From here, it follows that $\|T(t)^n\| \leq C \sqrt{n}$, where $C$ is a constant independent of $n$, and therefore the spectral radius $r(T(t)) \leq 1$ for any $t$. Therefore, $s(A) \leq 0$; and therefore $1 \in \rho(A)$.

The next result consists of a particular instance of [11, Theorem 1], where $C_{0}$-semigroups of hypercontractions are considered. We state it for $C_{0}$-semigroups of 2-isometries and include its proof for the sake of completeness.

**Proposition 2.2.** Let \( \{T(t)\}_{t \geq 0} \) be a $C_{0}$-semigroup on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$. Then the following conditions are equivalent:

(i) $T(t)$ is a 2-isometry for every $t \geq 0$.

(ii) The mapping $t \in \mathbb{R}_{+} \mapsto \|T(t)x\|^2$ is affine for each $x \in \mathcal{H}$.

(iii) $\text{Re}(A^2y, y) + \|Ay\|^2 = 0$ \quad $(y \in D(A^2))$.

(iv) The cogenerator $V$ of $\{T(t)\}_{t \geq 0}$ exists and is a 2-isometry.

**Proof.** (i) $\iff$ (ii): If each $T(t)$ is a 2-isometry, then for $t \geq 0$ and $\tau > 0$ we have
\[\langle T(t + 2\tau)x, T(t + 2\tau)x \rangle - 2\langle T(t + \tau)x, T(t + \tau)x \rangle + \langle T(t)x, T(t)x \rangle = 0,\]
so that
\[\|T(t + \tau)x\|^2 = \frac{1}{2}(\|T(t)x\|^2 + \|T(t + 2\tau)x\|^2).\]
Since $t \in \mathbb{R}_{+} \mapsto \|T(t)x\|^2$ is continuous, the mapping is affine.

Conversely, taking $t = 0$ we see that (iii) implies that $T(\tau)$ is a 2-isometry.

(ii) $\iff$ (iii): For $\tau > 0$ we calculate the second derivative of the function $g : t \mapsto \|T(t)y\|^2$ for $y \in D(A^2)$. We have
\[g''(t) = \frac{d^2}{dt^2}(T(t)y, T(t)y) = \langle A^2T(t)y, T(t)y \rangle + 2\langle AT(t)y, AT(t)y \rangle + \langle T(t)y, A^2T(t)y \rangle.\]
For $g$ affine, $g''$ is zero, and Condition (iii) follows on letting $t \to 0$. Conversely, Condition (iii) implies Condition (ii) for $y \in D(A^2)$, and hence for all $y$ by density.

(iii) $\iff$ (iv): We calculate
\[\langle (I - 2V^*V + V^*2V^2)x, x \rangle\]
for $x = (A - I)^2 y$ (note that $(A - I)^{-2} : H \to H$ is defined everywhere and has dense range). We obtain
\[
\langle (A - I)^2 y, (A - I)^2 y \rangle - 2\langle (A^2 - I)y, (A^2 - I)y \rangle + \langle (A + I)^2 y, (A + I)^2 y \rangle
= 4\langle A^2 y, y \rangle + 8\langle Ay, Ay \rangle + 4\langle y, A^2 y \rangle.
\]
Thus $V$ is a 2-isometry if and only if Condition (iii) holds. \qed

Before stating the main result of the section, let us introduce the following definition.

**Definition 2.3.** Let $\nu$ be a finite positive Borel measure supported on the imaginary axis. The Dirichlet space $\tilde{D}_{\mathbb{C}^+}(\nu)$ is defined as the space of analytic functions $F$ on right half-plane $\mathbb{C}^+$ such that
\[
\|F\|^2 = |F(1)|^2 + \frac{1}{\pi} \int_{\mathbb{C}^+} |F'(s)|^2 \left( x + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y - \tau)^2} \, d\nu(\tau) \right) \, dx \, dy < \infty
\]
where $s = x + iy$.

The spaces $\tilde{D}_{\mathbb{C}^+}(\nu)$ arise, in a natural way, when we analyze $C_0$-semigroups of analytic 2-isometries in Hilbert spaces, as it is stated in our main result:

**Theorem 2.4.** Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$ consisting of analytic 2-isometries for every $t > 0$ such that
\[
\dim \bigcap_{t > 0} \ker (T^*(t) - e^{-t} I) = 1.
\]
Then there exists a finite positive Borel measure $\nu$ supported on the imaginary axis such that $\{T(t)\}_{t \geq 0}$ is similar to the semigroup of multiplication operators induced by $\exp(-ts)$ acting on the space $\tilde{D}_{\mathbb{C}^+}(\nu)$. Moreover, if the multiplication operators induced by $\exp(-ts)$ act continuously for every $t > 0$ on a Dirichlet space $\tilde{D}_{\mathbb{C}^+}(\tilde{\nu})$ where $\tilde{\nu}$ is a finite positive Borel measure supported on the imaginary axis, then the corresponding semigroup consists of analytic 2-isometries and satisfies (2).

Before proceeding further, let us remark that our main result yields similarity for the semigroup $\{T(t)\}_{t \geq 0}$ because of the definition of the norm in $\tilde{D}_{\mathbb{C}^+}(\nu)$. In addition, as we shall see later, condition (2) is a way of expressing the property that $\dim \ker V^* = 1$, where $V$ is the cogenerator of the semigroup $\{T(t)\}_{t \geq 0}$.

In order to prove Theorem 2.4, we need the following auxiliary results.

**Proposition 2.5.** Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$ consisting of analytic 2-isometries. Then the cogenerator $V$ is an analytic 2-isometry.

**Proof.** First, we observe that $V$ is well-defined by Lemma 2.1 and, it is a 2-isometry by Proposition 2.2. So, we are required to show that $V$ is analytic.
The Wold Decomposition Theorem for 2-isometries (see [12], for instance), yields that $V$ can be decomposed as $V = S \oplus U$ with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $U$ is the unitary part on $\mathcal{H}_2 = \bigcap_n V^n \mathcal{H}$ and $S$ is an analytic 2-isometry. We will show that $U = 0$.

Let us assume, on the contrary, that $U \neq 0$.

First, we observe that since 1 is not an eigenvalue of $V$, the generator $A$ of the semigroup $\{T(t)\}_{t \geq 0}$ may be expressed as the (possibly) unbounded operator

$$(V + I)(V - I)^{-1}.$$ 

Moreover, since $T(t)$ commutes with $(A - I)^{-1}$ and hence with $V$, it holds that $\mathcal{H}_2$ is invariant under $T(t)$ for every $t \geq 0$. In addition, the generator $B$ of the restricted semigroup $\{T(t)|_{\mathcal{H}_2}\}_{t \geq 0}$ is the restriction of $A$ to the $D(A) \cap \mathcal{H}_2$ (see [6, Ch. 2, Sec. 2], for instance); and the cogenerator is $U$.

Now, taking into account the fact that $U$ is unitary, one deduces that $B$ is skew-adjoint (i.e., $B^* = -B$). Then the restriction of $T(t)$ to $\mathcal{H}_2$ is unitary for every $t \geq 0$ and, therefore, every vector in $\mathcal{H}_2$ is in the range of (powers of) $T(t)$. Since $T(t)$ is analytic, it follows that $\mathcal{H}_2 = \{0\}$, a contradiction. Hence, $U = 0$ and the proof is completed. □

**Lemma 2.6.** Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$ and $A$ its generator. The following conditions are equivalent:

1. $Ax_0 = -x_0$ for some $x_0 \in D(A)$.
2. $T(t)x_0 = e^{-t}x_0$ for all $t \geq 0$ and $x_0 \in D(A)$.

In addition, if $1 \in \rho(A)$ and $V$ is the cogenerator, any of the previous conditions is equivalent to

3. $Vx_0 = 0$ for some $x_0 \in D(A)$.

Note that the equivalence between (1) and (2) in Lemma 2.6 just follows from the relationship between the eigenspaces of $A$ and the semigroup $\{T(t)\}_{t \geq 0}$, that is,

$$\ker(\lambda I - A) = \bigcap_{t \geq 0} \ker(e^{\lambda t} - T(t)),$$

with $\lambda \in \mathbb{C}$ (see [6, Corollary 3.8, Section IV], for instance). The last statement follows from the definition of $V$.

We are now in position to prove Theorem 2.4.

**Proof of Theorem 2.4**. Assume that $\{T(t)\}_{t \geq 0}$ consists of analytic 2-isometries. Let $V$ denote its cogenerator; this is well-defined by Lemma 2.1 and it is an analytic 2-isometry by Proposition 2.5.

In addition, the hypotheses $\dim \bigcap_{t \geq 0} \ker (T^*(t) - e^{-t} I) = 1$ along with Lemma 2.6 applied to the adjoint semigroup $\{T^*(t)\}_{t \geq 0}$, yields that $\dim \ker V^* = 1$.

By means of Richter’s Theorem, it follows that $V$ is similar to $M_\mu$ acting on the space $D(\mu)$ for some finite non-negative Borel measure $\mu$ on $\mathbb{T}$ considered with the equivalent
norm
\[
\|f\|_{D(\mu)}^2 \approx |f(0)|^2 + \int_D |f'(z)|^2 \left( \int_{|\xi| = 1} \frac{1 - |z|}{|\xi - z|} \, d\mu(\xi) \right) \frac{dm(z)}{\pi}
\]
(3)
\[
= |f(0)|^2 + \int_D |f'(z)|^2 P_\mu(z) \frac{dm(z)}{\pi}.
\]
Observe that the similarity is the price paid when we consider the equivalent norm. Hence, for any \( t \geq 0 \), it follows that \( T(t) \) is unitarily equivalent to the multiplication operator induced by \( \exp(-t(1 + z)/(1 - z)) \) on \( D(\mu) \). Now, we migrate to the right half-plane \( \mathbb{C}_+ = \{ \Re s > 0 \} \) applying the change of variables \( s = (1 + z)/(1 - z) \), or \( z = (s - 1)/(s + 1) \).

First, we observe that
\[
P_\mu \left( \frac{s - 1}{s + 1} \right) = \int_{|\xi| = 1} \frac{1 - |\xi|^2}{|\xi - s|^2} \, d\mu(\xi) \quad (s \in \mathbb{C}_+)
\]
is a positive harmonic function in \( \mathbb{C}_+ \); so there exists a non-negative constant \( \rho \) and a finite positive Borel measure \( \nu \) supported on the imaginary axis such that
(4) \[
P_\mu \left( \frac{s - 1}{s + 1} \right) = \rho \left( -1 \right) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y - \tau)^2} \, d\nu(\tau), \quad (s = x + iy)
\]
(see [10] Exercise 6, p. 134], for instance).

We can express \( \nu \) in terms of \( \mu \), since with \( \xi = (u - 1)/(u + 1) \) for \( u = i\tau \in i\mathbb{R} \), we have
\[
P_\mu \left( \frac{s - 1}{s + 1} \right) = \mu(1) \left[ |s + 1|^2 - |s - 1|^2 \right] + \int_{\xi \in \mathbb{V}_-\{1\}} \frac{|(s + 1)^2 - (s - 1)^2|u + 1|^2}{(u - 1)(s + 1) - (u + 1)(s - 1)^2} \, d\mu(\xi)
\]
\[
= \mu(1) x + \int_{\xi \in \mathbb{V}_-\{1\}} \frac{x|u + 1|^2}{|u - s|^2} \, d\mu(\xi),
\]
\[
= \mu(1) x + \int_{\xi \in \mathbb{V}_-\{1\}} \frac{x(1 + \tau^2)}{x^2 + (y - \tau)^2} \, d\mu(\xi),
\]
where \( s = x + iy \in \mathbb{C}_+ \). So in (4) we have
\[
$\rho = \mu(1) \quad$ and $\frac{d\nu(\tau)}{\pi(1 + \tau^2)} = d\mu(\xi)$.
\]
Then, upon applying the change of variables \( s = (1 + z)/(1 - z) \) in (3), we deduce that \( T(t) \) is similar to the multiplication operator induced by \( \exp(-ts) \) acting on the space \( \mathcal{D}_{\mathbb{C}_+}(\nu) \) consisting of analytic functions \( F \) on \( \mathbb{C}_+ \) such that
(6) \[
\frac{1}{\pi} \int_{\mathbb{C}_+} \left| F'(s) \right|^2 \left( x + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y - \tau)^2} \, d\nu(\tau) \right) \, dx \, dy < \infty,
\]
where \( s = x + iy \) and \( F(s) = f(z) \). This proves the first half of Theorem 2.4.

In order to conclude the proof, let us assume that the multiplication operators induced by \( \exp(-ts) \) act continuously for every \( t > 0 \) on a Dirichlet space \( \mathcal{D}_{\mathbb{C}_+}(\tilde{\nu}) \) where \( \tilde{\nu} \) is a finite positive Borel measure supported on the imaginary axis. Reversing the steps above
and taking into account the fact that (5) defines a measure \( \tilde{\mu} \) on \( T \), where \( \tilde{\mu}(1) = \tilde{\nu}(0) = \rho \), we deduce that the given semigroup is similar to the semigroup of multiplication operators induced by \( \phi_t(z) = \exp(-t(1 + z)/(1 - z)) \) on \( D(\tilde{\mu}) \). Since the cogenerator of such a semigroup is \( M_z \), which is a 2-isometry, it follows by Proposition 2.2 that \( \{M_{\phi_t}\}_{t \geq 0} \) consists of 2-isometries.

It remains to show that \( M_{\phi_t} \) is analytic for every \( t > 0 \). If not, then there are a \( t_0 > 0 \) and a \( F \in \mathcal{D}_{C_+}(\tilde{\nu}) \) such that the function \( s \mapsto e^{nt_0 s} F(s) \) lies in \( \mathcal{D}_{C_+}(\tilde{\nu}) \) for \( n = 1, 2, 3, \ldots \).

In particular,

\[
\int_{C_+} |(F(s)e^{nt_0 s})'|^2 dx \, dy < \infty.
\]

Transferring to the disc by letting \( s = (1 + z)/(1 - z) \) and \( F(s) = f(z) \), we have

\[
\int_{D} |[f(z) \exp(nt_0(1 + z)/(1 - z))]'|^2 \frac{1 - |z|^2}{|1 - z|^2} dA(z) < \infty,
\]

so that the function \( z \mapsto f(z) \exp(nt_0(1 + z)/(1 - z)) \) lies in the weighted Dirichlet space \( D(\delta_t) \) corresponding to a Dirac measure at 1, and hence in \( H^2(D) \), by [7, Thm. 7.1.2].

We conclude that \( f \) is identically zero, since no nontrivial \( H^2 \) function can be divisible by an arbitrarily large power of a nonconstant inner function. Hence the analyticity is also established.

\[\square\]

A connection with the right-shift semigroup in weighted \( L^2(\mathbb{R}_+) \). Now, by means of the Laplace transform, we will establish a connection of \( C_0 \)-semigroups of analytic 2-isometries \( \{T(t)\}_{t \geq 0} \) acting on a Hilbert space \( \mathcal{H} \) and the the right shift semigroup \( \{S_t\}_{t \geq 0} \)

\[
S_t f(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq t, \\
 f(x - t) & \text{if } x > t,
\end{cases}
\]

acting on a weighted Lebesgue space on the half line \( \mathbb{R}_+ \).

First, let us begin by recalling a result asserting that for each \( \alpha > -1 \), a function \( G \) analytic in \( \mathbb{C}_+ \) belongs to the weighted Bergman space \( A^2_\alpha(\mathbb{C}_+) \), that is, the space consisting of analytic functions on \( \mathbb{C}_+ \) for which

\[
\|G\|^2_{A^2_\alpha(\mathbb{C}_+)} = \int_{\mathbb{C}_+} |G(x + iy)|^2 x^\alpha \, dx \, dy < \infty,
\]

if and only if it has the form

\[
G(s) := \mathcal{L} g(s) = \int_0^\infty e^{-st} g(t) \, dt, \quad s \in \mathbb{C}_+,
\]

where \( g \) is a measurable function on \( \mathbb{R}_+ \) with

\[
\int_0^\infty |g(t)|^2 t^{-1-\alpha} \, dt < \infty.
\]
Moreover,
\[ \|G\|^2_{A^2_\alpha(C_+)} = \frac{\pi \Gamma(1 + \alpha)}{2\alpha} \int_0^\infty |g(t)|^2 t^{-\alpha} \, dt, \]
(see [4] or [5, Theorem 1], for instance). In other words, the Laplace transform is an isometric isomorphism between \( A^2_\alpha(C_+) \) and \( L^2(\mathbb{R}_+, \left( \frac{\pi \Gamma(1 + \alpha)}{2\alpha} \right)^{1/2} t^{-\alpha} \, dt) \). Hence, by means of a density argument, and taking \( \alpha = 1 \), it follows that for any \( F \in \tilde{D}_{C_+}(\nu) \), there exists \( f \in L^2(\mathbb{R}_+) \) (unique in the usual sense of equivalence classes), such that
\begin{enumerate}[(i)]  
  \item \( \mathcal{L}(tf(t)) = F'(s) \),  
  \item \[ \frac{1}{\pi} \int_{C_+} |F'(s)|^2 \, dx \, dy = \frac{1}{2} \int_0^\infty |f(t)|^2 \, dt, \quad (s = x + iy), \]
  which corresponds to the first sum in [4]; and  
  \item \[ \frac{1}{\pi^2} \int_{C_+} |F'(s)|^2 \frac{x}{x^2 + (y - \tau)^2} \, dx \, dy = \frac{1}{2\pi} \int_0^\infty \left| \int_0^t u f(u)e^{-\imath \tau u} \, du \right|^2 dt, \quad (s = x + iy). \]
\end{enumerate}
These three items along with the fact that for any \( t \geq 0 \), \( T(t) \) is similar to the multiplication operator induced by \( \exp(-ts) \) acting on the space \( \tilde{D}_{C_+}(\nu) \), yields, by means of the Laplace transform, that \( \{T(t)\}_{t \geq 0} \) is transformed to the right-shift semigroup \( \{S_t\}_{t \geq 0} \) acting on the Hilbert space \( \mathcal{H} \) which consists of functions \( f \) defined on \( \mathbb{R}_+ \) such that
\[ \int_0^\infty |f(t)|^2 \, dt + \int_0^\infty \int_{-\infty}^\infty \left| \int_0^t u f(u)e^{-\imath \tau u} \, du \right|^2 \, d\nu(\tau) \frac{dt}{t^2} < \infty. \]

3. A FINAL REMARK ON INVARIANT SUBSPACES OF \( C_0 \)-SEMIGROUPS OF ANALYTIC 2-ISOMETRIES

As an application of our main result, we deal with the study of the lattice of the closed invariant subspaces of a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) of analytic 2-isometries.

Here we shall use the following result from [15 Thm. 7.1] and [16 Thm. 3.2].

**Theorem 3.1.** Let \( \mathcal{M} \) be a non-zero invariant subspace of \( (M_z, D(\mu)) \). Then \( \mathcal{M} = \phi D_{\mu_\phi} \) where \( \phi \in \mathcal{M} \ominus z \mathcal{M} \) is a multiplier of \( D(\mu) \) and \( d\mu_{\phi} = |\phi|^2 \, d\mu \).

In the continuous case we have the following result:

**Theorem 3.2.** Let \( \{T(t)\}_{t \geq 0} \) denote the semigroup of multiplication operators induced by \( \exp(-ts) \) on the space \( \tilde{D}_{C_+}(\nu) \), as in Theorem 2.4 and let \( \mathcal{M} \) be a non-zero closed subspace invariant under all the operators \( T(t) \). Then there is a function \( \psi \in \mathcal{M} \) such that \( \mathcal{M} = \psi \tilde{D}_{C_+}(\nu_\psi) \).
Proof. If $M$ is invariant under the semigroup, then it is also invariant under the cogen-
erator $V$, and after transforming to the disc as in the proof of Theorem 2.4, we may apply
Theorem 3.1.

Note that under the equivalence between $D(\mu)$ and $\tilde{D}_C(+\nu)$, as detailed in (4) and
(5), the subspace $\phi D_{\mu_\phi}$ maps to a space $\psi \tilde{D}_C(+\nu_\psi)$, where $\psi(s) = \phi((s-1)/(s+1))$ and $d\nu_\psi = |\psi|^2 d\nu$. □

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