MMS-type problems for Johnson scheme

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Abstract

In the current work we consider the minimization problems for the number of nonzero or negative values of vectors from the first and second eigenspaces of the Johnson scheme respectively. The topic is a meeting point for generalizations of the Manikam-Miklós-Singhi conjecture proven in [6] and the minimum support problem for the eigenspaces of the Johnson graph, asymptotically solved in [17].

Keywords: Eigenspace, equitable partition, MMS-conjecture, Johnson scheme, Eberlein polynomials.

1 Introduction

Let $V_i$ be $i$th eigenspace of a symmetric association scheme $(X, \{R_0, \ldots, R_d\})$, $i \in \{0, \ldots, d\}$. Following [3], given an eigenvector $v \in V_i$ denote by $X_+(v) = \{x \in X : v_x > 0\}$, $X_-(v) = \{x \in X : v_x < 0\}$, $X_0(v) = \{x \in X : v_x = 0\}$. A pair of $w$-subsets are in $i$th relation, if their intersection is of size $w - i$. The $w$-subsets of $\{1, \ldots, n\}$ together with $w + 1$ relations above define the Johnson scheme and the first relation defines the Johnson graph $J(n, w)$. The eigenvalues of the Johnson scheme are known as the values of the Eberlein polynomials $E_k(i, w, n) = \sum_{j=0}^{k} (-1)^j \binom{i}{j} \binom{w-i}{k-j} \binom{n-w-i}{k-j}$, $k, i \in \{0, 1, \ldots, w\}$. For this scheme (graph) by $V_i$, we denote the eigenspace corresponding to the eigenvalue $\lambda_i(n, w) = E_1(i, w, n) = (w - i)(n - w - i) - i$ for $i \in \{0, 1, \ldots, w\}$.

In the current correspondence we consider the following two characteristics for the Johnson scheme $J(n, w)$:

$$m_+^i(n, w) = \min_{v \in V_i, X_0(v) = \emptyset} |X_-(v)|,$$

$$m_-^i(n, w) = \min_{v \in V_i, X_0(v) = \emptyset} |X_+(v)|.$$

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When \( i = 1 \), the first number was suggested to be \( \binom{n-1}{w-1} \), for \( n \geq 4w \), which is known as the Manikam-Micklos-Singhi conjecture [13], [14]. The problem was completely solved in the paper of Blinovski [6] recently, after works with quadratic [1] and linear improvements [16].

For a vector \( v \) by the support of this vector we mean the value \( X_+(v) + X_-(v) \). The number \( m^0_i(n, w) \) was shown to be equal to \( 2^1 \binom{n-2i}{w-1} \) (along with the description of vectors attaining the bound) for sufficiently large \( n \) in [17]. In the paper we focus on the case when \( i = 1 \) and show that for any \( n \) and \( w \) the minimum of the support of vectors from the first eigenspace of \( J(n, w) \) is attained on the vectors from two classes having rather simple structure (see Section 3).

Bier and Delsarte [4] proposed to investigate the invariant \( \min_{v: v \in V_i, X_0(v)=\emptyset} |X_-(v)| \) for classical association schemes with further generalizations where \( v \) is from the direct sum of several eigenspaces. They obtained several bounds involving such well-known combinatorial concepts as coverings, completely regular codes, additive codes and designs. The current study is motivated by a recent progress in the area of completely regular codes and equitable partitions in Johnson graphs. In particular, the characterization of equitable 2-partitions of \( J(n, 3) \) in [10] for odd \( n \), completely regular codes in \( J(n, w) \) with eigenvalue \( \lambda_2 \) having nontrivial distance were characterized by Martin in [15].

An eigenvector \( u \) of the antipodal Johnson graph \( J(2w, w) \) corresponding to \( \lambda_i \) is such that its absolute values on the pairs of antipodal pairs are equal and signs are the same or opposite depending on the parity of \( i \) [7, p. 142-143]. So, in case of odd \( i \) we have that \( m^-_i(2w, w) = \binom{2w-1}{w-1} \).

In [4] it was shown that

\[
\frac{n!}{D!} \leq m^-_i(n, w) \leq |C|,
\]

where \( C \) and \( D \) are codes (subsets of the vertices of \( J(n, w) \)), whose characteristic functions belong to \( V_i \oplus V_0 \) and \( \bigoplus_{j \in \{0, \ldots, w\} \setminus i} V_j \) respectively. Eliminating a constant vector from the characteristic function of \( C \), we see that there is a two-valued eigenvector \( v \) from \( V_i \) such that \( v_x \neq v_y \) iff \( x \in C \), \( y \notin C \). In other words, \((C, \overline{C})\) is an equitable 2-partition of \( J(n,w) \). Suppose that there is a \((w-1)-(n, w, 1)\)-design \( C \) then its size is the value for \( m^-_w(n, w) \). Indeed, such a design produces an equitable 2-partition \((C, \overline{C})\) of \( J(n,w) \) with eigenvalue \( \lambda_w(n, w) \), see [15]. On the other hand the "anticode" \( D \) could be chosen to be the set \( \{x : y \subset x\} \) where \( y \) is a \( w-1 \)-element subset. The set \( D \) is a Delsarte clique in the Johnson graph which is a completely regular code with eigenvalues \( \lambda_0, \ldots, \lambda_{w-1} \), so the characteristic function of \( D \) is orthogonal to \( V_w \) [5]. The smallest open case is \( i = 2, w = 3 \), since \( m^-_1(n, 2) \) was shown to be \( \lceil n/2 \rceil \) in [5]. Again, for \( n = 1, 3(\text{mod } 6) \), \( w = 3 \) the best known "anticode" \( D \) from \( V_0 \oplus V_1 \oplus V_3 \) is a Steiner triple system, so from [11] we have that
\[ n - 2 \leq m_2^{-}(n, 3). \] (2)

The bound (2) could be tightened up to \(2n - 9\) by considering a modification of the weight distribution lower bound [12] with a generalization for arbitrary \(w\), which we discuss in Section 4.1. The choice of \(C\) in (1) is generalized to be a part of an equitable partition with appropriate eigenvalue. This gives an upper bound in case of \(J(n, 3)\) for odd \(n\) and \(i = 2\) (see Section 4.2), where no equitable 2-partitions exist [10]. For even \(n\) the upper bound (1) from equitable 2-partitions of \(J(n, 3)\) is \(n(n - 2) / 2\).

2 Definitions and Preliminaries

2.1 Equitable partitions

Let \(G\) be an undirected graph. An equitable \(r\)-partition with parts \(C_1, \ldots, C_r\) of the vertex set of \(G\) is called equitable if for any \(i, j \in \{1, \ldots, r\}\) a vertex from \(C_i\) has exactly \(A_{ij}\) neighbors in \(C_j\). The matrix \(A = (A_{ij})_{i,j\in\{1,\ldots,r\}}\) is called the quotient matrix. An eigenvalue of the quotient matrix \(A\) is called an eigenvalue of the partition. Given an eigenvector \(u\) of \(A\) corresponding to an eigenvalue \(\lambda\) define \(u_G\) to be the vector, indexed by the vertices of \(G\), such that \(u_G(x) = u_i\), if \(x \in C_i\). The vector \(u_G\) is an eigenvector of the adjacency matrix of \(G\) corresponding to \(\lambda\) [8][§4.5].

**Proposition 1.** Let \(u\) be an eigenvector without zero entries of the quotient matrix of an equitable partition of the Johnson graph \(J(n, w)\) with parts \(C_1, \ldots, C_r\). Then

\[ m_i^{-}(n, w) \leq \sum_{j: u_j < 0} |C_j|. \]

2.2 The first eigenspace of \(J(n, w)\)

Consider the eigenvectors of the complete graph \(K_n = J(n, 1)\) indexed by integers from \(\{1, \ldots, n\}\). The graph has two eigenvalues: \(n - 1\) and \(-1\). An eigenvector \(a = (\alpha_1, \ldots, \alpha_n)\) of the graph corresponding to the eigenvalue \(-1\) could be characterized as a solution for the equation: \(\alpha_1 + \ldots + \alpha_n = 0\). The first eigenspace of the Johnson graph \(J(n, w)\) could be obtained from that of the complete graph using the isomorphism established by the inclusion mapping \(I\) (see [9]): the image \(I(a)\) is such that \((I(a))_x = \sum_{i \in x} \alpha_i\).

Consider the following two equitable 2-partitions of \(J(n, w)\): (\(\{x : 1 \in x\}, \{x : 1 \notin x\}\)) and (\(\{x : 2 \in x\}, \{x : 2 \notin x\}\)) [15]. Denote by \(v^{1,2}\) the difference of two eigenvectors of \(J(n, w)\) arising from these partitions, i.e. \(v^{1,2}_1 = 0\), if \(1, 2\) are simultaneously in or are not in \(x\), \(v^{1,2}_x = 1\), if \(1 \in x \), \(2 \notin x\), \(v^{1,2}_x = -1\), if \(1 \notin x\), \(2 \in x\). In [17] it was shown that the minimum support eigenvectors from the first eigenspace are exactly \(v^{1,2}\) up to appropriate permutation of coordinate.
positions starting with large enough \( n \) (as well as a generalization of the result for any eigenspace). It is easy to see that \( v^{1,2} \) is \( I(e_1 - e_2) \), where \( e_1, e_2 \) are 1-st and 2-nd vectors of the standard basis.

In Section 3 we extend results from \([17]\) in further details. We show that for any \( n \) the minimum support eigenvector is either \( v^{1,2} \) or \( I(a) \), where \( a \) is a two-valued \((-1)\)-eigenvector of \( J(n, 1) \).

### 3 Minimum support \( \lambda_1 \)-eigenvectors

**Theorem 1.** Let \( v \) be \( \lambda_1 \)-eigenvector of \( J(n, w) \), \( n \geq 2w, w \geq 2 \) with minimum support. Then \( v \) is \( I(e_1 - e_2) \) or \( I(\sum_{i=1}^{k} e_i - \frac{k}{n} \sum_{i=k+1}^{n} e_i) \) for some \( k \in \{2, 3, \ldots, n - 2\} \) such that \( \frac{kw}{n} \in \mathbb{N} \) up to a permutation of coordinate positions and the multiplication by a scalar. In particular,

\[
m^0_1(n, w) = \min \left( 2 \left( \frac{n - 2}{w - 1} \right), \frac{n}{w} \right) - \max_{k \in \{2, 3, \ldots, n - 2\}, \frac{kw}{n} \in \mathbb{N}} \left( \frac{k}{n} j \right) \left( \frac{n - k}{n} j \right)
\]

**Proof**

As it was mentioned above every \( \lambda_1 \)-eigenvector equals \( I(a) \) for some vector \( a = (\alpha_1, \ldots, \alpha_n) \) such that \( \alpha_1 + \ldots + \alpha_n = 0 \). Our next goal is to determine values \( \alpha_1, \ldots, \alpha_n \) for which the support of \( I(a) \) is minimal. Let \( v = I(a) \) be a \( \lambda_1 \)-eigenvector with minimum support. Since the vector \( I(e_1 - e_2) \) has the size of the support equal \( 2 \left( \frac{n - 2}{w - 1} \right) \), we shall assume that the size of the support of \( I(a) \) is not more than \( 2 \left( \frac{n - 2}{w - 1} \right) \). Let us denote by \( m \) the size of \( \{\alpha_1, \ldots, \alpha_n\} \). There are two different cases:

\( m \geq 3 \). Without loss of generality we can assume that \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are pairwise different. Take arbitrary subsets \( A_1, A_2 \) of the set \( \{4, \ldots, n\} \) of cardinalities \( w - 1 \) and \( w - 2 \) respectively. Clearly, there are at least 2 nonzero values among \( I(a)_{i \cup A_1} = \alpha_i + \sum_{k \in A_1} \alpha_k, i = 1, 2, 3 \) and at least 2 nonzero values among \( I(a)_{i \cup A_2} = \alpha_i + \sum_{k \in A_2} \alpha_k, i, j \in \{1, 2, 3\}, i \neq j \). So the support of \( v \) is at least \( 2 \left( \frac{n - 3}{w - 1} \right) + 2 \left( \frac{n - 3}{w - 3} \right) = 2 \left( \frac{n - 2}{w - 1} \right) \). By hypothesis the vector \( I(a) \) has minimal size of the support, so we conclude that \( I(a)_{x} = 0 \) for any \( w \)-subset \( x \) of \( \{4, \ldots, n\} \). In other words, \( I'(a') \) is the zero vector, where \( a' \) is obtained from \( a \) by removing its first 3 entries, \( I' \) is the inclusion mapping from \( J(n - 3, 1) \) to \( J(n - 3, w) \).

We have that \( \sum_{i=4, \ldots, n} \alpha_i = \sum_{x \subset \{4, \ldots, n\}, |x| = w} I'(a'_x) = 0 \). Therefore the vector \( a' = (\alpha_4, \ldots, \alpha_n) \) belongs to \( V_1(n - 3, 1) \) and is the zero vector because \( I'(a') \) is the zero vector and \( I' \) is an isomorphism from \( V_1(n - 3, 1) \) to \( V_1(n - 3, w) \).

From the above there are exactly 2 nonzero values among \( \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3 \). Consequently, we can consider \( \alpha_3 = 0 \) and \( \alpha_1 = -\alpha_2 \), which means that \( v \) is equal to \( cI(e_1 - e_2) \) for some constant \( c \).
Without loss of generality we can take \( \alpha_1 = \alpha_2 = \ldots = \alpha_k = \hat{\alpha} \) and \( \alpha_{k+1} = \alpha_{k+2} = \ldots = \alpha_n = \hat{\beta} \) for some integer \( k \) such that \( 2 \leq k \leq n - 2 \). Using the equality \( \alpha_1 + \ldots + \alpha_n = 0 \) we have \( \hat{\beta} = -\hat{\alpha} \frac{k}{n} \). Let us take an arbitrary vertex \( x \) of \( J(n, w) \). It is easy to see that \( I(a)_x = 0 \) if and only if \( x \) has exactly \( \frac{kw}{n} \) ones in first \( k \) coordinate positions and \( w - \frac{kw}{n} \) ones in the rest \( n - k \) coordinate positions. Particularly, \( \frac{kw}{n} \) must be an integer. Therefore, the support of \( v \) equals \( \binom{n}{w} - \binom{k}{w} \binom{n-k}{w} \). Taking the minimum of this expression over all admissible \( k \) we obtain the statement of the theorem.

In the case \( m = 1 \) we automatically obtain the all-zero eigenvector \( v \), which is not possible.

The theorem 1 reduces the problem of minimizing \( m_1^0(n, w) \) to the comparison of two expressions containing binomial coefficients.

### 4 Bounds on \( m_i^-(n, w) \)

#### 4.1 A lower bound on \( m_i^-(n, w) \)

Let \( v \) be an eigenvector of the Johnson graph \( J(n, w) \) without zero entries, \( x \) be the vertex such that \( v_x \) is negative and takes maximum absolute value over all negative entries of \( v \). Consider the distance partition \( (C_0, \ldots, C_w) \) of the vertices of \( J(n, w) \) with respect to the vertex \( x \). It is well-known that the sum of the entries of \( v \) on \( C_k \) is expressed using the Eberlein polynomials and the value \( v_x \):

\[
\sum_{y \in C_k} v_y = v_x E_k(i, w, n).
\]

Let \( E_k(i, w, n) \) be non-negative. Then by the choice of \( v_x \) with the maximum absolute value we see that there are at least \( |E_k(i, w, n)| \) negative values for \( v_y \) in \( C_k \). Moreover, there are more than \( |E_k(i, w, n)| \) negative \( v_y \)'s not less than \( v_x \), because there is at least one positive \( v_y \) in \( C_k \), since obviously \( |E_k(i, w, n)| < |C_k| \) for \( k > 0 \). Thus we obtain the following bound.

**Theorem 2.** \( m_i^-(n, w) \geq 1 + \sum_{k>0: E_k(i,w,n) \geq 0} (|E_k(i, w, n)| + 1) \).

The consideration for the proof above is similar to the one for the weight distribution bound on the number of nonzeros for the eigenvector of distance-regular graph, see [12]. The values of the Eberlein polynomials for \( i = 2 \) and \( w = 3 \) are as follows \( E_0(2) = 1, E_1(2) = n - 7, E_2(2) = 11 - 2n, E_3(2) = n - 5 \). Therefore, we have the bound below.

**Corollary 1.** \( m_2^-(n, 3) \geq 2n - 9 \).
4.2 An upper bound on $m_2(n, 3)$

Let $n$ be $2r$. The following construction could be found in [11] (see also [2]). Consider the complement of a perfect matching on vertices labeled with $\{1, \ldots, 2k\}$ to a complete bipartite graph. Then the triples of vertices are parted into three orbits $C_1, C_2, C_3$ with respect to the action of the automorphism group of the graph. The triples of $C_1$ consist of vertices belonging to the same part, the triples of $C_2$ induce a walk of length 2 in the graph, the triples of $C_3$ contain exactly one pair of adjacent vertices. Any two parts could be merged and result in equitable 2-partition of triples, e.g. the Johnson graph $J(2r, 3)$ [2]. In particular, the partition $(C_1' = C_1 \cup C_2, C_2' = C_3)$ has the following quotient matrix:

$$
\begin{pmatrix}
3(2r - 5) & 6 \\
4(r - 2) & 2r - 1
\end{pmatrix}
$$

whose eigenvalues are $\lambda_0(n, 3)$ and $\lambda_2(n, 3)$. The parts are in $4(r - 2)$ to 6 ratio, so in view of the Proposition 1 we see that

$$m_2(n, 3) \leq n(n - 2)/2.$$

Let $n$ be $2r + 1$. Consider the graph $G$ with $2r + 1$ vertices which is a union of an isolated vertex and the graph $G'$ consisting the complement of a perfect matching on vertices of size $r$ to a complete bipartite graph. We have the following orbits of triples of vertices:

- **C1**: The vertices of the triple are in one part of $G'$
- **C2**: The vertices of the triple induce a walk of length 2 in $G'$
- **C3**: The vertices of the triple belong to $G'$ and contain only two adjacent vertices
- **C4**: Two nonadjacent vertices belong to different parts of $G'$ and the third one is isolated
- **C5**: Two vertices are in one part of $G'$ and the third one is isolated
- **C6**: Two vertices are adjacent and the third one is isolated

The equitable partition $(C_1, \ldots, C_6)$ of $J(2r+1, 3)$ has the following quotient matrix:

$$
\begin{pmatrix}
3(r - 3) & 3(r - 2) & 6 & 0 & 3 & 0 \\
r - 2 & 5r - 13 & 6 & 0 & 1 & 2 \\
r - 2 & 3(r - 2) & 2r - 1 & 1 & 1 & 1 \\
0 & 0 & 2(r - 1) & 0 & 2(r - 1) & 2(r - 1) \\
r - 2 & r - 2 & 2 & 2 & 2(r - 2) & 2(r - 1) \\
0 & 2(r - 2) & 2 & 2 & 2(r - 1) & 2(r - 2)
\end{pmatrix}
$$

The matrix has eigenvector $(3, 3, 4 - 2r, 2 - 2r, 1, 1)$ corresponding to eigenvalue $\lambda_2(2r + 1, 3) = 2r - 6$. By Proposition 1 we see that

$$m_2(n, 3) \leq |C_3| + |C_4| = 2r(r - 1) + r = (n - 1)(n - 2)/2.$$
Thus we obtain

**Theorem 3.**

\[
m^-(n, 3) \leq \begin{cases} 
\frac{n(n - 2)}{2}, & \text{if } n \text{ is even;} \\
\frac{(n - 1)(n - 2)}{2}, & \text{if } n \text{ is odd.}
\end{cases}
\]

5 Conclusion

Theorem 1 reduces the problem of finding \(m_1^0(n, w)\) to the determination which one of values

\[
\left(\binom{n}{w} - \max_{k \in \{2, 3, \ldots, n-2\}, \frac{k}{\frac{kw}{n}} \in \mathbb{N}} \left(\binom{n - k}{\frac{n-k}{w}}\right)\right) \text{ or } 2\left(\frac{n - 2}{w - 1}\right)
\]

is smaller. In [17] it was shown that the second one is the answer starting from some value \(n_0(w)\). We have compared these values for \(6 \leq n \leq 600\) and \(3 \leq w \leq \frac{n}{2}\) and consequently found corresponding \(m_1^0(n, w)\). Based on these computational results we state the following conjecture:

**Conjecture 1.** For \(w \geq 5\) and \(n \geq 2w + 1\) the following identity holds

\[
m_1^0(n, w) = 2\left(\frac{n - 2}{w - 1}\right).
\]

For \(w < 5\) we have found several curious examples:

1. \(m_1^0(6, 2) = 6\) is attained on the vector \(v = I(e_1 + e_2 + e_3 - e_4 - e_5 - e_6)\),
2. \(m_1^0(8, 2) = 12\) is attained on vectors \(v = I(e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8)\) and \(u = (e_1 - e_2)\),
3. \(m_1^0(9, 3) = 39\) is attained on the vector \(v = I(2e_1 + 2e_2 + 2e_3 - e_4 - e_5 - e_6 - e_7 - e_8 - e_9)\),
4. \(m_1^0(10, 4) = 110\) is attained on the vector \(v = I(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 - e_8 - e_9 - e_{10})\).

Let us notice that it is not hard to show using theorem 1 and basic properties of binomial coefficients that

\[
m_1^0(2w, w) = \left(\binom{2w}{w}\right) - 2\left(\binom{2w - 2}{w - 1}\right)
\]

which is attained on the vector \(I((w - 1)(e_1 + e_2) - \sum_{i=3}^{2w} e_i)\).

In the theorem 3 we described a construction providing a quadratic on \(n\) upper bound for the characteristic \(m^-_2(n, 3)\). At the same time, the corollary 1 gives us a lower bound which is linear on \(n\). The real behaviour of the growth rate of \(m^-_2(n, 3)\) remains to be an intriguing open problem.

The characteristic \(\min_{v:v \in V, X_0(v) = \emptyset} |X_{-}(v)|\) considered by Bier and Delsarte [4] requires that \(v\) does not have zero entries. It may be interesting in the future research to remove this condition and try to find this value in this case for classical association schemes.
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