Kaonic atoms and in-medium $K^-N$ amplitudes II:
Interplay between theory and phenomenology

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Abstract

A microscopic kaonic-atom optical potential $V^{(1)}_{K^-}$ is constructed, using the Ikeda-Hyodo-Weise NLO chiral $K^-N$ subthreshold scattering amplitudes constrained by the kaonic hydrogen SIDDHARTA measurement, and incorporating Pauli correlations within the Waas-Rho-Weise generalization of the Ericson-Ericson multiple-scattering approach. Good fits to kaonic atom data over the entire periodic table require additionally sizable $K^-NN$-motivated absorptive and dispersive phenomenological terms, in agreement with our former analysis based on a post-SIDDHARTA in-medium chirally-inspired NLO separable model by Cieplý and Smejkal. Such terms are included by introducing a phenomenological potential $V^{(2)}_{K^-}$ and coupling it self-consistently to $V^{(1)}_{K^-}$. Properties of resulting kaonic atom potentials are discussed with special attention paid to the role of $K^-$-nuclear absorption and to the extraction of density-dependent amplitudes representing $K^-$ multi-nucleon processes.

Keywords: meson-nuclear multiple scattering, meson-baryon coupled channel chiral models, kaonic atoms

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1. Introduction

The wealth of strong-interaction data deduced from kaonic atoms provides invaluable information on the $K^-$-nuclear interaction at threshold [1, 2]. Recent studies of $K^-$ atoms focused on constructing self-consistently a density-dependent $K^-$-nuclear potential:

$$2\omega_K V^{(1)}_{K^-}(\rho) = -4\pi \tilde{F}_{K^-N}(p_\rho, \sqrt{s_\rho}) \rho .$$ (1)
Here $\omega_K \approx m_K$ at threshold, $\vec{p}_\rho$ is the in-medium relative $K^-N$ momentum and $s_\rho = (E_K + E_N)^2 - (\vec{p}_K + \vec{p}_N)^2$ is the in-medium Lorentz invariant Mandelstam energy-squared variable, both of which depend on the density $\rho$. The in-medium scattering amplitude $\tilde{F}_{K^-N}(p_\rho, \sqrt{s_\rho})$ reduces in the low-density limit to $\tilde{f}_{K^-N}(p=0, \sqrt{s}=m_K+m_N)$, the free-space $K^-N$ scattering length in the $K^-$ nucleus c.m. system. The dependence of $\tilde{F}_{K^-N}(p_\rho, \sqrt{s_\rho})$ on its kinematical variables was transformed in these studies, using a self-consistent procedure, into a density-dependent $\tilde{F}_{K^-N}(\rho)$. The resulting density-dependent optical potential $V^{(1)}_{K^-}(\rho)$ accounts for single-nucleon $\bar{K}N \to \bar{K}N$ elastic and $\bar{K}N \to \pi Y$ reaction processes. Empirically, $\bar{K}N$ amplitudes at about 40 MeV below threshold are involved in pinning down $V^{(1)}_{K^-}$ in kaonic atoms at half nuclear-matter density.

This microscopic construction of $V^{(1)}_{K^-}$, however, did not provide any reasonable reproduction of the experimental values of strong-interaction level shifts and widths in kaonic atoms. In particular, because of the rapid decrease of the underlying absorptivity $\text{Im} \tilde{f}_{K^-N}(p, \sqrt{s})$ when the free-space amplitudes $\tilde{f}_{K^-N}$ are evaluated further below the $K^-N$ threshold, $\text{Im} V^{(1)}_{K^-}$ was unable to account for the strong absorptivity content of kaonic atoms (i.e., their level widths). Thus, in addition to the single-nucleon terms on the r.h.s. of Eq. (1), a sizable phenomenological absorptive term together with a strong dispersive term appeared necessary in order to achieve reasonable fits to the data. This has been demonstrated recently in Refs. [3, 4, 5] where in-medium $K^-N$ amplitudes $\tilde{F}_{K^-N}$ constructed within the chirally motivated separable-interaction models of Cieply and Smejkal (CS) [6, 7] were used to evaluate $V^{(1)}_{K^-}$. In particular, models TW1 and NLO30 from Ref. [7], accounting for the recently measured SIDDHARTA values of kaonic hydrogen $1s$ shift and width [8, 9], have been used in Refs. [4] and [5] respectively. It was found in these works that the added $V^{(2)}_{K^-}$ part consisting of dispersive and absorptive terms was as important as $V^{(1)}_{K^-}$.

The present work is a natural extension of our recent work [3] which may be considered part I of ongoing studies of kaonic atoms and in-medium $K^-N$ scattering amplitudes. The emphasis of the present work is on the interplay between theory and phenomenology that emerges in kaonic atom studies. Starting from the free-space NLO chiral $K^-p$ and $K^-n$ $s$-wave amplitudes constructed by Ikeda, Hyodo and Weise (IHW) [10, 11] which account for the SIDDHARTA data, we arrive at similar conclusions to those.
reached in part I outlined above. We note that the IHW construction is free of any phenomenologically adjusted momentum-space form factors which in the CS separable-model construction are not directly guided by a systematic chiral hierarchy. The IHW free-space charge-averaged $K^-N$ c.m. scattering amplitude $f_{K^-N}(\sqrt{s})$ is shown in Fig. 1 below the $K^-N$ threshold. One notes its strong energy dependence, with $\text{Re} f_{K^-N}(\sqrt{s})$ mostly rising in going subthreshold and $\text{Im} f_{K^-N}(\sqrt{s})$ sharply dropping below $\sqrt{s} = 1415$ MeV as one approaches the $\pi \Sigma$ threshold ($\sqrt{s} = 1330$ MeV).

Figure 1: The $K^-N$ c.m. scattering amplitude $f_{K^-N}(\sqrt{s}) = \frac{1}{2}(f_{K^-p}(\sqrt{s}) + f_{K^-n}(\sqrt{s}))$ below threshold, constructed from the IHW free-space $K^-p$ and $K^-n$ s-wave scattering amplitudes [11]. To convert from the two-body c.m. to the lab system, which for $A \gg 1$ coincides with the $K^-$ nucleus c.m. system, apply $\tilde{f} = (\sqrt{s}/m_N)f$.

To generate in-medium amplitudes $\tilde{F}_{K^-N}(\sqrt{s}_\rho)$ from the IHW free-space $K^-p$ and $K^-n$ s-wave amplitudes, we apply the multiple-scattering (MS) approach of Waas, Rho and Weise (WRW) [12] focusing on Pauli correlation effects, as described in Section 2. Charge (isospin) degrees of freedom are incorporated in this MS approach which determines, under a straightforward generalization of Eq. (1), the single-nucleon density-dependent potential $V_{K^-}^{(1)}(\rho)$. To represent multi-nucleon dispersive and absorptive processes, we add a phenomenological density-dependent interaction potential $V_{K^-}^{(2)}(\rho)$. Both $V_{K^-}^{(1)}$ and $V_{K^-}^{(2)}$ are coupled implicitly within a self-consistent cycle built
into the kaonic atom fitting procedure. Properties of the resulting kaonic atom potentials are discussed in Section 3 with special emphasis placed on the role of $K^-$ nuclear absorption. The paper ends with a brief summary in Section 4.

2. Methodology

Several subjects are introduced and discussed in this section. In Section 2.1 we briefly review the multiple-scattering procedure applied by WRW \[12\] to incorporate nuclear-correlation contributions, particularly Pauli correlations in the construction of a $K^-$ nucleus potential $V_{K^-}^{(1)}$ at low energies in terms of in-medium $K^-N$ amplitudes $\tilde{F}_{K^-N}(\sqrt{s})$. In Section 2.2 we introduce a phenomenological potential $V_{K^-}^{(2)}$ to represent multi-nucleon processes outside of the underlying meson-baryon chiral model on which $V_{K^-}^{(1)}$ is based. Although no explicit coupling between $V_{K^-}^{(1)}$ and $V_{K^-}^{(2)}$ is practised in the present calculations, in agreement with the spirit of the original Ericson-Ericson MS procedure \[14\], possible alternatives are considered in Appendix A to this work. In Section 2.3 we focus attention on the self-consistent procedure of relating subthreshold energies to densities through functional dependence involving both $V_{K^-}^{(1)}$ and $V_{K^-}^{(2)}$, thereby coupling implicitly $V_{K^-}^{(1)}$ to $V_{K^-}^{(2)}$. Finally, in Section 2.4 we discuss the effect of the $\Lambda(1405)$ subthreshold resonance on the low-density behavior of our in-medium amplitudes $\tilde{F}_{K^-N}(\rho)$.

2.1. Overview of WRW

In this MS procedure one starts by relating the meson wavefunction $\phi(\vec{r})$ generated by the incident plane wave $\exp(i\vec{k} \cdot \vec{r})$ to the effective meson wavefunctions $\phi_\text{eff}^I(\vec{r})$ at the scattering point $\vec{r}$:

$$\phi(\vec{r}) = \exp(i\vec{k} \cdot \vec{r}) + \sum_{I=0,1} \int d^3r' \frac{\exp(ik|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} \tilde{f}_I \rho_I(\vec{r}) \phi_\text{eff}^I(\vec{r}) .$$

(2)

Here, $\tilde{f}_I$ are free-space $\bar{K}N$ s-wave scattering amplitudes with good isospin in the lab system and $\rho_I = (2I + 1)\rho/4$ for isospin-symmetric nuclear density $\rho$. Operating with $(\Delta + k^2)$ on both sides of (2) one obtains the wave equation

$$(\Delta + k^2) \phi(\vec{r}) = -4\pi \sum_{I=0,1} \tilde{f}_I \rho_I(\vec{r}) \phi_\text{eff}^I(\vec{r}) .$$

(3)
The MS procedure relates effective wavefunctions $\phi_{I}^{\text{eff}}(\vec{r})$ at $\vec{r}$ to effective wavefunctions $\phi_{I'}^{\text{eff}}(\vec{r'})$ at $\vec{r}'$ as follows:

$$
\phi_{I}^{\text{eff}}(\vec{r}) = \phi(\vec{r}) + \sum_{I'=0,1} \int d^3r' \frac{\exp(ik|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} c_{II'}(\vec{r}, \vec{r'}) \tilde{f}_{I'} \rho_{I'}(\vec{r'}) \phi_{I'}^{\text{eff}}(\vec{r'}) ,
$$

where $c_{II'}(\vec{r}, \vec{r'})$ are isospin projected $NN$ correlation functions which in nuclear matter depend on $t = |\vec{r} - \vec{r}'|$ only. Replacing in the long-wavelength limit the argument $\vec{r}'$ by $\vec{r}$ in $\rho_{I'}(\vec{r'})$ and $\phi_{I'}^{\text{eff}}(\vec{r'})$, Eq. (4) reduces to

$$
\phi_{I}^{\text{eff}}(\vec{r}) = \phi(\vec{r}) - \sum_{I'=0,1} \xi_{II'} \tilde{f}_{I'} \rho_{I'}(\vec{r}) \phi_{I'}^{\text{eff}}(\vec{r}) ,
$$

where

$$
\xi_{II'} = -4\pi \int_{0}^{\infty} dt \exp(ikt) tc_{I}(t) .
$$

The following discussion is limited to Pauli correlation contributions to $c_{II'}$ which were found by WRW (and also confirmed by us) to overshadow contributions from dynamical short-range correlations. For isospin symmetric matter, the Pauli $\xi_{II'}$ are diagonal in isospin, see Ref. [12]. Solving Eqs. (5) for $\phi_{I}^{\text{eff}}(\vec{r})$ in terms of $\phi(\vec{r})$ and substituting on the r.h.s. of Eq. (4), the latter assumes the form of $2\omega_{K} V_{K^-}^{(1)}(\omega_{K}; \rho) \phi(\vec{r})$, with

$$
2\omega_{K} V_{K^-}^{(1)}(\omega_{K}; \rho) = -4\pi \sum_{I} \frac{2I + 1}{4} \frac{\tilde{f}_{I}}{1 + \frac{4}{3} \xi_{k}\tilde{f}_{I} \rho(\vec{r})} \rho(\vec{r}) ,
$$

where $\xi_{k}$ is given by

$$
\xi_{k} = \frac{9\pi}{p_{F}^{2}} \left( 4 \int_{0}^{\infty} \frac{dt}{t} \exp(iqt) j_{1}^{2}(t) \right) , \quad q = k/p_{F} .
$$

For kaonic atoms, $k = (\omega_{K}^{2} - m_{K}^{2})^{1/2} \approx 0$ and $\xi_{k=0} = 9\pi/p_{F}^{2}$, where the Fermi momentum $p_{F}$ is given by $p_{F} = (3\pi^{2}\rho/2)^{1/3}$. We note that the density dependence of $V_{K^-}^{(1)}(\omega_{K}; \rho)$ in Eq. (7) is not limited to the explicit dependence on $\rho$ in its right-hand side, but arises as well from the $\sqrt{s}$ energy argument of the free-space amplitudes $\tilde{f}_{I}$ which for simplicity was disregarded in the derivation above and which gives rise in the nuclear medium to a density-dependent subthreshold energy argument $\sqrt{s}_{\rho}$. The precise definition of $\sqrt{s}_{\rho}$ and the
self-consistent procedure by which the $\sqrt{s}_p$ dependence of $\tilde{f}_I$ is converted into a density dependence of $\tilde{F}_{K^-}$ and of $V_{K^-}^{(1)}$ are relegated to Section 2.3.

Eq. (7) was derived assuming implicitly equal proton and neutron density distributions, $\rho_p(r) = \rho_n(r) = \rho(r)/2$. Relaxing this constraint, primarily for use in kaonic atom applications where $\omega_K \approx \mu_K$ with $\mu_K$ the reduced mass of the kaon, one obtains to leading order:

$$2\mu_K V_{K^-}^{(1)}(\rho) = -4\pi \left[ \frac{\left(2\tilde{f}_{K^-p} - \tilde{f}_{K^-n}\right)}{1 + \frac{1}{2}\xi_{k=0}\tilde{f}_0(\rho)} \rho_p + \frac{\tilde{f}_{K^-n}(\rho)\rho_p + \rho_n}{1 + \frac{1}{2}\xi_{k=0}\tilde{f}_1(\rho)} \right].$$

This form of the MS summation is used in all of the numerical applications reported in the present work. Eq. (9) may be rewritten as

$$2\mu_K V_{K^-}^{(1)}(\rho) = -4\pi \left[ 1 + \frac{A - 1}{A} \frac{\mu_K}{m_N} \right] \left[ \mathcal{F}_{K^-p}(\rho)\rho_p(\rho) + \mathcal{F}_{K^-n}(\rho)\rho_n(\rho) \right],$$

which defines the in-medium amplitudes $\mathcal{F}_{K^-p}(\rho)$ and $\mathcal{F}_{K^-n}(\rho)$. Here, $A$ is the atomic mass number. Finally, to bring Eq. (10) into the more compact form (11), we define an effective amplitude $\mathcal{F}_{K^-N}^{\text{eff}}(\rho)$ through

$$\mathcal{F}_{K^-N}^{\text{eff}}(\rho) = \mathcal{F}_{K^-p}(\rho)\rho_p(\rho) + \mathcal{F}_{K^-n}(\rho)\rho_n(\rho),$$

with the standard conversion from two-body c.m. to the $K^-$-nuclear c.m. frame given by $\mathcal{F}_{K^-N}^{\text{eff}}(\rho) = (1 + \frac{A - 1}{A} \frac{\mu_K}{m_N}) \mathcal{F}_{K^-N}^{\text{eff}}(\rho)$. Eq. (11) may serve for defining similarly an effective free-space subthreshold amplitude $f_{K^-N}^{\text{eff}}(\rho)$, with density dependence arising from the underlying energy dependence $\sqrt{s} \rightarrow \sqrt{s}_p \rightarrow \rho$.

2.2 Adding $V_{K^-}^{(2)}(\omega_K; \rho)$

We wish now to account for multi-nucleon dispersive and absorptive processes by adding a phenomenological term $V_{K^-}^{(2)}(\omega_K; \rho)$ to the MS single-nucleon potential $V_{K^-}^{(1)}(\omega_K; \rho)$, Eq. (7). Traditionally, these processes are not iterated within the MS expansion in which scattering occurs on single nucleons via amplitudes $\tilde{f}$, and $V_{K^-}^{(2)}$ is simply added to the MS $V_{K^-}^{(1)}$ specified in Eq. (7) by $\xi_{k=0} = 9\pi/p_F^2$ for kaonic atoms. Alternative ways of introducing $V_{K^-}^{(2)}$, by letting it affect explicitly the MS procedure for deriving the in-medium $V_{K^-}^{(1)}$, or by redefining the splitting of $V_{K^-}$ into its $V_{K^-}^{(1)}$ and $V_{K^-}^{(2)}$ components, are discussed briefly in Appendix A to this work. None of these alternatives was found in the present work to offer advantage over the straightforward addition of $V_{K^-}^{(2)}$. 
2.3. Energy vs. density

Here we outline a basic difference between the present work and our previous ones [3, 4, 5] in which a model of using $K^-N$ amplitudes below threshold was introduced. In these calculations, energies and momenta of the $K^-$ meson and a bound nucleon were determined, independently, by the nuclear environment, and the Mandelstam energy variable $\sqrt{s}$ was evaluated in the nuclear medium, thereby becoming density dependent, $\sqrt{s_\rho}$. In particular, the in-medium $K^-$ momentum was assigned locally to the real part of the single-nucleon $K^-$-nucleus potential $\text{Re} V_{K^-}^{(1)}$, leading to

$$\sqrt{s_\rho} \approx E_{th} - B_N - \xi_N B_K - 15.1(\rho/\rho_0)^{2/3} + \xi_K (\text{Re} V_{K^-}^{(1)} + V_c) \text{ (MeV)} \quad (12)$$

with $E_{th} = m_N + m_K$, $\xi_N = m_N/(m_N + m_K)$, $\xi_K = m_K/(m_N + m_K)$, and with $V_c$ for the $K^-$ Coulomb potential, $B_N$ for a nucleon average binding energy and $\rho_0$ for nuclear matter density. The atomic binding energy $B_K$ of the $K^-$ is relatively small and, hence, it is safe to neglect it in Eq. (12). With a fixed value for $B_N$ it is evident that for $\rho \to 0$ the energy approaches $E_{th} - B_N$, thus violating the low-density limit whereby the $1N$ amplitude should approach the free amplitude at threshold, $E_{th}$. In the present work we have therefore replaced the fixed average nucleon binding energy $B_N$ by a density-dependent one,

$$B_N(\rho) = B_N \rho/\bar{\rho}, \quad (13)$$

where the average density is given by

$$\bar{\rho} = \frac{1}{A} \int \rho^2 \, d^3r \quad (14)$$

with $A$ the atomic mass number. For $B_N$ we used the value of 8.5 MeV, the same as in our earlier work. Furthermore, in order to enhance the very slow convergence of $\sqrt{s_\rho}$ to $E_{th}$ due to the Coulomb potential, $V_c$ was multiplied in Eq. (12) by $(\rho/\rho_0)^{1/3}$, based on dimensional arguments, thereby ensuring that the low-density limit is respected in the calculations.

Since $V_{K^-}^{(1)}$ is proportional locally according to Eq. (11) to $\tilde{F}_{K^-N}(\sqrt{s_\rho})$, and $\sqrt{s_\rho}$ according to Eq. (12) depends on $\text{Re} V_{K^-}^{(1)}$, a self-consistent (SC) procedure was applied, with 5–6 iterations proving more than adequate for convergence. In this way a simple algorithm for constructing $V_{K^-}^{(1)}$ from any
given model for \( \tilde{F}_{K^-N}(\sqrt{s\rho}) \) was formulated. As mentioned in the Introduction, the SC \( V_{K^-}^{(1)} \) potentials constructed thereby were characterized by ignoring multi-nucleon absorption processes and by an imaginary part that goes rapidly to zero towards the \( \pi\Sigma \) threshold. There were no free parameters at this phase of the calculation and there was no coupling between the real and imaginary parts of the potential beyond that provided by the input \( K^-N \) amplitudes. In carrying out global fits to strong-interaction shifts and widths data across the periodic table, a phenomenological potential \( V_{K^-}^{(2)} \) was added to the SC \( V_{K^-}^{(1)} \), with parameters determined in a \( \chi^2 \) fit search without perturbing the prior SC determination of \( V_{K^-}^{(1)} \). However, no compelling reason was given why \( V_{K^-}^{(2)} \) was excluded from the SC procedure.

In the present work, a phenomenological term \( V_{K^-}^{(2)} \) is included from the outset in the SC procedure. Eq. \( \text{(12)} \) is thus modified, replacing \( V_{K^-}^{(1)} \) by \( V_{K^-} = V_{K^-}^{(1)} + V_{K^-}^{(2)} \):

\[
(\sqrt{s\rho})_{\text{atom}} \approx E_{\text{th}} - B_N\rho/\bar{\rho} - 15.1(\rho/\rho_0)^{2/3} + \xi_K(\text{Re} V_{K^-} + V_c(\rho/\rho_0)^{1/3}) \quad \text{(MeV)},
\]

(15)

where the subscript \( \text{atom} \) indicates the limitation to kaonic atoms, thereby also setting \( B_K=0 \). The first effect of this modification is to introduce \textit{implicit coupling} between \( V_{K^-}^{(1)} \) and \( V_{K^-}^{(2)} \), since varying parameters of \( V_{K^-}^{(2)} \) affects the \( \sqrt{s\rho} \) energy argument of the in-medium \( \tilde{F}_{K^-N}(\sqrt{s\rho}) \) and that of the underlying free-space \( \tilde{f}_{K^-N}(\sqrt{s\rho}) \) amplitudes. Early tests of this approach using the NLO30 amplitudes revealed \[5\] that this coupling is non-negligible when the imaginary part of the single-nucleon amplitude drops sharply below threshold, which was particularly the case with the ‘SE’ in-medium version of model NLO30. The increased flexibility due to this coupling leads in the present work, building on the IHW free-space amplitudes, to good fits to the data, well beyond what was achieved in our earlier works \[3, 4, 5\].

Fig. \[2\] shows the functional dependence generated by the SC relation Eq. \( \text{(15)} \) between subthreshold \( K^-N \) energies and nuclear densities for Ni and Pb, calculated using the IHW-based global fit to kaonic atoms detailed in Section \[3\]. Compared with the earlier version \[3\] where only \( V_{K^-}^{(1)} \) appeared in the SC expression, Eq. \( \text{(12)} \), lower energies are now probed at the higher densities and higher energies are probed at lower densities. Consequently, for the more relevant region of 50% of the central nuclear density, the energy downward shift remains unchanged at 40 MeV below threshold.
Having introduced by Eq. (15) a relationship between the subthreshold energy argument of $K^-N$ amplitudes and their implied density dependence, it is instructive to demonstrate the density dependence generated by the WRW renormalization Eq. (9). Shown in Fig. 3 is the ratio $R$ of the in-medium effective amplitude $f_{K^-N}^{\text{eff}}$, Eq. (11), to the free effective amplitude $f_{K^-N}^{\text{eff}}$, calculated as a function of nuclear density in Ni for the IHW amplitudes in the absence of the $V_{K^-}^{(2)}$ potential term. Note the logarithmic density scale used to highlight the slow convergence of $R$ to its low-density limit. For example, Re $R=1.14$ and Im $R=0.07$ near 0.1% of the Ni central density $\rho_0$, still far from their limiting values Re $R=1$ and Im $R=0$, respectively. This slow convergence is caused by the predominance of the $\Lambda(1405)$ resonance for densities roughly below 6% of $\rho_0$ where Re $R$ exhibits hump structure with values exceeding 1, owing to the large negative values assumed by Re $\tilde{f}_{I=0}$ near threshold; the position of the $\Lambda(1405)$ is marked here by the vanishing of Re $\tilde{f}_{I=0}$ at $\sqrt{s}\approx1415$ MeV. At densities above 6% of $\rho_0$, Re $R\leq1$, decreasing monotonically with density owing to the rapid increase of Re $\tilde{f}_{I=0}$ below 1415 MeV and leveling off when Re $\tilde{f}_{I=0}$ has reached its (positive) maximum value.
A similar analysis shows that the Λ(1405) is directly responsible for Im $\mathcal{R}$ becoming slightly positive below a fraction of 1% of $\rho_0$ and slowly converging to its low-density limiting value of 0. However, it is not clear whether the isospin-based WRW method of introducing medium effects into the free-space IHW amplitudes, as applied here, is valid at such low densities where the $K^-$ Coulomb potential $V_c$ becomes comparable with and even exceeds $V_{K^-}^{(1)}$. In order to check sensitivities, $\chi^2$ fits were made also by imposing a transition to the free-space amplitudes at densities of 0.5, 1.0 and 2.0% of $\rho_0$, while retaining $V_{K^-}$ down to densities $\approx 10^{-5}\rho_0$ as normally done in kaonic atom calculations. Below $10^{-5}\rho_0$, only $V_c$ is retained, along with changing the radial integration scheme from nuclear to atomic dimensions. Weak sensitivities were found to the way the amplitudes were handled at very low densities. Most of the results in this paper are for transition to free amplitudes at 2% relative density.

3. Results and discussion

Here we report and discuss results of fits to kaonic atom data, using the methodology outlined in the previous section. A phenomenological potential $V_{K^-}^{(2)}$ representing $K^-$ multi-nucleon processes is defined in Section 3.1.
combined amplitudes that result from the self-consistent fitting procedure are shown and discussed, and the emergence of deep $K^-$ nucleus potentials is noted. The present IHW-based $K^-$ nuclear potential is compared in Section 3.2 with the NLO30 potentials of Ref. [5] and, in view of the significance of the resulting $V_{K^-}^{(2)}$ component, its signatures are discussed in Section 3.3 with special emphasis on its absorptive part.

3.1. IHW-based potentials

Fits to 65 data points for strong-interaction observables in kaonic atoms were made by introducing a phenomenological multi-nucleon term

$$V_{K^-}^{(2)} = -(4\pi/2\mu_K)[b \rho + B(\rho/\rho_0)^{\alpha}],$$

where \(\mu_K\) is the kaon-nucleus reduced mass, \(b\) and \(B\) are energy-independent complex parameters and the real exponent \(\alpha\) satisfies \(\alpha > 0\). The linear density term with coefficient \(b\) could be viewed as compensating for uncertainties in the IHW free-space amplitude input. The combined potential \(V_{K^-}^{(1)} + V_{K^-}^{(2)}\) was calculated according to the prescriptions of Section 2 and fits to the data were made varying the parameters of \(V_{K^-}^{(2)}\). The real part of the full potential \(V_{K^-}^{(1)} + V_{K^-}^{(2)}\) was used in the present SC procedure Eq. (15). Since good convergence was limited to varying up to four parameters, we varied the parameters \(b\) and \(B\) while griding on the parameter \(\alpha\). Strong correlations were observed between the parameters Im \(b\) and \(\alpha\) and the lowest \(\chi^2\) of 107 for 65 data points was obtained for Im \(b = -1.11\pm0.12\) fm and \(\alpha = 0.3\). However, this magnitude of |Im \(b\)| is too large to absorb into the uncertainties allowed by the IHW determination of the free-space \(KN\) scattering amplitudes at threshold [11]. Constraining Im \(V_{K^-}^{(2)}\) to be non-negative at all values of \(\rho\), we set Im \(b = 0\) resulting in \(\chi^2 = 118\) for \(\alpha = 1.2\), Re \(b = -0.34\pm0.07\) fm, Re \(B = 1.94\pm0.16\) fm and Im \(B = 0.83\pm0.16\) fm. This solution is used in the rest of this work. We note that the value of Re \(b\) could be related to a decrease of the poorly known Re \(a_{K^-n}\) by about 0.45 fm, twice as much as the downward uncertainty assigned in the IHW NLO determination.

Table 1 summarizes the results of fits to the data. The DD potential listed in the table is a purely phenomenological potential of a form similar to Eq. (16) and is included as a ‘benchmark’ for what could be an ultimate fit to 65 data points across the periodic table, where the data come from several experiments at different laboratories. The value of \(\chi^2 = 103\) for 65 data points (\(\chi^2/\text{dof}\) of 1.7) is most acceptable. The entry for the NLO30 model
Table 1: Characteristic quantities of $K^{-}$-Ni potentials from global fits to kaonic atom data. DD is a purely phenomenological (density-dependent) model, NLO30 is the model used in our previous work \cite{5} and IHW is the present model based on the free-space input from Ref. \cite{11}. Values of potentials are in MeV, values of r.m.s. radii are in fm. The r.m.s. radius of the point-proton distribution in Ni is 3.69 fm.

| model  | $\chi^2$ (N=65) | $V_R(0)$ | $V_I(0)$ | $r_R$ | $r_I$ | $\alpha$ |
|--------|----------------|----------|----------|-------|-------|---------|
| DD     | 103            | -199     | -76      | 3.48  | 3.71  | 0.25    |
| IHW    | 118            | -191     | -79      | 3.34  | 3.73  | 1.2     |
| NLO30  | 148            | -179     | -71      | 3.42  | 3.70  | 1       |

\cite{5} is typical of results obtained in the recent works \cite{3,4,5}. As was shown already in Ref. \cite{15} and re-iterated recently, the r.m.s. radii of the potentials are also characteristic quantities that reflect the density dependence of the potentials.

Fig. 4 shows the full effective $K^{-}N$ amplitudes $F_{\text{eff}}(\rho)$ and the corresponding in-medium IHW-based single-nucleon ($1N$) amplitudes for a typical nucleus, $^{58}$Ni. The increase or decrease of the amplitude with density leads to decrease or increase, respectively, of the radii of the corresponding part of the potential relative to the nuclear radius, in agreement with the r.m.s. radii listed in Table 1. The monotonic increase with density of the real part of the full effective amplitude $F_{\text{eff}}(\rho)$ is reflected in the relatively small r.m.s. radius of the real potential. This increase contrasts with the decrease with density of the in-medium $1N$ real amplitude above 10% of central nuclear density. On the other hand, for the imaginary part of the full effective amplitude $F_{\text{eff}}(\rho)$, the combined effect of decrease followed by increase with density causes the r.m.s. radius of the imaginary potential to be very close to 3.69 fm which is the radius of the proton distribution. In contrast, the imaginary part of the in-medium $1N$ amplitude practically decreases with density. The figure shows how the difference between the imaginary parts of the full amplitude and its $1N$ component increases steadily with density, providing solid evidence for the dominance of multi-nucleon $K^{-}$-nuclear absorption at $\rho/\rho_0 \geq 0.5$.

Finally, Figs. 5 and 6 show the full IHW-based potential for Ni and Pb, respectively, where the NLO30 in-medium potential (plus a phenomenological term) \cite{3} is also shown for Ni. Note the sizable attraction of order 150–
200 MeV at central densities. These potential depths obviously reflect a smooth extrapolation provided by the density dependence of the IHW-based $V_{K^{-}}^{(1)}$ and that of $V_{K^{-}}^{(2)}$, Eq. (16). However, depths of order 80–90 MeV around the half-density radius and down to 1 fm inward are reliably determined in kaonic atom fits [16], and the depths deduced in the present work are fully consistent with those deduced in earlier studies, beginning with the DD phenomenological analysis from 1994 [15] and ending with the very recent NLO30 theory-based analysis [5].

3.2. Comparison with previous results

Fig. 5 shows a comparison of the present IHW-based $K^{-}$-Ni potential with the corresponding potential for the NLO30 model. The real potentials are quite similar in the two cases whereas the imaginary potentials are different. It is instructive to trace the origin of the very significant difference of 30 units in $\chi^2$ values between the two models as shown in Table 1. Inspecting the fits for the different target nuclei reveals that most of the reduction in $\chi^2$ in the present work comes from species where strong-interaction observables
were measured for more than a single kaonic atom level.

Shifts and widths are usually measured for a ‘lower’ level and in favourable cases a relative yield of the transition from the next higher level to the lower level can also be measured. That quantity provides the width of the ‘upper’ level, based on the known radiation width for the electromagnetic transition. The smaller values of $\chi^2$ achieved in the present work are due to better agreement, in most cases, between calculation and experiment for widths of lower and upper levels in the same kaonic atom. Note that strong-interaction effects in kaonic atoms are dominated by absorption processes $[1, 2]$. Indeed, strong-interaction level widths in kaonic atoms are up to one order of magnitude larger than the corresponding strong-interaction level shifts. Furthermore, these shifts are almost universally repulsive, although the real potentials required to fit kaonic atom data are attractive. It is therefore not surprising that the differences between the imaginary potentials are the origin of the significantly better fits achieved in the present work.
Figure 6: $K^-$ nuclear potentials for $K^-$ atoms of Pb, derived from global fits based on in-medium IHW amplitudes (plus a phenomenological term).

3.3. On separating $V^{(2)}_{K^-}$ from $V^{(1)}_{K^-}$

In view of the dominance of absorption processes, it is instructive to study the relative importance of multi-nucleon ($mN$) absorption, described here by an added phenomenological potential $V^{(2)}_{K^-}$, and single-nucleon ($1N$) absorption encoded in the input $K^-N$ amplitudes. The present approach introduces coupling between the corresponding potentials $V^{(2)}_{K^-}$ and $V^{(1)}_{K^-}$, and thus it may offer an interesting point of view on this topic.

Fig. 7 shows the combined IHW-based potential for Ni together with its two components. The $1N$ component is as obtained when there is no $mN$ component present, but in the final (combined) potential the $1N$ component is affected by the $mN$ term due to the implicit coupling between the two in the SC procedure within the present approach. The non-additivity of the two terms is seen, particularly for the imaginary potential. Of greater interest is the radial extent of the two terms. At 4.13 fm in this example the two terms are equal for the imaginary part of the potential (3.95 fm for the real part), with the $1N$ part clearly dominating at larger radii whereas the $mN$ part dominates at smaller radii.

Widths of lower and upper levels in the same kaonic atom could display
Figure 7: Components of the $K^-$ nuclear potentials for $K^-$ atoms of Ni, with the $1N$ component derived from IHW amplitudes, see text.

different sensitivities to the $1N$ and $mN$ absorption terms due to differences between the overlaps of the radial atomic wavefunctions and the imaginary potential. Table 2 shows, as an example, calculated widths for the upper $5g$ and for the lower $4f$ levels in kaonic Ni, calculated by using several variants of $1N$ and $mN$ potential terms of the present model. The first row is for the IHW-based potential as obtained from global fits to 65 data points across the periodic table. Some idea on the relative importance of the $1N$ and $mN$ absorption terms may be gained by removing these in turn. However, one must recall that strong-interaction effects in kaonic atoms are extremely

Table 2: Widths of the ‘upper’ $5g$ state and ‘lower’ $4f$ state in kaonic Ni, calculated in the best-fit IHW-based model, see text.

| Re$(1N)$ | Im$(1N)$ | Re$(mN)$ | Im$(mN)$ | $\Gamma_{\text{upper}}$ (eV) | $\Gamma_{\text{lower}}$ (keV) |
|---------|---------|---------|---------|-----------------|-----------------|
| +       | +       | +       | +       | 2.7             | 0.97            |
| +       | -       | +       | +       | 1.1             | 0.72            |
| +       | +       | +       | -       | 2.0             | 1.54            |
non-perturbative. In particular, widths of lower states are highly saturated, meaning that the addition of absorption causes a reduction of widths, due to the repulsive effect on the radial wavefunction [17, 18]. Starting with the upper level, expecting that the local distortions are not too strong, then the second row shows that removing the $1N$ absorption depletes the width by 60% and the third row shows that removing the $mN$ absorption depletes it by only 27%. Additivity of the two terms is approximately respected for the width of the upper level. The reverse holds true for the lower level. Removing the longer-range $1N$ absorption reduces the width by only 26% but removing the shorter-range $mN$ absorption almost doubles the calculated width in this strongly saturated situation.

Fig. 8 shows the effective amplitude of the $mN$ term for Ni compared with the $1N$ amplitudes. This $mN$ amplitude could serve as a guide to what a more fundamental theory of multi-nucleon $K^-$ absorption and dispersion processes should yield. We note that, based on the IHW amplitude, it is possible to obtain a $mN$ phenomenological term whose imaginary part is absorptive throughout the nuclear volume. This was not the case for the NLO30 model.

The absorptive $mN$ amplitude shown in Fig. 8 is rather strong, exceeding
the absorptive IHW-based in-medium $1N$ amplitude for densities higher than $\approx 0.4\rho_0$. A possible comparison with experiment is provided by evidence from old emulsion data \cite{19} for roughly $1:3$ ratio of $mN:1N$ absorptivities. This ratio is seen in Fig. 8 to occur at a density of $\approx 0.2\rho_0$ which is representative of the density range in kaonic atoms where one expects absorption to occur. In contrast, the recent chiral-model calculations by Sekihara et al. \cite{20} reach a $1:3$ ratio for non-mesonic vs. mesonic absorptions at a considerably higher density close to $\rho_0$, implying thus that a substantial contribution to multi-nucleon absorption is missing in their model.

4. Summary

We have presented fits to kaonic atom data across the whole periodic table based on the IHW free-space NLO chiral $K^-N$ amplitudes below threshold \cite{11}. The WRW MS procedure \cite{12} was used to form in-medium $K^-N$ amplitudes in terms of which a $1N$ potential $V^{(1)}_{K^-}$ is constructed. The strong energy dependence of the free-space subthreshold $K^-N$ amplitudes induces substantial density dependence in $V^{(1)}_{K^-}$ within the SC calculation of the energy parameter $\sqrt{s}$. This dependence is enhanced further by the implicit coupling to a phenomenological $V^{(2)}_{K^-}$ term and good fits to the data were reached in this way. It was found, in full agreement with part I of this work \cite{5} which was based on in-medium NLO30 amplitudes due to CS \cite{7}, that a sizable empirical $mN$ potential was required, both for the imaginary part as well as for the real part. By considering in some detail the contribution of $\text{Im } V^{(2)}_{K^-}$ to the width of ‘upper’ and ‘lower’ states in Ni, we have demonstrated how its relative importance develops as one enters the denser regions of the nuclear surface and further inward. With a theoretically-based $1N$ term coupled to a phenomenological $mN$ term within a self-consistent subthreshold approach, the latter could guide more theoretical work to derive the origin of such a strong multi-nucleon $V^{(2)}_{K^-}$ component. Finally, new precision measurements of strong-interaction observables for more than a single level on a given target could greatly enhance our understanding of the various nuclear absorption processes of stopped $K^-$ mesons.

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Appendix A. On coupling $V^{(2)}_{K^-}$ explicitly to $V^{(1)}_{K^-}$

Here we wish to present alternative approaches in which the dispersive and absorptive nuclear processes are viewed as forming a background potential $V^{(2)}_{K^-}$ that affects meson propagation in between scatterings. In the first approach, the incident meson wave $\psi_{\text{inc}}$ is no longer a free-space plane wave $\exp(i\vec{k} \cdot \vec{r})$. It now satisfies the wave equation

$$(\Delta + k^2) \psi_{\text{inc}} = 2\omega_K V^{(2)}_{K^-}(\omega_K; \rho) \psi_{\text{inc}} ,$$

which for constant density $\rho$ admits a solution $\psi_{\text{inc}} = \exp(i\vec{k}_{\rho} \cdot \vec{r})$ with

$$k_{\rho}^2 = k^2 - 2\omega_K V^{(2)}_{K^-}(\omega_K; \rho) .$$

It is straightforward then to revise the MS exposition Eqs. (2) –(8) in Section 2.1, replacing the incident free-space meson wave $\exp(i\vec{k} \cdot \vec{r})$ by the in-medium incident wave $\exp(i\vec{k}_{\rho} \cdot \vec{r})$ where $k_{\rho}$ is defined by (A.2). The outcome is that an overall potential $V_{K^-} = V^{(1)}_{K^-} + V^{(2)}_{K^-}$ is identified, with Eq. (7) for $V^{(1)}_{K^-}$ re-derived under the replacement $\xi_k \to \xi_{k_{\rho}}$:

$$2\omega_K V^{(1)}_{K^-}(\omega_K; \rho) = -4\pi \sum_{I} \frac{2I + 1}{4} \frac{\tilde{f}_I(\sqrt{s\rho})}{1 + \frac{1}{4}\xi_{k_{\rho}} \tilde{f}_I(\sqrt{s\rho})} \rho ,$$

where $\xi_{k_{\rho}}$, in analogy with (8), is given by

$$\xi_{k_{\rho}} = \frac{9\pi}{p_F^2} \left[ 4 \int_0^{\infty} \frac{dt}{t} \exp(\frac{i q_{\rho} t}{2}) j_1^2(t) \right] , \quad q_{\rho} = k_{\rho}/p_F ,$$

with $k_{\rho}$ defined by (A.2) through the input $V^{(2)}_{K^-}$. The integral in (A.4) can be evaluated analytically, resulting in:

$$\xi_{k_{\rho}} = \frac{9\pi}{p_F^2} \left[ 1 + \frac{i q_{\rho}}{2} \left( (1 - \frac{q_{\rho}^2}{4}) \ln(1 - \frac{4}{q_{\rho}^2}) - 1 \right) \right] .$$

$V^{(1)}_{K^-}$, Eq. (A.3), is now explicitly coupled to $V^{(2)}_{K^-}$ which enters via (A.2) and the r.h.s. expression of $\xi_{k_{\rho}}$ in Eq. (A.5).
An alternative form of considering explicit coupling between $V_{K^-}^{(2)}$ and $V_{K^-}^{(1)}$ is motivated by observing that the single-nucleon $\bar{K}N$ scattering amplitudes $\tilde{f}_I$ which enter the r.h.s. of Eqs. (7) and (A.3) for $V_{K^-}^{(1)}$ are evaluated below the $\bar{K}N$ threshold. Their imaginary parts are related there exclusively to $K^-N \to \pi Y$ reaction processes which do not require imposing the Pauli principle in the final $\pi Y$ states. This suggests that the MS $V_{K^-}^{(1)}$ is constructed from the real parts of $\tilde{f}_I$, while the imaginary parts are attached to the background potential $V_{K^-}^{(2)}$. We denote the modified $V_{K^-}^{(j)}$ by $U_{K^-}^{(j)}$, so that $V_{K^-} = U_{K^-}^{(1)} + U_{K^-}^{(2)}$ as follows:

$$2\omega_K U_{K^-}^{(1)}(\omega_K; \rho) = -4\pi \sum_I \frac{2I + 1}{4} \frac{\text{Re} \tilde{f}_I(\sqrt{s}_\rho)}{1 + \frac{1}{4} \xi_k \text{Re} \tilde{f}_I(\sqrt{s}_\rho) \rho},$$

(A.6)

$$U_{K^-}^{(2)}(\omega_K; \rho) = V_{K^-}^{(2)}(\omega_K; \rho) - \frac{4\pi}{2\omega_K} \sum_I \frac{2I + 1}{4} i \text{Im} \tilde{f}_I(\sqrt{s}_\rho) \rho,$$

(A.7)

with $\xi_k$ from Eqs. (A.4) and (A.5), and $k_\rho$ now defined by

$$k_\rho^2 = k^2 - 2\omega_K U_{K^-}^{(2)}(\omega_K; \rho).$$

(A.8)

Similarly to the discussion in Section 2.2 it is optional to ignore any explicit coupling between $U_{K^-}^{(1)}$ and $U_{K^-}^{(2)}$ by using $\xi_{k=0} = 9\pi/p_F^2$ for $k_\rho$ on the r.h.s. expression (A.6) for $U_{K^-}^{(1)}$.

Before closing we demonstrate briefly the additional density dependence implied by using $\xi_k$, Eq. (A.3), in the MS renormalization on the r.h.s. of Eqs. (A.3) and (A.6), compared to using $\xi_{k=0} = 9\pi/p_F^2$ in the main text. To this end we assume a density-independent background (bgd) absorptive amplitude $\tilde{F}_{\text{bgd}}^{\text{eff}} = i0.5$ fm so that $k_\rho^2 = i2\pi\rho$ in fm-derived units. Fig. A.1 shows the MS complex expansion parameter $\frac{1}{4} \xi_k \rho$ along with the real parameter $9\pi\rho/4p_F^2$ which holds in the limit $k_\rho = 0$. We observe that the imaginary part of the expansion parameter is negligible with respect to the real part, also in absolute terms, and that the real part is considerably quenched relative to its $k_\rho = 0$ limit. Consequently, the WRW renormalization of the single-nucleon amplitudes $\tilde{f}$ is somewhat weaker than when using $\xi_{k=0} = 9\pi/p_F^2$. This conclusion holds true also when the balance between real and imaginary parts of the expansion parameter $\frac{1}{4} \xi_k \rho$ varies, depending on the precise complex structure of the assumed background amplitude $\tilde{F}_{\text{bgd}}^{\text{eff}}$. 

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Figure A.1: The density dependence of the MS complex expansion parameter $\frac{1}{4} \xi_{k,\rho}$, with $\xi_{k,\rho}$ from Eq. (A.5), for a background absorptive amplitude $\tilde{F}_{\text{bgd}} = i 0.5 \text{ fm}$. The upper curve shows, for comparison, the WRW MS real expansion parameter for no background amplitude, i.e. $9\pi\rho/4\rho_0^2$.

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