Symbolic dynamical scales: modes, orbitals, and transversals

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(Received 6 September 2020; accepted 19 June 2021)

We study classes of musical scales obtained from shift spaces in symbolic dynamics through the first symbol rule, which yields scales in any $n$-TET tuning system. The modes are thought as elements of orbit equivalence classes of cyclic shift actions on languages, and we study their orbitals and transversals. We present explicit formulations of the generating functions that allow us to deduce the orbital and transversal dimensions of classes of musical scales generated by vertex shifts, for all $n$, in particular for the 12-TET tuning system.

Keywords: symbolic dynamics; scales; classification; analytic combinatorics; generating functions

2010 Mathematics Subject Classifications: 37B10; 05A15; 00A65

1. Introduction and main result

A symbolic dynamical scale is a musical scale obtained from an infinite sequence of symbols, according to a certain coding rule. For example, the Thue-Morse scales (Gómez and Nasser 2020), and the Fibonacci and Feigenbaum scales (Gómez Aíza 2021), are defined in terms of the Thue-Morse, Fibonacci, and Feigenbaum binary sequences, with coding rule the binary representation of scales. This rule can be generalized to sequences over larger alphabets, for example, as the first symbol rule, that likewise generates scales on every $n$-TET tuning system, and it essentially consists on declaring the return times to a distinguished symbol as the number of tone measures within two consecutive pitch classes in a scale. The coding rules can be applied to sets of symbolic sequences. Our goal here is to study symbolic dynamical scales arising from vertex shifts, with the number of pitch classes on each scale as a parameter, together with their modes. We will establish the framework that leads us to a decomposition theorem of the generating functions of the musical scales that these dynamical systems define. The algorithmic constructions and the exact counting methods presented here can be used to implement and test applications aimed as tools for musicians to work out compositions and arrangements, and also for designing systematic methodologies of study and practice.

A vertex shift is an instance of a shift space, which is a set of (bi)infinite sequences of symbols that avoid a given set $\mathcal{F}$ of forbidden finite configurations, and they come with a natural $\mathbb{Z}$-action $\sigma$ by translation. A shift space is characterized by its language, which is defined as the union of the admissible configurations that occur in its sequences. Then a shift space $X$ can be constructed by specifying either its language $\mathcal{L}(X)$ (e.g. as the admissible blocks in a sequence

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like the Thue-Morse sequence), or a forbidden set $\mathcal{F}$. A shift of finite type is a shift space that can be defined by a finite forbidden set, and vertex shifts are shifts of finite type that can be defined by forbidden sets whose elements are 2-blocks. Shifts of finite type, in particular vertex shifts, are well-understood symbolic dynamical systems: they possess matrix representations that provide algebraic and analytic tools to study dynamical properties like entropy, periodic points and their zeta functions, etc. (Lind and Marcus 1995; Kitchens 1998).

The modes of musical scales can be thought as instances of orbit equivalence classes of a cyclic shift action $\alpha$ on finite sequences over some (countable) alphabet $S$. Orbitals are unions of $\alpha$-orbit equivalence classes, and they are the subsets upon which $\alpha$ acts. In general, arbitrary subsets of the full language $S^*$ are not orbitals, like the language of a shift space. Likewise, if the first symbol rule is formalized as a block function $\varphi : \mathcal{L}(X) \rightarrow S^*$ valued on finite symbolic sequences over some alphabet $S$ that represent musical scales in a way that the $\alpha$-orbits correspond to the modes of the scales (for example, see equation (6)), then $\varphi(\mathcal{L}(X))$ is not, in general, an orbital. Thus we consider orbitals generated by subsets $B \subseteq S^*$ as unions of the $\alpha$-orbit equivalence classes of their elements, and a transversal is a set of representatives of the generated orbital.

**Remark 1.1** If $B$ is a set of musical scales, like $\varphi(\mathcal{L}(X))$, then the cardinality of a transversal is the number of “essentially different” scales an instrumentalist would have to learn to play any scale in $B$, together with all its modes, for a total number of scales that corresponds to the cardinality of its generated orbital.

We will refer to these cardinalities as transversal and orbital dimensions. Since any set $B$ decomposes into a sequence $(B_n)_{n \geq 0}$ with $B_n \doteq B \cap S^n$, there are transversal and orbital generating functions $\dim_T^B(z)$ and $\dim_O^B(z)$, respectively. Thus we aim to find transversal and orbital generating functions of classes of musical scales generated by shift spaces. We use integer compositions as the main combinatorial model for the class of all musical scales, in particular because their $\alpha$-orbits represent the modes of the scales (as wheels). Integer compositions are represented as sequences of positive integers, their generating functions, including bivariate versions marking several parameters like the number of summands, are well known (Flajolet and Sedgewick 2009). With them (see Theorem 2.1), and the interplay of the $\sigma$-action on sequences and the $\alpha$-action on languages, it is possible to deduce transversal and orbital generating functions of musical scales generated by vertex shifts (with reference to Remark 1.1):

**Theorem 1.2** Let $X \subseteq \mathcal{A}^\mathbb{Z}$ be an irreducible vertex shift and choose a symbol $s \in \mathcal{A}$. Then there is a set $\mathcal{K}(s) \subseteq \mathbb{N}_{>0}$ of positive integers (see equation (13)) that yields decompositions of the transversal and orbital generating functions of all musical scales in $\varphi(\mathcal{L}(X, s))$, where $\mathcal{L}(X, s)$ is the language of all admissible words in $X$ that start with $s$. These decompositions are

$$\dim_T^{\varphi(X, s)}(z) = W^{\mathcal{K}(s)}(z) + a^{\mathcal{K}(s)}(z), \quad (1)$$

$$\dim_O^{\varphi(X, s)}(z) = C^{\mathcal{K}(s)}(z) + b^{\mathcal{K}(s)}(z), \quad (2)$$

where the four generating functions on the right-hand sides above are as follows:

1. $C^{\mathcal{K}(s)}(z)$ and $W^{\mathcal{K}(s)}(z)$ are the generating functions of integer compositions and wheels, respectively, both with summands in $\mathcal{K}(s)$ (see equations (4) and (5) evaluated at $u = 1$).
2. $a^{\mathcal{K}(s)}(z)$ is the generating function of aperiodic compositions, also with summands in $\mathcal{K}(s)$, except for the last one that belongs to the complement $\mathcal{K}(s)^c$ and is bounded above by an element of $\mathcal{K}(s)$ (see equation (14)).

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1 In fact, the language of a shift space is an $\alpha$-orbital if and only if the space is a full shift.
Figure 1. Number of pitch classes versus number of essentially different scales for the 12-TET tuning system. This is in general represented for every \( n \) by the coefficient \( [z^n]W(z,u) \), where \( W(z,u) \), given in equation (5), is the bivariate version of \( W(z) \) (when \( K = \mathbb{N}_{>0} \)). Thus, for example, 

\[
[z^{12}]W(z,u) = u + 6u^2 + 19u^3 + 43u^4 + 66u^5 + 80u^6 + 66u^7 + 43u^8 + 19u^9 + 6u^{10} + u^{11} + u^{12}.
\]

The limiting distribution of the number of pitch classes is Gaussian as \( n \rightarrow \infty \).

(3) *The orbital generating function* \( b^{K(s)}(z) \) *associated to the class represented by* \( a^{K(s)}(z) \) *is, in fact, the corresponding cumulative generating function with respect to the number of pitch classes (see equation (17)).*

The proof follows from the definitions and the formulations in the rest of the paper, which is organized as follows. In Section 2, we define the class of all musical scales as combinatorially isomorphic to the integer compositions, and also define their modes, orbitals, transversals, and their dimensions. In Section 3, we address shift spaces and the classes of musical scales they define. We recall periodic points and zeta functions and then focus on vertex shifts and first return loop systems. Finally, we settle the decompositions in Theorem 1.2. To illustrate our methods, in Section 4, we develop in detail one example: the *golden mean shift*. Our formalism is general for all \( n \)-TET tuning systems and adapts for finite values of \( n \) to numerical procedures for exact computations, for example, when \( n = 12 \), which is of most interest from the musical point of view. Our results can serve to construct and test applications aimed at many different aspects of music theory, composition, education, and so forth (for example, Figure 2 was generated with the “Scalescope”, a software application developed by the author to study random fields of scales). In Section 5, we make final remarks and conclusions, with respect to other works, possible generalizations, and further applications.

2. Scales, modes, orbitals, and transversals

2.1. Musical scales and integer compositions

A musical scale in 12-TET tuning system can be coded by a sequence of integers in which each term is the number of half tones within two consecutive pitch classes of the scale that are ordered in ascending order. For example, the chromatic and major scales, that are coded with a binary alphabet \( B \triangleq \{0,1\} \) as 111111111111 and 101011010101, correspond to \((1,1,1,1,1,1,1,1,1,1,1,1)\) and \((2,2,1,2,2,2,1)\) in the integer sequence representation, respectively. Observe that in both cases the sums of the entries yields 12. This is a general phenomenon that, though elementary by definition, we state formally:

**Theorem 2.1** *In* \( n \)-TET *tuning system, the musical scales are in bijective correspondence with the set of ordered sequences of positive integers that add up to* \( n \). *In other words, the set of all musical scales (in any tuning system) is combinatorially isomorphic to the combinatorial class of integer compositions.*
Let $\mathbb{N}_{>0} \triangleq \{1, 2, 3, \ldots\}$ denote the set of positive integers. Let

$$C \triangleq \text{SEQ}(\mathbb{N}_{>0}) = \bigcup_{k=0}^{\infty} \mathbb{N}^k_{>0}$$

denote the class of integer compositions and henceforth think of its elements as musical scales. For every integer $n \geq 0$, let $C_n \subseteq C$ be the compositions of $n$, i.e. $C_n$ denotes the set of scales in $n$-TET tuning system. Then $C_n \triangleq \#C_n = 2^{n-1}$, which is consistent with the binary representation of musical scales. Thus, the ordinary generating function (OGF) of all musical scales is the rational function

$$C(z) \triangleq \sum_{n=0}^{\infty} C_n z^n = \frac{1-z}{1-2z}.$$ 

For any arbitrary sequence $x = (x_1, \ldots, x_k)$, its length is denoted by $\ell(x) \triangleq k$ (e.g. when coded by integer compositions, the major and chromatic scales have lengths 7 and 12, respectively, but on the other hand, both have length 12 in binary code). More generally, for any $K \subseteq \mathbb{N}_{>0}$, the class $C^K \subseteq C$ of all integer compositions with summands in $K$, together with the length $\ell: C^K \rightarrow \mathbb{N}$ as a parameter, has bivariate generating function (BGF)

$$C^K(z, u) \triangleq \sum_{n,m \geq 0} C^K_{n,m} z^n u^m = \frac{1}{1-u \sum_{k \in K} z^k},$$

where $C^K_{n,m} \triangleq \#\{w \in C^K_n : \ell(w) = m\}$ and $C^K_n \triangleq \#(C^K \cap C_n)$. In particular, the OGF of $C^K$ is

$$C^K(z) \triangleq \sum_{n=0}^{\infty} C^K_n z^n = C^K(z, 1).$$

### 2.2. Modes, orbitals, transversals, and their dimensions

Let $A$ be a countable alphabet and then let $A^* \triangleq \bigcup_{k \geq 0} A^k$, where $A^k \triangleq A \times \cdots \times A$. Let $\alpha: \mathbb{Z} \rhd A^*$ be the cyclic left shift action induced by the combinatorial isomorphism $\alpha: A^* \rightarrow A^*$ defined for every $w = (w_1, \ldots, w_k) \in A^k$ by $\alpha(w) \triangleq (w_2, \ldots, w_k, w_1) \in A^k$, for all $k \geq 1$. The $\alpha$-orbit of $w \in A^*$ is $O_\alpha(w) \triangleq \{\alpha^j(w) : \forall j \in \mathbb{Z}\}$, and $\#O_\alpha(w)$ is its period. Then $\#O_\alpha(w) | \ell(w)$ and $w$ is primitive if $\#O_\alpha(w) = \ell(w)$. The set of $\alpha$-orbits forms a partition of $A^*$ induced by the $\alpha$-orbit equivalence relation $\sim$. The representation of musical scales by integer compositions is such that the $\alpha$-orbit equivalence class of an integer composition $w \in C$, i.e. the elements of its $\alpha$-orbit $O_\alpha(w)$, are the modes of the corresponding scale,\(^2\) and thus, in this case, we write $\text{modes}(w) \triangleq O_\alpha(w)$. For any subset $B \subseteq A^*$, let $O_\alpha(B) \triangleq \bigcup_{w \in B} O_\alpha(w)$, and similarly, if $B \subseteq C$, then we write $\text{modes}(B) \triangleq O_\alpha(B)$. Now, since $A^*/\overset{\sim}{\sim}$ is the combinatorial class of cycles of elements of $A$, the class of all musical scales, modulo their modes, is the class $W$ of cyclic compositions of positive integers, the so-called wheels. For example, the diatonic wheel $(2, 2, 1, 2, 2, 2, 1)$ has size 12, length 7, it is aperiodic, thus it consists of 7 modes. Two musical scales are essentially different (or independent) if they are different as wheels. Therefore, the

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\(^2\) This is not generally the case. For example, in the binary representation of musical scales, the $\alpha$-orbits do not always correspond to the modes of the scales.
OGF of all musical scales, modulo their modes, is

\[
W(z) \triangleq \sum_{k=1}^{\infty} W_k z^n = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \left( 1 - \frac{z^k}{1-z^k} \right)^{-1} = z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 13z^6 + \ldots,
\]

with \( \phi : \mathbb{N}_{>0} \to \mathbb{N}_{>0} \) the Euler totient function, that is, \( \phi(n) \triangleq \# \{ k \leq n : \gcd(n, k) = 1 \} \). More generally, the BGF of the class \( \mathcal{W}_K \) of wheels with summands in \( K \), with \( u \) marking the length of the wheels (i.e. the number of pitch classes in the scales), is

\[
W^K(z, u) \triangleq \sum_{n,m=1}^{\infty} W^K_{n,m} z^n T^m = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \frac{1}{1-\sum_{j \in K} u^j z^j}. \tag{5}
\]

For example, in 12-TET tuning system, there are 351 essentially different musical scales, and their distribution according to the number of notes is illustrated in Figure 1.

A set \( A \subseteq A^* \) is independent if any pair of distinct elements of \( A \) belong to distinct \( \alpha \)-orbit equivalence classes, i.e. \( O_{\alpha}(v) \cap O_{\alpha}(w) = \emptyset \) for all \( v, w \in A \) with \( v \neq w \). Two subsets \( A, B \subseteq A^* \) are mutually independent if \( O_{\alpha}(A) \cap O_{\alpha}(B) = \emptyset \) (each set \( A \) and \( B \) may or may not be independent). A transversal of \( A \) is a maximal independent subset \( T_A \subseteq A \). Any nonempty set \( A \neq \emptyset \) possesses at least one transversal \( T_A \subseteq A \), and any two transversals of \( A \) have the same cardinality, the transversal dimension \( \dim_T(A) \triangleq \# T_A \). Clearly, \( A \subseteq O_{\alpha}(A) = O_{\alpha}(T) \) and \( O_{\alpha}(T') \subseteq O_{\alpha}(T) \) for any transversal \( T \subseteq A \) and any proper (independent) subset \( T' \subset T \). Hence, if \( A \subseteq \mathcal{C} \) is a set of integer compositions, then the transversal dimension is the number of essentially different scales an instrumentalist would have to learn to play any scale in \( \text{MODES}(A) \) (see Remark 1.1). The orbital dimension of \( A \) is \( \dim_\alpha(A) \triangleq \# O_{\alpha}(A) \). Again for subsets \( A \subseteq \mathcal{C} \) of integer compositions, the orbital dimension \( \dim_\alpha(A) = \# \text{MODES}(A) \) is the total number of scales that an instrumentalist can play with the elements of \( A \). Thus, the orbital dimension \( \dim_\alpha(w) \triangleq \# O_{\alpha}(w) \) of \( w \in A^* \) is bounded above by \( \ell(w) \). Moreover, the former divides the latter, i.e. \( \dim_\alpha(w) | \ell(w) \), thus there is an integer \( k = k(w) \geq 1 \) such that \( \dim_\alpha(w) \cdot k = \ell(w) \), \( w \) is primitive if \( k = 1 \). The orbital dimension of \( A \) is computed for any \( \alpha \)-transversal \( T \subseteq A \) as \( \dim_\alpha(A) = \sum_{w \in T} \dim_\alpha(w) \).

Figure 2. The set \( O_{\alpha}^{(X, \bullet)} \) decomposes into the set of scales that do not have 2nd minors (above, there are 89), and the set of scales that have exactly one second minor, located at the end of the scale (below, there are 55). Here, a scale is represented as a polygon in a circle of 2nd minors.
3. Symbolic dynamical scales

3.1. Shift spaces and their musical scales

A shift space $X \subseteq A^\mathbb{Z}$ is determined by a set of forbidden blocks $\mathcal{F} \subseteq A^*$, that is, $X = X_\mathcal{F}$ where $X_\mathcal{F} \triangleq \{ x = (x_n)_{n \in \mathbb{Z}} \in A^\mathbb{Z} : \forall w \in \mathcal{F}, \ \forall k \in \mathbb{Z}, \ x_{[k,k+(w)-1]} \neq w \}$ (here and henceforth, for any sequence $x, x_{[i,j]} \triangleq x_i \ldots x_j \triangleq (x_i, \ldots, x_j)$), and is accompanied by the left shift $\mathbb{Z}$-action $\sigma : \mathbb{Z} \curvearrowright X$ induced by the automorphism $\sigma(x)_n \triangleq x_{n+1} \ \forall x = (x_n)_{n \in \mathbb{Z}} \in X, \ \forall n \in \mathbb{Z}$. The language of a shift space $X$ is $\mathcal{L}(X) \triangleq \bigcup_{n \geq 0} \mathcal{L}_n(X) \subseteq A^*$, where $\mathcal{L}_n(X) \triangleq \{ x_{[1,n]} \in A^n : x \in X \}$, and also, for every symbol $s \in A$, let $\mathcal{L}(X,s) \triangleq \bigcup_{n \geq 1} \mathcal{L}_n(X,s)$, where $\mathcal{L}_n(X,s) \triangleq \{ x_{[1,n]} \in \mathcal{L}_n(X) : x_1 = s \}$. $X$ is irreducible if for every $u, w \in \mathcal{L}(X)$, there exists $v \in \mathcal{L}(X)$ such that $uvw \in \mathcal{L}(X)$.

The first symbol rule $\varphi : \mathcal{L}(X) \rightarrow C$ is defined for each $w = w_1 \ldots w_n \in \mathcal{L}_n(X)$ as follows. Let $s \triangleq w_1$ and then let $1 = n_1 < n_2 < \ldots < n_{r(w)} \leq n$ be the coordinates where $s$ occurs in $w$, that is, $w_j = s$ if and only if $j = n_i$ for some $i = 1, \ldots, r(w)$. Then $\varphi(w)$ is a composition of $n$, has length $\ell(\varphi(w)) = r(w)$, and is defined by

$$\varphi(w) \triangleq (n_2 - n_1, n_3 - n_2, \ldots, n_{r(w)} - n_{r(w) - 1}, n - n_{r(w) + 1}).$$

(6)

For every $s \in A$, let $C^{(X,s)} \triangleq \varphi(\mathcal{L}(X,s))$ and $C^{(X)} \triangleq \varphi(\mathcal{L}(X))$, and for every $n \geq 1$, also let $C^{(X,s)}_n \triangleq \varphi(\mathcal{L}_n(X,s))$ and $C^{(X)}_n \triangleq \varphi(\mathcal{L}_n(X))$. Then we define the OGFs

$$C^{(X,s)}(z) \triangleq \sum_{n \geq 0} C^{(X,s)}_n z^n \text{ and } C^{(X)}(z) \triangleq \sum_{n \geq 0} C^{(X)}_n z^n,$$

where $C^{(X,s)}_n \triangleq \#C^{(X,s)}_n$ and $C^{(X)}_n \triangleq \#C^{(X)}_n$. We are concerned with the transversal and orbital BGFs

$$\dim^{(X)}_T(z,u) \triangleq \sum_{n,m \geq 0} \dim_T(C^{(X)}_{n,m}) z^n u^m, \quad \dim^{(X,s)}_T(z,u) \triangleq \sum_{n,m \geq 0} \dim_T(C^{(X,s)}_{n,m}) z^n u^m,$$

$$\dim^{(X)}_O(z,u) \triangleq \sum_{n,m \geq 0} \dim_O(C^{(X)}_{n,m}) z^n u^m, \quad \dim^{(X,s)}_O(z,u) \triangleq \sum_{n,m \geq 0} \dim_O(C^{(X,s)}_{n,m}) z^n u^m.$$

3.2. Periodic points and zeta functions

The $\sigma$-orbit of $x \in X$ is $O_\sigma(x) \triangleq \{ \sigma^n(x) : n \in \mathbb{Z} \} \subseteq X$. A point $x \in X$ is $n$-periodic for some $n \geq 1$ if $\sigma^n(x) = x$, that is, if $O_\sigma(x)$ and $\#O_\sigma(x) \mid n$, and in this case let $\#O_\sigma(x)$ be the minimal period of $x$. Let $P_n(X) \triangleq \{ x \in X : \sigma^n(x) = x \}$ and $Q_n(x) \triangleq \{ x \in P_n(X) : \#O_\sigma(x) = n \}$ be the sets of $n$-periodic points and minimal $n$-periodic points, respectively, and also let $p_n(X) \triangleq \#P_n(X)$ and $q_n(X) \triangleq \#Q_n(X)$. Recall that the relationship between $p_n(X)$ and $q_n(X)$ is through Mőbius inversion, namely

$$p_n(X) = \sum_{k \mid n} q_k(X) \quad \text{and} \quad q_n(X) = \sum_{k \mid n} \mu \left(\frac{n}{k}\right) p_k(X),$$

(7)

where $\mu : \mathbb{N}_0 \rightarrow \{-1,0,1\}$ is the Mőbius function defined by

$$\mu(n) \triangleq \begin{cases} 
0 & \text{if there exists } p \geq 2 \text{ such that } p^2 \mid n, \\
(-1)^r & \text{if } n = p_1 \cdots p_r \text{ with } p_1, p_2, \ldots, p_r \geq 2 \text{ distinct prime numbers}.
\end{cases}$$

The dynamic zeta function of $X$ is defined by

$$\zeta_X(z) \triangleq \exp \left( \sum_{n=1}^{\infty} \frac{p_n(X)}{n} z^n \right),$$
and it is an exercise to show that
\[ \zeta_X(z) = \prod_{n \geq 1} \frac{1}{(1 - z^n)^{q_n(X)/n}} \]
(see section 6.4 in Lind and Marcus 1995). The shift spaces that the Thue-Morse, Fibonacci, and Feigenbaum sequences define have no periodic points and thus they have the trivial zeta function 1. Vertex shifts, however, have lots of periodic points in general.

3.3. Vertex shifts and loop systems

A vertex shift is a shift space \( X = X_\mathcal{F} \) that can be defined by a set of forbidden 2-blocks \( \mathcal{F} \subseteq A^2 \). Let \( A \) be the square \( \{0, 1\} \)-matrix indexed by \( A \) and defined by the rule \( A(i,j) = 1 \) if and only if \( ij \notin \mathcal{F} \). Then \( X = \hat{X}_A \), where \( \hat{X}_A \triangleq \{ x = (x_n)_{n \in \mathbb{Z}} \in A^\mathbb{Z} : \forall n \in \mathbb{Z}, A(x_n, x_{n+1}) \neq 0 \} \). The matrix representation of vertex shifts yields expressions that can be useful to study transversal and orbital dimensions. For example, the dynamic zeta function of the vertex shift \( \hat{X}_A \) is obtained by the following formula (see, e.g. Theorem 6.4.6 in Lind and Marcus 1995):
\[ \zeta_{\hat{X}_A}(z) = \frac{1}{\det(I - zA)} \]
(observe that \( \det(I - zA) = z^{#A} \chi_A(z^{-1}) \), where \( \chi_A(z) \) is the characteristic polynomial of the matrix \( A \)). From here we can get
\[ p_n(\hat{X}_A) = \frac{1}{(n-1)!} \left. \frac{d^n}{dz^n} \log \zeta_{\hat{X}_A}(z) \right|_{z=0} = \text{trace}(A^n), \]
and then we can compute \( q_n(\hat{X}_A) \) using Möbius inversion (see equation (7)). The following result follows.

**Theorem 3.1** **The nth transversal dimension of the language of a vertex shift is**

\[ \dim_T(L_n(\hat{X}_A)) = \sum_{i,j \in A, A_{ij}=0} A_{ij}^{n-1} + \sum_{k|n} q_k(\hat{X}_A) \]
and the corresponding nth orbital dimension is

\[ \dim_O(L_n(\hat{X}_A)) = n \sum_{i,j \in A, A_{ij}=0} A_{ij}^{n-1} + p_n(\hat{X}_A). \]

Now, for studying musical scales arising from languages of vertex shift spaces through the distinguished symbol rule, consider the first return loop system to a given symbol \( s \in A \), defined by the OGF \( f^{(s)}(z) \triangleq \sum_{k=1}^{\infty} f_k^{(s)} z^k \) with coefficients
\[ f_k^{(s)} \triangleq \# \{ w = w_0 \ldots w_k \in L_{k+1}(X) : w_0 = s, w_k = s, \text{ and } w_j \neq s, \forall j \neq 0, k \}, \]
and that is obtained through the equation
\[ 1 - f^{(s)}(z) = \frac{\zeta_{\hat{X}_s}(z)}{\zeta_{\hat{X}_A}(z)}, \]
where \( B \) is the square \( \{0, 1\} \)-matrix indexed by \( A \setminus \{s\} \) and obtained from \( A \) by removing the row and column indexed by \( s \).
3.4. Generating functions for distinguished symbol rule on vertex shifts

Here we prove Theorem 1.2. Let

\[ \mathcal{K}(s) \triangleq \{ k \geq 1 : f_k^{(s)} \neq 0 \} \]  

and also denote its complement by \( \mathcal{K}(s)^c \triangleq \mathbb{N}_{>0} \setminus \mathcal{K}(s) \). According to equations (6) and (11), if \( w \in \mathcal{L}(X, s) \), then \( \varphi(w) = (k_1, k_2, \ldots, k_{\ell(\varphi(w))}) \) is a composition of \( \ell(w) \), with summands in \( \mathcal{K}(s) \), except perhaps for the last summand \( k_{\ell(\varphi(w))} \). Suppose that this is the case, that is, \( k_{\ell(\varphi(w))} \in \mathcal{K}(s)^c \). Since \( X \) is irreducible, there exists \( v \in \mathcal{L}(X, s) \) such that \( v \) also ends in \( s \) and \( w \) is a prefix of \( v \), that is, \( \nu(\nu(v)) = s \) and \( v = w\nu(\nu(w)+1, \nu(v)) \), thus \( k_{\ell(\varphi(w))} \) is bounded above by an element of \( \mathcal{K}(s) \). Let \( a^{\mathcal{K}(s)}_n(z) \triangleq \sum_{n \geq 1} a^{\mathcal{K}(s)}_n z^n \) be the OGF of this subclass which is described in item (2) of Theorem 1.2. Then

\[ a^{\mathcal{K}(s)}_n = \sum_{k \in \mathcal{K}(s)^c} \left( C^{\mathcal{K}(s)}_n \sum_{k' \in \mathcal{K}(s)} C^{\mathcal{K}(s)}_{n-k'} \right) \]  

(for any subset \( \mathcal{K} \neq \emptyset \) of positive integers, \( C^{\mathcal{K}}_0 \triangleq 1 \). If we also let \( b^{\mathcal{K}(s)}_n(z) \triangleq \sum_{n \geq 1} b^{\mathcal{K}(s)}_n z^n \) be the OGF of the corresponding orbital, then, by independence, there is a decomposition of the form given in equations (1) and (2), as described in Theorem 1.2, we just need to justify that \( b^{\mathcal{K}(s)}_n(z) \) is the cumulative generating function of the subclass represented by \( a^{\mathcal{K}(s)}_n(z) \), with respect to the number of pitch classes. This follows from the fact that the elements represented by \( a^{\mathcal{K}(s)}_n(z) \) are aperiodic. To be explicit, write the bivariate coefficients

\[ \dim_T(C^{(X,s)}_{n,m}) = W^{\mathcal{K}(s)}_{n,m} + d^{\mathcal{K}(s)}_{n,m} \quad \text{and} \quad \dim_0(C^{(X,s)}_{n,m}) = C^{\mathcal{K}(s)}_{n,m} + b^{\mathcal{K}(s)}_{n,m}. \]  

Then, for every \( n, m \geq 1 \), we have

\[ a^{\mathcal{K}(s)}_{n,m} = \sum_{k \in \mathcal{K}(s)^c} C^{\mathcal{K}(s)}_{n-k,m-1}. \]  

By aperiodicity, the corresponding orbital dimension is \( b^{\mathcal{K}(s)}_{n,m} = m \cdot a^{\mathcal{K}(s)}_{n,m} \). Thus, if we define

\[ a^{\mathcal{K}(s)}(z, u) \triangleq \sum_{n,m \geq 1} a^{\mathcal{K}(s)}_{n,m} z^n u^m \]  

and

\[ b^{\mathcal{K}(s)}(z, u) \triangleq \sum_{n,m \geq 1} b^{\mathcal{K}(s)}_{n,m} z^n u^m, \]  

then we observe that

\[ b^{\mathcal{K}(s)}(z) = \frac{\partial}{\partial u} a^{\mathcal{K}(s)}(z, u)|_{u=1}, \]  

and in fact

\[ b^{\mathcal{K}(s)}(z, u) = u \frac{\partial}{\partial u} a^{\mathcal{K}(s)}(z, u). \]  

Hence \( b^{\mathcal{K}(s)}(z) \) is the cumulative generating function of the number of summands in the class of compositions represented by \( a^{\mathcal{K}(s)}(z) \). This settles Theorem 1.2, and also gives decompositions.
of the transversal and orbital BGFs,
\[ \dim_t \varphi^{(X,s)}(z,u) = W^{(s)}(z,u) + a^{(s)}(z,u) \]
and
\[ \dim_o \varphi^{(X,s)}(z,u) = C^{(s)}(z,u) + b^{(s)}(z,u) \].

To determine the transversal and orbital dimensions of the whole set of vertex shift scales \( C(X) \triangleq \varphi(\mathcal{L}(X)) = \bigcup_{s \in A} C^{(X,s)} \), it is required to take into account the intersections between each pair of symbols, otherwise multiple counting may occur. The analysis can be done one symbol at the time, adding only new contributions to the cumulative counting.

4. Examples

Example 4.1 (Gómez Aíza (2021)) The Thue-Morse, Fibonacci, and Feigenbaum binary sequences are cataloged in the Online Encyclopedia of Integer Sequences (Sloane 2016), they can be defined by morphisms as follows (Kurka 2003; Allouche and Shallit 2003):

\[ \varphi \triangleq \{ \circ \mapsto \bullet, \bullet \mapsto \circ \} \]
\[ \phi \triangleq \{ \circ \mapsto \bullet, \bullet \mapsto \bullet \} \]
\[ \psi \triangleq \{ \circ \mapsto \bullet, \bullet \mapsto \circ \} \]

\[ m \triangleq \lim_{n \to \infty} \varphi^n(\circ) \]
\[ f \triangleq \lim_{n \to \infty} \phi^n(\circ) \]
\[ g \triangleq \lim_{n \to \infty} \psi^n(\circ) \]

The sets of scales they define under the first symbol rule have transversal and orbital dimensions \((8, 49)\), \((10, 66)\), and \((6, 28)\), respectively, and the following are transversals:

(1) Thue-Morse:
- \( \ell = 6 \) \((3, 2, 1, 3, 1, 2)\)
- \( \ell = 7 \) \((1, 3, 1, 2, 3, 1, 1)\)

(2) Fibonacci:
- \( \ell = 5 \) \((3, 3, 2, 3, 1)\)
- \( \ell = 7 \) \((2, 2, 1, 2, 1, 2, 2)\)
- \( \ell = 8 \) \((1, 2, 1, 2, 1, 2, 1)\)

(3) Feigenbaum:
- \( \ell = 3 \) \((4, 4, 4)\)
- \( \ell = 5 \) \((2, 2, 4, 2)\)
- \( \ell = 9 \) \((2, 1, 1, 2, 1, 1, 2, 1, 1)\)

The three classes of scales in Example 4.1 are mutually independent. The shift spaces that each of the sequences \( m, f, \) and \( g \) define are all zero entropy systems. Their admissible words, and hence the scales as their images under the first symbol rule, are found exhaustively. We know no explicit formulas for the generating functions in these cases. For vertex shifts, we have explicit formulations and will work out the example of the golden mean shift, which is probably the most well-known non-trivial vertex shift space.

Definition 4.2 For every \( k \geq 1 \), let \( \{F_n^{(k)}\}_{n \geq 0} \) be the \( k \)-Fibonacci sequence defined by
\[ F_0^{(k)} \triangleq 1, \quad F_1^{(k)} \triangleq k, \quad \text{and} \quad F_{n+2}^{(k)} \triangleq F_n^{(k)} + F_{n+1}^{(k)}. \]

For example, \( (\ell(\phi^n(w_n)))_{n \geq 0} = (1, 2, 3, 5, 8, \ldots) \) is 1-Fibonacci.

3 Also known as the “period doubling” sequence, see A096268 (Sloane 2016).
Example 4.3 (Golden mean scales) Consider the golden mean shift $X \triangleq X_\mathcal{F} \subseteq \mathcal{B}^\mathbb{Z}$ that is defined as the shift space over the binary alphabet $\mathcal{B} = \{\circ, \bullet\}$ that results from the forbidden set of blocks $\mathcal{F} = \{\bullet\cdot\}$. Thus $X = \hat{X}_{\mathcal{A}}$, where the matrix $A = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ is the adjacency matrix of the following directed graph:

\[
\begin{array}{c}
\circ \\
\bullet \\
\end{array} 
\]

First we will do the analysis of the generating functions of the language of $X$, and then we will turn into the scales that this language generates.

4.1. Transversals and orbitals of the golden mean language

First let us illustrate the use of Theorem 3.1. The counting sequence of the language of the golden mean shift is 2-Fibonacci, that is, $\# \mathcal{L}_n(X) = F_n^{(2)}$ for all $n \geq 0$. On the other hand, the zeta function

\[ \zeta_X(z) = \frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} F_n^{(1)} z^n \]

is the OGF of the 1-Fibonacci sequence, and the periodic counting sequence $(p_n(X) = F_{n-1}^{(3)})_{n \geq 1}$ is 3-Fibonacci. For transversals dimensions, with reference to equation (9), we see that

\[ \sum_{i,j \in \mathcal{A}} A_{ij} = A_{\bullet,\bullet}^{n-1} \]

represents return loops of length $n$ that begin and end at $\bullet$, but these are in fact sequences of first return loops to $\bullet$. Using equations (12) and (8), we get the OGF of the system of first return loops to $\bullet$, namely

\[ f^{(\bullet)}(z) = \frac{z^2}{1-z}, \]

and deduce that

\[
\sum_{n=1}^{\infty} A_{\bullet,\bullet}^{n-1} z^n = z + z f^{(\bullet)}(z) = \frac{z(1-z)}{1-z-z^2} = z + z^3 + z^4 + 2z^5 + 3z^6 + 5z^7 + 8z^8 + 13z^9 + \cdots ,
\]

in particular, for $n \geq 3$, $A_{\bullet,\bullet}^n = F_{n-3}^{(2)}$ is 2-Fibonacci. Next, the sequence $(q_n(X))_{n \geq 1}$, obtained from $(p_n(X) = F_{n-1}^{(3)})_{n \geq 1}$ by Möbius inversion, defines a minimal periodic OGF

\[ q_n^{(X)}(z) \triangleq \sum_{n \geq 1} q_n(X) z^n = z + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 4z^7 + 5z^8 + 8z^9 + \cdots \]
that corresponds to A006206\(^4\) (Sloane 2016). The coefficients of the OGF

\[
\bar{q}^{(X)}(z) \doteq \sum_{n \geq 1} \left( \sum_{k|n} \frac{q_k(X)}{k} \right) z^n
\]

\[
= z + 2z^2 + 2z^3 + 3z^4 + 3z^5 + 5z^6 + 5z^7 + 8z^8 + 10z^9 + \cdots
\]

(19)
correspond to A000358\(^5\) (Sloane 2016). Thus the transversal OGF of the language of \(X\) has the form

\[
\dim_{t}^{(X)}(z) = \frac{z(1-z)}{1-z-z^2} + \bar{q}^{(X)}(z).
\]

For orbital dimensions, with reference to equation (10), first we see that

\[
\sum_{n=1}^{\infty} n A_{\bullet, \bullet}^{n-1} z^{n-1} = \frac{d}{dz} \left( \frac{z(1-z)}{1-z-z^2} \right) = \frac{1-2z+2z^2}{(1-z-z^2)^2}
\]

\[
= 1 + 3z^2 + 4z^3 + 10z^4 + 18z^5 + 35z^6 + 64z^7 + 117z^8 + 210z^9 + \cdots,
\]

which corresponds to A006490 (Sloane 2016). We already know that \(p_n(X)\) is the 3-Fibonacci sequence \(F_n^{(3)}\). Thus

\[
\dim_{o}^{(X)}(z) = \frac{1-2z+2z^2}{(1-z-z^2)^2} + z \frac{1+2z}{1-z-z^2} = \frac{z(2-z-z^2-2z^3)}{(1-z-z^2)^2}
\]

\[
= 2z + 3z^2 + 7z^3 + 11z^4 + 21z^5 + 36z^6 + 64z^7 + 117z^8 + 193z^9 + \cdots
\]

and this sequence of coefficients now corresponds to A339610 (Sloane 2016).

### 4.2. Admissible golden mean scales

Now we look at the image \(\varphi(\mathcal{L}(X)) \subseteq \mathcal{C}\) that corresponds to the admissible golden mean scales. First, using again equations (12) and (8), we get the OGFs of the first return loop systems, namely

\[
f^{(\circ)}(z) = z + z^2, \quad (20)
\]

\[
f^{(\bullet)}(z) = \frac{z^2}{1-z} = z^2 + z^3 + z^4 + \ldots. \quad (21)
\]

Then the \(\circ\)-admissible golden mean scales \(C^{(X, \circ)} \doteq \varphi(\mathcal{L}(X, \circ))\) are all the integer compositions with summands in \(\mathcal{K}(\circ) = \{1, 2\}\) (see equation (20)), that is, all the scales with no more than two tone measures of difference between consecutive pitch classes, as expected (in this case we have \(a^{\mathcal{K}(\circ)}(z, u) = 0\), and thus also \(b^{\mathcal{K}(\circ)}(z, u) = 0\), because any element in \(\mathcal{K}(\circ)^6\) is not bounded above by any element of \(\mathcal{K}(\circ)\)). The corresponding OGF for this class of integer compositions, according to equation (4), is

\[
C^{(X, \circ)}(z) = C^{\mathcal{K}(\circ)}(z) = \frac{1}{1-z-z^2}, \quad (22)
\]

thus \(C^{(X, \circ)}_n = F^{(1)}_n\) is 1-Fibonacci. For example, for 12-TET tuning system, the set \(C^{(X, \circ)}_{12}\) consists of all the \(C^{(X, \circ)}_{12} = F^{(1)}_{12} = 233\) scales with only half and whole tones between consecutive pitch

---

\(^4\) Counts the number of *aperiodic* binary necklaces with no subsequence \(\bullet\bullet\), excluding the necklace \(\bullet\).  
\(^5\) Counts the number of necklaces with no subsequence \(\bullet\bullet\), excluding the necklace \(\bullet\).
classes. This set is too large to list here, but we will see that its transversal dimension is 31 (see equation (28)), and in fact any transversal \( T \) of \( C_{12}^{(X_0)} \) possesses only three elements of length 7, e.g. we may choose \( T \) to contain the major scale \((2,2,1,2,2,2,1)\), the melodic minor scale \((2,1,2,2,2,1,2)\), and the Neapolitan major scale \((1,2,2,2,2,1,2)\) (all the modes of these three independent scales also belong to \( C_{12}^{(X_0)} \), and all together form the set of scales in \( C_{12}^{(X_0)} \) of length 7). In a 5-TET tuning system, for example, we would have

\[
\begin{align*}
C_{5}^{(X_0)} = F_{5}^{(1)} = 8, \quad C_{5}^{(X_0)} &= \left\{ (1,1,1,1), (1,1,1,1), (1,1,1,1), (1,2,1,1), (2,1,1,1), (1,2,2), (2,1,2), (2,2,1) \right\}
\end{align*}
\]

(23)

(we have indicated the corresponding integer compositions above each \( \circ \)-admissible binary word, note that they are all the compositions of 5 with summands in \( K(\circ) = \{1,2\} \)). We also have the bivariate version of equation (22), with \( u \) marking the number of notes, namely

\[
\begin{align*}
C^{(X_0)}(z,u) &= C^{K_0}(z,u) = \frac{1}{1 - zu - uz^2}.
\end{align*}
\]

For the \( \bullet \)-admissible golden mean scales \( C^{(X_0)} \triangleq \varphi(\mathcal{L}(X_0)) \), first observe that the class of integer compositions with summands in \( K(\bullet) = \{2,3,4,\ldots\} \) (see equation (21)) has OGF

\[
C^{K(\bullet)}(z) = \frac{1 - z}{1 - z - z^2}.
\]

The bivariate version of \( C^{K(\bullet)}(z) \), with the variable \( u \) marking the number of notes, is

\[
C^{K(\bullet)}(z,u) = \frac{1 - z}{1 - z - uz^2}.
\]

In addition, in this case, the last summand in the elements of \( C^{(X_0)} \) is allowed to be \( 1 \not\in K(\bullet) \) (for example, in 12-TET tuning system, the binary admissible 12-block \( \bullet \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \) yields the \( \bullet \)-admissible golden mean scale \((2,2,2,2,2,1)\)). Therefore, the corresponding generating function is

\[
\begin{align*}
C^{(X_0)}(z) &= C^{K(\bullet)}(z) + zC^{K(\bullet)}(z) = \frac{1 - z^2}{1 - z - z^2},
\end{align*}
\]

and we also get its bivariate version

\[
\begin{align*}
C^{(X_0)}(z,u) &= C^{K(\bullet)}(z,u) + uzC^{K(\bullet)}(z,u) = \frac{(1 + uz)(1 - z)}{1 - z - uz^2}.
\end{align*}
\]

Thus \( C_{0}^{(X_0)} = 1 \) and for \( n \geq 1 \) the sequence of coefficients \( C_{n}^{(X_0)} = F_{n-1}^{(1)} \) is 1-Fibonacci. For the 5-TET tuning systems we get \( C_{5}^{(X_0)} = F_{4}^{(1)} = 5 \) scales conforming the set

\[
C_{5}^{(X_0)} = \left\{ \bullet \circ \circ \circ \circ \circ, \bullet \circ \circ \circ \circ \circ, \bullet \circ \circ \circ \circ \circ \circ \right\} \cup \left\{ \bullet \circ \circ \circ \circ \circ, \bullet \circ \circ \circ \circ \circ \circ \right\}
\]

that decomposes into two parts: the first corresponds to the scales with no consecutive pitch classes with one tone measure of difference between them, and the second results from the first by forming all possible compositions that can result by replacing the last summand \( k > 2 \) by \((k - 1, 1)\). For the 12-TET we get \( C_{12}^{(X_0)} = F_{11}^{(1)} = 144 \) scales conforming the set \( C_{12}^{(X_0)} \), it is not too large and we can illustrate it in Figure 2.
Remark 4.4 The only elements that are both \(\circ\)-admissible and \(\bullet\)-admissible are the compositions of even \(n = 2d\) and odd \(n = 2d + 1\) integers of the form

\[
(2, 2, \ldots, 2) \quad \text{and} \quad (2, 2, \ldots, 2, 1). \tag{25}
\]

Thus, combining equations (22) and (24), we conclude that the OGF of admissible golden mean scales is

\[
C^{(X)}(z) = C^{(X, \circ)}(z) + C^{(X, \bullet)}(z) - \frac{1 + z}{1 - z^2} = \frac{1 - z + z^3}{1 - 2z + z^3}
\]

\[
= 1 + z + 2z^2 + 4z^3 + 7z^4 + 12z^5 + 20z^6 + 33z^7 + 54z^8 + 88z^9 + 143z^{10} + \cdots
\]

(the corresponding coefficients are, essentially, A000071 (Sloane 2016)), and we also get its bivariate version,

\[
C^{(X)}(z, u) = C^{(X, \circ)}(z, u) + C^{(X, \bullet)}(z, u) - \frac{1 + uz}{1 - uz^2}.
\]

Then, for every \(n \geq 1\) we have \(C_n^{(X)} = F_n^{(1)} + F_{n-1}^{(1)} - 1\) (for example, the total numbers of admissible golden means scales in 12-TET and 5-TET tuning systems are \(C_{12}^{(X)} = 233 + 144 - 1 = 376\) and \(C_5^{(X)} = 8 + 5 - 1 = 12\)).

4.3. Transversal and orbital dimensions of golden mean scales

First look at the case when \(s = \circ\). With the ordinary form \(W^K(z) \triangleq W^K(z, 1)\) of equation (5), we obtain the first summand in the right-hand side of equation (1),

\[
W^K(\circ)(z) = z + 2z^2 + 2z^3 + 3z^4 + 3z^5 + 5z^6 + 5z^7 + 8z^8 + 10z^9 + 15z^{10} + 19z^{11} + 31z^{12} + \ldots
\]

with coefficients forming again the sequence A000358 (Sloane 2016) (that is, in this case, we have \(W^K(\circ)(z) = \tilde{q}(X)(z)\), see equation (19)). Since \(a^{K(\circ)}(z) = b^{K(\circ)}(z) = 0\),

\[
\dim_T^{\circ(X, \circ)}(z) = W^{K(\circ)}(z) \tag{26}
\]

and

\[
\dim_{\circ}^{\circ(X, \circ)}(z) = C^{K(\circ)}(z) \tag{27}
\]

(see equation (22)). In particular, for the 12-TET and 5-TET tuning system, the transversal dimensions are

\[
\dim_T\bigl(\varphi\bigl(L_{12}(X, \circ)\bigr)\bigr) = W_{12}^{K(\circ)}(z) = 31 \tag{28}
\]

\[
\dim_T\bigl(\varphi\bigl(L_5(X, \circ)\bigr)\bigr) = W_5^{K(\circ)}(z) = 3, \quad \varphi^{(X, \circ)}(z) \triangleq \left\{ \begin{array}{c} (1,1,1,1,1,1) \ 0 0 0 0 0 0 0 0 0 0 0 0 \ 0 0 0 0 0 0 0 0 0 0 0 0 \ (1,2,2) \end{array} \right\} \tag{29}
\]

and the corresponding orbital dimensions are

\[
\dim_{\circ}\bigl(\varphi\bigl(L_{12}(X, \circ)\bigr)\bigr) = C_{12}^{K(\circ)}(z) = 233
\]

\[
\dim_{\circ}\bigl(\varphi\bigl(L_5(X, \circ)\bigr)\bigr) = C_5^{K(\circ)}(z) = 8 \quad \text{(see again equation (23))}.
\]

Now suppose that \(s = \bullet\) and proceed similarly. For the first summand in the right-hand side of equation (15), use equation (5) to obtain the OGF

\[
W^{K(\bullet)}(z) = z^2 + z^3 + 2z^4 + 2z^5 + 4z^6 + 4z^7 + 7z^8 + 9z^9 + 14z^{10} + 18z^{11} + 30z^{12} + \ldots
\]
with coefficients essentially forming the sequence A032190 \(^6\) (Sloane 2016). For example, for the 12-TET and 5-TET tuning system, we have

\[ W_{12}^{K(*)} = 30, \]
\[ W_5^{K(*)} = 2, \quad W_5^{K(*)} = \{5, (3,2)\}, \]

Next, from equation (14) we get

\[ a_{n}^{K(*)} \triangleq \sum_{k \in K(*)} C_{n-k}^{K(*)} = C_{n-1}^{K(*)} \]

and thus \( a_{1}^{K(*)} = 1, a_{2}^{K(*)} = 0 \) and \( a_{n+3}^{K(*)} = F_{n}^{(1)} \) is the 1-Fibonacci sequence. Hence

\[ a^{K(*)}(z) = \frac{z - z^2}{1 - z - z^2} = z + z^3 + z^4 + 2z^5 + 3z^6 + 5z^7 + 8z^8 + 13z^9 + 21z^{10} + 34z^{11} + 55z^{12} + \ldots. \]

For instance, in 12-TET and 5-TET tuning systems, we have

\[ a_{12}^{K(*)} = 55, \]
\[ a_{5}^{K(*)} = 2, \quad \{5, (3,2)\}. \]

Thus equation (1) in Theorem 1.2 yields the transversal OGF

\[ \dim_{T}^{\psi(X,*)}(z) = W^{K(*)}(z) + \frac{z - z^2}{1 - z - z^2}. \]

For example, in 12-TET and 5-TET tuning system, we have

\[ \dim_{T}(L_{12}(X,\bullet)) = 30 + 55 = 85, \]
\[ \dim_{T}(L_{5}(X,\bullet)) = 2 + 2 = 4, \quad T_{5}^{(X,*)} = \{5, (3,2), (4,1), (2,2,1)\}. \]

For orbital dimensions, first we have

\[ C^{K(*)}(z) = \frac{1 - z}{1 - z - z^2} = 1 + z^2 + z^3 + 2z^4 + 3z^5 + 5z^6 + 8z^7 + 13z^8 + 21z^9 + 34z^{10} + \ldots. \]

For example, in 12-TET and 5-TET tuning systems, we get

\[ C_{12}^{K(*)} = 89, \]
\[ C_{5}^{K(*)} = 3, \quad C_{5}^{K(*)} = \{5, (3,2), (2,3)\}. \]

Next, according to item (3) in Theorem 1.2, we need the cumulative generating function of the class represented by \( a^{K(*)}(z) \), with respect to the number of notes. From equation (16),

\[ \tilde{a}_{n,m}^{K(*)} = \sum_{k \notin K(*)} C_{n-k,m}^{K(*)} = C_{n-1,m}^{K(*)}, \]

\(^6\) Counts the number of cyclic compositions of \( n \) into parts \( \geq 2 \).
thus, using equation (3), we get
\[ a^{K(*)}(z, u) = uz C^{K(*)}(z, u) = \frac{uz - uz^2}{1 - z - uz^2}. \]

Hence, from equation (18) we get
\[ b^{K(*)}(z, u) = \frac{u}{1} a^{K(*)}(z, u) = \frac{uz(1 - z)^2}{(1 - z - uz^2)^2}, \]

and from equation (17) we obtain
\[ b^{K(*)}(z) = \frac{z(1 - z)^2}{(1 - z - z^2)^2} = z + 2z^3 + 2z^4 + 5z^5 + 8z^6 + 15z^7 + 26z^8 + 46z^9 + 80z^{10} + \cdots \]  
(36)

(the coefficients are A006367 (Sloane 2016)). For example, in the 12-TET and 5-TET tuning system (for the latter see equation (30)), we get
\[ b_{12}^{K(*)} = 240, \]
\[ b_5^{K(*)} = 5, \quad \{4, 1\} \cup \{2, 2, 1\}. \]  
(37)
(38)

Hence, equation (2) in Theorem 1.2, together with equations (33) and (36), yield
\[ \dim_0^{\varphi(X, *)}(z) = C^{K(*)}(z) + b^{K(*)}(z) = \frac{(1 - z)(1 - 2z^2)}{(1 - z - z^2)^2} = 1 + z + z^2 + 3z^3 + 4z^4 + 8z^5 + 13z^6 + 23z^7 + 39z^8 + 67z^9 + \cdots, \]  
(39)

which corresponds to A206268 (Sloane 2016) and that is described as the number of compositions with at most one 1. For example, in the 12-TET and 5-TET tuning system (see equations (34), (35), (37), and (38)),
\[ \dim_0(\varphi(L_{12}(X, \bullet))) = 89 + 240 = 329, \]
\[ \dim_0(\varphi(L_5(X, \bullet))) = 3 + 5 = 8, \quad \text{MODES}(\varphi(L_5(X, \bullet))) = \left\{ \begin{array}{l} (5), \quad (3, 2), \quad (2, 3), \\ (4, 1), \quad (1, 4), \\ (2, 2, 1), \quad (2, 1, 2), \quad (1, 2, 2) \end{array} \right\}. \]

For a global transversal, according to Remark 4.3, with equations (26) and (31) we get
\[ \dim_T^{\varphi(X)}(z) = \dim_T^{\varphi(X, \circ)}(z) + \dim_T^{\varphi(X, *)}(z) - \dim_T^{\varphi(X, \circ) \cap \varphi(X, *)}(z) \]
\[ = W^{K(*)}(z) + W^{K(*)}(z) + \frac{z - z^2}{1 - z - z^2} - \frac{z}{1 - z} \]
\[ = z + 2z^2 + 3z^3 + 5z^4 + 6z^5 + 11z^6 + 13z^7 + 22z^8 + 31z^9 + \cdots \]

(we found no record in Sloane (2016) for the corresponding sequence of coefficients). For example, for the 12-TET and 5-TET tuning system, the transversal dimensions of the golden mean
scales are (for the latter see equations (29) and (32))
\[
\dim_{\tau}(\varphi(\mathcal{L}_{12}(X))) = 115 \\
\dim_{\tau}(\varphi(\mathcal{L}_{5}(X))) = 6, \quad T^{(X,\circ)}_5 \cup T^{(X,\bullet)}_5 \triangleq \left\{ \begin{array}{l}
(1,1,1,1), \circ \circ \circ \circ \circ \circ \circ 
(1,1,2), \circ \circ \circ \circ \circ \circ \circ 
(1,2,2), \circ \circ \circ \circ \circ \circ \circ \\
(5), \circ \circ \circ \circ \circ \circ \circ 
(13), \circ \circ \circ \circ \circ \circ \circ 
(2,2,1), \circ \circ \circ \circ \circ \circ \circ 
(3,2), \circ \circ \circ \circ \circ \circ \circ 
(4,1), \circ \circ \circ \circ \circ \circ \circ 
\end{array} \right\}
\]
((1, 2, 2) and (2, 2, 1) are equal as wheels).

Finally, for the global orbital, now we use equations (27) and (39), but first observe that, according to Remark 4.3, in addition to the empty composition, there are two kinds of compositions in the intersection (see equation (25)): one kind is formed by compositions of even integers \( n = 2d \) of length \( d \) and period 1, and the other kind is formed by compositions of odd integers \( n = 2d + 1 \) of length \( d + 1 \) that are aperiodic. We conclude that
\[
\dim_{\varphi(X)}(z) = \dim_{\varphi(X,\circ)}(z) + \dim_{\varphi(X,\bullet)}(z) - \dim_{\varphi(X,\circ) \cap \varphi(X,\bullet)}(z) - 1
\]
\[
= \frac{1}{1 - z - z^2} + \frac{(1-z)(1-2z^2)}{(1-z-z^2)^2} - \frac{z^2}{1-z^2} - \sum_{d=0}^{\infty} (d + 1)z^{2d+1} - 1
\]
\[
= 1 - z - 3z^2 + 3z^3 + 4z^4 - 5z^5 - 2z^6 + 2z^7
\]
\[
= \frac{1 - z - 3z^2 + 3z^3 + 4z^4 - 5z^5 - 2z^6 + 2z^7}{(1 - z - z^2)^2}
\]
\[
= 1 + z + 2z^2 + 4z^3 + 8z^4 + 13z^5 + 25z^6 + 40z^7 + \cdots .
\]

For example, in the 12-TET and 5-TET tuning system, we get
\[
\dim_{\varphi(X)}(\mathcal{L}_{12}(X))) = 561 \\
\dim_{\varphi(X)}(\mathcal{L}_{5}(X))) = 13, \quad \text{MODES}(\varphi(\mathcal{L}_{5}(X))) = \left\{ (1, 1, 1, 1, 1, 1, 1, 2, 1), (1, 1, 2, 1), (1, 1, 2, 1, 1), (1, 1, 2, 2), (1, 1, 2, 3), (1, 2, 1), (2, 1, 1, 1, 1), (1, 2, 2), (2, 2, 1), (2, 1, 2), (5), (3, 2), (2, 3), (4, 1), (1, 4) \right\}
\]

5. Related works and conclusion

We have seen a general method to deduce transversal and orbital generating functions of classes of musical scales induced by vertex shift spaces through the distinguished symbol rule. We can state the same problem for shifts of finite type, and it can be shown to be equivalent to the “distinguished set of symbols rule” for vertex shifts. Enumeration problems in music go back at least to the works of Reiner (1985), Read (1997), and Fripertinger (1999), they form part of the mathematics of music (Jedrzejewski 2006; Benson 2007; Hook 2007). Our methods can serve to complement and generalize several other works that address characterizations and classification of musical scales (Nuño 2020; Kozyra 2014), also octave subdivisions (Hearne, Milne, and Dean 2019), optimal spelling of pitches of musical scales (Bora, Tekin Tezel, and Vahaplar 2019), tuning systems other that 12-TET (Hearne, Milne, and Dean 2019), scales and constraint programming (Hooker 2016), modular arithmetic sequences and scales (Amiot 2015), algebras of periodic rhythms and scales (Amiot and Sethares 2011), formalisms to generate pure-tone systems that best approximate modulation/transposition properties of equal-tempered scales (Krantz and Douthett 2011), tuning systems and modes (Garmendia Rodríguez and Navarro González 1995), etc. Moreover, other combinatorial classes, such as non-crossing configurations (Flajolet
like dissections of polygons (Gómez Aíza 2015), can be incorporated to complement works that address constructions of balanced scales (Milne et al. 2015). There are many references that address the theory of musical scales that are relevant to our work (Forte 1973; Slonimsky 1947), from the point of view of mathematics inclusive (Isola 2016; Lindley and Turner-Smith 1993), several of which are related to combinatorics on words (Allouche and Johnson 2018; Clampitt and Noll 2018; Brlek, Chemillier, and Reutenauer 2018; Abdallah, Gold, and Marsden 2016).

In our arguments, a key ingredient has been the use of first return loop systems, which arise in the study of classification problems of Markov shifts (Gómez 2003; Boyle, Buzzi, and Gómez 2006). In fact, studying music theory in contexts of dynamical systems has been an active area of research (Pinto 2012; Amiot 2020; Tymoczko and Yust 2019). Shift spaces are used in statistical mechanics as “hard square” models, and our results can serve to complement other models of statistical mechanics that have been used in music (Berezovsky 2019).

Acknowledgments

We thank Doug Lind for pointing out to us one reference (Benson 2007). We are grateful to the anonymous referee for his suggestions and for pointing out to us that the Feigenbaum sequence is also known as the period-doubling sequence.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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