ABSTRACT. We give an operator space characterization of sub-
algebras of $C(\Omega, M_n)$. We also describe injective subspaces of $C(\Omega, M_n)$ and then give applications to sub-TROs of $C(\Omega, M_n)$. Finally, we prove an ‘$n$-minimal version’ of the Christensen-Effros-
Sinclair representation theorem.

1. INTRODUCTION AND PRELIMINARIES

Let $n \in \mathbb{N}^*$. An operator space $X$ is called $n$-minimal if there
exists a compact Hausdorff space $\Omega$ and a completely isometric map
$i : X \to C(\Omega, M_n)$. The readers are referred to [13] and [7] for details
on operator space theory. Recall that the $C^*$-algebra $C(\Omega, M_n)$ can
be identified $\ast$-isomorphically with $C(\Omega) \otimes_{\min} M_n$ or $M_n(C(\Omega))$ (see
[12] Proposition 12.5] for details). Obviously, in the case $n = 1$, we
just deal with the well-known class of minimal operator spaces. Smith
noticed that any linear map into $M_n$ is completely bounded and its cb
norm is achieved at the $n^{th}$ amplification i.e. $\|u\|_{cb} = \|id_{M_n} \otimes u\|$ (see
[12] Proposition 8.11]). Clearly, this property remains true for maps
into $C(\Omega, M_n)$. In fact, Pisier showed that this property characterized
$n$-minimal operator spaces. More precisely, if $X$ is an operator space
such that any linear map $u$ into $X$ is necessarily completely bounded and $\|u\|_{cb} = \|id_{M_n} \otimes u\|$, then $X$ is $n$-minimal (see
[14] Theorem 18]).

We now recall a few facts about injectivity (see [7], [12] or [2] for
details). A Banach space $X$ is injective if for any Banach spaces
$Y \subset Z$, each contractive map $u : Y \to X$ has a contractive exten-
sion $\tilde{u} : Z \to X$. Since the 50’s, it is known that a Banach space
is injective if and only if it is isometric to a $C(K)$-space with $K$ a
Stonean space and dual injective Banach spaces are exactly $L^\infty$-spaces
(see [6] for more details). More recently, injectivity has also been stud-
ied in operator spaces category. Analogously, an operator space $X$ is
said to be injective if for any operator spaces $Y \subset Z$, each completely
contractive map $u : Y \to X$ has a completely contractive extension
$\tilde{u} : Z \to X$. Note that a Banach space is injective if and only if it is

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injective as a minimal operator space. Let $X$ be an operator space, $(Y, i)$ is an injective envelope of $X$ if $Y$ is an injective operator space, $i : X \to Y$ is a complete isometry and for any injective operator space $Z$ with $i(X) \subset Z \subset Y$, then $Z = Y$. Sometimes, we may forget the completely isometric embedding. In fact, any operator space admits a unique injective envelope (up to complete isometry) and we write $I(X)$ the injective envelope of $X$. See [7, Chapter 6] for a proof of this construction.

Obviously, an $\ell^\infty$-direct sum of $n$-minimal operator spaces is again $n$-minimal. In the next proposition, we give some other easy properties of $n$-minimal operator spaces:

**Proposition 1.1.** Let $X$ be an $n$-minimal operator space.

i) Then its bidual $X^{**}$ and its injective envelope $I(X)$ are $n$-minimal too.

ii) If moreover, $X$ is a dual operator space, then there is a set $I$ and a $w^*$-continuous complete isometry $i : X \to \ell^\infty I(M_n)$.

**Proof.** The first assertion of i) follows from $C(\Omega, M_n)^{**} = M_n(C(\Omega))^{**} = M_n(C(\Omega)^{**})$ *-isomorphically. For the second, suppose $X \subset C(\Omega, M_n)$ completely isometrically. From the description of injective Banach spaces, $I(C(\Omega)) = C(\Omega')$ with $\Omega'$ Stonean. Then $X \subset C(\Omega', M_n)$ and this last $C^*$-algebra is injective, so $I(X) \subset C(\Omega', M_n)$ completely isometrically.

Suppose that $W$ is an operator space predual of $X$. Then $X = CB(W, C)$ and if $I = \cup_n \text{Ball}(M_n(W))$, we have a $w^*$-continuous complete isometry $\psi : X \to \oplus_{n\in I} M_{n_w}$ (where $n_w = m$ if $w \in M_m(W)$) defined by $\psi(x) = ([x(\omega_{ij})])_{w\in I}$. Let $x \in M_k(X) = CB(W, M_k)$. As $X$ is $n$-minimal, by [12, Proposition 8.11], $\|x^*\|_{cb} = \|id_{M_k} \otimes x^*\|$, where $x^* : M_k^* \to X$ denotes the adjoint map. However, for any $l$, $\|id_{M_l} \otimes x\| = \|id_{M_l} \otimes x^*\|$. Hence, $\|x\|_{cb} = \|id_{M_n} \otimes x\|$ and so, in the definition of $\psi$, we can majorize the $n_w$'s by $n$ and obtain a complete isometry.

We reviewed that an injective minimal operator space is a $C^*$-algebra, but this property is lost for $n$-minimal operator spaces (as soon as $n \geq 2$). Generally, an injective operator space only admits a structure of ternary ring of operators. We recall that a closed subspace $X$ of a $C^*$-algebra is a ternary ring of operators (TRO in short) if $XX^* X \subset X$, here $X^*$ denotes the adjoint space of $X$. And a $W^*$-TRO is $w^*$-closed subspace of a von Neumann algebra stable under the preceding ‘triple product’. TROs and $W^*$-TROs can be regarded as generalization of $C^*$-algebras and $W^*$-algebras. For instance, The Kaplansky density
Theorem and the Sakai Theorem remain valid for TROs (see e.g. [6]). A triple morphism between TROs is a linear map which preserves their ‘triple products’. This category enjoys some ‘rigidity properties’ like $C^*$-algebras category (see e.g. [6] or [2] Section 8.3) for details.

So far we have seen that certain properties of the minimal case ‘pass’ to the $n$-minimal situation. Therefore, the basic idea of this paper is to extend valid results in the commutative case to the more general $n$-minimal case.

A first commutative result that can be extended to the $n$-minimal case is a theorem on operator algebras due to Blecher. We recall that an operator algebra is a closed subalgebra of $B(H)$, see [2] or [12] for some backgrounds and developments. And an operator algebra is said to be approximately unital if it possesses a contractive approximate identity. In [1], Blecher showed that an approximately unital operator algebra which is minimal is in fact a uniform algebra (i.e a subalgebra of a commutative $C^*$-algebra). So here, let $A$ be an approximately unital operator algebra and assume that $A$ is $n$-minimal. Then we can obtain a completely isometric homomorphism from $A$ into a certain $C(\Omega, M_n)$ (see Corollary 2.3). Of course, we can ask this type of question in various categories of operator spaces. More precisely, let $\mathcal{C}$ denote a certain subcategory of the category of operator spaces with completely contractive maps. Let $X$ be an object of $\mathcal{C}$ which is $n$-minimal (as an operator space), can we obtain a completely isometric morphism of $\mathcal{C}$ from $X$ into a $C^*$-algebra of the form $C(\Omega, M_n)$ ? For example in Proposition 1.1 we answered this question in the category of dual operator spaces and $w^*$-continuous completely contractive maps. We will also give a positive answer in the category of:

- $C^*$-algebras and $*$-homomorphisms (see Theorem 2.2);
- von Neumann algebras and $w^*$-continuous $*$-homomorphisms (see Remark 2.4);
- approximately unital operator algebras and completely contractive homomorphisms (see Corollary 2.3);
- operator systems and completely positive unital maps (see Corollary 3.3);
- TRO and triple morphisms (see Proposition 4.1);
- $W^*$-TRO and $w^*$-continuous triple morphisms (see Corollary 4.5).

It means that, in any of the previous categories, the $n$-minimal operator space structure encodes the additional structure. Since the injective envelope of an $n$-minimal operator space is $n$-minimal too (see Proposition 1.1), passing to the injective envelope will be a useful technique to answer these preceding questions. In any case, the description of
\[\text{n-minimal injective operator spaces (established in Theorem 3.5) will be of major importance.}\]

The Christensen-Effros-Sinclair theorem (CES-theorem in short) is a second example of theorem that could be treated in the \(n\)-minimal case. Let \(A\) be an operator algebra (or more generally a Banach algebra endowed with an operator space structure) and let \(X\) be an operator space which is a left \(A\)-module. Then following [2, Chapter 3], we say that \(X\) is a left \(h\)-module over \(A\) if the action of \(A\) on \(X\) induces a completely contractive map from \(A \otimes_h X\) in \(X\) (where \(\otimes_h\) denotes the Haagerup tensor product). The CES-theorem states that if \(X\) is a non-degenerate \(h\)-module over an approximately unital operator algebra \(A\) (i.e. \(AX\) is dense in \(X\)), then there exists a \(C^*\)-algebra \(C\), a complete isometry \(i : X \to C\) and a completely contractive homomorphism \(\pi : A \to C\) such that \(i(a \cdot x) = \pi(a)i(x)\) for any \(a \in A\), \(x \in X\).

We will prove that if \(X\) is \(n\)-minimal, we can choose \(C\) to be \(n\)-minimal too. This leads to an ‘\(n\)-minimal version’ of the CES-theorem. The case \(n = 1\) has been treated (see [3]) in a Banach space framework; here we will use an operator space approach based on the multiplier algebra of an operator space.

2. SUBALGEBRAS OF \(C(\Omega, M_n)\)

Recall that a \(C^*\)-algebra is subhomogeneous of degree \(\leq n\) if it is contained \(*\)-isomorphically in a \(C^*\)-algebra of the form \(C(\Omega, M_n)\), where \(\Omega\) is compact Hausdorff space. Hence \(n\)-minimality could be seen as an operator space analog of subhomogeneity of degree \(\leq n\). We also recall the well-known characterization of subhomogeneous \(C^\ast\)-algebras in terms of representations. Indeed, a \(C^\ast\)-algebra \(A\) is subhomogeneous of degree \(\leq n\) if and only if every irreducible representation of \(A\) has dimension no greater than \(n\). The ‘if part’ is easily obtained taking a separating family of irreducible representations. Conversely, if \(A\) is contained \(*\)-isomorphically in \(C(\Omega, M_n)\), then every irreducible representation of \(A\) extends to one on \(C(\Omega, M_n)\) (because irreducible representations correspond to pure states). And as any irreducible representation of \(C(\Omega, M_n)\) has dimension no greater than \(n\), we can conclude (the author thanks Roger Smith for these explanations).

**Lemma 2.1.** Let \(k \in \mathbb{N}^\ast\), \(\Omega\) a compact Hausdorff space and \(t_k\) the transpose mapping

\[t_k : C(\Omega, M_k) \rightarrow C(\Omega, M_k), \quad [f_{ij}] \mapsto [f_{ji}]\]
Moreover in adapting the assumption (i) is of the form \( \sum C \). Hence the assumption (ii) holds. Then for any \( l \in \mathbb{N}^* \), \( \| id_{M_l} \otimes t_k \| = \inf(k, l) \). Thus \( t_k \) is completely bounded and \( \| id_{M_k} \otimes t_k \| = \| t_k \|_{cb} = k \).

**Proof.** The equality \( \| t_k \|_{cb} = k \) is obtained in adapting the proof of [7, Proposition 2.2.7]. Hence in the case \( k \leq l \), by [12, Proposition 8.11] we obtain \( \| id_{M_l} \otimes t_k \| = \inf(k, l) \). Next we prove \( \| id_{M_l} \otimes t_k \| \leq l \). Let \( \pi \) be the cyclical permutation matrix

\[
\pi = \begin{pmatrix}
0 & 0 & \cdots & 0 & I_k \\
I_k & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I_k & 0
\end{pmatrix} \in M_l(C(\Omega, M_k)).
\]

Let \( D_l : M_l(C(\Omega, M_k)) \rightarrow M_l(C(\Omega, M_k)) \) be the diagonal truncation of \( M_l \) i.e. \( D_l(\epsilon_{ij} \otimes y) = \delta_{ij} \epsilon_{ij} \otimes y \) where \( \epsilon_{ij} \) (\( i, j \leq l \)) denotes the matrix units of \( M_l \) and \( y \in C(\Omega, M_k) \). Let \( x = [x_{ij}]_{i,j \leq l} \in M_l(C(\Omega, M_k)) \) and for simplicity of notation, we wrote \( t(x) = id_{M_l} \otimes t_k(x) \in M_l(C(\Omega, M_k)) \).

Then \( t(x) = \sum_{i=0}^{l-1} D_l(t(x) \pi^i) \pi^{-i} \), and so \( \| t(x) \| \leq \sum_{i=0}^{l-1} \| D_l(t(x) \pi^i) \| \) (because \( \pi \) is unitary). To conclude it suffices to majorize each terms of the previous sum by the norm of \( x \). However, for any \( i \), \( D_l(t(x) \pi^i) \) is of the form \( \sum_{j=1}^{l} \epsilon_{jj} \otimes t_k(x_{p,jq}) \) and we can majorize its norm,

\[
\| \sum_{j=1}^{l} \epsilon_{jj} \otimes t_k(x_{p,jq}) \|^2 = \| \sum_{j=1}^{l} \epsilon_{jj} \otimes t_k(x_{p,jq} x_{p,jq}^*) \| = \max_j \{ \| t_k(x_{p,jq} x_{p,jq}^*) \| \}
\]

but \( x_{p,jq} x_{p,jq}^* \) is a selfadjoint element of \( C(\Omega, M_k) \), so its norm is unchanged by \( t_k \) and \( \| t_k(x_{p,jq} x_{p,jq}^*) \| = \| x_{p,jq} \|^2 \leq \| x \|^2 \). Finally, for any \( i \), \( \| D_l(t(x) \pi^i) \| \leq \| x \| \) which enable us to conclude.

Moreover in adapting [7, Proposition 2.2.7], we have easily \( \| id_{M_l} \otimes t_k \| = l \), if \( l \leq k \).

In the next theorem, we denote by \( A^{op} \) the opposite structure of a C*-algebra \( A \) (see e.g. [13, Paragraph 2.10] or [2, Paragraph 1.2.25] for details). More generally, if \( X \) is an operator space, \( X^{op} \) is the same vector space but with the new matrix norms defined by

\[
\| [x_{ij}] \|_{M_n(X^{op})} = \| [x_{ij}] \|_{M_n(X)} \quad \text{for any} \quad [x_{ij}] \in M_n(X).
\]

Hence the assumption (iii) in the next theorem is equivalent to

\[
\| id_A \otimes t_k \| \leq n \quad \text{for any} \quad k \in \mathbb{N}^*,
\]

where \( t_k \) denotes the transpose mapping from \( M_k \) to \( M_k \) discussed above.

**Theorem 2.2.** Let \( A \) be a C*-algebra. Then the following are equivalent:

\[ \text{...} \]
(i) $A$ is subhomogeneous of degree $\leq n$.
(ii) $A$ is $n$-minimal.
(iii) $\|id : A \to A^{op}\|_{cb} \leq n$.

**Proof.** (i) $\Rightarrow$ (ii) is obvious and (ii) $\Rightarrow$ (iii) follows from the first equality in the previous lemma. Suppose (iii). Let $\pi : A \to B(H)$ be an irreducible representation and $k \in \mathbb{N}^*$ such that $M_k \subset B(H)$; from the first paragraph of this section, we must prove that $k \leq n$. Using the previous lemma (with a singleton as $\Omega$), there is $x \in M_k(M_k) \subset M_k(B(H))$ satisfying

$$
k = \|id_{M_k} \otimes t_k(x)\| \quad \text{and} \quad \|x\| \leq 1.
$$

The representation $\pi_k = id_{M_k} \otimes \pi$ is also irreducible so the commutant $\pi_k(M_k(A))^\prime = C I_{H^k}$, thus by the von Neumann’s double commutant theorem

$$
M_k(\pi(A))^{so} = M_k(B(H)).
$$

Then by the Kaplansky density theorem, there exists a net $(x_\lambda)_{\lambda \in \Lambda} \subset M_k(\pi(A))$ converging to $x$ in the $\sigma$-strong operator topology and such that $\|x_\lambda\| \leq 1$. Therefore $id_{B(H)} \otimes t_k(x_\lambda)$ tends to $id_{M_k} \otimes t_k(x)$ in the $w^*$-topology and by the semicontinuity of the norm in the $w^*$-topology, we have

$$
k = \|id_{M_k} \otimes t_k(x)\| \leq \limsup_{\lambda} \|id_{B(H)} \otimes t_k(x_\lambda)\|.
$$

Let $\epsilon > 0$. For any $\lambda$, there exists $y_\lambda \in M_k(A)$ such that $x_\lambda = \pi_k(y_\lambda)$ and $\|y_\lambda\| \leq 1 + \epsilon$. By assumption,

$$
\|id_A \otimes t_k\| \leq n
$$

Moreover $(id_{B(H)} \otimes t_k) \circ \pi_k = \pi_k \circ (id_A \otimes t_k)$. Combining these arguments we finally obtain

$$
k = \|id_{M_k} \otimes t_k(x)\| \leq \limsup_{\lambda} \|id_{B(H)} \otimes t_k(\pi_k(y_\lambda))\|
$$

$$
\leq \limsup_{\lambda} \|\pi_k(id_A \otimes t_k(y_\lambda))\|
$$

$$
\leq \|id_A \otimes t_k\|(1 + \epsilon)
$$

$$
\leq n(1 + \epsilon).
$$

Hence $k \leq n$. \hfill \blacksquare

Now we extend $(i) \Leftrightarrow (ii)$ of the previous theorem, which concerns $C^*$-algebras, to the larger category of operator algebras and completely contractive homomorphisms.

**Corollary 2.3.** Let $A$ be an approximately unital operator algebra. Then the following are equivalent:

1. There exists a compact Hausdorff space $\Omega$ and a completely isometric homomorphism $\pi : A \to C(\Omega, M_n)$.  

(ii) $A$ is $n$-minimal.

**Proof.** $(i) \Rightarrow (ii)$ is obvious. Suppose $(ii)$. We know that the injective envelope $I(A)$ is a $C^*$-algebra and there is a completely isometric homomorphism from $A$ into $I(A)$ (see [2, Corollary 4.2.8]). Since $A$ is $n$-minimal, $I(A)$ is $n$-minimal too, by Proposition 1.1. Applying Theorem 2.2 to $I(A)$, we can conclude.

**Remark 2.4.** Using the well-known description of subhomogeneous $W^*$-algebras, we easily obtained that, if $M$ is a $W^*$-algebra and $M$ is $n$-minimal, then

$$M = \bigoplus_{i \in I} L^\infty(\Omega_i, M_{n_i})$$

via a normal $*$-isomorphism. Here $\Omega_i$ is a measure space and $n_i \leq n$, for any $i \in I$. This result will be extended to the category of $W^*$-TROs (see Corollary 4.5).

### 3. Injective $n$-Minimal Operator Spaces

Before describing injective $n$-minimal operator spaces, we can treat the more ‘rigid’ case of injective $n$-minimal $C^*$-algebras as an easy consequence of [16].

**Proposition 3.1.** Let $A$ be an $n$-minimal $C^*$-algebra. Then the following are equivalent:

(i) $A$ is injective.

(ii) There exists a finite family of Stonean compact Hausdorff spaces $(\Omega_i)_{i \in I}$ such that $A = \bigoplus_{i \in I} C(\Omega_i; M_{n_i})$ $*$-isomorphically with $n_i \leq n$, for any $i \in I$.

**Proof.** As $A$ is injective, $A$ is monotone complete (see [7, Theorem 6.1.3]). Thus $A$ is an $AW^*$-algebra. Moreover, by [16, Proposition 6.6], $A$ either contains $M_\infty = \bigoplus_k^\infty M_k$ or $A$ is of the desired form. The first alternative is impossible because $A$ is $n$-minimal, which ends the ‘only if’ part. The converse is clear, since each $\Omega_i$ is Stonean.

**Remark 3.2.** This theorem enables us to give a short proof of $(ii) \Rightarrow (i)$ in Theorem 2.2. If $A$ is an $n$-minimal $C^*$-algebra, its injective envelope $I(A)$ is $n$-minimal too (by Proposition 1.1). $I(A)$ is a $C^*$-algebra and contains $A$ $*$-isomorphically (see [7, Theorem 6.2.4]). Applying the previous proposition to $I(A)$, we obtain that

$$I(A) = \bigoplus_{i \in I}^\infty C(\Omega_i, M_{n_i})$$

$*$-isomorphically

with $n_i \leq n$, for any $i \in I$. And now it is not difficult to construct a $*$-isomorphism from $A$ into $C(\Omega, M_n)$ where $\Omega$ denotes the (finite) disjoint union of the $\Omega_i$'s.
We recall that an operator space $X$ is unital if there exists $e \in X$ and a complete isometry from $X$ into a certain $B(H)$ which sends $e$ on $I_H$. From the result below, an $n$-minimal operator system can embed into a $C^*$-algebra of the form $C(\Omega, M_n)$ via a unital complete order isomorphism.

**Corollary 3.3.** Let $X$ be a unital operator space. Then the following are equivalent:

(i) There exists a compact Hausdorff space $\Omega$ and a completely isometric unital map $\pi : X \to C(\Omega, M_n)$.

(ii) $X$ is $n$-minimal.

**Proof.** $(i) \Rightarrow (ii)$ is obvious. Suppose $(ii)$. We know that the injective envelope $I(X)$ is a $C^*$-algebra and there is a unital complete isometry from $X$ into $I(X)$ (see [2, Corollary 4.2.8]). As $X$ is $n$-minimal, $I(X)$ is $n$-minimal too (by Proposition [1]). By the previous theorem

$$I(X) = \bigoplus_{i \in I} C(\Omega_i, M_{n_i}) \ast$$-isomorphically.

Next we show that for any $i$ there exists a unital complete isometry $\varphi_i : M_{n_i} \to M_n$. By iteration, we only need to prove that for any $k \in \mathbb{N}^*$, there exists a unital complete isometry from $M_k$ into $M_{k+1}$. The map

$$i_k : M_k \to M_{k+1} \quad x \mapsto x \oplus tr_k(x)$$

(where $tr_k$ denotes the normalized trace on $M_k$) is a unital complete order isomorphism and thus a unital complete isometry. We can define a unital complete isometry

$$\psi : \bigoplus_{i \in I} C(\Omega_i, M_{n_i}) \to C(\Omega, M_n)$$

$$(f_i \otimes x_i)_i \mapsto \sum_i \tilde{f}_i \otimes \varphi_i(x_i)$$

where $\Omega$ denotes the disjoint union of $\Omega_i$’s and $\tilde{f}_i$ the continuous extension by 0 of $f_i$ on $\Omega$. Finally, we have

$$X \subset I(X) \subset C(\Omega, M_n)$$

via unital complete isometries.

**Remark 3.4.** This last corollary cannot be extended to the category of operator algebras and completely contractive homomorphisms. In fact, if $\pi : M_p \to C(\Omega, M_q)$ is a unital completely contractive homomorphism then $\pi$ is positive so it is a $\ast$-homomorphism. Therefore (composing by an evaluation) we can obtain a unital $\ast$-homomorphism from $M_p$ in $M_q$ and thus $p$ divides $q$ (see [12, Exercise 4.11]).
We must recall a crucial construction of the injective envelope of an operator space $X$ which will be useful in this paper (see [2, Paragraph 4.4.2] for more details on this construction). Assume that $X \subset B(H)$, we can consider its Paulsen system

$$S(X) = \left( \begin{array}{cc} \mathbb{C} & X \\ X^* & \mathbb{C} \end{array} \right) \subset M_2(B(H))$$

where $X^*$ denotes the adjoint space of $X$. The injective envelope of $S(X)$ is the range of a completely contractive projection $\varphi : M_2(B(H)) \to M_2(B(H))$ which leaves $S(X)$ invariant. By [7, Theorem 6.1.3], $I(S(X))$ admits a $C^*$-algebraic structure but it is not necessarily a sub-$C^*$-algebra of $M_2(B(H))$. However

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 - p$$

(which are invariant by $\varphi$) are still orthogonal projections (i.e. selfadjoint idempotents) of the new $C^*$-algebra $I(S(X))$. Since they satisfy $p + q = 1$ and $pq = 0$, we can decompose $I(S(X))$ in $2 \times 2$ matrices, as follow :

$$I(S(X)) = \begin{pmatrix} I_{11}(X) & I_{12}(X) \\ I_{21}(X) & I_{22}(X) \end{pmatrix}$$

where $I_{11}(X) = pI(S(X))p$ and $I_{22}(X) = qI(S(X))q$ are injective $C^*$-algebras, $I_{12}(X) = pI(S(X))q$ and $I_{21}(X) = qI(S(X))p$ coincide with $I_{12}(X)^*$. Therefore, we obtain the Hamana-Ruan Theorem i.e. an injective operator space is an ‘off-diagonal’ corner of an injective $C^*$-algebra (see [7, Theorem 6.1.6]). It links the study of injective operator spaces to injective $C^*$-algebras (and, by the way, it proves that an injective operator space is a TRO).

**Theorem 3.5.** Let $X$ be an $n$-minimal operator space. Then the following are equivalent :

(i) $X$ is injective.

(ii) There exists a finite family of Stonean compact Hausdorff spaces $(\Omega_i)_{i \in I}$ such that $X = \bigoplus_{i \in I}^\infty C(\Omega_i, M_{r_i,k_i})$ completely isometrically with $r_i, k_i \leq n$, for any $i \in I$.

**Proof.** (ii) $\Rightarrow$ (i) is obvious. Let $X$ be an injective $n$-minimal operator space. By the discussion above, we know that there exists an injective $C^*$-algebra $A$ and a projection $p \in A$ such that

$$X = pA(1 - p)$$

completely isometrically

In fact $A$ is the injective envelope of $S(X)$ the Paulsen system of $X$ (see above). As $X$ is $n$-minimal, $S(X)$ is $2n$-minimal, so is $A$ (by
Proposition 1.1. From Proposition 3.1,
\[ A = \bigoplus_{i \in I} C(\Omega_i, M_{m_i}) \text{ \textit{*-isomorphically}} \]
where \( m_i \leq 2n \). For simplicity of notation, we will assume momentarily that the cardinal of \( I \) is equal to 1 and so
\[ X = pC(\Omega, M_m)(1-p) \text{ \textit{completely isometrically}}, \]
for some projection \( p \in C(\Omega, M_m) \). Using [5, Corollary 3.3] or [8, Theorem 3.2], there is a unitary \( u \) of \( C(\Omega, M_m) \) such that for any \( \omega \in \Omega \), \( upu^*(\omega) \) is of the form \( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \). So we may assume that for any \( \omega \in \Omega \), \( p(\omega) \) is a diagonal matrix of the form given above. For any \( k \leq m \), we define
\[ \Omega_k = \{ \omega \in \Omega : \text{rg}(p(\omega)) = k \} \]
which is a closed subset of \( \Omega \) (because the rank and the trace of a projection coincide) and the family \( (\Omega_k)_{k \leq m} \) forms a partition of \( \Omega \). Hence, any \( \Omega_k \) is open (and closed) in \( \Omega \), so \( \Omega_k \) is still Stonean. We have the completely isometric identifications
\[ X = pC(\Omega, M_m)(1-p) = \bigoplus_{k \leq m} C(\Omega_k, M_{k,m-k}) = \bigoplus_{1 \leq k \leq m-1} C(\Omega_k, M_{k,m-k}). \]
Moreover, for any \( 1 \leq k \leq m-1 \), we have the completely isometric embeddings
\[ M_{k,m-k} \subset C(\Omega_k, M_{k,m-k}) \subset X \]
and as \( X \) is \( n \)-minimal, it forces \( k \leq n \) and \( m-k \leq n \); if not, at least the row Hilbert space \( R_{n+1} \) or the column Hilbert space \( C_{n+1} \) would be \( n \)-minimal. Thus \( X \) has the announced form. In general, \( I \) is a finite set and
\[ X = p \bigoplus_{i \in I} C(\Omega_i, M_{m_i})(1-p) = \bigoplus_{i \in I} p_i C(\Omega_i, M_{m_i})(1-p_i) \]
where \( p_i \) is a projection in \( C(\Omega_i, M_{m_i}) \) and \( p = \bigoplus p_i \). Applying the preceding argument to each terms \( p_i C(\Omega_i, M_{m_i})(1-p_i) \), we can conclude.

Corollary 3.6. Let \( X \) be an \( n \)-minimal dual operator space. Then the following are equivalent:

(i) \( X \) is injective.

(ii) There exists a finite family of measure spaces \( (\Omega_i)_{i \in I} \) such that \( X = \bigoplus_{i \in I} L^\infty(\Omega_i, M_{r_i,k_i}) \) via a completely isometric \( w^* \)-homeomorphism with \( r_i, k_i \leq n \), for any \( i \in I \).

Proof. From the previous theorem, \( X = \bigoplus_{i} C(K_i, M_{r_i,k_i}) \) completely isometrically, where \( K_i \) is Stonean. Since \( X \) is a dual operator space, it forces \( C(K_i) \) to be a dual commutative \( C^* \)-algebra i.e. \( C(K_i) = L^\infty(\Omega_i) \) (via a normal \( * \)-isomorphism) for some measure space \( \Omega_i \).
4. Application to \(n\)-minimal TROs

In this section, we will use the description of injective \(n\)-minimal operator spaces to obtain results on \(n\)-minimal TROs. First, we will see that the \(n\)-minimal operator structure of a TRO determines its whole triple structure. See e.g. [6] or [2, Section 8.3] for details on TROs.

**Proposition 4.1.** Let \(X\) be a TRO. The following are equivalent :

(i) There exists a compact Hausdorff space \(\Omega\) and an injective triple morphism \(\pi : X \to C(\Omega, M_n)\).

(ii) \(X\) is \(n\)-minimal.

**Proof.** \((i) \Rightarrow (ii)\) follows from the fact that an injective triple morphism is necessarily completely isometric (see e.g. [6, Proposition 2.2] or [2, Lemma 8.3.2]).

Suppose \((ii)\). By [2, Remark 4.4.5 (1)], the injective envelope of \(X\) admits a TRO structure and \(X\) can be viewed as a sub-TRO of \(I(X)\). From Theorem 3.5, we can describe this injective envelope as a direct sum,

\[
I(X) = \bigoplus_{i \in I} C(\Omega_i, M_{r_i,k_i}) \quad \text{completely isometrically.}
\]

But the right hand side of the equality admits a canonical TRO structure and it is known (see e.g. [2, Corollary 4.4.6]) that a surjective complete isometry between TROs is automatically a triple morphism. In addition, for any \(i\), the embedding \(\varphi_i : M_{r_i,k_i} \to M_n\) into the ‘up-left’ corner of \(M_n\) is an injective triple morphism. As in the end of the proof of Corollary 3.3, we finally obtain

\[
X \subset I(X) = \bigoplus_{i \in I} C(\Omega_i, M_{r_i,k_i}) \subset C(\Omega, M_n)
\]

as TROs.

For details on \(C^*\)-modules theory, the readers are referred to [11] or [2, Chapter 8] for an operator space approach. We must recall the construction of the linking \(C^*\)-algebra of a \(C^*\)-module. If \(X\) is a \(C^*\)-module over a \(C^*\)-algebra \(A\) then its conjugate vector space \(\overline{X}\) is a right \(C^*\)-module over \(A\) with the action \(\overline{x} \cdot a = a^*\overline{x}\) and inner product \(\langle \overline{x}, \overline{y} \rangle = \langle x, y \rangle\), for any \(a \in A, x, y \in X\). We denote by \(\mathcal{L}(X)\) the \(C^*\)-algebra of ‘compact’ adjointable maps of \(X\) and then

\[
\mathcal{L}(X) = \begin{pmatrix} A & X \\ \overline{X} & \mathcal{L}(X) \end{pmatrix}
\]
is a $C^*$-algebra too which is called the linking $C^*$-algebra of $X$. If $X$ is an equivalence bimodule (see [2] Paragraph 8.1.2) over two $C^*$-algebras $A$ and $B$, we define

$$\mathcal{L}(X) = \left( \begin{array}{cc} A & X \\ X & B \end{array} \right)$$ and $$\mathcal{L}^1(X) = \left( \begin{array}{cc} A^1 & X \\ X & B^1 \end{array} \right)$$

(where $A^1$ and $B^1$ denote the unitizations of $A$ and $B$) which are also $C^*$-algebras (see [2] Paragraph 8.1.17 for details on linking $C^*$-algebra). We can notice that $X$ is an ‘off-diagonal’ corner of a $C^*$-algebra i.e. $X = p\mathcal{L}^1(X)(1-p)$ for some projection $p \in \mathcal{L}^1(X)$. Hence a $C^*$-module admits a TRO structure. The converse will be seen later on, which will make the correspondence between $C^*$-modules, equivalence bimodules and TROs (see [2] Paragraph 8.1.19, 8.3.1). Thus the next corollary is a reformulation of the previous proposition in the $C^*$-modules language. However, this corollary on representation of module action can be compared with Theorem 5.4.

**Corollary 4.2.** Let $X$ be a full left $C^*$-module over a $C^*$-algebra $A$. Then the following are equivalent:

(i) There exists a compact Hausdorff space $\Omega$, a complete isometry $i : X \to C(\Omega, M_n)$ and a $\ast$-isomorphism $\sigma : A \to C(\Omega, M_n)$ such that for any $a \in A$, $x, y \in X$

$$i(a \cdot x) = \sigma(a)i(x)$$

$$\sigma(\langle x, y \rangle) = i(x)i(y)^*$$

(ii) $X$ is $n$-minimal and $A$ is subhomogeneous of degree $\leq n$.

(iii) $X$ is $n$-minimal.

**Proof.** Only (iii) $\Rightarrow$ (i) needs a proof. Since $X$ is a $C^*$-module, it’s also a TRO (see above). From Proposition 4.1 there exists a compact Hausdorff space $\Omega$ and an injective triple morphism $i : X \to C(\Omega, M_n)$. By [2], Corollary 8.3.5], we can construct a corner preserving $\ast$-isomorphism $\pi : \mathcal{L}(X) \to M_2(C(\Omega, M_n))$ such that $i = \pi_{12}$. Choosing $\sigma = \pi_{11}$, we obtain the desired relations.

An equivalence bimodule version of the previous corollary could be stated. In the previous result we transfer $n$-minimality from $X$ to $A$. We can treat the ‘reverse’ question ; let $X$ be an equivalence bimodule over two $n$-minimal $C^*$-algebras, we will prove that $X$ is $n$-minimal. But first, let us translate this proposition in the TROs language. Let $X$ be a TRO contained in a $C^*$-algebra $B$ via an injective triple morphism. As in the notation of the second section of [15], we define $C(X)$ (resp. $D(X)$) the norm closure of $\text{span}\{xy^*, \ x, y \in X\}$ (resp.
span\{x^*y, \; x, y \in X\}). As X is a sub-TRO of B, C(X) and D(X) are sub-C*-algebras of B and

\[ A(X) = \begin{pmatrix} C(X) & X \\ X^* & D(X) \end{pmatrix} \]

is a sub-C*-algebras of \( M_2(B) \). Hence a TRO can be regarded as an ‘off-diagonal’ corner of a C*-algebra which prove totally the correspondence between C*-modules, equivalence bimodules and TROs. And A(X) is also called the linking C*-algebra of X. Analogously, in \( W^* \)-TROs category, let X be a \( W^* \)-TRO contained in a \( W^* \)-algebra B via a \( w^* \)-continuous injective triple morphism. We define \( M(X) \) (resp. \( N(X) \)) the \( w^* \)-closure of \( \text{span}\{xy^*, \; x, y \in X\} \) (resp. \( \text{span}\{x^*y, \; x, y \in X\} \)).

As X is a sub-\( W^* \)-TRO of B, M(X) and N(X) are sub-\( W^* \)-algebras of B and

\[ R(X) = \begin{pmatrix} M(X) & X \\ X^* & N(X) \end{pmatrix} \]

is a sub-\( W^* \)-algebras of \( M_2(B) \). It is called the linking von Neumann algebra of X. In fact, the linking algebras do not depend on the embedding of X into a C*-algebra.

Obviously, if X is an equivalence bimodule over two C*-algebras A and B, C(X) and D(X) play the roles of A and B in the correspondence between equivalence bimodules and TROs. Hence in the TROs language, we obtain (in the dual case):

**Proposition 4.3.** Let X be a \( W^* \)-TRO such that \( M(X) \) and \( N(X) \) are n-minimal von Neumann algebras. Then X is n-minimal and

\[ X = \oplus_i^{\infty} L^\infty(\Omega_i) \otimes M_{r_i,k_i} \]

where \( \Omega_i \) is a measure space, \( r_i, k_i \leq n \), for any i.

**Proof.** We write \( R(X) \) the linking von Neumann of X. From [9, Theorem 6.5.2], there exist \( p_1, p_2 \) and \( p_3 \) three central projections of \( R(X) \) such that

\[ R(X) = p_1 R(X) \oplus^{\infty} p_2 R(X) \oplus^{\infty} p_3 R(X) \]

and for \( i = 1, 2, 3 \), \( p_i R(X) \) is a von Neumann algebra of type \( i \) or \( p_i = 0 \). Since \( M(X) \) is n-minimal, \( M(X) \) is of type I. However, \( M(X) = p R(X) p \) for some projection \( p \) in \( R(X) \) and for any i,

\[ p_i M(X) = pp_i M(X) pp_i p \]

As the type is unchanged by compression (see [9, Exercise 6.9.16]), \( p_i M(X) \) is of type I or \( p_i M(X) = 0 \). On the other hand, for any i,

\[ p_i M(X) = p_i p R(X) = pp_i R(X) p_i p \]
so \( p_i M(X) \) has the same type as \( p_i R(X) \) or \( p_i M(X) = 0 \). Thus \( p_i M(X) = 0 \) for \( i = 2, 3 \) i.e. \( p_i p = 0 \) for \( i = 2, 3 \). Symmetrically, using our assumption on \( N(X) \), we have \( p_i (1 - p) = 0 \) for \( i = 2, 3 \). Hence \( p_i = 0 \) for \( i = 2, 3 \) i.e. \( R(X) \) is of type \( I \). Using \cite{15} Theorem 4.1],
\[
X = \bigoplus_k^\infty L^\infty(\Omega_k) \otimes M_{I_k, J_k}
\]
where \( \Omega_k \) is a measure space, \( I_k, J_k \) are sets and \( M_{I_k, J_k} = B(\ell^2_k, \ell^2_k) \). Since \( M(X) \) (resp. \( N(X) \)) is \( n \)-minimal, it forces the cardinal of \( I_k \) (resp. \( J_k \)) to be no greater than \( n \), for any \( k \). So \( X \) is \( n \)-minimal and has the desired form.

**Remark 4.4.** In the next two results, we will use that the multiplier algebra of an \( n \)-minimal \( C^* \)-algebra is \( n \)-minimal too. It is due to Proposition \cite{1}.

The next corollary on \( W^* \)-TROs extends Remark \cite{2}.

**Corollary 4.5.** Let \( X \) be a \( W^* \)-TRO. The following are equivalent :

(i) \( X \) is \( n \)-minimal.

(ii) There exists a measure space \( \Omega \) and a \( w^* \)-continuous injective triple morphism \( \pi : X \to L^\infty(\Omega, M_n) \).

(iii) There exists a finite family of measure spaces \( (\Omega_i)_{i \in I} \) such that \( X = \bigoplus_{i \in I} L^\infty(\Omega_i, M_{r_i, k_i}) \) with \( r_i, k_i \leq n \), for any \( i \in I \).

**Proof.** Only (i) \( \Rightarrow \) (iii) needs a proof. Suppose (i). From Proposition \cite{4.1}, we can see \( X \) as a sub-TRO of \( C(\Omega, M_n) \), hence by construction \( C(X) \) and \( D(X) \) are \( n \)-minimal \( C^* \)-algebras. By \cite{10}, \( M(X) \) (resp. \( N(X) \)) is the multiplier algebra of \( C(X) \) (resp. \( D(X) \)), so \( M(X) \) and \( N(X) \) are \( n \)-minimal \( W^* \)-algebras (by Remark 4.4). The result follows from the previous proposition.

Finally, we can generalize (ii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (v) of \cite{2} Proposition 8.6.5] on minimal TROs to the \( n \)-minimal case.

**Theorem 4.6.** Let \( X \) be a TRO, the following are equivalent :

(i) \( X \) is \( n \)-minimal.

(ii) \( X^{**} \) is an injective \( n \)-minimal operator space (see Corollary \cite{3.6}).

(iii) \( C(X) \) and \( D(X) \) are \( n \)-minimal \( C^* \)-algebras.

**Proof.** (ii) \( \Rightarrow \) (i) and (i) \( \Rightarrow \) (iii) are obvious. Suppose (iii). From \cite{10} Proposition 2.4], we know that the multiplier algebra of \( C(X^{**}) \) is \( C(X)^{**} \) and this \( C^* \)-algebra is \( n \)-minimal by our assumption on \( C(X) \) and Remark 4.4]. Moreover by \cite{15}, \( M(X^{**}) \) is also the multiplier algebra of \( C(X^{**}) \), so \( M(X^{**}) \) is \( n \)-minimal too. The same argument works for \( N(X^{**}) \) and we can apply Proposition \cite{4.3} to \( X^{**} \).
5. An \( n \)-minimal version of the CES-theorem

To prove the ‘\( n \)-minimal’ version the CES-Theorem we need the notion of left multiplier algebra of an operator space \( X \). A left multiplier of an operator space \( X \) is a map \( u : X \rightarrow X \) such that there exist a \( C^* \)-algebra \( A \) containing \( X \) via a complete isometry \( i \) and \( a \in A \) satisfying \( i(u(x)) = ai(x) \) for any \( x \in X \). Let \( \mathcal{M}_l(X) \) denote the set of left multipliers of \( X \). And the multiplier norm of \( u \) is the infimum of \( \|a\| \) over all possible \( A, i, a \) as above. In fact Blecher-Paulsen proved that any left multiplier can be represented in the embedding of \( X \) into the \( C^* \)-algebra (discussed in section 3)

\[
I(S(X)) = \left( \begin{array}{cc} I_{11}(X) & I(X) \\ I(X)^* & I_{22}(X) \end{array} \right)
\]

More precisely, for any left multiplier \( u \) of norm no greater than 1, there exists a unique \( a \in I_{11}(X) \) of norm no greater than 1 such that \( u(x) = ax \) for any \( x \in X \) (see [2] Theorem 4.5.2). This result enables us to consider \( M_l(X) \) as an operator subalgebra of \( I_{11}(X) \) (see the proof of [2] Proposition 4.5.5 and [2] Paragraph 4.5.3 for more details) and

\[
\mathcal{M}_l(X) = \{ a \in I_{11}(X), \ aX \subset X \}
\]

as operator algebras. The product used in the preceding centered formula is the one on the \( C^* \)-algebra \( I(S(X)) \). And the operator algebra \( \mathcal{M}_l(X) \) is called the multiplier algebra of \( X \). We let \( \mathcal{A}_l(X) = \Delta(\mathcal{M}_l(X)) \) denote the diagonal (see [2] Paragraph 2.1.2) of \( \mathcal{M}_l(X) \), this \( C^* \)-algebra is called the left adjointable multiplier algebra of \( X \).

**Lemma 5.1.** Let \( X \) be an operator space and \( I(X) \) its injective envelope. Then there exists a completely contractive unital homomorphism \( \theta : \mathcal{M}_l(X) \rightarrow \mathcal{M}_l(I(X)) \) such that \( \theta(u)|_X = u \), for any \( u \in \mathcal{M}_l(X) \). And thus, \( \theta_{\mathcal{A}_l(X)} : \mathcal{A}_l(X) \rightarrow \mathcal{A}_l(I(X)) \) is a \( * \)-isomorphism. Moreover, the same results hold for right multipliers.

**Proof.** Let \( u \in \mathcal{M}_l(X) \), then \( u \) can be represented by an element \( a \) in \( \{ a \in I_{11}(X), \ aX \subset X \} \). And using the multiplication inside \( I(S(X)) \),
$aI(X) \subset I(X)$, so $a$ can be seen as an element of $\mathcal{M}_I(I(X))$ which will be written $\theta(u)$. Therefore, $\theta$ is an injective unital completely contractive homomorphism. The rest of the proof follows from [2, Paragraph 2.1.2].

In the next lemma, we use the $C^*$-envelope of a unital operator space, see [2, Theorem 4.3.1] for details. And we write $R_n$ (resp. $C_n$) the row (resp. column) Hilbert space of dimension $n$. If $X$ is an operator space, we let $C_n(X)$ be the minimal tensor product of $C_n$ and $X$ or equivalently

$$C_n(X) = \left\{ \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 0 \end{pmatrix}, x_i \in X \right\} \subset M_n(X).$$

The definition of $R_n(X)$ is similar using a row instead of a column. Adapting the proof of the first example of the third section of [17], we can obtain:

**Lemma 5.2.** Let $A$ be an injective $C^*$-algebra and $k \in \mathbb{N}^*$. Then

(1) $\mathcal{M}_l(R_k(A)) = A$ *-isomorphically and the action is given by :

$$a \cdot (x_1, \ldots, x_k) = (ax_1, \ldots, ax_k), \text{ for any } a, x_i \in A$$

(2) $\mathcal{M}_r(C_k(A)) = A$ *-isomorphically and the action is given by :

$$\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \cdot a = \begin{pmatrix} x_1a \\ \vdots \\ x_ka \end{pmatrix}, \text{ for any } a, x_i \in A$$

**Proof.** We only prove (1), the proof of (2) is similar. Since $R_n = B(\ell^2_n, \mathbb{C})$, the Paulsen system $S$ of $R_n(A)$ is

$$S = \left\{ \begin{pmatrix} \alpha 1_A \\ y^* \\ \beta I_n \otimes 1_A \end{pmatrix}, \alpha, \beta \in \mathbb{C}, x, y \in R_n(A) \right\} \subset M_{n+1}(A).$$

Clearly the $C^*$-algebra $C^*(S)$ generated by $S$ (inside $M_{n+1}(A)$) coincides with $M_{n+1}(A)$. Next we show that the $C^*$-envelope $C^*_e(S)$ of $S$ is $M_{n+1}(A)$. By the universal property of $C^*_e(S)$, there is a surjective $*$-homomorphism $\pi : C^*(S) \twoheadrightarrow C^*_e(S)$ such that the following commutative diagram holds

$$\begin{array}{ccc}
C^*(S) & \xrightarrow{\pi} & C^*_e(S) \\
\downarrow & & \\
S & \to & C^*_e(S)
\end{array}$$
We let 
\[ p = \pi\left( \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \right) \quad \text{and} \quad q = \pi\left( \begin{pmatrix} 0 & 0 \\ 0 & I_n \otimes 1_A \end{pmatrix} \right). \]

Then \( p \) and \( q \) are projections of \( C_\pi^*(S) \) satisfying \( p + q = 1 \) and \( pq = 0 \). Thus we can decompose \( C_\pi^*(S) \) in ‘2 \times 2’ matrix corners. Hence \( \pi \) is corner preserving and there exist \( \pi_1, \pi_2, \pi_3, \pi_4 \) such that for any \( a \in A, b \in M_n(A), x, y \in R_n(A), \)

\[ \pi\left( \begin{pmatrix} a & x \\ y & b \end{pmatrix} \right) = \begin{pmatrix} \pi_1(a) & \pi_2(x) \\ \pi_3(y) & \pi_4(b) \end{pmatrix}. \]

The (1,2) corners of \( S \) and of \( C^*(S) \) coincide so \( \pi_2 \) is injective (because \( \pi \) extends to \( C^*(S) \) the inclusion \( S \subset C^*(S) \)). Similarly \( \pi_3 \) is injective. On the other hand, for any \( a \in A, x \in R_n(A), \)

\[ \pi_2(ax) = \pi_1(a)\pi_2(x). \]

Thus choosing ‘good \( x \)’, it shows that \( \pi_1 \) is injective too. Analogously, using

\[ \pi_2(xb) = \pi_2(x)\pi_4(b), \quad \text{for any } b \in M_n(A), x \in R_n(A), \]

the previous argument works to prove the injectivity of \( \pi_4 \).

Finally, \( \pi \) is injective and so \( C^*(S) = M_{n+1}(A) \). By assumption on \( A, M_{n+1}(A) \) is an injective \( C^* \)-algebra. Therefore

\[ I(S) = M_{n+1}(A) \quad \ast \text{-isomorphically} \]

and

\[ I_{11}(R_n(A)) = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} I(S) \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} = A. \]

This proves (1).

\[ \square \]

**Remark 5.3.** We acknowledge that after the paper was submitted, D. Blecher pointed out to the author a more general result: let \( X \) be an operator space, then for any \( p, q \in \mathbb{N}^* \),

\[ \mathcal{M}_l(M_{p,q}(X)) = M_p(\mathcal{M}_l(X)). \]

We outline the proof. As in [2, Paragraph 4.4.11], we can define the \( C^* \)-algebra \( \mathcal{C}(X) = I(X)I(X)^* \). Using [2, Corollary 4.6.12], we note that

\[ \mathcal{C}(M_{p,q}(X)) = M_p(\mathcal{C}(X)). \]

Moreover, from [4], the multiplier algebra of \( \mathcal{C}(X) \) coincides with \( I_{11}(X) \) i.e.

\[ \mathcal{M}(\mathcal{C}(X)) = I_{11}(X). \]
Hence, using the two previous facts, we can compute
\[
\mathcal{M}_l(M_{p,q}(X)) = \{ a \in I_{11}(M_{p,q}(X)), aM_{p,q}(X) \subset M_{p,q}(X) \}
\]
\[
= \{ a \in \mathcal{M}(C(M_{p,q}(X))), aM_{p,q}(X) \subset M_{p,q}(X) \}
\]
\[
= \{ a \in \mathcal{M}(M_p(C(X))), aM_{p,q}(X) \subset M_{p,q}(X) \}
\]
\[
= \{ a \in M_p(C(X)), a_{ij}X \subset X, \forall i,j \}
\]
\[
= \{ a \in M_p(I_{11}(X)), a_{ij}X \subset X, \forall i,j \}
\]
\[
= M_p(\mathcal{M}_l(X)).
\]

The next theorem enables to represent completely contractively a module action on an \(n\)-minimal operator space into a \(C^*\)-algebra of the form \(C(\Omega, M_n)\). It constitutes the main result of this section and generalizes \((i) \iff (iii)\) of [3, Theorem 2.2].

**Theorem 5.4.** Let \(A\) be a Banach algebra endowed with an operator space structure (resp. a \(C^*\)-algebra). Let \(X\) be an \(n\)-minimal operator space which is also a left Banach \(A\)-module. Assume that there is a net \((e_t)_t \subset \text{Ball}(A)\) satisfying \(e_t \cdot x \to x\), for any \(x \in X\). The following are equivalent :

(i) \(X\) is a left \(h\)-module over \(A\).

(ii) There exists a compact Hausdorff space \(\Omega\), a complete isometry \(i : X \to C(\Omega, M_n)\) and a completely contractive homomorphism (resp. \(*\)-homomorphism) \(\pi : A \to C(\Omega, M_n)\) such that

\[
i(a \cdot x) = \pi(a)i(x), \quad \text{for any } a \in A, x \in X
\]

**Proof.** Suppose (i). We first treat the Banach algebra case. By Blecher's oplication Theorem (see [2, Theorem 4.6.2]), we know that there is a completely contractive homomorphism \(\eta : A \to \mathcal{M}_l(X)\) such that \(\eta(a)(x) = a \cdot x\), for any \(a \in A, x \in X\). Using \(\theta\) obtained in Lemma 5.1, we have a completely contractive homomorphism \(\sigma = \theta \circ \eta : A \to \mathcal{M}_l(I(X))\) satisfying

\[
\sigma(a)(x) = a \cdot x, \quad \text{for any } a \in A, x \in X.
\]

Moreover, \(I(X)\) is an injective \(n\)-minimal operator space, so

\[
I(X) = \bigoplus_{i \in I}^\infty C(\Omega_i, M_{r_i, k_i})\quad \text{completely isometrically}
\]

where the \(\Omega_i\)'s are Stonean and \(r_i, k_i \leq n\), for any \(i \in I\). We have the completely isometric unital isomorphisms

\[
\mathcal{M}_l(I(X)) = \bigoplus_{i \in I}^\infty \mathcal{M}_l(C(\Omega_i, M_{r_i, k_i}))
\]
\[
= \bigoplus_{i \in I}^\infty \mathcal{M}_l(C_{r_i} \otimes_{\min} R_{k_i} \otimes_{\min} C(\Omega_i))
\]
\[
= \bigoplus_{i \in I}^\infty M_{r_i}(\mathcal{M}_l(R_{k_i} \otimes_{\min} C(\Omega_i)))
\]
\[
= \bigoplus_{i \in I}^\infty M_{r_i}(C(\Omega_i)) \quad \text{(by Lemma 5.2)}
\]
and via these last identifications, the action of $\mathcal{M}_l(I(X))$ on $I(X)$ is the one inherited from the obvious left action of $M_{r_i}$ on $M_{r_i,k_i}$. More precisely for any $u = (f_i \otimes y_i)_i \in \mathcal{M}_l(I(X))$ and $x = (g_i \otimes x_i)_i \in I(X)$,

$$u(x) = (f_ig_i \otimes y_ix_i)_i.$$  

For each $i$, let $\varphi_i : M_{r_i} \to M_n$ (resp. $\varphi_i : M_{r_i,k_i} \to M_n$) be the embedding of $M_{r_i}$ (resp. $M_{r_i,k_i}$) in the ‘up-left corner’ of $M_n$. Hence, as in the end of the proof of Corollary 3.3, we have now a $*$-isomorphism

$$\psi : \mathcal{M}_l(I(X)) \to C(\Omega, M_n)$$

$$(f_i \otimes y_i)_i \mapsto \sum_i \tilde{f}_i \otimes \varphi(y_i)$$

and a complete isometry

$$j : I(X) \to C(\Omega, M_n)$$

$$(g_i \otimes x_i)_i \mapsto \sum_i \tilde{g}_i \otimes \varphi(x_i)$$

which verify

$$j(u(x)) = \psi(u)j(x) \text{ for any } u \in \mathcal{M}_l(I(X)), x \in I(X)$$

Finally $\Omega, i = j_\mathcal{X}$ and $\pi = \psi \circ \sigma$ satisfy the desired relations. If $A$ is a $C^*$-algebra, we conclude using the fact that a contractive homomorphism between $C^*$-algebras is necessarily a $*$-homomorphism. 

**Remark 5.5.** (1) From the previous result, a $C^*$-algebra which acts ‘suitably’ on an $n$-minimal operator space is necessarily an extension of a subhomogeneous $C^*$-algebra of degree $\leq n$.

(2) Suppose that $A$ is unital and its action too (i.e. $1 \cdot x = x$ for any $x$ in $X$). In the previous result, we cannot expect to obtain a unital completely contractive homomorphism $\pi$. Because when $A$ is an operator algebra and $A = X$, the assumption $(i)$ is verified (see the BRS theorem [2, Theorem 2.3.2]). Hence this particular case leads back to the Remark 3.4.

The theorem below could be considered as an ‘$n$-minimal version’ of the CES-theorem (see [2, Theorem 3.3.1]). It is the bimodule version of Theorem 5.4 and its proof is ‘symmetrically’ the same using the two lemmas above.

**Theorem 5.6.** Let $A$ and $B$ be two Banach algebras endowed with an operator space structure (resp. two $C^*$-algebras). Let $X$ be an $n$-minimal operator space which is also a Banach $A$-$B$-bimodule. Assume that there is a net $(e_t)_t \subset Ball(A)$ (resp. $(f_s)_s \subset Ball(B)$) satisfying $e_t \cdot x \to x$ (resp. $x \cdot f_s \to x$), for any $x \in X$. The following are equivalent :

(i) $X$ is an $h$-bimodule over $A$ and $B$. 

Corollary 5.7. Let $A$, $B$ and $X$ be three $n$-minimal operator spaces such that $A$ and $B$ are approximately unital operator algebras and $X$ is a Banach $A$-$B$-bimodule. Assume that there is a net $(e_i)_i \subset \text{Ball}(A)$ (resp. $(f_s)_s \subset \text{Ball}(B)$) satisfying $e_i \cdot x \to x$ (resp. $x \cdot f_s \to x$), for any $x \in X$. The following are equivalent:

(i) $X$ is a left $h$-module over $A$.

(ii) There exists a compact Hausdorff space $\Omega$, a complete isometry $i : X \to C(\Omega, M_n)$ and two completely isometric homomorphisms $\pi : A \to C(\Omega, M_n)$ and $\theta : B \to C(\Omega, M_n)$ such that

$$i(a \cdot x \cdot b) = \pi(a)i(x)\theta(b), \quad \text{for any } a \in A, \ b \in B, \ x \in X.$$ 

The next result states that if $A$ and $B$ are originally $n$-minimal operator algebras, then $\pi$ and $\theta$ can be chosen completely isometric. This corollary generalizes [3 Corollary 2.10].

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SUBALGEBRAS OF $C(\Omega, M_n)$ AND THEIR MODULES

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