Electromagnetic field with constraints and Papapetrou equation

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February 23, 2022

Abstract

It is shown that geometric optical description of electromagnetic wave with account of its polarization in curved space-time can be obtained straightforwardly from the classical variational principle for electromagnetic field. For this end the entire functional space of electromagnetic fields must be reduced to its subspace of locally plane monochromatic waves. We have formulated the constraints under which the entire functional space of electromagnetic fields reduces to its subspace of locally plane monochromatic waves. These constraints introduce variables of another kind which specify a field of local frames associated to the wave and contain some congruence of null-curves. The Lagrangian for constrained electromagnetic field contains variables of two kinds, namely, a congruence of null-curves and the field itself. This yields two kinds of Euler-Lagrange equations. Equations of first kind are trivial due to the constraints imposed. Variation of the curves yields the Papapetrou equations for a classical massless particle with helicity 1.

1 Introduction

Quantum mechanics provides exhaustive description of motion of a particle in limited scales, when typical length of run is comparable with the wavelength, normally, in atomic and sub-atomic ones. When considering motion of a particle in scales which are apparently non-comparable with typical wavelength, that exhaustive quantum mechanical picture becomes less convenient, and to use it, one passes to asymptotical behavior of incident and scattered waves. In astrophysical scales, particularly, when studying deflection of light in gravitational field, classical mechanics is evidently more convenient and one prefers to
consider photon as a classical massless particle drawing a null-geodesic in the space-time. This approach to light propagation in curved space-time is quite satisfactory while photon is considered as a scalar particle. If, however, its polarization becomes important the question arises, how to include it into the classical mechanical description.

If the wavelength is big enough, i.e., photon momentum is relatively small, the commonplace classical mechanical description becomes incomplete. The point is that under some conditions spin of the particle becomes comparable with some components of its orbital momentum, which usually are regarded as zero. This may happen, for example, when describing a light beam incident to a Schwarzschild black hole. Due to the classical mechanical considerations each photon of the beam has zero longitudinal component of orbital momentum. However, its spin is also longitudinal and, hence, contributes the total longitudinal component of angular momentum. Since, on one hand, spin is always collinear to momentum of photon, and, on the other hand, gravitational deflection of the beam changes the momentum, this description contradicts the conservation law of total angular momentum. It is clear that in order to have the correct picture one should take spin of photon into account such a way that the total angular momentum conserves. If it is done the sum of longitudinal components of orbital and inner momenta is constant, thus, the earlier is not zero after deflecting. Therefore, if photon momentum is small enough, this effect can change the shape of scattered beam. Thus, there exist situations when the well-known corpuscular theory does not work.

While in case of spinless particle one can make a choice between the usual (corpuscular) and wave mechanics due to scales under consideration, in case of particle with helicity the earlier does not work, so, the only possibility is to use the latter. The latter provides description in terms of special functions, say, Legendre polynomials, which are convenient while their powers are not very high. However, in some real situations powers of the polynomials are of the order of astronomical distances measured in wavelength units. In these situations corpuscular description in which spin of the particle is taken into account properly, would be much more convenient. The goal of the present work is construct a model of photon with given momentum and helicity ±1 in a curved space-time.

2 Statement of the problem

The desired model is assumed to provide certain world line of a massless particle which has helicity 1 and, at the same time, description of electromagnetic field which everywhere draws a locally plane wave. The main difficulty is that spin of the particle is quantized, thus, the desired construction must contain both classical and quantum degrees of freedom. The notion of classical particle with quantum spin seems to be one of the simplest systems of mixed classical-quantum nature, and attempts to built correct theory of this object are lasting for decades [1]. Our task is somewhat wider, because we try not only to obtain equation of the particle world line, but also a locally plane wave, in other words, to describe propagation of circularly polarized photon in terms of both geometric and wave optics.

An attempt to build such a model was made in the works [2] due to the problem of light propagation in Schwarzschild space-time. Since electromagnetic field presents in the
construction a new question arises, how to combine the field in the space-time and helicity which must be attached to the unknown world line. In the work [2] electromagnetic field was removed by a vector field which was defined on the world lines. This substitution made it possible to combine wave and world line under assumption that if the lines are found properly then the fields attached to them constitute the entire electromagnetic field in the space-time.

In the present work we revise this approach. Instead of specifying a functional space of curves and attaching a vector field to each curve, we assume that the same result could be obtained from the pure electromagnetic Lagrangian. The main idea of this work is that, after all, both geometric and wave optics should follow from pure electromagnetic theory, therefore, the model should be built in the framework of the theory of electromagnetic field. To do it, we start with the well-known form of Lagrangian of electromagnetic field and restrict the functional space of the field variables with its subspace of the fields behaving as locally plane waves. Since restrictions of this sort are known as constraints we assume that putting relevant constraints on the Lagrangian leads to special Euler-Lagrange equations for the waves in question, and the desired description containing both geometric and wave optics follows from it.

3 Locally plane wave and associated orthonormal frame

The notion of plane wave in space-time differs from that in space where the wave vector is orthogonal to the hyperplanes, cannot be tangent to them. In space-time the wave vector is orthogonal to the hyperplanes and, at the same time tangent to them. To see this consider flat space-time and Cartesian coordinates \( \{t, x, y, z\} \) in it which are chosen such a way that the wave has phase \( \phi = \omega(t - z) \). The phase takes constant values on luminal hyperplanes \( t = z \) and the wave vector \( e_- = 2^{-1/2}(\partial_t + \partial_z) \) tangent to the hyperplanes:

\[
e_- \circ \phi = 0.
\]  \(\text{(1)}\)

At the same time the wave vector is orthogonal to the hyperplanes because, due to pseudo-Euclidean metric any null-vector is orthogonal to itself. The wave propagates along this vector, therefore \( e_- \) must be identified with the vector of velocity and plays the role of velocity of photon in corpuscular model. Now we use these two objects to construct an orthonormal frame associated with the wave.

By construction, there exists an isotropic vector \( e_- \) tangent everywhere to the surfaces of constant phase, and, therefore, orthogonal to them. We introduce one more isotropic vector \( e_\pm \) whose direction is arbitrary, requiring only that its scalar product with the vector \( e_- \) is equal to one. As it is done we can introduce two unit space-like vectors \( e_\alpha \) which are orthogonal to each other and to \( e_\pm \). The four vectors defined this way constitute a local orthonormal frame.

In Minkowski space-time and Cartesian coordinates considered above the space-like vectors \( e_\alpha \), \( \alpha = 1, 2 \) are defined as follows:

\[
e_1 = \partial_x, \ e_2 = \partial_y.
\]
They are also tangent to the wave fronts and orthogonal to them. These vectors are used for specifying polarization of the wave. The four vectors $e_\alpha, e_\pm$ where coordinates are chosen such that $e_+ = 2^{-1/2}(\partial_t - \partial_z)$ form an orthonormal frame with metric

\[
\begin{align*}
<e_+, e_+> &= <e_-, e_-> = <e_\pm, e_\alpha> = 0, \\
<e_-, e_+> &= 1, <e_\alpha, e_\beta> = -\delta_{\alpha\beta}, \alpha, \beta = 1, 2.
\end{align*}
\]  

(2)

The frame of 1-forms dual to it, is

\[
\begin{align*}
\theta^- &= dt + dz, \quad \theta^+ = dt - dz \\
\theta^1 &= dx, \quad \theta^2 = dy
\end{align*}
\]

Now, let us pass to a curved space-time and account main features of an electromagnetic wave which can be called locally plane and monochromatic. We start with a wave which possesses locally a scalar function $\phi$ called phase. Gradient of this function is an isotropic 1-form, and its (hyper-)surfaces of level are wave fronts, i.e., it is possible to introduce local Cartesian coordinates in which the wave can be represented locally the way just discussed. This is possible under some special condition, due to which the wavelength is much less than typical scales specified by the space-time curvature. Hereafter we assume that this condition is satisfied. The field of orthonormal frames associated with the wave can be constructed similarly.

As this is done it remains to fix one detail. The point is that the metric of this frame given by the equations (2) remains unchanged if one of isotropic vectors is multiplied by an arbitrary factor and another one is divided by it. Employing this operation we can fix action of the vector $e_-$ on the phase:

\[
e_+ \circ \phi = \omega, \quad \omega = \text{const}
\]

(3)

while, by construction, the equation (1) remains in force. Now, as the orthonormal vector frame associated with the wave $\{e_\pm, e_\alpha\}$ is defined, one can find out the orthonormal covector frame $\{\theta^\pm, \theta^\alpha\}$ dual to it and the connection 1-form $\omega^{\alpha}_{\beta} = \gamma^{\alpha}_{\beta} \theta^c$ for the frame. Hereafter we assume that this is done, and pass to considering constraints whose imposed on electromagnetic field leaves only locally plane waves.

### 4 Electromagnetic field with constraints

If the field in question draws everywhere a locally plane wave there exists a congruence of null-curves $\{x^i(s)\}$ which are integral lines of the vector $e_-$. Since this vector is identified with the velocity the wave vector is collinear to it, and potential of the field $\alpha$ has no $\pm$ components, so, we have a constraint given by

\[
\alpha = A_\beta \theta^\beta, \quad \beta = 1, 2.
\]

(4)

The vectors $e_1, e_2$ are chosen to specify polarization of the wave, hence, their span does locally the instant (two-dimensional space-like) wave front. Therefore we can neglect changes of the potential in their directions. This constraint can be written as

\[
e_1 \circ \alpha = e_2 \circ \alpha = 0
\]

(5)
where the operator $e^\beta \circ$ stands for differentiation along the vector $e_\beta$.

By analogy with plane waves in Cartesian coordinates the amplitudes can be represented in the form

$$A_\beta = a_\beta e^{i\phi}$$

(6)

where the constants $a_\beta$ are complex numbers chosen such a way that the wave has left or right circular polarization and the function $\phi$, specifies the phase of the wave. In these denotions the constraint (5) coincides with the equation (1). Finally, all the constraints imposed above (1, 3-7) reduce to the following: the 1-form of field potential has only one non-zero derivative

$$e^+ \circ \alpha = i\omega \alpha, \quad e^- \circ \alpha = e^\beta \circ \alpha = 0$$

(7)

where $\phi$ and $\omega$ do not depend on the parameter $s$.

5 Action principle for constrained fields

The action functional for electromagnetic field is given by the well-known integral

$$A = \int d\alpha \wedge *d\alpha = \int <d\alpha, d\alpha> \varepsilon,$$

(8)

where $\alpha$ is 1-form of potential of the field, $<,>$ stands for scalar product and $\varepsilon$ denotes the unit 4-form: $\varepsilon = \varepsilon_{ijkl}/4! \theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^l$ that corresponds to four-dimensional integration in the space-time. Straightforward computation of variation of the action (8) yields Maxwell equations, valid for all possible shapes of electromagnetic field. Our goal is to restrict the functional space of the field with subspace of fields which draw everywhere locally plane waves by imposing constraints (1, 3-7).

For this end we expand the quadric form $<d\alpha, d\alpha>$ in the frame $\{\theta\}$ as follows:

$$<d\alpha, d\alpha> = (e_a \circ A_b) (e_c \circ A_d) \langle \theta^a \wedge \theta^b, \theta^c \wedge \theta^d \rangle$$

$$= \sum_{a, c = \pm} (e_a \circ A_b) (e_c \circ A_d) \langle \theta^a \wedge \theta^b, \theta^c \wedge \theta^d \rangle$$

$$= \sum_{a, c = \pm} (e_a \circ A_b) (e_c \circ A_d) \langle \theta^a, \theta^b \rangle \langle \theta^c, \theta^d \rangle$$

$$= \sum_{a, c = \pm} \langle e_a \circ A, e_c \circ A \rangle \langle \theta^a, \theta^c \rangle,$$

where $A = A^\beta e_\beta$.

Though most of terms of the expansion are zero, some of them have non-zero variations. Substituting the constrained Lagrangian into the action integral (8) yields:

$$A = \frac{1}{2} \int \left\{ \langle e_+ \circ A, e_+ \circ A \rangle \langle \theta^+, \theta^+ \rangle + \langle e_- \circ \bar{A}, e_- \circ A \rangle \langle \theta^-, \theta^- \rangle + \langle \bar{e}_- \circ A, e_+ \circ A \rangle + \text{C.C.} \right\} \varepsilon.$$

(9)

where we take into account the fact that for convenience we use complex valued field components. As usual, the components $A$ and their complex conjugates $\bar{A}$ are regarded
as independent variables. Due to the constraints (1) the Lagrangian under integral can be transformed as follows:

\[
\begin{align*}
\{ & \langle e_+ \circ \bar{A}, e_+ \circ A \rangle \langle \theta^+, \theta^+ \rangle + \langle e_- \circ \bar{A}, e_- \circ A \rangle \langle \theta^-, \theta^- \rangle + \\
& + \left( \langle e_- \circ \bar{A}, e_+ \circ A \rangle + \text{C.C.} \right) \langle \theta^-, \theta^+ \rangle \} = \omega^2 \langle \bar{A}, A \rangle \langle \theta^+, \theta^+ \rangle + i \omega \langle A, \dot{\bar{A}} \rangle + \text{C.C.}.
\end{align*}
\]

The third term can be ignored because, as will be shown below, action of the vector \(e_-\) annihilates the field, consequently, the expression \(\langle e_- \circ \bar{A}, e_- \circ A \rangle\) is product of two zero factors, hence, both the third term itself and its variation are identically zero. Though, due to the constraint (1) the term \(\dot{\bar{A}}\) (and \(\dot{A}\)) is equal to zero, we do not ignore it because its variation plays important role in the action principle. Thus, finally, the action integral has the form

\[
A = \int \mathcal{L} \, \varepsilon, \quad \mathcal{L} = \frac{1}{2} \omega \left\{ \omega \langle \bar{A}, A \rangle \langle \dot{x}, \dot{x} \rangle + (i \langle A, \dot{\bar{A}} \rangle + \text{C.C.}) \right\}.
\]

\section{Helicity of constrained fields}

The action integral is evidently invariant under rotations of the frame in the plane formed by the vectors \(e_1\) and \(e_2\). This invariance yields some conservation law due to the Noether theorem. To find the law we consider the change of the form of the action integral under rotations of the local frames specified by an infinitesimal matrix \(\delta\eta_{ab}(s)\) defined as a function of the parameter \(s\) on each curve. Rotation of the frame changes components of the vectors \(A, \bar{A}\) but does not change the vectors. Thus, the first term of the Lagrangian containing scalar product \(<A, \bar{A}>\) does not contribute variation. At the same time the rotation changes derivative of \(\dot{\bar{A}}\):

\[
\delta \dot{\bar{A}}^a = \delta \left( \frac{dA^a}{ds} + \gamma^{a}_{bc} x^b A_c \right) = \left( \frac{D\delta A^a}{ds} \right) + \delta \omega^a_c (\dot{x}) A^c e_a,
\]

\[
\delta \omega^a_c = D_b (\delta \eta_{cb}) \theta^b
\]

where the covariant derivative \(D_b (\delta \eta_{cb})\) is exactly variation of the connection 1-form.

The first term in the variation of \(\dot{\bar{A}}\) does not contribute variation of the action due to field equations. By the result, variation of the action is

\[
\int \delta \mathcal{L} \varepsilon = \int \left\{ i \omega \left( A^a \frac{D\delta \eta_{ab}}{ds} \bar{A}^b e_b \right) \right\} \varepsilon = \int \left\{ i \omega (\delta \eta_{ab}) A^a \bar{A}^b + \text{C.C.} \right\} \varepsilon = \int d[\ldots] + \int 2 \omega \delta \eta_{ab} \frac{D}{ds} \left( \bar{A}^a A^b - \bar{A}^b A^a \right) \varepsilon.
\]

Finally, we have:

\[
\frac{DS_{ab}}{ds} = 0.
\]

So, due to the Noether theorem we obtain conserved spin current with single non-zero component

\[
J^a_{ab} = J_{+ab} = 2 S_{ab}.
\]
where
\[ S^{ab} = \frac{\omega}{2i} \left( \bar{A}^a A^b - \bar{A}^b A^a \right) \]
is the spin tensor of the wave. Due to the constraint (6) it is zero for linearly polarized waves, and for circularly polarized waves has single non-zero element
\[ S^{12} = \pm \omega |a^1 a^2| , \]
whose sign depends only on helicity. The only consequence of this result we need is that the spin has single non-zero component, and as for its magnitude, we accept its quantum value 1 in dimensionless units. Note that the spin is always pointed along the vector of velocity, so no special equation is needed for it. This fact provides implementation of the Tulczyjew constraint which requires that conversion of the particle momentum with its spin is zero [5].

7 Field equations

Now we return to the action functional (10) and compute its variations only under small variations of the field which has two components \( A_1 \) and \( A_2 \). Variation of the first term in the constrained Lagrangian is identically zero because it contains the factor \( \langle \theta^+, \theta^+ \rangle \) which is not varied, therefore we ignore it. Variation of the rest part of the action is
\[
\delta A = \delta \int \left\{ \langle e_+ \circ A, e_- \circ \bar{A} \rangle + C.C. \right\} \varepsilon = \\
\int \left[ \langle \delta (e_+ \circ A), e_- \circ \bar{A} \rangle + \langle e_+ \circ A, \delta (e_- \circ \bar{A}) \rangle + C.C. \right] \varepsilon = \\
\int \left[ \langle e_+ \circ \delta A, e_- \circ \bar{A} \rangle + \langle e_+ \circ A, e_- \circ \delta \bar{A} \rangle + C.C. \right] \varepsilon
\]
where action of vectors \( e_\pm \) is considered as differentiation along the vectors. The next step is to extract total derivatives:
\[
\delta A = \int \left[ e_+ \circ \left\{ \delta A, e_- \circ \bar{A} \right\} + e_- \circ \left\{ e_+ \circ A, \delta \bar{A} \right\} + C.C. \right] \varepsilon + \\
\int \langle \delta \bar{A}, e_+ \circ (e_- \circ A) + e_- \circ (e_+ \circ A) \rangle \varepsilon + \\
\int \langle \delta A, e_+ \circ (e_- \circ \bar{A}) + e_- \circ (e_+ \circ \bar{A}) \rangle \varepsilon.
\]
Note that due to the metric of the null-frame (2) combinations like \( e_+ \circ V_- \) are parts of divergence of a vector \( V \), and if the vector has only one component ‘−’ this expression coincides with its divergence. In particular, the combination \( e_{(+} \circ \left( e_{-} \circ f \right) \) is exactly Dalembert operator applied to a function \( f \) which is constant on the wave fronts. Consequently, the first term in the right-hand side of the equation above is exactly a divergence, hence the integral can be taken by parts and variation of the action integral becomes:
\[
= \int d[...] - \int \left\{ \langle \delta \bar{A}, e_{(+} \circ \left( e_{-} \circ A \right) \rangle + C.C. \right\} \varepsilon
\]
where the first term reduces to a surface integral and figure brackets at subscripts mean symmetrization. As usual, the surface integral vanishes at infinity and all the rest reduces to the following Euler-Lagrange equation:

\[ e_+ \circ (e_- \circ A) = 0. \]

Evidently, this covariant equation reduces to Dalembert equation in local Cartesian coordinates provided that the curves \( x(s) \) are locally null straight lines. In fact, any vector whose covariant derivative along the curve is zero:

\[ e_- \circ A = 0, \quad (12) \]

and so for the complex conjugate, satisfies this equation. The constrained fields satisfy this equation due to the equations (11), consequently, this part of the entire variation of the action integral is zero due to the constraints. Though the field equation leaves \( \omega \) an arbitrary function of the phase we restrict our analysis with monochromatic waves for which this value is constant.

8 Papapetrou equations

It remains to consider the second part of variation of the action integral, produced by variation of local frames under fixed field variables. Since the local frames are defined as co-moving frames on the congruence of null-curves, it is possible to introduce variation of the congruence and derive variation of the frames from it. Small change of shape of a curve \( x(s) \) causes small change of the tangent vector without change of its length. Consequently, variation of the vector \( e_- \) is orthogonal to it and to the complementary null-vector \( e_+ \), hence, belongs to the span of the two polarization vectors \( e_\beta \). Since the vector \( e_+ \) is also orthogonal to the span, it suffers no change. Therefore, variations of both \( e_+ \) and the corresponding covector \( \theta^- \) are zero. Thus, only variations of the vector \( e_- \) and the corresponding 1-form \( \theta^+ \) contribute variation of the action integral (10).

The first term in the Lagrangian (10) contains the factor \( \langle \theta^+ , \theta^+ \rangle = \langle e_- , e_- \rangle \equiv 0 \), therefore its contribution is predetermined only by variation of the vector \( e_- = \dot{x} \). Consider variation of four-dimensional integral

\[ \frac{\omega^2}{2} \int |A|^2 \langle \dot{x}, \dot{x} \rangle \varepsilon \]

with respect to variation of the curves. Variation of the factor \( \langle \dot{x}, \dot{x} \rangle \) is well-known from the variation principle for geodesics [6]. Here we can use the fact that variation of the integrand reduces to the scalar product of the vector of variation of the curve \( \delta x(s) \) and the covariant acceleration \( \frac{D \dot{x}}{ds} \):

\[ \delta \int \frac{\omega^2}{2} |A|^2 \langle \dot{x}, \dot{x} \rangle \varepsilon = \omega^2 \int |A|^2 \langle \delta x(s), \frac{D \dot{x}}{ds} \rangle. \quad (13) \]

The second term in the Lagrangian (10) has single zero factor \( \dot{\bar{A}} \) in the scalar product, consequently, non-zero contribution to the variation of the action integral appears only
when varying this factor. The zero factor to be varied is $\dot{\bar{A}}$: $$\dot{\bar{A}} = \left( \frac{d\bar{A}^a}{ds} + \gamma_{bc}^a \dot{x}^b \bar{A}^c \right) e_a.$$ Variation of the covariant derivative contains derivative on $s$ and the term containing the connection $\gamma_{bc}^a$. Variation of the first of them does not contribute variation of the action integral because neither polarization vectors $e_a$ nor components of the field change under varying the congruence of curves. The only term suffering some change is connection $\gamma_{bc}^a$. Therefore we can write down contribution of this term as follows:

$$\frac{1}{2} \delta \int \omega \left( i \langle A, \dot{\bar{A}} \rangle + \text{C.C.} \right) \varepsilon = -\frac{1}{2} \int \left( i A^b \delta \gamma_{bc}^a \bar{A}^c + \text{C.C.} \right) \varepsilon$$

where the sign minus appears due to the lower index at the field component. Thus, the next task is to find variation of the connection $\delta \gamma_{bc}^a$.

It is easier to find variation of the 1-form $\omega_{ab}$ because the variation is to be taken in a fixed point under changing the field of frames which is dragged by the vector $\delta x$. This variation is, by definition, Lie derivative of the connection 1-form with respect to this vector. So, to find variation of the connection 1-form it suffices to take its Lie derivative with respect to the vector $\delta x$. Lie derivative of a 1-form $\lambda$ with respect to a vector $\vec{v}$ is

$$\mathcal{L}_{\vec{v}} \lambda = d\lambda(\vec{v}) + d(\lambda(\vec{v}))$$

Unlike ordinary 1-form the form of connection has components referred to local frames. Since the frames are built on the vector of velocity on the congruence its variation causes some infinitesimal rotation of the frames. Denote the corresponding matrix of rotation $\eta_{ac}$. This rotation transforms components of the connection 1-form and must be taken into account. To do it it is necessary to obtain explicit form of this matrix from the dragging vector $\delta x$.

Consider a point in the space-time a curve passing through it and the local frame and small variation of congruence of curves given by small vector $\delta x$ which drags the congruence. This dragging replaces the curve passed through this point with the curve dragged by the vector and the local frame built in this point is also to be replaced by the frame dragged by this vector from a neighboring point. Since, on one hand both the frames are orthonormal variation of the frames is small rotation. On the other hand, since this rotation is specified by dragging orthonormal frame from a neighboring point, this transformation is given by the connection itself, in other words the matrix of rotation $\eta_{ac}$ is exactly the value of the form of connection on the dragging vector: $\eta_{ab} = \omega_{ab}(\delta x)$.

Thus, Lie derivative of the connection 1-form with respect to the vector $\delta x$ is

$$\mathcal{L}_{\delta x} \omega_{ab} = d\omega_{ab}(\delta x) + d(\omega_{ab}(\delta x)) - \eta_c^b \omega_{ac}^b + \eta_c^b \omega_{ab}^c.$$ 

Substituting the matrix of rotation we obtain the desired Lie derivative:

$$\mathcal{L}_{\delta x} \omega_{ab} = d\omega_{ab}(\delta x) - \omega_{ac}^b(\delta x)\omega_{bc}^a + \omega_{bc}^b(\delta x)\omega_{ab}^c + d(\omega_{ab}(\delta x)) =$$

$$(d\omega_{ab}(\delta x) + \omega_{ac}^b \wedge \omega_{ab}^c)(\delta x) + d(\omega_{ab}(\delta x)) = \Omega_{ab}(\delta x) + d(\omega_{ab}(\delta x))$$
where we have obtained the curvature 2-form \( \Omega_{ab} \equiv R_{cdab} \theta^c \wedge \theta^d \). Substituting now this into variation of \( \dot{\mathbf{A}} \) gives:

\[
\delta \dot{\mathbf{A}} = R_{dbc} \delta x^d \dot{x}^b \dot{\mathbf{A}}^c_{\mathbf{e}a} + \dot{x}^d \partial_d (\gamma_{bc}^a \delta x^b \ddot{\mathbf{A}}^c_{\mathbf{e}a}) = R_{dbc} \delta x^d \dot{x}^b \dot{\mathbf{A}}^c_{\mathbf{e}a} + (\gamma_{bc}^a \delta x^b \ddot{\mathbf{A}}^c_{\mathbf{e}a}) \cdot \mathbf{e}_a.
\]

Variation of \( \dot{\mathbf{A}} \) is similar, so after composing the total variation of the second term in the action integral we obtain the two terms. One is total derivative of \( \omega(A_a \gamma_{bc}^a \delta x^b \ddot{\mathbf{A}}^c) \) on \( s \), which vanishes on the endpoints of the curves. Thus, the whole of variation of this part of Lagrangian is given by another term which is

\[
\frac{i \omega}{2} R_{dbc} \delta x^d \dot{x}^b (\ddot{\mathbf{A}}^c_{\mathbf{e}a} - \ddot{A}_a \mathbf{e}^c) = -R_{dbc} \delta x^d \dot{x}^b S^c_{\mathbf{e}a},
\]

where we introduce spin by its only component \( S_{12} \). The Euler-Lagrange equation for the curves \( x(s) \) coincides with Papapetrou equation:

\[
\omega^2 |A|^2 \frac{D\dot{x}^a}{ds} = R_{db-c} \dot{x}^c S^{db}.
\] (14)

9 Conclusion

Geometric-optical description of electromagnetic wave in curved space-time, with account of its polarization is obtained straightforwardly from the classical variational principle for electromagnetic field. For this end the entire functional space of electromagnetic fields is reduced to its subspace of locally plane monochromatic waves. Therefore, first of all, the notion of locally plane monochromatic wave in curved space-time should be defined. It turns out that waves of this sort exist provided that their wavelengths are small compared with scales under consideration. Assuming this, we have formulated the constraints under which the entire functional space of electromagnetic fields reduces to its subspace of locally plane monochromatic waves and imposed these constraints. These constraints not only reduce field variables but also introduce variables of another kind which specify a field of local frames associated to the wave and contain some congruence of null-curves \( x^i(s) \). These curves become the main object in the construction because it specifies the field of local frames and the field variables \( \dot{\mathbf{A}} \) and \( \ddot{\mathbf{A}} \) are referred to this frame.

Returning to the action principle for the constrained electromagnetic field we have Lagrangian \((10)\) which contains variables of two kinds, namely, a congruence of curves \( x^i(s) \) and the field itself and have two kinds of Euler-Lagrange equations. Equations of first kind reduce to local Dalembert equation for the field components \( \dot{\mathbf{A}} \) and \( \ddot{\mathbf{A}} \) which are trivial due to the constraints imposed. Variation of the curves yields all the rest equations which contain the main result of this investigation. It turns out that the Euler-Lagrange equations they yield are exactly the Papapetrou equations for a classical massless particle with helicity 1. This equation determines the shape of the 0-curves which, by construction, can be considered as world lines of photons with the same wavelengths and helicities. They apparently differ from null-geodesics and, thereby manifest influence of spin-gravitational...
interaction on propagation of electromagnetic waves in gravitational fields. Effect of this interaction is proportional to the wavelength $\lambda$, therefore this fact can be observed in radioastronomy.

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