A combinatorial proof of the Gaussian product inequality beyond the MTP$_2$ case

Abstract: A combinatorial proof of the Gaussian product inequality (GPI) is given under the assumption that each component of a centered Gaussian random vector $X = (X_1, \ldots, X_d)$ of arbitrary length can be written as a linear combination, with coefficients of identical sign, of the components of a standard Gaussian random vector. This condition on $X$ is shown to be strictly weaker than the assumption that the density of the random vector $|X| = (|X_1|, \ldots, |X_d|)$ is multivariate totally positive of order 2, abbreviated MTP$_2$, for which the GPI is already known to hold. Under this condition, the paper highlights a new link between the GPI and the monotonicity of a certain ratio of gamma functions.

Keywords: complete monotonicity, gamma function, Gaussian product inequality, Gaussian random vector, moment inequality, multinomial, multivariate normal, polygamma function

MSC 2020: Primary 60E15, Secondary 05A20, 33B15, 62E15, 62H10, 62H12

1 Introduction

The Gaussian product inequality (GPI) is a long-standing conjecture, which states that for any centered Gaussian random vector $X = (X_1, \ldots, X_d)$ of dimension $d \in \mathbb{N} = \{1, 2, \ldots\}$ and every integer $m \in \mathbb{N}$, one has

$$E \left( \prod_{i=1}^{d} X_i^{2m} \right) \geq \prod_{i=1}^{d} E(X_i^{2m}).$$

(1)

This inequality is known to imply the real polarization problem conjecture in functional analysis [14], and it is related to the so-called $U$-conjecture to the effect that if $P$ and $Q$ are two nonconstant polynomials on $\mathbb{R}^d$ such that the random variables $P(X)$ and $Q(X)$ are independent, then there exist an orthogonal transformation $L$ on $\mathbb{R}^d$ and an integer $k \in \{1, \ldots, d-1\}$ such that $P \circ L$ is a function of $(X_1, \ldots, X_k)$ and $Q \circ L$ is a function of $(X_{k+1}, \ldots, X_d)$; see, e.g., [7,14] and the references therein.

Inequality (1) is well known to be true when $m = 1$; see, e.g., Frenkel [5]. Karlin and Rinott [8] also showed that it holds when the random vector $|X| = (|X_1|, \ldots, |X_d|)$ has a multivariate totally positive density of order 2, denoted MTP$_2$. As stated in Remark 1.4 of their paper, the latter condition is verified, among others, in dimension $d = 2$ for all nonsingular Gaussian random pairs.

Interest in the problem has recently gained traction when Lan et al. [11] established the inequality in dimension $d = 3$. Hope that the result might be true in general is also fueled by the fact, established by Wei [25], that for any reals $\alpha_0, \ldots, \alpha_d \in (-1/2, 0)$, one has...
Li and Wei [13] have further conjectured that the latter inequality holds for all reals $a_1, \ldots, a_d \in [0, \infty)$ and any centered Gaussian random vector $X$.

The purpose of this paper is to report a combinatorial proof of inequality (2) in the special case where the reals $a_1, \ldots, a_d$ are nonnegative integers and when each of the components $X_1, \ldots, X_d$ of the centered Gaussian random vector $X$ can be written as a linear combination, with coefficients of identical sign, of the components of a standard Gaussian random vector. A precise statement of this assumption is given as Condition (III) in Section 2, and the proof of the main result, Proposition 2, appears in Section 3. It is then shown in Section 4, see Proposition 3, that this condition is strictly weaker than the assumption that the random vector $|X|$ is MTP$_2$.

Coincidentally, shortly after the first version of the present paper was posted on arXiv, inequality (2) for all nonnegative integers $a_1, \ldots, a_d \in \mathbb{N}_0 = \{0, 1, \ldots\}$ was established under an even weaker assumption, stated as Condition (IV) in Section 2. The latter condition states that up to a change of sign, the components of the Gaussian random vector $X$ are all nonnegatively correlated; see Lemma 2.3 of Russell and Sun [22]. Therefore, the present paper's main contribution resides in the method of proof using a combinatorial argument closely related to the complete monotonicity of multinomial probabilities previously shown by Ouimet [16] and Qi et al. [17].

All background materials required to understand the contribution and put it in perspective are provided in Section 2. The statements and proofs of the paper's results are then presented in Sections 3 and 4. This article concludes with a brief discussion in Section 5. For completeness, a technical lemma due to Ouimet [16], which is used in the proof of Proposition 2, is included in the Appendix.

## 2 Background

First, recall the definition of multivariate total positivity of order 2 (MTP$_2$) on a set $S \subseteq \mathbb{R}^d$.

**Definition 1.** A density $f : \mathbb{R}^d \to [0, \infty)$ supported on $S$ is said to be multivariate totally positive of order 2, denoted MTP$_2$, if and only if, for all vectors $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in S$, one has

$$f(x \cup y)f(x \land y) \geq f(x)f(y),$$

where $x \cup y = (\max(x_1, y_1), \ldots, \max(x_d, y_d))$ and $x \land y = (\min(x_1, y_1), \ldots, \min(x_d, y_d))$.

Densities in this class have many interesting properties, including the following result, which corresponds to equation (1.7) of Karlin and Rinott [8].

**Proposition 1.** Let $Y$ be an MTP$_2$ random vector on $S$, and let $\varphi_1, \ldots, \varphi_r$ be a collection of nonnegative and (component-wise) nondecreasing functions on $S$. Then

$$E\left(\prod_{i=1}^r \varphi_i(Y)\right) \geq \prod_{i=1}^r E\{\varphi_i(Y)\}.$$

In particular, let $X = (X_1, \ldots, X_d)$ be a $d$-variate Gaussian random vector with zero mean and nonsingular covariance matrix $\text{var}(X)$. Suppose that the following condition holds.

(i) The random vector $|X| = (|X_1|, \ldots, |X_d|)$ belongs to the MTP$_2$ class on $[0, \infty)^d$. 
Under Condition (I), the validity of the GPI conjecture (2) for all reals $a_1, \ldots, a_d \in [0, \infty)$ follows from Proposition 1 with $r = d$ and maps $\varphi_1, \ldots, \varphi_d$ defined, for every vector $y = (y_1, \ldots, y_d) \in [0, \infty)^d$ and integer $i \in \{1, \ldots, d\}$, by

$$
\varphi_i(y) = y_i^{2a_i}.
$$

When $X = (X_1, \ldots, X_d)$ is a centered Gaussian random vector with covariance matrix $\text{var}(X)$, Theorem 3.1 of Karlin and Rinott [8] finds an equivalence between Condition (I) and the requirement that the off-diagonal elements of the inverse of $\text{var}(X)$ are all nonpositive up to a change of sign for some of the components of $X$. The latter condition can be stated more precisely as follows using the notion of signature matrix, which refers to a diagonal matrix whose diagonal elements are $\pm 1$.

(II) There exists a $d \times d$ signature matrix $D$ such that the covariance matrix $\text{var}(DX)^{-1}$ only has nonpositive off-diagonal elements.

Two other conditions of interest on the structure of the random vector $X$ are as follows.

(III) There exist a $d \times d$ signature matrix $D$ and a $d \times d$ matrix $C$ with entries in $[0, \infty)$ such that the random vector $DX$ has the same distribution as the random vector $ZC$, where $Z \sim N_d(0_d, I_d)$ is a $d \times 1$ Gaussian random vector with zero mean vector $0_d$ and identity covariance matrix $I_d$.

(IV) There exists a $d \times d$ signature matrix $D$ such that the covariance matrix $\text{var}(DX)$ has only nonnegative elements.

Recently, Russell and Sun [22] used Condition (IV) to show that, for all integers $d \in \mathbb{N}$, $n_1, \ldots, n_d \in \mathbb{N}_0$ and $k \in \{1, \ldots, d - 1\}$, and up to a change of sign for some of the components of $X$, one has

$$
E \left( \prod_{i=1}^{d} X_i^{2n_i} \right) \geq E \left( \prod_{i=1}^{k} X_i^{2n_i} \right) E \left( \prod_{i=k+1}^{d} X_i^{2n_i} \right). \quad (3)
$$

This result was further extended by Edelmann et al. [4] to the case where the random vector $(X_1^2, \ldots, X_d^2)$ has a multivariate gamma distribution in the sense of Krishnamoorthy and Parthasarathy [10]. See also [3] for a use of Condition (IV) in the context of the Gaussian correlation inequality (GCI) conjecture.

In the following section, it will be shown how Condition (III) can be exploited to give a combinatorial proof of a weak form of inequality (3). It will then be seen in Section 4 that Condition (II) implies Condition (III), thereby proving the implications illustrated in Figure 1 between Conditions (I)–(IV). That the implications Condition (II) $\Rightarrow$ Condition (III) and Condition (III) $\Rightarrow$ Condition (IV) are strict can be checked using, respectively, the covariance matrices

![Figure 1: Implications between Conditions (I)–(IV) for a nonsingular centered Gaussian random vector $X$, with references.](image-url)
and
\[
\var(X) = \begin{pmatrix}
0 & 1/4 & 1/2 & 1/2 \\
1/4 & 1 & 1/2 & 1/2 \\
1/2 & 1/2 & 1 & 1/2 \\
1/2 & 1/2 & 1/2 & 1
\end{pmatrix} + \varepsilon
\]
for some appropriate \( \varepsilon \in (0, \infty) \).

In the first example, the matrix \( \var(X) \) is completely positive (meaning that it can be written as \( CC^\top \) for some matrix \( C \) with nonnegative entries) and positive definite by construction. Furthermore, the matrix \( D \var(X)^{-1}D \) has at least one positive off-diagonal element for any of the eight possible choices of \( 3 \times 3 \) signature matrix \( D \). Another way to see this is to observe that if \( A = \var(X)^{-1} \), then the cyclic product \( a_1 a_2 a_3 \), which is invariant to \( A \mapsto DA \), is strictly positive in the aforementioned example, so that the off-diagonal elements of \( \var(X)^{-1} \) cannot all be nonpositive. This shows that \( (\text{III}) \implies (\text{II}) \). This example was adapted from ideas communicated to the authors by Thomas Royen.

For the second example, when \( \varepsilon = 0 \), it is mentioned by Maxfield and Minc [15], using a result from Hall [6], that the matrix is positive semidefinite and has only nonnegative elements but that it is not completely positive. Using the fact that the set of \( 5 \times 5 \) completely positive matrices is closed, there exists \( \varepsilon \in (0, \infty) \) small enough that the matrix \( \var(X) \) is positive definite and has only nonnegative elements but is not completely positive. More generally, given that the elements of \( \var(X) \) are all nonnegative, the matrix \( D \var(X) D \) is not completely positive for any of the \( 32 \) possible choices of \( 5 \times 5 \) signature matrix \( D \), which shows that \( (\text{IV}) \implies (\text{III}) \). This idea was adapted from comments by Stein [24].

### 3 A combinatorial proof of the GPI conjecture

The following result, which is this paper’s main result, shows that the extended GPI conjecture of Li and Wei [13] given in (2) holds true under Condition (III) when the reals \( a_1, \ldots, a_d \) are nonnegative integers. This result also follows from inequality (3), due to Russell and Sun [22], but the argument below is completely different from the latter authors’ derivation based on Condition (IV).

**Proposition 2.** Let \( X = (X_1, \ldots, X_d) \) be a \( d \)-variate centered Gaussian random vector. Assume that there exist a \( d \times d \) signature matrix \( D \) and a \( d \times d \) matrix \( C \) with entries in \( \{0, \infty\} \) such that the random vector \( DX \) has the same distribution as the random vector \( CZ \), where \( Z \sim N_d(O_d, I_d) \) is a \( d \)-dimensional standard Gaussian random vector. Then, for all integers \( n_1, \ldots, n_d \in \mathbb{N}_0 \),

\[
E\left( \prod_{i=1}^{d} X_i^{2n_i} \right) \geq \prod_{i=1}^{d} E(X_i^{2n_i}).
\]

**Proof.** In terms of \( Z \), the claimed inequality is equivalent to

\[
E\left( \prod_{j=1}^{d} \left( \sum_{i=1}^{d} c_{ij} Z_j \right)^{2n_j} \right) \geq \prod_{j=1}^{d} E\left( \left( \sum_{i=1}^{d} c_{ij} Z_j \right)^{2n_j} \right).
\]

For each integer \( j \in \{1, \ldots, d\} \), set \( k_j = k_{ij} + \cdots + k_{dj} \) and \( L_j = \ell_{ij} + \cdots + \ell_{dj} \), where \( k_j \) and \( \ell_j \) are nonnegative integer-valued indices to be used in expressions (5) and (6).
By the multinomial formula, the left-hand side of inequality (4) can be expanded as follows:

$$E\left(\prod_{i=1}^{d} \sum_{k_i \in \mathbb{N}_0: k_i \geq 2n_i, k_i + \cdots + k_d = 2m} \frac{2n_i^{k_i} d^{k_i}}{k_i! \cdots k_d! \prod_{j=1}^{d} c_{ij}^{k_j}}\right).$$

Calling on the linearity of expectations and the mutual independence of the components of the random vector $Z$, one can then rewrite this expression as follows:

$$E\left(\sum_{k_i \in \mathbb{N}_0: k_i \geq 2n_i, k_i + \cdots + k_d = 2m} \frac{2n_i^{k_i} d^{k_i}}{k_i! \cdots k_d! \prod_{j=1}^{d} c_{ij}^{k_j}} \prod_{j=1}^{d} E(Z_j^{n_j}) \prod_{j=1}^{d} \prod_{i \neq j} c_{ij}^{k_{ij}}\right).$$

Given that the coefficients $c_{ij}$ are all nonnegative by assumption, and exploiting the fact that, for every integer $j \in \{1, \ldots, d\}$ and $m \in \mathbb{N}_0$,

$$E(Z_j^{2m}) = \frac{(2m)!}{2^m m!},$$

one can bound the left-hand side of inequality (4) from below by

$$\sum_{\epsilon_1 \in \mathbb{N}_0: \epsilon_1 + \cdots + \epsilon_d = m} \cdots \sum_{\epsilon_d \in \mathbb{N}_0: \epsilon_1 + \cdots + \epsilon_d = m} \left(\prod_{j=1}^{d} E(Z_j^{2\epsilon_j})\right) \left(\prod_{i=1}^{d} \left(2\epsilon_i, \ldots, 2\epsilon_d\right)\right) \prod_{j=1}^{d} c_{ij}^{2\epsilon_j}. \tag{5}$$

The right-hand side of (4) can be expanded in a similar way. Upon using the fact that $E(Y^{2m}) = (2m)! \sigma^{2m} / (2^m m!)$ for every integer $m \in \mathbb{N}_0$ when $Y \sim N(0, \sigma^2)$, one finds

$$\prod_{i=1}^{d} E\left(\left(\sum_{j=1}^{n_i} c_{ij} Z_j\right)^2\right) = \prod_{i=1}^{d} \frac{(2n_i)!}{2^{n_i} n_i!} \sum_{\epsilon_1 + \cdots + \epsilon_d = n_i} \left(\prod_{j=1}^{d} \epsilon_j\right) \prod_{j=1}^{d} c_{ij}^{\epsilon_j}. \tag{6}$$

Next, compare the coefficients of the corresponding powers $c_{ij}^{2\epsilon_j}$ in expressions (5) and (6). To prove inequality (4), it suffices to show that, for all integer-valued vectors $\epsilon_1, \ldots, \epsilon_d \in \mathbb{N}_0$ satisfying $\epsilon_1 + \cdots + \epsilon_d = n_i$ for every integer $i \in \{1, \ldots, d\}$, one has

$$\left(\prod_{j=1}^{d} \frac{(2\epsilon_j)!}{2^{\epsilon_j} \epsilon_j!}\right) \left(\prod_{i=1}^{d} \left(2\epsilon_i, \ldots, 2\epsilon_d\right)\right) \geq \prod_{i=1}^{d} \frac{(2n_i)!}{2^{n_i} n_i!} \left(\prod_{j=1}^{d} \epsilon_j\right).$$

Taking into account the fact that $2\epsilon_1 + \cdots + 2\epsilon_d = 2n_1 + \cdots n_d$, and after cancelling some factorials, one finds that the aforementioned inequality reduces to
\[ \prod_{j=1}^{d} \frac{(2L_j)!}{\prod_{k=1}^{L_j}(2\ell_{i_k})!} \geq \prod_{j=1}^{d} \frac{L_j!}{\prod_{k=1}^{L_j}(q_{i_k})!}. \]  

(7)

Therefore, the proof is complete if one can establish inequality (7). To this end, one can assume without loss of generality that the integers \(L_1, \ldots, L_d\) are all nonzero; otherwise, inequality (7) reduces to a lower dimensional case. For any given integers \(L_1, \ldots, L_d \in \mathbb{N}\) and every integer \(j \in \{1, \ldots, d\}\), define the function

\[ a \mapsto g_j(a) = \frac{\Gamma(aL_j + 1)}{\prod_{k=1}^{L_j} \Gamma(at_{i_k} + 1)}, \]

on the interval \((-1/L_j, \infty)\), where \(\Gamma\) denotes Euler’s gamma function.

To prove inequality (7), it thus suffices to show that, for every integer \(j \in \{1, \ldots, d\}\), the map \(a \mapsto \ln(g_j(a))\) is nondecreasing on the interval \([0, \infty)\). Direct computations yield, for every real \(a \in [0, \infty)\),

\[ \frac{d}{da} \ln(g_j(a)) = L_j \psi(aL_j + 1) - \sum_{i=1}^{d} \ell_{i_j} \psi(at_{i_j} + 1), \]

\[ \frac{d^2}{da^2} \ln(g_j(a)) = L_j^2 \psi'(aL_j + 1) - \sum_{i=1}^{d} \ell_{i_j}^2 \psi'(at_{i_j} + 1), \]

where \(\psi = (\ln \Gamma)'\) denotes the digamma function. Now call on the integral representation [1, p. 260]

\[ \psi'(z) = \int_{0}^{\infty} \frac{te^{-(z-1)t}}{e^t - 1} \, dt, \]

valid for every real \(z \in (0, \infty)\), to write

\[ \frac{d^2}{da^2} \ln(g_j(a)) = \int_{0}^{\infty} \left( \frac{L_j}{e^t - 1} \right) \cdot L_j dt - \sum_{i=1}^{d} \int_{0}^{\infty} \left( \frac{\ell_{i_j}}{e^t - 1} \right) \cdot \ell_{i_j} dt = \int_{0}^{\infty} \left( \frac{1}{a^{L_j} - 1} - \sum_{i=1}^{d} \frac{1}{(e^{\ell_{i_j}}L_j)_{L_j} - 1} \right) \, ds. \]  

(8)

Given that \((\ell_{i_j} + \cdots + \ell_{i_d})/L_j = 1\) by construction, the quantity within braces in equation (8) is always nonnegative by Lemma 1.4 of Ouimet [16]; this can be checked upon setting \(y = e^{\ell_{i_j}}\) and \(u_t = \ell_{i_j}/L_j\) for every integer \(i \in \{1, \ldots, d\}\) in that paper’s notation. Alternatively, see p. 516 in the study by Qi et al. [17]. Therefore,

\[ \forall a \in [0, \infty), \quad \frac{d^2}{da^2} \ln(g_j(a)) \geq 0. \]  

(9)

In fact, the map \(a \mapsto d^2 \ln(g_j(a))/da^2\) is even completely monotonic. Moreover, given that

\[ \frac{d}{da} \ln(g_j(a)) \bigg|_{a=0} = L_j \psi(1) - \sum_{i=1}^{d} \ell_{i_j} \psi(1) = 0 \times \psi(1) = 0, \]

one can deduce from inequality (9) that

\[ \forall a \in [0, \infty), \quad \frac{d}{da} \ln(g_j(a)) \geq 0. \]

Hence, the map \(a \mapsto \ln(g_j(a))\) is nondecreasing on \([0, \infty)\). This concludes the argument. \(\Box\)

### 4 Condition (II) implies Condition (III)

This paper’s second result, stated below, shows that Condition (II) implies Condition (III). In view of Figure 1, one may then conclude that Condition (I) also implies Conditions (III) and (IV), and hence, also Condition (II).
Proposition 3. Let $\Sigma$ be a symmetric positive definite matrix with Cholesky decomposition $\Sigma = CC^\top$. If the off-diagonal entries of $\Sigma^{-1}$ are all nonpositive, then the elements of $C$ are all nonnegative.

Proof. The proof is by induction on the dimension $d$ of $\Sigma$. The claim trivially holds when $d = 1$. Assume that it is verified for some integer $n \in \mathbb{N}$, and fix $d = n + 1$. Given the assumptions on $\Sigma$, one can write

$$\Sigma^{-1} = \begin{pmatrix} a & v^\top \\ v & B \end{pmatrix}$$

in terms of a real $a \in (0, \infty)$, an $n \times 1$ vector $v$ with nonpositive components, and an $n \times n$ matrix $B$ with nonpositive off-diagonal entries.

Given that $\Sigma$ is symmetric positive definite by assumption, so is $\Sigma^{-1}$, and hence so are $B$ and $B^{-1}$. Moreover, the off-diagonal entries of $B = (B^{-1})^{-1}$ are nonpositive, and hence by induction, the factor $L$ in the Cholesky decomposition $B^{-1} = LL^\top$ has nonnegative entries. Letting $w = a - v^\top LL^\top v$ denote the Schur complement, which is strictly positive, one has

$$\Sigma^{-1} = \begin{pmatrix} a & v^\top \\ v & (LL^\top)^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{w} & v^\top L \\ 0_n & (L^\top)^{-1} \end{pmatrix} \begin{pmatrix} \sqrt{w} & 0_n \\ 0_n & (L^\top)^{-1} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{w} & 0_n \\ -LL^\top v/\sqrt{w} & L \end{pmatrix} \begin{pmatrix} 1/\sqrt{w} & 0_n \\ -LL^\top v/\sqrt{w} & L \end{pmatrix} = CC^\top.$$

where $0_n$ is an $n \times 1$ vector of zeros. Accordingly,

$$\Sigma = \begin{pmatrix} \sqrt{w} & 0_n \\ L^\top v & L^{-1} \end{pmatrix} \begin{pmatrix} \sqrt{w} & v^\top L \\ 0_n & (L^\top)^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1/\sqrt{w} & 0_n \\ -LL^\top v/\sqrt{w} & L \end{pmatrix} \begin{pmatrix} 1/\sqrt{w} & 0_n \\ -LL^\top v/\sqrt{w} & L \end{pmatrix} = CC^\top.$$

Recall that $w$ is strictly positive and all the entries of $L$ and $-v$ are nonnegative. Hence, all the elements of $C$ are nonnegative, and the argument is complete. \qed

5 Discussion

This paper shows that the Gaussian product inequality (2) holds under Condition (III) when the reals $a_1, \ldots, a_d$ are nonnegative integers. This assumption is further seen to be strictly weaker than Condition (II). It thus follows from the implications in Figure 1 that when the reals $a_1, \ldots, a_d$ are nonnegative integers, inequality (2) holds more generally than under the MTP$_2$ condition of Karlin and Rinott [8].

Shortly after the first draft of this article was posted on arXiv, extensions of Proposition 2 were announced by Russell and Sun [22] and Edelmann et al. [4]; see Lemma 2.3 and Theorem 2.1, respectively, in their manuscripts. Beyond priority claims, which are nugatory, the originality of the present work lies mainly in its method of proof, and in the clarification it provides of the relationship between various assumptions made in the relevant literature, as summarized by Figure 1.

Beyond its intrinsic interest, the approach to the proof of the GPI presented herein, together with its link to the complete monotonicity of multinomial probabilities previously shown by Ouimet [16] and Qi et al. [17], hints to a deep relationship between the MTP$_2$ class for the multivariate gamma distribution of Krishnamoorthy and Parthasarathy [10], the range of admissible parameter values for their infinite divisibility, and the complete monotonicity of their Laplace transform; see the work by Royen on the GCI conjecture [18–21] and Theorems 1.2 and 1.3 of Scott and Sokal [23]. These topics, and the proof or refutation of the GPI in its full generality, provide interesting avenues for future research.
Appendix

Technical lemma

The following result, used in the proof of Proposition 2, extends Lemma 1 of Alzer [2] from the case $d = 1$ to an arbitrary integer $d \in \mathbb{N}$. It was already reported by Ouimet [16], see his Lemma 4.1, but its short statement and proof are included here to make the article more self-contained.

**Lemma A.1.** For every integer $d \in \mathbb{N}$, and real numbers $y \in (1, \infty)$ and $u_1, \ldots, u_{d+1} \in (0, 1)$ such that $u_1 + \cdots + u_{d+1} = 1$, one has

$$\frac{1}{y-1} > \sum_{i=1}^{d+1} \frac{1}{y^{1/u_i} - 1}.$$  

(A.1)

**Proof.** The proof is by induction on the integer $d$. The case $d = 1$ is the statement of Lemma 1 of Alzer [2]. Fix an integer $d \geq 2$ and assume that inequality (A.1) holds for every smaller integer. Fix any reals $y \in (1, \infty)$ and $u_1, \ldots, u_d \in (0, 1)$ such that $\|u\|_1 = u_1 + \cdots + u_d < 1$. Write $u_{d+1} = 1 - \|u\|_1 > 0$. Calling on Alzer’s result, one has

$$1 > \frac{1}{y^{1/\|u\|_1} - 1} + \frac{1}{y^{1/(1-\|u\|_1)} - 1}.$$  

Therefore, the conclusion follows if one can show that

$$\frac{1}{y^{1/\|u\|_1} - 1} > \sum_{i=1}^{d} \frac{1}{y^{1/u_i} - 1}.$$  

Upon setting $z = y^{1/\|u\|_1}$ and $v_i = u_i/\|u\|_1$, one finds that the aforementioned inequality is equivalent to

$$\frac{1}{z - 1} > \sum_{i=1}^{d} \frac{1}{z^{1/v_i} - 1},$$  

which is true by the induction assumption. Therefore, the argument is complete. \qed

**Acknowledgments:** The authors are grateful to Donald Richards and Thomas Royen for their comments on earlier versions of this note. The authors also thank the referees for their comments.

**Funding information:** C. Genest’s research is funded in part by the Canada Research Chairs Program, the Trottier Institute for Science and Public Policy, and the Natural Sciences and Engineering Research Council of Canada. F. Ouimet received postdoctoral fellowships from the Natural Sciences and Engineering Research Council of Canada and the Fond québécois de la recherche – Nature et technologies (B3X supplement and B3XR).

**Conflict of interest:** The authors state no conflict of interest.

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