On The M(atrix)-Model For M-Theory On $T^6$

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We study consistency conditions on a M(atrix)-model which would describe M-theory on $T^6$. We argue that there is a limit in moduli space for which it becomes a 6+1D theory and study the low-energy description of extended objects in the decompactified limit. We discuss the requirements from a M(atrix)-model which would describe such an $E_{6(6)}$ theory. We suggest that it could be a 5+1D theory and that a 1+1D theory with $(0,4)$ supersymmetry might be the M(atrix)-model for the M(atrix)-model of the $E_{6(6)}$ theory.

September, 1997
1. Introduction

One of the standing problems in M(atrix)-theory [1,2] is to understand the compactification of M-theory on compact manifolds. Toroidal compactification on $T^d$ for $d \leq 3$ has been defined as $U(N)$ SYM on a dual $\hat{T}^d$ [3,4] (for more discussion on SYM and toroidal compactifications of M-theory see [4,5,6,7,8,9,10,11,12,13,14]). The model for $T^4$ has been studied in [15] and in [16] and is given by the $(2,0)$ theory in 5+1D. The “state of the art” in toroidal compactification is M-theory on $T^5$ [17]. The moduli space of M-theory on $T^5$ is

$$\mathcal{M}_5 = SO(5,5,\mathbb{Z}) \backslash SO(5,5,\mathbb{R}) / (SO(5) \times SO(5))$$

and the theory defined in [17] can be thought of as a Hilbert space with an unknown 0+1D Hamiltonian which depends on 25 external parameters, namely the point in $\mathcal{M}_5$. At certain limiting points in $\mathcal{M}_5$ the spectrum of the Hamiltonian can be interpreted as a spectrum of a 5+1D field theory. However, unlike ordinary compactified field theories there are two different limits in $\mathcal{M}_5$ for which the spectrum looks like a 5+1D decompactified theory. In one limit the supersymmetry is $(2,0)$ and in the other it is $(1,1)$. Moreover, in appropriate limits in $\mathcal{M}_5$ one recovers also the previous theories for $T^d$ with $d < 5$.

The model for M-theory on $T^6$ remains elusive. Some suggestions have been studied in [18,19,20] but a large class of possibilities has been ruled out in [21] because one cannot decouple a subset of degrees of freedom of M-theory as in [17].

The purpose of these notes is to explore the assumption that there exists a Hilbert space and Hamiltonian with 16 supersymmetries which depends on parameters in $\mathcal{M}_6 = E_{6(6)}(\mathbb{Z}) \backslash E_{6(6)}(\mathbb{R}) / Sp(4)$, the moduli space of M-theory on $T^6$ [22]. For $q \in \mathcal{M}_6$, let us call the resulting theory $X(q)$.

The questions that we will ask are:

a. What is the maximal number of possible uncompactified dimensions of $X$?
b. What kind of extended objects does the non-compact $X$ accommodate and what is the low energy description of these?
c. What is the M(atrix) description of $X$?

To answer the first question, we will use the tools developed in [23] for the study of degenerations of type-II string theory. The $E_{6(6)}(\mathbb{Z})$ U-duality will then imply the existence of extended brane-like objects in the maximally decompactified $X$.

To construct a M(atrix) model of $X$ (which we denote by $MX$) one can try to compactify it on a small $S^1$ and find an appropriate description for multiple Kaluza-Klein
states. Using the $E_{6(6)}$ duality of $X$ we can map the KK states to extended branes in $X$. However, the theory on the branes will not decouple from the 6+1D bulk. Nevertheless, we can still ask questions similar to (a)-(c) about $MX$ and about $M^2X$ – the M(atrix)-model for $MX$. Altogether, we conjecture that a $M^3(atrix)$ model for M-theory on $T^6$ (i.e. a M(atrix) model for the M(atrix)-model of the M(atrix) model) is given by a 1+1D theory with $\mathcal{N} = (0, 4)$ supersymmetry. We will suggest only the $p_\parallel = 1$ version of $M^2X$ (and only for $p_\parallel = 1$ versions of $MX$ and $X$) where $p_\parallel$ is the longitudinal momentum in the light-like direction (see [2]).

These notes are organized as follows. Section (2) is a review of the theories involved in compactification of M(atrix) theory on $T^4$ and $T^5$. These are the $(2, 0)$ theories in 5+1D (which we denote $T(N)$) and the new 5+1D massive theory (which we denote $S(N)$). In section (3) we review the tools of [23] for analyzing degenerations and, using results of [14] and [13], we apply them to the case at hand. We find that the maximal degeneration is a 6+1D theory and we argue that it could have a local energy momentum tensor. In section (4) we explain the peculiar behaviour of the theory under U-duality which involves a rescaling of units. We explain how the 5+1D theory of [17] and its spectrum arise in limiting cases. In section (5) we study the low-energy descriptions of long extended objects in 6+1D. These are 2-branes and 5-branes. We argue that when two 5-branes coincide an interacting theory with $(1, 0)$ SUSY in 5+1D and a low-energy with tensor multiplets appears. Similarly, we study two coinciding 2-branes and argue that a 2-brane can end on a 5-brane. We argue that reparameterization anomalies could be canceled. In section (6) we study the required properties of a M(atrix)-model for $X$ and for its M(atrix)-model.

2. Review

There are two kinds of SUSY algebras with 16 supersymmetries in 5+1D. The spinor representations $4$ and $4'$ of $SO(5, 1)$ are pseudo-real, each with 8 real components. The first kind of SUSY algebra has SUSY charges one in the $4$ and one in the $4'$ and is referred to as $(1, 1)$. $U(1)$ Super-Yang-Mills theory in 5+1D has this kind of symmetry. The other kind is $(2, 0)$ and has two SUSY charges in the $4$ of $SO(5, 1)$. A free tensor multiplet is a realization of such a SUSY algebra. It comprises of an anti-self-dual tensor field-strength $G_{\mu\nu\tau}^{(-)}$ (satisfying $G_{\mu\nu\tau}^{(-)} = -\epsilon_{\mu\nu\tau\mu'\nu'\tau'}G_{\mu'\nu'\tau'}^{(-)}$ and $\partial_\sigma G_{\mu\nu\tau}^{(-)} = 0$) together with five real scalars $\Phi^I$ and two fermionic spinor fields. In the introduction we denoted this free theory by $T(1)$.

The two examples above, $U(1)$ and $T(1)$ are free.

What about interacting theories in 5+1D? Generalizing $U(1)$ to $U(N)$ Yang-Mills is possible at the classical level, but this leads to a non-renormalizable field theory. However,
a recent conjecture [17] has been put forward for an interacting 5+1D theory which at low-energies can be approximated by 5+1D $U(N)$ SYM (which is a free theory of $N^2$ gluons with IR-irrelevant cubic and quartic interactions). To be more precise, this new theory has two kinds of low-energy limits. Taking the limit in one direction of the parameter space one obtains a low-energy of massless vector multiplets. In another limit, one obtains massless tensor multiplets.

We will review the arguments of [17] momentarily, but as a preliminary we will discuss the generalization of $T(1)$ to an interacting theory $T(N)$.

2.1. Review of $T(N)$

The interacting generalization of $T(1)$ was discovered in [24] by studying type-IIB on an $A_N$ singularity. Another realization of the theory is the low-energy description of $N$ coincident 5-branes of M-theory [25]. $T(N)$ has a moduli space corresponding to separation of the 5-branes. The low-energy at a generic point in the moduli space is described by $N$ free tensor multiplets. The 5 scalar fields of each multiplet parameterize the moduli space. In uncompactified $\mathbb{R}^{5,1}$, $T(N)$ has no BPS particles but it has BPS 1-branes. They are charged under the field strength of the tensor multiplet and their tension is given by the square root of the sum of squares of the scalars which are the superpartners of the corresponding 3-form field strength [26]. In the 5-brane pictures these 1-branes are membranes with a boundary on a pair of 5-branes [25]. One of the discoveries in [24] was that when $T(N)$ is compactified on $T^2$ and when the size of the $T^2$ is much smaller than the scale of $T(N)$ (derived from the VEVs of the scalars) the resulting low-energy description in the uncompactified 3+1 dimensions is 3+1D $U(N)$. Thus $U(N)$ can be derived as a limit of $T(N)$.

2.2. Review of $S(N)$

The theory $S(N)$ is defined [17] as the degrees of freedom which describe $N$ coincident NS5-branes in type-IIB in the limit $\lambda_s \to 0$ while keeping $M_s$ (or equivalently, $\alpha'$) fixed. More precisely, this limit is described by a tower of perturbative non-interacting string states which live in the 9+1D bulk supplemented by interacting degrees of freedom on the 5+1D NS5-brane world-volume [17]. The uncompactified $S(N)$ theory has a low-energy description of $N$ free vector-multiplets on $\mathbb{R}^{5,1}$. The $S(N)$ theories are not scale invariant but have a scale $M_s$. $S(N)$ has a moduli space parameterized by the $4N$ scalars of the vector-multiplets. It was argued in [17] that $S(N)$ has BPS 1-brane excitations which can be thought of as bound states of elementary strings and the NS5-branes. By this we mean
that they are charged under the bulk NS-NS 2-form when one turns on $O(\lambda_s)$ corrections. These excitations have tension $M_s^2$.

All that has been said above is about $S(N)$ on the non-compact $\mathbb{R}^{5,1}$. One of the fascinating properties of $S(N)$ is that the space-time interpretation of the theory is not unique! Once one compactifies the theory on $T^5$ the theory has an $SO(5,5,\mathbb{Z})$ T-duality which means that the Hamiltonian and Hilbert space describing $S(N)$ on $T^5$ can be mapped in a 1-to-1 way to the Hamiltonian and Hilbert space of $S(N)$ on another $T^5$ related by an $SO(5,5,\mathbb{Z})$ T-duality transformation to the original one. In this map, momentum states become extended states. The existence of such a theory has been previously conjectured in [27,28] where the relation with the NS5-brane has also been suspected. The limit of $\lambda_s \to 0$ and the decoupling argument in [17] give a concrete realization of such a theory.

By studying the limit of $S(N)$ on a $T^5$ with all sides shrunk to zero one obtains a theory which can be given another space-time interpretation with a different SUSY structure. This theory is realized as the limit of $N$ NS5-branes of type-IIA when $\lambda_s \to 0$ keeping $M_s$ fixed [17]. This theory has a low-energy description of $N$ free tensor multiplets on $\mathbb{R}^{5,1}$. The moduli space is parameterized by the $5N$ scalars of these tensor-multiplets. However, $N$ of the scalars have a compact moduli space $S^1$ of radius $M_s^2$ [13]. The full moduli space is $(\mathbb{R}^4 \times S^1)^N/S_N$.

2.3. Locality and low-energy limits of $S(N)$

We will adopt the following point of view in this paper. $S(N)$ compactified on $T^5$ is to be thought of as some unknown Hamiltonian $H$ acting on some unknown Hilbert space $\mathcal{H}$. The Hamiltonian depends on the geometric parameters of the $T^5$, namely the 5 radii $R_1, \ldots, R_5$, the angles and the NS-NS background B-fields. They span the space

$$\mathcal{M}_5 = SO(5,5,\mathbb{Z})\backslash SO(5,5,\mathbb{R})/SO(5) \times SO(5).$$

These parameters are to be thought of as external constant parameters. At a generic point of $\mathcal{M}$ the theory is completely compactified and, like any compactified field theory, $H$ has a discrete spectrum. At various limits of the parameters of $\mathcal{M}_5$ one can discover that the low-energy spectrum of $H$ can be approximated by a field theory on a very large space. There are two such low-energy limits. One limit is to take all radii $R_i \to \infty$. In this limit the low-energy excitations correspond to $N$ free tensor-multiplets on $\mathbb{R}^{5,1}$. (To be more precise, the Hilbert space is approximated by super-selection sectors parameterized by the VEVs of the tensor fields.)

There is another limit of parameters, $R_1 \to 0$ and $R_2, \ldots R_5 \to \infty$, in which the low-energy excitations correspond to $N$ free vector-multiplets. For generic values of the
parameters there is no unique space-time interpretation but this doesn’t mean that there is no local interpretation at all. On the contrary, there are probably many good local descriptions of the theory. (By locality, I mean that there is a set of local operators, the Hamiltonian can be written as an integral of a local operator and the commutation relations are polynomial in derivatives).

However, there is no “canonical” local description. There is an $SO(5,5,\mathbb{Z})$ isometry of the Hilbert space which acts on the Hamiltonian as a T-duality on the parameters of the $T^5$. Local operators in one description are not local in another description.

Over the rest of this paper we will assume that there are good local descriptions. We must mention, however, one reason that might lead one to doubt that statement. The theory has a Hagedorn-like spectrum of states (this can be seen by counting BPS states of wound strings [13]) there might be a problem in defining short-distance Operator-Product-Expansions for distances shorter than the inverse of the Hagedorn temperature $M_s$.

3. Maximal decompactification limits

The purpose of this paper is to explore a possible M(atrix) description for M-theory on $T^6$. We start with the assumption that there exists a Hilbert space with a Hamiltonian $H$ depending on external parameters in

$$\mathcal{M}_6 = E_6(6)(\mathbb{Z}) \backslash E_6(6)(\mathbb{R}) / Sp(4)$$

and having 16 supersymmetries. Our first question will be what is the limit in $\mathcal{M}$ for which a low-energy description with a maximum number of dimensions is attained.

The way to settle this question is to follow an analysis similar to that given in [23]. There, the BPS formula for BPS excitations was used to determine the degenerations with a maximal number of states becoming light. These states were then interpreted as KK states. Let us recall the details of the analysis in [23]. For a moduli space of $G/K$ with $G = E_{d(d)}$ and $K$ a maximal compact subgroup, the central charge $Z^{ij}$ transforms in a certain representation $R_K$ of $K$. Denote the space of $Z^{ij}$ as $W$. The space of physical charges is in a representation $R_G$ of $G$. Denote it by $V$. The BPS mass formula is (in $(11 - d)$-dimensional Einstein units):

$$M_E = \|Z\| = \|Tg^{-1}\psi\|$$

for $\psi$ a vector of charges $g \in G/K$ and $T$ some fixed map $T : V \rightarrow W$.

The representations for different dimensions are (see [11] and [13]):
In our case, we wish to determine the maximal number of light states of the $M(atrix)$-model and not of space time itself. The difference will be that the representations $R_K$-s and $R_G$-s change. One way to think about it is that the space-time KK states are electric and magnetic fluxes in the $M(atrix)$-model. The representations for the KK states of the $M(atrix)$ model are (see [11] and [13]):

\[
\begin{array}{|c|c|c|c|}
\hline
| d & 4 & 5 & 6 |
\hline
G & SL(5) & SO(5,5) & E_6 |
\hline
R_G & 10 & 16 & 27 |
\hline
K & Sp(2) & Sp(2) \times Sp(2) & Sp(4) |
\hline
R_K & 10 & 16 & 27 |
\hline
\end{array}
\]

These are the same as the representations for longitudinal branes in $M$-theory [29,30].

Now we proceed to analyze the degeneration with a maximal number of massless particles. It is sufficient to restrict to diagonal matrices $g \in G$.

3.1. Calculation for $T^4$

For $M$-theory on $T^4$, the generic diagonal matrix of $5'$ has eigenvalues

\[
V, r_1^{-1}, r_2^{-1}, r_3^{-1}, r_4^{-1},
\]

where $V = \prod_1^4 r_i$. For $n_1, \ldots, n_5 \in \mathbb{Z}$ there are BPS states with squared masses

\[
\sum_1^4 (n_i r_i^{-1})^2 + (n_5 V)^2.
\]

These states are interpreted as the KK states of a theory in 5+1D compactified on $T^5$ with radii $\mu r_1, \ldots, \mu r_4, \mu V^{-1}$ where $\mu$ is a parameter that stands for a possible rescaling of the units of energy. Indeed, we know that we expect the $(2,0)$ 5+1D theory which is conformally invariant [15,16].
3.2. Calculation for $T^5$

For M-theory on $T^5$, the generic diagonal matrix of $10$ has eigenvalues

$$r_i^{-1}, \quad i = 1 \ldots 5,$$
$$r_i, \quad i = 1 \ldots 5,$$

(3.2)

For every $n_1, \ldots, n_5$ there exists a BPS state with mass squared

$$\sum_{i=1}^{5} (n_ir_i^{-1})^2.$$

(3.3)

There are also BPS states with masses squared

$$(n_1r_1)^2 + \sum_{i=2}^{5} (n_ir_i^{-1})^2, \quad (n_1r_1)^2 + (n_2r_2)^2 + \sum_{i=3}^{5} (n_ir_i^{-1})^2, \quad \ldots, \quad \sum_{i=1}^{5} (n_ir_i)^2.$$

We see that there are several low-energy limits. For example, taking $r_i \to \infty$ we find 5 decompactified dimensions. The states in (3.3) are KK states of this theory. There are also BPS states with masses squared

$$(r_1)^2 + \sum_{i=2}^{5} (n_ir_i^{-1})^2.$$

The existence of these states can be deduced entirely from U-duality. The fact that only directions $2 \ldots 5$ appear in $\sum_{i=2}^{5} (n_ir_i^{-1})^2$ indicates that these states can be given momentum only in 4 directions out of the five (and still be BPS). This gives them up as extended strings which can have momenta in 4 transverse directions. Their tension is constant and is set to 1. The fact that the tension is set (and does not depend on $r_i$) is consistent with a local physics. In fact, the existence of a local energy momentum tensor can also be deduced from the construction of [17]. Turning on $O(\lambda)$ corrections to the system of an NS5-brane in weakly coupled type-IIA one learns that for a consistent coupling to the bulk gravity there has to be a local energy momentum tensor on the 5-brane.

Another low energy limit is obtained when $r_i \to 0$ and the analysis is similar.

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1 I learned that this argument has also been given by L. Susskind.
3.3. Calculation for $T^6$

For M-theory on $T^6$, the generic diagonal matrix of $27'$ has eigenvalues

$$V^{-1/3}r_i^{-1}, \quad i = 1 \ldots 6,$$
$$V^{-1/3}r_ir_j, \quad 1 \leq i < j \leq 6,$$
$$V^{-1/3} \prod_{j \neq i} r_j \quad i = 1 \ldots 6,$$  \hspace{1cm} (3.4)

where $V = \prod_1^6 r_i$. The parameters $r_i$ have been chosen so as to exhibit the subgroup $SL(6) \times SL(2)$ of $E_6(6)$. The six values $r_i/V^{1/6}$ form an $SL(6)$ matrix and the remaining parameter $V$ is related to the $SL(2)$ generator. By applying $E_6(6)(\mathbb{Z})$ we see that all combinations

$$V^{-1/3} \sqrt{\sum_1^6 (n_ir_i^{-1})^2}, \quad n_i \in \mathbb{Z}$$

exist as masses of BPS states. There are also states with masses

$$V^{-1/3} \sqrt{(r_ir_j)^2 + \sum_{k \neq i,j} (n_kr_k^{-1})^2},$$

and states with masses

$$V^{-1/3} \sqrt{\left(\prod_{j \neq i} r_j\right)^2 + (n_ir_i^{-1})^2}.$$  

We will interpret these formulae in the next section.

4. The 6+1D limit and its compactification

In this section we will argue that $X$ has a 6+1D limit. We will study its properties and recover the properties of the 5+1D theory of [17] from compactification on $S^1$.

The assumption that a M(atrix) model for M-theory on $T^6$ exists lead us to the conclusion that there is a 0+1D Hamiltonian $H(r_1, \ldots, r_6)$ where $r_i$ are just external parameters for now. From this Hamiltonian we can define a new Hamiltonian

$$\tilde{H}(r_1, \ldots, r_6) = V^{1/3}H(r_1, \ldots, r_6), \quad V = r_1 \cdots r_6.$$  \hspace{1cm} (4.1)

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2 I am grateful to E. Witten for pointing this out.
From the BPS formula we learn that the BPS particles of $\tilde{H}$ have masses

$$m\{n_k\}^2 = \sum_{i=1}^{6} (n_i r_i^{-1})^2, \quad n_i \in \mathbb{Z},$$

$$m_{i,j}\{n_k\}^2 = (r_i r_j)^2 + \sum_{k \neq i,j} (n_k r_k^{-1})^2,$$

$$m_{i,n_i}^2 = (\prod_{j \neq i} r_j)^2 + (n_i r_i^{-1})^2. \quad (4.2)$$

All these states are related by the conjectured U-duality of the theory. In the limit $r_i \to \infty$ the first line constitutes the lightest particles. We see that the first line is consistent with an interpretation of momentum states of a Lorenz invariant field theory on $T^6$. The second line represents objects which can have momentum only in 4 directions. Thus, we must interpret them as 2-branes. We see that their tension is constant. In fact, the rescaling factor $V^{-1/3}$ in (4.1) was chosen such that the tension of the 2-branes will be independent of the $r_i$-s. The third line of (4.2) represents objects which can have momentum only in one transverse direction. They must be 5-branes and in the units (4.1), the tension of these is constant.

Thus, formula (4.2) is consistent with a Lorenz invariant 6+1D interpretation.

The 6+1D low-energy of the theory would then have to be $N^2$ free vector multiplets since there are no interacting conformal theories in 6+1D [31].

We are studying the $N = 1$ version of the theory so the low-energy must be $U(1)$ 6+1D Yang-Mills. Let us see what the coupling constant of the Yang-Mills would be. The M(atrix) conjecture requires us to identify $V^{-1/3} r_i^{-1}$ with the tension of a longitudinal object in M-theory on $T^6$. The tension must be measured in Einstein units in 4+1D and so we find

$$R_i = V^{-2/9} r_i$$

where $R_i$ are the radii of the $T^6$ in 10+1D units. The energy of an electric flux in direction $i$ is (see also [13]):

$$g^2 r_i^2 \frac{1}{V}.$$

This is to be identified with a square of the mass of a KK state of M-theory on $T^6$ and we find that $g$ is a constant as well.

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3 This formula was calculated together with N. Seiberg and S. Sethi.
4.1. Another IR limit

Like the theory of [17], $X$ has more than one IR limit. However, a novel feature of $X$ is that to describe the other IR limits we must change the units of energy.

Let us take the limit $r_i \to 0$ in (4.2). In this limit, the lightest BPS states will be the wrapped 5-branes so we will have to identify them with KK states of the new IR limit. Thus we will write

$$R_i^{-1} = \frac{V}{r_i},$$

where the RHS is the mass of a wrapped 5-brane transverse to the $i$-th direction. $R_i$ are the radii of the new IR limit. However, the 2-branes now have tension

$$r_i r_j = \left( \prod R_i \right)^{-2/5} R_i R_j.$$

Their tension is not constant and this is not a good local description of physics. The resolution is that there was no reason why the new IR limit should keep the same unit of time as before. After all, the energy direction is not in a Lorenz multiplet with the 5-brane charge. It is only in a multiplet with a momentum charge. Thus, we choose to rescale the energy by a factor of $V^{-2}$ and define

$$R'_i = V^{-3} r_i.$$

The new Hamiltonian

$$\tilde{H}(R'_1, \ldots, R'_6) = V^{-2} H(r_1, \ldots, r_6)$$

defines a local 6+1D theory compactified on radii $R'_i$.

4.2. Recovery of lower dimensional theories

We can recover the $SO(5,5)$ theory described in [17] by compactifying the 6+1D theory on a small $r_6$.

We first find a theory with a low-energy of 5+1D $U(1)$ SYM and coupling constant

$$\frac{1}{g^2} \sim r_6$$

since the 6+1D coupling constant was 1. This theory has string states (which are wrapped membranes) of tension proportional to $r_6$. It also has membranes with a constant tension, 4-branes with a tension of $r_6$ and 5-branes with a constant tension. The standard analysis shows that as $r_6 \to 0$ the strings are the lowest lying states. We should now change units (both time and space) so that the coupling constant will be 1 in the new units. This also sets the tension of strings to 1 and all the higher states are pushed to infinite energy as $r_6 \to 0$. 
5. Extended objects

We have seen in the previous section that there must be a limit in moduli space for which the spectrum can be interpreted as a 6+1D local theory. The 6+1D low-energy of that theory would then have to be $N^2$ free vector multiplets since there are no interacting conformal theories in 6+1D.

However, we can ask other “low-energy questions” for which the answer is nontrivial. We have seen that the theory has extended BPS 5-branes and extended BPS 2-branes. Note that because the 5-branes are not charged in this theory there are no IR divergences even when there is only one transverse direction.

5.1. 5-branes

What is the low-energy description of a theory with an extended 5-brane?

If the theory is local, the description has to be in terms of extra degrees of freedom “living” on the 5-brane and interacting with the bulk low-energy fields. The 5+1D theory has $\mathcal{N} = (1, 0)$ supersymmetry and its moduli space is $\mathbb{R}$, the motion in the single transverse direction. Thus, it can only be a free tensor multiplet.

What happens when two 5-branes coincide? We expect to find a 5+1D conformal theory with a moduli space of $(\mathbb{R})^2/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ exchanges the two $\mathbb{R}$-s. After compactification on $\mathbb{T}^3$ this theory should have a moduli space of $(\mathbb{T}^4)^2/\mathbb{Z}_2$. This is because when compactified on another $\mathbb{T}^2$, the system of two 5-branes can be mapped by the $E_6(6)(\mathbb{Z})$ U-duality to a system of two KK states and two KK states have $(\mathbb{T}^6)^2/\mathbb{Z}_2$ as their “moduli” space. Let us first rule out a few possibilities for such a theory. Various 5+1D theories with $(1, 0)$ SUSY and tensor multiplets in the low-energy have been studied in [32,33]. These theories contained in general also hyper-multiplets in the low-energy and so we believe that the theory we are looking for is not in that list. Another possibility is to separate the tensor multiplet corresponding to the center of mass motion of the two 5-branes and be left with a moduli space of $\mathbb{R}/\mathbb{Z}_2$. There exists a 5+1D theory with a low-energy comprising of a single tensor multiplet with moduli space $\mathbb{R}/\mathbb{Z}_2$. This is the theory associated with small $E_8$ instantons [34,26]. However, this theory cannot be the one we are looking for either. In fact, the mere assumption that the moduli space in 5+1D is $\mathbb{R}/\mathbb{Z}_2$ strongly suggests that the theory has an $E_8$ symmetry. The line of reasoning is as follows [35]. When one compactifies such a theory down to 3+1D on a torus one finds an $\mathcal{N} = 2$ theory with a vector multiplet. The moduli space at infinity is determined from the fact that we reduced a 5+1D tensor multiplet. This restricts the form of the Seiberg-Witten curve. The $H_2(\mathbb{Z})$ cohomology of the total space (moduli space together with the
elliptic fibers) is related to global symmetries of the theory. It can be shown that $H_2(\mathbb{Z})$ contains an $E_8$ lattice. It is unreasonable to expect a global $E_8$ symmetry from the system just described. M-theory on $T^6$ doesn’t have an $E_8$ global symmetry. Furthermore, the BPS excitations of the small $E_8$ instanton theory are strings which carry an $E_8$ chiral current algebra. In our case, these strings should be identified with 2-branes with boundaries on the two 5-branes (we will elaborate on this later on) and it is not clear from where the $E_8$ chiral current algebra would come. We see that the $E_8$ theory has to be ruled out. In fact, after compactification on $T^3$ the moduli space has to be $(T^4)^2/\mathbb{Z}_2$ and not include half a $K_3$.

Thus, we conclude that if the theory $X$ exists, there must be a new 5+1D theory with $(1,0)$ SUSY and a low energy of $k$ tensor multiplets with moduli space $(I\mathbb{R}^k)/S_k$. For the moment we will assume that such theories exist.

5.2. 2-branes

An extended 2-brane has 4 transverse directions and it is natural to expect that the position in these directions is related to the VEVs of 4 scalars. When $k$ 2-branes coincide we find a 2+1D theory with a moduli space of $(I\mathbb{R}^4)^k/S_k$. Let us restrict to the case of a single 2-brane. The 4 scalars could be super-partners of either a vector multiplet or a hypermultiplet of $\mathcal{N} = 4$ in 2+1D. Locally these two possibilities are identical (2+1D mirror symmetry replaces vector multiplets with hypermultiplets). Globally, there is a difference. If the world-volume of the 2-brane is a genus-$g$ Riemann surface, a vector multiplet will produce $2g$ global modes (the Wilson lines).

When the directions transverse to the 2-brane are compact we can use U-duality to map the 2-brane to a 5-brane. Since the world-volume of a 5-brane is described by a tensor field which upon dimensional reduction becomes a vector field, we see that the global modes do exist.

These global modes affect the counting of $\frac{1}{2}$-BPS states in $X$. Let $X$ be compactified on $T^6$ and let $k$ 5-branes be in directions $1 \ldots 5$ and let $k'$ units of momentum be on the 5-branes in direction 5. When the 5-th direction is small we can think of the 5-brane theory as 4+1D $U(k)$ Yang-Mills compactified on $T^4$. The $k'$ units of momentum become instantons in $U(k)$ Yang-Mills with 8 supersymmetries. The configuration is $\frac{1}{2}$-BPS and the multiplicities and quantum numbers of states are found by quantizing this moduli space (similarly to [4,11,42,43]). On the other hand, we can use $E_6(6)$ U-duality to map

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4 This setting is very similar to a situation discussed together with S. Sethi in a different context.
this system to a system of $k$ 2-branes wrapped on directions 1, 2 and $k'$ 2-branes wrapped on directions 3, 4. There are two compact transverse directions. In general the $k$ and $k'$ branes form a sort of bound state which is a single holomorphic curve in $T^4$ which wraps $k$ times around the directions 1, 2 and $k'$ times around the directions 3, 4. The moduli space of instantons in $T^4$ is related to the moduli space of holomorphic curves in the dual torus $\hat{T}^4$ via the spectral curve construction (see [44] and [45]). In general the moduli space of $U(k)$ instantons with instanton number $k'$ can be constructed as the moduli space of holomorphic curves as above (the “spectral-curve”) but in addition we need to specify a flat line-bundle on the curve (the “spectral-bundle”). Thus, we conclude that there must be a $U(1)$ vector-field living on the 2-brane with the global Wilson line modes.

5.3. 2-branes ending on 5-branes

In M-theory 2-branes can end on 5-branes and the end-point is a source for the tensor field-strength on the 5-branes [25]. Can a 2-brane of $X$ end on a 5-brane of $X$? We will bring supporting evidence that it can.

Let us compactify $X$ on a very large $T^5$ and put in a single 5-brane that wraps $T^5$. This is a BPS state of the Hamiltonian. Now let us put in a flux for the anti self-dual field-strength $G_{0ij}$ of the tensor field that lives on the 5-brane. Such a flux breaks the SUSY by another half. Since the final state is BPS the flux $G_{0ij}$ must correspond to a central charge. The only central charges of $X$ are those in the $27'$ of $Sp(4)$ so we must identify the flux-charge with the central charge of 2-branes. Now, let the 5-brane be stretched in directions 1...5 and let a 2-brane be stretched in directions 1, 6. The 1+1D endpoint of the membrane on the 5-brane acts as a source for $G_{01j}$ so that integrating the flux through a sphere inside the 5-brane which surrounds the endpoint of the 2-brane

$$\int_{S^3} G_{01j} dn^j$$

we get on the one hand the charge of the string inside the 5-brane theory and on the other hand we get the number of 2-branes according to the identification of $G_{0ij}$ with the 2-brane charge. I do not know for sure from this setting if one unit of flux corresponds to a single 2-brane or more.

5.4. A 2-brane stretched between two 5-branes

The 5+1D theory which describes two 5-branes has a low energy of two $(1, 0)$ tensor multiplets. It was explained in [26] that the scalar components of tensor multiplets are the central charges for strings which are charged under the tensor field. This leads one
to wonder whether there actually exist BPS strings in the 5+1D theory that are charged under the difference of the two tensor fields. From the discussion above it seems that the answer is positive. 2-branes with boundaries on both 5-branes will be charged under the difference of tensor fields just like the analogous situation in M-theory [25].

What is the low-energy description of such a string in 5+1D?

For a single membrane stretched between two 5-branes of M-theory the low-energy description is given by a 1+1D theory with 4 free scalars and $\mathcal{N} = (4, 4)$ supersymmetry. It can be thought of as the dimensional reduction of either a 5+1D hypermultiplet or a 5+1D vector multiplet of $\mathcal{N} = (1, 0)$ down to 1+1D. In our case, a standard analysis of the unbroken SUSY charges reveals that a single 2-brane is described by a 1+1D field theory with 4 scalars and $\mathcal{N} = (0, 4)$ supersymmetry.

5.5. 5-branes with a transverse $S^1$

Let us compactify $X$ on a small circle of radius $r \ll g^{2/3}$ and let us put in $k$ 5-branes which fill the uncompactified 5+1D directions. At energies $E \ll r^{-1}$ the theory looks like a 5+1D theory. The 6+1D bulk reduces to the theory of [14] with the energy scale $g^{-1} r^{1/2}$. At low-energies it looks like a 5+1D Yang-Mills with coupling constant $g^2/r$. The $k$ 5-branes give $k$ tensor multiplets at low-energy but this time the moduli space is $(S^1)^k / S_k$. This theory has $SO(5,5)$ T-duality and looks like a $\mathcal{N} = (1, 0)$ “cousin” of the theory of [14]. The “bulk” theory does not decouple from the theory of the $k$ 5-branes.

5.6. Remarks on reparameterization anomalies

The low-energy description of the extended 5-brane as well as the description of the 2-brane stretched between two 5-branes are chiral theories and the issue of gravitational anomalies arises.

Let us start with the 5-brane. Since $X$ does not include gravity, the gravitational anomaly of the tensor multiplet is not a problem when the 5-brane world-volume is flat.

When the 5-brane world-volume is a curved 5+1D manifold the anomaly under reparameterization is a problem even before coupling to gravity. As explained in [16], there is no canonical coordinate frame for the 5+1D world-volume when it is curved and we need to provide an expression for the action which is independent of the particular coordinate system.

The anomaly of a tensor multiplet has been calculated in [17] and does not vanish (Our situation is even simpler since the normal bundle to the 5-brane is trivial).

The present setting allows for a unique solution to the anomaly problem. Since the 5-brane fills a co-dimension 1 oriented manifold the 5-brane divides 6+1D space-time into
disconnected regions. For simplicity, let us assume that the 5-brane bounds a 6+1D region \( \Sigma \). Let \( J_7 \) be the 7-form such that

\[
dJ_7 = I_8(R),
\]

where \( I_8(R) \) is the 8-form from which the 6-form anomaly is derived. We can add a term

\[
\int_\Sigma J_7(\omega), \quad \partial \Sigma = \text{[fivebrane worldvolume]}
\]

(5.1)

to the action which will cancel the anomaly. If the ambient space were in dimension higher than 6+1D, there would not have been a canonical \( \Sigma \). For flat space, different choices of \( \Sigma \) would give the same result since \( dJ_7 = I_8 = 0 \), but the form (5.1) suggests that the anomaly could be canceled on curved spaces as well. There are situations for which the region \( \Sigma \) cannot be well defined, for example for a single 5-brane wrapping a face of a \( T^6 \). However, in this case the total homology class of all the 5-branes is a good quantum number and the relative phase of the wave function has to be well-defined only within the homology sector. We hope that the difference in the actions (5.1) for two configurations in the same class can still be well defined, though we have not checked this.

The situation of a 2-brane ending on a 5-brane can have a different solution. The reparameterization anomaly is supported at the 1+1D boundary of the 2-brane on the 5-brane. It is derived by descent equations from \( \text{tr}\{R^2\} \). Part of the anomaly could be canceled by adding a term \( \int \omega_3 \) where \( \omega_3 \) is the Chern-Simons 3-form such that \( d\omega_3 = \text{tr}\{R^2\} \) and the integral is over the 2-brane world-volume.

The anomaly could also be canceled if there is an extra chiral matter coming from the 1+1D boundary of the 2-brane on the 5-brane. The fact that the 5-brane is immersed in 6+1D is again important. If the codimension of the 5-brane in its ambient space-time were higher than 1 we could have argued by Lorenz invariance that there is no preferred direction “left” or “right” on the 1+1D boundary. In our case, since the 5-brane has an orientation and the 6+1D bulk has its own orientation there is a distinction between “left” or “right” of the 5-brane. Since the 2-brane is oriented as well there is also a distinction between “left” and “right” on the 1+1D boundary. This also suggests that under a parity transformation the 5-brane transforms into an anti-5-brane.

I do not know what is the composition of the required chiral matter. In section (6.2) we will encounter a related situation which will require fermions in a chiral representation of the transverse space but that suggestion is incomplete.

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5 I am grateful to W. Taylor for pointing this out.
Finally, we note that the anomaly cannot be canceled by an infl ow mechanism \([48]\), i.e. by adding a term

\[
\int_{5\text{-brane}} G_3 \wedge \omega_3
\]

where \(G_3\) is the anti-self-dual field strength on the 5-brane (in analogy with the \(\int C_3 \wedge J_8(R)\) term of M-theory \([49,50,51]\)). The term (5.2) has opposite signs depending on whether the 2-brane is on one side of the 5-brane or the other whereas the anomaly itself does not change sign.

6. A M(atrix) model for \(X\)

In this section we will assume that \(X\) is well defined and study a possible M(atrix) model which defines \(X\) compactified on \(T^5\). Note that this is the maximal compactification that can be achieved in the IMF. We will denote the \(p_\parallel = k\) DLCQ sector of \(X\) by \(MX(k)\).

When \(X\) is formulated on \(T^5 \times \mathbb{R}^{1,1}\) part of the \(E_{6(6)}(\mathbb{Z})\) U-duality is preserved. This part is the \(SO(5,5,\mathbb{Z})\) acting as T-duality on the \(T^5\) (including the \(B\)-fields). The M(atrix)-model for \(X\) should preserve that \(SO(5,5,\mathbb{Z})\) T-duality.

Another requirement is that \(MX\) will have 8 supersymmetries and the fermionic zero modes should, upon quantization, give the states of a vector-multiplet which will become the \(U(1)\) photon quantum numbers.

Finally, we expect \(MX\) fluxes which will correspond to the KK states and 2-branes of \(X\) similarly to \([5]\).

6.1. Maximal decompactification limits of \(MX(k)\)

Assuming that \(X\) exists and that a Hilbert space description of \(X\) compactified on a light-like direction exists requires the existence of a theory \(MX(k)\). We will now study it in a similar fashion as we studied \(X\).

Let us restrict for simplicity to \(MX(1)\). This theory has 8 supersymmetries and our first task is to find the formula for its BPS excitations. Since \(MX(1)\) describes \(X\) compactified on \(T^5 \times \mathbb{R}^{1,1}\) its Hamiltonian depends on 25 external parameters which describe the shape size and \(B\)-fields of the \(T^5\). Thus \(MX(1)\) depends on external parameters

\(q \in SO(5,5,\mathbb{Z})\backslash SO(5,5,\mathbb{R})/(SO(5) \times SO(5))\).

The BPS particles of \(MX(1)\) correspond to longitudinal objects of \(X\). These are either membranes wrapped on one cycle of \(T^5\) or 5-branes wrapped on 4 cycles. They are in

\[6\] The first idea of looking for a M(atrix) model for a M(atrix) model was presented in [52].
the $10$ of $SO(5,5,\mathbb{R})$. The analysis of the BPS mass formula is identical to the case of M-theory on $T^5$ analyzed above. Thus $MX(1)$ has to have particles of masses

$$
\sqrt{(n_1 r_1)^2 + \sum_{i=2}^{5} (n_i r_i^{-1})^2}, \quad \sqrt{(n_1 r_1)^2 + (n_2 r_2)^2 + \sum_{i=3}^{5} (n_i r_i^{-1})^2}, \quad \ldots, \quad \sqrt{\sum_{i=1}^{5} (n_i r_i)^2}.
$$

We see that $MX(1)$ has two kinds of 5+1D low-energy limits. Both have $\mathcal{N} = (1,0)$ supersymmetry. One limit has a low-energy of one vector multiplet and the other has a low-energy of one tensor multiplet with moduli space $S^1$. Both limits have BPS strings with fixed tension.

### 6.2. Low-energy description of BPS objects

We will proceed to analyze the low-energy description of strings in $MX(1)$. We assume that the string is infinite in the 5th direction. Let the transverse rotation group in directions $1\ldots4$ be denoted by $SO(4)_T$. We will write representations of $SO(4)_T$ as representations of $SU(2) \times SU(2)$.

Since T-duality ($SO(5,5,\mathbb{Z})$) along the 5th direction (when compactified) maps the string to a KK state, the $SO(4)_T$ quantum numbers of the ground state of the string must be the same as those of its T-dual KK state.

For $\mathcal{N} = (1,0)$ supersymmetry, the supersymmetry charges transform in the

$$(2, 4)$$

of $SU(2)_R \times SO(5,1)$, where $SU(2)_R$ is the R-symmetry. This R-symmetry is related to the $SO(3)$ R-symmetry of $X$ which rotates the 3 scalars. Note that both $2$ and $4$ are pseudo-real and that one can impose a reality condition on $(2, 4)$. Under $SU(2)_R \times SO(4)_T$ the charges transform as

$$(2, 2, 1) + (2, 1, 2).$$

Let the charges be $Q^{i\alpha}, Q^{i\dot{\alpha}}$ the reality condition is

$$(Q^{i\alpha})^\dagger = \epsilon_{ij} \epsilon_{\alpha\beta} Q^{j\beta}, \quad (Q^{i\dot{\alpha}})^\dagger = \epsilon_{ij} \epsilon_{\dot{\alpha}\dot{\beta}} Q^{j\dot{\beta}}.$$

Putting a KK state in the 5th direction leaves only the $Q^{i\dot{\alpha}}$ unbroken. The $Q^{i\alpha}$ generate zero-modes. The irreducible representation of the Clifford algebra of $Q^{i\alpha}$ decomposes under $SU(2)_R \times SO(4)_T$ as

$$(2, 1, 1) + (1, 2, 1).$$
The KK state is in a reducible representation. There are two cases to distinguish. When the 5+1D low-energy is a free vector-multiplet, the KK state is in

\[(1, 1, 2) \times \{(2, 1, 1) + (1, 2, 1)\} = (2, 1, 2) + (1, 2, 2). \quad (6.1)\]

These are the 4 states of a vector and 4 states of gluinos. When the 5+1D low-energy is a free tensor-multiplet the KK state is in

\[(1, 2, 1) \times \{(2, 1, 1) + (1, 2, 1)\} = (2, 2, 1) + (1, 3, 1) + (1, 1, 1). \quad (6.2)\]

These are the states of an anti-self-dual tensor field, a scalar and 4 fermions. By T-duality we deduce that the ground states of the extended strings are in the opposite representations, i.e. in (6.1) when the low-energy is a tensor multiplet and in (6.2) when the low-energy is a vector-multiplet.

How can this multiplicity be realized? We can expect that the string has a low-energy description with \(\mathcal{N} = (0, 4)\) supersymmetry. For a single string in \(MX(1)\), the low-energy will contain 4 scalars in the \((2, 2)\) of \(SO(4)_T\) and 2 complex right-moving fermions in the \((2, 1)\) of \(SO(4)_T\). The ground states of the Clifford algebra of the zero modes will then be in the representation

\[2(1, 1) + (2, 1).\]

This means that we need some left-moving fermionic zero modes. For the case of (6.2) their zero-mode algebra should have two states in the \((2, 1)\) and for the case of (6.1) the zero-mode algebra should have two states in the \((1, 2)\). It seems that the unique solution to the problem is to have three real left-moving fermions which in the case of (6.2) are in the \((3, 1)\) of \(SO(4)_T\) and in the case of (6.1) they are in the \((1, 3)\). Denote these by \(\zeta^{\alpha\beta}\) or \(\zeta^{\dot{\alpha}\dot{\beta}}\) respectively. These are real fermions, symmetric in the spinor indices. The Clifford algebra is then

\[\zeta_0^{\alpha\beta} |\gamma\rangle = \epsilon^{\alpha\gamma} |\beta\rangle + \epsilon^{\beta\gamma} |\alpha\rangle,\]

or

\[\zeta_0^{\dot{\alpha}\dot{\beta}} |\dot{\gamma}\rangle = \epsilon^{\dot{\alpha}\dot{\gamma}} |\dot{\beta}\rangle + \epsilon^{\dot{\beta}\dot{\gamma}} |\dot{\alpha}\rangle.\]

Altogether the low-energy Lagrangian will be for (6.2):

\[L = \int d^2\sigma \{\partial_+ \phi^a \partial_- \phi^a + \psi^{i\alpha} \partial_+ \psi_{i\alpha} + \zeta^{\alpha\beta} \partial_- \zeta_{\alpha\beta}\}, \quad (6.3)\]

and for (6.1):

\[L = \int d^2\sigma \{\partial_+ \phi^a \partial_- \phi^a + \psi^{i\alpha} \partial_+ \psi_{i\alpha} + \zeta^{\dot{\alpha}\dot{\beta}} \partial_- \zeta_{\dot{\alpha}\dot{\beta}}\}, \quad (6.4)\]
where \( a = 1 \ldots 4 \) is a vector index for \( SO(4)_T \), \( \alpha = 1, 2 \) and \( \dot{\alpha} = \dot{1}, \dot{2} \) are spinor indices and \( i = 1, 2 \) is an R-symmetry index.

The extra left-moving fermions will have an effect on the counting of \( \frac{1}{2} \)-BPS states in \( MX(1) \). A \( \frac{1}{2} \)-BPS state can be constructed by starting with a string wound around direction 1 and exciting some left-moving oscillators \([10, 11, 12, 43]\). These are the 4 left-moving scalars and the 3 left-moving fermions. Thus we expect the multiplicity and \( SO(4)_T \) quantum numbers of \( \frac{1}{2} \)-BPS states with winding number \( w = 1 \) along direction 1 and momentum \( p > 0 \) along direction 1 to be according to the states at level \( p \) of the Fock space of states of 4 bosons in the \((2, 2)\) of \( SO(4)_T \) and 3 fermions in the \((3, 1)\) of \( SO(4)_T \). On the other hand, these \( \frac{1}{2} \)-BPS states of \( MX(1) \) describe bound states of one longitudinal 5-brane of \( X \) which is wrapped on the DLCQ direction and on directions 2…5 together with \( p \) longitudinal 2-branes wrapped on the DLCQ direction and on direction 1. It might be interesting to attempt to quantize that system in the DLCQ and check this statement.

When there are \( k \) extended strings the low-energy description should be an interacting generalization of (6.3),(6.4). This should be a 1+1D conformal theory with \( \mathcal{N} = (0, 4) \) supersymmetry which in a certain limit looks like \( k \) copies of (6.3) or (6.4) with an \( S_k \) orbifold that acts on all the fields \( \phi, \psi, \zeta \). Although the target space of a 1+1D theory is not an invariant notion in general, the statement could be made more rigorous if we recast it in terms of wave-packets. The interacting theory that we are looking for does not seem to be easily defined as a perturbation of the orbifold theory. Unlike the cases of \([1, 13]\), The twist operator of this orbifold is of weight \((1, \frac{7}{8})\) and is not only relevant but even not 1+1D Lorenz invariant.

Nevertheless, we will assume that there is an interacting generalization of (6.3),(6.4).

It is perhaps interesting to note that we could have asked a similar question about the strings of the theory \( S(1) \) (of \([17]\)). The low-energy description of \( k \) strings is a \( \mathcal{N} = (4, 4) \) 1+1D theory which is very likely to be the theory discussed in \([54]\). This was the strongly coupled \( \sigma \)-model obtained from the orbifold \( (\mathbb{R}^4)^k/S_k \) by turning the \( \theta \)-angle at the \( \mathbb{Z}_2 \) fixed-points to \( \theta = 0 \) rather than \( \theta = \pi \). This is defined by resolving the \( \mathbb{Z}_2 \) fixed point locus with an exceptional divisor and setting the Kähler class of that divisor to zero and the \( B \)-field to zero as well.

### 6.3. Reparameterization anomalies

The 1+1D Lagrangians (6.3) and (6.4) have reparameterization anomalies.

The index for the gravitational anomaly for a chiral spinor in the representation \( r \) of a vector-bundle is given by:

\[
\hat{I}_r(F, R) = \text{tr}_r\{e^{iF}\}(1 + \frac{1}{48}\text{tr}\{R^2\} + \cdots),
\]
where $R$ is the world-sheet curvature and $F$ is the field-strength of the vector-bundle. In our case the spinors transform as sections of the normal bundle and $F$ satisfies:

$$\text{tr}_{(2,2)}\{F\wedge F\} = -\text{tr}\{R\wedge R\}$$

For the spinors $\psi$ the representation $r$ is $(2, 1)$ and we find

$$\hat{I}_{(2,1)}(F, R) = \frac{1}{24}\text{tr}\{R^2\} - \frac{1}{8}\text{tr}_{(2,2)}\{F^2\} = \frac{1}{6}\text{tr}\{R^2\}.$$ 

For the spinors $\zeta$ we find

$$\hat{I}_{(3,1)}(F, R) = \frac{1}{16}\text{tr}\{R^2\} - \frac{1}{2}\text{tr}_{(2,2)}\{F^2\} = \frac{9}{16}\text{tr}\{R^2\}.$$ 

Altogether the anomaly is

$$\left(\frac{9}{16} - \frac{2}{6}\right)\text{tr}\{R^2\} = \frac{11}{48}\text{tr}\{R^2\}.$$ 

Since the string is not charged under any of the bulk fields, we cannot cancel the anomaly by an inflow mechanism.

I do not know how to cancel this anomaly. In flat space one can add $\int_{\Sigma} \omega_3$ where $\omega_3$ is the 3-form such that $d\omega_3 = \text{tr}\{R^2\}$ and $\Sigma$ is any 3-manifold whose boundary is the string. This extra term will be a $c$-number but will make sure that the action is independent of the particular coordinate system in which we chose to write the action down. However, for a curved 5+1D space, it is impossible to add a canonical term like that, since $\int_{\Sigma} \omega_3$ depends on $\Sigma$. Maybe this means that we cannot put $MX$ on a curved background or, perhaps this is an indication that one has to add more chiral matter to the string. In fact, without adding more chiral matter (6.3),(6.4) have a non-vanishing Casimir energy. Since the Casimir energy can be calculated using only the low-energy Lagrangian its correction to the mass of a long wound string can be trusted and that would lead to a contradiction with T-duality. Nevertheless, in what follows we will assume that $MX$ is well defined.

6.4. The limit of M(atrix)-theory on $T^5$

When $X$ is compactified on a very small $S^1$, the resulting 5+1D theory is $S(1)$ – the theory discovered in [17]. A M(atrix) model for $S(1)$ has been given in [53,54]. It is a 1+1D theory with $\mathcal{N} = (4, 4)$. The $p_{\parallel} = 1$ sector is given by a free $\mathcal{N} = (4, 4)$ $\sigma$-model with target space $T^4$. The $p_{\parallel} = k$ is given by an interacting $\mathcal{N} = (4, 4)$ $\sigma$-model with target space $(T^4)^k/S_k$ with the singularities resolved by putting the $\theta$-angle to zero [54]. The limit of small $S^1$ for $X$ corresponds to the limit of a large $S^1$ for $MX(k)$. Thus, $MX(k)$ compactified on $T^4 \times \mathbb{R}^{1,1}$ must reduce in the low-energy limit to the $\mathcal{N} = (4, 4)$ $\sigma$-model found in [53,54]. This seems to be the case for $MX(1)$. 

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6.5. Properties Of A M(atrix)-model for $MX(k)$

If $MX(k)$ exists we can compactify one light-like direction and look for a Hamiltonian which describes the $p_{||} = N$ sector. We will denote this theory by $M^2X(k,N)$.

This has to be a theory with 4 supersymmetries. The large $N$ limit of it will be a M(atrix)-model for $MX(k)$ on $T^4 \times \mathbb{R}^{1,1}$ (in one of its low-energy limits). The theory has to depend on 16 external parameters

$$q \in SO(4,4,\mathbb{Z}) \backslash SO(4,4,\mathbb{R})/(SO(4) \times SO(4)).$$

For simplicity, we will proceed with $MX(1)$. Since the longitudinal strings of $MX(1)$ are unwrapped they are singlets of $SO(4,4,\mathbb{R})$ and their mass is independent of $q$. Since they are related by the $SO(5,5,\mathbb{Z})$ T-duality of $MX(k)$ to KK states of $MX(k)$, they must have the multiplicity of KK states. Thus the BPS mass formula for KK states in the M(atrix)-model for $MX(1)$ (which we denote $M^2X(1,1)$) is

$$n, \quad n \in \mathbb{Z}^+.$$ 

This means that $M^2X(1,1)$ has a 1+1D limit. This limit must be a conformal theory with 4 supersymmetries. Since the quantization of $M^2X(1,1)$ must describe the KK states of $MX(1)$ we assume that $M^2X(1,1)$ has a target space $T^4$.

We believe that the theory $M^2X(1,1)$ is the $\mathcal{N} = (0,4)$ theory given by (6.4) (or (6.3) for the other low-energy limit of $MX$) and $M^2X(k,1)$ is the unknown theory that describes $k$ strings of $MX(1)$ and was discussed in section (6.2).

Recently, 1+1D theories with $\mathcal{N} = (0,4)$ have been implemented [56,57] for the description of the $\mathcal{N} = (1,0)$ 5+1D theory of [17]. These theories are different from the 5+1D theories $MX(k)$ which do not contain hypermultiplets. Indeed the left-moving fermions of [56,57] are not in the strange $(3,1)$ representation as $\zeta^{\alpha\beta}$ of (6.3). It might be that the interacting theories $M^2X$ could be defined by modifying the actions of [56,57].

6.6. A comment on a different approach

One can try another approach to derive $MX(k)$. Let us compactify $X$ on $T^5 \times S^1$ such that the $S^1$ is in a light-like direction. We need to find a description for $N$ KK states along $S^1$. If $S^1$ were space-like we could have used the $E_{6(6)}(\mathbb{Z})$ U-duality to map the KK states to $N$ 5-branes wrapped on $T^5$. However, we have seen in section (5.5) that the theory of $N$ 5-branes does not decouple from the bulk $U(1)$. It would be interesting to know if one can derive $MX(k)$ from such a setup.
6.7. Summary of the conjectures

Let us summarize the M(atrix) conjectures:

**Conjecture 1:**
For every $N$ there exists a 5+1D theory $MX(N)$ with 8 supersymmetries and a low-energy of $N$ tensor multiplets with moduli space $(S^1)^N/S_N$. When compactified on $T^3$ the theory has a moduli space of $(T^4)^N/S_N$. When compactified on $T^5$ the theory has an $SO(5,5,\mathbb{Z})$ T-duality group. The large $N$ limit of this theory is the M(atrix) description of $X$ compactified on $T^5 \times \mathbb{R}^{1,1}$. However, I could not resolve the problem of anomalies and of the zero-point energy of strings in $MX$.

**Conjecture 2:**
For every $k, N$ there exists a 1+1D theory $M^2X(k,N)$ with $\mathcal{N} = (0,4)$ supersymmetry and an $SO(4,4,\mathbb{Z})$ T-duality group. The large $k$ limit of this theory is the M(atrix) description of $MX(k)$ compactified on $T^4 \times \mathbb{R}^{1,1}$.

7. Discussion

We have argued that if there exists a M(atrix) model which describes the DLCQ sector with $p_{\parallel} = 1$ of M-theory on $T^6$ it must have a 6+1D decompactified limit. This will be a local 6+1D theory without gravity, whose low-energy is $U(1)$ SYM. In units for which the $U(1)$ coupling constant is of order one, this theory has 2-branes and 5-branes with finite tension. Existence of such a local theory implies the existence of 5+1D conformal theories with $\mathcal{N} = (1,0)$ supersymmetry and $k$ tensor multiplets with a moduli space of $\mathbb{R}^k/S_k$.

We have briefly discussed the possible cancelation of reparameterization anomalies for the chiral theories which arise in the low-energy description of curved branes. We suggested that the 5-brane anomaly might be canceled by a 6+1D bulk term $\int J_7$ – a mechanism which, for curved 6+1D space-time, is only possible when the world-volume is of co-dimension one, as in our case.

We have explored the possibility that $X$ could be described by a M(atrix) model of its own – denoted $MX$. We argued that such a theory should be a 5+1D theory with 8 supersymmetries and have a $SO(5,5,\mathbb{Z})$ T-duality. Its M(atrix) model should then be a 1+1D theory with $\mathcal{N} = (0,4)$ supersymmetry. We presented a conjectured free theory for the $p_{\parallel} = 1$ sector of the DLCQ.
We showed that T-duality implies that there are macroscopic string states in $MX$, but we had a problem with their reparameterization anomaly and the zero-point energy. I do not know how it is canceled.

If a theory such as $X$ really exists, it might be useful for studying field-theories since it has higher dimensional branes with only 8 supersymmetries.

Can $X$ and $MX$ be realized in M-theory? First we point out that since M-theory is not fully understood we do not know the full extent of the notion “realized in M-theory”. Up until recently, the only method to extract field theories out of M-theory was via low-energy questions. In [17] a new method for extracting a subset of degrees of freedom has been presented. Perhaps the anomaly cancelation mechanism discussed in (5.6) might provide a clue on how to embed $X$ in M-theory or could be a starting point for a No-Go theorem. In any case, if $X$ exists it is realized in at least one way in M-theory – by compactifying a light-like direction! It would seem that the M(atrix) principle, namely that theories should have a DLCQ Hamiltonian is more important than the hope that all theories should be realized in a geometrical setting of what is now known about M-theory.

Acknowledgments

I am indebted to D. Kutasov, L. Susskind and E. Witten and in particular to N. Seiberg for very helpful discussions and comments. I am especially grateful to S. Sethi for collaborating in early stages of this work and for many useful discussions.

This research was supported by a Robert H. Dicke fellowship and by DOE grant DE-FG02-91ER40671.
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