Detectability of cosmic topology in almost flat universes

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Abstract

Recent observations suggest that the ratio of the total density to the critical density of the universe, $\Omega_0$, is likely to be very close to one, with a significant proportion of this energy being in the form of a dark component with negative pressure. Motivated by this result, we study the question of observational detection of possible non-trivial topologies in universes with $\Omega_0 \sim 1$, which include a cosmological constant. Using a number of indicators we find that as $\Omega_0 \to 1$, increasing families of possible manifolds (topologies) become either undetectable or can be excluded observationally. Furthermore, given a non-zero lower bound on $|\Omega_0 - 1|$, we can rule out families of topologies (manifolds) as possible candidates for the shape of our universe. We demonstrate these findings concretely by considering families of manifolds and employing bounds on cosmological parameters from recent observations. We find that given the present bounds on cosmological parameters, there are families of both hyperbolic and spherical manifolds that remain undetectable and families that can be excluded as the shape of our universe. These results are of importance in future search strategies for the detection of the shape of our universe, given that there are an infinite number of theoretically possible topologies and that the future observations are expected to put a non-zero lower bound on $|\Omega_0 - 1|$ which is more accurate and closer to zero.

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1 Introduction

A number of recent observations suggest that the ratio of the baryonic (plus dark) matter density to the critical density, $\Omega_{m0}$, is significantly less than unity, being of the order of $\Omega_{m0} \sim 0.2 - 0.3$ (see e.g. [1] and references therein). On the other hand, recent measurements of the position of the first acoustic peak in the angular power spectrum of CMBR anisotropies, by BOOMERANG-98 and MAXIMA-I experiments, seem to provide strong evidence that the corresponding ratio for the total density to the critical density, $\Omega_0$, is close to one $[4] - [7]$. Added to this is the evidence from the spectral and photometric observations of Type Ia Supernovae $[8, 9]$ which seem to suggest that the universe is undergoing an accelerated expansion at the present epoch $[10, 11]$. This rather diverse set of observations has led to an evolving consensus among cosmologists that the total density of the universe, $\Omega_0$, is likely to be very close to unity, a significant part of which resides in the form of a dark component which is smooth on cosmological scales and which possesses negative pressure. A candidate that satisfies these criteria is the vacuum energy corresponding to a cosmological constant.

Somewhat parallel to these developments, and to some extent unrelated to them, a great deal of work has also recently gone into studying the possibility that the universe may possess compact spatial sections with a non-trivial topology (see, for example, $[12] - [17]$), including the construction of different topological indicators (see e.g. $[18] - [33]$). A fair number of these studies have concentrated on cases where the densities corresponding to matter and vacuum energy are substantially smaller than the critical density. This was reasonable because until very recently observations used to point to a low density universe. In addition an important aim of most of these previous works has often been to produce examples where the topology of the universe has strong observational signals and can therefore be detected and even determined.

The main aims of this paper, which are complementary to these earlier works, are twofold. Firstly, to study the question of detectability of a possible non-trivial compact topology in locally homogeneous and isotropic universes whose total density parameter is taken to be close to one ($\Omega_0 \sim 1$). Secondly, to show how given a non-zero lower bound on $|\Omega_0 - 1|$, we can exclude certain families of compact manifolds as viable candidates for the shape of the universe. We shall do this by employing two indicators, namely the ratios of the so called injectivity radius ($r_{inj}$), and maximal inradius ($r_{max}$) to the depth $\chi_{obs}$ of given catalogues. We study almost flat models with both $\Omega_0 > 1$ and $\Omega_0 < 1$, and our results apply to any method of detection of topology based on observations of multiple images of either cosmic objects or spots of microwave background radiation. We note that even though the idea that non-trivial topologies become harder to detect as $\Omega_0 \rightarrow 1$ is implicitly present in some other works $[22, 27]$, it has, however, often been passed over as an uninteresting limit concerning non-trivial topologies, given the observations at the time. Here, in addition to studying this limit in detail by considering concrete families of manifolds (topologies), rather than individual examples as has often been done before, we treat it as the most relevant limit concerning the geometry of the universe today in view of the recent observations $[4] - [7]$. 

2
The structure of the paper is as follows. In Section 2 we give an account of the cosmological models employed. Section 3 contains a brief account of our topological setting. In Section 4 we discuss the question of detectability of cosmic topology using a number of indicators. Section 5 contains a detailed discussion of the question of detection of cosmic topology in universes with $|\Omega_0 - 1| \ll 1$, with the help of concrete examples, and finally Section 6 contains summary of our main results and conclusions.

2 Cosmological Setting

Let us begin by assuming that the universe is modelled by a 4-manifold $\mathcal{M}$ which allows a $(1 + 3)$ splitting, $\mathcal{M} = R \times M$, with a locally isotropic and homogeneous Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = -c^2dt^2 + R^2(t)\left[d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)\right], \quad (2.1)$$

where $t$ is a cosmic time, the function $f(\chi)$ is given by $f(\chi) = \chi, \sin \chi, \text{or sinh} \chi,$ depending on the sign of the constant spatial curvature ($k = 0, \pm 1$), and $R(t)$ is the scale factor. Furthermore, we shall assume throughout this article that the 3-space $M$ is a multiply connected compact quotient manifold of the form $M = \tilde{M}/\Gamma$, where $\Gamma$ is a discrete group of isometries of $\tilde{M}$ acting freely on the covering space $\tilde{M}$ of $M$, where $\tilde{M}$ can take one of the forms $E^3, S^3$ or $H^3$ [corresponding, respectively, to flat ($k = 0$), elliptic ($k > 0$) and hyperbolic ($k < 0$) spaces]. The group $\Gamma$ is called the covering group of $M$, and is isomorphic to its fundamental group $\pi_1(M)$.

For non-flat models ($k \neq 0$), the scale factor $R(t)$ is identified with the curvature radius of the spatial section of the universe at time $t$, and thus $\chi$ can be interpreted as the distance of any point with coordinates $(\chi, \theta, \phi)$ to the origin of coordinates (in $\tilde{M}$), in units of curvature radius, which is a natural unit of length and suitable for measuring areas and volumes. Throughout this paper we shall use this natural unit.

In the light of current observations, we assume the current matter content of the universe to be well approximated by dust (of density $\rho_m$) plus a cosmological constant $\Lambda$. The Friedmann equation is then given by

$$H^2 = \frac{8\pi G \rho_m}{3} - \frac{k c^2}{R^2} + \frac{\Lambda}{3}, \quad (2.2)$$

where $H = \dot{R}/R$ is the Hubble parameter and $G$ is the Newton’s constant, which upon introducing $\Omega_m = \frac{8\pi G \rho_m}{3 H^2}$ and $\Omega_\Lambda \equiv \frac{8\pi G \rho_\Lambda}{3 H^2} = \frac{\Lambda}{3 H^2}$ and letting $\Omega = \Omega_m + \Omega_\Lambda$, it can be rewritten as

$$R^2 = \frac{k c^2}{H^2 (\Omega - 1)}, \quad (2.3)$$

For universe models having compact spatial sections with non-trivial topology, which we shall be concerned with in this article, it is clear that any attempt at the discovery of such a topology through observations must start with the comparison of the curvature radius and
the horizon radius (distance) \( d_{\text{hor}} \) at the present time. To calculate the latter, we recall the redshift-distance relation in the above FLRW settings can be written in the form

\[
d(z) = \frac{c}{H_0} \int_0^z \left[ (1 + x)^3 \Omega_m + \Omega_\Lambda - (1 + x)^2 (\Omega_0 - 1) \right]^{-1/2} dx ,
\]

where the index 0 denotes evaluation at present time. The horizon radius \( d_{\text{hor}} \) is then defined as

\[
d_{\text{hor}} = \lim_{z \to \infty} d(z) ,
\]

which, by using (2.3) evaluated at the present time, can be expressed in units of the curvature radius in the form

\[
\chi_{\text{hor}} = \frac{d_{\text{hor}}}{R_0} = \sqrt{|1 - \Omega_0|} \int_0^\infty \left[ (1 + x)^3 \Omega_m + \Omega_\Lambda - (1 + x)^2 (\Omega_0 - 1) \right]^{-1/2} dx .
\]  

(2.5)

We note that in practice there are basically three types of catalogues which can be used in order to search for repeated patterns in the universe and hence the nature of the cosmic topology: namely, clusters of galaxies, containing clusters with redshifts of up to \( z \approx 0.3 \); active galactic nuclei (mainly QSO’s and quasars), with a redshift cut-off of \( z_{\text{max}} \approx 4 \); and the CMBR from the decoupling epoch with a redshift of \( z \approx 3000 \). It is expected that future catalogues will increase the first two cut-offs to \( z_{\text{max}} \approx 0.7 \) [34] and \( z_{\text{max}} \approx 6 \) (by 2005 [35]) respectively. Thus, instead of \( \chi_{\text{hor}} \) (for which \( z \to \infty \)) it is observationally more appropriate to consider the largest distance \( \chi_{\text{obs}} = \chi(z_{\text{max}}) \) explored by a given astronomical survey.

### 3 Topological prerequisites

Let us start by recalling that, contrary to the hyperbolic and elliptic cases, there is no natural unit of length for the Euclidean geometry, since in this case the curvature radius is infinite. Also, it is unlikely that astrophysical observations can fix the density parameter \( \Omega_0 \) to be exactly equal to one. Therefore, in this section we shall confine ourselves to nearly flat hyperbolic and spherical spaces and briefly consider some relevant facts about manifolds with constant non-zero curvature, which will be used in the following sections.

Regarding the hyperbolic case, there is at present no complete classification of these manifolds. A number of important results are, however, known about them, including the two important theorems of Mostow [36] (see also Prasad [37]) and Thurston [38]. According to the former, geometrical quantities of orientable hyperbolic manifolds, such as their (finite) volumes and the lengths of their closed geodesics, are topological invariants. Moreover, it turns out that there are only a finite number of compact hyperbolic manifolds with the same volume. According to the latter, there is a countable infinity of sequences of compact orientable hyperbolic manifolds, with the manifolds of each sequence being ordered in terms of their volumes [38]. Moreover there exists a hyperbolic 3-manifold with a minimum volume [38], whose volume is shown to be greater than 0.28151 [39].

Hyperbolic manifolds can be easily constructed and studied with the publicly available software package SnapPea [40] (see e.g. [41]). In this way, Hodgson and Weeks have compiled
a census of 11031 closed hyperbolic manifolds, ordered by increasing volumes (see \[40, 42\]). As examples, we show in Table 1 the first 10 manifolds from this census with the lowest volumes, together with their volumes, as well as the lengths $l_M$ of their smallest closed geodesics and their inradii $r_-$, which are discussed below. The first two entries (rows) of Table 1 are known as Weeks’ and Thurston’s manifolds respectively, with the former conjectured to be the smallest (minimum volume) closed hyperbolic manifold.

| $M$          | $Vol(M)$ | $l_M$    | $r_-$    |
|--------------|----------|----------|----------|
| m003(-3,1)   | 0.942707 | 0.584604 | 0.519162 |
| m003(-2,3)   | 0.981369 | 0.578082 | 0.535437 |
| m007(3,1)    | 1.014942 | 0.831443 | 0.527644 |
| m003(-4,3)   | 1.263709 | 0.575079 | 0.550153 |
| m004(6,1)    | 1.284485 | 0.480312 | 0.533500 |
| m004(1,2)    | 1.398509 | 0.366131 | 0.548345 |
| m009(4,1)    | 1.414061 | 0.794135 | 0.558355 |
| m003(-3,4)   | 1.414061 | 0.364895 | 0.562005 |
| m003(-4,1)   | 1.423612 | 0.364895 | 0.566475 |
| m004(3,2)    | 1.440699 | 0.361522 | 0.556475 |

Table 1: First ten manifolds in the Hodgson-Weeks census of closed hyperbolic manifolds, together with their corresponding volumes, length $l_M$ of their smallest closed geodesics and their inradii $r_-$. 

Regarding the spherical manifolds, we recall that the isometry group of the 3-sphere is $O(4)$. However, since any isometry of $S^3$ that has no fixed points is orientation-preserving, to construct multiply connected spherical manifolds it is sufficient to consider the subgroups of $SO(4)$, because for these subgroups the orientation is necessarily preserved. Thus any spherical 3-manifold of positive constant curvature is of the form $S^3/\Gamma$, where $\Gamma$ is a finite subgroup of $SO(4)$ acting freely on the 3-sphere. The classification of spherical 3-dimensional manifolds is well known and can be found, for instance in \[44, 45, 46\] (see also \[47\] for a description in the context of cosmic topology). In the remainder of this section we shall focus our attention on the important subset of such spaces, referred to as lens spaces, in order to examine the question of detectability of cosmic topology of nearly flat spherical universes.

Briefly, there are an infinite number of 3-dimensional lens spaces that are globally homogeneous and also an infinite number that are only locally homogeneous. In both cases the lens spaces are quotient spaces of the form $S^3/Z_p$ where the covering groups are the cyclic groups $Z_p$ ($p \geq 2$). The cyclic groups $Z_p$ can act on $S^3$ in different ways parameterized by an integer parameter, denoted by $q$, such that $p$ and $q$ are relatively prime integers such that $1 \leq q < p/2$. These quotients are the lens spaces $L(p,q)$. The action of $Z_p$ on $S^3$ gives rise to globally homogeneous lens spaces only if $q = 1$; in all other cases the quotient $S^3/Z_p$ gives rise to lens spaces that are globally inhomogeneous.

One can give a very simple description of the actions of $Z_p$ on $S^3$ that give rise to lens
spaces. In fact, the 3-sphere can be described as the points of the bi-dimensional complex space $\mathbb{C}^2$ with modulus 1, thus

$$S^3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \right\},$$

and any action of $\mathbb{Z}_p$ on $S^3$ giving rise to a lens space is generated by an isometry of the form

$$\alpha_{(p,q)} : S^3 \rightarrow S^3,$$

$$(z_1, z_2) \mapsto \left(e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2\right), \quad (3.6)$$

where $p$ and $q$ are given as above.

To close this section it is important to recall that since any covering group of spherical space forms is of finite order, all of its elements are cyclic. As a consequence, any 3-dimensional spherical manifold is finitely covered by a lens space. Thus lens spaces play a special role in studying the topology of spherical manifolds, which motivates their employment below in our study of the problem of detectability of the cosmic topology in nearly flat spherical universes.

## 4 Detectability of cosmic topology

Unless there are fundamental laws that restrict the topology of the universe, its detection and determination is ultimately expected to be an observational problem, at least at a classical level. As a first step, according to eq. (2.5), one would expect the topology of nearly flat compact universe to be detectable from cosmological observations, provided the bound $\chi_{\text{hor}} \geq 1$ ($d_{\text{hor}} \geq R_0$) holds, since the typical lengths of the simplest hyperbolic and spherical manifolds are of the order of their curvature radii (see, for example, the values of $l_M$ and $r_-$ in Table 1).

This is, however, a rough estimate and a more appropriate bound will crucially depend upon the detailed shape of the universe, our position in it, and on the cosmological parameters. This is particularly true in view of the fact that a crucial feature of generic 3-manifolds is their complicated shapes. Moreover, the fact that most 3-manifolds with non-zero curvature are globally inhomogeneous introduces the possibility of observer dependence (or location dependence) in the topological indicators, in the sense made precise below, which makes it likely for these bounds to be location dependent. What is therefore called for are indicators which are more sensitive (accurate) than $\chi_{\text{hor}}$ and which at the same time take into account the uncertainty in our location in such compact universes.

A natural way to characterize the shape of compact manifolds is through the size of their closed geodesics. More precisely, for any $x \in M$, the distance function $\delta_g(x)$ for a given isometry $g \in \Gamma$ is defined by

$$\delta_g(x) = d(x, gx), \quad (4.7)$$
which gives the length of the closed geodesic associated with the isometry $g$ that passes through $x$. So one can define the length of the smallest closed geodesic that passes through $x \in M$ as
\[
\ell(x) = \min_{g \in \tilde{\Gamma}} \{\delta_g(x)\},
\]
where $\tilde{\Gamma}$ denotes the covering group without the identity map, i.e. $\tilde{\Gamma} = \Gamma \setminus \{id\}$.

In a globally homogeneous manifold, the distance function for any covering isometry $g$ is constant, as is the length of the closed geodesic associated with it. However, this is not the case in a locally, but non-globally, homogeneous manifold, where in general the length of the smallest closed geodesic in $M$ is given by
\[
l_M = \inf_{x \in M} \{\ell(x)\} = \inf_{(g,x) \in \tilde{\Gamma} \times M} \{\delta_g(x)\},
\]
where $\inf_{x \in M} \{\ell(x)\}$ denotes the absolute (global) minimum of $\ell(x)$, and $l_M$ clearly satisfies $l_M \leq \ell(x)$. The injectivity radius, which is the radius of the smallest sphere inscribable in $M$ (terminology used in the SnapPea package [40] and in refs. [14, 22]) is then given by $r_{inj} = l_M/2$. One can similarly define the injectivity radius at any point $x$ as $r_{inj}(x) = \ell(x)/2$.

Furthermore, the maximal inradius $r_{max}^-$, can be defined as the radius of the largest sphere embeddable in $M$, and is given by
\[
r_{max}^- = \frac{1}{2} \sup_{x \in M} \{\ell(x)\},
\]
where $\sup_{x \in M} \{\ell(x)\}$ indicates the absolute (global) maximum of $\ell(x)$. The maximal inradius $r_{max}^-$ is half of the largest distance any point in $M$ can be from its closest image. Clearly in the covering space $\tilde{M}$, the maximal inradius $r_{max}^-$ is also the radius of the largest sphere inscribable in any fundamental polyhedron (FP) of the set of all possible fundamental polyhedra of $M$.

An indicator that has been utilized in most studies regarding searches for topological multiple images, is the ratio of the injectivity radius $r_{inj}$ to $\chi_{hor}$ (see, for example, [14] and reference therein and also [21] for a similar measure). At first sight this seems to be a very accurate indicator, since it defines the minimum scale required for multiple images to be in principle observable in a given multiply connected universe. This can be made more practical from an observational point of view by taking instead a different indicator, defined as the ratio of the injectivity radius to the largest distance explored by some astronomical survey $\chi_{obs}$ [12], namely
\[
T_{inj} = \frac{r_{inj}}{\chi_{obs}} .
\]

It turns out, however, that such individual indicators based on topological invariants do not encode all the information required to fix the topology uniquely. Thus, more than one indicator is often necessary in practice. This is the case for the above indicator $T_{inj}$, an

\footnote{We are assuming that the topology of the spacelike section $M$ of our universe is compact, so for any possible manifold a closed geodesic of non-zero minimum length always exists.}
important limitation of which arises from the fact that generic (globally inhomogeneous) manifolds are likely to have complicated shapes, making it unlikely for the smallest closed geodesic to pass precisely through our location in the universe. There is also the fact that in the set of all compact manifolds there is no lower bound to the length of the smallest closed geodesics (or equivalently on \( r_{inj} \)), even though each given manifold does have a lower bound \( l_M \).

To partially remedy the first shortcoming above, one can consider, for an observer \( O(x) \) situated at \( x \in M \), the analogous location dependent measure

\[
T_{inj}(x) = \frac{r_{inj}(x)}{\chi_{obs}}.
\]  

(4.10)

However, there is still a major problem associated with this indicator, since our uncertainty regarding our location in the universe makes \( r_{inj}(x) \) uncertain.

As another indicator, we take the ratio of the length of the maximal inradius to \( \chi_{obs} \), namely

\[
T_{max} = \frac{r_{\max}}{\chi_{obs}}.
\]  

(4.11)

A crucial feature of this indicator is that it provides a bound which holds for all observers, regardless of their location and in this sense, therefore, circumvents the problem of our ignorance regarding our location in the universe. We note also that assuming a fixed astronomical survey for a given compact universe, \( T_{inj}(x) \) is bounded by the values of \( r_{inj} \) and \( r_{\max} \), thus

\[
T_{inj} \leq T_{inj}(x) \leq T_{max}.
\]  

(4.12)

It should be noted that one cannot guarantee that the inradius (given in the fourth column of Table 1) is the maximal inradius for the corresponding hyperbolic manifolds. This is so because the inradius is calculated for a specific Dirichlet domain, which depends on the basepoint used for its construction. Thus, the same manifold can have different Dirichlet domains and so different inradii. The maximal inradius \( r_{\max} \) of a given manifold \( M \), however, is unique [43] and corresponds to the largest sphere inscribable in any possible fundamental polyhedra of \( M \). The package SnapPea chooses the basepoint \( x \) at a local maximum of the inradius, which may or may not be the global (absolute) maximum \( r_{\max} \), but the relation \( r_{\max} \geq r_-(x) \) clearly holds for all compact hyperbolic (and elliptic) manifolds. We shall show in the next section that the indicator \( T_{max} \) will enable us to exclude certain families of manifolds (topologies) as the shape of our universe.

Let us close this section by computing \( r_{inj} \) and \( r_{\max} \) for the lens spaces which will be used in the following sections. Recall that the distance between two points \( z = (z_1, z_2) \) and \( w = (w_1, w_2) \) on \( S^3 \) is the angle between these points viewed from the origin of complex bi-dimensional space \( C^2 \),

\[
\cos (d(z, w)) = \langle z, w \rangle = Re(z_1 w_1^* + z_2 w_2^*),
\]
where $\langle \cdot, \cdot \rangle$ indicates the inner product in $\mathbb{C}^2$, and $Re$ takes the real part of any complex expression. Taking $w = \alpha(p,q)z$, with $\alpha(p,q)$ as in (3.6), one has
\[
\cos(d(z,w)) = \cos \frac{2\pi}{p} + \left( \cos \frac{2\pi q}{p} - \cos \frac{2\pi}{p} \right) |z|^2. \quad (4.13)
\]
If $q = 1$, then $d(z,w) = 2\pi/p$ is position independent, showing that $L(p,1)$ is globally homogeneous, as anticipated in the previous section. In the general case of a lens space $L(p,q)$ one obtains
\[
\nu_{\text{inj}} = \frac{\pi}{p} \quad \text{and} \quad \nu_{\text{max}} = \frac{\pi q}{p}, \quad (4.14)
\]
since the coefficient of $|z|^2$ in (4.13) is always non-positive.

5 Detectability of topology of nearly flat universes

In this section we study how the bounds provided by recent cosmological observations can constrain the set of detectable topologies and how given a non-zero lower bound on $|\Omega_0 - 1|$, we can exclude certain families of manifolds as viable candidates for the shape of our universe.

To study the constraints on detectability as a function of $\Omega_0$, we begin by considering the horizon radius (2.5). Figures 1a and 1b show the behaviour of $\chi_{\text{hor}}$ as a function of $\Omega_m$ and $\Omega_\Lambda$. Since in $\Omega_m - \Omega_\Lambda$ plane (hereafter referred to as the parameter space) the flat universes are obviously characterized by the straight line $\Omega_m + \Omega_\Lambda = 1$, these figures clearly demonstrate that as $|\Omega_0 - 1| \to 0$ then $\chi_{\text{hor}} \to 0$, hence showing that, for a given manifold $M$ with non-zero curvature, there are values of $|\Omega_0 - 1|$ below which the topology of the universe is undetectable ($\chi_{\text{hor}} < 1$) for any mix of $\Omega_m$ and $\Omega_\Lambda$. A crucial feature of this result is the rapid way $\chi_{\text{hor}}$ drops to zero in a narrow neighbourhood of the $\Omega_0 = 1$ line. From the observational point of view, this is significant since it shows that the detection of the topology of the nearly flat models ($\Omega_0 \sim 1$) becomes more and more difficult as $\Omega_0 \to 1$, a limiting value favoured by recent observations.

To show concretely how $T_{\text{inj}}$ and $T_{\text{max}}$ can be used to set constraints on the topology of the universe, we recall two recent sets of bounds on cosmological parameters [4, 5], namely

(i) $\Omega_0 = 1.08 \pm 0.06$, and $\Omega_\Lambda = 0.66 \pm 0.06$, obtained by combining the data from CMBR and galaxy clusters; and

(ii) $\Omega_0 = 1.04 \pm 0.05$, and $\Omega_\Lambda = 0.68 \pm 0.05$, obtained by combining the CMBR, supernovae and large scale structure observations.

The bound (i) exclusively implies positive curvatures for the spatial 3-surfaces, while (ii) allows negative curvatures as well. We note that even though the precise values of these bounds are likely to be modified by future observations, the closeness of $\Omega_0$ to 1 is expected to be confirmed. We have chosen these bounds here as concrete examples of how recent observations may be employed in order to constrain the topology of the universe, and clearly similar procedures can be used for any modified future bounds on cosmological parameters.
Let us return to the question of detectability of the topology in the neighbourhood $\Omega_0 \sim 1$ and assume that a particular catalogue which covers the entire sky up to a redshift cut-off $z_{max}$. For a given universe [with fixed $(\Omega_m, \Omega_\Lambda)$] this assumption fixes the values of $\chi_{obs}$ used in (4.9) – (4.11).

Suppose that the universe is compact and let $r_{inj}$ denote its injectivity radius. Now if $\chi_{obs} < r_{inj}$, then the topology of the universe is undetectable by any survey of depth up to the corresponding $z_{max}$. Note that this assertion holds regardless of our particular position in the universe, as $r_{inj} \leq r_{inj}(x)$, for all $x$. Also, since $\chi_{hor}$ is the farthest distance at which events are causally connected to us, one can obtain limits on undetectability by considering $\chi_{hor}$ rather than $\chi_{obs}$. For example, for a given universe [fixed $(\Omega_m, \Omega_\Lambda)$] any manifold $M$ whose $r_{inj}$ has a value that lies above the bird-like surface in Figure 1 (thus ensuring $T_{inj} > 1$) is undetectable. The crucial point here is that as we approach the line $\Omega_0 = 1$, the rapid decrease in the allowed values of $\chi_{obs}$ will result in a rapid elimination of families of detectable manifolds (topologies). This is important given that recent observations seem to restrict the cosmological parameters to small regions near the $\Omega_0 = 1$ line.

As concrete examples of constraints set by the bound (i), we consider universes that possess lens space $L(p, q)$ topologies. Using (4.14), and recalling that the topology cannot be detected if $T_{inj} > 1$, one finds that for this family of universes the topology will be undetectable if

$$p < \frac{\pi}{\chi_{obs}} \leq p^* , \quad (5.15)$$

where $p^*$ is the smallest integer larger than $\pi/\chi_{obs}$. Using (2.5) and (5.13), we have compiled in Table 2 the values of $p^*$ for different sets of values of $(\Omega_m, \Omega_\Lambda)$ contained in bounds (i) and four catalogues with distinct values of $z_{max}$.

As concrete examples, we note that given $\Omega_0 = 1.08$ and $\Omega_\Lambda = 0.66$, then according to Table 2, it would be impossible to detect any multiple images if the universe turns out to be the projective 3-space $L(2, 1)$ (for which $r_{inj} = 1.57080$), or the lens space $L(3, 1)$ (for which $r_{inj} = 1.04712$), as the entire observable universe would lie inside some fundamental polyhedron of $M$. Similarly, if the universe turns out to be either of the lens spaces $L(4,1)$, $L(5,1)$, $L(5,2)$, or $L(6,1)$, it would be impossible to detect its topology using catalogues of quasars up to $z_{max} = 6$. It is important to note that from Table 2 one can see that as $\Omega_0$ approaches unity, $p^*$ increases, which implies that an increasing subset of lens space topologies are undetectable, in agreement with Figures 1. Note also that the greater the depth $z_{max}$ of survey, the smaller is the number of undetectable topologies, as expected. But even when $z_{max} \to \infty$ there is still a subset of lens space topologies that remains undetectable. The key point here is that proceeding in a similar way, one can translate bounds on cosmological parameters to constraints on allowed topologies (here taken to be lens topologies).

So far we have considered the question of detectability of families of manifolds (topologies) in universes with $\Omega_0 \to 1$. Alternatively, we may ask what is the region of the parameter space for which a given topology is undetectable. To this end, we note that for a given
Table 2: For each value of $z_{\text{max}}$ the first and third rows correspond to the smallest and largest values of $\chi_{\text{obs}}$ in the parameter space given by bounds (i), while the second entry corresponds to the central value. From this table one can see that the projective space $L(2,1)$, and the lens space $L(3,1)$ are undetectable in a universe with $\Omega_0 = 1.08$, and $\Omega_{\Lambda_0} = 0.66$; while for the same values of the density parameters the lens spaces $L(4,1)$, $L(5,1)$, $L(5,2)$, and $L(6,1)$ are undetectable, using catalogues of depth up to $z_{\text{max}} = 6$.

| $z_{\text{max}}$ | $\Omega_0$ | $\Omega_{\Lambda_0}$ | $\chi_{\text{obs}}$ | $p_s$ |
|------------------|------------|---------------------|---------------------|------|
| $\infty$         | 1.02       | 0.60                | 0.40885             | 8    |
|                  | 1.08       | 0.66                | 0.82728             | 4    |
|                  | 1.14       | 0.72                | 1.10777             | 3    |
| 3000             | 1.02       | 0.60                | 0.40088             | 8    |
|                  | 1.08       | 0.66                | 0.81135             | 4    |
|                  | 1.14       | 0.72                | 1.10777             | 3    |
| 6                | 1.02       | 0.60                | 0.24375             | 13   |
|                  | 1.08       | 0.66                | 0.49596             | 7    |
|                  | 1.14       | 0.72                | 0.66796             | 5    |
| 1                | 1.02       | 0.60                | 0.10278             | 31   |
|                  | 1.08       | 0.66                | 0.20879             | 16   |
|                  | 1.14       | 0.72                | 0.28073             | 12   |

topology (fixed $r_{\text{inj}}$) and for a given catalogue cut-off $z_{\text{max}}$, one can solve the equation

$$\chi_{\text{obs}} = r_{\text{inj}}, \quad (5.16)$$

which is an implicit function of $\Omega_m_0$ and $\Omega_{\Lambda_0}$. Equivalently using (5.5), equation (5.16) can be rewritten as the following implicit function of $\varepsilon_0 \equiv 1 - \Omega_0$ and $\varepsilon_{\Lambda} \equiv 1 - \Omega_{\Lambda_0}$ :

$$\chi_{\text{obs}} \equiv \sqrt{|\varepsilon_0|} \int_0^{z_{\text{max}}} \left[(\varepsilon_{\Lambda} - \varepsilon_0)(1+x)^3 + 1 - \varepsilon_{\Lambda} + \varepsilon_0(1+x)^2\right]^{-1/2} dx = r_{\text{inj}}. \quad (5.17)$$

Now, note that any point of the parameter space for which $\chi_{\text{obs}} < r_{\text{inj}}$, will lie below the graph of the solution of eq. (5.17) in the plane ($\varepsilon_{\Lambda}, |\varepsilon_0|$). Thus the points below the solution curves correspond to universe models for which any topology with injectivity radius larger than that given by the fixed $r_{\text{inj}}$ are undetectable.

As concrete examples of the constraints set by the bound (ii), we consider the subinterval of this bound with $\Omega_0 \leq 1$, corresponding to nearly flat hyperbolic universes. We will take $r_{\text{inj}}$ as the largest value of $\chi_{\text{obs}}$ in this region of parameter space, which can be easily calculated to be $\chi_{\text{obs}} = 0.20125$ for $z_{\text{max}} = 6$, and $\chi_{\text{obs}} = 0.34211$ for $z_{\text{max}} = 3000$. We solved equation (5.17) with these two specific extreme values of $\chi_{\text{obs}}$ as the values for $r_{\text{inj}}$, and the results are shown in Fig. 2 as plots of $\varepsilon_0$ against $\varepsilon_{\Lambda}$. As can be seen the allowed hyperbolic region of the parameter space, given by $\varepsilon_0 \in (0, 0.01]$ and $\varepsilon_{\Lambda} \in [0.27, 0.37]$, lies below both curves, showing that using quasars up to $z_{\text{max}} = 6$, nearly flat FLRW hyperbolic
universes, with the density parameters in this region, will have undetectable topologies, if their corresponding injectivity radii satisfy $r_{inj} \geq 0.20125$. Similarly, using CMBR, the topology of nearly flat FLRW hyperbolic universes with $r_{inj} \geq 0.34211$ and the density parameter in the hyperbolic range of the bound (ii) are undetectable.

In Table 3 we have summarized the restrictions on detectability imposed by the hyperbolic subinterval of the bounds (ii) on the first seven manifolds of Hodgson-Weeks census of closed orientable hyperbolic manifolds (there is no restrictions for the last three in the set of the first ten smallest manifolds). In Table 3, $U$ denotes that the topology is undetectable by any survey of depth up to the redshifts $z_{max} = 6$ (quasars) or $z_{max} = 3000$ (CMBR) respectively. Thus using quasars, the topology of the five known smallest hyperbolic manifolds, as well as m009(4,1), are undetectable within the hyperbolic region of the parameter space given by (ii), while only topologies m007(3,1) and m009(4,1) remain undetectable even if CMBR observations are used.

| $M$         | $r_{inj}$  | QUASARS | CMBR |
|-------------|------------|---------|------|
| m003(-3,1)  | 0.292302   | $U$     | ---  |
| m003(-2,3)  | 0.289041   | $U$     | ---  |
| m007(3,1)   | 0.415721   | $U$     | $U$  |
| m003(-4,3)  | 0.287539   | $U$     | ---  |
| m004(6,1)   | 0.240156   | $U$     | ---  |
| m004(1,2)   | 0.183065   | ---     | ---  |
| m009(4,1)   | 0.397067   | $U$     | $U$  |

Table 3: Restrictions on detectability imposed by the hyperbolic interval of the recent bounds (ii) for the first seven manifolds of Hodgson-Weeks census. The capital $U$ stands for undetectable using catalogues of quasars (up to $z_{max} = 6$) or CMBR (up to $z_{max} = 3000$).

Hitherto we have used $T_{inj}$ together with observational bounds on cosmological parameters in order to set bounds on detectability. We shall now employ the indicator $T_{max}$ as a way of excluding certain families of manifolds (topologies) for the universe. Recalling that for a hyperbolic or spherical compact manifold $M$, the maximal inradius $r_{max}$ is the radius of the largest sphere embeddable in $M$, then any catalogue of depth $z_{max}$, such that $\chi_{obs} > r_{max}$, may contain multiple images of cosmic sources or CMBR spots. Thus if we can be confident that multiple images do not exist up to a certain depth $z_{max}$, then one can claim that $T_{max} > 1$ ($r_{max} > \chi_{obs}$), and therefore any topology not satisfying this inequality (i.e. those for which $r_{max} < \chi_{obs}$) can be excluded by such observations.\(^2\)

As an illustration for the case of spherical manifolds, suppose again that our universe

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\(^2\)Obviously it is possible that even exploring a region of the universe deeper than a ball of radius $r_{max}$ with catalogues of cosmic sources, we do not observe any multiple images. This may be the case when the selection rules used to build the catalogues do not allow the recording of enough multiple images to have a detectable topological signal (see \(^2\) for a more detailed discussion on this point).
has a lens space topology. If using a catalogue of cosmic objects with redshift cut-off up to \( z_{\text{max}} \) yields no multiple images, then any lens space satisfying \( T_{\text{max}} < 1 \) \( (r_{\text{max}} < \chi_{\text{obs}}) \) can be discarded as a model for the shape of our universe, while those with \( T_{\text{max}} > 1 \) would be either detectable or undetectable. Using (4.14) one can see that the discarded lens spaces \( L(p, q) \) are those satisfying

\[
p \geq p_q^*,
\]

where \( p_q^* \) is the smallest integer larger than \( q\pi/\chi_{\text{obs}} \). As concrete examples, assume that a catalogue of cluster of galaxies with redshift cut-off \( z_{\text{max}} = 1 \) was constructed and no multiple images exist. Using the values of \( \chi_{\text{obs}} \) from Table 2, corresponding to the extremes and the central values of \( \Omega_0 \) and \( \Omega_{\Lambda 0} \) of the bounds (i), we can obtain values of \( p_q^* \) corresponding to \( q \) ranging from 1 to 7, as shown in Table 4. Equation (5.18) together with this table

\[
\begin{array}{cccccccc}
\Omega_0 & \Omega_{\Lambda 0} & q \rightarrow & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1.14 & 0.72 & p_q^* & 12 & 23 & 34 & 45 & 56 & 68 & 79 \\
1.08 & 0.66 & p_q^* & 16 & 31 & 46 & 61 & 76 & 91 & 106 \\
1.02 & 0.60 & p_q^* & 31 & 62 & 93 & 123 & 154 & 185 & 216 \\
\end{array}
\]

Table 4: Lens spaces \( L(p, q) \) with \( p \geq p_q^* \) are discarded on the basis of observations if there are no multiple images up to \( z_{\text{max}} = 1 \) in universes with three different set of \( (\Omega_0, \Omega_{\Lambda 0}) \) in the bound (i). For a given universe \( [\text{fixed} \ (\Omega_0, \Omega_{\Lambda 0})] \) and for each value of \( q \) the corresponding value of \( p_q^* \) is given, thus for example, any lens space of the family \( L(p, 3) \) with \( p \geq 93 \) could not be the shape of a universe with \( \Omega_0 = 1.02 \) and \( \Omega_{\Lambda 0} = 0.60 \). Make it clear that, for example, any lens space belonging to the family \( L(p, 4) \), with \( p \geq 61 \), cannot be the shape of an elliptic universe with the central values of the density parameters. Furthermore, it is clear from Table 4 that as \( \Omega_0 \) approaches unity, \( p_q^* \) increases for any fixed \( q \), which implies that a decreasing subset of lens space topologies are excluded.

Alternatively, we can ask how fixing the cosmological parameters results in the exclusion of a given topology as a possible shape of our universe. To this end, given a fixed value of \( r_{\text{max}} \), one can solve the equation

\[
\chi(z_{\text{max}}) = r_{\text{max}}^0,
\]

which is analogous to (5.19).

We recall that if one can be certain that there is no multiple images up to a certain depth \( z_{\text{max}} \) then one can claim that \( r_{\text{max}}^0 > \chi_{\text{obs}} \), and therefore any topology for which \( r_{\text{max}}^0 < \chi_{\text{obs}} \) can be excluded by such observations. Thus, for a fixed catalogue depth \( z_{\text{max}} \), the points above the graph of the solution of (5.19) in the plane \( \varepsilon_{\Lambda} - \varepsilon_0 \) correspond to cosmological models for which any topology with \( r_{\text{max}}^0 < \chi_{\text{obs}} \) can be excluded.

As an interesting example of application of the indicator \( T_{\text{max}} \), we consider the hyperbolic manifolds and recall the important recent mathematical result according to which any closed orientable hyperbolic 3–manifold contains a ball of radius \( r_0 = 0.24746 \ [39] \). It therefore follows that the absolute lower bound to \( r_{\text{max}}^0 \) for any compact hyperbolic manifold is given
by $r_0$. We solved Eq. (5.19) for $r^{\max} = r_0$ and with catalogue depths $z_{\max} = 6$ and 3000, and the results are shown in Fig. 3. This figure shows that the allowed observational hyperbolic range of the cosmological parameters given by bounds (ii) intersects the lowest solution curve, corresponding to the catalogue with $z_{\max} = 3000$. Thus, given the current observational bounds (ii), if no repeated patterns exist up to $z_{\max} = 3000$, there are families of hyperbolic manifolds that can be excluded as the shape of our universe. As concrete examples note that in such cases the topology of the first ten closed orientable hyperbolic manifolds with smallest volumes given in Table 1 would not be excluded, as they all possess values of $r^{\max} \geq 0.519162$, and therefore the corresponding solution curves would lie substantially above the hyperbolic range of the observational bounds (ii).

Interestingly Fig. 3 also shows that if only quasars are used (redshift up to $z_{\max} = 6$) and no multiple images exist, then no hyperbolic manifolds can be excluded as the shape of our universe, as the corresponding solution curve lies above the current bounds (ii) on the cosmological parameters.

The above considerations provide examples of the importance of considering families of manifolds (topologies), rather than arbitrary individual examples, in looking for the shape of our universe in the infinite set of distinct possible topologies.

6 Final Remarks

We have made a detailed study of the question of detectability of the cosmic topology in nearly flat universes ($\Omega_0 \sim 1$), which are favoured by recent cosmological observations. Most studies so far have concentrated on individual manifolds. Given the infinite number of theoretically possible topologies, we have instead concentrated on how to employ current observations and a number of indicators in order to find families of possible undetectable manifold (topologies) as well as families of manifolds (topologies) that can be excluded.

We have found that as $\Omega_0 \to 1$, increasing families of possible manifolds (topologies) become undetectable observationally. In this sense the topology of the universe can be said to become more difficult to detect through observations of multiple images of either cosmic objects or spots of microwave background radiation. We have also found that for any given manifold $M$ with non-zero curvature, there are values of $|\Omega_0 - 1|$ below which its topology is undetectable (using pattern repetition) for any mix of $\Omega_{m0}$ and $\Omega_{\Lambda 0}$.

We have made a detailed study of the constraints that the most recent estimates of the cosmological parameters place on the detectable and allowed topologies. Considering concrete examples of both spherical and hyperbolic manifolds, we find that, given the present observational bounds on cosmological parameters, there are families of both hyperbolic and

3Clearly we are assuming here that the universe is multiply connected, and no multiple images exist. These multiply connected universes are therefore indistinguishable (using pattern repetition) from simply-connected universes with the same covering space, equal radius, and identical distribution of cosmic sources. In these cases the scale of multiply-connectedness is greater than the observations depth $\chi_{\text{obs}}(z_{\max})$ and no sign of multiply-connectedness will arise.
spherical manifolds that remain undetectable and families that can be excluded as the shape of our universe. We also demonstrate the importance of considering families of possible manifolds (topologies), rather than arbitrary individual examples, in search strategies for the detection of the shape of our universe.

Finally we note that even though the precise values of the recent bounds on the density parameter used in this paper are likely to be modified by future observations, the closeness \( \Omega_0 \sim 1 \) is expected to be confirmed. We have chosen these bounds in this paper as concrete examples of how recent observations may be employed in order to constrain the topology of the universe, and clearly similar procedures can be used for any modified future bounds on cosmological parameters.

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Figure captions

Figure 1. The behaviour of the horizon radius \( \chi_{\text{hor}} \) in units of curvature radius, for FLRW models with dust and cosmological constant, given by Eq. (2.5), as a function of the cosmological density parameters \( \Omega_\Lambda \) and \( \Omega_m \). These figures (1a & 1b) show clearly the rapid way \( \chi_{\text{hor}} \) (as well as \( \chi_{\text{obs}} \)) falls off to zero for nearly flat (hyperbolic or elliptic) universes, as \( \Omega_0 = \Omega_m + \Omega_\Lambda \to 1 \). In both figures the vertical axes represent \( \chi_{\text{hor}} \), while the horizontal axes are given respectively (and anti–clockwise), by \( \Omega_\Lambda \) and \( \Omega_m \) (in Fig. 1a) and \( \Omega_m \) and \( \Omega_\Lambda \) (in Fig. 1b).

Figure 2. The solutions of Eq. (5.17) as plots of \( \epsilon_0 = 1 - \Omega_\Lambda 0 \) (vertical axis) versus \( \epsilon_\Lambda = 1 - \Omega_\Lambda 0 \) (horizontal axis), with \( r_{\text{inj}} \) taken as the largest values of the \( \chi_{\text{obs}} \) in the range of cosmological parameters given by the bounds (ii). The upper and the lower solutions curves correspond to the catalogues with \( z_{\text{max}} = 3000 \) and \( z_{\text{max}} = 6 \), respectively. Note that the allowed hyperbolic region of the parameter space given by \( (\epsilon_0 \in (0.01, 0.0]) \) & \( \epsilon_\Lambda \in [0.27, 0.37] \), and represented by the dashed rectangular box, lies below both curves, showing that using quasars up to \( z_{\text{max}} = 6 \) nearly flat FLRW hyperbolic universes, with the density parameters in this region, will have undetectable topologies, if their corresponding inradii satisfy \( r_{\text{inj}} \geq 0.20125 \). Similarly, using CMBR, with \( z_{\text{max}} = 3000 \), the topology of nearly flat hyperbolic FLRW universes with \( r_{\text{inj}} \geq 0.34211 \) will be undetectable.
Figure 3. The solutions of Eq. (5.19) for $r_{\text{max}} = 0.24746$, as plots of $\epsilon_0$ (vertical axis) versus $\epsilon_{\Lambda}$ (horizontal axis), for the catalogues with $z_{\text{max}} = 6$ and $z_{\text{max}} = 3000$. As can be seen, the allowed observational hyperbolic range of the cosmological parameters given by bounds (ii), and represented by the dashed rectangular box, intersects the lowest solution curve, corresponding to the catalogue with $z_{\text{max}} = 3000$, thus showing that given the current observational bounds, there are families of hyperbolic manifolds that remain undetectable and families that can be excluded as the shape of our universe, if there are no repeated patterns up to $z_{\text{max}} = 3000$. On the other hand if quasars ($z_{\text{max}} = 6$) are used, then no hyperbolic manifolds can be excluded as the shape of our universe, if there are no repeated patterns up to that depth, as the corresponding solution curve lies above the dashed rectangle.

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Figure 2

$z_{\text{max}} = 3000$

$z_{\text{max}} = 6$
