A new look at the theory uncertainty of $\epsilon_K$

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ABSTRACT: The observable $\epsilon_K$ is sensitive to flavor violation at some of the highest scales. While its experimental uncertainty is at the half percent level, the theoretical one is in the ballpark of 15%. We explore the nontrivial dependence of the theory prediction and uncertainty on various conventions, like the phase of the kaon field. In particular, we show how such a rephasing allows to make the short-distance contribution of the box diagram with two charm quarks, $\eta_{cc}$, purely real. Our results allow to slightly reduce the total theoretical uncertainty of $\epsilon_K$, while increasing the relative impact of the imaginary part of the long distance contribution, underlining the need to compute it reliably. We also give updated bounds on the new physics operators that contribute to $\epsilon_K$.

KEYWORDS: CP violation, Kaon Physics

ArXiv ePrint: 1602.08494
1 Introduction

The study of mixing and CP violation in the $K^0$-$\bar{K}^0$ system was crucial for the development of the standard model (SM). The comparison of the measurement of the CP violating parameter in $K^0-\bar{K}^0$ mixing, $\epsilon_K$, with its SM calculation provides important constraints on the CKM matrix. The observable $\epsilon_K$ also probes some of the highest new physics (NP) scales, and it gives severe constraints on explicit models of flavor. Moreover, to distinguish between possible NP interpretations of flavor anomalies, it is particularly important to know the level of consistency between the constraints on the flavor sector from $K$ and $B$ decay measurements.

What are the current limiting factors of the $\epsilon_K$ sensitivity to NP? How can we possibly improve them, now and in the future? The level to which we can answer these questions will have a major impact on our understanding of flavor. These limiting factors have to be looked for in the SM prediction of $\epsilon_K$, whose uncertainty is more than an order of magnitude above the half percent precision of the experimental measurement. Part of the SM uncertainty in the $\epsilon_K$ prediction is parametric, i.e., due to the relatively poor knowledge of some of the CKM parameters, most notably $A$ (or equivalently $|V_{cb}|$). This knowledge will be substantially improved by future measurements at Belle II and LHCb [1, 2], which will hopefully also resolve tensions between inclusive and exclusive determinations of $|V_{cb}|$ and $|V_{ub}|$ [3].
Besides $|V_{cb}|$, the largest uncertainty in the SM prediction for $\epsilon_K$ originates from the calculation of $\epsilon_{cc}$, the QCD correction to the box diagram with two charm quarks. The NNLO calculation of this quantity [4] found a large correction and a poorly behaved perturbation series, $1, 1.38, 1.87$, at leading, next-to-leading, and next-to-next-to-leading orders, respectively, and thus quoted $\epsilon_{cc} = 1.87 \pm 0.76$, and $|\epsilon_K| = (1.81 \pm 0.28) \times 10^{-3}$. Thus, to what extent $\epsilon_{cc}$ is determined by short distance physics may be questioned. This resulted in different groups treating $\epsilon_{cc}$ differently. For example CKMfitter [5, 6] uses $\epsilon_{cc}$ quoted in ref. [4], whereas UTfit [7, 8] uses the NLO calculation of $\epsilon_{cc}$ [9]. This contributes to the visibly different $\epsilon_K$ regions in CKMfitter and UTfit plots. Ref. [10] instead argued that $\epsilon_{cc} = 1.70 \pm 0.21$ was a reasonable estimate, assuming the dominance of $\Delta m_K$ by the SM contribution, and using an estimate of the long-distance contribution to $\Delta m_K$. Note also that the behavior of the perturbation series, which matters for the uncertainty estimate of $\epsilon_{cc}$, is scheme dependent. The perturbative QCD calculations of the $\eta_{ct} = 0.496(47)$ [11] and $\eta_{tt} = 0.5765(65)$ [12] correction factors to the box diagrams with internal $tt$ and $ct$ quarks, respectively, appear to be better behaved.

In this paper we show that one can eliminate $\epsilon_{cc}$ from the theoretical prediction of $\epsilon_K$, by setting the contribution of that term to the mixing amplitude, $M_{12}$, purely real. While physical results are independent of such conventions, numerically some dependence remains (similar to other scheme dependences), because $M_{12}$ and $\Gamma_{12}$ are calculated using different methods. We discuss the implications of this choice on the SM uncertainty of $\epsilon_K$ and on the resulting constraints on NP, both at present and in the future.

This paper is organized as follows: in section 2 we review some definitions and formalism, making clear the approximations and phase-dependences involved. In section 3 we show how to remove the $\epsilon_{cc}$ contribution from $\epsilon_K$, and discuss the resulting modified predictions for $\epsilon_K$. In section 4 we comment on implications for constraints on new physics. In section 5 summarize our findings, and conclude.

2 The state of the $\epsilon_K$ art

2.1 Definitions

The neutral kaon mass eigenstates are linear combinations of $|K^0\rangle = |d\bar{s}\rangle$ and $|\bar{K}^0\rangle = |\bar{d}s\rangle$. The time evolution of these states is described by the Schrödinger equation,

$$i \frac{d}{dt} \left( \begin{array}{c} K^0 \\ \bar{K}^0 \end{array} \right) = \left( \begin{array}{cc} M - i \frac{\Gamma}{2} & \langle K^0 | \bar{K}^0 \rangle \\ \langle \bar{K}^0 | K^0 \rangle & -M + i \frac{\Gamma}{2} \end{array} \right) \left( \begin{array}{c} K^0 \\ \bar{K}^0 \end{array} \right),$$

(2.1)

where the mass ($M$) and the decay ($\Gamma$) mixing matrices are $2 \times 2$ Hermitian matrices. The mass eigenstates are usually labeled with their lifetimes

$$|K_{S,L}\rangle = p|K^{0}\rangle \pm q|\bar{K}^{0}\rangle,$$

(2.2)

$^1$The sign of $q$ is a convention, degenerate with the choice of the phase $\theta = 0$ or $\pi$ in eq. (2.6). Setting the coefficients of $|K^{0}\rangle$ identical in $|K_{L}\rangle$ and $|K_{S}\rangle$, as done in eq. (2.2), sets another non-physical overall phase to zero.
and they are the eigenvectors of $M - i\Gamma/2$. To write eq. (2.2) we have assumed $CPT$ symmetry, as we do in the rest of this paper. The correspondence between the long/short lived and the heavy/light states is

$$K_L = K_{\text{heavy}}, \quad K_S = K_{\text{light}}.$$  \hfill (2.3)

Let us define

$$\Delta m = m_L - m_S > 0,$$  \hfill (2.4)

and

$$\Delta \Gamma = \Gamma_L - \Gamma_S \approx -\Gamma_S < 0.$$  \hfill (2.5)

Throughout this paper we keep explicitly the $CP$ transformation phase

$$CP|K^0\rangle = e^{i\theta}|\overline{K}^0\rangle, \quad CP|\overline{K}^0\rangle = e^{-i\theta}|K^0\rangle,$$  \hfill (2.6)

since both the $\theta = 0$ and $\theta = \pi$ choices are often used in the literature, and the cancellation of this is interesting to follow. The choice of the phase $\theta$ is not to be confused with the phase convention for the kaon and quark fields.

Let us define the decay amplitudes

$$A_f = \langle f|\mathcal{H}|K^0\rangle = |A_f|e^{i(\phi_f + \delta_f)}, \quad \tilde{A}_f = \langle f|\mathcal{H}|\overline{K}^0\rangle = |A_f|e^{i(-\phi_f + \delta_f - \theta)},$$  \hfill (2.7)

where $\phi_f$ and $\delta_f$ are the weak and strong phases respectively, and the amplitude ratios\footnote{The definition $\eta_f = \langle f|\mathcal{H}|K_L\rangle/\langle f|\mathcal{H}|K_S\rangle$ is often used in the literature, and measured magnitudes and phases are quoted. However, there is an arbitrary unphysical relative phase between $|K_L\rangle$ and $|K_S\rangle$. Effectively eq. (2.8) is measured in the interference of $|K_L\rangle$ and $|K_S\rangle$ decays in regeneration experiments.}

$$\eta_f \equiv \frac{\langle f|\mathcal{H}|K_L\rangle \langle K^0|K_S\rangle}{\langle f|\mathcal{H}|K_S\rangle \langle K^0|K_L\rangle} = \frac{1 - (q/p)(\tilde{A}_f/A_f)}{1 + (q/p)(\tilde{A}_f/A_f)}.$$  \hfill (2.8)

In terms of $\eta_f$ for $f = \pi^+\pi^-$ and $\pi^0\pi^0$, $\epsilon_K$ and $\epsilon'$ are defined as

$$\epsilon_K = \frac{2\eta_{+-} + \eta_{00}}{3}, \quad \epsilon' = \frac{\eta_{+-} - \eta_{00}}{3}.$$  \hfill (2.9)

It is $\eta_{+-}$ and $\eta_{00}$ which are measured (and $\epsilon'/\epsilon$ is extracted from $|\eta_{00}/\eta_{+-}|^2 \approx 1 - 6 \text{Re}(\epsilon'/\epsilon)$, valid for $|\epsilon'/\epsilon| \ll 1$).

For a theoretical discussion, since $K \to \pi\pi$ decays are dominated by the isospin $I = 0$ two-pion state over $I = 2$, it is convenient to define

$$\eta_I = \frac{\langle (\pi\pi)_I|\mathcal{H}|K_L\rangle \langle K^0|K_S\rangle}{\langle (\pi\pi)_I|\mathcal{H}|K_S\rangle \langle K^0|K_L\rangle}, \quad \omega = \frac{\langle (\pi\pi)_{I=2}|\mathcal{H}|K_S\rangle}{\langle (\pi\pi)_{I=0}|\mathcal{H}|K_S\rangle}.$$  \hfill (2.10)

The $CP$ violating quantities $\epsilon_K$ and $\epsilon'$ can also be defined as

$$\epsilon_K = \eta_0, \quad \epsilon' = \frac{\omega}{\sqrt{2}} (\eta_2 - \eta_0).$$  \hfill (2.11)

The definitions in eqs. (2.9) and (2.11) are equivalent up to differences of order $|\omega\epsilon'| \sim 10^{-7}$, i.e., to a relative error of $10^{-4}$ for $\epsilon_K$, and 1/22 for $\epsilon'$ (see table 1, and use $|\omega| = |A_2/A_0| |1 + \mathcal{O}(|\epsilon_K|)| \approx 1/22$). Neglecting isospin violation, we can further write

$$\eta_{+-} = \frac{\eta_0 + \eta_2 \omega/\sqrt{2}}{1 + \omega/\sqrt{2}}, \quad \eta_{00} = \frac{\eta_0 - \sqrt{2} \eta_2 \omega}{1 - \sqrt{2} \omega}.$$  \hfill (2.12)
2.2 $\epsilon_K$, phase convention independently

We summarize here how to express $\epsilon_K$ in terms of the off-diagonal elements of the mass and width mixing matrices, $M_{12}$ and $\Gamma_{12}$ (see refs. [13, 14] for more details). We pay attention to write expressions that are independent of the phase conventions for the kaon and quark fields, and we state explicitly the approximations used.

The semileptonic CP asymmetry

$$\delta_L = \frac{\Gamma(K_L \to \pi^- \ell^+ \nu) - \Gamma(K_L \to \pi^+ \ell^- \bar{\nu})}{\Gamma(K_L \to \pi^- \ell^+ \nu) + \Gamma(K_L \to \pi^+ \ell^- \bar{\nu})} = (3.32 \pm 0.06) \times 10^{-3} [3], \tag{2.13}$$

measures CP violation in mixing, in the limit when $A_{\pi^+\ell^-\bar{\nu}} = A_{\pi^-\ell^+\nu} = 0$ and $|A_{\pi^-\ell^+\nu}| = |A_{\pi^+\ell^-\bar{\nu}}|$. Note that these assumptions, valid in the SM to great accuracy, are not precisely tested yet, as the ratio $x_+ = A(K^0 \to \pi^- \ell^+\nu)/A(K^0 \to \pi^+ \ell^-\bar{\nu})$ is only constrained at the $10^{-3}$ level [3].\(^3\) In this limit, the definition in eq. (2.13), and solving the eigenvalue equations imply

$$\delta_L = \frac{1 - |q/p|^2}{1 + |q/p|^2} = \frac{2 \text{Re}(\epsilon_K)}{1 + |\epsilon_K|^2} = \frac{2 \text{Im}(M'_{12}\Gamma_{12})}{4|M_{12}|^2 + |\Gamma_{12}|^2}, \tag{2.14}$$

where we neglected relative higher orders in $|\omega\epsilon'/\epsilon|$. The expressions for the mass and width differences that follow from the eigenvalue equations are

$$\Delta m = 2|M_{12}|, \quad \Delta \Gamma = -2|\Gamma_{12}|, \tag{2.15}$$

and are valid up to relative orders $\delta_L^2$. The relative phase between $M_{12}$ and $\Gamma_{12}$ is $\pi + O(\delta_L)$, since eq. (2.14) implies that its sine is small, and the eigenvalue equation $4 \text{Re}(M'_{12}\Gamma_{12}) = \Delta m \Delta \Gamma < 0$ implies that its cosine is negative.

Equations (2.14) and (2.15) exhaust the information regarding kaon mixing, and $\text{Im}(\epsilon_K)$ is related to CP violation in interference of decay with and without mixing. Still, $\epsilon_K$ is the observable used to constrain CP violation in $K^0$ mixing. The reason is that $\epsilon_K$ is measured with about 4 times smaller relative uncertainty than $\delta_L$, and the phase of $\epsilon_K$ also depends only on mixing parameters. Indeed, the following relation for the phase $\phi_\epsilon$,

$$\phi_\epsilon \simeq \arctan \frac{2|M_{12}|}{|\Gamma_{12}|}, \tag{2.16}$$

is valid up to relative orders $\delta_L^2$ and $|\omega\epsilon'/\epsilon|$, and up to ratios of amplitudes to more than two-body final states, that do not exceed a relative contribution of $10^{-2}$ to $\phi_\epsilon$ (see ref. [15] and the updated measurements in ref. [3] for details). The quantity $\arctan(-2\Delta m/\Delta \Gamma) = 43.52^\circ$ is often referred to as “superweak phase”, and differs from the measured value of $\phi_\epsilon$ by one part in $10^4$, so that the error of eq. (2.16) neither exceeds that level. Using eq. (2.14) for $\text{Re}(\epsilon_K)$ and eq. (2.16) for $\phi_\epsilon$ we obtain

$$\epsilon_K = \frac{e^{i\phi_\epsilon}}{2} \sin \phi_\epsilon \arg \left( -\frac{M_{12}}{\Gamma_{12}} \right) = e^{i\phi_\epsilon} \sin \phi_\epsilon \frac{\text{Im}(-M_{12}/\Gamma_{12})}{2|M_{12}|/|\Gamma_{12}|} = e^{i\phi_\epsilon} \cos \phi_\epsilon \text{Im}(-M_{12}/\Gamma_{12}). \tag{2.17}$$

\(^3\)This is historically called the $\Delta s = \Delta Q$ rule. In the SM it is only violated by higher orders in the weak interaction; when we discuss NP scenarios below, we neglect the impact of NP on tree-level SM processes.
Clearly, $\epsilon_K$ only depends on $M_{12}/\Gamma_{12}$, which is physical, while the phases of $M_{12}$ and $\Gamma_{12}$ separately are not. The neglected higher order terms in eq. (2.17) are also independent of phase conventions.

The standard model predictions for $M_{12}$ and $\Gamma_{12}$ are calculated separately, using different methods, resulting in intermediate steps that depend on phase conventions. (In contrast, in the case of $B^0$ and $B^0_s$ mixing, both $M_{12}$ and $\Gamma_{12}$ are computed by perturbative QCD methods, hence the cancellations of conventions is more apparent. In $K^0$ mixing, the use of chiral perturbation theory, and the separate estimation of short and long distance contributions obscure the cancellations.) The conventions that lead to the “usual” $\epsilon_K$ formula is reviewed in the rest of this section. In section 3 we use the freedom of this choice to study and minimize the uncertainties of $\epsilon_K$.

2.3 $\epsilon_K$ in the standard phase convention

To connect the phase convention independent manifestly physical expressions in eq. (2.17) to actual calculations, we need to consider how $M_{12}$ and $\Gamma_{12}$ are computed. They are given by

$$M_{12} = \frac{1}{2m_K} \langle K^0|\mathcal{H}|\bar{K}^0\rangle, \quad \Gamma_{12} = \sum_f A^*(K^0 \to f) A(\bar{K}^0 \to f),$$

(2.18)

where $f$ denote common final states of $K^0$ and $\bar{K}^0$ decay. Usually $M_{12}$ is written as the short-distance calculation combined with the matrix element of the four-quark operator $O_1 = (\bar{d}_L\gamma_\mu s_L)^2$ in the vacuum insertion approximation, times a “bag parameter”, $B_K$, plus corrections. The definition of $B_K$ involves $\theta$ via [13, 16]

$$\langle K^0|(\bar{d}_L\gamma_\mu s_L)(\bar{d}_L\gamma_\mu s_L)|\bar{K}^0\rangle = -e^{-i\theta} \frac{2}{3} B_K(\mu) f_K m_K^2,$$

(2.19)

where $B_K(\mu)$ is the usual positive real quantity. One further defines $\tilde{B}_K$, to remove the $\mu$-dependence of $B_K(\mu)$. The width mixing, $\Gamma_{12}$, is dominated by

$$A^*_0 A_0 = e^{-i\theta} |A_0|^2 e^{-2i\phi_0},$$

(2.20)

while the subleading contributions are suppressed by $|A_2/A_0|^2 \simeq 2 \times 10^{-3}$ and $\mathcal{B}(K_S \to f \neq \pi \pi) < 10^{-3}$. Equations (2.19) and (2.20) show that $\theta$ drops out of $M_{12}/\Gamma_{12}$, as it must.

In an often used $CP$ phase convention which we also use hereafter, $\theta = \pi$ [17], and then with the usual CKM phase conventions [3], $M_{12}$ is near the positive real axis and $\Gamma_{12}$ is near the negative real axis. The weak phase, $\phi_0$, of the isospin-zero amplitude, $A_0$, depends on hadronic matrix elements of several operators in the effective Hamiltonian. It is convenient and customary to define

$$\xi = \frac{\text{Im}(A_0 e^{-i\phi_0})}{\text{Re}(A_0 e^{-i\phi_0})}.$$ 

(2.21)

Without specifying phase conventions, $\xi$ can take any values between $-\infty$ and $+\infty$, because $\phi_0$ is convention dependent. In phase conventions in which $|\xi| < 1$ and $\text{Re}(A_0 e^{-i\phi_0}) > 0$,
one has $\xi = \arg(A_0 e^{-i\delta_0}) = -\frac{1}{2} \arg(-\Gamma_{12})$ up to relative orders $\xi^2$ (in addition to the phase-independent relative orders $B(K_S \rightarrow f \neq \pi \pi)$ and $|A_2/A_0|^2$). Then
\begin{equation}
\arg(-M_{12}/\Gamma_{12}) = \arg(M_{12}) - \arg(-\Gamma_{12}) \simeq \frac{2 \text{Im} M_{12}}{|M_{12}|} + 2\xi, \tag{2.22}
\end{equation}
is valid to the required accuracy in phase conventions satisfying $\{\arg M_{12}, \arg \Gamma_{12}\} = \mathcal{O}(\delta_L) \ll 1 \pmod{\pi}$. Thus, starting from the manifestly convention independent eq. (2.17), choosing $\theta = \pi$ and weak phases such that $|\xi| \ll 1$, we recover the often quoted expression,
\begin{equation}
\epsilon_K = e^{i\phi} \sin \phi_{\epsilon} \left( \frac{\text{Im} M_{12}^{SD}}{\Delta m} + \xi \right) = e^{i\phi_{\epsilon}} \sin \phi_{\epsilon} \left( \frac{\text{Im} M_{12}^{SD}}{\Delta m} + \xi + \frac{\text{Im} M_{12}^{LD}}{\Delta m} \right) = \frac{\kappa_{\epsilon} e^{i\phi_{\epsilon}} \text{Im} M_{12}^{SD}}{\sqrt{2} \Delta m}. \tag{2.23}
\end{equation}
We have explicitly separated the short-distance $\Delta s = 2$ contribution, $M_{12}^{SD}$, from $\kappa_{\epsilon}$, and from the long-distance contribution, $M_{12}^{LD}$. The last term implicitly defines $\kappa_{\epsilon}$, which is often written as [18, 19]
\begin{equation}
\kappa_{\epsilon} = \sqrt{2} \sin \phi_{\epsilon} \left( 1 + \rho \frac{\xi}{\sqrt{2} |\epsilon_K|} \right). \tag{2.24}
\end{equation}

### 2.4 Estimating $\xi$ and $\rho$

Currently available estimates of $\xi$ use either lattice QCD calculations, or the measured value of the direct $CP$-violating quantity, $\epsilon'$, or a combination of the two. It must be emphasized that using $\epsilon'$ as an input is only valid assuming that it is determined by the SM. (As discussed below, it is possible that $\epsilon'$ is affected by NP but $\epsilon_K$ is not, and vice versa.)

One can write $\epsilon'$ as
\begin{equation}
\epsilon' = \frac{i}{\sqrt{2}} \frac{|A_2|}{|A_0|} e^{i(\delta_2 - \delta_0)} \sin(\phi_2 - \phi_0), \tag{2.25}
\end{equation}
valid up to relative orders $|A_2/A_0|$ and $|\epsilon_K|$. This expression is phase convention independent, as $\phi_2 - \phi_0$ and $\delta_2 - \delta_0$ are physical, and correctly implies $\phi_{\epsilon'} = \pi/2 + \delta_2 - \delta_0 = (42.3 \pm 1.5)^\circ$. In phase conventions in which $\phi_0$ and $\phi_2$ are both tiny,
\begin{equation}
\epsilon' = \frac{e^{i\phi_{\epsilon'}}}{\sqrt{2}} \frac{|A_2|}{|A_0|} \left[ \frac{\text{Im}(A_2 e^{-i\delta_2})}{|A_2|} - \xi \right]. \tag{2.26}
\end{equation}
This yields
\begin{equation}
\xi = \frac{\text{Im}(A_2 e^{-i\delta_2})}{|A_2|} - \sqrt{2} |\epsilon_K| \frac{|A_0|}{|A_2|} \left| \frac{\epsilon'}{|\epsilon_K|} \right|. \tag{2.27}
\end{equation}
where the relative errors in both eqs. (2.26) and (2.27), which depend on the phase convention, are of order $\xi^2$. The second term in eq. (2.27) is well-known experimentally, and this expression allows using lattice calculations of $A_2$ instead of $A_0$ to estimate $\xi$.

Using the lattice QCD result $\text{Im}(A_2 e^{-i\delta_2}) = -6.99(0.20)(0.84) \times 10^{-13}$ GeV [20], we obtain
\begin{equation}
\xi = -(1.65 \pm 0.17) \times 10^{-4} \quad \text{(input from $\epsilon'/\epsilon$ measurement)}. \tag{2.28}
\end{equation}
In contrast, the lattice calculation \( \text{Im}(A_0 e^{-i\delta_0}) = 1.90(1.22)(1.04) \times 10^{-11} \text{ GeV} \) [21], using eq. (2.21), yields
\[
\xi = - (0.57 \pm 0.48) \times 10^{-4} \quad \text{(no input from \( \epsilon'/\epsilon \) measurement).} \tag{2.29}
\]
This difference is equivalent to the statement that the lattice QCD calculations [20, 21] show about a 2.5\( \sigma \) tension with \( \epsilon' \), which can be further sharpened using additional inputs [22].

From eqs. (2.23) and (2.24), the parameter \( \rho \) is defined as
\[
\rho = 1 + \frac{1}{\xi} \frac{\text{Im}(M_{12}^{LD})}{\Delta m} . \tag{2.30}
\]
Without a lattice computation of \( M_{12}^{LD} \), \( \rho \) can be estimated in the framework of chiral perturbation theory (\( \chi \)PT) [18] (see also [23–25]). First, one argues that all important dispersive diagrams share the same phase [18, 23], so that the phase of the absorptive and dispersive parts are related via
\[
\frac{\text{Im}M_{12}^{LD}}{\text{Re}M_{12}^{LD}} \simeq \frac{\text{Im}F_{12}^{LD}}{\text{Re}F_{12}^{LD}} \simeq -2\xi(1 \pm 0.5) . \tag{2.31}
\]
Here we keep using the 50\% uncertainty quoted in ref. [18] to account for the non-aligned contributions. The dominant contribution to \( \text{Re}M_{12}^{LD} \) comes from the \( \pi\pi \) loop, which has been estimated as [18]
\[
\frac{\text{Re}M_{12}^{LD}}{\Delta m} \simeq \frac{\text{Re}M_{12}^{(\pi\pi)}}{\Delta m} \simeq 0.2 \pm 0.1 . \tag{2.32}
\]
(Preliminary lattice calculations [26] hint at a smaller role for the 2\( \pi \) state than the \( \chi \)PT estimate; refining this is important.) Equations (2.31) and (2.32) finally imply
\[
\rho = 1 - 2(0.2 \pm 0.14) = 0.6 \pm 0.3 . \tag{2.33}
\]

2.5 Short distance contribution and usual evaluation of \( \epsilon_K \)

Given eqs. (2.23) and (2.24) and estimates of \( \xi \) and \( \rho \), the only remaining ingredient in making a SM prediction for \( \epsilon_K \) is the expression for the short-distance contribution to \( M_{12} \) for \( \theta = \pi \),
\[
M_{12}^{SD} = \frac{\Delta m}{\sqrt{2}} \hat{C}_\epsilon \left[ \lambda_\ell^2 \eta_{\ell \ell} S_0(x_t) + 2 \lambda_\ell \lambda_\eta \eta_{\ell \ell} S_0(x_t, x_\ell) + \lambda_\ell^2 \eta_{\ell \ell} x_\ell \right] , \tag{2.34}
\]
where \( \lambda_q = V_{qd}V_{qs}^* \), \( x_q = [m_q(m_q)/m_W]^2 \), the Inami-Lim functions \( S_0 \) can be found, e.g., in ref. [17], and\(^4\)
\[
\hat{C}_\epsilon = \frac{G_F^2}{6\sqrt{2}\pi^2} \frac{m_K m_{\pi}^2}{\Delta m} f_K^2 \hat{B}_K = (2.806 \pm 0.049) \times 10^4 . \tag{2.35}
\]
\(^4\)The uncertainty of \( \hat{C}_\epsilon(= C_\epsilon \hat{B}_K) \) is dominated by those of \( f_K^2 \) and \( \hat{B}_K \). Their contributions are now comparable, making the past separation of \( C_\epsilon \) and \( \hat{B}_K \) less motivated.
Taking the imaginary part of $M_{12}^{SD}$, we obtain from eq. (2.23)
\[
\epsilon_K = \kappa e^{i\phi} \hat{C}e |V_{cb}|^2 \lambda^2 \eta \left\{ |V_{cb}|^2 [(1 - \rho) - \lambda^2 (\bar{\rho} - \bar{\rho}^2 - \bar{\eta}^2)] \eta t S_0(x_t) + \eta s^2 S_0(x_t, x_c) - \eta cc x_c \right\},
\]
where we neglected $O(\lambda^{14})$ terms in the CKM expansion.\(^5\) As is usually done, we replaced $\lambda^4 A^2$ by $|V_{cb}|^2$, which is valid in the SM, as $V_{cb} = A\lambda^2 + O(\lambda^6)$ [5, 6]. The $O(\lambda^2)$ correction to the leading order result, proportional to $(\bar{\rho} - \bar{\rho}^2 - \bar{\eta}^2)$, is severely suppressed accidentally, because $\bar{\rho}/(\bar{\rho}^2 - \bar{\eta}^2) = \sin^2 \alpha - \frac{1}{2} \sin 2\alpha \cot \beta$ ($\alpha$ and $\beta$ being the standard CKM angles) and $\alpha$ is near $90^\circ$.

Below we refer to the expression for $\epsilon_K$ in eq. (2.36) as the “usual evaluation”. We discuss its central values and error budget together with that of our evaluation of $\epsilon_K$, in section 3.2.

3 Removing $\eta_{cc}$ from $\epsilon_K$

3.1 Rephasering the evaluation of $\epsilon_K$

With respect to the “standard” phase convention that lead to eq. (2.36), one can rephase the kaon fields as
\[
|K^0\rangle \rightarrow |K^0\rangle' = e^{i\lambda_c/|\lambda_c|}|K^0\rangle, \quad \overline{|K^0\rangle} \rightarrow \overline{|K^0\rangle}' = e^{-i\lambda_c/|\lambda_c|}\overline{|K^0\rangle},
\]
which has the effect of multiplying the expression for $M_{12}^{SD}$ in eq. (2.34) by $\lambda_c^2/|\lambda_c|^2$, thus making the $\eta_{cc}$ contribution purely real.\(^9\) Since $|\Im(\lambda_c)/\Re(\lambda_c)| < 10^{-3}$, this rephasing has a negligible impact on the short distance contribution to $\Delta m$. However, the impact on $\epsilon_K$ is significant, which we study next.

All the results of section 2.2 are still valid, being independent of phase conventions. The results of section 2.3 and eq. (2.23) in particular are valid as well, since despite the $O(1)$ changes in arg $M_{12}$ and arg $\Gamma_{12}$, their orders of magnitude are unchanged. In fact, in every step the phase-dependent errors never exceed a relative amount of $O(\xi^2)$, and in the new phase convention $\xi' < 10^{-3}$ still holds (see below).

The consequences of the rephasing defined in eq. (3.1) are
\[
\text{Im} M_{12} \rightarrow \text{Im} M_{12}' = \text{Im} M_{12} \frac{\Re \lambda_c^2}{|\lambda_c^2|} + \Re M_{12} \frac{\Im \lambda_c^2}{|\lambda_c^2|}, \quad \text{Re} M_{12} \approx \text{Im} M_{12} + 2\lambda^4 A^2 \bar{\eta} \Re M_{12},
\]

\[
\xi \rightarrow \xi' = -\frac{1}{2} \frac{\text{Im}(\Gamma_{12} \lambda_c^2)}{\text{Re}(\Gamma_{12} \lambda_c^2)} \simeq -\frac{1}{2} \left( \frac{\text{Im} \Gamma_{12}}{\text{Re} \Gamma_{12}} + \frac{\text{Im} \lambda_c^2}{\text{Re} \lambda_c^2} \right) \approx \xi - \lambda^4 A^2 \bar{\eta}.
\]

\(^5\) We use the expansion of the CKM matrix valid to all orders [5, 6], which implies
\[
\lambda_c = -\lambda \left[ 1 - \frac{\lambda^2}{2} + O(\lambda^4) \right] - i\bar{\eta} A^2 \lambda^5 \left[ 1 + \frac{\lambda^2}{2} + O(\lambda^4) \right],
\]
\[
\lambda_t = -A^2 \lambda^5 \left[ 1 - \rho + \frac{\lambda^2}{2} (1 - 3\bar{\rho} + 2\bar{\rho}^2 + 2\bar{\eta}) + O(\lambda^4) \right] + i\bar{\eta} A^2 \lambda^5 \left[ 1 + \frac{\lambda^2}{2} + O(\lambda^4) \right].
\]

\(^6\) The definition of kaons in terms of quarks introduces two further non-physical arbitrary phases $\alpha$ and $\tilde{\alpha}$ ($|K^0\rangle = e^{i\alpha}|d\bar{s}\rangle$, $|\overline{K}^0\rangle = e^{i\tilde{\alpha}}|\bar{d}s\rangle$). If they are set to zero, then eq. (3.1) can also be obtained by choosing a CKM matrix convention where $V_{ud}^\ast V_{us}^\prime$ is real, e.g., $V_{\text{CKM}}^\prime = V_{\text{CKM}} \times \text{diag}(1, \lambda_c/|\lambda_c|, 1)$.
Both in $\text{Im}M'_{12}$ and in $\xi'$, the uncertainties due to neglected terms are negligible. Thus, the short-distance contribution to $M_{12}$ becomes

$$M_{12}^{\text{SD}} = \frac{\Delta m}{\sqrt{2}} C_\epsilon \left[ \frac{\lambda_2^2 \lambda_3^2}{|\lambda_c|^2} \eta_{tt} S_0(x_t) + 2 \lambda_c \lambda_7^* \eta_{ct} S_0(x_t, x_c) + |\lambda_c|^2 \eta_{cc} x_c \right], \tag{3.4}$$

and the $\eta_{cc}$ term does not contribute to the imaginary part.

For the long-distance contribution to $M_{12}$, we can use the same estimate as in ref. [18] to obtain

$$\text{Im}M_{12}^{\text{LD}} = -2 \left[ \xi (1 \pm 0.5) - \lambda^4 A^2 \eta \right] \text{Re}M_{12}^{\text{LD}} = -2 (\xi' \pm 0.5 \xi) \text{Re}M_{12}^{\text{LD}}, \tag{3.5}$$

where in the first equality we used eqs. (2.31) and (3.2), and in the second equality eq. (3.3). For simplicity, we define

$$\kappa' = \sqrt{2} \sin \phi_\epsilon \times \left( 1 + \rho' \frac{\xi'}{\sqrt{2}|\xi|} \right), \tag{3.6}$$

with

$$\rho' = 1 + \frac{1}{\xi'} \frac{\text{Im}(M_{12}^{\text{LD}'})}{\Delta m} = 1 - 2 \left( 1 \pm 0.5 \frac{\xi}{\xi'} \right) (0.2 \pm 0.1), \tag{3.7}$$

where in the second equality we used eqs. (3.5) and (2.32). Numerically, we find

$$\rho' = 0.6 \pm 0.2, \tag{3.8}$$

where the uncertainty of $\rho'$ coming from the CKM inputs (contained in $\xi'$) is negligible.

Thus, we finally obtain

$$\epsilon_K = \kappa' e^{i \phi_\epsilon} C_\epsilon |V_{cb}|^2 \lambda_3^2 \eta \left[ |V_{cb}|^2 \left( (1 - \bar{\rho}^2) - \lambda^2 (\bar{\rho}^2 - \bar{\eta}^2) \right) \eta_{tt} S_0(x_t) + \eta_{ct} S_0(x_t, x_c) \right], \tag{3.9}$$

to which we refer below as “our evaluation”. For convenience, we report our evaluation in a ready-to-use form in eqs. (5.2)–(5.4) in section 5.

### 3.2 Numerical results and discussion

We collect in table 1 the inputs used from experimental measurements, as well as from perturbative and lattice computations. Concerning CKM parameters, the SM prediction of $\epsilon_K$ is obtained using the parameters that result from the full CKM fit. In fact, their best-fit values are practically unaffected by the exclusion of $\epsilon_K$ from the fit inputs [30]. If one wants instead to account for possible NP contributions in the CKM fit, and obtain a prediction for $\epsilon_K$ that is as independent as possible of such NP, then one should use the values of the CKM parameters that come from a fit to tree-level observables only. In this second approach, the only assumption about NP is that it affects negligibly observables that are dominated by tree-level processes in the SM. We show the values of the CKM parameters in these two cases in table 2.\(^7\) The increased uncertainty in $|V_{cb}|$ and $\bar{\eta}$, when not determined from the CKM fit, reflects the tension between exclusive and inclusive determinations of $|V_{cb}|$ and $|V_{ub}|$.

\(^7\)CKMfit [6] performs several fits, using only tree-level observables to determine $\bar{\eta}$ and $\bar{\rho}$. Conservatively, we use the one where the only angle measurement included is $\gamma(DK)$, and that combines the measured values of $|V_{ub}|$, for consistency with our treatment of $|V_{cb}|$. CKMfit plots the fit results, without quoting numerical results. The values in table 2 are read off from the plot, which is sufficient for our purposes, given the large uncertainties.
| Parameter | value | source |
|-----------|-------|--------|
| $\Delta m$ | $3.484(6) \times 10^{-12}$ MeV | [3] |
| $m_{K^0}$ | 497.614(24) MeV | [3] |
| $\Delta \Gamma$ | $7.3382(33) \times 10^{-12}$ MeV | [3] |
| $|\epsilon_K|$ | $(2.228 \pm 0.011) \times 10^{-3}$ | [3] |
| $\phi_c$ | $(43.52 \pm 0.05)^0$ | [3] |
| $|\epsilon'/\epsilon|$ | $(1.66 \pm 0.23) \times 10^{-3}$ | [3] |
| $|A_0/A_2|$ | 22.45(6) | [3, 27] |
| $|A_0|$ | $3.32(2) \times 10^{-7}$ GeV | [3, 27] |
| $\eta_{cc}$ | 1.87(76) | [4] |
| $\eta_{ct}$ | 0.496(47) | [11] |
| $\eta_{tt}$ | 0.5765(65) | [12] |
| $\overline{m}_t(\overline{m}_t)$ | 162.3(2.3) GeV | [28] |
| $\overline{m}_c(\overline{m}_c)$ | 1.275(25) GeV | [3] |
| $\hat{B}_K$ | 0.7661(99) | [29] |
| $f_K$ | 156.3(0.9) MeV | [29] |
| $\text{Im}(A_2 e^{-i\delta_2})$ | $-6.99(0.20)(0.84) \times 10^{-13}$ GeV | [20] |
| $\text{Im}(A_0 e^{-i\delta_0})$ | $-1.90(1.22)(1.04) \times 10^{-11}$ GeV | [21] |

Table 1. Inputs used for the calculation of $\epsilon_K$.

| CKM parameters | SM CKM fit [6] | tree-level only |
|----------------|----------------|-----------------|
| $\lambda$     | $0.22543 \pm 0.00037$ | $0.2253 \pm 0.0008$ | [3] |
| $|V_{cb}| (=A\lambda^2)$ | $(41.80 \pm 0.51) \times 10^{-3}$ | $(41.1 \pm 1.3) \times 10^{-3}$ | [3] |
| $\tilde{\eta}$ | $0.3540 \pm 0.0073$ | $0.38 \pm 0.04$ | [6] |
| $\tilde{\rho}$ | $0.1504 \pm 0.0091$ | $0.115 \pm 0.065$ | [6] |

Table 2. The CKM parameters used as inputs. Using the SM CKM fit results assumes that the SM determines all observables. The tree-level inputs are applicable even if TeV-scale new physics affects the loop-mediated processes.

Thus, the usual evaluation eq. (2.36) and our evaluation eq. (3.9) lead to the SM predictions for $\epsilon_K$ shown in table 3. When interested in the SM prediction for $\epsilon_K$, we use the more precise value of $\xi$, determined using the measured value of $\epsilon'/\epsilon$ as an input (in line with the assumption that the SM accounts for all flavor measurements). In the determination where we allow for NP, instead, we use the lattice value of $\text{Im}(A_0)$ to determine $\xi$, instead of the measured $\epsilon'/\epsilon$. For convenience, we also report in table 3 the values of $\xi$, $\kappa_\epsilon$ and $\xi'$, $\kappa'_\epsilon$ in our evaluation that correspond to these choices. Finally, the various sources of uncertainties in $\epsilon_K$ and their relative impacts are shown in table 4. The total error of $\epsilon_K$ is obtained by adding all contributions in quadrature.
CKM inputs & $|\epsilon_K| \times 10^3$ & $\kappa^{(t)}_c$ & $\xi^{(t)} \times 10^3$

| Usual evaluation | tree-level | 2.30 ± 0.42 & 0.963 ± 0.010 & −0.57 ± 0.48 |
| SM CKM fit | 2.16 ± 0.22 & 0.943 ± 0.016 & −1.65 ± 0.17 |

| Our evaluation | tree-level | 2.38 ± 0.37 & 0.844 ± 0.044 & −6.99 ± 0.92 |
| SM CKM fit | 2.24 ± 0.19 & 0.829 ± 0.049 & −7.83 ± 0.26 |

Table 3. Present value of $\epsilon_K$ in the usual evaluation (upper part) and in our evaluation (lower part). For convenience, we also show the values of the quantities $\kappa_c$ and $\xi$ defined in eqs. (2.24) and (2.21) in the upper part, and $\kappa^{(t)}_c$ and $\xi^{(t)}$ defined in eqs. (3.6) and (3.3) in the lower part.

| CKM inputs | $\eta_{cc}$ & $\eta_{ct}$ & $\kappa^{(t)}_c$ & $m_t$ & $m_c$ & $f_{K^0B_K}$ & $|V_{cb}|$ & $\bar{\eta}$ & $\bar{\rho}$ & $\Delta\epsilon_K/\epsilon_K|_{\text{tot.}}$ |
|------------|--------|--------|------------|------|------|-------------|--------|-------|--------|-----------------|
| Usual evaluation | tree-level | 7.3% & 4.0% & 1.1% & 1.7% & 0.8% & 1.7% & 11.1% & 10.4% & 5.4% & 18.4% |
| SM CKM fit | 7.4% & 4.0% & 1.7% & 1.7% & 0.8% & 1.7% & 4.2% & 2.0% & 0.8% & 10.2% |
| Our evaluation | tree-level | — & 3.4% & 5.2% & 1.5% & 1.2% & 1.7% & 9.5% & 8.9% & 4.5% & 15.6% |
| SM CKM fit | — & 3.4% & 5.9% & 1.5% & 1.3% & 1.7% & 3.6% & 1.7% & 0.7% & 8.4% |

Table 4. The present error budget of $\epsilon_K$ in the usual evaluation (upper part) and using our evaluation (lower part). The parameters with a corresponding uncertainty above 1% are shown.

As expected, the central values of $\epsilon_K$ in table 3 vary according to the strategy used to compute $\epsilon_K$ (our vs. usual evaluation, and SM CKM fit vs. tree-level inputs). The central values are actually all within 1σ of each other, and of the experimental central value $|\epsilon_K|^{\text{exp}} = 2.228 \times 10^{-3}$. Note that the latest determination of $V_{cb}$ from $B \to D\ell\bar{\nu}$, $|V_{cb}| = 40.8(1.0) \times 10^{-3}$ [31], reduces the tension with its inclusive determination (however, that from $B \to D^*\ell\bar{\nu}$ remains lower; see, e.g., ref. [32] for more discussions). Table 3 also shows that in our evaluation the uncertainty in the long distance contribution to $\epsilon_K$ (i.e., $\kappa^{(t)}_c$) is relatively more important than in the usual evaluation. In the latter case, the $\eta_{cc}$ term contributes to $\epsilon_K$ with a negative sign, and its removal in our evaluation is compensated by an increase in the imaginary part of the long-distance contribution. Table 4 makes the usefulness of our evaluation of $\epsilon_K$ manifest:

- Given state-of-the-art inputs, our evaluation eq. (3.9) slightly reduces the relative uncertainties of $\epsilon_K$ with respect to the usual one in eq. (2.36);

- The gain in relative uncertainty from the removal of $\eta_{cc}$ is partially compensated by an increase in the uncertainty from $\kappa_c$, which is dominated by the uncertainty of the long-distance contribution $\text{Im}(M^{LD}_{12})$. (See sections 2.4 and 3.1 for its estimate, in the usual and in our evaluation respectively.)

These observations highlight the importance of achieving a better theoretical control of the long-distance contribution to $M_{12}$. While some progress could already be attained with tools like $\chi$PT, a significant step forward probably requires an effort from lattice QCD (recent attempts in this direction have appeared in refs. [26, 33, 34]). The importance of such an effort is even greater considering future prospects for the $\epsilon_K$ uncertainty, which, with the removal of $\eta_{cc}$, is dominated by the CKM parameters. Within the next decade it
should be possible to measure $|V_{cb}|$ with an uncertainty of about $0.3 \times 10^{-3}$ [1, 35], to be compared with $1.3 \times 10^{-3}$ in table 2. This would correspond to a reduced contribution to the $\epsilon_K$ error budget,

$$\frac{\Delta \epsilon_K}{\epsilon_K} \frac{\Delta |V_{cb}|=0.3 \times 10^{-3}}{1}$$.}

\[ (3.10) \]

in our evaluation of $\epsilon_K$ (2.6% in the usual one). Similarly, tree-dominated measurements will determine $\gamma$ and $|V_{ub}|$ with much better precision [1, 2, 35], which will translate to an uncertainty of $\epsilon_K$ due to CKM elements comparable to the current SM CKM fit in table 4. Finally, different lattice QCD calculations of $\hat{B}_K$ obtain different results for its uncertainty [36–38], which, however, do not exceed the 2–3% percent level and are thus subdominant in the error budget of $\epsilon_K$. (A more acute tension is present for the bag parameters of non-SM operators, see section 4.)

3.3 Further comments on the rephasing

We collect here some remarks that are not strictly necessary to the previous discussion, but that might help to make it clearer.

- Looking at table 3, it may appear counterintuitive that larger $\xi$ uncertainties correspond to more precise values of $\kappa_u$. That is the case because, when the $\xi$ uncertainty is larger, the $\xi$ central value is accidentally smaller. The larger impact on the $\kappa_u$ uncertainty comes from $\rho$, which multiplies $\xi$, and so its central value also impacts the error budget.

- The rephasing of kaon and quark fields is independent of the freedom to remove the charm or up (or top) contribution, via unitarity of the CKM matrix. The standard choice is to eliminate the $u$-quark contribution, $\lambda_u = -\lambda_t - \lambda_c$, which we also followed. The possibility to use CKM unitarity to remove $\lambda_c$, instead of $\lambda_u$, has been emphasized in ref. [33] (see appendix A of that paper). With that choice, $M_{12}^{\text{SD}}$ contains terms proportional to $\lambda_t^2 \lambda_u^* \lambda_t^* \lambda_u$, and the second one will not contribute to $\epsilon_K$, since $\lambda_u$ is real in the standard phase convention.

However, the expression for $\epsilon_K$ obtained using $\lambda_c = -\lambda_t - \lambda_u$ cannot yet be used to make precise predictions, since the coefficients analogous to $\eta_{tt}$ and $\eta_{ct}$ have not been computed. Ref. [33] argued that they would not have large uncertainties, and that the related lattice calculations would become more accurate, due to the suppression of the perturbative contribution for momenta smaller than $m_c$. While this could justify pursuing that path, using $\lambda_c = -\lambda_t - \lambda_u$ renders the top contribution sensitive to the $m_c$ scale, which is generically associated with larger uncertainties. Our evaluation relies instead on well established results, and allows immediate quantitative predictions.

- One may wonder if a rephasing other than that in eq. (3.1) could reduce the $\epsilon_K$ uncertainty even further. Instead of eq. (3.1), an optimal choice might reduce but not eliminate the $\eta_{tc}$ contribution to $\text{Im} M_{12}^{\text{SD}}$, and the combined uncertainty due to
etalks, and $K_{ee}$ may decrease. To explore this, let us define the general rephasing

$$|K^0 \rangle \to |K^0 \rangle' = e^{i a \lambda_e / |\lambda_e|} |K^0 \rangle, \quad |\overline{K}^0 \rangle \to |\overline{K}^0 \rangle' = e^{-i a \lambda_e / |\lambda_e|} |\overline{K}^0 \rangle,$$

where the usual evaluation corresponds to $a = 0$, and our evaluation to $a = 1$. We can choose $a$ to minimize the total uncertainty of $\epsilon_K$. We find that the optimal values are $a \approx 1.0$ and $a \approx 0.7$ for the cases of tree-level and SM CKM fit inputs, respectively. The resulting total uncertainties for the latter case is $|\Delta \epsilon_K / \epsilon_K|_{\text{total}} = 7.9\%$, to be compared with 8.4\% of the case $a = 1$ in table 4. The corresponding central $\epsilon_K$ value is $2.23 \times 10^{-3}$.

4 Constraints on new physics

If a pattern of deviations from the SM is given, like in a specific model of flavor, then the correct strategy to study flavor and CP constraints would be to perform a fit to the SM + NP parameters (see, e.g., ref. [35]). Here we would like to derive consequences for NP that are of a more general validity, and do not need the specification of a model. Therefore, we take an effective field theory (EFT) approach, and comment on explicit NP models at the end of this section. We parametrize the NP contribution to $K^0_0$ mixing in terms of dimension-six operators, suppressed by a mass scale squared, $\Lambda^2$. The operator basis we consider consists of $O_1$, defined before eq. (2.19), and

$$O_2 = \langle \overline{d}_R s_L \rangle^2, \quad O_3 = \langle \overline{d}_R s_L \rangle^2 \langle \overline{d}_L s_R \rangle, \quad O_4 = \langle \overline{d}_R s_L \rangle^2 \langle \overline{d}_L s_R \rangle, \quad O_5 = \langle \overline{d}_R s_L \rangle^2 \langle \overline{d}_L s_R \rangle,$$

where $\alpha, \beta$ are color indices, that are implicit when their contraction is between Lorentz-contracted fields. The observable most sensitive to $O_1, \ldots, 5$ is $\epsilon_K$, so our procedure is consistent ($\Delta m$, also sensitive to NP in $K^0$ mixing, suffers from larger long-distance and $\eta_{cc}$ uncertainties).

To derive bounds on the operators in eq. (4.1), we need both their matrix elements between two kaon states at a certain low scale $\mu$, and the running of their Wilson coefficients from $\Lambda$ down to that scale. The matrix elements are defined in terms of the bag parameters, with $B_1 = B_K$ of eq. (2.19), and

$$\langle K^0 | O_i(\mu) | \overline{K}^0 \rangle = \frac{a_i}{4} B_i(\mu) \frac{m_K^4 J_K^2}{[m_s(\mu) + m_d(\mu)]^2}, \quad i = 2, \ldots, 5,$$

with $a_i = \{-5/3, 1/3, 2, 2/3\}$. Recent calculations obtained partly consistent results [37, 39–41], while a 30–40% tension between calculations of $B_4$ and $B_5$ remains (as it was already the case nearly a decade ago [42, 43]). For definiteness, we use here the values obtained in ref. [37] (in the MS scheme), shown in table 5, together with the quark masses used.

We assume that only one operator deviates from the SM at the high scale $\Lambda$, with a purely imaginary coefficient. We run it down to the scale $\mu = 3$ GeV, at which the matrix elements are given. Because of the large uncertainties of the bag parameters $B_i$, we use the LO running and mixing of the Wilson coefficients of $O_1, \ldots, 5$ [44, 45] (see refs. [46, 47] for a consistent treatment of the Wilson coefficients together with the bag parameters at NLO).
Table 5. Inputs used for setting bounds on NP from $\epsilon_K$. Both the bag parameters [37] and the quark masses are in the $\overline{\text{MS}}$ scheme; the latter are obtained by NLO running from the values at 2 GeV given in ref. [3].

| Quark masses (at 3 GeV) | Bag parameters (at 3 GeV) |
|------------------------|--------------------------|
| $m_s$                  | $m_d$                     |
| 86.5 MeV               | 4.4 MeV                  |
| $B_1$                  | $B_2$                     |
| 0.506                  | 0.46                      |
| $B_3$                  | $B_4$                     |
| 0.79                   | 0.78                      |
| $B_5$                  |                          |
| 0.49                   |                          |

Table 6. Upper bounds from $\epsilon_K$ on the imaginary parts of the Wilson coefficients of the operators $O_1, \ldots, O_5$, run down to 3 GeV from a scale of 3 TeV. For each operator we give the bound both from the tree-level CKM inputs and from the SM CKM inputs.

| $\text{Im}(C_i) \frac{(3 \text{ TeV})^2}{\Lambda^4} < X$ | $O_1$ | $O_2$ | $O_3$ | $O_4$ | $O_5$ |
|---------------------------------------------------------|-------|-------|-------|-------|-------|
| tree-level                                              | $2.4 \times 10^{-8}$ | $3.3 \times 10^{-10}$ | $1.2 \times 10^{-9}$ | $7.5 \times 10^{-11}$ | $2.4 \times 10^{-10}$ |
| SM CKM fit                                              | $1.2 \times 10^{-8}$ | $1.7 \times 10^{-10}$ | $6.2 \times 10^{-10}$ | $3.9 \times 10^{-11}$ | $1.2 \times 10^{-10}$ |

We then express the constraints from $\epsilon_K$ as lower bounds on $\Lambda$, requiring the NP contribution to the experimental measured value of $\epsilon_K$ to be less than twice the theoretical uncertainties in table 4, i.e., 31% for tree-level inputs and 16% for SM CKM fit inputs (keeping in mind the last point of section 3.3). We ignore the differences between the experimental central value of $\epsilon_K$ and the theoretical predictions, because it is small and depends anyway on the CKM parameters resulting from a specific fit, and because this way the constraint on NP is independent of its sign.

The results are shown in figure 1, both for the SM CKM fit and for tree-level inputs, as darker (right) and lighter (left) histograms, respectively. From the point of view of NP, the former case assumes $\epsilon_K$ to be the most sensitive observable to flavor violation, and the second one is more conservative and only requires NP not to substantially affect processes that are tree-level in the SM. The operator most constrained by $\epsilon_K$ is $O_4$, which probes scales near $10^6$ TeV.

In addition we show, in table 6, the resulting bounds on the imaginary part of the Wilson coefficients $C_i$ of the operators $O_i$, for a fixed scale $\Lambda = 3$ TeV. That is useful for the reader interested in models with new degrees of freedom not too far from the TeV scale. In fact, the running from the scales shown in figure 1 down to 3 TeV is a sizable effect, which yields differences of order 50% or larger in the constraints on the Wilson coefficients. The same differences are, instead, below the 10% level if the running is performed from 3 TeV to, say, 1 or 10 TeV.

We end this section with comments concerning the sensitivity of $\epsilon_K$ to explicit and widely studied NP flavor models:

- Composite Higgs models with partial compositeness (see, e.g., [48]) constitute a case where $\epsilon_K$ is the most sensitive observable to flavor and $CP$ violation [49, 50], unless a flavor symmetry is imposed on the strong sector [50–53]. Then it is reasonable to derive bounds from $\epsilon_K$ using inputs from a CKM fit that assumes the SM, and corre-
对应于近似8%的理论误差在表4。这一过程意味着例如，就语言的参考文献[50]和与强相互作用中没有混沌的味结构，\(\epsilon_K\)能限制复合夸克共振子的质大于约30 TeV。

其他受启发的案例是模型实现一个“CKM-like”味和CP违反模式，与SM类似抑制作用的即在SM中，且消失的即在第二代中的5个。就论据的参考文献[54]，它们都包括在\(U(3)^3\) [55–57]，或在\(U(2)^3\) [52, 58]模型，所有其他对称性等价于它们。在这些模型中不是没有明显的观测值在对NP敏感的层次。因此，正确的过程是分析SM+NP味参的拟合，使用理论预测的\(\epsilon_K\)(见公式(5.2)–(5.4)代入可使用的表达)。

更具体地说，虽然在一般情况下由\(\epsilon_K\)探测的标度高于那些由\(\epsilon'/\epsilon\)探测的标度，在CKM-like模型中一个EFT分析显示[59]\(\epsilon'/\epsilon\)是比\(\epsilon_K\)更敏感的NP的。然而，在具体实现中它不是很难逆这个结论，例如在超对称性中第一和第二代比第三代重[59]。

5 Conclusions and outlook

在没有任何明确的偏差从CKM味和CP违反的图景，是难以，如果不是不可能，揭示出一个更基本的味理论。在所有可观察的，\(\epsilon_K\)探测某些最高能量，且对显式味模型施加某些最严重的约束。因此，改善其SM预测是重要的，其不确定性比其实验测定的大得多。
The theory uncertainty of $\epsilon_K$ depends on the uncertainty of CKM parameters, most notably on that of $A$ (or equivalently $|V_{cb}|$). The largest non-parametric uncertainty until now has been due to the perturbative QCD correction to the box diagram with two charm quarks, $\eta_{cc}$. We showed that the dependence of $\epsilon_K$ on $\eta_{cc}$ can be removed via a rephasing of the kaon fields, which makes this contribution to $M_{12}$ purely real. In other words, in our phase convention, the contribution to $\epsilon_K$ from dimension-six operators always contains the top mass scale. The resulting uncertainty of the SM prediction of $\epsilon_K$ is slightly reduced and, perhaps more importantly, the largest source of non-parametric error now comes from the long distance contribution to $M_{12}$. Thus, our formulation highlights the importance to achieve a better theoretical control of the latter, possibly using lattice QCD. The case is further strengthened by the precision with which the CKM inputs are expected to be measured at Belle II and LHCb.

In section 2, we reviewed the derivation of the SM prediction for $\epsilon_K$, explicitly exhibiting the phase convention dependences and the approximations used. Our evaluation is presented in section 3, together with its numerical consequences for the central values and uncertainties of $\epsilon_K$ summarized in table 3. The detailed error budget of $\epsilon_K$, in our evaluation, is compared with the conventional one in table 4.

Finally, we provided updated constraints on new physics contributions to $\epsilon_K$ in section 4, taking full advantage of the rephasing freedom. We also discussed how they apply to CKM-like models, and to composite Higgs models with an anarchic flavor structure. The constraints in figure 1 and table 6 provide a well-defined quantification of the $\epsilon_K$ sensitivity to NP, and are obtained from imposing

$$|\epsilon_K|^{(NP)} < \begin{cases} 0.31 \, |\epsilon_K|^{(exp)} & \text{(tree-level inputs)}, \\ 0.16 \, |\epsilon_K|^{(exp)} & \text{(SM CKM fit inputs)}, \end{cases}$$

as discussed in section 4.

Such an analysis ignores the pattern and correlations typical of specific NP realizations. For the convenience of the reader interested in such an analysis, that needs the CKM parameters coming from its own SM + NP fit, we report here our ready-to-use expression for $\epsilon_K$ without $\eta_{cc}$.

$$\epsilon_K = \kappa_e' e^{i \phi} \hat{C}_e |V_{cb}|^2 \lambda^2 \bar{\eta} \left| |V_{cb}|^2 [(1 - \bar{\rho}) - \lambda^2 (\rho - \bar{\rho}^2)] \eta_{1S} S_0(x_1) + \eta_{1T} S_0(x_t, x_c) \right|,$$

where $\kappa_e'$ is given using either the measured $\epsilon'/\epsilon$ value as an input or using only SM lattice inputs by

$$\kappa_e' = \begin{cases} 0.834 - 0.11\Delta \pm (0.047 + 0.036\Delta), & (\epsilon'/\epsilon \text{ and lattice } \text{Im}(A_2) \text{ input}), \\ 0.854 - 0.11\Delta \pm (0.041 + 0.035\Delta), & (\text{lattice } \text{Im}(A_0) \text{ input}), \end{cases}$$

and

$$\Delta = \frac{\bar{\eta}}{0.35} \left( \frac{|V_{cb}|}{41 \times 10^{-3}} \right)^2 - 1.$$

Equations (5.2) and (5.3), and the inputs in table 1 (which imply $\hat{C}_e = (2.806 \pm 0.049) \times 10^4$), allow making predictions for $\epsilon_K$ for the preferred values of CKM parameters $\lambda$, $|V_{cb}| = A\lambda^2$, $\bar{\eta}$, and $\bar{\rho}$.
Acknowledgments

We thank Joachim Brod, Robert Cahn, Nicolas Garron, Diego Guadagnoli, Gino Isidori, Michael Luke, Aneesh Manohar, Michele Papucci, and Gilad Perez for helpful conversations. ZL was supported in part by the Office of Science, Office of High Energy Physics, of the U.S. Department of Energy under contract DE-AC02-05CH11231. FS is supported by the European Research Council (ERC) under the EU Seventh Framework Programme (FP7/2007-2013)/ERC Starting Grant (agreement n. 278234 — ‘NEWDARK’ project).

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