(Near) Optimal Parallelism Bound for Fully Asynchronous Coordinate Descent with Linear Speedup

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Abstract

When solving massive optimization problems in areas such as machine learning, it is a common practice to seek speedup via massive parallelism. However, especially in an asynchronous environment, there are limits on the possible parallelism. Accordingly, we seek tight bounds on the viable parallelism in asynchronous implementations of coordinate descent.

We focus on asynchronous coordinate descent (ACD) algorithms on convex functions $F : \mathbb{R}^n \to \mathbb{R}$ of the form

$$F(x) = f(x) + \sum_{k=1}^{n} \Psi_k(x_k),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function, and each $\Psi_k : \mathbb{R} \to \mathbb{R}$ is a univariate and possibly non-smooth convex function.

Our approach is to quantify the shortfall in progress compared to the standard sequential stochastic gradient descent. This leads to a truly simple yet optimal analysis of the standard stochastic ACD in a partially asynchronous environment, which already generalizes and improves on the bounds in prior work. We also give a considerably more involved analysis for general asynchronous environments in which the only constraint is that each update can overlap with at most $q$ others, where $q$ is at most the number of processors times the ratio in the lengths of the longest and shortest updates. The main technical challenge is to demonstrate linear speedup in the latter environment. This stems from the subtle interplay of asynchrony and randomization. This improves Liu and Wright’s [16] lower bound on the maximum degree of parallelism almost quadratically, and we show that our new bound is almost optimal.

Keywords. Asynchronous Coordinate Descent, Asynchronous Optimization, Asynchronous Iterative Algorithm, Coordinate Descent, Amortized Analysis.
1 Introduction

We consider the problem of finding an (approximate) minimum point of a convex function \( F : \mathbb{R}^n \to \mathbb{R} \) of the form

\[
F(x) = f(x) + \sum_{k=1}^{n} \Psi_k(x_k),
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is a smooth convex function, and each \( \Psi_k : \mathbb{R} \to \mathbb{R} \) is a univariate convex function, but may be non-smooth. Such functions occur in many data analysis and machine learning problems, such as linear regression (e.g., the Lasso approach to regularized least squares [27]) where \( \Psi_k(x_k) = |x_k| \), logistic regression [20], ridge regression [25] where \( \Psi_k(x_k) \) is a quadratic function, and Support Vector Machines [10] where \( \Psi_k(x_k) \) is often a quadratic function or a hinge loss (essentially, \( \max\{0, x_k\} \)).

Gradient descent is the standard solution approach for the prevalent massive problems of this type. Broadly speaking, gradient descent proceeds by moving iteratively in the direction of the negative gradient of a convex function. Coordinate descent is a commonly studied version of gradient descent. It repeatedly selects and updates a single coordinate of the argument to the convex function. Stochastic versions are standard: at each iteration the next coordinate to update is chosen uniformly at random.

Due to the enormous size of modern problems, there has been considerable interest in parallel versions of coordinate descent in order to achieve speedup, ideally in proportion to the number of processors or cores at hand, called linear speedup.

One important issue in parallel implementations is whether the different processors are all using up-to-date information for their computations. To ensure this requires considerable synchronization, locking, and consequent waiting. Avoiding the need for the up-to-date requirement, i.e. enabling asynchronous updating, was a significant advance. The advantage of asynchronous updating is it reduces and potentially eliminates the need for waiting. At the same time, as some of the data being used in calculating updates will be out of date, one has to ensure that the out-of-datedness is bounded in some fashion.

Modeling asynchrony The study of asynchrony in parallel and distributed computing goes back to Chazan and Miranker [6] for linear systems and to Bertsekas and Tsitsiklis for a wider range of computations [3]. They obtained convergence results for both deterministic and stochastic algorithms along with rate of convergence results for deterministic algorithms. The first analyses to prove rate of convergence bounds for stochastic asynchronous computations were those by Avron, Druinsky and Gupta [1] (for the Gauss-Seidel algorithm), and by Liu et al. [17] and Liu and Wright [16] (for coordinate descent). Liu et al. [17] imposed a “consistent read” constraint on the asynchrony; the other two works considered a more general “inconsistent read” model.

Subsequent to Liu and Wright’s work, several overlooked issues were identified by Mania et al. [19] and Sun et al. [26]; we call them Undoing of Uniformity (UoU)\(^3\) and No-Common-Value.\(^2\)

In brief, as the asynchrony assumptions were relaxed, the bounds that could be shown, particularly in terms of achievable speedup, became successively weaker. In this work we ask the following question:

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\(^1\)There are also versions in which different coordinates can be selected with different probabilities.

\(^2\)“Consistent read” mean that all the coordinates a core read may have some delay, but they must appear simultaneously at some moment. Precisely, the vector of \( \tilde{x} \) values used by the update at time \( t \) must be \( x^{t-c} \) for some \( c \geq 1 \). “Inconsistent reads” mean that the \( \tilde{x} \) values used by the update at time \( t \) can be any of the \( (x^{t-c_1}_1, \ldots, x^{t-c_n}_n) \), where each \( c_j \geq 1 \) and the \( c_j \)'s can be distinct.

\(^3\)In [1], the authors also raised a similar issue about their asynchronous Gauss-Seidel algorithm.
Can we achieve both linear speedup and full asynchrony when applying coordinate descent to non-smooth functions $F$?

Our answer to this question is “yes”, and we obtain the maximum possible parallelism while maintaining linear speedup (up to at most $\sqrt{\log n}$ factor). Our results match the best speedup, namely linear speedup with $\sqrt{n}$ processors as in [17], but with no constraints on the asynchrony, beyond a requirement that unlimited delays do not occur. Specifically, as in [16], we assume there is a bounded amount of overlap between the various updates. We now state our results for strongly convex functions informally.

**Theorem 1** (Informal). Let $q$ be an upper bound on how many other updates a single update can overlap. $L_{res}$, $L_{max}$ and $L_{ii}$ are Lipschitz parameters defined in Section 2.

(i) Let $F$ be a strongly convex function with strongly convex parameter $\mu_F$. If $q = O\left(\frac{\sqrt{n}L_{max}}{L_{res}}\right)$ and if $\Gamma \geq L_{max}$, then $E\left[F(x^T) - F^*\right] = \left((1 - \frac{1}{3}\frac{L_{res}}{n})^T\right) \cdot (F(x^0) - F^*)$.

(ii) This bound is tight up to a $\sqrt{\log n}$ factor: there is a family of functions, with $L_{max}$, $L_{res} = O(1)$, such that for $q = \Theta(\sqrt{n} \ln n)$, with probability $1 - O\left(\frac{1}{n}\right)$, after $n^{c-1}$ updates, the current point is essentially the starting point.

Standard sequential analyses [18, 24] achieve similar bounds with the $\frac{1}{3}$ replaced by 1; i.e. up to a factor of 3, this is the same rate of convergence.

**Asynchronicity assumptions** The Uniformity assumption states that the start time ordering of the updates and their commit time ordering are identical. Undoing of Uniformity (UoU) arises because while each core initiates an update by choosing a coordinate uniformly at random, due to the possibly differing lengths of the different updates, and also due to various asynchronous effects, the commit time ordering of the updates may be far from uniformly distributed. In an experimental study, Sun et al. [26] showed that iteration lengths in coordinate descent problem instances varied by factors of 2–10, demonstrating that this effect is likely.

The Common Value assumption states that regardless of which coordinate is randomly selected to update, the same values are read in their gradient computation. If coordinates are not being read on the same schedule, as seems likely for sparse problems, it would appear that this assumption too will be repeatedly violated.

**Related work** Coordinate Descent is a method that has been widely studied; see Wright for a recent survey [31]. There have been multiple analyses of various asynchronous parallel implementations of coordinate descent [17, 16, 19, 26]. We have already mentioned the results of Liu et al. [17] and Liu and Wright [16]. Both obtained bounds for both convex and “optimally” strongly convex functions, attaining linear speedup so long as there are not too many cores. Liu et al. [17] obtained bounds similar to ours (see their Corollary 2 and our Theorem 2), but the version they analyzed is more restricted than ours in two respects: first, they imposed the strong assumption of consistent reads, and second, they considered only smooth functions (i.e., non-smooth univariate components $\Psi_k$). The version analyzed by Liu and Wright [16] is the same as ours, but their result requires both the Uniformity and Common Value assumptions. Our bound has a similar flavor but with a limit of $\Theta(n^{1/2})$. The analysis by Mania et al. [19] removed the Uniformity assumption. However, the maximum parallelism was much reduced (to at most $n^{1/6}$), and their results applied only to smooth strongly convex functions. We note that a major focus of their work concerned the analysis of Hogwild, a coordinate descent algorithm used for functions of the form $\sum_i f_i(x)$, where each of the $f_i$ is convex, and the bounds there were optimal. The analysis in Sun et al. [26].

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4This is a weakening of the standard strong convexity.
removed the Common Value assumption and partially removed the Uniformity assumption. However, this came at the cost of achieving no parallel speedup. [26] also noted that a hard bound on the parameter $q$ could be replaced by a probabilistic bound, which in practice is more plausible.

Asynchronous methods for solving linear systems have been studied at least since the work of Chazan and Miranker [6] in 1969. See [1] for an account of the development. Avron, Druiisky and Gupta [1] proposed and analyzed an asynchronous and randomized version of the Gauss-Seidel algorithm for solving symmetric and positive definite matrix systems. They pointed out that in practice delays depend on the random choice of direction (which corresponds to coordinate choice in our case), which is indeed one of the sources leading to Undone Uniformity. Their analysis bypasses this issue with their Assumption A-4, which states that delays are independent of the coordinate being updated, but the already mentioned experimental study of Sun et al. indicates that this assumption does not hold in general.

Another widely studied approach to speeding up gradient and coordinate descent is the use of acceleration. Very recently, attempts have been made to combine acceleration and parallelism [14, 12, 9]. But at this point, these results do not extend to non-smooth functions.

Organization of the Paper In Section 2 we describe our model of coordinate descent and state our main results, focusing on the SACD algorithm applied to strongly convex functions. In Section 3 we give a high-level sketch of the structure of our analyses. Then, in Section 4 we show that with the Common Value assumption we can obtain a truly simple analysis for SACD; this analysis achieves the maximum possible speedup (i.e. linear speedup with up to $\sqrt{n}$ processors). Note that this is the same assumption as in Mania et al.’s result [19] and less restrictive than the assumptions in Liu and Wright’s analysis [16]. We follow this with a discussion of some of the obstacles that need to be overcome in order to remove the Common Value assumption, and some comments on how we achieve this. The full analysis of SACD is deferred to the appendix.

2 Model and Main Results

Recall that we are considering convex functions $F : \mathbb{R}^n \to \mathbb{R}$ of the form $F(x) = f(x) + \sum_{k=1}^{n} \Psi_k(x_k)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function, and each $\Psi_k : \mathbb{R} \to \mathbb{R}$ is a univariate and possibly non-smooth convex function. We let $x^*$ denote a minimum point of $F$ and $X^*$ denote the set of all minimum points of $F$. Without loss of generality, we assume that $F^*$, the minimum value of $F$, is 0.

We recap a few standard terminologies. Let $\vec{e}_j$ denote the unit vector along coordinate $j$.

**Definition 1.** The function $f$ is $L$-Lipschitz-smooth if for any $x, \Delta x \in \mathbb{R}^n$, $\|\nabla f(x + \Delta x) - \nabla f(x)\| \leq L \cdot \|\Delta x\|$. For any coordinates $j, k$, the function $f$ is $L_{jk}$-Lipschitz-smooth if for any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$, $|\nabla_k f(x + r \cdot \vec{e}_j) - \nabla_k f(x)| \leq L_{jk} \cdot |r|$; it is $L_{res}$-Lipschitz-smooth if $|\nabla f(x + r \cdot \vec{e}_j) - \nabla f(x)| \leq L_{res} \cdot |r|$. Finally, $L_{\max} := \max_{j, k} L_{jk}$ and $L_{\text{res}} := \max_k \left(\sum_{j=1}^{n} (L_{kj})^2\right)^{1/2}$.

The difference between $L_{res}$ and $L_{\text{res}}$ In general, $L_{\text{res}} \geq L_{res}$. $L_{\text{res}} = L_{\text{res}}$ when the rates of change of the gradient are constant, as for example in quadratic functions such as $x^T Ax + bx + c$. We need $L_{\text{res}}$ because we do not make the Common Value assumption. We use $L_{\text{res}}$ to bound terms of the form $\sum_j |\nabla_j f(y^j) - \nabla_j f(x^j)|^2$, where $|y^j_k - x^j_k| \leq |\Delta_k|$, and for all $h, i$, $|y^h_k - y^i_k|$, $|x^h_k - x^i_k| \leq |\Delta_k|$, whereas in the analyses with the Common Value assumption, the term being bounded is $\sum_j |\nabla_j f(y) - \nabla_j f(x)|^2$, where $|y_k - x_k| \leq |\Delta_k|$; i.e., our bound is over a sum of gradient differences along the coordinate axes for pairs of points which are all nearby, whereas the other sum is over gradient differences along the coordinate axes for the same pair of nearby points. Finally, if the
convex function is $s$-sparse, meaning that each term $\nabla_k f(x)$ depends on at most $s$ variables, then $L_{\text{res}} \leq \sqrt{s}L_{\text{max}}$. When $n$ is huge, this would appear to be the only feasible case.

By a suitable rescaling of variables, we may assume that $L_{jj}$ is the same for all $j$ and equals $L_{\text{max}}$. This is equivalent to using step sizes proportional to $L_{jj}$ without rescaling, a common practice.

Next, we define strong convexity.

**Definition 2.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function. $f$ is strongly convex with parameter $\mu_f > 0$, if for all $x, y$, $f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle + \frac{1}{2} \mu_f ||y - x||^2$.

**The update rule** Recall that in a standard coordinate descent, be it sequential or parallel and synchronous, the update rule, applied to coordinate $j$, first computes the accurate gradient $g_j := \nabla_j f(x^{t-1})$, and then performs the update given below.

$$x^t_j \leftarrow x^{t-1}_j + \arg \min_d \{g_j \cdot d + \Gamma d^2 / 2 + \Psi_j(x^{t-1}_j + d) - \Psi_j(x^{t-1}_j)\} \equiv x^{t-1}_j + \tilde{d}_j(g_j, x^{t-1}_j)$$

and $\forall k \neq j$, $x_k^t \leftarrow x^{t-1}_k$, where $\Gamma \geq L_{\text{max}}$ is a parameter controlling the step size.

However, in an asynchronous environment, an updating core (or processor) might retrieve outdated information $\tilde{x}$ instead of $x^{t-1}$, so the gradient the core computes will be $\tilde{g}_j \equiv \tilde{g}_j := \nabla_j f(\tilde{x})$, instead of the accurate value $\nabla_j f(x^{t-1})$. Our update rule, which is naturally motivated by its synchronous counterpart, is

$$x^t_j \leftarrow x^{t-1}_j + \tilde{d}_j(\tilde{g}_j, x^{t-1}_j) = x^{t-1}_j + \Delta x^t_j \quad \text{and} \quad \forall k \neq j, x^t_k \leftarrow x^{t-1}_k. \quad (1)$$

We let

$$\tilde{W}_j(g, x) := - [g \cdot \tilde{d}_j(g, x) + \Gamma \cdot \tilde{d}_j(g, x)^2 / 2 + \Psi_j(x + \tilde{d}_j(g, x)) - \Psi_j(x)].$$

It is well known that in the synchronous case, $\tilde{W}_j(\nabla_j f(x^{t-1}), x^t_j)$ is a lower bound on the reduction in the value of $F$, which we treat as the progress. We let $k_t$ denote the coordinate being updated at time $t$.

### 2.1 The SACD Algorithm

**Algorithm 1: SACD Algorithm.**

**INPUT:** The initial point $x^0 = (x^0_1, x^0_2, \cdots, x^0_n)$.

Multiple cores use a shared memory. Each core iteratively repeats the following six-step procedure, with no global coordination among them:

1. Choose a coordinate $j \in [n]$ uniformly at random.
2. Retrieve coordinate values $\tilde{x}$ from the shared memory.
3. Compute the gradient $\nabla_j f(\tilde{x})$.
4. Request a lock on the memory that stores the value of the $j$-th coordinate.
5. Retrieve the most updated $j$-th coordinate value, then update it using rule (1).
6. Release the lock acquired in Step 4.

*Even if the core had retrieved the value of the $j$-th coordinate from the shared memory in Step 2, the core needs to retrieve it again here, because it needs the most updated value when applying update rule (1).

The coordinate descent process starts at an initial point $x^0 = (x^0_1, x^0_2, \cdots, x^0_n)$. Multiple cores then iteratively update the coordinate values. We assume that at each time, there is exactly one coordinate value being updated. In practice, since there will be little coordination between cores,
it is possible that multiple coordinate values are updated at the same moment; but by using an arbitrary tie-breaking rule, we can immediately extend our analyses to these scenarios.

In Algorithm 1, we provide the complete description of SACD. The retrieval times for Step 2 plus the gradient-computation time for Step 3 can be non-trivial, and also in Step 4 a core might need to wait if the coordinate it wants to update is locked by another core. Thus, during this period of time other coordinates are likely to be updated. For each update, we call the period of time spent performing the six-step procedure the span of the update. We say that update \( A \) interferes with update \( B \) if the commit time of update \( A \) lies in the span of update \( B \).

In Appendix F we discuss why locking is needed and when it can be avoided; we also explain why the random choice of coordinate should be made before retrieving coordinate values.

**Timing Scheme and the Undoing of Uniformity** Before stating the SACD result formally, we need to disambiguate our timing scheme. In every asynchronous iterative system, including our SACD algorithm, each procedure runs over a span of time rather than atomically. Generally, these spans are not consistent — it is possible for one update to start later than another one but to commit earlier. To create an analysis, we need a scheme that orders the updates in a consistent manner.

Using the commit times of the updates for the ordering seems the natural choice, since this ensures that future updates do not interfere with the current update. This is the choice made in many prior works. However, this causes uniformity to be undone. To understand why, consider the case when there are three cores and four coordinates, and suppose that the workload for updating \( x_1 \) is three times greater than those for updating \( x_2, x_3, x_4 \). If \( k_t = 1 \) for some \( t \), then the probability distribution which the random variable \( k_{t+1} \) follows is biased away from coordinate 1; precisely, \( \mathbb{P}[k_{t+1} = 1 | k_t = 1] < 1/4 \). When there are many more cores and coordinates than the simple case we just considered, and when the other asynchronous effects are taken into account, it is highly uncertain what is the exact or even an approximate distribution for \( k_{t+1} \) conditioned on knowledge of the history of \( k_1, \ldots, k_t \). However, all prior analyses apart from [19] and [26] proceeded by making the idealized assumption that the conditional probability distribution remains uniform, while in fact it may be far from uniform. While it seems plausible that without conditioning, the \( t \)-th update to commit is more or less uniformly distributed, the prior analyses needed this property with the conditioning, and they needed it for every update without fail.

To bypass the above issue, as in [19], we use the starting times of updates for the ordering — then clearly the history has no influence on the choice of \( k_{t+1} \). However, this raises a new issue: future updates can interfere with the current update. Here the term future is used w.r.t. the update ordering, which is by starting time; recall that an update \( U_1 \) with an earlier starting time can commit later than another later starting update \( U_2 \), and therefore \( U_2 \) could interfere with \( U_1 \).

We discuss the Common Value assumption further in Appendix E.

### 2.2 Selected Results

We assume that our algorithms are run until exactly \( T \) coordinates are selected and then updated for some pre-specified \( T \), with the commit times constrained by the following assumption.

**Assumption 1.** There exists a non-negative integer \( q \) such that for any update at time \( t \), the only updates that can interfere with it are those at times \( t - 1, t - 2, \ldots, t - q \) and \( t + 1, t + 2, \ldots, t + q \).

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5E.g., communication delays, interference from other computations (say due to mutual exclusion when multiple cores commit updates to the same coordinate), interference from the operating system and CPU scheduling.
When asynchronous effects are moderate, and if the various gradients have a similar computational cost, the parameter \( q \) will typically be bounded above by a small constant times the number of cores.

**Theorem 2 (SACD Upper Bound).** Given initial point \( x^0 \), Algorithm 1 is run for exactly \( T \) iterations by multiple cores. Suppose that Assumption 1 holds, \( \Gamma \geq L_{\max}, n \geq 2^{10} \), and \( q \leq \frac{\Gamma \sqrt{n}}{\mu F + \Gamma} \). If \( F \) is strongly convex with parameter \( \mu_F \), then

\[
\mathbb{E} \left[ F(x^T) \right] \leq \left[ 1 - \frac{1}{3n} \cdot \frac{\mu_F}{\mu_F + \Gamma} \right]^T \cdot F(x^0) \tag{2}
\]

In the main body of the paper, we focus on the case of strongly convex \( F \). The full result (including the plain convex case) is in Appendix B. In Appendix C we show that the bound on the degree of parallelism given in Theorem 2 is essentially optimal. Theorem 2 states that w.h.p., for the first \( n^c \) updates, each point oscillates around its starting position in a range of one of \( \pm [\frac{2}{3}, \frac{4}{3}] \), i.e. there is no progress toward the optimum.

**Theorem 3 (SACD Lower Bound).** For any constant \( c \geq 2 \) and \( \Gamma \geq 2 \), for all large enough \( n \), there exists a convex function \( f : \mathbb{R}^n \to \mathbb{R} \) with \( L_{\max} = 1 \), \( L_{\max} \leq 2 \), minimum point \( x^* = 0 \), an initial point \( x^0 \) satisfying \( x_j^0 \in \pm 1 \) for every coordinate \( j \), and an asynchronous schedule for Algorithm 1 such that with \( q = \Theta(\sqrt{cn \ln n}) \), at every one of the first \( n^{c-1} \) updates, \( |x_j - x_j^o| \leq \frac{2}{3} \) for all but \( O(\log n) \) coordinates, with probability \( 1 - O(1/n) \).

## 3 The Basic Framework

Let \( k_t \) denote the index of the coordinate that is updated at time \( t \), \( g_{k_t}^t := \nabla_k f(x^{t-1}) \) denote the value of the gradient along coordinate \( x_{k_t} \), computed at time \( t \) using up-to-date values of the coordinates, and \( g_{k_t}^t \) denote the actual value computed, which may use some out-of-date values.

The classical analysis of stochastic (synchronous) coordinate descent proceeds by first showing that for any chosen \( k_t \), \( F(x^{t-1}) - F(x^t) \geq \bar{W}_{k_t}(g_{k_t}^t, x_{k_t}^{t-1}, \Gamma, \Psi_{k_t}) \). Taking the expectation yields

\[
\mathbb{E} \left[ F(x^{t-1}) - F(x^t) \right] \geq \frac{1}{n} \sum_{j=1}^{n} \bar{W}_{j}(g_{j}^t, x_{j}^{t-1}, \Gamma, \Psi_{j}). \tag{3}
\]

By [24, Lemmas 4,6], the RHS of the above inequality is at least \( \frac{1}{n} \cdot \frac{\mu_F}{\mu_F + 1 - \mu_F} \cdot F(x^{t-1}) \); for completeness, we provide a proof of this result in Appendix A. Let \( \alpha := \frac{\mu_F}{\mu_F + 1 - \mu_F} \). Then

\[
\mathbb{E} \left[ F(x^t) \right] \leq (1 - \frac{\alpha}{n}) \cdot \mathbb{E} \left[ F(x^{t-1}) \right]; \text{ iterating this inequality yields } \mathbb{E} \left[ F(x^t) \right] \leq (1 - \frac{\alpha}{n})^t \cdot F(x^0).
\]

To handle the case where inaccurate gradients are used, we employ the following two lemmas.

**Lemma 1.** If \( \Gamma \geq L_{\max} \), \( F(x^{t-1}) - F(x^t) \geq \bar{W}_{k_t}(g_{k_t}^t, x_{k_t}^{t-1}, \Gamma, \Psi_{k_t}) - \frac{1}{\Gamma} \cdot (g_{k_t}^t - \hat{g}_{k_t}^t)^2 \).

**Lemma 2.** If \( \Gamma \geq L_{\max} \), \( F(x^{t-1}) - F(x^t) \geq \frac{1}{2} \Gamma \left( \Delta x_{k_t}^t \right)^2 - \frac{1}{\Gamma} \cdot (g_{k_t}^t - \hat{g}_{k_t}^t)^2 \).

Proving these results for smooth functions is straightforward. The version for non-smooth functions is less simple. It follows from Lemma 3 in Appendix A.

Combining Lemmas 1 and 2 yields

\[
F(x^{t-1}) - F(x^t) \geq \frac{1}{2} \cdot \bar{W}_{k_t}(g_{k_t}^t, x_{k_t}^{t-1}, \Gamma, \Psi_{k_t}) + \frac{\Gamma}{8} \cdot (\Delta x_{k_t}^t)^2 - \frac{1}{\Gamma} \cdot (g_{k_t}^t - \hat{g}_{k_t}^t)^2. \tag{4}
\]

\( ^6 \)Necessarily, \( \Gamma \geq \mu_F \).
The first term on the RHS of the above inequality, after taking the expectation, is more or less what is needed in order to demonstrate progress. To complete the analysis we need to show that

\[
\sum_{t=1}^{T} \frac{1}{2} \left( \nabla \pi_{k,t}^t \right)^2 (1 - \frac{\alpha}{2n})^{T-t} \geq \sum_{t=1}^{T} \frac{1}{2} \left( g_{k,t}^t - q_{k,t}^t \right)^2
\]

(5)

for then we can conclude that \( \mathbb{E} \left[ F(x^t) \right] \leq (1 - \frac{\alpha}{2n})^T \cdot F(x^0) \).

4 A Truly Simple Analysis with the Common Value Assumption

Suppose there are a total of \( T \) updates. We view the whole stochastic process as a branching tree of height \( T \). Each node in the tree corresponds to the moment when some core randomly picks a coordinate to update, and each edge corresponds to a possible choice of coordinate. We use \( \pi \) to denote a path from the root down to some leaf of this tree. A superscript of \( \pi \) on a variable will denote the instance of the variable on path \( \pi \). A double superscript of \( (\pi, t) \) will denote the instance of the variable at time \( t \) on path \( \pi \), i.e. following the \( t \)-th update. Finally \( \pi(k, t) \) will denote the path with the time \( t \) coordinate on path \( \pi \) replaced by coordinate \( k \). Note that \( \pi(k, t) = \pi \).

In this section, we give a simple proof which shows that the error term when reading out-of-date values, \( \mathbb{E}_k [(g_k^{\pi(k, t), t} - q_k^{\pi(k, t), t})^2] \), can be bounded by \( \frac{2g_L^2}{n} \sum_{s \in [t-q,t+q] \setminus \{q\}} |\Delta x_{k,s}^{\pi(s)}|^2 \), where \( \Delta x_{k,s}^{\pi(s)} \) denotes the update on path \( \pi \) at time \( s \), and \( k_s \) is the index of the coordinate chosen at time \( s \). (Note that by the Common Value assumption, the values \( \Delta x_{k,s}^{\pi(k, s)} \) are the same for all \( n \) paths \( \pi(k, s) \).

**Lemma 3.** With the Common Value assumption,

\[
\mathbb{E}_k [(g_k^{\pi(k, t), t} - q_k^{\pi(k, t), t})^2] \leq \frac{2g_L^2}{n} \sum_{s \in [t-q,t+q] \setminus \{q\}} |\Delta x_{k,s}^{\pi(s)}|^2.
\]

**Proof.** By definition, \( g_k^{\pi(k, t), t} = \nabla_k f(x^{\pi(k, t), t}) \), the gradient of up-to-date point \( x^{\pi(k, t), t} \), and \( q_k^{\pi(k, t), t} = \nabla_k f(x^{\pi(k, t), t}) \), the gradient of the point actually read from memory, out-of-date point \( \hat{x}^{\pi(k, t), t} \). By the definition of \( q \), we see that the difference between \( x^{\pi(k, t), t} \) and \( \hat{x}^{\pi(k, t), t} \) is a subset of the updates in the time interval \( [t-q, t+q] \setminus \{t\} \). We denote this subset by \( U \):

\[
U = \{t_1, t_2, \ldots, t_{|U|}\}.
\]

Viewing \( \Delta x_{k,s}^{\pi(k, t), t} \) as an \( n \)-vector with a non-zero entry for coordinate \( t_i \) and no other, we have:

\[
x_{k,t}^{\pi(k, t), t} = \hat{x}_{k,t}^{\pi(t), t} + \sum_{i=1}^{|U|} \begin{cases} \Delta x_{k,t}^{\pi(k, t), t} & \text{if } t_i < t; \\ -\Delta x_{k,t}^{\pi(k, t), t} & \text{if } t_i > t. \end{cases}
\]

For simplicity, we define

\[
x_{k,t}^{\pi(k, t), t}[j] = \hat{x}_{k,t}^{\pi(t), t} + \sum_{i=1}^j \begin{cases} \Delta x_{k,t}^{\pi(k, t), t} & \text{if } t_i < t; \\ -\Delta x_{k,t}^{\pi(k, t), t} & \text{if } t_i > t. \end{cases}
\]

(10)

Assumption \([\text{I}]) states that the updates before time \( t-q \) have been written into memory before the update at time \( t \) starts.
Then, \( x^{\pi(k,t),t}[0] = \tilde{x}^{\pi(k,t),t} \) and \( x^{\pi(k,t),t}[U] = x^{\pi(k,t),t} \). By the definition of \( L_{\text{res}} \) and the triangle inequality, we obtain:

\[
\| \nabla f(x^{\pi(k,t),t}) - \nabla f(x^{\pi(k,t),t}) \|^2 \leq \left( \sum_{j=0}^{U-1} \| \nabla f(x^{\pi(k,t),t}[j+1]) - \nabla f(x^{\pi(k,t),t}[j]) \| \right)^2 \\
\leq \left( \sum_{i=1}^{U} L_{\text{res}} |\Delta x^{\pi(k,t),t} \| \right)^2 \leq 2q \sum_{s \in \{t-q,t+q\}\{t\}} L_{\text{res}}^2 (\Delta x^{\pi(k,t),t})^2.
\]

(6)

The last inequality followed from applying the Cauchy-Schwarz inequality to the RHS, relaxing \( U \) to \( [t-q, t+q] \} \{t\}. Note that, for any \( k \) and \( k' \), \( \| \nabla_{k'} f(\tilde{x}^{\pi(k',t),t}) - \nabla_{k'} f(x^{\pi(k',t),t}) \|^2 = \| \nabla_{k'} f(\tilde{x}^{\pi(k,t),t}) - \nabla_{k'} f(x^{\pi(k,t),t}) \|^2 \) as \( \tilde{x}^{\pi(k,t),t} = \tilde{x}^{\pi(k,t),t} \) and \( x^{\pi(k',t),t} = x^{\pi(k,t),t} \) by the Common Value assumption. So,

\[
E_k \{ (g_k - \tilde{g}_k)^2 \} = \frac{1}{n} E_k \left[ \| \nabla f(\tilde{x}^{\pi(k,t),t}) - \nabla f(x^{\pi(k,t),t}) \|^2 \right].
\]

The result follows on applying (6). \( \square \)

To demonstrate the bound in (5), it suffices that \( \frac{2q^2 L_{\text{res}}^2}{n} (1 - \frac{m}{2n})^q \leq \frac{t^2}{8} \). The bound in Theorem 1 then follows readily.

5 Comments on Achieving the Full Result

The SACD analysis Although the analysis in the previous section is simple, it is not obvious how to obtain a similar bound without the Common Value assumption. We want to have a similar relationship between \( E_k \{ (g_k - \tilde{g}_k)^2 \} \) and \( \sum_{s \in \{t-q,t+q\}\{q\}} (\Delta x^{\pi,s}_{k_{\text{res}}})^2 \). We mention several of the challenges we face when we drop the Common Value assumption.

1. Without the Common Value assumption, \( \tilde{x}^{\pi,t} \) may depend on the coordinate being updated at time \( t \). The reason is that the updates to two different coordinates may read different subsets of coordinates and as a result their reads of a common coordinate may occur at different times, and as a result may be reads of different updates of this common coordinate. We write \( \tilde{x}^{t,k_t} \) for the value of \( x \) read by the update when coordinate \( k_t \) is chosen. Now we need to bound \( \sum_{k_t} [\nabla f(\tilde{x}^{t,k_t}) - \nabla f(x_t)]^2 \), and the first inequality in (6) may no longer apply.

2. In addition, \( x_t \) may also depend on the coordinate being updated at time \( t \). Suppose the updates to coordinates \( i \) and \( j \) at time \( t \) have different read schedules and this affects the timing of an earlier update to coordinate \( k \) (because the update has to be atomic and so may be slightly delayed if there is a read). Then a read of coordinate \( k \) by an update to coordinate \( l \) may occur before \( k \)'s update in the scenario with the time \( t \) update to coordinate \( i \) and after in the scenario with the time \( t \) update to coordinate \( j \). If the update to coordinate \( l \) occurs before time \( t \) then \( x_t \) will depend on the coordinate chosen at time \( t \). While this may seem esoteric, to rule it out imposes unclear limitations on the asynchronous schedules. So actually, we need to bound \( \sum_{k_t} [\nabla f(\tilde{x}^{t,k_t}) - \nabla f(x_t,k_t)]^2 \).
3. Without the assumption, a simple bound is that
\[(g_k^t - \tilde{g}_k^t)^2 \leq 2q \sum_{s \in [t-q,t+q]\setminus \{q\}} L_{k,s,k}^2 (\Delta x_k^t)^2 \]
\[\leq 2q L_{\max}^2 \sum_{s \in [t-q,t+q]} \{q\} (\Delta x_s^t)^2.\] This is essentially the bound in Sun et al. [26] (except that they use \(L\) rather than \(L_{\max}\)). But this bound does not enable any parallel speedup because of the \(q\) term.

4. Without the assumption, the RHS of (5) becomes
\[E_k\left[\hat{W}_k(g_k^t, x_{k,t-1}^t, \Gamma, \Psi_k)\right] = E_k\left[\hat{W}_k(g_k^t, x_{k,t-1}^t, \Gamma, \Psi_k)\right],\]
where \(g_k^t\) indicates the value of \(g_k\) at time \(t\) had coordinate \(j\) rather than coordinate \(k\) been selected, and similarly for \(x_{k,t-1}^t\). The two expectations would be the same if the Common Value assumption held. Our remedy is to devise new shifting lemmas to bound the cost of changing the arguments in \(\hat{W}_k\).

To handle the first three issues, roughly speaking, for each path \(\pi\), we bound the difference between the maximum and minimum possible updates over all possible asynchronous schedules. By considering the directed acyclic graph induced by the read dependencies, we show these differences are equal for many paths given some constraints on the asynchronous schedules. With this, we can average over these paths, and by amortizing with these values, we can achieve an \(O(qL_{\max}^2)\) bound.

We emphasize that our analysis considers all possible asynchronous schedules, but the averaging is done over subsets of these schedules.

**The lower bound** Our construction applies to the functions \(\sum_{i=1}^n x_i^2 + \epsilon \sum_{i \neq j} x_i x_j\), where \(\epsilon = \Theta(1/\sqrt{cn \ln n})\). The idea is that w.h.p., for each time step, after the first \(\Theta(\sqrt{cn \ln n})\) updates, among the most recent \(q = \Theta(\sqrt{cn \ln n})\) updates, a constant fraction will have had a \(\Theta(1)\) increment and a constant fraction will have had a \(\Theta(1)\) decrement. Then, for the next update, the asynchronous scheduling allows us to designate whether positive or negative updates are read and hence to maintain the property. The reason this can last for only \(\Theta(n^c)\) steps, is that each coordinate needs to alternate the direction of its update, and so eventually too large a fraction of the \(q\) most recent updates may be in one direction or the other. We organize this as a novel ball in urns problem.

**Problems with large \(L_{\max}\) and \(L_{\text{res}}\)**. Both \(L_{\text{res}}\) and \(L_{\text{res}}\) can be as large as \(\sqrt{n}\). For problem instances of this type, the bound on \(q\) becomes \(O(1)\); i.e. they do not demonstrate any parallel speedup. We conjecture that this is inherent. Even if the conjecture holds, it is still conceivable that parallel speedup will occur in practice, but to provide a confirming analysis would require new assumptions on the asynchronous behavior, and we leave the devising of such assumptions as an open problem.

**Acknowledgment**

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Index for the Appendices

For the reader’s convenience, we provide an index for the sections in the appendix and also brief descriptions of the topics.

1. **Appendix A: Basic Lemmas and Handling No Common Write.**  
   Page: 11
   We show some basic lemmas needed to measure progress and to bound the gradient error; these will be used later in the convergence analysis. Also, Appendix A.5 provides the lemmas that bound the costs for shifting \( \tilde{W} \)'s arguments; they are needed to handle the No Common Write setting (and elsewhere for that matter).

2. **Appendix B: Full SACD Analysis.**  
   Page: 20
   Here, we give the complete analysis of SACD. Beyond the analysis given in the main part, we show how to deal with \( \Psi \) when bounding the difference \( (g - \tilde{g})^2 \) (to do this we need to create a new ordering that lies between the start and commit time orderings); we then show how to perform the amortization.

3. **Appendix C: Lower bound on SACD.**  
   Page: 37
   We show a family of functions that yields the lower bound for SACD, demonstrating that our analysis of SACD is tight.

4. **Appendix D: Further Related Work.**  
   Page: 43

5. **Appendix E: Common Value Assumption.**  
   Page: 44
   We explain how the Common Value assumption is violated due to different retrieval schedules for different choices of coordinates, and due to varying iteration length.

6. **Appendix F: Comments on Locking.**  
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7. **References**  
   Page: 46
A Some Basic Lemmas and Facts

We now consider the general form of the function, i.e. \( F(x) = f(x) + \sum_k \Psi_k(x_k) \). Recall that the update rule is

\[
x'_j \leftarrow x''_j - \arg\min_d \{ g_j \cdot d + \Gamma d^2/2 - \Psi_j(x_j) + \Psi_j(x_j + d) \} \equiv x''_j + \hat{d}(g, x, \Gamma, \Psi_j) \quad \text{and} \quad \forall k \neq j, x_k' \leftarrow x_k'' - 1.
\]

We define \( W(d, g, x, \Gamma, \Psi_j) = -[g_j \cdot d + \Gamma d^2/2 - \Psi_j(x_j) + \Psi_j(x_j + d)] \), and we set \( \hat{W}(g, x, \Gamma, \Psi_j) = \arg\max_d W(d, g, x, \Gamma, \Psi_j) = -[g_j \cdot \hat{d} + \Gamma \hat{d}^2/2 - \Psi_j(x_j) + \Psi_j(x_j + \hat{d})] \).

Recall we assume that at each time, there is exactly one coordinate value being updated. For any time \( \tau \), let \( k_{\tau} \) denote the coordinate which is updated at time \( \tau \), and let \( \Delta x_{k_{\tau}, \tau} = x'_{k_{\tau}} - x''_{k_{\tau}} \).

Since, for each coordinate \( j \), the parameter \( \Gamma_j \) and the function \( \Psi_j \) remain unchanged throughout the ACD process, to avoid clutter, we use the shorthand

\[
\hat{W}_j(g, x) := \hat{W}(g, x, \Gamma_j, \Psi_j) \quad \text{and} \quad \hat{d}_j(g, x) := \hat{d}(g, x, \Gamma_j, \Psi_j).
\]

Note that \( W_j(0, g, x) = 0 \); thus \( \hat{W}_j(g, x) \geq 0 \).

Let \([n]\) denote the set of coordinates \( \{1, 2, \ldots, n\} \). In this proof, \( \Psi \) will always denote a function \( \mathbb{R} \to \mathbb{R} \) which is univariate, proper, convex and lower semi-continuous. Recall the definition of \( L_{kk} \) in Definition (\ref{def:4}). As is conventional, we write \( L_k \equiv L_{kk} \).

It is well-known that for any \( k \in [n], x \in \mathbb{R}^n \) and \( r \in \mathbb{R} \),

\[
f(x + r\hat{e}_k) \leq f(x) + \nabla_k f(x) \cdot r + \frac{L_k}{2} r^2.
\]

A.1 Some Lemmas about the Functions \( \hat{W} \) and \( \hat{d} \)

We state a number of technical lemmas concerning the functions \( \hat{W} \) and \( \hat{d} \). The following fact, which follows directly from the definition of \( \hat{W} \), will be used multiple times:

\[
\text{If } 0 < \Gamma < \Gamma', \text{ then } \forall g, x \in \mathbb{R}, \quad \hat{W}(g, x, \Gamma, \Psi) \geq \hat{W}(g, x, \Gamma', \Psi). \quad (8)
\]

**Lemma 4** (Three-Point Property, \( \hat{W} \)). For any proper, convex and lower semi-continuous function \( Y : \mathbb{R} \to \mathbb{R} \) and for any \( d^- \in \mathbb{R} \), let \( d^+ := \arg\max_{d \in \mathbb{R}} \{-Y(d) - \Gamma(d - d^-)\} \). Then for any \( d' \in \mathbb{R} \),

\[
Y(d') + \Gamma(d' - d^-)^2 \geq Y(d^+) + \Gamma(d^+ - d^-)^2 + \Gamma(d' - d^+)^2.
\]

**Proof:** We apply Lemma (\ref{lem:4}) with \( d^- = d' = 0 \) and \( Y(d) = gd - \Psi(x) + \Psi(x + d) \). Then \( W(d, g, x, \Gamma, \Psi) = -Y(d) - \Gamma d^2/2 \), and hence \( d^+ \), as defined in Lemma \( \hat{W} \), equals \( \hat{d}(g, x, \Gamma, \Psi) \). These yield

\[
Y(0) \geq Y(\hat{d}(g, x, \Gamma, \Psi)) + \Gamma \cdot \left( \hat{d}(g, x, \Gamma, \Psi) \right)^2.
\]

Since \( Y(0) = 0 \) and \( -Y(\hat{d}(g, x, \Gamma, \Psi)) = \hat{W}(g, x, \Gamma, \Psi) + \frac{\Gamma}{2} \left( \hat{d}(g, x, \Gamma, \Psi) \right)^2 \), we are done. \( \square \)

**Lemma 6** (\( \hat{W} \) Shifting on x parameter). Let \( \hat{W}(g, x_1, \Gamma, \Psi) = W(\hat{d}_1, g, x_1, \Gamma, \Psi) \) and \( \hat{W}(g, x_2, \Gamma, \Psi) = W(\hat{d}_2, g, x_2, \Gamma, \Psi) \). Then

\[
\hat{W}(g, x_1) + \Psi(x_2) - \Psi(x_1) \geq \hat{W}(g, x_2) - g(x_2 - x_1) - \Gamma \hat{d}_2(x_2 - x_1) - \frac{\Gamma}{2} \cdot (x_2 - x_1)^2.
\]
**Proof:** RJC: I think we should remove the $\Gamma$ from the $\hat{W}$ or include the $\Psi$. RJC: Right. We use Lemma 3 with $d' = 0$, and $Y(d) = gd - \Psi(x) + \Psi(x + d)$. Then we have
\[
Y(d') + \frac{\Gamma}{2} \cdot (d')^2 \geq -\hat{W}(g, x_1) + \frac{\Gamma}{2} \cdot (d' - \hat{d}_1)^2.
\]
The above inequality holds for any $d'$. In particular, we pick $d' = x_2 - x_1 + \hat{d}_2$, yielding
\[
\hat{W}(g, x_1) \geq -g(x_2 - x_1 + \hat{d}_2) + \Psi(x) - \Psi(x + \hat{d}_2) - \frac{\Gamma}{2} \cdot (x_2 - x_1 + \hat{d}_2)^2 + \frac{\Gamma}{2} \cdot (x_2 - x_1 + \hat{d}_1 - \hat{d}_2)^2.
\]
By adding $\Psi(x_2) - \Psi(x_1)$ to both sides, we have
\[
\hat{W}(g, x_1) + \Psi(x_2) - \Psi(x_1) \geq -g(x_2 - x_1 + \hat{d}_2) + \Psi(x_2) - \Psi(x + \hat{d}_2) - \frac{\Gamma}{2} \cdot (x_2 - x_1 + \hat{d}_2)^2 + \frac{\Gamma}{2} \cdot (x_2 - x_1 + \hat{d}_1 - \hat{d}_2)^2.
\]

**Lemma 7 (Ψ Shifting).** Let $\hat{W}(g_1, x_1) = W(\hat{d}_1, g_1, x_1)$ and $\hat{W}(g_2, x_2) = W(\hat{d}_2, g_2, x_2)$. Then
\[
\Psi(x_2 + \hat{d}_2) - \Psi(x_1 + \hat{d}_1) \leq g_2(x_1 - x_2 + \hat{d}_1 - \hat{d}_2) + \frac{\Gamma}{2} \cdot (x_1 - x_2 + \hat{d}_1)^2.
\]
**Proof:** By the definition of $\hat{d}_2$, we have the following inequality, which directly implies the one stated in the lemma.
\[
-g_2\hat{d}_2 - \frac{\Gamma}{2} \cdot (\hat{d}_2)^2 - \Psi(x_2 + \hat{d}_2) \geq -g_2(x_1 - x_2 + \hat{d}_1) - \frac{\Gamma}{2} \cdot (x_1 - x_2 + \hat{d}_1)^2 - \Psi(x_1 + \hat{d}_1).
\]

**Lemma 8 (\hat{W} Shifting on g parameter).** For any $g_i, g'_j$, $\hat{W}(g_j, x_j) \geq \frac{2}{3} \hat{W}(g'_j, x_j) - \frac{4}{3} (g_j - g'_j)^2$.

**Proof:** To avoid clutter, we use the shorthand $\hat{d}(g_i) := \hat{d}(g_i, x, \Gamma, \Psi)$ for $i = 1, 2$.
\[
\hat{W}(g_i, x, \Gamma, \Psi) = \max_{d \in \mathbb{R}} W(d, g_i, x, \Gamma, \Psi) \geq W(\hat{d}(g_i), g_i, x, \Gamma, \Psi) = -g_i \cdot \hat{d}(g_i) - \Gamma \cdot (\hat{d}(g_i))^2/2 + \Psi(x) - \Psi(x + \hat{d}(g_i)) = -g_i \cdot \hat{d}(g_i) - \Gamma \cdot (\hat{d}(g_i))^2/2 + \Psi(x) - \Psi(x + \hat{d}(g_i)) + (g_i - g_i) \cdot [\hat{d}(g_1) + (\hat{d}(g_2) - \hat{d}(g_1))] \geq \hat{W}(g_i, x, \Gamma, \Psi) - |g_1 - g_2| \cdot |\hat{d}(g_1)| - |g_1 - g_2| \cdot |\hat{d}(g_2) - \hat{d}(g_1)| \geq \hat{W}(g_i, x, \Gamma, \Psi) - |g_1 - g_2| \cdot |\hat{d}(g_1)| - \frac{1}{\Gamma} (g_1 - g_2)^2 \quad (\text{By Lemma 11}) \geq \hat{W}(g_i, x, \Gamma, \Psi) - \frac{1}{\Gamma} (g_1 - g_2)^2 - \frac{1}{\Gamma} (\hat{d}(g_1))^2 - \frac{1}{\Gamma} (g_1 - g_2)^2 \quad (\text{AM-GM inequ}) \geq \hat{W}(g_i, x, \Gamma, \Psi) - \frac{2}{\Gamma} (g_1 - g_2)^2 - \frac{1}{2} \hat{W}(g_1, x, \Gamma, \Psi). \quad (\text{By Lemma 5}) \]
A.2 Proofs of Lemmas 1 and 2

Lemmas 1 and 2 follow directly from the lemma below.

Lemma 9. Suppose there is an update to coordinate $j$ at time $t$ according to rule (1), and suppose that $\Gamma \geq L_{\text{max}}$. Let $g_j = \nabla_j f(x^{t-1})$ and $\tilde{g}_j = \nabla_j f(\tilde{x})$. Then

$$F(x^{t-1}) - F(x^t) \geq \frac{\Gamma}{4}(\tilde{d}_j(g_j, x_j^{t-1}))^2 - \frac{1}{\Gamma} \cdot (g_j - \tilde{g}_j)^2$$

and

$$F(x^{t-1}) - F(x^t) \geq \tilde{W}_j(g_j, x_j^{t-1}) - \frac{1}{\Gamma} \cdot (g_j - \tilde{g}_j)^2.$$  

Proof: To avoid clutter, we use the shorthands $d_j := \tilde{d}_j(g_j, x_j^{t-1})$ and $\tilde{d}_j := \tilde{d}_j(g_j, x_j^{t-1})$. By update rule (11), $\tilde{d}_j = \Delta x_{j,t}$.

$$F(x^t) = f(x^t) + \Psi_j(x_j^t) + \sum_{k \neq j} \Psi_k(x_k^t)$$

$$\leq f(x^{t-1}) + g_j \tilde{d}_j + \frac{\Gamma}{2}(\tilde{d}_j)^2 + \Psi_j(x_j^{t-1} + \tilde{d}_j) + \sum_{k \neq j} \Psi_k(x_k^{t-1})$$

(By (7), (1), and the assumption $\Gamma \geq L_{\text{max}} \geq L_j$)

$$= F(x^{t-1}) + \tilde{g}_j \tilde{d}_j + \frac{\Gamma}{2}(\tilde{d}_j)^2 - \Psi_j(x_j^{t-1}) + \Psi_j(x_j^{t-1} + \tilde{d}_j) + (g_j - \tilde{g}_j)\tilde{d}_j$$

$$= F(x^{t-1}) - \tilde{W}_j(\tilde{g}_j, x_j^{t-1}) + (g_j - \tilde{g}_j)\tilde{d}_j.$$ 

Hence,

$$F(x^{t-1}) - F(x^t) \geq \tilde{W}_j(\tilde{g}_j, x_j^{t-1}) - (g_j - \tilde{g}_j)\tilde{d}_j.$$ 

Then we can apply Lemma 5 to prove the first inequality in Lemma 9

$$F(x^{t-1}) - F(x^t) \geq \tilde{W}_j(\tilde{g}_j, x_j^{t-1}) - (g_j - \tilde{g}_j)\tilde{d}_j$$

$$\geq \frac{\Gamma}{2}(\tilde{d}_j)^2 - |g_j - \tilde{g}_j| \cdot |\tilde{d}_j|$$

$$\geq \frac{\Gamma}{2}(\tilde{d}_j)^2 - \frac{1}{2} \left[\frac{2}{\Gamma} \cdot (g_j - \tilde{g}_j)^2 + \frac{\Gamma}{2}(\tilde{d}_j)^2\right]$$

(by the AM-GM ineq.)

$$= \frac{\Gamma}{4}(\tilde{d}_j)^2 - \frac{1}{\Gamma} \cdot (g_j - \tilde{g}_j)^2.$$ 

We prove the second inequality in Lemma 9 as follows:

$$F(x^{t-1}) - F(x^t) \geq \tilde{W}_j(\tilde{g}_j, x_j^{t-1}) - (g_j - \tilde{g}_j)\tilde{d}_j$$

$$\geq \tilde{W}_j(d_j, \tilde{g}_j, x_j^{t-1}) - (g_j - \tilde{g}_j)\tilde{d}_j$$

$$= W_j(d_j, g_j, x_j^{t-1}) + (g_j - \tilde{g}_j)d_j - (g_j - \tilde{g}_j)\tilde{d}_j$$

$$= \tilde{W}_j(g_j, x_j^{t-1}) + (g_j - \tilde{g}_j)(d_j - \tilde{d}_j)$$

$$\geq \tilde{W}_j(g_j, x_j^{t-1}) - |g_j - \tilde{g}_j| \cdot |d_j - \tilde{d}_j|$$

$$\geq \tilde{W}_j(g_j, x_j^{t-1}) - \frac{1}{\Gamma}(g_j - \tilde{g}_j)^2.$$ (By Lemma 11)

Combining Lemmas 1 and 2 yields

$$F(x^{t-1}) - F(x^t) \geq \frac{1}{2} \tilde{W}_k(g_{k_i}, x_{k_i}^{t-1}, \Gamma, \Psi_{k_i}) + \frac{1}{8} \Gamma (\Delta x_{k_i}^t)^2 - \frac{1}{\Gamma}(g_{k_i} - \tilde{g}_{k_i})^2$$

or

$$F(x^{t-1}) - F(x^t) \geq \frac{1}{2} \tilde{W}_k(g_{k_i}, x_{k_i}^{t-1}, \Gamma, \Psi_{k_i}) + \frac{1}{4} \Gamma (\Delta x_{k_i}^t)^2$$

if $g_{k_i}^t = \tilde{g}_{k_i}^t$. (9)
A.3 Proof of Lemma 10
Lemma 10. For any \( g_1, g_2, x \in \mathbb{R} \) and \( \Gamma \in \mathbb{R}^+ \), \( |\hat{d}(g_1, x, \Gamma, \Psi) - \hat{d}(g_2, x, \Gamma, \Psi)| \leq |x_1 - x_2| + \frac{1}{\Gamma} \cdot |g_1 - g_2| \), and hence
\[
\left( \hat{d}(g_1, x, \Gamma, \Psi) - \hat{d}(g_2, x, \Gamma, \Psi) \right)^2 \leq 2(x_1 - x_2)^2 + \frac{2}{\Gamma^2} \cdot (g_1 - g_2)^2.
\]
If \( \Psi \) is the zero function, the upper bound on \( |\hat{d}(g_1, x, \Gamma, \Psi) - \hat{d}(g_2, x, \Gamma, \Psi)| \) can be improved to \( \frac{1}{\Gamma} \cdot |g_1 - g_2| \).

Lemma 10 is a simple corollary of the following two lemmas.

Lemma 11 ([28, Lemma 4]). For any \( g_1, g_2, x \in \mathbb{R} \) and \( \Gamma \in \mathbb{R}^+ \), \( |\hat{d}(g_1, x, \Gamma, \Psi) - \hat{d}(g_2, x, \Gamma, \Psi)| \leq \frac{1}{\Gamma} \cdot |g_1 - g_2| \).

Lemma 12. For any \( g, x_1, x_2 \in \mathbb{R} \) and \( \Gamma \in \mathbb{R}^+ \), \( |\hat{d}(g, x_1, \Gamma, \Psi) - \hat{d}(g, x_2, \Gamma, \Psi)| \leq |x_1 - x_2| \).

Proof: For \( i = 1, 2 \), let \( d_i := \hat{d}(g, x_i, \Gamma, \Psi) \). By the definition of \( \hat{d} \), for \( i = 1, 2 \), there exists a subgradient \( \Psi'(x_i + d_i) \) such that
\[
g + \Gamma \cdot d_i + \Psi'(x_i + d_i) = 0.
\]
If \( d_1 = d_2 \), we are done. If \( d_1 > d_2 \), then \( \Psi'(x_1 + d_1) < \Psi'(x_2 + d_2) \). Since \( \Psi \) is convex, \( x_1 + d_1 \leq x_2 + d_2 \) and hence \( 0 < d_1 - d_2 \leq x_2 - x_1 \).

If \( d_2 > d_1 \), by the same argument as above we have \( 0 < d_2 - d_1 \leq x_1 - x_2 \).

A.4 The Progress Lemma
The following lemma is key to the demonstration of progress in both the strongly convex and convex cases.

For any \( t \geq 1 \), we define the following:
\[
PRG(t-1) := \sum_{k=1}^{n} \hat{W}_k(\nabla_k f(x^{t-1}), x^{t-1}, \Gamma, \Psi_k).
\]

We will use the following lemma from [24, Lemmas 4,6]. We provide a proof here for completeness.

Lemma 13 ([24, Lemmas 4,6]).
(a) Suppose that \( f, F \) are strongly convex with parameters \( \mu_f, \mu_F > 0 \) respectively, and suppose that \( \Gamma \geq \mu_f \). Then
\[
PRG(t-1) \geq \frac{\mu_F}{\mu_F + \Gamma - \mu_f} \cdot F(x^{t-1}).
\]
(b) Suppose that \( f, F \) are convex functions. Suppose that \( R := \min_{x^* \in X^*} \|x^{t-1} - x^*\| < \infty \). Then
\[
PRG(t-1) \geq \min \left\{ \frac{1}{2}, \frac{F(x^{t-1})}{2\Gamma R^2} \right\} \cdot F(x^{t-1}).
\]
**Proof:** First of all, we show a lower bound for \(\text{PRG}(t - 1)\), which will be used to prove both (a) and (b).

\[
\text{PRG}(t - 1) = \sum_{k=1}^{n} \max_{d_k \in \mathbb{R}} \left\{ -\nabla_k f(x^{t-1}) \cdot d_k - \Gamma \cdot (d_k)^2/2 + \Psi_k(x^{t-1} - \Psi_k(x^{t-1} + d_k)) \right\}
\]

\[
\geq \max_{d \in \mathbb{R}^n} \left\{ \sum_{k=1}^{n} \left[ -\nabla_k f(x^{t-1}) \cdot d_k - \Gamma \cdot (d_k)^2/2 + \Psi_k(x^{t-1} - \Psi_k(x^{t-1} + d_k)) \right] \right\}.
\]

When \(f\) is strongly convex with parameter \(\mu_f\), for any \(d \in \mathbb{R}^n\),

\[
f(x^{t-1} + d) \geq f(x^{t-1}) + \sum_{k=1}^{n} \nabla_k f(x^{t-1}) \cdot d_k + \frac{\mu_f}{2} \sum_{k=1}^{n} (d_k)^2.
\]

Thus

\[
\text{PRG}(t - 1) \geq \max_{d \in \mathbb{R}^n} \left\{ f(x^{t-1}) - f(x^{t-1} + d) - \frac{\Gamma - \mu_f}{2} \sum_{k=1}^{n} (d_k)^2 + \sum_{k=1}^{n} \left[ \Psi_k(x^{t-1} - \Psi_k(x^{t-1} + d_k)) \right] \right\}
\]

\[
= \max_{d \in \mathbb{R}^n} \left\{ F(x^{t-1}) - F(x^{t-1} + d) - \frac{\Gamma - \mu_f}{2} \sum_{k=1}^{n} (d_k)^2 \right\}
\]

\[
\geq \max_{0 \leq \beta \leq 1} \left\{ F(x^{t-1}) - F(\beta x^* + (1 - \beta)x^{t-1}) - \frac{(\Gamma - \mu_f)^2}{2} \sum_{k=1}^{n} (x_k^{t-1} - x_k^*)^2 \right\}. \quad (11)
\]

To prove (a), we apply the following characterization of strong convexity of \(F\): for any \(0 \leq \beta \leq 1\),

\[
F(\beta x^* + (1 - \beta)x^{t-1}) \leq \beta \cdot F(x^*) + (1 - \beta) \cdot F(x^{t-1}) - \frac{\mu_f \beta (1 - \beta)}{2} \sum_{k=1}^{n} (x_k^{t-1} - x_k^*)^2.
\]

Note that \(F(x^*) = F^* = 0\). By (11),

\[
\text{PRG}(t - 1) \geq \max_{0 \leq \beta \leq 1} \left\{ \beta \cdot F(x^{t-1}) + \frac{\mu_f \beta (1 - \beta) - (\Gamma - \mu_f)^2}{2} \sum_{k=1}^{n} (x_k^{t-1} - x_k^*)^2 \right\}
\]

\[
\geq \left( \beta \cdot F(x^{t-1}) + \frac{\mu_f \beta (1 - \beta) - (\Gamma - \mu_f)^2}{2} \sum_{k=1}^{n} (x_k^{t-1} - x_k^*)^2 \right) \big|_{\beta = \frac{\mu_f}{\mu_f + (\Gamma - \mu_f)}}
\]

\[
= \frac{\mu_f}{\mu_f + (\Gamma - \mu_f)} \cdot F(x^{t-1}).
\]

Note that the constraint \(\beta \leq 1\) forces \(\Gamma \geq \mu_f\).

To prove (b), let \(x^*\) denote a point in \(X^*\) such that \(\|x^{t-1} - x^*\| \leq R\), where \(X^*\) is the set of minimum points for \(F\). By the convexity of \(F\), when \(0 \leq \beta \leq 1\), \(F(\beta x^* + (1 - \beta)x^{t-1}) \leq (1 - \beta) \cdot F(x^{t-1})\). Since \(f\) is convex, \(\mu_f \geq 0\) always. From (11),

\[
\text{PRG}(t - 1) \geq \max_{0 \leq \beta \leq 1} \left\{ \beta \cdot F(x^{t-1}) - \frac{\Gamma \beta^2}{2} R^2 \right\}.
\]

The R.H.S. of the above inequality is a maximization of a quadratic function of \(\beta\), which can be easily solved to yield the lower bound in (b). \(\Box\)
The basis form of all our convergence results is given by the following meta-theorem (or the same result in expectation for the stochastic case).

**Theorem 4.** Let $\Gamma$ be a sufficiently large step size for the update rule, and let $r, q$ be two fixed integer parameters. Let $A(t), B(t)$ be non-negative functions with $A(0) = 0$, $B(0) = 0$, and let $H(t) := F(x^t) + A(t) - B(t)$. Suppose that

- for all $t \geq 1$, $H(t) \leq H(t-1)$, i.e., $H(t)$ is a decreasing function of $t$;
- there exists constants $\alpha, \beta > 0$ such that for any $t \geq 2r$,

\[
\sum_{i=t-2r+1}^{t} [H(i-1) - H(i)] \geq \sum_{i=t-2r+1}^{t-r} \left[ \frac{\alpha}{n} \sum_{k=1}^{n} \hat{W}_k(\nabla_k f(x^{i-1}), x^{i-1}, \Gamma, \Psi_k) + \frac{\beta}{q} \cdot A(i-1) \right].
\]

(i) Then, if $F$ is strongly convex with parameter $\mu_F$ and $f$ has strongly convex parameter $\mu_f$, $H(t) \leq \left[ 1 - \min \left\{ \frac{\alpha}{2n} \cdot \frac{\mu_F}{\mu_F + \Gamma - \mu_f}, \frac{\beta}{2q} \right\} \right]^{t-2r+1} \cdot F(x^0)$.

(ii) Now suppose that $F$ is convex. Let $R$ be the radius of the level set for $x^0$. Formally, let $X = \{ x \mid F(x) \leq F(x^0) \}$; then $R = \sup_{x \in X} \min_{x^* \in X^*} \| x - x^* \|$. Then, for $t \geq r$,

\[
H(t) \leq \frac{F(x^0)}{1 + \min \left\{ \frac{\beta}{4q F(x^0)}, \frac{\alpha}{8n F(x^0)}, \frac{1}{8n} \right\} \cdot H(2r-1) \cdot (t-2r+1)}.
\]

This result also holds in expectation.

For $t = T$, we will ensure $B(T) = 0$, and then $F(T) \leq H(T)$.

**Proof:** We begin by showing (i). By the second assumption and Lemma 13,

\[
\sum_{i=t-2r+1}^{t} [H(i-1) - H(i)] \geq \sum_{i=t-2r+1}^{t-r} \left[ \frac{\alpha}{n} \cdot \text{PRG}(i-1) + \frac{\beta}{q} \cdot A(i-1) \right]
\]

\[
\geq \sum_{i=t-2r+1}^{t-r} \left[ \frac{\alpha}{n} \cdot \frac{\mu_F}{\mu_F + \Gamma - \mu_f} F(x^{i-1}) + \frac{\beta}{q} \cdot A(i-1) \right]
\]

\[
\geq \sum_{i=t-2r+1}^{t-r} \delta \cdot H(i-1),
\]

where $\delta := \min \left\{ \frac{\alpha}{n} \cdot \frac{\mu_F}{\mu_F + \Gamma - \mu_f}, \frac{\beta}{q} \right\}$.

By the assumption that $H$ is decreasing,

\[
\sum_{i=t-2r+1}^{t-r} \delta \cdot H(i-1) \geq \frac{\delta}{2} \cdot \sum_{i=t-2r+1}^{t} H(i-1).
\]

Combining all the above yields

\[
\sum_{i=t-2r+1}^{t} H(i) \leq \left( 1 - \frac{\delta}{2} \right) \cdot \sum_{i=t-2r+1}^{t} H(i-1) = \left( 1 - \frac{\delta}{2} \right) \cdot \sum_{i=t-2r}^{t-1} H(i).
\]

\[\text{i.e., for all } x, y \in \mathbb{R}^n \text{ and } F'(x) \text{ which is any subgradient of } F \text{ at } x, \ F(y) \geq F(x) + (F'(x), y - x) + \frac{1}{2} \mu_F \| y - x \|^2.\]
For any $t \geq 2r$, iterating the above inequality $(t - 2r + 1)$ times yields

$$\sum_{i=t-2r+1}^{t} H(i) \leq \left(1 - \frac{\delta}{2}\right)^{t-2r+1} \sum_{i=0}^{2r-1} H(i).$$

Since $H$ is decreasing, the summation in LHS is at least $2r \cdot H(t)$, while the summation in RHS is at most $2r \cdot H(0)$. Thus,

$$H(t) \leq \left(1 - \frac{\delta}{2}\right)^{t-2r+1} \cdot H(0).$$

To finish the proof note that since $A(0) = 0$, $H(0) = F(x^0)$.

Now we show (ii). By the second assumption and Lemma 13,

$$\sum_{i=t-2r+1}^{t} [H(i) - H(i)] \geq \sum_{i=t-2r+1}^{t-r} \left[ \frac{\alpha}{n} \text{PRG}_{i-1} + \frac{\beta}{q} \cdot A(i-1) \right] \geq \sum_{i=t-2r+1}^{t-r} \left[ \frac{\alpha}{n} \cdot \min \left\{ \frac{1}{2}, \frac{F(x^i - 1)}{2 \Gamma R^2} \right\} \cdot F(x^i - 1) + \frac{\beta}{q} \cdot A(i-1) \right].$$

For each $i$, there are two possible cases:

- If $F(x^i - 1) \leq A(i-1)$, then $A(i-1) \geq \frac{H(i-1)}{2}$, thus
  $$\frac{\alpha}{n} \cdot \min \left\{ \frac{1}{2}, \frac{F(x^i - 1)}{2 \Gamma R^2} \right\} \cdot F(x^i - 1) + \frac{\beta}{q} \cdot A(i-1) \geq \frac{\beta}{2q} \cdot H(i-1).$$

- If $F(x^i - 1) > A(i-1)$, then $F(x^i - 1) > \frac{H(i-1)}{2}$, thus
  $$\frac{\alpha}{n} \cdot \min \left\{ \frac{1}{2}, \frac{F(x^i - 1)}{2 \Gamma R^2} \right\} \cdot F(x^i - 1) + \frac{\beta}{q} \cdot A(i-1) > \frac{\alpha}{2n} \cdot \min \left\{ \frac{1}{2}, \frac{H(i-1)}{4 \Gamma R^2} \right\} \cdot H(i-1).$$

Since $H$ is a decreasing function, $H(i-1) \leq H(0) = F(x^0)$. Thus, unconditionally, we have

$$\frac{\alpha}{n} \cdot \min \left\{ \frac{1}{2}, \frac{F(x^i - 1)}{2 \Gamma R^2} \right\} \cdot F(x^i - 1) + \frac{\beta}{q} \cdot A(i-1) \geq \min \left\{ \frac{\beta}{2q}, \frac{\alpha}{4n}, \frac{H(i-1)}{4 \Gamma R^2} \right\} \cdot H(i-1) \geq \min \left\{ \frac{\beta}{2q}, \frac{\alpha}{4n}, \frac{H(i-1)}{4 \Gamma R^2} \right\} \cdot H(i-1)^2.$$

Note that the term $\min \left\{ \frac{\beta}{2q} F(x^0), \frac{\alpha}{4n} F(x^0), \frac{1}{4 \Gamma R^2} \right\}$ is independent of $i$. We let $\varepsilon$ denote it.

Now, we have

$$\sum_{i=t-2r+1}^{t} [H(i) - H(i)] \geq \sum_{i=t-2r+1}^{t-r} \varepsilon H(i-1)^2$$

$$\geq \frac{\varepsilon}{r} \left( \sum_{i=t-2r+1}^{t-r} H(i-1) \right)^2 \quad \text{(by the Power Mean Inequality)}$$

$$\geq \frac{\varepsilon}{r} \left( \frac{1}{2} \sum_{i=t-2r+1}^{t} H(i-1) \right)^2 \quad \text{as $H$ is a decreasing function}$$

$$= \frac{\varepsilon}{4r} \left( \sum_{i=t-2r+1}^{t} H(i-1) \right)^2.$$
For brevity, let $S_t := \sum_{i=t-2r+1}^{t} H(i)$. Then the above inequality translates to

$$S_{t-1} - S_t \geq \frac{\varepsilon}{4r} (S_{t-1})^2.$$  

Note that $S_{t-1} \geq S_t \geq 0$. Dividing both sides by $S_{t-1} \cdot S_t$ yields

$$\frac{1}{S_t} - \frac{1}{S_{t-1}} \geq \frac{\varepsilon}{4r} \frac{S_{t-1}}{S_t} \geq \frac{\varepsilon}{4r}.$$  

Iterating the above inequality $t - 2r + 1$ times yields

$$\frac{1}{S_t} - \frac{1}{S_{2r-1}} \geq \frac{\varepsilon}{4r} (t - 2r + 1)$$  

and hence

$$S_t \leq \frac{1}{\frac{1}{S_{2r-1}} + \frac{\varepsilon}{4r} (t - 2r + 1)} = \frac{S_{2r-1}}{1 + \frac{\varepsilon}{4r} (t - 2r + 1)}.$$  

Since $H$ is a decreasing function, $S_t \geq 2r \cdot H(t)$, while $2r \cdot H(2r - 1) \leq S_{2r-1} \leq 2r \cdot H(0) = 2r$. Thus,

$$H(t) \leq \frac{F(x^o)}{1 + \frac{\varepsilon}{4r} (2r \cdot H(2r - 1)) \cdot (t - 2r + 1)} = \frac{F(x^o)}{1 + \frac{\varepsilon}{4r} (2r \cdot H(2r - 1)) \cdot (t - 2r + 1)}.$$  

It is straightforward to check that taking expectations leaves the proof unchanged. □

A.5 Handling No Common Write

Recall that at the beginning of the $t$-th update, $x_j^o := x_j^{t-q-1}$ is already fixed when a core chooses $k_t$. However, if there are some updates to coordinate $j$ over the time interval $[t-q, t-1]$, the value of $x_j$ will be modified during this time interval. A subtle observation is that how $x_j$ is modified might depend on the choice of $k_t$ (and also of $k_{t+1}, k_{t+2}, \ldots, k_{t+q-1}$). More concretely, depending on the choices of the coordinates in times $[t, t + q - 1]$, and depending on various unpredictable asynchronous effects, the value of $x_j^{t-1}$ might not be the same. This is in contrast to the classical stochastic and synchronous case, where $x_j^{t-1}$ is already fixed by the history when bounding the relevant conditional expectation.

Let $x_j^{\pi(j), t-1}$ denote the value of $x_j^{t-1}$ when $j$ is chosen to be coordinate $k_t$. To avoid algebraic clutter, we write $x_j^{\pi(j), t-1} \equiv x_j^{\pi(j), t-1}$ in this subsection.

To remedy the situation, for each fixed $k \in [n]$, and for all $j \in [n]$, we “shift” the progress made when $j$ is chosen to be $k_t$, which is of the form $\tilde{W}_j(\tilde{g}, x_j^{\pi(j), t-1})$, to a progress term of the form $\tilde{W}_j(\nabla_j f(x_\pi(k), t-1), x_j^{\pi(k), t-1})$. We accomplish this by using Lemmas 6 and 7.

Suppose there are $\ell \geq 1$ updates to coordinate $j$ over the time interval $[t-q, t-1]$, and suppose the latest update to coordinate $j$ occurred at time $\hat{t}$. Suppose that

- the changes to $x_j$ from $x_j^0$ to $x_j^{\pi(j), t-1}$ are $d_{11}, d_{12}, \ldots, d_{1\ell}$;
- the changes to $x_j$ from $x_j^0$ to $x_j^{\pi(k), t-1}$ are $d_{21}, d_{22}, \ldots, d_{2\ell}$; furthermore, let

$$g_j^0 := \nabla_j f(x_\pi(k), t-1) \quad \text{and} \quad \hat{d} := \arg \max_d W(d, g_j^0, x_j^{\pi(k), t-1}, \Gamma).$$
In other words, \( x_j^{\pi(j),t-1} = x_j^\circ + \sum_{r=1}^\ell d_{1r} \) and \( x_j^{\pi(k),t-1} = x_j^\circ + \sum_{r=1}^\ell d_{2r} \).

By Lemma 6 we have
\[
\hat{W}(g_j^a, x_j^{\pi(j),t-1}) + \Psi(x_j^{\pi(k),t-1}) - \Psi(x_j^{\pi(j),t-1}) \\
\geq \hat{W}(g_j^a, x_j^{\pi(k),t-1}) - g_j^a \cdot (x_j^{\pi(k),t-1} - x_j^{\pi(j),t-1}) \\
- \Gamma \bar{d} \cdot (x_j^{\pi(k),t-1} - x_j^{\pi(j),t-1}) - \frac{\Gamma}{2} \cdot (x_j^{\pi(k),t-1} - x_j^{\pi(j),t-1})^2.
\]

On the other hand, let \( g_j^b \) be the gradient used to compute the update \( d_{2\ell} \). By Lemma 7 on setting \( x_2 = x_j^{\pi(k),t-1} - d_{2\ell} \) and \( x_1 = x_j^{\pi(j),t-1} - d_{1\ell} \), we have
\[
\Psi(x_j^{\pi(k),t-1}) - \Psi(x_j^{\pi(j),t-1}) \leq g_j^b (x_j^{\pi(j),t-1} - x_j^{\pi(k),t-1}) + \frac{\Gamma}{2} (x_j^{\pi(j),t-1} - x_j^{\pi(k),t-1} + d_{2\ell})^2.
\]
Combining the above two inequalities, and letting \( \delta := x_j^{\pi(k),t-1} - x_j^{\pi(j),t-1} \), yields
\[
\hat{W}(g_j^a, x_j^{\pi(j),t-1}) \\
\geq \hat{W}(g_j^a, x_j^{\pi(k),t-1}) - \frac{3}{2} \cdot \Gamma \delta^2 - \frac{3}{2} \cdot (d_{2\ell} - \delta)^2
\]

By Lemma 10
\[
\frac{\Gamma}{2} \cdot (\bar{d} - d_{2\ell})^2 \leq \Gamma \cdot (d_{2\ell})^2 + \frac{1}{\Gamma} \cdot (g_j^a - g_j^b)^2.
\]

Thus, we have
\[
\hat{W}(g_j^a, x_j^{\pi(j),t-1}) \geq \hat{W}(g_j^a, x_j^{\pi(k),t-1}) - \frac{3}{2\Gamma} \cdot (g_j^b - g_j^a)^2
\]
(12)

The key message from the above inequality is: \( \hat{W}(g_j^a, x_j^{\pi(j),t-1}) \) is as large as \( \hat{W}(g_j^a, x_j^{\pi(k),t-1}) \), modulo a few “error terms”. We might hope to balance these error terms by part of the \( \Gamma(\Delta x_j)^2 \) terms. First, we can bound \( \delta^2 \) as follows:
\[
\delta^2 \leq \sum_{1 \leq i \leq \ell} \left( \Delta_{\max} x_j^{\pi(k),t_i} - \Delta_{\min} x_j^{\pi(k),t_i} \right)^2 + \left( \Delta_{\max} x_j^{\pi(j),t_i} - \Delta_{\min} x_j^{\pi(j),t_i} \right)^2
\]
\[
\leq 2q \sum_{1 \leq i \leq \ell} \left( \Delta_{\max} x_j^{\pi(k),t_i} - \Delta_{\min} x_j^{\pi(k),t_i} \right)^2 + \left( \Delta_{\max} x_j^{\pi(j),t_i} - \Delta_{\min} x_j^{\pi(j),t_i} \right)^2,
\]
where \( t_1 < t_2 < \ldots < t_\ell \) are the times of the updates to \( x_j \) in time interval \([t-q,t-1]\). Note that the terms in the summation on the RHS above are similar to the terms on the LHS of (14).

The final step is to use Lemma 8 to “shift” the gradient in the \( \hat{W} \) term that we actually have, namely \( \hat{W}(\bar{g}, x_j^{\pi(j),t-1}) \), to \( \hat{W}(g_j^a, x_j^{\pi(j),t-1}) \). This will be done within the analyses of SACD.
B Full SACD Analysis

We use the starting time of updates as reference points, and thus future updates in this ordering might interfere with the current update. However, in any standard stochastic analysis, the progress is analyzed conditioning on the current information. Our high-level approach to achieve this is: with the current information in hand, give a worst-case estimate on how, in expectation, the future updates can interfere with the current update. While the above high-level approach seems natural, its implementation is quite non-trivial. We think it is plausible that our approach may also be effective in analyzing other asynchronous stochastic iterative systems.

Suppose there are a total of $T$ updates. We view the whole stochastic process as a branching tree of height $T$. Each node in the tree corresponds to the moment when some core randomly picks a coordinate to update, and each edge corresponds to a possible choice of coordinate. We use $\pi$ to denote a path from the root down to some node of the tree.

At this point, it is helpful to clarify the concept of a history. Suppose $\pi$ is a path of length $t$, and let $\mathcal{N}$ be $\pi$’s final node. What had really happened before $\mathcal{N}$, or in other words, what is the history before $\mathcal{N}$? Our timing scheme and Assumption 1 ensure that all updates starting strictly before time $t - q$ have committed before $\mathcal{N}$, and thus all information about such updates belongs to the history. Also, the coordinates $k_s$ for $s \in [t - q, t - 1]$ were already chosen, so their identities belong to the history; however, some or all of their updated values might not yet belong to the history.

The main novelty in our analysis is to achieve a bound on the difference between the computed gradient and the “up-to-date” gradient. The resulting bound is given in Equation (14) below. To fully understand this bound, new notation is needed, as defined below. We then use this bound to obtain the desired amortized progress, but as this is more standard, albeit non-trivial, we defer this part of the analysis to Appendix B.

Gradient Error Bounds

To analyze the update to $x^t_{k_t}$, we begin by fixing a path $\pi$ from the root to a leaf; to indicate its dependence of $\pi$, we write the coordinate as $x^\pi_{k_t}$; likewise, we write $g^\pi_{k_t}$ for the gradient component. We will seek to bound the range of values for the update to $x^\pi_{k_t}$. This range is maximized if $x^\pi_{k_t}$ is the last coordinate to commit among those updates that start in the time range $[t - q, t + q]$. This is reflected in the following notation.

$$\Delta^t_{\text{max}} x^\pi_{k_s, R} := \text{the maximum value that } \Delta x^\pi_{k_s, R} \text{ can assume when the first } (t - q - 1) \text{ updates on path } \pi \text{ have been fixed, the update does not read any of the updates at times in } R, \text{ nor any of the variable values updated at times } v > t + q.$$  

In our proof, $R$ is either a singleton set or the empty set $\emptyset$.

Let $\Delta^t_{\text{min}} x^\pi_{k_s, R}$ denote the analogous minimum value.

Note that for $u > t$, $\Delta^u_{\text{max}} x^\pi_{k_t} \leq \Delta^t_{\text{max}} x^\pi_{k_t}$ and $\Delta^u_{\text{min}} x^\pi_{k_t} \geq \Delta^t_{\text{min}} x^\pi_{k_t}$, since the update does not read any of the variable values updated at times $v > t + q$. Also, $\Delta^u_{\text{max}} x^\pi_{k_t} \leq \Delta^u_{\text{max}} x^\pi_{k_t, \emptyset}$ and $\Delta^u_{\text{min}} x^\pi_{k_t} \geq \Delta^u_{\text{min}} x^\pi_{k_t, \emptyset}$, for all $R$. Let

$$\Sigma^t_{\text{max}} x^\pi_{k_t} := \max_{t - q \leq u \leq t} \Delta^u_{\text{max}} x^\pi_{k_t, \emptyset} \text{ and } \Sigma^t_{\text{min}} x^\pi_{k_t} := \min_{t - q \leq u \leq t} \Delta^u_{\text{min}} x^\pi_{k_t, \emptyset}.$$
Let $g_{\Delta_{\max}, k_t}$ (resp. $g_{\Delta_{\min}, k_t}$) denote the value of $g_{k_t}^{\pi, t}$ used to evaluate $\Delta_{\max} x_{k_t}^{\pi, t}$ (resp. $\Delta_{\min} x_{k_t}^{\pi, t}$).

Let $g_{\Delta_{\max}, k_t}$ (resp. $g_{\Delta_{\min}, k_t}$) denote the maximum (resp. minimum) value of $g_{k_t}^{\pi, t}$ when the update for $x_{k_t}^{\pi, u}$ is the last to commit among updates with start time in $[u - q, u + q]$ (resp. $\Delta_{\min} x_{k_t}^{\pi, t}$).

Let $g_{\Delta_{\max}, k_t} = \max_{t - q \leq u \leq t} g_{\Delta_{\max}, k_t}$, $g_{\Delta_{\min}, k_t} = \min_{t - q \leq u \leq t} g_{\Delta_{\min}, k_t}$.

For simplicity, we first give an analysis for smooth functions. The key recursive bound on the gradient error follows.

**Lemma 14.** For smooth functions,

$$
E \left[ (\Delta_{\max} x_{k_t}^{\pi, t} - \Delta_{\min} x_{k_t}^{\pi, t})^2 \right] \leq \frac{12q L^2}{n1^2} \sum_{s \in \{t - 2q, t + q\} \setminus \{t\}} \left[ (\Delta_{\max} x_{k_s}^{\pi, s} - \Delta_{\min} x_{k_s}^{\pi, s})^2 + (\Delta x_{k_s}^{\pi, s})^2 \right].
$$

(14)

In Appendix B.3 we unwind recursive inequality (14) to bound $E_{\pi} \left[ \frac{1}{2} (g_{k_t}^{\pi, t} - \tilde{g}_{k_t}^{\pi, t})^2 \right]$ in terms of $E_{\pi} \left[ (\Delta x_{k_s}^{\pi, s})^2 \right]$. Following this, the remaining task is to design a potential function to demonstrate progress.

**Proof:** First, we review which updates can create differences in the values of $g_{\Delta_{\max}, k_t}$ and $g_{\Delta_{\min}, k_t}$. The computation of these gradients may differ due to reading different values for $x_{k_s}^{\pi, s}$, or reading an older value of the coordinate, for some or all of $t - 2q \leq s \leq t + q$, and only for this range of $s$; for all relevant $u$, the first $(u - q - 1)$ updates are already fixed, so the first $(t - q) - (u - q) - 1 = t - 2q - 1$ updates are fixed; also, an update to $x_{k_t}^{\pi, t}$ will only consider updates up to time $t + q$. Further, for each $s$, the change to the previous value of the variable due to the update yielding $x_{k_s}^{\pi, s}$ will lie in the range $[\Delta_{\min} x_{k_s}^{\pi, s, \{t\}}, \Delta_{\max} x_{k_s}^{\pi, s, \{t\}}]$ for a suitable span of $u$ values, specified next. First, we are concerned only with $u \in [t - 2q, t + q]$ because this is the range of $u$ for $\Delta_{\max} x_{k_s}^{\pi, t}$ and $\Delta_{\min} x_{k_s}^{\pi, t}$. Second, as for $s < u$, $\Delta_{\max} x_{k_s}^{\pi, s, \{t\}} \leq \Delta_{\max} x_{k_s}^{\pi, s, \{t\}}$, and similarly $\Delta_{\min} x_{k_s}^{\pi, s, \{t\}} \geq \Delta_{\max} x_{k_s}^{\pi, s, \{t\}}$, we can safely assume that $u \leq s$. Finally, as no variables updated at times $v > u + q$ are read here, we have that $s \leq u + q$, or $s - q \leq u$. So the range for $u$ is $T_{s,t} := \max\{s - q, t - q\}, \min\{s, t\}$.

Then by Lemma 10 in Appendix A (to show the first inequality), and by a use of the Cauchy-Schwarz inequality (for the fourth inequality),

$$
\left( \Delta_{\max} x_{k_t}^{\pi, t} - \Delta_{\min} x_{k_t}^{\pi, t} \right)^2 \leq \frac{2}{1^2} \left( g_{\Delta_{\max}, k_t}^{\pi, t} - g_{\Delta_{\min}, k_t}^{\pi, t} \right)^2 \leq \frac{2}{1^2} \left( g_{\Delta_{\max}, k_t}^{\pi, t} - g_{\Delta_{\min}, k_t}^{\pi, t} \right)^2 \leq \frac{6q L^2}{1^2} \sum_{s \in \{t - 2q, t + q\} \setminus \{t\}} \left[ \max_{u \in T_{s,t}} \Delta_{\max} x_{k_s}^{\pi, s, \{t\}} - \min_{u \in T_{s,t}} \Delta_{\min} x_{k_s}^{\pi, s, \{t\}} \right]^2
$$

(15)

$$
\leq \frac{6q L^2}{1^2} \sum_{s \in \{t - 2q, t + q\} \setminus \{t\}} L_{k_s, k_t} \cdot \max \left\{ \left[ \max_{u \in T_{s,t}} \Delta_{\max} x_{k_s}^{\pi, s, \{t\}} - \min_{u \in T_{s,t}} \Delta_{\min} x_{k_s}^{\pi, s, \{t\}} \right]^2, \left( \max_{u \in T_{s,t}} \Delta_{\max} x_{k_s}^{\pi, s, \{t\}} \right)^2, \left( \min_{u \in T_{s,t}} \Delta_{\min} x_{k_s}^{\pi, s, \{t\}} \right)^2 \right\}.
$$

(16)
Now, we average over all \( n \) choices of \( k_t \). Using the definition of \( L_{\text{res}} \) yields

\[
E \left[ \left( \bar{\Delta}_{\max}^{x,\pi,t} - \bar{\Delta}_{\min}^{x,\pi,t} \right)^2 \right]
\leq \frac{6q}{n 1^2} \sum_{s \in [t-2q,t+q] \setminus \{t\}} 1^2 \cdot \max \left\{ \left( \max_{u \in T_s,t} \Delta_{\max}^{u,s} - \min_{u \in T_s,t} \Delta_{\min}^{u,s} \right)^2, \left( \max_{u \in T_s,t} \Delta_{\max}^{u,s} \right)^2, \left( \min_{u \in T_s,t} \Delta_{\min}^{u,s} \right)^2 \right\}. \tag{17}
\]

This averaging is legitimate because on the RHS the paths \( \pi \) being considered in the averaging all have the same values for \( \Delta_{\max}^{\pi,s,t} \) and for \( \Delta_{\min}^{\pi,s,t} \) as their computation does not involve the update to \( x_t^{k_t} \), and because at most the first \((t-q-1)\) updates have been fixed in any of these terms, none of the updates that could affect the update to \( x_t^{k_t} \) have been fixed.

The summation in the RHS of (17) is bounded by

\[
\frac{6q 1^2}{n 1^2} \sum_{s \in [t-2q,t+q] \setminus \{t\}} \max \left\{ \left( \bar{\Delta}_{\max}^{\pi,s} - \bar{\Delta}_{\min}^{\pi,s} \right)^2, \left( \bar{\Delta}_{\max}^{\pi,s} \right)^2, \left( \bar{\Delta}_{\min}^{\pi,s} \right)^2 \right\}.
\]

\( \bar{\Delta}_{\pi,s} \in [\bar{\Delta}_{\min}^{\pi,s}, \bar{\Delta}_{\max}^{\pi,s}] \); thus

\[
|\bar{\Delta}_{\min}^{\pi,s}|, |\bar{\Delta}_{\max}^{\pi,s}| \leq |\Delta_{\max}^{\pi,s}| + (\bar{\Delta}_{\max}^{\pi,s} - \bar{\Delta}_{\min}^{\pi,s}).
\]

It follows that

\[
\left( \bar{\Delta}_{\min}^{\pi,s} \right)^2, \left( \bar{\Delta}_{\max}^{\pi,s} \right)^2 \leq 2 \left( \Delta_{\max}^{\pi,s} \right)^2 + 2 \left( \bar{\Delta}_{\max}^{\pi,s} - \bar{\Delta}_{\min}^{\pi,s} \right)^2.
\]

The result follows. \( \square \)
B.1 Analyzing Non-Smooth Functions

We need to rederive the bound in Lemma 14 to take account of non-trivial $\Psi$ terms. This creates difficulties if there are updates to the same coordinate whose start and commit time orderings differ. However, to fix this, for each path $\pi$, we simply rearrange the update ordering for each coordinate separately, so that they are in commit order, while collectively occupying the same places in the start ordering. e.g. if coordinate $x_1$ has updates $x_1^2, x_1^8, x_1^{11}$ that start at times 2, 8, and 11, respectively, but they finish at times 9, 18 and 12, in the new ordering $x_1^2$ will be in position 2, $x_1^8$ will be in position 8, and $x_1^{11}$ will be in position 11. We call this the Single Coordinate Consistent Ordering, the SCC ordering for short. It maintains the property that each coordinate has an equal probability of occurring at each time.

This does affect the range of coordinates that might be read by an update at time $t$; it is now $[t - 2q, t + q]$ — the upside does not increase, for if a coordinate has moved earlier to time $t$, it must commit before the coordinate that had moved later from time $t$, and by assumption the latter coordinate was no later than the $t + q$-th coordinate to commit.

By Lemma 13 we have

$$\left( \Delta_{\max}^{\pi,t} x_{k\ell} - \Delta_{\min}^{\pi,t} x_{k\ell} \right)^2 \leq 2 \left( \sum_{s \in [t - 3q, t - 1] \setminus \{t\}, k_s = k_t} \Delta_{\max}^{T_{\pi,s},t} x_{k_s} - \Delta_{\min}^{T_{\pi,s},t} x_{k_s} \right)^2 \leq 6q \left( \sum_{s \in [t - 3q, t - 1] \setminus \{t\}, k_s = k_t} \left( \Delta_{\max}^{\pi,s,t} x_{k_s} - \Delta_{\min}^{\pi,s,t} x_{k_s} \right)^2 \right)$$

$$+ \frac{2}{G^2} \left[ \sum_{s \in [t - 3q, t + q] \setminus \{t\}} L_{k_s, k_t} \cdot \max \left\{ \left( \max_{u \in T_{s,t}} \Delta_{\max}^{u,s,t} x_{k_s} - \min_{u \in T_{s,t}} \Delta_{\min}^{u,s,t} x_{k_s} \right), \left( \max_{u \in T_{s,t}} \Delta_{\max}^{u,s,t} x_{k_s} - \min_{u \in T_{s,t}} \Delta_{\min}^{u,s,t} x_{k_s} \right) \right\} \right]^2$$

$$\leq 6q \left( \sum_{s \in [t - 3q, t - 1] \setminus \{t\}, k_s = k_t} \left( \Delta_{\max}^{T_{\pi,s},t} x_{k_s} - \Delta_{\min}^{T_{\pi,s},t} x_{k_s} \right)^2 \right) + \frac{8q}{G^2} \sum_{s \in [t - 3q, t + q] \setminus \{t\}} L_{k_s, k_t} \cdot \max \left\{ \left( \max_{u \in T_{s,t}} \Delta_{\max}^{u,s,t} x_{k_s} - \min_{u \in T_{s,t}} \Delta_{\min}^{u,s,t} x_{k_s} \right)^2 \right\} \cdot \left( \min_{u \in T_{s,t}} \Delta_{\min}^{u,s,t} x_{k_s} \right)^2$$

Now we average over all $n$ choices of $k_t$. We obtain
\[
\mathbb{E} \left[ \left( \Delta_{\max}^{\pi,t} - \Delta_{\min}^{\pi,t} \right)^2 \right] \\
\leq \frac{6q}{n} \sum_{s\in[t-3q,t-1]\setminus\{t\}} \left( \Delta_{\max}^{\pi,s,t} - \Delta_{\min}^{\pi,s,t} \right)^2 \\
+ \frac{8q}{n \Gamma^2} \sum_{s\in[t-3q,t+q]\setminus\{t\}} L^2_{\text{res}} \cdot \max \left\{ \left( \max_{u \in T_{s,t}} \Delta_{\max}^{u,s,t} - \min_{u \in T_{s,t}} \Delta_{\min}^{u,s,t} \right)^2 , \left( \max_{u \in T_{s,t}} \Delta_{\max}^{u,s,t} \right)^2 , \left( \min_{u \in T_{s,t}} \Delta_{\min}^{u,s,t} \right)^2 \right\} .
\] 

(20)

Following the previous argument, we obtain

\[
\mathbb{E} \left[ \left( \Delta_{\max}^{\pi,t} - \Delta_{\min}^{\pi,t} \right)^2 \right] \\
\leq \frac{6q}{n} \sum_{s\in[t-3q,t-1]\setminus\{t\}} \left( \Delta_{\max}^{\pi,s,t} - \Delta_{\min}^{\pi,s,t} \right)^2 \\
+ \frac{16q L^2_{\text{res}}}{n \Gamma^2} \sum_{s\in[t-3q,t+q]\setminus\{t\}} \left[ \left( \Delta_{\max}^{\pi,s,t} - \Delta_{\min}^{\pi,s,t} \right)^2 + \left( \Delta_{\max}^{\pi,s,t} \right)^2 \right] .
\] 

(21)

Now, we extend the expectation to all paths \( \pi \). We let \( (\Delta_t^{\text{FE}})^2 := \mathbb{E}_{\pi} \left[ \left( \Delta_{\max}^{\pi,t} - \Delta_{\min}^{\pi,t} \right)^2 \right] \) denote the resulting expectation for level \( t \). Also, we let \( (E_s^x)^2 := \mathbb{E}_{\pi} \left[ \left( \Delta_{\max}^{\pi,s} \right)^2 \right] \).

Let \( \nu_1 := \frac{6q^2}{n} \) and \( \nu_2 = \frac{16q L^2_{\text{res}}}{n \Gamma^2} \). We have

\[
\Gamma \cdot (\Delta_t^{\text{FE}})^2 \leq \left( \frac{\nu_1}{q} + \frac{\nu_2}{q} \right) \sum_{s\in[t-3q,t+q]\setminus\{t\}} \Gamma \cdot (\Delta_s^{\text{FE}})^2 + \frac{\nu_2}{q} \sum_{s\in[t-3q,t+q]\setminus\{t\}} \Gamma \cdot (E_s^x)^2
\] 

(22)

We have also shown:

\[
\frac{2}{\Gamma} \left( g_{\max,k_t}^{\pi,t} - g_{\min,k_t}^{\pi,t} \right)^2 \leq \frac{\nu_2}{q} \sum_{s\in[t-3q,t+q]\setminus\{t\}} \Gamma \cdot \left[ (\Delta_s^{\text{FE}})^2 + (E_s^x)^2 \right]
\] 

(23)
B.2 Gradient Bounds

We will need to bound \( (g_{k,t}^{\pi,t} - \bar{g}_{k,t})^2 \). Unfortunately, \( g_{k,t}^{\pi,t} \) might not be in \([g_{\min,k,t}^{\pi,t}, g_{\max,k,t}^{\pi,t}]\). To handle this difficulty we introduce \( g_{k,t}^{S,\pi,t} \), the gradient of the point if the updates from time \( t-q \) to \( t \) were synchronous. Clearly,

\[
(g_{k,t}^{\pi,t} - g_{k,t}^{S,\pi,t})^2 \leq 2 \left( g_{k,t}^{\pi,t} - g_{k,t}^{S,\pi,t} \right)^2 + 2 \left( g_{k,t}^{S,\pi,t} - \bar{g}_{k,t} \right)^2. 
\] \( (24) \)

Also, it is clear that \( (g_{k,t}^{S,\pi,t} - \bar{g}_{k,t})^2 \) is smaller than \( (\pi,t, g_{\min,k,t}^{\pi,t}, g_{\max,k,t}^{\pi,t})^2 \).

We upper bound \( (g_{k,t}^{\pi,t} - g_{k,t}^{S,\pi,t})^2 \) by \( \sum_{l_0 \in [t-2q,t-1]} L_{k_{l_0},k_t}^2 \left( \Delta_{\min}^{\pi,t} x_{l_0,k_0} - \Delta_{\min}^{t} x_{l_0,k_0} \right)^2 \). In order to bound this term, we first show the following bound.

**Lemma 15.**

\[
\mathbb{E} \left[ \sum_{l_0 \in [t-2q,t-1]} \left( \sum_{l_s \in R_{m-1} \setminus S_{m-1}} \left( \prod_{l_s \in R_{m-1} \setminus S_{m-1}} \frac{L_{k_{l_s},k_{l_s+1}}^2}{\Gamma^2} \right) L_{k_{l_0},k_t}^2 \right) \right]
\]

\[
\leq \mathbb{E} \left[ 4 \cdot 2q \sum_{l_0 \in [t-2q,t-1]} \left( \sum_{l_s \in R_{m-1} \setminus S_{m-1}} \left( \prod_{l_s \in R_{m-1} \setminus S_{m-1}} \frac{L_{k_{l_s},k_{l_s+1}}^2}{\Gamma^2} \right) L_{k_{l_0},k_t}^2 \right) \right]
\]

\[
+ \mathbb{E} \left[ 4 \cdot 2q \sum_{l_0 \in [t-2q,t-1]} \left( \sum_{l_s \in R_{m-1} \setminus S_{m-1}} \left( \prod_{l_s \in R_{m-1} \setminus S_{m-1}} \frac{L_{k_{l_s},k_{l_s+1}}^2}{\Gamma^2} \right) L_{k_{l_0},k_t}^2 \right) \right]
\]

\[
+ \mathbb{E} \left[ \sum_{s \in [t-4q,t+q] \setminus \{t\}} 4(3q)2\Gamma^2 \max \left\{ \frac{t^2}{t^2}, 1 \right\}^{m+1} (3q)^{m-1}(2q) \left( \Delta_{\max}^{t} x_{s,R_{m-1}} - \Delta_{\min}^{t} x_{s,R_{m-1}} \right)^2 \right]
\]

\[
+ \mathbb{E} \left[ 112 L_{\max}^2 \max \left\{ \frac{t^2}{t^2}, 1 \right\}^m (3q)^{m-1}(2q) \left( \Delta_{\max}^{t} x_{t,R_{m-1}} - \Delta_{\min}^{t} x_{t,R_{m-1}} \right)^2 \right]
\]

\[
+ \mathbb{E} \left[ 64 L_{\max}^2 \max \left\{ \frac{t^2}{t^2}, 1 \right\}^m (3q)^{m-1}(2q) \left( \Delta_{t}^{t} x_{t} \right)^2 \right].
\]
Proof. Let’s first expand the term \((\Delta_{\text{max}}^t x_{k_{l_m-1} - 1} R_{l_m - 1} \setminus \{l_m - 1\}) - \Delta_{\text{min}}^t x_{k_{l_m-1} - 1} R_{l_m - 1} \setminus \{l_m - 1\})^2\) for \(l_m \in [t - 2q, t + q]\). Similarly to the previous analysis,

\[
\left( \Delta_{\text{max}}^t x_{k_{l_m-1} - 1} R_{l_m - 1} \setminus \{l_m - 1\} - \Delta_{\text{min}}^t x_{k_{l_m-1} - 1} R_{l_m - 1} \setminus \{l_m - 1\} \right)^2
\]

\[
\leq 2 \left[ \sum_{l_m \in \{\text{max}(l_m - 1 - 2q, t - 2q), l_m - 1\} \setminus \{t\}, k_{l_m} = k_{l_m - 1}} \left( \Delta_{\text{max}}^t x_{k_{l_m}} R_{l_m - 1} - \Delta_{\text{min}}^t x_{k_{l_m}} R_{l_m - 1} \right) \right]^2
\]

\[
+ \frac{2}{\Gamma^2} \left( \sum_{l_m \in \{\text{max}(l_m - 1 - 2q, t - 2q), l_m - 1\} \setminus \{t\}, k_{l_m} = k_{l_m - 1}} L_{k_{l_m}, k_{l_m - 1}} \max \left\{ \left| \Delta_{\text{max}}^t x_{k_{l_m}} R_{l_m - 1} - \Delta_{\text{min}}^t x_{k_{l_m}} R_{l_m - 1} \right|\right\}^2
\]

By the Cauchy-Schwarz inequality,

\[
2 \left( \sum_{l_m \in \{\text{max}(l_m - 1 - 2q, t - 2q), l_m - 1\} \setminus \{t\}, k_{l_m} = k_{l_m - 1}} \left( \Delta_{\text{max}}^t x_{k_{l_m}} R_{l_m - 1} - \Delta_{\text{min}}^t x_{k_{l_m}} R_{l_m - 1} \right) \right)^2
\]

\[
\leq 4 \cdot \left( \sum_{l_m \in \{\text{max}(l_m - 1 - 2q, t - 2q), l_m - 1\} \setminus \{t\}, k_{l_m} = k_{l_m - 1}} \left( \Delta_{\text{max}}^t x_{k_{l_m}} R_{l_m - 1} - \Delta_{\text{min}}^t x_{k_{l_m}} R_{l_m - 1} \right) \right)^2
\]

\[
+ 4 \cdot 1_{k_{l_m - 1} = k_{l_m}} \left( \Delta_{\text{max}}^t x_{k_{l_m}} R_{l_m - 1} - \Delta_{\text{min}}^t x_{k_{l_m}} R_{l_m - 1} \right)^2
\]

Also, by the Cauchy-Schwarz inequality,

\[
\frac{2}{\Gamma^2} \left( \sum_{l_m \in \{\text{max}(l_m - 1 - 2q, t - 2q), l_m - 1\} \setminus \{t\}, k_{l_m} = k_{l_m - 1}} \left( \Delta_{\text{max}}^t x_{k_{l_m}} R_{l_m - 1} - \Delta_{\text{min}}^t x_{k_{l_m}} R_{l_m - 1} \right) \right)^2
\]

\[
\leq 4(3q) \frac{2}{\Gamma^2} \left( \sum_{l_m \in \{\text{max}(l_m - 1 - 2q, t - 2q), l_m - 1\} \setminus \{t\}, k_{l_m} = k_{l_m - 1}} L_{k_{l_m}, k_{l_m - 1}}^2 \max \left\{ \left| \Delta_{\text{max}}^t x_{k_{l_m}} R_{l_m - 1} - \Delta_{\text{min}}^t x_{k_{l_m}} R_{l_m - 1} \right|\right\}^2
\]

\[
+ \frac{4}{\Gamma^2} L_{k_{l_m - 1}, k_{l_m - 1}}^2 \max \left\{ \left| \Delta_{\text{max}}^t x_{k_{l_m}} R_{l_m - 1} - \Delta_{\text{min}}^t x_{k_{l_m}} R_{l_m - 1} \right|\right\}^2.
\]
We know that for any \( s \leq t + q \),
\[
\max \left\{ \left( \Delta_{\max_{k_s}}^{t \pi,s,R_{m-1}} - \Delta_{\min_{k_s}}^{t \pi,s,R_{m-1}} \right), \Delta_{\max_{k_s}}^{t \pi,s,R_{m-1}}, \Delta_{\min_{k_s}}^{t \pi,s,R_{m-1}} \right\}^2 \leq 2 \left( \Delta_{\max_{k_s}}^{t \pi,s,R_{m-1}} - \Delta_{\min_{k_s}}^{t \pi,s,R_{m-1}} \right)^2 + 2 \left( \Delta_{\max_{k_s}}^{t \pi,s,R_{m-1} \cup \{t\}} \right)^2.
\]
Additionally, for any \( s < t - 2q \),
\[
\max \left\{ \left( \Delta_{\max_{k_s}}^{t \pi,s,R_{m-1}} - \Delta_{\min_{k_s}}^{t \pi,s,R_{m-1}} \right), \Delta_{\max_{k_s}}^{t \pi,s,R_{m-1}}, \Delta_{\min_{k_s}}^{t \pi,s,R_{m-1}} \right\}^2 = \left( \Delta_{\max_{k_s}}^{t \pi,s,R_{m-1} \cup \{t\}} \right)^2.
\]
So, in summary,
\[
\left( \Delta_{\max_{k_{l_{m-1}}}}^{t \pi,l_{m-1},R_{m-1} \setminus \{l_{m-1}\}} - \Delta_{\min_{k_{l_{m-1}}}}^{t \pi,l_{m-1},R_{m-1} \setminus \{l_{m-1}\}} \right)^2 \leq 4 \cdot (2q) \sum_{l_{m} \in [t-2q,t+q] \setminus \{t\}} \left( \Delta_{\max_{x_{k_{l_{m}}}}}^{t \pi,l,R_{m-1}} - \Delta_{\min_{x_{k_{l_{m}}}}}^{t \pi,l,R_{m-1}} \right)^2 + 4 \cdot \mathbf{1}_{k_{l_{m-1}} = k_{t}} \left( \Delta_{\max_{x_{k_{t}}}}^{t \pi,t,R_{m-1}} - \Delta_{\min_{x_{k_{t}}}}^{t \pi,t,R_{m-1}} \right)^2 + 4 \cdot (3q) \frac{2}{T^2} \sum_{l_{m} \in [t-2q,t+q] \setminus \{t\}} \sum_{l_{m-1} = t} \left( \Delta_{\max_{x_{k_{l_{m-1}}}}}^{t \pi,l,R_{m-1}} - \Delta_{\min_{x_{k_{l_{m-1}}}}}^{t \pi,l,R_{m-1}} \right)^2 + 4 \cdot (3q) \frac{2}{T^2} \sum_{s \in [t-2q,t+q] \setminus \{t\}} \sum_{l_{m-1} = s} \left( \Delta_{\max_{x_{k_{l_{m-1}}}}}^{t \pi,l,R_{m-1} \cup \{t\}} \right)^2 + 4 \cdot \frac{4}{T^2} \sum_{l_{m-1} = s} \left( \Delta_{\max_{x_{k_{l_{m-1}}}}}^{t \pi,l,R_{m-1}} - \Delta_{\min_{x_{k_{l_{m-1}}}}}^{t \pi,l,R_{m-1}} \right)^2 + 2 \left( \Delta_{\max_{x_{k_{t}}}}^{t \pi,t,R_{m-1} \cup \{t\}} \right)^2.
\]
Next we want to multiply this expression by a series of terms \( \frac{L_{k_{i_0},k_{i_1}}^2}{\Gamma^2} \) on the both sides and sum up over all choices of \( l_1, l_2, \cdots, l_{m-1} \in [t-2q,t+q] \). Note that
\[
\mathbb{E} \left[ \sum_{l_0 \in [t-2q,t-1]} \left( \sum_{l_1 \in [t-2q,t+q]} \cdots \sum_{l_{m-1} \in [t-2q,t+q]} \left( \prod_{l_s \in R_{m-1} \setminus S_{m-1}} \frac{L_{k_{l_s},k_{l_{s-1}}}}{\Gamma^2} \right) \right) \right]
\]
for any \( l_1, l_2, \cdots, l_{m-1} \in [t-2q,t+q] \) are distinct and not equal to \( t, l \) and \( R_{m-1} = \{l_1, l_2, \cdots, l_{m-1}\} \), let \( S_{m-1} = \{l_i | l_{i_{l_0}} = k_{l_{i_{l_1}} - 1}\} \) for any \( S_{m-1} = \{l_i | l_{i_{l_0}} = k_{l_{i_{l_1}}} - 1\} \).\n
\[
4 \cdot \mathbf{1}_{k_{l_{m-1}} = k_{t}} \left( \Delta_{\max_{x_{k_{l_{m-1}}}}}^{t \pi,l,R_{m-1}} - \Delta_{\min_{x_{k_{l_{m-1}}}}}^{t \pi,l,R_{m-1}} \right)^2 \leq \mathbb{E} \left[ 4 \cdot 4 L_{\max}^2 \left( \frac{t^{2m} \Gamma^2}{1}, 1 \right)^{m-1} (3q)^{m-1} \frac{2}{n^m} \left( \Delta_{\max_{x_{k_{t}}}}^{t \pi,t,R_{m-1}} - \Delta_{\min_{x_{k_{t}}}}^{t \pi,t,R_{m-1}} \right)^2 \right]
\]
as \( L_{k_{i_0},k_{t}}^2 \) can be bounded by \( 4L_{\max}^2 \) and we can then average, in turn, over \( k_{i_0}, k_{l_1}, \cdots, k_{l_{m-1}} \), each of which provides a \( \frac{1}{n} \) or \( \frac{L_{\max}^2}{n^m} \) term, depending on whether \( l_i \) is in \( S_{m-1} \) or not.
Note that several other terms can be bounded similarly, as \( (\Delta^t_{\max} x_{\pi, R_{m-1} \cup \{t\}})^2 \) is fixed over all paths \( \pi \) obtained by varying \( k_t \). We obtain

\[
\mathbb{E} \left[ \sum_{l_0 \in [t-2q,t-1]} \left( \sum_{l \in R_{m-1} \setminus S_{m-1}} \left( \prod_{l_i \in R_{m-1} \setminus S_{m-1}} \frac{L^2_{k_{l_i} k_{l_{i-1}}}}{\Gamma^2} \right) L^2_{k_{l_0} k_t} \right) \right]
\]

\[
\leq \mathbb{E} \left[ 4 \cdot 2q \sum_{l_0 \in [t-2q,t-1]} \left( \sum_{l \in R_{m-1} \setminus S_{m-1}} \left( \prod_{l_i \in R_{m-1} \setminus S_{m-1}} \frac{L^2_{k_{l_i} k_{l_{i-1}}}}{\Gamma^2} \right) L^2_{k_{l_0} k_t} \right) \right]
\]

\[
+ \mathbb{E} \left[ 4 \cdot 4L^2_{\max} \max \left\{ \frac{L^2_{k_{l_0} k_{l_{-1}}}}{\Gamma^2}, 1 \right\} \frac{m-1}{n^m} (3q)^{m-1} (2q) \left( \Delta^t_{\max} x_{k_{l_0}, R_{m-1}} - \Delta^t_{\min} x_{k_{l_0}, R_{m-1}} \right)^2 \right]
\]

\[
+ \mathbb{E} \left[ (3q)^2 \sum_{l_0 \in [t-2q,t-1]} \left( \sum_{l \in R_{m-1} \setminus S_{m-1}} \left( \prod_{l_i \in R_{m-1} \setminus S_{m-1}} \frac{L^2_{k_{l_i} k_{l_{i-1}}}}{\Gamma^2} \right) L^2_{k_{l_0} k_t} \right) \right]
\]

\[
+ \mathbb{E} \left[ 4(3q)^2 \frac{m+1}{n^{m+1}} (3q)^{m-1} (2q) \left( \Delta^t_{\max} x_{k_{l_0}, R_{m-1}} - \Delta^t_{\min} x_{k_{l_0}, R_{m-1}} \right)^2 \right]
\]

\[
+ \mathbb{E} \left[ 8 \cdot 4L^2_{\max} \max \left\{ \frac{L^2_{k_{l_0} k_{l_{-1}}}}{\Gamma^2}, 1 \right\} \frac{m}{n^m} (3q)^{m-1} (2q) \left( \Delta^t_{\max} x_{k_{l_0}, R_{m-1}} - \Delta^t_{\min} x_{k_{l_0}, R_{m-1}} \right)^2 \right]
\]

\[
+ \mathbb{E} \left[ 8 \cdot 4L^2_{\max} \max \left\{ \frac{L^2_{k_{l_0} k_{l_{-1}}}}{\Gamma^2}, 1 \right\} \frac{m}{n^m} (3q)^{m-1} (2q) \left( \Delta^t_{\max} x_{k_{l_0}, R_{m-1}} \right)^2 \right]
\]

Note that \( (\Delta^t_{\max} x_{k_{l_0}, R_{m-1}})^2 \leq 2 \left( \Delta^t_{\max} x_{k_{l_0}, R_{m-1}} - \Delta^t_{\min} x_{k_{l_0}, R_{m-1}} \right)^2 + 2 \left( \Delta^t x_{k_{l_0}, t} \right)^2 \); the result now follows. \( \square \)

Note that the first two terms on the RHS of Lemma 15 will be similar to the LHS of Lemma 15.
when we increase $m$ by 1. Let

\[ V_{m-1} = \mathbb{E} \left[ \sum_{l_0 \in [t-2q,t-1]} \left( \sum_{l_s \in R_{m-1} \setminus S_{m-1}} \prod_{l_s \in R_{m-1} \setminus S_{m-1}} \left( \frac{L_{k_{l_s},k_{l_s-1}}^2}{\Gamma^2} \right) L_{k_{l_0},k_t}^2 \right) \right]. \]

Note that $V_{3q} = 0$ and $V_0 = \mathbb{E} \left[ \sum_{l_0 \in [t-2q,t-1]} L_{k_{l_0},k_t}^2 \left( \Delta t^{\max x_{k_{l_0}}} - \Delta t^{\min x_{k_{l_0}}} \right)^2 \right].$

Thus, Lemma 13 proved that

\[ V_{m-1} \leq (4 \cdot 2q) V_m + \mathbb{E} \left[ \sum_{s \in [t-4q,t+q] \setminus \{t\}} \left( \frac{L_{s,t}^2}{n^2} \right)^{m-1} \left( 3q \right)^{m-1} \left( 2q \right) \left( \Delta t^{\max x_{k_s} R_{m-1}} - \Delta t^{\min x_{k_s} R_{m-1}} \right)^2 \right] \]

By recursively applying this bound, and since $\Delta t^{\max x_{k_s} R_{m-1}} \leq \Delta t^{\max x_{k_s}}$ and $\Delta t^{\min x_{k_s} R_{m-1}} \geq \Delta t^{\min x_{k_s}}$, we obtain

\[ \mathbb{E} \left[ q \sum_{l_0 \in [t-2q,t-1]} L_{k_{l_0},k_t}^2 \left( \Delta t^{\max x_{k_{l_0}}} - \Delta t^{\min x_{k_{l_0}}} \right)^2 \right] = q \cdot V_0 \leq \mathbb{E} \left[ (1 + r + r^2 + \cdots) \right]. \]

\[ \left( \sum_{s \in [t-4q,t+q] \setminus \{t\}} 48q^2 \Gamma^2 \max \left\{ \frac{L_{s,t}^2}{n^2} \right\} \left( \left( \Delta t^{\max x_{k_s}} - \Delta t^{\min x_{k_s}} \right)^2 + (\Delta t^{x_{k_s}})^2 \right) \right] + \left( 224q^2 \max \left\{ \frac{L_{s,t}^2}{n^2} \right\} L_{\max}^2 \left( \left( \Delta t^{\max x_{k_t}} - \Delta t^{\min x_{k_t}} \right)^2 + (\Delta t^{x_{k_t}})^2 \right) \right). \] (25)

where $r = \frac{8q(3q) \max \left\{ \frac{L_{s,t}^2}{n^2} \right\}}{n^2} = \frac{24q^2 \max \left\{ \frac{L_{s,t}^2}{n^2} \right\}}{n^2}$.

So long as $q^2 \leq \frac{n^2}{48L_{\max}^2}$ and $q^2 \leq \frac{n^2}{48}$, and since $\Gamma \geq L_{\max}$,
\[
E \left[ 2q \sum_{l_0 \in [t-2q,t-1]} L_{k_0}^{2} \left( \Delta_{\text{max}x_{k_0}}^{t} - \Delta_{\text{min}x_{k_0}}^{t} \right)^2 \right] \\
\leq \frac{\Gamma^2}{1-r} \cdot E \left[ 2 \left( \frac{1}{q} \sum_{s \in [t-4q,t+q]\{t\}} \frac{r^2}{12} \left( (\Delta_{\text{max}x_{k_s}}^{\pi,s} - \Delta_{\text{min}x_{k_s}}^{\pi,s})^2 + (\Delta x_{k_s}^{\pi,s})^2 \right) \\
+ 10r \left( (\Delta_{\text{max}x_{k_t}}^{\pi,t} - \Delta_{\text{min}x_{k_t}}^{\pi,t})^2 + (\Delta x_{k_t}^{\pi,t})^2 \right) \right) \right] \\
\leq \frac{r^2\Gamma^2}{6(1-r)q} \sum_{s \in [t-4q,t+q]\{t\}} \left[ (\Delta_{s}^{\text{FE}})^2 + (E_{s}^{x})^2 \right] + \frac{20r\Gamma^2}{1-r} \left[ (\Delta_{t}^{\text{FE}})^2 + (E_{t}^{x})^2 \right].
\]

Thus, by (24) and (23),
\[
\left( g_{k_t}^{\pi,t} - \tilde{g}_{k_t}^{\pi,t} \right)^2 \leq \frac{\nu_3 \Gamma^2}{q} \sum_{s \in [t-3q,t+q]\{t\}} \left[ (\Delta_{s}^{\text{FE}})^2 + (E_{s}^{x})^2 \right] + \frac{\nu_3 \Gamma^2}{q} \sum_{s \in [t-4q,t+q]\{t\}} \left[ (\Delta_{s}^{\text{FE}})^2 + (E_{s}^{x})^2 \right] \\
+ \nu_4 \cdot \Gamma^2 \left( \frac{\nu_1}{q} + \frac{\nu_2}{q} \right) \sum_{s \in [t-3q,t+q]\{t\}} \left[ (\Delta_{s}^{\text{FE}})^2 + (E_{s}^{x})^2 \right] + \nu_4 \cdot \Gamma^2 (E_{t}^{x})^2,
\]

where \( \nu_3 = \frac{2r^2}{6(1-r)} \) and \( \nu_4 = \frac{40r}{1-r} \).

(26)
B.3 Amortization for the Stochastic Analyses

Our amortized analyses take the following general form. We suppose that on each path \( \pi \) the update at time \( t \) causes a charge \( C^\pi(t, s) \) to the update at time \( s \), for \( t - d_1 \leq s \leq t + d_2 \) (\( s \neq t \) of course). Thus, the update at time \( t \) will receive \( R^\pi(t) \) charge, and will send out charge \( S^\pi(t) \), where

\[
R^\pi(t) = \sum_{s=t-d_1}^{t-1} C^\pi(t, s) \quad \text{and} \quad S^\pi(t) = \sum_{v=t+1}^{t+d_2} C^\pi(t, v).
\]

In order to account for these charges, and in order to be able to show progress from one round to the next by a factor of \((1 - \frac{1}{2n})\), we will incur a payment \( P^\pi(t) \) at time \( t \), where

\[
P^\pi(t) = \sum_{v=t+1}^{t+d_2} \frac{1}{1 - \frac{1}{2n}} C^\pi(v, t) + \sum_{s=t-d_1}^{t-1} C^\pi(s, t). \tag{27}
\]

We will also use the potential functions \( A_+^\pi(t) \) and \( A_-^\pi(t) \), given by

\[
A_+^\pi(t) = \sum_{s=t-d_1+1}^{t} \sum_{v=t+1}^{s+d_1} \frac{1}{1 - \frac{1}{2n}} C^\pi(v, s), \quad A_-^\pi(t) = \sum_{s=t-d_2+1}^{t} \sum_{v=t+1}^{s+d_2} C^\pi(s, v).
\]

**Lemma 16.**

\[
[A_+^\pi(t-1) - A_+^\pi(t)] - [A_-^\pi(t-1) - A_-^\pi(t)] = -P^\pi(t) + R^\pi(t) + S^\pi(t) + \frac{1}{2n} A_+^\pi(t-1).
\]

**Proof:** This is a straightforward calculation.

\[
A_+^\pi(t-1) - A_+^\pi(t) = \sum_{s=t-d_1}^{t-1} \frac{1}{1 - \frac{1}{2n}} C^\pi(t, s) - \sum_{v=t+1}^{t+d_1} \frac{1}{1 - \frac{1}{2n}} C^\pi(v, t)
\]

\[
= \frac{1}{2n} A_+^\pi(t-1) + \sum_{s=t-d_1}^{t-1} C^\pi(t, s) - \sum_{v=t+1}^{t+d_1} \frac{1}{1 - \frac{1}{2n}} C^\pi(v, t)
\]

\[
= \frac{1}{2n} A_+^\pi(t-1) + R^\pi(t) - \sum_{v=t+1}^{t+d_1} \frac{1}{1 - \frac{1}{2n}} C^\pi(v, t).
\]

\[
A_-^\pi(t) - A_-^\pi(t-1) = \sum_{v=t+1}^{t+d_2} C^\pi(t, v) - \sum_{s=t-d_2}^{t-1} C^\pi(s, t)
\]

\[
= S^\pi(t) - \sum_{s=t-d_2}^{t-1} C^\pi(s, t).
\]

Thus

\[
[A_+^\pi(t-1) - A_+^\pi(t-1)] - [A_-^\pi(t-1) - A_-^\pi(t)] = -P^\pi(t) + R^\pi(t) + S^\pi(t) + \frac{1}{2n} A_+^\pi(t-1). \square
\]
B.4 SACD Amortized Analysis

We use the potential function

\[ H(t) = \mathbb{E}_\pi \left[ F(x^t) + A^\pi_{t,1}(t) - A^\pi_{t,2}(t) \right], \quad \text{(28)} \]

where \( C^\pi(t, s) = \left( \frac{4\nu_1}{9q} + \frac{\nu_3}{2n} + \frac{10\nu_2}{3q} + \frac{10\nu_3}{3q} + \frac{5r(\nu_1 + \nu_2)}{q} + \gamma_1 \left( \frac{\nu_1}{q} + \frac{\nu_2}{q} \right) \right) \Gamma (\Delta^\pi_{s,1})^2 \]

\[ + \left( \frac{1}{2n} + \frac{\nu_3}{2n} + \frac{10\nu_2}{3q} + \frac{10\nu_3}{3q} + \frac{5r(\nu_1 + \nu_2)}{q} + \gamma_1 \frac{\nu_2}{q} \right) \cdot \Gamma (E^\pi_{s,1})^2, \]

and \( \gamma_1 > 0 \) is a suitable constant. The rationale for this choice of \( C \) will emerge in the proof of the next lemma.

**Lemma 17.** If \( r \leq \frac{1}{3\nu_1}, \Gamma \leq L_{\text{res}}, n \geq 2^{10}, \) and \( q \leq \frac{1}{15} \sqrt{n}, \) then

\[ H(t - 1) - H(t) \geq \frac{1}{9} \mathbb{E}_\pi \left[ n \sum_{k=1}^{n} \tilde{W}_k(\pi_{k,t}, \pi_{k,t-1}, \Gamma, \Psi_k, W) \right] + \frac{1}{2n} \mathbb{A}_t^\pi(t - 1) \geq 0, \]

and \( H(T) \geq \mathbb{E}_\pi \left[ F(x^T) \right]. \)

**Proof:** Recall that \( k_t \) denotes the index of the coordinate being updated on the \( t \)-th edge of \( \pi. \) We write \( \pi(k, t) \) to denote the path in which coordinate \( k_t \) at time \( t \) is replaced by coordinate \( k, \) and to reduce clutter we abbreviate this as \( \pi(k) \). Note that \( \pi(k_t) = \pi. \) We let \( \text{prev}(t, k) \) denote the time of the most recent update to coordinate \( k, \) if any, in the time range \([t - 2q, t - 1]\); otherwise, we set it to \( t. \) From (23),

\[ \mathbb{E}_\pi \left[ F(x^t - 1) - F(x^t) \right] \geq \frac{1}{2n} \mathbb{E}_\pi \left[ n \sum_{k=1}^{n} \tilde{W}_k(\pi_{k,t}, \pi_{k,t-1}, \Gamma, \Psi_k, W) \right] + \frac{\Gamma (E^\pi_{t,1} + 1)}{8} \mathbb{E}_\pi \left[ (g_{k_t}^\pi - g_{k_t}^\pi)^2 \right] \]

\[ \geq \frac{1}{2n} \mathbb{E}_\pi \left[ n \sum_{k=1}^{n} \tilde{W}_k(\pi_{k,t}, \pi_{k,t-1}, \Gamma, \Psi_k, W) \right] + \frac{\Gamma (E^\pi_{t,1} + 1)}{8} \mathbb{E}_\pi \left[ (g_{k_t}^\pi - g_{k_t}^\pi)^2 \right] \]

(by Lemma [5])

\[ \geq \frac{1}{3n^2} \mathbb{E}_\pi \left[ n \sum_{k=1}^{n} \tilde{W}_k(\pi_{k,t}, \pi_{k,t-1}, \Gamma, \Psi_k, W) \right] \]

\[ - \frac{1}{3n^2} \mathbb{E}_\pi \left[ \sum_{t-2q \leq s < t} \sum_{\text{s = prev}(t, k_s)}^{n} \frac{3}{2} \left( g_{k_s}^\pi - g_{k_s}^\pi \right)^2 \right] \]

\[ + 2\Gamma \left( x_{k_s}^\pi - x_{k_s}^\pi \right)^2 \]

\[ - \mathbb{E}_\pi \left[ \frac{1}{n^2} \sum_{k=1}^{n} \sum_{k_t=1}^{n} \frac{2}{3\Gamma} \left( g_{k_t}^\pi - g_{k_t}^\pi \right)^2 \right] + \frac{\Gamma (E^\pi_{t,1} + 1)}{8} \mathbb{E}_\pi \left[ (g_{k_t}^\pi - g_{k_t}^\pi)^2 \right] \]

(by (12)).
From (13), and noting that the range of \( \ell \) values which can affect the update at time \( t \) has been increased from \([t - q, t + q]\) to \([t - 2q, t + q]\), we obtain:

\[
\mathbb{E}_t \left[ \frac{2}{3n^2} \sum_{t-2q \leq s < t} \sum_{k=1}^{n} \Gamma \cdot \left( x^{\pi(k_s,t-1)}_{k_s} - x^{\pi(k_t,t-1)}_{k_s} \right)^2 \right] \\
\leq \frac{\Gamma \cdot 8q}{3n^2} \cdot \mathbb{E}_t \left[ \sum_{t-2q \leq s < t-1} \sum_{k=1}^{n} \left( (\Delta_{\text{max}}^{t} x^{\pi(k_s),s}_{k_s} - \Delta_{\text{min}}^{t} x^{\pi(k_t),s}_{k_s})^2 + (\Delta_{\text{max}}^{t} x^{\pi(k_s),s}_{k_s} - \Delta_{\text{min}}^{t} x^{\pi(k_t),s}_{k_s})^2 \right) \right] \\
\leq \frac{16q}{3n} \sum_{s \in [t-2q,t-1]} \Gamma \cdot (\Delta_{\text{FE}}^s)^2 = \frac{8\nu_1}{9q} \sum_{s \in [t-2q,t-1]} \Gamma \cdot (\Delta_{\text{FE}}^s)^2
\]

(see (22) and the definition of \( \nu_1 \) immediately above it).

Next, we bound \( \mathbb{E}_t \left[ \sum_{t-2q \leq s < t} \sum_{k=1}^{n} \frac{1}{2n^2} \cdot \left( g^{\pi(k_t),\text{prev}(t,k_s)}_{k_s} - g^{\pi(k_t),t}_{k_s} \right)^2 \right] \). But

\[
|g^{\pi(k_t),\text{prev}(t,k_s)}_{k_s} - g^{\pi(k_t),t}_{k_s}| \leq g^{\pi,t}_{\text{max},k_s} - g^{\pi,t}_{\text{min},k_s},
\]

where \( g^{\pi,t}_{\text{max},k_s} \) and \( g^{\pi,t}_{\text{min},k_s} \) are defined analogously to \( g^{\pi,t}_{\text{max},k_t} \) and \( g^{\pi,t}_{\text{min},k_t} \), respectively. By an analysis essentially identical to the one leading to (23), we obtain

\[
\mathbb{E}_t \left[ \frac{1}{2n^2\Gamma} \sum_{t-2q \leq s < t} \sum_{k=1}^{n} \left( g^{\pi(k_t),\text{prev}(t,k_s)}_{k_s} - g^{\pi(k_t),t}_{k_s} \right)^2 \right] \leq \frac{\nu_2}{q} \sum_{s \in [t-2q,t+q]\setminus\{t\}} \Gamma \cdot (\Delta_{\text{FE}}^s)^2 + (E_{\text{ FE}}^x)^2.
\]

Next, we consider the term \( \mathbb{E}_t \left[ \frac{2}{3n^2} \sum_{k=1}^{n} \sum_{s=1}^{n} \cdot \left( g^{\pi(k_t),t}_{k_s} - g^{\pi(k_t),t}_{k_t} \right)^2 \right] \). We will use the term \( g^{S,p(k),t}_k \) as an intermediary to allow us to compare values on two different paths, as follows.

\[
\left( g^{\pi(k_t),t}_k - g^{\pi(k_t),t}_k \right)^2 \leq 2 \left( g^{\pi(k_t),t}_k - g^{S,p(k),t}_k \right)^2 + 2 \left( g^{S,p(k),t}_k - g^{\pi(k_t),t}_k \right)^2 \leq 2 \left( g^{\pi(k_t),t}_k - g^{S,p(k),t}_k \right)^2 + 2 \left( g^{S,p(k),t}_k - g^{\pi(k_t),t}_k \right)^2,
\]

(29)

As \( g^{S,p(k),t}_k = g^{S,p(k),t}_k \) since the gradients evaluated in synchronous order do not depend on the time \( t \) update. And the expression on the RHS of (29) is bounded in the exact same way as the expression being bounded by (20). This yields

\[
\mathbb{E}_t \left[ \frac{2}{3n^2} \sum_{k=1}^{n} \sum_{s=1}^{n} \cdot \left( g^{\pi(k_t),t}_k - g^{\pi(k_t),t}_k \right)^2 \right] \\
\leq \frac{2\nu_2}{3q} \sum_{s \in [t-3q,t+q]\setminus\{t\}} \left( (\Delta_{\text{FE}}^s)^2 + (E_{\text{ FE}}^x)^2 \right) + \frac{2\nu_3}{3q} \sum_{s \in [t-4q,t+q]\setminus\{t\}} \left( (\Delta_{\text{FE}}^s)^2 + (E_{\text{ FE}}^x)^2 \right) \\
+ \nu_4 \cdot \Gamma \left( \frac{\nu_1}{q} + \frac{\nu_2}{q} \right) \sum_{s \in [t-3q,t+q]\setminus\{t\}} \left( (\Delta_{\text{FE}}^s)^2 + (E_{\text{ FE}}^x)^2 \right) + \nu_4 \cdot \Gamma \left( E_{\text{ FE}}^x \right)^2.
\]

(30)
Finally the term \( \frac{1}{n} \mathbb{E}_\pi \left[ \left( g_{k_t}^{(k_t)} - g_{k_t}^{\pi(k_t), t} \right)^2 \right] \), is bounded by (26), which \( \frac{3}{2} \) times the bound in (30).

Finally,

\[
\mathbb{E}_\pi \left[ \frac{\Gamma}{2n^2} \sum_{t-2q \leq s \leq t} \sum_{k_t=1}^n \left( \Delta x_{k_t}^{\pi(k_t), s} \right)^2 \right] \leq \frac{\Gamma}{2n} \sum_{t-2q \leq s \leq t-1} (E_s^x)^2.
\]

Thus

\[
\mathbb{E}_\pi \left[ F(x^{t-1}) - F(x^t) \right] \geq \frac{1}{3n^2} \mathbb{E}_\pi \left[ \sum_{k_t=1}^n \sum_{k_t=1}^n \hat{W}_k(g_{k_t}^{(k_t), t}, x_k^{\pi(k_t), t-1}, \Gamma, \Psi_k) \right] + \frac{\Gamma}{8} (E_t^-)^2 - \frac{5 \nu_4}{2} \Gamma (E_t^x)^2.
\]

So as to obtain a single common constraint on \( q \), we assume \( \Gamma \leq L_{\pi \pi} \) (note there is no good reason to make \( \Gamma \) larger than necessary), and we set \( \nu_1 = \frac{1}{r} \) and \( \nu_2 = \frac{2}{3} \). We will also choose constraints on \( q \) and \( n \) so that \( \frac{8 \nu_1}{q} \geq \frac{q}{2n} \). Now the above bound simplifies to

\[
\mathbb{E}_\pi \left[ F(x^{t-1}) - F(x^t) \right] \geq \frac{1}{3n^2} \mathbb{E}_\pi \left[ \sum_{k_t=1}^n \sum_{k_t=1}^n \hat{W}_k(g_{k_t}^{(k_t), t}, x_k^{\pi(k_t), t-1}, \Gamma, \Psi_k) \right] + \frac{\Gamma}{8} (E_t^-)^2 - \frac{1}{q} \left[ \frac{31r}{6} + \frac{34r^2}{9(1-r)} \right] \Gamma \sum_{s \in \{t-4q \leq s \leq t+q\} \setminus \{t\}} \left[ \left( \Delta s^{\pm} \right)^2 + \Gamma \cdot (E_s^x)^2 \right] - \frac{10r}{1-r} \Gamma (E_t^x)^2.
\]

By Lemma 16,

\[
H(t-1) - H(t) \geq \frac{1}{3n^2} \mathbb{E}_\pi \left[ \sum_{k_t=1}^n \sum_{k_t=1}^n \hat{W}_k(g_{k_t}^{(k_t), t}, x_k^{\pi(k_t), t-1}, \Gamma, \Psi_k) \right] + \frac{\Gamma}{8} (E_t^-)^2 - \frac{1}{q} \left[ \frac{31r}{6} + \frac{34r^2}{9(1-r)} \right] \Gamma \sum_{s \in \{t-4q \leq s \leq t+q\} \setminus \{t\}} \left[ \left( \Delta s^{\pm} \right)^2 + \Gamma \cdot (E_s^x)^2 \right] - \frac{10r}{1-r} \Gamma (E_t^x)^2 - P^\pi(t) + R^\pi(t) + S^\pi(t)
\]

\[
\geq \frac{1}{3n^2} \sum_{k_t=1}^n \sum_{k_t=1}^n \hat{W}_k(g_{k_t}^{(k_t), t}, x_k^{\pi(k_t), t-1}, \Gamma, \Psi_k) + \frac{1}{2n} A_+^\pi(t-1),
\]

if

\[
\Gamma \left( \frac{1}{8} - \frac{10r}{1-r} \right) (E_t^-)^2 + P^\pi(t) + S^\pi(t)
\]

\[
\geq \frac{1}{q} \left[ \frac{31r}{6} + \frac{34r^2}{9(1-r)} \right] \sum_{s \in \{t-4q \leq s \leq t+q\} \setminus \{t\}} \Gamma \cdot \left[ \left( \Delta s^{\pm} \right)^2 + \Gamma \cdot (E_s^x)^2 \right] + P^\pi(t). \quad (31)
\]

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Suppose that \( \mathbb{E}_\pi[P^\pi(t)] = \Gamma [\gamma_1 (\Delta_t^{FE})^2 + \gamma_2 (E_t^x)^2] \). \( \gamma_2 \) is set to \( \frac{1}{8} - \frac{10r}{1 - r} \) so that the term \( \gamma_2 \cdot \Gamma (E_t^x)^2 \) in \( \mathbb{E}_\pi[P^\pi(t)] \) is exactly covered by the term \( \Gamma (\frac{1}{8} - \frac{10r}{1 - r}) (E_t^x)^2 \) in (31). The charges \( R^x(t) \) and \( S^\pi(t) \) are chosen so as to exactly cover the terms \( \gamma_1 \cdot \Gamma (\Delta_t^{FE})^2 \) in \( \mathbb{E}_\pi[P^\pi(t)] \) and the remaining \( \frac{1}{q} \left[ \frac{31r}{6} + \frac{34r^2}{9(1 - r)} \right] \cdot \Gamma \cdot \left[ (\Delta_s^{FE})^2 + (E_s^x)^2 \right] \) on the RHS of (31), which by (22), implies that

\[
\mathbb{E}_\pi[C^\pi(t, s)] = \left( \frac{31r}{6} + \frac{34r^2}{9(1 - r)} + \gamma_1 \left( \frac{\nu_1}{q} + \frac{\nu_2}{q} \right) \right) \Gamma (\Delta_s^{FE})^2
+ \left( \frac{31r}{6} + \frac{34r^2}{9(1 - r)} + \gamma_1 \frac{\nu_2}{q} \right) \Gamma (E_s^x)^2
\]
suffices.

(26) also implies that \( d_1 = 4q \) and \( d_2 = q \). Substituting in (27), it suffices that

\[
\gamma_1 \geq \frac{1}{q} \left[ \frac{31r}{6} + \frac{34r^2}{9(1 - r)} + \gamma_1 \left( \nu_1 + \nu_2 \right) \right] \cdot \left[ 2n \left( \frac{1}{(1 - \frac{1}{2^n})} \right) \right] + d_2
= \frac{1}{q} \left[ \frac{31r}{6} + \frac{34r^2}{9(1 - r)} + \gamma_1 \frac{11r}{12} \right] \cdot \left[ d_2 + d_1 + \frac{(d_1 + 1)(d_1 + 2) - 1}{2(2n)} + \ldots \right]
\]

and

\[
\gamma_2 \geq \left[ \frac{31r}{6} + \frac{34r^2}{9(1 - r)} + \gamma_1 \frac{r}{4} \right] \cdot \left[ d_2 + d_1 + \frac{(d_1 + 1)(d_1 + 2) - 1}{2(2n)} + \ldots \right].
\]

If \( n \geq 2^{10} \) and \( q \leq \frac{1}{10} \sqrt{n} \),

\[
\gamma_1 \geq \frac{1}{q} \left[ \frac{31r}{6} + \frac{34r^2}{9(1 - r)} + \gamma_1 \frac{11r}{12} \right] (5q + 1)
\]

and

\[
\gamma_2 \geq \frac{1}{q} \left[ \frac{31r}{6} + \frac{34r^2}{9(1 - r)} + \gamma_1 \frac{r}{4} \right] (5q + 1)
\]
suffice.

Recall that \( \gamma_2 = \frac{1}{8} - \frac{10r}{1 - r} \). One choice of values that suffices is \( \gamma_1 = \frac{1}{10} \) and \( r = \frac{1}{36} \).

It is easy to verify that \( H(t) = F(x^t) \), for \( A^\pi(T) \) and \( A^\pi(T) \) are equal to 0 because there is no charge between the update at time \( T \) and updates at times \( v \), for \( v > T \).

We restate Theorem 2 to include the convex case.

**Theorem 2** Suppose that given initial point \( x^0 \), Algorithm 1 is run for exactly \( T \) iterations by multiple cores. Also suppose that Assumption 1 holds, \( \Gamma \geq \Gamma_{\max} \), \( n \geq 2^{10} \), and \( q \leq \min\left\{ \frac{\sqrt{n}}{90}, \frac{\sqrt{m}}{90\Gamma_{\max}} \right\} \).

(i) If \( F \) is strongly convex with parameter \( \mu_F \), and \( f \) has strongly convex parameter \( \mu_f \), then

\[
\mathbb{E} \left[ F(x^T) \right] \leq \left[ 1 - \frac{1}{3n} \cdot \frac{\mu_F}{\mu_F + \Gamma - \mu_f} \right]^T \cdot F(x^0).
\]

(ii) Now suppose that \( F \) is convex. Let \( R \) be the radius of the level set for \( x^0 \), \( \text{Level}(x^0) = \{ x | f(x) \leq f(x^0) \} \). Then

\[
\mathbb{E} \left[ F(x^T) \right] \leq \left[ 1 + \min\left\{ \frac{1}{12n}, \frac{F(x^0)}{12nR^2} \right\} \cdot T \right]^{-1} \cdot F(x^0).
\]
Proof: By Lemma 17 if $r \leq \frac{1}{336}$ and $n \geq 2^{10}$, the conditions for applying Theorem 4 hold. We apply it with $r = 1$, $\frac{\beta}{q} = \frac{1}{n}$, $\alpha = 1$, which yields the stated results. Recall that $r = \frac{24q^2L^2}{n^2}$. Thus, to achieve $r \leq \frac{1}{336}$ it suffices to have $q \leq \frac{\Gamma \sqrt{n}}{90L}$.

Note that we have not sought to fully optimize the constants.
C  Lower Bound on \( q \)

In this appendix, we show that if the degree of parallelism \( q \) is too high, for an appropriately chosen function \( f \) (in this appendix, all \( \Psi_j \equiv 0 \)) and a suitable starting point, then the effects of asynchrony might force the ACD process to remain near the starting point for a long time with high probability.

We first give a high-level description on our construction. Let \( L_{\text{max}}^{\text{off}} := \max_{k \neq j} L_{kj} \). We want that whenever a coordinate is chosen to be updated, by taking the advantage of flexibility to choose suitable asynchrony effects, the update can change the coordinate value by either of \( \pm \Delta \), for some \( \Delta = \Theta(1) \) (\( \Delta \leq \frac{1}{\epsilon} \) in our construction). If we can keep doing so, then every coordinate \( j \) can be kept in \( x_j^0 \pm \Delta \) forever.

To have the flexibility of choosing the change to be either of \( \pm \Delta \), we will need to make sure that the value \( g_j \) can take either \( g_j = \Gamma \Delta \) (the true gradient) or \( -g_j = -\Gamma \Delta \). One way to attain this flexibility is to ensure that among the previous \( q \) updates, there are at least \( 2g_j/(L_{\text{max}}^{\text{off}} \Delta) = 2\Gamma/L_{\text{max}}^{\text{off}} \) of updates with change \( +\Delta \), and the other \( 2\Gamma/L_{\text{max}}^{\text{off}} \) updates with change \( -\Delta \).

Intuitively, this is plausible when \( q \) is sufficiently large.

In our construction, \( L_{\text{max}}^{\text{off}} = \Theta(1/\sqrt{n}) \), \( L_{\text{res}} = \Theta(1) \) and \( \Gamma = \Theta(1) \). Thus, we need \( q \geq 4\Gamma/L_{\text{max}}^{\text{off}} = \Theta(\sqrt{n}) \); note that this is matching with the upper bound on the maximum degree of parallelism in Theorem 2 up to a constant factor.

Our construction will be based on a balls in urns problem which we describe next.

C.1 A Balls in Urns Problem

Version 1. We suppose there are \( n \) balls in three urns, called the Minus (M), Zero (Z), and Plus (P) urns. The reason for the names will become clear when we describe the asynchronous schedule. Initially all the balls are in Urn Z. We repeat the following process for \( T \) steps. At each step a ball is selected uniformly at random. If it is in Urn M or P, it is moved to Urn Z. If it is in Urn Z it is moved with equal probability to one of Urns M or P.

Let \( U \) be the event that over \( T \) moves the number of balls in Urn Z is always at least \( \frac{n}{2} - 2\sqrt{6(2c+1)n} \ln n \), and the number of balls in Urns M and P differ by at most \( 2\sqrt{12(2c+1)n} \ln n \).

We first present a lemma which is a simple consequence of the Chernoff bound.

Lemma 18. Let \( I_1, I_2, \cdots, I_T \) be independent random variables, and for each \( 1 \leq t \leq T \), let \( S_t = \sum_{j=1}^t I_j \). Suppose that for some \( 1/4 > \epsilon > 0 \), at each time \( t \geq 1 \), \( I_t \) has value +1 with probability at most \( 1/2 - \epsilon \), and it has value −1 otherwise. Then for any \( d > 0 \), with probability at least \( 1 - T \cdot \exp(-\epsilon d/6) \), all of \( S_1, S_2, \cdots, S_T \) are less than or equal to \( d \).

Proof: For each \( t \) satisfying \( d < t \leq T \), by the Chernoff bound,

\[
\mathbb{P}[S_t > d] \leq \mathbb{P}\left[ \sum_{j=1}^t \frac{I_j + 1}{2} > \frac{t}{2} + \frac{d}{2} \right] \leq \mathbb{P}\left[ \sum_{j=1}^t \frac{I_j + 1}{2} > \left(\frac{t}{2} - t\epsilon\right) \cdot \left(1 + \max\left\{ 2\epsilon, \frac{d}{t}\right\} \right) \right]
\]

\[
\leq \exp\left(-\max\left\{ 4\epsilon^2, \frac{d^2}{t^2}\right\} \cdot (t/2 - t\epsilon) \right)
\]

\[
\leq \exp\left(-\max\left\{ \epsilon^2 t, \frac{d^2}{(4t)}\right\} \right) / 3 \leq \exp(-\epsilon d/6) .
\]

\(^9\)It might be helpful to think of the asynchronous effects as an adversary, as is standard in online algorithm analysis.

\(^{10}\)For the purpose of exposition, view all \( L_{kj} \)'s as being identical to \( L_{\text{max}}^{\text{off}} \).
The claim follows from a simple union bound. □

Lemma 19. With probability at least $1 - 3T \cdot n^{-c}$, $\mathcal{U}$ holds.

Proof: Let $|Z|$ denote the number of balls in urn $Z$. Initially, $|Z| = n$. For some $d > 0$, we analyze the probability that $|Z|$ drops below $n/2 - 2d$ at some time on or before time $T$. Let $t'$ denote the first time this occurs. Then at some earlier time, $|Z| = \lfloor n/2 - d \rfloor$; Let $t''$ denote the latest time this occurs. Note that for each time $\tau$ which is between times $t', t''$, the probability that a ball in $Z$ is selected at time $\tau$ is at most $(n/2 - d)/n = 1/2 - d/n$. Thus,

$$\Pr[|Z| < n/2 - 2d \text{ at time } t'] \leq \sum_{t''=1}^{T-d} \sum_{t'=t''+d}^{T} \exp\left(-d^2/(6n)\right).$$

(By the Claim, with $\epsilon = d/n$.)

$$\leq T^2 \exp\left(-d^2/(6n)\right).$$

By picking $d = \sqrt{6(2c+1)n \ln n}$ and $T = n^c$, this probability is $1/n$.

Next, for some $d > 0$, we analyze the probability that $|P| - |M|$ exceeds $2d$ at some time on or before time $T$. The analysis is almost identical to the above, except that $\epsilon = d/(2n)$, the probability bound becomes $T^2 \exp\left(-d^2/(12n)\right)$. By picking $d = \sqrt{12(2c+1)n \ln n}$ and $T = n^c$, this probability is $1/n$. After accounting the symmetric probability that $|P| - |M|$ drops below $2d$, we are done. □

Version 2. We suppose there are $n$ balls in three urns, called the Minus (M), Zero (Z), and Plus (P) urns. Initially all the balls are in Urn Z. We repeat the following process for $T$ steps. At each step a ball is selected uniformly at random. If it is in Urn M or P, it is moved to Urn Z. If the selected ball is in Urn Z, if $|P| - |M| \geq 1$, the ball is moved to either of Urn M or P which has fewer balls, and we call this ball a $Z_B$ ball; if there are equal number of balls in Urns M and P, then the selected ball is moved to either of them randomly uniformly; then we call this ball a $Z_R$ ball. (The subscripts B and R stand for “Balancing” and “Random”.)

First, we show that at time $\tau$, with high probability, $|Z|$ is at least $49n/100$, and $|P| - |M| \leq d$. Then in the next $q$ steps, $|Z| \geq 12n/25$ with certainty. By the Chernoff bound, in the next $q$ steps, at least $47q/100$ selected balls are in Urn Z.

Next, we consider the following matching procedure. Step 1: for each $Z_B$ ball at time $\tau$, suppose that the deficit just before time $\tau$ is $w \geq 1$, we match the ball which occurs at the latest time $t \leq \tau' < \tau$ that brings the deficit from $w - 1$ to $w$. If such ball does not exist, then the $Z_B$ ball is left unmatched. Step 2: Then for each remaining unmatched $Z_R$ ball at time $\tau$, find the ball after time $\tau$ which is improving. If it is not yet matched, match it with the $Z_R$ ball.

It is easy to observe that each pair of matched balls are moving in the opposite direction, and the earlier one is worsening, when the latter one is improving.

We claim that the number of matched pairs is at least $|Z|/2 - d$. Let $r$ count the number of $Z_R$ balls, and $b$ count the number of $Z_B$ balls. Note that $r + b = |Z|$. It is not hard to prove that all matched pairs are disjoint; at most $d$ of the $Z_R$ balls are left unmatched (due to the original deficit). For each of the unmatched $Z_R$ balls in Step 2, it is either matched eventually, or when it is left unmatched but then implies a matched $Z_B$ ball (except the last $Z_R$ ball).

Thus, the number of pairs is at least $\max\{b - d, r - b - 1\}$. So we have the unconditional bound of $|Z|/3 - d/3$. 

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C.2 The Function and the Asynchronous Schedule

**Function $f$ and a Starting Point** Let $L$ be an $n \times n$ matrix, such that the value of every diagonal entry is 1, and the value of every off-diagonal entry is $\epsilon := \frac{1}{\sqrt{cn \ln n}}$. Note that $L$ can be rewritten as $\epsilon \cdot M_n + (1 - \epsilon) I_n$, where $M_n$ is the $n \times n$ matrix with every entry being 1, and $I_n$ is the $n \times n$ identity matrix. Since $M_n, I_n$ are both positive semi-definite, so is $L$. For simplicity, we suppose that $n$ is an even integer.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the convex quadratic function

$$f(x) = \frac{1}{2} x^T L x = \frac{1}{2} \sum_{j=1}^{n} (x_j)^2 + \sum_{1 \leq j < k \leq n} \epsilon x_j x_k.$$  

Note that the parameters $L_{jk}$ as defined in Definition 1 are identical to the corresponding entries of the matrix $L$ defined above. Thus, $L_{\text{sym}} = 1 + \frac{n-1}{cn \ln n}$. Also, note that the minimum value of $f$ is attained when $x = (0, 0, \cdots, 0)$. The gradient along coordinate $j$ is given by the following formula:

$$\frac{\partial F}{\partial x_j} = x_j + \epsilon \cdot \sum_{k \neq j} x_k.$$  

We will choose $\Gamma \geq 2$.

We pick the starting point randomly as follows: uniformly at random, we choose exactly half the coordinates $j$ to have value $x_j^0 = 1$, and the other half to have value $x_j^0 = -1$. Let $J^+$ denote the set of coordinates $j$ with $x_j^0 = +1$, and $J^-$ its complement.

**Asynchrony Leads to Stalling** Now we have a fixed $f$ and a fixed starting point. Let $q = C\sqrt{cn \ln n}$, where $C$ is a sufficiently large positive constant to be determined.

The first $q$ updates are all computed using the initial data (thus they do not read any of the updated values among these first $q$ updates). We call the coordinates that are updated two or more times during the initial $q$ updates the *special* coordinates. The next lemma bounds the total number of updates to the special coordinates during the first $q$ updates.

**Lemma 20.** With probability at least $1 - n^{-c}$ there are at most $C^2 \frac{q^2}{n} + \sqrt{3c \ln n} = (C^2 + C \sqrt{3})c \ln n$ special coordinates, and with probability at least $1 - n^{-c}$ there are at most $2(C^2 + C \sqrt{3})c \ln n$ updates to the special coordinates during the first $q$ updates.

**Proof:** The probability that an update among the first $q$ updates is a repeat update to a special coordinate is at most $q/n$. Thus, the expected number of such updates is at most $C^2 q^2/n$. By a Chernoff bound, the probability that the number of updates is greater than or equal to $C^2 q^2/n (1 + \delta)$ is at most

$$\exp \left(-\frac{C^2 q^2 \delta^2}{3n}\right).$$  

This probability is at most $n^{-c}$ if $\delta \geq \frac{1}{q} \sqrt{3c \ln n}$.

Thus, with probability $1 - n^{-c}$, there are at most $C^2 \frac{q^2}{n} + \sqrt{3c \ln n}$ repeat updates and hence at most twice this number of updates to the special coordinates.

Note that the coordinate $j_t$ updated at time $t \leq q$ satisfies $\left| \sum_{k \neq j_t} x_k^0 \right| = 1$. Thus the computed gradient, if $x_{j_t} = 1$, has value $1 - \epsilon$ and hence the updated $x_{j_t}$ has value $1 - \frac{1}{C}(1 - \epsilon)$. Analogously, if $x_{j_t} = -1$, the updated $x_{j_t}$ has value $-1 + \frac{1}{C}(1 - \epsilon)$.

Next we explain how the coordinates are assigned to urns. We want the movement of the coordinates to correspond exactly to the urn process described in the previous section. Initially, the coordinates are all in Urn Z. Intuitively, when a coordinate in Urn Z is reduced in value it
is moved to Urn M and if increased in value it is moved to Urn P. Coordinates in Urns M or P, when selected, are moved back to Urn Z. However, we need the property that if a coordinate in Z is selected there be exactly a one half probability that it go to each of M or P, and this is only approximately true for the first $q$ updates. Accordingly, we modify the urn assignment as follows.

Suppose that at time $t$ there are $r$ coordinates with value $+1$ and $s$ coordinates with value $-1$ that have not yet been updated. Updating a coordinate with value $+1$ would move it to Urn M and updating a coordinate with value $-1$ would move it to Urn P. If $r \neq s$ these are not equal probability events, and we need them to be equal to mimic the urn process. So suppose $s > r$. Then if we choose a non-updated coordinate with value $-1$, with probability $\frac{s-r}{2s}$ we put it in Urn M, and we call this a misallocated coordinate. We proceed analogously if $r > s$. This then makes sure that the urn process is being followed, in that if a non-updated coordinate in Z is selected, then it is equally likely that the choice goes to P or M (with the probability being over all possible selections of non-updated coordinates).

**Lemma 21.** With probability $1 - 2n^{-c}$ there are at most $8c$ misallocated coordinates, if $n \geq \lceil C^{3/2} e^{1/4} \ln^{5/4} n \rceil^8$.

**Proof:** We begin by bounding the difference in the number of non-updated coordinates with values $-1$ and $+1$. By a Chernoff bound, with probability $1 - n^{-c}$, this is at most $\sqrt{6cq \ln n}$ (recall that the expected number of $+1$s selected is at most $q/2$, as is the number of selected $-1$s). For the rest of the proof we condition on this event.

Then the probability that a single update results in a misallocated coordinate is at most $\frac{1}{n} \sqrt{6cq \ln n}$. Thus the probability that there are $b$ misallocated coordinates is at most

$$\binom{q}{b} \left( \frac{q}{n} \right)^b \leq \left( \frac{\sqrt{6cq \ln n} \cdot q^{3/2}}{bn} \right)^b \leq \left( \frac{e \sqrt{6c} \cdot C^{3/2} e^{1/4} \ln^{5/4} n}{n^{1/4}} \right)^b \leq n^{-b/8} \leq n^{-c} \quad \text{if } b = 8c.$$  

Suppose that $x_j^0 = 1$. Then the computed gradient in direction $j$ during the first $q$ updates has value $x_j - \epsilon$, and this is negative when $x_j < \epsilon$. As $\Gamma \geq 1$, it follows that during these initial $q$ updates, $x_j \geq \epsilon$ always. Similarly, if $x_j^0 = -1$, during the initial $q$ updates, $x_j \leq \epsilon$ always. Later updates will also maintain this property.

We note that the motion of the coordinates for the first $q$ updates corresponds exactly to the urn process. We will ensure that this property continues to hold henceforth. For the remainder of this section, we condition on event $\mathcal{U}$ holding, which it does with probability at least $1 - 3Tn^{-c}$ by Lemma 19.

We proceed to show how to manage the updates at times $t > q$.

All non-special coordinates with initial value $+1$ are maintained so that they have values in one of the following ranges: $1 - \Delta \pm \Delta \cdot \epsilon$, $1 \pm \Delta \cdot \epsilon$, $1 + \Delta \pm \Delta \cdot \epsilon$, and analogously for the non-special coordinates with initial value $-1$, where $0 < \Delta < 1$ is a suitably small value. Furthermore, each of these coordinate values will have been chosen from two values (specific to that update). For example, in the middle case of values near to 1, the possible values are $1 - a$ and $1 + b$, where $a, b \geq 0$ and $a + b \leq \Delta \cdot \epsilon$. We call the chosen value among $a$ and $b$ the deviation for that update.
We now consider an update to a non-special coordinate at time $t > q$.

Suppose WLOG that the coordinate currently has value $v$ in the range $1 \pm \Delta \epsilon$. If all the other coordinates were non-special and non-misallocated, if they all had zero deviation, and if the number of coordinates in Urns M and P were equal, then the gradient would have value $v - \Delta \epsilon$, assuming the most up-to-date coordinate values were read. We now bound the range of values for this gradient with high probability.

**Lemma 22.** Conditioned on $U$, With probability $1 - 5n^{-c}$, the above gradient lies in the range $1 \pm \Delta \epsilon \pm (1 + \Delta) (C^2 + C \sqrt{3}) \epsilon \ln n \pm \epsilon (2\Delta + 2\Delta \epsilon) \epsilon \pm (3 + \sqrt{2(c + 1) \ln n}) \sqrt{\epsilon \Delta(2\Delta \epsilon) \epsilon \pm n \Delta \epsilon^2} = 1 \pm \Lambda$.

**Proof:** The term $1 \pm \Delta \epsilon$ reflects the range of $v$.

By Lemma 20, with probability at least $1 - n^{-c}$ there are at most $(C^2 + C \sqrt{3}) \epsilon \ln n$ special coordinates. Each special coordinate lies in the range $[\epsilon, 1]$ if its original value is $+1$, and in the range $[-1, -\epsilon]$ if its original value is $-1$; thus its contribution to the gradient differs by at most $\pm (1 + \Delta) \epsilon$ from the contribution it would make if it were non-special.

By Lemma 21 with probability at least $1 - 2n^{-c}$ there are at most $8\epsilon$ misallocated coordinates. These each have a contribution to the gradient that differs by at most $\pm 2(\Delta + \Delta \epsilon) \epsilon$ from the contribution it would make if it were not misallocated.

As $U$ holds, the difference in the number of coordinates in Urns M and P is bounded by $(3 + \sqrt{2(c + 1) \ln n}) \sqrt{\epsilon \Delta(2\Delta \epsilon) \epsilon \pm n \Delta \epsilon^2} = 1 \pm \Lambda$.

Finally, the deviations each change the contribution to the gradient by at most $\Delta \epsilon^2$, and there are at most $n - 1$ of these contributions.

We will want to move this coordinate to each of Urns P and M with probability $\frac{1}{\epsilon}$. To go to Urn M we want the computed gradient to be $-\Gamma \Delta \epsilon \ln n$. To obtain this value we will ignore up to $\alpha q$ of the most recent updates to non-special, non-misallocated coordinates that occurred during the previous $q$ updates, and for which the coordinates are currently in $P$, where $\alpha > 0$ is a suitably sized constant.

This will guarantee we can reduce the value of the gradient by an amount up to $\alpha q (1 - \Delta - \Delta \epsilon) \epsilon$. It suffices to have

$$\alpha q (1 - \Delta - \Delta \epsilon) \epsilon = C \alpha (1 - \Delta - \Delta \epsilon) \geq 1 + \Lambda + \Lambda \Delta (1 + \epsilon).$$  (34)

In addition, as each ignored update contributes are most $(1 + \Delta + \Delta \epsilon) \epsilon$ to the gradient, this means that the error to the updated value is at most $\frac{1}{\epsilon} (1 + \Delta + \Delta \epsilon) \epsilon$. Thus we want

$$\Delta \epsilon \geq \frac{1}{\Gamma} (1 + \Delta + \Delta \epsilon) \epsilon;$$

$\Delta = \frac{1}{\Gamma} - \frac{1}{\epsilon}$ suffices.

It remains to argue that with high probability there are indeed $\alpha q$ such coordinates. We will choose $\alpha = \frac{1}{4}$.

**Lemma 23.** Conditioned on event $U$, during the interval $[t - q, t - 1]$, with probability at least $1 - 4n^{-c}$, at least $\frac{q}{4}$ non-special, non-misallocated coordinates are updated exactly once and end up in Urn M, if $\sqrt{n} \geq 4[3 + \sqrt{(2c + 1) \ln n}] + \frac{4\sqrt{2\epsilon(n \ln n)^{1/4}}}{\sqrt{C}} + 16(C + \sqrt{3}) \epsilon \sqrt{\ln n} + \frac{64\epsilon}{C \sqrt{\ln n}}$.

**Proof:** As $U$ holds, at each of the $q$ previous updates, there are at least $\frac{q}{4} - (3 + \sqrt{(2c + 1) \ln n}) \sqrt{n}$ coordinates in Urn Z. By a Chernoff bound, during these $q$ updates, with probability $1 - n^{-c}$, at least $\frac{q}{4} - \frac{q}{2n}(3 + \sqrt{(2c + 1) \ln n}) \sqrt{n} - \frac{1}{2} \sqrt{2eq \ln n}$ updates end up in Urn M.
By Lemma 20, with probability at least $1 - n^{-c}$, no more than $(C^2 + C\sqrt{3})c\ln n$ are repeat updates, and with probability at least $1 - n^{-c}$, no more than $(C^2 + C\sqrt{3})c\ln n$ are special coordinates.

by Lemma 21, with probability at least $1 - 2n^{-c}$, no more that $8c$ are misallocated coordinates. In total, with probability $1 - 4n^{-c}$, there are at least $\frac{2}{3} - \frac{2c}{\ln n}(3 + \sqrt{(2c + 1)\ln n})\sqrt{n} - \frac{1}{\sqrt{2}}2cq\ln n - 2(C^2 + C\sqrt{3})c\ln n - 8c$ coordinates that are non-special, non-misallocated, updated exactly once in the interval of $q$ updates, ending up in urn M. This is at least $\frac{9}{8}$ if the stated bound on $n$ holds.

Every other type of update is handled analogously.

Next, we show how to satisfy (34).

Lemma 24. (34) will hold if $\Gamma \geq 2$, $\Delta(\Gamma - 1 - \epsilon) = 1$, $\Delta + 2\Delta\epsilon \leq \frac{1}{3}$, $n \geq 100\max\{(C^2 + C\sqrt{3})c\ln n, c\}$, $c \geq 2$, and $C \geq 60$.

Proof: Note that $\epsilon = \frac{1}{\sqrt{cn\ln n}} \leq \frac{1}{30}$ and thus $\Gamma\Delta(1 + \epsilon) = (1 + \Delta + \Delta\epsilon)(1 + \epsilon) \leq \frac{5}{3} \cdot \frac{31}{30} \leq 2$. As $\alpha(1 - \Delta - \Delta\epsilon) \geq \frac{1}{12}$ and $\Gamma\Delta(1 + \epsilon) \leq 2$, it suffices that $\frac{C}{\alpha} \geq 3 + \Delta$. Recall the definition of $\Lambda$ in Lemma 22:

$$
\Lambda = \Delta + (1 + \Delta)(C^2 + C\sqrt{3})c\ln n \cdot \epsilon + 8c(2\Delta + 2\Delta\epsilon) + (3 + \sqrt{2(c + 1)\ln n})\sqrt{2n}(\Delta + 2\Delta\epsilon) + n\Delta^2
$$

$$
\leq \frac{1}{\sqrt{cn\ln n}} + \frac{4(C^2 + C\sqrt{3})c\ln n}{3\sqrt{n}} + \frac{16c}{\sqrt{c\ln n}} + \frac{2\sqrt{c + 1}}{3\sqrt{c}} + \frac{1}{3c\ln n}
$$

$$
\leq \frac{1}{30} + \frac{1}{7} + \frac{1}{4} + \frac{1}{2} + \frac{9}{10} + \frac{1}{30} \leq 2.
$$

Thus it suffices that $C \geq 60$.

Finally, we note that non-special misallocated coordinates, at their first update, can be restored to the correct value for Urn Z, and then can be treated as normal non-special coordinates. And even for the special coordinates, we can show that it takes at most $O(c)$ updates to the coordinate, following those in the initial sequence of $q$ updates, to restore them to a correct value (but this needs additional omitted analysis).

We have shown:

Lemma 25. With probability at least $1 - n^{-c}[12T + 4]$ the above process keeps all non-special, non-misallocated coordinates at values close to their original values during the first $T$ updates, and there are at most $(C^2 + C\sqrt{3})c\ln n + 8c$ of these latter coordinates, assuming $n \geq [C^{3/2}c^{1/4}\ln^{5/4} n]^8$, together with the conditions specified in Lemma 24.

Proof: By Lemma 19, with probability at least $1 - 3Tn^{-c}$ event $U$ holds. By Lemma 20, with probability at least $1 - 2n^{-c}$, there are at most $(C^2 + C\sqrt{3})c\ln n$ special coordinates and at most $2(C^2 + C\sqrt{3})c\ln n$ updates to these coordinates during the first $q$ updates. By Lemma 21, with probability at least $1 - 2n^{-c}$ there are at most $8c$ misallocated coordinates. By Lemma 22, with probability at least $1 - 5Tn^{-c}$, each computed gradient lies in the range specified there. Finally, by Lemma 23, with probability at least $1 - 4Tn^{-c}$, there are sufficiently many recently updated coordinates in the right urn to enable the desired update to be made.

Lemma 25 yields Theorem 3 on noting that for all non-special, non-misallocated coordinates, $|x_j - x_j^*| \leq \Delta(1 + \epsilon) = \frac{1 + \epsilon}{1 - 1 - \epsilon}$. 

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D Further Related Work

Related Work on Coordinate Descent  Convex optimization is one of the most widely used methodologies in applications across multiple disciplines. Unsurprisingly, there is a vast literature studying convex optimization, with various assumptions and in various contexts. We refer readers to Nesterov’s text [21] for an excellent overview of the development of optimization theory. Distributed and asynchronous computation has a long history in optimization, starting with the already mentioned work of Chazan and Miranker [6], with subsequent milestones in the work of Baudet [2], and of Tsitsiklis, Bertsekas and Athans [30, 3]; more recent results include [5, 4]. See Frommer and Szyld [13] for a fairly recent review, and Wright [31] for a recent survey on coordinate descent.

Stochastic (synchronous) coordinate descent, in which coordinates are updated in random order, has recently attracted attention. Relevant works include Nesterov [22], Richtárik and Takáč [24] and Lu and Xiao [18].

Other Related Work  In statistical machine learning, the objective functions to be minimized typically have the form $\sum_i \ell(x, Z_i)$, where the $Z_i$ are samples; updates are sample-wise but not coordinate-wise, so our model will not cover these update algorithms. It is an interesting problem to investigate if there is an adaption of our model and amortized analysis for these algorithms. Many of the assumptions made for asynchronous sample-wise updates share similarities with ours. For instance, Tsianos and Rabbat [29] extended the analysis of Duchi, Agarwal and Wainwright [11] to analyze distributed dual averaging (DDA) with communication delay; the same authors [23] studied DDA with heterogeneous systems, i.e., distributed computing units with different query and computing speeds. Langford, Smola and Zinkevich [15] also studied problems with bounded communication delay.

In a similar spirit to our analysis, Cheung, Cole and Rastogi [8] analyzed asynchronous tatonnement in certain Fisher markets. This earlier work employed a potential function which drops continuously when there is no update and does not increase when an update is made.
E Common Value Assumption

Since the retrievals of coordinate values is performed after choosing the coordinate \( k_t \) to update, and since the schedule of retrievals depends on the choice of \( k_t \), in general it is possible that the retrieved value \( \tilde{x} \) in Step 2 of Algorithm 1 varies with \( k_t \).

Also, a later starting update (update A) can affect the updates by with earlier starts (updates in B) if update A commits earlier than some of the updates in B. One likely scenario for this to happen is due to varying iteration lengths. Suppose that at time \( \tau \) a core B chooses coordinate \( k_\tau \) to update, and this update will take \( 2d \) time units to commit (where \( d \geq 3 \)). Also, suppose it schedules to read the value of coordinate \( j \) at some time after \( \tau + d + 2 \). At time \( \tau + 1 \), core A chooses a random coordinate to update. If it chooses coordinate \( j \), and if it takes \( d \) time units to commit, then core B will read the updated value by core A. On the other hand, if coordinate \( j \) is not chosen recently, then core B will surely read another value of coordinate \( j \).

More subtly, even if update A commits after all updates in B, it can still affect the updates in B due to differential delays coming from the operating environment (see Footnote 5 for examples on how these delay occur).

In [16], Liu and Wright made the Common Value assumption. This is the reason they can use the parameter \( L_{res} \) to bound gradient differences. To avoid the use of the Common Value assumption, we introduce a new but similar parameter \( \overline{L_{res}} \).
F A Few Remarks about Locking in the SACD Algorithm

We make the following remarks about the SACD algorithm.

In many optimization problems, e.g., those involving sparse matrices, the number of coordinate values needed for computing the gradient in Step 3 of Algorithm 1 is much smaller than $n$, i.e., in Step 2 the core needs to retrieve only a tiny portion of the full set of coordinate values. Also, the set of coordinate values needed for computing the gradients along different coordinates can be very different. Therefore, the random choice of coordinate (in Step 1) has to be made ahead of the process of retrieving required information from the shared memory.

If the convex function $F$ does not have the univariate non-smooth components, each update simply adds a number, which depends only on the computed gradient, to the current value in the memory. Then the update can be done atomically (e.g., by fetch-and-add$^{11}$), and no lock is required.

However, for general scenarios with univariate non-smooth components, the update to $x_j$ must depend on the value of $x_j$ in memory right before the update (see Equation (1)). Then the update cannot be done atomically, and a lock is necessary. We note that when the number of cores is far fewer than $n$, say when it is $\epsilon \sqrt{n}$ for some $\epsilon < 1$, delays due to locking can occur, but are unlikely to be significant.$^{12}$

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$^{11}$The fetch-and-add CPU instruction atomically increments the contents of a memory location by a specified value.

$^{12}$The standard birthday paradox result states that if $\epsilon \sqrt{n}$ cores each chooses a random coordinate among $[n]$ uniformly, the probability for a collision to occur is $\Theta(\epsilon^2)$. 

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