SIMPLICIAL COMPLEXES OF SMALL CODIMENSION

MATTEO VARBARO AND RAHIM ZAARE-NAHANDI

Abstract. We show that a Buchsbaum simplicial complex of small codimension must have large depth. More generally, we achieve a similar result for CM_t simplicial complexes, a notion generalizing Buchsbaum-ness, and we prove more precise results in the codimension 2 case. Along the paper, we show that the CM_t property is a topological invariant of a simplicial complex.

1. Introduction

In [11], Hartshorne proposed his tantalizing conjecture concerning smooth varieties of small codimension in some projective space. Precisely, if \( R = K[x_1, \ldots, x_n] \) is the polynomial ring in \( n \) variables over a field \( K \), the conjecture declares:

**Conjecture 1.1.** (Hartshorne) If \( I \subseteq R \) is a homogeneous ideal of height \( h \) less than \( (n - 1)/3 \) such that \( \text{Proj}R/I \) is nonsingular, then \( I \) is a complete intersection.

If \( h = 2 \), then the condition \( h < (n - 1)/3 \) is equivalent to \( n > 7 \). In this case, by a result of Evans and Griffith [6, Theorem 3.2], the conjecture is equivalent to:

**Conjecture 1.2.** If \( I \subseteq R \) is a homogeneous ideal of height 2 such that \( \text{Proj}R/I \) is nonsingular, and \( n > 7 \), then \( R/I \) is Cohen-Macaulay.

The present article has no pretension to give new insights on the conjecture of Hartshorne: the only result in this direction is Corollary 3.6, stating that \( R/I \) has depth larger than \( n - 2h \) if furthermore \( I \) admits a square-free initial ideal. Rather, this paper brings the philosophy of the conjecture to the world of combinatorial commutative algebra, as it had already been done, to some extent, in [3].

If \( \Delta \) is a simplicial complex in \( n \) variables, \( \text{Proj}K[\Delta] \) is almost never smooth, so Hartshorne’s conjecture is not interesting when stated for \( \text{Proj}K[\Delta] \). The notion of Cohen-Macaulay-ness in codimension \( t \) was introduced, independently and with the sole difference concerning a purity matter, in [16] and in [9]. In [16] this concept was suggested as the right one to measure the singularities of a simplicial complex: \( \Delta \) is Cohen-Macaulay in codimension \( t \) (according to [9]) if and only if \( \Delta \) is pure of singularity dimension less than \( t - 1 \) (according to [16]). In particular, if \( \Delta \) has negative singularity dimension, it is Buchsbaum. So, somehow Buchsbaum-ness plays the role of ‘smooth-ness’ for simplicial complexes. This way of thinking is also supported from the results in the recent paper [2], which imply that, if the ideal defining a smooth projective variety has a square-free Gröbner degeneration, then

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Emails: varbaro@dima.unige.it, rahimzn@ut.ac.ir.
the associated simplicial complex is Buchsbaum. With this definition in mind, the same philosophy that led Hartshorne to make his conjecture brings one to expect the following: If $\Delta$ is a Buchsbaum simplicial complex with small codimension, then $K[\Delta]$ should have large depth.

In this note, we show that if $\Delta$ is a $(d-1)$-dimensional Buchsbaum simplicial complex on $d+2$ vertices, then depth $K[\Delta] \geq d-1$. Moreover, in this case $K[\Delta]$ is not Cohen-Macaulay if and only if $\Delta$ is the Alexander dual of (the clique complex of) the $(d+2)$-cycle (Proposition 4.2). More generally, if $\Delta$ is a $(d-1)$-dimensional Buchsbaum simplicial complex on $n$ vertices, then depth $K[\Delta] \geq 2d-n+1$. Even more generally, if $\Delta$ is Cohen-Macaulay in codimension $t$, then $K[\Delta]$ satisfies the condition of Serre $S_{2d-n-t+2}$ (Corollary 3.5). Along the way, we also prove that being Cohen-Macaulay in codimension $t$ is a topological invariant (Theorem 2.5).

The paper is structured as follows: a brief review of some preliminaries and conventions is given in Section 2, where the topological invariance of Cohen-Macaulayness in an arbitrary codimension is also proved. Section 3 is devoted to the connection between Cohen-Macaulayness of a simplicial complex in some codimension with linearity of the Stanley-Reisner ideal of the Alexander dual of the simplicial complex up to a certain step. This leads to a connection between Cohen-Macaulayness in a certain codimension with the $S_r$ condition of Serre. Some corollaries and relevant examples are also given. In Section 4, the case of codimension 2 simplicial complexes is analyzed in more detail, and a combinatorial proof of the main result of Section 3 in the codimension 2 case is provided.

2. Preliminaries and conventions

Let $R = K[x_1, \ldots, x_n]$ be the ring of polynomials over a field $K$, equipped with the standard grading. For integers $p \geq 1$ and $d \geq 2$, we say that a simplicial complex $\Delta$ on $n$ vertices satisfies the Green-Lazarsfeld property $N_{d,p}$ if $I_\Delta$ is generated in degree $d$ and the first $p$ steps of the minimal graded free resolution

$$
\cdots \to F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_1} F_0 \to I_\Delta \to 0
$$

of $I_\Delta$ are linear, in the sense that $\varphi_1, \ldots, \varphi_{p-1}$ are represented by matrices of linear forms.

A simplicial complex $\Delta$ is said to satisfy the Serre’s condition $S_r$ if $\tilde{H}_i(\text{link}_\Delta F; K)$ vanishes for all $F \in \Delta$ and for all $i < \min\{r-1, \dim(\text{link}_\Delta F)\}$, where $\tilde{H}_i(\Delta; K)$ is the $i$th reduced homology group of $\Delta$ over the field $K$. This is equivalent to the usual definition of the condition $S_r$ on $K[\Delta]$.

By a CM$_t$ simplicial complex, we mean a pure simplicial complex $\Delta$ which is Cohen-Macaulay in codimension $t$, namely a simplicial complex such that $\text{link}_\Delta F$ is Cohen-Macaulay for all $F \in \Delta$ with $|F| \geq t$.

**Remark 2.1.** Let $\Delta$ be a pure simplicial complex of dimension $d-1$. It follows by the definition that $\Delta$ satisfies the $S_r$ condition $\implies$ $\Delta$ is CM$_{d-r}$. The vice versa is false, just think to a disconnected Buchsbaum simplicial complex $\Delta$ (such a $\Delta$ is CM$_1$ but does not even satisfy $S_2$). On the other hand, we will show in Corollary 3.5 that $\Delta$ is CM$_t$ on $n$ vertices $\implies$ $\Delta$ satisfies the $S_{2d-n-t+2}$ condition.

**Remark 2.2.** The notion of singularity dimension has been considered in [16] as follows: a simplicial complex $\Delta$ has singularity dimension less than $m$ if $\text{link}_\Delta F$ is
Cohen-Macaulay for all $F \in \Delta$ with $\dim F \geq m$ (by convention, $\dim \emptyset = -1$). So a simplicial complex $\Delta$ is CM if and only if it is pure and has singularity dimension less than $t - 1$.

**Remark 2.3.** The phrase “Cohen-Macaulay in codimension $t$” in the present paper has a different meaning from the phrase “Cohen-Macaulay in codimension $c$” considered in [16]. In fact, according to [16] Definition 3.6, even if $\Delta$ is a pure simplicial complex of dimension $d - 1$, then in [16] “$\Delta$ Cohen-Macaulay in codimension $c$” means that $\text{link}_\Delta F$ is Cohen-Macaulay for all $F \in \Delta$ with $|F| = d - 1 - c$.

For an $R$-module $M$ we write $\dim M$ for the Krull dimension of $M$; when $M = 0$ we write by convention $\dim M = -\infty$.

**Remark 2.4.** Notice that $\Delta$ is a pure $(d - 1)$-dimensional simplicial complex if and only if

$$\dim \text{Ext}^n_R(K[\Delta], R) < i \quad \forall i < d.$$ 

On the other hand, it has been proved in [16] Corollary 7.4 that $\Delta$ has singularity dimension $< m$ if and only if

$$\dim \text{Ext}^n_R(K[\Delta], R) \leq m \quad \forall i < d.$$ 

So, if $\Delta$ has singularity dimension $< m$ and $\text{depth} K[\Delta] > m$, then $\Delta$ is pure. In particular, since $\text{depth} K[\Delta] > 0$ for any simplicial complex $\Delta$, the following are equivalent:

1. $\Delta$ is Buchsbaum.
2. $\Delta$ has singularity dimension $< 0$.
3. $\Delta$ is CM.

A property of a simplicial complex $\Delta$ is a topological invariant of $\Delta$ if it holds for any simplicial complex whose geometric realization is homeomorphic to the one of $\Delta$. Next we prove that the properties of satisfying $S_r$, being CM, and having singularity dimension $< m$ are topological invariants. This fact has essentially been proved by Yanagawa in [22]. We report his result in our context for the convenience of the reader. We keep the same notations used in [22].

**Theorem 2.5.** Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex on $n$ vertices. Then, for all $i \in \mathbb{N}$,

$$\dim \text{Ext}^n_R(K[\Delta], R)$$

is a topological invariant of $\Delta$. In particular, satisfying $S_r$, being CM, and having singularity dimension $< m$ are topological invariants.

**Proof.** Let $X$ be a topological realization of $\Delta$. If $\dim \text{Ext}^n_R(K[\Delta], R) \leq 0$, then $\dim \text{Ext}^n_R(K[\Delta], R) = 0$ if and only if $\text{Ext}^n_R(K[\Delta], R) \neq 0$ if and only if $\check{H}^{-i}(X; K) \neq 0$, so we can assume that $\dim \text{Ext}^n_R(K[\Delta], R) > 0$.

Notice that $\text{Ext}^n_R(K[\Delta], R) = 0$ for $i > d$ or $i \leq 0$, and that $\text{Ext}^n_R(K[\Delta], R)$ is always $d$-dimensional. Therefore we will assume that $0 < i < d$. In this situation, [22] Theorem 4.1 yields that $\dim \text{Ext}^n_R(K[\Delta], R) - 1$ is equal to the dimension of the support of the sheaf $K^{-i+1}(\mathcal{D}^\bullet_X)$ on $X$, where $\mathcal{D}^\bullet_X$ is the Verdier dualizing complex of $X$ with coefficients in $K$. So we have that $\dim \text{Ext}^n_R(K[\Delta], R)$ is a topological invariant of $\Delta$.

For the last part, notice that being pure is obviously a topological invariant and:

1. $\Delta$ satisfies $S_r$ (for $r \geq 2$) $\iff$ $\dim \text{Ext}^n_R(K[\Delta], R) \leq i - r \forall i < d$. 

(2) $\Delta$ has singularity dimension $< m \iff \dim \text{Ext}^n_{\mathbb{R}}(K[\Delta], R) \leq m \forall i < d$.

(3) $\Delta$ is CM $\iff$ $\Delta$ is pure and $\dim \text{Ext}^n_{\mathbb{R}}(K[\Delta], R) < t \forall i < d$.

For further concepts and notations on simplicial complexes and combinatorial commutative algebra we refer to the standard books [19], [12] and [17].

3. The CM$_t$ property of simplicial complexes versus the Serre condition $S_r$

In this section, for a simplicial complex $\Delta$ of dimension $d - 1$ on $n$ vertices, applying a subadditivity result of Herzog and Srinivasan to the Betti diagram of the Stanley-Reisner ideal of $\Delta$, it is shown that if $\Delta$ satisfies CM$_t$ for some $t \geq 0$, then $\Delta^\vee$ satisfies the $N_{n-d, 2d-n-t+2}$ condition. In other words, the minimal graded free resolution of $I_{\Delta^\vee}$ is linear on the first $2d - n - t + 2$ steps. This leads to the implication that if $\Delta$ is CM$_t$ for some $t \geq 0$, then the Stanley-Reisner ring of $\Delta$ satisfies the $S_{2d-n-t+2}$ condition of Serre.

First we recall a generalization of the Eagon-Reiner’s theorem given in [8].

**Theorem 3.1.** [8, Theorem 3.1]. Let $\Delta$ be a simplicial complex on on $n$ vertices, $\Delta^\vee$ its Alexander dual and $I_\Delta \subset R$ the Stanley-Reisner ideal of $\Delta$. Then the following are equivalent:

(i) $\Delta^\vee$ is a CM$_t$ simplicial complex of dimension $d - 1$.

(ii) $\beta_{0,j}(I_\Delta) = 0 \forall j > n - d$ and $\beta_{i,i+j}(I_\Delta) = 0 \forall j > n - d$ and $i + j \leq n - t$.

I.e., the Betti diagram $\beta_{i,i+j}(I_\Delta)$ looks like in Figure 1.

![Betti diagram](image)

**Figure 1.** The shape of the Betti diagram of $I_\Delta$ when $\Delta^\vee$ is CM$_t$.

On the other hand, Herzog and Srinivasan [13] proved the following “subadditivity” result on the Betti numbers of monomial ideals.
**Theorem 3.2.** [13, Corollary 4]. Let $I = (u_1, \ldots, u_m)$ be a monomial ideal of $R$, and let $e = \max \{\deg(u_i)\}$. Then for all $j_0 \in \mathbb{Z}$:

$$\beta_{i,j}(I) = 0 \quad \forall \ j > j_0 \implies \beta_{i+1,j}(I) = 0 \quad \forall \ j > j_0 + e.$$  

Now we prove the main result of the paper.

**Theorem 3.3.** Let $\Delta$ be a $(d-1)$-dimensional CM$_t$ simplicial complex on $n$ vertices. Then $\Delta^\vee$ satisfies the $N_{n-d,2d-n-t+2}$ condition.

**Proof.** Notice that $I_{\Delta^\vee}$ is generated in degree $n-d$. Hence the assertion is trivially valid for $2d-n-t+2 \leq 1$. Therefore, we may assume that $2d-n-t \geq 0$. Then, (3.1) gives us

$$\beta_{i,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > j_0 \implies \beta_{i+1,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > j_0 + n - d.$$  

By Theorem 3.1, we know that, for all $i \in \mathbb{N}$,

$$\beta_{i,j}(I_{\Delta^\vee}) = 0 \quad \forall \ i+n-d < j \leq n-t,$$

and

$$\beta_{0,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > n-d.$$  

Now, suppose that $1 \leq i \leq 2d-n-t+1$, and assume we have already proved that

$$\beta_{i-1,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > i-1+n-d.$$  

By (3.4) together with (3.1) we have $\beta_{i,j}(I_{\Delta^\vee}) = 0$ for all $j > i-1+2n-2d$. In particular, we have $\beta_{i,j}(I_{\Delta^\vee}) = 0$ for $i = 2d-n-t+1$, $j > (2d-n-t+1)-1+2n-2d = n-t$. On the other hand, (3.2) guarantees us that $\beta_{i,j}(I_{\Delta^\vee}) = 0$ for all $i+n-d < j \leq n-t$. Putting all together we get

$$\beta_{i,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > i+n-d.$$  

□

In [20] and, independently, in [23], the following refinement of the result of Herzog and Srinivasan is proved:

**Theorem 3.4.** [20, Theorem 6.2, the $\mathbb{Z}$-graded part]. With the notation of Theorem 3.3, one has:

$$\beta_{i,k}(I) = 0, \forall k = j_0, \ldots, j_0 + e - 1 \implies \beta_{i+1,j_0+\epsilon}(I) = 0.$$  

This result can be applied to study the Betti numbers of $\Delta^\vee$ (inferring analogous results to Theorem 3.3) when $\Delta$ has singularity dimension less than $m$.

For $r \geq 2$, by a result of Yanagawa [21, Corollary 3.7], for a simplicial complex $\Delta$ of codimension $c$, $K[\Delta]$ satisfies the $S_r$ condition of Serre if and only if $I_{\Delta^\vee}$ satisfies the $N_{c,r}$ condition. Therefore, an interesting consequence of Theorem 3.3 is the following:

**Corollary 3.5.** Let $\Delta$ be a simplicial complex of dimension $d-1$ on $n$ vertices. Assume that $\Delta$ is CM$_t$ for some $t \geq 0$. Then $\Delta$ satisfies the $S_{2d-n-t+2}$ condition. In particular, if $\Delta$ is Buchsbaum, then $\text{depth} K[\Delta] \geq 2d-n+1$.  

\[\square\]
The following corollary is in the spirit of Hartshorne’s conjecture and goes in the direction of a question raised in [2, Question 4.2].

**Corollary 3.6.** Let \( I \subseteq R \) be a homogeneous ideal of height \( h \) such that \( \text{Proj} R/I \) is nonsingular. If \( I \) has a square-free initial ideal with respect to some term order, then \( \text{depth} R/I > n - 2h \).

**Proof.** Let \( J \) be a square-free initial ideal of \( I \). Since \( R/I \) is generalized Cohen-Macaulay, \( R/J \) is Buchsbaum by [2, Corollary 2.11]. By Corollary 3.5 then, \( \text{depth} R/J \geq n - 2h + 1 \). We conclude since the depth cannot go up by taking the initial ideal. \( \square \)

Another consequence, interestingly related to the result of Brehm and Kühlner [1, Theorem B], is the following:

**Corollary 3.7.** Let \( \Delta \) be a \((d-1)\)-dimensional Buchsbaum simplicial complex on \( n \) vertices such that \( H_i(\Delta; K) \neq 0 \) for some \( i \geq 1 \). Then \( n \geq 2d - i \).

**Remark 3.8.** Being the combinatorial manifolds a very special case of Buchsbaum simplicial complexes, even if the conclusion of Corollary 3.7 is slightly weaker than the one in [1, Theorem B], it applies to a much larger class of simplicial complexes.

**Example 3.9.** Since Theorem 3.3 and Corollary 3.5 are trivial for \( t \geq 2d - n + 1 \), it is natural to ask for examples of CM\( t \) simplicial complexes that are not CM\( t-1 \) for \( 1 \leq t \leq 2d - n \). Murai and Terai [18, Example 3.5] considered the following simplicial complex:

\[
\Delta = \{ \{1,2,3,5\}, \{1,2,4,6\}, \{1,3,4,5\}, \{1,3,4,6\}, \{1,3,5,6\}, \{2,3,4,6\}, \{2,3,5,6\}, \{2,4,5,6\} \},
\]

where \( \Delta \) satisfies \( S_3 \) but is not Cohen-Macaulay. Thus \( \Delta \) is Buchsbaum and the condition \( 1 \leq t \leq 2d - n \) is satisfied. Now if \( v \) is a new vertex, by [10, Theorem 3.1 (ii)], the cone on \( \Delta \) with vertex \( v \) is CM\( 2 \) but not Buchsbaum, and again we have \( 1 \leq t \leq 2d - n \). Taking further cones, one gets a family of CM\( t \) simplicial complexes which are not CM\( t-1 \) and we have \( 1 \leq t \leq 2d - n \).

**Remark 3.10.** Often, the minimal number of vertices necessary for triangulating a given \((d-1)\)-dimensional combinatorial manifold is more than \( 2d \). An exception is an 8 dimensional combinatorial manifold, the so called “Brehm and Kühlner manifold”, which has 6 combinatorially different triangulations on 15 vertices (see [1], [15, Proposition 48] and [14]).

### 4. The CM\( t \) Property and Minimal Chord-less Cycles of Graphs

In this section, we focus on pure \((d-1)\)-dimensional simplicial complexes on \( d + 2 \) vertices, i.e. pure codimension two simplicial complexes. If \( \Delta \) is such a simplicial complex, then its Alexander dual is flag, i.e., \( \Delta^\vee \) is the clique complex of a graph \( G \). In general, the clique complex and the independence complex of a graph \( H \) will be denoted by \( \Delta(H) \) and \( \Delta_H \), respectively. Also, by \( \Delta_T \) we will denote the complementary graph of \( H \).

**Theorem 4.1.** Let \( \Delta \) be a pure \((d-1)\)-dimensional codimension two simplicial complex. Then the following are equivalent:

(i) \( \Delta \) is CM\( t \),

(ii) \( \Delta^\vee \) satisfies the \( N_{2,d-t} \) condition,
(iii) $\Delta$ satisfies the $S_{d-t}$ condition,
(iv) Every cycle of the 1-skeleton $G$ of $\Delta'$ of length at most $d - t + 2$ has a chord.

Proof. The equivalence of (i), (ii) and (iii) is simply an application of Theorem 3.3.
Corollary 3.5 in the case $n = d + 2$, and Remark 2.1. The equivalence of (ii) and (iv) follows by [5, Theorem 2.1].

Proposition 4.2. If $\Delta$ is a codimension two Buchsbaum simplicial complex, then
$\text{depth} K[\Delta] \geq \dim \Delta$. Furthermore, $K[\Delta]$ is Cohen-Macaulay if and only if the
1-skeleton $G$ of $\Delta'$ is not the $(d + 2)$-cycle.

Proof. Notice that $\Delta$ being Buchsbaum is equivalent to $\Delta$ being CM. So the first
part of the statement follows by Theorem 4.1. If $K[\Delta]$ is not $\text{CM}$, again Theorem
4.1 implies that $G$ has an induced chord-less $(d + 2)$-cycle (in those notations, so
$d = \dim \Delta + 1$). Since the number of vertices is $d + 2$, $G$ is actually the $(d + 2)$-

Remark 4.3. In particular, if $\Delta$ is a codimension two Buchsbaum simplicial complex
which is not Cohen-Macaulay, then $\text{projdim} K[\Delta] = 3$. One might expect that,
in general, if $\Delta$ is a codimension 2 simplicial complex which is CM but not $\text{CM}$, then
$\text{projdim} K[\Delta] = t + 2$. This is false: a simple example is the Alexander dual
of $\Delta = \langle \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}, \{5, 6\} \rangle$ which has dimension $d - 1$ where
d = 4 = 6 - 2 = n - 2. Then $\Delta$ is CM$_2$ but not CM$_1$. Nevertheless, the projective
dimension of the Stanley-Reisner ring of $\Delta$ is 3.

For the sake of documenting a different method, we give an alternative proof,
more combinatorial, for the equivalence of (i) and (iv) in Theorem 4.1.

Theorem 4.4. Let $G$ be a simple graph on $[n] = \{1, \ldots, n\}$ with no isolated vertices.
Let $\Delta = \Delta(G)$ be the clique complex of $G$. Let $r \geq 3$ be an integer. Then $\Delta'$ is
CM$_{n-r}$ if an only if every cycle of $G$ of length at most $r$ has a chord.

Proof. The “if” direction follows by [5, Theorem 2.1], [21, Corollary 3.7] and Re-
mrk 2.1 so we focus on the “only if” part.

Assume that $\Delta'$ is CM$_{n-r}$. We prove by induction on $r$ that every cycle of
$G$ of length at most $r$ has a chord. The first case $r = 3$ is trivial. Assume
that $r \geq 4$. Since $\Delta'$ is CM$_{n-r+1}$, every cycle of $G$ of length at most $r - 1$
has a chord. So it is enough to show that $G$ has no chord-less $r$-cycles. Assume
that, on the contrary, $G$ has a chord-less $r$-cycle $C$. Let $V(C) = \{v_1, \ldots, v_r\}$ and
$E(C) = \{\{v_1, v_2\}, \ldots, \{v_{r-1}, v_r\}, \{v_r, v_1\}\}$ be the vertex set and the edge set of $C$, respectively. Then the induced subgraph of $\overline{C}$ on $V(C)$ is the graph $K_r \setminus E(C)$, where $K_r$ is the complete graph on $V(C)$. Clearly, $K_r \setminus E(C)$ has $(\binom{r}{2}) - r = r(r-3)/2$ edges. Let $F$ be the simplex on $V(G) \setminus V(C)$. Then, $|F| = n - r$ and $F$ is a face of
$\Delta'$ because $V(C) \not\subseteq \Delta$. Thus $\Gamma = \text{link}_{\Delta'} F$ should be Cohen-Macaulay. We prove
that this is not the case. Observe that the only facets of $\Delta'$ which contain $F$ are
$F \cup (V(C) \setminus \{v_i, v_j\})$ for some $\{v_i, v_j\} \in \overline{C}$. Therefore,

$$
\Gamma = \text{link}_{\Delta'} F = \langle V(C) \setminus \{v_i, v_j\} : \{v_i, v_j\} \in \overline{C} \rangle.
$$

In particular, $\dim \Gamma = r - 3$. We determine $h_{r-2}$ by computing the $f$-vector of $\Gamma$;
to this purpose, notice that every subset of the vertex set of $\Gamma$ of cardinality $\leq r - 3$
is also a face of Γ. To see this, let \( E = V(C) \setminus \{v_i, v_j, v_k\} \) be a subset of the vertex set of Γ of cardinality \( r - 3 \). Choose \( 1 \leq l \leq r \) such that \( l \notin \{i, j, k\} \). Then at least one of the pairs \((i, l)\), \((j, l)\) and \((k, l)\) will be a non-consecutive pair modulo \( r \). Let \((i, l)\) be such a pair. Then, \( \mathcal{E} = (V(G) \setminus \{v_i, v_j\}) \), i.e., \( E \) is a face of Γ. Therefore we got:

\[
    f_{-1} = 1, f_i = \binom{r}{i+1}, i = 0, \ldots, r - 4 \quad \text{and} \quad f_{r-3} = r(r - 3)/2.
\]

Consequently,

\[
    h_{r-2} = \sum_{i=0}^{r-2} (-1)^{r-2-i} f_{i-1} = \left( \sum_{i=0}^{r-3} (-1)^{r-i} \binom{r}{i} \right) + r(r - 3)/2 = (1 - 1)^r + \binom{r}{r-1} - \binom{r}{r-2} - 1 + r(r - 3)/2 = -1.
\]

Hence Γ is not Cohen-Macaulay. This completes the proof. \( \square \)

**Corollary 4.5.** With the assumptions of Theorem 4.4, assume that \( G \) is \( r \)-chordal, i.e., it has no chord-less cycles of length greater than \( r \). Then \( \Delta^v \) is CM \( n-r \) if and only if \( I_{\Delta} = I(G) \) has a linear resolution.

**Proof.** The assertion follows by Theorems 4.1 and 4.4 and Fröberg’s result that \( I_{\Delta} = I(G) \) has a linear resolution if and only if \( G \) is chordal \( \square \).

**Remark 4.6.** It is easy to see that if \( G \) is a bipartite graph or a chordal graph, then \( \overline{G} \) can only have chord-less four cycles (e.g., see [8, Lemma 4.2 and Lemma 4.6]). Assume that \( G \) is a graph on \( n \) vertices which is either bipartite or chordal. If the Alexander dual of \( \Delta(\overline{G}) = \Delta_G \) is CM \( n-4 \), then by Corollary 4.3 \( I(G) \) has a linear resolution.

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Dipartimento di Matematica, Universita’ di Genova, Via Dodecaneso 35, Genova 16146, Italy

Rahim Zaare-Nahandi, School of Mathematics, Statistics & Computer Science, University of Tehran, Tehran, Iran.