Preperiodic dynatomic curves for $z \mapsto z^d + c$

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May 11, 2014

Abstract

We study the preperiodic dynatomic curves $X_{n,p}$, the closure of set of $(c, z) \in \mathbb{C}^2$ such that $z$ is a preperiodic point of $f_c$ with preperiod $n$ and period $p$ ($n, p \geq 1$).

We prove that each $X_{n,p}$ has exactly $d - 1$ irreducible components, these components are all smooth and intersect pairwise transversally at the singular points of $X_{n,p}$. For each component, we calculate the genus of its compactification and then give a complete topological description of $X_{n,p}$. We also calculate the Galois group of the defining polynomial of $X_{n,p}$.

1 Introduction

Fix $d \geq 2$. For $c \in \mathbb{C}$, set $f_c(z) = z^d + c$. For $p \geq 1$, define

$$\tilde{X}_{0,p} := \{(c, z) \in \mathbb{C}^2 \mid f_c^p(z) = z \text{ and for all } 0 < k < p, \; f_c^k(z) \neq z\}.$$ 

$X_{0,p} :=$ the closure of $\tilde{X}_{0,p}$ in $\mathbb{C}^2$.

It is known that each $X_{0,p}$ is an affine algebraic curve. It is called the periodic dynatomic curve. It has been the subject of several studies in algebraic and holomorphic dynamical systems.

In case $d = 2$, Douady-Hubbard proved the smoothness of $X_{0,p}$ by the technique of parabolic implosion. Bousch [B] proved the irreducibility of $X_{0,p}$ and calculated the Galois group of its defining polynomial by using combination of algebraic and dynamical method. Bousch [B] also calculates the genus of the compactification of $X_{0,p}$. Recently, Buff and Tan Lei re proves the smoothness and irreducibility of $X_{0,p}$ with a different method ([BT]).

In the general case $d \geq 2$, the result of the irreducibility of $X_{0,p}$ and the Galois group of its defining polynomial have been obtained by Morton [Mo] using a combinatorics of algebraic arguments, and by Lau and Schleicher [LS] using dynamical arguments only. Yan Gao and Yafei Ou ([GO]) generalized the method in [BT] to give an elementary proof of the smoothness and the irreducibility of $X_{0,p}$.

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**Definition 1.1.** For \( n \geq 0, \ p \geq 1, \) a point \( z \) is called a \((n,p)\)-preperiodic point of \( f_c \) if 
\[
f_c^{n+p}(z) = f_c^n(z) \quad \text{and} \quad f_c^{l+k}(z) \neq f_c^l(z) \quad \text{for any} \ 0 \leq l \leq n, \ 0 \leq k \leq p \ \text{with} \ (l,k) \neq (n,p).
\]

Now, for any \( n \geq 1, \ p \geq 1, \) define
\[
\tilde{X}_{n,p} = \{(c, z) \in \mathbb{C}^2 | z \text{ is a } (n,p)\)-preperiodic point of } f_c \}
\]
\[
X_{n,p} := \text{the closure of } \tilde{X}_{n,p} \text{ in } \mathbb{C}^2.
\]

In fact, as we shall see below, each \( X_{n,p} \) is also an affine algebraic curve. These curves are called preperiodic dynatomic curves. There are much less studies about them. The known results include the connectivity of \( \tilde{X}_{n,p} \) and the computation of the Galois group of its defining polynomial (\[B\], \[S\]) in case \( d = 2 \).

In this work, we will give a more detailed description of \( X_{n,p} \) (for any degree \( d \)) from both algebraic and topology point of view. We summarize our main result below. This result is to be compared with results on periodic dynatomic curves. For \( \nu_d(p) \) the unique sequence of positive integers satisfying the recursive relation
\[
d^p = \sum_{k | p} \nu_d(k)
\]
and for \( \varphi(m) \) the Euler totient function (i.e. the number of positive integers less than \( m \) and co-prime with \( m \)), set
\[
g_p(d) = 1 + \frac{(d-1)(p-1)}{2d} \nu_d(p) - \frac{d-1}{2d} \sum_{k | p, k < p} \varphi \left( \frac{p}{k} \right) k \cdot \nu_d(k).
\]
\[
g_{n,p}(d) = 1 + \frac{1}{2} \nu_d(p) d^{n-2} [(d-1)(n+p) - 2d] - \frac{1}{2} d^{n-2} (d-1) \sum_{k | p, k < p} \varphi \left( \frac{p}{k} \right) k \cdot \nu_d(k)
\]

**Theorem 1.2.** For any \( d \geq 2, \ n, p \geq 1, \) the preperiodic dynatomic curve \( X_{n,p} \) has the following properties:

1. The set \( X_{n,p} \) is an affine algebraic curve. It has \( d-1 \) irreducible components and each one is smooth. Moreover, every pair of these components intersect transversally at the singular points of \( X_{n,p} \). The set \( \tilde{X}_{n,p} \) has \( d-1 \) connected components.

2. In particular, if \( d = 2 \), the curve \( X_{n,p} \) is smooth and irreducible, and the set \( \tilde{X}_{n,p} \) is connected.

3. The genus of the compactification of every irreducible component of \( X_{n,p} \) is \( g_{n,p}(d) \). Furthermore, all irreducible components are mutually homeomorphic.

4. The Galois group of the defining polynomial of \( X_{n,p} \) consists of all permutations on its roots which commute with \( f_c \) and \( d \)-th rotation.
Here is a tableau comparing these various curves:

| periodic $\mathcal{X}_{0,p}$ | $d = 2$ | $d > 2$ |
|-------------------------------|---------|---------|
| irreducible                   | irreducible |
| smooth                        | smooth |
| $\#$ ideal points             | $\nu_2(p)/2$ | $\nu_d(p)/d$ |
| genus                         | $g_p(2)$ | $g_p(d)$ |
| Galois group                  | $\text{sym}(\nu_2(p)/p) \ltimes \mathbb{Z}_p^{\nu_2(p)/p}$ | $\text{sym}(\nu_d(p)/p) \ltimes \mathbb{Z}_p^{\nu_d(p)/p}$ |

| preperiodic $\mathcal{X}_{n,p}$, $n \geq 1$ | $d = 2$ | $d > 2$ |
|---------------------------------------------|---------|---------|
| irreducible                                 | $d - 1$ irreducible components |
| smooth                                      | each component is smooth |
| component-wise genus                        | $g_{n,p}(2)$ | $g_{n,p}(d)$ |
| Galois group                                | $G_{n,p}(2)$ | $G_{n,p}(d)$ |
| component-wise Galois group                 | $G_{n,p}(2)$ | $G_{n,p}^j$ |
| pairwise intersection                       | empty | $C_{n,p,4}$ |

This manuscript is organized as follows:

In section 2, we summarize some results about periodic dynatomic curve that will be used in this paper.

In section 3, we recall some results about the filled-in Julia sets and the Multibrot set for the family of polynomials $z \mapsto z^d + c$.

In section 4, we will prove that every $\mathcal{X}_{n,p}$ is an affine algebraic curve and find its defining polynomial $Q_{n,p}(c, z)$.

In section 5, we give the irreducible factorization of $Q_{n,p}(c, z)$ and prove that each irreducible factor is smooth. Then each irreducible component of $\mathcal{X}_{n,p}$ is a Riemann surface. We will show that these Riemann surfaces intersect pairwise transversally at the singular points of $Q_{n,p}(c, z)$.

In section 6, we will give a kind of compactification for each irreducible component of $\mathcal{X}_{n,p}$ by adding some ideal boundary points such that it becomes a compact Riemann surface and then calculate the genus of this compact Riemann surface.

In section 7, we will describe $\mathcal{X}_{n,p}$ from the algebraic point of view by calculating the Galois group of $Q_{n,p}(c, z)$.

Acknowledgement. I thanks Tan Lei for helpful discussions and suggestions.
2 Periodic dynatomic curves

Some of our proofs and statements rely on results of the periodic curves $X_{0,p}$. We summarize related results in the following theorem. Its proof can be found in [B], [BT], [LS], [S], [GO].

Theorem 2.1. Let $X_{n,p}$ be defined at the beginning of introduction. Then

1. There exists a unique sequence of monic polynomials $\{Q_{0,p}(c, z)\}$ (monic about $z$) such that

$$\Phi_{0,p}(c, z) := f_{c}^{\circ p}(z) - z = \prod_{k \mid p} Q_{0,k}(c, z) \text{ with degree}_{(z,c)}(Q_{0,k}) = \nu_{d}(k).$$

2. Let $c_{0}$ be an arbitrary parameter. Then a point $z_{0}$ is a root of $Q_{0,p}(c_{0}, z) \in \mathbb{C}[z]$ if and only if one of the three exclusive conditions is satisfied:

(1) $z_{0}$ is a periodic point of $f_{c_{0}}$ of period $p$ and $[f_{c_{0}}^{\circ p}]'(z_{0}) \neq 1$
(2) $z_{0}$ is a periodic point of $f_{c_{0}}$ of period $p$ and $[f_{c_{0}}^{\circ p}]'(z_{0}) = 1$
(3) $z_{0}$ is a periodic point of $f_{c_{0}}$ of period $m$, where $m$ is a proper factor of $p$, and $[f_{c_{0}}^{\circ m}]'(z_{0})$ is a primitive $\frac{p}{m}$-th root of unity

3. $Q_{0,p}$ is smooth and irreducible for all $p \geq 1$ and

$$X_{0,p} = \{(c, z) \in \mathbb{C} \mid Q_{0,p}(c, z) = 0\}.$$

4. $\pi_{0,p} : X_{0,p} \rightarrow \mathbb{C}$ defined by $\pi_{0,p}(c, z) = c$ is a degree $\nu_{d}(p)$ branched covering with two kinds of critical points:

(1) $C_{0,p,2} = \{(c, z) \in X_{0,p} \mid (c, z) \text{ satisfies condition (2) in 2}\}$. In this case, $(c, z)$ is a simple critical point.
(2) $C_{0,p,3} = \{(c, z) \in X_{0,p} \mid (c, z) \text{ satisfies condition (3) in 2}\}$. In this case, the multiplicity of the critical point $(c, z)$ is $\frac{p}{m} - 1$.

The critical value set of $\pi_{0,p}$ consists of roots and co-roots (see definition below) of all hyperbolic components of period $p$, where $K$ is some algebraic closure of $K$.

5. The Galois group $G_{0,p}$ for polynomial $Q_{0,p}(c, z) \in K[z]$ (where $K = \mathbb{C}(c)$ is the field of rational functions about $c$), consists of the permutations on roots of $Q_{0,p}(c, z)$ in $K$ that commute with $f_{c}$.
3 Filled-in Julia sets and the Multibrot set

Let us recall some results about the filled-in Julia set for the polynomial $f_c : z \mapsto z^d + c$ and the Multibrot set in the parameter space. Their proofs can be found in [DH] and [DE].

For $c \in \mathbb{C}$, we denote by $K_c$ the filled-in Julia set of $f_c$, that is the set of points $z \in \mathbb{C}$ whose orbit under $f_c$ is bounded. We denote by $M_d$ the Multibrot set in the parameter plane, that is the set of parameters $c \in \mathbb{C}$ for which the critical point $0$ belongs to $K_c$.

Assume $c \in M_d$. Then $K_c$ is connected. There is a conformal isomorphism $\phi_c : \mathbb{C} \setminus K_c \to \mathbb{C} \setminus \overline{\mathbb{D}}$ satisfying $\phi_c \circ f_c = (\phi_c)^d$ and $\phi_c'(\infty) = 1$ (i.e. $\frac{\phi_c(z)}{z} \xrightarrow{z \to \infty} 1$). The dynamical ray of angle $\theta \in \mathbb{T}$ is defined by

$$R_c(\theta) := \{ z \in \mathbb{C} \setminus K_c \mid \arg(\phi_c(z)) = 2\pi \theta \}.$$ 

It is known that if $\theta$ is rational, then as $r \searrow 1$, the point $\phi_c^{-1}(r e^{2\pi i \theta})$ converges to a point in $K_c$.

Assume $c \notin M_d$. Then $K_c$ is a Cantor set. There is a conformal isomorphism $\phi_c : U_c \to V_c$ between neighborhoods of $\infty$ in $\mathbb{C}$, which satisfies $\phi_c \circ f_c = (\phi_c)^d$ on $U_c$. We may choose $U_c$ so that $U_c$ contains the critical value $c$ and $V_c$ is the complement of a closed disk. For each $\theta \in \mathbb{T}$, there is an infimum $r_c(\theta) \geq 1$ such that $\phi_c^{-1}$ extends analytically along $R_0(\theta) \cap \{ z \in \mathbb{C} \mid r_c(\theta) < |z| \}$. We denote by $\psi_c$ this extension and by $R_c(\theta)$ the dynamical ray

$$R_c(\theta) := \psi_c\left(R_0(\theta) \cap \{ z \in \mathbb{C} \mid r_c(\theta) < |z| \} \right).$$

As $|z| \searrow r_c(\theta)$, the point $\psi_c(\theta e^{2\pi i \theta})$ converges to a point $x \in \mathbb{C}$. If $r_c(\theta) > 1$, then $x \in \mathbb{C} \setminus K_c$ is an iterated preimage of $0$ and we say that $R_c(\theta)$ bifurcates at $x$. If $r_c(\theta) = 1$, then $x$ belongs to $K_c$ and we say that $R_c(\theta)$ lands at $x$.

The Multibrot set is connected. The map

$$\phi_{M_d} : \mathbb{C} \setminus M_d \ni c \mapsto \phi_c(c) \in \mathbb{C} \setminus \overline{\mathbb{D}}$$

is a conformal isomorphism. For $\theta \in \mathbb{T}$, the parameter ray $R_{M_d}(\theta)$ is

$$R_{M_d}(\theta) := \{ c \in \mathbb{C} \setminus M_d \mid \arg(\phi_{M_d}(c)) = 2\pi \theta \}.$$ 

It is known that if $\theta$ is rational, then as $r$ tends to 1 from above, $\phi_{M_d}^{-1}(r e^{2\pi i \theta})$ converges to a point of $M_d$. We say that $R_{M_d}(\theta)$ lands at this point. A dynamical ray or parameter ray is called $(n,p)$-preperiodic if its angle is $(n,p)$-periodic under $\tau : \theta \to d\theta \mod \mathbb{Z})$. There are three kinds of important parameters:

- $c$ is called a parabolic parameter if there exists a periodic point of $f_c$ with multiplier a primitive root of unity. Furthermore, if the multiplier is 1, the parameter $c$ is called a primitive parabolic parameter, otherwise $c$ is called a satellite parabolic parameter.
• $c$ is called a hyperbolic parameter if $f_c$ has an attracting periodic point, in other words a periodic point whose multiplier has an absolute value less than 1.

• $c$ is called a Misiurewicz parameter if $c$ is a $(n, p)$-preperiodic point of $f_c$ for some $n, p \geq 1$.

Parabolic parameters and Misiurewicz parameters lie on the boundary of $M_d$.

The set of hyperbolic parameters forms an open and closed subset of the interior of $M_d$. Each connected component is called a hyperbolic component. Within a hyperbolic component, the period of the attracting periodic point for any parameter is the same and this number is called the period of the hyperbolic component.

Let $H$ be a $p$-periodic ($p \geq 1$) hyperbolic component. For every parameter $c \in H$, the polynomial $f_c$ has an attracting periodic orbit $\{ z(c), \ldots, f_c^{p-1}(z(c)) \}$. Its multiplier defines a map

$$\lambda_H : H \to D, \quad c \mapsto \frac{\partial}{\partial z} f_c^{\cdot n}(z) \bigg|_{z=z(c)}.$$  

Then $\lambda_H : H \to D$ is a branched covering of degree $d - 1$ with only one branched point which is the preimage of 0. This branched point is called the center of $H$. The map $\lambda_H$ can be extended continuously to the closure $\overline{H}$. Considering parameter $c \in \partial H$ such that $\lambda_H(c) = 1$, Eberlein proved the following results:

• For $p \geq 2$, among these points, there is exactly one $c$ which is the landing point of two parameter rays of period $p$, this point is called the root of $H$. Any one of the other $d - 2$ points is the landing point of only one parameter ray of period $p$. They are called the co-roots of $H$. The component $H$ is said of primitive or satellite type according to whether its root is a primitive or satellite parabolic parameter. All co-roots of $H$ are primitive parabolic parameters.

• For $p = 1$, any one of these $d - 1$ points is the landing point of only one fixed parameter ray and hence a primitive parabolic parameter.

## 4 The defining polynomial for $X_{n,p}$

The objective of this section is to show that $X_{n,p}$ is an affine algebraic curve and find its defining polynomial.

Let $\Phi_{n,p}(c, z) = f_c^{n+p}(z) - f_c^{\cdot n}(z)$ ($n \geq 1$, $p \geq 1$). Then the solutions of the equation $\Phi_{n,p}(c, z) = 0$ consist of all $(c, z) \in \mathbb{C}^2$ such that $z$ is a preperiodic point of $f_c$ with preperiod $l$ and period $k$ where $0 \leq l \leq n$ and $k|p$. By abuse of notation, we will consider a polynomial in $\mathbb{C}[c, z]$ as a polynomial in $\mathbb{K}[z]$ where $\mathbb{K} = \mathbb{C}(c)$ is the field of rational functions about $c$.

**Definition 4.1.** A polynomial $g(c, z) \in \mathbb{C}[c, z]$ is called squarefree if it can’t be division by $h(c, z)^2$ for any non-constant $h(c, z) \in \mathbb{C}[c, z]$. 
Lemma 4.2. There exists a unique double indexed sequence of squirefree polynomials \( \{Q_{n,p}(c, z)\}_{n \geq 1, p \geq 1} \subset \mathbb{C}[c, z] \subset K[z] \) monic about \( z \) such that

\[
\Phi_{n,p}(c, z) = \Phi_{n-1,p}(c, z) \prod_{k | p} Q_{n,k}(c, z) \quad \text{for all } n \geq 1, \ p \geq 1.
\]

Proof. Fix any \( n \geq 1 \). We claim: for any \( c_0 \in \mathbb{C} \setminus M_d \), \( p \geq 1 \), all roots of \( \Phi_{n,p}(c_0, z) \) are simple. (Demonstration: In this case, all periodic points of \( f_{c_0} \) are repelling and the critical orbit escapes to \( \infty \). Then for any root \( z_0 \) of \( \Phi_{n,p}(c_0, z) \), \( (\partial \Phi_{n,p}/\partial z)(c_0, z_0) = [f_{c_0}^n]'(z_0)([f_{c_0}^p]'(z_0) - 1) \neq 0 \). From this claim and the fact that \( \Phi_{n,p}(c, z) \) is monic about \( z \), it deduce that if we can find a sequence of polynomials \( \{Q_{n,p}(c, z)\}_{n \geq 1, p \geq 1} \) which satisfy the equation in the lemma, they are naturally squirefree.

Let \( c_0 \in \mathbb{C} \setminus M_d \) be arbitrarily. The fact that \( z_0 \) is a root of \( \Phi_{n-1,p}(c_0, z) \) implies \( z_0 \) is a root of \( \Phi_{n,p}(c_0, z) \). By the claim above, we have \( \Phi_{n-1,p}(c_0, z) | \Phi_{n,p}(c_0, z) \in \mathbb{C}[z] \). Since \( c_0 \) is any point of \( \mathbb{C} \setminus M_d \), we also have \( \Phi_{n-1,p}(c, z) | \Phi_{n,p}(c, z) \in K[z] \).

We proceed by induction on \( p \). For \( p = 1 \), we define \( Q_{n,1} = \Phi_{n,1}(c, z)/\Phi_{n-1,1}(c, z) \). It satisfies the requirement of the lemma.

Assume now that for every \( 1 \leq k < p \), the polynomial \( Q_{n,k}(c, z) \) is defined and satisfies the requirement of the lemma. Let \( c_0 \) be any parameter in \( \mathbb{C} \setminus M_d \). Note that for \( 1 \leq k < p \), the two polynomials \( (en \ z) \Phi_{n-1,k}(c_0, z) \) and \( Q_{n,k}(c_0, z) \) don’t have a common root (by the claim above). Thus, if \( z_0 \) is a root of \( Q_{n,k}(c_0, z) \), then it is a preperiodic point of \( f_{c_0} \) with preperiod \( n \) and period \( m \) (and \( m|k \)). In fact, \( m \) must be equal to \( k \). (Otherwise, \( Q_{n,k}(c_0, z) \cdot \prod_{m'|m} Q_{n,m'}(c_0, z) \) would have a double root at \( z_0 \) by induction, but would at the same time divide \( \Phi_{n,k}(c_0, z) \), a contradiction to the claim above). Then we can conclude that any two polynomials among \( \{ \Phi_{n-1,p}(c_0, z), Q_{n,k}(c_0, z) \}_{1 \leq k < p} \) have no common roots and any one among \( \{ \Phi_{n-1,p}(c_0, z), Q_{n,k}(c_0, z) \}_{1 \leq k < p} \) divides \( \Phi_{n,p}(c_0, z) \). Hence \( \Phi_{n-1,p}(c_0, z) \cdot \prod_{k | p} Q_{n,k}(c_0, z) \) divides \( \Phi_{n,p}(c_0, z) \) in \( \mathbb{C}[z] \). As \( c_0 \) is any point of \( \mathbb{C} \setminus M_d \), the polynomial \( \Phi_{n-1,p}(c, z) \cdot \prod_{k | p} Q_{n,k}(c, z) \) divides \( \Phi_{n,p}(c, z) \) in \( K[z] \). We can then define

\[
Q_{n,p}(c, z) = \Phi_{n,p}(c, z)/[\Phi_{n-1,p}(c, z) \cdot \prod_{k | p} Q_{n,k}(c, z)].
\]

It satisfies the requirement of the lemma. \( \square \)

Recall that \( \{\nu_d(p)\}_{p \geq 1} \) is the unique sequence of positive integers satisfying the recursive relation \( d^p = \sum_{k | p} \nu_d(k) \). It is easy to see that the degree of \( Q_{0,p} \) is \( \nu_d(p) \) and the degree of \( Q_{n,p} \) is \( \nu_d(p)(d - 1)d^{n-1} \) for \( n \geq 1 \).

Remark 4.3. Note that \( \Phi_{n,p}(c, z) = \Phi_{n-1,p}(c, f_c(z)) \) for any \( n \geq 1, \ p \geq 1 \). By the definition of \( Q_{n,p} \), we have

\[
\begin{align*}
\prod_{k | p} Q_{n-1,p}(c, f_c(z)) &= \prod_{k | p} Q_{n,p}(c, z) \quad n \geq 2 \\
\prod_{k | p} Q_{0,p}(c, f_c(z)) &= \prod_{k | p} Q_{0,p}(c, z) \prod_{k | p} Q_{1,p}(c, z) \quad n = 1
\end{align*}
\]
for any \( p \geq 1 \). By induction on \( p \), it follows

\[
\begin{align*}
Q_{n-1,p}(c, f_c(z)) &= Q_{n,p}(c, z) & n \geq 2 \\
Q_{0,p}(c, f_c(z)) &= Q_{0,p}(c, z)Q_{1,p}(c, z) & n = 1
\end{align*}
\]

for any \( p \geq 1 \). This equation implies that we can obtain the properties of \( Q_{n,p} \) by induction on \( n \).

**Proposition 4.4.** Let \( n \geq 1 \), \( p \geq 1 \) be any pair of integers and \( c_0 \in \mathbb{C} \) be any parameter. Then \( z_0 \in \mathbb{C} \) is a root of \( Q_{n,p}(c_0, z) \) if and only if one of the following 5 mutually exclusive conditions holds:

1. \( z_0 \) is a \((n, p)\)-preperiodic point of \( f_{c_0} \) such that \( f_{c_0}^l(z_0) \neq 0 \) for any \( 0 \leq l < n \) and \([f_{c_0}^m]'(f_{c_0}^n(z_0)) \neq 1\).
2. \( z_0 \) is a \((n, p)\)-preperiodic point of \( f_{c_0} \) such that \( f_{c_0}^l(z_0) = 0 \) for some \( 0 \leq l < n \).
3. \( z_0 \) is a \((n, m)\)-preperiodic point of \( f_{c_0} \) and \([f_{c_0}^m]'(f_{c_0}^n(z_0)) = 1\).
4. \( z_0 \) is a \((n-1, p)\)-preperiodic point of \( f_{c_0} \) and \( f_{c_0}^{(n-1)}(z_0) = 0\).

**Proof.** The proof goes by induction on \( n \). As \( n = 1 \), \( Q_{0,p}(c, f_c(z)) = Q_{0,p}(c, z) \cdot Q_{1,p}(c, z) \). For any \( c_0 \in \mathbb{C} \), \( z_0 \) is a multiplier root of \( Q_{0,p}(c_0, f_{c_0}(z)) \iff (c_0, f_{c_0}(z_0)) \in C_{0,p,2} \cup C_{0,p,3} \) (case 1) or \( z_0 = 0 \) and \( c_0 \) is the center of hyperbolic component with period \( p \) (case 2).

In case 1, if \( z_0 \) is periodic, then \( z_0 \) is a root of \( Q_{0,p}(c_0, z) \). Moreover, \( Q_{0,p}(c_0, f_{c_0}(z)) \) and \( Q_{0,p}(c_0, z) \) have the same multiplicity at \( z_0 \), so \( z_0 \) is not a root of \( Q_{1,p}(c_0, z) \). If \( z_0 \) is not periodic, by 3 of Theorem 2.1, \( z_0 \) is not a root of \( Q_{0,p}(c_0, z) \). So \( z_0 \) is not a common root of \( Q_{0,p}(c_0, z) \) and \( Q_{1,p}(c_0, z) \) in case 2, by 4 of Theorem 2.1, \( z_0 \) is simple root of \( Q_{0,p}(c_0, z) \), hence \( z_0 \) is also a root of \( Q_{0,p}(c_0, f_{c_0}(z)) \). In any other situation, \( Q_{0,p}(c_0, f_{c_0}(z)) \) only has simple root. Then \( z_0 \) is a common root of \( Q_{0,p}(c_0, z) \) and \( Q_{1,p}(c_0, z) \) if and only if \( z_0 = 0 \) and \( c_0 \) is the center of some hyperbolic component with period \( p \) (condition (4)). Except this case, \( z_0 \) is a root of \( Q_{1,p}(c_0, z) \iff f_{c_0}(z_0) \) is a root of \( Q_{0,p}(c_0, z) \) but \( z_0 \) is not a root of \( Q_{0,p}(c_0, z) \). Then 2 of Theorem 2.1 implies that \( z_0 \) satisfies one of the conditions (0), (1), (2), (3) in Proposition 4.4.

Assume that the proposition is established for \( 1 \leq l < n \). At this time, \( Q_{n,p}(c, z) = Q_{n-1,p}(c, f_c(z)) \). So for any \( c_0 \in \mathbb{C} \), \( z_0 \) is a root of \( Q_{n,p}(c_0, z) \) if and only if \( f_{c_0}(z_0) \) is a root of \( Q_{n-1,p}(c_0, z) \). Then by the inductive assumption, the point \( z_0 \) satisfies one of the 5 exclusive conditions in Proposition 4.4.

Now we set

\[
\mathcal{X}_{n,p,0} = \{(c, z) \in \mathbb{C}^2 | (c, z) \text{ satisfies Condition } (0) \text{ in Proposition 4.4} \}.
\]
and for $1 \leq \alpha \leq 4$, set

$$C_{n,p,\alpha} = \{(c, z) \in \mathbb{C}^2 | (c, z) \text{ satisfies Condition } (\alpha) \text{ in Proposition } 4.4\}.$$ 

It is easy to see that $\mathcal{X}_{n,p,0} \cup C_{n,p,1} \cup C_{n,p,2} = \mathcal{X}_{n,p}$ and $C_{n,p,1} \cup C_{n,p,2} \cup C_{n,p,3} \cup C_{n,p,4}$ is a finite set. Then we have

$$\mathcal{X}_{n,p} = \{(c, z) | Q_{n,p}(c, z) = 0\}.$$ 

## 5 The irreducible factorization of $Q_{n,p}$

In the periodic case ($n = 0$), we know that $Q_{0,p}$ is smooth and irreducible (Proposition 2.1). But in the preperiodic case ($n \geq 1$), the polynomial $Q_{n,p}$ displays a very different behavior: for $d = 2$, it is smooth and irreducible as the periodic case, however, for $d \geq 3$, it is neither smooth nor irreducible. In this section, we will find its irreducible factorization and prove the smoothness for each irreducible component. We will also show that these components pairwise intersect transversally at the singular points of $\mathcal{X}_{n,p}$.

### 5.1 Affine algebraic curves, singular points and tangents

The objective here is to give some definition and notation about affine algebraic curve that will be used later. The material can be found in any book about algebraic curves, for example [G].

**Definition 5.1.** An affine algebraic curve over $\mathbb{C}$ is defined as the set

$$\mathcal{C} = \{(a, b) \in \mathbb{C}^2 | f(a, b) = 0\}$$

for a non-constant squarefree polynomial $f(a, b) \in \mathbb{C}[x,y]$.

The polynomial $f$ is called the defining polynomial of $\mathcal{C}$, the degree of $f$ is called the degree of $\mathcal{C}$. We assume that all polynomials appearing in this section are squarefree.

If $f = \prod_{i=1}^{m} f_i$, where $f_i$ are the irreducible factor of $f$, we say that the affine curve defined by $f_i$ is a component of $\mathcal{C}$. Furthermore, the curve $\mathcal{C}$ is said to be irreducible if the defining polynomial is irreducible.

**Definition 5.2.** Let $\mathcal{C}$ be an affine algebraic curve for $\mathbb{C}$ defined by $f \in \mathbb{C}[x,y]$, and let $P = (a,b) \in \mathcal{C}$. The multiplicity of $\mathcal{C}$ at $P$, denoted by $\text{mult}_P(\mathcal{C})$, is defined as the order of the first non-vanishing term in the Taylor expansion of $f$ at $P$, i.e.

$$f(x,y) = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{t=0}^{s} \binom{s}{t} (x-a)^t (y-b)^{s-t} \frac{\partial^sf}{\partial x^t \partial y^{s-t}}(a,b).$$

If $\text{mult}_P(\mathcal{C}) = 1$, the point $P$ is called a smooth point of $\mathcal{C}$. If $\text{mult}_P(\mathcal{C}) = r > 1$, then we say that $P$ is a singular point of multiplicity $r$. We say that $\mathcal{C}$ or $f$ is smooth if any point on $\mathcal{C}$ is smooth.
The following theorem provides a topological interpretation of the irreducibility of polynomials.

**Theorem 5.3.** The polynomial $f$ is irreducible if and only if the set of smooth points of $f$ is connected.

Let $P = (a, b) \in \mathbb{C}$ be a point of multiplicity $r$ ($r \geq 1$). Then the first non-vanishing term in the Taylor expansion of $f$ at $P$ is

$$T_r(x, y) = \sum_{t=0}^{r} \binom{r}{t} (x-a)^t (y-b)^{r-t} \frac{\partial^r f}{\partial x^t \partial y^{r-t}} (a, b).$$

Note that $T_r(c, z)$ is a homogeneous polynomial about $x-a$ and $y-b$, so all its irreducible factors are linear and they will be called the tangents of $C$ at $P$.

**Definition 5.4.** Let $C$, $P$, $T_r(c, z)$ be as above. Then the tangents of $C$ at $P$ are the linear irreducible factors of $T_r(x, y)$ and the multiplicity of a tangent is the multiplicity of the corresponding factor.

For analyzing a singular point $P$ on a curve $C$ we need to know its multiplicity but also the multiplicities of the tangents. If all the $r$ tangents at the point $P$ are different, then this singularity is of well-behaved type.

**Definition 5.5.** A singular point $P$ of multiplicity $r$ on an affine plane curve $C$ is called ordinary if the $r$ tangents to $C$ at $P$ are distinct, and non-ordinary otherwise.

**Example.** Considering the following three curves in $\mathbb{C}^2$. The pictures represent the projection of these curves to $\mathbb{R}^2$. $(0, 0)$ is the unique singular point of $A$, $B$, $C$. For $A$, the origin $\text{mult}_{(0,0)}(A) = 2$ and there are two different tangents $x \pm y = 0$ at $(0, 0)$. So $(0, 0)$ is a double ordinary singular point of $A$, called a “node”. For $B$, $\text{mult}_{(0,0)}(B) = 2$ and there is only one tangent $y = 0$ at $(0, 0)$ with multiplicity 2. So $(0, 0)$ is a double non-ordinary singular point of $B$, called a “cusp”. For $C$, $\text{mult}_{(0,0)}(C) = 4$. There are two different tangents $x = 0, y = 0$ at $(0, 0)$ with multiplicity 2 both. So $(0, 0)$ is a non-ordinary singular point of $C$ with multiplicity 4. It can be seen as a mixing of node and cusp.

**Definition 5.6.** Let $C_1$, $C_2$ be any two affine algebraic curves and $P$ is an intersecting point of the two curves. We say that $C_1$, $C_2$ intersect transversally at $P$ if $P$ is a smooth point for both $C_1$ and $C_2$ and the tangents of $C_1$, $C_2$ at $P$ are distinct.
5.2 Factorization of $Q_{n,p}$ and the behavior at its singular points

Fix any $n \geq 1$, $p \geq 1$. We have the following factorization result.

**Lemma 5.7. (Algebraic version)** There exists a unique sequence of monic polynomials 
\[ \{q^j_{n,p}(c, z)\}_{1 \leq j \leq d-1} \subset \mathbb{C}[c, z] \subset K[z] \] such that 
\[ Q_{n,p}(c, z) = \prod_{j=1}^{d-1} q^j_{n,p}(c, z). \]

The degree of $q^j_{n,p}$ is $d^{n-1} \nu_d(p)$. All points in $C_{n,p,4}$ are the roots of $q^j_{n,p}$ for any 
$1 \leq j \leq d-1$, and there are no other common roots for $q^i_{n,p}$ and $q^j_{n,p}$ with $1 \leq i \neq j \leq d-1$.

**Topological version** Let $\mathcal{V}^j_{n,p} = \{(c, z) \in \mathbb{C}^2 | q^j_{n,p}(c, z) = 0\} (1 \leq j \leq d-1)$. Then
\[ C_{n,p,4} \subset \mathcal{V}^j_{n,p} \text{ for any } 1 \leq j \leq d-1 \text{ and } \{\mathcal{V}^j_{n,p} \setminus C_{n,p,4}\}_{1 \leq j \leq d-1} \text{ are pairwise disjoint.} \]

**Proof.** Let $\overline{K}$ be a fixed algebraic closure of $K$. Set $\omega = e^{2\pi i/d}$. Let $\Delta$ be a root of $Q_{0,p}(c, z)$ in $\overline{K}$. Then $\omega\Delta, \ldots, \omega^{d-1}\Delta$ are roots of $Q_{1,p}(c, z)$ in $\overline{K}$.

Let us factorize $Q_{0,p}(c, z)$ in $\overline{K}$ by 
\[ Q_{0,p}(c, z) = \prod_{s=1}^{\nu_d(p)} (z - \Delta_s) \]

($\Delta_s \neq \Delta_s$ for $s_1 \neq s_2$). Then $Q_{1,p}(c, z)$ can be expressed as 
\[ Q_{1,p}(c, z) = \prod_{s=1}^{\nu_d(p)} (z - \omega \Delta_s) \cdots (z - \omega^{d-1}\Delta_s) = \prod_{j=1}^{d-1} \prod_{s=1}^{\nu_d(p)} (z - \omega^j \Delta_s) = \prod_{j=1}^{d-1} (\omega^j)^{\nu_d(p)} \prod_{s=1}^{\nu_d(p)} (\omega^{-j} z - \Delta_s) \]

Note that $d | \nu_d(p)$ so $(\omega^j)^{\nu_d(p)} = 1$. For $1 \leq j \leq d - 1$, set 
\[ q^j_{1,p}(c, z) = \prod_{s=1}^{\nu_d(p)} (z - \omega^j \Delta_s) = Q_{0,p}(c, \omega^{-j} z) \in \mathbb{C}[c, z] \subset K[z]. \]

Then $q^j_{1,p}(c, z)$ is a polynomial in $(c, z)$ and is monic in $K[z]$, satisfying 
\[ Q_{1,p}(c, z) = \prod_{j=1}^{d-1} q^j_{1,p}(c, z). \]

This gives a factorization of $Q_{1,p}$ in $K[z]$. For $n \geq 2$, we can define $q^j_{n,p}(c, z)$ inductively by 
$q^j_{n,p}(c, z) = q^j_{n-1,p}(c, f_c(z))$. As $Q_{n,p}(c, z) = Q_{n-1,p}(c, f_c(z))$, we have 
\[ Q_{n,p}(c, z) = \prod_{j=1}^{d-1} q^j_{1,p}(c, z). \]
This is a factorization of $Q_{n,p}(c,z)$ in $K[z]$.

We are left to prove that each $q_{n,p}^j(c,z)$ satisfies the remaining properties announced in the lemma. For $n = 1$, since $q_{1,p}^1(c,z) = Q_{0,p}(c,\omega^{-j}z)$, then $(c_0,z_0)$ is a common root of $q_{1,p}^1(c,z)$ and $q_{1,p}^j(c,z)$ for some $1 \leq i \neq j \leq d - 1 \iff (c_0,\omega^{-i}z_0)$ and $(c_0,\omega^{-j}z_0)$ are all roots of $Q_{0,p}(c,z) \iff (c_0,z_0) = (c_0,0) \in C_{1,p,4}$. For $n \geq 2$, the conclusion can be deduced from that of case $n = 1$ and the definition of $q_{n,p}(c,z)$.

From Lemma 5.7, we can see that $Q_{n,p}$ is reducible and non-smooth ($C_{n,p,4}$ belongs to the set of singular points of $Q_{n,p}$) for $d \geq 3$. Let us now concentrate our study on the factors $q_{n,p}^j(c,z)$. For $n \geq 2$, $q_{n,p}^j(c,z)$ is defined by $q_{n,p}^j(c,z) = q_{n-1,p}^j(c,f_c(z))$. Interpret these equations by topological view, we obtain a sequence of maps

$$
\left\{ \varphi_{n,p}^j : \mathcal{V}_{n,p}^j \rightarrow \mathcal{V}_{n-1,p}^j, \quad (c,z) \mapsto (c,f_c(z)) \right\} | n \geq 2, p \geq 1, 1 \leq j \leq d - 1
$$

Note that for $n = 1$, we can also define a map $\varphi_{1,p}^j : \mathcal{V}_{1,p}^j \rightarrow \mathcal{X}_{0,p}$ by $\varphi_{n,p}^j(c,z) = (c,f_c(z))$.

**Lemma 5.8.** For any $p \geq 1$, $1 \leq j \leq d - 1$,

- the map $\varphi_{1,p}^j : \mathcal{V}_{1,p}^j \rightarrow \mathcal{X}_{0,p}$ is a homeomorphism.
- for $n \geq 2$, the map $\varphi_{n,p}^j : \mathcal{V}_{n,p}^j \rightarrow \mathcal{V}_{n-1,p}^j$ is of degree $d$ with critical set

$$
D_{n,p}^j = \{(c,0) \in \mathcal{V}_{n,p}^j | q_{n,p}^j(c,0) = 0\}.
$$

Moreover, each critical point has multiplicity $d - 1$.

**Proof.** It can be deduced directly from the definition of $q_{n,p}^j(c,z)$, $n,p \geq 1, 1 \leq j \leq d - 1$.

The following proposition is the core of this section.

**Proposition 5.9.** For any $n,p \geq 1, 1 \leq j \leq d - 1$, the polynomial $q_{n,p}^j(c,z)$ is smooth and irreducible.

The proof of this proposition will be postponed to 5.3.

By Proposition 5.9, we can restate Lemma 5.8 as follows.

**Lemma 5.10.** For any $p \geq 1, 1 \leq j \leq d - 1$,

- the maps $\varphi_{1,p}^j : \mathcal{V}_{1,p}^j \rightarrow \mathcal{X}_{0,p}$ is a conformal homeomorphism.
• For $n \geq 2$, the map $\psi_{n,p}^j : \mathcal{V}_{n,p}^j \rightarrow \mathcal{V}_{n-1,p}^j$ is a holomorphic branched covering of degree $d$ with critical set

$$D_{n,p}^j = \{(c,0) \in \mathcal{V}_{n,p}^j | q_{n,p}^j(c,0) = 0\}$$

Moreover, each critical point has multiplier $d - 1$.

**Remark 5.11.** By the definition of $q_{1,p}^j(c,z)$, we have $q_{1,p}^j(c,z) = q_{1,p}^j(c,\omega^{i-j}z)$ for any $1 \leq i \neq j \leq d - 1$. Then we obtain a “rotation” $r_{ij}$ between $\mathcal{V}_{1,p}^j$ and $\mathcal{V}_{1,p}^j$ for any $1 \leq i \neq j \leq d - 1$, defined by $r_{ij}(c,z) = (c,\omega^{i-j}z)$. It is obviously a conformal map, then all $\mathcal{V}_{1,p}^j_{1\leq j \leq d-1}$ are conformal equivalent (also equivalent to $\mathcal{X}_{0,p}$). But unfortunately, the map $r_{ij}$ can’t be lifted along $\psi_{2,p}^j$ and $\psi_{2,p}^j$ because $r_{ij}$ doesn’t map the critical values of $\psi_{2,p}^j$ to that of $\psi_{2,p}^j$, so we can not prove that $\mathcal{V}_{2,p}^j$ is conformal equivalent to $\mathcal{V}_{2,p}^j$ by simply lifting $r_{ij}$.

**Q:** Are $\left\{\mathcal{V}_{n,p}^j\right\}_{1 \leq j \leq d-1}$ conformal equivalent for fixed $n \geq 2$, $p \geq 1$?

Now, we will provide some discussion about the singular points of on $\mathcal{X}_{n,p}$. Following the definition and notation in [5.1], we have the proposition below.

**Proposition 5.12.** For $n, p \geq 1$, each singular point of $\mathcal{X}_{n,p}$ is ordinary and has multiplicity $d - 1$.

**Proof.** For any $(c_0,0) \in C_{1,p,4}$, 0 is a simple root of $Q_{0,p}(c_0, z)$, then $\partial Q_{0,p}/\partial z(c_0,0) \neq 0$. By the Implicit Function theorem, there exists a local holomorphic function $z(c)$ in a neighborhood of $c_0$ such that $z(c_0) = 0$ and $z(c)$ is the attracting $p$ periodic point of $f_c$. Then as illustrated in section [3], we have a local holomorphic function

$$\mu(c) = [f_c^{\circ p}]'(z(c)) = f_c'(f_c^{\circ (p-1)}(z(c))) \cdots f_c'(f_c^{\circ (p-1)}(z(c)))$$

(5.1)

which has local degree $d - 1$ at $c_0$.

Let $z(c) = a_k(c - c_0)^k + O((c - c_0)^{k+1})$ be the Taylor expansion of $z(c)$ at $c_0$. Note that $f_c^{\circ (p-1)}(0), \ldots, f_c(0)$ are all distinct from 0. Then substitute the expansion of $z(c)$ into (5.1), we have

$$\mu(c) = \lambda (c - c_0)^{(d-1)k} + \text{higher order terms}$$

in a small neighborhood of $c_0$ with $\lambda \neq 0$. Thus $k$ must be 1. Then $Q_{0,p}(c,z)$ can be expressed as

$$Q_{0,p}(c,z) = a_{0,p}(c - c_0) + b_{0,p}z + \text{higher order terms}$$

at $(c_0,0)$ with $a_{0,p} \neq 0$, $b_{0,p} \neq 0$. And hence

$$q_{1,p}^j(c,z) = Q_{0,p}(c,\omega^{-j}z) = a_{0,p}(c - c_0) + b_{0,p}\omega^{-j}z + \text{higher order terms}$$

Therefore the tangents of $\left\{\mathcal{V}_{1,p}^j\right\}_{1 \leq j \leq d-1}$ at $(c_0,0)$ are pairwise distinct.
Now assume that for $1 \leq l < n$, the tangents of $\{V_{l,p}^j\}_{1 \leq j \leq d-1}$ are pairwise different at each point of $C_{l,p,4}$. Let $(c_0, z_0)$ be any point in $C_{n,p,4}$, then $(c_0, f_{c_0}(z_0)) = (c_0, w_0) \in C_{n-1,p,4}$. Denote the Taylor expansion of $q_{n-1,p}^l(c, z)$ at $(c_0, w_0)$ by

$$q_{n-1,p}^l(c, z) = a_{n-1,p}^j(c - c_0) + b_{n-1,p}^j(w - w_0) + \text{higher order terms} \quad (5.2)$$

where $b_{n-1,p}^j \neq 0 \ (w_0$ is a simple root of $q_{n-1,p}^l(c_0, z)$). Since $[\partial f_c/\partial c](c_0, z_0) = 1$, $[\partial f_c/\partial z](c_0, z_0) = dz_0^{d-1}$, the Taylor expansion of $f_c(z)$ at $(c_0, z_0)$ is

$$f_c(z) = w_0 + (c - c_0) + dz_0^{d-1}(z - z_0) + \text{higher order terms} \quad (5.3)$$

Substituting (5.3) into (5.2), we obtain

$$q_{n,p}^l(c, z) = (a_{n-1,p}^j + b_{n-1,p}^j)(c - c_0) + dz_0^{d-1} \cdot b_{n-1,p}^j(z - z_0) + \text{higher order terms}$$

By the inductive assumption, the tangents of $\{V_{n,p}^j\}_{1 \leq j \leq d-1}$ at $(c_0, z_0)$ are pairwise distinct.

So for $n, p \geq 1$ and any point $(c_0, z_0) \in C_{n,p,4}$, the first non vanishing term of $Q_{n,p}^l(c, z)$ at $(c_0, z_0)$ is

$$\prod_{j=1}^{d-1} (a_{n,p}^j(c - c_0) + b_{n,p}^j(z - z_0))$$

with $\{(a_{n,p}^j, b_{n,p}^j) \neq (0, 0)\}_{1 \leq j \leq d-1}$ pairwise different. Then $(c_0, z_0)$ is an ordinary singular point with multiplicity $d - 1$.

\[\square\]

Remark 5.13. Lemma 5.7, Proposition 5.9, Proposition 5.12 provide us a clear topological picture of $X_{n,p}$:

- $C_{n,p,4}$ is exactly the set of singular points of $Q_{n,p}$;
- $X_{n,p}$ is the union of the Riemann surfaces $\{V_{n,p}^j\}_{1 \leq j \leq d-1} \text{ and two of them intersect transversally at the singular points of } X_{n,p}$.

### 5.3 Proof of the smoothness and the irreducibility of $q_{n,p}^j$

The objective here is to prove Proposition 5.9.

The approach to prove the smoothness is similar to that in [BT]. The idea is to prove that some partial derivative of $q_{n,p}^j$ is non vanishing. Following A. Epstein, we will express this derivative as the coefficient of a quadratic differential of the form $(f_c)_* Q - Q$. Thurston’s contraction principle gives $(f_c)_* Q - Q \neq 0$, therefore the non-nullness of our partial derivative.

The approach to the irreducibility is based on the connectivity of periodic curve $X_{0,p}$. Then we will show the connectivity of $V_{n,p}^j$ using the branched covering $\varphi_{n,p}^j$ by induction on $n$. 

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Here we list some definitions and results about quadratic differentials and Thurston’s contraction principle. All their proofs can be found in [BT] and [LG].

We use \( Q(C) \) to denote the set of meromorphic quadratic differentials on \( C \) whose poles (if any) are all simple. If \( Q \in Q(C) \) and \( U \) is a bounded open subset of \( C \), the norm

\[
\|Q\|_U := \int_U |q|
\]

is well defined and finite.

For \( f : C \to C \) a non-constant polynomial and \( Q = q \, dz^2 \) a meromorphic quadratic differential on \( C \), the pushforward \( f_*Q \) is defined by the quadratic differential

\[
f_*Q := Tq \, dz^2 \quad \text{with} \quad Tq(z) := \sum_{f(w) = z} \frac{q(w)}{f'(w)^2}.
\]

If \( Q \in Q(C) \), then \( f_*Q \in Q(C) \) also. The following lemma is a weak version of Thurston’s contraction principle.

**Lemma 5.14.** If \( f : C \to C \) is a polynomial and if \( Q \in Q(C) \), then \( f_*Q \neq Q \).

The formulas below appeared in [LG] chapter 2, we write them together as a lemma.

**Lemma 5.15 (Levin).** For \( f = f_c \), we have

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{dz^2}{z} & = 0 \\
\frac{dz^2}{z - a} & = \frac{1}{f'(a)} \left( \frac{dz^2}{z - f(a)} - \frac{dz^2}{z - c} \right) \text{ if } a \neq 0
\end{array} \right.
\]

(5.4)

**Proof of Proposition 5.9** The proof goes by induction on \( n \).

For \( n = 1 \), as \( q^1_{1,p}(c, z) = Q_{0,p}(c, \omega^{-j}z) \) and \( Q_{0,p}(c, z) \) is smooth and irreducible, we know that \( q^1_{1,p}(c, z) \) is smooth and irreducible for \( 1 \leq j \leq d - 1 \). Assume that for \( 1 \leq l < n \), \( 1 \leq j \leq d - 1 \), the polynomial \( q^l_{1,p}(c, z) \) is smooth and irreducible. Then we will show that \( q^0_{n,p}(c, z) \) is smooth and irreducible. Now fix any \( j_0 \in [1, d - 1] \).

**Smoothness of \( q^{j_0}_{n,p} \):** As \( q^{j_0}_{n,p}(c, z) = q^{j_0}_{n-1,p}(c, f_c(z)) \), for any \( (c_0, z_0) \) a root of \( q^{j_0}_{n,p}(c, z) \), we have

\[
\begin{align*}
\frac{\partial q^{j_0}_{n,p}}{\partial c}(c_0, z_0) & = \frac{\partial q^{j_0}_{n-1,p}}{\partial c}(c_0, w_0) + \frac{\partial q^{j_0}_{n-1,p}}{\partial z}(c_0, w_0) \\
\frac{\partial q^{j_0}_{n,p}}{\partial z}(c_0, z_0) & = \frac{\partial q^{j_0}_{n-1,p}}{\partial z}(c_0, w_0) \cdot f'_c(z_0)
\end{align*}
\]

(5.5)

where \( w_0 = f_{c_0}(z_0) \). Then if \( z_0 \neq 0 \), by assumption of induction of smoothness, \( [\partial q^{j_0}_{n,p}/\partial c](c_0, z_0) \) and \( [\partial q^{j_0}_{n,p}/\partial c](c_0, z_0) \) cannot equal to 0 simultaneously, it follows that \( q^{j_0}_{n,p}(c, z) \) is smooth at \((c_0, z_0)\). So we are left to prove that \( q^{j_0}_{n,p}(c, z) \) is smooth at \((c_0, 0)\). In
undetermined coefficients (note that $y_Q$ be a quadratic differential in this case, $c_0$ is a Mişurewicz parameter with preperiod $n - 1$ and period $p$. Note that $[\partial q_{n,p}^{0}/\partial z](c_0, 0) = 0$, then we have to show $[\partial q_{n,p}^{0}/\partial c](c_0, 0) \neq 0$. Since

$$\frac{\partial Q_{n,p}}{\partial c}(c_0, 0) = \prod_{1 \leq j \neq j_0 \leq d-1} q_{n,p}^{j}(c_0, 0) \cdot \frac{\partial q_{n,p}^{0}}{\partial c}(c_0, 0)$$

and by Lemma 5.17, the point $(c_0, 0)$ is not a root of $\prod_{j \neq j_0} q_{n,p}^{j}(c, z)$, we only have to show $[\partial Q_{n,p}/\partial c](c_0, 0) \neq 0$. Furthermore,

$$\frac{\partial \Phi_{n,p}}{\partial c}(c_0, 0) = \frac{\Phi_{n-1,p}(c_0, 0)}{\prod_{k|p,k<p} Q_{n,k}(c_0, 0)} \cdot \frac{\partial Q_{n,p}}{\partial c}(c_0, 0)$$

and it is known that $\Phi_{n-1,p}(c_0, 0) \cdot \prod_{k|p,k<p} Q_{n,k}(c_0, 0) \neq 0$. So we only have to show $[\partial \Phi_{n,p}/\partial c](c_0, 0) \neq 0$. We shall choose a meromorphic quadratic differential with simple poles such that

$$(f_{c_0})_*, Q = \Phi_{n-1,p}(c_0, 0) = \prod_{k|p,k<p} Q_{n,k}(c_0, 0) \frac{dz^2}{z - c_0}.$$

Then with Lemma 5.14 we obtain $[\partial \Phi_{n,p}/\partial c](c_0, 0) \neq 0$.

We shall use the following notations:

$$z_k := f_{c_0}^{\frac{\alpha n}{p} + k}(0), \quad \delta_k := f_{c_0}'(z_k) = dz_k^{d-1}, \quad 0 \leq k \leq p - 1$$

$$y_l := f_{c_0}^{l}(0), \quad \varepsilon_l := f_{c_0}'(y_l) = dy_l^{d-1}, \quad 1 \leq l \leq n - 1$$

With these notations and a bit of calculations, we get

$$\frac{\partial \Phi_{n,p}}{\partial c}(c_0, 0) = \frac{\partial f_{c_0}^{\frac{\alpha n}{p}}}{\partial c}(c_0, 0) - \frac{\partial f_{c_0}^{\alpha n}}{\partial c}(c_0, 0)$$

$$= (\delta_0 \cdots \delta_{p-1} - 1)(\varepsilon_{n-1} \cdots \varepsilon_1 + \cdots + \varepsilon_{n-1} \varepsilon_{n-2} + \varepsilon_{n-1} + 1)$$

$$+ \lambda_{p-1} \cdots \lambda_1 + \cdots + \lambda_{p-1} + 1$$

Denote $(\delta_0 \cdots \delta_{p-1} - 1)(\varepsilon_{n-1} \cdots \varepsilon_1 + \cdots + \varepsilon_{n-1} \varepsilon_{n-2} + \varepsilon_{n-1} + 1)$ by $\alpha$. Let

$$Q = \sum_{k=0}^{p-1} \frac{\rho_k}{z - z_k} dz^2 + \sum_{l=1}^{n-1} \frac{\lambda_l}{z - y_l} dz^2$$

be a quadratic differential in $Q(\mathbb{C})$. Here $\rho_k$ $(0 \leq k \leq p - 1)$, $\lambda_l$ $(1 \leq l \leq n - 1)$ are undetermined coefficients (note that $y_1 = c_0$). Applying Lemma 5.15 and writing $f$ for $f_{c_0}$, we have

$$f_* Q = \sum_{k=0}^{p-1} \rho_k \left( \frac{dz^2}{z - z_{k+1}} - \frac{dz^2}{z - c_0} \right) + \sum_{l=1}^{n-2} \lambda_l \left( \frac{dz^2}{z - y_{l+1}} - \frac{dz^2}{z - c_0} \right) + \lambda_{n-1} \left( \frac{dz^2}{z - z_0} - \frac{dz^2}{z - c_0} \right)$$

$$= \left( \frac{\rho_{p-1}}{\delta_{p-1}} + \frac{\lambda_{n-1}}{\varepsilon_{n-1}} \right) \frac{dz^2}{z - z_0} + \rho_0 \frac{dz^2}{\delta_0 \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{n-1}}$$

$$+ \left( \kappa - \sum_{l=1}^{n-1} \frac{\lambda_l}{\varepsilon_l} \right) \frac{dz^2}{z - y_1} + \lambda_1 \frac{dz^2}{\varepsilon_1 \varepsilon_2} \cdots + \lambda_{n-2} \frac{dz^2}{\varepsilon_{n-2} \varepsilon_{n-1}} - \left( \alpha + \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k} \right) \frac{dz^2}{z - c_0}$$

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We want to choose \( Q \) so that

\[
f_* Q - Q = - \left( \alpha + \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k} \right) \frac{dz^2}{z - c_0}
\]

It amounts then to solve the following linear system on the unknown coefficient vector \((\rho_0, \ldots, \rho_{p-1}, \lambda_1, \ldots, \lambda_{n-1})\):

\[
\begin{pmatrix}
\frac{1}{\delta_0} & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \ddots & \ddots & \ddots & \ddots & \ddots \\
\cdot & \cdot & \frac{1}{\delta_{p-2}} & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \ddots & \ddots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \frac{1}{\delta_{p-1}} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\delta_0}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\rho_1 \\
\vdots \\
\rho_{p-2} \\
\lambda_1 \\
\lambda_{n-1}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{pmatrix}
\]

Denote by \( A \) the coefficient matrix, we have

\[
\det(A) = \frac{(-1)^{n-1} \alpha}{\delta_0 \cdots \delta_{p-1} \cdot \varepsilon_1 \cdots \varepsilon_{n-1}}
\]

Then whether \( \kappa = 0 \) or not, this linear system has non-zero solutions, and one of its solutions is

\[
\begin{align*}
\rho_0 &= \delta_0 \cdots \delta_{p-1} \\
\rho_1 &= \delta_1 \cdots \delta_{p-1} \\
\vdots \\
\rho_{p-1} &= \delta_{p-1} \\
\lambda_1 &= (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1} \cdots \varepsilon_1 \\
\vdots \\
\lambda_{n-2} &= (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1} \varepsilon_{n-2} \\
\lambda_{n-1} &= (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1} 
\end{align*}
\]

Therefore, for \((\rho_0, \ldots, \rho_{p-1}, \lambda_1, \ldots, \lambda_{n-1})\) satisfies [5.6], we have

\[
f_* Q - Q = - \left( \alpha + \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k} \right) \frac{dz^2}{z - c_0} = - \frac{\partial \Phi_{n,p}}{\partial c}(c_0, 0) \cdot \frac{dz^2}{z - c_0}
\]

As a consequence \([\partial \Phi_{n,p}/\partial c](c_0, 0) \neq 0\).

**Irreducibility of** \( \bar{a}^j_{n,p} \): By the smoothness of \( \bar{a}^j_{n,p}(c, z) \) \((n, p \geq 1, 1 \leq j \leq d - 1)\) and the inductive assumption of irreducibility, we know that \( \varphi_{n,p} : \mathcal{V}_{n,p} \rightarrow \mathcal{V}_{n-1,p} \) is a branched
covering of degree $d$, $\mathcal{V}^j_{n-1,p}$ and each connected component of $\mathcal{V}^j_{n,p}$ is a Riemann surface. Then it is easy to prove that the restriction of $\psi^j_{n,p}$ to any connected component of $\mathcal{V}^j_{n,p}$ is also a branched covering. Since the set of critical points of 

$$D^j_{n,p} = \{(c,0)\mid q^j_{n,p}(c,0) = 0\}$$

is non-empty and each critical point has multiplicity $d - 1$, the set $\mathcal{V}^j_{n,p}$ must be connected. By Theorem 5.3 and the smoothness of $q^j_{n,p}$, we conclude that $q^j_{n,p}(c,z)$ is irreducible in $\mathbb{C}[c,z]$. □

6 Genus of the compactification of $\mathcal{V}^j_{n,p}$

In the previous section, we have seen that $\mathcal{X}_{n,p}$ is the union of $d - 1$ Riemann surfaces $

\{\mathcal{V}^j_{n,p}\}_{1 \leq j \leq d-1}$ and any two of them intersect transversally at the singular points of $\mathcal{X}_{n,p}$. In order to give a complete topological description of $\mathcal{X}_{n,p}$, we also need the topological characterization of each $\mathcal{V}^j_{n,p}$.

In fact, by adding an ideal boundary point at each end of $\mathcal{V}^j_{n,p}$, we obtain a compactification of $\mathcal{V}^j_{n,p}$, denoted by $\hat{\mathcal{V}}^j_{n,p}$, such that $\hat{\mathcal{V}}^j_{n,p}$ is a compact Riemann surface (in section 6.2). We will also calculate the genus of $\hat{\mathcal{V}}^j_{n,p}$ (in section 6.3). Topologically, $\mathcal{X}_{n,p}$ is completely determined by the number of its singular points, the genus of $\hat{\mathcal{V}}^j_{n,p}$ and the number of ideal boundary points added on $\mathcal{V}^j_{n,p}$ (or the number of ends of $\mathcal{V}^j_{n,p}$ for $1 \leq j \leq d - 1$).

So we can give a complete topological description of $\mathcal{X}_{n,p}$ (Lemma 6.5).

6.1 Itineraries outside the Multibrot set

Here we give a symbolic description of dynamics on the filled in Julia set for parameter outside Multibrot set. This will be used in next section.

If $c \in \mathbb{C}\setminus M_d$, the Julia set of $f_c$ is a Cantor set. If $c \in R_{M_d}(\theta)$ with $\theta \neq 0$ not necessarily periodic, the dynamical rays $R_c(\theta/d) \cdots R_c((\theta + d - 1)/d)$ bifurcate on the critical point. The set $R_c(\theta/d) \cup \ldots \cup R_c((\theta + d - 1)/d) \cup \{0\}$ decomposes the complex plane into $d$ connected components. We denote by $U_0$ the component containing the dynamical ray $R_c(0)$ and by $U_1, \ldots, U_{d-1}$ the other components in counterclockwise order (see Figure 1). The orbit of a point $x \in K_c$ has an itinerary with respect to this partition. In other words, to each $x \in K_c$, we can associate a sequence in $\mathbb{Z}^N_d$ whose $j$-th entry is equal to $k$ if $f_c^{*j-1}(x) \in U_k$. This gives a map $\iota_c : K_c \to \mathbb{Z}^N_d$. Moreover, $\iota_c$ is a bijective for any $c \in \mathbb{C}\setminus M_d$.

In $\mathbb{Z}^N_d$, we can define a shift which maps $e_1e_2e_3 \cdots$ to $e_2e_3e_4 \cdots$. A sequence in $\mathbb{Z}^N_d$ is called $(n,p)$-preperiodic if it is preperiodic under shift with preperiod $n$ and period $p$. It is known that for $c$ outside Multibrot set $M_d$, the dynamic of $f_c$ on $K_c$ is conjugate to
shift on $\mathbb{Z}_d^N$ via the map $\iota_c$. In particular, $z$ is a $(n,p)$-preperiodic point of $f_c$ if and only if $\iota_c(z)$ is a $(n,p)$-preperiodic sequence in $\mathbb{Z}_d^N$.

6.2 Compactification of $\mathcal{V}_{n,p}^j$

Denote by $\pi_{n,p}^j : \mathcal{V}_{n,p}^j \to \mathbb{C}$ the projection from $\mathcal{V}_{n,p}^j$ to the parameter plane. It is easy to see

$$\pi_{n,p}^j = \pi_{0,p} \circ \varphi_{1,p}^j \circ \cdots \circ \varphi_{n-1,p}^j \circ \varphi_{n,p}^j$$

where $\pi_{0,p}$ is the projection from $\mathcal{X}_{0,p}$ to the parameter plane. By Theorem 2.1 and Lemma 5.10, the map $\pi_{n,p}^j : \mathcal{V}_{n,p}^j \to \mathbb{C}$ is a degree $\nu_d(p)d^{n-1}$ branched covering whose set of critical points equals to $C_{n,p,1}^j \cup C_{n,p,2}^j \cup C_{n,p,3}^j$, where $C_{n,p,\alpha}^j := C_{n,p,\alpha} \cap \mathcal{V}_{n,p}^j$ for all $n, p \geq 1$, $1 \leq j \leq d - 1$, $0 \leq \alpha \leq 4$ (note that $C_{1,p,1}^j = \emptyset$). Let $V_{n,p,\alpha}^j := \pi_{n,p}^j(C_{n,p,\alpha}^j)$ ($\alpha = 1, 2, 3$). Then the critical value set of $\pi_{n,p}^j$, denoted by $V_{n,p,1}^j$, is equal to $V_{n,p,1}^j \cup V_{n,p,2}^j \cup V_{n,p,3}^j$.

Lemma 6.1. (1) For any $1 \leq i \neq j \leq d - 1$, we have $V_{n,p,1}^i \cap V_{n,p,1}^j = \emptyset$. The set $\bigcup_{j=1}^{d-1} V_{n,p,1}^j$ consists of all Misiurewicz parameters such that $c$ is $(l,p)$-preperiodic point of $f_c$ for some $0 < l \leq n - 1$.

(2) For any $1 \leq j \leq d - 1$, $V_{n,p,2}^j \cup V_{n,p,3}^j$ consists of roots and co-roots of all hyperbolic components of period $p$. 

Figure 1: The regions $U_0$, $U_1$, $U_2$, $U_3$ for a parameter $c$ belonging to $R_{M_4}(1/15)$. 

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Proof. (1) By Proposition 4.4, the set $\bigcup_{j=1}^{d-1} V^j_{n,p,1}$ consists of all Misiurewicz parameters such that $c$ is $(l,p)$-preperiodic point of $f_c$ for some $0 < l \leq n - 1$. If $c_0 \in V^i_{n,p,1} \cap V^j_{n,p,1}$ for some $1 \leq i \neq j \leq d - 1$, then $c_0$ is a $(l,p)$-preperiodic point of $f_{c_0}$ for some $l \in [1, n - 1]$ and there are two points $(c_0, z_1)$, $(c_0, z_2)$ belonging to $V^i_{n,p}$ and $V^j_{n,p}$ respectively. It follows $(c_0, 0) \in V^i_{l+1,p} \cap V^j_{l+1,p}$, a contradiction to Lemma 5.7.

(2) follows directly from $\pi^j_{n,p} = \pi_{0,p} \circ \varphi^j_{n,p} \circ \cdots \circ \varphi^j_{n-1,p} \circ \varphi^j_{n,p}$ and Theorem 2.1.

Set $\bigcup_{i=1}^{l_{n,p}} E_{n,p,i} := (\pi^j_{n,p})^{-1}(\mathbb{C} \setminus M_d)$, where $E_{n,p,i}$ is a connected component of $(\pi^j_{n,p})^{-1}(\mathbb{C} \setminus M_d)$, called an end of $V^j_{n,p}$. Fix any $i_0 \in [1, m_{n,p}]$. Since $\mathbb{C} \setminus M_d$ contains no critical values of $\pi^j_{n,p}$, the map

$$\pi^j_{n,p}|_{E_{n,p,i_0}} : E_{n,p,i_0} \to \mathbb{C} \setminus M_d$$

is a covering whose degree is denoted by $d^j_{n,p,i_0}$. Note that $\mathbb{C} \setminus M_d$ is conformal to $\hat{\mathbb{C}} \setminus \mathbb{D}$, it follows that $E_{n,p,i_0}$ is also conformal to $\hat{\mathbb{C}} \setminus \mathbb{D}$. So if we add a point $\infty_{n,p,i_0}$ to the infinite far boundary of $E_{n,p,i_0}$, then we get a new set $\hat{E}_{n,p,i_0} := E_{n,p,i_0} \cup \{\infty_{n,p,i_0}\}$ and it is conformal to $\hat{\mathbb{C}} \setminus \mathbb{D}$. The point $\infty_{n,p,i_0}$ is called the ideal boundary point of $E_{n,p,i_0}$ and $d^j_{n,p,i_0}$ is called the multiplicity of $E_{n,p,i_0}$. In this case, $\pi^j_{n,p}|_{E_{n,p,i_0}}$ can be extended to

$$\hat{\pi}^j_{n,p}|_{\hat{E}_{n,p,i_0}} : \hat{E}_{n,p,i_0} \to \hat{\mathbb{C}} \setminus M_d$$

by setting $\hat{\pi}^j_{n,p}(\infty_{n,p,i_0}) = \infty$. This map becomes a branched covering of degree $d^j_{n,p,i_0}$ with a unique branched point $\infty_{n,p,i_0}$.

Adding the ideal boundary point at each end of $V^j_{n,p}$, we obtain a compact Riemann surface $\hat{V}^j_{n,p} := V^j_{n,p} \cup \{\infty_{n,p,i_0}\}_{i=1}^{m_{n,p}}$ and an extended branched covering $\hat{\pi}^j_{n,p} : \hat{V}^j_{n,p} \to \hat{\mathbb{C}}$. We are left to calculate the number $m^j_{n,p}$ of ends for $V^j_{n,p}$ and the multiplicity $d^j_{n,p,i}$ of each end of $V^j_{n,p}$.

**Definition 6.2.** Let $X$, $Y$ be two topological spaces, $X$ is connected. Let $f : Y \to X$ be a covering. Fix any base point $x_0 \in X$, then the monodromy action of the elements in $\pi_1(X, x_0)$ gives a group morphism

$$\Phi_f : \pi_1(X, x_0) \to \text{sym}(f^{-1}(x_0))$$

The image of $\pi_1(X, x_0)$ under $\Phi_f$ is called the monodromy group of $f$, denoted by $\text{Mon}(f)$.

**Lemma 6.3.** For any $n, p \geq 1$, $1 \leq j \leq d - 1$, $1 \leq i \leq m^j_{n,p}$, we have $d^j_{n,p,i} = d$ and $m^j_{n,p} = \nu_d(p)d^{n-2}$.

Proof. The map $\pi^j_{n,p}|_{E_{n,p,i}} : E_{n,p,i} \to \mathbb{C} \setminus M_d$ is a covering. Fix $c_0 \in \mathbb{C} \setminus (M_d \cup R_M(0))$, $d^j_{n,p,i} = \#(\pi^j_{n,p}|_{E_{n,p,i}})^{-1}(c_0)$. Since $E_{n,p,i}$ is connected, $\text{Mon}(\pi^j_{n,p}|_{E_{n,p,i}})$ acts
on \( (\pi^j_{n,p}|_{\mathcal{E}^j_{n,p,i}})^{-1} (c_0) \) transitively. Then fixing any point \((c_0, z_0) \in (\pi^j_{n,p}|_{\mathcal{E}^j_{n,p,i}})^{-1} (c_0)\), the set \( (\pi^j_{n,p}|_{\mathcal{E}^j_{n,p,i}})^{-1} (c_0) \) is exactly the orbit of \((c_0, z_0)\) under \( \text{Mon} \left( \pi^j_{n,p}|_{\mathcal{E}^j_{n,p,i}} \right) \).

Let \( c_t : [0,1] \rightarrow \mathbb{C} \setminus M_d \) be a oriented simple closed curve based at \( c_0 \) such that \( c_t \) intersects \( R_{M_d}(0) \) at only one point \( c_{t_0} \). Let \( z_t \) be the \((n, p)\)-preperiodic point of \( f_{c_t} \) obtained from the analytic continuation of \( z_0 \) along \( c_t \). Note that as \( c \) varies in \( \mathbb{C} \setminus (M_d \cup R_{M_d}(0)) \), the \((n, p)\)-preperiodic points of \( f_c \), the dynamical rays \( R_c(0) \) and \( R_c((\theta_c + s)/d) \) \((s \in \mathbb{Z}_d)\) move continuously. Consequently, we have

\[
\begin{align*}
\nu_c(z_t) &= \nu_c(z_0) \quad \text{for } t \in [0, t_0) \\
\nu_c(z_t) &= \nu_c(z_1) \quad \text{for } t \in (t_0, 1]
\end{align*}
\]

Furthermore, on one hand, \( z_t \) and \( R_{c_t}(0) \) move continuously for \( t \in [0, 1] \). On the other hand, when \( c_t \) passes through \( R_{M_d}(0) \), the dynamical rays \( R_{c_t}((\theta_t + s)/d) \) \((s \in \mathbb{Z}_d)\) move discontinuously and jump from \( R_{c_{t-}}((\theta_{t-} + s)/d) \) to \( R_{c_{t+}}((\theta_{t+} + s + 1)/d) \), \( t_- < t_0 < t_+ \).

So if \( \nu_c(z_0) = c_n \ldots c_1 \ldots c_p \), then

\[
\nu_c(z_1) = (\beta_n + 1) \ldots (\beta_1 + 1) s_1 \ldots (\epsilon_p + 1)
\]

Hence the map \( \Phi_{\pi_{n,p}}(c_t) \) maps \((c_0, z_0)\) to \((c_0, z_1)\) with \( z_1 \) satisfying (6.1). Since \( \pi_1(\mathbb{C} \setminus M_d, c_0) = \langle c_t \rangle \), then we have

\[
\left( \pi^j_{n,p}|_{\mathcal{E}^j_{n,p,i}} \right)^{-1} (c_0) = \left\{ (c_0, z) | \nu_c(z) = (\beta_n + 1) \ldots (\beta_1 + 1) s_1 \ldots (\epsilon_p + 1) \right\}
\]

As a consequence, \( d^j_{n,p,i} = d \) and \( m^j_{n,p} = \nu_d(p)d^{n-2} \).

We can also use the itinerary to label the ends of \( \mathcal{V}_{n,p}^j \). The open set \( W := \mathbb{C} \setminus (M_d \cup R_{M_d}(0)) \) is simply connected. Let \( \mathcal{W}_{n,p}^j \subset \mathcal{W}_{n,p} \) be the preimage of \( W \) by \( \pi^j_{n,p} : \mathcal{W}_{n,p}^j \rightarrow \mathbb{C} \).

Since \( W \) is simply connected, each connected component of \( \mathcal{W} \) maps isomorphically to \( W \) by \( \pi^j_{n,p} \) (so there are \( \nu_d(p)d^{n-1} \) components in \( \mathcal{W}_{n,p}^j \)).

Define \( \iota^j_{n,p} : \mathcal{W}_{n,p}^j \rightarrow \mathbb{Z}_d^j \) by \( \iota^j_{n,p}(c, z) = \iota_c(z) \). As \( c \) varies in \( W \), the \((n, p)\)-preperiodic points of \( f_c \), the dynamical rays \( R_c(0) \) and \( R_c((\theta_c + s)/d) \) \((s \in \mathbb{Z}_d)\) move continuously. As a consequence, the map \( \iota^j_{n,p} : \mathcal{W}_{n,p}^j \rightarrow \mathbb{Z}_d^j \) is locally constant, whence constant on each connected component of \( \mathcal{W}_{n,p}^j \).

Since \( \iota_c : K_c \rightarrow \mathbb{Z}_d^j \) is bijective, distinct components have distinct itineraries, so each connected component of \( \mathcal{U}_{n,p,t}^j \) of \( \mathcal{W}_{n,p}^j \) can be labelled by its itinerary \( \iota^j_{n,p}(\mathcal{U}_{n,p,t}) \).

According to the proof of Lemma 6.3, each end of \( \mathcal{V}_{n,p}^j \) contains \( d \) components of \( \mathcal{W}_{n,p}^j \) and they are labelled by

\[
\left\{ (\beta_n + 1) \ldots (\beta_1 + 1) s_1 \ldots (\epsilon_p + 1) \right\}_{s \in \mathbb{Z}_d}
\]

for some \((n, p)\)-sequence \( \beta_n \ldots \beta_1 \epsilon_2 \ldots \epsilon_p s_1 \in \mathbb{Z}_d^j \). We define an equivalence relationship in all \((n, p)\)-preperiodic sequences in \( \mathbb{Z}_d^j \) such that \( \beta_n \ldots \beta_1 \epsilon_2 \ldots \epsilon_p s_1 \sim \beta'_n \ldots \beta'_1 \epsilon'_2 \ldots \epsilon'_p s' \) if and only if

\[
\beta_n \ldots \beta'_1 \epsilon'_2 \ldots \epsilon'_p s' = (\beta_n + s) \ldots (\beta_1 + s) s'(\epsilon_2 + s) \ldots (\epsilon_p + s)(\epsilon_1 + s)
\]
for some \( s \in \mathbb{Z}_d \). The equivalence class of \( \beta_n \ldots \beta_1 \epsilon_2 \ldots \epsilon_p \epsilon_1 \) is denoted by \([\beta_n \ldots \beta_1 \epsilon_2 \ldots \epsilon_p \epsilon_1]\).

Then each \( E_{n,p}^{j} \) can be labelled by \([\iota_{n,p}(U_{n,p,i,s})]\) where \( U_{n,p,i,s} \) is a component of \( W_{n,p}^{j} \) contained in \( E_{n,p}^{j} \).

**Proposition 6.4.** All ends of \( V_{n,p}^{j} \) can be labelled by \([\beta_{n} \ldots \beta_{2}(\epsilon_1 + j) \epsilon_2 \ldots \epsilon_p \epsilon_1]\), where \( \beta_m \in \mathbb{Z}_d \) for \( 2 \leq m \leq n \) and \( \epsilon_2 \ldots \epsilon_p \epsilon_1 \in \mathbb{Z}_d^N \) is any \( p \)-periodic sequence under shift.

**Proof.** Let \( E_{1,p}^{j} \) be any end of \( V_{1,p}^{j} \). Let \( (c_0, w) \) be a point of \( E_{1,p}^{j} \) with \( \iota_{c_0}(w) = \beta_{2}(\epsilon_2 \ldots \epsilon_p \epsilon_1) \). By the following commutative graph, We have \( \beta_1 = \epsilon_1 + j \). Then for \( (c_0, z) \) belonging to any end of \( V_{n,p}^{j} \),

\[
\iota_{c_0}(z) = \beta_{n} \ldots \beta_{2}(\epsilon_1 + j) \epsilon_2 \ldots \epsilon_p \epsilon_1
\]

which is some \((n, p)\)-sequence in \( \mathbb{Z}_d^N \). So each end of \( V_{n,p}^{j} \) can be labelled by \([\beta_{n} \ldots \beta_{2}(\epsilon_1 + j) \epsilon_2 \ldots \epsilon_p \epsilon_1]\) for some \((n, p)\)-preperiodic sequence \( \beta_{n} \ldots \beta_{2}(\epsilon_1 + j) \epsilon_2 \ldots \epsilon_p \epsilon_1 \). Besides, the number of all equivalence classes with the form \([\beta_{n} \ldots \beta_{2}(\epsilon_1 + j) \epsilon_2 \ldots \epsilon_p \epsilon_1]\) is \( \nu_d(p)d^n-2 \), the same with the number of ends on \( V_{n,p}^{j} \) (Lemma 6.3). So all ends of \( V_{n,p}^{j} \) are labelled by

\[
\{[\beta_{n} \ldots \beta_{2}(\epsilon_1 + j) \epsilon_2 \ldots \epsilon_p \epsilon_1]\mid \beta_{n} \ldots \beta_{2}(\epsilon_1 + j) \epsilon_2 \ldots \epsilon_p \epsilon_1 \text{ is a } (n, p)\text{-preperiodic sequence}\}
\]

\( \square \)

### 6.3 Calculation of the genus of \( \hat{V}_{n,p}^{j} \)

Now, for any \( n, p \geq 1 \), \( 1 \leq j \leq d-1 \), we have obtained a branched covering \( \hat{\pi}_{n,p}^{j} : \hat{V}_{n,p}^{j} \rightarrow \hat{C} \) of degree \( \nu_d(p)d^{n-1} \) between two compact Riemann surface. By the Riemann-Hurwitz formula, we have

\[
2 - 2g_{n,p}^{j} + \text{total number of critical points of } \hat{\pi}_{n,p}^{j} = 2\nu_d(p)d^{n-1}.
\]
where $g^j_{n,p}$ denotes the genus of $\hat{\mathcal{V}}^j_{n,p}$. So in order to calculate the genus of $\hat{\mathcal{V}}^j_{n,p}$, we only need to count the number of critical points of $\pi^j_{n,p}$ counting with multiplicity. It is known that the set of critical points for $\hat{\pi}^j_{n,p}$ is

$$C^j_{n,p,1} \cup C^j_{n,p,2} \cup C^j_{n,p,3} \cup \{ \infty^j_{n,p,i} \}_{i=1}^{\nu_d(p)d^{n-2}}$$

We will count the critical points class by class.

- Counting the points in $C^j_{n,p,1}$.

  By the definition of $C^j_{n,p,1}$, we have

  $$C^j_{n,p,1} = \bigcup_{s=2}^{n} (\varphi^j_{n,p})^{-1} \circ \cdots \circ (\varphi^j_{s+1,p})^{-1}(D^j_{s,p}).$$

  Recall that $D^j_{s,p} = \{(c,0) \in \mathbb{C}^2 | q^j_{s,p}(c,0) = 0\}$ is the set of critical points of $\varphi^j_{s,p}$. Fix any $s \in [2, n]$. Firstly, we claim $\#D^j_{s,p} = \nu_d(p)d^{s-2}$: on one hand, $q^j_{s-1,p}(c,0) = 0 \iff q^j_{s-1,p}(c,c) = 0$. So $\deg(q^j_{s-1,p}(c,c)) = \nu_d(p)d^{s-2}$ implies $\#C^j_{s,p} \leq \nu_d(p)d^{s-2}$. On the other hand, by the smoothness of $\mathcal{V}^j_{s,p}$ at $(c,0) \in D^j_{s,p}$, we have $[\partial q^j_{s,p}/\partial c](c,0) \neq 0$. It follows that each root of $q^j_{s,p}(c,0)$ is simple, and $\#D^j_{s,p} = \nu_d(p)d^{s-2}$.

  Next, consider the map

  $$h^j_{n,s,p} := \varphi^j_{s+1,p} \circ \cdots \circ \varphi^j_{n,p} : \mathcal{V}^j_{n,p} \to \mathcal{V}^j_{s,p}$$

  It is easy to see that the set of critical points of $h^j_{n,s,p}$ is disjoint from $(h^j_{n,s,p})^{-1}(D^j_{s,p})$. It follows that $\#(h^j_{n,s,p})^{-1}(D^j_{s,p}) = \nu_d(p)d^{s-2} \cdot d^{n-s}$ and each point in $(h^j_{n,s,p})^{-1}(D^j_{s,p})$ is a critical point of $\hat{\pi}^j_{n,p}$ with multiplicity $d-1$. Therefore the total number of critical points of $\hat{\pi}^j_{n,p}$ in $C^j_{n,p,1}$ is equal to

  $$\sum_{s=2}^{n} \nu_d(p)d^{s-2} \cdot d^{n-s} \cdot (d-1) = (n-2)\nu_d(p)d^{n-2}(d-1).$$

- Counting the points in $C^j_{n,p,3}$

  By definition, $C^j_{n,p,3} = (\varphi^j_{n,p})^{-1} \circ \cdots \circ (\varphi^j_{1,p})^{-1}(C_{0,p,3}) = (h^j_{n,0,p})^{-1}(C_{0,p,3})$. It is obvious that $C_{0,p,3}$ is disjoint from the set of critical values for $h^j_{n,0,p}$, then each point $P$ in $C_{0,p,3}$ has exactly $d^{n-1}$ pre-image under $h^j_{n,0,p}$ and each pre-image of $P$, considered as critical point of $\hat{\pi}^j_{n,p}$, has the same multiplicity as that of $P$ considered as critical point of $\pi_{0,p}$. So the total number of critical points of $\hat{\pi}^j_{n,p}$ in $C^j_{n,p,3}$ is $d^{n-1}$ times the number of critical points of $\pi_{0,p}$ in $C_{0,p,3}$. Now we are only left to count the number of critical points of $\pi_{0,p}$ in $C_{0,p,3}$.

  Let $C_{0,p,3,k} = \{(c,z) \in C_{0,p,3} | z \text{ is } k \text{ periodic point of } f_c\}$, then $C_{0,p,3}$ is the disjoint union of all $C_{0,p,3,k}$ with $k|p, k < p$. Fix any $k_0$ satisfying $k_0|p, k_0 < p$. Note that there are $\nu_d(k_0)/d$ hyperbolic components in $M_d$ of period $k_0$ and on the boundary of any such component, there are $(d-1)\varphi(p/k_0)$ parameters with their parabolic
periodic points satisfying the property of \( C_{0,p,3,k_0} \). (For any \( t \in \mathbb{N} \), \( \varphi(t) \) is the number of all numbers among \( \{1, \ldots, t - 1\} \) which are co-prime with \( t \).) Then \( \#C_{0,p,3,k_0} = (\nu_d(k_0)/d)(d-1)\varphi(p/k_0)k_0 \). Moreover, each point in \( C_{0,p,3,k_0} \) has multiplicity \((p/k_0) - 1\) as a critical point of \( \pi_{0,p} \). So the number of critical points of \( \pi_{0,p} \) in \( C_{0,p,3} \) is equal to
\[
\sum_{k|p,k<p} (\nu_d(k)/d)(d-1)\varphi(p/k)k(k/k-1)
\]
Hence, the number of critical points of \( \pi_{n,p} \) in \( C_{n,p,3} \) is
\[
d^{n-1} \sum_{k|p,k<p} (\nu_d(k)/d)(d-1)\varphi(p/k)k(k/k-1).
\]

- **Counting the points in \( C_{n,p,2}^j \)**

By definition, \( C_{n,p,2}^j = (\varphi_{n,p}^j)^{-1} \circ \cdots \circ (\varphi_{n,p}^j)^{-1}(C_{0,p,2}) = (h_{n,0,p}^j)^{-1}(C_{0,p,2}) \). It is very similar to the case above. With the same reason, we can also conclude that the total number of critical points of \( \pi_{n,p}^j \) in \( C_{n,p,2}^j \) is \( d^{n-1} \) times the number of critical points of \( \pi_{0,p} \) in \( C_{0,p,2} \).

Now we begin to count the number of critical points of \( \pi_{0,p} \) in \( C_{0,p,2} \). By definition of \( C_{0,p,2} \), the parameter set \( \pi_{0,p}(C_{0,p,2}) \) consists of co-roots and primitive roots of all hyperbolic components of period \( p \). From section 3, the number of co-roots and roots for all hyperbolic component of period \( p \) is \((d-1)\nu_d(p)/d\). Moreover, in the calculation of number of critical points in \( C_{0,p,3} \), we have actually got the number of root of the satellite components with period \( p \), that is \( \sum_{k|p,k<p} (\nu_d(k)/d)(d-1)\varphi(p/k) \). Then
\[
\#C_{0,p,2} = p[(d-1)\nu_d(p)/d - \sum_{k|p,k<p} (\nu_d(k)/d)(d-1)\varphi(p/k)].
\]
By Theorem 2.1, each critical point of \( \pi_{0,p} \) in \( C_{0,p,2} \) is simple, then the number of all critical points of \( \pi_{0,p} \) in \( C_{0,p,2} \) is
\[
p[(d-1)\nu_d(p)/d - \sum_{k|p,k<p} (\nu_d(k)/d)(d-1)\varphi(p/k)].
\]
Hence the number of all critical points of \( \pi_{n,p}^j \) in \( C_{n,p,2}^j \) is
\[
d^{n-1}p[(d-1)\nu_d(p)/d - \sum_{k|p,k<p} (\nu_d(k)/d)(d-1)\varphi(p/k)].
\]

- **Counting the points in \( \{\infty_{n,p,i}^j\}_{i=1}^{\nu_d(p)d^{n-2}} \)**

By Lemma 6.3, there are \( \nu_d(p)d^{n-2} \) ideal boundary points on \( \hat{\mathbb{V}}_{n,p}^j \) and each one is a critical point of \( \hat{\mathbb{V}}_{n,p}^j \) with multiplicity \( d - 1 \). So the number of critical points of \( \pi_{n,p}^j \) in \( \{\infty_{n,p,i}^j\}_{i=1}^{\nu_d(p)d^{n-2}} \) is equal to \( \nu_d(p)d^{n-2}(d - 1) \).
By the Riemann-Hurwitz formula, we have
\[ g_{n,p}^j = 1 + \frac{1}{2} \nu_d(p)d^{n-2}[(d - 1)(n + p) - 2d] - \frac{1}{2} d^{n-2}(d - 1) \sum_{k \mid p, k < p} k \nu_d(k) \varphi(p/k). \]

From the formula of genus and Lemma 6.3, it is known that both \( g_{n,p}^j \) (genus of \( \hat{V}_{n,p}^j \)) and \( m_{n,p}^j \) (the number of ends of \( \hat{V}_{n,p}^j \)) are independent of \( j \). So we can omit \( j \) for simplicity.

The following lemma implies a complete topological description of \( V_{n,p}^j \) (\( j \in [1, d-1] \)) and \( X_{n,p} \).

**Lemma 6.5.** (1) \( S_1, S_2 \) are two compact Riemann surface with the same genus. \( X_1 \subset S_1, X_2 \subset S_2 \) are two finite set with \( \#X_1 = \#X_2 \). Then there exists a homeomorphism \( h : S_1 \to S_2 \) such that \( h(X_1) = h(X_2) \).

(2) \( S \) is a compact Riemann surface and \( X \subset S \) is a finite set. Then for any \( \sigma \in \text{sym}(X) \), there exist a homeomorphism \( h : S \to S \) such that \( h(X) = X \) and \( h|_X = \sigma \).

They are classical results of topology of surface, then we omit the proof.

**Corollary 6.6** (Topological description of \( X_{n,p} \)). Topologically, \( V_{n,p}^j \) is determined by \( g_{n,p} \) and \( m_{n,p} \). \( X_{n,p} \) is determined by \( g_{n,p}, m_{n,p} \) and \( \#C_{n,p}\).

**Proof.** It follows directly from Lemma 6.5 and remark 5.13.

\[ \square \]

7 The Galois group of \( Q_{n,p}(c, z) \)

The objective here is to study \( Q_{n,p}(c, z) \) from the algebraic point of view by calculating its Galois group.

7.1 The Galois group of a polynomial

Let \( M \) be a field, \( f(x) \in M[x] \). If all roots of \( f \) are simple, then the splitting field of \( f \) over \( M \), denoted by \( L \), is called the Galois extension over \( M \). The Galois group of \( f \) is

\[ \text{Gal}(f) = \{ \sigma \in \text{Aut}(L) \mid \sigma|_M = id_M \}. \]

Note that each \( \sigma \in \text{Gal}(f) \) can be seen as a permutation on the roots of \( f \) and it is completely determined by this permutation, then \( \text{Gal}(f) \) can be seen as a subgroup of \( \text{sym}(\text{roots of } f) \). Usually, \( \text{Gal}(f) \) doesn’t equal to the symmetric group on all roots of \( f \), so we can see some intrinsic structure and symmetry of polynomial \( f \) from its Galois group.

Now let \( M = K = \mathbb{C}(c) \), and let \( P(c, z) \in K[z] \) be a polynomial about \( c \) and \( z \). If \( P(c, z) \) has no multiple roots in \( K[z] \), applying the statement above to \( P(c, z) \), we obtain
a Galois group \( \text{Gal}(P) \) of \( P(c, z) \). In fact, the Galois theory admits an interpretation in terms of the covering theory. Here we only state what we need as a theorem below.

Let \( P(c, z) \in \mathbb{C}[c, z] \subset K(z) \) be a polynomial, monic in \( z \), and \( \mathcal{C} \) be the affine algebraic curve defined by \( P(c, z) = 0 \). Denote by \( \pi_P : \mathcal{C} \to \mathbb{C}, (c, z) \mapsto c \) be the projection to the first parameter and by \( C_P = \{(c, z) \in \mathcal{C} : \frac{\partial P}{\partial z}(c, z) = 0\} \) the set of critical points of \( \pi_P \). If \( C_P \neq \mathcal{C} \), then \( \mathcal{C} \setminus \pi_P^{-1}(\pi_P(C_P)) \) is a covering of \( \mathcal{C} \setminus \pi_P(C_P) \). So for any \( c_0 \in \mathbb{C} \setminus \pi_P(C_P) \), we obtain a group morphism

\[
\Phi_P : \pi_1(\mathcal{C} \setminus \pi_P(C_P), c_0) \longrightarrow \text{sym}(\pi_P^{-1}(c_0))
\]

whose image is the monodromy group \( \text{Mon}(P) \) (see Definition 6.2).

In fact, the monodromy action of \( \pi_1(\mathcal{C} \setminus \pi_P(C_P), c_0) \) on \( \pi_P^{-1}(c_0) \) under the covering map \( \pi_P \) is induced by analytic continuations. By the Implicit Function Theorem, for \( c_0 \in \mathbb{C} \setminus \pi_P(C_P) \), there exist \( \deg(P(c_0, z)) \) local holomorphic solutions for \( P(c, z) = 0 \) at \( c_0 \). These solutions accept analytic continuations along any curve in \( \mathcal{C} \setminus \pi_P(C_P) \). So analytic continuations along the closed curves based on \( c_0 \) give a group morphism

\[
\phi_P : \pi_1(\mathcal{C} \setminus \pi_P(C_P), c_0) \longrightarrow \text{sym}(Z_P)
\]

where \( Z_P \) is the set of roots of \( P(c_0, z) = 0 \) (note that \( \pi_P^{-1}(c_0) = \{(c_0, z) \in \mathcal{C} | z \in Z_P\} \)).

The monodromy action of \( \gamma \in \pi_1(\mathcal{C} \setminus \pi_P(C_P), c_0) \) under \( \pi_P : \mathcal{C} \setminus \pi_P^{-1}(\pi_P(C_P)) \to \mathcal{C} \setminus \pi_P(C_P) \) is induced by the analytic continuation of local solutions of \( P(c, z) = 0 \) at \( c_0 \) along \( \gamma \), that is

\[
\Phi_P(\gamma)(c_0, z) = (c_0, \phi_P(\gamma)(z))
\]

for all \( (c_0, z) \in \pi_P^{-1}(c_0) \). So \( \text{Mon}(P) \) is isomorphic to \( AC(P) := \phi_P(\pi_1(\mathcal{C} \setminus \pi_P(C_P), c_0)) \).

**Theorem 7.1.** The Galois group of \( P(c, z) \) is isomorphic to the monodromy group of \( \pi_P : \mathcal{C} \setminus \pi_P^{-1}(\pi_P(C_P)) \to \mathcal{C} \setminus \pi_P(C_P) \), that is, \( \text{Gal}(P) \cong \text{Mon}(P) \).

This proposition is due to the correspondence between the Galois theory and the covering theory. One can refer to [H] for its proof.

Now we can compute the Galois group of \( Q_{n,p} \) with the help of Theorem 7.1.

### 7.2 The Galois group of \( Q_{n,p} \)

We apply the discussion in 7.1 to \( Q_{n,p} \) \((n, p \geq 1)\). In proof of Lemma 5.7 we have seen that \( Q_{n,p}(c, z) \) has no multiple roots as a polynomial in \( K[z] \). So the splitting field of \( Q_{n,p}(c, z) \) over \( K \) is a Galois extension over \( K \) and then we obtain the Galois group of \( Q_{n,p} \), denoted by \( G_{n,p} \).

Let \( \pi_{n,p} : X_{n,p} \to \mathbb{C} \) be the projection from \( X_{n,p} \) to parameter space. From the previous content, we know that \( \pi_{n,p} = \bigcup_{j=1}^{d-1} \pi_{n,p}^j \) and the set of critical points is

\[
C_{n,p} := C_{n,p,1} \cup C_{n,p,2} \cup C_{n,p,3} \cup C_{n,p,4}.
\]
The set of critical values $V_{n,p} = \pi_{n,p}(C_{n,p})$ is equal to the union of $\bigcup_{j=1}^{d-1} V_{n,p}^{j}$ together with the center of the hyperbolic components of period $p$ (Lemma 6.1).

According to the discussion in 7.1 and Theorem 7.1 fixing any $c_0 \in \mathbb{C} \setminus M_d \subset \mathbb{C} \setminus V_{n,p}$, we have two group morphisms:

$$\Phi_{n,p} : \mathbb{C} \setminus V_{n,p} \to \text{sym}(\pi_{n,p}(c_0)) \quad \text{and} \quad \phi_{n,p} : \mathbb{C} \setminus V_{n,p} \to \text{sym}(Z_{n,p}).$$

($Z_{n,p}$ consists of all $(n,p)$-preperiodic points of $f_{c_0}$) and three kinds of expressions of the Galois group of $Q_{n,p} \in K[z]$:

$$G_{n,p} \cong Mon_{n,p} \cong AC_{n,p}.$$

We will compute the Galois group of $Q_{n,p}(c, z)$ in terms of the expression $AC_{n,p}$.

Firstly, we will find some necessary properties that $AC_{n,p}$ should satisfy.

$$AC_{n,p} = \{ \sigma_n^\gamma | \sigma_n^\gamma \text{ is the permutation on } Z_{n,p} \text{ induced by } \gamma \in \pi_1(\mathbb{C} \setminus V_{n,p}) \}.$$

Choose any $\sigma_n^\gamma \in AC_{n,p}$. Note that $\gamma$ can be seen as the element of $\pi_1(\mathbb{C} \setminus V_{l,p})$ for any $0 \leq l \leq n$. Then by the monodromy theorem of analytic continuation, we have

$$f_{c_0} \circ \sigma_n^\gamma = \sigma_n^{\gamma-1} \circ f_{c_0} \text{ on } Z_{n,p} \quad (7.1)$$

Now we turn to the expression $G_{n,p}$. For $\sigma \in G_{n,p}$, let $\Delta$ be any root of $Q_{n,p}$ in $\bar{K}$. If $\omega \Delta$ is also a root of $Q_{n,p}$, then $\sigma(\omega \Delta) = \omega \sigma(\Delta)$ is a root of $Q_{n,p}$, that is, $\sigma$ commutes with $d$-th rotation. (In case $n \geq 2$, if $\Delta$ is a root of $Q_{n,p}$, then $\omega \Delta$ is always a root of $Q_{n,p}$. But this fails in case $n = 1$). Interoperating of this property in term of the expression $AC_{n,p}$, we have

$$\text{For any } \sigma_{n,p}^\gamma \in AC_{n,p}, \ z \in Z_{n,p}, \text{ if } \omega z \in Z_{n,p}, \text{ then } \sigma_{n,p}^\gamma(\omega z) = \omega \sigma_{n,p}^\gamma(z) \quad (7.2)$$

So we have had two necessary properties that $AC_{n,p}$ should satisfy. What we would like to prove is that no other restrictions are imposed on $AC_{n,p}$. Set $H_{0,p} := AC_{0,p}$, consisting of all permutations of $Z_{0,p}$ which commute with $f_{c_0}$ (5 of Theorem 2.1). For $n \geq 1$, inductively define by $H_{n,p}$, the subgroup of $\text{sym}(Z_{n,p})$ consisting of all permutations satisfying (7.1) and (7.2), that is

- For each $\sigma \in H_{n,p}$, there is an unique $\sigma' \in H_{n-1,p}$ such that $f_{c_0} \circ \sigma = \sigma' \circ f_{c_0} \quad (7.1)$
- For each $\sigma \in H_{n,p}$, $z \in Z_{n,p}$, if $\omega z \in Z_{n,p}$, then $\sigma(\omega z) = \omega \sigma(z) \quad (7.2)$

$\sigma'$ is called the restriction of $\sigma$ on $Z_{n-1,p}$, denoted by $\sigma|_{z_{n-1,p}}$.

**Proposition 7.2.** For $n \geq 1$, $AC_{n,p} = H_{n,p}$, or equivalently, $G_{n,p}$ consists of all permutations on roots of $Q_{n,p}$ which commute with $f_c$ and the $d$-th rotation.
Proof. The inclusion “⊆” is obvious. We will show “⊇” by induction on $n$.

As $n = 1$, suppose $Z_{0,p} = \{z_s\}_{1 \leq s \leq \nu_d(p)}$ then $Z_{1,p} = \{\omega^j z_s\}_{1 \leq j \leq d-1}$. Choose any $\sigma_1 \in H_{1,p}$, properties $[7.1]$ and $[7.2]$ imply that

$$\sigma_1(\omega^j z_s) = \omega^j \sigma_0(z_s). \quad (7.3)$$

where $\sigma_0 := \sigma|_{z_{0,p}}$, $j \in [1, d-1]$, $s \in [1, \nu_d(p)]$. Since $H_{0,p} = AC_{0,p}$, there exists $\gamma_0 \in \pi_1(C \setminus V_{0,p}, c_0)$ with $\sigma_0^\gamma = \sigma_0$. We can find $\gamma_1 \in \pi_1(C \setminus V_{1,p}, c_0)$ such that $\gamma_1|_{C \setminus V_{0,p}} = \gamma_0$. By the following commutative diagram

\[
\begin{tikzpicture}
  \node (X0p) at (0,0) {$X_{0,p}$};
  \node (X1p) at (2,2) {$X_{1,p}$};
  \node (V1p) at (4,4) {$V_{1,p}$};
  \node (C) at (2,-2) {$C$};
  \draw[->] (X0p) -- node[above] {$\langle c, \omega^j \rangle$} (X1p);
  \draw[->] (X0p) -- node[below] {$\langle c, f(z) \rangle$} (C);
  \draw[->] (X1p) -- node[above] {$\langle c, f(z) \rangle$} (V1p);
  \draw[->] (C) -- node[below] {$\pi_{n,p}$} (V1p);
\end{tikzpicture}
\]

we have $\sigma_{1,p}^\gamma = \sigma_1$, and hence $G_{1,p} = H_{1,p}$. From $[7.2]$, we can also see $AC_{1,p} \cong AC_{0,p}$.

Assume now $AC_{l,p} = H_{l,p}$ for $1 \leq l < n$ ($n \geq 2$). Denote $\kappa_n := \nu_d(p)(d-1)d^{n-2}$, $Z_{n-1,p} = \{w_i\}_{1 \leq i \leq \kappa_n}$. Then $Z_{n,p} = \{\omega^j z_i\}_{1 \leq j \leq d-1}$, where $f_{c_0}(\omega^j z_i) = w_i$. Let $\sigma_n$ be any element of $H_{n,p}$. By property $[7.1]$ and assumption of induction,

$$\sigma_{n-1} := \sigma_n|_{Z_{n-1,p}} \in H_{n-1,p} = AC_{n-1,p}$$

Then there exists $\gamma_{n-1} \in \pi_1(C \setminus V_{n-1,p}, c_0)$ with $\sigma_{n-1}^\gamma = \sigma_{n-1}$. Consider $\gamma_n \in \pi_1(C \setminus V_{n,p})$ such that $\gamma_n|_{C \setminus V_{n-1,p}} = \gamma_{n-1}$, then we have

$$\sigma_n|_{Z_{n-1,p}} = \sigma_{n-1}.$$

Set $\delta = (\sigma_n^\gamma)^{-1} \circ \sigma_n$, then $\delta|_{Z_{n-1,p}} = id$ and by properties $[7.2]$, $[7.2]$, $[7.2], [7.2]$

$$\delta = \prod_{i=1}^{\kappa_n} (j_i (j_i + 1) \cdots (d-1) 1 \cdots (j_i - 1))$$

where $(j_i (j_i + 1) \cdots (d-1) 1 \cdots (j_i - 1))$ is the cyclical permutation on $\{z_i, \omega z_i, \ldots, \omega^{d-1} z_i\}$ mapping $z_i$ to $\omega^{j_i+1} z_i$. To finish the proof of Proposition $[7.2]$, we only need to find an element $\lambda \in \pi_1(C \setminus V_{n,p}, c_0)$ such that $\sigma_n^\lambda = \delta$. In fact, we will show a stronger result: for any $i \in [1, \kappa_n]$, we can find $\lambda_i \in \pi_1(C \setminus V_{n,p}, c_0)$ with $\sigma_{n,p}^\lambda = (j_i (j_i + 1) \cdots (j_i - 1))$. 28
Fix any $i_0 \in [1, \kappa_n]$. Suppose \( \{z_{i_0}, \omega z_{i_0}, \ldots, \omega^{d-1} z_{i_0}\} \subset \mathcal{V}_{n,p}^{j_0} \) for some $j_0 \in [1, d-1]$. Let $(\hat{c}, 0) \in \mathcal{V}_{n,p}^{j_0}$ be a critical point of $\pi_{n,p}$. Then by the Implicit Function Theorem, a neighborhood of $(\hat{c}, 0)$ in $\tilde{X}_{n,p}$ can be written as
\[
\{(c, z_c) \cup (c, \omega z_c) \cup \cdots \cup (c, \omega^{d-1} z_c) | |c - \hat{c}| < \epsilon \}
\]
where $z_c$ is a $(n, p)$ preperiodic point of $f_c$ nearby 0 for $c \neq \hat{c}$ and $z_{\hat{c}} = 0$. The map $\pi_{n,p}$ is a degree $d$ branched covering in a neighborhood of $(\hat{c}, 0)$ with only one branched point $(\hat{c}, 0)$. As $c$ make a small turn around $\hat{c}$, the set \( \{z_c, \omega z_c, \ldots, \omega^{d-1} z_c\} \) gets a cyclical permutation with $\omega z_c$ mapped to $\omega^{j+1} z_c$ and other $(n, p)$ preperiodic points of $f_c$ get fixed. It follows that for $\gamma \in \pi_1(\mathcal{C} \setminus V_{n,p}, c_0)$ homotopic to $\hat{c}$, $\sigma_{\pi_{n,p}}^\gamma = (2 \cdots d) 1$ acts on some \( \{z_{i_0}, \ldots, \omega^{d-1} z_{i_0}\} \) such that \( \{(c_0, \omega^j z_{i_0}) | 0 \leq j \leq d-1\} \subset \mathcal{V}_{n,p}^{j_0} \). Now we connect $(c_0, z_{i_0})$ and $(c_0, z_{i_0})$ by a curve from $(c_0, z_{i_0})$ to $(c_0, z_{i_0})$ on $\mathcal{V}_{n,p}^{j_0} \setminus \pi_{n,p}^{-1}(V_{n,p})$ and denote its projection under $\pi_{n,p}$ by $\beta$. With an abuse of notation of curves and their homotopy classes, we have $\beta \in \pi_1(\mathcal{C} \setminus V_{n,p}, c_0)$. Then
\[
\lambda_i = \beta \cdot \gamma^{j_0-1} \cdot \beta^{-1}
\]
satisfies our requirement.

By Theorem 2.1, we have known the Galois group $G_{0,p}$. Then with Proposition 7.2, we can calculate $G_{n,p}$ by induction on $n$. In the proof of this proposition, we obtain $G_1,p = G_{0,p}$ and a short exact sequence
\[
0 \longrightarrow \mathbb{Z}_d^\kappa_n \longrightarrow G_{n,p} \longrightarrow G_{n-1,p} \longrightarrow 0 \quad n \geq 2 \quad (\kappa_n = \nu_d(p)(d-1)d^{n-2})
\]
We will show that $G_{n,p}$ can be expressed as the wreath product of $\mathbb{Z}_d^\kappa_n$ and $G_{n-1,p}$ for $n \geq 2$.

**Definition 7.3.** Let $G$ be a group and $\Sigma$ be a subgroup of $\text{sym}(\mathbb{Z}_d)$. Denote by $\Sigma \ltimes G^d$ the wreath product of $G$ and $\Sigma$. As a set, it consists of $g = \sigma_g(g_1, \ldots, g_d)$ where $g_i \in G$ and $\sigma_g \in \Sigma$. The multiplication is defined by
\[
g \cdot h = \sigma_g(g_1, \ldots, g_d) \cdot \sigma_h(h_1, \ldots, h_d) = \sigma_g \circ \sigma_h(g_{\sigma(g_1)} \cdot h_1, \ldots, g_{\sigma(h_d)} \cdot h_d).
\]
Under this multiplicity, $\Sigma \ltimes G^d$ is a group with $g^{-1} = \sigma_g^{-1}(g_{\sigma(g_1)}^{-1}, \ldots, g_{\sigma(h_d)}^{-1})$ and unit element $(0, \ldots, 0)$.

**Corollary 7.4.** For $n \geq 2$, $G_{n,p} \cong G_{n-1,p} \ltimes \mathbb{Z}_d^{\nu_d(p)(d-1)d^{n-2}}$.

**Proof.** A nice way to visualize the action of $G_{n,p}$ on points $Z_{n,p}$ is to consider the following table:

| $w_1$ | $w_2$ | \cdots | $w_{\kappa_n-1}$ | $w_{\kappa_n}$ |
|-------|-------|--------|----------------|---------------|
| $z_1$ | $z_2$ | \cdots | $z_{\kappa_n-1}$ | $z_{\kappa_n}$ |
| $\omega z_1$ | $\omega z_2$ | \cdots | $\omega z_{\kappa_n-1}$ | $\omega z_{\kappa_n}$ |
| \vdots | \vdots | \cdots | \vdots | \vdots |
| $\omega^{d-1} z_1$ | $\omega^{d-1} z_2$ | \cdots | $\omega^{d-1} z_{\kappa_n-1}$ | $\omega^{d-1} z_{\kappa_n}$ |

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where $Z_{n-1,p} = \{w_i\}_{i=1}^{\kappa_n}$, $\omega^j z_i$ $(0 \leq j \leq d - 1)$ are the preimages of $w_i$ under $f_{c_i}$. An element of $G_{n,p}$ permutes the columns and within each column, it performs a cyclic shift. Algebraically, let $\sigma$ be any element of $G_{n,p}$, by Proposition 7.2 for $1 \leq i \leq \kappa_n$,

$$
\sigma(z_i) = \omega^{c_i} z_{s\sigma(i)} \quad \text{where} \quad c_\sigma \in \mathbb{Z}_d^\kappa \text{ and } s_\sigma \in G_{n-1,p}
$$

and $\sigma$ is completely determined by $c_\sigma$ and $s_\sigma$. We obtain a map

$$
\psi : G_{n,p} \to G_{n-1,p} \rtimes \mathbb{Z}_d^\kappa \text{ with } \psi(\sigma) = s_\sigma(c_\sigma(1), \cdots, c_\sigma(\kappa_n))
$$

For any $\sigma, \tau \in G_{n,p}$, $i \in [1, \kappa_n]$,

$$
\begin{align*}
\sigma \cdot \tau(z_i) &= \sigma(\tau(z_i)) = \sigma(\omega^{c_\tau(i)} \cdot z_{s_\tau(i)}) = \omega^{c_\sigma \cdot \tau(i)} \cdot \sigma(z_{s_\tau(i)}) \\
&= \omega^{c_\sigma \cdot \tau(i)} \cdot \omega^{c_\tau(s_\tau(i))} \cdot z_{s_\sigma(s_\tau(i))} = \omega^{c_\sigma(s_\tau(i)) + c_\tau(i)} \cdot z_{s_\sigma \cdot s_\tau(i)}
\end{align*}
$$

Then we have

$$
\begin{align*}
\psi(\sigma \cdot \tau) &= s_\sigma \cdot s_\tau \cdot (c_\sigma(s_\tau(1)) + c_\tau(1), \cdots, c_\sigma(s_\tau(\kappa_n)) + c_\tau(\kappa_n)) \\
&= s_\sigma(c_\sigma(1), \cdots, c_\sigma(\kappa_n)) \cdot s_\tau(c_\tau(1), \cdots, c_\tau(\kappa_n)) \\
&= \psi(\sigma) \cdot \psi(\tau).
\end{align*}
$$

Thus $\psi$ is a group isomorphism. The injectivity is obvious and subjectivity is ensured by proposition 7.2.

To end this manuscript, we will provide some simple remarks on the relationship between the Galois group of $Q_{n,p}$ and the Galois group of $q^j_{n,p}$ $(1 \leq j \leq d - 1)$. For $n \geq 1$, $p \geq 1$, denote by $G^j_{n,p}$ the Galois group of $q^j_{n,p}$. Note that the splitting field of $q^j_{n,p}$ are all the same as that of $Q_{0,p}$, then

$$
G_{0,p} = G^j_{1,p} = G_{1,p} \quad \text{for } 1 \leq j \leq d - 1.
$$

For $n \geq 2$, by the same reason as that of Proposition 7.2 and corollary 7.4, we have $G^j_{n,p} \cong G^j_{n-1,p} \rtimes \mathbb{Z}_d^{\nu_3(p)(d^n-2)}$. There are two natural group morphisms:

$$
G_{n,p} \xrightarrow{s^i_{n,p}} G^j_{n,p} \twoheadrightarrow 0 \quad \text{such that} \quad s^i_{n,p}(\sigma^\gamma_{n,p}) = \sigma^\gamma_{n,p}
$$

where $\gamma_j$ is the image of $\gamma$ under the canonical map from $\pi_1(\mathbb{C} \setminus V_{n,p})$ to $\pi_1(\mathbb{C} \setminus V^j_{n,p})$, and

$$
0 \xrightarrow{i_{n,p}} G_{n,p} \xrightarrow{\nu_3(p)(d^n-2)} G^1_{n,p} \times \cdots \times G^{d-1}_{n,p} \quad \text{satisfy } \quad i_{n,p}(\sigma^\gamma_{n,p}) = (\sigma^\gamma_{n,p}, \cdots, \sigma^\gamma_{n,p}).
$$

However, we have $G_{n,p} \not\cong G^1_{n,p} \times \cdots \times G^{d-1}_{n,p}$ for $n \geq 1$, $d \geq 3$. Note that for any $n \geq 2$,

$$
G^1_{n,p} \times \cdots \times G^{d-1}_{n,p} \cong \prod_{j=1}^{d-1} (G^j_{n-1,p} \rtimes \mathbb{Z}_d^{\nu_3(p)(d^n-2)}) \cong (G^1_{n-1,p} \times \cdots \times G^{d-1}_{n-1,p}) \rtimes \mathbb{Z}_d^{\nu_3(p)(d-1)d^{n-2}}
$$

and

$$
G_{n,p} \cong G^1_{n-1,p} \rtimes \mathbb{Z}_d^{\nu_3(p)(d-1)d^{n-2}}
$$

By an induction on $n$, it reduces to show $G_{1,p} \cong G^1_{1,p} \times \cdots \times G^{d-1}_{1,p}$. This is obvious because

$$
G_{1,p} \cong G^1_{1,p} \cong \cdots \cong G^{d-1}_{1,p} \cong G_{0,p}.
$$
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