Resonant resurgent asymptotics from quantum field theory

Michael Borinsky a,*, David Broadhurst b

a Institute for Theoretical Studies, ETH Zürich, 8092 Zürich, Switzerland
b School of Physical Sciences, Open University, Milton Keynes MK7 6AA, UK

Received 21 February 2022; received in revised form 26 April 2022; accepted 3 June 2022
Available online 9 June 2022
Editor: Hubert Saleur

Abstract

We perform an all-order resurgence analysis of a quantum field theory renormalon that contributes to an anomalous dimension in six-dimensional scalar \( \phi^3 \) theory and is governed by a third-order nonlinear differential equation. We augment the factorially divergent perturbative expansion associated to the renormalon by asymptotic expansions to all instanton orders, in a conjectured and well-tested formula. A distinctive feature of this renormalon singularity is the appearance of logarithmic terms, starting at second-instanton order in the trans-series. To highlight this and to illustrate our methods, we also analyze the trans-series for a closely related second-order nonlinear differential equation that exhibits a similarly resonant structure but lacks logarithmic contributions.

© 2022 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP3.

1. Introduction

1.1. Renormalons and resurgence

Perturbative computations in quantum field theories require the treatment of various kinds of divergences. For instance, renormalization is needed at each order in the perturbative expansion to bring UV-singularities under control. By now, the physical and mathematical reasoning behind this procedure is very well understood. A much less explored source of concern is the divergent...
nature of the perturbative expansion itself [1]. In this article we shall illuminate some of the technicalities which are necessary to make sense of such a divergent expansion via resurgence.

Roughly, we can distinguish two types of sources for a divergence of the perturbative expansion: a non-perturbative contribution that stems from a nontrivial solution of the classical field equations, which is referred to as an instanton solution. This type of divergence can also be observed in quantum mechanical models [2–6] and entirely combinatorial or topological models [7,8] (see also [9–11] for exhaustive introductions into such semi-classical instanton phenomena and calculations).

A further type of divergence is generated by renormalization subtraction terms of a certain set of Feynman diagrams. Such a divergence is called a renormalon [12–16] and can often be explicitly associated with a specific large-$N$ limit of the underlying theory [17–25]. In this article we will deal with a specific UV-renormalon, i.e. a renormalon that originates from the UV-subtraction terms of a specific class of diagrams. Recently, Mariño and Reis showed that also super-renormalizable QFTs and integrable models can contain infrared-renormalon singularities [26,27] while making use of technology from [18] (see also [28]). A direct approach to renormalon singularities based on renormalization group considerations has also been recently put forward [29–33]. Moreover, it has been shown recently that also quantum mechanical models can feature renormalon singularities [34]. Another recent development is the observation that the usually applied resurgence framework which works with Gevrey-1 sequences might have to be generalized to be able to handle certain QFT renormalon singularities [35]. In this work, we will only deal with Gevrey-1 singularities. The role of the renormalization scheme in the scope of renormalon divergences was recently studied in [36].

1.2. A renormalon in $\phi^3$ theory in six-dimensional spacetime

We will analyze the resurgence structure of a renormalon that contributes to the perturbative solution of six-dimensional scalar $\phi^3$ theory. This model is especially interesting as it is believed to be asymptotically free [37], its renormalization group functions are known up to a high perturbative order [38–42], the structure of its semi-classical instanton solutions is relatively well-studied [43–47] and, augmented with a bi-adjoint group structure, it is believed to be BCJ dual to Yang–Mills theory [48–50]. Moreover, $\phi^3$ theory in six dimensions has direct application to percolation theory [38,39,51,52] and the Lee–Yang edge singularity [38,39,53].

The Feynman diagrams that contribute to this $\phi^3$ theory renormalon are the one-loop self-energy correction together with all possible recursive insertions of this diagram into itself. This set of diagrams can be depicted via the Dyson–Schwinger equation,

$$\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1}\end{array}
\end{align*}
= \frac{1}{2} \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram2}\end{array} + \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3}\end{array} + \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram4}\end{array} + \cdots
\end{align*}
(1)

After renormalization the contribution from this set of diagrams turns out to yield a factorially divergent power series with highly nontrivial properties. Using a Hopf-algebraic momentum subtraction renormalization procedure [54], Broadhurst and Kreimer were able to deduce a third-order nonlinear differential equation for the associated contribution to the field anomalous dimension of $\phi^3$ theory in six spacetime dimensions [55, Equation (52)]. Remarkably, this equation can be brought into a simpler, factored form (see [55, Equation (53-57)] and [56, Equation (4.1)]). It reads,
\[(g(x)P - 1)(g(x)P - 2)(g(x)P - 3)g(x) = -3, \quad \text{where} \quad P = x \left(2x \frac{d}{dx} + 1\right). \quad (2)\]

As a third-order ODE, this equation is expected to have a three-dimensional solution space. Surprisingly, due to an irregular singular point at the origin, the equation has only one unique formal power series or perturbative solution around \(x = 0:\)

\[g_0(x) = A(x) = \sum_{n=0}^{\infty} A_n x^n = \frac{356789}{10358} x^3 + \frac{2249}{384} x^2 + \frac{11}{24} x + 1 + O(x^6). \quad (3)\]

This formal power series solution was developed to 500 terms by Broadhurst and Kreimer [55], who observed that the expansion is factorially divergent and therefore constitutes a renormalon contribution to the \(\phi^3\) theory field anomalous dimension. The three-dimensional family of other solutions to (2) is hidden behind exponentially suppressed non-perturbative corrections to the solution (3).

### 1.3. Resurgence

Resurgence is the mathematical theory of such factorially divergent power series which allows to associate concrete functions to them. It has been developed by Écalle [57] (see [58, Part II] for a mathematics focused introduction and [11,59–61] for reviews focused on applications in physics). Resurgence provides a promising approach to make non-perturbative predictions in quantum field theory and string theory (see for instance [62–71]). Moreover, by its intricate relationship to quantization, resurgence suggests various interesting connections of quantum theory to wall-crossing phenomena, topological recursion, three-manifolds, knot theory, combinatorics and other parts of mathematics [72–77].

In this article we shall investigate the detailed resurgence properties of the renormalon described by (2) and its perturbative solution (3). The singular structure of this renormalon features the complete set of divergences, including logarithmic terms, that are expected from a general QFT solution. These aspects make this renormalon a particularly interesting and instructive instance of this type of QFT divergence. Related studies of renormalons based on analysis of Dyson–Schwinger equations have been performed for the Wess–Zumino model [78–80], for six-dimensional scalar \(\phi^3\) theory in a more general context which also includes vertex corrections [81,82] and for Yukawa theory [83], where an all-order trans-series solution of the Dyson–Schwinger equation could be achieved.

### 1.4. Resonant resurgence phenomena and logarithmic terms

In [55,84], the expansion (3) occurs at \(x < 0\) and hence is amenable to Padé–Borel summation. On first sight this is the most relevant case as

\[
\gamma(\lambda) = -\frac{1}{3} \frac{\lambda^2}{(4\pi)^3} \cdot \left(-\frac{1}{3} \frac{\lambda^2}{(4\pi)^3}\right)^{\frac{3}{2}}
\]

is the contribution of (3) to the field anomalous dimension of six-dimensional \(\phi^3\)-theory with real coupling constant \(\lambda\) in the momentum subtraction scheme. Recently, the first author, Dunne and Meynig [56] considered the case with \(x > 0\) of (3) which corresponds to the analysis of a QFT with an imaginary coupling constant such that \(\lambda^2 < 0\). Physically this is a relevant case as
well, as it describes the Lee–Yang edge singularity [53]. From a resurgence perspective, the sign choice $x > 0$ is more interesting as naïve Padé–Borel resummation is not sufficient to make sense of the perturbative expansion. The series is not Borel-resummable and the resulting ambiguities have to be dealt with explicitly. This case will also serve as our entry point into the resurgence analysis of the renormalon described by (1).

The very first step for such a resurgence analysis of a highly nonlinear ODE is to consider its linearized variant. Solving this linearized equation (see [56, Eq. (5.3)]) which is homogeneous, one encounters three non-perturbative solutions $h_1, h_2, h_3$. These solutions can be obtained by setting

$$g(x) = g_0(x) + \sigma_k \left( x^{\frac{35}{12}} e^{-\frac{x}{x}} \right)^k h_k(x) + \mathcal{O}(\sigma_k^2), \quad k \in \{1, 2, 3\},$$

in (2) and discarding terms of order $\sigma_k^2$. The parameters $\sigma_1, \sigma_2, \sigma_3$ are the integration constants of the linearized homogeneous ODE. Eventually they will parameterize the three-dimensional non-perturbative solution space that is expected from a third-order ODE (2).

From the first-order contributions in the $\sigma_1, \sigma_2, \sigma_3$ parameters in (4), one can anticipate a trans-series solution of (2) of the form

$$g(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sigma_i^j \sigma_2^j \sigma_3^k \left( x^{\frac{35}{12}} e^{-\frac{x}{x}} \right)^{i+2j+3k} G_{i,j,k}(x)$$

with $G_{0,0,0}(x) = g_0(x), G_{1,0,0}(x) = h_1(x), G_{0,1,0}(x) = h_2(x)$ and $G_{0,0,1}(x) = h_3(x)$. It was observed in [56] that the expansions $G_{i,j,k}(x)$ start to contain logarithmic terms from the second order in $\sigma_1$ on. The first occurrence of a logarithmic contribution is found in the $\sigma_1^2$ term at order $x^5$, with

$$G_{2,0,0}(x) = -2 + \frac{49}{6} x + \frac{13235}{1728} x^2 + \ldots + \frac{21265}{2304} x^5 \log \left( \frac{x}{c} \right) \left( -1 + \frac{151}{24} x + \ldots \right)$$

where $c$ is an ambiguous constant, undetermined by the ODE. We will discuss the character of this ambiguity in detail in Section 4. Key to our analysis of (2) will be the representation (7)–(9) of its trans-series solution that accounts for the logarithmic terms in a particularly compact way.

The fact that no logarithmic terms appear in the closely related renormalon in four-dimensional Yukawa theory [55,83,84] is only slightly surprising, as this simpler four-dimensional renormalon is described by a nonlinear first-order ODE. A prominent nonlinear ODE whose solution features logarithmic terms is the Painlevé I equation [85,86]. This equation is relevant for matrix models and string theory [87].

A plausible explanation of the appearance of these logarithmic terms is an analogy to quantum mechanics where resonances between classical instanton solutions result in logarithmic terms which themselves lie in correspondence to ambiguities of the resummation procedure [63,64,88,89]. See also [90] for a recent application of this correspondence.

The exponents 2 and 3 of $e^{-\frac{x}{x}}$ in (4), which are associated in the truncated trans-series solution (4) to the solutions $h_2$ and $h_3$ of the linearized ODE, are integer multiples of the exponent 1 for the $h_1$ solution. This means, there is a resonance between the different solutions to the linearized equation, but this resonant ratio does not fully explain the appearance of logarithmic terms and the intuition gained from quantum mechanics unfortunately fails here. To prove this point and to illustrate the peculiarity of the appearance of logarithmic terms we will analyze the closely related second-order ODE (27) in Section 3 which features a resonant ratio of trans-series
exponential powers between the solutions to its linearized avatar, but is completely free of logarithmic terms. This second-order ODE will also serve as a suitable warm-up exercise before tackling the full complexity of (2) in Section 4.

1.5. All-order trans-series analysis

Our first main result is the following compact representation of the trans-series solution of (2):

\[ g(x) = \sum_{m=0}^{\infty} g_m(x) y^m, \quad y = x^{-\frac{35}{12}} e^{-\frac{1}{x}}, \tag{7} \]

\[ g_m(x) = \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor (m-2i)/3 \rfloor} \sigma_1^{m-2i-3j} \sigma_2^{i} \sigma_3^{j} x^{5(i+j)} \sum_{n \geq 0} a_{i,j}^{(m)}(n) x^n, \tag{8} \]

\[ \hat{\sigma}_2 = \sigma_2 + \frac{21265}{2304} \sigma_1^2 \log(x), \quad \hat{\sigma}_3 = \sigma_3 + \frac{21265}{2304} \sigma_1^3 \log(x), \tag{9} \]

with \( \log(x) \) neatly absorbed by (9). In analogy to instanton expansions of path integrals, we will call the coefficient \( g_m(x) \) in front of \( y^m \) the \( m \)-th instanton. The instanton action is reflected by the integer \( m \) in the exponent of the exponential in \( y \) which dictates the magnitude of the exponential suppression of the respective term for \( x \to 0^+ \).

The compact absorption of logarithmic terms in (9), which involves another interesting constant \( \frac{21265}{2304} \), provides strong hints for a cancellation mechanism. Such a mechanism is observed in quantum mechanical models \([5,91]\) where it ensures the reality of the trans-series solution in the presence of non-perturbative ambiguities as required by physical constraints.

Even after this convenient absorption of logarithms the rational coefficients \( a_{i,j}^{(m)}(n) \) are not uniquely determined by the ODE (2). There remain three intrinsic ambiguities. To understand their nature, we can think of Equations (7)–(9) as an Ansatz for \( g(x) \) that is parameterized by \( \sigma_1, \sigma_2, \sigma_3 \) and the coefficients \( a_{i,j}^{(m)}(n) \) for all \( 0 \leq i, j, m, n \) with \( 2i + 3j \leq m \). This Ansatz over-parameterizes the function \( g(x) \) as we can observe by inspection of (7)–(9): For instance, an arbitrary rescaling of \( \sigma_1 \to C_1 \sigma_1 \) can be compensated by a corresponding change of \( \sigma_2 \to C_2 \sigma_2 \), \( \sigma_3 \to C_3 \sigma_3 \) and \( a_{i,j}^{(m)}(n) \to C_1^{-m} a_{i,j}^{(m)}(n) \). The remaining two ambiguities result from the shifts \( \sigma_2 \to \sigma_2 + C_2 \sigma_1^2 \) and \( \sigma_3 \to \sigma_3 + C_3 \sigma_1^3 \) that can be compensated by redefinitions of the \( a_{i,j}^{(m)}(n) \) coefficients as well. We proceed to resolve all three ambiguities systematically by fixing specific \( a_{i,j}^{(m)}(n) \) coefficients. The scaling ambiguity is resolved by fixing

\[ a_{0,0}^{(1)}(0) = -1. \tag{10} \]

The choice of a negative sign has the effect that \( a_{0,0}^{(1)}(n) \) will eventually be positive for large \( n \). We resolve the remaining, more intricate shift ambiguities by fixing

\[ \frac{1}{2} a_{0,0}^{(2)}(5) = \frac{1}{6} a_{0,0}^{(3)}(5) = r_1 = \frac{32642693907919}{36691771392}. \tag{11} \]

Any other choice for the coefficients \( a_{0,0}^{(2)}(5) \) and \( a_{0,0}^{(3)}(5) \) would have fulfilled the purpose of resolving the overparameterization of \( g(x) \) and of producing a unique trans-series solution for the ODE (2). Our seemingly ad-hoc choices, which involve the large fraction \( r_1 \), are particularly favourable because they result in a simple asymptotic behaviour of the coefficients \( a_{i,j}^{(m)}(n) \) for large \( n \).
We emphasize that the three ambiguities above are of different nature than the usual three dimensional solution space of a third-order ODE. In our trans-series (7)–(9) of (2) this solution space is parameterized by the trans-series parameters $\sigma_1, \sigma_2$ and $\sigma_3$. The ODE (2) and the choices (10)–(11) only fix the values of the $a_{i,j}^{(m)}(n)$ coefficients while leaving the trans-series parameters arbitrary.

Our second main result is Conjecture 1 which completely describes this asymptotic behaviour (including subleading contributions to all orders). As expected from the general theory of resurgence the operation of taking the $n \to \infty$ asymptotic limit of a set of coefficients $a_{i,j}^{(m)}(n)$ closes among the sequences $a_{i,j}^{(m)}$, in the sense that the coefficients of the asymptotic expansion of $a_{i,j}^{(m)}(n)$, for large $n$, can be expressed using a linear combination of other coefficients $a_{i',j'}^{(m')}(k)$, beginning at small $k$. We explicitly determine these linear combinations and the associated connection constants, the Stokes constants, numerically.

The formulation of Conjecture 1 in its compact form was only possible with the specific choices made in (10)–(11). Generic or arguably more canonical choices such as, $a_{0,0}^{(2)}(5) = a_{0,0}^{(3)}(5) = 0$, would have led to the appearance of additional terms in this asymptotic behaviour, which would have involved the large fraction $r_1$ explicitly. We will illustrate our reasoning that led us to the specific choices (11) in detail in Section 4.

Before stating Conjecture 1 in the next section, we will introduce a few basic notions from the theory of resurgence.

2. All-order resurgent asymptotics

2.1. Asymptotic notation

Most power series considered in this article are factorially divergent. This means that the coefficients such as $A_n$ from (3) grow as a shifted $\Gamma$ function modulated by an exponential,

$$\limsup_{n \to \infty} \left| A_n/\alpha^n \Gamma(n + \beta) \right| = C,$$

with some constants $\alpha, \beta \in \mathbb{R}$ where $\alpha \neq 0$. For this reason, the formal power series $A(x) = \sum_{n=0}^{\infty} A_n x^n$ has a vanishing radius of convergence. Expressions such as $\sum_{n=0}^{\infty} A_n x^n$ are therefore supposed to be interpreted formally as objects in the ring of power series $\mathbb{R}[[x]]$. While working with these power series it is convenient to use following notation: If $A_n, B_n$ and $C_n$ are sequences of real numbers, then the $O$-notation $A_n = B_n + O(C_n)$ is a shorthand way to write $\limsup_{n \to \infty} |(A_n - B_n)/C_n| < \infty$. The $O$-notation requires us to specify the limit that is taken. In this article, asymptotic expansions that involve the integer variable $n$ refer to the limit $n \to \infty$. Taylor expansions of expressions with continuous variables such as $x$ refer to the limit $x \to 0$.

If $A_n$ and $B_n$ are sequences of real numbers and $M_{n,k}$ is a family of real number sequences indexed by $k$, then the asymptotic expansion notation $A_n \sim \sum_{k=0}^{\infty} M_{n,k} B_k$ is shorthand for the infinite family of $O$-statements, $A_n = \sum_{k=0}^{R-1} M_{n,k} B_k + O(M_{n,R})$ for all $R \geq 0$, which all shall hold in the $n \to \infty$ limit.

2.2. Intuition from the Borel transform

The coefficients $A_n$ in (3) grow factorially. Therefore, $g_0(x) = \sum_{n=0}^{\infty} A_n x^n$ does not converge for any non-zero value of $x$. This is a typical phenomenon in perturbative QFT computations,
which first has been properly appreciated by Dyson [1]. In general, it is a difficult task to reconstruct a function from a factorially divergent power series and there exist many different approaches and frameworks that aim to tackle this problem (see for instance [57,58,92–94]). A pragmatic solution is to analyze the Borel transform $\mathcal{B}[A](z)$ of the factorially divergent power series $A(x)$:

$$
\mathcal{B}[A](z) = \sum_{n=0}^{\infty} \frac{A_n}{n!} z^n. \tag{13}
$$

If (12) holds, then $\mathcal{B}[A](z)$ is a function of $z$ which is analytic in a non-vanishing domain around the origin. For resurgence in its basic form to be applicable, we have to assume that the function $\mathcal{B}[A](z)$ can be analytically continued to the whole complex plane with the exception of a countable number of isolated singular points. After this analytic continuation has been performed, an avatar function $\hat{A}(x)$ of the formal power series $A(x)$ can be constructed by applying the Laplace transform which is inverse to the Borel transform,

$$
\hat{A}(x) = \int_{0}^{\infty} dz e^{-z} \mathcal{B}[A](xz). \tag{14}
$$

In [84], the case with $x < 0$ case was considered, in which there are no singularities on the line of integration and which is therefore Borel-resummmable. Here, we wish to go beyond this simple situation and consider the $x > 0$ case where the line of integration meets singularities of $\mathcal{B}[A](z)$. Due to these singularities, it is a nontrivial problem to evaluate the Laplace transform of $\mathcal{B}[A](z)$. This problem is solved by Laplace–Borel–Écalle resummation (see [58, Ch. 5] for the details of this procedure). Resolving the ambiguities of the resummation in a situation with multiple trans-series parameters is known to require the full resurgence machinery and knowledge of all Stokes constants [95]. Here, we shall focus on a specific highly nontrivial step in this procedure: the analytic continuation of the function $\mathcal{B}[A](z)$.

We will assume that the function $\mathcal{B}[A](z)$ only has singularities on the real axis at the evenly spaced locations $z \in \{1, 2, \ldots\}$. This situation is typical for quantum field theory applications and does not imply a serious restriction of the applicability of our methods. Under this assumption, the power series representation (13) provides an analytic expression for $\mathcal{B}[A](z)$ which is valid inside the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Convergence in a larger domain is obstructed by the first singularity of $\mathcal{B}[A](z)$ at $z = 1$. For the Borel transforms of the power series under consideration in this article, the local expansions of $\mathcal{B}[A](z)$ in the vicinity of its singularities are of the form,

$$
\mathcal{B}[A](z) \sim \sum_{m \geq 0} \left(1 - \frac{z}{k}\right)^{m-\beta_k} c_{k,m} \quad \text{as} \quad z \to k^-, \tag{15}
$$

where $\beta_k$ are non-integer rational numbers and $c_{k,m}$ arbitrary coefficients. By Darboux’s theorem the large-order behaviour of the expansion coefficients $\frac{A_n}{n!}$ in (13) is governed by the singular behaviour of $\mathcal{B}[A](z)$ for $z \to 1$. To sketch the complete argument that establishes this relationship, we can assume that $z$ is close to 1. In this case, we can match both expansions,

$$
\mathcal{B}[A](z) = \sum_{n=0}^{\infty} \frac{A_n}{n!} z^n \sim \sum_{m \geq 0} (1 - z)^{m-\beta_1} c_{1,m} \quad \text{as} \quad z \to 1^-, \tag{16}
$$

and compare coefficients for $n \to \infty$.  

\[
\frac{A_n}{n!} \sim \sum_{m \geq 0} \binom{n - m + \beta_1 - 1}{n} c_{1,m} \quad \text{as } n \to \infty.
\]

(17)

It follows that

\[
A_n \sim \sum_{m \geq 0} \Gamma(n - m + \beta_1) \frac{c_{1,m}}{\Gamma(-m + \beta_1)}.
\]

(18)

Singular cases can arise if \(\beta_1\) is an integer and the \(\Gamma\) function on the right hand side can develop a pole. This would lead to logarithmic terms in the expansions but does not apply to our cases, where all \(\beta_k\) are non-integer rational numbers.

From this argument, it is evident that the factorially divergent large-order behaviour of the \(A_n\) coefficients can be translated into the singular behaviour of the Borel transform at the first singularity and vice versa. The power series \(\sum_{m \geq 0} (1 - z)^{m-\beta_1} c_{1,m}\) provides an expansion of the function \(B[A](z)\) which converges in the unit disc around the point \(z = 1\). A larger radius of convergence is obstructed by the singularity of \(B[A](z)\) at \(z = 2\). The asymptotic behaviour of the \(c_{1,m}\) coefficients for \(m \to \infty\) is in one-to-one correspondence with the singular expansion around the \(z = 2\) singularity in (15) by the same reasoning as above. Repeating this process for all further singularities along the real line, we can perform the analytic continuation procedure by iteratively computing the large-order expansion of the previous expansion.

We will apply this procedure to the power series solution (3) of the differential equation (2). A practically useful observation is that not only the explicit values of the initial coefficients are determined by the differential equation, but also information on the various higher-order expansions of the coefficients can be deduced. In fact, all the coefficients of the higher-order expansions of the initial sequence can be determined from the differential equation except for certain overall normalization constants. These normalization constants, the Stokes constants, provide the explicit connection coefficients between low-order and large-order behaviour of the various sequences.

An interesting phenomenon arises at the expansion of the Borel transform \(B[A](z)\) around the second singularity at \(z = 2\). The large-order behaviour of the coefficients is determined by two singularities of \(B[A](z)\) in the complex plane: the one at \(z = 1\) and the one at \(z = 3\). The large-order behaviour of the coefficients around the point \(z = 2\) will therefore be governed by two contributions: one alternating (backwards looking from 2 to 1) and one non-alternating (forwards looking from 2 to 3). We will observe and discuss this phenomenon in our detailed analysis in Sections 3 and 4.

2.3. All-order resurgent asymptotics for a resonant renormalon

Instead of working with the Borel transform of factorially divergent power series directly, it is usually more convenient to keep working with the initial factorially divergent power series and augment it by exponentially suppressed trans-series contributions as we did in (7)–(9). A higher-order trans-series term of the form \(x^{-\beta_k}e^{-k/x}\) can directly be associated to a expansion of the Borel transform at the singular point \(z = k\).

For our concrete trans-series solution (7) of the ODE (2) this means that the \(m\)-th instanton coefficient \(g_m(x)\) in front of \(\left(x^{-\frac{35}{12}}e^{-\frac{1}{x}}\right)^m\) encodes the local expansion around the \(m\)-th pole of the Borel transform of the perturbative solution \(g_0(x) = A(x)\).

An advantage of this trans-series based approach is that the low- and large-order correspondence works transparently with a minimal number of transcendental prefactors and that the
trans-series Ansatz, which is substituted into the ODE together with suitable choices for all ambiguities, yields a mechanical way to generate the respective higher-order expansions. Moreover, the trans-series approach is compatible with alien calculus, which constitutes a corner stone of resurgence theory. We will illustrate this compatibility in the scope of the analytic continuation process in Section 3.6.

With these mathematical tools at hand we are able to formulate our second main result: A conjectured though well-tested solution of the analytic continuation problem for ODE (2) at all orders. This result has been obtained by employing a mixture of empirical and analytical methods.

**Conjecture 1.** The asymptotic expansion of the coefficients in the trans-series (8) associated to the third-order problem in (2) is

\[
d_{i,j}^{(m)}(n) \sim -(s+1)S_1 \sum_{k \geq 0} a_{i,j}^{(m+1)}(k) \Gamma(n + \frac{35}{12} - k)
\]

\[
+ S_1 \sum_{k \geq 0} \left(4(i+1)a_{i+1,j}^{(m+1)}(k) + 6(j+1)a_{i,j+1}^{(m+1)}(k)\right) \Gamma(n - \frac{25}{12} - k)
\]

\[
\times \left(\frac{21265}{4608} \psi(n - \frac{25}{12} - k) - k\right) + d_1)
\]

\[
+ \frac{1}{4} S_3 \sum_{k \geq 0} \left(4(s+1)a_{i+1,j}^{(m-1)}(k) + 6(j+1)a_{i,j+1}^{(m-1)}(k)\right) (-1)^{n-k} \Gamma(n + \frac{25}{12} - k)
\]

\[
- 2(s-2i-1)S_3 \sum_{k \geq 0} a_{i,j}^{(m-1)}(k) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k)
\]

\[
\times \left(\frac{21265}{4608} \psi(n - \frac{35}{12} - k) + f_1\right)
\]

\[
- S_3 \sum_{k \geq 0} \left(8(i+1)a_{i+1,j}^{(m-1)}(k) + 6(j+1)a_{i,j+1}^{(m-1)}(k)\right) (-1)^{n-k} \Gamma(n - \frac{95}{12} - k) Q(n - \frac{95}{12} - k)
\]

\[
- (f_1 - c_1) S_3 \sum_{k \geq 0} \left(2(i+1)a_{i+1,j-1}^{(m-1)}(k) + 6(i+j)a_{i,j}^{(m-1)}(k)\right) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k)
\]

\[
(19)
\]

for all integers \(m, i, j \geq 0\) with \(s = m - 2i - 3j \geq 0\), on the understanding that \(a_{i',j'}^{(m')}(n)\) vanishes if \(m' - 2i' - 3j' < 0\) or any of \(m', i', j'\) is negative, \(\psi(z)\) is the derivative of \(\log \Gamma(z)\) and

\[
Q(z) = \left(\frac{21265}{4608}\right)^2 \left(\psi^2(z) + \psi'(z)\right) + 2c_1 \left(\frac{21265}{4608}\right) \psi(z) + c_2.
\]

(20)

Note that each term in \(Q(z)\) corresponds to a derivative of \(\Gamma\) as \(\psi^2(z) + \psi'(z) = \Gamma''(z)/\Gamma(z)\).

There are six indeterminate Stokes constants in (19). These constants are invariants of the ODE (2) and they encode the solution of the connection problem to get from a low-order perturbative expansion to a large-order asymptotic expansion as discussed in Section 2.2.

We evaluated 1000 digits of each constant. To 50 digits, their values are given by

\[
S_1 = 0.087595552909179124483795447421262990627388017406822\ldots
\]

(21)

\[
d_1 = -43.332634728250755924500717390319380703460728022278\ldots
\]

(22)

\[
S_3 = 2.17178531405909902116086012277903892302479464193027\ldots
\]

(23)

\[
f_1 = -40.903692509228515003814479126901354785263669553014\ldots
\]

(24)
\[ c_1 = -41.031956764302710583921068101545509453704897898188 \ldots \]  
\[ \frac{c_2}{c_1^2} = 1.0002016472131992595822805380838324188011572304276 \ldots \]  

The Stokes constant \( S_1 \) in the first line of \( (19) \) governs forwards resurgence of non-logarithmic terms. In the second line, \( d_1 S_1 \approx -3.79575 \) appears under the logarithmic contribution from \( \psi(z) = \log(z) + \mathcal{O}\left(\frac{1}{z}\right) \). Backwards resurgence, with alternating signs, is governed by \( S_3 \) in the third line. In the fourth line, \( f_1 S_3 \) appears, under \( \psi(z) \). The fifth line involves \( c_1 \) and \( c_2 \), via the abbreviation \( Q(z) \), defined in \( (20) \). The closeness of \( c_2 \) to \( c_1^2 \), noted in \( (26) \), gives a rather good quadratic approximation \( Q(z) \approx \left(\frac{21265}{4609}\right) \log(z) + c_1^2 \) at large \( z \). In the final line of \( (19) \), the small combination \( (f_1 - c_1) S_3 \approx 0.278562 \) governs the backwards resurgence of the second instanton in the asymptotic expansion for the coefficients of the third. The value of \( S_3 \) matches an estimate of this constant by Gerald Dunne [96], who obtained the first five of its digits by making use of a uniformization map combined with Borel–Padé-resummation methods [97–99].

The evidence for Conjecture 1 will be discussed in detail in Section 4. Before that, we will introduce our methods by applying them to a nonlinear ODE that is slightly simpler than \( (2) \). This simplified ODE will feature a resonance between the two solutions of its linearized ODE, but no logarithmic terms. This establishes that the logarithmic terms in \( (2) \) and the similarity to structures that appear in WKB analyses of quantum mechanical models cannot be explained merely by the resonant ratio of exponential prefactors in solutions to linearized ODEs.

3. Log-free resurgence of a second-order ODE

3.1. A second-order nonlinear ODE and its trans-series solution

The second-order differential equation

\[ (\tilde{g}(x)P - 1)(\tilde{g}(x)P - 2)\tilde{g}(x) = 1, \quad \text{where} \quad P = x \left( 2x \frac{d}{dx} + 1 \right), \]  

with cubic nonlinearity is arguably a natural simplification of \( (2) \). The ODE \( (27) \) is the intermediate case between \( (2) \) and the even simpler first-order ODE \( (\tilde{g}(x)P - 1)\tilde{g}(x) = -\frac{1}{2} \), where \( \tilde{g}(x) = C(x/2)/x \) and \( C(x) \) was the subject of [83]. Even though the ODE \( (27) \) is simpler than \( (2) \), it is obviously of similar structure. The trans-series solution and the all-order asymptotic resurgence analysis of \( (27) \) will be the subject of this section.

As before, we start by solving the linearized homogeneous differential equation for non-perturbative solutions. We obtain two such solutions:

\[ \tilde{g}(x) = \tilde{g}_0(x) + \tilde{h}_k \left( x^{-\frac{9}{4}} e^{-\frac{1}{4}} \right)^k, \quad k \in \{1, 2\}, \]  

where \( \tilde{g}_0(x) \) is the unique perturbative Frobenius-type solution of \( (27) \)

\[ \tilde{g}_0(x) = \frac{1}{2} + \frac{3}{8} x + \frac{15}{16} x^2 + \frac{483}{128} x^3 + \ldots \]  

Again, the two non-perturbative solutions \( \tilde{h}_1(x) \) and \( \tilde{h}_2(x) \) to the linearized version of \( (27) \) are resonant, as the action of \( \tilde{h}_2(x) \) is twice the action of \( \tilde{h}_1(x) \). In spite of this resonance, we do not observe logarithmic terms in the trans-series solution of \( (27) \) as we did in \( (7)–(9) \). This log-free trans-series solution of \( (27) \) reads
Table 1
Table of the coefficients in the trans-series solution (30)–(31) of (27). The peculiar choice for the coefficient \( \tilde{a}^{(2)}_0 (3) \) (in bold and green) leads to a particularly simple form of Conjecture 2. (For interpretation of the colours in the table(s), the reader is referred to the web version of this article.)

| n   | 0      | 1      | 2      | 3      | 4      | 5      |
|-----|--------|--------|--------|--------|--------|--------|
| \( \tilde{a}^{(0)}_0 (n) \) | 1      | \( \frac{8}{3} \) | 15     | 482    | 5157   | 134123 |
| \( \tilde{a}^{(1)}_0 (n) \) | -1     | 16     | 1047   | 70227  | 2553009| 41513422731 |
| \( \tilde{a}^{(2)}_0 (n) \) | -2     | 3      | 1512   | 8192   | 252488 | 1419409040 |
| \( \tilde{a}^{(3)}_0 (n) \) | -6     | \( \frac{8}{3} \) | -256   | 64     | 3127   | 655360  |
| \( \tilde{a}^{(4)}_0 (n) \) | -6     | \( \frac{5}{3} \) | 165    | \( \frac{3127}{4} \) | 32768  | 1310720  |
| \( \tilde{a}^{(5)}_0 (n) \) | -24    | 188    | -91    | \( \frac{1}{2} \) | 157613 | 209114347 |
| \( \tilde{a}^{(6)}_0 (n) \) | -2     | 35     | 2731   | 576    | 1024   | 184320  |
| \( \tilde{a}^{(7)}_0 (n) \) | -250   | 7475   | 854095| 69603643 | 675041215 | 11156692102565 |
| \( \tilde{a}^{(8)}_0 (n) \) | -350   | 9425   | 3034175| 321421075 | 25392960 | 113246208 |
| \( \tilde{a}^{(9)}_0 (n) \) | -45    | 7337   | 2820071| 390504835 | 6305133175 | 15960623177125 |
| \( \tilde{a}^{(10)}_0 (n) \) | -1728  | 23544  | 225189 | 4205583 | 129518877 | 15136299041 |
| \( \tilde{a}^{(11)}_0 (n) \) | -576   | 7164   | 64029  | 2654373 | 762031359 | 78271944319 |
| \( \tilde{a}^{(12)}_0 (n) \) | -180   | 4485   | 341941 | 127558889 | 5361308917 | 5439389708747 |
| \( \tilde{a}^{(13)}_0 (n) \) | -6     | \( \frac{5}{3} \) | 192    | 24080  | 172800  | 147556000 |

\[
\tilde{g}(x) = \sum_{m=0}^{\infty} \tilde{g}_m(x)\tilde{y}^m, \quad \tilde{y} = x^{-\frac{1}{2}} e^{-\frac{i}{2}}, \quad (30)
\]

\[
\tilde{g}_m(x) = \sum_{|j|=0}^{\lfloor m/2 \rfloor} \sigma_1^{m-2j} (\sigma_2 x^3)^j \tilde{T}_{m,j}(x), \quad \tilde{T}_{m,j}(x) = \sum_{n=0}^{\infty} \tilde{a}^{(m)}_j(n) x^n. \quad (31)
\]

As was the case in (7)–(9), the second-order ODE (27) does only fix the values of the coefficients \( \tilde{a}^{(m)}_j(n) \) up to a few ambiguities. Here, we find three such ambiguities. We resolve two of them by

\[
\tilde{a}^{(1)}_0(0) = -1, \quad \tilde{a}^{(2)}_0(0) = -1, \quad (32)
\]

which normalize \( \sigma_1 \) and \( \sigma_2 \). The third ambiguity comes from the freedom to shift \( \sigma_2 \) by an arbitrary multiple of \( \sigma_1^2 \). We resolve this by

\[
\tilde{a}^{(2)}_0(3) = \frac{9855}{512}, \quad (33)
\]
to simplify the asymptotic expansion of the coefficients \( \tilde{a}^{(m)}_j(n) \) for large \( n \) as much as possible. We will discuss this choice in detail in the next section.

Table 1 gives expansions up to \( x^5 \) of the 16 terms in (30)–(31) up to \( \tilde{y}^6 \) using this normalization.
3.2. Resolution of ambiguities via asymptotic analysis

The choices of signs in (32) ensure that the coefficients $\tilde{a}_j^{(m)}(n)$ are eventually positive, at large $n$, for all $m \geq 2j \geq 0$. The choice for the remaining ambiguity $\tilde{a}_0^{(2)}(3) = \frac{9855}{312}$ in (33) is harder to come by. It is justified by the especially simple asymptotic behaviour of the coefficients $\tilde{a}_j^{(m)}(n)$ obtained with it. We will discuss the line of thought that leads to this choice in the remainder of this section. Analogous arguments will lead to the ambiguity fixing choices in (11) for the full complexity of the solution in (7)–(9) of ODE (2), which we will discuss in Section 4.

We start by considering the asymptotic behaviour of the coefficients $\tilde{a}_0^{(0)}(n)$ of the perturbative solution $\tilde{g}_0(x) = \sum_{n \geq 0} \tilde{a}_0^{(0)}(n)x^n$:

$$
\tilde{a}_0^{(0)}(n) = \tilde{S}_1 \Gamma(n + \frac{9}{4}) \left( 1 - \frac{33}{16n} + \mathcal{O}\left( \frac{1}{n^2} \right) \right), \quad \tilde{S}_1 = 0.17595473991964250815209678804264889548688592517722 \ldots
$$

By the low- and large-order correspondence in the trans-series which has been discussed in Section 2.2, the asymptotic expansion that begins with (34) may be extended, ad libitum, using the first-instanton coefficients, $\tilde{a}_0^{(1)}(k)$, in

$$
\tilde{a}_0^{(0)}(n) \sim -\tilde{S}_1 \sum_{k \geq 0} \tilde{a}_0^{(1)}(k) \Gamma(n + \frac{9}{4} - k). \quad \text{(36)}
$$

To fix the remaining ambiguity, we have to consider the asymptotic expansion of the first-instanton coefficients $\tilde{a}_0^{(1)}(n)$ which is governed itself by the second-instanton contribution. In the second instanton, the second term of $\tilde{g}_2 = \sigma_1^2 \tilde{T}_{2,0} + \sigma_2 \tilde{T}_{2,1}x^3$, is suppressed by $x^3$. Thus the term of order $x^3$ in $\tilde{T}_{2,0}$ is ambiguous, since we may add to the inhomogeneous solution $\tilde{T}_{2,0}$ any multiple of the homogeneous instanton solution $\tilde{T}_{2,1}x^3$. We resolved this ambiguity by setting $\tilde{a}_0^{(2)}(3) = \frac{9855}{312}$ in (33). Ex post facto, this ensures that the asymptotic expansion for the first instanton

$$
\tilde{a}_0^{(1)}(n) \sim -2\tilde{S}_1 \sum_{k \geq 0} \tilde{a}_0^{(2)}(k) \Gamma(n + \frac{9}{4} - k) \quad \text{(37)}
$$

does not involve the coefficients $\tilde{a}_1^{(2)}(k)$ of the second instanton. We achieved this empirically, lacking an a priori method to decouple the instantons, by numerically computing the asymptotic expansion of $\tilde{a}_0^{(1)}(n)$ up to sufficiently high order. This numerical computation revealed the value $\frac{9855}{312}$ at the third order of the asymptotic expansion (37).

3.3. Patterns of resurgence

We proceed to analyze the asymptotic expansions of higher-order instantons. The asymptotic expansions for the terms in $\tilde{g}_2 = \sigma_1^2 \tilde{T}_{2,0} + \sigma_2 \tilde{T}_{2,1}x^3$ are

$$
\tilde{a}_0^{(2)}(n) \sim -3\tilde{S}_1 \sum_{k \geq 0} \tilde{a}_0^{(3)}(k) \Gamma(n + \frac{9}{4} - k) + \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_0^{(1)}(k)(-1)^{n-k} \Gamma(n - \frac{9}{4} - k) \quad \text{(38)}
$$

$$
\tilde{a}_1^{(2)}(n) \sim -\tilde{S}_1 \sum_{k \geq 0} \tilde{a}_1^{(3)}(k) \Gamma(n + \frac{9}{4} - k) + 16\tilde{S}_2 \sum_{k \geq 0} \tilde{a}_0^{(1)}(k)(-1)^{n-k} \Gamma(n + \frac{3}{4} - k) \quad \text{(39)}
$$
which look forwards, to \( \tilde{g}_3 = \sigma_1^3 \tilde{T}_{3,0} + \sigma_1 \sigma_2 \tilde{T}_{3,1}x^3 \), and also backwards, to \( \tilde{g}_1 \) = \( \sigma_1 \tilde{T}_{1,0} \), with alternating signs and an empirical constant
\[
\tilde{S}_2 = 0.11097873354795693645043942852479413454973815476146 \ldots
\]
which also appears in the backwards looking terms of
\[
\tilde{a}_0^{(3)}(n) \sim -4 \tilde{S}_1 \sum_{k \geq 0} \tilde{a}_0^{(4)}(k) \Gamma(n + \frac{9}{4} - k) + 2 \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_0^{(2)}(k)(-1)^{n-k} \Gamma(n - \frac{9}{4} - k)
\]
\[
\tilde{a}_1^{(3)}(n) \sim -2 \tilde{S}_1 \sum_{k \geq 0} \tilde{a}_1^{(4)}(k) \Gamma(n + \frac{9}{4} - k) + 32 \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_0^{(2)}(k)(-1)^{n-k} \Gamma(n + \frac{3}{4} - k)
\]
\[
+ 4 \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_1^{(2)}(k)(-1)^{n-k} \Gamma(n - \frac{9}{4} - k).
\]
The asymptotic expansions for the terms in \( \tilde{g}_4 \) = \( \sigma_1^4 \tilde{T}_{4,0} + \sigma_1^2 \sigma_2 \tilde{T}_{4,1}x^3 + \sigma_2^2 \tilde{T}_{4,2}x^6 \) are:
\[
\tilde{a}_0^{(4)}(n) \sim -5 \tilde{S}_1 \sum_{k \geq 0} \tilde{a}_0^{(5)}(k) \Gamma(n + \frac{9}{4} - k) + 3 \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_0^{(3)}(k)(-1)^{n-k} \Gamma(n - \frac{9}{4} - k)
\]
\[
\tilde{a}_1^{(4)}(n) \sim -3 \tilde{S}_1 \sum_{k \geq 0} \tilde{a}_1^{(5)}(k) \Gamma(n + \frac{9}{4} - k) + 48 \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_0^{(3)}(k)(-1)^{n-k} \Gamma(n + \frac{3}{4} - k)
\]
\[
+ 5 \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_1^{(3)}(k)(-1)^{n-k} \Gamma(n - \frac{9}{4} - k)
\]
\[
\tilde{a}_2^{(4)}(n) \sim - \tilde{S}_1 \sum_{k \geq 0} \tilde{a}_2^{(5)}(k) \Gamma(n + \frac{9}{4} - k) + 16 \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_1^{(3)}(k)(-1)^{n-k} \Gamma(n + \frac{3}{4} - k).
\]

3.4. All-order resurgent asymptotics

The previous observations lead us to the following conjecture about the large-order behaviour of all sets of coefficients \( \tilde{a}_j^{(m)}(n) \) for \( n \to \infty \).

**Conjecture 2.** The asymptotic expansion of the coefficients in the trans-series (30)–(31) associated to the second-order problem in (27) is:
\[
\tilde{a}_j^{(m)}(n) \sim (m - 2j + 1) \tilde{S}_1 \sum_{k \geq 0} \tilde{a}_j^{(m+1)}(k) \Gamma(n + \frac{9}{4} - k)
\]
\[
+ 16(m - 2j + 1) \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_j^{(m-1)}(k)(-1)^{n-k} \Gamma(n + \frac{3}{4} - k)
\]
\[
+ (m + 2j - 1) \tilde{S}_2 \sum_{k \geq 0} \tilde{a}_j^{(m-1)}(k)(-1)^{n-k} \Gamma(n - \frac{9}{4} - k)
\]

for all \( m, j \) with \( 0 \leq 2j \leq m \) on the understanding that \( \tilde{a}_j^{(m')}(n) \) vanishes if \( j' < 0 \) or \( m' < 2j' \).

Here, in contrast to Conjecture 1, only two Stokes constants are sufficient to completely capture the \( n \to \infty \) asymptotic behaviour of the coefficients \( \tilde{a}_j^{(m)}(n) \) for all \( 0 \leq 2j \leq m \).

To test Conjecture 2, we extended Table 1 to order \( \tilde{Y}^9 \) and the 30 rows of Tables 1 and 2 to \( n > 100 \). Using exact rational values of \( \tilde{a}_j^{(m)}(n) \), we checked the conjecture at 100 digits of precision for \( m \leq 4 \), at 70 digits for \( m = 5 \) and at 30 digits for \( m = 6, 7, 8 \).
In Section 3.6 we will augment some of the empirical observations that were made in previous sections to justify Conjecture 2 by an interpretation in terms of alien calculus. Before that we will briefly discuss further empirical results which follow from our analysis of (27) and which enable a partial trans-asymptotic resummation.

3.5. Trans-asymptotics: all-instanton-order results

Additionally to Conjecture 2 we empirically deduced for the leading coefficients of the trans-series in (30)–(31) that

\[
\tilde{a}_j^{(m)}(0) = -(m + 2j) \frac{2^{m-2j-1} m^{m-j-2}}{(m-2j)!} j!
\]

(47)

for all \( m \geq 2j \geq 0 \).

Setting \( j = 0 \), we have \( \tilde{a}_0^{(m)}(0) = -(−2)^{m−1} w_m, \) for \( m > 0 \), where \( w_m = (−m^{m−1}/m! \) is the coefficient of \( x^m \) in the expansion of the Lambert-W function on its principal branch. The Lambert function \( W = \sum_{m=1}^{\infty} w_m x^m = x − x^2 + \frac{3}{2} x^3 + O(x^4) \) solves \( We^W = x \) for \( |x| < e^{-1} \) and appears prominently in a trans-asymptotic analysis of the four-dimensional Yukawa theory renormalon \([55,83]\), which is closely related to the renormalon (1), and also while evaluating topological invariants using renormalization methods \([8]\) as well as in other expansions of importance with connection to renormalization \([100–103]\).

3.6. Alien calculus analysis

Alien calculus is an integral part of the resurgence framework \([58, \text{Ch. 6}] \) (see also \([77,104]\) for a simplified version of this technology that works without intricate Borel transform considerations and \([77]\) for a direct application to zero-dimensional quantum field theories). In this section we will use this calculus to explain the shapes of equations (36)–(38).
We can define a family of differential operators $\mathcal{A}_\omega$ that is indexed by complex numbers $\omega \in \mathbb{C} \setminus \{0\}$. These operators, the alien derivatives, linearly map a formal power series $f(x) \in \mathbb{C}[[x]]$ to another formal power series $(\mathcal{A}_\omega f)(x) \in \mathbb{C}[[x]]$ that encodes the local expansion of the Borel transform $B[A](z)$ near the point $z = \omega$.

If a factorially divergent power series $f(x) = \sum_{n=0}^{\infty} f_n x^n$ has a nontrivial alien derivative $(\mathcal{A}_\omega f)(x) \neq 0$ and $\omega$ is the position of the closest singularity to the origin in the Borel plane of $f(x)$ as discussed in Section 2.2, then we can translate the knowledge of the alien derivative into information on the large-order behaviour of the coefficients $f_n$ and vice-versa. In fact, alien calculus can be seen as an explicit mathematical realization of the low- and large-order correspondence that has been discussed repeatedly in this article. For instance, if we have the following power series expansion for $\mathcal{A}_\omega f$,

$$ (\mathcal{A}_\omega f)(x) = \sum_{k=0}^{\infty} c_k x^{k-\beta} ,$$

where $\omega$ is the ordinate of the dominant singularity of $B[A](z)$, then the asymptotic behaviour of the coefficients of $f(x)$ is given by

$$ f_n \sim \sum_{k \geq 0} c_k \omega^{-n+k-\beta} \Gamma(n-k+\beta) \text{ for } n \to \infty. $$

In other words, the coefficients $c_k$ encode both the series representation of the alien derivative $\mathcal{A}_\omega f$ and the large-order behaviour of its coefficients $f_n$. The utility of this definition of $\mathcal{A}_\omega f$ comes from the fact that the linear alien derivative operators fulfil product and chain rules,

$$ (\mathcal{A}_\omega f \cdot g)(x) = g(x)(\mathcal{A}_\omega f)(x) + f(x)(\mathcal{A}_\omega g)(x) $$

(50)

$$ (\mathcal{A}_\omega f \circ g)(x) = f'(g(x))(\mathcal{A}_\omega g)(x) + e^{-\omega(\frac{1}{\pi \text{i}} - \frac{1}{2})}(\mathcal{A}_\omega f)(g(x)), $$

(51)

where we have to assume $g(x) = x + \mathcal{O}(x^2)$ for the chain rule to hold.

The alien derivative does not commute with the ordinary derivative. The commutator reads,

$$ [\mathcal{A}_\omega, \partial_x] = \frac{\omega}{x^2} \mathcal{A}_\omega. $$

(52)

These identities are proven in [58, Ch. 6]. See also [77] for proofs and derivations in a simplified context.

We can apply the alien derivative operator $\mathcal{A}_\omega$ to the ODE (27) from the left and commute it to the right using the product and the commutation rule. We obtain a linear ODE for $(\mathcal{A}_\omega \tilde{g})(x)$ of the form,

$$ p_0(x, \tilde{g}, \tilde{g}', \tilde{g}'', \omega)(\mathcal{A}_\omega \tilde{g})(x) + p_1(x, \tilde{g}, \tilde{g}', \omega) \partial_x (\mathcal{A}_\omega \tilde{g})(x) + p_2(x, \tilde{g}) \partial_x^2 (\mathcal{A}_\omega \tilde{g})(x) = 0, $$

(53)

where $p_0, p_1, p_2$ are polynomials. The $\omega$-dependence appears due to the commutation rule (52). This ODE turns out to only have a power series solution of the form $(\mathcal{A}_\omega \tilde{g})(x) = x^{-\beta}(1 + \mathcal{O}(x))$ if $\omega \in \{1, 2\}$. We can conclude that $\mathcal{A}_\omega \tilde{g} = 0$ for all $\omega \notin \{1, 2\}$. The solutions for $(\mathcal{A}_\omega \tilde{g})(x)$ are the formal power series

$$ (\mathcal{A}_1 \tilde{g})(x) = \tilde{m}_1 x^{-\frac{3}{4}} \tilde{f}_{1,0}(x) \quad \text{and} $$

$$ (\mathcal{A}_2 \tilde{g})(x) = \tilde{m}_2 x^{\frac{3}{4}} \tilde{f}_{2,1}(x), $$

(54)

(55)
where $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are undetermined integration constants. For the asymptotic behaviour of the original expansion coefficients $\tilde{a}_0^{(\mu)}(n)$ only the solution for $\omega = 1$ is relevant, as it is closest to the origin. The differential equation only fixes the alien derivative $(A_1 g)(x)$ up to the overall constant $\mu_1$. We determined the value of this constant numerically. From equation (36) and the relationship between asymptotics and alien derivatives (48)–(49) it is evident that

$$ (A_1 g)(x) = -\tilde{S}_1 x^{-\frac{9}{2}} \tilde{T}_{1,0} (x) = -\tilde{S}_1 \sum_{n=0}^{\infty} \tilde{a}_0^{(1)}(n) x^{-\frac{n}{2}} $$

(56)

is the correct expression for the first $A_1$ derivative of $g$.

We can repeat the application of alien derivatives. Multiple $A_\omega$ derivatives with different values of $\omega$ do not commute; they form a *free algebra*. For example, we can apply the operator $A_1$ twice from the left to the ODE (27). We get an equation of the form

$$ p_0(x, g, g', g'') (A_1^2 g)(x) + p_1(x, g, g', g'') \partial_x (A_1^2 g)(x) + p_2(x, g) \partial_x^2 (A_1^2 g)(x) $$

$$ = p_1(x, g, g', g'', (A_1 g), (A_1 g)'(A_1 g)''), $$

(57)

which, up to an inhomogeneity that is captured by the polynomial $p_1$, is the same ODE for $(A_1^2 g)(x)$ as (53) for $(A_\omega g)(x)$ in the $\omega = 2$ case. The reason is that $[A_1^2, \partial_x] = \frac{2}{x^2} A_1^2$. Therefore $A_1^2$ behaves similarly to the operator $A_2$ for which $[A_2, \partial_x] = \frac{2}{x^2} A_2$. This leads to a one-parameter family of possible explicit expressions for $(A_1^2 g)(x)$. Again, the differential equation does not provide sufficient information to completely fix $(A_1^2 g)(x)$. Empirically, we determined the second alien derivative $(A_1^2 g)(x)$ in (37). The additional piece of information that $\tilde{a}_0^{(2)}(3) = \frac{9855}{312}$ leads to

$$ (A_1^2 g)(x) = A_1 (A_1 g)(x) = -\tilde{S}_1 A_1 \left( x^{-\frac{9}{2}} \tilde{T}_{1,0}(x) \right) = -\tilde{S}_1 x^{-\frac{9}{4}} (A_1 \tilde{T}_{1,0})(x) $$

$$ = 2\tilde{S}_1 x^{-\frac{9}{4}} \tilde{T}_{2,0}(x). $$

(58)

Note that the operator $A_1$ can be interpreted as the *forwards looking* alien derivative operator as in the illustrative argument from Section 2.2. Analogously we can interpret $A_{-1}$ as the *backwards looking* alien derivative. An instructive example is the application of the operator $A_{-1} A_1$ to both sides of the ODE (27). We get

$$ p_0(x, g, g', g'', 0) (A_{-1} A_1 g)(x) + p_1(x, g, g', 0) \partial_x (A_{-1} A_1 g)(x) $$

$$ + p_2(x, g) \partial_x^2 (A_{-1} A_1 g)(x) = 0, $$

(59)

where we have used that $A_{-1} g = 0$ by (53) and $-1 \notin \{1, 2\}$. This equation, which is of the form (53) has no power series solution for $A_{-1} A_1 g$ and we can infer that $A_{-1} A_1 g = 0$. This is in accordance with the missing alternating contribution to the asymptotics of the coefficients $\tilde{a}_0^{(1)}(n)$ in (37).

Acting with $A_{-1}$ on (57), we obtain

$$ p_0(x, g, g', g'', 1) (A_{-1} A_1^2 g)(x) + p_1(x, g, g', 1) \partial_x (A_{-1} A_1^2 g)(x) $$

$$ + p_2(x, g) \partial_x^2 (A_{-1} A_1^2 g)(x) = 0, $$

(60)

where we have used that $A_{-1} g = A_{-1} A_1 g = 0$ together with the commutator rule for ordinary and alien derivatives. This homogeneous equation is exactly the ODE in (53) with $\omega = 1$. Therefore,
where the overall constant for the homogeneous solution stays undetermined. Translating this into an alien derivative of \( \tilde{T}_{2,0}(x) \) by using (58) results in

\[
(A_{-1}\tilde{T}_{2,0})(x) = \frac{1}{2S_1^2} \tilde{\mu}_{-1,1} x^{\frac{9}{4}} \tilde{T}_{1,0}(x),
\]

where we established that indeed the \( \tilde{T}_{1,0}(x) \) sequence reappears in the alternating part of the \( \tilde{T}_{2,0}(x) \) asymptotic expansion in (38). The exponent of the prefactor \( x^{\frac{9}{4}} \) explains the negative shift of \( \frac{9}{4} \) in the \( \Gamma \) function of the alternating part of (38) due to the alien derivative and asymptotics relation (48)–(49).

We can fix the undetermined number \( \tilde{\mu}_{-1,1} \) numerically by comparing it to the large-order computation in (38) which implies that

\[
(A_{-1}\tilde{T}_{2,0})(x) = \tilde{S}_2(-x)^{\frac{9}{4}} \tilde{T}_{1,0}(x)
\]

where we accounted for the sign \((-1)^{n-k}\) in (38) using the definition (48)–(49) of \( A_\omega \). Therefore, we have \( \tilde{\mu}_{-1,1} = 2\tilde{S}_1^2 \tilde{S}_2 (-1)^{\frac{9}{4}} \), where the branch of the fourth root in this expression stays undetermined from our simple considerations.

It seems plausible that a complete proof of Conjecture 2 as well as for Conjecture 1 is achievable using alien calculus methods, but such a proof lies beyond the scope of this article. With this remark we finish our discussion of the alien calculus viewpoint.

4. Resonant resurgence from a Dyson–Schwinger equation

After this digression on the log-free ODE (27), we will return to the analysis of (2), which is associated to the six-dimensional \( \phi^3 \) theory renormalon singularity described by the Dyson–Schwinger equation (1) and which features the full complexity that is expected from a QFT renormalon.

In this section we will present the striking evidence for the validity of Conjecture 1 and our reasoning for the peculiar ambiguity fixing choices in (11), which involves the large fraction \( r_1 = \frac{326426293907919}{366917711392} \). The analysis and arguments follow similar lines as our analysis of the ODE (27) in the previous section.

4.1. Resolution of ambiguities via asymptotic analysis

To illustrate our process, we introduce a new convention for the \( m \)-th instanton order expansions in the trans-series solution (7) for the third-order problem (2). In (7), \( g_m \) has contributions proportional to \( \sigma_1 \sigma_2^2 \sigma_3^3 \) with \( m = s_1 + 2s_2 + 3s_3 \). For \( m \leq 5 \) there are 16 such monomials, which we label as follows:

\[
\begin{array}{ccccccccccccccc}
A & B & C & D & E & F & G & H & I & J & K & U & V & W & X & Y \\
\hline
s_1 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 3 & 2 & 4 & 0 & 1 & 2 & 3 & 5 \\
s_2 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 \\
s_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
s_1 + 2s_2 + 3s_3 & 0 & 1 & 2 & 3 & 3 & 2 & 4 & 4 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \\
\end{array}
\]

(64)
The lexicographic ordering in dictionary (64) corresponds to the order in which we investigated progressively more demanding expansions. To each letter, we associate a formal power series in \( x \).

The perturbative solution \( A(x) \) is given by (3). For the linearized non-perturbative solutions in (4), we use the notation \( h_1(x) = B(x) \), \( h_2(x) = x^5 C(x) \) and \( h_3(x) = x^5 D(x) \), with

\[
B(x) = -1 + \frac{97}{48} x + \frac{53917}{13824} x^2 + \frac{3026443}{221184} x^3 + \frac{32035763261}{352205952} x^4 + \frac{11517422581511}{18345885696} x^5 + O(x^6)
\]

\[C(x) = -1 + \frac{151}{24} x - \frac{63727}{3456} x^2 + \frac{7112963}{82944} x^3 - \frac{7975908763 x}{23887872} x^4 + \frac{517065181955}{191102976} x^5 + O(x^6)
\]

\[D(x) = -1 + \frac{227}{48} x + \frac{1399}{4608} x^2 + \frac{814211}{73728} x^3 + \frac{344654437}{42467328} x^4 + \frac{46933940993}{679477248} x^5 + O(x^6).
\]

Thus \{ \( A(x) \), \( B(x) \), \( C(x) \), \( D(x) \) \} in (3), (65)–(67) refer to the perturbative term \( g_0 = A \) and the instanton terms \( \sigma_1 Y B \), \( \sigma_2 Y^2 x^5 C \), \( \sigma_3 Y^3 x^5 D \). For each letter in the dictionary we follow the convention of (3), where \( A_n \) is the coefficient of \( x^n \) in \( A(x) \).

In the trans-series we encounter powers of

\[
L = \frac{21265}{2304} x^5 \log(x)
\]

with the largest power of \( L \) in \( g_m \) no greater than \( m/2 \). Specifically,

\[
g_0 = A, \quad g_1 = \sigma_1 B, \quad g_2 = \sigma_2 x^5 C + \sigma_1^2 (F + CL),
\]

\[
g_3 = \sigma_3 x^5 D + \sigma_1 \sigma_2 x^5 E + \sigma_1^2 (I + (D + E)L),
\]

\[
g_4 = \sigma_1 \sigma_3 x^5 G + \sigma_2^2 x^{10} H + \sigma_1^2 \sigma_2 x^5 (J + 2HL) + \sigma_1^4 (K + (G + J)L + HL^2),
\]

\[
g_5 = \sigma_2 \sigma_3 x^{10} U + \sigma_1 \sigma_2^2 x^{10} V + \sigma_1^2 \sigma_3 x^5 (W + UL) + \sigma_1^3 \sigma_2 x^5 (X + (U + 2V)L) + \sigma_1^5 (Y + (W + X)L + (U + V)L^2).
\]

We already resolved one of the three ambiguities by the normalization \( B(0) = -1 \). Two ambiguities remain: we may add to \( g_2 \) an arbitrary multiple of \( \sigma_1^2 x^5 C \) and we may add to \( g_3 \) an arbitrary multiple of \( \sigma_1^3 x^5 D \).

To resolve these ambiguities we again analyze the large-order behaviour of the coefficients of the power series that appear in the trans-series solution. As in Section 3, the six terms in the six lines of (19) of Conjecture 1 revealed themselves successively after the analysis of more and more asymptotic expansions of the trans-series coefficients.

We start with the asymptotic expansion of the perturbative coefficients \( A_n \) of \( g_0 \) developed in [55]. One readily finds that [56]

\[
A_n = S_1 \Gamma(n + \frac{35}{12}) \left( 1 - \frac{97}{48n} + O \left( \frac{1}{n^2} \right) \right),
\]

which is the \( m = 0 \) case of a general trans-asymptotic result

\[
a_{0,0}^{(m)}(n) = \frac{(2m + 2)^m}{m!} S_1 \Gamma(n + \frac{35}{12}) \left( 1 - \frac{120m^2 + 175m + 97}{48(m + 1)n} + O \left( \frac{1}{n^2} \right) \right),
\]

which we conjecture to hold for all fixed \( m \) and which we have checked up to \( m = 8 \). As in (47) we recover the coefficients of the Lambert-W function with a trivial shift. It would be interesting to study the summation of these coefficients over all \( m \), but such a study lies beyond the scope of the present article.
The coefficients of $B(x) = \sum_{k \geq 0} B_k x^k$, determine the asymptotic expansion
\begin{equation}
A_n \sim -S_1 \sum_{k \geq 0} \Gamma(n + \frac{35}{12} - k) B_k,
\end{equation}

enabling us to determine 3000 digits of $S_1$, by developing 10000 terms of (3) and 5000 terms of (65).

For the asymptotic expansion of the second-instanton coefficients, we found
\begin{equation}
C_n \sim -S_1 \sum_{k \geq 0} E_k \Gamma(n + \frac{35}{12} - k) + S_3 \sum_{k \geq 0} B_k (-1)^{n-k} \Gamma(n + \frac{25}{12} - k).
\end{equation}

The term with alternating signs in the second sum, which looks backwards to the first instanton, is suppressed by a factor of $\frac{1}{n^{5/6}}$ relative to the first sum and is multiplied by the empirically determined constant $S_3$. The first sum looks forwards, to terms of order $y^3$ in the trans-series, where
\begin{equation}
E(x) = -4 + \frac{371}{12} x - \frac{11785}{1152} x^2 + \frac{820607}{18432} x^3 - \frac{18251431003}{10616832} x^4 + \frac{2356471056847}{169869312} x^5 + \mathcal{O}(x^6)
\end{equation}

occurs in the $\sigma_1 \sigma_2 x^5 E$ term of $g_2$. It is notable that the third-instanton coefficients $D_k$ are absent from (76). This is a consequence of the form of (7)–(9).

By developing 5000 terms of $C(x)$, we obtained almost 1500 decimal digits of $S_3$, using about 2500 terms of $E(x)$ and $B(x)$ in optimal truncations of the forwards and backwards looking terms in (76).

We continue with the resolution of the two ambiguities to explain our choices in (11). We will have to combine information from multiple asymptotic expansions to obtain a clear picture.

We start with the expansion for the ambiguous log-free term $\sigma_1^2 F$ in $g_2$, which we can write with $\alpha$ parameterizing the ambiguity as,
\begin{equation}
F(x) = -2 + \frac{49}{6} x + \frac{13235}{1728} x^2 + \frac{43049}{1728} x^3 + \frac{2496477497}{1728} x^4 + 2\alpha x^5 + \mathcal{O}(x^6)
\end{equation}

bearing in mind that for $n > 4$ only the combination $F_n + 2\alpha C_{n-5}$ is unambiguous. Then the asymptotic expansion for the first instanton has the form
\begin{equation}
B_n \sim -2S_1 \sum_{k \geq 0} F_k \Gamma(n + \frac{35}{12} - k) + 4S_3 \sum_{k \geq 0} C_k \Gamma(n + \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right),
\end{equation}

where $C_k$ multiplies the digamma $\psi$ function
\begin{equation}
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \log(z) - \frac{1}{2z} + \mathcal{O} \left( \frac{1}{z^2} \right).
\end{equation}

Experiment then assigns a large value to $\alpha + d_1 \approx 846.31$, which has already been observed in [56], but offers, as yet, no way of apportioning that number between $\alpha$ and $d_1$.

The asymptotic expansion for the third instanton has a backwards constant $S_4$ in
\begin{equation}
D_n \sim -S_1 \sum_{k \geq 0} G_k \Gamma(n + \frac{35}{12} - k) - S_3 \sum_{k \geq 0} C_k (-1)^{n-k} \Gamma(n + \frac{25}{12} - k)
\end{equation}

\begin{equation}
G(x) = -\frac{20}{3} + \frac{368}{9} x - \frac{9421}{1248} x^2 + \frac{61483}{1296} x^3 + \frac{622009525}{1119744} x^4 + \frac{17261921677}{3359232} x^5 + \mathcal{O}(x^6)
\end{equation}

\begin{equation}
S_4 = 0.55712485097773646632802466946834574964057746422381\ldots
\end{equation}
A second large constant occurs, modulo $\alpha$, in the backwards-looking part of
\[
E_n \sim -2S_1 \sum_{k \geq 0} J_k \Gamma(n + \frac{35}{12} - k) + 8S_1 \sum_{k \geq 0} H_k \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \\
+ 2S_3 \sum_{k \geq 0} F_k (-1)^{n-k} \Gamma(n + \frac{25}{12} - k) \\
+ 4S_3 \sum_{k \geq 0} C_k (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + e_1 \right),
\]
where
\[
H(x) = -2 + \frac{53}{2} x - \frac{72395}{432} x^2 + \frac{4651117}{5184} x^3 - \frac{3511918891}{746496} x^4 + \frac{259237318621}{8957952} x^5 + O(x^6),
\]
with $\alpha + e_1 \approx 848.55$.

We expand the ambiguous log-free term $\sigma_1^3 I$ in $g_3$, with $\beta$ parameterizing the ambiguity, as
\[
I(x) = -6 + \frac{309}{8} x - \frac{8821}{768} x^2 + \frac{454379}{36864} x^3 + \frac{1344528799}{2359296} x^4 + 2(4\alpha - \beta)x^5 + O(x^6)
\]
bearing in mind that for $n > 4$ only the combination $I_n + 2(\alpha E_{n-5} - \beta D_{n-5})$ is unambiguous. The dependence on $\alpha$ comes from an inhomogeneous term, involving products of $g_1$, $g_2$ and their derivatives. Then $\beta$ parameterizes an undetermined homogeneous contribution to $I(x)$.

We investigated the asymptotic behaviours of $F_n$ and $I_n$ with $\alpha = \beta = 0$. Our empirical results involved a large rational number
\[
r_1 = \frac{32642693907919}{36691771392} \approx 889.646\ldots
\]
whose several appearances were confirmed at more than 200 decimal digits of precision. We found that the explicit appearance of $r_1$ was removed by choosing $\alpha = \beta = r_1$. This gives $a_{0,0}^{(2)}(5) = 2r_1$ and $a_{0,0}^{(3)}(5) = 6r_1$, in (11), and removes $r_1$ from
\[
F_n \sim -3S_1 \sum_{k \geq 0} I_k \Gamma(n + \frac{35}{12} - k) \\
+ 2S_1 \sum_{k \geq 0} (3D_k + 2E_k) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \\
- 2S_3 \sum_{k \geq 0} B_k (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + f_1 \right)
\]
with an empirical relation
\[
3S_4 = 4(f_1 - e_1)S_3.
\]
By this method, we arrived at a resolution of ambiguities that produces comparatively simple asymptotic expansions in Conjecture 1, which do not explicitly involve $r_1$.

We proceed to collect further evidence for the validity of Conjecture 1.

4.2. Patterns of resurgence

Asymptotic expansions of the coefficients in $g(x) = \sum_{m \geq 0} g_m(x) y^m$ at order $y^3$ involve the expansions of
\[ J(x) = -\frac{52}{3} + \frac{1567}{9} x - \frac{442379}{2592} x^2 + \frac{7216345}{119744} x^3 - \frac{12483431383}{119744} x^4 + O(x^5) \]  
\[ K(x) = -\frac{64}{3} + \frac{1702}{9} x - \frac{188909}{648} x^2 - \frac{214877}{559872} x^3 + \frac{1175872537}{559872} x^4 + O(x^5) \]

which appear in

\[ I_n \sim -4S_1 \sum_{k \geq 0} K_k \Gamma(n + \frac{35}{12} - k) \]

\[ + 2S_1 \sum_{k \geq 0} (3G_k + 2J_k) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \]

\[ - 4S_3 \sum_{k \geq 0} F_k (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + f_1 \right) \]

\[ - 8S_3 \sum_{k \geq 0} C_k (-1)^{n-k} \Gamma(n - \frac{95}{12} - k) Q(n - \frac{95}{12} - k), \]

\[ Q(z) = \left( \frac{21265}{4608} \right)^2 \left( \psi^2(z) + \psi'(z) \right) + 2c_1 \left( \frac{21265}{4608} \right) \psi(z) + c_2, \]

which is also free of \( r_1 \). Moreover, we found the empirical relation

\[ S_4 = 2S_3 (c_1 - f_1). \]

At order \( y^2 \), we developed 200 terms of each of the 5 expansions

\[ U(x) = -\frac{13}{3} + \frac{7529}{144} x - \frac{10635103}{41472} x^2 + \frac{2075033425}{1990656} x^3 - \frac{505590564695}{1146617856} x^4 + O(x^5) \]

\[ V(x) = -\frac{109}{6} + \frac{77129}{288} x - \frac{150627967}{82944} x^2 + \frac{3935029505}{3981312} x^3 - \frac{119208170587255}{22932357212} x^4 + O(x^5) \]

\[ W(x) = -\frac{110}{3} + \frac{21493}{72} x - \frac{2613617}{20736} x^2 + \frac{3848375}{36864} x^3 + \frac{168395379395}{573308928} x^4 + O(x^5) \]

\[ X(x) = -80 + \frac{18073}{18} x - \frac{2117255}{432} x^2 + \frac{4766702935}{248832} x^3 - \frac{1833416477245}{238873728} x^4 + O(x^5) \]

\[ Y(x) = -\frac{250}{3} + \frac{67925}{72} x - \frac{54524255}{20736} x^2 + \frac{265397983}{331776} x^3 + \frac{6083187427417}{573308928} x^4 + O(x^5) \]

in order to have good control of the forwards looking non-alternating parts of the asymptotic expansions of \( \{G_n, H_n, J_n, K_n\} \), from order \( y^4 \). We found that

\[ G_n \sim -2S_1 \sum_{k \geq 0} W_k \Gamma(n + \frac{35}{12} - k) \]

\[ + 4S_1 \sum_{k \geq 0} U_k \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \]

\[ - S_3 \sum_{k \geq 0} (3D_k + E_k) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \]

\[ H_n \sim -S_1 \sum_{k \geq 0} V_k \Gamma(n + \frac{35}{12} - k) \]

\[ + \frac{1}{2} S_3 \sum_{k \geq 0} (3D_k + 2E_k) (-1)^{n-k} \Gamma(n + \frac{25}{12} - k) \]
Table 3

Coefficients $a_{i,j}^{(6)}(n)$ in the trans-series solution (8).

| n   | 0      | 1      | 2      | 3      | 4      | 5      |
|-----|--------|--------|--------|--------|--------|--------|
| $a_{3,0}^{(6)}(n)$ | −6     | 487    | 4      | 229217 | 6575923 | −600452444947 | 457959210595637 |
| $a_{2,0}^{(6)}(n)$ | −126   | 8463   | 4      | 5127107 | 3486217751 | −8669377115513 | 600844499892388373 |
| $a_{1,1}^{(6)}(n)$ | −52    | 4229   | 6      | 1440   | 172800  | −9216000     | 99532800000         |
| $a_{0,2}^{(6)}(n)$ | −192   | 5796   | 6      | 24     | 2880   | 6144000     | 199965600000         |
| $a_{0,1}^{(6)}(n)$ | −192   | 1950   | 5      | 24     | 2880   | 1576000     | 125918348441         |
| $a_{0,0}^{(6)}(n)$ | −1728  | 23832  | 3      | 384519 | 1599631 | 52207793    | 1670028654245         |

\[
J_n \sim -3S_1 \sum_{k \geq 0} X_k \Gamma(n + \frac{35}{12} - k)
\]

\[
+ 2S_1 \sum_{k \geq 0} (3U_k + 4W_k) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right)
\]

\[
+ 3S_3 \sum_{k \geq 0} I_k (-1)^n-k \Gamma(n + \frac{35}{12} - k)
\]

\[
+ 2S_3 \sum_{k \geq 0} E_k (-1)^n-k \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + 2e_1 - f_1 \right)
\]

(102)

\[
K_n \sim -5S_1 \sum_{k \geq 0} Y_k \Gamma(n + \frac{35}{12} - k)
\]

\[
+ 2S_1 \sum_{k \geq 0} (3W_k + 2X_k) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right)
\]

\[
- 6S_3 \sum_{k \geq 0} I_k (-1)^n-k \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + f_1 \right)
\]

\[
- 2S_3 \sum_{k \geq 0} (3D_k + 4E_k)(-1)^n-k \Gamma(n - \frac{95}{12} - k) Q(n - \frac{95}{12} - k)
\]

(103)

with \(c_1, c_2\) appearing in the final line, which uses the abbreviation (20).

In conclusion, no new constant emerges from the asymptotic expansion of coefficients at order \(y^4\), neither looking forwards, to order \(y^5\), nor backwards, to order \(y^3\).

With these findings we may return to the convention for the coefficients in (8) and conclude that the number of monomials in (7) at order \(y^m\) is the integer nearest to \((m + 3)^2/12\), which is 7 in the case of \(m = 6\), with 7 formal power series in

\[
g_6 = \frac{\beta_3^3}{3} T_{3,0}^{(6)} x^{15} + \left( \sigma_1^2 \sigma_2^2 T_{2,0}^{(6)} + \sigma_1 \sigma_2 \sigma_3 T_{1,1}^{(6)} + \sigma_3^2 T_{0,2}^{(6)} \right) x^{10}
\]

\[
+ \left( \sigma_1^2 \sigma_2 T_{1,0}^{(6)} + \sigma_1^2 \sigma_3 T_{0,1}^{(6)} \right) x^5 + \sigma_1^6 T_{0,0}^{(6)}
\]

(104)

and \(T_{i,j}^{(6)} = \sum_{n \geq 0} a_{i,j}^{(6)}(n)x^n\) expanded to order \(x^5\) in Table 3.

We expanded (104) to order \(x^{150}\), finding the asymptotic expansions

\[
a_{1,1}^{(5)}(n) \sim -S_1 \sum_{k \geq 0} a_{1,1}^{(6)}(k) \Gamma(n + \frac{35}{12} - k)
\]
\[ + S_3 \sum_{k \geq 0} a_{0,1}^{(4)}(k) (-1)^{n-k} \Gamma(n + \frac{25}{12} - k) \]
\[ - 2S_4 \sum_{k \geq 0} a_{2,0}^{(4)}(k) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k), \]  
\[ (105) \]

\[ a_{2,0}^{(5)}(n) \sim - 2S_1 \sum_{k \geq 0} a_{2,0}^{(6)}(k) \Gamma(n + \frac{35}{12} - k) \]
\[ + 12S_1 \sum_{k \geq 0} a_{3,0}^{(6)}(k) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \]
\[ + \frac{1}{2} S_3 \sum_{k \geq 0} \left( 4a_{1,0}^{(4)}(k) + 3a_{0,1}^{(4)}(k) \right) (-1)^{n-k} \Gamma(n + \frac{25}{12} - k) \]
\[ + 8S_3 \sum_{k \geq 0} a_{2,0}^{(4)}(k) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + e_1 \right), \]  
\[ (106) \]

\[ a_{0,1}^{(5)}(n) \sim - 3S_1 \sum_{k \geq 0} a_{0,1}^{(6)}(k) \Gamma(n + \frac{35}{12} - k) \]
\[ + 4S_1 \sum_{k \geq 0} \left( a_{1,1}^{(6)}(k) + 3a_{0,2}^{(6)}(k) \right) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \]
\[ - 2S_3 \sum_{k \geq 0} a_{0,1}^{(4)}(k) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + 3f_1 - 2e_1 \right) \]
\[ - S_4 \sum_{k \geq 0} a_{1,0}^{(4)}(k) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k), \]  
\[ (107) \]

\[ a_{1,0}^{(5)}(n) \sim - 4S_1 \sum_{k \geq 0} a_{1,0}^{(6)}(k) \Gamma(n + \frac{35}{12} - k) \]
\[ + 2S_1 \sum_{k \geq 0} \left( 4a_{2,0}^{(6)}(k) + 3a_{1,1}^{(6)}(k) \right) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \]
\[ + 4S_3 \sum_{k \geq 0} a_{0,0}^{(4)}(k) (-1)^{n-k} \Gamma(n + \frac{25}{12} - k) \]
\[ - 16S_3 \sum_{k \geq 0} a_{2,0}^{(4)}(k) (-1)^{n-k} \Gamma(n - \frac{95}{12} - k) Q(n - \frac{95}{12} - k) \]
\[ - 3S_4 \sum_{k \geq 0} a_{1,0}^{(4)}(k) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k), \]  
\[ (108) \]

\[ a_{0,0}^{(5)}(n) \sim - 6S_1 \sum_{k \geq 0} a_{0,0}^{(6)}(k) \Gamma(n + \frac{35}{12} - k) \]
\[ + 2S_1 \sum_{k \geq 0} \left( 2a_{1,0}^{(6)}(k) + 3a_{0,1}^{(6)}(k) \right) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \]
\[ - 8S_3 \sum_{k \geq 0} a_{0,0}^{(4)}(k) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \]
\[ - 2S_3 \sum_{k \geq 0} \left( 4a_{1,0}^{(4)}(k) + 3a_{0,1}^{(4)}(k) \right) (-1)^{n-k} \Gamma(n - \frac{95}{12} - k) Q(n - \frac{95}{12} - k). \]  
\[ (109) \]
At this stage, we sought to consolidate our findings, arriving at the tentative Ansatz (19), in which the factors \((s + 1), (i + 1)\) or \((j + 1)\) appear when the values of \(s = m - 2i - 3j\), \(i\) or \(j\) on the left-hand side are increased by unity in the coefficients on the right-hand side. There remained some doubt about the factors multiplying \(a^{(m-1)}_{i,j}(k)\), which appears in both the fourth and sixth lines of (19). To strengthen the case for Conjecture 1, we checked it at \(m = 6, 7, 8\), as follows.

To control resurgence from order \(y^6\) to order \(y^7\), we extended our coefficient database to include Table 4 and obtained all 8 terms in \(g_7(x)\) up to order \(x^{100}\). Then we checked Conjecture 1 in the 7 cases with \(m = 6\), at precisions ranging between 38 and 42 decimal digits.

To control resurgence from order \(y^7\) to order \(y^8\), we extended our coefficient database to include Table 5 and obtained all 10 terms in \(g_8(x)\) up to order \(x^{55}\). Then we checked Conjecture 1 in the 8 cases with \(m = 7\), at precisions ranging between 22 and 27 decimal digits.

Finally, we extended all 12 terms in \(g_9(x)\) up to order \(x^{100}\) and checked Conjecture 1 in the 10 cases with \(m = 8\), at precisions ranging between 7 and 17 decimal digits.

After this labour, we are well persuaded by the data that the conjecture holds.

Before we conclude, we will provide some details on our computational methodology for the explicit computation of the coefficients \(a^{(m)}_{i,j}(n)\) up to sufficiently large values of \(n\).

### 4.3. Efficient calculation of expansion coefficients

To investigate asymptotic behaviour of the coefficients \(a^{(m)}_{i,j}(n)\) in (8), we developed at least 4000 terms of each of the 11 rational sequences involved up to order \(m = 4\), 200 terms of 5
sequences at order $m = 5$, 150 terms of 7 sequences at order $m = 6$, 100 terms of 8 sequences at order $m = 7$, 55 terms of 10 sequences at order $m = 8$ and 30 terms of 12 sequences at order $m = 9$.

To do this, we had to solve linearized inhomogeneous third-order differential equations, with coefficients that are cubic in $g_0$ and its derivatives. The inhomogeneous terms are quartics in lower terms of the trans-series and their derivatives. At order $m = 8$, they depend quartically on $\log(x)$. This complexity is compounded by the differing actions of the operator $D = x \frac{d}{dx}$ on $x$, $y$ and $\log(x)$.

In this work, we were greatly aided by the $\text{diffop}$ operator of Pari-GP [105], which enabled us to prepare problems symbolically, before performing series expansions and solving recursions. When one has a pair of functions in play, the GP process

$$\begin{align*}
\text{fvec} &= \{x, y, \log(x), u_0, u_1, u_2, v_0, v_1, v_2\}; \\
\text{dvec} &= \{x, (1/x - 35/12) \times y, 1, u_1, u_2, u_3, v_1, v_2, v_3\}; \\
Dz &= \text{diffop}(z, \text{fvec}, \text{dvec});
\end{align*}$$

will perform the action of $D$, symbolically, on any rational function $z$ of $\{x, y, \log(x), u_0(x)\}$ and its derivatives, $v_0(x)$ and its derivatives, on the understanding that $u_{k+1}$ and $v_{k+1}$ represent the actions of $D$ on $u_k$ and $v_k$ and that $D$ is applied no more than three times to $u_0$ and $v_0$.

After obtaining these symbolic equations, we then took care to process products of power series in a way that minimizes the many multiplications of series resulting from quartic nonlinearity. To handle logarithmic dependence, we exploited simplifications that arose from having solved previous problems with lesser powers of logs, following the example of Frobenius. This preparation was particularly intricate in the case of the $\sigma_4^4$ term in (7)–(9). Thereafter, we were able to develop 4000 exact terms of the sequence $a^{(4)}_{0,0}(n)$ in 6 hours on a laptop.

5. Conclusion

We performed an exhaustive resurgence analysis of the nonlinear ODE (2) which describes the renormalon contribution (1) to the $\phi^3$ theory field anomalous dimension in six dimensions. The coefficients $A_n = a^{(0)}_{0,0}(n)$ of the perturbative solution of this differential equation have a large-order behaviour which is encoded by the first-instanton contribution to the trans-series solution (7)–(9) with coefficients $a^{(1)}_{0,0}(n)$. The large-order behaviour of these coefficients $a^{(1)}_{0,0}(n)$ is encoded by a combination of the sequences $a^{(2)}_{0,0}(n)$ and $a^{(2)}_{1,0}(n)$, the next terms in the trans-series (8). This low- and large-order relationship persists for all higher-order terms in the trans-series solution to (2). Our analysis was greatly facilitated by the discovery of the compact form of the trans-series (7)–(9), which tamed the dependence on the logarithmic terms, and by ambiguity fixing choices, which the asymptotic expansions and were determined by our asymptotic analysis. The compact form of the trans-series solution (7)–(9) for (2) was the key to go beyond the previous work on this renormalon [56] and constitutes our first main result.

The role of the ambiguities that appear due to resonances of the various expansions turned out to be quite intricate. By comparing multiple large-order expansions, we were able to empirically find the particularly favourable ambiguity resolving constant $r_1 = \frac{32642693907939}{36691771392}$. This numerically determined value reduced the size of our expressions and the magnitude of the presumably transcendental constants significantly. We remark that the existence of the compact representation (7)–(9), which hints for a cancellation mechanism phenomenon that ensures reality after
resummation, and the role of the rather peculiar constant \(r_1\) deserve an explanation that lies beyond the scope of this article.

Our second main result is Conjecture 1: the well-tested formula (19) for the large-\(n\) asymptotics of any sequence \(a_{i,j}^{(m)}(n)\) that appears in the trans-series solution of (2). This large-\(n\) behaviour is entirely determined by a linear combination of nearby sequences \(a_{i',j'}^{(m')}\) which also contribute to the trans-series expansion. This linear combination involves six Stokes constants which appear to be transcendental and which we determined numerically. We therefore solved the connection problem of matching the low-order with the large-order expansion for the ODE (2).

To illustrate our process and to showcase the special role of the logarithmic terms in the initial problem, we also analyzed the related but simpler ODE (27). This simpler ODE’s trans-series solution (30)–(31) lacks the intricate logarithmic terms that feature in the original ODE’s trans-series solution (7)–(9), but inherits the property of a resonance in the linearized non-perturbative solution space. The observation that the simpler ODE (27) does not feature logarithmic terms in spite of such resonances proves that the origin of the logarithmic contributions is quite peculiar and that such resonances do not necessarily imply the existence of logarithmic terms in the trans-series solution. It would be highly beneficial to refine the notion of resonant instanton structure and characterize ODEs in whose trans-series solution logarithmic terms appear.

With Conjecture 2 we gave the simpler counterpart of Conjecture 1: a well-tested formula for the large-\(n\) behaviour of the coefficients of the trans-series solution of the simpler ODE (27).

In Section 3.6 we illuminated the case for Conjecture 2 using alien calculus. It might be possible to prove Conjectures 1–2 and to explain the compact trans-series solutions (7)–(9) and (30)–(31) by a more sophisticated application of similar alien calculus reasoning. We leave this for a future work.

The analysis of a similar renormalon in Yukawa theory [83] was greatly facilitated by the fact that the associated perturbative solution has a sound combinatorial interpretation as the generating function of connected chord diagrams. This interpretation led to an explicit all-order trans-series solution of the associated ODE in terms of a certain generating function. The combinatorial interpretation and its relation to the asymptotic expansion was studied further in [106–109]. Unfortunately, it seems unlikely that a similar combinatorial interpretation exists for the solution of (2). Assuming the validity of Conjecture 1, we were able to obtain almost the same level of understanding of the \(\phi^3\) theory renormalon (1) as [83] achieved for the simpler Yukawa renormalon. We gained complete control over the asymptotic connections between the different trans-series coefficients, even though we do not have an explicit formula for six Stokes constants, which had to be determined numerically.

For both ODEs (2) and (27) we found trans-asymptotic results that involve coefficients of the Lambert-W function. These might be of help in evaluating the function associated to the initial perturbative factorially divergent power series explicitly via numerical methods. See for instance [110] for an explicit application of this method. The Lambert-W function previously appeared in the trans-asymptotic resummation of the similar Yukawa theory renormalon [83] and in many other problems that are associated to renormalization and Dyson–Schwinger equations [8,100–103]. Its ubiquitous appearance in Dyson–Schwinger type contexts deserves further explanation which lies beyond the scope of this article.

Our analysis is limited inherently by the set of diagrams that are captured by the Dyson–Schwinger equation (1). For instance, it would be desirable to repeat it for a renormalon which also includes vertex-corrections. A promising route out of this limitation is a combinatorial framework of Dyson–Schwinger equations [81,82,100,111–116] which allows for larger sets
of diagrams to be included and which recently has been shown to allow for the extraction of asymptotic large-order information [117].

An illuminating comparison of instanton and renormalon phenomena based on available data in $\phi^4$ theory was recently put forward in [118]. As there exists data on the large-order behaviour of $\phi^3$ theory in the minimal subtraction scheme [44,119], a similar study that compares renormalon effects, as studied in this article, and instanton effects in $\phi^3$ theory would be another feasible and interesting future endeavour.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Michael Borinsky reports financial support was provided by Dr. Max Rössler. Michael Borinsky reports financial support was provided by Walter Haefner Foundation. Michael Borinsky reports financial support was provided by ETH Zürich Foundation.

Acknowledgements

We thank Gerald Dunne, Dirk Kreimer and Max Meynig, for joint work that set up this problem, Gerald Dunne for valuable comments on an early version of this article and the Isaac Newton Institute in Cambridge, for remotely hosting the programme Applicable Resurgent Asymptotics (ARA) that encouraged us to undertake this work. Moreover, we thank the anonymous referee for valuable comments that caused significant improvements of the manuscript. MB was supported by the NWO Vidi grant 680-47-551 “Decoding Singularities of Feynman graphs”, Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zürich Foundation.

References

[1] F.J. Dyson, Divergence of perturbation theory in quantum electrodynamics, Phys. Rev. 85 (1952) 631.
[2] C.S. Lam, Behavior of very high order perturbation diagrams, Nuovo Cimento A 55 (1968) 258.
[3] C.M. Bender, T.T. Wu, Anharmonic oscillator. 2: A study of perturbation theory in large order, Phys. Rev. D 7 (1973) 1620.
[4] L.N. Lipatov, Divergence of the perturbation theory series and the quasiclassical theory, Sov. Phys. JETP 45 (1977) 216.
[5] J. Zinn-Justin, U.D. Jentschura, Multi-instantons and exact results I: conjectures, WKB expansions, and instanton interactions, Ann. Phys. 313 (2004) 197, arXiv:quant-ph/0501136.
[6] J. Zinn-Justin, U.D. Jentschura, Multi-instantons and exact results II: specific cases, higher-order effects, and numerical calculations, Ann. Phys. 313 (2004) 269, arXiv:quant-ph/0501137.
[7] M. Borinsky, Renormalized asymptotic enumeration of Feynman diagrams, Ann. Phys. 385 (2017) 95, arXiv: 1703.00840.
[8] M. Borinsky, K. Vogtmann, The Euler characteristic of Out($F_n$), Comment. Math. Helv. 95 (2020) 703, arXiv: 1907.03543.
[9] J. Zinn-Justin, Perturbation series at large orders in quantum mechanics and field theories: application to the problem of resummation, Phys. Rep. 70 (1981) 109.
[10] J.-C. Le Guillou, J. Zinn-Justin, Large-Order Behaviour of Perturbation Theory, Elsevier, 2012.
[11] M. Mariño, Instantons and Large N: an Introduction to Non-perturbative Methods in Quantum Field Theory, Cambridge University Press, 2015.
[12] G. ‘t Hooft, Can we make sense out of quantum chromodynamics?, Subnucl. Ser. 15 (1979) 943.
[13] B.E. Lautrup, On high order estimates in QED, Phys. Lett. B 69 (1977) 109.
[14] G. Parisi, Singularities of the Borel transform in renormalizable theories, Phys. Lett. B 76 (1978) 65.
[15] M. Beneke, Renormalons, Phys. Rep. 317 (1999) 1, arXiv:hep-ph/9807443.
[16] M. Shifman, New and old about renormalons: in memoriam Kolya Uraltsev, Int. J. Mod. Phys. A 30 (2015) 1543001, arXiv:1310.1966.

[17] A. Palameres-Mestre, P. Pascual, The $1/N_f$ expansion of the $\gamma$ and $\beta$ functions in QED, Commun. Math. Phys. 95 (1984) 277.

[18] D.J. Broadhurst, Large $N$ expansion of QED: asymptotic photon propagator and contributions to the muon anomaly, for any number of loops, Z. Phys. C 58 (1993) 339.

[19] M. Beneke, V.I. Zakharov, Improving large order perturbative expansions in quantum chromodynamics, Phys. Rev. Lett. 69 (1992) 2472.

[20] M. Beneke, V.I. Zakharov, The first infrared renormalon in QED, Phys. Lett. B 312 (1993) 340.

[21] J.A. Gracey, The QCD beta function at $O(1/N_f)$, Phys. Lett. B 373 (1996) 178, arXiv:hep-ph/9602214.

[22] N.A. Dondi, G.V. Dunne, M. Reichert, F. Sannino, Towards the QED beta function and renormalons at $1/N^2_f$ and $1/N^3_f$, Phys. Rev. D 102 (2020) 035005, arXiv:2003.08397.

[23] N. Dondi, I. Kalogerakis, D. Orlando, S. Reffert, Resurgence of the large-charge expansion, J. High Energy Phys. 05 (2021) 035, arXiv:2102.12488.

[24] L. Di Pietro, M. Mariño, G. Sberveglieri, M. Serone, Resurgence and $1/N$ expansion in integrable field theories, J. High Energy Phys. 10 (2021) 166, arXiv:2108.02647.

[25] T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, T. Yoda, Quantum phase transition and resurgence: lessons from three-dimensional $\mathcal{N} = 4$ supersymmetric quantum electrodynamics, PTEP 2021 (2021) 103B04, arXiv:2103.13654.

[26] M. Mariño, T. Reis, Renormalons in integrable field theories, J. High Energy Phys. 04 (2020) 160, arXiv:1909.12134.

[27] M. Mariño, T. Reis, A new renormalon in two dimensions, J. High Energy Phys. 07 (2020) 216, arXiv:1912.06228.

[28] M. Mariño, R. Miravitllas, T. Reis, New renormalons from analytic trans-series, arXiv:2111.11951.

[29] A. Maiezza, J.C. Vasquez, Renormalons in a general quantum field theory, Ann. Phys. 394 (2018) 84, arXiv:1802.06022.

[30] O. Antipin, A. Maiezza, J.C. Vasquez, Resummation in QFT with Meijer G-functions, Nucl. Phys. B 941 (2019) 72, arXiv:1807.05060.

[31] A. Maiezza, J.C. Vasquez, Non-Wilsonian ultraviolet completion via transseries, Int. J. Mod. Phys. A 36 (2021) 2150016, arXiv:2007.01270.

[32] A. Maiezza, J.C. Vasquez, Resurgence of the QCD Adler function, Phys. Lett. B 817 (2021) 136338, arXiv:2104.03095.

[33] I. Caprini, Conformal mapping of the Borel plane: going beyond perturbative QCD, Phys. Rev. D 102 (2020) 054017, arXiv:2006.16605.

[34] C. Pazaraş, D. Van Den Bleeken, Renormalons in quantum mechanics, J. High Energy Phys. 08 (2019) 096, arXiv:1906.07198.

[35] E. Cavaletti, Renormalons beyond the Borel plane, Phys. Rev. D 103 (2021) 025019, arXiv:2011.11175.

[36] P.-H. Balduf, Dyson–Schwinger equations in minimal subtraction, arXiv:2109.13684.

[37] A.J. Macfarlane, G. Woo, $\phi^3$ theory in six dimensions and the renormalization group, Nucl. Phys. B 77 (1974) 91.

[38] J.A. Gracey, Four loop renormalization of $\phi^3$ theory in six dimensions, Phys. Rev. D 92 (2015) 025012, arXiv:1506.03357.

[39] M. Borinsky, J.A. Gracey, M.V. Kompaniets, O. Schnetz, Five-loop renormalization of $\phi^3$ theory with applications to the Lee–Yang edge singularity and percolation theory, Phys. Rev. D 103 (2021) 116024, arXiv:2103.16224.

[40] M. Borinsky, O. Schnetz, Graphical functions in even dimensions, arXiv:2105.05015.

[41] M. Kompaniets, A. Pikelnier, Critical exponents from five-loop scalar theory renormalization near six-dimensions, Phys. Lett. B 817 (2021) 136331, arXiv:2101.10018.

[42] M. Borinsky, O. Schnetz, Recursive computation of Feynman periods, 2022, in preparation.

[43] E. Brezin, J.C. Le Guillou, J. Zinn-Justin, Perturbation theory at large order. 1. The $\phi^{2N}$ interaction, Phys. Rev. D 15 (1977) 1544.

[44] A.J. McKane, Vacuum instability in scalar field theories, Nucl. Phys. B 152 (1979) 166.

[45] A. Houghton, J.S. Reeve, D.J. Wallace, High order behavior in $\phi^3$ field theories and the percolation problem, Phys. Rev. B 17 (1978) 2956.

[46] G. Álvarez, Coupling-constant behavior of the resonances of the cubic anharmonic oscillator, Phys. Rev. A 37 (1988) 4079.

[47] G. Álvarez, Bender–Wu branch points in the cubic oscillator, J. Phys. A, Math. Gen. 28 (1995) 4589.

[48] Z. Bern, J.J.M. Carrasco, H. Johansson, New relations for gauge-theory amplitudes, Phys. Rev. D 78 (2008) 085011, arXiv:0805.3993.
[80] P.J. Clavier, Borel-Écalle resummation of a two-point function, Ann. Henri Poincaré 22 (2021) 2103, arXiv:1912.03237.
[81] M.P. Bellon, E.I. Russo, Resurgent analysis of Ward–Schwinger–Dyson equations, SIGMA 17 (2021) 075, arXiv:2011.13822.
[82] M.P. Bellon, E.I. Russo, Ward–Schwinger–Dyson equations in $\phi^3_6$ quantum field theory, Lett. Math. Phys. 111 (2021) 42, arXiv:2007.15675.
[83] M. Borinsky, G.V. Dunne, Non-perturbative completion of Hopf-algebraic Dyson–Schwinger equations, Nucl. Phys. B 957 (2020) 115096, arXiv:2005.04265.
[84] D. Broadhurst, D. Kreimer, Combinatorial explosion of renormalization tamed by Hopf algebra: thirty loop Padé–Borel resummation, Phys. Lett. B 475 (2000) 63, arXiv:hep-th/9912093.
[85] S. Gourevitch, A. Its, A. Kapaev, M. Mariño, Asymptotics of the instantons of Painlevé I, Int. Math. Res. Not. 2012 (2012) 561, arXiv:1002.3634.
[86] I. Aniceto, R. Schiappa, M. Vonk, The resurgence of instantons in string theory, Commun. Number Theory Phys. 6 (2012) 339, arXiv:1106.5922.
[87] M. Mariño, Open string amplitudes and large order behavior in topological string theory, J. High Energy Phys. 03 (2008) 060, arXiv:hep-th/0612127.
[88] J.c.v. Cizek, R.J. Damborg, S. Graffi, V. Grecchi, E.M. Harrell, J.G. Harris, et al., $1/R$ expansion for $H^+_2$: calculation of exponentially small terms and asymptotics, Phys. Rev. A 33 (1986) 12.
[89] J. Zinn-Justin, Expansion around instantons in quantum mechanics, J. Math. Phys. 22 (1981) 511.
[90] C. Paziarbaş, M. Ünsal, Cluster expansion and resurgence in Polyakov model, arXiv:2110.05612.
[91] G.V. Dunne, M. Ünsal, WKB and resurgence in the Mathieu equation, in: Resurgence, Physics and Numbers, Springer, 2017, pp. 249–298, arXiv:1603.04924.
[92] R.B. Dingle, Asymptotic Expansions: Their Derivation and Interpretation, vol. 521, Academic Press, London, 1973.
[93] M.V. Berry, C.J. Howls, Hyperasymptotics for integrals with saddles, Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. 434 (1991) 657.
[94] O. Costin, Asymptotics and Borel Summability, CRC Press, 2008.
[95] I. Aniceto, R. Schiappa, Nonperturbative ambiguities and the reality of resurgent transseries, Commun. Math. Phys. 335 (2015) 183, arXiv:1308.1115.
[96] G. Dunne, Resurgent trans-series analysis of Hopf algebraic renormalization, Talk at IHES conference Algebraic Structures in Perturbative Quantum Field Theories Video: https://www.youtube.com/watch?v=29ENkKqUdVl#t=37m00s, 19 Nov. 2020.
[97] E. Caliceti, M. Meyer-Hermann, P. Ribeca, A. Surzhykov, U.D. Jentschura, From useful algorithms for slowly convergent series to physical predictions based on divergent perturbative expansions, Phys. Rep. 446 (2007) 1, arXiv:0707.1596.
[98] O. Costin, G.V. Dunne, Uniformization and constructive analytic continuation of Taylor series, arXiv:2009.01962.
[99] O. Costin, G.V. Dunne, Physical resurgent extrapolation, Phys. Lett. B 808 (2020) 135627, arXiv:2003.07451.
[100] G. van Baalen, D. Kreimer, D. Uminsly, K. Yeats, The QED beta-function from global solutions to Dyson–Schwinger equations, Ann. Phys. 324 (2009) 205, arXiv:0805.0826.
[101] L. Klaczynski, D. Kreimer, Avoidance of a Landau pole by flat contributions in QED, Ann. Phys. 344 (2014) 213, arXiv:1309.5061.
[102] E. Panzer, R. Wulkenhaar, Lambert-W solves the noncommutative $\varphi^4$-model, Commun. Math. Phys. 374 (2019) 1935, arXiv:1807.02945.
[103] G. Sberveglieri, M. Serone, G. Spada, Self-dualities and renormalization dependence of the phase diagram in 3d $O(N)$ vector models, J. High Energy Phys. 02 (2021) 098, arXiv:2010.09737.
[104] M. Borinsky, Graphs in Perturbation Theory: Algebraic Structure and Asymptotics, Springer, 2018.
[105] The PARI Group, Univ. Bordeaux, PARI/GP version 2.13.1, http://pari.math.u-bordeaux.fr/, 2021.
[106] A.A. Mahmoud, K. Yeats, Connected chord diagrams and the combinatorics of asymptotic expansions, arXiv:2010.06550.
[107] A.A. Mahmoud, An asymptotic expansion for the number of 2-connected chord diagrams, arXiv:2009.12688.
[108] A.A. Mahmoud, On Enumerative Structures in Quantum Field Theory, Ph.D. thesis, U. Waterloo, 2020, arXiv:2008.11661.
[109] A.A. Mahmoud, New prospects of enumeration in quantum electrodynamics, arXiv:2011.04291.
[110] I. Aniceto, D. Hasenbichler, C.J. Howls, C.J. Lustri, Capturing the cascade: a transseries approach to delayed bifurcations, Nonlinearity 34 (2021) 8248, arXiv:2012.09779.
[111] D. Kreimer, K. Yeats, An étude in non-linear Dyson–Schwinger equations, Nucl. Phys. B, Proc. Suppl. 160 (2006) 116, arXiv:hep-th/0605096.
[112] L. Foissy, Faà di Bruno subalgebras of the Hopf algebra of planar trees from combinatorial Dyson–Schwinger equations, Adv. Math. 218 (2008) 136, arXiv:0707.1204.
[113] G. van Baalen, D. Kreimer, D. Uminsky, K. Yeats, The QCD beta-function from global solutions to Dyson–Schwinger equations, Ann. Phys. 325 (2010) 300, arXiv:0906.1754.
[114] N. Marie, K. Yeats, A chord diagram expansion coming from some Dyson–Schwinger equations, Commun. Number Theory Phys. 07 (2013) 251, arXiv:1210.5457.
[115] O. Krüger, D. Kreimer, Filtrations in Dyson–Schwinger equations: next-to\textsuperscript{\textminus}leading log expansions systematically, Ann. Phys. 360 (2015) 293, arXiv:1412.1657.
[116] O. Krüger, Log expansions from combinatorial Dyson–Schwinger equations, Lett. Math. Phys. 110 (2020) 2175, arXiv:1906.06131.
[117] J. Courtiel, K. Yeats, Next-to\textsuperscript{\textminus}leading log expansions by chord diagrams, Commun. Math. Phys. 377 (2020) 469, arXiv:1906.05139.
[118] G.V. Dunne, M. Meynig, Instantons or renormalons? Remarks on $\phi^4_{\text{MS}}$ theory in the MS scheme, Phys. Rev. D 105 (2022) 025019, arXiv:2111.15554.
[119] A.J. McKane, Perturbation expansions at large order: results for scalar field theories revisited, J. Phys. A 52 (2019) 055401, arXiv:1807.00656.