Asymptotic minimization of expected time to reach a large wealth level in an asset market game

Mikhail Zhitlukhin

9 July 2020

Abstract

We consider a stochastic game-theoretic model of a discrete-time asset market with short-lived assets and endogenous asset prices. We prove that the strategy which invests in the assets proportionally to their expected relative payoffs asymptotically minimizes the expected time needed to reach a large wealth level. The result is obtained under the assumption that the relative asset payoffs and the growth rate of the total payoff during each time period are independent and identically distributed.

Keywords: asset market game, crossing time, survival strategy, martingales.

MSC 2010: 91A25, 91B55. JEL Classification: C73, G11.

1. Introduction

One of classical problems in mathematical finance consists in finding an investment strategy which reaches a given wealth level as quick as possible. It is generally known that in market models with exogenously specified asset prices log-optimal strategies asymptotically minimize the expected time of reaching a large wealth level, at least when asset returns are specified by i.i.d. random variables; see e.g. the seminal paper [7] for a result in discrete time, or a more recent work [16] for a continuous-time model with Lévy processes. In the present paper we obtain an analogous result for a game-theoretic model of a market with endogenous prices – an asset market game.

We consider a discrete-time model in which assets yield random payoffs that are divided between agents (investors) proportionally to the number of shares of each asset held by an investor. Asset prices are determined endogenously by an equilibrium of supply and demand and depend on investors’ strategies. As a result, the evolution of investors’ wealth depends not only on their own strategies and realized asset payoffs but also on strategies of the other investors in the market. Our goal is to identify an investment strategy that allows an investor to reach a large wealth level asymptotically not slower, on average, than any other investor in the market. We show that there exists a strategy with this property, and, moreover, it does not depend on the strategies used by the other investors.

This research was motivated by results in evolutionary finance – the field which studies financial markets from a point of view of evolutionary dynamics and investigates

Steklov Mathematical Institute of the Russian Academy of Sciences. 8 Gubkina St., Moscow, Russia. Email: mikhailzh@mi-ras.ru. The research was supported by the Russian Science Foundation, project no. 18-71-10097.
properties of investment strategies like survival, extinction, dominance, and how they affect the distribution of wealth; recent reviews of this direction can be found in [10, 14]. We will work within the evolutionary model of a market with short-lived assets proposed by Amir et al. [3] (among earlier models of a similar structure one can mention, e.g., [6, 11, 13]). Short-lived assets can be purchased by investors at time $t$, yield payoffs at $t + 1$, and then the cycle repeats. They have no liquidation value, so investors can get profit or loss only by receiving asset payoffs and paying for buying new assets. Certainly, such a model is a simplification of a real stock market, however models with short-lived assets have been widely studied in the literature because they are more amenable to mathematical analysis and ideas developed for them may be transferred to more realistic models.

The main results of the present paper are related to the strategy $\lambda^*$ of [3], which splits an investment budget between assets proportionally to their expected payoffs. In that paper, it was shown that $\lambda^*$ is a survival strategy in the sense that it allows investors using it to keep their relative wealth (the share in the total market wealth) bounded away from zero on the whole infinite time interval with probability 1. As observed in [3], in view of such a structure, this strategy is analogous to the Kelly rule of “betting one’s beliefs” in markets with exogenous asset prices (see [17]; a collection of papers on the Kelly rule can be found in [20]). Moreover, the key step to show that $\lambda^*$ is a survival strategy was to prove that it makes the logarithm of the relative wealth of an investor who uses it a submartingale, which is analogous to the log-optimality property in markets with exogenous prices (see, e.g., [1], or later literature where such strategies are often called growth-optimal, benchmark, or numéraire portfolios, [15, 18, 21]).

Our first main result shows that the expected time needed for an investor using $\lambda^*$ to reach a wealth level $l$ is asymptotically, as $l \to \infty$, not greater than the same time for any other investor in the market, i.e. if we denote these times by $\tau^*_l$ and $\tau_l$, then $\xi := \limsup_{l \to \infty} \frac{E\tau^*_l}{E\tau_l} \leq 1$. Compared to [3], where no conditions on the distribution of asset payoffs are imposed, we require that the payoffs are generated by sequences of i.i.d. random variables in a certain way, which is a usual assumption in various settings of time minimization problems in asset market models (cf., e.g., [7, 16]), as well as in earlier works in evolutionary finance.

The second main result states that, under some additional conditions, the strict inequality $\xi < 1$ takes place if the strategy of the other investor is essentially different from $\lambda^*$ in some sense, and we find an upper bound for $\xi$ which is strictly less than 1. It is interesting to note that, among its assumptions, this results requires the payoffs to be strictly random, and we provide a counterexample with non-random asset payoffs where $\xi = 1$. In other words, volatility, which we associate here with randomness of payoffs, helps $\lambda^*$ to beat other strategies (see further discussion in Section 3).

The paper is organized as follows. Section 2 describes the model, Section 3 states the main results, and Section 4 contains their proofs.

2. The model

The model we use is essentially equivalent to that of [3], but it will be more convenient to formulate it using the notation which is more common in stochastic analysis.

For ease of exposition, let us first briefly describe the structure of the model in plain language. The market consists of $M \geq 2$ agents (investors) and $N \geq 2$ assets. At each moment of time $t = 1, 2, \ldots$, the assets yield random payoffs, which are divided between the investors proportionally to the number of shares of each asset purchased.
by an investor at time $t - 1$. The supply of each asset is exogenously (and,
without loss of generality, is normalized to 1), while the demand depends on actions
of the investors, i.e. their investment strategies. An investment strategy consists of
an investor’s decisions, made at every moment of time simultaneously with the other
investors and independently of them, on what proportion of wealth to spend on buying
each asset. Asset prices are determined by means of the market clearing mechanism,
i.e. they are set in such a way that the demand becomes equal to the supply. Then,
at the next moment of time, the assets purchased by the investors yield payoffs and
the cycle repeats. The important simplifying modeling assumption consists in that the
assets bought at time $t - 1$ cannot be sold at $t$, i.e. they disappear after yielding payoffs
without any liquidation value and are replaced by their “copies”. Hence, we can say
that they live for just one period and call them short-lived.

To define the model formally, introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration
$\mathbb{F} = (\mathcal{F}_t)_{t=0}^{\infty}$. Payoffs of asset $n = 1, \ldots, N$ are specified by a sequence of random
variables $X^n_t \geq 0$, $t \geq 1$, which is $\mathbb{F}$-adapted (i.e. $X^n_t$ are $\mathcal{F}_t$-measurable). It is assumed
that $X^n_t$ are given exogenously, i.e. do not depend on actions of the investors, and that
$\sum_n X^n_t > 0$ for all $t \geq 1$ (hereinafter all equalities and inequalities for random variables
are assumed to hold with probability 1).

The wealth of investor $m = 1, \ldots, M$ is specified by an adapted random sequence
$Y^m_t \geq 0$. The initial wealth $Y^m_0$ of each investor is non-random and strictly positive.
Further evolution of wealth depends on the investors’ actions and the asset payoffs.
Actions of investor $m$ are represented by a sequence of vectors of investment proportions
$\lambda^m_t = (\lambda^m_n, 1, \ldots, \lambda^m_n, N, t \geq 1$, according to which this investor allocates the available
budget $Y^m_{t-1}$ for buying assets at time $t - 1$. Short sales are not possible and the whole
wealth is reinvested, so the vectors $\lambda^m_t$ belong to the standard $N$-simplex $\Delta = \{\lambda \in \mathbb{R}_+^N : \sum_n \lambda^n = 1\}$.

To allow dependence on a random outcome and the history of the market, we define
a strategy of an investor as a sequence of functions
$$
\Lambda_t(\omega, y_0, \lambda_1, \ldots, \lambda_{t-1}) : \Omega \times \mathbb{R}_+^M \times \Delta^{M(t-1)} \to \Delta, \quad t \geq 1,
$$
which are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^M \times \Delta^{M(t-1)})$–measurable ($\mathcal{B}$ stands for the Borel $\sigma$-algebra). The argument $y_0 \in \mathbb{R}_+^M$ corresponds to the vector of initial wealth $Y^m_0 = (Y^1_0, \ldots, Y^M_0)$. The arguments $\lambda_s = (\lambda^m_s, n, m = 1, \ldots, M, n = 1, \ldots, N, s = 1, \ldots, t - 1$, are investment proportions selected by the investors at the past moments of time (for $t = 1$, the function
$\Lambda_t(\omega, y_0)$ does not depend on $\lambda_s$). If this strategy is used by investor $m$, then the
value of the function $\Lambda_t$ corresponds to the vector of investment proportions $\lambda^m_t$. The
measurability of $\Lambda_t$ in $\omega$ with respect to $\mathcal{F}_t$ means that future payoffs are not known
to the investors at a moment when they decide upon their actions.

After the investors have chosen their investment proportions at time $t - 1$, the equi-
librium asset prices $p^n_{t-1}$ are determined from the market clearing condition that the
aggregate demand of each asset is equal to the aggregate supply, which is normalized
to 1. Since investor $m$ can buy $x^{m,n}_t = \lambda^{m,n}_t Y^m_{t-1}/p^n_{t-1}$ units of asset $n$, we must have
$$
p^n_{t-1} = \sum_{m=1}^M \lambda^{m,n}_t Y^m_{t-1}.
$$
If $\sum_m \lambda^{m,n}_t = 0$, i.e. no one invests in asset $n$, we put $x^{m,n}_t = 0$ for all $m$; in this case the
price $p^n_{t-1} = 0$ can be defined in an arbitrary way with no effect on the investors’ wealth,
so we will put \( p_{t-1}^m = 0 \) for convenience. At the next moment of time \( t \), the total payoff received by investor \( m \) from the assets in the portfolio will be equal to \( \sum_n x_{t,n}^m X_t^n \). Consequently, the wealth sequence \( Y_t^m \) is defined by the recursive relation

\[
Y_t^m(\omega) = \sum_{n=1}^{N} \frac{\lambda_{t,n}^m(\omega)Y_{t-1}^m(\omega)}{\sum_k \lambda_{k,n}^m(\omega)Y_{t-1}^k(\omega)} X_t^n(\omega),
\]

where \( \lambda_{t,n}^m(\omega) \) denotes the realization of investor \( m \)'s strategy in this market, which is defined recursively as the sequence

\[
\lambda_{t,n}^m(\omega) = \Lambda_{t,n}^m(\omega, Y_0^m, \lambda_1^m(\omega), \ldots, \lambda_{t-1}(\omega)).
\]

Equation (1) expresses the wealth dynamics of an individual investor in the market. Observe that \( Y_t^m \) implicitly depends on the strategies of the other investors. At the same time, if some investor \( m \) uses a fully diversified strategy, i.e. \( \lambda_{t,n}^m > 0 \) for all \( t, n \), then \( Y_t^m > 0 \) for all \( t \) and the total market wealth, which we will denote by \( W_t = \sum_m Y_t^m \), does not depend on the investors’ strategies and is equal to \( \sum_n X_t^n \).

**Remark 1** (On extensions of the model). The main features of the model which considerably simplify its mathematical analysis are that (a) the assets are short-lived, (b) the whole wealth is reinvested and there is no risk-free asset, (c) there are no short-sales, and (d) the time runs discretely. There is a number of papers where these assumptions are relaxed. Among them, one can mention, for example, [2], [5, 9], [4], [23] which address, respectively, the limitations (a), (b), (c), (d). However, to my knowledge, there is no general model which would combine all these extensions together.

### 3. Main results

For a number \( l > 0 \), let \( \tau_l^m \) denote the stopping time when the wealth of investor \( m \) reaches or exceeds the level \( l \) for the first time, i.e.

\[
\tau_l^m = \min\{t \geq 0 : Y_t^m \geq l\},
\]

where \( \min \emptyset = \infty \). We are interested in finding a strategy which makes \( \mathbb{E}\tau_l^m \) small compared to other strategies asymptotically as \( l \to \infty \).

Our first result, Theorem 1 below, provides such a strategy in an explicit form. We will prove it under the following assumption on the payoff sequences.

**Assumption (A).** The sequences \( X_t^n \) can be represented in the form

\[
X_t^n = \rho_t R_t^n \sum_{m=1}^{M} Y_t^m, \quad X_t^n = \rho_t R_t^n \sum_{i=1}^{N} X_{t-1}^i, \quad t \geq 2,
\]

where

(A.1) \( \rho_t > 0 \) is an adapted sequence of identically distributed random variables such that \( \mathbb{E}(\ln \rho_t)^2 < \infty \), \( \ln \rho_t > 0 \), and \( \rho_t \) are independent of \( \mathcal{F}_{t-1} \) for all \( t \);

(A.2) \( R_t = \{R_{t1}^1, \ldots, R_{tN}^N\} \) is an adapted sequence of random vectors with values in \( \Delta \) and there exists \( \varepsilon > 0 \) such that \( \mathbb{E}(R_t^n | \mathcal{F}_{t-1}) \geq \varepsilon \) for all \( n \) and \( t \).
The sequences \( \rho_t \) and \( R_t \) have a rather clear interpretation. Indeed, \( \rho_t \) expresses the growth rate of the total payoff (\( \sum_n X^a_t = \rho_t \sum_n X^a_{t-1} \)), and \( R^a_t = X^a_t / \sum_i X^i_t \) are the relative payoffs of the assets. Observe that if (3) holds, then \( W_t = \sum_n X^a_t = \rho_t W_{t-1} \), and (A.1) implies that \( \lim_{t \to \infty} W_t = \infty \) by the strong law of large numbers.

Introduce the following strategy \( \Lambda^* \), which depends only on \( t \) and \( \omega \), and has the components

\[
\Lambda^*_t,n \equiv \lambda^*_t,n = E(R^a_t \mid \mathcal{F}_{t-1}).
\]

Note that this is the same strategy as \( \lambda^* \) in [3].

**Theorem 1.** Let Assumption (A) hold and suppose investor 1 uses the strategy \( \Lambda^* \). Then \( E \tau^1_t < \infty \) for any \( l > 0 \), and for any other investor \( m \in \{2, \ldots, M\} \)

\[
\limsup_{t \to \infty} \frac{E \tau^1_t}{E \tau^m_t} \leq 1.
\]  

(4)

This theorem shows that no investor can reach a wealth level \( l \) faster asymptotically (as \( l \to \infty \)) than an investor who uses the strategy \( \Lambda^* \). The next theorem strengthens inequality (4) if the other investor uses an essentially different strategy. We will establish it for the case of two investors in the market and when the following additional assumption holds.

**Assumption (B).** The sequence of vectors \( R_t \) from Assumption (A) is such that

(B.1) \( R_t \) are identically distributed and independent of \( \mathcal{F}_{t-1} \) for all \( t \);

(B.2) \( R_t \) have linearly independent components, i.e. if \( \sum_n c^n R^a_t = 0 \) a.s. for \( c \in \mathbb{R}^N \), then \( c = 0 \).

Observe that if this assumption holds, then the strategy \( \Lambda^* \) is constant and \( \lambda^*_t,n = E(R^a_t) > 0 \) for all \( t \) and \( n \), where the inequality follows from (A.2). For \( a > 0 \), introduce the function

\[
f(a) = \sup \left\{ E \ln \sum_{n=1}^N \frac{\lambda^n R^a_t}{\lambda^*} \mid \lambda \in \Delta \text{ and } \| \lambda - \lambda^* \| \geq a \right\},
\]

where we put \( f(a) = -1 \) if the set under the supremum is empty. If this set is non-empty (i.e. \( a \) is small enough), then it is compact, so the supremum is attained since the above expectation is upper semicontinuous in \( \lambda \) as follows from the Fatou lemma. Moreover, by Jensen’s inequality, for any \( \lambda \in \Delta \), we have \( E \ln \sum_n \lambda^n R^a_t / \lambda^* < 0 \). The inequality is strict because the logarithm is strictly concave and \( \sum_n \lambda^n R^a_t / \lambda^* \) is non-constant as follows from (B.2). Consequently, \( f(a) < 0 \) for any \( a > 0 \).

**Theorem 2.** Let \( M = 2 \) and Assumptions (A), (B) hold. Suppose investor 1 uses the strategy \( \Lambda^* \) and investor 2 uses a strategy \( \tilde{\Lambda}_t \) such that its realization \( \tilde{\lambda}_t = \tilde{\lambda}_t(\omega) \) satisfies the inequality \( \| \tilde{\lambda}_t - \lambda^* \| \geq a \) a.s. for all \( t \geq 1 \) with some \( a > 0 \), and \( \tilde{\lambda}_t \) are uniformly bounded away from zero (i.e. \( \tilde{\lambda}_t > \tilde{\varepsilon} \) for all \( t,n \) and some \( \tilde{\varepsilon} > 0 \)).

Then, with \( \theta = E \ln \rho_1 > 0 \), we have

\[
\limsup_{t \to \infty} \frac{E \tau^1_t}{E \tau^2_t} \leq 1 - \frac{|f(a)| \wedge \theta}{\theta}.
\]

(5)

Note that the assumption about only two investors, \( M = 2 \), is not too restrictive. In the case \( M \geq 3 \), the above theorem can be used if one replaces investors \( m = 2, \ldots, M \).
with the representative investor and let \( \tilde{\lambda} \) be the realization of the strategy of this new investor (see (8)–(10) below). Since the time needed for an individual investor \( m \geq 2 \) to reach a given wealth level is not less than the same time for the representative investor, inequality (5) will remain valid if \( \tau_2^m \) is replaced by \( \tau_1^m \).

Inequality (4) generally cannot be improved if Assumption (B.2) does not hold. In Example 1 below, we demonstrate this for the case when \( R_t \) are non-random. This fact can be compared with the known phenomenon of volatility-induced growth in models with exogenous asset prices, which consists in that a constant proportions strategy can achieve a growth rate strictly greater than the growth rate of any asset, if the relative prices are non-constant. If the relative prices are constant this effect disappears, which may seem counter-intuitive since usually randomness (or volatility) is regarded as an impediment to financial growth. A popular intuitive explanation of this phenomenon consists in that a constant proportions strategy “buys low and sells high” (see, e.g., [12] or Chapter 15 in [19]), but such an explanation have known flaws [8].

Example 1. Suppose \( W_0 = 1 \) and the asset payoffs are non-random and given by

\[
X^n_t = R^n \rho t, \tag{6}
\]

where \( \rho > 1 \), \( R \in \Delta \) with \( R^n \geq \varepsilon \) for all \( n \) and some \( \varepsilon > 0 \). Clearly, this model satisfies Assumptions (A) and (B.1), and the strategy \( \Lambda^* \) is of the form \( \Lambda^*_t = R \) for all \( t \).

Proposition 1. Suppose in model (6) investor 1 uses the strategy \( \Lambda^* \) and investor 2 uses some constant strategy \( \tilde{\Lambda}_t = \tilde{\lambda} \). Then

\[
\lim_{l \to \infty} \frac{\tau_1^l}{\tau_2^l} = 1.
\]

4. Proofs

We begin with two simple lemmas, which will be needed in the proofs. Throughout this section, for a vector \( x \in \mathbb{R}^N \) we will denote its \( L^1 \) and \( L^2 \) norms by \( |x| = \sum_n |x^n| \) and \( \|x\| = \left( \sum_n (x^n)^2 \right)^{1/2} \).

Lemma 1. Suppose \( x, y \in \mathbb{R}^N \) have strictly positive coordinates and \( |x| = 1 \). Then

\[
\sum_{n=1}^N x^n (\ln x^n - \ln y^n) \geq \frac{1}{4} \left\| x - \frac{y}{|y|} \right\|^2 - \ln |y|. \tag{7}
\]

One can see that this lemma follows from a known inequality for the Kullback-Leibler divergence if \( x \) and \( y/|y| \) are considered as probability distributions on a set of \( N \) elements. A short direct proof of (7) can be found in [3] (see there Lemma 2, which is proved for \( |y| = 1 \), but easily implies our case as well).

Lemma 2. Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) be a filtered probability space. Suppose \( X_t, t \geq 1 \), is an adapted sequence of identically distributed random variables such that \( X_t \) is independent of \( \mathcal{F}_{t-1} \) and \( E X_t^2 < \infty \) for all \( t \geq 1 \). Denote \( \mu = E X_t, \sigma^2 = \text{Var} X_t \). Then for any stopping time \( \tau \geq 1 \)

\[
E X_\tau \leq \mu + 2\sigma \sqrt{E \tau}.
\]

6
Proof. Without loss of generality we may assume \( \mu = 0 \) and \( E \tau < \infty \). Introducing the martingale \( M_t = \sum_{s \leq t} X_s \), we obtain

\[
E X_\tau \leq E(M_\tau - \min_{s \leq \tau} M_s I(\tau > 1)) \leq \| \min_{s \leq \tau} M_s \|_{L^2} \leq 2 \left( \sum_{s \leq \tau} X_s^2 \right)^{1/2} = 2\sigma \sqrt{E \tau},
\]

where in the second inequality we applied Wald’s identity \( E M_\tau = 0 \), in the next one the Burkholder–Davis–Gundy inequality, and then Wald’s identity again (see, e.g., Chapter 7 of [22] for these results).

Proof of Theorem 1. Let us first show that the proof can be reduced to the case when \( M = 2 \) by replacing investors 2, \ldots, \( M \) with a representative investor. Let \( r^m_t = Y^m_t / W_t \) denote the relative wealth of the investors and define

\[
\tilde{Y}_t = \sum_{m=2}^{M} Y^m_t, \quad \tilde{\lambda}^n_t = \sum_{m=2}^{M} \frac{r^{m-1}_t}{1 - r^{m-1}_t} \lambda^{m,n}_t,
\]

where \( \lambda^{m,n}_t = \lambda^{m,n}_t(\omega) \) are the realizations of the strategies defined in (2), and we put \( \tilde{\lambda}^n_0 = 0 \) when \( r^{1-1}_t = 1 \). Since \( \lambda^{m,n}_t > 0 \) for all \( n \) by Assumption (A.2), we have \( W_t = \sum_n X^n_t > 0 \), so \( r^m_t \) are well-defined. Denote

\[
\tilde{\tau}_t = \min\{ t \geq 0 : \tilde{Y}_t \geq l \}
\]

and observe that \( \tilde{\tau}_t \leq \tau^m_t \) for any \( m \geq 2 \). Also, it is straightforward to check that the wealth sequence of investor 1 satisfies the relation

\[
Y^1_t = \sum_{n=1}^{N} \lambda^{n,1}_t Y^{1-1}_n + \lambda^n_t \tilde{Y}^{1-1}_n X^n_t,
\]

which is precisely relation (1) in the case of two investors who have the wealth \( Y^1_t, \tilde{Y}_t \) and use the strategies \( \Lambda^*_t, \tilde{\lambda}_t = \tilde{\lambda}_t(\omega) \), while \( \tilde{\tau}_t \) is the first moment when the wealth of the second investor reaches or exceeds \( l \). Consequently, to prove the theorem, it would be enough to show that

\[
\limsup_{t \to \infty} \frac{E \tau^1_t}{E \tilde{\tau}_t} \leq 1.
\]

So, from now on we will deal with the case \( M = 2 \). For brevity of notation, we will denote the realizations of strategies and the wealth of the first and the second investors, respectively, by \( \lambda_t = \lambda^*_t \), \( Y_t \) and \( \tilde{\lambda}_t, \tilde{Y}_t \); their relative wealth will be denoted by \( r_t \) and \( \tilde{r}_t = 1 - r_t \), and the moments of reaching or exceeding a wealth level \( l \) by \( \tau_t \) and \( \tilde{\tau}_t \).

From (10), we find

\[
\frac{r_t}{r^{1-1}_t} = \sum_{n=1}^{N} \frac{\lambda^n_t}{r^{n+1}_t} R^n_t.
\]

Denoting \( \beta^n_t = r^{n+1}_t \lambda^n_t + \tilde{r}^{n+1}_t \tilde{\lambda}^n_t \), we obtain the relation

\[
E(\ln r_t \mid \mathcal{F}_{t-1}) - \ln r^{1-1}_t \geq E \left( \sum_{n=1}^{N} R^n_t \ln \frac{\lambda^n_t}{\beta^n_t} \right) = \sum_{n=1}^{N} \lambda^n_t (\ln \lambda^n_t - \ln \beta^n_t) \geq 0,
\]

where in the first inequality we used the concavity of the logarithm, in the second one that \( \lambda^n_t = E(R^n_t \mid \mathcal{F}_{t-1}) \), and in the last inequality applied Lemma 1 to the vectors \( \lambda_t \).
and $\tilde{\beta}_t$. Inequality (12) implies that $\ln r_t$ is a submartingale (the integrability of $\ln r_t$ follows from that $r_t \geq r_{t-1} \min_n \lambda_t^n \geq \varepsilon r_{t-1}$, which can be seen from (11)). In passing, observe that since this submartingale is non-positive, with probability 1 there exists the finite limit $\lim_{t \to \infty} r_t$, so $\inf_{t \geq 0} r_t > 0$. This property allows to call $\lambda^*$ a survival strategy, i.e. an investor “survives” in the market by keeping a share of wealth bounded away from zero. This result was proved in [3] for general payoff sequences (note that in earlier papers, e.g. [6], the term “survival” has a somewhat different meaning).

If $E\tilde{\tau}_1 < \infty$, we find

$$\ln l \leq E\ln Y_{\tilde{\tau}_1} = E(\ln \tilde{\tau}_1 + \ln W_{\tilde{\tau}_1}) \leq E \ln W_{\tilde{\tau}_1} = \theta E\tilde{\tau}_1 + \ln W_0,$$  

where in the last equality we applied Wald’s identity to the sequence of i.i.d. random variables $\ln(W_t/W_{t-1}) = \ln \rho_t$. Here, as in Section 3, $\theta = E\ln \rho_1$.

Inequality (13) gives us the lower bound $E\tilde{\tau}_1 \geq \theta^{-1} \ln(l/W_0)$. Then we would like to obtain an upper bound for $E\tilde{\tau}_1$ of the same order. To do that, we will work with a slightly altered sequence $\tilde{\lambda}_t$, which we will define now.

Let $\varepsilon > 0$ be the constant from Assumption (A.2) and put $\delta = \varepsilon^2/256$. Define recursively the sequences $r'_t$, $t \geq 0$, and $\tilde{\lambda}'_t$, $t \geq 1$, by the relations

$$r'_0 = r_0,$$

$$\tilde{\lambda}'_t = \tilde{\lambda}_t + \delta \lambda_t I(\min_n \tilde{\lambda}_t^n \leq \varepsilon/2, r'_{t-1} \leq 1/2), \quad t \geq 1,$$

$$r'_t = \sum_{n=1}^N \frac{\lambda^n_t r'_{t-1}}{\tilde{\lambda}'_t^n (1 - r'_{t-1})} R^n_t, \quad t \geq 1.$$  

(14)

By induction, one can check that $r'_t \leq r_t$. Put $Y'_t = r'_t W_t$ and $\tau' = \min\{t \geq 0 : Y'_t \geq l\}$. Then we have $\tau_1 \leq \tau'$, so we will look for an upper bound for $E\tau'$.

Similarly to (12), we can show that $\ln r'_t$ is a submartingale. Indeed, let $\beta'_t = r'_{t-1} \lambda_t + (1 - r'_{t-1}) \tilde{\lambda}_t$. Then

$$E(\ln r'_t \mid \mathcal{F}_{t-1}) - \ln r'_{t-1} \geq \sum_{n=1}^N \lambda^n_t (\ln \lambda^n_t - \ln \beta'^m_t) \geq \frac{1}{4} \|\lambda_t - \beta'_t\|^2 - \ln |\beta'_t|. \quad (15)$$

On the event $\{\tilde{\lambda}'_t = \tilde{\lambda}_t\}$ we have $|\beta'_t| = 1$, so the right-hand side of (15) is non-negative.

On the event $\{\tilde{\lambda}'_t = \lambda_t + \delta \lambda_t\}$, there exists a coordinate $n = n(\omega)$ such that $\tilde{\lambda}_t^n \leq \varepsilon/2$, so, using that $|\beta'_t| = 1 + \delta(1 - r'_{t-1}) \leq 1 + \delta$, we can estimate $\ln |\beta'_t| \leq \delta$ and

$$\|\lambda_t - \beta'_t\|^2 \geq \frac{1 - r'_{t-1}}{|\beta'_t|} (\lambda^n_t - \tilde{\lambda}_t^n) \geq \frac{\varepsilon}{8}.$$  

Then the choice of $\delta$ implies that the right-hand side of (15) is non-negative on the event $\{\tilde{\lambda}'_t = \lambda_t + \delta \lambda_t\}$ as well. Thus, $\ln r'_t$ is a non-negative submartingale, and, in particular, $\inf_{t \geq 0} r'_t > 0$. Since $W_t \to \infty$, we also have $\tau' < \infty$.

From now on, assume that $l > Y_0$ (since we take $t \to \infty$). Applying Fatou’s lemma, we obtain

$$\ln l \geq \limsup_{t \to \infty} E \ln Y'_{\tau'\wedge t},$$

By Wald’s identity, $E \ln W'_{\tau'\wedge t} = \theta E(\tau' \wedge t) + \ln W_0$. From Lemma 2, $E \ln \rho_{\tau'\wedge t} \leq \theta + 2\sigma \sqrt{\rho_{\tau'\wedge t}}$, where $\sigma^2 = \var{\ln \rho_1}$. Since $\ln r'_t$ is a submartingale, $E \ln r'_{\tau'\wedge t} \geq \ln r_0$.

8
Finally, for all \( t \geq 1 \) we have \( r_t' \leq r_{t-1}'/(\delta \varepsilon) \). Indeed, if \( r_{t-1}' > 1/2 \) this is obvious, while if \( r_{t-1}' \leq 1/2 \) we can use (14) and the inequalities \( \lambda_t^n \geq \varepsilon \geq \delta \varepsilon \) and \( \tilde{\lambda}_t^n \geq \min(\varepsilon/2, \delta \lambda_t^n) \geq \delta \varepsilon \) to find
\[
r_t' \leq \sum_{n=1}^{N} \frac{r_{t-1}'}{\lambda_t^n + \lambda_t^n(1 - r_{t-1}')} R^n_t \leq \frac{r_{t-1}'}{\delta \varepsilon}.
\]
Consequently, we obtain the inequality
\[
\ln l > \limsup_{t \to \infty} \left( \theta E(\tau' \wedge t) - 2\sigma \sqrt{E(\tau' \wedge t)} \right) - \theta + \ln W_0 + \ln r_0 + \ln(\delta \varepsilon).
\]
Applying the monotone convergence theorem, we can see that \( E \tau' \) should be finite, and hence
\[
\ln l \geq \theta E \tau' - 2\sigma \sqrt{E \tau'} - \theta + \ln(Y_0 \delta \varepsilon). \tag{16}
\]
Now the claim of the theorem follows from (13), (16), and the relation \( E \tau_l \leq E \tau' \). □

In the following proofs of Theorem 2 and Proposition 1, we will use the same notation for investors 1 and 2 as in the proof of Theorem 1, i.e. without and with tilde, respectively.

**Proof of Theorem 2.** We can assume that \( E \tilde{\tau}_l < \infty \) for all \( l > 0 \), as otherwise the proof becomes trivial. Let us begin with an auxiliary estimate. For \( c \in [0, 1) \) we define \( \eta_c = \sum_{t \geq 0} I(r_t < c) \) and will now show that \( E \eta_c < \infty \). As was shown in the proof of Theorem 1, \( \ln r_t \) is a non-positive submartingale. If we denote by \( C_t \) its compensator, i.e. the non-negative and non-decreasing sequence
\[
C_t := \sum_{s=1}^{t} E \left( \ln \frac{r_s}{r_s-1} \mid F_{s-1} \right),
\]
then \( C_t \) a.s.-converges to a limit \( C_\infty \) with \( E C_\infty < \infty \). This follows from the monotone convergence theorem since \( E C_t = E \ln(r_t/r_0) \leq -\ln r_0 \). Using Lemma 1, similarly to (12) and (15), we see that (with the same \( \beta_t \) as in (12))
\[
C_\infty \geq \frac{1}{4} \sum_{i=1}^{\infty} \|\lambda - \beta_t\|^2 = \frac{1}{4} \sum_{i=1}^{\infty} (1 - r_{i-1})^2 \|\lambda - \tilde{\lambda}_t\|^2 \geq \frac{\sigma^2}{4} \sum_{i=1}^{\infty} (1 - r_{i-1})^2. \tag{17}
\]
Therefore, \( \eta_c \leq 4C_\infty/(a(1-c))^2 \), so \( E \eta_c < \infty \).

From (11), we find
\[
\tilde{r}_t = \sum_{n=1}^{N} \frac{\tilde{\lambda}_t^n}{\lambda^n r_{t-1} + \lambda^n \tilde{r}_{t-1}} R^n_t,
\]
which implies
\[
E \left( \ln \frac{\tilde{r}_t}{\tilde{r}_{t-1}} \mid F_{t-1} \right) \leq E \left( \ln \sum_{n=1}^{N} \frac{\tilde{\lambda}_t^n R^n_t}{\lambda^n r_{t-1}} \mid F_{t-1} \right) \leq f(a) - \ln r_{t-1}.
\]
Since \( r_t \) is a submartingale, we have \( E(\ln(\tilde{r}_t/\tilde{r}_{t-1}) \mid F_{t-1}) \leq 0 \) by Jensen’s inequality, so \( \ln \tilde{r}_t \) is a supermartingale (its integrability follows from that \( \tilde{\varepsilon} \leq \tilde{r}_t/\tilde{r}_{t-1} \leq \min(\varepsilon, \tilde{\varepsilon})^{-1} \)). Consequently, using that \( E \tilde{\tau}_l < \infty \) and applying Doob’s stopping theorem, we obtain
\[
E \ln \tilde{\tau}_l \leq E \sum_{s=1}^{l} \min(f(a) - \ln r_{s-1}, 0). \tag{18}
\]
The possibility of applying Doob’s theorem can be justified by first applying it to the bounded stopping times \( \tilde{\tau}_l \wedge t \), then passing to the limit \( t \to \infty \) using Fatou’s lemma in the left-hand side of (18) (note that \( \ln \tilde{r}_{\tilde{\tau} \land t} \) is bounded from below by the integrable random variable \( \tilde{\tau}_l \ln \tilde{r} + \ln \tilde{r}_0 \)), and using the monotone convergence theorem in the right-hand side.

Now, similarly to (13), for any \( c \in [e^{\rho}, 1) \) we find

\[
\ln \frac{l}{W_0} \leq E \ln \tilde{r}_{\tilde{\tau}_l} + \Theta E \tilde{r}_l \leq E \sum_{s=1}^{\tilde{r}_l} (f(a) - \ln(c)) I(r_{s-1} \geq c) + \Theta E \tilde{r}_l
\]

\[
\leq (\Theta + f(a) - \ln(c)) E \tilde{r}_l + (\ln c - f(a)) E \eta_c.
\]

Note that since we consider the case \( E \tilde{r}_l < \infty \) for all \( l > 0 \), we necessarily have \( \Theta + f(a) - \ln(c) > 0 \). Together with (16), this implies

\[
\limsup_{l \to \infty} E \frac{\tau_l}{\tilde{r}_l} \leq \frac{\Theta + f(a) - \ln(c)}{\Theta}.
\]

Taking \( c \to 1 \), we obtain the claim of the theorem. \( \square \)

**Proof of Proposition 1.** We will assume that \( \tilde{\lambda} \neq \lambda \), as otherwise the claim of the proposition is obvious. Since \( W_t = \rho^t \), for investor 1 we have \( \tau_l \geq \theta^{-1} \ln l \), where \( \theta = \ln \rho \).

Therefore, it will be enough to show that for investor 2 we have

\[
\tilde{\tau}_l \leq \frac{\ln l}{\theta} (1 + o(1)). \tag{19}
\]

Using the wealth equation (1) and that \( \lambda^n = R^n \), we obtain

\[
\tilde{Y}_t = \rho \sum_{n=1}^{N} \frac{\tilde{\lambda}^n \lambda^n}{\lambda^n r_{t-1} + \lambda^n \tilde{r}_{t-1}}. \tag{20}
\]

Inequality (17) implies that \( r_t \to 1 \), so the right-hand side of (20) is strictly greater than 1 for \( t \) large enough. Hence \( \tilde{Y}_t \to \infty \), which implies that \( \tilde{\tau}_l < \infty \) for all \( l \). Consequently,

\[
\ln l > \ln \tilde{Y}_{\tilde{\tau}_{l-1}} = \ln W_{\tilde{\tau}_{l-1}} + \ln \tilde{r}_{\tilde{\tau}_{l-1}} = \theta (\tilde{\tau}_l - 1) + \ln \tilde{r}_{\tilde{\tau}_{l-1}}. \tag{21}
\]

From (20), using the concavity of the logarithm and the inequality \( \ln x \geq 1 - x^{-1} \), we obtain the bound

\[
\ln \frac{\tilde{r}_l}{\tilde{r}_{t-1}} \geq \sum_{n=1}^{N} \tilde{\lambda}^n \ln \frac{\lambda^n}{\lambda^n r_{t-1} + \lambda^n \tilde{r}_{t-1}} \geq \ln \tilde{r}_{t-1} \sum_{n=1}^{N} \frac{(\tilde{\lambda}^n)^2 \tilde{r}_{t-1}}{\lambda^n r_{t-1}} \geq \tilde{r}_{t-1} \left( 1 - \sum_{n=1}^{N} \frac{(\tilde{\lambda}^n)^2}{\lambda^n r_{0}} \right),
\]

where in the last inequality we estimated \( r_{t-1} \geq r_0 \) since \( r_t \) is a non-decreasing sequence (in the proof of Theorem 1, we showed that it is a submartingale). Since \( \tilde{r}_t \to 0 \) and \( \tilde{\tau}_l \to \infty \), we get \( \tilde{\tau}_l^{-1} \ln \tilde{r}_{\tilde{\tau}_l} \to 0 \). Then relation (21) implies (19), which is what is needed.

**References**

[1] P. H. Algoet and T. M. Cover. Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. *The Annals of Probability*, 16(2):876–898, 1988.
[2] R. Amir, I. V. Evstigneev, T. Hens, and L. Xu. Evolutionary finance and dynamic games. *Mathematics and Financial Economics*, 5(3):161–184, 2011.

[3] R. Amir, I. V. Evstigneev, and K. R. Schenk-Hoppé. Asset market games of survival: a synthesis of evolutionary and dynamic games. *Annals of Finance*, 9(2):121–144, 2013.

[4] R. Amir, S. Belkov, I. V. Evstigneev, and T. Hens. An evolutionary finance model with short selling and endogenous asset supply. *Economic Theory*, published online, 2020.

[5] S. Belkov, I. V. Evstigneev, and T. Hens. An evolutionary finance model with a risk-free asset. *Annals of Finance*, published online, 2020.

[6] L. Blume and D. Easley. Evolution and market behavior. *Journal of Economic Theory*, 58(1):9–40, 1992.

[7] L. Breiman. Optimal gambling systems for favorable games. In *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, volume 1, pages 63–68, 1961.

[8] M. A. H. Dempster, I. V. Evstigneev, and K. R. Schenk-Hoppé. Volatility-induced financial growth. *Quantitative Finance*, 7(2):151–160, 2007.

[9] Ya. Drokin and M. Zhitlukhin. Relative growth optimal strategies in an asset market game. *Annals of Finance*, published online, 2020.

[10] I. Evstigneev, T. Hens, and K. R. Schenk-Hoppé. Evolutionary behavioral finance. In E. Haven et al., editors, *The Handbook of Post Crisis Financial Modelling*, pages 214–234. Palgrave Macmillan UK, 2016.

[11] I. V. Evstigneev, T. Hens, and K. R. Schenk-Hoppé. Market selection of financial trading strategies: global stability. *Mathematical Finance*, 12(4):329–339, 2002.

[12] R. Fernholz and B. Shay. Stochastic portfolio theory and stock market equilibrium. *The Journal of Finance*, 37(2):615–624, 1982.

[13] T. Hens and K. R. Schenk-Hoppé. Evolutionary stability of portfolio rules in incomplete markets. *Journal of Mathematical Economics*, 41(1-2):43–66, 2005.

[14] T. Holtfort. From standard to evolutionary finance: a literature survey. *Management Review Quarterly*, 69(2):207–232, 2019.

[15] I. Karatzas and C. Kardaras. The numéraire portfolio in semimartingale financial models. *Finance and Stochastics*, 11(4):447–493, 2007.

[16] C. Kardaras and E. Platen. Minimizing the expected market time to reach a certain wealth level. *SIAM Journal on Financial Mathematics*, 1(1):16–29, 2010.

[17] J. L. Kelly, Jr. A new interpretation of information rate. *Bell System Technical Journal*, 35(4):917–926, 1956.

[18] J. B. Long. The numéraire portfolio. *Journal of Financial Economics*, 26(1):29–69, 1990.
[19] D. Luenberger. *Investment Science*. Oxford University Press, 1998.

[20] L. C. MacLean, E. O. Thorp, and W. T. Ziemba, editors. *The Kelly Capital Growth Investment Criterion: Theory and Practice*, volume 3 of *Handbook in Financial Economic*. World Scientific, Singapore, 2011.

[21] E. Platen and D. Heath. *A Benchmark Approach to Quantitative Finance*. Springer-Verlag, Berlin, 2006.

[22] A. N. Shiryaev. *Probability-2*. Springer, 3rd edition, 2019.

[23] M. Zhitlukhin. Survival investment strategies in a continuous-time market model with competition. *arXiv:1811.12491*, 2018.