Dynamical invariance for random matrices

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We consider a general Langevin dynamics for the one-dimensional N-particle Coulomb gas with confining potential $V$ at temperature $\beta$. These dynamics describe for $\beta = 2$ the time evolution of the eigenvalues of $N \times N$ random Hermitian matrices. The equilibrium partition function – equal to the normalization constant of the Laughlin wave function in fractional quantum Hall effect – is known to satisfy an infinite number of constraints called Virasoro or loop constraints. We introduce here a dynamical generating function on the space of random trajectories which satisfies a large class of constraints of geometric origin. We focus in this article on a subclass induced by the invariance under the Schrödinger-Virasoro algebra.

Keywords: random matrices, Coulomb gas, quantum Hall effect, Virasoro constraints, loop constraints, Schrödinger-Virasoro algebra, dynamical invariance.

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0 Introduction

Let us start with a short preliminary discussion of the model (§0.1) and of the well-known equilibrium Virasoro constraints (§0.2). For a presentation of our results, the reader familiar with these may skip directly to §0.3.

0.1 Dyson’s Brownian motion

We consider the following Langevin dynamics [3, 4, 14] for $N$ particles confined to a line with positions $\{\lambda_i\}, i = 1, \ldots, N$

$$d\lambda_i = dB_i - \frac{\partial W}{\partial \lambda_i} dt = dB_i + \left( \sum_{j \not= i} \frac{\beta}{\lambda_i - \lambda_j} - V'(\lambda_i) \right) dt$$

(0.1)

where:

(i) the noises $(B_1, \ldots, B_N) = (B_1(t), \ldots, B_N(t))$ are $N$ independent Brownian motions;

(ii) $W(\lambda_i) = -\frac{\beta}{2} \sum_{i,j \not= i} \log |\lambda_i - \lambda_j| + \sum_i V(\lambda_i)$ is the sum of the electrostatic energy of a system of $N$ identically charged particles and of a one-body confining potential $V$.

In our convention, $d\langle B_i, B_i \rangle_t = 2 dt$. Then the probability distribution function $P(\lambda_i; t)$ for the positions of the particles satisfies the Fokker-Planck equation,

$$\partial_t P = \Delta P + \sum_i \frac{\partial}{\partial \lambda_i} \left( \frac{\partial W}{\partial \lambda_i} P \right)$$

$$= \sum_i \frac{\partial^2 P}{\partial \lambda_i^2} - \beta \sum_{i,j \not= i} \frac{\partial}{\partial \lambda_i} \left( \frac{P}{\lambda_i - \lambda_j} \right) + \sum_i \frac{\partial}{\partial \lambda_i} \left( V'(\lambda_i) P \right)$$

(0.2)

(with the usual normalization of Brownian motion one would get $\frac{\beta}{2} \Delta$ in the above expression).

For $V$ growing sufficiently fast at $\infty$, the unique stationary measure is the Gibbs measure

$$P_{eq}(\{\lambda_i\}) = \frac{1}{Z_N(V)} e^{-W(\lambda_i)} d\lambda = \frac{1}{Z_N(V)} \frac{1}{N!} \prod_{i=1}^N e^{-V(\lambda_i)} \prod_{i,j > i} (\lambda_j - \lambda_i)^{\beta} d\lambda$$

(0.3)

which may also be interpreted as a normalization constant for the celebrated fractional quantum Hall effect Laughlin wave-function [9].

For $\beta = 2$, the normalization constant is the partition function of the Hermitian ensemble with potential $V$, $Z_N(V) = \int dM e^{-\text{Tr} V(M)}$, for a suitable normalization of the measure $dM$ on the space of $N \times N$ Hermitian matrices. In fact, the measure $\frac{1}{Z_N(V)} e^{-\text{Tr} V(M)} dM$ projects down by conjugation invariance to a measure on the spectrum $\{\lambda_i\}$ of $M$ which is none other...
than $\mathcal{P}_{eq}(\{\lambda_i\})$. Following Dyson [3] who originally introduced this model, we may consider our dynamics to be the projection to the spectrum of a conjugation invariant random walk on the space of Hermitian matrices, $d\mathbf{M} = d\mathbf{B} - V'(\mathbf{M})dt$, where by assumption the linearly independent entries $(\{\mathbf{B}_{ij}(t)\}_{i<j}; \{\mathbf{B}_{ij}(t)\}_{i=j})$ are independent Brownian motions; in other words, $d\mathbf{B}(t), t \geq 0$ are independent infinitesimal increments drawn from GUE distribution.

In order to highlight the connection with the equilibrium case, we also use two equivalent reformulations of (0.1). First we have a measure $Q$ on the space of trajectories $(\{\lambda_i(t)\})_{t \geq 0}$, absolutely continuous with respect to the Wiener measure $D\mathcal{W}$, formally,

$$Q(\{\lambda_i\}) = D\lambda \exp \left(-\int dt \sum_i (\dot{\lambda}_i + \frac{\partial W}{\partial \lambda_i})^2 \right)$$

$$= D\lambda \exp \left(-\int dt \sum_i (\dot{\lambda}_i - \sum_{j \neq i} \frac{\beta}{\lambda_i - \lambda_j} + V'(\lambda_i))^2 \right).$$  (0.4)

This can be made rigorous by using a Girsanov transformation [12], namely,

$$Q(\{\lambda_i\}) = D\mathcal{W}(\lambda) \exp \left(-2 \sum_i \int dt \frac{\partial W}{\partial \lambda_i}(t) d\lambda_i(t) - \int dt \sum_i (\frac{\partial W}{\partial \lambda_i})^2(t) \right),$$  (0.5)

where the integral $\int \frac{\partial W}{\partial \lambda_i}(t) d\lambda_i(t)$ is an Itô integral.

Then, by an elementary Hubbard-Stratonovich transformation, we obtain a (mathematically ill-defined) complex Gibbs measure on the space of trajectories $(\{\lambda_i(t)\})_{t \geq 0}$, $(\{\mu_i(t)\})_{t \geq 0}$ of the particles and of associated virtual particles with positions $\{\mu_i\}$, also confined on the line,

$$\mathcal{Q}(\{\lambda_i\}, \{\mu_i\}) = \mathcal{Q}(\{\lambda_i(t)\})_{t \geq 0}, \{\mu_i(t)\}_{t \geq 0}$$

$$= D\lambda D\mu \exp \left(-\int dt \sum_i \mu_i(\dot{\lambda}_i + \frac{\partial W}{\partial \lambda_i}) - \int dt \sum_i \mu_i^2 \right)$$

$$= D\lambda D\mu \exp \left(-\int dt \sum_i \mu_i(\dot{\lambda}_i - \sum_{j \neq i} \frac{\beta}{\lambda_i - \lambda_j} + V'(\lambda_i)) - \int dt \sum_i \mu_i^2 \right).$$  (0.6)

The above formula is a reformulation of the initial coupled stochastic differential equations in the Martin-Siggia-Rose formalism [11]. The quantities $\mathcal{L}, \mathcal{L}$ in the exponentials,

$$Q(\{\lambda_i\}) \equiv D\lambda e^{-\int dt \mathcal{L}(\lambda_i(t))}, \quad \mathcal{Q}(\{\lambda_i\}, \{\mu_i\}) \equiv D\lambda D\mu e^{-\int dt \mathcal{L}(\lambda_i, \mu_i)}$$  (0.7)

may be interpreted as a space-time action.

The kernel $K$ of the quadratic form $(\mu, \mu) = \frac{1}{2} \int dt dt' \sum_{i,j} K_{ij}(t-t')\mu_i(t)\mu_j(t')$ appearing in the action is in this formalism equal to the covariance of the noise in the original Langevin equation, here $\langle \dot{\mathbf{B}}_i(t)\dot{\mathbf{B}}_j(t') \rangle = K_{ij}(t-t') = 2\delta_{i,j}\delta(t-t')$. 


0.2 Equilibrium Virasoro constraints: a reminder

The so-called loop (or Virasoro) constraints are a well-known invariance statement in the equilibrium theory which has proved extremely useful in obtaining formulas in various asymptotic regimes for \( N \to \infty \) (see e.g. [2] or [6]). We prove them using some elementary integration-by-parts trick on the equilibrium measure. Letting \( F \) for the sums of powers of the eigenvalues. Take \( N \) asymptotic regimes for equilibrium theory which has proved extremely useful in obtaining formulas in various regimes. The so-called loop (or Virasoro) constraints are a well-known invariance statement in the equilibrium theory, restricting to \( \beta = 2 \). Consider a potential \( V_0(\lambda) \) such that \( V_k^0(\lambda) \equiv \sum_{k \geq 1} b_k \lambda^k \), where all but a finite number of the coefficients \( (b_k)_{k \geq 0} \) are zero. Perturb it formally by letting \( V(\lambda) \equiv V[\tau](\lambda) := V_0(\lambda) + \sum_{k=1}^{\infty} \tau_k \lambda^k \) depend on a set of parameters \( \tau = \{\tau_k\}_{k \geq 0} \), and write accordingly \( Z_N(V) = Z[\tau], \mathcal{P} = \mathcal{P}[\tau], \mathcal{P}_{eq} = \mathcal{P}_{eq}[\tau], \mathcal{Q} = \mathcal{Q}[\tau], \overline{\mathcal{Q}} = \overline{\mathcal{Q}}[\tau]. \) Introduce the notation

\[
\pi_k \equiv \sum_i \lambda_i^k, \quad k \geq 0
\]

for the sums of powers of the eigenvalues. Take \( F(\{\lambda_i\}) = \sum_i \lambda_i^{n+1} \) in (0.8). Then

\[
\langle \sum_i (n+1)\lambda_i^n \rangle = (n+1)\langle \pi_n \rangle = \langle \lambda_{i+1}^n \frac{\partial W}{\partial \lambda_i} \rangle = \sum_{k=0}^{+\infty} b_k \langle \pi_{k+n+1} \rangle + \sum_{k=0}^{+\infty} k\tau_k \langle \pi_{k+n} \rangle - \beta \sum_{i,j \neq i} \langle \frac{\lambda_{i+1}^{n+1} - \lambda_j^n}{\lambda_i - \lambda_j} \rangle = \sum_{k=0}^{+\infty} b_k \langle \pi_{k+n+1} \rangle + \sum_{k=0}^{+\infty} k\tau_k \langle \pi_{k+n} \rangle - \beta \sum_{k=0}^{n} \langle (\pi_k \pi_{n-k}) - \langle \pi_n \rangle \rangle.
\]

(0.10)

Introduce the Fock representation of the oscillator algebra [7]

\[
\hat{a}_n = \beta^{1/2} \frac{\partial}{\partial \tau_n} (n \geq 1), \quad 0(n = 0), \quad \beta^{-1/2} |n| \tau_n (n \leq -1)
\]

(0.11)

and the associated free boson \( \hat{a}(z) \) and energy-momentum tensor \( \hat{L}(z) \),

\[
\hat{a}(z) = \sum_{n \in \mathbb{Z}} \hat{a}_n z^{-n-1}, \quad \hat{L}(z) = \sum_{n \in \mathbb{Z}} \hat{L}_n z^{-n-2}.
\]

(0.12)

Noting that

\[
\hat{L}_n = \sum_{k=0}^{+\infty} k\tau_k \frac{\partial}{\partial \tau_{k+n}} + \beta \sum_{k=0}^{n} \frac{\partial}{\partial \tau_{k}} \frac{\partial}{\partial \tau_{n-k}}, \quad n \geq -1
\]

(0.13)

we see that (0.10) amounts to

\[
\mathcal{L}_{eq}^\beta Z[\tau] = 0
\]

(0.14)
with
\[ L_n^{eq} = \hat{L}_n + \beta^{-1/2} \sum_{k=0}^{+\infty} b_k \hat{a}_{n+k+1} + (\frac{\beta}{2} - 1)(n + 1)\hat{a}_n. \quad (0.15) \]

This two-line derivation has its interest, but the spirit of these constraints is really of geometric origin, see e.g. [11] or [10]: they reflect the way that the potential \( V \) is transformed under generators of conformal transformations \( L_n = -\lambda^{n+1}\partial_\lambda \).

### 0.3 Results of the article

The aim of the article is to prove the existence dynamical constraints in the same spirit as the equilibrium Virasoro constraints discussed in the previous subsection.

As pointed out just above, the conventional way to prove Virasoro constraints, see e.g. [11] or [10], is to consider the transformation of the equilibrium measure under a conformal transformation of the eigenvalues, \( \lambda \mapsto \lambda + \varepsilon \lambda^{n+1} \).

In the dynamical case we miss a straightforward analogue of (i) conformal transformations; (ii) the equilibrium measure. Let us discuss these two points.

(i) Our first claim is the following. The analogue of the group of conformal transformations in the dynamical case is the group of noise-preserving transformations, briefly introduced in section 1 and discussed in full details in section 2.1, see in particular Definition 2.2 for the Lie algebra of this group. The corresponding infinitesimal transformations are "causality-preserving" transformations of the set of trajectories \( \{\lambda(t), t \geq 0\} \), with a condition called noise-invariance condition, see (1.7) or (2.10), ensuring that these preserve the strength of the noise for trajectories satisfying a Langevin equation. This group contains in particular as a subgroup the Schrödinger-Virasoro group, an infinite-dimensional group of coordinate transformations studied in details in the book [13], see also [5]. Briefly said, these are coupled space- and time-transformations which are affine in space, thus defining an infinite-dimensional extension of the two Virasoro generators \( L_{-1,0} \). In 1D the Lie algebra is generated by

\[ X_f := -f(t)\partial_t - \frac{1}{2}\dot{f}(t)\lambda\partial_\lambda, \quad Y_g := -g(t)\partial_\lambda. \quad (0.16) \]

While \( (Y_g) \) is simply the time-current generated by \( L_{-1} \), the \( (X_f) \) are local space-time transformations generalizing the infinitesimal scaling transformation \(-t\partial_t - \frac{1}{2}\lambda\partial_\lambda\) with dynamical exponent \( z = 2 \), which generates the parabolic scaling transformation \((t, \lambda) \mapsto (a^2 t, a\lambda)\). This scaling originates from the transformation properties of white noise. Noise-preserving infinitesimal transformations not belonging to the Schrödinger-Virasoro algebra may be seen as the sum of a very general transformation of the \( \lambda \)-coordinate, \( \lambda \mapsto \lambda + \varepsilon \delta \lambda \), with \( \delta \lambda(t) = f(\lambda(t))\Phi(t, \lambda) \), where \( \Phi(t, \cdot) \) is some time-integrated functional of the past of the trajectory, and of a time-transform depending on \( \lambda \), which suggests to introduce the notion of a proper time (see section 3). Though we are here in space dimension 1, the extension to \( d \) space-dimensions is more or less straightforward; using the conformal invariance of Brownian motion, it is enough to require that the function \( f \) in factor in the \( \lambda \)-coordinate transform should define a conformal transformation.
(ii) Turning to the second point, it is not clear to us if there is a straightforward analogue of the equilibrium measure. Naively, the partition function should be replaced by the measure on trajectories, which is automatically normalized, and thus cannot be used as a generating functional. However, perturbing the measure in the way of Adler-Van Moerbeke (see previous subsection), one is led very naturally to an un-normalized perturbed measure on the trajectories, $Q^{\text{lin}}[\tau] = Q[\{\lambda_i\}]$ (see Definition 4.1) whose integral $Z^{\text{lin}}[\tau] := \int dQ^{\text{lin}}[\tau](\{\lambda_i\})$ may serve as generating functional. The upper index "lin" stands for "linear", since $Q[\tau]$ is obtained from the original measure on the trajectories by linearizing in the $\tau$-parameters and then throwing away quadratic terms produced by the two-body potential.

Our main result is then Theorem 4.1, stating the invariance of the generating functional $Z^{\text{lin}}[\tau]$ under Schrödinger-Virasoro transformations. The action of these transformations is similar in aspect to the action of Virasoro transformations on the partition in the equilibrium measure, see (0.15), with the considerable difference though that we restricted ourselves to indices $n = -1, 0$, but on the other hand we have an infinite number of constraints because of the arbitrary time-dependence. It exhibits a sum of linear and of quadratic expressions in terms of a static free boson $\hat{\phi}(z, t)$ — the free boson of usual conformal field theory, with an extra, trivial time-dependence — and of a dynamical free boson $\hat{\psi}(z, t)$ defined via a kernel $K$ depending on $V_0$ (see Definition 4.2, Definition 4.3 and Definition 4.4). The kernel $K = K(z^{-1}, w)$, one of the main ingredients in the computations, is the Green function of the operator $D := \partial_t + (\frac{\delta}{\partial z} - 1) \frac{d^2}{dz^2} - \frac{d}{dz} \pi(z)$ acting on formal series $a_0 + a_1 z + a_2 z^2 + \ldots$

Just as equilibrium Virasoro constraints may be used to compute the $n$-point functions of the first few so-called linear statistics, $\pi_k := \sum_{i=1}^{\lambda_i} \lambda_i^k$, formula (4.24), which we reproduce here,

$$\left\langle \left( \int dt f_1(t) \pi_{k_1}(t) \right) \ldots \left( \int dt f_p(t) \pi_{k_p}(t) \right) \right\rangle_0 = \prod_{q=1}^{p} \left( -\int dt f_g(t)(K * \hat{\partial}/\hat{\partial} \tau)k_q(t) \right) Z^{\text{lin}}[\tau]_{\tau=0}$$

shows that $n$-point functions may be obtained from $Z^{\text{lin}}[\tau]$ by the differentiation "trick" $\pi_k \equiv (K * \hat{\partial}/\hat{\partial} \tau)k(t)$ or (in terms of generating series) $\pi(z) \equiv -(K * \hat{\partial}/\hat{\partial} \tau)(z, t)$. Note that if one had not linearized the generating functional, we would have to solve instead a complex Burgers equation, $D\pi(z) + (\pi^2)'(z) = -\hat{\partial}/\hat{\partial} \tau(z)$.

### 0.4 Perspectives

In a future article, we plan to extend Schrödinger-Virasoro constraints to a much more general class of constraints, one per generator of the Lie algebra of noise-preserving transformations. Formulas in Theorem 4.1 being readily generalized to arbitrary $n \neq -1, 0$, it seems very likely that some of the conformal field theoretic structure uncovered for $n = -1, 0$ will survive. As can be expected, such an extension is however far from obvious. Preliminary computations (some of them presented already in the computations of §4.3, which often hold for arbitrary $n$, and also ) show that the action of more general transformations on the two-body potential produces cubic terms, plus an infinite-number of new terms due to the particle-dependent time-shifts with unresolved singularity, typically $\int_0^t ds \frac{\lambda_i(s) - \lambda_j(s)}{(\lambda_i(t) - \lambda_j(t))^2}$. 

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(i \neq j) \text{ (see §1.2 B)}, which are however amenable to analysis provided one retains only a finite number of terms in some perturbative expansion, either short-time or large $N$, with foreseeable applications to the study of the limit $N \to \infty$ in the microscopic regime.

0.5 Outline of the article

Section 1 is an appetizer for the reader willing to understand the objective of the paper and to have a flavour of the computations. The noise invariance condition is introduced right from the beginning in (1.7), but we postpone the general discussion of this condition and consider only elementary transformations such as (1.9), which do not close under Lie brackets. The transformation of the force term under these transformations is given in (1.10) for $N = 1$ and (1.16, 1.23) for general $N$. The very complicated term (1.23) fortunately vanishes for Schrödinger-Virasoro transformations, for which all time shifts are equal.

We present the noise invariance condition in whole generality in the strictly algebraically-minded section 2 and define the Lie algebra of noise-preserving transformations $\mathcal{F}_{NP}$ in Definition 2.2. We also compute the Lie brackets of elementary transformations in the natural basis of iterated integrals.

The very short Section 3 lies a general geometric foundation to these sets of transformations. It is also the occasion to introduce the Schrödinger-Virasoro transformations.

The main section is Section 4. We present the key ingredients in §4.1: the generating functional (Definition 4.1); generating series for the linear statistics and the parameters; the equation of motion for the linear statistics (Definition 4.2); the algebra of static and dynamic free bosons (Definition 4.3) and its commutators (Definition 4.4). Then comes the statement of our main result, Theorem 4.1, yielding dynamical constraints parallel to the equilibrium constraints of §0.2. The rest of the section is devoted to the proof of Theorem 4.1.

Finally, we collected some technical lemmas used in the proofs in an appendix (section 5).

1 A short computational introduction

The purpose in this section is to introduce and motivate the fundamental noise-invariance condition in a simplified setting, and to show some preliminary computations in the case $N = 1$ and in the general case, paving the way to the more involved computations of section 4.

1.1 The case $N = 1$

For pedagogical reasons we start from the case $N = 1$, a one-dimensional general Langevin equation,

$$d\lambda_t = dB_t - V'(\lambda_t)dt. \quad (1.1)$$

We look for infinitesimal transformations of the set of trajectories,

$$\{\lambda(t), t \geq 0\} \mapsto \{\lambda(t) + \varepsilon (\delta \lambda)(t), t \geq 0\} \quad (1.2)$$
that preserve the general structure of the equation. We actually restrict to causality-preserving, first-order transformation, namely, we assume that

\[ \varepsilon(\hat{\delta}\lambda)(t) = \varepsilon \left( \phi(t, \lambda) - \psi(t, \lambda)\hat{\lambda}(t) \right) \]  \hspace{1cm} (1.3)

where \( \phi(t, \lambda), \psi(t, \lambda) \) depend only on the values of \( (\lambda_s)_{s \leq t} \). Since the trajectory \( t \mapsto \lambda(t) \) is not differentiable, this should be understood (to order one in \( \varepsilon \)) as the composition of two transformations,

\[ \lambda \mapsto \lambda + \varepsilon\delta\lambda, \quad t \mapsto t + \varepsilon\delta t \]  \hspace{1cm} (1.4)

where

\[ (\delta\lambda)(t) = \phi(t, \lambda), \quad \delta t = \psi(t, \lambda). \]  \hspace{1cm} (1.5)

Put in another way, we look for the dynamical law satisfied by the transformed trajectory \( \tilde{\lambda}(t + \varepsilon\delta t) := (\lambda + \varepsilon\delta\lambda)(t) \), or (to order 1 in \( \varepsilon \)) \( \tilde{\lambda}(t) = (\lambda + \varepsilon\delta\lambda)(t - \varepsilon\delta t) \). Since \( dB_{t-\varepsilon\delta t} = (1 - \varepsilon(\delta t))^{1/2}dB_t \) where \( \tilde{B} \) has the same law as \( B \), we get to order 1 in \( \varepsilon \), taking into account the Itô correction written as "Itô" in the following formula,

\[ d\tilde{\lambda} = \varepsilon \left( \frac{\partial(\delta\lambda)}{\partial t} + \text{Ito} \right) dt + (1 + \varepsilon \frac{\partial(\delta\lambda)}{\partial \lambda}) \left\{ -(1 - \varepsilon(\delta t))V'(\lambda)dt + (1 - \frac{\varepsilon}{2}(\delta t))dB \right\}. \]  \hspace{1cm} (1.6)

Under the fundamental noise invariance condition

\[ \frac{\partial(\delta\lambda)}{\partial \lambda} = \frac{1}{2}(\delta t). \]  \hspace{1cm} (1.7)

(1.1) is turned into a similar Langevin equation with transformed force \(-(V' + \varepsilon\delta V')\) defined to order one in \( \varepsilon \) by

\[ (V' + \varepsilon\delta V')(\lambda + \varepsilon\delta\lambda) = -\varepsilon \frac{\partial(\delta\lambda)}{\partial t} + \left[ 1 + \varepsilon \left( \frac{\partial(\delta\lambda)}{\partial \lambda} - (\delta t) \right) \right] V'(\lambda) - \varepsilon \text{Ito} \]

\[ = V'(\lambda + \varepsilon\delta\lambda) - \varepsilon \left\{ \frac{\partial(\delta\lambda)}{\partial t} + \frac{\partial(\delta\lambda)}{\partial \lambda} V'(\lambda + \varepsilon\delta\lambda) + V''(\lambda + \varepsilon\delta\lambda)\delta\lambda + \text{Ito} \right\} \]

\[ = V'(\lambda + \varepsilon\delta\lambda) - \varepsilon \left\{ \frac{\partial(\delta\lambda)}{\partial t} + \frac{\partial}{\partial \lambda} (\delta\lambda V'(\lambda + \varepsilon\delta\lambda)) + \text{Ito} \right\} \]  \hspace{1cm} (1.8)

Looking for specific examples, we now specialize to the transformations where \( \phi(t, \lambda) \) depends only on the value of \( \phi \) at time \( t \), namely (for \( n \geq -1 \))

\[ \delta\lambda(t) = -\lambda^{n+1}(t) \hat{a}(t), \quad \delta t = 2 \int_0^t ds \frac{\partial(\delta\lambda)}{\partial \lambda}(s) = -2(n + 1) \int_0^t ds \hat{a}(s)\lambda^n(s). \]  \hspace{1cm} (1.9)

For \( n = -1, 0 \) these infinitesimal trajectory transformations may be seen as simple coordinate transformations; they generate the Schrödinger-Virasoro algebra introduced in section 3. For \( n \geq 1 \) however, these transformations act on the whole trajectory, and commutators generate a much larger class of transformations studied in the next section.

With an Itô correction \( \text{Ito} = \frac{\partial^2(\delta\lambda(t))}{\partial \lambda(t)^2} = -(n + 1)n\lambda^{n-1}(t) \hat{a}(t) \) in this specific case, we get our first important formula,
\((N = 1\ \text{force change})\)
\[
\delta V' = \left(\lambda^{n+1} \dot{a} + \left\{ \sum_{k \geq 0} b_k (n + 1 + k) \lambda^{n+k} + (n + 1)n \lambda^{n-1} \right\} \dot{a} \right).
\] (1.10)

All these terms extend trivially to the case of \(N\) particles when \(\beta = 0\), i.e. in absence of two-body potential, yielding \(N\) terms, \(\delta V'_i\), where
\[
V'_i := -\frac{\partial W}{\partial \lambda_i} = \sum_{j \neq i} \frac{\beta}{\lambda_i - \lambda_j} - V'(\lambda_i)
\] (1.11)
is the force felt by the \(i\)-th particle. The first term in (1.10) reflects the time-dependence of the transformation. The last term is the Itô’s correction. The second term expresses simply the action of the Virasoro vector field \(\dot{a}(\lambda^{n+1} \frac{\partial}{\partial \lambda} + (n + 1) \lambda^n)\) on the confining force \(-V'\).

1.2 The \(N\)-particle model

General transformations leaving invariant the form of the equation for \(N \geq 1\) will imply different time-changes for the \(N\) particles located at \(\{\lambda_i\}\), as is immediately seen from the noise invariance condition (1.7); see section 3 for general geometric considerations. This makes in general the transformation of the two-body force more complicated, though (as we shall see later on) the change in the action remains surprisingly simple.

Generally speaking the change of the force felt by the \(i\)-th particle (see (1.11)) or simply force change, \(\delta V'_i\), is the sum of two terms. The first one (thereafter called simultaneous), \(\delta_{\text{simul}} V'_i\), is the more or less straightforward of the \(N = 1\) force change written in the previous subsection, taking also into account the action of the coordinate change on the two-body force. The second (called delayed), \(\delta_{\text{delay}} V'_i\), takes into account the difference of time-shifts between two trajectories \((\lambda_i(t))_{t \geq 0}\) and \((\lambda_j(t))_{t \geq 0}\), \(i \neq j\).

A. Simultaneous force change

Compared with the \(N = 1\) case, we must now write down the effect on the dynamics of \(\lambda_i\) of the coordinate change. In addition to the term (1.9), one has an extra term due to the transformation of the two-body force, which must also take into account the transformation of the other eigenvalues \(\{\lambda_j\}_{j \neq i}\),
\[
-\dot{a} \left( \sum_{\eta' = 1}^{N} \lambda_i^{\eta+1} \frac{\partial}{\partial \lambda_{\eta'}} + (n + 1) \lambda_i^n \right) \left( \sum_{j \neq i} \frac{\beta}{\lambda_i - \lambda_j} \right) = -\beta \left\{ \sum_{j \neq i} \frac{\lambda_i^{n+1} - \lambda_j^{n+1}}{(\lambda_i - \lambda_j)^2} \right\} \dot{a}(t) \\
-(n + 1) \left\{ \sum_{j \neq i} \frac{\lambda_i^n}{\lambda_i - \lambda_j} \right\} \dot{a}(t) \equiv - \left[ \sum_{k = 0}^{n} A_{n,k} \right] \dot{a}(t),
\] (1.12)
where
\[
A_{n,k} := \beta \sum_{j \neq i} \frac{\lambda_i^k \lambda_j^{n-k} - \lambda_i^n}{\lambda_i - \lambda_j} = \beta \lambda_i^k \sum_{p=0}^{n-1-k} \lambda_j^p \sum_{j \neq i} \lambda_j^{n-1-k-p},
\] (1.13)
from which
\[ \sum_{k=0}^{n} A_{n,k} = \beta \sum_{q=0}^{n-1} (q+1)\lambda_i^q \sum_{j\neq i} \lambda_j^{n-q} = \beta \left[ \sum_{q=0}^{n-1} (q+1)\lambda_i^q \pi_{n-1-q} - \frac{1}{2} (n+1)n\lambda_i^{n-1} \right]. \tag{1.14} \]

Changing sign, we may interpret \ref{eq:1.14} as an additive contribution to $\delta V'_i$. Combining with the Itô term, see third term in \ref{eq:1.9}, we get
\[ \sum_{k=0}^{n} A_{n,k} + (n+1)n\lambda_i^{n-1} = \beta \sum_{q=0}^{n-1} (q+1)\lambda_i^q \pi_{n-1-q} + (1 - \frac{\beta}{2})(n+1)n\lambda_i^{n-1}. \tag{1.15} \]

The other terms in the action transform as in section 1 (compare with \ref{eq:1.9}), yielding a total variation

\[ \delta_{simul} V'_i = \lambda_i^{n+1}\dot{\alpha} + \left\{ \sum_{l=0}^{+\infty} b_l (n+l+1)\lambda_i^{n+l} \right. \]
\[ \left. + \left[ \beta \sum_{q=0}^{n-1} (q+1)\lambda_i^q \pi_{n-1-q} - \frac{\beta}{2} - 1 \right](n+1)n\lambda_i^{n-1} \right\} \dot{\alpha}. \tag{1.16} \]

We return to these computations in section 4 after a more detailed discussion of the noise-preserving condition.

**B. Delayed force change**

Consider only the part of the variation $\delta \lambda$ due to the time-shifts,
\[ \delta t_i := 2(n+1) \int_0^t ds \dot{\alpha}(s)\lambda_i^n(s). \tag{1.17} \]

Letting $\tilde{\lambda}_i(t) := \lambda_i(t - \varepsilon \delta t_i)$, the system of coupled equations for the particles becomes (to first order in $\varepsilon$)
\[ d\tilde{\lambda}_i(t) = \left( 1 - \varepsilon (n+1)\dot{\alpha}(t)\tilde{\lambda}_i^n(t) \right) d\tilde{B}_i(t) - \left( 1 - 2\varepsilon (n+1)\dot{\alpha}(t)\tilde{\lambda}_i^n(t) \right) V'(\tilde{\lambda}_i(t)) dt \]
\[ + \left( 1 - 2\varepsilon (n+1)\dot{\alpha}(t)\tilde{\lambda}_i^n(t) \right) \sum_{j\neq i} \frac{\beta}{\lambda_i(t) - \lambda_j(t - \varepsilon \delta t_i)} dt \tag{1.18} \]

where $\frac{1}{\sqrt{2}}(\tilde{B}_i)_i$ are standard Brownian motions. Adding to this variation the one due to $\delta \lambda$ compensates the change of noise strength due to the noise invariance condition. The last term in the r.-h.s. of \ref{eq:1.18} brings to light a new effect due to the different time-shift. Since $\lambda_j(t - \varepsilon \delta t_i) = \lambda_j(t + \varepsilon (\delta t_j - \delta t_i))$, we have to order 1 in $\varepsilon$
\[ \frac{\beta}{\lambda_i(t) - \lambda_j(t - \varepsilon \delta t_i)} = \frac{\beta}{\lambda_i(t) - \lambda_j(t)} + \varepsilon \frac{\beta}{(\lambda_i(t) - \lambda_j(t))^2} \cdot \frac{d\tilde{\lambda}_j(t)}{dt} (\delta t_j - \delta t_i) \tag{1.19} \]
Thus (combining with the effect of the $\delta \lambda$-variation studied in A.), $\lambda'_i = \lambda_i + (\delta \lambda)_i$, $i = 1, \ldots, N$ follow the modified system of equations to first order in $\varepsilon$

$$\frac{d\lambda'_i}{dt} = d\tilde{B}_i(t) - (V'_i(t, \lambda') + \varepsilon \delta_{\text{simul}} V'_i(t, \lambda'))dt + \varepsilon \sum_{j \neq i} (\delta t_j - \delta t_i) \frac{\beta}{(\lambda'_i(t) - \lambda'_j(t))^2} \frac{d\lambda'_j}{dt}. \quad (1.20)$$

Replacing $\frac{d\lambda'_i}{dt}$ by the 0-th order term $d\tilde{B}_i(t) - \frac{\partial W}{\partial \lambda'_i}(t)dt$ in the right-hand side of (1.20), we get

$$\frac{d\lambda'_i}{dt} = dB'_i(t) - (V'_i(t, \lambda') + \varepsilon \delta_{\text{simul}} V'_i(t, \lambda'))dt - \varepsilon \beta \sum_{j \neq i} \frac{\partial W}{\partial \lambda'_j}(t) \frac{\delta t_j - \delta t_i}{(\lambda'_i(t) - \lambda'_j(t))^2} dt, \quad (1.21)$$

where

$$dB'_i(t) := d\tilde{B}_i(t) + \varepsilon \sum_{j \neq i} (\delta t_j - \delta t_i) \frac{\beta}{(\lambda'_i(t) - \lambda'_j(t))^2} d\tilde{B}_j(t), \quad i = 1, \ldots, n \quad (1.22)$$

have same law as the original Brownians since the $\varepsilon$-term defines an infinitesimal rotation and white noise is invariant by rotation. Thus we have found

$$\delta_{\text{delay}} V'_i(t) = -\beta \sum_{j \neq i} \frac{\partial W}{\partial \lambda'_j}(t) \frac{\delta t_j - \delta t_i}{(\lambda'_i(t) - \lambda'_j(t))^2}. \quad (1.23)$$

### C. Change of measure

Let us finally discuss the change of measure on the trajectories – we shall return to this in section [4] with a modified, $\tau$-dependent measure.

Comparing the measure $Q(V')$, resp. $Q(V' + \delta V') \equiv Q(V') + \delta Q$ on the space of trajectories of (0.1) with confining forces $\{-V'(\lambda_i)\}$, resp. $\{-V'(\lambda_i) + \delta V'_i\}$, we see from (0.4) or rather from the rigorous Girsanov formula (0.5) that

$$\delta Q(\{\lambda_i\}) = Q(\{\lambda_i\}) \sum_i \int \delta V'_i dB_i(t)$$

$$= Q(\{\lambda_i\}) \sum_i \left( \int \delta V'_i d\lambda_i(t) + \int \delta V'_i \frac{\partial W}{\partial \lambda_i}(t) dt \right). \quad (1.24)$$

The main technical task in section 4 is to compute the terms appearing in (1.24) in the case of Schrödinger-Virasoro transformations, for which $\delta V'_i = \delta_{\text{simul}} V'_i$ simply.

## 2 Higher-order extension

We shall now construct the Lie algebra generated by the transformations (1.9).

**Definition 2.1** Let $\mathcal{F}$ be the space of functionals $\Phi = \Phi(t, \lambda)$ generated (as as vector space) by functionals of the form

$$\dot{a}(t) \int_0^t ds_1 \dot{a}_1(s_1) \lambda^{k_1}(s_1) \int_0^{s_1} ds_2 \dot{a}_2(s_2) \lambda^{k_2}(s_2) \cdots \int_0^{s_{p-1}} ds_p \dot{a}_p(s_p) \lambda^{k_p}(s_p) \quad (p \geq 0, k_1, \ldots, k_p \geq 0) \quad (2.1)$$

where $\dot{a}, \dot{a}_1, \ldots, \dot{a}_p$ are smooth functions of time.
Integrals

\[ \Phi(k_1,\ldots,k_p)(\hat{a}_1,\ldots,\hat{a}_p;\lambda)(t) := \int_0^t ds_1 \hat{a}_1(s_1)\lambda^{k_1}(s_1) \int_0^{s_1} ds_2 \hat{a}_2(s_2)\lambda^{k_2}(s_2) \cdots \int_0^{s_{p-1}} ds_p \hat{a}_p(s_p)\lambda^{k_p}(s_p) \]

(2.2)

are called \textit{iterated integrals}. As a prominent example, a completely factorized functional

\[ \prod_{i=1}^p \left( \int_0^t ds_i \hat{a}_i(s_i)\lambda^{k_i}(s_i) \right) \]

is a sum of \(p\) iterated integrals since (denoting by \(\Sigma_p\) the group of permutations of a set of \(p\) elements)

\[ \int_0^t ds_1 \int_0^t ds_2 \cdots \int_0^t ds_p (\cdots) = \sum_{\sigma \in \Sigma_p} \int_0^t ds_{\sigma(1)} \int_0^{s_{\sigma(1)}} ds_{\sigma(2)} \cdots \int_0^{s_{\sigma(p-1)}} ds_{\sigma(p)} (\cdots) . \]

(2.3)

The class \(\mathcal{F}\) is stable by multiplication because of the shuffle relation,

\[ \left[ \int_0^t ds_1 \int_0^t ds_2 \cdots \int_0^t ds_p (\cdots) \right] \left[ \int_0^t ds_1 \int_0^t ds_2 \cdots \int_0^t ds_q (\cdots) \right] = \]

\[ = \sum_{\sigma} \int_0^t ds_{\sigma(1)} \int_0^{s_{\sigma(1)}} ds_{\sigma(2)} \cdots \int_0^{s_{\sigma(p+q)} (\cdots)} \]

(2.4)

where \(\sigma\) ranges over shuffles of the lists \((1,\ldots,p)\), \((\bar{1},\ldots,\bar{q})\), i.e. over all re-orderings of the compound list \((1,\ldots,p,\bar{1},\ldots,\bar{q})\) preserving the orderings of the two sub-lists. In particular, the prefactor \(\hat{a}(t)\) in (2.1) may be interpreted as a multiplication by \(\int_0^t ds_0 \hat{a}(s_0)\) and absorbed into an iterated integral of order \(p+1\). Finally, ”polarizing” a \(p\)-th iterated integral by replacing \(\lambda\) with \(p\) independent copies \(\lambda_1,\ldots,\lambda_p\), namely,

\[ \Phi(k_1,\ldots,k_p)(\hat{a}_1,\ldots,\hat{a}_p;\lambda_1,\ldots,\lambda_p)(t) := \int_0^t ds_1 \hat{a}_1(s_1)\lambda_1^{k_1}(s_1) \int_0^{s_1} ds_2 \hat{a}_2(s_2)\lambda_2^{k_2}(s_2) \cdots \int_0^{s_{p-1}} ds_p \hat{a}_p(s_p)\lambda_p^{k_p}(s_p) \]

(2.5)

and permuting the order of integration by use of Fubini’s theorem, we obtain after some computations (see [15] or [16])

\[ \Phi(k_1,\ldots,k_p)(\hat{a}_1,\ldots,\hat{a}_p;\lambda_1,\ldots,\lambda_p)(t) = \]

\[ = \sum_{\sigma \in \Sigma_p} \varepsilon(\sigma) \int_0^t ds_1 \hat{a}_\sigma(1)(s_1)\lambda_{\sigma(1)}^{k_1}(s_1) \int_0^{s_1} ds_2 \hat{a}_\sigma(2)(s_2)\lambda_{\sigma(2)}^{k_2}(s_2) \cdots \int_0^{s_{p-1}} ds_p \hat{a}_\sigma(p)(s_p)\lambda_{\sigma(p)}^{k_p}(s_p) \]

(2.6)

for some universal coefficients \(\varepsilon(\sigma) \in \mathbb{Z}\). Alternatively, if \(\Phi(\lambda) \equiv \Phi(k_1,\ldots,k_p)(\hat{a}_1,\ldots,\hat{a}_p;\lambda)\), then we get

\[ \Phi(\lambda_1,\ldots,\lambda_p)(t) = \sum_{\sigma \in \Sigma_p} \varepsilon(\sigma)\Phi(\lambda_\sigma(1),\ldots,\lambda_\sigma(p))(t) \]

(2.7)

by defining \(\Phi(\lambda_1,\ldots,\lambda_p)(t)\) to be the polarization of \(\Phi(\lambda)(t) := \Phi(k_{\sigma(1)},\ldots,k_{\sigma(p)})(\hat{a}_{\sigma(1)},\ldots,\hat{a}_{\sigma(p)};\lambda)(t)\). This polarization trick will allow us later on to evaluate \(\frac{d}{d\varepsilon} \big|_{\varepsilon=0} \Phi(\lambda + \varepsilon\delta\lambda,\lambda,\ldots,\lambda)(t)\)

\[ \sum_{\sigma \in \Sigma_p} \varepsilon(\sigma) \frac{d}{d\varepsilon} \big|_{\varepsilon=0} \Phi(\lambda + \varepsilon\delta\lambda,\lambda,\ldots,\lambda)(t) . \]
Note that, by completing the tensor product, we may also choose to replace \( \hat{a}_1(s_1)\hat{a}_2(s_2) \cdots \hat{a}_p(s_p) \) with a general time coefficient \( g(s_1, \ldots, s_p) \) in Definition 2.1.

Now comes our main definition.

**Definition 2.2 (noise-preserving transformations)** Let \( \mathcal{F}_{NP} \) be the Lie algebra generated (as a vector space) by infinitesimal trajectory transformations of the type

\[
(\tilde{\delta}\lambda)(t) = \lambda(t)^{n+1}\Phi(t, \lambda) - \dot{\lambda}(t)\Psi(t, \lambda)
\]

with \( \Phi \in \mathcal{F} \) and

\[
\Psi(t, \lambda) = 2(n + 1) \int_0^t ds \lambda(s)^n\Phi(s, \lambda).
\]

Replacing as in the previous section the infinitesimal transformation \( \lambda \mapsto \lambda + \varepsilon \tilde{\delta}\lambda \) by the composition of \( \lambda \mapsto \lambda + \varepsilon \delta\lambda = \lambda + \varepsilon \lambda^{n+1}\Phi(\cdot, \lambda) \) with the time-transformation \( t \mapsto t + \varepsilon \Psi(t, \lambda) \), we see that (2.10) generalizes (1.7) in an obvious way to transformations depending on the past of the trajectory. For the sequel we note that:

\[
\frac{1}{2} \partial_t \Psi(t, \lambda) = (n + 1)\lambda(t)^n\Phi(t, \lambda) = \frac{\partial}{\partial \lambda(t)}\Phi(t, \lambda),
\]

where the partial derivative \( \frac{\partial}{\partial \lambda(t)} \) (to be distinguished from the functional derivative \( \delta \delta(\lambda(t)) \)) acts on the function \( \lambda(t)^{n+1} \) but vanishes on the integrated functional \( \Phi(t, \lambda) \).

We want to compute the Lie bracket of two noise-preserving transformations (2.9) and check that it is still a noise-preserving transformation. We start by specializing to the case when

\[
(\tilde{\delta}_i\lambda)(t) = \lambda(t)^{n_i+1}\dot{a}_i(t) - 2(n_i + 1)\dot{\lambda}(t)\int_0^t ds \dot{a}_i(s)\lambda(s), \quad i = 1, 2.
\]

Then

\[
([\tilde{\delta}_1, \tilde{\delta}_2]\lambda)(t) = \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}\bigg|_{\varepsilon_1 = \varepsilon_2 = 0} \left[ (\lambda + \varepsilon_1 \tilde{\delta}_1\lambda) + \varepsilon_2 \tilde{\delta}_2(\lambda + \tilde{\delta}_1\lambda) - (\lambda + \varepsilon_2 \tilde{\delta}_2\lambda) - \varepsilon_1 \tilde{\delta}_1(\lambda + \tilde{\delta}_2\lambda) \right]
\]

\[
= \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}\bigg|_{\varepsilon_1 = \varepsilon_2 = 0} \left[ \varepsilon_2 \left( \lambda(t) + \varepsilon_1 \lambda(t)^{n_1+1}\dot{a}_1(t) - 2\varepsilon_1(n_1 + 1)\dot{\lambda}(t)\int_0^t ds \dot{a}_1(s)\lambda^{n_1}(s) \right)^{n_2+1}\dot{a}_2(t) 
\right.
\]

\[
- 2(n_2 + 1)\dot{\lambda}(t) \left( \lambda(t) + \varepsilon_1 \lambda(t)^{n_1+1}\dot{a}_1(t) - 2\varepsilon_1(n_1 + 1)\dot{\lambda}(t)\int_0^t ds \dot{a}_1(s)\lambda^{n_1}(s) \right) \cdot 
\]

\[
\left. \cdot \int_0^t ds \dot{a}_2(s) \left( \lambda(s) + \varepsilon_1 \lambda(s)^{n_1+1}\dot{a}_1(s) - 2\varepsilon_1(n_1 + 1)\dot{\lambda}(s)\int_0^s ds' \dot{a}_1(s')\lambda^{n_1}(s') \right)^{n_2} \right] - (1 \leftrightarrow 2)
\]

(2.13)

Easy computations give

\[
([\tilde{\delta}_1, \tilde{\delta}_2]\lambda)(t) = \left[ F_I + F_{II} + (F_I + F_2 + F_3 + F) \dot{\lambda}(t) \right] (n_1, \dot{a}_1; n_2, \dot{a}_2) - 
\]

\[
\left[ F_I + F_{II} + (F_1 + F_2 + F_3 + F) \dot{\lambda}(t) \right] (n_2, \dot{a}_2; n_1, \dot{a}_1), \text{ with (abbreviating } F.(n_1, \dot{a}_1; n_2, \dot{a}_2) \text{ to } F_): \]

\[
F_I := \dot{a}_2(n_2 + 1)\lambda(t)^{n_1+n_2+1}\dot{a}_1(t);
\]

(2.14)
\[ F_{11} := -2(n_2 + 1)\lambda(t)^{n_1+1} \bar{a}_1(t) \int_0^t ds \dot{a}_2(s) \lambda(s)^{n_2}; \]  
\[ F_1 := -2\dot{a}_2(t)(n_2 + 1)(n_1 + 1)\lambda(t)^{n_2} \int_0^t ds \dot{a}_1(s) \lambda(s)^{n_1}; \]  
\[ F_2 := 2(n_2 + 1)(n_1 + 1) \left( \int_0^t ds \dot{a}_2(s) \lambda(s)^{n_2} \right) \lambda(t)^{n_1} \dot{a}_1(t); \]  
\[ F_3 := -2(n_2 + 1)n_2 \int_0^t ds \dot{a}_2(s) \lambda(s)^{n_1+n_2} \dot{a}_1(s); \]
and (integrating by parts)
\[ F := 4(n_2+1)n_2(n_1+1) \int_0^t ds \dot{\lambda}(s) \dot{a}_2(s) \lambda(s)^{n_2-1} \int_0^s ds' \dot{a}_1(s') \lambda(s')^{n_1} \equiv 4(n_1+1)(n_2+1)(F_4+F_5+F_6), \] 
with
\[ F_4 := \lambda(t)^{n_2} \dot{a}_2(t) \int_0^t ds' \dot{a}_1(s') \lambda(s')^{n_1}, \quad F_5 := -\int_0^t ds \lambda^{n_2}(s) \ddot{a}_2(s) \int_0^s ds' \dot{a}_1(s') \lambda^{n_1}(s') \]  
\[ F_6 := -\int_0^t ds \lambda^{n_2+n_1}(s) \dot{a}_2(s) \dot{a}_1(s) \]  

There is also a term \( F^\ast \) in \( \ddot{\lambda} \), but due to symmetry \( F^\ast(n_1, \dot{a}_1; n_2, \dot{a}_2) - F^\ast(n_2, \dot{a}_2; n_1, \dot{a}_1) = 0 \). For the same reason, the two \( F_6 \)-terms cancel. Finally, one remarks that \( F_1, F_4, F_2 \) are proportional and sum up to 0, while \( \frac{1}{2} \partial_t F_5 = -\frac{\partial}{\partial \lambda(t)} F_{11} \) and \( \frac{1}{2} \partial_t F_3 = -\frac{\partial}{\partial \lambda(t)} F_I \).

Concluding,
\[
([\delta_1, \delta_2] \lambda)(t) = \left( \lambda(t)^{n_1+n_2+1} \Phi^{12}_{[\delta_1, \delta_2]}(t, \lambda) - \ddot{\lambda}(t) \Psi^{12}_{[\delta_1, \delta_2]}(t, \lambda) \right) \\
+ \left( \lambda(t)^{n_1+1} \Phi^{2}_{[\delta_1, \delta_2]}(t, \lambda) - \ddot{\lambda}(t) \Psi^{2}_{[\delta_1, \delta_2]}(t, \lambda) \right) - \left( \lambda(t)^{n_2+1} \Phi^{1}_{[\delta_1, \delta_2]}(t, \lambda) - \ddot{\lambda}(t) \Psi^{1}_{[\delta_1, \delta_2]}(t, \lambda) \right)
\]  
(2.22)
is a noise-preserving transformation, with
\[ \Phi^{12}_{[\delta_1, \delta_2]}(t, \lambda) = (n_2 - n_1)\dot{a}_2(t)\dot{a}_1(t); \]
\[ \Phi^{2}_{[\delta_1, \delta_2]}(t, \lambda) = -2(n_2+1)\ddot{a}_1(t) \int_0^t ds \ddot{a}_2(s) \lambda(s)^{n_2}, \quad \Phi^{1}_{[\delta_1, \delta_2]}(t, \lambda) = -2(n_1+1)\ddot{a}_2(t) \int_0^t ds \ddot{a}_1(s) \lambda(s)^{n_1} \]  
(2.24)
and \( \Psi^{12}_{[\delta_1, \delta_2]}, \Psi^{2}_{[\delta_1, \delta_2]}, \Psi^{1}_{[\delta_1, \delta_2]} \) associated to \( \Phi^{12}_{[\delta_1, \delta_2]}, \Phi^{2}_{[\delta_1, \delta_2]}, \Phi^{1}_{[\delta_1, \delta_2]} \) by (2.10).

The above formulas for \( \Phi^{2}_{[\delta_1, \delta_2]}, \Phi^{1}_{[\delta_1, \delta_2]} \) show clearly the necessity to extend the set of noise-preserving transformations by allowing iterated integrals.

Consider now two general noise-preserving transformations \( \delta_i \) with
\[ \bar{\delta}_i \lambda(t) = \lambda(t)^{n_i+1} \Phi_i(t, \lambda) - 2(n_i + 1)\dot{\lambda}(t) \int_0^t ds_1 \lambda(s_1)^{n_i} \Phi_i(s_1, \lambda), \]  
(2.25)
with
\[
\Phi_1(t, \lambda) = \int_0^t ds_2 \dot{a}_2(s_2) \lambda(s_2)^{k_2} \int_0^{s_2} ds_3 \dot{a}_3(s_3) \lambda(s_3)^{k_3} \cdots \int_0^{s_{p-1}} ds_p \dot{a}_p(s_p) \lambda(s_p)^{k_p},
\]
\[
\Phi_2(t, \lambda) = \int_0^t ds_2 \dot{a}_2(s_2) \lambda(s_2)^{k_2} \int_0^{s_2} ds_3 \dot{b}_3(s_3) \lambda(s_3)^{k_3} \cdots \int_0^{s_{p'-1}} ds_{p'} \dot{b}_{p'}(s_{p'}) \lambda(s_{p'})^{k_{p'}}. \tag{2.26}
\]

Let \( \Phi_\sigma^1(\lambda_{\sigma(2)}, \ldots, \lambda)(t) = \int_0^t ds_2 \dot{a}_\sigma(s_2)(\lambda_{\sigma(2)}(s_2))^{k_{\sigma(2)}} B(\Phi_\sigma^1)(s_2, \lambda) \) and similarly, \( \Phi_\sigma^2(\lambda_{\sigma(2)}, \ldots, \lambda)(t) = \int_0^t ds_2 \dot{b}_\sigma(s_2)(\lambda_{\sigma(2)}(s_2))^{k_{\sigma(2)}} B(\Phi_\sigma^2)(s_2, \lambda) \). Then we get
\[
([\tilde{\delta}_1, \tilde{\delta}_2]\lambda)(t) = \left( \lambda(t)^{n_1+n_2+1} \Phi_{[\tilde{\delta}_1, \tilde{\delta}_2]}^1(t, \lambda) - \dot{\lambda}(t) \Psi_{[\tilde{\delta}_1, \tilde{\delta}_2]}^1(t, \lambda) \right) + \left( \lambda(t)^{n_1+1} \Phi_{[\tilde{\delta}_1, \tilde{\delta}_2]}^2(t, \lambda) - \dot{\lambda}(t) \Psi_{[\tilde{\delta}_1, \tilde{\delta}_2]}^1(t, \lambda) \right) + \left[ F_{I'} + F_{II'} + (F_{I'} + F_{2'}) \dot{\lambda}(t) \right] (n_1, \Phi_1; n_2, \Phi_2) - \left[ F_{I'} + F_{II'} + (F_{I'} + F_{2'}) \dot{\lambda}(t) \right] (n_2, \Phi_2; n_1, \Phi_1), \tag{2.27}
\]

where (substituting \( \dot{a}_\sigma(s, \lambda) \) to \( \dot{a}_\sigma(s) \) with respect to the previous computations)
\[
\Phi_{[\tilde{\delta}_1, \tilde{\delta}_2]}^1(t, \lambda) = (n_2 - n_1) \Phi_2(t, \lambda) \Phi_1(t, \lambda); \tag{2.28}
\]
\[
\Phi_{[\tilde{\delta}_1, \tilde{\delta}_2]}^2(t, \lambda) = -2(n_2 + 1) \partial_\tau \Phi_1(t, \lambda) \int_0^t ds \Phi_2(s, \lambda) \lambda(s)^{n_2}, \tag{2.29}
\]
\[
\Phi_{[\tilde{\delta}_1, \tilde{\delta}_2]}^3(t, \lambda) = -2(n_1 + 1) \partial_\tau \Phi_2(t, \lambda) \int_0^t ds \Phi_1(s, \lambda) \lambda(s)^{n_1} \tag{2.30}
\]
and \( F_{I'}(n_1, \Phi_1; n_2, \Phi_2) \equiv F_{I'}, F_{II'}(n_1, \Phi_1; n_2, \Phi_2) \equiv F_{II'}, F_{I'}(n_1, \Phi_1; n_2, \Phi_2) \equiv F_{I'}, F_{2'}(n_1, \Phi_1; n_2, \Phi_2) \equiv F_{2'} \) are new terms obtained by letting the derivative \( \frac{\partial}{\partial \lambda} |_{\lambda_1=0} \) act on the \( \lambda \)-dependent terms \( \Phi_2(t, \lambda), \Phi_2(s, \lambda) \) found instead of \( \dot{a}_\sigma(t) \), resp. \( \dot{a}_\sigma(s) \) in the straightforward generalization of \( \Phi_\sigma^1 \). The polarization trick \( (2.8) \) applied to \( \Phi_2 \) yields \( F_{\tau} = \sum_{\sigma \in \Sigma_{\mu-1}} \epsilon(\sigma) F_{\tau}(\sigma) \), with
\[
F_{I'}(\sigma) = \lambda(t)^{n_2+1} \int_0^t ds_2 \dot{b}_\sigma(s_2) k_{\sigma(2)}' \lambda(s_2)^{k_{\sigma(2)}'} \Phi_1(s_2, \lambda) B(\Phi_\sigma^2)(s_2, \lambda) \tag{2.31}
\]
\[
F_{II'}(\sigma) = -2(n_1+1) \lambda(t)^{n_2+1} \int_0^t ds_2 \dot{b}_\sigma(s_2) B(\Phi_\sigma^2)(s_2, \lambda) k_{\sigma(2)}' \lambda(s_2)^{k_{\sigma(2)}'} \lambda(s_2) \int_0^{s_2} ds_3 \lambda(s_3)^{n_1} \Phi_1(s_3, \lambda) \tag{2.32}
\]
\[
F_{I'}(\sigma) = -2(n_2+1) \int_0^t ds_1 \lambda(s_1)^{n_2} \int_0^{s_1} ds_2 \dot{b}_\sigma(s_2) B(\Phi_\sigma^2)(s_2, \lambda) k_{\sigma(2)}' \lambda(s_2)^{k_{\sigma(2)}'} \Phi_1(s_2, \lambda) \tag{2.33}
\]
\[
F_{2'}(\sigma) = 4(n_2+1)(n_1+1) \int_0^t ds_1 \lambda(s_1)^{n_2} \int_0^{s_1} ds_2 \dot{b}_\sigma(s_2) B(\Phi_\sigma^2)(s_2, \lambda) k_{\sigma(2)}' \lambda(s_2)^{k_{\sigma(2)}'-1} \lambda(s_2) \int_0^{s_2} ds_3 \lambda(s_3)^{n_1} \Phi_1(s_3, \lambda) \tag{2.34}
\]
One checks straightforwardly that \( \frac{1}{2} \partial_t F_{1'} = \frac{\partial}{\partial x^i} F_{1'} \) and \( \frac{1}{2} \partial_t F_{2'} = \frac{\partial}{\partial x^i} F_{1'1'} \). Hence \([\delta_1, \delta_2]\) is indeed a noise-preserving transformation.

The next task is obviously to express the above Lie brackets in some appropriate basis. The natural basis here is \( \{ L_{n_{a_1}, \ldots, n_{a_p}} \} \) where \( n \geq -1 \) (or more generally \( n \in \mathbb{Z} \)), \( n_1, \ldots, n_p \geq 0 \) are positive integers, and \( a, a_1, \ldots, a_k \) are chosen in some fixed basis of time functions (say \( t^i, l = 0, 1, \ldots \)). By definition, \( L_{n_{a_1}, \ldots, n_{a_p}} \) acts on the trajectories \( (\lambda(t))_{t \geq 0} \) as the infinitesimal transformation \( \lambda \mapsto \lambda + \varepsilon \delta \lambda \), with

\[
(\delta \lambda)(t) := \dot{\lambda}(t)\lambda(t)^{n+1} \Phi^{(n_1, \ldots, n_p)}(\dot{a}_1, \ldots, \dot{a}_p; \lambda)(t) - 2(n+1)\dot{\lambda}(t)\Phi^{(n,n_1, \ldots, n_p)}(\dot{a}_0, \dot{a}_1, \ldots, \dot{a}_p; \lambda)(t) \tag{2.35}
\]

(see noise-preserving condition in Definition 2.2) where \( \dot{a}_0 \) is the constant function \( \equiv 1 \).

We refrain from computing \([L_{n_{a_1}, \ldots, n_{a_p}}, L_{n'_{a'_1}, \ldots, n'_{a'_p}}]\) for general indices \( p, p' \) and correspond only elementary transformations \( L_{n_{a_1}} \) of the type \((2.12)\), corresponding to \( p, p' = 0 \).

Computing the bracket in the above basis yields

\[
[L_{n_{a_1}}, L_{n_{a_2}}] = (n_2 - n_1)F_{n_1+n_2}^{a_1,a_2} - 2 \{ (n_2 + 1)F_{n_1,n_2}^{a_1,a_2} - (n_1 + 1)F_{n_2,n_1}^{a_2,a_1} \}. \tag{2.36}
\]

The second and last terms in the above equation become very simple for \( n_1, n_2 = -1, 0 \) since \( L_{n,0} = L_n \), which explains why the Schrödinger-Virasoro algebra is closed under brackets. For \( n_1, n_2 \geq 1 \), on the other hand, we get iterated integrals of higher order and general formulas become very involved, exhibiting sums over shuffles and permutations. Let us simply remark at this point that the linear span of the \( L_{n,(-)} \) with \( n = -1, 0, 1 \) is a Lie subalgebra, just as the three-dimensional Lie algebra of finite conformal transformations span\( (L_{-1}, L_0, L_1) \) is.

### 3 The space-time geometry of the problem

We consider in this article space-time transformations such as \((1.3)\) whose form is dictated by the noise invariance condition \((1.7)\). Briefly said, these are obtained by integrating time-dependent infinitesimal conformal transformations and considering an associated transformation of the time parameter. As shown in the previous section, such transformations do not constitute a group for the composition of space-time transformations, except if one restricts to indices \( n = -1, 0 \), obtaining in this way the Schrödinger-Virasoro group \((13)\). Let us introduce the latter smoothly in a pleasant geometric framework.

It turns out that these transformations may be described in a coordinate-independent setting on an arbitrary manifold \( \mathbb{R}_+ \times \mathcal{M} \), where \( (\mathcal{M}, g) \) is any Riemannian manifold with its metric two-form \( g \). The applications we have in view in the context of random matrices are \( (\mathcal{M}, g) = (\mathbb{R}, d\lambda^2) \), resp. \( (\mathcal{M}, g) = (\mathbb{C} \cup \{ \infty \}, d\lambda d\bar{\lambda}) \), where \( \lambda \) is an eigenvalue of a Hermitian, resp. normal matrix. Let \( \Phi_t : \mathcal{M} \to \mathcal{M} (t \geq 0) \) be a \( C^1 \) family of conformal diffeomorphisms of \( \mathcal{M} \): by definition, the Jacobian matrix \( J(\Phi_t(m)) = \frac{\partial \Phi_t(m)}{\partial m} \) is scalar. Restricting to transformations \( (\Phi_t)_{t \geq 0} \) such that \( \Phi_0 = I_d \), we get a set \( C_0 = C_0^1(\mathbb{R}_+, \text{Conf}(\mathcal{M})) \). Consider now a world-line \((m(t), t) = (\phi_t(m), t) \) in \( \mathcal{M} \times \mathbb{R}_+ \). A non-relativistic particle living on this world-line has proper time

\[
T_m(t) = T(m, t) = \int_0^t ds \left| J(\Phi_s(m)) \right|^\alpha, \tag{3.1}
\]
where the dynamical scaling exponent $\alpha$ equals 2 if we want \(1.7\) to be satisfied. This relation defines an extended space-time diffeomorphism $\Phi : \mathcal{M} \times \mathbb{R}_+ \to \mathcal{M} \times \mathbb{R}_+$, $\Phi(m,t) = (T_m(t), \Phi_t(m))$ such that

$$
\Phi(m,0) = (m,0), \quad \frac{d}{dt} T(m,t) = \frac{(\Phi_t)_* (g_{ij}dx^i dx^j)(m)}{g_{ij}dx^i dx^j(m)} = |J(\Phi_t(m))|^2.
$$

(3.2)

Alternatively, $(\Phi_t)_{t \geq 0}$ is characterized by the velocity field $v(m,t) \equiv \frac{d\Phi_t(m,t)}{dt}$, a time-dependent vector field on $\mathcal{M}$ such that $v(\cdot, t)$ belongs for every fixed $t$ to the conformal Lie algebra $\text{conf}(\mathcal{M})$.

Keeping to the one-dimensional case, let us now introduce the Schrödinger-Virasoro group in this context. Let $\mathcal{M}$ be flat space $\mathbb{R}^d$ for some $d \geq 1$. Global conformal transformations are simply affine transformations, i.e. compositions of rotations, scale changes $x \mapsto ax$ ($a \in \mathbb{R}$) and translations, $x \mapsto x + v$ ($v \in \mathbb{R}^d$).

(i) Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a $C^1$-diffeomorphism with $\phi(0) = 0$. Define $a(t) \equiv \sqrt{\frac{1}{2}\dot{\phi}(t)}$ and $v(x,t) := a(t)x$. Then

$$
\Phi_t(x) = \left(e^{\int_0^t ds a(s)x}, \int_0^t ds e^{2\int_0^s ds' a(s')}x\right) = \left(\sqrt{\phi(t)x}, \phi(t)\right).
$$

(3.3)

(ii) Let $v(x,t) := \dot{b}(t)$ for some function $b : \mathbb{R}_+ \to \mathbb{R}^d$. Then

$$
\Phi_t(x) = \left(x + \int_0^t ds b(s), t\right).
$$

(3.4)

(iii) Let $v(x,t) := R(t)x$ where $R(t) \in \mathfrak{so}(d)$ is an antisymmetric matrix (infinitesimal rotation). Then

$$
\Phi_t(x) = \left(t, \text{exp}\left(\int_0^t ds R(s)\right)x\right)
$$

(3.5)

where $\text{exp}$ is the time-ordered exponential.

Composing these transformations one obtains a zero-mass representation of the so-called Schrödinger-Virasoro group. Considering infinitesimal transformations and restricting to the one-dimensional case for simplicity, one gets

$$
\frac{d}{d\varepsilon} F\left(e^{-\frac{\varepsilon}{2}\int_0^t ds \tilde{f}(s)x} \int_0^t ds e^{-\varepsilon \int_0^s ds' \tilde{f}(s')x}\right)\bigg|_{\varepsilon=0} = (X_f F)(x,t)
$$

(3.6)

and

$$
\frac{d}{d\varepsilon} F\left(x - \varepsilon g(t), t\right)\bigg|_{\varepsilon=0} = (Y_g F)(x,t)
$$

(3.7)

where the vector fields

$$
X_f := -f(t)\partial_t - \frac{1}{2} \dot{f}(t)x\partial_x, \quad Y_g := -g(t)\partial_x
$$

(3.8)
make up a zero mass representation of the Schrödinger-Virasoro algebra, namely,

\[ [X_f, X_g] = X_{f g - f \dot{g}} \]  
\[ [Y_f, X_g] = Y_{f g - \frac{1}{2} f \dot{g}}, \quad [Y_f, Y_g] = 0. \]  

One recognizes elementary transformations as in (2.12), with \( X_f \equiv \dot{L}^0_f \) and \( Y_g \equiv \dot{L}^0_g \) (see end of §2 for the notation).

4 Dynamical constraints

4.1 Introduction and statement of main result

If we search for a dynamical analogue of the equilibrium constraints \( L^n_{eq} Z[\tau] = 0 \), we must choose a dynamical functional replacing the partition function. Clearly a substitute for the equilibrium measure is the measure \( Q \) on trajectories. However (whatever its precise dependence on the parameters of the potential), \( Q \) is always normalized, viz. \( Q[1] = 1 \). Using the trivial identity \( \delta Q[1] = 0 \) does give non-trivial identities, but \( Q[1] \) is not a generating functional. Instead we consider a perturbed evolution,

\[ d\lambda_i = dB_i - \frac{\partial W[\tau]}{\partial \lambda_i} dt, \]  

where \( W[\tau](\{\lambda_i\}) \equiv W(\{\lambda_i\}) + \frac{1}{2} \sum_{k \geq 1} \tau_k \sum_i \lambda_i^k \). Copying the change-of-measure leading to (0.5), and throwing away the second-order term in \( \tau \) completing the square, \( e^{-\frac{1}{4} \sum_i (\partial \lambda_i (\sum_{k \geq 0} \tau_k \lambda_i^k))^2} \), we get a new measure

\[ Q[\tau](\{\lambda_i\}) = Q(\{\lambda_i\}) e^{-S[\tau]}, \]  

where

\[ S[\tau] \equiv \sum_{k \geq 0} \int S_k(t) \tau_k(t) dt, \]  

\[ S_k(t) = \pi_k(t) + \left( \frac{\beta}{2} - 1 \right) k(k-1) \pi_{k-2}(t) + k \sum_{l \geq 0} b_l \pi_{l+k-1}(t) - \frac{\beta}{2} k \sum_{q=0}^{k-2} \pi_q(t) \pi_{k-2-q}(t). \]
The action $S[\tau]$ is the sum of a linear term (linearized action)

$$S^{\text{lin}}[\tau] \equiv \sum_{k \geq 0} \int S_k^{\text{lin}}(t) \pi_k(t) \, dt, \quad S_k^{\text{lin}}(t) = \dot{\pi}_k(t) + \left(\frac{\beta}{2} - 1\right)k(k-1)\pi_{k-2}(t) + k \sum_{l \geq 0} b_l \pi_{l+k-1}(t)$$

(4.8)

and of a quadratic term, $S^{\text{quad}}[\tau]$.

Throwing this quadratic term in turn, we finally get a linearized functional.

**Definition 4.1 (generating functional)** Let

$$Q^{\text{lin}}[\tau](\{\lambda_i\}) := Q(\{\lambda_i\}) e^{-S^{\text{lin}}[\tau]}$$

(4.9)

and

$$Z^{\text{lin}}[\tau] := Q^{\text{lin}}[\tau][1] = \int dQ^{\text{lin}}[\tau](\{\lambda_i\}).$$

(4.10)

For simplicity we shall from now on write $\langle \cdot \rangle_0$ instead of $Q(\cdot)$, viz. $\langle \cdot \rangle_\tau$ instead of $Q^{\text{lin}}[\tau](\cdot)$.

Differentiating with respect to $\tau$ yields

$$\frac{\delta}{\delta \tau_k(t)} e^{-S[\tau]} = -e^{-S[\tau]} \left( \dot{\pi}_k(t) + \left(\frac{\beta}{2} - 1\right)k(k-1)\pi_{k-2}(t) + k \sum_{l \geq 0} b_l \pi_{l+k-1}(t) + \frac{\beta}{2} k \sum_{q=0}^{k-2} \pi_q(t) \pi_{k-2-q}(t) \right),$$

(4.11)

formally,

$$\dot{\pi}_k(t) + \left(\frac{\beta}{2} - 1\right)k(k-1)\pi_{k-2}(t) + k \sum_{l \geq 0} b_l \pi_{l+k-1}(t) - k \frac{\beta}{2} \sum_{q=0}^{k-2} \pi_q(t) \pi_{k-2-q}(t) = -\frac{\delta}{\delta \tau_k(t)}$$

(4.12)

in average (i.e. when inserted into an expectation value).

We need to invert the linearized version of this equation,

$$\dot{\pi}_k(t) + \left(\frac{\beta}{2} - 1\right)k(k-1)\pi_{k-2}(t) + k \sum_{l \geq 1} b_l \pi_{l+k-1}(t) = -\frac{\delta}{\delta \tau_k(t)}, \quad k \geq 1$$

(4.13)

with solution

$$\pi_k(t) = -\sum_{l} \int_{0}^{t} ds K_{kl}(t-s) \frac{\delta}{\delta \tau_l(s)},$$

(4.14)

or, in a mixed operator/convolutional notation, $\pi_k(t) = -(K * \frac{\delta}{\delta \tau}(t))$. Note that $K(t-s) = (K_{kl}(t-s))_{kl}$ is an upper-triangular matrix, so the sum in (4.14) really ranges over $l \geq k$.

At this point we introduce the generating series

$$\frac{\partial}{\partial \tau}(z, t) \equiv \sum_{k \geq 1} z^{-k-1} \frac{\partial}{\partial \tau_k(t)}, \quad \pi(z, t) = \sum_{k \geq 0} \pi_k(t) z^{-k-1},$$

(4.15)
\[\tau(z,t) = \sum_{k \geq 1} k\tau_k(t)z^{k-1}, \quad b(z) = \sum_{l \geq 1} b_l z^l \]  \hfill (4.16)

Note that the zero mode \(\pi_0(t)\) of the field \(\pi(z,t)\) is a constant, \(\pi_0(t) = N\). In somewhat abstract terms, we use the canonical splitting of the formal series algebra \(\mathbb{C}[[z,z^{-1}]]\) into \(\mathcal{A}_+ \oplus \mathcal{A}_- \equiv \mathbb{C}[[z]] \oplus z^{-1}\mathbb{C}[[z^{-1}]]\); each of these two subalgebras is isotropic for the scalar product

\[[u,v] = \int_C u(z)v(z)dz := \frac{1}{2\pi i} \int_C u(z)v(z)dz \]  \hfill (4.17)

given by the residue integral, where \(C\) is any counterclockwise simple contour circling around 0, and \(\mathcal{A}_+ = \mathcal{A}_+^*\). Then \(\partial/\partial \tau(z), \pi(z) \in \mathcal{A}_-\), while \(b(z) \in \mathcal{A}_+\). We write quite generally

\[u_-(z) = (\mathcal{P}_- u)(z) := \sum_{n \leq -1} u_n z^n, \quad u_+(z) = (\mathcal{P}_+ u)(z) := \sum_{n \geq 0} u_n z^n \]  \hfill (4.18)

if \(u(z) = \sum_{n \in \mathbb{Z}} u_n z^n \in \mathbb{C}[[z,z^{-1}]]\) (see Appendix B). Also, we write : \(u(z) = v(z) \mod \mathcal{A}_+\) if \(\mathcal{P}_- u = \mathcal{P}_- v\). With these notations, letting \(\hat{\pi} = \partial \hat{\pi}/\partial \tau, \pi' = \partial \pi/\partial z\), we see that (4.13) is equivalent to the following

**Definition 4.2 (equation of motion)**

\[\hat{\pi}(z) + (\frac{\beta}{2} - 1)\pi'(z) - (bn)'(z) = -\hat{\partial}/\hat{\partial \tau}(z) \mod \mathcal{A}_+. \]  \hfill (4.19)

As mentioned in the Introduction, the original non-linearized equation of motion (4.12) contributes to (4.19) an additive term \((\pi^2)'(z)\) which turns it into a Burgers equation, but we shall not pursue along this road.

The solution of (4.19) is

\[\pi(z,t) \equiv -z^{-1} \int_0^t ds \oint dw K_{t-s}(z^{-1},w)\hat{\partial}/\hat{\partial \tau}(w,s)\]  \hfill (4.20)

or more schematically,

\[\pi(z,t) \equiv -(K \ast \hat{\partial}/\hat{\partial \tau})(z,t), \]  \hfill (4.21)

where (comparing with (4.14))

\[K_{t-s}(z^{-1},w) = \sum_{k,l \geq 0} z^{-k} w^l K_{kl}(t-s). \]  \hfill (4.22)

When \(t \to 0\) we must get \(K_t(z^{-1},w) \to \frac{1}{1-w/z}\), so that \(z^{-1} \oint dw K_0(z^{-1},w)f(w^{-1}) = f(z^{-1})\) for every \(f = f(w^{-1}) = a_0 + a_1 w^{-1} + a_2 w^{-2} + \ldots\).

When \(\beta = 2\), the equation of motion (4.19) is a transport equation which may be solved explicitly in terms of the characteristics; as proved in Appendix B,

\[K_t(z^{-1},w) = \frac{1}{1 - w(t)/z} \]  \hfill (4.23)

where \(w(t) \in \mathbb{C}[[w]]\) is the solution at time \(t\) of the ordinary differential equation \(\dot{w}_t = -b(w_t)\) with initial condition \(w_0 \equiv w\). When \(\beta \neq 2\), semi-explicit but complicated formulas for \(K_t\)
may be obtained by composing the semi-group generated by $\partial_z^2 : \pi \mapsto -\pi''$ with the semi-group generated by the transport equation $B : \pi \mapsto (b\pi)'(z)$ through the use of Trotter’s formula, $\exp t \left( -\frac{\beta}{n} \partial_z^2 + B \right) = \lim_{n \to \infty} \left( \exp \left( -\frac{t}{n} \left( \frac{\beta}{2} - 1 \right) \partial_z^2 \right) \exp \left( \frac{t}{n} B \right) \right)^n$, resulting in a Feynman-Kac type formula which looks awful. Hence we do not write it down, but the reader should be able to reproduce it by looking at the computations in Appendix B.

As mentioned in the introduction, we see that n-point functions of the functions $\{\pi_k(t)\}$ may be obtained by differentiating $Z$, 

$$\left\langle \left( \int dt f_1(t)\pi_k_1(t) \right) \cdots \left( \int dt f_p(t)\pi_k_p(t) \right) \right\rangle_0 = \prod_{q=1}^p \left( -\int dt f_q(t)(K \ast \partial/\partial\tau)_{q}(t) \right) Z_{lin}[\tau]_{|\tau=0}. \quad (4.24)$$

The kernel $K$ satisfies the semi-group properties,

$$\sum_{l \geq 0} K_{kl}(t - t')K_{lm}(t' - t'') = K_{lm}(t - t''), \quad t > t' > t'' \tag{4.25}$$

or equivalently

$$\oint dz' K_{t-t'}(z^{-1}, z')K_{t'-t''}((z')^{-1}, z'') = K_{t-t''}(z^{-1}, z''). \quad (4.26)$$

letting $t' \to t$ or $t' \to t''$ and differentiating we get the following formulas,

$$\frac{\partial}{\partial t}K_{km}(t) = -\sum_{l \geq 0} kb_{l-k+1}K_{lm}(t) = -\sum_{l \geq 0} K_{kl}(t)b_{m-l+1} \tag{4.27}$$

or equivalently

$$\frac{\partial}{\partial t}K_t(z^{-1}, z'') = -\oint \frac{dz'}{z'} b(z') \frac{1}{(1-z'/z)^2} K_t((z')^{-1}, z'') \tag{4.28}$$

$$= -\oint \frac{dz'}{z'} K_t(z^{-1}, z') \frac{b(z'')}{z'} \frac{1}{(1-z''/z')^2}. \tag{4.29}$$

In the last two equalities we used the following expression for the generator, \(\sum_{k,l \geq 0} z^{-k}w^lkb_{l-k+1} = \frac{b(w)}{z} \frac{1}{(1-w/z)^2}\). Following the probabilists’ convention we shall refer to (4.28), resp. (4.29) as the forward, resp. backward Kolmogorov equation.

We now define the two bosonic fields.

**Definition 4.3 (static and dynamic free bosons)**

(i) (static free boson) Let, for $k \geq 1$,

$$\hat{\phi}_{-k}(t) := \beta^{-1/2}k\tau_k(t), \quad \hat{\phi}_k(t) := \beta^{1/2}\frac{\delta}{\delta\tau_k(t)} \quad (4.30)$$

and $\hat{\phi}_0 := 0$,

$$\hat{\phi}(z, t) := \sum_{k \in \mathbb{Z}} \hat{\phi}_k(t)z^{-k-1}. \quad (4.31)$$
(ii) (dynamic free boson) Let, for \( k \geq 1 \),
\[
\hat{\psi}_k(t) := \beta^{-1/2} k \tau_k(t), \quad \hat{\psi}_0(t) := \beta^{1/2} (K * \frac{\delta}{\delta \tau})_k(t) \tag{4.32}
\]
and \( \hat{\psi}_0 := -\beta^{1/2} N \),
\[
\hat{\psi}(z, t) := \sum_{k \in \mathbb{Z}} \hat{\psi}_k z^{-k-1}(t). \tag{4.33}
\]
Since \( (K * \frac{\delta}{\delta \tau})_k(t) \) identifies with \(-\pi_k(t)\) for \( k \geq 1 \), and \( \pi_0(t) = \sum_i 1 \equiv N \), the definition of the zero mode \( \hat{\psi}_0 \) is coherent. Then
\[
\hat{\psi}(z, t) \equiv \hat{\psi}_+(t, z) + \hat{\psi}_-(t, z) \tag{4.34}
\]
where
\[
\hat{\psi}_+(z, t) = \beta^{-\frac{k}{2}} \sum_{k \geq 1} k \tau_k(t) z^{-k-1} = \beta^{-1/2} \tau(z, t) \in \mathcal{A}_+, \tag{4.35}
\]
\[
\hat{\psi}_-(z, t) = -\beta^{1/2} \left\{ N z^{-1} - \sum_{k \geq 1} (K * \frac{\delta}{\delta \tau})_k(t) z^{-k-1} \right\} = \beta^{1/2} \left\{ -N z^{-1} + (K * \frac{\partial}{\partial \tau})(z, t) \right\} \tag{4.36}
\]
\[
\hat{\phi}(z, t) \equiv \hat{\phi}_+(t, z) + \hat{\phi}_-(z, t) \tag{4.37}
\]
where
\[
\hat{\phi}_+ \equiv \hat{\psi}_+, \quad \hat{\phi}_-(z, t) = \beta^{1/2} \frac{\partial}{\partial \tau}(z, t). \tag{4.38}
\]
For further use we write down a formula regarding the time-derivative of \( \hat{\psi} \),
\[
\partial_t(\hat{\psi}_-(z, t)) = \beta^{1/2} \left\{ \frac{\partial}{\partial \tau}(z, t) + \frac{1}{z} \int_0^t \int d\zeta K_{t-s}(z^{-1}, \zeta) \frac{\partial}{\partial \tau}(\zeta, s) \right\}. \tag{4.39}
\]
Alternatively, from (5.25),
\[
\partial_t(\hat{\psi}_-(z, t)) = \hat{\phi}_-(z, t) - \beta^{1/2} \int_0^t \int d\zeta b(\zeta) G_{t-s}^+(z^{-1}, \zeta) \frac{\partial}{\partial \tau}(\zeta, s). \tag{4.40}
\]
Taking commutators, we get:

**Definition 4.4 (Dynamic free boson algebra)** Let \( G^+(t, z^{-1}, t'; w) = G^+_{t-w}(z^{-1}, w) \) be the retarded propagator,
\[
G^+_{t-w}(z^{-1}, w) := \text{1}_{t>\nu} \frac{1}{z} \frac{\partial}{\partial w} K_{t-w}(z^{-1}, w), \tag{4.41}
\]
\( G^-(t', z; t, w) := G^+(t, w^{-1}, t', z) \) the advanced propagator, and
\[
G_0^+(z^{-1}, w) := \lim_{t \to 0, t>0} G^+_t(z^{-1}, w) = \frac{1}{z^2(1-w/z)^2}. \tag{4.42}
\]
\[
G_0^-(z, w^{-1}) := \lim_{t \to 0, t < 0} G_t^-(z, w^{-1}) = \frac{1}{w^2(1 - z/w)^2}. \tag{4.43}
\]

Then
\[
[\hat{\phi}(z, t), \hat{\phi}(w, t')] = \delta(t - t') \left\{ G_0^+(z^{-1}, w) - G_0^-(z, w^{-1}) \right\} \tag{4.44}
\]
\[
[\hat{\psi}(z, t), \hat{\psi}(w, t')] = G_{t-t'}^+(z^{-1}, w) - G_{t-t'}^-(z, w^{-1}) \tag{4.45}
\]
\[
[\hat{\psi}(z, t), \hat{\phi}(w, t')] = G_{t-t'}^+(z^{-1}, w) - \delta(t - t')G_0^-(z, w^{-1}). \tag{4.46}
\]

Let us give a sketchy proof. First, if \(k, l \geq 0\),
\[
[\hat{\psi}_k(t), \hat{\psi}_{-l}(t')] = 1_{t > t'} lK_{kl}(t - t'). \tag{4.47}
\]

Summing over Fourier components yields for \(t > t'\)
\[
[\hat{\psi}(z, t), \hat{\phi}(w, t')] = [\hat{\psi}(z, t), \hat{\psi}(w, t')] = \sum_{k,l \geq 0} lw^{-1}z^{-k-1}K_{kl}(t - t') = G_{t-t'}^+(z^{-1}, w) \tag{4.48}
\]

Other commutators either vanish identically or involve a \(\delta\)-function.

**Example (Hermite polynomials).** Assume \(b_1 = 1/\sigma^2\) and \(b_1 = 0, i \neq 1\). Then the equation \(\Box \pi + (\frac{\beta}{2} - 1)\pi'' - \hat{\phi} z^0 = \frac{1}{\sigma^2}(\pi + z \frac{dw}{ds}) - \frac{\partial}{\partial \tau}(z)\).

Consider first the case \(\beta = 2\). The equation is diagonal when written in Fourier modes, \(\hat{\pi}_k = -k \hat{\pi}_k - \frac{\partial}{\partial n_k}\). The solution is
\[
\pi(z, t) = \int_0^t ds \sum_{k \geq 0} e^{-k(t-s)/\sigma^2} \frac{\partial}{\partial \tau(s)} z^{-k-1} = \int_0^t ds \int dw \sum_{k \geq 0} (e^{-(t-s)/\sigma^2})^{w/z} \frac{\partial}{\partial \tau}(w, s).
\]

Hence for \(t \geq 0\)
\[
K_t(z^{-1}, w) = \frac{1}{1 - e^{-t/\sigma^2} w/z}, \quad G_t^+(z^{-1}, w) = 1_{t > 0} \frac{e^{-t/\sigma^2}}{z^2(1 - e^{-t/\sigma^2} w/z)^2}. \tag{4.49}
\]

Note that \(G_t^+(z^{-1}, w) = \frac{1}{z^2(1 - w/z)^2}\); one retrieves the equal-time, equilibrium OPE \(\hat{\phi}(z, t)\hat{\phi}(w, t') \sim \delta(t - t') \frac{1}{(z - w)^2}\).

When \(\beta \neq 2\), an explicit expression for \(K_t\), loosely related to the Mehler kernel, is given in Appendix B, see \([5, 13]\).

We may now state our main result. We denote by \(C_c^\infty(\mathbb{R}^+_z)\) the space of smooth functions with compact support \(\subset (0, +\infty)\). (In particular, a function in \(C_c^\infty(\mathbb{R}^+_z)\) vanishes to arbitrary order at 0).
Theorem 4.1 (dynamical constraints) Let, for \( a \in C^\infty_c (\mathbb{R}^*_+) \),
\[
L^a_{-1} := \beta^{-1/2} \int dt \left\{ \ddot{a}(t) \int z \hat{\psi}_t(z) \, dz - a(t) \int ((\frac{\beta}{2} - 1)b'' + b'b)\hat{\psi}_t(z) \, dz \right\} 
- \int dt \left[ \frac{1}{2} \dddot{a}(t) \int \hat{\psi}_t(z)^2 \, dz - \frac{1}{2} a(t) \left\{ \int b'(z) \cdot (\dot{\hat{\psi}}(z,t))^2 \, dz + \int z \cdot (\dot{\dot{\phi}}(z,t))^2 \, dz \right\} \right]
\]
and
\[
L^0_0 := -a(t) \partial_t + \frac{1}{2} \beta^{-1/2} \int dt \left\{ \frac{1}{2} \dddot{a}(t) \int z^2 \hat{\psi}_t(z) \, dz - \dot{a}(t) \int ((\frac{\beta}{2} - 1)(zb(z))' + (zb(z))'b(z))\hat{\psi}_t(z) \, dz \right\} 
- \frac{1}{2} \int dt \left[ \frac{1}{2} \dddot{a}(t) \int \hat{\psi}_t(z)^2 \, dz - \ddot{a}(t) \left\{ \int (zb(z))^' \cdot (\dot{\hat{\psi}}(z,t))^2 \, dz + \int z \cdot (\ddot{\phi}(z,t))^2 \, dz \right\} \right].
\]

(4.50)

Then
\[
L^a_n Z[\tau] = 0, \quad n = -1, 0.
\]

(4.52)

The proof is elementary but somewhat lengthy. It will take up the rest of the section. As it happens, see (1.9), the Schrödinger-Virasoro transformation \( Y_a \),
\[
\delta \lambda(t) = -\dot{a}(t), \quad \delta t = 0
\]
generates \( L^a_{-1} \), while the transformation \( X_a \),
\[
\delta \lambda(t) = -\lambda \dot{a}(t), \quad \delta t = -a(t)
\]
generates \( L^a_0 \).

The action of the time derivation \( a(t) \partial_t \) is made explicit in the Appendix.

4.2 Preliminary computations

We show here that the variation \( \delta Q^{lin}[\tau] := \delta_n Q^{lin}[\tau] \) of the action under the change of coordinates (1.9) is the sum of four terms, \( \delta_{(i)} Q^{lin}[\tau], \ldots, \delta_{(iv)} Q^{lin}[\tau] \) which we evaluate one by one.

First, a straightforward extension of Girsanov’s formula yields
\[
\delta Q^{lin}[\tau] = Q^{lin}[\tau] \left( \int \delta V'_i(t) \, d\lambda_i(t) + \int \delta V'_i(t) \frac{\partial W}{\partial \lambda_i}(t) \, dt \right) - Q^{lin}[\tau] \delta \left( \sum_{k=0}^{+\infty} \tau_k(t)S_k(t) \right)
= \delta_{(i)} Q^{lin}[\tau] + \delta_{(ii)} Q^{lin}[\tau] + \delta_{(iii)} Q^{lin}[\tau] + \delta_{(iv)} Q^{lin}[\tau]
\]
with
\[
\delta_{(i)} Q^{lin}[\tau] = Q^{lin}[\tau] \int \delta V'_i(t) \, d\lambda_i(t), \quad \delta_{(ii)} Q^{lin}[\tau] = Q^{lin}[\tau] \int \delta V'_i(t) \frac{\partial W}{\partial \lambda_i}(t) \, dt.
\]
(4.53)
\[ \delta_{(iii)}Q^{lin}[\tau] = -Q^{lin}[\tau] \int \delta V^i(t) \sum_{j \neq i} \frac{\beta}{\lambda_i(t) - \lambda_j(t)} \, dt, \quad \delta_{(iv)}Q^{lin}[\tau] = -Q^{lin}[\tau] \delta \left( \sum_{k=0}^{+\infty} \tau_k(t) S_k(t) \right). \] (4.55)

We consider separately each of these four terms.

(i) The quantity \( \delta V^i \) is given by the total variation formula as a sum of four terms. Though the third and the fourth one vanish for \( n = -1, 0 \), we evaluate them to some point and shall use those computations in another article. By Itô’s formula,

\[ \lambda_i^{n+1}(t) d\lambda_i(t) = \frac{1}{n+2} d(\lambda_i^{n+2})(t) - (n+1) \lambda_i^n(t) dt \] (4.56)

hence (by integration by parts)

\[ \sum_i \int \dot{a}(t) \lambda_i^{n+1}(t) d\lambda_i(t) = -\frac{1}{n+2} \int \ddot{a}(t) \pi_{n+2}(t) dt - (n+1) \int \dot{a}(t) \pi_n(t) dt \]

\[ = -\frac{1}{n+2} \ddot{a}(t) \int \pi(z) z^{n+2} \, dz + \dot{a}(t) \int \pi'(z) z^{n+1} \, dz. \] (4.57)

The second and fourth terms are similar,

\[ \sum_i \int \dot{a} \left[ \sum_{l \geq 0} b_l(n+l+1) \lambda_i^{n+l} - \left( \frac{\beta}{2} - 1 \right)(n+1) n \lambda_i^{n-1} \right] d\lambda_i \]

\[ = -\int \dot{a} \left[ \sum_{l \geq 0} b_l \pi_{n+l+1} - \left( \frac{\beta}{2} - 1 \right)(n+1) \pi_n \right] dt + \]

\[ -\int \dot{a} \left[ \sum_{l \geq 0} b_l(n+l+1) \pi_{n+l+1} - \left( \frac{\beta}{2} - 1 \right)(n+1) n \pi_{n-1} \right] dt \]

\[ = -\dot{a} \left( \int b(z) \pi(z) z^{n+1} \, dz + \left( \frac{\beta}{2} - 1 \right) \int z^{n+1} \pi'(z) \, dz \right) \]

\[ -\dot{a} \left( \int b(z) \pi''(z) z^{n+1} \, dz + \left( \frac{\beta}{2} - 1 \right) \int z^{n+1} \pi''(z) \, dz \right) \] (4.58)

For the third term, we remark similarly that

\[ d \left( \sum_{q=0}^{n-1} \pi_q \pi_{n-q} \right) = 2 \sum_{q=0}^{n-1} \sum_i d\lambda_i (q+1) \lambda_i^q \pi_{n-q-1} + \sum_{q=0}^{n-1} \sum_i \frac{\partial^2}{\partial \lambda_i^2} (\pi_q \pi_{n-q}) dt. \] (4.59)

We compute

\[ \sum_{q=0}^{n-1} \sum_i \frac{\partial^2}{\partial \lambda_i^2} (\pi_q \pi_{n-q}) = 2 \left( \sum_{q=1}^{n-1} q(q-1) \pi_{q-2} \pi_{n-q} + q(n-q) \pi_{q-1} \pi_{n-q-1} \right) \]

\[ = 2(n+1) \sum_{q=1}^{n-1} (q-1) \pi_{q-2} \pi_{n-q} \] (4.60)
Hence

\[ \beta \sum_{q=0}^{n-1} \int \dot{a}(t)(q + 1) \lambda_i^q \pi_{n-q} d\lambda_i(t) = -\frac{\beta}{2} \int \dot{a}(t) \sum_{q=0}^{n-1} \pi_q(t) \pi_{n-q}(t) dt - \\
-\beta(n + 1) \int \dot{a}(t) \sum_{q=1}^{n-1} (q - 1) \pi_{q-2}(t) \pi_{n-q}(t) dt \\
= -\frac{\beta}{2} \int \dot{a}(t) \pi^2(z) z^{n+1} dz - \beta \int (\pi(z)^2) z^{n+1} dz. \]  \hspace{1cm} (4.61)

(ii) We now evaluate \( \int dt \delta V_i' V_{ij}'(\lambda_i) \). It is a linear combination of terms of the type (with \( \phi(t) = \dot{a}(t) \) or \( \ddot{a}(t) \)) \( b_i \int dt \phi \sum_i \lambda_i^{m+l} = b_i \int dt \phi \pi_{m+l} \) and \( b_i \int dt \phi \sum_i \lambda_i^{k+l} \pi_{q-k} = b_i \int dt \phi \pi_{q+l} \). Summing up all four terms, we get a \( \beta \)-independent contribution,

\[ \ddot{a} \sum_{l \geq 0} b_l \pi_{l+n+1} + \dot{a} \sum_{l,l' \geq 0} b_l b_{l'} (l' + n + 1) \pi_{l+l'+n} + \beta \dot{a} \sum_{l \geq 0} b_l \sum_{q=0}^{n-1} (q + 1) \pi_{n+l-1} \\
- \ddot{a} \sum_{l \geq 0} b_l \cdot (\frac{\beta}{2} - 1)(n + 1) n \pi_{n+l-1} \\
= \ddot{a} \sum_{l \geq 0} b_l \pi_{l+n+1} + \dot{a} \sum_{l,l' \geq 0} b_l b_{l'} (l' + n + 1) \pi_{l+l'+n} + \frac{\beta}{2} (n + 1) n \sum_{l \geq 0} b_l \pi_{n+l-1} \\
= \ddot{a} \int b(z) \pi(z) z^{n+1} dz + \dot{a} \left( - \int b(z) (b \pi)'(z) z^{n+1} dz + \frac{\beta}{2} \int (b \pi)'(z) z^{n+1} dz \right). \]  \hspace{1cm} (4.62)

Comparing (4.62) with (4.58), we see that the first terms sum up to zero.

(iii) We now evaluate \( -\sum_{i,j \neq i} \int dt \delta V_i' \frac{\beta}{\pi_i - \pi_{i'}} \). First

\[ -\frac{\beta}{2} \sum_{i,j \neq i} \int dt \phi \frac{\lambda_i^m - \lambda_j^m}{\lambda_i - \lambda_j} = -\frac{\beta}{2} \int dt \phi(t) \sum_{q=0}^{m-1} (\pi_q(t) \pi_{m-q}(t) - \pi_{m-1}(t)) \\
= \frac{\beta}{2} \int dt \phi(t) \left( m \pi_{m-1}(t) - \sum_{q=0}^{m-1} \pi_q(t) \pi_{m-1-q}(t) \right) \right); \]  \hspace{1cm} (4.63)

summing up the contributions of the terms 1,2,4 in (1.16), we get
Finally, we must compute the variation

\[
\frac{\beta}{2} \dot{a} \left( (n + 1)\pi_n - \sum_{q=0}^{n} \pi_q \pi_{n-q} \right) + \frac{\beta}{2} \dot{a} \sum_{l \geq 0} b_l (l + n + 1) \left( (l + n)\pi_{l+n-1} - \sum_{q=0}^{l+n-1} \pi_q \pi_{l+n-1-q} \right) - \\
- \frac{\beta}{2} \dot{a} \cdot \left( \frac{\beta}{2} - 1 \right)(n+1) \left( (n - 1)\pi_{n-2} - \sum_{q=0}^{n-2} \pi_q \pi_{n-2-q} \right)
\]

\[
= -\frac{\beta}{2} \dot{a} \int (\pi'(z) + \pi^2(z))z^{n+1} \, dz + \frac{\beta}{2} \dot{a} \left( \oint b(z)(\pi''(z) + (\pi^2(z))')z^{n+1} \, dz \\
+ \frac{\beta}{2} \left( \frac{\beta}{2} - 1 \right) \oint (\pi'''(z) + (\pi^2(z))'')z^{n+1} \, dz \right).
\]

The third term contributes

\[
- \frac{\beta^2}{2} \int dt \sum_{q=1}^{l+n-1} (q+1) \sum_{i,j,i' \neq i} \frac{(\lambda_i^q - \lambda_i'^q)\lambda_j^{n-1-q}}{\lambda_i - \lambda_i'}
\]

\[
= -\frac{\beta^2}{2} \int dt \dot{\alpha} \left\{ 2 \sum_{q+r+s=n-2} \sum_{p=0}^{q-1} \sum_{i,j,i' \neq i} \lambda_i^p \lambda_i'^{q-1-p} \lambda_j^{n-1-q} \right\}
\]

\[
= -\frac{\beta^2}{2} \int dt \dot{\alpha} \left\{ -2 \oint \pi'(z)\pi^2(z)z^{n+1} \, dz - \oint \pi''(z)\pi(z)z^{n+1} \, dz \right\},
\]

including a term of order 3 (which does not appear for \(n \leq 1\) however).

(iv) Finally, we must compute the variation \(-\delta_n \left( \int_0^{+\infty} dt \sum_{k=0}^{+\infty} \tau_k(t)S_k(t) \right) = -\delta S^\text{lin} \) under the change of coordinates [14.9] for \(n = -1, 0\). First we have

\[
\delta_n(\pi_k(t)) = -\dot{\alpha}(t)k\pi_{k+n}(t), \quad \delta_n(\tilde{\pi}_k) = \frac{d}{dt}(\delta_n(\pi_k)) = -\ddot{\alpha}k\pi_{k+n} - \dddot{\alpha}k\pi_{k+n}.
\]

Recall we have defined \(\tau(z) \equiv \sum_{k \geq 1} k\tau_k z^{k-1}\). Thus, for \(n = -1\)

\[
-\delta_{-1}S^\text{lin} = \dot{\alpha} \left\{ \sum_{k \geq 0} k\tau_k \sum_{l \geq 0} b_l (l + k - 1)\pi_{l+k-2} + \sum_{k \geq 0} k\tau_k \tilde{\pi}_{k-1} + \left( \frac{\beta}{2} - 1 \right) \sum_{k \geq 0} k(k - 1)(k - 2)\tau_k \tilde{\pi}_{k-3} \right\}
\]

\[
+ \ddot{\alpha} \sum_{k \geq 0} k\tau_k \pi_{k-1}
\]

\[
= \dot{\alpha} \left\{ -\oint \tau(z)b(z)\pi'(z) \, dz + \oint \tau(z)\tilde{\pi}(z) + \left( \frac{\beta}{2} - 1 \right) \oint \tau(z)\pi''(z) \, dz \right\} + \ddot{\alpha} \oint \tau(z)\pi(z) \, dz.
\]
When \( n = 0 \) we must add to a term similar to (4.67) a contribution \( \delta^{t}_{0}(S) \) due to the time change; letting \( \bar{t} = t - \varepsilon \delta t = t + 2\varepsilon a \) be the new time coordinate, we get

\[
-\varepsilon \delta^{t}_{0}(\int dt \tau_{k}(t)S^{lin}_{k}(t)) = -\varepsilon \delta^{t}_{0} \left( \int dt \tau_{k}(t) \left( \dot{\pi}_{k}(t) + \left( \frac{\beta}{2} - 1 \right) k(k-1)\pi_{k-2}(t) \right) \right) + k \sum_{l \geq 0} b_{l} \int dt \tau_{k}(t)\pi_{l+k-1}(t) \right) = \int dt \tau_{k}(t)S^{lin}_{k}(t) - \int d\bar{t}(\tau_{k}(\bar{t}) - 2\varepsilon a\dot{\tau}_{k}(\bar{t})) \frac{d\pi_{k}}{d\bar{t}}(\bar{t})
\]

\[
- \int d\bar{t} \left( 1 - 2\varepsilon \dot{a}(\bar{t})))(\tau_{k}(\bar{t}) - 2\varepsilon a\dot{\tau}_{k}(\bar{t}) \right) \left\{ k \sum_{l \leq 0} b_{l}\pi_{l+k-1}(\bar{t}) + \left( \frac{\beta}{2} - 1 \right) k(k-1)\pi_{k-2}(\bar{t}) \right\}
\]

\[
= 2\varepsilon \left( \int dt \ a(t)\dot{\tau}_{k}(t)\dot{\pi}_{k}(t) + \int dt (\dot{a}(t)\tau_{k}(t) + a(t)\dot{\tau}_{k}(t)) \cdot \left\{ k \sum_{l \geq 0} b_{l}\pi_{l+k-1}(t) + \left( \frac{\beta}{2} - 1 \right) k(k-1)\pi_{k-2}(t) \right\} \right) .
\]

(4.68)

Hence

\[
-\delta^{t}_{0}S^{lin} = \dot{a} \left\{ \sum_{k \geq 0} k\tau_{k} \sum_{l \geq 0} b_{l}(l + k - 1)\pi_{l+k-1} + \sum_{k \geq 0} k\tau_{k}\dot{\pi}_{k} + \left( \frac{\beta}{2} - 1 \right) \sum_{k \geq 0} k(k-1)(k-2)\tau_{k}\pi_{k-2} \right\}
\]

\[
+ \ddot{a} \left\{ \sum_{k \geq 0} k\tau_{k}\pi_{k} + 2\dot{a} \left\{ \sum_{k \geq 0} k\tau_{k} \sum_{l \geq 0} b_{l}\pi_{l+k-1} + \left( \frac{\beta}{2} - 1 \right) \sum_{k \geq 0} k\tau_{k}\pi_{k-2} \right\} \right\}
\]

\[
+ 2a \left\{ \sum_{k \geq 0} \dot{\tau}_{k}(\dot{\pi}_{k} + \left( \frac{\beta}{2} - 1 \right) k(k-1)\pi_{k-2}) + \sum_{k \geq 0} \dot{\tau}_{k} \sum_{l \geq 0} b_{l}\pi_{l+k-1} \right\}
\]

\[
= \dot{a} \int \tau(z)\pi(z)dz + \ddot{a} \left\{ - \int \tau(z)b(z)(z\pi(z))' dz + \int \tau(z)(\dot{\pi}(z) + \left( \frac{\beta}{2} - 1 \right)\pi''(z))dz \right\} + 2 \int \tau(z)b(z)\pi(z)dz \right\} - 2a\partial_{t}
\]

(4.69)

where \( a\partial_{t} = a(t)\partial_{t} \) is the time derivation acting on the coefficients of the functional \( Z[\tau] \) (see Appendix B). The term \( \int \tau(z)\pi''(z)dz \) in (4.69) is equal to the sum \( \sum_{k \geq 0} k(k-1)(k-2)\tau_{k}\pi_{k-2} + 2\sum_{k \geq 0} k(k-1)\tau_{k}\pi_{k-2} \).

### 4.3 Schrödinger-Virasoro transformations

We may finally collect all contributions to obtain generators of transformations, denoted by \( L^{-1}_{-\lambda} \) and \( L^{2a}_{0} \). Recall \( \pi(z, t) = -\beta^{-1/2}\dot{\phi}_{-}(z, t) \).
(i) \((n = -1)\) Collecting all terms in \(\delta_{(i)}, \delta_{(ii)}, \delta_{(iii)}\) and \(\delta_{(iv)}\), we get a term \(L_{-1,\text{lin}}\) linear in \((\tau, \partial/\partial \tau)\), plus a term \(L_{-1,\text{quadr}}\) which is quadratic,

\[
L_{-1}^\hat{a} = \int dt \left\{ L_{-1,\text{lin}}^\hat{a}(t) + L_{-1,\text{quadr}}^\hat{a}(t) \right\}, \tag{4.70}
\]

\[
L_{-1,\text{lin}}^\hat{a}(t) = -\ddot{\alpha} \pi_1 + \left(\frac{\beta}{2} - 1\right) \dot{\alpha} \int b(z) \pi''(z) \, dz - \dot{\alpha} \int b(z)(b\pi)'(z) \, dz
\]

\[
= \beta^{-1/2} \left\{ \ddot{\alpha} \int z\hat{\psi}(z) \, dz - \dot{\alpha} \int \left(\frac{\beta}{2} - 1\right)b''(z) + b(z)b'(z)\hat{\psi}(z) \, dz \right\}
\]

\[
(4.71)
\]

\[
L_{-1,\text{quadr}}^\hat{a}(t) = \dot{\alpha} \left\{ \frac{\beta}{2} \int b(z)(\pi^2)'(z) \, dz - \int \tau(z)b(z)\pi'(z) \, dz + \int \tau(z)\pi(z) \, dz\right\}
\]

\[
+ \left(\frac{\beta}{2} - 1\right) \int \tau(z)\pi''(z) \, dz \right\} + \ddot{\alpha} \int \tau(z)\pi(z) \, dz.
\]

\[
(4.72)
\]

Using (4.19) and taking into account the fact that the subalgebras \(A_+, A_-\) are isotropic, we get

\[
L_{-1,\text{quadr}}^\hat{a}(t) = \dot{\alpha} \left\{ \frac{\beta}{2} \int b(z)(\pi^2)'(z) \, dz - \int \tau(z)b(z)\pi'(z) \, dz + \frac{\beta}{2} \int \tau(z)\pi''(z) \, dz\right\}
\]

\[
+ \left(\frac{\beta}{2} - 1\right) \int \tau(z)\pi''(z) \, dz \right\} + \ddot{\alpha} \int \tau(z)\pi(z) \, dz
\]

\[
= \dot{\alpha} \left\{ - \int b'(z)(\beta/2)(\pi^2)(z) - \tau(z)\pi(z)) \, dz - \int \tau(z)\dot{\pi}(\tau(z)) \, dz\right\}
\]

\[
+ \ddot{\alpha} \int \tau(z)\pi(z) \, dz
\]

\[
= -\dot{\alpha} \left\{ \frac{1}{2} \int b'(z) (\beta^{1/2} \pi(z) - \beta^{-1/2} \tau(z))^2 \, dz
\]

\[
+ \frac{1}{2} \int (\beta^{1/2}\dot{\tau}(z) + \beta^{-1/2}\tau(z))^2 \, dz\right\}
\]

\[
- \frac{1}{2} \ddot{\alpha} \int (\beta^{1/2} \pi(z) - \beta^{-1/2} \tau(z))^2 \, dz
\]

\[
= -\dot{\alpha} \left\{ \frac{1}{2} \int b'(z) : (\hat{\psi}(z))^2 : \, dz + \frac{1}{2} \int : (\hat{\phi}(z))^2 : \, dz\right\}
\]

\[
- \frac{1}{2} \ddot{\alpha} \int (\hat{\psi}(z))^2 \, dz
\]

\[
(4.73)
\]

(ii) \((n = 0)\) One finds

\[
L_{0}^{2a} = -2a\partial_t + \int dt \left\{ L_{0,\text{lin}}^{2a}(t) + L_{0,\text{quadr}}^{2a}(t) \right\}, \tag{4.74}
\]

\[
L_{0,\text{lin}}^{2a}(t) = \left(\frac{\beta}{2} - 1\right) \ddot{\alpha} \pi_0 - \frac{1}{2} \ddot{\alpha} \pi_2 + \dot{\alpha} \left\{ \left(\frac{\beta}{2} - 1\right) \int b(z)\pi''(z) \, dz - \int b(z)(b\pi)'z \, dz\right\}
\]

\[
= \left(\frac{\beta}{2} - 1\right) N\ddot{\alpha} + \beta^{-1/2} \left\{ \frac{1}{2} \ddot{\alpha} \int z^2\hat{\psi}(z) \, dz - \dot{\alpha} \int ((\beta/2 - 1)(zb(z))' + (zb(z))')\hat{\psi}(z) \, dz\right\}
\]

\[
(4.75)
\]

29
(note that the first term, a total derivative, disappears after integration in (4.51));

\[
L_{0,\text{quad}r}^{2\beta}(t) = \tilde{a} \int z(\tau - \frac{\beta}{2}\pi^2(z) dz + \dot{a} \left\{ \frac{\beta}{2} \int zb(z)(\pi^2)'(z) dz + (\frac{\beta}{2} - 1) \int \tau(z)\pi''(z) dz + \int z \tau(z)((\beta - 1)\pi''(z) dz - \frac{\beta}{2} \int z \tau(z)b(z)\pi'(z) dz + \int \tau(z)b(z)\pi(z) dz \right\}
\]

\[
= \tilde{a} \int z(\tau - \frac{\beta}{2}\pi^2(z) dz + \dot{a} \left\{ \int zb(z)((\tau - \frac{\beta}{2}\pi^2(z) dz - \int z \tau(z)\partial(\partial \tau(z) dz) \right\}
\]

\[
= -\frac{1}{2}\tilde{a} \int (\dot{\psi}(z))^2 : z dz - \frac{1}{2} \int z(\dot{\psi}(z))^2 : dz + \int z : (\ddot{\phi}(z))^2 : dz \right\}.
\]

(4.76)

4.4 Commutators: the quadratic part

We prove in this paragraph that \((L_{-\text{quad}r}^g L_{0,\text{quad}r}, L_{0,\text{quad}r}^g)_{a \in C^\infty(\mathbb{R}^*_+)}\) provide a zero mass representation of the Schrödinger-Virasoro algebra, see (3.10):

**Theorem 4.2**

\[
\left[ \int L_{0,\text{quad}r}^f \, dt, \int L_{0,\text{quad}r}^g \right] = \int L_{0,\text{quad}r}^{\hat{f} - \hat{g}} \, dt, \tag{4.77}
\]

\[
\left[ \int L_{-1,\text{quad}r}^f \, dt, \int L_{0,\text{quad}r}^g \right] = \int L_{-1,\text{quad}r}^{\hat{f} - \hat{g}} \, dt, \left[ \int L_{-1,\text{quad}r}^f \, dt, \int L_{-1,\text{quad}r}^g \, dt' \right] = 0. \tag{4.78}
\]

In order to keep computations to a reasonable length, we consider commutators of the functionals

\[
A_{\text{quad}r}^v(f) := -2v'(z)f(t)\partial_t - \frac{1}{2} \int dt \hat{f}(t) \int (\dot{\psi}_t(z))^2 : v(z) dz
\]

\[
-\frac{1}{2} \int dt \hat{f}(t) \left\{ \int (v(z)b(z))' : (\dot{\psi}_t(z))^2 : dz + \int v(z) : (\ddot{\phi}_t(z))^2 : dz \right\}, \tag{4.79}
\]

with \(v(z) = 1\) or \(z\). Note that \(A_{\text{quad}r}^v(f) = L_{-1,\text{quad}r}^f\) for \(v(z) = 1\), and \(A_{\text{quad}r}^v(f) = L_{0,\text{quad}r}^{2f}\) for \(v(z) = z\). The non-differential part of \(A_{\text{quad}r}^v(f)\), \(\hat{A}_{\text{quad}r}^v(f) := A_{\text{quad}r}^v(f) + 2v'(z)f(t)\partial_t\), is by definition \(A_{\text{quad}r}^v(f)\) shorn of its differential part \(-2v'(z)f(t)\partial_t\). The first computations are valid for an arbitrary function \(v \in C[[z]]\), but at some point we must restrict to \(v(z) = 1\) or \(z\), which are the only cases needed. We want to prove:

\[
[A_{\text{quad}r}^v(f), A_{\text{quad}r}^g(g)] = 4 \int L_{0,\text{quad}r}^{\hat{f} - \hat{g}} \, dt, \tag{4.80}
\]

\[
[A_{\text{quad}r}^1(f), A_{\text{quad}r}^2(g)] = 2 \int L_{-1,\text{quad}r}^{\hat{f} - \frac{1}{2}\hat{g}} \, dt, \quad [A_{\text{quad}r}^1(f), A_{\text{quad}r}^1(g)] = 0. \tag{4.81}
\]

30
Using the commutation relations of the boson algebra, we find for \( u, v \in \mathbb{C}[[z]] \)

\[
\left[ \frac{1}{2} \oint u(z) : (\hat{\psi}(z))^2 : dz, \frac{1}{2} \oint v(w) : (\hat{\phi}(w))^2 : dw \right]
\]

\[
= \oint dzdw \ u(z)v(w) : \hat{\psi}(z)\hat{\phi}(w) : G^+_{t-t'}(z^{-1}, w) - G^-_{t-t'}(z, w^{-1});
\]  

(4.82)

\[
\left[ \frac{1}{2} \oint u(z) : (\hat{\psi}(z))^2 : dz, \frac{1}{2} \oint v(w) : (\hat{\phi}(w))^2 : dw \right]
\]

\[
= \oint dzdw \ u(z)v(w) : \hat{\psi}(z)\hat{\phi}(w) : (G^+_{t-t'}(z^{-1}, w) - G^-_{t-t'}(z, w^{-1}))
\]  

(4.83)

\[
\left[ \frac{1}{2} \oint u(z) : (\hat{\phi}(z))^2 : dz, \frac{1}{2} \oint v(w) : (\hat{\phi}(w))^2 : dw \right]
\]

\[
= \oint dzdw \ u(z)v(w) : \hat{\phi}(z)\hat{\phi}(w) : (G^+_{t-t'}(z^{-1}, w) - G^-_{t-t'}(z, w^{-1}))
\]  

(4.84)

hence

\[
[A_u^{\text{quad}}, A_v^{\text{quad}}] = \sum_{i=1}^{8} C_i(f, g),
\]

(4.85)

where:

(i) (contribution of the commutator \([\psi, \psi]\))

\[
C_1(f, g) = \oint dzdw \int dt \int_0^t dt' (\hat{f}(t)(ub)^t(z) + u(z)\hat{f}(t))v(w)\hat{g}(t') : \hat{\psi}(z)\hat{\psi}(w) : G^+_{t-t'}(z^{-1}, w)
\]

\[
- \oint dzdw \int dt' \int_0^t dt u(z)\hat{f}(t)(\hat{g}(t')(vb)^t(w) + v(w)\hat{g}(t')) : \hat{\psi}(z)\hat{\psi}(w) : G^-_{t-t'}(z, w^{-1})
\]

\[
= C_{1,1}(f, g) + C_{1,2}(f, g) + C_{1,3}(f, g)
\]

(4.86)

where (by integrating by parts with respect to \( t' \) or \( t \), and using the fundamental relations (5.41.25.43))

(1) \( C_{1,1}(f, g) = \)

\[
= -\oint dz \int dt (\hat{f}(t)(ub)^t(z) + u(z)\hat{f}(t))\hat{g}(t)\hat{\psi}(z)(P_-(vw\hat{\psi}))'(z)
\]

\[
+ \oint dw \int dt' \hat{f}(t')(\hat{g}(t')(vb)^t(w) + v(w)\hat{g}(t'))\hat{\psi}(w)(P_-(vw\hat{\psi}))'(w)
\]  

(4.87)

31
If \( u = v \) then the two terms in \( \dot{f}(t)\dot{g}(t) \) cancel each other, and there remains only

\[
C_{1,1}(f, g) = -\int dz \int dt \left( \dot{f}(t)\dot{g}(t) - \dot{f}(t)\dot{g}(t) \right)(P_+(u\dot{\psi}_t))(z)(P_-(w\dot{\psi}_t))'(z).
\]

(4.88)

Otherwise we may assume that \( u(z) = 1, v(w) = w, \) from which \( (P_-(v\dot{\psi}_t))'(z) = (vP_-(\dot{\psi}_t))'(z) = z(P_-(\dot{\psi}_t))'(z) + (P_t\dot{\psi}_t)(z) \) (observe that the first equality is wrong if \( v(w) = w^{n+1}, n \geq 1 \) and

\[
C_{1,1}(f, g) = -\int dz \int dt \left( \dot{f}(t)\dot{g}(t) - \dot{f}(t)\dot{g}(t) \right)(P_+(\dot{\psi}_t))(z)(P_-(\dot{\psi}_t))'(z)
\]

(4.89)

\[
-\int dt \int dz \left\{ \int dz b'(z)\dot{\psi}_t(z)(P_-(\dot{\psi}_t))'(z) - \int dz b(z)\dot{\psi}_t(z)(P_-(\dot{\psi}_t))'(z) \right\}
\]

(4.90)

In the last line we have used:

\[
\int dz : (\dot{\psi}_t(z))^2 : = \int dz (P_+(\dot{\psi}_t))(z)(P_-(\dot{\psi}_t))(z) = \int dz \dot{\psi}_t(z)(P_-(\dot{\psi}_t))(z).
\]

(4.91)

a contribution due to the first term in the right-hand side of (4.39); ",-sym". indicates, here as in the following computations, a similar term with the kernel \( G_{t-v}^+ \) in factor;

\[
C_{1,2}(f, g) = -\beta^{1/2} \int dz dw \int dt \int_0^t dt' (\dot{f}(t)(ub)'(z) + u(z)\ddot{f}(t))v(w)\dot{g}(t')\dot{\psi}_t(z)G_{t-v}^+(z^{-1}, w)\partial/\partial t(w, t') - \text{sym.}
\]

(4.92)

\[
= -\int dz dw \int dt \int_0^t dt' (\dot{f}(t)(ub)'(z) + u(z)\ddot{f}(t))v(w)\dot{g}(t')\dot{\psi}_t(z)G_{t-v}^+(z^{-1}, w)\dot{\psi}_w(w) - \text{sym.,}
\]

(4.93)

by (5.13);
(ii) (contribution of the commutator $[\psi,\psi]$, continued)

\[
C_2(f, g) = \int dt \int_0^t dt' \int d\zeta f(t)(ub)'(z) + u(z)f(t)g(t) : \dot{\psi}_t(z)\dot{\psi}_t(w) : G_{t-t'}^{+}(z^{-1}, w) \]
\[- \int dt' \int_0^t dt \int d\zeta f(t)(ub)'(z)(g(t')vb)'(w) + v(w)g(t') : \dot{\psi}_t(z)\dot{\psi}_t(w) : G_{t-t'}^{-}(z, w^{-1})
\]

(4.93)

(iii) (non-$\delta$ contribution of the commutator $[\psi, \phi]$ for $t \neq t'$)

\[
C_3(f, g) = \int dt \int_0^t dt' \int d\zeta f(t)(ub)'(z) + u(z)f(t)v(w)g(t') : \dot{\psi}_t(z)\dot{\phi}_t(w) : G_{t-t'}^{+}(z^{-1}, w) \]
\[- \int dt' \int_0^t dt \int d\zeta f(t)u(z)(g(t')vb)'(w) + v(w)g(t') : \dot{\phi}_t(z)\psi_t(w) : G_{t-t'}^{-}(z, w^{-1})
\]

(4.94)

(iv) ($\delta$-contribution of the commutator $[\psi, \phi]$)

\[
C_4(f, g) = - \int dt \int d\zeta f(t)(ub)'(z) + u(z)f(t)g(t)(P_+(v\dot{\psi}_t))'(z)G_{t-t'}^{-}(z, w^{-1}) \]
\[+ \int dt \int d\zeta u(z)f(t)g(t)(vb)'(w) + v(w)g(t)\dot{\psi}_t(z)(P_+(\dot{\phi}_t))'(z)\dot{\psi}_t(z) \]
\[= - \int dt \int d\zeta f(t)(ub)'(z) + u(z)f(t)g(t)(P_+(v\dot{\psi}_t))'(z)\dot{\psi}_t(z) \]
\[+ \int dt \int d\zeta f(t)(vb)'(w) + v(w)g(t)(P_+(u\dot{\phi}_t))'(w)\dot{\psi}_t(w)
\]

(4.95)

This term is parallel to the term $C_{1,1}(f, g)$, to the analysis of which we refer. In the following expressions, we use the fact that $P_+\dot{\phi}_t = P_+\dot{\psi}_t$. When $u = v$ we find

\[
C_4(f, g) = - \int dt \int d\zeta f(t)(\dot{g}(t) - \dot{\phi}(t))(P_+(u\dot{\psi}_t))'(z)(P_+(u\dot{\psi}_t))(z).
\]

(4.96)

Otherwise we may assume that $u(z) = 1$, $v(w) = w$, from which

\[
C_4(f, g) = \]
\[- \int d\zeta z \int dt (\dot{f}(t)\dot{g}(t) - \dot{f}(t)\dot{\phi}(t))(P_+(u\dot{\psi}_t))'(z)(P_-(u\dot{\psi}_t))(z)
\]
\[- \int dt \dot{f}(t)\dot{g}(t) \left\{ \int dz b'(z)(P_+(\dot{\psi}_t))'(z)\dot{\psi}_t(z) - \int dz b(z)(P_+(\dot{\psi}_t))'(z)\dot{\psi}_t(z) \right\}
\]
\[- \frac{1}{2} \int dt \dot{f}(t)\dot{g}(t) \int dz : (\dot{\psi}_t(z))^2 : \]

(4.97)

(v) ($\delta$-contribution due to the commutator $[\phi, \phi]$)
This term clearly vanishes when \( u = v \). Hence we may assume that \( u(z) = 1, v(w) = w \), in which case

\[
C_5(f, g) = \int dt \int dz \int dw w : \hat{\phi}_t(z) \hat{\phi}_t(w) : (G^+_0(z^{-1}, w) - G^-_0(z, w^{-1}))
\]

\[
= \int dt \int dz \int dw w : \hat{\phi}_t(w) \left\{ (P_+ \hat{\phi}_t)'(w) + (P_- \hat{\phi}_t)'(w) \right\} :
\]

\[
= -\frac{1}{2} \int dt (\dot{f} \cdot \dot{g})(t) \int dw : (\hat{\phi}_t(w))^2 :
\]

\[
(4.99)
\]

(vi) assume \( v(w) = w \): by the results of Appendix B, in particular, \([5.20, 5.22, 5.23, 5.26]\)

\[
C_6(f, g) = 2 \left[ A^{quad}_u(f), -g(t) \partial_t \right]
\]

\[
= -2 \int dt \int dz (\dot{f}(t)(ub)'(z) + u(z) \ddot{f}(t))
\]

\[
: \hat{\psi}_t(z) \cdot \left\{ (g(t) \partial_t \cdot \hat{\psi}_-(z, t)) + (g(t) \partial_t \cdot \hat{\psi}_+(z, t)) \right\} :
\]

\[
= -2 \left( \int dt \int dz u(z) \ddot{f}(t) \hat{\psi}_t(z) \left\{ (g(t) \partial_t \cdot \hat{\psi}_-(z, t)) + (g(t) \partial_t \cdot \hat{\psi}_+(z, t)) \right\} :
\]

\[
= -2 \left( \int dt \int dz \hat{\psi}_t(z) \ddot{f}(t) (\partial_t \hat{\psi}_-(z, t)) + (\partial_t \hat{\psi}_-(z, t) g(t)(\partial_t \hat{\psi}_+(z, t))
\]

\[
+ \int dt \int dz (\ddot{f}(t)(ub)'(z) + u(z) \dddot{f}(t)) \hat{\psi}_t(z) \int_0^t dt' \dot{g}(t') \int dw b(w) G^+_t w(z^{-1}, w) \hat{\psi}_t(w)
\]

\[
-2 \left( -\int dt \int dz g(t)(\ddot{f}(t)(ub)'(z) + u(z) \dddot{f}(t)) \hat{\psi}_t(z) (\partial_t \hat{\psi}_-(z, t))
\]

\[
- \int dt \int dz g(t) u(z) \dddot{f}(t)(\partial_t \hat{\phi}_t)(z, t) \hat{\phi}_-(z, t)
\]

\[
- \int dt \int dz g(t) u(z) \dddot{f}(t)(\partial_t \hat{\phi}_t)(z, t) \hat{\phi}_-(z, t)
\]

\[
= C^{nonloc}_6(f, g) + C^{loc}_6(f, g)
\]

\[
(4.100)
\]

where

\[
C^{nonloc}_6(f, g) := -2 \int dt \int dz (\dot{f}(t)(ub)'(z) + u(z) \dddot{f}(t)) \hat{\psi}_t(z)
\]

\[
\int_0^t dt' \dot{g}(t') \int dw b(w) G^+_t w(z^{-1}, w) \hat{\psi}_t(w)
\]

\[
(4.101)
\]
is a non-local term, and

\[
C_6^{\text{loc}}(f, g) = \int dt \int dz \left\{ \frac{d}{dt}(g\dot{f})(t)(ub)'(z) + u(z)\frac{d}{dt}(g\ddot{f})(t) \right\} : (\dot{\psi}_t(z))^2 :
\]

\[ - \int dt \int dz u(z)\frac{d}{dt}(g\dot{f})(t) : (\dot{\phi}_t(z))^2 :
\]

\[ + 2 \int dt \int dz g(t)\dot{f}(t)u(z) : (\dot{\phi}_t(z))^2 :
\]

\[ + 2 \int dt \int dz u(z)\dot{g}(t)\hat{f}(t)\dot{\phi}_+(z, t)\dot{\phi}_-(z, t)
\]

\[ = \int dt \int dz \left\{ \frac{d}{dt}(g\dot{f})(t)(ub)'(z) + u(z)\frac{d}{dt}(g\ddot{f})(t) \right\} : (\dot{\psi}_t(z))^2 :
\]

\[ + \int dt \int dz u(z)(g\ddot{f})(t) : (\dot{\phi}_t(z))^2 :
\]

(4.102)

(vii) if \( u(z) = z \), \( C_7(f, g) := 2[-f(t)\partial_t, \bar{A}_u^{\text{quadr}}(g)] \) is ”sym.” of (vi);

(viii) finally, assuming \( u(z) = z \) and \( v(w) = w \),

\[ C_8(f, g) = 4[-f(t)\partial_t, -g(t)\partial_t] = 4(f(t)\dot{g}(t) - \dot{f}(t)g(t))\partial_t. \quad (4.103)
\]

Let us now sum up the different contributions. We leave out the differential term (viii) which is as expected. Note that \( C_{1,2} + C_3 \equiv 0 \) in all cases. Also,

\[
C_{1,3}(f, g) + C_2(f, g) = 2 \int dt \int_0^t dt' \int dzd\psi (\dot{f}(t)(ub)'(z) + u(z)\dot{f}(t))\dot{\psi}(t')(v'b)(w)
\]

\[ : \dot{\psi}_t(z)\dot{\psi}_t'(w) : G_{t-t'}^{+}(z^{-1}, w) - \text{sym.}
\]

(4.104)

is exactly compensated by \( C_{6}^{\text{nonloc}}(f, g) + C_{7}^{\text{nonloc}}(f, g) \).

Now all remaining terms (\( C_{1,1}, C_4, C_5, C_6^{\text{loc}}, C_7^{\text{loc}} \)) are local functionals of the fields \( \hat{\psi}, \hat{\phi} \), i.e. are expressed as some integral \( \int dt \int dz dz\psi(z, t, \hat{\psi}(z, t), \hat{\phi}(z, t)) \).

Assume first \( u = v \). Then \( C_5 = 0, C_{1,1} + C_4 = 0 \) (see (4.88), (4.96)). Thus \( [A_1^{\text{quadr}}(f), A_1^{\text{quadr}}(g)] = 0 \),

\[
[A_2^{\text{quadr}}(f), A_2^{\text{quadr}}(g)] = C_6^{\text{loc}}(f, g) + C_7^{\text{loc}}(f, g)
\]

\[ = \int dt \int dz \left\{ \frac{d}{dt}(g\dot{f} - f\dot{g})(t)(zb)'(z) + z\frac{d}{dt}(g\ddot{f} - f\ddot{g})(t) \right\} : (\dot{\psi}_t(z))^2 :
\]

\[ + \int dt \int dz z(g\ddot{f} - f\ddot{g})(t) : (\dot{\phi}_t(z))^2 :
\]

\[ = 4 \int dt L_{0, \text{quadr}}(t). \quad (4.105)
\]
Assume now \( u(z) = 1, v(w) = w \). Then \((4.89, 4.97)\) sum up to

\[
- \int dt \left( \dot{f}(t) \dot{g}(t) - \dot{f}(t) \dot{g}(t) \right) \oint dz z : \left\{ \hat{\psi}_t(z)((P_+ \hat{\psi}_t)'(z) + (P_+ \hat{\psi}_t)'(z)) \right\} : \\
= \frac{1}{2} \int dt \left( \dot{f}(t) \dot{g}(t) - \dot{f}(t) \dot{g}(t) \right) \oint dz : (\hat{\psi}_t(z))^2 : \\
= \frac{1}{2} \int dt \frac{d}{dt} \left( \dot{f} \dot{g}(t) \right) : (\hat{\psi}_t(z))^2 : \\
- \int dt \frac{d}{dt} \left( \dot{f} \dot{g}(t) \right) \left\{ \oint dz b'(z) : (\hat{\psi}_t(z))^2 : - \int dz b(z) : \hat{\psi}_t(z)(\hat{\psi}_t)'(z) : \right\}
\]

Adding this to \((4.90, 4.98)\) yields

\[
- \frac{1}{2} \int dt \frac{d}{dt} \left( \dot{f} \dot{g}(t) \right) : (\hat{\psi}_t(z))^2 : \\
- \int dt \frac{d}{dt} \left( \dot{f} \dot{g}(t) \right) \left\{ \oint dz b'(z) : (\hat{\psi}_t(z))^2 : - \int dz b(z) : \hat{\psi}_t(z)(\hat{\psi}_t)'(z) : \right\}
= - \frac{1}{2} \int dt \frac{d}{dt} \left( \dot{f} \dot{g}(t) \right) : (\hat{\psi}_t(z))^2 : - \frac{3}{2} \int dt \frac{d}{dt} \left( \dot{f} \dot{g}(t) \right) \oint dz b'(z) : (\hat{\psi}_t(z))^2 : (4.107)
\]

To the latter expression we must still add

\[
C_5(f, g) = - \frac{1}{2} \int dt \oint dz (\dot{f} \dot{g}(t) : (\hat{\phi}_t(z))^2 : (4.108)
\]

and

\[
C_6^{loc}(f, g) = \int dt \oint dz \left\{ \frac{d}{dt}(g \dot{f}(t)b'(z) + \frac{d}{dt}(g \ddot{f}(t)) \right\} : (\hat{\psi}_t(z))^2 : \\
+ \int dt \oint dz (g \dot{f}(t)) : (\hat{\phi}_t(z))^2 :
\]

Adding all terms yields as expected

\[
2L^{\hat{f}g - \frac{1}{2} \hat{f} \hat{g}}_{-1,quadr} = - \int dt \oint dz \left\{ (\ddot{f} \dot{g}(t)b'(z) + \frac{d}{dt}(\ddot{f} \dot{g} - \frac{1}{2} \ddot{f} \dot{g}) \right\} : (\hat{\psi}_t(z))^2 : \\
- \int dt \oint dz (\ddot{f} \dot{g} - \frac{1}{2} \ddot{f} \dot{g}(t) : (\hat{\phi}_t(z))^2 :
\]

\[(4.110)\]

### 4.5 Commutators: the linear contribution

Copying what we did in the last subsection, we set

\[
A_u^{lin}(f) := \beta^{-1/2} \int dt \left\{ \dot{f}(t) \int u(z) \hat{\psi}_t(z) dz - \dot{f}(t) \int ((\frac{\beta}{2} - 1)(ub)'(z) + (ub)'(z)b(z)) \hat{\psi}_t(z) dz \right\} \\
(4.111)
\]

for \( u(z) = 1, z \), with \((\int u)(z) = z \), resp. \( \frac{z^2}{2} \) when \( u(z) = 1, \) resp. \( z \). In coherence with the quadratic parts, \( A_u^{lin}(f) = L_{-1,quadr}^f \) for \( u(z) = 1 \), and \( A_u^{lin}(f) = L_{0,quadr}^{2f} \) for \( u(z) = z \). Since obviously \([A_u^{lin}(f), A_u^{lin}(g)] = 0\), we must prove:

\[
[A_z^{lin}(f), A_z^{quadr}(g)] - (f \leftrightarrow g) = 4 \int L_{0,lin}^{\hat{f}g - \hat{f} \hat{g}}(t) dt, \\
(4.112)
\]
\[ [A_{\text{lin}}^1(f), A_z^{\text{quadr}}(g)] - [A_{\text{lin}}^1(f), A_{\text{lin}}^1(g)] = 2 \int_{t-1,\text{lin}}^t \dot{f}(t) dt, \quad [A_{\text{lin}}^1(f), A_z^{\text{quadr}}(g)] - (f \leftrightarrow g) = 0. \]  

For \( u(z) = 1, z, v(w) = 1, w \), we find in general:

\[ [A_{u_i}^{\text{lin}}(f), A_v^{\text{quadr}}(g)] = \sum_{i=1}^4 D_i(f, g), \]  

with:

(i)

\[ D_1(f, g) = \left[ A_{u_i}^{\text{lin}}(f), -\frac{1}{2} \int dt \int dw \psi(t) : (\dot{\psi}(w, t'))^2 : \right] \]
\[ = -\beta^{-1/2} \int dt \int_0^t dt' \int dw \left( \int u(z) \ddot{f}(t) - \left( \frac{\beta}{2} - 1 \right) (ub)^"(z) + (ub)'(z)b(z) \right) \dot{f}(t) \]
\[ = -\int dt \int_0^t dt' \int_0^t ds \int dw \ddot{f} \left( z\left( \int u(z) \ddot{f}(t) - \left( \frac{\beta}{2} - 1 \right) (ub)^"(z) + (ub)'(z)b(z) \right) \right) \]
\[ = D_{1,1}(f, g) + D_{1,2}(f, g) + D_{1,3}(f, g), \]  

where (by integrating by parts)

(1)

\[ D_{1,1}(f, g) = -\beta^{-1/2} \int dt \int dz dw \left( \int u(z) \ddot{f}(t) - \left( \frac{\beta}{2} - 1 \right) (ub)^"(z) + (ub)'(z)b(z) \right) \dot{f}(t) \]
\[ \dot{g}(t)(\dot{v}(w)) (z) \]
\[ = \beta^{-1/2} \int dt \int dz \left( \int u(z) \ddot{f}(t) - \left( \frac{\beta}{2} - 1 \right) (ub)^"(z) + (ub)'(z)b(z) \right) \dot{f}(t) \]
\[ \dot{g}(t) \left( \mathcal{P}_-(v(w))^"(z) \right) \]

(4.116)

(2)

\[ D_{1,2}(f, g) = \int dt \int_0^t ds \int dz dw \ddot{f} \left( \int u(z) \ddot{f}(t) - \left( \frac{\beta}{2} - 1 \right) (ub)^"(z) + (ub)'(z)b(z) \right) \dot{f}(t) \]
\[ \dot{g}(s)v(w)G_{t-s}^+(z-1, w) \frac{1}{w} K_0(w^{-1}, \zeta) \frac{\partial}{\partial \tau}(\zeta, s) \]
\[ = \int dt \int_0^t ds \int dz \int d\zeta \ddot{f} \left( \int u(z) \ddot{f}(t) - \left( \frac{\beta}{2} - 1 \right) (ub)^"(z) + (ub)'(z)b(z) \right) \dot{f}(t) \]
\[ \dot{g}(s)v(\zeta)G_{t-s}^+(z-1, \zeta) \frac{\partial}{\partial \tau}(\zeta, s); \]  

(4.117)
where:

\[
D_3(f, g) = \left[ A_u^{\text{lin}}(f), -\frac{1}{2} \right] \int_0^t dt' \int dw \, \hat{g}(t') v(w) : (\hat{\psi}(w, t'))^2 : \\
= -\int dt \int_0^t dt' \int dz \, dw \, \hat{g}(t') v(w) \left( (\int u(z) \dddot{f}(t) - ((\frac{\beta}{2} - 1)(ub)'(z) + (ub)'(z)b(z)) \hat{f}(t) \right) v(w) \\
\hat{g}(t') v(w) G_{t-t'}^+(z^{-1}, w) \frac{1}{w} \hat{\psi}_v(w); \tag{4.119}
\]

(iv)

\[
D_4(f, g) = \left[ A_u^{\text{lin}}(f), -2g(t) \partial_t \right] =: D_{4,1}^{\text{loc}}(f, g) + D_{4,2}(f, g) + D_{4}^{\text{nonloc}}(f, g), \tag{4.121}
\]

where:

\[
D_{4,1}^{\text{loc}}(f, g) = 2\beta^{-1/2} \int dt \dddot{f}(t) \int (\int u(z) g(t)(\partial_t \hat{\psi}_-) (z, t) dz \\
= -2\beta^{-1/2} \int dt \left[ \frac{d}{dt}(\dddot{f})(t) \int (\int u(z) \hat{\psi}_-(z, t) dz; \tag{4.122}
\]

using (5.15);
\[ D_{4,2}^{\text{loc}}(f, g) = -2\beta^{-1/2} \int dt \, \ddot{f}(t) \phi((\frac{\beta}{2} - 1)(ub)^\nu(z) + (ub)'(z)b(z))g(t)(\partial_t \hat{\psi}_-)(z, t) \, dz \]
\[ = 2\beta^{-1/2} \int dt \, \frac{d}{dt}(g\dot{f})(t) \phi((\frac{\beta}{2} - 1)(ub)^\nu(z) + (ub)'(z)b(z))\hat{\psi}_-(z, t) \, dz; \quad (4.123) \]

\[ D_{4}^{\text{nonloc}}(f, g) = 2 \int dt \int_0^t dt' \int dz \int dw \left\{ \ddot{f}(t)(\int u)(z) - \dot{f}(t)(\frac{\beta}{2} - 1)(ub)^\nu(z) + (ub)'(z)b(z) \right\} \]
\[ \hat{g}(t')b(w)G_{t-v}(z^{-1}, w)\hat{\psi}_v(w). \quad (4.124) \]

Let us now add the different contributions. First, \( D_{1,2} + D_3 \equiv 0, \ D_{1,3} + D_2 + D_4^{\text{nonloc}} \equiv 0, \) leaving out only local contributions, \( D_{1,1}(f, g), D_{1}^{\text{loc}}(f, g) \) and their symmetric counterparts.

Assume first \( u(z) = v(z) = 1 \) or \( z \). Then

\[ D_{1,1}(f, g) - (f \leftrightarrow g) = \beta^{-1/2} \int dt \, (\ddot{f}(t)\dot{g}(t) - \dot{f}(t)\ddot{g}(t)) \phi \left( \int u)(z)(\mathcal{P}_-(u\hat{\psi}_t))'(z) \right) \]
\[ = -\beta^{-1/2} \int dt \, (\ddot{f}(t)\dot{g}(t) - \dot{f}(t)\ddot{g}(t)) \phi \left( \int dz \, u^2(z)\hat{\psi}_t(z) \right). \quad (4.125) \]

In particular, if \( u(z) = v(z) = 1 \), this is equal to

\[ -N \int dt \, (\ddot{f}(t)\dot{g}(t) - \dot{f}(t)\ddot{g}(t)) = -N \int dt \, \frac{d}{dt}(\ddot{f}(t)\dot{g}(t) - \dot{f}(t)\ddot{g}(t)) = 0. \quad (4.126) \]

Since there is not \( D_4 \)-term in that case, we have proved: \( [A_1^{\text{lin}}(f), A_1^{\text{quad}}(g)] - (f \leftrightarrow g) = 0. \)
The reader may easily check that one also gets the correct formula for \( [A_2^{\text{lin}}(f), A_2^{\text{quad}}(g)] - (f \leftrightarrow g). \)

There remains the case \( u(z) = 1, v(w) = w \). Then

\[ D_{1,1}(f, g) - \text{sym.} = \beta^{-1/2} \int dt \left\{ (\ddot{f} \dot{g})(t) \phi \left( z(\mathcal{P}_-(z\hat{\psi}_t))'(z) \right) dz - (\ddot{f} \ddot{g})(t) \phi \left( \frac{z^2}{2} (\mathcal{P}_-\hat{\psi}_t)'(z) \right) dz \right\} \]
\[ -\beta^{-1/2} \int dt \, (\ddot{f} \ddot{g})(t) \left\{ \left( \frac{\beta}{2} - 1 \right)b''(z) + b'(z)b(z)(\mathcal{P}_-(z\hat{\psi}_t))'(z) \right. \]
\[ - \left. ((\frac{\beta}{2} - 1)(z^2b(z))'' + (z^2b(z))'b(z)(\mathcal{P}_-\hat{\psi}_t)'(z) \right\} \]
\[ = -\beta^{-1/2} \int dt \, (\ddot{f} \dot{g} - \dot{f} \ddot{g})(t) \phi \left( z\hat{\psi}_t(z) \right) dz \]
\[ - 3\beta^{-1/2} \int dt \, (\ddot{f} \ddot{g})(t)((\frac{\beta}{2} - 1)b''(z) + b'(z)b(z))\hat{\psi}_t(z) \, dz, \quad (4.127) \]

from which the reader may easily check the remaining bracket, \( [A_1^{\text{lin}}(f), A_2^{\text{quad}}(g)] - [A_2^{\text{lin}}(f), A_1^{\text{quad}}(g)]. \)
4.6 A detailed example: the Hermite case

We compute once again commutators for the sake of the reader in a simple case (Hermite polynomials, $\beta = 2$) using Fourier modes. Assume as in Example 1 that $b_1 = 1/\sigma^2$ and $b_i = 0$, $i \neq 1$ and let $\beta = 2$. Then

$$L_{a_1} = \int dt \left\{ \frac{\dot{\alpha}}{\sigma^2} - \ddot{\alpha} \right\} + \frac{\dot{\alpha}}{\sigma^2} + \dot{\alpha} \right\}$$

and

$$L_0^a = \int dt \left\{ \frac{\dot{\alpha}}{\sigma^2} - \frac{1}{2} \ddot{\alpha} \right\} + \frac{4N}{\sigma^2} - 2a(t)\partial_t - \frac{1}{2} \ddot{\alpha} + \ddot{\alpha} \right\}$$

with $\frac{1}{2} \ddot{\phi}^2(z,t) = \sum_{k \geq 2} k \tau_k(t) \frac{\delta}{\sigma t_k-1(t)}$ and

$$\frac{1}{2} \int \ddot{\phi}^2(z,t) dz = -N \tau_1(t) + \sum_{k \geq 2} k \tau_k(t) \int_0^t ds \; e^{-(k-1)(t-s)/\sigma^2} \frac{\delta}{\sigma \tau_{k-1}(s)}$$

We first compute Lie brackets and prove that $(L_{a_1}, L_0^a)_{a \in \mathbb{C}}$ provide a zero mass representation of the Schrödinger-Virasoro algebra. Let $L_{a_1,lin}(a), L_{a_1,quad}(a)$, resp. $L_0,lin(a), L_0,quad(a)$ be the linear and quadratic parts of $L_{a_1}$, resp. $L_0^a$ as in the previous paragraph. Using the

relations in the dynamic boson algebra, we find

$$\left[ \int \ddot{\phi}^2(z,t) dz, \int \ddot{\phi}^2(w,t') dw \right] = \frac{1}{2} \int \ddot{\phi}^2(z,t) dz, \int \ddot{\phi}^2(w,t') dw$$

for $t > t'$,

$$\left[ \int \ddot{\phi}^2(z,t) dz, \int \ddot{\phi}^2(w,t) dw \right] = -2N \tau_2(t) e^{-(t-t')/\sigma^2} +$$

$$\sum_{k \geq 2} k(k-1) \tau_k(t) e^{-(k-1)(t-t')/\sigma^2} \int_0^t ds \; e^{-(k-2)(t-s)/\sigma^2} \frac{\delta}{\sigma \tau_{k-1}(s)}.$$

From this we get

$$[L_{a_1,quad}(f), L_{a_1,quad}(g)] = \sum_{i=1}^6 (C_i(f,g) - C_i(g,f)),$$

with (following the same scheme as in the previous subsection):

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(i) (contribution of the commutator \([\psi, \psi]\))

\[
C_1(f, g) = \int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t) \right) \sum_k k(k-1) \tau_k(t)e^{-(k-1)t/\sigma^2} \int_0^t dt'e^{-(k-1)t'/\sigma^2} \hat{g}(t') \left( \int_0^{t'} ds e^{-(k-2)(t'-s)/\sigma^2} \frac{\delta}{\delta \tau_{k-2}(s)} \right)
\]

\[\equiv C_{1,1}(f, g) + C_{1,2}(f, g) + C_{1,3}(f, g)\]

where (by integration by parts)

\[
C_{1,1}(f, g) = \int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t) \right) \sum_k k(k-1) \tau_k(t) \hat{g}(t) \int_0^t ds e^{-(k-2)(t-s)/\sigma^2} \frac{\delta}{\delta \tau_{k-2}(s)}
\]

\[
C_{1,2}(f, g) = -\int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t) \right) \sum_k k(k-1) \tau_k(t) \int_0^t dt' e^{-(k-1)(t-t')/\sigma^2} \hat{g}(t') \frac{\delta}{\delta \tau_{k-2}(t')}
\]

\[
C_{1,3}(f, g) = -\int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t) \right) \sum_k k(k-1) \tau_k(t) \int_0^t dt' e^{-(k-1)(t-t')/\sigma^2} \frac{\delta}{\delta \tau_{k-2}(t')}
\]

(ii) (contribution of the commutator \([\psi, \psi]\), continued)

\[
C_2(f, g) = \int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t) \right) \sum_k k(k-1) \tau_k(t) e^{-(k-1)t/\sigma^2} \int_0^t dt' e^{-(k-1)t'/\sigma^2} \frac{\delta}{\delta \tau_{k-2}(t')}
\]

(iii) (contribution of the commutator \([\psi, \phi]\) for \(t \neq t'\))

\[
C_3(f, g) = \int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t) \right) \sum_k k(k-1) \tau_k(t) \int_0^t dt' e^{-(k-1)(t-t')/\sigma^2} \frac{\delta}{\delta \tau_{k-2}(t')}
\]

(iv) (\(\delta\)-contribution)

\[
C_4(f, g) = -\int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t) \right) \sum_k k(k-1) \tau_k(t) \int_0^t ds e^{-(k-2)(t-s)/\sigma^2} \frac{\delta}{\delta \tau_{k-2}(s)}
\]
(v) (zero-momentum contribution)

\[ C_5(f, g) = -2N \int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t)\right) \tau_2(t) = -2N \int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t)\right) \tau_2(t) \]

by integration by parts;

(vi) (zero-momentum contribution, continued)

\[ C_6(f, g) = 2N \int dt \left( \frac{\dot{f}(t)}{\sigma^2} + \ddot{f}(t)\right) \tau_2(t). \]

Then one sees that \( C_{1,1} + C_4 = 0, \ C_{1,2} + C_3 = 0, \ C_{1,3} + C_2 = 0, \ C_5 + C_6 = 0. \) Consequently, \( [L_{-1,\text{quad}}, f, g] = 0. \)

The contribution of \( L_{-1,\text{lin}} \) to the bracket \( [L_{-1}^f, L_{-1}^g] \) is easily computed, \( [L_{-1,\text{lin}}, f, g] = 0 \) clearly while by integration by parts

\[ [L_{-1,\text{lin}}, f, g] - (f \leftrightarrow g) \]

\[ = \left[-\int dt \left( \frac{\dot{f}(t)}{\sigma^2} - \ddot{f}(t)\right) \int_0^t ds e^{-(t-s)/\sigma^2} \frac{\delta}{\delta \tau_1(s)} \right] - (f \leftrightarrow g) \]

\[ = -\int dt \left( \frac{\dot{f}(t)}{\sigma^2} - \ddot{f}(t)\right) \int_0^t ds e^{-(t-s)/\sigma^2} \frac{\delta}{\delta \tau_1(s)} \]

\[ = \frac{N}{2} \int dt \left( \frac{\dot{f}(t)}{\sigma^2} - \ddot{f}(t)\right) \int_0^t ds e^{-(t-s)/\sigma^2} \left( \frac{\dot{g}(s)}{\sigma^2} - \ddot{g}(s)\right) \]

Thus finally: \( [A(f), A(g)] = 0. \)

5 Appendix

We prove here a certain number of explicit expressions given in section 4. At some point we use in the proofs the following fundamental relations,

\[ \mathcal{P}_- f(z) = \frac{1}{z} \int \frac{dw}{1 - w/z} f(w), \quad \mathcal{P}_+ f(z) = \int \frac{dw}{w(1 - z/w)} f(w) \]

if \( f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathbb{C}[[z^{-1}, z]], \) where \( \mathcal{P}_- f, \) resp. \( \mathcal{P}_+ f, \) is the projection onto \( \mathcal{A}_- \) parallel to \( \mathcal{A}_+ \), resp. onto \( \mathcal{A}_+ \) parallel to \( \mathcal{A}_- \), namely, \( \mathcal{P}_- f(z) = \sum_{n \leq 0} a_n z^n, \) \( \mathcal{P}_+ f(z) = \sum_{n \geq 0} a_n z^n. \)

Differentiating with respect to \( z \) we also get

\[ (\mathcal{P}_- f)'(z) = -\int \frac{dw}{z^2(1 - w/z)^2} f(w) = -\oint G_0^+ (z^{-1}, w) f(w), \]

\[ (\mathcal{P}_+ f)'(z) = \int \frac{dw}{w^2(1 - z/w)^2} f(w) = \oint G_0^- (z, w^{-1}) f(w). \]

Note that, by construction,

\[ (\mathcal{P}_+ \hat{\psi}_t)(z) = \hat{\psi}_+(z, t), \quad (\mathcal{P}_- \hat{\phi}_t)(z) = \hat{\phi}_-(z, t). \]
5.1 Explicit solution of equation of motion when $\beta = 2$

We prove here formula (4.23). Let $\tilde{w} \equiv w(t) \in \mathbb{C}[[w]]$, resp. $z(t) \in \mathbb{C}[[z]]$ be the solution at time $t \in \mathbb{R}$ of the ODE $\dot{w}_t = -b(w(t))$, resp. $\dot{z} = -b(z(t))$ with initial condition $w(0) = w$, resp. $z(0) = z$. Let $K_t(z^{-1}, w) := \frac{1}{1 - w(t)/z}$. Then

$$\frac{\partial}{\partial t} \left( \frac{1}{z} \oint dw \tilde{K}_t(z^{-1}, w) \pi_0(w) \right) = \frac{1}{z^2} \oint dw \frac{\tilde{w}(t)}{(1 - w(t)/z)^2} \pi_0(w)$$

$$= - \oint dw \frac{b(w(t))}{z^2(1 - w(t)/z)^2} \pi_0(w)$$

$$= - \oint \frac{d\tilde{w}}{z^2(1 - \tilde{w}/z)^2} \frac{b(\tilde{w}) \pi_0(\tilde{w}(t))}{\partial \tilde{w}/\partial w}$$

$$= \mathcal{P}_- \left( \left( \frac{b(z) \pi_0(z(-t))}{\partial z/\partial z(-t)} \right)'(z) \right) \quad (5.5)$$

We used (5.1) in the last step. Similarly,

$$\mathcal{P}_- \left( \frac{\partial}{\partial z} \left( \frac{b(z)}{z} \oint \frac{dw}{1 - w(t)/z} \pi_0(w) \right) \right) = \mathcal{P}_- \left( b'(z) \oint \frac{\pi_0(w)}{z(1 - w(t)/z)} - b(z) \oint \frac{\pi_0(w)}{z^2(1 - w(t)/z)^2} \right)$$

$$= \mathcal{P}_- \left( b'(z) \mathcal{P}_- \left( \frac{\pi_0(z(-t))}{\partial z/\partial z(-t)} \right) \right) + b(z) \mathcal{P}_- \left( \left( \frac{\pi_0(z(-t))}{\partial z/\partial z(-t)} \right)'(z) \right)$$

$$= \mathcal{P}_- \left( \left( \frac{b(z) \pi_0(z(-t))}{\partial z/\partial z(-t)} \right)'(z) \right) \quad (5.6)$$

5.2 Solution of equation of motion when $\beta \neq 2$

Let us now consider the equation of motion for $\beta \neq 2$. To start with, let $\pi(z, t) := \exp(t\partial_z^2)\pi(z, 0) \equiv \sum_{k \geq 0} \pi_k(t) z^{-k-1}$ be the image of $\pi(z, 0) \equiv \sum_{k \geq 0} \pi_k z^{-k-1}$ by the semi-group generated by $\partial_z^2$. One may check by inspection that

$$\pi(z, t) = \frac{1}{z} \sum_{l=0}^{\infty} \frac{\pi_{l+1}}{l!} \sum_{m=0}^{\infty} \frac{(l + 2m)!}{(2m)!} \left( \frac{t}{z^2} \right)^{2m} \quad (5.7)$$

in Fourier modes one gets

$$\pi_k(t) = \sum_{m=0}^{[k/2]} \binom{k}{2m} \pi_{k-2m} t^{2m} \quad (5.8)$$

Next we compute $K_t(z^{-1}, w)$ in the Hermite case (see example in Section 1). By definition $\pi(z, t) = \exp(t\mathcal{D})\pi(z, 0) = \frac{1}{2} \oint dw \tilde{K}_t(z^{-1}, w) \pi(w, 0)$, where $(\mathcal{D}\pi)(z) := \frac{1}{\pi'}(\pi(z) + z\pi'(z)) - (\frac{\beta}{2} - 1)\pi''(z)$. In order to exponentiate the semi-group $\mathcal{D}$, we consider the formal series

$$\rho(\zeta, t) := \sum_{k \geq 0} \frac{\pi_k(t)}{k!} t^k \quad (5.9)$$

related to $\pi(\zeta, t)$ by a Mellin transform. Through this non-local transform $\partial_z$ becomes the multiplication by $-\zeta$, and the multiplication by $z$ becomes the derivative $\partial_\zeta$, hence
\[ D \equiv -\left(\frac{\beta}{2} - 1\right)\zeta^2 - \frac{1}{\sigma^2} \zeta \partial_\zeta \] is a first-order operator. Looking for a solution of the form 
\[ \rho(\zeta, t) \equiv e^{f(t)} \rho(g(t)\zeta, 0), \]
we get by identification
\[ \dot{g} = -\frac{1}{\sigma^2} \dot{f}, \quad \dot{f} = -\frac{\beta - 2}{\sigma^2} f - \left(\frac{\beta}{2} - 1\right) \] (5.10)
which can be solved straightforwardly, yielding
\[ \rho(\zeta, t) = e^{f(t)} \rho(e^{-t/\sigma^2} \zeta, 0) \] (5.11)
with \( f(t) = -\frac{\sigma^2}{2}(1 - e^{-(\beta - 2)t/\sigma^2}). \) Inverting now the Mellin transform, we remark that \( \rho(\zeta, t) \) is given in (5.11) as the image by \( \exp(f(t)\zeta^2) \equiv \exp(f(t)\partial_\zeta^2) \) of a transformed initial condition \( \tilde{\rho}(e^{-t/\sigma^2} \zeta) = \sum_{k \geq 0} e^{-kt/\sigma^2} \frac{\pi_k}{k!} \zeta^k \), associated to \( \tilde{\pi}(z, 0) = \sum_{k \geq 0} e^{-kt/\sigma^2} \pi_k z^{-k-1} \).

Hence we get
\[ \pi(z, t) = \frac{1}{z} \sum_{k \geq 0} e^{-kt/\sigma^2} \frac{\pi_k}{k!} z^{-k} \sum_{m \geq 0} \left( \frac{k + 2m}{2m} \right) \left( \frac{f(t)}{z^2} \right)^{2m} \] (5.12)
from which we finally obtain an explicit formula for \( K_t \) in the Hermite case,
\[ \hat{K}_t(z^{-1}, w) = \sum_{k \geq 0} (e^{-t/\sigma^2} w/z)^k \sum_{m \geq 0} \left( \frac{k + 2m}{2m} \right) \left( \frac{f(t)}{z^2} \right)^{2m} \] (5.13)
extending (4.49). In Fourier modes this is
\[ \pi_k(t) = \sum_{m=0}^{\lfloor k/2 \rfloor} e^{-(k-2m)t/\sigma^2} \left( \frac{k}{2m} \right) (f(t))^{2m} \pi_{k-2m}. \] (5.14)

### 5.3 A technical lemma

Let \( u \in \mathcal{A}_+ \). We prove here the following result,
\[ \partial_\nu \left( \oint \frac{dw}{w} u(w) \partial_w (K_{t-v}(z^{-1}, w)) K_{v-s}(w^{-1}, \zeta) \right) = \]
\[ = \oint \frac{dw}{w} (u(w)b'(w) - u'(w)b(w)) \partial_w (K_{t-v}(z^{-1}, w)) K_{v-s}(w^{-1}, \zeta) \] (5.15)
Namely,
\[ \partial_\nu \left( \partial_w (K_{t-v}(z^{-1}, w)) K_{v-s}(w^{-1}, \zeta) \right) = \]
\[ = \partial_w (\partial_\nu (K_{t-v}(z^{-1}, w)) K_{v-s}(w^{-1}, \zeta) + \partial_\nu (K_{t-v}(z^{-1}, w))) \partial_\nu (K_{v-s}(w^{-1}, \zeta)) \] (5.16)
is the sum of two terms. We use the second Kolmogorov formula (4.29) for the first one, and the first Kolmogorov formula (4.28) for the second one; using the fundamental relations
and

\[
\frac{dw}{w} u(w) \partial_w (\partial_{t^\prime} (K_{t^\prime - t^\prime} (z^{-1}, w))) \cdot K_{t^\prime - s} (w^{-1}, \zeta) = \int \frac{dw}{w} u(w) \partial_w \left( b(w) \int \frac{d\alpha}{\alpha^2 (1 - w/\alpha)^2} K_{t^\prime - t^\prime} (z^{-1}, \alpha) \right) \cdot K_{t^\prime - s} (w^{-1}, \zeta) = \int \frac{dw}{w} u(w) \partial_w \left( b(w) \partial_{w} (K_{t^\prime - t^\prime} (z^{-1}, w))) \right) \cdot K_{t^\prime - s} (w^{-1}, \zeta) (5.17)
\]

Hence the result.

### 5.4 Time derivations

One finds in the formula (4.51) for \( L^\prime_t \) the time-derivation \( f(t) \partial_t \). By definition, it acts on local functionals of \( \{ (\tau_k(t))_{t \geq 0} \} \) as an infinitesimal change of coordinates,

\[
f(t) \partial_t \cdot \int g(s) \gamma_k(s) ds := \int f(s) \frac{\partial}{\partial y_k} F(s, (y_t)_t) \bigg|_{y = \tau(s)} \gamma_k(s) ds.
\]

In particular, for a linear functional, one finds

\[
f(t) \partial_t \cdot \int g(s) \gamma_k(s) ds = \int g(s) f(s) \gamma_k(s) ds = - \int \frac{d}{ds} (f(s) g(s)) \gamma_k(s) ds.
\]

This action of \( f(t) \partial_t \) extends in a natural way (by duality) to an action on local functionals of \( (\tau_k(\cdot))_{k \geq 0} \) and \( \gamma_k(\cdot)_{k \geq 0} \). Restricting to local functionals, we impose

\[
0 \equiv f(t) \partial_t \cdot \left( \left\langle \int g(s) \partial/\partial \tau_k(s) ds, \int g(s) \gamma_k(s) ds \right\rangle \right)
\]

\[
= \left\langle f(t) \partial_t \cdot \int g(s) \partial/\partial \tau_k(s) ds, \int g(s) \gamma_k(s) ds \right\rangle + \left\langle \int g(s) \partial/\partial \tau_k(s) ds, f(t) \partial_t \cdot \int g(s) \gamma_k(s) ds \right\rangle
\]

(5.21)

so

\[
f(t) \partial_t \cdot \int g(s) \partial/\partial \tau_k(s) ds = - \int f(s) \gamma(s) \partial/\partial \tau_k(s) ds.
\]

(5.22)
In particular,

\[ f(t)\partial_t \cdot \hat{\psi}_-(z, t) = -\beta^{1/2} \cdot \frac{1}{z} \int ds f(s) \frac{\partial}{\partial s} \left( 1_{[0,t]}(s)K_{t-s}(z^{-1}, \zeta) \right) \frac{\partial}{\partial \tau}(\zeta, s) \]

\[ = \beta^{1/2} \left\{ f(t)\partial_t \cdot \hat{\psi}_-(z, t) + \frac{1}{z} \int_0^t ds f(s) \int d\zeta \partial_t(K_{t-s}(z^{-1}, \zeta))\frac{\partial}{\partial \tau}(\zeta, s) \right\}, \]

(compare with the straightforward time-derivative formula (4.39). Obviously the two formulas coincide when \( f = 1 \).

Using the semi-group property of the kernel \( K \), we may express (5.23) somewhat differently. First

\[ f(t)\partial_t \cdot \hat{\psi}_-(z, t) = f(t)(\partial_t \hat{\psi}_-)(z, t) + \beta^{1/2} \frac{1}{z} \int_0^t ds(f(s) - f(t)) \int d\zeta \partial_t(K_{t-s}(z^{-1}, \zeta))\frac{\partial}{\partial \tau}(\zeta, s). \]

Then, by (4.39),

\[ (\partial_t \hat{\psi}_-)(z, t) = \hat{\phi}_-(z, t) + \beta^{1/2} \frac{1}{z} \int_0^t ds \int d\zeta \partial_t(K_{t-s}(z^{-1}, \zeta))\frac{\partial}{\partial \tau}(\zeta, s) \]

\[ = \hat{\phi}_-(z, t) - \beta^{1/2} \frac{1}{z} \int_0^t ds \int d\zeta \int d\alpha \frac{d\alpha}{\alpha^2(1 - \zeta/\alpha)^2} K_{t-s}(z^{-1}, \alpha)b(\zeta)\frac{\partial}{\partial \tau}(\zeta, s) \text{ using (4.29)} \]

\[ = \hat{\phi}_-(z, t) + \beta^{1/2} \frac{1}{z} \int_0^t ds d\alpha K_{t-s}(z^{-1}, \alpha) \left( \mathcal{P}_-(b(\zeta))\frac{\partial}{\partial \tau}(\zeta, s) \right)'(\zeta = \alpha) \]

\[ = \hat{\phi}_-(z, t) - \beta^{1/2} \int_0^t ds \int d\zeta G_{t-s}^+(z^{-1}, \zeta)b(\zeta)\frac{\partial}{\partial \tau}(\zeta, s) \]

(5.25)

and

\[ \frac{1}{z} \beta^{1/2} \int_0^t ds \int d\zeta \partial_t(K_{t-s}(z^{-1}, \zeta))\frac{\partial}{\partial \tau}(\zeta, s) \]

\[ = -\frac{1}{z} \beta^{1/2} \int_0^t ds \int_0^s dt' \int d\zeta \partial_t(K_{t-t'}(z^{-1}, \zeta))\frac{\partial}{\partial \tau}(\zeta, t') \]

\[ = -\frac{1}{z} \beta^{1/2} \int_0^t ds \int_0^s dt' \int d\zeta \int d\xi \frac{d\xi}{\xi} \partial_t(K_{t-s}(z^{-1}, \xi))K_{s-t'}(\xi^{-1}, \zeta)\frac{\partial}{\partial \tau}(\zeta, t') \text{ using (4.26)} \]

\[ = -\frac{1}{z} \int_0^t ds \int d\xi \partial_t(K_{t-s}(z^{-1}, \xi))\mathcal{P}_-\hat{\psi}_s(\xi) \]

\[ = -\frac{1}{z} \int_0^t ds \int d\xi \partial_t(K_{t-s}(z^{-1}, \xi))\hat{\psi}_s(\xi) \]

\[ = \frac{1}{z} \int_0^t ds \int d\xi \hat{\psi}_s(\xi) \int d\zeta \frac{d\zeta}{\zeta^2(1 - \xi/\zeta)^2} b(\zeta)K_{t-s}(z^{-1}, \zeta) \text{ using (4.29)} \]

\[ = -\frac{1}{z} \int_0^t ds \int d\zeta K_{t-s}(z^{-1}, \zeta)\mathcal{P}_-(b\hat{\psi}_s)'(\zeta) \]

\[ = \int_0^t ds \int d\zeta b(\zeta)G_{t-s}^+(z^{-1}, \zeta)\hat{\psi}_s(\zeta). \]

(5.26)
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