On the average behavior of coefficients related to triple product L-functions

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Abstract

In this paper, we study the average behaviour of the coefficients of triple product L-functions and some related L-functions corresponding to normalized primitive holomorphic cusp form \( f(z) \) of weight \( k \) for the full modular group \( SL(2, \mathbb{Z}) \). Here we call \( f(z) \) a primitive cusp form if it is an eighenfunction of all Hecke operators simultaneously.

1 Introduction

For an even integer \( k \geq 2 \), denote \( H_k^* \) the set of all normalized Hecke primitive cusp forms of weight \( k \) for the full modular group \( SL(2, \mathbb{Z}) \). Throughout this paper we call the function \( f(z) \) as a primitive cusp form

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if it is an eighenfunction of all Hecke operators simultaneously. It is known that \( f(z) \) has a Fourier expansion at cusp \( \infty \), write it as

\[
f(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}
\]

for \( \Im(z) > 0 \).

Rewrite the Fourier expansion as

\[
f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{(k-1)/2}e^{2\pi inz}
\]

for \( \Im(z) > 0 \), where \( \lambda_f(n) = \frac{c(n)}{n^{(k-1)/2}} \).

Then by Deligne [B], we have, for any prime number \( p \), there exists two complex numbers \( \alpha_f(p) \) and \( \beta_f(p) \), such that

\[
\alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1
\]

and

\[
\lambda_f(p) = \alpha_f(p) + \beta_f(p).
\]

For a normalized primitive cusp form \( f(z) \) of weight \( k \), the triple product \( L \)-function \( L(f \otimes f \otimes f) \) is defined as

\[
L(f \otimes f \otimes f) = \prod_p \left(1 - \frac{\alpha_p^3}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p}{p^s}\right)^{-3} \left(1 - \frac{\beta_p^3}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-3}
\]

for \( \Re(s) > 1 \). The \( j^{th} \) symmetric power \( L \)-function attached to \( f \) is defined by

\[
(1) \quad L(\text{sym}^j f, s) = \prod_p \prod_{m=0}^{j} (1 - \alpha_p^{-m}\beta_p^m p^{-s})^{-1}
\]

for \( \Re(s) > 1 \). We may express it as a Derichlet series: for \( \Re(s) > 1 \),

\[
L(\text{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^j f}(n)}{n^s}
\]

\[
= \prod_p \left(1 + \frac{\lambda_{\text{sym}^j f}(p)}{p^s} + \ldots + \frac{\lambda_{\text{sym}^j f}(p^k)}{p^{ks}} + \ldots \right).
\]
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It is well known that \( \lambda_{\text{sym}\, j\, f}(n) \) is a real multiplicative function. The Rankin-Selberg \( L \)-function \( L(\text{sym}\, i\, f \otimes \text{sym}\, j\, f, \ s) \) attached to \( \text{sym}\, i\, f \) and \( \text{sym}\, j\, f \) is defined as

\[
L(\text{sym}\, i\, f \otimes \text{sym}\, j\, f, \ s) = \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \left( 1 - \frac{\alpha_p^{i-m} \beta_p^m \alpha_p^{j-m'} \beta_p^{m'}}{p^s} \right)^{-1}
\]

\[
= \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}\, i\, f \otimes \text{sym}\, j\, f}(n)}{n^s}.
\]

For \( \Re(s) > 1 \), define

\[
L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}\, 2\, f \otimes \text{sym}\, f}(n)^2}{n^s}
\]

and for \( \Re(s) > 1 \), define

\[
D_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}\, 2\, f \otimes \text{sym}\, f}(n)^2}{n^s}.
\]

2 Theorems etc.

Here, after \( \epsilon \) and \( \delta \) denote any small positive constants and implied constants will depend at most only on the form \( f \) and \( \epsilon \).

In the paper [C], the following theorems are established.

**Theorem A.** For any \( \epsilon > 0 \), we have

\[
\sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = xP(\log x) + O_{f, \, \epsilon}(x^{(175/181) + \epsilon})
\]

where \( P(t) \) is a polynomial of degree 4.

**Theorem B.** For any \( \epsilon > 0 \), we have

\[
\sum_{n \leq x} \lambda_{\text{sym}\, 2\, f \otimes f}(n)^2 = xQ(\log x) + O_{f, \, \epsilon}(x^{(17/18) + \epsilon})
\]

where \( Q(t) \) is a polynomial of degree 1.

The aim of this article is to improve Theorems A and B.

More precisely we prove:

**Theorem 2.1.** For any \( \epsilon > 0 \) and \( f \in H_k^* \), we have

\[
\sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = xP(\log x) + O_{f, \, \epsilon}(x^{(605/719) + \epsilon})
\]

where \( P(t) \) is a polynomial of degree 4.
Theorem 2.2. For any $\epsilon > 0$ and $f \in H_k^*$, we have
\[
\sum_{n \leq x} \lambda_{\text{sym}^2 f \otimes f}(n)^2 = xQ(\log x) + O_{f, \epsilon}(x^{(2729/2897)+\epsilon})
\]
where $Q(t)$ is a polynomial of degree 1.

Remark 1. It should be noted that the theorem of K. Ramachandra and A Sankaranarayanan [Lemma 3.5] plays a vital role in the proofs of Theorems 2.1 and 2.2. It is easy to check that $\frac{695}{119} < \frac{175}{181}$ and $\frac{2729}{2897} < \frac{17}{18}$. Thus Theorem 2.1 and Theorem 2.2 are unconditional improvements to the Theorem A and Theorem B respectively. Under the assumption of Lindelöf Hypothesis the error terms of Theorem 2.1 and Theorem 2.2 can slightly be improved for which we refer to section 4.

3 Lemmas

Lemma 3.1. Suppose that $\mathcal{L}(s)$ is a general $L$-function of degree $m$. Then, for any $\epsilon > 0$, we have
\[
\int_T^{2T} |\mathcal{L}(\sigma + it)|^2 \, dt \ll T^{\max\{m(1-\sigma),1\}+\epsilon}
\]
uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $T > 1$; and
\[
\mathcal{L}(\sigma + it) \ll (|t| + 1)^{\frac{m}{2}(1-\sigma)+\epsilon}
\]
uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ and $|t| \geq 1$.

For some $L$-functions with small degrees, we invoke either individual or average subconvexity bounds.

Lemma 3.2. For any $\epsilon > 0$, we have
\[
\int_0^T |\zeta\left(\frac{5}{7} + it\right)|^{12} \, dt \ll \epsilon T^{1+\epsilon}
\]
uniformly for $T \geq 1$.

Proof. See, Theorem 8.4 and (8.87) of [D].

Lemma 3.3. For $f \in H_k^*$ and $\epsilon > 0$, we have
\[
L(\text{sym}^2 f, \sigma + it) \ll_{f, \epsilon} (|t| + 1)^{\max\{\frac{11}{7}(1-\sigma),0\}+\epsilon}
\]
uniformly for $\frac{1}{2} \leq \sigma \leq 2$ and $|t| \geq 1$. 

Proof. For $\mathcal{L}(s)$ is a general $L$-function of degree $m$. Then, for any $\epsilon > 0$, we have
\[
\int_T^{2T} |\mathcal{L}(\sigma + it)|^2 \, dt \ll T^{\max\{m(1-\sigma),1\}+\epsilon}
\]
uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $T > 1$; and
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\int_T^{2T} |\mathcal{L}(\sigma + it)|^2 \, dt \ll T^{\max\{m(1-\sigma),1\}+\epsilon}
\]
uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $T > 1$; and
\[
\mathcal{L}(\sigma + it) \ll (|t| + 1)^{\frac{m}{2}(1-\sigma)+\epsilon}
\]
uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ and $|t| \geq 1$.

For some $L$-functions with small degrees, we invoke either individual or average subconvexity bounds.
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**Proof.** See, Corollary 1.2 of [F]. \( \square \)

**Lemma 3.4.** For $|t| \geq 10$, we have

\[
\zeta(\sigma + it) \ll (|t| + 10)^{2\kappa(1-\sigma)+\epsilon}
\]

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ and for some $\kappa \geq 0$.

**Proof.** Follows from [A] and Maximum modulus principle with $\kappa = \frac{13}{84}$. \( \square \)

**Lemma 3.5 (KR+AS).** For $\frac{1}{2} \leq \sigma \leq 2$, $T$-sufficiently large, there exist a $T^* \in [T, T + T^{\frac{1}{3}}]$ such that the bound

\[
\log \zeta(\sigma + iT) \ll (\log \log T^*)^2 \ll (\log \log T)^2
\]

holds uniformly and we have

\[
|\zeta(\sigma + iT)| \ll \exp((\log \log T)^2) \ll, T^\epsilon
\]

on the horizontal line with $T = T^*$ and $\frac{1}{2} \leq \sigma \leq 2$.

**Proof.** See, Lemma 1 of [G]. \( \square \)

**Lemma 3.6.** For $\Re(s) > 1$, define

\[
L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_{f \otimes f \otimes f}(n)^2}{n^s}.
\]

Then we have

\[
L_f(s) = \zeta(s)^5 L(\text{sym}^2 f, s)^8 L(\text{sym}^4 f, s)^4 L(\text{sym}^4 f \otimes \text{sym}^2 f, s)U(s),
\]

where the function $U(s)$ is a Derichlet series which converges absolutely for $\Re(s) > \frac{1}{2}$ and $U(s) \neq 0$ for $\Re(s) = 1$.

**Proof.** Since we have $\lambda_{f \otimes f \otimes f}(n)^2$ is a multiplicative function and the trivial upper bound $O(n^\epsilon)$, we have that, for $\Re(s) > 1$,

\[
L_f(s) = \prod_p \left(1 + \frac{\lambda_{f \otimes f \otimes f}(p)^2}{p^s} + \frac{\lambda_{f \otimes f \otimes f}(p^2)^2}{p^{2s}} + \ldots \right).
\]

In the half-plane $\Re(s) > 1$, the corresponding coefficients of the term $p^{-s}$ determine the analytic properties of $L_f(s)$. By Lemma 2.1 of [C] we easily find the identity

\[
\lambda_{f \otimes f \otimes f}(p)^2 = (\lambda_{\text{sym}^2 f}(p) + 2\lambda_f(p))^2
\]

\[
= \lambda_{\text{sym}^2 f}(p)^2 + 4\lambda_{\text{sym}^2 f}(p)\lambda_f(p) + 4\lambda_f(p)^2.
\]
Consider $\lambda_{\text{sym}^2 f \otimes \text{sym}^4 f}(p)$, the coefficient of $p^{-s}$ in the Euler product of $L(\text{sym}^2 f \otimes \text{sym}^4 f, s)$,

$$
\lambda_{\text{sym}^2 f \otimes \text{sym}^4 f}(p) = 3 + 3\alpha_f(p)^2 + 2\alpha_f(p)^4 + \alpha_f(p)^6 + 3\beta_f(p)^2 + 2\beta_f(p)^4 + \beta_f(p)^6.
$$

Now, consider $\lambda_{\text{sym}^3 f}(p)$, the coefficient of $p^{-s}$ in the Euler product of the $L$–function $L(\text{sym}^3 f, s)$,

$$
\lambda_{\text{sym}^3 f}(p) = \alpha_f(p)^3 + \alpha_f(p) + \beta_f(p)^3 + \beta_f(p).
$$

We have

$$
\lambda_{\text{sym}^3 f}(p)^2 = \left(\alpha_f(p)^3 + \alpha_f(p) + \beta_f(p)^3 + \beta_f(p)\right)^2
= 2(\alpha_f(p)^2 + \beta_f(p)^2 + \alpha_f(p)^4 + \beta_f(p)^4 + 2) \\
+ \alpha_f(p)^2 + \beta_f(p)^2 + \alpha_f(p)^6 + \beta_f(p)^6
$$

(13)

$$
= 1 + \lambda_{\text{sym}^2 f \otimes \text{sym}^4 f}(p).
$$

Now, consider

$$
\lambda_{\text{sym}^3 f}(p)\lambda_f(p) = \left(\alpha_f(p)^3 + \beta_f(p)^3 + \alpha_f(p) + \beta_f(p)\right)\left(\alpha_f(p) + \beta_f(p)\right)
= \alpha_f(p)^4 + \beta_f(p)^4 + 2\alpha_f(p)^2 + 2\beta_f(p)^2 + 2.
$$

By the coefficients of $p^{-s}$ in the Euler products of $L(\text{sym}^2 f, s)$ and $L(\text{sym}^4 f, s)$, we have

$$
\lambda_{\text{sym}^2 f}(p) + \lambda_{\text{sym}^4 f}(p) = \alpha_f(p)^4 + \beta_f(p)^4 + 2\alpha_f(p)^2 + 2\beta_f(p)^2 + 2
$$

(14)

$$
= \lambda_{\text{sym}^3 f}(p)\lambda_f(p).
$$

Consider $\lambda_{\text{sym}^2 f}(p)$, the coefficient of $p^{-s}$ in the Euler product of $L(\text{sym}^2 f, s)$,

$$
\lambda_{\text{sym}^2 f}(p) = \alpha_f(p)^2 + \beta_f(p)^2 + 1.
$$

Now, consider

$$
\lambda_f(p)^2 = \left(\alpha_f(p) + \beta_f(p)\right)^2
= \alpha_f(p)^2 + \beta_f(p)^2 + 2
$$

(15)

$$
= 1 + \lambda_{\text{sym}^2 f}(p).
$$

By using (13), (14) and (15), we have

$$
\lambda_{\otimes f \otimes f}(p)^2 = \left(1 + \lambda_{\text{sym}^2 f \otimes \text{sym}^4 f}(p)\right) + 4\left(\lambda_{\text{sym}^2 f}(p) + \lambda_{\text{sym}^4 f}(p)\right)
+ 4\left(1 + \lambda_{\text{sym}^2 f}(p)\right)
= 5 + 8\lambda_{\text{sym}^2 f}(p) + 4\lambda_{\text{sym}^4 f}(p) + \lambda_{\text{sym}^2 f \otimes \text{sym}^4 f}(p).
$$

Now the lemma follows by standard arguments. \qed
Lemma 3.7. For $\Re(s) > 1$, define

$$D_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^2 f \otimes f}(n)^2}{n^s}. $$

Then we have

$$D_f(s) = \zeta(s)^2 L(\text{sym}^2 f, s)^3 L(\text{sym}^4 f, s)^2 L(\text{sym}^4 f \otimes \text{sym}^2 f, s)V(s), $$

where the function $V(s)$ is a Dirichlet series which converges absolutely for $\Re(s) > \frac{1}{2}$ and $V(s) \neq 0$ for $\Re(s) = 1$.

Proof. By (6.2) of [C], we have

$$\lambda_{\text{sym}^2 f \otimes f}(p) = \lambda_{\text{sym}^3 f}(p) + \lambda_f(p) $$

for $\Re(s) > 1$.

Now the lemma follows in a similar manner as the proof of Lemma 3.6.

Proof of Theorem 2.1. Firstly, recall that

$$L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_{f \otimes f \otimes f}(n)^2}{n^s} $$

for $\Re(s) > 1$ and by (11), we have

$$L_f(s) = \zeta(s)^5 L(\text{sym}^2 f, s)^8 L(\text{sym}^4 f, s)^4 L(\text{sym}^4 f \otimes \text{sym}^2 f, s)U(s), $$

where the function $U(s)$ is a Dirichlet series which converges absolutely for $\Re(s) > \frac{1}{2}$ and $U(s) \neq 0$ for $\Re(s) = 1$.

By applying Perron formula to $L_f(s)$, we have

$$\sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L_f(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\epsilon}}{T}\right) $$

where $b = 1 + \epsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later.

Now, we make the special choice $T = T^*$ of Lemma 3.5 which satisfies (10) and shifting the line of integration to $\Re(s) = \frac{5}{7}$, we have by Cauchy residue theorem

$$\sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = \frac{1}{2\pi i} \left\{ \int_{\frac{5}{7}-iT}^{\frac{5}{7}+iT} + \int_{\frac{5}{7}+iT}^{\frac{5}{7}-iT} + \int_{b-iT}^{b+iT} \right\} L_f(s) \frac{x^s}{s} ds $$

$$\quad + xP(\log x) + O\left(\frac{x^{1+\epsilon}}{T}\right) $$

$$= J_1 + J_2 + J_3 + xP(\log x) + O\left(\frac{x^{1+\epsilon}}{T}\right) $$

(17)
where $P(t)$ is a polynomial of degree 4 and the main term $xP(\log x)$ is coming from the residue of $L_f(s)\frac{x^s}{s}$ at the pole $s = 1$ of order 5.

For $J_1$, we have

$$J_1 \ll x^{5/7+\epsilon} \int_1^T \left| \zeta \left( \frac{5}{7} + it \right) \right|^2 dt + x^{5/7+\epsilon}$$

(18) $$\ll x^{5/7+\epsilon} \sup_{1 \leq T_1 \leq T} I_1(T_1)^{5/12} I_2(T_1)^{1/2} I_3(T_1)^{1/12} T_1^{-1},$$

where

$$I_1(T_1) = \int_{T_1}^{2T_1} \left| \zeta \left( \frac{5}{7} + it \right) \right|^2 dt,$$

$$I_2(T_1) = \int_{T_1}^{2T_1} \left| L(sym^2 f, \frac{5}{7} + it) \right|^2 \left| L(sym^4 f, \frac{5}{7} + it) \right|^4 dt$$

and

$$I_3(T_1) = \int_{T_1}^{2T_1} \left| L(sym^4 f \otimes sym^2 f, \frac{5}{7} + it) \right|^2 dt.$$

Then by Lemmas 3.1, 3.2 and 3.3, we have

$$I_1(T_1) \ll T_1^{1/2}, \quad I_3(T_1) \ll T_1^{180/7+\epsilon}$$

and

$$I_2(T_1) \ll T_1^{16 \times \frac{5}{7} + \frac{5}{7} + \epsilon} \int_{T_1}^{2T_1} \left| L(sym^4 f, \frac{5}{7} + it) \right|^4 dt$$

$$\ll T_1^{12+\epsilon}.$$ 

Hence, we have

$$J_1 \ll x^{5/7+\epsilon} \sup_{1 \leq T_1 \leq T} I_1(T_1)^{5/12} I_2(T_1)^{1/2} I_3(T_1)^{1/12} T_1^{-1}$$

(19) $$\ll x^{(5/7)+\epsilon} T^{(635/84)+\epsilon}.$$ 

For the integrals over horizontal segments, by using (5), (7) and (10), we have

$$J_2 + J_3 \ll \max_{\frac{5}{7} \leq \sigma \leq b} x^{\sigma} T^{10\epsilon + \left(8 \times \frac{11}{8} + \frac{20}{7}\right) (1 - \sigma) - 1}$$

$$= T^{10\epsilon} \max_{\frac{5}{7} \leq \sigma \leq b} \left( \frac{x}{T^{2\sigma}} \right)^{\sigma} T^{\frac{55}{16} + \epsilon}$$

$$\ll x^{\frac{5}{7} + \epsilon} T^{\frac{55}{16} + 10\epsilon} + x^{1+15\epsilon} T.$$

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From (17), (19) and (20), we have

\[ \sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = xP(\log x) + O\left( \frac{x^{1+15\epsilon}}{T} \right) + o\left( x^{(5/7) + \epsilon T(635/84) + \epsilon} \right). \]

By taking \( T = \frac{x^{24}}{719} \) in (21), we have

\[ \sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = xP(\log x) + O\left( x^{(695/719) + \epsilon} \right). \]

This completes the proof of Theorem 2.1.

**Proof of Theorem 2.2.** Recall that

\[ D_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^2 f \otimes f}(n)^2}{n^s} \]

for \( \Re(s) > 1 \) and by Lemma 3.7, we have

\[ D_f(s) = \zeta(s)^2 L(\text{sym}^2 f, s)^3 L(\text{sym}^4 f, s)^2 L(\text{sym}^4 f \otimes \text{sym}^2 f, s)V(s), \]

where the function \( V(s) \) is a Dirichlet series which converges absolutely for \( \Re(s) > \frac{1}{2} \) and \( V(s) \neq 0 \) for \( \Re(s) = 1 \).

Now, by applying Perron formula to \( D_f(s) \), we have

\[ \sum_{n \leq x} \lambda_{\text{sym}^2 f \otimes f}(n)^2 = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} D_f(s) \frac{x^s}{s} ds + O\left( \frac{x^{1+\epsilon}}{T} \right) \]

where \( b = 1 + \epsilon \) and \( 1 \leq T \leq x \) is a parameter to be chosen later.

Now, we make the special case \( T = T^{**} \) of Lemma 3.5 which satisfies (10) and shifting the line of integration to \( \Re(s) = \frac{1}{2} + \epsilon \), we have by Cauchy residue theorem

\[ \sum_{n \leq x} \lambda_{\text{sym}^2 f \otimes f}(n)^2 = \frac{1}{2\pi i} \left\{ \int_{\frac{1}{2}+\epsilon-iT}^{\frac{1}{2}+\epsilon+iT} + \int_{b-iT}^{b+iT} + \int_{\frac{1}{2}+\epsilon+iT}^{b+\epsilon+iT} \right\} D_f(s) \frac{x^s}{s} ds \]

\[ + xQ(\log x) + O\left( \frac{x^{1+\epsilon}}{T} \right) \]

\[ = J_1 + J_2 + J_3 + xQ(\log x) + O\left( \frac{x^{1+\epsilon}}{T} \right). \]

(22)

Where \( Q(t) \) is a polynomial of degree 1 and the main term \( xQ(\log x) \) is coming from the residue of \( D_f(s) \frac{x^s}{s} \) at the pole \( s = 1 \) of order 2.

For \( J_1 \), we have

\[ J_1 \ll x^{\frac{1}{2}+\epsilon} \int_{1}^{T} |\zeta(\frac{1}{2} + \epsilon + it)^2 L(\text{sym}^2 f, \frac{1}{2} + \epsilon + it)^3 L(\text{sym}^4 f, \frac{1}{2} + \epsilon + it)^2 L(\text{sym}^4 f \otimes \text{sym}^2 f, \frac{1}{2} + \epsilon + it)V(\frac{1}{2} + \epsilon + it)|^{-1} dt + x^{\frac{1}{2}+\epsilon}. \]
By Lemma 3.4 and Cauchy-Schwarz inequality, we have
\[ J_1 \ll x^{\frac{1}{2}+\epsilon} \sup_{1 \leq T_i \leq T} T_1^{2x+2k+1+33/16} \left( \int_{T_1}^{2T_1} \left| L(\text{sym}^4 f, \frac{1}{2} + it) \right|^2 dt \right)^{\frac{1}{2}}. \]

By (1) of Lemma 3.1, we have
\[ J_1 \ll x^{\frac{1}{2}+\epsilon} + x^{\frac{1}{2}+\epsilon} \sup_{1 \leq T_i \leq T} T_1^{2k+11/16 + \frac{10}{24} + \frac{15}{8} + \frac{1}{2} - 1+\epsilon} \]
\[ \ll x^{(1/2)+\epsilon} T^{2k+(117/16)+\epsilon}. \]

For the integrals over the horizontal segments, by using (5), (7) and (10), we have
\[ J_2 + J_3 \ll \max_{\frac{1}{2}+\epsilon \leq \sigma \leq b} x^\sigma T^{(3\times \frac{11}{8} + 25/7)(1-\sigma)-1+2\epsilon} \]
\[ = T^{2\epsilon} \max_{\frac{1}{2}+\epsilon \leq \sigma \leq b} \left( \frac{x}{T^{\frac{1}{133+32k}}} \right)^\sigma T^{\frac{125}{133+32k}} \]
\[ \ll x^{(1/2)+\epsilon} T^{(117/16)+2\epsilon} + \frac{x^{1+10\epsilon}}{T}. \]

From (22), (24) and (25), we have
\[ (26) \quad \sum_{n \leq x} \lambda_{\text{sym}^2 f \otimes f}(n)^2 \ll xQ(\log x) + x^{(1/2)+\epsilon} T^{2k+(117/16)+\epsilon} + O\left( \frac{x^{1+\epsilon}}{T} \right). \]

By taking \( T = x^{\frac{8}{133+32k}} \) in (26), we have
\[ (27) \quad \sum_{n \leq x} \lambda_{\text{sym}^2 f \otimes f}(n)^2 = xQ(\log x) + O\left( x^{(1/2)+\epsilon} \right). \]

Now Theorem 2.2 follows by taking \( \kappa = \frac{13}{84} \) and we obtain
\[ \sum_{n \leq x} \lambda_{\text{sym}^2 f \otimes f}(n)^2 = xQ(\log x) + O\left( x^{(2729/2897)+\epsilon} \right). \]

\[ \square \]

4 \quad Some conditional results

**Lindelöf Hypothesis for \( \zeta(s) \):**

This states that for \( |t| \geq 10 \)
\[ (28) \quad \zeta(s + it) \ll (|t| + 10)^{\epsilon} \]
for all \( \epsilon > 0 \) uniformly for \( \frac{1}{2} \leq \sigma \leq 2 \). [Refer [H], pp. 328-335].
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**Theorem 4.1.** Assuming Lindelöf Hypothesis for $\zeta(s)$. For any $\epsilon > 0$ and $f \in H_k^*$, we have

$$\sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = xP(\log x) + O_f, \epsilon(x^{(55/57) + \epsilon})$$

where $P(t)$ is a polynomial of degree 4.

**Theorem 4.2.** Assuming Lindelöf Hypothesis for $\zeta(s)$. For any $\epsilon > 0$ and $f \in H_k^*$, we have

$$\sum_{n \leq x} \lambda_{\text{sym}^2 f \otimes f}(n)^2 = xQ(\log x) + O_f, \epsilon(x^{(125/133) + \epsilon})$$

where $Q(t)$ is a polynomial of degree 1.

**Proof of Theorem 4.1.** From the proof of Theorem 2.1, recall that

$$\sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = J_1 + J_2 + J_3 + xP(\log x) + O\left(\frac{x^{1+\epsilon}}{T}\right)$$

Where $P(t)$ is a polynomial of degree 4.

For $J_1$ by (7), (28) and Cauchy-Schwarz inequality, we have

$$J_1 \ll x^{5/7 + \epsilon} + x^{5/7 + \epsilon} \int_1^T \left| \zeta\left(\frac{5}{7} + \epsilon + it\right)\right|^5 L(\text{sym}^2 f, \frac{5}{7} + \epsilon + it)^8 L(\text{sym}^4 f, \frac{5}{7} + \epsilon + it)^4 L(\text{sym}^4 f \otimes \text{sym}^2 f, \frac{5}{7} + \epsilon + it) |t^{-1} dt$$

$$\ll x^{5/7 + \epsilon} + x^{5/7 + \epsilon} \sup_{1 \leq T_1 \leq T} \left\{ \left\{ \max_{T_1 \leq t \leq 2T_1} \left| \zeta\left(\frac{5}{7} + \epsilon + it\right)\right|^5 L(\text{sym}^2 f, \frac{5}{7} + \epsilon + it)^8 \right| \right\} \right\} \left\{ \int_{T_1}^{2T_1} |L(\text{sym}^4 f, \frac{5}{7} + \epsilon + it)^4 L(\text{sym}^4 f \otimes \text{sym}^2 f, \frac{5}{7} + \epsilon + it)| t^{-1} dt \right\}$$

$$\ll x^{5/7 + \epsilon} + x^{5/7 + \epsilon} \sup_{1 \leq T_1 \leq T} T_1^{-5/7 + 2\epsilon - 1} \left( \int_{T_1}^{2T_1} |L(\text{sym}^4 f, \frac{5}{7} + \epsilon + it)^4|^2 dt \right)^{1/2} \left( \int_{T_1}^{2T_1} |L(\text{sym}^4 f \otimes \text{sym}^2 f, \frac{5}{7} + \epsilon + it)|^2 dt \right)^{1/2}.$$

By (11), we have

$$J_1 \ll x^{5/7 + \epsilon} + x^{5/7 + \epsilon} \sup_{1 \leq T_1 \leq T} T_1^{-5/7 + 2\epsilon - 1} \left( \int_{T_1}^{2T_1} |L(\text{sym}^4 f, \frac{5}{7} + \epsilon + it)^4|^2 dt \right)^{1/2} \left( \int_{T_1}^{2T_1} |L(\text{sym}^4 f \otimes \text{sym}^2 f, \frac{5}{7} + \epsilon + it)|^2 dt \right)^{1/2} \ll x^{(5/7) + \epsilon} T^{(50/7) + 10\epsilon}.$$
For the integrals over horizontal segments, by (20), we have

\[ J_2 + J_3 \ll x^{(5/7) + \epsilon} T^{(50/7) + 10\epsilon} + \frac{x^{1+15\epsilon}}{T}. \]

Hence by (29), we have

\[ \sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = xP(\log x) + O\left(\frac{x^{1+15\epsilon}}{T}\right) + O\left(x^{(5/7) + \epsilon} T^{(50/7) + \epsilon}\right). \]

By taking \( T = x^{\frac{5}{57}} \) in (30), we have

\[ \sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = xP(\log x) + O\left(x^{(55/57) + \epsilon}\right). \]

**Proof of Theorem 4.2.** From the asymptotic formula in (27), by assuming Lindelöf Hypothesis for \( \zeta(s) \), we have \( \kappa = \epsilon \), where \( \epsilon \) is any positive constant and we obtain

\[ \sum_{n \leq x} \lambda_{\text{sym}^2 f \otimes f}(n)^2 = xQ(\log x) + O\left(x^{(125/133) + \epsilon}\right). \]

**Remark 2.** It is easy to check that \( \frac{55}{57} < \frac{605}{697} < \frac{175}{181} \) and \( \frac{125}{133} < \frac{2729}{2897} < \frac{17}{18} \) (pertain into Theorem A and Theorem B).

**Concluding Remark.** If one has the Lindelöf Hypothesis bound for the \( L \)-function \( L_f(s) \), namely

\[ L_f(\sigma + it) \ll (|t| + 10)^\epsilon \]

holds for all \( \epsilon > 0 \) uniformly for \( \frac{1}{2} \leq \sigma \leq 2 \) and \( |t| \geq 10 \), then it is not difficult to see that the asymptotic formula

\[ \sum_{n \leq x} \lambda_{f \otimes f \otimes f}(n)^2 = xP(\log x) + O\left(x^{\frac{1}{2} + \epsilon}\right) \]

holds, where \( P(t) \) is a polynomial of degree 4. Of course such an expected improvement is far away.

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