ASSOCIATIVE SUBMANIFOLDS OF A $G_2$ MANIFOLD

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Abstract. We study deformations of associative submanifolds $Y^3 \subset M^7$ of a $G_2$ manifold $M^7$. We show that the deformation space can be perturbed to be smooth, and it can be made compact and zero dimensional by constraining it with an additional equation. This allows us to associate local invariants to associative submanifolds of $M$. The local equations at each associative $Y$ are restrictions of a global equation on a certain associated Grassmann bundle over $M$.

0. Introduction

McLean showed that, in a $G_2$ manifold $(M^7, \varphi)$, the space of associative submanifolds near a given one $Y^3$, can be identified with the harmonic spinors on $Y$ twisted by a certain bundle $E$ (the kernel of a twisted Dirac operator) [M]. But since we cannot control the cokernel of the Dirac operator (it has index zero), the dimension of its kernel might vary. This is the obstruction to smoothness of the moduli space of associative submanifolds. This problem can be remedied either by deforming the ambient $G_2$ structure (i.e. by deforming $\varphi$) or by deforming the connection in the normal bundle [AS]. The first process might move $\varphi$ to a non-integrable $G_2$ structure. If we are to view $(M, \varphi)$ as an analogue of a symplectic manifold and $\varphi$ a symplectic form, and view the associative submanifolds as analogues of holomorphic curves, deforming $\varphi$ would be too destructive. In the second process we use the connections as auxiliary objects to deform the associative submanifolds in a larger space, just like deforming the holomorphic curves by using almost complex structures (pseudo-holomorphic curves). By this approach we obtain the smoothness of the moduli space. We get compactness by relating the deformation equation to the Seiberg-Witten equations.

In this paper we summarize the results of [AS] where we introduced complex associative submanifolds of $G_2$ manifolds; they are associative submanifolds whose normal bundles carry a $U(2)$ structure. This is no restriction, since every associative submanifold has this structure, but if we require that their deformations be compatible with the background connection we must have an integrability condition, i.e. the condition that the connection on their normal bundles (induced by the $G_2$ background metric)

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reduce to a $U(2)$ connection. We call such manifolds \emph{integrable complex associative submanifolds}. The prototype of these manifolds are holomorphic curves crossed by circles inside of $CalabiYau \times S^1$. We don’t know how abundant these manifolds are in general. One natural way to remove this restriction is to use a generalized version of the Seiberg-Witten theory \cite{FL}, the drawback of this is that it is a harder theory where we don’t have automatic compactness. There is a more general class of submanifolds, namely the associative submanifolds whose normal bundles carry a $\text{Spin}^c(4)$ structure. This structure allows us to break the normal bundle into pair of complex bundles, define necessary Dirac operator between them, and perform the deformations in these bundles via the standard Seiberg-Witten theory. In this case we don’t have a connection compatibility problem, because we perform the deformation in a larger space by using the whole background connection with the help of an auxiliary connection. More naturally, we can define $U(2)$ or $\text{Spin}^c(4)$ structures on a $G_2$ manifold itself, and induce them to all its associative submanifolds. We can also define global equations on the 15-dimensional associative Grassmannian bundle over a $G_2$ manifold $(M, \varphi)$, such a way that they restrict to the above mentioned Seiberg-Witten equations on each of its associative submanifold $Y^3 \subset M$.

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1. Preliminaries

Here we give some basic definitions and known facts concerning the manifolds with special holonomy, the reader can find all these interesting facts about them in \cite{B2}, \cite{B3}, \cite{HL}. The exceptional Lie group $G_2$ can be defined as the subgroup of $GL(7, \mathbb{R})$ fixing a particular 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$. Denote $\epsilon^{ijk} = dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7)$, then

$$G_2 = \{ A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0 \}$$

$$\varphi_0 = \epsilon^{123} + \epsilon^{145} + \epsilon^{167} + \epsilon^{246} - \epsilon^{257} - \epsilon^{347} - \epsilon^{356}$$

\textbf{Definition 1.} A smooth 7-manifold $M^7$ has a $G_2$ structure if its tangent frame bundle reduces to a $G_2$ bundle. Equivalently $M^7$ has a $G_2$ structure if it has a 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$ the pair $(T_x(M), \varphi(x))$ is isomorphic to $(\mathbb{R}^7, \varphi_0)$.

In fact $G_2$ is a 14-dimensional subgroup of $SO(7)$. Since $GL(7, \mathbb{R})$ acts on $\Omega^3(\mathbb{R}^7)$ with stabilizer $G_2$, its orbit $\Omega^3_+(\mathbb{R}^7)$ is open for dimension reasons, so the choice of $\varphi_0$ in the above definition is generic. In fact the action of $GL^+(7, \mathbb{R})$ has two orbits containing $\pm \varphi_0$. The set of smooth 7-manifolds with $G_2$-structures coincides with the set of 7-manifolds with spin structure, though this correspondence is not $1 \rightarrow 1$. This is because $\text{Spin}(7)$ acts on $S^7$ with stabilizer $G_2$ inducing the fibrations

$$G_2 \rightarrow \text{Spin}(7) \rightarrow S^7 \rightarrow BG_2 \rightarrow B\text{Spin}(7)$$
and hence there is no obstruction to lifting maps $M^7 \to B\text{Spin}(7)$ to $BG_2$. The cotangent frame bundle $P^\ast(M) \to M$ of a manifold with $G_2$ structure $(M, \varphi)$ can be expressed as $P^\ast(M) = \bigcup_{x \in M} P_x^\ast(M)$, where each fiber is given by:

$$(1) \quad P_x^\ast(M) = \{ u \in \text{Hom}(T_x(M), \mathbb{R}^7) \mid u^\ast(\varphi_0) = \varphi(x) \}$$

It turns out that any $G_2$ structure $\varphi$ on $M^7$ gives an orientation $\mu \in \Omega^7(M)$ on $M$, and in turn $\mu$ determines a metric $g = \langle , \rangle$ on $M$, and a cross product structure $\times$ on its tangent bundle of $M$ as follows: Let $i_v$ denote the interior product with a vector $v$, then

$$(2) \quad g(u, v) = \langle i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi \rangle / \mu$$

$$(3) \quad \varphi(u, v, w) = g(u \times v, w)$$

To emphasize the dependency on $\varphi$ sometimes $g$ is denoted by $g_\varphi$. There is a notion of a $G_2$ structure $\varphi$ on $M^7$ being integrable, which corresponds to $\varphi$ being harmonic.

**Definition 2.** A manifold with $G_2$ structure $(M, \varphi)$ is called a $G_2$ manifold if the holonomy group of the Levi-Civita connection (of the metric $g_\varphi$) lies inside of $G_2$. Equivalently $(M, \varphi)$ is a $G_2$ manifold if $d\varphi = d^\ast\varphi = 0$ (i.e. $\varphi$ harmonic).

In short one can define a $G_2$ manifold to be any Riemannian manifold $(M^7, g)$ whose holonomy group is contained in $G_2$; then $\varphi$ and the cross product $\times$ come as a consequence. It turns out that the condition $\varphi$ being harmonic is equivalent to the condition that at each point $x \in M$ there is a chart such that $\varphi(x) = \varphi_0 + O(|x|^2)$.

For example if $(X, \omega, \Omega)$ is a complex 3-dimensional Calabi-Yau manifold with Kähler form $\omega$ and a nowhere vanishing holomorphic 3-form $\Omega$, then $X^6 \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a $G_2$ manifold. In this case $\varphi = \text{Re } \Omega + \omega \wedge dt$.

**Definition 3.** A 3-dimensional submanifold $Y$ of a $G_2$ manifold $(M, \varphi)$ is called associative submanifold if $\varphi|_Y \equiv \text{vol}(Y)$. This condition is equivalent to $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by the identity:

$$(4) \quad \langle \chi(u, v, w), z \rangle = \ast\varphi(u, v, w, z)$$

The equivalence of these conditions follows from the ‘associator equality’ of [HL]

$$\varphi(u, v, w)^2 + |\chi(u, v, w)|^2 / 4 = |u \wedge v \wedge w|^2$$

We will denote the tangent bundle valued 3-form in $\mathbb{R}^7$ corresponding to $\varphi_0$ by $\chi_0$. Throughout this paper we will denote the sections of a bundle $\xi \to Y$ by $\Omega^p(Y, \xi)$ or simply by $\Omega^p(\xi)$, and the bundle valued $p$-forms by $\Omega^p(\xi) = \Omega^p(\Lambda^p T^\ast Y \otimes \xi)$. We will denote the coframe bundle by $P^\ast(M) \to M$ and its adapted frame bundle by $P(M)$. 

They can be $G_2$ or $SO(7)$ frame bundles; when needed we will specify them by the notations $\mathcal{P}_{SO(7)}(M)$ or $\mathcal{P}_{G_2}(M)$.

2. Grassmann bundles

Let $G(3, 7)$ be the Grassmann manifold of oriented 3-planes in $\mathbb{R}^7$. Let $M^7$ be any smooth 7-manifold, and let $\tilde{M}$ be the 3-planes in $T(M)$, i.e. $\tilde{M} \to M$ is the bundle of Grassmannians over $M$ defined by

$$\tilde{M} = \mathcal{P}_{SO(7)}(M) \times_{SO(7)} G(3, 7) \to M$$

Let $\xi \to G(3, 7)$ be the universal $\mathbb{R}^3$ bundle, and $\nu = \xi^\perp \to G(3, 7)$ be the dual $\mathbb{R}^4$ bundle. Hence $Hom(\xi, \nu) = \xi^* \otimes \nu \to G(3, 7)$ is just the tangent bundle $TG(3, 7)$. Let $\xi$, $\nu$ extend fiberwise to give bundles $\Xi \to \tilde{M}$, $V \to \tilde{M}$ respectively, and let $\Xi^*$ denote the dual of $\Xi$. Notice that $Hom(\Xi, V) = \Xi^* \otimes V \to \tilde{M}$ is just the bundle of vertical vectors $T_{vert}(\tilde{M})$ of $T(\tilde{M}) \to M$, i.e. it is the bundle of tangent vectors to the fibers of $\pi : \tilde{M} \to M$. Hence

$$T\tilde{M} = T_{vert}(\tilde{M}) \oplus \Xi \oplus V = (\Xi^* \otimes V) \oplus \pi^*TM$$

Let $\mathcal{P}(V) \to \tilde{M}$ be the $SO(4)$ frame bundle of the vector bundle $V$, and identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, and identify $SU(2)$ with the unit quaternions $Sp(1)$. Recall that $SO(4)$ is the equivalence classes of pairs $[q, \lambda]$ of unit quaternions

$$SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$$

Hence $V \to \tilde{M}$ is the associated vector bundle to $\mathcal{P}(V)$ via the $SO(4)$ representation

$$x \mapsto qx\lambda^{-1}$$

There is a pair of $\mathbb{R}^3 = im(\mathbb{H})$ bundles over $\tilde{M}$ corresponding to the left and right $SO(3)$ reductions of $SO(4)$, so they are given by the $SO(3)$ representations

$$\lambda_+(V) : x \mapsto qxq^{-1}$$
$$\lambda_-(V) : y \mapsto \lambda y\lambda^{-1}$$

The map $x \otimes y \mapsto xy$ gives actions $\lambda_+(V) \otimes V \to V$ and $V \otimes \lambda_-(V) \to V$; by combining we can think of it as one conjugation action

$$(\lambda_+(V) \otimes \lambda_-(V)) \otimes V \to V$$

If the $SO(4)$ bundle $V \to \tilde{M}$ lifts to a $Spin(4) = SU(2) \times SU(2)$ bundle (locally it does), we get two additional bundles over $\tilde{M}$

$$S : y \mapsto qy$$
$$E : y \mapsto y\lambda^{-1}$$
which gives $V$ as a tensor product of two quaternionic line bundles $V = S \otimes H^E$. In particular $\lambda_+(V) = \text{ad}(S)$ and $\lambda_-(V) = \text{ad}(E)$, i.e. they are the $SO(3)$ reductions of the $SU(2)$ bundles $S$ and $E$. In particular there is multiplication map $S \otimes E \to V$.

3. Associative Grassmann bundles

Now consider the Grassmannian of associative 3-planes $G^\varphi(3, 7)$ in $\mathbb{R}^7$, consisting of elements $L \in G(3, 7)$ with the property $\varphi_0|_L = \text{vol}(L)$ (or equivalently $\chi_0|_L = 0$). $G_2$ acts on $G^\varphi(3, 7)$ transitively with the stabilizer $SO(4)$, giving the the identification $G^\varphi(3, 7) = G_2/SO(4)$. Identify the imaginary octonions $\mathbb{R}^7 = \text{Im}(O) \cong \text{im}(H) \oplus \mathbb{H}$, then the action of the subgroup $SO(4) \subset G_2$ on $\mathbb{R}^7$ is given by:

$$\begin{pmatrix} A & 0 \\ 0 & \rho(A) \end{pmatrix}$$

where $\rho : SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2 \to SO(3)$ is the projection of the first factor [HL]. Now let $M^7$ be a $G_2$ manifold. As above, this time we can construct the bundle of associative Grassmannians over $\tilde{M}$

$$\tilde{M}_\varphi = \mathcal{P}_{G_2}(M) \times_{G_2} G^\varphi(3, 7) \to M$$

which is just the quotient bundle $\tilde{M}_\varphi = \mathcal{P}(M)/SO(4) \to \mathcal{P}(M)/G_2 = M$. As in the previous section, the universal bundles $\xi, \nu = \xi^\pm \to G^\varphi(3, 7)$ induce 3 and 4 plane bundles $\Xi \to \tilde{M}_\varphi$ and $\mathbb{V} \to \tilde{M}_\varphi$ (i.e. restricting the universal bundles from $\tilde{M}$). Also

$$T\tilde{M}_\varphi = T_{\text{vert}}(\tilde{M}_\varphi) \oplus \Xi \oplus \mathbb{V}$$

So, in this associative case, we have an important identification $\Xi = \lambda_+(\mathbb{V})$ (as bundles over $\tilde{M}_\varphi$). For simplicity we will denote $Q = \lambda_-(\mathbb{V})$. As before we have an action $\mathbb{V} \otimes Q \to \mathbb{V}$, and the similar action of $\lambda_+(\mathbb{V})$ on $\mathbb{V}$. We will break the dual of this last action into a pair of Clifford multiplications on $\mathbb{V}^\pm = \mathbb{V}$

$$\Xi^* \otimes \mathbb{V}^\pm \to \mathbb{V}^\mp$$

given by $x \otimes y \mapsto -\bar{y}$ and $x \otimes y \mapsto xy$, on $\mathbb{V}^+$ and $\mathbb{V}^-$ respectively. By writing $x_1 \wedge x_2 = \frac{1}{2}(x_1 \otimes x_2 - x_2 \otimes x_1)$ we can extend this to an action of the whole exterior algebra $\Lambda^*(\Xi^*)$. For example $(x_1 \wedge x_2) \otimes y \mapsto \frac{1}{2}(-x_1 \bar{x}_2 + x_2 \bar{x}_1)y = \text{Im}(x_2 \bar{x}_1)y$ gives

$$\Lambda^2(\Xi^*) \otimes \mathbb{V} \to \mathbb{V}$$

Recall $T_{\text{vert}}(\tilde{M}) = \Xi^* \otimes \mathbb{V}$ is the the subbundle of vertical vectors of $T(\tilde{M}) \to M$. The total space $E(\nu_\varphi)$ of the normal bundle of the imbedding $\tilde{M}_\varphi \subset \tilde{M}$ should be thought of an open tubular neighborhood of $\tilde{M}_\varphi$ in $\tilde{M}$, and it has a nice description:
Lemma 1. \((M)\) The normal bundle \(\nu_\varphi\) of \(\tilde{M}_\varphi \subset \tilde{M}\) is isomorphic to \(\mathbb{V}\), and the bundle of vertical vectors \(T_{\text{vert}}(\tilde{M}_\varphi)\) of \(T(\tilde{M}_\varphi) \rightarrow M\) is the kernel of the Clifford multiplication \(c : \Xi^* \otimes \mathbb{V} \rightarrow \mathbb{V}\).

Hence we have \(\Xi^* \otimes \mathbb{V}|_{\tilde{M}_\varphi} = T_{\text{vert}}(\tilde{M}_\varphi) \oplus \nu_\varphi\); from the exact sequence over \(\tilde{M}_\varphi\)
\[
T_{\text{vert}}(\tilde{M}_\varphi) \rightarrow \Xi^* \otimes \mathbb{V}|_{\tilde{M}_\varphi} \rightarrow c \rightarrow \mathbb{V}|_{\tilde{M}_\varphi}
\]
the quotient bundle, \(T_{\text{vert}}(\tilde{M})/T_{\text{vert}}(\tilde{M}_\varphi)\) is isomorphic to \(\nu_\varphi\). Also, if \(\{e^i\}\) is a local orthonormal basis for \(\Xi^*\), then the projection \(\pi_\varphi : T_{\text{vert}}(\tilde{M})|_{\tilde{M}_\varphi} \rightarrow T_{\text{vert}}(\tilde{M}_\varphi)\) is given by
\[
\pi_\varphi(a \otimes v) = a \otimes v + (1/3) \sum_{j=1}^3 e^j \otimes e^j(a.v)
\]

Remark 1. The lemma holds since the Lie algebra inclusion \(g_2 \subset so(7)\) is given by
\[
\begin{pmatrix}
a \\
-\beta^t \\
\rho(a)
\end{pmatrix}
\]
where \(a \in so(4)\) is \(y \mapsto qy - y\lambda\), and \(\rho(a) \in so(3)\) is \(x \mapsto qx - xq\). So the tangent space inclusion of \(G_2/so(4) \subset SO(7)/SO(4) \times SO(3)\) is given by the matrix \(\beta\). And if we write \(\beta\) as column vectors \(\beta = (\beta_1, \beta_2, \beta_3)\), then \(\beta_1i + \beta_2j + \beta_3k = 0\) \((M, Mc)\)

4. Associative submanifolds

Any imbedding of a 3-manifold \(f : Y^3 \hookrightarrow M^7\) induces a bundle \(\tilde{Y} \rightarrow Y\) with fibers \(G(3, 7)\), and the Gauss map of \(f\) canonically lifts to an imbedding \(\tilde{f} : Y \hookrightarrow \tilde{M}\):
\[
\begin{array}{c}
\tilde{M} \supset \tilde{M}_\varphi \\
\downarrow \tilde{f} \\
Y \overset{f}{\rightarrow} M
\end{array}
\]
Also, the pull-backs \(\tilde{f}^*\Xi = T(Y)\) and \(\tilde{f}^*\nu = \nu(Y)\) give the tangent and normal bundles of \(Y\). Furthermore, if \(f\) is an imbedding as an associative submanifold in a \(G_2\) manifold \((M, \varphi)\), then the image of \(\tilde{f}\) lands in \(\tilde{M}_\varphi\). We will denote this canonical lifting of any 3-manifold \(Y \subset M\) by \(\tilde{Y} \subset \tilde{M}\).

\(\tilde{M}_\varphi\) can be thought of as a universal space parametrizing associative submanifolds of \(M\). In particular, if \(\tilde{f} : Y \hookrightarrow \tilde{M}_\varphi\) is the lifting of an associative submanifold, by pulling back we see that the principal \(SO(4)\) bundle \(\mathcal{P}(\mathbb{V}) \rightarrow \tilde{M}_\varphi\) induces an
SO(4)-bundle \( \mathcal{P}(Y) \rightarrow Y \), and gives the following vector bundles over \( Y \) via the representations:

\[
\begin{align*}
\nu(Y) & : y \mapsto qy\lambda^{-1} \\
T(Y) & : x \mapsto qx q^{-1} \\
Q & : y \mapsto \lambda y\lambda^{-1}
\end{align*}
\]

where \([q, \lambda] \in SO(4)\). Note that \( T(Y) = \lambda_+(\nu) \) and \( Q = \lambda_-(\nu) \), where \( \nu = \nu(Y) \).

Also we can identify \( T^*Y \) with \( \lambda - \nu \) and \( Q = \lambda - \nu \), where \( \nu = \nu(Y) \).

Also we can identify \( T^*Y \) with \( \lambda^* \) by the induced metric. We have the actions \( T^*Y \otimes \nu \rightarrow \nu, \Lambda^*(T^*Y) \otimes \nu \rightarrow \nu \), \( \nu \otimes Q \rightarrow \nu \) from above. By combining them we can define an action of the bundle valued differential forms

\[
\Omega^*(Y; Q) \otimes \Omega^0(Y, \nu) \rightarrow \Omega^0(Y, \nu)
\]

For example, if \( F = (x_1 \wedge x_2) \otimes y \), and \( y \in \Omega^0(Y, Q) \), and \( z \in \Omega^0(Y, \nu) \) then

\[
F \otimes z \mapsto \text{Im}(x_2 \bar{x}_1) z y
\]

Let \( L = \Lambda^3(\Xi) \rightarrow \tilde{M} \) be the determinant (real) line bundle. Note that the definition \( (4) \) implies that \( \chi \) maps every oriented 3-plane in \( T_x(M) \) to its complementary subspace, so \( \chi \) gives a bundle map \( L \rightarrow \mathcal{V} \) over \( \tilde{M} \), which is a section of \( L^* \otimes \mathcal{V} \rightarrow \tilde{M} \). Also, \( \Xi \) being an oriented bundle implies \( L \) is trivial so \( \chi \) actually gives a section

\[
\chi \in \Omega^0(\tilde{M}, \mathcal{V})
\]

Clearly \( \tilde{M}_\varphi \subset \tilde{M} \) is the codimension 4 submanifold given as the zeros of this section. So associative submanifolds \( Y \subset M \) are characterized by the condition \( \chi|_Y = 0 \), where \( \tilde{Y} \subset \tilde{M} \) is the canonical lifting of \( Y \). Similarly \( \varphi \) defines a map \( \varphi : \tilde{M} \rightarrow \mathbb{R} \).

5. Dirac operator

In general, the normal bundle \( \nu = \nu(Y) \) of any orientable 3-manifold \( Y \) in a \( G_2 \) manifold \( (M, \varphi) \) has a Spin(4) structure (e.g. [12]). Hence we have SU(2) bundles \( S \) and \( E \) over \( Y \) (the pull-backs of \( S \) and \( E \) in (9)) such that \( \nu = S \otimes_H E \), with \( SO(3) \) reductions \( \text{ad} S = \lambda_+(\nu) \) and \( \text{ad} E = \lambda_-(\nu) \). Note that \( \text{ad} E \) is also the bundle of endomorphisms \( \text{End}(E) \). If \( Y \) is associative, then the SU(2) bundle \( S \) reduces to \( TY \), i.e. \( S \) becomes the spinor bundle of \( Y \), and so \( \nu \) becomes a twisted spinor bundle on \( Y \).

The Levi-Civita connection of the \( G_2 \) metric of \( (M, \varphi) \) induces connections on the associated bundles \( \mathcal{V} \) and \( \Xi \) on \( \tilde{M} \). In particular it induces connections on the tangent and normal bundles of any submanifold \( Y^3 \subset M \). We will call these connections the background connections. Let \( A_0 \) be the induced connection on the normal bundle \( \nu = S \otimes E \). From the Lie algebra decomposition \( \text{so}(4) = \text{so}(3) \oplus \text{so}(3) \), we can write \( A_0 = A_0 \oplus B_0 \), where \( A_0 \) and \( B_0 \) are connections on \( S \) and \( E \) respectively.
Let $\mathcal{A}(S)$ and $\mathcal{A}(E)$ be the set of connections on the bundles $S$ and $E$. Hence connections $A \in \mathcal{A}(S)$, $B \in \mathcal{A}(E)$ are in the form $A = A_0 + a$, $B = B_0 + b$, where $a \in \Omega^1(Y, ad S)$ and $b \in \Omega^1(Y, ad E)$. So $\Omega^1(Y, \lambda_\pm(\nu))$ parametrizes connections on $S$ and $E$, and the connections on $\nu$ are in the form $\mathbb{A} = \mathbb{A} \oplus B$. To emphasize the dependency on $a$ and $b$ we will sometimes denote $\mathbb{A} = \mathbb{A}(a, b)$, and $\mathbb{A}_0 = \mathbb{A}(0, 0)$.

Now, let $Y^3 \subset M$ be any smooth manifold. We can express the covariant derivative $\nabla_{\mathbb{A}} : \Omega^0(Y, \nu) \to \Omega^1(Y, \nu)$ on $\nu$ by $\nabla_A = \sum e^i \otimes \nabla_{e^i}$, where $\{e_1\}$ and $\{e^1\}$ are orthonormal tangent and cotangent frame fields of $Y$, respectively. Furthermore if $Y$ is an associative submanifold, we can use the Clifford multiplication (12) to form the twisted Dirac operator $\mathcal{D}_A : \Omega^0(Y, \nu) \to \Omega^0(Y, \nu)$(21)

$$\mathcal{D}_A = \sum e^i \cdot \nabla_{e^i}$$

The sections lying in the kernel of this operator are usually called harmonic spinors twisted by $(\bar{E}, \bar{A})$. Elements of the kernel of $\mathcal{D}_{A_0}$ are called the harmonic spinors twisted by $E$, or just the twisted harmonic spinors.

6. Complex associative submanifolds

There is an interesting class of associative submanifolds which we call complex associative, they are defined as follows: The subgroups $U(2) \subset SO(4) \subset G_2$, from

$$(SU(2) \times S^1)/\mathbb{Z}_2 \subset (SU(2) \times SU(2))/\mathbb{Z}_2 \subset G_2$$

give a $U(2)$-principal bundle $\mathcal{P}_{G_2}(M) \to \widetilde{M}_\varphi = \mathcal{P}_{G_2}(M)/U(2)$. Note that $\widetilde{M}_\varphi$ is the total space of an $S^2$ bundle $\widetilde{M}_\varphi \rightarrow \widetilde{M}_\varphi = \mathcal{P}(M)/SO(4)$. This is just the sphere bundle of the $\mathbb{R}^3$-bundle $\lambda_-(\mathbb{V}) \rightarrow \widetilde{M}_\varphi$. So we can identify the unit sections $j \in \lambda_-(\mathbb{V})$ with complex structures on $\mathbb{V}$, i.e. the right reductions of $SO(4)$ to $U(2)$, where

$$U(2) = \text{Spin}^c(3) = (SU(2) \times S^1)/\mathbb{Z}_2$$

Just as $SO(4)$ is the stabilizer subgroup of the action of $G_2$ on $G^\varphi(3, 7)$, the subgroup $U(2) \subset SO(4)$ is the stabilizer of the action of $G_2$ on the "framed" version of $G^\varphi(3, 7)$. That is, if $\nu \to G^\varphi(3, 7)$ is the dual universal bundle, then the action of $G_2$ on $G^\varphi(3, 7)$ extends to an action to the sphere bundle of $\lambda_-(\nu)$ with stabilizer $U(2)$.

By the representations of $U(2)$ induced from $SO(4)$ we can form the same vector bundles $\mathbb{V}, \Xi$ on $\widetilde{M}_\varphi$ as in Sections (2) and (3). Except in this case we have $\lambda \in S^1$ and hence $\lambda_-(\mathbb{V})$ becomes the trivial bundle, and $\mathbb{V}$ becomes a $\mathbb{C}^2$ bundle. The reduction $j$ gives an extra line bundle $\mathbb{L}$ via the representation $\rho_2(q, \lambda)(y) = y\lambda^{-2}$ (the determinant line bundle of $\mathbb{V}$). Similarly we have the same actions of (12), (13).
Since \( \lambda_-(\nabla) \) is trivial its action becomes the multiplication with complex numbers. In this case we have a useful quadratic bundle map \( \sigma : \nabla \otimes \nabla \to \Lambda^2(\Xi^*)_C \) given by
\[
(22) \quad \sigma(x, y) = -\frac{1}{2}(x\bar{y})i
\]
In particular if \( x = z + jw \in \mathcal{O} \) (octonions), we have \( \sigma(x, x) = \frac{|z|^2 - |w|^2}{2} + j\bar{z}w \).

So if \( f : Y^3 \hookrightarrow M \) is an associative submanifold, and \( \tilde{f} : Y^3 \hookrightarrow \tilde{M}_\varphi \) is the lifting of its Gauss map, then the normal bundle of \( f \) has a \( U(2) \) structure if and only if \( \tilde{f} \) lifts
\[
(23) \quad \tilde{f} \quad \downarrow \\
Y \quad \tilde{f} \quad M_\varphi
\]

As before we get \( \tilde{f}^*\Xi = T(Y) \) and \( \tilde{f}^*\nabla = \nu(Y) \), plus a line bundle \( L = \tilde{f}^*(L) \), and \( \nu \) becomes a complex \( U(2) \) bundle, which we will call \( W \). So \( W \) is the spinor bundle \( W(L) \to Y \) of a \( Spin^c(3) \) structure on \( Y \), and \( L \) is its determinant bundle. If this \( Spin^c(3) \) structure comes from a \( Spin(3) \) structure, then \( W = S \otimes L^{1/2} \) where \( S \) is the spinor bundle on \( Y \). So in concrete terms:

**Definition 4.** A complex associative submanifold \((Y, j)\) of a \( G_2 \) manifold \( M \) is an associative manifold in \( M \) with a unit section \( j \in \Omega^0(\lambda_-(\nu)) \), where \( \nu = \nu(Y) \).

Note that the normal bundle \( \nu(Y) \) of any associative submanifold \( Y \) has a \( U(2) \) structure (in fact it is trivial \[\text{[BSH]}\]), so liftings to \( \tilde{M}_\varphi \) always exists. We will call any \( Y \) with such a lifting a complex associative submanifold. But for such a lifting the background \( SO(4) \) connection \( \mathbb{A}_0 \) on \( \nu = \nu(Y) \) may not reduce to a \( U(2) \) connection. Let \( j \in \lambda_-(\nu) \) be the unit section describing the \( U(2) \) structure on \( \nu \). Then \( \mathbb{A}_0 \) reduces to a \( U(2) \) connection if \( \nabla_{\mathbb{A}_0}(j) = 0 \), where \( \mathbb{A}_0 \) is the connection on \( \lambda_-(\nu) \) induced from the background connection \( \mathbb{A}_0 \), and \( \nabla_{\mathbb{A}_0} \) is the covariant differentiation.

\[
(24) \quad \Omega^0(Y, \lambda_-(\nu)) \xrightarrow{\nabla_{\mathbb{A}_0}} \Omega^1(Y, \lambda_-(\nu))
\]

**Definition 5.** An integrable complex associative submanifold \((Y, j)\) of a \( G_2 \) manifold \( M \) is an associative manifold in \( M \) with a unit section \( j \in \Omega^0(\lambda_-(\nu)) \) with \( \nabla_{\mathbb{A}_0}(j) = 0 \).

A natural way to obtain complex associative submanifolds is by pulling back a universal complex structure from the ambient \( G_2 \) manifold \((M, \varphi)\):

**Definition 6.** \((M, \varphi, J)\) is called a complex \( G_2 \) manifold if \( J : \tilde{M}_\varphi \to \tilde{M}_\varphi \) is a lifting.

If \( Y \) be an integrable complex associative submanifold of a \( G_2 \) manifold \( M \), and \( \mathbb{A}_0 \) be the background \( U(2) \) connection on its normal bundle \( W \) induced by the \( G_2 \)
metric. Then \( A_0 = S_0 \oplus A_0 \) where \( S_0 \) is a connection on \( \lambda_+ (W) = T(Y) \) and \( A_0 \) is a connection on the line bundle \( L \). We can deform the background connection \( A_0 = A(0) \mapsto A(a) \), where \( A(a) = S_0 \oplus (A_0 + a) \), where \( a \in \Omega^1(Y) \) (e.g. Sec 8).

**Remark 2.** Recall that if \((X, \omega, \Omega)\) is a complex 3-dimensional Calabi-Yau manifold then \((X^6 \times S^1, \varphi)\) is a \( G_2 \) manifold, where \( \varphi = \text{Re} \Omega + \omega \wedge dt \). The submanifolds of the form \( C \times S^1 \) where \( C \subset X \) a holomorphic curve, are examples of integrable complex associative submanifolds of \( X \times S^1 \).

7. Deformations

In [M] McLean showed that, in a neighborhood of an associative submanifold \( Y \) in a \( G_2 \) manifold \((M, \varphi)\), the space of associative submanifolds can be identified with the harmonic spinors on \( Y \) twisted by \( E \). Since the cokernel of the Dirac operator can vary, we cannot determine the dimension of its kernel (it always has zero index on \( Y \)). We will remedy this by deforming \( Y \) in a larger class of submanifolds. To motivate our approach let us sketch a proof of McLean’s theorem by adapting the explanation in [B3]. Let \( Y \subset M \) be an associative submanifold, \( Y \) will determine a lifting \( \tilde{Y} \subset \tilde{M}_\varphi \). Let us recall that the \( G_2 \) structure \( \varphi \) gives a metric connection on \( M \), hence it gives a covariant differentiation in the normal bundle \( \nu(Y) = \nu \). We want the normal vector fields \( v \) of \( Y \subset M \) to always move \( Y \) in the class of associative submanifolds of \( M \), i.e. we want the liftings \( \tilde{Y}_v \) of the nearby copies \( Y_v \) of \( Y \) (pushed off by the vector field \( v \)) to lie in \( T(\tilde{M}_\varphi) \subset T(\tilde{M}) \) upstairs, i.e. we want the component of \( \tilde{v} \) in the direction of the normal bundle \( \tilde{M}_\varphi \subset \tilde{M} \) to vanish. By Lemma 1, this means \( \nabla_{\tilde{A}_0}(v) \) should be in the kernel of the Clifford multiplication \( c : \Omega^0(T^*(Y) \otimes \nu) \rightarrow \Omega^0(\nu) \), i.e.

\[
(25) \quad \mathcal{D}_{\tilde{A}_0}(v) = c(\nabla_{\tilde{A}_0}(v)) = 0
\]

Here \( \mathcal{D}_{\tilde{A}_0} \) is the Dirac operator induced by the background connection \( \tilde{A}_0 \), which is the composition

\[
(26) \quad \Omega^0(Y, \nu) \xrightarrow{\nabla_{\tilde{A}_0}} \Omega^0(Y, T^*Y \otimes \nu) \xrightarrow{c} \Omega^0(Y, \nu)
\]

To sum up what we have done: Let \( f : Y^3 \hookrightarrow M \) be an associative manifold in a \( G_2 \) manifold, and \( \tilde{f} : Y^3 \hookrightarrow \tilde{M}_\varphi \) be its canonical lifting. By using the metric connection,
the map on the space of immersions $\Phi : \text{Im}(Y, M) \rightarrow \text{Im}(Y, \tilde{M})$ given by $f \mapsto \tilde{f}$ induces a map on the tangent spaces

$$d\Phi : T_f \text{Im}(Y, M) \rightarrow T_f \text{Im}(Y, \tilde{M}) \supset T_f \text{Im}(Y, \tilde{M}_\varphi)$$

and the condition that $d\Phi(v)$ lie in $T_f \text{Im}(Y, \tilde{M}_\varphi)$ is given by $\Psi_{A_0}(v) = 0$.

Now $\Phi$ may not be transversal to $\text{Im}(Y, \tilde{M}_\varphi)$, i.e. the cokernel of $\Psi_{A_0}$ may be non-zero, but we can make it zero by deforming the connection in the normal bundle (i.e. in $E$ or $S$ as indicated in Section 5) $A_0 = A_0(0,0) \rightarrow A = A(a,b)$, where $a \in \Omega^1(Y, \lambda_+(\nu))$ and $b \in \Omega^1(Y, \lambda_-(\nu))$, and $\nu = \nu(Y)$. Then (26) becomes parametrized

$$\Omega^0(\nu) \times \Omega^1(\lambda_\pm(\nu)) \xrightarrow{\partial_+} \Omega^0(\nu)$$

that is, (25) becomes a twisted Dirac equation $\partial_+(v) = c(\nabla_+(v)) = \partial_+(v) + \alpha v = 0$, where $\alpha = (a,b)$. This gives a smooth moduli space parametrized by $\Omega^0(\lambda_\pm(\nu))$. Then by a generic choice of $\alpha \in \Omega^0(\lambda_\pm(\nu))$ we get a zero-dimensional perturbed moduli space, whose elements perhaps should be called $A$-associative submanifolds (to make an analogy with $J$-holomorphic curves).

**Theorem 2.** In a $G_2$ manifold $(M, \varphi)$, for a generic $A$, the space of $A$-associative submanifolds in a neighborhood of an associative manifold is a zero dimensional oriented smooth manifold.
In [AS] it was also shown that by deforming the $G_2$ structure $\varphi$ we can also obtain smoothness. To get compactness we will couple the Dirac equation with another equation to get Seiberg-Witten equations. We will first treat the complex associatives.

8. Deforming Complex Associative Submanifolds

Let $\mathcal{M}_C(M, \varphi)$ be the set of complex associative submanifolds $(Y, j)$ of a $G_2$ manifold $(M, \varphi)$. We wish to study the local structure of $\mathcal{M}_C(M, \varphi)$ near a particular $Y$. The normal bundle of $Y$ is a $U(2)$ bundle $\nu(Y) = W$, and $L \to Y$ is its determinant line bundle. If $(Y, j)$ is integrable we may assume $A_0 = S_0 \oplus A_0$ is the background connection on $W$, where $S_0$ is a connection on $\lambda_+(W) = T(Y)$, and $A_0$ is a connection on $L$. As before we will denote $A = A(a) = S_0 \oplus (A_0 + a)$, where $a \in \Omega^1(Y) = T_{\mathcal{A}_0} \mathcal{A}(L)$ (the tangent space to the space of connections). Now (28) becomes

\begin{equation}
\Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \xrightarrow{\mathcal{D}_A} \Omega^0(Y, W)
\end{equation}

which can be thought of the derivative of a similarly defined map

\begin{equation}
\Omega^0(Y, W) \times \mathcal{A}(L) \to \Omega^0(Y, W)
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2.}
\end{figure}

In each slice $A$, we are deforming along normal vector fields by the connection $A$ (which is a perturbation of the background connection $A_0$). To get compactness we can cut down the moduli space by an additional equation as in Seiberg-Witten theory, i.e.

\begin{equation}
\Phi : \Omega^0(Y, W) \times \mathcal{A}(L) \to \Omega^0(Y, W) \times \Omega^2(Y, i\mathbb{R})
\end{equation}

\begin{equation}
\mathcal{D}_A(v) = 0 \\
F_A = \sigma(v, v)
\end{equation}
where $F_A$ is the curvature of the connection $A = A_0 + a$ in $L$. For convenience we will replace the second equation above by $F_A = \mu(v, v)$, where $*$ is the star operator on $Y$ and $\mu(v, v) = \ast\sigma(v, v)$. Now we proceed exactly as in the Seiberg-Witten theory of 3-manifolds (e.g. [C], [Ma], [W]). To obtain smoothness on the zeros of $\Phi$, we perturb the equations by 1-forms $\delta \in \Omega^1(Y)$. This gives a new parametrized equation $\Phi = 0$

\begin{equation}
\phi : \Omega^0(Y, W) \times A(L) \times \Omega^1(Y) \rightarrow \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})
\end{equation}

\begin{equation}
*F_A + i\delta = \mu(v, v)
\end{equation}

It follows that in particular $\delta$ must be coclosed. $\Phi$ has a linearization:

\begin{equation}
D\Phi_{(v_0, A_0, 0)} : \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \rightarrow \Omega^1(Y, i\mathbb{R})
\end{equation}

\begin{equation}
D\Phi_{(v_0, A_0, 0)}(v, a, \delta) = (\partial A_0(v) + a.v_0, *da + i\delta - 2\mu(v_0, v))
\end{equation}

We see that $\Phi^{-1}(0)$ is smooth and the projection $\Phi^{-1}(0) \rightarrow \Omega^1(Y)$ is onto, so by Sard’s theorem for a generic choice of $\delta$ we can make $\Phi^{-1}(0)$ smooth, where $\Phi(v, A) = \Phi(v, A, \delta)$. The normal bundle of $Y$ has a complex structure, so the gauge group $G(Y) = \text{Map}(Y, S^1)$ acts on the solution set $\Phi^{-1}(0)$, and makes the quotient $\Phi^{-1}(0)/G(Y)$ a smooth zero-dimensional manifold. This is because the infinitesimal action of $G(Y)$ on the complex $\Phi_{\delta} : \Omega^0(Y, W) \times A(L) \rightarrow \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})$ is given by the map

$$
\Omega^0(Y, i\mathbb{R}) \xrightarrow{\partial} \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})
$$

where $G(f) = (fv_0, df)$. So after dividing by $G$, tangentially the complex $\Phi_{\delta}$ becomes

\begin{equation}
\Omega^0(Y, i\mathbb{R}) \xrightarrow{G} \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \rightarrow \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})/G
\end{equation}

Hence the index of this complex is the sum of the indicies of the Dirac operator $\partial A_0 : \Omega^0(Y, W) \rightarrow \Omega^0(Y, W)$ (which is zero), and the index of the following complex

\begin{equation}
\Omega^0(Y, i\mathbb{R}) \times \Omega^1(Y, i\mathbb{R}) \rightarrow \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})
\end{equation}

given by $(f, a) \mapsto (d^*(a), df + *da)$, which is also zero, since $Y^3$ has zero Euler characteristic. Furthermore $\Phi_{\delta}^{-1}(0)/G(Y)$ is compact and oriented (by Seiberg-Witten theory). Hence we can assign a number $\lambda_{\delta}(Y, j)$ to a complex associative submanifold $Y$ in a $G_2$ manifold. Here we don’t worry about metric dependence of $\lambda_{\delta}(Y, j)$, since we have a fixed background metric induced from the $G_2$ structure. In particular such a $Y$ moves in an unobstructed way after perturbation, not in the sections of its normal bundle $\Omega^0(Y, W)$, but in $\Omega^0(Y, W) \times A(L)$ that lies over it. So we have:
**Theorem 3.** Let \((Y, j)\) be a complex associative submanifold of a \(G_2\) manifold \((M, \varphi)\). Deformation space of \(Y\), given by equations (32), can be perturbed to a zero dimensional compact smooth oriented manifold, so we can associate a number \(\lambda_\varphi(Y, j) \in \mathbb{Z}\).

9. \(\text{Spin}^c(4)\) Structures (general case)

We can generalize the deformations in Section 8 without the “complex associative” hypothesis (i.e. without the assumption that \(E\) reduces to a line bundle). We can do this is by coupling the Dirac equation (25) with a curvature equation similar to the second equation of (32) we get a generalized version of Seiberg-Witten theory [FL] (so called \(SO(3)\)-monopoles), in which case we get a similar deformation picture as in Section 8 (Figure 2), but since technically this theory is much more complicated, instead we will put a slight topological restriction on \(M\), which will bring \(\text{Spin}^c(4)\) structures into the picture and allow us to do the deformations in the associated bundles in an easier way by using only the standard Seiberg-Witten theory. Also, we will first work globally on \(\tilde{M}_\varphi\) rather than studying deformations locally at each submanifold. Then at each given associative submanifold \(Y \subset M\), we can localize this global deformations to get local deformations.

Let \((M, \varphi)\) be a \(G_2\) manifold. Now suppose the \(SO(4)\) bundle \(\mathcal{P}(\mathcal{V}) \to \tilde{M}\) lifts to a \(\text{Spin}^c(4)\) bundle \(\mathcal{V}\), call such a \((M, \varphi)\) a \(G_2\) manifold with \(\text{Spin}^c(4)\) structure. Since

\[
\text{Spin}^c(4) = (SU(2) \times SU(2) \times S^1)/\mathbb{Z}_2
\]

a lifting exists if the Stiefel-Whitney class \(w_2(\mathcal{V}) \in H^2(\tilde{M}, \mathbb{Z}_2)\) is an integral class, and the number of different liftings, i.e. \(\text{Spin}^c\) structures, on \(\mathcal{V}\) is given by \(H^2(M, \mathbb{Z})\).

These facts follow from the fibrations:

\[
S^1 \to \text{Spin}^c(4) \to SO(4) \to K(\mathbb{Z}, 2) \to B\text{Spin}^c(4) \to BSO(4) \to K(\mathbb{Z}, 3)
\]

\(\mathcal{V}\) gives the following vector bundles by the representations, for \([q, \lambda, t] \in \text{Spin}^c(4)\):

\[
\begin{align*}
\mathcal{V} : & \quad y \mapsto qy\lambda^{-1} \\
\mathcal{V}^+ : & \quad y \mapsto qyt^{-1} \\
\mathcal{V}^- : & \quad y \mapsto \lambda yt^{-1} \\
ad(\mathcal{V}^+) : & \quad y \mapsto qyq^{-1} \\
ad(\mathcal{V}^-) : & \quad y \mapsto \lambda y\lambda^{-1} \\
L : & \quad y \mapsto yt^2
\end{align*}
\]

\(\mathcal{V}^\pm\) are pair of \(\mathbb{C}^2\) bundles that reduce to \(\mathbb{R}^3\) bundles \(ad(\mathcal{V}^\pm)\), and \(L\) is a complex line bundle, and \(\mathcal{V}_C = \mathcal{V}^+ \otimes_\mathbb{C} \mathcal{V}^-\). Clearly \(\lambda_+(\mathcal{V}) = ad(\mathcal{V}^+)\), and \(\lambda_-(\mathcal{V}) = ad(\mathcal{V}^-)\). Recall that if we restrict to \(\tilde{M}_\varphi\) we get an additional identifications \(\Xi = ad(\mathcal{V}^+)\). Also the \(\text{Spin}^c\) structure on \(\mathcal{V}\) gives a \(\text{Spin}^c\) structure on the \(\mathbb{R}^3\) bundle \(ad(\mathcal{V}^+)\), which can
be identified as an equivalence class of a non-vanishing vector field \( \xi_0 \) on \( ad(\nabla^+) \), call this the basic section of \( ad(\nabla^+) \).

Recall that \( T(\tilde{M}_\varphi) = T_{\text{vert}}(\tilde{M}_\varphi) \oplus \Xi \oplus \nabla \), and \( T_{\text{vert}}(\tilde{M}_\varphi) \subset \Xi^* \otimes \nabla \). Let \( \xi_0 \in \Xi \) be the basic section. We can define an action \( \rho : T(\tilde{M}_\varphi) \to \text{Hom}(\nabla^+ \oplus \nabla^-) \), i.e.

\[
T(\tilde{M}_\varphi) \otimes \nabla^\pm \to \nabla^\pm
\]

For \( w = a \otimes v + x + y \in T(\tilde{M}_\varphi) \), \( a \otimes v \in \Xi^* \otimes \nabla \), \( x \in \Xi \), \( y \in \nabla \), let \( v_0 = a(\xi_0)v \) and

\[
w.(z_1, z_2) = (\bar{a}vz_2 + \bar{x}v_0z_2 + yz_2, -\bar{v}az_1 - \bar{v}_0xz_1 - \bar{y}z_1)
\]

where \( (z_1, z_2) \in \nabla^+ \oplus \nabla^- \). From (38) it is easy to check that this is a Dirac action, i.e. \( \rho(w) \circ \rho(w) = -|w|^2I \). As usual we can dualize and extend this to an action

\[
\Lambda^2(T^*(\tilde{M}_\varphi)) \otimes \nabla^+ \to \nabla^+
\]

Also as in (22) we have the quadratic bundle map \( \sigma : \nabla^+ \otimes \nabla^+ \to \Lambda^2(\Xi^*)_c \) defined by

\[
\sigma(x, y) = -\frac{1}{2}(xiy)i
\]

Now let \( \mathcal{A}(L) \) be the space of connections on \( L \). Let \( A_0 \) be a given background connection on \( \nabla \) (e.g. the one induced from the \( G_2 \) metric on \( M \)). Then any \( A \in \mathcal{A}(L) \) along with \( A_0 \) determines a connection on \( \nabla \) and hence connections all the above associated bundles. Also if we fix \( A_0 \in \mathcal{A}(L) \), any other \( A \in \mathcal{A}(L) \) can be written as \( A = A_0 + a \) where \( a \in \Omega^1(\tilde{M}_\varphi) \). Hence for each \( A \in \mathcal{A}(L) \) we get the corresponding Dirac operator \( \mathcal{D}_A(v) = \mathcal{D}_{A_0}(v) + a.v \) on bundles \( \nabla^\pm \to \tilde{M}_\varphi \) which is the composition:

\[
\Omega^0(\tilde{M}_\varphi, \nabla^+) \xrightarrow{\Sigma} \Omega^0(\tilde{M}_\varphi, T^*(\tilde{M}_\varphi) \otimes \nabla^+) \xrightarrow{c} \Omega^0(\tilde{M}_\varphi, \nabla^-)
\]

We can now write generalized Seiberg-Witten equations on \( \tilde{M}_\varphi \) in the usual way

\[
\phi : \Omega^0(\tilde{M}_\varphi, \nabla^+) \times \mathcal{A}(L) \to \Omega^0(\tilde{M}_\varphi, \nabla^-) \times \Omega^2(\tilde{M}_\varphi)
\]

(40)

\[
\mathcal{D}_A(v) = 0
F_A = \sigma(v, v)
\]

and proceed as in Section 8. Equations (40) imply \( F_A - \sigma(v, v) \) is closed. By perturbing the equations \( \phi = 0 \) by closed 2-forms \( \Omega^2(\tilde{M}_\varphi) \) of \( \tilde{M}_\varphi \), i.e. by changing \( \phi \) to

\[
\Phi : \Omega^0(\tilde{M}_\varphi, \nabla^+) \times \mathcal{A}(L) \times \Omega^2(\tilde{M}_\varphi) \to \Omega^0(\tilde{M}_\varphi, \nabla^-) \times \Omega^2(\tilde{M}_\varphi)
\]

where \( \Phi(A, v, \delta) = (F_A - \sigma(v, v) + i\delta, \mathcal{D}_{A_0}(v) + a.v) \) we can make \( \Phi^{-1}(0) \) smooth. That is, by applying Sard’s theorem to the projection \( \Phi^{-1}(0) \to \Omega^2(\tilde{M}_\varphi) \) for generic choice of \( \delta \) we can make \( \Phi^{-1}_\delta(0) \) smooth, where \( \Phi_\delta(v, A) = \Phi(v, A, \delta) \). The normal bundle of
Y has a complex structure, so the gauge group $G(\tilde{M}_\varphi) = Map(\tilde{M}_\varphi, S^1)$ acts on the solution set $\Phi^{-1}(0)$, and makes quotient $\Phi^{-1}(0)/G(\tilde{M}_\varphi)$ smooth. The linearization of $\Phi$ (modulo the gauge group) is given by the complex

$$\Omega^0(\tilde{M}_\varphi) \to \Omega^1(\tilde{M}_\varphi) \times \Omega^0(\tilde{M}_\varphi V^+) \to \Omega^2(\tilde{M}_\varphi) \times \Omega^0(\tilde{M}_\varphi, V^-)$$

which is the sum of the Dirac operator $\mathcal{D}/A$ with zero index, and a complex which is not elliptic:

$$\Omega^0(\tilde{M}_\varphi) \xrightarrow{d} \Omega^1(\tilde{M}_\varphi) \xrightarrow{d} \Omega^2(\tilde{M}_\varphi)$$

where the first map is the derivative of the gauge group action. So we have:

**Theorem 4.** Normal bundle of any associative $f : Y^3 \hookrightarrow M$ in a $G_2$ manifold with a Spin$^c(4)$ structure $(M, \varphi, c)$, pulls back the Spin$^c(4)$ structure $c$ by the canonical lifting $\tilde{f} : Y \to \tilde{M}$. The map $\tilde{f}$ also transforms all the bundles and their Clifford multiplication information to $Y$. So we get a pair of $\mathbb{C}$ bundles $V^\pm = \tilde{f}^*(V^\pm)$ and a line bundle $L \to Y$ describing a Spin$^c(4)$ structure on $\nu(Y) = \tilde{f}^*(\nu)$ with $\nu(Y)_C = V^+ \otimes V^-$. Also, $T^*(Y) = \tilde{f}^*ad(V^+)$, and any $A \in A(L)$ with the help of the background connection $A_0$ gives connections on $V^\pm$, and so we can transform equations (40) to $Y$ and as in Section 8 assign an invariant $\lambda_\varphi(Y, c)$ to $Y$.

**Remark 3.** We can also write global version of the equations given by (37)

\[
\begin{align*}
  d^*(a) &= 0 \\
  da + d^*(f\tilde{\varphi}) &= 0
\end{align*}
\]

where $\tilde{\varphi} \in \Omega^3(\tilde{M})$ is the pullback of the 3-form $\varphi$ by the projection $\tilde{M}_\varphi \to M$.

**References**

[A] S. Akbulut, *Lectures on Seiberg-Witten Invariants*, Turkish Jour. of Math 20 (1996) 1329-1355.

[AS] S. Akbulut and S. Salur, *Calibrated manifolds and gauge theory*, GT/0402368.

[B1] R.L. Bryant, *Metrics with exceptional holonomy*, Ann of Math 126 (1987) 525-576.

[B2] R.L. Bryant, *Some remarks on G_2-structures*, DG/0305124 v3

[B3] R.L. Bryant *Manifolds with G_2- holonomy* (Duke Lectures).

[BSh] R. L. Bryant and E. Sharpe, *D-Branes and Spin^c structures* hep-th/9812084

[BSa] R.L. Bryant, R.L. and M.S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Mathematical Journal, vol. 58, no 3 (1989), 829-850.

[C] W. Chen, *The Seiberg-Witten theory of homology spheres*, arXiv: dg-ga/9703009

[FL] P.M.N. Feehan and T.G. Leness, *On Donaldson and Seiberg-Witten invariants*, arXiv: DG/0106221

[HL] F.R. Harvey, and H.B. Lawson, *Calibrated geometries*, Acta. Math. 148 (1982), 47-157.

[J] D.D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, Oxford, 2000.
[Li-Liu] T-J. Li and A-K. Liu, *Family of Seiberg-Witten invariants and wall crossing formulas*, GT/0107211.

[LM] H.B. Lawson and M.L. Michelson, *Spin geometry*, Princeton University Press (1989)

[Ma] M. Marcolli, *Seiberg-Witten Floer homology and Heegaard splittings*, arXiv: dg-ga/9601011

[M] R.C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. 6 (1998), 705-747.

[Mc] B. McKay, *G₂ Manifolds of cohomogeneity two*, DG/0311441

[W] Bai-Liang Wang, *Seiberg-Witten-Floer homology for homology 3-spheres*, arXiv: dg-ga/9602003

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