Euler Totient Function And The Largest Integer Function
Over The Shifted Primes

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Abstract: Let \( x \geq 1 \) be a large number, let \( \lfloor x \rfloor = x - \{x\} \) be the largest integer function, and let \( \varphi(n) \) be the Euler totient function. The asymptotic formula for the new finite sum over the primes \( \sum_{p \leq x} \varphi(\lfloor x/p \rfloor) = (6/\pi^2)x \log \log x + c_1 x + O(x(\log x)^{-1}) \), where \( c_1 \) is a constant, is evaluated in this note.

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1 Introduction

The average order \( \sum_{n \leq x} d(n) \) of the divisors counting function \( d(n) = \# \{d \mid n \} \) has a direct representation in term of the average order of the largest integer function \( \lfloor x \rfloor = x - \{x\} \) as
\[
\sum_{n \leq x} d(n) = \sum_{n \leq x} \lfloor \frac{x}{n} \rfloor .
\] (1)

Following this equivalence, the average orders \( \sum_{n \leq x} f(n) \) of some arithmetic functions were extended to \( \sum_{n \leq x} f(\lfloor x/n \rfloor) \) in [2], [3], [8], et alii. Continuing these equivalences, the average order \( \sum_{n \leq x} \omega(n) \) of the prime divisors counting function \( \omega(n) = \# \{p \mid n\} \) has a direct representation in term of the average order of largest integer function as
\[
\sum_{n \leq x} \omega(n) = \sum_{p \leq x} \lfloor \frac{x}{p} \rfloor .
\] (2)
Further, this equivalence introduces a new phenomenon, it changes the indices from the integer domain to the prime domain. The closely related Titchmarsh divisor problem seems to have a complicated representation in term of the average order of largest integer function as
\[
\sum_{p \leq x} d(p-1) = \sum_{p \leq x} \left\lceil \frac{x}{p-1} \right\rceil f(p),
\]
where \( f(n) \) is a function.

Let \( f, g, h : \mathbb{N} \rightarrow \mathbb{C} \) be multiplicative functions. Some analytic techniques for evaluating the finite sums \( \sum_{p \leq x} f([x/p]) \) for multiplicative functions defined by Dirichlet convolutions \( f(n) = \sum_{d|n} g(d)h(n/d) \), and having fast rates of growth approximately \( f(n) \gg n(\log n)^b \), for some \( b \in \mathbb{Z} \), are introduced here. These elementary methods used within are simpler and about four fold more efficient than the analytic methods used in the current literature as [8], et alii, to evaluate the simpler sums \( \sum_{n \leq x} f([x/n]) \). As a demonstration, the asymptotic formula for the new finite sum \( \sum_{p \leq x} \phi([x/p]) \) is determined in Theorem 2.1.

2 Euler Totient Function Over The Shifted Primes

The Euler totient function is defined by \( \phi(n) = n \sum_{d|n} \mu(d)/d \), and other identities. It is multiplicative and satisfies the growth condition \( \phi(n) \gg n/\log \log n \). The average order over the integer domain has the form
\[
\sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x),
\]
see [1, Theorem 3.7], and similar references, and average order over the prime domain has the form
\[
\sum_{p \leq x} \phi(p-1) = \text{li}(x^2) \prod_{p \geq 2} \left( 1 - \frac{1}{p(p-1)} \right) + O \left( \frac{x^2}{(\log x)^B} \right),
\]
where \( B > 1 \) is an arbitrary constant, see [6]. The new finite sum
\[
\sum_{n \leq x} \phi \left( \left\lceil \frac{x}{n} \right\rceil \right) = \frac{6}{\pi^2} x \log x + O(x \log \log x),
\]
is evaluated in [3], and a weaker version in [8]. Standard analytic techniques are used here to assemble the first proof of the asymptotic formula for the new finite sum \( \sum_{p \leq x} \phi([x/p]) \).

**Theorem 2.1.** If \( x \geq 1 \) is a large number, then,
\[
\sum_{p \leq x} \phi \left( \left\lceil \frac{x}{p} \right\rceil \right) = c_0 x \log \log x + c_1 x + O(\text{li}(x)),
\]
where \( c_0 = 6/\pi^2 \), and \( c_1 > 0 \) are constants.

**Proof.** Use the identity \( \phi(n) = n \sum_{d|n} \mu(d)/d \) to rewrite the finite sum, and switch the order of summation:
\[
\sum_{p \leq x} \phi \left( \left\lceil \frac{x}{p} \right\rceil \right) = \sum_{p \leq x} \left\lceil \frac{x}{p} \right\rceil \sum_{d \mid [x/p]} \frac{\mu(d)}{d}
\]
\[
= \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{p \leq x} \left\lceil \frac{x}{p} \right\rceil,\]
Apply Lemma 3.1 to remove the congruence on the inner sum index, and break it up into two subsums. Specifically,

\[
\sum_{d \leq x} \frac{\mu(d)}{d} \sum_{p \leq x \mid \left\lfloor \frac{x}{p} \right\rfloor} e^{i2\pi s\left\lfloor \frac{x}{p} \right\rfloor /d} = \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{p \leq x} \frac{x}{p} \sum_{0 \leq s \leq d-1} e^{i2\pi s\left\lfloor \frac{x}{p} \right\rfloor /d}
\]

\[
= \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{p \leq x} \frac{x}{p} \sum_{0 \leq s \leq d-1} e^{i2\pi s\left\lfloor \frac{x}{p} \right\rfloor /d}
\]

\[
= M(x) + E(x).
\]

The main term \(M(x)\) is computed in Lemma 3.3 and the error term \(E(x)\) is computed in Lemma 3.4. Summing these expressions yields

\[
\sum_{p \leq x} \phi \left( \left\lfloor \frac{x}{p} \right\rfloor \right) = M(x) + E(x)
\]

\[
= c_0 x \log \log x + c_1 x + c_2 \text{li}(x) + O \left( x e^{-c\sqrt{\log x}} \right) + O \left( \text{li}(x) \right)
\]

where \(c_0 = 6/\pi^2, c_1, c_2,\) and \(c > 0\) are constants.

The finite sum \(\sum_{p \leq x} \phi \left( \left\lfloor \frac{x}{p} \right\rfloor \right)\) is the canonical representative of this class of finite sums associated with the totient function. The leading term in the asymptotic formula for the more general sum \(\sum_{p \leq x} \phi \left( \left\lfloor \frac{x}{(p + a)} \right\rfloor \right)\), where \(a \neq -p\) is an integer parameter, is independent of the small parameter \(a \in \mathbb{Z}\). However, the parameter \(a\) does contribute to the secondary terms. Other classes of arithmetic functions having the same leading terms in the asymptotic formulas of the average orders are described in [4, Section 3.1].

Under the RH, the optimal evaluation is expected to be of the form

\[
\sum_{p \leq x} \phi \left( \left\lfloor \frac{x}{p + a} \right\rfloor \right) = c_0 x \log \log x + c_1 x + c_2 \text{li}(x) + O \left( x^{1/2} (\log x)^2 \right),
\]

where \(c_0 = 6/\pi^2, c_1 = c_1(a)\) and \(c_2 = c_2(a)\) are constants, depending on the parameter \(a\).

3 Foundation Results for the Phi Function

The detailed and elementary proofs of the preliminary results required in the proof of Theorem 2.1 concerning the Euler phi function \(\phi(n) = \sum_{d \mid n} \mu(d)d\) are recorded in this section.

3.1 Preliminary Results

Lemma 3.1. Let \(x \geq 1\) be a large number, and let \(1 \leq d, m, n \leq x\) be integers. Then,

\[
\frac{1}{d} \sum_{0 \leq s \leq d-1} e^{i2\pi ms/d} = \begin{cases} 1 & \text{if } d \mid m, \\ 0 & \text{if } d \nmid m, \end{cases}
\]

(12)
Lemma 3.2. Let $m \leq x$ be a fixed integer, and let $x \geq 1$ be a large number. Then,

1. $\sum_{n \leq x \atop n|m} \frac{\mu(n)}{n} = O(1)$.

2. $\sum_{n \leq x \atop n|m} \frac{\mu(n)}{n^2} = A_0(m) + O\left(\frac{1}{x}\right)$,

3. $\sum_{n \leq x \atop n|m} \frac{\mu(n)}{n^2} = A_1(m) + O\left(\frac{1}{x}\right)$,

where $|A_0(m)| < 2$, and $|A_1(m)| < 2$ are constants depending on $m \geq 1$.

Proof. (i) Since $m \leq x$, the upper bound of the first finite sum is as follows.

$$\sum_{n \leq x \atop n|m} \frac{\mu(n)}{n} = \sum_{n \leq x \atop n|m} \mu(n) = \prod_{p \leq x \atop r|m} \left(1 - \frac{1}{r}\right) = O(1),$$

where $r \geq 2$ is prime. The other relations (ii) and (iii) are routine calculations. ■

3.2 The Main Term $S(x)$

Lemma 3.3. If $x \geq 1$ is a large number, then,

$$\sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{p \leq x} \left\{\frac{x}{p}\right\} = \frac{6}{\pi^2} x \log \log x + c_1 x + c_2 \log x + O\left(x e^{-c \sqrt{\log x}}\right),$$

where $c_1, c_2$, and $c > 0$ are constants.

Proof. Expand the bracket and evaluate the two subsums. Specifically, the first subsum is

$$x \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{p \leq x} \left\{\frac{x}{p}\right\} = x \left(\frac{6}{\pi^2} + O\left(\frac{1}{x}\right)\right) \left(\log \log x + B_1 + O\left(\frac{1}{x}\right)\right)$$

$$= \frac{6}{\pi^2} x \log \log x + c_1 x + O\left(\log \log x\right),$$

and the second subsum is

$$- \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{p \leq x} \left\{\frac{x}{p}\right\} = - \left(\frac{6}{\pi^2} + O\left(\frac{1}{x}\right)\right) \left(1 - \gamma\right) \log x + O\left(x e^{-c \sqrt{\log x}}\right)$$

$$= c_2 \log x + O\left(x e^{-c \sqrt{\log x}}\right),$$

where $\log x$ is the logarithm integral, $c_1, c_2$, and $B_1$ are constants, and $c > 0$ is an absolute constant. Summing (15) and (16) completes the verification. ■

The optimal error term in (14) is the same as the optimal error term in the prime number theorem, see [7, Corollary 1]. This is a direct consequence of the optimal evaluation of the basic sum of fractional parts

$$\sum_{p \leq x} \left\{\frac{x}{p}\right\} = (1 - \gamma) \log x + O\left(x e^{-c \sqrt{\log x}}\right),$$

where $\gamma$ is Euler constant, see [5, Theorem 0].
3.3 The Error Term $E(x)$

**Lemma 3.4.** Let $x \geq 1$ be a large number. Then,
\[
\sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{p \leq x} \left[ \frac{x}{p} \right] \sum_{0 < a < d-1} e^{i2\pi a[x/p]/d} = ax + O \left( \frac{x}{\log x} \right),
\]
where $a \neq 0$ is a constant.

**Proof.** Let $\pi(x) = \# \{ \text{prime } p \leq x \}$ be the primes counting function, let $\text{li}(x)$ be the logarithm integral, and let $p_k$ be the $k$th prime in increasing order. The sequence of values
\[
\left[ \frac{x}{p_k} \right] = \left[ \frac{x}{p_{k+1}} \right] = \cdots = \left[ \frac{x}{p_{k+r}} \right]
\]
(19)
arises from the sequence of primes $x/(n+1) \leq p_k, p_{k+1}, \ldots, p_{k+r} \leq x/n$. Therefore, the value $m = [x/p] \geq 1$ is repeated $\pi \left( \left[ \frac{x}{n} \right] \right) - \pi \left( \left[ \frac{x}{n+1} \right] \right)$ times as $p$ ranges over the prime values in the interval $[x/(n+1), x/n]$. Hence, substituting (20) into the triple sum $E(x)$, and reordering it yield
\[
E(x) = \sum_{p \leq x} \left( \frac{\text{li}(x)}{n(n+1)} + O \left( \frac{x}{n} e^{-c\sqrt{\log x}} \right) \right) \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{0 < a < d-1} e^{i2\pi a m/d}
\]
\[
= \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{0 < a < d-1} e^{i2\pi a m/d}
\]
\[
+ O \left( x e^{-c\sqrt{\log x}} \sum_{n \leq x} \frac{1}{n} \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{0 < a < d-1} e^{i2\pi a m/d} \right)
\]
\[
= E_0(x) + E_1(x).
\]
The finite subsums $E_0(x)$, estimated in Lemma 3.5, and $E_1(x)$, estimated in Lemma 3.6, correspond to the subsets of integers $p \leq x$ such that $d \mid [x/p]$, and $d \nmid [x/p]$, respectively. Summing yields
\[
E(x) = E_0(x) + E_1(x)
\]
\[
= O \left( \text{li}(x) \right) + O \left( x e^{-c\sqrt{\log x}} \right)
\]
\[
= O \left( \text{li}(x) \right),
\]
where $c > 0$ is an absolute constant. ■

3.4 The Sum $E_0(x)$

**Lemma 3.5.** Let $x \geq 1$ be a large number, let $[x] = x - \{x\}$ be the largest integer function, and $m = [x/p] \leq [x/n] \leq x$. Then,
\[
\text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{0 < a < d-1} e^{i2\pi a m/d} = O(\text{li}(x)).
\]
\[
(23)
\]
Proof. The set of values \( m = \lfloor x/p \rfloor \leq \lfloor x/n \rfloor \leq x \) such that \( d \mid m \). Evaluating the incomplete function returns

\[
E_0(x) = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x \mid m} \frac{\mu(d)}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi am/d}
\]

(24)

\[
= \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x \mid m} \frac{\mu(d)}{d^2} \cdot (d - 1)
\]

\[
= \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x \mid m} \frac{\mu(d)}{d} - \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x \mid m} \frac{\mu(d)}{d^2}
\]

\[
= E_{00}(x) + E_{01}(x).
\]

The first term has the upper bound

\[
E_{00}(x) = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \mid m} \frac{\varphi(m)}{m} \leq 1,
\]

(25)

This follows from the trivial inequality

\[
\sum_{d \leq x \mid m} \frac{\mu(d)}{d} = \sum_{d \mid m} \frac{\mu(d)}{d} = \frac{\varphi(m)}{m} \leq 1,
\]

(26)

where \( m = \lfloor x/p \rfloor \leq \lfloor x/n \rfloor \leq x \). The second term has the upper bound

\[
E_{01}(x) = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x \mid m} \frac{\mu(d)}{d^2}
\]

(27)

\[
\sum_{d \mid [x/n]} \frac{\mu(d)}{d^2} \leq \text{li}(x).
\]

Summing yields \( E_0(x) = E_{00}(x) + E_{01}(x) = O(\text{li}(x)) \).

\[\blacksquare\]

3.5 The Sum \( E_1(x) \)

**Lemma 3.6.** If \( x \geq 1 \) is a large number, then,

\[
xe^{-c_0 \sqrt{\log x}} \sum_{n \leq x} \frac{1}{n} \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi am/d} = O \left( xe^{-c \sqrt{\log x}} \right),
\]

where \( c_0 > 0 \) and \( c > 0 \) are absolute constants.
Proof. The absolute value provides an upper bound:

\[ |E_1(x)| = \left| xe^{-c_0\sqrt{\log x}} \sum_{n \leq x} \frac{1}{n} \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{0 < a \leq d-1} e^{2\pi i a n/d} \right| \]

\[ \leq xe^{-c_0\sqrt{\log x}} \sum_{n \leq x} \frac{1}{n} \sum_{d \leq x} \frac{1}{d} \]

\[ = O \left( (x \log^2 x) e^{-c_0\sqrt{\log x}} \right) \]

\[ = O \left( xe^{c_0\sqrt{\log x}} \right), \]

where \( c_0 > 0 \) and \( c > 0 \) are absolute constants.

4 Numerical Data

Small numerical tables were generated by an online computer algebra system, the range of numbers \( x \leq 10^6 \) is limited by the wi-fi bandwidth. For each parameter \( a \), the error term for the finite sum over the shifted primes is defined by

\[ E(x, a) = \sum_{p \leq x} \varphi \left( \left\lfloor \frac{x}{p + a} \right\rfloor \right) - 6\pi^2 x \log \log x. \]

The prime values \( a = -p \) are not used since (29) has a singularity at this point. The tables are for the parameters \( a = 0, a = 4, \) and \( a = -4 \) respectively. All the calculations are within the predicted ranges \( E(x, a) = O(x) \).

Table 1: Numerical Data For \( \sum_{p \leq x} \varphi([x/p]). \)

| \( x \)  | \( \sum_{p \leq x} \varphi([x/p]) \) | \( 6\pi^2 x \log \log x \) | Error \( E(x, 0) \) |
|--------|---------------------------------|-----------------|-------------|
| 10     | 8                               | 5.07            | -2.93       |
| 100    | 94                              | 92.84           | 1.96        |
| 1000   | 1115                            | 1174.91         | -59.91      |
| 10000  | 12891                           | 13497.97        | -606.97     |
| 100000 | 147771                          | 148545.18       | -774.20     |
| 1000000| 1526405                         | 1596290.10      | -69885.10   |

Table 2: Numerical Data For \( \sum_{p \leq x} \varphi([x/(p - 4))]. \)

| \( x \)  | \( \sum_{p \leq x} \varphi([x/(p - 4)]) \) | \( 6\pi^2 x \log \log x \) | Error \( E(x, -4) \) |
|--------|---------------------------------|-----------------|-------------|
| 10     | 14                               | 5.07            | 8.93        |
| 100    | 167                              | 92.84           | 74.16       |
| 1000   | 1868                            | 1174.91         | 693.09      |
| 10000  | 20537                           | 13497.97        | 7039.03     |
| 100000 | 224901                          | 148545.18       | 76355.81    |
| 1000000| 2244876                         | 1596290.10      | 648585.93   |
Table 3: Numerical Data For $\sum_{p \leq x} \varphi([x/(p + 4)])$.

| $x$  | $\sum_{p \leq x} \varphi([x/(p + 4)])$ | $6\pi^{-2}x \log \log x$ | Error $E(x, 4)$ |
|------|----------------------------------------|--------------------------|----------------|
| 10   | 3                                      | 5.07                     | −2.07          |
| 100  | 58                                     | 92.84                    | −34.84         |
| 1000 | 791                                    | 1174.91                  | −383.91        |
| 10000| 8956                                   | 13497.97                 | −4541.97       |
| 100000| 113334                                 | 148545.18                | −35211.19      |
| 1000000| 1225300                               | 1596290.10               | −370990.07     |

5 Open Problems

Exercise 5.1. Let $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$ be the Mobius function, and let $q \geq 1$ be a fixed integer. Prove or disprove the following asymptotic formula.

$$\sum_{n \leq x, n \mid q} \mu(n) = O \left( xe^{-c\sqrt{\log x}} \right),$$

where $c > 0$ is an absolute constant. A weaker estimate is cited in Lemma 3.4.

Exercise 5.2. Let $\lambda : \mathbb{N} \rightarrow \{-1, 1\}$ be the Liouville function, and let $q \geq 1$ be a fixed integer. Prove or disprove the following asymptotic formula.

$$\sum_{n \leq x, n \mid q} \lambda(n) = O \left( xe^{-c/\sqrt{\log x}} \right),$$

where $c > 0$ is a constant.

Exercise 5.3. Let $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ be the sum of divisors function, and let $q \geq 1$ be a fixed integer. Prove the following explicit upper bound.

$$\sum_{n \leq x, n \mid q} \sigma(n) \leq 2q \log \log q.$$

Exercise 5.4. Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be the totient function, and let $x \geq 1$ be a large number. Determine an asymptotic formula for the following finite sum.

$$\sum_{p \leq x} \left[ \frac{x}{p} \right] \varphi\left( \left[ \frac{x}{p} \right] \right).$$

Exercise 5.5. Let $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ be the Carmichael function, and let $x \geq 1$ be a large number. Determine an asymptotic formula for the following finite sum.

$$\sum_{p \leq x} \lambda\left( \left[ \frac{x}{p} \right] \right).$$

Exercise 5.6. Let $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ be the Carmichael function, let $\{x\}$ be the fractional function, and let $x \geq 1$ be a large number. Verify the asymptotic formula for the following finite sum.

$$\sum_{n \leq x} \left\{ \frac{\lambda(n + 1)}{n} \right\} = \frac{x}{\log x} e^{b \log \log x/\log \log \log x} - \text{li}(x) + O \left( e^{-c\sqrt{\log x}} \right),$$

where $\text{li}(x)$ is the logarithm integral, and $b > 0$, and $c > 0$ are constants. Hint: $\{x\} = x - [x]$. 


**Exercise 5.7.** Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be the totient function, let $\{x\}$ be the fractional function, and let $x \geq 1$ be a large number. Verify the asymptotic formula for the following finite sum.

$$\sum_{n \leq x} \left\{ \frac{\varphi(n + 1)}{n} \right\} = \frac{3}{\pi^2} x - \text{li}(x) + O\left(e^{-c\sqrt{\log x}}\right),$$

where $\text{li}(x)$ is the logarithm integral, and $c > 0$ is a constant.

**Exercise 5.8.** Let $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ be the sum of divisors function, let $\{x\}$ be the fractional function, and let $x \geq 1$ be a large number. Verify the asymptotic formula for the following finite sum.

$$\sum_{n \leq x} \left\{ \frac{n}{\sigma(n) - 1} \right\} = ax - \text{li}(x) + O\left(e^{-c\sqrt{\log x}}\right),$$

where $\text{li}(x)$ is the logarithm integral, and $a > 0$ and $c > 0$ are constants.

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