Abstract

Nowozin et al showed last year how to extend the GAN principle to all $f$-divergences. The approach is elegant but falls short of a full description of the supervised game, and says little about the key player, the generator: for example, what does the generator actually converge to if solving the GAN game means convergence in some space of parameters? How does that provide hints on the generator’s design and compare to the flourishing but almost exclusively experimental literature on the subject?

In this paper, we unveil a broad class of distributions for which such convergence happens — namely, deformed exponential families, a wide superset of exponential families — and show tight connections with the three other key GAN parameters: loss, game and architecture. In particular, we show that current deep architectures are able to factorize a very large number of such densities using an especially compact design, hence displaying the power of deep architectures and their concinnity in the $f$-GAN game. This result holds given a sufficient condition on activation functions — which turns out to be satisfied by popular choices. The key to our results is a variational generalization of an old theorem that relates the KL divergence between regular exponential families and divergences between their natural parameters. We complete this picture with additional results and experimental insights on how these results may be used to ground further improvements of GAN architectures, via (i) a principled design of the activation functions in the generator and (ii) an explicit integration of proper composite losses’ link function in the discriminator.
## 1 Introduction

In a recent paper, Nowozin et al. [47] showed that the GAN principle [27] can be extended to the variational formulation of all $f$-divergences. In the GAN game, there is an unknown distribution $P$ which we want to approximate using a parameterized distribution $Q$. $Q$ is learned by a **generator** by finding a saddle point of a function which we summarize for now as $f$-GAN($P, Q$), where $f$ is a convex function (see eq. (14) below for its formal expression). A part of the generator’s training involves as a subroutine a supervised adversary — hence, the saddle point formulation — called **discriminator**, which tries to guess whether randomly generated observations come from $P$ or $Q$. Ideally, at the end of this supervised game, we want $Q$ to be close to $P$, and a good measure of this is the $f$-divergence $I_f(P \parallel Q)$, also known as Ali-Silvey distance [1, 20]. Initially, one choice of $f$ was considered [27]. Nowozin et al. significantly grounded the game and expanded its scope by showing that for any $f$ convex and suitably defined, it actually holds that [47, Eq. 4]:

$$f\text{-GAN}(P, Q) \leq I_f(P \parallel Q). \quad (1)$$

Furthermore, the inequality is an equality if the discriminator is powerful enough: so, solving the $f$-GAN game can give guarantees on how $P$ and $Q$ are distant to each other in terms of $f$-divergence. This elegant characterization of the supervised game unfortunately falls short of justifying or elucidating all parameters of the supervised game [47, Section 2.4].

The paper is also silent regarding a key part of the game: the link between distributions in the variational formulation and the **generator**, the main player which learns a parametric model of a density. In doing so, the $f$-GAN approach and its members remain within an information theoretic framework that relies on divergences between distributions only [47]. In the GAN world at large, this position contrasts with other prominent approaches that explicitly optimize geometric distortions between the parameters or support of distributions [36]: moment matching methods optimize distortions between expected parameters [35], Wasserstein-1 method and optimal transport methods (regularized or not) optimize transportation costs between supports [7, 28, 24]. This problem of connecting the information theoretic and (information) geometric understanding of GANs is not just a theoretical question: there is growing experimental evidence that a careful geometric optimization, either on the support of the distributions [7, 28] or directly on these parameters [55] (which is related to the $f$-GAN framework) improves further GANs.

So, how can we link the $f$-GAN approach to any sort of information geometric optimization? The variational formulation of the GAN game in eq. (1) hints on a specific direction of research to answer this question: the identity between information-theoretic distortions on **distributions** and information-geometric distortions on their **parameterization** [5]. One such identity is well known: The Kullback-Leibler (KL) divergence between two distributions of the same (regular) exponential family equals a Bregman divergence $D$ between their natural parameters [2, 5, 10, 14, 56], which we can summarize for now (the complete statement is in Theorem 2 below) as:

$$I_{\text{fKL}}(P \parallel Q) = D(\theta \parallel \vartheta). \quad (2)$$

Here, $\theta$ and $\vartheta$ are respectively the natural parameters of $P$ and $Q$. Hence, distributions are represented by points on a manifold on the right-hand side, which is a powerful geometric
statement [5]; however, being restricted to KL divergence or "just" exponential families, it certainly falls short of the power to explain the GAN game. To our knowledge, there is no previously known "GAN-amenable" generalization of this identity above exponential families. Related identities have recently been proven for two generalizations of exponential families [6, Theorem 9], [23, Theorem 3], but fall short of the $f$-divergence formulation and are not amenable to the variational GAN formulation.

Our first contribution is such an identity that connects the general $I_f$-divergence formulation in eq. (1) to the general $D$ (Bregman) divergence formulation in eq. (2). We now briefly state it, postponing the details to Section 3:

$$f\text{-GAN}(\mathbb{P}, \text{escort}(\mathbb{Q})) = D(\theta\|\vartheta) + \text{Penalty}(\mathbb{Q}),$$

for $\mathbb{P}$ and $\mathbb{Q}$ (with respective parameters $\theta$ and $\vartheta$) which happen to lie in a superset of exponential families called deformed exponential families, that have received extensive treatment in statistical physics and differential information geometry over the last decade [3, 41]. The right-hand side of eq. (3) is the information geometric part [5], in which $D$ is a Bregman divergence. Therefore, whenever the Penalty is small, solving the $f$-GAN game solves a geometric optimization problem [5], like for the Wasserstein GAN and its variants [7], but with the difference that the geometric part is essentially implicit. Notice also that $\mathbb{Q}$ appears in the game in the form of an escort: its density is obtained from $\mathbb{Q}$’s density through a mapping (in general non-linear) completed with a simple normalization [6]. These differences vanish only for exponential families: the mapping is the identity and thus $\text{escort}(\mathbb{Q}) = \mathbb{Q}$; also, $\text{Penalty}(\mathbb{Q}) = 0$ and $f = \text{KL}$. This raises questions as to how eq. (3) and these differences relate to GAN architectures and the common understanding and implementation of the general $(f)$-GAN game [27, 47].

Our second contribution answers several of these questions via several independent results. A subset is relevant to the $f$-GAN game at large:

(a) we completely specify the parameters of the supervised game, unveiling a key parameter left arbitrary in [47] (explicitly incorporating the link function of proper composite losses [53]);

(b) we develop a novel min-max game interpretation of eq. (3) in the context of the expected utility theory [13];

(c) we show that relevant choices for escorts yield explicit upper bounds on the Penalty which vanish with the normalization coefficient of the escort.

Another subset dwells on deep architectures:

(d) we show that typical deep generator architectures are indeed powerful at modelling complex escorts of any deformed exponential family, factorising a number of escorts in order of the total inner layers’ dimensions; this provides theoretical support for the widespread empirical speculations that deep architectures may be powerful at modeling highly multimodal densities, which is a hot topic in the field [9];
(e) we show that this factorisation happens on an especially compact model design, compared e.g. to shallow architectures;

(f) we derive a connection between the parameters of the deformed exponential families and those of the generator. Quite notably, the activation function gives the deformed exponential family.

The connection between the generator and escorts via eq. (3) supports the use of geometric parameter based optimisation in the GAN game [55]. It suggests the existence of a large class of activation functions for which the factorisation in deformed exponential families holds as described in (d-f). In a field where such functions have been the subject of intensive research [19, 37, 39] and face numerous constraints in their design [7, 49], the study of this class is not just important for the theory at hand: it is also of high practical relevance.

Our last contribution studies this class and details several theoretical and experimental findings. We show that a simple sufficient condition on the activation function guarantees the escort modelling in (d), (f). Such a condition still allows for properties in activation that handle sparsity, gradient vanishing, gradient exploding and/or Lipschitz continuity [7, 25, 49]. In fact, this condition is satisfied, exactly or in a limit sense, by most popular activation functions (ELU, ReLU, Softplus, ...). We also provide experiments that display the uplift that can be obtained through tuning the activations (generator), or the link function (discriminator).

The rest of this paper is as follows. Section §2 presents definition, §3 formally presents eq. (3), §4 completes the supervised game picture of [47], §5 derives a number of consequences for deep learning, including distributions achieved by deep architectures for the generator. Section §6 presents experiments and a last Section concludes. An appendix contains all proofs and complementary experiments. Since our paper drills down into the four components of the GAN game (loss, distribution, game and architectures = models), we summarize for clarity in appendix (Section §8) our main notations, the objects they refer to and their relationships through some of our key results.

Code availability — the code used for our experiments is available through

https://github.com/qulizhen/fgan_info_geometric

2 Definitions

Throughout this paper, the domain X of observations is a measurable set. We begin with two important classes of distortion measures, f-divergences and Bregman divergences.

Definition 1 For any two distributions P and Q having respective densities P and Q absolutely continuous with respect to a base measure µ, the f-divergence between P and Q, where f : R_+ → R is convex with f(1) = 0, is

\[ I_f(P||Q) = \mathbb{E}_{x \sim Q} \left[ f \left( \frac{P(x)}{Q(x)} \right) \right] = \int_X Q(x) \cdot f \left( \frac{P(x)}{Q(x)} \right) d\mu(x) \, . \] (4)
For any convex differentiable $\varphi : \mathbb{R}^d \to \mathbb{R}$, the (\(\varphi\)-)Bregman divergence between $\theta$ and $\varphi$ is:

$$D_{\varphi}(\theta \parallel \varphi) \equiv \varphi(\theta) - \varphi(\varphi) - (\theta - \varphi)^\top \nabla \varphi(\varphi),$$

(5)

where $\varphi$ is called the generator of the Bregman divergence.

\(f\)-divergences are the key distortion measure of information theory. Under mild assumptions, they are the only distortions that satisfy the data processing inequality \cite{30, 48}. Bregman divergences are the key distortion measure of information geometry. Under mild assumptions, they are the only distortions that elicitate the sample average as a population minimizer \cite{5, 11, 46, 58}.

A distribution $P$ from a (regular) exponential family with cumulant $C : \Theta \to \mathbb{R}$ and sufficient statistics $\phi : X \to \mathbb{R}^d$ has density

$$P_C(x \mid \theta, \phi) \equiv \exp(\phi(x) \top \theta - C(\theta)),$$

(6)

where $\Theta$ is a convex open set, $C$ is convex and ensures normalization on the simplex (we leave implicit the associated dominating measure \cite{3}). A fundamental Theorem ties Bregman divergences and \(f\)-divergences.

**Theorem 2** \cite{3, 14} Suppose $P$ and $Q$ belong to the same exponential family, and denote their respective densities $P_C(x \mid \theta, \phi)$ and $Q_C(x \mid \varphi, \phi)$. Then,

$$I_{KL}(P \parallel Q) = D_C(\varphi \parallel \theta).$$

(7)

Here, $I_{KL}$ is Kullback-Leibler (KL) \(f\)-divergence ($f = x \mapsto x \log x$).

Remark that the arguments in the Bregman divergence are permuted with respect to those in eq. \cite{2} in the introduction. This also holds if we consider $f_{kl}$ in eq. \cite{2} to be the Csiszár dual of $f$ in Theorem \cite{2, 14}, namely $f_{kl} : x \mapsto -\log x$, since in this case $I_{f_{kl}}(P \parallel Q) = I_{KL}(Q \parallel P) = D_C(\varphi \parallel \theta)$. We made this choice in the introduction for the sake of readability in presenting eqs. \cite{1, 3}. Theorem \cite{2} is useful because it shows that distributions can be replaced by their parameterisation (and \textit{vice versa}) to tackle a problem — we just need to pick the right distortion for the objects at hand. There is analytic convenience in this: for example, the Bregman divergence bypasses sampling issues to estimate the integral in the \(f\)-divergence — at the expense of the estimation of the parameters, though. In fact, Theorem \cite{2} is so important that we state and prove a generalization of it in appendix, Section \cite{9}, showing that dropping the "same family" constraint does not change the \(f\)-divergence (information-theoretic) vs Bregman divergence (information-geometric) picture.

We now define generalizations of exponential families, following \cite{6, 23}. Let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing \cite[Chapter 10]{11}. We define the $\chi$-logarithm, $\log_\chi$, as

$$\log_\chi(z) \equiv \int_1^z \frac{1}{\chi(t)} \, dt.$$  

(8)

The $\chi$-exponential is

$$\exp_\chi(z) \equiv 1 + \int_0^z \lambda(t) \, dt,$$

(9)

where $\lambda$ is defined by $\lambda(\log_\chi(z)) \equiv \chi(z)$. In the case where the integrals are improper, we consider the corresponding limit in the argument / integrand.
Definition 3 [6] A distribution $P$ from a $\chi$-exponential family (or deformed exponential family, $\chi$ being implicit) with convex cumulant $C : \Theta \to \mathbb{R}$ and sufficient statistics $\phi : X \to \mathbb{R}^d$ has density given by:

$$P_{\chi,C}(x|\theta, \phi) = \exp(\chi(\phi(x)^\top \theta - C(\theta))) ,$$

(10)

with respect to a dominating measure $\mu$. Here, $\Theta$ is a convex open set and $\theta$ is called the coordinate of $P$. The escort density (or $\chi$-escort) of $P$ is

$$\tilde{P}_{\chi,C} = \frac{1}{Z} \cdot \chi(P_{\chi,C}) ,$$

(11)

where

$$Z = \int_X \chi(P_{\chi,C}(x|\theta, \phi)) d\mu(x)$$

(12)

is the escort’s normalization constant.

We leaving implicit the dominating measure and denote $\tilde{P}$ the escort distribution of $P$ whose density is given by eq. (11). We shall name $\chi$ the signature of the deformed (or $\chi$-)exponential family, and sometimes drop indexes to save readability without ambiguity, noting e.g. $\tilde{P}$ for $\tilde{P}_{\chi,C}$. Notice that normalization in the escort is ensured by a simple integration [6, Eq. 7]. For the escort to exist, we require that $Z < \infty$ and therefore $\chi(P)$ is finite almost everywhere. Such a requirement would naturally be satisfied in the GAN game.

There is another generalization of regular exponential families, known as generalized exponential families [23] (appendix, Section 9). Their densities are defined from the subdifferential of a convex function, but involves an inner product similar to eq. (10). There is no known strict equivalent of Theorem 2 for whichever of the generalizations. For example, [23, Theorem 3] provides a generalization of Theorem 2 but replaces KL by a Bregman divergence

The closest result appears for deformed exponential families [6, Theorem 9][59].

Theorem 4 [6][59] for any two $\chi$-exponential distributions $P$ and $Q$ with respective densities $P_{\chi,C}, Q_{\chi,C}$ and coordinates $\theta, \vartheta$,

$$D_C(\theta||\vartheta) = \mathbb{E}_{X \sim \tilde{Q}}[\log(\chi(Q_{\chi,C}(X))) - \log(\chi(P_{\chi,C}(X)))] .$$

(13)

Theorem 4 is a generalization of Theorem 2 for $\chi(z) = z$, in which case $\log_\chi = \log, \exp_\chi = \exp$ and escorts disappear: $\tilde{Q} = Q$. There are two important things to notice in eq. (13):

- the expectation is computed over the escort of $Q$;
- the difference of two $\chi$-logarithms is in general not the $\chi$-logarithm of the density ratio.

The $f$-GAN game relies on distortions being formulated via convex functions over density ratios. As such, Theorem 4 is not amenable to the variational $f$-GAN formulation [47, Section 2.2]. In the following Section, we show how to achieve this goal, but before, we briefly frame the now popular ($f$-)GAN adversarial learning [27, 47].

1Under mild assumptions on support and functions, $\{KL\} = f$-divergences $\cap$ Bregman divergences [30].
We have a true unknown distribution \( P \) over a set of objects, e.g. 3D pictures, which we want to learn. In the GAN setting, this is the objective of a *generator*, who learns a distribution \( Q_{\theta} \) parameterized by vector \( \theta \). \( Q_{\theta} \) works by passing (the support of) a simple, uninformed distribution, e.g. standard Gaussian, through a possibly complex function, e.g. a deep net whose parameters are \( \theta \) and maps to the support of the objects of interest. Fitting \( Q_{\theta} \) involves an *adversary* (the discriminator) as subroutine, which fits classifiers, e.g. deep nets, parameterized by \( \omega \). The generator’s objective is to come up with \( \arg\min_{\theta} L_f(\theta) \) with \( L_f(\theta) \) the discriminator’s objective:

\[
L_f(\theta) = \sup_{\omega} \{ \mathbb{E}_{X \sim P} [T_{\omega}(X)] - \mathbb{E}_{X \sim Q_{\theta}} [f^*(T_{\omega}(X))] \},
\]

where \( \star \) is Legendre conjugate [15] and \( T_{\omega} : X \to \mathbb{R} \) integrates the classifier of the discriminator and is therefore parameterized by \( \omega \). \( L_f \) is a variational approximation to a \( f \)-divergence [47]; the discriminator’s objective is to segregate true (\( P \)) from fake (\( Q_{\theta} \)) data. The original GAN choice, [27]

\[
f_{\text{gan}}(z) = z \log z - (z+1) \log(z+1) + 2 \log 2
\]

(the constant ensures \( f(1) = 0 \)) can be replaced by any convex \( f \) meeting mild assumptions.

### 3 A variational information geometric identity for the \( f \)-GAN game

We now make a series of Lemmata and Theorems that will bring us to formalize eq. (3), in two main steps: first, we show that the right-hand side of eq. (13) in Theorem 4 can be reformulated using a new set of distortion measures which is amenable to the variational \( f \)-GAN formulation. Second, we connect this variational formulation to the classical \( f \)-GAN game [47] by showing that, modulo finiteness conditions that make sense to the GAN game, this new set of distortion measures essentially coincides with \( f \)-divergences.

**\( KL_\chi \) divergences** — First, we define this new set of distortion measures, that we call \( KL_\chi \) divergences.

**Definition 5** For any \( \chi \)-logarithm and distributions \( P, Q \) having respective densities \( P \) and \( Q \) absolutely continuous with respect to base measure \( \mu \), the \( KL_\chi \) divergence between \( P \) and \( Q \) is defined as:

\[
KL_\chi(P \| Q) = \mathbb{E}_{X \sim P} \left[ - \log_\chi \left( \frac{Q(X)}{P(X)} \right) \right].
\]

Since \( \chi \) is non-decreasing, \( - \log_\chi \) is convex and so any \( KL_\chi \) divergence is an \( f \)-divergence. When \( \chi(z) = z \), \( KL_\chi \) is the KL divergence. In what follows, base measure \( \mu \) and absolute continuity are implicit, as well as that \( P \) (resp. \( Q \)) is the density of \( P \) (resp. \( Q \)). In the same way as \( f \) divergences are invariant to specific affine translations (see the proof of Theorem 7), \( KL_\chi \) divergences satisfy an interesting invariance.
Lemma 6  For any \( \chi \)-logarithm, distributions \( P, Q \) and constant \( k \in \mathbb{R}_+ \),
\[
KL_\chi(P\|Q) = KL_{\frac{1}{1+k}(P\|Q)}. \tag{17}
\]
(Proof in appendix, Section \textbf{10}) Hence, we can in fact assume that any \( KL_\chi \) divergence is obtained for a signature which is bounded.

\( KL_\chi \) divergences vs \( f \)-divergences — Let \( \partial f \) be the subdifferential of convex \( f \) and \( I_{P,Q} = \inf_x P(x)/Q(x), \sup_x P(x)/Q(x) \subseteq \mathbb{R}_+ \) denote the range of density ratios of \( P \) over \( Q \). Our first result states that if there is an element of the subdifferential which is upperbounded on \( I_{P,Q} \), the \( f \)-divergence \( I_f(P\|Q) \) is equal to a \( KL_\chi \) divergence.

Theorem 7  Suppose that \( P, Q \) are such that \( \exists \xi \in \partial f \) with \( \sup \xi(I_{P,Q}) < \infty \). Then \( \exists \chi : \mathbb{R}_+ \to \mathbb{R}_+ \) non decreasing such that \( I_f(P\|Q) = KL_\chi(Q\|P) \).

Remark.  Notice that because the constraint relies on the subdifferential, it actually does not prevent the \( f \)-divergence to diverge. Also, Theorem \textbf{7} essentially covers most if not all relevant GAN cases, as the assumption has to be satisfied in the GAN game for its solution not to be vacuous up to a large extent (eq. (14)). Indeed, if the subdifferential diverges on a finite ratio, then the optimal \( \omega \) makes \( T_\omega \) explode on some \( x \) \cite[Eq. 5]{17}. If it diverges on an infinite ratio, then \( \lim_{z \to \infty} f(z) = +\infty \) and essentially \( I_f(P\|Q) \) is unbounded. In this case, \( Q \) vanishes in the neighborhood of some \( x \in X \) for which \( P > 0 \). We can make \( L_f(\theta) \) artificially large by just picking \( \omega \) such that \( T_\omega \) is as large as necessary in such a neighborhood: the discriminator only focuses on one "pit" of \( Q \) (relative to \( P \)) to detect natural examples, which is not an appealing solution to the GAN game.

The proof of Theorem \textbf{7} (in appendix, Section \textbf{11}) is constructive: it shows how to pick \( \chi \) which satisfies all requirements. It brings the following interesting corollary: under mild assumptions on \( f \), there exists a \( \chi \) that fits for all densities \( P \) and \( Q \). A prominent example of \( f \) that fits is the original GAN choice for which we can pick
\[
\chi_{\mathrm{GAN}}(z) = \frac{1}{\log (1 + \frac{1}{z})}. \tag{18}
\]

Corollary 8  Suppose \( \exists \xi \in \partial f \) with \( \sup \xi(\text{intdom} f) < \infty \). Then \( \exists \chi : \mathbb{R}_+ \to \mathbb{R}_+ \) increasing such that for any distributions \( P, Q \), \( I_f(P\|Q) = KL_\chi(Q\|P) \).

Remark.  Even when \( f \) does not satisfy Corollary \textbf{8} it may well be the case that its Csiszár dual does \cite{13}, or equivalently, that Corollary \textbf{8} holds if we permute the arguments in one of the distortions. Let \( f_\circ(z) = z \cdot f(1/z) \). We have \( I_f(P\|Q) = I_{f_\circ}(Q\|P) \). Then, for example, picking \( f(z) = z \log z \) (KL) does not fit to Corollary \textbf{8} but picking \( f_\circ(z) = -\log z \) (reverse KL) does. Picking Pearson \( \chi^2 \) (\( f(z) = (z - 1)^2 \)) does not fit to Corollary \textbf{8} but picking \( f_\circ(z) = (1/z) \cdot (z - 1)^2 \) (Neyman \( \chi^2 \)) does.

We now show that when the subdifferential diverges (but \( I_f \) is finite), it it still possible to approximate \( I_f(P\|Q) \) by some \( KL_\chi \) divergence, up to any required precision.
Theorem 9 Suppose that $P, Q$ are such that $\sup \xi(\mathbb{1}_{P,Q}) = +\infty, \forall \xi \in \partial f$, but $I_f(P||Q) < +\infty$, then $\forall \delta > 0, \exists \chi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing such that

$$KL_{\chi}(Q||P) \leq I_f(P||Q) \leq KL_{\chi}(P||Q) + \delta .$$

(Proof in appendix, Section 12)

A $KL_{\chi}$ divergences formulation for Theorem 4 — To connect $KL_{\chi}$-divergences and Theorem 4, we need a slight generalization of $KL_{\chi}$-divergences and allow for $\chi$ in eq. (16) to depend on the choice of the expectation’s $X$, granted that for any of these choices, it will meet the constraints to be $\mathbb{R}_+ \to \mathbb{R}_+$ and also increasing, and therefore define a valid signature. For any $f : \mathcal{X} \to \mathbb{R}_+$, we denote

$$KL_{\chi_f}(P||Q) = \mathbb{E}_{X \sim P} \left[ - \log_{\chi_f(x)} \left( \frac{Q(X)}{P(X)} \right) \right] ,$$

(20)

where for any $p \in \mathbb{R}_+$,

$$\chi_p(t) = \frac{1}{p} \cdot \chi(tp) .$$

(21)

Whenever $f = 1$, we just write $KL_{\chi}$ as we already did in Definition 5. We note that for any $x \in \mathcal{X}$, $\chi_f(x)$ is increasing and non negative because of the properties of $\chi$ and $f$, so $\chi_f(x)(t)$ defines a $\chi$-logarithm. We also note that the invariance of Lemma 6 holds as well for $KL_{\chi_f}(P||Q)$. With this generalization of $KL_{\chi}$, we are ready to state a Theorem that connects $KL_{\chi}$-divergences and Theorem 4.

Theorem 10 Letting $P = P_{\chi_C}$ and $Q = Q_{\chi_C}$ for short in Theorem 4, we have:

$$\mathbb{E}_{X \sim \tilde{Q}}[\log_{\chi} (Q(X)) - \log_{\chi} (P(X))] = KL_{\chi_{\tilde{Q}}} (\tilde{Q}||P) - J(Q) ,$$

(22)

with

$$J(Q) = KL_{\chi_{\tilde{Q}}} (\tilde{Q}||Q) .$$

(23)

(Proof in appendix, Section 13) To summarize, we know that under mild assumptions relatively to the GAN game, $f$-divergences coincide with $KL_{\chi}$ divergences (Theorems 7, 9). We also know from Theorem 10 that $KL_{\chi}$ divergences quantify the geometric proximity between the coordinates of generalized exponential families (Theorem 4). Hence, finding a geometric (parameter-based) interpretation of the variational $f$-GAN game as described in eq. (14) can be done via a variational formulation of the $KL_{\chi}$ divergences appearing in Theorem 10.

A variational formulation for $KL_{\chi}$ divergences — Since penalty $J(Q)$ does not belong to the GAN game (it does not depend on $P$), it reduces our focus on $KL_{\chi_{\tilde{Q}}} (\tilde{Q}||P)$.

Theorem 11 $KL_{\chi_{\tilde{Q}}} (\tilde{Q}||P)$ admits the variational formulation

$$KL_{\chi_{\tilde{Q}}} (\tilde{Q}||P) = \sup_{T \in \mathbb{R}_{++}^\chi} \left\{ \mathbb{E}_{X \sim \tilde{P}}[T(X)] - \mathbb{E}_{X \sim \tilde{Q}}[(-\log_{\chi_{\tilde{Q}}})^*(T(X))] \right\} ,$$

(24)
with $\mathbb{R}^+ = \mathbb{R} \setminus \mathbb{R}^+$. Furthermore, letting $Z$ denote the normalization constant of the $\chi$-escort of $Q$, the optimum $T^* : \mathcal{X} \to \mathbb{R}^+$ to eq. (24) is

$$T^*(x) = -\frac{1}{Z} \cdot \frac{\chi(Q(x))}{\chi(P(x))}.$$  \hspace{1cm} (25)

(Proof in appendix, Section 14) Hence, the variational $f$-GAN formulation can be captured in an information-geometric framework by the following identity using Theorems 4, 7, 10, 11.

**Corollary 12 (the variational information-geometric $f$-GAN identity)** Using notations from Theorems 10, 11, we have

$$\sup_{T \in \mathbb{R}^+} \left\{ \mathbb{E}_{X \sim P}[T(X)] - \mathbb{E}_{X \sim \tilde{Q}}[(-\log\chi_{\tilde{Q}})^*(T(X))] \right\} = D_C(\theta \parallel \vartheta) + J(Q),$$  \hspace{1cm} (26)

where $\theta$ (resp. $\vartheta$) is the coordinate of $P$ (resp. $Q$).

We shall also name for short vig-$f$-GAN the identity in eq. (26). Even when it is not needed to understand the high-level picture of the identity, we can reduce the Legendre conjugate $(-\log\chi_{\tilde{Q}})^*$ to an equivalent "dual" (negative) $\chi^*$-logarithm in the variational problem.

**Theorem 13** The variational formulation of $KL_{\chi_{\tilde{Q}}}(\tilde{Q} \parallel P)$ (Theorem 11) satisfies:

$$\sup_{T \in \mathbb{R}^+} \left\{ \mathbb{E}_{X \sim P}[T(X)] - \mathbb{E}_{X \sim \tilde{Q}}[(-\log\chi_{\tilde{Q}})^*(T(X))] \right\} = \sup_{T \in \mathbb{R}^+} \left\{ \mathbb{E}_{X \sim P}[T(X)] - \mathbb{E}_{X \sim \tilde{Q}}[- \log(\chi^*)_{\frac{1}{\tilde{Q}}}(T(X))] \right\} - K(Q),$$  \hspace{1cm} (27)

where $K(\cdot)$ is a function of $Q$ only and

$$\chi^*(t) = \frac{1}{\chi^{-1}\left(\frac{1}{t}\right)}.$$  \hspace{1cm} (28)

(Proof in appendix, Section 15) Since only the "sup" part is of interest in the supervised discriminator-generator game, the main interest of Theorem 13 is to give a more precise shape to the losses involved in the supervised game (See Section 4).

**Remark.** The left hand-side of Eq. (26) has the exact same overall shape as the variational objective of [47, Eqs 2, 6], in which we would have equivalently $f = -\log_{\chi_{\tilde{Q}}}$, $f^* = -\log_{(\chi^*)_{1/\tilde{Q}}}$, eq. (14). However, it tells the formal story of GANs in significantly greater details, in particular for what concerns the generator. For example, eq. (26) yields a new characterization of the generators’ convergence: because $D_C$ is a Bregman divergence, it satisfies the identity of the indiscernibles. So, up to the proximity of $Q$ to its escort (to have $J(Q)$ small), solving the $f$-GAN game [47] guarantees convergence in the parameter space ($\vartheta$ vs $\theta$). In the realm of GAN applications, it makes sense to consider that $P$ (the true distribution) can be extremely complex. Therefore, even when deformed exponential families are significantly more expressive than regular exponential families [41], extra care should be put before arguing that
complex applications comply with such a geometric convergence in the parameter space. One way to circumvent this problem is to build distributions in \( Q \) that factorize many deformed exponential families. This is one strong point of deep architectures that we shall prove in Section 5.

We also remark two key components of the \( f \)-GAN identify in deformed exponential families which are absent from Theorem 2:

1. the generator \( (Q) \) appears in the form of an escort in the variational component — this distinction vanishes for exponential families, where \( \tilde{Q} = Q \);
2. an information theoretic penalty appears in the identity \( (J(Q)) \) — this penalty vanishes for exponential families, for which \( J(Q) = 0 \).

These two components are crucial to link the \( f \)-GAN variational optimization to the geometric convergence in the parameter space. We shall drill down into both in Section 5.

4 A complete proper loss picture of the supervised GAN game

In their generalization of the GAN objective, Nowozin et al. [47] leave untold a key part of the supervised game: they split in eq. (14) the discriminator’s contribution in two, \( T = g_f \circ V_\omega \), where \( V_\omega : X \rightarrow \mathbb{R} \) is the actual discriminator, and \( g_f \) is essentially a technical constraint to ensure that \( V_\omega(.) \) is in the domain of \( f^* \). They leave the choice of \( g_f "some-what arbitrary" [47, Section 2.4] \). We now show that if one wants the supervised loss to have the desirable property to be proper composite [53], then \( g_f \) is not arbitrary. We proceed in three steps, first unveiling a broad class of proper \( f \)-GANs that deal with this property.

**Proper \( f \)-GANs** — The initial motivation of eq. (14) was that the inner maximisation may be seen as the \( f \)-divergence between \( P \) and \( Q_\theta \), \( L_f(\theta) = I_f(P \parallel Q_\theta) \). In fact, this variational representation of an \( f \)-divergence holds more generally: by [54, Theorem 9], we know that for any convex \( f \), and invertible link function \( \Psi : (0,1) \rightarrow \mathbb{R} \), we have:

\[
\inf_{T : X \rightarrow \mathbb{R} \times \{\text{fake}, \text{real}\} \sim D} E_{D} [\ell_\Psi(Y, T(X))] = -\frac{1}{2} \cdot I_f(P \parallel Q) \tag{29}
\]

where \( D \) is the distribution over (observations \( \times \{\text{fake, real}\} \)) and the loss function \( \ell_\Psi \) is defined by:

\[
\ell_\Psi(+1, z) = -f'(\frac{\Psi^{-1}(z)}{1 - \Psi^{-1}(z)}) \quad ; \quad \ell_\Psi(-1, z) = f^* \left( f'(\frac{\Psi^{-1}(z)}{1 - \Psi^{-1}(z)}) \right) \tag{30}
\]

assuming \( f \) differentiable. Note now that picking \( \Psi(z) = f'(z/(1 - z)) \) with \( z = T(x) \) and simplifying eq. (29) with \( P[Y = \text{fake}] = P[Y = \text{real}] = 1/2 \) in the GAN game yields eq. (14).

For other link functions, however, we get an equally valid class of losses whose optimisation

\footnote{Informally, Bayes rule realizes the optimum and the loss accommodates for any real valued predictor.}
will yield a meaningful estimate of the $f$-divergence. The losses of eq. (30) belong to the class of proper composite losses with link function $\Psi$\ref{33}. Thus (omitting parameters $\theta, \omega$), we rephrase eq. (14) and refer to the proper $f$-GAN formulation as $\inf_Q L_\Psi(Q)$ with ($\ell$ is as per eq. (30)):

$$L_\Psi(Q) = \sup_{T: X \to \mathbb{R}} \left\{ \mathbb{E}_{X \sim \mathcal{P}}[-\ell_\Psi(+1, T(X))] + \mathbb{E}_{X \sim \mathcal{Q}}[-\ell_\Psi(-1, T(X))] \right\}. \tag{31}$$

Note also that it is trivial to start from a suitable proper composite loss, and derive the corresponding generator $f$ for the $f$-divergence as per eq. (29). Finally, our proper composite loss view of the $f$-GAN game allows us to elicitate $g_f$ in [47]: it is the composition of $f'$ and $\Psi$ in eq. (30).

Proper $f$-GANs and density ratios — The use of proper composite losses as part of the supervised GAN formulation sheds further light on another aspect the game: the connection between the value of the optimal discriminator, and the density ratio between the generator and discriminator distributions. Instead of the optimal $T^*(x) = f'(P(x)/Q(x))$ for eq. (14)\ref{47, Eq. 5}, we now have with the more general eq. (31) the result $T^*(x) = \Psi((1 + Q(x)/P(x))^{-1})$.

Proper vig-$f$-GANs — We now show that proper $f$-GANs can easily be adapted to eq. (26).

**Theorem 14** For any $\chi$, define $\ell_x(-1, z) = -\log(\chi \cdot \frac{1}{\bar{Q}(x)}) (z)$, and let $\ell(+1, z) = -z$. Then $L_\Psi(Q)$ in eq. (31) equals eq. (26). Its link in eq. (31) is

$$\Psi_x(z) = -\frac{1}{\chi \bar{Q}(x) \left( \frac{z}{1-z} \right)}. \tag{32}$$

(Proof in appendix, Section \ref{16}) Hence, in the proper composite view of the vig-$f$-GAN identity, the generator rules over the supervised game: it tempers with both the link function and the loss — but only for fake examples. Notice also that when $z = -1$, the fake examples loss satisfies $\ell_x(-1, -1) = 0$ regardless of $x$ by definition of the $\chi$-logarithm.

## 5 Consequences for deep learning

In this Section, we highlight a number of consequences of our results, from the standpoint of deep learning. Eq. (26) shows the importance for the generator to be able to model escorts — and complex ones, in the realm of the GAN applications. We start here with a proof that, when used for the generator, mainstream deep architectures\ref{34} are amenable to such complex factorizations of escorts using an especially compact design.

### 5.1 Deep architectures and escorts in the vig-$f$-GAN game

In the GAN game, distribution $Q$ in eq. (26) is built by the generator (call it $Q_g$), by passing the support of a simple distribution (e.g., uniform, standard Gaussian), $Q_{in}$, through a series...
Figure 1: Deep architecture for the generator; it takes as input a simple distribution ($Q_{in}$) and outputs a complex distribution ($Q_g$) through a (deep) series of non-linear transformations (best viewed in color, see text).

of non-linear transformations (Figure 1). Letting $Q_{in}$ denote the corresponding density, we now compute $Q_g$. Our generator $g: \mathcal{X} \to \mathbb{R}^d$ consists of two parts: a deep part and a last layer. The deep part is, given some $L \in \mathbb{N}$, the computation of a non-linear transformation $\phi_L: \mathcal{X} \to \mathbb{R}^{d_L}$ as

$$\mathbb{R}^{d_L} \ni \phi_l(x) = v(W_l \phi_{l-1}(x) + b_l), \forall l \in \{1, 2, \ldots, L\}, \quad (33)$$

$$\phi_0(x) = x \in \mathcal{X}. \quad (34)$$

$v$ is a function computed coordinate-wise, such as (leaky) ReLUs, ELUs \cite{19, 29, 37, 39}, $W_l \in \mathbb{R}^{d_l \times d_{l-1}}, b_l \in \mathbb{R}^{d_l}$. The last layer computes the generator’s output from $\phi_L$:

$$g(x) = v_{out}(\Gamma \phi_L(x) + \beta), \quad (35)$$

with $\Gamma \in \mathbb{R}^{d \times d_L}, \beta \in \mathbb{R}^{d}$; in general, $v_{out} \neq v$ and $v_{out}$ fits the output to the domain at hand, ranging from linear \cite{7, 34} to non-linear functions like tanh \cite{47}. Our generator, sketched in Figure 1 captures the high-level features of some state of the art generative approaches \cite{52, 60, 62}.

To carry our analysis, we make the assumption that the network is reversible, which is going to require that $v_{out}, \Gamma, W_l (l \in \{1, 2, \ldots, L\})$ are invertible. Since $v_{out}$ would be in many experimental cases (identity, tanh, etc.), we essentially assume that dimensions match like in Figure 1 and so the simple input density is in fact of dimension $d$ (e.g. uniform over $\mathcal{X} = a$ hypercube). At this reasonable price, we get in closed form the generator’s density and it shows the following: for any continuous signature $\chi_{net}$, there exists an activation function $v$ such that the deep, most important part in the network (Figure 1) can factor exactly as escorts for the $\chi_{net}$-exponential family. Let $1_i$ denote the $i^{th}$ canonical basis vector.

**Theorem 15** $\forall v_{out}, \Gamma, W_l$ invertible ($l \in \{1, 2, \ldots, L\}$), for any continuous signature $\chi_{net}$, there exists activation $v$ and $b_l \in \mathbb{R}^{d}$ ($\forall l \in \{1, 2, \ldots, L\}$) such that for any output $z$, letting
\[ x = g^{-1}(z), \ Q_g(z) \text{ factorizes as:} \]

\[ Q_g(z) = \frac{Q_m(x)}{Q_{\text{deep}}(x)} \cdot \frac{1}{H_{\text{out}}(x) \cdot Z_{\text{net}}}, \]  \hspace{1cm} (36) 

with \( Z_{\text{net}} > 0 \) a constant, \( H_{\text{out}}(x) = \prod_{i=1}^{d} |v'_{\text{out}}(\gamma_i^\top \phi_L(x) + \beta_i)|, \gamma_i \equiv \Gamma^\top 1_i, \) and (letting \( w_{l,i} = W_l 1_i)): 

\[ \tilde{Q}_{\text{deep}}(x) = \prod_{l=1}^{L} \prod_{i=1}^{d} \tilde{P}_{\chi_{\text{net}},b_{l,i}}(x|w_{l,i}, \phi_{l-1}). \]  \hspace{1cm} (37) 

(Proof in appendix, Section 17) The relationship between the inner layers of a deep net and deformed exponential families (Definition 3) follows from the Theorem:

- rows in \( W_l \)'s define coordinates;
- \( \phi_l \) define "deep" sufficient statistics;
- \( b_l \) are cumulants;
- the crucial part, the \( \chi \)-family, is given by the activation function \( v \).

Notice also that the \( b_l \)'s are learned, and so the deformed exponential families’ normalization is in fact learned and not specified. The proof of the Theorem comments on a simplification of the constant when we also suppose that the escorts’ normalization is not specified. The proof of the Theorem also comments on two additional keypoints:

(i) how \( Q_g(z) \) may factor as a likelihood on a graphical model defined by the inner layers of \( g \);

(ii) how the "twist" introduced by \( H_{\text{out}}(x) \) can be absorbed in a "det(.)" volume element with general sigmoid activations [47, 60, 52, 62], which is standard to the change of variable formula [21]. We also note that with linear activation [7, 34], \( H_{\text{out}}(x) \) is constant.

We see that \( \tilde{Q}_{\text{deep}} \) factors escorts, and in number, which is good news with respect to the power of deep architectures and their adequation to the GAN framework. What is remarkable is the compactness achieved by the deep representation: the total dimension of all deep sufficient statistics in \( \tilde{Q}_{\text{deep}} \) (eq. (37)) is \( L \cdot d \). To handle this, a shallow net with a single inner layer would require a matrix \( W \) of space \( \Omega(L^2 \cdot d^2) \). The deep net \( g \) requires only \( O(L \cdot d^2) \) space to store all \( W_l \)’s.

5.2 Escort-compliant design of inner activations in the generator

The proof of Theorem 15 is constructive: it builds \( v \) as a function of \( \chi \). In fact, the proof also shows how to build \( \chi \) from the activation function \( v \) in such a way that \( \tilde{Q}_{\text{deep}} \) factors \( \chi \)-escorts. The following Lemma essentially says that this is possible for all strongly admissible activations \( v \).
Figure 2: Convergence of the signature $\chi$ for $\mu$-ReLU to that of ReLU (dashed pink at the back, also displayed in Figure 3).

**Definition 16** Activation function $v$ is strongly admissible iff $\text{dom}(v) \cap \mathbb{R}^+ \neq \emptyset$ and $v$ is $C^1$, lowerbounded, strictly increasing and convex.

**Lemma 17** For any strongly admissible $v$, there exists signature $\chi$ such that Theorem 15 holds.

(proof in appendix, Section 18) $(\gamma, \gamma)$-ELU (for any $\gamma > 0$), Softplus are strongly admissible, which leaves open the status of more general ELUs, leaky ReLU and, or course, ReLU [19, 22, 37, 39]. We note that these latter activations satisfy parts of the constraints already, as they are increasing, convex and meet the domain requirement. We shall analyze them through the property that they can be arbitrarily closely approximated by a strongly admissible activation, a property that we define as weak admissibility.

**Definition 18** Activation $v$ is weakly admissible iff for any $\epsilon > 0$, there exists $v_\epsilon$ strongly admissible such that $\|v - v_\epsilon\|_{L_1} < \epsilon$, where $\|f\|_{L_1} = \int |f(t)|dt$.

Notice that the constraint is stronger than just controlling $\sup_z |v(z) - v_\epsilon(z)|$. Nevertheless, we can prove the following.

**Lemma 19** ReLU is weakly admissible.

(proof in appendix, Section 19) The trick is simple: approximate the function by a strongly admissible smooth activation, to get rid of the fact that ReLU is not differentiable everywhere and not strictly increasing. For this reason, this trick can easily be repeated for $(\alpha, \beta)$-ELU. For leaky-ReLU, we need to add the constraint that the domain is lowerbounded, and then
This is happening when  

Table 1: Some (strongly or weakly) admissible couples \((v, \chi)\). (§): 1. is the indicator function; (†): \(\delta \leq 0, 0 < \epsilon \leq 1\) and \(\text{dom}(v) = [\delta/\epsilon, +\infty)\). (\(\vee\)): \(\beta \geq \alpha > 0\); (\(\blacklozenge\)): \(\star\) is Legendre conjugate; (\(\lozenge\)): \(\mu \in [0, 1)\). Shaded: prop-\(\tau\) activations; \(k\) is a constant (e.g. such that \(v(0) = 0\)); (\(\bullet\)): LSU = Least Square Unit (see text).

the trick is the same. Table 1 presents several couples \((v, \chi)\) for which \(v\) is (strongly or weakly) admissible. In the case where \(v\) is strongly admissible, we give the signature \(\chi\) that would be obtained through Lemma 17. If it is weakly admissible, we give the limit \(\chi\) for the sequence of strong admissible activations in Definition 18. Figure 2 gives an example of such a sequence for the \(\mu\)-ReLU activation. Table 1 includes a wide class of so-called "prop-\(\tau\) activations", where \(\tau\) is negative a concave entropy, defined on \([0, 1]\) and symmetric around \(1/2\). Softplus is a prop-\(\tau\) activation. We also remark that \(\text{ReLU} = \lim_{\mu \to 1} \mu\)-ReLU (in the sense that \(\lim_{\mu \to 1} \sup_z |\text{ReLU}(z) - \mu\)-ReLU\((z)| = 0\)). One property of prop-\(\tau\) activations is especially handy for Wasserstein GANs [7, Eq. 3]: prop-\(\tau\) activations are Lipschitz (proof in Section 3). Finally, the LSU activation should in theory be constrained to domain \([-1, 1]\), so we have linearly extended it to \(\mathbb{R}\) by linearity, keeping convexity and differentiability. Figure 3 plots several choices of signatures \(\chi\), corresponding to different choices of activation functions, distributions or \(f\)-divergences (Figure 9 in appendix provides the correspondence from the choice of \(\chi\)).

5.3 \(J(Q)\) vs not \(J(Q)\)

By focusing on the left hand side of eq. 26, the usual \(f\)-GAN approaches guarantee convergence in the parameter spaces which is all the better as \(J(Q)\) is small after convergence. This is happening when \(\chi\) is (close enough to) identity because in this case \(\hat{Q} \to Q\), but this is not really interesting in the context of deep learning where non-linear transformations imply \(\chi\) is not going to comply (Theorem 15). For several interesting cases, we show an upperbound on \(J(Q)\) which is decreasing with \(Z\), the normalization parameter of the escort (Definition 3). Recall that \(J(Q) = K_{\chi Q}(\hat{Q}||Q)\), so there needs to be two components to specify \(J\): \(\chi\) and...
Figure 3: Choices of $\chi$ corresponding to various activation functions (LSU, Softplus, $(\alpha, \beta)$-ELU, ReLU, see Table 1), distributions (exp. fam. = exponential families) or $f$-divergences (GAN, see eq. (18)).

$Q$. In theory, there is no need for $Q$ to belong to the $\chi$-family for $J(Q)$ to be measurable, so our results will be general in the sense that we shall make no assumption about $Q$; $\chi$ will be fixed either directly (original $f$-GAN choice) or as a function of the activation function (e.g. Table 1).

For any predicate $\pi : X \rightarrow \{\text{false}, \text{true}\}$, $M(\pi) = \int_{x : \pi(x) = \text{true}} d\mu(x)$ denotes the total measure of the support satisfying $\pi$.

**Theorem 20** The following bounds on $J(Q)$ and $Z$ hold, for any $Q$:

(i) for the original GAN choice of $\chi$, we have $Z > 1$ and

$$J(Q) \leq \frac{1}{Z} \cdot M\left(Q(.) < \frac{1}{Z-1}\right).$$

(ii) for $\mu$-ReLU activation, letting $L = 1/(1 - \mu)$, we have $Z \leq L$ and

$$J(Q) \leq \frac{1}{Z} \cdot \left(1 + \frac{L}{Z}\right).$$

(iii) for the $(\gamma, \gamma)$-ELU activation with $\gamma \geq 1$, we have

$$J(Q) \leq \frac{\log \gamma}{Z} + \frac{1 - Z}{Z^2} + \frac{H_*(Q)}{Z},$$

where $H_*(Q) = \mathbb{E}_{X \sim q}[\max\{0, -\log Q(X)\}]$.

Proof in appendix, Section 20. These results seems to display the pattern that reducing $J(.)$ can be obtained via maximizing $Z$, the normalization coefficient for the escort. How $Z$
depends in fine on $\chi, v$ is non trivial. It seems that picking $\chi$ that augments the "contrast" (blows up high density regions) is a good idea. Figure 4 presents some examples of density shapes (not normalized) obtained from a simple density passed through various $\chi$, showing how one can control such a contrast. Figure 5 does the same for a standard Gaussian, where the resulting densities (in color) are normalized. Since Lemma 17 is very general, we can engineer very specific $\chi$s for this objective: inspired from the leaky-ReLU activation, the example of Figure 5 uses such leaky-$\chi$ escorts when $\chi$ is that of the $\mu$-ReLU (Table 1, $\delta > 0$, small $\epsilon > 0$):

$$
\chi_{\delta, \epsilon}(z) = 1_{z < \delta} \cdot (\epsilon z) + 1_{z \geq \delta} \cdot (\epsilon \delta + \chi(z - \delta)) .
$$

(41)

5.4 How to play the proper-GAN game

In [50], the density ratio connection was used to modify the GAN training procedure as follows: first, one trains the discriminator to solve the inner maximisation in eq. (14) for convex $f$; next, one estimates the density ratio $r(x) = P(x)/Q(x)$ by

$$
r(x) = (f')^{-1}(T^*(x)) ,
$$

(42)

finally, one trains the generator to minimise the $f$-divergence $I_\varphi(P\|Q) = \mathbb{E}_{X \sim P}\varphi(r(X'))$ for convex $\varphi$. In terms of proper composite losses, the first two steps can be generalised as
follows: first, one trains the discriminator to solve the inner maximisation in eq. (31) for convex $f$ and link function $\Psi$; next, one estimates the density ratio $r(x) = P(x)/Q(x)$ by $r(x) = \Psi^{-1}(T^*(x))/(1 - \Psi^{-1}(T^*(x)))$. Note that this allows us e.g. to use the logistic loss, for which $\Psi(z) = \log(z/(1 - z))$ and $f(z) = z \cdot \log z - (z + 1) \cdot \log(z + 1) + 2 \log 2 = f_{\text{gan}}(z)$ (eq. (15)).

5.5 A more complete picture of geometric optimization in GANs

Any Bregman divergence is locally Mahalanobis’, i.e. a squared distance with a particular metric [5, Section 3]. For eq. (26), is means when $C$ is strictly convex that $\forall \theta_P, \theta_Q$, there exists Symmetric Positive Definite (SPD) matrix $M$ such that

$$D_C(\theta_P \parallel \theta_Q) = D_{C^*}(\mu_Q \parallel \mu_P) = \| \mu_Q - \mu_P \|^2_M,$$

where $\mu = \nabla C(\theta) = \mathbb{E}[\phi]$ [12, Section 4]. Inner layers in the generator’s deep net are sufficient statistics ($\phi$, Theorem 15 and Subsection 5.1). We see that the parameterization chosen for the geometric optimization of [55, Section 3] looks like such a divergence, with $M = I$. The only difference with the vig-$f$-GAN identity is that the optimization occurs on the statistics $\mu_Q, \mu_P$ of the discriminator and not the generator, but it turns out that the $f$-divergences involved in the supervised game (Section 4 and [54]) also admit a formulation in terms of Bregman divergences [15] and therefore can be approximated using eq. (43). Hence, our results support the feature matching technique of Salimans et al. [55, Section 3.1].

5.6 The generator can accommodate complex multimodal densities

This is currently a hot topic in GAN architectures, with some concerns raised about the capacity of the networks to capture multimodal densities [8, 9]. More specifically, whenever the discriminator is too "small", then the generator may be trapped in densities with very small support, thereby preventing it to capture the many modes of highly multi-modal densities. This is the so-called "mode collapse" problem, and it is crucial since the modes of a density being its local maxima, they locally represent the most natural objects to model. Because GAN applications are complex, one works with the objective to capture numerous modes [17]. We consider the problem from the generator’s side and ask, at first hand, whether it is amenable to model such complex densities — if it were not, then GAN architectures would be doomed beyond the training concerns raised by [8, 9].

Such a question can be answered in the affirmative via Theorem 15 (See appendix, Section 21), yet it requires specific signatures tailor made for the generator’s density to capture all modes. It is therefore more a theoretical result than a proof of validity for current architectures, yet using such signatures can accommodate as many as $\Omega(d \cdot L)$ modes.

5.7 Playing the (vig-)$f$-GAN game in the expected utility theory

To play the GAN game at its fullest extent, we need to understand it in extenso. Most of the game-theoretic focus on GANs has been focused on the convergence and/or its Nash equilibrium [8, 26], around the idea that the generator tries to "fool" the discriminator. The
expected utility theory allows to better qualify the quotes directly in the context of the vig-f-GAN game. This requires some background which we now briefly state [16].

In an insurance market, a portfolio is a function $\Upsilon : \mathcal{X} \to \mathbb{R}$ such that $\Upsilon(x)$ is the amount of cash $\Upsilon$ pays to whomever holds it under the state of the world $x \in \mathcal{X}$ (negative payoffs are interpreted as costs to the asset holder). Portfolio management for a Decision Maker (DM) works in two steps: first, DM purchases the portfolio $\Upsilon$ with market prices $P$, for a cost $\kappa = \mathbb{E}_{X \sim P}[\Upsilon(X)]$. Then DM receives a payoff $\Upsilon(x)$ upon the revelation of the state of the world $x \in \mathcal{X}$. In the expected utility theory [16], assuming DM (i) has a quasilinear utility function and (ii) maximises expected utility according to subjective beliefs $Q$. Then there exists utility $u_{\text{dm}} : \mathbb{R} \to \mathbb{R}$ increasing and concave such that DM achieves maximal utility $U(Q)$:

$$U(Q) \doteq \sup_{\Upsilon : \mathcal{X} \to \mathbb{R}} \{ \mathbb{E}_{X \sim Q}[u_{\text{dm}}(\Upsilon(X)) + \kappa] \} = \sup_{\Upsilon : \mathcal{X} \to \mathbb{R}} \{ \mathbb{E}_{X \sim P}[-\Upsilon(X)] + \mathbb{E}_{X \sim Q}[u_{\text{dm}}(\Upsilon(X))] \} \quad (44)$$

Suppose now that subjective beliefs $Q$ are in the hand of another player, $G$, distinct from DM, and whose objective is to minimize $U(Q)$, the game being the horizon of of min-max optimization iterations. The following Lemma sheds light on the key parameters of the game.

**Lemma 21** The DM vs G game is equivalent to the (vig-f)-GAN game (eq. (26)) in which $\text{DM} = \text{discriminator}$, $G = \text{generator}$, the set of portfolios $\{\Upsilon\} = \{T\}$, the subjective beliefs $Q = \tilde{Q}$ and the utility $u_{\text{om}}(z) = \log(\chi \cdot 1_{\tilde{Q}}(z))$. \hfill (45)

(Proof in appendix, Section 22) Hence, $G$ tampers with the utility function of $\text{DM}$ in this game — we note, amounts for $G$ to learn the true market prices $P$. There is more to drill from the game in terms of risk aversion, as shown below.

**Lemma 22** Let the Arrow-Pratt coefficient of absolute risk aversion [51] $a_u(z) \doteq -\frac{u''(z)}{u'(z)}$, and the Arrow-Pratt coefficient of relative risk aversion, $r_u(z) \doteq z \cdot a_u(z)$. Suppose $\chi$ differentiable. Then, in the DM vs G game, (i) $\text{DM}$ is always risk averse. Furthermore, (ii) $r_u(z)$ is also indexed by $X \sim Q$ and we have

$$r_u(z) = g \left( \frac{z}{\chi(P(x))} \right), \quad (46)$$

$$g(z) \doteq z \cdot \frac{(\chi^{-1})'(z)}{\chi^{-1}(z)} \quad (47)$$

Finally, (iii) at the optimum $\Upsilon^*$, we have

$$r_u(\Upsilon^*(x)) = g \left( \frac{1}{\chi(P(x))} \right). \quad (48)$$

(Proof in appendix, Section 23) Hence, $\text{DM}$ is always risk averse and his relative risk aversion depends on subjective beliefs with the notable exception of the optimum $T^*$ for which it depends on market prices only. Everything is like if $\text{DM}$ was getting rid of $G$’s influenced subjective beliefs to come up with the optimal solution.
Figure 6: Summary of our results on MNIST, on experiment A, comparing different values of $\mu$ for the $\mu$-ReLU activation in the generator (ReLU = 1-ReLU, see text). Thicker horizontal dashed lines present the ReLU average baseline: for each color, points above the baselines represent values of $\mu$ for which ReLU is beaten on average.

6 Experiments

Two of our theoretical contributions are:

(A) the fact that on the generator’s side, there exists numerous activation functions $v$ that comply with the design of its density as factoring escorts (Lemma 17), and

(B) the fact that on the discriminator’s side, the so-called output activation function $g_f$ of [47] aggregates in fact two components of proper composite losses, one of which, the link function $\Psi$, should be a fine knob to operate (Theorem 14).

We have tested these two possibilities with the idea that an experimental validation should provide substantial ground to be competitive with mainstream approaches, leaving space for a finer tuning in specific applications. Also, in order not to mix their effects, we have treated (A) and (B) separately.

Architectures and datasets — We provide in appendix (Section 23) the detail of all experiments. To summarize, we consider two architectures in our experiments: DCGAN [52] and the multilayer feedforward network (MLP) used in [47]. Our datasets are MNIST [33] and LSUN tower category [61].

Comparison of varying activations in the generator (A) — We have compared $\mu$-ReLUs with varying $\mu$ in $[0, 0.1, ..., 1]$ (hence, we include ReLU as a baseline for $\mu = 1$), the Softplus and the LSU activation (Figure 1). For each choice of the activation function, all
inner layers of the generator use the same activation function. We evaluate the activation functions by using both DCGAN and the MLP used in [37] as the architectures. As training divergence, we adopt both GAN [27] and Wasserstein GAN (WGAN, [7]). Results are shown in Figure 6. Three behaviours emerge when varying $\mu$: either it is globally equivalent to ReLU (GAN DCGAN) but with local variations that can be better ($\mu = 0.7$) or worse ($\mu = 0$), or it is almost consistently better than ReLU (WGAN MLP) or worse (GAN MLP). The best results were obtained for GAN DCGAN, and we note that the ReLU baseline was essentially beaten for values of $\mu$ yielding smaller variance, and hence yielding smaller uncertainty in the results.

The comparison between different activation functions (Figure 7) reveals that ($\mu$)-ReLU performs overall the best, yet with some variations among architectures. We note in particular that, in the same way as for the comparisons intra $\mu$-ReLU (Figure 6), ReLU performs relatively worse than the other criteria for WGAN MLP, indicating that there may be different best fit activations for different architectures, which is good news. Visual results on LSUN (appendix, Table 7) also display the quality of results when changing the $\mu$-ReLU activation.

**Comparison of varying link functions in the discriminator (B) —** We have compared the replacement of the sigmoid function by a link which corresponds to the entropy which is theoretically optimal in boosting algorithms, Matsushita entropy [31, 44], for which $\Psi_{\text{MAT}}(z) = (1/2) \cdot (1 + z/\sqrt{1 + z^2})$ and the entropy (Table 1) is $-\tau_{\text{MAT}}(z) = 2\sqrt{z(1-z)}$. Figure 8 displays the comparison Matsushita vs "standard" (more specifically, we use sigmoid in the case of GAN [47], and none in the case of WGAN to follow current implementations [7]). We evaluate with both DCGAN and MLP on MNIST (same hyperparameters as for generators, ReLU activation for all hidden layer activation of generators). Experiments tend
to display that tuning the link may indeed bring additional uplift: for GANs, Matsushita is indeed better than the sigmoid link for both DCGAN and MLP, while it remains very competitive with the no-link (or equivalently an identity link) of WGAN, at least for DCGAN.

7 Conclusion

It is hard to exaggerate the success of GAN approaches in modelling complex domains, and with their success comes an increasing need for a rigorous theoretical understanding [55]. In this paper, we complete the supervised understanding of the generalization of GANs introduced in [47], and provide a theoretical background to understand its unsupervised part. We show in particular how deep architectures can be powerful at tackling the generative part of the game, and can factor densities known to be far more general than exponential families, both in terms of the available densities (e.g. Cauchy, Student) or physical phenomena that can be modeled [6, 40, 41]. Our contribution therefore improves the understanding of both players in the GAN game. Experiments display that the tools we develop may help to improve further the state of the art. Among the most prominent avenues for future work relies the integration of penalty $J(Q)$ directly in the GAN game. It turns out that a recent paper has precisely displayed that the introduction of a mutual information regularizer in the GAN game improves results and helps in disentangling representations [18].

8 Acknowledgments

The authors wish to thank Shun-ichi Amari, Giorgio Patrini and Frank Nielsen for numerous comments.
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Figure 9: Summary of the main parameters notations with respect to the GAN game, according to the four main components of the game (loss, distribution, game, model = deep generator). Plain (black / blue) arcs denote formal relationships between parameters that we show. The distribution learned by a deep generator decomposes in three parts, one which depends on the simple input distribution, one which depends on the very last layer and one which incorporates all the deep architecture components, $\tilde{Q}_{\text{deep}}$ (Theorem 15).

— Summary of the paper’s notations

Figure 9 summarizes the main notations with respect to our contributions on the four components of a GAN "quadrangle": loss, distribution, game and architecture = model (= deep generator). Blue arcs identify some key parameters as a function of the signature of the deformed exponential family, $\chi$, to match several quantities of interest:

- the arc $\chi \rightarrow f$ identifies the $f$ from $\chi$ which allows to prove the identity between vig-$f$-GAN and the variational $f$-GAN identity in [47, Eq. 4] (Theorems 11, 13);
- the arc $\chi \rightarrow v$ identifies the activation function $v$ from $\chi$ for which the inner deep part of the generator in Theorem 15 factors with $\chi$-escorts ($\tilde{Q}_{\text{deep}}$);
- the arc $\chi \rightarrow u$ identifies the utility function of the discriminator / decision maker such
that the decision maker’s utility $U(\Omega)$ maximization (eq. (44)) matches vig-$f$-GAN (Lemma [21]).

Name are as follows:

**LOSS** $f = \text{generator of the } f\text{-divergence}; \ell = \text{loss function(s) for the supervised game}; \Psi = \text{link function for the supervised loss};$

**DISTRIBUTION** $\chi = \text{signature of the deformed exponential family}; \phi = \text{sufficient statistics}; C = \text{cumulant};$

**GAME** $u = \text{utility function}; r = \text{Arrow-Pratt coefficient of relative risk aversion};$

**MODEL** $W = \text{inner layer matrices}; b = \text{inner layers bias vectors}; v = \text{inner layers activation function}; \phi = \text{inner layers vectors / "deep" sufficient statistics};$
— Appendix on proofs and formal results

9 Generalization of Theorem 2

In this Section, we adopt notations of [23]. When dealing with exponential families, it will be convenient to rewrite \( \phi^\top \theta \) as the output of a function \( \phi^\top \theta : X \to \mathbb{R} \) with \( \phi^\top \theta(x) = \langle \phi(x), \theta \rangle \) — remark that \( \theta \) is implicitly fixed. Hence, the definition of the density of a (regular) exponential family with cumulant \( C : \Theta \to \mathbb{R} \) and sufficient statistics \( \phi : X \to \mathbb{R}^d \) now becomes equivalently:

\[
P_C(x|\theta, \phi) = \exp((\phi^\top \theta)(x) - C(\theta)).
\]  

(49)

If we fix \( \theta \), then the sufficient statistics uniquely determines the cumulant (and therefore the exponential family) and \textit{vice-versa}. Let us fix such a vector \( \theta \) and adopt the concise formulation of Generalized exponential families of [23], which we now introduce. Let \( \Delta \) denote a set of probability measures over \( X \) [23], and \( \star \) denotes the Legendre transform [15].

**Definition 23** [23] Let \( F : \Delta \to \mathbb{R} \) be convex, lower semi-continuous and proper. The \( F \)-Generalized exponential family (GEF) of distributions is the set \( \{ P_F(x|\theta, \phi) \in \partial F^\star (\phi^\top \theta) : \theta \in \Theta \} \), where \( \phi : X \to \mathbb{R}^d \) is called the statistic.

Notice that \( \phi \) does not necessarily bear the properties of sufficient statistics, and we can also define a cumulant, \( C(\theta) = F^\star (\phi^\top \theta) \) [23] and we have \( \Theta = \text{dom}(C) \). Deformed and generalized exponential families emerged from two different grounds, thermostatistics and information geometry for the former, convex optimization for the latter. So, they are known for very different properties, yet regular exponential families belong to both sets (\( F \) is negative Shannon entropy for regular exponential families in generalized exponential families). For the sake of readability we now assume that the cumulant is differentiable, so that the density \( P_F(x|\theta, \phi) = \nabla F^\star (\phi^\top \theta) \) in Definition [23]. For any pairs of cumulants statistics \( \phi_a, \phi_b \), we define the Bregman divergence with generator \( F^\star \),

\[
D_\theta(C_a\|C_b) = F^\star (\phi_a^\top \theta) - F^\star (\phi_b^\top \theta) - \langle \phi_a^\top \theta - \phi_b^\top \theta, \nabla F^\star (\phi_b^\top \theta) \rangle.
\]  

(50)

A key point of the bilinear form \( \langle \cdot, \cdot \rangle \) is that it has the fundamental property to transfer inner products from/to supports to/from distribution parameters [23, Section 2]:

\[
\langle \phi^\top \theta, \phi \rangle = \langle \mathbb{E}_P[\phi], \theta \rangle,
\]  

(51)

and in fact the inner product appearing in eq. [50] is also an inner product on parameters in disguise, a fact that will be key to our result. We note that \( D_\theta(C_a\|C_b) \) is indeed a Bregman divergence [23, Theorem 3], which we can unambiguously formulate over sufficient statistics or generators. Being a Bregman divergence, it satisfies the identity of the indiscernibles: \( C_a = C_b \) iff \( D_\theta(C_a\|C_b) = 0 \). Notice also that the definition makes implicitly that the dimension of the sufficient statistics is the same for both families defined by cumulants \( C_a, C_b \).

---

\(^3\text{Notice the slight abuse of notation: this definition makes in fact the cumulant to be a function } C : \mathbb{R}^X \to \mathbb{R}, \text{ but it does not affect our results.}\)
With this notion of divergence between cumulants, we can now formulate and prove our generalization of Theorem 2: if we alleviate the membership constraint, then the KL divergence is equal to the sum of two divergences, one between parameters (indexed by cumulants), and one between cumulants (indexed by parameters).

**Theorem 24** Consider any two GEF distributions \( P \) and \( Q \) having respective natural parameters \( \theta_p \) and \( \theta_q \), cumulants \( C_p \) and \( C_q \) and densities \( P \) and \( Q \) absolutely continuous with respect to base measure \( \mu \). Then

\[
KL(P \parallel Q) = D_{C_p}(\theta_q \parallel \theta_p) + D_{\theta_q}(C_q \parallel C_p). \tag{52}
\]

**Proof** We have:

\[
KL(P \parallel Q) = \int x P(x) \log \frac{P(x)}{Q(x)} d\mu(x)
\]

\[
= \int x P(x) \cdot (C_q(\theta_q) - C_p(\theta_p) + \theta_p^\top \phi_p(x) - \theta_q^\top \phi_q(x)) d\mu(x)
\]

\[
= C_q(\theta_q) - C_p(\theta_p) - (\theta_q^\top \mathbb{E}_P[\phi_q(x)] - \theta_p^\top \nabla C_p(\theta_p))
\]

\[
= C_q(\theta_q) - C_p(\theta_p) - (\theta_q - \theta_p)^\top \nabla C_p(\theta_p)
\]

\[
= C_p(\theta_q) - C_p(\theta_p) - (\theta_q - \theta_p)^\top \nabla C_p(\theta_p)
\]

\[
+ C_q(\theta_q) - C_p(\theta_q) - (\mathbb{E}_P[\phi_q(x)] - \mathbb{E}_P[\phi_p(x)])^\top \theta_q
\]

\[
= D_{C_p}(\theta_q \parallel \theta_p) + A. \tag{54}
\]

In eq. (53), we use the fact that \( \nabla C_p(\theta_p) = \mathbb{E}_P[\phi_p(x)] \). Now, we remark that \( C_a(\theta_q) = F^*(\phi_a^\top \theta_q) \) [23 Definition 2, Lemma 2], and

\[
(\mathbb{E}_P[\phi_p(x)] - \mathbb{E}_P[\phi_q(x)])^\top \theta_q = \langle P_{\theta_p}, \phi_p^\top \theta_q \rangle - \langle P_{\theta_q}, \phi_q^\top \theta_q \rangle
\]

\[
= \langle (\phi_p - \phi_q)^\top \theta_q, P_{\theta_p} \rangle
\]

\[
= \langle (\phi_p - \phi_q)^\top \theta_q, \nabla F^*(\phi_p^\top \theta) \rangle , \tag{55}
\]

using definitions of \( \langle \cdot, \cdot \rangle \) and \( F^* \) in [23] (see also eq. (51)). There remains to identify \( A \) in eq. (54) and \( D_{\theta_q}(C_q \parallel C_p) \) from eq. (50). This ends the proof of Theorem 24 \( \blacksquare \)
10 Proof of Lemma 6

We have by definition of $KL$ divergences and properties of the integration,

$$KL_{\chi\chi}(P\|Q) = -\mathbb{E}_{X \sim P} \left[ -\log_{\chi\chi} \left( \frac{Q(X)}{P(X)} \right) \right]$$

$$= -\mathbb{E}_{X \sim P} \left[ -\int_1^{\frac{Q(X)}{P(X)}} \frac{1 + k\chi(t)}{\chi(t)} dt \right]$$

$$= -\mathbb{E}_{X \sim P} \left[ -\int_1^{\frac{Q(X)}{P(X)}} \left( \frac{1}{\chi(t)} + k \right) dt \right]$$

$$= -\mathbb{E}_{X \sim P} \left[ -\log \left( \frac{Q(X)}{P(X)} \right) - \int_1^{\frac{Q(X)}{P(X)}} k dt \right]$$

$$= -\mathbb{E}_{X \sim P} \left[ -\log \left( \frac{Q(X)}{P(X)} \right) - k \cdot \int_1^{\frac{Q(X)}{P(X)}} dt \right]$$

$$= -\mathbb{E}_{X \sim P} \left[ \log \left( \frac{Q(X)}{P(X)} \right) + k \cdot \left[ \frac{Q(X)}{P(X)} \right]_1 \right]$$

$$= -\mathbb{E}_{X \sim P} \left[ \log \left( \frac{Q(X)}{P(X)} \right) + k \cdot \mathbb{E}_{X \sim P} \left[ \frac{Q(X)}{P(X)} - 1 \right] \right]$$

$$= -\mathbb{E}_{X \sim P} \left[ \log \left( \frac{Q(X)}{P(X)} \right) + k \cdot (\mathbb{E}_{X \sim Q} [1] - \mathbb{E}_{X \sim P} [1]) \right]$$

$$= KL_{\chi}(P\|Q) ,$$

as claimed. We finally check that $z \mapsto z/(1 + kz)$ is increasing and so is $t \mapsto \chi(t)/(1 + k\chi(t))$ because $\chi$ is increasing, which is also non negative and defined over $\mathbb{R}_+$ since $k \geq 0$, and so defines a signature and a valid $\chi$-logarithm.

11 Proof of Theorem 7

Our basis for the proof of the Theorem is the following Lemma.

**Lemma 25** [43, Proposition 1.6.1] Let $f : I \to \mathbb{R}$ be continuous convex and let $\xi : I \to \mathbb{R}$ such that $\xi(z) \in \partial f(z), \forall z \in \text{int} I$. Then for any $a < b$ in $I$, it holds that:

$$f(b) = f(a) + \int_a^b \xi(t) dt .$$

(57)

Suppose that $b < a$. Then Lemma 25 says that we have $f(a) = f(b) + \int_b^a \xi(t) dt$, that is, after reordering, $f(b) = f(a) - \int_b^a \xi(t) dt = f(a) + \int_a^b \xi(t) dt$, so in fact the requested ordering between the integral’s bounds can be removed. Also, we can suppose that the integral may
not be proper, in which case we compute it as a limit of a proper integral for which Lemma 25 therefore holds.

We now prove Theorem 7. Suppose there exists $M \in \mathbb{R}$ such that $\sup \xi(I_{P,Q}) \leq M$, for some $\partial f \ni \xi : \text{int dom}(f) \to \mathbb{R}$. For any constants $k$, letting $f_k(z) = f(z) - k(z - 1)$, which is convex since $f$ is, we note that

$$E_{X \sim Q} \left[ f_k \left( \frac{P(X)}{Q(X)} \right) \right] = E_{X \sim Q} \left[ f \left( \frac{P(X)}{Q(X)} \right) \right] - k \cdot E_{X \sim Q} \left[ \frac{P(X)}{Q(X)} - 1 \right]$$

$$= E_{X \sim Q} \left[ f \left( \frac{P(X)}{Q(X)} \right) \right] - k \cdot \left( \int P(X)d\mu(X) - \int Q(X)d\mu(X) \right)$$

$$= E_{X \sim Q} \left[ f \left( \frac{P(X)}{Q(X)} \right) \right]. \quad (58)$$

Let $\xi_k = \xi - k \in \partial f_k$. Since $f_k$ is convex continuous, it follows from [43, Proposition 1.6.1] (Lemma 25) that:

$$f_k \left( \frac{P(x)}{Q(x)} \right) = f_k(1) + \lim_{\rho \to P(x)} \int_1^\rho \xi_k(t)dt$$

$$= - \lim_{\rho \to P(x)} \int_1^\rho (-\xi(t) + M + \epsilon)dt. \quad (59)$$

The second identity comes from the assumption that $f(1) = 0 = f_k(1)$. The limit appears to cope with a subdifferential that would diverge around a density ratio. Fix some constant $\epsilon > 0$ and let

$$\chi(t) = \begin{cases} \frac{1}{\xi(t) + M + \epsilon} & \text{if } t < \sup I_{P,Q} \\ \epsilon & \text{if } t \geq \sup I_{P,Q} \end{cases}, \quad (60)$$

which, since $\sup \xi(I_{P,Q}) \leq M$, guarantees $\chi \geq 0$ and $\chi$ is also increasing since $\xi$ is increasing ($f$ is convex). We then check, using eqs. (58) and (60) that:

$$KL_{\chi}(Q\parallel P) = E_{X \sim Q} \left[ - \log \chi \left( \frac{P(X)}{Q(X)} \right) \right]$$

$$= E_{X \sim Q} \left[ - \lim_{\rho \to P(x)} \int_1^\rho \frac{1}{\chi(t)}dt \right]$$

$$= E_{X \sim Q} \left[ - \lim_{\rho \to P(x)} \int_1^\rho (-\xi(t) + M + \epsilon)dt \right]$$

$$= E_{X \sim Q} \left[ f_{M+\epsilon} \left( \frac{P(X)}{Q(X)} \right) \right]$$

$$= E_{X \sim Q} \left[ f \left( \frac{P(X)}{Q(X)} \right) \right] = I_f(P\parallel Q). \quad (61)$$

This ends the proof of Theorem 7.
12 Proof of Theorem 9

Without loss of generality we can assume that $\sup I_{P,Q} < +\infty$. Otherwise, when $\sup I_{P,Q} = +\infty$, requesting $\sup \xi(I_{P,Q}) = +\infty$ $(\forall \xi \in \partial f)$ implies, because $f$ is convex, that $\lim_{\sup I_{P,Q}} f(z) = +\infty$, and so the constraint $I_f(\mathbb{P}\|\mathbb{Q}) < +\infty$ essentially enforces zero measure over all infinite density ratios.

We make use of [43, Proposition 1.6.1] (Lemma 25), now with a subdifferential which is not Riemann integrable in $M = \sup I_{P,Q}$. Notice that we can assume without loss of generality that $M > 1$ since otherwise, since it is convex, $f$ would not be defined for $z > 1$ and $I_f(\mathbb{P}\|\mathbb{Q})$ would essentially be infinite unless $Q \geq P$ almost everywhere (i.e. $P$ dominates $Q$ only on sets of zero measure).

For any constants $\epsilon$ and $t^* < M$ such that $\xi(t^*) < +\infty$, let

$$g_{t^*,\epsilon}(z) = \int_{1}^{z} (-\xi(t) + \xi(t^*) + \epsilon)dt,$$

where $z \in \mathbb{R}_+$ is any real such that the integral in $g_{t^*,\epsilon}$ is not improper (therefore, $z < M$). Let

$$\chi_{t^*,\epsilon}(t) = \begin{cases} 
1 & \text{if } t < t^* \\
\frac{1}{-\xi(t) + \xi(t^*) + \epsilon} & \text{if } t \geq t^*
\end{cases},$$

(63)

which, if $\epsilon > 0$, is non negative and also increasing since $\xi$ is increasing. Consider any fixed $z^* \in I_{P,Q} \cap (1, \infty)$ with $0 < \xi(z^*) < \infty$ and let $t^* = \sup \{z : \xi(z) \leq \xi(z^*)\}$. We have:

$$g_{t^*,\epsilon}(z) = \int_{1}^{z} (-\xi(t) + \xi(t^*) + \epsilon)dt$$

$$= \int_{1}^{z} \frac{1}{\chi_{t^*,\epsilon}(t)} dt + 1_{[z \geq t^*]} \cdot \int_{t^*}^{z} (-\xi(t) + \xi(t^*))dt$$

$$= \int_{1}^{z} \frac{1}{\chi_{t^*,\epsilon}(t)} dt - 1_{[z \geq t^*]} \cdot \int_{t^*}^{z} (\xi(t) - \xi(t^*))dt$$

$$= \int_{1}^{z} \frac{1}{\chi_{t^*,\epsilon}(t)} dt - 1_{[z \geq t^*]} \cdot D_{f,\xi}(z \| t^*) .$$

(64)

The last identity comes from [43, Proposition 1.6.1] (Lemma 25) and the fact that $\xi(t) - \xi(t^*)$ belongs to the subdifferential of the Bregman divergence whose generator is $f$ (being convex in its left parameter we can apply Lemma 25). We extend hereafter the definition of Bregman divergences to non-differentiable functions, and let $D_{f,\xi}$ denote the Bregman divergence with generator the (convex) $f$ in which we replace the gradient by $\xi \in \partial f$. We
We can split the limits in eq. (68) because each term in the expectation of eq. (69)
where we have used ineq. (71) in the last inequality. Since we get the upperbound on \( \xi \)
which is indeed finite. Figure 10 provides an illustration of this bound. It then comes by remarking that \( R(t^*) \) is non-decreasing for \( x \geq t^* \)
and
\[
t^* \leq \rho \leq M \quad \Rightarrow \quad D_{f,\xi}(\rho \| t^*) \leq D_{f,\xi}(M \| t^*) \leq f(M) - f(t^*)
\]
which is indeed finite. Figure 10 provides an illustration of this bound. It then comes

\[
R(t^*) = \mathbb{E}_{X \sim Q} \left[ \lim_{\rho \rightarrow \rho(X)} \left[ 1_{[\rho \geq t^*]} \cdot D_{f,\xi}(\rho \| t^*) \right] \right] = \mathbb{E}_{X \sim Q} \left[ \lim_{\rho \rightarrow \rho(X)} \left[ 1_{[\rho(X) \geq t^*]} \cdot D_{f,\xi} \left( \frac{P(X)}{Q(X)} \right) \| t^* \right] \right]
\leq \mathbb{E}_{X \sim Q} \left[ D_{f,\xi}(M \| t^*) \right]
\leq f(M) - f(t^*)
\]

where we have used ineq. (71) in the last inequality. Since \( f \) is continuous, we get the upperbound on \( I_f(P \| Q) \) by choosing \( t^* < M \) as close as desired to \( M \). We get the lowerbound by remarking that \( R(t^*) \geq 0 \) (a Bregman divergence cannot be negative).
13 Proof of Theorem 10

We have

\[ \mathbb{E}_{\mathcal{X} \sim \tilde{Q}}[-(\log \chi(P(X)) - \log \chi(Q(X)))] \]
\[ = \mathbb{E}_{\mathcal{X} \sim \tilde{Q}}[-(\log \chi(P(X)) - \log \chi(\tilde{Q}(X)))] + \mathbb{E}_{\mathcal{X} \sim \tilde{Q}}[-(\log \chi(\tilde{Q}(X)) - \log \chi(Q(X)))] \]
\[ = \mathbb{E}_{\mathcal{X} \sim \tilde{Q}}[-(\log \chi(P(X)) - \log \chi(\tilde{Q}(X)))] - \mathbb{E}_{\mathcal{X} \sim \tilde{Q}}[-(\log \chi(Q(X)) - \log \chi(\tilde{Q}(X)))] . \]

Consider some fixed \( x \in \mathcal{X} \). We have

\[ \log \chi(P(x)) - \log \chi(\tilde{Q}(x)) = \int_{1}^{P(x)} \frac{1}{\chi(t)} \cdot dt - \int_{1}^{\tilde{Q}(x)} \frac{1}{\chi(t)} \cdot dt \]
\[ = \int_{Q(x)}^{P(x)} \frac{1}{\chi(t)} \cdot dt \]
\[ = \int_{1}^{\tilde{Q}(x)} \frac{\tilde{Q}(x)}{\chi(t\tilde{Q}(x))} \cdot dt \]
\[ = \int_{1}^{\tilde{Q}(x)} \frac{1}{\chi\tilde{Q}(x)(t)} \cdot dt \]
\[ = \log_{\chi\tilde{Q}(x)} \left( \frac{P(x)}{\tilde{Q}(x)} \right) , \] (73)
with
\[ \chi_{\tilde{Q}(x)}(t) := \frac{1}{Q(x)} \cdot \chi(t\tilde{Q}(x)) . \] (74)

To cope with the case where any of the integrals is improper, we derive the limit expression:
\[ (\log\chi(P(x)) - \log\chi(\tilde{Q}(x))) = \lim_{(p,q) \to (P(x),\tilde{Q}(x))} \log_{\chi_{q}} \left( \frac{p}{q} \right) , \] (75)
so we get in all cases,
\[ E_{x \sim \tilde{Q}}[-(\log\chi(P(X)) - \log\chi(\tilde{Q}(X)))] = KL_{\chi_{\tilde{Q}}} (\tilde{Q}||P) . \] (76)
We also note that
\[ \log\chi(Q(X)) - \log\chi(\tilde{Q}(X)) = \lim_{(q,q') \to (Q(x),\tilde{Q}(x))} \log_{\chi_{q'}} \left( \frac{q}{q'} \right) \]
\[ = \log_{\chi_{Q(x)}} \left( \frac{Q(x)}{\tilde{Q}(x)} \right) \] (77)
(if the limit exists) so we get
\[ E_{x \sim \tilde{Q}}[-(\log\chi(P(X)) - \log\chi(\tilde{Q}(X)))] = KL_{\chi_{\tilde{Q}}} (\tilde{Q}||P) , \] (78)
\[ E_{x \sim \tilde{Q}}[-(\log\chi(Q(X)) - \log\chi(\tilde{Q}(X)))] = KL_{\chi_{\tilde{Q}}} (\tilde{Q}||Q) , \] (79)
and
\[ E_{x \sim \tilde{Q}}[\log\chi(Q(X)) - \log\chi(P(X))] = KL_{\chi_{\tilde{Q}}} (\tilde{Q}||P) - KL_{\chi_{\tilde{Q}}} (\tilde{Q}||Q) , \] (80)
as claimed.

14 Proof of Theorem 11

Let us denote \( \mathcal{F}_{\tilde{Q}} \subseteq \mathbb{R}^X \) denote the subset of functions : \( X \to \mathbb{R} \) whose values are constrained as follows:
\[ \mathcal{F}_{\tilde{Q}} \doteq \left\{ T \in \mathbb{R}^X : T(x) \in \text{dom} \left( -\log_{\chi_{Q(x)}} \right)^* \right\} . \] (81)
Since \( -\log_{\chi_{Q(x)}} \) is convex for any \( x \), it follows from Legendre duality,
\[ KL_{\chi_{\tilde{Q}}} (\tilde{Q}||P) = E_{x \sim \tilde{Q}} \left[ -\log_{\chi_{Q(x)}} \left( \frac{P(X)}{Q(X)} \right) \right] \]
\[ = E_{x \sim \tilde{Q}} \left[ \sup_{T(X) \in \text{dom}\left( \log_{\chi_{Q(x)}} \right)^*} \left\{ T(X) \cdot \frac{P(X)}{Q(X)} - (-\log_{\chi_{Q(x)}})^*(T(X)) \right\} \right] \]
\[ = \sup_{T \in \mathcal{F}_{\tilde{Q}}} \left\{ E_{x \sim \tilde{Q}} \left[ T(X) \cdot \frac{P(X)}{Q(X)} - (-\log_{\chi_{Q(x)}})^*(T(X)) \right] \right\} \]
\[ = \sup_{T \in \mathcal{F}_{\tilde{Q}}} \left\{ E_{x \sim P} [T(X)] - E_{x \sim \tilde{Q}} [(-\log_{\chi_{Q(x)}})^*(T(X))] \right\} . \] (82)
Now, we know that \(- \log_{\chi_Q(x)}(z)\) is proper lower-semicontinuous and therefore \((- \log_{\chi_Q(x)})^\bullet = - \log_{\chi_Q(x)}\). Being closed, the domain of the derivative of \((- \log_{\chi_Q(x)})^\star\) is the image of the derivative of \(- \log_{\chi_Q(x)}\), given by \(- \tilde{Q}(x)/\chi(\tilde{Q}(x)t)\). If \(\chi : \mathbb{R}_+ \to \mathbb{R}_+\), then \(- \tilde{Q}(x)/\chi(\tilde{Q}(x)t) \in \mathbb{R}_+\) and so \(\mathcal{F}_\tilde{Q} = \{ T \in \mathbb{R}_+^\star \}\).

A pointwise differentiation of eq. (82) yields that at the optimum, we have

\[
P(x) - \tilde{Q}(x) \cdot (- \log_{\chi_Q(x)})^\star(T(x)) = P(x) - \tilde{Q}(x) \cdot (- \log_{\chi_Q(x)})^\star(T(x)) = 0,
\]

that is, exploiting the fact that \((- \log_{\chi_Q(x)})^\prime = - \tilde{Q}(x)/\chi(\tilde{Q}(x)t),\)

\[
T^\star(x) = (\log_{\chi_Q})' \left( \frac{P(x)}{\tilde{Q}(x)} \right)
\]

\[
= - \frac{\tilde{Q}(x)}{\chi(\tilde{Q}(x))} \frac{P(x)}{\tilde{Q}(x)}
\]

\[
= - \frac{\tilde{Q}(x)}{\chi(P(x))}
\]

\[
= - \frac{1}{Z} \chi(Q(x)) \frac{P(x)}{\chi(\tilde{Q}(x))}.
\]

15 Proof of Theorem 13

We now elicitate \((- \log_{\chi_q})^\star\) for \(q \in \mathbb{R}_+,\) under the conditions of Theorem 11. By definition,

\[
(- \log_{\chi_q})^\star(z) = \sup_{z' \in \mathbb{R}_+} \left\{ zz' - (- \log_{\chi_q}(z')) \right\}
\]

\[
= \sup_{z' \in \mathbb{R}_+} \left\{ zz' + \int_1^{z'} \frac{q}{\chi(qt)} \, dt \right\}
\]

\[
= \sup_{z' \in \mathbb{R}_+} \left\{ z + \int_1^{z'} \left( t + \frac{q}{\chi(qt)} \right) \, dt \right\}
\]

\[
= z + \sup_{z' \in \mathbb{R}_+} \left\{ \int_1^{z'} \left( z + \frac{q}{\chi(qt)} \right) \, dt \right\}
\]

Because \(\text{dom} \log_{\chi_q} \subseteq \mathbb{R}_+\) and \(q/\chi(qt) \geq 0\), the sup is unbounded if \(z > 0\). If \(z = 0\), it is bounded iff

\[
\sup_z \log_{\chi_q}(z) < \infty.
\]

Otherwise, when \(z < 0\), it reaches its maximum when \(z'\) belongs to the integrand’s zeroes, \(\{ t : z + q/\chi(qt) = 0 \}\), or equivalently, when \(z'\) satisfies:

\[
\chi(qz') = \frac{-q}{z}.
\]
Figure 11: Explanation of eq. (90).

Let us denote
\[ h(t) = \frac{q}{\chi(qt)} \]  
for short \((q \geq 0)\), noting that \(h\) is non increasing. The set of reals for which eq. (88) holds is \(Z = h^{-1}(-z) = \{z' : q/\chi(qz') = -z\}\), which may not be a singleton if \(\chi\) is not invertible. For any \(z^* \in Z\), letting \(h(t) = q/\chi(qt)\) for short, we get:

\[
(- \log \chi_q)^*(z) = zz^* + \int_{1}^{zz^*} \frac{q}{\chi(qt)} dt
\]

\[
= zz^* + \int_{-z}^{h(1)} h^{-1}(t) dt - 1 \cdot (h(1) - (-z)) + (-z) \cdot (z^* - 1)
\]

\[
= -h(1) + \int_{-z}^{h(1)} h^{-1}(t) dt
\]

\[
= -h(1) - \int_{h(1)}^{-z} h^{-1}(t) dt
\]

\[
= -h(1) + \int_{1}^{h(1)} h^{-1}(t) dt + \int_{1}^{-z} -h^{-1}(t) dt .
\]

The derivation in eq. (90) is explained in Figure 11. We remark that \(k\) depends only on \(q\), so it is not affected by the choice of \(T\). Concerning \(B(z)\), we have

\[
B(z) = \int_{1}^{-z} \left( \frac{q}{\chi(qt)} \right)^{-1} dt
\]

\[
= \int_{1}^{-z} \left( \frac{1}{\chi(qt)} \right)^{-1} dt
\]

\[
= - \int_{1}^{-z} (\chi_q)^{-1} \left( \frac{1}{t} \right) dt ,
\]

(92)
and finally, letting
\[
\chi^\bullet(t) = \frac{1}{\chi^{-1}\left(\frac{1}{t}\right)}, \tag{93}
\]
we remark that
\[
(\chi^\bullet)_q(t) = \frac{1}{q}\chi^\bullet(qt)
\]
\[
= \frac{1}{q\chi^{-1}\left(\frac{1}{qt}\right)}
\]
\[
= \frac{1}{\left(\chi^{-1}\right)_q^{-1}\left(\frac{1}{t}\right)}
\]
\[
= \frac{1}{\left(\chi^{-1}\right)_q^{-1}\left(\frac{1}{t}\right)}, \tag{94}
\]
so
\[
(\chi^\bullet)_q(t) = \frac{1}{(\chi_q)^{-1}\left(\frac{1}{t}\right)}, \tag{95}
\]
and finally
\[
B(z) = -\log(\chi^\bullet)^\frac{1}{q}(-z). \tag{96}
\]
We can check that whenever \(\chi^\bullet\) is differentiable,
\[
(\chi^\bullet)'(t) = \frac{1}{t^2 \cdot \chi'\left(\chi^{-1}\left(\frac{1}{t}\right)\right) \cdot \left(\chi^{-1}\left(\frac{1}{t}\right)\right)^2} \geq 0, \tag{97}
\]
so that \(\chi^\bullet\) is non decreasing and since it is positive, it defines a \(\chi^\bullet\)-logarithm. We end up with
\[
\sup_{T \in \mathbb{R}^+} \left\{ \mathbb{E}_{X \sim P}[T(X)] - \mathbb{E}_{X \sim \tilde{Q}}[(-\log_{\chi_{Q(x)}})\chi^\bullet(T(X))] \right\}
\]
\[
= \sup_{T \in \mathbb{R}^+} \left\{ \mathbb{E}_{X \sim P}[T(X)] - \mathbb{E}_{X \sim \tilde{Q}}[k(\tilde{Q}(X)) - \log_{\chi^\bullet_{Q(x)}}(-T(X))] \right\}
\]
\[
= \sup_{T \in \mathbb{R}^+} \left\{ \mathbb{E}_{X \sim P}[T(X)] - \mathbb{E}_{X \sim \tilde{Q}}[-\log_{\chi^\bullet_{Q(x)}}(-T(X))] \right\} - \mathbb{E}_{X \sim \tilde{Q}}[k(\tilde{Q}(X))]
\]
\[
= \sup_{T \in \mathbb{R}^+} \left\{ \mathbb{E}_{X \sim P}[T(X)] - \mathbb{E}_{X \sim \tilde{Q}}[-\log_{\chi^\bullet_{Q(x)}}(-T(X))] \right\} - K(\tilde{Q}), \tag{98}
\]
as claimed. We finally remark that it is clear from Figure 11 that \(\chi\) being used to compute integrals, it does not need to be strictly monotonic for this to be possible: we just have to break the continuity in \(\chi^{-1}(y)\) whenever the set \(\mathbb{I}\) defined by \(\chi(\mathbb{I}) = y\) is of non-zero Lebesgue measure taking care that \(\chi^{-1}(y)\) be still defined in \(y\). This does not change the integral values.
16 Proof of Theorem 14

The proof of the Theorem mainly follows from identifying the parameters of eq. (31) with the variational part of eq. (26). Recall from eq. (95) that
\[
(\chi^\ast_q)(t) = \frac{1}{(\chi_q)^{-1}(\frac{1}{t})}, \quad (99)
\]
so, exploiting eq. (27) (Theorem 13) and the fact that \( K(Q) \) does not depend on \( T \), we get:
\[
\ell^\prime_x(-1, z) = \frac{d}{dz}(-\log_{\chi\hat{Q}(x)})(-z)
= \frac{d}{dz} - \log_{(\chi^\ast_{\hat{Q}(x)})^{-1}}(-z)
= (\chi\hat{Q}(x))^{-1} \left( -\frac{1}{z} \right). \quad (100)
\]
Since \( \ell^\prime(+1, z) = -1 \), we deduce that the loss is proper composite with inverse link function [53, Corollary 12] given by:
\[
\Psi^{-1}_x(z) = \frac{\ell^\prime(-1, z)}{\ell^\prime(-1, z) - \ell^\prime(+1, z)}
= \frac{(\chi\hat{Q}(x))^{-1} \left( -\frac{1}{z} \right)}{\chi\hat{Q}(x)^{-1} \left( -\frac{1}{z} \right) + 1}, \quad (101)
\]
so that the link is
\[
\Psi_x(z) = -\frac{1}{\chi\hat{Q}(x) \left( \frac{z}{1-z} \right)}. \quad (102)
\]
Remark. We easily retrieve the optimal discriminator (Theorem 11) but this time from the proper composite loss, since (the first line is a general property of \( \Psi_x \), see Section 4):
\[
T^\ast(x) = \Psi_x \left( \frac{P(x)}{P(x) + \hat{Q}(x)} \right)
= -\frac{1}{\chi\hat{Q}(x) \left( \frac{P(x)}{P(x) + \hat{Q}(x)} \right)}.
= -\frac{1}{\chi\hat{Q}(x) \left( \frac{P(x)}{\hat{Q}(x)} \right)}
= -\frac{1}{Z} \cdot \frac{\chi(Q(x))}{\chi(P(x))}.
\]
The last identity follows from eqs. (84) — (85).
17 Proof of Theorem 15

In the context of the proof, we simplify notations and replace signature $\chi_{\text{net}}$ by $\chi$ and output activation $v_{out}$ by $v_2$. Let us call $z \in \mathbb{R}^d$ the output of $g$. We revert the transformation and check:

$$\phi_{l-1}(z) = W_l^{-1}(v^{-1}(\phi_l(z)) - b_l), \forall l \in \{1, 2, ..., L\}, \quad (103)$$
$$\phi_L(z) = \Gamma^{-1}(v_2^{-1}(z) - \beta), \quad (104)$$

For the sake of readability, we shall sometimes remove the dependence in $z$. Letting $a_i$ denote coordinate $i$ in vector $a$, $(A)_{ij}$ the coordinate in row $i$ and column $j$ of matrix $A$, for any $i, j \in [d]$, and $a_{l,i}$ coordinate $i$ in vector $a_l$, we have

$$\frac{\partial \phi_{l-1,i}}{\partial \phi_{l,j}} = (W_l^{-1})_{ij} \cdot \frac{1}{v_l'(v^{-1}(\phi_{l}))}, \quad (105)$$

and furthermore

$$\frac{\partial \phi_{L,i}}{\partial z_j} = (\Gamma^{-1})_{ij} \cdot \frac{1}{v_2'(v_2^{-1}(z))}. \quad (106)$$

Let us denote vector $\tilde{a}$ as the vector whose coordinates are the inverses of those of $a$, namely $\tilde{a}_i = 1/a_i$. From eqs. $(105)$ and $(106)$, the layerwise Jacobians are:

$$\frac{\partial \phi_{l-1}}{\partial \phi_{l}} = W_l^{-1} \odot v'(v^{-1}(\phi_l))1^T, \forall l \in \{1, 2, ..., L\}, \quad (107)$$
$$\frac{\partial \phi_{L}}{\partial z} = \Gamma^{-1} \odot v_2'(v_2^{-1}(z))1^T, \quad (108)$$

where $\odot$ is Hadamard (coordinate-wise) product. These Jacobians have a very convenient form, since:

$$\det \left( \frac{\partial \phi_{l-1}}{\partial \phi_{l}} \right) = \sum_{\sigma \in S_d} \text{sign}(\sigma) \cdot \prod_{i=1}^{d} \left( W_l^{-1} \odot v'(v^{-1}(\phi_l))1^T \right)_{i,\sigma_i}$$
$$= \sum_{\sigma \in S_d} \text{sign}(\sigma) \cdot \prod_{i=1}^{d} (W_l^{-1})_{i,i,\sigma_i} \left( v'(v^{-1}(\phi_l))1^T \right)_{i,\sigma_i}$$
$$= \sum_{\sigma \in S_d} \left( \prod_{i=1}^{d} v'(v^{-1}(\phi_l)) \right) \cdot \text{sign}(\sigma) \cdot \prod_{i=1}^{d} (W_l^{-1})_{i,i,\sigma_i}$$
$$= \left( \prod_{i=1}^{d} v'(v^{-1}(\phi_l)) \right) \cdot \sum_{\sigma \in S_d} \text{sign}(\sigma) \cdot \prod_{i=1}^{d} (W_l^{-1})_{i,i,\sigma_i}$$
$$= \left( \prod_{i=1}^{d} v'(v^{-1}(\phi_l)) \right) \cdot \det (W_l^{-1})$$
$$= \left( \prod_{i=1}^{d} v'(v^{-1}(\phi_l)) \right) \cdot (\det (W_l))^{-1}, \forall l \in \{1, 2, ..., L\},$$
and, using the same derivations,

\[ \det \left( \frac{\partial \phi_L}{\partial z^T} \right) = \left( \prod_{i=1}^{d} v_{2i}(v_2^{-1}(z)) \right) \cdot (\det(G))^{-1} . \]  

(109)

The change of variable formula \[21\] yields:

\[
Q_g(z) = Q_{in}(g^{-1}(z)) \cdot \left| \det \left( \frac{\partial g^{-1}}{\partial z^T} \right) \right| \\
= Q_{in}(g^{-1}(z)) \cdot \left| \det \left( \frac{\partial \phi_0}{\partial z^T} \right) \right| \\
= Q_{in}(g^{-1}(z)) \cdot \left| \det \left( \prod_{l=1}^{L} \frac{\partial \phi_{l-1}}{\partial \phi_l} \cdot \frac{\partial \phi_L}{\partial z^T} \right) \right| \\
= Q_{in}(g^{-1}(z)) \cdot \prod_{l=1}^{L} \det \left( \frac{\partial \phi_{l-1}}{\partial \phi_l} \right) \cdot \left| \det \left( \frac{\partial \phi_L}{\partial z^T} \right) \right| \\
= Q_{in}(g^{-1}(z)) \cdot \prod_{l=1}^{L} \prod_{i=1}^{d} |v'(v^{-1}(\phi_{l,i}))| \cdot \prod_{i=1}^{d} |v_2'(v_2^{-1}(z))| \cdot \left| \det \left( \prod_{l=1}^{L} W_l \right) \right|^{-1} \\
= Q_{in}(g^{-1}(z)) \cdot \prod_{l=1}^{L} \prod_{i=1}^{d} |v'(v^{-1}(\phi_{l,i}))| \cdot \prod_{i=1}^{d} |v_2'(v_2^{-1}(z))| \cdot \left| \det \left( \prod_{l=1}^{L} W_l \right) \right|^{-1} \\
= Q_{in}(g^{-1}(z)) \cdot \prod_{l=1}^{L} \prod_{i=1}^{d} |v'(v^{-1}(\phi_{l,i}))| \cdot \prod_{i=1}^{d} |v_2'(v_2^{-1}(z))| \cdot |\det(N)| . \\
\]

because \(v\) and \(v_2\) are coordinatewise. We have let

\[ N = \Gamma \cdot \prod_{l=1}^{L} W_l \]  

(110)

and also \(\phi_{l,i} = v(w_{l,i}^\top \phi_{l-1} + b_{l,i}),\) where \(w_{l,i} = W_l^\top 1_i\) is the (column) vector built from row \(i\) in \(W_l\) and similarly \(z_i \equiv v_2(\gamma_i^\top \phi_L + \beta_i)\) with \(\gamma_i = \Gamma^\top 1_i\). Notice that we can also write

\[ \prod_{l=1}^{L} \prod_{i=1}^{d} |v'(v^{-1}(\phi_{l,i}))| = \prod_{l=1}^{L} \prod_{i=1}^{d} |v'(w_{l,i}^\top \phi_{l-1} + b_{l,i})| . \]  

(111)

So, letting \(\tilde{Q}_{\text{deep}}^* = \prod_{l=1}^{L} \prod_{i=1}^{d} |v'(w_{l,i}^\top \phi_{l-1} + b_{l,i})|,\) \(H_{\text{out}} = \prod_{i=1}^{d} |v_{\text{out}}'(\gamma_i^\top \phi_L(x) + \beta_i)|\) (with \(x = g^{-1}(z)\)), and dropping the determinant which does not depend on \(z\), we get:

\[ Q_g(z) \propto \frac{Q_{in}(g^{-1}(z))}{\tilde{Q}_{\text{deep}}^*} \cdot \frac{1}{H_{\text{out}}} . \]  

(112)
To finish up the proof, we are going to identify $\tilde{Q}^*_\text{deep}$ to (a constant times) the product of escorts in eq. (37). To do so, we are first going to design the general activation function $v$ as a function of $\chi$, and choose:

$$v(z) = k + k' \cdot \exp_\chi(z) ,$$

(113)

for $k \in \mathbb{R}, k' > 0$ constants, which can be chosen e.g. to ensure that zero signal implies zero activation ($v(0) = 0$). Our choice for $v$ has the following key properties.

**Lemma 26** $v$ is $C^1$, invertible and we have $v'(z) = k' \cdot \chi(\exp_\chi(z))$.

**Proof** The derivative comes from [6, Eq. 84]. Notice that $\exp_\chi$ is continuous as an integral, $\chi$ is continuous by assumption and so $v'$ is continuous, implying $v$ is $C^1$. We prove the invertibility. Because of the expression of $v'$, $v$ is increasing, and in fact strictly increasing with the sole exception when $\exp_\chi(z) \in \chi^{-1}(0)$. However, note that $\chi^{-1}(0) \not\subset \text{dom}(\log_\chi)$ because of the definition of $\log_\chi$. Since $\exp_\chi$ is the inverse of $\log_\chi$ [41, Section 10.1], it follows that $\chi^{-1}(0) \not\subset \text{im}(\exp_\chi)$ and so $\exp_\chi(z) \not\in \chi^{-1}(0), \forall z \in \text{dom}(\exp_\chi)$, which implies $v$ invertible.

What the Lemma shows is that we can plug $v$ as in eq. (113) directly in $\tilde{Q}^*_\text{deep}$. To do so, let us now define strictly positive constants $Z_{li}$ that shall be fixed later. We directly get from eq. (111)

$$\tilde{Q}^*_\text{deep} = \left( \prod_{l=1}^{L} \prod_{i=1}^{d} Z_{li} \right) \cdot \left( \prod_{l=1}^{L} \prod_{i=1}^{d} \frac{1}{Z_{li}} \cdot |v'(w_{l,i}^\top \phi_{l-1} + b_{l,i})| \right)$$

(114)

(we can remove the absolute values since $\chi$ is non-negative). We now ensure that $\tilde{Q}_{\text{deep}}$ is indeed a product of escorts: to do so, we just need to ensure that (i) $b_{l,i}$ normalizes the deformed exponential family, i.e. defines (negative) its cumulant (Definition 3), and (ii) $Z_{li}$ normalizes its escort as in eq. (12). To be more explicit, we pick $b_{l,i}$ the solution of

$$\int_{\phi} \exp_\chi(w_{l,i}^\top \phi - b_{l,i}) d\nu_{l-1}(\phi) = 1 ,$$

(115)

where $d\nu_{l-1}(\phi) \doteq \int_{\phi_{l-1}(x)=\phi} d\mu(x)$ is the pushforward measure, and

$$Z_{li} = \int_{x} \chi(P_{\chi,b_{l,i}}(x|w_{l,i}, \phi_{l-1})) d\mu(x) .$$

(116)

We get

$$\tilde{Q}_{\text{deep}} = \prod_{l=1}^{L} \prod_{i=1}^{d} \tilde{P}_{\chi,b_{l,i}}(x|w_{l,i}, \phi_{l-1}) ,$$

(117)
and finally,

\[ Q_g(z) = \frac{Q_{in}(x)}{Q_{deep}(x)} \cdot \frac{1}{H_{out}(x) \cdot |\det(N)|} \]

\[ = \frac{Q_{in}(x)}{Q_{deep}(x)} \cdot \frac{1}{H_{out}(x) \cdot Z_{net}}, \tag{118} \]

with

\[ Z_{net} = \left( k^{Ld} \cdot \prod_{l=1}^{L} \prod_{i=1}^{d} Z_{li} \right) \cdot |\det(N)| \tag{119} \]

a constant. We get the statement of Theorem 15.

**Remark.** (unnormalized densities) since in practice all \( b \)s are learned, we in fact work with deformed exponential families with unspecified normalization. We may also consider that the normalization of escorts is unspecified and therefore drop all \( Z_{li} \)s, which simplifies \( Z_{net} \) to \( Z_{net} = |\det(N)| \).

**Remark.** (completely factoring \( Q_g \) as an escort) Denote for short \( z_p = \phi_L(x) \) the penultimate layer of \( g \), and \( g_p \) the net obtain from eliminating the last layer of \( g \), which allows us to drop \( H_{out}(\cdot) \) from \( Q_{gp}(z) \) and we have \( Q_g(z) \propto Q_r = Q_{in}(g^{-1}_p(z_p)) / Q_{deep}(g^{-1}_p(z_p)) \). One can factor \( Q_r \) as a proper likelihood over escorts of \( \chi_{net} \)-exponential families: for this, replace all \( Ld \) inner nodes of \( g_p \) in Figure 1 by random variables, say \( \Phi_{l,i} \) (for \( l \in \{0, 1, ..., L - 1\}, i \in \{1, 2, ..., d\} \)), treat the deep net \( g_p \) as a directed graphical model whose connections are the dashed arcs. Now, if we let, say, \( Q_{in}(g^{-1}_p(z_p)) = Q_b(\cap_{l>0,i} \Phi_{l,i}) \) and \( Q_{deep}(g^{-1}_p(z_p)) = Q_b(\cap_{l>0,i} \Phi_{l,i}) \), and if we use as \( Q_{in} \) an uninformed escort (i.e. with constant coordinate, say for example \( \theta = 1 \), Definition 3), then assuming correct factorization one may obtain \( Q_r = Q_c(g^{-1}_p(z_p)) \mid \cap_{l>0,i} \Phi_{l,i} \) for some escort \( Q_c \) that we can plug directly in eq. 26. To properly understand the relationships between \( \chi, Q_a, Q_b \) and how the escorts factor in \( Q_c \) requires a push of the state of the art: conjugacy in deformed exponential families is less understood than for exponential families; it is also unknown how product of deformed exponential families factor within the same deformed exponential families [4]; some factorizations are known but only on subsets of deformed exponential families and rely on particular notions of independence [38];

**Remark.** (twist introduced by the last layer) We return to the twist introduced by the last layer of \( g \):

\[ H_{out}(x) = \prod_{i=1}^{d} \left| v_2^T (\gamma_i^T \phi_L(x) + \beta_L) \right|. \tag{120} \]

It is clear that when \( v_2 \) is the identity, \( H_{out}(x) \) is constant; so deep architectures, as experimentally carried out e.g. in Wasserstein GANs [7] or analyzed theoretically e.g. in [34] exactly fit to the escort factoring — notice that one can choose as input density one from some particular deformed exponential family, as e.g. done experimentally for [47, Section 2.5] (standard Gaussian), so that in this case \( Q_g(z) \) factors completely as escorts.
Suppose now that \( v_2 \) is not the identity but chosen so that, for some couple \((\chi, g)\) where \( \chi \) is differentiable and \( g : \mathbb{R}_+ \to \mathbb{R} \) is invertible,

\[
(v'_2 \circ g)(z) = \frac{d}{dz} (\log \chi \circ \chi)(z) = \frac{\chi'(z)}{\chi(z)} ,
\]

which is equivalent, after a variable change, to having \( v_2 \) satisfy

\[
v'_2(t) = \frac{\chi' \circ g^{-1}}{\chi \circ g^{-1}}(t) .
\]

In addition, suppose that \( g \) is chosen so that \( \sum_i g^{-1}(\gamma_i^T \phi_L(x) + \beta_i) = 1 \). Call \( D = \{p_1, p_2, ..., p_d\} \) this discrete distribution, removing reference to \( x \). We then have:

\[
H_{out}(x) = \prod_{i=1}^d \frac{\chi'(g^{-1}(\gamma_i^T \phi_L(x) + \beta_i))}{\chi(g^{-1}(\gamma_i^T \phi_L(x) + \beta_i))}
\]

\[
= \prod_{i=1}^d \frac{\chi'(p_i)}{\chi(p_i)}
\]

\[
= \prod_{i=1}^d \chi(p_i) \cdot \frac{\chi'(p_i)}{\chi(p_i)}
\]

\[
= \prod_{i=1}^d ((\exp \chi)' \circ \log \chi)(p_i) \cdot (\log \chi)''(p_i)
\]

\[
\propto |\det(H)| .
\]

Here, \( H \) is the \( \chi \)-Fisher information metric of \( D \) [6, Theorem 12, eqs 119, 120]. In other words, \( H_{out}(x) \) can be absorbed in the volume element in eq. (36).

As an example, pick a prop-\( \tau \) activation (Table 1), for which \( \log \chi = (\tau^*)^{-1}(\tau^*(0)z) \) and

\[
\chi(t) = \frac{(\tau^*)' \circ (\tau^*)^{-1}(\tau^*(0)z)}{\tau^*(0)} .
\]

Now, pick \( g(z) = \log \chi(K \cdot z) \), where \( K = \sum_i \exp \chi(\gamma_i^T \phi_L(x) + \beta_i) \) guarantees:

\[
\sum_i g^{-1}(\gamma_i^T \phi_L(x) + \beta_i) = \frac{1}{K} \cdot \sum_i \exp \chi(\gamma_i^T \phi_L(x) + \beta_i) = 1 .
\]

Condition in eq. (121) becomes

\[
(v'_2 \circ (\tau^*)^{-1})(\tau^*(0)Kz) = \frac{\chi' \circ g^{-1}}{\chi \circ g^{-1}}(t)
\]

\[
= \tau^*(0) \cdot \frac{(\tau^*)'' \circ (\tau^*)^{-1}(\tau^*(0)Kz)}{(\tau^*)'(\tau^*)^{-1}(\tau^*(0)Kz))^2} ,
\]

and we obtain after a variable change,

\[
v_2 = \tau^*(0) \cdot \int_{t} \frac{(\tau^*)''(t)}{(\tau^*)'(t)^2} dt ,
\]
which does not depend on $K$ and, if $\tau^*$ is strictly convex, is strictly increasing. Notice that we can carry out the integration, $v_2(z) = K' - (\tau^*(0)/(\tau^*)'(z))$ for some constant $K'$. To make a parallel with a popular activation for the last layer, consider the sigmoid, $v_2 = v_s(z) = 1/(1 + \exp(-z))$, for which

$$v'_s(z) = \frac{\exp(z)}{(1 + \exp(z))^2}. \quad (128)$$

Fitting it to eq. (127),

$$\frac{\exp(z)}{(1 + \exp(z))^2} = \tau_s^*(0) \cdot \frac{(\tau_s^*)''(t)}{((\tau_s^*)')^2(t)} \quad (129)$$

reveals that we can pick $\tau_s^*(z) = z + \exp(z)$ (we control that $\tau_s^*(0) = 1$). Such a $\tau^*$ analytically fits to the prop-$\tau$ definition and in fact corresponds to a $\chi$-exponential family, but it does not correspond to an entropy $\tau$. This would be also true for affine scalings (argument and function) of the sigmoid of the type $v_2 = a + bv_s(c + dz)$.

18 Proof of Lemma 17

Define function

$$h(z) = \frac{v(z) - \inf v(z)}{v(0) - \inf v(z)}, \quad (130)$$

and let $g(z) = h^{-1}(z)$. Since $\text{dom}(v) \cap \mathbb{R}_+ \neq \emptyset$, $v(0) - \inf v(z) > 0$, so $h(z)$ bears the same properties as $v$. We first show that $g$ is a valid $\chi$-logarithm. Since $v$ is convex increasing, $g(z)$ is concave increasing and $-g$ is convex decreasing. Therefore, since $g$ is $C^1$ as well, letting $\xi = g'$, we get:

$$g(z) = \int_1^z \frac{1}{\xi(t)} \, dt. \quad (131)$$

We also check that $g(1) = 0$ since $h(0) = 1$. If we let $\chi = 1/\xi$, then because $\xi(z) \geq 0$, $\chi(z) \geq 0$ and also because $\xi$ is decreasing, $\chi$ is increasing. Finally, $\chi : \mathbb{R}_+ \to \mathbb{R}_+$. Summarizing, we have shown that $\chi$ defines a valid signature and $g(z) = \log \chi(z)$. Therefore, $h(z) = \exp \chi(z)$ and it comes that

$$v(z) = k + k' \cdot \exp \chi(z), \quad (132)$$

for $k = \inf v(z) \in \mathbb{R}$ and $k' = v(0) - \inf v(z) > 0$, so $v$ matches the analytic expression in eq. (113), which allows to complete the proof of the Lemma.

19 Proof of Lemma 19

We use a scaled perspective transform of the Softplus function and let:

$$v_\mu(z) = (1 - \mu) \cdot \log \left(1 + \exp \left(\frac{z}{1 - \mu}\right)\right), \quad (133)$$

with $\mu \in [0, 1]$. It is clear that $v_\mu$ is strongly admissible for any $\mu \in [0, 1]$.
Lemma 27 For any \( z \geq 0, \mu \in [0, 1], \)
\[
(1 - \mu) \cdot \log \left( \frac{1 + \exp \left( \frac{z}{1 - \mu} \right)}{1 + \exp(z)} \right) \leq \mu z . \tag{134}
\]

Proof Equivalently, we want
\[
1 + \exp \left( \frac{z}{1 - \mu} \right) \leq \exp \left( \frac{\mu z}{1 - \mu} \right), \tag{135}
\]
or, equivalently,
\[
1 + \exp \left( \frac{z}{1 - \mu} \right) \leq \exp \left( \frac{\mu z}{1 - \mu} \right) + \exp(z) \cdot \exp \left( \frac{\mu z}{1 - \mu} \right)
= \exp \left( \frac{\mu z}{1 - \mu} \right) + \exp \left( \frac{z}{1 - \mu} \right), \tag{136}
\]
which, after simplification, is equivalent to \( \mu z / (1 - \mu) \geq 0, \) which indeed holds when \( z \geq 0, \mu \in [0, 1] \).

We now have \( v_\mu(z) \geq \max\{0, z\}, \forall \mu \in [0, 1], \) and we can also check that Lemma 27 implies
\[
(1 - \mu) \cdot \log \left( 1 + \exp \left( \frac{z}{1 - \mu} \right) \right) - z
\leq (1 - \mu) \cdot (\log (1 + \exp(z)) - z ), \forall z \geq 0, \mu \in [0, 1] . \tag{137}
\]

Let us denote, for any \( z \geq 0, \)
\[
I_\mu(z) \doteq \int_0^z |v_\mu(t) - \max\{0, t\}| \, dt
= \int_0^z |v_\mu(t) - t| \, dt
= \int_0^z (v_\mu(t) - t) \, dt . \tag{138}
\]
Since \( \max\{0, -t\} = \max\{0, t\} - t \) and \( v_\mu(-t) = v_\mu(t) - t, \) we have \( \|v_\mu - \text{ReLU}\|_{L_1} = 2 \lim_{z \to +\infty} I_\mu(z). \) It also comes from ineq. (137) that
\[
I_\mu(z) \leq (1 - \mu) I_0(z) , \forall z \leq 0 , \tag{139}
\]
furthermore, it can be shown by numerical integration that \( \lim_{z \to +\infty} I_0(z) = \pi^2 / 6, \) so we get
\[
\|v_\mu - \text{ReLU}\|_{L_1} \leq \frac{(1 - \mu) \pi^2}{3} , \forall \mu \in [0, 1] , \tag{140}
\]
and to have the right hand side smaller than \( \epsilon > 0, \) it suffices to take
\[
\mu > 1 - \frac{3 \epsilon}{\pi^2} , \tag{141}
\]
which yields the statement of the Lemma.
20 Proof of Theorem 20

We split the proof of the Theorem in several Lemmata.

Lemma 28 Suppose $f$ satisfies Corollary 8, and let

\[ \chi(t) = \frac{1}{-\xi(t) + k}, \quad (142) \]

where $k \geq \sup_{\mathbb{R}_+} \xi$. Let $\mathbb{R}_+ \supseteq Q \equiv \{Q : f(\tilde{Q}) < f(Q)\}$ and $\mathbb{R}_+ \supseteq Q' \equiv \{Q : \tilde{Q} < Q\}$. Suppose the following property (A) holds: there exists $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that

\[ f(Q(x)) - f(\tilde{Q}(x)) \leq g\left(\frac{Q(x)}{Q(x)}\right), \quad \forall x : Q(x) \in \Omega. \quad (143) \]

Then,

\[ KL_{\chi_\tilde{Q}}(\tilde{Q}||Q) \leq (-k) \cdot \sup Q' + \int_{x : Q(x) \in \Omega} \tilde{Q}(x)g\left(\frac{Q(x)}{Q(x)}\right) d\mu(x). \quad (144) \]

Proof It follows from the definition of $KL_\chi$ that:

\[ KL_{\chi_\tilde{Q}}(\tilde{Q}||Q) = \mathbb{E}_{X \sim \tilde{Q}} \left[ -\log_{\chi_{\tilde{Q}}} \left( \frac{Q(X)}{\tilde{Q}(X)} \right) \right] \]

\[ = \int_{x} \tilde{Q}(x) \int_{Q(x)}^{\tilde{Q}(x)} \frac{1}{\chi(t)} dt d\mu(x) \]

\[ = \int_{x} \tilde{Q}(x) \left[ -f(z) + k\tilde{Q}(x) \right] d\mu(x) \]

\[ = \int_{x} \tilde{Q}(x)(f(Q(x)) - f(\tilde{Q}(x))) - k \cdot (Q(x) - \tilde{Q}(x)) d\mu(x) \]

\[ = A(f(\tilde{Q}) \leq f(Q)) + A(f(\tilde{Q}) > f(Q)) \]

\[ + (-k) \cdot \int_{x} \tilde{Q}(x)(Q(x) - \tilde{Q}(x)) d\mu(x). \quad (146) \]

where, for any predicate $\pi : X \to \{\text{false, true}\}$,

\[ A(\pi) = \int_{x : \pi(x) = \text{true}} \tilde{Q}(x)(f(Q(x)) - f(\tilde{Q}(x)) d\mu(x). \quad (147) \]

Let $\mathbb{R}_+ \supseteq \Omega \equiv \{Q : f(\tilde{Q}) < f(Q)\}$ and $\mathbb{R}_+ \supseteq Q' \equiv \{Q : \tilde{Q} < Q\}$. Remark that $A(f(\tilde{Q}) > f(Q)) \leq 0$ and

\[ f(Q(x)) - f(\tilde{Q}(x)) \leq g\left(\frac{Q(x)}{\tilde{Q}(x)}\right), \quad \forall x : Q(x) \in \Omega, \quad (148) \]
from property (A) and, densities being non-negative,
\[
\int \tilde{Q}(x)(Q(x) - \tilde{Q}(x))d\mu(x) \leq \int_{x:Q(x) \in \Omega'} \tilde{Q}(x)(Q(x) - \tilde{Q}(x))d\mu(x) \\
\leq \int_{x:Q(x) \in \Omega'} Q^2(x)d\mu(x) \\
\leq \left(\sup \Omega'\right) \cdot \int_{x:Q(x) \in \Omega'} Q(x)d\mu(x) \leq \sup \Omega', \tag{149}
\]
where eq. (149) follows from the definition of $\Omega'$. Putting this altogether, we get
\[
KL_{\chi_{\tilde{Q}}} (\tilde{Q} \parallel Q) \leq (-k) \cdot \sup \Omega' + \int_{x:Q(x) \in \Omega} \tilde{Q}(x)g\left(\frac{Q(x)}{\tilde{Q}(x)}\right) d\mu(x), \tag{151}
\]
as claimed. 

We now check that Lemma 28 is optimal in the sense that we recover $KL_{\chi_{\tilde{Q}}} (\tilde{Q} \parallel Q) = 0$ for all exponential families.

**Lemma 29** Suppose $Q$ is an exponential family. Then the bound in eq. (144) is zero.

**Proof** The KL divergence admits the following form, for $h(z) = z \log z$:
\[
I_{KL}(P \parallel Q) = E_{X \sim Q} \left[ h\left(\frac{P(X)}{Q(X)}\right) \right] \\
= E_{X \sim P} \left[ -\log \chi\left(\frac{Q(X)}{P(X)}\right) \right], \tag{152}
\]
with $\chi(z) = z$ (and $f(z) = -\log z$, yielding $k = 0$ in Lemma 28), $Q = \mathbb{R}_+$, $Q' = \{0\}$. We also have
\[
f(u) - f(v) = -\log \frac{u}{v}, \tag{153}
\]
and so we can pick $g(z) = -\log(z)$ for property (A). After remarking that $Z = 1$, and using the convenient choice $k = 0$, we get from ineq. (151),
\[
KL_{\chi_Q} (\tilde{Q} \parallel Q) \leq 0 \cdot 0 + \int_{x:Q(x) \geq 0} \tilde{Q}(x)g\left(\frac{Q(x)}{\tilde{Q}(x)}\right) d\mu(x) \\
\leq 0 + \sup_{Q(x) \geq 0} (-\log) (1) \\
= 0, \tag{154}
\]
as claimed. We now treat all cases of Theorem 20, starting with point (i).

**Lemma 30** For the original GAN choice of $f$, $Z > 1$ and $J(Q) \leq (1/Z)\cdot M (Q(.)) < 1/(Z - 1)$. 

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Proof. In this case, we choose
\[ f(z) = f_{\text{gan}}(z) = z \log z - (1 + z) \log(1 + z) + 2 \log 2. \] (155)

We also remark that for any \( 0 \leq u \leq v, \)
\[ f_{\text{gan}}(u) - f_{\text{gan}}(v) \leq -\log \frac{u}{v}, \] (156)
We can show this by analyzing function \( f_{\text{gan}}(\varepsilon z) - f_{\text{gan}}(z) \) for any fixed \( \varepsilon \in [0, 1] \), which is increasing on \( z \in \mathbb{R}_+ \) and converges to \(-\log(\varepsilon)\). So we can pick \( g(z) = -\log(z) \) for assumption (A) and since \( f_{\text{gan}} \) is strictly decreasing, \( \mathcal{Q} = \{ Q : \tilde{Q} > Q \} \). Using \( k = 0 \) in Lemma 28, we obtain
\[ \chi(z) = -\frac{1}{f'_{\text{gan}}(z)} = \frac{1}{\log \left( 1 + \frac{1}{z} \right)} . \] (157)

We have \( \chi(z) > z, \forall z > 0 \), so \( Z > 1 \). We also obtain
\[ \sup_{\mathcal{Q}} \left( Z \cdot \frac{Q(x)}{\chi(Q(x))} \right) = \sup_{\mathcal{Q}} \log \left( \frac{1}{Z} \cdot \frac{1}{r(z)} \right) , \] (158)
\[ r(z) = z \cdot \log \left( 1 + \frac{1}{z} \right) . \] (159)

Because \( \chi(z) \) is strictly increasing and satisfies \( \chi(z) \in [z, z + 1/2] \) with \( \lim_0 \chi(z) = 0 \), we have \( Z > 1 \) and
\[
\int_{x:Q(x)\in\mathcal{Q}} \tilde{Q}(x) g \left( \frac{Q(x)}{\tilde{Q}(x)} \right) d\mu(x) = \int_{x:Q(x)\in\mathcal{Q}} \tilde{Q}(x) \log \left( \frac{\tilde{Q}(x)}{Q(x)} \right) d\mu(x)
\] (160)
\[
= \int_{x:Q(x)\in\mathcal{Q}} \frac{\chi(Q(x))}{Z} \log \left( \frac{\chi(Q(x))}{ZQ(x)} \right) d\mu(x)
\] (161)
\[
+ \int_{x:Q(x)\in\mathcal{Q}} \frac{\chi(Q(x))}{Z} \log \left( \frac{1}{Z} \right) d\mu(x)
\]
\[
\leq \log \left( \frac{1}{Z} \right) + \frac{1}{Z} \cdot \int_{x:Q(x)\in\mathcal{Q}} s(Q(x)) d\mu(x) \] (160)
\[
\leq \frac{1}{Z} \cdot \int_{x:Q(x)\in\mathcal{Q}} s(Q(x)) d\mu(x) , \] (161)

with
\[ s(z) = \frac{1}{\log \left( 1 + \frac{1}{z} \right)} \cdot \log \left( \frac{1}{z \log \left( 1 + \frac{1}{z} \right)} \right) \in \left[ \frac{1}{2}, 1 \right] . \] (162)

In eq. (160), we have exploited the choice of \( \chi \) in eq. (157). We remark that
\[ \frac{\chi(Q)}{Q} = \frac{1}{h(Q)} , \] (163)

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with
\[ h(z) = z \cdot \log \left( 1 + \frac{1}{z} \right) , \quad (164) \]

which is strictly increasing on \( \mathbb{R}_+ \), satisfies \( \text{im} h = [0, 1) \), and so \( \tilde{Q}/Q \) is a strictly decreasing function of \( Q \) and \( \sup Q \) is strictly smaller than the solution of \( h(z) = 1/Z \) (equivalently, \( Q = \tilde{Q} \)), call it \( q(Z) \). We get, since \( s(z) \leq 1 \),
\[
\int_{x:Q(x)\in Q} s(Q(x))d\mu(x) \leq M(Q < q(Z)) , \quad (165)
\]

where \( M(Q < z) = \int_{x:Q(x)\leq z} d\mu(x) \) is the total measure of the support with "small" density (i.e. upperbounded by \( q(Z) \)). We get
\[
KL_{\chi_{\tilde{Q}}}(\tilde{Q}\|Q) \leq \frac{M(Q < q(Z))}{Z} , \quad (166)
\]

and to make this bound further readable, it can be shown that \( h(z) \geq z/(1 + z) \) and so \( h^{-1}(z) \leq z/(1 - z) \) for \( z \in [0, 1) \). It follows
\[
q(Z) = h^{-1}\left( \frac{1}{Z} \right) \leq \frac{1}{Z - 1} , \quad (167)
\]

and so \( M(Q < q(Z)) \leq M(Q < 1/(Z - 1)) \), and we obtain:
\[
J_{\text{aux}}(Q) = KL_{\chi_{\tilde{Q}}}(\tilde{Q}\|Q) \leq \frac{1}{Z} \cdot M\left( Q < \frac{1}{Z - 1} \right) , \quad (168)
\]

as claimed. \( \blacksquare \)

We now treat point (ii) in Theorem 20.

**Lemma 31** Consider the \( \mu\text{-ReLU} \) choice for which
\[
\chi(z) = \frac{4z^2}{(1 - \mu)^2 + 4z^2} \quad (169)
\]

with \( \mu \in [0, 1) \). Then the associated normalization constant of the escort, \( Z \), satisfies
\[
Z \leq \frac{1}{1 - \mu} , \quad (170)
\]

and penalty \( J(Q) \) satisfies:
\[
J(Q) \leq \frac{1}{Z} \cdot \left( 1 + \frac{3\sqrt{3}}{8Z(1 - \mu)} \right) . \quad (171)
\]
**Proof**  We first remark that
\[
\max_{R_+} \frac{4z}{(1 - \mu)^2 + 4z^2} = \frac{1}{1 - \mu}, \tag{172}
\]
from which we derive
\[
Z = \int_x \chi(Q(x))d\mu(x) = \int_x \frac{4Q^2(x)}{(1 - \mu)^2 + 4Q^2(x)}d\mu(x) = \int_x \frac{4Q(x)}{(1 - \mu)^2 + 4Q^2(x)} \cdot Q(x)d\mu(x) \leq \frac{1}{1 - \mu} \cdot \int_x Q(x)d\mu(x) = \frac{1}{1 - \mu} \tag{173}
\]
Then, we remark that
\[
\log \chi(z) = \int_1^z \frac{dt}{\chi(t)} = z - \frac{(1 - \mu)^2}{4z} - K, \tag{174}
\]
with \(K = 1 - (1 - \mu)^2/4\) and it comes from eq. (145) that
\[
KL\chi_Q(\tilde{Q} \parallel Q) = \int_x \tilde{Q}(x) \int_{Q(x)} \frac{1}{\chi(t)}dtd\mu(x) = \int_x g(Q(x))d\mu(x) \tag{175}
\]
with
\[
g(z) = \frac{1}{Z} \cdot \frac{4z^2}{(1 - \mu)^2 + 4z^2} \cdot \left( \frac{1}{Z} \cdot \frac{4z^2}{(1 - \mu)^2 + 4z^2} \right) = z \cdot \left( \frac{16z^3}{(1 - \mu)^2 + 4z^2} + \frac{(1 - \mu)^2}{4z^2} \right). \tag{176}
\]
We then remark that
\[
\max_{R_+} \frac{16z^3}{((1 - \mu)^2 + 4z^2)^2} = \frac{3\sqrt{3}}{8(1 - \mu)}, \tag{176}
\]
Table 2: Bounds on $\tilde{Q}$ and $-\tilde{Q}$ as a function of $Q$, as used for the proof of Lemma 32, using the fact that $\gamma \geq 1$.

| $\tilde{Q}$ | $[0, Z)$ | $[Z, \gamma Z)$ | $[\gamma Z, \gamma)$ | $[\gamma, +\infty)$ |
|-------------|---------|----------------|----------------|----------------|
| $\tilde{Q} \log_\gamma \tilde{Q}$ | $\frac{\gamma}{Z} \log_\gamma \frac{\gamma}{Z}$ | $\frac{\gamma}{Z} \log_\gamma \frac{\gamma}{Z}$ | $\frac{\gamma}{Z} \left(\log \gamma + \frac{\gamma}{Z} - 1\right)$ | $\frac{\gamma}{Z} \left(\log \gamma + \frac{1}{Z} - 1\right)$ |
| $-\tilde{Q} \log_\gamma \tilde{Q}$ | $-\frac{\gamma}{Z} \log_\gamma \frac{\gamma}{Z}$ | $-\frac{\gamma}{Z} \log_\gamma \frac{\gamma}{Z}$ | $-\frac{\gamma}{Z} \log_\gamma \frac{\gamma}{Z}$ | $-\frac{\gamma}{Z} \left(\log \gamma + \frac{1}{Z} - 1\right)$ |

so that

$$g(z) \leq z \cdot \left(\frac{3\sqrt{3}}{8Z^2(1-\mu)} + \frac{1}{Z} \cdot \frac{(1-\mu)^2}{(1-\mu)^2 + 4z^2}\right) \leq z \cdot \left(\frac{3\sqrt{3}}{8Z^2(1-\mu)} + \frac{1}{Z}\right),$$

and finally

$$J(Q) = KL_{\chi_Q}(\tilde{Q}||Q) \leq \left(\frac{3\sqrt{3}}{8Z^2(1-\mu)} + \frac{1}{Z}\right) \cdot \int_x Q(x) d\mu(x) = \frac{3\sqrt{3}}{8Z^2(1-\mu)} + \frac{1}{Z} ,$$

as claimed.

We complete point (ii) by remarking that $3\sqrt{3}/8 \approx 0.65 < 1$. We now treat point (iii) in Theorem 20. We are going to show a more complete statement.

**Lemma 32** Consider the $(\alpha, \beta)$-ELU choice for which

$$\chi(z) = \begin{cases} \beta & \text{if } z > \alpha \\ z & \text{if } z \leq \alpha \end{cases} .$$

Then the associated normalization constant of the escort, $Z$, satisfies

$$Z \leq \frac{\beta}{\alpha} ,$$

so that for the choice $\beta = \alpha = \gamma$, we have $Z \leq 1$. Furthermore, whenever $\gamma \geq 1$, penalty $J(Q)$ satisfies:

$$J(Q) \leq \frac{\log \gamma}{Z} + \frac{1 - Z}{Z^2} + \frac{1}{Z} \cdot H_*(Q) ,$$

where $H_*(Q) \doteq \mathbb{E}_{X \sim Q} \max\{0, -\log Q(X)\}$. This bound is tight.
Proof We obtain directly \( \chi(z) \leq (\beta/\alpha) \cdot z \), from which

\[
Z = \int_x \chi(Q(x)) \, d\mu(x) \\
\leq \frac{\beta}{\alpha} \cdot \int_x Q(x) \, d\mu(x) = \frac{\beta}{\alpha}.
\]

Then, we remark that if \( \beta = \alpha = \gamma \geq 1 \), \( Z \leq 1 \) and

\[
\log_\chi(z) = \int_1^z \frac{dt}{\chi(t)} = \begin{cases} 
\log \gamma + \frac{z}{\gamma} - 1 & \text{if } z > \gamma \\
\log z & \text{if } z \leq \gamma
\end{cases}, 
\]

and we finally obtain,

\[
KL_{\chi Q}(\tilde{Q} \parallel Q) = \int_x \tilde{Q}(x) \int_{\tilde{Q}(x)} \frac{1}{\chi(t)} \, dt \, d\mu(x) \\
= \int_x \tilde{Q}(x) \log_\chi \tilde{Q}(x) \, d\mu(x) - \int_x \tilde{Q}(x) \log_\chi Q(x) \, d\mu(x) \\
\leq \left( \frac{\log \gamma}{Z} + \frac{1-Z}{Z^2} \right) \cdot \int_x Q(x) \, d\mu(x) + \frac{1}{Z} \cdot H_*(Q) \\
= \frac{\log \gamma}{Z} + \frac{1-Z}{Z^2} + \frac{1}{Z} \cdot H_*(Q),
\]

with \( H_*(Q) \equiv \mathbb{E}_{X \sim Q}[\max\{0, -\log Q(X)\}] \) is a clipping of Shannon’s entropy (which prevents it from being negative). The inequality follows from bounding the two terms in the integral depending on the value of \( Q \) following Table 2.

For tightness, consider the "square" uniform distribution with support an interval \([a, a+1]\) with \( a \leq 0 \), and fix \( \gamma = 1 \), which brings \( J(Q) = 0 \) and \( \log \gamma = 0 \), \( Z = 1 \) and \( H_*(Q) = 0 \), so both bounds in eq. (180) match.

We end up this Section with three additional results related to Theorem 20:

(iv) bounding \( J(Q) \) when \( \chi \) is the signature of \( q \)-exponential families, also displaying that \( J(Q) = O(1/Z) \);

(v) computing exactly \( J(Q) \) for a particular \( \chi \)-family and a member of the \( \chi \)-family for \( Q \), displaying that \( J(Q) = \theta(1/Z) \);

(vi) showing how a particular choice for \( \chi \) that blows up large density regions for some \( Q \) can yield \( Z \) arbitrarily large.

We focus now on (v) and pick \( \chi \) as the signature of popular deformed exponential families, the \( q \)-exponential families.

Lemma 33 Consider \( \chi(z) = \chi_q(z) = z^q \) for \( q > 1 \). Then for any \( Q \),

\[
J_q(Q) \leq \frac{1}{(q-1)Z}.
\]
Proof. We get directly

\[ J_q(Q) = KL_{\chi_\varrho}(\varrho \parallel Q) = \int_x \tilde{Q}(x) \int_{Q(x)} t^{-q} d\mu(x) \]

\[ = \frac{1}{1-q} \cdot \int_x \left( \tilde{Q}^2(x) - \tilde{Q}(x)Q^{1-q}(x) \right) d\mu(x) \]

\[ = \frac{1}{1-q} \cdot \int_x \left( \frac{1}{Z^2} \cdot Q^{q(2-q)}(x) - \frac{1}{Z} \cdot Q(x) \right) d\mu(x) \]

\[ = \frac{q-1}{q-1} \cdot \left( \frac{1}{Z} - \frac{1}{Z^2} \cdot \int_x Q^{q(2-q)}(x) d\mu(x) \right) \]

(185)

\[ \leq \frac{1}{(q-1)Z} \]

(186)

since \( q > 1 \).

We continue with (v) and pick a particular case for which \( Q \) belongs to the \( \chi \)-family, with an exact computation of \( J(Q) \). We choose the \( 1/2 \)-Gaussian.

Lemma 34 Consider the \( 1/2 \)-Gaussian on the real interval \([-\sigma, \sigma]\), for some \( \sigma > 0 \), whose density is given by

\[ Q(x) = \frac{A}{\sigma} \cdot [1 - (x^2/\sigma^2)]^2_+ , \]

(187)

with \( A = 15\sqrt{2}/32 \) and \([z]_+ = \max\{0, z\} \). Then, for \( \chi \) being the one of the \( 1/2 \)-Gaussian, we have:

\[ J(Q) = \frac{3^3}{2\sqrt{\sigma}} \cdot \left( \frac{3\pi}{16} \cdot \frac{1}{\sqrt{15\sqrt{2}}} \right) = \theta \left( \frac{1}{Z} \right) \cdot \]

(188)

Proof. The \( 1/2 \)-Gaussian arises from the more general class of \( q \)-exponential families, for which \( \chi \) is given in Lemma 33 [6], [41, Chapter 7]. We start at eq. (185):

\[ J(Q) = KL_{\chi_\varrho}(\varrho \parallel Q) = \frac{1}{1-q} \cdot \left( -\frac{1}{Z} + \frac{1}{Z^{2-q}} \cdot \int_x Q^{q(2-q)}(x) d\mu(x) \right) \]

(189)

Now, consider more specifically the \( q = 1/2 \)-Gaussian defined on the real line, for which \( Q(x) = (A/\sigma)[1 - (x^2/\sigma^2)]^2_+ \). In this case one can obtain that \( Z = B\sqrt{\sigma}, B = 4\sqrt{A}/3 \), and so

\[ \int_x Q^{q(2-q)}(x) d\mu(x) = A^\frac{3}{2} \cdot \sigma^{-\frac{3}{2}} \cdot \int_{-\sigma}^{\sigma} \left[ 1 - \frac{x^2}{\sigma^2} \right]_+^\frac{3}{2} dx \]

\[ = A^\frac{3}{2} \cdot \sigma^{-\frac{3}{2}} \cdot \int_{-1}^{1} \left[ 1 - x^2 \right]_+^\frac{3}{2} dx \]

\[ = \frac{3\pi}{8} \cdot A^\frac{3}{2} \cdot \sigma^{-\frac{3}{2}} \]

(190)
Figure 12: Consider an initial density $Q$ given by two adjacent squares (in red). Using $\chi$ as defined in eq. (193), the largest values can be blown up $\beta$ as large as desired so that $Z$ (in blue) is in turn as large as desired (see text; Figure best seen in colour).

since

$$\int_{-1}^{1} [1 - x^2] \frac{3}{2} \, dx = \frac{3\pi}{8} (> 1) . \quad (191)$$

So we obtain, taking into account that $A \approx 15\sqrt{2}/32$,

$$J(Q) = \frac{3\pi}{4B^{\frac{3}{2}} \cdot \sigma^\frac{3}{4}} \cdot A^\frac{3}{4} \sigma^\frac{3}{4} - \frac{3}{2\sqrt{A} \sqrt{\sigma}}$$

$$= \frac{1}{\sqrt{\sigma}} \cdot \left( 3^{\frac{3}{2}} \cdot \frac{3\pi}{2 \cdot 16} - \frac{3^{\frac{3}{2}}}{2\sqrt{15\sqrt{2}}} \right)$$

$$= \frac{3^{\frac{3}{2}}}{2\sqrt{\sigma}} \cdot \left( \frac{3\pi}{16} - \frac{1}{\sqrt{15\sqrt{2}}} \right) \approx \frac{0.9663}{\sqrt{\sigma}}$$

$$= \theta \left( \frac{1}{\sqrt{\sigma}} \right), \quad (192)$$

and we can conclude for the proof of Lemma 34.

We finish with (vi) and an example on how picking an escort that blows up large values for a density can indeed make $Z$ very large.
Lemma 35 Fix $K > 0$ and, for some $0 < \epsilon < 1/4$, let

$$\chi(z) = \begin{cases} 
\epsilon + \frac{1}{\log K} \cdot (K^{z-\epsilon} - 1) & \text{if } z \leq \epsilon \\
\epsilon + 1 & \text{if } z > \epsilon 
\end{cases} \quad (193)$$

Consider the density $Q$ given in Figure 12. Then, letting $Z(Q)$ denote the normalization of the escort of $Q$, it holds that $\lim_{K \to +\infty} Z(Q) = +\infty$.

**Proof** It follows that

$$Z = \epsilon^2 + \sqrt{1-\epsilon^2} \cdot \left( \epsilon + \frac{1}{\log K} \cdot (K^{\sqrt{1-\epsilon^2}-\epsilon} - 1) \right)$$

$$= \epsilon^2 + \sqrt{1-\epsilon^2} \cdot \left( \epsilon - \frac{1}{\log K} \right) + \frac{\sqrt{1-\epsilon^2}}{\log K} \cdot K^{\sqrt{1-\epsilon^2}-\epsilon} \quad (194)$$

Provided $K \geq \exp(4) \geq \exp(1/\epsilon)$, $g_1(K,\epsilon) \geq 0$; since $\epsilon < 1/4$, $\sqrt{1-\epsilon^2} - \epsilon \geq 1/2$ and $\sqrt{1-\epsilon^2} \geq 1/2$. Hence, if $0 < \epsilon < 1/4$ and $K \geq \exp(4)$, we have

$$Z \geq \frac{\sqrt{K}}{2 \log K} \quad (195)$$

and we indeed have $\lim_{K \to +\infty} Z = +\infty$.

21 Many modes for GAN architectures

In Section 5, we claim that a deep architecture working under the general model specified in Section 5 can accommodate a number of modes of the order of the total dimension of deep sufficient statistics, $\Omega(d \cdot L)$. To develop a simple argument, assume that the last layer is the identity function so we do not have to care for $H_{\text{out}}(x)$ in Theorem 15. A simple argument for this consists in three steps. First, we pick a $\chi$ like in Figure 13, whose derivative is going to zero as many times as necessary. Then,

- $(\Omega(d \cdot L)$ critical points at the modes) first computing the critical points using the gradient $\nabla Q_g(z)$ from Theorem 15, which yields:

$$\nabla Q_g(z) \propto \zeta(z) + \sum_{l=1}^L \sum_{i=1}^d \zeta_{l,i} \cdot \chi_{l,i}^*(P_{\chi,b_l,i}) \cdot \nabla_z (w_{l,i}^\top \phi_{l-1}) \quad (196)$$

where the $\zeta$ functions are not important, since (i) $\zeta(z)$ is always the null vector for $Q_{\text{in}}$ uniform, and (ii) $\zeta_{l,i}$ depends on $z$ but we can assume it never zeroes (it factors the escort’s density with a non zero constant). Then, using the $\chi$ as defined before, we choose the modes of the inner deformed exponential families (for which $\nabla_z (w_{l,i}^\top \phi_{l-1}) = 0$) in such a way that they are located at the critical points of $\chi$, and a different one for each of them. We obtain a $Q_g(z)$ which has up to (an order of) $d \cdot L$ critical points, exactly at all modes, as claimed;
Figure 13: Taking a $\chi$ whose derivatives zeroes as many times as needed, and then putting the modes of each inner deformed exponential family where it zeroes makes it easy to accommodate as many modes as needed for a deep architecture.

- (modes at all critical points) since each critical point is located at a mode for one of the densities, it is sufficient to ensure that the influence of all other densities in the curvature of the density is sufficiently small: for this, it is sufficient to then control the second derivative of $\chi$ in the neighborhood its critical points, making sure it does not exceed a small threshold in absolute value.

Notice that this property is independent from the one which allows to craft escorts that blow high density regions (and may yield large $Z$, Theorem 20, see also Lemma 35 and Figure 12), so we can combine both properties and obtain densities for the deep net with both high contrast around the modes and a large number of modes.

Of course, the $\chi$ we choose is very artificial and corresponds to an activation which would be almost piecewise linear, a sort of generalization of the ReLU activation with a large number of segments or half lines instead of two. Yet, it gives some simple intuition as to how fitting multimodal densities can indeed happen.

22 Proof of Lemma 21

The proof directly comes from Theorem 13, the variational part in the vig-$f$-GAN identity is:

$$
\sup_{T \in \mathbb{R}_{++}^X} \left\{ \mathbb{E}_{X \sim P}[T(X)] - \mathbb{E}_{X \sim Q} \left[ \frac{1}{\lambda} \log \chi \left( \frac{1}{\lambda} T(X) \right) \right] \right\}
$$

$$
= \sup_{T \in \mathbb{R}_{++}^X} \left\{ \mathbb{E}_{X \sim P}[-T(X)] + \mathbb{E}_{X \sim Q} \left[ \log \chi \left( \frac{1}{\lambda} T(X) \right) \right] \right\},
$$

(197)
and the DM vs G game in which DM’s objective is to realize:

$$\sup_{\gamma, x \to \mathbb{R}} \{ \mathbb{E}_{X \sim P}[-\gamma(X)] + \mathbb{E}_{X \sim Q}[u_{\text{dm}}(\gamma(X))] \}.$$  \hspace{1cm} (198)

Making the correspondence between (198) and the right hand-side of (197) gives the statement of the Lemma.

\section{Proof of Lemma 22}

We recall the utility $u_{\text{dm}} = u$ (for the sake of readability), and note that it depends on the state of the world / observation $x$,

$$u_x(z) = \log_{(\chi^*)} \frac{1}{Q(x)} (z).$$  \hspace{1cm} (199)

It follows from Lemma 21 and the definition of $\chi$-logarithms,

$$u'_x(z) = \frac{1}{\chi^*} \left( \frac{Q(x)}{z} \right),$$  \hspace{1cm} (200)

$$u''_x(z) = -\frac{1}{z^2} \left( \chi^{-1} \right)' \left( \frac{Q(x)}{z} \right)$$

$$= -\frac{1}{z^2} \left( \chi^{-1} \right)' \left( \frac{Q(x)}{z} \right) \left( \leq 0 \right).$$  \hspace{1cm} (201)

Putting back all parameters, we obtain the following Arrow-Pratt measure of absolute risk aversion:

$$a_{ux}(z) = - \frac{u''_x(z)}{u'_x(z)}$$

$$= \frac{\frac{Q(x)}{z} \left( \chi^{-1} \right)' \left( \frac{Q(x)}{z} \right)}{z^2}$$

$$= \frac{1}{z} \left( \chi^{-1} \right)' \left( \frac{Q(x)}{z} \right) \left( \leq 0 \right).$$  \hspace{1cm} (202)

Since $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ and is non decreasing, we see that

$$a_{ux}(z) \geq 0, \forall \chi, z, x,$$  \hspace{1cm} (203)

and therefore player DM is always risk-averse (this proves point (i)). Define

$$g(z) = \frac{z \cdot (\chi^{-1})'(z)}{\chi^{-1}(z)},$$  \hspace{1cm} (204)
Figure 14: Illustration of the optimality conditions for the insurance problem (44), wherein we pick the risky asset $\Upsilon$ such that for every $x \in X$, marginal utility is equal to the odds ratio. That is, the tangent with slope $P(x)/Q(x)$. (First Diagram) If DM suddenly believes a certain state state $x \in X$ is more likely, this pushes the optimal function $\Upsilon^*$ to the right. (Second Diagram) If we compare two decision makers utilities $u_1$ and $u_2$ such that $u_2$ is more risk averse than $u_1$. We see that for $u_2$ to consume at the same level as $u_1$, $u_2$ must believe $x \in X$ is far more likely.

so that

$$a_{ux}(z) = \frac{1}{z} \cdot g\left(\frac{z}{Q(x)}\right), \quad (205)$$

and

$$r_{ux}(z) = z \cdot a_{ux}(z) = g\left(\frac{z}{Q(x)}\right). \quad (206)$$

This proves point (ii). Notably, at the optimum, the dependency on the subjective beliefs disappears since the optimum (Theorem 11) of $\Upsilon^* = -T^*$ yields:

$$r_{ux}(\Upsilon^*(x)) = g\left(\frac{\chi(Q(x))}{Z \cdot \chi(P(x))} \cdot \frac{1}{Q(x)}\right) = g\left(\frac{1}{\chi(P(x))}\right). \quad (207)$$

**Remark.** We can make a connection with a more traditional view of portfolio allocation. The first order conditions for (44) gives us

$$u'_\text{out}(\Upsilon^*(x)) \cdot Q(x) - P(x) = 0 \implies u'_\text{out}(\Upsilon^*(x)) = \frac{P(x)}{Q(x)}. \quad (208)$$

From (208) we see that at optimality (equilibrium), Decision Maker picks a portfolio such that his marginal utility over the risky asset under each state $x \in X$ is equal to the corresponding odds ratio. If Decision Maker (for whatever reason) suddenly believes a certain state $x_0 \in X$ is more likely ($Q(x_0)$ goes up), then with usual assumptions about decreasing marginal utility, it’s intuitive that he will respond by consuming more of $\Upsilon(x_0)$. Similarly if compare decision makers with differing risk aversion in the risky asset, the more risk averse decision maker must hold much stronger beliefs to consume at the same level as the less risk averse decision maker. The optimality condition (208) is illustrated in Figure 14.
Appendix on experiments

24 Architectures

We consider two architectures in our experiments: DCGAN [52] and the multilayer feedforward network (MLP) used in [17]. Suppose the size of input images is isize-by-isize, the details of architectures are given as follows:

**Generator of DCGAN**: 
ConvTranspose(input=100, output=8×isize, stride=1) → BatchNorm → Activation → Conv(input=8×isize, output=4×isize, stride=2, padding=1) → BatchNorm → Activation → ConvTranspose(input=4×isize, output=2×isize, stride=2, padding=2) → BatchNorm → Activation → ConvTranspose(input=2×isize, output=isize, stride=2, padding=1) → BatchNorm → Activation → Conv(isize, number of channel, stride=2, padding=1) → Last Activation

**Discriminator of DCGAN**: 
Conv(1, 2×isize, stride=2) → BatchNorm → LeakyReLU → Conv(input=2×isize, output=4×isize, stride=2, padding=1) → BatchNorm → LeakyReLU → Conv(input=4×isize, output=8×isize, stride=2, padding=2) → BatchNorm → LeakyReLU → Conv(input=8×isize, output=1, stride=2, padding=1) → Link function

**Generator of MLP**: 
z → Linear(100, 1024) → BatchNorm → Activation → Linear(1024, 1024) → BatchNorm → Activation → Linear(1024, isize×isize) → last Activation

**Discriminator of MLP**: 
x → Linear(isize×isize, 1024) → ELU → Linear(1024, 1024) → ELU → Linear(1024, 1) → Link function

25 Experimental setup for varying the activation function in the generator

**Setup.** We train adversarial networks with varying activation functions for the generators on the MNIST [33] and LSUN [61] datasets. In particular, we compare ReLU, Softplus, Least Square loss as an example of prop-μ, and μ-ReLU with varying μ in [0, 0.1, ..., 1] by using them as the activation functions in all hidden layers of the generators. For all models, we fix the learning rate to 0.0002 and batch size to 64 throughout all experiments after tuning on a hold-out set.

**MNIST.** We evaluate the activation functions by using both DCGAN and the MLP used in [17] as the architectures. As training divergence, we adopt both GAN and Wasserstein distance (WGAN) because GAN belongs to variational f-divergence formulation while WGAN does not. The link function of the discriminators is specific to the respective divergence, which
is sigmoid for GAN and linear for WGAN. We sample random noise $z \in \text{Uniform}_{100}(0, 1)$ for MLP and $z \in \text{Gaussian}(0, 1)$ for DCGAN, which is found slightly better than sampling from $\text{Uniform}_{100}(-1, 1)$. As the best practice, we apply Adam \cite{DBLP:journals/corr/KingmaB14} to optimize models with GAN and RMSprop \cite{DBLP:journals/corr/Tieleman12} to optimize WGAN based models. For GAN, we train one batch for discriminator and one batch for generator iteratively during training. For WGAN, we apply weight clipping with 0.01 and train five batches for discriminator and one batch for generator interchangeably during training.

We train all models on the full MNIST training data set and evaluate the performance on the test set by using the kernel density estimation (KDE). Since the size of images accepted by DCGAN should be n-fold of 16, all images are rescaled to 32-by-32 for all models. Following \cite{9}, we apply three-fold cross validation to find optimal bandwidth for the isotropic Gaussian kernel of KDE on a hold-out set. To estimate the log probability of the test set, we sample 16k images from the models in the same way as \cite{9}. We observe that the initialization of model parameters has significant influence on performance. Therefore, we conduct three runs with different random seeds for each experimental setting and report the mean and standard deviation of the results.

**LSUN.** We also evaluate all activation functions in consideration for the generator on LSUN natural scene images. We train DCGAN with GAN as the divergence on the *tower* category of images, which are rescaled and center-cropped to 64-by-64 pixels, as in \cite{9}. Due to the center-cropped images, we apply tanh as last activation of generators instead of sigmoid for GAN based models.

## 26 Visual results on MNIST
Table 3: MNIST results for GAN_DCGAN at varying $\mu$ ($\mu = 1$ is ReLU).
| µ        | µ        |
|----------|----------|
| 0        | 0.1      |
| 0.2      | 0.3      |
| 0.4      | 0.5      |
| 0.6      | 0.7      |
| 0.8      | 0.9      |
| 1        |          |

Table 4: MNIST results for WGAN_DCGAN at varying µ (µ = 1 is ReLU).
Table 5: MNIST results for WGAN_MLP at varying $\mu$ ($\mu = 1$ is ReLU).
| $\mu$ | $\mu$ |
|---|---|
| 0 | 0.1 |
| 0.2 | 0.3 |
| 0.4 | 0.5 |
| 0.6 | 0.7 |
| 0.8 | 0.9 |
| 1 | |

Table 6: MNIST results for GAN_MLP at varying $\mu$ ($\mu = 1$ is ReLU).
| $\mu$ | $\mu$ |
|-------|-------|
| 0     | 0.1   |
| 0.2   | 0.3   |
| 0.4   | 0.5   |
| 0.6   | 0.7   |
| 0.8   | 0.9   |
| 1     |       |

Table 7: LSUN results for GAN_DCGAN at varying $\mu$ ($\mu = 1$ is ReLU).