Some comments on a recently derived approximated solution of the Einstein equations for a spinning body with negligible mass

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Abstract

Recently, an approximated solution of the Einstein equations for a rotating body whose mass effects are negligible with respect to the rotational ones has been derived by Tartaglia. At first sight, it seems to be interesting because both external and internal metric tensors have been consistently found, together an appropriate source tensor; moreover, it may suggest possible experimental checks since the conditions of validity of the considered metric are well satisfied at Earth laboratory scales. However, it should be pointed out that reasonable doubts exist if it is physically meaningful because it is not clear if the source tensor related to the adopted metric can be realized by any real extended body. Here we derive the geodesic equations of the metric and analyze the allowed motions in order to disclose possible unphysical features which may help in shedding further light on the real nature of such approximated solution of the Einstein equations.
1 Introduction

As it is well known, the Kerr metric is a stationary solution of the Einstein field equations in the vacuum which is endowed with axial symmetry and is characterized by two independent parameters: the asymptotic mass and the asymptotic angular momentum per unit mass. Unfortunately, it would probably be incorrect to adopt such metric in order to describe the motion of light rays and test masses in the gravitational field of any real material rotating body because, at present, the problem of finding an internal mass-energy distribution corresponding to the Kerr metric has not yet received a complete general answer. Many efforts have been dedicated to the construction of sources which could represent some plausible models of stars. In [1] the authors have tried to connect the external Kerr metric to the multipolar structure of various types of stars assumed to be rigidly and slowly rotating. Terms of greater than the second order in the angular velocity were neglected. In [2] a physically reasonable fluid source for the Kerr metric has been obtained. For previous attempts to construct sources for the Kerr metric see [3, 4]. In [5] a general class of solutions of Einstein’s equations for a slowly rotating fluid source, with supporting internal pressure, is matched to the Kerr metric up to and including first order terms in angular speed parameter. In [6] counterrotating thin disks of finite mass, consisting of two streams of collisionless particles, circulating in opposite directions with different velocities are examined. The authors show how such disks can act as exact sources of all types of the Kerr metric.

Recently, in [7] an approximated solution of the Einstein field equations for a rotating, weakly gravitating body has been found. It seems promising because both external and internal metric tensors have been consistently found, together an appropriate source tensor. Moreover, the mass and the angular momentum per unit mass are assumed to be such that the mass effects are negligible with respect to the rotation effects: this is a situation which is rather easy to obtain, especially at laboratory scale [8]. As a consequence, only quadratic terms in the body’s angular velocity has been retained. One main concern is that it is not clear if there is a real mass distribution able to generate the found source tensor.

In this paper we wish to derive some consequences which could allow to shed light on the physical nature of the considered metric. Of course, if some exotic effects in apparent contrast
with the experience will be found, this would represent a further, strong sign that the examined metric is not physically reasonable in describing a real rotating source. There are well developed theoretical methods to exclude pathological metrics from further considerations, but we think that fully deriving some consequences which may turn out to be unphysical is less formal and more vivid.

The paper is organized as follows. In Section 2 we briefly review the metric. In Section 3 we calculate the geodesic equations of the metric both in spherical and in cartesian coordinates and consider free fall and weighing scenarios in a terrestrial laboratory context. Section 4 is devoted to the conclusions.

2 An approximated solution of the Einstein equations for a weakly gravitating, spinning body

Here we wish to investigate the case of a rotating body of mass $M$, radius $R$ and proper angular momentum $J$ with negligible mass with respect to the rotation in the sense that

$$a = \frac{J}{Mc} >> R_s = \frac{2GM}{c^2},$$

(1)

where $G$ is the Newtonian gravitational constant. This is an interesting situation which is quite common at laboratory scales and also in some astronomical situations [8]. For, say, a rotating sphere with\(^1\) $\omega = 4.3 \times 10^4$ rad s\(^{-1}\), $R = 2.5 \times 10^{-2}$ m, $M = 1.11 \times 10^{-1}$ kg we would have $a = 3.6 \times 10^{-8}$ m and $R_s = 1.6 \times 10^{-28}$ m, while the Earth has $a_{\oplus} = 3.3$ m and $R_{\oplus} = 8.86 \times 10^{-3}$ m. An approximated solution of the Einstein equations for a body which satisfies the condition of eq. (1) has been recently obtained in [7].

In a frame whose origin coincides with the center of the spinning mass let us adopt the spherical coordinates $(r, \theta, \phi)$ with the $\theta$ angle counted from the axis of rotation of the body; then, the external solution is [7]

$$(ds)^2 \sim c^2(1 + h_{\theta\theta})(dt)^2 - (1 + h_{rr})(dr)^2 - r^2(1 + h_{\theta\theta})(d\theta)^2 - r^2 \sin^2 \theta(1 + h_{\phi\phi})(d\phi)^2,$$

(2)

\(^1\)These critical values for the mechanical parameters of the sphere can be obtained with the cutting-edge technologies available today or in the near future [9], cited in [8].
with

\[ h_{00} = C_0 \frac{a^2}{r^2} \cos \theta, \]
\[ h_{rr} = h_{\theta\theta} = h_{\phi\phi} = -C_0 \frac{a^2}{r^2} \cos \theta, \]

where, for a rotating homogeneous sphere, \( a = \frac{2\omega R^2}{c^2} \) and the value \( C_0 = \frac{50}{3} \pi \) can be obtained only on the base of an analogy [7]; however, it should be of the order of unity. Eqs. (3)-(4) has been obtained by assuming \( \varepsilon = \frac{R}{2r} \ll \alpha = \frac{a}{r} \). This is the reason why the off-diagonal gravitomagnetic term, linear in \( \omega \), has been neglected.

The interesting point is that, contrary to the Kerr solution which cannot be extended to the interior of a generic material body, it has an internal counterpart. On the other hand, it is questionable if such metric is really able to describe any material body: indeed, the source tensor for a rotating homogeneous sphere yielded by the Tartaglia’s metric is

\[ T_{00} = -\frac{6\omega^2}{25\pi c^2} C_0 \cos \theta, \]
\[ T_{rr} = T_{\theta\theta} = T_{\phi\phi} = 0. \]

Note that \( T_{00} \) is antisymmetric for reflections with respect to the equatorial plane and its integral over the entire volume of the sphere vanishes. Now the question is: what kind of real mass distribution could give rise to such an energy-momentum tensor as that of eqs. (5)-(6)? Note also that an elastic energy-momentum tensor should be added to it in order to account for the elastic force needed to keep the whole body together against the centrifugal forces and, from a mathematical point of view, to insure the continuity of the radial derivatives of the metric tensor at the boundary of the body. The metric that Tartaglia gives does not contain the influence of such additional parts of the energy-momentum tensor and its significance must therefore be investigated in terms of the source that it does represent.

The observable consequences derived in the following section could shed some light to this problem.

3 The geodesic motion of a test particle

Let us investigate the free motion of a point mass in the metric of eqs. (3)-(4).
By defining \( k = C_0 a^2 = \frac{4}{25} C_0 \omega^2 R^4 \), from eqs. (3)–(4) the determinant \( g \) of the metric tensor \( g_{\mu\nu} \) is

\[
g = -r^4 \sin^2 \theta \left( 1 + \frac{k \cos \theta}{r^2} \right) \left( 1 - \frac{k \cos \theta}{r^2} \right)^3.
\]

Then, the inverse of \( g_{\mu\nu} \) is

\[
g^{00} = \frac{1}{\left( 1 + \frac{k \cos \theta}{r^2} \right)},
\]

\[
g^{rr} = -\frac{1}{\left( 1 - \frac{k \cos \theta}{r^2} \right)},
\]

\[
g^{\theta\theta} = -\frac{1}{r^2 \left( 1 - \frac{k \cos \theta}{r^2} \right)},
\]

\[
g^{\phi\phi} = -\frac{1}{r^2 \sin^2 \theta \left( 1 - \frac{k \cos \theta}{r^2} \right)}.
\]

The geodesic equations of the motion of a test particle are

\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0,
\]

where \( \tau \) is the proper time of the particle and the \( \Gamma^\mu_{\nu\rho} \) are the Christoffel symbols

\[
\Gamma^\mu_{\nu\rho} = \frac{g^{\mu\alpha}}{2} \left( \frac{\partial g_{\alpha\nu}}{\partial x^\rho} + \frac{\partial g_{\alpha\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\rho}}{\partial x^\alpha} \right).
\]

From eqs. (3)–(4) and eqs. (8)–(11) it turns out that the only non vanishing Christoffel symbols are

\[
\Gamma^0_{0r} = -\frac{k \cos \theta}{r^3 \left( 1 + \frac{k \cos \theta}{r^2} \right)},
\]

\[
\Gamma^0_{0\theta} = -\frac{k \sin \theta}{2r^2 \left( 1 + \frac{k \cos \theta}{r^2} \right)},
\]

\[
\Gamma^r_{00} = -\frac{k \cos \theta}{r^3 \left( 1 - \frac{k \cos \theta}{r^2} \right)},
\]

\[
\Gamma^r_{rr} = \frac{k \cos \theta}{r^3 \left( 1 - \frac{k \cos \theta}{r^2} \right)},
\]

\[
\Gamma^r_{\theta\theta} = -\frac{r}{\left( 1 - \frac{k \cos \theta}{r^2} \right)},
\]

\[
\Gamma^r_{\phi\phi} = -\frac{r \sin^2 \theta}{\left( 1 - \frac{k \cos \theta}{r^2} \right)}.
\]
\[
\begin{align*}
\Gamma^r_{r\theta} &= -\frac{k \sin \theta}{2r^2 \left(1 - \frac{k \cos \theta}{r^2}\right)^\prime}, \\
\Gamma^\theta_{00} &= -\frac{k \sin \theta}{2r^4 \left(1 - \frac{k \cos \theta}{r^2}\right)^\prime}, \\
\Gamma^\theta_{rr} &= -\frac{k \sin \theta}{2r^4 \left(1 - \frac{k \cos \theta}{r^2}\right)^\prime}, \\
\Gamma^r_{\theta\theta} &= \frac{k \sin \theta}{2r^2 \left(1 - \frac{k \cos \theta}{r^2}\right)^\prime}, \\
\Gamma^\phi_{\phi\phi} &= \frac{-2r^2 \sin \theta \cos \theta + k \sin \theta(3 \cos^2 \theta - 1)}{2r^2 \left(1 - \frac{k \cos \theta}{r^2}\right)}, \\
\Gamma^r_{r\phi} &= \frac{1}{r \left(1 - \frac{k \cos \theta}{r^2}\right)}, \\
\Gamma^\phi_{r\phi} &= \frac{1}{r \left(1 - \frac{k \cos \theta}{r^2}\right)}, \\
\Gamma^\phi_{\theta\phi} &= \frac{2r^2 \sin \theta \cos \theta - k \sin \theta(3 \cos^2 \theta - 1)}{2r^2 \sin^2 \theta \left(1 - \frac{k \cos \theta}{r^2}\right)}.
\end{align*}
\]

Then, the geodesic equations for \(t, r, \theta\) and \(\phi\) are, in explicit form\(^2\)

\[
\begin{align*}
\ddot{r} \left(1 + \frac{k \cos \theta}{r^2}\right) &= \frac{2k \cos \theta}{r^3} \dot{r} + \frac{k \sin \theta}{r^2} \dot{\theta}, \\
\ddot{r} \left(1 - \frac{k \cos \theta}{r^2}\right) &= r (\dot{\theta})^2 + r \sin^2 \theta (\dot{\phi})^2 + \frac{c^2 k \cos \theta}{r^3} (\dot{r})^2 - \frac{k \cos \theta}{r^3} (\dot{r})^2 - \frac{k \sin \theta}{r^2} r \dot{\theta}, \\
\ddot{\theta} \left(1 - \frac{k \cos \theta}{r^2}\right) &= -\frac{2 \dot{r} \dot{r}^\prime}{r} + \sin \theta \cos \theta (\dot{\phi})^2 + \frac{c^2 k \sin \theta}{2r^4} (\dot{r})^2 + \frac{k \sin \theta}{2r^4} (\dot{r})^2 - \frac{k \sin \theta}{2r^2} (\dot{\theta})^2 - \frac{k \sin \theta}{2r^2} \frac{3 \cos^2 \theta - 1}{(\dot{\phi})^2}, \\
\ddot{\phi} \left(1 - \frac{k \cos \theta}{r^2}\right) &= -\frac{2 \dot{r} \dot{\phi}^\prime}{r} - \frac{2 \cos \theta}{r} \sin \theta \dot{\theta} \dot{\phi} + \left[\frac{k}{r^2 \sin \theta} \frac{3 \cos^2 \theta - 1}{(\dot{\phi})^2}\right] \dot{\phi}. 
\end{align*}
\]

Note that in such equations the terms \(1 \pm \frac{k \cos \theta}{r^2}\) on the left-hand sides can be considered almost equal to 1 because \(\frac{k}{r^2}\) is of the order of \(10^{-11}\) or less. For the same reason, since 

\(d\tau = \sqrt{g_{00}} \, dt = \sqrt{1 + \frac{k \cos \theta}{r^2}} \, dt\) we can safely assume \(\dot{r} = 1\). Note also that for \(k = 0\) eqs.\(^{29}-\)\(^{31}\) reduce to the equations of motion in a flat spacetime in spherical coordinates, i.e.

\[
a_c \equiv \ddot{r} - r (\dot{\theta})^2 - r \sin^2 \theta (\dot{\phi})^2 = 0,
\]

\(^2\)Of course, they can also be derived from the Lagrangian \(\mathcal{L} = \frac{1}{2} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu\) and the Lagrange equations \(\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0\).
\[
a_\theta \equiv r \ddot{\theta} + 2\dot{\theta} \dot{r} - r \sin \theta \cos (\dot{\theta})^2 = 0, \tag{33}
\]
\[
a_\phi \equiv r \sin \theta \ddot{\phi} + 2 \sin \theta \dot{r} \dot{\phi} + 2 r \cos \theta \dot{\theta} \dot{\phi} = 0, \tag{34}
\]
where \(a_r, a_\theta, a_\phi\) are the three components of the particle’s acceleration in spherical coordinates.

Let us analyze some possible types of motion for a point particle obeying eqs. (29)-(31), which are, as it can be seen, highly nonlinear and difficult to resolve without some simplifying assumptions. A purely radial motion is not allowed. Indeed, for \(\theta\) and \(\phi\) constant eq. (30) yields
\[
c^2 + (\dot{r})^2 = 0 \tag{35}
\]
which does not admit solution.

Consider now the case of a motion with \(r = r_0\) and \(\theta = \theta_0\). The geodesic equations reduce to
\[
\ddot{t} \left(1 + \frac{k \cos \theta_0}{r_0^2}\right) = 0, \tag{36}
\]
\[
\frac{k c^2 \cos \theta_0}{r_0^3} (\dot{t})^2 + r_0 \sin^2 \theta_0 (\dot{\phi})^2 = 0, \tag{37}
\]
\[
\frac{k c^2}{r_0^2} (\dot{t})^2 + [-2 r_0^2 \cos \theta_0 + k (3 \cos^2 \theta_0 - 1)] (\dot{\phi})^2 = 0, \tag{38}
\]
\[
\ddot{\phi} \left(1 - \frac{k \cos \theta_0}{r_0^2}\right) = 0. \tag{39}
\]
For \(\theta_0 = \frac{\pi}{2}\) eq. (37) yields \(r_0 (\dot{\phi})^2 = 0\), i.e. \(\phi = \phi_0 = \text{constant}\). This condition satisfies eq. (39) but, from eq. (38), it would imply \(\frac{k c^2}{r_0^2} = 0\). So, the equations for \(r = r_0\), \(\theta = \theta_0 = \frac{\pi}{2}\) do not admit solutions.

Would it be possible a circular motion along a parallel for \(r = r_0\) and \(\theta = \theta_0 \neq \frac{\pi}{2}\)? If \(\cos \theta_0 > 0\), eq. (37) can never be satisfied. On the other hand, for \(\cos \theta_0 < 0\) eq. (37) can be satisfied, while eq. (38) does not admit any solution because it can be shown that \(2 r_0^2 \cos \theta_0 + k (1 - 3 \cos^2 \theta_0) > 0\) cannot be satisfied for \(\frac{\pi}{2} < \theta_0 < \pi\). This means that motions with \(r\) and \(\theta\) constant cannot occur even outside the equatorial plane of the source.

The motion along a meridian with \(r = r_0\) and \(\phi = \phi_0\) is governed by the equations
\[
\ddot{t} \left(1 + \frac{k \cos \theta}{r_0^2}\right) - \frac{k \sin \theta}{r_0^2} \dot{t} \dot{\phi} = 0, \tag{40}
\]
\[ c^2 k \cos \theta \left( \frac{\dot{i}}{r_0^3} \right)^2 + r_0 (\dot{\theta})^2 = 0, \]  
(41)

\[ \dot{\theta} \left( 1 - \frac{k \cos \theta}{r_0^3} \right) - \frac{c^2 k \sin \theta}{2r_0^4} (\dot{i})^2 + \frac{k \sin \theta}{2r_0^4} (\dot{\theta})^2 = 0. \]  
(42)

The geodesic equation for \( \phi \) identically vanishes. From eq. (41) it follows that a motion for \( 0 < \theta < \frac{\pi}{2} \), i.e. \( \cos \theta > 0 \), is not allowed.

The "spherical" motion with \( r = r_0 \) is described by

\[ \ddot{t} \left( 1 + \frac{k \cos \theta}{r_0^3} \right) = \frac{k \sin \theta}{r_0^2} \dot{\theta}, \]  
(43)

\[ c^2 k \cos \theta \left( \frac{\dot{i}}{r_0^3} \right)^2 = -r_0 (\dot{\theta})^2 - r_0 \sin^2 \theta (\dot{\phi})^2, \]  
(44)

\[ \dot{\theta} \left( 1 - \frac{k \cos \theta}{r_0^3} \right) = + \frac{c^2 k \sin \theta}{2r_0^4} (\dot{i})^2 - \frac{k \sin \theta}{2r_0^4} (\dot{\theta})^2 - \frac{\sin \theta}{2r_0^2} \left[ -2r_0 \cos \theta + k (3 \cos^2 \theta - 1) \right] (\dot{\phi})^2, \]  
(45)

\[ \ddot{\phi} \left( 1 - \frac{k \cos \theta}{r_0^3} \right) = \left[ -\frac{2 \cos \theta}{\sin \theta} - \frac{k}{r_0^2 \sin \theta} (3 \cos^2 \theta - 1) \right] \dot{\theta} \dot{\phi}. \]  
(46)

From eq. (44) it follows that a motion for \( 0 < \theta < \frac{\pi}{2} \), i.e. \( \cos \theta > 0 \), is not allowed.

Finally, after such rather pathological situations, let us investigate the case \( r = r_0, \theta = \theta_0, \phi = \phi_0, \dot{r} = 0, \dot{\theta} = 0, \dot{\phi} = 0 \). We have

\[ \ddot{t} = 0, \]  
(47)

\[ \ddot{r} = \frac{kc^2 \cos \theta_0}{r_0^3}, \]  
(48)

\[ \ddot{\theta} = \frac{kc^2 \sin \theta_0}{2r_0^4}, \]  
(49)

\[ \ddot{\phi} = 0. \]  
(50)

Such results can describe the relativistic acceleration experienced by a material sample suspended over the rotating central body. For example, for \( \theta_0 = 0 \) we have that the sample would be acted upon by an acceleration of magnitude \( \frac{kc^2}{r_0^3} \) directed upwards along the local vertical which would tend to counteract the Earth’s gravitational acceleration \( g \) in a laboratory experiment. This topic will be treated in more details in the following subsection.

In order to derive some more vivid consequences of the geodesic equations of motion and visualize them it is helpful to adopt the cartesian coordinates.
By using
\[
    dr = dx \sin \theta \cos \phi + dy \sin \theta \sin \phi + dz \cos \theta, 
\]
\[
    rd\theta = dx \cos \theta \cos \phi + dy \cos \theta \sin \phi - dz \sin \theta, 
\]
\[
    r \sin \theta d\phi = -dx \sin \phi + dy \cos \phi, 
\]
\[
    \cos \theta = \frac{z}{r}, 
\]
in the metric in spherical coordinates it is possible to express the \( g_{\mu\nu} \) in cartesian coordinates
\[
    h_{00} = \frac{kz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, 
\]
\[
    h_{11} = h_{22} = h_{33} = -\frac{kz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}. 
\]

From eqs. (54)-(56) the Lagrangian of a particle with mass \( m \)
\[
    \mathcal{L} = \frac{m}{2} \left\{ \left[ 1 + \frac{kz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] c^2 \dot{t}^2 - \left[ 1 - \frac{kz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] \left[ (\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 \right] \right\}. 
\]

From it the geodesic equations of motions are
\[
    \ddot{x} \left[ 1 + \frac{kz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] = \frac{3kz\dot{t}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} - \frac{k\dot{z}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, 
\]
\[
    \ddot{y} \left[ 1 - \frac{kz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] = \frac{k\dot{y}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} - \frac{k\dot{z}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, 
\]
\[
    \ddot{z} \left[ 1 - \frac{kz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] = \frac{k\dot{z}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} - \frac{k\dot{z}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, 
\]

Note that eqs. (59)-(61) for \( k = 0 \) reduce to the ordinary \( \ddot{x} = \ddot{y} = \ddot{z} = 0 \) of the flat spacetime case.
It is interesting to note that the relativistic acceleration induced by the rotating sphere is non-central, has some terms which are quadratic in the velocity $v$ of the test particle and depends on the inverse of the third power of the distance. Such features could be important in view of a possible experimental investigation. Notice that the classical Newtonian acceleration of the sphere turns out to be far smaller than the relativistic acceleration. For a sphere with $\omega = 4.33 \times 10^4$ rad s$^{-1}$, $R = 2.5 \times 10^{-2}$ m, $M = 1.11$ kg and particles with $v = 3.9 \times 10^3$ m s$^{-1}$, as for, say, thermal neutrons, we have that the ratio of the relativistic acceleration $\sim \frac{kv^2}{r^3}$ to the Newtonian one $\frac{GM}{r^2}$ is
\begin{equation}
\frac{kv^2}{GM} = \frac{1.4 \times 10^5}{r} \text{ m.}
\end{equation}
Moreover, while $\frac{GM}{r^2} \sim 10^{-9}$ m s$^{-2}$ for $r = 5 \times 10^{-2}$ m, $\frac{kv^2}{r^3} \sim 8 \times 10^{-4}$ m s$^{-2}$ for $r = 5 \times 10^{-2}$ m. Of course, if we would think about some experiments on the Earth’s surface, its acceleration of gravity $g = 9.86$ m s$^{-2}$ should be accounted for.

3.1 An Earth laboratory free fall scenario

In order to examine a concrete scenario at Earth laboratory scale, we have numerically solved eqs. (59)-(61) with the software MATHEMATICA for two different scenarios. In the first one we have assumed $k = 3.67 \times 10^{-9}$ m$^2$, which corresponds to a spinning sphere with angular velocity $\omega = 6.28 \times 10^3$ rad s$^{-1}$ and radius $R = 1$ m, with the initial conditions $x(0) = y(0) = 2$ m, $z(0) = 120$ m, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$. In order to account for the Earth’s gravity acceleration $g$ we have added $-g$ to the right-hand side of eq. (61). The interesting results are plotted in Figure 1-Figure 3. It can be noted that, while in the classical case of free fall $x$ and $y$ remain constant and $z$ vanishes after $t = 4.94$ s, in this case the falling body starts going upwards acquiring a displacement from the vertical of a large amount, after almost 15 s it inverts its motion downwards and finally it reaches the reference plane $z = 0$ in 33 s. It seems a very strange behavior.

In the second scenario, for a simple sphere with just $\omega = 10^2$ rad s$^{-1}$ and $R = 2 \times 10^{-1}$ m

\begin{itemize}
\item Notice that the radius of curvature of the particle trajectory induced by the Newtonian acceleration of gravity of the spinning sphere would be $4.86 \times 10^{19}$ m, i.e. the particles would fly along straight lines.
\item In doing so we have posed $(1 \pm \frac{kz}{r^3}) = 1$ and $t = 1$. We have also checked that, as it could be expected, the results of the integration do not change if the full expressions of the geodesic equations are retained.
\item It is exactly the result it can be obtained by putting $k = 0$ in solving the equations.
\end{itemize}
Figure 1: Time evolution of the $x$ coordinate of a point mass freely falling in the Earth’s gravitational field and in the local gravitational field of a spinning sphere with $\omega = 6.28 \times 10^3 \text{ rad s}^{-1}$, $R = 1 \text{ m}$. The initial conditions are $x(0) = y(0) = 2 \text{ m}$, $z(0) = 120 \text{ m}$, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$. The origin of the coordinates is the center of the sphere and the $z = 0$ plane is its equatorial plane.

m, corresponding to $k = 1.5 \times 10^{-15} \text{ m}^2$, and $x(0) = y(0) = 5 \times 10^{-1} \text{ m}$, $z(0) = 1 \text{ m}$ and $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$ it would be possible to observe a notable deviation with respect to the vertical and an increment of the time required to reach the $z = 0$ plane, as shown in Figure 4 and Figure 5. It should be noted that the mechanical parameters of the spinning sphere of this scenario are quite common and easy to obtain, so that it should be possible to observe the exotic effects described here in many ordinary situations.

3.2 A gravitational shielding effect?

Let us consider now the case of a body suspended upon the rotating source at $x = y = 0$, $z = h$. It can be shown that the weight of the suspended body is reduced by the local gravitational field of the spinning disk. Indeed, from eqs. (59)-(61) it can be noted that there are no in-plane, tangential components of the relativistic acceleration while the Earth’s gravity acceleration

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6It can be shown that in both the cases considered here the speeds reached by the freely falling bodies are nonrelativistic.
Figure 2: Time evolution of the $y$ coordinate of a point mass freely falling in the Earth’s gravitational field and in the local gravitational field of a spinning sphere with $\omega = 6.28 \times 10^3$ rad s$^{-1}$, $R = 1$ m. The initial conditions are $x(0) = y(0) = 2$ m, $z(0) = 120$ m, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$. The origin of the coordinates is the center of the sphere and the $z = 0$ plane is its equatorial plane.

along the vertical is reduced according to

$$\frac{d^2z}{dt^2} = -g + \frac{kc^2}{h^3}.$$  \hspace{1cm} (63)

For a generic position of the sample there is also a tangential part of the acceleration: indeed we have

$$\frac{d^2x}{dt^2} = \frac{3kc^2 x_0 z_0}{2(x_0^2 + y_0^2 + z_0^2)^{3/2}};$$ \hspace{1cm} (64)

$$\frac{d^2y}{dt^2} = \frac{3kc^2 y_0 z_0}{2(x_0^2 + y_0^2 + z_0^2)^{3/2}};$$ \hspace{1cm} (65)

$$\frac{d^2z}{dt^2} = -g + \frac{kc^2}{2} \left[ \frac{3z_0^2}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} - \frac{1}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} \right].$$ \hspace{1cm} (66)

For a spinning sphere with $\omega = 10^2$ rad s$^{-1}$ and $R = 2 \times 10^{-1}$ m, i.e. $k = 1.5 \times 10^{-15}$ m$^2$ and a body suspended over the disk with $x = y = 0$ and $z = h = 3$ m, the weight-reducing contribution would be $4.96$ m s$^{-2}$, which is a very large figure. Note also that the relativistic additional forces of eqs. (63)–(66) are independent of the speed of light $c$. 
Figure 3: Time evolution of the $z$ coordinate of a point mass freely falling in the Earth’s gravitational field and in the local gravitational field of a spinning sphere with $\omega = 6.28 \times 10^3 \text{ rad s}^{-1}$, $R = 1 \text{ m}$. The initial conditions are $x(0) = y(0) = 2 \text{ m}$, $z(0) = 120 \text{ m}$, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$. The origin of the coordinates is the center of the sphere and the $z = 0$ plane is its equatorial plane. Note that, according to Newtonian gravity, $t(z = 0) = 4.9 \text{ s}$.

Let us consider now the case of a disk with inner radius $R_1$, outer radius $R_2$, thickness $l$ and uniform density $\rho$ placed in rotation around an axis orthogonal to its plane assumed as $z$ axis. In this case

$$a = \frac{(R_2^4 - R_1^4) \omega}{2c(R_2^2 - R_1^2)} \quad (67)$$

For $\omega = 5 \times 10^3 \text{ rpm}=5.23 \times 10^2 \text{ rad s}^{-1}$, $R_1 = 4 \times 10^{-2} \text{ m}$, $R_2 = 1.375 \times 10^{-1} \text{ m}$ we have $k = C_0 a^2 = C_0 \times 3.2 \times 10^{-16} \text{ m}^2$. For a body suspended over the disk with $x = y = 0$ and $z = h = 3 \text{ m}$ the correction to the Earth’s gravity acceleration amounts to $C_0 \times 1.06 \text{ m s}^{-2}$, i.e. $C_0 \times 10\%$ of $g$.

It is interesting a comparison with the famous and controversial antigravity experiment by E.E. Podkletnov [10]. In it it was reported that a high-temperature $YBa_2Cu_3O_{7-x}$ bulk ceramic superconductor with composite structure has revealed weak shielding properties against gravitational force while in a levitating state at temperatures below 70 K. A toroidal disk with an outer diameter of $2.75 \times 10^{-1} \text{ m}$ and a thickness of $1 \times 10^{-2} \text{ m}$ was prepared using conventional ceramic technology in combination with melt-texture growth. Two solenoids were
Figure 4: Time evolution of the $z$ coordinate of a point mass freely falling in the Earth's gravitational field and in the local gravitational field of a spinning sphere with $\omega = 10^2 \text{ rad s}^{-1}$, $R = 2 \times 10^{-1} \text{ m}$. The initial conditions are $x(0) = y(0) = 5 \times 10^{-1} \text{ m}$, $z(0) = 1 \text{ m}$, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$. The origin of the coordinates is the center of the sphere and the $z = 0$ plane is its equatorial plane. Note that, according to Newtonian gravity, $t(z = 0) = 0.4 \text{ s}$.

placed around the disk in order to initiate the current inside it and to rotate the disk around its central axis. Material bodies placed over the rotating disk initially demonstrated a weight loss of 0.3-0.5%. Moreover, the air over the cryostat in which the apparatus was enclosed began to rise slowly toward the ceiling. Particles of dust and smoke in the air made the effect clearly visible. Interestingly, the boundaries of the flow could be seen clearly and corresponded exactly to the shape of the toroid. When the angular velocity of the disk was slowly reduced from 5,000 rpm to 3,500 rpm by changing the current in the solenoids, the shielding effect became considerably higher and reached 1.9-2.1% at maximum. Moreover, the effective weight loss turned out to be independent of the height of the suspended bodies over the disk. Finally, the shielding effect was present even in absence of rotation ranging from 0.05% to 0.07%.

It is evident that the phenomenology described in [10] cannot be accounted for by the general relativistic phenomena considered here.

It could be interesting to mention that a sort of genuine antigravitational effect does exist in General Relativity. It is related to the behavior of a test particle which moves along the
Figure 5: Spatial trajectory followed by a point mass freely falling in the Earth’s gravitational field and in the local gravitational field of a spinning sphere with $\omega = 10^2$ rad $s^{-1}$, $R = 2 \times 10^{-1}$ m. The initial conditions are $x(0) = y(0) = 5 \times 10^{-1}$ m, $z(0) = 1$ m, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$. The origin of the coordinates is the center of the sphere and the $z = 0$ plane is its equatorial plane.

rotational axis of a naked singularity in the Kerr metric: it is the so called rotational paradox \[11, 12, 13\].

4 Conclusions

In this paper we have derived some features of the motion of test particles in a spacetime metric represented by an approximated solution of the Einstein equations for a weakly gravitating spinning object in which the mass effects are negligible with respect to the rotational effects. These conditions are well satisfied at laboratory scale. Since it is doubtful that such metric could really describe the gravitational field of a material body, such calculations could also be viewed as an attempt to shed light on its validity by looking for possible strange or non existent observable consequences.
Then, we have derived the geodesic equations of motion for a massive test particle. Some rather puzzling features have been found in the allowed geodesic motions. For example, neither circular motions in the equatorial plane of the source nor spherical motions would be permitted. We have numerically solved the geodesic equations for a pair of particular choices of initial conditions representing, in classical Newtonian mechanics, a vertical free-fall motion in a possible laboratory scenario. For a realistic choice of the mechanical parameters of the sphere the investigated effects would be quite measurable. We have assumed $C_0 = \frac{50}{3} \pi$. We have found that the rotation of the central source would induce a sort of antigravity effect with an increment of the time required to reach the ground and also a deviation of the trajectory from the local vertical. We have also noted that the weight of a massive body turns out to be reduced by the acceleration considered here. The crucial point is that for a realistic and rather common choice of the mechanical parameters of the central rotating sphere all such effects seem to be very large in magnitude: nothing similar to them has ever been observed.

As a conclusion, the obtained results in this paper might be considered as a further insight against any real physical significance of the approximated metric considered here.

**Appendix: The second order powers of the rotation in the PPN equations of motion**

The equations of motion for a particle orbiting a finite sized spherical extended body with constant density in the PPN formalism can be found in [14]. From them the contribution of the square of the angular velocity of the central body to the particle’s acceleration can be extracted [15]: it consists of two radial terms and a third term directed along the spin axis of the rotating mass. For a spherical rotating body with the $z$ axis directed along its spin axis General Relativity yields

\[
\ddot{x} = \frac{G}{c^2} M R^4 \omega^2 \left[ \frac{6 x z^2}{7(x^2 + y^2 + z^2)^{7/2}} - \frac{6 x}{35(x^2 + y^2 + z^2)^{3/2}} - \frac{3 x}{5 R^2(x^2 + y^2 + z^2)^{5/2}} \right],
\]

\[
\ddot{y} = \frac{G}{c^2} M R^4 \omega^2 \left[ \frac{6 y z^2}{7(x^2 + y^2 + z^2)^{7/2}} - \frac{6 y}{35(x^2 + y^2 + z^2)^{3/2}} - \frac{3 y}{5 R^2(x^2 + y^2 + z^2)^{5/2}} \right],
\]

\[
\ddot{z} = \frac{G}{c^2} M R^4 \omega^2 \left[ \frac{6 z^3}{7(x^2 + y^2 + z^2)^{7/2}} - \frac{18 z}{35(x^2 + y^2 + z^2)^{3/2}} - \frac{3 z}{5 R^2(x^2 + y^2 + z^2)^{5/2}} \right].
\]
The second order powers of the rotation have here been interpreted as arising from the rotational energy of the central body. These accelerations come from the potentials evaluated for the $g_{00}$ portion of the metric and as such originate from the curvature of the spacetime. They are, therefore, related to the energy and do not arise from a second order power of the angular momentum of the body.

Note that in eqs. (69)-(70) the factor $\frac{G_\odot}{c^2} = 7.42 \times 10^{-28} \text{ m kg}^{-1}$ is present, so that their effects at laboratory scale are completely negligible.

**Acknowledgements**

I am grateful to L. Guerriero for his support while at Bari. Special thanks to B. Mashhoon for useful and important discussions and clarifications and to F. de Felice for the explanations of the rotational paradox.

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