Strong Converse Bounds for Compression of Mixed States

Zahra Baghali Khanian
Munich Center for Quantum Science and Technology & Zentrum Mathematik, Technical University of Munich, Germany

We consider many copies of a general mixed-state source $\rho^{AR}$ shared between an encoder and an inaccessible reference system $R$. We obtain a strong converse bound for the compression of this source. This immediately implies a strong converse for the blind compression of ensembles of mixed states since this is a special case of the general mixed-state source $\rho^{AR}$. Moreover, we consider the visible compression of ensembles of mixed states. For a bipartite state $\rho^{AR}$, we define a new quantity $E_{\alpha,p}(A : R)$ as the $\alpha$-Rényi generalization of the entanglement of purification $E_p(A : R)$. For $\alpha = 1$, we define $E_{1,p}(A : R) := E_p(A : R)$. We show that for any rate below the regularization $\lim_{\alpha \to 1^+} E_{\alpha,p}(A : R) := \lim_{\alpha \to 1^+} \lim_{n \to \infty} E_{\alpha,p}(A_n : R_n) \equiv 0$, the fidelity for the visible compression of ensembles of mixed states exponentially converges to zero. We conclude that if this regularized quantity is continuous with respect to $\alpha$, namely, $\lim_{\alpha \to 1^+} E_{\alpha,p}(A : R) = E_p(A : R)$, then the strong converse holds for the visible compression of ensembles of mixed states.

I. INTRODUCTION AND COMPRESSION MODEL

Quantum source compression was pioneered by Schumacher in [1] where he provided two definitions for a quantum source: 1. a quantum system together with correlations with a purifying reference system, and 2. an ensemble of pure states. For both models, he showed that the optimal compression rate is equal to the von Neumann entropy of the quantum system. The compression of an ensemble source is respectively called visible or blind compression if the identity of the state is known to the encoder or not. For ensemble of pure states, both blind and visible compression models lead to the same optimal compression rate [1, 2]. Horodecki generalized the ensemble model of Schumacher to mixed states and found a multi-letter optimal rate considering the visible scheme [3, 4]. Later, Hayashi showed that this optimal rate is equal to the regularized entanglement of the purification of the source [5]. Contrary to ensemble of pure states, the blind and visible compression of the ensemble of mixed states do not lead to the same compression rate. In the blind model, the optimal rate is characterized by a single-letter rate formula in terms of a decomposition of the ensemble, called the Koashi-Imoto decomposition [6, 7].

Recently, a general mixed-state source is considered in [8–10] where a quantum source is defined as a quantum system together with correlations with a general quantum reference system. It is shown that the models defined by Schumacher as well as the ensemble of mixed states (blind scheme) are special cases of this general mixed-state model. The optimal compression rate of this source is characterized by a single-letter entropic quantity in terms of a decomposition of the source which is a generalization of the Koashi-Imoto decomposition [11].

Proving the optimality of the compression rates requires establishing converse bounds. In so-called weak converse proofs, the compression rate is lower bounded in the limit of the error converging to zero. However, weak converses leave open whether there is a trade-off between the compression rate and the error, that is if a smaller compression rate can be achieved at the expense of having a larger error. A strong converse, on the
other hand, states that such a trade-off does not hold in the asymptotic limit of many copies of the source, that is for any rate below the optimal rate the error converges to 1. For ensembles of pure states assuming that the decoder is a unitary, a strong converse was proved in [2]. The assumption of the unitary decoder was lifted in [12] where a strong converse bound is obtained using trace inequalities. However, despite all the progress in establishing strong converse bounds in various information-theoretic tasks, which deal with processing ensembles of pure states or quantum systems with purifying reference systems e.g. [12, 13], it was an open problem so far whether the strong converse holds for the compression of mixed states.

In this paper, we first prove a strong converse for compression of a general mixed-state source as defined in [8–10]. This immediately implies a strong converse for the blind compression of the ensemble of mixed states since this ensemble source is a special case of the general mixed-state source where the reference system has a classical structure. To deal with the visible compression case, we define a new quantity $E_{\alpha,p}(A : R)_{\rho}$ in Definition 7 as an $\alpha$-Rényi generalization of the entanglement of purification and obtain a bound on the fidelity with respect to the regularization of this quantity i.e. $E_{\alpha,p}^{\infty}(A : R)_{\rho}$. We conclude that continuity of $E_{\alpha,p}^{\infty}(A : R)_{\rho}$ at $\alpha = 1$ in the limit of $\alpha \to 1^{+}$ implies that the strong converse holds for the visible compression of mixed states. It remains an open question if $\lim_{\alpha \to 1^{+}} E_{\alpha,p}^{\infty}(A : R)_{\rho} = E_{p}^{\infty}(A : R)_{\rho}$ holds true.

The organization of the paper is as follows. We introduce some notation and convention at the end of this section. In Section II we define the asymptotic compression task which unifies the visible and blind model. In Section III and Section IV, we address respectively the blind and visible schemes. Finally, we discuss our results in Section V.

Notation. In this paper, quantum systems are associated with finite dimensional Hilbert spaces $A$, $R$, etc., whose dimensions are denoted by $|A|$, $|R|$, respectively. Since it is clear from the context, we slightly abuse the notation and let $Q$ denote both a quantum system and a quantum rate. The von Neumann entropy and the $\alpha$-Rényi entropy for $\alpha \in (0, 1) \cup (1, \infty)$ are defined respectively as

\[
S(\rho) := -\text{Tr} \rho \log \rho \\
S_{\alpha}(\rho) := \frac{1}{1-\alpha} \log \text{Tr} \rho^{\alpha}.
\]

Throughout this paper, $\log$ denotes by default the binary logarithm.

The fidelity between two states $\rho$ and $\xi$ is defined as $F(\rho, \xi) = \|\sqrt{\rho} \sqrt{\xi}\|_1 = \text{Tr} \sqrt{\rho^{1/2} \xi \rho^{1/2}}$, with the trace norm $\|X\|_1 = \text{Tr} |X| = \text{Tr} \sqrt{X^\dagger X}$.

II. COMPRESSION MODEL

An ensemble source can be defined as a classical-quantum state where preserving the fidelity with the classical reference system $R$ is equivalent to preserving the fidelity by taking the average over the states of the ensemble, as shown in Eq. (1). Therefore, we consider the following classical-quantum state as the source

\[
\rho^{ACR} := \sum_{x} p(x) \rho^A_x \otimes |c_x\rangle\langle c_x|^C \otimes |x\rangle\langle x|^R,
\]

where $\{|x\rangle^R\}_x$ is an orthonormal basis for system $R$. We let systems $A$, $C$ and $R$ denote the system to be compressed, the side information of the encoder and the reference system, respectively. The compression is called blind or visible if $c_x = 0$ for all $x$ or $c_x = x$ for all $x$, respectively. Hence, depending on the side
information system this model includes both blind and visible schemes. We will consider the information theoretic limit of many copies of the source $\rho_{ACR}$, i.e., $\rho^{AC^nR^n} = (\rho_{ACR})^\otimes n = \sum_{x^n} p(x^n)\rho^n_{x^n} \otimes |x^n\rangle\langle x^n|^{R^n}$. The encoder, Alice, performs the encoding compression operation $\mathcal{C} : A^nC^n \rightarrow M$ on the systems $A^nC^n$ which is a quantum channel, i.e. a completely positive and trace preserving (CPTP) map. Notice that as functions CPTP maps act on the operators (density matrices) over the respective input and output Hilbert spaces, but as there is no risk of confusion, we will simply write the Hilbert spaces when denoting a CPTP map. Alice’s encoding operation produces the state $\sigma^{MC^n} = \sum_{x^n} p(x^n)\sigma^n_{x^n} \otimes |x^n\rangle\langle x^n|^{C^n}$ with $M$ as the compressed system of Alice. The system $M$ is then sent to Bob via a noiseless quantum channel, who performs a decoding operation $\mathcal{D} : M \rightarrow \hat{A}^n$ on the compressed system $M$. The output system $\hat{A}^n$ is the reconstruction of the system $A^n$. Bob’s decoding operation produces the state $\xi^{\hat{A}^nR^n} = \sum_{x^n} p(x^n)\xi^{\hat{A}^n}_{x^n} \otimes |x^n\rangle\langle x^n|^{R^n}$ with system $\hat{A}^n$ as the reconstruction of system $A^n$. We say the encoding-decoding scheme has fidelity $1 - \epsilon$, or error $\epsilon$, if

$$\mathcal{F} = \sum_{x^n} p(x^n) \mathcal{F} (\rho^n_{x^n},\xi^n_{x^n}) = \mathcal{F} (\rho^{AN^nR^n},\xi^{\hat{A}nR^n}) \geq 1 - \epsilon,$$

where the second equality is due to the definition of the fidelity. In section III where the side information system $C$ is trivial, $\mathcal{F}_b$ denote the fidelity of the blind scheme. Similarly, in section IV where we assume $C = R$, $\mathcal{F}_v$ denotes the fidelity of the visible scheme. For the above compression task, $\frac{1}{n} \log |M|$ is called the compression rate. Moreover, a rate $Q$ is called an (asymptotically) achievable rate if for all $n$ there is a sequence of encoders and decoders such that the fidelity converges to 1 and the compression rate converges to $Q$. The optimal rate is the infimum of all achievable rates.

According to Stinespring’s theorem [14], a CPTP map $\mathcal{T} : A \rightarrow \hat{A}$ can be dilated to an isometry $U : A \leftrightarrow \hat{A}E$ with $E$ as an environment system, called an isometric extension of a CPTP map, such that $\mathcal{T}(\rho_A) = \text{Tr}_E (U\rho_A U^\dagger)$. Therefore, the encoding and decoding operations can in general be viewed as isometries $U_C : A^n \leftrightarrow MW$ and $U_D : M \leftrightarrow \hat{A}^nV$, respectively, with the systems $W$ and $V$ as the environment systems of Alice and Bob, respectively.

### III. BLIND COMPRESSION CASE

In this section, we prove a strong converse bound for the compression of a general mixed state source $\rho^{AR}$ where $A$ is the system to be compressed, and $R$ is the reference system. The ensemble source $\{p(x),\rho^n_x\}$ considered by Koashi and Imoto in [6, 7] is a special case of $\rho^{AR}$ where the reference is a classical system as follows:

$$\rho^{AR} = \sum_x p(x)\rho^n_x \otimes |x\rangle\langle x|^{R^n},$$

and $\{|x\rangle^{R^n}\}_x$ is the orthonormal basis for the classical system $R$. To be more specific about the role of the reference system, let the CPTP map $\mathcal{N} : A \rightarrow \hat{A}$ denote the combined action of the encoder and the decoder. Then, preserving the average fidelity, as considered by Koashi and Imoto, is equivalent to preserving the classical-quantum state $\rho^{AR}$ in Eq. (2):

$$\sum_x p(x)\mathcal{F}(\rho^n_x,\mathcal{N}(\rho^n_x)) = \mathcal{F}(\rho^{AR},(\mathcal{N} \otimes \text{id}_R)\rho^{AR}).$$
In this section, we consider a general mixed-state source $\rho^{AR}$, which is not necessarily a classical-quantum state as in Eq. (2), and prove a strong converse bound for it. This immediately implies a strong converse for the source defined in Eq. (2).

We consider the same encoding-decoding model as in Section II. The encoder performs the encoding isometry $V_C: A^n \mapsto MW$ to obtain the compressed system $M$ and an environment system $W$:

$$(V_C \otimes \mathbb{1}_{R^n})(\rho^{AR})^\otimes n (V_C \otimes \mathbb{1}_{R^n})^\dagger =: \sigma^{MW^n R^n}.$$ 

Receiving $M$, the decoder performs the decoding isometry $V_D: M \mapsto \hat{A}^n V$ to obtain the reconstructed system $\hat{A}^n$ and the environment system $V$:

$$(V_D \otimes \mathbb{1}_{R^n})\sigma^{MW^n R^n} (V_D \otimes \mathbb{1}_{R^n})^\dagger =: \xi^{\hat{A}^n W V^n R^n}.$$ 

The fidelity for this blind scheme is defined as follows:

$$F_b = F(\rho^{A^n R^n}, \xi^{\hat{A}^n R^n}),$$  

(3)

where $\xi^{\hat{A}^n R^n} = \text{Tr}_{V W} \xi^{\hat{A}^n W V^n R^n}$.

As shown in [8–10], the optimal compression rate of a general mixed-state source $\rho^{AR}$ is equal to $S(CQ)_{\omega}$, i.e the von Neumann entropy of the classical and quantum systems in a decomposition of this state introduced in [11], which is a generalization of the decomposition introduced by Koashi and Imoto in [7]. Namely, for any set of quantum states $\{\rho^A_j\}$, there is a unique decomposition of the Hilbert space describing the structure of CPTP maps which preserve the set $\{\rho^A_j\}$. This idea was generalized in [11] for a general mixed state $\rho^{AR}$ describing the structure of CPTP maps acting on system $A$ which preserve the overall state $\rho^{AR}$. This was achieved by showing that any such map preserves the set of all possible states on system $A$ which can be obtained by measuring system $R$, and conversely any map preserving the set of all possible states on system $A$ obtained by measuring system $R$, preserves the state $\rho^{AR}$, thus reducing the general case to the case of classical-quantum states

$$\rho^{AY} = \sum_y q(y)\rho^A_y \otimes |y\rangle \langle y|^Y = \sum_y \text{Tr}_R \rho^{AR}(\mathbb{1}_A \otimes M^R_y) \otimes |y\rangle \langle y|^Y,$$

which is the ensemble case considered by Koashi and Imoto. The properties of this decomposition are stated in the following theorem.

**Theorem 1 ([7, 11]).** Associated to the state $\rho^{AR}$, there are Hilbert spaces $C$, $N$ and $Q$ and an isometry $U_{KI}: A \mapsto CNQ$ such that:

(i) The state $\rho^{AR}$ is transformed by $U_{KI}$ as

$$(U_{KI} \otimes \mathbb{1}_R)\rho^{AR}(U_{KI}^\dagger \otimes \mathbb{1}_R) = \sum_j p_j |j\rangle \langle j|^C \otimes \omega^N_j \otimes \rho_{QR}^j =: \omega^{CNQR},$$  

(4)

where the set of vectors $\{|j\rangle^C\}$ form an orthonormal basis for Hilbert space $C$, and $p_j$ is a probability distribution over $j$. The states $\omega^N_j$ and $\rho_{QR}^j$ act on the Hilbert spaces $N$ and $Q \otimes R$, respectively.

(ii) For any CPTP map $\Lambda$ acting on system $A$ which leaves the state $\rho^{AR}$ invariant, that is $(\Lambda \otimes \mathbb{1}_R)\rho^{AR} =$
\( \rho^{AR} \), every associated isometric extension \( U : A \hookrightarrow AE \) of \( \Lambda \) with the environment system \( E \) is of the following form

\[
U = (U_{KI} \otimes I_E)^\dagger \left( \sum_j |j\rangle \langle j| \otimes U_j^N \otimes I_j^Q \right) U_{KI},
\]

where the isometries \( U_j : N \hookrightarrow NE \) satisfy \( \text{Tr}_E[U_j \omega_j U_j^\dagger] = \omega_j \) for all \( j \). The isometry \( U_{KI} \) is unique (up to a trivial change of basis of the Hilbert spaces \( C, N \) and \( Q \)). Henceforth, we call the isometry \( U_{KI} \) and the state \( \omega_{CNQR} \) defined by Eq. (4) the Koashi-Imoto (KI) isometry and KI-decomposition of the state \( \rho^{AR} \), respectively.

(iii) In the particular case of a tripartite system \( CNQ \) and a state \( \omega_{CNQR} \) already in Koashi-Imoto form (4), property 2 says the following: For any CPTP map \( \Lambda \) acting on systems \( CNQ \) with \( (\Lambda \otimes \text{id}_R)\omega_{CNQR} = \omega_{CNQR} \), every associated isometric extension \( U : CNQ \hookrightarrow CNQE \) of \( \Lambda \) with the environment system \( E \) is of the form

\[
U = \sum_j |j\rangle \langle j| \otimes U_j^N \otimes I_j^Q,
\]

where the isometries \( U_j : N \hookrightarrow NE \) satisfy \( \text{Tr}_E[U_j \omega_j U_j^\dagger] = \omega_j \) for all \( j \).

To prove a strong converse bound, we use the following lemma where the fidelity between two states is upper bounded by the Rényi entropies of the corresponding states. This lemma, which was initially proved in [15], was later generalized in [13] to bound the fidelity by conditional Rényi entropies.

**Lemma 2** ([15]). For \( \beta \in \left( \frac{1}{2}, 1 \right) \) and any two states \( \rho \) and \( \sigma \) on a finite-dimensional quantum system the following holds:

\[
S_\beta(\rho) \geq S_\alpha(\sigma) + \frac{2\beta}{1-\beta} F(\rho, \sigma),
\]

where \( \alpha = \frac{\beta}{2\beta - 1} \).

In the following theorem, we obtain a bound on the fidelity \( F_b \) which is with respect to the Rényi entropy. Below, in Proposition 5 we state the strong converse bound.

**Theorem 3.** For any \( \alpha > 1 \), the fidelity for the blind compression of mixed states with the rate \( Q_b \) is bounded as:

\[
F_b \leq 2^{-n \frac{\alpha}{\alpha-1}(S_\alpha(CQ) - Q_b)}.
\]

**Proof.** Consider the following state which is obtained by applying the Koashi-Imoto isometry on the decoded system \( \hat{A}_n \):

\[
\zeta^{CNQ^{\omega_WVR^n}} := (U_{KI}^{\otimes n} \otimes I_R^n)\zeta^{\hat{A}_n^{WVR^n}} (U_{KI}^{\otimes n} \otimes I_R^n)^\dagger.
\]

Recall that the Koashi-Imoto decomposition of \( \rho^{\otimes n} \) is equal to the tensor product of Koashi-Imoto states...
where the second line follows because the decoding isometry $\Gamma : \Omega \to \hat{A}^n V$ does not change the Rényi entropy. The third line is obtained by applying the Koashi-I moto isometry $\hat{\zeta}$ on system $\nu$. By rearranging the terms in the above inequality we obtain the second inequality and the first inequality is due to the monotonicity of the fidelity under partial trace on system $\nu$.

In order to prove the desired bound, we assume that the state $\tau^{C^n N^n Q^n R^n}$ is the state that minimizes the entropy $S(\tau^{C^n N^n Q^n R^n})$ in the minimization. The last line follows from Lemma 2 for $\alpha = \frac{\beta}{1 + \beta}$. We note that this lemma holds for any two states $M$ and $\tau$, and the fidelity between these two states appear in the lower bound. In order to prove the desired bound, we assume that the state $\tau^{C^n N^n Q^n R^n} := (\Gamma \otimes \text{id}_{R^n}) \omega^{C^n N^n Q^n R^n}$ where $\Gamma : C^n N^n Q^n \to C^n N^n Q^n E$ is a CPTP map with the following properties:

1. $\text{Tr}_E \tau^{C^n N^n Q^n R^n} = \omega^{C^n N^n Q^n R^n}$
2. $\tau^{C^n N^n Q^n R^n} = \sum_{j^n} p_j^n |j^n\rangle \langle j^n| \otimes |\omega_{j^n}\rangle \langle \omega_{j^n}| N^n E \otimes \rho^n_{j^n} R^n$.

By Theorem 1, the first assumption implies that $\Gamma$ only acts on system $N^n$. The second assumption implies that given the classical system $C^n$ the state on systems $N^n E$ is pure. Due to these assumptions, Lemma 14 implies the following:

$$S_\alpha(C^n N^n Q^n E_\tau) = S_\alpha(C^n Q^n)_\tau = n S_\alpha(CQ). \tag{10}$$

Hence, from Eq. (9) and Eq. (10) we obtain

$$n Q_b \geq n S_\alpha(CQ)_\omega + \frac{2 \beta}{1 - \beta} \log F(\tau^{C^n N^n Q^n E}, \nu^{C^n \hat{N}^n \hat{Q}^n E^n}).$$

By rearranging the terms in the above inequality we obtain the second inequality

$$F(\tau^{C^n N^n Q^n E^n}, \nu^{C^n \hat{N}^n \hat{Q}^n E^n}) \leq F(\tau^{C^n N^n Q^n E^n}, \nu^{C^n \hat{N}^n \hat{Q}^n E^n}) \leq 2^{-n \frac{1}{\min(S_\alpha(CQ) - Q_b)}} \tag{11}$$

and the first inequality is due to the monotonicity of the fidelity under partial trace on system $R^n$.

In the above bound, we note that $\nu^{C^n \hat{N}^n \hat{Q}^n E^n}$ is the state that minimizes the entropy $S(\tau^{C^n N^n Q^n R^n})$ in Eq. (8). The bound in Eq. (11) holds for any state $\nu^{C^n \hat{N}^n \hat{Q}^n E^n}$ which is obtained by applying a CPTP map on systems
Lemma 4. The dimension of system $E$ in Eq. (8) is bounded as $|E| \leq (|C| \cdot |N| \cdot |Q|)^n$.

Proof. The Rényi entropy for $\alpha \in (0, 1)$ is a concave function of quantum states [16]. Therefore, the minimum in Eq. (8) can be attained by extremal CPTP maps which are maps that cannot be expressed as convex combination of other CPTP maps. Moreover, extremal CPTP maps with input dimension $d$ have at most $d$ operators in Kraus representation [17]. Therefore, the dimension of the output systems in Eq. (8) is bounded by $(|C| \cdot |N| \cdot |Q|)^n$. This implies that the dimension of system $E$ is bounded by $(|C| \cdot |N| \cdot |Q|)^n$.

Proposition 5. For any rate $Q_b < S(CQ)$, there is a $K > 0$ such that the following strong converse holds for the blind compression of mixed states:

$$ F_b \leq 2^{-nK}. $$

Proof. The Rényi entropy $S_\alpha(CQ)$ is continuous and decreasing in $\alpha > 1$ and $\lim_{\alpha \to 1^+} S_\alpha(CQ) = S(CQ)$ [18]. Therefore, for any $Q_b < S(CQ)$, there is a $\alpha > 1$ such that $Q_b < S_\alpha(CQ) < S(CQ)$. Hence, the claim follows from Theorem 3 by letting $K := \frac{a-1}{2\alpha}(S_\alpha(CQ) - Q_b)$. ■

IV. VISIBLE COMPRESSION CASE

In this section, we consider the visible compression case. The encoding-decoding model is defined in Section II, and we assume that the states of the side information system are $|c_x^C x^C = |x^C$, that is the encoder have access to the identity of the states in the ensemble. The optimal compression rate for this source is equal to the regularized entanglement of purification $E^\infty_p(A : R) := \lim_{n \to \infty} E_p(A^n : R^n)_{\rho A^n}^{\otimes n}$ of the source [5], where $E_p(\cdot, \cdot)$ is defined as follows:

Definition 6 ([19]). The entanglement of purification of a bipartite state $\rho^{AR}$ is defined as

$$ E_p(A : R)_{\rho} := \inf_{N : B \to E} S(AE)_{\sigma}, $$

where the infimum is over all CPTP maps $N$ such that $\sigma^{AE} = (\text{id}_A \otimes N)\rho^{AB}$ and $\rho^{AB} = \text{Tr}_R |\psi(\psi)^{ABR}$ for a purification $|\psi)^{ABR}$ of $\rho^{AR}$.

In the above definition, there is dimension bound on system $E$, hence, the infimum is attainable [19]. We define an $\alpha$-Rényi generalization of the entanglement of purification. We use this quantity to obtain a strong converse bound.

Definition 7. For $\alpha \in (0, 1) \cup (1, \infty)$, the $\alpha$-Rényi entanglement of purification of a bipartite state $\rho^{AR}$ is defined as

$$ E_{\alpha,p}(A : R)_{\rho} := \inf_{N : B \to E} S_\alpha(AE)_{\sigma}, $$
where the infimum is over all CPTP maps $N$ such that $\sigma^{AE} = (\text{id}_A \otimes N)\rho^{AB}$ and $\rho^{AB} = \text{Tr}_R |\psi\rangle\langle\psi|^{ABR}$ for a purification $|\psi\rangle^{ABR}$ of $\rho^{AR}$. For $\alpha = 1$, define $E_{1,p}(A : R)_\rho := E_p(A : R)_\rho$.

The regularized $\alpha$-Rényi entanglement of purification is defined as:

$$E_{\infty, p}^{\alpha}(A : R)_\rho := \lim_{n \to \infty} \frac{E_{\alpha,p}(A^n : R^n)_{\rho^{\otimes n}}}{n},$$

where the right hand side is evaluated with respect to the state $(\rho^{AR})^{\otimes n}$.

**Lemma 8.** The dimension of the system $E$ is Definition 7 is bounded by $|A|^2 \cdot |R|^2$.

**Proof.** For $\alpha > 1$ we have

$$E_{\alpha,p}(A : R)_\rho = \inf_{N : B \to E} S_\alpha(\sigma^{AE})_\sigma,$$

$$= \frac{1}{\alpha - 1} \inf_{N : B \to E} - \log \text{Tr} (\sigma^{AE})^\alpha$$

$$= -\frac{1}{\alpha - 1} \log \sup_{N : B \to E} \text{Tr} (\sigma^{AE})^\alpha$$

the last equality is due to the fact that $-\log(\cdot)$ is a decreasing function. The map $\rho \mapsto \text{Tr} \rho^\alpha$ is convex [18, 20]. Hence, the supremum in the last line can be attained by extremal CPTP maps which are maps that cannot be expressed as convex combination of other CPTP maps. Moreover, extremal CPTP maps with input dimension $d$ have at most $d$ operators in Kraus representation [17]. Therefore, the dimension of the output system $E$ is bounded by $|B|^2 = |A|^2 \cdot |R|^2$.

For $\alpha \in (0, 1)$, the Rényi entropy $S_\alpha(\sigma^{AE})_\sigma$ is a concave function of quantum states, hence, the infimum in the definition of $E_{\alpha,p}(A : R)_\rho$ can be attained by extremal CPTP maps. As discussed above, the dimension of the output system $E$ is bounded by $|B|^2 = |A|^2 \cdot |R|^2$.

**Corollary 9.** The infimum in the definition of the $\alpha$-Rényi entanglement of purification is attainable.

In the following lemma, we state some useful properties of the $\alpha$-Rényi entanglement which we apply in the subsequent statements. Further properties of this quantity is discussed in [21].

**Lemma 10.** The $\alpha$-Rényi entanglement of purification $E_{\alpha,p}(A : R)_\rho$ and its regularization $E_{\infty, \alpha,p}(A : R)_\rho$ have the following properties:

(i) They are both decreasing with respect to $\alpha$ for $\alpha > 0$.

(ii) $E_{\alpha,p}(A : R)_\rho$ is continuous with respect to $\alpha$ at any $\alpha \geq 1$.

(iii) $E_{\infty, \alpha,p}(A : R)_\rho$ is continuous with respect to $\alpha$ at any $\alpha > 1$.

**Proof.** (i) The Rényi entropy $S_\alpha(\sigma^{AE})_\sigma$ is decreasing with $\alpha$, hence, for $\alpha' > \alpha$ we obtain

$$E_{\alpha,p}(A : R)_\rho = \min_{N : B \to E} S_\alpha(\sigma^{AE})_\sigma$$

$$\geq S_{\alpha'}(\sigma^{AE})_\sigma$$

$$\geq \min_{N : B \to E} S_{\alpha'}(\sigma^{AE})_\sigma$$

$$= E_{\alpha',p}(A : R)_\rho.$$
This implies that $E_{\alpha,p}^\infty (A : R)_\rho$ is decreasing with $\alpha$ as well.

(ii) For a bipartite state $\rho^{AR}$ define $Z_\alpha(A : R)_\rho := \max_{N:B \rightarrow E} \log \text{Tr} (\sigma^{AE})^\alpha$ where $\sigma^{AE} = (\text{id}_A \otimes N) \rho^{AB}$ and $\rho^{AB} = \text{Tr}_R |\psi\rangle \langle \psi|^{ABR}$ for purification $|\psi\rangle^{ABR}$ of $\rho^{AR}$. We obtain

$$E_{\alpha,p}(A : B)_\rho = \min_{N:R \rightarrow E} S_\alpha(\sigma^{AE})_\sigma$$
$$= \min_{N:R \rightarrow E} \frac{\log \text{Tr} (\sigma^{AE})^\alpha}{1 - \alpha}$$
$$= \frac{1}{1 - \alpha} \max_{N:R \rightarrow E} \log \text{Tr} (\sigma^{AE})^\alpha$$
$$= \frac{1}{1 - \alpha} Z_\alpha(A : R)_\rho,$$

where the third follows for $\alpha > 1$. In the following we show that $Z_\alpha(A : R)_\rho$ is a continuous function of $\alpha$ at any $\alpha > 1$. It follows that $E_{\alpha,p}(A : B)_\rho$ is a continuous function of $\alpha$ for $\alpha > 1$ because $\frac{1}{1 - \alpha}$ is continuous. To this end, we first show that $\alpha \mapsto Z_\alpha(A : R)_\rho$ is convex. Let $\alpha = p\alpha_1 + (1 - p)\alpha_2$ for $p \in [0, 1]$. We obtain

$$Z_\alpha(A : R)_\rho = \log \text{Tr} (\sigma^{AE})^\alpha$$
$$= \log \text{Tr} (\sigma^{AE})^{p\alpha_1 + (1 - p)\alpha_2}$$
$$\leq p \log \text{Tr} (\sigma^{AE})^{\alpha_1} + (1 - p) \log \text{Tr} (\sigma^{AE})^{\alpha_2}$$
$$\leq p \max_{N:B \rightarrow E} \log \text{Tr} (\sigma^{AE})^{\alpha_1} + (1 - p) \max_{N:B \rightarrow E} \log \text{Tr} (\sigma^{AE})^{\alpha_2}$$
$$= pZ_{\alpha_1}(A : R)_\rho + (1 - p)Z_{\alpha_2}(A : R)_\rho,$$

where in the first line the state $\sigma^{AE}$ is the optimal state, and the third line follows from Lemma 15, i.e. $\alpha \mapsto \log \text{Tr} (\sigma)^\alpha$ is convex. Furthermore, note that $Z_\alpha(A : R)_\rho$ is decreasing with $\alpha$, that is for $\alpha_1 < \alpha_2$ we obtain:

$$Z_{\alpha_2}(A : R)_\rho = \log \text{Tr} (\sigma^{AE})^{\alpha_2}$$
$$\leq \log \text{Tr} (\sigma^{AE})^{\alpha_1}$$
$$\leq \max_{N:B \rightarrow E} \log \text{Tr} (\sigma^{AE})^{\alpha_1}$$
$$= Z_{\alpha_1}(A : R)_\rho,$$

where in the first line $\sigma^{AE}$ is the optimal state, and the second line follows from Lemma 15, namely $\log \text{Tr} (\sigma)^\alpha$ is decreasing in $\alpha$. Hence, $Z_{\alpha_2}(A : R)_\rho$ is continuous at any $\alpha > 1$.

To show the continuity for $\alpha \rightarrow 1^+$, note that the Rényi entropy is continuous with $\alpha$, i.e. $\lim_{\alpha \rightarrow 1^+} S_\alpha(\rho) = S(\rho)$, as well as the quantum states. Also the minimization is over a compact set, therefore, $\lim_{\alpha \rightarrow 1^+} E_{\alpha,p}(A : R)_\rho = E_p(A : R)_\rho$.

(iii) In point (ii), we show that the function $\alpha \mapsto Z_\alpha(A : R)_\rho$ is decreasing and convex, therefore, its regularized version defined in Eq. (12) is decreasing and convex therefore continuous at any $\alpha > 1$

$$Z_\alpha^\infty(A : R)_\rho := \lim_{n \rightarrow \infty} Z_\alpha(A^n : R^n)^\rho_n = E_{\alpha,p}(A : R)_\rho.$$

This implies that $E_{\alpha,p}(A : R)_\rho = \frac{1}{1 - \alpha} Z_\alpha^\infty(A : R)_\rho$ is continuous for $\alpha > 1$. 

$\blacksquare$
Theorem 11. For any $\alpha > 1$, the fidelity for the visible compression of mixed states with the rate $Q_v$ is bounded as

$$F_v \leq 2^{-n \frac{\alpha}{2\alpha - 1} (S_{\alpha}^n(A^v) - Q_v)}.$$

Proof. We bound the compression rate for $\beta \in (\frac{1}{2}, 1)$ as follows:

$$nQ_v \geq S_{\beta}(M)$$

$$= S_{\beta}(A^n V) \xi$$

$$\geq S_\alpha(A^n V) + \frac{2\alpha}{1-\alpha} \log F(\tau A^n V, \xi A^n V)$$

$$\geq S_\alpha(A^n V) + \frac{2\alpha}{1-\alpha} \log F_v$$

(13)

where the equality follows because the decoding isometry does not change the Rényi entropy. The second inequality follows from Lemma 2 for $\alpha = \frac{2\beta}{2\beta - 1}$. This lemma holds for arbitrary states $\tau$ and $\xi$. Here we let $\tau A^n V = \sum_x p(x^n) \tau_x A^n V$ where for all $x^n$ the state $\tau_x A^n V$ is any extension of the state $\rho_x A^n$ such that $F(\tau A^n V, \xi A^n V) \geq F_v$. We can construct $\tau_x A^n V$ as follows: let $|\xi_x A^n V\rangle$ and $|\psi_x A^n V\rangle$ be purifications of $\xi A^n$ and $\rho_x A^n$, respectively. By Uhlmann’s theorem, for any $x^n$ there is a unitary $U_{x^n} : V V' \rightarrow V' V'$ such that

$$F(\rho_x A^n, \xi_x A^n) = F((\mathbb{I}_{A^n} \otimes U_{x^n})|\psi_x A^n V V'\rangle (\mathbb{I}_{A^n} \otimes U_{x^n})\dagger, \xi_x A^n V V') \leq F(\tau_x A^n V, \xi_x A^n V),$$

where the inequality follows from the monotonicity of the fidelity under partial trace on system $V'$. By taking the average on the both sides of the above inequality, we obtain

$$F_v = \frac{1}{2^n} \sum_x p(x^n) F(\rho_x A^n, \xi_x A^n)$$

$$\leq \sum_x p(x^n) F(\tau_x A^n V, \xi_x A^n V).$$

$$\leq F(\tau A^n V, \xi A^n V),$$

where the last inequality follows from concavity of the fidelity.

Next, we bound the Rényi entropy $S_\alpha(A^n V)_\tau$ in Eq. (13) where we defined the state $\tau_x A^n V$ as an extension of the state $\rho_x A^n$. Consider a purification $|\psi\rangle A^n R^n R'^n R'' n$ of the source $\rho A^n R^n$. Then, any extended state $\tau A^n V R'' n = \sum_x p(x^n) \tau_x A^n V \otimes |x^n\rangle |R\rangle |R'' n\rangle$ is obtained by applying a CPTP map $\Lambda : R^n R'' \rightarrow V$ as follows: $\tau A^n V R'' n = (\text{id}_{A^n} \otimes \Lambda)\rho A^n R^n R'' n$ where $\rho A^n R^n R'' n = \text{Tr}_{R^n} |\psi\rangle \langle \psi| A^n R^n R'' n$. Hence, we obtain the inequality

$$S_\alpha(A^n V)_\tau \geq \min_{\Lambda : R^n R'' \rightarrow V} S_\alpha(A^n E)_\nu$$

$$= E_{\rho,\alpha}(A^n : R^n)_{\rho \otimes \mu},$$

(14)

where the second line is due to the definition of the Rényi entanglement of purification for the state $(\rho A^n)_{\rho \otimes \mu}$. 


From Eq. (13) and Eq. (14), we obtain

\[
Q_v \geq \frac{1}{n} E_{\alpha, p}(A^n : R^n)_{\rho^n} + \frac{2\beta}{(1 - \beta)n} \log F_v \\
\geq E_{\alpha, p}^\infty (A : R)_{\rho} + \frac{2\beta}{(1 - \beta)n} \log F_v,
\]

where the second inequality follows because the definition of the Rényi entanglement of purification implies that \( m E_{\alpha, p}(A^n : R^n)_{\rho^n} \geq E_{\alpha, p}(A^{nm} : R^{nm})_{\rho^{nm}} \) holds for any \( m \). Hence, we have

\[
\frac{1}{n} E_{\alpha, p}(A^n : R^n)_{\rho^n} \geq \lim_{m \to \infty} \frac{1}{nm} E_{\alpha, p}(A^{nm} : R^{nm})_{\rho^{nm}} = E_{\alpha, p}^\infty (A : R)_{\rho},
\]

where the equality follows because every subsequence of a convergent sequence converges to the same limit as the original sequence. The theorem follows by rearranging the terms in Eq. (15).

In Lemma 10, we show that \( E_{\alpha, p}^\infty (A : B)_{\rho} \) is continuous for \( \alpha > 1 \), however, it is not known if it is continuous at \( \alpha = 1 \). We prove that for any rate below \( \lim_{\alpha \to 1^+} E_{\alpha, p}^\infty (A : B)_{\rho} \) the fidelity converges to zero.

**Proposition 12.** For any rate \( Q_v < \lim_{\alpha \to 1^+} E_{\alpha, p}^\infty (A : B)_{\rho} \), there is a \( K > 0 \) such that the following bound holds for the visible compression of mixed states:

\[
F_v \leq 2^{-nK}.
\]

**Proof.** From Lemma 10 we know that the regularized Rényi entanglement of purification \( E_{\alpha, p}^\infty (A : B)_{\rho} \) is continuous and decreasing with respect to \( \alpha \) for \( \alpha > 1 \). Therefore, for any \( Q_v < \lim_{\alpha \to 1^+} E_{\alpha, p}^\infty (A : B)_{\rho} \), there is a \( \alpha > 1 \) such that \( Q_v < E_{\alpha, p}^\infty (A : B)_{\rho} < \lim_{\alpha \to 1^+} E_{\alpha, p}^\infty (A : B)_{\rho} \). Hence, the claim follows from Theorem 11 by letting \( K := \frac{2\beta}{2\alpha} (E_{\alpha, p}^\infty (A : B)_{\rho} - Q_v) \).

**Remark 13.** If \( \lim_{\alpha \to 1^+} E_{\alpha, p}^\infty (A : B)_{\rho} = E_{p}^\infty (A : B)_{\rho} \), then the above bound is the strong converse for the visible compression of mixed states.

V. DISCUSSION

We prove a strong converse bound for the compression of a general mixed-state source \( \rho^{AR} \) shared between an encoder and an inaccessible reference system as defined in [8–10]. This implies a strong converse for the blind compression of ensembles of mixed states since it is a special case of the general mixed-state source when the reference system \( R \) is classical. We apply Lemma 2 to establish an upper bound on the fidelity between the decoded output systems, including the environment of the decoding isometry, and a specific extension of the source with their the Rényi entropy differences. This implies that if the aforementioned fidelity is bounded for all such extensions of the source, then the fidelity criterion for the blind compression of ensembles of mixed states is bounded as well (Lemma 14).

The second part of the paper is dedicated to the visible compression of ensembles of mixed states. We define a new quantity in Definition 7 which is similar to the entanglement of purification, however, instead of von Neumann entropy, the minimization of the Rényi entropy is taken into account. We obtain an upper bound on the fidelity criterion in terms of the regularization of the \( \alpha \)-Rényi entanglement of purification \( E_{\alpha, p}^\infty (A : R)_{\rho} \) and show that for any rate below \( \lim_{\alpha \to 1^+} E_{\alpha, p}^\infty (A : R)_{\rho} \) the fidelity exponentially converges to
Then, for $V$ where $\nu$ moreover, for a given state $R$ where $\omega$.

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**APPENDIX**

In Lemma 4, we establish dimension bound on system $E$ which is defined in Eq. (8). A similar system $E$ is defined in the following lemma which we use to prove Theorem 3. The only difference is that the following lemma is based on a one-shot setting. Therefore, we assume that systems $E$ and $E'$ have dimensions bounded by $|C| \cdot |N| \cdot |Q|$ and $(|C| \cdot |N| \cdot |Q|)^2$ respectively.

**Lemma 14.** Let $\omega^{CNQR}$ be the corresponding Koashi-Imoto state to $\rho^{AR}$ with a purification $|\omega\rangle^{CNQRR'} = \sum_j \sqrt{p_j} |j\rangle^C |j\rangle^C |\omega_j\rangle^{NN'} |\rho_j\rangle^{QQ'R}$, where $R' := C'N'Q'$ is a purifying system. Also, $\text{Tr}_{N'}(\omega_j^{NN'}) = \omega_j^N$ and $\text{Tr}_{Q'}(\rho_j^{QQ'R}) = \rho_j^{QR'}$. Define the following states:

$$|\nu\rangle^{CNQEEE''R'} := (V_1 \otimes \mathbb{I}_{RR'}) |\omega\rangle^{CNQRR'}$$

$$|\tau\rangle^{CNQEEE''R'} := (V_2 \otimes \mathbb{I}_{RR'}) |\omega\rangle^{CNQRR'}$$

where $V_1 : CNQ \to C\hat{N}\hat{Q}EE'$ and $V_2 : CNQ \to CNQEE'$ are isometries, and $|\tau\rangle^{CNQEEE''R'}$ satisfies the following conditions:

(i) $\text{Tr}_{EE''R'} |\nu\rangle^{CNQEEE''R'} = |\omega\rangle^{CNQRR'}$ and

(ii) $\text{Tr}_{CQQR'R'} |\tau\rangle^{CNQEEE''R'} = \sum_j p_j |j\rangle^C \otimes |\tau_j\rangle^{NE} \otimes |\eta_j\rangle^{E'N'}$.

Then, for $\alpha \in (0, 1) \cup (1, \infty)$, the following holds:

$$S_{\alpha}(CNQE)_{\tau} = S_{\alpha}(CQ)_{\omega}.$$

Moreover, for a given state $\nu^{\hat{C}\hat{N}\hat{Q}ER} = \text{Tr}_{E'R'} |\nu\rangle |\nu\rangle$ obtained as above, if $F_1 := F(\nu^{\hat{C}\hat{N}\hat{Q}ER}, \tau^{\hat{C}\hat{N}\hat{Q}ER}) \leq c$ for $c > 0$ and any state $\tau^{\hat{C}\hat{N}\hat{Q}ER}$ obtained as above satisfying conditions 1 and 2, then $F_2 := F(\nu^{\hat{C}\hat{N}\hat{Q}ER}, \omega^{\hat{C}\hat{N}\hat{Q}R}) \leq c$.

**Proof.** First we show how we can construct the state $\tau$ and isometry $V_2$. Due to Theorem 1, the first condition of the lemma implies that the isometry $V_2$ acts only on system $N$. We define $V_2$ as follows:

$$V_2 = (\sum_j |j\rangle^C \otimes |\tau_j\rangle^{NE} \otimes V_2) \otimes \mathbb{I}_Q$$
where \( V_2 : N \to E' \) is an isometry such that \((V_2 \otimes \mathbb{1}_N') |\omega_j\rangle^N_N' = |\eta_j\rangle^{E'N'} \) and \( \text{Tr}_E |\tau_j\rangle^{|\tau_j\rangle^{NE}} = N_j^N \) for all \( j \). By applying \( V_2 \) we obtain the following state which satisfies conditions 1 and 2:

\[
|\tau\rangle^{|CNQE\rangle^{E'RR'}} = \sum_j \sqrt{p_j} |j\rangle^C |j\rangle^C |\tau_j\rangle^{|\tau_j\rangle^{NE}} |\eta_j\rangle^{E'N'} |\rho_j\rangle^{QQ'R},
\]

where \( R = C'N'Q' \).

Now, we prove that \( S_\alpha(CNQE)_{\tau} = S_\alpha(CQ)_{\omega} \) holds for the following reduced state

\[
\tau_{CNQE} = \text{Tr}_{E'RR'} |\tau\rangle^{|CNQE\rangle^{E'RR'}} = \sum_j p_j |j\rangle^C \otimes |\tau_j\rangle^{|\tau_j\rangle^{NE}} \otimes \rho_j^Q.
\]

This state has a classical-quantum structure, hence the following holds:

\[
(c^{CNQE})^\alpha = \sum_j p_j^\alpha |j\rangle^C \otimes |\tau_j\rangle^{|\tau_j\rangle^{NE}} \otimes (\rho_j^Q)^\alpha.
\]

Using this we obtain that

\[
S_\alpha(CNQE)_{\tau} = \frac{1}{1 - \alpha} \log \text{Tr}(c^{CNQE})^\alpha
\]

\[
= \frac{1}{1 - \alpha} \log \sum_j p_j^\alpha \text{Tr}(\rho_j^Q)^\alpha
\]

\[
= \frac{1}{1 - \alpha} \log \text{Tr}(c^{CQ})^\alpha
\]

\[
= S_\alpha(CQ)_{\tau}.
\]

Now, we prove the second statement of the lemma. By the definition of the fidelity and the assumptions of the lemma, the following holds:

\[
F_1 = \max_{U_{E'RR'}} \left| \langle \nu | \tilde{C}^{CNQE\rangle^{E'RR'}} (1_{CNQER} \otimes U_{E'RR'}) |\tau\rangle^{|CNQE\rangle^{E'RR'}} \right|^2 \leq c,
\]

\[
F_2 = \max_{U_{EE'RR'}} \left| \langle \nu | \tilde{C}^{CNQE\rangle^{E'RR'}} (1_{CNQR} \otimes U_{EE'RR'}) |\tau\rangle^{|CNQE\rangle^{E'RR'}} \right|^2
\]

where \( U_{E'RR} \) and \( U_{EE'RR} \) are unitaries acting on systems \( E'R' \) and \( EE'R' \), respectively. We can construct any unitary \( U_{EE'RR} \) as follows. Let \( \{U_E^i\}_i \) and \( \{U_{E'RR}^j\}_j \) be orthogonal basis for the space of Hilbert–Schmidt operators on \( E \) and \( E'R' \), respectively. The basis are orthogonal in the sense that \( \text{Tr}(U_{E}^i U_{E}^j) = 0 \) if and only if \( i \neq j \). Then, any unitary \( U_{EE'RR} \) is obtained as the linear combination of the basis for the spaces \( EE'R' \):

\[
U_{EE'RR} = \sum_{i,j} \mu_{ij} U_{E}^i \otimes U_{E'RR}^j.
\]

The orthogonality of the basis implies that

\[
\text{Tr}(1_{E} \otimes 1_{EE'RR}) = \text{Tr}(U_{EE'RR}^i U_{EE'RR}) = \sum_{i,j} |\mu_{ij}|^2 \text{Tr}(1_{E}) \text{Tr}(1_{E'RR}),
\]
by canceling out the trace terms from both sides, so we conclude that \( \sum_{i,j} |\mu_{i,j}|^2 = 1 \). Now, we rewrite the fidelity \( F_2 \) as follows:

\[
F_2 = \max_{U_{E'W}} |\langle \nu | \hat{\mathcal{N}}_{QEE'RR'} (\mathbb{I}_{CNRQ} \otimes \sum_{i,j} \mu_{i,j} U_{E}^i \otimes U_{E'}^j) |\tau \rangle |^{CNQEE'RR'}^2
\]

\[
= |\langle \nu | \hat{\mathcal{N}}_{QEE'RR'} (\mathbb{I}_{CNRQ} \otimes \sum_{i,j} \mu_{i,j}^* U_{E}^i \otimes U_{E'}^j) |\tau \rangle |^{CNQEE'RR'}^2
\]

\[
\leq \sum_{i,j} |\mu_{i,j}|^2 |\langle \nu | \hat{\mathcal{N}}_{QEE'RR'} (\mathbb{I}_{CNRQ} \otimes \sum_{i,j} \mu_{i,j}^* U_{E}^i \otimes U_{E'}^j) |\tau \rangle |^{CNQEE'RR'}^2
\]

\[
= \sum_{i,j} |\mu_{i,j}|^2 |\langle \nu | \hat{\mathcal{N}}_{QEE'RR'} (\mathbb{I}_{CNRQ} \otimes \sum_{i,j} \mu_{i,j}^* U_{E}^i \otimes U_{E'}^j) |\tau \rangle |^{CNQEE'RR'}^2
\]

\[
= \sum_{i,j} |\mu_{i,j}|^2 F(\hat{\mathcal{N}}_{QER}, \tau_i^{CNQER}),
\]

\[
\leq c.
\]

where in the second line \( \sum_{i,j} \mu_{i,j}^* U_{E}^i \otimes U_{E'}^j \) is the optimal unitary. The third line is due to the triangle inequality. In the forth line, we define \( |\tau_i|^{CNQEE'RR'} := (\mathbb{I}_{CNRQ} \otimes U_{E}^i) |\tau \rangle |^{CNQEE'RR'} \). Here note that, similar to \( \tau \), \( \tau_i \) satisfies the conditions 1 and 2 of the lemma, therefore the last line follows because by the assumption of the lemma the upper bound holds for any state \( \tau \) that satisfies the conditions. The penultimate line follows because \( \sum_{i,j} |\mu_{i,j}|^2 = 1 \).

We apply the following lemma to discuss continuity for the \( \alpha \)-Rényi entanglement of purification and its regularization.

**Lemma 15.** Let \( \{\sigma^n = T_n(\rho^{\otimes n})\}_{n} \) be a sequence of states for \( n \geq 1 \) where \( \{T_n\}_{n} \) is a sequence of CPTP maps. For \( \alpha > 1 \) define \( g_{\alpha,n}(\sigma^n) := \frac{(1-\alpha)S(\sigma^n)}{n} = \frac{\log \text{Tr}(\sigma^n^\alpha)}{n} \). Then, \( g_{\alpha,n}(\sigma^n) \) has the follows properties for \( \alpha > 1 \):

(i) It is a decreasing function of \( \alpha \).

(ii) It is a convex function of \( \alpha \).

(iii) It is a continuous function of \( \alpha \).

**Proof.** (i) Consider the spectral decomposition \( \sigma^n = \sum_{i=1}^{d^n} \lambda_{i,n} |v_{i,n}\rangle \langle v_{i,n}| \) where \( d \) is the dimension of a single output system. Define \( \gamma_{\alpha,n} := \sum_{i=1}^{d^n} \lambda_{i,n}^\alpha \). Then, the derivative of \( g_{\alpha,n}(\sigma^n) \) with respect to \( \alpha \) is always
negative:
\[
g'_{\alpha,n}(\sigma^n) = \frac{1}{n} \sum_{i=1}^{d^n} \lambda^\alpha_{i,n} \log \lambda_{i,n} \leq \frac{1}{n} \log \left( \sum_{i=1}^{d^n} \lambda^\alpha_{i,n} \right) \leq 1
\]
\[
g''_{\alpha,n}(\sigma^n) = \frac{1}{n} \sum_{i=1}^{d^n} \lambda^\alpha_{i,n} (\log \lambda_{i,n})^2 \leq \frac{1}{n} \left( \sum_{i=1}^{d^n} \lambda^\alpha_{i,n} \log \lambda_{i,n} \right)^2 \leq 1
\]
where in the first line the denominator is equal to \(\gamma_{\alpha,n}\). In the second line note that \(\left\{ \frac{\lambda^\alpha_{i,n}}{\gamma_{\alpha,n}} \right\}_i\) is a probability distribution, therefore, the third line follows from the concavity of \(\log(\cdot)\). The penultimate line follows because \(0 \leq \frac{\lambda^\alpha_{i,n}}{\gamma_{\alpha,n}} \leq 1\). The last line follows because the eigenvalues sum up to one.

(ii) The second derivative with respect to \(\alpha\) is always positive:
\[
g''_{\alpha,n}(\sigma^n) = \frac{1}{n} \sum_{i=1}^{d^n} \lambda^\alpha_{i,n} (\log \lambda_{i,n})^2 \leq \frac{1}{n} \left( \sum_{i=1}^{d^n} \lambda^\alpha_{i,n} \log \lambda_{i,n} \right)^2 \leq 1
\]
where in the second line \(\left\{ \frac{\lambda^\alpha_{i,n}}{\gamma_{\alpha,n}} \right\}_i\) is a probability distribution, therefore, the third line follows from the concavity of \(f(x) = -x^2\).

(iii) The continuity follows because \(g_{\alpha,n}(\sigma^n)\) is decreasing and convex for \(\alpha > 1\).

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