Inhomogeneous global minimizers to the one-phase free boundary problem

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ABSTRACT

Given a global 1-homogeneous minimizer $U_0$ to the Alt-Caffarelli energy functional, with $\text{sing}(F(U_0)) = \{0\}$, we provide a foliation of the half-space $\mathbb{R}^n \times [0, +\infty)$ with dilations of graphs of global minimizers $U \leq U_0 \leq \hat{U}$ with analytic free boundaries at distance 1 from the origin.

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1. Introduction

1.1. Background

In this article, we are interested in minimizers to the energy functional,

$$J(u) = J_\Omega(u) = \int_\Omega \left( |\nabla u|^2 + \chi_{\{u > 0\}} \right) dx,$$

where $\Omega$ is a domain in $\mathbb{R}^n$ ($n \geq 2$) and $u \geq 0$. Minimizers of $J$ were first investigated systematically by Alt and Caffarelli. Two fundamental properties are proved in the pioneering article [1], namely, the Lipschitz regularity of minimizers and the regularity of “flat” free boundaries. These in turn, give the almost-everywhere regularity of minimizing free boundaries. The viscosity approach to the regularity of the associated free boundary problem

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega^+(u) := \Omega \cap \{u > 0\} \\
|\nabla u| = 1 & \text{on } F(u) := \partial \{u > 0\} \cap \Omega,
\end{cases}$$

was later developed by Caffarelli in [2, 3]. There is a wide literature on this problem and the corresponding two-phase problem, and for a comprehensive treatment we refer the reader to [4] and the references therein.

Here, we are interested in one-phase global minimizers, that is, minimizers $u \geq 0$ of $J$ over all balls $B_R \subset \mathbb{R}^n$ among functions that agree with $u$ on $\partial B_R$. In dimension $n = 2$, Alt and Caffarelli [1] showed that (up to rotation) the only global minimizer of $J$ is $x^+_n$. 

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The same result was obtained by Caffarelli, Jerison, and Kenig in dimension \( n = 3 \) \[5\],
and by Jerison and Savin in dimension \( n = 4 \) \[6\]. The classification of global minimizers
implies the smoothness of minimizing free boundaries in both the one-phase and two-
phase problem in dimension \( n \leq 4 \). These results rely on the flatness theorem and on
the Weiss Monotonicity formula (see Section 2 for the precise statement) which allows
one to consider only the case of 1-homogeneous minimizers. It is conjectured that the
results above remain true up to dimension \( n \leq 6 \). On the other hand, De Silva and
Jerison provided in \[7\] an example of a singular 1-homogeneous minimal solution in
dimension \( n = 7 \). In \[8\], Hong studied the stability of Lawson-type cones for (1.2) in
low dimensions and showed that in dimension \( n = 7 \) there is another stable cone besides
the minimizer in \[7\].

### 1.2. Main result

Let \( U_0 \geq 0 \) be a global energy minimizer to \( J \), and assume that \( U_0 \) is homogeneous of
degree 1, and \( F(U_0) \setminus \{0\} \) is an analytic hypersurface, while 0 is a singular point (hence,
by the discussion above, \( n \geq 5 \)). In this article, we construct smooth inhomogeneous
global minimizers asymptotic to \( U_0 \) at infinity. In fact, \( U_0 \) is trapped between two global
smooth inhomogeneous minimizers obtained by perturbing away the singularity, and
whose dilations foliate the whole space. The existing theory of one-phase minimizers
establishes a strong resemblance between the theory of free boundaries and the theory
of minimal surfaces. Our result reaffirms this similarity, providing an analogue of Hardt
and Simon’s result in the context of area minimizing cones \[9\].

In order to state our main theorem precisely, we recall some known facts and refer
to \[6\] for further details on the linearized problem (see also Section 3). Let \( H > 0 \)
denote the mean curvature of \( \partial \{ U_0 > 0 \} \) oriented toward the complement of the con-
ected set \( \{ U_0 > 0 \} \), and consider the linearized problem associated to \( U_0 \):

\[
\begin{cases}
\Delta w = 0 & \text{in } \{ U_0 > 0 \} \\
\partial_n w + H w = 0 & \text{on } F(U_0) \setminus \{0\},
\end{cases}
\]

with \( \nu \) the interior unit normal to \( F(U_0) \). Let \( w(x) := |x|^{-\gamma} \tilde{v}(\theta) \) with \( \tilde{v} \) the first eigen-
function of the Laplacian on \( S^{n-1} \cap \{ U_0 > 0 \} \),

\[
\Delta_{S^{n-1}} \tilde{v} = \lambda \tilde{v}, \quad \text{on } S^{n-1} \cap \{ U_0 > 0 \},
\]

with Neumann boundary condition

\[
\partial_\nu \tilde{v} + H \tilde{v} = 0, \quad \text{on } F(U_0) \cap S^{n-1}.
\]

Then, \( \tilde{v} > 0 \) on \( S^{n-1} \cap \{ U_0 > 0 \} \) and \( \lambda > 0 \). It is easily computed (see Section 4) that \( w \)
solves (1.3) as long as \( \gamma = \gamma_\pm \) satisfy \( \gamma^2 - (n-2)\gamma + \lambda = 0 \). The stability of \( U_0 \) is
equivalent to the fact that this quadratic equation must have real roots, that is,
\( (n-2)^2 - 4\lambda \geq 0 \). Moreover, \( \lambda > 0 \), thus \( \gamma = \gamma_\pm \in \mathbb{R}, \gamma_+ \geq \gamma_- > 0 \).

If \( \gamma_- \neq \gamma_+ \), we call

\[
V_{\gamma_+}(x) := |x|^{-\gamma_+} \tilde{v}.
\]

By abuse of notation, if \( \gamma_- = \gamma_+ \) and we call \( \gamma_0 \) this common value, we set
Our main result reads as follow. As usual, \( B_R(x_0) \subset \mathbb{R}^n \) denotes the ball of radius \( R > 0 \) and center \( x_0 \), and when \( x_0 = 0 \) we drop the dependence on it. Also, given a function \( u \geq 0 \) defined on \( \mathbb{R}^n \), we define for \( t > 0 \)

\[
u_t(x) = \frac{1}{t} u(tx), \quad \Gamma(u_t) := \left\{ (x, u_t(x)) : x \in \left\{ u_t > 0 \right\} \right\}.
\]

Finally, constants that depend only on the ingredients, that is, \( n, U_0 \), are called universal.

**Theorem 1.1.** Let \( U_0 \) be as above. There exist \( \bar{U}, \underline{U} \) global minimizers to (1.1), such that

(i) \( \underline{U} \leq U_0 \leq \bar{U}, \) and \( \text{dist}(F(\bar{U}), \{0\}) = \text{dist}(F(U_0), \{0\}) = 1; \)

(ii) \( F(\bar{U}), F(U_0) \) are analytic hypersurfaces;

(iii) there exist universal constants \( R > 0 \) large, \( \gamma' > 0 \) small, such that if \( \gamma_- \neq \gamma_+ \),

\[
\bar{U}(x) = U_0(x) + \bar{a} V_{\gamma_-}(x) + O(|x|^{-\gamma_- - \gamma'}), \quad \text{in } \{ U_0 > 0 \} \setminus B_R,
\]

or

\[
\bar{U}(x) = U_0(x) + \bar{a} V_{\gamma_+}(x) + O(|x|^{-\gamma_+ - \gamma'}), \quad \text{in } \{ U_0 > 0 \} \setminus B_R,
\]

for some \( \bar{a} > 0 \) universal. The analogous statement for \( \underline{U} \) holds in \( \{ U > 0 \} \setminus B_R \) with \( \underline{a} < 0 \). If \( \gamma_- = \gamma_+ \), then

\[
\bar{U}(x) = U_0(x) + \underline{a} V_{\gamma_-}(x) + \underline{b} V_{\gamma_+} + O(|x|^{-\gamma_-}), \quad \text{in } \{ U_0 > 0 \} \setminus B_R,
\]

with \( \underline{a} \geq 0 \) and \( \max\{\underline{a}, \underline{b}\} > 0 \). The analogous statement for \( \underline{U} \) holds in \( \{ U > 0 \} \setminus B_R \) with \( \underline{a} \leq 0 \) and \( \min\{\underline{a}, \underline{b}\} < 0 \);

(iv) for any \( x \in \{ U_0 = 0 \}^c \), the ray \( \{ tx, \ t > 0 \} \), intersects \( F(\bar{U}) \) in a single point and the intersection is transverse; similarly, for any \( x \in \{ U_0 > 0 \} \), the ray \( \{ tx, \ t > 0 \} \), intersects \( F(U_0) \) in a single point and the intersection is transverse;

(v) the graphs \( \Gamma_t := \Gamma(\nu_t), \Gamma_t := \Gamma(u_t) \) foliate the half-space \( \mathbb{R}^n \times [0, +\infty) \), that is,

\[
\mathbb{R}^n \times [0, +\infty) = \bigcup_{t \in (0, \infty)} (\Gamma_t \cup \Gamma_t);
\]

(vi) if \( V \) is a global minimizer to \( J \) and \( V \geq U_0 \) (resp. \( V \leq U_0 \)), then

\[
V \equiv \bar{U}_t, \quad (\text{resp. } V \equiv \underline{U}_t),
\]

for some \( t > 0 \), unless \( V \equiv U_0 \).

In fact, the uniqueness property (vi) holds more generally for global critical points that satisfy a uniform nondegeneracy condition (see Section 2 for this notion). Furthermore, the expansion in (iii) implies the same expansion for the free boundaries \( F(\bar{U}), F(U_0) \), as graphs over the cone \( F(U_0) \) in the outer normal direction \(-\nu\) (see Remark 3.3).

### 1.3. Further background

In the context of critical points, Hauswirth et al. [10] discovered an explicit family of simply-connected planar regions (so-called exceptional domains)
\[ \Omega_a := \{ (x_1, x_2) \in \mathbb{R}^2 : |x_1/a| < \frac{\pi}{2} + \cosh(x_2/a) \}, \ a > 0 \]

whose boundary consists of two curves (hairpins), and a positive harmonic function \( H_a(x) = aH_1(x/a) \) on \( \Omega_a \) that satisfies the free boundary conditions \( H_a = 0 \) and \( |\nabla H_a| = 1 \) on \( \partial \Omega_a \). Extending \( H_a \) to be zero in the complement of \( \Omega_a \), we have a nontrivial entire solution to (1.2) (an explicit formula for \( H_a \) is given using conformal mappings). The graphs of the functions \( H_a \) give a foliation of space using dilates of an unstable critical point of the functional. In [11], Jerison and Kamburov characterized blow-up limits of classical solutions to (1.2) in the disk, with simply-connected positive phase, as either (a) half-plane, (b) two-plane, or (c) hairpin solutions. This relates to previous results of classifications of entire solutions with simply connected positive phase due to Khavinson et al. [12] and Traizet [13]. Traizet showed that classical entire solutions satisfying that no connected component of \( F(u) \) is compact in the open disk must be one of the forms (a), (b), or (c). Khavinson et al. (2013) showed that the same conclusion is true under a natural, weak regularity assumption on the free boundary known as the Smirnov property.

### 1.4. Organization of the article

The article is organized as follows. In Section 2, we provide some preliminary properties of minimizers. The following section is devoted to the proof of Theorem 1.1. The core of the proof is the analysis of the asymptotic behavior of a solution of a perturbation of the linearized problem (1.3), using appropriate families of subsolutions and supersolutions. This analysis is carried on in Section 4. Finally, the Appendix contains a technical result (relying on the Hodograph transform) needed in the proof of the main theorem. We also recall in the Appendix the standard Schauder estimates and the Harnack inequality for oblique derivative problems [14].

When we announced this result, M. Engelstein and coauthors informed us that they have obtained what appear to be essentially similar results to the ones in this article in a paper under preparation [15].

### 2. Preliminaries

In this section, we recall some basic properties of minimizers and we obtain a few preliminary results which we use in Section 3 toward the proof of Theorem 1.1. Throughout the article, a ball in \( \mathbb{R}^n \) of radius \( R \) and centered at \( x_0 \) will be denoted by \( B_R(x_0) \). The dependence on \( x_0 \) is dropped when \( x_0 = 0 \). For a non-negative function \( u \) on a domain \( \Omega \), we use the notation

\[ F(u) := \Omega \cap \partial \{ u > 0 \}. \]

We start by a quick recap of the classical theory for minimizers of (1.1), which we will be using throughout the paper (see [1]). First of all, we say that \( u \in H^1_{\text{loc}}(\Omega) \) minimizes \( J \) in \( \Omega \) if on any smooth compact set \( \Omega' \subset \Omega \),

\[ J(u) \leq J(u + v), \quad \forall v \in H^1_{0}(\Omega'). \]

For a given boundary data \( \varphi \geq 0 \) with finite energy, there exists a non-negative minimizer \( u \in H^1(\Omega) \) of \( J \) such that \( u - \varphi \in H^1_{0}(\Omega) \). Moreover, \( u \in C^{0,1}(\Omega) \) and satisfies
(1.2) in the viscosity sense (see [3] for the definition of viscosity solution and a proof of this claim). In fact, if $x_0 \in F(u)$ and $B_2(x_0) \subset \Omega$, then the Lipschitz norm of $u$ in $B_r(x_0)$ is bounded by a constant depending only on $n$. Finally, $u$ satisfies a strong non-degeneracy property, that is for any $x_0 \in F(u)$, $\sup_{B_r(x_0)} u \geq cr$ for all balls $B_r(x_0) \subset \Omega$. Lipschitz continuity and non-degeneracy are the key to the following compactness property (see Lemmas 4.7 and 5.4 in [1]).

**Lemma 2.1.** Let $\{u_k\} \subset H^1_{\text{loc}}(\Omega)$ be a sequence of minimizers to $J$ in $\Omega$ with $u_k \rightarrow u$ uniformly on compact subsets, then

- $\{u_k > 0\} \rightarrow \{u > 0\}$ and $F(u_k) \rightarrow F(u)$ locally in the Hausdorff distance;
- $\chi_{\{u_k > 0\}} \rightarrow \chi_{\{u > 0\}}$ in $L^1_{\text{loc}}(\Omega)$;
- $\nabla u_k \rightarrow \nabla u$ a.e. in $\Omega$.

Moreover, $u$ minimizes $J$ in $\Omega$.

We also recall Weiss Monotonicity Formula [16].

**Theorem 2.2.** If $u$ is a minimizer to $J$ in $B_R$, then

$$\Phi_u(r) := r^{-n} J_{B_r}(u) - r^{-n-1} \int_{\partial B_r} u^2, \quad 0 < r \leq R,$$

is increasing in $r$. Moreover $\Phi_u$ is constant if and only if $u$ is homogeneous of degree 1.

We now notice that,

$$J(\min\{u, w\}) + J(\max\{u, w\}) = J(u) + J(w),$$

from which we deduce that if $u, w$ minimize $J$ in $\Omega$ and $u \geq w$ on $\partial \Omega$, then $\min\{u, w\}, \max\{u, w\}$ minimize $J$ in $\Omega$ as well (with boundary data $w$ and $u$ respectively.). We can then provide a comparison principle.

**Proposition 2.3.** Let $u, w$ minimize $J$ in $B_1$ and assume that for $0 < \epsilon < 1$,

$$u \geq w \quad \text{on } B_1 \setminus B_1 - \epsilon.$$

Then, $u \geq w$ in $B_1$.

**Proof.** Since $u \geq w$ on $\partial B_1$, we have that $v := \min\{u, w\}$ minimizes $J$ among competitors with boundary value $w$. However, by our assumptions, $v \equiv w$ in $B_1 \setminus B_1 - \epsilon$, hence by the harmonicity of minimizers in their positivity set, we conclude that $v \equiv w$ in $D \cap B_1$ for any connected component $D$ of $\{w > 0\}$ which intersect the annulus. By the maximum principle, any connected component of $\{w > 0\}$ will intersect the annulus, hence $v \equiv w$ in $B_1$. \hfill $\square$

**Proposition 2.4.** Let $u, w$ minimize $J$ in $B_1$ and assume that

$$u \geq w \quad \text{on } \partial B_1, \quad u(x_0) > w(x_0) \quad \text{at } x_0 \in \{w > 0\} \cap \partial B_1.$$

If $\{w > 0\} \cap \partial B_1$ is connected, then $u \geq w$ in $B_1$.

**Proof.** Since $w(x_0) > 0$, we argue as in the proof of the lemma above, however in this case we only have that $\min\{u, w\} \equiv w$ in a neighborhood of $x_0$. Thus, they coincide in
the connected component of \( \{ w > 0 \} \) which includes this neighborhood, and the conclusion follows from the connectivity assumption. \( \Box \)

We now obtain a uniqueness result. Recall that \( \{ U_0 > 0 \} \) is connected (in fact NTA) \([4, 5]\). This can be easily deduced by the following observation. The restrictions of \( U_0 \) and \( H = x_n^+ \) (the half-plane) to the unit sphere solve the eigenvalue problem (because they are both 1 homogeneous),

\[
\Delta_{\mathbb{S}^{n-1}} u = -\lambda_1 u \quad \text{on} \quad \{ u > 0 \} \cap \mathbb{S}^{n-1},
\]

for \( \lambda_1 = n - 1 \). However, since \( \{ H > 0 \} \) is axisymmetric, it follows by a spherical rearrangement argument that each connected component \( D \) of \( \{ U_0 > 0 \} \cap \mathbb{S}^{n-1} \) satisfies the measure estimate \( |D| \geq |\{ H > 0 \} \cap \mathbb{S}^{n-1}| \), with equality only if \( U_0 \) is a half-plane solution on \( D \). This implies connectivity.

**Lemma 2.5** (Uniqueness). Let \( w \) be a minimizer of (1.1) in \( B_1 \) such that \( w = U_0 \) on \( \partial B_1 \). Then, \( w \equiv U_0 \) in \( B_1 \).

**Proof.** Let

\[
W = w \quad \text{in} \quad B_1, \quad W = U_0 \quad \text{in} \quad B_2 \setminus B_1.
\]

Since \( w - U_0 \in H_0^1(B_1) \), we have \( W - U_0 \in H_0^1(B_2) \) and \( W \) and \( U_0 \) both minimize \( J \) in \( B_2 \) among competitors with boundary data \( U_0 \). Hence,

\[
\Delta W = 0 \quad \text{in} \quad B_2 \cap \{ W > 0 \}, \quad \Delta U_0 = 0 \quad \text{in} \quad B_2 \cap \{ U_0 > 0 \}.
\]

In particular, since \( W = U_0 \) in \( B_2 \setminus B_1 \), by unique continuation \( w \) and \( U_0 \) must agree on \( B_1 \cap \{ U_0 > 0 \} \). If \( w(\bar{x}) > 0 \) at a point \( \bar{x} \in B_1 \cap \{ U_0 = 0 \} \), then \( J_{B_1}(w) > J_{B_1}(U_0) \) in \( B_1 \), contradicting minimality. \( \square \)

Finally, we prove a strict separation lemma.

**Lemma 2.6.** Let \( w \) be a minimizer to (1.1) in \( B_1 \) such that \( w \geq U_0 \) on \( \partial B_1 \). Then, \( w \geq U_0 \) in \( B_1 \). Moreover, \( w > U_0 \) in \( B_1 \cap \{ U_0 > 0 \} \), unless \( w \equiv U_0 \) in \( B_1 \cap \{ U_0 > 0 \} \).

**Proof.** Since \( w \geq U_0 \) on \( \partial B_1 \), \( \min \{ w, U_0 \} \) is also a minimizer to \( J \) in \( B_1 \) with boundary data \( U_0 \). By the previous uniqueness result, we deduce that \( \min \{ w, U_0 \} \equiv U_0 \) in \( B_1 \) and the first part of our lemma is proved. Assume \( w \neq U_0 \) in \( B_1 \cap \{ U_0 > 0 \} \). By the maximum principle, we immediately deduce that \( w > U_0 \) in \( B_2 \cap \{ U_0 > 0 \} \). Moreover, if \( w \) touches \( U_0 \) by above at a free boundary point \( x_0 \neq 0 \), since \( \partial_n U_0(x_0) = \partial_n w(x_0) = 1 \), we contradict Hopf's boundary point lemma. In particular,

\[
w > U_0 \quad \text{in} \quad (B_{3/4} \setminus B_{1/2}) \cap \{ U_0 > 0 \},
\]

implying that for \( \delta \) small,

\[
w(x) \geq U_0(x + y), \quad |y| \leq \delta, \quad \text{in} \quad B_{3/4} \setminus B_{1/2},
\]

and by Proposition 2.3 this is true in \( B_{1/4} \). Choosing \( y \) so that \( U_0(y) > 0 \), we get that \( w(0) > 0 \), concluding the proof of the desired claim. \( \square \)
3. The proof of Theorem 1.1

This section is devoted to the proof of our main Theorem 1.1. The first subsection deals with the existence, while the second one handles the regularity.

3.1. Existence and basic properties

In this subsection, we prove part (i) of Theorem 1.1, that is the existence of $\tilde{U}, \tilde{U}$, and provide some key technical lemmas about their vertical distance from $U_0$. We start with the existence.

**Proof of Theorem 1.1-(i).** First, we construct the global minimizer $\tilde{U}$. Let $x_0 \in \partial B_1 \cap \{U_0 > 0\}$ and let $g \geq 0$ be a smooth function compactly supported in a neighborhood of $x_0$ on $\partial B_1$. For $\epsilon > 0$, we define the boundary value $g_\epsilon := U_0 + \epsilon g \geq U_0$ on $\partial B_1$, $g_\epsilon(x_0) > U_0(x_0)$, and call $u_\epsilon$ a minimizer to $J_{B_1}$ with this boundary value. Notice that since $g_\epsilon$ is Hölder continuous, $u_\epsilon$ is uniformly Hölder continuous up to the boundary and it achieves the boundary data continuously. By Lemma 2.6, $0 \notin F(u_\epsilon)$ and $u_\epsilon \geq U_0$ for all $\epsilon > 0$. Let $\epsilon_j \to 0$ (as $j \to \infty$), and for each $j$, let $B_{r_j} \subset \{u_{\epsilon_j} > 0\}$ be the largest ball included in the positive phase of $u_{\epsilon_j}$. Notice that by Proposition 2.4, $u_{\epsilon_j}$ is a decreasing sequence and $u_{\epsilon_j} \geq U_0$ (and the proposition can be applied as $\{u_{\epsilon_j} > 0\} \cap \partial B_1 = \{U_0 > 0\} \cap \partial B_1$ which is connected, and $g_{\epsilon_j}(x_0) > U(x_0) > 0$). By the Lipschitz continuity and the compactness Lemma 2.1, we conclude that up to extracting a subsequence, $u_{\epsilon_j}$ converges to a minimizer $v$ of $J$ in $B_1$, and since $g_{\epsilon_j} \to U_0$, we conclude by the uniqueness Lemma 2.5 that $v = U_0$ and in particular $r_j \to 0$ and $j \to \infty$. Consider now for each $j$ the rescaling $\bar{U}_j(x) = u_{\epsilon_j}(r_j x)/r_j$. Then, $\bar{U}_j$ is a minimizer in $B_{1/r_j}$, $B_1 \subset \{\bar{U}_j > 0\}$, and $\partial B_1 \cap F(\bar{U}_j) \neq \emptyset$. Moreover, $\bar{U}_j(x) \leq C(1 + |x|)$ in $B_{1/r_j}$, where $C$ is independent of $j$, since (as recalled in the preliminaries) each $u_{\epsilon_j}$ is uniformly Lipschitz in $B_{1/2}$. Since the $\bar{U}_j$’s are uniformly Lipschitz, up to extracting a subsequence, $\bar{U}_j \to \bar{U}$ and $\bar{U}$ is a global minimizer with $B_1 \subset \{\bar{U} > 0\}$, $\partial B_1 \cap F(\bar{U}) \neq \emptyset$, and a universal Lipschitz bound. Finally by construction $\bar{U} \geq U_0$ as desired.

Similarly, $\bar{U}$ can be built starting from a family of minimizers with boundary data, $0 \leq g_\epsilon := U_0 - \epsilon g \leq U_0$.

Next, we prove the following asymptotic behavior for $\tilde{U}, \tilde{U}$. Recall that for a given function $u$, we denote by $u_t(x) := \frac{u(tx)}{t}$.

**Lemma 3.1.** $\tilde{U}, \tilde{U}$ are asymptotic to $U_0$ at infinity, that is,

$$\lim_{t \to \infty} \tilde{U}_t(x) = \lim_{t \to \infty} \tilde{U}_t(x) = U_0(x),$$

uniformly on compact sets in $\mathbb{R}^n$.

**Proof.** We prove the statement for $\tilde{U}$. The same argument also applies to $\tilde{U}$. For $t_k \to \infty$ as $k \to \infty$, consider the sequence of rescalings, $\tilde{U}_{t_k}(x) := \frac{\tilde{U}(tx)}{t_k}$ which in view of the equi-Lipschitz continuity and the compactness Lemma 2.1, will converge to a global minimizer $V \geq U_0$ (up to extracting a subsequence). On the other hand, again by the Lipschitz continuity,
Proof. Let thus, by the flatness implies regularity results \cite{3}, there exists

due activity set of one appearance to another in the body of the proofs. For notational simplicity, the posi-
tivity set of \( R \) for \( n \). However, since \( \{ U_0 > 0 \} \subset \{ V > 0 \} \) and eigenvalues are monotone
with respect to the inclusion of domains, we deduce from standard arguments that \( U_0 \equiv V \).

Finally, using Lemma 3.1 and Proposition 5.1 in the Appendix, we can deduce the
following result, which is key to our strategy. Here and henceforth, we denote by \( E^c := \mathbb{R}^n \setminus \bar{E} \). Also, constants depending on \( n, U_0 \) will be called universal and may vary from
one appearance to another in the body of the proofs. For notational simplicity, the posi-
tivity set of \( U_0 \) is called \( \Omega_0 := \{ U_0 > 0 \} \), and as pointed out in the introduction,
\(- (\partial_{vv} U_0) = H > 0 \) is the mean curvature of \( \partial \Omega_0 \) oriented toward the complement of
the connected set \( \Omega_0 \).

To fix ideas, from now on, we prove the statements involving \( \bar{U} \).

Lemma 3.2. There exists universally large \( R > 0 \), such that \( \nu := \bar{U} - U_0 > 0 \) satisfies

\[
\begin{align*}
\Delta \nu &= 0 \quad \text{in } \Omega_0 \cap B_R, \\
\partial_n \nu + H (1 + O \left( \frac{\nu}{|x|} \right)) \nu &= 0 \quad \text{on } \partial \Omega_0 \cap B_R, \\
\end{align*}
\]

(3.1)

with \( \nu \) the interior unit normal to \( \partial \Omega_0 \). Moreover,

\[
O \left( \frac{\nu}{|x|} \right) = o(1), \quad \text{as } |x| \to \infty.
\]

Proof. Let \( \bar{U}_s(x) := s^{-1} \bar{U}(sx), s \geq 1 \). We prove that \( \nu_s := \bar{U}_s - U_0 \) satisfies

\[
\begin{align*}
\Delta \nu_s &= 0 \quad \text{in } \Omega_0 \cap A, \\
\partial_n \nu_s + Hv_s &= O(|\nu_s|^2) \quad \text{on } \partial \Omega_0 \cap A, \\
\end{align*}
\]

(3.2)

with \( \nu \) the interior unit normal to \( \partial \Omega_0 \), the constant in \( O(|\nu_s|^2) \) universal, and \( A \) an open annulus included in \( B_2 \setminus \bar{B}_1 \). Below \( A_1 := B_{2-1} \setminus \bar{B}_{1+l}, l < 1/2 \). 

Since \( \bar{U}_s, U_0 \) are harmonic in their positivity set, and \( \bar{U}_s \geq U_0 \), the harmonicity of \( \nu_s \)
in \( \Omega_0 \) is obvious. In order to prove the boundary condition, we proceed as follows. First, in view of Lemma 3.1

\[
\| \bar{U}_s - U_0 \|_{L^\infty (B_2 \setminus B_1)} \to 0, \quad \text{as } s \to \infty,
\]

thus, by the flatness implies regularity results \cite{3}, there exists \( \epsilon_0 \) universal such that if
\( s = s(\epsilon_0) \) is large enough so that

\[
\| \bar{U}_s - U_0 \|_{L^\infty (B_2 \setminus B_1)} \leq \epsilon_0
\]

(3.3)

then in \( A_{1/5} \cap \{ \bar{U}_s > 0 \} \), \( \| \bar{U}_s \|_{C^1} \leq C \) with \( C > 0 \) universal. Moreover, in a sufficiently
small ball \( B_{\rho}(x), x \in \partial \Omega_0 \cap A_{1/5}, \bar{U}_s \) and \( U_0 \) satisfy Proposition 5.1 (in view of the
convergence of $\bar{U}_s$ to $U_0$ and non-degeneracy), hence by a covering argument we obtain that in $A_{2/5} \cap \Omega_0$,
\[ \|\bar{U}_s - U_0\|_{L^\infty} \leq C \|\bar{U}_s - U_0\|_{\infty}, \quad \bar{U}_s - U_0 \sim \epsilon := \|\bar{U}_s - U_0\|_{\infty}, \quad (3.4) \]
s large, and $\epsilon = \epsilon(s) \ll \epsilon_0$.

Let us define the function $\tilde{\psi}_s > 0$, such that
\[ x \in F(U_0) \cap A_{2/5} \rightarrow x - \psi_s(x) v_s \in F(\bar{U}_s). \quad (3.5) \]

In view of (3.4), since $\partial_\nu U_0(x) = 1$ on $F(U_0)$, we have that $\partial_\nu \bar{U}_s(x) = 1 + O(\epsilon)$ on $F(U_0) \cap A_{2/5}$. Hence, $F(\bar{U}_s)$ is a graph over $F(U_0)$ locally in $A_{2/5}$, $\psi_s$ is well defined, it is bounded by $C\epsilon$, and in fact
\[ \psi_s(x) = v_s(x) + O(\epsilon^2). \quad (3.6) \]

Then,
\[ \nabla \bar{U}_s(x - \psi_s(x) v_s) = \nabla \bar{U}_s(x) - v_s(x) D^2 \bar{U}_s(x) v_s + O(\epsilon^2) \]
\[ = v_s(x) D^2 \bar{U}_s(x) v_s + \nabla v_s(x) + O(\epsilon^2). \]

Hence, again using (3.4), and the free boundary condition for $\bar{U}_s$, we get
\[ 1 = |\nabla \bar{U}_s(x - \psi_s v_s)|^2 = 1 + 2(\partial_\nu v_s - v_s (\partial_\nu U_0)) + O(\epsilon^2), \]
which gives the second condition in (3.2). Rescaling back and using that $H$ is homogeneous of degree $-1$ and bounded below on the annulus, we get that $v$ satisfies (3.1) as desired. Moreover, by Lemma 3.1,
\[ O\left(\frac{\nu}{|x|}\right) = o(1), \quad \text{as } |x| \rightarrow \infty. \]

Lemma 3.2 remains valid if $\nu := U_0 - \bar{U}$, after extending $U$ analytically in $\Omega_0 \cap B_R^c$ (for $R$ large enough), and moreover $\nu > 0$ in this region. Indeed, in the proof above, (3.3) holds for $\bar{U}_s$. Thus $\bar{U}_s$ can be extended analytically in $\Omega_0 \cap (B_2 \setminus \bar{B}_1)$, and all the estimates following (3.3) remain valid in $\Omega_0$.

**Remark 3.3.** Notice that, from (3.6), we deduce the expansion ($R$ large)
\[ \psi(x) = v(x) + O\left(\frac{\nu^2}{|x|}\right), \quad x \in F(U_0) \cap B_R^c. \]

While $v$ solves the perturbed linearized equation above in a subset of $\Omega_0$, at times our analysis leads to estimates that must be extended outside of $\Omega_0$. For that purpose, we use the following remark. Here, $A_1$ is the annulus defined in the previous Lemma.

**Remark 3.4.** If $u_1$, $u_2$ are critical points to $J$ in $A_0$, with $u_1 \geq U_0$ and
\[ \|u_i - U_0\|_{L^\infty} \leq \epsilon, \quad u_1 \geq u_2 + c\epsilon \quad \text{in } \tilde{\Omega}_0 \cap A_0, \]
for some $c > 0$ and $\epsilon > 0$ small depending on $c$, then
\[ u_1 \geq u_2 \quad \text{in } A_{1/5}. \]
Indeed, if we argue as for $\bar{U}$ above, we obtain that if $\psi_i$ is the associated function defining the free boundary of $u_i$ as in (3.5), then for $x \in A_{1/\delta} \cap F(U_0)$,
\[ u_i(x - t\nu) = u_i(x) - t + O(\epsilon^2), \quad t \in [0, \psi_i(x)] \]
Hence, for all $t$’s for which this expansion holds and $\epsilon$ small,
\[ u_2(x - t\nu) \leq u_1(x) - \epsilon \epsilon - t + O(\epsilon^2) \leq u_1(x - t\nu) - \epsilon \epsilon + O(\epsilon^2) < u_1(x - t\nu), \]
which gives the desired result.

### 3.2. Regularity

This subsection is devoted to the proof of parts (ii) – (vi) of Theorem 1.1. It relies on the following asymptotic expansion, which will be derived in the next section. Here, we use $V_{\gamma,\delta}$, which have been defined in the introduction (see (1.4),(1.5)).

**Proposition 3.5.** Let $v > 0$ be a classical solution to (3.1). If $\gamma_- \neq \gamma_+$, there exist $\delta' > 0$ small universal and $R_0 > 0$ large universal, such that in $B_{R_0}^c \cap \Omega_0$, either
\[ v(x) = a_- V_{\gamma_-}(x) + O(|x|^{-\gamma_- - \delta'}) \quad a_- > 0, \quad \text{if} \quad \gamma_- 
\]
or
\[ v(x) = a_+ V_{\gamma_+}(x) + O(|x|^{-\gamma_+ - \delta'}) \quad a_+ > 0. \quad \text{if} \quad \gamma_+ = \gamma_- \]

If $\gamma_- = \gamma_+$, then in $B_{R_0}^c \cap \Omega_0$
\[ v(x) = \bar{a} V_{\gamma_-}(x) + \bar{b} V_{\gamma_+} + O(|x|^{-\gamma_0 - \delta'}) \quad \bar{a} \geq 0, \bar{b} \in \mathbb{R}, \max \{\bar{a}, \bar{b}\} > 0. \]

With this proposition at hands, we can now obtain (ii) in Theorem 1.1, that is the following result.

**Theorem 3.6.** $F(\bar{U})$ is analytic.

**Proof.** Toward the proof of analyticity, we wish to obtain the following claim.

**Claim 1.** There is a universal (large) $R_0 > 0$ such that
\[ \nabla \bar{U}(x) \cdot x - \bar{U}(x) < 0, \quad \text{in} \quad \{\bar{U} > 0\} \cap B_{R_0}^c. \]

In order to prove Claim 1, we set $v := \bar{U} - U_0$. Recall that, in view of Lemma 3.2, $v > 0$ satisfies (3.1), and hence the asymptotic expansion in Proposition 3.5 holds. Notice that since $U_0$ is homogeneous of degree 1,
\[ \nabla \bar{U} \cdot x - \bar{U} = \nabla \bar{v} \cdot x - \bar{v}, \quad \text{in} \quad \bar{\Omega}_0. \]

On the other hand, by Proposition 3.5,
\[ v = Z_{\gamma_0} + O(|x|^{-\gamma_0 - \delta'}) \quad \text{in} \quad B_{R_0}^c \cap \bar{\Omega}_0, \quad (3.10) \]
with $Z_{\gamma_0}$ given by
\[ Z_{\gamma_0} = a V_{\gamma_0} \quad \text{if} \quad \gamma_- \neq \gamma_+, \quad a > 0, \]
and $\gamma_0$ either $\gamma_-$ or $\gamma_+$, or
\[ Z_{\gamma_0} = aV_{\gamma_-} + bV_{\gamma_+} \quad \text{if} \quad \gamma_- = \gamma_+ = \gamma_0, \quad a \geq 0, b \in \mathbb{R}, \quad \max\{a, b\} > 0. \]

Now, let \( v_s(x) = v(sx)/s \) and similarly \((Z_{\gamma_0})_s(x) = Z_{\gamma_0}(sx)/s\). Then, for \( s \) large, according to (3.10),
\[ v_s(x) = (Z_{\gamma_0})_s(x) + s^{-\gamma_0 - x^{-1}}O(1), \quad \text{in} \ (B_2 \setminus B_1) \cap \Omega_0, \]
and from the formula for \( Z_{\gamma_0} \) (see also (1.4)--(1.5)),
\[
\epsilon = \epsilon(s) := \begin{cases} 
\| (Z_{\gamma_0})_s \|_{\infty} \sim s^{-\gamma_0 - 1}, & \text{if} \ \gamma_- \neq \gamma_+,
\| (Z_{\gamma_0})_s \|_{\infty} \sim s^{-\gamma_0 - 1} \ln s, & \text{if} \ \gamma_- = \gamma_+.
\end{cases}
\]

From this we deduce that, for \( \sigma := \frac{\epsilon^\prime}{2(\gamma_0 + 1)} \),
\[
v_s(x) = (Z_{\gamma_0})_s(x) + O(e^{1+\sigma}), \quad \text{in} \ (B_2 \setminus B_1) \cap \Omega_0,
\]
and hence for universally large \( s \), we have \( \| v_s \|_{\infty} \leq C \epsilon \) in the annulus \((B_2 \setminus B_1) \cap \Omega_0\).

Next, in view of the first inequality in (3.4) we have (with the notation for annuli in the Proof of Lemma 3.2),
\[
\left\| \frac{v_s(x) - (Z_{\gamma_0})_s(x)}{\epsilon} \right\|_{2,\varepsilon} \leq C, \quad \text{in} \ A_2/5 \cap \Omega_0,
\]
while from the expansion above,
\[
\left\| \frac{v_s(x) - (Z_{\gamma_0})_s(x)}{\epsilon} \right\|_{\infty} \leq C \epsilon^\sigma, \quad \text{in} \ A_2/5 \cap \Omega_0.
\]

We can interpolate to obtain (because \( \partial \Omega_0 \setminus \{0\} \) is smooth),
\[
\left\| \frac{v_s(x) - (Z_{\gamma_0})_s(x)}{\epsilon} \right\|_{0,1} \leq C \epsilon^\sigma', \quad \text{in} \ A_7/15 \cap \Omega_0.
\]

Combining (3.11) with the estimate above, we now compute (for \( s \) large)
\[
\nabla v_s \cdot x - v_s = \nabla (Z_{\gamma_0})_s \cdot x - (Z_{\gamma_0})_s + O(e^{1+\sigma})
\leq -C \epsilon (\gamma_0 + 1)(1 + o(1)) + O(e^{1+\sigma}) \leq -c \epsilon \leq 0 \quad \text{in} \ A_7/15 \cap \Omega_0,
\]
where the first inequality follows from the formula for \( Z_{\gamma_0} \) (see also (1.4)--(1.5)). Unraveling the scaling in the inequality above, we obtain Claim 1 in \( B_{4 \epsilon} \cap \Omega_0 \).

However, we need to extend this inequality to \( \{ \hat{U}_s > 0 \} \). Let us take a point \( x_0 \in F(U_0) \) and argue as in Lemma 3.2, that is, in view of the \( C^{2,\varepsilon} \) estimate on \( \hat{U}_s \) and the fact that \( \hat{U}_s(x_0) = O(\epsilon), \ \partial \hat{U}_s(x_0) = 1 + O(\epsilon), \)
we conclude that \( F(\hat{U}_s) \) is included in an \( \epsilon \)-neighborhood of \( F(U_0) \). On the other hand, using that the \( C^{2,\varepsilon} \) norm of \( v_s \) is controlled by its \( L^{\infty} \) norm, and that by homogeneity \( \nabla U_0 \cdot x - U_0 \equiv 0 \), we get
\[
(\nabla \hat{U}_s \cdot x - \hat{U}_s)(x_0) = (\nabla v_s \cdot x - v_s)(x_0) \leq -c \epsilon,
\]
\[
\partial \epsilon (\nabla \hat{U}_s \cdot x - \hat{U}_s)(x_0) = \partial \epsilon (\nabla v_s \cdot x - v_s)(x_0) = O(\epsilon),
\]
from which we obtain (using the fact that the \( C^{2,\varepsilon} \) norm of \( \hat{U}_s \) is universally bounded) that for \( s \) large universal,
\[ \nabla \bar{U}_s \cdot x - \bar{U}_s < 0 \text{ in } A_{7/15} \cap \{ \bar{U}_s > 0 \}, \]

as desired.

Claim 1 implies (on a compact set),
\[ \frac{d}{dt} \bar{U}_t(x) \leq -\frac{\kappa_1}{t^2} \text{ for } tx \in \{ \bar{U} > 0 \} \cap (\bar{B}_{4R_0} \setminus B_{R_0}), \quad (\kappa_1 = \kappa_1(\bar{U})). \quad (3.12) \]

Given \( x \in \{ \bar{U} > 0 \} \cap (B_{4R_0} \setminus \bar{B}_{R_0}) \) let \( \bar{t} = \bar{t}(x) \) be the smallest value for which the ray \( tx \in \{ \bar{U} > 0 \} \cap (B_{4R_0} \setminus \bar{B}_{R_0}) \) for all \( \bar{t} \leq t \leq 1 \), that is \([\bar{t}, 1]\) is the maximal interval \( I \) for which \( tx \in \{ \bar{U} > 0 \} \cap (B_{4R_0} \setminus \bar{B}_{R_0}) \) for all \( t \in I \). Then,
\[ \bar{U}_t(x) - \bar{U}(x) = -\int_0^1 \frac{d}{dt} \bar{U}_t(x) \, dt \geq \kappa_1(1 - \bar{t}). \quad (3.13) \]

From this we deduce that \( \bar{U}_t(x) > 0 \), hence \( \bar{t}x \in \partial B_{R_0} \) and \( \bar{U}_t(x) \) is strictly decreasing in \( t \in [1/2, 1] \) when \( x \in \{ \bar{U} > 0 \} \cap (\bar{B}_{3R_0} \setminus \bar{B}_{2R_0}) \) (since for such values of \( t, x \), then \( tx \) belongs to the region in which (3.12) holds). Now, for \( \delta \) small and \( x \in \{ \bar{U} > 0 \} \cap (B_{3R_0} \setminus \bar{B}_{2R_0}) \), the same computation as in (3.13) gives that
\[ \bar{U}_{1-\delta}(x) - \bar{U}(x) \geq \kappa_1 \delta. \]

On the other hand, for any unit vector \( \tau \), since \( \bar{U}_{1-\delta} \) is Lipschitz, with Lipschitz constant in the annulus independent of \( \delta \), it follows from the inequality above that for \( x \in \{ \bar{U} > 0 \} \cap (B_{3R_0} \setminus \bar{B}_{2R_0}) \), and \( \mu = \kappa_1/L \),
\[ \bar{U}_{1-\delta}(x + \mu \delta \tau) \geq \bar{U}_{1-\delta}(x) - L \mu \delta \geq \bar{U}(x). \]

Hence,
\[ \bar{U}_{1-\delta}(x + \mu \delta \tau) \geq \bar{U}(x), \quad 2R_0 \leq |x| \leq 3R_0, \]

and by the comparison principle Proposition 2.3, we conclude that this inequality holds for all \( x \)'s in \( B_{3R_0} \). In particular, if \( x \in \{ \bar{U} > 0 \} \) then \( (1 - \delta)(x + \mu \delta \tau) \in \{ \bar{U} > 0 \} \), which gives (for \( \mu \) possibly smaller),
\[ B_{\frac{\mu}{2}}((1 - \delta)x) \subset \{ \bar{U} > 0 \} \cap B_{3R_0} \text{ for all } \delta \text{'s small enough}. \]

This implies the existence of a cone with vertex at \( x \) which is contained in the positivity set of \( u \) near the vertex. Letting \( x \) approach \( F(\bar{U}) \) we conclude that \( F(\bar{U}) \) is a Lipschitz radial graph, and by standard arguments it is also locally Lipschitz. By the classical regularity theory for one phase free boundaries \([1, 2, 17]\), we conclude that \( F(\bar{U}) \) is analytic in \( B_{R_0} \) (outside the ball \( B_{R_0} \) analyticity is already guaranteed by the flatness result).

We can now deduce the proof of parts (iii) – (vi) in Theorem 1.1.

**Proof of Theorem 1.1**(iii)–(vi). Claim 1 in the proof of Theorem 3.6 and Proposition 3.5, provide the statements in part (iii) – (iv) of Theorem 1.1. The statement in (v) follows immediately from (iv). We are left with the proof of (vi), from which we also deduce the universality of the coefficient in the expansion in (iv). Assume \( V \) is not identically equal to \( U_0 \). Then, the same arguments in Lemma 3.1, Lemma 3.2, and Proposition 3.5
can be applied to \( V \) giving that \( V \) is asymptotic to \( U_0 \), and

\[
V - U_0 = V_{\gamma_1}(d + o(1)) \quad \text{in } B_{R_0}^c \cap \Omega_0,
\]

(3.14)

with \( \gamma_1 \) either \( \gamma_- \) or \( \gamma_+ \), and \( d > 0 \). Here we write the expansion in such a way that it is not necessary to distinguish whether the roots are distinct or not.

On the other hand,

\[
\bar{U} - U_0 = V_{\gamma_0}(a + o(1)) \quad \text{in } B_{R_0}^c \cap \Omega_0,
\]

(3.15)

with \( \gamma_0 \) either \( \gamma_- \) or \( \gamma_+ \), and \( a > 0 \).

If \( \bar{U} \) and \( V \) both have expansions in terms of \( V_{\gamma_-} \) (resp. \( V_{\gamma_+} \)), then we can argue as follows. Define \( t_0 \) by,

\[
at_0^{-\gamma_-^{-1}} = d.
\]

Using the expansions above for a given \( t > t_0 \), we conclude that for \( |x| \) large, \( x \in \Omega_0 \),

\[
\frac{V - \bar{U}}{V_{\gamma_-}} = (d - at_0^{-\gamma_-^{-1}} + o(1)), \quad d - at_0^{-\gamma_-^{-1}} > 0.
\]

Thus, for any \( t > t_0 \), there exists \( R_t \) large such that in an annulus \( (B_{2R} \setminus B_R) \cap \Omega_0 \) with \( R \geq R_t \),

\[
V \geq c(t) \bar{U} + \bar{U}_t, \quad \bar{U} = \| V_{\gamma_-} \|_{L^\infty((B_{2R} \setminus B_R) \cap \Omega_0)},
\]

where we have used that \( V_{\gamma_-} \sim \| V_{\gamma_-} \|_\infty \) in \( (B_{2R} \setminus B_R) \cap \Omega_0 \). By a rescaled version of Remark 3.4 (in which \( \epsilon = \bar{U} / R \) can be chosen small depending on \( c(t) \) as long as \( R \) is large) we get \( V \geq \bar{U}_t \) in the annulus \( B_{2R} \setminus B_R \), and the comparison Proposition 2.3 gives, (as \( R \to \infty \)),

\[
V \geq \bar{U}_t, \quad \forall t > t_0, \quad \text{in } \mathbb{R}^n.
\]

Similarly,

\[
V \leq \bar{U}_t, \quad \forall t < t_0, \quad \text{in } \mathbb{R}^n,
\]

and the claim follows letting \( t \to t_0 \). If the expansions are different, say \( \bar{U} \) and \( V_{\gamma_-} \) have a expansions in terms of \( V \), respectively \( V_{\gamma_+} \), then arguing as above we conclude that \( U_t \geq V \) for all \( t \)'s, and by letting \( t \to \infty \) we obtain \( U_0 \geq V \), hence \( U_0 \equiv V \) a contradiction.

\[\square\]

4. The perturbed linearized problem

This section is intended for the proof of Proposition 3.5. For convenience, we recall some notation from the introduction and refer the reader to [6] for further details on the discussion below. Consider the problem,

\[
\begin{cases}
\Delta w = 0 & \text{in } \Omega_0 \\
\partial_\nu w + Hw = 0 & \text{on } \partial \Omega_0 \setminus \{0\},
\end{cases}
\]

(4.1)

with \( \nu \) the interior unit normal to \( \partial \Omega_0 \), and \( -(\partial_\nu U_0) = H > 0 \) the mean curvature of \( \partial \Omega_0 \) oriented toward the complement of the connected set \( \Omega_0 \). Let \( w := f(r)\bar{v}(\theta) \), with
$f \geq 0$ a radial function, $r := |x|$, and $\tilde{v}$ the corresponding first eigenfunction of the Laplacian on $S^{n-1} \cap \Omega$, that is,
\[
\Delta_{S^{n-1}} \tilde{v} = \lambda \tilde{v}, \quad \text{on } S^{n-1} \cap \Omega_0,
\]
satisfying the Neumann condition,
\[
\partial_{\nu} \tilde{v} + H \tilde{v} = 0, \quad \text{on } \partial \Omega_0 \cap S^{n-1}.
\] (4.2)
Then, $\tilde{v} > 0$ on $S^{n-1} \cap \Omega_0$ and $\lambda > 0$. We compute,
\[
\Delta w = \tilde{v} \Delta f + 2 \nabla \tilde{v} \cdot \nabla f + f \Delta \tilde{v} = \tilde{v} \left( f'' + (n - 1) \frac{f'}{r} + \lambda \frac{f}{r^2} \right),
\]
thus, for $f = r^{-\gamma}$, we obtain that $w$ solves (4.1) as long as $\gamma = \gamma_{\pm}$ satisfy $\gamma^2 - (n - 2)\gamma + \lambda = 0$. The stability of $U_0$ is equivalent to the fact that this quadratic equation must have real roots i.e. $(n - 2)^2 - 4\lambda \geq 0$. Moreover, $\lambda > 0$, thus $\gamma = \gamma_{\pm} \in \mathbb{R}$, $\gamma_+ \geq \gamma_- > 0$.

If $\gamma_{-} \neq \gamma_{+}$, we call
\[
V_{\gamma_{-}}(x) := |x|^{-\gamma_{-}} \tilde{v},
\]
while by abuse of notation, if $\gamma_{-} = \gamma_{+}$ and we call $\gamma_0$ this common value, we set
\[
V_{\gamma_{-}}(x) := |x|^{-\gamma_0} (\ln |x| + 1) \tilde{v}, \quad V_{\gamma_{+}}(x) := |x|^{-\gamma_0} \tilde{v}. \quad (4.3)
\]
Next, we build the following special family of functions, which will play an essential role in the proof of Theorem 1.1. Call $u_0 := |x|^{-1} U_0$, and define for real numbers $\gamma, \beta$, $A = A(\gamma)$
\[
W_{\gamma}^\beta(x) = \begin{cases} 
|x|^{-\gamma}(A \tilde{v} + u_0) & A > 0 \text{ if } 0 < \gamma < \gamma_, \\
|x|^{-\gamma}(A \tilde{v} + u_0) & A < 0 \text{ if } \gamma_- < \gamma < \gamma_+, \\
|x|^{-\gamma}(A \tilde{v} + u_0) + \beta V_{\gamma_+} & A > 0 \text{ if } \gamma > \gamma_+. 
\end{cases} \quad (4.4)
\]
In what follows, when $\gamma < \gamma_+$, we drop the dependence on $\beta$. Notice that our definition also includes the case $\gamma_+ = \gamma_+$. Then, using the formula above,
\[
\Delta W_{\gamma}^\beta = |x|^{-\gamma-2} \left\{ A [\gamma(\gamma - n + 2) + \lambda] \tilde{v} + u_0 [\gamma(\gamma - n + 2) + (1 - n)] \right\}
\]
and for $|A|$ large enough,
\[
\Delta W_{\gamma}^\beta \geq 0, \quad \text{in } \Omega_0.
\]
Moreover, using that $\tilde{v}$ solves (4.2), while $u_0 = 0, (u_0)_\nu = 1/|x|$ on $\partial \Omega \setminus \{0\}$, we get
\[
\partial_{\nu} W_{\gamma}^\beta + HW_{\gamma}^\beta = |x|^{-\gamma-1} \quad \text{on } \partial \Omega \setminus \{0\}. \quad (4.5)
\]
We remark that $W_\gamma > 0$ in $\Omega$ when $\gamma < \gamma_-$, and choosing $|A|$ large, $W_\gamma < 0$ in $\Omega$ when $\gamma_- < \gamma < \gamma_+$. The sign of $W_\gamma^\beta$ depends on the choice of $\beta$.

Having introduced this family of functions, we can now provide the proof of Proposition 3.5. The proof relies on a comparison principle for solutions to (3.1) in an annulus $(B_R \setminus B_1) \cap \Omega$, which we prove in the next lemma. Consider the problem,
\[
 \begin{cases}
 \Delta V = 0 & \text{in } D, \\
 \partial_{\nu} V = h(x)V & \text{on } T \subset \partial D,
 \end{cases} \quad (4.6)
\]
with $D$ a Lipschitz domain in $\mathbb{R}^n$, $T$ a smooth open subset of $\partial D$, and $\nu$ the inward unit normal to $T$.

**Lemma 4.1.** If there exists a classical supersolution to (4.6) (continuous up to $\partial D$) such that $V > 0$ on $\bar{D}$, then the comparison principle holds, that is, if $z$, $w$ are respectively a classical supersolution and a classical subsolution to (4.6), with $z \geq \nu$ on $\partial D \setminus T$, then $z \geq \nu$ in $D$.

**Proof.** Let $w := v - z$, and set $M := \max_D \frac{w}{V} = \frac{w}{V}(x_0)$ for some $x_0 \in \bar{D}$. If $x_0 \in \partial D \setminus T$, then by our assumptions $M \leq 0$, which implies our claim. Similarly, if $x_0 \in D$, then $MV - w$ attains a minimum at $x_0$ and by the maximum principle $MV \equiv w$, which contradicts our assumptions. Finally, $x_0$ cannot occur on $T$. Indeed, again $MV - w$ attains a minimum at $x_0$ and by the Hopf Lemma, $\partial_{\nu}(MV - w)(x_0) > 0$. On the other hand,

$$0 < M\partial_{\nu}V(x_0) - \partial_{\nu}w(x_0) \leq h(x_0)(MV - w)(x_0) = 0,$$

a contradiction. $\square$

Besides the above comparison principle we also use the Harnack inequality for Neumann problems, see for example [14]. The precise statement is provided in the Appendix.

We are now ready to prove **Proposition 3.5.**

**Proof of Proposition 3.5.** The proof is divided into five steps. For simplicity, in each step we consider first the case when $\gamma_- \neq \gamma_+$, and then point out the modifications needed for the case $\gamma_- = \gamma_+$.

Since $U_0$ is homogeneous of degree 1, after a dilation $x \to \rho x$, $\rho = \rho(\delta)$ large, we may assume that

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \cap B_1^c \\ \partial_{\nu}v + H(1 + o(1))v = 0 & \text{on } \partial \Omega \cap B_1, \end{cases}$$

(4.7)

with

$$|o(1)| \leq \delta,$$

(4.8)

for $\delta > 0$ to be made precise later. By abuse of notation, in what follows all dilations of $v$ will still be denoted by $v$.

**Step 1:** Decay at infinity. Let $v$ satisfy (4.7)-(4.8). Given $0 < \gamma < \gamma_-$, there exists $M = M(\gamma, \nu) > 0$ large such that

$$v \leq M|x|^{-\gamma} \quad \text{on } \Omega \cap B_1^c.$$

For this step, we do not need to distinguish whether $\gamma_-$ and $\gamma_+$ coincide or not. Indeed, let $\gamma$ be given and the constants below possibly depend on $\gamma$. To prove the desired bound, we use the function $W_{\gamma}$ defined in (4.4) (we drop the dependence on $\beta$ in the regime $\gamma < \gamma_-$), and we construct a subsolution $W = -CV_{\gamma_-} + cW_{\gamma}$ to (4.7), such that for large $C$ and small $c$,

$$W \leq R^{-\gamma} \quad \text{on } \partial B_R \cap \Omega, \quad R \text{ large,} \quad W \leq 0 \quad \text{on } \partial B_1 \cap \Omega.$$  

(4.9)
This is possible because $\gamma < \gamma_-$, hence $W_\gamma > 0$ is the leading term. For the boundary condition to yield a subsolution, we need,
\[
\partial_\nu W + H(1 + o(1))W \geq 0 \quad \text{on } \partial\Omega_0 \cap B'_1,
\]
which in view of (4.5) and (4.8) holds as long as (recall that $V_{\gamma_-}$ solves (4.1))
\[-CH\delta V_{\gamma_-} + c|x|^{-\gamma} - c\delta HW_\gamma \geq 0, \quad \text{on } \partial\Omega_0 \cap B'_1.
\]
This can be achieved by choosing $\delta$ small.
Moreover, again since $W_\gamma$ is the leading term, we can pick $R < R$ large, such that
\[
R^{-\gamma_-} \leq W \quad \text{on } \partial B_R \cap \Omega_0.
\] (4.10)

Now, for $M$ large to be specified later, let us assume by contradiction that $v(\bar{x}) > MR^{-\gamma}$ at some point on $\partial B_R \cap \Omega_0$. Then by the Harnack inequality for the Neumann problem (3.1) (see Theorem 5.2), we have that $v \geq \bar{c}MR^{-\gamma}$ on $\partial B_R \cap \Omega_0$, with $\bar{c} > 0$ universal. Thus, using (4.9), and the fact that $v \geq 0$, we have that
\[
v \geq \bar{c}MW \quad \text{on } (\partial B_R \cup \partial B_1) \cap \Omega_0.
\]
By Lemma 4.1, which can be applied because $v$ itself is a solution to (4.7) which is strictly positive up to the boundary, we get
\[
v \geq \bar{c}MW \quad \text{in } (B_R \setminus B_1) \cap \Omega_0.
\]
In particular, by (4.10) above,
\[
v \geq c(R)M \quad \text{on } \partial B_R \cap \Omega_0,
\]
hence we reach a contradiction if $M$ is large enough, and Step 1 is proved.

Now, choosing $\gamma = \gamma_-/2$, for a universal $\varepsilon_0 = \gamma_-/2 + 9/10$ we have by Step 1 that,
\[
\frac{v(x)}{|x|} = o(|x|^{-\varepsilon_0}), \quad |x| \to \infty
\]
and hence, again after a dilation, we can assume that $v$ satisfies:
\[
\begin{cases}
\Delta v = 0 & \text{in } \Omega_0 \cap B_1^r,
\partial_\nu v + H(1 + o(|x|^{-\varepsilon_0}))v = 0 & \text{on } \partial\Omega_0 \cap B_1^r,
\end{cases}
\] (4.11)
with
\[
|o(|x|^{-\varepsilon_0})| \leq \delta|x|^{-\varepsilon_0},
\] (4.12)
and $\delta$ small universal to be chosen later.

**Step 2:** Improved Decay at infinity. Let $v$ satisfy (4.11)–(4.12). There exists $M_- > 0$ large, such that
\[
\frac{v}{V_{\gamma_-}} \leq M_- \quad \text{on } \Omega_0 \cap B_1^r.
\]
The argument follows the lines of Step 1, but we need to distinguish whether $\gamma_-$ and $\gamma_+$ coincide or not. If the roots are distinct, fix $\gamma \in (\gamma_-, \gamma_+)$. We construct a subsolution $W = V_{\gamma_-} + W_\gamma$ to (4.11), such that $(R = R(\gamma) \text{ large})$,
\[ W \leq \frac{3}{2} V_{\gamma_-} \text{ on } \partial B_R \cap \Omega_0, \quad W \leq 0 \text{ on } \partial B_1 \cap \Omega_0. \]  

(4.13)

The desired inequalities can be achieved as \( W/V_{\gamma_-} \to 1 \) as \( r \to \infty \) and by choosing \(|A|\) possibly larger. Moreover, for the same reason, we can pick a large \( R < R^* \) such that

\[ \frac{1}{2} V_{\gamma_-} \leq W \text{ on } \partial B_R \cap \Omega_0. \]  

(4.14)

Now, for the boundary inequality to be satisfied, we need to choose (see (4.12)),

\[ \gamma_- < \gamma \leq \gamma_- + \epsilon_0, \]

and \( \delta \) small enough, so that

\[ -\delta |x|^{-2\beta} V_{\gamma_-} + |x|^{-\gamma - 1} - \delta H |x|^{-2\beta} W_{\gamma} \geq 0 \text{ on } \partial \Omega_0 \cap B^1_1. \]  

(4.15)

The argument is now the same as in Step 1. Given \( M \) large, to be specified later, let us assume by contradiction that \( \frac{\nu}{V_{\gamma_-}} \geq M \) at some point at some point on \( \partial B_R \cap \Omega_0 \). Then by the Harnack inequality in Theorem 5.2, we have that \( \frac{\nu}{V_{\gamma_-}} \geq c M \) on \( \partial B_R \cap \Omega_0 \). This combined with (4.13) and Lemma 4.1 (again since \( v > 0 \)) gives that,

\[ \frac{\nu}{V_{\gamma_-}} \geq cM \frac{W}{V_{\gamma_-}} \text{ in } (B_R \setminus B_1) \cap \Omega_0. \]

Hence, in view of (4.14),

\[ \nu \geq c(\bar{R}) M \text{ on } \partial B_R \cap \Omega_0, \]

which is a contradiction if \( M \) is large enough.

In the case when \( \gamma_- = \gamma_+ \), we need to choose \( W = V_{\gamma_-} + W^\beta \) and let \( \beta \) be negative and \(|\beta|\) large, so that the second inequality in (4.13) holds (see (4.4) for the definition of \( W^\beta \)).

**Step 3:** Limit at infinity and expansion. Let \( v \) satisfy (4.11)–(4.12). Then,

\[ \lim_{|x| \to \infty} \frac{\nu}{V_{\gamma_-}} = a_- \geq 0. \]

Moreover, if \( \gamma_- \neq \gamma_+ \),

\[ v(x) = a_- V_{\gamma_-}(x) + O(|x|^{-\gamma_- - 2}), \quad \text{in } \Omega_0 \cap B^c_1, \]

(4.16)

while if \( \gamma_- = \gamma_+ \),

\[ v(x) = a_- V_{\gamma_-}(x) + O(V_{\gamma_+}), \quad \text{in } \Omega_0 \cap B^c_1. \]

(4.17)

We consider first that case when \( \gamma_- \neq \gamma_+ \). For \( \rho \geq 1 \), let

\[ a(\rho) := \sup \{ a \geq 0 \mid v \geq a V_{\gamma_-} \text{ in } B^c_\rho \cap \Omega_0 \}. \]

This is an increasing function which in view of Step 2 is bounded by \( M_- \). Thus, there exists

\[ a_- := \lim_{\rho \to \infty} a(\rho) \geq 0, \]

and by definition of \( a(\rho) \),
\[
\liminf_{|x| \to \infty} \frac{v}{V_{\gamma_-}} \geq a_-.
\] (4.18)

We wish to show that
\[
\limsup_{|x| \to \infty} \frac{v}{V_{\gamma_-}} \leq a_-,
\]
from which our claim will follow. Assume by contradiction that for a small \(\eta > 0\) along a sequence of points \(x_k \in \Omega_0\) with \(\rho_k := |x_k| \to \infty\),
\[
\frac{v}{V_{\gamma_-}}(x_k) \geq a_- + \eta,
\]
while in view of (4.18), for \(k\) large and \(\epsilon \ll \eta\),
\[
\frac{v}{V_{\gamma_-}} \geq a_- - \epsilon \quad \text{in} \quad B_{\rho_k/2} \cap \Omega_0.
\]
(4.20)

Thus, by the Harnack inequality (Theorem 5.2) applied to \(v - (a_- - \epsilon)V_{\gamma_-}\), we conclude that for \(k\) large,
\[
\frac{v}{V_{\gamma_-}} \geq a_- + c\eta, \quad \text{on} \quad \partial B_{\rho_k} \cap \Omega_0.
\]
(4.21)

Indeed, let us call \(w := v - (a_- - \epsilon)V_{\gamma_-}\) and consider for \(\frac{1}{2} < |x| < 2\) the functions
\[
w_k(x) = \rho_k^\gamma v(\rho_k x) - (a_- - \epsilon)V_{\gamma_-}(x).
\]
In view of (4.20), \(w_k \geq 0\), and by (4.19), \(w_k(\bar{x}_k) \geq c\eta\) at some point \(\bar{x}_k \in \partial B_1 \cap \Omega_0\). On the other hand, \(v_k(x) = \rho_k^\gamma v(\rho_k x)\) satisfies
\[
\begin{cases}
\Delta v_k = 0 & \text{in} \quad \Omega_0 \cap (B_2 \setminus B_{1/2}), \\
\partial_{\nu} v_k + H(1 + o_k(1)) v_k = 0 & \text{on} \quad \partial \Omega_0 \cap (B_2 \setminus B_{1/2}),
\end{cases}
\]
with \(o_k(1) \to 0\) as \(k \to \infty\). Furthermore, in view of Step 2, \(v_k\) is uniformly bounded since \(v_k \leq M \sup_{\gamma_-} v \leq C\) on \(\Omega_0 \cap (B_2 \setminus B_{1/2})\). Thus by standard regularity estimates, the \(v_k\)'s are uniformly Hölder continuous in \((B_{15/8} \setminus B_{5/8}) \cap \Omega_0\) and, up to extracting a subsequence, \(w_k\) converges uniformly to a nonnegative limiting function \(\bar{w}\) which solves the unperturbed problem (4.1) in \((B_{7/4} \setminus B_{3/4}) \cap \Omega_0\) and \(\bar{w}(\bar{x}) \geq c\eta\) at some point on \(\partial B_1 \cap \Omega_0\). By the Harnack inequality and the uniform convergence we get, \(w_k \geq c\eta\) on \(\partial B_1 \cap \Omega_0\) for \(c\) universal, and unraveling the scaling this gives the desired claim (4.21).

As in Step 2, we now build a subsolution
\[
W = \left( a_- + \frac{c}{2} \eta \right) V_{\gamma_-} + W_{\gamma}, \quad \gamma_- < \gamma \leq \gamma_0, \quad \gamma \leq \gamma_+,
\]
such that for \(k \geq k_0\) large
\[
W \leq (a_- + c\eta) V_{\gamma_-} \quad \text{on} \quad \partial B_{\rho_k} \cap \Omega_0.
\]
Thus by Lemma 4.1, for \(k\) large,
\[
v \geq W \quad \text{in} \quad (B_{\rho_k} \setminus B_{\rho_0}) \cap \Omega_0,
\]
hence outside a very large ball
\[ \frac{v}{V_{\gamma_-}} \geq a_- + \frac{\epsilon}{4} \eta, \quad \text{in } B_R^c \cap \Omega_0, \]
contradicting the definition of \( a_- \). Finally, in order to obtain the expansion, we use a similar argument as above to trap \( v \) in any annulus \( B_R \setminus B_1 \) with \( R \) large, between a sub-solution and a supersolution, respectively,
\[ (a_- - \epsilon)V_{\gamma_-} + CW_{\gamma}, \quad (a_- + \epsilon)V_{\gamma_-} - CW_{\gamma}, \tag{4.22} \]
by choosing \( C \) large enough. Letting \( \epsilon \) go to zero and rescaling back, we get the desired expansion with \( \gamma' = \gamma - \gamma_- \).

There are two modifications needed in the case \( \gamma_0 = \gamma_+ \). The first one is the definition of \( w_k \) and \( v_k \). Since \( V_{\gamma_-} \) is no longer homogeneous, we set (with \( C_{22} \) defined at the beginning of Section 4, see for example (4.2))
\[ v_k(x) = \frac{1}{\ln \rho_k + 1} \rho_k^{\gamma_-} v(\rho_k x), \]
and
\[ w_k(x) = v_k(x) - (a_- - \epsilon) \left[ V_{\gamma_-}(x) + \frac{|x|^{-\gamma_-} \ln |x|}{\ln \rho_k + 1} v \right]. \]

The second one is in the construction of the subsolution/supersolution in the last step of the previous argument. Indeed the subsolution and the supersolution in (4.22) must be replaced by
\[ (a_- - \epsilon)V_{\gamma_-} + CW_{\gamma}^{\beta}, \quad (a_- + \epsilon)V_{\gamma_-} - CW_{\gamma}^{\beta}, \tag{4.23} \]
with \( \beta < 0 \) and \( |\beta| \) large.

**Step 4:** The case \( \gamma_- \neq \gamma_+ \). (Comparison with \( V_{\gamma_-} \)) Let \( v \) satisfy (4.11)–(4.12) and assume that \( a_- = 0 \) (from Step 3) and
\[ \frac{v}{V_{\gamma_-}} = O(|x|^{-\gamma'}), \quad \text{in } B_1^c, \quad 0 < \gamma' \leq \alpha_0, \quad \gamma' < \gamma_+ - \gamma_- . \]
There exists \( M_+, R > 0 \) large, such that
\[ \frac{v}{V_{\gamma_+}} \leq M_+ \quad \text{on } B_R^c \cap \Omega_0. \]

Since \( a_- = 0 \), we can repeat the same arguments as above with \( \alpha_0 \) replaced by \( \alpha_1 = \gamma' + \gamma_- + 9/10 > \alpha_0 \), that is a dilation of \( v \) satisfies (4.11) with \( \alpha_0 \) replaced by \( \alpha_1 \). If \( \gamma_- + \alpha_1 < \gamma_+ \), we use Lemma 4.1 and compare \( v \) with the supersolution (choose \( \gamma \leq \gamma_- + \alpha_1 \) according to a similar computation as in (4.15))
\[ Z = \epsilon V_{\gamma_-} - CW_{\gamma}, \]
We check that
\[ v \leq Z \quad \text{on } \partial B_1 \cap \Omega_0, \quad \text{for } C \text{ large} \]
and
\( v \leq Z \) on \( \partial B_R \cap \Omega_0 \), for \( R \) large.

The first inequality holds as \( W_\gamma < -\tilde{C} < 0 \). The second inequality follows as \( a_- = 0 \) and \( \frac{W}{v \to 0} \) as \(|x| \to \infty\). As \( \epsilon \to 0 \) we get that

\[
v \leq -C W_\gamma \quad \text{in} \quad B_1^c \cap \Omega_0.
\]

We repeat the same argument with \( z_2 = \tilde{\gamma} + 9/10 > \alpha_i \), and we continue till we reach the first \( l \geq 1 \) for which \( \gamma_+ + \alpha_i > \gamma_+ \). Then, via Lemma 4.1, we compare \( v \) with the supersolution \( (\gamma_+ < \tilde{\gamma} \leq \gamma_+ + \alpha_i) \)

\[
Z = c V_\gamma - CW_\gamma^\beta,
\]

(again see (4.4) for the definition of \( W_\gamma^\beta \)) and choose \( \beta < 0 \) and \( |\beta| \) large enough to guarantee that \( v \leq Z \) on \( \partial B_1 \cap \Omega_0 \) (a similar computation as in (4.15) can be carried out again). Therefore, we obtain that

\[
v \leq -C W_\gamma^\beta \quad \text{in} \quad B_1^c \cap \Omega_0,
\]

and the claim follows from the definition of \( W_\gamma^\beta \).

**Step 5:** Separation from 0 and Expansion. Let \( v \) solve (4.11)–(4.12) and whenever \( \gamma_- \neq \gamma_+ \) replace \( \alpha_0 \) by \( \alpha_i \) from Step 4. Then,

\[
v \geq \tilde{c} V_{\gamma_+}, \quad \text{in} \quad B_2^c \cap \Omega_0, \quad \tilde{c} > 0.
\] (4.24)

To see this, we let \( a = v(\tilde{x}) > 0 \) for some \( \tilde{x} \in \partial B_2 \cap \Omega_0 \). By the Harnack inequality in \( (B_3 \setminus B_1) \cap \Omega_0 \), \( v \sim a \) on \( \partial B_2 \cap \Omega_0 \). Define (the subsolution) \( W := -\epsilon V_{\gamma_-} + W_1^\gamma \) (with \( \gamma > \gamma_+ ) \),\(^1\) such that \( W \leq C \) on \( \partial B_2 \cap \Omega_0 \) and \( W \leq 0 \) for \( R \) large. As usual, we are using the function \( W_\gamma^\beta \) defined in (4.4). Then, \( v \geq c a W \) in \( (B_R \setminus B_2) \cap \Omega_0 \) and by letting \( \epsilon \to 0 \) and using the definition of \( W_1^\gamma \), we get the desired statement.

To conclude the proof of the expansions (3.8)–(3.9), we prove the following improvement of oscillation. First we consider the case of distinct roots.

**Improvement of oscillation.** There exist positive sequences \( \{a_k\} \) increasing and \( \{b_k\} \) decreasing with

\[
b_{k+1} - a_{k+1} = (1 - \epsilon)(b_k - a_k)
\]

such that

\[
a_k \leq \frac{v}{V_{\gamma_+}} \leq b_k, \quad \text{in} \quad B_2^c \cap \Omega_0.
\] (4.25)

Let (4.25) hold for some \( k \geq 1 \), call \( \rho_k := 2^k \), and for \( \mu > 0 \) small, write \( b_k - a_k = \rho_k^{-\mu} \). Define,

\[
w(x) := \rho_k^{\mu+\gamma_+} (v - a_k V_{\gamma_+})(\rho_k x), \quad x \in B_1^c \cap \Omega_0.
\]

Then, \( w \) is harmonic in \( B_1^c \cap \Omega_0 \) and it satisfies the boundary condition

\[
w + Hw = o(|x|^{-\alpha_+ - 1}) \quad \text{on} \quad \partial \Omega_0 \cap B_1^c.
\]

\(^1\)In view of the iteration argument at the end of the previous step, this also works in the case of distinct roots.
with \(|o(|x|^{-\gamma_1-\gamma_2})| \leq \delta |x|^{-\gamma_1-\gamma_2} \). Moreover, in view of (4.25),
\[
0 \leq \frac{w}{V_{r_+}} \leq 1, \quad \text{in } \Omega_0 \cap B_1^r.
\]
Assume that \(\frac{w}{V_{r_+}} \geq \frac{1}{2} \) at some fixed point \(x_0 \in \partial B_{3/2} \cap \Omega_0\). Then by the Harnack inequality for a Neumann problem (see Theorem 5.2), \(\frac{w}{V_{r_+}} \geq c\) on \(\partial B_{3/2} \cap \Omega_0\). We build a sub-solution \(W\) (to the problem satisfied by \(w\)), such that
\[
W \leq c, \quad \text{on } \partial B_{3/2} \cap \Omega_0, \quad W \leq 0 \quad \text{on } \partial B_R \cap \Omega_0, R \text{ large.} \tag{4.26}
\]
Set
\[
W = c_1 V_{r_+} + c_2 W_0 - c V_{r_-}, \quad \gamma > \gamma_+.
\]
Then, the first inequality in (4.26) is satisfied for \(c_1, c_2\) small, while the second one follows as \(W/V_{r_+} \to -\infty\) as \(|x| \to \infty\). We need to verify that
\[
W_v + HW \geq \delta (|x|^{-\gamma_1-\gamma_2}) \quad \text{on } \partial \Omega_0 \cap B_{3/2}^r,
\]
or equivalently,
\[
c_2 |x|^{-\gamma - 1} \geq \delta |x|^{-\gamma_2-\gamma_1} \quad \text{on } \partial \Omega_0 \cap B_{3/2}^r.
\]
This holds for \(\delta\) small, as long as \(\gamma < \gamma_1 + \gamma_+\). By Lemma 4.1, which can be applied because \(V_{r_+}\) is a solution to (4.1), which is strictly positive on the closure of \(D := (B_R \setminus B_{3/2}) \cap \Omega_0\), while \(W - w\) is a subsolution to (4.1) in \(D\), non-positive on \((\partial B_R \cup \partial B_{3/2}) \cap \Omega_0\),
\[
w \geq W \quad \text{in } (B_R \setminus B_{3/2}) \cap \Omega_0
\]
and by letting \(\epsilon \to 0\) we deduce,
\[
w \geq c_1 V_{r_+} \quad \text{in } B_2^r \cap \Omega_0,
\]
which after unraveling what \(w\) is, gives the desired improvement.

When \(\gamma_- = \gamma_+\) the only modification is that, using the expansion in Step 3, (4.25) will be satisfied by \(v - a_- V_{r_-}\), and
\[
w := w(x) := p_k^{\mu_+ \gamma_{+}} (v - a_- V_{r_-} - a_k V_{r_+})(p_k x). \tag{4.27}
\]
The rest of the proof works in the same way.

The proof of the proposition is now complete. Indeed, in the case \(\gamma_- \neq \gamma_+\), (4.16) immediately gives the desired expansions (3.7) if \(a_- \neq 0\). Otherwise, by standard arguments, the improvement of oscillation in Step 5 leads to (3.8) (after rescaling back), with \(\bar{a} > 0\) in view of (4.24). If \(\gamma_- = \gamma_+\), again the improvement of oscillation in Step 5 and (4.24) give (3.9) (see (4.27)).

\[\square\]

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Appendix

In this Appendix, we provide a $C^{2,\alpha}$ estimate for the difference of two nearby smooth solutions of (1.2) in terms of the $L^\infty$ norm. This estimate is essential in our proof of Theorem 1.1, and it relies on the use of the hodograph transform. We refer to [17] for further details on this important tool. Here, universal constants only depend on dimension.

**Proposition 5.1.** Let $u_1, u_2$ be classical solutions to (1.2) in $B_1$, with
\[(x_n - \epsilon_1)^+ \leq u_1 \leq u_2 \leq (x_n + \epsilon_1)^+,\]for $\epsilon_1$ small universal. Then, $v := u_2 - u_1$ satisfies
\[\|v\|_{C^2(B_1/2)} \leq C\|v\|_{L^\infty(B_1)}, \quad \frac{v(x)}{v(y)} \leq C, \quad x, y \in B_1/2 \cap \{u_1 > 0\},\]with $C > 0$ universal.

**Proof.** Let $k = 1, 2$. In view of the flatness assumption (5.1), if $\epsilon_1$ is chosen small (depending on $n$) such that the improvement of flatness Theorem in [3] holds, then $\|u_k\|_{C^1(B_1/4)} \leq C$, with $C$ universal. Thus, for $c > 0$ universal, (again by (5.1)),
\[1 - c \leq \partial_n u_k \leq 1 + c, \quad \text{in } B_{3/4} \cap \{u_k > 0\}.
\]Then, we can perform the partial hodograph transform $y' = x', y_n = u_k(x)$, with $\mathcal{H}_{u_k}$ the inverse mapping given by the partial Legendre transform, and obtain
\[
\begin{cases}
\mathcal{F}(D^2\mathcal{H}_{u_k}, \nabla \mathcal{H}_{u_k}) = 0 & \text{in } B_{1/2} \cap \{y_n > 0\}, \\
g(\nabla \mathcal{H}_{u_k}) = 0, & \text{on } B_{1/2} \cap \{y_n = 0\}.
\end{cases}
\]The free boundary of $u_k$ is given by the graph of the trace of $\mathcal{H}_{u_k}$ on $\{y_n = 0\}$.

The difference $\psi := \mathcal{H}_{u_2} - \mathcal{H}_{u_1} \geq 0$ will then satisfy an equation of the form
\[a^{ij} \partial_i \partial_j \psi + b^i \partial_i \psi = 0 \quad \text{in } B_{1/2} \cap \{y_n > 0\}\]with boundary condition,
\[\psi_n = d^i \partial_i \psi \quad \text{on } B_{1/2} \cap \{y_n = 0\},\]and $a^{ij}, b^i$ Hölder continuous and $d^i \in C^{1,\alpha}$ with norms depending on the $C^{2,\alpha}$ norm of $\mathcal{H}_{u_k}$ which in turn depends on the $C^{2,\alpha}$ norm of $u_k$ (hence is bounded by a universal constant). By the standard regularity estimates in Theorem 5.2, the $C^{2,\alpha}$ norm of $\psi$ is controlled by its $L^\infty$ norm. Moreover by the Harnack inequality, Theorem 5.2, $\psi \sim \|\psi\|_{L^\infty} \in B_{1/4} \cap \{y_n > 0\}$. Thus,
\[u_1(x) = u_2(x', x_n - \psi(x', u_1(x))), \quad x \in B_{1/8} \cap \{u_1 > 0\},\]for $\psi$ as above, from which our claim follows. \qed

We conclude the Appendix by stating the Schauder estimates and the Harnack inequality that we need for boundary problems.

**Theorem 5.2.** Let $v$ satisfy,
\[a^{ij} \partial_i \partial_j v + b^i \partial_i v + cv = f \quad \text{in } B_1 \cap \Omega, \quad \beta \cdot \nabla v + hv = g \quad \text{on } B_1 \cap \partial \Omega,\]with $a^{ij}$ uniformly elliptic,
\[\Omega := \{x_n > \phi(x')\}, \quad \phi(0) = 0,\]$v$ the inner normal to $\partial \Omega$, and $\beta \cdot v > 0$.\[\]
(i) If $a^{ij}, b^i, c, f \in C^{0, \alpha}, \beta, h, g \in C^{1, \alpha}, \phi \in C^{1, \alpha}$, then
\[ \|v\|_{C^{2, \alpha}(B_{1/2}(\Omega))} \leq C(\|v\|_{\infty} + \|f\|_{0, \alpha} + \|g\|_{1, \alpha}); \]

(ii) If $v \geq 0$, $a^{ij}, b^i, c, \beta, h \in C^{0, \alpha}, \phi \in C^1$, then
\[ \sup_{B_{1/2} \cap \Omega} v \leq C \left( \inf_{B_{1/2} \cap \Omega} v + \|f\|_{\infty} + \|g\|_{\infty} \right), \]
for $C > 0$ depending on $n$, the coefficients, the ellipticity constants, and $\Omega$.

Theorem 5.2 is contained for example in [14]. However (ii) is often stated under a sign assumption on the lower order coefficients, that is $c, h \leq 0$, which is needed for the existence theory. We point out that this is not needed for the Harnack inequality, and the statement without sign assumption can be deduced from Corollary 3.5. in [18]. The reason why the sign does not play a role is that, after a dilation, the lower order coefficients are arbitrarily small and the standard arguments continue to hold.

We remark that in the Proof of Proposition 3.5 the estimate (ii)-Theorem 5.2 without the sign assumption is only needed in the form of Lemma 5.3 below, for which we briefly sketch the proof.

**Lemma 5.3.** Let $v \geq 0$ satisfy
\[ \Delta v = 0 \quad \text{in} \; B_1 \cap \Omega, \]
\[ \partial_n v + hv = g \quad \text{in} \; B_1 \cap \partial \Omega, \]
with $\Omega$ as in Theorem 5.2, and $v$ the inner unit normal to $\partial \Omega$. If $v(\bar{x}) \geq 1$ at $\bar{x} = \frac{1}{4}e_n$, then
\[ \inf_{B_{1/2} \cap \Omega} v \geq c - C\|g\|_{\infty}, \]
for $c, C > 0$ depending on $n, \|h\|_{\infty}, \Omega$.

**Proof.** After a dilation and a rotation, we may reduce to the case when $\|h\|_{\infty}, \|g\|_{\infty}, \|\phi\|_{C^{0, 1}} \leq \epsilon_0$ for some $\epsilon_0$ small, and we need to show that
\[ v \geq \frac{c}{2} \quad \text{in} \; B_{r_0} \cap \Omega, \]
for some small $r_0$.

By the interior Harnack inequality, $v \geq c_0$ on $B_{1/8}(\bar{x})$. Let
\[ V := c_1 \left( |x - \bar{x}|^{-2n} - \left( \frac{1}{2} \right)^{2-n} \right)^+, \quad \text{in} \; B_{1/2}(\bar{x}) \]
with $c_1 := c_0(1/8)^{-2n} - (1/2)^{2-n} \gamma$. We claim that,
\[ v \geq V \quad \text{on} \; B_{1/2}(\bar{x}) \cap \Omega, \]
from which the desired bound follows. Indeed, let $w := v - V$ and assume $w(x_0) := \min_{B_{1/2}(\bar{x}) \cap \Omega} w < 0$. By the maximum principle, $x_0$ cannot occur in $(B_{1/2}(\bar{x}) \setminus B_{1/8}(\bar{x})) \cap \Omega$. Thus, we only need to rule out that $x_0 \in B_{1/2}(\bar{x}) \cap \partial \Omega$. On the other hand, at such point $w_t(x_0) \geq 0$, hence $v_t(x_0) \geq V_t(x_0) \geq c_2 > 0$, if $\epsilon_0$ is chosen small. Therefore,
\[ \epsilon_0 + \epsilon_0 V(x_0) \geq \epsilon_0 + \epsilon_0 v(x_0) \geq g(x_0) - h(x_0)v(x_0) = \partial_n v(x_0) \geq c_2, \]
and we reach a contradiction by choosing $\epsilon_0$ small enough. \qed