The $S_3$ flavour symmetry: Neutrino masses and mixings

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Abstract

In this work, we discuss the neutrino masses and mixings as the realization of an $S_3$ flavour permutational symmetry in two models, namely the Standard Model and an extension of the Standard Model with three Higgs doublets. In the $S_3$ Standard Model, mass matrices of the same generic form are obtained for the neutrinos and charged leptons when the $S_3$ flavour symmetry is broken sequentially according to the chain $S_3L \otimes S_3R \supset S_3^{\text{diag}} \supset S_2$. In the minimal $S_3$-symmetric extension of the Standard Model, the $S_3$ symmetry is left unbroken, and the concept of flavour is extended to the Higgs sector by introducing in the theory three Higgs fields which are $SU(2)$ doublets. In both models, the mass matrices of the neutrino and charged leptons are reparametrized in terms of their eigenvalues, and exact, explicit analytical expressions for the neutrino mixing angles as functions of the masses of neutrinos and charged leptons are obtained. In the case of the $S_3$ Standard Model, from a $\chi^2$ fit of the theoretical expressions of the lepton mixing matrix to the values extracted from experiment, the numerical values of the neutrino mixing angles are obtained in excellent agreement with experimental data. In the $S_3$ extension of the Standard Model, if two of the right handed neutrinos masses are degenerate, the reactor and atmospheric mixing angles are determined by the masses of the charged leptons, yielding $\theta_{23}$ in excellent agreement with experimental data, and $\theta_{13}$ different from zero but very small. If the masses of the three right handed neutrinos are assumed to be different, then it is possible to get $\theta_{13}$ also in very good agreement with experimental data. We also show the branching ratios of some selected flavour changing neutral currents (FCNC) process as well as the contribution of the exchange of a neutral flavour changing scalar to the anomaly of the magnetic moment of the muon.
1 Introduction

The observation of flavour oscillations of solar, atmospheric, reactor, and accelerator neutrinos established that they have non-vanishing masses and mix among themselves, much like the quarks do [1, 20]. In these observations and experiments, the differences of the squared neutrino masses as well as the neutrino mixing angles are measured. These discoveries brought out very forcibly the need of extending the Standard Model (SM) in order to accommodate in the theory the new data on neutrino physics in a consistent way, that would allow for a unified and systematic treatment of the observed hierarchy of masses and mixings of all fermions. At the same time, the number of free parameters in the extended form of the SM had to be drastically reduced in order to give predictive power to the theory. These two seemingly contradictory demands are met by a flavour symmetry under which the families transform in a non-trivial fashion. The observed pattern of neutrino mixing and, in particular, the non vanishing and sizable value of the reactor mixing angle strongly suggests a flavour permutational symmetry $S_3$.

The result of a combined analysis of all available neutrino oscillation data, including the recent results from long-baseline $\nu_\mu \rightarrow \nu_e$ searches at the Tokai to Kamioka (T2K) [21] and Double CHOOZ experiments [22] as well as the Main Injector Neutrino Oscillation Search (MINOS) experiment [23], give the following values for the differences of the squared neutrino masses and the mixing angles in the lepton mixing matrix, $U_{PMNS}$, at 1$\sigma$ confidence level [24]:

$$\Delta m^2_{21} = 7.59^{+0.20}_{-0.18} \times 10^{-5} \text{ eV}^2, \quad \Delta m^2_{31} = \begin{cases} -2.40^{+0.08}_{-0.09} \times 10^{-3} \text{ eV}^2, \\ +2.50^{+0.09}_{-0.16} \times 10^{-3} \text{ eV}^2. \end{cases}$$

$$\sin^2 \theta_{12} = 0.312^{+0.017}_{-0.015}, \quad \sin^2 \theta_{23} = \begin{cases} 0.52 \pm 0.06 \\ 0.52^{+0.06}_{-0.07} \end{cases}, \quad \sin^2 \theta_{13} = \begin{cases} 0.016^{+0.008}_{-0.006} \\ 0.013^{+0.007}_{-0.005} \end{cases},$$

the upper (lower) row corresponds to inverted (normal) neutrino mass hierarchy, see also Gonzalez Garcia et al [25, 26] and J. F. W. Valle et al [27]. In fact, from the three angles needed to describe the mixing of the neutrinos, the least known is $\theta_{13}$. A global analysis of the T2K [21], MINOS [23] and CHOOZ [22] experiments yielded a non-vanishing value for the reactor mixing angle [27, 28]. Recently, the Daya Bay [29] and RENO experiments [30] found the following values for the reactor neutrino mixing angle: $\sin^2 2\theta_{13} = 0.092 \pm 0.016$ (stat) $\pm 0.005$ (syst) which is equivalent to $\theta_{13} \simeq 8.8^\circ \pm 0.8^\circ$ at 5.2 $\sigma$ level, and $\sin^2 2\theta_{13} = 0.113 \pm 0.013$ (stat) $\pm 0.019$ (syst) which is equivalent to $\theta_{13} \simeq 9.8^\circ$ at 4.9 $\sigma$ level [30].

In the last ten years, important theoretical advances have been made in the understanding of the mechanisms for fermion mass generation and flavour mixing. The imposition of flavour
symmetries in the Standard Model and its extensions strongly constrains the number of free parameters in the Yukawa couplings and gives rise to special forms of the fermions mass matrices with few free parameters and a number of texture zeroes \[31,34\]. For a recent review of flavour symmetry models see \[35,36\].

In the case of the Minimal $S_3$-Invariant Extension of the Standard Model \[37,43\], the concept of flavour and generations is extended to the Higgs sector in such a way that all the matter fields – Higgs, quarks, and lepton fields, including the right-handed neutrino fields—have three species and transform under the flavour symmetry group as the three dimensional representation $1 \oplus 2$ of the permutational group $S_3$. A model with more than one Higgs $SU(2)$ doublet has tree level flavour changing neutral currents whose exchange may give rise to lepton flavour violating processes and may also contribute to the anomalous magnetic moment of the muon. An effective test of the phenomenological success of the model is obtained by verifying that all flavour changing neutral current processes and the magnetic anomaly of the muon, computed in the $S_3$-Invariant extended form of the Standard Model, agree with the experimental values.

Another important application of the permutational group $S_3$ is the classification of physically equivalent mass matrices. Different mass matrices with texture zeroes located in different positions have exactly the same physical content if they are related by a similarity transformation in flavour space \[31,32,34,44\]. If the invariants of the mass matrix are to be preserved, the elements on the diagonal can only exchange positions in the diagonal while the off diagonal elements can only exchange positions off the diagonal. So, the transformations matrices that define the similarity classes are the six elements of the three dimensional real representation of the group of permutations $S_3$ \[33,45\].

The left-handed Majorana neutrinos naturally acquire their small masses through the type I seesaw mechanism

$$M_{\nu_L} = M_{\nu_D} M_{\nu_R}^{-1} M_{\nu_D}^T,$$

where $M_{\nu_D}$ and $M_{\nu_R}$ denote the Dirac and right handed Majorana neutrino mass matrices, respectively.

The mass matrices are diagonalized by bi-unitary transformations as

$$U_{iL}^\dagger M_i U_{iR} = \text{diag}(m_{i1}, m_{i2}, m_{i3}), \quad \text{and} \quad U_\nu^T M_{\nu_L} U_\nu = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}),$$

where $i = d, u, e$. The entries in the diagonal matrices may be complex, so the physical masses are their absolute values.

The lepton flavour mixing matrices $V_{PMNS}$ arise from the mismatch between diagonalization of the mass matrices of charged leptons and left-handed neutrinos,

$$V_{PMNS} = U_{eL}^\dagger U_\nu K,$$
where $\mathbf{K}$ is the diagonal matrix of the Majorana phase factors. Therefore, in order to obtain the unitary matrices appearing in eq. (5) and get predictions for the flavour mixing angles and CP violating phases, we should specify the mass matrices. Also, in the case of three neutrino mixing there are three CP violation rephasing invariants [46], associated with the three CP violating phases present in the $\mathbf{V}_{PMNS}$ matrix. The rephasing invariant related to the Dirac phase, analogous to the Jarlskog invariant in the quark sector, is given by:

$$J_I \equiv \Im \left[ V_{e1}^* V_{e3} V_{\mu 1} \right].$$

(6)

The rephasing invariant $J_I$ controls the magnitude of CP violation effects in neutrino oscillations and is a directly observable quantity. The other two rephasing invariants associated with the two Majorana phases in the $\mathbf{V}_{PMNS}$ matrix, can be chosen as: $S_1 \equiv \Im \left[ V_{e1} V_{e3}^* \right]$ and $S_2 \equiv \Im \left[ V_{e2} V_{e3}^* \right]$. These rephasing invariants are not uniquely defined, but $J_I$, $S_1$ and $S_2$ are relevant for the definition of the effective Majorana neutrino mass, $m_{ee}$, in the neutrinoless double beta decay.

In the standard PDG parametrization [47], the entries in the lepton mixing matrix are parametrized in terms of the mixing angles and phases. Thus, the mixing angles are related to the observable moduli of lepton $\mathbf{V}_{PMNS}$ through the relations:

$$\sin^2 \theta_{12} = \frac{|V_{e2}|^2}{1 - |V_{e3}|^2}, \quad \sin^2 \theta_{23} = \frac{|V_{e3}|^2}{1 - |V_{e3}|^2}, \quad \sin^2 \theta_{13} = |V_{e3}|^2. \quad (7)$$

The neutrino oscillations do not provide information about either the absolute mass scale or their nature, this is, if neutrinos are Dirac or Majorana particles [48]. Thus, one of the most fundamental problems in neutrinos physics is the question of the nature of massive neutrinos. A direct way to reveal the nature of massive neutrinos is to investigate processes in which the total lepton number is not conserved [49]. The matrix elements for these processes are proportional to the effective Majorana neutrino masses, whose magnitudes squared are

$$|\langle m_{ll} \rangle|^2 = \sum_{j=1}^3 m_{\nu_j}^2 |V_{ij}|^4 + 2 \sum_{j<k} m_{\nu_j} m_{\nu_k} |V_{ij}|^2 |V_{lk}|^2 \cos 2\left( w_{lj} - w_{lk} \right), \quad l = e, \mu, \tau, \quad (8)$$

where $m_{\nu_j}$ are the neutrino Majorana masses, $V_{ij}$ are the elements of the lepton mixing matrix, $w_{lj} = \arg \{V_{lj}\}$; this term includes phases of both types, Dirac and Majorana. The half-life of the neutrinoless double beta decay ($0\nu2\beta$ decay) can be expressed as

$$T_{1/2} = G_{0\nu} |M_{0\nu}|^2 |\langle m_{ee} \rangle|^2,$$

where $G_{0\nu}$ is a phase factor, $|M_{0\nu}|$ is the isotope specific nuclear decay matrix element and $|\langle m_{ee} \rangle|$ is the magnitude of the effective Majorana neutrino masses defined in eq. (8).

Current experiment data yield only an upper bound for this quantity $|\langle m_{ee} \rangle| < 0.3 \text{ eV}$ [50,51].

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2 An $S_3$ flavour symmetry in the SM

In analogy with the work of A. Mondragon and E. Rodriguez Jauregu i on the $S_3$ flavour symmetry in the quark sector of the Standard Model \[44, 52\], we will start by assuming the obvious, the one Higgs boson in the SM is an $SU(2)_L$ doublet and, since it has no flavour, it can only be accommodated in a singlet representation of $S_3$. The mass term in the Lagrangian, obtained by taking the vacuum expectation value of the Higgs field in the quark and lepton Yukawa couplings, gives rise to mass matrices $M_d$, $M_u$, $M_l$ and $M_\nu$ \[33, 34\];

$$\mathcal{L}_Y = \bar{q}_{d,L}M_d q_{d,R} + \bar{q}_{u,L}M_u q_{u,R} + \bar{L}_LM_l L_R + \bar{\nu}_LM_\nu (\nu_L)^c + h.c.$$ (9)

If it is assumed that $S_3$ is an exact symmetry of the model, these mass matrices give mass only to the one fermion in each family that is assigned to the singlet representation of $S_3$. Hence, in a symmetry adapted basis, all entries in these matrices should vanish except for the one at the third row and third column. There is no mismatch between the diagonalization of the mass matrices of the charged leptons and neutrinos, or $d$- and $u$- type quarks, and, consequently there is no mixing of the flavour indices. Therefore, we propose, along with many other authors \[31, 34, 37, 44, 53\], that the texture zeroes of the mass matrices of quarks and leptons are the result of a flavour permutational symmetry $S_3$ and its spontaneous or explicit breaking. In particular, the permutational $S_3$ flavour symmetry and its sequential explicit breaking allows us to justify using the same generic form for the mass matrices of all Dirac fermions \[33, 34\], this form is conventionally called a two texture zeroes form. Some reasons to propose the validity of a matrix with two texture zeroes as a universal form for the mass matrices of all Dirac fermions in the theory are the following:

1. The idea of $S_3$ flavour symmetry and its explicit breaking has been successfully realized as a mass matrix with two texture zeroes in the quark sector to describe the strong mass hierarchy of up and down type quarks \[44, 53−56\]. Also, the numerical values of the mixing matrices of the quarks determined in this framework are in good agreement with the experimental data \[34, 44\].

2. Since the mass spectrum of the charged leptons exhibits a hierarchy similar to the quark’s one, it would be natural to consider the same $S_3$ symmetry and its explicit breaking to justify the use of the same generic form with two texture zeroes for the charged lepton mass matrix.

3. As for the Dirac neutrinos, we have no direct information about the absolute values or the relative values of the Dirac neutrino masses, but the mass matrix with two texture zeroes can be obtained from an $SO(10)$ Grand Unified Theory which describes well the data on masses and mixings of Majorana neutrinos \[57, 59\]. Furthermore, from supersymmetry
arguments, it would be sensible to assume that the Dirac neutrinos have a mass hierarchy similar to that of the u-quarks and it would be natural to take for the Dirac neutrino mass matrix also a matrix with two texture zeroes.

2.1 Mass matrices from the breaking of $S_{3L} \otimes S_{3R}$

Some authors have pointed out that realistic Dirac fermion mass matrices result from the flavour permutational symmetry $S_{3L} \otimes S_{3R}$ and its spontaneous or explicit breaking according to the chain: $S_{3L} \times S_{3R} \supset S_3 \supset S_{2L} \times S_{2R} \supset S_2$. The group $S_3$ treats three objects symmetrically, while the structure $1 \oplus 2$ of its $3 \times 3$ matrix representations treats the generations differently and adapts itself readily to the hierarchical nature of the fermion mass spectra. As explained above, under exact $S_{3L} \otimes S_{3R}$ symmetry, the mass spectrum for either quark sector (up or down quarks) or leptonic sector (charged leptons or Dirac neutrinos) consists of one massive particle in a singlet irreducible representation and a pair of massless particles in a doublet irreducible representation of $S_{3L} \otimes S_{3R}$. In order to be more precise, in the case of exact $S_{3L} \otimes S_{3R}$ symmetry, and assuming that there is only one Higgs boson in the theory, this $SU(2)_L$ doublet can only be in a singlet representation of $S_3$, that is, it is a scalar with respect to the $S_3$ transformations. Hence, the corresponding mass matrices, $M_{i3}$, are invariant with respect to a permutation of the family (columns) and flavour (rows) indices, and take the form

$$M_{i3}^{(W)} = \frac{(1 - \Delta)}{3m_{i3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_{W}, \quad i = u, d, l, \nu_D$$

(10)

the subindex $W$ stands for weak basis. In order to make explicit the assignment of particles to irreducible representations of $S_3$, it will be convenient to make a change of basis from the weak basis to a symmetry adapted or hierarchical basis by means of the unitary matrix that diagonalizes the matrix $M_{i3}$,

$$M_{i3}^{(H)} = U^\dagger M_{i3}^{(W)} U$$

(11)

where

$$U = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{pmatrix}$$

and

$$M_{i3}^{(H)} = m_{i3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - \Delta_i \end{pmatrix}_H$$

(12)

In the Standard Model with the $S_3$ symmetry, masses for the first and second families are generated if we add the terms $M_{i2}$ and $M_{i1}$ to $M_{i3}$. The term $M_{i2}$ breaks the permutational symmetry $S_{3L} \otimes S_{3R}$ down to $S_{2L} \otimes S_{2R}$ and mixes the singlet and doublet representation of $S_3$. 

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while the term $M_{i1}$ transform as the mixed symmetry term of the doublet complex tensorial representation of the $S^3_{\text{diag}}$ diagonal subgroup of $S_{3L} \otimes S_{3R}$. Thus,

$$M^{(w)}_{i2} = \frac{m_{i3}}{3} \begin{pmatrix} \alpha_i & \alpha_i & \beta_i \\ \alpha_i & \alpha_i & \beta_i \\ \beta_i & \beta_i & -2\beta_i \end{pmatrix}_W$$

and

$$M^{(w)}_{i1} = \frac{m_{i3}}{\sqrt{3}} \begin{pmatrix} A_{i1} & iA_{i2} & -A_{i1} - iA_{i2} \\ -iA_{i2} & \alpha_i & \beta_i \\ \beta_i & \beta_i & -2\beta_i \end{pmatrix}_W.$$ (13)

Finally, adding the mass matrices eqs. (10)-(13), we get the mass matrix $M_i$ in the weak basis. Then, in a symmetry adapted basis $M_i$ takes the form

$$M_i = m_{i3} \begin{pmatrix} 0 & A_i & 0 \\ A_i^* & B_i & C_i \\ 0 & C_i & D_i \end{pmatrix}_H, \quad i = u, d, l, \nu_D. \quad (14)$$

where $A_i = |A_i|e^{i\phi_i}$, $B_i = -\triangle_i + \delta_i$ and $D_i = 1 - \delta_i$. From the strong hierarchy of the masses of the Dirac fermions, $m_{i3} >> m_{i2} > m_{i1}$, we expect $1 - \delta_i$ to be very close to unity. The Hermitian mass matrix (14) may be written in terms of a real symmetric matrix $\overline{M}_i$ and a diagonal matrix of phases $P_i \equiv \text{diag}[1, e^{i\phi_i}, e^{i\phi_i}]$ as follows:

$$M_i = P_i^\dagger \overline{M}_i P_i, \quad i = u, d, l, \nu_D. \quad (15)$$

Each possible symmetry breaking pattern is now characterized by the flavour symmetry breaking parameter $Z^1/2_i$, which is defined as the ratio $Z^1/2_i = \frac{(M_i)_{22}}{(M_i)_{22}}$. This ratio measures the mixing of the singlet and doublet irreducible representations of $S_3$. The small parameter $\delta_i$ is a function of the flavour symmetry breaking parameter $Z^1/2_i$ [33, 34, 44].

Thus, we obtain a universal form for the mass matrices of all Dirac fermions in the theory. But in the Standard Model and its extensions considering a mass term for left-handed neutrinos purely of Dirac nature is not theoretically favored, because it cannot explain naturally why neutrinos are much lighter than the charged leptons. Thus, we assume that the neutrinos have Majorana masses and acquire their small masses through the type I seesaw mechanism.

2.2 Classification of mass matrices with texture zeroes in equivalence classes

In this section we make a classification of mass matrices with texture zeroes in terms of similarity classes [33]. The similarity classes are defined as follows: Two matrices $M$ and $M'$ are similar if there exists an invertible matrix $T$ such that $M' = TMT^{-1}$ or $M' = T^{-1}MT$. The equivalence classes associated with a similarity transformation are called similarity classes. Another way to see the similarity classes is that the matrices that satisfy the similarity transformation have the same invariants: trace, determinant and $\chi \equiv \frac{1}{2} (\text{Tr} \{M^2\} - \text{Tr} \{M\}^2)$. Therefore, all matrices in
a class of similarity have the same eigenvalues, since all have the same characteristic polynomial, given by \( \lambda^3 - \text{Tr} \{ M^2 \} \lambda^2 - \chi \lambda - \det \{ M \} = 0 \).

Now, from the most general form of the symmetric and Hermitian mass matrices of \( 3 \times 3 \):

\[
M^s = \begin{pmatrix} g & a & e \\ a & b & c \\ e & c & d \end{pmatrix} \quad \text{and} \quad M^h = \begin{pmatrix} g & a & e \\ a^* & b & c \\ e^* & c^* & d \end{pmatrix},
\]

we can see that only six of the nine elements of these matrices are independent of each other. Therefore, the similarity transformation is realized as the permutation of the six independent elements in the nine entries of the mass matrices. But if we want to preserve the invariants, the elements on the diagonal can only exchange positions on the diagonal, while the off-diagonal elements can only exchange positions outside the diagonal. Thus, all these operations reduce to the permutations of three objects. So it is natural to propose as transformation matrices \( T \) in the similarity classes, the six elements of the real representation of the group of permutations \( S_3 \) which are:

\[
T(A_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ T(A_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ T(A_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

\[
T(A_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ T(A_4) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ T(A_5) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Then, we get the classification of mass matrices with texture zeroes, which is shown in the table 1. In this table, the "\( \ast \)" and "\( \times \)" denote an arbitrary non-vanishing matrix element on the diagonal and off-diagonal entries, respectively. We recall the rule for counting the texture zeroes in a mass matrix: two texture zeroes off-diagonal counts as one zero, while one on the diagonal counts as one \[31\].

### 2.3 The mass matrix for left-handed neutrinos

The left-handed Majorana neutrinos acquire their small masses through the type I seesaw mechanism, eq. (3). The form of \( M_{\nu_D} \) is given in eq. (13), which is a Hermitian matrix with two texture zeroes of class I. From the conjecture of a universal \( S_3 \) flavour symmetry in a unified treatment of all fermions, it is natural to take for \( M_{\nu_R} \) also a matrix with two texture zeroes of class I, non Hermitian but symmetric. Let us further assume that the phases in the entries of \( M_{\nu_R} \) may be factorized out as \[33, 34\]: \( M_{\nu_R} = R \bar{M}_{\nu_R} R \), where \( R \equiv \text{diag} \{ e^{-i \phi_e}, e^{i \phi_e}, 1 \} \) with
\[ \phi_c \equiv \arg \{ c_{\nu_R} \} \]

\[ \bar{\mathbf{M}}_{\nu_R} = \begin{pmatrix} 0 & a_{\nu_R} & 0 \\ a_{\nu_R} & |d_{\nu_R}| & |c_{\nu_R}| \\ 0 & |c_{\nu_R}| & |d_{\nu_R}| \end{pmatrix} \]  

(18)

Then, the mass matrix of the left-handed Majorana neutrinos has also the same generic form with two texture zeroes of class I:

\[ \mathbf{M}_{\nu_L} = \begin{pmatrix} 0 & a_{\nu_L} & 0 \\ a_{\nu_L} & b_{\nu_L} & c_{\nu_L} \\ 0 & c_{\nu_L} & d_{\nu_L} \end{pmatrix} \]  

(19)

where

\[ a_{\nu_L} = \frac{|a_{\nu_D}|^2}{d_{\nu_R}^2}, \quad d_{\nu_L} = \frac{d_{\nu_D}^2}{d_{\nu_R}^2}, \quad c_{\nu_L} = \frac{c_{\nu_D}^2}{d_{\nu_R}^2} + \frac{|a_{\nu_R}|^2}{d_{\nu_R}^2} \left( c_{\nu_D} e^{-i\phi_{\nu_D}} - \frac{|c_{\nu_D}|^2}{d_{\nu_R}} e^{i(\phi_c \pm \phi_{\nu_D})} \right) \]

\[ b_{\nu_L} = \frac{c_{\nu_D}^2 + 2 |a_{\nu_D}|^2 + 2 |a_{\nu_D}|^2}{d_{\nu_R}^2} e^{i(\phi_c \pm \phi_{\nu_D})} + 2 \frac{|a_{\nu_D}|^2}{d_{\nu_R}^2} \left( b_{\nu_D} e^{-i\phi_{\nu_D}} - \frac{|c_{\nu_D}|^2}{d_{\nu_R}} e^{i(\phi_c \pm \phi_{\nu_D})} \right) \]  

(20)

The elements \( a_{\nu_L} \) and \( d_{\nu_L} \) are real, while \( b_{\nu_L} \) and \( c_{\nu_L} \) are complex. Therefore, the form of mass matrices with two texture zeroes is invariant under the action of the seesaw mechanism of type I \[31,32,34,61\].

It may also be noticed that, if we set \( b_{\nu_R} = 0 \) or/and \( c_{\nu_R} = 0 \), the resulting expression for \( \mathbf{M}_{\nu_L} \) still has two texture zeroes. Therefore, \( \mathbf{M}_{\nu_L} \) also has two texture zeroes when \( \mathbf{M}_{\nu_R} \) has
two, three or four texture zeroes (the last two cases are called Fritzsch textures) \[33, 34\]. In fact, the information on the total number of texture zeroes present in the mass matrix \(M_{\nu R}^c\), can be obtained from the elements \((2, 2)\) and \((2, 3)\) of the left-handed neutrinos mass matrix.

From the previous analysis, the matrix \(M_{\nu L}^c\) has two non-ignorable phases which are \(\phi_1 \equiv \arg \{b_{\nu L}\}\) and \(\phi_2 \equiv \arg \{c_{\nu L}\}\). However, to describe the phenomenology of neutrinos masses and mixing, only one phase in \(M_{\nu L}^c\) is required. Therefore, without loss of generality, we may choose \(\phi_1 = 2\phi_2 = 2\phi\) \[33, 34\]. In this case the analysis simplifies, since the phases in \(M_{\nu L}^c\) may be factorized out as

\[M_{\nu L} = Q\tilde{M}_{\nu L} Q,\]

where \(Q\) is a diagonal matrix of phases \(Q \equiv \text{diag} \{e^{-i\phi}, e^{i\phi}, 1\}\) and \(\tilde{M}_{\nu L}\) is a real symmetric matrix. Then, the matrix \(M_{\nu L}^c\), can be diagonalized by a unitary matrix of the form \(U_{\nu} \equiv QO_{\nu}\), where \(O_{\nu}\) is an real orthogonal matrix that diagonalizes the real symmetric matrix \(\tilde{M}_{\nu L}\).

### 2.4 Mass matrix as function of the fermion masses

The real symmetric matrix \(\tilde{M}_i\), eqs. \((15)\) and \((21)\), may be brought to diagonal form by the transformation, \(\tilde{M}_i = O_i \text{diag} \{m_{i1}, m_{i2}, m_{i3}\} O_i^T\), where the \(m_i\)'s are the eigenvalues of \(M_i\) and \(O_i\) is a real orthogonal matrix, with \(i = u, d, l, \nu_L\). Now, computing the invariants of the real symmetric matrix \(\tilde{M}_i\), we may express the real parameters \(a_i, b_i, c_i\) and \(d_i\) occurring in eqs. \((15)\) and \((21)\) in terms of the mass eigenvalues \[33, 34\] as:

\[a_i^2 = -\frac{m_{i1}m_{i2}m_{i3}}{d_i}, \quad b_i = m_{i1} + m_{i2} + m_{i3} - d_i, \quad c_i^2 = \frac{(d_i - m_{i1})(d_i - m_{i2})(m_{i3} - d_i)}{d_i}.\]  

From the condition that \(a_i, b_i, c_i\) and \(d_i\) are real, we determine the allowed region of \(d_i\) \[62\]. In other words, all elements of the matrix \(\tilde{M}_i\) must be real and depending on which eigenvalue is chosen as negative, the parameter \(d_i\) must satisfy one of the following conditions:

- **Normal hierarchy**
  - \(m_{i3} > d_i > m_{i2}\), for \(m_{i1} = -|m_{i1}|\),
  - \(m_{i3} > d_i > m_{i1}\), for \(m_{i2} = -|m_{i2}|\),
  - \(m_{i2} > d_i > m_{i1}\), for \(m_{i3} = -|m_{i3}|\).

- **Inverted hierarchy**
  - \(m_{i1} > d_i > m_{i2}\), for \(m_{i3} = -|m_{i3}|\),
  - \(m_{i2} > d_i > m_{i3}\), for \(m_{i1} = -|m_{i1}|\),
  - \(m_{i3} > d_i > m_{i1}\), for \(m_{i2} = -|m_{i2}|\).

In this way, for a normal hierarchy and taking \(m_{i2} = -|m_{i2}|\), we get the \(\tilde{M}_i\) matrix \((i = u, d, l, \nu_L)\), reparametrized in terms of its eigenvalues and the parameter \(\delta_i\) as

\[
\tilde{M}_i = \begin{pmatrix}
0 & \sqrt{\frac{m_{i1}^2m_{i2}}{1-\delta_i}} & 0 \\
\sqrt{\frac{m_{i1}m_{i2}}{1-\delta_i}} & \tilde{m}_{i1} - \tilde{m}_{i2} + \delta_i & \sqrt{\frac{\delta_i}{(1-\delta_i)f_{i1}f_{i2}}} \\
0 & \sqrt{\frac{\delta_i}{(1-\delta_i)f_{i1}f_{i2}}} & 1 - \delta_i
\end{pmatrix},
\]  

\[(23)\]
where $\tilde{m}_{i1} = \frac{m_{i1}}{m_{i3}}$, $\tilde{m}_{i2} = \frac{|m_{i2}|}{m_{i3}}$, $f_{i1} = 1 - \tilde{m}_{i1} - \delta_i$ and $f_{i2} = 1 + \tilde{m}_{i2} - \delta_i$.

The small parameters $\delta_i$, which are also functions of the mass ratios and the flavour symmetry breaking parameters $Z_i^{1/2}$, are obtained as the solution of the cubic equation $(1 - \delta_i)(\tilde{m}_{i1} - \tilde{m}_{i2} + \delta_i)^2 Z_i = \delta_i f_{i1} f_{i2}$. The solution of the cubic equation that vanishes when $Z_i$ vanishes may be written as

$$\delta_i = \frac{Z_i}{Z_i + 1} \left( \frac{\tilde{m}_{i2} - \tilde{m}_{i1}}{W_i(Z_i)} \right)^2,$$

where $W_i(Z)$ is the product of the two roots of the cubic equation that do not vanish when $Z_i$ vanishes

$$W_i(Z) = \left[ p_i^3 + 2q_i^2 + 2q \sqrt{p_i^3 + q_i^2} \right]^{1/4} - |p_i| + \left[ p_i^3 + 2q_i^2 - 2q \sqrt{p_i^3 + q_i^2} \right]^{1/4} + \frac{1}{3} \left( \left[ q_i + \sqrt{p_i^3 + q_i^2} \right]^{1/4} + \left[ q_i - \sqrt{p_i^3 + q_i^2} \right]^{1/4} \right) (Z_i (2 (\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + (\tilde{m}_{i2} - \tilde{m}_{i1} + 2) Z_i (2 (\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + (\tilde{m}_{i2} - \tilde{m}_{i1} + 2)^2, \right. \quad (25)$$

with

$$p_i = -\frac{1}{3} \frac{Z_i (Z_i (2 (\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + (\tilde{m}_{i2} - \tilde{m}_{i1} + 2))^2}{Z_i + 1} + \frac{Z_i (\tilde{m}_{i2} - \tilde{m}_{i1}) (\tilde{m}_{i2} - \tilde{m}_{i1} + 2) (1 + \tilde{m}_{i2}) (1 - \tilde{m}_{i1})}{Z_i + 1}$$

and

$$q_i = \frac{1}{6} \frac{Z_i (Z_i (2 (\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + (\tilde{m}_{i2} - \tilde{m}_{i1} + 2))^3}{(Z_i + 1)^3}.$$

The allowed values for the parameters $\delta_i$ are in the following range $0 < \delta_i < 1 - \tilde{m}_{i1}$.

Now, the entries in the real orthogonal matrix $O$ that diagonalize the matrix $\tilde{M}_i$, may be expressed as

$$O_i = \begin{pmatrix}
\left[ \tilde{m}_{i1} f_{i1} \right]^{1/2} & -\left[ \tilde{m}_{i3} \tilde{m}_{i2} \right]^{1/2} & \left[ \tilde{m}_{i1} \tilde{m}_{i2} \delta_i \right]^{1/2} \\
\left[ \tilde{m}_{i1} (1 - \delta_i) f_{i1} \right]^{1/2} & \left[ \tilde{m}_{i2} (1 - \delta_i) f_{i2} \right]^{1/2} & \left[ (1 - \delta_i) \delta_i \right]^{1/2} \\
-\left[ \tilde{m}_{i1} f_{i2} \right]^{1/2} & -\left[ \tilde{m}_{i2} f_{i2} \right]^{1/2} & \left[ f_{i1} f_{i2} \right]^{1/2}
\end{pmatrix}, \quad (26)$$

where, $D_{i1} = (1 - \delta_i)(\tilde{m}_{i1} + \tilde{m}_{i2})(1 - \tilde{m}_{i1})$, $D_{i2} = (1 - \delta_i)(\tilde{m}_{i1} + \tilde{m}_{i2})(1 + \tilde{m}_{i2})$, and $D_{i3} = (1 - \delta_i)(1 - \tilde{m}_{i1})(1 + \tilde{m}_{i2})$.

### 2.5 Mixing Matrices as Functions of the Fermion Masses

The unitary matrices $U_{\nu, i}$ occurring in the definition of $V_{PMNS}$, eq. (3), may be written in polar form as $U_{\nu, i} = P_{\nu, i} O_{\nu, i}$. In this expression, $P_{\nu, i}$ is the diagonal matrix of phases appearing in
the two texture zeroes mass matrix \[15\]. Then, the lepton mixing matrix takes the form

\[
V_{PMNS}^{th} = O_{PMNS}^T P^{(\nu-l)} O_{PMNS} \nu K,
\]

where \(P^{(\nu-l)} = \text{diag}[1, e^{i\Phi_1}, e^{i\Phi_2}]\) is the diagonal matrix of the Dirac phases, with \(\Phi_1 = 2\varphi - \phi_1\) and \(\Phi_2 = \varphi - \phi_1\). The real orthogonal matrices \(O_{\nu,l}\) are defined in eq. \[26\]. Thus, the mixing matrix \(V_{PMNS}^{th}\) whose entries are explicit function of the masses of the leptons has the following form \[34\]:

\[
V_{PMNS}^{th} = \begin{pmatrix}
V_{e1}^{th} & V_{e2}^{th} e^{i\beta_1} & V_{e3}^{th} e^{i\beta_2} \\
V_{\mu1}^{th} & V_{\mu2}^{th} e^{i\beta_1} & V_{\mu3}^{th} e^{i\beta_2} \\
V_{\tau1}^{th} & V_{\tau2}^{th} e^{i\beta_1} & V_{\tau3}^{th} e^{i\beta_2}
\end{pmatrix},
\]

where

\[
\begin{align*}
V_{e1}^{th} &= \sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_1} f_{11} f_{v1}^2}{D_{11}^3 D_{v1}^3}} + \sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_1} (1 - \delta_1)(1 - \delta_\nu) f_{11} f_{v1} e^{i\Phi_1} + \delta_1 f_{12} f_{v1} e^{i\Phi_2}}{D_{11}^3 D_{v1}^3}}, \\
V_{e2}^{th} &= -\sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_2} f_{11} f_{v2}^2}{D_{11}^3 D_{v2}^3}} + \sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_2} (1 - \delta_1)(1 - \delta_\nu) f_{11} f_{v2} e^{i\Phi_1} + \delta_1 f_{12} f_{v2} e^{i\Phi_2}}{D_{11}^3 D_{v2}^3}}, \\
V_{e3}^{th} &= \sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_3} f_{13} f_{v3}^2}{D_{13}^3 D_{v3}^3}} + \sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_3} (1 - \delta_1)(1 - \delta_\nu) f_{13} f_{v3} e^{i\Phi_1} + \delta_3 f_{12} f_{v3} e^{i\Phi_2}}{D_{13}^3 D_{v3}^3}}, \\
V_{\mu1}^{th} &= -\sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_1} f_{11} f_{v1}^2}{D_{11}^3 D_{v1}^3}} + \sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_1} (1 - \delta_1)(1 - \delta_\nu) f_{11} f_{v1} e^{i\Phi_1} + \delta_1 f_{12} f_{v1} e^{i\Phi_2}}{D_{11}^3 D_{v1}^3}}, \\
V_{\mu2}^{th} &= -\sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_2} f_{11} f_{v2}^2}{D_{11}^3 D_{v2}^3}} + \sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_2} (1 - \delta_1)(1 - \delta_\nu) f_{11} f_{v2} e^{i\Phi_1} + \delta_1 f_{12} f_{v2} e^{i\Phi_2}}{D_{11}^3 D_{v2}^3}}, \\
V_{\mu3}^{th} &= \sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_3} f_{13} f_{v3}^2}{D_{13}^3 D_{v3}^3}} + \sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_3} (1 - \delta_1)(1 - \delta_\nu) f_{13} f_{v3} e^{i\Phi_1} + \delta_3 f_{12} f_{v3} e^{i\Phi_2}}{D_{13}^3 D_{v3}^3}}, \\
V_{\tau1}^{th} &= -\sqrt{\frac{\tilde{m}_\tau \tilde{m}_{\nu_1} f_{11} f_{v1}^2}{D_{11}^3 D_{v1}^3}} + \sqrt{\frac{\tilde{m}_\tau \tilde{m}_{\nu_1} (1 - \delta_1)(1 - \delta_\nu) f_{11} f_{v1} e^{i\Phi_1} + \delta_1 f_{12} f_{v1} e^{i\Phi_2}}{D_{11}^3 D_{v1}^3}}, \\
V_{\tau2}^{th} &= -\sqrt{\frac{\tilde{m}_\tau \tilde{m}_{\nu_2} f_{11} f_{v2}^2}{D_{11}^3 D_{v2}^3}} + \sqrt{\frac{\tilde{m}_\tau \tilde{m}_{\nu_2} (1 - \delta_1)(1 - \delta_\nu) f_{11} f_{v2} e^{i\Phi_1} + \delta_1 f_{12} f_{v2} e^{i\Phi_2}}{D_{11}^3 D_{v2}^3}}, \\
V_{\tau3}^{th} &= \sqrt{\frac{\tilde{m}_\tau \tilde{m}_{\nu_3} f_{13} f_{v3}^2}{D_{13}^3 D_{v3}^3}} + \sqrt{\frac{\tilde{m}_\tau \tilde{m}_{\nu_3} (1 - \delta_1)(1 - \delta_\nu) f_{13} f_{v3} e^{i\Phi_1} + \delta_3 f_{12} f_{v3} e^{i\Phi_2}}{D_{13}^3 D_{v3}^3}},
\end{align*}
\]

in these expresions the \(f_{\nu}'s\), and \(D_{\nu}'s\) are

\[
\begin{align*}
f_{\nu(l)1} &= (1 - \tilde{m}_{\nu_1}(e) - \delta_{\nu(l)}), & D_{\nu(l)1} &= (1 - \delta_{\nu(l)})(\tilde{m}_{\nu_1}(e) + \tilde{m}_{\nu_2}(\mu))(1 - \tilde{m}_{\nu_1}(e)), \\
f_{\nu(l)2} &= (1 + \tilde{m}_{\nu_2}(\mu) - \delta_{\nu(l)}), & D_{\nu(l)2} &= (1 - \delta_{\nu(l)})(\tilde{m}_{\nu_1}(e) + \tilde{m}_{\nu_2}(\mu))(1 + \tilde{m}_{\nu_2}(\mu)), \\
D_{\nu(l)3} &= (1 - \delta_{\nu(l)})(1 - \tilde{m}_{\nu_1}(e))(1 + \tilde{m}_{\nu_2}(\mu)),
\end{align*}
\]

where \(\tilde{m}_{\nu_1}(e) = \frac{m_{\nu_1}(e)}{m_{\nu_2}(e)}\), and \(\tilde{m}_{\nu_2}(\mu) = \frac{|m_{\nu_2}(\mu)|}{m_{\nu_2}(\tau)}\). In the quark sector, the elements of the mixing matrix \(V_{CKM}\) may also be expressed as functions of the quark masses ratios, the resulting expressions are similar to the expressions obtained above for the elements of the matrix \(V_{PMNS}^{th}\), for more details see \[34\].
2.6 The $\chi^2$ fit for the Lepton Mixing Matrix

We made a $\chi^2$ fit of the theoretical expressions for the modulii of the entries of the lepton mixing matrix $|\langle V_{PMNS}^{th} \rangle_{ij}|$ given in eq. (29) to the values extracted from experiment as given by Gonzalez-Garcia and Maltoni [14]. The computation was made using the following values for the charged lepton masses [47]:

$$m_e = 0.51099 \text{ MeV}, \quad m_\mu = 105.6583 \text{ MeV}, \quad m_\tau = 1776.82 \text{ MeV}. \quad (31)$$

We took for the masses of the left-handed Majorana neutrinos a normal hierarchy. This allows us to write the left-handed Majorana neutrino mass ratios in terms of the neutrino squared mass differences and the neutrino mass $m_{\nu_3}$ in the following form:

$$\tilde{m}_{\nu_1} = \sqrt{1 - \left(\frac{\Delta m_{32}^2 + \Delta m_{21}^2}{m_{\nu_3}^2}\right)^2}, \quad \tilde{m}_{\nu_2} = \sqrt{1 - \frac{\Delta m_{32}^2}{m_{\nu_3}^2}}.$$  \quad (32)

The numerical values of the neutrino squared mass differences were obtained from the global analysis of the experimental data on neutrino oscillations given in [14], and we left the mass $m_{\nu_3}$ as a free parameter of the $\chi^2$ fit. Also, the parameters $\delta_e$, $\delta_\nu$, $\Phi_1$ and $\Phi_2$ were left as free parameters to be varied. Hence, in this $\chi^2$ fit we have four degrees of freedom.

From the best values obtained for $m_{\nu_3}$ and the experimental values of $\Delta m_{23}^2$ and $\Delta m_{21}^2$, we obtained the following best values for the neutrino masses at 1$\sigma$:

$$m_{\nu_1} = (3.22^{+0.67}_{-0.39}) \times 10^{-3} \text{ eV}, \quad m_{\nu_2} = (9.10^{+0.25}_{-0.13}) \times 10^{-3} \text{ eV}, \quad m_{\nu_3} = (4.92^{+0.21}_{-0.22}) \times 10^{-2} \text{ eV}. \quad (33)$$

The resulting best values of the symmetry breaking parameters are $\delta_e = (6 \pm 2.98) \times 10^{-2}$ and $\delta_\nu = 0.522^{+0.09}_{-0.12}$, and the best values of the Dirac CP violating phases are $\Phi_1 = (270 \pm 15)^\circ$ and $\Phi_2 = (180 \pm 10)^\circ$. The best values obtained for the modulii of the entries of the $PMNS$ mixing matrix are given in the following expression

$$\left| V_{PMNS}^{th} \right|_{1\sigma} = \begin{pmatrix}
0.8204^{+0.008}_{-0.010} & 0.5616^{+0.012}_{-0.014} & 0.1181^{+0.017}_{-0.011} \\
0.3748^{+0.018}_{-0.031} & 0.6280^{+0.010}_{-0.010} & 0.6819 \pm 0.025 \\
0.4345^{+0.024}_{-0.020} & 0.5388^{+0.022}_{-0.024} & 0.7216^{+0.024}_{-0.027}
\end{pmatrix}. \quad (34)$$

The value of the rephasing invariant related to the Dirac phase is $J_1^{th} = (1.8 \pm 0.6) \times 10^{-2}$. In the absence of experimental information about the Majorana phases $\beta_1$ and $\beta_2$, the two rephasing invariants $S_1$ and $S_2$ associated with the two Majorana phases in the $V_{PMNS}$ matrix, cannot be determined from experimental values. Therefore, in order to make a numerical estimate of the Majorana phases, we maximized the rephasing invariants $S_1$ and $S_2$, thus obtaining a numerical value for the Majorana phases $\beta_1$ and $\beta_2$. Then, the maximum values of the rephasing invariants are $S_1^{\max} = -4.9 \times 10^{-2}$ and $S_2^{\max} = 3.4 \times 10^{-2}$, with $\beta_1 = -1.4^\circ$ and $\beta_2 = 77^\circ$. In this
numerical analysis, the minimum value of the $\chi^2$, corresponding to the best fit, is $\chi^2 = 0.288$ and the resulting value of $\chi^2$ for degree of freedom is $\frac{\chi^2_{min}}{d.o.f.} = 0.075$. All numerical results of the fit are in very good agreement with the values of the moduli of the entries in the matrix $V_{PMNS}$ as given in ref. [14].

2.7 The Mixing Angles

The theoretical expressions for the neutrino mixing angles as functions of the charged lepton and neutrino mass ratios are obtained from eqs. (7), when the theoretical expressions for the modulii of the entries in the $PMNS$ mixing matrix are substituted for $|V_{ij}|$ in the right hand side of eqs. (7). If we keep only terms at leading order, we obtain:

$$\sin^2 \theta_{12}^{\text{th}} \approx f_{\nu 2} \left\{ \frac{m_{\nu 1}}{m_{\nu 2}} + \frac{m_{\nu e}}{m_{\mu}} (1 - \delta_{\nu}) + 2 \sqrt{\frac{m_{\nu 1}}{m_{\nu 2}} \frac{m_{\nu e}}{m_{\mu}} (1 - \delta_{\nu}) \cos \Phi_1} \right\},$$  

$$\sin^2 \theta_{23}^{\text{th}} \approx \delta_{\nu} + \delta_{e} f_{\nu 2} - \sqrt{\delta_{\nu} \delta_{e} f_{\nu 2}} \cos (\Phi_1 - \Phi_2),$$  

$$\sin^2 \theta_{13}^{\text{th}} \approx \delta_{\nu} \left\{ \frac{m_{\nu e}}{m_{\mu}} + \frac{m_{\nu 1}}{m_{\nu 2}} - \frac{m_{\nu 1} m_{\nu 2}}{m_{\mu} (1 - \delta_{\nu}) \cos \Phi_1} \right\},$$

From eqs. (30) we have that $f_{\nu 2} = 1 + \tilde{m}_{\nu 2} - \delta_{\nu}$. The expressions quoted above are written in terms of the ratios of the lepton masses, defined in eq. (23). When the well known values of the charged lepton masses, the values of the neutrino masses, eq. (33), the values of the delta parameters and the Dirac CP violating phases obtained from the $\chi^2$ fit in the lepton sector, are inserted in eqs. (35)-(37), we obtain the following numerical values for the mixing angles

$$\theta_{12}^{\text{th}} = (34.43^{+0.85}_{-0.98})^\circ, \quad \theta_{23}^{\text{th}} = (43.60^{+1.97}_{-2.22})^\circ, \quad \theta_{13}^{\text{th}} = (6.80^{+0.95}_{-0.66})^\circ,$$

which are in very good agreement with the latest experimental data [14, 21–27]. We may conclude that:

1. The strong mass hierarchy of the Dirac fermions produces small and very small mass ratios of charged leptons. Then, the Dirac fermions mass hierarchy is reflected in a similar hierarchy of small and very small Dirac fermion flavour mixing angles [34].

2. The normal seesaw mechanism type I which gives very small masses to the left-handed Majorana neutrinos with relatively large values of the neutrino mass ratio $m_{\nu 1}/m_{\nu 2}$ allows for large $\theta_{12}$ and $\theta_{23}$ mixing angles (see eqs. (35)-(37)).
In the quark sector, the mixing angles may also be expressed as functions of the quark masses ratios, the resulting expressions are similar to the expressions obtained above for the lepton mixing angles, for more details see [34].

2.8 The effective Majorana masses

The theoretical expression for the squared magnitude of the effective Majorana neutrino mass of the electron neutrino is:

$$|\langle m_{\text{ee}} \rangle|^2 = m_{\nu_1}^2|V_{e1}|^4 + m_{\nu_2}^2|V_{e2}|^4 + m_{\nu_3}^2|V_{e3}|^4 + 2m_{\nu_1}m_{\nu_2}|V_{e1}|^2|V_{e2}|^2 \cos(w_{e1} - w_{e2}) + 2m_{\nu_1}m_{\nu_3}|V_{e1}|^2|V_{e3}|^2 \cos(w_{e1} - w_{e3}) + 2m_{\nu_2}m_{\nu_3}|V_{e2}|^2|V_{e3}|^2 \cos(w_{e2} - w_{e3}), \quad (39)$$

where $w_{ei} = \arg\{V_{ei}\}$. In a similar way, the theoretical expression for the squared magnitude of the effective Majorana neutrino mass of the muon neutrino is:

$$|\langle m_{\mu\mu} \rangle|^2 = m_{\nu_1}^2|V_{\mu1}|^4 + m_{\nu_2}^2|V_{\mu2}|^4 + m_{\nu_3}^2|V_{\mu3}|^4 + 2m_{\nu_1}m_{\nu_2}|V_{\mu1}|^2|V_{\mu2}|^2 \cos(w_{\mu1} - w_{\mu2}) + 2m_{\nu_1}m_{\nu_3}|V_{\mu1}|^2|V_{\mu3}|^2 \cos(w_{\mu1} - w_{\mu3}) + 2m_{\nu_2}m_{\nu_3}|V_{\mu2}|^2|V_{\mu3}|^2 \cos(w_{\mu2} - w_{\mu3}), \quad (40)$$

where $w_{\mu i} = \arg\{V_{\mu i}\}$. Expressions for the squared magnitude of effective Majorana masses $|\langle m_{ee} \rangle|^2$ and $|\langle m_{\mu\mu} \rangle|^2$ with only terms of leading order are given in Barranco et al [34]. From these expressions and the numerical values of the neutrinos masses given in eq. (33), we obtain the following expressions for the effective Majorana masses with the phases as free parameters:

$$|\langle m_{ee} \rangle|^2 \approx \{9.41 + 8.29 \cos(1^0 - 2\beta_1) + 4.3 \cos(1^0 - 2w_{e3}) + 4.31 \cos 2(\beta_1 - w_{e3})\} \times 10^{-6} \text{ eV}^2, \quad (41)$$

where $w_{e3} = \arctan\{0.15 \tan\beta_2 + 0.013(0.15 + 0.013 \tan\beta_2)\}$. Similarly,

$$|\langle m_{\mu\mu} \rangle|^2 \approx \{4.8 + 0.17 \cos 2(44^0 - w_{\mu2}) + 1.8 \cos 2(w_{\mu2} - w_{\mu3})\} \times 10^{-4} \text{ eV}^2, \quad (42)$$

where $w_{\mu2} \approx \arctan\{0.65 \tan\beta_1 + 0.13(0.65 + 0.13 \tan\beta_1)\}$ and $w_{\mu3} \approx \arctan\{\frac{\tan\beta_1 - 0.13}{1 + 0.13 \tan\beta_1}\}$.

In order to make a numerical estimate of the effective Majorana neutrinos masses $|\langle m_{ee} \rangle|$ and $|\langle m_{\mu\mu} \rangle|$, we used the following values for the Majorana phases $\beta_1 = -1.4^0$ and $\beta_2 = 77^0$ obtained by maximizing the rephasing invariants $S_1$ and $S_2$. Then, the numerical value of the effective Majorana neutrino masses are: $|\langle m_{ee} \rangle|^2 \approx 4.6 \times 10^{-3} \text{ eV}$, $|\langle m_{\mu\mu} \rangle|^2 \approx 2.1 \times 10^{-2} \text{ eV}$. The numerical value of $|\langle m_{ee} \rangle|$ obtained in this theoretical scheme is at least two orders of magnitude smaller than the current experimental upper bound. Therefore, it will be very difficult to determine with good precision the numerical value of the effective mass of Majorana neutrinos, since, at present, this value could be of the same order magnitude as the error bars of the nuclear matrix elements.
3 A Minimal invariant Extension of the Standard Model with three Higgs bosons

In the Standard Model analogous fermions in different generations have identical couplings to all gauge bosons of the strong, weak and electromagnetic interactions. Prior to electroweak symmetry breaking, the Lagrangian is chiral and invariant with respect to permutations of the left and right fermionic fields, that is, it is invariant under the action of the group of permutations acting on the flavour indices of the matter fields.

Since the Standard Model has only one Higgs $SU(2)_L$ doublet, which can only be an $S_3$ singlet, it can only give mass to the quark or charged leptons in the $S_3$ singlet representation, one in each family, without breaking the $S_3$ symmetry. Hence, in order to impose $S_3$ as a fundamental symmetry, unbroken at the Fermi scale, we are led to extend the Higgs sector of the theory [37]. The quark, lepton and Higgs fields are

$$Q^T = (u_L, d_L), u_R, d_R, L^T = (\nu_L, e_L), \nu_R, e_R, H,$$

in an obvious notation. All these fields have three species, and we assume that each one forms a reducible representation $1_{S_3} \oplus 2$ of the $S_3$ group. The doublets carry capital indices $I$ and $J$, which run from 1 to 2 and the singlets are denoted by $Q_3, u_3, d_3, L_3, e_3, \nu_3$ and $H_S$. Note that the subscript 3 denotes the singlet representation and not the third generation.

The most general renormalizable Yukawa interactions for the lepton sector of this model are given by

$$L_Y = L_{Y_E} + L_{Y_\nu},$$

where

$$L_{Y_E} = -Y_{1E} \mathcal{I}_1 H_S e_{1R} - Y_{2E} \mathcal{T}_3 H_S e_{3R} - Y_{4E} \mathcal{I}_I \kappa_{IJ} H_{1R} e_{JR} + \mathcal{I}_J \eta_{IJ} H_{2R} e_{JR} + \text{h.c.,}$$

$$L_{Y_\nu} = -Y_{1\nu} \mathcal{I}_1 (i\sigma_2) H_S^* \nu_{1R} - Y_{3\nu} \mathcal{T}_3 (i\sigma_2) H_S^* \nu_{3R} - Y_{4\nu} \mathcal{T}_3 (i\sigma_2) H_{1R}^* \nu_{1R} - Y_{5\nu} \mathcal{I}_I \kappa_{IJ} (i\sigma_2) H_{1R}^* \nu_{JR} + \mathcal{I}_J \eta_{IJ} (i\sigma_2) H_{2R}^* \nu_{JR} + \text{h.c.,}$$

and

$$\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ (45)

Furthermore, we add to the Lagrangian the Majorana mass terms for the right-handed neutrinos,

$$L_M = -\nu_R^T C M_{\nu_R} \nu_R,$$ (46)

where $C$ is the charge conjugation matrix and $M_{\nu_R} = \text{diag} \{M_1, M_2, M_3\}$ is the mass matrix for the right-handed neutrinos. In the following we will consider two cases: i) when the first two masses are degenerate, $M_1 = M_2 \neq M_3$, ii) when all masses are different.

Due to the presence of three Higgs fields, the Higgs potential of the $S_3$ invariant extension of the Standard Model is more complicated than that of the Standard Model [38, 63, 64]. This potential was first analyzed by Pakvasa and Sugawara [54] who found that in addition to the $S_3$
Table 2: $Z_2$ assignment in the leptonic sector.

| $H_S$, $\nu_{3R}$ | $H_1$, $L_3$, $e_{3R}$, $\nu_{IR}$ |
|---------------------|------------------------------------|

symmetry, it has an accidental permutational symmetry $S'_2$: $H_1 \leftrightarrow H_2$, which is not a subgroup of the flavour group $S_3$. In this communication, we will assume that the vacuum respects the accidental $S'_2$ symmetry of the Higgs potential and that $\langle H_1 \rangle = \langle H_2 \rangle$. With these assumptions, the Yukawa interactions, eqs. (43)-(44) yield mass matrices, for all fermions in the theory, of the general form [37]

$$M = \begin{pmatrix} 
\mu_1 + \mu_2 & \mu_2 & \mu_5 \\
\mu_2 & \mu_1 - \mu_2 & \mu_5 \\
\mu_4 & \mu_4 & \mu_3 
\end{pmatrix}.$$  \hspace{1cm} (47)

In principle, all entries in the mass matrices can be complex since there is no restriction coming from the flavour symmetry $S_3$. Therefore, there are $4 \times 5 = 20$ complex parameters in the mass matrices, which should be compared with $4 \times 9 = 36$ of the SM, including the Majorana masses of the left-handed neutrinos.

3.1 The mass matrices in the leptonic sector and $Z_2$ symmetry

A further reduction of the number of parameters in the leptonic sector may be achieved by means of an Abelian $Z_2$ symmetry [37]. A possible set of charge assignments of $Z_2$, compatible with the experimental data on masses and mixings in the leptonic sector is given in Table 2. These $Z_2$ assignments forbid the following Yukawa couplings $Y^e_1 = Y^e_3 = Y^{\nu}_1 = Y^{\nu}_5 = 0$. Therefore, the corresponding entries in the mass matrices vanish, i.e., $\mu^e_1 = \mu^e_3 = 0$ and $\mu^\nu_1 = \mu^\nu_5 = 0$.

The mass matrix of the charged leptons takes the form

$$M_e = m_\tau \begin{pmatrix} 
\tilde{\mu}_2 & \tilde{\mu}_2 & \tilde{\mu}_5 \\
\tilde{\mu}_2 & -\tilde{\mu}_2 & \tilde{\mu}_5 \\
\tilde{\mu}_4 & \tilde{\mu}_4 & 0 
\end{pmatrix}.$$  \hspace{1cm} (48)

The resulting expression for $M_e$, reparametrized in terms of its eigenvalues and written to order
\((m_\mu m_e/m_\tau^2)^2\) and \(x^4 = (m_e/m_\mu)^4\), is

\[
M_e \approx m_\tau \begin{pmatrix}
\frac{1}{\sqrt{2}} \frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}} \frac{\tilde{m}_\tau}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}} \sqrt{\frac{1+x^2-\tilde{m}_e^2}{1+x^2}} \\
\frac{1}{\sqrt{2}} \frac{\tilde{m}_l}{\sqrt{1+x^2}} & -\frac{1}{\sqrt{2}} \frac{\tilde{m}_\tau}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}} \sqrt{\frac{1+x^2-\tilde{m}_e^2}{1+x^2}} \\
\frac{\tilde{m}_e (1+x^2)}{\sqrt{1+x^2-\tilde{m}_e^2}} e^{i\delta_e} & \frac{\tilde{m}_e (1+x^2)}{\sqrt{1+x^2-\tilde{m}_e^2}} e^{i\delta_e} & 0
\end{pmatrix}.
\]

(49)

This approximation is numerically exact up to order \(10^{-9}\) in units of the \(\tau\) mass. Notice that this matrix has no free parameters other than the Dirac phase \(\delta_e\).

The unitary matrix \(U_{eL}\) that enters in the definition of the mixing matrix, \(V_{PMNS}\), is calculated from \(U_{eL}^\dagger M_e M_L^\dagger U_{eL} = \text{diag}(m_e^2, m_\mu^2, m_\tau^2)\), where \(m_e\), \(m_\mu\) and \(m_\tau\) are the masses of the charged leptons, written to the same order of magnitude as \(M_e\) is \(U_{eL} = P^e O^e\), where \(P^e = \text{diag}\{1,1, e^{i\delta_e}\}\) and

\[
O^e \approx \begin{pmatrix}
\frac{1}{\sqrt{2} x} \sqrt{1+5\tilde{m}_e^4+4x^2+\tilde{m}_e^4+2\tilde{m}_\tau^2} & \frac{1}{\sqrt{2} \sqrt{1-4\tilde{m}_e^2+x^2+6\tilde{m}_\mu^2-4\tilde{m}_\tau^2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2} x} \frac{1}{\sqrt{1+5\tilde{m}_e^4+4x^2+\tilde{m}_e^4+2\tilde{m}_\tau^2}} & \frac{1}{\sqrt{2} \sqrt{1-4\tilde{m}_e^2+x^2+6\tilde{m}_\mu^2-4\tilde{m}_\tau^2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2} x} \frac{1}{\sqrt{1+5\tilde{m}_e^4+4x^2+\tilde{m}_e^4+2\tilde{m}_\tau^2}} & \frac{1}{\sqrt{2} \sqrt{1-4\tilde{m}_e^2+x^2+6\tilde{m}_\mu^2-4\tilde{m}_\tau^2}} & \frac{1}{\sqrt{2}} \\
\end{pmatrix},
\]

(50)
in this expression, as before, \(\tilde{m}_\mu = m_\mu/m_\tau\), \(\tilde{m}_e = m_e/m_\tau\) and \(x = m_e/m_\mu\) [39].

### 3.1.1 Neutrino masses and mixings with two degenerate masses of the right-handed neutrinos

According to the \(Z_2\) selection rule the mass matrix of the Dirac neutrinos takes the form

\[
M_{\nu D} = \begin{pmatrix}
\mu_2' & \mu_2' & 0 \\
\mu_2' & -\mu_2' & 0 \\
\mu_3' & \mu_3' & \mu_3'
\end{pmatrix},
\]

(51)

and considering the following form to the mass matrix of right-handed neutrinos \(M_{\nu R} = \text{diag}\{M_1, M_1, M_3\}\), the mass matrix for the left-handed Majorana neutrinos, \(M_{\nu L}\), obtained from the seesaw mechanism type I is,

\[
M_{\nu L} = \begin{pmatrix}
2(\rho_2')^2 & 0 & 2\rho_2'\rho_4' \\
0 & 2(\rho_2')^2 & 0 \\
2\rho_2'\rho_4' & 0 & 2(\rho_4')^2 + (\rho_3')^2 \\
\end{pmatrix},
\]

(52)
where $\rho_2^\nu = (\mu_2^\nu)/M_1^{1/2}$, $\rho_4^\nu = (\mu_4^\nu)/M_1^{1/2}$ and $\rho_3^\nu = (\mu_3^\nu)/M_3^{1/2}$; $M_1$ and $M_3$ are the masses of the right handed neutrinos. The complex symmetric neutrino mass matrix $M_{\nu L}$ may be brought to a diagonal form by the transformation $U^{T}_\nu M_{\nu L} U_\nu = \text{diag} \left( |m_{\nu_1}| e^{i \delta_{\nu}}, |m_{\nu_2}| e^{i \delta_{\nu}}, |m_{\nu_3}| e^{i \delta_{\nu}} \right)$, where $U_\nu$ is the matrix that diagonalizes the matrix $M_{\nu L}^T M_{\nu L}$. This allows us to reparametrize the matrices $M_{\nu L}$ and $U_\nu$ in terms of the complex neutrino masses, which have the following form:

$$
M_{\nu L} = \begin{pmatrix}
 m_{\nu_3} & 0 & \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_3} - m_{\nu_2})} e^{-i \delta_{\nu}} \\
 0 & m_{\nu_3} & 0 \\
 \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_2} - m_{\nu_3})} e^{-i \delta_{\nu}} & 0 & (m_{\nu_1} + m_{\nu_2} - m_{\nu_3}) e^{-2i \delta_{\nu}} 
\end{pmatrix}
$$

(53)

and $U_\nu = P_\nu O_\nu$ where $P_\nu = \text{diag} \left\{ 1, 1, e^{i \delta_{\nu}} \right\}$ and

$$
O_\nu = \begin{pmatrix}
 \cos \eta & \sin \eta & 0 \\
 0 & 0 & 1 \\
 -\sin \eta & \cos \eta & 0 
\end{pmatrix} = \begin{pmatrix}
 \sqrt{\frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_3} - m_{\nu_2}}} & \sqrt{\frac{m_{\nu_3} - m_{\nu_2}}{m_{\nu_3} - m_{\nu_1}}} & 0 \\
 0 & 1 & 0 \\
 -\sqrt{\frac{m_{\nu_2} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}}} & \sqrt{\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_2} - m_{\nu_1}}} & 0 
\end{pmatrix}.
$$

(54)

The unitarity of $U_\nu$ constrains $\sin \eta$ to be real and thus $|\sin \eta| \leq 1$, this condition fixes the phases $\phi_1$ and $\phi_2$ as $|m_{\nu_1}| \sin \phi_1 = |m_{\nu_2}| \sin \phi_2 = |m_{\nu_3}| \sin \phi_\nu$. The only free parameters in the matrices $M_\nu$ and $U_\nu$ are the phase $\phi_\nu$, implicit in $m_{\nu_1}$, $m_{\nu_2}$ and $m_{\nu_3}$, and the Dirac phase $\delta_\nu$.

The neutrino mixing matrix $V_{PMNS}^{th}$ is the product $U^{T}_\nu K U_\nu$, where $K = \text{diag} \left( 1, e^{i \alpha}, e^{i \beta} \right)$ is the diagonal matrix of the Majorana phase factors. Therefore, the theoretical mixing matrix $V_{PMNS}^{th}$, is given by

$$
V_{PMNS}^{th} = \begin{pmatrix}
 O_{11}^\nu \cos \eta + O_{31}^\nu \sin \eta e^{i \delta_{\nu}} & O_{11}^\nu \sin \eta - O_{31}^\nu \cos \eta e^{i \delta_{\nu}} & -O_{21}^\nu \\
 -O_{12}^\nu \cos \eta + O_{32}^\nu \sin \eta e^{i \delta_{\nu}} & -O_{12}^\nu \sin \eta - O_{32}^\nu \cos \eta e^{i \delta_{\nu}} & O_{22}^\nu \\
 O_{13}^\nu \cos \eta - O_{33}^\nu \sin \eta e^{i \delta_{\nu}} & O_{13}^\nu \sin \eta + O_{33}^\nu \cos \eta e^{i \delta_{\nu}} & O_{23}^\nu 
\end{pmatrix} K,
$$

(55)

where $\cos \eta$ and $\sin \eta$ are given in eq. (54), $O_{ij}^\nu$ are given in eq. (50), and $\delta_{\nu} = \delta_{\nu}^L - \delta_{\nu}^e$.

The mixing angles

The magnitudes of the reactor and atmospheric mixing angles, $\theta_{13}^{th}$ and $\theta_{23}^{th}$, are determined by the masses of the charged leptons only. Keeping only terms of order $(m_2^e/m_\mu^2)$ and $(m_\mu/m_\tau)^4$, we get

$$
\sin \theta_{13}^{th} \approx \frac{1}{\sqrt{2}} \frac{1 + 4x^2 - \tilde{m}_\mu^4}{\sqrt{4 + 45x^2 + 5x^2 - \tilde{m}_\mu^4}}, \quad \sin \theta_{23}^{th} \approx \frac{1}{\sqrt{2}} \frac{1 + 4x^2 - 2\tilde{m}_\mu^2 + \tilde{m}_\tau^2}{\sqrt{2 - 4\tilde{m}_\mu^2 + x^2 + 6\tilde{m}_\mu^4}},
$$

(56)
The magnitude of the solar angle depends on the charged lepton and neutrino masses as well as the Dirac and Majorana phases

\[
|\tan \theta_{12}^{th}|^2 = \frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}} \left( 1 - 2 \frac{O_{31}}{O_{33}} \cos \delta_l \sqrt{\frac{m_{\nu_2} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}} + \left( \frac{O_{31}}{O_{33}} \right)^2 \frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}}} \right). \tag{57}
\]

The dependence of \( \tan \theta_{12}^{th} \) on the Dirac phase \( \delta_l \), see (57), is very weak, since \( O_{11} \sim 1/\sqrt{2}(m_e/m_\mu) \). Hence, we may neglect it when comparing (57) with the data on neutrino mixings. The dependence of \( \tan \theta_{12}^{th} \) on the phase \( \phi_\nu \) and the physical masses of the neutrinos enters through the ratio of the neutrino mass differences, it can be made explicit with the help of the unitarity constraint on \( U_\nu \) as

\[
\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}} = \frac{(|m_{\nu_2}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_\nu)^{1/2} - |m_{\nu_3}| \cos \phi_\nu}{(|m_{\nu_1}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_\nu)^{1/2} + |m_{\nu_3}| \cos \phi_\nu}. \tag{58}
\]

Similarly, the Majorana phases are given by \( \sin 2\alpha = \sin(\phi_1 - \phi_2) \) and \( \sin 2\beta = \sin(\phi_1 - \phi_\nu) \) where

\[
\sin 2\alpha = \frac{|m_{\nu_3}| \sin \phi_\nu}{|m_{\nu_1}| m_{\nu_2}} \left( \sqrt{|m_{\nu_2}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_\nu} + \sqrt{|m_{\nu_1}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_\nu} \right),
\]

\[
\sin 2\beta = \frac{\sin \phi_\nu}{|m_{\nu_1}|} \left( |m_{\nu_3}| \sqrt{1 - \sin^2 \phi_\nu} + \sqrt{|m_{\nu_1}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_\nu} \right). \tag{59}
\]

A more complete and detailed discussion of the neutrino mixing matrix \( V_{PMNS} \) and the Majorana phases obtained in this model is given in refs. [37,39,40,43].

In the present \( S_3 \)-invariant extension of the Standard Model, the experimental restriction \( |\Delta m_{12}^2| < |\Delta m_{13}^2| \) implies an inverted neutrino mass spectrum \( |m_{\nu_3}| < |m_{\nu_1}| < |m_{\nu_2}| \). In this model, the reactor and atmospheric mixing angles, \( \theta_{13}^{th} \) and \( \theta_{23}^{th} \), are determined by the masses of the charged leptons only as

\[
\sin^2 \theta_{13}^{th} = 1.1 \times 10^{-5}, \quad \sin^2 \theta_{23}^{th} = 0.5. \tag{60}
\]

Even in this simplified analysis it is clear that the \( S_3 \) symmetry gives a non-vanishing reactor mixing angle, within the bounds of MINOS [23], albeit small. The atmospheric angle is in very good agreement with the recent experimental data. As can be seen from equations (57) and (58), the solar angle is sensitive to the differences of the squared neutrino masses and the phase
\( \phi \nu \) but is only weakly sensitive to the charged lepton masses. If the small terms proportional to \( O_{11} \) and \( O_{12} \) are neglected in (57), we obtain

\[
\tan^2 \theta_{12}^{th} = \frac{(\Delta m^2_{12} + \Delta m^2_{13} + |m_{\nu_3}|^2 \cos^2 \phi_{\nu})^{1/2} - |m_{\nu_3}| \cos \phi_{\nu}}{(\Delta m^2_{13} + |m_{\nu_3}|^2 \cos^2 \phi_{\nu})^{1/2} + |m_{\nu_3}| \cos \phi_{\nu}}. \tag{61}
\]

From this equation, we may readily derive expressions for the neutrino masses in terms of \( \tan \theta_{12}^{th} \), \( \cos \phi_{\nu} \) and the differences of the squared neutrino masses

\[
|m_{\nu_1}| = \sqrt{|m_{\nu_1}|^2 + \Delta m^2_{13}}, \quad |m_{\nu_2}| = \sqrt{|m_{\nu_3}|^2 + \Delta m^2_{13}(1 + r^2)}, \quad r^2 = \Delta m^2_{12} / \Delta m^2_{13} \approx 3 \times 10^{-2}. \quad \text{As } r^2 << 1, \text{ the sum of the neutrino masses is}
\]

\[
\sum_{i=1}^{3} |m_{\nu_i}| \approx \frac{\sqrt{\Delta m^2_{13}}}{2 \cos \phi_{\nu} \tan \theta_{12}^{th}} \left( 1 + 2 \sqrt{1 + 2 \tan^2 \theta_{12}^{th} \cos 2 \phi_{\nu} + \tan^4 \theta_{12}^{th} - \tan^2 \theta_{12}^{th}} \right). \tag{63}
\]

The most restrictive cosmological upper bound for this sum is \( \sum |m_{\nu}| \leq 0.17 \text{eV} \) \([18]\). This upper bound and the experimentally determined values of \( \tan \theta_{12} \) and \( \Delta m^2_{ij} \), give a lower bound for \( \cos \phi_{\nu} \geq 0.55 \) or \( 0 \leq \phi_{\nu} \leq 57^\circ \). The neutrino masses \( |m_{\nu_i}| \) assume their minimal values when \( \cos \phi_{\nu} = 1 \). When \( \cos \phi_{\nu} \) takes values in the range \( 0.55 \leq \cos \phi \leq 1 \), the neutrino masses change very slowly with \( \cos \phi_{\nu} \). In the absence of experimental information we will assume that \( \phi_{\nu} \) vanishes. Hence, setting \( \phi_{\nu} = 0 \) in our formula, we find

\[
m_{\nu_1} = 0.052 \text{ eV} \quad m_{\nu_2} = 0.053 \text{ eV} \quad m_{\nu_3} = 0.019 \text{ eV}. \tag{64}
\]

The computed sum of the neutrino masses is \( \left( \sum_{i=1}^{3} |m_{\nu_i}| \right)^{th} = 0.13 \text{ eV} \), which is consistent with the cosmological upper bound \([18]\), as expected, since we used the cosmological bound to fix the bound on \( \cos \phi_{\nu} \). The effective Majorana mass in neutrinoless double beta decay \( \langle m_{ee} \rangle \), is defined in eq. (8). The most stringent bound on \( \langle m_{ee} \rangle \), obtained from the analysis of the data collected by the Heidelberg-Moscow experiment on the process neutrinoless double beta decay in enriched Ge \([50]\), is \( \langle m_{ee} \rangle < 0.3 \text{ eV} \). In this framework for a preliminary analysis we are assuming that the Majorana phases vanish, thus we get

\[
\langle m_{ee} \rangle^{th} = 0.053 \text{ eV} \tag{65}
\]

well below the experimental upper bound.
Deviation of the mixing matrix $V_{PMNS}^{th}$ from the tri-bimaximal form

The previous results on neutrino masses and mixings weakly depend on the Dirac phase $\delta$, for simplicity we will assume in this work that $\delta = \pi/2$. We may write the mixing matrix as follows, $V_{PMNS}^{th} = V_{PMNS}^{tri} + \Delta V_{PMNS}^{tri}$, where the tri-bimaximal form $V_{PMNS}^{tri}$ [65] is

$$
V_{PMNS}^{tri} = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \sqrt{\frac{1}{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix},
$$

$\Delta V_{PMNS}^{tri} = \Delta V_e + \delta t_{12} \frac{\sqrt{2} + \delta t_{12}}{g(\delta t_{12})} \Delta V_\nu,$

where $g(\delta t_{12}) = 1 + \frac{3}{2} \delta t_{12} (\sqrt{2} + \delta t_{12})$. All entries in $\Delta V_e$ are proportional to $(m_e/m_\mu)^2$ except $(\Delta V_e)_{13}$ that is proportional to $(m_e/m_\mu)$.

The value for $\delta t_{12}$, which is a small parameter, fixes the scale and the origin of the neutrino mass matrix. If we take for $\delta t_{12}$ the experimental central value $\delta t_{12} \approx -0.04$, we obtain $|m_{\nu_2}| \approx 0.056 \text{ eV}$, $|m_{\nu_3}| \approx 0.055 \text{ eV}$, and $|m_{\nu_3}| \approx 0.022 \text{ eV}$ [40]. When we take for $\delta t_{12}$ the tri-bimaximal value $\delta t_{12} = 0$, the neutrino masses are $m_{\nu_1} = 0.0521 \text{ eV}$, $m_{\nu_2} = 0.0528 \text{ eV}$, and $m_{\nu_3} = 0.0178 \text{ eV}$. For a detailed discussion on this subject see [42].

In both cases the $S_3$ invariant extension of the SM predicts an inverted hierarchy. Since the tri-bimaximal value of $\delta t_{12}$ differs from the experimental central value by less than 6% of $\tan \theta_{12}$, the difference in the corresponding numerical values of the neutrino masses are not significant within the present experimental uncertainties.

3.1.2 Neutrino masses and mixings when the masses of the right-handed neutrinos are non-degenerate

In the minimal $S_3$-invariant extension of Standard Model, the Yukawa interactions and the $S_3 \times Z_2$ flavour symmetry yield a mass matrix for the Dirac neutrinos of the form (51). The masses of the left-handed Majorana neutrinos, $M_{\nu L}$, are generated by the seesaw mechanism type I, eq. (3), where $M_{\nu R}$ is the mass matrix of the right-handed neutrinos, which we take to be real and diagonal but non-degenerate $M_{\nu R} = \text{diag}(M_1, M_2, M_3)$. Then, the mass matrix $M_{\nu L}$ takes the form

$$
M_{\nu L} = \begin{pmatrix}
\frac{2(\mu_1^\nu)^2}{M} & \frac{2\lambda(\mu_2^\nu)^2}{M} & \frac{2\mu_2^\nu \mu_3^\nu}{M} \\
\frac{2\lambda(\mu_2^\nu)^2}{M} & \frac{2(\mu_2^\nu)^2}{M} & \frac{2\mu_2^\nu \mu_3^\nu \lambda}{M} \\
\frac{2\mu_2^\nu \mu_3^\nu}{M} & \frac{2\mu_2^\nu \mu_3^\nu \lambda}{M} & \frac{(\mu_3^\nu)^2}{M}
\end{pmatrix}, \quad \lambda = \left( \frac{M_2-M_1}{M_1+M_2} \right), \quad \text{and} \quad \overline{M} = 2 \frac{M_1 M_2}{M_2+M_1}.
$$

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When the first two right-handed neutrino masses are equal, the parameter $\lambda$ vanishes and we recover the expression for $M_{\nu L}$ given in Kubo et al. [37], eq. (52) in the present paper, which leads to the results presented in the previous section [39–43].

Since we assumed the right-handed neutrino mass matrix $M_{\nu R}$ to be real, the complex symmetric neutrino mass matrix $M_{\nu L}$ has only three independent phase factors that come from the parameters $\mu_2$, $\mu_3$ and $\mu_4$. Here, to simplify the analysis we will consider the case when $\arg \{\mu_4\} = \arg \{\mu_3^*\}$ or $2 \arg \{\mu_4\} = \arg \{2(\mu_4^* M^{-1} + (\mu_3^*)^2 M_3^{-1})\}$. The general case, with three independent phase factors, will be considered in detail elsewhere.

In the case considered here the diagonalization of $M_{\nu L}$ may be reduced to the diagonalization of a mass matrix with two texture zeroes discussed in section 2.3. The phase factors may be factored out of $M_{\nu L}$ as

$$M_{\nu L} = Q^\nu U_+ \left( \mu_0 I_{3 \times 3} + \tilde{M} \right) U_+^\dagger Q^\nu,$$

where $Q^\nu = e^{i\phi_2} \text{diag} \{1, 1, e^{i\delta_\nu}\}$ with $\delta_\nu = \phi_4 - \phi_2 = \arg \{\mu_4^*\} - \arg \{\mu_2^*\}$,

$$U_+ = \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{array} \right), \quad \mu_0 = \frac{2 |\mu_3|^2}{|M|} (1 - |\lambda|), \quad \text{and} \quad \tilde{M} = \begin{pmatrix} 0 & A & 0 \\ A & B & C \\ 0 & C & 2d \end{pmatrix}$$

with $A = \sqrt{2} \frac{|\mu_4^*||\mu_3^*|}{|M|} (1 - |\lambda|)$, $B = \frac{2|\mu_3|^2}{|M|} + \frac{|\mu_3|^2}{M_3} - \frac{2|\mu_4|^2}{|M|} (1 - |\lambda|)$, $C = \sqrt{2} \frac{|\mu_4^*||\mu_3^*|}{|M|} (1 + |\lambda|)$ and $d = \frac{2|\lambda||\mu_2|^2}{|M|}$. As mentioned before, the diagonalization of $M_{\nu L}$ is reduced to the diagonalization of the real symmetric matrix $\tilde{M}$, which is a matrix with two texture zeroes of class I [33]. Hence the matrix $M_{\nu L}$ is diagonalized by a unitary matrix

$$U_\nu = Q^\nu U_+ 	ilde{Q}_\nu^{N[1]}.$$}

In the literature, these similarity transformations are also known as weak basis transformations, since they leave invariant the gauge currents [45].

As in the case of the charged leptons, the matrices $M_{\nu L}$ and $U_\nu$ can be reparametrized in terms of the neutrino masses. For this we use the information that we already have about the diagonalization of a matrix with two texture zeroes of class I [33, 34, 44, 52]. Then, the mass
matrix $M^N_{\nu L}$ for a normal [inverted] hierarchy in the mass spectrum takes the form

$$
M^N_{\nu L} = \begin{pmatrix}
\mu_0 + d & d & \frac{1}{\sqrt{2}} (C^N + A^N) \\
d & \mu_0 + d & \frac{1}{\sqrt{2}} (C^N - A^N) \\
\frac{1}{\sqrt{2}} (C^N + A^N) & \frac{1}{\sqrt{2}} (C^N - A^N) & m_{\nu_1} + m_{\nu_2} + m_{\nu_3} - 2(\mu_0 + d)
\end{pmatrix}
$$

(71)

with $C^N = \sqrt{\frac{2d + \mu_0 - m_{\nu_1}}{2d}} \left( m_{\nu_2}^2 - m_{\nu_1}^2 \right)$ and $A^N = \sqrt{\frac{2d - \mu_0}{2d}} \left( m_{\nu_1} - m_{\nu_3} \right)$. The values allowed for the parameters $\mu_0$ and $2d + \mu_0$ are in the following ranges: $m_{\nu_2} > \mu_0 > m_{\nu_{1,3}}$ and $m_{\nu_{2,3}} > 2d + \mu_0 > m_{\nu_{1,3}}$. The orthogonal matrix $O^N_{\nu l}$ reparametrized in terms of the neutrino masses is given by

$$
\begin{pmatrix}
\sqrt{-1} \left( m_{\nu_1} - \mu_0 \right) f_1 & \sqrt{-1} \left( m_{\nu_3} - \mu_0 \right) \nu_2 - \nu_2 f_2 & \sqrt{-1} \left( m_{\nu_2} - \mu_0 \right) f_3 \\
\sqrt{-1} \left( m_{\nu_2} - \mu_0 \right) f_1 & \sqrt{-1} \left( m_{\nu_3} - \mu_0 \right) f_2 & \sqrt{-1} \left( m_{\nu_1} - \mu_0 \right) f_3 \\
-\sqrt{-1} \left( m_{\nu_1} - \mu_0 \right) f_1 f_2 & -\sqrt{-1} \left( m_{\nu_3} - \mu_0 \right) f_1 f_3 & -\sqrt{-1} \left( m_{\nu_2} - \mu_0 \right) f_2 f_3
\end{pmatrix}
$$

(72)

where:

$$
D_1^N = 2d \left( m_{\nu_2} - m_{\nu_1} \right) \left( m_{\nu_{3,2}} - m_{\nu_{1,3}} \right),
D_2^N = 2d \left( m_{\nu_2} - m_{\nu_1} \right) \left( m_{\nu_{3,2}} - m_{\nu_{1,3}} \right),
D_3^N = 2d \left( m_{\nu_{3,2}} - m_{\nu_{1,3}} \right) \left( m_{\nu_{3,2}} - m_{\nu_{1,3}} \right),
$$

$$
f_1 = (2d + \mu_0 - m_{\nu_1}),

f_2 = [-1] (2d + \mu_0 - m_{\nu_2}),

f_3 = [-1] (m_{\nu_3} - \mu_0 - 2d).
$$

The superscripts $N$ and $I$ denote the normal and inverted hierarchies respectively.

The neutrino mixing matrix

The neutrino mixing matrix $V_{PMNS}^{th}$, is the product $U^* e^i K e^j$, where $K$ is the diagonal matrix of the Majorana phase factors, defined by $K = \text{diag}(1, e^{i\alpha}, e^{i\beta})$. Now, we obtain the theoretical expression of the elements for the lepton mixing matrix, $V_{PMNS}^{th}$, which is:

$$
V_{PMNS}^{th} = \begin{pmatrix}
V_{\tau e}^{th} e^{i\alpha} & V_{\tau\mu}^{th} e^{i\beta} & V_{\tau\nu}^{th} e^{i\beta} \\
V_{\mu e}^{th} e^{i\alpha} & V_{\mu\mu}^{th} e^{i\beta} & V_{\mu\nu}^{th} e^{i\beta} \\
V_{\nu e}^{th} e^{i\alpha} & V_{\nu\mu}^{th} e^{i\beta} & V_{\nu\nu}^{th} e^{i\beta}
\end{pmatrix}
$$

(73)
one Higgs $SU(2)$ doublet may be written in a flavour labelled, symmetry adapted weak basis as \([40, 41, 43]\) currents. In the Minimal Models with more than one Higgs the values given in the analysis by Gonzalez-Garcia \([14]\) will be considered in detail elsewhere.

The theoretical expression for the lepton mixing angles as functions of the lepton mass ratios are readily obtained when the theoretical expressions for the modulii of the entries in the $PMNS$ mixing matrix, given in eqs. \((74)\), are substituted for $|V_{ij}|$ in the right hand side of eqs. \((7)\). In a first, preliminary analysis for the reactor mixing angle $\theta_{13}$ and for an inverted neutrino mass hierarchy ($m_{\nu_2} > m_{\nu_1} > m_{\nu_3}$) we obtain:

$$\sin^2 \theta_{13}^{th} \approx \frac{(\mu_0 + 2d - m_{\nu_3}) (\mu_0 - m_{\nu_3})}{(m_{\nu_1} - m_{\nu_3}) (m_{\nu_2} - m_{\nu_3})}. \quad (75)$$

Now, with the following values for the neutrino masses $m_{\nu_2} = 0.056$ eV, $m_{\nu_1} = 0.053$ eV and $m_{\nu_3} = 0.048$ eV, and the parameter values $\delta_l = \pi/2$, $\mu_0 = 0.049$ eV and $d = 8 \times 10^{-5}$ eV, we get $\sin^2 \theta_{13}^{th} \approx 0.029 \rightarrow \theta_{13}^{th} \approx 9.8^\circ$, in good agreement with experimental data \([29, 30]\). A more complete analysis, from a $\chi^2$ fit of the exact theoretical expressions for the modulii of the entries of the lepton mixing matrix of the $|V_{P M N S}^{th}\rangle_{ij}$ to the experimental values (for example the values given in the analysis by Gonzalez-Garcia \([13]\)) will be considered in detail elsewhere.

4 Flavour Changing Neutral Currents (FCNC) and g-2

Models with more than one Higgs $SU(2)$ doublet have tree level flavour changing neutral currents. In the Minimal $S_3$-invariant Extension of the Standard Model considered here, there is one Higgs $SU(2)$ doublet per generation coupling to all fermions. The flavour changing Yukawa couplings may be written in a flavour labelled, symmetry adapted weak basis as \([40, 41, 43]\)

$$L_Y^{FCNC} = (\tilde{E}_a L Y_0^{E S} E_{bR} + \tilde{U}_{aL} Y_{bS} U_{bR} + \tilde{D}_{aL} Y_{D S} D_{bR}) H^0_S + (\tilde{E}_a L Y_{bE} E_{bR} + \tilde{U}_{aL} Y_{bU} U_{bR}$$

$$+ \tilde{D}_{aL} Y_{bD} D_{bR}) H^0 + (\tilde{E}_a L Y_{bE2} E_{bR} + \tilde{U}_{aL} Y_{bU2} U_{bR} + \tilde{D}_{aL} Y_{bD2} D_{bR}) H^2_H + h.c. \quad (76)$$

The Yukawa couplings of immediate physical interest in the computation of the flavour changing neutral currents are those defined in the mass basis, according to $Y_{m}^{E1} = U_{eL}^{\dagger} Y_{w}^{E1} U_{eR}$, where $U_{eL}$. 

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and $U_{eR}$ are the matrices that diagonalize the charged lepton mass matrix defined in eqs. [4]. We obtain [40]

$$
\tilde{Y}_m^E \approx \frac{m_\tau}{v_1} \begin{pmatrix}
2\tilde{m}_e & -\frac{1}{2}\tilde{m}_e & \frac{1}{2}x \\
-\tilde{m}_\mu & \frac{1}{2}\tilde{m}_\mu & -\frac{1}{2} \\
\frac{1}{2}\tilde{m}_\mu x^2 & -\frac{1}{2}\tilde{m}_\mu & \frac{1}{2}
\end{pmatrix} \text{ and } \tilde{Y}_m^E \approx \frac{m_\tau}{v_2} \begin{pmatrix}
-\tilde{m}_e & \frac{1}{2}\tilde{m}_e & -\frac{1}{2}x \\
\tilde{m}_\mu & \frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \\
\frac{1}{2}\tilde{m}_\mu x^2 & \frac{1}{2}\tilde{m}_\mu & \frac{1}{2}
\end{pmatrix},
$$

where $\tilde{m}_\mu = 5.94 \times 10^{-2}$, $\tilde{m}_e = 2.876 \times 10^{-4}$ and $x = m_e/m_\mu = 4.84 \times 10^{-3}$. All the non-diagonal elements are responsible for tree-level FCNC processes. If the $S'_2$ symmetry in the Higgs sector is preserved [54, 63], $\langle H^0_1 \rangle = \langle H^0_2 \rangle = v$.

The amplitude of the flavour violating process $\mu \rightarrow 3e$, is proportional to $\tilde{Y}_{\mu e}^E \tilde{Y}_{ee}^E$ [66]. Then, the leptonic branching ratio,

$$
Br(\mu \rightarrow 3e) \approx \frac{\Gamma(\mu \rightarrow 3e)}{\Gamma(\mu \rightarrow e\nu\bar{\nu})} \text{ and } \Gamma(\mu \rightarrow 3e) \approx \frac{m_\mu^5}{3 \times 2^{10} \pi^3} \left(\frac{Y_{\mu e}^1 Y_{ee}^1}{M_{H_{1,2}}^4}\right)^2,
$$

which is the dominant term, and the well known expression for $\Gamma(\mu \rightarrow e\nu\bar{\nu})$ [67], give

$$
Br(\mu \rightarrow 3e) \approx 2(2 + \tan^2 \beta)^2 \left(\frac{m_\mu m_\mu}{m_\tau^2}\right)^2 \left(\frac{m_\tau}{M_H}\right)^4,
$$

where $M_H$ is the neutral Higgs involved in the process, whose mass we take as $M_H \approx 120$ GeV, and $\tan \beta = 1$. We obtain $Br(\mu \rightarrow 3e) = 2.53 \times 10^{-16}$, well below the experimental upper bound for this process, which is $1 \times 10^{-12}$ [68].

Similar computations give the numerical estimates of the branching ratios for some others flavour violating processes in the leptonic sector. These results, and the corresponding experimental upper bounds are shown in Table 3. In all cases considered, the theoretical estimations made in the framework of the minimal $S_3$-invariant extension of the SM are well below the experimental upper bounds [40, 41, 43].

### 4.1 Muon anomalous magnetic moment

In the minimal $S_3$-invariant extension of the Standard Model we are considering here, the $Z_2$ symmetry decouples the charged leptons from the Higgs boson in the $S_3$ singlet representation. Therefore, at leading order of perturbation theory there are two neutral scalars and two neutral pseudoscalars whose exchange will contribute to the anomalous magnetic moment of the muon [40]. Since the heavier generations have larger flavour-changing couplings, the largest
contribution comes from the heaviest charged leptons coupled to the lightest of the neutral Higgs bosons.

After a straightforward calculation, with the help of (77), we may write $\delta a_{\mu}^{(H)}$ as

$$\delta a_{\mu}^{(H)} = \frac{Y_{\mu \tau} Y_{\tau \mu}}{16\pi^2} \frac{m_{\mu} m_{\tau}}{M_H^2} \left( \log \left( \frac{M_H^2}{m_{\mu}^2} \right) - \frac{3}{2} \right) = \frac{m_{\mu}^2}{(246 \text{ GeV})^2} \frac{(2 + \tan^2 \beta)}{32\pi^2} \frac{m_{\tau}^2}{M_H^2} \left( \log \left( \frac{M_H^2}{m_{\tau}^2} \right) - \frac{3}{2} \right).$$

Taking again $M_H = 120 \text{ GeV}$ and the upper bound for $\tan \beta = 14$ gives an estimate of the largest possible contribution of the FCNC to the anomaly of the muon’s magnetic moment $\delta a_{\mu}^{(H)} \approx 1.7 \times 10^{-10}$. This number has to be compared with the difference between the experimental value and the Standard Model prediction for the anomaly of the muon’s magnetic moment $\Delta a_{\mu} = a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}}$ [73], whose numerical value is $\Delta a_{\mu} = (28.7 \pm 9.1) \times 10^{-10}$, which means $\delta a_{\mu}^{(H)}/\Delta a_{\mu} \approx 0.06$. Hence, the contribution of the flavour changing neutral currents to the anomaly of the magnetic moment of the muon is smaller than or of the order of 6% of the discrepancy between the experimental value and the Standard Model prediction.

5 Conclusions

We have discussed the theory of the neutrino masses and mixings as the realization of an $S_3$ flavour permutational symmetry in two models, the Standard Model with an $S_3$ flavour symmetry and the minimal $S_3$-symmetric extension of the Standard Model, with three Higgs doublets.

In the Standard Model the imposition of the non-abelian permutational symmetry $S_3$ as a broken symmetry of flavour, leads to a unified treatment of masses and mixings of quarks and leptons in which the left-handed Majorana neutrinos acquire their masses via the type-I seesaw mechanism. The explicit sequential breaking of the $S_3$ flavour group according to the chain
$S_3R \otimes S_{3L} \supset S_3^{\text{diag}} \supset S_2^{\text{diag}}$, is a sufficient condition to define a generic form for the mass matrices of all fermions in the theory. In a symmetry adapted or hierarchical basis, this generic form is characterized as a mass matrix with two texture zeroes of class I. All mass matrices are, then, reparametrized in terms of their eigenvalues \([33,34,41,52]\). After analytically diagonalizing the mass matrices, explicit analytical expressions for all entries in the neutrino mixing matrix are obtained as functions of the masses of the charged leptons and neutrinos and one CP-violating Dirac phase in very good agreement with all available experimental data including the recent measurements of the reactor angle $\theta_{13}$ made by the T2K, Daya Bay and RENO experiments.

In the minimal $S_3$-invariant extension of the Standard Model, $S_3$ is imposed as a fundamental, exact symmetry in the matter sector. This assumption leads to extend the concept of flavour and generations to the Higgs sector. The fermion sector of the Standard Model is left unaltered. Hence, going to the irreducible representation of $S_3$, the model has one $SU(2)_L$ Higgs doublet in the $S_3$-singlet representation plus two $SU(2)$ Higgs doublets in the two components of the $S_3$-doublet representation. In this way, all the matter fields, quarks and lepton fields, the right-handed neutrino fields, and the Higgs fields, belong to the three dimensional $1_s \oplus 2$ representation of the group $S_3$. A well defined structure of the Yukawa couplings is obtained which permits the calculation of mass and mixing matrices as functions of the charged leptons and neutrino masses \([38, 40]\). The magnitudes of the three mixing angles are determined by the interplay of the flavour $S_3$ symmetry, the charged lepton and neutrino mass hierarchies, and the seesaw mechanism. The solar mixing angle is almost insensitive to the value of the masses of the charged leptons, but its experimental value allowed us to fix the scale and origin of the neutrino mass spectrum. The numerical value of the atmospheric mixing angle, $\theta_{23}^{\text{th}}$, depends strongly on the masses of the charged leptons and is in very good agreement with the experiment. In this model the magnitude of the reactor mixing angle, $\theta_{13}^{\text{th}}$, is sensitive to the difference of the values of the masses of the first and second right-handed neutrinos. In the case where two of the neutrino masses are degenerate, $\theta_{13}$ is different from zero but very small \([40, 42]\). Allowing for the masses to be non-degenerate gives a values for $\theta_{13}$ in very good agreement with recent experimental data. Explicit expressions for the matrices of the Yukawa couplings of the leptonic sector, parameterized in terms of the leptons masses and the VEV’s of the neutral Higgs bosons in the $S_3$- doublet representation, can be obtained in this model. Taking for the neutral Higgses $M_{H_1,2}$ a very conservative value ($M_{H_1,2} \approx 120 \text{ GeV}$), it is found that the numerical values of the branching ratios of the FCNC’s in the leptonic sector are well below the corresponding experimental upper bounds by many orders of magnitude. The contribution of the flavour changing neutral currents to the anomaly of the magnetic moment of the muon is small (6%) but non-negligible \([40, 41, 43]\).
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