ANALYTIC CONTINUATION OF MULTIPLE POLYLOGARITHMS IN POSITIVE CHARACTERISTIC

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Abstract. Our aim of this paper is to propose a method of analytic continuation of Carlitz multiple (star) polylogarithms to the whole space by using Artin-Schreier equation and present a treatment of their branches by introducing the notion of monodromy modules. As applications of this method, we obtain (1) a method of continuation of the logarithms of higher tensor powers of Carlitz module, (2) the orthogonal property (Chang-Mishiba functional relations), (3) a branch independency of the Eulerian property.

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0. Introduction

It is said that the history of study of the polylogarithm goes back to the the correspondence of Leibniz with Bernoulli in 1696. The polylogarithm is the complex function defined by the following series:

$$\text{Li}_n(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^n}$$

with a positive integer $n \geq 1$. The case for $n = 1$ gives $\text{Li}_1(z) = -\log(1-z)$ and that for $n = 2$ gives the dilogarithm. Though it converges on $|z| < 1$, it can be analytically continued to a bigger region, in precise a covering of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$,
by iterated path integrals. It is significant in number theory that its special value at $z = 1$, that is, its limit value $z \to 1$, attains the Riemann zeta value $\zeta(n) = \sum_{i=1}^{\infty} \frac{1}{i^n}$ ($n > 1$). The function is generalized to the *multiple polylogarithm* which is defined by the following series:

$$L_{i_1, \ldots, i_d}(z_1, \ldots, z_d) = \sum_{0 < i_1 < \cdots < i_d} \frac{z_1^{i_1} \cdots z_d^{i_d}}{i_1^{n_1} \cdots i_d^{n_d}}$$

with $n_1, \ldots, n_d \geq 1$. Though it converges when $|z_k| < 1$ for $k = 1, \ldots, d$, it can be analytically continued to a bigger region by iterated integrals (cf. [Z]). It is remarkable that its special value at $z_1 = \cdots = z_d = 1$ gives the multiple zeta value

$$\zeta(n_1, \ldots, n_d) = \sum_{0 < i_1 < \cdots < i_d} \frac{1}{i_1^{n_1} \cdots i_d^{n_d}}$$

when $n_d > 1$ (the condition to converge).

While in the case of the global function field in positive characteristic, Carlitz introduced Carlitz zeta value $\zeta_C(n)$ ($n \geq 1$) around 1935, which is regarded to be an analogue of the Riemann zeta value $\zeta(n)$. Anderson and Thakur [AT90] considered the *Carlitz polylogarithm* (denoted $Li_n(z)$ by abuse of notation) as an analogue of the above polylogarithm, which is defined by the series

$$Li_n(z) = \sum_{i=0}^{\infty} \frac{z^q^i}{L_i^q} \in \mathbb{C}_\infty[[z]]$$

(consult §1.1 for these symbols). The function converges on $|z|_\infty < q^{\frac{n}{q-1}}$. They showed that $\zeta_A(n)$ is given by a certain linear combination of its special value at some algebraic numbers lying on the region of convergence. Thakur [T] introduced an analogue $\zeta_A(n_1, \ldots, n_d)$ ($n_1, \ldots, n_d \geq 1$) of multiple zeta value which generalizes the Carlitz zeta value. Chang [C] generalized the Carlitz polylogarithm to the *Carlitz multiple polylogarithm* (denoted $Li_{n_1, \ldots, n_d}(z_1, \ldots, z_d)$ by abuse of notation) which is defined by the series

$$Li_{n_1, \ldots, n_d}(z_1, \ldots, z_d) = \sum_{0 < i_1 < \cdots < i_d} \frac{z_1^{q^{i_1}} \cdots z_d^{q^{i_d}}}{L_{i_1}^q \cdots L_{i_d}^q} \in \mathbb{C}_\infty[[z_1, \ldots, z_d]]$$

in the region of convergence $\mathbb{D}$ (cf. [1.3]) and he further showed that $\zeta_C(n_1, \ldots, n_d)$ is given by a certain linear combination of its special value at some algebraic numbers lying on $\mathbb{D}$. Its star variant [1.4] was introduced and discussed in Chang-Mishiba [CM]. Its relationship with Anderson dual $t$-motives and $t$-modules is developed in [CPY] [CGM] [GN].

The aim of this paper is to extend the regions of convergence of ($t$-motivic) Carlitz multiple (star) polylogarithms by using Artin-Schreier equations which serve as a substitute of iterated path integrals. In §1 we extend the functions by using Artin-Schreier equations and explain a manipulation of their associated branches by introducing the notion of monodromy modules. In §2 by exploiting this method, we give a method of continuation of the logarithms of $t$-modules associated with higher tensor powers of Carlitz module, analytic continuation of Chang-Mishiba functional relations, and a branch independency of the Eulerian property.
1. Analytic continuation of Carlitz multiple polylogarithms

We explain a method of analytic continuation of the Carlitz multiple (star) polylogarithm by using Artin-Schreier equation. In §1.1, we prepare the notations to be used and also present a key lemma (Lemma 1.1.1) related to Artin-Schreier equation. In §1.2, we recall the definition of the (t-motivic) Carlitz multiple (star) polylogarithm. In §1.3, we explain a method of continuation of the Carlitz polylogarithm. By extending the method, we give an analytic continuation of the Carlitz multiple polylogarithm in §1.4 and the Carlitz multiple star polylogarithm in §1.5 both as one variable functions.

1.1. Preparation. In this paper the following notation is employed.

- \( N \): the set of positive integers
- \( \mathbb{F}_q \): the field with \( q \) elements, for \( q \) a power of a prime number \( p \)
- \( A = \mathbb{F}_q[\theta] \): the polynomial ring in the variable \( \theta \) over \( \mathbb{F}_q \)
- \( A_+ \): the set of monic polynomials in \( A \), which is an analogue of the set of positive integers \( \mathbb{N} = \mathbb{Z}_{>0} \)
- \( K \): the fraction field of \( A \)
- \( \infty \): the infinite place of \( K \) with an associated absolute value \( | \cdot |_\infty \) such that \( |\theta|_\infty = q \)
- \( K_\infty = \mathbb{F}_q((1/\theta)) \): the \( \infty \)-adic completion of \( K \)
- \( \mathbb{C}_\infty \): the \( \infty \)-adic completion of the algebraic closure \( \bar{K}_\infty \)
- \( T \): the Tate algebra with respect to another parameter \( t \), the ring of formal power series \( f = \sum a_i t^i \in \mathbb{C}_\infty[[t]] \) convergent on \( |t|_\infty \leq 1 \), encoded with the Gauss norm given by \( ||f||_\infty := \max_i \{|a_i|_\infty\} \)
- \( T_r (r \in \mathbb{Q}_+^2) \): the subalgebra of \( \mathbb{C}_\infty[[t]] \) which converges on \( |t|_\infty \leq r \), so \( T = T_1 \).
- \( T(\infty) \): the intersection of \( T_r \) for all \( r \in \mathbb{Q}_+^2 \), the set of formal power series \( \sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}_\infty[[t]] \) such that \( \lim_{n \to \infty} \sqrt[n]{|a_n|_\infty} = 0 \)
- \( \mathcal{E} \): the ring of entire functions, that is, formal power series \( f = \sum_{i=0}^{\infty} a_i t^i \in \bar{K}[[t]] \) such that \( f \in T(\infty) \) and \( [K_\infty(a_0, a_1, a_2, \ldots) : K_\infty] < \infty \).
- The \( n \)-fold Frobenius twisting \( (n \in \mathbb{Z}) \) on the field \( \mathbb{C}_\infty((t)) \) is defined by \( f = \sum a_i t^i \in \mathbb{C}_\infty((t)) \mapsto f^{(n)}(x) = \sum a_i^q t^i \in \mathbb{C}_\infty((t)) \)
- \( \varphi : T \to T \) is the \( \mathbb{F}_q[t] \)-linear map sending \( f \mapsto f - f^{(1)} \)

The following lemma plays an essential role in this section.

Lemma 1.1.1. (1) The map \( \varphi : T \to T \) is surjective and the inverse \( \varphi^{-1}(h) \) for each \( h \in T \) is given by \( h' + \mathbb{F}_q[t] \) for some \( h' \in T \).
(2) For any \( f \in T \), \( f \) and \( \varphi(f) \) have a same radius of convergence.
(3) If \( V \) is an \( \mathbb{F}_q[t] \)-submodule of \( T \), then so is \( \varphi^{-1}(V) \).
(4). \( \varphi^{-1}(\mathcal{E}) = \mathcal{E} \).

Proof. For \( f = \sum a_i t^i \in T \) with \( a_i \in \mathbb{C}_\infty \), we calculate its inverse image \( g = \sum b_i t^i \) by solving the following Artin-Schreier type equation

\[
(1.1) \quad b_i - b_i^q = a_i
\]
for each \( i \). Though solutions of the above equation are unique modulo \( \mathbb{F}_q \) for each \( i \), we see that \( g \) is uniquely determined modulo \( \mathbb{F}_q[t] \) because we impose the condition \( g \in T \). It is immediate to see that \( g \) belongs to \( T \) because we have

\[
(1.2) \quad |b_i|_\infty = |a_i|_\infty
\]
for all sufficiently large \( i \)'s by the above Artin-Schreier equation and \( \sum_i a_i t^i \in \mathbb{T} \). Whence (1) is proved. (2) follows from (1.2). (3) is immediate because \( \wp \) is \( \mathbb{F}_q[t] \)-linear.

Suppose that \( f \) is in \( \mathcal{E} \). Then by (1.2), we see that the inverse image \( g \) satisfies the first condition of \( \mathcal{E} \). Put \( K'_\infty := K_\infty(a_0, a_1, a_2, \ldots) \). Then \( K'_\infty \) is presented as the field of Laurent series \( \mathbb{F}_q((\frac{1}{\theta'})\big) \) with a finite extension \( \mathbb{F}_q \) and an element \( \theta' \in \mathbb{C}_\infty \). Since all the solutions of the equation (1.1) lie in \( K'_\infty \) whenever \( a_i \) lies in a maximal ideal of \( K'_\infty \), we see that \( K'_\infty(b_0, b_1, b_2, \ldots) \) is a finite extension of \( K_\infty(a_0, a_1, a_2, \ldots) \). Thus \( g \) is in \( \mathcal{E} \). (4) is proved. □

1.2. Carlitz multiple (star) polylogarithms. We recall the definition of Carlitz multiple (star) polylogarithms and also their \( t \)-motivic variants.

Throughout this paper we fix a \((q - 1)\)-th root of \(-\theta \). We consider the function

\[
\Omega = \Omega(t) := (-\theta)^{-\frac{1}{(q-1)}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^q}ight) \in \mathbb{C}_\infty[[t]].
\]

It is an entire function, namely it belongs to \( \mathcal{E} \) and \( \mathcal{T} \times \), and satisfies the difference equation

\[
\Omega^{(-1)}(t) = (t - \theta)\Omega(t).
\]

The value \( \bar{\pi} := \frac{1}{\Omega(\theta)} \) is a period of Carlitz module (cf. \[AT90\] T).

The \emph{Carlitz multiple polylogarithm} (CMPL) and \emph{Carlitz multiple star polylogarithm} (CMSPL), introduced in \[C\ CM\], are defined by the following power series respectively

\[
\text{Li}_{n_1, \ldots, n_d}(z_1, \ldots, z_d) = \sum_{0 \leq i_1 < \cdots < i_d} \frac{z_1^{q^{i_1}} \cdots z_d^{q^{i_d}}}{L_{n_1}^{i_1} \cdots L_{n_d}^{i_d}} \in \mathbb{C}_\infty[[z_1, \ldots, z_d]]
\]

and

\[
\text{Li}^*_{n_1, \ldots, n_d}(z_1, \ldots, z_d) = \sum_{0 \leq i_1 < \cdots < i_d} \frac{z_1^{q^{i_1}} \cdots z_d^{q^{i_d}}}{L_{n_1}^{i_1} \cdots L_{n_d}^{i_d}} \in \mathbb{C}_\infty[[z_1, \ldots, z_d]]
\]

for \( n_1, \ldots, n_d, d \in \mathbb{N} \), where \( L_0 := 1 \) and \( L_i := (\theta - \theta') \cdots (\theta - \theta^{q^i}) \in K \) for \( i \geq 1 \). When \( d = 1 \), they coincide and recover the Carlitz polylogarithm of Anderson-Thakur \[AT90\ \S2.1\]. By \[C\ \S5.1\], CMPL converges in the region

\[
\mathbb{D} = \left\{ (z_i) \in \mathbb{C}_\infty^d \mid |z_1/\theta^{m_1}|_\infty \cdots |z_d/\theta^{m_d}|_\infty \to 0 \text{ as } 0 \leq i_1 < \cdots < i_d \to \infty \right\}
\]

and CMSPL converges in the similar region \( \mathbb{D}' \) replacing \( < \) with \( \leq \), both of which contain the polydisk \( \mathbb{D}' = \{ (z_i) \in \mathbb{C}_\infty^d \mid |z_i|_\infty < q^{-\frac{1}{m_i}} \} \).
For a fixed \( d \)-tuple of \((Z_1, \ldots, Z_d) \in \mathbb{T}^d \), the \textit{t-motivic CMSPL} and \textit{t-motivic CMSPL} (cf. [CGM]) are defined by the following series respectively

\[
\mathfrak{L}_{n_1,\ldots,n_d}(Z_1, \ldots, Z_d) = \Omega^{-n_1-\cdots-n_d} \sum_{0 \leq i_1 < \cdots < i_d} (\Omega^{n_1} Z_1)^{(i_1)} \cdots (\Omega^{n_d} Z_d)^{(i_d)}
\]

\[
= \sum_{0 \leq i_1 < \cdots < i_d} \frac{Z_1^{(i_1)} \cdots Z_d^{(i_d)}}{\prod_{1 \leq i_1 < \cdots < i_d}},
\]

\[
\mathfrak{L}_{n_1,\ldots,n_d}^*(Z_1, \ldots, Z_d) = \Omega^{-n_1-\cdots-n_d} \sum_{0 \leq i_1 \leq \cdots \leq i_d} (\Omega^{n_1} Z_1)^{(i_1)} \cdots (\Omega^{n_d} Z_d)^{(i_d)}
\]

\[
= \sum_{0 \leq i_1 \leq \cdots \leq i_d} \frac{Z_1^{(i_1)} \cdots Z_d^{(i_d)}}{\prod_{1 \leq i_1 \leq \cdots \leq i_d}},
\]

where \( \mathbb{L}_0 = 1 \) and \( \mathbb{L}_i = (t - \theta^i) \cdots (t - \theta^i) \in K[t] \) for \( i \geq 1 \). They coincide when \( d = 1 \). The t-motivic CMSPL converges with respect to the Gauss norm when

\[
(\|Z_1\|_\infty / |\theta|^{-1}) \cdots (\|Z_d\|_\infty / |\theta|^{-1}) \rightarrow 0 \quad \text{as} \quad 0 < i_1 < \cdots < i_d \rightarrow \infty.
\]

Similarly the t-motivic CMSPL converges in the same situation replacing \( \leq \) with \( \leq \). We remind that the substitution \( t = \theta \) gives (1.3) and (1.4). We have

\[
\mathfrak{L}_{n_1,\ldots,n_d}(Z_1, \ldots, \Omega Z_k, \ldots, Z_d) = \Omega \cdot \mathfrak{L}_{n_1,\ldots,n_k+1,\ldots,n_d}(Z_1, \ldots, Z_d)
\]

\[
\mathfrak{L}_{n_1,\ldots,n_d}^*(Z_1, \ldots, \Omega Z_k, \ldots, Z_d) = \Omega \cdot \mathfrak{L}_{n_1,\ldots,n_k+1,\ldots,n_d}^*(Z_1, \ldots, Z_d)
\]

for \( k \) with \( 1 \leq k \leq d \) by definition.

1.3. \textbf{Continuation of Carlitz polylogarithms.} We explain a method of continuation of the Carlitz polylogarithm to \( \mathbb{C}_\infty \) and a treatment of branches, which is an initial step for continuation of the Carlitz multiple polylogarithm (explained in §1.4) and the star version (explained in §1.5). Our method consists of three steps.

1.3.1. \textbf{Algebraic step.} We introduce the following series for \( Z \in \mathbb{T} \):

\[
\mathfrak{L}_0(Z) = \sum_{i=0}^{\infty} Z^{(i)}
\]

which is ‘a \((d, n_d) = (1, 0)\) version’ of (1.10). When \( \|Z\|_\infty < 1 \), it converges and is \( F_q[t] \)-linear with respect to \( Z \). We have

\[
\mathfrak{L}_0(Z) - \mathfrak{L}_0(Z)^{(1)} = Z,
\]

that is,

\[
\varphi(\mathfrak{L}_0(Z)) = Z.
\]

By (1.11) we remark that \( \mathfrak{L}_0(Z) \) converges to an algebraic function when \( Z \in \mathbb{C}_\infty \) with \( |Z| < 1 \). Lemma 1.4.4 enables us to associate each \( Z \in \mathbb{T} \) with \( \mathfrak{L}_0(Z) \) in the quotient \( F_q[t] \)-module \( \mathbb{T}/F_q[t] \) by keeping the above equation, which yields the extended \( F_q[t] \)-linear map

\[
\mathfrak{L}_0 : \mathbb{T} \rightarrow \mathbb{T}/F_q[t].
\]

A branch \( \mathfrak{L}_0^g : \mathbb{T} \rightarrow \mathbb{T} \) means an \( F_q[t] \)-linear lift of \( \mathfrak{L}_0 \).
We note that \( \tilde{\mathcal{L}}_0(Z) \) is congruent to \( \mathcal{L}_0^n(Z) \) modulo \( \mathbb{F}_q[t] \) when \( \|Z\|_\infty < 1 \). ByLemma 1.1.1 any \( Z \in \mathbb{T} \) and its any branch \( \mathcal{L}_0^n(Z) \) have a same radius of convergence.

1.3.2. **Analytic step.** We consider the continuation of the \( t \)-motivic Carlitz polylogarithm by making use of the equality

\[
(1.13) \quad \mathcal{L}_n(Z) = \Omega^{-n} \mathcal{L}_0(\Omega^n Z)
\]
deduced from (1.9).

**Definition 1.3.1.** For \( n \in \mathbb{N} \), we define the \( \mathbb{F}_q[t] \)-linear map

\[
\tilde{\mathcal{L}}_n : \mathbb{T} \to \mathbb{T}/\Omega^{-n} \mathbb{F}_q[t]
\]
by sending \( Z \in \mathbb{T} \) to

\[
(1.14) \quad \tilde{\mathcal{L}}_n(Z) := \Omega^{-n} \cdot \tilde{\mathcal{L}}_0(\Omega^n Z)
\]
(N.B. \( \Omega^{-n} \mathbb{T} = \mathbb{T} \)). A branch \( \mathcal{L}_n^o : \mathbb{T} \to \mathbb{T} \) means an \( \mathbb{F}_q \)-linear lift of \( \tilde{\mathcal{L}}_n \).

By (1.13), it is congruent to \( \mathcal{L}_n(Z) \) modulo \( \Omega^{-n} \mathbb{F}_q[t] \) when (1.8) holds. The following properties will be used later.

**Lemma 1.3.2.** Let \( n \geq 1 \) and \( Z \in \mathbb{T} \). Let \( \mathcal{L}_n^o(Z) \) be a branch. Then
1. \( \Omega^n \cdot \mathcal{L}_n^o(Z) \in \mathcal{E} \) when \( \Omega^n Z \in \mathcal{E} \).
2. \( \Omega^n \mathcal{L}_n^o(Z) - (\Omega^n \mathcal{L}_n^o(Z))^{(1)} = \Omega^n Z \) when \( Z \in \mathbb{T} \).
3. \( (\Omega^n \cdot \mathcal{L}_n^o(Z)) (\theta^{\ell^k}) = (\Omega^n \cdot \mathcal{L}_n^o(Z)) (\theta)^{\ell^k} \) for \( k \geq 1 \) when \( \Omega^n Z \in \mathcal{E} \).

**Proof.** (1). It follows from Lemma 1.1.1 and (1.14) because we have \( \Omega \in \mathcal{E} \).

(2). Put \( \mathcal{L}_0^o(\Omega^n Z) = \Omega^n \cdot \mathcal{L}_0^o(Z) \). By (1.11), we have

\[
\mathcal{L}_0^o(\Omega^n Z) - \mathcal{L}_0^o(\Omega^n Z)^{(1)} = \Omega^n Z,
\]
which implies the equality.

(3) By Lemma 1.1.1(4) and (1.11), we have \( \mathcal{L}_0^o(\Omega^n Z) \in \mathcal{E} \) when \( \Omega^n Z \in \mathcal{E} \). By evaluating \( t = \theta^{h+1} \) to the above equality, we obtain

\[
\mathcal{L}_0^o(\Omega^n Z)(\theta^{h+1}) - \mathcal{L}_0^o(\Omega^n Z)(\theta)^{h+1} = 0
\]
for \( h \geq 0 \) because we have \( F^{(1)}(t^q) = F(t)^q \) for any \( F \in \mathbb{C}_\infty[[t]] \) and \( \Omega(\theta^{h+1}) = 0 \).

Thus we obtain the formula. \( \square \)

1.3.3. **Evaluation step.** By the evaluation of \( t = \theta \), we carry out the continuation of the Carlitz polylogarithm.

**Definition 1.3.3.** For \( n \in \mathbb{N} \), we define the \( A \)-linear map

\[
\tilde{\mathcal{L}}_n : \mathbb{C}_\infty \to \mathbb{C}_\infty /\pi^n A
\]
by a restriction of \( \tilde{\mathcal{L}}_n \) to \( Z = z \in \mathbb{C}_\infty \subset \mathbb{T} \) and a substitution of \( t = \theta \) there (we note that \( t = \theta \) is inside a region of convergence of \( \tilde{\mathcal{L}}_n(z) \) because \( \Omega^n \mathcal{L}_n^o(Z) \) and \( \Omega^n Z \) have a same radius of convergence by Lemma 1.1.1(2)). A branch \( \mathcal{L}_n^o : \mathbb{C}_\infty \to \mathbb{C}_\infty \) means an \( \mathbb{F}_q \)-linear lift of \( \tilde{\mathcal{L}}_n \).

The following proposition ensures that \( \tilde{\mathcal{L}}_n \) is an analytic continuation of \( \mathcal{L}_n \).
Proposition 1.3.4. \(1\). \(\bar{L}_n(z) \equiv L_n(z) \mod \tilde{\pi}^n A \) when \(z\) lies in \(\mathbb{D}\).

\(2\). \(\bar{L}_n\) locally admits an analytic lift, that is, for each given point \(z \in \mathbb{C}_\infty\), there is a small disk \(U\) centered at \(z\) in \(\mathbb{C}_\infty\) and a lift \(\bar{L}_n|_U : U \to \mathbb{C}_\infty\) which is given by a converging power series.

Proof. \(1\). It follows from our construction.

\(2\). By our construction we have \(\bar{L}_n(z + w) = \bar{L}_n(z) + \bar{L}_n(w)\). We have \(\bar{L}_n(w) \equiv L_n(w)\) for \(w \in \mathbb{D}\) and \(L_n(w)\) is rigid analytic on an appropriately smaller closed disk centered at 0. Then our claim follows because \(\tilde{\pi}^n A\) is discrete in \(\mathbb{C}_\infty\).

Remark 1.3.5. By definition, difference of any two branches of the Carlitz polylogarithm \(L_n(z)\) is given by

\[\alpha \cdot \tilde{\pi}^n \quad (\alpha \in A).\]

While it is worthy to recall in the complex case (characteristic 0 case), difference of any two branches of (analytically continued) polylogarithm \(L_n(z)\) is given by a \(\mathbb{Q}\)-linear combination of

\[(2\pi i)^a \zeta(b)(\log c)^c\]

with \(a + b + c = n\).

Here is an application of our method to evaluate an analytic continuation at a point outside of the region of convergence of \(L_n(z)\):

Remark 1.3.6. Let \(n \in \mathbb{N}\) and \(\alpha \in \mathbb{C}_\infty\). Let \(\sum_{i=0}^{\infty} a_i t^i\) be the power series expansion of \(\alpha \Omega^n \in \mathbb{C}_\infty[[t]]\). Take a series \(\{b_i\}_{i=0}^{\infty}\) such that \(\varphi(b_i) = a_i\) for all \(i\) and \(|b_i|_\infty = |a_i|_\infty\) for all sufficiently large \(i\)’s. Then \(\tilde{\pi}^n \sum_{i=0}^{\infty} b_i \theta^i \in \mathbb{C}_\infty\) gives a representative of lifts of \(\bar{L}_n(\alpha) \in \mathbb{C}_\infty/\tilde{\pi}^n A\).

1.4. Continuation of Carlitz multiple polylogarithms. By exploiting the method of continuation of Carlitz polylogarithm developed in \([1.3]\) we extend the Carlitz multiple polylogarithm to \(\mathbb{C}_\infty\) with a treatment of branches, that is, a monodromy module by three steps in a similar fashion.

1.4.1. Algebraic step. We denote \(\{0\}^d\) to be the multi-index where 0 is repeated \(d\)-times and consider the following series for \(Z_1, \ldots, Z_d \in \mathbb{T}\):

\[\mathfrak{L}_{\{0\}^d}(Z_1, \ldots, Z_d) = \sum_{0 \leq i_1 < \cdots < i_d} Z_{1}^{(i_1)} \cdots Z_{d}^{(i_d)}\]

which is ‘an \((n_1, \ldots, n_d) = \{0\}^d\) version’ of \((1.6)\). When \((1.8)\) holds for \((n_1, \ldots, n_d) = \{0\}^d\), it converges and is \(F_q[t]\)-linear with respect to \(Z_1, \ldots, Z_d\). We observe the following system of difference equations

\[(1.15) \quad \varphi(\mathfrak{L}_{\{0\}^d}(Z_{d-i+1}, \ldots, Z_d)) = Z_{d-i+1} \cdot \mathfrak{L}_{\{0\}^{d-i+1}}(Z_{d-i+2}, \ldots, Z_d)^{(1)}\]

which they satisfy for \(1 \leq i \leq d\). Here we put \(\mathfrak{L}_{\{0\}^0} = 1\). Again by \((1.15)\) we remark that \(\mathfrak{L}_{\{0\}^d}(Z_1, \ldots, Z_d)\) converges to an algebraic function when \(Z_1, \ldots, Z_d\) are in \(\mathbb{C}_\infty\) and \(|Z_1|_\infty, \ldots, |Z_d|_\infty\) are enough small.

We note that, by Lemma \(1.1.1(1)\), for any \(Z_1, \ldots, Z_d \in \mathbb{T}\), there always exists a solution of the above system \((1.15)\), denoted by

\[(1.16) \quad \mathfrak{L}_{\{0\}^d}(Z_1, \ldots, Z_d) := (\mathfrak{L}_{\{0\}^1}(Z_1), \ldots, \mathfrak{L}_{\{0\}^{d-1}}(Z_{d-1}, \ldots, Z_d), \mathfrak{L}_{\{0\}^d}(Z_1, \ldots, Z_d))^T \in \mathbb{T}^d,\]
and all the solutions of the above system (1.15) are described as linear combinations

\[(1.17) \quad \tilde{\mathcal{L}}_1^\circ(Z_1, \ldots, Z_d) + \sum_{k=0}^{d-1} \alpha_k \cdot \tilde{\mathcal{L}}_1^\circ(Z_1, \ldots, Z_k)\]

with \(\alpha_k \in \mathbb{F}_q[t]\) and

\(\tilde{\mathcal{L}}_1^\circ(Z_1, \ldots, Z_k) := (\{0\}^{d-k-1}, 1, \mathcal{L}_1^\circ(Z_k), \ldots, \mathcal{L}_1^\circ(Z_2, \ldots, Z_k), \mathcal{L}_1^\circ(Z_1, \ldots, Z_k))^T\)

in \(\mathbb{T}^d\) whose last \(k\) components are solutions of (1.15) with \(d = k\). When \(k = 0\), it means \((0, \ldots, 0, 1)^T\).

We put \(M_{Z_1, \ldots, Z_{d-1}}^\circ\) to be the \(\mathbb{F}_q[t]\)-submodule of \(\mathbb{T}^d\) generated by the \(d\) elements:

\(M_{Z_1, \ldots, Z_{d-1}}^\circ := (\tilde{\mathcal{L}}_1^\circ(Z_1, \ldots, Z_k) \mid 0 \leq k \leq d - 1, \mathbb{F}_q[t]).\)

It follows from (1.17) that \(M_{Z_1, \ldots, Z_{d-1}}^\circ\) is free from any choice of branches. For fixed \(Z_1, \ldots, Z_{d-1} \in \mathbb{T}\), we obtain a well-defined \(\mathbb{F}_q[t]\)-linear map

\(\tilde{\mathcal{L}}_1^\circ(Z_1, \ldots, Z_{d-1}, -) : \mathbb{T} \to \mathbb{T}^d/M_{Z_1, \ldots, Z_{d-1}}^\circ.\)

A branch \(\tilde{\mathcal{L}}_1^\circ(Z_1, \ldots, Z_{d-1}, -) : \mathbb{T} \to \mathbb{T}^d\) means an \(\mathbb{F}_q[t]\)-linear lift of \(\tilde{\mathcal{L}}_1^\circ(Z_1, \ldots, Z_{d-1}, -).\)

We note that the vector (1.15) is congruent to its ‘non-\(\circ\)’ version

\(\tilde{\mathcal{L}}_1(Z_1, \ldots, Z_{d-1}) := (\mathcal{L}_1(Z_k), \ldots, \mathcal{L}_1(Z_2, \ldots, Z_k), \mathcal{L}_1(Z_1, \ldots, Z_k))^T\)

modulo \(M_{Z_1, \ldots, Z_{d-1}}\) when all components converge.

### 1.4.2. Analytic step.

For \(n_1, \ldots, n_d \geq 1, Z_1, \ldots, Z_d \in \mathbb{T}\), we put

\(M_{n_1, \ldots, n_d}^\circ := \Omega^{-n_1 \cdots n_d} M_{Z_1, \ldots, Z_{d-1}}^\circ\)

which is an \(\mathbb{F}_q[t]\)-submodule of \(\mathbb{T}^d\) because \(\Omega^{-1} \in \mathbb{T}\). Then the continuation of the \(t\)-motivic Carlitz multiple polylogarithm is carried out as follows:

**Definition 1.4.1.** Let \(n_1, \ldots, n_d \in \mathbb{N}\). For fixed \(Z_1, \ldots, Z_{d-1} \in \mathbb{T}\), we define the \(\mathbb{F}_q[t]\)-linear map

\(\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_{d-1}, -) : \mathbb{T} \to \mathbb{T}^d/M_{n_1, \ldots, n_d}\)

sending \(Z_d \in \mathbb{T}\) to

\(\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d) := \Omega^{-n_1 \cdots n_d} \tilde{\mathcal{L}}_{1}^\circ(\Omega^{n_1} Z_1, \ldots, \Omega^{n_d} Z_d).\)

A branch \(\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_{d-1}, -) : \mathbb{T} \to \mathbb{T}^d\) means an \(\mathbb{F}_q[t]\)-linear lift of \(\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_{d-1}, -)\) and, for each \(Z_d \in \mathbb{T}\), we denote

\(\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d) = (\Omega^{-n_1 \cdots n_d-1} \tilde{\mathcal{L}}_{n_d}(Z_d), \ldots, \Omega^{-n_1} \tilde{\mathcal{L}}_{n_2, \ldots, n_d}(Z_2, \ldots, Z_d), \tilde{\mathcal{L}}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d))^T \in \mathbb{T}^d.\)

It turns that the module \(M_{n_1, \ldots, n_d}^\circ\) is generated by \(d\) elements, in precise,

\(M_{n_1, \ldots, n_d}^\circ = (\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_k) \mid 0 \leq k \leq d - 1, \mathbb{F}_q[t],\)
with
\[ \mathfrak{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_k) := \Omega^{-n_1-\cdots-n_d} \cdot \mathfrak{L}_{\{0\}^d}(\Omega^{n_1} Z_1,\ldots,\Omega^{n_d} Z_k) = \Omega^{-n_{k+1}-\cdots-n_d} \cdot \left( \{0\}^{d-k-1}, \Omega^{-n_1-\cdots-n_k}, \Omega^{-n_1-\cdots-n_{k-1}} \mathfrak{L}_{n_k}(Z_k), \ldots, \Omega^{-n_1} \mathfrak{L}_{n_2,\ldots,n_k}(Z_2,\ldots,Z_k), \mathfrak{L}_{n_1,\ldots,n_k}(Z_1,\ldots,Z_k) \right)^T \in \mathbb{T}^d \]

with \(0 \leq k \leq d - 1\). The definition of \(M_{n_1,\ldots,n_d}^{Z_i,\ldots,Z_{d-1}}\) is independent of any choice of branches.

Again we note that the vector \(\mathfrak{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_d)\) is congruent to its non-\(o\) version modulo \(M_{n_1,\ldots,n_d}^{Z_i,\ldots,Z_{d-1}}\) when all components converge.

The following properties will be used in our later sections.

**Proposition 1.4.2.** Put \(n_1,\ldots,n_d \geq 1\) and \(Z_1,\ldots,Z_d \in \mathbb{T}\). Let \(\mathfrak{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_d)\) be a branch as above. Then we have

1. A congruence with the tuple
\[ (\Omega^{-n_1-\cdots-n_d-1} \mathfrak{L}_{n_1}(Z_1),\ldots,\Omega^{-n_1} \mathfrak{L}_{n_2,\ldots,n_d}(Z_2,\ldots,Z_d),\mathfrak{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_d) )^T \in \mathbb{T}^d \]

of \([1,4]\) modulo \(M_{n_1,\ldots,n_d}^{Z_i,\ldots,Z_{d-1}}\) when \([1.9]\) holds.

2. \(\Omega^{n_1+\cdots+n_d} \mathfrak{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_k) \in \mathbb{T}(\infty)^d\) (resp. \(\mathcal{E}^d\)) for \(k = 1,\ldots,d\) when \(\Omega^{n_1} Z_1,\ldots,\Omega^{n_d} Z_d \in \mathbb{T}(\infty)\) (resp. \(\mathcal{E}\)).

3. \(\Omega^{n_1+\cdots+n_d} \mathfrak{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_d) = \Omega^{n_1} Z_1 (\Omega^{n_2+\cdots+n_d} \mathfrak{L}_{n_2,\ldots,n_d}(Z_2,\ldots,Z_d))\) \((1)\).

4. \(\Omega^{n_1+\cdots+n_d} \mathfrak{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_d) (\theta^m) = (\Omega^{n_1+\cdots+n_d} \mathfrak{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_d)) (\theta^m)\) \(\text{for } k \geq 1\) when \(\Omega^{n_1} Z_1,\ldots,\Omega^{n_d} Z_d \in \mathcal{E}\).

**Proof.** The proof can be done in the same way to that of Lemma \(1.3.2\)

1. It can be deduced from \([1.9]\).

2. By Lemma \(1.1.1\) we have \(\mathfrak{L}_{\{0\}^d}(Z_1,\ldots,Z_d) \in \mathbb{T}(\infty)^d\) (resp. \(\mathcal{E}^d\)) for \(Z_1,\ldots,Z_d \in \mathbb{T}(\infty)\) (resp. \(\mathcal{E}\)), which implies the claim.

3. Put
\[ \mathfrak{L}_{\{0\}^d}(\Omega^{n_1} Z_1,\ldots,\Omega^{n_d} Z_d) = \Omega^{n_1+\cdots+n_d} \mathfrak{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_d). \]

By \([1.1.5]\), we have
\[ \mathfrak{L}_{\{0\}^d}(\Omega^{n_1} Z_1,\ldots,\Omega^{n_d} Z_d) - \mathfrak{L}_{\{0\}^d}(\Omega^{n_1} Z_1,\ldots,\Omega^{n_d} Z_d) \]

which proves the claim.

4. By Lemma \(1.1.1\) and \(1.1.6\), we inductively obtain \(\mathfrak{L}_{\{0\}^d}(\Omega^{n_1} Z_1,\ldots,\Omega^{n_d} Z_d) \in \mathcal{E}\). Evaluation of \(t = \theta^{m+1}\) to the above equation yields
\[ \mathfrak{L}_{\{0\}^d}(\Omega^{n_1} Z_1,\ldots,\Omega^{n_d} Z_d) (\theta^{m+1}) - \mathfrak{L}_{\{0\}^d}(\Omega^{n_1} Z_1,\ldots,\Omega^{n_d} Z_d) (\theta^{m}) = 0 \]

by the same reason to the proof of Lemma \(1.3.2\). \(\square\)

We remark that (2) and (4) are shown in [C] Lemma 5.3.1 and 5.3.5 under the convergence condition for \((Z_1,\ldots,Z_d) \in (K^\times)^d \cap \mathbb{D}\) and in [CPY] Proposition 2.3.3 under the condition \((Z_1,\ldots,Z_d) \in (K[t])^d \cap \mathbb{D}\).
1.4.3. Evaluation step. By the evaluation of $t = \theta$, we carry out the continuation of the Carlitz multiple polylogarithm.

**Definition 1.4.3.** Let $n_1, \ldots, n_d \in \mathbb{N}$ and $z_1, \ldots, z_{d-1} \in \mathbb{C}_\infty$.

1. The monodromy module

$$M_{n_1, \ldots, n_d}^{z_1, \ldots, z_{d-1}}$$

is defined to be the $\mathbb{F}_q[t]$-submodule of $\mathbb{C}_\infty^d$ given by the evaluation of $t = \theta$ to $M_{n_1, \ldots, n_d}^{z_1, \ldots, z_{d-1}}$.

2. We define the $\mathbb{F}_q$-linear map

$$\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(z_1, \ldots, z_{d-1}, -) : \mathbb{C}_\infty \to \mathbb{C}_\infty^d / M_{n_1, \ldots, n_d}^{z_1, \ldots, z_{d-1}}$$

by a restriction of $\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_{d-1}, -)$ to $Z_i = z_i \in \mathbb{C}_\infty \subset \mathbb{T}$ and a substitution of $t = \theta$ there (we note again that $t = \theta$ is inside its region of convergence by $\Omega(\theta) \neq 0$, the entirety of $\Omega$ and the above proposition).

3. A branch $\tilde{\mathcal{L}}_{n_1, \ldots, n_d}^\theta(z_1, \ldots, z_{d-1}, -)$ means an $\mathbb{F}_q$-linear lift of $\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(z_1, \ldots, z_{d-1}, -)$.

For each $z_d \in \mathbb{C}_\infty$, we denote

$$\tilde{\mathcal{L}}_{n_1, \ldots, n_d}^\theta(z_1, \ldots, z_d) := (\tilde{\mathcal{L}}^\theta_{n_1, \ldots, n_d}(z_1), \tilde{\mathcal{L}}^\theta_{n_1, \ldots, n_d}(z_2), \ldots, \tilde{\mathcal{L}}^\theta_{n_1, \ldots, n_d}(z_d)) \in \mathbb{C}_\infty^d.$$

The definition of the monodromy module $M_{n_1, \ldots, n_d}^{z_1, \ldots, z_{d-1}}$ is independent of any choice of branches. It is the $A$-submodule of $\mathbb{C}_\infty^d$ generated by $d$ elements, in precise,

$$M_{n_1, \ldots, n_d}^{z_1, \ldots, z_{d-1}} = \langle \tilde{\mathcal{L}}_{n_1, \ldots, n_d}^\theta(z_1, \ldots, z_k) \mid 0 \leq k \leq d - 1 \rangle_A,$$

with

$$\tilde{\mathcal{L}}_{n_1, \ldots, n_d}^\theta(z_1, \ldots, z_k) := \tilde{\mathcal{L}}_{n_1, \ldots, n_d}^\theta(z_1, \ldots, z_k)|_{t=\theta}$$

$$= \tilde{\mathcal{L}}^\theta_{n_1, \ldots, n_d}(z_1, \ldots, z_k) \equiv \begin{pmatrix} 0^{d-k-1}, n_1 + \cdots + n_k, 0, \ldots, 0 \end{pmatrix} \left( \begin{array}{c} \tilde{\mathcal{L}}^\theta_{n_1, \ldots, n_d}(z_1), \ldots, \tilde{\mathcal{L}}^\theta_{n_1, \ldots, n_d}(z_k) \end{array} \right)^T \in \mathbb{C}_\infty^d$$

with $0 \leq k \leq d - 1$. In other word, it is the $A$-submodule of $\mathbb{C}_\infty^d$ generated by $d$ columns of the following matrix:

$$\begin{pmatrix} n_1 + \cdots + n_d & 0 & 0 & \cdots & 0 \\ \cdots & \tilde{\mathcal{L}}_{n_1, \ldots, n_d}^\theta(z_1, \ldots, z_{d-1}) & 0 & \cdots & 0 \\ 0 & \cdots & \tilde{\mathcal{L}}_{n_1, \ldots, n_d}^\theta(z_1, \ldots, z_{d-1}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \tilde{\mathcal{L}}_{n_1, \ldots, n_d}^\theta(z_1, \ldots, z_{d-1}) \
\end{pmatrix}$$

**Theorem 1.4.4.**

1. $\tilde{\mathcal{L}}_{n_1, \ldots, n_d}^\theta(z_1, \ldots, z_d)$ is congruent to the tuple

$$(\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(z_1), \ldots, \tilde{\mathcal{L}}_{n_2, \ldots, n_d}(z_2), \ldots, \tilde{\mathcal{L}}_{n_1, \ldots, n_d}(z_1, \ldots, z_d) \in \mathbb{C}_\infty^d$$

of $M_{n_1, \ldots, n_d}^{z_1, \ldots, z_{d-1}}$ modulo $M_{n_1, \ldots, n_d}^{z_1, \ldots, z_{d-1}}$ when $(z_1, \ldots, z_d)$ lies in $D$. 

2. $\tilde{\mathcal{L}}_{n_1, \ldots, n_d}(z_1, \ldots, z_{d-1}, -)$ locally admits an analytic lift (as a function on $z_d$) in the sense of Proposition [1.3.3].

**Proof.** The proof can be done in the same way to that of Proposition [1.3.3]. We note that $M_{n_1, \ldots, n_d}^{z_1, \ldots, z_{d-1}}$ is discrete in $\mathbb{C}_\infty^d$ because the above matrix forms a lower triangular matrix with invertible diagonals. \qed
Remark 1.4.5. Let $\Li^{o}_{n_1\ldots,n_d}(z_1,\ldots,z_d)$ and $\Li^{o'}_{n_1\ldots,n_d}(z_1,\ldots,z_d)$ be any branches and denote their last coordinates by $\Li^{o}_{n_1\ldots,n_d}(z_1,\ldots,z_d)$ and $\Li^{o'}_{n_1\ldots,n_d}(z_1,\ldots,z_d)$ respectively. By definition, the difference between them is given by an integral combination of the last row of the above matrix:

$$
\Li^{o}_{n_1\ldots,n_d}(z_1,\ldots,z_d)-\Li^{o'}_{n_1\ldots,n_d}(z_1,\ldots,z_d) = \sum_{i=0}^{d-1} \alpha_i \cdot \tilde{\pi}^{n_{i+1}+\ldots+n_d} \Li^{o}_{n_1\ldots,n_i}(z_1,\ldots,z_i)
$$

with $\alpha_i \in A$.

1.5. Continuation of Carlitz multiple star polylogarithms. By exploiting the method of continuation of Carlitz polylogarithm in §1.3 and imitating the arguments in §1.4, we now extend the Carlitz multiple star polylogarithm to $C_\infty$ with a treatment of branches, that is, a monodromy module.

1.5.1. Algebraic step. We consider the series for $Z_1,\ldots,Z_d \in \mathbb{T}$:

$$
\mathcal{L}_{1}(0)_{\bar{\cdot}}(Z_1,\ldots,Z_d) = \sum_{0 \leq i_1 \leq \cdots \leq i_d} Z_{i_1}^{(1)} \cdots Z_{i_d}^{(d)}.
$$

We observe the following system of difference equations

$$
(1.18) \quad \varphi(\mathcal{L}_{1}(0)_{\bar{\cdot}})(Z_{d-i+1},\ldots,Z_d) = Z_{d-i+1} \cdot \mathcal{L}_{1}(0)_{\bar{\cdot}}(Z_{d-i+2},\ldots,Z_d)
$$

which satisfy for $1 \leq i \leq d$.

We note that, by Remark 1.1.1.1, for any $Z_1,\ldots,Z_d \in \mathbb{T}$, there always exists a solution of the above system (1.18) in $\mathbb{T}^d$, denoted by $\tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_d)$.

$$
(1.19) \quad \tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_d) := \left( \mathcal{L}_{1}(0)_{\bar{\cdot}}(Z_1,\ldots,Z_d), \mathcal{L}_{1}(0)_{\bar{\cdot}}(Z_2,\ldots,Z_d), \ldots, \mathcal{L}_{1}(0)_{\bar{\cdot}}(Z_d) \right)^T,
$$

and all the solutions of the above system (1.18) are described as linear combinations

$$
\tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_d) + \sum_{k=0}^{d-1} \alpha_k \cdot \tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_k)
$$

with $\alpha_k \in \mathbb{F}_q[t]$ and

$$
\tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_k) := \left( \mathcal{L}_{1}(0)_{\bar{\cdot}}(Z_1,\ldots,Z_k), \mathcal{L}_{1}(0)_{\bar{\cdot}}(Z_2,\ldots,Z_k), \ldots, \mathcal{L}_{1}(0)_{\bar{\cdot}}(Z_k) \right)^T
$$

in $\mathbb{T}^d$ whose first $k$ components are solutions of (1.18) with $d = k$. When $k = 0$, it means $(0,\ldots,0,1)^T$.

Put $\mathcal{M}^{*}_{\{0\}^d}_{Z_{1},\ldots,Z_{d-1}}$ to be the $\mathbb{F}_q[t]$-submodule of $\mathbb{T}^d$ generated by the $d$ elements:

$$
\mathcal{M}^{*}_{\{0\}^d}_{Z_{1},\ldots,Z_{d-1}} := \langle \tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_k) \mid 0 \leq k \leq d-1 \rangle \mathbb{F}_q[t],
$$

which is actually independent of any choice of branches. Then for a fixed $Z_1,\ldots,Z_{d-1} \in \mathbb{T}$, we obtain a well-defined $\mathbb{F}_q[t]$-linear map

$$
\tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_{d-1},-) : \mathbb{T}^d / \mathcal{M}^{*}_{\{0\}^d}_{Z_{1},\ldots,Z_{d-1}}.
$$

A branch $\tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_{d-1},-) : \mathbb{T} \to \mathbb{T}^d$ means an $\mathbb{F}_q$-linear lift of $\tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_{d-1},-)$.  

\footnote{For our convenience in the next section, we reverse here the order of coordinate to that of $\tilde{\mathcal{L}}_{1}(0)^{\bar{\cdot}}_{\bar{\cdot}}(Z_1,\ldots,Z_d)$ in the previous subsection.}
We note that the vector $1^{1.19}$ is congruent to its ‘non-o’ version modulo $M^*_{q(t)}$ when all components converge.

1.5.2. Analytic step. For $n_1, \ldots, n_d \geq 1$, $Z_1, \ldots, Z_d \in T$, we put

$$M^*_{n_1, \ldots, n_d} := \Omega^{-n_1-\cdots-n_d}M^*_{q(t)}Z_1, \ldots, \Omega^{n_d}Z_d,$$

which is the $F_q[t]$-submodule of $T^d$. Then the continuation of the $t$-motivic Carlitz star multiple polylogarithm is carried out as follows:

**Definition 1.5.1.** Let $n_1, \ldots, n_d \in \mathbb{N}$. For fixed $Z_1, \ldots, Z_{d-1} \in T$, we define the $F_q[t]$-linear map

$$\hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_{d-1}, -) : T \to T^d/M^*_{n_1, \ldots, n_d}$$

sending $Z_d \in T$ to

$$\hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d) := \Omega^{-n_1-\cdots-n_d} \hat{\mathcal{L}}^*_{n_1, n_d}(Z_1, \ldots, Z_d).$$

A branch $\hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_{d-1}, -) : T \to T^d$ means an $F_q$-linear lift of $\hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_{d-1}, -)$ and, for each $Z_d \in T$, we denote

$$\hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d) := (\mathcal{L}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d), \Omega^{-n_1} \mathcal{L}_{n_1, n_d}(Z_2, \ldots, Z_d), \ldots, \Omega^{-n_1-\cdots-n_{d-1}} \mathcal{L}_{n_d}(Z_d))^T \in T^d.$$

It turns out that the module $M^*_{n_1, \ldots, n_d}$ is the $F_q[t]$-submodule of $T^d$ generated by $d$ elements, in precise,

$$M^*_{n_1, \ldots, n_d} = \langle \hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d) \mid 0 \leq k \leq d-1 \rangle_{F_q[t]}.$$

with

$$\hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_k) := \Omega^{-n_1-\cdots-n_d} \hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_k) = \Omega^{-n_1-\cdots-n_{k+1}}(\mathcal{L}_{n_1, \ldots, n_k}(Z_1, \ldots, Z_k), \Omega^{-n_1} \mathcal{L}_{n_1, n_k}(Z_2, \ldots, Z_k), \ldots, \Omega^{-n_1-\cdots-n_{k-1}} \mathcal{L}_{n_k}(Z_k), \Omega^{-n_1-\cdots-n_k}, \langle 0 \rangle^d)^T \in T^d$$

with $0 \leq k \leq d-1$. Again the definition of $M^*_{n_1, \ldots, n_d}$ is independent of any branches. The following properties will be used in our later sections.

**Proposition 1.5.2.** Put $n_1, \ldots, n_d \geq 1$ and $Z_1, \ldots, Z_d \in T$. Let $\hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d)$ be a branch as above. Then we have

1. A congruence with the tuple given by

$$(\mathcal{L}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d), \Omega^{-n_1} \mathcal{L}_{n_1, n_d}(Z_2, \ldots, Z_d), \ldots, \Omega^{-n_1-\cdots-n_{d-1}} \mathcal{L}_{n_d}(Z_d))^T \in T^d$$

of (1.7) modulo $M^*_{n_1, \ldots, n_d}$ when it converges.

2. $\Omega^{n_1+\cdots+n_d} \hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d) \in T(\infty)^d$ (resp. $E^d$) for $k = 1, \ldots, d$ when $\Omega^{n_1} Z_1, \ldots, \Omega^{n_d} Z_d \in T(\infty)$ (resp. $E$).

3. $\varphi(\Omega^{n_1+\cdots+n_d} \hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d)) = \Omega^{n_1} Z_1 \Omega^{n_2+\cdots+n_d} \hat{\mathcal{L}}^*_{n_1, n_d}(Z_2, \ldots, Z_d)$.

4. $(\Omega^{n_1+\cdots+n_d} \hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d)) (\theta^k) = (\Omega^{n_1+\cdots+n_d} \hat{\mathcal{L}}^*_{n_1, \ldots, n_d}(Z_1, \ldots, Z_d)) (\theta)^k$ for $k \geq 1$ when $\Omega^{n_1} Z_1, \ldots, \Omega^{n_d} Z_d \in E$.

**Proof.** The proof can be done in the same way to that of Proposition 1.4.2. □
1.5.3. Evaluation step. By the evaluation of \( t = \theta \), we carry out the continuation of the Carlitz star multiple polylogarithm.

**Definition 1.5.3.** Let \( n_1, \ldots, n_d \in \mathbb{N} \) and \( z_1, \ldots, z_{d-1} \in \mathbb{C}_\infty \).

1. The monodromy module \( M_{n_1, \ldots, n_d}^* \) is defined to be the \( \mathbb{F}_q[t] \)-submodule of \( \mathbb{C}_\infty^d \) given by the evaluation of \( t = \theta \) to \( M_{n_1, \ldots, n_d}^* \).
2. We define the \( \mathbb{F}_q \)-linear map
   \[
   \tilde{\Pi}_{n_1, \ldots, n_d}^*(z_1, \ldots, z_{d-1}, -): \mathbb{C}_\infty \to \mathbb{C}_\infty^d / M_{n_1, \ldots, n_d}^*
   \]
   by a restriction of \( \tilde{\Pi}_{n_1, \ldots, n_d}(Z_1, \ldots, Z_{d-1}, -) \) to \( Z_i = z_i \in \mathbb{C}_\infty \subset \mathbb{T} \) and a substitution of \( t = \theta \) there (we note again that \( t = \theta \) is inside its region of convergence by \( \Omega(\theta) \neq 0 \), the entireness of \( \Omega \) and the above proposition).
3. A branch \( \tilde{\Pi}_{n_1, \ldots, n_d}^*(Z_1, \ldots, Z_{d-1}, -) : \mathbb{C}_\infty \to \mathbb{C}_\infty^d \) means an \( \mathbb{F}_q \)-linear lift of
   \[
   \tilde{\Pi}_{n_1, \ldots, n_d}^*(z_1, \ldots, z_d) \quad \text{and, for each} \quad z_d \in \mathbb{C}_\infty, \text{we denote}
   \]
   \[
   \tilde{\Pi}_{n_1, \ldots, n_d}^*(z_1, \ldots, z_d) = (\Pi_{n_1, \ldots, n_d}(z_1, \ldots, z_d), \tilde{n}^1 \Pi_{n_2, \ldots, n_d}(z_2, \ldots, z_d), \ldots, \tilde{n}^1 \Pi_{n_2, \ldots, n_d}(z_{d-1}, z_d), \tilde{n}^1 \Pi_{n_2, \ldots, n_d}(z_{d-1}, z_d)) \in \mathbb{C}_\infty^d.
   \]

It turns out that the monodromy module \( M_{n_1, \ldots, n_d}^* \) is the \( A \)-submodule of \( \mathbb{C}_\infty^d \) generated by \( d \) elements
\[
\tilde{\Pi}_{n_1, \ldots, n_d}^*(z_1, \ldots, z_k) := \tilde{n}^{k+1} + \cdots + n_d \cdot \tilde{\Pi}_{n_1, \ldots, n_k}^*(z_1, \ldots, z_k),
\]
\[
\ldots, \tilde{n}^{k+1} + \cdots + n_d \cdot \tilde{\Pi}_{n_1, \ldots, n_k}^*(z_1, \ldots, z_k), \tilde{n}^{k+1} + \cdots + n_k \cdot \tilde{\Pi}_{n_1, \ldots, n_k}^*(z_k), \tilde{n}^{k+1} + \cdots + n_k \cdot \{0\}^{d-k-1} \rangle \in \mathbb{C}_\infty^d
\]
with \( 0 \leq k \leq d - 1 \). Actually it is independent of any choice of branches.

**Theorem 1.5.4.**

1. \( \tilde{\Pi}_{n_1, \ldots, n_d}^*(z_1, \ldots, z_d) \) is congruent to the tuple
   \[
   (\Pi_{n_1, \ldots, n_d}(z_1, \ldots, z_d), \Pi_{n_2, \ldots, n_d}(z_2, \ldots, z_d), \ldots, \Pi_{n_2, \ldots, n_d}(z_{d-1}, z_d)) \in \mathbb{C}_\infty^d
   \]
   of \( \Pi(\mathbb{C}) \) modulo \( M_{n_1, \ldots, n_d}^* \) when \( (z_1, \ldots, z_d) \) lies in \( \mathbb{D}^* \).
2. \( \tilde{\Pi}_{n_1, \ldots, n_d}^*(z_1, \ldots, z_{d-1}, -) \) locally admits an analytic lift (as a function on \( z_d \)) in the sense of Proposition 1.3.4.

**Proof.** The proof can be done in the same way to that of Theorem 1.4.3 \( \square \)

### 2. Applications

By exploiting the techniques of the continuation of multiple polylogarithms developed in the previous section, we explain how the logarithms associated with the tensor power of Carlitz module are extended to the whole space in \( \S 2.4 \). We present the orthogonal property of \( t \)-motivic CMPL and CMSPL which extends the functional relations of Chang-Mishiba in \( \S 2.2 \). We show that Eulerian property is independent of any choice of branches in \( \S 2.3 \).
2.1. Logarithms of tensor powers of Carlitz module. We explain a method of
continuation of the logarithms associated with tensor powers of the Carlitz module.

We begin with the review of the definition of \( t \)-modules (cf. [BP]). Let \( \mathbb{C}_\infty \{ \tau \} \)
be the twisted polynomial algebra in the variable \( \tau \) over \( \mathbb{C}_\infty \) with the relation
\[
\tau \alpha = \alpha^q \tau \quad \text{for} \quad \alpha \in \mathbb{C}_\infty.
\]

An \( n \)-dimensional \( t \)-module \( E \) over \( \mathbb{C}_\infty \) is an \( \mathbb{F}_q \)-algebra homomorphism \( \rho_E : \mathbb{F}_q[t] \to \operatorname{Mat}_n(\mathbb{C}_\infty \{ \tau \}) \) such that for each \( \alpha \in \mathbb{F}_q[t] \),
\[
\rho_E(\alpha) = \sum_i E_{a,i} \tau^i
\]
with \( E_{a,i} \in \operatorname{Mat}_n(\mathbb{C}_\infty) \) and \( d\rho_E(a) - a \cdot I_n \) (where \( d\rho_E(a) \) mean \( \rho_E(\alpha) \) is a nilpotent matrix. We denote the \( t \)-module whose action is given by \( d\rho_E \) by \( \operatorname{Lie}_E \). One can show that there exists a unique \( \mathbb{F}_q \)-linear \( n \)-variable power series of the form
\[
\operatorname{Exp}_E = \tau^0 + \sum_{i=1}^{\infty} \alpha_i \tau^i \text{ with } \alpha_i \in \operatorname{Mat}_n(\mathbb{C}_\infty) \text{ such that } \operatorname{Exp}_E \circ d\rho_E(a) = \rho_E(a) \circ \operatorname{Exp}_E.
\]

The logarithm \( \log_E \) is defined to be the formal power series which is inverse to \( \operatorname{Exp}_E \) and has the property
\[
(2.1) \quad \log_E \circ \rho_E(a) = d\rho_E(a) \circ \log_E.
\]

We note that \( \operatorname{Exp}_E \) converges everywhere on \( \mathbb{C}_\infty \) while \( \log_E \) converges on a certain
multi-disk centered at the origin (cf. [AT90] Proposition/Definition 2.4.3).

For a positive integer \( n \) we denote by \( \mathbb{C}^\otimes n \) to be the \( n \)-th tensor power of the
Carlitz module \( C \) (cf. [AT90]). It is given by an \( \mathbb{F}_q \)-algebra homomorphism \( \rho_n : \mathbb{F}_q[t] \to \operatorname{Mat}_n(\mathbb{C}_\infty \{ \tau \}) \) determined by \( \rho_n(t) = \theta I_n + N + E \tau \) with
\[
N = \begin{pmatrix} 0 & 1 & \ldots & \ldots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & \ldots & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix}.
\]

The corresponding \( \log_{\mathbb{C}^\otimes n} \) is an \( \mathbb{F}_q \)-linear map which satisfies
\[
(2.2) \quad \log_{\mathbb{C}^\otimes n} \circ (\theta I + N + E \tau)((z_1, \ldots, z_n)^T) = (\theta I + N) \circ \log_{\mathbb{C}^\otimes n}((z_1, \ldots, z_n)^T)
\]
in the region where both the sides converge. Here \( T \) stands for the transpose.

In [AT90] Proposition/Definition 2.4.3, it is shown that the formal power series
\( \log_{\mathbb{C}^\otimes n}((z_1, \ldots, z_n)^T) \) converges when
\[
(2.3) \quad |z_i|_\infty < |\theta|^{-n+i+a} \quad (1 \leq i \leq n).
\]

The continuation of \( \log_{\mathbb{C}^\otimes n} \) can be done as follows: For any map \( F : T \to T \) we define \( L(F) : T \to T \) by
\[
L(F)(Z) = tF(Z) - F(\theta Z)
\]
for \( Z \in T \). Since we have \( t\tilde{\mathcal{L}}_0(1) = \tilde{\mathcal{L}}_0(t) \), \( \tilde{\mathcal{L}}_0(Z^{(1)}) = \tilde{\mathcal{L}}_0(Z) \) and \( \tilde{\mathcal{L}}_0(Z + Z') = \tilde{\mathcal{L}}_0(Z) + \tilde{\mathcal{L}}_0(Z') \mod \mathbb{F}_q[t] \) for \( Z, Z' \in T \) by our construction in [IL3]
we have
\[
L(\tilde{\mathcal{L}}_n)(z) = t\tilde{\mathcal{L}}_n(z) - \tilde{\mathcal{L}}_n(\theta z) = \tilde{\mathcal{L}}_n(tz) - \tilde{\mathcal{L}}_n(\theta z) = \tilde{\mathcal{L}}_n((t-\theta)z) \mod \Omega^{-n}\mathbb{F}_q[t],
\]
for \( z \in \mathbb{C}_\infty \). Since
\[
L^i(\tilde{\mathcal{L}}_n)(z) \equiv \tilde{\mathcal{L}}_n((t-\theta)^i z) \mod \Omega^{-n}\mathbb{F}_q[t],
\]
we have
\[ L^n(L_n(z)) = L_n((t - \theta)^n z) = \Omega^{-n} \tilde{L}_0(\Omega^n (t - \theta)^n z) = \Omega^{-n} \tilde{L}_0((\Omega^{(1)})^n z). \]

By (1.11), we have
\[ \Omega^{-n} \{ L_0(\Omega^n z^{(1)}) + (\Omega^{(1)})^n z \} \]
(2.4)
\[ \equiv L_n(z^{(1)}) + (t - \theta)^n z \mod \Omega^{-n} \mathbb{F}_q[t]. \]

By following [CGM, GN], we consider the map for \( r \geq q \)
\[ \delta^n_0 : \mathbb{T}_r \to \mathbb{C}_\infty ( = \text{Mat}_{n \times 1} \mathbb{C}_\infty) \]
sending each \( f = \sum_{i \geq 0} c_i (t - \theta)^i \in \mathbb{T}_r \) to \((c_{n-1}, \ldots, c_1, c_0)^T\). By [AT90] Proposition 2.5.5, we have
(2.5)
\[ \delta^n_0(\Omega^{-n} \mathbb{F}_q[t]) = \Lambda_n \]
where \( \Lambda_n \) is the \( A \)-module under the \( \delta^n_0 \)-action given by \( \ker \text{Exp}_{\mathbb{C}_\infty^n} \). It induces the \( \mathbb{C}_\infty \)-linear map
\[ \delta^n_0 : \mathbb{T}_r / \Omega^{-n} \mathbb{F}_q[t] \to \mathbb{C}_\infty / \Lambda_n. \]
We have \( \tilde{L}_n(z) \in \mathbb{T}_r / \Omega^{-n} \mathbb{F}_q[t] \) for each \( z \in \mathbb{C}_\infty \) and whence \( L^i(\tilde{L}_n(z)) \in \mathbb{T}_r / \Omega^{-n} \mathbb{F}_q[t] \) for \( i = 1, \ldots, n - 1 \).

**Definition 2.1.1.** Let \( \tilde{e}_k \) be the unit vector of \( \mathbb{C}_\infty^n \) whose \( k \)-th coordinate is 1. We define the \( \mathbb{C}_\infty \)-linear map
\[ \tilde{\text{Log}}_n : \mathbb{C}_\infty^n \to \mathbb{C}_\infty^n / \Lambda_n \]
by sending \((z_1, \ldots, z_n)^T = \sum_{k=1}^n z_k \tilde{e}_k\) to
\[ \sum_{k=1}^n \delta_0^n \left( L^{n-k} (\tilde{L}_n(z_k)) \right) \equiv \delta_0^n \circ \tilde{L}_n \left( \sum_{k=1}^n (t - \theta)^{n-k} z_k \right) \mod \Lambda_n. \]

The following is an extension of the property (2.2).

**Proposition 2.1.2.** For \((z_1, \ldots, z_n) \in \mathbb{C}_\infty^n\), we have
(2.6)
\[ \tilde{\text{Log}}_n((\theta I_n + N + E \tau)^T(z_1, \ldots, z_n)^T) \equiv (\theta I_n + N) \tilde{\text{Log}}_n((z_1, \ldots, z_n)^T) \mod \Lambda_n. \]

**Proof.** The right hand side is well-defined because we have \((\theta I_n + N) \Lambda_n \subset \Lambda_n \) by \( \Lambda_n = \ker \text{Exp}_{\mathbb{C}_\infty^n} \). Put
\[ \tilde{e}_k(z) = (\ell_{k,1}(z), \ldots, \ell_{k,n}(z))^T := \tilde{\text{Log}}_n(z \tilde{e}_k) = \delta_0^n \left( L^{n-k} (\tilde{L}_n(z)) \right) \]
in \( \mathbb{C}_\infty^n / \Lambda_n \) for \( 1 \leq k \leq n \). Since \( L(F)(z) = \{ \theta + (t - \theta) \} F(z) - F(\theta z) \) for any map \( F : \mathbb{T} \to \mathbb{T} \), we have
\[ \tilde{e}_i(z) = \left( \ell_{i,1}(z), \ldots, \ell_{i,n}(z) \right)^T = \delta_0^n \left( L^{n-i} (\tilde{L}_n(z)) \right) = \delta_0^n L^{n-i-1} \left( L(\tilde{L}_n(z)) \right) \]
\[ = \delta_0^n L^{n-i-1} \left( \{ \theta + (t - \theta) \} \tilde{L}_n(z) - \tilde{L}_n(\theta z) \right) \]
\[ \equiv (\theta \ell_{i+1,1}(z) + \ell_{i+1,2}(z) - \ell_{i+1,1}(\theta z), \ldots, \theta \ell_{i+1,n-1}(z) + \ell_{i+1,n}(z) - \ell_{i+1,n-1}(\theta z), \theta \ell_{i+1,n}(z) - \ell_{i+1,n}(\theta z))^T \]
\[ \equiv (\theta I_n + N) \left( \ell_{i+1,1}(z), \ldots, \ell_{i+1,n}(z) \right)^T - \left( \ell_{i+1,1}(\theta z), \ldots, \ell_{i+1,n}(\theta z) \right)^T \]
\[ \equiv (\theta I_n + N) \tilde{e}_{i+1}(z) - \tilde{e}_{i+1}(\theta z) \]
for $1 \leq i < n$. Actually the equation holds for $i = 0$. By \[2.3\], we also obtain
\[
(\ell_{11}(z^{(1)}), \ldots, \ell_{nn}(z^{(1)}))^T \equiv (\theta I_n + N)(\ell_{11}(z), \ldots, \ell_{1n}(z))^T - (\ell_{11}(\theta z), \ldots, \ell_{1n}(\theta z))^T,
\]
that is
\[
\bar{\ell}_n(z^{(1)}) \equiv (\theta I_n + N)\bar{l}_1(z) - \bar{l}_1(\theta z).
\]

Therefore
\[
\tilde{\Log}_n((\theta I_n + N + E\tau)(z_1, \ldots, z_n)^T) = \Log_n((\theta z_1 + z_2, \ldots, \theta z_{n-1} + z_n, \theta z_n + z_1^{(1)})^T)
\]
\[
= \sum_{i=1}^{n-1} \tilde{\ell}_i(\theta z_i + z_{i+1}) + \bar{\ell}_n(\theta z_n + z_1^{(1)}) = \sum_{i=1}^{n} \bar{\ell}_i(\theta z_i) + \sum_{i=1}^{n-1} \tilde{\ell}_i(z_{i+1}) + \bar{\ell}_n(z_1^{(1)})
\]
\[
= \sum_{i=1}^{n-1} \ell_{i+1}(\theta z_i + z_{i+1}) + \sum_{i=1}^{n-1} \bar{\ell}_i(z_{i+1}) + (\theta I_n + N)\bar{l}_1(z_1) = \sum_{i=1}^{n} (\theta I_n + N)\bar{l}_i(z_i)
\]
\[
= (\theta I_n + N)\tilde{\Log}_n((z_1, \ldots, z_n)^T).
\]

Thus we obtain the claim.

The following theorem assures that $\tilde{\Log}_n$ is an analytic continuation of $\Log_{C^{\otimes \infty}}$.

**Theorem 2.1.3.** (1) $\Log_{C^{\otimes \infty}}((z_1, \ldots, z_n)^T) \equiv \Log_n((z_1, \ldots, z_n)^T) \mod \Lambda_n$ when $(z_1, \ldots, z_n)$ is in the convergence region of \[2.3\].

(2) Let $\Exp_n : C^{\infty}_\infty / \Lambda_n \to C^{\infty}_\infty$ be the induced map from $\Exp_{C^{\otimes \infty}} : C^{\infty}_\infty \to C^{\infty}_\infty$. Then $\Log_n$ is the inverse of $\Exp_n$.

**Proof.** (1). We put
\[
\tilde{l}_i(z) = (l_{i1}(z), \ldots, l_{in}(z))^T := \Log_{C^{\otimes \infty}}(z\hat{e}_i)
\]
for each $i$. By \[2.2\], we have
\[
\bar{\ell}_i(z) = (\theta I_n + N)\bar{l}_{i+1}(z) - \bar{l}_{i+1}(\theta z)
\]
for $1 \leq i < n$. Hence to show $\hat{\ell}_i(z) = \bar{\ell}_i(z)$ for all $i$, it is enough to prove $\tilde{l}_n(z) = \bar{l}_n(z)$. By \[CCM\] Theorem 3.3.5 and \[GN\] Theorem 4.14, we have
\[
\Log_{C^{\otimes \infty}}((0, \ldots, 0, z)^T) = \delta_0^\infty(\mathcal{L}_n(z))
\]
for $|z|_\infty < |\theta|_\infty^{-1}$, which means $\tilde{l}_n(z) = \bar{l}_n(z)$. Hence our claim is proved.

(2). Let $\mathfrak{z} \in C^{\infty}_\infty$. Since the sequence $\mathfrak{z}_k = (\theta I + N)^{-k}\mathfrak{z}$ ($k = 0, 1, 2, \ldots$) goes to $0 \in C^{\infty}_\infty$ and $\Exp_{C^{\otimes \infty}}$ is continuous, there is a $\mathfrak{z}_m$ such that $\Exp_{C^{\otimes \infty}}(\mathfrak{z}_m)$ lies in the region defined by \[2.3\]. Then we have
\[
\Log_n \circ \Exp_{C^{\otimes \infty}}(\mathfrak{z}) = \Log_n \circ \Exp_{C^{\otimes \infty}}((\theta I + N)^m\mathfrak{z}_m)
\]
\[
= \Log_n \circ (\theta I + N + E\tau)^m \circ \Exp_{C^{\otimes \infty}}(\mathfrak{z}_m)
\]
\[
= (\theta I + N)^m \circ \Log_n \circ \Exp_{C^{\otimes \infty}}(\mathfrak{z}_m)
\]
\[
= (\theta I + N)^m \circ \Log_{C^{\otimes \infty}} \circ \Exp_{C^{\otimes \infty}}(\mathfrak{z}_m)
\]
\[
= (\theta I + N)^m(\mathfrak{z}_m) = \mathfrak{z}.
\]

Since $\Exp_{C^{\otimes \infty}} : C^{\infty}_\infty \to C^{\infty}_\infty$ is a surjection with ker $\Exp_{C^{\otimes \infty}} = \Lambda_n$, we get that $\Log_n$ is the inverse of $\Exp_n$. \[\square\]
Remark 2.1.4. The logarithms of $t$-modules associated with Anderson-Thakur dual $t$-motive \cite{AT90} are discussed in \cite{CGM, CM, GN}. They described a certain special value of their logarithms in terms of CMSPL's. The above logarithm $\log_{\mathcal{C}_{\Omega_{\ast,0}}}$ is regarded as the simplest case. The author expects that their logarithms could be also analytically continued to the whole space by elaborate description of the technical lemma in \cite[Lemma 4.2.1]{CM} in terms of CMSPL's and some sort of their relatives.

2.2. Orthogonality. The following functional relation was shown in \cite{GN}:

$$\mathcal{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_d) = \sum_{i=2}^{d} (-1)^{i} \mathcal{L}_{n_1,\ldots,n_1,Z_1,\ldots,Z_1}^\ast \mathcal{L}_{n_1,\ldots,n_d}(Z_1,\ldots,Z_d)$$

$$+ (-1)^{d+1} \mathcal{L}_{n_1,\ldots,n_1}(Z_d,\ldots,Z_1)$$

for $n_1,\ldots,n_d \in \mathbb{N}$ and $Z_1,\ldots,Z_d \in \mathbb{T}$ belonging to all the regions of convergence of each term. The orthogonal property below is an extension of the above relation to all branches:

Theorem 2.2.1. Let $n_1,\ldots,n_d \in \mathbb{N}$ and $Z_1,\ldots,Z_d \in \mathbb{T}$. For any branch $\mathcal{L}_{n_1,\ldots,n_d}^\ast(Z_1,\ldots,Z_d) \in \mathbb{T}^d$ and $\mathcal{L}_{n_1,\ldots,n_d}^\ast(Z_d,\ldots,Z_1) \in \mathbb{T}^d$, we have

\[
\begin{pmatrix}
\mathcal{L}_{n_1,\ldots,n_1}^\ast(-Z_d,\ldots,-Z_1) \\
\mathcal{L}_{n_1,\ldots,n_d}^\ast(Z_1,\ldots,Z_d)
\end{pmatrix}^T \begin{pmatrix}
\Omega^{-n_1-\cdots-n_d} \\
\Omega^{-n_1-\cdots-n_d-1}
\end{pmatrix} \equiv 0 \text { mod } \mathbb{F}_q[[t]] \cdot \Omega^{-2(n_1+\cdots+n_d)}.
\]

Here $\mathcal{L}_{n_1,\ldots,n_1}^\ast(-Z_d,\ldots,-Z_1)$ means the vector putting $\mathcal{L}_{n_1,\ldots,n_1}^\ast(-Z_i,\ldots,-Z_1) = (-1)^i \mathcal{L}_{n_1,\ldots,n_1}^\ast(Z_i,\ldots,Z_1)$ for each $i$.

Proof. Our proof is influenced by \cite[§4.2]{GN}. Hereafter we fix $d$ generators $\mathcal{L}_{n_1,\ldots,n_d}^\ast(Z_1,\ldots,Z_k)$ $(0 \leq k \leq d - 1)$ of $\mathbb{M}_{n_1,\ldots,n_d}^{Z_1,\ldots,Z_{d-1}}$ as in Definition \cite{AT90}. By using their coordinates, we define the matrix $\Psi \in \text{GL}_{d+1}(\mathbb{T})$ by

\[
\begin{pmatrix}
\Omega^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\
\Omega^{n_1+\cdots+n_d} \mathcal{L}_{n_d}^\ast(Z_d) & \Omega^{n_1+\cdots+n_d-1} & 0 & \cdots & : \\
\vdots & \vdots & \ddots & \ddots & : \\
\Omega^{n_1+\cdots+n_d-1} \mathcal{L}_{n_{d-1}}^\ast(Z_{d-1}) & : & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\Omega^{n_1} & \ddots & \ddots & \ddots & \ddots \\
\mathcal{L}_{n_1,\ldots,n_1}^\ast(Z_1,\ldots,Z_d) & \mathcal{L}_{n_1,\ldots,n_{d-1}}^\ast(Z_1,\ldots,Z_{d-1}) & \cdots & \cdots & \mathcal{L}_{n_1}^\ast(Z_1)
\end{pmatrix}
\]

Similarly we also fix $d$ generators $\mathcal{L}_{n_{d-1},\ldots,n_1}^\ast(Z_d,\ldots,Z_{d-k})$ $(1 \leq k \leq d)$ of $\mathbb{M}_{n_{d-1},\ldots,n_1}^{Z_d,\ldots,Z_2}$ and define $\Psi_\ast \in \text{GL}_{d+1}(\mathbb{T})$ by

\[
\begin{pmatrix}
\Omega^{-n_1-\cdots-n_d} & 0 & 0 & \cdots & 0 \\
\Omega^{-n_1-\cdots-n_d-1} \mathcal{L}_{n_d}^\ast(-Z_d) & \Omega^{-n_1-\cdots-n_d-1} & 0 & \cdots & : \\
\vdots & \vdots & \ddots & \ddots & : \\
\Omega^{-n_1-\cdots-n_d-2} \mathcal{L}_{n_{d-1}}^\ast(-Z_{d-1}) & : & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\Omega^{-n_1} & \ddots & \ddots & \ddots & \ddots \\
\mathcal{L}_{n_1,\ldots,n_1}^\ast(-Z_1,\ldots,-Z_1) & \mathcal{L}_{n_1,\ldots,n_{d-1}}^\ast(-Z_1,\ldots,-Z_1) & \cdots & \cdots & \mathcal{L}_{n_1}^\ast(-Z_1)
\end{pmatrix}
\]
Here we note that $\mathcal{L}_{-Z_1, \ldots, -Z_d} = (-1)^d \mathcal{L}_{Z_1, \ldots, Z_d}$ by linearity. We consider the matrix $\Phi = \text{Mat}_{d+1}(\mathbb{T})$ given by

$$
\Phi = \begin{pmatrix}
(t-\theta)^{n_1+n_2-n_d} & 0 & \ldots & 0 \\
Z_{d}^{(-1)}(t-\theta)^{n_1+n_2-n_d} & (t-\theta)^{n_1+n_2-n_d-1} & \ldots & \vdots \\
0 & Z_{d-1}^{(-1)}(t-\theta)^{n_1+n_2-n_d-1} & \ddots & \vdots \\
\vdots & \vdots & \ddots & (t-\theta)^{n_1} \\
0 & \ldots & 0 & Z_{1}^{(-1)}(t-\theta)^{n_1} 1
\end{pmatrix}.
$$

Then by Proposition [1.4.2] we have

$$
\Psi = \Phi^{(1)} \Psi^{(1)}.
$$

While by Proposition [1.5.2] we also have

$$
\Psi^{(1)} = \Psi^{(1)} \Phi^{(1)}.
$$

Therefore

$$
(\Psi, \Psi)^{(1)} = \Psi^{(1)} \Psi^{(1)} = \Psi^{(1)} \Phi^{(1)} = \Psi, \Psi.
$$

Thus $\Psi, \Psi \in \text{GL}_{d+1}(\mathbb{F}_q[[t]])$. By calculating its $(d+1, 1)$-component, we obtain the claim. \hfill \Box

Chang-Mishiba functional relation ([CM, Lemma 4.2.1]) is

$$
\text{Li}_{n_1, \ldots, n_d}(z_1, \ldots, z_d) = \sum_{i=2}^{d} (-1)^i \text{Li}_{n_{i-1}, \ldots, n_1}(z_{i-1}, \ldots, z_1) \text{Li}_{n_i, \ldots, n_d}(z_i, \ldots, z_d)
$$

$$
+ (-1)^{d+1} \text{Li}_{n_{d-1}, \ldots, n_1}(z_{d-1}, \ldots, z_1)
$$

for $n_1, \ldots, n_d \in \mathbb{N}$ and $z_1, \ldots, z_d \in \mathbb{C}_\infty$ belonging to all the regions of convergence of all terms. It is extended to all branches as follows:

**Corollary 2.2.2.** Let $n_1, \ldots, n_d \in \mathbb{N}$ and $z_1, \ldots, z_d \in \mathbb{C}_\infty$. For any branch $\text{Li}_{n_1, \ldots, n_d}(z_1, \ldots, z_d)$ and $\text{Li}_{n_{i-1}, \ldots, n_1}(z_{i-1}, \ldots, z_1) \in \mathbb{C}_\infty$, we have

$$
\left( \begin{array}{c}
\text{Li}_{n_{i-1}, \ldots, n_1}(z_{i-1}, \ldots, z_1) \n
\text{Li}_{n_{i-1}, \ldots, n_1}(z_{i-1}, \ldots, z_1)
\end{array} \right)^\top \left( \begin{array}{c}
\frac{\pi}{\sqrt{n_1+\cdots+n_d}} 

\frac{\pi}{\sqrt{n_1+\cdots+n_d}}
\end{array} \right) \equiv 0 \mod \pi^{2(n_1+\cdots+n_d)} A.
$$

**Proof.** By Lemma [1.1.1](2), Proposition [1.4.2](3), Proposition [1.5.2](3) and $\Omega(\theta) \neq 0$, $t = \theta$ is inside the regions of convergence of $\text{Li}_{n_1, \ldots, n_1}(z_1, \ldots, z_j)$ and $\text{Li}_{n_{i-1}, \ldots, n_1}(z_{i-1}, \ldots, z_j)$ $(1 \leq i \leq j \leq d)$. So we have $\Psi, \Psi \in \text{GL}_{d+1}(\mathbb{F}_q[[t]])$, which implies

$$
\left( \begin{array}{c}
\text{Li}_{n_{i-1}, \ldots, n_1}(z_{i-1}, \ldots, z_1) \n
\text{Li}_{n_{i-1}, \ldots, n_1}(z_{i-1}, \ldots, z_1)
\end{array} \right)^\top \left( \begin{array}{c}
\Omega^{-n_1-\cdots-n_d} 

\Omega^{-n_1-\cdots-n_d}
\end{array} \right) \in \Omega^{-2(n_1+\cdots+n_d)} \mathbb{F}_q[[t]].
$$

By evaluating $t = \theta$, we obtain the claim. \hfill \Box
2.3. Eulerian property. We discuss Eulerian properties of the special values of multiple polylogarithm at algebraic points. We show that Eulerian property for CMPL and CMSPL is independent of any choice of branches.

**Definition 2.3.1.** Let \( n_1, \ldots, n_d \in \mathbb{N} \) and \( z_1, \ldots, z_d \in \mathbb{C}_\infty \). Put \( L_{n_1, \ldots, n_d}(z_1, \ldots, z_d) \in \mathbb{C}_\infty \) be an branch, that is, the last coordinate of an appropriate branch \( \tilde{L}_{n_1, \ldots, n_d}^o(z_1, \ldots, z_d) \in \mathbb{C}_d^d \). It is called "Eulerian" when \( \tilde{L}_{n_1, \ldots, n_d}^o(z_1, \ldots, z_d)/\theta^{n_1+\cdots+n_d} \in K \). We may say the same thing for \( \tilde{L}_{n_1, \ldots, n_d}(z_1, \ldots, z_d) \).

**Theorem 2.3.2.** Put \( n_1, \ldots, n_d \in \mathbb{N} \) and \( z_1, \ldots, z_d \in \tilde{K} \). Let \( \tilde{L}_{n_1, \ldots, n_d}(z_1, \ldots, z_d) \) be an branch with coordinates satisfying
\[
(2.10) \quad \tilde{L}_{n_1, \ldots, n_d}(z_1, \ldots, z_d) := L_{n_1, \ldots, n_d}(z_1, \ldots, z_d)(\theta) \neq 0
\]
for all \( i = 1, 2, \ldots, d \). If \( L_{n_1, \ldots, n_d}(z_1, \ldots, z_d) \) is Eulerian, then so is any other branch \( L_{n_1, \ldots, n_d}(z_1, \ldots, z_d) \).

**Proof.** Though the proof goes in the same way as that of [CPY Theorem 2.5.2], we repeat here for the proof of Theorem 2.3.2. We consider the matrix \( \Phi \in \text{Mat}_{d+1}(\tilde{K}[t]) \) given in (2.9) with \( z_i = z_i \in \tilde{K} \) and the vector
\[
\psi = (1, \Omega^{n_1+\cdots+n_d}, \Omega^{n_1+\cdots+n_d}L_{n_2, \ldots, n_d}(z_2, \ldots, z_d), \ldots, \Omega^{n_1+\cdots+n_d}L_{n_1, \ldots, n_d}(z_1, \ldots, z_d)) \in \text{Mat}_{d+2}(\mathbb{T}).
\]
We have the difference equation
\[
(2.11) \quad \psi^{(-1)} = \begin{pmatrix} 1 \\ \Phi \end{pmatrix} \psi.
\]
Proposition 1.4.2 assures that \( \psi \) is in \( \text{Mat}_{d+2}(\mathcal{E}) \) for \( z_1, \ldots, z_d \in \tilde{K} \). By the ABP criteria ([ABP Theorem 3.1.1]), there exists \( (f_0, \ldots, f_{d+1}) \in \text{Mat}_{1,d+2}(\tilde{K}[t]) \) such that
\[
(f_0, \ldots, f_{d+1}) \psi = 0
\]
and whose specialization at \( t = \theta \) yields Eulerian property of \( L_{n_1, \ldots, n_d}(z_1, \ldots, z_d) \). Particularly we have \( f_0(\theta) \neq 0, f_{d+1}(\theta) \neq 0 \) and \( f_1(\theta) = \cdots = f_d(\theta) = 0 \). Put
\[
(B_0, B_1, \ldots, B_d, 0) := \begin{pmatrix} f_0/f_{d+1}, \ldots, f_d/f_{d+1}, 1 \end{pmatrix}^{(-1)} \begin{pmatrix} 1 \\ \Phi \end{pmatrix}
\]
in \( \text{Mat}_{1,d+2}(\tilde{K}[t]) \).

The equation \( (f_0, \ldots, f_{d+1}) \psi = 0 \) implies
\[
(B_0, B_1, \ldots, B_d, 0) \psi = 0,
\]
that means
\[
B_0 + B_1 \Omega^{n_1+\cdots+n_d} + B_2 \Omega^{n_1+\cdots+n_{d-1}}L_{i_1}(\Omega^{n_d}z_d) + \cdots + B_d \Omega^{n_1}L_{i_d}(\Omega^{n_2}z_2, \ldots, \Omega^{n_d}z_d) = 0.
\]
While we have
\[
L_{i_d}(\Omega^{n_1}z_1, \ldots, \Omega^{n_d}z_d)(\theta^{n_1}) \neq 0
\]
for all $i = 1, 2, \ldots, d$ and $n \geq 1$ by our assumption (2.10), Proposition 1.4.2 (4) and $\Omega(\theta) \neq 0$. By combining it with $\Omega(\theta^n) = 0$ for all $n$ and $B_i \in \bar{K}(t)$, we recursively obtain

$$B_0 = B_1 = \cdots = B_d = 0.$$

Put $D = \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ \delta_1 & \ldots & \delta_d & 1 \end{pmatrix}$ in $\text{Mat}_{1,d+1}(\bar{K}(t))$ with $\delta_i = \frac{f}{f_{a+1}} \in \bar{K}(t)$ ($i = 1, \ldots, d$). Then we have

$$D^{-1} \Phi = \begin{pmatrix} \Phi' \\ 1 \end{pmatrix} D$$

where $\Phi'$ is the upper left $d \times d$-part of $\Phi$.

Hereafter we fix $d$ generators $\bar{L}_{i_1,\ldots,i_d}^0(z_1, \ldots, z_k)$ ($0 \leq k \leq d - 1$) of $\mathbb{M}_{n_1,\ldots,n_d}^{z_1,\ldots,z_{d-1}}$ as in Definition 1.4.1. By using their coordinates, we define the matrix $\Psi \in \text{Mat}_{d+1}(\mathcal{E}) \cap \text{GL}_{d+1}(\mathbb{F}_q)$ given in (2.7) with $Z_i = z_i \in \bar{K}$. It satisfies

$$\Psi^{-1} = \Phi \Psi.$$ 

Thus we have

$$(D \Psi)^{-1} = \begin{pmatrix} \Phi' \\ 1 \end{pmatrix} D \Psi.$$ 

While we have the difference equation

$$\left( \begin{pmatrix} \Psi' \\ 1 \end{pmatrix} \right)^{-1} = \begin{pmatrix} \Phi' \\ 1 \end{pmatrix} \left( \begin{pmatrix} \Psi' \\ 1 \end{pmatrix} \right),$$

where $\Psi'$ means the upper left $d \times d$-part of $\Psi$ since $\Phi$ is a lower triangular matrix. Then by [P] 4.1.6 there exist $\nu_1, \ldots, \nu_d \in \mathbb{F}_q(t)$ such that

$$D \Psi = \begin{pmatrix} \Psi' \\ 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \ddots \\ & \nu_1 & \ldots & \nu_d & 1 \end{pmatrix}.$$ 

The equation implies on the last row

$$\nu_1 = \delta_1 \Omega^{n_1} + \cdots + \delta_d \Omega^{n_d} \bar{L}_{i_0}^0(\Omega^{n_d} z_d) + \cdots + \delta_d \Omega^{n_1} \bar{L}_{(0)_{d-1}}(\Omega^{n_d} z_{d-1}) + \bar{L}_{(0)_{d}}(\Omega^{n_1} z_1, \ldots, \Omega^{n_d} z_d),$$

$$\nu_2 = \delta_2 \Omega^{n_1} + \cdots + \delta_d \Omega^{n_2} \bar{L}_{i_0}^0(\Omega^{n_d} z_d) + \cdots + \delta_d \Omega^{n_1} \bar{L}_{(0)_{d-2}}(\Omega^{n_d} z_{d-2}) + \bar{L}_{(0)_{d-1}}(\Omega^{n_1} z_1, \ldots, \Omega^{n_d} z_{d-1}),$$

$$\vdots$$

$$\nu_d = \delta_d \Omega^{n_1} + \bar{L}_{(0)_{d}}(\Omega^{n_1} z_1).$$

By $\nu_i \in \mathbb{F}_q(t)$, we have $\nu_i(\theta^n) = \nu_i(\theta)q^n$ for all $i = 1, \ldots, d$ and $n \geq 0$. By Proposition 1.4.2 (4) and $\Omega(\theta^n) = 0$ for $n \geq 1$, we obtain

$$\nu_i(\theta)q^n = \bar{L}_{(0)_{i}}(\Omega^{n_1} z_1, \ldots, \Omega^{n_i} z_i)(\theta)q^n$$

for infinitely many $n$. By taking $q^n$-th root of both hand sides, we see that $\nu_i(\theta) = \bar{L}_{(0)_{i}}(\Omega^{n_1} z_1, \ldots, \Omega^{n_i} z_i)(\theta) = \bar{\pi}^{-n_1-\cdots-n_i} \bar{L}_{i_1,\ldots,i_d}(z_1, \ldots, z_i)$ is in $\bar{K}$. 

By Remark 2.3.3, any other branch $\text{Li}^{o}_{n_1,\ldots,n_d}(z_1,\ldots,z_d)$ is given by the form

$$\text{Li}^{o}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) + \sum_{i=0}^{d-1} \alpha_i \cdot \Xi^{n_{i+1}+\cdots+n_d} \text{Li}^{o}_{n_1,\ldots,n_{i}}(z_1,\ldots,z_i)$$

with $\alpha_i \in \mathbb{A}$. Whence we get that $\text{Li}^{o'}_{n_1,\ldots,n_d}(z_1,\ldots,z_d)$ is also Eulerian. $\square$

**Remark 2.3.3.** In [CPY] Theorem 4.3.2], $z_1,\ldots,z_d$ are assumed to be in $\overline{K} \times \mathbb{D}$. And [CPY] Theorem 2.5.2] is shown for $z_1,\ldots,z_d \in \overline{K}[t]$ satisfying (1.8) and (2.10) for $(n_i,\ldots,n_j)$ with $1 \leq i \leq j \leq d$.

The branch independence also holds for the star version.

**Theorem 2.3.4.** Let $n_1,\ldots,n_d \in \mathbb{N}$ and $z_1,\ldots,z_d \in \overline{K}$. Let $\mathcal{L}^{*}_{n_1,\ldots,n_d}(z_1,\ldots,z_d)$ be a branch with coordinates satisfying

$$(2.12) \quad \text{Li}^{*}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) := \mathcal{L}^{*}_{n_1,\ldots,n_d}(z_1,\ldots,z_d)(\theta) \neq 0.$$ 

If $\text{Li}^{*}_{n_1,\ldots,n_d}(z_1,\ldots,z_d)$ is Eulerian, then so is any other branch $\text{Li}^{*'}_{n_1,\ldots,n_d}(z_1,\ldots,z_d)$.

**Proof.** The proof goes in the same way to that of Theorem 2.3.2. We work over the matrices of the star dual $t$-motives constructed in [CN].

Fix $d$ generators $\mathcal{L}^{*}_{n_1,\ldots,n_d}(z_1,\ldots,z_k)$ $(0 \leq k \leq d-1)$ of $M^{*,z_1,\ldots,z_{d-1}}$ as in Definition 1.4.1. By using their coordinates, we define the matrix $\Psi^{*} \in \text{Mat}_{d+1}(\mathcal{E}) \cap \text{GL}_{d+1}(\mathbb{T})$ given in (2.7) with $\mathcal{L}_{n_1,\ldots,n_j}(Z_1,\ldots,Z_j)$ replaced with $\mathcal{L}^{*}_{n_1,\ldots,n_j}(-z_1,\ldots,-z_j)$, and the matrix $\Phi^{*} \in \text{Mat}_{d+1}(\overline{K}[t])$ given by

$$\begin{pmatrix}
(t-\theta)^{n_1+\cdots+n_d} & \cdots & 0 \\
-(z_1^{(-1)}-\theta)^{n_1+\cdots+n_{d-1}} & \cdots & \\
\vdots & \ddots & \vdots \\
-(z_{d}^{(-1)}-\theta)^{n_1+\cdots+n_{d-1}} & \cdots & 0
\end{pmatrix}$$

Then we have

$$\Psi^{*(-1)} = \Phi^{*} \Psi^{*}$$

(cf. [CN] Remark 5.1].) Put the vector

$$\psi^{*} = (1, \Omega^{n_1+\cdots+n_d}, \Omega^{n_1+\cdots+n_d} \mathcal{L}^{*}_{n_1,\ldots,n_d}(-z_d), \ldots, \Omega^{n_1+\cdots+n_d} \mathcal{L}^{*}_{n_2,\ldots,n_d}(-z_d), \ldots, \Omega^{n_1+\cdots+n_d} \mathcal{L}^{*}_{n_1,\ldots,n_d}(-z_1,\ldots,-z_d))^{T}$$

in $\text{Mat}_{d+2,1}(\mathbb{T})$. The ABP criteria ([ABP] Theorem 3.1.1]) assures the existence of $(f_{0}^{*},\ldots,f_{d+1}^{*}) \in \text{Mat}_{1,d+2}(\overline{K}[t])$ such that

$$(f_{0}^{*},\ldots,f_{d+1}^{*}) \psi^{*} = 0.$$ 

It follows

$$(\delta_{0}^{*},\ldots,\delta_{d}^{*},1) - (\delta_{0},\ldots,\delta_{d},1)^{(-1)}, \begin{pmatrix} 1 \\ \Phi^{*} \end{pmatrix} = (0,\ldots,0)$$
in $\text{Mat}_{1,d+2}(\bar{K}(t))$ with $\delta_i^* = f_i^*/f_{i+1}^*$. Then by [P §4.1.6] there exists $\nu^*_1, \ldots, \nu^*_d \in \mathbb{F}_q(t)$ such that

$$
\begin{pmatrix}
1 & 1 & \cdots \\
\delta_1^* & \cdots & \delta_d^* \\
\end{pmatrix}
\Psi^* = 
\begin{pmatrix}
\Psi'/1 \\
\nu_1^* & \cdots & \nu_d^* \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & \cdots \\
\end{pmatrix}.
$$

The equation on the last row implies that $\nu_i^*(\theta) = Li_{n_1}^*\ldots Li_{n_d}^*(-n_1z_1, \ldots, -n_dz_d) = (-1)^i\pi^{-n_1-\cdots-n_d}Li_{n_1}^*\ldots Li_{n_d}^*(z_1, \ldots, z_d)$ is in $K$.

Since any other branch $Li_{n_1}^*\ldots Li_{n_d}^*(z_1, \ldots, z_d)$ is given by the form

$$Li_{n_1}^*\ldots Li_{n_d}^*(z_1, \ldots, z_d) + \sum_{i=0}^{d-1} \alpha_i \cdot \pi^{-n_1-\cdots-n_d}Li_{n_1}^*\ldots Li_{n_d}^*(z_1, \ldots, z_d)$$

with $\alpha_i \in A$, we get that $Li_{n_1}^*\ldots Li_{n_d}^*(z_1, \ldots, z_d)$ is also Eulerian.

$\Box$

Remark 2.3.5. We note that Theorems 2.3.2 and 2.3.4 hold without the assumption (2.10) and (2.12) respectively when $d = 1$. One could hope that both theorems hold without imposing the assumptions though they are required to apply [CPY, Theorem 2.5.2] in the proofs of the theorems.

Remark 2.3.6. In $p$-adic situation (in characteristic 0 case), Coleman established the theory of $p$-adic iterated integrals in [Cd] by making essential use of crystalline Frobenius automorphisms and his theory enables us to carry out an analytic continuation of $p$-adic multiple polylogarithms (cf.[E]). While in positive characteristic case, a theory of iterated integrals does not seem to have been established yet, however the author expects that there would exist such a theory where Artin-Schreier equations alternatively play a crucial role.

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