BOUNDNESS AND COMPACTNESS OF COMMUTATORS ASSOCIATED WITH LIPSCHITZ FUNCTIONS

WEICHAO GUO, JIANXUN HE, HUOXIONG WU, AND DONGYONG YANG

Abstract. Let \( \alpha \in (0, 1] \), \( \beta \in [0, n) \) and \( T_{\Omega, \beta} \) be a singular or fractional integral operator with homogeneous kernel \( \Omega \). In this article, a CMO type space \( \text{CMO}_\alpha (\mathbb{R}^n) \) is introduced and studied. In particular, the relationship between \( \text{CMO}_\alpha (\mathbb{R}^n) \) and the Lipschitz space \( \text{Lip}_\alpha (\mathbb{R}^n) \) is discussed. Moreover, a necessary condition of restricted boundedness of the iterated commutator \( (T_{\Omega, \beta})_m^b \) on weighted Lebesgue spaces via functions in \( \text{Lip}_\alpha (\mathbb{R}^n) \), and an equivalent characterization of the compactness for \( (T_{\Omega, \beta})_m^b \) via functions in \( \text{CMO}_\alpha (\mathbb{R}^n) \) are obtained. Some results are new even in the unweighted setting for the first order commutators.

1. Introduction and preliminaries

Let \( \beta \in [0, n) \). The singular or fractional integral operator with homogeneous kernel is defined by

\[
T_{\Omega, \beta} f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\beta}} f(y) dy,
\]

where \( \Omega \) is a homogeneous function of degree zero and satisfies the following mean value zero property when \( \beta = 0 \):

\[
\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0.
\]

Let \( b \) be a locally integrable function on \( \mathbb{R}^n \), and let \( T \) be a linear operator. The commutator \( [b, T] \) is defined by

\[
[b, T] f(x) := b(x) T(f)(x) - T(bf)(x)
\]

for suitable functions \( f \). The iterated commutator \( T_m^b \) with \( m \geq 2 \) is defined by

\[
T_m^b f := [b, T_{m-1}]^b f,
\]

where we also write \( T_m^b f := [b, T] f \). In 1976, Coifman, Rochberg and Weiss [5] proved that if a function \( b \in \text{BMO}(\mathbb{R}^n) \), then the commutator \( [b, T_{\Omega, 0}] \) is bounded on \( L^p(\mathbb{R}^n) \) for any \( p \in (1, \infty) \); via a spherical harmonics expansion argument, they also proved that, if \( [b, R_j] \) is bounded on \( L^p(\mathbb{R}^n) \) for every Riesz transform \( R_j, j = 1, 2, \cdots, n \), then \( b \in \text{BMO}(\mathbb{R}^n) \). In 1978, using a Fourier expansion technique, Janson [13] first proved that for \( 0 < \alpha < 1 \), \( b \in \text{Lip}_\alpha(\mathbb{R}^n) \) if and only if \( [b, T_{\Omega, 0}] \) with smooth kernel \( \Omega \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for \( 1 < p < q < \infty \) with \( 1/q = 1/p - \alpha/n \). Later on, Paluszyński [22] established the corresponding result for the commutator of Riesz potential \( [b, I_{\beta}] \). In 2008, Hu-Gu [10] provided the equivalent characterizations between \( [b, T_{\Omega, \beta}] \) with smooth kernel \( \Omega \) and weighted Lipschitz spaces. Recently, the necessity of boundedness of iterated commutators \( (T_{\Omega, \beta})_m^b \)

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was proved for a rather wide class of operators, by Lerner-Ombrosi-Rivera-Ríos [16], by a technique in terms of the local mean oscillation.

The study of equivalent characterization on $L^p(\mathbb{R}^n)$-compactness of commutators $[b, T]$ of singular integral operators $T$ was initiated by Uchiyama in his remarkable work [25], in which he showed that the commutator $[b, T_{\Omega,0}]$ is bounded (compact resp.) on $L^p(\mathbb{R}^n)$ if and only if the symbol $b$ is in $\text{BMO}(\mathbb{R}^n)$ (CMO$(\mathbb{R}^n)$ resp.). Here and in what follows, CMO$(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in the $\text{BMO}(\mathbb{R}^n)$ topology. Moreover, Uchiyama [25] also established the following equivalent characterization of CMO$(\mathbb{R}^n)$ in terms of mean value oscillation of functions, which plays a key role in the compactness characterization of $[b, T_{\Omega,0}]$ on $L^p(\mathbb{R}^n)$.

**Theorem A** Let $f \in \text{BMO}(\mathbb{R}^n)$. Then $f \in \text{CMO}(\mathbb{R}^n)$ if and only if the following three conditions hold:

1. $\lim_{r \to 0} \sup_{|Q|=r} \mathcal{O}(f;Q) = 0$,
2. $\lim_{r \to \infty} \sup_{|Q|=r} \mathcal{O}(f;Q) = 0$,
3. $\lim_{d \to \infty} \sup_{Q \cap (-d,d)^n \neq \emptyset} \mathcal{O}(f;Q) = 0$.

Here and in what follows, the symbol $Q$ means closed cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes,

$$f_Q := \frac{1}{|Q|} \int_Q f(y) \, dy \quad \text{and} \quad \mathcal{O}(f;Q) := \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx.$$  

In [26], by applying the characterization of CMO$(\mathbb{R}^n)$ in [25], Wang showed that the fact $b \in \text{CMO}(\mathbb{R}^n)$ is also sufficient and necessary for the compactness of the commutator $[b, I_{\beta}]$ with fractional integral operator $I_{\beta}$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $\beta \in (0, n)$, $p, q \in (1, \infty)$ with $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}$ and

$$I_{\beta} f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} \, dy.$$  

Since then, the work on compactness of commutators of singular and fractional integral operators and its applications to PDE’s have been paid more and more attention; see, for example, [12, 15, 3, 24, 4, 1, 2] and the references therein. We only mention that very recently, inspired by the method developed in [16], an equivalent characterization of the weighted Lebesgue space compactness for iterated commutator $(T_{\Omega,\beta})^n$ via CMO$(\mathbb{R}^n)$ was obtained in [8], where in the necessity part (see [8, Theorem 1.4]) the author only assume that $\Omega \in L^\infty(S^{n-1})$ and does not change sign and is not equivalent to zero on some open subset of $S^{n-1}$.

Let $\alpha \in (0,1]$ and $\text{Lip}_\alpha(\mathbb{R}^n)$ be the Banach space of all continuous functions on $\mathbb{R}^n$ such that $\|f\|_{\text{Lip}_\alpha(\mathbb{R}^n)} < \infty$, where for a continuous function $f$ on $\mathbb{R}^n$, the (homogeneous) $\alpha$-order Lipschitz norm is defined by

$$\|f\|_{\text{Lip}_\alpha(\mathbb{R}^n)} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$  

One of the main purposes of this article is to consider the compactness characterization of iterated commutators $(T_{\Omega,\beta})^n$ on weighted Lebesgue spaces, where $b \in \text{Lip}_\alpha(\mathbb{R}^n)$ and $\Omega \in L^r(S^{n-1})$ with $r \in (1, \infty)$. To this end, we first recall the following characterization of $\text{Lip}_\alpha(\mathbb{R}^n)$ by Meyers [18].
**Definition 1.1.** Let $\alpha \in [0, 1]$. The space of functions with bounded fractional mean oscillation, denoted by $\text{BMO}_\alpha(\mathbb{R}^n)$, consists of all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\text{BMO}_\alpha(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \mathcal{O}_\alpha(f; Q) < \infty,$$

where

$$\mathcal{O}_\alpha(f; Q) := \frac{1}{|Q|^{1 + \frac{n}{\alpha}}} \int_Q |f(x) - f_Q|dx.$$

In [8], Meyers established the following equivalent characterization of $\text{Lip}_\alpha(\mathbb{R}^n)$ in terms of $\text{BMO}_\alpha(\mathbb{R}^n)$.

**Lemma 1.2.** Let $\alpha \in (0, 1]$. Then

$$\text{Lip}_\alpha(\mathbb{R}^n) = \text{BMO}_\alpha(\mathbb{R}^n).$$

Moreover, if $f \in \text{Lip}_\alpha(\mathbb{R}^n)$, $p \in [1, \infty]$, we have

$$\|f\|_{\text{Lip}_\alpha(\mathbb{R}^n)} \sim \sup_Q \mathcal{O}_\alpha(f, Q) \sim \sup_Q \frac{1}{|Q|^{\frac{1}{p}} \left( \frac{1}{|Q|} \int_Q |f(y) - f_Q|^pdy \right)^{1/p}} ,$$

where and in what follows, the symbol $f \lesssim g$ represents that $f \leq Cg$ for some positive constant $C$ and $f \sim g$ represents $f \lesssim g$ and $g \lesssim f$.

The following class $A_p$ was introduced by Muckenhoupt [19] to study the weighted norm inequalities of Hardy-Littlewood maximal operators, and $A_{p, q}$ was introduced by Muckenhoupt–Wheeden [20] to study the weighted norm inequalities of fractional integrals, respectively.

**Definition 1.3.** For $1 < p < \infty$, the Muckenhoupt class $A_p$ is the set of locally integrable weights $\omega$ such that

$$[\omega]^{1/p}_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x)dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \omega(x)^{1-\frac{1}{p}}dx \right)^{1'/p'} < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. For $1 < p, q < \infty$, $1/q = 1/p - \alpha/n$ with $0 < \alpha < n$, a weight function $\omega$ is called an $A_{p, q}$ weight if

$$[\omega]^{1/q}_{A_{p, q}} := \sup_Q \left( \frac{1}{|Q|} \int_Q \omega^q(x)dx \right)^{1/q} \left( \frac{1}{|Q|} \int_Q \omega^{-\frac{1}{p'}}(x)dx \right)^{1'/p'} < \infty.$$

Now, we are in the position to state our first main result.

**Theorem 1.4.** Let $1 < p, q < \infty$, $0 < \alpha \leq 1$, $0 \leq \beta < n$, $\alpha + \beta < n$, $1/q = 1/p - (m\alpha + \beta)/n$, $m \in \mathbb{Z}^+$ and $\omega \in A_{p, q}$. Let $\Omega$ be a measurable function on $\mathbb{S}^{n-1}$, which does not change sign and is not equivalent to zero on some open subset of $\mathbb{S}^{n-1}$. If there is $C > 0$ such that for every bounded measurable set $E \subset \mathbb{R}^n$,

$$\|(T_{\Omega, \beta})_E^m(\chi_E)\|_{L^q(\omega^n)} \leq C(\omega^p(E))^{1/p},$$

then $b \in \text{BMO}_\alpha(\mathbb{R}^n)$.

**Remark 1.5.** One can see that the method of proof of Theorem 1.4 is also valid for the case $\alpha = 0$, and for the weighted cases (see [10, 16]). In particular, this method implies another approach of [16, Theorem 1.1 (ii)].
Denote by $\widehat{CMO}_\alpha(\mathbb{R}^n)$ the $C_c^\infty(\mathbb{R}^n)$ closure in $BMO_\alpha(\mathbb{R}^n)$. Two natural questions arise: Is it true if we replace $BMO(\mathbb{R}^n)$ by $BMO_\alpha(\mathbb{R}^n)$, $O$ by $O_\alpha$, and $CMO(\mathbb{R}^n)$ by the $\widehat{CMO}_\alpha(\mathbb{R}^n)$? And what is the relation between $\widehat{CMO}_\alpha(\mathbb{R}^n)$ and the compactness of commutators? We consider the following example for a first glimpse.

**An example.** Let $\alpha \in (0, 1]$. Take
\[
\varphi_1(t) := sgn(t)|t|^\alpha, \quad t \in \mathbb{R}, \quad \varphi(x) := \varphi(x_1), x \in \mathbb{R}^n.
\]
We first claim that $\varphi_1 \in Lip_\alpha(\mathbb{R})$. Indeed, since $\alpha \in (0, 1]$, for $t, s \in \mathbb{R}$,
\[
|t + s|^\alpha \leq |t|^\alpha + |s|^\alpha, \quad |t|^\alpha \leq |t + s|^\alpha + |s|^\alpha,
\]
which implies that
\[
|t + s|^\alpha - |t|^\alpha \leq |s|^\alpha.
\]
Thus, the function $| \cdot |^\alpha$ belongs to $Lip_\alpha(\mathbb{R})$. To prove that $\varphi_1 \in Lip_\alpha(\mathbb{R})$, we only need to verify
\[
\frac{|\varphi_1(t+s) - \varphi_1(t)|}{|s|^\alpha} \leq C.
\]
Obviously, the above inequality is valid for $t(t+s) \geq 0$ by $| \cdot |^\alpha \in Lip_\alpha(\mathbb{R})$. If $t(t+s) < 0$, we have $|s| \geq |t|$. So
\[
\frac{|\varphi_1(t+s) - \varphi_1(t)|}{|s|^\alpha} \leq \frac{|\varphi_1(t+s) + |\varphi_1(t)||}{|s|^\alpha} = \frac{|t + s|^\alpha + |t|^\alpha}{|s|^\alpha} \leq \frac{(2\alpha + 1)|s|^\alpha}{|s|^\alpha} = 2\alpha + 1.
\]
For every $x, z \in \mathbb{R}^n$,
\[
|\varphi(x + z) - \varphi(x)| = |\varphi_1(x_1 + z_1) - \varphi_1(x_1)| \leq \|\varphi_1\|_{Lip_\alpha(\mathbb{R})}|z_1|^\alpha \leq \|\varphi_1\|_{Lip_\alpha(\mathbb{R})}|z|^\alpha.
\]
Next, we will see that the condition in (1) of Theorem A fails. By a direct calculation, for $Q_0 := [-1/2, 1/2]^n$ and any $a > 0$, we have
\[
\int_{aQ_0} \varphi(x)dx = a^{n-1} \int_{a/2}^{a/2} sgn(x_1)|x_1|^\alpha dx_1 = 0,
\]
and
\[
\int_{aQ_0} |\varphi(y)|dy = a^{n-1} \int_{a/2}^{a/2} |x_1|^\alpha dx_1 \sim a^{n+\alpha}.
\]
Thus,
\[
O_\alpha(\varphi, aQ_0) = \frac{1}{|aQ_0|^{1+\frac{\alpha}{n}}} \int_{aQ_0} |\varphi(y) - \varphi_{aQ_0}|dy = \frac{1}{|aQ_0|^{1+\frac{\alpha}{n}}} \int_{aQ_0} |\varphi(y)|dy \sim 1.
\]
From the above example, we have two observations:

1. For $\alpha \in (0, 1)$, since a $\widehat{CMO}_\alpha(\mathbb{R}^n)$ function must satisfy (1) in Theorem A with $O_\alpha$, we see that $\varphi \in Lip_\alpha(\mathbb{R}^n) \setminus \widehat{CMO}_\alpha(\mathbb{R}^n)$. Hence, $\widehat{CMO}_\alpha(\mathbb{R}^n)$ is a non-trivial and proper subspace of $Lip_\alpha(\mathbb{R}^n)$.
2. For $\alpha = 1$, since the function $\varphi \chi_{B(0,1)}$ can be a part of certain $C_c^\infty(\mathbb{R}^n)$ function in $B(0,1)$, we see that there exists a $C_c^\infty(\mathbb{R}^n)$ function (belongs to $\widehat{CMO}_\alpha(\mathbb{R}^n)$) that does not satisfy (1) in Theorem A with $O_1$. Hence, Lemma 1 fails if we replace $BMO(\mathbb{R}^n)$ by $Lip_1(\mathbb{R}^n)$, $O$ by $O_1$, and $CMO(\mathbb{R}^n)$ by the $\widehat{CMO}_1(\mathbb{R}^n)$. 

Now, we introduce another function space, $CMO_\alpha(\mathbb{R}^n)$, associated with $BMO_\alpha(\mathbb{R}^n)$. One will see that when $\alpha = 1$, the following $CMO_\alpha(\mathbb{R}^n)$ is the right function space for the equivalent characterization of compact commutators. We will use $CMO_\alpha(\mathbb{R}^n)$ to give an answer of the second question posed above, see Theorem 1.8 below.

**Definition 1.6.** Let $\alpha \in [0, 1]$. A $BMO_\alpha(\mathbb{R}^n)$ function $f$ belongs to $CMO_\alpha(\mathbb{R}^n)$ if it satisfies the following three conditions:

1. $\lim_{r \to 0} \sup_{|Q|=r} \mathcal{O}_\alpha(f; Q) = 0,$
2. $\lim_{r \to \infty} \sup_{|Q|=r} \mathcal{O}_\alpha(f; Q) = 0,$
3. $\lim_{d \to \infty} \sup_{Q \cap [-d,d]^n} \mathcal{O}_\alpha(f; Q) = 0.$

Observe that $CMO_0(\mathbb{R}^n) = CMO(\mathbb{R}^n)$ by Theorem A. In the following, we give our second main result corresponding to Theorem A. As in the $CMO(\mathbb{R}^n)$ case, this theorem is also a key tool for the equivalent characterization of compact commutators.

**Theorem 1.7.** When $\alpha \in [0, 1)$, we have

$$\widetilde{CMO}_\alpha(\mathbb{R}^n) = CMO_\alpha(\mathbb{R}^n).$$

When $\alpha = 1$, $CMO_\alpha(\mathbb{R}^n) \subseteq \widetilde{CMO}_\alpha(\mathbb{R}^n)$. In fact, $CMO_1(\mathbb{R}^n)$ is equal to the constant space $\mathbb{C}$ containing all complex numbers with usual norm.

Based on Theorems 1.4 and 1.7, we further have the following result on compactness characterization of iterated commutator $(T_{\Omega,\beta})_b^m$.

**Theorem 1.8.** Let $1 < p, q < \infty$, $0 < \alpha \leq 1$, $0 \leq \beta < n$, $0 < m \alpha + \beta < n$, $1/q = 1/p - (m \alpha + \beta)/n$, $m \in \mathbb{Z}^+$. Suppose $r' \in [1, p)$, $\omega \in A_{\frac{q}{p}, \frac{q}{p'}}$. Let $\Omega \in L^r(S^{n-1})$, which does not change sign and is not equivalent to zero on some open subset of $S^{n-1}$. The following two statements is equivalent:

1. $(T_{\Omega,\beta})_b^m$ is a compact operator from $L^p(\omega^p)$ to $L^q(\omega^q)$;
2. $b \in CMO_\alpha(\mathbb{R}^n)$.

We remark that, while we were putting the finishing touches on this manuscript, we learned that a similar result concerning Theorem 1.7 have been obtained independently in [21], where the authors characterize the compactness of the commutators generated by Lipschitz functions and fractional integral operators on Morrey spaces. Our new contribution of the above three theorems is the following.

1. For $m = 1$, Theorem 1.4 can be applied to a much wider class of operators, compared to the known results, even for the unweighed cases. For $m \geq 2$, Theorems 1.4 is new even for $\Omega \equiv 1$ and $\omega \equiv 1$.
2. The proof of Theorem 1.4 is also valid for the case $\alpha = 0$, and for the weighted cases (see [10] [16]). We mention that we prove Theorem 1.4 by using some idea from [16] (see also [8]). It is known that $BMO(\mathbb{R}^n)$ and $CMO(\mathbb{R}^n)$ can be characterized by local mean oscillation of functions (see [23] and [8] respectively), and [16, Proposition 3.1] plays a crucial role in the study of two weighted $L^p$-boundedness of iterated commutator via weighted BMO function therein, which is on the domination of the local mean oscillation of a function $b$ on any given cube $Q$ by the quantity $|b(x) - b(y)|$ pointwise on a subset $G$ of $Q \times P$, where $P$ is a cube having comparable volume to $Q$; see also [8, Proposition 4.1]. However, since the loss of local mean oscillation in
Lip}_α(\mathbb{R}^n) with \( \alpha \in (0, 1] \), the method in \cite{16} can not be applied to our case. A novelty of this article lies in that we obtain a version of such domination via the so-called median value of \( b \) on two subsets of \( Q \times P \), see Proposition 2.4 below. In this sense, our proof can be applied to the function spaces without local mean oscillation.

(3) For \( \alpha = 1 \), Theorem 1.7 is new, and for \( \alpha \in (0, 1) \), our proof of Theorem 1.7 is totally different from \cite{21}.

(4) For \( m = 1 \), Theorem 1.8 is the first result of equivalent characterization of compact commutators with rough kernel. For \( m \geq 2 \), Theorem 1.8 is new even for \( \Omega \equiv 1 \) and \( \omega \equiv 1 \). Since the loss of local mean oscillation in Lip}_α(\mathbb{R}^n), our method is quite different from \cite{8}.

The rest of the paper is organized as follows. In Section 2, we first obtain a domination result via the median value of a given function \( b \) and cube \( Q \) by the quantity \(|b(x) - b(y)|\) pointwise on two subsets of \( Q \times P \), see Proposition 2.4 below. By making use of Proposition 2.4, for any given cube \( Q \) and real-valued measurable function \( b \), we further construct two functions \( f_i (i = 1, 2) \) related to \( Q \), and obtain a lower bound for the sum of weighted \( L^q \) norm of \((T_{\Omega, \alpha})_b^m (f_i)\) over \( Q \) in terms of \( O_\alpha (b, Q) \); see Proposition 2.5 below. Using this lower bound, we present the proof of Theorem 1.4.

Section 3 contains the proof of Theorem 1.7. In Section 3.1, we first introduce and study a kind of Lipschitz functions \( \{ F_Q \} \) associated with a finite family \( Q \) of dyadic cubes. In Subsection 3.2 for any given function \( f \in \text{CMO}_\alpha(\mathbb{R}^n) \) with \( \alpha \in (0, 1) \), we further construct a Lip}_1(\mathbb{R}^n) function via the features of Lip}_α(\mathbb{R}^n) and \( \{ F_Q \} \) as an approximate function of \( f \) in the topology of Lip}_α(\mathbb{R}^n). Using this key approximate function, we complete the proof of Theorem 1.7.

Section 4 is devoted to the proof of the (1) \( \implies \) (2) part of Theorem 1.8 and is divided into three subsections. In Subsection 4.1, we establish a further lower bounded estimate closely related to compact commutators. Here, although the local mean oscillation is lost, the continuity of \( b \in \text{Lip}_\alpha(\mathbb{R}^n) \) provides enough information for the distribution of the values of \( b \). This observation makes it possible for us to get the lower bound for the weighted \( L^q \) norm of \((T_{\Omega, \alpha})_b^m (f)\) over certain measurable set associated with \( Q \) in terms of \( \min\left\{ \left( O_\alpha (b, Q) \right)^{2n/\alpha}, 1 \right\} \), when \( f \) is a suitable function related to \( Q \). In Subsection 4.2 for \( b \in \text{Lip}_\alpha(\mathbb{R}^n) \), \( \Omega \in L^\infty(\mathbb{S}^{n-1}) \) and any cube \( Q \), we also obtain an upper bound of the weighted \( L^q \) norm of \((T_{\Omega, \alpha})_b^m (f)\) over the annulus \( 2^{d+1}Q \backslash 2^d Q \) in terms of \( 2^{-\delta d n/p} d^m \), where \( d \in \mathbb{N} \) large enough, \( \delta \) is a positive constant depending on \( w \in A_{p,q} \) and \( f \) is aforementioned. Using Theorem 1.7, the upper and further lower bounds, and a reduction of \( \Omega \), we further present the proof of (1) \( \implies \) (2) part in Theorem 1.8 via a contradiction argument in Subsection 4.3.

In Section 5, we give the proof for the (2) \( \implies \) (1) part in Theorem 1.8. Using a classical boundedness result of fractional integral, we reduce the proof to the case of "good kernel".

It would be helpful to clarify that in this paper, the kernel \( \Omega \) is assumed to be real-valued. For \( m \geq 2 \), we only consider the real-valued symbol \( b \). This restriction is actually implied in all the previous results of this topic. Here, we emphasize this to avoid possible misunderstanding.

Finally, we make some conventions on notation. Throughout the paper, for a real number \( a \), \( \lfloor a \rfloor \) means the biggest integer no more than \( a \). By \( C \) we denote a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts do not change in different occurrences. For a given cube \( Q \), we use \( c_Q \), \( l_Q \), \( Q \) and \( \chi_Q \) to denote the center, side length, interior and characteristic function of \( Q \), respectively.
Moreover, we denote $Q_0 := [-1/2, 1/2]^n$. For any point $x_0 \in \mathbb{R}^n$ and sets $E, F \subset \mathbb{R}^n$, $E + x_0 := \{y + x_0 : y \in E\}$ and $E - F := \{x - y : x \in E, y \in F\}$.

2. Necessity of boundedness of commutators

This section is devoted to the proof of Theorem 1.4. Because of the loss of local mean oscillation in $\text{Lip}_\alpha(\mathbb{R}^n)$ with $\alpha \in (0, 1]$, the methods used in [8] and [16] can not be applied here. By using the median value of $b$, we obtain the useful lower bound associated with $\Omega$ and $b$. Then, for a real-valued measurable function $b$, we construct two functions $f_i(i = 1, 2)$ related to $Q$, and obtain a lower bound for the sum of weighted $L^q$ norm of $(T_{\Omega, \alpha})^n(f_i)$ over $Q$ in terms of $O_\alpha(b, Q)$. Using this lower bound, Theorem 1.4 is proved.

We first recall some useful properties of $A_p$ and $A_{p,q}$ weights; see [20, 7, 11, 9]. Define the $A_\infty$ class of weights by $A_\infty := \cup_{p > 1} A_p$, and recall the Fujii-Wilson $A_\infty$ constant

$$[\omega]_{A_\infty} := \sup_Q \frac{1}{\omega(Q)} \int_Q M(\chi_Q \omega) \, dx,$$

where $M$ is the Hardy-Littlewood maximal operator.

**Lemma 2.1.** Let $p \in (1, \infty)$ and $w \in A_p$.

(i) For every $0 < \alpha < 1$, there exists $0 < \beta < 1$ such that for every $Q$ and every measurable set $E \subset Q$ with $|E| \geq \alpha |Q|$, 

$$\omega(E) \geq \beta \omega(Q).$$

(ii) For all $\lambda > 1$, and all cubes $Q$,

$$\omega(\lambda Q) \leq \lambda^{np} [\omega]_{A_p} \omega(Q).$$

(iii) $[\omega]_{A_\infty} \leq c_n [\omega]_{A_p}$.

(iv) There exists a constant $\epsilon_n$ only depending on $n$, such that if $0 < \epsilon \leq \epsilon_n/[\omega]_{A_\infty}$, then $\omega$ satisfies the reverse Hölder inequality that for any cube $Q$,

$$\left(\frac{1}{|Q|} \int_Q \omega(x)^{1+\epsilon} \, dx\right)^{\frac{1}{1+\epsilon}} \leq 2 \frac{1}{|Q|} \int_Q \omega(x) \, dx.$$

(v) There exists a small positive constant $\epsilon$ depending only on $n, p$ and $[\omega]_{A_p}$ such that

$$\omega^{1+\epsilon} \in A_p, \quad \omega \in A_{p-\epsilon}.$$

**Lemma 2.2.** Let $1 < p, q < \infty, 1/q = 1/p - \alpha/n$ with $0 < \alpha < n$ and $w \in A_{p,q}$.

(i) $\omega^p \in A_p(\mathbb{R}^n)$, $\omega^q \in A_q$ and $\omega^{-p^\prime} \in A_{p^\prime}$

(ii) $w \in A_{p,q} \iff \omega^q \in A_{p+\alpha/n} \iff \omega^q \in A_{1+\delta/n} \iff \omega^{-p^\prime} \in A_{1+p^\prime}^\prime$.

In this part, one can see that median value studied by Journé [14] plays a key role in this type of lower estimates. Compared to [16], our method does not use the so-called local mean oscillation, and in this sense, can be used to a much wider class of function spaces.

**Definition 2.3.** By a median value of a real-valued measurable function $f$ over $Q$ we mean a possibly nonunique, real number $m_f(Q)$ such that

$$|\{x \in Q : f(x) > m_f(Q)\}| \leq |Q|/2$$

and

$$|\{x \in Q : f(x) < m_f(Q)\}| \leq |Q|/2.$$
Proposition 2.4. Let $b$ be a real-valued measurable function. Suppose that $\Omega$ satisfies the assumption in Theorem 1.4. For every $\gamma \in (0, 1)$, there exist $\epsilon_0 > 0$ and $k_0 > 10\sqrt{n}$ depending only on $\Omega$, $\gamma$, and $n$ such that the following holds. For every cube $Q$ with the same side length of $Q$ satisfying $|c_Q - c_P| = k_0|Q|$, and measurable sets $E_1, E_2 \subset Q$ with $Q = E_1 \cup E_2$, and $F_1, F_2 \subset P$ with $|F_1| = |F_2| = \frac{1}{2}|Q|$, such that

1. $b(x) - b(y)$ do not change sign for all $(x, y)$ in $E_i \times F_i$, $i = 1, 2$,
2. $|b(x) - m_b(P)| = |b(x) - b(y)|$ for all $(x, y)$ in $E_i \times F_i$, $i = 1, 2$;
3. $\Omega(x - y)$ does not change sign for all $(x, y)$ in $Q \times P$;
4. $|N_x \cap P| \leq \gamma|Q|$ for all $x \in Q$, where $N_x := \{y \in \mathbb{R}^n : |\Omega(x - y)| < \epsilon_0\}$.

Proof. Without loss of generality, assume $\Omega$ is nonnegative on an open set of $S^{n-1}$. By the assumption of $\Omega$, there exists a point $\theta_0$ of approximate continuity of $\Omega$ such that $\Omega(\theta_0) = 2\epsilon_0$ for some $\epsilon_0 > 0$ (see [6, pp.46-47] for the definition of approximate continuity). It follows from the definition of approximate continuity that for every $\beta \in (0, 1)$, there exists a small constant $r_\beta$ such that

$$\sigma(\{\theta \in B(\theta_0, r_\beta) \cap S^{n-1} : \Omega(\theta) \geq \epsilon_0\}) \geq (1 - \beta)\sigma(B(\theta_0, r_\beta) \cap S^{n-1})$$

Let $\Gamma_\beta$ be the cone containing all $x \in \mathbb{R}^n$ such that $x' \in B(\theta_0, r_\beta) \cap S^{n-1}$.

There exists a vector $v_\beta := \frac{c_\beta \theta_0}{r_\beta}$, such that

$$2Q_0 + v_\beta \in \Gamma_\beta.$$

For a fixed cube $Q$, we set $P_{\beta} := Q - l_Qv_\beta$. Thus, $|c_Q - c_{P_\beta}| = \frac{c_n}{r_\beta}l_Q$. Observing that $Q - P_{\beta} \subset 2l_QQ_0 + l_Qv_\beta \subset \Gamma_\beta$, for $x \in Q$ we obtain that

$$|P_{\beta} \cap N_x| = |(P_{\beta} - x) \cap N_0|$$

$$\leq l_Q^2|2Q_0 + v_\beta| \cap (-N_0)|$$

$$\leq l_Q^2 \cdot c_n \cdot |v_\beta|^{n-1} \cdot \beta r_\beta^{-n} \leq c_n|Q|.$$  

Take $\beta = \beta_0$ sufficiently small such that

$$|P_{\beta_0} \cap N_x| \leq \gamma|Q|, \quad k_0 := \frac{c_n}{r_{\beta_0}} > 10\sqrt{n}.$$  

Set

$$E_1 := \{x \in Q : b(x) \geq m_b(P_{\beta_0})\}, \quad E_2 := \{x \in Q : b(x) \leq m_b(P_{\beta_0})\},$$

and

$$F_1 := \{y \in P_{\beta_0} : b(y) \leq m_b(P_{\beta_0})\}, \quad F_2 := \{y \in P_{\beta_0} : b(y) \geq m_b(P_{\beta_0})\},$$

such that $|F_1| = |F_2| = \frac{|P_{\beta_0}|}{2} = \frac{|Q|}{2}$. We get the desired conclusion by

$$|b(x) - b(y)| = |b(x) - m_b(P_{\beta_0})| + |m_b(P_{\beta_0}) - b(y)| \geq |b(x) - m_b(P_{\beta_0})|$$

for $(x, y) \in E_i \times F_i, i = 1, 2$. \hfill \Box

The proof of Theorem 1.4 is reduced to the following proposition.

Proposition 2.5. Let $1 < p, q < \infty$, $0 < \alpha \leq 1$, $0 \leq \beta < n$, $m_\alpha + \beta < n \ 1/q = 1/p - (m_\alpha + \beta)/n$, $m \in \mathbb{Z}^+$. Let $\Omega$ be a measurable function on $S^{n-1}$, which does not change sign and is not equivalent to zero on some open subset of $S^{n-1}$. Let $w \in A_{p,q}$ and $b$ be a real-valued measurable function. For a given cube $Q$, let $P, E_i, F_i, i = 1, 2$, be the sets associated with $Q$. 
mentioned in Proposition 2.4 with $\gamma = \frac{1}{4}$. Then there exists a positive constant $C$ independent of $Q$ and $b$ such that for $f_i := [\omega^p(F_i)]^{-1/p} \chi_{F_i}$, $i = 1, 2$, 
\[
\sum_{i=1,2} \| (T_{\Omega, \beta})^m_b f_i \|_{L^q(\omega^q, Q)} \geq C O_\alpha(b; Q)^m.
\]

Now assume the conclusion of Proposition 2.5 for the moment, we present the proof of Theorem 1.4. For any cube $Q$, take $P$, $f_i$ as in Proposition 2.5. We then have 
\[
\sum_{i=1,2} \| (T_{\Omega, \beta})^m_b f_i \|_{L^q(\omega^q, Q)} \geq O_\alpha(b; Q)^m,
\]
Hence, by (1.4), 
\[
O_\alpha(b; Q)^m \leq \sum_{i=1,2} \| (T_{\Omega, \beta})^m_b f_i \|_{L^q(\omega^q, Q)} \leq \sum_{i=1,2} \| f_i \|_{L^p(\omega^p)} \leq 2.
\]
This yields $b \in \text{BMO}_\alpha(\mathbb{R}^n)$ and finishes the proof of Theorem 1.4.

**Proof of Proposition 2.5** Using Hölder’s inequality, we obtain 
\[
\int_Q |(T_{\Omega, \beta})^m_b f_i(x)| dx \leq \left( \int_Q |(T_{\Omega, \beta})^m_b f_i(x)|^q \omega^q(x) dx \right)^{1/q} \left( \int_Q \omega^{-q'}(x) dx \right)^{1/q'}.
\]
By the Hölder inequality, the definition of $f_i$ and Proposition 2.4, for $x \in E_i$, 
\[
|(T_{\Omega, \beta})^m_b f_i(x)| \gtrsim \frac{|\omega^p(F_i)|^{-1/p}}{|Q|^{1-\beta/n}} |b(x) - m_b(P)|^m \int_{F_i} |\Omega(x - y)| dy \\
\geq \frac{|\omega^p(F_i)|^{-1/p}}{|Q|^{1-\beta/n}} |b(x) - m_b(P)|^m \int_{F_i \setminus (N_x \cap P)} |\Omega(x - y)| dy \\
\geq \epsilon_0 \frac{|\omega^p(F_i)|^{-1/p}}{|Q|^{1-\beta/n}} |b(x) - m_b(P)|^m : |F_i \setminus (N_x \cap P)| \\
\geq \epsilon_0 \frac{|\omega^p(F_i)|^{-1/p}}{|Q|^{1-\beta/n}} |b(x) - m_b(P)|^m \cdot \frac{|Q|}{4} \\
\sim |\omega^p(F_i)|^{-1/p} |Q|^{\frac{\beta}{n}} |b(x) - m_b(P)|^m \\
\gtrsim |\omega^p(P)|^{-1/p} |Q|^{\frac{\beta}{n}} |b(x) - m_b(P)|^m,
\]
where we use the facts $F_i \subset P$, $|F_i| = \frac{|P|}{2}$ and $|N_x \cap P| \leq \frac{|P|}{4}$. This and that fact $Q = E_1 \cup E_2$ yield that for $x \in Q$, 
\[
|(T_{\Omega, \beta})^m_b f_i(x)| + |(T_{\Omega, \beta})^m_b f_2(x)| \gtrsim |\omega^p(P)|^{-1/p} |Q|^{\frac{\beta}{n}} |b(x) - m_b(P)|^m \chi_Q(x).
\]
Hence, 
\[
\sum_{i=1,2} \left( \int_Q |(T_{\Omega, \beta})^m_b f_i(x)|^q \omega^q(x) dx \right)^{1/q} \left( \int_Q \omega^{-q'}(x) dx \right)^{1/q'} \\
\geq \sum_{i=1,2} \int_Q |(T_{\Omega, \beta})^m_b f_i(x)| dx \\
\geq |\omega^p(P)|^{-1/p} |Q|^{\frac{\beta}{n}} \int_Q |b(x) - m_b(P)|^m dx.
\]
\[ \geq |\omega^p(P)|^{-1/p}|Q|^{\frac{m}{n}} \left( \frac{1}{|Q|} \int_Q |b(x) - m_b(P)| \, dx \right)^m |Q|^{1-m} \]
\[ \geq |\omega^p(P)|^{-1/p}|Q|^{1+\frac{m\alpha}{n}} \tilde{\Omega}_\alpha(b; Q)^m = |\omega^p(P)|^{-1/p} |Q|^{1/p+1/q} \tilde{\Omega}_\alpha(b; Q)^m. \]

By the fact \( P \subset 4k_0Q \), we use the definition of \( A_{p,q} \) and the Hölder inequality to deduce that
\[ \left( \frac{1}{|Q|} \int_P \omega^p(x) \, dx \right)^{1/p} \left( \frac{1}{|Q|} \int_P \omega^{-q'}(x) \, dx \right)^{1/q'} \lesssim 1. \]
\[ \left( \frac{1}{|Q|} \int_Q \omega^p(x) \, dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \omega^{-q'}(x) \, dx \right)^{1/q'} \lesssim 1. \]

The above two estimates yield that
\[ \sum_{i=1,2} \| (T_{\Omega, \beta})_{b_i}^{\alpha} (f_i) \|_{L^p(\omega^q)} \gtrsim \tilde{\Omega}_\alpha(b; Q)^m. \]

\[ \square \]

3. Characterization of \( \text{CMO}_\alpha(\mathbb{R}^n) \) by fractional mean oscillation

In this section, we present the proof of Theorem 1.7. Different from the BMO(\( \mathbb{R}^n \)) case in [25], since a \( \text{Lip}_\alpha(\mathbb{R}^n) \) function is continuous, the simple functions can no longer be used to approximate other functions in the topology of \( \text{Lip}_\alpha(\mathbb{R}^n) \). In order to fix this situation, we find a kind of \( \text{Lip}_1(\mathbb{R}^n) \) functions which is a suitable replacement for the simple functions; see \( \{F_Q\} \) in Proposition 3.1 below. This kind of \( \text{Lip}_1(\mathbb{R}^n) \) functions can be constructed by some functions \( \{\psi^Q\} \) defined on the vertexes of cubes. Thanks to the \( \text{Lip}_1(\mathbb{R}^n) \) functions mentioned above, we prove Theorem 1.7 by the geometric part of arguments in [25] with some careful technical modifications fitting our cases. More precisely, for any given function \( f \in \text{CMO}_\alpha(\mathbb{R}^n) \) and \( \epsilon > 0 \), we first choose a finite family \( Q \) of dyadic cubes, define the functions \( \{\psi^Q\} \) and \( \{F_Q\} \) for all \( Q \in Q \), and further construct a function \( h_\epsilon \in C^\infty(\mathbb{R}^n) \) via \( \{F_Q\} \) which approximates to \( f \) in the norm of \( \text{BMO}_\alpha(\mathbb{R}^n) \).

3.1. Lipschitz function associated with cubes. In this section, we use \( V_Q \) to denote the set of all vertexes of a given cube \( Q \). By a weighted cube we mean that there exists a vertex mapping:
\[ \psi^Q : V_Q \rightarrow \mathbb{C}. \]

The oscillation of a weighted cube \( Q \) is defined by
\[ \mathcal{M}_{Q} := \inf_{c \in \mathbb{C}} \sum_{a \in V_Q} |\psi^Q(a) - c|. \] (3.1)

For a point \( x := (x_1, \ldots, x_n) \in \mathbb{R}^n \), we define the product function by
\[ x_\Pi := \prod_{j=1}^n x_j. \]

**Proposition 3.1.** For a given weighted cube \( Q \subset \mathbb{R}^n \) with vertex mapping \( \psi^Q \), we have following properties:

1. There is a unique function \( F_Q \) defined on \( \mathbb{R}^n \) satisfying:
(a) $F_Q = \psi^Q$ on $V_Q$, \\
(b) $F_Q$ is linear for each component $x_j$, $j = 1, 2, \cdots, n$, when the other $n - 1$ components are fixed. \\
Moreover, this unique function associated with weighted cube $Q$ can be expressed by \\
\[ F_Q(x) = \sum_{a \in V_Q} \frac{(2cQ - x - a)_i}{(2cQ - 2a)_i} \psi^Q(a) \chi_Q(x). \tag{3.2} \]

(2) Let $\tilde{Q}$ be a cube on $\mathbb{R}^n$, or on the $m$-dimensional hyperplane $P_m$ with $m < n$. Set $\psi^Q := F_Q|_{V_Q}$. Then \\
\[ F_{\tilde{Q}} = F_Q|_{P_m}. \]

(3) $F_Q$ is smooth with the following Lipschitz bound: \\
\[ \left| \frac{\partial F_Q(x)}{\partial x_i} \right| \leq |Q|^{-1/n} \cdot M_{Q}, \quad x \in Q, \quad i = 1, 2, \cdots, n. \]

Proof. We first prove property (1). By a standard translation and dilation argument, we only need to deal with $Q := [0,1]^n$. Since $F_Q$ is linear for each component mentioned above, we decompose $F_Q$ by \\
\[ F_Q(x_1, x_2, \cdots, x_n) = (1 - x_1)F_Q(0, x_2, \cdots, x_n) + x_1F_Q(1, x_2, \cdots, x_n) \]
\[ = \sum_{\epsilon_1 = 0,1} \frac{1 - x_1 - \epsilon_1}{1 - 2\epsilon_1} F_Q(\epsilon_1, x_2, \cdots, x_n) \]
\[ = \sum_{\epsilon_1 = 0,1} \frac{1 - x_1 - \epsilon_1}{1 - 2\epsilon_1} \sum_{\epsilon_2 = 0,1} \frac{1 - x_2 - \epsilon_2}{1 - 2\epsilon_2} \cdot F_Q(\epsilon_1, \epsilon_2, x_3, \cdots, x_n) \]
\[ = \sum_{(\epsilon_1, \epsilon_2) \in \{0,1\}^2} \frac{1 - x_1 - \epsilon_1}{1 - 2\epsilon_1} \cdot \frac{1 - x_2 - \epsilon_2}{1 - 2\epsilon_2} \cdot F_Q(\epsilon_1, \epsilon_2, x_3, \cdots, x_n) \]
\[ = \cdots = \sum_{\epsilon \in \{0,1\}^n} \frac{(1 - x - \epsilon)_i}{(1 - 2\epsilon)_i} F_Q(\epsilon) = \sum_{\epsilon \in \{0,1\}^n} \frac{(1 - x - \epsilon)_i}{(1 - 2\epsilon)_i} \psi^Q(\epsilon), \]
where we use property (a) in the last equality.

Next, we verify property (2). Note that both $F_{\tilde{Q}}$ and $F_Q$ are linear for each component on $P_m$. Moreover, they share the same value at each vertex in $V_Q$. By property (1), they must be equal.

Finally, we proceed to the proof of (3). By a standard translation and dilation argument, we only need to deal with $Q := [0,1]^n$. In this case, write \\
\[ F_Q(x) = \sum_{\epsilon \in \{0,1\}^n} \frac{(1 - x - \epsilon)_i}{(1 - 2\epsilon)_i} \psi^Q(\epsilon) = \sum_{\epsilon \in \{0,1\}^n} \prod_{j=1}^n \frac{1 - x_j - \epsilon_j}{1 - 2\epsilon_j} \psi^Q(\epsilon). \]

Note that \\
\[ \sum_{\epsilon \in \{0,1\}^n} \prod_{j=1}^n \frac{1 - x_j - \epsilon_j}{1 - 2\epsilon_j} = \prod_{j=1}^n \left( \frac{1 - x_j - 0}{1 - 0} + \frac{1 - x_j - 1}{1 - 2} \right) = 1. \tag{3.3} \]

Hence, for any $x \in \mathbb{R}^n$ and $c \in \mathbb{C}$, \\
\[ \frac{\partial F_Q(x)}{\partial x_i} = \frac{\partial (F_Q(x) - c)}{\partial x_i} = \frac{\partial (\sum_{\epsilon \in \{0,1\}^n} \prod_{j=1}^n \frac{1 - x_j - \epsilon_j}{1 - 2\epsilon_j} (\psi^Q(\epsilon) - c))}{\partial x_i}, \]
which yields that
\[
\left| \frac{\partial F(x)}{\partial x_i} \right| \leq \sum_{\epsilon \in \{0,1\}^n} |\psi^Q(\epsilon) - c|, \quad x \in Q.
\]
The desired conclusion follows by taking infimum over \(c\).

The following is a useful regularity proposition in the proof of Theorem 1.7.

**Proposition 3.2.** Let \(\alpha \in (0, 1]\), \(Q, \tilde{Q}\) be two cubes satisfying that \(\tilde{Q} \subset Q\). Suppose that \(O_\alpha(f; P) < \epsilon\) for all cubes \(P \subset Q\) with \(|P| \geq |\tilde{Q}|\). Then
\[
|f_Q - f_{\tilde{Q}}| \leq C|Q|^{\frac{\alpha}{2}}\epsilon,
\]
where the constant \(C\) is independent of \(Q, \tilde{Q}\) and \(f\).

**Proof.** Take \(N := \lceil \log_2 \frac{|Q|}{|\tilde{Q}|} \rceil\). We can find a sequence of cubes \(\{Q_i\}_{i=1}^N\) satisfying
\[
Q \supset Q_1 \supset Q_2 \supset \cdots \supset Q_N \supset \tilde{Q}, \quad |Q_i| = (1/2)^i|Q|, \quad (i = 1, 2, \ldots, N)
\]
and
\[
(1/2)^N|Q| \geq |\tilde{Q}| > (1/2)^{N+1}|Q|.
\]
By the assumption of Proposition 3.2 and the choice of \(Q_1\),
\[
|f_Q - f_{Q_1}| \leq \frac{1}{|Q_1|} \int_{Q_1} |f(y) - f_Q|dy \leq \frac{|Q_1|^{1+\alpha}}{|Q_1|^{1+\alpha}} \int_{Q} |f(y) - f_Q|dy \leq 2|Q|^{\frac{\alpha}{2}}\epsilon.
\]
A similar argument yields that
\[
|f_{Q_j} - f_{Q_{j+1}}| \leq 2|Q_j|^{\frac{\alpha}{2}}\epsilon \quad \text{for} \quad j = 1, 2, \cdots, N - 1,
\]
and
\[
|f_{Q_N} - f_{\tilde{Q}}| \leq 2|Q_N|^{\frac{\alpha}{2}}.
\]
Hence,
\[
|f_Q - f_{\tilde{Q}}| \leq |f_Q - f_{Q_1}| + \sum_{j=1}^{N-1} |f_{Q_j} - f_{Q_{j+1}}| + |f_{Q_N} - f_{\tilde{Q}}| \\
\leq 2 \sum_{j=1}^{N} |Q_j|^{\frac{\alpha}{2}} + 2|Q|^{\frac{\alpha}{2}} \epsilon \leq C|Q|^{\frac{\alpha}{2}}\epsilon.
\]

**3.2. Proof of Theorem 1.7** The case of \(\alpha = 0\) is due to Uchiyama [25], see Theorem A in Section 1.

Now, we start our proof for \(\alpha \in (0, 1]\). Obviously, the conditions (1), (2) and (3) in Definition 1.6 are valid if \(f \in C^\infty_c(\mathbb{R}^n)\). For a fixed function \(f \in \mathcal{CMO}_\alpha(\mathbb{R}^n)\) and every \(\epsilon > 0\), we take a function \(g \in C^\infty_c(\mathbb{R}^n)\) such that
\[
\|f - g\|_{\text{BMO}_\alpha(\mathbb{R}^n)} < \epsilon.
\]
Then,
\[
\limsup_Q O_\alpha(f; Q) \leq \limsup_Q O_\alpha(f - g; Q) + \limsup_Q O_\alpha(g; Q) \leq \|f - g\|_{\text{BMO}_\alpha(\mathbb{R}^n)} + \limsup_Q O_\alpha(g; Q) \leq \epsilon,
\]
where the cubes $Q$ vary in the way of that mentioned in (1)-(3). This completes the proof of $\text{CMO}_\alpha(\mathbb{R}^n) \subset \text{CMO}_\alpha(\mathbb{R}^n)$.

For the proof of $\text{CMO}_\alpha(\mathbb{R}^n) \subset \text{CMO}_\alpha(\mathbb{R}^n)$, roughly speaking, the geometric part of the argument in [25] still works in our case. However, the analytical part of argument in [25] does not work again. More precisely, since a $\text{Lip}_\alpha(\mathbb{R}^n)$ function is continuous, it is clear that a simple function such as in [25] can not be used as an approximation function in our case.

Now, we start our proof for $\text{CMO}_\alpha(\mathbb{R}^n) \subset \text{CMO}_\alpha(\mathbb{R}^n)$. Assume that $f$ satisfies conditions (1)-(3) of Theorem 1.7. We will show that for a fixed small number $\epsilon > 0$, there exists a function $g_\epsilon \in \text{Lip}_\alpha(\mathbb{R}^n) \cap \text{Lip}_1(\mathbb{R}^n)$ and $h_\epsilon \in C_\infty^c(\mathbb{R}^n)$, such that

$$\|f - g_\epsilon\|_{\text{Lip}_\alpha(\mathbb{R}^n)} \leq C\epsilon$$

(3.4)

and

$$\|g_\epsilon - h_\epsilon\|_{\text{Lip}_\alpha(\mathbb{R}^n)} \leq C\epsilon,$$

(3.5)

where $C$ is independent of $f$, $g_\epsilon$ and $h_\epsilon$.

We prove (3.4) by the following two steps.

**Step I** To define the function $g_\epsilon$, we first introduce a finite family $Q$ of closed dyadic cubes and define the vertex mappings $\{\psi^Q\}$ for all $Q \in \mathcal{Q}$ and auxiliary functions $\{g_{\epsilon,m}\}$ as follows. By conditions (1)-(3), there exist three integers depending on $\epsilon$, denoted by $i_\epsilon$, $j_\epsilon$ and $k_\epsilon$ respectively, satisfying $i_\epsilon + 3 \leq k_\epsilon$,

$$\sup\{\mathcal{O}_\alpha(f; Q) : |Q| \leq 2^{(i_\epsilon+2)n}\} < \epsilon, \quad \sup\{\mathcal{O}_\alpha(f; Q) : |Q| \geq 2^{i_\epsilon n}\} < \epsilon,$$

and

$$\sup\{\mathcal{O}_\alpha(f; Q) : Q \cap R_{k_\epsilon} = \emptyset\} < \epsilon,$$

where for $i \in \mathbb{Z}$, $R_i := [-2^i, 2^i]^n$. Moreover, let

$$d_1 := d_1(\epsilon) := k_\epsilon + 1$$

and for any integer $m \geq d_1$,

$$P_m := R_{i_\epsilon + m - d_1 - 1} = \left[ -2^{i_\epsilon + m - d_1 - 1}, 2^{i_\epsilon + m - d_1 - 1} \right]^n.$$

Using condition (2), we can find a sufficient large integer $d_2 \geq j_\epsilon$ such that for all cubes $Q$ such that $|Q| \geq |P_m| = 2^{(i_\epsilon + m - d_1)n}$ with $m \geq d_2$,

$$\mathcal{O}_\alpha(f; Q) \leq \left( \frac{|P_m|}{|R_m|} \right)^{\frac{1}{n}} \epsilon.$$  

(3.6)

Now we consider a finite family $Q$ of closed dyadic cubes in $R_{d_2+1}$ as follows. For $x \in R_{d_1}$, $Q_x$ means the closed dyadic cubes of side length $2^{i_\epsilon}$ that contain $x$. If $x \in R_m \setminus R_{m-1}$ for $m > d_1$, $Q_x$ means the closed dyadic cubes with side length $2^{i_\epsilon + m - d_1}$ that contain $x$. Observe that for any $x \in \mathbb{R}^n$, the number of cubes $Q_x$ is not more than $2^n$. Take

$$Q := \{Q_x : Q_x \subset R_{d_2+1}\}, \quad Q_{d_1} := \{Q \in Q : Q \subset R_{d_1}\}$$

and

$$Q_m := \{Q \in \mathcal{Q} : Q \subset R_m \setminus R_{m-1}\}, \quad d_1 + 1 \leq m \leq d_2 + 1.$$

Then by the definitions of $i_\epsilon$ and $k_\epsilon$, we see that for any cube $\tilde{Q} \subset 4Q$ with $Q \in \mathcal{Q}$,

$$\mathcal{O}(f; \tilde{Q}) < \epsilon,$$

(3.7)

which via Proposition 3.2 further implies that for any $Q \in Q_m$ and $a \in (V_Q \setminus \partial R_m)$,

$$|f_{4Q} - f_Q|, |f_{4Q} - f_{P_m+a}| \lesssim \mathcal{O}(f; 4Q)|Q|^\frac{1}{n} \lesssim \epsilon |Q|^\frac{1}{n},$$

(3.8)
Next, we add weights on the vertexes of \( Q_x \in \mathcal{Q} \). For \( Q \in \mathcal{Q}_{d_2+1} \) and \( a \in V_Q \), set
\[
\psi^Q(a) := A_{d_2} := \frac{\sum_{b \in V_Q} f_{P_{d_2+b}}}{|Q|}.
\]
where \(|\bigcup_{Q \in \mathcal{Q}_{d_2}} V_Q|\) is the cardinality of \( \bigcup_{Q \in \mathcal{Q}_{d_2}} V_Q \).

For \( y \in \mathbb{R}^n \), define
\[
g_{\epsilon,d_2+1}(y) := \begin{cases} F_Q(y), & \exists Q \in \mathcal{Q}_{d_2+1} \text{ such that } Q \ni y, \\ 0, & \text{others}, \end{cases}
\]
where \( F_Q \) is the function associated with \( \psi^Q \) as in Proposition 3.1. For any \( y \in R_{d_2+1} \setminus \tilde{R}_{d_2} \), if there exist \( Q, \tilde{Q} \in \mathcal{Q}_{d_2+1} \) such that \( y \in Q \cap \tilde{Q} \), then (3.9) and property (2) in Proposition 3.1 imply that \( F_Q(y) = F_{\tilde{Q}}(y) \). Thus, the function \( g_{\epsilon,d_2+1} \) is well-defined.

Next, for \( m := d_2, d_2 - 1, d_2 - 2, \ldots, d_1 \) and \( Q \in \mathcal{Q}_m \), we set
\[
\psi^Q(a) := \begin{cases} g_{\epsilon,m+1}(a), & a \in V_Q \cap \partial R_m, \\ f_{P_{m+a}}, & a \in V_Q \setminus \partial R_m, \end{cases}
\]
and
\[
g_{\epsilon,m}(y) := \begin{cases} F_Q(y), & \exists Q \in \mathcal{Q}_m \text{ such that } Q \ni y, \\ 0, & \text{others}. \end{cases}
\]

By the definition of \( g_{\epsilon,m} \), we further set
\[
g_{\epsilon}(y) := \begin{cases} g_{\epsilon,m}(y), & \exists Q \in \mathcal{Q}_m, d_1 \leq m \leq d_2 + 1, \text{ such that } Q \ni y, \\ A_{d_2}, & y \notin R_{d_2+1}. \end{cases}
\]

Thanks to the property (2) in Proposition 3.1, the functions \( g_{\epsilon,m}(d_1 \leq m \leq d_2 + 1) \) and \( g_{\epsilon} \) are well-defined. Moreover, to show \( g_{\epsilon} \in \text{Lip}_\alpha(\mathbb{R}^n) \), we first claim that for all \( Q \in \mathcal{Q} \),
\[
\mathcal{M}_Q \lesssim |Q|^{\alpha} \epsilon,
\]
where \( \mathcal{M}_Q \) is as in 3.1.

In fact, for any \( Q \in \mathcal{Q}_{d_2+1} \), by 3.1 and 3.9, it is trivial that
\[
\mathcal{M}_Q \leq \sum_{a \in V_Q} |\psi^Q(a) - A_{d_2}| = 0.
\]

On the other hand, for any \( Q \in \mathcal{Q}_{d_2} \) and \( a, b \in V_Q \), by 3.6, 3.7 and Proposition 3.2, we see that
\[
|f_{P_{d_2+b}} - f_{P_{d_2+a}}| \lesssim \mathcal{O}_{\alpha}(f; 4Q)|Q|^{\frac{n}{\alpha}} \lesssim \left( \frac{|Q|}{|R_{d_2}|} \right)^{\frac{1}{\alpha}} \epsilon |Q|^{\frac{n}{\alpha}}.
\]
Now for any \( Q \in \mathcal{Q}_{d_2} \), from 3.15, it follows that for all \( a \in V_Q \),
\[
|f_{P_{d_2+a}} - A_{d_2}| \lesssim \left( \frac{|Q|}{|R_{d_2}|} \right)^{\frac{1}{\alpha}} \epsilon |Q|^{\frac{n}{\alpha}} \left( \frac{|R_{d_2}|}{|Q|} \right)^{\frac{1}{\alpha}} \sim |Q|^{\frac{n}{\alpha}} \epsilon.
\]
We see that $g \in Q$ belong to a cube $Q$. Then to show (3.4), it suffices to show that

$$\sum \text{points } x \in Q = 1$$

where in the equality, $\tilde{a} = \sum_{j=0}^{N-1} |x_j - x_{j+1}|$, and for any fixed $j = 0, 1, \cdots, N - 1$, each pair of points $x_j, x_{j+1}$ both belong to a cube $Q \in Q$ or $R_{d_2+1}$. Since by (3.19), $g_{\epsilon}$ is a Lipschitz function on every cubes $Q \in Q$, we obtain

$$\left| \frac{\partial F_Q(y)}{\partial y_j} \right| \leq |Q|^{-1/n} M_Q \lesssim |Q|^{a-1} \epsilon.$$

Now we show that $g_{\epsilon} \in Lip_1(\mathbb{R}^n)$. For any two points $x, y \in \mathbb{R}^n$, one can choose finite points $x_1, x_2, \cdots, x_N$ where $N$ is independent of $x, y$, such that $x_N := y$, $x_N = \sum_{j=0}^{N-1} |x_j - x_{j+1}|$, and for any fixed $j = 0, 1, \cdots, N - 1$, each pair of points $x_j, x_{j+1}$ both belong to a cube $Q \in Q$ or $R_{d_2+1}$. Since by (3.19), $g_{\epsilon}$ is a Lipschitz function on every cubes $Q \in Q$, we obtain

$$|g_{\epsilon}(x) - g_{\epsilon}(y)| \leq \sum_{j=0}^{N-1} |g_{\epsilon}(x_j) - g_{\epsilon}(x_{j+1})| \leq \sup_{Q \in Q} \sup_{x \in Q} |\nabla F_Q(x)| \sum_{j=0}^{N-1} |x_j - x_{j+1}|$$

$$= \sup_{Q \in Q} \sup_{x \in Q} |\nabla F_Q(x)| \cdot |x_0 - y_0|.$$ 

We see that $g_{\epsilon}$ is a bounded function in $Lip_1(\mathbb{R}^n)$. This implies that $g_{\epsilon} \in Lip_\alpha(\mathbb{R}^n)$ for all $\alpha \in (0, 1)$.

**Step II** We now show (3.4). For any cube $Q$, define

$$\tilde{\mathcal{O}}_\alpha(f; Q) := \inf_{c \in \mathbb{C}} \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |f(x) - c|dx.$$ 

It is easy to see that

$$\tilde{\mathcal{O}}_\alpha(f; Q) \leq \mathcal{O}_\alpha(f; Q) \leq 2\tilde{\mathcal{O}}_\alpha(f; Q).$$

Then to show (3.21), it suffices to show that

$$\tilde{\mathcal{O}}_\alpha(f - g_{\epsilon}; Q) \lesssim \epsilon$$

(3.21)
for all cubes $Q$ on $\mathbb{R}^n$. This part is divided into following three cases.

**Case 1.** $Q \subset R_{d_2+1}$. Let $\mathcal{D}_Q := \{Q_x : Q_x \cap Q \neq \emptyset\}$. We further consider the following two cases.

Subcase (i) $\max \{l_{Q_x} : Q_x \in \mathcal{D}_Q\} \geq 2l_Q$. In this case, the number of cubes $Q_x \in \mathcal{D}_Q$ is not more than $2^n$ and $l_{Q_x} \geq l_Q$ for any $Q_x \in \mathcal{D}_Q$. If $Q \cap R_{d_1} \neq \emptyset$, then $|Q| \leq 2^{n_3}$. From this and the definition of $d_1$, we obtain $\mathcal{O}_\alpha(f; Q) < \epsilon$. By (3.19), we deduce that

$$
\frac{1}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |g(x) - g(y)|dy \lesssim \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \sum_{Q_x \in \mathcal{D}_Q} |Q_x|^\frac{\alpha-1}{n} \cdot \int_{Q \cap Q_x} |y-c_Q|dy
$$

$$
\lesssim \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \sum_{Q_x \in \mathcal{D}_Q} |Q_x|^\frac{\alpha-1}{n} \cdot |Q|^{1+\frac{1}{n}}
$$

$$
\sim \epsilon \sum_{Q_x \in \mathcal{D}_Q} \left( \frac{|Q|}{|Q_x|} \right)^\frac{1-\alpha}{n} \lesssim \epsilon.
$$

Thus,

$$
\mathcal{O}_\alpha(f - g; Q) \leq \mathcal{O}_\alpha(f; Q) + \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |g(x) - g(y)|dy \lesssim \epsilon.
$$

If $Q \cap R_{d_1} = \emptyset$, by the definition of $d_1$ we have $\mathcal{O}_\alpha(f; Q) < \epsilon$. By a similar argument, we also have

$$
\mathcal{O}_\alpha(f - g; Q) \lesssim \epsilon.
$$

Subcase (ii) $\max \{l_{Q_x} : Q_x \in \mathcal{D}_Q\} < 2l_Q$. In this subcase, we first show that for any $Q_x \in \mathcal{D}_Q$ and $y \in Q_x$,

$$
|f_{Q_x} - \mathcal{F}_{Q_x}(y)| \lesssim |Q_x|^{\frac{\alpha}{n}} \epsilon. \tag{3.22}
$$

In fact, for any $Q_x \subset R_{d_2}$ and $y \in Q_x$, we take $a \in (v_{Q_x} \setminus \partial R_{d_2})$ and write

$$
|f_{Q_x} - \mathcal{F}_{Q_x}(y)| \leq |f_{Q_x} - f_{Q_x}| + |f_{Q_x} - \mathcal{F}_{Q_x}(a)| + |\mathcal{F}_{Q_x}(a) - \mathcal{F}_{Q_x}(y)|
$$

$$
= |f_{Q_x} - f_{Q_x}| + |f_{Q_x} - f_{|_{Q_x}|_{d_2+a}}| + |\mathcal{F}_{Q_x}(a) - \mathcal{F}_{Q_x}(y)|
$$

$$
\lesssim |Q_x|^{\frac{\alpha}{n}} \epsilon + |a - y| \cdot |Q_x|^{\frac{\alpha-1}{n}} \lesssim |Q_x|^{\frac{\alpha}{n}} \epsilon,
$$

where we use (3.3), (3.19) and the fact $\mathcal{F}_{Q_x}(a) = \psi|_{Q_x}(a) = f|_{Q_x}|_{d_2+a}$.

For any $Q_x \subset Q_{d_2+1}$ and $y \in Q_x$, arguing as (3.16), we also have

$$
|f_{Q_x} - \mathcal{F}_{Q_x}(y)| = |f_{Q_x} - A_{d_2}| \lesssim |Q_x|^{\frac{\alpha}{n}} \epsilon. \tag{3.23}
$$

Thus, (3.22) holds.

By (3.12), (3.11), (3.7) and (3.22), we further write

$$
\int_Q |f(y) - g(y)| dy \leq \sum_{Q_x \in \mathcal{D}_Q} \left( \int_{Q_x} |f(y) - f_{Q_x}| dy + \int_{Q_x} |f_{Q_x} - \mathcal{F}_{Q_x}(y)| dy \right)
$$

$$
\leq \sum_{Q_x \in \mathcal{D}_Q} \left( |Q_x|^{1+\frac{\alpha}{n}} \epsilon + \int_{Q_x} |f_{Q_x} - \mathcal{F}_{Q_x}(y)| dy \right)
$$

$$
\lesssim \sum_{Q_x \in \mathcal{D}_Q} |Q_x|^{1+\frac{\alpha}{n}} \epsilon \lesssim \left( \sum_{Q_x \in \mathcal{D}_Q} |Q_x| \right)^{1+\frac{\alpha}{n}} \epsilon \lesssim |Q|^{1+\frac{\alpha}{n}} \epsilon.
$$
This implies that
$$\widetilde{O}_\alpha(f - g_\epsilon; Q) \leq \frac{1}{|Q|^{1 + \frac{n}{n}}} \int_Q |f(y) - g_\epsilon(y)|dy \lesssim \epsilon.$$ 

**Case 2.** $Q \subset R^c_{d_2}$. Note that $g_\epsilon(y) = A_{d_2}$ for any $y \in R^c_{d_2}$. Then
$$\widetilde{O}_\alpha(f - g_\epsilon; Q) \leq \frac{1}{|Q|^{1 + \frac{n}{n}}} \int_Q |f(y) - f_Q|dy \lesssim \epsilon.$$ 

**Case 3.** $Q \cap R_{d_2} \neq \emptyset$, $Q \cap R^c_{d_2 + 1} \neq \emptyset$. Note that $l_Q \geq \frac{1}{2}l_{R_{d_2}}$. By this, (3.6) and the definition of $g_\epsilon$, we write
$$\int_Q |f(y) - g_\epsilon(y) - f_Q + A_{d_2}|dy$$
$$\leq \int_{Q \setminus R_{d_2}} |f(y) - g_\epsilon(y) - f_Q + A_{d_2}|dy + \int_{Q \cap R_{d_2}} |f(y) - g_\epsilon(y) - f_Q + A_{d_2}|dy$$
$$\leq \int_Q |f(y) - f_Q|dy + \int_{Q \cap R_{d_2}} |f(y) - g_\epsilon(y) - f_Q + A_{d_2}|dy$$
$$\leq |Q|^{1 + \frac{n}{n}} \epsilon + \int_{Q \cap R_{d_2}} |f(y) - g_\epsilon(y) - f_Q + A_{d_2}|dy.$$ 

Take a cube $Q_{x_0} \subset (R_{d_2 + 1} \setminus R_{d_2}) \cap Q$. By Proposition 3.2, (3.7) and (3.16), we obtain
$$|f_Q - A_{d_2}| \lesssim |f_Q - f_{Q_{x_0}}| + |f_{Q_{x_0}} - A_{d_2}| \lesssim |Q|^\frac{n}{n} \epsilon + C|Q_{x_0}|^\frac{n}{n} \epsilon \lesssim |Q|^\frac{n}{n} \epsilon.$$ 

This implies that
$$\int_{Q \cap R_{d_2}} |f_Q - A_{d_2}|dy \lesssim |Q|^{1 + \frac{n}{n}} \epsilon.$$ 

On the other hand, let $D_Q$ be as in Case 1. A similar argument as in Case 1 yields that
$$\int_{Q \cap R_{d_2}} |f(y) - g_\epsilon(y)|dy \leq \sum_{Q_x \in D_Q} \int_{Q_x} |f(y) - g_\epsilon(y)|dy \lesssim \sum_{Q_x \in D_Q} |Q_x|^{1 + \frac{n}{n}} \epsilon \lesssim |Q|^{1 + \frac{n}{n}} \epsilon.$$ 

Combining the above two estimates yields that
$$\int_Q |f(y) - g_\epsilon(y) - f_Q + A_{d_2}|dy$$
$$\leq |Q|^{1 + \frac{n}{n}} \epsilon + \int_{Q \cap R_{d_2}} |f(y) - g_\epsilon(y) - f_Q + A_{d_2}|dy \lesssim |Q|^{1 + \frac{n}{n}} \epsilon.$$ 

This implies that
$$\widetilde{O}_\alpha(f - g_\epsilon; Q) \leq \frac{1}{|Q|^{1 + \frac{n}{n}}} \int_Q |f(y) - g_\epsilon(y) - f_Q + A_{d_2}|dy \lesssim \epsilon.$$ 

Thus, (3.21) holds.

Now we show (3.5). Take a positive $C^\infty_c(\mathbb{R}^n)$ function $\varphi$ supported on $B(0, 1)$, satisfying $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$. Set $\varphi_t(x) := \frac{1}{t^n} \varphi(\frac{x}{t})$, $t \in (0, \infty)$. Recall that $g_\epsilon \in Lip_1(\mathbb{R}^n)$. Take sufficiently small $r$ such that
$$2\|g_\epsilon\|_{Lip_1(\mathbb{R}^n)} r^{1 - \alpha} < \epsilon.$$ 

Thus, for $|z| < r$,
$$|(g_\epsilon * \varphi_t)(x + z) - g_\epsilon(x + z) - ((g_\epsilon * \varphi_t)(x) - g_\epsilon(x))|$$
\[
\begin{aligned}
&= \left| \int_{B(0,1)} (g_\epsilon(x + z - ty) - g_\epsilon(x - ty))\varphi(y)dy \right| + |g_\epsilon(x + z) - g_\epsilon(x)| \\
&\leq \|g_\epsilon\|_{Lip_1(\mathbb{R}^n)} \cdot \left( \int_{B(0,1)} |z|\varphi(y)dy + |z| \right) \\
&= 2\|g_\epsilon\|_{Lip_1(\mathbb{R}^n)}|z| = 2\|g_\epsilon\|_{Lip_1(\mathbb{R}^n)} |z|^{1-\alpha} |z|^\alpha < \epsilon |z|^\alpha.
\end{aligned}
\] (3.24)

On the other hand, \( g_\epsilon \) is uniformly continuous, so
\[(g_\epsilon * \varphi_\epsilon)(x) \to g_\epsilon(x) \] uniformly for all \( x \in \mathbb{R}^n \).

Thus, one can choose sufficiently small \( t \) such that

\[
|(g_\epsilon * \varphi_\epsilon)(x + z) - g_\epsilon(x + z) - ((g_\epsilon * \varphi_\epsilon)(x) - g_\epsilon(x))| < \epsilon r^\alpha
\]
uniformly for all \( x, z \in \mathbb{R}^n \). From this, for \( |z| \geq r \),

\[
\frac{|(g_\epsilon * \varphi_\epsilon)(x + z) - g_\epsilon(x + z) - ((g_\epsilon * \varphi_\epsilon)(x) - g_\epsilon(x))|}{|z|^\alpha} < \epsilon.
\]

Combining this and (3.24), we actually get

\[
\frac{|(g_\epsilon * \varphi_\epsilon)(x + z) - g_\epsilon(x + z) - ((g_\epsilon * \varphi_\epsilon)(x) - g_\epsilon(x))|}{|z|^\alpha} < \epsilon.
\]

for all \( |z| \neq 0 \). This implies that

\[
\|g_\epsilon * \varphi_\epsilon - g_\epsilon\|_{Lip_\alpha(\mathbb{R}^n)} < \epsilon.
\]

Note that \( h_\epsilon := g_\epsilon * \varphi_\epsilon - A_{d_\omega} \in C^\infty_c(\mathbb{R}^n) \), and \( \|h_\epsilon - g_\epsilon\|_{Lip_\omega(\mathbb{R}^n)} < \epsilon \). We have now completed the proof of (3.25), which together with (3.4) implies \( \text{CMO}_\alpha(\mathbb{R}^n) \subset \widehat{\text{CMO}_\alpha}(\mathbb{R}^n) \) for \( \alpha \in (0, 1) \).

Next, we deal with the case \( \alpha := 1 \). By condition (1) of Definition 3.2 for any fixed \( \epsilon > 0 \), there is a constant \( a > 0 \) such that for any \( Q \) with \( l_Q < 2a \),

\[
\mathcal{O}_1(f; Q) < \epsilon.
\]

For a fixed point \( x \) and a variable point \( y \) with \( |x - y| < a \), one can choose a suitable cube \( Q \) with side length \( l_Q \leq 2|x - y| \), such that

\[
Q_{x,y,b} := x + \frac{bl_Q}{10}Q_0 \subset Q, \quad Q_{y,x,b} := y + \frac{bl_Q}{10}Q_0 \subset Q, \quad b \in (0, 1].
\]

It follows from Proposition 3.2 and \( \alpha = 1 \) that

\[
|f_{Q_{x,y,b}} - f_Q| \lesssim |Q|^{1-\epsilon} |x - y|, \quad |f_{Q_{y,x,b}} - f_Q| \lesssim |Q|^{1-\epsilon} |x - y|.
\]

Letting \( b \to 0 \), we obtain

\[
|f(x) - f_Q| \lesssim |x - y|, \quad |f(y) - f_Q| \lesssim |x - y|.
\]

This implies that

\[
|f(x) - f(y)| \lesssim |x - y|.
\]

Hence,

\[
limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} \lesssim \epsilon.
\]

By the arbitrariness of \( \epsilon \), we actually get

\[
\frac{\partial f(x)}{\partial x_j} \equiv 0 \quad \text{for all } x \in \mathbb{R}^n, \, j = 1, 2, \cdots, n.
\]
This completes the proof for \( \alpha = 1 \).

4. Necessity of compactness of commutators

In this section, in order to deal with the necessity of compact commutators, the lower bound in Proposition 2.5 is not enough. So, based on Proposition 2.5, we establish a further lower bounded estimate providing that \( b \in Lip_\alpha (\mathbb{R}^n) \). Next, for \( b \in Lip_\alpha (\mathbb{R}^n) \), \( \Omega \in L^\infty (S^{n-1}) \) and any cube \( Q \), we also obtain an upper bound of the weighted \( L^q \) norm of \((T_{\Omega, b})_Q^n(f)\) over the annulus \( 2^{d+1}Q \setminus 2^dQ \). Using Theorem 1.7, the upper and further lower bounds, and a reduction of \( \Omega \), we further present the proof of \( (1) \implies (2) \) part in Theorem 1.8 via a contradiction argument in Subsection 4.3.

4.1. Further lower estimates. In this subsection, we further establish the lower bound fitting for compactness. Here, although the local mean oscillation is lost, the continuity of \( b \in Lip_\alpha (\mathbb{R}^n) \) provides enough information for the distribution of the values of \( b \). This observation makes it possible for us to get the lower bound for the \( L^q \) norm over certain measurable set associated with \( Q \).

**Proposition 4.1.** Let \( \eta_0 > 0 \), \( 1 < p, q < \infty \), \( 0 < \alpha \leq 1 \), \( 0 \leq \beta < n \), \( 1/q = 1/p - (m \alpha + \beta)/n \) and \( m \in \mathbb{Z}^+ \). Let \( w \in A_{p, q} \), \( b \in Lip_\alpha (\mathbb{R}^n) \) be a real-valued function and \( \Omega \) be a measurable function on \( S^{n-1} \), which does not change sign and is not equivalent to zero on some open subset of \( S^{n-1} \). For any given cube \( Q \) with \( \overline{O}_\alpha (b; Q) \geq \eta_0 \), let \( \tilde{P} := 2P \) be the set associated with \( \tilde{Q} := 2Q \) and \( \gamma := \left( \min \left\{ \left( \frac{\eta_0}{\| b \|_{Lip_\alpha (\mathbb{R}^n)}} \right)^{1/\alpha}, \frac{1}{\sqrt{n}} \right\} \right)^n \) as mentioned in Proposition 2.2. There are cubes \( E \subset 2Q \) and \( F \subset 2P \) with \( |E| = |F| \geq \tilde{C} \min \left\{ (O_\alpha (b; Q))^{n/\alpha}, 1 \right\} |Q| \), where the constant \( \tilde{C} \) is independent of \( Q \). For \( f := \left( \int_F \omega (x)^p dx \right)^{-1/p} \chi_F \) and any measurable set \( B \) with \( |B| \leq \frac{|E|}{2} \), we have

\[
\| (T_{\Omega, \beta})_b^m (f) \|_{L^q(E \setminus B, \omega^n)} \geq C \min \left\{ (O_\alpha (b; Q))^{2n/\alpha}, 1 \right\} O_\alpha (b; Q)^n,
\]

where the constant \( C \) is independent of \( Q \).

**Proof.** Assume \( \overline{O}_\alpha (b; Q) \geq \eta_0 \). By the continuity of \( b \), there exist \( x_0 \in Q \) and \( y_0 \in P \), such that

\[
|b(x_0) - b(y_0)| = \frac{1}{|Q|} \int_Q |b(x) - b_P| dx \geq |Q| \hat{\overline{O}}_\alpha (b; Q).
\]

Set

\[
L_Q := \min \left\{ \left( \frac{\overline{O}_\alpha (b; Q)}{4 \| b \|_{Lip_\alpha (\mathbb{R}^n)}} \right)^{1/\alpha} \frac{l_Q}{\sqrt{n}}, \frac{l_Q}{2} \right\},
\]

and

\[
E := x_0 + L_QQ_0, \quad F := y_0 + L_QQ_0.
\]

We have

\[
E \subset 2Q, \quad F \subset 2P.
\]
For any $x \in E$, $y \in F$,
\[
|b(x) - b(y)| \geq |b(x_0) - b(y_0)| - |b(x) - b(x_0)| - |b(y) - b(y_0)|
\geq |Q|^{2/n} \tilde{O}_\alpha(b; Q) - |x - x_0|^{\alpha} \|b\|_{\text{Lip}_n(\mathbb{R}^n)} - |y - y_0|^{\alpha} \|b\|_{\text{Lip}_n(\mathbb{R}^n)}
\geq |Q|^{2/n} \tilde{O}_\alpha(b; Q) - \frac{\tilde{O}_\alpha(b; Q)}{4\|b\|_{\text{Lip}_n(\mathbb{R}^n)}} \|b\|_{\text{Lip}_n(\mathbb{R}^n)} - \frac{\tilde{O}_\alpha(b; Q)}{4\|b\|_{\text{Lip}_n(\mathbb{R}^n)}} \|b\|_{\text{Lip}_n(\mathbb{R}^n)}
\geq |Q|^{2/n} \tilde{O}_\alpha(b; Q).
\]
Again, by the continuity of $b$, $b(x) - b(y)$ does not change sign in $E \times F$.

On the other hand, by the above estimate of $b$ and the fact for $x \in 2Q$,
\[
|N_x \cap 2P| \leq \gamma |2P| \leq \frac{1}{2} \left( \min \left\{ \left( \frac{\eta_0}{4\|b\|_{\text{Lip}_n(\mathbb{R}^n)}} \right)^{1/\alpha}, \frac{1}{\sqrt{n}}, \frac{1}{2} \right\} \right)^n |P| \leq \frac{1}{2} L^n = \frac{1}{2} |F|,
\]
we obtain that
\[
\int_{E \setminus B} |(T_{\Omega, \beta})_b^m (f)(x)| dx
\leq \left( \int_{E \setminus B} \left( \int_{F} \omega(x)^p dx \right)^{-1/p} \mid \int_{F} (b(x) - b(y))^m \frac{\Omega(x - y)}{|x - y|^{n - \beta}} dy \mid dx \right.
\leq \left( \int_{F} \omega(x)^p dx \right)^{-1/p} \cdot \int_{E \setminus B} \int_{F} |b(x) - b(y)|^m \frac{\Omega(x - y)}{|x - y|^{n - \beta}} dy dx
\geq \left( \int_{F} \omega(x)^p dx \right)^{-1/p} \cdot \int_{E \setminus B} \int_{F \setminus (N_x \cap 2P)} |b(x) - b(y)|^m \frac{\Omega(x - y)}{|x - y|^{n - \beta}} dy dx
\geq \left( \int_{F} \omega(x)^p dx \right)^{-1/p} \cdot \frac{L^{2n}}{2} \cdot |Q|^\frac{m + \beta}{n} \tilde{O}_\alpha(b; Q)^m
\geq \left( \int_{F} \omega(x)^p dx \right)^{-1/p} \cdot |Q|^\frac{m + \beta}{n} + 1 \tilde{O}_\alpha(b; Q)^m \min \left\{ \left( \tilde{O}_\alpha(b; Q) \right)^{2n/\alpha}, 1 \right\}.
\]

Using Hölder’s inequality, we obtain
\[
\int_{E \setminus B} |(T_{\Omega, \beta})_b^m (f)(x)| dx \leq \left( \int_{E \setminus B} \left( (T_{\Omega, \beta})_b^m (f)(x) \right)^q \omega^q(x) dx \right)^{1/q} \left( \int_{F} \omega^{-q'}(x) dx \right)^{1/q'}.
\tag{4.1}
\]

The above two estimates yield that
\[
\left( \int_{E \setminus B} \left( (T_{\Omega, \alpha})_b^m (f)(x) \right)^q \omega^q(x) dx \right)^{1/q}
\geq \left( \int_{F} \omega^{-q'}(x) dx \right)^{-1/q'} \cdot \left( \int_{F} \omega(x)^p dx \right)^{-1/p}
\times |Q|^\frac{m + \beta}{n} + 1 \tilde{O}_\alpha(b; Q)^m \min \left\{ \left( \tilde{O}_\alpha(b; Q) \right)^{2n/\alpha}, 1 \right\}
\geq \min \left\{ \left( \tilde{O}_\alpha(b; Q) \right)^{2n/\alpha}, 1 \right\} \tilde{O}_\alpha(b; Q)^m.
\]
\[\square\]
4.2. Upper estimates. This part follows by the approach of \([8]\) with some technique modifications fitting for \(\text{Lip}_\alpha(\mathbb{R}^n)\).

**Proposition 4.2.** Let \(1 < p, q < \infty\), \(0 < \alpha \leq n\), \(0 \leq \beta < n\), \(m\alpha + \beta < n\), \(1/q = 1/p - (m\alpha + \beta)/n\), \(m \in \mathbb{Z}^+\). Suppose that \(b \in \text{Lip}_\alpha(\mathbb{R}^n)\), \(\Omega \in L^\infty(\mathbb{S}^{n-1})\), \(\omega \in A_{p,q}\). For any cube \(Q\) with \(\overline{O}_\alpha(b,Q) \geq \gamma_0 > 0\), denote by \(P, F\) the sets associated with \(Q\) mentioned in Proposition 4.1. Let \(f := (\int_F \omega(x)p^pdx)^{-1/p}\chi_f\). Then, there exists a positive constant \(\delta\) such that

\[
\| (\Omega, \alpha)_b^m (f) \|_{L^p(2^{d+1}Q \setminus 2^dQ, \omega^q)} \lesssim 2^{-\delta n^{1/p}}.
\]

for sufficient large positive constant \(d\), where the implicit constant is independent of \(d\), \(f\) and \(Q\).

**Proof.** Since \(\omega^p \in A_p\) and \(F \subset 2P\) with \(|F| \gtrsim |P|\), we have \(\omega^p(F) \gtrsim \omega^p(2P) \gtrsim \omega^p(P)\). Then

\[
f(x) = \left( \int_F \omega(x)p^pdx \right)^{-1/p} \chi_F(x) \lesssim \left( \int_P \omega(x)p^pdx \right)^{-1/p} \chi_{2P}(x).
\]

A direct calculation yields that

\[
\| (\Omega, \alpha)_b^m (f) \| \lesssim \left( \int_P \omega(y)p^pdy \right)^{-1/p} \int_{2P} |b(x) - b(y)|^m \frac{\Omega(x - y)}{|x - y|^{n-\beta}} dy
\]

\[
= \left( \int_P \omega(y)p^pdy \right)^{-1/p} \int_{2P} |b(x) - b_{2P} + b_{2P} - b(y)|^m \frac{\Omega(x - y)}{|x - y|^{n-\beta}} dy
\]

\[
\leq \left( \int_P \omega(y)p^pdy \right)^{-1/p} \sum_{i+j=m} C_m |b(x) - b_{2P}|^i \int_{2P} |b_{2P} - b(y)|^j \frac{\Omega(x - y)}{|x - y|^{n-\beta}} dy. \quad (4.2)
\]

For sufficiently large \(d\), observe that \(|x - y| \sim 2^d|Q|\) for \(x \in 2^{d+1}Q \setminus 2^dQ\) and \(y \in 2P\). By Lemma 1.2, we deduce that

\[
\int_{2P} |b_{2P} - b(y)|^j \frac{\Omega(x - y)}{|x - y|^{n-\beta}} dy \lesssim \frac{\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}}{2^{d(n-\beta)}|P|^{1-\beta/n}} \int_{2P} |b_{2P} - b(y)|^j dy
\]

\[
= \frac{\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}}{2^{d(n-\beta)}|P|^{1-\beta/n}} \frac{1}{|P|} \int_{2P} |b_{2P} - b(y)|^j dy
\]

\[
\lesssim \frac{\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}}{2^{d(n-\beta)}|P|^{1-\beta/n}} \|b\|_{\text{Lip}_\alpha(\mathbb{R}^n)}^j. \quad (4.3)
\]

Since \(\omega^q \in A^q\), there exists a small positive constant \(\epsilon \leq \epsilon_n/|\omega^q|_{A_{\infty}}\), such that

\[
\left( \frac{1}{|Q|} \int_{\overline{Q}} \omega^{q(1+\epsilon)}(x) dx \right)^{1/(1+\epsilon)} \leq \frac{2}{|Q|} \int_{\overline{Q}} \omega^q(x) dx \quad \text{for all cubes } \overline{Q}.
\]

From this and the Hölder inequality, we obtain

\[
\| |b - b_{2P}|^j \|_{L^q(2^{d+1}Q \setminus 2^dQ, \omega^q)} \lesssim \| |b - b_{2P}|^j \|_{L^q(2^{d+1}Q, \omega^q)} \leq \left( \int_{2^{d+1}P} |b(x) - b_{2P}|^{jq} \omega^q(x) dx \right)^{1/q}
\]

\[
\lesssim \left( \int_{2^{d+1}P} |b(x) - b_{2P}|^{jq} \omega^q(x) dx \right)^{1/q}
\]
By Lemma 2.1 (v) and (ii), there exists a small constant \( \delta > 0 \) such that

\[
|b(x) - b_2P| \lesssim (2^{d+}P)^{\alpha} = 2^{d\alpha}|P|^{\frac{\alpha}{n}}, \quad x \in 2^{d+}P,
\]
we have

\[
\left( \frac{1}{|2^{d+}P|} \int_{2^{d+}P} |b(x) - b_2P|^{q(1+\epsilon)'} dx \right)^{\frac{1}{q(1+\epsilon)'}} \lesssim 2^{d\alpha}|P|^{\frac{\alpha}{n}}.
\]

Combining this with (4.2), (4.3), (4.4) yields that

\[
\| (T_{\Omega, \beta})^m_\alpha (f) \|_{L^q(2^{d+}Q \setminus 2^dQ, \omega^q)} \lesssim \sum_{i+j=m} \frac{|2^{d+}P|^{1/q} |P|^{\frac{\alpha}{n}} 2^{d\alpha}|P|^{\frac{\alpha}{n}}}{2^{d(n-\beta)}|P|^{-n/\beta}} \left( \frac{1}{|2^{d+}P|} \int_{2^{d+}P} \omega(x)^q dx \right)^{\frac{1}{q}} \left( \int_P \omega(x)^p dx \right)^{-1/p}
\]

\[
\lesssim 2^{dn(1/q-1+(\beta+\alpha)/n)} \left( \frac{1}{|2^dP|} \int_{2^dP} \omega(x)^q dx \right)^{\frac{1}{q}} \left( \frac{1}{|P|} \int_P \omega(x)^p dx \right)^{-1/p}.
\]

By Lemma 2.1 (v) and (ii), there exists a small constant \( \delta > 0 \), such that \( \omega^p \in A_{p-\delta} \) and

\[
\int_{2^dP} \omega(x)^p dx \leq 2^{dn(p-\delta)} \left[ \omega^p \right]_{A_{p-\delta}} \int_P \omega(x)^p dx,
\]
which implies

\[
\left( \frac{1}{|P|} \int_P \omega(x)^p dx \right)^{-1/p} \lesssim 2^{-dn/p} 2^{dn(1-\delta/p)} \left( \frac{1}{|2^dP|} \int_{2^dP} \omega(x)^p dx \right)^{-1/p}.
\]

Thus,

\[
\| (T_{\Omega, \alpha})^m_\alpha (f) \|_{L^q(2^{d+}Q \setminus 2^dQ, \omega^q)} \lesssim 2^{dn(1/q-1+(\beta+\alpha)/n)} 2^{-dn/p} 2^{dn(1-\delta/p)}
\]

\[
\times \left( \frac{1}{|2^dP|} \int_{2^dP} \omega(x)^q dx \right)^{\frac{1}{q}} \left( \frac{1}{|2^dP|} \int_{2^dP} \omega(x)^p dx \right)^{-1/p}
\]

\[
\lesssim 2^{-\delta dn/p} \left( \frac{1}{|2^dP|} \int_{2^dP} \omega(x)^q dx \right)^{\frac{1}{q}} \left( \frac{1}{|2^dP|} \int_{2^dP} \omega(x)^p dx \right)^{-1/p}.
\]

By the definition of \( A_{p,q} \), we obtain

\[
\left( \frac{1}{|2^dP|} \int_{2^dP} \omega(x)^q dx \right)^{1/q} \left( \frac{1}{|2^dP|} \int_{2^dP} \omega(x)^{-p'} dx \right)^{1/p'} \lesssim 1.
\]
This together with the following inequality
\[ 1 \lesssim \left( \frac{1}{|2^d P|} \int_{2^d P} \omega(x)^{-\theta'} dx \right)^{1/\theta'} \left( \frac{1}{|2^d P|} \int_{2^d P} \omega(x)^{\theta} dx \right)^{1/\theta} \]
yields that
\[ \frac{1}{|2^d P|} \int_{2^d P} \omega(x)^{\frac{n}{\theta}} dx \lesssim 1. \]

From this and (4.5), we get the desired estimate
\[ \| (T_{\Omega, \beta})_b^m (f) \|_{L^\theta(2^{d+1}Q, 2^d Q, \omega^n)} \lesssim 2^{-\delta dn/p}. \]

4.3. Proof of (1) \implies (2) in Theorem 1.8. This part follows by the approach of [8]. Noting that \( \Omega \in L^\infty(S^{n-1}) \) is not assumed in Theorem 1.8, a reduction of \( \Omega \) is needed.

First, we need the following proposition for reduction.

**Proposition 4.3.** Let \( 1 < p, q < \infty, 0 < \alpha \leq 1, 0 \leq \beta < n, m\alpha + \beta < n, 1/q = 1/p - (m\alpha + \beta)/n \) and \( m \in \mathbb{Z}^+ \). Suppose \( \tau' \in (1, p), \Omega \in L^{\tau'}(S^{n-1}), \omega' \in A_{\frac{r'}{p}, \frac{1}{\tau'}}. \) For a vector-valued function \( \vec{b} := (b_1, b_2, \ldots, b_m), b_j \in \text{BMO}_\alpha(\mathbb{R}^n) \), we have
\[ \| (T_{\Omega, \beta})_b^m f \|_{L^\theta(\omega^n)} \lesssim \| \Omega \|_{L^{\tau'}(S^{n-1})} \prod_{j=1}^m \| b_j \|_{\text{BMO}_\alpha(\mathbb{R}^n)} \| f \|_{L^p(\omega^n)}, \]
where for suitable function \( f \),
\[ (T_{\Omega, \beta})_b^m f(x) := \int_{\mathbb{R}^n} \prod_{j=1}^n [b_j(x) - b_j(y)] \frac{\Omega(x - y)}{|x - y|^{n-\beta}} f(y) dy. \]

**Proof.** By Lemma 1.2,
\[ |(T_{\Omega, \beta})_b^m (f)(x)| \lesssim \prod_{j=1}^m \| b_j \|_{\text{BMO}_\alpha(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Omega(x - y)|}{|x - y|^{n-\beta - m\alpha}} |f(y)| dy \]
\[ = \prod_{j=1}^m \| b_j \|_{\text{BMO}_\alpha(\mathbb{R}^n)} \cdot T_{|\Omega|, \beta + m\alpha}(|f|)(x). \]

Then the desired conclusion follows by a classical result of \( T_{|\Omega|, \beta + m\alpha} \) (see [17, Theorem 3.4.2]).

To prove (1) \implies (2) in Theorem 1.8, we only need to deal with the case that \( b \) is real-valued. If \( (T_{\Omega, \beta})_b^m \) is a compact operator from \( L^p(\omega^p) \) to \( L^q(\omega^q) \), then from Theorem 1.7 \( b \in \text{BMO}_\alpha(\mathbb{R}^n) \). To show \( b \in \text{CMO}_\alpha(\mathbb{R}^n) \), we use a contradiction argument. Observe that if \( b \notin \text{CMO}_\alpha(\mathbb{R}^n) \), \( b \) does not satisfy at least one of (1)-(3) in Definition 1.6. We further consider the following three cases.

First suppose that \( b \) does not satisfy condition (1) in Definition 1.6. There exist \( \theta_0 \in (0, 1) \) and a sequence of cubes \( \{Q_j\}_{j=1}^\infty \) with \( |Q_j| \searrow 0 \) as \( j \to \infty \), such that
\[ O_\alpha(b; Q_j) \geq \theta_0. \]
Given a cube $Q$ with $\mathcal{O}_\alpha(b; Q) \geq \theta_0$, let $E, F$ with $|E| = |F| \geq \bar{C} \min \left\{ (\mathcal{O}_\alpha(b; Q))^{2n/\alpha}, 1 \right\} |Q|$ be the cubes mentioned in Proposition 4.1 with $\eta_0 = \theta_0$. Let $f := (\int_{E} \omega(x)^p dx)^{-1/p} \chi_{F}$. Since $\Omega \in L^r(S^{n-1})$, for any $\epsilon > 0$ there exists a function $\Omega_\epsilon$ on $S^{n-1}$ such that

$$
\Omega_\epsilon \in L^\infty(S^{n-1}) \quad \text{and} \quad \|\Omega - \Omega_\epsilon\|_{L^r(S^{n-1})} < \epsilon.
$$

Applying Propositions 4.1 there exists a positive constant $C_0$ independent of $Q$, such that

$$
\| (T_{\Omega, \beta})^m_b (f) \|_{L^q(E \cap B_{\omega})} \geq 2C_0 \min \left\{ (\mathcal{O}_\alpha(b; Q))^{2n/\alpha}, 1 \right\} \mathcal{O}_\alpha(b; Q)^m \quad \text{for} \quad |B| \leq \frac{|E|}{2}.
$$

(4.6)

Next, by $\Omega - \Omega_\epsilon \in L^r(S^{n-1})$, Proposition 4.3 and $b \in \text{BMO}_\alpha(\mathbb{R}^n)$, we can choose sufficiently small constant $\epsilon_0 > 0$ such that

$$
\| (T_{\Omega_\epsilon, \beta})^m_b (f) - (T_{\Omega_\epsilon, \beta})^m_b (f) \|_{L^q(\mathbb{R}^n, \omega)} \leq \bar{C} \|\Omega - \Omega_\epsilon\|_{L^r(S^{n-1})} \|b\|_{\text{BMO}_\alpha(\mathbb{R}^n)} \|f\|_{L^p(\omega^n)} \leq \bar{C} \|\Omega - \Omega_\epsilon\|_{L^r(S^{n-1})} \|b\|_{\text{BMO}_\alpha(\mathbb{R}^n)} \frac{\epsilon_0^{m + \frac{2n}{\alpha}}}{2}.
$$

Then, applying Propositions 4.2 with $\eta_0 = \theta_0$ for $(T_{\Omega_\epsilon, \beta})^m_b$, there exists a positive constant $d_0$ independent of $Q$, such that

$$
\| (T_{\Omega_\epsilon, \beta})^m_b (f) \|_{L^q(\mathbb{R}^n \setminus 2d_0 Q, \omega)} \leq \frac{C_0 \epsilon_0^{m + \frac{2n}{\alpha}}}{2} = \frac{C_0 \min \left\{ \theta_0^{2n/\alpha}, 1 \right\} \theta_0^m}{2}.
$$

The above two estimates yield that

$$
\| (T_{\Omega, \beta})^m_b (f) \|_{L^q(\mathbb{R}^n \setminus 2d_0 Q, \omega^n)} \leq \| (T_{\Omega_\epsilon, \beta})^m_b (f) \|_{L^q(\mathbb{R}^n \setminus 2d_0 Q, \omega^n)} + \| (T_{\Omega_\epsilon, \beta})^m_b (f) - (T_{\Omega_\epsilon, \beta})^m_b (f) \|_{L^q(\mathbb{R}^n, \omega^n)} \leq C_0 \theta_0^m + \frac{2n}{\alpha}.
$$

(4.7)

Take a subsequence of $\{Q_j\}_{j=1}^\infty$, also denoted by $\{Q_j\}_{j=1}^\infty$, such that

$$
\frac{|Q_{j+1}|}{|Q_j|} \leq \min \left\{ \bar{C}^2 \theta_0^{\frac{2n}{\alpha}} / 4, 2^{-2d_0 n} \right\}.
$$

Denote $B_j := \left( \frac{|Q_j-1|}{|Q_j|} \right)^{\frac{1}{2n}} Q_j$, \quad $j \geq 2$. It is easy to check

$$
\left( \frac{|Q_j - 1|}{|Q_j|} \right)^{\frac{1}{2n}} \geq 2^{d_0}, \quad |B_{j+1}| = \left( \frac{|Q_{j+1}|}{|Q_j|} \right)^{\frac{1}{2}} Q_j \leq \bar{C} \theta_0^{\frac{2n}{\alpha}} |Q_j| / 2 \leq |E_j| / 2,
$$

where $E_j$ is the corresponding cube associated with $Q_j$ as mentioned in Proposition 4.1. Moreover, for any $k > j$, we have

$$
2^{d_0} Q_k \subseteq B_k, \quad |B_k| \leq |E_j| / 2.
$$

Denote by $F_j$ the set associated with $Q_j$ as mentioned in Proposition 4.1. Let

$$
f_j := \left( \int_{E_j} \omega(x)^p dx \right)^{-1/p} \chi_{F_j}.
$$

Again, from (4.6) and (4.7), for any $k > j \geq 1$, we obtain

$$
\| (T_{\Omega, \beta})^m_b (f_j) \|_{L^q(E_j \cap B_k, \omega^n)} \geq 2C_0 \min \left\{ (\mathcal{O}_\alpha(b; Q_j))^{2n/\alpha}, 1 \right\} \mathcal{O}_\alpha(b; Q_j)^m \geq 2C_0 \theta_0^m + \frac{2n}{\alpha}
$$
and
\[ \| (T_{\Omega, \beta})_b^m (f_k) \|_{L^q(E \setminus B_{k, \omega}^q)} \leq \| (T_{\Omega, \beta})_b^m (f_k) \|_{L^q(\mathbb{R}^n \setminus 2^{d_0} Q_k, \omega^q)} \leq C_0 \theta_0^{m + \frac{2m}{\alpha}}. \]
Hence,
\[ \| (T_{\Omega, \beta})_b^m (f_j) - (T_{\Omega, \beta})_b^m (f_k) \|_{L^q(\mathbb{R}^n, \omega^q)} \geq \| (T_{\Omega, \beta})_b^m (f_j) - (T_{\Omega, \beta})_b^m (f_k) \|_{L^q(E \setminus B_{k, \omega}^q)} \]
\[ \geq \| (T_{\Omega, \beta})_b^m (f_j) \|_{L^q(E \setminus B_{k, \omega}^q)} - \| (T_{\Omega, \beta})_b^m (f_k) \|_{L^q(E \setminus B_{k, \omega}^q)} \geq C_0 \theta_0^{m + \frac{2m}{\alpha}}, \]
which leads to a contradiction to the compactness of \((T_{\Omega, \beta})_b^m\).

A similar contradiction argument is valid for the proof of condition (2), we omit the details here. It remains to prove \(b\) satisfies condition (3) of Definition 1.6.

Assume that \(b\) satisfies (2) but does not satisfy (3). Hence, there exist \(\theta_1 \in (0, 1)\) and a sequence of cube \(\{\tilde{Q}_j\}_{j=1}^\infty\) with \(|\tilde{Q}_j| \lesssim 1\) such that
\[ \tilde{Q}_j \cap R_j = \emptyset, \quad \mathcal{O}_\alpha(b, \tilde{Q}_j) \geq \theta_1, \]
where \(R_j := [2^{-j}, 2^j]^n\). Denote by \(\tilde{E}_j, \tilde{F}_j\) the sets associated with \(\tilde{Q}_j\) as mentioned in Proposition 2.6 with \(\eta_0 = \theta_1\). Let
\[ \tilde{f}_j := \left( \int_{\tilde{E}_j} \omega(x)^p dx \right)^{-1/p} \chi_{\tilde{F}_j}. \]
Then, there exists a positive constant \(C_1\) independent of \(Q\), such that
\[ \| (T_{\Omega, \beta})_b^m (\tilde{f}_j) \|_{L^q(\tilde{E}_j, \omega^q)} \geq 2C_1 \min \left\{ \left( \mathcal{O}_\alpha(b; \tilde{Q}_j) \right)^{2n/\alpha}, 1 \right\} \mathcal{O}_\alpha(b; \tilde{Q}_j)^m. \quad (4.8) \]
By the similar method as above, there exists a positive constant \(d_1\) independent of \(\tilde{Q}_j\), such that
\[ \| (T_{\Omega, \beta})_b^m (\tilde{f}_j) \|_{L^q(\mathbb{R}^n \setminus 2\tilde{Q}_j, \omega^q)} \leq C_1 \min \left\{ \theta_1^{2n/\alpha}, 1 \right\} \theta_1^m = C_1 \theta_1^{m + \frac{2m}{\alpha}}. \quad (4.9) \]
Take \(d_2 \geq d_1\) such that \(\tilde{E}_j \subset 2^{d_2} \tilde{Q}_j\), and a subsequence of \(\{\tilde{Q}_j\}_{j=1}^\infty\), still denoted by \(\{\tilde{Q}_j\}_{j=1}^\infty\), such that
\[ 2^{d_2} \tilde{Q}_i \cap 2^{d_2} \tilde{Q}_j = \emptyset, \quad i \neq j. \]
For any \(k \neq j\), note that \(2^{d_1} \tilde{Q}_k \cap \tilde{E}_j \subset 2^{d_2} \tilde{Q}_k \cap 2^{d_2} \tilde{Q}_j = \emptyset\), then (4.8) implise
\[ \| (T_{\Omega, \beta})_b^m (\tilde{f}_j) \|_{L^q(\tilde{E}_j \setminus 2^{d_1} \tilde{Q}_k, \omega^q)} = \| (T_{\Omega, \beta})_b^m (\tilde{f}_j) \|_{L^q(\tilde{E}_j, \omega^q)} \]
\[ \geq 2C_1 \min \left\{ \left( \mathcal{O}_\alpha(b; \tilde{Q}_j) \right)^{2n/\alpha}, 1 \right\} \mathcal{O}_\alpha(b; \tilde{Q}_j)^m \geq 2C_1 \theta_1^{m + \frac{2m}{\alpha}}. \]
From this and (4.9) we get
\[ \| (T_{\Omega, \beta})_b^m (\tilde{f}_j) - (T_{\Omega, \beta})_b^m (\tilde{f}_k) \|_{L^q(\mathbb{R}^n, \omega^q)} \]
\[ \geq \| (T_{\Omega, \beta})_b^m (\tilde{f}_j) - (T_{\Omega, \beta})_b^m (\tilde{f}_k) \|_{L^q(\tilde{E}_j \setminus 2^{d_1} \tilde{Q}_k, \omega^q)} \]
\[ \geq \| (T_{\Omega, \beta})_b^m (\tilde{f}_j) \|_{L^q(\tilde{E}_j \setminus 2^{d_1} \tilde{Q}_k, \omega^q)} - \| (T_{\Omega, \beta})_b^m (\tilde{f}_k) \|_{L^q(\tilde{E}_j \setminus 2^{d_1} \tilde{Q}_k, \omega^q)} \geq C_1 \theta_1^{m + \frac{2m}{\alpha}}, \]
which leads to a contradiction to the compactness of \((T_{\Omega, \beta})_b^m\).
and

From this and Proposition \(4.3\), we obtain

Thus, in order to verify the set

Theorem 1.8. We follow some approach in \([8]\), see also \([15]\). Here, since the kernel \(\Omega\) is rough, we will use Proposition \(4.3\) to give a reduction for \(\Omega\), regaining the smoothness of kernel in the further proof.

Firstly, we recall the following weighted Fréchet-Kolmogorov theorem obtained in \([4]\).

**Lemma 5.1.** Let \(p \in (1, \infty)\) and \(\omega \in A_p\). A subset \(E\) of \(L^p(\omega)\) is precompact (or totally bounded) if the following statements hold:

(a) \(E\) is bounded, i.e., \(\sup_{f \in E} \|f\|_{L^p(\omega)} \leq 1\);
(b) \(E\) uniformly vanishes at infinity, that is,

\[
\lim_{N \to \infty} \int_{|x| > N} |f(x)|^p \omega(x) dx \to 0, \text{ uniformly for all } f \in E.
\]

(c) \(E\) is uniformly equicontinuous, that is,

\[
\lim_{\rho \to 0} \sup_{y \in B(0, \rho)} \int_{\mathbb{R}^n} |f(x + y) - f(x)|^p \omega(x) dx \to 0, \text{ uniformly for all } f \in E.
\]

**Proof of (2) \(\implies\) (1) in Theorem 1.8.** For \(\alpha = 1\), we have \((T_{\Omega, \beta})_b^m = 0\) by Theorem 1.7.

For \(\alpha \in (0, 1)\), by the definition of compact operator, we will verify the set

\[
A(\Omega, b) := \{(T_{\Omega, \beta})_b^m(f) : \|f\|_{L^p(\omega^p)} \leq 1\}
\]

is precompact.

Suppose \(b \in \text{CMO}_\alpha(\mathbb{R}^n)\). If \(r < \infty\), for any \(\varepsilon > 0\), there exist \(b \in C_\infty(\mathbb{R}^n)\) and \(\Omega \in \text{Lip}_1(\mathbb{S}^{n-1})\) such that

\[
\|b - b\|_{\text{BMO}_\alpha(\mathbb{R}^n)} < \varepsilon \quad \text{and} \quad \|\Omega - \Omega\|_{L^r(\mathbb{S}^{n-1})} < \varepsilon.
\]

From this and Proposition \(4.3\) we obtain

\[
\|(T_{\Omega, \beta})_b^m - (T_{\Omega, \beta})_b^m\|_{L^p(\omega^p) \to L^q(\omega^q)} \lesssim \varepsilon \|\Omega\|_{L^r(\mathbb{S}^{n-1})} \|b\|_{\text{BMO}_\alpha(\mathbb{R}^n)}^{m-1}
\]

and

\[
\|(T_{\Omega, \beta})_b^m - (T_{\Omega, \beta})_b^m\|_{L^p(\omega^p) \to L^q(\omega^q)} \lesssim \varepsilon \|\Omega\|_{L^r(\mathbb{S}^{n-1})} \|b\|_{\text{BMO}_\alpha(\mathbb{R}^n)}^{m-1}.
\]

If \(r = \infty\), then \(r' = 1\) and hence \(\omega \in A_{p, q}\). One can choose a constant \(\tilde{r} < \infty\) such that

\[
\tilde{r}' \in (1, p) \quad \text{and} \quad \omega \tilde{r}' \in A_{p, \tilde{r}}.\]

For any \(\tilde{\varepsilon} > 0\) there exist \(b \in C_\infty(\mathbb{R}^n)\) and \(\Omega \in \text{Lip}_1(\mathbb{S}^{n-1})\) such that

\[
\|b - b\|_{\text{BMO}_\alpha(\mathbb{R}^n)} < \tilde{\varepsilon} \quad \text{and} \quad \|\Omega - \Omega\|_{L^\tilde{r}(\mathbb{S}^{n-1})} < \tilde{\varepsilon}.
\]

Then, Proposition \(4.3\) yields that

\[
\|(T_{\Omega, \beta})_b^m - (T_{\Omega, \beta})_b^m\|_{L^p(\omega^p) \to L^q(\omega^q)} \lesssim \tilde{\varepsilon} \|\Omega\|_{L^\tilde{r}(\mathbb{S}^{n-1})} \|b\|_{\text{BMO}_\alpha(\mathbb{R}^n)}^{m-1}
\]

and

\[
\|(T_{\Omega, \beta})_b^m - (T_{\Omega, \beta})_b^m\|_{L^p(\omega^p) \to L^q(\omega^q)} \lesssim \|\Omega - \Omega\|_{L^r(\mathbb{S}^{n-1})} \|b\|_{\text{BMO}_\alpha(\mathbb{R}^n)}^{m-1}.
\]

Thus, in order to verify the set \(A(\Omega, b)\) is precompact, or equivalently, totally bounded on \(L^q(\omega^q)\), we only need to consider the case of \(b \in C_\infty(\mathbb{R}^n)\) and \(\Omega \in \text{Lip}_1(\mathbb{S}^{n-1})\).

Next, take \(\varphi \in C_\infty(\mathbb{R}^n)\) supported on \(B(0, 1)\) such that \(\varphi = 1\) on \(B(0, 1/2), 0 \leq \varphi \leq 1\). Let \(\varphi_\beta(x) := \varphi_\beta(x)\), \(K_\beta(x) := \frac{\varphi(x)}{\|\varphi\|_{L^\infty}} \cdot (1 - \varphi_\beta(x))\),

\[
A(K_\beta, b) := \{(T_{K_\beta})_b^m(f) : \|f\|_{L^p(\omega^p)} \leq 1\}, \quad T_{K_\beta} := \int_{\mathbb{R}^n} K_\beta(x - y)f(y)dy \quad \text{for all } x \notin \text{supp}f.
\]
Since \( b \in C_c^\infty(\mathbb{R}^n) \) and \( \Omega \in L^\infty(\mathbb{S}^{n-1}) \), we have
\[
\|(T_{K^\delta_{\beta}})^m f(x) - (T_{\Omega, \beta})_b^m f(x)\| \leq \left| \int_{\mathbb{R}^n} (b(x) - b(y))^m \varphi_\beta(x - y) \frac{\Omega(x - y)}{|x - y|^{n-\beta}} f(y) dy \right|
\leq \int_{|x - y| \leq \delta} \frac{|f(y)|}{|x - y|^{n-\beta-m}} dy.
\]
By the usual dyadic decomposition method, we get
\[
\int_{|x - y| \leq \delta} \frac{|f(y)|}{|x - y|^{n-\beta-m}} dy = \sum_{j=0}^{\infty} \int_{2^{-j+1}\delta \leq |x - y| \leq 2^{-j}\delta} \frac{|f(y)|}{|x - y|^{n-\beta-m}} dy
\leq \sum_{j=0}^{\infty} (2^{-j}\delta)^m (1-\alpha) \int_{2^{-j+1}\delta \leq |x - y| \leq 2^{-j}\delta} \frac{|f(y)|}{|x - y|^{n-\beta-m\alpha}} dy
\leq \sum_{j=0}^{\infty} 2^{-jm(1-\alpha)} \delta^m (1-\alpha) M_{\beta+m\alpha} (f)(x) \lesssim \delta^m (1-\alpha) M_{\beta+m\alpha} (f)(x),
\]
where for \( \gamma \in (0, n) \), \( M_\gamma f(x) \) is the fractional maximal function defined by
\[
M_\gamma f(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\gamma/n}} \int_Q |f(y)| dy.
\]
From the above two estimates we obtain
\[
\|(T_{K^\delta_{\beta}})^m f - (T_{\Omega, \beta})_b^m f\|_{L^q(\omega^\delta)} \lesssim \delta^m (1-\alpha) \|M_{\beta+m\alpha} f\|_{L^q(\omega^\delta)} \lesssim \delta^m (1-\alpha) \|f\|_{L^p(\omega^\delta)}.
\]
Since \( \delta \) can be chosen arbitrarily small, we only need to verify \( A(K^\delta, b) \) is totally bounded, where \( \delta > 0 \), \( b \in C_c^\infty(\mathbb{R}^n) \). Now, we finish the reduction argument for \( b \) and \( \Omega \).

Since \( \Omega \in Lip_1(\mathbb{S}^{n-1}) \), one can see that
\[
|K_{\beta}^\delta (x) - K_{\beta}^\delta (x')| \lesssim \frac{|x - x'|}{|x|^{n-\beta+1}}, \quad 2|x - x'| \leq |x|.
\]
We only need to check the conditions (a)-(c) of Lemma 5.1 for \( A(K^\delta, b) \).

Without loss of generality, we assume that \( b \) is supported in a cube \( Q \) centered at the origin. By the boundedness of \( (T_{K^\delta_{\beta}})^m \), \( A(K^\delta, b) \) is a bounded set in \( L^q(\omega^\delta) \), which verifies condition (a).

For \( x \in (2Q)^c \),
\[
|(T_{K^\delta_{\beta}})^m f(x)| = \left| \int_{\mathbb{R}^n} (b(y))^m K_{\beta}^\delta (x - y) f(y) dy \right|
\leq \|b\|_{L^\infty}^m \int_Q |f(y)| dy
\leq \|b\|_{L^\infty}^m \|f\|_{L^p(\omega^\delta)} \left( \int_Q \omega^{-p'} (x) dx \right)^{1/p'}.
\]
Take \( N > 2 \),
\[
\left( \int_{(2^N Q)^c} |(T_{K^\delta_{\beta}})^m f(x)|^q \omega(x)^q dx \right)^{1/q}
\]
\[
\lesssim \left( \int_{(2^N Q)^c} \frac{\omega(x)^q}{|x|^{q(n-\beta)}} \, dx \right)^{1/q} \left( \int_Q \omega^{-p'}(x) \, dx \right)^{1/p'}.
\]

Since \( \omega^q \in A_{(n-\beta-m\alpha)} \), we obtain
\[
\int_{2^d Q} \omega(x)^q \, dx \leq 2^{dq(n-\beta-m\alpha)} [\omega^q]_{A_{(n-\beta-m\alpha)}} \int_Q \omega(x)^q \, dx,
\]
which implies
\[
\int_{2^{d+1} Q \setminus 2^d Q} \frac{\omega(x)^q}{|x|^{q(n-\beta)}} \, dx \lesssim \frac{2^{dq(n-\beta-m\alpha)}}{2^{dq(n-\beta)}} = 2^{-dqma}.
\]

This and (5.1) yield that
\[
\left( \int_{(2^N Q)^c} |(T_{K_\alpha})^m(f)(x)|^q \omega(x)^q \, dx \right)^{1/q} \lesssim \left( \sum_{j=0}^\infty \int_{2^{N+j+1} Q \setminus 2^{N+j} Q} \frac{\omega(x)^q}{|x|^{q(n-\beta)}} \, dx \right)^{1/q}
\]
\[
\lesssim \left( \sum_{j=0}^\infty 2^{-\gamma(N+j)qma} \right)^{1/q} = 2^{-Nma} \left( \sum_{j=0}^\infty 2^{-jqma} \right)^{1/q},
\]
which tends to zero as \( N \) tends to infinity. This proves condition (b).

It remains to prove that \( A(K^\delta, b) \) is equicontinuous in \( L^q(\omega^q) \). Assume that \( \|f\|_{L^q_{w_p}(\mathbb{R}^n)} = 1 \) and take \( z \in \mathbb{R}^n \) with \( |z| \leq \frac{\delta}{4} \). Then
\[
(T_{K_\alpha})^m(f)(x + z) - (T_{K_\alpha})^m(f)(x)
\]
\[
= \int_{\mathbb{R}^n} \left( (b(x + z) - b(y))^m(K^\delta_\beta(x + z - y) - K^\delta_\beta(x - y)) f(y) \right) dy
\]
\[
+ \int_{\mathbb{R}^n} \left( (b(x + z) - b(y))^m - (b(x) - b(y))^m \right) K^\delta_\beta(x - y) f(y) dy
\]
\[
=: I_1(x, z) + I_2(x, z).
\]

We start the estimate of the first term. Observing that \( K^\delta_\beta(x + z - y) \) and \( K^\delta_\beta(x - y) \) both vanish when \( |x - y| \leq \frac{\delta}{4} \), then
\[
|I_1(x, z)| \lesssim \int_{|x-y| \geq \delta/4} |b(x + z) - b(y)|^m |K^\delta_\beta(x + z - y) - K^\delta_\beta(x - y)| \cdot |f(y)| \, dy
\]
\[
\lesssim \int_{|x-y| \geq \delta/4} \frac{|z|}{|x-y|^{n-\beta-m\alpha+1}} |f(y)| \, dy
\]
\[
\lesssim \sum_{j=0}^\infty \int_{2^{-j-2} \delta \leq |x-y| \leq 2^{-j-1} \delta} \frac{|z|}{|x-y|^{n-\beta-m\alpha+1}} |f(y)| \, dy
\]
\[
\lesssim \sum_{j=0}^\infty \frac{2^{2j} |z|}{\delta} \int_{2^{-j-2} \delta \leq |x-y| \leq 2^{-j-1} \delta} \frac{1}{|x-y|^{n-\beta-m\alpha}} |f(y)| \, dy
\]
\[
\lesssim \sum_{j=0}^\infty \frac{2^{2j} |z|}{\delta} M_{\beta+m\alpha}(f)(x) \lesssim \frac{|z|}{\delta} M_{\beta+m\alpha}(f)(x).
\]
Hence,
\[ \|I_1(\cdot, z)\|_{L^q(\omega^q)} \lesssim \frac{|z|}{\delta} \|M_{\beta + m\alpha}f\|_{L^q(\omega^q)} \lesssim \frac{|z|}{\delta} \|f\|_{L^p(\omega^p)} \leq \frac{|z|}{\delta}. \]

Finally,
\[
|I_2(x, z)| = \left| \int_{\mathbb{R}^n} \left( (b(x + z) - b(y))^m - (b(x) - b(y))^m \right) K_{\beta}^\delta(x - y)f(y)\,dy \right|
\lesssim |z| \int_{\mathbb{R}^n} |K_{\beta}^\delta(x - y)| \cdot |f(y)|\,dy \lesssim \int_{\mathbb{R}^n} |f(y)|\left|\frac{y}{|x - y|^{n-\beta-m\alpha}}\right|\,dy.
\]

Hence,
\[ \|I_2(\cdot, z)\|_{L^q(\omega^q)} \lesssim |z| \|I_{\beta + m\alpha}f\|_{L^q(\omega^q)} \lesssim |z| \|f\|_{L^p(\omega^p)} \leq |z|. \]

It follows from above estimates of $I_1$, $I_2$ that
\[ \|(T_{K_{\beta}^\delta})_m^0(\cdot + z) - (T_{K_{\beta}^\delta})_m^0(\cdot)\|_{L^q(\omega^q)} \to 0, \]
as $|z| \to 0$, uniformly for all $f$ with $\|f\|_{L^p(\omega^p)} \leq 1$.  \hfill $\Box$

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