Abstract

We investigate the non-elementary computational complexity of a family of substructural logics without contraction. With the aid of the technique pioneered by Lazić and Schmitz (2015), we show that the deducibility problem for full Lambek calculus with exchange and weakening ($FL_{ew}$) is not in ELEMENTARY (i.e., the class of decision problems that can be decided in time bounded by an elementary recursive function), but is in PR (i.e., the class of decision problems that can be decided in time bounded by a primitive recursive function). More precisely, we show that this problem is complete for TOWER, which is a non-elementary complexity class forming a part of the fast-growing complexity hierarchy introduced by Schmitz (2016). The same complexity result holds even for deducibility in BCK-logic, i.e., the implicational fragment of $FL_{ew}$. We furthermore show the TOWER-completeness of the provability problem for elementary affine logic, which was proved to be decidable by Dal Lago and Martini (2004).

1 Introduction

The term “substructural logic” [14, 31, 34] is an umbrella term for a family of logics that limit the use of some of the structural rules. Substructural logics encompass a wide range of non-classical logics (e.g., intuitionistic, classical, relevance, paraconsistent and multi-valued logics), and thus are discussed in several distinct areas close to mathematical logic such as philosophy, linguistics and computer science. In any of those research fields, one of the major topics is to settle the computational complexity of the provability problem for a logic, i.e., the problem of whether a given formula is provable in the logic. There are many seminal papers concerning this subject; see, e.g., [3, 21, 26, 28, 33, 36, 37, 41].

It is no surprise that a more general problem can be considered for a given logic. The deducibility problem for the logic asks for a given finite set $\Phi$ of formulas and a given formula $A$ whether $A$ is provable in the logic augmented with $\Phi$ as a set of non-logical axioms. In the setting of classical and intuitionistic logic, the notion of deducibility is reduced to provability via the deduction theorem. As a result, the deducibility problem for intuitionistic (resp. classical) propositional logic is complete for $PSPACE$ (resp. coNP) as with the provability problem. On the other hand, since most of substructural logics do not admit the deduction theorem, there is no guarantee that these two problems are inter-reducible to each other. For
this reason, it is important to distinguish them in the framework of substructural logic. In fact, some substructural logics have a critical “gap” between the complexity of provability and the complexity of deducibility. The so-called Lambek calculus is a striking example of such logics. Buszkowski [6] showed that its deducibility is undecidable, but later on Pentus [33] proved that its provability is NP-complete.

**Main contribution**

This paper aims at clarifying the non-elementary computational complexity of deducibility in contraction-free substructural logics. So far, such a topic has not been sufficiently explored while some earlier papers investigated a non-primitive recursive complexity of weakening-free substructural logics (i.e., relevance logics); see [26, 36, 41].

Full Lambek calculus with exchange and weakening ($\text{FL}_{\text{ew}}$), i.e., intuitionistic logic without the rule of contraction, is one of the most basic contraction-free logics. The deducibility problem for this logic is known to be decidable, thanks to the finite embeddability property of $\text{FL}_{\text{ew}}$-algebras shown by Blok and van Alten [4, 5]. However, its exact complexity remained open. Hence the following natural questions arise:

- Is there a primitive recursive algorithm — i.e., one whose runtime is bounded by a primitive recursive function — for the deducibility problem for $\text{FL}_{\text{ew}}$?
- If so, is there an elementary recursive algorithm — i.e., one whose runtime is bounded by a tower of exponentials of fixed height — for the problem?

We answer “yes” to the first question, but provide a negative answer to the second question. To be precise, we show that the problem is actually completely for the class $\text{Tower}$ (Corollary 23). This class forms a part of the fast-growing complexity hierarchy introduced by Schmitz [35], and roughly speaking, is located between ELEMENTARY (i.e., the class of problems decidable in elementary time) and PR (i.e., the class of problems that can be solved in time bounded by a primitive recursive function). As a consequence, it turns out that the aforementioned “gap” also lies between provability and deducibility in $\text{FL}_{\text{ew}}$; the provability problem for $\text{FL}_{\text{ew}}$ is PSPACE-complete, cf. [21].

We stress that the same holds even when almost all the logical connectives are removed from $\text{FL}_{\text{ew}}$. We also prove that the deducibility problem for BCK-logic [22, 32], i.e., the implicational fragment of $\text{FL}_{\text{ew}}$, is $\text{Tower}$-complete (Corollary 23). This is in sharp contrast to the NP-completeness of provability in BCK-logic (Corollary 3).

**Proof overview**

To show the $\text{Tower}$-membership of deducibility in $\text{FL}_{\text{ew}}$, we prove that there are reductions:

1. from deducibility in $\text{FL}_{\text{ew}}$ to provability in a variant of intuitionistic affine logic (denoted by $\text{ILZW}'$),
2. from the provability problem for $\text{ILZW}'$ to the lossy reachability problem for alternating branching vector addition systems with states (ABVASS, for short).

The first reduction is quite similar to the one used in the famed proof of the undecidability of propositional linear logic by Lincoln, Mitchell, Scedrov and Shankar [28]. The second reduction is substantially inspired by Lazić and Schmitz [26]. Due to the $\text{Tower}$-completeness of lossy reachability in ABVASS, shown in [26], we obtain the membership in $\text{Tower}$ of deducibility in $\text{FL}_{\text{ew}}$.

In order to show the $\text{Tower}$-hardness, we introduce the notion of $!$-prenex implicational sequent. It is a slight modification of $!$-prenex sequents which Terui introduced in his PhD thesis [38]. We prove the $\text{Tower}$-hardness of a restricted version of the provability problem...
for intuitionistic affine logic, which asks whether a given !-prenex implicational sequent is provable in intuitionistic affine logic. We obtain the desired result by showing that this problem can be reduced into deducibility in FL_{ew}.

**Provability (type inhabitation) in elementary affine logic**

As a by-product resulting from our methods, we provide the precise complexity of provability (not of deducibility) in propositional elementary affine logic [1]. Its name comes from the fact that it characterizes elementary recursive computation in the paradigm of proofs-as-programs; see also [11, 17]. Although this logic is seemingly just an extension of BCK-logic (and FL_{ew}) by a sort of modal storage operator, it is exploited for a variety of purposes, e.g., to characterize the class P and the exponential time hierarchy [2], to formulate a consistent naive set theory with a rich computational power [39].

In most situations, elementary affine logic is treated as a type system rather than a purely logical system. Accordingly, as with many other type systems, some decision problems can be considered, i.e., typability, type checking and type inhabitation (provability). For instance, Coppola and Martini [8] showed the decidability of typability in the \{\neg, !\}-fragment of intuitionistic elementary affine logic.

On the other hand, of particular interest to us is the provability problem for elementary affine logic. Dal Lago and Martini [10] showed that provability in a classical variant of elementary affine logic is decidable. However, there are no known upper and lower bounds for that problem. We refine and extend the existing decidability result by showing the Tower-completeness of some variants of elementary affine logic (Section 6.2). It should be noted that such a non-elementary aspect of elementary affine logic does not conflict with its elementary character that comes from the proofs-as-programs correspondence.

**Organization of this paper**

In the next section, we review various contraction-free logics in a step-by-step manner, and define a useful translation from classical affine logic into intuitionistic affine logic. A large portion of Sections 3 and 4 is taken from [26, Section 3]. In Section 3, we summarize some basic notions involved in ABVASSs. Section 4 is devoted to a short discussion about the existing complexity results which are crucial in proving the main claims in Sections 5 and 6. We prove the main results in Sections 5 and 6. In Section 7, we conclude the paper with some remarks on the complexity status of other substructural logics.

Proofs omitted due to space limitations appear in the full version of this paper (https://arxiv.org/abs/2201.02496).

### 2 Contraction-free substructural logics

#### 2.1 Sequent calculi for contraction-free substructural logics

For convenience, we start with the formal definition of a sequent calculus for intuitionistic affine logic with bottom, denoted by ILZW. It is merely the extension of Troelstra’s ILZ by the rule of left-weakening, cf. [26, 40]. The language \( \mathcal{L} \) of ILZW contains logical connectives \&, \oplus, \otimes, \neg of arity 2, ! of arity 1, and \( 1, \top, \bot, 0 \) of arity 0. We fix a countable set of propositional variables \( V = \{p, q, r, \ldots\} \). An intuitionistic \( \mathcal{L} \)-formula is built from propositional variables using connectives in \( \mathcal{L} \). For brevity, parentheses in formulas are omitted when confusion is unlikely. Throughout this paper, metavariables \( A, B, C, \ldots \) range over formulas and \( \Gamma, \Delta, \Sigma, \ldots \) over finite multisets of formulas. By abuse of notation, we
simply write \( A \) for the singleton of a formula \( A \). The multiset sum of \( \Gamma \) and \( \Delta \) is denoted by \( \Gamma, \Delta \). We write \( !\Gamma \) for the multiset obtained by prefixing each formula in \( \Gamma \) with exactly one \( ! \). An intuitionistic \( \mathcal{L} \)-sequent is an expression of the form \( \Gamma \vdash \Pi \), where \( \Gamma \) is a finite multiset of intuitionistic \( \mathcal{L} \)-formulas, and \( \Pi \) is a stoup, i.e., either an intuitionistic \( \mathcal{L} \)-formula or the empty multiset \( \varepsilon \). We always denote an intuitionistic \( \mathcal{L} \)-sequent of the form \( \varepsilon \vdash \Pi \) (resp. \( \Gamma \vdash \varepsilon \)) by \( \vdash \Pi \) (resp. \( \vdash \Gamma \)). The sequent calculus for \( \text{ILZW} \) consists of the inference rules depicted in Figure 1. A proof of a sequent \( \Gamma \vdash \Pi \) in \( \text{ILZW} \) is defined in the usual manner. We furthermore define another variant of intuitionistic affine logic by adding the following right-weakening rule (W') to \( \text{ILZW} \):

\[
\Gamma \vdash A \quad (W')
\]

The resulting system is denoted by \( \text{ILZW}' \).

Let \( \mathcal{K} \) be a non-empty subset of \( \mathcal{L} \). An intuitionistic \( \mathcal{K} \)-formula is an intuitionistic formula containing only logical connectives from \( \mathcal{K} \). An intuitionistic \( \mathcal{K} \)-sequent is an intuitionistic sequent consisting only of intuitionistic \( \mathcal{K} \)-formulas. The \( \mathcal{K} \)-fragment of \( \text{ILZW} \) (resp. \( \text{ILZW}' \)) is the sequent calculus obtained from \( \text{ILZW} \) (resp. \( \text{ILZW}' \)) by dropping all the inference rules concerning connectives not in \( \mathcal{K} \).

Each of the logical systems within the scope of this paper is a fragment of \( \text{ILZW} \) or \( \text{ILZW}' \). We list them below:

- **Full Lambek calculus with exchange and left-weakening** (\( \text{FL}_{\text{el}} \)) is the \( \{ \otimes, \rightarrow, \& , \oplus, 1, \perp \} \)-fragment of \( \text{ILZW} \).
- **Full Lambek calculus with exchange and weakening** (\( \text{FL}_{\text{ew}} \)) is the \( \{ \otimes, \rightarrow, \& , \oplus, 1, \perp \} \)-fragment of \( \text{ILZW}' \).
- The positive fragment (\( \text{FL}_{\text{el}}^+ \)) of \( \text{FL}_{\text{el}} \) is the \( \{ \otimes, \rightarrow, \& , \oplus, 1 \} \)-fragment of \( \text{ILZW} \).
- **BCK logic** (\( \text{BCK} \)) is the implicational fragment (i.e., \( \{ \rightarrow \} \)-fragment) of \( \text{ILZW} \).

Unfortunately, our notation is considerably different from the notation widely employed in the substructural logic community; we refer the reader to [14, Table 2.1] for the notational correspondence between linear logic and substructural logic.
We next formulate right-hand sided sequent calculi for classical affine logic \((LLW)\), cf. [16, 24]. For our purpose here, we again employ the countable set \(V\) of propositional variables, and introduce their duals \(V^\perp = \{p^\perp, q^\perp, r^\perp, \ldots\}\). Elements in \(V \cup V^\perp\) are often referred to as literals. The language \(\mathcal{L}_C\) consists of binary operation symbols \(\&, \oplus, \otimes, \Y\), and constants \(\bot, \top, \bot, 0\), and unary operation symbols \(!, ?\). Given a sublanguage \(\mathcal{K}\) of \(\mathcal{L}_C\), \textit{classical} \(\mathcal{K}\)-\textit{formulas} are built up from literals using operation symbols in \(\mathcal{K}\). For each classical \(\mathcal{L}_C\)-formula \(A\), we inductively define the formula \(A^\perp\) by the de Morgan duality:

\[
\begin{align*}
(p^\perp)^\perp & = p, (A \& B)^\perp = A^\perp \oplus B^\perp, (A \oplus B)^\perp = A^\perp \& B^\perp, (A \otimes B)^\perp = A^\perp \Y B^\perp, (A \Y B)^\perp = A^\perp \otimes B^\perp, (!A)^\perp = ?A^\perp, (?A)^\perp = !A^\perp, \bot^\perp = \top, \top^\perp = \bot, 0^\perp = 0, 0^\perp = 0.
\end{align*}
\]

It is easy to see that \(A = A^\perp\) for any classical \(\mathcal{L}_C\)-formula \(A\). A \textit{classical} \(\mathcal{K}\)-\textit{sequent} is an expression of the form \(\Gamma \vdash \Gamma\), where \(\Gamma\) is a finite multiset of classical \(\mathcal{K}\)-formulas. The inference rules of \(LLW\) are presented in Figure 2.

As with the intuitionistic sequent systems discussed earlier, various fragments of \(LLW\) can be defined in the usual manner. Among such fragments, of importance to us is the \(\{\otimes, \Y, \&, \oplus, 1, 1\}\)-fragment of \(LLW\), called \textit{involutive full Lambek calculus with exchange and weakening} \((\text{Infl}_{\text{ew}})\).

In [17] Girard proposed a logic which captures elementary recursive computation, called \textit{elementary linear logic} \((ELL)\). We review here some affine variants of \(ELL\), following [2, 8, 10]. \textit{Intuitionistic elementary affine logic with bottom} \((\text{IEZW})\) is obtained from \(ILZW\) by dropping the rules of \((\text{ID})\) and \((\text{IP})\) and by adding the following functorial promotion rule:

\[
\Gamma \vdash A \quad \frac{}{\Gamma \vdash !A} \quad \text{(IF)}
\]

Similarly, \textit{classical} \textit{elementary affine logic} \((ELLW)\) is obtained from \(LLW\) by deleting the rules of \((?)\) and \((!)\) and by adding the following rule:

\[
\Gamma \vdash A \quad \frac{}{\Gamma \vdash ?A, !A} \quad \text{(F)}
\]

It is easy to see that \(IEZW\) (resp. \(ELLW\)) is a subsystem of \(ILZW\) (resp. \(LLW\)), i.e., every sequent that is provable in \(IEZW\) (resp. \(ELLW\)) is provable in \(ILZW\) (resp. \(LLW\)). Henceforce, we write \(\mathcal{L}^+_E\) for the language \(\mathcal{L} \setminus \{\bot\}\). The notation \(ILZW\) (resp. \(IELW\)) is used to denote the \(\mathcal{L}^+_E\)-fragment of \(ILZW\) (resp. \(IELW\)). Every sequent in such \(\bot\)-free logical systems is of the form \(\Gamma \vdash A\).

Let \(L\) be one of the sequent calculi described so far and \(\Phi\) a set of formulas in \(L\). We write \(L[\Phi]\) for the sequent calculus obtained from \(L\) by adding \(\vdash B\) as an initial sequent for every \(B \in \Phi\). In this paper, we consider the following two types of decision problems for \(L\).
Problem (Provability in L).
Instance: A formula \( F \) in \( L \).
Question: Is the sequent \( \vdash F \) provable in \( L \)?

Problem (Deducibility in \( L \)).
Instance: A finite set \( \Phi \cup \{ F \} \) of formulas in \( L \).
Question: Is the sequent \( \vdash F \) provable in \( L[\Phi] \)?

Our argument depends heavily on the following cut-elimination theorem, as we will see in the remaining sections:

\[ \text{Theorem 1} \ (\text{cf. [10, 24, 30]}) \]

The sequent calculi for \( \text{ILLW} \), \( \text{ILZW} \), \( \text{ILZW}' \), \( \text{LLW} \), \( \text{IELW} \), \( \text{IEZW} \), and \( \text{ELLW} \) all enjoy cut-elimination.

As far as we know, for instance, the cut-elimination for \( \text{ILZW}' \) has not been settled. The reader can however show this without great difficulty, using a proof-theoretic or algebraic manner; see also [28, 29, 40] for technical details on cut-elimination in linear logic. We thus omit the proof in this paper.

Of course, the cut-elimination theorem also holds for various fragments of the systems stated in Theorem 1, e.g., \( \text{BCK} \), \( \text{FL}_{el} \), the \( \{\to, !\} \)-fragment of \( \text{ILZW} \).

2.2 Translation from classical affine logic to intuitionistic affine logic

In this subsection, we present an efficient (i.e., polynomial-time) translation from \( \text{LLW} \) (resp. \( \text{ELLW} \)) into \( \text{ILLW} \) (resp. \( \text{IELW} \)). It is a modification of Laurent’s \textit{parametric negative translation} from classical linear logic to intuitionistic linear logic; see [25, Definition 2.2].

Let us fix an intuitionistic \( L \)-formula \( F \). Given a classical \( L_C \)-formula \( A \), we inductively define the intuitionistic \( L \)-formula \( A[F] \) as follows:

\[
\begin{align*}
P[F] &:= \neg F p \\
P^\perp[F] &:= p \\
1[F] &:= \neg F 1 \\
\top[F] &:= 0 \\
(B \otimes C)[F] &:= \neg F B[F] \to C[F] \\
(B \& C)[F] &:= B[F] \oplus C[F] \\
(!B)[F] &:= \neg F !F B[F] \\
(B \& C)[F] &:= \neg F (B[F] \to \neg F C[F]) \\
(B \oplus C)[F] &:= \neg F (\neg F B[F] \to \neg F C[F]) \\
(?B)[F] &:= B[F]
\end{align*}
\]

where \( \neg_F A \) is an abbreviation for \( A \to F \). We have the following theorem. The proof can be found in the full version.

Theorem 2. Let \( \vdash \Gamma \) be a classical \( L_C \)-sequent and \( x \) a fresh propositional variable not occurring in \( \Gamma \).

1. \( \vdash \Gamma \) is provable in \( \text{LLW} \) if and only if \( \Gamma[x]\vdash x \) is provable in \( \text{ILLW} \).
2. \( \vdash \Gamma \) is provable in \( \text{ELLW} \) if and only if \( \Gamma[x]\vdash x \) is provable in \( \text{IELW} \).

This translation is convenient to show the complexity of the contraction-free logics that we deal with, e.g., the NP-completeness of the provability problem for \( \text{BCK} \).

Corollary 3. The provability problem for \( \text{BCK} \) is NP-complete.

Proof. Membership in NP is an immediate consequence of cut elimination for \( \text{BCK} \). The proof is based on that of [28, Lemma 5.3]. In any cut-free proof in the system, the only applicable rules are (Init), (\( \to \text{L} \)), (\( \to \text{R} \)) and (W). Thus each subformula occurring in the
endsequent is analyzed at most once in such a proof-tree. This means that, the size of a cut-free proof in the system is polynomially bounded in the size of the endsequent. Hence the problem is in NP.\(^1\)

For the hardness, we construct a polynomial-time reduction from provability in the \(\{\otimes, \forall\}\)-fragment of LLW (i.e., the constant-free fragment of multiplicative classical affine logic) into provability in BCK. The NP-completeness of the former is shown by Lincoln-Mitchell-Seedrov-Shankar [28], and Kanovich [23]. Let \(A\) be a classical \(\{\otimes, \forall\}\)-formula and \(x\) a fresh variable not in \(A\). Our goal is to show that \(\vdash A\) is provable in the \(\{\otimes, \forall\}\)-fragment of LLW if and only if \(\vdash \neg \gamma A[x]\) is provable in BCK. As a consequence of cut elimination for LLW, we can easily show that LLW is conservative over its \(\{\otimes, \forall\}\)-fragment. That is, \(\vdash A\) is provable in the \(\{\otimes, \forall\}\)-fragment of LLW if and only if \(\vdash A\) is provable in LLW. By Theorem 2, \(\vdash A\) is provable in LLW if and only if \(A[x] \vdash x\) is provable in LLW. Here \(A[x] \vdash x\) is an intuitionistic \(\neg \to\)-sequent. Again, by the cut elimination theorem for ILLW, we know that ILLW is conservative over BCK; hence \(A[x] \vdash x\) is provable in ILLW if and only if \(A[x] \vdash x\) is provable in BCK. By the invertibility of \((\neg \to R)\), \(A[x] \vdash x\) is provable in BCK if and only if \(\vdash \neg \gamma A[x]\) is provable in BCK; hence we conclude that \(\vdash A\) is provable in the \(\{\otimes, \forall\}\)-fragment of LLW if and only if \(\vdash \neg \gamma A[x]\) is provable in BCK.

In particular, the translation \((\_)[\L_0]\) is a sort of standard negative translation from LLW (resp. ELLW) to ILZW (resp. IEZW). One can also show the following:

\textbf{Theorem 4.} Let \(\vdash \Gamma\) be a classical \(L_0\)-sequent. \(\vdash \Gamma\) is provable in LLW (resp. ELLW) if and only if \(\Gamma[\L_0]\vdash \Gamma\) is provable in ILZW (resp. IEZW).

### 3 Alternating branching VASS

The whole content of this section is taken from [26, Section 3]. Let \(d\) be in \(\mathbb{N}\). The symbols \(\bar{v}_1, \bar{v}_2, \ldots\) are used to denote \(d\)-dimensional vectors. In particular, we write \(\bar{e}_i\) for the \(i\)-th unit vector in \(\mathbb{N}^d\) (i.e., the vector with a one in the \(i\)-th coordinate and zeros elsewhere), and \(\bar{0}\) for the vector whose every coordinate is zero.

An alternating branching vector addition system with states and full zero tests (ABVASS\(\bar{0}\), for short) is a structure of the form \(A = (Q, d, T_u, T_s, T_f, T_z)\) where:

- \(Q\) is a finite set,
- \(d\) is in \(\mathbb{N}\),
- \(T_u\) is a finite subset of \(Q \times \mathbb{Z}^d \times Q\),
- \(T_s\) and \(T_f\) are subsets of \(Q^3\), and
- \(T_z\) is a subset of \(Q^2\).

We call \(Q\) a state space, \(d\) a dimension, and \(T_u\) (resp. \(T_s\), \(T_f\), \(T_z\)) the set of unary (resp. split, fork, full zero test) rules of \(A\). For readability, we always write \(q \rightarrow q'\) for \((q, \bar{u}, q') \in T_u\), \(q \rightarrow q_1 \land q_2\) for \((q, q_1, q_2) \in T_f\), \(q \rightarrow q_1 + q_2\) for \((q, q_1, q_2) \in T_s\), and \(q \vdash_\bar{0} q'\) for \((q, q') \in T_z\).

A configuration of \(A\) is an element of \(Q \times \mathbb{N}^d\).

Given an ABVASS\(\bar{0}\) \(A = (Q, d, T_u, T_s, T_f, T_z)\), the operational semantics for \(A\) is given by a deduction system over configurations in \(Q \times \mathbb{N}^d\). It consists of the deduction rules depicted in Figure 3. Given a subset \(Q_\ell\) of \(Q\), a \(Q_\ell \times \{\bar{0}\}\)-leaf-covering deduction tree in \(A\) is a finite tree labeled by configurations, where leaves are all in \(Q_\ell \times \{\bar{0}\}\) and each other node is obtained

\(^1\) This was already pointed out in [20, p. 71].
Tower-Complete Problems in Contraction-Free Substructural Logics

Figure 3 The deduction rules of an ABVASS\(_0\) \(\mathcal{A} = (Q, d, T_u, T_s, T_f, T_z)\), where \(q \xrightarrow{\overline{0}} q'\) \(\in T_s\), \(q \rightarrow q_1 + q_2 \in T_z\), and \(q \rightarrow q_1 \land q_2 \in T_f\). The symbol + stands for componentwise addition and \(\overline{v} + \overline{u}\) must be in \(\mathbb{N}^d\).

from its children by applying one of the deduction rules derived from \(T_u \cup T_s \cup T_f \cup T_z\). A deduction tree \(T\) whose root configuration is \(q, \overline{v}\) is denoted by the following figure:

\[
\frac{q, \overline{v}}{q_1, \overline{v}} \quad \frac{q, \overline{v}}{q_2, \overline{v}} \quad \frac{q, \overline{v}}{q_1 + q_2, \overline{v}_{1} + \overline{v}_{2}} \quad \frac{q, \overline{v}}{q, \overline{v} + \overline{u}}
\]

We write \(\mathcal{A}, Q_e \triangleright q, \overline{v}\) if there exists a \(Q_e \times \{\overline{0}\}\)-leaf-covering deduction tree whose root configuration is \(q, \overline{v}\).

In addition, we also give an account of another semantics for ABVASS\(_0\)s. Given an ABVASS\(_0\) \(\mathcal{A} = (Q, d, T_u, T_s, T_f, T_z)\), the lossy semantics for \(\mathcal{A}\) is given by the aforementioned deduction system of \(\mathcal{A}\) augmented with the following additional deduction rules:

\[
\frac{q, \overline{v} + e_i}{q, \overline{v}} \quad \text{(loss)}
\]

for every \(q \in Q\) and every \(i \in \{1, \ldots, d\}\). In the natural way, we define the notion of \(Q_e \times \{\overline{0}\}\)-leaf-covering lossy deduction tree in \(\mathcal{A}\). We write \(\mathcal{A}, Q_e \triangleright q, \overline{v}\) if there exists a \(Q_e \times \{\overline{0}\}\)-leaf-covering lossy deduction tree with root \(q, \overline{v}\).

## 4 Some Tower-complete problems

Following the terminology of [26, 35], we define

\[
\text{TOWER} := \bigcup_{f \in \text{FELEMENTARY}} \text{DTIME}(2^{-f(n)} \text{ times})
\]

where \(\text{FELEMENTARY}\) denotes the set of elementary functions. This class of problems is closed under elementary many-one reductions (and elementary Turing reductions), i.e., for any two languages \(X\) and \(Y\), if there is an elementary reduction from \(X\) to \(Y\) and \(Y\) is in \(\text{TOWER}\), then \(X\) is in \(\text{TOWER}\). The notion of \text{TOWER-completeness} is defined with respect to elementary reductions in the usual manner. For an elaborate discussion on the fast-growing complexity hierarchy, we refer the reader to [35].

For later use, we summarize here some Tower-complete problems. Given an ABVASS\(_0\) \(\mathcal{A} = (Q, d, T_u, T_s, T_f, T_z)\), \(Q_e \subseteq Q\), and \(q_e \in Q\), the reachability problem (resp. lossy reachability problem) asks whether it holds that \(\mathcal{A}, Q_e \triangleright q_e, \overline{0}\) (resp. \(\mathcal{A}, Q_e \triangleright q_e, \overline{0}\)).

- **Theorem 5** (Lazić and Schmitz [26], Theorem 3.6). The lossy reachability problem for ABVASS\(_0\) is Tower-complete.

In contrast, the reachability problem is undecidable for ABVASS\(_0\)s. In fact, the same holds even for alternating VASSs, which are ABVASS\(_0\)s with only unary rules and fork rules; see [28, Section 3.4] and [26, Section 3.3.1] for details.
We show that there exists an elementary reduction from the provability problem for ordinary BVASSs to Tower-complete.

Theorem 6 (Lazić and Schmitz [26], Lemma 3.5 and Theorem 3.6). The provability problem for ordinary BVASSs is Tower-complete.

Theorem 7 (Lazić and Schmitz [26]; Fact 4.2, Corollary 5.4 and Corollary 6.3). The provability problems for ILZW and LLW are Tower-complete.

5 Membership in Tower of contraction-free logics

This section consists of two parts. We first show that provability in elementary affine logic is to the lossy reachability problem for ABVASSs, which are defined as in Figure 4.

5.1 Tower upper bound for provability in elementary affine logic

We show that there exists an elementary reduction from the provability problem for IEZGW to provability in ILZW to the lossy reachability problem for ABVASSs. Our reduction is a slightly modified version of the (polynomial-space) reduction, given in [26, Section 4.1.2], from provability in ILZW to lossy reachability in ABVASSs.

Let $F$ be an intuitionistic $\mathcal{L}$-formula. Let $S$ be the set of subformulas of $F$, $S_l$ the set of formulas in $S$ of the form $\exists B$, and $\bullet$ a fresh symbol not in $S$. For $\Pi \in S \cup \{\varepsilon\}$, we define $\Pi^\varepsilon = A$ if $\Pi = A$, and $\Pi^\varepsilon = \bullet$ otherwise. A multiset over $S$ is just a map from $S$ to $\mathbb{N}$, i.e., an element of $\mathbb{N}^S$. Given a multiset $m$ over $S$, $m(B)$ denotes the multiplicity of a formula $B$ in $m$.

Fix an enumeration $F_1, \ldots, F_d$ of all the formulas in $S$. A multiset $m$ over $S$ can be expressed as $L^{m(F_1)}, \ldots, L^{m(F_d)}$. For each multiset $m$ over $S$, we write $\nu_m$ for the vector $\langle m(F_1), \ldots, m(F_d) \rangle$ in $\mathbb{N}^d$. In particular, we write $\nu_B$ for the vector $\nu_B$ corresponding to a formula $B$ in $S$. Note that $\nu_m = \emptyset$ if $m$ is the empty multiset. We write $\sigma(m)$ for the support of a multiset $m$, i.e., $\{B \in S \mid m(B) > 0\}$. We then construct an ABVASS $\mathcal{A}_F^E$ as follows:

- The dimension of $\mathcal{A}_F^E$ is $d $ (i.e., $|S|$).
- The state space of $\mathcal{A}_F^E$ contains $P(S) \times (S \cup \{\bullet\})$, a distinguished leaf state $q_0$, and several intermediate states which are needed for defining the rules for $(\neg L)$, $(\& L)$, $(\oplus L)$, $(\otimes L)$, $(0 L)$, $(T R)$ and $(f u n c)$ in $\mathcal{A}_F^E$ (cf. Figure 4).
- The rules and intermediate states of $\mathcal{A}_F^E$ are defined as in Figure 4.

The construction of $\mathcal{A}_F^E$ is quite similar to that of the ABVASS $\mathcal{A}_F$ defined in [26, Section 4.1.2]. We stress that $\mathcal{A}_F^E$ has the rules for $(f u n c)$ instead of the rules for $(\Diamond L)$ and $(\Diamond P)$ (see Figure 5 in Section 5.2), whereas $\mathcal{A}_F^E$ has the rules for $(\Pi L)$ and $(\Diamond P)$ instead of the rules for $(f u n c)$. Notice that $\mathcal{A}_F^E$ does not have the rules corresponding to the inference rule of $\mathcal{W}$ in ILZW. The left-weakening rule is implemented by loss rules derived from the lossy semantics for $\mathcal{A}_F^E$.

Let $\Theta, \Gamma \vdash \Pi$ be an intuitionistic $\mathcal{L}$-sequent such that $\Theta$ is a multiset of formulas in $S$, $\Gamma$ is a multiset of formulas in $S \setminus S_l$, and $\Pi$ is in $S \cup \{\varepsilon\}$. It is translated as the configuration $\sigma(\Theta), \Pi^\varepsilon, \nu_\Pi$ in $P(S) \times (S \cup \{\bullet\}) \times \mathbb{N}^d$. We have the key theorem of this subsection; see the full version for a detailed proof.
We first describe the notion of $5.2$ Tower upper bound for deducibility in FLew and related systems $K$

to intuitionistic $\forall \Gamma$ $\vdash \exists$ $\Theta$ if and only if $\exists$ $\Pi$ $\vdash \exists \sigma(\Theta)$.

In particular, Theorem 8 guarantees that for any intuitionistic $\Gamma$-formula $F$, $F$ is provable in $\text{IEZW}$ if and only if $A_{\text{FL}}^C$, $\{q_1\} \vdash \emptyset$, $F$. By Theorems 5 and 4:

$\textbf{Corollary 9.}$ The provability problems for $\text{IEZW}$ and $\text{ELLW}$ are in Tower.

We stress that the Tower upper bound also holds for fragments of $\text{IEZW}$ and $\text{ELLW}$, e.g., the $\{\neg, \forall\}$-fragment of $\text{IEZW}$.

### 5.2 Tower upper bound for deducibility in FLew and related systems

We first describe the notion of $\forall$-prenex sequent which originates in $[38, \text{Section 2}]$. Let $K$ be a language such that $\bot \in K \subseteq L$. A $\forall$-prenex $K$-sequent is an intuitionistic sequent of the form $\forall \Gamma, \Delta \vdash \Pi$, where $\Gamma$ and $\Delta$ are finite multisets of intuitionistic $K$-formulas, and $\Pi$ is an intuitionistic $K$-formula or the empty multiset. Similarly, let $K$ be a sublanguage of $L_C$. $A$
?-prefix $\mathcal{K}$-sequent is a right-hand sided sequent of the form $\vdash ?\Gamma, \Delta$, where $\Gamma$ and $\Delta$ are finite multisets of classical $\mathcal{K}$-formulas. We write $\Gamma^n$ for the multiset sum of $n$ copies of $\Gamma$ for each $n \geq 0$.

**Lemma 10.** Let $\Gamma, \Delta \vdash \Pi$ be a $?!$-prefix $\{\otimes, \neg, \&\oplus, 1, \bot\}$-sequent.

1. If $\Gamma, \Delta \vdash \Pi$ is provable in ILZW, then $\Gamma^n, \Delta \vdash \Pi$ is provable in FL$_{el}$ for some $n \geq 0$.
2. If $\Gamma, \Delta \vdash \Pi$ is provable in ILZW, then $\Gamma^n, \Delta \vdash \Pi$ is provable in FL$_{ew}$ for some $n \geq 0$.

**Proof.** The proof of Statement (1) proceeds by induction on the size of the cut-free proof of $\Gamma, \Delta \vdash \Pi$ in ILZW. We perform a case analysis, depending on which inference rule is applied last.

We consider only the case of $(\neg L)$. If $\Gamma, !\Sigma, A \rightarrow B, \Delta, \Xi \vdash \Pi$ is obtained from $\Gamma, \Delta \vdash A$ and $!\Sigma, B, \Xi \vdash \Pi$ by an application of $(\neg L)$, then by the induction hypothesis, $\Gamma^\prime, \Delta \vdash A$ is provable in FL$_{el}$ for some $n^\prime$, and $\Sigma^\prime, B, \Xi \vdash \Pi$ is provable in FL$_{el}$ for some $n^\prime$. Applying $(\neg L)$ we obtain a proof of $\Gamma, \Sigma^\prime, \Sigma^\prime \rightarrow B, \Delta, \Xi \vdash \Pi$ in FL$_{el}$. Note that $n^\prime$ is not always equal to $n^\prime$. However, we may unify them using the rule of (W); thus $(\Gamma, \Sigma^{n^\prime} \rightarrow B, \Delta, \Xi \vdash \Pi$ is provable in FL$_{el}$. The remaining cases are similar.

One can show Statement (2) similarly.

Similarly to the above theorem, one can also show the following:

**Lemma 11.** Let $\vdash ?\Gamma, \Delta$ be a $?!$-prefix $\{\otimes, \nabla, \&\oplus, 1, \bot\}$-sequent. If $\vdash ?\Gamma, \Delta$ is provable in LLW, then $\vdash \Gamma^n, \Delta$ is provable in InfFL$_{ew}$ for some $n \geq 0$.

Lemmas 10 and 11 are affine analogues of [38, Proposition 2.6]. Interestingly, a similar idea is found in the proof of the local deduction theorem for FL$_{ew}$; see [15, Corollary 2.15]. The following lemma, which is inspired by [28, Lemmas 3.2 and 3.3], provides a very simple reduction from FL$_{el}$ deducibility (resp. FL$_{ew}$ deducibility) into ILZW provability (resp. ILZW' provability).

**Lemma 12.** Let $\Phi$ be a finite set of intuitionistic $\{\otimes, \neg, \&\oplus, 1, \bot\}$-formulas and $\Gamma \vdash \Pi$ an intuitionistic $\{\otimes, \neg, \&\oplus, 1, \bot\}$-sequent.

1. $\Gamma \vdash \Pi$ is provable in FL$_{el}[\Phi]$ if and only if $\Gamma, \Gamma \vdash \Pi$ is provable in ILZW.
2. $\Gamma \vdash \Pi$ is provable in FL$_{ew}[\Phi]$ if and only if $\Gamma, \Gamma \vdash \Pi$ is provable in ILZW'.

**Proof.** We only prove the first statement, the proof of the second one being similar. For the if part, let us suppose that $\Phi, \Gamma \vdash \Pi$ is provable in ILZW. Since $\Phi, \Gamma \vdash \Pi$ is a 1-prefix $\{\otimes, \neg, \&\oplus, 1, \bot\}$-sequent, $\Phi^n, \Gamma \vdash \Pi$ is provable in FL$_{el}$ for some $n$ by Lemma 10. Since $\vdash B$ is an initial sequent of FL$_{el}[\Phi]$ for each $B \in \Phi$, we can construct a proof of $\Gamma \vdash \Pi$ in FL$_{el}[\Phi]$ by several applications of (Cut). The only-if part follows by induction on the height of the proof of $\Gamma \vdash \Pi$ in FL$_{el}[\Phi]$.

In the same way as before, the deducibility problems for FL$_{el}$ and BCK are also reduced to the provability problem for ILZW. Furthermore, one can show that there is also a straightforward reduction from InfFL$_{ew}$ deducibility to LLW provability, using Lemma 11.

**Lemma 13.** Let $\Phi$ be a finite set of classical $\{\otimes, \nabla, \&\oplus, 1, \bot\}$-formulas and $\vdash \Gamma$ a classical $\{\otimes, \nabla, \&\oplus, 1, \bot\}$-sequent. $\vdash \Gamma$ is provable in InfFL$_{ew}[\Phi]$ if and only if $\vdash ?\Phi^-, \Gamma$ is provable in LLW.

Recall that the provability problems for ILZW and LLW are both in Tower by Theorem 7. We obtain:
Corollary 14. The following decision problems are in TOWER.

1. the deducibility problem for BCK,
2. the deducibility problem for FL_{el},
3. the deducibility problem for FL_{elI},
4. the deducibility problem for InFL_{lew}.

At the end of this section, we show the membership in TOWER of the deducibility problem for FL_{lew}. It suffices by Lemma 12 to show that the provability problem for ILZW' is in TOWER.

Let $F$ be an instance of the provability problem for ILZW'. As before, $S$ denotes the set of subformulas of $F$, $S_{\text{I}}$ the set of formulas in $S$ of the form $!B$, and $\bullet$ a distinguished symbol. We then construct an ABVASS$_0$, $A^F_0$ of dimension $|S|$, by modifying the construction of $A^F_0$ given in the previous subsection. The state space of $A^F_0$ contains $P(S_{\text{I}}) \times (S \cup \{\bullet\})$, a distinguished leaf state $q_e$, and intermediate states that are needed for defining the rules for $(\neg L), (\&L), (\oplus L), (\&L), \text{ and } (\top R)$, cf. Figure 4. The rules of $A^F_0$ are $\text{(init1), (init2), (store), (IL1), (IR1), (L1, L2), (R1, R2), (R1, R2), (R1, R2), (R1, R2), (R1, R2), (R1, R2), (R1, R2), (R1, R2), (R1, R2), (R1, R2), (R1, R2)}$, all of which except for $A^F_0$ are depicted in Figure 4. The rules for $(W'), (!D)$ and $(!P)$ are defined as in Figure 5. Clearly, these three types of ABVASS$_0$ rules correspond to the inference rules of $(W'), (!D)$ and $(!P)$ in ILZW', respectively. Note that $A^F_0$ is not equipped with the rule for $(\text{func})$. We can show that $A^F_0$ simulates the proof search of $F$ in ILZW' with the lossy semantics; see the full version for a proof.

Theorem 15. Let $\Pi$ be in $\Sigma \cup \{\varepsilon\}$, $\Theta$ a multiset of formulas in $S_{\text{I}}$, $\Gamma$ a multiset of formulas in $S \setminus S_{\text{I}}$. $\Theta, \Gamma \vdash \Pi$ is provable in ILZW' if and only if $A^F_0, \{q_e\} \vdash (\sigma(\Theta)), \Pi^\uparrow, \varphi_R$.

Specifically, $A^F_0, \{q_e\} \vdash (\sigma(\Theta)), \Pi^\uparrow, \varphi_R$ if and only if $\vdash F$ is provable in ILZW'; thus Lemma 12 and Theorem 5 provide:

Corollary 16. The provability problem for ILZW' is in TOWER.

Corollary 17. The deducibility problem for FL_{lew} is in TOWER.

6 Tower-hardness of contraction-free logics

For our purposes, we recall from [26, Section 4.2] a log-space reduction from the lossy reachability problem for ordinary BVASSs to the provability problem of $?-\text{prenex } \{\&, \top\}$-sequents in LLW.

Let $(B, Q_{\ell}, q_{\ell})$ be an instance of the lossy reachability problem for ordinary BVASSs, where $B = (Q, d, T_u, T_s, 0, 0)$. We fix a set $Q \cup \{e_i \mid 1 \leq i \leq d\}$ of propositional variables. Given $(q, \bar{v}) \in Q \times \mathbb{N}^d$, we define $\theta(q, \bar{v}) = q^{+\bar{v}}, (e_1^{\bar{v}(1)}, \ldots, e_d^{\bar{v}(d)})$, where $\bar{v}(i)$ stands for the $i$-th coordinate of $\bar{v}$. We write $T$ for the set of the three types of non-logical axioms, each of which is derived from $T_u \cup T_s$ as follows:

- $\vdash q^{+\bar{v}}, \forall_i e_i$ for $\vdash q^{+\bar{v}}, q_1 \in T_u$,
- $\vdash q^{+\bar{v}}, e_i^{+\bar{v}}, q_1$ for $\vdash q^{+\bar{v}}, q_1 \in T_u$,
\[ \vdash q^\perp, q \not\vdash q_1 + q_2 \] for \( q \rightarrow q_1 + q_2 \in T_s. \]

Each sequent in \( T \) is of the form \( \vdash q_1^\perp, \ldots, q_n^\perp, C \), where \( q_1^\perp, \ldots, q_n^\perp \) are negative literals and \( C \) is a classical \( \{\otimes, \exists\} \)-formula. For any sequent \( t = \vdash q_1^\perp, \ldots, q_n^\perp, C \) in \( T \), we define \( \langle t \rangle = q_1 \otimes \cdots \otimes q_n \otimes C^\perp \). Given a finite set \( T = \{t_1, \ldots, t_n\} \) of sequents, \( \langle T \rangle \) denotes the multiset \( \langle t_1 \rangle, \ldots, \langle t_n \rangle \). It then holds that, for any \( (q, \bar{v}) \in Q \times \mathbb{N}^d \), \( B, Q \vdash q_1 q \) if and only if \( \vdash \langle T \rangle, Q q \) is provable in \( LLW \). In particular, the following holds:

**Theorem 18** (Lazić and Schmitz [26], Section 4.2.3). \( B, Q \vdash q, \bar{v} \) if and only if \( \vdash \langle T \rangle, Q q \) is provable in \( LLW \).

The key observation here is that \( \vdash \langle T \rangle, Q q \) forms a \( \neg\)-sequent \( \{\otimes, \exists\} \)-sequent. Thus in conjunction with Theorem 6, we obtain:

**Corollary 19** (Lazić and Schmitz [26], Section 4.2.3). The problem of determining if a given \( \neg\)-sequent \( \{\otimes, \exists\} \)-sequent is provable in \( LLW \) is \( TOWER\)-hard.

### 6.1 Tower-hardness of deducibility in \( FLew \) and related systems

A \( \neg\)-sequent is an intuitionistic \( \{\neg, \not\} \)-sequent of the form \( \not\vdash \not\vdash \Gamma, \Delta \vdash C \) where the only connective occurring in \( \Gamma, \Delta, C \) is \( \neg \). We prove:

**Theorem 20.** The problem of determining if a given \( \neg\)-sequent \( \{\neg, \not\} \)-sequent is provable in the \( \{\neg, \not\} \)-fragment of \( ILLW \) is \( TOWER\)-hard.

**Proof.** In view of Corollary 19, we reduce from the problem of whether a given \( \not\)-sequent \( \{\otimes, \exists\} \)-sequent is provable in \( LLW \). Let \( \vdash \not\vdash \Gamma, \Delta \vdash A \) be a \( \not\)-sequent \( \{\otimes, \exists\} \)-sequent and \( x \) a new propositional variable not occurring in \( \not\vdash \not\vdash \Gamma, \Delta \vdash C \). Theorem 2 guarantees that \( \vdash \not\vdash \Gamma, \Delta \vdash A \) is provable in \( LLW \) if and only if \( \vdash \not\vdash \Gamma, \Delta \vdash [x] \gamma \vdash x \) is provable in \( ILLW \). Clearly, the latter sequent forms a \( \neg\)-sequent. Due to the fact that \( ILLW \) admits cut-elimination, \( ILLW \) is a conservative extension of its \( \{\neg, \not\} \)-fragment; hence \( \vdash \not\vdash \Gamma, \Delta \vdash [x] \gamma \vdash x \) is provable in \( ILLW \) if and only if it is provable in the \( \{\neg, \not\} \)-fragment of \( ILLW \). We conclude that \( \vdash \not\vdash \Gamma, \Delta \vdash A \) is provable in \( LLW \) if and only if \( \vdash \not\vdash \Gamma, \Delta \vdash [x] \gamma \vdash x \) is provable in the \( \{\neg, \not\} \)-fragment of \( ILLW \). Hence the problem is hard for \( TOWER \).

We furthermore show the following lemma:

**Lemma 21.** Let \( \not\vdash \Gamma, \Delta \vdash A \) be a \( \neg\)-sequent. The following statements are mutually equivalent:

1. \( \not\vdash \Gamma, \Delta \vdash A \) is provable in the \( \{\neg, \not\} \)-fragment of \( ILLW \),
2. \( \Delta \vdash A \) is provable in \( BCK[\sigma(\Gamma)] \),
3. \( \Delta \vdash A \) is provable in \( FLew[\sigma(\Gamma)] \),
4. \( \Delta \vdash A \) is provable in \( FLew[\sigma(\Gamma)] \).

Here \( \sigma(\Gamma) \) stands for the support of \( \Gamma \), i.e., the set of formulas that are contained in \( \Gamma \) at least once.

**Proof.** For starters, observe that the following claim holds:

(a) Let \( \not\vdash \Sigma \vdash C \) be a \( \neg\)-sequent. If \( \not\vdash \Sigma \vdash C \) is provable in the \( \{\neg, \not\} \)-fragment of \( ILLW \), then \( \not\vdash \Sigma^n \vdash C \) is provable in \( BCK \) for some \( n \).

Similarly to Lemma 10, this is shown by induction on the size of cut-free proofs in the \( \{\neg, \not\} \)-fragment of \( ILLW \). To show that Statement (1) implies Statement (2), suppose that \( \not\vdash \Gamma, \Delta \vdash A \) is provable in the \( \{\neg, \not\} \)-fragment of \( ILLW \). By Claim (a), \( \Gamma^n, \Delta \vdash A \) is provable
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in BCK for some n. For each formula B in Γ, B is also in σ(Γ). Hence we obtain a proof of Δ ⊢ A in BCK[σ(Γ)], applying (Cut) several times. The implications (2 ⇒ 3), (3 ⇒ 4) and (4 ⇒ 5) trivially hold.

We now embark on the proof of the remaining implication (5 ⇒ 1). By straightforward induction on the length of derivations, we can show:

(b) Let Ξ be a finite multiset of formulas in FLew and Σ ⊢ Π a sequent of FLew. If Σ ⊢ Π

is provable in FLew[σ(Ξ)], then Ξ, Σ ⊢ Π is provable in ILZW′.

Let us assume that Δ ⊢ A is provable in FLew[σ(Γ)]. By Claim (b), !Γ, Δ ⊢ A is provable in ILZW′. By the cut elimination theorem for ILZW′, we can easily check that ILZW′ is conservative over the \{-ο, !\}-fragment of ILLW. Hence !Γ, Δ ⊢ A is provable in the \{-ο, !\}-fragment of ILLW.

The above lemma provides a polynomial time reduction from the provability problem

of \{-prenex \{-ο\}-\}-sequents in the \{-ο, !\}-fragment of ILLW to deducibility in BCK, FLel+, FLel, and FLew; hence by Corollary 20 we obtain the TOWER-hardness of deducibility in these systems. In the same way as before, we can show:

► Lemma 22. Let \vdash \Gamma, \Delta be a ?-prenex \{-ο, \forall, \& & 1, \bot\}-sequent. \vdash \Gamma, \Delta is provable in LLW if and only if \vdash \Delta is provable in InFLew[σ(Γ⊥)].

By Corollary 19 and Lemma 22, the deducibility problem for InFLew is also hard for TOWER. The deducibility problems for BCK, FLel+, FLel, FLew, and InFLew are all in TOWER by Corollaries 14 and 17. We thus conclude:

► Corollary 23. Each of the following decision problems is complete for TOWER:

- the deducibility problem for BCK,
- the deducibility problem for FLel+,
- the deducibility problem for FLel,
- the deducibility problem for FLew,
- the deducibility problem for InFLew.

6.2 Tower-hardness of provability in elementary affine logic

► Lemma 24. Let Γ be a finite multiset of classical \{-ο, \forall, \& & 1, \bot\}-formulas and A a classical \{-ο, \forall, \& & 1, \bot\}-formula. \vdash \Gamma, A is provable in LLW if and only if \vdash \Gamma, !A is provable in ELLW.

Proof. Suppose that \vdash \Gamma, A is provable in LLW. By Lemma 11, \vdash Γ^n, A is provable in InFLew for some n. Obviously, \vdash Γ^n, A is provable in ELLW for some n. Using the rule of (F) and structural rules, we obtain a proof of \vdash \Gamma, !A in ELLW. For the other direction, assume that \vdash \Gamma, !A is provable in ELLW. It is provable in LLW since every sequent that is provable in ELLW is provable in LLW. Recall that \vdash \Gamma, A is provable in ELLW whereas it is not provable in ELLW. We therefore obtain a proof of \vdash \Gamma, A in LLW by a single application of (Cut).

It follows from Theorem 18 and Lemma 24 that B, Q, q, q, q is provable in ELLW. Here the sequent \vdash \Gamma, !Q, q is built from literals using only connectives in \{-ο, \forall, !, ?\}. We therefore obtain:

► Corollary 25. The problem of determining if a given classical \{-ο, \forall, !, ?\}-sequent is provable in ELLW is TOWER-hard, and hence so is the provability problem for ELLW.
Combining this with Corollary 9, we obtain:

**Corollary 26.** *The provability problem for ELLW is Tower-complete.*

Clearly, provability in the multiplicative-exponential fragment of ELLW is also complete for Tower. The same holds for a very small fragment of IELW:

**Theorem 27.** *The provability problem for the \{\text{	extordmasculine},!\}-fragment of IELW is Tower-complete.*

**Proof.** Similar to the proof of Theorem 20. The problem is clearly in Tower due to Corollary 9. Let \( \vdash \Gamma \) be a classical \{\text{	extordmasculine}, ?, !, ?\}-sequent and \( x \) a fresh variable not occurring in this sequent. By Theorem 2, \( \vdash \Gamma \) is provable in ELLW if and only if \( \Gamma[^x] \vdash x \) is provable in IELW. By cut elimination for IELW, we know that IELW is a conservative extension of its \{\text{	extordmasculine}, !\}-fragment. Thus \( \Gamma[^x] \vdash x \) is provable in IELW if and only if it is provable in the \{\text{	extordmasculine}, !\}-fragment of IELW. Consequently, \( \vdash \Gamma \) is provable in ELLW if and only if \( \Gamma[^x] \vdash x \) is provable in the \{\text{	extordmasculine}, !\}-fragment of IELW. Hence there is a polynomial time reduction from the problem of whether a given classical \{\text{	extordmasculine}, ?, !, ?\}-sequent is provable in ELLW to provability in the \{\text{	extordmasculine}, !\}-fragment of IELW. By Corollary 25, provability in the \{\text{	extordmasculine}, !\}-fragment of IELW is hard for Tower.

\( \blacksquare \)

### 7 Concluding remarks

We have shown the Tower-completeness of deducibility in some contraction-free substructural logics without modal operators. We hope that our work sheds new light on computational aspects of fuzzy logic. This is because FLew forms a theoretical basis for a wide range of fuzzy logics. In fact, one can construct from FLew various fuzzy logics (such as monoidal t-norm based logic, basic logic, weak nilpotent minimum logic, \L ukasiewicz logic and product logic) by adding some new axioms, cf. [13, 18]. Remarkably, the latter four of the above examples are shown to be coNP-complete with respect to both provability and deducibility; see [19] for a detailed survey. Together with those facts, our result suggests that there is a critical difference between FLew and its fuzzy extensions with respect to computational complexity.

We summarize the known complexity results in Tables 1 and 2. We adopt the notation of combinatory logic in Table 1:

- **BCI** = the implicational fragment of intuitionistic linear logic,
- **BCIW** = the extension of BCI by the rule of contraction,
- **BCK** = the extension of BCI by the rule of weakening.

BCIW is usually denoted by \( R \to \) in the relevance logic community. **IL\( \to \)** stands for the extension of BCI by weakening and contraction. It is nothing but the implicational fragment of intuitionistic propositional logic (IL). In Table 2, \( \text{FL}_{\text{ew}} \) (resp. \( \text{FL}_{\text{ewc}} \)) denotes the extension of BCI (resp. BCIW) by the connectives \( \otimes, \& , \oplus, 1, \bot \).

Below we comment on some results not covered in the main sections.

**Ackermannian complexity and deducibility in BCI.** In [35] Schmitz also defined a non-primitive recursive complexity class by:

\[
\text{ACKERMANN} := \bigcup_{f \in \text{FPR}} \text{DTIME}(\text{Ack}(f(n)))
\]
Table 1 Complexity results of extensions of $BCI$.

|                | Provability                  | Deducibility                     |
|----------------|-----------------------------|---------------------------------|
| $BCI$          | NP-complete [7]             | open (ACkERMANN-hard, cf. Corollary 31) |
| $BCIW$         | 2-ExpTime-complete [36]     | decidable (2-ExpTime-hard, cf. [36]) |
| $BCK$          | NP-complete (Corollary 3)   | TOWER-complete (Corollary 23)    |
| $IL_{\rightarrow}$ | PSPACE-complete [37]  | PSPACE-complete [37]            |

Table 2 Complexity results of extensions of $FL_{eq}$.

|                | Provability                  | Deducibility                     |
|----------------|-----------------------------|---------------------------------|
| $FL_{eq}$      | PSPACE-complete [28]        | undecidable [28]                |
| $FL_{eqc}$     | ACKERMANN-complete, cf. [26, 41] | ACKERMANN-complete, cf. [26, 41] |
| $FL_{eqi}$     | PSPACE-complete, cf. [21]   | TOWER-complete (Corollary 23)    |
| $FL_{ew}$      | PSPACE-complete, cf. [21]   | TOWER-complete (Corollary 23)    |
| $FL_{ewc}$ (= $IL$) | PSPACE-complete [37]  | PSPACE-complete [37]            |

where $Ack$ is the Ackermannian function and FPR denotes the set of primitive recursive functions. The class contains the class PR of primitive recursive problems, and is closed under primitive recursive reductions. As in [26, 35], we define the notion of ACKERMANN-hardness using primitive recursive reductions.

The decidability of deducibility in $BCI$ is a long-standing open problem. However, one reviewer pointed out that the ACKERMANN-hardness of deducibility in $BCI$ follows from the recent breakthrough by Czerwiński and Orlikowski [9], and Leroux [27], who independently proved that the reachability problem for VASSs is ACKERMANN-hard.

Let us sketch the proof of the Ackermannian lower bound for deducibility in $BCI$. Clearly, by the aforementioned result in [9, 27], reachability in BVASSs is also hard for ACKERMANN. As is the case of lossy reachability for ABVASSs, the BVASS-reachability problem is reduced to reachability in ordinary BVASSs (cf. Section 4); see [26, Lemma 3.5] for details. Furthermore, reachability in ordinary BVASSs can be efficiently encoded to the problem of whether classical linear logic (LL), i.e., LLW without the rule of (W), proves a given $?$-prenex $\{\otimes, \exists\}$-sequent; see [26, Section 4.2.2]. To sum up, the following holds:

- **Theorem 28** ([9, 26, 27]). The problem of determining if a given $?$-prenex $\{\otimes, \exists\}$-sequent is provable in LL is ACKERMANN-hard.

The translation in Section 2.2 also holds between LL and ILL (i.e., ILLW without the unrestricted left-weakening); see the full version for a sketch of a proof.

- **Theorem 29.** Let $\vdash \Gamma$ be a classical $L_C$-sequent and $x$ a fresh propositional variable not occurring in $\Gamma$. $\vdash \Gamma$ is provable in LL if and only if $\Gamma^{[2]} \vdash x$ is provable in ILL.

Using this, we show the intuitionistic version of Theorem 28.2

- **Corollary 30** ([9, 12, 27]). The problem of determining if a given $!$-prenex $\{\neg \to\}$-sequent is provable in ILL is ACKERMANN-hard.

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2 In [12] de Groote, Guillaume and Salvati provided a reduction from BVASS-reachability to provability of $!$-prenex $\{\neg \to\}$-sequents in ILL. Thus one can also show Corollaries 30 and 31, using the results in [9, 12, 27].
Proof. By Theorem 28, it suffices to show that there is a reduction from the provability of \(\exists\)-prenex \(\{\otimes, \exists\}\)-sequents in \(\mathbf{LL}\) to the provability of \(!\)-prenex \(\{-\}\)-sequents in \(\mathbf{ILL}\). The proof is essentially the same as that of Theorem 20, but we use Theorem 29 instead of Theorem 2. Let \(\vdash \Sigma, \Delta\) be a \(\exists\)-prenex \(\{\otimes, \exists\}\)-sequent and \(x\) a fresh propositional variable not in the sequent. By Theorem 29, \(\vdash \Sigma, \Delta\) is provable in \(\mathbf{LL}\) if and only if \((\exists \Sigma, \Delta)[x] \vdash x\) is provable in \(\mathbf{ILL}\). Obviously, the latter sequent is a \(!\)-prenex implicational sequent.

\(\triangleright\) Corollary 31. The deducibility problem for \(\mathbf{BCI}\) is Ackermann-hard.

Proof. In view of Corollary 30, our goal is to show that: for any \(!\)-prenex \(\{\neg\}\)-sequent \(\Sigma, \Delta \vdash C\), \(\Delta \vdash C\) is provable in \(\mathbf{BCI}[\sigma(\Gamma)]\) if and only if \(\forall \Sigma, \Delta \vdash C\) is provable in \(\mathbf{ILL}\).

The proof of the \(\text{if}\) part goes by induction on the length of the cut-free proof of \(\forall \Sigma, \Delta \vdash C\) in \(\mathbf{ILL}\). We only analyze the case where the rule of \((\forall D)\) is applied last. If \(\forall \Sigma, \forall A, \Delta \vdash C\) is obtained from \(\forall \Sigma, A, \Delta \vdash C\) by an application of \((\forall D)\), then by the induction hypothesis, \(A, \Delta \vdash C\) is provable in \(\mathbf{IFF}[\sigma(\Gamma)]\); thus it is also provable in \(\mathbf{ILL}[\sigma(\Gamma, A)]\). Since \(\vdash A\) is a non-logical axiom of \(\mathbf{ILL}[\sigma(\Gamma, A)]\), we may construct a derivation of \(\Delta \vdash C\) in \(\mathbf{ILL}[\sigma(\Gamma, A)]\) by using (Cut). The remaining cases are similar.

The converse direction is shown by induction on the length of the derivation for \(\Delta \vdash C\) in \(\mathbf{BCI}[\sigma(\Gamma)]\). All cases are straightforward.

Tower-hardness of light affine logic. We have also shown the Tower-completeness of provability in elementary affine logic. On the other hand, one can define another subsystem of affine logic, called intuitionistic light affine logic (\(\mathbf{ILAL}\)) \([1, 38]\), which characterizes polynomial time functions. We obtain it from the \(\{\neg, !\}\)-fragment of \(\mathbf{ILL}\) by dropping the rules for \((\forall D)\) and \((\exists P)\), and by adding a new unary connective \(\langle\rangle\) and the following inference rules:

\[
\begin{align*}
E \vdash A & \quad (\langle\rangle) \\
\forall E \vdash \forall A & \quad (\langle\rangle)
\end{align*}
\]

where \(E\) is a formula or the empty multiset. It is known that provability in \(\mathbf{ILAL}\) is decidable because Terui proved that it has the finite model property; see \([38, \text{Corollary 7.45}]\). We show that there is no elementary recursive algorithm for solving the provability problem for \(\mathbf{ILAL}\):

\(\triangleright\) Lemma 32. Let \(\Sigma, \Delta \vdash A\) be a \(\exists\)-prenex \(\{\neg\\}\)-sequent. \(\forall \Sigma, \Delta \vdash A\) is provable in the \(\{\neg, !\}\)-fragment of \(\mathbf{ILL}\) if and only if \(\forall \Sigma, \langle\rangle \Delta \vdash \langle\rangle A\) is provable in \(\mathbf{ILAL}\).

Proof. Our proof is inspired by the proof of the undecidability of provability in light linear logic by Terui; see \([38, \text{Section 2.3.2}]\). For any intuitionistic \(\{\neg, !, \langle\rangle\}\)-formula \(A\), we inductively define the intuitionistic \(\{\neg, !\}\)-formula \(A^{-}\) by \(p^{-} = p, (B \rightarrow C)^{-} = B^{-} \rightarrow C^{-}\), \((\langle\rangle B)^{-} = \langle\rangle B^{-}\), and \((\langle\rangle B)^{-} = \langle\rangle B^{-}\). It is easy to see that if \(\Sigma \vdash C\) is provable in \(\mathbf{ILAL}\) then \(\Sigma^{-} \vdash C^{-}\) is provable in the \(\{\neg, !\}\)-fragment of \(\mathbf{ILL}\), for any sequent \(\Sigma \vdash C\) of \(\mathbf{ILAL}\).

This is checked by induction on the size of proofs. Thus the \(\text{if}\) direction holds. The \(\text{only-if}\) direction follows from Claim (a) used in the proof of Lemma 21, the rule of \((\langle\rangle)\), and the structural rules.

By Theorem 20, the provability problem for \(\mathbf{ILAL}\) is hard for Tower. We strongly believe that this problem is in Tower.
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