Scaling cosmologies from duality twisted compactifications

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Abstract
Oscillating moduli fields can support a cosmological scaling solution in the presence of a perfect fluid when the scalar field potential satisfies appropriate conditions. We examine when such conditions arise in higher dimensional, nonlinear sigma models that are reduced to four dimensions under a generalized Scherk–Schwarz compactification. We show explicitly that scaling behaviour is possible when the higher dimensional action exhibits a global $SL(n, \mathbb{R})$ or $O(2, 2)$ symmetry. These underlying symmetries can be exploited to generate non-trivial scaling solutions when the moduli fields have non-canonical kinetic energy. We also consider the compactification of 11D vacuum Einstein gravity on an elliptic twisted torus.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Scaling solutions play a central role in cosmology [1–22]. They allow one to determine the asymptotic behaviour of various models and to establish whether such behaviour is stable or not. They also provide a framework for understanding the general dynamics of scalar fields in cosmological backgrounds. A scaling solution arises when the energy densities of the fields that contribute to the total energy density vary at the same rate with time. This implies that the field densities relative to the total density remain constant. For a spatially flat and isotropic Friedmann–Robertson–Walker (FRW) universe, this generates a power-law expansion (or contraction) of the scale factor.

A novel type of scaling behaviour in flat FRW cosmologies was recently uncovered by Karthauser, Saffin and Hindmarsh (KSH) [23]. These authors considered a multi-field scenario consisting of a perfect fluid with equation of state $\gamma \equiv (\rho + p)/\rho < 2$ and a set of canonical scalar fields interacting through a potential
\[
V = \frac{1}{2} c e^\phi \sum_a m_a^2 \chi_a^2 ,
\]

where \( c \) is a constant and \( m_a \) represent the masses of the moduli fields \( \chi_a \). When these fields oscillate about the potential minimum, there exists an approximate analytical solution where the perfect fluid and scalar fields maintain constant fractional energy densities given by

\[
\Omega_\gamma = 1 - \Omega_s \approx 1 - \frac{3(\gamma - 1)}{c^2} ,
\]

respectively [23].

Such behaviour may have important implications for resolving the moduli problem of the early universe in the sense that the fields will not necessarily dominate the energy density prior to the epoch of primordial nucleosynthesis if \( c \) is sufficiently small. More specifically, if the oscillating moduli were uncoupled to the \( \phi \)-field (i.e. if \( c = 0 \)), they would behave collectively as a pressureless fluid and would quickly come to dominate the cosmic dynamics [24]. On the other hand, massless moduli fields behave as a stiff perfect fluid and rapidly become subdominant to a fluid such as radiation with \( \gamma < 2 \). The coupling of the \( \phi \)-field exponentially suppresses the moduli masses in the limit \( c\phi \rightarrow -\infty \) in such a way that their energy density remains fixed relative to that of the background fluid.

It is therefore important to consider more general potentials inspired by unified field theory that support cosmological scaling behaviour driven by oscillating moduli fields. The purpose of the present paper is to investigate concrete examples that are derived under a generalized dimensional reduction scheme introduced by Scherk and Schwarz [25], which is referred to as ‘reduction with a duality twist’ [26]. In this scheme, a nonlinear sigma model with global duality symmetry \( G \) is reduced on a circle (or more generally a torus) with periodic coordinate \( y \sim y + 1 \) in such a way that the fields are ‘twisted’ over \( S^1 \) by an element of \( G \). The dimensionally reduced theory is independent of the compactification coordinate if the field dependence on \( y \) is of the form \( \Phi(x, y) = \exp(Cy)\phi(x) \), where \( C \) is a generator of \( G \) in an appropriate representation [26–29, 31].

The structure of the paper is as follows. In section 2, we review the derivation of the scaling solution and identify the necessary conditions that a multi-exponential potential must satisfy for such behaviour to arise. Section 3 develops a compactification scheme where a nonlinear sigma model is compactified on a circle with a \( G \) duality twist and then reduced to four dimensions by a standard Kaluza–Klein ansatz. We proceed in sections 4 and 5 to investigate nonlinear sigma models that exhibit global \( \text{SL}(2, R) \) and \( \text{O}(2, 2) \) symmetries, respectively, and demonstrate that such models can result in cosmological scaling. These symmetry groups are motivated by the compactification of pure Einstein gravity and the Neveu–Schwarz/Neveu–Schwarz (NS–NS) sector of the string effective action on a 2-torus.

We then discuss the \( \text{S}(n, R) \) model in section 6 and employ our results in section 7 to resolve an ambiguity of [23]. Finally, we conclude in section 8.

Units are chosen such that \( 16\pi G = 1 \).

2. Scaling cosmology with oscillating moduli fields

2.1. Approximate analytical scaling solution

The canonical scalar field equations for the system (1) take the form\(^1\)

\(^1\) The fields are described as canonical when they have standard kinetic terms in the action, i.e. when the metric on the target (field) space is trivial, \( \gamma_{ab} = \text{diag}(1, \ldots, 1) \). The fields are non-canonical when this condition is not satisfied.
\[ \ddot{\phi} + 3H \dot{\phi} + \frac{1}{2} c e^{c\phi} \sum_a m_a^2 \chi_a^2 = 0 \]  
(3)

\[ \ddot{\chi}_a + 3H \dot{\chi}_a + m_a^2 \chi_a e^{c\phi} = 0, \]
(4)

and these fields have a combined energy density
\[ \rho_s = \frac{1}{2} \dot{\phi}^2 + \sum_a \left( \ddot{\chi}_a^2 + m_a^2 \chi_a^2 e^{c\phi} \right). \]
(5)

It is assumed that the massive fields \( \chi_a \) are oscillating around the minimum of the potential with a period much shorter than the timescales characterized by \( H^{-1} \) or \( \dot{\phi}/\phi \). It is further assumed that the time average of these oscillations decays as \( \langle \chi_a^2 \rangle \propto t^{-2\sigma} \) for some constant \( \sigma \) and that the combined energy density in the scalar fields is tracking that of the perfect fluid. This implies that the universe expands as if it were sourced only by the perfect fluid, \( H = 2/(3\gamma t) \).

It can then be shown that an approximate solution to equations (3) and (4) is given by [23]
\[ \phi \simeq \frac{4(1 - \gamma)}{c\gamma} \ln t \]
(6)

\[ \chi_a \propto \text{Re} \left[ \int \frac{1}{t^{2\sigma}} \exp \left( -i \int t' \omega_a(t') \, dt' \right) \right], \]
(7)

where \( \omega_a \simeq m_a e^{c\phi/2}, \sigma = (2 - \gamma)/\gamma \) and a time average \( \langle \chi_a^2 \rangle = \omega_a^2 \langle \chi_a^2 \rangle \) has been employed to remove the effects of the oscillations. The time-averaged effective energy density of the fields therefore scales with cosmic time as
\[ \langle \rho_s \rangle \simeq \frac{8(\gamma - 1)}{c^2 \gamma^2} \frac{1}{t^2}. \]
(8)

and this implies that the fraction of the total energy density in the scalar fields is given by equation (2). Hence, the universe exhibits scaling behaviour where the energy densities of the fluid and scalar fields redshift at the same rate. It is important to emphasize that the ratio (2) is independent of the total number and masses of the oscillating fields. It is also worth remarking that the scaling solution only arises for \( \gamma > 1 \) and \( c^2 > 3(\gamma - 1) \). It is not clear how this solution would apply in a universe containing only pressureless matter.

### 2.2. Necessary conditions for scaling behaviour

Multi-exponential potentials typically arise in the dimensional reduction of supergravity theories. A suitable framework to consider, therefore, is given by a set of \( N \) canonically coupled scalar fields, \( \phi = (\phi_1, \ldots, \phi_N) \), which interact through \( M \) exponential terms such that
\[ V(\phi) = \sum_{a=1}^{M} \Lambda_a \exp(\vec{\alpha}_a \cdot \vec{\phi}), \]
(9)

where the constant vectors \( \vec{\alpha}_a \) parametrize the field couplings, \( \vec{\alpha}_a \cdot \vec{\phi} = \sum_{i=1}^{N} \alpha_{ai} \phi^i \), and the constants \( \Lambda_a \) may be positive or negative. We label the \( i \)th component of \( \vec{\alpha}_a \) by \( \alpha_{ai} \). The necessary conditions that the potential (9) must satisfy for the existence of the scaling solution (2) are that it be semi-positive definite, admit a Minkowski global vacuum and contain an overall exponential factor.
In general, the $\vec{\alpha}_a$ row vectors form an $M \times N$ matrix, whose rank $R$ determines the number of independent $\vec{\alpha}_a$. Models of the type (9) can be separated into two main classes, characterized by whether the $\vec{\alpha}_a$ vectors are linearly independent (type I) or linearly dependent (type II) [14]. We focus on the latter class of models in this subsection, since the potentials which arise in the following sections belong to this class. Furthermore, if $R < N$, one may perform an orthogonal rotation in field space to decouple $(N - R)$ of the fields from the potential. These degrees of freedom may then be consistently set to zero, and we assume that such a rotation and truncation have been performed in what follows.

A necessary condition for reducing the general type II potential (9) to the form (1) is that there should exist a rotation in field space, $\phi \rightarrow \vec{\phi}$, which transforms the potential into the separable form

$$V(\phi) = e^{c\phi_1} U(\phi_2, \ldots, \phi_N),$$

where $c$ is a constant and $U$ is a positive, semi-definite function. To establish when this is possible, we label the $R$ independent vectors $\vec{\alpha}_a$ with $a = 1, \ldots, R$ [15]. The remaining dependent vectors $\vec{\alpha}_b$ with $R + 1 \leq b \leq M$ can then be expressed as linear combinations of the $\vec{\alpha}_a$ vectors such that

$$\vec{\alpha}_b = \sum_{a=1}^{R} c_{ba} \vec{\alpha}_a,$$

where $c_{ba}$ are constant coefficients. Introducing a unit vector $\vec{n}$ that satisfies the condition $\vec{\alpha}_a \cdot \vec{n} = c$ for all $a \leq R$ and defining a new basis $\vec{\phi} = \phi_1 \vec{n} + \vec{\phi}_\perp$ then imply that $\alpha_{b1} = c$ in this basis for all $a \leq R$ [12, 14, 15]. It then follows that the 1-components of the $\vec{\alpha}_b$ vectors are given by

$$\alpha_{b1} = \sum_{a=1}^{R} c_{ba} \alpha_{a1}$$

and consequently, the potential (9) can be factorized into the separable form (10) if the coefficients $c_{ba}$ are related through the so-called ‘affine’ condition [12, 14, 15]:

$$\sum_{a=1}^{R} c_{ba} = 1 \quad \forall \quad b = R + 1, \ldots, M.$$  (12)

A further necessary condition for the potential (10) to reduce to equation (1) is that the function $U$ should admit a global minimum:

$$U = 0, \quad \partial_i U = 0, \quad \partial_i \partial_j U > 0$$

and depend quadratically on the fields $\phi_a$ for $a \geq 2$ in the neighbourhood of this critical point. Conditions (12) and (13) arise naturally in duality twisted compactifications of higher dimensional, nonlinear sigma models, and we focus on such compactifications in the following section.

3. Dimensional reduction with a duality twist

In the Kaluza–Klein compactification of a $(D + n + 1)$-dimensional supergravity theory to $(D + 1)$ dimensions, the full set of massless scalar (moduli) fields $\Phi^a$ typically parametrize a coset space $G/K$, where $K \subset G$ is the maximal compact subgroup of a non-compact Lie group $G$. (For a review, see, e.g., [32].) The graviton-moduli sector of the dimensionally reduced action is given by a nonlinear sigma model

$$S = \int d^{D+1}x \sqrt{|g|} \left[ \hat{R} + \frac{1}{4} \text{Tr}(\nabla \Phi^a \nabla \Phi^a) \right].$$

The covariant derivative on the $(D + 1)$-dimensional spacetime is denoted by $\nabla$ and the corresponding derivative on the $D$-dimensional spacetime is denoted by $\partial$. We distinguish $(D + 1)$-dimensional fields from $D$-dimensional ones by a hat. The coordinates of the $D$-dimensional spacetime are denoted by $x$ unless otherwise stated.
where the symmetric moduli matrix $\hat{M}(\hat{\Phi}) \in G$ determines the metric on the coset space, $\mathrm{d}s^2 = -\frac{1}{4} \text{Tr}(\mathrm{d}\hat{M}\mathrm{d}\hat{M}^{-1})$. Action (14) is invariant under the global $G$ symmetry transformation $\hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu}$ and $\hat{M} \to U^T \hat{M} U$, where $U \in G$ is a constant matrix.

The ansatz for a consistent dimensional reduction of action (14) on a circle with a $G$ duality twist is \[25–27, 30, 31, 33\]

\[\mathrm{d}s^2_{D+1} = e^{-2\alpha \psi} \mathrm{d}s^2_4 + e^{2(D-2)\alpha \psi} \mathrm{d}y^2\]  
\[\hat{M}(x, y) = \lambda^T(y) M(x) \lambda(y), \quad \lambda(y) \equiv \exp(Cy),\]  
\[\alpha \equiv \frac{1}{\sqrt{2(D-1)(D-2)}}.\]  

The matrix $C = \lambda^{-1} \nabla_y \lambda = - (\nabla_y \lambda^{-1}) \lambda$ is an element of the Lie algebra of $G$ and has dimensions of mass. The map $\lambda(y) = \exp(Cy)$ has a monodromy $C = \exp(C) \in G$, and the physically distinct reductions are characterized by the conjugacy classes of this monodromy \[34\]. The scalar kinetic term in equation (14) generates both a kinetic term and a scalar field potential in $D$ dimensions, and the resulting action is given by \[25–27, 31, 33\]

\[S = \int \mathrm{d}^Dx \sqrt{|g|} \left[ R - \frac{1}{2} (\partial \psi)^2 + \frac{1}{4} \text{Tr}(\partial M \partial M^{-1}) - V(\psi, \lambda) \right]\]  
\[V \equiv e^{-2(D-1)\alpha \psi} U(M), \quad U(M) \equiv \frac{1}{2} \text{Tr}(C^2 + C M^{-1} C^T M).\]  

When $D > 4$, we may perform a standard Kaluza–Klein dimensional reduction on an isotropic $(D-4)$-dimensional torus $T^{D-4}$ with periodic coordinates $z^a \sim z^a + 1$, where all the moduli fields on the torus are frozen with the exception of the breathing mode, $\sigma$, which parametrizes the volume of $T^{D-4}$. In this case, the metric ansatz is

\[\mathrm{d}s^2_{D} = e^{2\epsilon} \mathrm{d}s^2_4 + e^{2\omega} \delta_{ab} \mathrm{d}z^a \mathrm{d}z^b\]  
\[\epsilon \equiv -\frac{1}{2} \sqrt{\frac{D-4}{D-2}}, \quad \omega \equiv -\frac{2\epsilon}{D-4},\]  

and this results in the four-dimensional action

\[S = \int \mathrm{d}^4x \sqrt{|g|} \left[ R - \frac{1}{2} (\partial \psi)^2 - \frac{1}{2} (\partial \sigma)^2 + \frac{1}{4} \text{Tr}(\partial M \partial M^{-1}) - e^{-2(D-1)\alpha \psi} U(M) \right].\]  

An SO(2) rotation on the moduli fields $\{\psi, \sigma\}$ defined by

\[\varphi = -\frac{1}{\sqrt{3}} \left[ \sqrt{\frac{2(D-1)}{D-2}} \psi + \sqrt{\frac{D-4}{D-2}} \sigma \right]\]  
\[\gamma = \frac{1}{\sqrt{3}} \left[ \sqrt{\frac{D-4}{D-2}} \psi - \sqrt{\frac{2(D-1)}{D-2}} \sigma \right]\]  

then decouples the $\gamma$-field from the moduli potential (21). Consequently, the potential reduces to the separable form required for scaling cosmology:

\[V = \frac{e^{\sqrt{3} \psi}}{2} \text{Tr}(C^2 + C M^{-1} C^T M).\]
The question that now arises is whether the function $U(M)$ has a stable minimum at $U = 0$. To proceed, it proves convenient [26] to introduce the real vielbein matrix $V \in G$ defined by $M \equiv V^T Y V$ and the real, symmetric matrix $Y \equiv [\hat{C} + \hat{C}^T]$, where $\hat{C} \equiv VCV^{-1}$. The potential (23) can then be expressed as

$$V = \frac{e^{\sqrt{3} \phi}}{4} \text{Tr}(Y^2)$$ \hspace{1cm} (24)

and, since $Y$ is diagonalizable and has real eigenvalues, it follows that $\text{Tr}(Y^2)$ is the sum of the squares of these eigenvalues. Hence, the potential is semi-positive definite, $V \geq 0$, and can vanish at a point $\Phi_0$ in the moduli space if and only if $Y = 0$ [26]. This occurs when $\hat{C}$ is given by a rotation generator at that point, $\hat{C}_0 = -\hat{C}^T_0$, which implies that

$$M_0 C = -C^T M_0.$$ \hspace{1cm} (25)

The solution to equation (25) is given by $C = S_0^{-1} R S_0$ and $M_0 = S_0^T S_0$, where $S_0 \in G$ is a constant matrix and $R = -R^T$ is a generator of $O(n)$. In particular, when $C$ is itself an $n \times n$ antisymmetric matrix equation (25) is solved by $M_0 = I_n$, where $I_n$ is the identity matrix.

To summarize thus far, the potential (23) has a stable Minkowski minimum when equation (25) is satisfied. Moreover, the exponential coupling $c = \sqrt{3}$ of the $\phi$-field to the other moduli is independent of both the global symmetry group $G$ and the spacetime dimensionality $D$. However, in general the fields parametrized by the moduli matrix $M$ will not necessarily have a canonical kinetic energy term, as was assumed in the derivation of equation (2). Thus, we cannot yet conclude that scaling cosmologies in four dimensions will necessarily arise. On the other hand, if the canonical condition is satisfied, we arrive at the generic prediction that the scaling cosmologies will be characterized by the fractional energy densities

$$\Omega_\gamma = 2 - \gamma, \hspace{1cm} \Omega_\delta = \gamma - 1$$ \hspace{1cm} (26)

for the fluid and scalar fields, respectively.

Condition (25) implies that $\text{Tr} C = 0$, in which case $C$ belongs to the Lie algebra of $\text{SL}(n, R)$. In the following section, therefore, we investigate when scaling cosmologies may arise after compactification with a $G = \text{SL}(2, R)$ duality twist.

4. Scaling cosmologies from a $G = \text{SL}(2, R)$ duality twist

The Kaluza–Klein reduction of $(D + 3)$-dimensional pure gravity with an Einstein–Hilbert action on a 2-torus $T^2 \times \mathbb{R}^D$ with real periodic coordinates $z^a \sim z^a + 1$ and fixed volume results after truncation to the zero-mode sector in a $(D + 1)$-dimensional action of the form (14), where the scalar fields take values in the coset space $\text{SL}(2, R)/\text{SO}(2)$. This internal symmetry of the torus can be promoted to an external symmetry of the $(D + 1)$-dimensional theory, which can then be exploited to perform a further reduction onto a circle with an $\text{SL}(2, R)$ duality twist.

The $\text{SL}(2, R)/\text{SO}(2)$ nonlinear sigma model is parametrized by a complex ‘dilaton-axion’ field $\hat{\tau} \equiv \hat{\tau}_1 + i \hat{\tau}_2 = \hat{\chi} + i e^{-\phi}$, where the moduli matrix is given by

$$M_\hat{\tau} = \frac{1}{\hat{\tau}_2} \begin{pmatrix} 1 & \hat{\tau}_1 \\ \hat{\tau}_1 & |\hat{\tau}|^2 \end{pmatrix}. \hspace{1cm} (27)

The corresponding action (14) has the form

$$S = \int d^{D+1}x \sqrt{|g|} \left[ R - \frac{1}{2}(\nabla \phi)^2 - \frac{1}{2}e^{2\phi}(\nabla \hat{\chi})^2 \right]. \hspace{1cm} (28)

In general, a dimensional reduction with a duality twist is characterized by the conjugacy classes of the monodromy [34]. The group $\text{SL}(2, R)$ has three such classes and it can be
shown that the lower dimensional potential admits a stable Minkowski minimum only for the elliptic class represented by the monodromy matrix [26]

\[ C = \begin{pmatrix} \cos m & \sin m \\ -\sin m & \cos m \end{pmatrix}, \]

where \( m \) is a constant mass parameter and \( J \) is the generator of SO(2). In this case, the moduli potential (19) simplifies to [27, 35]

\[ U = \frac{m^2}{2} \text{Tr}(-I_2 + \mathcal{M}^2), \]

where the condition \( \mathcal{M}^{-1} = -JMJ \) has been employed.

The reduction of theory (28) on \( S^1 \) with an elliptic SL(2, R) duality twist, followed by the Kaluza–Klein toroidal compactification outlined in equations (20)–(22), therefore leads to the four-dimensional action

\[ S = \int d^4x \sqrt{\left| g \right|} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{2\phi} (\partial \chi)^2 - \frac{m^2}{2} e^{\sqrt{3}\phi} \sinh^2 \phi \right]. \]

The potential in equation (31) has a stable Minkowski minimum at \( \phi = \chi = 0 \) and, since the axion field \( \chi \) appears quadratically in the potential, it may consistently be set to zero. In this case, the potential simplifies to [3]

\[ V = 2m^2 e^{\sqrt{3}\phi} \sinh^2 \phi, \]

and equation (32) reduces to equation (1) in the limit where \( |\phi| \ll 1 \). Moreover, since the \( \phi \)- and \( \varphi \)-fields have canonical kinetic energy, we may conclude that they drive a scaling cosmology in the presence of a background fluid.

Figure 1 gives the results of a numerical calculation for the system (32) when a relativistic fluid is present. The full field equations are presented in appendix A and follow by specifying \( \chi = 0 \) in equations (A.1)–(A.5). The evolution of the fractional energy densities for the fluid and scalar fields is shown as a function of the number of e-foldings, \( N = \ln a \), in the left-hand figure. The dashed lines correspond to the values \( \Omega_\rho = 2/3 \) and \( \Omega_\chi = 1/3 \) that are predicted by the approximate analytic solution (6), and it is seen that the fractional densities asymptote to these constant values after a few e-foldings have elapsed. The right-hand figure illustrates the evolution of the respective fields.

The question that now arises is whether similar scaling behaviour is possible when the axion field \( \chi \) is non-trivial. The non-canonical coupling of this field to \( \phi \) may violate one or more of the necessary conditions that must be satisfied for cosmological scaling to proceed. We address this question by employing symmetry arguments. Although the kinetic sector of these fields is invariant under a general SL(2, R) transformation, the potential in the action (31) breaks this symmetry. On the other hand, the potential (30) is invariant under a global SO(2) symmetry. This implies that four-dimensional action (31) exhibits an SO(2) symmetry:

\[ \tilde{g}_{\mu\nu} = g_{\mu\nu}, \qquad \tilde{\phi} = \varphi, \qquad \tilde{\mathcal{M}} = \Sigma^T \mathcal{M} \Sigma, \]

3 The potential (32) can also be derived by compactifying vacuum Einstein gravity on the three-dimensional Bianchi-type VII\( \omega \) manifold corresponding to the group ISO(2) [11, 14, 31]. This follows because the elements of the mass matrix \( C \) in equation (29) correspond to the structure constants of the iso(2) algebra.
Figure 1. Illustrating the scaling solution for the four-dimensional cosmology derived from compactification of the SL(2, R) nonlinear sigma model with a duality twist in the elliptic conjugacy class. The axion field in action (31) has been set to zero and the background fluid has a relativistic equation of state, $\gamma = 4/3$. In the left-hand figure, the fractional energy densities of the fluid, $\Omega_{\gamma}$, and the scalar fields, $\Omega_s$, asymptote to the values predicted by the analytical solution (6). Initial conditions are chosen so that $\rho_{\gamma,i} = 0.15, \phi_i = 0.1$ and $\dot{\rho}_{\gamma,i} = \dot{\phi} = 0$ and the mass parameter $m = 1$. The right-hand figure shows the evolution of the fields with respect to the number of e-foldings, $N = \ln a$.

where

$$\Sigma \equiv \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

is an arbitrary, constant SO(2) matrix satisfying $\Sigma^T \Sigma = I_2$. The transformation (33) acts non-linearly on the scalar fields $\phi$ and $\chi$ such that

$$e^\tilde{\phi} = s^2 e^{-\phi} + (c + s \chi)^2 e^\phi$$
$$e^\tilde{\chi} e^\phi = cs e^{-\phi} - (s - c \chi)(c + s \chi) e^\phi,$$

where $c \equiv \cos \theta$ and $s \equiv \sin \theta$.

We may employ equation (35) to generate a cosmological solution involving a non-trivial axion field ($\tilde{\chi} \neq 0$) directly from a background where such a field is trivial ($\chi = 0$). Moreover, the spacetime metric is invariant under the action of equation (33). This implies that a perfect fluid source may be introduced into the system without breaking the symmetry of the field equations. Since the evolution of this fluid depends only on the cosmic scale factor, its energy density and pressure will remain invariant under a global SO(2) transformation. Hence, any scaling behaviour which arises in the absence of the axion field will also be possible when this field is dynamically non-trivial. In general, the evolution of the moduli fields $\tilde{\phi}$ and $\tilde{\chi}$ will be determined by equation (35), where $\chi = 0$ and the time dependence of $\phi$ is given by the analytic solution (6). Although the fractional energy densities of the individual fields will alter under the symmetry transformation (35), their combined fractional density will remain invariant and will be given by $\Omega_\gamma = \gamma - 1$.

Figure 2 shows a numerical integration of the spatially flat FRW field equations (A.1)–(A.5) derived from the full action (31) when a relativistic fluid is present. The solution asymptotes to a power-law expansion of the scale factor, $a \propto t^{1/2}$. Figure 3 shows the
Figure 2. Illustrating the scaling behaviour of the four-dimensional cosmology derived from the compactification of the $SL(2,\mathbb{R})$ nonlinear sigma model with a duality twist in the elliptic conjugacy class. The axion field in action (31) is non-trivial. The background fluid has a relativistic equation of state, $\gamma = 4/3$, and the initial conditions are the same as in figure 1 with $\chi = 0.1$ and $\chi = 0$. The product $Ht$, where $H$ denotes the Hubble parameter, tends to a constant, $Ht \to 1/2$, which implies that the scale factor grows as $a \propto t^{1/2}$.

Figure 3. Illustrating the evolution of the fractional energy densities of the fluid and moduli fields for the model shown in figure 2 where the axion field is non-trivial. The densities asymptote to the analytically predicted values $\Omega_\gamma = 2/3$ and $\Omega_\chi = 1/3$, respectively. The right-hand figure shows the evolution of the fields.

Thus far, we have considered the cosmological dynamics of a model derived from pure Einstein gravity in $(D + 3)$ dimensions compactified on a 2-torus. In the following section, we extend this analysis to consider the nonlinear sigma model that is motivated by the compactification of the NS–NS sector of the string effective action on $T^2$.

5. Scaling cosmologies from a $G = O(2,2)$ duality twist

The NS–NS sector of the ten-dimensional string effective action consists of the graviton, the dilaton and an antisymmetric two-form potential, $B$. After toroidal compactification on $T^2$, the resulting massless scalar fields parametrize the $O(2, 2)/O(2) \times O(2)$ coset when the value of the dilaton field is fixed [36]. In general, the group $O(n, n)$ is the non-compact,
pseudo-orthogonal group in $2n$ dimensions (see, e.g., [37]). Its representation is given by matrices $U$ that preserve the bilinear form $\eta$:

$$U^T \eta U = \eta, \quad \eta = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$  

(36)

where $I_n$ is the $n \times n$ identity matrix. Since $\eta^2 = I_{2n}$, the inverse of $U$ is given linearly by $U^{-1} = \eta U^T \eta$.

The group $G = O(2, 2)$ is isomorphic to $\text{SL}(2, R) \times \text{SL}(2, R)$ and is related to the $T$-duality group of the 2-torus [37]. This implies that the four degrees of freedom may be arranged into two complex coordinates defined by [38]

$$\tau = \tau_1 + i \tau_2 = \frac{\Gamma_{mn}}{\sqrt{\gamma}} + i \frac{\sqrt{\gamma}}{\sqrt{\gamma}} \rho \equiv \rho_1 + i \rho_2 = B_{mn} + i \sqrt{\gamma},$$

(37)

where the metric on $T^2$ is denoted by $\gamma_{ab} = \Gamma_{ab} dz^a d\bar{z}^b$ and has determinant $\gamma = \det \Gamma_{ab}$.

The corresponding nonlinear sigma model is given by equation (14), where the moduli matrix takes the form [37]

$$\mathcal{M} = \frac{1}{\tau_2 \rho_2} \begin{pmatrix} 1 & -\tau_1 & -\tau_1 \rho_1 & -\rho_1 \\ -\tau_1 & |\tau|^2 & \rho_1 |\tau|^2 & \tau_1 \rho_1 \\ -\tau_1 \rho_1 & \rho_1 |\tau|^2 & |\tau|^2 |\rho|^2 & \tau_1 |\rho|^2 \\ -\rho_1 & \tau_1 \rho_1 & \tau_1 |\rho|^2 & |\rho|^2 \end{pmatrix}.$$  

(38)

An element $C$ of the Lie algebra of $O(n, n)$ satisfies the constraint

$$\eta C + C^T \eta = 0.$$  

(39)

We are free to choose any form for $C$ when reducing the theory on a circle with a duality twist as long as it belongs to this Lie algebra. We specify

$$C = \begin{pmatrix} m_1 J & m_3 J \\ m_3 J & m_1 J \end{pmatrix},$$  

(40)

where $m_{1,3}$ are constant mass parameters and $J$ is the $\text{SL}(2, R)$ metric defined in equation (29) and satisfying $J^2 = -I_2$. After a twisted reduction with this mass matrix, the moduli potential takes the form

$$U = -2(m_1^2 + m_3^2) - \frac{1}{2} \text{Tr}(C \eta M \eta C \mathcal{M})$$  

(41)

and, since $C = -C^T$, this potential admits a global Minkowski minimum at $\mathcal{M}_0 = I_4$.

The $\text{SL}(2, R)$ subgroups of $O(2, 2)$ can be made more transparent by defining the $\text{SL}(2, R)$ matrices:

$$S = \frac{1}{\tau_2} \begin{pmatrix} 1 & -\tau_1 \\ -\tau_1 & |\tau|^2 \end{pmatrix}, \quad P = \frac{1}{\rho_2} \begin{pmatrix} 1 & -\rho_1 \\ -\rho_1 & |\rho|^2 \end{pmatrix}.$$  

(42)

The moduli matrix (38) can then be written in the block form

$$\mathcal{M} = \frac{1}{\rho_2} \begin{pmatrix} S & -\rho_1 SJ \\ \rho_1 JS & |\rho|^2 S^{-1} \end{pmatrix},$$  

(43)

and this implies that the moduli potential (41) simplifies to

$$U = m_1^2 \text{Tr}(-I_2 + S^2) + m_3^2 \text{Tr}(-I_2 + P^2).$$  

(44)

The potential (44) is manifestly invariant under two global $\text{SO}(2)$ transformations. The $O(2, 2)$ matrix that generates such transformations is given by [38]

$$\Omega \equiv \Omega_{\tau} \Omega_{\rho} = \begin{pmatrix} c_1 \Sigma^T & s_1 \Sigma^T J \\ s_1 \Sigma^T J & c_1 \Sigma^T \end{pmatrix},$$  

(45)
where
\[ \Omega_\mu \equiv \begin{pmatrix} c_1 I_2 & s_1 J \\ s_1 J & c_1 I_2 \end{pmatrix}, \quad \Omega_\tau \equiv \begin{pmatrix} \Sigma^T & 0 \\ 0 & \Sigma^T \end{pmatrix}, \]
and 
\[ \Sigma \equiv \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}, \]
and \( c_i \equiv \cos \theta_i, \) \( s_i \equiv \sin \theta_i. \) Matrix (45) satisfies the conditions \( \Omega_\mu \Omega_\tau^T = I_4 \) and \( \Omega_\tau^T \eta \Omega_\mu = \eta \) and these conditions imply that \( \eta \Omega = \Omega \eta. \) Moreover, it may be verified that \( C \Omega = \Omega C, \) which implies that \( C \eta = \Omega C \eta \Omega^T. \) Hence, the moduli potential (41) is invariant under the global symmetry transformation
\[ \mathcal{M} = \Omega^T \Omega \]
and the dimensionally reduced action will respect this symmetry, where the metric transforms as a singlet.

We may now employ the arguments of section 4 to deduce that the potential (41) admits cosmological scaling behaviour in four dimensions. Equation (44) implies that the real (axionic) components of the complex moduli (37) appear quadratically in the potential, and these degrees of freedom may therefore be set to zero. In this case, the potential reduces to
\[ V = 4 e^{\sqrt{3} \phi} \sum_{j=1,3} m_j^2 \sinh^2 \phi_j, \]
where we have labelled the canonical fields \( e^{-\phi_1} = \sqrt{\Gamma} / \Gamma_{11} \) and \( e^{-\phi_2} = \sqrt{\Gamma}, \) respectively. Equation (48) reduces to equation (1) in the limit \( |\phi_j| \ll 1 \) and the cosmological scaling solution (2) will therefore arise in the presence of a background fluid, since this behaviour is independent of the mass parameters \( m_j \) and the number of moduli fields \( \phi_j. \) Moreover, the symmetry transformation (47) may be used to generate non-trivial, real components for the complex moduli fields (37). Specifically, the matrix \( \Omega_\mu \) generates the transformation
\[ \tilde{\rho} = \frac{s_1 + c_1 \rho}{c_1 - s_1 \rho}, \quad \tilde{\tau} = \tau \]
that leaves the complex scalar field \( \tau \) invariant and the matrix \( \Omega_\tau \) generates the transformation
\[ \tilde{\tau} = \frac{c_2 \tau - s_2}{s_2 \tau + c_2}, \quad \tilde{\rho} = \rho \]
that leaves \( \rho \) invariant. Since the four-dimensional metric is invariant under these transformations, the scaling behaviour (2) will be preserved when the axion fields evolve non-trivially.

In the following section, we extend our discussion to consider models which exhibit a global SL\((n, R)\) symmetry.

6. Scaling cosmologies from a \( G = \text{SL}(n, R) \) duality twist

The SL\((n, R)\) nonlinear sigma model is motivated by the Kaluza–Klein compactification of pure Einstein gravity on an \( n \)-dimensional torus\(^4\). The dilatons which arise in such a reduction parametrize a geodesically complete submanifold in the coset space and this allows for a consistent truncation where the axion fields are frozen. We therefore consider the maximal Abelian subgroup of SL\((n, R)\), where the moduli matrix \( \mathcal{M} \) takes a diagonal form
\[ \mathcal{M} = \text{diag}(e^{-\tilde{\phi}_a}), \quad a = 1, \ldots, n, \]
\(^4\) The case \( n = 4 \) is of particular interest, since SL\((4, R)\) is isomorphic to SO\((3, 3)\). This latter group arises as the global symmetry of the dimensionally reduced NS–NS string effective action on \( T^3 \).
and the vectors $\vec{\beta}_a$ denote the weights of the $\text{SL}(n, R)$-algebra and satisfy the conditions
\[
\sum_a \beta_{ai} = 0
\]
\[
\sum_a \beta_{ai} \beta_{aj} = 2\delta_{ij}
\]
\[
\vec{\beta}_a \cdot \vec{\beta}_b = 2\delta_{ab} - \frac{2}{n}
\]
in the fundamental representation. It follows immediately from equation (52) that the kinetic sector of the $(n - 1)$ moduli fields takes the canonical form:
\[
\text{Tr} (\nabla \hat{M} \nabla \hat{M}^{-1}) = -2\delta_{ab} \nabla \hat{\phi}^a \nabla \hat{\phi}^b.
\]
The matrix $C$ should belong to the Lie algebra of $\text{SL}(n, R)$, i.e. it should be a real, traceless, $n \times n$ matrix. We consider the case where $C$ is given by
\[
C = \begin{pmatrix}
0 & m_1 & 0 & 0 & 0 \\
-m_1 & 0 & 0 & 0 & 0 \\
0 & 0 & m_3 & 0 & 0 \\
0 & 0 & 0 & 0 & m_5 \\
0 & 0 & 0 & 0 & -m_5
\end{pmatrix}
\]
and $m_a$ are constants. We reduce the $(D + 1)$-dimensional $\text{SL}(n, R)/\text{SO}(n)$ nonlinear sigma model on a circle with a duality twist by employing equations (51) and (54). When $n$ is even, we find that the action reduces to that of equation (18), where the potential (19) is given by
\[
V = 2 e^{-2(D-1)\alpha \phi} \sum_j m_j^2 \sinh^2 \left[ \frac{1}{2} (\vec{\beta}_{j+1} - \vec{\beta}_j) \cdot \vec{\phi} \right]
\]
and the sum is over $j = 1, 3, \ldots, (n - 1)$.

Let us consider the case $n = 6$ as a concrete example. A convenient basis for the weights of $\text{SL}(6, R)$ which satisfies the constraints (52) is given by
\[
\vec{\beta}_1 = \left(1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}\right), \quad \vec{\beta}_2 = \left(-1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}\right)
\]
\[
\vec{\beta}_3 = \left(0, -\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}\right), \quad \vec{\beta}_4 = \left(0, 0, -\frac{3}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}\right)
\]
\[
\vec{\beta}_5 = \left(0, 0, 0, -\frac{4}{\sqrt{10}}, \frac{1}{\sqrt{15}}\right), \quad \vec{\beta}_6 = \left(0, 0, 0, 0, -\frac{5}{\sqrt{15}}\right)
\]
and it follows from equation (56) that
\[
\vec{\beta}_2 - \vec{\beta}_1 = (-2, 0, 0, 0, 0), \quad \vec{\beta}_4 - \vec{\beta}_3 = \left(0, \frac{2}{\sqrt{3}}, -\frac{4}{\sqrt{6}}, 0, 0\right)
\]
\[
\vec{\beta}_6 - \vec{\beta}_5 = \left(0, 0, 0, \frac{4}{\sqrt{10}}, -\frac{6}{\sqrt{15}}\right).
\]
Consequently, the two $\text{SO}(2)$ field redefinitions
\[
\psi_3 = \frac{1}{2} \left(\frac{2}{\sqrt{3}} \phi_2 - \frac{4}{\sqrt{6}} \phi_3\right), \quad \gamma_3 = \frac{1}{2} \left(\frac{4}{\sqrt{6}} \phi_2 + \frac{2}{\sqrt{3}} \phi_3\right)
\]
\[
\psi_5 = \frac{1}{2} \left(\frac{4}{\sqrt{10}} \phi_4 - \frac{6}{\sqrt{15}} \phi_5\right), \quad \gamma_5 = \frac{1}{2} \left(\frac{6}{\sqrt{15}} \phi_4 + \frac{4}{\sqrt{10}} \phi_5\right)
\]
decouple the scalar degrees of freedom $\gamma_1$ and $\gamma_3$ from the potential (55). These fields may then be set to zero. A further toroidal compactification to four dimensions, as outlined in section 3, then results in an action consisting of four canonical fields which interact through the potential

$$V = 2e^{\phi_1} \sum_{j=1,3,5} m_j^2 \sinh^2 \psi_j,$$  \hspace{1cm} (59)

where we have labelled $\phi_1 = -\psi_1$.

The potential (59) is of the general form (9). It belongs to the type II class of models since the seven $\vec{\alpha}_a$ vectors are linearly dependent and satisfy the affine relation (12). Moreover, the rank of the $7 \times 4$ matrix formed from $\vec{\alpha}_a$ is 4, which implies that no further decoupling of the fields from the potential can be made. Since the potential (59) is in the separable form (10) and reduces to equation (1) in the small $\psi_j$ limit, we may conclude immediately that this type of twisted compactification admits the cosmological scaling solution (2).

It can be verified that since the weights of SL$(n, R)$ satisfy the constraints (52), similar potentials of the form (59) arise for all even values of $n$ and differ only by the number of hyperbolic terms in the summation. Scaling behaviour will therefore be generated in these cases, since such behaviour is independent of the number of oscillating fields and the magnitude of the mass parameters $m_j$.

Likewise, the scaling cosmologies are preserved in the presence of the axionic degrees of freedom. In general, when the mass matrix belonging to the Lie algebra of $G$ is antisymmetric, the monodromy $C = \exp(C)$ is an element of the rotation group O($n$) and therefore satisfies $CC^T = I_n$. This implies that a Minkowski minimum in the potential located at $M_0 = I_n$ in the moduli space is a fixed point under the action of the monodromy group, since $M_0 \rightarrow C^T M_0 C = C^T C = I_n$ [26]. Consequently, the reduced theory will be symmetric under the action of an SO($n$) transformation if $G' = SO(n)$ is a subgroup of $G$. In the case where $G = SL(n, R)$, such a transformation may be employed to generate a cosmological background with non-trivial axion fields, as was demonstrated explicitly in section 4 when $n = 2$. More generally, the moduli potential (19) is invariant under $M \rightarrow \Omega^T M \Omega$ if the mass matrix transforms as $\sigma \rightarrow \Omega^T \sigma \Omega$, where $\Omega \Omega^T = I_n$. In this case, $\Omega^T C \Omega$ belongs to the Lie algebra of SL$(n, R)$ if $\text{Tr} \, C = 0$.

In the following section, we use the above results to resolve an apparent ambiguity that arose in the work of [23].

7. Compactification on an elliptic twisted torus

When the duality symmetry has a geometrical origin, as is the case when $G = SL(n, R)$, a reduction with a duality twist is equivalent to compactification on a twisted torus [28]. In this latter scheme, the dependence of the fields on the internal coordinates $y'$ arises through a matrix $\sigma^m_i(y)$ with inverse $\sigma^n_i(y)$ [25]. The compactification is consistent if the matrices $\sigma^m_i(y)$ satisfy the constraint $f^m_{np} = -\sigma^m_i(y) (\delta^m_i \sigma^p_j - \delta^p_j \sigma^m_i)$, where $f^m_{np}$ are the structure constants for a Lie group $G$. The four-dimensional potential is then determined by these structure constants and the metric on the internal manifold.

The elements of the mass matrix (54) represent the structure constants of the elliptic twisted torus employed by KSH in the compactification of 11D pure Einstein gravity [23]. This manifold is related to the Lie group ISO$(n)$, corresponding to the group of isometries of $n$-dimensional Euclidean space. In the case where the internal manifold has a diagonal metric,
the potential in the twisted torus compactification was found to have the form [23]
\[ V(\phi_a) = e^{-3\sqrt{3}\psi} m_1^2 e^{-3\sqrt{3}\psi + \frac{3}{2} \psi_1 + \frac{3}{2} \psi_2} (e^{\psi_2} - e^{\sqrt{3}\phi})^2 + m_2^2 e^{\psi_1 + \frac{3}{2} \psi_1 + \frac{3}{2} \psi_2} (e^{\sqrt{3}\psi_1} - e^{\sqrt{3}\phi})^2 + m_3^2 e^{\psi_2 + \frac{3}{2} \psi_1 + \frac{3}{2} \psi_2 - 2\sqrt{3}\psi} (e^{\sqrt{3}\psi_2} - e^{\sqrt{3}\phi})^2. \] (60)

where the seven moduli fields \( \phi_a \) are related to the components of the internal metric and are canonically coupled to Einstein gravity. The field \( \phi_1 \) parametrizes the volume of the internal space.

The numerical results of [23] indicate that at late times the system oscillates around the vacuum \( \psi_0 = \sqrt{3}\psi_1, \psi_2 = \sqrt{3}/3\psi_5 \) and \( \psi_5 = \sqrt{7}/5\psi_7 \), where the modulus field \( \psi_i \) increases monotonically with time. This would imply that the potential should reduce to equation (1) in the neighbourhood of this critical point, where \( c = -3/\sqrt{7} \). Consequently, the scalar fields should contribute a fraction \( \Omega_\tau \simeq 7(y - 1)/3 \) of the total energy density during the scaling regime. However, although the numerical simulations of [23] indeed confirmed the existence of scaling behaviour, it was found that \( \Omega_\tau \simeq 1/3 \) in the presence of a radiation fluid, which corresponds to \( c = \sqrt{3} \). The results from section 6 imply that this is indeed the correct value for the coupling parameter.

To gain further insight, we rewrite the potential (60) in the more compact form
\[ V = \sum_{j=1}^{6} m_j^2 e^{\tilde{\alpha}_j \cdot \tilde{\psi}} - 2M^2 e^{\tilde{\alpha}_7 \cdot \tilde{\psi}}, \] (61)

where \( \tilde{\psi} = (\psi_1, \ldots, \psi_7) \), the constants \( m_1 = m_2, m_3 = m_4, m_5 = m_6 \) and \( M^2 = m_1^2 + m_2^2 + m_3^2 \). The constant vectors \( \tilde{\alpha}_j \) parametrizing the field couplings are given by
\[
\tilde{\alpha}_1 = \left(-\frac{3}{\sqrt{7}}, 2, -\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}\right), \\
\tilde{\alpha}_2 = \left(-\frac{3}{\sqrt{7}}, 0, \frac{4}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}\right), \\
\tilde{\alpha}_3 = \left(-\frac{3}{\sqrt{7}}, 1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}\right), \\
\tilde{\alpha}_4 = \left(-\frac{3}{\sqrt{7}}, 1, \frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}, \frac{18}{5}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}\right), \\
\tilde{\alpha}_5 = \left(-\frac{3}{\sqrt{7}}, 1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, -\frac{6}{\sqrt{21}}\right), \\
\tilde{\alpha}_6 = \left(-\frac{3}{\sqrt{7}}, 1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, -\frac{4}{\sqrt{15}}, \frac{8}{\sqrt{21}}\right), \\
\tilde{\alpha}_7 = \left(-\frac{3}{\sqrt{7}}, 1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}\right).
\] (62)

and satisfy the affine relations
\[ \tilde{\alpha}_j + \tilde{\alpha}_{j+1} = 2\tilde{\alpha}_7, \quad j = 1, 3, 5. \] (63)
Hence, the potential (60) is of the type II discussed in section 2.

The 7 \( \times \) 7 matrix formed from the \( \tilde{\alpha}_j \) vectors has rank \( R = 4 \), so three of the scalar fields can be decoupled from the potential. We now define new fields such that \( \tilde{\psi} = \phi \tilde{n} + \tilde{\psi}_\perp \), where \( \tilde{n} \) is a unit vector, \( \phi \) is a scalar function and \( \tilde{\psi}_\perp \) is perpendicular to \( \tilde{n} \), i.e. \( \tilde{n} \cdot \tilde{\psi}_\perp = 0 \) [21, 39].
The projections of the vectors $\vec{a}_j$ onto $\vec{n}$ must satisfy the constraint $\vec{a}_j \cdot \vec{n} = c$ for all $j$, where $c$ is a constant. The value of $c$ may be deduced by considering the symmetric matrix

$$A_{ij} \equiv \vec{a}_i \cdot \vec{a}_j = \begin{pmatrix} 7 & -1 & 3 & 3 & 3 & 3 \\ -1 & 7 & 3 & 3 & 3 & 3 \\ 3 & 3 & 7 & -1 & 3 & 3 \\ 3 & 3 & 3 & 7 & -1 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{pmatrix}. \quad (64)$$

Since $\vec{a}_j \cdot \vec{a}_7 = 3$ for all $j = 1, \ldots, 7$, we may specify $\vec{n} = \vec{a}_7/\sqrt{3}$ and $c = \sqrt{3}$.

The equations of motion for the $\vec{\varphi}$-fields are given by

$$\partial^2 \vec{\varphi} - \sum_{j=1}^6 m_j^2 \vec{a}_j e^{\vec{a}_j \cdot \vec{\varphi}} + 2M^2 \vec{a}_7 e^{\vec{a}_7 \cdot \vec{\varphi}} = 0. \quad (65)$$

Taking the dot product of equation (65) with respect to the unit vector $\vec{n}$ yields

$$\partial^2 \varphi - c e^{c\varphi} \left[ \sum_{j=1}^6 m_j^2 e^{\vec{a}_j \cdot \vec{\varphi}_\perp} - 2M^2 e^{\vec{a}_7 \cdot \vec{\varphi}_\perp} \right] = 0, \quad (66)$$

and it follows after substitution of equation (66) into the field equations (65) that

$$\partial^2 \vec{\varphi}_\perp - e^{c\varphi} \sum_{j=1}^6 m_j^2 (\vec{a}_j - \vec{a}_7) e^{\vec{a}_j \cdot \vec{\varphi}_\perp} = 0. \quad (67)$$

Employing equation (62) for each of the components $\varphi_{\perp i}$ of $\vec{\varphi}_\perp$ in equation (67) then implies that

$$\partial^2 \left( \varphi_{\perp 2} + \frac{\varphi_{\perp 3}}{\sqrt{3}} \right) = 0, \quad \partial^2 \left( \varphi_{\perp 4} + \frac{\varphi_{\perp 5}}{\sqrt{3}} \right) = 0, \quad \partial^2 \left( \frac{\varphi_{\perp 6}}{\sqrt{3}} + \frac{\varphi_{\perp 7}}{\sqrt{7}} \right) = 0. \quad (68)$$

These linear combinations of $\varphi_{\perp i}$ represent the three degrees of freedom that decouple from the potential.

The field equations for three of the remaining four interacting fields can be derived by considering the dot product $\vec{a}_i \cdot \vec{\varphi}_\perp$ with $i = 1, \ldots, 6$. Only three of the $\vec{a}_i \cdot \vec{\varphi}_\perp$ fields are independent, since the orthogonality constraint $\vec{a}_7 \cdot \vec{\varphi}_\perp = 0$ and the affine conditions (63) together imply that

$$\vec{a}_i \cdot \vec{\varphi}_\perp = -\vec{a}_{i+1} \cdot \vec{\varphi}_\perp, \quad i = 1, 3, 5. \quad (69)$$

Hence, taking the dot product of equation (67) with $\vec{a}_j$ results in the equations of motion:

$$\partial^2 (\vec{a}_i \cdot \vec{\varphi}_\perp) - e^{c\varphi} \sum_{j=1}^6 m_j^2 (\vec{a}_i \cdot \vec{a}_j - 3) e^{\vec{a}_j \cdot \vec{\varphi}_\perp} = 0 \quad (70)$$

and it follows from matrix (64) and constraints (69) that

$$\partial^2 (\vec{a}_i \cdot \vec{\varphi}_\perp) - 8m_i^2 e^{c\varphi} \sinh^2 (\vec{a}_i \cdot \vec{\varphi}_\perp) = 0. \quad (71)$$

Finally, the fourth interacting scalar is the function $\varphi$ and its field equation (66) reduces to

$$\partial^2 \varphi = 4c e^{c\varphi} \sum_{j=1,3,5} m_j^2 \sinh^2 \left( \frac{\vec{a}_j \cdot \vec{\varphi}_\perp}{2} \right) = 0. \quad (72)$$
The question that now arises is whether the $\vec{\alpha}_j \cdot \vec{\phi}_\perp$ fields are canonical. In general, the field rotation introduces off-diagonal terms into the metric of the field space. On the other hand, a consistent solution to equation (68) gives

$$\varphi_{\perp 2} = -\frac{1}{\sqrt{3}} \varphi_{\perp 3}, \hspace{1cm} \varphi_{\perp 4} = -\sqrt{\frac{2}{5}} \varphi_{\perp 5}, \hspace{1cm} \varphi_{\perp 6} = -\sqrt{\frac{2}{7}} \varphi_{\perp 7} \quad (73)$$

and, since equation (68) and the constraint $\vec{a}_7 \cdot \vec{\phi}_\perp = 0$ imply that $\varphi_{\perp 1} = 0$, equation (73) is equivalent to

$$\vec{a}_1 \cdot \vec{\phi}_\perp = 4 \varphi_{\perp 2}, \hspace{1cm} \vec{a}_3 \cdot \vec{\phi}_\perp = \frac{8}{\sqrt{6}} \varphi_{\perp 4}, \hspace{1cm} \vec{a}_5 \cdot \vec{\phi}_\perp = \frac{12}{\sqrt{15}} \varphi_{\perp 6}. \quad (74)$$

This implies that

$$\partial \vec{\phi} \cdot \partial \vec{\phi} = (\partial \varphi)^2 + \frac{1}{4} \sum_{j=1,3,5} \left[ \partial \left( \vec{a}_j \cdot \vec{\phi}_\perp \right) \right]^2 \quad (75)$$

and we conclude, therefore, that the field equations (71) and (72) can be derived from the effective Lagrangian

$$L = -\frac{1}{2} \partial \vec{\Psi} \cdot \partial \vec{\Psi} - 4 e^{c\varphi} \sum_{j=1,3,5} m_j^2 \sinh^2 \psi_j, \quad (76)$$

where $\vec{\Psi} = (\varphi, \psi_1, \psi_3, \psi_5)$, $\psi_j = \frac{1}{\sqrt{\rho}} \vec{a}_j \cdot \vec{\phi}_\perp$, and $c = \sqrt{3}$. The potential in equation (76) is precisely of the form given in equation (59), confirming that the radiation scaling solution is indeed characterized by $\Omega_r = 1/3$.

8. Conclusion

In this paper we have investigated the cosmological consequences of the Scherk–Schwarz compactification of higher dimensional nonlinear sigma models. If the $(D + 1)$-dimensional theory exhibits a global symmetry $G$, reduction on a circle with a $G$ duality twist, followed by a Kaluza–Klein reduction on an isotropic $(D - 4)$-torus, results in an effective four-dimensional potential of the form $V = e^{c\varphi} U(M)$, where $c = \sqrt{3}$ and $M \in G$ represents the matrix of moduli fields. The value of the coupling parameter $c = \sqrt{3}$ is independent of the symmetry group $G$ and the spacetime dimensionality $D$. When the scalar fields are canonically coupled and the function $U$ admits a Minkowski ground state, oscillations of the moduli around this vacuum support a cosmological scaling solution in the presence of a perfect fluid [23]. When the fluid has a relativistic equation of state, the fluid and scalar field energy densities are predicted to be $\Omega_F = 2\Omega_r = 2/3$ and this was confirmed by numerical simulations. The scaling solution (6) therefore provides a very good approximation to the cosmological dynamics in this regime.

These conclusions can be extended to non-canonical moduli when the dimensionally reduced action is symmetric under a global symmetry $G' \subset G$, where the metric transforms as a singlet under the action of $G'$. In the case where $G = SL(n, R)$, the moduli potential is invariant under $G' = SO(n)$. Since the perfect fluid source does not break the corresponding symmetry of the field equations, such a symmetry transformation may be employed to generate a cosmological scaling solution with dynamical, non-canonical fields from a background where such fields are trivial.

Finally, we have resolved an apparent ambiguity between the analytical and numerical results of [23] by demonstrating explicitly that the potential derived from the reduction of 11D

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5 The solution (6) is only an approximate solution to the field equations and does not represent a global attractor of the system. Nonetheless, our numerical results indicate that it accurately describes the cosmic dynamics over a large number of e-foldings.
pure Einstein gravity on an elliptic twisted torus reduces to a potential of the form (1) with \( c = \sqrt{3} \) after three of the moduli fields have been consistently decoupled from the potential.

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Appendix A. Cosmological field equations for the SL(2, R) model

The scalar field equations derived by extremizing the action (31) are given by

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{\sqrt{3}m^2}{2} e^{\sqrt{3}\phi}[e^{2\phi}(\chi^2 + 1)^2 + e^{-2\phi} + 2(\chi^2 - 1)] = 0, \tag{A.1}
\]

\[
\ddot{\phi} + 3H \dot{\phi} - m^2 e^{\sqrt{3}\phi}[e^{2\phi}(\chi^2 + 1)^2 - e^{-2\phi}] = 0, \tag{A.2}
\]

\[
\ddot{\chi} + 3H \dot{\chi} + 2\dot{\phi} + 2\chi m^2 e^{\sqrt{3}\phi}[e^{2\phi}(\chi^2 + 1) + e^{-2\phi}] = 0. \tag{A.3}
\]

The perfect fluid satisfies the standard conservation equation

\[
\dot{\rho}_\gamma + 3H(\rho_\gamma + p_\gamma) = 0, \tag{A.4}
\]

and the Friedmann constraint equation takes the form

\[
H^2 = \frac{1}{6} \left[ \rho_\gamma + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\chi}^2 + \frac{1}{2} m^2 e^{\sqrt{3}\phi}(e^{2\phi}(\chi^2 + 1)^2 + e^{-2\phi} + 2(\chi^2 - 1)) \right]. \tag{A.5}
\]

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