Reduction algorithm based on finding the maximum mutual information in incomplete information systems

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Abstract. Aiming at such drawbacks of the existing incomplete information system attribute reduction algorithms as high time complexity and insufficiency of the reduction result completeness, this paper proposes an attribute reduction algorithm based on finding the maximum mutual information via weakening the equivalence class in classical rough set theory into tolerance class and through combining the definition of information entropy with the tolerance class. Taking the mutual information of condition attribute and decision attribute as the iterative criterion, and the empty set as the initial reduction, the algorithm superposes the condition attribute set corresponding to the maximum value of the mutual information of condition attributes and decision attributes, thus equating the mutual information of condition attribute set and decision attribute with the information entropy of decision attribute, obtaining the relative reduction of the incomplete decision system and guaranteeing the completeness of the reduction result. In the solution of the attribute set tolerance class, an algorithm with lower time complexity is adopted, which can reduce the time complexity of the whole algorithm. The feasibility of the algorithm is illustrated by an example.

1. Introduction

Rough set theory [1] is a theory of data analysis and reasoning, which was proposed by Polish scientist Pawlak in 1982, and it has been widely used in many fields such as data mining, pattern recognition and artificial intelligence. Attribute reduction is the main problem in rough set theory. The main research objects in classical rough set theory are complete information systems. However, incomplete information systems exist widely in reality.

Attribute reduction is usually adopted in two ways for incomplete information systems: The first is to delete or complete the uncertainty values, thus transforming the incomplete information system into a complete information system. This method is simple and easy to operate, but to some extent, it might weaken the objectivity of information in information system. Besides, the method cannot be applied in some information systems where the attribute dispersion value is large; the second is to assign null values to the default values, and to process information in information system based on similar relationship. Yang Chengfu [2] proposed an attribute reduction algorithm, which takes the positive domain of decision attribute relative condition attribute as the iterative criterion and gradually reduces the redundant condition attribute to obtain the final reduction; Yang Jilin [4] designed a reduction algorithm based upon attribute partial order relationship. Two algorithms in the literatures [2,4] are feasible to find the relative reduction of incomplete decision information systems, however, the reduction results are not complete enough, and the algorithm time complexity needs to be reduced. Shu Ang [6] studied that the knowledge reduction under the definition of information view in incomplete information systems has stronger classification ability than the algebraic; Hu Feng [3]
proposed a knowledge reduction algorithm which is based on the importance of attributes of decision entropy; Huang Bing [5] used the condition information quantity to define the importance of attributes, and have proposed an attribute reduction algorithm based on conditional information quantity. The literatures [3,5] both consider the attribute reduction from the perspective of information theory, but do not consider the complexity of computing attribute tolerance classes. J.H.Dai[12]carried out attribute reduction in incomplete information systems from the perspective of conditional entropy; H.Zhao [8] constructed a heuristic feature selection algorithm on the basis of neighborhood tolerance condition entropy. However, the algorithm is only suitable for incompletely mixing data; Huang Weihua [7] carried out attribute reduction on the complete information system from the perspective of mutual information, and verified that the classification ability of attribute set can be measured by mutual information.

As for the existing algorithms, the reduction completeness is not strong, and the complexity of computing the attribute tolerance class is not considered, which causes the high time complexity of algorithm, and the knowledge entropy defined by the information view has the advantage that can reflect better the classification ability of knowledge. On the basis of the attribute reduction algorithm based on mutual information for complete information system in [7], this paper proposes an attribute reduction algorithm based on finding the maximum mutual information of condition attributes and decision attributes for incomplete information system , and the example shows that the algorithm can make the reduction result more complete.

2. Basic concept

Definition 1[5]. Quad \( S = (U, A = C \cup D, V, f) \) is called an information system. \( U \) is non-empty finite set of objects ,also is called universe; \( A = C \cup D \) is non-empty finite set of attributes, \( C \) is condition attributes set, \( D \) is decision attributes set, and \( C \cap D = \phi \); \( V = \bigcup_{a \in A} V_a \), \( V_a \) is the range of values of the attribute \( a \); \( f \) expresses an information function of \( U \times A \rightarrow V \),which is assigned to each object an information value on each attribute, that is \( a \in A, x \in U, f(x,a) \in V_a \), if \( D = \phi \), the information system is called data table, otherwise it is called decision table. If there exists \( x \in U, a \in C, f(x,a) \) is unknown (denoted as: \( f(x,a) = \ast \)), it is called incomplete information system; otherwise it is called complete information system.

Definition 2[2]. The tolerance relationship \( T \) is defined as:

\[
T(x, y) = \{ x \in U \land y \in U \land \forall c_e (c_e \in B) \Rightarrow (c_e(x) = c_e(y) \lor c_e(x) = \ast \land c_e(y) = \ast), \}
\]

\( I_B(x) \) is expressed the set of objects \( x \) that satisfy tolerance relationship with \( x \) on the attribute subset \( B \), namely \( I_B(x) = \{ y | y \in T(x, y) \} \).

For tolerance relationship \( T \), \( B \) is a subset of attributes, note: \( U_SM(B) = \{ T_B(x) \mid x \in U \} \). Then \( T_B(x) \) is expressed the tolerance class of object \( x \) on attribute subset \( B \).

Definition 3[12].Let \( S = (U, C \cup D, V, f) \) be an incomplete information system, where \( R \subseteq C \), \( U/R = \{ T_R(x_1), T_R(x_2), \ldots, T_R(x_m) \} \), \( U/D = \{ Z_1, Z_2, \ldots, Z_m \} \).

The entropy of \( R \) is defined as:

\[
H(R) = -\sum_{i=1}^{\|V\|} p(T_R(x_i)) \log_2 p(T_R(x_i))
\]

The conditional entropy of \( R \) relative to \( D \) is defined as:

\[
H(D|R) = -\sum_{i=1}^{\|V\|} p(T_R(x)) \sum_{j=1}^{\|V\|} p(Z_j \mid T_R(x)) \log_2 p(Z_j \mid T_R(x))
\]

where \( p(T_R(x)) = \|T_R(x)\|/\|U\| \), \( p(Z_j \mid X_i) = \|Z_j \cap T_R(x)\|/\|T_R(x)\| \).
the mutual information of $R$ and $D$ is defined as: $I(R;D) = H(D) - H(D|R)$.

According to mutual information the importance degree of attribute is defined as:
$$SIF(a,R;D) = I(R,U_{\{a\}};D) - I(R;D).$$

**Theorem 1**[10]. Let $S = (U,C \cup D,V,f)$ be a decision information system, Where $U$ is the universe, $C$ is condition attribute set, $D$ is decision attribute set, and $B \subseteq C$, then the necessary and sufficient conditions for that $B$ is the attribute reduction of $C$ relative to $D$ are:

a). $I(B;D) = I(C,D)$,

b). for any $p \in B$, there is $H(D|B) < H(D|B - \{p\})$.

Theorem 1 shows that the classification ability of attribute sets can be measured by mutual information. However, if there is attribute reduction for information system according to two conditions of Theorem 1, it is necessary to calculate the mutual information of the decision attributes and all condition attribute sets which are composed of all condition attributes. The calculation amount is large and efficiency is not high. Based on the definition of mutual information, this paper proposes a theorem that reflects the relationship between mutual information and decision information entropy in consistent incomplete information system.

3. **Mutual information theorem for consistent incomplete decision information systems**

**Theorem 2**. Let $S = (U,C \cup D,V,f)$ be a consistent incomplete decision information system, Where $U$ is universe, $C$ is condition attribute set, $D$ is decision attribute set, then $I(C;D) = H(D)$.

**Proof.** Let $\partial_B : U \rightarrow V_d$ be the generalized decision function, defined as: $\partial_B(x) = [\{ i | f(d,y) \land y \in T_B(x) \}]$, since $S$ is consistent, it can be obtained for any $x \in U$, there is $\partial_B(x) = 1$ [12], then there exist $Z_j \in U/D, T_c(x) \subseteq Z_j$, then $T_c(x_j) \cap Z_i = \phi$, which $Z_i \in U/D, Z_i = Z_j$, that is
$$\sum_{j=1}^{\infty} \frac{|P_c(x_j) \cap Z_i|}{|P_c(x_j)|} = 0 + 0 + \cdots + \log_2 1 + 0 + \cdots = 0,$$
then $H(D|C) = \sum_{i=1}^{F} P(TC(x_i)) \sum_{j=1}^{\infty} \frac{|P_c(x_j) \cap Z_i|}{|P_c(x_j)|} = 0$,
therefore $I(C;D) = H(D) - H(D|C) = H(D) - 0 = H(D)$.

Theorem 2 shows that in the incomplete information system, only mutual information of condition attribute set and decision attribute is equal with information entropy of decision attribute, and then can obtain the relative reduction of the information system. Therefore, according to the calculation formula of mutual information, in the process of attribute reduction, only the condition attribute set corresponding to the maximum mutual information can be found. This avoids calculating mutual information of the redundant condition attribute set and decision attribute, thereby the amount of calculation is reduced, and the calculation efficiency is improved.

4. **Reduction algorithm in incomplete information system based on finding maximum mutual information**

4.1. **Attribute reduction algorithm**

For a complete information system, Huang Weihua [7] proposed an attribute reduction algorithm with mutual information as heuristic information, which only needs to satisfy $I(R;D) = I(C;D)$, so $R$ is the attribute reduction of the information system. The algorithm verifies that mutual information can be used to measure the classification ability of attribute sets, but it is necessary to calculate the mutual information of the decision attributes and all condition attribute sets which are composed of all
condition attributes. The calculation amount is relatively large and the efficiency is relatively low. On the basis of attribute reduction algorithm based on mutual information in complete information system [7], the equivalence relation is extended to the tolerance relationship. According to the definition of information entropy in incomplete information system, firstly, calculating the mutual information of each condition attribute \( c_i \) and decision attribute \( D \), then finding the corresponding condition attribute when the mutual information is the maximum value, which is selecting the condition attribute with the greatest importance measurement, and the mutual information is used as the iterative criterion, the corresponding condition attribute is superimposed to satisfy \( I(R;D) = H(D) \), then \( R \) is the relative reduction of the information system.

Obtain relative reduction algorithm in incomplete information system.

Input: incomplete information system \( S = (U, A, V, f) \)

Output: relative reduction \( R \) in \( S \)

a) Let \( R = \emptyset \), \( U/D = \{Y_1, Y_2, \cdots, Y_m\} \);

b) Solve the information entropy \( H(D) \) of the decision attribute \( D \);

c) Let \( R = R \cup \{c_i\} \), calculate tolerance class \( T_R(x_i) \), where \( x_i \in U, i = 1, \cdots, |U| \), calculate the conditional entropy \( H(D|R) \) of \( R \) relative to the decision attribute \( D \), according to the formula \( I(R;D) = H(D) - H(D|R) \), the mutual information \( I(R;D) \) of \( R \) and \( D \) is obtained;

d) Select the maximum value of mutual information, then \( I(R;D) = \max I(R \cup \{c_i\};D) \), which is selecting the attribute with the greatest importance measure \( SIF(a,R,D) \);

e) \( R = R \cup \{c_i\} \), if \( I(R;D) = H(D) \) is satisfied, then \( R \) is the relative reduction of \( C \), and \( R \) is directly output, otherwise turn c).

Solving tolerance class of attribute sets is a key step in the attribute reduction of incomplete information systems. The complexity of calculating attribute tolerance class will affect the time complexity of the whole algorithm. The usual way to solve the tolerance class is selecting two objects in the universe, then comparing whether they satisfy the tolerance relationship for each attribute in the attribute set. If the tolerance relationship is satisfied, the two objects belong to a tolerance class. In the least ideal case, since \( O(|U|^2) \) comparisons are required, the time complexity is \( O(|U|^2) \). The tolerance class of the attribute set is calculated by using algorithm 3 in [6], where the value range of the attribute \( c_i \in R \), \( V_{c_i} = \{v_1^{c_i}, v_2^{c_i}, \cdots, v_m^{c_i}\} \), where the null value "*" is treated as a special value, \( n_j^{c_i} \) \( (j = 1, 2, \cdots, m) \)

indicates that the condition attribute value is the number of \( v_j^{c_i} \), the value corresponding to \( c_i \) in all condition attributes is the minimum value, and the time complexity is \( O(\sum_{j=1}^{m} |R| (n_j^{c_i})^2) \).

Compared with general methods, it has lower time complexity, which is helpful for reducing the time complexity of the overall attribute reduction algorithm. The detailed calculation of tolerance algorithm can be found in [6].

4.2. Analysis of time complexity

The time complexity for calculating the tolerance class of all objects on attribute set \( R \) is \( O(\sum_{j=1}^{m} |R| (n_j^{c_i})^2) \), due to the number \( m \) of different attribute values of the condition attribute \( R \) in the
information table, in general, exists \( m > 1 \), then there is \( O\left( \sum_{j=1}^{m} |R| (n_j')^2 \right) < O\left( |R| |U|^2 \right) \).

It is known from [11] that the time complexity of calculating \( DU \) in step a) is \( O(|U||V|) \). For each object \( x_i \in U \), the time complexity for calculating the tolerance class \( T_k(x_i) \) is \( O\left( \sum_{j=1}^{m} |R| (n_j')^2 \right) < O\left( |R| |U|^2 \right) \), and due to \( |R| \leq |C| \), \( m \leq |V| \), the time complexity is \( O(|U||V|) + O\left( |R| \left( |n_j'|^2 \right) \right) \). c) to d) calculate the mutual information of the condition attribute set and the decision attribute and obtain the maximum mutual information of attribute set, the complexity is \( O(|C||V|) \). c) to e) in the least ideal state, it needs to be repeated \( |C|^{-1} \) times, thus the time complexity is \( O(|C||V| |C|^{-1} \left( |n_j'|^2 \right) \).

5. Case analysis

In order to detect the effectiveness of the algorithm, an example analysis is performed according to the incomplete information decision table in [12]. The information system \( S = (U,A,V,f) \), where the domain \( U = \{1,2,3,4,5,6\} \), the attribute set \( C = \{\text{price}, \text{mileage}, \text{size}, \text{max-speed}\} \), \( D \) is the decision attribute, * represents the unknown value, and the relative reduction of the information table is sought.

Table 1. Incomplete information decision table.

| Car | Price | Mileage | Size | Max-Speed | D   |
|-----|-------|---------|------|-----------|-----|
| 1   | High  | High    | Full | Low       | Good|
| 2   | High  | *       | Full | High      | Good|
| 3   | *     | *       | Compact | High | Poor|
| 4   | *     | *       | Full | Low       | Good|
| 5   | Low   | *       | Full | High      | Excellent|
| 6   | *     | High    | Full | Low       | Good|

6. The generalized information table is shown in Table 2.

Table 2. Generalized information table.

|   | U   | c1 | c2 | c3 | c4 | D   |
|---|-----|----|----|----|----|-----|
| 1 | 2   | 2  | 2  | 1  | 2  |
| 2 | 2   | *  | 2  | 2  | 2  |
| 3 | *   | *  | 1  | 2  | 1  |
| 4 | *   | *  | 2  | 1  | 2  |
| 5 | 1   | *  | 2  | 2  | 3  |
| 6 | *   | 2  | 2  | 1  | 2  |

The decision table is reduced by using the above algorithm, and the specific calculation steps are the following:

\[
U/D = \{Y_1, Y_2, \ldots, Y_n\}, \text{ where } Y_1 = \{1,2,4,6\}, \ Y_2 = \{3\}, \ Y_3 = \{5\}; \quad p(Y_1) = \frac{2}{3}, \quad p(Y_2) = \frac{1}{6}, \quad p(Y_3) = \frac{1}{6} \text{, then }
\]

\[
H(D) = - \sum_{j=1}^{3} p(Y_j) \log_2 p(Y_j) = 0.918;
\]

When \( R = \phi \), the mutual information of each condition attribute \( c_i \) and decision attribute \( D \) is calculated. First calculate the attribute tolerance class as follows:
\[ T_1(1) = T_1(2) = (1,2,3,4,6) \quad T_1(3) = T_1(4) = T_1(6) = (1,2,3,4,5,6) \quad T_1(5) = (3,4,5,6) \]
\[ T_2(1) = T_2(2) = T_2(3) = T_2(4) = T_2(5) = T_2(6) = (1,2,3,4,5,6) \]
\[ T_3(1) = T_3(2) = T_3(4) = T_3(5) = T_3(6) = (1,2,4,5,6) \quad T_3(3) = (3) \]
\[ T_4(1) = T_4(3) = T_4(5) = T_4(6) = (1,2,4,5,6) \quad T_4(2) = T_4(4) = (2,4,5,6) \]
\[ T_5(1) = T_5(2) = (4,6) \quad T_5(3) = (3) \quad T_5(4) = (2,5,6) \quad T_5(5) = (2,3,5) \]

Calculating mutual information:
\[ I(\{c_1\}; D) = H(D) - H(D|c_1)) = 0.918 - 5.9581 = -5.0401 \]
\[ I(\{c_2\}; D) = H(D) - H(D|c_2)) = 0.918 - 7.5098 = -6.5918 \]
\[ I(\{c_3\}; D) = H(D) - H(D|c_3)) = 0.918 - 3.008 = -2.09 \]
\[ I(\{c_4\}; D) = H(D) - H(D|c_4)) = 0.918 - 2.3774 = -1.4594 \]

It can be concluded that the mutual information of \( c_4 \) and \( D \) is the maximum value, and then the mutual information of \( \{c_1, c_4\}, \{c_2, c_4\}, \{c_3, c_4\} \) and \( D \) is calculated separately. Again, follow the steps above to first calculate the tolerance class:
\[ T_{\{c_1,c_4\}}(1) = T_{\{c_1,c_4\}}(2) = T_{\{c_1,c_4\}}(6) = (1,4,6) \quad T_{\{c_1,c_4\}}(3) = (2,3) \quad T_{\{c_1,c_4\}}(5) = (2,3,5) \]
\[ T_{\{c_2,c_4\}}(1) = T_{\{c_2,c_4\}}(2) = T_{\{c_2,c_4\}}(3) = T_{\{c_2,c_4\}}(5) = (2,3,5) \]
\[ T_{\{c_3,c_4\}}(1) = T_{\{c_3,c_4\}}(2) = T_{\{c_3,c_4\}}(3) = T_{\{c_3,c_4\}}(5) = (2,3,5) \]

Calculating mutual information:
\[ I(\{c_1, c_4\}; D) = H(D) - H(D|\{c_1, c_4\}) = 0.918 - 1.4591 = -0.5411 \]
\[ I(\{c_2, c_4\}; D) = H(D) - H(D|\{c_2, c_4\}) = 0.918 - 2.3774 = -1.4594 \]
\[ I(\{c_3, c_4\}; D) = H(D) - H(D|\{c_3, c_4\}) = 0.918 - 0.6667 = 0.2513 \]

From the above calculation results, it can be obtained that the mutual information is the maximum mutual information when \( R = \{c_3, c_4\} \). Next solve the mutual information of \( \{c_1, c_3, c_4\}, \{c_2, c_3, c_4\} \) and \( D \).

Tolerance class:
\[ T_{\{c_1,c_3,c_4\}}(1) = T_{\{c_1,c_3,c_4\}}(4) = T_{\{c_1,c_3,c_4\}}(6) = (1,4,6) \quad T_{\{c_1,c_3,c_4\}}(2) = (2) \quad T_{\{c_1,c_3,c_4\}}(3) = (3) \quad T_{\{c_1,c_3,c_4\}}(5) = (5) \]
\[ T_{\{c_2,c_3,c_4\}}(1) = T_{\{c_2,c_3,c_4\}}(4) = T_{\{c_2,c_3,c_4\}}(6) = (1,4,6) \quad T_{\{c_2,c_3,c_4\}}(2) = T_{\{c_2,c_3,c_4\}}(3) = T_{\{c_2,c_3,c_4\}}(5) = (2,3,5) \]

Mutual information:
\[ I(\{c_1, c_3, c_4\}; D) = H(D) - H(D|\{c_1, c_3, c_4\}) = 0.918 \]
\[ I(\{c_2, c_3, c_4\}; D) = H(D) - H(D|\{c_2, c_3, c_4\}) = 0.918 - 0.6667 = 0.2513 \]

Since \( I(\{c_1, c_3, c_4\}; D) = H(D) = 0.918 \), \( R = \{c_1, c_3, c_4\} \) is the relative attribute reduction of the incomplete information table.

The algorithm is compared with the algorithms in [2], [5] and [12] in terms of reduction results and algorithm time complexity in this paper, as shown in Table 3 and Table 4.

Table 3. Three algorithmic reduction results.

| Algorithms          | Reduction results           |
|---------------------|-----------------------------|
| Literature [5]      | \{c_1, c_3, c_4\}           |
| Literature 12       | \{c_1, c_3, c_4\}           |
| Literature [2]      | \{c_3, c_4\}                |
| The algorithm in the paper | \{c_1, c_3, c_4\} |

Table 4. Comparison of time complexity.

| Algorithms          | Inspirational information | Time complexity       |
|---------------------|---------------------------|-----------------------|
| Literature [5]      | Conditional information quantity | \( O(|I|^0|U|^1) \) |
By comparing and analyzing the algorithm reduction result and time complexity, we can get: the reduction result of the algorithm in the paper is the same as those in [5] and [12], which indicates that the attribute reduction algorithm based on the maximum mutual information is useful to find relative reduction in incomplete information systems, but compared with the algorithm in [2], the algorithmic reduction result is not simple enough. However, in terms of algorithm time complexity, it is obviously superior to the algorithms in [2], [5] and [12] in this paper. The algorithm in this paper and in [5] and [12] all carry out attribute reduction in incomplete information systems from the perspective of information theory, this paper uses a simple algorithm for setting up queues to obtain tolerance class, and in order to satisfy $I(R;D) = H(D)$, in the calculation of the mutual information between the condition attribute set and the decision attribute, the algorithm only selects the attribute set superimposed on the condition attribute corresponding to the maximum value of the mutual information value, which improves the computational efficiency, and compared with [5] and [12], the algorithm has a lower time complexity, that embodies the practicality of the algorithm based on finding the maximum mutual information in this paper.

7. Conclusion
The attribute reduction of the information system is one of the core problems in the field of rough set research. On the basis of the attribute reduction method based on mutual information in complete information system, according to the definition of information entropy of incomplete information system, this paper proposes an attribute reduction algorithm based on finding the maximum mutual information for incomplete information systems. In the solution of the attribute set tolerance class, we apply a time-complexity algorithm to ensure that the time complexity of the overall algorithm is reduced. The next step is to improve and optimize the algorithm for the simplicity of reduction results.

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