A four-mean theorem and its application to pseudospectra

Thomas Ransford$^1$ • Nathan Walsh$^1$

Received: 20 September 2021 / Accepted: 5 July 2022 © The Author(s), under exclusive licence to Springer Nature Switzerland AG 2022

Abstract
Let $N$ be an integer with $N \geq 4$. We show that, if $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ are $N$-tuples of strictly positive numbers whose arithmetic, geometric and harmonic means agree, then

$$\max_j x_j < (N - 2) \max_j y_j \quad \text{and} \quad \min_j x_j < (N - 2) \min_j y_j.$$

A generalized version of this result (where some of $x_j$ and $y_j$ are allowed to be zero) is used to show that, if $N \geq 4$ and $A, B$ are $N \times N$ matrices with super-identical pseudospectra, then, for every polynomial $p$, we have

$$\|p(A)\| < \sqrt{N - 2}\|p(B)\|,$$

unless $p(A) = p(B) = 0$. This improves a previously known inequality to the point of being sharp, at least for $N = 4$.

Keywords Arithmetic mean • Geometric mean • Harmonic mean • Pseudospectrum • Singular value

Mathematics Subject Classification Primary 26E20; Secondary 15A18 • 26D15

1 Introduction and statement of results

Our main result is the following theorem.
Theorem 1.1 Let $N \geq 4$ and let $x_1, \ldots, x_N, y_1, \ldots, y_N \geq 0$. If

$$\sum_{j=1}^{N} x_j = \sum_{j=1}^{N} y_j, \quad (1)$$

$$\prod_{j=1}^{N} x_j = \prod_{j=1}^{N} y_j, \quad (2)$$

and

$$\sum_{k=1}^{N} \prod_{j=1, j \neq k}^{N} x_j = \sum_{k=1}^{N} \prod_{j=1, j \neq k}^{N} y_j, \quad (3)$$

then

$$\max_j x_j \leq (N - 2) \max_j y_j. \quad (4)$$

Equality holds in (4) if and only if there exists $c \geq 0$ such that, after rearranging the $x_j$ and $y_j$ in decreasing order, we have

$$(x_1, \ldots, x_N) = c(N - 2, 0, \ldots, 0) \quad \text{and} \quad (y_1, \ldots, y_N) = c(1, \ldots, 1, 0, 0).$$

Corollary 1.2 Let $N \geq 4$. If $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ are $N$-tuples of strictly positive numbers whose arithmetic, geometric and harmonic means agree, then

$$\max_j x_j < (N - 2) \max_j y_j \quad \text{and} \quad \min_j x_j < (N - 2) \min_j y_j. \quad (5)$$

Proof If all the $x_j$ and $y_j$ are strictly positive, then, under condition (2), the condition (3) is equivalent to the statement that $\sum_{j=1}^{N} 1/x_j = \sum_{j=1}^{N} 1/y_j$, in other words, that the harmonic means agree. Therefore the inequality between maxima in (5) is a consequence of Theorem 1.1.

The inequality between minima follows by applying the one for maxima to the $N$-tuples $(1/y_1, \ldots, 1/y_N)$ and $(1/x_1, \ldots, 1/x_N)$, whose arithmetic, geometric and harmonic means also agree.

Remarks (i) Corollary 1.2 shows that, if the $x_j$ and $y_j$ are constrained so that three of their means (arithmetic, geometric, harmonic) agree, then the fourth (maximum or minimum) satisfies the inequality (5). For this reason, we have dubbed it (and by extension Theorem 1.1) a ‘four-mean theorem’.

(ii) The more complicated formulation of the condition (3) allows for the possibility that some of the $x_j$ and $y_j$ are zero. This is useful since it contains the case of equality.

(iii) Theorem 1.1 is only interesting if $N \geq 4$ since, if $N = 3$, then the conditions (1), (2) and (3) already imply that the $x_j$ are a permutation of the $y_j$. Indeed, they tell
us that the polynomials \((t - x_1)(t - x_2)(t - x_3)\) and \((t - y_1)(t - y_2)(t - y_3)\) have the same coefficients, and therefore the same roots.

(iv) There are also two-mean and three-mean theorems. In fact it is obvious that if just (1) holds, then (4) is true with \((N - 2)\) replaced by \(N\), and this is optimal. Also, if (1) and (2) both hold, then (4) is true with \((N - 2)\) replaced by \((N - 1)\), and again this is optimal. However, this is less obvious, and indeed this three-mean theorem is a step on the route to establishing Theorem 1.1.

Our motivation for studying constraints and inequalities such as those in Theorem 1.1 arises from a problem in the theory of pseudospectra of matrices, which we now describe. For background on pseudospectra, we refer to the book [1] and the survey article [2].

It is known that the pseudospectra of a matrix \(A\), namely the level sets of \(\|(A - zI)^{-1}\|\), do not suffice to determine the operator norm of polynomials of \(A\), see e.g. [1, Section 47] and [2, Section 2], in particular [2, Theorem 2.3]. In an attempt to overcome this problem, the authors of [3] proposed looking at not only the lowest singular values of \(A - zI\) for \(z \in \mathbb{C}\) (namely the reciprocals of \(\|(A - zI)^{-1}\|\)), but all the singular values. They showed that this information is sufficient to determine the operator norm of any polynomial of \(A\), up to a factor depending only on the dimension.

Here is a more precise formulation of their result. We say that two complex \(N \times N\) matrices \(A, B\) have super-identical pseudospectra if

\[s_j(A - zI) = s_j(B - zI)\quad (j = 1, 2, \ldots, N, \ z \in \mathbb{C}),\]

where \(s_1, \ldots, s_N\) denote the singular values, ordered so that \(s_1 \geq \cdots \geq s_N\). It was shown in [3, Theorem 1.3] that, if \(A, B\) are \(N \times N\) matrices with super-identical pseudospectra and \(p\) is a polynomial, then

\[\|p(A)\| \leq \sqrt{N} \|p(B)\|.\] (6)

It is not obvious, a priori, whether the constant \(\sqrt{N}\) can be replaced by 1 in this inequality. In fact it can if \(N = 1, 2, 3\) (see [2, Section 4]), but an example constructed in [3] shows that this is no longer the case when \(N \geq 4\). However, the authors of [3] mentioned that they did not know whether the bound \(\sqrt{N}\) in (6) is optimal. The following result shows that it is not, and replaces it with a bound that is sharp, at least when \(N = 4\).

**Theorem 1.3** Let \(N \geq 4\) and let \(A, B\) be complex \(N \times N\) matrices with super-identical pseudospectra. Then, for any polynomial \(p\), we have

\[\|p(A)\| < \sqrt{N - 2} \|p(B)\|,\] (7)

unless \(p(A) = p(B) = 0\). The constant \(\sqrt{N - 2}\) is sharp at least if \(N = 4\).

The proof of (7) is based on the four-mean theorem, Theorem 1.1. The proof of sharpness in the case \(N = 4\) is based on the example from [3] mentioned above,
which will be described in detail in Sect. 5 below. The same example also permits us to deduce a related result, which we now describe.

Armentia, Gracia and Velasco [4] showed that, if $A, B$ have super-identical pseudospectra, then they are similar, in other words, there exists an invertible matrix $W$ such that

$$ B = W^{-1} AW. \quad (8) $$

In this case, $p(B) = W^{-1} p(A) W$ for every polynomial $p$, and so

$$ \|p(B)\| \leq \|W^{-1}\| \|p(A)\| \|W\|. $$

It is thus tempting to believe that (7) and (8) can be subsumed in a single result in which (8) holds with $\|W\|\|W^{-1}\| \leq \sqrt{N - 2}$. Even if this is false, one might hope that, at the very least, $W$ may be chosen so that $\|W\|\|W^{-1}\| \leq C(N)$ for some constant $C(N)$ depending only on $N$. The following theorem shows that, perhaps surprisingly, there

is no such result.

**Theorem 1.4** Given $M > 0$, there exist $4 \times 4$ matrices $A, B$ with super-identical pseudospectra such that

$$ \inf \left\{ \|W\|\|W^{-1}\| : B = W^{-1} AW \right\} > M. \quad (9) $$

The rest of the paper is organized as follows. In Sect. 2 we establish the three-mean theorem mentioned earlier, which is then used in Sect. 3 to prove the four-mean theorem, Theorem 1.1. Theorem 1.3 is deduced in Sect. 4, except for the sharpness statement, which is established in Sect. 5, where Theorem 1.4 is also proved.

## 2 Three-mean theorem

Our goal in this section is to establish the following theorem.

**Theorem 2.1** Let $N \geq 3$ and let $x_1, \ldots, x_N, y_1, \ldots, y_N \geq 0$. If

$$ \sum_{j=1}^{N} x_j = \sum_{j=1}^{N} y_j \quad \text{and} \quad \prod_{j=1}^{N} x_j = \prod_{j=1}^{N} y_j, $$

then

$$ \max_j x_j \leq (N - 1) \max_j y_j. \quad (10) $$

Equality holds in (10) if and only if there exists $c \geq 0$ such that, after rearranging the $x_j$ and $y_j$ in decreasing order, we have

$$(x_1, \ldots, x_N) = c(N - 1, 0, \ldots, 0) \quad \text{and} \quad (y_1, \ldots, y_N) = c(1, \ldots, 1, 1, 0).$$
We shall prove this theorem by reformulating it as an optimization result. Since the result obviously holds if all the numbers $y_j$ are equal to zero, we can suppose that at least one of them is non-zero. Normalizing so that $\max_j x_j = x_1$ and $\max_j y_j = y_1 = 1$, we are led to consider the following problem.

**Problem 2.2** Let $N \geq 3$. Maximize $x_1$ subject to the following constraints:

$$
\begin{align*}
 &\begin{cases}
x_j \geq 0 (j = 1, \ldots, N), \\
y_1 = 1, \ 0 \leq y_j \leq 1 (2 \leq j \leq N), \\
\sum_{j=1}^{N} x_j = \sum_{j=1}^{N} y_j, \\
\prod_{j=1}^{N} x_j = \prod_{j=1}^{N} y_j.
\end{cases} \\
\text{(11)}
\end{align*}
$$

Here is the solution to Problem 2.2, which establishes Theorem 2.1.

**Theorem 2.3** Let $N \geq 3$. The maximum value of $x_1$ subject to the constraints (11) is $N - 1$, attained uniquely when $x_2 = \cdots = x_N = 0$ and all but one of the $y_j$ are equal to 1, the remaining one being equal to 0.

**Proof** Let $S$ be the set of $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ that satisfy the constraints (11). Clearly the numbers $y_j$ lie in $[0, 1]$. Also it is easy to see that the numbers $x_j$ lie in $[0, N]$. Therefore $S$ is a compact set. The function $(x, y) \mapsto x_1$ is continuous, so it attains its maximum on $S$, say at $(x^*, y^*)$.

If $x = (N - 1, 0, \ldots, 0)$ and $y = (1, \ldots, 1, 0)$, then $(x, y) \in S$ and $x_1 = N - 1$. Thus we certainly have $x_1^* \geq N - 1$.

We shall show by contradiction that at least one of the terms $x_j^*$ or $y_j^*$ is equal to zero. Suppose, if possible, that $x_j^* > 0$ and $y_j^* > 0$ for all $j$. Applying the standard Lagrange-multiplier argument to

$$
\begin{align*}
x_1 + \alpha \left( \sum_{j=1}^{N} x_j - \sum_{j=1}^{N} y_j \right) + \beta \left( \sum_{j=1}^{N} \log x_j - \sum_{j=1}^{N} \log y_j \right),
\end{align*}
$$

we see that

$$
\begin{align*}
&\begin{cases}
1 + \alpha + \beta/x_1^* = 0, \\
\alpha + \beta/x_j^* = 0, \ (2 \leq j \leq N), \\
\alpha + \beta/y_j^* = 0, \ (1 \leq j \leq N, \text{ if } y_j^* < 1).
\end{cases}
\end{align*}
$$

From the first equation, $\alpha$ and $\beta$ cannot both be zero, and from the second, they are in fact both non-zero. Writing $u := -\beta/\alpha$, we deduce that

$$
\begin{align*}
&\begin{cases}
x_j^* = u \ (2 \leq j \leq N), \\
y_j^* = u \text{ or } 1 \ (1 \leq j \leq N).
\end{cases}
\end{align*}
$$
There are now two possibilities. The first is that all the \( y_j^* \) are equal to 1. In this case, the arithmetic and geometric means of the \( y_j^* \) are equal to 1, so by (11) the arithmetic and geometric means of the \( x_j^* \) are also equal to 1. By the case of equality in the AM-GM inequality, this forces all the \( x_j^* \) to be equal to 1. This contradicts the fact that \( x_1^* \geq N - 1 \).

The second possibility is that some \( y_j^* = u < 1 \). This implies that

\[
N - 1 + (N - 1)u \leq \sum_{j=1}^{N} x_j^* = \sum_{j=1}^{N} y_j^* \leq N - 1 + u.
\]

Since \( N \geq 3 \) and \( u > 0 \), this is impossible.

Thus both possibilities lead to contradictions. We conclude that at least one of the terms \( x_j^* \) or \( y_j^* \) is equal to zero, as claimed.

Since \( \prod_{j=1}^{N} x_j^* = \prod_{j=1}^{N} y_j^* = 0 \), at least one \( y_j^* = 0 \), so \( \sum_{j=1}^{N} y_j^* \leq N - 1 \). It follows that

\[
N - 1 \leq x_1^* \leq \sum_{j=1}^{N} x_j^* = \sum_{j=1}^{N} y_j^* \leq N - 1.
\]

Therefore we have equality throughout, which shows that \( x_1^* = N - 1 \) and \( x_j^* = 0 \) for all \( j \geq 2 \), and also that all but one of the \( y_j^* \) satisfy \( y_j^* = 1 \). This concludes the proof. \( \square \)

## 3 Four-mean theorem

Following the idea of the preceding section, we shall prove Theorem 1.1 by formulating it as the solution to an optimization problem. Here is the problem:

**Problem 3.1** Let \( N \geq 4 \). Maximize \( x_1 \) subject to the following constraints:

\[
\begin{align*}
  x_j &\geq 0 \ (j = 1, \ldots, N), \\
  y_1 &= 1, \ 0 \leq y_j \leq 1 \ (2 \leq j \leq N), \\
  \sum_{j=1}^{N} x_j &= \sum_{j=1}^{N} y_j, \\
  \prod_{j=1}^{N} x_j &= \prod_{j=1}^{N} y_j, \\
  \sum_{k=1}^{N} \prod_{j=1 \atop j \neq k}^{N} x_j &= \sum_{k=1}^{N} \prod_{j=1 \atop j \neq k}^{N} y_j.
\end{align*}
\]

(12)
And here is the solution.

**Theorem 3.2** Let $N \geq 4$. The maximum value of $x_1$ subject to the constraints (12) is $N - 2$, attained uniquely when $x_2 = \cdots = x_N = 0$ and all but two of the $y_j$ are equal to 1, the remaining ones being equal to 0.

Before embarking upon the main proof, it will be convenient to separate out some algebraic results needed in the course of the argument.

**Lemma 3.3** Let $N \geq 4$ and let $r \in \{1, 2, \ldots, N - 1\}$.

(i) If $f(t) := t^{2N-2} - (N - 1)t^N + (N - 1)t^{N-2} - 1$,

then $f(t) \neq 0$ for all $t > 0$ except $t = 1$.

(ii) If $g(u, v) := (N - 1)u^N - (N - r)u^{N-1} - rvu^{N-1} + v^r$,

then $g(u, v) \neq 0$ in $0 < u < v < 1$.

**Proof** (i) A direct calculation shows that $f(1) = f'(1) = f''(1) = 0$, in other words, that $f$ has a triple zero at $t = 1$. By Descartes’ rule of signs, $f$ has at most three zeros in $(0, \infty)$, counted according to multiplicity. It follows that $f$ has no zeros in $(0, \infty)$ other than 1.

(ii) On the diagonal $u = v$, we have

$$g(u, u) = (N - 1 - r)u^N - (N - r)u^{N-1} + u^r.$$ 

This is identically zero if $r = N - 1$. Suppose that $r < N - 1$. Then, by Descartes’ rule of signs again, the right-hand side has at most two zeros in $(0, \infty)$. On the other hand, a direct verification shows that the right-hand side has a double zero at $u = 1$. Therefore $g(u, u) \neq 0$ for all $u \in (0, 1)$. Since $g(u, u) = u^r(1 + o(u))$ as $u \to 0$, it follows that $g(u, u) > 0$ for all $u \in (0, 1)$. Putting together the cases $r = N - 1$ and $r < N - 1$, we obtain

$$g(u, u) \geq 0 \quad (0 < u < 1). \quad (13)$$

Now, a simple computation gives

$$\frac{\partial g}{\partial v}(u, v) = -ru^{N-1} + rv^{r-1} > 0 \quad (0 < u < v < 1). \quad (14)$$

Finally, (13) and (14) together imply that $g(u, v) > 0$ in $0 < u < v < 1$. \qed
Proof of Theorem 3.2  The proof follows the same general lines as that of Theorem 2.1, though the details are a bit more involved.

Let $S$ be the set of $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ obeying the constraints (12). Then $S$ is a compact set, so the function $(x, y) \mapsto x_1$ attains its maximum on $S$, say at $(x^*, y^*)$.

If $x = (N - 2, 0, \ldots, 0)$ and $y = (1, \ldots, 1, 0, 0)$, then $(x, y) \in S$ and $x_1 = N - 2$, so we certainly have $x_1^* \geq N - 2$.

We claim that at least one of the $x_j^*$ or $y_j^*$ is equal to zero. To prove the claim, we proceed by contradiction. So, let us suppose, if possible, that $x_j^* > 0$ and $y_j^* > 0$ for all $j$.

In this situation, the final constraint in (12) is equivalent to the condition that

$$\sum_{j=1}^N 1/x_j^* = \sum_{j=1}^N 1/y_j^*.\tag{12}$$

We may therefore apply the standard Lagrange-multiplier argument to

$$x_1 + \alpha \left( \sum_{j=1}^N x_j - \sum_{j=1}^N y_j \right) + \beta \left( \sum_{j=1}^N \log x_j - \sum_{j=1}^N \log y_j \right) + \gamma \left( \sum_{j=1}^N 1/x_j - \sum_{j=1}^N 1/y_j \right)$$

to obtain

$$\begin{align*}
1 + \alpha + \beta/x_1^* - \gamma/(x_1^*)^2 &= 0, \\
\alpha + \beta/x_j^* - \gamma/(x_j^*)^2 &= 0, \quad (2 \leq j \leq N), \\
\alpha + \beta/y_j^* - \gamma/(y_j^*)^2 &= 0, \quad (1 \leq j \leq N, \text{ if } y_j^* < 1).
\end{align*}$$

The first equality shows that the constants $\alpha, \beta, \gamma$ are not all zero, and the remaining equalities show that

$$\begin{align*}
x_j^* &= u \text{ or } v \quad (2 \leq j \leq N), \\
y_j^* &= u \text{ or } v \text{ or } 1 \quad (1 \leq j \leq N),
\end{align*}$$

where $u$ and $v$ are the roots of $\alpha t^2 + \beta t - \gamma = 0$.

If some $x_j^*$ is equal to some $y_k^*$, then the vectors $x', y' \in \mathbb{R}^{N-1}$ formed by the remaining components satisfy the hypotheses of Theorem 2.1 (with $N$ replaced by $N - 1$). By that theorem, we deduce that $x_1' \leq N - 2$. Since $x_1' = x_1^* \geq N - 2$, we actually have equality throughout. By the case of equality in Theorem 2.1, all the remaining $x_j' = 0$, which contradicts the supposition that $x_j^* > 0$ for all $j$. We are thus led to conclude that, in fact,

$$\begin{align*}
x_j^* &= u \quad (2 \leq j \leq N), \\
y_j^* &= v \text{ or } 1 \quad (1 \leq j \leq N).
\end{align*}$$

If all the numbers $y_j^*$ are equal to 1, then the arithmetic and geometric means of the $y_j^*$ are equal to 1, so by (12) the arithmetic and geometric means of the $x_j^*$ are also equal to 1, which forces all the $x_j^*$ to be equal to 1. This contradicts the fact that $x_1^* \geq N - 2$. We conclude that there exists an integer $r$ with $1 \leq r \leq N - 1$ such
that exactly $r$ of the $y_j^*$ are equal to $v$ and the remaining $(N - r)$ are equal to 1. The constraints \( (12) \) then become

\[
x_1^* + (N - 1)u = N - r + rv, \quad (15)
\]
\[
x_1^* u^{N-1} = v^r, \quad (16)
\]
\[
\frac{1}{x_1^*} + \frac{N - 1}{u} = N - r + \frac{r}{v}. \quad (17)
\]

We also have $u \neq v$ and $v < 1$.

The argument now subdivides into two cases, according to whether $u < v$ or $u > v$.

**Case I: $u < v$.** Eliminating $x_1^*$ from \( (15) \) and \( (16) \) gives

\[
v^r + (N - 1)u^N = (N - r)u^{N-1} + rvu^{N-1}. \quad (18)
\]

In other words, $g(u, v) = 0$, where $g$ is the function in Lemma 3.3 (ii). But by that lemma, $g(u, v) \neq 0$ for $0 < u < v < 1$. This contradiction concludes the argument for Case I.

**Case II: $u > v$.** In this case we must have $r = 1$. Indeed, by \( (15) \), we have

\[
N - r + rv = x_1^* + (N - 1)u \geq N - 2 + (N - 1)v,
\]

which, after simplification, leads to

\[
(N - 1 - r) v \leq 2 - r.
\]

Since the left-hand side is non-negative, we must have $r \leq 2$. Also, if $r = 2$, then the left-hand side is zero, which implies that $r = N - 1$, contradicting the fact that $N \geq 4$. The only remaining possibility is that $r = 1$, as claimed.

Multiplying together \( (15) \) and \( (17) \), and recalling that $r = 1$, we obtain

\[
1 + (N - 1)^2 + (N - 1) \frac{x_1^*}{u} + (N - 1) \frac{u}{x_1^*} = (N - 1)^2 + 1 + \frac{N - 1}{v} + (N - 1)v,
\]

which, after simplification, becomes

\[
\frac{x_1^*}{u} + \frac{u}{x_1^*} = v + \frac{1}{v}.
\]

Since the function $t \mapsto (t + 1/t)$ is 2-to-1 on $t > 0$, it follows that either $x_1^*/u = v$ or $x_1^*/u = 1/v$. In the first case, \( (16) \) implies that $vu^N = v$, which in turn implies that $u = 1$ and $x_1^* = v < 1$, a contradiction. So we must have $x_1^* = u/v$. Substituting this information into \( (16) \), we find that $u^N = v^2$ and $x_1^* = u^{1-N/2}$. Substituting this into \( (15) \), and rearranging, we obtain

\[
u^{N-1} - (N - 1)u^{N/2} + (N - 1)u^{N/2-1} - 1 = 0,
\]
in other words, \( f(u^{1/2}) = 0 \), where \( f \) is the polynomial in Lemma 3.3 (i). By that lemma, \( f(t) \neq 0 \) for all \( t > 0 \) except \( t = 1 \). We conclude that \( u = 1 \), and hence that \( x_1^* = 1 \), contradicting the fact that \( x_1^* \geq N - 2 \). This concludes the argument for Case II.

Thus, whichever case we are in, we arrive at a contradiction. This shows that, as claimed, at least one of the \( x_j^* \) or \( y_j^* \) is equal to zero.

Because their geometric means are equal, both vectors \( x^*, y^* \) contain a component equal to zero. The vectors \( x', y' \in \mathbb{R}^{N-1} \) formed by the remaining components then satisfy the hypotheses of Theorem 2.1 (with \( N \) replaced by \( N - 1 \)). By that theorem, we deduce that \( x_1' \leq N - 2 \). Since \( x_1' = x_1^* \geq N - 2 \), we actually have equality throughout. By the case of equality in Theorem 2.1, all the remaining \( x_j' \) are equal to 0 and all but one of the \( y_j' \) are equal to 1, the remaining one being equal to zero. We conclude that \( x_1^* = N - 2 \), all the remaining \( x_j^* \) are equal to 0 and all but two of the \( y_j^* \) are equal to 1, the remaining ones being equal to zero. This completes the proof of Theorem 3.2, and with it, that of Theorem 1.1. \( \square \)

4 Super-identical pseudospectra

The proof of Theorem 1.3 is based on Theorem 1.1 and the following lemma. As before, we write \( s_1, \ldots, s_N \) to denote the singular values of an \( N \times N \) matrix, ordered so that \( s_1 \geq \cdots \geq s_N \).

Lemma 4.1 Let \( N \geq 4 \) and let \( A, B \) be complex \( N \times N \) matrices with super-identical pseudospectra. Then, for every polynomial \( p \),

\[
\sum_{j=1}^{N} s_j(p(A))^2 = \sum_{j=1}^{N} s_j(p(B))^2, \tag{19}
\]

\[
\prod_{j=1}^{N} s_j(p(A))^2 = \prod_{j=1}^{N} s_j(p(B))^2, \tag{20}
\]

and

\[
\sum_{k=1}^{N} \prod_{\substack{j=1 \atop j \neq k}}^{N} s_j(p(A))^2 = \sum_{k=1}^{N} \prod_{\substack{j=1 \atop j \neq k}}^{N} s_j(p(B))^2. \tag{21}
\]

Proof The equality (19) was already obtained in the course of the proof of (6), see [3, pp. 516–517]. We do not repeat the argument here.

The equality (20) is a consequence of the result of Armentia, Gracia and Velasco mentioned earlier, according to which matrices with super-identical pseudospectra are always similar. Thus \( A \) and \( B \) are similar, hence also \( p(A) \) and \( p(B) \). In particular, \( \det(p(A)) = \det(p(B)) \). Since the absolute value of the determinant is the product of the singular values, we deduce that (20) holds.
To establish (21), let us first consider the case when \( p(A) \) and \( p(B) \) are invertible. Then \( p \) has no common zeros with the characteristic polynomial \( f \) of \( A \), so there exist polynomials \( q, r \) such that \( pq + fr = 1 \). Since \( f(A) = 0 \), it follows that \( p(A)q(A) = I \). As \( A, B \) are similar, \( f(B) = 0 \) as well, and so also \( p(B)q(B) = I \).

By (19), with \( p \) replaced by \( q \), we have

\[
\sum_{j=1}^{N} s_j(q(A))^2 = \sum_{j=1}^{N} s_j(q(B))^2.
\]

But also we have \( q(A) = p(A)^{-1} \), and the singular values of \( p(A)^{-1} \) are \( 1/s_N(p(A)), \ldots, 1/s_1(p(A)) \). Likewise for \( B \). It follows that

\[
\sum_{j=1}^{N} \frac{1}{s_j(p(A))^2} = \sum_{j=1}^{N} \frac{1}{s_j(p(B))^2}.
\]

Multiplying this equation by Eq. (20), we obtain (21). This proves (21) in the case when \( p(A) \) and \( p(B) \) are invertible. The general case follows by replacing \( p(z) \) by \( p(z) + \epsilon \), where \( \epsilon \neq 0 \), and then letting \( \epsilon \to 0 \).

**Proof of Theorem 1.3** Let \( A, B \) have super-identical pseudospectra, and let \( p \) be a polynomial. Then Lemma 4.1 implies that the non-negative numbers \( x_j := s_j(p(A))^2 \) and \( y_j := s_j(p(B))^2 \) satisfy the relations (1), (2) and (3) of Theorem 1.1. By that theorem, it follows that

\[
\max_j s_j(p(A))^2 \leq (N - 2) \max_j s_j(p(B))^2,
\]

in other words, that the operator norms of \( p(A) \) and \( p(B) \) satisfy

\[
\|p(A)\|^2 \leq (N - 2)\|p(B)\|^2.
\]

(22)

If we have equality in (22), then, by the case of equality in Theorem 1.1, there exists \( c \geq 0 \) such that

\[
(s_1(A)^2, \ldots, s_N(A)^2) = c(N - 2, 0, \ldots, 0),
\]

\[
(s_1(B)^2, \ldots, s_N(B)^2) = c(1, \ldots, 1, 0, 0).
\]

However, since \( A, B \) are similar, \( p(A) \) and \( p(B) \) have the same rank, so they have exactly the same number of non-zero singular values. This can only happen if \( c = 0 \). Thus equality holds in (22) if and only if \( p(A) = p(B) = 0 \).

This completes the proof of Theorem 1.3 except for the sharpness statement, which will be treated in the next section. □
5 Sharpness results

Both the sharpness statement in Theorem 1.3 and the negative result about quantitative similarity, Theorem 1.4, are consequences of the example contained in the following proposition.

**Proposition 5.1** Let $\alpha, \beta \in (0, \pi/4]$, and let

$$A = \begin{pmatrix} 0 & \sec \alpha & 0 & 1 \\ 0 & 0 & \sec \beta \csc \beta & 0 \\ 0 & 0 & 0 & \csc \alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \sec \beta & 0 & 1 \\ 0 & 0 & \sec \alpha \csc \alpha & 0 \\ 0 & 0 & 0 & \csc \beta \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

Then $A, B$ have super-identical pseudospectra, and

$$s_1(A^2) = \frac{\cos \alpha}{\cos \beta} \quad \text{and} \quad s_2(A^2) = \frac{\sin \alpha}{\sin \beta}.$$  \hspace{1cm} (23)

**Proof** The matrices $A, B$ are taken from [3, Theorem 5.1], where it is shown that they have super-identical pseudospectra. It remains to establish (23).

A calculation gives

$$A^2 = \begin{pmatrix} 0 & 0 & \sec \alpha \sec \beta \csc \beta & 0 \\ 0 & 0 & 0 & \csc \alpha \sec \beta \csc \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

Since $\alpha \in (0, \pi/4]$, we have $\sec \alpha \leq \csc \alpha$, whence

$$s_1(A^2) = \csc \alpha \sec \beta \csc \beta \quad \text{and} \quad s_2(A^2) = \sec \alpha \sec \beta \csc \beta.$$  

Similarly

$$s_1(B^2) = \csc \beta \sec \alpha \csc \alpha \quad \text{and} \quad s_2(B^2) = \sec \beta \sec \alpha \csc \alpha.$$  

The result follows. \hfill $\square$

**Proof of the sharpness statement in Theorem 1.3** Taking $\beta = \pi/4$ and $\alpha$ close to 0 in Proposition 5.1, we see that, given $\epsilon > 0$, there exist $4 \times 4$ matrices $A, B$ with super-identical pseudospectra such that $\|A^2\|/\|B^2\| > \sqrt{2} - \epsilon$. This demonstrates that, if $N = 4$, then the constant $\sqrt{N - 2}$ in (7) is sharp. \hfill $\square$

**Proof of Theorem 1.4** Given $M > 0$, choose $\beta := \pi/4$ and $\alpha > 0$ sufficiently small so that $\sin \beta / \sin \alpha > M$. Let $A, B$ be the $4 \times 4$ matrices with super-identical pseudospectra furnished by Proposition 5.1.
If $W$ is an invertible $N \times N$ matrix such that $B = W^{-1}AW$, then $p(B) = W^{-1}p(A)W$ for every polynomial $p$. It follows that

$$s_j(p(B)) \leq \|W^{-1}\|s_j(p(A))\|W\|$$

for all polynomials $p$ and all $j \in \{1, 2, 3, 4\}$. In particular, we have

$$\|W\|\|W^{-1}\| \geq \frac{s_2(B^2)}{s_2(A^2)} = \frac{\sin \beta}{\sin \alpha} > M.$$  

This proves (9) and establishes the result.  

Acknowledgements  The authors are grateful to the referee for a careful reading of the manuscript.

Author Contributions Not applicable.

Funding  Ransford supported by grants from NSERC and the Canada Research Chairs program. Walsh supported by an NSERC Undergraduate Student Research Award and an FRQNT Supplement.

Availability of data and materials  Not applicable.

Declarations

Conflict of interest  None.

Ethics approval  Not applicable.

Consent to participate  Not applicable.

Consent for publication  Not applicable.

Code availability  Not applicable.

References

1. Trefethen, L.N., Embree, M.: Spectra and Pseudospectra. Princeton University Press, Princeton, NJ (2005)
2. Ransford, T.: Pseudospectra and matrix behaviour. In: Banach Algebras 2009. Banach Center Publ., vol. 91, pp. 327–338. Polish Acad. Sci. Inst. Math., Warsaw (2010)
3. Fortier Bourque, M., Ransford, T.: Super-identical pseudospectra. J. Lond. Math. Soc. 79(2), 511–528 (2009)
4. Armentia, G., Gracia, J.-M., Velasco, F.E.: Identical pseudospectra of any geometric multiplicity. Linear Algebra Appl. 436(6), 1683–1688 (2012)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.