Sequential Channel Synthesis

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Abstract

The channel synthesis problem has been widely investigated over the last decade. In this paper, we consider the sequential version in which the encoder and the decoder work in a sequential way. Under a mild assumption on the target joint distribution we provide a complete (single-letter) characterization of the solution for the point-to-point case, which shows that the canonical symbol-by-symbol mapping is not optimal in general, but is indeed optimal if we make some additional assumptions on the encoder and decoder. We also extend this result to the broadcast scenario and the interactive communication scenario. We provide bounds in the broadcast setting and a complete characterization of the solution under a mild condition on the target joint distribution in the interactive communication case. Our proofs are based on a Rényi entropy method.

I. INTRODUCTION

The study of the synthesis of distributions can be traced back to the seminal work by Wyner [1] where the problem studied was to characterize the smallest rate, in bits per symbol, at which common randomness needs to be provided to two agents, Alice and Bob, each having an arbitrary amount of private randomness, such that each of them can separately generate a sequence of random variables from respective finite sets, with the joint distribution being close to that of an i.i.d. sequence with a desired joint distribution at each symbol time (the notion of approximation in [1] is based on relative entropy). Wyner used this framework to define a notion of the common information of two dependent sources (the ones being synthesized by Alice and Bob respectively), which is known nowadays as Wyner’s common information. This formulation is of considerable interest for problem of distributed control and game theory with distributed agents [2] because of the need to randomize for strategic reasons. It was generalized to the context of networks by Cuff et al. [3] where, in particular, the formulation allows for communication between the agents attempting to create i.i.d. copies of a target joint distribution, with the communication occurring at the level of blocks of symbols, see also [4]. For instance, for two agents, one can seek to find the minimum communication rate required for a pair of sender and receiver to synthesize a channel with a given input distribution in a distributed way. Specifically, the sender and receiver share a sequence of common random variables \( W^n \). After observing a source \( X^n \sim \pi^n_X \) and the common randomness \( W^n \), the sender generates bits and send them to the receiver, who generates another source \( Y^n \) according to the common randomness \( W^n \) and the bits that he/she receives. They cooperate in such a way so that the channel induced by the code \( P_{Y^n|X^n} \) is close to a target channel \( \pi^n_{Y|X} \). If the closeness here is measured by the total variation (TV) distance between \( \pi^n_{Y^n|X^n} \) and the target joint distribution \( \pi^n_{X,Y} \), this channel synthesis problem was investigated in [5]–[8] and the minimum communication rate was completely characterized by Cuff [7]. The exact synthesis of such a channel was considered in [9]–[12] where exact synthesis here means that the synthesized channel \( P_{Y^n|X^n} \) is exactly equal to the target channel \( \pi^n_{Y|X} \). The characterization of the minimum communication rate for exact synthesis (given the shared randomness rate) is an interesting but hard problem. It is still open until now except for some cases: the exact synthesis for symmetric binary erasure source (completely characterized by Kumar, Li, and El Gamal [10]) and the doubly symmetric binary source (completely characterized by Yu and Tan [12]).

In this paper, we consider an arguably more natural variant of the channel synthesis problem, which we call the sequential channel synthesis problem, in which the encoder and the decoder work in a sequential way. Under a mild assumption on the target joint distribution we provide a complete (single-letter) characterization for the point-to-point case, which shows that the canonical symbol-by-symbol mapping is not optimal in general (but we also show that it is indeed optimal if we make an additional assumption on the encoder and decoder). We also extend this result to the broadcast scenario and the interactive communication scenario, where we provide bounds in the former case and a complete solution in the latter case under a mild assumption on the target joint distribution. Our proofs in this paper are based on a Rényi entropy method.

A. Problem Formulation

Let \( W, X, Y \) and \( B \) be finite sets. Alice and Bob share a sequence of i.i.d. random variables \( \{W_i\} \) taking values in \( W \), with each \( W_i \sim P_W \). Let \( \{X_i\} \) be a sequence of i.i.d. random variables taking values in \( X \), with each \( X_i \sim \pi_X \). We assume that \( \{X_i\} \) and \( \{W_i\} \) are independent. \( \{X_i\} \) is called the source sequence.

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Consider the following sequential channel synthesis problem. At the epoch $k$, upon observing the common random sequence $W^k$, the source sequence $X^k$, and previous communication random variables $B^{k-1}$, Alice generates $B_k \in \mathcal{B}$ by using a random mapping with conditional distribution $P_{B_k|W^kX^kB^{k-1}}$, and then sends $B_k$ to Bob. At the epoch $k$, upon observing $W^k$, $B^k$, and the previous outputs $Y^{k-1}$, Bob generates $Y_k$ taking values in $\mathcal{Y}$, by using a random mapping with conditional distribution $P_{Y_k|W^kB^kY^{k-1}}$. Given a target channel $\pi_{Y|X}$, the goal for Alice and Bob is to cooperate in this sequential manner to minimize the Kullback-Leibler (KL) divergence $D\left(\mathbf{P}_n|\mathbf{Y}_n|\mathbf{X}_n|\pi_{Y|X}\right)$ of the synthesized joint distribution $\pi_{Y|X}^n$ with respect to the target joint distribution $\pi_{Y|X}^n\pi_{Y|X}$, where $\pi_{Y|X}^n(x^n) := \prod_{i=1}^n \pi_{Y|X}(y_i|x_i)$ and $\pi_{Y|X}^n(x^n) := \prod_{i=1}^n \pi_{Y|X}(y_i|x_i)$. Here the conditional KL divergence for two conditional distributions $\mathbf{P}_U|\mathbf{V}$ and $\mathbf{P}_V|\mathbf{V}$ conditioned on the marginal distribution $\pi_{\mathbf{V}}$ is defined as

$$D\left(\mathbf{P}_U|\mathbf{V}|\mathbf{P}_V|\mathbf{V}\right) := D\left(\mathbf{P}_U|\mathbf{V}\pi_{\mathbf{V}}|\mathbf{P}_U|\mathbf{V}\pi_{\mathbf{V}}\right).$$

The channel synthesized by Alice and Bob can be expressed as

$$P_{Y^n|X^n}(y^n|x^n) := \sum_{b^n} \sum_{w^n} P_{W^n}(w^n) P_{B_k|W^kX^kB^{k-1}}(b_k|w^k,x^k,b^{k-1}) \prod_{k=1}^n P_{B_k|W^kX^kB^{k-1}}(b_k|w^k,b^{k-1}y^{k-1}) P_{Y_k|W^kxB^{k-1}}(y_k|w^k,b^k),$$

where $P_{W^n}(w^n) := \prod_{i=1}^n P_{W}(w_i)$. We are interested in characterizing

$$\Gamma (\pi_{XY}, P_W) := \lim_{n \to \infty} \frac{1}{n} \Gamma^{(n)}(\pi_{XY}, P_W),$$

where

$$\Gamma^{(n)}(\pi_{XY}, P_W) := \inf_{\left\{(P_{B_k|W^kX^kB^{k-1}}P_{Y_k|W^kxB^{k-1}})\right\}_{k=1}^n} D\left(P_{Y^n|X^n}||P_{Y^n|X^n}\pi_{n|X}^n\right).$$

and $\pi_{XY}(x,y) := \pi_X(x)\pi_Y(y|x)$. The limit in (1) exists since $\Gamma^{(n)}(\pi_{XY}, P_W)$ is subadditive in $n$.

When $P_W$ is degenerate, i.e., $W_i$ is constant for all $i$, then this corresponds to the case in which there is no common randomness. The optimal asymptotic KL divergence for this case is denoted by $\Gamma_0(\pi_{XY})$.

We assume throughout that $|\mathcal{B}| \geq 2$, where $|\mathcal{B}|$ denotes the cardinality of $\mathcal{B}$, since otherwise the problem is of no interest.

### B. Notation

We use upper-case letters, e.g., $X$, to denote a random variable on a finite alphabet $\mathcal{X}$. We use the lower-case letter $x$ to denote a realization of $X$. We denote the distribution or the probability mass function of $X$ as $P_X$, and use $Q_X$ to denote the distribution of another r.v. on the same alphabet $\mathcal{X}$. For brevity, the probability values $P_X(x)$ are sometimes written as $P(x)$, when the subscript and the parameter are the same except that the subscript is upper-case, and the parameter is lower-case.

We use $X := (X_1, X_2, \ldots, X_N)$ to denote a random vector. We use the notation $A \leftrightarrow C \leftrightarrow B$ for a triple of random variables $(A, B, C)$ to denote that $A$ and $B$ are conditionally independent given $C$. We will also use notations $H_Q(X)$ or $H(Q_X)$ to denote the entropy of $X \sim Q_X$. If the distribution is denoted by $P_X$, we sometimes write the entropy as $H(X)$ for brevity. We use $\text{supp}(P_X)$ to denote the support of $P_X$. The logarithm is taken to the natural base. Note that, as is the case for many other information-theoretic results, the results in this paper can be viewed as independent of the choice of the base of the logarithm as long as exponentiation is interpreted as being with respect to the same base. Also, for notational convenience, we will write $\text{Unif}\left[1 : e^{NR}\right]$ for a probability distribution that is uniform on $[e^{NR}]$, where for a positive integer $n$ the notation $[n]$ denotes the set $\{1, \ldots, n\}$.

Since there are different notions of conditional Rényi divergence in the literature, we give a detailed description of the notion we use. Fix distributions $P_X, Q_X$ on the same alphabet $\mathcal{X}$. For $s > 0$ the relative entropy and the Rényi divergence of order $1 + s$ are respectively defined as

$$D(P_X||Q_X) := \sum_{x \in \text{supp}(P_X)} P_X(x) \log \frac{P_X(x)}{Q_X(x)} \tag{3}$$

$$D_{1+s}(P_X||Q_X) := \frac{\log}{s} \sum_{x \in \text{supp}(P_X)} P_X(x)^{1+s}Q_X(x)^{-s}. \tag{4}$$

These are standard notions, see e.g. [13]. The conditional versions are respectively defined as

$$D(P_Y||Q_Y|X) := D(P_X P_Y|X||P_X Q_Y|X) \tag{5}$$

$$D_{1+s}(P_Y||Q_Y|X) := D_{1+s}(P_X P_Y|X||P_X Q_Y|X), \tag{6}$$

the first of these being of course standard. It is known that $D_1(P_X||Q_X) := \lim_{s \to 0} D_{1+s}(P_X||Q_X) = D(P_X||Q_X)$ so a special case of the Rényi divergence (or the conditional version) is the usual relative entropy (respectively the conditional

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1 Throughout this paper, for any sequence $(z_k, k \geq 1)$, we use the notation $z^k := (z_1, \ldots, z_k)$ for $k \geq 1$. 
version). It can be checked that the data processing inequality for relative entropy extends to the Rényi divergence, i.e. for \( s \geq 0 \),
\[
D_{1+s}(P_{XY}\|Q_{XY}) \geq D_{1+s}(P_{X}\|Q_{X})
\]

The entropy of a random variable \( X \) on a finite alphabet \( \mathcal{X} \) with probability distribution \( P_X \) can be written as
\[
H(X) := H(P_X) = \log |\mathcal{X}| - D(P_X\|U_X),
\]
where \(|\mathcal{X}|\) denotes the cardinality of \( \mathcal{X} \) and \( U_X \) denotes the uniform distribution on \( \mathcal{X} \). Thus for \( s > 0 \) the Rényi entropy of order \( 1+s \) is defined as
\[
H_{1+s}(X) := H_{1+s}(P_X) := \log |\mathcal{X}| - D_{1+s}(P_X\|U_X) = -\frac{1}{s} \log \sum_{x \in \text{supp}(P_X)} P_X(x)^{1+s}.
\]
This is a standard notion. Note that if \( X \) and \( Y \) are independent then for all \( s > 0 \) we have
\[
H_{1+s}(XY) = H_{1+s}(X) + H_{1+s}(Y).
\]

For the conditional versions, for conditional entropy we have
\[
H(Y|X) = H(XY) - H(X) = H(P_{Y|X}|P_X) = \log |\mathcal{Y}| - D(P_{Y|X}\|U_Y|P_X).
\]
Thus for \( s > 0 \) the conditional Rényi entropy of order \( 1+s \) is defined as
\[
H_{1+s}(Y|X) := H_{1+s}(P_{Y|X}|P_X) := \log |\mathcal{Y}| - D_{1+s}(P_{Y|X}\|U_Y|P_X) = -\frac{1}{s} \log \sum_{x \in \text{supp}(P_X)} P_X(x) \sum_{y \in \text{supp}(P_Y)} P_Y(y|x)^{1+s}.
\]
It can be checked that we have \( \lim_{s \to 0} H_{1+s}(X) = H(X) \) and \( \lim_{s \to 0} H_{1+s}(Y|X) = H(Y|X) \).

With these definitions, as a caveat we note that while it is true that
\[
H(Y|X) = \sum_{x \in \text{supp}(P_X)} P_X(x) H(Y|X = x),
\]
where \( H(Y|X = x) \) denotes the entropy of the probability distribution \( (P_{Y|X}(y|x), y \in \mathcal{Y}) \), for \( s > 0 \) we have in general that
\[
H_{1+s}(Y|X) \neq \sum_{x \in \text{supp}(P_X)} P_X(x) H_{1+s}(Y|X = x),
\]
where we will use the notation \( H_{1+s}(Y|X = x) \) to denote the Rényi entropy of the probability distribution \( (P_{Y|X}(y|x), y \in \mathcal{Y}) \) (similarly, for instance, \( H_{1+s}(Z|Y, X = x) \) will denote the conditional Rényi entropy under the joint probability distribution \( (P_{Z|Y,X}(y,z|x), (y,z) \in \mathcal{Y} \times \mathcal{Z}) \)). Note that \( \lim_{s \to 0} H_{1+s}(Y|X = x) = H(Y|X = x) \).

Similarly the chain rule does not hold for Rényi entropy, i.e. for \( s > 0 \) in general we have
\[
H_{1+s}(YZ|X) \neq H_{1+s}(Y|X) + H_{1+s}(Z|XY).
\]

On the other hand, if \( Z \) is independent of \( (X, Y) \) then we have
\[
H_{1+s}(YZ|X) = H_{1+s}(Y|X) + H_{1+s}(Z).
\]

In this document we do not need the Rényi divergence and related notions for \( s < 0 \).

II. THE POINT-TO-POINT CASE

For the sequential channel synthesis problem, in this section we provide a single-letter characterization of \( \Gamma(\pi_{XY}, P_W) \) in Theorem 1, which is one of our main results. Define \( \Psi : \mathbb{R} \to [0, +\infty) \) by
\[
\Psi(t) := \min_{P_{UX}, P_{Y|UX}, P_W: \text{H}(U|V) < H(B(U|X|Y)V) + t} D(P_{Y|XV}\|\pi_{Y|X}\|\pi_{X}P_{Y|V}),
\]
where \( B \in \mathcal{B} \), and all the entropies in (7) are evaluated at the joint distribution \( \pi_{XY}P_{UV}P_{B|XUV}P_{Y|BUV} \) and the distribution \( P_{Y|XV} \) is also induced by this joint distribution. Note that the minimum in (7) is achieved because a nonnegative lower semicontinuous function achieves its minimum on a compact set. Denote \( t_{\min} \) as the infimum of \( t \in \mathbb{R} \) such that \( \Psi(t) < +\infty \).

**Lemma 1.** 1) \( \Psi(t) \) is convex and nonincreasing on \( \mathbb{R} \). Moreover, \( \Psi(t) \) is equal to \( +\infty \) on \((-\infty, t_{\min})\), and continuous on \([t_{\min}, +\infty)\).

2) A sufficient condition for \( t_{\min} < 0 \) is the following assumption.

**Assumption 1:** \( |B| \geq 2 \) and there is at least one \( y \) such that \( \pi_{Y|X}(y|x) > 0 \) for all \( x \) such that \( \pi_X(x) > 0 \).
Proof: We first prove Statement 1. Since both the objective function and constraint functions are linear in $P_Y$ given $(P_{U|V}, P_B|X|Y, P_{Y|BUV})$, $\Psi(t)$ is in fact a convex function. This can be shown by the standard argument that for two tuples of r.v.'s $(X_1, V_1, U_1, B_1, Y_1)$ and $(X_2, V_2, U_2, B_2, Y_2)$, we can define a new r.v. $(X, U, B, Y) := (X_J, U_J, B_J, Y_J)$ and $V := (V_J, J)$ where $J \sim \text{Bern}(p)$ is independent of $V_1, V_2$. Then, the resultant objective function and constraint functions are the averages (with respect to $\text{Bern}(p)$) of those for $(X_1, V_1, U_1, B_1, Y_1)$ and $(X_2, V_2, U_2, B_2, Y_2)$. By the convexity, $\Psi(t)$ is continuous on $(t_{\min}, +\infty)$.

We next prove Statement 2. If we choose $U, V$ as constants, $B \sim \text{Unif}(B)$, $X$ are mutually independent, and $Y = y$ as constant as well (here $y$ is the element given in the lemma), then this set of distributions is feasible if $t > -\log|B|$ and the resultant value is finite. Hence, $t_{\min} < 0$.

Denote $\Delta(\pi_{XY}, P_W) := \Psi(\mathcal{H}(W))$. The proof of the following theorem is provided in Appendix A.

**Theorem 1.** Under Assumption 1, we have

$$\Gamma(\pi_{XY}, P_W) = \Delta(\pi_{XY}, P_W). \tag{8}$$

Furthermore, it suffices to restrict the cardinality of $U$ and $V$ in the calculation of $\Delta(\pi_{XY}, P_W)$ such that $|U| \leq 2$ and $|V| \leq 2 |X| |Y|$.

Remark 1. Note that $\Delta(\pi_{XY}, P_W)$ depends on $P_W$ only through its entropy $\mathcal{H}(W)$.

We next consider the case in which the stochastic encoder $P_{B_k|X^k Y^{k-1}}$ and decoder $P_{Y_k|X^k Y^{k-1}}$ are respectively replaced by $P_{B_k|X^k}$ and $P_{Y_k|X^k Y^{k-1}}$. In other words, in this case, it is not allowed to extract common randomness for the communication at the $k$-th epoch from the previous communication bits $B^{k-1}$ and there is no externally provided common randomness. We next show that a symbol-by-symbol mapping suffices to achieve the optimal KL divergence for this case, which we denote by $\Gamma_0(\pi_{XY})$, as shown in the following result.

Remark 2. Note that $\Gamma_0(\pi_{XY})$ is a priori smaller than $\Gamma_0(\pi_{XY})$, because the latter allows for stochastic encoders of the form $P_{B_k|X^k Y^{k-1}}$ which, with decoders of the form $P_{Y_k|X^k Y^{k-1}}$, allows for the possibility of extracting common randomness from the communication.

**Theorem 2.** If the stochastic encoder $P_{B_k|X^k Y^{k-1}}$ and decoder $P_{Y_k|X^k Y^{k-1}}$ are respectively replaced by $P_{B_k|X^k}$ and $P_{Y_k|X^k Y^{k-1}}$, then

$$\Gamma_0(\pi_{XY}) = \Delta(\pi_{XY}) := \inf_{P_{B_k|X^k}, P_{Y_k|X^k Y^{k-1}}} D(\pi_{X|Y} \parallel \pi_{X|Y})$$

where $B \in \mathcal{B}$ and $P_{Y_k|X}$ is induced by the joint distribution $\pi_{XY}$.

Proof: It is easy to see that $\Gamma_0(\pi_{XY}) \leq \Delta(\pi_{XY})$ since $\Delta(\pi_{XY})$ is achievable by a communication scheme consisting of symbol-by-symbol mappings.

On the other hand,

$$D(\pi_{X^n|X^n} \parallel \pi_{Y^n|X^n})$$

$$= D\left(\prod_{k=1}^n P_{Y_k|X^n Y^{k-1}} \parallel \pi^n_{Y|X^n} \right)$$

$$= \sum_{k=1}^n D(P_{Y_k|X^n Y^{k-1}} \parallel \pi_{Y|X} P_{Y^{k-1}|X})$$

$$\geq \sum_{k=1}^n D(\pi_k|X) \geq \sum_{k=1}^n \min_{x^{k-1}} D\left(P_{Y_k|X^{x^{k-1}}=x^{k-1}} \parallel \pi_{Y|X} \right) \tag{9}$$

where (9) follows from the convexity of $D(p||q)$ in the pair $(p, q)$, and (10) follows since

$$D(\pi_{Y_k|X^n} \parallel \pi_{Y^n|X^n}) = \mathbb{E}_{X^{k-1} \sim \pi_{X^{k-1}}} D\left(P_{Y_k|X^{x^{k-1}}=x^{k-1}} \parallel \pi_{Y|X} \right)$$

$$\geq \min_{x^{k-1}} D\left(P_{Y_k|X^{x^{k-1}}=x^{k-1}} \parallel \pi_{Y|X} \right). \tag{10}$$

It is easy to verify that (note that the following does not hold if we consider encoder $P_{B_k|X^k Y^{k-1}}$)

$$P_{X^n B_k Y_k}(x^k, b_k, y_k) = \sum_{b_{k-1}^{k-1}, x_{k-1}^{k-1}} \pi^k_X(x^k) P_{B_k-1|X^{k-1}} (b_{k-1}|x_{k-1}^{k-1}) P_{Y_{k-1}|B^{k-1}X^{k-1}} (y_{k-1}|b_{k-1}, x_{k-1}^{k-1})$$

$$\times P_{B_k|X^k} (b_k|x^k) P_{Y_k|B^k Y^{k-1}} (y_k|b_k, y_{k-1}^{k-1})$$

$$= \pi^k_X(x^k) P_{B_k|X^k} (b_k|x^k) P_{Y_k|B_k X^{k-1}} (y_k|b_k, x_{k-1}^{k-1}). \tag{11}$$

$$= \pi^k_X(x^k) P_{B_k|X^k} (b_k|x^k) P_{Y_k|B_k X^{k-1}} (y_k|b_k, x_{k-1}^{k-1}). \tag{12}$$
Let $\hat{x}^{k-1}$ be the optimal sequence that attains the minimum in (10). Then given $X^{k-1} = \hat{x}^{k-1}$,

$$
P_{X_k B_k Y_k | X^{k-1}} (x_k, b_k, y_k | \hat{x}^{k-1}) = \pi_X (x_k) P_{B_k | X^k} (b_k | x_k, \hat{x}^{k-1}) P_{Y_k | B_k, X^{k-1}} (y_k | b_k, \hat{x}^{k-1}).$$

By identifying $P_{B | X} = P_{B_k | X_k, X^{k-1} = \hat{x}^{k-1}}, P_{Y | B} = P_{Y_k | B_k, X^{k-1} = \hat{x}^{k-1}},$ we have $\Gamma_0 (\pi_{XY}) \geq \widetilde{\Delta} (\pi_{XY}).$

### III. The Broadcast Case

We now consider the sequential channel synthesis problem over a noiseless broadcast channel. Let $\mathcal{W}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and $\mathcal{B}$ be finite sets. Assume that $|\mathcal{B}| \geq 2$. Assume that a sender Alice and two receivers Bob and Charles share a common random sequence $W^k$; in addition to this, Alice and Bob also share another common random sequence $\hat{W}^k$. Here $\{W_i\}$ is an i.i.d. sequence of random variables taking values in $\mathcal{W}$ with each $W_i \sim P_W$ and $\{\hat{W}_i\}$ is an i.i.d. sequence of random variables taking values in $\hat{W}$ with each $\hat{W}_i \sim \hat{P}_W$. There is also a sequence of random variables $\{X_i\}$ taking values in $\mathcal{X}$, with $X_i \sim P_X$. We assume that $\{X_i\}, \{W_i\}$ and $\{\hat{W}_i\}$ are mutually independent. The sequence $\{X_i\}$ is called the source sequence and is observed only by Alice.

At the epoch $k$, upon observing the random sequences $(W^k, \hat{W}^k)$, the source sequence $X^k$, and previous communication random variables $B^{k-1}$, Alice generates $B_k \in \mathcal{B}$ by using a random mapping with conditional distribution $P_{B_k | W^k, \hat{W}^k, X^k, B^{k-1}}$, and then sends $B_k$ to Bob and Charles. Upon observing $W^k, \hat{W}^k, B^k$, and previous outputs $Y^{k-1}$, Bob generates $Y_k$ by using a random mapping with conditional distribution $P_{Y_k | W^k, \hat{W}^k, B^k, Y^{k-1}}$. Bob observes $Y_k$ by using a random mapping with conditional distribution $P_{Z_k | W^k, \hat{W}^k, X^k, B^{k-1}, Y^{k-1}, Z^{k-1}}$. Given a target broadcast channel $\pi_{Y X | Z}$, the goal is for Alice, Bob, and Charles to cooperate in this sequential manner to minimize the KL divergence

$$D \left( P_{Y^k Z^k | X^k} || \pi^n_{Y | X} | \pi^n_{Z | X} \right)$$

between the synthesized joint distribution $\pi^n_{Y^k Z^k | X^k}$ and the target joint distribution $\pi^n_{Y | X} \pi^n_{Z | X}$. Here the broadcast channel from Alice to Bob and Charles that has been synthesized is

$$P_{Y^k Z^k | X^k} (y^n, z^n | x^n) := \sum_{w^n, u^n, b^n} P_W (w^n) P_{\hat{W}} (u^n) \prod_{k=1}^n P_{B_k | W^k, \hat{W}^k, X^k, B^{k-1}} (b_k | w^k, u^k, x^k, b^{k-1}) \times \prod_{k=1}^n P_{Y_k | W^k, \hat{W}^k, B^k, Y^{k-1}} (y_k | w^k, u^k, b^k, y^{k-1}) \prod_{k=1}^n P_{Z_k | W^k, B^k, Z^{k-1}} (z_k | w^k, b^k, z^{k-1}).$$

We are interested in characterizing

$$\Gamma (\pi_{XYZ}, P_W P_{\hat{W}}) := \lim_{n \to \infty} \inf \left\{ \left( P_{B_k | W^k, \hat{W}^k, X^k, B^{k-1}, P_{Y_k | W^k, \hat{W}^k, B^k, Y^{k-1}} \right) \right\}^{n-k=1} \frac{1}{n} D \left( P_{Y^k Z^k | X^k} || \pi^n_{Y | X} | \pi^n_{Z | X} \right).$$

For this sequential broadcast channel synthesis problem, we prove the following result. The proof is provided in Appendix B.

**Theorem 3.** Assume $|\mathcal{B}| \geq 2$ and there is at least one pair $(y, z)$ such that $\pi_{Y | X} (y, z | x) > 0$ for all $x$ such that $\pi_X (x) > 0$. Then we have

$$\Delta (\pi_{XYZ}, P_W P_{\hat{W}}) \leq \Gamma (\pi_{XYZ}, P_W P_{\hat{W}}) \leq \breve{\Delta} (\pi_{XYZ}, P_W P_{\hat{W}})$$

where

$$\Delta (\pi_{XYZ}, P_W P_{\hat{W}}) := \min_{P_{UV \cdot P_{B | X U V} P_{Y | B U V} P_{Z | B U V} : H(U | Y) \leq H(W) + H(B | X U V), \ H(U \hat{U} | V) \leq H(W) + H(U \hat{U} | X U V) \ Y Z V \ Y Z V} D \left( P_{Y^k Z^k | X^k} || \pi^n_{Y | X} | \pi^n_{Z | X} P_Y \right).$$

and $\breve{\Delta} (\pi_{XYZ}, P_W P_{\hat{W}})$ is defined as the expression identical to $\Delta (\pi_{XYZ}, P_W P_{\hat{W}})$ except that $I (B; \hat{U} | X Y Z U V)$ is additionally added to the LHS in the first constraint. Here all the entropies are evaluated at the joint distribution

$$\pi_{X P_{U U} P_B \hat{U} U V P_Y \hat{U} U V P_{Z | B U V} P_{Z | B U V}}$$

and the distribution $P_{Y Z | X V}$ is also induced by this joint distribution. Furthermore, it suffices to restrict the cardinality of $V, U$ and $\hat{U}$ in the calculation of $\Delta (\pi_{XYZ}, P_W P_{\hat{W}})$ such that $|V| \leq 3$, $|U| \leq 3(|X| |Y| |Z| + 1)$, and $|\hat{U}| \leq 3(|X| |Y| |Z| + 1)(|B| |X| |Y| |Z| + 1)$. Similarly, it suffices to restrict the cardinality of $V, U$ and $\hat{U}$ in the calculation of $\Delta (\pi_{XYZ}, P_W P_{\hat{W}})$ such that $|V| \leq 3$, $|U| \leq 3(|X| |Y| |Z| + 1)$, and $|\hat{U}| \leq 3(|X| |Y| |Z| + 1)(|B| |X| |Y| |Z| + 1)$. 

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IV. THE INTERACTIVE COMMUNICATION CASE

We now consider the sequential channel synthesis problem over a noiseless two-way channel. Let \( (S_k, X_k) \) be a memoryless source with \( (S_k, X_k) \sim \pi_{SX} \) for all \( k \). At epoch \( k \), upon observing the common random sequence \( W^k \), the source sequence \( S^k \), previous communication random variables \( (A^{k-1}, B^{k-1}) \), and the previous output \( Y^{k-1} \), Alice generates \( A_k \in A \) by using a random mapping \( P_{A_k | S_k, A^{k-1}, B^{k-1}, Y^{k-1}, W^k} \), and then sends it to Bob. At the same epoch, upon observing the common random sequence \( W^k \), previous communication random variables \( (A^{k-1}, B^{k-1}) \), and the previous output \( Z^{k-1} \), Bob generates \( B_k \in B \) by using a random mapping \( P_{B_k | X^k, A^{k-1}, B^{k-1}, Z^{k-1}, W^k} \), and then sends it to Alice.

Also at epoch \( k \), upon observing \( W^k, A^k, B^k \), source sequence \( S^k \), and previous outputs \( Y^{k-1} \), Alice generates \( Y_k \) by using a random mapping \( P_{Y_k | A^k, B^k, S^k, Y^{k-1}} \). Upon observing \( W^k, A^k, B^k \), source sequence \( X^k \), and previous outputs \( Z^{k-1} \), Bob generates a r.v. \( Z_k \) by using a random mapping \( P_{Z_k | X^k, A^k, B^k, Z^{k-1}, X^{k-1}} \). Given a target channel \( \pi_{YZ|SX} \), Alice and Bob cooperate to minimize the KL divergence \( D \left( P_{Y^n Z^n | S^n X^n} \parallel \pi_{YZ|SX}^{n} \right) \) between the synthesized channel and the target channel. We are interested in characterizing

\[
\Gamma (\pi_{XYZ}, P_W) := \lim_{n \to \infty} \inf \left\{ \left( P_{A_k | S_k, A^{k-1} B^{k-1} Y^{k-1} W^k}, P_{B_k | X^{k} A^k B^{k-1} Y^{k-1} W^k}, P_{Z_k | X^k A^k B^{k-1} Z^{k-1} W^k} \right): \right. \\
\left. H(U|V) \leq H(ABU|SXVY) + H(W) \right\}
\]

Let

\[
\Delta (\pi_{XYZ}, P_W) := \min_{P_{UV}, P_{BU}, P_{BV}, P_{BY}, P_{Z}} D \left( P_{Y^n Z^n | S^n X^n} \parallel \pi_{YZ|SX}^{n} \right) \]

where \( A \in A, B \in B \) and all the entropies above are evaluated at the joint distribution \( \pi_{XY} P_{UV} P_{BU} P_{BV} P_{BY} P_{Z} \) and the distribution \( P_{Y^n Z^n | S^n X^n} \) is also included by this joint distribution. Note that these expressions depend on \( P_W \) only through \( H(W) \).

For this interactive version of sequential channel synthesis problem, we prove the following result. The proof is provided in Appendix C.

Theorem 4. Assume that \(|A|, |B| \geq 2\) and there is at least one \((y, z)\) such that \( \pi_{YZ|SX}(y, z|s, x) > 0 \) for all \((s, x)\) such that \( \pi_{SX}(s, x) > 0 \). We have

\[
\Gamma (\pi_{XYZ}, P_W) = \Delta (\pi_{XYZ}, P_W).
\]

Furthermore, in the calculation of \( \Delta (\pi_{XYZ}, P_W) \) it suffices to restrict the cardinality of \( U \) and \( V \) such that \(|V| \leq 2 |X| |Y| |Z| \).

APPENDIX A

PROOF OF THEOREM 1

A. Cardinality Bounds

To prove the claimed cardinality bounds for \( \Delta (\pi_{XY}, P_W) \), it suffices to prove that the same cardinality bounds hold for \( \Psi(t) \) with \( t \geq 0 \). Note that the constraint in (7) can be rewritten as \( H(XY|V) - H(BXY|UV) \leq t \). By the support lemma in [14, Appendix C], the cardinality of \( V \) can be upper bounded by 2, without changing the constraint function and the objective function (both of which are linear in \( P_V \)).

Applying the support lemma in [14, Appendix C] again, for each \( v \), we can restrict the size of the support of \( P_{U|V = v} \), no larger than \(|X||Y| \) without changing the linear functionals \( P_{XY|V = v} \) and \( H(BXY|UV = v) \), and hence also without changing the constraint function and the objective function. Therefore, the cardinality of \( U \) can be upper bounded by \( 2|X||Y| \).

B. Achievability

To prove the achievability part, i.e., \( \Gamma (\pi_{XY}, P_W) \leq \Delta (\pi_{XY}, P_W) \), we first prove

\[
\Gamma (\pi_{XY}, P_W) \leq \Delta (\pi_{XY}, P_W) := \inf_{P_{U}, P_{BU}, P_{BY}} D \left( P_{Y|X} \parallel \pi_{Y|X} \right).
\]

Let \( (Q_U, Q_{BU}, Q_{BY}) \) be a tuple that satisfy the constraints in the expression on the RHS of (18). For the achievability proof we will adopt block-by-block codes. For brevity, for a sequence of r.v.'s \( Z_i \), we denote \( Z := Z^N \) and \( Z_k := Z_{(k-1)N+1}^{kN} \). We will also use the notation \( Z_k \) for \( Z_{(k-1)N+1} \) for \( k \geq 1 \) and \( 1 \leq i \leq N \) when \( N \) is known from the context. Let \( C := \{ M(b, w) : (b, w) \in B^N \times W^N \} \) be a random binning codebook where \( M(b, w) \sim \text{Unif} \left[ 1 : e^{NR} \right] \) are generated independently. Let \( C_k = \{ M(b, w) : (b, w) \in B^N \times W^N \} \) be independent copies of \( C \). The codebook \( C_k \) will be used to generate a nearly uniform r.v. from the previous block of communication bits \( B_{k-1} \) and the common randomness \( W_{k-1} \). Let \( \hat{C} := \{ U(m) : m \in \left[ 1 : e^{NR} \right] \} \)
be another random codebook, where \( \mathbf{U} (i) \sim \tilde{Q}_U \) are generated independently with \( \tilde{Q}_U \) denoting the following truncated product distribution:

\[
\tilde{Q}_U = \frac{Q_U^{N \mathbf{1}_{T_e^{(N)} (Q_U)}}}{Q_U^{N \left( T_e^{(N)} (Q_U) \right)}}.
\]

(Here \( T_e^{(N)} (Q_U) \) denotes the set of \( \epsilon \)-typical sequences of length \( N \) with respect to the marginal distribution \( Q_U \).) Let \( \hat{C}_k, k = 1, 2, \ldots \) be independent copies of \( C \). The codebook \( \hat{C}_k \) will be used to generate a nearly i.i.d. r.v. from the output of \( C_k \). The codebook sequences \( \{ C_k \}, \{ \hat{C}_k \} \) are shared by both the terminals. We choose the rate \( R \) in these two sequence of codebooks such that

\[
I_Q (U; XY) < R < H (W) + H_Q (B | XY U).
\]

It can be checked that this is feasible because \( (Q_U, Q_B | XY U, Q_V | BY U) \) satisfies the constraints in the expression on the RHS of (18).

We now describe our scheme in detail. Consider the following sequence of block codes with each block consisting of \( N \) symbols. For the first block (from epoch 1 to epoch \( N \)), the encoder sends a sequence of i.i.d. uniform r.v.’s \( B_t \sim \text{Unif} (B) \) to the decoder, where \( B_1 \) is independent of \( X_1 \). The decoder generates \( Y_1 \) with a fixed distribution \( \hat{Q}_Y \) where \( \hat{Q}_Y \) is an optimal distribution attaining \( \Delta := \min_{Q_Y} D (Q_Y \| \pi_Y | X | \pi_X) \). Note that \( \Delta \) is finite by assumption. Furthermore, \( M_1, U_1 \) are set to be constant. Obviously, \( B_1, X_1, Y_1 \) are independent of \( C_1, \hat{C}_1 \).

For the \( k \)-th block (from epoch \( (k - 1) N + 1 \) to epoch \( k N \)) with \( k \geq 2 \), the encoder and decoder adopt the following strategy. First the encoder and decoder extract common randomness \( M_k \) from the previous block of communication bits \( B_{k-1} \) and common randomness \( W_{k-1} \), by using random binning based on the codebook \( C_k \). That is, the encoder and decoder generate \( M_k = M (B_{k-1}, W_{k-1}) \), where \( M (b, w) \) is the codeword indexed by \( (b, w) \) in \( C_k \). Next, the encoder and decoder generate \( U_k = U (M_k) \) based on the codebook \( C_k \), where \( U (m) \) is the codeword indexed by \( m \) in \( C_k \). Then by using \( (X_k, U_k) \), the encoder generates \( B_k \) according to the product conditional distribution \( Q_B^{N} | X U \). In fact, the random binning code in the encoder forms a privacy amplification code with \( (B_{k-1}, W_{k-1}) \) as public sources and \( (X_{k-1}, Y_{k-1}, U_{k-1}) \) as private sources. (The target in privacy amplification is to maximize the alphabet size of the output r.v. \( M_k \), generated from the public sources, under the condition that \( M_k \) is nearly uniform and nearly independent of the private sources.) At the decoder side, upon observing \( (B_k, U_k) \) the decoder generates \( Y_k \) according to the product conditional distribution \( Q_Y^{N} | B U \). Note that this corresponds to the channel resolvability problem for the channel \( Q_{XY | U} \) with \( M_k \) considered as the input. (The target in a channel resolvability problem is to synthesize a target output distribution of a channel over a block by inputting an input block that is a function of a uniform r.v., usually with the least alphabet size.)

The distribution for the first \( K \) blocks in this code can be expressed as

\[
P_{C^K \hat{C}^K W^K M^K U^K X^K B^K Y^K} = P_{C^K} P_{\hat{C}^K} P_{W^K} P_{M^K} P_{U^K} (P_{Y_1} \tilde{Q}_Y) \prod_{k=2}^{K} (P_{M_k | B_{k-1}, W_{k-1}, C_k} P_{U_k | M_k, \hat{C}_k} Q_{B^{N} | X U} Q_{Y^{N} | B U}),
\]

where \( P_{M_1}, P_{U_1} \) are some Dirac measures, \( P_{B_1} \) is as described above, \( P_{M_k | B_{k-1}, W_{k-1}, \hat{C}_k} \) corresponds to the deterministic function \( M_k = M (B_{k-1}, W_{k-1}) \) with \( M (b, w) \) denoting the codeword indexed by \( (b, w) \) in \( C_k \), and \( P_{U_k | M_k, \hat{C}_k} \) corresponds to the deterministic function \( U_k = U (M_k) \) with \( U (m) \) denoting the codeword indexed by \( m \) in \( \hat{C}_k \).

Although the code above is random (since the codebooks are random), we next show that for this random code,

\[
\frac{1}{K N} D \left( P_{Y^K | X^K \hat{C}^K} \| \pi_{Y^K | X}^{KN} P_{\hat{C}^K} P_{C^K} \right) \rightarrow D \left( Q_{Y | X} \| \pi_{Y | X} \right)
\]

as \( K \to \infty \) and \( N \to \infty \) along an appropriately chosen sequence, which implies that there is a sequence of deterministic codebooks \( (\hat{c}^K, \hat{c}^K) \) satisfying

\[
\frac{1}{K N} D \left( P_{Y^K | X^K, \hat{C}^K = c^K, \hat{C}^K = \hat{c}^K} \| \pi_{Y^K | X}^{KN} \right) \rightarrow D \left( Q_{Y | X} \| \pi_{Y | X} \right),
\]

as \( K \to \infty \) and \( N \to \infty \) along the same sequence.

For the random code above, we have the following lemma.

**Lemma 2.** For the random code above,

\[
D \left( P_{M_k | X_{k-1}, Y_{k-1}, U_{k-1}, C_k} \| \text{Unif} \left[ 1 : e^{NR} \right] \right) \rightarrow 0 \quad (20)
\]

\[
D \left( P_{Y_k | X_k, \hat{C}_k} \| Q_{Y_k | X}^{N} \right) \rightarrow 0 \quad (21)
\]

uniformly for all \( k \geq 2 \) as \( N \to \infty \).

The proof of this lemma is given in Appendix A-D.
For the first $K$ blocks induced by the code above we have

$$
D \left( P_{Y|X}^{K} || \pi^X \pi^Y \pi^K \right) \\
= \sum_{x^K, y^K \in \mathcal{X}^K} P(x^K) P(y^K) \log \frac{P(y^K|x^K)}{\pi(y^K)}
$$

(22)

where $y^K = (y_1, y_2, \ldots, y_K)$ and $x^K = (x_1, x_2, \ldots, x_K)$. Hence, combining (20), (26), and (28), for $k \geq 2$, we have $I \left( Y_k; X^{k-1} Y^{k-1} | X_k C^k \right) \rightarrow 0$.

We next consider the second term in (24) for $k \geq 2$.

$$
D \left( P_{Y_k|X}^{K} || \pi^X \pi^Y \pi^K \right) = D \left( P_{Y_k|X}^{K} || \pi^X \pi^Y \pi^K \right) + \sum_{x_k, y_k, c_k, \hat{c}_k} P(c_k, \hat{c}_k) \log \frac{P(y_k|x_k, c_k, \hat{c}_k)}{\pi(y_k|x_k)}
$$

(23)

By (21), $D \left( P_{Y_k|X}^{K} || \pi^X \pi^Y \pi^K \right) \rightarrow 0$. Denote $J$ as a random time index, which is independent of all other r.v.’s involved in the system. Observe that $\pi_x P_{Y_j|X}^{K}$ and $\pi_x Q_{Y_j|X}$ are respectively the output distributions of the channel $(X_j, Y_j) \mapsto (X, Y)$ with input distributions $\pi^X P_{Y_j|X}$ and $\pi^X Q_{Y_j|X}$. Hence by the data processing inequality concerning relative entropy, we have for $k \geq 2$,

$$
D \left( P_{Y_j|X}^{K} || \pi^X \pi^Y \pi^K \right) \leq D \left( P_{Y_k|X}^{K} || \pi^X \pi^Y \pi^K \right) \rightarrow 0.
$$

(24)
By Pinsker's inequality, this further implies that \( P^k_c P^k_c \pi_X P_{Y|X}X_c^k \) converges to \( P^k_c P^k_c \pi_X Q_{Y|X} \) under the total variation distance, which further implies that
\[
\frac{1}{N} \sum_{x_k, y_k, c_k} P(c_k, \hat{c}_k) \pi(x_k) P(y_k|x_k, c_k, \hat{c}_k) \log \frac{Q(y_k|x_k)}{\pi(y_k|x_k)} - D(Q_{Y|X}||\pi_{Y|X}|\pi_X) 
\]
\[
= \sum_{x, y, c, \hat{c}} P(c_k, \hat{c}_k) \pi_X(x) P_{Y|X}X_c^k(y|x, c, \hat{c}_k) - Q_{Y|X}(y|x) \log \frac{Q_{Y|X}(y|x)}{\pi_{Y|X}(y|x)} 
\]
\[
\leq \sum_{x, y, c, \hat{c}_k} P(c_k, \hat{c}_k) \pi_X(x) \left| P_{Y|X}X_c^k(y|x, c, \hat{c}_k) - Q_{Y|X}(y|x) \right| \log \frac{Q_{Y|X}(y|x)}{\pi_{Y|X}(y|x)} 
\]
\[
\leq \left| P^k_c P^k_c \pi_X P_{Y|X}X_c^k \right|_{TV} \times \max_{x, y} \left| \log \frac{Q_{Y|X}(y|x)}{\pi_{Y|X}(y|x)} \right| 
\]
\[
\to 0. 
\]
In the last inequality above, the max term is finite since \( D(Q_{Y|X}||\pi_{Y|X}|\pi_X) \) is finite and \( \pi_X \) is fully supported. Hence, for \( k \geq 2 \), \( \frac{1}{N} \Delta \left( P_{Y|X}X_c^k \right) \to D(Q_{Y|X}||\pi_{Y|X}|\pi_X) \).

Hence combining the two points above for any given \( N \) and noting that both the summands for \( k = 1 \) in the first and second summations in \( (24) \) are finite, we have that \( \frac{1}{K \cdot N} D \left( P_{Y|X}X_c^k \pi_{Y|X}|\pi_X \right) \to D \left( Q_{Y|X}||\pi_{Y|X}|\pi_X \right) \) as \( K \to \infty \), which implies that there is a sequence of deterministic codebooks \( (c^K, \hat{c}^K) \) satisfying
\[
\frac{1}{K \cdot N} D \left( P_{Y|X}X_c^{K+\epsilon}c^{K+\epsilon} \pi_{Y|X}|\pi_X \right) \to D \left( Q_{Y|X}||\pi_{Y|X}|\pi_X \right). 
\]

Hence, \( \Gamma(\pi_{XY}, P_W) \leq \Delta(\pi_{XY}, P_W) \).

We next extend the code above to the \( m \)-letter version by substituting
\[
(P_{W|C}^{m}, \pi_{C}^{m}, \pi_{X|C}, \pi_{Y|C}) \leftrightarrow (P_{W|C}^{m}, \pi_{C}^{m}, \pi_{U|C}, \pi_{V|C}) 
\]

into the code above. In this \( m \)-letter code, the basic unit is the supersymbol which consists of \( m \) successive original letters. Even so, by definition, the random mappings \( Q_{B|X}^{m}|\pi_{X|C}|\pi_{X}^{m} \) are in fact still symbol-by-symbol way, which means that the \( m \)-letter code is also a feasible code for the single-letter scenario. In other words, the encoder and decoder of this \( m \)-letter code are still a special case of \( (P_{B|X}^{m}, \pi_{X|C}, \pi_{Y|C}) \).

where \( \Gamma(\pi_{XY}, P_W) \leq \Delta(\pi_{XY}, P_W) \).

By the conditional typicality lemma \([14]\), we have that with high probability,
\[
(W^m, X^m, U^m, B^m, Y^m) \sim P_{W}^{m} \pi_{X}^{m} P_{U}^{m} (|v^m) P_{B|UXV}^{m} (|v^m) P_{Y|UXV}^{m} (|v^m) 
\]
is jointly \( \epsilon \)-typical with \( v^m \) (with respect to the distribution \( P_{W}^{m} \pi_{X}^{m} P_{U}^{m} (|v^m) P_{B|UXV}^{m} (|v^m) P_{Y|UXV}^{m} (|v^m) \) for some \( \epsilon > \epsilon' \) and sufficiently large \( m \). Hence, \( \frac{1}{m} H(U^m|V^m = v^m) = H(U|V) + o(1), \frac{1}{m} H(B^m|U^m, X^m, Y^m, V^m = v^m) = H(B|UXY) + o(1) \), \( \frac{1}{m} H(W^m|V^m = v^m) = H(W) \), and \( \frac{1}{m} D \left( P_{Y|UXV}^{m} (|v^m) \pi_{Y|X}^{m}|\pi_{X}^{m} \right) = D \left( P_{Y|UXV}^{m} (|v^m) \pi_{Y|X}^{m} \right) + o(1), \) where \( o(1) \) denotes a generic term vanishing as \( m \to \infty \). This implies the claim above.

By the claim above, we have \( \Gamma(\pi_{XY}, P_W) \leq \lim_{t \to H(W)} (t) \). Since \( H(W) \neq t_{\min}, \Psi(t) \) is continuous at \( t = H(W) \). We have \( \Gamma(\pi_{XY}, P_W) \leq \Psi(H(W)) = \Delta(\pi_{XY}, P_W) \). This completes the proof of the achievability part.

C. Converse

We next consider the converse part. Observe that
\[
D \left( P_{Y|X}^{m} (|v^m) \pi_{Y|X}^{m} \right) = D \left( \prod_{k=1}^{n} P_{Y|X}^{m} (|v^m) \pi_{Y|X}^{m} \right) 
\]
\[
= D \left( P_{Y|X}^{m} (|v^m) \pi_{Y|X}^{m} \right) P_{Y|X}^{m} (|v^m) \pi_{Y|X}^{m} \right). 
\]
Denote $K \sim \text{Unif } [1:n]$ as a random time index, which is independent of all other r.v.'s involved in the system. Define $U := (B^{K-1}, W^K), V := (X^{K-1}, Y^{K-1}, K), B := B_K, X := X_K, Y := Y_K$. Then

$$\frac{1}{n^2} D \left( P_{Y^n|X^n} \| \pi^n_{Y|X} \| \pi^n_X P_Y \right) = D \left( P_{Y|X} \| \pi_Y \| \pi_X P_Y \right).$$

(32)

It is easy to verify that

$$P_{UV, BXY}(u, v, b, x, y) = P_W(w^k) P_K(k) P_{X^{k-1}Y^{k-1}|W^k}(x^{k-1}, y^{k-1}|w^k) P_{B^{k-1}|X^{k-1}Y^{k-1}W^k}(b^{k-1}|x^{k-1}, y^{k-1}, w^k) \times \pi_X(x) P_{B|X,Y,W}(b|x, y) P_{Y|X,Y,W}(y|x, u, v) P_{Y|B,U,V}(y|b, u, v).$$

Hence it remains to show $H(U|V) \leq H(BU|XYV) + H(W)$. This can be easily verified as follows:

$$H(BU|XYV) - H(U|V)$$

$$= \frac{1}{n} \sum_{k=1}^{n} \left\{ H(B^k W^k|X^k Y^k) - H(B^{k-1} W^k|X^{k-1} Y^{k-1}) \right\}$$

(33)

$$= \frac{1}{n} \sum_{k=1}^{n} \left\{ H(B^k W^k|X^k Y^k) - H(B^{k-1} W^k-1|X^{k-1} Y^{k-1}) - H(W_k|X^{k-1} Y^{k-1} B^{k-1} W^{k-1}) \right\}$$

(34)

$$= \frac{1}{n} H(B^n W^n|X^n Y^n) - H(W)$$

$$\geq -H(W),$$

(35)

where (34) follows since $W_k$ is independent of $X^{k-1}Y^{k-1}B^{k-1}W^{k-1}$ and has entropy $H(W)$.

D. Proof of Lemma 2

We now prove Lemma 2 by using a Rényi entropy method. Recall that the rate $R$ is chosen such that

$$I_Q(U; XY) < R < H(W) + H_Q(B|XYU).$$

(36)

The condition can be relaxed to

$$(1 + \epsilon) D_{1+s}(Q_{XY|U} \| Q_{XY|Q_U}) \leq R$$

(37)

$$< (1 - \epsilon) \sum_n Q_U(u) H_{1+s}(B|XY, U = u) + H_{1+s}(W)$$

(38)

for some $\epsilon, s > 0$, since both the expressions in (37) and (38) are continuous in $\epsilon$ and $s$ and we have $H_Q(B|XYU) = \sum_u Q_U(u) H(B|XY, U = u)$. We first prove that if the upper bound on $R$ given by (38) holds, then we have (20). To show this, we need the following lemma on one-shot privacy amplification.

Lemma 3. [15, Equation (29)] Consider a random mapping $f_C : X \rightarrow M := \{1, \ldots, e^R\}$. We set $C = \{M(x) \mid x \in X \}$ with $M(x), x \in X$ drawn independently for different $x$'s and according to the uniform distribution Unif $[1 : e^R]$, and set $f_C(x) = M(x)$. This forms a random binning code. For this random code, we have for $s \in (0, 1]$ and any distribution $P_{XY}$,

$$e^{sD_{1+s}(P_{f_C(X|Y) \| \text{Unif}[1:e^R]|P_{C})} \leq 1 + e^{-s(H_{1+s}(X|Y) - R)}.$$  

(39)

Note that the codebook in this lemma is generated in the same way as the codebook $C_k$ in our scheme. By applying the lemma above with substitution $X \leftarrow (B_{k-1}, W_{k-1}), Y \leftarrow (X_{k-1}, Y_{k-1}, U_{k-1})$, we have

$$D_{1+s}(P_{M_k|X_{k-1}Y_{k-1}U_{k-1}C_k} \| \text{Unif}[1:e^R]|P_{X_{k-1}Y_{k-1}U_{k-1}} P_C) \leq \frac{1}{s} \log \left[ 1 + e^{-s(H_{1+s}(B_{k-1}|W_{k-1}X_{k-1}Y_{k-1}U_{k-1}) - NR)} \right]$$

$$\leq \frac{1}{s} e^{-s(H_{1+s}(B_{k-1}|W_{k-1}X_{k-1}Y_{k-1}U_{k-1}) - NR)}.$$  

(40)

Note that $W_{k-1}$ is in fact independent of $(B_{k-1}, X_{k-1}, Y_{k-1}, U_{k-1})$, since in the first $k-1$ blocks, only $W_1, W_2, \ldots, W_{k-2}$ are used in the encoding process. Hence,

$$H_{1+s}(B_{k-1}W_{k-1}X_{k-1}Y_{k-1}U_{k-1}) = H_{1+s}(B_{k-1}|X_{k-1}Y_{k-1}U_{k-1}) + NH_{1+s}(W).$$

(41)
On the other hand, for \( k \geq 2 \),
\[
\frac{1}{N} H_{1+s}(B_{k-1} | X_{k-1} Y_{k-1} U_{k-1}) \\
= \frac{1}{sN} \log \left[ \mathbb{E}_{U_{k-1}} \sum_{x,y} Q_{XY}^{N}(x,y | U_{k-1}) \sum_{b} Q_{B|XYU}^{N}(b | x,y, U_{k-1}) \right]^{1+s} \\
= \frac{1}{sN} \log \left[ \sum_{m} P_{M_{k-1}}(m) \prod_{i=1}^{N} \left( \sum_{x,y} Q_{XY}^{N}(x,y | U_{i}(m)) \sum_{b} Q_{B|XYU}^{N}(b | x,y, U_{i}(m)) \right)^{1+s} \right] \\
= \frac{1}{sN} \log \left[ \sum_{m} P_{M_{k-1}}(m) e^{sN \sum_{u} T_{U(m)}(u) H_{1+s}(B|XY,U=U_{i}(m))} \right] \\
\geq \frac{1}{sN} \log \left[ \sum_{m} P_{M_{k-1}}(m) e^{(1-\epsilon)sN \sum_{u} Q_{U}(u) H_{1+s}(B|XY,U=U_{i}(m))} \right] \\
= (1-\epsilon) \sum_{u} Q_{U}(u) H_{1+s}(B|XY,U=U_{i}(m)) ,
\] (45)

where \( T_{U(m)} \) in (43) denotes the empirical distribution of the sequence \( U(m) \), and (44) follows by combining the typical average lemma on p. 26 of [14] and the fact that by the construction of the codebook, all codewords \( U(m) \) come from \( T_{e^{(n)}}(Q_{U}) \). In fact, for \( k = 2 \), (45) still holds since in this case \( B_{1} \) is uniform and independent of \( X_{1}, Y_{1} \) and \( U_{1} \) is set to a constant. Substituting (41) and (44) into (45), we have (20), i.e.,
\[
D_{1+s}(P_{M_{k}|X_{k-1}Y_{k-1}U_{k-1}C_{k}} || \text{Unif } [1 : e^{NR}] | P_{X_{k-1}Y_{k-1}U_{k-1}C_{k}}) \rightarrow 0
\] (46)
uniformly for all \( k \) as \( N \rightarrow \infty \).

We next prove that if the inequality in (37) holds, then we have (21). First, by the data processing inequality,
\[
D_{1+s}(P_{M_{k}|X_{k-1}Y_{k-1}U_{k-1}C_{k}} || \text{Unif } [1 : e^{NR}] | P_{X_{k-1}Y_{k-1}U_{k-1}C_{k}} P_{C_{k}^{k-1}}) \\
\geq D_{1+s}(P_{M_{k}|C_{k}^{k-1}} || \text{Unif } [1 : e^{NR}] | P_{C_{k}^{k-1}}) .
\]

In fact, the LHS above is identical to the LHS of (20) (or (46)), since \( (X^{k-2}, Y^{k-2}, C^{k-1}, \hat{C}^{k}) \leftrightarrow (U_{k-1}, C_{k}) \leftrightarrow (X_{k-1}, Y_{k-1}, M_{k}) \) holds under the distribution \( P \) (see the reasoning around (25)). Combining this with (20), we have
\[
D_{1+s}(P_{M_{k}|C_{k}^{k-1}} || \text{Unif } [1 : e^{NR}] | P_{C_{k}^{k-1}}) \rightarrow 0
\]
uniformly for all \( k \) as \( N \rightarrow \infty \). That is,
\[
\frac{1}{N} H_{1+s}(P_{M_{k}|C_{k}^{k-1}} | P_{C_{k}^{k-1}}^{k-1}) \rightarrow R
\] (47)
\[
> (1 + \epsilon) D_{1+s}(Q_{XY}^{U} || Q_{XY}^{U} Q_{U}) .
\] (48)

Now we need the following lemma on one-shot channel resolvability. This can be proved by a technique similar to that used in Lemma 7, for which we have given a complete proof.

**Lemma 4.** [16, Lemma 1] Consider a random mapping \( f_{C} : W \rightarrow X \). We set \( C = \{ X(w) \}_{w \in W} \) with \( X(w) \), \( w \in W \) drawn independently for different \( w \)’s and according to a same distribution \( P_{X} \), and set \( f_{C}(w) = X(w) \). This forms a random code. For this random code, we have for \( s \in (0,1] \) and any distributions \( P_{W} \), \( P_{Y|X} \) and \( Q_{Y} \),
\[
e^{sD_{1+s}(P_{Y|C} || Q_{Y|C} P_{C})} \\
\leq e^{sD_{1+s}(P_{Y|X} || Q_{Y|X} P_{X}) - sH_{1+s}(P_{W})} + e^{sD_{1+s}(P_{Y} || Q_{Y|C} P_{C})},
\] (49)

where the distribution \( P_{Y|C} \) is induced by the “true” joint distribution \( P_{C} P_{W} P_{Y|X = f_{C}(W)} \), and the distribution \( P_{Y} \) is induced by the “ideal” joint distribution \( P_{X} P_{Y|X} \).

This lemma immediately implies the following conditional version.

**Lemma 5.** Under the same assumptions as in Lemma 4, for \( s \in (0,1] \) and any distributions \( P_{AW} P_{B} P_{Y|XB} \) and \( Q_{Y|B} \), we have
\[
e^{sD_{1+s}(P_{Y|AB} || Q_{Y|B} P_{B} P_{A} P_{B})} \\
\leq e^{sD_{1+s}(P_{Y|XB} || Q_{Y|B} P_{X} P_{B}) - sH_{1+s}(P_{AW} P_{A})} + e^{sD_{1+s}(P_{Y|B} || Q_{Y|B} P_{B})},
\] (50)
where the distribution $P_Y|_{ABC}$ is induced by the “true” joint distribution $P_CP_AWP_BP_XP_Y|_{X,B=X(\text{W})}$, and the distribution $P_Y|_{B}$ is induced by the “ideal” joint distribution $P_BP_XP_Y|_{X,B}$. 

Proof of Lemma 5: Applying Lemma 4 with substitution $P_W \leftarrow P_{W|a}, P_Y|_{X} \leftarrow P_{Y|X,b}$ and $Q_Y \leftarrow Q_{Y|b}$, we obtain that

$$e^{sD_{1+s}}(P_{Y|a}P_WP_Y|_{X,B=X(W)}) \leq e^{sD_{1+s}}(P_{Y|X,b}Q_Y|_{X,B}) - sH_{1+s}(P_{W|a}) + e^{sD_{1+s}}(P_{Y|B=b}Q_Y|_{B=b}).$$

(51)

Taking expectation with respect to $(A,B) \sim P_A P_B$ for the two sides above, we obtain (50).

Recall that

$$\tilde{Q}_U = \frac{Q_U^{N}1_{T_e(N)}(Q_U)}{Q_U^{N}(T_e(N)(Q_U))}$$

Note that the codebook in Lemmas 4 and 5 is generated in the same way as the codebook $\hat{C}_k$ in our scheme. Applying Lemma 5 with substitution $A \leftarrow (C^k, \hat{C}^{k-1}), B \leftarrow X_k, W \leftarrow M_k, X \leftarrow U_k, Y \leftarrow Y_k, C \leftarrow \hat{C}_k$ and the corresponding distributions $P_{AW} \leftarrow P_kP_k^{k-1}P_{M_k}|_{C^{k-1}}, P_B \leftarrow \pi_X P_X, P_Y \leftarrow \tilde{Q}_U, P_{Y|X} \leftarrow Q_Y|_{X,U}, Q_Y \leftarrow Q_{Y|X}$ (which induces $P_Y|_{ABC} = P_Y|_{X_kC^k\hat{C}_k}$), we have

$$e^{sD_{1+s}}(P_{Y|X_kC^k\hat{C}_k}Q_{Y|X_kC^k\hat{C}_k}) \leq e^{sD_{1+s}}(\tilde{Q}_Y|_{X,U}Q_{Y|X_kC^k\hat{C}_k}) - sH_{1+s}(P_{M_k}|_{C^{k-1}}P_k^{k-1}) + e^{sD_{1+s}}(\tilde{Q}_Y|_{X,U}Q_{Y|X_kC^k\hat{C}_k}).$$

(52)

where $\tilde{Q}_Y|_{X,U} := Q_{Y|X,U}$ and $\tilde{Q}_Y|_{X}$ are induced by the “ideal” joint distribution

$$\tilde{Q}_{UXY} := \tilde{Q}_U\pi_X Q_Y|_{X,U}.$$  

(53)

Note that here $P_kP_k^{k-1}P_{M_k}|_{C^{k-1}}$ and $P_{Y|X_kC^k\hat{C}_k}$ correspond to the “true” conditional distributions induced by our scheme. Moreover, according to the process of encoding, $X_k, \hat{C}_k, (C^k, \hat{C}^{k-1}, M_k)$ are mutually independent.

On one hand, $Q_U$ is not far from the product version $Q_U^{N}$, as shown in the following equations:

$$D_{1+s}(\tilde{Q}_U||Q_U^{N}) = e^{sD_{1+s}}(\tilde{Q}_U||Q_U^{N})$$

(54)

$$= \frac{1}{s} \log \sum_{u} \left( \frac{Q_U^{N}(u)}{Q_U^{N}(T_e(N)(u))} \right)^{1+s} (Q_U^{N}(u))^{-s}$$

$$= \frac{1}{s} \log \sum_{u \in T_e(N)(u)} (Q_U^{N}(u))$$

$$= \log \left( \frac{1}{Q_U^{N}(T_e(N))} \right)$$

$$\rightarrow 0,$$  

(57)

where (57) follows from the fact that $Q_U^{N}(T_e(N)) \rightarrow 1$. By the data processing inequality for Rényi divergence [13] and by the definition of the distribution $\tilde{Q}$ in (53), we have

$$D_{1+s}(\tilde{Q}_Y|_{X}|Q_{Y|X}^{N}|\pi_X^{N}) \leq D_{1+s}(\tilde{Q}_{UX}|_{X}|Q_{UX|X}^{N}|\pi_X^{N}) = D_{1+s}(\tilde{Q}_U||Q_U^{N}).$$  

(58)

Hence $D_{1+s}(\tilde{Q}_Y|_{X}|Q_{Y|X}^{N}|\pi_X^{N}) \rightarrow 0$ as well.

On the other hand, by a derivation similar to the steps from (42) to (45), we have

$$\frac{1}{N} D_{1+s}(\tilde{Q}_Y|_{X,U}|Q_{Y|X}^{N}|\pi_X^{N}) \leq (1 + \epsilon) D_{1+s}(Q_{XY|U}||Q_{XY|U}),$$

since $\tilde{Q}_Y|_{X,U} = Q_{Y|X,U}$ and any sequences $u$ such that $\tilde{Q}_U(u) > 0$ have a type close to $Q_U$.

By (52), $D_{1+s}(P_{Y|X_kC^k\hat{C}_k}||Q_{Y|X_kC^k\hat{C}_k}P_k^{k-1}P_k^{k-1}) \rightarrow 0$ since the conditions in (47) and (48) hold.
A. Cardinality Bounds

We first prove the cardinality bounds for the calculation of $\Delta \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$. Note that the constraints in (15) can be rewritten as $H(XYZ|V) - H(BXYZ|UV) \leq H(W)$ and $H(XYZ|V) - H(BXYZ|UUU) \leq H(W) + H(W)$. By the support lemma in [14, Appendix C], the cardinality of $V$ can be upper bounded by 3, without changing the constraints and the objective function. Applying the support lemma in [14, Appendix C] again, for each $v$, we can restrict the size of the support of $P_{U|V=v}$ no larger than $|X||Y||Z| + 1$ without changing the linear functionals $P_{XYZ|V=v}$ and $H(BXYZ|U=U, V=v)$, and hence also without changing the constraints and the objective function. Therefore, the cardinality of $U$ can be upper bounded by $3(|X||Y||Z| + 1)$.

Applying the support lemma in [14, Appendix C] again, for each $(u, v)$, we can restrict the size of the support of $P_{U|U=U, V=v}$ no larger than $|B||X||Y||Z|$ without changing the linear functionals $P_{BXYZ|U=U, V=v}$ and $H(BXYZ|U=U, V=v)$, and hence also without changing the constraints and the objective function (since $P_{XYZ|V=v}$ remains unchanged as well). Therefore, the cardinality of $U$ can be upper bounded by $3(|X||Y||Z| + 1)$.

We next prove the cardinality bounds for the calculation of $\Delta \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$. The constraints for this case can be rewritten as $H(XYZ|V) - H(XYZ|UV) - H(BXYZ|UUU) \leq H(W)$ and $H(XYZ|V) - H(BXYZ|UUU) \leq H(W) + H(W)$. By the support lemma in [14, Appendix C], we can restrict $|V| \leq 3$. Applying the support lemma in [14, Appendix C] again, for each $v$, we can restrict the size of the support of $P_{U|V=v}$ no larger than $|X||Y||Z| + 1$ without changing the linear functionals $P_{XYZ|V=v}$ and $H(BXYZ|U=U, V=v)$, and hence also without changing the constraints and the objective function. Therefore, the cardinality of $U$ can be upper bounded by $3(|X||Y||Z| + 1)$.

Applying the support lemma in [14, Appendix C] again, for each $(u, v)$, we can restrict the size of the support of $P_{U|U=U, V=v}$ no larger than $|B||X||Y||Z|$ without changing the linear functionals $P_{BXYZ|U=U, V=v}$ and $H(BXYZ|U=U, V=v)$, and hence also without changing the constraints and the objective function. Therefore, the cardinality of $U$ can be upper bounded by $3(|X||Y||Z| + 1)$.

B. Upper Bound

We first prove the upper bound, i.e., $\Gamma \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right) \leq \Delta \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$, by using a proof similar to that of Theorem 1. In order to do this, we prove that $\Gamma \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right) \leq \Delta^{+} \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$, where $\Delta^{+} \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$ is defined like $\Delta \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$ except that the strict inequalities in the constraints are replaced by weak inequalities and min is replaced by inf. This suffices because under the assumption that there is at least one pair $(y, z)$ such that $\pi_{Y|Z}(y, z|x) > 0$ for all $x$ such that $\pi_{X}(x) > 0$ we can show that $\Delta^{+} \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$ equals $\Delta \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$ by using an argument similar to that in Lemma 1.

Let $\overline{\Delta} \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$ be defined like $\Delta^{+} \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$ but with $V$ replaced with a constant. Let $\left( Q_{UUU}, Q_{B|XUUU}, Q_{Y|BUU}, Q_{Y|BU} \right)$ be a tuple that satisfies the constraints under the infimum in the definition of $\overline{\Delta} \left( \pi_{XYZ}, P_{W}P_{W}^{\prime} \right)$. Let

$\mathcal{C} := \left\{ \mathbf{M}(\mathbf{b}, \mathbf{w}) : (\mathbf{b}, \mathbf{w}) \in \mathcal{B}^{N} \times \mathcal{W}^{N} \right\}$

$\mathcal{C}^{'} := \left\{ \hat{\mathbf{M}}(\hat{\mathbf{w}}) : \hat{\mathbf{w}} \in \hat{\mathcal{W}}^{N} \right\}$

be two random binning codebooks where $\mathbf{M}(\mathbf{b}, \mathbf{w}) \sim \text{Unif} \left[ 1 : e^{NR} \right], \hat{\mathbf{M}}(\hat{\mathbf{w}}) \sim \text{Unif} \left[ 1 : e^{NR} \right]$ are respectively generated independently. Let $\mathcal{C}_{k}, k = 1, 2, \ldots$ be independent copies of $\mathcal{C}$ and $\mathcal{C}_{k}^{'}$, $k = 1, 2, \ldots$ be independent copies of $\mathcal{C}^{'}$. The codebook sequences $\{\mathcal{C}_{k}\}, \{\mathcal{C}_{k}^{'}\}$ are shared by all the terminals, Alice, Bob, and Charles (although $\{\mathcal{C}_{k}^{'}\}$ will not be used by Charles). Let

$\hat{\mathcal{C}} := \left\{ \left( \mathbf{U}(m), \hat{\mathbf{U}}(m, \hat{m}) \right) : m \in [1 : e^{NR}], \hat{m} \in [1 : e^{NR}] \right\}$

be another random codebook where $\mathbf{U}(m) \sim \hat{Q}_{U}, \hat{\mathbf{U}}(m, \hat{m}) \sim \hat{Q}_{U|U} (\cdot | \mathbf{U}(m))$ are generated independently. Here $\hat{Q}_{U}$ and $\hat{Q}_{U|U}$ are the following truncated product distributions:

$\hat{Q}_{U} = \frac{Q_{U}^{N}1_{T_{U}^{(N)}(Q_{U})}}{Q_{U}^{N}\left( T_{e}^{(N)}(Q_{U}) \right)},$

$\hat{Q}_{U|U} (\cdot | \mathbf{u}) = \frac{Q_{U|U}^{N} (\cdot | \mathbf{u})1_{T_{e}^{(N)}(Q_{U|U} | \mathbf{u})}}{Q_{U|U}^{N} \left( T_{e}^{(N)}(Q_{U|U} | \mathbf{u}) \right)}, \forall \mathbf{u} \in \mathcal{U}^{N}.$
Let \( \hat{C}_k, k = 1, 2, \ldots \) be independent copies of \( \hat{C} \). The codebook sequence \( \{ \hat{C}_k \} \) is also shared by all the terminals (Alice, Bob, and Charles). We choose rates \( R, \hat{R} \) such that

\[
I_Q(U; XYZ) < R < H(W) + H_Q(B|XYZU\hat{U}), \tag{59}
\]

\[
\hat{R} < H(W), \tag{60}
\]

\[
I_Q(U\hat{U}; XYZ) < R + \hat{R}.
\]

Such \( (R, \hat{R}) \) exists if and only if

\[
I_Q(U; XYZ) < H(W) + H_Q(B|XYZU\hat{U}),
\]

\[
I_Q(U\hat{U}; XYZ) < H(W) + H(W) + H_Q(B|XYZU\hat{U}),
\]

or equivalently,

\[
H_Q(U) < H(W) + H_Q(BU|XYZ) - I_Q(B; \hat{U}|XYZU),
\]

\[
H_Q(U\hat{U}) < H(W) + H(W) + H_Q(BU\hat{U}|XYZ),
\]

which are satisfied by the tuple \( (Q_U; Q_{BU|XYZ}; Q_{XYZ}; Q_{YBU}; Q_{Y|BU}) \) by assumption.

Consider the following superposition sequence. For the first block (from epoch 1 to epoch \( N \)), Alice sends a sequence of i.i.d. uniform r.v.'s \( B_t \sim \text{Unif}(B) \) to Bob and Charles, where \( B_1 \) is independent of \( X_1 \). Bob and Charles respectively generate \( Y_1 \) with a fixed distribution \( Q_{XY} \) and \( Z_1 \) with a fixed distribution \( Q_{XZ} \) where \( (Q_{XY}, Q_{XZ}) \) is an optimal distribution attaining \( \Delta := \min_{Q_{XY}, Q_{XZ}} D(Q_{XY}Q_{XZ}\|\pi_{Y|X}) \). Note that \( \Delta \) is finite by assumption. Furthermore, \( M_1, M_1, U_1, \hat{U}_1 \) are set to be constant. Obviously, \( B_1, X_1, Y_1 \) are independent of \( C_1, C'_1, \hat{C}_1 \).

For the \( k \)-th block (from epoch \((k-1)N+1\) to epoch \( kN \)) with \( k \geq 2 \), the encoder and decoder adopt the following strategy. All the terminals (Alice, Bob, and Charles) extract common randomness \( M_k \) from the previous block of communication bits \( B_{k-1} \) and common randomness \( W_{k-1} \), by using random binning based on \( C_k \). That is, they generate \( M_k = M(B_{k-1}, W_{k-1}) \), where \( M(b, w) \) is the codeword indexed by \((b, w)\) in \( C_k \). Besides, Alice and Bob also generate \( \hat{M}_k = \hat{M}(W_k) \), where \( \hat{M}(w) \) is the codeword indexed by \( w \) in \( \hat{C}_k \). Next, Alice and Bob generate \( (U_k, \hat{U}_k) = (U(M_k), \hat{U}(M_k, \hat{M}_k)) \), where \( (U(m), \hat{U}(m, \hat{m})) \) the codeword indexed by \((m, \hat{m})\) in \( \hat{C}_k \). Moreover, \( U_k \) is also available at Charles since he knows \( M_k \). Then by using \( (X_k, U_k, \hat{U}_k) \), the encoder Alice generates \( B_k \) by the product distribution \( Q_{N|BU}^{X|BU} \). At the decoder sides, upon observing \( (B_k, U_k, \hat{U}_k) \) Bob generates \( Y_k \) by the product distribution \( Q_{N|BU}^{Y|BU} \), and upon observing \( (B_k, U_k) \) Charlie generates \( Z_k \) by the product distribution \( Q_{N|BU}^{Z|BU} \).

**Lemma 6.** For this code,

\[
D\left(P_{M_k,M_k|X_k-1,Y_k-1,Z^{k-1}U_k-1,\hat{U}_k-1,\hat{C}_k}_k\|\text{Unif} \left[ 1 : e^{NR} \right] \right) \Unif \left[ 1 : e^{NR} \right] \left| P_{X_k-1,Y_k-1,Z^{k-1}U_k-1,\hat{U}_k-1,\hat{C}_k}_k \right| \rightarrow 0 \tag{61}
\]

\[
D\left(P_{Y_k|X_k\hat{C}^{k}c_k}^{k}\|Q_{X|Z}^{N}\|\pi_{X}^{k} F_{C}^{k} P_{C}^{k} P_{C}^{k} \rightarrow 0 \tag{62}
\]

uniformly for all \( k \geq 2 \) as \( N \rightarrow \infty \).

The convergence in (61) follows since on one hand,

\[
P_{M_k,M_k|X_k-1,Y_k-1,Z^{k-1}U_k-1,\hat{U}_k-1,\hat{C}_k}_k = P_{M_k|X_k-1,Y_k-1,Z^{k-1}U_k-1,\hat{U}_k-1,\hat{C}_k}_k P_{M_k|\hat{C}_k}_k
\]

and hence, the divergence in (61) can be written as the sum of the following two divergences

\[
D\left(P_{M_k|X_k-1,Y_k-1,Z^{k-1}U_k-1,\hat{U}_k-1,\hat{C}_k}_k \| \text{Unif} \left[ 1 : e^{NR} \right] \mid P_{X_k-1,Y_k-1,Z^{k-1}U_k-1,\hat{U}_k-1,\hat{C}_k}_k \right), \tag{63}
\]

\[
D\left(P_{M_k|\hat{C}_k}_k \| \text{Unif} \left[ 1 : e^{NR} \right] \mid P_{\hat{C}_k}_k \right), \tag{64}
\]

and on the other hand, by Lemma 3, the divergences in (63) and (64) vanish as \( N \rightarrow \infty \) once the upper bounds on \( R, \hat{R} \) in (59) and (60) hold.

In order to prove (62), we need the following lemmas, which are generalizations of Lemma 4 and Lemma 5 respectively to superposition codes.
Lemma 7. Let $P_{X\hat{X}}$ be a probability distribution. Consider a random mapping $f_C : \mathcal{W} \times \hat{\mathcal{W}} \rightarrow \mathcal{X} \times \hat{\mathcal{X}}$. We set $\mathcal{C} = \{ \left( X(w), \hat{X}(w, \hat{w}) \right) \}_{w \in \mathcal{W}}$ with $X(w), w \in \mathcal{W}$ drawn independently for different $w$’s and according to the same distribution $P_X$ and given $w$, $X(w, \hat{w}) \in \mathcal{W}$ drawn independently for different $\hat{w}$’s and according to the same distribution $P_{|X}(\cdot | X(w))$, and set $f_C(w, \hat{w}) = \left( X(w), \hat{X}(w, \hat{w}) \right)$. This forms a random superposition code. For this code, we have for $s \in (0, 1]$ and any distributions $P_{W\hat{W}}, P_{Y|X\hat{X}}$ and $Q_Y$,

$$e^{sD_{1+s}(P_{Y|C}||Q_Y|P_C)} \leq e^{sD_{1+s}(P_{Y|X\hat{X}}||Q_Y|P_{X\hat{X}})-sH_{1+s}(P_{W\hat{W}})} + e^{sD_{1+s}(P_{Y|X}||Q_Y|P_X)-sH_{1+s}(P_W)} + e^{sD_{1+s}(P_Y||Q_Y|P_Y)}$$

where the distribution $P_{Y|C}$ is induced by the “true” joint distribution $P_C P_{W\hat{W}} P_{Y|(X, \hat{X})}=f_C(W, \hat{W})$, and the distribution $P_Y$ is induced by the “ideal” joint distribution $P_{X\hat{X}} P_{Y|X\hat{X}}$.

Lemma 8. Let $P_{X\hat{X}}$ be a probability distribution. Consider a random mapping $f_C : \mathcal{W} \times \hat{\mathcal{W}} \rightarrow \mathcal{X} \times \hat{\mathcal{X}}$. We set $\mathcal{C} = \{ \left( X(w), \hat{X}(w, \hat{w}) \right) \}_{w \in \mathcal{W}}$ with $X(w), w \in \mathcal{W}$ drawn independently for different $w$’s and according to the same distribution $P_X$ and given $w$, $X(w, \hat{w}) \in \mathcal{W}$ drawn independently for different $\hat{w}$’s and according to the same distribution $P_{|X}(\cdot | X(w))$, and set $f_C(w, \hat{w}) = \left( X(w), \hat{X}(w, \hat{w}) \right)$. This forms a random superposition code. For this code, we have for $s \in (0, 1]$ and any distributions $P_{AW\hat{W}} P_B, P_{Y|X\hat{X}}$, and $Q_{Y|B}$,

$$e^{sD_{1+s}(P_{Y|ABC}||Q_{Y|B}|P_{ABP_BP_C})} \leq e^{sD_{1+s}(P_{Y|X\hat{X}B}||Q_{Y|B}|P_{X\hat{X}B})-sH_{1+s}(P_{W\hat{W}|A}|P_A)} + e^{sD_{1+s}(P_{Y|X}||Q_{Y|B}|P_X)-sH_{1+s}(P_{W|A}|P_A)} + e^{sD_{1+s}(P_Y||Q_{Y|B}|P_Y)}$$

where the distribution $P_{Y|ABC}$ is induced by the “true” joint distribution $P_C P_{AW\hat{W}} P_B P_{Y|(X, \hat{X})}=f_C(W, \hat{W})$, and the distributions $P_{Y|B}$ and $P_{Y|X\hat{X}}$ are induced by the “ideal” joint distribution $P_{X\hat{X}} P_{Y|X\hat{X}}$.

Lemma 8 can be seen as a conditional version of Lemma 7. The proof of Lemma 7 is provided in Appendix B-D. The extension of Lemma 7 to Lemma 8 follows similarly to the extension of Lemma 4 to Lemma 5.

By proof steps similar to that of Lemma 2 except for replacing Lemma 5 with Lemma 8, one can prove (62). Specifically, consider the following substitution in Lemma 8: $A \leftarrow (C^k, C^{\Omega k}, C^{\Omega k, -1}), B \leftarrow X_k, W \leftarrow M_k, W \leftarrow M_k, X \leftarrow U_k, \hat{X} \leftarrow \overline{U}_k, Y \leftarrow (Y_{\overline{U}}, \overline{Z}), k \leftarrow \overline{\Theta}_k$, and the corresponding distributions $P_{AW\hat{W}} \leftarrow P_C^k P_{\overline{U}}^k P_{\overline{Z}}^k P_{\overline{M}} P_{\overline{M}}^k$, $P_B \leftarrow \pi_B^N, P_X \leftarrow \overline{Q}_{U}, P_{\overline{X}} \leftarrow \overline{Q}_{\overline{U}}, P_{Y|X\hat{X}B} \leftarrow Q_{Y|U}^N Q_{\overline{B}X} Q_{Z|U} Q_{Z|\overline{X}}$, $Q_{Y|B} \leftarrow Q_{Y|\overline{Z}X}$. Furthermore, by proof steps similar to those from (22) to (29), one can show that $\Gamma (\pi_{XYZ}, P_{W|P_W}) \leq \Delta (\pi_{XYZ}, P_{W|P_W})$. The random variable $V$ can be added by an argument similar to that at the end of achievability proof of Theorem 1 to conclude that $\Gamma (\pi_{XYZ}, P_{W|P_W}) \leq \Delta^+(\pi_{XYZ}, P_{W|P_W})$. Since $\Delta^+(\pi_{XYZ}, P_{W|P_W}) = \Delta (\pi_{XYZ}, P_{W|P_W})$ under our assumptions, this completes the proof of the achievability part of Theorem 3. Here we omit the detailed proofs.

C. Lower Bound

The lower bound follows similarly to the converse to Theorem 1. Denote $K \sim \text{Unif}[1 : n]$ as a random time index, which is independent of all other r.v.’s involved in the system. Define $U := (B^{K-1}, W^K), \overline{U} := W^K, V := (X^{K-1}, Y^{K-1}, Z^{K-1}, K) \}, B := B_K, X := X_K, Y := Y_K, Z := Z_K$. Then, following derivations similar to the ones for the converse of Theorem 1, we have

$$\frac{1}{n} D \left( \frac{P_{Y^n Z^n|X^n} \| P_{Y^n Z^n|X^n}}{P_X^n} \right) = D \left( (P_{YZ}|X) \| (P_X^n) \right), H(U|V) \leq H(W) + H(BU|XYZV), H(U|\overline{U}|V) \leq H(W) + H(BU|XYZV).$$

Moreover,

$$P_{U\overline{U}VXXYZ}(u, \overline{u}, v, b, x, y, z) = P_W^b(w^k) P_W^k(\hat{w}^k) P_K(k) P(b^{k-1}, x^{k-1}, y^{k-1}, z^{k-1}|w^k, \hat{w}^k) \times \pi_X(x) P(b_k|b^k, x^{k-1}, w^k, \hat{w}^k) P(y_k|b^k, y^{k-1}, w^k, \hat{w}^k) P(z_k|b^k, z^{k-1}, w^k) = P_{U\overline{U}V}(u, \overline{u}) \pi_X(x) P_B(b|x, u, \overline{u}) P_{Y|BU\overline{U}}(y|b, u, \overline{u}) P_{Z|BU\overline{U}}(z|b, u, \overline{u}).$$

Combining all the above yields the lower bound $\Delta (\pi_{XYZ}, P_{W|P_W})$. 
D. Proof of Lemma 7

Observe that

\[ e^{sD_{1+s}(P_{Y'C}||Q_Y \times P_c)} = \mathbb{E}_C \sum_{y} P^{1+s}(y|C) Q^{-s}(y) \]

\[ = \mathbb{E}_C \sum_{y} \sum_{w, \hat{w}} P(w, \hat{w}) P(y|f_C(w, \hat{w})) \left( P(w) P(y|f_C(w, \hat{w}) \right) \]

\[ + \sum_{\hat{w}' \neq \hat{w}} P(w, \hat{w}') P(y|f_C(w, \hat{w}')) + \sum_{w' \neq w} \sum_{\hat{w}'} P(w', \hat{w}') P(y|f_C(w', \hat{w}')) \right)^s Q^{-s}(y) \]  

Then using the inequality \((a + b + c)^s \leq a^s + b^s + c^s\) for \(a, b, c \geq 0\) and \(0 < s \leq 1\) we get

\[ e^{sD_{1+s}(P_{Y'C}||Q_Y \times P_c)} \leq L_1 + L_2 + L_3, \]

where

\[ L_1 := \sum_{y} \sum_{w, \hat{w}} P^{1+s}(w, \hat{w}) \mathbb{E}_C \left[ P^{1+s}(y|f_C(w, \hat{w}) \right] Q^{-s}(y) \]

\[ L_2 := \mathbb{E}_C \sum_{y} \sum_{w, \hat{w}} P(w, \hat{w}) P(y|f_C(w, \hat{w})) \]

\[ \times \left( \sum_{\hat{w}' \neq \hat{w}} P(w, \hat{w}') P(y|f_C(w, \hat{w}')) \right)^s Q^{-s}(y) \]

\[ L_3 := \mathbb{E}_C \sum_{y} \sum_{w, \hat{w}} P(w, \hat{w}) P(y|f_C(w, \hat{w})) \]

\[ \times \left( \sum_{w' \neq w} \sum_{\hat{w}'} P(w', \hat{w}') P(y|f_C(w', \hat{w}')) \right)^s Q^{-s}(y). \]

Furthermore, \(L_1, L_2,\) and \(L_3\) can be respectively expressed or upper bounded as follows.

\[ L_1 = \sum_{y} \sum_{w, \hat{w}} P^{1+s}(w, \hat{w}) \sum_{x, \hat{x}} P(x, \hat{x}) P^{1+s}(y|x, \hat{x}) Q^{-s}(y) \]

\[ = e^{sD_{1+s}(P_{Y'|X}||Q_Y\|P_{X'})} - sH_{1+s}(W\hat{W}), \]

\[ L_2 = \sum_{y} \sum_{w, \hat{w}} P(w, \hat{w}) \mathbb{E}_{X(w)} \mathbb{E}_{\hat{X}(w, \hat{w})} \left[ P_{Y'|X} \left( y|X(w), \hat{X}(w, \hat{w}) \right) \right] \]

\[ \times \mathbb{E}_{\hat{X}(w, \hat{w})} \left( \sum_{\hat{w}' \neq \hat{w}} P(w, \hat{w}') P_{Y'|X} \left( y|X(w), \hat{X}(w, \hat{w}') \right) Q^{-s}(y) \right) \]

\[ \leq \sum_{y} \sum_{w, \hat{w}} P(w, \hat{w}) \mathbb{E}_{X(w)} \sum_{\hat{x}} P_{\hat{X}|X}(\hat{x}|X(w)) P_{Y'|X} \left( y|X(w), \hat{x} \right) \]

\[ \times \left( \sum_{w'} \sum_{\hat{w}'} P(w', \hat{w}') \mathbb{E}_{\hat{X}(w, \hat{w}, \hat{w}')} P_{Y'|X} \left( y|X(w), \hat{X}(w, \hat{w}') \right) \right)^s Q^{-s}(y) \]

\[ = \sum_{y} \sum_{w, \hat{w}} P(w, \hat{w}) \mathbb{E}_{X(w)} P_{Y'|X} \left( y|X(w) \right) \]

\[ \times \left( P(w) P_{Y'|X} \left( y|X(w) \right) \right)^s Q^{-s}(y) \]

\[ = \sum_{w} P(w)^{1+s} \sum_{y} P(x) P(y|x)^{1+s} Q^{-s}(y) \]

\[ = e^{sD_{1+s}(P_{Y'|X}||Q_Y\|P_{X'})} - sH_{1+s}(W), \]
and

$$L_3 = \sum_{y} \sum_{w, \hat{w}} P(w, \hat{w}) \mathbb{E}_C[P(y|f_C(w, \hat{w}))]$$

$$\times \mathbb{E}_C \left[ \left( \sum_{w' \neq w} \sum_{\hat{w}'} P(w', \hat{w}') P(y|f_C(w', \hat{w}')) \right)^s \right] Q^{-s}(y)$$

$$\leq \sum_{y} \sum_{w, \hat{w}} P(w, \hat{w}) \mathbb{E}_C[P(y|f_C(w, \hat{w}))]$$

$$\times \left( \sum_{w', \hat{w}'} P(w', \hat{w}') \mathbb{E}_C[P(y|f_C(w', \hat{w}'))] \right)^s \sum_{w} Q^{-s}(y)$$

$$= \sum_{y} \sum_{x, \hat{x}} P(x, \hat{x}) P(y|x, \hat{x})$$

$$\times \left( \sum_{x, \hat{x}} P(x, \hat{x}) P(y|x, \hat{x}) \right)^s \sum_{y} Q^{-s}(y)$$

$$= \sum_{y} P^{1+s}(y) Q^{-s}(y)$$

$$= e^{sD_{1+s}(P_Y||Q_Y)}.$$  

(85)

where (82) follows since $x \mapsto x^s$ is a concave function for $0 < s \leq 1$ and we relax the summation $\sum_{w' \neq w}$ to $\sum_{w'}$.

**APPENDIX C**

**PROOF OF THEOREM 4**

The proof of the cardinality bound is similar to the one for Theorem 1. We next prove the equality in (17).

**A. Achievability**

We first prove the achievability part, i.e., $\Gamma(\pi_{SXYZ}, P_W) \leq \Delta(\pi_{SXYZ}, P_W).$ We first prove that

$$\Gamma(\pi_{SXYZ}, P_W) \leq \Delta(\pi_{SXYZ}, P_W):= \inf_{P_{U_1}P_{A_1|SU}P_{B_1|XU}P_{Y_1|ABU}: \text{H}(U) < H(ABU|SXYZ) + H(W)} D\left( P_{Y_{1|SU}}||\pi_{Y_{1|SU}} \right).$$

(86)

Let $(Q_{U}, Q_{A|SU}, Q_{B|XU}, Q_{Y|ABU}, Q_{Z|ABU})$ be any tuple of joint distributions that satisfy the constraints on the right hand side of (86). Both Alice and Bob adopt a coding scheme as in the point-to-point setting. Specifically, for the first block, Alice sends a sequence of i.i.d.

$$A_1 \sim \text{Unif}(A)$$

to Bob, and Bob sends a sequence of i.i.d.

$$B_1 \sim \text{Unif}(B)$$

to Alice, where $A_1, B_1$ are independent of $S_1, X_1.$ Alice generates $Y_1$ as a constant sequence equal to $y$ and Bob generates

$$Z_1$$

as a constant sequence equal to $z$ where $(y, z)$ are such that $\pi_{Y_1|S_1}(y, z|x, s) > 0$ for all $(s, x)$ (the existence of such a pair $(y, z)$ was assumed in the statement of the theorem). Note that $D(\delta_{(y, z)}||\pi_{Y_1|S_1})$ is finite, where $\delta_{(y, z)}$ denotes the probability distribution concentrated at $(y, z).$ Furthermore, $M_1, U_1$ are set to be constant. For $k$-th block with $k \geq 2,$ Alice and Bob individually generate $M_k = M(A_{k-1}, B_{k-1}, W_{k-1})$ and $U_k = U(M_k)$ by using the common codebooks, the previous communication bits $A_{k-1}, B_{k-1},$ and the common randomness $W_{k-1}$ and then by using $(S_k, U_k),$ Alice generates

$$A_k$$

according to the product conditional distribution $Q^N_{A|SU},$ and then sends it to Bob. By using $(X_k, U_k),$ Bob generates

$$B_k$$

according to the product conditional distribution $Q^N_{B|XU},$ and then sends it to Alice. Upon observing $(A_k, B_k, U_k),$ Alice generates

$$Y_k$$

according to the product conditional distribution $Q^N_{Y|ABU}.$ Upon observing $(A_k, B_k, U_k),$ Bob generates

$$Z_k$$

according to the product conditional distribution $Q^N_{Z|ABU}.$

The distribution synthesized by the code above is exactly the one synthesized by the following code in the point-to-point setting. Consider a new scenario in which Alice is a server and Bob is a receiver. Specifically, for the first block, the encoder sends a sequence of i.i.d.

$$A_1 \sim \text{Unif}(A \times B)$$

to the decoder, where $A_1, B_1$ are independent of $S_1, X_1.$ The decoder generates

$$Y_1 \sim Q^N_{Y|SU},$$

and $Z_1 \sim Q^N_{Z|SU}$ independently. For the $k$-th block with $k \geq 2,$ as in the interactive setting above, Alice and Bob can individually generate $M_k = M(A_{k-1}, B_{k-1}, W_{k-1})$ and $U_k = U(M_k)$ by using the common codebooks, the previous communication bits $A_{k-1}, B_{k-1},$ and the common randomness $W_{k-1}.$ Alice observes $(S_k, X_k),$ generates bits $(A_k, B_k)$ according to the distribution $Q^N_{A|SU}, Q^N_{B|XU},$ and then sends these bits to Bob. After receiving these bits, Bob generates $(Y_k, Z_k)$ according to the product conditional distribution $Q^N_{Y|ABU}, Q^N_{Z|ABU}.$ By the achievability part of the proof of Theorem 1, the KL divergence induced by this code is bounded above by the term in the infimum on the RHS of (86) corresponding to the choice that was made of $(Q_{U}, Q_{A|SU}, Q_{B|XU}, Q_{Y|ABU}, Q_{Z|ABU}).$ This proves (86).
The random variable $V$ can be added into the optimization in the definition of $\Delta^+(\pi_{SYXZ}, P_W)$ by the argument given at the end of achievability proof of Theorem 1. This shows that $\Gamma(\pi_{SYXZ}, P_W) \leq \Delta^+(\pi_{SYXZ}, P_W)$, where

$$\Delta^+(\pi_{SYXZ}, P_W) := \inf_{P_{UV}, P_{A|SV}, P_B|XUV, P_Y|ABUV, P_Z|ABUV: H(U|V) < H(AB|SYXZV)+H(W)} D(P_{YZ|SYXV}||P_{YZ|SYX}P_YP_V).$$

But under the assumption that there is some $(y, z)$ such that $\pi_{YZ|SX}(y, z|s, x) > 0$ for all $(s, x)$ one can show by an argument similar to that of Lemma 1 that $\Delta^+(\pi_{SYXZ}, P_W) = \Delta(\pi_{SYXZ}, P_W)$.

**B. Converse**

We next consider the converse part. Observe that

$$D\left(P_{Y^nZ^n|S^nX^n}||\pi_{Y|Z|S|X}^n\right) = \sum_{k=1}^{\infty} D\left(P_{Y^n_kZ^n_k|S^n_kX^n_k}||\pi_{Y^n_kZ^n_k|S^n_kX^n_k}\right)$$

Denote $K \sim \text{Unif}[1:n]$ as a random time index, which is independent of all other r.v.’s involved in the system. Define

$$U := (A^{K-1}, B^{K-1}, W^K), V := (S^{K-1}, X^{K-1}, Y^{K-1}, Z^{K-1}, K), A := A_K, B := B_K, S := S_K, X := X_K, Y := Y_K, Z := Z_K.$$

Then

$$\frac{1}{n} D\left(P_{Y^nZ^n|S^nX^n}||\pi_{Y|Z|S|X}^n\right) = D\left(P_{YZ|SYXV}||\pi_{Y|Z|S|X}P_Y\right).$$

It is easy to verify that

$$P_{UV|ABSXYZ} = P_K(k)P_W(k)\pi_{S}^{k-1}P_{A^{k-1}B^{k-1}Y^{k-1}Z^{k-1}|S^{k-1}X^{k-1}W^{k-1}}\times \pi_{S}P_{A|SV}P_{B|XUV}P_{Y|ABUV}P_{Z|ABXUV}.$$

Hence it remains to show $H(U|V) \leq H(AB|SYXZV) + H(W)$. This can be easily verified similarly to (33)-(35).

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