An inverse problem for determination of the right part of parabolic equation by conjugate gradient method

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Abstract. In the paper for multidimensional parabolic equation we consider a problem of the right-hand side identification which is a product of two functions, one of which depends on time and the other one depends on space variables. For numerical solution of the posed inverse initial-boundary value problem, the conjugate gradient method is used in combination with the method of finite differences with a purely implicit time approximation. The authors discuss the results of the computational experiment for model problems with quasi-real solutions.

1. Introduction

Mathematical modeling used in many current applied problems of science and technology underlines the need for numerical solution of inverse problems in Mathematical Physics. Different research issues on correct formulation of inverse problems, development of effective methods for their numerical solution are summarized in monographs of Lions J.-L., Magenes E. [1], Isakov V. [2], Vogel C.R. [3], Samarskii A.A. and Vabishchevich, P.N. [4], Kabanikhin S.I. [5]. In terms of applications the inverse problems for source identification in heat equation are of great interest. Johansson and Lesnik in works [6] – [7] proposed an iterative algorithm for stable iterative source recovery using a boundary element method and a finite difference method. In these methods, a correct direct problem is solved at each iteration. Numerical methods used for inverse problems with the right-hand side depending on spatial variables and time are provided in works of Erdem A., Lesnic D., Hasanov A. [8] D’haeyer S., Johansson, B. T., Slodicka, M.[9], Huntul M, Lesnic D, Johansson T.[10]. P.N. Vabishchevich and his colleagues in works [11] – [12] proposed a method for determination of the desired coefficient by decomposition of solution at each time layer for numerical solution of problems of the lower coefficients identification in parabolic equation and the right-hand side, depending on the time. In his work [13] he also suggested two special iterative processes of refinement of the right part, which do not contain regularization parameters. In work [14] the authors have suggested an iterative method for solving the inverse problem to determine the right-hand part with separating variables on spatial variables and time.

In this paper we consider various cases of numerical solution for source identification, which is a product of two functions, one of which depending on time and the second depending on spatial variables. It is proposed to use an iterative method of conjugate gradients in combination with decomposition method for refinement on each time layer. Solution examples for model test problem are given.
2. Problem statement
Let \( x = (x_1, x_2, \ldots, x_p) \in \Omega = \prod_{\alpha=1}^{p} [0, l_\alpha] \) — be a parallelepiped of \( R^p \). We set a direct problem for a linear parabolic equation with variable coefficients. A function to be defined is \( u(x, t), \quad x = (x_1, x_2, \ldots, x_p) \in \Omega, \quad 0 < t \leq T, \quad T > 0, \) satisfying a second-order parabolic equation:
\[
\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{p} \frac{\partial}{\partial x_{\alpha}} \left( a_{\alpha}(x, t) \frac{\partial u}{\partial x_{\alpha}} \right) = f(x), \quad x \in \Omega, \quad 0 < t \leq T,
\]
and boundary and initial conditions:
\[
u(x, t) = 0, \quad x \in \partial \Omega, \quad 0 < t \leq T,
\]
\[
u(x, 0) = u_0(x), \quad x \in \Omega.
\]

If the equation coefficients (1) represent sufficiently smooth functions and satisfy the conditions \( 0 < c_1 \leq k_{\alpha}(x, t) \leq c_2 < \infty, \alpha = 1, 2, \ldots, p, \) the right side and the initial condition are also sufficiently smooth and bounded functions, and the initial condition vanishes at the boundary of the domain \( \Omega, \) then we can consider the direct problem (1) – (3) to be correctly set.

Further, we will consider the inverse problem, when the functions \( \psi(t) \) and \( f(x) \) from the additional conditions are subject to determination:
\[
u(x, t) = \eta(t), \quad x \in \Omega, \quad t \in (0, T],
\]
\[
u(x, T) = \phi(x), \quad x \in \Omega.
\]

The inverse problem in finding \( u(x, t), \) \( \psi(t), \) \( f(x) \) out of conditions (1) – (3) and conditions of redefinition (4) – (5) is conditionally correct under the corresponding conditions of existence and unambiguous solvability. In this paper, we focus on problems of numerical solution of the inverse problem, skipping theoretical issues of convergence of the approximate solution to the exact one.

3. Difference analog of the problem
Numerical solution of the parabolic problem (1) – (3) is carried out through the finite-difference method [15] on a grid uniform in spatial variables and time \( \omega_{ht} = \omega_h \times \omega_t \) with the step \( h_\alpha = l_\alpha/N_\alpha, \alpha = 1, \ldots, p, \quad \tau = T/M. \)

Under assumption of sufficient smoothness of coefficients \( k_{\alpha}(x, t) \) for all internal nodes \( x \in \omega \) let us indicate a grid analogue of multidimensional elliptic operator \( A(t) \) in the following form:
\[
A(t) = \sum_{\alpha=1}^{p} A_{\alpha}(t), \quad x \in \omega_h,
\]
where \( A_{\alpha}(t) \) is a discrete analogue of the elliptic part of differential operator in the original problem (1) – (2) on \( \alpha \)-th direction \( \alpha = 1, \ldots, p \) having the form:
\[
A(t)_{\alpha} y = -(a_{\alpha}(x)y_{x_{\alpha}})_{x_{\alpha}}, \quad \alpha = 1, \ldots, p, \quad x \in \omega_h.
\]

Here, in accordance with the integro-interpolation method of discrete analogue construction [15], the following formulas for coefficients can be used:
\[
a_{\alpha}(x, t) = k_{\alpha}(x_1, \ldots, x_\alpha + 0.5h_\alpha, \ldots, x_p, t), \quad \alpha = 1, \ldots, p, \quad x \in \omega_h.
\]

In the set of grid functions vanishing on the set of boundary nodes \( \partial \omega, \) the constructed operator \( A(t) \) is self-adjoint, positively definite and limited:
\[
A(t) = A^*(t), \quad B_{c_1} \sum_{\alpha=1}^{p} \frac{1}{h_\alpha^2} \leq \| A(t) \| \leq 4c_2 \sum_{\alpha=1}^{p} \frac{1}{h_\alpha^2}.
\]

Let us assign the Cauchy problem for differential operator equation to inverse problem (1) – (5):
\[
\begin{cases}
\frac{dy}{dt} + A(t)y = \psi(t)f(x), & x \in \omega_h, \quad 0 < t \leq T, \\
y(x, 0) = u_0(x), & x \in \partial \omega_h
\end{cases}
\]
with discrete analogues of specified additional conditions:
\[
y(x, t) = \eta(t), \quad x \in \omega_h, \quad t \in (0, T],
\]
\[
y(x, T) = \phi(x), \quad x \in \partial \omega_h.
\]

We denote a finite-difference solution on the time period \( t^n = n\tau \) by \( y^n, \) where \( \tau > 0 \) is a step on time \( N\tau = T. \) We construct a purely implicit two-layer difference scheme:
\[ \frac{y^n - y^{n-1}}{\tau} + A^n y^n = \psi^n f(x), \quad x \in \omega_h, \quad n = 1, 2, \ldots, N, \]  
\[ y_0 = u_0(x), \quad x \in \omega_h. \]  
\[
(10) \quad (11)
\]

It is known that the purely implicit difference scheme (10) – (11) is unconditionally stable on the right side and on the initial condition. For its solution the following a priori assessment is valid due to positive certainty of the operator \( A(t) > 0: \)
\[
\| y^N \| < \| u_0 \| + T \max_{n=1}^M |\psi^n| \| f \|. \]  
\[ (12) \]

Here, while solving the inverse problem of identification of discrete analogs of the time-dependent source \( \psi^n, \quad t \in \omega_T \) and spatially distributed source \( f(x), \quad x \in \omega \) we use grid analogues of additional conditions (8) – (9):
\[
y(x, t) = \eta(t), \quad x \in \omega_h, \quad t \in \omega, \]  
\[ (13) \]
\[
y^N(x) = \varphi(x), \quad x \in \omega_h. \]  
\[ (14) \]

4. Iterative method

A numerical implementation of the difference inverse problem (10) – (11) and (13) – (14) is carried out with computational identification of the right side, depending only on spatial variables from the additional condition (14). And this leads to the need for using iterative methods. We will use the most rapidly converging iterative method of conjugate gradients described by A.A.Samarskii and E.S.Nikolaev [16]:

1. We set \( k = 0, \alpha_k = 1, \) an initial approximation of the sought function \( f_0(x), \quad x \in \omega_h. \)
2. We find the grid function \( y_N(x) \) on the given \( f_k(x) \) implementing a difference scheme (10) – (11) and (14):
\[
y_0 = u_0(x), \quad x \in \omega_h, \]
\[
\frac{y^n - y^{n-1}}{\tau} + A^n y^n = \psi^n f_k(x), \quad x \in \omega_h, \quad n = 1, 2, \ldots, N. \]  
\[ (10) \quad (11) \]

Here to identify an unknown function \( \psi(t_n) \) on each time layer, we use the idea of decomposition suggested in work [11]:
\[
y^n = \nu^n + \psi^n w^n, \quad x \in \omega_h. \]  
\[ (15) \]

Then in order to define the function \( \nu(x, t_n) \) we obtain the following system of linear algebraic equations:
\[
\frac{\nu^n - \nu^{n-1}}{\tau} + A^n \nu^n = 0, \quad x \in \omega_h, \]  
and to determine the grid function \( w(x, t_n) \) we have a system
\[
\frac{w^n - w^{n-1}}{\tau} + A^n w^n = f_k(x), \quad x \in \omega_h. \]

A calculation of the discrete analogue \( \psi^n \) of the unknown function \( p(t_n) \) on the given time layer is performed by means of an additional condition (13):
\[
\psi^n = \frac{\eta(t_n) - \nu^n(x_1)}{w^n(x_1)}. \]

3. Using the formula (15) we calculate the grid function \( y^n, \quad x \in \omega_h. \)
4. We calculate the residual \( r_k(x) = \varphi(x) - y^n(x), \quad x \in \omega_h. \)
5. Then we define an auxiliary grid function \( z_k(x) \) which is a solution of the difference scheme:
\[
\frac{z^n - z^{n-1}}{\tau} + A^n z^n = \psi^n r_k(x), \quad x \in \omega_h, \quad n = 1, 2, \ldots, N. \]  
\[ (12) \]
6. We calculate iterative parameters \( \tau_{k+1} \) and \( \alpha_{k+1} \) by formula:
\[
\tau_{k+1} = \frac{(r_k + r_k)}{(x_k, r_k)}, \]
\[ \alpha_{k+1} = \left( 1 - \frac{\tau_{k+1}}{\tau_k} \right) \left( \frac{r_k r_k}{(x_k, r_k)} \right) a_k^{-1} \]  
if \( k > 0. \)
7. Another approach \( f_{k+1}(x), \quad x \in \omega_h \) is calculated by formula:
\[ f_{k+1}(x) = \begin{cases} f_k(x) - \tau_{k+1} r_k(x), & \text{at } k = 0; \\ \alpha_{k+1} f_k(x) + (1 - \alpha_{k+1}) f_{k-1}(x) - \alpha_{k+1} \tau_{k+1} r_k(x), & \text{at } k > 0. \end{cases} \]

8. Then we check an exit condition from the iterative cycle \( \| r \| < \varepsilon \), increasing an iteration counter by one \( k = k + 1 \) and return to step 2. Otherwise, we exit the iterative cycle.

5. Computational experiment

To check an efficiency of the proposed iterative method in solving the linear inverse problem to determine the right side of the two–dimensional parabolic equation (1) – (3) with homogeneous Dirichlet boundary conditions, we construct a “quasi-exact solution of the inverse problem” as follows. Let us assume that multipliers of the right side of equation (1) are given in the form of:

\[ p(t) = e^{-t}, \quad 0 < t \leq T, \quad f(x) = e^{-\sigma \| x - x_0 \|_2^2}, \quad x \in \Omega, \quad \sigma = 150. \]

First, using a finite-difference method, we solve the direct problem (1) – (3) with the given right-hand side in the form of product of functions depending only on time and on spatial variables at the following values of the input data:

\[ \Omega = [0, l_1] \times [0, l_2], \quad l_1 = l_2 = 1, \quad T = 1, \quad x_0 = (l_1/2, l_2/2), \]

Fig. 1: Identifiable functions \( A = p(t) \), \( B = f(x) \) and functions of additional conditions \( C = \eta(t) \), \( D = \varphi(x) \).

Fig. 2: A solution error of identifiable functions \( A = p(t) \), \( B = f(x) \). \( C, D \) – logarithms of errors depending on iteration number at \( f_0(x) = e^{-2000 \| x - x_0 \|_2^2} \).
Fig. 3: Solution error of identifiable functions $A - p(t)$, $B - f(\mathbf{x})$. $C$, $D$ are logarithms of errors depending on iteration number at $f_0(\mathbf{x}) = e^{-300\|\mathbf{x} - \mathbf{x}_0\|^2}$.

$$k_d(\mathbf{x}, t) \equiv 1, \quad \alpha = 1, 2, \quad \mathbf{x} \in \Omega, \quad 0 < t \leq T,$$

$$u_0(\mathbf{x}) = \sin(\pi x_1/l_1)\sin(\pi x_2/l_2), \quad \mathbf{x} \in \Omega.$$

Space-time grid parameters: $N_1 = N_2 = 20; \quad M = 40$.

Fig. 1A presents a time-dependent graph of the right-hand side multiplier function $p(t)$, $0 < t \leq T$ (left), in Fig. 1B there is a right-hand side multiplier function that depends on spatial variables $f(\mathbf{x})$, $\mathbf{x} \in \Omega$.

From the solution of the direct problem we take a redefinition condition for the grid inverse problem – the grid functions $\eta(t_m)$, $t \in \omega_\tau$ in Fig. 1C and $\varphi(\mathbf{x})$, $\mathbf{x} \in \omega$ in Fig. 1D. The problem of identification of the right part will be carried out while giving three different initial approximations for the desired function $f_0(\mathbf{x})$.

In the first case, the initial approximation is given in the form of $f_0(\mathbf{x}) = e^{-200\|\mathbf{x} - \mathbf{x}_0\|^2}$. For numerical implementation of its finite-difference analogue we use the above iterative method when $\varepsilon = 10^{-9}$.

In Fig. 2A we can see errors in identification of a multiplier of the right side $p(t)$ and in 2B – an identification error of the multiplier of the right side $f_0(\mathbf{x})$ when exiting an iterative cycle. Fig. 2C and 2D present decimal logarithms of error rates $p(t)$ and $f_0(\mathbf{x})$ depending on iteration, respectively.

In Fig. 3A we can see an identification error of the multiplier of the right side $p(t)$ and Fig. 3B shows an identification error of the right-hand side multiplier $f_0(\mathbf{x})$ when exiting an iterative cycle by specifying an initial approximation $f_0(\mathbf{x}) = e^{-300\|\mathbf{x} - \mathbf{x}_0\|^2}$. In Fig. 3C and 3D there are decimal logarithms of error norms depending on number of iteration.

Fig. 4A presents an error of identification of the right-hand part multiplier $p(t)$ and 4B is an identification error of the right-hand side multiplier $f_0(\mathbf{x})$ while exiting the iterative cycle at the given initial approximation $f_0(\mathbf{x}) = e^{-500\|\mathbf{x} - \mathbf{x}_0\|^2}$. In Fig. 4C and 4D decimal logarithms of the error rate depending on the iteration number are presented.
6. Conclusion
The given results of the computational experiment have confirmed a sufficiently high efficiency of the iterative conjugate gradient method in combination with decomposition method for numerical solution of the inverse problem of identification of the right part in parabolic equation, which is a product of function of time to function of spatial variables.

Acknowledgments
The work is supported by the mega-grant by the Russian Federation Government (No 14.Y26.31.0013) and grant by the RFBR (No 17-01-00732).

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