A sign of the times

João Magueijo

Theoretical Physics Group, The Blackett Laboratory, Imperial College, Prince Consort Rd., London, SW7 2BZ, United Kingdom

(Dated: October 13, 2021)

We examine in greater detail the recent proposal that time is the conjugate of the constants of nature. Fundamentally distinct times are associated with different constants and we should select the one related to the constant dominating the dynamics in each region or epoch. We show in detail how in regions dominated by a single constant the Hamiltonian constraint can be reframed as a Schrodinger equation in the corresponding time, solved in the connection representation by outgoing-only monochromatic plane waves moving in a “space” that generalizes the Chern-Simons functional. We pay special attention to the issues of unitarity and the measure employed for the inner product. Normalizable superpositions can be built, including solitons, “light-rays” and coherent/squeezed states saturating a Heisenberg uncertainty relation between constants and their times. A healthy classical limit is obtained for factorizable coherent states, with classical cosmology seen through the prism of the connection (the comoving Hubble length) rather than the more conventional expansion factor (metric). A brief discussion of the arrow of time within this framework is included. In this multi-time setting we show how to deal with transition regions, where one is passing on the baton from one time to to another, and investigate the fate of the subdominant clock. For this purpose minisuperspace is best seen as a dispersive medium, with packets moving with a group speed distinct from the phase speed. We show that the motion of the packets’ peaks reproduces the classical limit even during the transition periods, and for subdominant clocks once the transition is over. Strong deviations from the coherent/semi-classical limit are expected in these cases, however. Could these be a “sign of the times”, accessible notably in the transition period (from matter to Lambda domination) we live in?

I. INTRODUCTION

The problem of time in General Relativity \([1–4]\) and the mystery of the origin and value of the constants of nature \([5]\) are two well-known nuisances embedded in the foundations of physics. In \([6]\) we suggested that they could be inextricably intertwined. A priori this should certainly be the case. Physical time concerns relational change. In contrast, the constants of nature, if true to their name, are the hallmarks of immutability. We could therefore expect that time and the constants are conjugate dynamical variables, or at least complementary in a quantum sense. Should we, therefore, promote the constants to observables, with their complementaries providing physical definitions of time?

Naturally, the devil is in the detail, and a great many questions pour in. Foremost, given the plethora of constants (some more fundamental than others \([2]\)), we have to settle on whether we should contend with different physical times, or if, instead, a select constant is the progenitor of a single time. In \([6]\) we suggested following the first route. A clock is crafted with what is at hand. Depending on the constant(s) dominating the dynamics, different phase space regions should employ different times. Thus, we are led to times variously conjugate to the cosmological constant, \(\Lambda\), the gravitational constant, \(G_N\), or even the speed of light, \(c\). Within such a pragmatic democracy we need to know how to pass on the baton from one clock to another. This adjustment of clocks should be seen as a physical feature of our world.

The roots of this proposal go back to well-known literature. In the context of \(\Lambda\) it resonates with findings for uni-modular gravity \([8, 9]\), where the conjugate of \(\Lambda\) is identified with a time variable (which turns out to be Misner’s volume time \([10]\)). A further point of contact is the concept of Chern-Simons time \([11]\), related to York time \([12]\). The proposal in \([6]\) amounts to reinterpreting these ideas and extending them to constants other than \(\Lambda\). The implications will be investigated further here.

Schematically, in this paper we posit the following:

- Time is the conjugate of the constants of nature and vice versa.

- Given the multitude of options, the constant chosen for the progenitor of time in a given region of phase space is the one that allows for a clear separation of space and time in that region. This is a dynamical issue.

- Dealing with multi-time situations, or settings where one must handover from one time to another, is part of physics.

These precepts will emerge from an examination of minisuperspace quantum gravity in the connection representation. As explained later (and in more detail elsewhere \([13]\)) our constructions often generalize beyond minisuperspace. When they do not (due to the choice of constant), we should replace “time” by “cosmological time” and assume that locality and Lorentz invariance have been broken.

The plan of this paper is as follows. A formal and aspirational general prescription will first be proposed in Section \([11]\) in the form of 3 precepts. These will be purposefully of great generality, because we do not wish to wed our proposal to a particular theory (even though we will be using Einstein-Cartan gravity for the rest of the paper). Putting the Hamiltonian constraint in a given format, and extending the action and

---

\(\text{j.maguejo@imperial.ac.uk}\)
phase space, transforms the Wheeler-DeWitt (WDW) equation into a Schrodinger equation, with the conjugate of a constant playing the role of "time". The WDW equation then appears as the condition for defining a monochromatic (fixed constant) wave.

In Section III we apply this procedure to pure $\Lambda$, showing that in the connection representation the monochromatic plane waves move in a time conjugate to $1/\Lambda$ (seen as a "frequency") and a space which is the Chern-Simons (CS) functional (so that the spatial part of the wave is the real CS state $|1\rangle$). Crucially, we can superpose the monochromatic plane waves into normalizable solutions, as we show in Section IV where we identify solitons, light rays and coherent/squeezed states. In Section V we present a straightforward generalization to Universes dominated by radiation and fluids with generic equation of state. In each of these the constant of choice is different, so that the chosen time is different.

Having a quantum time variable and the ability to find peaked wave-packets is instrumental in finding the correct classical limit. We explain how this is achieved by coherent states in Section VII once the reader is taught how to do cosmology in the connection representation and with constant-times.

The rest of the paper is spent formalizing what to do in multi-time situations and how to change the "time zone", as already sketched in [6] and reviewed in Section VII. For this purpose, we need to generalize the admittedly "aspirational" constructions in Section II typically possible only when one fluid dominates the Universe. This is done in Section VIII where we show how minisuperspace can be seen as a dispersive medium, with wave-numbers that can be position and frequency dependent. By examining the associated group speed we can then find the equations of motion of the peak of suitable wave functions. Using this technique, and assuming that the wave function remains peaked, in Section IX we prove that the correct semi-classical limit is still obtained in cross-over regions. We also illuminate the fate of the minority clock once the handover of clocks is completed. In Section X we formalize the reasons why a clock should indeed be built with "what is at hand", exposing the limitations of minority clocks. In a concluding Section we summarize our findings and speculate on their ultimate implications.

## II. CLOCKS AND RODS IN AN IDEAL MINISUPERWORLD

We first propose a number of aspirational principles for the definition of clocks derived from constants. These clocks naturally lead to a measure in connection space (the associated "rod") which generalizes the CS invariant. We stress that some of these principles are too restrictive once we go beyond the ideal situation of minisuperspace and single fluids

A. Precept 1: constants as the generators of clocks

Let a generic gravity-plus-matter Hamiltonian depend on variables $q_i$ and momenta $p_i$, as well as "constants" $\alpha_j$. The idea is to put the standard Hamiltonian constraint in minisuperspace:

$$H(q_i, p_i; \alpha_j) = 0$$

into the form:

$$H_0 - \alpha = 0$$

where we have singled out one of the constants, $\alpha$. We then expand phase space by promoting $\alpha$ to a d.o.f. and adding to the minisuperspace action a new term:

$$S \to S + \int dt \dot{\alpha} p_\alpha,$$

so that $\alpha$ has conjugate momentum $p_\alpha$, with:

$$\{\alpha, p_\alpha\} = 1.$$  

The momentum $p_\alpha$ does not appear anywhere else (in particular, in $H$), and so one of Hamilton's equations is:

$$\dot{\alpha} = \{\alpha, H\} = 0$$

implying that $\alpha$ is now not a "constitutional" constant or parameter of the system, but a constant of motion. Quantization therefore leads to a Schrodinger equation:

$$\left[ H_0(p_i, q_i) - i\hbar \frac{\partial}{\partial T_\alpha} \right] \psi = 0$$

where before there was the WDW equation. Its time variable is $T_\alpha \equiv p_\alpha$. The conjugate of the no-longer-fundamental-constant $\alpha$ is the quantum time variable we were missing.

The following comments are in order:

- Not all constants are amenable to this treatment. Singling out $G_N$ or $\Lambda$ is easier than targeting the electron charge $e$ or the elementary particles' masses. Still, setting $\alpha$ to $c$ or even $\hbar$ is not as outlandish as it might appear, and raises important foundational questions [14].

- Not in all models or regions of phase space can this prescription for a clock be applied. For example, it may be hard to implement for a scalar field with a potential, even if this is just a mass potential. In general it may be cumbersome to implement the precept for multi-fluids exchanging energy even in a stable scaling regime.

- Even when, for a given constant and model, the prescription can be applied in one or several regions, it may not be doable everywhere. An example is a multi-fluid situation, where in regions where one fluid dominates a clock emerges, but not in the transition regions, where more than one fluid dominates.
All of this leads up to the view that clocks are at best effective and tied to the circumstances. Maybe one can find many ideal clocks, one per region/epoch. Perhaps in some regions one can even find more than one ideal clock\(^1\). Or perhaps there are no ideal clocks at all in some other regions. All of this is OK. It is better than nothing.

In addition the following structural points can be made:

- Not in all cases does the stated prescription generalize beyond minisuperspace. As we will describe in more detail elsewhere\(^2\) we can lift the equivalent procedure for unimodular gravity

\[
S_0 \rightarrow S = S_0 + \int d^4x \Lambda \partial_\mu T^{\mu}_\Lambda
\]

(7)
to achieve a covariant and local version of our proposal (for example, targeting is \(G_N\)). However, this is by no means always possible (for example if the target constant is \(c\)) or even necessary. In that case we may still accept the minisuperspace model and state that we are breaking Lorentz invariance: indeed we can regard such models as a class of Lorentz symmetry breaking models.

- Eq. (2) must be equivalent to (1), but off-shell the relation between \(H\) and \(H_0 - \alpha\) can be as complicated as wanted. Sometimes the two will be proportional (and so the two conditions amount to nothing but a redefinition of lapse function; see the case of radiation). Other times the relation can be very complicated, so that the theories are expected to be very different off-shell (as in the case of \(\Lambda\) in Chern-Simons theory, where (2) is obtained inverting identities implied by (1)).

## B. Precept 2: “time” independence of the Hamiltonian \(H_0\)

Precept 1 typically leaves us with an embarrassment of riches. Furthermore, numerous ambiguities remain: the choice of constant (and function of constant) to be taken as \(\alpha\), the choice of the proportionality constant for \(p_\alpha\), etc. Some of these ambiguities should be seen as a fact of life (e.g. in \([13]\) we will investigate the implications of concurrent multi-times); others should be fixed. Hence (3) should be seen as schematic before the next two precepts are introduced.

As a rule of thumb we should target the constant which is most “at hand” in a given context (model and epoch). One makes a clock with what there is. The next 2 precepts help us further select the appropriate time/constant. They depend on the concrete dynamics.

First, we would like \(H_0\) to NOT depend on \(\alpha\). This may be a strong criterium for selecting a constant in a given context (model and/or phase space region), and for abandoning it in other, as we shall see. If \(H_0\) depends on \(\alpha\) we have a situation where the Hamiltonian depends on the “energy” rather than being time-dependent. This will be unavoidable in some situations (see Section \([X]\)), but when it can be avoided one should cherish the chance. As we will see later in this paper, revisiting this precept will be unavoidable where one sort of time passes over the baton to another.

## C. Precept 3: rod-clock segregation

Furthermore, we would like to see a clear separation of space and time. What good is a clock, if it is also a rod? The existence of monochromatic (plane, with regards to some measure) waves in superspace is a criterium for such a separation. This will resolve another ambiguity (besides the choice of \(\alpha\)): the choice of the function of \(\alpha\) to be taken as spanning phase space. If \(\alpha\) is a constant, then so is any function of \(\alpha\), but this affects the definition of the conjugate momentum. The introduction of this precept removes this ambiguity\(^3\).

The concrete implementation of this principle depends on the classical dynamics and on the representation and ordering chosen for the quantum theory. Let us illustrate it with the Einstein-Cartan action reduced to homogeneity and isotropy\(^4\)\(^5\)\(^6\):

\[
S_\theta = 6\kappa V_c \int dt \left( a^2 b + N\alpha (b^2 + k) \right).
\]

(8)

where \(\kappa = 1/(16\pi G_N)\), \(k = 0, \pm 1\) is the normalized spatial curvature, \(a\) is the expansion factor (the only metric variable), and the connection variable \(b\) is the off-shell version of the Hubble parameter (since \(b = \dot{a}\) on-shell, if there is no torsion). The Lagrange multiplier \(N\) is the lapse function and \(V_c = \int d^3x\) is the comoving volume of the region under study (which could be the whole manifold, should this be compact). To this action one should add a matter or Lambda action, as appropriate. We may also need to define \(\kappa\) with a fixed fiducial \(G_{N0}\)\(^6\)\(^7\) or not\(^8\).

Hence, the Poisson bracket is:

\[
\{b, a^2\} = \frac{1}{6\kappa V_c},
\]

(9)

leading to commutator:

\[
[b, a^2] = \frac{l_P^2}{3V_c},
\]

(10)

where \(l_P = \sqrt{8\pi G_N h}\) is the reduced Planck length. In the \(b\) representation this implies:

\[
\dot{a}^2 = -\frac{l_P^2}{3V_c} \frac{\partial}{\partial b}.
\]

(11)

What follows is tied to the connection representation and it would need to be adapted for a metric representation counterpart.

\(^1\) Strictly speaking this ambiguity amounts to a canonical transformation in \((\alpha, p_\alpha)\), leading to equivalent classical theories, but different quantum theories (for example the measure is different; coherent states in one variable will not be coherent in any other, etc).

\(^2\) See Section \([X]\).

\(^3\) For example, in Chern-Simons theory, where (2) is obtained inverting identities implied by (1).

\(^4\) For example, in Chern-Simons theory, where (2) is obtained inverting identities implied by (1).

\(^5\) For example, in Chern-Simons theory, where (2) is obtained inverting identities implied by (1).

\(^6\) For example, in Chern-Simons theory, where (2) is obtained inverting identities implied by (1).

\(^7\) For example, in Chern-Simons theory, where (2) is obtained inverting identities implied by (1).

\(^8\) For example, in Chern-Simons theory, where (2) is obtained inverting identities implied by (1).
In the light of Precept 3, we now show that it is desirable to put the Hamiltonian constraint \( h_\alpha(b) a^2 - \alpha = 0 \), \(^{(12)}\)
that is replace whichever constant \( \alpha \) we started from by a function thereof, for the sake of obtaining \( H_0 = h_\alpha(b) a^2 \), if possible.\(^{(3)}\) This may require more than a redefinition of lapse function. We will also normalize the momentum \( p_\alpha \) (to be identified with time \( T_\alpha \)) according to:
\[
S \rightarrow S + 6\kappa V_c \int dt \dot{\alpha} p_\alpha,
\]
\(^{(13)}\)
implying the refined version of \(^{(4)}\):
\[
\{ \alpha, p_\alpha \} = \frac{1}{6\kappa V_c}.
\]
\(^{(14)}\)
The factor of \( V_c \) is there to avoid introducing a “time per unit volume”\(^{(4)}\). The factor of \( 6\kappa \) is pure convenience (and amounts to setting to one the “speed of light” in minisuperspace in “linearizing” variables \(^{(25)}\)). Thus:
\[
[\alpha, p_\alpha] = \frac{i \hbar P}{3 V_c},
\]
\(^{(15)}\)
leading to a Schroedinger equation of the form \(^{(6)}\) with “Planck’s constant”\(^{(1)}\):
\[
\hbar \rightarrow \hbar \ = \ \frac{i \hbar P}{3 V_c},
\]
\(^{(16)}\)
and \( T_\alpha = p_\alpha \), that is:
\[
[H_0(b) - i\hbar \frac{\partial}{\partial T_\alpha}] \psi = 0.
\]
\(^{(17)}\)
Its monochromatic solutions are:
\[
\psi = \psi_s(b; \alpha) \exp \left[ \frac{i}{\hbar} \alpha p_\alpha \right]
\]
\(^{(18)}\)
with the “spatial” \( \psi_s \) satisfying
\[
(H_0 - \alpha) \psi_s = 0.
\]
\(^{(19)}\)
The advantage of \( H_0 = h_\alpha(b) a^2 \) as in \(^{(12)}\) (and of its implied ordering in the quantum theory) is now obvious. Given \(^{(11)}\), it translates in the \( b \) representation into:
\[
\left[ -i \frac{\hbar P}{3 V_c} h_\alpha(b) \frac{\partial}{\partial b} - \alpha \right] \psi_s(b; \alpha) = 0.
\]
\(^{(20)}\)
Setting:
\[
X_\alpha(b) = \int \frac{db}{h_\alpha(b)}
\]
\(^{(21)}\)
this becomes a plane-wave equation in \( X_\alpha \):
\[
\left( -i \frac{\hbar P}{3 V_c} \frac{\partial}{\partial X_\alpha} - \alpha \right) \psi_s = 0,
\]
\(^{(22)}\)
with solution:
\[
\psi_s(b; \alpha) = \mathcal{N} \exp \left[ \frac{3 V_c}{\hbar P} \alpha X_\alpha(b) \right].
\]
\(^{(23)}\)
The monochromatic solutions are therefore plane-waves in \( X_\alpha \) moving at fixed speed (set to 1):
\[
\psi(b; \alpha) = \mathcal{N} \exp \left[ \frac{3 V_c}{\hbar P} \alpha (X_\alpha(b) - T_\alpha) \right].
\]
\(^{(24)}\)
with the choice
\[
\mathcal{N} = \frac{1}{\sqrt{2\pi\hbar}}
\]
\(^{(25)}\)
to be justified later. Notice that \( X_\alpha(b) \) is like a linearizing variable in DSR: the speed of propagation is variable if measured in terms of the more physically available \( b \), rather than \( X_\alpha \).

In summary, under Precept 3 the WDW equation becomes a free Schroedinger equation, which in fact is a wave equation accepting only retarded waves:
\[
\left( \frac{\partial}{\partial X_\alpha} + \frac{\partial}{\partial T_\alpha} \right) \psi = 0.
\]
\(^{(26)}\)
Its associated conserved current is:
\[
j^0 = j^1 = |\psi|^2
\]
\(^{(27)}\)
i.e. all the waves are outgoing (or retarded time) solutions. The general solution takes the form:
\[
\psi(b) = F(T_\alpha - X_\alpha),
\]
\(^{(28)}\)
where \( F \) can be any function. This can be written as a superposition of monochromatic solutions indexed by \( \alpha \) as is:
\[
\psi(b) = \int \frac{d\alpha}{\sqrt{2\pi\hbar}} A(\alpha) \exp \left[ \frac{i}{\hbar} \alpha (X_\alpha(b) - T_\alpha) \right].
\]
\(^{(29)}\)

III. PURE LAMBDA AND A REINTERPRETATION OF CHERN-SIMONS TIME

We first illustrate these principles with the cosmological constant \( \Lambda \), showing that the implications are a twist on both unimodular gravity \(^{(8, 9)}\) (specifically the time variable defined in \(^{(9)}\)), and the concept of Chern-Simons time \(^{(11)}\). Indeed our proposal leads to a hybrid between these works, with a significant reinterpretation of Chern-Simons “time”.

---

\(^2\) Although \( \hbar \) and its related functions should not explicitly depend on \( \alpha \), and be only a function of the connection \( b \), this function depends on the \( \alpha \) selected as the progenitor of time, as we shall see. Hence we will keep \( \alpha \) as a subscript.

\(^3\) One may wonder, why not?, but that would be another paper. It is always assumed that time is intensive, but it could be extensive \(^{(22)}\) in some regimes, or even assume any fractal dimension.

\(^4\) Throughout the paper we will use the shorthand \( \hbar \) or not depending on convenience. Its interpretation as the actual Planck constant \(^{(23)}\) will not be relevant here.
We start by reviewing some standard results. For a pure \( \Lambda \) we have in minisuperspace (ignoring torsion \([18,19]\)):
\[
H = 6\kappa V_c N a \left( -b^2 + k \right) + \frac{\Lambda}{3} a^2. 
\]
(30)
A direct solution to the quantum Hamiltonian constraint in the \( b \) representation:
\[
\left[ -(b^2 + k) - i \frac{\Lambda l_P^2}{9 V_c} \frac{\partial}{\partial b} \right] \psi = 0
\]
is given by the (real) Chern-Simons state \([13,17]\) reduced to minisuperspace \([18,20]\):
\[
\psi_{CS} = \mathcal{N} \exp \left[ i \frac{3V_c}{l_P^2} \frac{b^3}{3} + bk \right].
\]
(32)
As is well known, this is a pure phase, which is the product of a “frequency” proportional to \( 1/\Lambda \), and the “Chern-Simons time” as proposed by Smolin and Soo \([11]\).

Something similar can be obtained from our prescription. We can put the Hamiltonian constraint associated with \( \psi \) in the form:
\[
\mathcal{H} = \frac{1}{b^2 + k} a^2 - \frac{3}{\Lambda} = 0.
\]
(33)
By doing this the Hamiltonian constraint acquires the form \([12]\) required by Precept 3:
\[
\mathcal{H} = h_\alpha(b) a^2 - \alpha = 0
\]
in this case with:
\[
h_\alpha(b) = \frac{1}{b^2 + k} \quad \alpha = \frac{3}{\Lambda}.
\]
(35)
(36)
A Schrodinger equation \([17]\) follows, with a time variable \( T_\phi \) identified with \( p_\phi \), normalized such that:
\[
[\phi, p_\phi] = i \frac{l_P^2}{3 V_c}.
\]
(37)
(cf. \([13,15]\)). Its monochromatic solutions are:
\[
\psi = \psi_s(b; \phi) \exp \left[ -i \frac{3V_c}{l_P^2} \phi p_\phi \right] = \psi_s(b; \phi) \exp \left[ -i \frac{9V_c}{l_P^2} T_\phi \right].
\]
(38)
with the “spatial” factor of the wave-function satisfying:
\[
\left[ -i \frac{l_P^2}{3 V_c} h_\alpha(b) \frac{\partial}{\partial b} - \frac{3}{\Lambda} \right] \psi_s = 0.
\]
(39)
This is the point of bringing the Hamiltonian to form \([34]\) and choosing the ordering we chose. As in \([24]\), the \( \psi_s \) are plane waves:
\[
\psi_s(b; \phi) = \mathcal{N} \exp \left[ i \frac{3V_c}{l_P^2} \phi X_\phi(b) \right].
\]
(40)
in “spatial” variable:
\[
X_\phi(b) = \int \frac{db}{h_\alpha(b)} = \frac{b^3}{3} + kb.
\]
(41)
This is nothing but the Chern-Simons functional (in minisuperspace) and \([40]\) is the Chern-Simons state \([32]\).

However, our interpretation of “Chern-Simons time” is different from that of Smolin and Soo. The full monochromatic solution is:
\[
\psi(b; \phi) = \mathcal{N} \exp \left[ i \frac{3V_c}{l_P^2} \phi (T_\phi - X_\phi(b)) \right]
\]
(42)
with \( T_\phi \equiv p_\phi \). Hence the (unitary) time evolution happens in terms of a time which is not the Chern-Simons functional, but the momentum conjugate to \( 1/\Lambda \) (up to a conventional proportionality constant). Here \( X_t = \Im(Y_{CS}) \) is not a time, but a spatial variable. Time, instead, is the conjugate of \( \phi = 3/\Lambda \). The waves, however, move at constant speed (set to one by the conventional proportionality factors) in terms of \( X_\phi \) and \( T_\phi \). This is true of the phase speed and also of the group speed if we construct wave packets, as we shall do in the next Section. Hence the spatial \( X_\phi \) and the time \( T_\phi \) can be loosely confused if \( \psi \) is peaked. Its peak moves along the outgoing “light-ray” \( T_\phi = X_\phi \), and hence confusing the two may be harmless for some purposes.

**IV. MORE GENERAL STATES**

By denoting \( \Lambda \) to a circumstantial constant we gain more than a time variable in the quantum theory: we enlarge the space of solutions. Instead of being restricted to \([42]\) we can now admit the most general superposition of these monochromatic plane waves:
\[
\psi(b) = \int \frac{d\phi}{\sqrt{2\pi \hbar}} A(\phi) \exp \left[ i \frac{\phi}{\hbar} (X_\phi(b) - T_\phi) \right],
\]
(43)
with the probability for the cosmological constant given from:
\[
P(\phi) = |A(\phi)|^2
\]
(44)
with measure \( d\phi \) (fast-forward to the end of this Section for more details; also see \([21]\) for alternatives).

Everything we state in this Section about general states for \( \Lambda \), parameterized by \( \phi \), works for any other \( \alpha \), with suitable modifications (i.e. \( \phi \rightarrow \alpha \) in all relevant quantities).

**A. Extreme cases**

Two limiting cases are of interest. At one extreme we may have a completely undetermined \( \phi \):
\[
A(\phi) = \epsilon
\]
(45)
leading to:
\[
\psi = \sqrt{2\pi \hbar} \epsilon \delta(T_\phi - X_\phi).
\]
(46)
This is the conformal constrain present in the parity-even branch of the quasi-topological theories of [13, 19, 21], where \( \Lambda \) is allowed to vary by virtue of multiplying a Gauss-Bonnet topological term. In such a theories \( 1/\Lambda \) has a conjugate momentum which is forced to equal the Chern-Simons functional by a primary constraint. Here we see that this constraint is interpreted as a “light-ray” in minisuperspace: of the many waves generally acceptable only a delta function ray is possible in this theory. Time \( T_\phi \) is fully fixed by the position \( X_\phi \) along this ray. Whereas in standard Relativity any state is a solution, in these quasi-topological theories one is forced to have an infinitely sharp clock, with total delocalization in “constant” \( \phi \).

At the opposite extreme we may completely fix \( \phi \):

\[
A(\phi) = \delta(\phi - \phi_0)
\]

leading to:

\[
\psi = N \exp \left[ -i \frac{3V_0 \phi_0}{l_p^2} (T_\phi - X_\phi(b)) \right].
\]

This is the Chern-Simons state in the usual Einstein-Cartan theory, where Lambda is fully fixed. It implies a uniform distribution in \( X_\phi \). The crests of this infinite plane wave still move at the speed light, but its “location” does not, because it is not localized. Hence time effectively disappears, since the wave function is fully smeared in \( X \) and \( T \). This is an example of a more general fact: infinitely sharp constants are failed clocks. They imply complete delocalization in time. This clarifies and reinterprets the discussion on a time-like tower of turtles in [22]. Obviously these states are not strictly speaking normalizable.

### B. Coherent/squeezed states

In between these two extremes we can build coherent/squeezed states centred at \( \phi_0 \):

\[
A(\phi) = \sqrt{N(\phi_0, \sigma_1)} = \exp \left[ -\frac{(\phi - \phi_0)^2}{4\sigma^2} \right],
\]

(where \( N \) denotes a normal distribution). Evaluating the integral we get:

\[
\psi(b, T) = N' \exp \left[ -\frac{\sigma^2(X_\phi - T_\phi)^2 + i \phi_0(X_\phi - T_\phi)}{b^2} \right] = N' \psi(b, T; \phi_0) \exp \left[ -\frac{\sigma^2(X_\phi - T_\phi)^2}{b^2} \right].
\]

The last expression relates the infinite norm Chern-Simons state for \( \phi_0 \) to the finite norm wave packet built around a fixed \( \Lambda \). We see that it is dressed by a Gaussian, which regularizes it. This is just a Gaussian distribution in \( X - T \), with variance:

\[
\sigma_T^2 = \sigma_X^2 = \frac{b^2}{4\sigma^2}.
\]

A Heisenberg uncertainty principle can therefore be established, with: \( \sigma_X = \sigma_T \) and

\[
\sigma_T \sigma_\phi \geq \frac{l_p^2}{6V_c}.
\]

For a coherent state:

\[
\sigma_T^2 = \sigma_\phi^2 = \frac{l_p^2}{6V_c}.
\]

### C. Solitons

Note also that we do not need to use monochromatic waves to build states. As already explained, any function of the form:

\[
\psi = F(X_\phi - T_\phi)
\]

would work, as we can see from [see before; this is repetition]. Namely \( F \) can be just a Gaussian, without the plane wave factor, as for \( 50 \):

\[
\psi(b, T) = N'' \exp \left[ -\frac{(X_\phi - T_\phi)^2}{4\sigma^2} \right]
\]

Such “solitons” could be interesting. We stress that unlike coherent states, such solitons are not well localized in \( \phi \): for that we need the internal beats of a plane-wave, for which this \( F \) would be the envelope.

### D. Normalizability and measure

All of these solutions are normalizable with the “naive” inner product. No longer do we need to blame the trivial inner product for the non-normalizability of the monochromatic solutions. Of course monochromatic solutions are strictly speaking non-normalizable by themselves; their superpositions, on the other hand, are normalizable in the standard sense.

Specifically, mimicking the procedure in [27] for the simpler current [27], we can infer the inner product:

\[
\langle \psi_1 | \psi_2 \rangle = \int dX_\phi \psi_1^* (b, T_\phi) \psi_2 (b, T_\phi)
\]

with unitarity:

\[
\frac{\partial}{\partial T_\phi} \langle \psi_1 | \psi_2 \rangle = 0
\]

enforced by current conservation:

\[
\frac{\partial}{\partial T_\phi} \langle \psi_1 | \psi_2 \rangle = \int dX_\phi \frac{\partial}{\partial X_\phi} (\psi_1^* (b, T_\phi) \psi_2 (b, T_\phi)) = 0.
\]

(This vanishes only with suitable boundary conditions, with subtleties like the ones highlight in [24], e.g. singularities, etc.) We can also swap \( X_\phi \) and \( T_\phi \) in this definition:

\[
\langle \psi_1 | \psi_2 \rangle = \int dT_\phi \psi_1^* (b, T_\phi) \psi_2 (b, T_\phi)
\]
with the two definitions equivalent and amounting to:
\[ \langle \psi_1 | \psi_2 \rangle = \int d\phi A_1^* (\phi) A_2 (\phi), \]  
(60)

in view of (43). The normalizability condition \(|\langle \psi | \psi \rangle| = 1\) therefore supports (44) identifying the probability of \(\dot{\phi}\).

We stress that this argument is valid for each dominating fluid \(i\), adopting its associated \(\alpha, X_\alpha\) and \(T_\alpha\). The argument for unitarity is far more complicated in the transition regions for multi-fluids, or for the minority clock, i.e. clocks corresponding to sub-dominant components, as we shall see later in this paper.

V. RADIATION AND GENERAL PERFECT FLUIDS

Our approach can be used to find the equivalent of the Chern-Simons state for Universes filled with a single fluid with an equations of state other than \(w = -1\). To this purpose we add to the gravitational action \(S\) the “matter” action:
\[ S_m = -6\kappa V_c \int dt Na \frac{m}{a^{1+3w}} \]  
(61)

valid for a perfect fluid with equation of state \(w = p/\rho\) (see[24] and refs therein). The factor of \(6\kappa\) puts \(m\) on the same footing as \(\Lambda\) (at least dimensionally), and it may need to be adapted if we target \(G_N\) for our procedure, as outlined in [3] [add to this].

Hence, the total Hamiltonian (matter and gravity) is:
\[ H = 6\kappa V_c Na \left( -(b^2 + k) + \frac{m}{a^{1+3w}} \right). \]  
(62)

so that the Hamiltonian constraint can be put in the equivalent form (34) (i.e. \(H = h_\alpha (a) a^2 - \alpha = 0\), cf. Precept 3), with:
\[ h_\alpha (a) = (b^2 + k + \frac{m}{a^{1+3w}}). \]  
(63)

Having identified the function of \(m\) best suited to producing a Schrödinger equation, we can then expand the phase space according to (13), so that the total matter action is:
\[ S_m = 6\kappa V_c \int dt (\dot{a} \dot{p}_\alpha - Na \frac{m}{a^{1+3w}}). \]  
(65)

We now have commutation relations of the form (15) between \(\alpha\) and its momentum, \(p_\alpha\), leading to a Schrödinger equation of the form (17), just as before. Time \(T_\alpha\) is now identified with \(p_\alpha\). Its monochromatic solutions, as before, are (18) with extra condition (19), that is, we find that the WDW equation for a given fixed value of \(m\) (or equivalently \(\alpha\)) is:
\[ \left[ -i \frac{\hbar^2}{3V_c} h_\alpha (a) \frac{\partial}{\partial b} - \alpha \right] \psi (b; \alpha) = 0. \]  
(66)

By construction we therefore find plane waves in a “spatial” variable which, instead of the Chern-Simons functional, has general form:
\[ X_\alpha (b) = \int \frac{db}{h_\alpha (b)} = \int \frac{db}{(b^2 + k + \frac{m}{a^{1+3w}})}. \]  
(67)

Notice how this reduces to the Chern-Simons functional (41) when \(w = -1\). All we need to do is change \(X\) and the WDW equation still is:
\[ \left( -i \frac{\hbar^2}{3V_c} \frac{\partial}{\partial X_\alpha} - \alpha \right) \psi = 0, \]  
(68)

with solution:
\[ \psi_s (b; \alpha) = N \exp \left[ \frac{3V_c}{\hbar^2} \alpha (X_\alpha (b) - T_\alpha) \right]. \]  
(69)

This is the generalization of the (monochromatic) Chern-Simons state for equations of state other than \(w = -1\). Adding to it the unitary time evolution (13) we have:
\[ \psi (b; \alpha) = N \exp \left[ \frac{3V_c}{\hbar^2} \alpha (X_\alpha (b) - T_\alpha) \right], \]  
(70)

with \(T_\alpha = p_\alpha\). The most general solution superposes these as:
\[ \psi (b) = \int \frac{d\alpha}{\sqrt{2\pi \hbar}} A(\alpha) \exp \left[ \frac{i}{\hbar} \alpha (X_\alpha (b) - T_\alpha) \right], \]  
(71)

or:
\[ \psi (b) = F(T_\alpha - X_\alpha). \]  
(72)

A. The example of radiation

A radiation dominated Universe \((w = 1/3)\) is a particularly simple case. Then the Hamiltonian is:
\[ H = 6\kappa V_c \frac{Na}{\alpha} \left( -(b^2 + k) a^2 + m \right) \]  
(73)

already compliant with Precept 3, up to a redefinition of the lapse function:
\[ \tilde{N} = 6\kappa \frac{Na}{\alpha} V_c \]  
(74)

even off-shell. Time, therefore, is the conjugate momentum to \(m\) and this can be identified with conformal time [more on this later]. The monochromatic solutions to the time-dependent Sch equations are:
\[ \psi = \psi_s (b; m) \exp \left[ -\frac{i}{\hbar} m \right], \]  
(75)

with:
\[ \left[ i \hbar (b^2 + k) \frac{\partial}{\partial b} + m \right] \psi_s = 0. \]  
(76)

so that we have plane waves in terms of:
\[ X_r (b) = \int \frac{db}{b^2 + k} = \frac{1}{\sqrt{|k|}} \arctan \left( \frac{\sqrt{|k|}}{k} \right) \quad \text{if } k > 0 \]
\[ = -\frac{1}{\sqrt{|k|}} \quad \text{if } k = 0 \]
\[ = -\frac{1}{\sqrt{|k|}} \arctanh \left( \frac{\sqrt{|k|}}{k} \right) \quad \text{if } k < 0 \]
to be seen as the equivalent of the Chern-Simons functional for a radiation dominated Universe. The plane wave solutions at a generic time therefore are:

$$\psi(b, T_r; m) = \mathcal{N} \exp \left[ \frac{i}{\hbar} m (X_r - T_r) \right].$$

(77)

(with $T_r = p_m$).

B. One exception: Milne or curvature domination

The general solution (63) breaks down for $w = -1/3$, an equation of state degenerate with spatial curvature $k$ (or $kc^2$, to put it suggestively). Backtracking to (62) we see that the problem is that we lose the spatial differential operator contained in $a^2$. The spatial solution then becomes $\psi = \delta(b^2 - m)$, where $m$ can include, or indeed be just $-kc^2$. The monochromatic solution is:

$$\psi(b, T_m; m) = N \delta(b^2 - m) e^{-\frac{\hbar}{m} T_m}$$

(78)

with $T_m = p_m$ as usual. Hence in this case there is no time evolution, since for any superposition we have:

$$\psi(b, T_m) = \int dm A(m) \delta(b^2 - m) e^{-\frac{\hbar}{m} T_m}$$

$$= A(b^2) e^{-\frac{\hbar}{m} b^2 T_m}$$

(79)

so that $|\psi|^2 = |A(b^2)|^2$. The reason why this happens will be made clear in the next Section.

VI. THE CLASSICAL LIMIT

Given that “time evolution” is the most obvious feature of classical cosmology, it is obvious that any quantum cosmology scheme lacking a “time” will have trouble connecting with the classical world. Reciprocally, the discovery of a quantum time should be used in the first instance to make sure that the classical limit is sound, before exploring possible quantum departures/corrections.

In this Section we first present the format in which the classical results are best presented so that they can be recovered by appropriately (semi)-classical wave-functions, within our scheme. We then prove that coherent states reproduce the classical limit.

A. The classical “time-formula”

We first find a classical expression for our physical times $T_\alpha$ as a function of the non-physical coordinate time $t$ associated with lapse $N$. From the second Hamilton equation (using 62, 14 and 64) we can derive the “time-formula”:

$$\dot{T}_\alpha = \dot{p}_\alpha = \{p_\alpha, H\} = -\frac{1 + 3w}{2} Na^{-3w} m^{\frac{3w-1}{1+w}}$$

$$= -\frac{1 + 3w}{2} Na^{-3w} \alpha^{\frac{3w-1}{1+w}}.$$  

(80)

Note that we have used the original Hamiltonian, and not $\mathcal{H}$, to work out the relation between $T_\alpha$ and time. We can now set $N = 1$ to derive the relation between $T_\alpha$ and cosmological proper time $t$:

$$\frac{dT_\alpha}{dt} = -\frac{1 + 3w}{2} a^{-3w} m^{\frac{3w-1}{1+w}}$$

(81)

or else set

$$N = N_\alpha = -\frac{2}{1 + 3w} a^{3w} m^{\frac{3w}{1+w}}$$

(82)

to ensure we are using a time coordinate coincident with the physical time $T$.

Within this scheme (but note [13]), we highlight the following facts:

- Radiation is unique in that its time does not depend on $m$, so when this goes to zero its time is still well defined.
- Specifically, radiation time is minus conformal time, $T_r = -\eta$, since:

$$\dot{T}_r = -N/a.$$  

(83)

This is in agreement with [24].
- Dust time is proportional to minus proper cosmological time, with:

$$T_m = -\frac{t}{2m}.$$  

(84)

- Our Lambda time is proportional to Misner’s volume time [10] (as well as unimodular time [8]):

$$\dot{T}_\phi = N \frac{a^3}{\phi^2} = N \frac{A^2}{9} a^3.$$  

(85)

A canonical transformation relates the two: this is responsible for linearizing the dispersion relations.
- The only degenerate case in (80) is $w = -1/3$, but this case is exceptional, as already discussed in Section VI. In this case we should not transform from $m$ to $\alpha$, so that $\dot{T}_m = -Na$.

The sign in the time-formula (80) is important and we note that it changes from $w > -1/3$ to $w < -1/3$. This is a key feature of our formalism and we will comment further on this below.

B. The classical trajectory: a connection space picture

All classical descriptions are equivalent, so we may select whichever makes better contact with our quantum theory. In our case we pick a description which is unusual in that:

- Instead of the expansion factor $a$, we take for dependent variable the minisuperspace connection variable $b$. Recall that when torsion is zero, on-shell this is the comoving inverse Hubble length $\dot{a} = b$. Quantum mechanically $b$ is an independent and complementary variable to the metric (or rather, the densitized inverse triad $a^2$).
Instead of using some coordinate time $t$ as independent variable, we use the physical time(s) $T$; These are classically (on-shell) a function of $t$, as just calculated in \[80\]. Fundamentally, and quantum mechanically, there can be many $T$, but in the classical limit they all become functions of $t$ (so that there is only one time classically, but there are several quantum mechanical times).

Hence, the classical description we are aiming for takes the form $b = b(T)$, possibly in the parametric form:

$$b = b(t)$$

$$T_\alpha = T(t),$$

rather than the textbook $a = a(t)$.

Then, we can show that the classical trajectory for a single fluid system is given by:

$$X_\alpha = T_\alpha.$$  \[88\]

Indeed the full content of the classical equations can be obtained from the first Friedman equation (equivalent to the Hamiltonian constraint $H = 0$):

$$b^2 + k = \frac{m}{a^{1+3w}}$$  \[89\]

which should be assumed throughout, as well as the two dynamical Hamilton equations:

$$\dot{a} = \{a, H\} = Nb$$  \[90\]

$$\dot{b} = \{b, H\} = -\frac{1 + 3w}{2} Na \frac{m}{a^{3(1+w)}}$$  \[91\]

(where we have used \[89\] in the second equation). Together, these two dynamical equations imply the Raychaudhuri (second Friedman) equation (for $N = 1$):

$$\dot{a} = -\frac{1 + 3w}{2} a \frac{m}{a^{3(1+w)}}.$$  \[92\]

It is easy to see that \[88\] implies:

$$\frac{\dot{b}}{h_\alpha(b)} = -\frac{1 + 3w}{2} a^{-3w} m^{\frac{3w-1}{3}}$$  \[93\]

which upon some manipulations reproduces the dynamical equation \[91\] (assuming the constraint \[89\] throughout).

Using this unconventional description (i.e. \[88\]) may take some getting used to, even though it is classically equivalent to the $a = a(t)$ description. Points of note include:

- Expanding and contracting Universes correspond to $b > 0$ and $b < 0$, with $b = 0$ representing a static Universe (and its vicinity the loitering model).

- Hence, the ekpyrotic \[28\], or any such similar “bouncing” models, will see $b$ go through zero. Tunnelling between branches with different signs may also be possible.

- For a given matter content, $b$ can either only increase or only decrease in parameter time $t$; the first if $w < -1/3$, the second if $w > -1/3$. For $w = -1/3$ (or for the Milne Universe, for example) $b$ does not change (this starts shedding light one the anomaly found in Section [V.B].)

- Hence, the equivalent in of a “bounce” in $b$ space is a Universe undergoing a transition from decelerated to accelerated expansion, such as we have seemingly undergone recently. At the end of inflation the reverse happens, the equivalent of a “turn-around” in $b$ space.

The fact that in this picture we have recently emerged from a $b$-bounce must have quantum mechanical implications: quantum reflection always leaves its traces. The matter will be studied in more detail in \[26\].

C. Parenthesis on the “arrow of time”

In view of what we said, the issue of the arrow of “time” merits a parenthesis. In the metric representation flipping the time arrow inter-converts expanding (increasing $a$) and contracting (decreasing $a$) Universes. There are always two branches of solutions, as required by time-reversal symmetry.

In the $b = b(T)$ description, face value, there is only one solution, for which $b$ must increase with $T_\alpha$:

$$\frac{db}{dT_\alpha} > 0,$$  \[94\]

as implied by \[88\]. This is reflected in the quantum mechanical solutions (cf. \[28\]): there can only be outgoing waves. Thus, there is a sense in which there is only one arrow of time in the connection representation and using the times $T_\alpha$. A Feynman absorber is not needed to set the physical arrow of time.

This fact is actually an expression of the horizon (and ultimately also the flatness) problem, as well as its standard solution, as we now show. Let us first assume expanding Universes ($b > 0$). Then, the horizon problem is that for $w > -1/3$ the comoving Hubble length $1/b$ increases in time, whereas its solution follows from that it decreases if $w < -1/3$. In our description this is expressed by \[94\] and the fact that, using $t$ as the auxiliary arbitrary time arrow, the sign in the time-formula \[80\] changes from $w > -1/3$ to $w < -1/3$. Converting this into $b(T_\alpha)$ we then get that \[94\] is a statement of the horizon problem for $w > -1/3$ and its solution for $w < -1/3$.

The actual arrow of $t$ does not matter, because it cancels in its double effect on $b = \dot{a}$ and on $b$ (note the invariance of the Raychaudhuri equation \[91\] under time reversal). Solutions to the horizon problem based on a contracting phase (e.g. the ekpyrotic scenario \[28\]) can be understood in our description from \[94\] being independent of the sign of $b$; however the statement of the problem and its solution involves $|b|$, not $b$. So the criteria for problem and solution are reversed for models in a contracting phase ($b < 0$), and this is still expressed by \[94\].
D. Coherent states and the classical limit

Having a quantum time variable and a larger space of solutions (as described in Sec. IV) are the two reasons why contact with the classical limit is possible. Monochromatic waves, such as those solving the fixed constant theory, imply a uniform distribution (in $X(b)$), hardly a prediction, but they are also not immediately physical. One needs both a time variable and the ability to superpose plane waves into normalized peaked distributions to recover something minimally physical.

In fact having a peak is not enough. For example, since $X_\alpha = \hat{T}_\alpha$ represents the classical trajectory (as just discussed), one might think that the light-ray, $\psi \propto \delta(X_\alpha - T_\alpha)$, described in Section IV.B would be perfectly classical. But such a state would have a totally undefined $\alpha$, and so the $T_\alpha = \hat{T}_\alpha(t)$ part of the argument could not be true (note that $\alpha$ generally appears in the RHS of (80)).

The semi-classical limit is only recovered for the coherent states $\psi(b, T_\alpha)$ described in Section IV.B. For these, the second Hamilton equation (80) is true not only on average (an expression of Ehrenfest’s theorem), but with minimal and balanced uncertainties in the complementary $\alpha$ and $T_\alpha$ appearing on the two sides of (80). Both sides of the argument implying that $X_\alpha = \hat{T}_\alpha$ represents the classical trajectory can now be reproduced, and so we have a truly semi-classical state.

We can also understand the result in Section IV.B. We do not have propagating waves in this case. However, the classical equation of motion is $\dot{b} = 0$, i.e. the Universe is static in $b$, as already explained. Any coherent state in $m$ therefore reproduces this result (as well as the time formula in terms of $m$).

VII. MULTITIME

Naturally, for all our pains, we now end up with the “curse of time” in Quantum Gravity: either there is no time, or, if we succeed in defining one, we are left spoil for choice. The proposal in [6] was to accept a democracy of times, with the adjustment of clocks across different “time zones” to be seen as a physical feature of our world. We now examine further how the handover between clocks can be made seamless for some states. But there are also alternative states, which might be of great predictive relevance, considering that the Universe is currently busy passing on the baton from a matter (or $G_N$) clock to a $\lambda$ clock.

How do we deal with multiple times and multiple fluids, even in situations where there are epochs where one fluid dominates? Let $\alpha$ be a vector representing the whole set of relevant constants and $T$ their conjugates. The various components of $T$ are a priori independent variables, so we have a plethora of times instead of a single one. The statements made in this paper imply that there is not one “Schroedinger” equation (our examples being merely limiting cases) but a PDE in multiple times running concurrently, obtained by taking the Hamiltonian in

$$S = \frac{3V_c}{8\pi G_0} \int dt \left(\alpha^2 b - Na \left[\left(\dot{b}^2 + k c^2\right) + \sum_i \frac{m_i}{a^{1+3w_i}}\right]\right),$$

and applying the replacements:

$$H \left[b, a^2; \alpha \rightarrow i \frac{\partial}{3V_c \partial T}\right] \psi = 0. \quad (96)$$

Its general solutions are:

$$\psi(b, T) = \int d\alpha A(\alpha) \exp \left[-i \frac{3V_c}{T_P} \alpha T\right] \psi_s(b; \alpha), \quad (97)$$

where $\psi_s(b; \alpha)$ solves the WDW equation with constant $\alpha$.

As outlined in [6], for a Hamiltonian carving up phase space into regions dominated by a single constant, the readjustment of quantum clocks across such regions is seamless if we assume coherent states in all $\alpha$ and factorization:

$$A(\alpha) = \prod_i \sqrt{N(\alpha_{0i}, T_P^2/6V_c)}. \quad (98)$$

Then, the $\psi_s(b; \alpha)$ is a piecewise plane wave in the $X_i(b)$ associated with each dominant $\alpha_i$. Each piece engages with the phase associated with the corresponding $T_i$ (hence, the approximate single-time Schrodinger equation), producing a wave-packet describing the correct classical limit, as above. What happens to the minority clocks in each phase will be studied in more detail later in this paper.

Indeed the rest of this paper will be devoted to making this argument more explicit, whilst dealing with the unpleasant truth that we are not living in a Universe where one component dominates, but quite the opposite. In order to deal with multi-fluid situations we need to first re-examine the structure of minisuperspace from a different angle.

VIII. MINISUPERSPACE AS A DISPERSIVE MEDIUM

The arguments in Section III for variables $\alpha$, $X_\alpha$, and $T_\alpha$ are persuasive, but realistic only in idealized situations, such as when there is only one fluid filling the Universe. Unfortunately the real world does not oblige. In some cases such variables cannot be found. For example, at the crossover between 2 epochs dominated by different fluids, we will find an $X$ variable which is a function not only of $b$ but also of $\alpha$ (see next Section). In such cases it is more fruitful to revert to the original variables (e.g. $b$ instead of any $X(b)$) and regard minisuperspace as a dispersive medium. From this point of view the variables $\alpha$, $X_\alpha$, and $T_\alpha$, when they exist, are the “linearizing variables” of the dispersive medium, to use the terminology of [25]. They can and should be used where they exist, but more generally we should face the dispersive nature of minisuperspace head on.

In the general case we can still define times $T$ for the various $\alpha$, impose the monochromatic ansatz [147], and find the spatial solutions $\psi_s$, which in general will not be plane waves.
in any $X(b)$ variable independent of $\alpha$. The monochromatic solutions can still be superposed into peaked wave-packets, as in (147). However it is important to realize that, as with any other dispersive medium, the envelope of such packets moves with a group speed that should not be confused with the phase speed.

Specifically, writing:

$$\psi_a(b, \alpha) = \exp \left[ \frac{3V}{T_p} P(b, \alpha) \right]$$  \hspace{1cm} (99)

we identify dispersion relations:

$$\alpha \cdot T - P(b, \alpha) = 0.$$  \hspace{1cm} (100)

Assuming that the amplitude $A(\alpha)$ is factorizable and sufficiently peaked around $\alpha_0$, we can expand:

$$P(b, \alpha) = P(b; \alpha_0) + \sum_i \frac{\partial P}{\partial \alpha_i} (\alpha_i - \alpha_0) + \ldots$$  \hspace{1cm} (101)

to find that the wave function factorizes as:

$$\psi \approx e^{\frac{i}{\hbar} \int P(b; \alpha_0) - \alpha_0 \cdot T_i} \prod_i \psi_i(b, T_i).$$  \hspace{1cm} (102)

The first factor is the monochromatic (generally non-plane) wave centered on $\alpha_0$. The other factors describe envelopes of the form:

$$\psi_i(b, T_i) = \int d\alpha_i A(\alpha_i) e^{-\frac{i}{\hbar} \int (\alpha_i - \alpha_0)(T_i - \frac{\partial P}{\partial \alpha_i})}$$  \hspace{1cm} (103)

which therefore move according to:

$$T_i = \frac{\partial P(b)}{\partial \alpha_i} \bigg|_{\alpha_0}.$$  \hspace{1cm} (104)

We can also dot this equation, to find the group speed on \{b, T_i\} space:

$$c_g \equiv \frac{db}{dT} \bigg|_{\text{peak}} = \frac{\dot{b}}{T} \bigg|_{\text{peak}} = \frac{1}{\frac{\partial P}{\partial \alpha_i}}.$$  \hspace{1cm} (105)

The motion of these envelopes (and so of the peak of the distribution) should agree with the classical equations of motion. We will show in the rest of this paper that indeed it does so, for coherent states, in a number of non-trivial situations (such as for mixtures of fluids during transition periods when none of them dominates, or for the sub-dominant clock).

This obviously generalizes the construction for single fluids, for which a variable $X(b)$ can be found such that $P = \alpha X(b)$ for some $\alpha$. Then, with a suitable choice of $\alpha$ (and canonical $T_\alpha$) we can always make the first term in the dispersion relations $\alpha T_\alpha$ (for example, in the case of Lambda by $\Lambda \to \phi = 3/\Lambda, T_\Lambda \to T_\phi = -T_\Lambda/\phi^2$). They are “linearizing” variables because $c_{lin} = \dot{X}/\dot{T} = 1$.

**IX. DEALING WITH CROSSOVER REGIONS**

As it happens we are sitting right on a bounce in $b$. How do we deal with such transitions? In this Section we show that the correct semi-classical limit is still obtained assuming the wave function remains sharply peaked. What actually happens to the wave function is left to future work [23]. We also investigate the fate of the minority clock (i.e. the radiation and Lambda clocks in the Lambda and radiation epochs) once the handover of clocks is completed. We will use as a working model a mixture of radiation and Lambda, because the algebra is clearer, but generalizations to the more relevant case of dust and Lambda behave in the same way.

**A. Mono-chromatic solutions**

Our working model has classical Hamiltonian:

$$H = Na \left( -(b^2 + k) + \frac{a^2}{\phi} + \frac{m}{a^2} \right)$$  \hspace{1cm} (106)

spanning a two-dimensional constant space with

$$\alpha = \left( \phi = \frac{3}{\Lambda} m \right).$$  \hspace{1cm} (107)

Its multi-time “Schroedinger” equation (96) has solutions of the form (147). One way to find the spatial $\psi_a$ is to put $H = 0$ in the form (112) with $\alpha = \phi$, aware that $h_n(b)$ will then not satisfy precept 2. To this end we solve the quadratic in $a^2$ equivalent to $H = 0$ to find:

$$a^2 = \frac{g \pm \sqrt{g^2 - 4m/\phi}}{2/\phi}$$  \hspace{1cm} (108)

with $g(b) = b^2 + k$. Since $a^2$ must be real (although not necessarily positive) we have:

$$g^2 \geq g_0^2 = \frac{4m}{\phi} = \frac{4}{3} \Lambda m.$$  \hspace{1cm} (109)

The plus branch contains Lambda domination when $g^2 \gg g_0^2$; the minus branch contains radiation domination, also with $g^2 \gg g_0^2$. The transition happens when $g^2 \approx g_0^2$ (with $g^2 > g_0^2$). Thus, we have a “bounce” in $b$ space at $g = g_0$, i.e. a transition from decelerated expansion (decreasing $b$) to accelerated expansion (increasing $b$). The Hamiltonian constraint is therefore equivalent to two constraints of the required form:

$$H_\pm = h_\pm(b, \phi, m)a^2 - \phi = 0$$  \hspace{1cm} (110)

with the important novelty that $h$ (and so $H_0$) is “energy”-dependent (dependent on the conjugate of time; i.e. the constants):

$$h_\pm = \frac{2}{g \pm \sqrt{g^2 - 4m\phi}}.$$  \hspace{1cm} (111)
This is of course irrelevant for the $\psi_s$, which is given by:

$$\psi_{s,\pm}(b; \phi, m) = N \exp \left[ i \frac{3Vc}{l_p^2} \phi X_{\pm}(b; \phi, m) \right]. \quad (112)$$

with:

$$X_{\pm}(b; \phi, m) = \int db \frac{1}{2} \left( g \pm \sqrt{g^2 - 4m/\phi} \right). \quad (113)$$

We see that for $g^2 \gg m/\phi$ the $+/-$ branches have:

$$X_+(b; \phi, m) \approx X_\phi = \frac{b^3}{3} + kb \quad (114)$$

$$X_-(b; \phi, m) \approx \frac{m}{\phi} X_r \quad (115)$$

leading to the correct limits:

$$\psi_{s,+}(b; \phi, m) \approx N \exp \left[ i \frac{3Vc}{l_p^2} \phi X_\phi(b) \right] \quad (116)$$

$$\psi_{s,-}(b; \phi, m) \approx N \exp \left[ i \frac{3Vc}{l_p^2} m X_r(b) \right]. \quad (117)$$

This illustrates with a concrete example the comments made just after Eq. (128): the $\psi_s(b; \alpha)$ is a piece-wise plane wave in the relevant $\alpha$ and $X_\alpha$ in each region of single fluid domination. To leading order it might seem that if $A(\alpha)$ factorizes, then all the other times factorize, too, and stop describing the $b$ evolution since they became $b$-independent phases. However this is not the case, as we now show by considering the next to leading order in the expansion.

### B. What happens to the minority clock(s)?

Before addressing the handover region itself, we first examine in more detail what happens to the “minority” clock once the handover is finished. Expanding (113) to the next order we find:

$$X_+(b) \approx X_\phi - \frac{m}{\phi} X_r + ... \quad (118)$$

$$X_-(b) \approx \frac{m}{\phi} X_r + \frac{m^2}{\phi^2} \int db \frac{db}{g^2} + ... \quad (119)$$

1. The radiation clock in the Lambda epoch

Including the next order term in $X_+$ we find that deep in the Lambda epoch the monochromatic wave function is:

$$\psi(b, T; \alpha) = N \exp \left[ -i \frac{3Vc}{l_p^2} (\phi(T_\phi - X_\phi) + m(T_r + X_r)) \right].$$

Inserting into (124) we find for any factorizable amplitude:

$$\psi(b, T) = F_1(X_\phi - T_\phi) F_2(X_r + T_r). \quad (120)$$

In particular we could choose factorizable Gaussian amplitudes for $\phi$ and $m$ leading to coherent $F_1$ and $F_2$ (of the form (50)). As we will see this is more the exception than the rule.

We see that both factors reproduce the classical equations of motion. These amount to the first and second Friedmann equations:

$$b^2 + k = \frac{a^2}{\phi} + \frac{m^2}{a^2} \quad (121)$$

$$\dot{b} = \frac{a}{\phi} - \frac{m}{a^2}. \quad (122)$$

In addition, the times formulae (replicated quantum mechanically by coherent factorizable states, just as before) are:

$$\dot{T}_\phi = \frac{a^3}{\phi^2} \quad (123)$$

$$\dot{T}_r = -\frac{1}{a}. \quad (124)$$

Evaluating:

$$\dot{X}_\phi = b(b^2 + k) \quad (125)$$

$$\dot{X}_r = \frac{b}{b^2 + k} \quad (126)$$

we can then recover the mono-fluid equations of motion in the appropriate epochs with $X_\alpha = \bar{T}_\alpha$, for $\alpha = \phi, m$. But some more algebra also reveals that deep in the Lambda era we can write the classical trajectory as:

$$\dot{X}_r \approx -\dot{T}_r \quad (128)$$

and this is equivalent to $\dot{X}_\phi \approx \dot{T}_\phi$. Hence, the peak of both factors in (120) describes the classical trajectory.

This sheds light on what happens to our quantum “multi-time” in semi-classical situations, given that classically only one time can exist. Quantum mechanically the two $T_i$ are independent variables and fundamentally remain so, even for semi-classical states. There is never a constraint between the different $T_i$. What happens is that for peaked states the peak of the joint distribution maps out a trajectory of $b$ in 2D space $T$ (in this case $X_\phi(b) = T_\phi$ and $X_r(b) = -T_r$). This implies a constraint between the two times at the peak of the joint distribution, so classically only one time exists. The quantum fluctuations of these different times, on the other hand, would remain independent.

2. The Lambda clock in the radiation epoch

Deep in the radiation epoch we have instead

$$\psi(b, T; \alpha) =$$

$$N \exp \left[ -i \left( m(T_r - X_r) + \phi \left( T_\phi - \frac{m^2}{\phi^2} \int db \frac{db}{g^2} \right) \right) \right]. \quad (129)$$

with the novelty that the factor associated with the subdominant component (Lambda) now depends on $m$ as well. As a result, the wave packets never factorizes into separate radiation and Lambda factors, even if the amplitudes $A(\alpha)$ do. In addition the minority factor no longer is a plane wave in the
original $X(b)$ and $\alpha = \phi$. Hence, even if the original amplitudes were a diagonal Gaussian, the wave functions will be very distorted. The simple arguments for unitarity for single fluids also break down for a minority clock in this situation. This will be discussed further in the next Section.

Nonetheless, we can show that for a peaked second factor, the motion of the peak still reproduces the correct classical limit (the first one obviously does). Using (105) and

$$c_g^{-1} = \frac{\partial^2}{\partial \phi \partial b} \int db \frac{m^2}{g^3} = -\frac{m^2}{\phi^2 g^3}$$

(130)

we find that for the peak:

$$c_g = \frac{b}{T_0} \bigg|_{\text{peak}}$$

(131)

implies

$$\dot{b} = -\frac{m}{\omega^3}$$

(132)

which is nothing but approximately (122) in the radiation epoch.

Notice that within the same approximations used in Section VIII to derive (105), the wave-function effectively factorizes as:

$$\psi(b, T; \alpha) \approx \psi_1(T_r - X_r)(b), T_0; m_0$$

(133)

Hence to this order, the arguments at the end of Section IX B.1 in support of a single classical time still apply.

C. Sitting on the fence

We can also show that if the distribution remains peaked (26), then the peak follows the classical trajectory even during the $b$-bounce, for both branches $\pm$. In this case the $P$ function defined in Section VIII is given by:

$$P_{\pm}(b, m, \phi) = \phi X_{\pm}(b; m/\phi)$$

$$= \phi \int db \frac{1}{2} \left( g \pm \sqrt{g^2 - 4m/\phi} \right).$$

(134)

Within the approximations of Section VIII (see (102) in particular) the wave function must be approximately given by the monochromatic solution times the product of two envelopes $\psi_1(b, T_0)\psi_2(b, T_r)$. The latter move with group speeds:

$$c_{g1} = \frac{\dot{b}}{T_0 \bigg|_{\text{peak}}} = \frac{1}{\partial \phi \partial b}$$

(135)

$$c_{g2} = \frac{\dot{b}}{T_r \bigg|_{\text{peak}}} = \frac{1}{\partial m \partial b}.$$  

(136)

It is now a matter of algebra to shown that in both branches ($\pm$) these are equivalent to the classical equation of motion (122) (with (121) assumed throughout).

Indeed, for the Lambda wave packet factor we have:

$$\frac{\partial^2 P}{\partial \phi \partial b} = \frac{1}{\hbar} \pm \frac{m}{\phi} \frac{1}{\sqrt{g^2 - g_0^2}}.$$  

(137)

Using:

$$\pm \sqrt{g^2 - g_0^2} = \frac{a^2}{\phi} - \frac{m}{a^2}$$

(138)

and $h = \phi/a^2$ we have:

$$\frac{\partial^2 P}{\partial \phi \partial b} = \frac{a^2/\phi^2}{\phi - a^2/\phi}$$

(139)

implying that the peak moves along:

$$\dot{b} = \frac{T_0}{a^2 / \phi^2} \left( \frac{a^2}{\phi} - \frac{m}{a^2} \right) = \frac{a}{\phi} - \frac{m}{a^3}$$

(140)

i.e. (122), as required.

Likewise, for the $m$ wave packet factor we have:

$$\frac{\partial^2 P}{\partial m \partial b} = \mp \frac{1}{\sqrt{g^2 - g_0^2}}$$

$$= -\frac{a^2}{\phi} - \frac{m}{a^3}$$

(141)

leading to:

$$\dot{b} = -\frac{T_r}{a^2 / \phi^2} \left( \frac{a^2}{\phi} - \frac{m}{a^2} \right)$$

(142)

or (122). 

Hence the correct classical limit is always obtained, assuming the wave functions remain peaked. Whether this is a good approximation remains to be seen [26]. In addition there are other issues regarding the semi-classical limit, as we now explain.

X. WHY SWAP CLOCKS?

In this Section we explain better why “a clock [should be] crafted with what is at hand”, as proposed in [6]. This is not just common sense: It affects the semi-classical limit. As we have just seen in detail, the probability peak’s motion has the correct classical limit (assuming Ehrenfest’s theorem) even for the minority clock, but this hides the fact that typically the state will not be coherent in such a set up, and so departures from the semi-classical regime are expected.

The Lambda clock in the radiation epoch is a good illustration of this. As we saw in Section IX B.2 (cf. Eq. (129)), to leading order (in the saddle approximation of Section VIII), the Lambda factor in the radiation epoch changes its dependence on $b$ from $X_\phi$ to:

$$X = \int \frac{db}{g^3}$$

(143)
and its $\alpha$ from $\phi$ to:

$$\alpha = \frac{m_0}{\phi^2}, \quad (144)$$

In Section [X B 2] we studied in detail the peak of the wave function, but the semi-classical limit requires also the arguments in Section [V I] for the correct representation of the time-formula, and so we need more than a peaked distribution: we should have a coherent state in $m_0/\phi^2$. But if we chose a Gaussian amplitude in $\phi$, then this will not be Gaussian in $m_0/\phi^2$, quite the contrary: strong distortions are expected. This is representative of what usually happens to minority clocks. Indeed, the radiation clock in the Lambda epoch (see Section [X B 1]) is the exception to this rule. It is a rare case where a coherent majority clock remains coherent in the sub-dominant phase.

We can also add the issue of unitarity and inner product to the discussion. We can define a conserved inner product as in Section [IV D] but it only leads to a simple conserved current and re-expression in terms of a measure in $b$ in the dominant epoch. As we saw, in mono-fluid situations (and so using the dominant clock in a multi-fluid situation) there is a range of options for setting up the inner product and conserved current. These are all ultimately equivalent: we can use Eq. (56), leading to general expression:

$$d\mu(b) = dX_\alpha = \frac{db}{(b^2 + k)^{1/2}}, \quad (145)$$

we can trade $X_\alpha$ for $T_\alpha$ as in (59), or we can use (146) (with $\phi$ replaced by the applicable $\alpha$).

However, only

$$\langle \psi_1 | \psi_2 \rangle = \int d\alpha A^\ast (\alpha_1) A(\alpha_2). \quad (146)$$

generalizes to multi-fluid situations. Bearing in mind that the general solution now is:

$$\psi(b, T) = \int \frac{d\alpha}{\sqrt{2\pi}} A(\alpha) \exp \left[-i\frac{3V_c}{2p} \alpha T \right] \psi_\alpha(b; \alpha), \quad (147)$$

where $\psi_\alpha(b; \alpha)$ solves the WDW equation with constant $\alpha$, it is obvious that (146) reduces to (56) and (59) for single fluids. For multi-fluids we recover (59) iff $\psi_\alpha(b; \alpha)$ is a pure phase (so in cases where there is no $b$ bounce), but not (56). For multi-times this is more complicated. Whatever the case (146) appears to be a general time-independent definition for the inner product. But only when the dominant clock and its $X$ are used does this lead to a simple inner product in terms of $b$.

**XI. CONCLUSIONS**

In summary, in this paper we started off by proposing a number of aspirational principles for defining clocks and rods in quantum cosmology, based on an amplification of the standard theory allowing the constants of nature to be non-constant off-shell. Classically, the constants remain good old constants, but quantum mechanically it all changes. Each constant generates a space of quantum states composed of superpositions of waves indexed by the value of the constant (these can be seen as monochromatic partial waves). The waves propagate in a fundamentally dispersive medium where the “space” is the connection, the “time” is the momentum conjugate to the constant, and the “energy” and “momentum” are functions of the constant. In some regions (or “epochs”) we can find simple linearizing variables $(X_\alpha, \alpha$ and $T_\alpha)$, in terms of which the partial waves are plane waves moving at fixed speed, conventionally set to 1. These implement our aspirational principles. In such regions, for a given constant, our construction provides a good clock and rod, leading to a simple inner product and definition of unitarity, with coherent states providing the perfect definition of a semi-classical states. However, this construction is never global, and hence the need to change clocks at different “time zones” in our Universe.

Specifically, we showed that the dominant fluid always generates a good clock, but the “minority clock” is generally problematic in that the linearizing variables are not always available, and so the dispersive nature of the medium has to be faced. Even if approximate linearizing variables can be found, they will generally be different functions of the constant than in the dominant phase. If the wave function was coherent in the original variable, it will not be in the second. If the wave function is peaked, the peak follows the classical equations of motion; however the quantum nature of the system can never be erased. Unitarity can be defined in a universal way, but it becomes cumbersome when written in terms of the minority clock. Changing clocks is advisable.

The interesting point remains that in transition regions we may expect anomalies, and so interesting phenomenology. To make matters more poignant, we happen to be loosely sitting on the fence separating matter and Lambda domination. Our clocks, therefore, have been likely non-aspirational for the past few billion years. The fact that the transition from deceleration to acceleration is a quantum bounce, with its inevitable ringing, only compounds the issue. We should not regard this as an ontological tragedy but as an epistemological opportunity. Is the Universe going quantum?

To this question we should attach another, more fundamental one: How would we see a Universe going quantum? We are used to quantum systems as microscopical sub-systems living inside larger classical macroscopical systems; but this is just the opposite. What if our local classical world were encased in Universe which on the very largest scales is behaving quantum mechanically? How would we see it?

Even ignoring this million dollar question, other interesting problems remain. Clock swapping is essential for keeping the classical description, but why would the Universe choose to swap clocks? A selection principle seems to be at play, and this may shed light on other fine tuning problems, such as the cosmological constant problem (e.g. [29-33]). Also, what implications are there for the constants of nature if they are not allowed to be infinitely sharp in a classical world? In
this framework, and infinitely sharp constant implies total de-localization in the conjugate time. In this sense, a perfect constant is a failed classical clock\[5\]. Would this have implications, either for our local physics, or for our description of the large-scale Universe (the two being essentially complementary)? Finally, one may ask what happens if more than one dominant clock is at play? Preliminary work suggests that they would get in each other’s way regarding classicality \[13\], but how problematic is this? Is this, instead, yet another observational window of opportunity?

Acknowledgments. I would like to thank Bruno Alexandre, Steffen Gielen, Chris Isham and Tony Padilla for discussions and advice in relation to this paper. This work was supported by the STFC Consolidated Grant ST/L00044X/1.

[1] C. J. Isham, “Canonical quantum gravity and the problem of time,” NATO Sci. Ser. C 409, 157-287 (1993).
[2] K. Kuchar, “Time and interpretations of quantum gravity,” in Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics’, World Scientific, Singapore, 1992.
[3] M. Cortês and L. Smolin, Phys. Rev. D 90, no.8, 084007 (2014); Phys. Rev. D 90, no.4, 044035 (2014).
[4] C. Rovelli and M. Smerlak, Class. Quant. Grav. 28, 075007 (2011).
[5] J. D. Barrow and F. J. Tipler, “The Anthropic Cosmological Principle,” Oxford University Press, 1988.
[6] J. Magueijo, Phys. Lett. B 820, 136487 (2021) doi:10.1016/j.physletb.2021.136487 [arXiv:2104.11529 [gr-qc]].
[7] M. J. Duff, L. B. Okun and G. Veneziano, JHEP 03, 023 (2002) doi:10.1088/1126-6708/2002/03/023 [arXiv:hep-th/0110060 [physics]].
[8] W. G. Unruh, Phys. Rev. D40, 1048 (1989).
[9] M. Henneaux and M. Teitelboim, Physics Letters B 222, 195 (1989).
[10] C. W. Misner, Phys. Rev. 186, 1328 (1969); Phys. Rev.1 86, 1319 (1969).
[11] L. Smolin and C. Soo, Nucl. Phys. B 449, 289-316 (1995).
[12] J. W. York, Phys. Rev. Lett. 26, 1656 (1971); 28, 1082 (1972).
[13] B. Alexandre and J. Magueijo, “When one time is better than two”, to be submitted.
[14] J. Magueijo and C. Isham, “Quantizing the Planck constant: a self-referential construction”, in preparation.
[15] S. S. Chern and J. Simons, Ann. Math. 99, 48 (1974); G. V. Dunne, “Aspects of Chern-Simons theory,” arXiv:hep-th/9902115 [hep-th].
[16] R. Jackiw, Conf. Proc. C 830627, 221-331 (1983) MIT-CTP-1108.
[17] H. Kodama, Phys. Rev. D 42, 2548-2565 (1990); J. Magueijo, arXiv:2012.05847 [gr-qc]. ADD PUBLISHED REF

5 One may also appeal to Borges’ quote, ”[Eternity can be defined as] the simultaneous and lucid possession of all the instants of time”. With this definition, an exact constant of Nature is equivalent to eternity in one of the many possible times (viz. the one dual to the infinitely sharp constant).