Weighted sparsity regularization for source identification for elliptic PDEs

Ole Løseth Elvetun* and Bjørn Fredrik Nielsen†

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Abstract

This investigation is motivated by PDE-constrained optimization problems arising in connection with electrocardiograms (ECGs) and electroencephalography (EEG). Standard sparsity regularization does not necessarily produce adequate results for these applications because only boundary data/observations are available for the identification of the unknown source, which may be interior. We therefore study a weighted \(\ell^1\)-regularization technique for solving inverse problems when the forward operator has a significant null space. In particular, we prove that a sparse source, regardless of whether it is interior or located at the boundary, can be exactly recovered with this weighting procedure as the regularization parameter \(\alpha\) tends to zero. Our analysis is supported by numerical experiments for cases with one and several local sources. The theory is developed in terms of Euclidean spaces, and our results can therefore be applied to many problems.

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1 Introduction

Consider the challenge of identifying a sparse source in an elliptic PDE from Dirichlet boundary data:

\[
\min_{(f,u) \in F_h \times H^1(\Omega)} \left\{ \frac{1}{2} \| u - d \|^2_{L^2(\partial \Omega)} + \alpha \sum_i w_i |(f, \phi_i)_{L^2(\Omega)}| \right\}
\]

\[ (1) \]

*Faculty of Science and Technology, Norwegian University of Life Sciences, P.O. Box 5003, NO-1432 Ås, Norway. Email: ole.elvetun@nmbu.no.

†Faculty of Science and Technology, Norwegian University of Life Sciences, P.O. Box 5003, NO-1432 Ås, Norway. Email: bjorn.f.nielsen@nmbu.no. Nielsen’s work was supported by The Research Council of Norway, project number 239070.
subject to
\[-\Delta u + \epsilon u = f \quad \text{in } \Omega,\]
\[\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega,\]
where we employ weighted \(\ell^1\)-regularization in (1). Here, \(\{\phi_1, \phi_2, \ldots, \phi_n\}\) is an \(L^2\)-orthonormal basis for \(F_h\), \(\{w_i\}\) are positive weights, \(\alpha > 0\) is a regularization parameter, \(d\) represents the Dirichlet boundary data, \(\epsilon\) is a parameter, \(n\) denotes the outwards pointing unit normal vector of the boundary \(\partial \Omega\) of the bounded domain \(\Omega\), and \(f\) is the unknown source.

Note that we, for the sake of simplicity, consider a finite dimensional control/source space \(F_h\). Since the goal is to recover spatially sparse solutions, the basis functions \(\{\phi_1, \phi_2, \ldots, \phi_n\}\) should have small and local support. In the infinite dimensional setting one would thus typically search for point-sources or Dirac measures. As shown in [5], this leads to a number of subtle mathematical issues, even in the unweighted case.

Variants of (1)-(2) appear in many applications, such as in crack determination, in the inverse ECG problem and in EEG investigations. Consequently, it has received much attention from researchers, see, e.g., [1, 7, 20, 22, 24, 28, 34, 35]. However, in these studies the authors did not use \(\ell^1\)-regularization. A more detailed description of previous investigations is presented in [10].

Regularization of inverse problems with sparsity promoting methods has increased in popularity in recent years [8, 18, 25, 26, 30, 31, 14]. In this context, the notion of sparsity, with respect to a given basis \(\{\phi_i\}\) for the control space, means that the inverse solution \(f^*\) can be represented using very few of the basis functions. That is, if we expand the inverse solution \(f^*\) as \(f^*(x) = \sum_i f_i^* \phi_i(x)\), then \(f_i^* = (f^*, \phi_i)_{L^2(\Omega)} \neq 0\) only for very few of the basis functions. If we have \(s\) such non-zero components, we say that the solution \(f^*\) is \(s\)-sparse.

In compressed sensing the study of \(\ell^1\)-regularization for the exact recovery of a sparse source from noise free data has been studied in detail [2, 3, 9]. To elaborate the main findings in [3], let us assume that \(A\) is a matrix with a significant null space. If \(b^\dagger\) is generated from a sparse source \(x^\dagger\), i.e., \(b^\dagger = Ax^\dagger\), then the minimizer of

\[\min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{subject to } Ax = b^\dagger,\]

is the true sparse solution \(x^\dagger\) when a certain assumption on \(A\), known as the restricted isometry property, is fulfilled.

In [19] the authors unified this result with the theory developed by the inverse problem community. In particular, they showed that the commonly used range condition, combined with an additional restricted injectivity condition on the forward operator, are weaker assumptions than the previously mentioned restricted isometry property, and they proved that the former conditions are the weakest which admit linear convergence rates for the regularized problem

\[\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|Ax - b\|^2_2 + \alpha \|x\|_1 \right\}.\]
Numerical experiments indicate that the identification criteria, mentioned in the previous paragraph, are not fulfilled by the forward/transfer matrix $A$ associated with (1)-(2), see Figure 1. (Further details about the matrix $A$ are presented in the next section and in Appendix B.) This is the main motive for the present study. Observe also that the inverse solution in panel (b) in Figure 1 only is nonzero close to where observations are made, i.e., close to the boundary $\partial \Omega$ of $\Omega$. This type of ”behaviour” was not only observed with $\alpha = 10^{-4}$, but in every experiment we performed with standard sparsity regularization. In view of the analysis presented in Appendix A, see also Proposition 2.3 in [5], this is not surprising.

Figure 1: Panel (b) shows the outcome of attempting to use (1)-(2), with $w_i = 1$ for $i = 1, 2, \ldots, n$ and $\alpha = 10^{-4}$, to recover the true interior source depicted in panel (a). That is, the Dirichlet boundary data $d$ in (1) was generated by the true source (a).

Many researchers have explored the use of sparsity regularization in connection with PDEs, see, e.g., [5, 6, 17, 21, 37, 29, 15, 36]. However, as far as the authors know, the use of weighted $\ell_1$-regularization to solve inverse source problems for PDEs, employing only boundary measurements, has not previously been attempted. In [27] the authors apply an iterative reweighted $\ell_1$-regularization technique [4] to recover sources, but their approach differs significantly from ours and involve data recorded at equidistant locations inside the domain.

We will start with a brief motivation for developing weighted sparsity promoting regularization for ECG and EEG applications in Section 3. In Section 4 we prove that our methodology can recover, without any errors or blurring, a single local source, regardless of whether it is interior or at the boundary. We also analyze a problem for which it is impossible to recover two separate sources. This implies that we can not guarantee, in general, that our approach can recover multiple local sources. However, in Section 5 many numerical experiments are presented, for both single and multiple sources, and we observe that several sources often can be successfully recovered.

We would like to emphasize that, even though our motivation originates from inverse source problems for elliptic PDEs, the general theory presented in
this paper is applicable to any linear finite-dimensional inverse problem where
the forward operator has a non-trivial null space.

One may regard this paper to be follow-up work to [10]: In [10] weighted
Tikhonov regularization is proposed and analyzed, and in this text we em-
ploy the same weight-matrix in connection with sparsity regularization. How-
ever, since both the results and the analysis of the sparsity approach differs
significantly from the investigation of the quadratic regularization, a separate
study is needed. More precisely, weighted Tikhonov regularization yields a
"blurred/smooth" reconstruction of internal sources and the cost-functional is
differentiable, whereas weighted sparsity regularization enables perfect recovery
of a single source but the objective function is not differentiable.

2 Preliminaries

To make the forthcoming results applicable to more general finite-dimensional
problems, we will present our analysis in terms of Euclidean spaces. First,
however, we will study (1)-(2) in greater detail and derive the associated fully
discretized problem.

We interpret (2) in the following weak sense: Let \( \Omega \subset \mathbb{R}^\nu, \nu = 1, 2, 3, \) be
a Lipschitz domain and assume that \( f \in F_h \subset L^2(\Omega) \) is given. A function
\( u \in H^1(\Omega) \) is a solution of (2) if
\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx + \epsilon \int_{\Omega} uv \, dx = \int_{\Omega} fv \, dx, \quad \forall v \in H^1(\Omega).
\]
(3)

From standard elliptic PDE-theory [13] it follows that there exists a unique
solution to (3) which depends continuously on \( f \in F_h \subset L^2(\Omega) \). Since the trace
operator \( T : H^1(\Omega) \to L^2(\partial \Omega), u \mapsto u|_{\partial \Omega} \), is continuous, we can conclude that
the forward operator
\[
K_h : F_h \to L^2(\partial \Omega), \quad f \mapsto u(f)|_{\partial \Omega},
\]
(4)

associated with (1)-(2), is well-defined and continuous.

We can now formulate the problem
\[
\min_{f \in F_h} \left\{ \frac{1}{2} \|K_h f - d\|^2_{L^2(\partial \Omega)} + \alpha \sum_i w_i |(f, \phi_i)_{L^2(\Omega)}| \right\},
\]
(5)

which is equivalent to (1)-(2). Continuity of the cost functional \( G \) follows im-
mEDIATELY from the continuity of the norm and, since we assume that \( w_i > 0, \quad i = 1, 2, \ldots, n, \) \( G \) is also coercive, provided that \( \alpha > 0 \). Standard optimization
theory thus yields that (5) has a global minimizer, see, e.g., [33].

Since \( K_h \) is a linear mapping from a finite-dimensional space onto a finite
dimensional subspace of \( L^2(\partial \Omega) \), it can be represented by its standard matrix
K. By expanding $f$ in the orthonormal basis $\{\phi_1, \phi_2, \ldots, \phi_n\}$ for $F_h$, we obtain the Euclidean approximation of (5):

$$
\min_{f \in \mathbb{R}^n} \left\{ \frac{1}{2} \| M_\partial^\frac{1}{2} K f - M_\partial^\frac{1}{2} d \|^2_2 + \alpha \sum_i w_i | f_i | \right\},
$$

(6)

where $f$ and $d$ are the Euclidean vectors associated with $f$ and $d$, respectively, $M_\partial$ represents the (so-called) boundary mass matrix, and $K$ is a numerical approximation of $K$ obtained from a discretization of (3), see Appendix B.

3 Motivation

The purpose of EEG is to recover electrical activity in the brain from voltage recordings on the scalp. If one suspects that the true signal is spatially local, e.g., for focal epileptic seizures, it is natural to search for sparse solutions.

Similarly, in the inverse problem of ECG, the aim can be to locate an ischemic region of the heart. This area will have an electrical potential which is different from the voltage in healthy tissue. The difference in the potential can be interpreted as the source in an elliptic PDE. If we assume that the ischemic region is small, with a sharp transition between ischemic and healthy tissue, it is reasonable to search for a sparse inverse solution.

Solving inverse source problems using optimization procedures is challenging. For example, now employing standard Tikhonov regularization instead of the weighted sparsity approach, the minimizer $f_\alpha \in F_h \subset L^2(\Omega)$ of

$$
\min_{(f,u) \in F_h \times H^1(\Omega)} \left\{ \frac{1}{2} \| u - d \|^2_{L^2(\partial\Omega)} + \frac{1}{2} \alpha \| f \|^2_{L^2(\Omega)} \right\}
$$

subject to

$$
-\Delta u + \epsilon u = f \quad \text{in} \quad \Omega,
$$

$$
\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega,
$$

is not, in general, a good approximation of a true interior source: Even if the data $d$ is generated from a single basis function $\phi_j \in F_h$, representing an interior local source, the minimum $L^2$-norm least-squares solution $f^* = \lim_{\alpha \to 0} f_\alpha$ will attain its maximum at the boundary $\partial\Omega$ of the domain $\Omega$, see [10]. However, by employing a carefully chosen “diagonal” regularization operator $W$, i.e.,

$$
W \phi_i = w_i \phi_i, \quad i = 1, 2, \ldots, n,
$$

with suitable weights $\{w_i\}$, we proved in [10] that $W^{-1} f^*$ attains its maximum at the position of the true local source $\phi_j$. On the other hand, $W^{-1} f^*$ is very

\footnote{Ischemia is a precursor of a heart infarct.}
smooth and its magnitude will typically not be close to the magnitude of the true source.

The need for sparse solutions in many applications, as exemplified above, thus motivates the study of $\ell^1$-regularization for inverse source problems. We now present a brief overview of the main results of our forthcoming analysis.

Assume that the Dirichlet boundary data $d \in L^2(\partial \Omega)$ in (1) is generated from a basis function $\phi_j \in F_h$, i.e.,
\[ d = K_h \phi_j, \]
where $K_h$ is the forward operator (4) associated with (1)-(2). Provided that $W$ denotes the above-mentioned regularization operator introduced in [10], we will prove that $\phi_j$ is the unique solution of
\[
\min_{f \in F_h} \sum_i w_i |(f, \phi_i)_{L^2(\Omega)}| \quad \text{subject to} \quad K_h f = K_h \phi_j.
\]
We will further show, using a slightly modified fidelity term in (1)-(2), that the associated minimizer of the weighted $\ell^1$-regularized problem is
\[ f_\alpha = \gamma_\alpha \phi_j, \]
where $\gamma_\alpha = 1 - c \alpha$, and $c$ is a positive constant. That is, the correct basis function is recovered without any blurring for all values of $\alpha < \bar{\alpha}$, albeit with an error in magnitude equal to $c \alpha$. The constants $\bar{\alpha}$ and $c$ can be computed from $W$ and a projection operator.

Ideally, we would like to analyze the recovery of more general composite sources. We have not been able to do so, but we address this issue numerically in Section 5.

4 Analysis

As mentioned earlier, we will present our theoretical results in terms of Euclidean spaces, employing the standard inner product and the standard basis vectors. It should be noted that analogous results can be established for linear operators acting on finite dimensional vector spaces, provided that an orthonormal basis is employed for the domain of the operators.

Consider the problem
\[
\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|Ax - b\|^2_2 + \alpha \|Wx\|_1 \right\},
\]
where $A \in \mathbb{R}^{m \times n}$ has a non-trivial null space $\mathcal{N}(A)$. Associated with $A$ is the orthogonal projection matrix
\[ P : \mathbb{R}^n \to \mathcal{N}(A)^\perp. \]

Our analysis also holds for matrices with linearly independent columns, but this is the trivial case.
That is, $P = A^t A$, where $A^t$ denotes the Moore-Penrose inverse of $A$. Note that we can write (6) in the form (7) by putting $A = M_*^T \tilde{K}$, $b = M_*^T d$, $W = \text{diag}(w_1, w_2, \ldots, w_n)$ and $x = f$.

Throughout this paper, the diagonal regularization matrix $W \in \mathbb{R}^{n \times n}$ is defined as

$$W e_i = \|P e_i\|_2 e_i, \quad i = 1, 2, \ldots, n,$$

where we assume that $Pe_i \neq 0$, $i = 1, 2, \ldots, n$. The definition of this matrix can be motivated by the classical theory for the minimum norm least squares solution of linear systems and Tikhonov regularization, see [10] for further details. Furthermore, the beneficial mathematical properties of this operator in connection with quadratic regularization are studied in [10, 11]. Below it will become clear that (9) also plays an important role for developing sparsity promoting regularization techniques.

It can be CPU demanding to compute $Pe_i$ for large systems because $P$ involves the Moore-Penrose inverse $A^t$ of $A$. On the other hand, if the underlying problem is ill posed, such as (1)-(2), then one would typically not use a very fine mesh for the discretization of the source, and the regularization matrix $W$ is applicable. Below we will approximate $A^t$ using either truncated SVD (Subsection 4.2) or standard Tikhonov regularization (Subsection 5.3).

The main purpose of this section is to analyze whether the use of our weighted regularization technique enables the recovery of a standard basis vector $e_j \in \mathbb{R}^n$ from the exact data $b^t = Ae_j$. Our starting point is thus the following optimization problem.

- **Problem 0:**

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|Ax - b^t\|_2^2 + \alpha \|Wx\|_1 \right\}.$$  

4.1 Weighted basis pursuit

Our first result concerns the solution of Problem 0 in the limit $\alpha \to 0$. Recalling that $b^t = Ae_j$, it is well-known that this limit problem can be formulated as

- **Problem I:**

$$\min_{x \in \mathbb{R}^n} \|Wx\|_1 \quad \text{subject to} \quad Ax = Ae_j.$$  

We also introduce an equivalent formulation of Problem I, which reads

- **Problem II:**

$$\min_{x \in \mathbb{R}^n} \|Wx\|_1 \quad \text{subject to} \quad Px = Pe_j.$$  

Problems I and II are equivalent because the null spaces of $A$ and $P$ coincide, i.e., $\mathcal{N}(A) = \mathcal{N}(P)$. Hence, either problem can be reformulated as

$$\min_{q \in \mathcal{N}(A)} \|W(e_j + q)\|_1.$$  

We will assume that no two columns of $A$ are parallel in order to ensure that the solutions of our minimization problems are unique.
Assumption 4.1. We assume that $A \in \mathbb{R}^{m \times n}$ is such that $A e_j \neq c A e_i$ for all $i, j \in \{1, 2, ..., n\}$, $i \neq j$, and all $c \in \mathbb{R}$.

Note that this assumption implies that $e_i \notin \mathcal{N}(A)$, $i \in \{1, 2, ..., n\}$. Furthermore, invoking the orthogonal decomposition $e_i = Pe_i + (e_i - Pe_i)$, where $(e_i - Pe_i) \in \mathcal{N}(A)$, it follows that $A e_i = A P e_i$. Consequently, if $A$ obeys Assumption 4.1 then $P$ must also satisfy

$$P e_j \neq c P e_i \text{ for all } i, j \in \{1, 2, ..., n\}, i \neq j, \text{ and all } c \in \mathbb{R}.$$  \hfill (11)

We can now formulate our first result. Let us remark that the study of sparsity promoting regularization techniques usually leads to very involved mathematical analysis. However, with the very specific choice of the weight matrix $W$ in $\text{(9)}$ and the data $b^\dagger = A e_j$, our analysis becomes rather "transparent" because we only consider the recovery of a 1-sparse solution.

Theorem 4.2 (Exact recovery of a basis vector). Assume that $A \in \mathbb{R}^{m \times n}$ satisfies Assumption 4.1 and let $P$ and $W$ be the matrices defined in $\text{(8)}$ and $\text{(9)}$, respectively. Then $e_j$ is the unique solution of problems I and II.

Proof. Problem I and Problem II are equivalent. We choose to consider the latter, i.e.,

$$\min_{x \in \mathbb{R}^n} \|Wx\|_1 \text{ subject to } P x = P e_j.$$  

Define $X_j = \{x \in \mathbb{R}^n : P x = P e_j\}$. Let $x = \sum_i c_i e_i \in X_j, x \neq e_j$ be arbitrary. Then, see $\text{(9)}$,

$$\|W e_j\|_1 = \|P e_j\|_2 e_j \|_1$$
$$= \|P e_j\|_2$$
$$= \|P \left( \sum_i c_i e_i \right)\|_2$$
$$= \|\sum_i c_i P e_i\|_2$$
$$\leq \sum_i |c_i| \|P e_i\|_2$$
$$= \|\sum_i c_i \|P e_i\|_2 e_i\|_1$$
$$= \|W x\|_1.$$  

Using $\text{(11)}$, which is a consequence of Assumption 4.1, we get strict inequality in the third to last step, and the result follows. $\square$

Theorem 4.2 states that a single basis vector $e_j$ can be exactly recovered from the data $b^\dagger = A e_j$ by solving either Problem I or Problem II. If Assumption 4.1 does not hold, $e_j$ would still be a minimizer, but the uniqueness is not assured. (Similar statements hold for the remaining results presented in this paper.)
4.2 Alternative optimization problems

Inspired by Problem II, we now suggest an alternative to (7). Since $P = A^\dagger A$, it follows that

$$Pe_j = A^\dagger Ae_j = A^\dagger b^\dagger,$$

where $b^\dagger = Ae_j$. Consequently, for a general right-hand-side $b$, Problem II motivates the following alternative to (7)

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|Px - A^\dagger b\|_2^2 + \alpha \|Wx\|_1 \right\}. \quad (12)$$

If $A$ has very small singular values, it may not be advisable to apply $A^\dagger$ in practice. Therefore we want to approximate $A^\dagger$ with a more well-behaved matrix. This can, e.g., be accomplished by employing the truncated SVD to get an approximation $A_k$ of $A$. Here, $k \leq m$ represents the number of singular values that are unchanged in the truncation: Assuming that the singular values of $A$ are sorted in decreasing order $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m$, $A_k$ will have the singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \geq 0 = 0 = \ldots = 0$.

Below we need the orthogonal projection onto the orthogonal complement of the null space of $A_k$,

$$P_k : \mathbb{R}^n \to \mathcal{N}(A_k)^\perp. \quad (13)$$

And, analogously to (9), we define

$$W_k e_i = \|P_k e_i\|_2 e_i, \quad i = 1, 2, \ldots, n. \quad (14)$$

Replacing $A^\dagger$ in (12) with $A_k^\dagger$, keeping in mind that $P = A^\dagger A$, leads to the following alternative to (7)

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|A_k^\dagger Ax - A_k^\dagger b\|_2^2 + \alpha \|W_k x\|_1 \right\}. \quad (15)$$

Note that we have also replaced $W$ with $W_k$.

From the SVD of $A = U \Sigma V^T$, we find the SVD of $A_k$,

$$A_k = U_k \Sigma_k V^T.$$ 

Observe that

$$P_k = A_k^\dagger A_k = V_k \Sigma_k^\dagger \Sigma_k V^T,$$

and that

$$A_k^\dagger A = V \Sigma_k^\dagger \Sigma V^T.$$ 

This yields, since $\Sigma_k^\dagger \Sigma_k = \Sigma_k^\dagger \Sigma$,

$$P_k = A_k^\dagger A, \quad (16)$$

and we can write (15) in the form

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|P_k x - A_k^\dagger b\|_2^2 + \alpha \|W_k x\|_1 \right\}. \quad (17)$$

In the next subsections we will analyze (12) and (17) when $b = Ae_j$ and $b = Ae_j + \eta$, respectively, where $\eta$ represents noise.
4.3 Analysis of regularized problems

In this subsection we will make use of the following maximum property derived in [10] and [11]:

\[ j = \arg \max_{i \in \{1, 2, \ldots, n\}} |[W^{-1}P e_j]_i|, \]  

(18)

where \([W^{-1}P e_j]_i\) denotes the \(i\)’th component of the vector \(W^{-1}P e_j\), and \(P\) and \(W\) are defined in (8) and (9), respectively. More precisely, the proof of Theorem 4.2 in [10] reveals that

\[ W^{-1}P e_j = \|P e_j\| \sum_{i=1}^{n} \left( \frac{P e_j}{\|P e_j\|} \right) \epsilon_i, \]  

(19)

which combined with Assumption 4.1 yields (18), see [11] for further details.

With \(b = A e_j\), (12) reads

• Problem III:

\[ \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|P x - P e_j\|_2^2 + \alpha \|W x\|_1 \right\}. \]  

(20)

We will now see that the maximum property (18) allows an analysis of this problem which only involves classical convex optimization theory.

**Theorem 4.3.** Assume that the matrix \(A \in \mathbb{R}^{m \times n}\) satisfies Assumption 4.1, and let \(P\) and \(W\) be the matrices defined in (8) and (9), respectively. Then

\[ x^*_\alpha = \gamma_{j, \alpha} e_j \]

is the unique solution of Problem III, where

\[ \gamma_{j, \alpha} = 1 - \frac{\alpha}{[W^{-1}P e_j]_j}, \]  

for \(0 < \alpha < [W^{-1}P e_j]_j\).  

(21)

**Proof.**

Existence: Let us define the cost-functional \(J : \mathbb{R}^n \to \mathbb{R}\) associated with (20),

\[ J(x) = \frac{1}{2} \|P x - P e_j\|_2^2 + \alpha \|W x\|_1. \]  

(22)

where \(g(\cdot)\) and \(h(\cdot)\) represent the fidelity and regularization terms, respectively. According to standard convex optimization theory, \(x\) is a minimizer of \(J\) if and only if

\[ 0 \in \partial J(x) = \nabla g(x) + \alpha W^T \partial h(Wx), \]

where "\(\partial\)" denotes the subgradient. Since \(W^T = W\), we can multiply with \(W^{-1}\) to obtain

\[ -W^{-1} \nabla g(x) \in \alpha \partial h(Wx), \]
and from the expression for \( g \) we find, keeping in mind that \( P^T P = PP = P \),
\[
W^{-1} P(e_j - x) \in \alpha \partial h(Wx).
\]  
We also observe, using the fact that \( h(y) = \|y\|_1 \) and that \( W \) is a diagonal matrix with positive entries at its diagonal,
\[
[\partial h(Wx)]_i = [\partial h(W[x_1 x_2 \ldots x_n]^T)]_i = \begin{cases} 
1, & x_i > 0, \\
-1, & x_i < 0, \\
[-1,1], & x_i = 0.
\end{cases}
\]  
We will now investigate whether there exists a scalar \( \gamma \) such that \( x = \gamma e_j \) satisfies the optimality criterion (23). Note that, for \( \gamma > 0 \),
\[
[\partial h(W \gamma e_j)]_i = \begin{cases} 
1, & i = j, \\
-1, & i \neq j.
\end{cases}
\]  
and the condition (23), with \( x = \gamma e_j \), becomes
\[
(1 - \gamma)[W^{-1} P e_j]_i \in \alpha \begin{cases} 
1, & i = j, \\
[-1,1], & i \neq j.
\end{cases}
\]  
Setting \( \gamma = \gamma_{j,\alpha} = 1 - \frac{\alpha}{[W^{-1} P e_j]_j} \),
we observe from (18) that
\[
(1 - \gamma_{j,\alpha})[W^{-1} P e_j]_i = \alpha \frac{[W^{-1} P e_j]_i}{[W^{-1} P e_j]_j} \in \alpha \begin{cases} 
1, & i = j, \\
(-1,1), & i \neq j.
\end{cases}
\]  
and we conclude that (25) holds for the particular choice (26) of \( \gamma \).

This argument shows that \( x^*_\alpha = \gamma_{j,\alpha} e_j \) is a minimizer of \( J \). The next step is to use the property that \( (1 - \gamma_{j,\alpha})[W^{-1} P e_j]_i \) is contained in the open interval \( (-\alpha, \alpha) \), \( i \neq j \), to prove the uniqueness.

**Uniqueness:** We have determined a minimizer \( x^*_\alpha = \gamma_{j,\alpha} e_j \) of \( J \). Let \( y \in \mathbb{R}^n, y \neq x^*_\alpha \), be arbitrary. We will show that
\[
J(y) > J(x^*_\alpha).
\]  
If \( y = c x^*_\alpha, c \neq 1 \), it follows from the analysis presented above that this is not a minimizer of \( J \). Consequently, for the remaining part of the proof we assume that \( y \neq c x^*_\alpha \). In particular, this implies that at least one of the components, say \( y_k, k \neq j \), of \( y \) is such that \( y_k \neq 0 \).

Recall the definition (22) of \( J \), \( g \) and \( h \) and that, using the definition of the subgradient,
\[
h(Wy) - h(Wx^*_\alpha) \geq z^T(Wy - Wx^*_\alpha) \quad \text{for all } z \in \partial h(Wx^*_\alpha).
\]
Therefore
\[
\mathcal{J}(y) - \mathcal{J}(x^*_\alpha) = g(y) + \alpha h(Wy) - g(x^*_\alpha) - \alpha h(Wx^*_\alpha) \\
\geq \frac{1}{2} \|Py - Pe_j\|_2^2 - \frac{1}{2} \|Px^*_\alpha - Pe_j\|_2^2 \\
+ \alpha z^T W(y - x^*_\alpha) \quad \text{for all } z \in \partial h(Wx^*_\alpha).
\] (28)

Since \(x^*_\alpha = \nu_{j,\alpha} e_j\), we can write (27) as
\[
[W^{-1} P(e_j - x^*_\alpha)]_i \in \alpha \begin{cases} 
{1}, & i = j \\
(-1, 1), & i \neq j
\end{cases}
\] (29)

see (24), i.e.,
\[
\alpha \left( \frac{1}{\alpha} W^{-1} P(e_j - x^*_\alpha) \right) \in \partial h(Wx^*_\alpha).
\] (30)

Hence, we could choose \(z = \frac{1}{\nu} W^{-1} P(e_j - x^*_\alpha)\) in (28), but then we do not (directly) get a strict inequality. Recall that \(y\) has a component \(y_k \neq 0\), where \(k \neq j\). Without loss of generality, we may assume that \([Wy - Wx^*_\alpha]_k > 0\).

Define \(\tilde{z} = [\tilde{z}_1 \tilde{z}_2 \ldots \tilde{z}_n]^T\) as follows\(^3\)
\[
\tilde{z}_i = \begin{cases} 
1, & i = k, \\
\frac{1}{\alpha} W^{-1} P(e_j - x^*_\alpha)_i, & i \neq k.
\end{cases}
\]

Since (29) implies that \(\left| \frac{1}{\alpha} W^{-1} P(e_j - x^*_\alpha) \right|_k < 1\), we find that
\[
\tilde{z}^T [Wy - Wx^*_\alpha] > \frac{1}{\nu} \left[ W^{-1} P(e_j - x^*_\alpha) \right]^T [Wy - Wx^*_\alpha].
\]

Due to (29) and (24), \(\tilde{z} \in \partial h(Wx^*_\alpha)\), and therefore (28) yields
\[
\mathcal{J}(y) - \mathcal{J}(x^*_\alpha) \geq \frac{1}{2} \|Py - Pe_j\|_2^2 - \frac{1}{2} \|Px^*_\alpha - Pe_j\|_2^2 \\
+ \alpha \tilde{z}^T W(y - x^*_\alpha) \\
> \frac{1}{2} \|Py - Pe_j\|_2^2 - \frac{1}{2} \|Px^*_\alpha - Pe_j\|_2^2 \\
+ \alpha \left[ W^{-1} P(e_j - x^*_\alpha) \right]^T [Wy - Wx^*_\alpha] \\
= \frac{1}{2} \|Py - Pe_j\|_2^2 - \frac{1}{2} \|Px^*_\alpha - Pe_j\|_2^2 \\
+ [P(e_j - x^*_\alpha)]^T [y - x^*_\alpha].
\]

\(^3\)If \([Wy - Wx^*_\alpha]_k < 0\), define \(\tilde{z}_k = -1\), etc.
The gradient of $g$, see (22), is $\nabla g(x) = P(x - e_j)$. Consequently, the convexity of $g$ implies that

$$J(y) - J(x^*_\alpha) > \frac{1}{2} \| Py - Pe_j \|^2 - \frac{1}{2} \| Px^*_\alpha - Pe_j \|^2$$

$$- \nabla g(x^*_\alpha)^T [y - x^*_\alpha]$$

$$\geq \frac{1}{2} \| Py - Pe_j \|^2 - \frac{1}{2} \| Px^*_\alpha - Pe_j \|^2$$

$$- [g(y) - g(x^*_\alpha)]$$

$$= \frac{1}{2} \| Py - Pe_j \|^2 - \frac{1}{2} \| Px^*_\alpha - Pe_j \|^2$$

$$- \left[ \frac{1}{2} \| Py - Pe_j \|^2 - \frac{1}{2} \| Px^*_\alpha - Pe_j \|^2 \right]$$

$$= 0,$$

which finishes the proof.

This theorem shows that $x^*_\alpha = \gamma_{j,\alpha} e_j$ is the unique minimizer of (20). The solution of (20) is thus obtained by only changing the magnitude of the true source $e_j$, where the scaling factor $\gamma_{j,\alpha} \to 1$ as $\alpha \to 0$.

4.3.1 Noisy observation data

As mentioned in Subsection 4.2, it may not be advisable to apply the pseudo-inverse $A^\dagger$ in practical computations. We therefore now want to study (15) in more detail. Setting $b = Ae_j + \eta$ in (17), where $\eta \in \mathbb{R}^m$ represents noise, leads to

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \| P_k x - (A_k^\dagger \eta) \|_2^2 + \alpha \| W_k x \|_1 \right\}.$$  

Since $A_k^\dagger A = P_k$, see (16), this problem can also be written in the form

- **Problem IV:**

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \| P_k x - (P_k e_j + A_k^\dagger \eta) \|_2^2 + \alpha \| W_k x \|_1 \right\}. \tag{31}$$

**Theorem 4.4.** Assume that $A_k \in \mathbb{R}^{m \times n}$ satisfies Assumption 4.1 and let $P_k$ and $W_k$ be the matrices defined in (13) and (14), respectively. Then

$$x^*_{\alpha,\eta} = \gamma_{j,\alpha,\eta} e_j$$

is the unique solution of Problem IV, where

$$\gamma_{j,\alpha,\eta} = 1 - \frac{\alpha + |W_k^{-1}A_k^\dagger \eta|_j}{|W_k^{-1}P_k e_j|_j}. \tag{32}$$
In order for this to hold, \( \alpha \) must obey

\[
\max_{i \neq j} \frac{1 + |\tau_{ij}|}{1 - |\tau_{ij}|} \max_i \left| \frac{[W_k^{-1}A_k^i \eta]_i}{[W_k^{-1}P_ke_j]_j} \right| < \alpha < \frac{[W_k^{-1}P_k^{-1}e_j]_j - [W_k^{-1}A_k^i \eta]_i}{[W_k^{-1}P_k^{-1}e_j]_j},
\]

(33)

where

\[
\tau_{ij} = \frac{[W_k^{-1}P_k^{-1}e_j]_i}{[W_k^{-1}P_k^{-1}e_j]_j} \in (-1, 1), \quad i \neq j.
\]

**Proof.** Following the same reasoning as in the proof of Theorem 4.3, we derive that \( x^*_a = \gamma e_j \) is the unique minimizer of (31) if

\[
(1 - \gamma)[W_k^{-1}P_k e_j]_i - [W_k^{-1}A_k^i \eta]_i \in \alpha \begin{cases} \{1\}, & i = j \\ (-1, 1), & i \neq j, \end{cases}
\]

(34)

where we have used the fact that \( P_k^T = P_k = A_k^i A_k^j \), and consequently

\[
P_k^T A_k^\dagger = A_k^i A_k^j = A_k^j.
\]

The criterion (34) holds for \( i = j \) if

\[
\gamma = \gamma_{j,a,\eta} = 1 - \frac{\alpha + [W_k^{-1}A_k^i \eta]_i}{[W_k^{-1}P_k e_j]_j} > 0.
\]

Consequently, \( \alpha \) must satisfy the upper bound

\[
\alpha < \frac{[W_k^{-1}P_k e_j]_j - [W_k^{-1}A_k^i \eta]_i}{[W_k^{-1}P_k e_j]_j}.
\]

Furthermore, setting \( \gamma = \gamma_{j,a,\eta} \) in (34), the condition (34) for \( i \neq j \) reads

\[
\left( \alpha + [W_k^{-1}A_k^i \eta]_j \right) \frac{[W_k^{-1}P_k^{-1}e_j]_i}{[W_k^{-1}P_k^{-1}e_j]_j} \in \left( -\alpha + [W_k^{-1}A_k^i \eta]_i, \alpha + [W_k^{-1}A_k^i \eta]_i \right) .
\]

(35)

Recall that \( \tau_{ij} := \frac{[W_k^{-1}P_k e_j]_i}{[W_k^{-1}P_k^{-1}e_j]_j} \in (-1, 1) \) for \( i \neq j \), see\(^4\) (18), and (35) can thus be expressed as the two inequalities

\[
\begin{align*}
(1 + \tau_{ij}) & \alpha > [W_k^{-1}A_k^i \eta]_i - \tau_{ij} [W_k^{-1}A_k^i \eta]_j, \\
(1 - \tau_{ij}) & \alpha > \tau_{ij} [W_k^{-1}A_k^i \eta]_i - [W_k^{-1}A_k^i \eta]_j.
\end{align*}
\]

(36)

(37)

It turns out that both of these inequalities are satisfied, for \( i \neq j \), if

\[
\max_{i \neq j} \frac{1 + |\tau_{ij}|}{1 - |\tau_{ij}|} \max_i \left| \frac{[W_k^{-1}A_k^i \eta]_i}{[W_k^{-1}P_k e_j]_j} \right| < \alpha.
\]

(38)

\(^4\)The property (18) holds for any matrix \( A \) satisfying Assumption 4.1, where \( P \) and \( W \) are defined in (8) and (9), respectively. Consequently, (18) also holds for \( A_k \), replacing \( W \) with \( W_k \) and \( P \) with \( P_k \), provided that \( A_k \) also satisfies Assumption 4.1.
Let us end the proof by verifying that (36) holds if (38) is satisfied. Since
\((1 + \tau_{ij}) > 0\), the requirement (36) can be written in the form
\[ \alpha > \frac{|W_k^{-1}A_k^i\eta_i| - \tau_{ij}|W_k^{-1}A_k^j\eta_j|}{1 + \tau_{ij}}. \]

We derive the following inequalities, considering the case \(i \neq j\),
\[ \frac{|W_k^{-1}A_k^i\eta_i| - \tau_{ij}|W_k^{-1}A_k^j\eta_j|}{1 + \tau_{ij}} \leq \frac{|W_k^{-1}A_k^i\eta_i| - \tau_{ij}|W_k^{-1}A_k^j\eta_j|}{|1 + \tau_{ij}|}, \]
\[ \leq \frac{|W_k^{-1}A_k^i\eta_i| + |\tau_{ij}| |W_k^{-1}A_k^j\eta_j|}{1 - |\tau_{ij}|}, \]
\[ \leq \max_{i \neq j} \frac{1 + |\tau_{ij}|}{1 - |\tau_{ij}|} \max_i |W_k^{-1}A_k^i\eta_i|, \]
and we conclude that: If \(\alpha\) satisfies (38), then (36) holds. Similarly, one verifies that (38) implies (37). This finishes the proof. \(\square\)

Roughly, the left inequality in (33) ensures that the error amplification caused by the inverse solution procedure does not become too dominate, and the right inequality prevents the regularization term from becoming too “strong” and thereby yielding a poor recovery of the source.

We note that, in the zero-noise-limit \(\|\eta\|_{\infty} \to 0\), the lower and upper bounds in (33) become 0 and \(|W_k^{-1}P_k\eta_j|\), respectively. From (19) we find that \(|W_k^{-1}P_k\eta_j| > 0\) and a similar argument reveals that also \(|W_k^{-1}P_k\eta_j| > 0\), cf. definitions (8), (9), (13) and (14) of \(P, W, P_k\) and \(W_k\), respectively. Hence, provided that the noise level is sufficiently small, one can in principle always choose the size of \(\alpha\) such that (33) holds. On the other hand, as the degree of noise increases, (33) may not hold for any \(\alpha > 0\).

One can also use standard Tikhonov regularization to obtain an approximation of the Moore-Penrose inverse \(A^\dagger\) of \(A\). We will explore this approach numerically in Subsection 5.3. It is, however, an open problem how to modify the proof of Theorem 4.4 to Tikhonov based approximations of \(A^\dagger\).

Remark 4.5 (Several sources). Let us mention that the methods introduced in this paper can not, in general, guarantee the recovery of multiple sources. To show this, assume that the exact data is \(b^\dagger = Ae_m + Ae_n\) and that there exist a constant \(c\) and an index \(j\) such that \(Ae_m + Ae_n = cAe_j\).

Recall that \(P = A^\dagger A\). Hence, multiplying \(Ae_m + Ae_n = cAe_j\) with \(A^\dagger\) yields
\[ Pe_m + Pe_n = cPe_j. \]

The weighted basis pursuit problem, cf. Problem II, then reads
\[ \min_{x \in \mathbb{R}^n} \|Wx\|_1 \quad \text{subject to} \quad Px = Pe_m + Pe_n. \]

(39)
Following the proof of Theorem 4.2 we get, see (9),

\[ \|cW e_j\|_1 = |c| \|P e_j\|_2 = |c| \left\| \frac{P e_m + P e_n}{c} \right\|_2 = \|P e_m + P e_n\|_2 < \|P e_m\|_2 + \|P e_n\|_2 = \|W (e_m + e_n)\|_1, \]

where the strict inequality is a consequence of Assumption 4.1, see also (11). This argument shows that the two true sources \(e_m\) and \(e_n\) will not be recovered by solving (39). See Figure 2 for an illustration.

On the other hand, if there do not exist an index \(j\) and a constant \(c\) such that \(A e_m + A e_n = c A e_j\), it is an open problem whether the weighted basis pursuit formulation can recover a 2-sparse vector. Or, more generally, can we recover a \(s\)-sparse solution if its image under \(A\) is not equal to the image under \(A\) of any \(s'\)-sparse solution with \(s' < s\)?

![Figure 2: Panel (b) shows the solution \(f^*\) of (1)-(2) when \(\Omega = (0, 1)\), \(\epsilon = 1\) and \(\alpha = 0.001\). The observation data \(d\) was in this case generated by the two true local sources depicted in Panel (a).](image)

5 Numerical experiments

In order to illuminate our theoretical work, we committed the so-called "inverse crime" in Example 1 below: The same grid was used to both generate the boundary observation data \(d\) in (1) and for solving the inverse problem. Consequently, the assumptions needed in Theorem 4.3 are (in principle\(^5\)) satisfied, provided that we consider the single-source-case. In all the other experiments, the data \(d\) was generated using a finer grid for the state \(u\) than was used in

\(^5\)Disregarding round-off errors.
the inverse computations. More specifically, $h_{\text{forward}} = 0.5 h_{\text{inverse}}$, and we performed experiments on the unit square with $129 \times 129$ and $65 \times 65$ nodes for the forward and inverse computations of the state $u$, respectively. The source $f \in F_h$ was discretized in terms of a $16 \times 16$ mesh in all the simulations presented in Examples 1-2 and 4. In Example 3, however, the true source was discretized using a finer $129 \times 129$ grid. Note that we employed the basis functions

$$\phi_i = \frac{1}{\|\chi_{\Omega_i}\|_{L^2(\Omega)}} \chi_{\Omega_i}, \quad i = 1, 2, \ldots, n,$$  \hspace{1cm} (40)$$

for $F_h$, where $\Omega_1, \Omega_2, \ldots, \Omega_n$ are uniformly sized disjoint grid cells and $\chi_{\Omega_i}$ denotes the characteristic function of $\Omega_i$.

We employed the FEniCS software, discretizing the state $u$ in terms of first order Lagrange elements, to generate the matrices involved in our experiments. Thereafter, the matrices were exported to MATLAB, where the optimization problems were solved with the split-Bregman algorithm [16]. Some details about the forward/transfer matrix $A$ is presented in Section 2 and Appendix B. We do not present a detailed description of the well-known mappings between the finite element spaces arising from the discretization of (1)-(2) to the Euclidean spaces used in (7), (12) and (17). Note, however, that $e_j \in \mathbb{R}^n$ is associated with the FE basis function $\phi_j \in F_h \subset L^2(\Omega)$.

In all the simulations $\epsilon = 1$, see (2), and no noise was added to the data $d$, except in Example 2.

Figure 3 contains visualizations of the entries of the weight matrix $W_k$ defined in (14). More specifically, each panel shows a plot of $\|P_k e_i\|$, $i = 1, 2, \ldots, n$. We observe that the weights are largest for indexes associated with basis functions positioned close to the boundary of the domain $\Omega$.

![Figure 3: Visualizations of the matrix $W_k$. Panels (a) and (b) show plots of $\|P_k e_i\|$, $i = 1, 2, \ldots, n$, for two different choices of the truncation parameter $k$.](image)

(a) $k = 7$.  \hspace{2cm} (b) $k = 70$.  

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5.1 Example 1: Exact recovery of a single source

Figures 4 and 5 show numerical solutions of (12). In these problems we recover a single source and \( b = A e_j \). The theory developed for Problem III is therefore applicable, see Theorem 4.3. Figure 6 contains a comparison of the size of \( \gamma_{j,\alpha} \), cf. (21), and the maximum value, \( \max_i |x^\alpha_i| \), of the solution \( x^\alpha \) of Problem III. We observe that the outcome of these experiments is as one could have anticipated from Theorem 4.3.

![Figure 4: Comparison of a true interior source and the inverse solution computed by solving (12), Example 1. The size of the regularization parameter was \( \alpha = 10^{-4} \).](image)

![Figure 5: Comparison of a true source located at the boundary and the inverse solution computed by solving (12), Example 1. The size of the regularization parameter was \( \alpha = 10^{-3} \).](image)

5.2 Example 2: Noise

If the observation data contains noise, it is natural to solve (17). Throughout this example, \( k = 7 \) in the truncated SVD employed to obtain the approximation \( A_k \) of \( A \), see Subsection 4.2. The experiment was executed as follows:
Figure 6: Example 1. The red curve shows the size of $\gamma_{j, \alpha}$, see (21), as a function of the regularization parameter $\alpha$, and the asterisks represent $\max_i [x^*_\alpha]_i$, where $x^*_\alpha$ is the solution of (12).

1. Generate the data

$$b = \tilde{b} + \eta = A e_j + \eta, \quad \eta = \delta \rho,$$

where $\delta$ is a scalar and $\rho$ is a vector containing normally distributed numbers with zero mean and standard deviation equal to 1. See [10, Example 6] for a thorough discussion of the noise.

2. Compute $x^*_k = A_k^\dagger b = A_k^\dagger (A e_j + \eta)$, see (17).

3. Set

$$\tilde{\alpha} = \max_{i \neq j} \frac{1 + |\tau_{ij}|}{1 - |\tau_{ij}|} \max_i \left| W_k^{-1} A_k^\dagger \eta \right|_i,$$

cf. Theorem 4.4

4. Compute, see (17),

$$\alpha^*_\alpha = \arg \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \| P_k x - x^*_k \|_2^2 + \alpha \| W_k x \|_1 \right\}$$

for both $\alpha = 0.3 \tilde{\alpha}$ and $\alpha = 3 \tilde{\alpha}$.

This "setup" is such that the theory, presented in Theorem 4.4 for Problem IV is applicable.

Note that the problem is regularized with both standard truncated SVD (i.e., the choice of the truncation parameter $k$), and $\ell^1$-regularization (i.e., the choice of $\alpha$). How to optimally choose these parameters in relation to each other is a complicated matter and left for future research.
Figure 7: Example 2, true source.

Figure 8 compares over-regularized ($\alpha = 3\bar{\alpha}$) and under-regularized ($\alpha = 0.3\bar{\alpha}$) solutions of (17) with observation data containing 5% noise. Similar comparisons are presented in Figures 9 and 10 for 10% and 15% noise, respectively. The true source is displayed in Figure 7.

When $\alpha = 3\bar{\alpha}$, the plots of $x^*_\alpha$, displayed in panels 8a), 9a) and 10a), show that the true source is successfully recovered in all three cases, albeit with an underestimated magnitude. We can, however, as a post-processing step, improve the magnitude of the solutions using (32): Assuming $W^{-1}A_k\eta_j << W^{-1}P_k e_j$, we compute

$$x^*_\alpha, \text{SCALED} = x^*_\alpha \left(1 - \frac{x^*_\alpha}{W^{-1}P_k e_j}ight),$$

where the index $j$ in the denominator is known since $\arg \max_i x^*_\alpha = j$, see Theorem 4.4. By re-scaling the solution displayed in panel (a) of Figures 8-10 the magnitude of the rescaled source becomes 0.88, 1.0 and 0.93, respectively. This post-processing step can only be mathematically justified if the assumptions needed in Theorem 4.4 hold, but it can "always" be applied in practice. (We have not explored its success when the assumptions in Theorem 4.4 are violated.)

When the problem is under-regularized, i.e., $\alpha = 0.3\bar{\alpha}$, the inverse solutions are still good visual approximations of the true source, see Figures 8b), 9b) and 10b). However, the inverse solutions also contain small contributions from other basis vectors than $e_j$. Both the magnitude and the number of incorrect active basis vectors appear to increase as the noise level increases.
In most applications, $\bar{\alpha}$ is not available because it requires full knowledge of the noise $\eta$. We therefore performed a numerical study under the common assumption that only (an estimate of) $\|\eta\|_2$ is known. This allows the use of
Morozov’s discrepancy principle \cite{32,12} for choosing the truncation parameter $k$ for each fixed size of $\alpha$: We did not attempt to estimate appropriate values for both $k$ and $\alpha$ simultaneously, which must thus be regarded as an open problem. Figure 11 compares solutions of \eqref{17} for different choices of $k$ and $\alpha$, where the choice $k = 5$ is the outcome of applying Morozov’s discrepancy principle with the threshold $1.05\|\eta\|_2$. The values $k = 3$ and $k = 15$ are chosen simply to compare the choice $k = 5$ with a smaller and larger truncation parameter.

In this particular example, we observe that using a relatively strong regularization in the truncated SVD step, i.e., choosing $k = 3$, gives good reconstruction of the source for all the tested values of $\alpha$. If the regularization by truncated SVD is reduced, i.e., when $k$ increases, it appears that $\alpha$ must be chosen more carefully to obtain a good reconstruction.

Figure 11: Example 2, 10% noise. Comparison of inverse solutions computed with different choices of the regularization parameters $k$ and $\alpha$. The choice $k = 5$ was the outcome of using Morozov’s discrepancy principle, for each given size of $\alpha$, with the threshold $1.05\|\eta\|_2$. For this test problem, the discrepancy principle lead to the same value $k = 5$ for $\alpha = 10^{-2}, 10^{-3}, 10^{-4}$. Figure 7 shows the true source.
5.3 Example 3: Large circular source

So far we have considered examples covered by our analysis. We will now depart from this and explore more involved cases: The true source depicted in Figure 12(a) does not belong to the finite element space associated with the coarse mesh used to represent the source in the inverse computations.

We first note that Figure 12(b) shows that classical Tikhonov regularization

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|Ax - b\|_2^2 + \zeta \|x\|_2^2 \right\},$$

with $$\zeta = 10^{-4}$$, fails to yield an adequate solution to this problem: Compare panels (a) and (b) of Figure 12. The mathematical explanation for this is presented in [10].

Next, we observe in Figure 13, left column, that the sparsity structure of the inverse solution computed by solving (17) deteriorates as the $$\ell^1$$-regularization parameter $$\alpha$$ tends to zero. However, for all three values of $$\alpha$$, the position of the source is quite well recovered. In these simulations, the truncated SVD employed to obtain the approximation $$A_k^\dagger$$ of the pseudo inverse $$A^\dagger$$ in [17], was obtained by choosing the truncation parameter $$k = 5$$.

As an alternative to the truncated SVD, we also used standard Tikhonov regularization to obtain an approximation of $$A^\dagger$$: Employing the singular value decomposition $$A = U\Sigma V^T$$ of $$A$$, the solution of the minimization problem

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|A\hat{x} - b\|_2^2 + \frac{1}{2} \beta \|\hat{x}\|_2^2 \right\},$$

can be expressed as

$$\hat{x}_\beta = V(\Sigma^2 + \beta I)^{-1}V^Tb = S_\beta b,$$

where

$$S_\beta = V(\Sigma^2 + \beta I)^{-1}V^T \approx A^\dagger.$$ (42)

Replacing $$A^\dagger$$ in (12) with $$S_\beta$$, keeping in mind that $$P = A^\dagger A$$, leads to the following alternative to (17)

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|S_\beta Ax - S_\beta b\|_2^2 + \alpha \|Wx\|_1 \right\}. $$ (43)

We also mention that, so far, we have not been able to modify the proof of Theorem 4.4 to Tikhonov based approximations of $$A^\dagger$$, i.e., to (43) with $$b = Ae_j + \eta.$$
Figure 12: Example 3. Panel (a) shows the true source, and panel (b) displays the numerical solution of (41)-(2) using Tikhonov regularization $\zeta \|f\|_{L^2(\Omega)}$ instead of the weighted $\ell^1$-regularization $\alpha \sum_i w_i |(f, \phi_i)_{L^2(\Omega)}|$.

The right column in Figure 13 contains the results obtained by solving (43). We observe, particularly for the two largest values of the $\ell^1$-regularization parameter $\alpha$, that truncated SVD and Tikhonov regularization yield visually rather similar results.

5.4 Example 4: Multiple sources

Our last examples concern several sources. To solve the problems, we did the following:

1. For $N$ true sources, we computed

$$b^\dagger = A \left( \sum_{i=1}^{N} e_{q_i} \right).$$

2. Thereafter, we solved the problem

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|S_\beta Ax - S_\beta b^\dagger\|_2^2 + \alpha \|Wx\|_1 \right\},$$

where $S_\beta$ is defined in (42).

Figure 14 contains results obtained with 2, 4 and 8 true sources. Panel b) shows that the two sources are nearly perfectly recovered. For the case with 4 sources, the inverse solution recovers three of the sources almost perfectly, while the fourth is ‘split’ into two adjacent sources with less magnitude, cf. Panel d). Panel f) shows the inverse solution for the case with 8 true sources, where we observe that the three sources located at the boundary are recovered very well (note the color bar), whereas the interior sources are merged somewhat into two clusters in the inverse solution.
6 Conclusions

If the exact data is generated from a single basis vector $e_j$, our weighted $\ell^1$-regularization technique is able to exactly recover the true solution. When noise is present, we have obtained estimates for the size of the regularization parameter $\alpha$ which yield an inverse solution in the form $\gamma_{j,\alpha,\eta}e_j$, where $\gamma_{j,\alpha,\eta}$ is a positive scalar. Numerical experiments suggest that our method also can, in many cases, identify several local sources, but we do not have a thorough mathematical understanding of this. We only know with certainty that it is possible to construct scenarios for which our scheme will fail to recover two sources.

The computation of our weight matrix involves the pseudo inverse, which in practice must be approximated by a more "well-behaved" operator. In the analysis we accomplished this by employing a truncated SVD approach, and we observed numerically that also Tikhonov based approximations of the Moore-Penrose inverse work well. For the latter, however, it remains to develop a rigorous mathematical analysis.

Concerning the practical use of our weighted $\ell^1$-regularization method, it seems reasonable to expect that the method can recover two or three well-separated sources. Nevertheless, the definition of "well-separated" is problem dependent since it must depend on the smoothing properties of the involved forward operator and the geometry of the solution domain $\Omega$. This means that one should, for each concrete application, perform a simulation study to explore which source patterns that can be identified by the weighted $\ell^1$-regularization method.

We defined our regularization operator and presented our analysis in terms of Euclidean spaces. Consequently, the methodology can be applied whenever a discrete version of a source identification task can be formulated in terms of a transfer matrix with a significant null space. For example, the use is not restricted to PDE-constrained optimization problems with elliptic state equations, but can also be applied when the state equation is parabolic or hyperbolic.

This study was motivated by inverse problems arising in connection with EEG and ECG recordings. In principle, our scheme can be applied to these problems, but a number of challenging engineering issues must be handled: one must construct suitable geometrical models (using, e.g., MR images), obtain EEC or ECG recordings, handle noisy data, construct suitable basis functions for the source term which enables the incorporation of dipoles, etc. We intend to explore the EEG and ECG applications in forthcoming investigations.

In this paper we search for a source in a finite dimensional space. From a pure mathematical perspective, an interesting problem would be to develop an analogous theory using an infinite dimensional source space. The authors believe that this might be possible following the approach presented in [5].
Figure 13: Example 3, employing weighted $\ell^1$-regularization $\alpha \| W x \|_1$. The inverse solutions were computed by solving (17) (approximating $A^\dagger$ with truncated SVD, $k = 5$) and (43) (approximating $A^\dagger$ with Tikhonov regularization, $\beta = 10^{-6}$). The true source is displayed in Figure 12(a).
A Standard sparsity regularization

We observed in panel b) of Figure 1 that standard sparsity regularization failed to recover an interior source. We will now explore this issue in some detail.

Recall the definition of the forward operator $K_h : F_h \to L^2(\partial \Omega)$ and consider the problem

$$\min_{f \in F_h} \sum_i |(f, \phi_i)| \ \text{subject to} \quad K_h f = K_h \phi_j. \quad (44)$$
This is the basis pursuit problem associated with (1)-(2) when \( d = K_h \phi_j \), provided that \( w_i = 1 \) for \( i = 1, 2, \ldots, n \), i.e., with standard unweighted sparsity regularization. We assume in this appendix that the basis functions \( \phi_1, \phi_2, \ldots, \phi_n \) for \( F_h \) satisfy
\[
\|\phi_1\|_\infty = \|\phi_2\|_\infty = \ldots = \|\phi_n\|_\infty.
\] (45)

Let us define the orthogonal projection \( P_h : F_h \to \mathcal{N}(K_h) \perp \), where \( \mathcal{N}(K_h) \) denotes the null space of \( K_h \) and we employ the standard \( L^2 \)-inner product on \( F_h \subset L^2(\Omega) \). Since \( P_h \) and \( K_h \) have the same null space, we can reformulate the basis pursuit problem (44) as
\[
\min_{f \in F_h} \sum_i |(f, \phi_i)| \quad \text{subject to} \quad P_h f = P_h \phi_j.
\] (46)

The associated Lagrangian \( \mathcal{L} : F_h \times F_h \to \mathbb{R} \) reads
\[
\mathcal{L}(f, \lambda) = \sum_i |(f, \phi_i)| + (\lambda, P_h \phi_j - P_h f),
\]
and the Lagrange conditions become
\[
P_h \lambda \in \partial_f \left( \sum_i |(f, \phi_i)| \right),
\] (47)
\[
P_h f = P_h \phi_j.
\]

Since \( f \to |(f, \phi_i)| \) is convex for \( i = 1, 2, \ldots, n \), it follows that
\[
\partial_f \left( \sum_i |(f, \phi_i)| \right) = \sum_i \partial_f |(f, \phi_i)|,
\]
provided that one interprets the right-hand-side in terms of the Minkowski sum of sets. According to the definition of the subgradient, \( q \in \partial_f |(f, \phi_i)| \) if
\[
|(g, \phi_i)| \geq |(f, \phi_i)| + (q, g - f), \quad \forall g \in F_h.
\] (48)

By expanding \( f, g \) and \( q \) in the orthonormal \( F_h \)-basis, i.e.,
\[
f(x) = \sum_k f_k \phi_k(x),
g(x) = \sum_k g_k \phi_k(x),
q(x) = \sum_k q_k \phi_k(x),
\]
we get from (48) that
\[ |g_i| \geq |f_i| + \sum_k q_k(\phi_k, g - f), \forall g \in F_h. \]
Note that this implies that \( q_k = 0 \) for \( k \neq i \). Consequently, \( q(x) = q_i \phi_i(x) \), and \( q_i \) must obey the inequality constraint
\[ |g_i| \geq |f_i| + q_i(g_i - f_i), \forall g \in F_h, \]
which implies that
\[ q_i \in \begin{cases} 
{1}, & f_i > 0, \\
{-1}, & f_i < 0, \\
[-1,1], & f_i = 0.
\end{cases} \]
Thus, we can write (47) as
\[ [P_h \lambda]_i \in \begin{cases} 
{1}, & f_i > 0, \\
{-1}, & f_i < 0, \\
[-1,1], & f_i = 0,
\end{cases} \]
where we use the notation \([P_h \lambda]_i = (P_h \lambda, \phi_i)\).
Assume that
\[ f^*(x) = \sum_i f_i^* \phi_i(x) \]
is a solution of (46) with associate Lagrange multiplier \( \lambda^* \). Then \( f^* \) and \( \lambda^* \) satisfy (47) and from (49) we find that:
(a) \(-1 \leq [P_h \lambda^*]_i \leq 1, \forall i = 1, 2, \ldots, n.\)
(b) If a basis function \( \phi_i \) with support strictly in the interior of \( \Omega \) is present in the solution (50) and \( f_i^* > 0 \), then \([P_h \lambda^*]_i = 1\). That is, \( P_h \lambda^* \) attains its maximum in the interior region associated with \( \phi_i \), provided that (45) holds.
(c) On the other hand, from the analysis presented in Section 2 in [10], we know that the infinite-dimensional counterpart\(^6\) \( P\lambda^* \) to \( P_h \lambda^* \) satisfies
\[ -\Delta P\lambda^* + \epsilon P\lambda^* = 0. \]
It thus follows from classical maximum principles that \( P\lambda^* \) can not attain a (non-negative) maximum in the interior of \( \Omega \). This is not compatible/consistent with \( P_h \lambda^* \) attaining its maximum in the interior which, according to (b), would be case if the solution \( f^* \) is positive in an interior region. Hence, we expect that \( f^* \) only can be positive close to the boundary \( \partial \Omega \) of \( \Omega \), cf. panel b) in Figure [1].

\(^6\)That is, the orthogonal projection when \( F_h \) is replaced with \( L^2(\Omega) \).
B  Discretization of the state equation

We will briefly explain how (3) can be discretized, using the finite element method, and thereby obtain an expression for the matrix \( \tilde{K} \) in (6). As mentioned in the numerical experiments section, the state \( u(x) = \sum_k u_k N_k(x) \) and the source \( f(x) = \sum_i f_i \phi_i(x) \) were discretized in terms of first order Lagrange elements and the characteristic functions \([40]\), respectively.

The discrete matrix-vector version of (3) reads

\[
Lu + \epsilon Mu = \tilde{M}f,
\]

where \( L \) and \( M \) denote the standard stiffness and mass matrices, respectively, and

\[
\tilde{M} = [\tilde{m}_{ki}], \quad \tilde{m}_{ki} = (\phi_i, N_k)_{L^2(\Omega)}.
\]

Hence,

\[
u = [L + \epsilon M]^{-1} \tilde{M} f,
\]

and, if we discretize the fidelity term in (1) and combine it with this expression for \( u \), we get

\[
\frac{1}{2} (u - d)^T M_\beta (u - d) = \frac{1}{2} \left( [L + \epsilon M]^{-1} \tilde{M} f - d \right)^T M_\beta \left( [L + \epsilon M]^{-1} \tilde{M} f - d \right).
\]

Since the ”boundary mass matrix” \( M_\beta \) is symmetric and positive semi-definite, we can take the square root of it to obtain the following Euclidean form of the fidelity term

\[
\frac{1}{2} \left\| M_\beta^{\frac{1}{2}} [L + \epsilon M]^{-1} \tilde{M} f - M_\beta^{\frac{1}{2}} d \right\|^2 = \frac{1}{2} \left\| \tilde{K} f - M_\beta^{\frac{1}{2}} d \right\|^2,
\]

where

\[
\tilde{K} := [L + \epsilon M]^{-1} \tilde{M}.
\]

References

[1] A. Ben Abda, F. Ben Hassen, J. Leblond, and M. Mahjoub. Sources recovery from boundary data: A model related to electroencephalography. Mathematical and Computer Modelling, 49:2213–2223, 2009.

[2] E. J. Candes, J. Romberg, and T. Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. IEEE Transactions on Information Theory, 52(2):489–509, 2006.

[3] E. J. Candes and T. Tao. Decoding by linear programming. IEEE Transactions on Information Theory, 51(12):4203–4215, 2005.
[4] E. J. Candes, M. B. Wakin, and S. P. Boyd. Enhancing sparsity by reweighted $\ell_1$ minimization. *Journal of Fourier analysis and applications*, 14(5):877–905, 2008.

[5] E. Casas, C. Clason, and K. Kunisch. Approximation of elliptic control problems in measure spaces with sparse solutions. *SIAM Journal on Control and Optimization*, 50(4):1735–1752, 2012.

[6] E. Casas, C. Clason, and K. Kunisch. Parabolic control problems in measure spaces with sparse solutions. *SIAM Journal on Control and Optimization*, 51(1):28–63, 2013.

[7] X. Cheng, R. Gong, and W. Han. A new Kohn-Vogelius type formulation for inverse source problems. *Inverse Problems and Imaging*, 9(4):1051–1067, 2015.

[8] I. Daubechies, M. Defrise, and C. De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Communications on Pure and Applied Mathematics*, 57(11):1413–1457, 2004.

[9] D. L. Donoho and M. Elad. Optimally sparse representation in general (nonorthogonal) dictionaries via $\ell_1$ minimization. *Proceedings of the National Academy of Sciences*, 100(5):2197–2202, 2003.

[10] O. L. Elvetun and B. F. Nielsen. A regularization operator for source identification for elliptic PDEs. *Inverse Problems and Imaging*, 15(4):599–618, 2021.

[11] O. L. Elvetun and B. F. Nielsen. Modified Tikhonov regularization for identifying several sources. *International Journal of Numerical Analysis and Modeling*, 18(6):740–757, 2021.

[12] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems*. Kluwer Academic Publishers, 1996.

[13] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, 1998.

[14] J. Flemming. Convergence rates for $\ell^1$-regularization without injectivity-type assumptions. *Inverse Problems*, 32(9), 2016.

[15] S. Ghosh and Y. Rudy. Application of L1-norm regularization to epicardial potential solution of the inverse electrocardiography problem. *Annals of biomedical engineering*, 37(5):902–912, 2009.

[16] T. Goldstein and S. Osher. The split Bregman method for l1-regularized problems. *SIAM Journal on Imaging Sciences*, 2:323–343, 2009.
[17] A. Golmohammadi, M. R. M. Khaninezhad, and B. Jafarpour. Exploiting Sparsity in Solving PDE-Constrained Inverse Problems: Application to Subsurface Flow Model Calibration, pages 399–434. Springer New York, 2018.

[18] M. Grasmair, M. Haltmeier, and O. Scherzer. Sparse regularization with $l^q$ penalty term. Inverse Problems, 24(5):055020, 2008.

[19] M. Grasmair, O. Scherzer, and M. Haltmeier. Necessary and sufficient conditions for linear convergence of $\ell^q$-regularization. Communications on Pure and Applied Mathematics, 64(2):161–182, 2011.

[20] M. Hanke and W. Rundell. On rational approximation methods for inverse source problems. Inverse Problems and Imaging, 5(1):185–202, 2011.

[21] E. Herman, A. Alexanderian, and A. K. Saibaba. Randomization and reweighted $\ell_1$-minimization for A-optimal design of linear inverse problems. SIAM Journal on Scientific Computing, 42(3):A1714–A1740, 2020.

[22] F. Hettlich and W. Rundell. Iterative methods for the reconstruction of an inverse potential problem. Inverse Problems, 12:251–266, 1996.

[23] M. Hinze, B. Hofmann, and T. N. T. Quyen. A regularization approach for an inverse source problem in elliptic systems from single Cauchy data. Numerical Functional Analysis and Optimization, 40(9):1080–1112, 2019.

[24] V. Isakov. Inverse Problems for Partial Differential Equations. Springer-Verlag, 2005.

[25] B. Jin and P. Maaß. Sparsity regularization for parameter identification problems. Inverse Problems, 28(12):123001, 2012.

[26] B. Jin, P. Maaß, and O. Scherzer. Sparsity regularization in inverse problems. Inverse Problems, 33(6):060301, 2017.

[27] R. Khodayi-mehr, W. Aquino, and M. M. Zavlanos. Model-based sparse source identification. In 2015 American Control Conference (ACC), pages 1818–1823, 2015.

[28] K. Kunisch and X. Pan. Estimation of interfaces from boundary measurements. SIAM J. Control Optim., 32(6):1643–1674, 1994.

[29] C. Li and G. Stadler. Sparse solutions in optimal control of PDEs with uncertain parameters: The linear case. SIAM Journal on Control and Optimization, 57(1):633–658, 2019.

[30] D. A. Lorenz. Convergence rates and source conditions for Tikhonov regularization with sparsity constraints. Journal of Inverse and Ill-posed Problems, 16(5):463–478, 2008.
[31] Z.-R. Lu, T. Pan, and L. Wang. A sparse regularization approach to inverse heat source identification. *International Journal of Heat and Mass Transfer*, 142:118430, 2019.

[32] V. A. Morozov. On the solution of functional equations by the method of regularization. In *Doklady Akademii Nauk*, volume 167, pages 510–512. Russian Academy of Sciences, 1966.

[33] A. L. Peressini, F. E. Sullivan, and J. J. Uhl, Jr. *The Mathematics of Nonlinear Programming*. Springer-Verlag, 1988.

[34] W. Ring. Identification of a core from boundary data. *SIAM Journal on Applied Mathematics*, 55(3):677–706, 1995.

[35] S. J. Song and J. G. Huang. Solving an inverse problem from bioluminescence tomography by minimizing an energy-like functional. *J. Comput. Anal. Appl.*, 14:544–558, 2012.

[36] L. Wang. *Applications of Sparse Regularization to Inverse Problem of Electrocardiography*. PhD thesis, 2012.

[37] X. Xiang and H. Sun. Sparse reconstructions of acoustic source for inverse scattering problems in measure space. *Inverse Problems*, 36(3), 2020.