QCD Soft Function from Large-Momentum Effective Theory on Lattice

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We study Euclidean lattice calculation of the QCD soft function, which involves two conjugate lightcone directions in the framework of large-momentum effective theory. We find that the transverse momentum dependent (TMD) soft function required by TMD factorization can be formulated as the form factor of a pair of color sources traveling with nearly-lightlike velocities, and thus can be calculated using lattice heavy-quark effective theory. A simple generalization shows that the factorization of a large-momentum light-meson form factor combining with quasi-TMD wave function can also be used to extract the soft function on lattice.

Introduction.—Radiation of soft gluons from fast-moving charged particles is a ubiquitous phenomenon of high energy processes in quantum chromodynamics (QCD). Soft radiation usually cancels in total cross section such as inclusive deep inelastic scattering (DIS). However, for certain processes where a small transverse momentum is measured, such cancellation can be incomplete and result in measurable consequences. The soft radiation, from arbitrary numbers of gluons at small momentum below \( \Lambda_{QCD} \), usually involves formidable nonperturbative physics. Fortunately, in some cases soft radiations factorizes from other energy scales of the process which leads to surprising simplification. In such cases, a universal and process-independent function \( S(Y,\gamma_\perp,\mu) \), called soft function, emerges to capture soft gluon effects.

The soft function appears in factorization theorems for the Drell-Yan (DY) process [3,4], semi-inclusive DIS (SIDIS) [5,6], and other processes [1,7,8]. Intuitively, soft gluons have no impact on the velocity of the fast-moving color charged partons, and the propagators of partons eikonalize to straight gauge links along their moving trajectory. Therefore, the soft function \( S(Y,\gamma_\perp,\mu) \) is defined as vacuum expectation value of a Wilson loop composed of nearly-lightlike gauge links, and usually depends on three variables: the rapidity regulator \( Y \), the transverse separation \( \gamma_\perp \) (conjugate to transverse momentum), and the renormalization scale \( \mu \) associated to cusps of Wilson links. This function has been studied in perturbation theory in many aspects such as structure of rapidity divergences [9,10] and high order calculations [10,11]. However, despite its universality and important role in factorization theorems, a first-principle calculation of soft function in the nonperturbative region at large \( \gamma_\perp \) remains an open question.

Recently, there were attempts to calculate the soft function on lattice [12,13], however, the problem was not completely resolved. In this paper, we solve this long-standing problem by observing that the soft function can be interpreted as a form factor independent of Minkowski time. We introduce a soft function equivalent to the one defined by Collins [15,16] and show that it is equal to the form factor of a boosted heavy-quark pair

\[
S(Y,Y',\gamma_\perp,\mu) = \left\langle \langle QQ,\gamma_\perp,\mu | J(Y',\gamma_\perp,\mu) | QQ,\gamma_\perp,\mu \rangle \right\rangle
\]

where \( \mu = \gamma(1,\beta,\gamma_\perp) \) and \( \mu' = \gamma'(1,-\beta',\gamma_\perp) \) are two opposite nearly-lightlike velocities in \((t,z,\perp)\) coordinates: \( | QQ,\gamma_\perp,\mu \rangle \) is a heavy quark-anti-quark ground state with fixed transverse separation \( \gamma_\perp \) and boosted to velocity \( v \); \( Q \) and \( \bar{Q} \) are color sources in heavy quark effective theory (HQET); the rapidity \( Y \) and the speed \( \beta \) are related through \( \beta = \tanh Y \); \( J \) is the time-independent transition current, the cusp anomalous dimension of which is the same as for the Wilson lines [17,20]. The soft function actually depends on relative rapidity, \( S(Y,Y',\gamma_\perp,\mu) = S(Y+Y',\gamma_\perp,\mu) \), due to Lorentz invariance. We choose to keep the variable \( Y \) and \( Y' \) in \( S \) separately to distinguish the shape of the soft function on different sides. Equation (1) paves the way for direct lattice calculations using HQET [21,22]. This generalizes the large-momentum effective theory (LaMET) [23,24] to the case of two lightcones: by boosting two hadrons to large-momenta in opposite directions, one generates gauge links in two conjugate lightlike directions.

We also find that the soft function can be extracted from a form factor of a light-meson with opposite large momenta. However in this case, the form factor also contains nonperturbative collinear contributions, which can be subtracted using the quasi transverse-momentum-dependent wave function (quasi-TMDWF). The result is a combination of three soft functions,

\[
S(Y,Y',\gamma_\perp,\mu) \rightarrow S_I(\gamma_\perp,\mu)
\]

which is independent of the rapidity regulator in the large rapidity limit, and reflects intrinsic properties of soft gluon radiation [20].

With this intrinsic soft functions \( S_I \) and quasi-TMD parton distributions [12,13,24], the cross section of the DY process [3] becomes entirely predictable from first principle.
A more common definition of the universal soft function was proposed Refs. [15, 16]. The spacelike vectors \( n^\mu = \gamma(\beta, 1, 0, 0) \) and \( n'^\mu = \gamma'(-\beta', 1, 0, 0) \) were chosen instead of timelike \( v \) and \( v' \) to define the soft function for the DY process. After similar subtraction as in Eq. (3), we denote this subtracted soft function as \( S_C(Y, Y', b_\perp) \).

While \( S \) and \( S_C \) are defined differently, we can show that

\[
S(Y, Y', b_\perp) = S_C(Y, Y', b_\perp)
\]

using analyticity property [20]. In this paper, however, we focus on \( S \) in Eq. (3), which has a simple Euclidean realization.

**Soft function as form factor of color source pair.**—In HQET, the propagator of a color source is equivalent to a gauge link along its moving direction. Thus \( W(Y, Y', T, T', b_{\perp}) \) can be expressed by fields in HQET with the Lagrangian

\[
\mathcal{L}_{HQET} = \psi^\dagger_v(x)(iv \cdot D)\psi_v(x) + \eta^\dagger_v(x)(iv \cdot D)\eta_v(x)
\]

where \( \psi_v \) and \( \eta_v \) are quark and anti-quark in the fundamental and anti-fundamental representations, respectively; \( v^\mu = \gamma(1, \beta, \vec{0}_\perp) \) is the four velocity; \( D \) is the covariant derivative. Note that quarks in HQET can be viewed as color sources. If the gluon soft function is considered, the heavy quarks should be in adjoint representation.

In HQET, a color-singlet heavy-quark pair separated by \( \vec{b} \) generates a heavy quark potential \( V(\vec{b}) \) in the ground state, and the spectrum includes a gapped continuum above it. The state can also have a residual momentum \( \delta \vec{P} \), which is arbitrary due to reparameterization invariance [28–29] and for simplicity we always consider \( \delta \vec{P} = 0 \). When the sources move with a velocity \( v \), the ground state can be labeled by \( |\vec{Q}Q, \vec{b}, \delta \vec{P} \rangle_v \), where \( \delta \vec{P} = \vec{P}_{\text{total}} - 2m_Q \gamma \vec{\beta} \). The residual energy of the state is \( E = \gamma^2 V(\vec{b}) + \vec{\beta} \cdot \delta \vec{P} \).

Consider a process with incoming and outgoing states being heavy-quark pairs separated by \( \vec{b}_{\perp} \) and at velocity \( v \) and \( v' \), respectively. Such a state is created by the interpolating fields

\[
\mathcal{O}_v(t, \vec{b}_{\perp}) = \int d^3\vec{r}\psi^\dagger_v(t, \vec{r})U(\vec{r}, \vec{r}', t)\eta^\dagger_v(t, \vec{r}')
\]

where \( \vec{r}' = \vec{r} + \vec{b}_{\perp} \); \( U(\vec{r}, \vec{r}', t) \) is a gauge link connecting \( \vec{r}' \) to \( \vec{r} \) at time \( t \). The heavy-quark pair created by \( \mathcal{O}_v \) is forced to be at relative separation \( \vec{b}_{\perp} \) and to have vanishing residual momentum \( \delta \vec{P} = 0 \). Between the incoming and outgoing states, a product of two local equal-time operators

\[
J(v, v', \vec{b}_{\perp}) = \eta^\dagger_v(\vec{b}_{\perp})\eta_v(\vec{b}_{\perp})\psi^\dagger_v(0)\psi_v(0)
\]

is inserted at \( t = 0 \). Then \( W \) can be expressed, up to an
Through HQET, we find a time-independent formulation of the subtracted soft function, and open up the possibility of direct lattice calculations.

**Soft function from large-momentum light-meson form factor.**—Consider a similar large-momentum form factor as in Eq. (11) for pseudoscalar light-meson states with constituents $\psi_i$.

$$\Pi(P, P', b_\perp) = \langle P'|\bar{\psi}(\tilde{b}_\perp) \Gamma^i \psi(0) \Gamma^j(0)|P\rangle$$

(15)

where $\psi$ and $\eta$ are light quark fields of different flavors; $P^\mu = (P^t, P^z, 0_\perp)$ and $P'^\mu = (P'^t, P'^z, 0'_\perp)$ are two large momenta which approach to two lightlike directions in the limit $P^z \rightarrow \infty$; $\Gamma$ and $\Gamma'$ are Dirac gamma matrices. At large $P^z$, similar to the situation for the DY process, $\Pi$ can be factorized [26]

$$\Pi(P, P', b_\perp) = \int d\tilde{x} \tilde{d} \tilde{y} H_{\Pi}(x, x', P \cdot P')$$

(16)

$$\frac{\tilde{\phi}(x', x, P, P')}{S(Y', Y, b_\perp)} \frac{S(Y, \tilde{Y}, b_\perp)}{S(Y, Y, b_\perp)} S(Y', Y', b_\perp)$$

where $H_{\Pi}$ is the hard kernel; $S$ is defined in Eq. (4). $\phi$ is the lightcone TMDWF with self-interaction of staple shaped gauge links subtracted,

$$\frac{\langle P|\tilde{\psi}(\xi_2 - \xi_1, \tilde{b}_\perp) \tilde{\psi}(\xi_1, 0, \tilde{b}_\perp)|0\rangle}{\sqrt{Z_E(Y, 2L, b_\perp)}}$$

(17)

where $\xi^{-} = (t - z)/\sqrt{2}$ and $P^+ = (P^t + P^z)/\sqrt{2}$; the link $U_\phi$ consists of straight gauge links along the contour: $0 \rightarrow \tilde{n} L/\gamma \rightarrow \tilde{n} L/\gamma + b \rightarrow (\xi_1^-, \tilde{b}_\perp)$ and $\tilde{n}^\mu = \gamma(\beta, -1, \tilde{0}_\perp)$ in $(t, z, \tilde{L})$ coordinates; $Z_E(Y, 2L, b_\perp)$ is a spacelike Wilson loop which is the analytic continuation of $Z$ defined in Eq. (4). $\sqrt{Z_E}$ cancels the self-interaction of the staple shaped gauge link. The same lightcone TMDWF also appears in factorization for electromagnetic pion form factor in Ref. [30].

In order to extract soft functions in Eq. (16), we need to remove collinear contributions. Therefore, we consider the quasi-TMDWF with self-interaction subtraction

$$\tilde{\phi}(x, P, b_\perp)$$

(18)

$$\tilde{\phi}(x, P, b_\perp) = H_\phi(x, P^\perp) \frac{\phi(x, \tilde{Y}, P, b_\perp)}{S(Y, \tilde{Y}, b_\perp)} S(Y, 0, b_\perp)$$

(19)
where $H_\delta$ is the hard kernel; $S$ is defined in Eq. [1].

While deriving factorization theorem, we naturally obtain $S_C$ whose subscript $C$ is suppressed since $S_C = S$. $S(Y,0,b_\perp)$ captures the soft gluon contribution to quasi-TMDWF with nearly-lightlike and $z$ directions associated to the outgoing fast-moving hadron and staple shaped gauge links, respectively.

Combining Eqs. [10] and [19], we have

$$S_I(b_\perp) = \frac{S(Y,Y',b_\perp)}{S(Y,0,b_\perp)S(0,Y',b_\perp)} = \Pi(P,P',b_\perp) \int dx dx' H(x,x',P,P') \phi(x',P',b_\perp)\phi(x,P,b_\perp)$$

where $H(x,x',P,P') = \frac{H_\delta(x,x',P,P')}{H_\delta(x,x',P,P')H_\delta(x',P,P')}$. Since every term on the right hand side of Eq. (20) is calculable either on lattice or in perturbation theory, this ratio of three soft functions can be determined from first principle.

Notice that for large rapidities, the renormalized soft function can be expressed as [11]

$$S(Y,Y',b_\perp,\mu) \rightarrow \frac{S(Y,Y',b_\perp)}{Y+Y'\rightarrow \infty} e^{(Y+Y')D(b_\perp,\mu)+D_I(b_\perp,\mu)}$$

where $D_I(b_\perp,\mu)$ is the Collins-Soper kernel, and $D_I(b_\perp,\mu)$ is related to intrinsic soft function through $S_I(b_\perp,\mu) = e^{-D_I(b_\perp,\mu)}$. We therefore can extract the intrinsic soft contribution using a light-meson form factor. To further obtain Collins-Soper kernel, we require the form factor of heavy-quark pair in HQET or calculations along the lines of [31].

The soft function $S(Y,Y',b_\perp)$ depends on a specific regulator of rapidity divergience. However, this regulator dependence cancels in the physical cross section. Similar to Eq. (20), we can show that the cross section of DY can be factorized by quasi-TMDPDF [20]

$$\frac{d\sigma_{DY}}{d^2b_\perp dx dx'} = \hat{\sigma}(x,x',P\cdot P') \tilde{f}(x,P,b_\perp) \tilde{f}(x',P',b_\perp)$$

where $\hat{\sigma}(x,x',P\cdot P')$ is the hard kernel, and

$$\tilde{f}(x,P,b_\perp) = \lim_{L\rightarrow \infty} \int \frac{dz}{4\pi} e^{izxP^z} \langle \bar{\psi}(0,z,b_\perp)\Gamma \tilde{U}_{\delta}\psi(0)|P\rangle \sqrt{Z_E(0,2L,b_\perp)}$$

is a quasi-TMD parton distribution, whose definition is similar to the quasi-TMDWF in Eq. (18). With Eq. (22), the DY process becomes predictable from first-principle calculations.

**Discussion and conclusion.—** Another definition of soft function for the SIDIS process [9] [10] involves a Wilson loop similar to $U$ in Eq. (3), but with time and anti-time order. We call this soft function with mixed time-order prescription $S_{mix}$. However, it is actually equal to $S$ and $S_C$ [9] [20]. Note that the $S_C$ is insensitive to time-order because the distance between any two points on the Wilson loop is spacelike. For $S_C$ in mixed time-order prescription, the Feynman integral implies analyticity in rapidity regulator, then $S_{mix}$ is related to $S_C$ through analytic continuation which leads to $S_{mix} = S_C$ in the lightcone limit. On the other hand, $S_C$ in single time-order prescription also defines an analytic continuation of the rapidity regulator into the SIDIS region. Since the two prescriptions for $S_C$ are equal, after analytic continuation the results are equal up to a phase factor associated to branches of logarithm in the lightcone limit. Thus, it indicates that $S_{mix} = S$ up to an overall phase factor. Since both $S_{mix}$ and $S$ are real, the phase factor is unity.

We have shown that the soft function in the off-lightcone regularization can be calculated on lattice. It is worth to mention that the relation to other regularization schemes in which the lightcone direction is not modified (lightcone schemes), such as $\delta$ regularization [10] or exponential regularization [11], is not so straightforward. At one-loop level, soft function in off-lightcone schemes involve only single logarithm $\ln^2 \mu^2 b_\perp^2$, while in lightcone schemes double logarithms $\ln^2 \mu^2 b_\perp^2$ do appear even at one-loop level. Thus, in general one would expect that nonperturbative matching is required to connect these two type of schemes. However, one should notice that once we combine the soft function together with the TMD collinear factors, the result will be independent to the regularization scheme.

There are other efforts to propose soft functions on lattice connecting quasi-TMDPDF to lightcone TMD-PDF [12] [14]. However, with close examinations, we found that the soft function is controlled by cusp anomalous dimension at large hyperbolic angle, while other proposed soft functions are composed by Euclidean gauge links with circular angle which cannot be arbitrarily large. Although, the cusp anomalous dimension of the bent soft functions in Ref. [12] [14] coincide with $S_I$ at one-loop level, in general it is different beyond one-loop contribution [17]. The difference in the cusp anomalous dimension will leads to different logarithmic structure in $\ln^2 \mu^2 b_\perp^2$ which is not properly controlled by perturbation theory at large $b_\perp$. As we have emphasized, $S_I$ can not be represented as a Wilson-loop composed of a finite number of Euclidean Wilson lines.

In conclusion, we have extended the formulation of LaMET to incorporate observables defined with two lightlike directions. The subtracted soft function in tilted angle regularization allows an interpretation as a form factor of a fast-moving heavy-quark pair in HQET. Thus it opens an opportunity to calculate soft functions on lattice. The form factor of light meson combined with quasi-TMDWF can also be used to extract the intrinsic soft function. The regularization independent factorization into quasi-TMD parton distribution together with intrinsic soft function allows the TMD cross section of the DY process to be predicted by first-principle lattice calculation.
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