LOCAL-GLOBAL PRINCIPLE FOR THE
BAUM-CONNES CONJECTURE WITH COEFFICIENTS

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Abstract. We establish the Hasse principle (local-global principle) in the context of the Baum-Connes conjecture with coefficients. We illustrate this principle with the discrete group $GL(2, F)$ where $F$ is any global field.

1. Introduction

Let $G$ be a second countable locally compact Hausdorff topological group. We shall say that $G$ satisfies $BCC$, or $BCC$ is true for $G$, if the Baum-Connes conjecture with coefficients in an arbitrary $G - C^*$-algebra is true for $G$.

Our first result is the following permanence property:

Theorem 1.1. Let $G$ be the ascending union of open subgroups $G_n$, and let $A$ be a $G - C^*$-algebra. If each subgroup $G_n$ satisfies the Baum-Connes conjecture with coefficients $A$, then $G$ satisfies the Baum-Connes conjecture with coefficients $A$.

Our main application is to the following new permanence property.

Theorem 1.2. Let $F$ be a global field, $A$ its ring of adeles, $G$ a linear algebraic group defined over $F$. Let $F_v$ denote a place of $F$. If $BCC$ is true for each local group $G(F_v)$ then $BCC$ is true for the adelic group $G(A)$.

Another application of Theorem 1.1 is the proof of the Baum-Connes conjecture for reductive adelic groups [2].

To derive Theorem 1.2 from Theorem 1.1 we note that the adelic group $G(A)$ admits an ascending union of open subgroups. We then make use of a crucial permanence property due to Chabert-Echterhoff [8], namely that $BCC$ is stable under direct product of finitely many groups.

If $G$ satisfies $BCC$, then any closed subgroup of $G$ also satisfies $BCC$ [2, Theorem 2.5]. Since $G(F)$ is a discrete subgroup of $G(A)$, we have the following result:
Theorem 1.3. If BCC is true for each local group $G(F_v)$ then BCC is true for the discrete group $G(F)$.

There is, at present, a limited supply of local groups for which BCC is known to be true. Nevertheless, some examples are known.

For the group $SO(n, 1)$, Kasparov [12] proved that $\gamma = 1_G$ in the Kasparov representation ring $R(G) = KK_G(C, C)$. For the group $SU(n, 1)$, Julg-Kasparov [11] and Higson-Kasparov [8] proved that $\gamma = 1_G$ in the ring $R(G)$. For the group $Sp(n, 1)$, Julg [10] has recently proved that, for any $G - C^*$-algebra $A$, the image of $\gamma$ via the map:

$$R(G) \rightarrow KK_G(A, A) \rightarrow KK_\times(A \rtimes_r G, A \rtimes_r G) \rightarrow End_{K_\times}(A \rtimes_r G)$$

is the identity element even though $\gamma \neq 1$ in the ring $R(G)$. Therefore the following rank-one Lie groups satisfy BCC:

$$SO(n, 1), SU(n, 1), Sp(n, 1).$$

Let $F$ be a global field and consider the group $GL(2, F)$. At each place $v$ of $F$ we have the local group $GL(2, F_v)$. The Lie group $SL(2, \mathbb{R})$ acts properly and isometrically on the Poincaré disc, and so has the Haagerup property [9, p.159]. The Lie group $SL(2, \mathbb{C})$ acts properly and isometrically on the Poincaré ball [7, p.11], and so has the Haagerup property [4, p.159]. If $v$ is a finite place then the totally disconnected group $SL(2, F_v)$ acts properly and isometrically on its tree [13], admits a proper 1-cocycle, and so has the Haagerup property [4, p.155].

For each place $v$, the determinant map creates a short exact sequence

$$1 \rightarrow SL(2, F_v) \rightarrow GL(2, F_v) \rightarrow F_v^\times \rightarrow 1$$

in which $SL(2, F_v)$ is a closed normal subgroup with the Haagerup property, and the locally compact abelian group $F_v^\times$ certainly satisfies BCC. Therefore, by [4, Corollary 3.14] each local group $GL(2, F_v)$ satisfies BCC.

Let $\mathbb{A}_F$ be the adele ring attached to $F$. By Theorem 1.2, we have

Theorem 1.4. The adelic group $GL(2, \mathbb{A}_F)$ satisfies BCC.

By Theorem 1.3, we have

Theorem 1.5. The discrete group $GL(2, F)$, and each of its subgroups, satisfies BCC.

Our method is therefore an example of the Hasse principle (local-global principle) applied to the local groups $G(F_v)$ and the discrete group $G(F)$. 
It is worth noting that if the Baum-Connes conjecture fails for the discrete group $SL(n, \mathbb{Z})$, then BCC fails for $SL(n, \mathbb{R})$.

The discrete group $SL(2, F)$, with $F$ a global field, is discussed in the Bourbaki seminar by Pierre Julg [9, p. 160]. He proves that $SL(2, F)$ has the Haagerup property. It then follows from a theorem of Higson-Kasparov [8] that $SL(2, F)$ satisfies the Baum-Connes conjecture, see [9, p. 152].

In the course of our work, we find it necessary to use a model $Pc(G)$ of the universal example for proper actions of $G$ which is itself a direct limit of compact spaces. This model $Pc(G)$ is paracompact, Hausdorff and separable, but not metrizable, and so falls outside the discussion of proper actions in [1]. We have therefore to choose a different starting point for the theory of proper actions: we use the definition of Bourbaki [4].

This paper is a sequel to our Note [2]. We thank Siegfried Echterhoff, Pierre Julg, Ryszard Nest and Richard Sharp for valuable conversations.

2. Adelic groups

A local field is a non discrete locally compact topological field. It is shown in [15] that a local field $F$ must be of the following form. If char($F$) = 0, then $F = \mathbb{R}, \mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ for some prime $p$. If char($F$) = $p > 0$, then $F$ is the field $\mathbb{F}_q((X))$ of formal Laurent series (with finite tail) in one variable with coefficients in a finite field $\mathbb{F}_q$.

The fields $\mathbb{R}, \mathbb{C}$ are known as archimedean fields. All other local fields are known as nonarchimedean fields. The topology on a nonarchimedean field is always totally disconnected.

Let $\mathbb{F}_p(t)$ denote the field of fractions of the polynomial ring $\mathbb{F}_p[t]$. A global field is a finite extension of $\mathbb{Q}$, or a finite extension of the function field $\mathbb{F}_p(t)$. A completion $(v,F_v)$ of $F$ is a dense isomorphic embedding $v$ of $F$ into a local field $F_v$. Two completions $(v,F_v), (u,F_u)$ are said to be equivalent if there is an isomorphism $\rho$ of $F_v$ onto $F_u$ such that $u = \rho \circ v$. A place of $F$ is an equivalence class of completions. We say the place $(v,F_v)$ is infinite if $F_v$ is an archimedean field and finite otherwise. If char($F$) = $p > 0$, then $F$ has countably many finite places and no infinite places. If char($F$) = 0 then $F$ has countably many finite places and finitely many (but at least one) infinite places.

Suppose we have an ascending sequence of topological spaces

$$X_1 \subset X_2 \subset X_3 \subset \ldots$$
Then we can give the union $X = \bigcup X_n$ the direct limit topology: that is a set is open in $X$ if and only if it has open intersection with each $X_n$.

The following proposition admits a standard proof in point-set topology.

**Lemma 2.1.** Let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of $T_1$ (=points are closed) topological spaces and give $X = \bigcup_{n=1}^{\infty} X_n$ the direct limit topology. Then any compact subset of $X$ lies entirely within some $X_n$.

Suppose that for each $n = 1, 2, 3, \ldots$ we have a locally compact, second countable and Hausdorff topological group $G_n$, such that $G_m$ is an open subgroup of $G_n$ for $m \leq n$:

$$G_1 \subset G_2 \subset G_3 \subset \ldots$$

Let $G = \bigcup_{n=1}^{\infty} G_n$ and furnish this with the direct limit topology. Then $G$ is a topological group which is locally compact, second countable and Hausdorff.

Any nonarchimedean local field $F_v$ contains a unique maximal compact open subring $O_v$. Let $S$ denote any finite set of places of $F$ which contains all the infinite places. By an adele we mean an element $a = (a)_v$ of the product $\prod_v F_v$ such that $a \in A_S = \prod_{v \in S} F_v \times \prod_{v \notin S} O_v$ for some $S$. The adeles of $F$ form a ring $A_F$, addition and multiplication being defined componentwise. Each $A_S$ has its natural topology and $A_F = \bigcup S A_S$ is topologized as the inductive limit with respect to $S$. There is an obvious embedding of $F$ in $A_F$, by means of which we identify $F$ with a subring of $A_F$. The field $F$ is a discrete cocompact subfield of the non-discrete locally compact semisimple commutative ring $A_F$.

Now suppose $G$ is a linear algebraic group defined over $F$. We shall be interested in the adelic group $G(\mathbb{A}_F)$ of $\mathbb{A}_F$-rational points of $G$. For a finite place $v$ of $F$ let $G(O_v)$ denote the group $G(F_v) \cap GL_n(O_v)$. We set

$$G_S = \prod_{v \in S} G(F_v) \times \prod_{v \notin S} G(O_v)$$

Then $G(\mathbb{A})$ is equal, by definition, to the direct limit of the groups $G_S$. We now equip $G(\mathbb{A})$ with the direct limit topology, following Weil [14, p.2]. Then $G(\mathbb{A})$ is a locally compact second countable Hausdorff group. The map $x \mapsto (x, x, x, \ldots)$ embeds $G(F)$ as a discrete subgroup of $G(\mathbb{A})$.

### 3. Proper actions and universal examples

We recall that a topological space $X$ is completely regular if it satisfies the following separation axiom: If $B$ is a closed subset of $X$ and $p \in$
$X \setminus B$ then there exists a continuous function $f : X \to [0, 1]$ such that $f(p) = 0$ and $f(B) = \{1\}$. Let $G$ be a locally compact, Hausdorff, second countable group. A $G$-{}space is a topological space $X$ with a given continuous action of $G$ such that

- $X$ is completely regular and any compact subset of $X$ is metrizable.
- The quotient space $X/G$ is paracompact and Hausdorff.

Note that the definition of a $G$-{}space in [1] uses the slightly more restrictive conditions that $X$ and $X/G$ be metrizable. Nothing is altered in [1] if this is replaced throughout with the above relaxed conditions.

The following definition may be found in Bourbaki [4, Definition 1, p.250, Prop.7, p.255].

**Definition 3.1.** The action of $G$ on a $G$-{}space $X$ is proper if given any two points $x, y \in X$ there are open neighbourhoods $U_x, U_y$ of $x$ and $y$ respectively such that the set

$$\{g \in G : gU_x \cap U_y \neq \emptyset\}$$

has compact closure in $G$. A proper $G$-{}space $X$ is said to be $G$-compact if the quotient $X/G$ is compact.

In appendix A we give the full statement of Theorem 3.8 in Biller [3]. This theorem, on the existence of slices, reconciles the Bourbaki definition of proper actions with the definition in [1]. In particular, if $X$ is a proper $G$-{}space such that $X$ and $X/G$ are metrizable then $X$ satisfies the condition which was taken in [1] to be the definition of proper. The reverse implication is also valid. If $X$ is a metrizable $G$-{}space which is proper in the sense of [1], then $X$ is a proper $G$-{}space.

Note that if $X$ is locally compact then this is equivalent to the following condition: if $K_1, K_2$ are compact subsets of $X$ then the set

$$\{g \in G : gK_1 \cap K_2 \neq \emptyset\}$$

is compact.

It is easy to see any locally compact Hausdorff group acts properly on itself. Also note if $X$ is a proper $G$-{}space and $Y$ is any $G$-{}invariant subset of $X$ then $Y$ is a proper $G$-{}space when equipped with the subspace topology and obvious action of $G$. We shall want to define $KK$ groups for the algebra $C_0(X)$ of a $G$-compact proper $G$-{}space and the following ensures that these algebras are separable.

**Proposition 3.2.** Suppose $G$ acts properly on $X$ and $X/G$ is compact. Then $X$ is locally compact and second countable.

**Proof.** The space $X$ is locally compact by [4, (e), p.310]. We show that $X = GS$ for some compact $S \subset X$. 

For each \( x \in X \) let \( U_x \) be an open neighbourhood of \( x \) with compact closure. If \( \pi : X \to X/G \) denotes the quotient map then the collection \( \pi(U_x) \) cover \( X/G \). We know \( X/G \) is compact and so get a finite sub-cover \( \pi(U_{x_1}), \ldots, \pi(U_{x_n}) \). Now let \( S = \bigcup_{i=1}^n U_{x_i} \). This is a compact set with \( GS = X \).

By assumption any compact subset of \( X \) is metrizable, and we may deduce that \( S \) is second countable when given the subspace topology from \( X \).

The action of \( G \) on \( X \) gives rise to a map

\[
G \times S \to GS = X,
\]

which is continuous, open and surjective. By assumption \( G \) is second countable and so clearly \( X \) must be second countable. \( \square \)

**Definition 3.3.** Let \( X \) be a proper \( G \)-space. A **cutoff function** on \( X \) is a function \( c : X \to \mathbb{R}_+ \) such that the support of \( c \) has compact intersection with \( GK \) for any compact subset \( K \) of \( X \) and

\[
\int c(g^{-1}x)dg = 1, \quad \text{for each } x \in X
\]

Note the set of all such functions is convex.

**Proposition 3.4.** Let \( X \) be a \( G \)-compact proper \( G \)-space. Then there exists a cutoff function on \( G \).

**Proof.** By Proposition 3.2 \( X \) is locally compact and \( X = GS \) for some compact subset \( S \subseteq X \). Take an open neighbourhood \( V \) of \( S \) with compact closure and let \( f \) be a continuous function from \( X \) into the interval \([0, 1]\) which is equal to 1 on \( S \) and zero on the complement of \( V \), hence compactly supported.

Now for any \( x \in X \) the function

\[
f_x : G \to [0, 1], \quad f_x(g) = f(g^{-1}x)
\]

is continuous. Because \( GS = X \) we may find a \( g \in G \) with \( g^{-1}x \in S \). Recall \( f = 1 \) on \( S \) and so \( f_x(g) = f(g^{-1}x) = 1 \). As \( f_x \) is continuous we can find a neighbourhood of \( g \) on which \( f_x \) is non zero. Thus we may conclude

\[
0 < \int f_x(g)dg
\]

The action of \( G \) on \( X \) is proper so the set

\[
\{ g \in G : g\{x\} \cap supp(f) \neq \emptyset \}
\]
is compact for each \( x \in X \). This implies that \( f_x \) is compactly supported and so
\[
\int f_x(g)dg < \infty.
\]
Thanks to the above we may define
\[
c(x) = \frac{f(x)}{\int f_x(g)dg}
\]
This is compactly supported and has the property that \( G \cdot \text{supp}(c) = X \) and
\[
\int c(g^{-1}x)dg = 1
\]
for each \( x \in X \). Clearly \( c \) is a cutoff function on \( X \). \( \square \)

**Definition 3.5.** Let \( X, Y \) be proper \( G \)-spaces. A continuous map \( \varphi : X \to Y \) is a called a \( G \)-map if
\[
\varphi(gx) = g\varphi(x) \quad \text{for all } g \in G, x \in X
\]
Two \( G \)-maps are \( G \)-homotopic if they are homotopic through \( G \)-maps.

**Definition 3.6.** A universal example for proper actions of \( G \), denoted \( \mathbb{E}G \), is a proper \( G \)-space with the following property: If \( X \) is any proper \( G \)-space, then there exists a \( G \)-map \( f : X \to \mathbb{E}G \), and any two such maps are \( G \)-homotopic.

Let \( K \) be any compact subset of \( G \). The set of all probability measures on \( K \) — denoted \( \mathcal{P}(K) \) — is a separable compact Hausdorff space in the topology induced from the weak* topology on \( C(K) \). Recall \( \mu_i \to \mu \) in the weak* topology if and only if
\[
\int f d\mu_i \to \int f d\mu \quad \text{for any } f \in C(K).
\]
If \( K_1 \subseteq K_2 \) are compact subgroups of \( G \) then define the following map
\[
\iota : \mathcal{P}(K_1) \to \mathcal{P}(K_2), \quad (\iota \mu)(U) = \mu(U \cap K_1) \quad \text{for each Borel set } U \subseteq K_2
\]
This is clearly injective and continuous. So \( \iota \) is an injective map from a compact space to a Hausdorff space and hence gives rise to a homeomorphism between \( \mathcal{P}(K_1) \) and its image in \( \mathcal{P}(K_2) \) with the subspace topology. To simplify notation we identify \( \mathcal{P}(K_1) \) with its image in \( \mathcal{P}(K_2) \).

As \( G \) is locally compact and second countable it is clearly \( \sigma \)-compact and by [3, p.94, Prop. 15 and Cor. 2] we may find compact subsets...
$K_1 \subset K_2 \subset K_3 \subset \ldots$ with $G = \bigcup_{i=1}^{\infty} K_i$ such that any compact subset of $G$ is contained within some $K_i$. Now define

$$P_c(G) = \bigcup_{i=1}^{\infty} P(K_i).$$

Clearly $P_c(G)$ consists of all compactly supported probability measures on $G$. We topologize $P_c(G)$ by giving it the direct limit topology with respect to the sequence $P(K_i)$. This topology is independent of the sequence $K_i$.

The group $G$ acts on $P_c(G)$ by setting

$$(g\mu)(U) = \mu(g^{-1}U)$$

for any Borel set $U \subseteq G$.

**Lemma 3.7.** The action of $G$ on $P_c(G)$ is continuous.

**Proof.** Let $(g_\alpha, \mu_\alpha) \to (g, \mu)$, $\alpha \in A$, be a convergent net in $G \times P_c(G)$. By definition of the direct limit topology on $P_c(G)$, we must have $\mu_\alpha \to \mu$ in $P(K_i)$ for some $i$. Given $f \in C(K_i)$ and $\epsilon > 0$ we can find $\alpha_0$ with the property that

$$\alpha > \alpha_0 \Rightarrow \left| \int f \, d\mu_\alpha - \int f \, d\mu \right| \leq \epsilon/2$$

Furthermore $g_\alpha \to g$ and so we may choose $\alpha_0$ in such a way that we also have

$$\alpha > \alpha_0 \Rightarrow \|f_{g_\alpha} - f_g\| \leq \epsilon/2$$

where $f_g$ denotes the function $x \mapsto f(g^{-1}x)$. Finally note

$$\left| \int f \, d(g_\alpha \mu_\alpha) - \int f \, d(g \mu) \right| = \left| \int f_{g_\alpha} \, d\mu_\alpha - \int f_g \, d\mu \right|$$

$$\leq \left| \int (f_{g_\alpha} - f_g) \, d\mu_\alpha \right| + \left| \int f_g \, d\mu_\alpha - \int f_g \, d\mu \right|$$

$$\leq \epsilon/2 + \epsilon/2$$

This shows that $g_\alpha \mu_\alpha \to g\mu$ in $P(K_i)$ and therefore $g_\alpha \mu_\alpha \to g\mu$ in $P_c(G)$. \qed

**Lemma 3.8.** The action of $G$ on $P_c(G)$ is proper.

**Proof.** Take $\mu \in P_c(G)$ and let $f_\mu$ be a continuous compactly supported function $f_\mu : G \to [0, 1]$ with $f_\mu \equiv 1$ on $\text{supp}(\mu)$.

Then the set

$$U_\mu = \left\{ \lambda \in P_c(G) : \left| \int f_\mu \, d\mu - \int f_\mu \, d\lambda \right| \leq 1/2 \right\}$$

$$= \left\{ \lambda \in P_c(G) : \int f_\mu \, d\lambda > 1/2 \right\}$$

is an open neighbourhood of $\mu$ in $P_c(G)$. \qed
Now take any $\mu_1, \nu_1 \in \mathcal{P}_c(G)$ and assume $gU_\mu \cap U_\nu \neq \emptyset$ for some $g \in G$. Indeed let $\lambda \in gU_\mu \cap U_\nu$. Then

$$\int g^{-1} f_\mu \, d\lambda > 1/2 \quad \text{and} \quad \int f_\nu \, d\lambda > 1/2$$

If $\text{supp} (g^{-1} f_\mu)$ and $\text{supp} (f_\nu)$ are disjoint then we have $0 \leq g^{-1} f_\mu + f_\nu \leq 1$. However by the above

$$\int g^{-1} f_\mu + f_\nu \, d\lambda > 1$$

clearly contradicting the fact that $\lambda$ is a probability measure. Hence we may conclude that $g \text{supp} (\mu_1) \cap \text{supp} (\nu_1) \neq \emptyset$ and so

$$\{ g \in G : gU_\mu \cap U_\nu \neq \emptyset \} \subset \{ g \in G : g \text{supp} (\mu_1) \cap \text{supp} (\nu_1) \neq \emptyset \}.$$ 

Now both $\mu_1$ and $\nu_1$ are compactly supported and since any group acts properly on itself the larger set here is compact. □

**Theorem 3.9.** The space $\mathcal{P}_c(G)$ is a universal example for proper actions of $G$.

**Proof.** Let $X$ be any proper $G$-space, we aim to show there exists a $G$-equivariant map $X \to \mathcal{P}_c(G)$. Take any $x \in X$. By [3, Theorem 3.8] there is a $G$-invariant open neighbourhood $U_x$ of $x$, a compact subgroup $H$ of $G$, and a $G$-equivariant map

$$\rho : U_x \to G/H.$$ 

Let $\mu_H$ denote Haar measure on the compact subgroup $H$, normalized to have total mass 1. There is an obvious $G$-equivariant map

$$\Psi : G/H \to \mathcal{P}_c(G), \quad gH \mapsto g \cdot \mu_H$$

and let

$$\theta_x = \Psi \circ \rho : U_x \to \mathcal{P}_c(G).$$

Now $X$ may be covered by such neighbourhoods and if $\pi$ denotes the quotient map $X \to X/G$ then $\{ \pi(U_x) \}$ is an open cover of $X/G$. Recall that by definition $X/G$ is paracompact and Hausdorff and so there is a locally finite partition of unity subordinate to the cover $\{ \pi(U_x) \}$. Precisely there are continuous maps

$$\omega_x : X/G \to [0, 1], \quad \text{for each } x \in X$$

with $\text{supp} (\omega_x) \subseteq \pi(U_x)$ and for each $[y] \in X/G$ we have

$$\omega_x([y]) = 0 \text{ for almost all } x \in X \text{ and } \sum_{x \in X} \omega_x([y]) = 1.$$
Now
\[ \Xi : X \to \mathcal{P}_c(G), \quad \Xi(y) = \sum_{x \in X} \omega_x(\pi(x))\theta_x(y) \]
is the required map. Note that (as remarked above) \( \mathcal{P}_c(G) \) is a convex set and this convexity is being used in the construction of \( \Xi \).

Finally, if \( \varphi_1, \varphi_2 : X \to \mathcal{P}_c(G) \) are \( G \)-equivariant maps then they are \( G \)-homotopic via
\[ t\varphi_1 + (1 - t)\varphi_2, \quad t \in [0, 1]. \]
So \( \mathcal{P}_c(G) \) is a universal example for \( G \).

**Baum-Connes conjecture with coefficients.** If \( A \) is any \( G \)-C*-algebra then we may define
\[ K^\text{top}_* (G, A) = \lim_{\longrightarrow} K^G_* (C_0(Z), A) \]
for \( G \)-invariant \( G \)-compact \( Z \subseteq E_G \).

We say a group \( G \) satisfies the Baum-Connes conjecture with coefficients if for every \( G \)-C*-algebra \( A \) the map
\[ \mu_A : K^\text{top}_* (G, A) \to K_* (A \rtimes_r G) \]
is an isomorphism.

**4. \( K \)-theory for ascending unions of groups**

**Theorem 4.1.** Let \( H \) be an open subgroup of \( G \). Then the inclusion of \( H \) in \( G \) determines a homomorphism of abelian groups
\[ \mathcal{T}_H^G : K_* (A \rtimes_r H) \to K_* (A \rtimes_r G). \]
Furthermore suppose \( G_1 \subset G_2 \subset G_3 \subset \ldots \) is an ascending sequence of open subgroups, then there is an inductive system of abelian groups
\[ K_* (A \rtimes_r G_1) \xrightarrow{\mathcal{T}_{G_1}^{G_2}} K_* (A \rtimes_r G_2) \xrightarrow{\mathcal{T}_{G_2}^{G_3}} K_* (A \rtimes_r G_3) \xrightarrow{\mathcal{T}_{G_3}^{G_4}} \ldots \]
If \( G = \cup G_n \) we have
\[ K_* (A \rtimes_r G) = \lim_{\longrightarrow} K_* (A \rtimes_r G_n). \]

**Proof.** Any open subgroup \( H \) of \( G \) is also closed. Let \( H \) be an open subgroup of \( G \) and fix a Haar measure \( dg \) on \( G \). Since \( H \) is an open subgroup of \( G \), the restriction of \( dg \) to \( H \) is a Haar measure on \( H \). To simplify notation we write \( dg \) to signify this choice of Haar measure on both \( G \) and \( H \).

For \( f \in C_c(H, A) \) define
\[ \iota f : G \to A, \quad (\iota f)(x) = \begin{cases} f(x) & \text{if } x \in H \\ 0 & \text{otherwise.} \end{cases} \]

Then \( \iota \) defines a map from \( C_c(H, A) \) into \( C_c(G, A) \) as \( H \) is clopen in \( G \). \( \iota \) is clearly additive and furthermore for \( f, g \in C_c(H, A) \)

\[
\iota f \ast \iota g(x) = \int_G \iota f(y) \alpha_y (\iota g(y^{-1} x)) dy \\
= \int_H f(y) \alpha_y (\iota g(y^{-1} x)) dy \\
= \begin{cases} \int_H f(y) \alpha_y (g(y^{-1} x)) dy & \text{if } x \in H \\ 0 & \text{otherwise.} \end{cases} \\
= \iota(f \ast g)(x).
\]

Also for \( f \in C_c(H, A) \)

\[
(\iota f)^*(x) = \begin{cases} \Delta_G(x^{-1}) \alpha_x (f(x^{-1})^*) & \text{if } x \in H \\ 0 & \text{otherwise} \end{cases} = (\iota f^*)(x)
\]

This follows from the observation that \( \Delta_H(x) = \Delta_G(x) \) for \( x \in H \).

We now follow [6, II.C, p.172]. Let \( \| \cdot \|_{A \rtimes_r G} \) signify the norm obtained on \( C_c(G, A) \) from the left regular representation on the \( C^* \)-module

\[ E = L^2(G, dg) \otimes A. \]

We shall also need the \( C^* \)-module

\[ F = L^2(H, dg) \otimes A. \]

If \( f \in C_c(H, A) \) then

\[
\| \iota f \|_{A \rtimes_r G} = \sup \{ \| \iota f \ast \xi \|_E : \xi \in E, \| \xi \|_E \leq 1 \} \\
= \sup \{ \| f \ast \xi \|_F : \xi \in F, \| \xi \|_F \leq 1 \}
\]

So \( \iota \) is an isometric \(*\)-algebra map from \( C_c(H, A) \) to \( C_c(G, A) \). We can then complete this map to obtain an injective \( C^* \)-algebra morphism

\[ \iota : A \rtimes_r H \to A \rtimes_r G. \]

The map \( T^G_H \) is then defined by functoriality:

\[ T^G_H = \iota_* : K_* (A \rtimes_r H) \to K_* (A \rtimes_r G). \]

Now let \( G_1 \subset G_2 \subset G_3 \subset \ldots \) be an open ascending sequence of groups and let \( G = \bigcup G_n \). Fixing a Haar measure on \( G \) prescribes a Haar measure on each \( G_n \). For any \( m < n \), \( G_m \) is an open subgroup of \( G_n \) and so by the above we may define a map

\[ \iota^m_n : A \rtimes_r G_m \to A \rtimes_r G_n. \]
Indeed we may form an inductive system of \( C^* \)-algebras \( (A \rtimes_r G_n, \iota_n^m) \),

\[
A \rtimes_r G_1 \xrightarrow{\iota_1^2} A \rtimes_r G_2 \xrightarrow{\iota_2^3} A \rtimes_r G_3 \xrightarrow{\iota_3^4} \ldots
\]

We aim to show that the \( C^* \)-inductive limit \( \lim_{\rightarrow} A \rtimes_r G_n \) of this inductive system is equal to the algebra \( A \rtimes_r G \). For any \( n \), \( G_n \) is an open subgroup of \( G \) and let \( \iota_n : A \rtimes_r G_n \to A \rtimes_r G \) be defined as above. Since each \( \iota_n \) is injective, all that remains to be demonstrated is that

\[
\bigcup_n \iota_n(A \rtimes_r G_n)
\]

is dense in \( A \rtimes_r G \).

Taking \( \varepsilon > 0 \) and \( x \in A \rtimes_r G \) we can find \( f \in C_c(G, A) \) with \( \|f - x\|_{A \rtimes_r G} < \varepsilon \). The support of \( f \) is compact and by Lemma 2.1 must lie totally within some \( G_n \). Hence \( f \in C_c(G_n, A) \) for some \( n \) and so \( \bigcup_n \iota_n(A \rtimes_r G_n) \) is dense in \( A \rtimes_r G \). Hence \( A \rtimes_r G \) is the \( C^* \)-inductive limit of the \( C^* \)-algebras \( A \rtimes_r G_n \) and we therefore have

\[
K_*(A \rtimes_r G) = \lim_{\rightarrow} K_*(A \rtimes_r G_n).
\]

\( \square \)

5. Equivariant \( K \)-homology for ascending sequences of groups

**Theorem 5.1.** Let \( H \) be an open subgroup of \( G \). Then the inclusion of \( H \) in \( G \) determines a homomorphism of abelian groups

\[
\mathcal{R}_H^G : K^\text{top}_*(H, A) \to K^\text{top}_*(G, A).
\]

Furthermore suppose \( G_1 \subset G_2 \subset G_3 \subset \ldots \) is an open ascending sequence of groups, then there is an inductive system of abelian groups

\[
K^\text{top}_*(G_1, A) \xrightarrow{\mathcal{R}_{G_1}^{G_2}} K^\text{top}_*(G_2, A) \xrightarrow{\mathcal{R}_{G_2}^{G_3}} K^\text{top}_*(G_3, A) \xrightarrow{\mathcal{R}_{G_3}^{G_4}} \ldots
\]

If \( G = \bigcup G_n \) then we have

\[
K^\text{top}_*(G, A) = \lim_{\rightarrow} K^\text{top}_*(G_n, A)
\]

In the course of proving this result, we shall make use of the reciprocity isomorphism \([5, \text{p.157}]\) in equivariant KK-theory. Let \( H \) be an open subgroup of \( G \), let \( A \) be an \( H - C^* \)-algebra, let \( \text{Ind}_H^G A \) denote the induced algebra and let \( B \) be a \( G - C^* \)-algebra. Then we have the reciprocity isomorphism:

\[
\text{Inf}_H^G : KK_*(H, A) \cong KK_*(\text{Ind}_H^G A, B).
\]
Lemma 5.2. Let $H$ be an open subgroup of $G$ and let $X$ be any $H$-compact subset of $\mathcal{P}_c(H) \subset \mathcal{P}_c(G)$ then $G \times_H X \cong G \cdot X$ as $G$-spaces.

Proof. Recall the definition of the space $G \times_H X$. The group $H$ acts on the product $G \times X$ by setting $h \cdot (g, x) = (gh^{-1}, hx)$ and $G \times_H X$ is the quotient. The action of $G$ on $G \times_H X$ is given by $g' \cdot [g, x] = [g'g, x]$, where $[g, x]$ is the equivalence class of the pair $(g, x)$.

The map $F_X$ is defined as follows:

$$F_X : G \times_H X \rightarrow G \cdot X, \; [g, x] \mapsto gx.$$  

The map $F_X$ is clearly surjective and $G$-equivariant. To show this map is injective take $[g_1, x_1], [g_2, x_2] \in G \times_H X$ with $g_1 x_1 = g_2 x_2$. Recall that $x_1, x_2$ are in fact probability measures on $H$ and so

$$1 = x_1(H) = g_1^{-1}g_2 x_2(H) = x_2(g_2^{-1}g_1 H).$$

As $x_2$ is a measure in $\mathcal{P}_c(H)$ it will clearly be equal to zero on the coset $g_2^{-1}g_1 H$ unless $g_2^{-1}g_1 \in H$. Now $g_2^{-1}g_1 \cdot (g_1, x_1) = (g_1 g_1^{-1}g_2, g_2^{-1}g_1 x_1) = (g_2, x_2)$ as required.

It now remains to show this map is a homeomorphism. Let $\pi$ denote the quotient map $G \times X \rightarrow G \times_H X$ and $\theta : G \times X \rightarrow G \cdot X$ denote the map given by the action of $G$ on $X$. Both $\pi$ and $\theta$ are open, continuous maps. Since $\theta = F_X \circ \pi$, this implies that $F_X$ is open and continuous. \hfill $\square$

Let $\mathcal{R}^G_{H,X}$ denote the composition

$$KK_*^H(C_0(X), A) \xrightarrow{\Inf_X} KK_*^G(C_0(G \times_H X), A) \xrightarrow{(F_X)_*} KK_*^G(C_0(G \cdot X), A)$$

If $X$ and $Y$ are $H$-compact subsets of $\mathcal{P}_c(H)$ with $X \subset Y$ then the following diagram commutes,

$$
\begin{array}{ccc}
KK_*^H(C_0(X), A) & \longrightarrow & KK_*^H(C_0(Y), A) \\
\mathcal{R}^G_{H,X} & \downarrow & \mathcal{R}^G_{H,Y} \\
KK_*^G(C_0(G \cdot X), A) & \longrightarrow & KK_*^G(C_0(G \cdot Y), A)
\end{array}
$$

where each horizontal map is given by inclusion of the spaces involved.
Now $\mathcal{P}_c(H), \mathcal{P}_c(G)$ are universal examples for $H, G$ respectively. If $X$ is an $H$-compact subset of $\mathcal{P}_c(H)$ then $G \cdot X$ is a $G$-compact subset of $\mathcal{P}_c(G)$. Due to the above the following map is well defined

$$R^G_H : K^\text{top}_*(H, A) \to K^\text{top}_*(G, A).$$

Now let $G_1 \subset G_2 \subset G_3 \subset \ldots$ be an ascending sequence of open subgroups and let $G = \bigcup G_n$. There is then an inductive system of abelian groups

$$K^\text{top}_*(G_1, A) \xrightarrow{R^G_{G_1}} K^\text{top}_*(G_2, A) \xrightarrow{R^G_{G_2}} K^\text{top}_*(G_3, A) \xrightarrow{R^G_{G_3}} \ldots$$

and the maps

$$R^G_{G_n} : K^\text{top}_*(G_n, A) \to K^\text{top}_*(G, A).$$

**Lemma 5.3.** There exist compact sets $\Delta_1 \subset \Delta_2 \subset \Delta_3 \subset \ldots$ such that $\Delta_n \subset G_n$ and $\bigcup \text{Interior}(\Delta_n) = G$. Set $Z^{n,m} = G_n \cdot P_c(G_n \cap \Delta_m)$. Then

- $Z^{n,m} \subset P_c(G_n)$
- $Z^{n,m}$ is preserved by $G_n$ and is $G_n$-compact
- $Z^{n,m} \subset Z^{n,m+1}$
- $\bigcup_m Z^{n,m} = P_c(G_n)$
- $G_{n+1} \cdot Z^{n,m} \subset Z^{n+1,m}$

**Proof.** Let $W_1, W_2, W_3, \ldots$ be a countable basis for the topology of $G$. From this list $W_1, W_2, W_3, \ldots$ delete all $W_j$ such that $\overline{W_j}$ is not compact. Call the remaining list $V_1, V_2, V_3, \ldots$. Let $U$ be any non-empty open set in $G$ and choose $p \in U$. Since $G$ is locally compact there exists an open set $\Lambda$ in $Y$ with $p \in \Lambda$ and $\overline{\Lambda}$ compact. Consider $\Lambda \cap U$. Now $\Lambda \cap U$ is a union of $W_j$'s. But any $W_j$ which is contained in $\Lambda \cap U$ must in fact be a $V_k$ because $\Lambda \cap U$ has compact closure. $\Lambda \cap U$ is a union of $V_k$'s (now let $p$ vary throughout $U$).

Now let $\Sigma_n = \overline{V_1 \cup V_2 \cup \cdots \cup V_n}$ and $\Delta_n = G_n \cap \Sigma_n$.

Any compact set in $G$ is contained in some $\Delta_n$.

□

**Lemma 5.4.** Let $\{A^{m,n}\}$ be a commutative diagram of abelian groups in which the typical commutative square is

$$\begin{array}{ccc}
A^{n,m} & \longrightarrow & A^{n,m+1} \\
\downarrow & & \downarrow \\
A^{n+1,m} & \longrightarrow & A^{n+1,m+1}
\end{array}$$
with $n, m = 1, 2, 3, \ldots$. Then there is a canonical isomorphism of abelian groups:

$$\lim_{m \to \infty}(\lim_{n \to \infty} A^{m,n}) \cong \lim_{n \to \infty}(\lim_{m \to \infty} A^{m,n}).$$

**Proof.** Each side is canonically isomorphic to the direct limit $\lim \to A^{n,n}$ of the directed system $\{A^{n,n}\}$. \hfill \square

**Theorem 5.5.** $\lim_{n \to \infty} K^{{\text{top}}}_j(G_n, A) = K^{{\text{top}}}_j(G, A)$.

**Proof.** Consider the commutative diagram of abelian groups in which the typical commutative square is:

$$\begin{array}{ccc}
K^G_j(Z^{n,m}, A) & \longrightarrow & K^G_j(Z^{n,m+1}, A) \\
\rho_n \downarrow & & \rho_n \downarrow \\
K^G_{j+1}(Z^{n+1,m}, A) & \longrightarrow & K^G_{j+1}(Z^{n+1,m+1}, A)
\end{array}$$

with $n, m = 1, 2, 3, \ldots$. Each horizontal arrow

$$K^G_j(Z^{n,m}, A) \to K^G_j(Z^{n,m+1}, A)$$

is the map of abelian groups determined by the inclusion $Z^{n,m} \to Z^{n,m+1}$. Each vertical map $\rho_n : K^G_j(Z^{n,m}, A) \to K^G_{j+1}(Z^{n+1,m}, A)$ is the map $R^G_{G_n}$ followed by the map of abelian groups induced by the inclusion $G_{n+1} \cdot Z^{n,m} \to Z^{n+1,m}$.

We will write

$$A^{n,m} = K^G_j(Z^{n,m}, A).$$

Each $Z^{n,m}$ is $G_n$-compact, $\cup_m Z^{n,m} = P_c(G_n)$ and any $G_n$-compact set in $P_c(G_n)$ is contained in some $Z^{n,m}$. Taking the direct limit along the $n$th row, we have

$$\lim_{m \to \infty} A^{m,n} = K^{{\text{top}}}_j(G_n, A).$$

If we now take the direct limit in a vertically downward direction, we obtain

$$\lim_{n \to \infty} K^{{\text{top}}}_j(G_n, A).$$

Now we fix attention on the $m$th column. If $n \geq m$ then $G_n \supset \Delta_m$ and so $G_n \cap \Delta_m = \Delta_m$. Then we have

$$A^{n,m} = K^G_j(G_n \cdot P_c(\Delta_m), A).$$

Now the $G_n$-saturation of $P_c(\Delta_m)$ is equal to the $G$-saturation of $P_c(\Delta_m)$. Now we apply the reciprocity isomorphism and we have

$$A^{n,m} \cong K^G_j(G \cdot P_c(\Delta_m), A).$$
if \( n \geq m \). Therefore the \( m \)th column stabilizes as soon as \( n \geq m \). Therefore the direct limit down the \( m \)th column is given by

\[
\lim_{n \to \infty} A_{m,n} = K^G_j(G \cdot P_c(\Delta_m), A).
\]

Now the sets \( G \cdot P_c(\Delta_m) \) are cofinal in \( G \)-compact sets in \( P_c(G) \). Taking the direct limit in the horizontal direction, we have

\[
\lim_{m \to \infty} \lim_{n \to \infty} A_{m,n} \sim K_{top}^j(G_n, A) \sim K_{top}^j(G, A).
\]

By Lemma 5.4 we have

\[
\lim_{n \to \infty} K_{top}^j(G_n, A) \sim K_{top}^j(G, A).
\]

\[\square\]

6. Adelic groups

For \( H \) an open subgroup of \( G \) we have constructed a homomorphism from \( K_*^{top}(H, A) \) to \( K_*^{top}(G, A) \), and likewise for the \( K \) theory of the reduced crossed product \( C^* \)-algebras. We wish to check that these maps are compatible with the Baum–Connes \( \mu \) map, i.e. that the following diagram commutes.

\[
\begin{array}{ccc}
K_*^{top}(H, A) & \xrightarrow{\mu_H} & K_*(A \rtimes_r H) \\
\parallel & \downarrow{\tau^G_H} & \parallel \\
K_*^{top}(G, A) & \xrightarrow{\mu_G} & K_*(A \rtimes_r G)
\end{array}
\]

As a first step, we prove that the reciprocity isomorphism \( \text{Inf}_X \) is compatible with the Baum-Connes \( \mu \) map.

**Lemma 6.1.** Let \( A \) be a \( G - C^* \)-algebra, let \( H \) be an open subgroup of \( G \) and let \( X \) be a locally compact proper \( H \)-compact \( H \)-space. Then the following diagram commutes:

\[
\begin{array}{ccc}
KK^H_*(C_0(X), A) & \xrightarrow{\mu_H} & K_*(A \rtimes_r H) \\
\downarrow{\text{Inf}_X} & \parallel & \downarrow{\tau^G_H} \\
KK^G_*(C_0(G \times_H X), A) & \xrightarrow{\mu_G} & K_*(A \rtimes_r G)
\end{array}
\]

**Proof.** The inverse of the reciprocity isomorphism \( \text{Inf} \) is the compression isomorphism [3, p. 157] and is given by the composition \( i_* \circ \text{Res}_X \), where \( \text{Res}_X \) is the obvious restriction map \( KK_*^G(C_0(X), A) \to KK_*^H(C_0(X), A) \) and \( i \) is the inclusion \( C_0(X) \hookrightarrow C_0(G \times_H X) \) given by

\[
i(f)[g, x] = \begin{cases} f(gx) & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases}
\]

Each of these maps is clearly functorial.
There is a commutative diagram

\[
\begin{array}{ccc}
KK^*_H(C_0(X), A) & \xrightarrow{j_H} & KK^*_s(C_0(X) \rtimes_r H, A \rtimes_r H) \\
i_* & & i_* \\
KK^*_H(C_0(G \times_H X), A) & \xrightarrow{j_H} & KK^*_s(C_0(G \times_H X) \rtimes_r H, A \rtimes_r H) \\
\text{Res}_X & & p \circ q_* \\
KK^*_F(C_0(G \times_H X), A) & \xrightarrow{j_G} & KK^*_s(C_0(G \times_H X) \rtimes_r G, A \rtimes_r G)
\end{array}
\]

in which \(j_H, j_G\) are descent homomorphisms. Here \(p\) denotes the map \(p : C_0(G \times_H X) \rtimes_r G \to C_0(G \times_H X) \rtimes_r H\) induced from the obvious restriction map \(C_c(G, A) \to C_c(H, A)\), with \(A = C_0(G \times_H X)\). And \(q\) denotes the map \(q : A \rtimes_r H \to A \rtimes_r G\)

of Theorem 4.1.

Now let \(c\) be a cutoff function on the proper \(H\)-space \(X\), we claim \(i(c) \in C_0(G \times_H X)\) is a cutoff function on the proper \(G\)-space \(G \times_H X\). To see this take any \([g_0, x] \in G \times_H X\). Then by definition we have

\[i(c)([g_0, x]) = 0\]

unless \(g_0 \in H\) in which case

\[i(c)[g_0, x] = c(g_0x)\]

Up to a normalizing factor between the Haar measures on \(H\) and \(G\)

\[
\int_G i(c)(g^{-1}[g_0, x]) \, dg = \int_G i(c)(g^{-1}g_0^{-1}[g_0, x]) \, dg
\]

\[= \int_G i(c)([g^{-1}, x]) \, dg
\]

\[= \int_H c(h^{-1}x) \, dh = 1.
\]

If \(K\) is any compact subset of \(G \times_H X\) and if \(F\) denotes the homeomorphism between \(G \times_H X\) and \(G \cdot X\) then \(F(GK) = G \cdot F(K)\) and \(F(K)\) is compact. Also note \(F(\text{supp} \, (i(c))) \subseteq H \cdot \text{supp} \, (c)\) and so

\[F(GK \cap \text{supp} \, (i(c))) \subseteq G \cdot F(K) \cap H \cdot \text{supp} \, (c) \subseteq H \cdot F(K) \cap \text{supp} \, (c)\]

which is compact and so \(GK \cap \text{supp} \, (i(c))\) is compact. So we have shown \(i(c)\) is a cutoff function on \(G \times_H X\).

Let \(\lambda_X\) denote the projection in the twisted convolution algebra \(C_c(H \times X)\) arising from the cutoff function \(c\):
\[ \lambda_Z(g, x) = c_Z(x)^{1/2}c_Z(g^{-1}x)^{1/2} \Delta(g)^{-1/2} \]

and let \( \lambda_{G \times H \times X} \) denote the projection in \( C_\ell(G \times G \times H \times X) \) arising from the cutoff function \( i(c) \).

Then \( p(\lambda_{G \times H \times X}) \) is simply the restriction of \( \lambda_{G \times H \times X} \) to \( H \times G \times H \times X \), and for any \( h \) in \( G \) and any \( [g, x] \in G \times H \times X \)

\[
p(\lambda_{G \times H \times X})(h, [g, x]) = i(c)[g, x]^{1/2}i(c)(h^{-1}[g, x])^{1/2} \Delta_G(h)^{-1/2} \]

\[
= \begin{cases} 
  c(gx)^{1/2}c(h^{-1}gx) \Delta_H(h)^{-1/2} & \text{if } g \in H \\
  0 & \text{otherwise.} 
\end{cases}
\]

So the following diagram commutes

\[
\begin{array}{ccc}
KK_*(C_0(X) \rtimes_r H, A \rtimes_r H) & \xrightarrow{[\lambda_X] \otimes} & KK_*(\mathbb{C}, A \rtimes_r H) \\
\uparrow \iota_* & & \downarrow |
\end{array}
\]

\[
\begin{array}{ccc}
KK_*(C_0(G \rtimes H \times X) \rtimes_r H, A \rtimes_r H) & \xrightarrow{\iota'_*([\lambda_{X}]) \otimes} & KK_*(\mathbb{C}, A \rtimes_r H) \\
p \circ q_* & & q_*
\end{array}
\]

\[
\begin{array}{ccc}
KK_*(C_0(G \rtimes H \times X) \rtimes_r G, A \rtimes_r G) & \xrightarrow{[\lambda_{G \rtimes H \times X}] \otimes} & KK_*(\mathbb{C}, A \rtimes_r G)
\end{array}
\]

We finish the proof by splicing together these two diagrams. \( \Box \)

**Lemma 6.2.** Let \( X \) be an \( H \)-compact subset of \( P_c(H) \). Then the following diagram commutes:

\[
\begin{array}{ccc}
KK_*^H(C_0(X), A) & \xrightarrow{\mu_H} & K_*(A \rtimes_r H) \\
\tau_{H, X}^G & & \tau_H^G
\end{array}
\]

**Proof.** By Lemma 3.2 we know that the induced space \( G \times H \times X \) is \( G \)-homeomorphic to the \( G \)-saturation \( G \cdot X \). We now apply Lemma 1.1. \( \Box \)

**Theorem 6.3.** Let \( G \) be a locally compact, second countable Hausdorff topological group and let \( A \) be a \( G \)–\( C^* \)-algebra. Let \( G \) be the union of open subgroups \( G_n \) such that the Baum-Connes conjecture with coefficients \( A \) is true for each \( G_n \). Then the Baum-Connes conjecture with coefficients \( A \) is true for \( G \).
Proof. We start with the commutative diagram in Lemma 6.2 and take the direct limit over all $H$-compact subsets of $P_c(H)$. We then obtain the commutative diagram

$$
\begin{array}{ccc}
K^\top_*(H, A) & \xrightarrow{\mu_H} & K_*(A \rtimes_r H) \\
\pi_H^G & & \downarrow \tau_H^G \\
K^\top_*(G, A) & \xrightarrow{\mu_G} & K_*(A \rtimes_r G)
\end{array}
$$

If $G$ is the union of open subgroups $G_i$ each of which satisfies $BCC$, then applying Theorem 4.1 and Theorem 5.1 along with the above commutative diagram is enough to show that $G$ satisfies $BCC$. □

**Theorem 6.4.** Let $F$ be a global field, $\mathfrak{A}$ its ring of adeles, $G$ a linear algebraic group over $F$. Let $F_v$ denote a place of $F$. If $BCC$ is true for each local group $G(F_v)$ then $BCC$ is true for the adelic group $G(\mathfrak{A})$.

Proof. Let $v_1, v_2, v_3, \ldots$ be an ordering of the finite places of $F$, let $S_\infty$ denote the finite set of all infinite places of $F$, and let $S(n) = \{v_1, v_2, \ldots, v_n\} \cup S_\infty$. Let $G_n = G_{S(n)}$ in the notation of section 2.

If $\Gamma$ is a compact group then $\mathbb{E}\Gamma$ is a point, and we have

$$K^\top_*(\Gamma, B) \cong KK^\top_*(B) \cong K_*(B \rtimes \Gamma)$$

by the Green-Julg theorem. Therefore $BCC$ is true for any compact group. Now $G_n$ is a product of finitely many local groups and one compact group. But $BCC$ is stable under the direct product of finitely many groups, by an important result of Chabert-Echterhoff [5, Theorem 3.17]. Therefore $BCC$ is true for each open subgroup $G_n$. Now the adelic group $G(\mathfrak{A})$ is the ascending union of the open subgroups $G_n$, therefore $BCC$ is true for $G(\mathfrak{A})$, by Theorem 6.3. □

**Appendix A. Biller’s theorem**

We give here the full statement of Theorem 3.8 in Biller [3]. The stabilizer of $x \in X$ is denoted $G_x$.

**Theorem (Existence of slices).** Let $G$ be a locally compact group acting properly on a completely regular space $X$, and choose $x \in X$. Then there is a convergent filter basis $\mathcal{N}$ that consists of compact subgroups of $G$ normalized by $G_x$ such that for every $N \in \mathcal{N}$, the coset space $G/G_xN$ is a manifold and $x$ is contained in a $G_xN$-slice for the action of $G$ on $X$. In particular, some neighbourhood of the orbit $G \cdot x$ is a locally trivial fibre bundle over the manifold $G/G_xN$.

The dimension of $G \cdot x$ is infinite if and only if $N \in \mathcal{N}$ may be chosen such that the dimension of $G/G_xN$ is arbitrarily high. If the
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dimension of \( G \cdot x \) is finite, then \( N \in \mathcal{N} \) may be chosen in such a way that \( \dim G/G_x N = \dim G \cdot x \).

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