Some Applications of the Overlap Formalism.

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The massless vector Schwinger model with $N_f = 1, 2, 3, 4$ number of flavors is studied on the lattice using the overlap formalism. A full Monte Carlo simulation yields values for the bilinear fermion condensate that are in agreement with the exact solution of the continuum Schwinger model with an added Thirring interaction.

The solution to the massless vector Schwinger model is known in the continuum. There the role of topology is clear: In the single flavor case, $\langle \Psi \Psi \rangle = 0$ if the gauge field has topological charge $q \neq \pm 1$ and $\langle \Psi \Psi \rangle \neq 0$ if $q = \pm 1$. No previous lattice regularization of chiral fermions could reproduce this result but the overlap formalism \cite{1} can. Here we present the results of a MC simulation of the massless vector Schwinger model on the lattice for $N_f = 1, 2, 3, 4$ flavors in the overlap formalism. Preliminary results were reported in \cite{2}.

The continuum action in Euclidean space is:

$$S[\varphi, \psi, A_\mu] = \frac{1}{2g^2} \int (\epsilon_{\mu\nu} \partial_\mu A_\nu)^2 - \int \bar{\psi}_L \sigma_\mu (\partial_\mu + i A_\mu) \psi_L - \int \bar{\psi}_R \sigma_\mu (\partial_\mu + i A_\mu) \psi_R$$

(1)

where $\mu, \nu = 1, 2$, $\sigma_1 = 1$ and $\sigma_2 = i$. $\varphi_{R,L}, \psi_{R,L}$ are Grassmann variables and $A_\mu$ is a $U(1)$ gauge field. The model is defined on a torus of fixed physical size $l \times l$ with periodic boundary conditions for the electric field, $E = \epsilon_{\mu\nu} \partial_\mu A_\nu$, and the fermions. The topological charge $q$ is an integer given by $q = \frac{1}{2\pi} \int E$. The expectation value of the fermion bilinear $\langle \bar{\Psi} \Psi \rangle$ is

$$\frac{\int [d\bar{\psi}d\psi dA_\mu] (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) e^{-S(\bar{\psi}, \psi, A_\mu)}}{\int [d\bar{\psi}d\psi dA_\mu] e^{-S(\bar{\psi}, \psi, A_\mu)}}$$

$$\equiv \frac{N}{D}$$

(2)

$N$ has contributions only when $q = \pm 1$ and $D$ has a contribution only when $q = 0$.

The continuum torus is replaced by a toroidal square lattice with $L$ sites in both directions. The lattice spacing $a$ is $l/L$. Gauge fields $U_\mu(x) = e^{iaA_\mu(x)}$ are associated with the links. In the overlap formalism, $N$ and $D$ are expressed as follows:

$$N = \frac{1}{L^2} \sum_{x,\alpha} \int \prod dU_\mu(x) e^{-S_e(U)}$$

$$[|v(L) - |a_{x,\alpha}|L+\rangle |U| + |v(L) - |a_{x,\alpha}|L+\rangle |U|^2]$$

(3)

$$D = \int \prod dU_\mu(x) e^{-S_e(U)} |v(L) - |L+\rangle |U|$$

(4)

The $a_{x,\alpha}$-term in $N$ will differ from zero only for gauge fields carrying topological charge $+1$ and is the overlap expression for $|\bar{\psi}_R \psi_L\rangle$ while the $a_{x,\alpha}$-term contributes only for charge $-1$ and represents $|\bar{\psi}_L \psi_R\rangle$. In our simulations we could have looked at only one of these terms; we took both in order to increase statistics. It is easy to prove that the two overlap matrix elements have exactly equal magnitudes. The CP violating angle $\theta$ is set to zero. The states $|L\pm\rangle$ are the ground states of many body Hamiltonians

$$H^\pm = \sum_{x,\alpha,\beta} a^\dagger_{x,\alpha,\beta} H^{\pm}(x, \alpha, \beta; U) a_{x,\beta}$$

$$\{a^\dagger_{x,\alpha,\beta}, a_{y,\mu,\nu}\} = \delta_{\alpha,\beta} \delta_{x,y}$$

(5)

$$\alpha, \beta = 1, 2, x = (x_1, x_2), x_\mu = 0, 1, \ldots, L - 1.$$
are given in [1]. We say that the topological charge \( q \) equals \( \pm 1 \) when the number of single particle states filled in \( [L+]_U \) and \( [L-]_U \) differ by \( \pm 1 \).

The gauge action appearing in the formulae for \( N \) and \( D \) is a single plaquette action:

\[
S_p(U) = \sum_p s_p(U_p),
\]

where \( U_p \) is the oriented product of the four link elements making up the plaquette \( p \). With such an action for a pure gauge theory the plaquettes become almost independent variables. To minimize the finite lattice spacing effects arising from \( S_p(U) \), we chose the "heat-kernel" form [3]:

\[
e^{-s_p(U_p)} = \sum_{m \in \mathbb{Z}} e^{-\frac{\mu^2}{2} |U_p|^m}
\]

This choice assures exact scaling of the string tension with the coupling \( g \) in the pure gauge theory. Other choices with similar properties are possible.

Conventional MC techniques would generate a gauge field distribution corresponding to the integral in \( D \) (c.f. (4)) and thus compute the fermion bilinear in (2). But this will not work here because the set of gauge fields that contribute to the integral in (3) \( q = \pm 1 \) are completely disconnected from the set of gauge fields that contribute to the integral in (4) \( q = 0 \). Since the overlap formalism maintains this separation of gauge fields into topological classes even on a finite lattice we cannot use the conventional MC techniques. Of course, it is precisely the ability of the overlap formalism to maintain this property that enables us to measure the fermion bilinear on a finite lattice directly. We compute \( N/D \) in the following manner: Measure \( N \) and \( D \) separately, as observables in a pure gauge theory, and calculate \( \langle \overline{\Psi} \Psi \rangle \) as

\[
\langle \overline{\Psi} \Psi \rangle = \frac{N}{D} = \frac{N}{Z} = \frac{N'}{D'} = \int [dU] e^{-S_p(U)}
\]

It is easy to generate statistically independent configurations for a single plaquette action: We choose a gauge and draw \( L^2 - 1 \) independent plaquette variables \( U_p \), each with probability \( e^{-s_p(U_p)} \). This fixes \( L^2 - 1 \) out of the \( L^2 + 1 \) link variables. The last two link variables represent Polyakov loops winding around the torus and since they do not affect the action we draw their values at random. The last plaquette is now fixed and comes with weight \( e^{-s_p(U_{\text{last}})} \). The whole configuration is kept with probability \( e^{-s_p(U_{\text{last}})} \).

Since the quantity we measure is strictly gauge invariant on a finite lattice any choice of the gauge will give the same result.

Because we treat the overlap factors as observables in a pure gauge theory we must choose our parameters so that these factors do not fluctuate too rapidly. Since it has been shown [1] that for smooth gauge fields the overlap agrees with the continuum we use the continuum result of [4] to estimate the parameters of our simulation.

We find that typical contributions are \( \sim e^{-\frac{\mu l^2}{4\pi^2}} \), where \( \mu = \frac{1}{g^2} \) is the photon mass, and therefore, \( 1 \leq \mu l \leq 3 \) is acceptable.

The simulation is done in double precision. The single particle hermitian hamiltonians \( H^k \) are diagonalized using the Jacobi algorithm (scales as \( L^8 \)). The determinants are computed using LU decomposition (also scales as \( L^8 \)). The Jacobi procedure has a larger prefactor and computer time for the whole computation scales a little worse than \( L^6 \) for \( L \approx 4 - 10 \) rather than the asymptotic \( L^8 \) (the last factor of \( L^2 \) comes because of the \( x \) summation in (3)).

We present our results in terms of the dimensionless quantity

\[
f_L(\mu l) = \frac{\langle \overline{\Psi} \Psi \rangle}{\mu} = \frac{N'}{\mu D'}
\]

Results for \( \mu l = 3 \) have appeared in [2]. Within statistical errors (five percent) we found an agreement with the continuum result of [4] for \( L > 6 \).

The error in [2] is dominated by the error in generating the correct distribution of the various topological sectors. We can isolate this contribution to the error by first rewriting (8) as

\[
\langle \overline{\Psi} \Psi \rangle = \frac{N'}{D'} = R \frac{N''}{D''}
\]

where \( N'' \) is the same as \( N' \) but evaluated over the restricted set of gauge field configurations with \( q = \pm 1 \), and \( D'' \) is the same as \( D' \) but
evaluated over the restricted set of gauge field configurations with \( q = 0 \). \( R \) is the ratio of the number of gauge field configurations with \( q = \pm 1 \) divided by the number of gauge field configurations with \( q = 0 \). Until now we have been using the overlap definition of the topological charge. With this definition a separate evaluation of \( R \) would be costly. However, we have found that the geometrical definition \( q = \sum_x \phi_x / \pi, -\pi < \phi_x < \pi, U_x = e^{i\phi_x} \) agrees with the overlap definition better than 99\% for lattices with \( L > 6 \) if we reduce the value of \( \mu l \) focusing on systems of smaller physical size, but with a larger ultraviolet cutoff. We can therefore evaluate \( R \) cheaply using the geometric definition or simply take it to have its continuum value from [4]. We choose the latter approach and this reduces the errors significantly. The \( \mu l = 1 \) results are the diamonds in fig. 1. In that figure the square represents the continuum result of [4]. Now the data deviates from the continuum prediction by about ten percent. The explanation for the deviation from the continuum result is simple and should have been anticipated: The “pure” Schwinger model does not include a marginal four-Fermi, Thirring, term. There is no symmetry that would turn this term off and any regularization will generate some unknown amount of it. Since it is marginal it does not vanish in the continuum and actually induces significant effects. The higher precision has allowed us to refine the conclusions of [2] and explicitly show the presence of a Thirring term in our regularization. \(^2\)

The effect of the Thirring term is expected to be enhanced if one studies a model with more flavors than one. This can be done in the overlap formalism at no extra cost: Simply raise each overlap factor that appears in eq. (3), (4) to a power given by the number of flavors \( N_f \). We obtain results similar to the ones in fig. 1, and, in particular, the points do not approach the continuum limit predicted by the exact solution of the continuum Schwinger model, providing further evidence for the presence of a Thirring term.

The continuum Schwinger model with a Thirring term has been solved analytically in[5]. We leave the Thirring coupling constant, \( g_T \), and an overall normalization factor (reflecting the infinite renormalization when \( g_T \neq 0 \)) as free parameters and we perform a global fit of the continuum expression, modified for a finite lattice, to our data. We have collected data for \( L = 4, 5, 6, 7, 8, 9, 10, N_f = 1, 2, 3, 4 \) and \( \mu l = 1, 1.5, 2, 3 \). The \( \mu l = 1, 1.5, L > 6 \) data for all \( N_f \) are closer to the scaling region and were used for the global fit. The rest are used to estimate the “true” scaling violations (subleading in the lattice spacing). The solid line in fig. 1, shows the fit for \( N_f = 1, \mu l = 1 \). The fit has a \( \chi^2/d.o.f. = 1.09 \).

\(^2\)As far as we know this issue never came up in previous lattice work concerning the Schwinger model. Maybe the accuracy was not good enough.

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