IS AN OPTIMAL QUANTUM CLONER THE BEST CHOICE FOR LOCAL COPYING IN BROADCASTING ENTANGLEMENT?

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Abstract

We point out that in broadcasting entanglement the use of a particular class of non-optimal but universal quantum cloners instead of an optimal one, to copy the qubits locally may be a better choice. We show that by giving up the optimality of the quantum cloner but not universality, a larger class of pure

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entangled states are made accessible for broadcast, making our cloning transformation more suitable than the optimal one when all operations are carried out locally. We also discuss the quality of the entangled pairs produced in the context of quantum teleportation and Bell’s inequality violation.

1. Introduction:

An optimal Quantum Cloning Machine (QCM) [1, 2] is a device which produces best permissible copies (one or more) of an unknown input state, since quantum mechanics forbids in making perfect copies [3,4]. Recently QCM has found interesting applications in quantum cryptography [5], quantum entanglement [5-7], and building quantum computer networks [8]. One may therefore ask that whether the natural choice of an optimal quantum cloner is always the best possible one in various possible scenario that one may look for. In this paper we particularly focus on one such application viz. broadcasting of entanglement via local copying discovered by Buzek et al [6]. Their scheme can also be viewed as cloning of entanglement since the procedure involves in starting from a given entangled pair to produce two entangled pairs. We briefly sketch the essentials of Buzek et al [6].

Suppose we have an entangled state of two quantum bits (qubits) given by

\[ |\Psi> = \alpha |00> + \beta |11> \]  

(1)

After copying locally each of the two qubits by two independent optimal quantum cloners (an optimal QCM is necessarily universal) it turns out that the two two-qubit clones are inseparable if [6]

\[ \frac{1}{2} - \frac{\sqrt{39}}{16} \leq \alpha^2 \leq \frac{1}{2} + \frac{\sqrt{39}}{16} \]  

(2)
Considering the potential applicability of broadcasting entanglement in the field of quantum communications, the range of $\alpha^2$ given by (2) becomes crucial. The motivation of this paper follows from the fact that if the sole purpose is to broadcast entanglement then one should use that cloning machine for local copying of the qubits so that the range of $\alpha^2$ is the maximum. Besides one may also judge other aspects of the entangled pairs being produced, for example one may look for the fidelity of quantum teleportation under the standard scheme, violation of Bell’s inequality etc.

In this Letter we point out that using a particular class of universal but non-optimal quantum copier (cloner) to copy the two qubits locally within the scheme of Buzek et al. one can have a larger range of $\alpha^2$ than given by (2). Though using optimal quantum cloning machine for the purpose of broadcast one can create entangled pairs which are useful for teleportation, we show that by using a suitable class of universal but non-optimal QCM one can produce entangled pairs whose fidelity for teleportation is better than that produced by using optimal QCM. We also show that the entangled pairs thus produced necessarily satisfy Bell’s inequality and even applying Gisin’s filtering method [9] the states cannot be made to violate Bell’s inequality. Besides the output states can always be recast into Werner form [10] if and only if one starts with a maximally entangled state irrespective of whether the universal cloning machine used is optimal or not.

2. UQCM and cloning of entanglement:

We follow Buzek and Hillary [2] to obtain our cloning transformation. In obtaining their universal cloning transformation Buzek and Hillary imposed two conditions,

(1) All input states should be copied equally well.

(2) The distance (norm) between the output density operator and the ideal output density operator is input state independent.

But in our cloning transformation we only require the first condition and relax the second, since it leads to the optimal quantum cloner. So our cloning
transformation is not optimal but universal in the sense that it copies all input states equally well. Therefore, it belongs to a restricted class of universal quantum cloners. The action of the universal quantum copier that we have used is given by,

\[
\left| 0 \right>_a \left| b \right>_b \left| Q \right>_x \rightarrow \sqrt{\eta} \left| 00 \right>_a \left| \uparrow \right>_b + \sqrt{2\xi} \left| \downarrow \right>_a \left| \uparrow \right>_b \right.
\]

\[\text{(3)}\]

\[
\left| 1 \right>_a \left| b \right>_b \left| Q \right>_x \rightarrow \sqrt{\eta} \left| 11 \right>_a \left| \downarrow \right>_b + \sqrt{2\xi} \left| \uparrow \right>_a \left| \downarrow \right>_b \right.
\]

\[\text{(4)}\]

where \( \left| Q \right>_x \) denotes the initial state of the quantum copier (ancilla), \( \left| \uparrow \right>_x \), \( \left| \downarrow \right>_x \) are the two orthonormal vectors of the Hilbert space of the copier and \( \eta, \xi \) being the cloning machine parameters. The bounds of \( \eta \) and \( \xi \) are given by [2],

\[
0 \leq \eta \leq 2^\xi^{1/2} (1 - 2\xi)^{1/2} \leq \frac{1}{\sqrt{2}} \text{ and } 0 \leq \xi \leq \frac{1}{2}
\]

\[\text{(5)}\]

Also

\[
\eta = 1 - 2\xi
\]

\[\text{(6)}\]

Subsequently one can rewrite the action of the cloning machine and the bound of the machine parameter \( \xi \) as,

\[
\left| 0 \right>_a \left| b \right>_b \left| Q \right>_x \rightarrow \sqrt{1 - 2\xi} \left| 00 \right>_a \left| \uparrow \right>_b + \sqrt{2\xi} \left| \downarrow \right>_a \left| \uparrow \right>_b \right.
\]

\[\text{(7)}\]

\[
\left| 1 \right>_a \left| b \right>_b \left| Q \right>_x \rightarrow \sqrt{1 - 2\xi} \left| 11 \right>_a \left| \downarrow \right>_b + \sqrt{2\xi} \left| \uparrow \right>_a \left| \downarrow \right>_b \right.
\]

\[\text{(8)}\]
\[ \frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \xi \leq \frac{1}{2} \] (9)

Note that the lower bound of the machine parameter \( \xi \) in (9) follows by using (5) and (6). Any cloning machine specified by some definite value of \( \xi \) lying within the range (9) is universal in the sense that all input states are copied equally well. The parameter \( \xi \) therefore specifies the type (nature) of the cloning machine used. Note that for \( \xi = \frac{1}{6} \) this is nothing but the Buzek-Hillery optimal UQCM [2]. The cloning machine for \( \xi \neq \frac{1}{6} \) is therefore a non-optimal universal cloning machine. Now we have a whole class of universal cloning machine defined by the range (9) at our disposal including the optimal one which can be used for the purpose of cloning entanglement. We first obtain the range of \( \alpha^2 \) as a function of \( \xi \). We will see that there are further restrictions on the values, that \( \xi \) can take so that the range of \( \alpha^2 \) is a valid one. Then we obtain the value of \( \xi \) for which one gets the largest possible range of \( \alpha^2 \). We emphasize that all the operations are carried out locally.

Two distant parties \( a_1 \) and \( a_2 \) share a pair of particles prepared in the state

\[ | \Psi > = \alpha | 00 > + \beta | 11 > \] (10)

where \( \alpha, \beta \) are real and \( \alpha^2 + \beta^2 = 1 \). The first qubit belongs to \( a_1 \) and the second qubit belongs to \( a_2 \). Now the two systems \( a_i (i = 1, 2) \) are locally copied by the copier \( X_i (i = 1, 2) \) according to the cloning transformations (7) and (8) to produce output two systems \( b_i (i = 1, 2) \). The local output state of the copier \( X_i \) is given by the density operator

\[ \hat{\rho}_{a_i b_i}^{(out)} = \alpha^2 (1 - 2\xi) | 00 >< 00 | + \beta^2 (1 - 2\xi) | 11 >< 11 | + 2\xi | + >< + | \] (11)
The nonlocal output is described by the density operator

\[ \hat{\rho}^{(\text{out})}_{a_i b_j} = (\alpha^2 (1 - 2\xi) + \xi^2) \mid 00 >\ 00 \rangle + (\beta^2 (1 - 2\xi) + \xi^2) \mid 11 >\ 11 \rangle \\
+ \xi (1 - \xi) (\mid 01 >\ 01 \rangle + \mid 10 >\ 10 \rangle) \\
+ \alpha \beta (1 - 2\xi)^2 (\mid 00 >\ 11 \rangle + \mid 11 >\ 00 \rangle) \ldots \ i \neq j; i, j = 1, 2 \]

(12)

It follows from the Peres-Horodecki theorem [11,12] that \( \hat{\rho}^{(\text{out})}_{a_i b_j} \) is inseparable if

\[ \frac{1}{2} - \left[ \frac{1}{4} - \frac{\xi^2 (1 - \xi)^2}{(1 - 2\xi)^2} \right]^{1/2} \leq \alpha^2 \leq \frac{1}{2} + \left[ \frac{1}{4} - \frac{\xi^2 (1 - \xi)^2}{(1 - 2\xi)^2} \right]^{1/2} \]

(13)

where,

\[ \frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \xi \leq \frac{1}{2} - \frac{1}{2\sqrt{3}} \]

(14)

The above restriction on \( \xi \) in particular the upper bound follows from the fact that \( \left[ \frac{1}{4} - \frac{\xi^2 (1 - \xi)^2}{(1 - 2\xi)^2} \right]^{1/2} \) has to be positive otherwise the domain of \( \alpha^2 \) would be meaningless.

Again applying the Peres-Horodecki theorem it is easy to obtain that \( \hat{\rho}^{(\text{out})}_{a_i b_i} \) is separable if

\[ \frac{1}{2} - \left[ \frac{1}{4} - \frac{\xi^2 (1 - \xi)^2}{(1 - 2\xi)^2} \right]^{1/2} \leq \alpha^2 \leq \frac{1}{2} + \left[ \frac{1}{4} - \frac{\xi^2 (1 - \xi)^2}{(1 - 2\xi)^2} \right]^{1/2} \]

(15)
As one can observe comparing the above two equations $\hat{\rho}_{a|b_i}^{(\text{out})}$ is separable if $\hat{\rho}_{a|b_j}^{(\text{out})}$ is inseparable. Clearly when the purpose is to broadcast entanglement our requirement is to maximize the range of $\alpha^2$. The question is how much one can achieve using only local operations? Recently Buzek and Hillery [13] achieved substantial increase of the range of $\alpha^2$ using nonlocal cloning. But their procedure is quite a difficult task to perform in reality. We note that using the OQCM ($\xi = 1/6$) and applying local operations the range of $\alpha^2$ is given by (2). But observe that this range can be increased if we take $\xi = \frac{1}{2} - \frac{1}{2\sqrt{2}}$ for which

$$\frac{1}{2} - \frac{\sqrt{3}}{4} \leq \alpha^2 \leq \frac{1}{2} + \frac{\sqrt{3}}{4}$$  \hspace{1cm} (16)$$

resulting in a substantial increase in the range of $\alpha^2$. In fact this is the maximum possible range of $\alpha^2$ one can obtain using local operations and requiring that cloning machine is universal. It follows that using any universal cloning machine for which $\frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \xi < \frac{1}{6}$ is bound to increase the range of $\alpha^2$, the maximum being attained for $\xi = \frac{1}{2} - \frac{1}{2\sqrt{2}}$. So an universal non-optimal cloning machine does the job better than the optimal one. Here one should note that the possibility of still having a larger range possibly using a state dependent quantum cloner is not excluded.

3. Violation of Bell’s Inequality and Quantum teleportation:

Our next objective is to analyze the nature of entanglement of the state described by the density operator $\hat{\rho}_{a|b_j}^{(\text{out})}$. It is known that violation of Bell’s inequality and the fidelity of teleportation provide two different criteria to study entanglement although the second one being a much stronger filter to test entanglement. First we discuss from the violation of Bell’s inequality point of view.
Applying the Horodecki theorem [14] one obtains that the state $\hat{\rho}^{(\text{out})}_{a,b_i}$ violates Bell’s inequality if

$$\frac{1}{2} - \left[\frac{1}{2} - \frac{1}{4(1-2\xi)^2}\right]^{1/2} \leq \alpha^2 \leq \frac{1}{2} + \left[\frac{1}{2} - \frac{1}{4(1-2\xi)^2}\right]^{1/2}$$

(17)

Note that the validity of the above domain of $\alpha^2$ requires that $\xi \leq \frac{1}{2} - \frac{1}{2\sqrt{2}}$. But $\frac{1}{2} - \frac{1}{2\sqrt{2}} < \frac{1}{2} - \frac{1}{2\sqrt{2}}$, which implies that using an universal cloning machine for the purpose of entanglement broadcast one comes up with those inseparable entangled pairs which necessarily do not violate Bell’s inequality. But we know that there is still a possibility to make these states violate Bell’s inequality by applying Gisin’s filtering method [9] which works for some states very well. However it can be shown that by performing the filtering operation defined by $M \otimes P$, where the matrices $M$ and $P$ are given by, $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$ and $\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ respectively, on state (12), the state $\hat{\rho}_{\text{new}}$ that emerges out of the local filter is given by

$$\hat{\rho}_{\text{new}} = \frac{1}{N} \hat{\rho}^t$$

where,

$$\hat{\rho}^t = (\alpha^2(1-2\xi) + \xi^2)m_1^2p_1^2 | 00 > < 00 | + (\beta^2(1-2\xi) + \xi^2)m_2^2p_2^2 | 11 > < 11 |$$

$$+ \xi(1-\xi)m_1^2p_2^2 | 01 > < 01 | + \xi(1-\xi)m_2^2p_1^2 | 10 > < 10 |$$

$$+ \alpha\beta(1-2\xi)^2m_1m_2p_1p_2(| 00 > < 11 | + | 11 > < 00 |)$$

$i \neq j; i,j = 1,2$

(18)

and
\[ N = Tr(\hat{\rho}) = [\alpha^2(1 - 2\xi) + \xi^2]m_1^2p_1^2 + [\beta^2(1 - 2\xi) + \xi^2]m_2^2p_2^2 + \xi(1 - \xi)(m_1^2p_2^2 + m_2^2p_1^2) \]

Applying the Horodecki theorem to the state (18) it can be shown that Bell’s inequality is not violated.

Another interesting aspect is that it is possible to recast the state (12) into Werner form

\[ \rho_W = (\frac{1-x}{4})I + xP(\Psi) \]

where \( x = (1 - 2\xi)^2 \), \( \Psi = \alpha |00\rangle + \beta |11\rangle \), if and only if \( \alpha = 1/\sqrt{2} \). This impure state consists of a single fraction \( x \) and a random fraction \( (1 - x) \).

Note that since \( x \) is a function of cloner parameter \( \xi \) it is possible to have a state for some particular choice of \( \xi \) where the contribution of the random part would be minimum which is a natural requirement. In fact for \( \frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \xi < \frac{1}{6} \) the contribution of the random part is less than that obtained by using an optimal quantum cloner, the minimum value 1/2 of the random fraction being attained for \( \xi = \frac{1}{2} - \frac{1}{2\sqrt{2}} \).

In this case also we find that an universal non-optimal quantum cloner is a better choice than the optimal one.

Lastly we discuss the usefulness entangled pairs produced for the purpose of quantum teleportation. In the standard scheme of quantum teleportation using the Horodeckis’ result [15] fidelity can be written as

\[ F_{max} = \frac{1}{2}(1 + \frac{1}{3}f(\xi)) \] (20)

where \( f(\xi) = Tr\sqrt{T^\dagger T} = (1 - 2\xi)^2(1 + 4\alpha\beta) \). The \( T \) matrix is obtained from the state (12) following the Horodecki prescription [13]. It turns out that for \( \xi = \frac{1}{6} \)

\[ F_{max} = \frac{1}{2}(1 + \frac{1}{27}(1 + 4\alpha\beta)) \] (21)
But observe that $F_{\text{max}}$ given by (20) is larger than that given by (21) when $\frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \xi < \frac{1}{6}$. In fact the highest fidelity is obtained for $\xi = \frac{1}{2} - \frac{1}{2\sqrt{2}}$ for which we have

$$F_{\text{max}} = \frac{1}{2}(1 + \frac{1}{6}(1 + 4\alpha \beta))$$  \hspace{1cm} (22)

If we compare the teleportation fidelity between two mixed entangled states which can be written in the Werner form, the state having a lower random fraction is more suitable for teleportation. We have seen earlier that if one starts only with a maximally entangled state the resulting output state can be recast into Werner form. Therefore putting $\alpha = \beta = \frac{1}{\sqrt{2}}$ in (21) and (22) we find that the values of $F_{\text{max}}$ are $\frac{13}{18}$ and $\frac{3}{4}$ respectively. Since the former value of $F_{\text{max}}$ corresponds to the use of an optimal quantum cloner we find that in this case also use of a suitable class of non-optimal universal quantum cloners (defined by the range of $\xi$, see the remark after Eq. 21) for local copying in broadcasting entanglement produces those inseparable output states with larger fidelity.

4. Conclusion:

In conclusion we want to stress that when carrying out certain scheme where one uses cloning machines as one of the tools needed, the optimal quantum cloner may not always be the best possible choice. In this letter we pointed out in the context of broadcasting entanglement, where giving up the optimality of the quantum cloner one can achieve better, in particular to obtain a larger range of $\alpha^2$ which we believe to be most crucial. We have also illustrated explicitly by analysing the nature of the inseparable output state in the context of quantum teleportation and generation of Werner type states why the choice of a non-optimal universal quantum cloner is better than the optimal one.
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