Non-Hermitian Hamiltonian approach to the microwave transmission through one-dimensional qubit chain

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We investigate the propagation of microwave photons in a one-dimensional waveguide interacting with a number of artificial atoms (qubits). Within the formalism of projection operators and non-Hermitian Hamiltonian approach we develop a one-photon approximation scheme for the calculation of the transmission and reflection factors of the microwave signal in a waveguide which contains an arbitrary number \( N \) of non-interacting qubits. It is shown that for identical qubits in the long-wave limit a coherent superradiance state is formed with the width being equal to the sum of the widths of spontaneous transitions of \( N \) individual qubits.

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I. INTRODUCTION

One-dimensional (1D) waveguide-QED systems are emerging as promising candidates for quantum information processing motivated by tremendous experimental progress in a wide variety of solid state systems with embedded artificially designed atoms. In recent years one of the basic type of these physical systems has been realized in solid state setups where qubits are coupled to microwave cavities. An important advantage of these systems is that the qubits can be placed in the microwave field confined in a microwave cavity at fixed positions at separations of the order of relevant wavelengths. Moreover, the excitation energy of every such qubit can easily be adjusted by external circuit. From practical point, it is important to study 1D waveguide systems having more than just one qubit. Recent experiments with superconducting qubits showed that the one-dimensional guided modes can be a useful tool to create two-qubit\textsuperscript{1} and many-qubit entanglement\textsuperscript{2–4}.

The experimental investigation of these systems is based on the measurements of the transmitted and reflected signals with their properties being dependent on the quantum states of every qubit in a waveguide. Theoretical calculations of the transmission and reflection factors in the qubit + waveguide systems are being performed in a configuration space\textsuperscript{5–9} or by the input-output formalism\textsuperscript{10,11}. These methods are physically sound but they become very cumbersome if we try to find solutions for several or more qubits in a waveguide. While the transmission for one qubit has long been known\textsuperscript{15}, the analytical expressions for the transmissions for two and three qubits have been obtained quite recently\textsuperscript{7,9}. In the latter case the expression for the transmission is known only for symmetrical arrangement of three identical qubits\textsuperscript{5}.

Here we propose a matrix formalism for the calculation of the transmission and reflection factors in the qubit + waveguide system. Similar, but not identical, matrix technique has been developed for the study of photon transport in the coupled resonator optical waveguides\textsuperscript{12}.

Our technique provide us with a powerful analytical tool, which allows us to investigate the arbitrary number of qubits with different parameters and arbitrary arrangement in the array. Our approach is based on the projection operators formalism and the method of the effective non-Hermitian Hamiltonian which has originally been developed for the description of nuclear reactions\textsuperscript{13,14} with many later applications for different open mesoscopic systems (see review paper\textsuperscript{15} and references therein). Recently this method has been applied for the description of electron transmission through 1D solid state nanostructures\textsuperscript{16,17}. With the aid of this technique we study in detail the one-photon microwave transport for one, two and three qubits imbedded in a waveguide. We show that for \( N \) identical qubits in the long-wavelength limit a coherent superradiance state is formed with the width being equal to the sum of the widths of spontaneous transitions of \( N \) individual qubits. The calculated transmission factors for one and two qubits coincide with those known from the literature\textsuperscript{5–9}. This technique can be straightforwardly applied for the investigation of the photon transport when arbitrary number of qubits interacts with photon field in a waveguide, and in more general case when additionally a direct interaction between qubits are switched on.

The paper is organized as follows. In the Section II we define the model Hamiltonian of \( N \) noninteracting qubits imbedded in a microwave resonator. In Section III we briefly describe projection formalism and effective non-Hermitian Hamiltonian approach in application to the photon transport in 1D waveguide. The application of the model Hamiltonian to the derivation of the general expressions for the transmission and reflection coefficients for \( N \) qubits in a waveguide is performed in the Section IV. The Section V is devoted to the detail investigation of the microwave transport for one, two, and three qubits in a waveguide. In the conclusion to this section we briefly analyze the general case of \( N \) qubits.

II. THE MODEL HAMILTONIAN

We consider a microwave 1D waveguide resonator with \( N \) qubits imbedded at the fixed positions \( x_i \). The Hamiltonian of the system reads:

\[
H = H_{ph} + H_{qb} + H_{int}
\]

where

\[
H_{ph} = \sum_k \hbar \omega_k a_k^+ a_k
\]
is the Hamiltonian of photon field,

$$ H_{qb} = \sum_{i=1}^{N} H_{qb}^i $$

(3)

is the Hamiltonian of $N$ noninteracting qubits, with

$$ H_{qb}^i = \frac{1}{2} h \Omega_i \sigma_x^{(i)} $$

(4)

the Hamiltonian of the individual $i$-th qubit with the excitation frequency $\Omega_i$.

The interaction of the qubit chain with the photon field is given by Hamiltonian:

$$ H_{\text{int}} = \sum_k \sum_{i=1}^{N} \lambda_i (a_k^* e^{-ikx_i} + a_k e^{ikx_i}) \sigma_x^{(i)} $$

(5)

where $\lambda_i$ is the qubit-photon interaction strength, $x_i$ are the qubit positions relative to the waveguide center, $x_0 = 0$.

In the one photon approximation there are two possibilities: either one photon is in the waveguide in the state $|1_k\rangle \equiv |k\rangle$ and all qubits are in their ground states $|g_i\rangle$ with a corresponding state vector $|g_1, g_2, \ldots, g_N, k\rangle$, or no photons in the waveguide, $|0_k\rangle \equiv |0\rangle$, with $i$-th qubit excited and $N-1$ qubits in the ground state. In this case the system is described by $N$ vectors of the type $|g_1, \ldots, g_{i-1}, e_i, g_{i+1}, \ldots, g_N, 0\rangle$.

In order to simplify the notations we will use throughout the paper the following concise forms for state vectors:

$$ |k\rangle \equiv |g_1, g_2, \ldots, g_N, k\rangle $$

$$ |n\rangle \equiv |g_1, \ldots, g_{n-1}, e_n, g_{n+1}, \ldots, g_N, 0\rangle $$

with the orthogonality relations

$$ \langle n|m \rangle = \delta_{nm} $$

$$ \langle n|k \rangle = 0 $$

$$ \langle k|k' \rangle = \frac{2\pi}{L} \delta(k - k') $$

where $L$ is the waveguide length.

### III. PROJECTION FORMALISM AND EFFECTIVE NON-HERMITIAN HAMILTONIAN

In order to proceed further, here we briefly describe the essence of non-Hermitian Hamiltonian approach.

It is always possible to formally subdivide the Hilbert space of a quantum system with the Hermitian Hamiltonian $H$ into two arbitrarily selected orthogonal projectors, $P$ and $Q$, where

$$ 1 = P + Q, PP = P, QQ = Q, PQ = QP = 0 $$

The solution of the stationary Schrödinger equation,

$$ H\Psi = E\Psi $$

(6)

in general consists of two parts,

$$ \Psi \equiv P\Psi + Q\Psi $$

(7)

Hamiltonian (6) acts within each subspace. If we eliminate one subspace of states, we can obtain an equation for a part of the wave function. For example, if we eliminate the $P$-subspace, the equation for the wave function in $Q$-subspace takes the form

$$ H_{\text{eff}}(E)\Psi = EQ\Psi $$

(8)

where the energy dependent effective Hamiltonian

$$ H_{\text{eff}}(E) = H_{QQ} + H_{QP} \frac{1}{E - H_{PP}} H_{QP} $$

(9)

with

$$ H_{QQ} = QHQ, H_{PP} = PHP, H_{QP} = QHP, H_{QP} = PHP $$

projects Hilbert space on the $Q$ subspace.

Below we restrict the Hilbert space to one photon approximation, hence we take the projection operators as follows:

$$ P = \sum_k |k\rangle \langle k| = \frac{L}{2\pi} \int_{-\infty}^{+\infty} dk |k\rangle \langle k| $$

(10)

$$ Q = \sum_{n=1}^{N} |n\rangle \langle n| $$

(11)

We assume that Hamiltonian $H_{PP}$ is diagonal in subspace $P$. In order to avoid the singularities emerging when $H_{PP}$ has eigenvalues at real energy $E$, it has to be considered as a limiting value from the upper half of the complex energy plane, $E^+ = E + i\epsilon$. With this rule, those states of subspace $Q$ which will turn out to be coupled to the states in subspace $P$ will acquire the outgoing waves and become unstable. Then the matrix elements of the effective Hamiltonian (9) in subspace $Q$ can be written as

$$ \langle m|H_{\text{eff}}|n\rangle = \langle m|H|n\rangle + $$

$$ \frac{L}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\langle m|H_{QP}|k\rangle \langle k|H_{QP}|n\rangle}{E - E_k + i\epsilon} $$

(12)

where $v_p$ is the photon phase velocity.

The resonance energies of the $Q$-system lie in the low half of the complex energy plane, $E^+ = \bar{E} - i\hbar\Gamma$ and are given by the roots of the equation

$$ \text{det}(E - H_{\text{eff}}) = 0 $$

(13)

In the framework of projection formalism we can find the wavefunction of the whole system, a solution of Shrödinger equation (6), in terms of the operator which acts on the wavefunction of incident photon, $|k\rangle$.
\[ |\Psi\rangle = |k\rangle + \left(1 + \frac{1}{E - H_{PP} + i\varepsilon} H_{QP}\right) \times \frac{1}{E - H_{PP} + i\varepsilon} H_{QP} |k\rangle \]  

(14)

In the basis of Q- subspace vectors the expression (14) can be written as

\[ |\Psi\rangle = |k\rangle + \sum_{n,m} |n\rangle R_{m,n} \langle m| H_{QP} |k\rangle + \frac{1}{2\pi} \sum_{n,m} \int dq \frac{e^{iqx}}{E_{n} - E_{q} + i\varepsilon} (q) H_{QP} |n\rangle R_{m,n} \langle m| H_{QP} |k\rangle \]  

(15)

where \( R_{m,n} \) is the matrix inverse to the matrix \( \langle m| (E - H_{eff}) |n\rangle \):

\[ R_{m,n} = \frac{1}{E - H_{eff}} \langle m| |n\rangle \]  

(16)

In fact, the photon wavefunction (15) is nothing more than a decomposition (7).

The photon wavefunction in configuration space is obtained by multiplying (15) from the left by the vector \( \langle x| \):

\[ \Psi_{N}(x) = \langle x|\Psi\rangle = e^{ikx} + \frac{2\pi}{\lambda_{L}} \sum_{n,m} \int dq \frac{e^{iqx}}{E_{n} - E_{q} + i\varepsilon} (q) H_{QP} |n\rangle R_{m,n} \langle m| H_{QP} |k\rangle \]  

(17)

where we have used the definitions \( \langle x|k\rangle = e^{ikx} \) and \( \langle x|n\rangle = 0 \).

The wavefunction (15) is the superposition of the incident wave and the wave which results from the virtual transitions between qubits and photon field in the resonator. We will see below that this superposition leads to the destructive interference when the frequency of incident photon is equal to the excitation frequency of any qubit. In this case the transmitted signal outside the qubit array is equal to zero.

From (15) we can also find the probability for the \( n\)-th qubit to be in excited state:

\[ \langle n|\Psi\rangle = \sum_{m} R_{n,m} \langle m| H_{QP} |k\rangle \]  

(18)

IV. THE APPLICATION OF PROJECTION FORMALISM TO THE MODEL HAMILTONIAN

Here we apply the model Hamiltonian (1) from the Section II to the general expressions found in Section III. First we calculate the matrix elements of the effective Hamiltonian (12). The first term in rhs of (12) reads:

\[ \langle m| H |n\rangle = \varepsilon_{m} \delta_{m,n} \]  

(19)

where

\[ \varepsilon_{m} = \frac{1}{2}\hbar \left( \Omega_{m} - \sum_{n \neq m} \Omega_{n} \right) \]  

(20)

It is also not difficult to calculate the matrix elements in rhs of equation (12):

\[ \langle m| H_{QP} |k\rangle = \lambda_{m} \exp(ikx_{m}) \]  

(21)

Then, the effective Hamiltonian (12) can be written as

\[ \langle m| H_{eff} |n\rangle = \varepsilon_{m} \delta_{m,n} + \left( \frac{\lambda_{m}\lambda_{n}L}{2\pi} \right) J(x_{m}, x_{n}) \]  

(22)

where

\[ J(x_{m}, x_{n}) = \int_{-\infty}^{+\infty} dk \frac{\exp(ik(x_{m} - x_{n}))}{E - E_{k} + i\varepsilon} \]  

(23)

It is shown in the Appendix that

\[ J(x_{m}, x_{n}) = -\frac{2\pi}{\hbar v_{g}} e^{ik|d_{mn}|} \]  

(24)

where \( k \) is the wave vector of incident photon, \( d_{mn} = x_{m} - x_{n} \).

Finally, the effective Hamiltonian (22) can be written as follows

\[ \langle m| H_{eff} |n\rangle = \varepsilon_{m} \delta_{m,n} - i\hbar(\Gamma_{m}\Gamma_{n})^{1/2} e^{ik|d_{mn}|} \]  

(25)

where we define the half width of spontaneous emission

\[ \Gamma_{m} = \frac{L\lambda_{L}^{2}}{\hbar v_{g}} \]  

(26)

Throughout the paper we will use \( \Gamma \) for the halfwidth of resonance line.

The photon wavefunction (17) for our model follows from (21) and (24):

\[ \Psi_{N}(x) = e^{ikx} - i\hbar \sum_{m,n=1}^{N} (\Gamma_{m}\Gamma_{n})^{1/2} e^{ikx_{m}} R_{m,n} e^{ik|x - x_{n}|} \]  

(27)

Finally, for our model we write down from (18) the probability for the \( n\)-th qubit to be in excited state:

\[ \langle n|\Psi\rangle = \sum_{m} \lambda_{m} R_{n,m} e^{ikx_{m}} \]  

(28)

We assume that all qubits are arranged in the array from left to right, so that \( x_{1} \) is the position of the qubit at the left end of the array and \( x_{N} \) is the qubit’s position at its right end. In this case the photon wavefunction (27) outside the array can be written as:

\[ \Psi_{N}(x) = \begin{cases} t_{N} e^{ikx} \ (x > x_{N}) \\ e^{ikx} + r_{N} e^{-ikx} \ (x < x_{1}) \end{cases} \]  

(29)

where the transmission and reflection coefficients are as follows
\begin{align}
    t_N &= 1 - i\hbar \sum_{m,n=1}^{N} (\Gamma_m \Gamma_n)^{1/2} e^{ikx_m} R_{m,n} e^{-ikx_n} \quad (30) \\
    r_N &= -i\hbar \sum_{m,n=1}^{N} (\Gamma_m \Gamma_n)^{1/2} e^{ikx_m} R_{m,n} e^{ikx_n} \quad (31)
\end{align}

The conservation of the energy flux requires the additional condition for \( t \) and \( r \):
\[ |t_N|^2 + |r_N|^2 = 1 \quad (32) \]

The expressions (30) and (31) are of general nature and they form the basis for the calculation of microwave transmission and reflection in particular cases.

V. THE CALCULATION OF MICROWAVE TRANSMISSION AND REFLECTION

A. One qubit in a waveguide

In this case there is the only vector in subspace Q, \(|1\rangle\). We also assume that the qubit is located at the point \( x = 0 \). From (30) and (31) we obtain:
\[ \begin{align*}
    t_1 &= 1 - i\hbar \Gamma R_{11} \\
    r_1 &= -i\hbar \Gamma R_{11}
\end{align*} \]

where
\[ R_{11} = \frac{1}{1} \langle 1 | E - H_{eff} | 1 \rangle = \frac{1}{E - \langle 1 | H_{eff} | 1 \rangle} \quad (33) \]

The running energy \( E \) in (33) is the energy of incident photon plus the energy of the qubit in the ground state, \( E = \hbar \omega - \hbar \Omega/2 \). From (25) we also have:
\[ \langle 1 | H_{eff} | 1 \rangle = \frac{\hbar \Omega}{2} - i\hbar \Gamma \]

Hence, for \( t \) and \( r \) we finally obtain:
\[ \begin{align*}
    t_1 &= \frac{\omega - \Omega}{\omega - \Omega + i\Gamma} \quad (34) \\
    r_1 &= \frac{-i\Gamma}{\omega - \Omega + i\Gamma} \quad (35)
\end{align*} \]

The plots of microwave transmission and reflection are shown in Fig. 1. At resonance the signal transmission is zero. The expressions (34), (35) coincide with those obtained in \( n \) where one qubit problem has been solved in a configuration space.

From (34) and (35) we see that \( t-r = 1 \). Unlike the general condition (32) it is valid only for one qubit and reflects the continuity of the wavefunction (29) at the point \( x = 0 \).

The probability for the qubit to be excited is given by (28) with \( n = 1 \):
\[ \langle 1 | \Psi \rangle = \lambda R_{11} = \frac{\lambda}{\hbar} \frac{1}{\omega - \Omega + i\Gamma} \quad (36) \]

B. Two qubits in a waveguide

1. Spectral properties of effective Hamiltonian

Here we consider the first nontrivial example that exhibits superradiant transition: the two noninteracting qubits in a waveguide. The qubits are positioned at the points \( x_1 = -d/2 \) and \( x_2 = +d/2 \), respectively, with a distance \( d \) between them. The Q- subspace is formed by two state vectors \(|1\rangle \equiv |e_1, g_2, 0\rangle\) and \(|2\rangle \equiv |g_1, e_2, 0\rangle\). According to (25) the matrix of effective Hamiltonian is as follows:
\[ H_{eff} = \left( \begin{array}{cccc}
    \varepsilon - i\hbar \Gamma_1 & -i\hbar \sqrt{\Gamma_1 \Gamma_2} e^{ikd} & -i\hbar \sqrt{\Gamma_1 \Gamma_2} e^{ikd} \\
    -i\hbar \sqrt{\Gamma_1 \Gamma_2} e^{ikd} & -\varepsilon + i\hbar \Gamma_2 \\
    -i\hbar \sqrt{\Gamma_1 \Gamma_2} e^{ikd} & -i\hbar \sqrt{\Gamma_1 \Gamma_2} e^{ikd} & -\varepsilon - i\hbar \Gamma_2 \\
    \end{array} \right) \quad (37) \]

where \( \varepsilon = \frac{\hbar}{\lambda} (\Omega_1 - \Omega_2) \), \( \Gamma_i (i = 1, 2) \) are defined in (26), and the wave vector \( k \) is related to the physical frequency \( \omega = \omega/\nu_g \). From the matrix (37) we can find the complex energies of the Q- system from the equation (13), where \( E = \hbar \tilde{\omega} - \frac{\lambda}{2} (\Omega_1 + \Omega_2) \). For two qubit case the equation (13) gives two poles in the complex \( \tilde{\omega} \) plane as the function of physical frequency \( \omega \).
\[ \tilde{\omega} = \frac{\Omega_1 + \Omega_2}{2} - i \frac{\Gamma_1 + \Gamma_2}{2} \pm \sqrt{\frac{1}{4} (\Omega_1 - \Omega_2)^2 - \frac{1}{2} (\Gamma_1 + \Gamma_2)^2} \quad (38) \]

From this expression it follows that the positions of resonances and their widths depend on the frequency \( \omega \) of incident photon \( (k = \omega/\nu_g) \). This is a common feature if the number of qubits is more than one.

For identical noninteracting qubits \( \Omega_1 = \Omega_2 = \Omega \), \( \Gamma_1 = \Gamma_2 = \Gamma \) we obtain from (38)
\[ \tilde{\omega} = \Omega - i\Gamma \pm i\Gamma e^{ikd} \quad (39) \]

In the complex \( \tilde{\omega} \) plane the roots are as follows
\[ \text{Re} \tilde{\omega} \equiv \tilde{E} = \Omega \mp \Gamma \sin kd \quad (40) \]
ant state corresponds to a coherent symmetric superposition, which is the inverse to the matrix \((H_{eff})_{m,n}\) is given in (37). Direct calculations yield for \(R_{m,n}\) the following result:

\[
R_{m,n} = \frac{1}{\hbar D_2(\omega)} \left( \frac{\omega - \Omega_2 + i\Gamma_2}{\omega - \Omega_1 + i\Gamma_1} + e^{i k d} \right) \tag{43}
\]

where

\[
D_2(\omega) = [\omega - \Omega_2 + i\Gamma_2] [\omega - \Omega_1 + i\Gamma_1] + \Gamma_1 e^{i k d} \tag{44}
\]

Finally, according to prescriptions in (30) and (31) we obtain \(t\) and \(r\) in terms of running frequency \(\omega\):

\[
t_2 = \frac{(\omega - \Omega_1)(\omega - \Omega_2)}{(\omega - \Omega_2 + i\Gamma_2)[\omega - \Omega_1 + i\Gamma_1] + \Gamma_1 e^{i k d}} \tag{45}
\]

\[
r_2 = -i \left\{ \frac{\epsilon^{i k d} \Gamma_1 [\omega - \Omega_2] + \epsilon^{-i k d} \Gamma_2 [\omega - \Omega_1 + i\Gamma_1]}{[\omega - \Omega_2 + i\Gamma_2][\omega - \Omega_1 + i\Gamma_1] + \Gamma_1 e^{i k d}} \right\} \tag{46}
\]

As it follows from the results of subsection V B 1 the denominator in (45) and (46) can be written as \([\omega - \omega_1(\omega)] [\omega - \omega_2(\omega)]\), where \(\omega_1, \omega_2\) are the roots of equation (38). Hence, the resonance frequencies of the incident photon are given by, in general nonlinear, equations \(\omega = \text{Re}[\omega_1(\omega)], \omega = \text{Re}[\omega_2(\omega)]\).

For identical qubits we obtain from (45), (46):

\[
t_2 = \frac{(\omega - \Omega)^2}{(\omega - \Omega + i\Gamma)^2 + \Gamma^2 e^{i k d}} \tag{47}
\]

\[
r_2 = -i \frac{2\Gamma [i(\omega - \Omega) \cos k d + \Gamma \sin k d]}{(\omega - \Omega + i\Gamma)^2 + \Gamma^2 e^{i k d}} \tag{48}
\]

As is seen from these expressions the form of the transmission and reflection spectra depend on the inter qubit distance \(d\). In the long wavelength limit we obtain from (47) and (48):

\[
t_2 = \frac{\omega - \Omega}{\omega - \Omega + i2\Gamma} \tag{49}
\]

\[
r_2 = \frac{-i2\Gamma}{\omega - \Omega + i2\Gamma} \tag{50}
\]

The expressions (49), (50) are identical to the one qubit case \((34), (35)\) with the only exception. For two identical qubits the resonance width is twice the resonance width for one qubit, which is clear signature of superradiance transition.

Below we show several plots of transmission and reflection amplitudes for different values of \(k_0 d\), where \(k_0 = \Omega / v_g\). The plots are calculated for two identical qubits from (47) and (48).
3. Photon wave function for two qubits in a waveguide

Photon wave function for two qubits is calculated from (27) with $t$ and $r$ from (45) and (46). Below we write the wavefunction for two qubits in the intermediate region $-d/2 < x < +d/2$:

$$\Psi_2(x) = \frac{(\omega - \Omega)}{D_2(\omega)} (e^{ikx}(\omega - \Omega_2 + i\Gamma_2) - i\Gamma_2 e^{ikd} e^{-ikx})$$

It can easily be verified that the wavefunction (51) is continuous at the points $x = \pm d/2$. At resonance with the first qubit ($\omega = \Omega_1$) photon is reflected from the first qubit and does not penetrate in the inter qubit region $x > -d/2$. However, at resonance with the second qubit ($\omega = \Omega_2$) the wave function $\Psi_2(x) \neq 0$ at inter qubit region $-d/2 < x < d/2$, but $\Psi_2(d/2) = 0$ as it follows from continuity condition.

From (28) we calculate the probability amplitude for the first or second qubit to be excited.

$$|\Psi_2 = e^{-ikd/2}\frac{1}{hD_2(\omega)} \left[ \lambda_1(\omega - \Omega_2 + i\Gamma_2) - i\lambda_2 \sqrt{1 + 2e^{2ikd}} \right]$$

$$|\psi_2 = e^{ikd/2}\frac{1}{hD_2(\omega)} \left[ \lambda_2(\omega - \Omega_1 + i\Gamma_1) - i\lambda_1 \sqrt{1 + 2e^{2ikd}} \right]$$

From the definition of $\Gamma$ (26) we may rewrite (53) as:

$$|\psi_2 = e^{ikd/2}\frac{1}{hD_2(\omega)} \left[ \lambda_2(\omega - \Omega_1) \right]$$

Hence, if the photon is in resonance with the first qubit, the second qubit remains unexcited. If the photon is in resonance with the second qubit, the first qubit is unexcited only if $\Omega_2d/\nu_g = \pi$.

C. Three qubits in a waveguide

1. Spectral properties of effective Hamiltonian

Here we consider three noninteracting qubits in a waveguide. The qubits are positioned at the points $x_1 = -d$, $x_2 = +d$ and $x_3 = 0$, respectively, with a distance $d$ between adjacent qubits. The Q-subspace is formed by three state vectors $|1\rangle \equiv |e_1, g_2, g_3, 0\rangle$, $|2\rangle \equiv |g_1, e_2, g_3, 0\rangle$ and $|3\rangle \equiv |g_1, g_2, e_3, 0\rangle$. Hence, the states $|1\rangle$ and $|2\rangle$ correspond to qubits located at the points $x = \pm d$, respectively. The state $|3\rangle$ is for the qubit placed at the point $x = 0$. The P-subspace is formed by the vectors $|k\rangle \equiv |g_1, g_2, g_3, k\rangle$. According to (28) the matrix of effective Hamiltonian is as follows:

$$H_{eff} = \begin{pmatrix} 
\varepsilon_1 - i\hbar\Gamma_1 & -i\hbar\sqrt{1 + 2e^{2ikd}} & -i\hbar\sqrt{1 + 2e^{2ikd}} \\
-i\hbar\sqrt{1 + 2e^{2ikd}} & \varepsilon_2 - i\hbar\Gamma_2 & -i\hbar\sqrt{1 + 2e^{2ikd}} \\
-i\hbar\sqrt{1 + 2e^{2ikd}} & -i\hbar\sqrt{1 + 2e^{2ikd}} & \varepsilon_3 - i\hbar\Gamma_3 
\end{pmatrix}$$

where $\varepsilon_i$ and $\Gamma_i, (i = 1, 2, 3)$ are defined in (20) and (26), respectively.

The roots of this Hamiltonian in the complex frequency plane are defined by the equation

$$\det \left( \tilde{\omega} - \frac{1}{2} (\Omega_1 + \Omega_2 + \Omega_3) - H_{eff}/\hbar \right) = 0$$

that can be expressed as:

$$\begin{align*}
(\tilde{\omega} - \Omega_1 + i\Gamma_1) (\tilde{\omega} - \Omega_2 + i\Gamma_2) (\tilde{\omega} - \Omega_3 + i\Gamma_3) \\
+ (\tilde{\omega} - \Omega_1 + i\Gamma_1) \Gamma_2 e^{2ikd} + (\tilde{\omega} - \Omega_2 + i\Gamma_2) \Gamma_1 e^{2ikd} \\
+ (\tilde{\omega} - \Omega_3 + i\Gamma_3) \Gamma_1 e^{2ikd} = 0
\end{align*}$$

We note that in general the energies and the widths of resonances depend on the physical frequency $\omega (k = \omega/v_g)$.

For identical qubits ($\Omega_1 = \Omega_2 = \Omega_3 = \Omega$, $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma$) we obtain from (55)

$$\begin{align*}
(\tilde{\omega} - \Omega + i\Gamma)^3 + 2(\tilde{\omega} - \Omega + i\Gamma) \Gamma^2 e^{2ikd} \\
+ (\tilde{\omega} - \Omega - i\Gamma) \Gamma^2 e^{-2ikd} = 0
\end{align*}$$

In the long wavelength limit we find from (56)

$$\begin{align*}
(\tilde{\omega} - \Omega)^3 + 3i\Gamma(\tilde{\omega} - \Omega)^2 = 0
\end{align*}$$

which gives two resonances with null width and one resonance which absorbs the widths of all three qubits:

$$\tilde{\omega}_{1,2} = \Omega; \quad \tilde{\omega}_3 = \Omega - 3i\Gamma$$
We obtain the same result if the running frequency \( \omega \) in (56) corresponds to \( kd = n\pi \), \( n = 1, 2, \ldots \).

The structure of the eigenfunction \( Q\hat{\Psi} \) of effective Hamiltonian \( H_{eff} \) for three identical qubits in the long wavelength limit \( kd << 1 \) can be found from the solution of the Schrödinger equation (3). For this case diagonal matrix elements in (54) are equal to \(-\hbar\Omega/2 - i\hbar\Gamma\), while all non diagonal elements are equal to \(-i\hbar\Gamma\). For the eigenfunction \( Q\hat{\Psi} = a|1\rangle + b|2\rangle + c|3\rangle \), where \( a, b, c \) are arbitrary constants we obtain two solutions with eigenenergies \( E = -\hbar\Omega/2 \) \((\omega = \Omega)\) and \( a + b + c = 0 \), and one superradiant state for \( E = -\hbar\Omega/2 - 3i\hbar\Gamma \) \((\omega = \Omega - 2i\hbar\Gamma)\) with \( a = b = c \). Therefore, the superradiant state corresponds to a coherent symmetric superposition \((Q\hat{\Psi})_S = a(|1\rangle + |2\rangle + |3\rangle)\).

The \( kd \)-dependence of real and imaginary part of three complex roots of equation (56), \( \bar{\omega} = \text{Re}\bar{\omega} - i\text{Im}\bar{\omega} \), is shown in Fig. 4 where \( x = (\text{Re}\bar{\omega} - \Omega)/\Gamma \), \( y = (\Gamma - \Gamma)/\Gamma \). In the projection to \( x, y \) plane these roots form two circles as shown in Fig. 5. Every point on this graph are merged from black and red points of Fig. 4 which belong to the same root.

**FIG. 4:** The \( kd \)-dependence of real and imaginary part of three complex roots of equation (56), \( \bar{\omega} = \text{Re}\bar{\omega} - i\text{Im}\bar{\omega} \), where \( x = (\text{Re}\bar{\omega} - \Omega)/\Gamma \), \( y = (\Gamma - \Gamma)/\Gamma \). Black dot lines correspond to \( x \), while red dot lines correspond to \( y \). The line numbers correspond to real and imaginary parts of three roots of Eq. (56).

**FIG. 5:** The projection of the three roots of Eq. (56) to \( x, y \) plane, where \( x = (\text{Re}\bar{\omega} - \Omega)/\Gamma \), \( y = (\Gamma - \Gamma)/\Gamma \). Every point on this graph are merged from black and red points of Fig. 4 which belong to the same root.

2. Transmission and reflection spectra for three qubits in a waveguide

Transmission and reflection factors are calculated from (30) and (31), where \( m, n = 1, 2, 3 \) and \( R_{mn} \) is given in Appendix. In the frequency picture with \( E = \hbar\omega - \hbar(\Omega_1 + \Omega_2 + \Omega_3)/2 \) we obtain for three qubits \( t \) and \( r \) the following expressions:

\[
t_3 = \frac{(\omega - \Omega_1)(\omega - \Omega_2)(\omega - \Omega_3)}{D_3(\omega)} \quad (57)
\]

\[
r_3 = -\frac{G(\omega)}{D_3(\omega)} \quad (58)
\]

where

\[
D_3(\omega) = (\omega - \Omega_1 + i\Gamma_1)(\omega - \Omega_2 + i\Gamma_2)(\omega - \Omega_3 + i\Gamma_3)
+ (\omega - \Omega_1 + i\Gamma_1)\Gamma_2\Gamma_3 e^{2ikd} + (\omega - \Omega_2 + i\Gamma_2)\Gamma_1\Gamma_3 e^{2ikd}
+ (\omega - \Omega_3 - i\Gamma_3)\Gamma_1\Gamma_2 e^{2ikd} \quad (59)
\]

As in the two qubit case, the resonance frequencies in (57) and (58) are given by the equations \( \omega = \text{Re}[\tilde{\omega}_1(\omega)] \), \( \omega = \text{Re}[\tilde{\omega}_2(\omega)] \), \( \omega = \text{Re}[\tilde{\omega}_3(\omega)] \), where \( \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3 \) are the roots of equation (55).

For identical qubits we obtain

\[
t_3 = \frac{(\omega - \Omega)^3}{D_{3id}(\omega)} \quad (61)
\]

\[
r_3 = -\frac{G_{id}(\omega)}{D_{3id}(\omega)} \quad (62)
\]

where \( D_{3id}(\omega) \) is calculated in the section 2 of Appendix.

\[
D_{3id}(\omega) = (\omega - \Omega + i\Gamma)^3 + 2(\omega - \Omega + i\Gamma)\Gamma^2 e^{2ikd}
+ (\omega - \Omega - i\Gamma)\Gamma^2 e^{2ikd} \quad (63)
\]

\[
G_{id}(\omega) = 2\Gamma(\omega - \Omega + i\Gamma)^2 \cos \text{2kd} + \Gamma(\omega - \Omega)^2 + 2\Gamma^3
- 4i\Gamma^3(\omega - \Omega)e^{2ikd} \quad (64)
\]

In the long wavelength limit \( kd << 1 \) we obtain from (61) and (62)

\[
t_3 = \frac{\omega - \Omega}{\omega - \Omega + 3i\Gamma} \quad (65)
\]

\[
r_3 = -\frac{3i\Gamma}{\omega - \Omega + 3i\Gamma} \quad (66)
\]
We note that the expressions (65), (66) are valid for a broad range of frequencies satisfying the condition k0d << 1. However, as k = ω/vg these expressions are also valid for k0 = nπ/νg, n = 1, 2, . . . but only at the fixed frequencies ωn = nπνg/d.

As in the two qubits case (49), (50) the expressions (65), (66) are similar to (29), where the transmission from (27) with the matrix

\[
T_{3}R_{3} = \frac{\lambda}{\hbar D_{3}^{de}(\omega)} \left[ e^{-\frac{1}{2}(\omega - \Omega)^{2}} e^{i\omega d} \right]
\]

for 0 < x < d, and

\[
\Psi_{3}(x) = \frac{\lambda}{\hbar D_{3}^{de}(\omega)} \left[ e^{i\omega x} \left( \omega - \Omega + i\Gamma \right) + e^{2ikd} \right]
\]

for x < 0 or d < x.<ref> Similar to the two qubit case, here at the exact resonance (ω = Ω) the photon wavefunction does not penetrate in the inter qubit region.

In the conclusion to this section we write the probability for the particular qubit in the array to be in excited state. From (28) we obtain:

\[
\langle 1|\Psi_{3} \rangle = \frac{\lambda}{\hbar D_{3}^{de}(\omega)} \left[ \left( \omega - \Omega + i\Gamma \right)^{2} e^{-i\omega d} \right] - i\Gamma e^{-i\omega d} \left[ \left( \omega - \Omega + i\Gamma \right) \left( \omega - \Omega - i\Gamma \right) e^{2ikd} \right]
\]

for x < 0 or d < x.

We see that at resonance the first qubit only is excited. This is consistent with the above conclusion that at resonance photon does not penetrate beyond the first qubit.

All qubit arrays considered above have a general property: if the photon frequency is equal to the resonance frequency of any qubit in the chain, the transmission signal is absent. We attributed this property to the destructive interference between the input wave and the wave which resulted from the virtual transitions between the qubit and the photon field in the resonator. However, it is not clear to what extent this property can be attributed to uniform distribution of the qubit in the chain with the equal distance between adjacent qubits. The simplest system where we can check this property is the three qubit chain. We made the calculation of the transmission for three different qubits which are positioned at the points x1 = −d1, x2 = +d2 and x3 = 0, respectively, with unequal distance between adjacent qubits. It turned out that in this unequal distance case the transmission is similar to (71) with the just the same numerator, but different denominator, which is given in the Appendix. Hence, we may assume that for nonuniform qubit array the transmission is also zero if the input photon is at resonance with any qubit in the chain.

3. Photon wave function for three identical qubits in a waveguide

Photon wave function for three identical qubits is calculated from (27) with the matrix \( R_{mn} \) defined in (A.9).

\[
\Psi_{3}(x) = e^{ikx} - i\hbar \Gamma e^{ik|x|} \left[ e^{ikd} R_{11} + e^{-ikd} R_{12} + R_{13} \right] - i\hbar \Gamma e^{ik|x+d|} \left[ e^{ikd} R_{12} + e^{-ikd} R_{11} + R_{13} \right] - i\hbar \Gamma e^{ik|x|} \left[ e^{ikd} R_{13} + e^{-ikd} R_{13} + R_{33} \right]
\]

(67)

Outside the qubit array x < −d and x > d the wavefunction in (67) is similar to (29), where the transmission \( t \) and reflection \( r \) are given in (61) and (62). Inside the array we obtain

\[
\Psi_{3}(x) = \frac{(\omega - \Omega)^{2}}{D_{3}^{de}(\omega)} \left[ e^{ikx} (\omega - \Omega + i\Gamma) - i\hbar e^{-ikx} e^{2ikd} \right]
\]

(68)

for 0 < x < d, and

\[
\Psi_{3}(x) = \frac{\lambda}{\hbar D_{3}^{de}(\omega)} \left[ e^{i\omega x} (\omega - \Omega + i\Gamma) + e^{2ikd} \right]
\]

for x < 0 or d < x.

\[
\langle 1|\Psi_{3} \rangle = \frac{\lambda}{\hbar D_{3}^{de}(\omega)} \left[ (\omega - \Omega + i\Gamma)^{2} e^{-i\omega d} \right] - i\Gamma e^{-i\omega d} \left[ (\omega - \Omega + i\Gamma) (\omega - \Omega - i\Gamma) e^{2ikd} \right]
\]

for x < 0 or d < x.

Similarly to the two qubit case, here at the exact resonance (ω = Ω) the photon wavefunction does not penetrate in the inter qubit region.

In the conclusion to this section we write the probability for the particular qubit in the array to be in excited state. From (28) we obtain:

\[
\langle 1|\Psi_{3} \rangle = \frac{\lambda}{\hbar D_{3}^{de}(\omega)} \left[ (\omega - \Omega + i\Gamma)^{2} e^{-i\omega d} \right] - i\Gamma e^{-i\omega d} \left[ (\omega - \Omega + i\Gamma) (\omega - \Omega - i\Gamma) e^{2ikd} \right]
\]

for x < 0 or d < x.

We see that at resonance the first qubit only is excited. This is consistent with the above conclusion that at resonance photon does not penetrate beyond the first qubit.

All qubit arrays considered above have a general property: if the photon frequency is equal to the resonance frequency of any qubit in the chain, the transmission signal is absent. We attributed this property to the destructive interference between the input wave and the wave which resulted from the virtual transitions between the qubit and the photon field in the resonator. However, it is not clear to what extent this property can be attributed to uniform distribution of the qubit in the chain with the equal distance between adjacent qubits. The simplest system where we can check this property is the three qubit chain. We made the calculation of the transmission for three different qubits which are positioned at the points x1 = −d1, x2 = +d2 and x3 = 0, respectively, with unequal distance between adjacent qubits. It turned out that in this unequal distance case the transmission is similar to (71) with the just the same numerator, but different denominator, which is given in the Appendix. Hence, we may assume that for nonuniform qubit array the transmission is also zero if the input photon is at resonance with any qubit in the chain.

D. N qubits in a waveguide

In principle the transmission and reflection for any number of qubits can be found from general expressions (50) and (31). While it is not easy to find analytical solutions for N qubits, nevertheless, from previous calculations of the transmission for one, (42), two, (43), and three (57) qubits, we may guess the general structure of the transmission for N qubits in a waveguide:
\[ t_N = \frac{\prod_{n=1}^{N} (\omega - \Omega_n)}{D_N(\omega)} \]  
\[
D_N(\omega) = \det \left( \omega - \frac{1}{2} \sum_{i=1}^{N} \Omega_i - H_{eff}/\hbar \right)
\]  

As for the spectral properties of the effective Hamiltonian, we can assert that the secular equation \( \det(E-H_{eff}) \) has \( N \) poles in the low half plain of the complex energy. Foridentical qubits in the long wavelength limit there are \( N-1 \) stable states with \( \tilde{\omega} = \Omega \) and one resonance \( \tilde{\omega} = \Omega - iN\Gamma \) which absorbs all the widths of individual qubits.

\[ t_N = \frac{\omega - \Omega}{\omega - \Omega + iN\Gamma} \]  
\[
t_N = -\frac{iN\Gamma}{\omega - \Omega + iN\Gamma}
\]

VI. CONCLUSION

In this paper we develop a new technique for the investigation of the photon transport through multiple qubit array in a 1D waveguide. The technique is based on the projection operators formalism and non Hermitian approach, which is known to be a successful tool in some fields of nuclear physics and condensed matter. We considered in detail the one photon transport for two and three qubits in a waveguide, and made some conclusions for \( N \) qubit case. We showed that in the long wavelength limit for uniformly distributed array of identical qubits a coherent superradiance state is formed with the width being equal to the sum of the widths of spontaneous transitions of \( N \) individual qubits. Within the framework of our method it is not difficult to account for the decay of the qubit states to the modes other than the waveguide continuum. It can be done by simply adding an imaginary term in the qubit energy level, \( \Omega_n \rightarrow \Omega_n - i\Gamma_n \).

The approach developed in the paper can be easily generalized to include the exchange interaction \( H_J \) between nearest neighbor qubits:

\[ H_J = \hbar \sum_{i=1}^{N} J_i \left( \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ \right) \]  

For this case it is necessary to change only the matrix of effective Hamiltonian (A.25):

\[ \langle m | H_{eff} | n \rangle = \varepsilon_m \delta_{m,n} + \hbar J_{n-1} \delta_{m,n-1} + \hbar J_n \delta_{m,n+1} - i\hbar (\Gamma_m \Gamma_n)^{1/2} e^{-ik|m-n|} \]  

The results obtained in the paper are of general nature and can be applied to any type of qubits. The specific properties of the qubit are encoded in only two parameters: the qubit energy \( \Omega \) and the rate of spontaneous emission \( \Gamma \). For example, for a superconducting qubit \( \Omega = \sqrt{\varepsilon^2 + \Delta^2} \) where \( \varepsilon \) is an external parameter which by virtue of external magnetic flux, \( \Phi_X \) controls the gap between ground and excited states, and the quantity \( \Delta \) is the qubit’s gap at the degeneracy point (\( \varepsilon = 0 \)). The rate of spontaneous emission \( \Gamma = g\Delta/\Omega^2 \), where \( g \) is the qubit-waveguide coupling.

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Appendix

1. The calculation of integral \( J(x_m, x_n) \)

In (23) the running energy \( E \) is the energy of incident photon \( |q\rangle + N \) qubits in the ground state. The energy \( E_k \) is the energy of the photon \( |k\rangle \) in the waveguide + \( N \) qubits in the ground state. Hence, \( E - E_k = \hbar (\omega_g - \omega_k) \). For (23) we, therefore, have

\[ J(x_m, x_n) = \frac{1}{\hbar} \int_{-\infty}^{+\infty} dk \frac{e^{ikd_{mn}}}{\omega_q - \omega_k + i\varepsilon} \]  

The main contribution to this integral comes from the region \( \omega_k \approx \omega_q \). Since \( \omega_k \) is the even function of \( k \), the poles of the integrand (A.1) in the \( k \) plane are located near the points \( k \approx \pm q \). For an arbitrary frequency \( \omega_q \) that is away from the cutoff of the dispersion, with the corresponding wave vector \( \pm q \), we approximate \( \omega_k \) around \( +q \) and \( -q \) as

\[ \omega_k \approx \omega_q + (k - q) \frac{d\omega_k}{dk} \bigg|_{k=q} = \omega_q + (k - q)v_q \]  

\[ \omega_k \approx \omega_q + (k + q) \frac{d\omega_k}{dk} \bigg|_{k=-q} = \omega_q - (k + q)v_q \]  

Near the poles the denominator in (A.1) takes the form:

\[ -v_q(k - q) + i\varepsilon \]  

\[ v_q(k + q) + i\varepsilon \]  

Therefore, one pole is located in the upper half of the \( k \) plane, \( k = q + i\varepsilon \), the other pole is located in the lower half of the \( k \) plane, \( k = -q - i\varepsilon \). For positive \( d_{mn} \), when calculating the integral (A.1) we must close the path in the upper plane. For
negative $d_{mn}$ the path should be closed in lower plane. Thus, we obtain:

$$J(x_m, x_n) = -\frac{2\pi i}{h\nu_g} e^{ik|d_{mn}|} \quad (A.6)$$

2. Calculation of the $R$ matrix for three qubits in a waveguide

The matrix $R_{m,n}$, $(m, n = 1, 2, 3)$ is calculated as the inverse to the matrix $(E - H_{eff})_{m,n}$, the elements of which are given in (21). The matrix $R_{m,n}$ is symmetric so that $R_{12} = R_{21}, R_{13} = R_{31}, R_{23} = R_{32}$. Direct calculations yield for $R_{m,n}$ the following result:

$$D_3(E)R_{11} = (E - \varepsilon_2 + i\hbar\Gamma_2) (E - \varepsilon_3 + i\hbar\Gamma_3) + \hbar^2 \Gamma_1 \Gamma_3 e^{2ikd}$$

$$D_3(E)R_{22} = (E - \varepsilon_1 + i\hbar\Gamma_1) (E - \varepsilon_2 + i\hbar\Gamma_2) (E - \varepsilon_3 + i\hbar\Gamma_3) + \hbar^2 \Gamma_1 \Gamma_3 e^{2ikd}$$

$$D_3(E)R_{33} = (E - \varepsilon_2 + i\hbar\Gamma_2) (E - \varepsilon_1 + i\hbar\Gamma_1) (E - \varepsilon_3 + i\hbar\Gamma_3) + \hbar^2 \Gamma_1 \Gamma_3 e^{2ikd}$$

$$D_3(E)R_{12} = - (E - \varepsilon_3 + i\hbar\Gamma_3) i\hbar\sqrt{\Gamma_1 \Gamma_2} e^{2ikd}$$

$$D_3(E)R_{13} = - (E - \varepsilon_2 + i\hbar\Gamma_2) i\hbar\sqrt{\Gamma_1 \Gamma_3} e^{2ikd}$$

$$D_3(E)R_{23} = - (E - \varepsilon_1 + i\hbar\Gamma_1) i\hbar\sqrt{\Gamma_2 \Gamma_3} e^{2ikd}$$

(A.7)

where $D_3(E) = \text{det} (E - H_{eff})_{m,n}$:

$$D_3(E) = (E - \varepsilon_1 + i\hbar\Gamma_1) (E - \varepsilon_2 + i\hbar\Gamma_2) (E - \varepsilon_3 + i\hbar\Gamma_3) + (E - \varepsilon_2 + i\hbar\Gamma_2) \hbar^2 \Gamma_1 \Gamma_3 e^{2ikd} + (E - \varepsilon_3 + i\hbar\Gamma_3) \hbar^2 \Gamma_1 \Gamma_2 e^{2ikd}$$

(A.8)

and the quantities $\varepsilon_i (i = 1, 2, 3)$ are defined in (20).

At the end of this subsection we write down from (A.7) the matrix $R_{m,n}$ for three identical qubits.

$$hD_3^{id}(\omega)R_{11} = (\omega - \Omega + i\Gamma)^2 + \Gamma^2 e^{2ikd}$$

$$hD_3^{id}(\omega)R_{22} = (\omega - \Omega + i\Gamma)^2 + \Gamma^2 e^{2ikd}$$

$$hD_3^{id}(\omega)R_{12} = -i\Gamma (\omega - \Omega) e^{2ikd}$$

$$hD_3^{id}(\omega)R_{13} = -i\Gamma (\omega - \Omega + i\Gamma) e^{ikd} - \Gamma^2 e^{3ikd}$$

$$R_{22} = R_{11}, \quad R_{23} = R_{13} = R_{31} = R_{32}, \quad R_{31} = R_{12}$$

(A.9)

3. The transmission for three qubit chain with unequal distance between each other

Here we consider three different qubits which are positioned at the points $x_1 = -d_1, x_2 = +d_2$ and $x_3 = 0$, respectively, with $d_1 \neq d_2$. The calculations yield the result:

$$t_3 = \frac{(\omega - \Omega_1)(\omega - \Omega_2)(\omega - \Omega_3)}{F(\omega)} \quad (A.11)$$

where

$$F(\omega) = U_1 U_2 U_3 + i\Gamma_1 \Gamma_2 \Gamma_3 \left( e^{2ikd_1} + e^{2ikd_2} - e^{2ik(d_1 + d_2)} - 1 \right) + U_3 \Gamma_1 \Gamma_2 \left( e^{2ikd_1 + 2ikd_2} - 1 \right) + U_1 \Gamma_2 \Gamma_3 \left( e^{2ikd_2} - 1 \right) + U_2 \Gamma_1 \Gamma_3 \left( e^{2ikd_1} - 1 \right) + iU_1 U_2 \Gamma_3 + iU_1 U_3 \Gamma_2 + iU_2 U_3 \Gamma$$

(A.12)

$$U_1 = \omega - \Omega_1; \quad U_2 = \omega - \Omega_2; \quad U_3 = \omega - \Omega_3$$

The equation (A.11) is similar to (77) with the same numerator but different denominator.
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