Note on Product-Form Monogamy Relations for Nonlocality and Other Correlation Measures

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Abstract: The monogamy relations satisfied by quantum correlation measures play important roles in quantum information processing. Generally they are given in the summation form. In this note, we study monogamy relations in product form. We present product-form monogamy relations for Bell nonlocality for three-qubit and multi-qubit quantum systems. We then extend our studies to other quantum correlations such as concurrence.

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INTRODUCTION

Quantum technologies radically change the landscape of modern communication and computation. Quantum correlations, such as quantum discord [1, 2], quantum entanglement [3, 4], quantum steering [5, 6], quantum nonlocality [7, 8] have shown to be useful resources in many quantum information processing tasks like quantum cryptography [9], quantum metrology [10], quantum illumination [11]. Various measures have been proposed to quantify these correlations [12]. Generally these quantifiers are very difficult to calculate, except for some specific cases like two-qubit states [13-17].

A successful and secure quantum network relies on quantum correlations distributed and shared over many sites [18]. Different kinds of multipartite quantum correlations have been considered as valuable resources for various applications in quantum communication tasks. A key property is that such quantum correlations cannot be freely shared among the multipartite systems. The monogamy relation states that the more two systems correlated, the less with the rest systems. It was first shown in [14] that the bipartite entanglement measure concurrence $C$ of the reduced states $\rho_{AB}$ and $\rho_{AC}$ of a three-qubit state $\rho_{ABC}$ satisfies the Coffman-Kundu-Wootters (CKW) relation, $C(\rho_{ABC}) \geq C(\rho_{AB}) + C(\rho_{AC})$, where $C(\rho_{A|BC})$ stands for the concurrence between subsystems $A$ and the remaining subsystems $BC$. The CKW relation means that the sum of the individual pairwise entanglement between $A$ and $B$ and $C$ cannot exceed the entanglement between $A$ and the remaining parties together. Since then there have been many papers focused on such monogamy or polygamy relations for quantum entanglement [13, 24]. In [24] the generalized summation-form monogamy relations for any valid entanglement measure have been investigated. Monogamy relations have also been studied for quantum discord [25, 26], quantum steering [27, 28], Bell nonlocality [29, 30], indistinguishability [31], coherence [32] and other nonclassical correlations [33].

For tripartite quantum systems, monogamy relations have the following general (trade-off) form,

$$Q^\alpha(\rho_{AB}) + Q^\alpha(\rho_{AC}) \leq Q^\alpha(\rho_{A|BC})$$ (1)

or

$$Q(\rho_{AB}) + Q(\rho_{AC}) \leq K$$ (2)

for some bipartite quantum correlation measure $Q$ and positive real numbers $\alpha$ and $K$, where $Q(\rho_{A|BC})$ stands for the correlation between subsystems $A$ and the remaining subsystems $BC$, the vertical bar is the familiar notation for the bipartite split, $Q(\rho_{AB})$ ($Q(\rho_{AC})$) represents the bipartite correlation the reduced state $\rho_{AB}$ ($\rho_{AC}$) of the tripartite state $\rho_{ABC}$.

Generally (1) does not hold for $\alpha = 1$ for many correlation quantifiers like the geometric measure of discord [25]. However, for any given $Q$, (1) holds for sufficient large $\alpha$ [22, 23]. For example, it has been shown that the $\alpha$th ($\alpha \geq 2$) power of discord for 3-qubit pure states [25, 26], the $\alpha$th ($\alpha \geq 2$) power of concurrence and the $\alpha$th ($\alpha \geq \sqrt{2}$) power of entanglement of formation for N-qubit states do satisfy the monogamy relations [20].

And for Eq (2), there are also some examples for entanglement [21] and Bell nonlocality [32] of three qubit states. The well-known CHSH-Bell [36] operator is given by,

$$B = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2,$$ (3)

where $A_i = \vec{a}_i \cdot \vec{\sigma}$, $B_j = \vec{b}_j \cdot \vec{\sigma}$, $\vec{a}_i$ and $\vec{b}_j$ are three-dimensional real unit vectors, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with $\sigma_1$, $\sigma_2$ and $\sigma_3$ the standard Pauli matrices, $i,j = 1,2$. The CHSH inequality says that $|\langle B(\rho) \rangle| \leq 2$, where $\langle B(\rho) \rangle = tr(\rho B)$ is the mean value of the Bell operator.
\[ \langle \mathcal{B}(\rho_{AB}) \rangle \langle \mathcal{B}(\rho_{AC}) \rangle + \langle \mathcal{B}(\rho_{AC}) \rangle \langle \mathcal{B}(\rho_{BC}) \rangle \leq 12. \]

It has been further shown that
\[ \langle \mathcal{B}(\rho_{AB}) \rangle^2 + \langle \mathcal{B}(\rho_{AC}) \rangle^2 \leq 8. \]  

Cheng et al. [35] also considered monogamy relation in summation form like \( \langle \mathcal{B}(\rho_{AB}) \rangle^\alpha + \langle \mathcal{B}(\rho_{AC}) \rangle^\alpha \leq \langle \mathcal{B}(\rho_{A|BC}) \rangle^\alpha \). However, it is not true in general. For example, given the state \(|000\rangle\), one has \( \langle \mathcal{B}(\rho_{AB}) \rangle = \langle \mathcal{B}(\rho_{AC}) \rangle = \langle \mathcal{B}(\rho_{A|BC}) \rangle = 2 \). The investigation on such relations for three-qubit pure states has applications in the study of bi-locality [38], and can be easily extended to other kinds of quantum correlations such as concurrence.

As for quantum uncertainty relations, there are both summation forms [33][41] and product forms [42], which have their own advantages. Since the monogamy relations of Bell nonlocality in summation form equation (1) or the CKW type do not exist, it is worthy of studying the possible product forms.

In this paper, we mainly investigate the product-form monogamy relations for nonlocal correlation measures. They can not be derived from the summation-form monogamy relations which even do not exist in some cases. Concerning the relations among the product-form monogamy relations we obtained can not be obtained from summation-form ones by arithmetic geometric mean inequalities. They are tighter than the ones derived from the summation-form relations (if they exist).

**PRODUCT-FORM MONOGAMY RELATIONS FOR BELL NONLOCALITY OF THREE QUBIT STATES**

A two-qubit state \( \rho \) can be expressed as
\[ \rho = \frac{1}{4}(I \otimes I + a \cdot \sigma \otimes I + I \otimes b \cdot \sigma + \sum_{n,m=1}^{3} t_{m,n} \sigma_{m} \otimes \sigma_{n}), \]
where \( I \) is the identity operator. Let \( T_{\rho} \) denote the real matrix with entries given by \( t_{nm} = Tr(\rho \sigma_{n} \otimes \sigma_{m}) \). Set \( U_{\rho} \equiv T_{\rho}^{T} T_{\rho} \). Denote \( \mu \) and \( \bar{\mu} \) the two greater eigenvalues of \( U_{\rho} \) and \( M_{\rho} \equiv \mu + \bar{\mu} \). Then \( \langle \mathcal{B}(\rho_{AB}) \rangle = 2 \sqrt{M_{\rho AB}} \). Then, \( \bar{\mu} \) has the following simple form,
\[ M_{\rho AB} + M_{\rho AC} \leq 2. \]  
Using the arithmetic geometric mean inequality, we have
\[ \sqrt{M_{\rho AB} M_{\rho AC}} \leq 1. \]  

By [44], one has \( \langle \mathcal{B}(\rho_{A|BC}) \rangle = 2 \sqrt{[1 + [2(c_{1}c_{2})^2]} \), namely, \( M_{\rho A|BC} \geq 1 + [2(c_{1}c_{2})^2] \geq 1. \)

Therefore, similar to the summation-form monogamy relation [11], for three-qubit pure state \( |\psi_{ABC}\rangle \) we have the product-form monogamy relation by using \( \bar{\mu} \).
\[ \sqrt{M_{\rho AB} M_{\rho AC}} \leq M_{\rho A|BC}. \]

In fact, \( \bar{\mu} \) can be further tightened without the use of \( \bar{\mu} \). We have the following result:

**Theorem** For three-qubit pure state \( |\psi_{ABC}\rangle \),
\[ M_{\rho AB} M_{\rho AC} \leq \frac{1}{1 + [C(\rho_{A|BC})]^2} M_{\rho A|BC}, \]
where \( C(\rho_{A|BC}) \) is the concurrence of \( |\psi_{A|BC}\rangle \).

**Proof:** It is direct to derive that [32], \( M_{\rho AB} = 1 + s_{3}^{AB} - s_{3}^{AC} - s_{3}^{BC} \), \( M_{\rho AC} = 1 + s_{3}^{AC} - s_{3}^{AB} - s_{3}^{BC} \), where \( s_{3}^{X} \) denotes the smallest eigenvalue of \( U_{\rho X} \). Thus,
\[ M_{\rho AB} M_{\rho AC} = [1 - s_{3}^{BC}]^2 - [s_{3}^{AB} - s_{3}^{AC}]^2. \]
Moreover, since \( s_A^{BC} - s_A^{AC} = |\bar{c}|^2 - |\bar{b}|^2 \), where \( \bar{b} = (2l_1^2 + 2l_2^1, -2l_2^1 \sin \phi, 1 - 2l_2^2 - 2l_2^2) \), \( \bar{c} = (2l_1^2 \cos \phi + 2l_2^1 \sin \phi, 1 - 2l_2^2 - 2l_2^2) \), we obtain

\[
M_{\rho AB}M_{\rho AC} = |1 - s_A^{BC}|^2 - |\bar{c}|^2 - |\bar{b}|^2|^2. \tag{9}
\]

Combining (9) and that \( M_{\rho AB} \geq 1 + 2|c_1c_2|^2 \), we get

\[
M_{\rho AB}M_{\rho AC} \leq \left| 1 - s_A^{BC} - |\bar{c}|^2 - |\bar{b}|^2 \right|^2 \left/ \left( 1 + 2|c_1c_2|^2 \right) \right. \quad \tag{10}
\]

By the definition of concurrence for a bipartite pure state \( |\psi_{AB}\rangle \), \( C(|\psi_{AB}\rangle) = \sqrt{2(1 - Tr(\rho_A^2))} \) with \( \rho_A = Tr_B(\rho_{AB}) \) the reduced density matrix of \( \rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}| \) by tracing over the subsystem \( B \). We can obtain \( C(|\psi_{AB}\rangle) = 2|c_1c_2| \). Theorem get a complete proof.

Interestingly, we see that the tightened product-form monogamy relation (8) for nonlocality depends also on the entanglement concurrence. In (38), the authors studied bi-locality (It can be understood as the simplest three-point quantum networks with one node in the middle) and showed that all possible pairs of entangled pure states can violate the so-called “bi-locality” inequality. For arbitrary pairs of mixed two-qubit states, the bi-locality inequality has the following form,

\[
S_{\text{biloc}}^{\text{max}} = 2\sqrt{\xi_1 \xi_1 + \xi_2 \xi_2} \leq 2,
\]

where, \( \xi_1, \xi_2 \) (\( \xi_1, \xi_2 \)) are the first and second large singular values of \( U_{\rho AB} \). Due to \( \langle \mathcal{B}(\rho_{AB}) \rangle = 2\sqrt{M_{\rho AB}} \)

\[
S_{\text{biloc}}^{\text{max}} \leq 2\sqrt{M_{\rho AB}M_{\rho AC}}.
\]

Hence, the violation of the bi-locality inequality implies that \( M_{\rho AB}M_{\rho AC} > 1 \). Therefore, from (10) all the pairs of reduced density matrices from three-qubit states cannot violate the bi-locality inequality.

Similar to the residual entanglement, we can also define

\[
M_{\rho ABC} = M_{\rho AB} - M_{\rho AC} \quad \text{to be the “residual” quantum nonlocality of three-qubit pure states. For the state } \alpha(000) + \beta(111), \text{ we have } M_{\rho AB} = M_{\rho AC} = 1, M_{\rho ABC} = 2 - (|\alpha|^2 - |\beta|^2)^2, \text{ and thus } M_{\rho ABC} = 1 - (|\alpha|^2 - |\beta|^2)^2 \text{. Therefore, when } \rho_{ABC} \text{ is the GHZ state } (\alpha = \beta), M_{\rho ABC} = 1. \text{ When } \rho_{ABC} \text{ is a fully separable state } (\alpha = 0 \text{ or } \beta = 0), M_{\rho ABC} = 0. \text{ Nevertheless, different from the tangle of entanglement, } \tau_{ABC} = C_{\rho ABC}^2 - C_{\rho AB}^2 - C_{\rho AC}^2 \text{, } M_{\rho ABC} \text{ is not invariant under the permutation of the qubits.}
\]

The monogamy relation (9) can be generalized to multi-qubit systems. It has been shown that (33),

\[
M_{\rho AB} + M_{\rho AC} + M_{\rho AD} + \cdots \leq n - 1, \tag{11}
\]

for any \( n \)-qubit pure or mixed state \( \rho_{ABCD} \). Using the generalized arithmetical geometric mean inequality \( \sqrt{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \), we have for any \( n \)-qubit state \( \rho_{ABCD} \)

\[
\sqrt{n}M_{\rho AB}M_{\rho AC}M_{\rho AD} \cdots \leq \frac{M_{\rho AB} + M_{\rho AC} + M_{\rho AD} + \cdots}{n - 1} \leq 1. \tag{12}
\]

In particular, an \( n \)-qubit pure states \( |\psi_{ABCD}\rangle \) can be viewed as a bipartite state under partition \( A \) and \( BCD \). Therefore, according to the previous discussions on CHSH Bell inequality violation of bipartite high-dimensional pure states, we can obtain

\[
\sqrt{n}M_{\rho AB}M_{\rho AC}M_{\rho AD} \cdots \leq M_{\rho ABCD} \tag{13}
\]

For multipartite quantum nonlocality, it is possible that the product-form monogamy, like three-qubit, can be applied to more complex quantum networks (48, 49).

**MONOGAMY RELATIONS FOR OTHER QUANTUM CORRELATIONS**

Our investigation on product-form monogamy relations for non-locality can be extended to other quantum correlations.

Let us consider the entanglement measure concurrence (45, 47). For three-qubit pure states \( |\psi_{ABC}\rangle \), the concurrence \( C_{AB} \) satisfies (14).

\[
C_{AB}^2 = Tr(\rho_{AB}^2) - 2\lambda_1 \lambda_2,
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the square roots of the greater eigenvalues of matrix \( \rho_{AB}^2 \). \( \rho_{AB}^2 \) is the reduced density matrix of the qubit pair \( AB \), \( \rho_{AB} = (\sigma_2 \otimes \sigma_2) \rho_{AB}^2 (\sigma_2 \otimes \sigma_2) \), with the asterisk denoting complex conjugation and \( \sigma_2 \) is the Pauli matrix

\[
\begin{pmatrix}
0 & -i \\
 i & 0
\end{pmatrix}.
\]

Similarly, the concurrence \( C_{AC} \) is given by

\[
C_{AC}^2 = Tr(\rho_{AC}^2) - 2\mu_1 \mu_2,
\]

Therefore,

\[
C_{AB}^2C_{AC}^2 = [Tr(\rho_{AB}^2) - 2\lambda_1 \lambda_2][Tr(\rho_{AC}^2) - 2\mu_1 \mu_2] = Tr(\rho_{AB}^2)Tr(\rho_{AC}^2) - 2[Tr(\rho_{AB}^2\rho_{AC}^2)\mu_1 \mu_2] + Tr(\rho_{AC}^2\rho_{AC}^2)\lambda_1 \lambda_2 + 4\lambda_1 \lambda_2 \mu_1 \mu_2. \tag{14}
\]

On other hand, the concurrence \( C_{A(BC)} \) between partition \( A \) and \( BC \) has the form, \( C_{A(BC)}^2 = Tr(\rho_{A(BC)}^2) + \)
Therefore, we have

\[
C_{ABC}^2 - 2 \sqrt{\text{Tr}(\rho_{AB} \rho_{AC})} \text{Tr}(\rho_{AB} \rho_{AC}) + \text{Tr}(\rho_{AC} \rho_{BC}) \\
\geq 2 \sqrt{\text{Tr}(\rho_{AB} \rho_{AC})} \text{Tr}(\rho_{AC} \rho_{BC}) \\
= 2(C_{AB}^2 C_{AC} + 2\text{Tr}(\rho_{AB} \rho_{AC}) \mu_1 \mu_2 \\
+ \text{Tr}(\rho_{AC} \rho_{BC}) \lambda_1 \lambda_2 - 4 \lambda_1 \lambda_2 \mu_1 \mu_2)^2 \\
= 2(C_{AB}^2 C_{AC} + 2(\lambda_1^2 + \lambda_2^2) \mu_1 \mu_2 \\
+ (\mu_1^2 + \mu_2^2) \lambda_1 \lambda_2 - 4 \lambda_1 \lambda_2 \mu_1 \mu_2)^2 \\
\geq 2(C_{AB}^2 C_{AC} + 4 \lambda_1 \lambda_2 \mu_1 \mu_2)^2 \\
= 2(C_{AB}^2 C_{AC} + \tau_{ABC}^2)^2,
\]

where both inequalities are based on the arithmetic geometric mean inequalities \( a^2 + b^2 \geq 2ab \) for \( a, b \geq 0 \). The second equality is due to [14], the third and last equalities are from the results in [14], and the tangle \( \tau_{ABC} = 4 \lambda_1 \lambda_2 = 4 \mu_1 \mu_2 \). Inequality (14) is tighter than the inequality \( C_{ABC}^2 \geq 2C_{AB}C_{AC} \) obtained by arithmetic geometric mean inequality from CKW type. The result consists with the one from the summation form only when \( \tau_{ABC} = 0 \). Therefore, inequality (15) is basically different from the summation-form monogamy inequalities.

For arbitrary quantum correlations \( Q \), the multiplication form of monogamy relations may not be as tight as [15]. However, when \( \alpha = 2 \) in [11], we can directly get

\[
\sqrt{Q(\rho_{AB})Q(\rho_{AC})} \leq \frac{\sqrt{2}}{2} Q(\rho_{ABC}) \leq Q(\rho_{ABC}).
\]

Namely, any monogamy inequalities [11] satisfied by a quantum correlation \( Q \) imply a product form [16]. This is also true for multipartite case. From a monogamy relation

\[
Q^2(\rho_{A|B_1B_2...B_{n-1}}) \geq \sum_{i=1}^{n-1} Q^2(\rho_{AB_i}),
\]

we obtain

\[
\frac{\prod_{i=1}^{n-1} Q^2(\rho_{AB_i})}{n-1} \leq \sum_{i=1}^{n-1} Q^2(\rho_{AB_i}) \leq Q^2(\rho_{A|B_1B_2...B_{n-1}}) \frac{1}{n-1},
\]

i.e.

\[
\sqrt{\frac{n-1}{\prod_{i=1}^{n-1} Q(\rho_{AB_i})}} \leq \frac{Q(\rho_{A|B_1B_2...B_{n-1}})}{\sqrt{n-1}}.
\]

In [54, 55], the authors studied some restrictive relationships for different quantum correlations such as contextually [54]. For instance, for general three-qubit states, the Bell nonlocality and three tangle have the complementarity relation \( \max\{M_{\text{Bell}}, M_{\text{MAC}}, M_{\text{ESR}}\} + \tau \leq 2 \) [51]. The internal entanglement of a bipartite system, and its correlations with an environment system, have the relation \( E(A_1 : A_2) + E(A_1A_2 : B) \leq E_{\text{max}} \) [52]. All these inequalities in summation form can be transformed into product form in a similar way.

**CONCLUSION AND DISCUSSION**

We have studied the monogamy relations in product form for Bell nonlocality and concurrence of multi-qubit states. Product-form monogamy relations can be obtained from the summation form by arithmetic geometric mean inequality, like inequalities (7) and (12). Such product-form monogamy relations can be tighten by tightening the summation-form monogamy inequalities or by using weighted arithmetic geometric mean inequalities. Nevertheless, the inequalities (8), (10) and (15) are not obtained from the arithmetic geometric mean inequalities. They are tighter than the ones derived from the summation-form relations. These product-monogamy relations may be of their own applications in quantum information processing. For instance, the inequality (8) can be used in three point quantum network. Thus, for multipartite quantum nonlocality the product-form monogamy relation may be applied to more complex quantum networks [48, 49]. Our results may highlight further researches on other quantum correlations like quantum discord and quantum steering.

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