Investigation and selection of a functional in the problem of synthesis of an optimal control law providing inspection motion

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Abstract. In this paper we consider the problem of keeping a nanosatellite near the nominal inspection ellipse, which is the Hill ellipse. The source of the disturbing effect, leading to the deformation of the nominal trajectory, is the drag of the atmosphere and the nonlinearity of the central attraction field. Using the traditional approach to finding the control law in the form of a linear-quadratic controller does not always meet the specified requirements, since it is based on the formation of a control law based on a linearized model of motion dynamics. The method of control law synthesis in the form of a linear-quadratic controller, using a quasilinear model of motion (SDRE-method), allows to increase the accuracy of the implementation of the inspection motion. In this work, a study was carried out to select the parameters of the quadratic quality criterion from the conditions to provide the requirements for energy costs and the quality of implementation of the inspection motion.

1. Introduction
At present, the formation flight technology is used to solve scientific and applied problems in space, when two or more spacecraft (SC) jointly solve the problem, being in the zone of direct radio visibility and flying in the required geometric formation. This technology is used to solve both scientific [1] and applied problems [2]. One of the possible types of formation is an inspection motion, when one spacecraft makes an inspection motion relative to another along the Hill ellipse [3]. This type of formation is used for complex multi-module objects in orbit, spacecraft refueling or space debris detection [4]. With the advent of a new class of spacecraft – nanosatellites (NS), it became possible to use them for the implementation of a formation flight. During orbital motion, the geometry of the formation will change. The main disturbing factors affecting the NS motion along the inspection ellipse in low Earth orbits will be aerodynamic perturbation, nonlinearity of the central attraction field [5] and the second zonal harmonic of the Earth's gravitational potential [6]. It is possible to reduce the rate of degradation of the nominal geometric structure of the formation caused by gravitational disturbances by selecting the initial conditions of motion [6]. This makes it possible to implement the inspection motion at a certain finite time interval without motion correction, but it is necessary to carry out a correction for maintaining the nominal trajectory throughout the entire mission. The robust control can be used to create control actions [7, 8]. If there are more than two spacecraft in the constellation, decentralized control is used [9, 10]. The control law can also be obtained using methods based on Bellman's dynamic programming [11] in which a linear-quadratic controller (LQR) is used. To take into account nonlinearities in models of relative motion, a method is used that uses the State Dependent Riccati
Equation (SDRE), which converts a nonlinear system into a quasilinear one [11]. In this paper, we study the capabilities of the SDRE method to provide inspection motion near the nominal ellipse and formulate recommendations for choosing the weight coefficients of the quadratic quality criterion based on the conditions for providing the required quality of implementation of the relative movement and admissible control costs.

2. Formulation of the problem

The paper considers the nanosatellite motion relative to the inspected object. The orbital coordinate system (OSC) was chosen as the reference system, the origin of which is connected with the center of mass of the inspected object, the Oy axis is directed along the radius vector, the Ox axis lies in the orbital plane and is directed towards the orbital motion, and the Oz axis complements the coordinate system to the right-hand. The assumption is made that the aerodynamic disturbance acts only in the direction of the incident flow. The motion of the NS in the OSC under the assumption of a central attraction field and a circular orbit of the inspected object will be described by the system of equations [12]:

$$\begin{align*}
\dot{x} + 2\omega \dot{y} - \omega^2 x + \frac{\omega^2 r^3}{\left(\sqrt{x^2 + (r+y)^2 + z^2}\right)^3} x &= P_x + u_x \\
\dot{y} - 2\omega \dot{x} - \omega^2 (r + y) + \frac{\omega^2 r^3}{\left(\sqrt{x^2 + (r+y)^2 + z^2}\right)^3} (r + y) &= u_y \\
\dot{z} + \frac{\omega^2 r^3}{\left(\sqrt{x^2 + (r+y)^2 + z^2}\right)^3} z &= 0
\end{align*}$$

where $P_x = \rho V^2 \Delta S$ – disturbing aerodynamic acceleration, $u_x, u_y$ – control acceleration; $\omega$ – angular speed of the inspected object.

Having made the assumption about the smallness of the distance between the NS and the inspected object in comparison with the radius vector and taking into account the linearization of the central attraction field, a linear analog of the system (1) is written [12]:

$$\begin{align*}
\dot{x} + 2\omega \dot{y} &= P_x + u_x \\
\dot{y} - 2\omega \dot{x} - 3\omega^2 y &= u_y \\
\dot{z} + \omega^2 z &= 0
\end{align*}$$

In the absence of control and perturbing accelerations in the right-hand side, the system (2) will have an analytical solution in which the motion in the orbital plane does not depend on the out-of-plane motion. Further in this paper, the plane case of orbital motion is considered. When using special initial conditions, the NS will move along the Hill ellipse, which is taken as the nominal inspection trajectory.

$$X_{\text{nom}} = \left(x_0, y_0, \dot{x}_0 = y_0 \omega, \dot{y}_0 = -2x_0 \omega, \dot{z}_0 = 0\right)^T$$

Traditionally, a mathematical model (2) is used to synthesize a control law with a quadratic quality criterion (LQR method). As shown in [11], to obtain the quality of control of the inspection motion, it is possible to use the SDRE method for the nonlinear model (1), which is interpreted as a quasilinear model.

The paper investigates the influence of the quadratic quality criterion parameters on the error in the implementation of the inspection motion and energy costs. The preliminary study is performed using
the less labor-intensive LQR method. Then the results are extended to the SDRE method. Numerical calculations confirmed the validity of the proposed approach and allowed to make a conclusion about the effectiveness of using the SDRE method.

3. Control law based on Bellman’s optimality principle

The quality criterion for the problem under consideration is written in the form of a quadratic functional:

\[ J(\Delta X, u_{opt}(\Delta X)) = K_1 \cdot (\Delta X(t_f))^T \cdot F \cdot \Delta X(t_f) + K_2 \int_{t_0}^{t_f} (\Delta X(t))^T \cdot Q \cdot (\Delta X(t)) \, dt + K_3 \int_{t_0}^{t_f} u_{opt}(\Delta X(t))^T \cdot R \cdot u_{opt}(\Delta X(t)) \, dt \rightarrow \min \] (4)

where: \( \Delta X \) – vector of the disturbed deviations trajectory from the nominal inspection ellipse; \( t_f \) – initial time of control law formation; \( t_0 \) – finite simulation time; \( T \) – time interval that determines the frequency of the control law formation; \( F, Q, R \) – given square weight matrices, \( F \geq 0, Q \geq 0 \) non-negative defining matrices; \( R > 0 \) positive definite matrix; \( K_1, K_2, K_3 \) – the coefficients of significance of the terms of the quadratic quality criterion;

It is known that the optimal control law when using (4) has the form [13]:

\[ u_{opt}(\Delta X) = -R^{-1} \cdot B^T \cdot k \cdot \Delta X \] (5)

where the matrix \( k \) is the solution to the matrix of differential equations of the Riccati type:

\[ \frac{dk}{dt} = -k \cdot A - A^T \cdot k + k \cdot B \cdot R^{-1} \cdot B^T \cdot k - Q \] (6).

The matrices included in (6) compiled for model (2) have the form:

\[
A = \begin{bmatrix} 0 & -2w & 0 & 0 \\ 2w & 0 & 0 & 3w^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad F = K_1 \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q = K_2 \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = K_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

The coefficients \( q \) and \( f \) in the matrices determine the significance of the velocity projection disturbances in comparison to the trajectory disturbances. To synthesize the control law, a traditional algorithm is used that simulates the disturbed motion using model (1) and calculates control law (5) with a periodicity \( T \) while simultaneously solving equation (6), which will compensate for deviations from the inspection nominal ellipse in the next time interval. The parameter \( T \) is selected from the condition of meeting the requirements for the accuracy of the implementation of the inspection motion.

4. Parametric study of the relative motion control law

The motion of the inspected object in a circular orbit in the range of altitudes from 350 to 450 km is considered. The difference in ballistic coefficients corresponds to the difference in the ballistic coefficients of the NS of CubeSat 3U format and the ISS. The OSC origin is associated with the ISS center of mass. For a preliminary analysis, the coefficients of significance of the terms of the quadratic quality criterion (4) were taken equal \( K_1 = K_2 = K_3 = f = 1 \), matrix \( Q \) is taken to be zero. The initial position of the nanosatellite has coordinates \( x_o = x(0), y_o = 0 \). Density of the atmosphere \( \rho \) is taken static and calculated according to GOST 4401-81. The motion is simulated on a time interval of one turn. The following initial conditions and parameters are used:

- \( r = 6371 \text{ km} \) – radius of the Earth; \( \mu = 398602 \text{ km}^2/\text{s}^2 \) – gravitational parameter of the earth;
- \( S_b^{\text{ISS}} = 0,03176 \text{ m}^2/\text{kg} \) – ballistic coefficient of the ISS [14]; \( S_b^{\text{NS}} = 0,0115 \text{ m}^2/\text{kg} \) – ballistic coefficient of NS (CubeSat 3U) [15]; \( \Delta S = S_b^{\text{ISS}} - S_b^{\text{NS}} \) – difference between ballistic coefficients of the ISS and NS. The variable parameters are the size of the inspection ellipse, which is specified through
the initial condition $x_0 = 500...5000 \text{m}$ and the frequency of recalculation of the control law in fractions of the period of orbital motion $T = 1/8, 1/16, 1/32, 1/5440$ of turn.

Figures 2-5 show the simulation results obtained for the ISS orbit altitude of 400 km, $x_0 = 1000 \text{ m}$, $T = 1/5440$ of turn (the control law is recalculated every second).
Table 1 shows the maximum deviations and maximum control actions obtained as a result of the relative motion simulation with varying ISS flight altitude $h$.

| Parameter $h$, km | 350  | 400  | 450  |
|-------------------|------|------|------|
| $\Delta x_{\text{max}}$ (m) | 15,06 | 6,18 | 2,78 |
| $\Delta y_{\text{max}}$ (m) | 10,23 | 4,37 | 2,01 |
| $\Delta V_{x,\text{max}}$ (m/s) | 0,0149 | 0,0062 | 0,0028 |
| $\Delta V_{y,\text{max}}$ (m/s) | 0,0089 | 0,0037 | 0,0016 |
| $u_{\text{max}}$ (m/s$^2$) | $12,68 \times 10^{-5}$ | $6,09 \times 10^{-5}$ | $3,33 \times 10^{-5}$ |

Table 1 shows that with an increase in the ISS flight altitude, the maximum deviations decrease due to a decrease in disturbing aerodynamic accelerations, which leads to a decrease in the values of control actions.

Table 2 shows the results showing the effect of the inspection ellipse size on the control process.

| Parameter $x_0$, m | 500  | 1000 | 2000 | 4000 | 5000 |
|-------------------|------|------|------|------|------|
| $\Delta x_{\text{max}}$ (m) | 5,99 | 6,18 | 6,97 | 10,03 | 12,31 |
| $\Delta y_{\text{max}}$ (m) | 4,35 | 4,37 | 4,47 | 5,06 | 5,69 |
| $\Delta V_{x,\text{max}}$ (m/s) | 0,0061 | 0,0062 | 0,0065 | 0,0088 | 0,0105 |
| $\Delta V_{y,\text{max}}$ (m/s) | 0,0037 | 0,0037 | 0,0037 | 0,0046 | 0,0054 |
| $u_{\text{max}}$ (m/s$^2$) | $4,99 \times 10^{-5}$ | $6,09 \times 10^{-5}$ | $6,68 \times 10^{-5}$ | $6,81 \times 10^{-5}$ | $9,85 \times 10^{-5}$ |

With an increase of the size of the Hill ellipse, the influence of the linearization error of the central attraction field increases and, accordingly, the control actions increase.

Table 3 shows the results of the influence of the control law recalculation frequency on the control process.
With an increase in the interval $T$, that is, with a less frequent recalculation of the control law, the quality of the implementation of the inspection motion deteriorates.

The study of the influence of the significance coefficients of the quadratic quality criterion on the error in the implementation of the inspection motion. The resulting maximum deviations and control actions are used as performance indicators. The first two diagonal elements of the matrices $Q$ and $F$ determine the weight of the velocity deviations, and the second two determine the weight of the coordinate deviations. From the condition of providing the approximate equivalence of all deviations, the following values are taken $f = q = 10^6$.

To reveal the sensitivity of the inspection motion control results to the change in the coefficients of the significance of the quadratic quality criterion, the interval of the frequency of recalculation of the control law $T = 1/8$ of turn, the ISS orbit altitude is 400 km, the size of the ellipse of the nominal inspection $x_m = 1000 m$. Table 4 shows the results of the study of the relative motion for various weights.

### Table 4

| Parameter | $K_1 = 0.01$ | $K_1 = 0.1$ | $K_1 = 0.01$ | $K_1 = 0.01$ |
|-----------|---------------|--------------|---------------|---------------|
| $\Delta x_{\text{max}}$ (m) | 16.75 | 13.92 | 24.19 | 3.54 |
| $\Delta y_{\text{max}}$ (m) | 7.58 | 6.85 | 9.40 | 4.56 |
| $\Delta V_{\text{x,max}}$ (m/s) | 0.0150 | 0.0074 | 0.0099 | 0.0255 |
| $\Delta V_{\text{y,max}}$ (m/s) | 0.0084 | 0.0074 | 0.0099 | 0.0255 |
| $u_{\text{max}}$ (m/s$^2$) | $1.39 \cdot 10^{-5}$ | $1.53 \cdot 10^{-5}$ | $1.22 \cdot 10^{-5}$ | $3.85 \cdot 10^{-5}$ |

The smallest deviations of the disturbed inspection motion trajectory from the nominal inspection ellipse are observed at $K_1 = 0.01$, $K_2 = 0$, $K_3 = 10^6$, which correspond to the case of approximately equal contribution of each item to the formation of the control quality criterion. However, this is accompanied by an increase in the values of the maximum control actions and leads, as a consequence, to high energy costs. Let us investigate the possibility of reducing energy costs by using a more accurate, but also more time consuming SDRE method.

5. **Synthesis of the control law based on the SDRE method**

The SDRE method takes into account the nonlinear motion model of the system (1), transforming it into a quasilinear one, which includes matrices with coefficients dependent on the state of the system (State-Dependent Coefficient - SDC). The Riccati equation using SDC matrices is integrated with motion simulation. The mathematical model in deviations for a nonlinear system (1) will be written in the form:
\[ \Delta X = A(\Delta X^*) \cdot \Delta X + B \cdot u_{opt}(\Delta X) \]  (7)

Unlike the previously used LQR method, instead the vector \( \Delta X \), when calculating the matrix \( A(\Delta X^*) \), \( \Delta X^* \) is used, which is formed without control over the entire simulation interval.

System (7) has a quasilinear structure, where \( A \) is the SDC matrix:

\[
A(\Delta X^*) = \begin{bmatrix}
0 & -2w & w^2 & -\frac{\omega^2 r^3}{\sqrt{\Delta x^* + (r + \Delta y^*)^2}} & 0 \\
2w & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]  (8)

Functional (4) is also used as a quality criterion. The Riccati equation using SDC matrices will have the form:

\[
\frac{dk(\Delta X^*)}{dt} = -k(\Delta X^*) \cdot A(\Delta X^*) - A(\Delta X^*)^T \cdot k(\Delta X^*) + k(\Delta X^*) \cdot B \cdot R^{-1} \cdot B^T \cdot k(\Delta X^*) - Q
\]  (9)

The quasilinear optimal control controller is written as a function of \( \Delta X, \Delta X^* \):

\[ u_{opt}(\Delta X, \Delta X^*) = -R^{-1} \cdot B^T \cdot k(\Delta X^*) \cdot \Delta X \]  (10)

The algorithm for calculating the control action is similar to the algorithm based on the Bellman optimality principle, with the exception of solving the Riccati equation. The matrix Riccati equation using SDC matrices includes deviations \( \Delta X^* \), so it must be integrated in conjunction with the simulation of the disturbed motion. Thus, the difference between the methods is in the structure of the matrices included in the Riccati equation.

6. Comparison of the effectiveness of the two methods for forming the control law

Comparison of the effectiveness of the inspection motion implementation in the formation of the control law using SDRE and LQR methods is performed. The coefficients of significance of the quality functional items correspond to the previously selected values \( K_1 = 0.01, K_2 = 10^{-6}, K_3 = 10^6 \). The initial data correspond to those previously accepted: \( T=1/8 \) of turn, \( x_o = 1000 \text{ m} \). Table 5 shows the results of the relative motion simulation for different flight altitudes of the ISS.

| Parameter         | 350       | 400       | 450       |
|-------------------|-----------|-----------|-----------|
| \( \Delta x_{max} \) (m) | 8.73      | 5.28      | 3.54      |
| \( \Delta y_{max} \) (m) | 11.52     | 6.12      | 4.56      |
| \( \Delta V_{x_{max}} \) (m/s) | 0.0286    | 0.0145    | 0.0109    |
| \( \Delta V_{y_{max}} \) (m/s) | 0.0681    | 0.0366    | 0.0255    |
| \( u_{x_{max}} \) (m/s²) | 9.10^{-5} | 6.7710^{-5} | 3.8510^{-5} |
| \( u_{y_{max}} \) (m/s²) | 14.9510^{-5} | 7.0510^{-5} | 5.7110^{-5} |
It can be concluded that the control law formed on the basis of the SDRE method makes it possible to achieve a twice reduction in the error in the implementation of the nominal inspection ellipse with an approximately twice decrease in energy consumption for control. This conclusion is confirmed by varying the size of the nominal inspection ellipse (Table 6).

| Parameter       | 500     | 1000    | 2000    |
|-----------------|---------|---------|---------|
| \( \Delta x_{\text{max}} \) (m) | 3.47    | 1.96    | 3.54    | 2.19    | 3.81    | 2.21    |
| \( \Delta y_{\text{max}} \) (m) | 4.46    | 2.51    | 4.56    | 2.53    | 4.88    | 2.64    |
| \( \Delta V_{x_{\text{max}}} \) (m/s) | 0.0104  | 0.0057  | 0.0109  | 0.0058  | 0.0126  | 0.0066  |
| \( \Delta V_{y_{\text{max}}} \) (m/s) | 0.0245  | 0.0136  | 0.0255  | 0.0138  | 0.0286  | 0.0146  |
| \( u_{x_{\text{max}}} \) (m/s²) | 3.71·10^{-5} | 2.70·10^{-5} | 3.85·10^{-5} | 2.74·10^{-5} | 4.24·10^{-5} | 2.91·10^{-5} |
| \( u_{y_{\text{max}}} \) (m/s²) | 5.49·10^{-5} | 2.61·10^{-5} | 5.71·10^{-5} | 2.68·10^{-5} | 6.36·10^{-5} | 2.86·10^{-5} |

The disadvantage of the SDRE method is a significant increase in the complexity of calculating the control law, which imposes high requirements on the performance of computing facilities if the considered approach to control formation is implemented on board a spacecraft.

7. Conclusion
The paper investigates the efficiency of applying the optimal control law based on the SDRE-method in the problem of providing inspection motion near the nominal ellipse. Parametric studies have confirmed the hypothesis that it is possible to select the coefficients of significance in the quadratic quality criterion using the LQR method, which is caused by the need to reduce the complexity of the computational process. This is also due to the savings in computing time. The analysis showed that the SDRE method provides a decrease in deviations of the disturbed trajectory from the nominal one with a significant decrease in the values of control actions, and therefore energy costs, compared to the LQR method, due to the nonlinearity of the mathematical model of relative motion. However, the application of this approach leads to an increase in intensity and increased requirements for onboard computing facilities during implementation.

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