Stability estimates for the regularized inversion of the truncated Hilbert transform

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Abstract
In limited data computerized tomography, the 2D or 3D problem can be reduced to a family of 1D problems using the differentiated backprojection method. Each 1D problem consists of recovering a compactly supported function \( f \in L^2(\mathcal{F}) \), where \( \mathcal{F} \) is a finite interval from its partial Hilbert transform data. When the Hilbert transform is measured on a finite interval \( \mathcal{G} \) that only overlaps but does not cover \( \mathcal{F} \), this inversion problem is known to be severely ill-posed (Alaifari et al 2015 SIAM J. Math. Anal. 47 797–824).

In this paper, we study the reconstruction of \( f \) restricted to the overlap region \( \mathcal{F} \cap \mathcal{G} \). We show that with this restriction and by assuming prior knowledge on the \( L^2 \) norm or on the variation of \( f \), better stability with Hölder continuity (typical for mildly ill-posed problems) can be obtained.

Keywords: limited data computerized tomography, truncated Hilbert transform, severely ill-posed, regularized inversion, stability estimate

1. Introduction

This paper presents new results on the stability of the inversion of the Hilbert transform with limited data. The main application concerns tomographic reconstruction from truncated projections where some 2D and 3D problems can be reduced to the inversion of the Hilbert transform on a family of line segments defined by the geometry of the scanners. The reduction of the 2D or 3D tomography problem to a 1D Hilbert transform inversion problem is achieved by backprojecting a derivative of the projection data over an angular range of 180°. This operation based on a result by Gelfand and Graev [9] is referred to as the differentiated...
backprojection (DBP). In the 2000s, the DBP triggered a remarkable evolution in tomography by allowing accurate reconstruction of regions of interest (ROI) from truncated projection data sets [15, 16, 21], which were previously believed to exclude any accurate reconstruction. A second method for accurate ROI reconstruction from limited data, the virtual fan-beam algorithm, was introduced in 2002; it is based on a different mathematical property of the radon transform and will not be studied in this paper. The reader is referred to [6] for a review on ROI reconstruction in tomography and the link between that problem and the Hilbert transform.

We work with a line integral model of the tomographic data, assuming continuous (infinitely fine) sampling of the 2D or 3D x-ray transform of the unknown function \( f(x) \) within some subset of the space of lines in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). In this context ‘accurate’ reconstruction means that the data uniquely and stably determine \( f(x) \) within a ROI under appropriate assumptions. These assumptions specify the functional spaces to which the object and the data belong, the nature and magnitude of the measurement noise, and, typically, involve prior constraints on the object. Loosely speaking, we call a reconstruction ‘stable’ with Hölder continuity if the norm of the error on the estimated \( f(x) \) (possibly within some limited ROI) can be bounded by some positive power of the norm of the measurement error [5]. In contrast, one speaks of logarithmic continuity when the error bound decreases only with the logarithm of the noise. The noise propagation properties of the DBP are well understood since they are similar to those of the usual filtered-backprojection (FBP) algorithm. This can be seen by observing that the two procedures involve the same backprojection and that the ramp filter \(|\nu|\) (for FBP) and the derivative filter \(2\pi i\nu\) (for DBP) similarly amplify high frequencies. Thus, the major question left is to analyze the stability of the inversion of the Hilbert transform with limited data. The rest of this paper focuses on that 1D inverse problem.

To date, much more is known on conditions guaranteeing uniqueness of the solution than on the stability in the presence of noise. Stability is easy to analyze when a closed form analytic inversion formula is known, such as the FBP algorithm for the radon transform with complete data. With the DBP, however, a closed form inversion exists only when the limited data allow calculating the Hilbert transform of \( f(x) \) on a line segment \( \mathcal{G} \) that contains the support \( \mathcal{F} \) of \( f(x) \) along that line. In this case, inversion is based on the finite inverse Hilbert transform and has very good stability (all singular values but one are equal to 1 in a weighted \( L^2 \) space [17]). Unfortunately, no closed form inversion is known for the two other configurations: the interior problem (\( \mathcal{G} \subset \mathcal{F} \)) and the truncated problem where \( \mathcal{G} \) partially overlaps \( \mathcal{F} \).

It has long been known [14] that unique inversion is not possible for the interior problem, even within the ROI \( \mathcal{G} \) where the data are known. However, uniqueness holds when additional information is available. One such instance is the case where \( f(x) \) is known in some segment \( \mathcal{K} \subset \mathcal{G} \) [7, 12, 20]. Other results show that uniqueness is restored when \( f(x) \) is assumed to belong to some function space (e.g., \( f(x) \) is piecewise constant, a polynomial function, or a generalized spline) [18, 19]. Numerical tests with discretized models of these problems suggest good stability, but these results may depend on the way the problem is discretized. To our knowledge, explicit stability estimates have only been obtained when \( f(x) \) is a polynomial function in the interior region [11].

For the truncated problem, uniqueness of the solution follows from simple analyticity arguments [8]. However, the stability bound obtained in [8] is difficult to interpret since it relates the reconstruction error to an upper bound on the error on an intermediate function and not to the error on the Hilbert transform. Also, the results in [8] do not pertain to a specific regularization method. Recently, a more explicit stability estimate has been obtained in [3]. It
guarantees logarithmic continuity for the reconstruction on the entire support $\mathcal{F}$ of $f(x)$ if prior knowledge on the total variation of $f$ is assumed.

This paper exploits recent results on the asymptotics of the singular value decomposition (SVD) of the truncated Hilbert transform with overlap [1]. With these, explicit $L^2$ stability estimates are obtained. They allow guaranteeing stable inversion of the corresponding inverse problem when an \textit{a priori} bound on the $L^2$ norm of the solution is assumed. We give specific examples for the truncated SVD and for Tikhonov’s method. By seeking to reconstruct only within a ROI, Hölder continuity is obtained. A similar result is shown for prior knowledge on the total variation of $f(x)$.

**Remark.** Stability estimates allow assessing the ill posedness of an inverse problem independent of the algorithm and independent of the discrete sampling of the data and solution. The inversion of the truncated Hilbert transform is a very specific problem, which is severely or mildly ill posed depending on which ROI is reconstructed. The stability estimates in this paper are derived in a deterministic framework, they give worst case error bounds, which are pessimistic and for that reason do not provide reliable absolute values for the regularization parameter. However, our results for the Hilbert transform determine the Hölder power of the reconstruction accuracy in terms of known constants depending on the geometry of the problem (i.e., on the intervals $\mathcal{F}$ and $\mathcal{G}$). This may provide guidance to adapt the regularization parameter to varying geometry or data error once it has been empirically estimated for one case. Future work will investigate application to image reconstruction from incomplete computerized tomography data.

2. The truncated Hilbert transform

Consider the truncated Hilbert transform $H_T : L^2(\mathcal{F}) \to L^2(\mathcal{G})$ with $\mathcal{F} = (a_2, a_4)$, $\mathcal{G} = (a_1, a_3)$, and $a_1 < a_2 < a_3 < a_4$,

$$(H_T f)(x) = \frac{1}{\pi} \text{p.v.} \int_{a_2}^{a_3} \frac{f(y) dy}{y-x}, \quad x \in (a_1, a_3).$$

(2.1)

We define the normalized singular function pair $u_n \in L^2(\mathcal{F})$, $v_n \in L^2(\mathcal{G})$ such that $H_T u_n = \sigma_n v_n$ for $n \in \mathbb{Z}$. The singular values $\sigma_n$ are ordered as usual by decreasing values, and the spectrum has two accumulation points, $\lim_{n \to -\infty} \sigma_n = 1$ and $\lim_{n \to \infty} \sigma_n = 0$. We will make use of the following properties of the singular system of $H_T$:

- For large $n > 0$, the asymptotic behavior of the singular values is given by

$$\sigma_n = 2e^{-nK_+/(2z)} (1 + O(n^{-1/2+\zeta})), \quad n \to \infty$$

(2.2)

for some small $\zeta > 0$ and with

$$K_- = \int_{a_1}^{a_2} \frac{dx}{\sqrt{-P(x)}}, \quad K_+ = \int_{a_3}^{a_4} \frac{dx}{\sqrt{P(x)}}.$$

(2.3)

Here, $P(x) = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$.

- For large negative $n$, the asymptotic behavior of the singular values is

$$\sigma_n = (1 - 2e^{-2|n|zK_-/(2z)}) (1 + O(|n|^{1/2+\zeta})), \quad n \to -\infty.$$  

(2.4)
The singular function \( v_n \) is bounded at \( a_1 \) and \( a_3 \) and has a logarithmic singularity at \( a_2 \). For large \( n > 0 \), it is an oscillatory function in the interval \((a_1, a_2)\) and a monotonically decreasing function on \((a_2, a_3)\).

The function \( u_n \) is bounded at \( a_2 \) and \( a_4 \) and has a logarithmic singularity at \( a_3 \). For large \( n > 0 \), it is a monotonically increasing function in the interval \((a_2, a_3)\) and an oscillatory function on \((a_3, a_4)\).

For sufficiently small \( \mu > 0 \), the norm of \( u_n \) over the segment \((a_2, a_3 - \mu)\) has the following asymptotic behavior:

\[
\left( \int_{a_2}^{a_3 - \mu} dx \ |u_n(x)|^2 \right)^{1/2} = \frac{1}{\sqrt{2\pi}} e^{-\beta_\mu n} (1 + O(n^{-1/2} + \epsilon)),
\]

where

\[
\beta_\mu = \frac{\pi}{K} \int_{a_2}^{a_3 - \mu} \frac{dr}{\sqrt{P(t)}} = \frac{2\pi}{K} \frac{\sqrt{\mu}}{\sqrt{-P'(a_3)}} (1 + O(\mu)).
\]

Except for the last property, which is proven in the appendix (see lemma 1), the proofs of all the above properties can be found in [1, 2]. However, the monotonicity of the singular functions \( u_n \) is shown more explicitly in the appendix (see lemma 2).

3. Inversion of the truncated Hilbert transform: regularization with an \( L^2 \) penalty

We consider the following inverse problem. Let \( f_{ex} \in L^2(F) \) be an object of interest, and let \( g_{ex} = H_T f_{ex} \in L^2(\mathcal{G}) \) be the corresponding noise-free data with \( H_T \) as the truncated Hilbert transform (2.1). Given a noisy measurement \( g \) such that \( \| g - g_{ex} \|_{L^2(\mathcal{G})} \leq \delta \), for some noise level \( \delta > 0 \), we seek an estimate of \( f_{ex} \) on some interval \((a_2, a_3 - \mu)\) with some small \( \mu > 0 \).

The rationale to restrict the reconstruction to a limited interval is that there is no hope of obtaining a Hölder stability estimate outside the segment \( \mathcal{G} = (a_1, a_3) \) where the Hilbert transform is known. Following a similar intuitive analysis by Natterer [14] for the exterior problem of tomography, this can be seen by considering an object \( f_{ex} \), which is supported in \((a_3, a_4)\) and has a discontinuity somewhere in that interval. For such an object, one sees, from (2.1), that \( H_T f_{ex} \) is \( C^\infty \) and, therefore, an inverse operator should map a \( C^\infty \) function onto a discontinuous function, which corresponds to a severely ill-posed problem. This severe ill posedness is also expected from the exponential decay of the singular values, equation (2.2). Thus, unless very restrictive prior knowledge is assumed, the quest for a stable reconstruction must be restricted to an interval, such as \((a_2, a_3 - \mu)\). By varying the parameter \( \mu \), one can study the degradation of the stability as one gets closer to the limit of the stably recoverable region \((a_2, a_3)\). We denote by \( \chi_\mu \) the characteristic function of that ROI interval \((a_2, a_3 - \mu)\).

We assume the following properties of the SVD:

- P1: 0 < \( \sigma_n < 1 \).
- P2: There are some \( N_0 \in \mathbb{N} \) and positive real \( A, \alpha \) such that \( \sigma_n \geq Ae^{-\alpha n} \) for \( n > N_0 \).
- P3: There are some \( N'_0 \in \mathbb{N} \) and positive real \( B_\mu, \beta_\mu \) such that \( \| \chi_\mu u_n \|_{L^2(F)} \leq B_\mu e^{-\beta_\mu n} \) for \( n > N'_0 \). The constants \( A \) and \( \alpha \) can be obtained from (2.2),

\[
A < 2, \quad \alpha = \pi K_+ / K_-, \tag{3.1}
\]
where $A = 2$ corresponds to the leading term of the asymptotic behavior of the singular values. Hence, any value of $A$ smaller than 2 provides an upper bound for sufficiently large $n$. We note that the index $N_n$ and the constants $B_\mu$, $\beta_\mu$ all depend on the choice of $\mu$. In particular, for P3 to hold, $N_n = O(\mu^{-1/2})$, so the smaller $\mu$ is chosen to be, the larger $N_n$ has to be taken. Throughout this paper, we assume $\mu > 0$ to be arbitrarily small but fixed. The results obtained in the following sections are no longer valid as $m \to 0$. This is due to the fact that $\mathcal{A}_{cm} \mathcal{A}_{cm}$ only decays as $O(n^{1/2})$ (see equation (8.6) in [1]). In what follows, we select $N_\mu$ such that

\[
\frac{1}{\sqrt{N_\mu \pi}}.
\]

3.1. An algorithm independent stability bound

We assume as regularizing prior knowledge on the solution, an upper bound $E > 0$ on the norm of the object $\mathcal{A}_{cm}$. The stability bound that we present follows the approach of Miller [13]. This formulation is independent of a specific algorithm but, instead, yields an upper bound on the $L^2$ distance (within the interval $(a_2, a_3 - \mu)$) between any pair of solutions that are compatible both with the data and with the prior knowledge.

We define the set of admissible solutions as

\[
S = \{ f \in L^2(\mathcal{F}) \mid \| Hf - g \|_{L^2(\mathcal{G})} \leq \delta \text{ and } \| f \|_{L^2(\mathcal{F})} \leq E \}.
\]

Proposition 1. Let $Hf_{ex} = g_{ex}$ and assume $\| f_{ex} \|_{L^2(\mathcal{F})} \leq E$. We consider the reconstruction of $f_{ex}$ from a noisy measurement $g$ for which $\| g - g_{ex} \|_{L^2(\mathcal{G})} \leq \delta$ is given. Then, any algorithm that, for given $\delta$, $E$, and $g$, computes a solution in $S$ is a regularization method for the reconstruction on $(a_2, a_3 - \mu)$. More precisely, given any two admissible solutions $f_a, f_b \in S$, one has for sufficiently small $\delta$, the bound,

\[
\| Hf_a - f_b \|_{L^2(\mathcal{F})} \leq 2\delta e^{2\beta_\mu} + 2EB_\mu \left\{ \frac{\beta_\mu}{A} \frac{\alpha}{V_\mu E} \right\},
\]

where $V_\mu$ is a constant that only depends on $\alpha$ and $\beta_\mu$.

Proof. Consider two admissible solutions $f_a, f_b \in S$. The corresponding Hilbert transforms $g_a = Hf_a$ and $g_b = Hf_b$ satisfy $\| g_a - g_b \|_{L^2(\mathcal{G})} \leq \| g_a - g \|_{L^2(\mathcal{G})} + \| g_b - g \|_{L^2(\mathcal{G})} \leq 2\delta$. Similarly, $\| f_a - f_b \|_{L^2(\mathcal{F})} \leq 2E$. We can rewrite $f_a - f_b$ using the SVD of $Hf$ and splitting the obtained expansion into three terms as follows:

\[
f_a - f_b = \sum_{n=-\infty}^{N} (g_a - g_b, v_n) \frac{1}{\sigma_n} u_n + \sum_{n=N+1}^{N} (g_a - g_b, v_n) \frac{1}{\sigma_n} u_n + \sum_{n=N+1}^\infty (f_a - f_b, u_n) u_n.
\]

(3.3)
Here, the SVD series is split at $N_\mu$ and at some—so far arbitrary—index $N$, and we used
\[ \langle g_a, v_n \rangle = (H f_a, v_n) = \sigma_n (f_a, u_n) \] and the same as before for $f_b$. Equation (3.3) holds for any $N > N_\mu$.

We seek to derive an upper bound on the error in the interval of interest in the form of
\[ \| \chi_\mu (f_a - f_b) \|_{L^2(\mathcal{F})} \leq I_1 + I_2 + I_3, \]
where $I_1$–$I_3$ correspond to the norms of the three terms in (3.3) restricted by $\chi_\mu$.

The error due to the large singular values is bounded as
\[
I_1^2 = \left\| \chi_\mu \sum_{n = -\infty}^N (g_a - g_b, v_n) \frac{1}{\sigma_n} u_n \right\|^2_{L^2(\mathcal{F})} \leq \left\| \sum_{n = -\infty}^N (g_a - g_b, v_n) \frac{1}{\sigma_n} u_n \right\|^2_{L^2(\mathcal{F})} \leq \frac{(2\delta)^2}{\sigma_{N_\mu}}. \tag{3.5}
\]

The contribution of the intermediate terms corresponding to $N_\mu < n \leq N$ to the error is bounded as
\[
I_2^2 = \left\| \sum_{n = N_\mu + 1}^N (g_a - g_b, v_n) \frac{1}{\sigma_n} \chi_\mu u_n \right\|^2_{L^2(\mathcal{F})} \leq \left\{ \sum_{n = N_\mu + 1}^N \left| \langle g_a - g_b, v_n \rangle \right|^2 \frac{1}{\sigma_n^2} \right\} \leq \frac{(2\delta)^2 B_{N_b}^2}{A^2} \sum_{n = N_\mu + 1}^N \frac{e^{2(\alpha - \beta_n)n}}{1 - e^{-2(\alpha - \beta_n)}} \tag{3.6}
\]

where we successively used the triangle inequality, Schwarz inequality, and majorized by extending the sum over $n'$ to all $\mathbb{Z}$. Recall from the definitions of these constants that $\alpha > \beta_\mu > 0$. The last inequality is not strictly needed but simplifies subsequent derivations, and it is expected to have a limited impact since, for small $\delta$, we will find that the optimal cutoff satisfies $N \gg N_\mu$.

Finally,
\[
I_3^2 = \left\| \sum_{n = N + 1}^\infty (f_a - f_b, u_n) \chi_\mu u_n \right\|^2_{L^2(\mathcal{F})} \leq \left\{ \sum_{n = N + 1}^\infty \left| \langle f_a - f_b, u_n \rangle \right| \right\}^2 \leq \sum_{n' = N + 1}^\infty \left| \langle f_a - f_b, u_n \rangle \right|^2 \sum_{n = N + 1}^\infty \left| \chi_\mu u_n \right|^2_{L^2(\mathcal{F})} \leq \sum_{n' = N + 1}^\infty \frac{e^{-2\beta_n}}{1 - e^{-2\alpha}} \leq (2E) \sum_{n = N + 1}^\infty \frac{e^{-2\beta_n}}{1 - e^{-2\alpha}} \leq (2E) \frac{B_\mu^2 e^{-2\beta_n}}{1 - e^{-2\alpha}}. \tag{3.7}
\]
Inserting equations (3.5), (3.6), and (3.7) into (3.3) leads to the following upper bound on the $L^2$ distance between $f_a$ and $f_b$ on the ROI interval $(a_2, a_3 - \mu)$,

$$
\|X_f(f_a - f_b)\|_{L^2(\mathcal{F})} \leq \frac{2\delta}{\sigma_n} + \frac{2\delta B_n}{A} \frac{e^{(\alpha - \beta_n)N}}{\sqrt{1 - e^{-2(\alpha - \beta_n)}}} + \frac{2EB_n e^{-\beta_n N}}{\sqrt{e^{2\beta_n} - 1}}.
$$

(3.8)

A quasioptimal splitting index $N(\delta)$ is obtained by treating $N$ as a real variable and by minimizing the convex function on the right-hand side (RHS) of (3.8) with respect to $N$,

$$
N(\delta) = \frac{1}{\alpha} \log \left( \frac{EAV_n}{\delta} \right),
$$

(3.9)

with the constant,

$$
V_n = \frac{\beta_n}{(\alpha - \beta_n)} \left\{ \left(1 - e^{-2(\alpha - \beta_n)}\right) \frac{1}{(e^{2\beta_n} - 1)} \right\}^{1/2}.
$$

Inserting $N(\delta)$ into (3.8) finally yields (3.2). This error bound is valid when $N(\delta) > N_\delta$, a condition that is always satisfied for sufficiently small noise level $\delta$.

To conclude the proof, consider any algorithm that produces, for each $d, E$, and $g$, a solution $f_a \in S$. The bound (3.2) can be applied with $f_b = f_x$ because $f_x \in S$, and since the RHS in (3.2) tends to zero as $\delta \to 0$, this algorithm is a regularizing method for the reconstruction on $(a_2, a_3 - \mu)$.

\[\square\]

**Remarks.**

- For small $\delta$, the second term in (3.2) eventually dominates. An intuitive interpretation of the stability estimate (3.2) is that the number of significant digits that can be reliably recovered when estimating the solution is roughly equal to a fraction $\beta_n/\alpha$ of the number of significant digits in the measured data.

- The first sum in (3.3) starts at $-\infty$ because the spectrum of $H_T$ has two accumulation points. We remark that the singular components up to $n \to -\infty$ can be stably recovered from the data because $\sigma_n \to 1$ as $n \to -\infty$. This holds only for the continuous–continuous model of the inverse problem, which is analyzed in this paper. In practice, only sampled data are available. Since the functions $u_n$ are increasingly oscillating functions on the interval $(a_2, a_3)$ when $n \to -\infty$ [1, 2], one would, in practice, start the SVD expansion at a large negative $N_{min} < 0$. This lower SVD cutoff would have an effect similar to a cutoff on the high spatial frequencies.

- Setting $g = f_b = 0$ leads to another equivalent formulation of stability: if $f \in L^2(\mathcal{F})$ is such that $\|H_T f\|_{L^2(\mathcal{G})} \leq \delta$ and $\|f\|_{L^2(\mathcal{F})} \leq E$, then,

$$
\|X_f f\|_{L^2(\mathcal{F})} \leq \frac{\delta e^{\alpha N_n}}{A} + EB f \left\{ \frac{\delta}{AV_n E} \right\}^{1/2} \left\{ \frac{\alpha}{(\alpha - \beta_n) \sqrt{e^{2\beta_n} - 1}} \right\}.
$$

(3.10)

The following corollaries of proposition 1 give explicit error bounds for two specific regularization algorithms. The proofs follow the same line as the proof of proposition 1 and are, therefore, omitted.

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5 We hereby neglect the fact that $N$ only takes integer values. This hardly changes the error estimate when $\delta$ is small.
Corollary 1. (Truncated SVD) The truncated SVD estimate,
\[ f_{N(\delta)} = \sum_{n=-\infty}^{N(\delta)} \langle g, v_n \rangle \frac{1}{\sigma_n} u_n, \]  
with the cutoff index \( N(\delta) \) defined by (3.9) guarantees regularization in the sense that
\[ \lim_{\delta \to 0} \|X_\mu(f_{N(\delta)} - f_{\text{ex}})\|_{L^2(F)} = 0. \]  
Specifically, if \( N(\delta) > N_M \),
\[ \|X_\mu(f_{N(\delta)} - f_{\text{ex}})\|_{L^2(F)} \leq \frac{\delta e^{\alpha N_M}}{A} + EB_\alpha \left\{ \frac{\delta}{AV_\mu E} \right\}^{\beta_\mu/\alpha} \left\{ \frac{\alpha}{(\alpha - \beta_\mu) \sqrt{e^{2\beta_\mu} - 1}} \right\}. \]  
Note that the quasioptimal number (3.9) of terms in the truncated SVD (3.11) increases only very slowly with decreasing noise level \( \delta \), which is a typical behavior of severely ill-posed inverse problems. The corresponding lower bound on the singular values that are included in the truncated SVD solution estimate,
\[ \sigma_{N(\delta)} \geq \frac{\delta}{E^2}, \]  
is proportional to the noise to signal ratio \( \delta/E \). This behavior of the cutoff singular value is also obtained for inverse problems with compact operators using Miller’s method [5, 13, 4, p. 258].

Corollary 2. (Tikhonov regularization) The Tikhonov estimate,
\[ f_\eta = \arg \min_{f \in L^2(F)} \Phi_\eta(f) \text{ with } \Phi_\eta(f) = \|Hf - g\|^2_{L^2(G)} + \eta \|f\|^2_{L^2(F)}, \]  
with \( \eta = \eta(\delta) > 0 \) such that \( \eta(\delta) \to 0 \) and that \( \delta^2/\eta(\delta) \) remains bounded as \( \delta \to 0 \), guarantees regularization in the sense that
\[ \lim_{\delta \to 0} \|X_\mu(f_{\eta(\delta)} - f_{\text{ex}})\|_{L^2(F)} = 0. \]  
In particular, the choice \( \eta = \delta^2/E^2 \) leads to
\[ \|X_\mu(f_{\eta(\delta)} - f_{\text{ex}})\|_{L^2(F)} \leq (1 + \sqrt{2}) \frac{\delta e^{\alpha N_M}}{A} \]
\[ + (1 + \sqrt{2}) EB_\alpha \left\{ \frac{\delta}{AV_\mu E} \right\}^{\beta_\mu/\alpha} \left\{ \frac{\alpha}{(\alpha - \beta_\mu) \sqrt{e^{2\beta_\mu} - 1}} \right\}. \]  
provided \( \delta \) is small enough such that \( N(\delta) > N_M \) in (3.9).

The exponential decay of the singular values of the truncated Hilbert transform (see equation (2.2)) is typical of severely ill-posed inverse problems. Equations (3.2, 3.13, 3.16), however, give a Hölder continuity expected for mildly ill-posed problems. This mild ill posedness is due to the fact that the error is only calculated over a restricted interval \((a_2, a_3 - \mu)\), which excludes a neighborhood of the limit \( a_3 \) of the segment where the Hilbert transform is known. The Hölder error bound does not hold in the limit \( \mu \to 0 \); the squared
norm \( \| \chi_{\mu=0}^B \|_{L^2(\mathcal{J})} \) decays as \( O(1/n) \) (equation (8.6) in [1]) too slowly to balance the exponential decay of \( \sigma_g \).

The error bounds obtained have the same dependence on \( \delta^{\beta_g/\alpha} \). This behavior is determined essentially by the exponential decay rates \( \alpha \) and \( \beta_g \) but also by the assumed regularizing prior knowledge \( \| f \|_{L^2(\mathcal{J})} \leq E \). Figure 1 illustrates the change in the rate \( \delta^{\beta_g/\alpha} \) as the amount of overlap between the two intervals \( \mathcal{F} \) and \( \mathcal{G} \) is varied. In the following section, we seek to find the Hölder exponent using a nonquadratic regularizing penalty or, more precisely, a total variation (TV) penalty. It turns out that this yields the same dependence on \( \delta^{\beta_g/\alpha} \).

4. Inversion of the truncated Hilbert transform: stability through a TV penalty

In this section, we drop the condition \( \| f_{ex} \|_{L^2(\mathcal{J})} \leq E \) and, instead, consider regularization of the inversion problem by assuming prior knowledge on the variation of \( f_{ex} \) in the form of the upper bound \( \| f_{ex} \|_{TV} \leq \kappa \) for some \( \kappa > 0 \). Additionally, we assume that \( f_{ex} \) vanishes at \( a_2 \) and \( a_4 \). This latter assumption is not too restrictive; for if \( f_{ex} \) does not vanish at the boundaries, we can always artificially enlarge the interval \( \mathcal{F} = (a_2, a_4) \) such that the support of \( f_{ex} \) is a strict subset of it.

We define the set of admissible solutions as

\[
S = \{ f \in BV(\mathcal{F}) : \| H_f f - g \|_{L^2(\mathcal{G})} \leq \delta, \| f \|_{TV} \leq \kappa \text{ and } f \text{ vanishes at } a_2, a_4 \}. \tag{4.1}
\]

**Proposition 2.** Let \( H_f f_{ex} = g_{ex} \) and assume \( \| f_{ex} \|_{TV} \leq \kappa \) and \( f_{ex}(a_2) = f_{ex}(a_4) = 0 \). We consider the reconstruction of \( f_{ex} \) from a noisy measurement \( g \) for which \( \| g - g_{ex} \|_{L^2(\mathcal{G})} \leq \delta \) is given. Then, any algorithm that, for given \( \delta, \kappa, \) and \( g \), computes a solution in \( S \) is a regularization method for the reconstruction on \( (a_2, a_3 - \mu) \). More precisely, given any two admissible solutions \( f_a, f_b \in S \) and for sufficiently small \( \delta \), one has the following bound:

\[
\| \chi_{\mu}(f_a - f_b) \|_{L^2(\mathcal{J})} \leq 2\delta A^{-1}e^{\omega\theta} + \frac{2e}{N_\mu B_\mu e^{\omega\theta}} \left( \frac{\delta}{AW_\mu} \right)^{1/2} \frac{(\alpha - \beta_g)(e^{\theta - 1})}{(\alpha - \beta_g)(e^{\theta - 1})}, \tag{4.2}
\]
Here, c is a constant only dependent on a_1, ..., a_4, and W_μ is a constant only dependent on α, β_μ, c, and N_μ.

**Proof.** Let g_a = H_μ f_a and g_b = H_μ f_b. Then, we can write the same identity as in (3.3), which is exact for any choice of N > N_μ. Lemma 6.1 in [3] implies the following decay rate for a sufficiently large index n: there exists a constant c only dependent on a_1, ..., a_4 such that for n > N_μ,

\[ |\langle f_a - f_b, u_n \rangle| \leq c \frac{|f_a - f_b|_{TV}}{n}. \]

Similar to the derivations in the previous section, we can obtain an upper bound on \( \|\chi_\mu(f_a - f_b)\|_{L^2(\mathcal{F})} \leq I_1 + I_2 + I_3 \), where \( I_1 \) and \( I_2 \) are bounded as in (3.5), (3.6), respectively. An upper bound on the last term is obtained as follows:

\[ I_3 = \| \sum_{n=N+1}^{\infty} \langle f_a - f_b, u_n \rangle \chi_\mu u_n \|_{L^2(\mathcal{F})} \leq \sum_{n=N+1}^{\infty} |\langle f_a - f_b, u_n \rangle| \|\chi_\mu u_n\|_{L^2(\mathcal{F})} \]
\[ \leq c |f_a - f_b|_{TV} \sum_{n=N+1}^{\infty} \frac{1}{n} \|\chi_\mu u_n\|_{L^2(\mathcal{F})} \leq c |f_a - f_b|_{TV} B_\mu \sum_{n=N+1}^{\infty} \frac{e^{-\beta_\mu n}}{n} \]
\[ \leq \frac{2cR_\mu e^{-\beta_\mu N}}{N_\mu (e^{\beta_\mu} - 1)}. \]

The last inequality in the above is not sharp but simplifies subsequent derivations. With this, we obtain

\[ \|\chi_\mu(f_a - f_b)\|_{L^2(\mathcal{F})} \leq \frac{2\delta}{\sigma_{N_\mu}} + \frac{2\delta B_\mu}{A} \frac{e^{(\alpha - \beta_\mu)N}}{1 - e^{-2(\alpha - \beta_\mu)N/2}} + \frac{2cR_\mu e^{-\beta_\mu N}}{N_\mu (e^{\beta_\mu} - 1)}. \]

The value \( N(\delta) \) that minimizes this upper bound is found to be

\[ N(\delta) = \frac{1}{\alpha} \log \left( \frac{\kappa W_\mu}{\delta} \right), \]

where

\[ W_\mu = \frac{\beta_\mu}{\alpha - \beta_\mu} \frac{(1 - e^{-2(\alpha - \beta_\mu)N/2}) e}{N_\mu (e^{\beta_\mu} - 1)}. \]

This choice yields (4.2). This bound is valid if \( N(\delta) > N_\mu \), which can be reformulated as an upper bound on the ratio \( \delta/\kappa \),

\[ \frac{\delta}{\kappa} < AW_{\mu} e^{-\alpha N_\mu}. \]  \hspace{1cm} (4.3)

To conclude the proof, consider any algorithm that, for each \( \delta, \kappa, \) and \( g \), produces a solution \( f_a \in \mathcal{S} \). The bound (4.2) can be applied with \( f_b = f_a \) because \( f_a \in \mathcal{S} \), and since the RHS in (4.2) tends to zero as \( \delta \to 0 \), this algorithm is a regularizing method for the reconstruction on \( (a_2, a_3 - \mu) \).

**Remark.** The bounds in (4.2) together with (4.3) imply that the error within the ROI tends to zero as \( \kappa \to 0 \) even for constant noise level \( \delta \). This is only due to the prior assumption that \( f_a \) vanishes at \( a_2 \) and \( a_4 \). Together with \( |f_a|_{TV} = 0 \), this assumption implies
that $f \equiv 0$. Any stability estimate that only requires an upper bound on $\|f^a\|_{TV}$ without assuming that $f^a$ vanishes at the boundaries will not tend to zero as $\kappa \to 0$.

4.1. Reconstruction on the entire interval

So far, we have found stability estimates for the reconstruction on the overlap region (or ROI), while stable reconstruction outside of the ROI has not been guaranteed. One might ask whether it is possible to guarantee stable reconstruction on all of $\Omega$. If the assumed prior knowledge is of the form $\|f^a\|_{L^2(\Omega)} \leq E$, such an estimate cannot hold, in general. This can be seen, for example, by taking the sequence of singular functions $u_n$ for which $\|u_n\|_{L^2(\Omega)} = 1$, while $\|H_f u_n\|_{L^2(\Omega)} \to 0$.

In [3], it has been shown that stability on all of $\Omega = (a_2, a_3)$ can be guaranteed if the variation of the solution is assumed to be bounded, i.e., $\|f\|_{TV} \leq \kappa$, for some $\kappa > 0$. The estimate obtained is of logarithmic continuity (as opposed to Hölder continuity). This is typical for severely ill-posed problems.

We seek to derive such an estimate employing the methods applied in the previous sections. More precisely, we find an upper bound on $\|f_a - f_b\|_{L^2(\Omega)}$ for $f_a, f_b \in S$ where $S$ is the set of admissible solutions in (4.1). For this, we write

$$\|f_a - f_b\|_{L^2(\Omega)} \leq I_1 + I_2,$$

where

$$I_1 = \| \sum_{n=-\infty}^N (g_n - g_b) \varpi_n \frac{1}{\sigma_n} u_n \|_{L^2(\Omega)},$$

$$I_2 = \| \sum_{n=N+1}^\infty (f_a - f_b) \varpi_n u_n \|_{L^2(\Omega)},$$

and $N > N_0$. As before, $I_1 \leq \frac{2\kappa}{\sigma_n}$. For an upper bound on $I_2$, we again apply lemma 6.1 in [3] to obtain

$$I_2^2 = \sum_{n=N+1}^\infty |\langle f_a - f_b, u_n \rangle|^2 \leq c^2(2\kappa)^2 \sum_{n=N+1}^\infty \frac{n^2}{n^2} \leq c^2(2\kappa)^2 \frac{1}{N}.$$ 

This yields

$$\|f_a - f_b\|_{L^2(\Omega)} \leq 2\delta A^{-1} e^{\alpha N} + 2\kappa c \frac{1}{\sqrt{N}}.$$ 

The value $N = N(\delta)$ that minimizes the RHS in the above satisfies

$$2\delta A^{-1} e^{\alpha N} = \kappa c N(\delta)^{-3/2} = 0,$$

and, therefore,

$$\alpha N(\delta) + \frac{3}{2} \log N(\delta) = \log \left( \frac{A\kappa c}{2\delta \alpha} \right).$$

As a consequence, the optimal $N(\delta)$ satisfies

$$(\alpha + \frac{3}{2}) N(\delta) \geq \log \left( \frac{A\kappa c}{2\delta \alpha} \right) \geq \alpha N(\delta).$$
and, hence,

\[ N(\delta)^{-\frac{1}{\alpha}} \leq \left( \alpha + \frac{3}{2} \right)^{\frac{1}{\alpha}} \left[ \log\left( \frac{A\kappa}{2\delta} \right) \right]^{-\frac{1}{2}}. \]

Thus,

\[
\|f_a - f_b\|_{L^2(\mathcal{X})} \leq \frac{\kappa C}{\alpha} N(\delta)^{-\frac{1}{2}} + 2\kappa C N(\delta)^{-\frac{1}{2}}
\]

\[
\leq \kappa C \left( \frac{1}{\alpha} + 2 \right) N(\delta)^{-\frac{1}{2}} \leq \kappa C \left[ \log\left( \frac{\kappa}{\delta} \right) + D \right]^{-\frac{1}{2}}
\]

(4.4)

for some constants \( C > 0 \) and \( D \) that depend on \( a_1-a_4 \). This bound holds for \( N(\delta) > N_0 \), for which

\[
\frac{\delta}{\kappa} \leq \frac{A\kappa}{2\alpha} \left( e^{-\frac{\alpha}{2}} \right)^{N_0}
\]

is a sufficient condition. Note that by this requirement, \( \log\left( \frac{\kappa}{\delta} \right) + D > 0 \).

5. Numerical validation of the asymptotic expressions in (2.5)

We have calculated the SVD of the truncated Hilbert transform with \( a_1 = 0, a_2 = 450, a_3 = 1350, a_4 = 1725 \). The problem has been discretized with sampling step equal to 1 and a shift of 1/2 between the object and the data samples. Mathematica (using 20 significant digits) finds 910 nonzero singular values for the discretized 1351 × 1276 Hilbert matrix \( \mathbf{H}_T \). As observed in [1], most singular values are very close to 1, and only ten singular values are smaller than 0.97; of those, the last nine are smaller than 0.01. These last singular values are well fitted by the asymptotic expression \( Ae^{-\alpha} \) (figure 2(a)) with the constants from equation (2.3).

For each singular function \( u_n \) corresponding to these last nine values, we have computed the norm \( \|\chi_{\mu} u_n\|_{L^2(\mathcal{X})} \) in the interval \((a_2, a_3 - \mu)\) for integer values of \( \mu \) between 1 and 400. For all \( \mu \), we find that \( \|\chi_{\mu} u_n\|_{L^2(\mathcal{X})} \) is well fitted by the asymptotic expression (2.5) as

![Figure 2. Left: log \( \sigma_n \) versus \( n \) for the discretized truncated Hilbert problem (dots) with \( a_1 = 0, a_2 = 450, a_3 = 1350, a_4 = 1725 \). The red line is the asymptotic expression (2.2). Right: log \( \|\chi_{\mu} u_n\|_{L^2(\mathcal{X})} \) versus \( n \) for \( \mu = 5 \) (red), \( \mu = 20 \) (blue), and \( \mu = 100 \) (green). The lines are the corresponding asymptotic expressions (2.5). The plots show the last nine singular components.](image-url)
illustrated in figure 2(b). The fits are obtained by associating $n = 1$ with the singular value $n' = 902$ of the discrete SVD.

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Appendix

This appendix gives the proof of equations (2.5) and (2.6), which characterize the asymptotic behavior as $n \to \infty$ of the integral,

$$\|X_n u_n\|_{L^2(\mathcal{F})} = \left( \int_{a_2}^{a_1} \frac{dx}{\sqrt{P(x)}} \right)^{1/2},$$

(6.1)

where $u_n$ is the $n$th singular function of the truncated Hilbert transform with overlap $HT$. The proof relies heavily on the results from [1, 2]. These findings rely on the fact that the operator $HT$ shares an intertwining property with second-order differential operators $L_S$ and $L_{\overline{S}}$ [2]. It implies that the singular functions $u_n$ and $v_n$ are specially constructed solutions to

$$P(x)f''(x) + Pf'(x)f'(x) = (\lambda_n - 2(x - \sigma)^3)f(x),$$

(6.2)

where $\lambda_n \to \infty$ as $n \to \infty$ and $\lambda_n \to -\infty$ as $n \to -\infty$ (see [1, 2] for details). Below, we omit detailed references to these papers to which the reader is referred for all properties stated without explanation.

**Lemma 1.** For small $\zeta > 0$, the asymptotic behavior as $n \to \infty$ of the singular function in $x \in (a_2 + O(\varepsilon^{1/2} \zeta), a_1 - O(\varepsilon^{1/2} \zeta))$ is

$$u_n(x) = \sqrt{\frac{2}{K_+}} \left( -1 \right)^{n+1} e^{-w_3(x)/\varepsilon} \left( 1 + O(\varepsilon^{1/2-\zeta}) \right),$$

(6.3)

with

$$w_3(x) = \int_x^{a_1} \frac{dt}{\sqrt{P(t)}},$$

(6.4)

and where $\varepsilon$ is related to $n$ by

$$\varepsilon = \frac{K_-}{n\pi} \left( 1 + O(n^{-1/2+\zeta}) \right).$$

(6.5)

**Proof.** In the interval $(a_1 + O(\varepsilon^{1/2} \zeta), a_2 - O(\varepsilon^{1/2} \zeta))$, where $\varepsilon$ is related to $n$ by (6.5), the Wentzel–Kramers–Brillouin (WKB) approximation of the singular function $v_n$ can be written as (see equations (5.12), (8.1) in [1])
\[ v_n(x) = \frac{1}{\sqrt{2K_+}} \frac{1}{(-P(x))^{1/4}} \left[ e^{i\Omega(x)/\epsilon - i\pi/4} + e^{-i\Omega(x)/\epsilon + i\pi/4} \right] \left( 1 + O(\epsilon^{1/2-\zeta}) \right) \]

(6.6)

with

\[ w_1(x) = \int_{\alpha_i}^{x} \frac{dt}{\sqrt{-P(t)}}. \]

Let \( v \) be the analytic continuation of \( v_n \) from the interval \((a_1, a_2)\) to \( \mathbb{C} \setminus [a_2, a_3] \). Employing the uniqueness of the solution to the Riemann–Hilbert problem and the Plemelj–Sokhotski formulas, one can show (cf. equations (4.2)–(4.8) in [1]) that

\[ u_n(x) = \frac{\sigma_0}{2i} \frac{v(x - i0) - v(x + i0)}{2}, \quad x \in (a_2, a_4) \setminus [a_3], \]

(6.7)

\[ v_n(x) = \frac{v(x + i0) + v(x - i0)}{2}, \quad x \in (a_1, a_3) \setminus [a_2]. \]

(6.8)

Furthermore, we have that (cf. equations (4.9), (4.10) in [1])

\[ u_n(x) = \sigma_0 \text{Im} \left( v(x + i0) \right), \quad x \in (a_2, a_4) \setminus [a_3], \]

(6.9)

\[ v_n(x) = \text{Re} \left( v(x + i0) \right), \quad x \in (a_1, a_3) \setminus [a_2]. \]

The analytic continuation of \(-P(x)\) from \((a_1, a_2)\) to \((a_2, a_3)\) via the upper half plane is given by \(-P(x) = e^{-\pi}P(x)\). With this, the analytic continuation of \( v \) via the upper half plane to \((a_2 + O(\epsilon^{1+2\zeta}), a_3 - \mu) \subset (a_2 + O(\epsilon^{1+2\zeta}), a_3 - O(\epsilon^{1+2\zeta}))\) can be expressed as (see section 5.1 and figure 2 in [1, 10] for the uniform accuracy of this continuation)

\[ v(x + i0) = \frac{1}{\sqrt{2K_+}} e^{i\Omega(x)/\epsilon} \left( 1 + O(\epsilon^{1/2-\zeta}) \right) \]

(6.9)

\[ \times \left( 1 + e^{iK_+/\epsilon - i\pi/4} e^{-i\Omega(x)/\epsilon} + e^{-iK_+/\epsilon + i\pi/4} e^{-i\Omega(x)/\epsilon} \right) \]

\[ \times \left( 1 + O(\epsilon^{1/2-\zeta}) \right) \]

\[ + (1/i) \left( 1 + e^{iK_+/\epsilon - i\pi/4} e^{-i\Omega(x)/\epsilon} + e^{-iK_+/\epsilon + i\pi/4} e^{-i\Omega(x)/\epsilon} \right) \left( 1 + O(\epsilon^{1/2-\zeta}) \right). \]

(6.10)

with

\[ w_2(x) = K_+ - w_3(x) = \int_{a_2}^{x} \frac{dt}{\sqrt{-P(t)}}. \]

Equation (6.9) can be rewritten as

\[ v(x + i0) = \frac{1}{\sqrt{2K_+}} \frac{1}{(-P(x))^{1/4}} \left[ e^{iK_+/\epsilon - w_3(x)/\epsilon} + e^{-iK_+/\epsilon + i\pi/2} e^{-w_3(x)/\epsilon} \right] \]

\[ \cdot \left( 1 + O(\epsilon^{1/2-\zeta}) \right) \]

\[ + (1/i) \left[ e^{iK_+/\epsilon - w_3(x)/\epsilon} + e^{-iK_+/\epsilon + i\pi/2} e^{-w_3(x)/\epsilon} \right] \left( 1 + O(\epsilon^{1/2-\zeta}) \right). \]

(6.10)

Noting that the modulus of all complex exponentials is equal to 1 and that \( e^{w_3(x)/\epsilon} > e^{-w_3(x)/\epsilon} \), one can regroup all terms in \( O(\epsilon^{1/2-\zeta}) \) in (6.10) to

\[ \frac{1}{(-P(x))^{1/4}} e^{w_3(x)/\epsilon} O(\epsilon^{1/2-\zeta}). \]

(6.11)
Therefore, (6.10) becomes

\[ v(x + i0) = \frac{1}{\sqrt{2K}} \frac{1}{(P(x))^{1/4}} (e^{-iK_{1}/e^{+i\pi/2}e^{w_{1}(x)/e}} + e^{w_{1}(x)/e}O(e^{1/2-\zeta})) \]

\[ + \frac{1}{\sqrt{2K}} \frac{1}{(P(x))^{1/4}} e^{w_{1}(x)/e} (e^{-iK_{1}/e^{+i\pi/2}e + O(e^{1/2-\zeta})), (6.12) \]

Combining (6.7) and (6.12) one obtains

\[ u_{a}(x) = \frac{\sigma_{n}}{\sqrt{2K}} (-1)^{n+1} e^{w_{1}(x)/e} (1 + O(e^{1/2-\zeta})), (6.13) \]

where we have used \( \cos(K_{1}/e) = (-1)^{n+1} (1 + O(e^{1/2-\zeta})) \) (see equation (5.36) in [1]).

Finally, using the explicit expression (see (5.45) in [1]) for the asymptotics of the singular values yields equation (6.3) with \( w_{3}(x) \) as in (6.4). This result holds for \( x \in (a_{2} + O(e^{1/2+\zeta}), a_{3} - O(e^{1/2+\zeta})) \).

**Lemma 2.** For sufficiently large positive index \( n \), the singular function \( u_{n} \) is strictly monotonic on \((a_{2}, a_{3})\).

**Proof.** The singular function \( u_{n}(x) \) is a solution of (6.2) for \( x \in (a_{2}, a_{3}) \) \( \setminus \{a_{3}\}, \) where \( \lambda_{n} \to \infty \) as \( n \to \infty \). By definition, \( P(x) > 0 \) for \( x \in (a_{2}, a_{3}) \) and \( P'(a_{2}) > 0 \). Since \( u_{n}(x) \) is bounded at \( x = a_{2} \) and \( u_{n}(a_{2}) = 0 \) (see, e.g. [2], section 2), we may assume without loss of generality that \( u_{n}(a_{2}) > 0 \). Then, if

\[ \lambda_{n} > \max_{a_{2} < x < a_{3}} 2(x - \sigma)^{2}, \quad (6.14) \]

the RHS of (6.2) is positive at \( a_{2} \) and since \( P(a_{2}) = 0, u_{n}''(a_{2}) \) is finite and \( P'(a_{2}) > 0 \), one concludes that \( u_{n}'(a_{2}) > 0 \). We will now show that \( u_{n}'(x) > 0 \) for \( x \in (a_{2}, a_{3}) \).

Let \( \tilde{x} \) be the first point after \( a_{2} \) for which \( u_{n}'(\tilde{x}) = 0 \). Then, by construction, we have \( u_{n}(\tilde{x}) > 0 \) and \( u_{n}''(\tilde{x}) \leq 0 \). With this, evaluating (6.2) at \( x = \tilde{x} \) results in a contradiction since the left-hand side is nonpositive, whereas the RHS is positive. Hence, \( u_{n}'(x) > 0 \) for all \( x \in (a_{2}, a_{3}) \), i.e., \( u_{n} \) is strictly monotonic.

**Proof of (2.5), (2.6).** Split the integral (6.1) into

\[ \| \chi_{\mu} u_{n} \|_{L^{2}(\mathbb{R})}^{2} = I_{a} + I_{b} = \int_{a_{2}+\rho}^{a_{2}+\mu} dx |u_{n}(x)|^{2} + \int_{a_{2}}^{a_{2}+\rho} dx |u_{n}(x)|^{2}, \quad (6.15) \]

with \( \rho = O(e^{1+2\zeta}) \) and \( \mu > 0 \) such that the integration interval for \( I_{a} \) is contained in the domain of validity of the WKB approximation for \( n > N_{\mu} \). Using lemma 1, the first term is
\[ I_a = \int_{a_2 + \rho}^{a_2 - \rho} dx \left| u_n(x) \right|^2 = \frac{\epsilon}{K} \left( e^{-2w_3(a_3 - \mu)/\epsilon} - e^{-2w_3(a_3 + \mu)/\epsilon} \right) (1 + O(\epsilon^{1/2 - \zeta})) \]

\[ = \frac{\epsilon}{K} e^{-2w_3(a_3 - \mu)/\epsilon} (1 - \frac{\epsilon^2}{\epsilon} (w_3(a_3 - \mu) - w_3(a_3 + \mu))) (1 + O(\epsilon^{1/2 - \zeta})) \]

\[ = \frac{\epsilon}{K} e^{-2w_3(a_3 - \mu)/\epsilon} (1 + O(\epsilon^{1/2 - \zeta})), \]

where, in the second line, the first factor in parentheses gets absorbed in the error term since \( w_3(a_3 - \mu) - w_3(a_3 + \mu) \) is negative and strictly bounded away from 0. Consider now the second term \( I_b \). For sufficiently small \( \epsilon = 1/\sqrt{n} \), \( u_n \) is monotonic on \((a_2, a_3)\) by lemma 2. Thus,

\[ I_b = \int_{a_2}^{a_2 + \rho} dx \left| u_n(x) \right|^2 \leq \frac{\rho}{a_3 - \mu - a_2 - \rho} \int_{a_2}^{a_2 + \rho} dx \left| u_n(x) \right|^2 \leq O(\epsilon^{1/2 + \zeta}) I_a, \]

so that the contribution of \( I_b \) can be absorbed in the error term of equation (3.5). We, finally, obtain

\[ \| \chi \mu u_n \|_{L^2(F)} = (I_a + I_b)^{1/2} = \left( \frac{\epsilon}{K} e^{-2w_3(a_3 - \mu)/\epsilon} (1 + O(\epsilon^{1/2 - \zeta})) \right)^{1/2} \]

\[ = \left( \frac{1}{\sqrt{n}} \right) e^{-\lambda n} (1 + O(n^{-1/2 + \zeta})). \]

\[ \square \]

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