A NEW INVARIANT AND DECOMPOSITIONS OF MANIFOLDS

EIJI OGASA

Abstract. We introduce a new topological invariant \( \in \mathbb{N} \cup \{0\} \) of compact manifolds with boundaries associated with a kind of decomposition of them. Let \( M \) and \( N \) be \( m \)-dimensional compact connected manifolds with boundaries. Let \( M \cup \partial N \) be a boundary union of \( M \) and \( N \). Let \( \nu(M) \) be the new invariant of \( M \). Then we have

\[
0 \leq \nu(M \cup \partial N) \leq \max\{\nu(M), \nu(N)\}.
\]

1. A PROBLEM

In order to state our problem we prepare a definition. We work in the smooth category.

Definition 1.1. Let \( M \) (resp. \( N \)) be an \( m \)-dimensional smooth connected compact manifold with boundary. Let \( \partial M = \bigsqcup_{i=1}^{\alpha} X_i \) and \( \partial N = \bigsqcup_{j=1}^{\beta} Y_j \), where \( \bigsqcup \) denotes a disjoint union and \( \alpha, \beta \in \mathbb{N} \). Suppose that each \( X_i \) (resp. \( Y_j \)) is a connected closed manifold. A boundary union \( M \cup \partial N \) is an \( m \)-manifold with boundary which is a union \( M \cup N \) with the following properties:

1. There are closed manifolds \( X_{\sigma_1} \cdots X_{\sigma_\mu} \subseteq \partial M \) and \( Y_{\tau_1} \cdots Y_{\tau_\mu} \subseteq \partial N \) which are diffeomorphic. Here, \( \sigma_s \) are different each other, \( \tau_s \) are different each other, \( 1 \leq \sigma_s \leq \alpha \), \( 1 \leq \tau_s \leq \beta \), \( \mu \in \mathbb{N} \), \( \mu \leq \alpha \), and \( \mu \leq \beta \).

2. We make \( M \cup \partial N \) by identifying \( X_{\sigma_1} \cdots X_{\sigma_\mu} \) with \( Y_{\tau_1} \cdots Y_{\tau_\mu} \) by a diffeomorphism map \( X_{\sigma_1} \cdots X_{\sigma_\mu} \rightarrow Y_{\tau_1} \cdots Y_{\tau_\mu} \).

(Hence, of course, \( M \cap N = X_{\sigma_1} \cdots X_{\sigma_\mu} = Y_{\tau_1} \cdots Y_{\tau_\mu} \).)

Let \( \rho \) be an integer \( \geq 2 \). Suppose that a boundary union \( L' \) of \( \rho \) ‘manifolds with boundaries’, \( L_1, \ldots, L_\rho \), is defined. Then a boundary union of ‘manifolds with boundaries’, \( L_{\rho+1} \) and \( L' \), is said to be a boundary union of \( (\rho + 1) \) ‘manifolds with boundaries’, \( L_1, \ldots, L_{\rho+1} \), be denoted by \( \bigcup_{i=1}^{\rho+1} L_i \). We say that \( M \) is a boundary union of one connected ‘manifold with boundary’, \( M \). We say that the disjoint union \( M \sqcup N \) is a boundary union of \( M \) and \( N \).

Keywords: decomposition of manifolds, a new invariant \( \nu(M) \), boundary union.

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Note. Not all unions are boundary unions. Let $B^3$ denote the 3-ball. Let $D^2$ denote the 2-disc. We can regard the solid torus as a union $B^3 \cup B^3$ such that $B^3 \cap B^3$ is a disjoint union $D^2 \amalg D^2$. This union $B^3 \cup B^3$ is not a boundary union because $B^3 \cap B^3$ is not a disjoint union of closed manifolds.

An example of boundary unions is the following: Let $T^2$ denote the torus. Let $A$ denote the annulus. Let $D^\circ$ denote the open 2-disc. Let $S^1$ denote the circle. We can regard $T^2 - D^\circ$ as a union of $A \cup (A - D^\circ)$ such that $A \cap (A - D^\circ)$ is a disjoint union $S^1 \amalg S^1$. This union $A \cup (A - D^\circ)$ is a boundary union.

Any connected sum is a boundary union. However, the converse is not true. A Heegaard splitting of a closed oriented 3-manifold gives a boundary union of two copies of a handle body, which consists of a single 3-dimensional 0-handle and 3-dimensional 1-handles. However, it is not a connected sum if the handle body is not the 3-ball.

We state our problem.

**Problem 1.2.** Let $m$ be a nonnegative integer. Is there a finite set $\mathcal{S}$ of (oriented) compact connected $m$-manifolds with boundaries with the following property ($\star$)?

($\star$) For an arbitrary closed connected $m$-manifold $M$, there are $m$-manifolds with boundaries $Z_1, \ldots, Z_\nu$, where $\nu \in \mathbb{N}$, such that $M$ is a boundary union of $Z_1, \ldots, Z_\nu$ and that $Z_i \in \mathcal{S}$. Note that $Z_i$ may be same as $Z_j$ if $i \neq j$.

Of course, we can make a problem in the $\partial M \neq \emptyset$ case if we impose an appropriate condition on $\partial M$.

It is trivial that the answer is affirmative if $m \leq 2$. String theory uses the fact that the $m = 2$ case has the affirmative answer, discussing the world sheet (see [2], [6] etc).

At least, to the author, a motivation of this paper is the following: In QFT, each Feynman diagram is made by the given fundamental parts. In string theory a world sheet (so to say, 2-dimensional Feynman diagram) is decomposed into a finite number of 2-manifolds as stated above. In $M$-theory we may need high dimensional Feynman diagrams (see [6], P. 607, 608 of [8] and so on) although, of course, there is an obstruction written in P60 of I of [2]. Considering high dimensional Feynman diagrams, we need to research a kind of decomposition of manifolds, for example, in Problem 1.2. If the answer to Problem 1.2 is negative, it might be a kind of obstruction for the existence of a consistent theory of high dimensional Feynman diagrams (resp. that of quantization of high dimensional objects). If such a theory does not exist, the obstruction is a reason why it does not. If the answer to Problem 1.2 is negative and if such a theory exists, we have to impose conditions on diffeomorphism type of high dimensional Feynman diagrams in order to avoid the negative answer.

In [5] we prove that the answer to Problem 1.2 is negative under the following condition ($\sharp$): Let $m \geq 3$. Suppose that each element of $\mathcal{S}$ has more than three connected boundary components (Note that each of the boundary components is a closed connected manifold).
2. A NEW INVARIANT

We introduce a new invariant in order to discuss the $m \geq 3$ case of Problem 1.2.

**Definition 2.1.** Let $M$ be an $m$-dimensional smooth connected compact manifold with boundary ($m \in \mathbb{N}$). Take a handle decomposition $A \times [0,1] \cup \text{handles}$ of $M$. Here, we recall the following. (See P.83 of [1], P.3 of [3], and [4, 7] for handle decompositions.)

1. The manifold $A$ is a closed $(m-1)$-manifold $\subset \partial M$. The manifold $A$ may be $\partial M$. The manifold $A$ may not be $\partial M$. We may have $A = \phi$.

2. The manifold $A$ may not be connected.

3. There may be no handle (then $M = A \times [0,1]$). If handles are attached to $A \times [0,1]$, all handles are attached to $A \times \{1\}$.

Let $H(M,A)$ denote this handle decomposition. An **ordered handle decomposition** $H_O(M,A)$ consists of

1. a handle decomposition $H(M,A)$ of $M$, and
2. an order of the handles in $H(M,A)$: If we give an order to the handles, let handles be called $h(\xi)$ ($\xi = 1, 2, 3, ..., \delta$).

The order satisfies the following: Let $\mu$ be a natural number $\leq \delta$. Let

$$(M,A)_\mu = A \times [0,1] \cup h(j) \subset M.$$  

Then this $A \times [0,1] \cup h(j)$ is a handle decomposition of $(M,A)_\mu$. (We sometimes abbreviate $(M,A)_\mu$ to $M_\mu$.)

For $\mu = 0$, we define $M_0 = A \times [0,1]$.

Take an ordered handle decomposition $H_O(M,A)$. Suppose that there are ordered handles $h(\xi)$ ($\xi = 1, 2, 3, ..., \delta$). Let $\mu \in \{0, 1, ..., \delta\}$.

Let $\partial M_\mu - A \times \{0\} = E_{\mu_1} \cup ... \cup E_{\mu_\delta}$, where each $E_{\mu_i}$ is a connected closed $(m-1)$-manifold. Note: If $A = \phi$, $\partial M_0 - A \times \{0\} = \phi$. Then we suppose that $\xi_0 = 1$ and $E_{01} = \phi$. We consider both the $A = \phi$ case and the $A \neq \phi$ case.

Let $\nu(H_O(M,A))$ be the maximum of $\sum_{s=0}^{s=m-1} \dim H_s(E_{\mu_i}; \mathbb{R})$ for all $i, \mu$.

Let $\nu(M,A)$ be the minimum of $\nu(H_O(M,A))$ for all ordered handle decompositions $H_O(M,A)$.

Let $\nu(M)$ be the maximum of $\nu(M,A)$ for all $A$.

**Note.** By the definition, $\nu(M)$ is an invariant of diffeomorphism type of $M$. If we consider $\nu(M)$ for all smooth structures on $M$, we get an invariant of homeomorphism type of $M$.

**Note.** $\sum_{s=0}^{s=m-1} \dim H_s(E_{\mu_i}; \mathbb{R})$ is not the Euler number of $E_{\mu_i}$. Their definitions are different.
**Note.** Suppose that we can attach an $m$-dimensional $k$-handle $h$ to an $m$-dimensional compact manifold $M$. The union $h \cup M$ is not a boundary union of $h$ and $M$ if $k < m$. It is a boundary union if $k = m$.

**Theorem 2.2.** Let $M$ and $N$ be $m$-dimensional compact connected manifolds with boundaries. Let $M \cup_\partial N$ be a boundary union of $M$ and $N$. Then we have

$$0 \leq \nu(M \cup_\partial N) \leq \max\{\nu(M), \nu(N)\}.$$ 

By the induction, we have a corollary.

**Corollary 2.3.** Let $L_1, \ldots, L_\rho$ be $m$-dimensional compact connected manifolds with boundaries. Let $\bigcup_{i=1}^\rho L_i$ be a boundary union of $L_1, \ldots, L_\rho$. Then we have

$$0 \leq \nu\left( \bigcup_{i=1}^\rho L_i \right) \leq \max\{\nu(L_1), \ldots, \nu(L_\rho)\}.$$ 

**Claim 2.4.** The answer to the $m \geq 3$ case of Problem 1.2 is negative if the $\partial X = \phi$ case of the following Problem 2.5 has the affirmative answer.

**Problem 2.5.** Let $m$ be an integer $\geq 3$. Suppose that there is an $m$-dimensional compact connected ‘manifold with boundary’ $X$. Take any natural number $N$. Then is there an $m$-dimensional compact connected ‘manifold with boundary’ $M$ such that $\partial M = \partial X$ and that

$$\nu(M) \geq N?$$

In particular, consider the $\partial X = \phi$ case.

**Note.** If we do not fix the diffeomorphism type of $\partial M$, it is easy to prove that there are ‘manifolds with boundaries’, $M$, such that $\nu(M) \geq N$. Because: Examples are ‘manifolds with boundaries’, $M$, made from one 0-handle $h^0$ and $N'$ copies of 1-handles $h^1$, where $N' \geq N$.

3. **Proof of Theorem 2.2 and Claim 2.4**

**Proof of Theorem 2.2.** By the definition of $\nu(M \cup_\partial N)$, there is a closed $(m - 1)$-manifold $P$ such that

$$\nu(M \cup_\partial N) = \nu(M \cup_\partial N, P). \quad \cdot \cdot \cdot [1]$$

Let $A = P \cap M$. Let $B = P \cap N$. Let $C = M \cap N$. [4]
Suppose that an ordered handle decomposition $H_O(M, A)$ gives $\nu(M, A)$. Hence 
$$\nu(M, A) = \nu(H_O(M, A)). \quad \cdots \cdot [[2]]$$

Let $H_O(M, A)$ consist of ordered handles $h(1), ..., h(\alpha)$.

Suppose that an ordered handle decomposition $H_O(N, B \sqcup C)$ gives $\nu(N, B \sqcup C)$. Hence 
$$\nu(N, B \sqcup C) = \nu(H_O(N, B \sqcup C)). \quad \cdots \cdot [[3]]$$

Let $H_O(N, B \sqcup C)$ consist of ordered handles $k(1), ..., k(\beta)$.

Let $H_O(M \cup \partial N, P)$ be an ordered handle decomposition to consist of $l(1), ..., l(\alpha + \beta)$, where the restriction of $H_O(M \cup \partial N, P)$ to 
$$\begin{cases} (M, A) \\ (N, B \sqcup C) \end{cases}$$

is 
$$\begin{cases} H_O(M, A) \\ H_O(N, B \sqcup C) \end{cases}.$$那

That is, we have an ordered handle decomposition

$$M \cup \partial N = (A \sqcup B) \cup l(1) \cup ... \cup l(\alpha) \cup l(\alpha + 1) \cup ... \cup l(\alpha + \beta)$$

$$P \hline h(1) \hline ... \hline h(\alpha) \hline k(1) \hline ... \hline k(\beta).$$

Here, note that $l(i) = h(i) (i = 1, ..., \alpha)$, and that $l(i) = k(i - \alpha) (i = \alpha + 1, ..., \alpha + \beta)$.

Recall $E_\mu$ in Definition $2.1$. Take $E_\mu$ for this 
$$\begin{cases} H_O(M \cup \partial N, P) \\ H_O(M, A) \\ H_O(N, B \sqcup C), \end{cases}$$

call it 
$$\begin{cases} E_{2*}^{M \cup \partial N} \\ E_{*}^{M} \hline E_{*}^{N} \hline E_{*}^{N}. \end{cases}$$
Then \( \{ E^{M\cup\emptyset N}_\ast \| \ast \text{ take all values} \} = \{ E^M_{\diamond} \| \diamond \text{ and } \delta \text{ take all values} \} \cup \{ E^N_{\lozenge} \| \lozenge \text{ and } \triangleright \text{ take all values} \} \).

**Note.** Furthermore we have the following.

If \( 0 \leq \mu \leq \alpha \), \( \{ E^{M\cup\emptyset N}_{\mu\ast} \| \ast \text{ takes all values} \} = \{ E^M_{\mu\diamond} \| \diamond \text{ takes all values} \} \cup \{ B_\beta \| \beta \text{ takes all values} \} \) holds. Here \( B_\beta = \{ E^N_{\alpha\lozenge} \| \lozenge \text{ and } \triangleright \text{ take all values} \} \) holds.

If \( \alpha + 1 \leq \mu \leq \alpha + \beta \), \( \{ E^{M\cup\emptyset N}_{\mu\ast} \| \ast \text{ takes all values} \} = \{ E^N_{\alpha\diamond} \| \diamond \text{ takes all values} \} \cup \{ D_\epsilon \| \epsilon \text{ takes all values} \} \) holds. Here, \( \partial M - A - C = D_1 \Pi \ldots \Pi D_\epsilon \), where each \( D_\epsilon \) is a closed connected manifold.

Suppose \( \nu(H_0(M \cup \emptyset N, P)) = \sum_{s=0}^{s=m-1} \dim H_s(E^{M\cup\emptyset N}_{\mu i}; \mathbb{R}) \) for an integer \( \mu \) and an integer \( i \). Then \( E^{M\cup\emptyset N}_{\mu i} \) is

\( E^M_{\sigma j} \) for an integer \( \sigma \) and an integer \( j \) \( \ldots \ldots \) [I]

or

\( E^N_{\tau k} \) for an integer \( \tau \) and an integer \( k \) \( \ldots \ldots \) [II]

Suppose [I] holds. Then \( \nu(H_0(M, A)) = \sum_{s=0}^{s=m-1} \dim H_s(E^M_{\sigma j}; \mathbb{R}) \) for the integer \( \sigma \) and the integer \( j \). Hence

\[ \nu(H_0(M \cup \emptyset N, P)) = \nu(H_0(M, A)). \quad \ldots \ldots \] [4]

Suppose [II] holds. Then \( \nu(H_0(N, B \Pi C)) = \sum_{s=0}^{s=m-1} \dim H_s(E^N_{\tau k}; \mathbb{R}) \) for the integer \( \tau \) and the integer \( k \). Hence

\[ \nu(H_0(M \cup \emptyset N, P)) = \nu(H_0(N, B \Pi C)). \quad \ldots \ldots \] [5]

Since [4] or [5] holds,

\[ \nu(H_0(M \cup \emptyset N, P)) \leq \max\{ \nu(H_0(M, A)), \nu(H_0(N, B \Pi C)) \}. \quad \ldots \ldots \] [6]

By [2], [3], and [6],

\[ \nu(H_0(M \cup \emptyset N, P)) \leq \max\{ \nu(M, A), \nu(N, B \Pi C) \}. \quad \ldots \ldots \] [7]

By the definition of \( \nu(M \cup \emptyset N, P) \),

\[ \nu(M \cup \emptyset N, P) \leq \nu(H_0(M \cup \emptyset N, P)). \quad \ldots \ldots \] [8]

By [7] and [8],

\[ \nu(M \cup \emptyset N, P) \leq \max\{ \nu(M, A), \nu(N, B \Pi C) \}. \quad \ldots \ldots \] [9]
By [1] and [9]

\[ \nu(M \cup \partial N) \leq \max\{\nu(M, A), \nu(N, B)\}. \]

By the definition of \( \nu(M) \) and \( \nu(N) \), we have

\[ \nu(M, A) \leq \nu(M) \quad \text{and} \quad \nu(N, B) \leq \nu(N). \]

By [10] and [11],

\[ \nu(M \cup \partial N) \leq \max\{\nu(M), \nu(N)\}. \]

By the definition of \( \nu(\cdot) \), \( 0 \leq \nu(M \cup \partial N) \). This completes the proof. \( \square \)

**Proof of Claim 2.4.** We suppose the following assumption and we deduce a contradiction.

Assumption: we have the affirmative answer to Problem 1.2.

By the above assumption there is a finite set \( S = \{S_1, \ldots, S_s\} \) as in Problem 1.2.

By Corollary 2.3, for any connected closed \( m \)-manifold \( L \), \( \nu(L) \leq \max\{\nu(S_1), \ldots, \nu(S_s)\} \).

If the \( \partial X = \phi \) case of Problem 2.5 has the affirmative answer, then there is a connected closed \( m \)-manifold \( M \) such that \( \nu(M) > \max\{\nu(S_1), \ldots, \nu(S_s)\} \).

We arrived at a contradiction. Hence Claim 2.4 is true. \( \square \)

4. Some results on our new invariant

Let \( M \neq \phi \). Let \( m \geq 2 \). Let \( M \) be a smooth closed oriented connected \( m \)-manifold. By the definition, \( \nu(M) \geq 2 \).

We prove the following.

**Theorem 4.1.** Let \( m \geq 2 \). Let \( S^m \) be diffeomorphic to the standard sphere. Then \( \nu(S^m) = 2 \).

**Proof of Theorem 4.1.** There is an ordered handle decomposition \( H_O = h(1) \cup h(2) \) such that \( h(1) = h^0, h(1) = h^m \). Then \( \nu(H_O) = 2 \). Hence \( \nu(S^m) \geq 2 \). By the definition, \( \nu(S^m) \geq 2 \). Hence \( \nu(S^m) = 2 \).

**Note.** Furthermore we have the following: If \( M \) has a handle decomposition \( h^0 \cup h^m \), then \( \nu(H) = 2 \).

It is natural to ask the following: Suppose \( M \) is a closed connected oriented manifold. Then does \( \nu(M) = 2 \) imply that \( M \) is PL homeomorphic to \( S^m \)?

We have the following theorem as an answer to this question.

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Theorem 4.2. Let \( n \geq 2 \). We have \( \nu \left( S^1 \times S^{n-1} \right) = 2 \), where \( \ast \) is any natural number and where \( \nu \left( S^1 \times S^{n-1} \right) = S^1 \times S^{n-1} \).

Proof of Theorem 4.2. There is an ordered handle decomposition
\[
h^0 \cup h^{n-1} \cup h^1 \cup h^{n-1} \cup \cdots \cup h^{n-1} \cup h^1 \cup h^n,\]
where this order is the order of this ordered handle decomposition. Recall \( E_{\mu_i} \) in Definition 2.1. We can suppose that each \( E_{\mu_i} \) is a sphere. □

We have more results on the \( \nu \) invariant.

Theorem 4.3. In Theorem 2.2, there is a pair of ‘manifolds with boundaries’ \( M, N \) such that
\[
\nu(M \cup \partial N) \neq \max\{\nu(M), \nu(N)\}.
\]

Proof of Theorem 4.3. Let \( M \cong N \). Let \( M \) be the 2-dimensional solid torus \( S^1 \times D^2 \). Note \( \partial M \) is the torus \( T^2 \). The closed manifold \( A \) in Definition 2.1 for \( S^1 \times D^2 \) is \( T^2 \) or \( \phi \). Consider all \( H_O(S^1 \times D^2, T^2) \) and \( H_O(S^1 \times D^2, \phi) \). Then all these \( H_O \) have \( \partial M = T^2 \) for an integer \( \mu \). Hence \( \nu(H) \geq 4 \).

There is a handle decomposition \( h^0 \cup h^1 \) such that \( \nu(H_O(S^1 \times D^2, \phi)) = 4 \).

There is a handle decomposition \( T^2 \times [0, 1] \cup h^2 \cup h^3 \) such that \( \nu(H_O(S^1 \times D^2, T^2)) = 4 \).

Hence \( \nu(M) = \nu(N) = 4 \).

Note that there is a boundary union \( S^3 = M \cup \partial N \). By Theorem 4.1, \( \nu(S^3) = 2 \).

Hence \( \nu(M \cup \partial N) = 2 < 4 = \max\{\nu(M), \nu(N)\} \).

Hence \( \nu(M \cup \partial N) \neq \max\{\nu(M), \nu(N)\} \). □

It is natural to ask the following question. Is there a closed connected oriented manifold \( M \) with \( \nu(M) > 2 \)?

We have the following theorems as answers to this question.

Theorem 4.4. Let \( M \) be a connected closed oriented 3-manifold. Suppose that \( \pi_1(M) \) is not the trivial group or any of the free groups \( \mathbb{Z} \ast \ldots \ast \mathbb{Z} \). Then \( \nu(M) > 2 \).

Theorem 4.5. Let \( M \) be one of the 3-dimensional lens spaces \( L(p, q) \), where \( L(p, q) \) is not \( S^3 \) or \( S^1 \times S^2 \) as usual. Then \( \nu(M) = 4 \).

Note to Theorem 4.5. \( \pi_1(M) = \mathbb{Z}_p \).

Proof of Theorem 4.4. We suppose the following assumption and we deduce a contradiction.

Assumption. \( \nu(M) \leq 2 \).
By the definition of the $\nu$-invariant, $\nu(M) \geq 2$. Hence $\nu(M) \leq 2$ and $\nu(M) \geq 2$. Hence $\nu(M) = 2$.

Take an ordered handle decomposition $h(1) \cup h(2) \cup \ldots h(\mu) \cup \ldots$ which determines $\nu(M) = 2$. Take $M_\mu$ and $E_{\mu i}$ in Definition 2.1 Let $\mu \geq 1$ because $M_\mu = \phi$ if $\mu = 0$. Since $M$ is a connected closed oriented 3-manifold, each $E_{\mu i}$ is a connected closed orientable surface. Hence $\sum_{j=0}^{j=2} \dim H_j(E_{\mu i} : \mathbb{R}) \geq 2$ for each $i$. By $\nu(M) = 2$ we have $\sum_{j=0}^{j=2} \dim H_j(E_{\mu i} : \mathbb{R}) \leq 2$ for each $i$. Hence each $E_{\mu i}$ is a 2-sphere for each $i$.

Let $M_\mu = M_{\mu_1} \amalg \ldots \amalg M_{\mu_\psi}$, where $\amalg$ denotes a disjoint union and where each $M_{\mu_*}$ is a connected compact manifold with boundary.

Let $\Pi(M_\mu)$ be a ‘finite sequence of groups’ $(\pi_1(M_{\mu_1}), \ldots, \pi_1(M_{\mu_\psi}))$.

Suppose that $h(\mu + 1)$ is a 3-handle. Note that $M_{\mu+1} = M_\mu \cup h(\mu + 1)$. By Van Kampen’s theorem, $\Pi(M_{\mu+1}) = \Pi(M_\mu)$.

Suppose that $h(\mu + 1)$ is a 2-handle. Then the core of the attaching part of $h(\mu + 1)$ is a circle. Since the attaching part is connected, the attaching part is included in one of the 2-spheres $E_{\mu i}$, call it $E_{\mu i}$. Hence it holds that

the attaching part of $h(\mu + 1) \subset$ the 2-sphere $E_{\mu i} \subset M_\mu$.

Hence the attaching part is contractible in $M_\mu$. By Van Kampen’s theorem, $\Pi(M_{\mu+1}) = \Pi(M_\mu)$.

Suppose that $h(\mu + 1)$ is a 1-handle. One of the following two conditions holds if we change the suffix * of $G_*$. 

(1) $\Pi(M_\mu) = (G_1, \ldots, G_{\psi-1}, G_\psi)$ and $\Pi(M_{\mu+1}) = (G_1, \ldots, G_{\psi-1}, G_\psi * \mathbb{Z})$.

(2) $\Pi(M_\mu) = (G_1, \ldots, G_{\psi-2}, G_{\psi-1}, G_\psi)$ and $\Pi(M_{\mu+1}) = (G_1, \ldots, G_{\psi-2}, G_{\psi-1} * G_\psi)$.

Suppose that $h(\mu + 1)$ is a 0-handle. The following condition holds if we change the suffix * of $G_*$. $\Pi(M_\mu) = (G_1, \ldots, G_\psi)$ and $\Pi(M_{\mu+1}) = (G_1, \ldots, G_\psi, 1)$. Here, 1 denotes the trivial group.

Note that $M_0 = \phi$ and that $M_1 = h(1)$ is a 0-handle. Hence $\pi_1(M)$ is the trivial group or one of the free groups $\mathbb{Z} * \ldots * \mathbb{Z}$. This is a contradiction.

This completes the proof. □
Proof of Theorem 4.5. By Theorem 4.4 and \( \pi_1 M = \mathbb{Z}_p \), we have \( \nu(M) > 2 \).

As in the Proof of Theorem 4.4, each \( E_{\mu i} \) is a connected closed orientable surface.

Hence \( \sum_{j=0}^{j=2} \dim H_j(E_{\mu i} : \mathbb{R}) \) is an even number for each \( \mu \geq 1 \) and each \( i \). Hence \( \nu(M) \) is an even number. By \( \nu(M) > 2 \), we have \( \nu(M) \geq 4 \).

There is an ordered handle decomposition \( h^0 \cup h^1 \cup h^2 \cup h^3 \) of \( M \) such that \( \partial M_1 = S^2 \), \( \partial M_2 = T^2 \), \( \partial M_3 = S^2 \), \( \partial M_4 = \phi \). Hence \( \nu(M) \leq 4 \).

Since \( \nu(M) \geq 4 \) and \( \nu(M) \leq 4 \), we have \( \nu(M) = 4 \).

This completes the proof. \( \square \)

Alternative proof of \( \nu(L(p, q)) \leq 4 \). Note that \( L(p, q) \) is a boundary union of two solid torus. In the proof of Theorem 4.3, we prove that the \( \nu \) invariant of the solid torus is four. By Theorem 2.2, \( \nu(L(p, q)) \leq 4 \).

5. The solution to Problem 1.2 in a special case

As a partial solution to Problem 1.2, we prove that the answer to Problem 1.2 is negative under the condition (\#\#) in the last paragraph of \( \S 1 \).

In this case we obtain the result without using the \( \nu \)-invariant.

We suppose that the following assumption is true, and deduce a contradiction.

**Assumption.** Then the answer to Problem 1.2 is affirmative under the condition (\#\#) in the last paragraph of \( \S 1 \).

Let \( W \) be an \( m \)-dimensional arbitrary compact connected manifold with boundary. Suppose that \( \partial W \) has \( z \) connected components. Note that we fix \( z \). We prove both the \( z = 0 \) case and the \( z \geq 1 \) case. Note that we suppose that \( z = 0 \) in Problem 1.2 and that \( z \geq 1 \) in the paragraph right under Problem 1.2.

By the assumption we can divide \( W \) into pieces \( W_i \in S \) and can regard \( W = W_1 \cup \ldots \cup W_w \).

Let \( W_i \cap W_i' \) have \( \rho \) connected components. Hence \( \rho \geq \frac{3w-z}{2} \).

Consider the Meyer-Vietoris exact sequence:

\[
H_j(\bigcup_{i,i'} \{W_i \cap W_i'\}; \mathbb{R}) \to H_j(\bigcup_{i=i'=w} \{W_i; \mathbb{R}\}) \to H_j(W; \mathbb{R})
\]

Here, \( \bigcup_{i,i'} \) denotes the disjoint union of \( W_i \cap W_i' \) for all \( (i, i') \). Consider

\[
H_1(W; \mathbb{R}) \to H_0(\bigcup_{i,i'} \{W_i \cap W_i'\}; \mathbb{R}) \to H_0(\bigcup W_i; \mathbb{R}) \to H_0(W; \mathbb{R}) \to 0
\]

Note \( H_0(\bigcup W_i; \mathbb{R}) \cong \mathbb{R}^w \) and \( H_0(W; \mathbb{R}) \cong \mathbb{R}^l \). We suppose \( H_1(W; \mathbb{R}) \cong \mathbb{R}^l \). Then we have the exact sequence:

\[
\mathbb{R}^l \to \mathbb{R}^\rho \to \mathbb{R}^w \to \mathbb{R} \to 0
\]

Hence \( l \geq \rho - w + 1 \). Hence \( l \geq \frac{w-z+2}{2} \). Hence \( (2l + z - 2) \geq w \).
We define an invariant \( h(\cdot) \). Let \( X \) be a compact manifold with boundary. Take a handle decomposition of \( X \). Consider the numbers of handles in the handle decompositions of \((X, A)\), where \( A \) is one as defined in Definition 2.1. Let \( h(X, A) \) be the minimum of such the numbers. Let \( h(X) \) be the maximum of \( h(X, A) \) for all \( A \).

Note that \( S \) is a finite set \( \{M_1, \ldots, M_\mu\} \). Suppose that \( M' \) is one of the ‘manifolds with boundaries’ \( M_i \) and that \( h(M') \geq h(M_i) \) for any \( i \). Then we have \( w \times h(M') \geq h(W) \). Hence \((2l + z - 2) \times h(M') \geq h(W)\). Note that the left side is constant.

For any natural number \( N \), there are countably infinitely many compact oriented connected \( m \)-manifolds \( W' \) with boundaries such that \( \partial W' = \partial W \), that \( H_1(W; \mathbb{R}) \cong \mathbb{R}^l \), and that \( h(W) \geq N \). Because: There is an \( n \)-dimensional closed manifold \( P \) such that \( H_1(P; \mathbb{R}) \cong \mathbb{R}^l \). There is an \( n \)-dimensional rational homology sphere \( Q \) which is not an integral homology sphere. Make a connected sum which is made from one copy of \( P \) and \( q \) copies of \( Q \) \( (q \in \mathbb{N} \cup \{0\}) \).

We arrived at a contradiction. This completes the proof.

Furthermore [5] pointed out the following.

(1) There is an \( n \)-dimensional connected Feynman diagram with three outlines (a compact connected \( n \)-manifold with boundary whose boundary has three connected components) with the following properties. Two copies of the diagram is made into countably infinitely many kinds of diagrams with four outlines.

(The idea of the proof: Let the diagram be \{the solid torus\}—two open 3-balls. Use the fact that all 3-dimensional lens spaces, \( S^3 \), and \( S^1 \times S^2 \) are made from two solid torus.)

(2) There is an infinite set \( S \) with the following properties.

(i) All \( m \)-dimensional Feynman diagrams (compact \( m \)-manifolds with boundaries) are boundary unions of finite elements of \( S \).

(ii) Each element of \( S \) is what is made by attaching an \( m \)-dimensional handle to (an \((m - 1)\)-dimensional connected closed manifold) \( \times [0, 1] \). Note it has one, two or three connected boundary components. (The idea of the proof: Use handle decompositions.)

6. Discussion

Take a group \( G = \{g_1, \ldots, g_N\} \)

\[
\begin{align*}
g_1 \cdot g_2 \cdots g_{N-1} \cdot g_N \cdot g_2^{-1} \cdots g_N^{-1} &= 1, \\
g_2 \cdot g_3 \cdots g_N \cdot g_1 \cdot g_3^{-1} \cdots g_N^{-1} \cdot g_1^{-1} &= 1, \\
g_N \cdot g_1 \cdots g_{N-2} \cdot g_{N-1} \cdot g_1^{-1} \cdots g_{N-1}^{-1} &= 1. 
\end{align*}
\]

In the \( m \geq 4 \) case, we can make a compact connected oriented manifold \( Z \) with boundary such that

(1) \( \pi_1(Z) = G \).
(2) $Z$ is made of one $0$-handle, $N$ copies of $1$-handles, and $N$ copies of $2$-handles. (Each of the generators $g_*$ corresponds to each of the $1$-handles. Each of the $N$ relations corresponds to each of the $2$-handles.)

Take the double of $Z$. Call it $W$. Note $\pi_1(W) = G$.

Thus we submit the following problem.

**Problem 6.1.** Do you prove $\nu(W) \geq N$?

If the answer to Problem 6.1 is affirmative, then the answer to Problem 2.5 is affirmative (in the closed manifold case, which would be extended in all cases).

By using a manifold with boundary whose fundamental group is so complicated as above, we may solve Problem 1.2, 2.5.

We explain the above strategy more.

Let $M$ be a finite dimensional smooth compact connected manifold with boundary. Take any handle decomposition $H$, $M = h^0 \cup h^1_1 \cup h^2_1 \cup \ldots$.

Suppose that there is only one $0$-handle $h^0$ in this handle decomposition $H$.

Suppose that the $1$-handles $h^1_1, \ldots, h^1_{\alpha_k}$ are all $1$-handles of this handle decomposition $H$.

Different $1$-handles $h^1_{\alpha_1}, \ldots, h^1_{\alpha_k} \in \{h^1_1, \ldots, h^1_{\ell_2}\}$ are called *brother-handles* if $h^1_{\alpha_1}, \ldots, h^1_{\alpha_k}$ satisfy the following. If a $2$-handle of the handle decomposition $H$ is attached to one of $h^1_{\alpha_1}, \ldots, h^1_{\alpha_k}$, then the $2$-handle is attached to all of $h^1_{\alpha_1}, \ldots, h^1_{\alpha_k}$.

Take each set of brother-handles of $H$. Take the order of each set. Take the maximum of the orders. Call it $b(H)$.

Consider each handle decomposition $E$ of $M$. Take $b(E)$ for each handle decomposition $E$. Take the minimum of all $b(E)$. Call it $b(M)$.

We submit the following problem.

**Problem 6.2.** Let $m$ be an integer $\geq 3$. Suppose that there is an $m$-dimensional compact connected manifold $X$ with boundary. Take any natural number $n$. Then is there an $m$-dimensional compact connected manifold $M$ with boundary such that $\partial M = \partial X$ and that $b(M) \geq n$?

In particular, consider the $\partial X = \phi$ case.

If Problem 6.2 has the affirmative answer, then Problem 2.5 may have the affirmative answer.

The author could prove that $b(S^n) = 0$, $b(\mathbb{RP}^n) = 1$, $b(\sharp^m \mathbb{RP}^n) = 1$, $b(T^2) = 2$ and some other cases. The author guesses that it might not be difficult to calculate the $b(\ )$ in the following Problem 6.3 or to evaluate it from the lower side.

**Problem 6.3.** For the closed manifold $W$ in Problem 6.1 is $b(W) \geq N$?
The author would think that these problems could be solved by group theoretic ways. The author tries to interpret Problem 6.2 in terminology of group theory.

Let $G$ be a finitely generated group. Let $\langle g_1, \ldots, g_\xi | r_1, \ldots, r_\zeta \rangle$ be a presentation $P$ of $G$. Different generators $g_{\alpha_1}, \ldots, g_{\alpha_k} \in \{g_1, \ldots, g_\xi\}$ are called brother-generators if $g_{\alpha_1}, \ldots, g_{\alpha_k}$ satisfy the following. If a relation $r_*$ of the presentation $P$ includes one of $g_{\alpha_1}, \ldots, g_{\alpha_k}$, then the relation $r_*$ includes all of $g_{\alpha_1}, \ldots, g_{\alpha_k}$. Here, if we say that $r_*$ includes $g_\rho$, then it means that $r_*$ includes $g_\rho$ or $(g_\rho)^{-1}$.

Take each set of brother-generators of $P$. Take the order of each set. Take the maximum of the orders. Call it $b(P)$.

Consider each presentation $E$ of $G$. Take $b(E)$ for each presentation $E$. Take the minimum of all $b(E)$. Call it $b(G)$.

We submit the following problems.

**Problem 6.4.** Let $n$ be any natural number. For a group $G$, is $b(G)$ greater than $n$?

The following problem may be connected with Problem 6.3.

**Problem 6.5.** For the closed manifold $W$ in Problem 6.1, do we have $b(\pi_1(W)) \geq N$?

We submit one more problem which seems easier than other problems in this paper.

**Problem 6.6.** Is the $\nu$ invariant of the Poincaré sphere six?

Use $Z_p$ coefficient homology groups instead in the definition of $\nu$. Use the order of $\text{Tor}H_*(\ ; \ Z)$ instead in the definition of $\nu$. Can we solve Problem 1.2?

Calculate $\nu$, $b$ of the knot complement. In particular, in the case of 1-dimensional prime knots. In this case, what kind of connection with the Heegaard genus?

If we replace $\sum_{*=-m}^{*=-1} \dim H_*(E_{\mu_i}; \mathbb{R})$ in Definition 2.1 with $\dim H_*(E_{\mu_i}; \mathbb{R})$ for the fixed integer $*$ or a nonnegative real number valued topological invariant (resp. diffeomorphism type invariant) of $E_{\mu_i}$, we obtain another invariant instead of $\nu$. It satisfies Theorem 2.2.

An example made by using such a nonnegative real number valued topological invariant: If we replace $\sum_{*=-m}^{*=-1} \dim H_*(E_{\mu_i}; \mathbb{R})$ in the $m=4$ case in Definition 2.1 with the absolute value of a quantum invariant $\tau_*(E_{\mu_i})$, we obtain another invariant instead of $\nu$. It satisfies Theorem 2.2.

This paper is based on the author’s preprints [5].

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Computer Science, Meijigakuin University, Yokohama, Kanagawa, 244-8539, Japan
pqr100pqr100@yahoo.co.jp, ogasa@mail.meijigkauin.ac.jp