On the combinatorics of last passage percolation in a quarter square and GOE\(^2\) fluctuations

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September 19, 2018

Abstract

In this note we give a(n)other combinatorial proof of an old result of Baik–Rains: that for appropriately considered independent geometric weights, the generating series for last passage percolation polymers in a \(2n \times n \times n\) quarter square (point-to-half-line-reflected geometry) splits as the product of two simpler generating series—that for last passage percolation polymers in a point-to-line geometry and that for last passage percolation in a point-to-point-reflected (half-space) geometry, the latter both in an \(n \times n \times n\) triangle. As a corollary, for iid geometric random variables—of parameter \(q\) off-diagonal and parameter \(\sqrt{q}\) on the diagonal—we see that the last passage percolation time in said quarter square obeys Tracy–Widom GOE\(^2\) fluctuations in the large \(n\) limit as both the point-to-line and the point-to-point-reflected geometries have known GOE fluctuations. This is a discrete analogue of a celebrated Baik–Rains theorem (the limit \(q \to 0\)) and more recently of results from Bisi’s PhD thesis (the limit \(q \to 1\)).

1 Introduction

Background and motivation. Discrete point-to-point last passage percolation with iid geometric weights of parameter \(q\) was introduced by Johansson [Joh00]. Using the Robinson–Schensted–Knuth [Knu70] correspondence, Johansson showed that the last passage time is a certain observable of a large class of determinantal measures called Schur measures [Oko01]. Using this together with orthogonal polynomial techniques, he then analyzed the last passage time asymptotically to find the celebrated KPZ \(n^{1/3}\) fluctuation scaling regime and the Tracy–Widom [TW94] GUE distribution as the limiting distribution for fluctuations. The result also holds in the \(q \to 0\) limit (the Poisson case) and the \(q \to 1\) limit (the case of iid exponential weights).

Other geometries have also been considered. Rains [Rai00], Baik–Rains [BR01a, BR01b], Ferrari [Fer04], Forrester–Rains [FR07] and Bisi–Zygouras [BZ17b] have considered the similar problem in a point-to-line geometry and here the last passage time is an observable of a pfaffian Schur measure (respectively symplectic measure for Bisi–Zygouras). The last passage percolation time (at least in the Poisson and exponential limits) has been shown to have Tracy–Widom GOE fluctuations in the works of Baik–Rains [BR01b] (Poisson limit), Ferrari and Bisi–Zygouras [Fer04, BZ17a] (exponential limit). Rains [Rai00], Baik–Rains [BR01a, BR01b], Forrester–Rains [FR07], Sasamoto–Imamura [SI04], Baik–Barraquand–Corwin–Suidan [BBCS17] and Betea–Bouttier–Nejjar–Vuletić [BBNV18] have further considered the point-to-point-reflected (sometimes dubbed half-space) geometry where again for geometric/Poisson/exponential weights, the fluctuations of the last passage time are Tracy–Widom GOE [TW96b].

In this note we consider last passage percolation in a quarter square which we call, inspired by the previous paragraph, point-to-half-line-reflected geometry—see Figure 1 (top)—with independent geometric weights and show that the distribution for the last passage percolation time is a product of two distributions: that of the time in a point-to-line geometry and that in a point-to-point-reflected (half-space) geometry—both of which were discussed in the previous paragraph. This fact leads to GEO\(^2\) fluctuations in the

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\(^1\)In the case, say, of independent geometric weights, we want the parameter for the weights on the diagonal be the square root of the off-diagonal parameter.
appropriate $n^{1/3}$ fluctuation regime when all weights are iid geometric random variables with parameter $q$ off-diagonal and $\sqrt{q}$ on diagonal. This model was considered and different formulas for the distribution first obtained by Rains [Rai00], Baik–Rains [BR01a] (for the Poisson limit $q \to 0$), Rains–Forrester [PR07], and Bisi–Zygouras [BZ17b] (for the exponential limit $q \to 1$). Asymptotics revealing GOE\(^2\) fluctuations were considered by Baik–Rains [BR01b] (for the Poisson limit, using Toeplitz+Hankel determinants and Riemann–Hilbert techniques) and Bisi [Bis18] (for the exponential limit, using classical steepest descent analysis).

One of the upshots of the present note is that, like in the case of Baik–Rains for the Poisson limit [BR01a, BR01b], no analysis is required to arrive at said Tracy–Widom GOE\(^2\) fluctuations. The mere fact that the distribution splits as a product of two known simpler distributions with known asymptotics is enough to pass to the limit in the original one. Another, more subtle, is that the result as proven combines bijective and representation theoretic techniques—notably Robinson–Schensted–Knuth correspondences and bounded Littlewood identities for various types of characters—which could hopefully be amenable to generalizations studying so-called “non-free fermionic models”.

**The main result.** To state the main result, let us fix some notation and vocabulary. Since one of the main ingredients in the proof is a discrete version of a result of Bisi–Zygouras [BZ17b], we will keep notation and terminology close to op. cit. whenever feasible. Throughout, $n$ and $u$ will denote positive integers, and we will assume—to simplify some formulas—that $u := 2v$ is even. Furthermore, we will use $x_1, \ldots, x_n$ as variables throughout.

Consider the following three discrete domains in the quarter plane $i,j \geq 0$ ($i$ the horizontal axis) built up from $1 \times 1$ unit squares: $D_n^{p2hlr}$ is the $2n \times 2n \times 2n$ discrete quarter square (triangle) consisting of the $n^2 + n$ unit squares on or above the diagonal $i = j$ and on or below the anti-diagonal $i = 2n - j + 1$; $D_n^{p2pr}$ is the $n \times n \times n$ triangle consisting of the $(n+1)/2$ unit squares on or above the diagonal $i = j$ and below the horizontal line $j = n$; and $D_n^{p2l}$ is the $n \times n \times n$ triangle consisting of the $(n+1)/2$ unit squares on or below the anti-diagonal $i = n - j + 1$. They are depicted in Figure 1 (top, bottom left and bottom right respectively). The meaning of the abbreviations stands for the type of geometry we will consider: $p2hlr$ stands for point-to-half-line-reflected, $p2pr$ stands for point-to-point-reflected, and $p2l$ stands for point-to-line.

Each unit square $(i,j)$ in each of the three types of triangles has a non-negative integer $w_{i,j}$ sitting inside it, and to such a filling $W := (w_{i,j})$ we associate a weight $\text{wt}(W)$ as follows. The total weight of a triangle is a product over weights of all individual unit squares inside it. A unit square $(i,j)$ contributes a weight $(x_i x_j)^{w_{i,j}}$ if it off-diagonal and—in the case of triangles from $D_n^{p2pr}$ and $D_n^{p2hlr}$ only—a weight $x_{i,j}^{w_{i,j}}$ if the square is on-diagonal. The parameters are assigned to each row and column inside the domains thusly: $x_i$ corresponds to column $i$ placed under the $i$-axes in Figure 1; while $x_j$ corresponds to row $j$ in the point-to-point-reflected (p2pr) geometry, to row $n - j + 1$ in the point-to-line geometry (p2l) and to rows $j$ and $2n - j + 1$ in the point-to-half-line. We are interested in: the set of all fillings $W$ of domain $D_n^{p2hlr}$ such that the longest up-right path—called a polymer—from the $(1,1)$ square to any square on the line $i = 2n - j + 1$ is $\leq u$; the set of all fillings $W$ of domain $D_n^{p2pr}$ such that the longest up-right path/polymer from the $(1,1)$ square to any square on the line $i = n - j + 1$ is $\leq v = u/2$; and the set of all fillings $W$ of domain $D_n^{p2l}$ such that the longest polymer from the $(1,1)$ square to any square on the line $i = n - j + 1$ is $\leq v = u/2$. Here the length of a polymer is the sum of all $w_{i,j}$ crossed by the associated path and the longest such polymer is usually called the last passage percolation (LPP) time for that geometry. We denote the latter by $L_n^{p2hlr}$, $L_n^{p2pr}$ and $L_n^{p2l}$. We are thus interested in the sets $\mathcal{W}_{n,u}^{p2hlr} = \{W \mid L_n^{p2hlr} \leq \ell\}$ where $x \in \{p2hlr, p2pr, p2l\}$ and $\ell = u$ if $x = p2hlr$ and $\ell = v = u/2$ otherwise. See Figure 2 for the geometry, sample polymers and parametrization of rows and columns—note the weights $w_{i,j}$ are replaced by bullets.

The main result is the following equality of generating series.

**Theorem 1.** We have:

$$
\sum_{W \in \mathcal{W}_{n,u}^{p2hlr}} \text{wt}(W) = \left( \sum_{W \in \mathcal{W}_{n,u}^{p2pr}} \text{wt}(W) \right) \cdot \left( \sum_{W \in \mathcal{W}_{n,u}^{p2l}} \text{wt}(W) \right)
$$

\(^2(i,j)\) is the cartesian coordinate of the top right corner of the square.
In other words and with parameters as stated above, the generating series for point-to-half-line-reflected polymers of length at most $u$ is a product of the two generating for point-to-point-reflected and point-to-line polymers of length at most $v = u/2$.

The result will be proven using purely combinatorial techniques and formulas pieced together from the works of Okada [Oka98], Stembridge [Ste90] and Bisi–Zygouras [BZ17b].

Remark 2. Theorem 1 is implicit already in the work of Baik–Rains [BR01a, Corollary 4.3]: one can multiply equations (4.21) and (4.23) to obtain equation (4.25) in op. cit., which certainly implies the result. We believe the non-intersecting lattice paths proof of Forrester–Rains [FR07, Section 6] can also be modified to produce the same outcome. The idea in both references is that the left-hand side in Theorem 1 can be written as a sum over semi-standard domino tableaux which are in bijection to pairs of semi-standard Young tableaux which then yield a product of two bounded Littlewood sums of Schur polynomials giving the product on the right-hand side. A similar idea is employed in this note except instead of passing through self-dual (in the Schützenberger sense) and domino tableaux, we go through symplectic and orthogonal tableaux and characters. We thus side-step the Schützenberger involution [Ful97].

Passing to probability, suppose $x_i = \sqrt{q}$ for all $i$ where $0 < q < 1$ is a fixed parameter and suppose $w_{i,j}$ are now independent geometric random variables of parameter $q$ off-diagonal and $\sqrt{q}$ on-diagonal for the p2hlr and p2pr geometries. Based on known $n^{1/3}$ asymptotics for the p2pr and p2l geometries—see Section 2.2 for details and references, we have the following corollary.

Corollary 3. We have:

$$\lim_{n \to \infty} \text{Prob} \left( \frac{L_n^{p2hlr} - 2c_1n}{2c_2n^{1/3}} \leq s \right) = F_1^2(s)$$

where

$$c_1 = \frac{2\sqrt{q}}{1 - \sqrt{q}}, \quad c_2 = \frac{q^{1/6}(1 + \sqrt{q})^{1/3}}{1 - \sqrt{q}}$$

and where $F_1(s)$ is the Tracy–Widom GOE distribution [TW96].

Remark 4. Corollary 3 appears as Theorem 4.2 equation (4.30) with $(w = 0, \beta = 0$ case) in the extended abstract [BR99], with the proof left to the reader but following along similar lines as the proof of the Poisson limit $q \to 0$. The latter limit of Corollary 3 was proven in Baik–Rains [BR01b] based on the combinatorics explained in Remark 2 and on Riemann–Hilbert techniques for the asymptotics of certain Toeplitz+Hankel determinants. The exponential limit of the corollary (i.e., $q \to 1$) has a more recent analytical proof by Bisi (private communication and [Bis18]) who uses classical steepest descent analysis.

Organization of the paper. The paper is organized as follows. In Section 2, we prove the main result. We first set up the machinery of Schur functions and symplectic/orthogonal characters along with the associated tableaux/Gelfand–Tsetlin patterns in Section 2.1 and then prove the main result in Section 2.2. We conclude in Section 3. As the heavy lifting involves variants of the Robinson–Schensted–Knuth correspondence, we recall all the required material in Appendix A using the well-established language of Fomin (corner) growth diagrams. Finally, to prevent the somewhat large figures from interrupting the flow of text, we have placed all figures at the end of the document.

Acknowledgements. The author acknowledges illuminating conversations with Nikos Zygouras and Elia Bisi regarding [BZ17b].

2 Proofs

2.1 Some preliminaries

An (integer) partition $\lambda$ is a non-increasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$, only finitely many of them non-zero. The non-zero integers are called parts. The length of $\lambda$, denoted $\ell(\lambda)$, is the number $\text{of parts}$. Or at least sweep it under a moderately thick rug.

$X$ is a geometric random variable of parameter $q$ on $\mathbb{N}$ if $\text{Prob}(X = k) = (1 - q)q^k$. 

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of its (non-zero) parts, while its size, denoted $|\lambda|$, is the sum of its parts $|\lambda| := \sum_{i \geq 1} \lambda_i$. The empty partition, denoted by $\emptyset$, is the one with no parts and has length 0. Partitions are usually written/thought of as Young diagrams, collections of square boxes, left aligned, containing $\lambda_i$ squares in row $i$, counting from the top. For two partitions $\mu \subset \lambda$—meaning $\mu_i \leq \lambda_i \ \forall i$, we write $\mu \subset \lambda$ (respectively $\mu \prec \lambda$) and say $\mu$ (upwards) interlaces with $\lambda$ (respectively $\mu$ dual interlaces with $\lambda$), if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ (respectively $\lambda_i - \mu_i \in \{0,1\}$) for all $i$.

A Gelfand–Tsetlin pattern of height $n$ is a triangular array $z = (z_{i,j})_{1 \leq i \leq j \leq n}$ with non-negative integer entries that satisfy the interlacing conditions $z_{i+1,j+1} \leq z_{i,j} \leq z_{i+1,j}$ for all meaningful $i,j$. Its shape is the bottom row partition $(z_{n,1}, \ldots, z_{n,n})$. Its type is the vector type($z$) defined by $\text{type}(z)_i = \sum_{j=1}^{i-1} z_{i,j} - \sum_{j=1}^{i-1} z_{j,i-1}$ for $1 \leq i \leq n$. A Gelfand–Tsetlin pattern $z$ of height $n$ and shape $\lambda$ can be equivalently viewed as an upwards interlacing sequence of $n+1$ partitions

$$z \leftrightarrow \Lambda = \left( \emptyset = \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} = \lambda \right)$$

via $\lambda^{(i)} := z_{i,i}$. Note $\ell(\lambda^{(i)}) \leq i$. Finally by placing the symbol $i$ ($1 \leq i \leq n$) in the skew Young diagram $\lambda^{(i)}/\lambda^{(i-1)}$—the boxes in $\lambda^{(i)}$ and not in $\lambda^{(i-1)}$, one obtains a third equivalent characterization: a semi-standard Young tableau (SSYT) $T$—a filling of $\lambda$ with symbols from $1 \leq 2 \leq \cdots \leq n$ strictly increasing down columns and weakly increasing down rows. Observe that in the equivalence $z \leftrightarrow \Lambda \leftrightarrow T$ we have $\text{type}(z)_i = |\lambda^{(i)}| - |\lambda^{(i-1)}|$ = number of $i$ in $T$.

Given an integer partition $\lambda$ with $\ell(\lambda) \leq n$, we denote by $GT_n(\lambda)$ the set of all Gelfand–Tsetlin patterns of height $n$ and shape $\lambda$.

A symplectic Gelfand–Tsetlin pattern of height $2n$ is a “half-triangular” array $z = (z_{i,j})_{1 \leq i \leq 2n, 1 \leq j \leq \lfloor i/2 \rfloor}$ with non-negative integer entries satisfying the interlacing conditions $z_{i+1,j+1} \leq z_{i,j} \leq z_{i+1,j}$ for $1 \leq i < 2n, 1 \leq j \leq \lfloor i/2 \rfloor$ with the convention that $z_{i,j} := 0$ when $j > \lceil i/2 \rceil$ (so that all its entries are non-negative). Its shape $\lambda$ is the bottom row partition $(z_{2n,1}, \ldots, z_{2n,n})$, and its type is the vector type($z$) with $\text{type}(z)_i := \sum_{j=1}^{\lfloor i/2 \rfloor} z_{i,j} - \sum_{j=1}^{\lfloor (i-1)/2 \rfloor} z_{i-1,j}$ for $1 \leq i \leq 2n$. A symplectic Gelfand–Tsetlin pattern $z$ of height $2n$ and shape $\lambda$ can be equivalently viewed as an upwards interlacing sequence of $2n+1$ partitions

$$z \leftrightarrow \Lambda = \left( \emptyset = \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(2n)} = \lambda \right)$$

via $\lambda^{(i)} := z_{i,i}$. Note $\ell(\lambda^{(i)}) \leq \lceil i/2 \rceil$. Finally by placing the symbol $i$ respectively $\tilde{i}$ ($1 \leq i \leq n$) in the skew Young diagram $\lambda^{(i)}/\lambda^{(i-1)}$ respectively $\lambda^{(2i)}/\lambda^{(2i-1)}$, one equivalently obtains a symplectic tableau (SpT) $T$ of King [Kin76, KES83]—a semi-standard Young tableau of shape $\lambda$ on $1 \leq T < 2 \leq 3 \leq \cdots < n < \pi$ having the symplectic property: elements in row $i$ are $\geq i$. In the equivalence $z \leftrightarrow \Lambda \leftrightarrow T$ we have $\text{type}(z)_{2i-1} = |\lambda^{(2i-1)}| - |\lambda^{(2i-2)}| = \text{number of } i \text{ in } T$ and $\text{type}(z)_{2i} = |\lambda^{(2i)}| - |\lambda^{(2i-1)}| = \text{number of } \tilde{i} \text{ in } T$.

An example is given in Figure 2 (bottom right).

A Sundaram [Sun90b] odd orthogonal tableau (OOT) of shape $\lambda$ and height $n$ is a filling of $\lambda$ with the alphabet $1 \leq \tilde{T} < 2 \leq 3 \leq \cdots < n < \pi$ satisfying the same conditions as a symplectic tableau and the extra condition that no two infinite symbols can appear in the same line of the Young diagram $\lambda$. As above, it can be put into a correspondence with a sequence of $2n+2$ partitions, the first $2n+1$ interlacing (and the last two dual interlacing) or an odd orthogonal Gelfand–Tsetlin pattern of height $2n$—though technically speaking having $2n+1$ rows. Since we will make no use of odd orthogonal tableaux in the sequel, we leave the details to the interested reader.

Given an integer partition $\lambda$ with $\ell(\lambda) \leq n$, we denote by $SpGT_{2n}(\lambda)$ (respectively OOGT$_{2n}(\lambda)$) the set of all symplectic (respectively odd orthogonal) Gelfand–Tsetlin patterns of height $2n$ and shape $\lambda$.

Let $h_k(x_1, x_2, \ldots)$ be the $k$-th complete symmetric function defined (among many possibilities) by its generating series $\sum_{k \geq 0} h_k(x_1, x_2, \ldots) z^k = \prod_{i=1}^{\infty} \frac{1}{1 - x_i z}$. The Schur polynomials [Mac95] and symplectic and odd orthogonal characters [FH91, Sun90a] can be defined by the following Jacobi–Trudi formulæ:

$$s_\lambda(x_1, \ldots, x_n) = \det[h_{\lambda_i-i+j}(x_1, \ldots, x_n)]_{1 \leq i,j \leq \ell(\lambda)},$$

$$sp_\lambda(x_1^\pm, \ldots, x_n^\pm) = \frac{1}{2} \det[h_{\lambda_i-i+j}(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) + h_{\lambda_i-i+j+2}(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})]_{1 \leq i,j \leq \ell(\lambda)},$$

$$so^{odd}_\lambda(x_1^\pm, \ldots, x_n^\pm) = \det[h_{\lambda_i-i+j}(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) - h_{\lambda_i-i-j}(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})]_{1 \leq i,j \leq \ell(\lambda)}.$$
The Schur polynomial is a manifestly symmetric polynomial in its variables, while the symplectic and orthogonal characters are Laurent polynomials having $BC$-symmetry—symmetry under permuting as well as inverting the variables—and this explains the notation $x_i^\pm$. An equivalent way to define them—see [FK97] for a combinatorial proof, and the way they will appear in this note, is as generating series of semi-standard (SSYT)s, symplectic (SpT)s and odd orthogonal (OOT)s tableaux (Gelfand–Tsetlin patterns):

$$s_\lambda(x_1,\ldots,x_n) = \sum_{T: \text{SSYT of shape } \lambda} \prod_{i=1}^n x_i^{\text{number of } i \text{ in } T} = \sum_{z \in GT_n(\lambda)} \prod_{i=1}^n x_i^{\text{type}(z)_i},$$

$$sp_\lambda(x_1^\pm,\ldots,x_n^\pm) = \sum_{T: \text{SpT of shape } \lambda} \prod_{i=1}^n x_i^{\text{number of } i \text{ in } T} = \sum_{z \in \text{SpGT}_{2n}(\lambda)} \prod_{i=1}^n x_i^{\text{type}(z)_{2i-1} - \text{type}(z)_{2i}},$$

$$so_\lambda(x_1^\pm,\ldots,x_n^\pm) = \sum_{T: \text{OOT of shape } \lambda} \prod_{i=1}^n x_i^{\text{number of } i \text{ in } T} = \sum_{z \in \text{OOGT}_{2n}(\lambda)} \prod_{i=1}^n x_i^{\text{type}(z)_{2i-1} - \text{type}(z)_{2i}}.$$

For a combinatorial proof of the above, see [FK97].

### 2.2 Proof of the main result

We first prove Theorem [1]. The proof consists of four steps, outlined below.

**Step 1.** The first step consists in showing that

$$\sum_{W \in \Lambda_{n,u}^{p2hlr}} \text{wt}(W) = \left(\prod_{i=1}^n x_i\right)^u \sum_{\lambda: \lambda_1 \leq u} sp_\lambda(x_1^\pm,\ldots,x_n^\pm).$$

This proof is a discrete adaptation of the (finite temperature) argument given by Bisi and Zygouras [BZ17b]. In can be summarized as follows: on a triangle $W \in \Lambda_{n,u}^{p2hlr}$ we inductively apply the bijective rowRSK local rule of Appendix A to obtain a generalized (semi-standard) oscillating tableau—the entries of which will all be $\leq u$—and we subtract the entries of this tableau from $u$ to obtain a triangle of integers (between 0 and $u$) which turns out to be our sought symplectic Gelfand–Tsetlin pattern. Its shape $\lambda$ has $\lambda_1 \leq u$. The transformation sends the weight $\text{wt}(W)$ to the proper weight appearing in the tableau definition of the desired symplectic character $sp_\lambda(x_1^\pm,\ldots,x_n^\pm)$. We note that the first part of the bijection is well-known and appears in e.g. the work of Krattenthaler [Kra16].

More precisely, we take $W$—its entries sitting on $(N + 1/2)^2$ as depicted in Figure 2—and to it we inductively attach partitions $\mu(i,j)$ sitting on the lattice points of $N^2 \cap D_{2n}$. We assign the empty partition $\emptyset$ on the axes: $\mu(0,j) = \mu(i,0) = \emptyset, i, j \in \{0,1\}, 0 \leq j \leq 2n$. To each other lattice point $(i,j)$—starting with $(1,1)$—we inductively assign the partition $\mu(i,j)$ which is the output $\nu$ (in the notation of Appendix A) of the rowRSK local rule from Appendix A applied on input $\alpha = \mu(i-1,j), \beta = \mu(i,j-1), \kappa = \mu(i-1,j-1), \gamma = w_{i,j}$. We call rowRSK the totality of steps applying the local rule rowRSK on $W$. We note that for a square $(i,j)$ on the $i = j$ diagonal (blue squares in Figure 2), once we have obtained $\kappa = \mu(i-1,j)$ and $\alpha = \mu(i,j)$ we set $\mu(i,j-1) = \mu(i-1,j)$ to act as our $\beta$ for the local rule rowRSK. At the end, the output of rowRSK is a sequence of up-down interlacing partitions

$$\emptyset = \mu(0,2n) \prec \mu(1,2n) \prec \mu(1,2n-1) \prec \cdots \prec \mu(n,n+1) \succ \mu(n,n)$$

depicted in bold in Figure 2 (top right, the north and east borders). We note by construction $\ell(\mu(i,j)) = \min(i,j)$ for all meaningful $i,j$. We then slide the partitions (except for the two empty ones) from equation (9) back into the triangular shape: each partition on the north-east border goes south-west into the diagonal squares from the diagonal it lies on—see Figure 2 (top right to bottom left). This is the generalized oscillating tableau (fig. cit., bottom left) and we note, due to Greene’s Theorem [7a], that all its entries are $\leq u$ so we can subtract all said entries from $u$ and obtain a triangular tableau of numbers which now has weakly decreasing rows (left to right) and weakly increasing columns (top to bottom)—see Figure 2 (bottom, middle.
This is our symplectic Gelfand–Tsetlin pattern $z$ of height $2n$: more precisely, the $k$-th row of $z$ is the $i = j - (2n - k)$-th diagonal of the tableau just described—bottom last two panels in Figure 2. Its shape $\lambda := z_{2n-1}$ is the partition sitting on the diagonal $i = j$.

Note that everything described so far is reversible: $\text{rowRSK}$ and thus $\text{rowRSK}$ is a bijection, as is subtraction from $u$. In terms of the sequence of partitions from (9) we have

$$
\mu_{[k/2],2n-\lfloor k/2\rfloor}^{k/2} = t_{i,2n-k+i}, \quad 1 \leq i \leq \lfloor k/2\rfloor, 1 \leq k \leq 2n. \quad (10)
$$

Let $\text{row}_i$ (respectively $\text{col}_i$) be the sum of the $i$-th row (respectively column) of $W$. Inductive (repeated) application of (24)—see Remark 8—implies

$$
\text{row}_{2n-i+1} = |\mu^{(i,2n-i+1)}| - |\mu^{(i,2n-i)}|, \quad \text{col}_i - w_{i,i} = |\mu^{(i,2n-i+1)}| - |\mu^{(i,2n-i+1)}|, \quad 1 \leq i \leq 2n. \quad (11)
$$

Furthermore Greene’s Theorem (7a) ensures $0 \leq t_{i,j} \leq u$ for all meaningful $i,j$. Let us change variables by setting

$$
z_{i,j} := u - t_{j,2n-i-j}, \quad 1 \leq j \leq \lfloor i/2\rfloor, 1 \leq i \leq 2n. \quad (12)
$$

These variables also satisfy $0 \leq z_{i,j} \leq u$ for all meaningful $i,j$. Observe that because of the up-down interlacing constraints satisfied by the diagonals of $T$ (the partitions in (9)) and the further entry-wise subtraction from $u$, $z$ satisfies the interlacing constraints of a symplectic Gelfand–Tsetlin pattern. Its shape $\lambda$ clearly satisfies $\lambda_1 \leq u$.

Denote by $|z_i| := \sum_{j=1}^{\lfloor i/2\rfloor} z_{i,j}$, i.e. the sum of the $i$-th row of $z$. The definition of $T$ and (12) implies that for all $1 \leq i \leq 2n$ we have

$$
|\mu^{(i,2n-i+1)}| - |\mu^{(i,2n-i+1)}| = u - |z_{2i-1}| + |z_{2i-2}|, \quad |\mu^{(i,2n-i+1)}| - |\mu^{(i,2n-i)}| = |z_{2i}| - |z_{2i-1}|. \quad (13)
$$

Putting it all together, we obtain that

$$
\sum_{W \in \mathcal{W}_n^{2khr}} \text{wt}(W) = \sum_{W \in \mathcal{W}_n^{2khr}} \prod_{i=1}^{n} x_i^{\text{row}_{2n-1-i} + \text{col}_i + \text{row}_i - w_{i,i}}
$$

$$
= \sum_{\text{oscillating tableau } T} \prod_{i=1}^{n} x_i^{2|\mu^{(i,2n-i+1)}| - |\mu^{(i,2n-i)}| - |\mu^{(i,2n-i+1)}|}
$$

$$
= \sum_{\lambda : \lambda_1 \leq u} \sum_{z \in \text{SpGT}_n(\lambda)} \prod_{i=1}^{n} x_i^{u - |z_{2i-1}| + |z_{2i-2}| - |z_{2i-1}| + |z_{2i}|}
$$

$$
= \left(\prod_{i=1}^{n} x_i^u\right) \sum_{\lambda : \lambda_1 \leq u} \sum_{z \in \text{SpGT}_n(\lambda)} \prod_{i=1}^{n} x_i^{-[\text{type}(z)_{2i-1} - \text{type}(z)_{2i}]}
$$

$$
= \left(\prod_{i=1}^{n} x_i^u\right) \sum_{\lambda : \lambda_1 \leq u} \sum_{\lambda \leq \lambda_1 \leq u} sp_{\lambda}(x_1^{\pm}, \ldots, x_n^{\pm}) \quad (14)
$$

where we note in the last equation we get the symplectic characters in the “inverse variables” $1/x_i$. Recalling $sp$ is symmetric under inversion of variables, we conclude the argument.

**Step 2.** The next step consists in observing the following formula due to Okada [Oka98]—see also [BKW16] Theorem 4.3] for a recent proof, notation similar to ours, and a wider context and use for formulas of this type:

$$
\sum_{\lambda : \lambda_1 \leq u} sp_{\lambda}(x_1^{\pm}, \ldots, x_n^{\pm}) = sp_{\nu_0}(x_1^{\pm}, \ldots, x_n^{\pm}) so_{\nu_0}(x_1^{\pm}, \ldots, x_n^{\pm}). \quad (15)
$$
Corollary 7.4: side above as bounded Littlewood sums of Schur polynomials, combinatorially proven by Stembridge [Ste90 Corollary 7.4]:

\[
\sum_{\lambda: \lambda|_1 \leq u} s_{\lambda}(x_1, \ldots, x_n) = sp_{\alpha v}(x_1^{\pm}, \ldots, x_n^{\pm}) = \text{odd}(x_1^{\pm}, \ldots, x_n^{\pm})
\]

with \((s,t) = ([u/2], [u/2])\) and indeed this implies our assumption that \(u\) was even was merely cosmetic to avoid floors and ceilings.

**Step 3.** The third step consists of rewriting the symplectic and odd orthogonal characters on the right-hand side above as bounded Littlewood sums of Schur polynomials, combinatorially proven by Stembridge [Ste90 Corollary 7.4]:

\[
\sum_{\lambda: \lambda|_1 \leq v} s_{\lambda}(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^v \text{so}_{\alpha v}(x_1^{\pm}, \ldots, x_n^{\pm}),
\]

\[
\sum_{\mu: \mu|_1 \leq u, \mu \text{ has even rows}} s_{\mu}(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^v \text{sp}_{\alpha v}(x_1^{\pm}, \ldots, x_n^{\pm}).
\]

**Step 4.** This fourth and final step consists of simply observing the following well-known identities [BR01a Fer04 FR07] equating the bounded Littlewood sums with the desired generating series of polymers:

\[
\sum_{W \in \mathcal{W}_{n,v}^{2\mu}} \text{wt}(W) = \sum_{\lambda: \lambda|_1 \leq v} s_{\lambda}(x_1, \ldots, x_n),
\]

\[
\sum_{W \in \mathcal{W}_{n,v}^{2\mu}} \text{wt}(W) = \sum_{\mu: \mu|_1 \leq u, \mu \text{ has even rows}} s_{\mu}(x_1, \ldots, x_n).
\]

The proofs of both are standard applications of the Robinson–Schensted–Knuth algorithms described in the appendix. We omit the proof of the first (see [BR01a, FR07] for details) but nevertheless prove the second equation using the seldom-used \(\text{colRSK}\) local growth rule from the appendix. While our proof is equivalent to those in [BR01a, Fer04, FR07], we chose to present because it uses a different local growth rule than in the aforementioned references. Furthermore, the proof shows that “the other RSK algorithm” (\(\text{colRSK}\)) should be treated on an equal footing with the classical one of Knuth [Knu70] (\(\text{rowRSK}\) in our language) and using it can lead to interesting observables in the theory of last passage percolation—in our language, to the generating series of (bounded by \(v\)) point-to-line polymers.

First, take a triangle \(W \in \mathcal{W}_{n,v}^{2\mu}\), flip it upside-down, double the numbers on the hypotenuse, and reflect the result across the hypotenuse to make it into a symmetric matrix we call \(M\). Let us call \(\text{colRSK}\) the inductive application of the \(\text{colRSK}\) local rule \(n^2\) times in the aforementioned way—see Figure 3 (middle) for an example. Clearly this step is bijective.

Second, on \(M\), perform the local rule \(\text{colRSK}\) inductively similarly to what was done above. The matrix entries \(M_{i,j}\) sit at half-integer lattice points \((i-1/2, j-1/2)\). On the integer points \(0 \leq i, j \leq n\), place partitions \(\mu^{(i,j)}\), starting the empty partition on the axes \(i = 0, 0 \leq j \leq n\) and \(j = 0, 0 \leq i \leq n\). For the other \(1 \leq i, j \leq n\), proceed inductively—starting with \((i = 1, j = 1)\)—as follows: place at \((i, j)\) the partition \(\mu^{(i,j)}\) which is the output/result \(\nu\) (in the notation of Appendix A) of applying \(\text{colRSK}\) on the input \(\alpha = \mu^{(i-1,j)}, \beta = \mu^{(i,j-1)}, \kappa = \mu^{(i-1,j-1)}\), \(G = M_{i,j}\). Let us call \(\text{colRSK}\) the inductive application of the \(\text{colRSK}\) local rule \(n^2\) times in the aforementioned way—see Figure 3 (middle) for an example. It produces as output (depicted in bold in fig. cit.) the sequence of interlacing partitions on the outer north and east boundary of \(M\):

\[
\emptyset = \mu^{(0,n)} \prec \mu^{(1,n)} \prec \ldots \prec \mu^{(n-1,n)} \prec \mu^{(n,n)} \succ \mu^{(n,n-1)} \succ \ldots \succ \mu^{(n,1)} \succ \mu^{(1,0)} = \emptyset.
\]

As the matrix \(M\) is symmetric and \(\text{colRSK}\) from Appendix A is manifestly symmetric in \(\alpha\) and \(\beta\), the sequence from \([19]\) is symmetric about its middle so the two Gelfand–Tsetlin patterns (equivalently SSYTs)
corresponding to it are the same. Let us call the resulting pattern \( z = (z_{i,j})_{1 \leq i \leq n, 1 \leq j \leq i} \): i.e., \( z_{i,j} = \mu_{i}^{(n,j)} \). The shape of \( z \), denoted \( \mu \), is \( \mu = \mu^{(n,n)} \).

This step is also bijective as given the output partitions in equation (19), we can inductively apply \( \text{colRSK}^{-1} \) from Appendix A to obtain the matrix \( M \) and the empty partitions on the axes (which of course we can then remove).

Thus the output of our procedure, starting from \( W \), is the GT pattern \( z: W \rightarrow z \). Due to Greene’s Theorem (b), all parts of \( \mu \)—the shape of \( z \)—are even as our matrix \( M \) is symmetric and has even diagonal. Moreover, the same theorem yields \( \mu_{i} \leq 2v = u \) as \( M \) comes from \( W \in W^{n} \) on which we have, by definition, imposed exactly the conditions making this true.

We finally argue that indeed the weights are preserved under the mapping \( W \rightarrow z \) when we pass to generating series. Let \( \text{col}^{W} \) (respectively \( \text{row}^{W} \)) be the sum of the integers in the \( i \)-th column (respectively row) of \( W \), and define \( \text{col}^{M} \), \( \text{row}^{M} \) similarly for \( M \). By symmetry of \( M \), \( \text{col}^{M}_{i} = \text{row}^{M}_{i} \). Our definition of \( M \) (recall the flipping that was involved) implies \( 2(\text{col}^{W}_{i} + \text{row}^{W}_{n-i+1}) = \text{col}^{M}_{i} + \text{row}^{M}_{i} = 2\text{col}^{M}_{i} \). Inductive application of (24)—see Remark 3—implies that for the symmetric interlacing sequence (19), we have

\[
\text{col}^{M}_{i} = |\mu^{(i,n)}_{i}| - |\mu^{(i-1,n+1)}_{i}| = \text{type}(z)_{i}
\]

Putting it all together using the notation \( \mu := \mu^{(n,n)} \) = shape of \( z \), we have:

\[
\sum_{W \in W^{n}_{u,v}} \text{wt}(W) = \sum_{W \in W^{n}_{u,v}} \prod_{i=1}^{n} x_{i}^{\text{col}^{W}_{i} + \text{row}^{W}_{n+1-i}} = \sum_{M} \prod_{i=1}^{n} x_{i}^{\text{col}^{M}_{i}} = \sum_{\mu : \mu_{i} \leq u} \sum_{z \in \text{GT}_{n}(\mu)} \prod_{i=1}^{n} x_{i}^{\text{type}(z)_{i}} = \sum_{\mu : \mu_{i} \leq u, \mu \text{ has even rows}} s_{\mu}(x_{1}, \ldots, x_{n})
\]

and the result follows.

This concludes the proof of Theorem 1. We now proceed to the proof of Corollary 2.

Suppose now that \( 0 < x_{i} < 1 \) for all \( i \) and that for each geometry the weights \( w_{i,j} \) are geometric random variables of parameter \( x_{i}x_{j} \) off-diagonal and \( x_{i} \) on-diagonal—the later only for the p2hlr and p2pr geometries. We have just proven that:

\[
\text{Prob}(L_{n}^{2hlr} \leq u) = \prod_{i=1}^{n} (1 - x_{i}) \prod_{1 \leq i < j \leq n} (1 - x_{i}x_{j}) \prod_{1 \leq i \leq j \leq n} (1 - x_{i}x_{j}) \sum_{W \in W_{n}^{2hlr}} \text{wt}(W)
\]

\[
= \left( \prod_{i=1}^{n} (1 - x_{i}) \sum_{W \in W_{n}^{2pr}} \text{wt}(W) \right) \left( \prod_{1 \leq i \leq j \leq n} (1 - x_{i}x_{j}) \sum_{W \in W_{n}^{2hlr}} \text{wt}(W) \right) = \text{Prob}(L_{n}^{2pr} \leq v) \cdot \text{Prob}(L_{n}^{2hlr} \leq u).
\]

Fix \( 0 < q < 1 \) and put all \( x_{i} = q \). Then if above on the right-hand side we take \( v = c_{1}n + c_{2}n^{1/3}s \) with \( c_{1} \) and \( c_{2} \) from (3), both probabilities have the same well-defined \( n \rightarrow \infty \) limit—see [BR01b, SI04, Fer04, BCRS17, BBNV18] for various proofs but note some are of the results only in the limit \( q \rightarrow 0 \) or \( q \rightarrow 1 \). The common limit is the Tracy–Widom GOE [TW96] distribution function \( F_{1}(s) \). The result follows.

### 3 Conclusion

We conclude with a few remarks. First, the proof of Theorem 1 is not completely and transparently bijective. The equations in steps 1 and 4 (8), (18) are consequences of RSK variants, and even the second equation in...
step 3 (17) can be seen as such. We think the first identity of (17) is also a consequence of (some) RSK but know of no proof. Nonetheless, we suspect proving (15) using an RSK variant is significantly harder, as we would be proving a Littlewood–Richardson-type rule (albeit one for very simple shapes).

Second, while formulas of type (8) and (15) can be characterized as “representation-theoretic happy accidents”, the question that arises is if there exist other such accidents that are useful for the analysis of different LPP variants (or other mathematical physics) models. A small catalogue of useful such formulas can be found in the recent works of Brent–Krattenthaler–Warnaar [BKW16] and Rains–Warnaar [RW18].

Third, one can presumably use the techniques of Betea–Bouttier–Nejjar–Vuletić [BBNV18] to obtain not just one-point distributional results and asymptotics for the p2hlr polymer but also full correlations, as double contour integrals, for the full pfaffian process—i.e., distributions for collections of non-intersecting longest up-right paths. One can additionally add extra parameters to the model, like an α parameter governing the diagonal—see [PR07, Section 6]. Are there any interesting asymptotic kernels and distributions coming from exploiting these two facts? One can already speculate about GOE × GSE kernels and fluctuations, or some GOE/GSE crossover distribution × GOE, etc. For the corresponding one-point functions (i.e., for the longest polymers), this was already treated in [BR01], again in the Poisson q → 0 limit.

Fourth, it is tempting to speculate whether Theorem 1 has a finite temperature (geometric/Whittaker) analogue. Indeed it was in this context that the Whittaker equivalent of equation (8) was discovered by Bisi and Zygouras [BZ17b]. The main obstruction we see is a finite temperature analogue of (15). However the existence of a Macdonald–Koornwinder lift to eq. cit. would go a long way as one could presumably take appropriate limits towards the desired goal. A finite temperature analogue of Corollary 3 seems further away as there are no known rigorous asymptotics for the logarithm of the partition function of the O’Connell–Yor point-to-line gamma polymer.

Last, some of the equations used in Section 2.2 along with both RSK correspondences themselves have t lifts (deformations) to the Hall–Littlewood level. For example, equation (15) (at least for \( v = \infty \)) can be recast as a partition function identity in an appropriate 6-vertex model [WZ10]; equation (17) has a plethora of Hall–Littlewood (even Macdonald or elliptic) analogues [RW18]; and the RSK correspondences from the appendix have t lifts as well, recently discovered by Bufetov–Matveev [BM17]. Can one piece together these facts to obtain distributional results for height functions in the appropriate 6-vertex model? Something similar has already been achieved in a simpler setting by Borodin–Bufetov–Wheeler [BBW16].

A Fomin growth and Greene’s theorem

We recall here, for the benefit of the reader and in the form of Fomin growth diagrams, the two Robinson–Schensted–Knuth bijections used throughout the note.

Fix two partitions \( \alpha, \beta \). We list two (max, +) local Fomin growth rules, dubbed rowRSK and colRSK, which provide bijections between the two sets

\[
\{\kappa \text{ a partition : } \alpha > \kappa < \beta\} \times \{G : G \in \mathbb{N}\} \leftrightarrow \{\nu \text{ a partition : } \alpha < \nu > \beta\}
\]

satisfying the condition

\[
|\kappa| + |\nu| = |\alpha| + |\beta| + G.
\]

The rule rowRSK is Fomin’s local rule for the Robinson–Schensted–Knuth (RSK) row insertion correspondence [Knu70], as first written down by Gessel [Ges93]. The second, colRSK, is a local growth rule for Burge’s column insertion RSK—see Appendix A.4 of [Ful97] and [Bun74, page 21, second paragraph, after “It can also be used...”]. It is, up to conjugating the partitions and other “minor” manipulations, the same rule as Krattenthaler’s third variation from [Kra06] and equivalent to the Hilman–Grassl bijection—see [HG76] but also the recent [MPP18] for an extended explanation. For a dynamical view of both rules in terms of particles hopping on a lattice, see [BPT16, MPT17]—in particular we arrived at the colRSK construction below by reverse-engineering the “column α” algorithm therein. The rules are listed below, taking as inputs pairs \((\kappa, G)\) with \(\kappa\) a partition and \(G \in \mathbb{N}\)—and of course the fixed partitions \(\alpha, \beta\)—and producing the partition \(\nu\).

---

5 We address this and its connection to last passage percolation and other combinatorial models in upcoming work in progress with Bisi and Zygouras.
That \textit{rowRSK} and \textit{colRSK} produce \( \nu \) satisfying \(|\kappa| + |\nu| = |\alpha| + |\beta| + G \) is immediate from their explicit form, as are the interlacing conditions satisfied by \( \nu \) given the ones satisfied by \( \kappa \). Moreover, \textit{rowRSK} can be manifestly inverted thus showing it is indeed a bijection. For \textit{colRSK} and the benefit of the reader, we list the inverse below, taking \( \nu \) and producing \((\kappa,G)\).

**Algorithm \textit{rowRSK}**

\begin{itemize}
\item \textbf{Input:} \( \alpha,\beta;\kappa,G \) satisfying \( \alpha \succ \kappa \prec \beta \)
\item \( \ell - 1 = \min(|\ell(\alpha)|,|\ell(\beta)|) \)
\item \( \nu_1 = \max(\alpha_1,\beta_1) + G \)
\item \textbf{FOR} \( s = 2,3,\ldots,\ell \)
\item \quad \( \nu_s = \max(\alpha_s,\beta_s) + \min(\alpha_{s-1},\beta_{s-1}) - \kappa_{s-1} \)
\item \textbf{ENDFOR}
\item \textbf{Output:} \( \nu \) satisfying \( \alpha \prec \nu \succ \beta \)
\end{itemize}

**Algorithm \textit{colRSK}**

\begin{itemize}
\item \textbf{Input:} \( \alpha,\beta;\kappa,G \) satisfying \( \alpha \succ \kappa \prec \beta \)
\item \( \ell - 1 = \min(|\ell(\alpha)|,|\ell(\beta)|) \)
\item \( G_\alpha = G \)
\item \textbf{FOR} \( s = \ell,\ell - 1,\ldots,1 \)
\item \quad \( \nu_s = \min(\max(\alpha_s,\beta_s) + G_s,\kappa_s) \)
\item \quad \( G_{s-1} = G_s - \min(G_s,\kappa_s - \max(\alpha_s,\beta_s)) + \min(\alpha_{s-1},\beta_{s-1}) - \kappa_{s-1} \)
\item \textbf{ENDFOR}
\item \textbf{Output:} \( \nu \) satisfying \( \alpha \prec \nu \succ \beta \)
\end{itemize}

Remark 6. The interlacing conditions on \( \kappa \) and \( \nu \) preserved by both \textit{rowRSK} and \textit{colRSK} bijections \([23]\) along with the size condition \([24]\) imply that both bijections give proofs of the skew Cauchy identity

\[
\sum_\nu s_{\nu/\alpha}(x)s_{\nu/\beta}(y) = \frac{1}{1-xy} \sum_\kappa s_{\alpha/\kappa}(y)s_{\beta/\kappa}(x)
\]

where \( x, y \) are variables, and \( s_{\lambda/\mu}(x) = x^{\lambda\setminus|\mu|} [\delta_{\mu<\lambda} \mu] \) is the skew Schur function specialized in a single variable. The same identity (in any number of variables) is also proven bijectively by Sagan and Stanley \([SS90]\) using \textit{rowRSK} and \textit{colRSK} along with the skew tableau insertion version of \textit{rowRSK}. The interested reader can easily modify their argument for a skew tableau insertion version of \textit{colRSK}.

Among the many bijections (local growth rules) that can be produced satisfying \([23]\) and \([24]\), these two seem (to the best of our knowledge) to be the only ones satisfying a Greene-type \([Gre74]\) theorem (see also \([Kra09]\) for the \textit{colRSK} version).

To set up the picture, suppose we have non-negative integers \((w_{i,j})_{1 \leq i,j \leq (m+n)/2}^2\) sitting at the half-integer points \((i-1/2,j-1/2)\) of \((\mathbb{N}+1/2)^2\) and consider the \( m \times n \) matrix \((w_{i,j})_{1 \leq i \leq m,1 \leq j \leq n} \) sitting inside the rectangle \( R := [0,m] \times [0,n] \)—see Figure 1. On the integer points inside \( R \) we put the empty partition on the axes and then put partitions \( \lambda^{(i,j)} \) (respectively \( \mu^{(i,j)} \)) on the other lattice points—starting with \((1,1)\)—which we define as the output \( \nu \) of inductive successive applications of the rule \textit{rowRSK} (respectively \textit{colRSK}) on input \( \alpha = \lambda^{(i-1,j)}, \beta = \lambda^{(i,j-1)}, \kappa = \lambda^{(i-1,j-1)}, G = w_{i,j} \) (respectively \( \alpha = \mu^{(i-1,j)}, \beta = \mu^{(i,j-1)}, \kappa = \mu^{(i-1,j-1)}, G = w_{i,j} \) for the case of \textit{colRSK}). By construction \( \ell(\lambda^{(i,j)}), \ell(\mu^{(i,j)}) \leq \min(i,j) \). Let us call \textit{rowRSK} this sequence of \( mn \) successive applications of \textit{rowRSK}, and similarly for \textit{colRSK} and \textit{colRSK}. Let \( \lambda := \lambda^{(m,n)} \) (respectively \( \mu := \mu^{(m,n)} \)) be the partition sitting at the outermost corner—see Figure 1—after applying \textit{rowRSK} (respectively \textit{colRSK}). Pick a non-negative integer \( k \leq \min(m,n) \). The following was proven by Greene \([Gre74]\) and connects the RSK bijections herein described with last passage percolation models.
Theorem 7 (Greene [Gre74]). (a) After application of \(\text{rowRSK}\) on input \((w_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}\), we have:

\[
\lambda_1 + \cdots + \lambda_k = \max_{\pi \in P_k} \sum_{(i,j) \in \pi} w_{i,j}
\]

(26)

where \(P_k\) is the collection of \(k\) non–intersecting up-right paths (i.e., having unit steps up or east from one square to its adjacent), starting from the south–west vertical edge of the matrix and ending on the north–east vertical edge. In particular \(\lambda_1\) is length of the the longest up-right path going from the south-west corner to the north-east corner of the rectangle \(R\).

(b) Moreover, after application of \(\text{colRSK}\) on input \((w_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}\), we have:

\[
\mu_1 + \cdots + \mu_k = \max_{\pi \in P'_k} \sum_{(i,j) \in \pi} w_{i,j}
\]

(27)

where \(P'_k\) is the collection of \(k\) non–intersecting down-right paths (i.e., having unit steps down or east from one square to its adjacent), starting from the north–west vertical edge of the matrix and ending on the south–east vertical edge. In particular \(\mu_1\) is length of the the longest down-right path going from the north-west to the south-east corners of \(R\).

Examples of \(k\) non-intersecting paths from \(P_k, P'_k\) appear in Figure 4. The proof that \(\lambda_1\) is as stated is immediate from the \(\text{rowRSK}\) local rule construction. We do not know of a direct proof for \(\mu_1\) and \(\text{colRSK}\), though one can of course reinterpret \(\text{colRSK}\) in terms of the Burge column insertion [Bur74] and then proceed as was done by Greene [Gre74] using the so-called Knuth relations [Knu70].

Remark 8. With the same setup as above, if \(\text{row}_k\) (respectively \(\text{col}_k\)) denote the sum of the \(k\)-th row (respectively column) of \(w\), inductive application of (24) yields

\[
\text{row}_k = |\lambda^{(m,k)}| - |\lambda^{(m,k-1)}| = |\mu^{(m,k)}| - |\mu^{(m,k-1)}|, \quad \text{col}_k = |\lambda^{(k,n)}| - |\lambda^{(k-1,n)}| = |\mu^{(k,n)}| - |\mu^{(k-1,n)}|.
\]

(28)

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Figure 1: The three geometries and types of polymers studied in this paper (depicted for $n = 8$): point-to-line-reflected (top), point-to-point-reflected (bottom left) and point-to-line (bottom right). The light blue squares act as reflecting boundaries. The bullets are independent non-negative integer valued random variables—with parameter $Geom(x_i, x_j)$ away from the diagonal (blue squares) and parameter $Geom(x_i)$ on the diagonal (reflecting boundaries, only in the first two cases). Example polymers (up-right paths) are depicted in dark blue, the weights of which are the sums of the numbers (bullets) collected.
Figure 2: An example for the Bisi–Zygouras bijective procedure proving equation (8) of Section 2.2, going from an input triangle $W \in W_{n,u}$ for $(n,u) = (4,18)$ (top left, with longest polymer highlighted in dark blue) via rowRSK Fomin growth diagrams (top right, with output in bold) to a generalized oscillating tableau (bottom left) and then via entry-wise subtraction from $u$ to a symplectic tableau (bottom, middle and right).

Figure 3: An example for the bijective procedure proving the second equation in (5), going from an input triangle $W \in W_{n,u}$ for $n = 4, u \geq 9$ (left, with longest polymer of length 9 highlighted in dark blue) via flipping/symmetrizing/doubling diagonal followed by the application of colRSK Fomin local growth rules (middle, with output in bold) to a semi-standard Young tableau (right) of shape the even partition $(18,10,8,2)$. 
\[
\lambda := \lambda(m,n)
\]

\[
\mu := \mu(m,n)
\]

Figure 4: Examples of \( k \) non-intersecting lattice paths appearing in the statement of Greene’s Theorem 7 for \( m = n = 5, k = 3 \). Left: paths appearing in Theorem 7(a) for \text{rowRSK}. Right: paths appearing in Theorem 7(b) for \text{colRSK}. All squares are filled with non-negative integers \( w_{i,j} \). The weight of any such collection of paths is the sum over all the numbers crossed by the paths. Starting/ending points are denoted by small disks.