Distillable entanglement in $d \otimes d$ dimension

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Abstract
Distillable entanglement ($E_d$) is one of the acceptable measures of entanglement of mixed states. Based on discrimination through local operation and classical communication, this paper gives $E_d$ for two classes of orthogonal multipartite maximally entangled states.

1 Introduction
Entanglement is believed to be a genuine resource for quantum computation and teleportation. Entanglement shows up in composite quantum systems where subsystems do not have pure states of their own. This is a strict quantum phenomena with no classical analogue. However it is highly nontrivial to keep such entangled states away from the effect of the environment. The environment destroys “part” of such entanglement. The residual entanglement could be purified or distilled to give maximally entangled states. The maximally entangled states (MES), among which there is singlet states, can be taken as the basic unit to quantifying entanglement. The distillation process uses only local operation (e.g. unitary transformation) and classical communication (e.g. phone calls) known as LOCC. Different LOCC processes could give different number of singlet MES or states that could be transformed into singlet state. The maximum number of singlet MES is the distillable entanglement. Distillable entanglement is of major importance in quantum information processing, the transmission of a pure state through a noisy channel introduces noise $\Pi$ to the initial pure state and the state becomes mixed. Therefore it is important to know the distillable
entanglement of the state after transmission.

For pure bipartite states there is a widely acceptable measure of entanglement, the Von Neumann entropy. Whereas for mixed states\(^1\) different quantities are proposed as a measure of the degree of entanglement. Few of these quantities survive nowadays, among the acceptable degrees of entanglement is the distillable entanglement \(E_d\) and the relative entropy of entanglement \(E_r\) \[^2\]. The later is an upper bound for the former \[^3\][^4\]. This upper bound will be used to calculate \(E_d\). Hence our strategy in evaluating the distillable entanglement will be

\[
\text{Number of singlet MES} \leq E_d \leq E_r.
\]

To saturate the lower bound we use discrimination of orthogonal MES by LOCC as a distillation procedure. It should be mentioned that any orthogonal states \(|\psi\rangle\) and \(|\phi\rangle\) could be discriminated perfectly by global measurement, since they satisfy the necessary condition for discrimination \(\langle\phi|\psi\rangle = 0\). Walgate et al. \[^5\] have shown that two orthogonal pure states can always be discriminated by LOCC without need for global measurements. However two orthogonal entangled states can be discriminated by LOCC if only one copy is provided. Recently, Ghosh et al. \[^6\] have generalized this result, by using the teleportation protocol of \[^7\], and proved that \(d\) pairwise orthogonal MES in \(d \otimes d\) (spin \((d-1)/2\) particles) can always be discriminated by LOCC with a single copy provided. Following the outlined strategy of saturating the lower and upper bound for \(E_d\) and based on the above results of discrimination in \(d \otimes d\), we will consider two classes of mixtures of orthogonal MES in \(d \otimes d\), and evaluate their \(E_d\).

## Four-party state

The first class of states we consider is the generalization of the state studied in \[^8\]

\[
\rho = \frac{1}{d} \sum_{i=1}^{d} |\Psi_i\rangle\langle\Psi_i|_{AB} \otimes |\Psi_i\rangle\langle\Psi_i|_{CD},
\]

(1)

where the states \(|\Psi_i\rangle\) are any \(d\) pairwise orthogonal MES chosen out of the \(d^2\) orthogonal MES, \(|\psi_{nm}\rangle; n, m = 0, 1, \ldots, d - 1\), in \(d \otimes d\) defined by \[^7\]

\[
|\psi_{nm}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{2\pi i j n/d} |j\rangle \otimes |(j + m) \mod d\rangle,
\]

(2)

where \(|\langle\rangle j; j = 0, 1, \ldots, d - 1\rangle\) is the standard orthogonal basis of the \(d\)-dimensional Hilbert space. \(\rho\) could be understood as a four party state where \(A\) and \(B\) share one of the \(d\) orthonormal MES, but don’t know which one (all terms are equally weighted by \(1/d\)), and \(C\) and \(D\) share the same MES, also not knowing which state they are sharing.

\(^1\)A mixed state is entangled if it cannot be represented as a mixture of unentangled pure states.
As in [9], in order to compute the distillable entanglement of \( \rho \) we will use the relative entropy of entanglement for an entangled quantum state \( \sigma \) defined as [2]

\[
E_r(\sigma) = \min_{\sigma^* \in D} S(\sigma || \sigma^*),
\]

where \( D \) is the set of all separable states on the Hilbert space on which \( \sigma \) is defined and \( S(\sigma || \sigma^*) = \text{Tr}\{\sigma(\log_2 \sigma - \log_2 \sigma^*)\} \) is the relative entropy of \( \sigma \) with respect to \( \sigma^* \).

The relative entropy of entanglement for \( \rho \) could be found by computing the relative entropy of \( \rho \) with respect to the equal combination of all the \( d^2 \) MES:

\[
\rho^S = \frac{1}{d^2} \sum_{i=1}^{d^2} |\Psi_i\rangle \langle \Psi_i|_{AB} \otimes |\Psi_i\rangle \langle \Psi_i|_{CD}.
\]

We will show that \( \rho^S \) minimizes \( S(\rho || \rho^*) \) over \( \rho^* \in D \). What is particular about \( \rho^S \) is that it is separable across all two party cuts, \( AC : BD, AD : BC \) and by construction across \( AB : CD \) (see [8] for the bipartite case, the generalization to multipartite case is straightforward). Hence the relative entropy of entanglement for \( \rho \) across \( AC : BD \) cut is

\[
E_r(\rho) \leq S(\rho || \rho^S) = \log_2 d.
\]

However the distillable entanglement is bounded above by \( E_d(\rho) \leq \log_2 d \). But it has been shown [6] that it is possible to distinguish between the \( d \) orthogonal MES by only LOCC if one copy of the state is provided. C and D can then know which state they have by LOCC and hence enabling A and B to know with certainty which Bell state they share, which could then be transformed by local operation into a singlet state, this corresponds to \( \log_2 d \) ebit\(^2\), hence the distillable entanglement of \( \rho \) in \( AC : BD \) cut is at least \( \log_2 d \) (\( E_d(\rho) \geq \log_2 d \)), from the lower and upper bound for \( E_d \) we deduce that

\[
E_d(\rho) = \log_2 d.
\]

This result is a generalization for the bipartite case

\[
\rho = \frac{1}{2} \sum_{i=1}^{2} |\Psi_i\rangle \langle \Psi_i|_{AB} \otimes |\Psi_i\rangle \langle \Psi_i|_{CD},
\]

where the state \( |\Psi_i\rangle \) is one of the four Bell states

\[
\psi_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),
\]

\[
\psi_{01} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),
\]

\(^2\)For the bipartite case a 1 ebit describes any quantum system which contains entanglement equivalent to that of a singlet. However a multipartite MES of two subsystems A and B has \( d \) equally weighted terms in its Schmidt decomposition, giving \( \log_2 d \) of entanglement.
ψ_{11} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),

ψ_{10} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),

(8)

and the distillable entanglement in this case is \( E_d(\rho) = 1 \) ebit as found in [9].

3 Multi-copy Bill state

The second class of state we will study is the multi-copy Bill state defined by

\[
\rho_n = \frac{1}{d^2} \sum_{i=1}^{d^2} |\Psi_i\rangle \otimes_n \langle \Psi_i|,
\]

(9)

where each biparty among the \( n \)-biparties\(^3\) shares one of the \( d^2 \) orthogonal MES, with an equal probability. To compute the distillable entanglement of \( \rho_n \) we will use the method described in [10] (for the \( 2 \otimes 2 \) case).

For \( n = 1, 2 \) the state \( \rho_n \) is separable [8] and hence the distillable entanglement is zero.

When \( n \) is even: \( n = 2m \) with \( m > 1 \) the separable state \( \rho_2 \otimes_m \) could be used to optimize the relative entropy of \( \rho_2 \). Indeed,

\[
E_d(\rho_2^m) \leq S(\rho_2^m || \rho_2^m) = (2m - 2) \log_2 d.
\]

(10)

However, it has been shown [6] that two copies are necessary and sufficient to discriminate between \( d^2 \) orthogonal MES and hence any two biparties from the \( 2m \) biparties can use their two copies of \( |\Psi_i\rangle \) to determine which state they have initially shared and hence distill \((2m - 2) \log_2 d \) ebit between the remaining \( 2m - 2 \) biparties. So the distillation entanglement is at least \((2m - 2) \log_2 d \).

Combining this with Eq. (10) we get

\[
E_d(\rho_2^m) = (2m - 2) \log_2 d.
\]

(11)

For odd \( n \): \( n = 2m + 1 \), an additional step is needed. We first use the fact that one can always decompose \( \rho_2^{2m+1} \) into two copies of the optimal decomposition of \( \rho_{2m+1} \), i.e. \( 2E_r(\rho_{2m+1}) \leq E_r(\rho_2^{2m+1}) \). Then the relative entropy of \( \rho_2^{2m+1} \) is evaluated with respect to \( \rho_2^{2m+1} \), which is separable:

\[
E_r(\rho_2^{2m+1}) \leq S(\rho_2^{2m+1} || \rho_2^{2m+1}) = (4m - 2) \log_2 d,
\]

(12)

hence, the distillable entanglement of \( \rho_2^{2m+1} \) is bounded above as

\[
E_d(\rho_2^{2m+1}) \leq E_r(\rho_{2m+1}) \leq \frac{1}{2} S(\rho_2^{2m+1} || \rho_2^{2m+1}) = (2m - 1) \log_2 d.
\]

(13)

\(^3\)not to be confused with bipartite
Again, using two copies, out of the $2m + 1$ copies, to distinguish between the $d^2$ $|\Psi_i\rangle$’s we get the lower bound

$$E_d(\rho_{2m+1}) \geq (2m + 1 - 2) \log_2 d = (2m - 1) \log_2 d.$$ 

The lower and upper bound for $E_d$ give

$$E_d(\rho_{2m+1}) = (2m - 1) \log_2 d. \quad (14)$$

Here we do not venture to conjecture any statement concerning the additivity of the relative entropy of entanglement for the multipartite case, this will be investigated further in a future work.

Finally, combining the even and odd $n$ cases, the distillable entanglement of $\rho_n$ in $d \otimes d$ is

$$E_d(\rho_n) = (n - 2) \log_2 d. \quad (15)$$

This is a generalization of the $d = 2$ case, studied in [10], where $E_{d=2}(\rho_n) = n - 2$. Again discrimination is shown to be an optimal distillation procedure.

4 Conclusion

In this note we have made a generalization to $d \otimes d$ of the results found in [9] and [10] for the distillable entanglement of the four-party state Eq. (1) and the multi-copy Bell state Eq. (9) respectively. Discrimination was used as distillation procedure and was shown to be optimal, for the specific studied classes of states. A natural extension to our work would be the study of more than $d$ states for the first class of states we used. For more than $d$ state mixtures discrimination is no longer a good candidate for distillation (a mixture of $d + 1$ states can not be discriminated with one copy provided [6]). Another important issue is the distillation entanglement of orthogonal partially entangled mixtures, this will be addressed in future work.

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