The dynamics of a qubit coupled strongly with a quantum oscillator

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The dynamics of a qubit coupled with a quantum oscillator is re-studied in the region of strong coupling. The non-degenerate perturbation is added to the usual degenerate one and new results are given.

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I. INTRODUCTION

A qubit coupled with a quantum oscillator is the simplest model in quantum system, yet is an abundance of applications both in theoretical and practical aspects. Its dynamics is described by the Rabi Hamiltonian [1]. However, its analytical solutions is still unavailable. Many approximations are applied. Among them, the rotation wave approximation (RWA) [2] and adiabatic approximation (AA) are major ones. Under weak coupling and near resonant condition, it has been solved analytically by the method of RWA by discarding the non-energy-conserving terms, and is also called the Jaynes-Cumming model in quantum optics [2]. In the opposite condition, that is, the strong coupling and large detuning, it is investigate by the adiabatic approximation (AA) and generalized RWA [3]-[5]. The AA method assumed large detuning and first only treat the quantum oscillator influenced by the qubit through the coupling, then solve the whole system by treating the qubit as a perturbation. Further assumption of the qubit must be made for its solutions in AA method, that is why sometimes called it adiabatic approximation. The adiabatic assumption requires that the qubit as a perturbation does not have influence on the transition of the quantum oscillator from its different number states and discards these transition terms concerning the displaced number states and solved it accordingly.

Because recent achievements in circuit QED make the strong or even the ultrastrong coupling regime of light matter interaction achievable [3]-[10], this in turn renews the theoretical interesting in qubit under adiabatic approximation. Further extensions are made by replacing the qubit by two qubit and N-qubits or linear coupling by non-linear ones, etc [3]-[10].

In the paper, we will restudy the qubit coupled strongly with quantum oscillator and focus on the rationality of the adiabatic assumption. Though physically the low energy of the qubit can not promotes the transition of the oscillator to its different displaced number states, these omitting terms sometimes are larger than the keepep terms of the transition of the same number states displaced differently. So we will make a scrutiny into the adiabatic approximation and a generalized approximation is given and its new results are discussed. Section 2 reviews the adiabatic approximation in the study of the qubit and quantum oscillator coupled system and section 3 gives new approximation. Final section will discuss and conclude the results.

II. THE BRIEF INTRODUCTION OF THE ADIABATIC APPROXIMATION

The Rabi Hamiltonian for a qubit interacting with a harmonic oscillator is [3], [8],

\[ H = \frac{1}{2} \hbar \omega | \sigma_z \rangle \langle \sigma_z | + \hbar \omega a^\dagger a + \hbar \beta (a + a^\dagger) \sigma_x, \]  

where \( \sigma_x, \sigma_z \) are the usual Pauli matrices with \( \sigma_z = \sigma_+ + \sigma_- \).

In the paper, we do not consider the RWA case where the non-rotation-terms \( a^\dagger \sigma_+ \), \( a \sigma_- \) are neglected. Under large detuning and strong coupling condition, the qubit is regarded as a perturbation. The so-called free hamiltonian is

\[ H_0 = \hbar \omega a^\dagger a + \hbar \beta (a + a^\dagger) \sigma_x, \]

whose solutions are

\[ H_0 | N_m, m \rangle = (N - \beta^2) \hbar \omega | N_m, m \rangle, \]

\[ | N_m, m \rangle = | m \rangle | N_m \rangle, \]

where |+\rangle, |−\rangle are the eigenvectors of the operator \( \sigma_z \) with \( \sigma_z |+\rangle = |+\rangle, \sigma_z |−\rangle = |−\rangle \), and |\( N_m \rangle \) are the displaced number states for the quantum oscillator, that is

\[ | N_{\pm} \rangle = D(\mp \beta) | N \rangle, \quad D(\beta) = e^{\beta (a^\dagger - a)}. \]

The Hamiltonian describes the quantum oscillator influenced by force from its coupling with with qubit, and its solutions show that the force of the qubit acting on the oscillator has make the oscillator displace its equilibrium position according to the states of the qubit, just as shown in the displaced number states |\( \psi_{n,m} \rangle \) in its solutions.
The displaced number states \(|N_{\pm}\rangle\) are not completely orthogonal due to the fact
\[
\langle N_{\mp}|N_{\pm}\rangle \neq 0 \text{ with } m \neq m'.
\] (6)
This will result in the complex situation when including the perturbation term \(\frac{i}{\hbar}\hbar\omega_0\sigma_z\) of the qubit to be treated. This term \(\frac{i}{\hbar}\hbar\omega_0\sigma_z\) of the qubit will produce the transition of the oscillator from its various eigenstates \(|N_{\pm}\rangle\). From the theoretical view, the large detuning \((\omega_0 \ll \omega)\) will make it reasonable to omit its transition among different numbers states. This means that the qubit mainly results in transition of the oscillator from the same number states which are displaced differently induced by the corresponding states of the qubit, that is, the transitions of the form \(|N_{\pm}\rangle \rightarrow |N_{\pm'}\rangle\). The other transitions \((|N_{\pm}\rangle \rightarrow |N_{\pm'}\rangle, N \neq N'\) will be omitted from the consideration. This approximation is called adiabatic. In this way, the whole Hamiltonian becomes block-diagonal as
\[
H = \hbar\omega \begin{pmatrix} H_0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & H_1 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & H_N & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}
\]
with the \(2 \times 2\) matrices \(H_N\) defined as
\[
H_N = \begin{pmatrix} N - \beta^2 & \Omega_N \\ \Omega_N & N - \beta^2 \end{pmatrix},
\] (8)
where
\[
\Omega_N = \frac{\omega_0}{2\omega} \langle +|\sigma_z| - \rangle \langle N_+|N_- \rangle
\]
\[
= \frac{\omega_0}{2\omega} \exp(-2\beta^2)L_N(4\beta^2).
\] (9)
\(L_N(x)\) in the above equation are the Laguerre polynomials. The new solutions are easy to obtain:
\[
E_{N,\pm}^{\text{ad}} = \hbar\omega (N - \beta^2 \pm \Omega_N),
\] (10)
\[
|E_{N,\pm}^{\text{ad}}\rangle = \frac{1}{\sqrt{2}} (|N_+,+\rangle \pm |N_-,+\rangle).
\] (11)
All eigen-values \(E_{N,\pm}^{\text{ad}}\) are influenced by the parameters \(\beta, \frac{i}{\hbar}\omega, N\), so are the eigenvectors \(|E_{N,\pm}^{\text{ad}}\rangle\).
The eigenvectors \(|E_{N,\pm}^{\text{ad}}\rangle\) are complete and orthogonal basis for the composite system of the qubit and the quantum oscillator. The evolution of the qubit will definitely depends on the initial states of the composite system. Here we only give some examples for showing the dynamics of the qubit. When the initial states of the composite system are \(E^{\text{ad}}(0) = |\Psi_{N,+}\rangle\), the possibility of the qubit remaining in its initial state are
\[
P(t) = \cos^2(\Omega_N \ast \omega t).
\] (12)
The general initial state will evolve by the principle of superposition, see details in Ref. [3].

FIG. 1: The dynamics of the system depends on the parameters \(n, \beta\). The thick, solid and dashed lines correspond to the figures of \(P(t)\) with \(N = 1, \beta = 0.2, N = 4, \beta = 0.2, N = 1, \beta = 0.7\) respectively.

III. THE PERTURBATION RESTUDIED IN DETAILS

As stated before, the adiabatic approximation omits various terms of the form
\[
|N_\pm\rangle \rightarrow |N_\pm'\rangle, N \neq N'
\] under the physical ground that \(\omega_0 \ll \omega\). Though physically the qubit seems not able to promote the oscillator to go from one displaced number state to a new one with different number as its energy gap \(n - n'\hbar\omega_0\) is far more less that \(\hbar\omega\) of the transition \(|N_+\rangle \rightarrow |N_+\rangle\), \(N \neq N'\) or \(|N_-\rangle \rightarrow |N_-\rangle\), \(N \neq N'\), however, this is not true for the transition of the type \(|N_+\rangle \rightarrow |N_-\rangle, n \neq n'\) or \(|N_-\rangle \rightarrow |N_+\rangle, N \neq N'\). Detailed quantities study also support the conclusion, and shows that the perturbation terms \(\langle N_{\pm}|\frac{\hbar\omega_0\sigma_z}{\hbar}\rangle|N_{\pm}'\rangle\) depend non-linearly on the coupling strength \(\beta\) and the parameters \(N, N'\). Given an arbitrarily strong coupling \(\beta\), it is generally true that \(\langle N_{\pm}|\frac{\hbar\omega_0\sigma_z}{\hbar}\rangle|N_{\pm}'\rangle\) will be larger than \(\langle N_{\pm}|\frac{\hbar\omega_0\sigma_z}{\hbar}\rangle|N_{\pm}'\rangle\). However, there are still several numbers \(N' = N_i, i = 1, 2, \ldots, N_k\) that \(\langle N_{\pm}|\frac{\hbar\omega_0\sigma_z}{\hbar}\rangle|N_{\pm}\rangle\) are much larger than \(\langle N_{\pm}|\frac{\hbar\omega_0\sigma_z}{\hbar}\rangle|N_{\pm}\rangle\). Quantity study shows the number \(N_k\) is finite and depends both the coupling strength \(\beta\) and the number \(N\), so it will change accordingly. For example, Fig. [2] shows \(\langle 13_+, +|\frac{\hbar\omega_0\sigma_z}{\hbar}|10_-, -\rangle\) is greater than \(\langle 10_+, +|\frac{\hbar\omega_0\sigma_z}{\hbar}|10_-, -\rangle\) when \(\beta = 0.7\). So the adiabatic approximation must add these terms to be more applicable.

There are two kinds of perturbations in quantum theory. The perturbation to the degenerate system and that to the non-degenerate ones. Let’s reconsider the perturbation \(\hbar\omega_0\sigma_z\) to the free Hamiltonian \(H_0\). The eigenstates are \(|N_+, +\rangle, |N_-, -\rangle, |N_+, +\rangle, |N_-, -\rangle, \ldots\). The states with the same numbers \(n\) are degenerate, while other states with different numbers are not. So the perturbation \(\frac{\hbar}{\hbar}\omega_0\sigma_z\) will relate the two kinds ones in quantum theory. Obviously, the adiabatic perturbation is the one...
only applied to the degenerate cases of the free Hamiltonian. So, it is not complete as there is the non-degenerate perturbation to need to be added.

Some observation is given here for convenience: whenever there are perturbations of combination of both the degenerate one and non-degenerate one, one must first treat the degenerate one, then treats the non-degenerate one. The reverse order can not work, as the condition of perturbation is small relative the original energy gap of the free Hamiltonian can not be met due to its degeneration. Further more, for the second perturbation, there are two bases to use: one is from the eigen-states of the the free hamiltonian, the other comes from the first degenerate perturbation. In our case, they are $|N_{+},\pm\rangle$, $|N_{-},\pm\rangle$, $N = 0, 1, 2, \cdots$, and $|E_{N,\pm}^{0}\rangle$, $N = 0, 1, 2, \cdots$, $i = +, -$. Of course, the basis $|E_{N,\pm}^{0}\rangle$, $N = 0, 1, 2, \cdots$, $i = +, -$ are more favorable for the calculation of the second perturbation.

By the use of the basis $|E_{N,\pm}^{0}\rangle$, $N = 0, 1, 2, \cdots$, the dynamic equation for the composite systems is

$$H|E_{N,+}\rangle = E_{N,+}|E_{N,+}\rangle, \ H|E_{N,-}\rangle = E_{N,-}|E_{N,-}\rangle. \quad (13)$$

Then the corresponding eigen-vectors and eigen-energies are

$$|E_{N,+}\rangle = |E_{N,+}^{0}\rangle + \sum_{I=1}^{N_{k}}(a_{N,I}^{|+|}E_{I,+}^{0}\rangle + b_{N,I}^{|+|}E_{I,-}^{0}\rangle), \quad (14)$$

$$|E_{N,-}\rangle = |E_{N,-}^{0}\rangle + \sum_{I=1}^{N_{k}}(a_{N,I}^{|-|}E_{I,+}^{0}\rangle + b_{N,I}^{|-|}E_{I,-}^{0}\rangle), \quad (15)$$

$$E_{N,+} = \hbar \omega \left[ (N - \beta^{2} + \Omega_{N}) + \sum_{I=1}^{N_{k}}\frac{|\langle N_{I}|I_{-}\rangle|^{2}\omega_{0}^{2}}{4(N - I) \omega^{2}} \right], \quad (16)$$

$$E_{N,-} = \hbar \omega \left[ (N - \beta^{2} - \Omega_{N}) + \sum_{I=1}^{N_{k}}\frac{|\langle N_{I}|I_{-}\rangle|^{2}\omega_{0}^{2}}{4(N - I) \omega^{2}} \right]. \quad (17)$$

where

$$a_{N,I}^{|+|} = -b_{N,I}^{|+|} = \frac{(1 + (-1)^{N-I})I_{+}|N_{-}\rangle}{4(N-I)\omega}, \quad (18)$$

$$b_{N,I}^{|-|} = -a_{N,I}^{|-|} = \frac{(1 - (-1)^{N-I})I_{+}|N_{-}\rangle}{4(N-I)\omega}. \quad (19)$$

The summation includes those terms of $\langle N_{+}\frac{1}{2}\hbar \omega_{0}\sigma_{z}|N_{\pm}\rangle$ that are comparable with the term $\Omega_{N} = \langle N_{+}\frac{1}{2}\hbar \omega_{0}\sigma_{z}|N_{\pm}\rangle$. See Appendix for details.

### IV. DISCUSSION AND CONCLUSION

From comparison of the results [15, 10, 17] with the old ones [11, 10], we could see the justification of the adiabatic approximation. The parameter $\frac{\hbar \omega_{0}}{\omega}$ is small, the new eigenstates [15] differ from the old one [11] only on first order of $\frac{\hbar \omega_{0}}{\omega}$. Though the new eigenvalues only has some extra terms in the second order of $\frac{\hbar \omega_{0}}{\omega}$, the old one [10] is already up to the first order of $\frac{\hbar \omega_{0}}{\omega}$. Some terms $\langle N_{+}\frac{1}{2}\hbar \omega_{0}\sigma_{z}|N_{\pm}\rangle$ may be comparable or even larger than $\langle N_{+}\frac{1}{2}\hbar \omega_{0}\sigma_{z}|N_{\pm}\rangle$, discarding these terms to make the whole Hamiltonian block-diagonal generally is feasible under the large detuning condition. However, because the term $\Omega_{N}$ depends non-linearly on the parameter $\beta$, in case of it being zero by some chosen $\beta$, that is at the critical points, we consider it is reasonable to refer the adiabatic approximation as the new results [15, 16, 17].

Here we give an example. Consider the composite system is initially in state $\Psi(0) = |+\rangle|0_{+}\rangle = |0_{+},+\rangle$. The quantum oscillator is in its displaced vacuum state or coherent state $|0_{+}\rangle = D(-\beta)|0\rangle$. Suppose the coupling strength $\beta = 0.2$, we will have

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}[e^{-\frac{\hbar \omega_{0}}{\omega}t}|0_{+},+\rangle + e^{-\frac{\hbar \omega_{0}}{\omega}t}|0_{-},-\rangle], \quad (20)$$

$$E_{0,+}^{0} = \hbar \omega(-\beta^{2} + \Omega_{0}), \quad E_{0,-}^{0} = \hbar \omega(-\beta^{2} - \Omega_{0}), \quad (21)$$

$$\Omega_{0} = \frac{\hbar \omega_{0}}{2\omega}|0_{-},0_{+}\rangle (22)$$
as the state of the system at time $t$ by the old adiabatic approximation method. From Fig. 3, we obtain that $\langle 0_+|0_+\rangle = 0.98099$, $\langle 0_-|1_+\rangle = 0.19604$, $\langle 0_-|2_+\rangle = 0.027724$, $\langle 0_-|3_+\rangle = 0.00320132$ and all others much smaller than 0.00320132. So, in the new results of Eqs. (15), we have

$$|E_{0, +}⟩ = |E_{0, +}^0⟩ + (a_{0, 2}^+|E_{2, +}^0⟩ + b_{0, 1}^+|E_{1, +}^0⟩ + b_{0, 3}^+|E_{3, +}^0⟩),$$

(23)

If we further choose $\frac{\omega_0}{\omega} = 0.3$, then

$$|E_{0, +}⟩ \approx |E_{0, +}^0⟩ + 0.0029406|E_{1, +}^0⟩.$$  (24)

Similarly, we have

$$|E_{0, -}⟩ \approx |E_{0, -}^0⟩ - 0.0029406|E_{1, +}^0⟩.$$  (25)

The interesting things of the above equations are that $|E_{0, +}⟩$ combines the two states $|E_{0, +}^0⟩$, $|E_{1, +}^0⟩$, and $|E_{0, -}⟩$ combines the two states $|E_{0, -}^0⟩$, $|E_{1, +}^0⟩$ with opposite coefficients. Of course, the coefficient in front of $|E_{1, +}^0⟩$ or $|E_{0, -}⟩$ very small. The change of the eigenvalues is ignorable in this case, that is, $E_{N, +} ≈ E_{N, +}^0$. Therefore, $Ψ(t) = e^{-\frac{3}{2}\hbar t}|0_+, +⟩$ becomes

$$Ψ(t) \approx \frac{1}{\sqrt{2}} \left( e^{-\frac{\omega_0}{\hbar}t}|E_{0, +}^0⟩ + e^{-\frac{\omega_0}{\hbar}t}|E_{0, -}^0⟩ \right)$$

$$+ \frac{0.0029406}{\sqrt{2}} \left[ (e^{-\frac{\omega_0}{\hbar}t} - e^{-\frac{\omega_0}{\hbar}t})|E_{1, +}^0⟩ \right]$$

$$- \frac{0.0029406}{\sqrt{2}} \left[ (e^{-\frac{\omega_0}{\hbar}t} - e^{-\frac{\omega_0}{\hbar}t})|E_{1, +}^0⟩ \right].$$

(26)

(27)

Eq. (27) has the extra term, its last two parts, compared with the old one (22). So, even the quantum oscillator is initially in its displaced vacuum states, it will have some possibility to evolve into its displaced one photon states. The result may be of some use for the application of qubits coupled with a quantum oscillator.

In summary, concerning the system of a qubit coupled strongly to a quantum oscillator, the adiabatic approximation method is restudied and is extended or modified to the second order of $\frac{3}{2}\hbar t$ for the eigenvalues and the first order for the eigenstates. Though the modification is small and the old adiabatic approximation is justified, the new results will have some application. We could also extend the study to the system of $N$-qubits coupled with a quantum oscillator, where the spectrum of the free Hamiltonian $H_0$ is $E_{N, m} = \hbar\omega(1 - m^2/2)$, $m = 0, ±1, \ldots$, $N = 0, 1, \ldots$. The smallest energy gap is $\Delta E_0 = \hbar\omega(1 - \Delta m^2/2)$. For example, $\Delta m^2 = m_2^2 - m_1^2 = 3$, the condition that the qubits energy $h\omega_0$ is much more smaller than the transition energy gap $\Delta E = h\omega(1 - 3\beta^2)$ can not be set for strong coupling (large $\beta$). So it is plausible the inclusion of the $N$-qubits will not induce the the transition of different displaced number states, that is, the adiabatic approximation might be put into doubt in this case. Our study in the paper must be used to see that the adiabatic approximation can still be justified. This will be investigated in ref. 11.

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V. APPENDIX

By the use of the perturbation method in quantum mechanics, we could solve Eq. (13) by the following

$$|E_{N, +}⟩ = |E_{N, +}^0⟩ + \sum_{I=1}^{N_k} a_{N, I}^+|E_{I, +}^0⟩ + \sum_{I=1}^{N_k'} b_{N, I}^+|E_{I, +}^0⟩,$$

$$|E_{N, -}⟩ = |E_{N, -}^0⟩ + \sum_{I=1}^{N_j} a_{N, I}^-|E_{I, +}^0⟩ + \sum_{i=1}^{n'_j} b_{N, I}^-|E_{I, +}^0⟩,$$  (28)

where

$$a_{N, I}^+ = \frac{\omega_0}{2(N - I)\omega} \langle E_{I, +}^0|\hat{σ}_z|E_{N, +}^0⟩ = \frac{\omega_0}{4(N - I)\omega}(⟨I_+|−N_−⟩ + ⟨I_−|N_+⟩),$$

(29)

$$b_{N, I}^+ = \frac{\langle E_{I, +}^0|\hat{σ}_z|E_{N, +}^0⟩}{(N - I)\hbar\omega} = \frac{\omega_0}{4(N - I)\omega}(⟨I_+|N_−⟩ - ⟨I_−|N_+⟩),$$

(30)

$$a_{N, I}^- = \frac{\langle E_{I, +}^0|\hat{σ}_z|E_{N, +}^0⟩}{(N - I)\hbar\omega} = \frac{\omega_0}{4(N - I)\omega}(−⟨I_+|N_−⟩ + ⟨I_−|N_+⟩),$$

(31)

$$b_{N, I}^- = \frac{\langle E_{I, +}^0|\hat{σ}_z|E_{N, +}^0⟩}{(N - I)\hbar\omega} = \frac{\omega_0}{4(N - I)\omega}(−⟨I_+|N_−⟩ − ⟨I_−|N_+⟩),$$

(32)
and

\[ E_{N,+} = E_{N,+}^0 + \frac{1}{4} \hbar \omega_0 \sum_{I=1}^{N_k} a_{N,I}^+ (\langle N_+ | I_- \rangle + \langle N_- | I_+ \rangle) + \frac{1}{4} \hbar \omega_0 \sum_{I=1}^{N_k} b_{N,I}^+ (\langle N_+ | I_- \rangle + \langle N_- | I_+ \rangle) \]

\[ = \hbar \omega \left( \langle n - \beta^2 - \Omega_N | \right) + \frac{1}{4} \hbar \omega_0 \sum_{I=1}^{N_k} \frac{\omega_0^2}{(N-I)16\omega^2} (\langle N_+ | I_- \rangle + \langle N_- | I_+ \rangle)^2 + \frac{N_k}{16(N-I)\omega^2} (\langle N_+ | I_- \rangle - \langle N_- | I_+ \rangle)^2 \right), \quad (33) \]

\[ E_{N,-} = E_{N,-}^0 + \frac{1}{4} \hbar \omega_0 \sum_{I=1}^{N_k} a_{N,I}^- (\langle N_+ | I_- \rangle - \langle N_- | I_+ \rangle) + \frac{1}{4} \hbar \omega_0 \sum_{I=1}^{N_k} b_{N,I}^- (\langle N_+ | I_- \rangle - \langle N_- | I_+ \rangle) \]

\[ = \hbar \omega \left( \langle n - \beta^2 - \Omega_N | \right) + \frac{1}{4} \hbar \omega_0 \sum_{I=1}^{N_k} \frac{\omega_0^2}{(N-I)16\omega^2} (\langle N_+ | I_- \rangle - \langle N_- | I_+ \rangle)^2 + \frac{N_k}{16(N-I)\omega^2} (\langle N_+ | I_- \rangle - \langle N_- | I_+ \rangle)^2 \right), \quad (34) \]

In all the above equations, the summations all mean the sum of the terms involved the quantities \( \langle N_+ | \frac{1}{2} \hbar \omega_0 \sigma_z | N_\pm \rangle \) that can not be ignorable. Usually, there are just several such terms need to calculate.

Because of the fact [3]

\[ \langle M_+ | N_- \rangle = e^{-2\beta^2} (-\beta)^{n-N} \sqrt{\frac{M!}{N!}} L_{M-N}^{N-M} (4\beta^2), \quad M \geq N \quad (35) \]

\[ \langle M_+ | N_- \rangle = e^{-2\beta^2} (-\beta)^{N-M} \sqrt{\frac{M!}{N!}} L_{M-N}^{N-M} (4\beta^2), \quad M < N \quad (36) \]

and \( \langle M_+ | N_- \rangle \) being real, there are the relations

\[ \langle M_+ | N_- \rangle = (-1)^{N-M} \langle M_- | N_+ \rangle \]

and

\[ \langle M_- | N_+ \rangle = (-1)^{N-M} \langle N_- | M_+ \rangle, \]

the above results concerning the coefficients \( a_{N,I}^+, a_{N,I}^-, b_{N,I}^+, b_{N,I}^- \) etc., could be simplified as Eqs. (19). Please note that \( L_N^\pm(x) \) in above is the associated Laguerre function.

The simple relation among the coefficients \( a_{N,I}^+, a_{N,I}^-, b_{N,I}^+, b_{N,I}^- \) shows that all upper limits of summation in Eqs (15-17) could be put to the same as \( N_k \) which depends on both \( N, I \). So, the results could be concisely written as Eqs (15-17).

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