COUNTING RISES, LEVELS, AND DROPS IN COMPOSITIONS

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Abstract
A composition of \( n \in \mathbb{N} \) is an ordered collection of one or more positive integers whose sum is \( n \). The number of summands is called the number of parts of the composition. A palindromic composition of \( n \) is a composition of \( n \) in which the summands are the same in the given or in reverse order. In this paper we study the generating function for the number of compositions (respectively palindromic compositions) of \( n \) with \( m \) parts in a given set \( A \subseteq \mathbb{N} \) with respect to the number of rises, levels, and drops. As a consequence, we derive all the previously known results for this kind of problem, as well as many new results.

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1. Introduction

A composition \( \sigma = \sigma_1 \sigma_2 \ldots \sigma_m \) of \( n \in \mathbb{N} \) is an ordered collection of one or more positive integers whose sum is \( n \). The number of summands, namely \( m \), is called the number of parts of the composition. A palindromic composition of \( n \in \mathbb{N} \) is a composition for which \( \sigma_1 \sigma_2 \ldots \sigma_m = \sigma_m \sigma_{m-1} \ldots \sigma_1 \). A Carlitz composition is a composition of \( n \in \mathbb{N} \) in which no two consecutive parts are the same. We will derive the generating functions for the number of compositions, number of parts, and number of rises (a summand followed by a larger summand), levels (a summand followed by itself), and drops (a summand followed by a smaller summand) in all compositions of \( n \) whose parts are in a given set \( A \). This unified framework generalizes earlier work by several authors.

Alladi and Hoggatt [1] considered \( A = \{1, 2\} \), and derived generating functions for the number of compositions, number of parts, and number of rises, levels and drops in compositions and palindromic compositions of \( n \). Chinn and Heubach [5] generalized to \( A = \{1, k\} \) and derived all the respective generating functions. Chinn, Grimaldi and Heubach [8] considered the case \( A = \mathbb{N} \), and derived generating functions for all quantities of interest. Grimaldi [7] studied \( A = \{m|m = 2k + 1, k \geq 0\} \), and derived generating functions for the number of such compositions, as well as the number of parts, but not for the number of rises, levels
and drops. In addition, he studied compositions without the summand 1 \[6\], which was

generalized by Chinn and Heubach \[4\], who looked at compositions without the summand

\( k \), i.e. \( A = \mathbb{N} - \{k\} \). In both cases, the authors only derived generating functions for the
total number of compositions and the number of parts, but not for the number of rises, levels
and drops. Finally, Hoggatt and Bricknell \[8\] looked at compositions with parts in a general
set \( A \), and gave generating functions for the number of compositions and the number of
parts. This work was generalized by Heubach and Mansour \[9\], which also considered Carlitz
compositions and gave additional generating functions for the number of compositions with
a given number of parts in a set \( B \subset A \).

We will present a unified framework which allows us to derive previous results by choosing
a specific set \( A \), as well as new results. We will therefore study the specific sets \( A = \mathbb{N} \),
\( A = \{1, 2\} \), \( A = \{1, k\} \), \( A = \mathbb{N} - \{k\} \), and \( A = \{m \mid m = 2k + 1, k \geq 0\} \). In the case of Carlitz
compositions, we will restrict ourselves to the sets \( A = \{1, 2\} \), \( A = \{1, k\} \) and \( A = \{a, b\} \).
The main result and its proof will be stated in Section 2, and in Section 3 we present several
applications on the set of compositions (see Subsection 3.1), palindromic compositions (see
Subsection 3.2), Carlitz compositions (see Subsection 3.3), Carlitz palindromic compositions
(see Subsection 3.4), and partitions (see Subsection 3.5) of \( \sigma \) with \( m \) parts in \( A \), respectively.
As a consequence, we derive all the previously known results for this kind of problem, as well
as many new results.

2. Main Result

Let \( \mathbb{N} \) be the set of all positive integers, and let \( A \) be any ordered (finite or infinite) set of
positive integers, say \( A = \{a_1, a_2, \ldots, a_k\} \), where \( a_1 < a_2 < a_3 < \cdots < a_k \), with the obvious
modifications in the case \( |A| = \infty \). In the theorems and proofs, we will treat the two cases
together if possible, and will note if the case \( |A| = \infty \) requires additional steps. For ease
of notation, “ordered set” will always refer to a set whose elements are listed in increasing
order.

For any ordered set \( A = \{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{N} \), we denote the set of all compositions (respec-
tively palindromic compositions) of \( n \) with parts in \( A \) by \( C_n^A \) (respectively \( P_n^A \)). For any
composition \( \sigma \), we denote the number of parts, rises, levels, and drops by \( \text{parts}(\sigma) \), \( \text{rises}(\sigma) \),
levels(\( \sigma \)), and \( \text{drops}(\sigma) \), respectively. We denote the generating function for the number of
compositions (respectively palindromic compositions) of \( n \) with parts(\( \sigma \)) parts in a set
\( A \) such that there are \( \text{rises}(\sigma) \) rises, \( \text{levels}(\sigma) \) levels, and \( \text{drops}(\sigma) \) drops by \( C_A(x; y; r, \ell, d) \)
(respectively \( P_A(x; y; r, \ell, d) \)), that is,

\[
C_A(x; y; r, \ell, d) = \sum_{n \geq 0} \sum_{\sigma \in C_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)}
\]

and

\[
P_A(x; y; r, \ell, d) = \sum_{n \geq 0} \sum_{\sigma \in P_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)}.
\]

The main result of this paper gives explicit expressions for the generating functions \( C_A(x; y; r, \ell, d) \)
and \( P_A(x; y; r, \ell, d) \).

**Theorem 2.1.** Let \( A = \{a_1, \ldots, a_k\} \) be any ordered subset of \( \mathbb{N} \).
The generating function $C_A(x; y; r, \ell, d)$ is given by

$$1 + (1 - d) \sum_{j=1}^{k} \left( \frac{x^{a_i} y}{1 - x^{a_i} y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i} y(\ell - r)}{1 - x^{a_i} y(\ell - d)} \right).$$

The generating function $P_A(x; y; r, \ell, d)$ is given by

$$1 + \sum_{i=1}^{k} \frac{x^{a_i} y + x^{2a_i} y^2 (\ell - d r)}{1 - x^{2a_i} y^2 (\ell^2 - d r)}.$$

2.1. **Proof of Theorem 2.1(i).** Our present aim is to find $C_A(x; y; r, \ell, d)$ explicitly, thus we need the following definition. For all $e \geq 1$, we define

$$C_A(s_1 s_2 \ldots s_e|x; y; r, \ell, d) = \sum_{n \geq 0} \sum \sum x^n y \text{parts(}\sigma\text{)} \cdot \text{rises(}\sigma\text{)} \cdot \text{levels(}\sigma\text{)} \cdot d \cdot \text{drops(}\sigma\text{)},$$

where the sum on the right side of the equation is over all the composition $\sigma \in C_n^A$ such that $s_j = s_j$ for all $j = 1, 2, \ldots, e$, i.e., the composition $\sigma$ starts with $s_1 s_2 \ldots s_e$.

Now, let us introduce two relations (Equation (2.1) and Lemma 2.2) between the generating functions $C_A(x; y; r, \ell, d)$ and $C_A(a_i|x; y; r, \ell, d)$. The first relation is given by

$$C_A(x; y; r, \ell, d) = 1 + \sum_{i=1}^{k} C_A(a_i|x; y; r, \ell, d),$$

which follows immediately from the definitions (note that the summand 1 covers the case $n = 0$). The second relation is given by the following lemma, and stems from a recursive creation of the compositions of $n$.

**Lemma 2.2.** Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. For all $i = 1, 2, \ldots, k$, the generating function $C_A(a_i|x; y; r, \ell, d)$ is given by

$$x^{a_i} y \left( 1 + d \sum_{j=1}^{i-1} C_A(a_j|x; y; r, \ell, d) + \ell C_A(a_i|x; y; r, \ell, d) + r \sum_{j=i+1}^{k} C_A(a_j|x; y; r, \ell, d) \right).$$

**Proof.** The compositions of $n$ starting with $a_i$ with at least two parts can be created recursively by prepending $a_i$ to a composition of $n - a_i$ which starts with $a_j$ for some $j$. This either creates a rise (if $i < j$), a level (if $i = j$), or a drop (if $i > j$), and in each case, results in one more part. Thus,

$$C_A(a_i a_j|x; y; r, \ell, d) = \begin{cases} r x^{a_i} y C_A(a_j|x; y; r, \ell, d), & i < j \\ \ell x^{a_i} y C_A(a_j|x; y; r, \ell, d), & i = j \\ d x^{a_i} y C_A(a_j|x; y; r, \ell, d), & i > j \end{cases}.$$

Summing over $j$ and accounting for the single composition with exactly one part, namely $a_i$, gives the stated result. \qed
We are now ready to prove Theorem 2.1 (i). Lemma 2.2 together with Equation 2.1 results in a system of \( k + 1 \) equations in \( k + 1 \) variables, where we define \( t_0 = C_A(x; y; r, \ell, d) \), \( t_i = C_A(a_i|x; y; r, \ell, d) \) and \( b_i = x^{a_i}y \), for \( i = 1, 2, \ldots, k \):

\[
\begin{align*}
& t_0 - t_1 - t_2 - t_3 \cdots - t_{k-1} - t_k = 1 \\
& (1 - b_1\ell)t_1 - b_1rt_2 - b_1rt_3 \cdots - b_1rt_{k-1} - b_1rt_k = b_1 \\
& -b_2dt_1 + (1 - b_2\ell)t_2 - b_2rt_3 \cdots - b_2rt_{k-1} - b_2rt_k = b_2 \\
& -b_3dt_1 - b_3dt_2 + (1 - b_3\ell)t_3 \cdots - b_3rt_{k-1} - b_3rt_k = b_3 \\
& \vdots \\
& -b_{k-1}dt_1 - b_{k-1}dt_2 - b_{k-1}t_3 \cdots + (1 - b_{k-1}\ell)t_{k-1} - b_{k-1}rt_k = b_{k-1} \\
& -b_kdt_1 - b_kdt_2 - b_kdt_3 \cdots - b_kdt_{k-1} + (1 - b_k\ell)t_k = b_k \\
\end{align*}
\]

(2.2)

Let \( M_k \) be the \((k+1) \times (k+1)\) matrix of the system of equations (2.2), i.e.,

\[
M_k = \begin{pmatrix}
1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
0 & 1 - b_1\ell & -b_1r & -b_1r & \cdots & -b_1r & -b_1r \\
0 & -b_2d & 1 - b_2\ell & -b_2r & -b_2r & \cdots & -b_2r & -b_2r \\
0 & -b_3d & -b_3d & 1 - b_3\ell & -b_3r & \cdots & -b_3r & -b_3r \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -b_{k-1}d & -b_{k-1}d & -b_{k-1} & 1 - b_{k-1}\ell & -b_{k-1}r & \cdots & -b_{k-1}r \\
0 & -b_kd & -b_kd & -b_kd & -b_kd & 1 - b_k\ell & & & \\
\end{pmatrix}.
\]

We also define the \((k+1) \times (k+1)\) matrix \( N_k \) which results from replacing the first column in \( M_k \) by the vector of the right-hand side of (2.2), i.e.,

\[
N_k = \begin{pmatrix}
1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
b_1 & 1 - b_1\ell & -b_1r & -b_1r & \cdots & -b_1r & -b_1r \\
b_2 & -b_2d & 1 - b_2\ell & -b_2r & -b_2r & \cdots & -b_2r & -b_2r \\
b_3 & -b_3d & -b_3d & 1 - b_3\ell & -b_3r & \cdots & -b_3r & -b_3r \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{k-1} & -b_{k-1}d & -b_{k-1}d & -b_{k-1}d & 1 - b_{k-1}\ell & -b_{k-1}r & \cdots & -b_{k-1}r \\
b_k & -b_kd & -b_kd & -b_kd & -b_kd & 1 - b_k\ell & & & \\
\end{pmatrix}.
\]

Then, by Cramer’s Rule, \( t_0 = C_A(x; y; r, \ell, d) = \frac{\det(N_k)}{\det(M_k)} \). We now derive formulas for these two determinants. Expanding down the first column of \( M_k \), we get that

\[
\det(M_k) = \left| \begin{array}{cccc}
1 - b_1\ell & -b_1r & -b_1r & \cdots & -b_1r \\
-2b_2d & 1 - b_2\ell & -b_2r & \cdots & -b_2r \\
-3b_3d & -b_3d & 1 - b_3\ell & \cdots & -b_3r \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-(k-1)b_{k-1}d & -b_{k-1}d & -b_{k-1}d & \cdots & 1 - b_{k-1}\ell \\
-kb_kd & -b_kd & -b_kd & \cdots & 1 - b_k\ell \\
\end{array} \right|.
\]

Subtracting the \((k-1)\)st column from \( k\)th column of the above matrix, then expanding down the resulting column gives that

\[
\det(M_k) = (1 - b_k(\ell - d)) \det(M_{k-1}) - b_kd(1 - b_{k-1}(\ell - r)) \det(E(b_1, b_2, \ldots, b_{k-2}))
\]

(2.3)
where

\[
E(b_1, b_2, ..., b_{k-2}) = 
\begin{pmatrix}
1 - b_1 \ell & -b_1 r & -b_1 r & \cdots & -b_1 r \\
-b_2 d & 1 - b_2 \ell & -b_2 r & \cdots & -b_2 r \\
-b_3 d & -b_3 d & 1 - b_3 \ell & \cdots & -b_3 r \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-b_{k-2} d & -b_{k-2} d & -b_{k-2} d & \cdots & -b_{k-2} r \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}.
\]

Adding \((b_1 r)\) times the last row to the first row in the matrix \(E(b_1, b_2, ..., b_{k-2})\), then expanding across the resulting first row gives

\[
det(E(b_1, b_2, ..., b_{k-2})) = (1 - b_1(\ell - r)) det(E(b_2, ..., b_{k-2})),
\]

and, since \(det(E(b_{k-2})) = (1 - b_{k-2}(\ell - r))\),

\[
(2.4) 
\quad det(E(b_1, b_2, ..., b_{k-2})) = \prod_{j=1}^{k-2}(1 - b_j(\ell - r)).
\]

Equations (2.3) and (2.4) result in

\[
det(M_k) = (1 - b_k(\ell - d)) det(M_{k-1}) - b_k d \prod_{j=1}^{k-1}(1 - b_j(\ell - r)).
\]

Thus, if we define \(det(M_0) = 1\) and use the fact that \(det(M_1) = 1 - b_1 \ell = 1 - b_1(\ell - d) - b_1 d\), then we can show by induction on \(k\) that for all \(k \geq 1\),

\[
(2.5) 
\quad det(M_k) = \prod_{j=1}^{k}(1 - b_j(\ell - d)) - d \sum_{j=1}^{k} b_j \prod_{i=1}^{j-1}(1 - b_i(\ell - r)) \prod_{i=j+1}^{k}(1 - b_j(\ell - d)).
\]

Similarly, by subtracting \((b_k d)\) times the last row from the \(k^{th}\) row in the matrix \(N_k\) and then expanding across the resulting \(k^{th}\) row we get

\[
(2.6) 
\quad det(N_k) = (1 - b_k(\ell - d)) det(N_{k-1}) + b_k (1 - d) det(D(b_1, b_2, ..., b_{k-1})),
\]

where \(D(b_1, b_2, ..., b_{k-1})\) agrees with \(E(b_1, b_2, ..., b_{k-1})\) except for the signs of the last row. Thus, \(det(D(b_1, b_2, ..., b_{k-1})) = - det(E(b_1, b_2, ..., b_{k-1}))\), which yields

\[
det(N_k) = (1 - b_k(\ell - d)) det(N_{k-1}) - b_k (1 - d) \prod_{j=1}^{k-1}(1 - b_j(\ell - r)).
\]

With \(det(N_0) = 1\) and \(det(N_1) = 1 - b_1 \ell + b_1 = 1 - b_1(\ell - d) + (1 - d) b_1\), we can show by induction on \(k\) that for all \(k \geq 1\),

\[
(2.7) 
\quad det(N_k) = \prod_{j=1}^{k}(1 - b_j(\ell - d)) + (1 - d) \sum_{j=1}^{k} b_j \prod_{i=1}^{j-1}(1 - b_i(\ell - r)) \prod_{i=j+1}^{k}(1 - b_j(\ell - d)).
\]

Substituting Equations (2.5) and (2.7) and \(b_i = x^{a_i} y^{b_i}\) into \(\frac{det(N_k)}{det(M_k)}\) completes the proof of Theorem 2.1(i). Note that if \(|A| = \infty\), then the result follows by taking limits as \(k \to \infty\). \(\Box\)
2.2. Proof of Theorem 2.1(ii). As in the proof of part (i), we need to find an explicit expression for \( P_A(x; y; r, \ell, d) \), thus we define for all \( e \geq 1 \)

\[
P_A(s_1s_2 \ldots s_e|x; y; r, \ell, d) = \sum_{n \geq 0} \sum_{\sigma} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)},
\]

where the sum on the right side of the equation is over all the palindromic compositions \( \sigma \in P_n^A \) such that \( \sigma_j = s_j \) for all \( j = 1, 2, \ldots, e \).

As before, we get two relations (Equation (2.8) and Lemma 2.3) between the generating functions \( P_A(x; y; r, \ell, d) \) and \( P_A(a_i|x; y; r, \ell, d) \). The first relation is given by

\[
(2.8) \quad P_A(x; y; r, \ell, d) = 1 + \sum_{i=1}^{k} P_A(a_i|x; y; r, \ell, d),
\]

which holds immediately from the definitions. The second relation is given by the following lemma.

**Lemma 2.3.** Let \( A = \{a_1, \ldots, a_k\} \) be any ordered subset of \( \mathbb{N} \). For all \( i = 1, 2, \ldots, k \), the generating function \( P_A(a_i|x; y; r, \ell, d) \) is given by

\[
\frac{x^{a_i} y + x^{2a_i} y^2 \ell}{1 - x^{2a_i} y^2 (\ell^2 - d r)} + \frac{x^{2a_i} y^2 d r}{1 - x^{2a_i} y^2 (\ell^2 - d r)} (P_A(x; y; r, \ell, d) - 1).
\]

**Proof.** First of all, for all \( e, m \geq 1 \) we define

\[
P_A(s_1s_2 \ldots s_e|m|x; r, \ell, d) = \sum_{n \geq 0} \sum_{\sigma} x^n r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)},
\]

where the sum on the right side of the equation is over all the palindromic compositions \( \sigma \in P_n^A \) with \( m \) parts such that \( \sigma_j = s_j \) for all \( j = 1, 2, \ldots, e \).

Now, by fixing \( i \) and using the definitions we have that

\[
P_A(a_i; 0|x; r, \ell, d) = 0, \quad P_A(a_i; 1|x; r, \ell, d) = x^{a_i}, \quad P_A(a_i; 2|x; r, \ell, d) = x^{2a_i} \ell.
\]

For \( m \geq 3 \), we create the palindromic compositions of \( n \) that start and end with \( a_i \) from those of \( n - 2a_i \) that start with \( a_j \) by prepending and appending \( a_i \). This results in two additional parts, and in one additional drop and rise when \( i \neq j \), and two additional levels when \( i = j \). (Note that the symmetry of the palindromic compositions, which distinguishes only the case \( i = j \), allows us to use a different proof technique, which does not work for compositions.)

Thus, for \( m \geq 3 \),

\[
P_A(a_i; m|x; r, \ell, d) = \sum_{j=1}^{k} P_A(a_i a_j; m|x; r, \ell, d) + P_A(a_i a_i; m|x; r, \ell, d)
\]

\[
= x^{2a_i} d r \sum_{j=1}^{k} P_A(a_j; m - 2|x; r, \ell, d) + x^{2a_i} \ell^2 P_A(a_i; m - 2|x; r, \ell, d)
\]

\[
= x^{2a_i} d r \sum_{j=1}^{k} P_A(a_j; m - 2|x; r, \ell, d) + x^{2a_i} (\ell^2 - d r) P_A(a_i; m - 2|x; r, \ell, d).
\]

Multiplying by \( y^m \) and summing over all \( m \geq 0 \), we get that

\[
P_A(a_i|x; y; r, \ell, d) = x^{a_i} y + x^{2a_i} y^2 \ell + x^{2a_i} y^2 d r \sum_{j=1}^{k} P_A(a_j|x; y; r, \ell, d)
\]

\[
+ x^{2a_i} y^2 (\ell^2 - d r) P_A(a_i|x; y; r, \ell, d),
\]
or, equivalently,

\[ P_A(a_i | x; y; r, \ell, d) = \frac{y^j}{1 - x^j} + \frac{x^j}{1 - x^j r} \sum_{j=1}^{k} P_A(a_j | x; y; r, \ell, d), \]

from which Lemma 2.3 follows by using Equation (2.8). □

Now we are ready to give the proof of Theorem 2.1(ii). Applying Lemma 2.3 for all \( i = 1, 2, \ldots, k \) together with using Equation (2.8), we get that the generating function \( P_A(x; y; r, \ell, d) \) is given by

\[
1 + \sum_{i=1}^{k} \frac{x^i + x^{2i} y^2 (\ell - d r)}{1 - x^{2i} y^2 (\ell^2 - d r)} + \sum_{i=1}^{k} \frac{x^{2i} y^2 d r}{1 - x^{2i} y^2 (\ell^2 - d r)} P_A(x; y; r, \ell, d).
\]

Equivalently,

\[
P_A(x; y; r, \ell, d) = \frac{1 + \sum_{i=1}^{k} \frac{x^i + x^{2i} y^2 (\ell - d r)}{1 - x^{2i} y^2 (\ell^2 - d r)}}{1 - \sum_{i=1}^{k} \frac{x^{2i} y^2 d r}{1 - x^{2i} y^2 (\ell^2 - d r)}},
\]

as claimed. □

3. Applications

In the following subsections we give several applications for both parts of Theorem 2.1.

3.1. Compositions with parts in \( A \). In this subsection we study the number of compositions of \( n \) as well as the number of rises, levels, and drops in the compositions of \( n \) with parts in \( A \). Applying Theorem 2.1(iii) for \( r = 1, \ell = 1, \) and \( d = 1 \), we get that the generating function for the number of compositions of \( n \) with \( m \) parts in \( A \) is given by

\[
\sum_{n \geq 0} \sum_{\sigma \in C_n^A} x^m = \frac{1}{1 - y \sum_{j=1}^{k} x^j}.
\]

Therefore, the generating function for the number of compositions of \( n \) with \( m \) parts in \( \mathbb{N} \) is given by

\[
\sum_{n \geq 0} \sum_{\sigma \in C_n^\mathbb{N}} x^m = \frac{1}{1 - y \sum_{j=1}^{\infty} x^j} = \frac{1}{1 - \frac{x}{1-x}} = \sum_{m \geq 0} \frac{x^m}{(1-x)^m} y^m.
\]

Furthermore, setting \( y = 1 \) in Equation (3.1) gives the generating function for the number of compositions of \( n \) with parts in \( A \) (see [9], Theorem 2.4):

\[
\frac{1}{1 - \sum_{j=1}^{k} x^j}.
\]

In particular, for \( A = \mathbb{N} \), the generating function for the number of compositions of \( n \) with parts in \( \mathbb{N} \) is given by (see [3], Theorem 6)

\[
\frac{1 - x}{1 - 2x}.
\]

Additional examples for specific choices of \( A \) are given in [9].
3.1.1. Number of rises and drops. Note that the number of rises always equals the number of drops in all compositions of n; for each non-palindromic composition there exists a composition in reverse order, thus the rises match the drops, and for palindromic compositions, symmetry matches up rises and drops within the composition. Thus, we will derive results only for rises, and the results for drops follow by interchanging the roles of r and d in the proofs.

Setting \( \ell = 1 \) and \( d = 1 \) in Theorem 2.1(i) gives

\[
C_A(x; y; r, 1, 1) = \frac{1}{1 - \sum_{j=1}^{m} \left( x^{a_j} y \prod_{i=1}^{j-1} (1 - x^{a_i} y (1 - r)) \right)}.
\]

Using Equation (3.2) together with the fact that for \( f_i(r) \neq 0 \)

\[
\frac{\partial}{\partial r} \prod_{i=1}^{m} f_i(r) = \left( \prod_{i=1}^{m} f_i(r) \right) \sum_{i=1}^{m} \frac{\partial f_i(r)}{f_i(r)},
\]

we get that

\[
\frac{\partial}{\partial r} C_A(x; y; r, 1, 1) \bigg|_{r=1} = \frac{y^2 \sum_{k \geq j \geq i \geq 1} x^{a_i + a_j}}{(1 - y \sum_{j=1}^{k} x^{a_j})^2}.
\]

Hence, expressing this function as a power series about \( y = 0 \), we get the following result.

**Corollary 3.1.** Let \( A = \{a_1, \ldots, a_k\} \) be any ordered subset of \( \mathbb{N} \). Then

\[
\sum_{n \geq 0} \sum_{\sigma \in C_A^n} \text{rises}(\sigma) x^n y^{\text{parts}(\sigma)} = \left( \sum_{k \geq j \geq i \geq 1} x^{a_i + a_j} \right) \sum_{m \geq 0} (m + 1) \left( \sum_{j=1}^{k} x^{a_j} \right)^m y^{m+2}
\]

and

\[
\sum_{n \geq 0} \sum_{\sigma \in C_A^n} \text{drops}(\sigma) x^n y^{\text{parts}(\sigma)} = \left( \sum_{k \geq j \geq i \geq 1} x^{a_i + a_j} \right) \sum_{m \geq 0} (m + 1) \left( \sum_{j=1}^{k} x^{a_j} \right)^m y^{m+2}.
\]

For example, letting \( A = \mathbb{N} \) and looking at the coefficient of \( y^m \) in Corollary 3.1 we get that the generating function for the number of rises (drops) in the compositions of \( n \) with a fixed number of parts, \( m \geq 2 \), in \( \mathbb{N} \) is given by

\[
\sum_{j > i \geq 1} x^{i+j} (m-1) (\sum_{j \geq 1} x^j)^{m-2} = \sum_{i \geq 1} \sum_{j \geq i+1} x^{i+j} (m-1) \left( \frac{x}{1-x} \right)^{m-2} = \sum_{i \geq 1} x^i \sum_{j \geq 1} x^{2j} \frac{(m-1)x^{m-2}}{(1-x)^{m-2}}
\]

\[
= \frac{x^3}{(1-x)(1-x^2)} \cdot \frac{(m-1)x^{m-2}}{(1-x)^{m-2}} = \frac{(m-1)x^{m+1}}{(1+x)(1-x)^m}.
\]

Furthermore, setting \( y = 1 \) and \( A = \mathbb{N} \) in Corollary 3.1 allows us to compute the generating function for the number of rises (drops) in all compositions of \( n \) with parts in \( \mathbb{N} \) (see [3].
Theorem 6) in a similar way:

$$
\sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{rises}(\sigma) x^n = \sum_{j > i \geq 1} x^{i+j} \sum_{m \geq 0} (m+1) \left( \frac{x}{1-x} \right)^m = \frac{x^3}{(1-x)(1-x^2)} \cdot \frac{1}{(1-x^2)}.
$$

In other words, as shown in [3, Theorem 3], the number of rises (drops) in the compositions of \( n \) with parts in \( \mathbb{N} \) is given by

$$
\frac{1}{9} \left( 2^{n-2}(3n-5) + (-1)^{n+1} \right) \quad \text{for} \quad n \geq 3.
$$

For \( A = \{1, k\} \) and \( y = 1 \), Corollary 3.1 gives the generating function for the number of rises (drops) in all compositions of \( n \) with parts in \( \{1, k\} \) as (see [5], Theorem 4)

$$
\frac{x^{k+1}}{(1-x-x^k)^2}.
$$

For \( A = \{m \mid m = 2k+1, k \geq 0\} \) and \( y = 1 \), and using that \( \sum_{0 \leq i < j} x^{(2i+1)+(2j+1)} = \sum_{i \geq 0} (x^2)^i \sum_{j \geq 1} (x^4)^j \), Corollary 3.1 yields a new result, namely that the generating function for the number of rises (drops) in compositions of \( n \) with odd parts is given by

$$
\frac{x^{k+1}}{(1-x-x^k)^2}.
$$

For \( A = \mathbb{N} - \{k\} \), and defining \( g(x, y; k) = \sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{rises}(\sigma) x^n y^{\text{parts}(\sigma)} \), Corollary 3.1 gives

$$
g(x, y; k) = \left( \frac{x^3}{(1-x)(1-x^2)} - \frac{x^{k+1}(1-x^{-k-1}) + x^{2k+1}}{1-x} \right) \sum_{m \geq 0} (m+1) \left( \frac{x}{1-x-x^k} \right)^m y^{m+2}.
$$

For \( k = 1 \), i.e., \( A = \mathbb{N} - \{1\} \) we get that

$$
g(x, y; 1) = \sum_{m \geq 2} (m-1) \frac{x^{2m+1}}{(1+x)(1-x)^m} y^m.
$$

Thus, the generating function for the number of rises (drops) in the compositions of \( n \) with a fixed number of parts, \( m \geq 2 \), in \( A = \mathbb{N} - \{1\} \) is given by

$$
(\ell-1) \frac{x^{2m+1}}{(1+x)(1-x)^m} = \sum_{n \geq 0} x^{n+2m-1} \sum_{j=0}^{n} (-1)^{n-j} \binom{j+m-1}{m-1}.
$$

3.1.2. Number of levels. Theorem 2.1(i) for \( r = 1 \) and \( d = 1 \) gives

$$
(3.5) \quad C_A(x; y; 1, \ell, 1) = \frac{1}{1 - \sum_{j=1}^{k} \frac{x^{\ell+j} y}{1-x^j y}(\ell-1)}.
$$
Therefore, using Equation (3.5) we have that
\[
\left. \frac{\partial}{\partial \ell} C_A(x; y; 1, \ell, 1) \right|_{\ell=1} = \frac{y^2 \sum_{j=1}^{k} x^{2a_j}}{1 - y \sum_{j=1}^{k} x^{a_j}}.
\]
Expressing the above function as a power series about \( y = 0 \), we get the following result.

**Corollary 3.2.** Let \( A = \{a_1, \ldots, a_k\} \) be any ordered subset of \( \mathbb{N} \). Then
\[
\sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{levels}(\sigma)x^n y^{\text{parts}(\sigma)} = \left( \sum_{j=1}^{k} x^{2a_j} \right) \sum_{m \geq 0} (m + 1) \left( \sum_{j=1}^{k} x^{a_j} \right)^m y^{m+2}.
\]

Using computations similar to those for rises and drops, by looking at the coefficient of \( y^n \), we get from Corollary 3.2 that the generating function for the number of levels in all compositions of \( n \) with a fixed number of parts \( m \) in \( \mathbb{N} \) is given by
\[
\frac{(m - 1)x^m}{(1 + x)(1 - x)^{m-1}}.
\]
In addition, by setting \( y = 1 \) and \( A = \mathbb{N} \) in Corollary 3.2 we obtain that the generating function for the number of levels in the compositions of \( n \) with parts in \( \mathbb{N} \) (see [3], Theorem 6) is given by
\[
\frac{x^2(1-x)}{(1+x)(1-2x)^2}.
\]
Thus, as shown in [3] Theorem 3], the number of levels in all compositions of \( n \) with parts in \( \mathbb{N} \) is given by
\[
\frac{1}{9} \left( 2^{n-2}(3n+1) + 2(-1)^n \right) \quad \text{for } n \geq 1.
\]
Applying Corollary 3.2 for \( A = \{1, 2\} \) and \( y = 1 \), we get the result given in Theorem 1.1 [1] for the generating function for the number of levels in all compositions with only 1’s and 2’s:
\[
\frac{x^2 + x^4}{(1 - (x + x^2))^2},
\]
and more generally, for \( A = \{1, k\} \) and \( y = 1 \), we get the result stated in Theorem 4 [5]:
\[
\frac{x^2 + x^{2k}}{(1 - (x^k + x^{2k}))^2}.
\]
If we apply Corollary 3.2 to \( A = \{m \mid m = 2k + 1, k \geq 0\} \), then we get a new result, namely that the generating function for the number of levels in the compositions of \( n \) with odd summands is given by
\[
\frac{x^2(1-x)}{(1 + x^2)(1 - x - x^2)^2}.
\]
Finally, we look at \( A = \mathbb{N} - \{k\} \) and define \( g(x; y; k) = \sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{levels}(\sigma)x^n y^{\text{parts}(\sigma)} \). Then Corollary 3.2 gives
\[
g(x; y; k) = \left( \frac{x^2}{1-x^2} - x^{2k} \right) \sum_{m \geq 0} (m + 1) \left( \frac{x}{1-x} - x^k \right)^m y^{m+2}.
\]
If we set $y = 1$, then we get a new result, namely that the generating function for the number of levels in the compositions of $n$ without $k$ is given by

$$\frac{(1 - x)x^2(1 - x^{2(k-1)} + x^{2k})}{(1 + x)(1 - 2x + x^k - x^{k+1})^2}.$$ 

### 3.2. Palindromic compositions with parts in $A$.

Applying Theorem 2.1(ii) for $r = 1$, $\ell = 1$, and $d = 1$ we get that the generating function for the number of palindromic compositions of $n$ with $m$ parts in $A$ is given by

$$1 + y \sum_{i=1}^{k} x^{a_i} \left( \sum_{i=1}^{k} x^{2a_i} \right)^2.$$ 

Setting $y = 1$ we get that the number of palindromic compositions of $n$ with parts in $A$ is given by (see [9], Theorem 3.2)

$$1 + \sum_{i=1}^{k} x^{a_i} \left( \sum_{i=1}^{k} x^{2a_i} \right)^2.$$ 

Using $A = N$ we get that the generating function for the number of palindromic compositions of $n$ with parts in $N$ is given by (see [3], Theorem 6)

$$1 + x \left( \sum_{i=1}^{k} x^{2a_i} \right)^2.$$ 

Therefore, the number of palindromic compositions of $n$ with parts in $N$ is given by $2^{\lfloor n/2 \rfloor}$ (see [3], Theorem 1).

### 3.2.1. Number of rises or drops.

As before, the number of rises equals the number of drops. Theorem 2.1(ii) for $\ell = 1$ and $d = 1$ gives

$$P_A(x; y; r, 1, 1) = \frac{1 + y \sum_{i=1}^{k} x^{a_i} \left( \sum_{i=1}^{k} x^{2a_i} \right)^2}{1 - y^2 \sum_{i=1}^{k} x^{2a_i} (1 - x^{2a_i} y^2)}.$$ 

Therefore, by finding $\frac{\partial}{\partial r} P_A(x; y; r, 1, 1)$ and setting $r = 1$ we obtain the following result.

**Corollary 3.3.** Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then the generating function $g_A(x; y) = \sum_{n \geq 0} \sum_{\sigma \in P_A} \text{rises}(\sigma) x^n y^{\text{parts}(\sigma)} = \sum_{n \geq 0} \sum_{\sigma \in P_A} \text{drops}(\sigma) x^n y^{\text{parts}(\sigma)}$ is given by

$$y^2 \left( 1 + y \sum_{i=1}^{k} x^{a_i} \right) \left( \sum_{i=1}^{k} x^{2a_i} \left( 1 - x^{2a_i} y^2 \right) \right) - y^2 \left( 1 - y^2 \sum_{i=1}^{k} x^{2a_i} \right) \sum_{i=1}^{k} x^{2a_i} \left( 1 + x^{a_i} y \right) \left( 1 - y^2 \sum_{i=1}^{k} x^{2a_i} \right)^2.$$ 

For example, if $A = \mathbb{N}$, then Corollary 3.3 gives that

$$g_{\mathbb{N}}(x; y) = \frac{y^2 \left( 1 + \frac{y^2}{1-x} \right) \left( \frac{x^2}{1-x^2} - \frac{x^4}{1-x^4} \right) - y^2 \left( 1 - \frac{y^2}{1-x} \right) \left( \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} \right)}{\left( 1 - \frac{y^2}{1-x} \right)^2}.$$
Thus, we can derive the generating function for the number of rises (drops) in the compositions of \( n \) with a given number of parts, \( m \), in \( \mathbb{N} \), by looking at the coefficient of \( y^m \) in \( g_n(x; y) \). To do so, we expand the numerator of \( g_n(x; y) \) and collect terms according to powers of \( y \):

\[
\frac{x^4y^3}{(1 - x)^2(1 + x)} \left( \frac{2x + 1}{(x^2 + x + 1)} + \frac{2x^2}{(x + 1)(x^2 + 1)}y - \frac{x^2}{(x^2 + x + 1)(x^2 + 1)y^2} \right).
\]

Furthermore,

\[
\frac{1}{(1 - \frac{y^2}{1 - x^2})^2} = \sum_{m \geq 0} \frac{(m + 1)x^{2m}}{(1 - x^2)^m} \cdot \frac{y^{2m+1}}{1 - x^2},
\]

so altogether,

\[
g_n(x; y) = \sum_{m \geq 0} \frac{(m + 1)x^{2m+4}}{(1 - x)^2(1 + x)(1 - x^2)}y^{2m+3}.
\]

We now have to distinguish between two cases, namely, \( m \) odd and \( m \) even. In the first case, only the summand with factor \( y \) needs to be taken into account, whereas in the second case, the summands with factors \( y^0 \) and \( y^2 \) need to be considered. Thus, the generating function for the number of rises (drops) in the compositions of \( n \) with a given number of parts, \( m \), in \( \mathbb{N} \) is given by

\[
\frac{(2m' - 2)x^{2m'+2}}{(1 + x^2)(1 - x^2)^{m'}} \quad \text{for } m = 2m',
\]

and

\[
\frac{x^{2m'}(1 - x)(1 + (2m' - 2)x + (2m' - 3)x^2 + (2m' - 2)x^3)}{(1 + x^2)(1 + x + x^2)^2(1 - x^2)^{m'}} \quad \text{for } m = 2m' - 1.
\]

Furthermore, setting \( y = 1 \) in Equation (4.37) and simplifying yields that the generating function for the number of rises (drops) in the compositions of \( n \) with parts in \( \mathbb{N} \) (see [3], Theorem 6) is given by

\[
g_n(x; 1) = \frac{x^4(4x^4 + 4x^3 + 4x^2 + 3x + 1)}{(1 + x^2)(1 + x + x^2)(1 - 2x^2)^2}.
\]

We now apply Corollary 5.33 for \( A = \{1, k\} \) and get that

\[
g_{1,k}(x; y) = \frac{x^{k+1}y^3(x + x^k + 2x^{k+1}y - y^2(x^3 + x^{3k} + x^{k+2} - x^{2k+1}))}{(1 - y^2(x^2 + x^{2k}))^2}.
\]

In particular, when setting \( y = 1 \) in the above expression we get that the generating function for the number of rises (drops) in the palindromic compositions of \( n \) with any number of parts in \( A = \{1, k\} \) is given by (see [5], Theorem 5)

\[
g_{1,k}(x; 1) = \frac{x^{k+1}(x - x^3 + x^3k + 2x^{k+1} + x^{k+2} + x^{2k+1})}{(1 - x^2 - x^{2k})^2}.
\]
If we let $A = \{m \mid m = 2k + 1, k \geq 0\}$ in Corollary 3.3, then we get that the generating function $g_A(x; y)$ is given by
\[
y^2 \left(1 + \frac{xy}{1-x^2}\right) \left(\frac{x^2}{1-x^2} - \frac{y^2 x^4}{1-x^6}\right) - y^2 \left(1 - \frac{x^2 y^2}{1-x^4}\right) \left(\frac{x^2}{1-x^4} + \frac{x^4 y}{1-x^8}\right).\]

Furthermore, if we let $y = 1$ in the above expression, then we get that the generating function for the number of rises (drops) in the palindromic compositions of $n$ with any number of odd parts is given by
\[
g_A(x; 1) = \frac{x^5(1 + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 3x^6 + 2x^7 + 2x^8)}{(1 + x^4)(1 - x^2 - x^4)^2(1 + x^2 + x^4)},
\]
which extends the work of Grimaldi [7].

Applying Corollary 3.3 to $A = \mathbb{N} \setminus \{k\}$ gives that
\[
g_{\mathbb{N} \setminus \{k\}}(x; y) = \frac{y^2 \left(1 + \frac{xy}{1-x^2} - yx^k\right) \left(\frac{x^2}{1-x^2} - x^{2k} - \frac{y^2 x^4}{1-x^6} + y^2 x^{4k}\right)}{\left(1 - \frac{y^2 x^2}{1-x^2} + y^2 x^{2k}\right)^2} - \frac{y^2 \left(1 - \frac{y^2 x^2}{1-x^2} + y^2 x^{2k}\right) \left(\frac{x^2}{1-x^2} - x^{2k} + \frac{yx^3}{1-x^3} - yx^{3k}\right)}{\left(1 - \frac{y^2 x^2}{1-x^2} + y^2 x^{2k}\right)^2}.
\]

In particular, when setting $y = 1$ in the above expression, we get that the generating function for the number of rises (drops) in the palindromic compositions of $n$ with any number of odd parts in $A = \mathbb{N} \setminus \{k\}$ is given by
\[
g_{\mathbb{N} \setminus \{k\}}(x; 1) = \frac{x^4(1 + 3x + 4x^2 + 4x^3 + 4x^4) + x^{2k+1}(x^4 - 1)(1 + 4x + 5x^2 + 4x^3)}{(1 + x^4)(1 + x + x^2)(1 - 2x^2 + x^{2k} - x^{2(k+1)}))} + \frac{(x^2 - 1)(x^{k+2} + x^{3k}(1 + x^2)(3x^2 - 2) + x^{4k}(1 + x)(x - 2))}{(1 + x^2)(1 - 2x^2 + x^{2k} - x^{2(k+1)}))}.
\]

This extends the work of Chinn and Heubach [4]. Likewise, we can extend the work of Grimaldi [6] by setting $k = 1$ to get that
\[
g_{\mathbb{N} \setminus \{1\}}(x; 1) = \frac{(x^5 + 3x^4 + 5x^3 + 3x^2 + 3x + 1)x^7}{(1 - x^2 - x^4)(1 + x + x^2)(1 + x^2)}.
\]

3.2.2. Number of levels. Theorem 2.1(ii) for $r = 1$ and $d = 1$ gives
\[
P_A(x; y; 1, \ell, 1) = \frac{1 + \sum_{i=1}^{k} x^{2i}y + x^{2i}y^2(\ell - 1)}{1 - x^{2i}y^2(\ell^2 - 1)}.\]

Therefore, finding $\frac{\partial}{\partial y} P_A(x; y; 1, \ell, 1)$ and setting $\ell = 1$ yields the following result.
Corollary 3.4. Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then the generating function $g_A(x; y) = \sum_{n \geq 0} \sum_{\sigma \in P_n^A} \text{levels}(\sigma)x^n$ is given by

$$
g_A(x; y) = \frac{y^2 \left(1 - y^2 \sum_{i=1}^{k} x^{2a_i}\right) \sum_{i=1}^{k} x^{2a_i}(1 + 2x^{a_i}y) + 2y^4 \left(1 + y \sum_{i=1}^{k} x^{a_i}\right) \sum_{i=1}^{k} x^{4a_i}}{\left(1 - y^2 \sum_{i=1}^{k} x^{2a_i}\right)^2}.
$$

For example, applying Corollary 3.4 with $A = \mathbb{N}$ gives that the generating function $g_n(x; y)$ for the number of levels in all palindromic compositions of $n$ with $m$ parts in $\mathbb{N}$ is given by (3.8)

$$
x^2y^2 \left(\frac{2x^4(x + 1)y^3 + x^2(1 - 3x^2)(1 + x + x^2)y^2 + (1 - x^4)(2x(1 + x)y + 1 + x + x^2)}{(1 + x^2)(1 + x + x^2)(1 - x^2 - x^2y^2)}\right).
$$

Rewriting $\frac{1}{(1 - x^2)(1 - x^2y^2)^2}$ as

$$
\frac{1}{(1 - x^2)^2 \left(1 - x^2y^2 - \frac{1 - x^2}{1 - x}\right)} = \frac{1}{(1 - x^2)^2} \sum_{m \geq 0} (m + 1) \frac{x^{2m}}{(1 - x^2)^m} y^{2m}
$$

allows us to compute the generating function $l_m(x)$ for the number of levels in palindromic compositions of $n$ with a given number of parts, $m$, by looking at the coefficient of $y^m$ in expression (3.8):

$$
l_m(x) = \begin{cases} 
\frac{x^2}{1 - x^2} & \text{for } m = 2 \\
\frac{(2m' - 1 - (2m' - 3)x^2)x^{2m'}}{(1 + x^2)(1 - x^2)^{m'}} & \text{for } m = 2m', m' \geq 2 \\
\frac{2(1 + x)(m' + (m' - 1)x) + m'x^2}{(1 + x^2)(1 + x + x^2)(1 - x^2)^{m'}} x^{2m' + 1} & \text{for } m = 2m' + 1, m' \geq 1 
\end{cases}
$$

In addition, setting $y = 1$ in (3.8) gives that the generating function for the number of levels in the palindromic compositions of $n$ with parts in $\mathbb{N}$ (see [3], Theorem 6) is given by

$$
g_n(x; 1) = \frac{x^2(1 + 3x + 4x^2 + x^3 - 4x^4 - 4x^5 - 6x^6)}{(1 + x^2)(1 + x + x^2)(1 - 2x^2)^2}.
$$

If we let $A = \{1, k\}$ in Corollary 3.4, then we get that $g_{\{1,k\}}(x; y)$ is given by

$$
y^2(x^2 + x^{2k}) + 2y^3(x^3 + x^{3k}) + y^4(x^4 + x^{4k} - 2x^{2(k+1)}) + 2y^5(x^{k+4} - x^{2k+3} - x^{3k+2} + x^{4k+1})
$$

$$(1 - y^2x^2 - y^2x^{2k})^2.
$$

Setting $y = 1$ in the above expression yields that the generating function for the number of levels in the palindromic compositions of $n$ with any number of parts in $\{1, k\}$ is given by

$$
g_{\{1,k\}}(x; 1) = \frac{x^2 + x^{2k} + x^3 + x^{3k} + x^4 + x^{4k} + 2(x^{k+4} - x^{2k+3} - x^{3k+2} + x^{4k+1})}{(1 - y^2x^2 - y^2x^{2k})^2}.
$$

This result was not explicitly stated in [3], but can be easily computed from the generating functions for other quantities given in [5].

We look next at $A = \{m \mid m = 2k + 1, k \geq 0\}$. Applying Corollary 3.4 for this case, we get that

$$
g_A(x; y) = \frac{y^2 \left(1 - \frac{x^2}{1 - x^2}\right) \left(\frac{x^2}{1 - x^2} + 2 \frac{x^3y}{1 - x^2} + 2 \frac{x^4y^2}{1 - x^2}\right) \left(1 + \frac{x}{1 - x}\right)}{\left(1 - \frac{x^2y^2}{1 - x^2}\right)^2}.
$$
Furthermore, if we set \( y = 1 \) in the above expression, then we get that the generating function for the number of levels in the palindromic compositions of \( n \) with any number of odd parts is given by
\[
g_A(x; 1) = \frac{x^2(1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 - 2x^6 + 2x^7 - 4x^8 - 2x^9 - 2x^{10} - 2x^{11} - x^{12})}{(1 + x^2)(1 - x^2 - x^4)^2(1 + x^2 + x^4)},
\]
which extends the work of Grimaldi [7].

Finally, applying Corollary 3.3 for \( A = \mathbb{N} - \{k\} \) gives that the generating function \( g_{\mathbb{N} - \{k\}}(x; y) \) is given by
\[
y^2 \left( 1 - \frac{y^2 x^2}{1 - x^2} + y^2 x^{2k} \right) \left( \frac{x^2}{1 - x^2} - x^{2k} + \frac{2y^2 x^3}{1 - x^2} - 2y x^{3k} \right) + 2y^4 \left( 1 + \frac{y x}{1 - x} - y x^k \right) \left( \frac{x^4}{1 - x^2} - x^{4k} \right)
\]
\[
\left( 1 - \frac{y^2 x^2}{1 - x^2} + y^2 x^{2k} \right)^2.
\]

In particular, when setting \( y = 1 \) in the above expression we get that the generating function for the number of levels in the palindromic compositions of \( n \) with any number of parts in \( A = \mathbb{N} - \{k\} \) is given by
\[
\frac{x^2(1 + 3x + 4x^2 + x^3 - x^4 - 4x^5 - 6x^6) + x^{2k}(x^4 - 1)(1 + x - 2x^2 - 5x^3 - 5x^4)}{(1 + x^2)(1 + x + x^2)(1 - 2x^2 + x^{2k} - x^{2k+1})^2}
\]
\[
+ \frac{(x^2 - 1)(2x^{k+1} + 2x^{3k}(1 + x^2)(1 - 2x^2) + x^{4k}(1 + x)(3 - x)(1 + x^2))}{(1 + x^2)^2(1 - 2x^2 + x^{2k} - x^{2k+1})^2}.
\]

This extends the work of Chinn and Heubach [4]. Likewise, we can extend the work of Grimaldi [6] by setting \( k = 1 \) to get that
\[
g_{\mathbb{N} - \{1\}}(x; 1) = \frac{(1 + x + 3x^2 + 2x^3 - 5x^6 - 3x^7 - x^8)x^4}{(1 - x^2 - x^4)^2(1 + x^2)(1 + x + x^2)}.
\]

3.3. Carlitz Compositions with parts in \( A \). A Carlitz composition of \( n \), introduced in [2], is a composition of \( n \) in which no adjacent parts are the same. In other words, a Carlitz composition \( \sigma \) is a composition with levels(\( \sigma \)) \( = 0 \). We will derive results on the set of Carlitz compositions of \( n \) with parts in \( A \), denoted by \( E_n^A \). In this section we study the generating functions for the number of Carlitz compositions of \( n \) with parts in \( A \) with respect to the number of rises and drops.

3.3.1. Number of Carlitz compositions. We denote the generating function for the number of Carlitz compositions of \( n \) with \( m \) parts in \( A \) with respect to the number of rises and drops by \( E_A(x; y; r, d) \), that is,
\[
E_A(x; y; r, d) = \sum_{n \geq 0} \sum_{\sigma \in E_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} d^{\text{drops}(\sigma)}.
\]

Note that \( E_A(x; y; r, d) = C_A(x; y; r, 0, d) \). Therefore, Theorem (2.1(i)) for \( \ell = 0 \) gives the following result.
Corollary 3.5. Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then

$$E_A(x; y; r, d) = 1 + \frac{\sum_{j=1}^{k} \left( \frac{x^{a_j} y}{1 + x^{a_j} y} \prod_{i=1}^{j-1} \frac{1 + x^{a_i} y r}{1 + x^{a_i} y d} \right)}{1 - \sum_{j=1}^{k} \left( \frac{x^{a_j} y d}{1 + x^{a_j} y d} \prod_{i=1}^{j-1} \frac{1 + x^{a_i} y r}{1 + x^{a_i} y d} \right)}.$$

For example, with $r = d = 1$, Corollary 3.5 gives that the generating function for the number of Carlitz compositions with $m$ parts in $A$ (for the case $A = \mathbb{N}$, see [2]) is given by

$$E_A(x; y; 1, 1) = \frac{1}{1 - \sum_{j=1}^{k} \frac{x^{a_j} y}{1 + x^{a_j} y}}.$$

Applying Corollary 3.5 for $A = \{a, b\}$ and $r = d = 1$ yields the generating function for the number of Carlitz compositions of $n$ with $m$ parts in $\{a, b\}$ is given by

$$\frac{(1 + x^a y)(1 + x^b y)}{1 - x^{a+b} y^2} = 1 + (x^a + x^b) y + \sum_{m \geq 1} x^{m(a+b)} (2 y^{2m} + (x^a + x^b) y^{2m+1}).$$

In particular, setting $y = 1$ in the expression above yields that the generating function for the number of Carlitz compositions of $n$ with parts in $\{a, b\}$ is given by

$$\frac{(1 + x^a)(1 + x^b)}{1 - x^{a+b}}.$$

Remark: In the case $A = \{a, b\}$, the requirement that no adjacent parts are to be the same restricts the compositions to those with alternating $a$’s and $b$’s. This results in the following possibilities:

$$\begin{align*}
n & \quad \text{Carlitz compositions of } n \\
n' (a+b) & \quad \text{abab...ab and baba...ba} \\
n' (a+b) + a & \quad \text{abab...aba} \\
n' (a+b) + b & \quad \text{babab...ab}
\end{align*}$$

Thus, the number of Carlitz compositions of $n > 0$ is 2 if $n = n'(a+b)$, 1 if $n = n'(a+b) + a$ or $n = n'(a+b) + b$, and 0 otherwise.

3.3.2. Number of Rises and Drops. We now study the number of rises (drops) in all Carlitz compositions of $n$ with $m$ parts in $A$. Once more, the number of rises equals the number of drops. Using Corollary 3.5 to find an explicit expression for $\frac{\partial}{\partial r} E_A(x; y; r, 1) \big|_{r=1}$ gives the following result.

Corollary 3.6. Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then the generating functions $\sum_{n \geq 0} \sum_{\sigma \in E_A^n} \text{rises}(\sigma) x^n y^{\text{parts}(\sigma)}$ and $\sum_{n \geq 0} \sum_{\sigma \in E_A^n} \text{drops}(\sigma) x^n y^{\text{parts}(\sigma)}$ are given
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by

\[
\sum_{j=1}^{k} \left( \frac{x^a_j y^{j-1}}{1 + x^a_j y} \sum_{i=1}^{j-1} x^a_i y \right) \left( 1 - \sum_{j=1}^{k} \frac{x^a_j y}{1 + x^a_j y} \right)^2.
\]

Setting \( A = \mathbb{N} \) and \( y = 1 \) in Corollary 3.6 yields that the generating function for the number of rises (drops) in the Carlitz compositions of \( n \) with parts in \( \mathbb{N} \) is given by

\[
\sum_{j \geq 1} \left( \frac{x^j y^{j-1}}{1 + x^j y} \sum_{i=1}^{j-1} x^i y \right) \left( 1 - \sum_{j \geq 1} \frac{x^j y}{1 + x^j y} \right)^2.
\]

Applying Corollary 3.6 for \( A = \{a, b\} \) gives that

\[
\sum_{n \geq 0} \sum_{\sigma \in E_n^A} \text{rises}(\sigma) x^n y^{\text{parts}(\sigma)} = \frac{x^{a+b} y^2 (1 + x^a y)(1 + x^b y)}{(1 - x^{a+b} y^2)^2} = \sum_{m \geq 1} x^{m(a+b)} \left( (2m - 1)y^{2m} + m(x^a + x^b)y^{2m+1}\right),
\]

where the second equation follows after collecting even and odd powers of \( y \).

In particular, setting \( y = 1 \) in the expression above yields that the generating function for the number of rises (drops) in the Carlitz compositions of \( n \) with parts in \( \{a, b\} \) is given by

\[
\frac{x^{a+b}(1 + x^a)(1 + x^b)}{(1 - x^{a+b})^2}.
\]

Thus, the number of rises (drops) in Carlitz compositions of \( n \geq (a + b) \) with parts in \( \{a, b\} \) is given by

\[ n' \text{ if } n = (a + b)n' + a \text{ or } n = (a + b)n' + b \text{ and } 2n' - 1 \text{ if } n = (a + b)n' \text{ for } n' \geq 1. \]

This follows immediately from \( \text{Corollary 3.6} \) since there is a rise for every occurrence of “ab”. If \( n = (a + b)n' \) and the composition starts with \( a \), then there are \( n' \) rises. For the composition that starts with \( b \), there is one less rise, for a total of \( 2n' - 1 \) rises. If \( n \) is not a multiple of \( a + b \), then the composition starts with \( r \), where \( n = (a + b)n' + r \). In either case, there are exactly \( n' \) rises, as there are \( n' \) occurrences of “ab” in the composition.

3.4. Carlitz palindromic compositions. A Carlitz palindromic composition of \( n \) is both a Carlitz composition and a palindromic composition. Let \( F_n^A = E_n^A \cap P_n^A \) be the set of all Carlitz palindromic compositions of \( n \) with parts in \( A \).
3.4.1. Number of Carlitz palindromic compositions. We denote the generating function for the number of Carlitz palindromic compositions of \( n \) with \( m \) parts in \( A \) with respect to the number of rises by \( F_A(x; y; r) \), that is,
\[
F_A(x; y; r) = \sum_{n \geq 0} \sum_{\sigma \in F_n^A} x^n y^{\text{parts}(\sigma)} y^{\text{rises}(\sigma)}.
\]
Note that \( F_A(x; y; r) = P_A(x; y; r, 0, 1) \). Using Theorem \ref{thm:generating}(ii) for \( \ell = 0 \) and \( d = 1 \) gives the following result.

**Corollary 3.7.** Let \( A = \{a_1, \ldots, a_k\} \) be any ordered subset of \( \mathbb{N} \). Then
\[
F_A(x; y; r) = 1 + \sum_{i=1}^{k} \frac{x^{a_i} y}{1 + x^{2n_i} y^{2r}}.
\]
Applying Corollary \ref{cor:generating} for \( A = \{a, b\} \) and \( y = r = 1 \) yields that the generating function for the number of Carlitz palindromic compositions of \( n \) with parts in \( \{a, b\} \) is given by
\[
\frac{1 + x^a + x^b - x^{a+b}}{1 - x^{a+b}}.
\]
Thus, the number of Carlitz palindromic compositions of \( n \) with parts in \( \{a, b\} \) is 1 if \( n = (a+b)n' + a \) or \( n = (a+b)n' + b \) for some \( n' \geq 0 \), and 0 otherwise. This follows immediately from \ref{eq:compositions}, since the Carlitz compositions for \( n = (a+b)n' \) are not symmetric.

3.4.2. Number of Rises and Drops. We now study the number of rises (drops) in all Carlitz palindromic compositions of \( n \) with \( m \) parts in \( A \). Using Corollary \ref{cor:generating} to compute \( \frac{\partial}{\partial r} F_A(x; y; r) \big|_{r=1} \) gives the following result.

**Corollary 3.8.** Let \( A = \{a_1, \ldots, a_k\} \) be any ordered subset of \( \mathbb{N} \). Then the generating function for the number of rises in all Carlitz palindromic compositions of \( n \) with \( m \) parts in \( A \) is given by
\[
\frac{\partial}{\partial r} F_A(x; y; r) \bigg|_{r=1} = \sum_{i=1}^{k} \frac{x^{a_i} y^3}{(1 + x^{2a_i} y^2)^{2r}} \left[ \sum_{i=1}^{k} \frac{x^{a_i} y^2}{1 + x^{2a_i} y^2} - 1 \right] + \sum_{i=1}^{k} \frac{x^{a_i} y}{1 + x^{2a_i} y^2} \sum_{i=1}^{k} \frac{x^{2a_i} y^2}{(1 + x^{2a_i} y^2)^{2r}} \left( 1 - \sum_{i=1}^{k} \frac{x^{2a_i} y^2}{1 + x^{2a_i} y^2} \right)^2.
\]
Applying Corollary \ref{cor:generating} for \( A = \{a, b\} \) gives that
\[
\sum_{n \geq 0} \sum_{\sigma \in F_n^A} \text{rises}(\sigma) x^n y^{\text{parts}(\sigma)} = \frac{x^{a+b} y^3(x^a + x^b)}{(1 - x^{a+b} y^2)^2} = (x^a + x^b) \sum_{m \geq 1} m x^m (a+b) y^{2m+1}.
\]
In particular, setting \( y = 1 \) in the expression above yields that the generating function for the number of rises (drops) in all Carlitz palindromic compositions of \( n \) with parts in \( \{a, b\} \) is given by
\[
\frac{x^{a+b} (x^a + x^b)}{(1 - x^{a+b})^2}.
\]
Thus, the number of rises (drops) in the Carlitz palindromic compositions of \( n \geq a + b \) with parts in \( \{a, b\} \) is given by

\[ n' \text{ if } n = (a + b)n' + a \text{ or } n = (a + b)n' + b \text{ for } n' \geq 1 \text{ and } 0 \text{ otherwise.} \]

This follows immediately from (3.9), as the Carlitz compositions for \( n = (a + b)n' + a \) and \( n = (a + b)n' + b \) are symmetric.

### 3.5. Partitions with parts in \( A \)

A partition \( \sigma \) of \( n \) is a composition of \( n \) with \( \text{rises}(\sigma) = 0 \). Let \( G_n^A \) be the set of all partitions of \( n \) with parts in \( A \).

#### 3.5.1. Number of partitions

We denote the generating function for the number of partitions of \( n \) with \( m \) parts in \( A \) with respect to the number of levels and drops by

\[ G_A(x; y; \ell, d) = \sum_{n \geq 0} \sum_{\sigma \in G_n^A} x^n y^{\text{parts}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)}. \]

Note that \( G_A(x; y; \ell, d) = C_A(x; y; 0, \ell, d) \). Using Theorem 2.1(i) for \( r = 0 \) we get the following result.

**Corollary 3.9.** Let \( A = \{a_1, \ldots, a_k\} \) be any ordered subset of \( \mathbb{N} \). Then the generating function \( G_A(x; y; \ell, d) \) is given by

\[ 1 + \frac{\sum_{j=1}^{k} \left( \frac{x^{a_j} y}{1-x^{a_j} y(\ell-d)} \prod_{i=1}^{j-1} \frac{1-x^{a_i} y}{1-x^{a_i} y(\ell-d)} \right)}{1 - d \sum_{j=1}^{k} \left( \frac{x^{a_j} y}{1-x^{a_j} y(\ell-d)} \prod_{i=1}^{j-1} \frac{1-x^{a_i} y}{1-x^{a_i} y(\ell-d)} \right)}. \]

For example, if we apply Corollary 3.9 for \( A = \mathbb{N} \) and \( \ell = d = 1 \) and use the identity

\[ \sum_{j=1}^{k} x^{a_j} \prod_{i=1}^{j-1} (1-x^{a_i}) = \frac{1}{\alpha} \left( 1 - \prod_{j=1}^{k} (1-x^{a_j}) \right), \]

then we get that the generating function for the number of partitions of \( n \) with \( m \) parts in \( A = \mathbb{N} \) is given by

\[ F_\mathbb{N}(x; y; 1, 1) = \prod_{j \geq 1} (1-x^j y)^{-1}. \]

Note that the identity in (3.10) follows from the fact that

\[ 1 - \alpha \sum_{j=1}^{k} x^{a_j} \prod_{i=1}^{j-1} (1-x^{a_i}) = \left( 1 - \prod_{j=1}^{k} (1-x^{a_j}) \right), \]

which can be easily proved by induction.

If we apply Corollary 3.9 to \( A = \{a, b\} \) and set \( y = \ell = d = 1 \), then we get that the generating function for the number of partitions of \( n \) with parts in \( A \) is given by

\[ \frac{1}{1-x^a - x^b (1-x^a)} = \frac{1}{(1-x^a)(1-x^b)}. \]

In particular, if \( A = \{1, k\} \) then we have that the number of partitions of \( n \) with parts in \( A \) is given by \( \lfloor (n+k)/k \rfloor \). This can be easily explained by the following observation. For \( n \in [n'k, (n'+1)k) \), the only partitions are those consisting of all 1’s, one \( k \) and all 1’s, . . . , \( n' \) \( k \)’s and all 1’s, for a total of \( n'+1 = \lfloor (n+k)/k \rfloor \) partitions.
Another interesting example, namely setting \( \ell = 0 \) and \( d = 1 \) in Corollary 3.9 gives that the generating function for the number of partitions of \( n \) with \( m \) parts in \( A \) in which no adjacent parts are the same is given by

\[
G_A(x; y; 0, 1) = \frac{1}{1 - \sum_{j=1}^{k} x^{a_j} y \prod_{i=1}^{j-1} (1 + x^{a_i} y) - 1} = \prod_{j=1}^{k} (1 + x^{a_j} y),
\]

where the second equality is easily proved by induction. In particular, the generating function for the number of partitions of \( n \) with parts in \( \mathbb{N} \) in which no adjacent parts are the same is given by \( \prod_{j \geq 1} (1 + x^{j}) \).

3.5.2. **Number of levels and drops.** We now study the number of levels and drops in all partitions of \( n \). Using Corollary 3.9 to compute \( \frac{\partial}{\partial \ell} G_A(x; y; \ell, 1) \big|_{\ell=1} \) and \( \frac{\partial}{\partial d} G_A(x; y; 1, d) \big|_{d=1} \), we get the following result.

**Corollary 3.10.** Let \( A = \{a_1, \ldots, a_k\} \) be any ordered subset of \( \mathbb{N} \). Then the generating function \( \sum_{n \geq 0} \sum_{\sigma \in G_A} \text{levels}(\sigma) x^n y^\text{parts}(\sigma) \) is given by

\[
\sum_{j=1}^{k} \left( \frac{x^{a_j} y^2 \prod_{i=1}^{j-1} (1 - x^{a_i} y)}{\prod_{j=1}^{k} (1 - x^{a_j} y)^2} \right) - \sum_{j=1}^{k} \left( \frac{x^{a_j} y \prod_{i=1}^{j-1} (1 - x^{a_i} y) \sum_{i=1}^{j-1} x^{a_i} y^2}{\prod_{j=1}^{k} (1 - x^{a_j} y)^2} \right),
\]

and the generating function \( \sum_{n \geq 0} \sum_{\sigma \in G_A} \text{drops}(\sigma) x^n y^\text{parts}(\sigma) \) is given by

\[
\left( 1 - \prod_{j=1}^{k} (1 - x^{a_j} y)^2 \right) - y^2 \sum_{j=1}^{k} \left( \frac{x^{a_j} y \prod_{i=1}^{j-1} (1 - x^{a_i} y) \sum_{i=1}^{j} x^{a_i}}{\prod_{j=1}^{k} (1 - x^{a_j} y)^2} \right).
\]

**Proof.** We give a sketch of the proof for the first generating function. Since \( G_A(x; y; \ell, 1) = \frac{S(\ell)}{1 - S(\ell)} \), where

\[
S(\ell) = \sum_{j=1}^{k} \left( \frac{x^{a_j} y}{1 - x^{a_j} y (\ell - 1)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i} y \ell}{1 - x^{a_i} y (\ell - 1)} \right) = \sum_{j=1}^{k} \left( g_j(\ell) \prod_{i=1}^{j-1} f_i(\ell) \right),
\]

we get that

\[
\frac{\partial}{\partial \ell} G_A(x; y; \ell, 1) = \frac{\partial}{\partial \ell} S(\ell) \frac{1}{(1 - S(\ell))^2} = \sum_{j=1}^{k} \frac{\partial}{\partial \ell} g_j(\ell) \prod_{i=1}^{j-1} f_i(\ell) + g_j(\ell) \frac{\partial}{\partial \ell} \prod_{i=1}^{j-1} f_i(\ell)}{(1 - S(\ell))^2}.
\]

Using Equation 3.10 gives that \( \frac{\partial}{\partial \ell} \prod_{i=1}^{j-1} f_i(\ell) = - \prod_{i=1}^{k} f_i(\ell) \sum_{j=1}^{k} \frac{a^{2a_j} y^2}{(1 - x^{a_j} y) (1 - x^{a_j} y (\ell - 1))} \). Computing \( \frac{\partial}{\partial \ell} g_j(\ell) \), setting \( \ell = 1 \) in the expression for \( \frac{\partial}{\partial d} G_A(x; y; \ell, 1) \), then using Equation 3.10 to simplify the denominator gives the stated result. \( \square \)

Applying Corollary 3.10 to \( A = \{a, b\} \) gives that the generating function for the number of levels in the partitions of \( n \) with \( m \) parts in \( \{a, b\} \) is given by

\[
\frac{x^{2a} y^2 (1 - x^b y) + x^{2b} y^2 (1 - x^a y)}{(1 - x^a y)^2 (1 - x^b y)^2}.
\]
In particular, the generating function for the number of levels in the partitions of \( n \) with parts in \( \{1, 2\} \) is given by

\[
\frac{x^2(1 - x^3)}{(1 - x)^4(1 + x)^2}.
\]

From the second part of Corollary 3.10 we get for \( A = \{a, b\} \) that the generating function for the number of drops in the partitions of \( n \) with \( m \) parts in \( \{a, b\} \) is given by

\[
x^{a+b}y^2 \frac{1}{(1-x^a y)(1-x^b y)}.
\]

In particular, setting \( y = 1 \) in the above expression yields that the generating function for the number of drops in all partitions of \( n \) with parts in \( \{1, k\} \) is given by

\[
x^{k+1} \frac{1}{(1-x)(1-x^k)}.
\]

Thus, the number of drops in the partitions of \( n \) with parts in \( \{1, k\} \) is \( \lfloor (n - 1)/k \rfloor \). This again follows from the specific structure of the partitions with parts in \( \{1, k\} \). A single drop occurs in all the partitions that do not consist of either all 1’s or all \( k \)’s. Thus, for \( n \in [n'k + 1, (n' + 1)k) \), there are exactly \( n' = \lfloor (n - 1)/k \rfloor \) drops.

4. Concluding Remarks

We have provided a very general framework for answering questions concerning the number of compositions, number of parts, and number of rises, levels and drops in all compositions of \( n \) with parts in \( A \). We have used this framework to investigate compositions, palindromic compositions, Carlitz compositions, Carlitz palindromic compositions and partitions of \( n \). Our results generalize work by several authors, and we have applied our results to the specific sets studied previously, which has led to several new results. In addition, our results can be applied to any set \( A \subseteq \mathbb{N} \), which will allow for further study of special cases.

In addition, the techniques used in this paper can be used to investigate products among the number of rises, levels and drops which show interesting connections to the Fibonacci sequence, one of the reasons Alladi and Hoggatt investigated the these quantities for compositions with summands 1 and 2. For example, by computing the derivative with respect to \( d \) twice in Theorem 1.1 (ii) and setting \( y = r = \ell = 1 \), we get that

\[
\sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{drops}(\sigma)(\text{drops}(\sigma) - 1)x^n = \frac{2x^6}{(1 - x - x^2)^3} = 2x^3 \sum_{n \geq 3} \left( \sum_{a+b+c=n} F_a F_b F_c \right) x^n,
\]

i.e., a convolution of three Fibonacci sequences. However, the formulas for the various products become more complicated, and not as easy to evaluate.

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