Emergent Quantum Coherence from Instabilities of a Perturbed Fractionalized Spin Liquid

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Intrinsically unconventional non-local order parameters and non-trivial ground state degeneracies with fractionalized excitations are hallmarks of states of matter with topological order. The fate of these features in their suitably perturbed counterpart(s) is even more intriguing: while small perturbations are not expected to affect a robust topological order, sizable perturbations can lead to novel routes to onset of apparently more conventional order(s). However, one of the major fundamental issues is that of the observability of such states in realistic condensed matter systems. Using a simplest perturbed honeycomb Kitaev model as a template, we present a specific instance of a rigorous featureless spin liquid giving way to a partial topological ordered state coexisting with a wide staircase of novel ‘excitonic’ orders of fractionalized entities. On the fundamental front, these emergent orders provide quasi-exact realizations for a range of strange features, from nodal spin-metals, fractionalization in selective-Mott states, to Luttinger surfaces and hidden coherence, all characteristics of the influential resonating- valence-bond as well as fractionalized-Fermi-Liquid views in the context of unique anomalies of cuprate superconductors. As a specific and novel application, we propose how a previously proposed and suitably engineered honeycomb lattice Josephson-junction array may exhibit such novel effects by flux tuning. Our study may thus aid in applications of such engineered systems to plasmonics, terahertz generators and emitters, and, possibly, to topological quantum computation.

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Topological excitations have fascinated physicists for many decades. While individual solitons and vortices have been observed as emergent objects in many areas of physics, their higher-dimensional cousins, such as magnetic monopoles (which arise in any grand unified theories) responsible for charge quantization, have remained mostly elusive. Remarkably, recent advances in condensed matter have reached a stage where the ability to observe and even manipulate such exotic excitations seem to be within the realm of possibility. Cold-atom setups, topological insulators and appropriately engineered Josephson junction arrays (JJA) are instances that seem poised to offer renewed insight into formation and dynamics of topological excitations arising from ground states with topological order (TO). In many instances, however, ensuring their stability is a potential hurdle: e.g., in extant cold-atom settings, skyrmions tend to expand or shrink, or to slip away from the atomic trap. Another, more exacting hurdle is to unearth specific examples of systems capable of exhibiting stable (symmetry protected) TO.

The birth of the Kitaev model (KM), an exactly solvable model of two-dimensional quantum spins on a honeycomb lattice, led to a spurt of proposals to ‘simulate’ its TO ground state in various contexts. Here, obvious obstacles present themselves in (i) the lack of real systems that can be realized by a pure KM, and (ii) theoretically, our relative lack of understanding of the effects of various perturbations applied to a KM. The instability of the TO state to a host of relevant perturbations has meanwhile been studied, predominantly by numerical means. A deeper understanding, and, in particular, a coherent picture of such instabilities in terms of changes in the spectrum of topological (elementary) excitations of the TO state has, however, remained elusive. Given that the KM hosts a topological fractionalized spin liquid phase, novel, unanticipated ordered states, depending upon specific nature of perturbations, are expected theoretically. Classifying such novel orders and inquiring about their observability in natural or engineered condensed matter systems holds the promise to offer invaluable insight into these issues, apart from the attractive motivation of application(s) to fault-tolerant quantum computing.

We consider these issues in a simplest version of a perturbed KM in detail. We are motivated by recent proposals for realization of topological excitations and their manipulation for fault-tolerant quantum computing in perturbed Kitaev models defined on suitably engineered JJA. In contrast to paucity of “conventional” spin systems exhibiting KM physics, there is hope that such a perturbed KM involving assemblies of two-level Cooper-pair (charge) qubits are realizable in JJA; however, for the realistic range of physical parameters, the JJA on a honeycomb lattice is always perturbed by ‘magnetic fields’ along x and/or z bonds. Remarkably therefore, such JJA offer attractive avenues to investigate the above issues in a realistic context. This may also emerge in cold-atom settings involving $p$-wave Bose-Einstein condensation.
densates if the topological excitations can be stabilized long enough. These issues are also of paramount importance in the context of renewed hope to facilitate observation of topological liquids in real multi-orbital Mott-Hubbard systems with built-in geometrical frustration (GF). Models that are variations of the KM and compass models are believed to apply even to pseudocubic and honeycomb lattice-based transition-metal oxides (TMO) though, in this context, combination of spin-orbit coupling, lattice distortions and real Heisenberg-like super-exchanges can inevitably make such descriptions untenable so that great care must be exercised on a case-by-case basis. Interesting issues in this context relate to conditions under which a fractionalized orbital liquid can emerge, and whether analogues of orbital-Kondo destroying QCPs exist in TMOs?

**Simplest Perturbed Kitaev Model**

Motivated by these issues, we study the simplest version of the perturbed KM: the Kitaev model in an ‘external’ Zeeman magnetic field:

\[
H = -\sum_\alpha J_\alpha \sum_{<i,j>} S^\alpha_i S^\alpha_j - h_z \sum_i S^z_i \tag{1}
\]

Kitaev’s famed solution relies on mapping spin-1/2s to bilinears of four \((b^\alpha, b^\alpha, b^\alpha, c)\) Majorana fermions along the three \((xx, yy, zz)\) bonds. We start with an alternative but equivalent approach due to Chen and H[11] and Nussinov[12] that exploits a duality mapping of the spins via a \(d = 2\) Jordan-Wigner (JW) fermionization of \(H\). Remarkably, since this affords an exact mapping between topological and classical orders in \(d = 2\) spin systems, studying TO(s), their excitations and instabilities is transmuted into the better-known language of well-known models hosting classical order(s) characterized by symmetry-breaking. Using

\[
\sigma^+_i = 2[\Pi_{j'<i,j}\sigma^+_{j'}][\Pi_{j'<i}\sigma^+_{j}]c^\dagger_{ij} \tag{2}
\]

and

\[
\sigma^z_i = (2c^\dagger_{ij}c_{ij} - 1) \tag{3}
\]

and defining Majorana fermions on the two (“black” and “white”) sublattice sites of the honeycomb or brickwall lattice as \(A_w = (c - c^\dagger)_w / i\), \(B_w = (c + c^\dagger)_w\) and \(A_b = (c + c^\dagger)_b\), \(B_b = (c - c^\dagger)_b / i\), followed by the introduction of fermions \(c = (A_w + iA_b)/2\), \(c^\dagger = (A_w - iA_b)/2\), we find that the Hamiltonian of the Kitaev model in an external Zeeman field can thus be brought to the following instructive form by the above \(d = 2\) non-local JW transformation to spinless fermions (this also bares the exact emergent low-dimensional gauge symmetries, as pointed out by Fradkin et al[11]):

\[
H_K = \sum_q (\varepsilon_q c^\dagger_q c_q + i\Delta_q (c^\dagger_q c_{-q} + h.c)] + \frac{J}{4} \sum_i (2n_{\alpha,i} - 1)(c^\dagger_i c_i - 1) \tag{4}
\]

and \(H_z = 2h_z \sum_i (c^\dagger_i \alpha_i + h.c).\) When \(h_z = 0\), this is just a spinless Falicov-Kimball model with a local hybridization \(h_z\) in \(d = 2\), but with an additional ‘spin’-triplet pairing term, whence topologically ordered ground state of \(H_K\) now appears as a spin-triplet BCS superfluid of JW fermions. Here, \(\alpha_i = iB_1B_2 = \pm 1\) is now a dynamic gauge field variable on the center of each \(zz\)-links of the honeycomb Kitaev model. This transforms the original spin model on a honeycomb lattice to a two-orbital spinless model on an effective square lattice. While \([\alpha_i, H] = 0\) for all \(i\) (reminiscent of the spinless FK model) ensures the exact solvability of the KM when \(h_z = 0\), a finite Zeeman field immediately spoils the beautiful integrability of \(H\). Interestingly, however, finite \(h_z\) simply translates to a local hybridization between the \(c, c^\dagger\) JW fermions. These observations are crucial for an ‘almost exact’ analysis of the KM in a Zeeman field as detailed below.

Before going over to this, we present an alternative way to motivate the instabilities of the TO phase of \(H_K\). In Majorana fermion language, a partial local \(Z_2\) symmetry associated with \([b^\alpha_i \alpha^\dagger_i, H_K] = 0\) for all \((ij)\) and \(\alpha = x, y\) still holds when \(h_z \neq 0\) (only \([b^\alpha_i \alpha^\dagger_i, H_K] \neq 0\) along \(zz\)-links anymore, since \(H_z = i \sum_i b^\dagger_i c_i\) mixes \(b^\alpha, c\) on each site). This directly implies a dimensional reduction and associated lower-d (here, \(d = 1\)) gauge-like symmetries (GLS) naturally emerge. The only order that survives the action of these GLS is a directional spin-nematic order[12], whose order parameter, \((n) = (\sigma^x_1 \sigma^x_2 - \sigma^x_1 \sigma^x_2)\) is the only one that remains invariant under the emergent \(d = 1\) GLS: thus, the TO of \(H_K\) is only partially “melted” by a Zeeman field. The ultra-short-ranged spin-liquid phase known for \(h_z = 0\) thus immediately becomes unstable to a deconfined critical liquid inherently unstable to incipient spin-nematic order (on \(xx, yy, zig-zag\) links) co-existing with a (with finite flux on \(zz\)-links) field-induced \(zz\)-spin order, for any \(h_z \neq 0\) at \(T = 0\!\) It also turns out that the \(zz\)-spin correlation function, \(\chi_{zz}(i-j) \sim (i-j)^{-4}\) (not shown, proof essentially identical to Tikhonov et al[12]) using a simple bubble composed of the \(G_{a,b}(i-j), G_{a,c}(i-j)\) propagators for \(H\) in Majorana language. Remarkably, however, field-induced magnetization in the JW-fermionized model now appears as an excitonic condensate of JW-fermions. This rigorous argument bares novel order arising from a topological state.

In addition, for special values of \(h_z\), the new dynamical \(Z_2\) flux can, by itself, self-organize into stripe-ordered patterns, whose origin is the competition between “Mott localization” of the flux and its hybridization-induced delocalization. This tendency is expected due to the fact
that \( n_{\text{loc}} = (1 + \alpha_i)/2 \) play the role of immobile fermions in the FK-like model for \( h_z = 0 \), where such crystal states are rigorously known. In the present case, such crystals occurring with a large range of periodic structures have a novel origin as condensation of topological kink dipoles. This is clear from the observation that ordered patterns of the \( \langle \alpha_i \rangle \) are nothing but those associated with a Bose-Einstein condensate of \( \langle iB_1B_2 \rangle \) on each zz-bond, i.e., of topological kink dipoles. On the other hand, field-induced metamagnetic transitions, now associated with ‘excitonic solid’ states, are also generically expected from a Falicov-Kimball model with hybridization.

We now proceed to analyze \( H = H_K + H_z \) within dynamical mean-field theory (DMFT). Our choice is motivated by features special for \( H_K : \) (i) when \( h_z = 0 \), the KM possesses ultra-short-range spin correlations \( \approx \), rigorously uncorrelated beyond nearest neighbors, (ii) a DMFT-inspired approach thus gives the first instance of an exact dynamical spin fluctuation spectrum in a \( d = 2 \) model, and (iii) the fact that spin correlations on all length scales along \( xx, yy \) bonds have been exactly subsumed into one-fermion hopping and \( p \)-wave BCs pairing terms (this is not the result of an approximation for the Kitaev model: both \( xx, yy \) spin correlations exactly transform into free fermion terms) obviates the usual difficulty of ignoring long-range spatial correlations, wherein any mean-field theory, including DMFT, is justifiably subjected to a critique in transitions, now associated with local Hubbard interaction. Finally, the above-discussed emergent lower \( d(= 1) \) GLS are exactly encoded in the non-interacting c-fermion part of \( H_K \). This novelty, again specific for the KM, thus makes the DMFT almost exact and allows for exhaustive analysis.

In the TO phase \( (h_z = 0, c) \)-fermions thus experience a strongly fluctuating \( (Z_2) \) potential, which ‘suddenly’ switches between \( \pm J_z/2 \) during every hop of the \( c \)-fermions. If the pairing term were absent, this would constitute a complete reconstruction of the entire Fermi sea due to the Anderson-Nozieres-de Dominics “orthogonality catastrophe” (OC). The resulting incoherence ensures that the \( c \)-fermions lose their phase coherence as they encounter spatially separated scattering centers on neighboring \( (i,i \pm e_a \) with \( a = x,y \)) zz-bond-centered sites: along with ultra-short spin correlation length, this is what makes the ‘local’ approximation exact for the case here. Actually, with \( h_z = 0, c \) has a Dirac-like form, which cuts off the infra-red singular behavior and the OC, since the DOS vanishes linearly at \( \omega = 0(= \epsilon_F) \); this is not true anymore when \( h_z \) is finite and Fermi pockets, corresponding to finite DOS at \( \omega = 0 \), are expected to appear in the c-fermion spectrum. However, this is still not enough, as the local hybridization also produces finite recoil of the \( \alpha \)-fermion, which, in principle must cut-off the X-ray-edge singularities in spectral functions. But since any finite \( h_z \) will directly lead to an excitonic condensate at \( T = 0 \), this provides an additional route to removal of infra-red singularities by coherence-restoring instability to field-induced order. This is precisely what we find (see below).

We have solved the two-band spinless-fermion Anderson lattice model within DMFT as detailed above. The impurity model of DMFT is solved by multi-orbital iterated perturbation theory as an impurity solver: its proven quantitative accuracy vis-a-vis exact QMC results for the spinful Anderson lattice model ensures its efficacy in this case. Local quantities like the \( c, \alpha \)-fermion spectral functions and field-induced magnetization, now simply equal to a local excitonic \( c,\alpha \) spectral functions and field-induced magnetization, now simply equal to a local excitonic spins, are very reliably captured by DMFT, even in \( d = 2 \). To facilitate discussion of results, we first show the evolution of \( m(h_z) \) vs. \( h_z \) in Fig. 1. When \( J_z << J_y = J_x, m(h_z) \) rises monotonically with \( h_z \), as expected of a field-polarized paramagnetic phase. However, the underlying topological origin of this phase is underscored by observing that \( m(h_z) \approx h_z^\eta \) with \( 1 > \eta > 0.78 \) for \( 0 \leq J_z \leq 0.25(=J_x/4) \): for larger \( J_z \), \( \eta(J_z) \) reverses its slope (Fig. 1 lower panel (inset)), increasing monotonically toward 1.0. The apparently special character of \( J_z/J_x = 1/4 \) is cleared...
upon observing that this is precisely the (mean-field) excitation energy of the three flat bands that arise in a mean-field theory (related to the original resonating valence bond (RVB) description in the cuprate context\textsuperscript{17} distinct to ours\textsuperscript{18}. In our language, it marks the point where the $\alpha$-fermion level and concomitant excitonic tendency begins to play a role for $J_z/J_x \geq 1/4$.

Additional evidence in favor of this interpretation comes by observing that $m(h_z)$ starts developing plateaus (or incompressible JW-excitonic solid phases) as a function of $h_z$ only for $J_z/J_x > 1/4$.

A remarkable finding is that for $J_z/J_x > 1/4$, DMFT results as a function of $h_z$ unearth an extensive sequence of magnetization plateaus, now associated with incompressible excitonic solid phases, separated by excitonic “fluid” phases. These are thus quantum phase transitions between sequences of “topological crystal” states, each sandwiched between topological Bose-Einstein condensates of excitonic kink dipoles. Remarkably, we find that these topological crystal phases occur (Fig. 2) for a range of odd-denominator valued magnetizations, as well as at certain even denomiators (relative to saturation, at $m(h_z)/m_{sat} = 1/16, 1/12, 1/11, 1/9, 1/8, 1/7, 2/13, 1/6, 2/11, 1/5, 2/9, 1/4, 2/7, 1/3, 3/8, 3/7, 2/5, 4/7, 5/8, 3/4,...). Odd denominator plateaus in the off-diagonal conductivity are well-known in the other famed instance, the fractional quantum Hall effect (FQHE), which is also the only other example of a real system rigorously exhibiting topological order. In spin systems, such exotica have also been found and widely discussed in frustrated Shastry-Sutherland models\textsuperscript{19}, where they correspond to triplon crystals emerging due to competition between frustration and field-induced magnetism (even-denominator plateaus are also visible there). So our plateaus are not related to FQHE-like physics, even though they arise from a perturbed (field-induced) partially TO phase. Nevertheless, in Fig. 2 we exhibit $m(h_z)$ and $\chi(h_z) = (d/dh_z)m(h_z)$ together to illustrate “analogies” with FQHE. We interpret the oscillations in $\chi(h_z)$ as “de Haas van-Alphen (dHvA)” or “Shubnikov de-Haas (SdH)” quantum oscillations of topological JW fermions in a partially magnetized spin liquid phase. This must reflect “hidden” coherence in a spin liquid: it is indeed interesting that the possibility of hidden off-diagonal long-range order (ODLRO) was conjectured by Anderson in the first RVB proposal\textsuperscript{20} in 1973. Here, a Zeeman field partially polarizes the spinons reflecting in magnetization plateaus, and these ordered phases “ride on top” of a partially topological ordered state (a remnant of the original field-free Kitaev model). This component has in-built coherence, thanks to the $p$-wave BCS term in $H_K$, manifesting as coherent quantum oscillations we unearth. Thus, in stark contrast to conventional paramagnets, the TO phase of the Kitaev model undergoes an intricate sequence of partial ordering “solid” instabilities, co-existing with the remnant of the TO fractionalized spin liquid (cf. emergent $d=1$ GLS left untouched by $h_z$ in the KM), in an external Zeeman field before reaching full saturation.

The DMFT spectral functions reveal deeper insights. Clear orbital selectivity is seen in $c$ and $\alpha$-fermion spectral functions, and this is a direct fall-out of the in-
built differentiation between these states in the JW-femtonized $H_K$ itself. While hybridization and local Hubbard correlations conspire to produce a JW “Kondo” insulator for $\alpha$-fermions, they produce large-scale spectral changes in the $c$-fermion sector. Focussing on the variations in $\rho_c(\omega)$, $\rho_\alpha(\omega)$ across representative samples where magnetization steps appear, it is clear that large-scale reshuffling of dynamical spectral weight across energy scales of order the band-width (equal to $2J_c$ here) accompany emergence of each plateau. Thus, the plateau structure has its origin in the ubiquitous competition between Mott localization (Hubbard term) and itinerance (hybridization and hopping) in $H$, and the field simply tunes the balance between these tendencies (thus, any simple non-interacting, or an effective, static Hartree-Fock interpretations that ignore the ubiquitous dynamical spectral weight transfer yield incorrect excitation spectra in general, and thus may miss out aspects found here), now by favoring the tendency to exciton formation. This is reminiscent of the state of affairs characterizing the FL* or fractionalized-Fermi liquid view proposed in the context of f-electron quantum criticality. By analogy, the topological liquid phase of $H_K$ is thus also an in-built topological FL* state where $S = 1/2$ degrees of freedom of $H_K$ fractionalize into $c, \alpha$ JW-fermions, followed by selective localization of the $\alpha$-fermions. An external magnetic field does give the $\alpha$-fermions quantum dynamics, but, remarkably enough, maintains “orbital” selectivity (we find a selective Kondo, rather than Mott, phase here).

More unanticipated features reveal themselves from analysis of the DMFT band structures. In Fig. 3 we exhibit the c-fermion dispersions and Luttinger surfaces obtained as usual from the DMFT by setting $G_{cc}^{-1}(k, \omega_k) = 0$ (dispersions) and evaluating the locus of ks corresponding to $\omega_k = 0 (= E_F)$ (“Fermi” surfaces). Remarkably, we find that the “Fermi surface” corresponds to zeros, rather than poles, of $G_{cc}(k, \omega)$: this is evident from the structure of $G_{cc}(k, \omega)$ itself, which features an anomalous “BCS”-like self-energy structure given by $\Sigma_{an}(k, \omega) = \frac{\Delta^2}{\omega + \epsilon_{c,k} - \epsilon_v}$, so that the poles of $\Sigma_{an}$ appear as zeros of $G_{cc}(k, \omega)$. Thus, this is a generalized Luttinger surface, rather than a conventional Fermi surface. A similar phenomenon is known to occur in correlation-driven selective-Mott transitions in f-electron and multi-orbital systems, where the same phenomenon signals an orbital-selective Mott phase with accompanying local non-FL behavior. In our case, clear changes in topology of the Luttinger surface occur at each magnetization plateau, and are thus intimately linked to correlation-driven Lifshitz-like transitions accompanying each field-induced magnetization plateau. In JW-fermion language, each of the excitonic solid-to-excitonic fluid quantum phase transitions is thence seen to exhibit a crucial aspect of topological criticality, manifested by proliferation of zeros to the poles of $G_{cc}(k, \omega)$. This also presents a concrete microscopic realization of the phenomenological YRZ ansatz used with some success in context of underdoped cuprates. Moreover, given the $p$-wave BCS $c$-JW pair term in $H_K$, the $c$-fermion spectral function is that of $p$-wave nodal Bogoliubov quasiparticles: this permits association of this state with a $p$-wave nodal spin semimetal. Ultimately, these features are novel signatures of the underlying partial TO surviving in the partially magnetized phases). A Zeeman field results in the transfer of dynamical spectral weight from the fractionalized spin liquid component to the excitonic solid, but it is indeed remarkable, thanks to the undisturbed emergent $d = 1$-GLS, that it partially preserves the original TO.

Finally, the large-$J_z$ limit of $H_K$ is known to be nothing but the Kitaev Toric Code (TC) model. First, we notice that this must imply a Mott-like insulating phase of the JW fermions. In this limit, the no double-occupancy constraint of a $c, \alpha$ fermion on each site is naturally implemented by a Gutzwiller projector, $P_G = \Pi_{\{1-n_{ic}\alpha_{ic}\}}$ acting on the $p$-wave BCS ground state. Explicitly, $|\Psi\rangle_{TC} = P_G|\Psi_{cc}(u_k + v_k^\dagger c_{-k}^\dagger)|\langle vac|$ is the $c$-fermion vacuum. But this is precisely the Gutzwiller-projected ($p$-wave) BCS state, long known as one of the best variational states describing a $(d$-wave) RVB spin liquid state in the context of the famed resonating valence bond (RVB) description of the cuprate. Thus, we find that the ground state of the TC model is just the Gutzwiller-projected $p$-wave BCS, or $p$-wave RVB state! Finally, this suggests the enticing possibility of investigating TC physics and its instabilities in future in terms hitherto developed for quantum spin liquids in quantum Heisenberg model. Alternatively, since an arbitrarily oriented magnetic field produces a single-fermion next-nearest-neighbor hopping term of $c$-fermions, the TC model and its non-abelian excitations can also be directly studied by our DMFT route.

We now detail a specific quasi-realistic scenario where our findings could aid finding of novel orders in a real-world set-up:

1. **Kitaev Josephson Junction Arrays** As advertised, there have been proposals for suitably engineering using Josephson charge qubits coupled in different ways along three directions of a honeycomb network. Interestingly, such proposals always require an external magnetic field along $xz$ and/or $zz$ bonds. Thus, our results are directly applicable to the engineered JJA of You et al. as long as we identify the Kitaev spins as pseudospin-1/2s associated with a pair charge qubit: defining $| \uparrow \rangle_i, \downarrow \rangle_i$ as two charge states with Cooper pair number $n_i = 1, 0$, the $S = 1/2$ representation permits a rigorous mapping, $n_i = (1 - S_i^z)/2, \cos \phi_i = S_i^z/2$ and $\sin \phi_i = -S_i^y/2$ with $S_i$ being the pair phase at $i$, conjugate to the number as implied by $\phi_i = -id\phi_i$. If each charge qubit is placed at an “optimal” point where $n_g = 1/2$, and $J_{x} = J_{y} < J_{z}$ is considered, one ends up with the anisotropic Kitaev model in an “external field” $h \sum_i S_i^z$ (not $hS_i^z$). However, this is of no consequence, as we obtain our model simply by relabelling $S_i^x \rightarrow S_i^y, S_i^y \rightarrow -S_i^z$. Further, we
then have $J_x < J_y (= J)$, which is precisely the limit we are concerned with. Remarkably, since $\hbar = E_{Jx}(\Phi_i) = 2E_{Jx}\cos(\pi\Phi_i/\Phi_0)$ with $\Phi_0 = \pi\hbar/e$ the flux quantum in the JJA, changing the Josephson coupling energy offers a subtle knob to tune the Zeeman field in the KM, and, in fact, in the JJA context, a real external magnetic field or adding non-magnetic impurities can change $E_{Jx}(\Phi_i)$.

A direct application of our results is now possible. We immediately see that varying the flux through a qubit leads to a generation of a whole host of ordered “JW-immediate see that varying the flux through a qubit
ODLRO$^{20}$ long believed to be a feature of RVB liquids. Here, they arise due to a combination of $p$-wave BCS features and their interplay with emergent solid ordered phases of topological kink-dipoles, which can be interpreted to be an “emergent” and novel ODLRO involving kink-dipole condensation. Other subtle manifestations of ODLRO, such as topological analogues of Little-Parks experiment, etc, could also be imagined in the JJA context. Thus, our work opens a door to investigate novel topological analogues of celebrated macroscopic quantum coherence effects like Josephson plasmonics well-known in the in the context of conventional symmetry-broken states in the Landau paradigm of phase transitions. Further, to the extent that it is difficult to generate single JW fermions by splitting up JW excitons, the states we find have a much more efficient topological protection (the topological liquid phases are inherently symmetry-protected topological (SPT) phases), these findings can be exploited for possible topological quantum computation (TQC) engineering, along lines proposed by You et al.$^3$

Finally, nematic order, the only order allowed under emergent $d = 1$-GLS, must obtain when $J_x \neq J_y$. This can obtain in the JJA context in the more general case when a combination of “magnetic” fields, $\sum_i (h_x S_i^x + h_z S_i^z)$ supplement $H_K$ and $h_x \neq h_z$ but $J_z > J_x, J_y >> h_x, h_z$. But when $J_x = J_y, h_z = 0$, thanks to the proliferation of topological defects associated with emergent $d = 1$ GLS, it will again be a topological (Mott) liquid of JW-fermions as found here.

Conclusion

To conclude, we have presented strong evidence linking a large sequence of “excitonic ordered solid” phases of the Zeeman-field driven KM to novel instabilities of a rigorously known TO phase. These could be touted as topological analogues of the famous ”Barkhausen steps” encountered in the magnetization process of a conventional field-driven magnet. Here, however, they correspond to partially magnetically ordered complex unit-cell patterns emerging as partial ordered states of a fractionalized spin liquid, betraying their highly unconventional nature. Suitably engineered JJAs as proposed$^2$ could unearth the structures we propose by flux tuning, with novel applications to novel quantum interference phenomena, plasmonics and TQC. Finally, if examples of such perturbed KMs to real transition-metal oxides could be found, manifestations of the underlying TO emerging as lower-$d(= 1)$ gauge-like symmetries, together with their intimate link to unconventional nematic order, could reveal themselves under appropriate external perturbations. This remains an enticing perspective for future work.

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1 A. Y. Kitaev, Ann. Phys. (NY), 303, 2, (2003).
2 A Buehler et al., Nature Comm., 5, 4054 (2014).
3 J-Q. You et al., Phys Rev B 81, 014505 (2010).
4 J. Reuther et al., Phys Rev B 84, 100406(R), (2011).
5 Z. Nussinov et al., arXiv:1303.5922 (to appear in Revs. Mod. Phys).
6 G. Jackeli et al., Phys Rev Lett. 102, 017205 (2009).
7 S. Bhattacharjee et al., New Journ Phys. 14, 073015 (2012).
8 Y. Yamaji et al., Phys Rev Lett. 113, 107201 (2014).
9 H-D Chen et al., Phys Rev B 76, 193101 (2007).
10 H-D Chen et al., J. Phys A: Math and Theor. 41, 075001 (2008).
11 Z. Nussinov et al., Phys Rev B 71, 195120 (2005).
12 K. Tikhonov et al., Phys Rev Lett 106, 067203 (2011).
13 T. Kennedy, Rev. Math. Phys. 06, 901 (1994).
14 G. Baskaran et al., Phys Rev Lett 98, 274201 (2007).
15 J. Knolle et al., Phys Rev Lett. 112, 207203 (2014).
16 N Vidhyadhiraja et al., Europhys Lett. 49, 459 (2000).
17 “The Theory of Superconductivity in the High-Tc Cuprate Superconductors”, P. W. Anderson, (Princeton Univ Press, 1997).
18 F. Burnell et al., Phys Rev B 84, 125125 (2011).
19 “Introduction to Frustrated Magnetism: Models, Experiment, Theory” eds. C. Lacroix et al; (Springer, Vol. 164 (2011)).
20 P. W. Anderson et al., J. Phys: Condens Matt; 16, R755 (2004).
21 T. Senthil et al., Phys Rev Lett. 90, 216403 (2003).
22 K-Y Yang et al., Europhys Lett. 86, 37002 (2009).
23 S. Yunoki et al., Phys Rev Lett. 92, 157003 (2004).
24 V. Gurarie et al., J. Low Temp. Phys. 135, 245 (2004).
25 Y. Joglekar et al., Phys Rev B 72, 205313 (2005).