Some Separate Quasi-Asymptotics Properties of Multidimensional Distributions and Application

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Abstract: Quasi-asymptotic behavior of functions as a method has its application in observing many physical phenomena which are expressed by differential equations. The aim of the asymptotic method is to allow one to present the solution of a problem depending on the large (or small) parameter. One application of asymptotic methods in describing physical phenomena is the quasi-asymptotic approximation. The aim of this paper is to look at the quasi-asymptotic properties of multidimensional distributions by extracted variable. Distribution \( T(x_0, x) \) from \( S'(\mathbb{R}^1 \times \mathbb{R}^n) \) has the property of the separability of variables, if it can be represented in form \( T(x_0, x) = \sum \phi_i(x_0) \psi_i(x) \) where distributions, \( \phi_i(x_0) \) from \( S'(\mathbb{R}^1) \) and \( \psi_i \) from \( S(\mathbb{R}^n) \), \( x_0 \) from \( \mathbb{R}^1 \) and \( x \) is element \( \mathbb{R}^n \) different values of do not depend on each other. Distribution \( T(x_0, x) \) the element \( S'(\mathbb{R}^1 \times \mathbb{R}^n) \) is homogeneous and of order \( \alpha \) at variable \( x_0 \) is element \( \mathbb{R}^1 \) and \( x = x_1, x_2, ..., x_n \) from \( \mathbb{R}^n \) if for \( k > 0 \) it applies that \( T(kx_0, kx) = k^\alpha T(x_0, x) \). The method of separating variables is one of the most widespread methods for solving linear differential equations in mathematical physics. In this paper, the results by V. S Vladimirov are used to present the proof of the basic theorems, regarding the quasi-asymptotic behavior of multidimensional distributions by a singular variable, with the application of quasi-asymptotics to the solution of differential equations.

Keywords: Distribution Spaces, Asymptotics, Separate Quasi-Asymptotics, Multidimensional Distributions

1. Introduction

We use \( S(\mathbb{R}^n) \) to mark the standard space of the Schwartz’s rapidly decreasing functions, and \( S'(\mathbb{R}^n) \) to mark the corresponding space of the slowly increasing distributions [1, 7].

If \( f(t) \in S' \) and \( g(k) \) is a positive and a continuous function for \( k > 0 \), distribution \( f(t) \) has quasi-asymptotics at infinity (at zero) with respect to positive function \( g(k) \), if the following is valid

\[
\lim_{k \to \infty} \frac{\phi(k)}{\rho(k)} = a^\alpha
\]

(2)

where by it converges evenly along \( a \) on each compact semi-axes \((0, \infty)\). Distribution \( f(t) \in S'_+ \) is asymptotically homogeneous with respect to function \( g(k) \) of order \( \alpha \) if

\[
\frac{1}{g(k)} f(kt) \to \mathcal{C} \cdot f_{\alpha+1}(t) \text{ in } S'_+
\]

(3)

where the nucleus of fractional differentiation and integration \( f_{\alpha}(t) \in S' \) is defined by

\[
f_{\alpha}(t) = \begin{cases} \frac{\theta(t)}{\Gamma(\alpha)} t^{\alpha-1}, & \alpha > 0 \\ \frac{d^{\alpha}}{dt^{\alpha}} f_{\alpha+N}(t), & \alpha \leq 0, \alpha + N > 0 \end{cases}
\]

(4)

with \( \Gamma(\alpha) \) being the gamma function, and \( \theta(t) \) being the Heaviside function [1-3, 6, 7].

The fractional derivative, [3-5, 7], of order \( \alpha \) and...
distribution of \( f(t) \in S'(\mathbb{R}) \) is defined by the formula

\[
f^{(-\alpha)}(t) = f_{\alpha}(t) * f(t)
\]

(5)

Distribution \( T(x_0,x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \) has the property of the separability of variables, if it can be represented in form \( T(x_0,x) = \sum_{\delta} \varphi(x_0) \psi(x) \) where distributions \( \varphi(x_0) \in S'(\mathbb{R}_+^4) \) and \( \psi \in S(\mathbb{R}^n) \), \( x_0 \in \mathbb{R}_+^4 \) and \( x \in \mathbb{R}^n \) for different values of \( i \) do not depend on each other, [15].

Distribution \( T(x_0,x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \) is homogeneous and of order \( \alpha \) at variable \( x_0 \in \mathbb{R}_+^4 \) and \( x = x_1, x_2, ..., x_n \in \mathbb{R}^n \) if for \( k > 0 \) it applies that \( T(kx_0, kx) = k^\alpha T(x_0,x) \), [1, 3, 7, 8].

In other words, distribution \( T(x_0,x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \) is homogeneous and of order \( \alpha \) at variable \( x_0 \) and \( x \) if for each test function \( \phi(x_0, x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \) the following is valid

\[
(T(x_0, x), \phi \left( \frac{x_0, x}{k^4, k^4} \right)) = k^{2\alpha + n + 1} (T(x_0, x), \phi(x_0, x)), \quad k > 0.
\]

(6)

Indeed,

\[
\begin{align*}
(T(kx_0, kx), \phi(x_0, x)) &= \\
\left\{ \begin{array}{l}
\text{shift } kx_0 = x_0' \rightarrow x_0 = \frac{x_0'}{k} \\
kx = x' \rightarrow x = \frac{x'}{k}
\end{array} \right.
\end{align*}
\]

\[
= \frac{1}{k^{n+1}} \left( T(x_0', x'), \phi \left( \frac{x_0', x'}{k^4, k^4} \right) \right) = \left( \frac{x_0'}{k} = x_0, \quad \frac{x'}{k} = x \right)
\]

(7)

For example, let it be that \( T(x_0, x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \) in the form of \( T(x_0, x) = f(x_0) \times g(x) \) with distributions \( f(x_0) \in S'([0, \infty))^4 \), \( g(x) \in S'(\mathbb{R}^n) \) being homogeneous and of order \( \alpha \). Then, there is a number of equations that are valid:

\[
(T(kx_0, kx), \phi(x_0, x)) = (f(kx_0) \times g(kx), \phi(x_0, x))
\]

\[
= (f(kx_0), g(kx), \phi(x_0, x)) = \left( f(kx_0) = k^\alpha f(x_0), \quad g(kx) = k^\alpha g(x) \right)
\]

\[
= k^{2\alpha} (f(x_0), g(x), \phi(x_0, x))
\]

\[
= k^{2\alpha} (T(x_0, x), \phi(x_0, x)).
\]

(7)

From (6) and (7) we can see that the following equation is valid

\[
\frac{1}{k^{n+1}} (T(x_0, x), \phi \left( \frac{x_0, x}{k^4, k^4} \right)) = k^{2\alpha} (T(x_0, x), \phi(x_0, x))
\]

and from here, there is

\[
(T(x_0, x), \phi \left( \frac{x_0, x}{k^4, k^4} \right)) = k^{2\alpha + n + 1} (T(x_0, x), \phi(x_0, x)).
\]

(6)

For example, distribution \( T(x_0, x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \) in the form of \( T(x_0, x) = f(x_0) \times g(x) \) with \( f(x_0) \in S'([0, \infty]^4), g(x) \in S'([0, \infty]^n) \), and \( f(x_0) \) being homogenous and of order \( \alpha \), and test function \( \phi(x_0, x) \) in the form of

\[
\phi(x_0, x) = \sum_i \varphi_i(x_0) \varphi_i^2(x)
\]

with \( \varphi_i(x_0) \in S'(\mathbb{R}_+^4), \varphi_i^2(x) \in S'(\mathbb{R}^n) \) and \( \varphi(x_0) \) is a homogenous distribution of order \( \alpha \) and then, for \( (\forall i) \) the following is true

\[
(T(kx_0, x), \phi(x_0, x)) = (f(kx_0) \times g(x), \phi(x_0, x)) = (f(kx_0), g(x), \phi(x_0, x))
\]

\[
= (f(kx_0), g(x), \sum_i \varphi_i(x_0) \varphi_i^2(x))
\]

\[
= \left\{ \begin{array}{l}
\varphi(x_0, x) = \varphi_i(x_0) \varphi_i^2(x)
\end{array} \right.
\]

\[
= \sum_i \left( f(kx_0), \varphi_i^2(x_0) \right) (g(x), \varphi_i^2(x))
\]

\[
= k^\alpha \sum_i \left( f(kx_0), \varphi_i^2(x_0) \right) (g(x), \varphi_i^2(x))
\]

(9)

Since the set of functions \( \sum_i \varphi_i(x_0) \varphi_i^2(x) \) is dense in \( S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \). This is followed by the claim, because it is valid in a dense set, and with its continuity, it extends to entire set in \( S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \). Homogeneity by the second variable is similarly defined [1, 6].

The homogeneity of distribution \( T(x_0, x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \) separable at variable \( x_0 \) assuming that distribution \( \varphi_i^2(x) \in S'(\mathbb{R}_+^4) \) is homogeneous and of order \( \alpha \) for each \( i \), then, form these relations, it follows that

\[
T(kx_0, x) = \sum_i \varphi_i(kx_0) \psi(x)
\]

\[
= k^\alpha \sum_i \varphi_i(x_0) \psi(x) = k^\alpha T(x_0, x).
\]

(8)

(2)

Homogeneity at variable \( x \) is similarly observed.

Let there be distribution \( T(x_0, x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \). Distribution \( T(x_0, x) \) with \( x_0 \in \mathbb{R}_+^4 \) and \( x \in \mathbb{R}^n \) has quasi-asymptotics at infinity at variable \( x_0 \) relative to auto-modal function \( g \), if there is distribution \( G(x_0, x) \neq 0 \) such that

\[
\lim_{k \to \infty} \frac{1}{k^{\rho(k)}} T(kx_0, x) = G(x_0, x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n).
\]

(9)

Quasi-asymptotics by the separated variable at zero is similarly defined [1, 7].

Let us suppose a distribution \( T(x_0, x) \in S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \). Distribution \( T(x_0, x) \), \( x_0 \in \mathbb{R}_+^4 \) and \( x \in \mathbb{R}^n \) has quasi-asymptotics at zero at variable \( x_0 \) with respect to auto-modal function \( g \), if, and only if there is distribution \( G(x_0, x) \neq 0 \) such that

\[
\lim_{k \to \infty} \frac{1}{k^{\rho(k)}} T \left( \frac{x_0}{k}, x \right) = G \left( \frac{x_0}{k}, x \right) \neq 0
\]

(10)

in \( S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \).

For distributions from \( D'(\mathbb{R}_+^4 \times \mathbb{R}^n) \) or \( S'(\mathbb{R}_+^4 \times \mathbb{R}^n) \) we define the fractional (rational) differentiation at variable \( x_0 \) as a convolution \( f_{\alpha}(x_0) \) with \( f(x_0) \) at \( x_0 \) by the following formula

\[
f^{(\alpha)(x_0)} = f_{\alpha}(x_0) * f(x_0)
\]

(11)
which belongs to $D'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ if $f \in D'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ that is, $S'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ if $f \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n)$, (more in [3-6, 8, 12, 13]).

2. Some Quasi-Asymptotics Properties of Multidimensional Distributions

We provide proof of some of the basic theorems that apply to multidimensional distributions, and their formulaic presentation can be seen in [1].

Theorem 1. If distribution $T(x_0, x) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ is asymptotically homogeneous with respect to positive function $\rho(k)$ at variable $x_0$ or if the following is true

$$\lim_{k \to \infty} \frac{1}{\rho(k)} T(kx_0, x) = G(x_0, x) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n) \quad (12)$$

then $\rho(k)$ is an auto-modal function.

Proof: Let (12) be true and let $\phi(x_0, x) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ test function such that $(G(x_0, x), \phi(x_0, x)) \neq 0$.

Then let the test function be of the following form

$$\phi(x_0, x) = \sum_i \phi_i(x_0) \psi_i^1(x),$$

so that $\psi_i^1(x_0) \in S'(\mathbb{R}_+^1)$, $\psi_i^1(x) \in S(\mathbb{R}^n)$, are continuous functions with the following feature:

$$\text{supp } \psi_i^1 \subset \mathbb{R}_+^1, \text{supp } \psi_i^2 \subset \mathbb{R}^n,$$

$$\text{supp } \phi = \text{supp } \psi_1^1 \times \text{supp } \psi_1^2 \subset (\mathbb{R}_+^1 \times \mathbb{R}^n), (\forall i),$$

$$K \subset \mathbb{R}_+^1 \text{ compact set.}$$

For $\phi(x_0, x)$ and $a \in K$ it applies that

$$\frac{1}{a} \phi \left(\frac{x_0}{a}, x \right) = \frac{1}{a} \sum_i \phi_i \left(\frac{x_0}{a}, \psi_i^1(x) \right).$$

Now, the following is valid for distribution $T(x_0, x) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ and test function

$$\phi(x_0, x) \in S(\mathbb{R}_+^1 \times \mathbb{R}^n);$$

$$\begin{align*}
\left( T(kx_0, x), \frac{1}{a} \phi \left(\frac{x_0}{a}, x \right) \right) & \to \left( G(x_0, x), \frac{1}{a} \phi \left(\frac{x_0}{a}, x \right) \right) \quad (k \to \infty, a \in K)
\end{align*}$$

For $a \in K$, and using $(x_0 = ax_0^a)$ the following is valid

$$\begin{align*}
\frac{\rho(ak)}{\rho(k)} \left( T(kx_0, x), \frac{1}{a} \phi \left(\frac{x_0}{a}, x \right) \right) & = \frac{\rho(ak)}{\rho(k)} \left( T(akx_0, x), \phi(x_0, x) \right) \\
& = \frac{\rho(ak)}{\rho(k)} \left( T(akx_0, x), \phi(x_0, x) \right) \\
& = \frac{\rho(ak)}{\rho(k)} \left( G(x_0, x), \frac{1}{a} \phi \left(\frac{x_0}{a}, x \right) \right) \quad (13)
\end{align*}$$

Further, if we replace $k$ with $ak, a \in K$, the following is valid

$$\begin{align*}
\frac{1}{\rho(ak)} \left( T(akx_0, x), \phi(x_0, x) \right) & \to \left( G(x_0, x), \phi(x_0, x) \right) \quad (k \to \infty)
\end{align*}$$

Using relations (12) and (13), we get the following relation

$$\rho(ak) \quad k \to \infty \quad G(x_0, x) \quad (\frac{1}{\rho(k)} \left( T(kx_0, x), \phi(x_0, x) \right) \quad (15)$$

From here, by inserting $(ax_0^a, x) \to x_0$ we get the following

$$\rho(ak) \quad k \to \infty \quad G(ax_0^a, x), (\phi(x_0, x)) \quad (G(x_0, x), \phi(x_0, x))$$

From here, we get the required relation

$$\rho(ak) \quad k \to \infty \quad G(ax_0^a, x), (\phi(x_0, x)) \quad = C(a).$$

From the existence of $\lim \frac{\rho(ak)}{\rho(k)} = C(a)$ following $C(a) = a^a$ and $\rho(a) = a^a L(a)$, and Karamata L function [16], it follows that function $\rho(k)$ is an auto-modal function, even in the case of multi-variable distributions.

Theorem 2. Let distribution $T(x_0, x) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ be asymptotically homogeneous with respect to positive function $\rho(k)\in K$ at variable $x_0$. In this case, if the order of auto-modal function $\rho(k)$ is equal to $a$, then distribution $G(x_0, x)$ in the following equation

$$\lim_{k \to \infty} \frac{1}{\rho(k)} T(kx_0, x) = G(x_0, x) \quad G(x_0, x) = C_f(a)(x_0) \times g(x), \quad C \text{ being the constant.}$$

Proof: It has already been shown in the case of distributions of one variable [1],[7], that distribution $G(x) \in S_+(\mathbb{R}^n)$ has the form of $G(x) = C_f(a)(x_0) \times g(x)$ being the nucleus of fractional differentiation.

For $G(x_0, x) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ let us suppose that

$$f(x_0) \in S_+(\mathbb{R}_+^1), \quad g(x) \in S_+(\mathbb{R}^n), \quad \text{and that distribution } f(x_0) \text{ is homogeneous and of order } a, \quad G(x_0, x) = f(x_0) \times g(x).$$

Since for the function in the form of $\phi(x_0) \in S_+(\mathbb{R}_+^1 \times \mathbb{R}^n)$ the following applies

$$\lim_{k \to \infty} \frac{1}{\rho(k)} \left( T(kx_0, x), \phi(x_0, x) \right) \quad (k \to \infty)$$

so distribution $g(x) = S_+(\mathbb{R}^n), G(x_0, x)$ is in the form of $G(x_0, x) = C_f(a)(x_0) \times g(x)$.

Theorem 3. If distribution $T(x_0, x)$ is separated at variable forms $x_0$, then it has the following form:

$$T(x_0, x) = T_1(x_0) \times T_2(x_0), \quad T_1(x_0) \times T_2(x_0) \quad \text{has the quasi-asymptotics of order } a \text{ in relation to function } k^a \rho(k) \text{ at a variable } x_0 \text{ if } T_1 \text{ and } T_2 \text{ have the same quasi-asymptotics in relation to function } \rho(k). \text{ The reverse of the theorem is not valid.}$$

Proof. Let us show that distribution $T(x_0, x)$ has quasi-asymptotics of order $a$ with respect to $\rho(k)$ if $T_1$ and $T_2$ have the same quasi-asymptotics. Let the test function $\phi(x_0, x)$ be in the form of $\phi(x_0, x) = \sum_i \phi_i(x_0) \psi_i(x)$. By the definition of quasi-asymptotics, the following applies:
\[
\frac{1}{\rho(k)} (T(kx_0, x), \phi(x_0, x)) = \left( \frac{T(kx_0, x)}{\rho(k)}, \sum_i \varphi_i(x_0) \psi_i(x) \right) = \frac{t_1(kx_0) g_1(x) + t_2(kx_0) g_2(x)}{k^a \rho(k)} \sum_i \varphi_i(x_0) \psi_i(x) \}
\]

\[
= \left( \frac{t_1(kx_0) g_1(x)}{\rho(k)}, \sum_i \varphi_i(x_0) \psi_i(x) \right) \} + \left( \frac{t_2(kx_0) g_2(x)}{\rho(k)}, \sum_i \varphi_i(x_0) \psi_i(x) \right) \}
\]

Since \( \rho(k) = k^a \Gamma(k) \) and \( T_1(kx_0) = k^a T_1(x_0) \) and \( T_2(kx_0) = k^a T_2(x_0) \) therefore

\[
= \frac{1}{k^a L(k)} (T_1(kx_0), g_1(x), \sum_i \varphi_i(x_0) \psi_i(x)) + \frac{1}{k^a L(k)} (T_2(kx_0), g_2(x), \sum_i \varphi_i(x_0) \psi_i(x))
\]

\[
= \frac{1}{k^a L(k)} (T_1(kx_0), g_1(x), \sum_i \varphi_i(x_0) \psi_i(x)) + \frac{1}{k^a L(k)} (T_2(kx_0), g_2(x), \sum_i \varphi_i(x_0) \psi_i(x))
\]

\[
= \frac{1}{k^a L(k)} (T_1(kx_0), g_1(x), \sum_i \varphi_i(x_0) \psi_i(x)) + \frac{1}{k^a L(k)} (T_2(kx_0), g_2(x), \sum_i \varphi_i(x_0) \psi_i(x))
\]

\[
= \frac{1}{k^a L(k)} (T_1(kx_0), g_1(x), \sum_i \varphi_i(x_0) \psi_i(x)) + \frac{1}{k^a L(k)} (T_2(kx_0), g_2(x), \sum_i \varphi_i(x_0) \psi_i(x))
\]

\[
= \frac{1}{k^a L(k)} (T_1(kx_0), g_1(x), \sum_i \varphi_i(x_0) \psi_i(x)) + \frac{1}{k^a L(k)} (T_2(kx_0), g_2(x), \sum_i \varphi_i(x_0) \psi_i(x))
\]

\[
= \frac{1}{k^a L(k)} (T_1(kx_0), g_1(x), \sum_i \varphi_i(x_0) \psi_i(x)) + \frac{1}{k^a L(k)} (T_2(kx_0), g_2(x), \sum_i \varphi_i(x_0) \psi_i(x))
\]

\[
= \frac{1}{k^a L(k)} (T_1(kx_0), g_1(x), \sum_i \varphi_i(x_0) \psi_i(x)) + \frac{1}{k^a L(k)} (T_2(kx_0), g_2(x), \sum_i \varphi_i(x_0) \psi_i(x))
\]

\[
= \frac{1}{k^a L(k)} (T_1(kx_0), g_1(x), \sum_i \varphi_i(x_0) \psi_i(x)) + \frac{1}{k^a L(k)} (T_2(kx_0), g_2(x), \sum_i \varphi_i(x_0) \psi_i(x))
\]

This shows that distribution \( T(x_0, x) = T_1 g_1(x) + T_2(x_0) g_2(x) \) has quasi-asymptotics of order \( \alpha \) with respect to function \( k^a \rho(k) \) at variable \( x_0 \) if distributions \( T_1 \) and \( T_2 \) have the same quasi-asymptotics.

The reverse of the theorem is not valid. To show this, it is enough to show that, for example, the following is not valid for distribution \( T_1(x_0) = x_0^\alpha + x_0^{\alpha+1} \) and \( T_2(x_0) = -x_0^{\alpha+1} + x_0^\alpha \) with respect to function \( k^a \rho(k) \). Indeed

\[
(\frac{1}{\rho(k)} (T(kx_0, x), \phi(x_0, x)) = \left( \frac{T(kx_0, x)}{\rho(k)}, \sum_i \varphi_i(x_0) \psi_i(x) \right) = \frac{k^\alpha x_0^\beta [kx_0 (g_1(x) - g_2(x)) + g_1(x) + g_2(x)]}{\rho(k)} \sum_i \varphi_i(x_0) \psi_i(x) \}
\]

\[
= \frac{k^\alpha x_0^\beta [kx_0 (g_1(x) - g_2(x)) + g_1(x) + g_2(x)]}{k^a L(k)} \sum_i \varphi_i(x_0) \psi_i(x) \}
\]

\[
= \frac{x_0^\beta [kx_0 (g_1(x) - g_2(x)) + g_1(x) + g_2(x)]}{L(k)} \sum_i \varphi_i(x_0) \psi_i(x) \}
\]

\[
= \frac{x_0^\beta [(kx_0 + 1) g_1(x) + (kx_0 - 1) g_2(x)]}{L(k)} \sum_i \varphi_i(x_0) \psi_i(x) \}
\]

From here, it can be seen that \( T(x_0, x) \) has no quasi-asymptotics when \( k \to \infty \).

Theorem 4. In order for distribution \( T(x_0, x) \in S'(\mathbb{R}^n \times \mathbb{R}^n) \) to be asymptotically homogeneous at infinity, with respect to auto-modal function \( \rho(k) \) at variable \( x_0 \), it is necessary, and it is also sufficient, that for each \( \beta \in \mathbb{R} \) its
fractional derivative $T^{(-\beta)}(x_0, x)$ is asymptotically homogeneous with respect to $k^\beta \rho(k)$.

Proof: We define fractional differentiation in $S'(\mathbb{R}^1_+ \times \mathbb{R}^n)$ with distribution $T(x_0, x)$ at $x_0$ as convolution of distribution $f_\beta(x_0) \in S'(\mathbb{R}^1_+)$ and distribution $T(x_0, x) \in S'(\mathbb{R}^1_+ \times \mathbb{R}^n)$ i.e. $T^{(-\beta)}(x_0, x) = T(x_0, x) * f_\beta(x_0)$. Using the property of distribution $f_\beta(x_0)$ to be homogeneous and of order $\beta - 1$, that is, using the validity of the following $f_\beta(x_0) = k^{\beta - 1} f_\beta(x_0)$, we get the following:

$$\lim_{k \to 0} \frac{1}{k^{\beta + 1} \rho(k)} (T^{(-\beta)}(x_0, x), \phi(x_0, x))$$

from here, if we put that $kx_0 = x'$, we get

$$= \lim_{k \to 0} \frac{1}{k^{\beta + 1} \rho(k)} (T^{(-\beta)}(x', x), \phi(x_0 \frac{x'}{k}, x))$$

$$= \lim_{k \to 0} \frac{1}{k^{\beta + 1} \rho(k)} (T^{(-\beta)}(x_0, x), \phi(x_0 \frac{x'}{k}, x))$$

By using the definition of convolution

$$T(x_0, x) * f_\beta(x_0) = \frac{1}{\Gamma(\beta)} \Theta(x_0) x_0^{\beta - 1} * T(x_0, x)$$

$$= \frac{1}{\Gamma(\beta)} \int_0^\infty (x_0 \beta = 1) T(t, x) dt = T^{(-\beta)}(x_0, x).$$

we can see that the last equation is precisely the $\beta$ primitive integral for $T(x_0, x)$. Based on this, we have that $T(x_0, x) \in S'(\mathbb{R}^1_+ \times \mathbb{R}^n), f_\beta(x_0) \in S'(\mathbb{R}^1_+).$

$$T(x_0, x) * f_\beta(x_0), \phi(x_0, x))$$

$$= \lim_{k \to 0} (T(x_0, x) * f_\beta(x_0), \phi(x_0, x))$$

$\eta_k$ being unit sequence. If there is a limes on the right-hand side for each series $\eta_k, k \to \infty$ then the function from $S'(\mathbb{R}^2)$ which converges to number one in $\mathbb{R}^2$ and this limit does not depend on the choice of series $\eta_k, k \to \infty$ then we have that $T(x_0, x) * f_\beta(x_0) \in S'(\mathbb{R}^{n+1})$. Based on this, the last equation transforms into

$$\lim_{k \to 0} \frac{1}{k^{\beta + 1} \rho(k)} (T(x_0, x) * f_\beta(x_0), \phi(x_0 \frac{x'}{k}, x))$$

$$= \lim_{k \to 0} \frac{1}{k^{\beta + 1} \rho(k)} (T(x_0, x) * f_\beta(x_0), \phi(x_0 \frac{x'}{k}, x))$$

Now, if we put that

$$\left( x_0 + \frac{\tau}{k} = x_0 + \frac{k\tau}{k} = \frac{x_0}{k} + \frac{\tau}{k} \right)$$

the last equation transforms into the following form:

$$\lim_{k \to 0} \frac{1}{k^{\beta + 1} \rho(k)} (T(x_0, x), (f_\beta(k\tau'), \phi(x_0 \frac{x'}{k}, \phi(x_0 \frac{x'}{k}, \tau, x)))$$

$$\lim_{k \to 0} \frac{1}{k^{\beta + 1} \rho(k)} (T(x_0, x), (f_\beta(k\tau'), \phi(x_0 \frac{x'}{k}, \tau, x)))$$

(since $f_\beta(k\tau') = k^{\beta - 1} f_\beta(\tau)$)

$$\lim_{k \to 0} \frac{k^{\beta - 1}}{k^{\beta} \rho(k)} (T(x_0, x), (f_\beta(k\tau), \phi(x_0 \frac{x'}{k} + \tau, x)))$$

From the last equation, using the shift $(x_0 = kx_0')$ we get the following:

$$\lim_{k \to 0} \frac{1}{k^{\beta} \rho(k)} (T(kx_0', x), (f_\beta(k\tau), \phi(x_0' + \tau, x)))$$

$$\lim_{k \to 0} \frac{1}{k^{\beta} \rho(k)} (T(x_0, x), (f_\beta(k\tau), \phi(x_0 + \tau, x)))$$

$$\lim_{k \to 0} \frac{1}{\rho(k)} (T(kx_0', x), (f_\beta(k\tau), \phi(x_0' + \tau, x)))$$

From this, we have that

$$\lim_{k \to 0} \frac{1}{\rho(k)} (T(kx_0', x), (f_\beta(k\tau), \phi(x_0 + \tau, x)))$$

$$\lim_{k \to 0} \frac{1}{\rho(k)} (T(kx_0', x), (f_\beta(k\tau), \phi(x_0 + \tau, x)))$$

3. Example of the Use of Quasi-Asymptotics to the Solutions of Differential Equations

Let $L$ be a differential operator with constant coefficients $a_\beta(x) = a_\beta$ and let $f \in \mathcal{D}'$, be such a distribution that convolution $E * f$ exists in $\mathcal{D}'$ where $E \in \mathcal{D}'$ is the fundamental solution of equation $L(D)E = \delta(x)$, [3, 6, 9, 11].

Then the solution $u = E * f$ of differential equation $L(D)u = f(x), f \in \mathcal{D}'$ has quasi-asymptotics of order $\alpha$ with respect to $\rho(k) = k^{\alpha} L(k)$ with $L(k)$ being the Karamata slow-varying function, if distribution $f \in \mathcal{D}'$ has such quasi-asymptotics, $\mathcal{D}'$-distribution space.

Proof: Let $f$ have the quasi-asymptotics with respect to $\rho(k) = k^{\alpha} L(k)$. Then the following is valid

$$\frac{1}{\rho(k)} (f(kx), \phi(x)) = \frac{1}{k^{\alpha} L(k)} (f(x), \phi(x))$$
\[
\int \frac{1}{k \rho(k)} \left( \delta(x) \ast f(x), \varphi \left( \frac{x}{k} \right) \right) = \int \frac{1}{k \rho(k)} \left( L(D) \ast f(x), \varphi \left( \frac{x}{k} \right) \right) \\
= \int \frac{1}{k \rho(k)} \left( \left( \sum_{|a|=0}^{m} a_{a}D^{a}E(x) \right) \ast f(x), \varphi \left( \frac{x}{k} \right) \right) \\
= \int \frac{1}{k \rho(k)} \left( \left( \sum_{|a|=0}^{m} a_{a}D^{a}(E \ast f)(x) \right), \varphi \left( \frac{x}{k} \right) \right) \\
= \int \frac{1}{k \rho(k)} \left( L(D)(E \ast f)(x), \varphi \left( \frac{x}{k} \right) \right) \\
= \int \frac{1}{k \rho(k)} \left( L(D)u(x), \varphi \left( \frac{x}{k} \right) \right) \\
= \int \frac{1}{k \rho(k)} \left( \left( \sum_{|a|=0}^{m} a_{a}D^{a}u(x), \varphi \left( \frac{x}{k} \right) \right) \right) \\
= \int \frac{1}{k \rho(k)} \left( \left( \sum_{|a|=0}^{m} (-1)^{|a|} u(x), D^{a} \left( a_{a} \varphi \left( \frac{x}{k} \right) \right) \right) \right) \\
= \int \frac{1}{k \rho(k)} \left( (-1)^{|a|} u(x), L^{*}(D) \varphi \left( \frac{x}{k} \right) \right) \\
= \frac{1}{\rho(k)} \left( \frac{u(kx)}{L(-D) \varphi(x)} \right) = \left( \frac{u(kx)}{\rho(k)}, L(-D) \phi(x) \right) \\
\]

Therefore, we have the following:

\[
\int \frac{1}{\rho(k)} \left( f(kx), \phi(x) \right) \frac{1}{\rho(k)} \left( \frac{u(kx)}{L(-D) \phi(x)} \right), \text{ and, as per assumption, } f \text{ has the quasi-asymptotics, thus, distribution } u \text{ has one also.}
\]

4. Conclusion

Most of the theorems proved in this paper on quasi-asymptotics of distributions at a separable variable have their analog in the case of one-dimensional distributions. In [1], Vladimirov showed a theorem that does not have a one-dimensional analog, the consequence of which is very important, and on the basis of which the application of separated quasi-asymptotics in to the solutions of differential equations.

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