Black hole mass and angular momentum in 2+1 gravity

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Abstract

We propose a new definition for the mass and angular momentum of neutral or electrically charged black holes in 2+1 gravity with two Killing vectors. These finite conserved quantities, associated with the $SL(2,R)$ invariance of the reduced mechanical system, are shown to be identical to the quasilocal conserved quantities for an improved gravitational action corresponding to mixed boundary conditions. They obey a general Smarr-like formula and, in all cases investigated, are consistent with the first law of black hole thermodynamics. Our framework is applied to the computation of the mass and angular momentum of black hole solutions to several field-theoretical models.

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1 Introduction

A long-standing problem in general relativity is that of a satisfactory definition of the total energy of a self-gravitating distribution of matter. The modern approach to this question uses the idea of quasilocal energy \[1, 2, 3, 4, 5, 6, 7, 8, 9\] (a more extensive bibliography is given in the last three references). From an action functional for the gravity-matter system, with boundary conditions on a given hypersurface, one derives canonically a Hamiltonian, given by the sum of a bulk integral, which vanishes on shell, and of a surface term. The quasilocal energy is defined as the on-shell value of the Hamiltonian. This is generically divergent in the limit where the spatial boundary goes to infinity, but can be made finite by subtracting the contribution of an appropriate background. From this Hamiltonian, one can also define in the rotationally symmetric case a quasilocal angular momentum. The various possible choices of boundary conditions (for instance Dirichlet or Neumann boundary conditions) and of backgrounds lead to different possible forms for these conserved quantities, which however give the same result in the case of asymptotically flat configurations with a Minkowski background. Investigations of conserved quantities in non-asymptotically flat spacetimes have concerned mainly asymptotically dS or AdS spacetimes \[10, 11, 12, 13\]. The definition of the total energy given in \[10\] has been shown to agree with the Dirichlet quasilocal energy \[5\]. The first application of the quasilocal formalism to AdS black holes was given in \[14\]. The quasilocal framework has also recently been applied to the computation of the mass and angular momentum of non-asymptotically flat, non-asymptotically AdS dilatonic black holes \[15\].

A testing ground for these various possible definitions of conserved quantities is provided by 2+1 gravity, as all known (2+1)-dimensional black hole spacetimes are non-asymptotically flat. Indeed, it can be argued that the departure from asymptotic flatness in the 2+1 case is such that it does not make sense to impose boundary conditions at infinity. Rather, boundary conditions should be imposed on some (arbitrarily chosen) timelike surface. It turns out that this is possible because of two peculiarities of general relativity in 2+1 dimensions. First, this theory is, as Newtonian gravity in four dimensions, dynamically trivial \[16\], so that (unlike the case of general relativity in 3+1 dimensions), the total mass and angular momentum inside a closed one-surface (i.e. a closed line) can in principle be written as some fluxes through that surface. Second, these black hole spacetimes are stationary and rotationally symmetric, i.e. they have two commuting Killing vectors. The resulting $SL(2, R) \sim SO(2, 1)$ invariance leads, by the usual
Noether mechanism, to three constants of the motion \[17\]. We shall argue, and check on a number of specific examples, that two of these constants may be identified with mass and angular momentum, which can therefore be computed on a circle of arbitrary finite radius. This identification will be supported by a quasilocal computation, our surface-independent energy corresponding to a well-defined finite part of the canonical quasilocal energy for Dirichlet boundary conditions, while our angular momentum coincides with the canonical quasilocal angular momentum.

In the next section, we recall the dimensional reduction of 2+1 Einstein gravity with two Killing vectors to a mechanical problem. The residual \(SL(2, \mathbb{R})\) invariance of the reduced theory leads to the conservation of a super angular momentum, two components of which are associated with mass and angular momentum. This association is checked in the case of the black-hole solutions to vacuum Einstein gravity with a negative cosmological constant. We also show, by evaluating our surface-independent energy and angular momentum on the horizon, that they satisfy a Smarr-like formula. We then relate in Sect. 3 our conserved quantities to the same quantities computed in the canonical quasilocal approach, and show that our energy and angular momentum may be derived canonically from an improved action, corresponding to mixed Dirichlet-Neumann boundary conditions. Our framework is applied in Sect. 4 to the computation of the mass of black hole solutions to two gravitating scalar field models. Sect. 5 is devoted to the extension of our approach to the case of matter gauge fields, which is then tested in Sect. 6 on the example of black hole solutions to Einstein-Maxwell gravity with a negative cosmological constant.

2 Conserved quantities from angular momentum in minisuperspace

We consider a 3-dimensional Lorentzian spacetime \(\mathcal{M}\) with metric \(g_{\mu\nu}\) (we use the Landau-Lifshitz conventions, except for the signature of the metric which is \(-++\)) and boundary \(\partial\mathcal{M}\). This boundary consists of initial and final spacelike surfaces (two-surfaces in the present case) \(\Sigma_{t_1}\) and \(\Sigma_{t_2}\), and a timelike surface \(\Sigma^\rho\) (not necessarily at spatial infinity), which we shall assume to be orthogonal to the \(\Sigma_t\). In the canonical 1+2 ADM decomposition \[18\], the metric on \(M\) is written as

\[
\text{d}s^2 = -N^2\text{d}t^2 + h_{ij}(\text{d}x^i + V^i \text{d}t)(\text{d}x^j + V^j \text{d}t),
\] (2.1)
where \( h_{ij} \) is the induced metric on \( \Sigma_t \). The two-surfaces \( \Sigma_t \) and \( \Sigma^\rho \) intersect on a one-surface \( \Sigma^\rho_t \), with induced metric \( \sigma_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu \), where \( n^i \) is the unit normal to \( \Sigma^\rho \). The action for general relativity with Dirichlet boundary conditions on \( \partial M \) is

\[
I_D = \int_M \left( \frac{1}{2\kappa} R + \mathcal{L}_m \right) + \frac{1}{\kappa} \oint_{\Sigma_t} K - \frac{1}{\kappa} \oint_{\Sigma^\rho} \Theta
\]

(\( \kappa = 8\pi G \)), where \( R \) is the Ricci scalar density, \( K \) and \( \Theta \) are the traces of the extrinsic curvature densities of \( \Sigma_t \) and \( \Sigma^\rho \), and \( \mathcal{L}_m \) is the matter Lagrangian density. We will first consider scalar matter described by a set of scalar fields, the case of vector (gauge) fields shall be treated in Sect. 5. The action (2.2) diverges for noncompact spatial geometries, in which case one considers rather the relative action, defined as the difference between the action evaluated on the configuration \( g, \phi \) (where \( \phi \) stands for the matter fields) and that evaluated on a background configuration \( g_0, \phi_0 \) (not necessarily flat empty space),

\[
I_R = I - I_0.
\]

We specialize to configurations with two commuting Killing vectors, one timelike and one spacelike, and recall the dimensional reduction procedure of \[19\] \[17\]. We may choose adapted coordinates on \( M \) such that

\[
ds^2 = \lambda_{ab} \, dx^a \, dx^b + \zeta^{-2} R^{-2} \, d\rho^2 \quad \text{ \( (R^2 = - \det \lambda) \)}
\]

(\( \lambda_{ab} \) spans a Minkowski space, i.e.

\[
R^2 = X^2 = \eta_{AB} X^A X^B = -T^2 + X^2 + Y^2.
\]

Stationary solutions correspond to “spacelike” paths \( X(\rho) \) with \( R^2 > 0 \). Intersections of these paths with the future light cone \( (R^2 = 0, \, T > 0) \) correspond to event horizons, while intersections with the past light cone

\[
S^\rho_t \text{, with induced metric } \sigma_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu, \text{ where } n^i \text{ is the unit normal to } \Sigma^\rho. \]
\( R^2 = 0, \; T < 0 \) correspond to naked singularities \[19\]. Our metric ansatz \[2.4\] may also be written in terms of null target space coordinates

\[
U = T + X, \quad V = T - X \quad (R^2 = Y^2 - UV), \quad (2.7)
\]
as

\[
ds^2 = U dt^2 + 2Y dt d\varphi + V d\varphi^2 + \zeta^{-2} R^{-2} d\rho^2. \quad (2.8)
\]
The canonical, 1 + 2 metric ansatz \[2.1\] is related to this 2 + 1 ansatz by

\[
N^2 = \frac{R^2}{V}, \quad V^\varphi = \frac{Y}{V}, \quad h_{\rho\rho} = n^2_\rho = \zeta^{-2} R^{-2}, \quad h_{\varphi\varphi} = \sigma_{\varphi\varphi} = V. \quad (2.9)
\]
The Christoffel symbols for the metric \[2.4\] are

\[
\Gamma^a_{b\beta} = \frac{1}{2} \chi^a_{b\beta}, \quad \Gamma^a_{\beta\gamma} = -\frac{1}{2} \zeta^2 R^2 (\lambda \chi)_{ab}, \quad \Gamma^2_{22} = -R^{-1} R' - \zeta^{-1} \zeta', \quad (2.10)
\]
where the prime stands for \(d/d\rho\), and

\[
\chi = \lambda^{-1} \lambda' = R^{-2} \left( \frac{R R' - \ell Y}{\ell T + \ell X} - \frac{\ell^T + \ell X}{R R' + \ell Y} \right), \quad (2.11)
\]
where

\[
\ell^A = \eta^{AB} \epsilon_{BCD} X^C X'^D \quad (2.12)
\]
(with \( \epsilon_{012} = +1 \)) are the contravariant components of the wedge product \( \ell = X \wedge X' \). The corresponding Ricci scalar density is

\[
R = \zeta \left( R^2 - \frac{1}{4} R^2 \operatorname{Tr}(\chi^2) \right) - 2(2 \zeta R R')' = \frac{1}{2} \zeta X'^2 - (2 \zeta R R')'. \quad (2.13)
\]
Using \( n^i = \delta^i_2 \zeta R \), we obtain for the trace of the extrinsic curvature density of the cylinder \( \Sigma^0 \)

\[
\Theta = -\sqrt{|\lambda|} \lambda^{ab} \nabla_a n_b = -\zeta R R', \quad (2.14)
\]
while the trace of the extrinsic curvature of \( \Sigma_t \) vanishes for these stationary configurations, so that the purely gravitational part of the action \[2.2\] reduces to \[19\]

\[
I_D = \int d^2 x \frac{1}{4\kappa} \int d\rho \zeta X'^2. \quad (2.15)
\]

\[1\]Note that the sign convention for \( X \) in Ref. \[19\] is opposite to that chosen here.
We thus have reduced 2+1 Einstein gravity with 2 Killing vectors to a “minisuperspace” mechanical problem on the Minkowski plane. The (reparametrization invariant) momentum conjugate to $X$ is

$$P = \frac{1}{2\kappa}\zeta X'.$$ (2.16)

The gravitational contribution (2.15) to the action is invariant under Lorentz transformations in target space. Assuming that the matter scalar fields depend only on the radial coordinate $\rho$, the matter part of the action depends only the metric component $g_{\rho\rho}$ and on $\sqrt{|g|} = \zeta^{-1}$, and so is also Lorentz invariant. It follows that the super angular momentum

$$L = X \wedge P = \frac{1}{2\kappa} \zeta \ell,$$ (2.17)

is a constant of the motion. However, only two conserved quantities are associated with this vectorial conservation law, because of the freedom to perform infinitesimal gauge transformations (transition to rotating frames) $\delta \varphi = -\delta \Omega t$, leading to infinitesimal rotations of the configuration vector

$$\delta X = \delta \Omega \wedge X$$ (2.18)

around the null direction $\delta \Omega = \delta \Omega (1,1,0)$. As we shall show in the next section, using the quasilocal approach, and check on a number of specific examples in the following, the two physical conserved quantities associated with $L$ are the energy and angular momentum,

$$E = -2\pi L^Y,$$ (2.19)

$$J = 2\pi (L^T - L^X).$$ (2.20)

Under an infinitesimal gauge transformation (2.18), these transform as

$$\delta E = J \delta \Omega, \quad \delta J = 0.$$ (2.21)

It is easily checked that this transformation law remains exact for finite gauge transformations. The preferred coordinate frame will as usual be defined relative to an asymptotic corotating observer, so that the asymptotic angular velocity

$$\Omega_\infty = \left. \frac{Y}{V} \right|_{(\rho \to \infty)}$$ (2.22)
vanishes. The mass $M$ is obtained from the energy (2.19) computed in this frame by subtracting the energy of an appropriate background$^2$,

$$M = E - E_0.$$  \hspace{1cm} (2.23)

The simplest example is that of the BTZ black hole [20], corresponding to the matter Lagrangian density $\mathcal{L}_m = - (\Lambda / \kappa) \sqrt{|g|}$ with a negative cosmological constant $\Lambda = - l^{-2}$. In the ADM parametrization (2.1), these solutions are given by

$$N^2 = h_{rr}^{-1} = \frac{r^2}{l^2} - M + \frac{J^2}{4 r^2}, \quad h_{\varphi\varphi} = r^2, \quad V^\varphi = - \frac{J}{2 r^2},$$ \hspace{1cm} (2.24)

in units such that $\kappa = \pi$. This may be transformed into the $2 + 1$ form (2.8) by the coordinate transformation $r^2 = 2 \rho + M l^2 / 2$. The BTZ black hole (2.24) corresponds to the spacelike geodesic

$$U = - 2 l^{-2} \rho + \frac{M}{2}, \quad V = 2 \rho + \frac{M l^2}{2}, \quad Y = - \frac{J}{2},$$ \hspace{1cm} (2.25)

with $\zeta = 1$. Using

$$l^T = Y X' - X Y'$$ \hspace{1cm} (2.26)

$$l^X = Y T' - T Y'$$ \hspace{1cm} (2.27)

$$l^Y = T X' - X T' = \frac{1}{2} (V U' - U V'),$$ \hspace{1cm} (2.28)

we check that the mass and angular momentum given by (2.19) (with the background corresponding to the BTZ vacuum $M = J = 0$) and (2.20) coincide with their BTZ values,

$$M = M, \quad J = J.$$ \hspace{1cm} (2.29)

We conclude this section by showing that equations (2.19) and (2.20) imply the validity of the Smarr-like formula [21]

$$M = - E_0 + \frac{1}{2} T_H S + \Omega_h J$$ \hspace{1cm} (2.30)

for any black hole configuration. In (2.30), $T_H$ is the Hawking temperature, defined as the inverse of the period in imaginary time,

$$T_H \equiv \frac{1}{2 \pi} n^\rho \partial_\rho N \big|_{(\rho=\rho_h)} = \frac{\zeta R R'}{2 \pi \sqrt{V}} \big|_{(\rho=\rho_h)}$$ \hspace{1cm} (2.31)

$^2$One could also in principle subtract from the angular momentum (2.20) the angular momentum of an appropriate background. In practice, the background is usually non-rotating, so that the angular momentum subtraction constant vanishes.
(where $\rho_h$ is the horizon radius, defined by $R^2(\rho_h) = 0$), $S$ is the black hole entropy \[22\],

$$S \equiv \frac{A}{4G} = \frac{4\pi^2 \sqrt{V}}{\kappa} \bigg|_{(\rho=\rho_h)}$$ \hspace{1cm} (2.32)

($A$ being the horizon perimeter), and $\Omega_h$ is the horizon angular velocity

$$\Omega_h = -\frac{Y}{V} \bigg|_{(\rho=\rho_h)}.$$ \hspace{1cm} (2.33)

The integral mass formula (2.30) is easily proven by collecting the above definitions and equations (2.19) and (2.20) for the energy and angular momentum evaluated on the horizon. It differs from the integral mass formula for 3+1 dimensions\[23\]

$$M = \int_{\Sigma_t} (2\tau^b_a - \tau \delta^b_a) u^a d\Sigma_b + 2T_H S + 2\Omega_h J$$ \hspace{1cm} (2.34)

(where $\tau^b_a$ is the matter energy-momentum, and $u^a$ is the unit normal to $\Sigma_t$) in two respects. First, the (2+1)-dimensional formula involves only geometric quantities evaluated on the horizon. Second, the numerical coefficients are different.

## 3 Quasilocal energy and angular momentum

We now recall the standard construction of quasilocal mass and angular momentum. We start again from the action \[24\], without any assumptions of symmetry. Introducing the canonical momenta $p^{ij}$ and $p$ conjugate to $h_{ij}$ and $\phi$, and rearranging the action, one arrives in the absence of matter gauge fields (the case of the Maxwell field shall be considered in Sect. 5) at the form

$$I_D = \int dt \left[ \int_{\Sigma_t} (p^{ij} \dot{h}_{ij} + p \dot{\phi} - N \mathcal{H} - V^i \mathcal{H}_i) - \oint_{S^0_t} (N \epsilon + 2 V_i \pi^{ij} n_j) \right],$$ \hspace{1cm} (3.1)

where $\mathcal{H}$ and $\mathcal{H}_i$ are the Hamiltonian and momentum constraints, $\epsilon = \epsilon_{(g)}$, where $\epsilon_{(g)}$ is the density

$$\epsilon_{(g)} = \frac{1}{\kappa} k \sqrt{|\sigma|},$$ \hspace{1cm} (3.2)

$k$ being the trace of the extrinsic curvature of $S^0_t$ in $\Sigma_t$,

$$k = -\sigma^{\mu\nu} D_{\mu} n_{\nu}.$$ \hspace{1cm} (3.3)
(with $D_\mu$ the covariant derivative on $\Sigma_t$), and the reduced momenta $\pi^{ij} = (\sqrt{|\sigma|}/\sqrt{|h|}) p^{ij}$ are related to the extrinsic curvature of $\Sigma_t$

$$K_{ij} = -\frac{1}{2N}(\dot{h}_{ij} - 2D_i V_j)$$

(3.4)

by

$$\pi^{ij} = \frac{1}{2\kappa} \sqrt{|\sigma|} (K h^{ij} - K^{ij}).$$

(3.5)

For a configuration solving the field equations, the constraints $\mathcal{H}$ and $\mathcal{H}_i$ vanish, and the Hamiltonian reduces to the one-surface integral

$$H_D = \oint_{S^\rho_i} (N \epsilon_{(g)} + 2V_i \pi^{ij} n_j).$$

(3.6)

The quasilocal energy or mass is the difference between the value of this Hamiltonian and that for the background evaluated with the same boundary data for the fields $\sigma$ and $N$. The quasilocal momenta are obtained from the Hamiltonian by carrying out an infinitesimal gauge transformation $\delta x^i = \delta \xi^i t$ and evaluating the response

$$P_i = \frac{\delta H}{\delta \xi^i} = -2 \oint_{S^\rho_i} \pi^{ij} n_j.$$

(3.7)

What are the values of these quasilocal quantities in the stationary rotationally symmetric case considered in Sect. 2? Using the relations between the 1+2 and 2+1 metric ansätze and the definitions we obtain

$$k = \frac{1}{2} \zeta R \frac{V'}{V}, \quad K_{12} = \sqrt{\frac{V}{2R}} \frac{V' Y' - Y' V'}{Y'},$$

leading to

$$\epsilon_{(g)} = -\frac{1}{2\kappa} \zeta R \sqrt{\frac{V'}{V}}, \quad \pi^{12} = -\frac{1}{4\kappa} \zeta^2 R \frac{V' Y' - Y' V'}{V}.$$ 

(3.9)

Evaluating the Hamiltonian and the momentum $P_\phi$ on a circle $\rho = \text{constant}$, we obtain the quasilocal energy and angular momentum

$$E_D = \frac{\pi}{\kappa} \zeta (V V' - Y Y'),$$

$$J = \frac{\pi}{\kappa} \zeta (V Y' - Y V').$$

(3.10)
Using Eq. (2.14), (3.10) and (3.11) may be rewritten as

\[
E_D = -2\pi L^Y + \frac{1}{2\kappa} \oint_{\Sigma^\rho} \Theta, \tag{3.12}
\]

\[
J = 2\pi (L^T - L^X). \tag{3.13}
\]

It follows that the quasilocal energy deriving from the “Dirichlet” action (2.2) on a one-surface \(\Sigma^\rho\) is the sum of a surface-independent term, the constant (2.19), and a purely geometric surface-dependent term proportional to the mean extrinsic curvature of this surface. In the BTZ case, this last term \((\Theta = -4\rho/l^2)\) diverges when the “radius” \(\rho\) is taken to infinity. As we shall see on specific examples in the next two sections, this divergence is generic. However it is well known \[7\] that the quasilocal energy is not uniquely defined, but depends on the boundary conditions. Modifying the Lagrangian density by the addition of a total divergence will not change the field equations, but will modify the boundary conditions and the value of the quasi-local energy-momentum. In the present case, the “improved” action which leads to the constant quasilocal energy and angular momentum (2.19) and (2.20) is

\[
I = I_D - \frac{1}{4\kappa} \int_\mathcal{M} \partial_\mu (\sqrt{|g|} k^\mu) - \frac{1}{2\kappa} \int_\Sigma \kappa - \frac{1}{2\kappa} \oint_{\Sigma^\rho} \Theta, \tag{3.14}
\]

where

\[
\sqrt{|g|} k^\mu = \sqrt{|g|} \left(g^{\mu\nu} \Gamma^\lambda_{\nu\lambda} - g^{\nu\lambda} \Gamma^\mu_{\nu\lambda}\right) = \frac{1}{\sqrt{|g|}} \partial_\nu \left(|g| g^{\mu
u}\right). \tag{3.15}
\]

Remembering that under a variation of the boundary data on \(\partial\mathcal{M}\) for classical solutions the variation of the Dirichlet action (2.2) is

\[
\delta I_D = \frac{1}{2\kappa} \oint_{\partial\mathcal{M}} n_\mu \left(\Gamma^\lambda_{\nu\lambda} \delta G^{\mu\nu} - \Gamma^\mu_{\nu\lambda} \delta G^{\nu\lambda}\right), \tag{3.16}
\]

where \(G^{\mu\nu} = \sqrt{|g|} g^{\mu\nu}\) is the contravariant metric density, we find that the variation of the improved action (3.14)

\[
\delta I = \frac{1}{4\kappa} \oint_{\partial\mathcal{M}} n_\mu \left(\Gamma^\lambda_{\nu\lambda} \delta G^{\mu\nu} - \Gamma^\mu_{\nu\lambda} \delta G^{\nu\lambda}\right) - \left(\Gamma^\mu_{\nu\lambda} \delta G^{\nu\lambda} - G^{\nu\lambda} \delta \Gamma^\mu_{\nu\lambda}\right) \tag{3.17}
\]

vanishes for mixed Dirichlet-Neumann boundary conditions, such as those discussed in [24].
4 Einstein-scalar black holes

In this section we apply equations (2.19) and (2.20) to the computation of the mass and angular momentum of black hole solutions to some specific gravitating scalar field models. Let us note that it follows from the Minkowski space identity

\[ L^2 = \frac{\zeta^2}{4\kappa^2} \left( - R^2 P^2 + (X \cdot P)^2 \right) \]  

(4.1)
evaluated on the horizon \( R^2 = 0 \) that the constant vector \( L \) is spacelike. Therefore we can always carry out a local coordinate transformation in (2.4) (corresponding to a pseudo-rotation in target space) such that \( L^T = L^X = 0 \), i.e. \( Y = 0 \). This means that without loss of generality we can restrict ourselves to static black holes. Rotating black holes can be generated from these by the local coordinate transformation \( t = \tilde{t} + \omega \tilde{\phi}, \phi = \tilde{\phi}, \rho = \tilde{\rho} \) (\( \omega \) constant), leading to the transformed metric components

\[ \tilde{U} = U, \quad \tilde{Y} = \omega U, \quad \tilde{V} = V + \omega^2 U, \]  

(4.2)
and to the transformed energy and angular momentum \( \tilde{E} = E, \tilde{J} = -2\omega E \).

As our first example we consider the HMTZ black holes [25]. The matter Lagrangian density is

\[ \mathcal{L}_m = \frac{1}{\pi} \sqrt{|g|} \left[ -\frac{1}{2}(\nabla \phi)^2 + \frac{1}{8l^2}(\cosh^6 \phi + \nu \sinh^6 \phi) \right], \]  

(4.3)
where \( l \) and \( \nu \) are coupling constants. The static black hole solutions with regular scalar field given in [25] are, for \( \nu > -1 \),

\[ ds^2 = -\left( \frac{H}{H + B} \right)^2 F dt^2 + \left( \frac{H + B}{H + 2B} \right)^2 \frac{dr^2}{F} + r^2 d\phi^2, \]  

(4.4)

\[ \phi = \arctanh \sqrt{\frac{B}{H + B}}, \]  

(4.5)
where

\[ H(r) = \frac{1}{2} \left( r + \sqrt{r^2 + 4Br} \right), \quad F(r) = \frac{H^2}{T^2} - (1 + \nu) \left( \frac{3B^2}{T^2} + \frac{2B^3}{T^2H} \right), \]  

(4.6)
with \( B \) a non-negative integration constant, and \( G = 1 \ (\kappa = 8\pi) \). This is of the form (2.8) with \( \rho = r \) and

\[ \zeta = \frac{H + 2B}{rH}. \]  

(4.7)
Using the identity
\[ H^2 = r(H + B), \]  
we obtain
\[ \frac{U}{V} = -\frac{F}{H^2}. \] (4.9)

Differentiating this last relation and using
\[ H' = \frac{H(H + B)}{r(H + 2B)}, \] (4.10)
we obtain the mass
\[ M = \frac{3(1 + \nu)B^2}{8l^2} \] (4.11)
(with the BTZ vacuum solution \( B = 0 \) as background). This value, which can be checked to agree with the first law of black hole thermodynamics
\[ dM = T_H dS + \Omega_H dJ, \] (4.12)
is identical to that obtained in [25] by a totally different approach, namely the computation of the total charge, which involves the knowledge of the asymptotic behaviors of the metric functions and of the scalar field, and yields a finite result only after substraction of the background contribution. Computation of the Dirichlet quasilocal energy (3.10) would also lead to a divergent result (before background substraction), as the metric (4.4) is asymptotically AdS.

Our second example will be the cold black hole solutions to the model of a gravitating massless scalar field with a **negative** gravitational constant \( \kappa \) [26] (because \( 2 + 1 \) gravity is dynamically trivial, both signs of the gravitational constant are allowed [16]). The matter Lagrangian density is
\[ \mathcal{L}_m = -\frac{1}{2} \sqrt{|g|} (\nabla \phi)^2. \] (4.13)
The static rotationally symmetric solutions [27, 26]
\[ ds^2 = -x^2 dt^2 + b^2 x^{2\alpha} (dx^2 + d\phi^2), \quad \phi = a \ln x \] (4.14)
(\( \alpha = \kappa a^2 / 2 \) and \( b \) integration constants) have a Killing horizon at \( x = 0 \). We shall show that the metric (4.14) can be extended across this horizon for a discrete set of values \( \alpha < 0 \) (implying \( \kappa < 0 \)). We transform to the conformal gauge
\[ ds^2 = \left( \frac{|\alpha|}{b} r \right)^{2/\alpha} (-dt^2 + dr^2) + \alpha^2 r^2 d\phi^2 \] (4.15)
\( r = (b/|\alpha|)x^\alpha \), and define for \( \alpha \neq 0 \) Kruskal-like null coordinates \( \bar{u} \) and \( \bar{v} \) by

\[
t + r = \bar{u}^{1-n}, \quad -t + r = -v = (\bar{v})^{1-n},
\]

(4.16)

with

\[
n = \frac{2}{\alpha + 2}.
\]

(4.17)

The metric (4.15) may be written in mixed coordinates \((\bar{u}, v)\) as

\[
ds^2 = (n-1)|\frac{n-1}{b n}|(1-v\bar{u}^{n-1})d\bar{u}dv + \left(\frac{n-1}{n}\right)^2 \bar{u}^{2(1-n)(1-v\bar{u}^{n-1})^2}d\phi^2.
\]

(4.18)

For \( n > 1 \) \((-2 < \alpha < 0)\), this may be extended across the future horizon \( \bar{u} = 0 \) provided \( n \) is a positive integer. Extension across the past horizon \( \bar{v} = 0 \) is likewise achieved under the same condition. So there is a discrete sequence of regular black hole solutions (4.15) with \( \alpha = 2(1-n)/n, n \) integer \[26\]. These black holes have infinite horizon area (infinite entropy) and vanishing surface gravity (vanishing temperature), so that their mass cannot be computed as usual from the first law of black-hole thermodynamics (4.12).

To compute this mass, we note that the metric (4.14) is of the form (2.8) with \( \rho = x \) and

\[U = -x^2, \quad V = b^2x^{2\alpha}, \quad \zeta = b^{-2}x^{-1-2\alpha}.
\]

(4.19)

The computation of the energy (2.19) gives straightforwardly

\[E = \frac{\pi}{\kappa}(1 - \alpha).
\]

(4.20)

The natural background here is the vacuum solution, (4.14) with \( \alpha = 0 \) \((n = 1)\), corresponding to the (flat) rotationally symmetric Rindler metric. So the black hole masses \( M_n = E_n - E_1 \) are

\[M_n = \frac{\pi}{\kappa \alpha} = \frac{2\pi}{|\kappa|} \left(1 - \frac{1}{n}\right).
\]

(4.21)

Note that while these masses are negative, they are bounded from below, \( M_n > -2\pi/|\kappa| \).

In this very special case, the difference \(- (\pi/2\kappa)\zeta(R^2)'\) between the Dirichlet quasilocal energy (3.10) and the energy (2.19) happens to be constant and dependent on the parameter \( \alpha \), so that the quasilocal (Brown–York) energy

\[E_D = -\frac{2\pi}{\kappa \alpha},
\]

(4.22)

would lead to a value for the mass larger by a factor two than the correct value given by (4.21).
5 Conserved quantities in Einstein-Maxwell theory

The approach of Sects. 2 and 3 must be modified in the case of matter gauge fields. To be specific we consider the case of one Maxwell gauge field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with the Lagrangian density

$$\mathcal{L}_m = \frac{1}{4} \sqrt{|g|} F^{\mu\nu} F_{\mu\nu}$$

(plus possible scalar field contributions), more general cases (such as that of a Maxwell-Chern-Simons gauge field \[^{28}\]) can be treated similarly. In this case, the stationary rotationally symmetric ansatz \(^{24}\) together with the ansatz for the potential $A_\mu$, $A_\mu dx^\mu = \psi_a(\rho) dx^a$, (5.2)

reduces the gravitational plus gauge part of the action to \[^{28}\]

$$I_D = \int d^2x \frac{1}{2} \int d\rho \zeta \left( \frac{1}{2\kappa} X^2 + \overline{\psi} \Sigma \cdot X \psi' \right),$$

(5.3)

where the real Dirac-like matrices are defined by

$$\Sigma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Sigma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(5.4)

$\psi$ ($\psi'$) is the column (row) matrix of elements $(\psi_0, \psi_1)$, and $\overline{\psi} \equiv \psi^t \Sigma^0$.

The reduced problem now involves, besides the three “vector” coordinates $X^A$, two “spinor” coordinates $\psi_a$ which may be eliminated in terms of their constant conjugate momenta $\Pi^a = -\zeta^{-1} F^{ra}$,

$$\Pi = \zeta \overline{\psi} \Sigma \cdot X.$$

(5.5)

The $SL(2, R)$ invariance of the reduced action \[^{53}\] again leads to the conservation of a super angular momentum vector, which is now the sum

$$\mathbf{J} = \mathbf{L} + \mathbf{S},$$

(5.6)

of the “orbital” contribution \[^{24,17}\] and of the “spin” contribution

$$\mathbf{S} \equiv \frac{1}{2} \Pi \Sigma \psi.$$
The natural generalisation of the assignments (2.19) and (2.20) is now to identify the total physical energy and angular momentum with the relevant components of $J$, i.e.

$$E = -2\pi J^Y, \quad J = 2\pi(J^T - J^X).$$

(5.8) \hspace{1cm} (5.9)

To check this in the quasilocal approach, we note that the Dirichlet action can again be written in the canonical form (3.1), where now

$$\epsilon = \epsilon_{(g)} + \epsilon_{(m)},$$

(5.10)

with

$$\epsilon_{(m)} = \frac{1}{N}A_0\pi^r,$$

(5.11)

where $\pi^r = \sqrt{|g|}F^{r0}$. The Hamiltonian (3.6) therefore becomes

$$H_D = \oint_{\Sigma_T} (N\epsilon_{(g)} + A_0\pi^r + 2V_i\pi^{ij}n_j).$$

(5.12)

Noting that $\pi^r = -\Pi^0$, we derive from (5.12) the quasilocal energy and angular momentum

$$E_D = \frac{\pi}{\kappa} (UV' - YY') - 2\pi\Pi^0\psi_0,$$

(5.13)

$$J = \frac{\pi}{\kappa} (VY' - YV') + 2\pi\Pi^0\psi_1.$$

(5.14)

Using Eq. (2.14), (3.10) and (3.11) may be rewritten as

$$E_D = -2\pi J^Y + \frac{1}{2} \oint_{\Sigma_T} \left( \frac{1}{\kappa} \Theta - \Pi \psi \right),$$

(5.15)

$$J = 2\pi(J^T - J^X).$$

(5.16)

So the quasilocal total angular momentum for a charged configuration is indeed given by (5.9), while the quasilocal total energy is again the sum of the surface-independent value (5.8) and of a surface-dependent term which may in principle be discarded by suitably modifying the boundary conditions.

### 6 Einstein-Maxwell black holes

Let us apply our formulas (5.8) and (5.9) to the computation of the mass and angular momentum of the charged black holes constructed in [19, 28].
These solutions to the Einstein-Maxwell equations with a negative cosmological constant \( \Lambda = -l^{-2} \) depend on three parameters \( Q, r_0 \) and \( \omega \), and are given by

\[
ds^2 = -N^2 \, dt^2 + K^2 \left( d\varphi + V^\varphi \, dt \right)^2 + \frac{r^2 \, dr^2}{K^2 \, N^2},
\]

with

\[
N^2 = \frac{r^2}{l^2 K^2} \left( r^2 - l^2 \kappa Q^2 \ln(r/r_0) \right), \quad V^\varphi = -\frac{\omega}{K^2} \kappa Q^2 \ln(r/r_0),
\]

\[
K^2 = r^2 + \omega^2 \kappa Q^2 \ln(r/r_0), \quad A_\mu \, dx^\mu = Q \ln(r/r_1) \left( dt - \omega d\varphi \right),
\]

where \( l^2 = l^2 - \omega^2 \), and the gravitational constant \( \kappa \) is assumed to be positive.

It follows from the inequality

\[
r^2 - l^2 \kappa Q^2 \ln(r/r_0) \geq \frac{l^2 \kappa Q^2}{2r_0^2} \left( 1 - \ln \left( \frac{l^2 \kappa Q^2}{2r_0^2} \right) \right)
\]

that the condition for the existence of horizons is

\[
l^2 \geq 2e r_0^2 / \kappa Q^2.
\]

There are then two horizons, the event horizon radius \( r_h \) being given by the largest root of \( N^2 \),

\[
r_h^2 - l^2 \kappa Q^2 \ln(r_h/r_0) = 0,
\]

with \( r_h/r_0 \geq e^{1/2} \). The angular velocity of the horizon is

\[
\Omega_h = \omega / l^2.
\]

In the expression for the gauge potentials we have allowed for an arbitrary additive parameter \( \ln(r_1) \) common to \( A_t \) and \( A_\varphi \), which does not modify the asymptotic potentials (the choice \( r_1 = r_0 \) was made in [28]). The electric and “magnetic” charges associated with the conjugate momenta \( \Pi^a \) are related to the black hole parameters by

\[
\Pi = Q \left( 1 - \omega / l^2 \right).
\]

The computation of the quasilocal mass of the charged black hole is not straightforward. The Hamiltonian (5.12) diverges as \( r^2 \ln r \) so that, even after subtraction of the background energy, there remains a logarithmic divergence. In [30] an \textit{ad hoc} renormalization procedure was used to cancel this divergence. On the other hand in [31], the isolated horizon framework
together with a Hamiltonian approach based on a dreibein rather than a metric formulation led to a finite result. As we shall now show, our super angular momentum approach also leads to finite results, which we shall compare with those of [31].

We first put the solution (6.1) in the form (2.8), with

\[ U = -\frac{r^2}{l^2} + \kappa Q^2 L, \quad V = r^2 + \omega^2 \kappa Q^2 L, \quad Y = -\omega \kappa Q^2 L, \quad \zeta = \frac{1}{r}, \quad (6.9) \]

where \( L(r) = \ln(r/r_0) \). Computation of the orbital and spin super angular momenta gives

\[ L^Y = \left(1 + \frac{\omega^2}{l^2}\right) \frac{Q^2}{4} (1 - 2L), \quad S^V = \left(1 + \frac{\omega^2}{l^2}\right) \frac{Q^2}{2} \hat{L}, \quad (6.10) \]

\[ L^T - L^X = -\frac{\omega Q^2}{2} (1 - 2L), \quad S^T - S^X = -\omega Q^2 \hat{L}, \quad (6.11) \]

with \( \hat{L}(r) = \ln(r/r_1) \), leading to

\[ M = -E_0 + \left(1 + \frac{\omega^2}{l^2}\right) \mu, \quad (6.12) \]

\[ J = 2\omega \mu, \quad (6.13) \]

with

\[ \mu = \pi Q^2 \left[ \ln(\frac{r_1}{r_0}) - \frac{1}{2} \right]. \quad (6.14) \]

It is easily checked that these values for the mass and angular momentum satisfy for all values of \( r_1 \) the generalized Smarr-like formula, which replaces (2.30) in the charged case

\[ M = -E_0 + \frac{1}{2} T_H S + \Omega_h J + \frac{1}{2} \Phi_h \overline{Q}. \quad (6.15) \]

In (6.15), \( \overline{Q} = 2\pi Q \) is the electric charge, \( \Phi_h \) is the horizon electric potential

\[ \Phi_h \equiv -(A_t + \Omega_h A_\varphi) = -\frac{l^2}{r_1^2} Q \ln(r_h/r_1), \quad (6.16) \]

and the Hawking temperature and black hole entropy defined in (2.31) and (2.32) are here given by

\[ T_H = \frac{l^3}{l^3} \frac{\kappa Q^2}{2\pi r_h} \left( \ln(\frac{r_h}{r_0}) - 1/2 \right), \quad S = \frac{4\pi^2 r_h}{l^3} \kappa. \quad (6.17) \]
Our mass and angular momentum (6.12) and (6.13) depend on two arbitrary parameters $r_1$ and $E_0$. As mentioned in [31], the gauge parameter $r_1$, or equivalently the boundary value of the electromagnetic potentials on the horizon, is constrained by the requirement that the first law of charged black hole thermodynamics

$$dM = T_H dS + \Omega_h dJ + \Phi_h dQ$$  

holds. We start by evaluating the difference

$$dM - \Omega_h dJ = -dE_0 + \frac{l^2}{l^2} d\mu$$

$$= -dE_0 + \frac{l^2}{l^2} \pi Q^2 \left[ 2(L_1 - 1/2) \frac{dQ}{Q} + \frac{dr_1}{r_1} - \frac{dr_0}{r_0} \right]$$  

(6.19)

with $L_1 = \ln(r_1/r_0)$. The variation of (6.6) and of the entropy (6.17) leads to

$$\frac{dr_0}{r_0} = -2(L_h - 1/2) \frac{dS}{S} + \frac{dl}{l} + 2L_h \frac{dQ}{Q}$$  

(6.20)

with $L_h = \ln(r_h/r_0)$. Finally we obtain

$$dM - \Omega_h dJ - T_H dS - \Phi_h dQ = -dE_0 + \frac{l^2}{l^2} \pi Q^2 \left( \frac{dr_1}{r_1} - \frac{dQ}{Q} - \frac{dl}{l} \right).$$  

(6.21)

We assume here

$$E_0 = 0.$$  

(6.22)

It then follows that the first law is satisfied provided

$$r_1 = e^\alpha \sqrt{\frac{\kappa}{2} Q l},$$  

(6.23)

where $\alpha$ is some fixed constant, leading to

$$\mu = \pi Q^2 \left[ \alpha + \frac{1}{2} \ln \left( \frac{\kappa Q^2 l^2}{2\pi r_0^2} \right) \right].$$  

(6.24)

For extreme black holes, which correspond to the minimum of (6.5), (6.24) reduces to $\mu_{ex} = \pi Q^2 \alpha$, so that the mass and angular momentum of extreme black holes are proportional to the arbitrary constant $\alpha$. A natural choice is

$$\alpha = 0,$$  

(6.25)

such that all extreme black holes have zero mass and angular momentum. Extreme black holes also have vanishing Hawking temperature and (for $\alpha =$
vanishing horizon electric potential, but nevertheless are classified by two parameters, the horizon angular velocity \( \Omega_h = \omega/l^2 \), and the horizon perimeter or entropy proportional to the electric charge, \( S_{ex} = (4\pi^2 l/\sqrt{2\kappa})Q_{ex} \). Thus, choosing the arbitrary parameters \( E_0 \) and \( r_1 \) to have the values \( \alpha = 0 \) amounts to choosing, for each one-parameter family of black holes with given values of \( Q \) and \( \omega \) (or \( l \)), the corresponding extreme black hole as background. It then follows from the inequality \( \mu > 0 \) that \( \mu \) is positive for all nonextreme black holes, ensuring the positivity of the mass \( M \).

The neutral limit to the uncharged BTZ black holes may be performed by fixing the horizon perimeter and angular velocity, i.e., the parameters \( r_h \) and \( \omega \), and taking the electric charge \( Q \) to zero. From the relations \( \mu \) we see that this is possible only if \( r_0 \) goes to zero so that \( \pi Q^2 \ln(r_h/r_0) \) and \( \mu \) converge (for any fixed value of \( \alpha \) in \( 6.23 \)) to a fixed limit \( \mu_0 \). It follows that the neutral limit is achieved by replacing in the various metric functions of \( \kappa Q^2 \ln(r/r_0) \) by its limiting horizon value \( \kappa \mu_0/\pi \). It is easily checked that this replacement, together with the radial coordinate transformation \( r^2 = r^2 - \omega^2 \kappa \mu_0/\pi \) leads, for \( \kappa = \pi \), to the BTZ metric \( 2.24 \). On account of \( 6.5 \) this neutral limit does not commute with the extreme limit \( l \to 0 \), which leads to charged, massless black holes.

In order to compare our expressions for the black hole mass and angular momentum with those obtained in \[31\] (AWD), we note that in \[31\] \( \kappa \) has been set to 1, while their charge parameter is related to ours by \( \kappa Q_{AWD}^2 = 2\pi Q^2 \). Furthermore, the gauge choice of \[31\],

\[
A_\mu dx^\mu = Q \ln(r/l)(dt - \omega d\varphi) - Q \frac{\omega^2}{2l^2} dt, \tag{6.26}
\]
differs from our choice \( 6.3 \). The transformation from \( 6.3 \) to \( 6.26 \) may be made in two steps: first, set \( r_1 = l \) in \( 6.3 \); second, translate the electric potential by \( A_t \rightarrow A_t - Q \omega^2/2l^2 \). Going back to \( 6.21 \), we see that the first step \( (r_1 \text{ constant}) \) is consistent with the first law only if our assumption \( 6.22 \) for the background energy is replaced by

\[
\tilde{E}_0 = -\frac{\pi Q^2}{2} \frac{l^2}{l^2}. \tag{6.27}
\]

Our expression \( 6.12 \) accordingly becomes

\[
\tilde{M} = \pi Q^2 \left[ \left( 1 + \frac{\omega^2}{l^2} \right) \ln(l/r_0) - \frac{\omega^2}{l^2} \right]. \tag{6.28}
\]
The second step then gives (using e.g. (6.15))

\[ M_{AWD} = \tilde{M} + \frac{\pi Q^2 \omega^2}{2l^2} = \pi Q^2 \left( 1 + \frac{\omega^2}{l^2} \right) \ln(l/r_0) - \frac{\omega^2}{2l^2} \], \quad (6.29)

which corresponds to the value of the energy given in Eq. (IV. 10) of [31].

The comparison of the angular momentum values is more straightforward.

Rewriting equation (6.14) for \( \mu \) as

\[ \mu = \pi Q^2 \ln(r_h/r_0) - \ln(r_h/r_1) - 1/2 \], \quad (6.30)

and using (6.6), we obtain

\[ J = \omega \left[ A^2 \left( \frac{2A^2}{2l^2 \kappa} - 2\pi Q^2 \ln \frac{Al}{2\\pi l r_1} - \pi Q^2 \right) \right] \] \quad (6.31)

(\( A = 2\pi r_h l/l \) being the horizon perimeter). This is equivalent to Eq. (V.13) of [31] for the choice \( r_1 = l \).

### 7 Conclusion

We have proposed a new definition for the mass and angular momentum of black holes in 2+1 gravity with two Killing vectors. These are associated with two components of the super angular momentum of the reduced mechanical system, which are finite and independent of the one-surface on which they are computed. We have compared our approach to the standard quasilocal approach, and showed that our mass and angular momentum were the quasilocal conserved quantities for an improved action corresponding to mixed boundary conditions. We have also shown that these quantities, together with the other black hole parameters, obey a general Smarr-like formula and, in all cases investigated, are consistent with the first law of black hole thermodynamics. Finally, we have tested our new definitions on the example of several models. In the case of gravitating scalar field models, our values for the mass and angular momentum agree with previous independent computations.

In the case of charged black holes, our values are also consistent with previous computations. However the situation in this case is (as in the case of four-dimensional non-asymptotically flat charged black holes) not fully satisfactory. For asymptotically flat charged black holes, the electric potential \( \Phi \) goes to a constant value at spatial infinity, and there is a natural gauge \( \Phi(\infty) = 0 \) in which to compute the quasilocal energy. When the
electric potential diverges at infinity, it is only defined up to an additive constant, which leads to an inherent one-parameter additive ambiguity in the electrostatic energy \[31\]. It has been suggested in \[29\] that the gauge should be fixed such that the electric potential be regular on the black hole horizon. However such a gauge-fixing is not consistent with the first law in the case of the (2+1)-dimensional charged black holes studied in \[31\] and here.

Let us also emphasize that our new definitions apply only to the case where the matter fields depend only on the radial coordinate. For instance, they do not apply to the conical spacetime \[32\] generated by a delta-function source \(\delta^2(\mathbf{x})\) (the quasi-local computation of the energy in this case has been given in \[5\]). Another example where our formalism does not apply is the regular asymptotically conical spacetime generated by a \(\sigma\)-model scalar field mapping the two-plane on the two-sphere \[33\].

Finally, let us comment on the comparison of our approach with the counterterm approach \[34, 35\]. In this procedure, inspired by the AdS/CFT correspondence, the quasilocal stress-energy of gravity is renormalized by adding to the action a finite number of boundary curvature invariants with coefficients fixed to ensure finiteness of the stress tensor when the boundary is sent to infinity. This approach was very recently extended to the case of 2+1 gravity with a minimally coupled scalar field, leading to a finite quasilocal energy involving a counterterm which depends explicitly on the scalar field \[36\]. The resulting value of the HMTZ black hole mass coincides with our result \(4.11\). The advantage of our procedure is that, in the neutral case, our mass and angular momentum are defined entirely in terms of boundary data (on an arbitrary boundary) of the metric tensor field.

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