Extrinsic curvatures of distributions of arbitrary codimension

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Abstract

In this article, using the generalized Newton transformations, we define higher order mean curvatures of distributions of arbitrary codimension and we show that they coincide with the ones from Brito and Naveira (Ann. Global Anal. Geom. 18, 371–383 (2000)). We also introduce higher order mean curvature vector fields and we compute their divergence for certain distributions and using this we obtain total extrinsic mean curvatures.

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1 Introduction

Using some special forms $\Gamma_r$ Brito and Naveira [7] defined higher order extrinsic curvatures of distributions and they computed the total $r$th mean curvature $S^T_r$ of certain distributions on closed spaces of constant curvature. They generalize the ones for foliations [3, 6, 11, 12, 15].

On the other hand, many authors (see, among the others, [2, 4, 9, 10, 12]) have recently investigated higher order mean curvatures and higher order mean curvature vector fields of hypersurfaces using the Newton transformations of the second fundamental form. Especially, the papers [9, 10] are devoted to submanifolds of codimension greater than one. In this paper we show that these methods can be also applied successfully for distributions of arbitrary codimension. Namely, using the generalized Newton transformation $T_r$ we define $r$th mean curvature $S_r$ and $(r+1)$th mean curvature vector field $S_{r+1}$ of a distribution $D$. We show that they agree with the ones from [7] (Theorem 3.1). Since most of the interesting and useful integral
formulae in Riemannian geometry are obtained by computing the divergence of certain vector fields and applying the divergence theorem, we compute the divergence of $(r+1)$th mean curvature vector field of a distribution which is orthogonal to a totally geodesic foliation in a manifold of constant sectional curvature (Theorem 3.7). Using this quantity we obtain a recurrence formula for the total mean curvatures (Corollary 3.8) and consequently we get another proof of the main theorem from [7] (Theorem 3.2).

The paper is organized as follows. Section 2 provides some preliminaries. The main results of the paper are contained in Section 3. Throughout the paper everything (manifolds, distribution, metrics and etc.) is assumed to be $C^\infty$-differentiable and oriented and we usually work with $S_r$ instead of its normalized counterpart $H_r$.

2 Preliminaries

Let $M$ be a $m$-dimensional oriented, connected Riemannian manifold. On $M$ we consider a distribution $D$, $n = \dim D$ and a distribution $F$ which is the orthogonal complement of $D$, $l = \dim F = m - n$. We assume that both are orientable and transversally orientable. Let $\langle \cdot, \cdot \rangle$ represent a metric on $M$ and $\nabla$ denote the Levi-Civita connection of the metric. Let $\Gamma(D)$ denote the set of all vector fields tangent to $D$. If $v$ is a vector tangent to $M$, then we write

$$v = v^\top + v^\perp,$$

where $v^\top$ is tangent to $D$ and $v^\perp$ to $F$. Define the second fundamental form $B$ of the distribution $D$, by

$$B(X, Y) = (\nabla_Y X)^\perp,$$

where $X, Y$ are vector fields tangent to $D$. The second fundamental form of $F$ is defined similarly.

Throughout this paper we will use the following index convention: $1 \leq i, j, \ldots \leq n$, $n + 1 \leq \alpha, \beta, \ldots \leq m$, and $1 \leq A, B, \ldots \leq m$. Repeated indices denote summation over their range. Let us take a local orthonormal frame $\{e_A\}$ adapted to $D, F$, i.e., $\{e_i\}$ are tangent to $D$ and $\{e_\alpha\}$ are tangent to $F$. Moreover, the frames $\{e_A\},\{e_i\}$ and $\{e_\alpha\}$ are compatible with the orientation of $M, D$ and $F$, respectively. Let $\{\theta^i\}$ and $\{\theta^\alpha\}$ be their dual frame and $\omega^{AB}(e_C) = -\langle \nabla_{e_C} e_A, e_B \rangle$

Define the second fundamental form (or the shape operator) $A^\alpha$ of $D$ with respect to $e_\alpha$, by

$$A^\alpha(X) = -\langle \nabla_X e_\alpha \rangle^\top,$$
for $X$ tangent to $D$. Then, using the notation

$$A^α_i e_i = A^{αj}_i e_j \quad \text{and} \quad B^i_j = B(e_i, e_j),$$

we have

$$B^i_j = A^{αj}_i e_α.$$ Note that, matrices $A^{αj}_i$ and $B^i_j$ are not symmetric with respect to $i, j$ if $D$ is not integrable. In spite of this, for even $r \in \{1, \ldots, n\}$, we can define $r$th mean curvature $S_r$ of the distribution $D$ by

$$S_r = \frac{1}{r!} \delta^{i_1 \ldots i_r}_{j_1 \ldots j_r} \langle B^{i_1}_{j_1}, B^{i_2}_{j_2} \rangle \cdots \langle B^{i_{r-1}}_{j_{r-1}}, B^{i_r}_{j_r} \rangle,$$

where the generalized Kronecker symbol $δ^{i_1 \ldots i_r}_{j_1 \ldots j_r}$ is +1 or −1 according as the $i$’s are distinct and the $j$’s are either even or odd permutation of the $i$’s, and is 0 in all other cases. By convention, we put $S_0 = 1$ and $S_{n+1} = 0$.

Moreover, for even $r \in \{0, \ldots, n-1\}$ we define $(r+1)$th mean curvature vector field $S_{r+1}$ of $D$ by

$$S_{r+1} = \frac{1}{(r+1)!} \delta^{i_1 \ldots i_{r+1}}_{j_1 \ldots j_{r+1}} \langle B^{i_1}_{j_1}, B^{i_2}_{j_2} \rangle \cdots \langle B^{i_{r+1}}_{j_{r+1}} \rangle.$$

We put $S_{n+1} = 0$. If $D$ is of codimension one, then $S_{r+1} = S_{r+1}N$ where $N$ is a unit vector field orthogonal to $D$, see [1]. The normalized $r$th mean curvature $H_r$ of a distribution $D$ is defined by

$$H_r = S_r \binom{n}{r}^{-1}.$$

Obviously, the functions $S_r$ ($H_r$ respectively) are smooth on the whole $M$. If the distribution $D$ is integrable, then for any point $p \in M$, $S_r(p)$ coincides with the $r$th mean curvature at $p$ of the leaf $L$ of foliations which passes through $p$ [2] [9].

Now, we introduce the operators $T_r : \Gamma(D) \rightarrow \Gamma(D)$ which generalizes the Newton transformations of the shape operator for hypersurfaces and foliations (see, among the others, [11, 14, 19, 10, 12]).

For every $r \in \{1, \ldots, n\}$, we set

$$T^i_r \vdash j = \frac{1}{r!} \delta^{i_1 \ldots i_{r+1}}_{j_1 \ldots j_{r+1}} \langle B^{i_1}_{j_1}, B^{i_2}_{j_2} \rangle \cdots \langle B^{i_{r+1}}_{j_{r+1}} \rangle,$$

and by convention $T_0 = I$. Note that $T_n = 0$. We also set for a fixed index $α$

$$T^{α}_{r-1} \vdash j = \frac{1}{(r-1)!} \delta^{i_1 \ldots i_{r-1} j}_{j_1 \ldots j_{r-1} i} \langle B^{i_1}_{j_1}, B^{i_2}_{j_2} \rangle \cdots \langle B^{i_{r-1}}_{j_{r-1}}, B^{i_{r-2}}_{j_{r-2}} \rangle A^{α}_{i_{r-1}}.$$

In the following lemma, we provide some relations between the $r$th mean curvature (vector field) and the operator $T_r$. 

Lemma 2.1 For any even integer $r \in \{1, \ldots, n\}$ we have

$$S_r = \frac{1}{r} \text{Tr}(T_{r-1}^\alpha A^\alpha),$$

$$S_{r+1} = \frac{1}{r+1} \text{Tr}(T_r A^\alpha) e_\alpha,$$

$$\text{Tr}(T_r) = (n - r) S_r,$$

$$T_r = S_r I - A^\alpha T_{r-1}^\alpha,$$

and when $r$ is odd, for each $\alpha$, we have

$$\text{tr}(T_r^\alpha) = \frac{n - r}{r} \text{Tr}(T_{r-1}^\alpha A^\alpha),$$

where $\text{Tr} = \text{Tr}_D = (\cdot)_i^i$.

Proof. The proof of lemma is quite similar to the one for submanifolds [9, 10], we must only be more careful because $B_j^i$ need not be a symmetric matrix. \(\square\)

On the other hand, Brito and Naveira [7] have introduced $n$-forms $\Gamma_r$ for even $r = 2s$ as follows:

$$\Gamma_r = \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma)(\omega^{(1)}_{\sigma(1)} \wedge \omega^{(2)}_{\sigma(2)}) \wedge \cdots \wedge (\omega^{(2s-1)}_{\sigma(2s-1)} \wedge \omega^{(2s)}_{\sigma(2s)}) \wedge$$

$$\wedge \theta^{(2s+1)} \wedge \cdots \wedge \theta^{(n)},$$

where $\Sigma_n$ is the group of permutations of the set $\{1, \ldots, n\}, \varepsilon(\sigma)$ stands for the sign of the permutation $\sigma$. Furthermore, they define the total $r$th extrinsic mean curvature $S_r^T$ of a distribution $D$ on a compact manifold $M$ as

$$S_r^T = \frac{1}{r!(n-r)!} \int_M \Gamma_r \wedge \nu,$$

where $\nu = \theta^{n+1} \wedge \cdots \wedge \theta^m$. This suggests that we should have

$$\frac{1}{r!(n-r)!} \Gamma_r \wedge \nu = S_r \Omega,$$

where $\Omega$ is volume element of $(M, \langle \cdot, \cdot \rangle)$. We will show this equality in the next section.

3 Main results

Using definitions and notations as in Preliminaries, we obtain the following theorem which states that, $S_r^T$ defined by Brito and Naveira [7] is indeed the total mean curvature of the distribution in our sense.
Theorem 3.1 If \( r = 2s \), \( S_r \) is the \( r \)th mean curvature of the distribution \( D \) and \( \Gamma_r \) is defined by (1), then we have

\[
\frac{1}{r!(n-r)!} \Gamma_r \wedge \nu = S_r \Omega.
\]

Proof. Using the following expression for the generalized Kronecker symbol

\[
\delta_{j_1 \ldots j_r}^{i_1 \ldots i_r} = \left| \begin{array}{ccc}
\delta_{j_1}^{i_1} & \cdots & \delta_{j_r}^{i_r} \\
\vdots & \ddots & \vdots \\
\delta_{j_1}^{i_1} & \cdots & \delta_{j_r}^{i_r}
\end{array} \right| = \sum_{\tau \in \Sigma_r} \varepsilon(\tau) \delta_{j_1}^{i_1} \cdots \delta_{j_r}^{i_r},
\]

we have

\[
S_r = \frac{1}{r!} \delta_{j_1 \ldots j_r}^{i_1 \ldots i_r} A^{\alpha_1 i_1} A^{\alpha_2 i_2} \cdots A^{\alpha_{2s-1} i_{2s-1}} A^{\alpha_{2s} i_{2s}}
\]

\[
= \frac{1}{r!} \sum_{j_1 \ldots j_r \text{ distinct}} \delta_{j_1 \ldots j_r}^{i_1 \ldots i_r} A^{\alpha_1 i_1} A^{\alpha_2 i_2} \cdots A^{\alpha_{2s-1} i_{2s-1}} A^{\alpha_{2s} i_{2s}}
\]

\[
= \frac{1}{r!} \sum_{j_1 \ldots j_r \text{ distinct}} \sum_{\tau \in \Sigma_r} \varepsilon(\tau) \delta_{j_1}^{i_1} \cdots \delta_{j_r}^{i_r} A^{\alpha_1 i_1} A^{\alpha_2 i_2} \cdots A^{\alpha_{2s-1} i_{2s-1}} A^{\alpha_{2s} i_{2s}}.
\]

On the other hand, by the definition of \( \omega^{i\alpha} \), we deduce

\[
\omega^{i\alpha}(e_j) = \langle e_i, \nabla_{e_j} e_\alpha \rangle = -A^{\alpha j}_i,
\]

thus

\[
\omega^{i\alpha} = -A^{\alpha j}_i \theta^j + X^{i\alpha}_\beta \theta^\beta.
\]
From (1) and (3), we have

\[ \Gamma_r \wedge \nu = \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma)(A^{\alpha_1 \sigma(1)}_{j_1} A^{\alpha_2 \sigma(2)}_{j_2} \cdots A^{\alpha_s \sigma(2s-1)}_{j_{2s-1}} A^{\alpha_s \sigma(2)}_{j_{2s}} \theta^{j_1} \wedge \cdots \wedge \theta^{j_{2s}}) \wedge \]

\[ \wedge \theta^{(2s+1)} \wedge \cdots \wedge \theta^{2s} \wedge \nu \]

\[ = \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma) \sum_{\tau \in \Sigma\{\sigma(1) \ldots \sigma(2s)\}} \left( \varepsilon(\tau) A^{\alpha_1 \sigma(1)}_{\tau(\sigma(1))} A^{\alpha_1 \sigma(2)}_{\tau(\sigma(2))} \cdots \right. \]

\[ \left. A^{\alpha_s \sigma(2s-1)}_{\tau(\sigma(2s-1))} A^{\alpha_s \sigma(2s)}_{\tau(\sigma(2s))} \right) \theta^{\sigma(1)} \wedge \cdots \wedge \theta^{\sigma(n)} \wedge \nu \]

\[ = \sum_{\sigma \in \Sigma_n} \left( \sum_{\tau \in \Sigma\{\sigma(1) \ldots \sigma(2s)\}} \varepsilon(\tau) A^{\alpha_1 \sigma(1)}_{\tau(\sigma(1))} A^{\alpha_1 \sigma(2)}_{\tau(\sigma(2))} \cdots \right. \]

\[ \left. A^{\alpha_s \sigma(2s-1)}_{\tau(\sigma(2s-1))} A^{\alpha_s \sigma(2s)}_{\tau(\sigma(2s))} \right) \Omega \]

\[ = (n-2s)! \sum_{\sigma: \{1 \ldots 2s\} \to \{1 \ldots n\}} \left( \sum_{\tau \in \Sigma\{\sigma(1) \ldots \sigma(2s)\}} \varepsilon(\tau) A^{\alpha_1 \sigma(1)}_{\tau(\sigma(1))} A^{\alpha_1 \sigma(2)}_{\tau(\sigma(2))} \cdots \right. \]

\[ \left. A^{\alpha_s \sigma(2s-1)}_{\tau(\sigma(2s-1))} A^{\alpha_s \sigma(2s)}_{\tau(\sigma(2s))} \right) \Omega \]

\[ = (n-2s)! \sum_{j_1 \ldots j_{2s} \text{ distinct}} \left( \sum_{\tau \in \Sigma_{2s}} \varepsilon(\tau) A^{\alpha_1 j_1}_{\tau(1)} A^{\alpha_1 j_2}_{\tau(2)} \cdots \right. \]

\[ \left. A^{\alpha_s j_{2s-1}}_{\tau(2s-1)} A^{\alpha_s j_{2s}}_{\tau(2s)} \right) \Omega. \]

Comparing the above with (2) we complete the proof of our theorem. \( \square \)

Brito and Naveira have also shown that in some special cases one can compute explicitly the total mean curvature \( S_r^F \) of the distribution \( D \) and it does not depend on \( D \). Indeed, we have the following theorem [7].

**Theorem 3.2** If \( M \) is a closed manifold of constant sectional curvature \( c \geq 0 \) and \( F = D^\perp \) is a totally geodesic distribution, then

\[ S^T_{2s} = \begin{cases} 
\left( \frac{n/2}{s} \right) \left( \frac{l + 2s - 1}{2s} \right) \left( \frac{(l + 2s - 1)/2}{s} \right)^{-1} c^s \text{vol}(M) \\
\text{if } n \text{ is even and } l \text{ is odd,} \end{cases} \]

\[ 2^{2s}(s!)^2((2s)!)^{-1} \left( \frac{1/2 + s - 1}{s} \right) \left( \frac{n/2}{s} \right) c^s \text{vol}(M) \]

\[ \text{if } n \text{ and } l \text{ are even,} \]

\[ 0, \text{ otherwise.} \]

**Remark 3.3** Since the distribution \( F \) determines a totally geodesic foliation \( \mathcal{F} \) on \( M \), the constant curvature \( c \) must be nonnegative; see [15].
The next part of this section will be devoted to the calculation of the divergence of the mean curvature vector field. Next, we will use this to find a recurrence formula for the total mean curvatures and consequently we will get an alternative proof of Theorem 3.2. In order to do this we need the following lemma.

**Lemma 3.4** Let \( p \in M \) and \( \{e_1, \ldots, e_m\} \) be a local orthonormal frame field adapted to \( D \) and \( F \), such that \((\nabla_X e_i)^	op(p) = 0\) and \((\nabla_X e_\alpha)^	op(p) = 0\) for any vector field \( X \) on \( M \). Then at the point \( p \)

\[
e\alpha(A^{\beta\gamma})_j = (A^{\beta\gamma})_j - \langle R(e_j, e_\alpha) e_i, e_\gamma \rangle + \langle (\nabla_{e_\alpha} e_\gamma)^	op, e_j \rangle \langle e_i, (\nabla_{e_\alpha} e_\beta)^	op \rangle - \langle \nabla_{e_j} (\nabla_{e_\alpha} e_\beta)^	op, e_i \rangle.
\]

**Proof.** Our proof starts with the observation that at \( p \) we have the following equality

\[
0 = \langle \nabla_{e_j} \nabla_{e_\alpha} e_\beta, e_i \rangle + \langle e_\beta, \nabla_{e_j} \nabla_{e_\alpha} e_i \rangle.
\]

Thus, we have also at \( p \)

\[
- e_\alpha(A^{\beta\gamma})_j + (A^{\beta\gamma})_j - \langle R(e_j, e_\alpha) e_i, e_\beta \rangle = (A^{\beta\gamma})_j - \langle \nabla_{e_j} \nabla_{e_\alpha} e_i, e_\beta \rangle + \langle (\nabla_{e_\alpha} e_\gamma)^	op, e_j \rangle \langle e_i, (\nabla_{e_\alpha} e_\beta)^	op \rangle - \langle \nabla_{e_j} (\nabla_{e_\alpha} e_\beta)^	op, e_i \rangle.
\]

This ends the proof. \( \square \)

**Remark 3.5** Note that, using parallel transport in \( D \) and \( F \) respectively, we can always construct the frame field from Lemma 3.4.

Now, for even \( r \), we introduce auxiliary notations as follows

\[
T^r_{j_{r+1}j_{r+2}} = \frac{1}{r!} s^{i_1 \ldots i_{r+2}} (B_{i_1}^{j_1}, B_{i_2}^{j_2}) \cdots (B_{i_{r+1}}^{j_{r+1}}, B_{i_{r+2}}^{j_{r+2}}),
\]

\[
T^r_{j_{r+1}j_{r+2}j_{r+3}} = \frac{1}{r!} s^{i_1 \ldots i_{r+2}j_{r+3}} (B_{i_1}^{j_1}, B_{i_2}^{j_2}) \cdots (B_{i_{r+1}}^{j_{r+1}}, B_{i_{r+2}}^{j_{r+2}}).
\]
Lemma 3.6

\[ T_{ijr+1}^{i+1} = \delta_{r+2}^{i+1}T_{ijr+1}^{i+1} - \delta_{r+2}^{i+1}T_{ijr+1}^{i+1} - \frac{1}{r-1}T_{ijr+1}^{i+1}T_{ijr+1}^{i+1}A_{ijr+1}^{i+1}A_{ijr+1}^{i+1}. \]

Proof. The proof is analogous to the one for submanifolds [9]. \( \square \)

Now, we are ready to find the divergence of \( S_{r+1} \).

Theorem 3.7 Let \( D \) be a distribution on a Riemannian manifold \( M \) with constant sectional curvature \( c \) and \( S_r(\alpha) \) its \( r \)th mean curvature (vector field), for even \( r \in \{0, 1, \ldots, n\} \). Assume that \( F \) is a totally geodesic distribution (equivalently a totally geodesic foliation) orthogonal to \( D \). Then

\[ \text{div}(S_{r+1}) = -(r+2)S_{r+1} + \frac{c(n-r)(l+r)}{r+1}S_r, \]

where \( n = \dim D, l = \dim F \).

Proof. Let \( \{e_1, \ldots, e_m\} \) be a frame in the neighbourhood of a point \( p \) as in Lemma 3.4. By Lemma 2.1, we have at \( p \)

\[ \text{div}(S_{r+1}) = \frac{1}{r+1}\langle \nabla_{e_1}(\text{Tr}(T_rA^\alpha)e_\alpha), e_i \rangle + \frac{1}{r+1}\langle \nabla_{e_\alpha}(\text{Tr}(T_rA^\alpha)e_\alpha), e_\beta \rangle \]

\[ = \frac{1}{r+1}\text{Tr}(T_rA^\alpha)\langle \nabla_{e_1}e_\alpha, e_i \rangle + \frac{1}{r+1}e_\alpha(\text{Tr}(T_rA^\alpha)) \]

\[ = -\frac{1}{r+1}\text{Tr}(T_rA^\alpha)\text{Tr}(A^\alpha) + \frac{1}{r+1}e_\alpha(\text{Tr}(T_rA^\alpha)). \] \( (4) \)

Using the definition of \( T_r \) and the symmetries of the generalized Kronecker symbol we obtain

\[ e_\alpha(\text{Tr}(T_rA^\alpha)) = e_\alpha(T_{ijr}^i)A_{ijr}^\alpha + T_{ijr}^i e_\alpha(A_{ijr}^\alpha) \]

\[ = \frac{r}{r!}\delta_{r+1}^{i+1i+1}(B_{i1}^iB_{i2}^i) \ldots (B_{i,r-2}^iB_{i,r-1}^iA_{i,r-1}^\alpha e_\alpha(A_{i,r}^\alpha) \]

\[ + T_{ijr}^i e_\alpha(A_{ijr}^\alpha) \]

\[ = \frac{1}{r-1}T_{r-2}^{i-1i-1i-1}A_{i,r-1}^\alpha e_\alpha(A_{i,r}^\alpha) + T_{ijr}^i e_\alpha(A_{ijr}^\alpha). \] \( (5) \)

Now let us compute the terms on the right hand side of \( (5) \) one by one. From Lemma 3.4 under our assumption \( \langle \nabla_{e_\alpha}e_\beta \rangle = 0 \), we obtain

\[ e_\alpha(A_{ijr}^\alpha) = (A_{ijr}^\alpha)_{ijr} + c\delta_{ijr}^\alpha \delta_{j}^\alpha. \] \( (6) \)
Using Lemma 2.1, Lemma 3.6 and (6), we see that the first term on the right hand side of (5) is of the form

\[
\frac{1}{r-1} Tr_{-2}^{-r} \alpha_i A_{i-1}^{-1} e_\alpha (A_t) A_{\alpha}^j
\]

\[
= \frac{1}{r-1} Tr_{-2}^{-r} \alpha_i A_{i-1}^{-1} A_{i-1}^\alpha A_k^\alpha A_{i-1}^{\beta j} + \frac{t}{r-1} Tr_{-2}^{-r} \alpha_i A_{i-1}^{-1} A_{i-1}^{\beta j} \delta_i^\beta \delta_j^\beta
\]

\[
= \frac{1}{r-1} Tr_{-2}^{-r} \alpha_i A_{i-1}^{-1} A_{i-1}^\alpha A_k^\alpha A_{i-1}^{\beta j} + \frac{t}{r-1} Tr_{-2}^{-r} \alpha_i A_{i-1}^{-1} A_{i-1}^{\beta j} + cr Tr(T_r)
\]

\[
= \left( \frac{-Tr_{-2}^{-r} \alpha_i A_{i-1}^{-1} A_{i-1}^\alpha A_k^\alpha A_{i-1}^{\beta j} + cr (n-r)S_r}{r-1} \right)
\]

\[
= -(r+1)(r+2)S_{r+2} + Tr(T_r) Tr(A^\beta) + Tr(T_r A^\beta) + cr(n-r)S_r.
\]

By the use of (6) and Lemma 2.1, we see that the second term on the right hand side of (5) is of the form

\[
T_r^j e_\alpha (A_\omega^j) = Tr(T_r A^\alpha) + c Tr(T_r)
\]

\[
= Tr(T_r A^\alpha) + c(n-r)S_r.
\]

Hence (5) is of the form

\[
e_\alpha (Tr(T_r A^\alpha)) = -(r+1)(r+2)S_{r+2} + Tr(T_r A^\alpha) Tr(A^\alpha) + c(r+1)(n-r)S_r.
\]

Inserting (5) into (6) we complete the proof of theorem. \hfill \Box

**Corollary 3.8** Let D be a distribution on a closed Riemannian manifold M with constant sectional curvature c ≥ 0 and (S^T r) S r its (total) rth mean curvature. Let us assume that F is a totally geodesic distribution orthogonal to D. Then

\[
\int_M S_{r+2} = \int_M \frac{c(n-r)(l+r)}{(r+1)(r+2)} S_r,
\]

equivalently

\[
S^T_{r+2} = \frac{c(n-r)(l+r)}{(r+1)(r+2)} S^T r.
\]

Finally, note that, we can use Corollary 3.8 to prove Theorem 3.2.

**Proof of Theorem 3.2** For even n using (6) and induction one gets S^T r as in Theorem 3.2. When n is odd, then c must be zero, because there is no totally geodesic foliation on a closed Riemannian manifold of constant positive...
curvature. Indeed, without loss of generality, we may assume that $M = S^n$. For the existence of foliations the sphere should have odd dimension. Since $n$ is odd, the foliation should be even dimensional and should not contain any compact spherical leaf. Otherwise, we might pull back the Euler class of the foliation to this spherical even dimensional leaf, proving that it has Euler number zero. On the other hand, totally geodesic foliations on round spheres should have spheres as leaves - contradiction. Consequently $c = 0$ and using again (9), we complete the proof of Theorem 3.2. □

For $r = 0$ there are known applications of Theorem 3.7 in different areas of differential geometry, analysis and mathematical physics; see - for example [5, 8, 14]. The reader is warmly invited to find them for other $r$.

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