Thomae Formulae for General Fully Ramified $Z_n$ Curves

Shaul Zemel*

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Introduction

The original Thomae formula is an assertion relating the theta constants on a hyper-elliptic Riemann surface $X$, presented as a double cover of $\mathbb{P}^1(\mathbb{C})$, to certain polynomials in the $\mathbb{P}^1(\mathbb{C})$-values of the fixed points of the hyper-elliptic involution on $X$. They were initially derived by Thomae in the 19th century (see [T1] and [T2]). After laying dormant for more than a 100 years, these formulae returned to active research, mainly due to the interest of the mathematical physics community. The first generalization of these formulae appears in [BR], who considered non-singular $Z_n$ curves, i.e., those compact Riemann surfaces which are associated with an equation of the type $w^n = \prod_{i=1}^{nr} (z - \lambda_i)$ for some $r \geq 1$, with $\lambda_i \neq \lambda_j$ for $i \neq j$. The proof for this case was simplified by [N], using the Szegő kernel function. A family of singular $Z_n$ curves has been treated in [EG]. A very elementary proof for the original formulae was found in [EiF], where the case of non-singular $Z_3$ curves was also seen to be covered by these elementary techniques. The idea is that certain quotients of powers of theta functions can be identified as simple meromorphic functions on the Riemann surface under consideration. These techniques were extended to arbitrary non-singular $Z_n$ curves in [EbF]. The book [FZ] presents in detail the proof of the Thomae formulae for several families of $Z_n$ curves, namely the non-singular ones, the families treated in [EG], as well as two other smaller families. The proofs in this book follow the elementary methods of [EiF] and [EbF]. On the other hand, formulae for the general case have been obtained in [K], again using the Szegő kernel function. Additional results on the Thomae formulae for $Z_3$ curves are presented in [M] and [MT].

An important step in the proof of the Thomae formulae in all the cases considered in [FZ] is the construction of non-special divisors of degree $g$ on $Z_n$ curves, which are supported on the branch points on the $Z_n$ curve. A characterization of these divisors in the case of prime $n$ is presented in [GD], using certain sums of residues modulo $n$. The first result of the present paper

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(Theorem 1.6) is a characterization of non-special divisors on fully ramified \( Z_n \) curves for arbitrary \( n \) in terms of cardinalities of certain sets which are based on the divisor in question. In our method, it is easier to work with the cardinalities than with the residues, and our result is equivalent to that of \[ GD \] in the case of prime \( n \). Next, certain operators defined in \[ EbF \] and \[ FZ \] are useful in the derivations. We show how to define these operators for general Riemann surfaces, and provide the formula to evaluate them in the case of a fully ramified \( Z_n \) curve and a divisor which is supported on the branch points (see Theorem 2.3 and Proposition 2.5). We remark that a result of \[ GDT \] yields \( Z_n \) curves with no non-special divisors of the sort required for us. However, it turns out that the Thomae formulae which we obtain, at least in the form presented here, are independent of the cardinality conditions required for non-speciality. Therefore they can be trivially extended to the cases considered in \[ GDT \].

We now indicate how the construction and proof of the Thomae formulae are established in this paper. The process follows \[ FZ \], as well as \[ EiF \] and \[ EbF \]. We begin by identifying quotients of powers of theta functions with characteristics as meromorphic functions on the \( Z_n \) curve in order to derive relations between pairs of theta constants whose characteristics are related by operators of the form \( T_{Q,R} \), where \( Q \) is the branch point we choose as the base point. Next we obtain, for every type of points (i.e., the power to which the point appears in the \( Z_n \) equation defining the \( Z_n \) curve), a quotient which is invariant under all the operator \( T_{Q,R} \) with our base point \( Q \) and \( R \) of the chosen type. A simple correction of the resulting denominator yields, for every base point \( Q \), a quotient which is invariant under the operators \( T_{Q,R} \) for all admissible points \( R \). This quotient is called, following \[ FZ \], the Poor Man’s Thomae, or PMT for short. The next step, which is technically more difficult, is to obtain a denominator for which the quotient is invariant also under the negation operator \( N_{Q} \) (or \( N_{\beta} \)). A “base point change operator” \( M \) is also introduced, and the quotient is invariant also under \( M \). If these operators act transitively on the set of divisors under consideration, a fact which we prove in several cases and should hold in general, then the quotient \( \frac{\theta^{2\pi n^2 \Xi((0,1))}}{n_\Xi} \) thus obtained is independent of the divisor \( \Xi \). This is the Thomae formulae for the \( Z_n \) curve.

The resulting Thomae formulae are based on certain integral-valued functions, denoted here \( f_{\beta,\alpha} \) and later \( f_{d}^{(n)} \). We investigate the properties of these functions, which have an interesting recursive definition (see Theorem 6.4). We remark again that \[ K \] obtained expressions for the Thomae formulae which include sums of certain fractional parts. Comparing with our results gives a tool for evaluating these sums, and it seems that our recursive relation yields a more efficient way to obtain the actual values of these powers in every particular case.

In Section 11 we describe the fully ramified \( Z_n \) curves, and characterize the non-special divisors of degree \( g \) supported on their branch points using the cardinality conditions. Section 12 defines the operators whose action is necessary for establishing the Thomae formulae. The first relations are obtained in Section 13 and the manipulations required in order to achieve the PMT are also performed.
The quotient which is invariant under the negation operator is given in Section 4, where certain properties of the functions $f_{\beta,\alpha}$ appearing in this quotient are also proved. Section 5 introduces the base point change operator $M$, proves some partial results about transitivity, and states the final Thomae formulae. The recursive relation which is required in order to evaluate the functions $f_{\beta,\alpha}$ is proved in Section 6, where explicit expressions for these functions are given in a few cases. Section 7 presents examples of Thomae formulae for two families of $\mathbb{Z}_n$ curves, including all the $\mathbb{Z}_n$ curves which are treated in [FZ]. Finally, in Section 8 we discuss some remaining open questions.

1 Non-Special Divisors on $\mathbb{Z}_n$ Curves

Let $X$ be a $\mathbb{Z}_n$ curve, namely a cyclic cover of order $n$ of $\mathbb{P}^1(\mathbb{C})$, with projection map $z$. We assume $n > 1$ throughout, since a $\mathbb{Z}_1$ curve is just $\mathbb{P}^1(\mathbb{C})$ with a chosen isomorphism $z$. Any such curve can be presented as the Riemann surface associated with an equation of the sort $w^n = f(z)$ (called a $\mathbb{Z}_n$-equation) for some meromorphic function $f \in \mathbb{C}(z)$ which is not a $d$th power in $\mathbb{C}(z)$ for any $d$ dividing $n$ (so that the equation is irreducible). We call such a defining equation for a $\mathbb{Z}_n$ curve normalized if $f$ is a monic polynomial which has no roots of order $n$ or more. Every $\mathbb{Z}_n$-equation can be made normalized by multiplying $w$ by an appropriate function of $z$, and this function is unique as long as we keep $w$ in a fixed component under the action of the cyclic Galois group (see the end of this paragraph). The field $\mathbb{C}(X)$ of meromorphic functions on $X$ decomposes as $\bigoplus_{r=0}^{n-1} \mathbb{C}(z)w^r$, or equivalently $\bigoplus_{r=0}^{n-1} \mathbb{C}(z) \cdot \frac{1}{w^r}$. The space of meromorphic differentials on $X$ is 1-dimensional over $\mathbb{C}(X)$, and is spanned by any non-zero differential on $X$. By choosing the differential to be $dz$ we obtain that this space decomposes as $\bigoplus_{r=0}^{n-1} \mathbb{C}(z)dz^r$. The cyclic Galois group of the map $z : X \to \mathbb{P}^1(\mathbb{C})$ acts on $\mathbb{C}(X)$ and on the space of meromorphic differentials on $X$, and the decompositions given here are precisely the decompositions according to the action of this Galois group. In particular, $w$ generates a 1-dimensional complex vector space which is invariant under the action of this group.

Let $\varphi : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Recall that for every point $x \in X$ there is a unique number $b_x \in \mathbb{N}$, called the ramification index or branching number of $x$, such that in local charts around $x$ and $\varphi(x)$ the map $\varphi$ looks like $z \mapsto z^{b_x+1}$. The number $b_x$ is 0 for all $x \in X$ except for a finite number of points, called the branch points of $\varphi$. The sum $\sum_{x \in \varphi^{-1}(y)} b_x$ gives the same value for every $y \in Y$, and this value equals the rank of the map $\varphi$. Thus, generically, the inverse image of a point in $Y$ consists of $d$ points in $X$. The only points of $Y$ whose inverse image has smaller cardinality are images of branch points of $\varphi$ on $X$. The degree and branching numbers appear in the Riemann–Hurwitz formula, relating the genus $g_X$ of $X$ with the genus $g_Y$ of $Y$ according to the equality

$$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X} b_x.$$
We will consider only the case where $X$ is a $Z_n$ curve, $Y = \mathbb{P}^1(\mathbb{C})$, and $\varphi$ is the meromorphic function $z$ (which has degree $n$ since the equation is irreducible). Since the genus of $Y$ is 0, we write simply $g$ for $g_X$, with no confusion arising. Now, assume that $X$ is given through a $Z_n$-equation $w^n = f(z)$. For every point $\lambda \in \mathbb{P}^1(\mathbb{C})$, we let $d$ be the greatest common divisor of $n$ and $\text{ord}_\lambda(f)$. For each such $\lambda$ there are $d$ points of $X$ lying over $\lambda \in \mathbb{P}^1(\mathbb{C})$, each of which has ramification index $\frac{n}{d} - 1$. In particular, no point outside the divisor of $f$ is a branch point. Let $u$ be another meromorphic function on $X$, which generates $\mathbb{C}(X)$ over $\mathbb{C}(z)$ and spans a complex vector space which is invariant under the Galois group of $z$. Replacing $w$ by $u$ leaves the set of branch points, as well as their ramification indices, invariant: Indeed, multiplying $w$ by an element of $\mathbb{C}(z)$ changes the order of $f$ at $\lambda \in \mathbb{P}^1(\mathbb{C})$ by a multiple of $n$, and replacing $w$ by $w^k$ for $k \in \mathbb{Z}$ which is prime to $n$ multiplies these orders by $k$ and leaves the greatest common divisors invariant. Thus we can assume that the $Z_n$-equation is normalized. Then the branch points on which $z$ is finite lie only over the roots of $f$ and, for each such root $\zeta$, we let $u_\zeta$ be another meromorphic function on $X$, which generates $\mathbb{C}(X)$ over $\mathbb{C}(z)$ and spans a complex vector space which is invariant under the Galois group of $z$. Replacing $w$ by $u$ leaves the set of branch points, as well as their ramification indices, invariant: Indeed, multiplying $w$ by an element of $\mathbb{C}(z)$ changes the order of $f$ at $\lambda \in \mathbb{P}^1(\mathbb{C})$ by a multiple of $n$, and replacing $w$ by $w^k$ for $k \in \mathbb{Z}$ which is prime to $n$ multiplies these orders by $k$ and leaves the greatest common divisors invariant. Thus we can assume that the $Z_n$-equation is normalized. Then the branch points on which $z$ is finite lie only over the roots of $f$ (and over all of them). Points lying over $\infty$ are branch points if and only if the degree of $f$ is not divisible by $n$.

Following [FZ], we call a $Z_n$ curve $X$ fully ramified if any branch point on $X$ has maximal ramification index (namely $n - 1$). Equivalently, the curve is fully ramified if for $\lambda \in \mathbb{P}^1(\mathbb{C})$, $z^{-1}(\lambda)$ consists either of $n$ points or of a unique branch point. Given a normalized $Z_n$-equation defining the $Z_n$ curve $X$, this property is equivalent to all the roots of $f$ appearing with orders which are prime to $n$, and the degree of $f$ is either divisible by $n$ or also prime to $n$. In this case we have the following

**Lemma 1.1.** Fix $0 \leq k \leq n - 1$, and fix a branch point $P \in X$. Every meromorphic function of the form $\frac{dz}{w^k}$ with $p \in \mathbb{C}(z)$ has the same order at $P$ modulo $n$, and for $l \neq k$ the classes modulo $n$ of the orders of $\frac{dz}{w^k}$ and $\frac{dz}{w^l}$ at $\lambda$ are distinct. The same assertion holds for differentials in $\mathbb{C}(z)\frac{dz}{w}$.

**Proof.** The first assertion (for both meromorphic functions and meromorphic differentials) follows from the fact the order of an element in $\mathbb{C}(z)$ at $P$ is divisible by $n$. The second assertion is established by taking the quotient of these functions or differentials, which is of the form $\psi(z)w^{k-l}$ with $\psi \in \mathbb{C}(z)$ and $k - l$ not divisible by $n$. Since the order of $\psi(z)$ at $\lambda$ is divisible by $n$, the order of $w$ at $\lambda$ is prime to $n$, and $n$ does not divide $k - l$, this completes the proof of the lemma. \hfill \Box

Let $\Delta$ be a divisor on $X$ (not necessarily integral). We recall from [FK] (or [FZ]) that the space of meromorphic functions whose divisor is at least $\Delta$, denoted $L(\Delta)$, is finite-dimensional, of dimension denoted $r(\Delta)$. Similarly, the space of meromorphic differentials with this property (denoted $\Omega(\Delta)$) is also finite-dimensional (with $i(\Delta)$ denoting this dimension). These numbers are related by the Riemann–Roch Theorem, stating that $r(\Delta) = \deg \Delta + 1 - g + i(\Delta)$. Using the decomposition of $\mathbb{C}(X)$ and $\mathbb{C}(X)dz$ from above, we denote $\mu_k(\Delta)$ the dimension of the space of meromorphic functions of the form $\frac{f(z)}{w^k}$ (with
$p \in \mathbb{C}(z)$) which lie in $L(\Delta)$, and $i_k(\Delta)$ denotes the dimension of the space of meromorphic differentials of the form $\frac{\varphi(z)}{n}$ lying in $\Omega(\Delta)$. The inequalities $\sum_{k=0}^{n-1} r_k(\Delta) \leq r(\Delta)$ and $\sum_{k=0}^{n-1} i_k(\Delta) \leq i(\Delta)$ are clear. For divisors supported on the branch points, Lemma 1.1 yields the following generalization of Lemma 2.7 of [FZ]:

**Proposition 1.2.** Let $\Delta$ be a divisor on $X$ (not necessarily integral) which is based on the branch points of $z$, and let $h$ and $\omega$ be a meromorphic function and a meromorphic differential on $X$ respectively. Decompose $h$ and $\omega$ as $\sum_{k=0}^{n-1} h_k$ with $h_k \in \mathbb{C}(z) \frac{1}{w}$ and $\sum_{k=0}^{n-1} \omega_k$ with $\omega_k \in \mathbb{C}(z) \frac{1}{w}$ respectively. Then $h \in L(\Delta)$ (resp. $\omega \in \Omega(\Delta)$) if and only if the same assertion holds for $h_k$ (resp. $\omega_k$) for all $0 \leq k \leq n - 1$. In particular $r(\Delta) = \sum_{k=0}^{n-1} r_k(\Delta)$ and $i(\Delta) = \sum_{k=0}^{n-1} i_k(\Delta)$.

Before we prove Proposition 1.2, we introduce, following [FZ] and others, the useful notation $e(t) = e^{2\pi i t}$ for $t \in \mathbb{C}$.

**Proof.** We prove the assertion only for functions, as the claim for differentials is established by replacing every $h$ by $\omega$ etc. Lemma 1.1 shows that the orders of the different functions $h_k$ at every branch point (hence at every point at the support of $\Delta$) are distinct. Vanishing components $h_k$, which have order $+\infty$ at every point, can be ignored in all considerations. It follows that the order of $h$ (or $\omega$) at every branch point $P$ equals $\min_k ord_P(h_k)$, so that if $ord_P(h) \geq ord_P(\Delta)$ then the same inequality holds with $h$ replaced by $h_k$ for any $k$. It remains to prove that if $h$ has no pole at any point $Q$ on $X$ which is not a branch point then neither do the functions $h_k$, $0 \leq k \leq n - 1$. Let $V$ be an open subset of $\mathbb{P}^1(\mathbb{C})$ containing no $z$-image of a branch point. Hence $z^{-1}(V)$ is a disjoint union of $n$ open sets $U_i \subseteq X$, $1 \leq i \leq n$, with each $U_i$ homeomorphic to $V$ via $z$. First assume $\infty \notin V$, and observe that $w$ does not vanish on $\bigcup_{i=1}^{n} U_i$. Moreover, we can choose the indices $i$ such that the action of the homeomorphism $(z|_{U_i})^{-1} \circ z|_{U_i}$ multiplies each $h_k$ by $e\left(\frac{(j-i)k}{n}\right)$. Let $\mu \in V$, and let $Q_i \in U_i$ be the point with $z(Q_i) = \mu$. If $h_k$ has a pole in some point $Q_j$ then it has a pole of the same order at all the points $Q_i$. Let now $a = -\min_k ord_Q(h_k)$ (this number is independent of $i$), so that $a > 0$ if and only if one of the functions $h_k$ has a pole at the points $Q_i$. In coordinates we have $h_k(P) = \frac{\psi_k(z(P))}{(z(P)-\mu)^a}$ for $P \in U_i$, with $\psi_k$ a function on $V$ which has no pole at $\mu$. Since $h = \sum h_k$ has no pole at none of the points $Q_i$ we have $\sum \psi_k(\mu) = 0$ for all $i$. But the fact that $h_k$ lies in $\mathbb{C}(z) \frac{1}{w}$ yields the relation $\psi_k = e\left(\frac{(j-i)k}{n}\right) \psi_k$, so that the equality $\sum \psi_k e\left(\frac{(j-i)k}{n}\right) \psi_k(\mu) = 0$ holds for every $i$ and $j$. As the Vandermonde matrix whose $jk$-entry is $e\left(\frac{(j-i)k}{n}\right)$, is non-singular, this implies $\psi_k(\mu) = 0$ for all $i$ and $k$, in contradiction to the choice of $a$. This implies $a \leq 0$, i.e., all the functions $h_k$ are holomorphic outside the branch points. The same argument but with $h_k(P) = z(P)^a \psi_k(z(P))$ proves the assertion for the case in which $\infty \in V$ as well. The equalities involving $r(\Delta)$ and $i(\Delta)$ follow directly from the previous assertions. 

\[\square\]
We remark that the same argument shows that Proposition 1.2 holds also if we let the divisor $\Delta$ contain points which are not branch points in its support, but insist that all the points with the same $z$-value appear to the same power in $\Delta$. In fact, in this case we can replace $\Delta$ by a linearly equivalent divisor which is supported only on the branch points, using a function from $\mathbb{C}(z)^*$. Using this method one can show that Proposition 1.2 holds for any such divisor on a $\mathbb{C}$ curve, provided that there exists at least one fully ramified branch point. However, we consider only divisors supported on branch points on fully ramified $\mathbb{C}$ curves in this paper.

The set of integers between 0 and $n - 1$ (inclusive) will play a prominent role in this paper. Hence we denote it $\mathbb{N}_n$. The set $\mathbb{N}_n$ is also a good set of representatives of $\mathbb{Z}/n\mathbb{Z}$ in $\mathbb{Z}$.

We now turn to bases for the holomorphic differentials on a fully ramified $\mathbb{C}$ curve. For simplicity and symmetry, we shall assume throughout that there is no branching over $\infty$ (this can always be obtained by composing $z$ with an automorphism of $\mathbb{P}^1(\mathbb{C})$). We thus write the normalized $\mathbb{C}$-equation defining $X$ as

$$w^n = \prod_{\alpha} \prod_{i=1}^{r_{\alpha}} (z - \lambda_{\alpha,i})^\alpha,$$

where $\alpha$ runs over the set of numbers in $\mathbb{N}_n$ which are prime to $n$. The assumption that no branch point lies over $\infty$ is equivalent to the assertion that $n$ divides $\sum_{\alpha} \alpha r_{\alpha}$. The genus $g$ of $X$ equals $(n - 1)(\sum_{\alpha} r_{\alpha} - 2)/2$ by the Riemann–Hurwitz formula. This number is always an integer: This is clear if $n$ is odd, and if $n$ is even then so is $\sum_{\alpha} \alpha r_{\alpha}$, and since we take only odd $\alpha$, the same assumption holds for $\sum_{\alpha} r_{\alpha}$. Proposition 1.2 implies that we can decompose the space $\Omega(1)$ of holomorphic differentials on $X$ as $\bigoplus_{k=0}^{n-1} \Omega_k(1)$. Now, $\Omega_0(1)$ is the space of holomorphic differentials in $\mathbb{C}(z)dz$, i.e., holomorphic differentials which are pullbacks of holomorphic differentials on $\mathbb{P}^1(\mathbb{C})$. As there are no such differentials, the decomposition is in fact $\bigoplus_{k=1}^{n-1} \Omega_k(1)$. It follows that for an integral divisor $\Delta$, Proposition 1.2 yields $i(\Delta) = \sum_{k=1}^{n-1} i_k(\Delta)$ (since $i_0(\Delta) = 0$).

We shall denote, here and throughout, the poles of $z$ on $X$ by $\infty_h$, $1 \leq h \leq n$. The (unique) branch point on $X$ lying over $\lambda_{\alpha,i}$ will be denoted $P_{\alpha,i}$. Then

$$\text{div}(z - \lambda_{\alpha,i}) = \frac{P_{\alpha,i}^n}{\prod_{h=1}^{n} \infty_h}, \quad \text{div}(w) = \prod_{\alpha} \prod_{i=1}^{r_{\alpha}} \frac{P_{\alpha,i}^n}{\prod_{h=1}^{n} \infty_h^{t_{\alpha,i}}}, \quad \text{div}(dz) = \frac{\prod_{\alpha} \prod_{i=1}^{r_{\alpha}} P_{\alpha,i}^{n-1}}{\prod_{h=1}^{n} \infty_h^{2}}$$

where $\text{div}$ denotes the divisor of a meromorphic function or differential and $nt_1 = \sum_{\alpha} \alpha r_{\alpha}$. We now introduce a convenient basis for the space $\mathbb{C}(X)dz$ over $\mathbb{C}(z)$. For any $k \in \mathbb{Z}$ and any $\alpha$ we define $s_{\alpha,k} = \lfloor \frac{\alpha k}{n} \rfloor$, where for a real number $x$ the symbol $\lfloor x \rfloor$ stands for the integral value of $x$, namely the maximal integer $m$ satisfying $m \leq x$. Then $s_{\alpha,k}$ satisfies $\frac{\alpha k + 1 - n}{n} \leq s_{\alpha,k} \leq \frac{\alpha k}{n}$. Moreover, the number $\alpha k - n s_{\alpha,k}$, which lies in $\mathbb{N}_n$, depends only on the class of $k$ modulo $n$. Let $\omega_k = \prod_{\alpha} \prod_{i=1}^{r_{\alpha}} (z - \lambda_{\alpha,i})^{s_{\alpha,k}} \frac{dz}{w}$. This differential is well-defined for $k \in \mathbb{Z}/n\mathbb{Z}$, though we usually assume $k \in \mathbb{N}_n$. We remark that for $k$ prime to $n$, the denominator under $dz$ in $\omega_k$ corresponds to the normalized $\mathbb{C}$-equation.
proves the proposition. Moreover, if the greatest common divisor of \( k \) and \( n \) is \( d \) then this denominator corresponds to the normalized \( Z_{n/d} \)-equation describing the quotient of \( X \) by the subgroup of order \( d \) of the Galois group, which is a \( Z_{n/d} \) curve. We evaluate

\[
\operatorname{div}(\omega_k) = \prod_{\alpha} \prod_{i=1}^{r_{\alpha}} P_{\alpha,i}^{n-1+n s_{\alpha,k} - \alpha k} \prod_{h=1}^{n} \infty_h^{t_h - 2}, \quad t_k = \sum_{\alpha} r_{\alpha} \left( \frac{\alpha k}{n} - s_{\alpha,k} \right). \tag{2}
\]

The numbers \( n s_{\alpha,k} - \alpha k - 1 + n \) lie also in \( \mathbb{N}_n \), and observe that \( t_k \in \mathbb{Z} \) (since \( n|\sum_{\alpha} r_{\alpha} \)). Moreover, \( t_k \) vanishes if \( n|k \) and is positive for every \( k \) not divisible by \( n \) (or \( 1 \leq k \leq n - 1 \)), since all the summands are positive (recall that we take only \( \alpha \) which is prime to \( n \)). In particular \( \omega_0 = dz \) with the divisor written above, and \( \omega_1 = \frac{dz}{\omega_0} \) since \( s_{\alpha,1} = 0 \) for all \( \alpha \). Hence the two formulae for \( t_1 \) coincide. We define, for every \( d \in \mathbb{Z} \) with \( d \geq -1 \), the space \( P_{\leq d}(z) \) of polynomials in \( z \) of degree not exceeding \( d \) (so that \( P_{\leq 0}(z) \) is the space of constant polynomials and \( P_{\leq -1}(z) = \{0\} \)). The dimension of \( P_{\leq d}(z) \) is \( d + 1 \) (also for \( d = -1 \)). We therefore obtain

**Proposition 1.3.** The space \( \Omega(1) \) of holomorphic differentials on \( X \) decomposes as \( \bigoplus_{k=1}^{n-1} P_{\leq t_k - 2}(z)\omega_k \).

**Proof.** Proposition 1.2 and the paragraph below Equation (1) show that \( \Omega(1) \) decomposes as \( \bigoplus_{k=1}^{n-1} \Omega_k(1) \). It therefore suffices to show that a differential in \( \mathbb{C}(z) \frac{dz}{\omega_k} \), or equivalently \( \mathbb{C}(z) \omega_k \), is holomorphic if and only if it lies in \( P_{\leq t_k - 2}(z)\omega_k \). Let \( \varphi \in \mathbb{C}(z) \) and assume that \( \varphi(z)\omega_k \) is holomorphic. The divisor of \( \omega_k \) is supported only on the branch points and poles of \( z \), and the former points appear to non-negative powers which are smaller than \( n \) in this divisor. It follows that \( \varphi \) cannot have any pole in \( \mathbb{C} \), hence it must be a polynomial of some degree \( d \). But then the order of \( \varphi(z)\omega_k \) at any point \( \infty_h \) is \( t_k - 2 - d \), so that \( \varphi(z)\omega_k \) is holomorphic precisely when \( d \leq t_k - 2 \) (and this implies \( \varphi = 0 \), i.e., there exists no such holomorphic differential, if \( t_k = 1 \)). This proves the proposition. \( \Box \)

When we evaluate \( \sum_{k=1}^{n-1} t_k = \sum_{\alpha,k} r_{\alpha} \left( \frac{\alpha k}{n} - \left\lfloor \frac{\alpha k}{n} \right\rfloor \right) \), we observe that for any \( \alpha \) which is prime to \( n \) the set of numbers \( \left\{ \frac{\alpha k}{n} - \left\lfloor \frac{\alpha k}{n} \right\rfloor \right\}_{k=1}^{n-1} \) (or equivalently \( \left\{ \frac{\alpha k - n s_{\alpha,k}}{n} \right\}_{k=1}^{n-1} \)) is precisely the set \( \left\{ \frac{1}{\pi} \right\}_{l=1}^{n-1} \). Hence the sum \( \sum_{k=1}^{n-1} (t_k - 1) \) of the dimensions of these spaces is indeed \( \sum_{\alpha,k} r_{\alpha} \frac{n(a-1)}{2n} - (n - 1) = g \), as required.

Lemma 1.1 and Proposition 1.3 imply that the set \( \bigcup_{k=1}^{n-1} \{ (z - \lambda_{\alpha,i})^i \omega_k \}_{l=0}^{t_k - 2} \) is a basis for \( \Omega(1) \) which is adapted to the point \( P_{\alpha,i} \) for any \( \alpha \) and \( i \). The gap sequence at \( P_{\alpha,i} \) can be read from this basis. However, it is not needed for finding non-special divisors or for the proof of the Thomae formulae. Moreover, in some of the examples in [FZ] some of the points \( P_i \) have the usual gap sequence and are not Weierstrass points. For these reasons we do not pursue this subject further in this work.

Using the notation \( |Y| \) for the cardinality of the finite set \( Y \), we prove
Corollary 1.4. Let $\Delta$ be an integral divisor on $X$ which is supported on the branch points, and assume that no branch point appears in $\Delta$ to a power $n$ or higher. For any $\alpha$ and any $1 \leq k \leq n-1$ denote $A_{\alpha,k}$ the set of indices $1 \leq i \leq r_\alpha$ such that $P_{\alpha,i}$ appears in $\Delta$ to a power larger than $ns_{\alpha,k} - \alpha k - 1 + n$. Then $i_\alpha(\Delta) = \max\{t_k - 1 - \sum\alpha |A_{\alpha,k}|, 0\}$ and $i(\Delta)$ is the sum of these numbers.

Proof. By Proposition 1.3, $\Omega_k(1)$ is a space of differentials of the form $p(z)\omega_k$, where $p$ is a polynomial of degree not exceeding $t_k - 2$. We claim that $\Omega_k(\Delta)$ consists of those differentials in which $p$ vanishes at all the values $\lambda_{\alpha,i}$ with $i \in A_{\alpha,k}$. Indeed, since no branch point appears in $\Delta$ to a power $n$ or higher, if $p(\lambda_{\alpha,i}) = 0$ then $\operatorname{ord}_{P_{\alpha,i}}(p(z)\omega_k) \geq n > \operatorname{ord}_{P_{\alpha,i}}(\Delta)$. Hence simple zeroes of $p$ suffice. For an index $i$ not lying in $A_{\alpha,k}$ we have $\operatorname{ord}_{P_{\alpha,i}}(\omega_k) \geq \operatorname{ord}_{P_{\alpha,i}}(\Delta)$, and multiplying by any polynomial in $z$ can only increase the order of the differential at $P_{\alpha,i}$. On the other hand, if $i \in A_{\alpha,k}$ then $p$ must vanish at $\lambda_{\alpha,i}$ in order for $\operatorname{ord}_{P_{\alpha,i}}(\varphi(z)\omega_k)$ to reach $\operatorname{ord}_{P_{\alpha,i}}(\Delta)$. This shows that $\Omega_k(\Delta)$ is indeed the asserted space. Since the conditions $p(\lambda_{\alpha,i}) = 0$ are linearly independent (unless we reach the 0 space), the assertion about $i_\alpha(\Delta)$ follows. The assertion about $i(\Delta)$ is now a consequence of Proposition 1.2. This proves the corollary. \[\square\]

The definition of $A_{\alpha,k}$ extends to arbitrary $k \in \mathbb{Z}$ by considering the image of $k$ in $\mathbb{Z}/n\mathbb{Z}$. For $k$ divisible by $n$ all the sets $A_{\alpha,k}$ are empty, and $i_0(\Delta) = 0$ since $t_0 = 0$.

The following argument has been used in several special cases in \[\text{[FZ]}\]. We include it here since it is simple, short, and general. Recall that an integral divisor of degree $g$ on a Riemann surface of genus $g$ is called special if $i(\Delta) > 0$, and is called non-special otherwise.

Lemma 1.5. Any integral divisor $\Delta$ of degree $g$ on a fully ramified $Z_n$ curve $X$ containing a branch point to power $n$ or higher is special.

Proof. Let $Q$ be a branch point on $X$. Apart from the constant functions, the space $L(1/Q^n)$ contains the meromorphic function $\frac{1}{z-z(Q)}$ if $z(Q) \in \mathbb{C}$ or the function $z$ if $z(Q) = \infty$ (by full ramification). It follows that $r(1/Q^n) \geq 2$, hence $r(1/\Delta) \geq 2$ if $Q^n$ divides $\Delta$. But then the Riemann–Roch Theorem implies $i(\Delta) \geq 1$ (since the degree of $\Delta$ is $g$), hence $\Delta$ is special. \[\square\]

We will be interested in non-special divisors supported on the branch points on a fully ramified $Z_n$ curve. In the quest for such divisors, Lemma 1.5 allows us to restrict attention to divisors in which the branch points appear only to powers at most $n-1$, without losing possibilities. Every such divisor $\Delta$ can be written as

$$\Delta = \prod_{\alpha} \prod_{l=0}^{n-1} C_{\alpha,l}^{n-1+l},$$

where for every $\alpha$ the sets $C_{\alpha,l}$, $l \in \mathbb{N}_n$, form a partition of the set of points $\{P_{\alpha,i}\}_{i=1}^{r_\alpha}$. 

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We can now characterize the non-special divisors of degree $g$ supported on the branch points on $X$ by appropriate cardinality conditions.

**Theorem 1.6.** Let $\Delta$ be an integral divisor of degree $g$ which is supported on the branch points on $X$. Then $\Delta$ is non-special if and only if it can be written as in Equation (3) and the cardinalities of the sets $C_{\alpha,l}$ satisfy the equality

$$\sum_{\alpha} \sum_{l=0}^{ak/ns_{\alpha,k} - 1} |C_{\alpha,l}| = t_k - 1$$

for every $1 \leq k \leq n-1$.

**Proof.** Lemma 1.5 allows us to restrict attention to divisors $\Delta$ in which the branch points appear to powers not exceeding $n - 1$. We can thus define, for every $l \in \mathbb{N}_n$, the set $C_{\alpha,l}$ to contain those branch points $P_{\alpha,i}$ appearing to the power $n - 1 - l$ in $\Delta$. Every point $P_{\alpha,i}$ must lie in some set $C_{\alpha,l}$. Thus $\Delta$ takes the form given in Equation (3), and the sets $C_{\alpha,l}$ form the required partitions.

As $l \leq ak - ns_{\alpha,k} - 1$ is equivalent to $n - 1 - l > n - 1 + ns_{\alpha,k} - ak$, it follows from Corollary 1.4 that $\Delta$ is non-special if and only if $\sum_{\alpha} \sum_{l=0}^{ak - ns_{\alpha,k} - 1} |C_{\alpha,l}| \geq t_k - 1$ for every $1 \leq k \leq n - 1$. But taking the sum over $k$ yields $g$ on the right hand side, and we claim that the sum of the left hand sides equals the degree of $\Delta$. Indeed, the set $\{ak - ns_{\alpha,k}\}_{k=1}^{n-1}$ consists precisely of the numbers between 1 and $n - 1$ (as $\alpha$ is prime to $n$), precisely $n - 1 - l$ of those numbers are larger than $l$. Since the degree of $\Delta$ is $g$, all these inequalities must hold as equalities, which completes the proof of the theorem. \qed

One can verify that Theorems 2.6, 2.9, 2.13, and 2.15 of [FZ], as well as the claim in Section A.7 of that reference, are special cases of Theorem 1.6. This verification requires some care: The sets $C_j$ and $D_j$ of [FZ] correspond to our sets $C_{1,j+1}$ and $C_{n-1,n-2-j}$ respectively, and the $j$th cardinality condition in these special cases is obtained by taking the difference of consecutive equalities in Theorem 1.6. It is also possible to verify that Theorems 6.3 and 6.13 of [FZ] follow from Theorem 1.6. As a point in $C_{\alpha,l}$ appears to the power $n - 1 - l$ in $\Delta$, we find that adding $ak$ to it and then taking the number in $\mathbb{N}_n$ which is congruent to the result yields

$$n - 1 - l + ak - ns_{\alpha,k} - n\chi(l < ak - ns_{\alpha,k}),$$

where $\chi$ of a given condition gives 1 if the condition is satisfied and 0 otherwise. It follows that the sum appearing in Theorem 1 of [GD] and Theorem 2 of [GDT] equals $g + nt_k - n \sum_{\alpha} \sum_{l=0}^{ak - ns_{\alpha,k} - 1} |C_{\alpha,l}|$, and for prime $n$ Theorem 1.6 is equivalent to the results given in these references. Moreover, this argument shows that the results of [GD] and [GDT] extend to arbitrary $n$, provided that the $Z_n$ curve is fully ramified (which is always the case when $n$ is prime). Note that it can happen that no divisors satisfying the conditions of Theorem 1.6 exist (see [GDT]).
2 Operators on Divisors

Let $X$ be a compact Riemann surface of genus $g > 0$. By taking a canonical basis for the homology of $X$, one obtains a symmetric matrix $\Pi \in M_g(\mathbb{C})$, the period matrix of $X$ with respect to this basis, whose imaginary part is positive definite. We identify the Jacobian variety $J(X)$ with the complex torus $\mathbb{C}^g/\mathbb{Z}^g \oplus \Pi \mathbb{Z}^g$. Let $\text{Div}(X)$ denote the group of divisors on $X$, and let $\text{Div}^0(X)$ be the subgroup of $\text{Div}(X)$ consisting of those divisors whose degree is 0. For a point $Q$ on $X$, we denote $\phi_Q$ the Abel–Jacobi map from $\text{Div}(X)$ to $J(X)$ with base point $Q$ (see Chapter 3 of [FK] or Chapter 1 of [FZ] for some properties of this map). It is related to the algebraic Abel–Jacobi map $\phi: \text{Div}^0(X) \to J(X)$ by $\phi_Q(\Delta) = \phi(\Delta + \frac{Q}{2\pi i})$. Hence on divisors of degree 0 the value of $\phi_Q$ is independent of the choice of the base point $Q$ (see also Equation (1.1) of [FZ]).

Given two vectors $\varepsilon$ and $\varepsilon'$ in $\mathbb{R}^g$, one defines the theta function with characteristics $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ and period matrix $\Pi$ as

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\zeta, \Pi) = \sum_{N \in \mathbb{Z}^g} e^{\frac{1}{2} \left( N + \frac{\varepsilon}{2} \right)^t \Pi \left( N + \frac{\varepsilon}{2} \right) + \left( N + \frac{\varepsilon}{2} \right)^t \left( \zeta + \frac{\varepsilon'}{2} \right)}.$$

For the properties of this function see Chapter 6 of [FK] or Section 1.3 of [FZ]. In particular, up to a non-zero factor, the characteristics correspond to translations of the variable $\zeta$ (see Equation (1.3) of [FZ]) in the classical theta function with $\varepsilon = \varepsilon' = 0$. We are interested in theta constants, i.e., the values of theta functions with rational characteristics at $\zeta = 0$. The original Thomae’s formula is a relation between these theta constants on a hyper-elliptic Riemann surface (or, in our language, a $\mathbb{Z}_2$ curve). Here we extend this formula to arbitrary fully ramified $\mathbb{Z}_n$ curves.

Take a point $e$ in $J(X)$ (or in $\mathbb{C}^g$), and consider the multi-valued function $f(P) = \theta(\phi_Q(P) - e, \Pi)$ on $X$. The Riemann Vanishing Theorem (see, e.g., Theorem 1.8 of [FZ]) states that $f$ either vanishes identically on $X$ or has precisely $g$ (well-defined) zeroes (counted with multiplicity). In the latter case the divisor $\Delta$ of zeroes of $f$ is non-special and satisfies $e = \phi_Q(\Delta) + K_Q$, where $K_Q$ is the vector of Riemann constants associated with $Q$. Moreover, any element of $e \in J(X)$ can be written as $\phi_Q(\Delta) + K_Q$ for some integral divisor $\Delta$ of degree $g$ on $X$ by the Jacobi Inversion Theorem. Proposition 1.10 of [FZ] shows that $f$ vanishes identically if and only if $\Delta$ is special. Observe that otherwise the presentation of $e$ as $\phi_Q(\Delta) + K_Q$ is unique: Indeed, applying the Riemann–Roch Theorem for a non-special integral divisor $\Delta$ of degree $g$ yields $r(\frac{1}{2}) = 1$. Hence $L(\frac{1}{2}) = \mathbb{C}$ (the constant functions), and there is no other integral divisor $\Xi$ of degree $g$ such that $\phi_Q(\Xi) = \phi_Q(\Delta)$ and $e = \phi_Q(\Xi) + K_Q$.

The following proposition about the vector of Riemann constants is very useful in the theory of Thomae formulae:

**Proposition 2.1.** $\phi_Q$ takes any canonical divisor on $X$ to $-2K_Q$.

*Proof.* See the theorem on page 298 of [FK], or page 21 of [FZ].
The dependence of $K_Q$ on the base point $Q$ is given through the fact that $\varphi_Q(\Delta) + K_Q$ is independent of $Q$ if $\Delta$ is a divisor of degree $g - 1$ on $X$ (see Theorem 1.12 of [FZ]).

The following property of the vector of Riemann constants, in case the base point is a branch point on a fully ramified $Z_n$ curve, has been obtained in a few special cases in [FZ] (see Lemma 2.4, Lemma 2.12, Lemma 6.2, and Lemma 6.12 of that reference). However, it turns out to hold in general:

**Lemma 2.2.** Let $Q$ be a branch point on a fully ramified $Z_n$ curve of genus $g \geq 1$. Then the vector $K_Q$ of Riemann constants associated with the base point $Q$ has order dividing $2n$ in $J(X)$.

**Proof.** Let $\mu = z(Q) \in \mathbb{C}$. Since $g \geq 1$, there exists some $1 \leq k \leq n - 1$ such that $t_k \geq 2$, and then the divisor of $\omega = (z - \mu)^{t_k - 2}\omega_k$ is supported only on the branch points. But the fact that $\frac{n^2}{z^2}$ is principal for any branch point $R$ (as the divisor of $\frac{z - z(R)}{z - \mu}$) implies that $n\varphi_Q(\text{div}(\omega)) = 0$. In case $z(Q) = \infty$, the divisor of every differential $\omega_k$ is supported on the branch points, and $\frac{R^n}{Q}$ is the divisor of $z - z(R)$. The conclusion $n\varphi_Q(\text{div}(\omega)) = 0$ follows also in this case. As $\varphi_Q(\text{div}(\omega)) = -2K_Q$ by Proposition 2.11, the assertion follows.

In all the cases considered in [FZ], the Thomae formulae have been proved using two types of operators, denoted $N$ and $T_R$ (with base point $P_0$), acting on the set of non-special divisors of degree $g$ which are supported on the branch points distinct from $P_0$. We now show that these operators exist in general (not only on $Z_n$ curves!). Let $X$ be an arbitrary Riemann surface of genus $g \geq 1$. We denote $v_Q(\Delta)$ the power to which the point $Q$ on $X$ appears in the divisor $\Delta$ on $X$.

**Theorem 2.3.** (i) Let $\Delta$ be a non-special integral divisor of degree $g \geq 1$ on $X$, and let $Q$ be a point on $X$ such that $v_Q(\Delta) = 0$. There exists a unique integral divisor $N_Q(\Delta)$ of degree $g$ on $X$ satisfying

$$\varphi_Q(N_Q(\Delta)) + K_Q = - (\varphi_Q(\Delta) + K_Q).$$

(4)

The divisor $N_Q(\Delta)$ is non-special, and satisfies $v_Q(N_Q(\Delta)) = 0$. The operator $N_Q$ is an involution on the set of non-special integral divisors of degree $g$ not containing $Q$ in their support. (ii) Given any point $R$ such that $v_R(N_Q(\Delta)) = 0$, there exists a unique integral divisor $T_{Q,R}(\Delta)$ of degree $g$ on $X$ such that the equality

$$\varphi_Q(T_{Q,R}(\Delta)) + K_Q = - (\varphi_Q(\Delta) + \varphi_Q(R) + K_Q)$$

(5)

holds. The divisor $T_{Q,R}(\Delta)$ is also non-special, and we have the equalities $v_Q(T_{Q,R}(\Delta)) = 0$ and $v_R(N_Q(T_{Q,R}(\Delta))) = 0$. The operator $T_{Q,R}$, which is defined on the set of non-special divisors on $X$ not containing $Q$ in their support and such that $R$ does not appear in $N_Q(\Delta)$, is an involution on this set of divisors.
Proof. Denote by $e \in J(X)$ the expression on the right hand side of Equation (4), and consider the (multi-valued) function $f(P) = \theta(\varphi_Q(P) + e, \Pi)$. Since $-e$ equals $\varphi_Q(\Delta) + K_Q$ and $i(\Delta) = 0$, we find that $f$ does not vanish identically, but rather vanishes only on points in the support of $\Delta$. The condition $v_Q(\Delta) = 0$ thus implies $\theta(e, \Pi) \neq 0$, and since $\theta$ is an even function, we deduce $\theta(-e, \Pi) \neq 0$. But this implies that $\psi(P) = \theta(\varphi_Q(P) - e, \Pi)$ does not vanish at $P = Q$, hence does not vanish identically. Thus $e = \varphi_Q(\Xi) + K_Q$ for some non-special divisor $\Xi$ representing the zeroes of $\psi$, so that in particular $v_Q(\Xi) = 0$. Since $\Xi$ and $N_Q(\Delta)$ are both integral of degree $g$ and have the same $\varphi_Q$-images, the fact that $\Xi$ is non-special implies $\Xi = N_Q(\Delta)$. The fact that $N_Q(\Delta)$ is non-special and $v_Q(N_Q(\Delta)) = 0$ yields the existence of a unique divisor $N_Q(N_Q(\Delta))$ satisfying Equation (1) with $\Delta$ replaced by $N_Q(\Delta)$. As $\Delta$ satisfies this equation, the equality $N_Q(N_Q(\Delta)) = \Delta$ follows, and $N_Q$ is an involution. This proves (i). In order to establish (ii) we denote the value on the right hand side of Equation (5) by $d$, and consider the multi-valued function $\varphi(\Delta) = \theta(\varphi_Q(P) + d, \Pi)$. As $\varphi(R) = f(Q)$ and the latter expression is non-vanishing, we find that $-d$ can be written as $\varphi_Q(\gamma) + K_Q$ where $\gamma$ is a non-special integral divisor of degree $g$ representing the zeroes of $\varphi$ (hence $v_R(\gamma) = 0$). Moreover, $\varphi(Q)$ equals $f(R)$ and is also non-vanishing by our assumption on $R$. This shows that $v_Q(\gamma) = 0$ as well, and we define $T_Q,R(\Delta) = N_Q(\gamma)$. The equality $\gamma = N_Q(T_Q,R(\Delta))$ (as $N_Q$ is an involution) and part (i) imply that $T_Q,R(\Delta)$ has the asserted properties. In particular, $T_Q,R(T_Q,R(\Delta))$ is defined, and since it is characterized by satisfying Equation (5) with $\Delta$ replaced by $T_Q,R(\Delta)$, we deduce that $T_Q,R$ is an involution as in part (i). This completes the proof of the theorem.

Note that part (ii) of Theorem 2.3 does not require that $R \neq Q$. However, if $R = Q$ then the right hand side of Equation (5) reduces to that of Equation (4), implying that $T_Q,Q(\Delta)$ is simply $N_Q(\Delta)$. We shall therefore always assume $R \neq Q$ in $T_Q,R$.

We are interested in the form of the operators $N_Q$ and $T_Q,R$ in the case where $X$ is a fully ramified $Z_n$ curve and $Q$ and $R$ are branch points on $X$. Assume that $X$ is associated with Equation (1), $\Delta$ is given by Equation (3), and $Q = P_{\beta,i}$ for some $\beta \in N_n$ which is prime to $n$ and some index $i$. Hence $\mu = z(Q)$ equals $\lambda_{\beta,i}$, but we keep the notation $\mu$. Let $k_{\beta}$ be an integer such that $n|\beta k_{\beta} - 1$ (hence $\beta k_{\beta} - ns_{\beta,k_{\beta}} = 1$). This characterizes the class of $k_{\beta}$ in $\mathbb{Z}/n\mathbb{Z}$ (we rather not impose the assumption $k_{\beta} \in \mathbb{N}_n$). The point $Q$ does not lie in the support of $\Delta$ if and only if $Q \in C_{\beta,n-1}$ in the notation of Equation (3). For any $\alpha$ and $l$ we denote by $a_{\beta,\alpha}(l)$ and $b_{\beta,\alpha}(l)$ the elements of $\mathbb{N}_n$ which are congruent modulo $n$ to $\alpha k_{\beta} - 1 - l$ and $2\alpha k_{\beta} - 1 - l$ respectively. These numbers are of course independent of the choice of $k_{\beta} \in \mathbb{Z}$. We consider $a_{\beta,\alpha}$ and $b_{\beta,\alpha}$ as functions on $\mathbb{N}_n$, and these functions are involutions. Two useful equalities concerning these involutions are given in the following

Lemma 2.4. The equality $a_{\beta,\alpha}(b_{\beta,\alpha}(l)) = n - 1 - a_{\beta,\alpha}(l)$ holds for every $\alpha$, $\beta$, and $l \in \mathbb{N}_n$. It is equivalent to the equality $a_{\beta,\alpha}[b_{\beta,\alpha}(a_{\beta,\alpha}(l))] = n - 1 - l$ holding for all such $\alpha$, $\beta$, and $l$.
Proof. The first equality follows from the fact that both expressions are elements of \( \mathbb{N}_n \) which are congruent to \( l - \alpha k_\beta \) modulo \( n \). The second equality is obtained from the first by replacing \( l \) by \( a_\beta,\alpha(l) \) and using the fact that \( a_\beta,\alpha \) is an involution. This proves the lemma.

It will turn our convenient to let the index \( l \) of \( C_{\alpha,l} \) to be any integer, while identifying \( C_{\alpha,l} \) with \( C_{\alpha,l+n} \) for every \( l \in \mathbb{Z} \). In this way we can consider the set \( C_{\alpha,\alpha k_\beta - n s_\alpha, k_\beta} \), for example, without having to write \( C_{\alpha,\alpha k_\beta - n s_\alpha, k_\beta} - ns_\alpha, k_\beta \).

We now assume that the \( \mathbb{Z}_n \) curve has genus \( g \) at least 1, for the theory of theta functions to be non-trivial. This means \( \sum r_\alpha \geq 3 \) by the expression for \( g \). The following proposition generalizes Definitions 2.16, 2.18, 6.4, and 6.14 of [FZ], as well as Propositions 2.17, 2.19, 6.5, and 6.15 there:

**Proposition 2.5.** If \( \Delta \) is given by Equation (3) and \( v_Q(\Delta) = 0 \) then the divisor \( N_Q(\Delta) \) is defined by the formula

\[
N_Q(\Delta) = \prod_{\alpha} \prod_{l=0}^{n-1} C_{\alpha,l}^{n-1-a_\beta,\alpha(l)} / Q^{n-2}.
\]

Moreover, assume that the branch point \( R \neq Q \) does not appear in the support of \( N_Q(\Delta) \) (this means \( R \in C_{\gamma,\gamma k_\beta} \) if \( R = P_{\gamma,m} \) for some index \( m \)). Then \( T_{Q,R}(\Delta) \) is given by

\[
T_{Q,R}(\Delta) = \prod_{\alpha} \prod_{l=0}^{n-1} C_{\alpha,l}^{n-1-b_\beta,\alpha(l)} / RQ^{n-3},
\]

and \( v_R(T_{Q,R}(\Delta)) = v_R(\Delta) \).

**Proof.** Denote the asserted values of \( N_Q(\Delta) \) and \( T_{Q,R}(\Delta) \) by \( \Xi \) and \( \Psi \) respectively. By Theorem 2.3 it suffices to prove that \( \Xi \) and \( \Psi \) are of degree \( g \) and satisfy Equation (4) and (5) respectively. The latter equations are equivalent to

\[
\varphi_Q(\Delta : \Xi) = \varphi_Q(\Delta : R : \Psi) = -2K_Q,
\]

so that by Proposition 2.1 it suffices to find differentials on \( X \) such that their divisors have the same \( \varphi_Q \)-images as \( \Delta \Xi \) or \( \Delta R \Psi \). Consider the differentials \( (z - \mu)^{t_{k_\beta}} \omega_{k_\beta} \) and \( (z - \mu)^{t_{k_\beta}} \omega_{k_\beta} \). Equation (2) shows that their divisors are

\[
\prod_{\alpha} \prod_{i=1}^{r_\alpha} P_{\alpha,i}^{n-1+ns_\alpha, k_\beta - ak_\beta} Q^{n(t_{k_\beta} - 2)} \quad \text{and} \quad \prod_{\alpha} \prod_{i=1}^{r_\alpha} P_{\alpha,i}^{n-1+ns_\alpha, 2k_\beta - 2ak_\beta} Q^{n(t_{k_\beta} - 2)}
\]

respectively (in fact, our choice of \( k_\beta \) shows that the total power of \( Q \) in these divisors are \( n(t_{k_\beta} - 1) - 2 \) and \( n(t_{k_\beta} - 1) - 3 \) respectively). Now, for any \( \alpha \) and \( l \) the equalities

\[
(n - 1 - l) + (n - 1 - a) = n - 1 + ns_\alpha, k_\beta - ak_\beta + n\chi(l < ak_\beta - ns_\alpha, k_\beta)
\]

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and

\[(n - 1 - l) + (n - 1 - b) = n - 1 + ns_{\alpha, 2k_\beta} - 2ak_\beta + n(\chi(l < 2ak_\beta - ns_{\alpha, 2k_\beta})
\]

hold, where \(a\) and \(b\) stand for \(a_{\beta, \alpha}(l)\) and \(b_{\beta, \alpha}(l)\) respectively. Indeed, both sides are congruent to \(-1 - \eta ak_\beta\) modulo \(n\) (with \(\eta\) being 1 for the first equation and 2 for the second one), and the two numbers on the left hand side and the number on the right hand side not involving the conditional expression are all elements of \(\mathbb{N}_n\). As an index \(i\) satisfies \(i \in A_{\alpha, \eta k_\beta}\) if and only if \(P_{i, \alpha}\) lies in a set \(C_{\alpha, l}\) with \(l < \eta ak_\beta - ns_{\alpha, \eta k_\beta}\), it follows that

\[
\Delta \cdot \Xi = \prod_{\alpha} \left( \prod_{l=1}^{r_{\alpha}} P_{\alpha, i}^{n_{\alpha, \beta} - 2ak_\beta} \right) / Q^{n-2}
\]

and

\[
\Delta \cdot R \cdot \Psi = \prod_{\alpha} \left( \prod_{l=1}^{r_{\alpha}} P_{\alpha, i}^{n_{\alpha, \beta} - 2ak_\beta} \right) / Q^{n-3}.
\]

The number of points \(P_{\alpha, i}\) appearing to the power \(n\) is \(\sum_{\alpha} \sum_{l=0}^{t_{k_\beta} - 1} \frac{n_{\alpha, \beta} - ns_{\alpha, 2k_\beta}}{|C_{\alpha, l}|}\) or \(\sum_{\alpha} \sum_{l=1}^{t_{k_\beta} - 1} \frac{n_{\alpha, \beta} - ns_{\alpha, 2k_\beta} - 1}{|C_{\alpha, l}|}\), These divisors equal \(t_{k_\beta} - 1\) and \(t_{2k_\beta} - 1\) respectively by Theorem 1.6 as \(\Delta\) is non-special. Hence \(\Delta \Xi\) is linearly equivalent to \(\prod_{\alpha} \prod_{l=1}^{r_{\alpha}} P_{\alpha, i}^{n_{\alpha, \beta} - 2ak_\beta} Q^{n(t_{k_\beta} - 1) - (n-2)}\), while \(\Delta R \Psi\) is linearly equivalent to \(\prod_{\alpha} \prod_{l=1}^{r_{\alpha}} P_{\alpha, i}^{n_{\alpha, \beta} - 2ak_\beta} Q^{n(t_{2k_\beta} - 1) - (n-3)}\). These divisors are \(Q^2\) times the divisor of \((z - \mu)^{t_{k_\beta} - 2}\omega_{k_\beta}\) and \(Q^3\) times the divisor of \((z - \mu)^{t_{2k_\beta} - 2}\omega_{2k_\beta}\) given above. Since the degree of a canonical divisor is \(2g - 2\) and \(\varphi_Q(Q) = 0\), this proves that \(\Xi\) and \(\Psi\) have the required properties. Hence \(\Xi = N_Q(\Delta)\) and \(\Psi = N_Q(\Delta)\) as desired. Observe that for \(l = \gamma k_\beta - ns_{\gamma, k_\beta}\) we have \(a_{\beta, \gamma}(l) = n - 1\) (as desired for \(v_R(N_Q(\Delta)) = 0\)) and \(b_{\beta, \gamma}(l)\) is congruent to \(\gamma k_\beta - 1\) modulo \(n\). It follows that \(v_R(T_Q, R(\Delta))\) coincides with \(v_R(\Delta)\) since the division by \(R\) covers for this difference of 1 between \(l\) and \(b_{\beta, \gamma}(l)\). This proves the proposition. \(\square\)

The part of Proposition 6.2 concerning \(N_Q\) relates to Proposition 6.2 of [K], with a simpler proof.

### 3 The Poor Man’s Thomae Formulae

Let \(Q\) be a point on a Riemann surface \(X\) with period matrix \(\Pi\) with respect to a canonical basis, and let \(\Delta\) be a divisor of degree \(g\) on \(X\). If \(\varphi_Q(\Delta) + \overline{K}_Q\) is the \(J(X)\)-image of \(\Pi^2 + I\overline{\Pi} \in \mathbb{C}^g\) then we denote, following Section 2.6 of [FZ], the theta function with characteristics \([\begin{array}{c} \varepsilon \\ \varepsilon' \end{array}]\) by \(\theta(Q, \Delta, (z, \Pi))\). This function depends on the choice of the lift (i.e., \(\varepsilon\) and \(\varepsilon'\) not up to \(2\mathbb{Z}^g\)). However, if \(X\) is a fully ramified \(Z_n\) curve, \(Q\) is a branch point, and \(\Delta\) is supported on the
branch points, then the vectors $\varepsilon$ and $\varepsilon'$ lie in $\frac{1}{n}\mathbb{Z}^g$ (see Lemma 2.3). In this case Equation (1.4) of [FZ] shows that changing the lift can only multiply the function by a constant which is a root of unity of order dividing $2n$. It follows that $\theta^{2n}(Q, \Delta)(z, \Pi)$ is independent of the lift. The same assertion thus holds for $\theta^{en^2}(Q, \Delta)(\varepsilon, \Pi)$ where $e$ is 1 for even $n$ and 2 for odd $n$. The arguments of Section 2.6 of [FZ] show that given two non-special divisors $\Delta$ and $\Xi$ of degree $g$ which are supported on the branch points distinct from $Q$, the quotient $\frac{\theta^{en^2}(Q, \Delta)(\varepsilon, \Pi)}{\theta^{en^2}(Q, \Xi)(\varepsilon, \Pi)}$ is a well-defined function on $X$, which is a constant multiple $\prod_{i=1}^n (z - \lambda_{\alpha,i})^{\nu_{P\alpha,i}}(N_Q(\Delta))^{-\nu_{P\alpha,i}}(N_Q(\Xi))$ (see Propositions 2.21, 6.6, and 6.16 for that reference for special cases). Moreover, if $R$ is some branch point such that $v_R(Q,\Delta) = v_R(Q,\Xi) = 0$ then the value of this function at $R$ equals $\frac{\theta^{en^2}(Q, T_{Q,R}(\Delta))(0, \Pi)}{\theta^{en^2}(Q, T_{Q,R}(\Xi))(0, \Pi)}$. This generalizes Equations (2.1) and (2.2) of [FZ] to the general setting considered here, and choosing $\Xi = T_{Q,R}(\Delta)$ (as we shall soon do) yields the corresponding generalization of Equations (2.3) and (2.4) of that reference.

We now obtain relations between theta constants on $X$, following the method used in all the special cases presented in [FZ]. By substituting $P = Q$ in the quotient given in the previous paragraph we obtain the value of the constant, so that this quotient equals

$$\frac{\theta^{en^2}(Q, \Delta)(0, \Pi)}{\theta^{en^2}(Q, \Xi)(0, \Pi)} \cdot \frac{\prod_{i=1}^n (\mu - \lambda_{\alpha,i})^{\nu_{P\alpha,i}}(N_Q(\Delta))}{\prod_{i=1}^n (\mu - \lambda_{\alpha,i})^{\nu_{P\alpha,i}}(N_Q(\Xi))} \cdot \frac{\prod_{i=1}^n (z - \lambda_{\alpha,i})^{\nu_{P\alpha,i}}(N_Q(\Delta))}{\prod_{i=1}^n (z - \lambda_{\alpha,i})^{\nu_{P\alpha,i}}(N_Q(\Xi))}.$$ 

By choosing $\Xi = T_{Q,R}(\Delta)$ and substituting $P = R$ we obtain the equality

$$\frac{\theta^{2en^2}(Q, \Delta)(0, \Pi)}{\prod_{i=1}^n (\mu - \lambda_{\alpha,i})^{\nu_{P\alpha,i}}(N_Q(\Delta)) \prod_{i=1}^n (\sigma - \lambda_{\alpha,i})^{\nu_{P\alpha,i}}(N_Q(\Xi))} = \frac{\theta^{2en^2}(Q, T_{Q,R}(\Delta))(0, \Pi)}{\prod_{i=1}^n (\mu - \lambda_{\alpha,i})^{\nu_{P\alpha,i}}(N_Q(T_{Q,R}(\Delta))) \prod_{i=1}^n (\sigma - \lambda_{\alpha,i})^{\nu_{P\alpha,i}}(N_Q(\Delta))},$$

where $\sigma = z(R)$.

Write $\Delta$ as in Equation (5) in order to express the latter equality using the sets appearing that Equation. The divisor $N_Q(\Delta)$ is given in Proposition 2.3 and using the fact that $b_{\beta,\alpha}$ is an involution on $N_\alpha$ we write the formula for $T_{Q,R}(\Delta)$ in Proposition 2.3 as $\prod_{i=1}^{n-1} C^{\alpha,i}_{\beta,\alpha}(l)/RQ^{n-3}$. Thus $N_Q(T_{Q,R}(\Delta))$ is $\prod_{I=0}^{n-1} C^{\alpha,i}_{\beta,\alpha}(l)/Q^{R^{n-1}}$, or equivalently $\prod_{I=0}^{n-1} C^{\alpha,i}_{\beta,\alpha}(l)/Q^{R^{n}}$ by the involutive property of $b_{\beta,\alpha}$ and Lemma 2.3. The powers of $R$ and $Q$ are determined by the condition that both points must not appear in the support of $N_Q(T_{Q,R}(\Delta))$. Let $S$ be a point in $X$ and let $Y$ and $Z$ be (finite) disjoint subsets of points on $X$. Following Definition 4.1 of [FZ], we introduce the notation

$$[S, Y] = \prod_{T \in Y \cap T \neq S} (z(S) - z(T)), \quad [Y, Z] = \prod_{S \in Y \cap T \in Z} (z(S) - z(T)).$$

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and

\[ [Y, Y] = \prod_{S < T \in Y} (z(S) - z(T)) \]

for some ordering on the points of \( Y \). When taken to an even power, the expression \([Y, Y]\) becomes independent of the order and \([Y, Z]\) coincides with \([Z, Y]\). In order to ease notation in some expressions below we shorthand the sets \( C_{\alpha, l} \setminus \{Q\} \) and \( C_{\alpha, l} \setminus \{Q, R\} \) to simply \( C_{\alpha, l}^Q \) and \( C_{\alpha, l}^{QR} \). The denominators under \( \theta^{2en^2}[Q, \Delta](0, \Pi) \) and \( \theta^{2en^2}[Q, T_{Q,R}(\Delta)](0, \Pi) \) in Equation (6) become

\[
\prod_{\alpha, l} [Q, C_{\alpha, l}^{QR}][\gamma_l[n-1-a_{\beta, \alpha}(l)] \prod_{\alpha, l} [R, C_{\alpha, l}^{QR}][\gamma_l[n-1-a_{\beta, \alpha}(l)]
\]

and

\[
\prod_{\alpha, l} [Q, C_{\alpha, l}^{QR}][\gamma_l[n-1-a_{\beta, \alpha}(l)] \prod_{\alpha, l} [R, C_{\alpha, l}^{QR}][\gamma_l[n-1-a_{\beta, \alpha}(l)]
\]

respectively. We prefer to use the sets \( C_{\alpha, l}^Q \) and \( C_{\alpha, l}^{QR} \) rather than evaluating the powers of \((\sigma - \mu)\) which have to be canceled since the symmetrization is easier in this way.

In order to free the denominator under \( \theta^{2en^2}[Q, \Delta](0, \Pi) \) in Equation (6) from its dependence on \( R \in C_{\gamma, \gamma k_\beta}^Q \) we would like to divide that Equation by the expression

\[
\prod_{(\alpha, l) \neq (\gamma, \gamma k_\beta)}\left[ C_{\gamma, \gamma k_\beta}^{QR}C_{\alpha, l}^{QR}\gamma_{l}^{n-1-a_{\beta, \alpha}(l)} : C_{\gamma, \gamma k_\beta}^{QR}C_{\gamma, \gamma k_\beta}^{QR}\gamma_{l}^{n-1} \right].
\]

The latter multiplier is obtained by setting \( \alpha = \gamma \) and \( j = \gamma k_\beta \), but as the behavior of \([Y, Z]\) for \( Y \cap Z = \emptyset \) is different from that of \([Y, Y]\), we prefer to separate this terms from the product. Now, Equation (6) is symmetric under interchanging \( \Delta \) and \( T_{Q,R}(\Delta) \), and we wish to preserve this symmetry. By writing the formula for \( T_{Q,R}(\Delta) \) from Proposition 2.3 as \( \prod_{\alpha, l} C_{\alpha, l}^{n-1-\ell} \) we obtain, after omitting the problematic points \( Q \) and \( R \), the equality \( \overline{C}_{\alpha, l}^{QR} = C_{\alpha, b_{\beta, \alpha}(l)}^{QR} \).

As \( b_{\beta, \gamma} \) subtracts 1 from \( \gamma k_\beta - n\kappa_{\gamma, k_\beta} \), the fact that \( b_{\beta, \alpha} \) is an involution for every \( \alpha \) and Lemma 2.4 allow us to write the expression by which we have divided Equation (6) as

\[
\prod_{(\alpha, l) \neq (\gamma, \gamma k_\beta-1)}\left[ \overline{C}_{\gamma, \gamma k_\beta-1}^{QR}C_{\alpha, l}^{QR}\gamma_{l}^{n-1-a_{\beta, \alpha}(l)} : \overline{C}_{\gamma, \gamma k_\beta-1}^{QR}C_{\gamma, \gamma k_\beta-1}^{QR}\gamma_{l}^{n-1} \right].
\]

In order to keep the symmetry, we must divide Equation (6) also by

\[
\prod_{(\alpha, l) \neq (\gamma, \gamma k_\beta-1)}\left[ C_{\gamma, \gamma k_\beta-1}^{QR}C_{\alpha, l}^{QR}\gamma_{l}^{n-1-a_{\beta, \alpha}(l)} : C_{\gamma, \gamma k_\beta-1}^{QR}C_{\gamma, \gamma k_\beta-1}^{QR}\gamma_{l}^{n-1} \right].
\]

The following observations help to simplify the result. First, as \( R \in C_{\gamma, \gamma k_\beta} \) and \( \gamma k_\beta - 1 \neq \gamma k_\beta \) in \( \mathbb{Z}/n\mathbb{Z} \), we can omit the superscript \( R \) from \( C_{\gamma, \gamma k_\beta}^{QR} \). The
same assertion holds for any set \( C_{\alpha,l} \) with \( (\alpha,l) \neq (\gamma,\gamma k_\beta) \). Second, the sets \( C_{\gamma,\gamma k_\beta-1} \) and \( C_{\gamma,\gamma k_\beta} \) do not contain \( Q \) (since \( Q \in C_{\beta,n-1} \) and if \( \gamma = \beta \) then neither \( \gamma k_\beta \in 1 + n\mathbb{Z} \) nor \( \gamma k_\beta - 1 \in n\mathbb{Z} \) are congruent to \( n - 1 \) modulo \( n \)), so that we can omit \( Q \) from its notation as well. Third, the set \( C^{Q,R}_{\gamma,\gamma k_\beta} \) (which is the only set \( C^{Q,l}_{\alpha,l} \) which really differs from \( C^{Q}_{\alpha,l} \)) appears, in the expression involving \( Q \) or \( C_{\gamma,\gamma k_\beta-1} \), to the power 0 (as \( a_{\beta,\gamma}(\gamma k_\beta - ns_{\gamma,k_\beta}) = n - 1 \)). Hence the superscript \( R \) can be omitted from the notation there as well. Corollary 4.3 of [FZ] now allows us, when considering the total product, to add \( R \) and \( Q \) to the appropriate sets. Let \( C^+_Q \) denote the set \( C_{\gamma,\gamma k_\beta-1} \cup \{Q\} \), and then the product of the denominator appearing under \( \theta^{2en^2}(Q,\Delta)|(0,\Pi) \) in Equation (6) and the correction terms considered above equals

\[
q^\Delta_\gamma = \prod_{(\alpha,l)\neq(\gamma,\gamma k_\beta)} [C_{\gamma,\gamma k_\beta}, C^{Q}_{\alpha,l}]^{en\alpha_{\beta,\alpha}(l)} \cdot [C^{*}_{\gamma,\gamma k_\beta}, C^{*}_{\gamma,\gamma k_\beta}]^{en(n-1)} \times \\
\times \prod_{(\alpha,l)\neq(\gamma,\gamma k_\beta-1)} [C^{+}_{\gamma,\gamma k_\beta-1}, C^{Q}_{\alpha,l}]^{en[n-1-a_{\beta,\alpha}(l)]} \cdot [C^{+*}_{\gamma,\gamma k_\beta-1}, C^{+Q}_{\gamma,\gamma k_\beta-1}]^{en(n-1)}.
\]

This is the required form of the denominator under \( \theta^{2en^2}(Q,\Delta)|(0,\Pi) \) which depends on \( \gamma \) but no longer on \( R \in C_{\gamma,\gamma k_\beta} \). As we preserved the symmetry of yielding the same equation from \( \Delta \) and from \( T_{Q,R}(\Delta) \), we have established

**Proposition 3.1.** The quotient \( \theta^{2en^2}(Q,\Delta)|(0,M) \) is invariant under the operators \( T_{Q,R} \) for all admissible branch points \( R \) of the form \( P_{\gamma,m} \).

As \( a_{\beta,\gamma} \) takes \( \gamma k_\beta - 1 - ns_{\gamma,k_\beta} \) to 0 and \( \gamma k_\beta - n s_{\gamma,k_\beta} \) to \( n - 1 \), the power to which the exceptional sets \( C_{\gamma,\gamma k_\beta-1} \) and \( C_{\gamma,\gamma k_\beta} \) appear in \( q^\Delta_\gamma \) is determined by the same rule as the other sets. Observe that Proposition 3.1 generalizes Propositions 4.4, 5.1 and 5.2 of [FZ], where the sets with superscript \( +Q \) are denoted \( C_{\gamma} \) for \( \gamma = \beta = 1 \) and \( H \) for \( \gamma = -1 \). We will ultimately express our formulae in terms of the set \( C_{\beta,Q} \), which corresponds to the divisor \( Q^{n-1}\Delta \) used for changing the base point below. We also remark that the fact that only products of the form \( \alpha k_\beta \) show up in our operators and denominators is not coincidental. Indeed, by replacing \( w \) by \( w^k \) (divided by the appropriate polynomial in \( z \)) for some \( k \) which is prime to \( n \), all the indices \( \alpha, \beta \), etc. are divided by \( k \) modulo \( n \), so that only such products are independent of the choice of the generator \( w \) of \( C(X) \) over \( \mathbb{C}(z) \).

We can now prove the Poor Man’s Thomae (PMT) for \( X \). Recall that the PMT is a formula which attaches, given a branch point \( Q \) as base point, an expression \( g^Q_\Delta \) to every non-special divisor \( \Delta \) supported on the branch points distinct from \( Q \), such that the quotient \( \theta^{2en^2}(Q,\Delta)|(0,M) \) remains invariant under all the operators \( T_{Q,R} \) for admissible \( R \). Our aim is to multiply \( g^Q_\Delta \) (hence divide Equation (6) further) by an expression which is invariant under all the
operators $T_{Q,R}$ with $R \in C_{\gamma,k_{\beta}}$, and obtain an expression which is independent of $\gamma$ as well. Consider the expression

$$
\prod_{\delta \neq \gamma} \left[ \prod_{\alpha,l \neq ak_{\beta}} [C_{\delta,\delta k_{\beta}}, C_{\alpha,l}^{Q}]^{en_{\alpha,l}(l)} \cdot [C_{\delta,\delta k_{\beta}}, C_{\delta,\delta k_{\beta}}^{n}]^{en(n-1)\times

\times \prod_{\alpha,l \neq ak_{\beta}} [C_{\delta,\delta k_{\beta}}^{n}, C_{\alpha,l}^{Q}]^{en(n-1)} \cdot [C_{\delta,\delta k_{\beta}}^{n}, C_{\delta,\delta k_{\beta}}^{n}]^{en(n-1)} \right] \times

\times \prod_{\{(\alpha,\delta)\alpha<\delta,\alpha,\gamma,\delta\neq\gamma\}} [C_{\alpha,ak_{\beta}}^{n}, C_{\delta,\delta k_{\beta}}^{n}]^{en(n-1)} \cdot [C_{\alpha,ak_{\beta}}, C_{\delta,\delta k_{\beta}}^{n}]^{en(n-1)}.
$$

We claim that this expression is invariant under $T_{Q,R}$ for all $R \in C_{\gamma,k_{\beta}}$. This follows from the considerations regarding the sets $C_{\alpha,l}^{Q}$ above, together with the fact that the only set in which $C_{\alpha,l}^{Q} \neq C_{\alpha,l}^{Q}$ is with $\alpha = \gamma$ and $l = \gamma k_{\beta}$ (modulo $n$). Since this set appears in our expression only once, with the power involving $a_{\beta,\gamma}(\gamma k_{\beta} - ns_{\gamma,k_{\beta}}) = n - 1$, and this power vanishes, we can replace every $C_{\alpha,l}^{Q}$ by the $R$-independent notation $C_{\alpha,l}$. Therefore multiplying $q_{\Delta}^{Q}$ by this expression gives a denominator $g_{\Delta}^{Q}$ such that $\frac{g_{\Delta}^{Q\times[\Delta(0,1)]}}{g_{\Delta}^{Q}}$ is invariant under $T_{Q,R}$ for all admissible points $R = P_{\gamma,m}$ (with our $\gamma$). In order to analyze $g_{\Delta}^{Q}$, we use the following generalization of Lemma 4.2 of [FZ].

**Lemma 3.2.** Assume the set $Y$ is the union of the finite sets $Z_{j}$, $1 \leq j \leq d$, and let $W$ be a finite set which is disjoint from $Y$. Then $[Y,W]$ is the product $\prod_{j=1}^{d}[Z_{j},W]$ up to sign, and $[Y,Y]$ equals $\prod_{j=1}^{d}[Z_{j},Z_{j}] \cdot \prod_{1 \leq i < j \leq d}[Z_{i},Z_{j}]$ up to sign.

**Proof.** The first assertion is clear. We prove the second assertion by induction. For $d = 2$ this is just Lemma 4.2 of [FZ]. Assume that the assertion holds for $d-1$. Considering the expressions with $Z_{d}$ and $Z_{d-1}$ we claim that we can replace the product $\prod_{j=1}^{d}[Z_{j},Z_{j}] \cdot \prod_{1 \leq i < j \leq d}[Z_{i},Z_{j}]$ by $\prod_{j=1}^{d-1}[U_{j},U_{j}] \cdot \prod_{1 \leq i < j \leq d-1}[U_{i},U_{j}]$ (up to sign), where $U_{j} = Z_{j}$ for $1 \leq j \leq d-2$ and $U_{d-1} = Z_{d-1} \cup Z_{d}$. Indeed, apply Lemma 4.2 of [FZ] to $[Z_{d-1},Z_{d-1}]/Z_{d-1}, Z_{d}/Z_{d}$, and the first assertion here establishes the claim. The induction hypothesis now completes the proof of the lemma. \qed

By taking even powers of the expressions appearing in Lemma 3.2 we obtain exact equalities there. Denote the set $\bigcup_{\delta} C_{\delta,\delta k_{\beta}} \cup \{Q\}$ by $E_{Q}$ and the set $\bigcup_{\delta} C_{\delta,\delta k_{\beta}}$ by $E_{Q}^{\beta}$. We decompose, in the expression for $q_{\Delta}^{Q}$, the product over the pairs $(\alpha,l) \neq (\gamma,k_{\beta})$ to the product over those pairs in which $l \neq ak_{\beta}$ and the product over $(\alpha,ak_{\beta})$ for $\alpha \neq \gamma$. Similarly, we split the product over $(\alpha,l) \neq (\gamma,k_{\beta} - 1)$ to the one over $l \neq ak_{\beta} - 1$ and the one over $(\alpha,ak_{\beta} - 1)$ with $\alpha \neq \gamma$. Using these considerations we find that

$$
g_{\Delta}^{Q} = \prod_{\alpha,l \neq ak_{\beta}} [C_{\alpha,l}^{\beta}, C_{\alpha,l}^{Q}]^{en_{\alpha,l}(l)} \cdot [C_{\alpha,l}^{\beta}, C_{\alpha,l}^{\beta}]^{en(n-1)\times

\times \prod_{\alpha,l \neq ak_{\beta}} [C_{\alpha,l}^{\beta}, C_{\alpha,l}^{Q}]^{en(n-1)} \cdot [C_{\alpha,l}^{\beta}, C_{\alpha,l}^{\beta}]^{en(n-1)} \cdot [C_{\alpha,l}^{\beta}, C_{\alpha,l}^{\beta}]^{en(n-1)}.
$$
\[ \times \prod_{\alpha,i \neq \alpha k_3 - 1} [E_Q, C^{Q_t}_{\alpha,i}]^{\gamma[n-1-a_{\beta\alpha}]} \cdot [E_Q, E_{Q_t}]^{\gamma(n-1)}. \]

Since \( g_\Delta^Q \) does not depend on \( \gamma \), this argument proves

**Proposition 3.3.** The quotient \( g^{2n+3}([Q, \Delta], (0,\Pi)) \) is invariant under all the admissible operators \( T_{Q,R} \), and it is the PMT of the \( Z_n \) curve \( X \).

One can check that the PMT appearing in Propositions 4.4, 5.3, and 6.7 of [FZ] are special cases of Proposition 3.3, except that the isolated divisor \( P_3^{-1} \) of Section 6.1 of [FZ] (on which no \( T_{R,P} \) can act) is now given the denominator \((\lambda_0 - \lambda_1)^{2n(n-3)}(\lambda_0 - \lambda_2)^{2n}(\lambda_0 - \lambda_3)^{2n} \) rather than 1. As for Propositions 6.17 and 6.19 of that reference, our formula for \( g_\Delta^Q \) multiplies the expression given there for the divisor \( P_i^{2s}P_j^{t} \) for \( t = 1 \) (resp. \( P_i^{2s+1}P_j^{t} \) for \( t = 2 \)) by the \( en^{th} \) power of the \( T_{R,P} \)-invariant (resp. \( T_{P_i,P_j} \)-invariant) expressions \((\lambda - \lambda_j)(\lambda - \lambda_k) \) and \((\lambda_i - \lambda_j)(\lambda_i - \lambda_k) \) (resp. \((\lambda - \lambda_j)(\lambda - \lambda_k) \) and \((\lambda_j - \lambda_i)(\lambda_j - \lambda_k) \)). Hence our results are compatible also in these cases.

As already remarked in Section 2.6 of [FZ], we can allow (full) ramification at \( \infty \) by assuming that \( \sum r_0 \) is prime to \( n \) in Equation (1). Then the integers \( t_k \) from Equation (2) (which are no longer integers) have to be replaced by their upper integral values. All our further results hold also in this setting, when we omit any meaningless expression involving \( \infty \). This holds also when we substitute \( \infty \) in a rational function, since every such substitution always yields the value 1. The same assertion applies for what follows as well.

We also observe that the formula for \( g_\Delta^Q \) (as well as the preceding expressions) is independent of the cardinality conditions on the set \( C_{\alpha,i} \). Therefore the form of the Thomae formulae is unrelated to the actual set of divisors needed in order to define the characteristics etc., but is only based on the general shape of a divisor supported on the branch points distinct from \( Q \) containing no \( n \)th powers or higher. In particular, the formulae are not connected to the question whether such divisors exist or not, and one might say that they hold in a trivial manner in the latter case.

We now turn to changing the base point \( Q \) (but leave the index \( \beta \) fixed). Although this change is not required at this stage, it helps to simplify the notation. In the proof of Proposition 3.1 we have encountered the sets \( C_{\gamma,k_3 - 1} \), namely \( C_{\gamma,k_3 - 1} \cup \{ Q \} \), for various \( \gamma \). Since \( Q = P_\beta,i \), it is natural to consider this set for \( \gamma = \beta \), namely \( C_{\beta,0} \cup \{ Q \} \). Omitting \( Q \) from its original set \( C_{\beta,n-1} \) (which stands for the fact that \( v_Q(\Delta) = 0 \)) and including it in \( C_{\beta,0} \) (the set of points \( P_{\beta,m} \) appearing to the power \( n-1 \) in \( \Delta \)) corresponds to replacing \( \Delta \) by the divisor \( \Xi = Q^{n-1}\Delta \) of degree \( g+n-1 \). This is the divisor appearing in the symmetric notation of the Thomae formulae in Chapters 3, 4, and 5 of [FZ], since the second statement in Corollary 1.13 there implies that for such divisors the element \( \varphi_P(\Xi) + K_P \) of \( J(X) \) is independent of the choice of the branch point \( P \). Its value coincides with \( \varphi_Q(\Delta) + K_Q \), as is easily seen by taking \( Q = P \). We denote the appropriate theta constant \( \theta(\Xi)(0,\Pi) \) (with no need to add the
base point), and it coincides with \( \theta(Q, \Delta) \). Taking \( D_{\alpha,l} \) to be \( C_{\beta,0} \cup \{Q\} \) if \( \alpha = \beta \) and \( l = 0 \) and \( C^Q_{\alpha,l} \), otherwise, we obtain from Equation 39 that

\[
\Xi = Q^{n-1} \Delta = \prod_{\alpha} \prod_{l=0}^{n-1} D_{\alpha,l}^{n-1-l}.
\]

Moreover, for every \( 1 \leq k \leq n-1 \) the value \( l = n-1 \) does not participate in the summation over \( 0 \leq l \leq \beta k - ns_{\beta,k} - 1 \), while the value \( l = 0 \) does participates in this summation. It follows that the point \( Q \) does not contribute to any of the cardinalities appearing in Theorem 15, but after replacing every set \( C_{\alpha,l} \) by \( D_{\alpha,l} \) it contributes to all of them. Therefore the divisors of degree \( g + n - 1 \) in which we are interested are those which take the form of Equation 7 with the sets \( D_{\alpha,l} \) satisfying

\[
\sum_{\alpha} \sum_{l=0}^{\alpha,k-n_{\alpha,k}-1} |D_{\alpha,l}| = t_k
\]

for every \( 1 \leq k \leq n - 1 \).

After multiplying the images of the operators from Proposition 2.5 by \( Q^{n-1} \) as well and using the fact that \( a_{\delta,\alpha} \) and \( b_{\beta,\alpha} \) are involutions, we can write these operators in terms of the divisors \( \Xi \) as

\[
N_{\beta}(\Xi) = \prod_{\alpha,l} D_{\alpha,l}^{n-1-a_{\beta,\alpha}(l)} = \prod_{\alpha,l} D_{\beta,\alpha}(l)
\]

(the notation \( N_{\beta}, \) rather than \( N_Q, \) can be used here since the effect of this operator depends only on \( \beta \) and not on the choice of \( Q \in D_{\beta,0} \)) and

\[
T_{Q,R}(\Xi) = Q \prod_{\alpha,l} D_{\alpha,l}^{n-1-b_{\beta,\alpha}(l)} / R = Q \prod_{\alpha,l} D_{\alpha,b_{\beta,\alpha}(l)} / R
\]

for \( Q \in D_{\beta,0} \) and \( R \in D_{\gamma,\gamma k_3} \). Moreover, the set \( C^Q_{\alpha,l} \) is just \( D_{\alpha,l} \) unless \( \alpha = \beta \) and \( l = 0 \). It follows that the set \( E_Q \) appearing in \( g^Q_{\delta} \) is simply \( \bigcup_{\delta} D_{\delta,\delta k_3} \) (since for \( \delta = \beta \) we already have \( Q \in D_{\beta,0} \)). This set depends on \( \beta \), but no longer on \( Q \in D_{\beta,0} \), just like \( F^\beta = \bigcup_{\delta} D_{\delta,\delta k_3} \). In addition, the set \( C^Q_{\alpha,l} \), which is the only choice of indices \( \alpha \) and \( l \) for which \( C^Q_{\alpha,l} \neq D_{\alpha,l} \), does not appear in the expression for \( g^Q_{\delta} \). With \( E_Q \) the index \( l = 0 \) is \( \alpha k_3 - 1 \) for \( \alpha = \beta \) (modulo \( n \)), and with \( F^\beta \) the power \( a_{\beta,\beta}(0) \) vanishes (as \( n \) divides \( \beta k_3 - 1 \)). In total \( g^Q_{\delta} \) does not depend on \( Q \in D_{\beta,0} \) in this setting, so we denote it \( g^\beta_{\Xi} \). Furthermore, as \( a_{\delta,\alpha} (\alpha k_3 - ns_{\alpha,k_3}) = n - 1 \) and \( a_{\beta,\alpha} (\alpha k_3 - 1 - ns_{\alpha,k_3}) = 0 \), the power \( en(n-1) \) to which the expressions \([D_{\delta,\delta k_3}, D_{\alpha,ak_3}]\) and \([D_{\delta,\delta k_3-1}, D_{\alpha,ak_3-1}]\) (coming from \([F^\beta, F^\beta] \) or \([E_Q, E_Q] \) respectively) appear in \( g^\beta_{\Xi} \) obeys the same rule as with the other expressions \([D_{\delta,\delta k_3}, D_{\alpha,l}]\) or \([D_{\delta,\delta k_3-1}, D_{\alpha,l}]\). Expanding the products using Lemma 32 we can write

\[
g^\beta_{\Xi} = \prod_{\{\delta,\alpha,l\} | \delta < \alpha \text{ if } l = \alpha k_3 - ns_{\alpha,k_3} \} [D_{\delta,\delta k_3}, D_{\alpha,l}]^{\text{en}_{\alpha,l}(l)} \times
\]
\[
\times \prod_{\{(\delta, \alpha, l) | \delta \leq \alpha \text{ if } l = \alpha k_\beta - 1 - n s_{\delta, k_\beta}\}} [D_{\delta, \delta k_\beta - 1}, D_{\alpha, l}]^{\epsilon n[\alpha - a_{\delta, \alpha}(l)]}
\]

(the condition \(\delta \leq \alpha\) for the appropriate value of \(l\) is imposed to avoid undesired repetitions), and Proposition 3.3 takes the form

**Proposition 3.4.** The quotient \(\theta_{\alpha, l}^{g_{\alpha}^{|\Xi|(0, \Pi)}}\) is invariant under all the operators \(T_{Q, R}\) with \(Q \in D_{\beta, 0}\) and \(R \in D_{\gamma, \gamma k_\beta}\) (with arbitrary \(\gamma\)), and it is the base-point-invariant form of the PMT of \(X\).

We remark that a divisor \(\Xi\) takes the form \(Q^{\alpha - 1} \Delta\) for some integral divisor \(\Delta\) of degree \(g\) and some base point \(Q\) only if some set \(D_{\alpha, 0}\) is not empty. In general, however, this condition might not be satisfied, and there exist divisors \(\Xi\) satisfying the cardinality conditions such that \(D_{\alpha, 0} = \emptyset\) for all \(\alpha\). These divisors cannot be presented as \(Q^{\alpha - 1} \Delta\) for any branch point \(Q\). The operators \(N_\beta\) act on these divisors, but no \(T_{Q, R}\) does so since \(Q \in D_{\beta, 0}\) (for the appropriate \(\beta\)) is required to define the action of these operators. Hence the assertion of Proposition 3.4 holds trivially for these divisors, at least at this point. More details will be given in Section 5.

### 4 Invariance Under \(N_\beta\)

Consider a \(Z_n\) curve \(X\), a branch point \(Q\) on \(X\), and a non-special divisor \(\Delta\) of degree \(g\) on \(X\) which is supported on the branch points distinct from \(Q\). The combination of Equation (1.5) of [FZ] and Equation (4) yields the equality

\[
\theta^N[Q, \Delta](0, \Pi) = \theta^N[Q, N_\beta(\Delta)](0, \Pi)
\]

for any \(N\) divisible by \(2n\). The condition \(2n|N\) is necessary to ensure independence of the lifts. Expressed in terms of the degree \(g + n - 1\) divisors \(\Xi\), the latter equality with \(N = 2en^2\) becomes

\[
\theta^{2en^2}[N_\beta(\Xi)](0, \Pi) = \theta^{2en^2}[\Xi](0, \Pi),
\]

holding for every \(\beta \in \mathbb{N}_n\) which is prime to \(n\) and for every divisor \(\Xi\) of the form presented above. Hence our goal is to divide the quotient from Proposition 3.4 (or equivalently, multiply \(g_{\Xi}^\beta\)) by an expression which is invariant under all the admissible operators \(T_{Q, R}\) considered in that Proposition, such that the product \(h_\Xi\) of \(g_{\Xi}^\beta\) with this expression will satisfy \(h_{N_\beta(\Xi)} = h_\Xi\). In case the expression \(h_\Xi\) is independent also of \(\beta\), the quotient \(\frac{2en^2[\Xi](0, \Pi)}{h_\Xi}\) will be invariant under all the operators \(T_{Q, R}\) as well as \(N_\beta\) for all \(\beta\).

To achieve this goal, we need to compare \(g_{\Xi}^\beta\) with \(g_{N_\beta(\Xi)}^\beta\). According to Equation (8), moving from \(\Xi\) to \(N_\beta(\Xi)\) is equivalent to replacing every set \(D_{\alpha, l}\) by \(D_{\alpha, a_{\beta, \alpha}(l)}\). Now, \(a_{\beta, \delta}(\delta k_\beta - 1 - n s_{\delta, k_\beta}) = 0\) and \(a_{\beta, \delta}(\delta k_\beta - n s_{\delta, k_\beta}) = n - 1\).
while $a_{\beta,\delta}$ is an involution. These considerations imply that $g_{N_\beta}(\Xi)$ equals

$$
\prod_{(\delta,\alpha,l) \delta \leq \alpha \text{ if } l = n-1} \left[D_{\delta,n-1}, D_{\alpha,l}\right]_{\text{en}l} \prod_{(\delta,\alpha,l) \delta \leq \alpha \text{ if } l = 0} \left[D_{\delta,0}, D_{\alpha,l}\right]_{\text{en}(n-1-l)}.
$$

Observe that this expression does not depend on $\beta$, which suggests that we might take it as the denominator under $\theta^{2\text{en}^2}\Xi(0,\Pi)$ in the PMT using Equation (10).

Nevertheless, we prefer to follow [FZ] and maintain the denominator $g_\beta^{\delta}$.

In order to motivate the following definition, we consider only those parts of $g_\beta^{\delta}$ and $g_{N_\beta}^{\delta}(\Xi)$ which involve the set $D_{\beta,0}$ (which remains invariant under $N_\beta$).

An expression of the form $[D_{\beta,0}, D_{\alpha,l}]$ appears to the power $\text{en}[n - 1 - a_{\beta,\alpha}(l)]$ in $g_\beta^{\delta}$. Assume that there exists an expression $h_\Xi$ with the properties stated in the previous paragraph. Write the power to which $[D_{\beta,0}, D_{\alpha,l}]$ appears in $h_\Xi$ as $\text{en}[c(\beta,\alpha) - f_{\beta,\alpha}(l)]$, where $c(\beta,\alpha) \in \mathbb{Z}$ and $f_{\beta,\alpha} : \mathbb{N} \rightarrow \mathbb{Z}$ is some function. By altering the constant $c(\beta,\alpha)$ if necessary, we can always assume $f_{\beta,\alpha}(0) = 0$. Then the $N_\beta$-invariance of $h_\Xi$ yields the equality

$$
f_{\beta,\alpha}(a_{\beta,\alpha}(l)) = f_{\beta,\alpha}(l)
$$

for every $l \in \mathbb{N}$. Moreover, the $T_{Q,R}$-invariance of $\frac{\Xi}{g_\Xi}$ implies that the equality

$$
f_{\beta,\alpha}(b_{\beta,\alpha}(l)) - a_{\beta,\alpha} (b_{\beta,\alpha}(l)) = f_{\beta,\alpha}(l) - a_{\beta,\alpha}(l)
$$

holds for every $l \in \mathbb{N}$. The latter property becomes easier to work with when we replace $l$ by $a_{\beta,\alpha}(l)$. Indeed, Lemma 2.4 the fact that $a_{\beta,\alpha}$ is an involution, and Equation (11) combine to show that the latter equality is equivalent to

$$
f_{\beta,\alpha}[b_{\beta,\alpha}(a_{\beta,\alpha}(l))] + l = f_{\beta,\alpha}(l) + n - 1 - l
$$

holding for every $l \in \mathbb{N}$. Moreover, the common value of the two sides in Equation (12) is left invariant under replacing $l$ by $n - 1 - l$, as follows from Equation (11) and the second assertion of Lemma 2.4. In particular, the normalization $f_{\beta,\alpha}(0) = 0$ implies $f_{\beta,\alpha}(n - 1) = n - 1$ for every $\alpha$ and $\beta$.

**Theorem 4.1.** For any $n$ and any $\alpha$ and $\beta$ in $\mathbb{N}$ which are prime to $n$ there exists a unique function $f_{\beta,\alpha} : \mathbb{N} \rightarrow \mathbb{Z}$ which satisfies Equations (11) and (12) for every $l \in \mathbb{N}$ and attains 0 on $l \in \mathbb{N}$.

**Proof.** We first prove that there is a unique function $f_{\beta,\alpha} : \mathbb{N} \rightarrow \mathbb{Z}$ satisfying $f_{\beta,\alpha}(0) = 0$ and Equation (12) for every $l \in \mathbb{N}$. Observe that $b_{\beta,\alpha} \circ a_{\beta,\alpha}$ adds $\alpha k_\beta$ to $l$ up to multiples of $n$, and that $\alpha k_\beta$ is prime to $n$. Hence multiple applications of $b_{\beta,\alpha} \circ a_{\beta,\alpha}$ takes any element of $\mathbb{N}$ to any other. Since Equation (12) presents $f_{\beta,\alpha}[b_{\beta,\alpha}(a_{\beta,\alpha}(l))]$ as $f_{\beta,\alpha}(l)$ plus another term, knowing the value of $f_{\beta,\alpha}$ on one element of $\mathbb{N}$ determines the values of $f_{\beta,\alpha}$ on all the elements of $\mathbb{N}$. Hence the normalization $f_{\beta,\alpha}(0) = 0$ determines $f_{\beta,\alpha}$ uniquely. Note that $n$ applications of $b_{\beta,\alpha} \circ a_{\beta,\alpha}$ takes every $l \in \mathbb{N}$ to itself. Applying Equation (12) $n$ times shows that while doing so we add to $f_{\beta,\alpha}(l)$ the values $n - 1 - 2j$.
for all possible values of \( j \in \mathbb{N}_n \). As this sum is \( n(n-1) - 2\frac{n(n-1)}{2} = 0 \), these equalities are consistent with one another, and the function \( f_{\beta,\alpha} \) indeed exists (and is unique).

It remains to show that the function \( f_{\beta,\alpha} \) thus obtained satisfies also Equation (11) for all \( l \in \mathbb{N}_n \). First, Equation (11) holds if \( l \) is a fixed point of \( a_{\beta,\alpha} \), and we claim that \( a_{\beta,\alpha} \) must have at least one fixed point. Indeed, we are looking for \( l \in \mathbb{N}_n \) such that \( 2l = a_k - 1 \) (mod \( n \)). For odd \( n \) such \( l \) exists and is unique. On the other hand, if \( n \) is even then so is \( a_k - 1 \), implying that there are two such values of \( l \). Assume that Equation (11) holds for some value of \( l \). We claim that Equation (11) holds also for \( b_{\beta,\alpha}(a_{\beta,\alpha}(l)) \). To see this, first substitute \( l = a_{\beta,\alpha}(b_{\beta,\alpha}(j)) \) in Equation (12). Since \( a_{\beta,\alpha} \) and \( b_{\beta,\alpha} \) are involutions, Lemma 2.4 shows that this substitution yields the equality

\[
f_{\beta,\alpha}(j) + n - 1 - a_{\beta,\alpha}(j) = f_{\beta,\alpha}(a_{\beta,\alpha}(b_{\beta,\alpha}(j))) + a_{\beta,\alpha}(j).
\]

Put now \( j = a_{\beta,\alpha}(l) \) and use the involutive property of \( a_{\beta,\alpha} \) again to obtain

\[
f_{\beta,\alpha}(a_{\beta,\alpha}(l)) + n - 1 - l = f_{\beta,\alpha}(a_{\beta,\alpha}(a_{\beta,\alpha}(l))) + a_{\beta,\alpha}(l).
\]

Equation (12) and the assumption that Equation (11) holds for \( l \) now yield Equation (11) for \( b_{\beta,\alpha}(a_{\beta,\alpha}(l)) \). Since we have shown that Equation (11) holds for some \( l \in \mathbb{N}_n \) and that multiple applications of \( b_{\beta,\alpha} \circ a_{\beta,\alpha} \) connect any two elements of \( \mathbb{N}_n \), this completes the proof of the theorem.

Observe that altering the constants \( c(\beta,\alpha) \) does not affect the invariance of the quotient \( \frac{\sum_{\beta,j=0}^{n-1} \Xi(\beta,j)}{h_2} \) under any operator, so one may choose these constants arbitrarily. It is natural to normalize the constants such that \( h_2 \) is a polynomial (i.e., excluding negative powers) and reduced (i.e., some \([D_{\beta,j},D_{\alpha,l}]\) appears with vanishing power). However, determining these constants depends much more delicately on the relations between \( n, \alpha, \) and \( \beta \). For example, such a normalizing constant \( c(\beta,\beta) \) depends on the parity of \( n \) while \( f_{\beta,\beta} \) does not (see the differences between the formulae for odd and even \( n \) in Chapters 4 and 5 of [FZ]). As another example, if \( n \) is odd and \( \alpha k_3 \) is 2 modulo \( n \) then the form of these constants depends on whether \( n \) is equivalent to 1 or to 3 modulo 4, while the form of the function \( f_{\beta,\alpha} \) does not depend on this congruence (see the example in Section 6.1 of [FZ]).

We now present several lemmas, which are needed to define the denominator \( h_2 \) and to establish its properties.

**Lemma 4.2.** Given three elements \( \alpha, \beta, \) and \( \delta \) of \( \mathbb{N}_n \) which are all prime to \( n \) and two elements \( l \) and \( r \) of \( \mathbb{N}_n \), let \( j \in \mathbb{N}_n \) be the element which is congruent to \( l + r\alpha k_3 \) modulo \( n \). Then the congruences

\[
a_{\beta,\alpha}(j) \equiv a_{\delta,\alpha}(l) + a_{\beta,\delta}(r)\alpha k_3 \pmod{n}, \quad b_{\beta,\alpha}(j) \equiv a_{\delta,\alpha}(l) + b_{\beta,\delta}(r)\alpha k_3 \pmod{n}
\]

hold.
Proof. As in the proof of Proposition 2.5 we take $\eta$ to be 1 when we work with $a$ and 2 when we work with $b$. By definition, the left hand side of our expressions is congruent to $n\alpha \beta - 1 - l - r\alpha k_\delta$ modulo $n$, while the right hand side is congruent to $\alpha k_\delta - 1 - l + \alpha k_\delta (\nu \delta k_\beta - 1 - r)$ modulo $n$. The latter expression contains $-1 - l - r\alpha k_\delta$, the two terms with $\alpha k_\delta$ cancel, and the terms including $\eta$ also coincide since $\delta k_\delta \equiv 1 (\text{mod } n)$. This proves the lemma.

Lemma 4.3. For every $l \in \mathbb{N}_n$ (given $\alpha$ and $\delta$) let $y_{\alpha,\delta,l}$ denote the number $-\delta k_\alpha - n s_{\delta, l, k_\alpha} \in \mathbb{N}_n$. Then the equality

$$f_{\alpha,\delta}(y_{\alpha,\delta,l} + n\delta 0 - 1) + l = f_{\alpha,\delta}(y_{\alpha,\delta,l}) + n - 1 - l$$

holds.

In this Lemma, $\delta 0$ denotes Kronecker’s symbol (namely 1 if $l = 0$ and 0 otherwise). It is included here to account for the fact that for $y_{\alpha,\delta,0} = 0$ the number $y_{\alpha,\delta,0} - 1 = -1$ is not in $\mathbb{N}_n$ but adding $n$ to it yields $n - 1 \in \mathbb{N}_n$. In this case the assertion of Lemma 4.3 reduces to the equality $f_{\beta,\alpha}(n - 1) = n - 1$ which we already obtained above.

Proof. We prove the asserted equality by decreasing induction on $l$. We begin by observing that $y_{\alpha,\delta,n-1} = a_{\alpha,\delta}(n - 1)$ while $y_{\alpha,\delta,n-1} - 1 = b_{\alpha,\delta}(a_{\alpha,\delta}(n - 1))$ (or alternatively, $y_{\alpha,\delta,n-1} = b_{\alpha,\delta}(a_{\alpha,\delta}(0))$ and $y_{\alpha,\delta,n-1} - 1 = a_{\alpha,\delta}(0)$). Hence the assertion for $l = n - 1$ follows directly from Equations (11) and (12). Now assume that the assertion holds for $0 < l \leq n - 1$, and we wish to prove it for $l - 1$. As $y_{\alpha,\delta,l-1}$ is $b_{\alpha,\delta}(a_{\alpha,\delta}(y_{\alpha,\delta,l}))$ and $y_{\alpha,\delta,l-1} + n \delta l - 1$ equals $b_{\alpha,\delta}(a_{\alpha,\delta}(y_{\alpha,\delta,l} - 1))$, Equation (12) shows that the left hand side and right hand side of the equation corresponding to $l - 1$ are

$$f_{\alpha,\delta}(y_{\alpha,\delta,l} - 1) + n - 2y_{\alpha,\delta,l} + l \quad \text{and} \quad f_{\alpha,\delta}(y_{\alpha,\delta,l}) + 2n - 1 - 2y_{\alpha,\delta,l} - l$$

respectively. But these expressions are obtained by adding $n - 2y_{\alpha,\delta,l}$ to both sides of the equality corresponding to $l$. Hence if the equality holds for $l$ it also holds for $l - 1$. This completes the proof of the lemma.

Lemma 4.4. The equality $f_{\beta,\alpha}(l) = f_{\alpha,\delta}(-\delta k_\alpha - n s_{\delta, -, k_\alpha})$ (namely $f_{\alpha,\delta}(y_{\alpha,\delta,l})$) in the notation of Equation (11) holds for every $\alpha$ and $\delta$ and every $l \in \mathbb{N}_n$.

Proof. As both sides attain 0 at $l = 0$, Theorem 4.1 reduces the assertion to verifying that the function of $l$ given on the right hand side satisfies Equations (11) and (12) with the parameters $\delta$ and $\alpha$. Substituting $a_{\delta,\alpha}(l)$ in place of $l$ yields an argument of $f_{\alpha,\delta}$ which lies between 0 and $n - 1$ and is congruent to $\delta k_\alpha - 1 + \delta k_\alpha$ modulo $n$ (recall that $a k_\alpha \equiv \delta k_\beta \equiv 1 (\text{mod } n)$). Since this number is (by definition) the $a_{\alpha,\delta}$-image of $-\delta k_\alpha - n s_{\delta, -, k_\alpha}$, Equation (11) for the latter function confirms that Equation (11) is satisfied also with the required argument. For Equation (12) we consider the right hand side as $f_{\alpha,\delta}(y_{\alpha,\delta,l})$. Applying $b_{\delta,\alpha} \circ a_{\alpha,\delta}$ to $l$ is the same as adding $\alpha k_\delta$ to it (modulo $n$), and after multiplying by $-\delta k_\alpha$ the argument of $f_{\alpha,\delta}$ becomes $y_{\alpha,\delta,l} + n \delta l - 1$ (recall that both $\delta k_\beta$ and $\alpha k_\alpha$ are 1 modulo $n$). The desired Equation (12) now follows from Lemma 4.3. This proves the lemma.
The following lemma is not a part of the proof of the Thomae formulae (Theorem 4.4 below), but it will turn out to be useful for deriving explicit expressions for the functions $f_{\beta,\alpha}$ in Section 6.

**Lemma 4.5.** The function $f_{n-\beta,\alpha}$ is related to the function $f_{\beta,\alpha}$ through the equality $f_{n-\beta,\alpha}(l) = 2l - f_{\beta,\alpha}(l)$ (holding for all $l \in \mathbb{N}_n$).

**Proof.** First we observe that the equalities $n - 1 - a_{n-\beta,\alpha}(l) = a_{\beta,\alpha}(n - 1 - l)$ and $n - 1 - b_{n-\beta,\alpha}(l) = b_{\beta,\alpha}(n - 1 - l)$ hold for every $\alpha$, $\beta$, and $l \in \mathbb{N}_n$. Indeed, all four numbers are in $\mathbb{N}_n$, the former two are congruent to $\alpha \beta + l$ modulo $n$, and the latter two are $2\alpha \beta + l$ up to multiples of $n$. Consider now the function $\psi_{n-\beta,\alpha}(l) = n - 1 - f_{\beta,\alpha}(n - 1 - l)$. Equation (11) for $f_{\beta,\alpha}$ implies

$$f_{\beta,\alpha}(n - 1 - a_{n-\beta,\alpha}(l)) = f_{\beta,\alpha}(a_{\beta,\alpha}(n - 1 - l)) = f_{\beta,\alpha}(n - 1 - l),$$

which yields Equation (11) for $\psi_{n-\beta,\alpha}$ with the parameters $n - \beta$ and $\alpha$. Using the equalities above and Equation (12) for $f_{\beta,\alpha}$ we also obtain

$$f_{\beta,\alpha}(a_{n-\beta,\alpha}(l)) + n - 1 - l = f_{\beta,\alpha}(a_{\beta,\alpha}(n - 1 - l)) + n - 1 - l = f_{\beta,\alpha}(n - 1 - l) + n - 1 - (n - 1 - l) = f_{\beta,\alpha}(n - 1 - l) + l.$$

Subtracting both sides from $2n - 2$ establishes (12) for $\psi_{n-\beta,\alpha}$ with the same parameters. As $f_{\beta,\alpha}(n - 1) = n - 1$ for all $\alpha$ and $\beta$, we deduce that $\psi_{n-\beta,\alpha}(0) = 0$. Hence $\psi_{n-\beta,\alpha} = f_{n-\beta,\alpha}$ by Theorem 4.4. As replacing $l$ by $n - 1 - l$ leaves the expression appearing in Equation (12) invariant, the expression defining $\psi_{n-\beta,\alpha}(l)$ can be written as $\psi_{n-\beta,\alpha}(l) = 2l - f_{\beta,\alpha}(l)$ for every $l \in \mathbb{N}_n$. This proves the lemma. 

Fix an order on the set of pairs $(\alpha, l)$ with $\alpha \in \mathbb{N}_n$ prime to $n$ and $l \in \mathbb{Z}/n\mathbb{Z}$. Choose, for every $\delta$ and $\alpha$, an integral constant $c(\delta, \alpha)$ such that $c(\delta, \alpha) = c(\alpha, \delta)$ for every $\alpha$ and $\delta$. Define, for any divisor $\Xi$ as in Equation (7), the expression

$$h_\Xi = \prod_{(\delta, \alpha) \leq (\alpha, l + \alpha k_n)} [D_{\delta, r}, D_{\alpha, l + \alpha k_n}]^{\epsilon_n[\varepsilon(\delta, \alpha) - f_{\beta, \alpha}(l)]}.$$

The inequality in the product is with respect to the chosen order. We now prove

**Theorem 4.6.** The expression $h_\Xi$ is independent of the order chosen. The quotient $h_\Xi^{2^\kappa} \Xi^{(0, 1)}$ is invariant under all the operators $N_{\beta}$ as well as under all the admissible operators $T_{Q, R}$.

**Proof.** Changing the order means that for some pairs, we write $[D_{\delta, r}, D_{\alpha, l + \alpha k_n}]$ as $[D_{\alpha, s}, D_{\delta, j + \delta k_n}]$ for appropriate $s$ and $j$. We need to see that the power to which this expression appears in $h_\Xi$ is the same. But $s \equiv l + r\alpha k_n (\mod n)$ and $j \equiv r - s\delta k_n (\mod n)$, so that $j \equiv -s\delta k_n (\mod n)$ since $\alpha k_n$ and $\delta k_n$ are congruent to 1 modulo $n$. Therefore the powers to which the two forms of this expression appear in $h_\Xi$, namely $c(\delta, \alpha) - f_{\beta, \alpha}(l)$ and $c(\alpha, \delta) - f_{\alpha, \beta}(j)$, coincide.
by Lemma 4.4 and the choice of the constants. This proves the independence of $h_{\Xi}$ of the order chosen on the set of pairs $(\alpha, l)$.

Proposition 3.4 and Equation (10) reduce the invariance assertions to the statements that $h_{\Xi}$ is invariant under any operator $N_\beta$, and for any $\beta$ the quotient $\frac{b_{\Xi}}{g_{\Xi}}$ is invariant under every admissible operator $T_{Q,R}$ with $Q \in D_{\beta,0}$ (for this $\beta$). Decompose $\frac{b_{\Xi}}{g_{\Xi}}$ into the product of expressions involving some set $D_{\alpha,\alpha k_\beta}$ or $D_{\alpha,\alpha k_\beta-1}$ and those which do not. The division by $g_{\Xi}^\beta$ affects only the powers appearing in the first part in this decomposition. We start with the invariance under $N_\beta$, as well as the $T_{Q,R}$-invariance of the second part of $\frac{b_{\Xi}}{g_{\Xi}}$ (or simply of $h_{\Xi}$). By Equations (8) and (9) this quotient reduces to verifying that together with any expression $[D_{\delta(r),D_{\alpha,j}}]$, the expressions $[D_{\delta,\alpha j(r)},D_{\alpha,\alpha_{j+1}(r)}]$ and $[D_{\delta,b_j,\alpha j(r)},D_{\alpha,b_j,\alpha_{j+1}(r)}]$ appear to the same power in $h_{\Xi}$. But if $j \equiv l + r\alpha k_\beta$ for some $l \in \mathbb{N}$, then the first expression appears to the power $e n[c(\delta,\alpha) - f_\alpha(\alpha)]$, and Lemma 4.2 implies that the other two expression must then appear to the power $e n[c(\delta,\alpha) - f_\beta(\alpha)]$. The two invariance assertions now follow from Equation (11) for $f_\beta,\alpha$.

It remains to prove the invariance of the first part of $\frac{b_{\Xi}}{g_{\Xi}}$ under the operators $T_{Q,R}$ (with $Q \in D_{\beta,0}$). We may choose the order such that the expressions we consider include only powers of $[D_{\delta,\delta k_\beta},D_{\alpha,j}]$ and $[D_{\delta,\delta k_\beta-1},D_{\alpha,j}]$ (either with $j$ taking the values $\alpha k_\beta$ or $\alpha k_\beta - 1$ or with $j$ taking other values). Since the operators $T_{Q,R}$ may mix $D_{\delta,\delta k_\beta}$ with $D_{\delta,\delta k_\beta-1}$, Equation (1) shows that for $T_{Q,R}$-invariance of $\frac{b_{\Xi}}{g_{\Xi}}$, this quotient must contain all the expressions $[D_{\delta,\delta k_\beta},D_{\alpha,j}]$, $[D_{\delta,\delta k_\beta-1},D_{\alpha,j}]$, $[D_{\delta,\delta k_\beta},D_{\alpha,b_j,\alpha_{j+1}(r)}]$, and $[D_{\delta,\delta k_\beta-1},D_{\alpha,b_j,\alpha_{j+1}(r)}]$ raised to the same power. Observe that this assertion holds regardless of whether $j$ is congruent to one of $\alpha k_\beta$ and $\alpha k_\beta - 1$ modulo $n$ or not, since in the former case, where additional mixing may appear, $b_{\beta,\alpha}$ interchanges the elements of $\mathbb{N}$ which are congruent to $\alpha k_\beta$ and $\alpha k_\beta - 1$ modulo $n$ with one another. We remark that in the former case with $\alpha = \delta$ the assertion refers to the three expressions $[D_{\delta,\delta k_\beta},D_{\delta,\delta k_\beta}]$, $[D_{\delta,\delta k_\beta-1},D_{\delta,\delta k_\beta}]$, and $[D_{\delta,\delta k_\beta-1},D_{\delta,\delta k_\beta-1}]$.

Now, $h_{\Xi}$ is given in terms of $[D_{\delta,r},D_{\alpha,l+r\alpha k_\beta}]$ while $g_{\Xi}^\beta$ is given in terms of $[D_{\delta,r},D_{\alpha,l}]$ for $r$ being either $\delta k_\beta$ or $\delta k_\beta - 1$. In the case $r = \delta k_\beta$ the index $l + r\alpha k_\beta$ coincides modulo $n$ with $\alpha k_\beta$ (as $\delta$ and $k_\delta$ cancel modulo $n$) hence with $b_{\delta,\alpha}(a_{\delta,\alpha}(l))$. We shall thus express $g_{\Xi}^\beta$ also in terms of this set. In addition, $D_{\delta,\delta k_\beta-1}$ is associated in $h_{\Xi}$ with $D_{\alpha,l+\alpha k_\beta-\alpha k_\beta}$, which we write as $D_{\alpha,\alpha_{\delta,\alpha}(\delta),(\delta,\alpha)(l)+\alpha k_\beta}$. By replacing $l$ by $b_{\delta,\alpha}(a_{\delta,\alpha}(l))$ in the expressions involving $D_{\delta,\delta k_\beta-1}$ in $h_{\Xi}$ we find that the part of $h_{\Xi}$ containing $D_{\delta,\delta k_\beta}$ or $D_{\delta,\delta k_\beta-1}$ is

$$\prod_{\{(\delta,\alpha,l)\mid l\neq n-1, \delta \leq \alpha \text{ if } l=0\}} [D_{\delta,\delta k_\beta},D_{\alpha,l+\alpha k_\beta}]_{en[c(\delta,\alpha)-f_\alpha(\alpha)]} \times \prod_{\{(\delta,\alpha,l)\mid \delta \leq \alpha \text{ if } l=n-1\}} [D_{\delta,\delta k_\beta-1},D_{\alpha,l+\alpha k_\beta}]_{en[c(\delta,\alpha)-f_\beta(\alpha)]} \times [b_{\delta,\alpha}(a_{\delta,\alpha}(l))]$$

(this form is based on an order in which $\delta \leq \alpha$ implies $(\delta,\delta k_\beta) \leq (\alpha, \alpha k_\beta)$ and $(\delta,\delta k_\beta-1) \leq (\alpha, \alpha k_\beta - 1)$ and in which $(\delta,\delta k_\beta-1) < (\alpha, \alpha k_\beta)$ for all $\alpha$ and
On the other hand, Lemma 2.4, the fact that $a_{\bar{\beta},\alpha}$ is an involution, and the congruence $l + \alpha k_\beta \equiv b_{\bar{\beta},\alpha}(a_{\bar{\beta},\alpha}(l))(\text{mod } n)$ allow us to write $g_Z^2$ as

$$
\prod_{\{\delta,\alpha,l\} \delta \leq \alpha \text{ if } l=0} [D_{\delta,\delta k_\beta,1}, D_{\alpha,l+\alpha k_\beta}]^{en(n-1-l)}, \prod_{\{\delta,\alpha,l\} \delta \leq \alpha \text{ if } l=n-1} [D_{\delta,\delta k_\beta-1,1}, D_{\alpha,l+\alpha k_\beta}]^{enl}
$$

(we can add the condition $l \neq n-1$ trivially to the first product, in order to resemble the expression arising from $h_Z$). The powers to which $[D_{\delta,\delta k_\beta,1}, D_{\alpha,l+\alpha k_\beta}]$ and $[D_{\delta,\delta k_\beta-1,1}, D_{\alpha,l+\alpha k_\beta}]$ appear in the quotient $\frac{h_Z}{g_Z}$ are now seen to be $en$ times $c(\delta,\alpha) - f_{\delta,\alpha}(l) - n + 1 + l$ and $c(\delta,\alpha) - f_{\delta,\alpha}[b_{\delta,\alpha}(a_{\delta,\alpha}(l))] - l$ respectively. These numbers are equal by Equation (13). Applying $b_{\bar{\beta},\alpha}$ to an element of $\mathbb{N}_n$ which is congruent to $l + \alpha k_\beta$ modulo $n$ yields the element of $\mathbb{N}_n$ which is congruent to $n - 1 - l + \alpha k_\beta$ modulo $n$. Hence the action of $b_{\bar{\beta},\alpha}$ in this setting takes $l$ to $n - 1 - l$. The invariance of the number appearing in Equation (12) under this operation now completes the proof of the theorem.

We remark that the assertion of Theorem 4.6 holds also for divisors $\Xi$ for which all the sets $D_{\bar{\beta},0}$ are empty. In this case it refers only to the action of the operators $N_\beta$. Moreover, the fact that the power to which an expression $[D_{\delta,\beta,1}, D_{\alpha,\lambda}]$ appears in $h_Z$ depends only on the index difference between $l$ and $j$ in some sense allows, in any particular case, for a pictorial description of these powers, in similarity to Chapters 4 and 5 of [FZ].

## 5 Transitivity and the Full Thomae Formulae

Another operation on the divisors $\Xi$ from Equation (17) is related to changing the base point. For any $k \in \mathbb{N}_n$ (and even $k \in \mathbb{Z}$), we let $w_k = \frac{\partial}{\partial k} = \prod_{i=1}^n \omega_{\alpha,i}^{k-n_{\alpha,i}} \omega_{\alpha,i}^{k}$ be the “normalized $k$th power of $w$”. $w_k$ is defined for $k \in \mathbb{Z}/n\mathbb{Z}$ and its divisor is $\prod_{i=1}^n \prod_{\lambda_{\alpha,i}} P_{\alpha,i}^{k-n_{\alpha,i}} / \prod_{\lambda_{\alpha,i}=1} \alpha_{\lambda_{\alpha,i}}$ (which is trivial if $n$ divides $k$). If $k$ is prime to $n$ then this function is the function whose $n$th power gives a normalized $Z_n$-equation for $X$ with the appropriate generator of $\mathbb{C}(X)$ over $\mathbb{C}(z)$. Multiplying $\Xi$ by $div(w_k)$ yields a divisor with the same $\varphi_Q$-image for any base point $Q$ (by Abel’s Theorem), and the same claim holds for multiplication by $div(\frac{\partial}{\partial k})$ for any polynomial in $z$. For every $k \in \mathbb{Z}$ we define $M^k$ to be the operator which takes a divisor $\Xi$ from Equation (17) and multiplies it by $div\left(\frac{\partial^{n_k}}{\partial k(z)}\right)$, where $p_{k,\Xi}(z) = \prod_{i \in A_{\alpha,k}} (z - \lambda_{\alpha,i})$. Recall from Corollary 1.4 that $A_{\alpha,k}$ denotes the set of indices $1 \leq i \leq r_{\alpha}$ such that $v_{P_{\alpha,i}}(\Xi) > n - 1 - ak + ns_{\alpha,k}$, or equivalently $P_{\alpha,i} \in D_{\alpha,l}$ for $0 \leq l < ak - ns_{\alpha,k}$ (so that for $k$ divisible by $n$ all the sets $A_{\alpha,k}$ are empty). Thus, dividing by the divisor of $p_{k,\Xi}(z)$ ensures that all the branch points appear in $M^k(\Xi)$ raised to powers from $\mathbb{N}_n$.

**Proposition 5.1.** $M^k$ defines an operator on the set of divisors $\Xi$ from Equation (17) satisfying the cardinality conditions. Moreover, this operator leaves the quotient $\frac{\omega^{\sum_{i=1}^n ns_{\alpha,i}}}{\prod_{\lambda_{\alpha,i}=1} \omega_{\alpha,i}^{\lambda_{\alpha,i}}}$ invariant.
Proof. By definition, the divisor $M^k(\Xi)$ contains no branch point to the power $n$ or higher. Moreover, the $k$th cardinality condition on $\Xi$ implies that the powers of the points $\infty_h$ arising from $w_k$ and from $p_{k,\Xi}(z)$ cancel, so that $M^k(\Xi)$ also takes the form given in Equation (7). Its degree is $g + n - 1$, since we multiplied $\Xi$ by the divisor of a rational function on $X$, which thus has degree 0. The set $D_{\alpha,l}$ now appears in $M^k(\Xi)$ to a power which lies in $\mathbb{N}_0$, and is congruent to $n - 1 - l + ak$ modulo $n$. Hence we can write $M^k(\Xi)$ in the form of Equation (7) as $\prod_{\alpha} \prod_{l=0}^{n-1} D_{\alpha,l+1}^{-1}$. As for the cardinality conditions, we may assume that $n$ does not divide $k$ (since otherwise $M^k(\Xi) = \Xi$ for which we know that the assertion holds). To see that $M^k(\Xi)$ satisfies the $j$th cardinality condition, we observe that for $\alpha$ such that $\alpha j - n s_{\alpha,j} + ak - ns_{\alpha,k} < n$ we take the cardinalities of the sets $D_{\alpha,l}$ with $ak - ns_{\alpha,k} \leq l \leq \alpha (k + j) - ns_{\alpha,k+j} - 1$, while if $\alpha j - n s_{\alpha,j} + ak - ns_{\alpha,k} \geq n$ then we consider the sets with $ak - ns_{\alpha,k} \leq l \leq n - 1$ together with those with $0 \leq l \leq \alpha (k + j) - ns_{\alpha,k+j} - 1$. Adding and subtracting $\sum_{\alpha} \sum_{l=0}^{ak-ns_{\alpha,k}} |D_{\alpha,l}|$ (which equals $t_k$) shows that the sum in question (which we need to be $t_j$) equals $t_{k+j} - t_k + \sum_{\alpha} (\alpha j - ns_{\alpha,j} + ak - ns_{\alpha,k}) r_{\alpha}$. The fact that $s_{\alpha,k} + \alpha j$ is $s_{\alpha,k+j}$ if $\alpha j - ns_{\alpha,j} + ak - ns_{\alpha,k} < n$ and equals $s_{\alpha,k+j}$ otherwise and the definition of the numbers $t_k$, $t_j$, and $t_{k+j}$ completes the proof of the cardinality conditions for $M^k(\Xi)$. Note that the proof works also if $n | k + j$, where $t_{k+j} = 0$ and the sum of $r_{\alpha}$ is taken over all $\alpha$. This proves the first assertion.

We now turn to the second assertion. Since $M^k$ multiplies $\Xi$ by a principal divisor, $M^k(\Xi)$ and $\Xi$ represent the same characteristic for all $k$ and $\Xi$. Hence we have to show that $h_{M^k(\Xi)} = h_{\Xi}$. According to the formula for the action of $M^k$, this assertion is equivalent to the statement that $[D_{\delta,\alpha}]_k$ appears in $h_{\Xi}$ to the same power as $[D_{\delta,\alpha} + nk]$ for every $\alpha$, $\delta$, $r$, and $j$. Write $j$ as $l + rk_{\delta} + ak$ modulo $n$, so that the first power is $c(\delta, \alpha) - f_{\delta,\alpha}(l)$. The congruence $l + r k_{\delta} = l + rk_{\delta} + ak (\text{mod } n)$ (since $n$ divides $k_{\delta} - 1$) shows that $[D_{\delta,\alpha} + nk]$ appears to the same power in $h_{\Xi}$. This completes the proof of the proposition.\hfill $\Box$

We remark that the proof of the cardinality conditions in Proposition 5.1 can be adapted to provide a direct proof of the appropriate assertions for $N_\beta(\Xi)$ (or $N_Q(\Delta)$) above, as well as for the images of $T_{Q,R}$. However, using Theorem 2.3 Proposition 2.4 and Theorem 1.6 we could establish these assertions without the need of direct evaluations.

From the formula for $M^k(\Xi)$ in the proof of Proposition 5.1, it is clear that $M^k$ is the $k$th power of the operator $M = M^1$, and that this operator is of order $n$ (recall that the index $l$ of $D_{\alpha,l}$ is considered in $\mathbb{Z}/n\mathbb{Z}$). This operator $M$ reduces to the operator denoted $M$ and given explicitly in Propositions 1.14 and 1.15 (as well as after Theorem A.2) and implicitly in Propositions 6.10 and 6.12 of [FZ] in the appropriate special cases.

The proof of Proposition 5.1 shows that rather than divisors of the form of Equation (7) defining characteristics, $M$-orbits of such divisors (which are sets of $n$ such divisors) provide better definitions for these characteristics. Moreover,
it follows from the formula for \( M^k(\Xi) \) that for any branch point \( Q \) and any \( M \)-orbit, any two divisors in that orbit contain \( Q \) raised to different powers in \( \mathbb{N}_n \). In particular, there is precisely one divisor \( \Xi \) in this orbit such that \( v_\Xi(Q) = n-1 \), and by Theorem 1.6 this divisor \( \Xi \) is of the form \( Q^{n-1} \Delta \) with \( \Delta \) non-special. Hence every \( M \)-orbit represents another characteristic (with a given base point). Hence every \( M \)-orbit represents another characteristic. In addition, this establishes the base point change formula: Let a non-special divisor \( \Delta \) supported on the branch points distinct from \( Q \) and another base point \( S \) be given. The non-special degree \( g \) divisor \( \Gamma \) not containing \( S \) in its support and satisfying the equality \( \varphi_S(\Gamma) + K_S = \varphi_Q(\Delta) + K_Q \) is \( \frac{\Gamma}{\mathbb{N}_k} \), where \( \Upsilon \) is the unique divisor in the \( M \)-orbit of \( \Xi = Q^{n-1} \Delta \) with \( v_S(\Upsilon) = n-1 \). This generalizes Propositions 1.14, 1.15, 6.10, and 6.21 of [FZ].

We remark at this point that given a base point \( Q \), we considered only those divisors from Theorem 1.6 whose support does not contain \( Q \) for characteristics. This is required for the theta constant \( \theta[\Delta, Q](0, \Pi) \) not to vanish. It follows that if a non-special divisor of degree \( g \) contains all the branch points in its support, then it cannot represent a non-vanishing theta constant with any branch point as base point. One such divisor shows up in Theorem 6.3 of [FZ].

We present an assertion about the operators which are defined on all the divisors \( \Xi \).

**Lemma 5.2.** The operator \( M \) and the operators \( N_\beta \) for the various \( \beta \) all lie in the same dihedral group \( G \) of order \( 2n \).

**Proof.** Given any \( \beta \) and \( k \), both compositions \( M^k(N_\beta(\Xi)) \) and \( N_\beta(M^{-k}(\Xi)) \) yield the divisor \( \prod_{\alpha,l} D_{\alpha,\alpha(k_\beta-k)}^{n-1-l} \). Hence the order \( 2 \) operator \( N_\beta \) and the elements \( M^k \) of the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) satisfy the relation \( M^k \circ N_\beta = N_\beta \circ M^{-k} \) defining the dihedral group of order \( 2n \). Moreover, replacing \( \beta \) by another element \( \delta \in \mathbb{N}_n \) which is prime to \( n \) and replacing \( k \) by \( j = k + k_\delta - k_\beta \) yields the same operator as above (namely \( M^k \circ N_\beta = N_\delta \circ M^{-j} \)). Hence the dihedral group generated by \( M \) and \( N_\beta \) is the same group for every \( \beta \). This proves the lemma.

Apart from the powers of \( M \), the dihedral group \( G \) from Lemma 5.2 consists of the operators taking \( \Xi \) from Equation (7) to \( \prod_{\alpha,l} D_{\alpha,\alpha(k_\beta-k)}^{n-1-l} \) for all \( k \in \mathbb{Z}/n\mathbb{Z} \). Let \( N \) be the operator with the simplest choice \( k_\beta = 0 \), whose action is \( N(\Xi) = \prod_{\alpha,l} D_{\alpha,n-1-l}^{n-1-l} \). Let \( G \) be generated by \( M \) and \( N \). Moreover, given any \( \beta \) as above, the operator mapping \( \Xi \) to \( \prod_{\alpha,l} D_{\alpha,\alpha(k_\beta-k)}^{n-1-l} \) lies in \( G \) (as \( N \circ M^{2k_\beta} \) or as \( M^{-2k_\beta} \circ N \)). Let \( \widehat{T}_{Q,R} \) be the composition of \( T_{Q,R} \) with \( N \circ M^{2k_\beta} = M^{-2k_\beta} \circ N \). The operators \( T_{Q,R} \) and \( \widehat{T}_{Q,R} \) are admissible on the same divisors \( \Xi \), but the action of \( \widehat{T}_{Q,R} \) is much simpler: It takes every such \( \Xi \) to \( R\Xi/Q \). Moreover, since we work with \( M \)-orbits rather than divisors, we can phrase the condition for admissibility of \( T_{Q,R} \) (or of \( \widehat{T}_{Q,R} \)) on some divisor \( \Xi \) as the requirement that if \( Q \in D_{\beta,j} \) for some \( j \in \mathbb{Z}/n\mathbb{Z} \) then the set containing
$R$ is $D_{\gamma,\gamma k\beta(j+1)}$. As the action of $M^k$ takes $D_{\beta,j}$ to $D_{\beta,j-\beta k}$ and $D_{\gamma,\gamma k\beta(j+1)}$ to $D_{\gamma,\gamma k\beta(j+1)-\gamma k}$, the congruence $\gamma k\beta(j+1) - \gamma k \equiv \gamma k\beta(j-\beta k + 1) \pmod{n}$ shows that this requirement is well-defined on $M$-orbits. This condition actually means that these operators are admissible on the (unique) divisor $\Xi$ in this orbit containing $Q$ to the power $n-1$. The action of $T_{Q,R}$ on such divisors is obtained using conjugation by the appropriate power of $M$. Explicitly, it takes $\Xi$ to $R\Xi/Q$, unless the index $j$ equals $n-1$ and the resulting divisor is $Q^{n-1}\Xi/R^{n-1}$. The operator $T_{R,Q}$ is applicable precisely on those divisors which are images of $T_{Q,R}$. Indeed, after this application $Q$ is taken to $D_{\beta,j+1}$ and $R$ to $D_{\gamma,\gamma k\beta(j+1)-1}$, and if $l = \gamma k\beta(j+1) - 1$ then $\beta k\gamma(l+1) \equiv j + 1 \pmod{n}$. The operator $T_{R,Q}$ is now seen to be the inverse of $T_{Q,R}$.

The final step of the proof establishment of the Thomae formulae depends on the following

**Conjecture 5.3.** The action of $G$ and the admissible operators $\hat{T}_{Q,R}$ relate any two operators $\Xi$ from Equation (7) satisfying the cardinality conditions.

We remark that the ordering (4.9) of [FZ] was implicitly based on the fact that given a base point $Q$ and an index $\gamma \in \mathbb{N}_n$ which is prime to $n$, one of each pair of the sets which are mixed by $T_{Q,R}$ is invariant under $N_{\beta}$. This happens for $\gamma = \beta$ and for $\gamma = n - \beta$ (the cases appearing in Chapters 4 and 5 and Appendix A of [FZ]), but in no other case. The cases studied in Chapter 6 of that book considered a small number of divisors with a simple behavior, so that ad-hoc considerations were sufficient to prove Conjecture 5.3 in these cases. Hence any proof of Conjecture 5.3 in general must involve new considerations, and cannot resemble any of these special cases. In fact, as some $Z_n$ curves do not carry any such divisors $\Xi$, finding an entirely general argument might be difficult.

We now prove several assertions, which together with the special cases given in [FZ], support Conjecture 5.3. Fix $\beta \in \mathbb{N}_n$ which is prime to $n$, and take $j \in \mathbb{Z}/n\mathbb{Z}$. If $j$ is neither 0 nor $n-1$ modulo $n$, then consider the difference between the cardinality conditions corresponding to $jk\beta$ and to $(j+1)k\beta$ (neither elements of $\mathbb{Z}/n\mathbb{Z}$ are 0 by our assumption on $j$, hence they both yield cardinality conditions). This difference yields a relation of the form

$$|D_{\beta,j}| = |D_{n-\beta,n-1-j}| + u_j,$$

where $u_j$ is $t_{(j+1)k\beta} - t_{jk\beta}$ plus the appropriate difference of the cardinalities of the sets $D_{n-\beta,n-1-j}$ with $\alpha$ different from $\beta$ and $n - \beta$. For $j = 0$ we get Equation (13) by subtracting $r_{n-\beta}$ from the $k\beta$th cardinality condition (with $u_0$ being $t_{k\beta}$ minus the appropriate cardinalities), and for $j = n-1$ the $-k\beta$th cardinality condition minus $r_{\beta}$ yields Equation (13) as well (where $u_{n-1}$ involves $-t_{-k\beta}$ and certain cardinalities). We remark that this is the form in which the cardinality conditions for the divisors $\Delta$ in Theorems 2.6, 2.9, 2.13, 2.15, and A.1 of [FZ] are given. We adopt from [FZ] the useful notation $(D_{n,1})_{\Xi}$ for the sets $D_{n,1}$ appearing in Equation (7) for the divisor $\Xi$. Observe that in contrast to the
numbers \( t_k \), the numbers \( u_j \) may be positive, negative or 0, and they depend on the divisor \( \Xi \) (and on \( \beta \)). In fact, given a divisor \( \Xi \), the number \( u_j \) arising from the index \( \beta \) is the additive inverse of the number \( u_{n-1-j} \) arising from \( n-\beta \).

We also remark that an argument similar to the proof of Proposition 5.1 shows that replacing \( \Xi \) by \( M^{l_k}(\Xi) \) takes the number \( u_j \) to \( u_{j-i} \).

**Proposition 5.4.** Let \( \Xi \) and \( \Upsilon \) be two divisors such that \((D_{\alpha,l})\Xi = (D_{\alpha,l})\Upsilon\) for all \( l \) wherever \( \alpha \) is neither \( \beta \) nor \( n-\beta \). Assume that either (i) \( D_{\beta,j} \neq \emptyset \) and \( D_{n-\beta,j} \neq \emptyset \) and \( D_{n-\beta,n-1-j} \) are non-empty. Then the operators \( \widehat{T_{Q,R}} \), with \( Q \) and \( R \) being either \( P_{\beta,i} \) or \( P_{n-\beta,i} \), are sufficient in order to reach from \( \Xi \) to \( \Upsilon \).

**Proof.** Note that as the numbers \( u_j \) depend on the sets \( D_{\alpha,l} \) for \( \alpha \) being neither \( \beta \) nor \( n-\beta \), the first hypothesis implies that they coincide for \( \Xi \) and \( \Upsilon \). Since in case (i) we have \( r_{n-\beta} = 0 \) and the sets \( D_{n-\beta,j} \) are empty for every divisor, Equation (19) implies \(|D_{\beta,j}| = |D_{\beta,j}|_{\Upsilon} \) for all \( j \). Moreover, Equation (13) shows that \( |D_{\beta,j}| + |D_{n-\beta,n-1-j}| \geq |u_j| \), with equality holding if and only if one of the sets in question is empty. Summing the latter inequality over \( j \in \mathbb{Z}/n\mathbb{Z} \) shows that \( \sum_{j=0}^{n-1}|u_j| \leq r_{\beta} + r_{n-\beta} \), and the hypothesis of case (ii) is satisfied precisely when this inequality is strict (in comparison, summing Equation (13) over \( j \in \mathbb{Z}/n\mathbb{Z} \) yields \( \sum_{j=0}^{n-1}u_j = r_{\beta} - r_{n-\beta} \)). Hence \( \Xi \) and \( \Upsilon \) satisfy the assumption of case (ii) simultaneously, and we can indeed use this assumption without referring to any of the divisors \( \Xi \) and \( \Upsilon \).

The first observation we make is that for \( Q = P_{\beta,i} \) lying in some set \( D_{\beta,j} \) and \( R = P_{\beta,m} \), the operator \( \widehat{T_{Q,R}} \) can act on \( \Xi \) if and only if \( R \in D_{\beta,j+1} \). In this case the operator interchanges these branch points. We can thus also interchange the point \( Q \) with a point \( S \) from \( D_{\beta,j+2} \), provided that \( D_{\beta,j+2} \) is not empty: Indeed, take \( R \in D_{\beta,j+1} \), and the combination \( \widehat{T_{R,S}} \circ \widehat{T_{Q,S}} \circ \widehat{T_{Q,R}} \) is a composition of operators, each one applicable on the divisor on which it is supposed to act, which has the desired effect. Easy induction now shows that if no set \( D_{\beta,l} \) is empty then we can interchange any two points \( P_{\beta,i} \) and \( P_{\beta,m} \) with one another using these operators, regardless of the sets in which they lie.

As in case (i) the divisor \( \Upsilon \) can be obtained from \( \Xi \) by a finite sequence of such transpositions, this proves the assertion in this case.

On the other hand, if \( R = P_{n-\beta,m} \) (and \( Q \) is as above) then the admissibility condition is \( R \in D_{n-\beta,n-1-j} \). Applying \( \widehat{T_{Q,R}} \) takes \( Q \) to \( D_{\beta,j+1} \) and \( R \) to \( D_{n-\beta,n-j} \). Hence \( \widehat{T_{Q,R}} \) is applicable again, so that we can move \( Q \) to \( D_{\beta,j+k} \) and \( R \) to \( D_{n-\beta,n-1-j+k} \) for any \( k \in \mathbb{Z}/n\mathbb{Z} \). Therefore if \( D_{\beta,j} \) and \( D_{n-\beta,n-1-j} \) are both not empty then we can take an arbitrary point from each set and move it to \( D_{\beta,l} \) and \( D_{n-\beta,n-1-l} \) respectively for any \( l \) of our choice. We now claim that also in this case the operators \( \widehat{T_{Q,R}} \) allow us to interchange any two points \( P_{\beta,i} \) and \( P_{\beta,m} \) with one another. Indeed, assume that one point \( Q \) lies in \( C_{\beta,j} \) and the other point \( S \) lies in \( C_{\beta,l} \), assume that both \( C_{\beta,k} \) and \( C_{n-\beta,n-1-k} \) are non-empty, and take \( P \in C_{\beta,k} \) and \( R \in C_{n-\beta,n-1-k} \). Using the operations just described, we can take \( P \) and \( R \) to \( D_{\beta,j} \) and \( D_{n-\beta,n-1-j} \) respectively, then \( Q \) and \( R \) to \( D_{\beta,l} \) and \( D_{n-\beta,n-1-l} \) respectively, followed by transferring \( S \) and \( R \) to \( D_{\beta,j} \) and
$D_{n-\beta,n-1-j}$ again. Sending $P$ and $R$ back to their original sets completes a combination of admissible operations which acts as the asserted interchange. In a similar manner we can replace any point from $C_{n-\beta,j}$ with any point from $C_{n-\beta,l}$ by admissible operations. Note that under the assumptions of case $(ii)$, the divisor $\Upsilon$ can be reached from $\Xi$ by a finite sequence of transfers of points from $D_{\beta,j}$ and $D_{n-\beta,n-1-j}$ to $D_{\beta,l}$ and $D_{n-\beta,n-1-l}$ followed by permutations of the points $P_{\beta,i}, 1 \leq i \leq r_{\beta}$ and of the points $P_{n-\beta,i}, 1 \leq i \leq r_{n-\beta}$. The proof of the proposition is therefore complete.

The validity of Proposition 5.4 extends to divisors $\Xi$ and $\Upsilon$ for which $D_{\beta,j}$ is empty and $D_{n-\beta,j}$ is not empty for all $j \in \mathbb{Z}/n\mathbb{Z}$. This is achieved either using symmetry or by a simple adaptation of the proof.

We note that by taking $\beta = 1$ in a $Z_{n}$ curve of the form considered in Section A.7 of [FZ], Proposition 5.4 can replace Theorem A.2 of that reference in the proof of the Thomae formulae for these $Z_{n}$ curves. The same statement holds for the special cases treated in Theorems 4.8 and 5.7 there. Indeed, if either $r_{1}$ or $r_{n-1}$ vanish then every two divisors satisfy the assumptions of case $(i)$ of Proposition 5.4. Otherwise the assumptions of case $(ii)$ hold for any $\Xi$ and $\Upsilon$. The proof here is simpler than those given in [FZ] because of the freedom to move the base point. This removes a basic obstacle in the argument, and deals with the non-singular case in a unified way, regardless of whether $r = 1$ or $r \geq 2$.

We proceed with the following

**Lemma 5.5.** Let $\Upsilon$ be a divisor satisfying the hypothesis of either case of Proposition 5.4 and let $\Xi$ be a divisor satisfying the usual cardinality conditions. Let $S$ be a branch point $P_{\gamma,m}$ for $\gamma$ which is neither $\beta$ nor $n-\beta$. Assume that the equality $v_{S}(\Xi) = v_{S}(\Upsilon)$ holds for every branch point $P_{\beta,i} \neq S$ in which $\delta$ equals neither $\beta$ nor $n-\beta$, and that the difference between $v_{S}(\Xi)$ and $v_{S}(\Upsilon)$ is 1. Then one can get from $\Xi$ to $\Upsilon$ using the operators $T_{Q,R}$.

**Proof.** Fix the sign of $\pm$ according to $v_{S}(\Upsilon) = v_{S}(\Xi) \pm 1$, and choose the element $j \in \mathbb{Z}/n\mathbb{Z}$ such that the set containing $S$ is $(D_{\gamma-k\delta(j+1)})_{\Xi}$ in case $\pm = +$ and is $(D_{\gamma-k\delta(j-1)})_{\Xi}$ if $\pm = -$. By denoting the difference $|(D_{\beta,i})_{\Upsilon} - (D_{n-\beta,n-1-l})_{\Upsilon}|$ by $\bar{u}_{i}$ we find that $\bar{u}_{i} = u_{j} - 1$, $\bar{u}_{j \pm 1} = u_{j \pm 1} + 1$, and $\bar{u}_{k} = u_{k}$ for any other $k \in \mathbb{Z}/n\mathbb{Z}$. Assume first that $\Xi$ satisfies the hypothesis of case $(ii)$ of Proposition 5.4. Then the proof of this proposition shows that there exist points $Q = P_{\beta,i}$ and $R = P_{n-\beta,p}$ such that an appropriate power of $T_{Q,R}$ takes $\Xi$ to a divisor $\Sigma$ such that $(D_{\beta,j})_{\Sigma}$ contains $Q$. Otherwise the cardinalities of the sets $(D_{\beta,k})_{\Xi}$ (as well as $(D_{n-\beta,k})_{\Xi}$) are determined by the cardinalities of the other sets corresponding to $\Xi$. Indeed, the proof of Proposition 5.4 shows that in this case the equality $\sum_{j} |u_{j}| = r_{\beta} + r_{n-\beta}$ holds and at least one of $D_{\beta,j}$ and $D_{n-\beta,n-1-j}$ is empty for all $j \in \mathbb{Z}/n\mathbb{Z}$. The cardinalities are thus determined by Equation 13. We claim that $(D_{\beta,j})_{\Xi} \neq \emptyset$ in this case. Indeed, if $D_{n-\beta,k} = \emptyset$ for all $k$ ($\Upsilon$ satisfying the hypothesis of case $(i)$ of Proposition 5.4) then $u_{j} \geq 0$ hence $u_{j} \geq 1$. On the other hand, the proof of Proposition 5.4 implies that the only
case in which \( \Upsilon \) satisfies the hypothesis of case (ii) of that proposition but \( \Xi \) does not is when
\[
\sum_k |u_k| = r_\beta + r_{n-\beta} \text{ and } \sum_k |\hat{u}_k| < r_\beta + r_{n-\beta}.
\]
Under the given relations between the \( u_k \)s and the \( \hat{u}_k \)s this can happen only if \( u_j \neq 0 < u_j \), so that in particular \( (D_{\beta,j})_\Xi \neq \emptyset \). Take \( Q \in (D_{\beta,j})_\Xi \), and choose \( \Sigma = \Xi \) in this case. Now, the location of \( Q \) and \( S \) implies that the operator \( \hat{T}_{Q,S} \) in case \( \pm = + \) and \( \hat{T}_{S,Q} \) if \( \pm = - \) is admissible on \( \Sigma \). Moreover, \( \Upsilon \) and the image of \( \Sigma \) under this operator satisfy the hypothesis of Proposition 5.4. An application of the aforementioned proposition now completes the proof.

Proposition 5.4 and Lemma 5.5 suggest another example for the validity of Conjecture 5.3. Consider a \( \mathbb{Z}_n \) curve \( X \) for which there is some \( \beta \) such that one of the following is satisfied: (i) \( r_\beta \) is much larger than \( r_\alpha \) for other \( \alpha \), or (ii) \( r_\beta + r_{n-\beta} \) is a sum of positive integers which is relatively large. Then, Conjecture 5.3 holds for \( X \), at least heuristically. This is so, since most divisors will satisfy the assumptions of Proposition 5.4, hence can be related to many “neighboring” divisors.

Combining Theorem 4.6 and Proposition 5.1 yields the main result of this paper, which is the Thomae formulae for the general fully ramified \( \mathbb{Z}_n \) curve \( X \):

**Theorem 5.6.** Assume that \( X \) satisfies Conjecture 5.3. Then the value of the quotient \( \frac{\vartheta^{n-1}}{h_\Xi} \) from Theorem 4.6 is independent of the choice of the divisor \( \Xi \).

Following [FZ] we would like to relate the characteristics to those arising from the non-special divisors \( \Delta \) from Theorem 1.6 with the choice of some branch point \( Q \) as base point. For every such \( Q \) and \( \Delta \) we define the denominator \( h^Q_\Delta \) to be \( h_\Xi \) for \( \Xi = Q^n - \Delta \). Theorem 5.6 then implies that if Conjecture 5.3 holds for the \( \mathbb{Z}_n \) curve \( X \) then the quotient \( \frac{\vartheta^{n-1}}{h_\Sigma} \) is independent of the choice of the divisor \( \Delta \). Moreover, this quotient yields the same constant for every base point \( Q \). The formulation of Theorem 5.6 as appears here corresponds to the “symmetric” Thomae formulae in [FZ]. The fact that [K] finds a constant for every such \( \mathbb{Z}_n \) curve also suggests that Conjecture 5.3 might be true.

### 6 The Functions \( f_{\beta,\alpha} \)

In this Section we investigate the functions \( f_{\beta,\alpha} \) further, and obtain explicit expressions to evaluate them in some cases. First observe that \( f_{\beta,\alpha} \) depends only on the number \( \alpha k_\beta \) modulo \( n \) (because \( a_{\beta,\alpha} \) and \( b_{\beta,\alpha} \) depend only on this number). Fix \( d \in \mathbb{N}_n \) which is prime to \( n \), and consider the functions \( f_{\beta,\alpha} \) for which \( \alpha \equiv \beta (\text{mod } n) \). Theorem 4.4 implies that all these functions coincide to a unique function, which we denote \( f_d^{(n)} \). Lemma 4.3 expresses \( f_{n-d}^{(n)} \) in terms of \( f_d^{(n)} \), so that we can restrict attention to those \( d \in \mathbb{N}_n \) satisfying \( d < \frac{n}{2} \). Moreover, Lemma 4.4 relates \( f_d^{(n)} \) to \( f_k^{(n)} \) (recall that \( k_d = 0 \) satisfies \( dk_d \equiv 1 \) (mod \( n \)), but here we do require \( k_d \in \mathbb{N}_n \)), or in fact shows that in the expression for \( h_\Xi \).
functions $f$ in some cases, to write the full expression for $h$ that it satisfies the second condition as well. These considerations allow us, in

Assume that $f$ (recall that $l$ is given in Theorem 4.6 and we can always use one instead of the other. Note that for $d = 1$ and for $d = n - 1$ we have $k_d = d$ (regardless of $n$). Lemma 4.3 reduces to Equation (11) for $d = 1$ (since $a_{\alpha,0}(l) = n - l - n\delta_0$) and is trivial for $d = n - 1$.

Wishing to preserve this symmetry, we choose the constants $c(\delta, \alpha)$ such that they depend only on the number $d \in \mathbb{N}_n$ satisfying $\alpha \equiv d\delta (\text{mod} \ n)$. Denoting the appropriate constant $c^{(n)}_d$, we must have $c^{(n)}_d = c^{(n)}_{k_d}$ by the assumption made when we defined $\Xi$. The normalization $c(\delta, \alpha) = \max_{l \in \mathbb{N}_n} f_{\delta, \alpha}(l)$ satisfies the first condition (so that $c^{(n)}_d = \max_{l \in \mathbb{N}_n} f^{(n)}_d(l)$), and Lemma 4.4 shows that it satisfies the second condition as well. These considerations allow us, in some cases, to write the full expression for $\Xi$ explicitly using just a few of the functions $f^{(n)}_d$.

Now fix $n$ and $d \in \mathbb{N}_n$ which is prime to $n$. We begin our analysis of the functions $f^{(n)}_d$ with

**Proposition 6.1.** Assume that $l \in \mathbb{N}_n$ equals $pd + q$ with some $q \in \mathbb{N}_d$. Then $f^{(n)}_d(l)$ is related to $f^{(n)}_d(q)$ through the formula stating that $f^{(n)}_d(l)$ equals

$$f^{(n)}_d(q) + p(n + d - 1 - q - l) = \frac{(n + d - 1 - l)}{d} + f^{(n)}_d(q) - \frac{q(n + d - 1 - q)}{d}.$$ 

**Proof.** Equation (12) translates to the equality $f^{(n)}_d(j + d) = f^{(n)}_d(j) + n - 1 - 2q$ for all $j \in \mathbb{N}_n$ such that $j + d < n$. By applying this for $j = di + q$ for all $i \in \mathbb{N}_p$ (with $j = q$ for $i = 0$ and $j + d = l$ for $i = p - 1$) we obtain

$$f^{(n)}_d(l) = f^{(n)}_d(q) + \sum_{i=0}^{p-1} [n - 1 - 2(di + q)] = f^{(n)}_d(q) + p(n - 1 - 2q) - dp(p - 1) = f^{(n)}_d(q) + \frac{l - q}{d} = f^{(n)}_d(q) + \frac{n + d - 1 - q - l}{d}$$

(recall that $l = pd + q$ hence $p = \frac{l - q}{d}$). This gives the first expression, and the second follows from simple arithmetics. This proves the proposition. \qed

Proposition 6.1 shows that knowing $f^{(n)}_d(q)$ only for $q \in \mathbb{N}_d$ is sufficient for evaluating $f^{(n)}_d(l)$ for all $l \in \mathbb{N}_n$. It turns out useful to write $n = sd + t$ for some $t \in \mathbb{N}_d$ in what follows. We now derive a few properties of the expression $f^{(n)}_d(q) - \frac{q(n + d - 1 - q)}{d}$ with $q \in \mathbb{N}_d$ appearing in Proposition 6.1. It turns out useful to multiply this expression by $-d$, and call the result $g^{(d)}_t(q)$ (this is an abuse of notation at this point, since we do not yet know that the value of $g^{(d)}_t(q)$ depends only on $t$ and not on $n = sd + t$, but we use it nonetheless). Recall that Equation (11) for $f^{(n)}_d$ compares the value which this function attains on $q \in \mathbb{N}_d$ with the value it takes on $d - 1 - q$, as well as the image of some $d \leq l \in \mathbb{N}_n$ under this function with the image of $n + d - 1 - l$. The following Lemma gives a similar assertion for $g^{(d)}_t(q)$.
Lemma 6.2. The expression $g^{(d)}_t(q)$ is invariant under replacing $q \in \mathbb{N}_d$ by $t - 1 - q$ and $t \leq q \in \mathbb{N}_d$ by $d + t - 1 - q$.

Proof. Express $f^{(n)}_d(l)$ for $l \geq d$ in terms of the formula from Proposition 6.1 and compare it to $f^{(n)}_d(n + d - 1 - l)$. These values coincide by Equation (11), as remarked above. The terms involving $-f^l$ have the remark after Theorem 6.4 below.

The terms involving $-f^l$ have the remark after Theorem 6.4 below. The equality $\text{Lemma 6.2}$.

Proof. Set $d$.

We now prove that the functions $f^{(n)}_d$ can be evaluated using a recursive process, based on Euclid’s algorithm for finding greatest common divisors using division with residue:

Theorem 6.4. If $n = sd + t$ and $l = pd + q$ for $t$ and $q$ in $\mathbb{N}_d$ then we can write $f^{(n)}_d(l)$ as $\frac{l(n + d - 1 - l)}{d} - \frac{n}{d} f^{(d)}_t(q)$. The value of $f^{(n)}_d(l)$ can be written as $\frac{d}{d} f^{(d)}_t(q) - \frac{l(n - d - 1 - l)}{d}$.

Proof. By Proposition 6.1 the first assertion boils down to the statement that $g^{(d)}_t(q) = n f^{(d)}_t(q)$ for every $q \in \mathbb{N}_d$. Lemma 6.2 shows that $g^{(d)}_t(q)$ satisfies Equation (11) for $d$ and $t$. We wish to prove that it satisfies also the appropriate Equation (12), namely that $g^{(d)}_t(q) + n(d - 1 - 2q)$ gives us $g^{(d)}_t(q + t)$ if $q \in \mathbb{N}_{d-t}$.
which reduces to the asserted value $g_t^{(d)}(q + t - d)$ if $q \geq d - t$. Recall that $s = \frac{n - t}{d}$. Take $q \in \mathbb{N}_{d-t}$ and apply part (i) of Lemma 6.3. This evaluates $g_t^{(d)}(q + t) = (q + t)(n + d - 1 - q - t) - df_d^{(n)}(q + t)$ as

$$q(n + d - 1 - q) + t(n + d - 1 - q - t) - tq - df_d^{(n)}(q) + (n - t)(d - t - 1 - 2q),$$

which reduces to the asserted value $g_t^{(d)}(q) + n(d - 1 - 2q)$. For $q \geq d - t$ we use part (ii) of Lemma 6.3 and write $s + 1 = \frac{n + d - t}{d}$ in order to find that the difference between

$$g_t^{(d)}(q + t - d) = (q + t - d)(n + 2d - 1 - q - t) - df_d^{(n)}(q + t - d)$$

and $g_t^{(d)}(q) = q(n + d - 1 - q) - df_d^{(n)}(q)$ is

$$(n + d - t)(2d - t - 1 - 2q) - (d - t)(n + 2d - 1 - q - t) + q(d - t) = n(d - 1 - 2q),$$

as desired. Since $g_t^{(d)}(0)$ clearly vanishes, the desired equality $g_t^{(d)}(q) = nJ_t^{(d)}(q)$ for all $q \in \mathbb{N}_d$ follows from Theorem 4.1, which proves the formula for $f_d^{(n)}$. The result for $f_{n-d}^{(n)}$ now follows from Lemma 4.5. This proves the theorem.

The proof of Theorem 6.4 justifies the notation $g_t^{(d)}$ a fortiori, since it indeed depends only on the residue $t$ of $n$ modulo $d$. We remark that as the proof of Theorem 4.1 requires only vanishing at 0 and Equation (12), Lemma 6.2 is not necessary for the proof of Theorem 6.4. Hence Theorem 6.4 holds for every $n$ and $d$ (without the assumption $d < \frac{n}{2}$), and Lemma 6.2 (for all $n$ and $d$) follows as a corollary of its proof. Lemma 6.2 in fact implies that the formula for $f_d^{(n)}(l)$ takes the same form for $l \equiv q\pmod{d}$ and for $l$ which is equivalent to $t - 1 - q$ or to $d + t - 1 - q$ modulo $d$.

For any $y \in \mathbb{Z}$, the expression $t(n + d - 1 - l) - ny$ can be written as $(l - y)(n + d - 1 - y - 1) - y(y - d + 1)$. Since for $y = f_d^{(d)}(q)$ the number $d - 1 - y$ is $f_d^{(d)}(d - 1 - q)$, we can write

$$f_d^{(n)}(l) = \frac{(l - f_d^{(d)}(q))(n + f_d^{(d)}(d - 1 - q) - l) + f_d^{(d)}(q)f_d^{(d)}(d - 1 - q)}{d}.$$ 

The latter formula presents an interesting symmetry between the formula for $f_d^{(n)}(l)$ with $n = sd + t$ and $l = pd + q$ and the one for $f_d^{(m)}(j)$ with $m = rd - t$ and $j = kd - 1 - q$. In particular, if $f_d^{(d)}(q) = d - 1$ then the expression for $f_d^{(n)}(l)$ given in Theorem 6.4 becomes just $\frac{(l-d+1)(n-l)}{d}$. Thus, Theorem 6.4 (or Proposition 6.1) and Lemma 6.2 combine with Lemma 4.5 to give

**Corollary 6.5.** If $l$ is divisible by $d$, or if $l$ is congruent to $t - 1$ modulo $d$, then $f_d^{(n)}(l) = \frac{(l-n+1-l)}{d}$ and $f_d^{(n)}(l) = \frac{(l-n+1-l)}{d}$. If $d$ divides $l + 1$, or if $l \equiv t\pmod{d}$, then $f_d^{(n)}(l) = \frac{(l-d+1)(n-l)}{d}$ and $f_d^{(n)}(l) = \frac{(d-1)n-l}{d}$. If $d$ divides $l + 1$, or if $l \equiv t\pmod{d}$, then $f_d^{(n)}(l) = \frac{(l-d+1)(n-l)}{d}$ and $f_d^{(n)}(l) = \frac{(d-1)n-l}{d}$.
Corollary 6.5 is already sufficient to evaluate \( f_d^{(n)} \) for some values of \( d \). For \( d = 1 \) all the assertions of Corollary yield the formulae \( f_1^{(n)}(l) = l(n-l) \) and \( f_{n-1}^{(n)}(l) = -l(n-l) \) for all \( l \in \mathbb{N} \). Equation (11) for these cases correspond to the fact that the value of \( f_1^{(n)} \) is invariant under sending \( l \) to \( n-l-n\delta_{00} \) and \( f_{n-1}^{(n)} \) attains the same value on \( l \) and on \( n-2-l+n\delta_{n-1} \). Observe that for even \( n \) the function \( f_1^{(n)} \) attains its maximal value \( \frac{n^2}{4} \) at \( l = \frac{n}{2} \), while for odd \( n \) the maximal value is \( \frac{n^2-1}{4} \), attained on \( l = \frac{n-1}{2} \) and on \( l = \frac{n+1}{2} \). Hence the expression for the function \( f_1^{(n)} \) does not depend on the parity of \( n \), but its normalizing constant \( c_1^{(n)} \) does depend on the parity of \( n \). This is the reason for the different formulae for odd and even \( n \) given in Chapter 4 of [PZ]. \( f_{n-1}^{(n)} \) attains its maximal value \( n-1 \) at \( l = n-1 \) (regardless of the parity of \( n \)), and \( n-1 - f_{n-1}^{(n)}(l) \) equals \((n-1)(n-l-1)\) for all \( l \in \mathbb{N} \). Note that the expression \([C_1, D_{i+k}]\) from Chapter 5 of [PZ] becomes \([D_{1,i+1}, D_{n-1,n-2-i-k}]\) in our notation, and the second index of the latter set is \( i+1 \) times \( \alpha k \equiv -1 \pmod{n} \) plus \( n-1-k \). Since \( k(n-k) \) agrees with our \( n-1 - f_{n-1}^{(n)}(l) \) for \( l = n-1-k \), we verify the results of this Chapter (and of Section A.7 there) as well.

Corollary 6.5 also yields the full formula for the case \( d = 2 \) (for odd \( n \)): \( f_2^{(n)} \) takes even \( l \in \mathbb{N} \) to \( \frac{l(n+1-l)}{2} \) and odd \( l \in \mathbb{N} \) to \( \frac{(l-1)(n-l)}{2} \), while \( f_{n-2}^{(n)}(l) \) is \(-l(n-3-l)\) if \( l \) is even and equals \( 2 - \frac{(l-1)(n-4-l)}{2} \) if \( l \) is odd. Equation (11) is satisfied since the involution for \( d = 2 \) takes \( l \geq 2 \) to \( n+1-l \) (and interchanges 0 and 1) while the one with \( d = n-2 \) maps \( l \leq n-3 \) to \( n-3-l \) (and interchanges \( n-1 \) and \( n-2 \)). The maximal value of \( f_2^{(n)}(l) \) is obtained for even \( l \), and depends on whether \( n \) is equivalent to 1 or to 3 modulo 4: It equals \( \frac{n^2+2n-3}{8} \) (for \( l \) being \( \frac{n-1}{2} \) or \( \frac{n+1}{2} \)) in the former case and \( \frac{n^2+2n+3}{8} \) (for \( l = \frac{n+1}{2} \)) in the latter case. \( f_{n-2}^{(n)} \) attains its maximal value \( n-1 \) at \( l = n-2 \) and at \( l = n-1 \) for every odd \( n \), and the difference \( n-1 - f_{n-2}^{(n)}(l) \) equals \( \frac{(t+2)(n-1-l)}{2} \) if \( l \) is even and \( \frac{(t+1)(n-2-l)}{2} \) if \( l \) is odd. Lemma 14 now shows that \( f_{n-2}^{(n)} \) takes \( l \in \mathbb{N} \) to \( l(n-1-2l) \) and maps \( \frac{n+1}{2} \leq l \in \mathbb{N} \) to \( n-l(2l+1-n) \), while the function \( f_{n-2}^{(n)} \) sends \( l \in \mathbb{N} \) to \(-l(n-3-2l)\) and takes \( n+1 \leq l \in \mathbb{N} \) to \( 2 - (n-2-l)(2l-1-n) \). These values suffice to determine \( h \) in the first example in Section 7, and in fact we can use the values of \( f_2^{(n)} \) and \( f_{n-2}^{(n)} \) alone for this purpose.

For \( d = 3 \) we encounter the dependence of the form of \( f_d^{(n)} \) on the residue \( t \) of \( n \) modulo \( d \). Corollary 6.5 again gives us the full answer: If \( t = 1 \) then \( f_d^{(n)}(l) \) is \( \frac{(n+2-l)}{3} \) and \( f_{n-3}^{(n)}(l) \) equals \( -\frac{(n-4-l)}{3} \) if \( 3 \) divides \( l \), while otherwise \( f_3^{(n)}(l) \) and \( f_{n-3}^{(n)}(l) \) attain \( \frac{(l-2)(n-l)}{3} \) and \( 4 - \frac{(l-2)(n-6-l)}{3} \) on \( l \) respectively. On the other hand, for \( t = 2 \) we find that \( f_d^{(n)}(l) = \frac{(l+2-2l)}{3} \) and \( f_{n-3}^{(n)}(l) = -\frac{(n-4-l)}{3} \) if \( l \) is congruent to 0 or 1 modulo 3, while the values \( \frac{(l-2)(n-l)}{3} \) and \( 4 - \frac{(l-2)(n-6-l)}{3} \) are attained only on \( l \equiv 2 \pmod{3} \). We also need the maximal values of these functions,
which depend on $t$ as well as on the parity of $n$. For $t = 2$ the maximal value of $f_3^{(n)}$ is $\frac{n^2 + 4n + 3}{12}$ for odd $n$ (with $l$ being $\frac{n+1}{2}$ or $\frac{n+3}{2}$) and $\frac{n^2 + 4n}{12}$ (arising from $l = \frac{n}{2}$ or from $l = \frac{n+4}{2}$) if $n$ is even. $f_{n-3}^{(n)}$ attains its maximal value $n - 1$ on $l = n - 1$ and on $l = n - 3$ in this case. On the other hand, if $t = 1$ then the maximal value which $f_3^{(n)}$ attains is $\frac{n^2 + 4n + 5}{12}$ (on $l = \frac{n+2}{2}$) if $n$ is even and is $\frac{n^2 + 4n}{12}$ for odd $n$ (with the value of $l$ being either $\frac{n+1}{2}$ or $\frac{n+3}{2}$). The function $f_{n-3}^{(n)}$ then attains $n - 1 + s = 4s$ on $l = 1$ (and no larger values). The functions for which we can now apply Lemma 5.4 depend themselves on the value of $t$: $k_3$ is $s + 1$ if $t = 2$ and is $2s + 1$ if $t = 1$, while $k_{n-3}$ equals $s$ for $t = 1$ and is $2s + 1$ for $t = 2$. Hence we shall not write the formulae for these functions, but only mention that their value on $l$ depends on whether $0 \leq l \leq s$, $s + 1 \leq l \leq n - s - 1$, or $n - s \leq l \leq n - 1$. Moreover, the formulae for these functions take the same form for $t = 1$ and for $t = 2$ except for $l$ in the middle interval. In any case, the values of $f_3^{(n)}$ and $f_{n-3}^{(n)}$ are sufficient to determine $h_\Xi$ explicitly in the second example below.

To give the flavor of how the functions $f_d^{(n)}$ look like for larger values of $d$, we use our knowledge of the function $f_1^{(d)}$ and $f_{d-1}^{(d)}$ from above in order to extract from Theorem 6.3 and Lemma 6.5 the following

**Corollary 6.6.** In the case $t = 1$ (which means that $d$ divides $n - 1$) $f_d^{(n)}(l)$ equals $\frac{l - q(d - q)(n - (q - 1)(d - 1 - q) - l) - q(q - 1)(d - q)(d - 1) - q}{d}$, or alternatively

\[
\frac{l - q(d - q)(n - (q - 1)(d - 1 - q) - l) - q(q - 1)(d - q)(d - 1 - q) - l}{d}
\]

for every $l \in \mathbb{N}_n$ which is congruent to $q \in \mathbb{N}_n$ modulo $d$. The function $f_{n-d}^{(n)}$ attains on such $l$ the value $\frac{nq(d - q)(n - (d - 1 - l) - l)}{d}$, which can also be written as

\[
\frac{n - (q + 1)(d - q)(d + 1 - q) - l - q(q - 1)(d - q)(d - 1 - q) - l}{d}
\]

For $t = d - 1$ (namely, if $d$ divides $n + 1$) and $l \in \mathbb{N}_n$ of the form $pd + q$ we find that $f_d^{(n)}(l)$ is $\frac{l - q(d - 2 - q)(n - (d - 1 - l) - l) - q(q + 1)(d - 1 - q)(d - 2 - q)}{d}$, namely

\[
\frac{l + q(d - 2 - q)(n + (q + 1)(d - 1 - q) - l) - q(q + 1)(d - 1 - q)(d - 2 - q)}{d}
\]

An element $l \in \mathbb{N}_n$ of this form is taken by the function $f_{n-d}^{(n)}$ to the number $\frac{-nq(d - 2 - q)(n - (d - 1 - l))}{d}$, which also equals

\[
\frac{q(q + 1)(d - 2 - q)(d - 1 - q) - l + q(d - 2 - q)(n + (q + 1)(d - 3 - q) - 4 - l)}{d} - 2q(d - 2 - q).
\]

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Corollary 6.6 reproduces the formula for \( f_2^{(n)} \), \( f_3^{(n)} \), and \( f_5^{(n)} \). It also gives the formula for \( f_4^{(n)} \) and \( f_6^{(n)} \) for all odd \( n \) and for \( f_5^{(n)} \) and \( f_{n-5}^{(n)} \) for every odd \( n \) which is not divisible by 3, since for both \( d = 4 \) and \( d = 6 \) (as well as for \( d = 3 \) considered above) the only two elements of \( \mathbb{N}_d \) which are prime to \( d \) are 1 and \( d - 1 \). For example, if \( n = 4s + 1 \) then \( f_4^{(n)} \) takes \( l \) which is divisible by 4 to \( \frac{k(n+3-l)}{4} \), other \( l \) to \( \frac{l-4(n-1-l)}{4} - 1 \), and odd \( l \) to \( \frac{l-4(n-1-l)}{4} - (l-3)(n-1-l) \). In this case if 4 divides \( l \) then \( f_{n-4}^{(n)}(l) = \frac{-4(n-5-l)}{4} \), the \( f_{n-4}^{(n)} \) image of other even \( l \) is \( 9 - \frac{4(n-5-l)}{4} \), and \( f_{n-4}^{(n)} \) sends any odd \( l \) to \( 6 - \frac{4(n-5-l)}{4} \). On the other hand, for \( n = 4s + 3 \), the function \( f_3^{(n)} \) takes any even \( l \) to \( \frac{k(n+3-l)}{4} \), \( l \) which is congruent to 3 modulo 4 to \( \frac{(l-3)(n-1-l)}{4} - 1 \), and \( l \) satisfying \( l \equiv 1 \pmod{4} \) to \( \frac{(l-3)(n-1-l)}{4} - 1 \). The function \( f_{n-4}^{(n)} \) here sends even \( l \) to \( \frac{-4(n-5-l)}{4} \), while \( l \) which satisfies \( l \equiv 3 \pmod{4} \) is taken to \( 6 - \frac{4(n-5-l)}{4} \) and \( l \) which is congruent to 1 modulo 4 is sent to \( - \frac{(l+1)(n-4-l)}{4} - 1 \). For \( d = 5 \) the formulae are based on the various functions \( f_5^{(5)} \) for \( 0 \neq t \in \mathbb{N}_5 \). The fact that \( f_1^{(5)} \) takes the numbers 0, 1, 2, 3, and 4 to 0, 4, 6, 6, and 4 respectively and \( f_4^{(5)} \) sends them to 0, -2, -2, 0, and 4 respectively gives the case \( d = 5 \) of Corollary 6.6. The values of \( f_2^{(5)} \) are 0, 4, 2, and 4 respectively, while \( f_3^{(5)} \) takes the values 0, 2, 0, 4, and 4 respectively. The values for \( f_5^{(5)}(l) \) and \( f_{n-5}^{(n)}(l) \) thus agree with the values given in Corollary 6.5. The additional values which we obtain are for \( l \equiv 3 \pmod{5} \) in case \( n = 5s + 2 \) and for \( l \equiv 1 \pmod{5} \) in the case \( n = 5s + 3 \), where \( f_5^{(5)}(l) \) and \( f_{n-5}^{(n)}(l) \) equal \( \frac{(l-2)(n+2-t)+4}{8} \) and \( 1+(l-2)(n+2-t)+4 \) respectively.

7 Examples of Thomae Formulae

In this Section we give examples of Thomae formulae for two families of \( Z_n \) curves, and discuss the divisors on \( Z_n \) curves belonging to a third family.

We begin with the \( Z_n \) curves associated with equations of the form

\[
w^n = \prod_{i=1}^{r}(z - \lambda_i) \prod_{i=1}^{p}(z - \sigma_i)^2 \prod_{i=1}^{q}(z - \tau_i)^{n-2} \prod_{i=1}^{m}(z - \mu_i)^{n-1}
\]

for odd \( n \), where \( r + 2p - 2q - m \) is divisible by \( n \) hence equals \( nu \) for some \( u \in \mathbb{Z} \). The number \( t_k \) is \( ku + q + m \) if \( k \leq \frac{n+1}{2} \) is \( s \) and equals \( ks + p + 2q + m \) if \( k \geq \frac{n+1}{2} = n - s \). The only non-trivial index \( d \) which we shall need here is \( d = 2 \), so that in \( n = sd + t \) we have \( t = 1 \) and \( s = \frac{n-1}{2} \). The expressions using \( s \) are given here in order to emphasize the similarity with the results of the following example. We switch to a notation similar to that of \([FZ]\), so that the branch points over \( \lambda_i \), \( \mu_i \), \( \sigma_i \), and \( \tau_i \) are denoted \( P_t \), \( Q_t \), \( R_t \), and \( S_t \) respectively. A divisor \( \Xi \) in Equation \([2]\) takes the form \( \prod_{t=0}^{n-1} C_t^{n-1-t} D_t^{n-1-t} E_t^{n-1-t} F_t^{l} \), where \( C_t \) contains points of the sort \( P_t \), \( D_t \) contains points of the type \( Q_t \), \( E_t \) contains points of the type \( R_t \), and \( F_t \) contains points of the sort \( S_t \). Hence \( C_t \) stands
for $D_{1,l}$, $D_{l}$ is the set $D_{n-1,n-1-l}$, $E_{l}$ represents $D_{2,l}$, and $F_{l}$ is our notation for $D_{n-1,n-1-l}$. Subtracting $m = \sum_{l} |D_{l}|$ and $q = \sum_{l} |F_{l}|$ from the first cardinality condition in Theorem 1.6 and subtracting the $k$th cardinality condition from the $(k + 1)$st condition yields the equality

$$|C_k| + |E_{2k}| + |E_{2k+1}| = |D_k| + |F_{2k}| + |F_{2k+1}| + u$$

for all $0 \leq k \leq \frac{n-3}{2} = s - 1$. A similar argument gives

$$|C_k| + |E_{2k-n}| + |E_{2k+1-n}| = |D_k| + |F_{2k-n}| + |F_{2k+1-n}| + u$$

for all $n - s = \frac{n+1}{2} \leq k \leq n - 2$, while subtracting $m = \sum_{l} |C_{l}|$ and $p = \sum_{l} |E_{l}|$ from the $(n-1)$st condition yields, using the value of $nu$, the same equality also for $k = n - 1$. Finally, the difference between the $s$th and $(s-1)$st conditions leads us to the equality

$$|C_{n-1}| + |E_0| + |E_{n-1}| = |D_{n-1}| + |F_0| + |F_{n-1}| + u,$$

corresponding to the remaining index $k = \frac{n+1}{2} = s = n - s - 1$. Indeed the left hand sides sum to $r + 2p$ and the sum of the right hand sides is $nu + 2q + m$, in agreement with the value of $u$.

In order to evaluate $h_{\Xi}$, we define $\varepsilon$ to be 1 if $n \equiv 1$ (mod 4) and 0 if $n \equiv 3$ (mod 4). Since the formulae for $f_{2}^{(n)}(l)$ and $f_{n-2}^{(n)}(l)$ depend on the parity of $l$, we separate the corresponding products into the product over $l = 2k$ with $0 \leq k \leq \frac{n-1}{2} = s$ and the product over $l = 2k+1$ for $0 \leq k \leq \frac{n-3}{2} = s - 1$. Taking the notational differences, the index shifts, Lemma 4.5, and the symmetry of $f_{2}^{(n)}$ under $l \mapsto n + 1 - l$ into consideration, we obtain

$$h_{\Xi} = \prod_{r=0}^{n-1} \prod_{l=0}^{n-1-r} \left( [C_r, C_{r+l}] [D_r, D_{r+l}] [E_r, E_{r+l}] [F_r, F_{r+l}] \right)^{2n[(n^2-1)/4-(n-l)]} \times$$

$$\times \prod_{l=0}^{n-1-r} \left( [C_r, D_{r+l}] [D_r, C_{r+l}] [E_r, F_{r+l}] [F_r, E_{r+l}] \right)^{2nl(n-l)} \times$$

$$\times \prod_{k=0}^{s} \left( [C_r, E_{2r+2k-\rho_n}] [D_r, F_{2r+2k-\rho_n}] \right)^{2n[(n^2+2n+1-4\varepsilon)/8-k(n+1-2k)]} \times$$

$$\times \prod_{k=0}^{s-1} \left( [C_r, E_{2r+2k+1-\rho_n}] [D_r, F_{2r+2k+1-\rho_n}] \right)^{2n[(n^2+2n+1-4\varepsilon)/8-k(n-1-2k)]} \times$$

$$\times \prod_{k=0}^{s} \left( [C_r, F_{2r+2k-\rho_n}] [D_r, E_{2r+2k-\rho_n}] \right)^{2nk(n+1-2k)} \times$$

$$\times \prod_{k=0}^{s-1} \left( [C_r, F_{2r+2k+1-\rho_n}] [D_r, E_{2r+2k+1-\rho_n}] \right)^{2nk(n-1-2k)} \right].$$

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Here \( \rho \) is the appropriate multiple of \( n \) which one needs for the resulting index to be in \( \mathbb{N}_n \).

Take \( r = p = q = m = 1 \) (hence \( u = 0 \)). Considering only divisors \( \Xi \) for which \( v_Q(\Xi) = n - 1 \) for some point \( Q \) (i.e., \( \Xi = Q^{n-1} \Delta \) for some base point \( Q \) and divisor \( \Delta \) of degree \( g = n - 1 \)), we obtain two types of divisors: First there are the simple solutions in which \( |C_l| = |D_l| \) and \( |E_l| = |F_l| \) for all \( l \in \mathbb{N}_n \), representing the divisors \( P_1^{n-1}R_1^{-1}S_1^j, Q_1^{n-1}R_1^{-1}S_1^j, R_1^{n-1}P_1^{-1}Q_1^j, \) or \( S_1^{n-1}P_1^{-1}Q_1^j \), with \( j \in \mathbb{N}_n \) (assuming the existence of order \( n - 1 \) points in \( \Xi \)). The second type of divisors are those divisors not of this form, and there are finitely many of them, given by the other solutions to the cardinality equations.

In case \( |C_0| = 1 \) these other solutions are with \( F_0, E_{n-1} \) and \( D_{n-1} \) being the non-empty sets (and \( \Xi = P_1^{n-1}Q_1^{n-1} \)) or with the points being in \( F_1, E_2 \) and \( D_1 \) (and \( \Xi = P_1^{n-1}Q_1^{n-1} \)). For \( |D_{n-1}| = 1 \) we obtain again the solution in which \( E_{n-1}, F_0, \) and \( C_0 \) are non-empty and \( \Xi = P_1^{n-1}Q_1^{n-1} \), and another solution is with the non-trivial sets \( E_{n-2}, F_{n-3}, \) and \( C_{n-2} \) and the divisor \( Q_1^{n-1}P_1R_1S_1^{n-3} \). If \( |E_0| = 1 \) then we obtain one possibility in which \( F_{n-1}, D_0, \) and \( C_{n-1} \) are the non-trivial sets (hence \( \Xi = R_1^{n-1}S_1^{n-1} \)) and another one with non-trivial \( F_1, D_{n-1}, \) and \( C_{n-1} \) and divisor \( R_1^{n-1}P_1^{-1}Q_1^{n-1}R_1 \). Finally, by taking \( |F_{n-1}| = 1 \) we get again the solution in which \( E_0, C_{n-1}, \) and \( D_0 \) contain points and \( \Xi = R_1^{n-1}S_1^{n-1} \), as well as the solution in which the points lie in \( E_{n-2}, C_{n-1}, \) and \( D_{n-2} \), representing the divisor \( S_1^{n-1}P_1^{-1}Q_1^{n-1}R_1 \). This gives us all the possibilities of \( \Xi = Q^{n-1} \Delta \) with some branch point \( Q \) and divisor \( \Delta \) from Theorem 6.3 of [FZ] not containing \( Q \). Observe that the remaining possibility for \( \Delta \) in that theorem, namely \( P_1^{n-1}Q_1^{n-1}R_1S_1 \), does not represent any theta constant since \( \theta[Q, \Delta](0, \Pi) = 0 \) with this \( \Delta \) for every possible \( Q \).

We now consider the form which \( h_\Xi \) takes in this case. If \( \Xi = P_1^{n-1}R_1^{n-1-l}S_1^l \) or \( \Xi = Q_1^{n-1}R_1^{n-1-l}S_1^l \) then \( h_\Xi \) becomes

\[
[(\lambda_1 - \sigma_1)(\mu_1 - \tau_1)]^{2n[(n^2 + 2n + 1 - 4l)/(8 - (n + 1 - l)/2)]}[(\lambda_1 - \tau_1)(\mu_1 - \sigma_1)]^{2n[(n^2 + 1 - 4l)/(8 - (n + 1 - 2l)/2)]}
\]

for even \( l \) and

\[
[(\lambda_1 - \sigma_1)(\mu_1 - \tau_1)]^{2n[(n^2 + 2n + 1 - 4l)/(8 - (l - 1)(n - l)/2)]}[(\lambda_1 - \tau_1)(\mu_1 - \sigma_1)]^{2n[(l - 1)(n - l)/2]}
\]

if \( l \) is odd (the contribution of the first two lines is an empty product). In case \( \Xi \) takes the form \( R_1^{n-1}P_1^{n-1-l}Q_1^l \) or \( S_1^{n-1}P_1^{n-1-l}Q_1^l \) the denominator \( h_\Xi \) takes the form

\[
[(\lambda_1 - \sigma_1)(\mu_1 - \tau_1)]^{2n[(n^2 + 2n + 1 - 4l)/(8 - (n - l)(2l - 1 - n)/2)]}[(\lambda_1 - \tau_1)(\mu_1 - \sigma_1)]^{2n[(n - l)(2l + 1 - n)/2]}
\]

if \( l \) satisfies \( 0 \leq l \leq n - 1/2 \) and

\[
[(\lambda_1 - \sigma_1)(\mu_1 - \tau_1)]^{2n[(n^2 + 2n + 1 - 4l)/(8 - (n - l)(2l + 1 - n)/2)]}[(\lambda_1 - \tau_1)(\mu_1 - \sigma_1)]^{2n[(n - l)(2l + 1 - n)/2]}
\]

for \( n + 1/2 \leq l \leq n - 1 \) (again, no contribution comes from the first two lines). For each of the remaining divisors, namely the 6 divisors \( P_1^{n-1}Q_1^{n-1}, R_1^{n-1}S_1^{n-1}, \ldots \),
\[ P^{n-1}O_1^1R_1^{n-3}S_1, Q_1^{n-1}F_1R_1S_1^{n-3}, R_1^{n-1}P_1^{n-3}O_1^{n-1}R_1 \]

called “of second type” above, the associated denominator \( h_2 \) equals

\[ [(\lambda_1 - \sigma_1)(\mu_1 - \tau_1)]^{2n(n^2 - 6n + 9 - 4\varepsilon)/8}[(\lambda_1 - \mu_1)(\sigma_1 - \tau_1)]^{2n(n-1)}. \]

Stating these results with the notation and index conventions of [FZ], these formulae reproduce the results of Section 6.1 and Appendix B of that reference.

Take now \( n \) which is not divisible by 3, and write \( n = 3s + t \). As a second example we consider the family of \( Z_n \) curves whose defining equation is

\[ w^n = \prod_{i=1}^{r}(z - \lambda_i) \prod_{i=1}^{p}(z - \sigma_i)^3 \prod_{i=1}^{q}(z - \tau_i)^{n-3} \prod_{i=1}^{m}(z - \mu_i)^{n-1}, \]

where \( n \) divides \( r + 3p - 3q - m \) with quotient \( u \). The points \( \lambda_i, \sigma_i, \tau_i, \) and \( \mu_i \) have the same meaning as above, and the same statement holds for the sets \( C_i, D_i, E_i, \) and \( F_i \). Thus the divisor \( \Xi \) from Equation (1) again becomes \( \prod_{i=0}^{n-1} C_i^{n-1-i} D_i^{n-1-i} F_i \), but now \( E_i \) stands for \( D_{3i} \) and \( F_i \) denotes \( D_{n-3s-1-i} \). Repeating the operations of the previous example on the cardinality conditions from Theorem [1,0] now yields the equalities

\[ |C_k| + |E_{3k}| + |E_{3k+1}| + |E_{3k+2}| = |D_k| + |F_{3k}| + |F_{3k+1}| + |F_{3k+2}| + u \]

for every \( 0 \leq k \leq s - 1 \),

\[ |C_{s}| + |E_{3s-2a}| + |E_{3s-1-2a}| + |E_{3s+2-2a}| = |D_{s}| + |F_{3s-2a}| + |F_{3s+1-2a}| + |F_{3s+2-2a}| + u \]

if \( s + 1 \leq k \leq n - s - 2 \), and

\[ |C_{s}| + |E_{3s-2a}| + |E_{3s+1-2a}| + |E_{3s+2-2a}| = |D_{s}| + |F_{3s-2a}| + |F_{3s+1-2a}| + |F_{3s+2-2a}| + u \]

in case \( n - s \leq k \leq n - 1 \). For the values \( k = s \) and \( k = n - s - 1 \) the resulting equalities depend on whether \( t = 1 \) or \( t = 2 \). If \( t = 1 \) then the equalities become

\[ |C_{s}| + |E_{0}| + |E_{1}| + |E_{n-1}| = |D_{s}| + |F_{0}| + |F_{1}| + |F_{n-1}| + u \]

and

\[ |C_{n-s-1}| + |E_{0}| + |E_{n-2}| + |E_{n-1}| = |D_{n-s-1}| + |F_{0}| + |F_{n-2}| + |F_{n-1}| + u, \]

while for \( t = 2 \) we get

\[ |C_{s}| + |E_{0}| + |E_{n-2}| + |E_{n-1}| = |D_{s}| + |F_{0}| + |F_{n-2}| + |F_{n-1}| + u \]

and

\[ |C_{n-s-1}| + |E_{0}| + |E_{n-1}| = |D_{n-s-1}| + |F_{0}| + |F_{n-1}| + u. \]

In any case, summing all the equalities yields \( r + 3p = m + 3q + nu \), as indeed follows from the current definition of \( u \).
In this case $n$ can be either odd or even, and the maximal value of $f_1^{(n)}$ can be written uniformly as $\frac{n^2 + 1 - \varepsilon}{4}$ for both cases (recall that $\varepsilon$ is 1 for even $n$ and 2 for odd $n$). The maximal value of $f_3^{(n)}$ can be written uniformly in terms of $t$ and $e$, but as the expression for $h_\geq$ depend on $t$ in other terms as well, we shall avoid doing so. In the expressions involving $f_3^{(n)}$ or $f_{n-3}^{(n)}$ we separate the product into $l = 3k$ for $0 \leq k \leq s$, $t = 3k + 1$ for $k$ between 0 and $s + t - 1$ (which includes $k = s$ if $t = 2$ since then $3s + 1 = n - 1 \in \mathbb{N}_n$, but excludes this value of $k$ if $t = 1$ as $3s + 1 = n$ no longer lies in $\mathbb{N}_n$ in this case), and $l = 3k + 2$ for $0 \leq k \leq s - 1$. The maximal value of $f_3^{(n)}$ is $\frac{n^2 + 4n + 4 - 9(\varepsilon - 1)}{12}$ if $t = 1$ and $\frac{n^2 + 4n + 3(\varepsilon - 1)}{12}$ if $t = 2$. Using Lemma 4.5 we can write $\frac{1}{n - 3} - f_3^{(n)}(l)$ as $f_3^{(n)}(n - 1 - l) + s$ if $t = 1$ and as $f_3^{(n)}(n - 1 - l)$ when $t = 2$. The value of $f_3^{(n)}(3k + 1)$ can be written as $k(n - 3k) - \frac{n - 1}{2}$ (i.e., $k(n - 3k) - s$) if $t = 1$ and as $k(n - 3k) + \frac{n - 1}{2}$ (namely $k(n - 3k) + s + 1$) when $t = 2$. The constant $\frac{n - 1}{2}$ is absorbed into the preceding coefficients in the powers appearing in the corresponding line. In total, $h_\geq$ is given by

$$
\prod_{r=0}^{n-1} \prod_{l=0}^{n-1-r} \left( \left[ C_r, C_{r+l} \right][D_r, D_{r+l}]\left[ E_r, E_{r+l} \right][F_r, F_{r+l}] \right)^{cn[(n^2+1-\varepsilon)/4] - l(n-l)} \times
$$

$$
\times \prod_{l=0}^{n-1-r} \left( \left[ C_r, D_{r+l} \right][D_r, C_{r+l}]\left[ E_r, F_{r+l} \right][F_r, E_{r+l}] \right)^{cn(l-l)} \times
$$

$$
\times \prod_{k=0}^{l} \left( \left[ C_r, E_{3r+3k}\cdot\rho_n \right][D_r, F_{3r+3k}\cdot\rho_n] \right)^{cn[(n^2+4n+13-9\varepsilon)/12] - k(n+2-3k)} \times
$$

$$
\times \prod_{k=0}^{l} \left( \left[ C_r, E_{3r+3k+1}\cdot\rho_n \right][D_r, F_{3r+3k+1}\cdot\rho_n] \right)^{cn[(n^2+8n+9-9\varepsilon)/12] - k(n-3k)} \times
$$

$$
\times \prod_{k=0}^{l} \left( \left[ C_r, E_{3r+3k+2}\cdot\rho_n \right][D_r, F_{3r+3k+2}\cdot\rho_n] \right)^{cn[(n^2+4n+13-9\varepsilon)/12] - k(n-2-3k)} \times
$$

$$
\times \prod_{k=0}^{l} \left( \left[ C_r, F_{3r+3k}\cdot\rho_n \right][D_r, E_{3r+3k}\cdot\rho_n] \right)^{cn[k(n+2-3k)+(n-1)/3]} \times
$$

$$
\times \prod_{k=0}^{l} \left( \left[ C_r, F_{3r+3k+1}\cdot\rho_n \right][D_r, E_{3r+3k+1}\cdot\rho_n] \right)^{cn[k(n-3k)]} \times
$$

$$
\times \prod_{k=0}^{l} \left( \left[ C_r, F_{3r+3k+2}\cdot\rho_n \right][D_r, E_{3r+3k+2}\cdot\rho_n] \right)^{cn[k(n-2-3k)+(n-1)/3]} \right]
$$

for $t = 1$ and

$$
\prod_{r=0}^{n-1} \prod_{l=0}^{n-1-r} \left( \left[ C_r, C_{r+l} \right][D_r, D_{r+l}]\left[ E_r, E_{r+l} \right][F_r, F_{r+l}] \right)^{cn[(n^2+1-\varepsilon)/4] - l(n-l)} \times
$$

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\[
\times \prod_{l=0}^{n-1-r} \left( [C_r, D_{r+l}][D_r, C_{r+l}][E_r, F_{r+l}][F_r, E_{r+l}] \right)^{e_{n1}(l-t)} \times \\
\times \prod_{k=0}^{s} \left( [C_r, E_{3r+3k-\rho n}][D_r, F_{3r+3k-\rho n}] \right)^{e_{n1}((n^2+4n+3e-3)/12-k(n+2-3k))} \times \\
\times \prod_{k=0}^{s-1} \left( [C_r, E_{3r+3k+1-\rho n}][D_r, F_{3r+3k+1-\rho n}] \right)^{e_{n1}((n^2+3e-7)/12-k(n-3k))} \times \\
\times \prod_{k=0}^{s-1} \left( [C_r, E_{3r+3k+2-\rho n}][D_r, F_{3r+3k+2-\rho n}] \right)^{e_{n1}((n^2+4n+3e-3)/12-k(n-2-3k))} \times \\
\times \prod_{k=0}^{s} \left( [C_r, F_{3r+3k-\rho n}][D_r, E_{3r+3k-\rho n}] \right)^{e_{nk}(n+2-3k)} \times \\
\times \prod_{k=0}^{s-1} \left( [C_r, F_{3r+3k+1-\rho n}][D_r, E_{3r+3k+1-\rho n}] \right)^{e_{nk}(n-3k)+(n+1)/3} \times \\
\times \prod_{k=0}^{s-1} \left( [C_r, F_{3r+3k+2-\rho n}][D_r, E_{3r+3k+2-\rho n}] \right)^{e_{nk}(n-2-3k)} \\
\] 

if \( t = 2 \), where subtracting \( \rho n \) takes the index to \( \mathbb{N}_n \).

The numerical example which we now consider is with \( r = 3 \), \( p = m = 0 \), and \( q = 1 \) (and again \( u = 0 \)). The divisors \( \Xi \) which are of the form \( S_1^{n-1} \Delta \), namely with \( |F_{n-1}| = 1 \), must satisfy also \( |C_{n-1}| = |C_s| = |C_{n-1-s}| = 1 \). This yields the 6 divisors \( S_1^{n-1} P_i^{n-1-s} P_j^{s} \), with \( i, j, \) and \( k \) being some choice for the indices 1, 2, and 3. On the other hand, if \( \text{ord}_{P_1}(\Xi) = n - 1 \) then \( |C_0| \) must be 1, so that the point \( S_1 \) must lie in either \( F_0, F_1, \) or \( F_2 \). If it lies in \( F_0 \) then the other non-empty sets must be \( C_s \) and \( C_{n-1-s} \), yielding the divisors \( P_i^{n-1} P_j^{n-1-s} P_j^{s} \). Having \( S_1 \) in \( F_2 \) assigns the other points to \( C_{s+1} \) and \( C_{n-s} \) and produces the divisors \( P_i^{n-1} P_i^{n-2-s} P_j^{2s} S_1^2 \). On the other hand, in the case where \( F_1 \) contains \( S_1 \) we obtain non-trivial sets which are \( C_s \) and \( C_{n-s} \) for \( t = 1 \) and \( C_{s+1} \) and \( C_{n-s-1} \) if \( t = 2 \). The divisors can thus be written uniformly as \( P_i^{n-1} P_i^{n-1-s} P_j^{s} \), replacing \( n \) by \( 3s + t \) shows that these results reproduce the divisors from Theorem 6.13 of [EZ], where each divisor from that theorem comes multiplied by the \( (n-1) \)st power of any branch point not appearing in it. As for the denominators \( h_\Xi \), we have contributions only from expressions of the form \([C_r, C_{r+l}]\) or \([C_r, F_{3r+l-\rho n}]\). We substitute \( n = 3s + t \), and write our expressions in terms of \( s \). We distinguish the cases \( t = 1 \) and \( t = 2 \), hence use the value of \( t \) explicitly in each formula. For \( t = 1 \) we find that if \( \Xi \) is \( S_1^{n-1} P_i^{n-1-s} P_j^{s}, P_i^{n-1} P_j^{n-1-s} P_j^{s}, P_j^{n-1} P_k^{2s} P_i^{n-2-s} S_1, \) or \( P_k^{n-1} P_i^{n-2-s} P_j^{s-1} S_2 \), then \( h_\Xi \) takes the form

\[
[(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)]^{e_{n1}(s^2+2s+2-e)/4}(\lambda_i - \lambda_k)^{e_{ns}(s^2-2s+2-e)/4}[(\tau_1 - \lambda_i)(\tau_1 - \lambda_k)]^{e_{ns}}.
\]
For $t = 2$ and $\Xi$ being $S_1^{n-1}P_1^{n-1-s}P_2^{\sigma}$, $P_2^{-1}P_2^{n-1-s}P_3^{\tau}$, $P_1^{n-1}P_2^{n-1-s}P_3^{\sigma}$, or $P_1^{n-1}P_2^{n-2-s}P_3^{s-1}S_1^{2}$, the expression for $h_\Xi$ becomes

$$(\lambda_j - \lambda_k)^{en(s^2+4s+5-c)/4}[\lambda_i - \lambda_j](\lambda_i - \lambda_k)]^{sn(s^2+1-c)/4}(\tau_1 - \lambda_i)^{en(s+1)}.$$ 

These expressions are not reduced, since the combination of $[C_t, C_{t+1}]$ having power 0 do not appear. The common factor is $[(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)]^{en}$ taken to a power which is $n^2-2s^2+2s-c$ for $t = 1$ and is $n^2+1-c$ if $t = 2$. After eliminating this common factor we obtain again the results of Section 6.2 of [FZ]. This shows that our general formula can indeed reproduce all the special cases considered in [FZ].

The first numerical example presented here involves a non-special divisor of degree $g$ which is supported on all the branch points, namely $P_1^{n-s}R_1S_1Q_1^{n-2}$ (see Theorem 6.3 of [FZ]). In this spirit we now give examples of fully ramified $Z_n$ curves admitting no $M$-orbits, though they do carry non-special divisors of degree $g$ which are supported on the branch points. This means that all the non-special degree $g$ divisors supported on the branch point on these curves contain all the branch points in their support. These examples relate to the families of curves considered in Section 3 of [GDT], and have only 3 branch points. Let $n$ be an odd number which is not divisible by 3, and consider the $Z_n$ curve defined by the equation

$$w^n = (z - \lambda)(z - \sigma)^2(z - \tau)^{n-3}.$$ 

As above, we denote the branch points lying over $\lambda$, $\sigma$, and $\tau$ by $P$, $R$, and $S$ respectively. In this case we can find the non-special divisors directly, rather than using Theorem [L6]. Indeed, we have only 3 points, the degree $g$ is $\frac{n-1}{2}$, and the rational equivalence of $PR^2$ and $S^3$ implies that in no such divisor can be a multiple of either of these degree 3 divisors (as in the proof of Lemma [L6]). For $n \geq 11$ this leaves us with only 9 divisors to consider: $P^{\frac{n-1}{2}}$, $P^{\frac{n-3}{2}}S^2$, $P^{\frac{n-5}{2}}R$, $P^{\frac{n-7}{2}}R^2$, $P^{\frac{n-7}{2}}S$, $P^{\frac{n-9}{2}}R^2$, $P^{\frac{n-9}{2}}R^2$, $R^{\frac{n-3}{2}}$, and $R^{\frac{n-3}{2}}S^2$. For $n = 7$ we have only 8 divisors, since the 6th and 8th divisors on this list coincide. Moreover, for $n = 5$ these are only 6 divisors, as the 6th divisor is not integral, the 3rd and 9th divisors coincide, and the 5th and 8th divisors coincide. The basis for the holomorphic differentials consists of those $\omega_k$ with $k$ satisfying either $\frac{n}{4} < k < \frac{3n}{4}$ or $\frac{3n}{4} < k < n$. Substituting $n = 3s + t$, these bounds become either $s + 1 \leq k \leq \frac{n-1}{2}$ or $n - s \leq k \leq n - 1$. Now, the holomorphic differentials $\omega_{s-3}$ and $\omega_{s+1}$ have the (canonical) divisors $P^{s-1}R^{2s-1}S^{t-1}$ and $P^{n-2-s}R^{s+1-t}S^2-t$ respectively. Moreover, for $s \geq 2$ (i.e., $n \geq 7$) the (integral) divisor $P^{s-2}R^{2s-3}S^{t+2}$ of $\omega_{s+1-s}$ is also canonical, and in case $s + t \geq 5$ (i.e., $n \geq 11$) the same assertion applies for the divisor $P^{n-3-s}R^{s+1-t}S^2-t$ of $\omega_{s+2}$. The inequality $s + t \geq 3$ implies $n - 2 - s \geq \frac{n-1}{4}$, and since $s \geq t + 1$ for odd $n \geq 7$ it follows that $2s - 1 \geq n-1$ as well in this case. Hence for $n \geq 7$ the divisors $P^{\frac{n-1}{2}}$, $R^{\frac{n-1}{2}}$ and $R^{\frac{n-3}{2}}S^2$ are special. Moreover, if $n \geq 11$ then $s + t \geq 5$, and we have either $s \geq t + 3$ (for $n \geq 13$) or $t = 2$ (for $n = 11$).
This implies that $P_{3,5}^2 S$, $P_{3,5}^2 S^2$, $P_{3,4}^2 R$, and $R_{3,5}^2 S$ are also special. The remaining divisors $P_{3,5}^2 RS$ and $P_{3,5}^2 RS^2$ are special for all $n \geq 17$ (since $s + t \geq 7$). Hence these curves with $n \geq 17$ admit no non-special divisors of degree $g$. In addition, for $n = 13$ we have $t = 1$, so that the divisor $P_{3,5}^2 RS$ (namely $P^4 RS$) is special also for this value of $n$. On the other hand, for $n = 13$ the divisor $P_{3,5}^2 RS^3 = P^3 RS^2$ is non-special: The divisors $P^3 R^7$, $P_3 R_5 S^3$, $P_7 R_2 S$, and $P^6 S^4$ of $\omega_9$, $\omega_{10}$, $\omega_5$, and $\omega_6$ already considered are not multiples of this divisor, and the same applies for the divisors $P R^3 S^6$ of $\omega_{11}$ and $RS^9$ of $\omega_{12}$. Similarly, in the case $n = 11$ the two divisors $P^3 RS$ and $P^2 RS^2$ are special: The differentials $\omega_8$, $\omega_9$, $\omega_4$, and $\omega_5$ have the divisors $P^2 R^5 S$, $PR^3 S^4$, $P^6 R^2$, and $P^5 S^3$ considered above, the remaining differential $\omega_{10}$ has divisor $RS^7$, and none of these canonical divisors is a multiple of any of the divisors in question. As in these examples the divisors contain all the branch points in their support, there are no $M$-orbits and non-vanishing characteristics, even though there exist non-special divisors of degree $g$. This example completes, in some sense, the families presented in [GDT], and shows that for $r = 1$ the bound $p > 12$ is not sufficient (see also another example discussed briefly in Section 8). Interestingly, for $n = 7$ there exists precisely one $M$-orbit (or characteristic) arising from the non-special divisors $PR^3$, $P^3 S$, and $RS^3$, and the divisor $PR$ is also non-special. Hence the Thomae formulae are trivial also in this case. For $n = 5$ we have the 4 non-special $PS$, $PR$, $R^2$, and $S^2$, yielding 2 $M$-orbits which are related by $N$.

8 Open Problems

We close this paper by a brief discussion of a few questions which arise from our considerations.

First, proving Conjecture 5.3 As mentioned above, [K] establishes Thomae formulae for our general setting, which points to the direction of the validity of Conjecture 5.3. Therefore, even if Conjecture 5.3 does not hold, the Thomae quotient should give the same value on every orbit of this action. Therefore, in case some counter-example to Conjecture 5.3 arises, one might look for additional operations which extend the action of $G$ and the operators $T_{Q,R}$ and leaving the Thomae quotient invariant.

Second, in the case $n = 2$ our divisors capture all the order $n = 2$ points in $J(X)$. As already remarked in the introduction to [FZ], this statement does not extend to any case with $n > 2$. Indeed, the set of $n$-torsion points in $J(X)$ is a free module of rank $2g = (n - 1)(\sum_\alpha r_\alpha - 2)$ over $\mathbb{Z}/n\mathbb{Z}$, and the points $P_{i,\alpha}$ generate (with the fixed base point $Q$), a subgroup of rank at most $\sum_\alpha r_\alpha - 2$ (one relation comes from the vanishing of the base point or from the degree 0 condition, the other one arising from the vanishing of $\text{div}(w)$). In fact, the observation that the only relations between the divisors $\Xi$ in Section 5 are given by powers of the operator $M$ (where a relation means that two divisors represent the same characteristic) suggests that these are the only relations holding between the points $\varphi_Q(P_{\alpha,i})$ in $J(X)$. Hence the rank is precisely
Therefore it is interesting to ask what kind of divisors represent the other $n$-torsion points in $J(X)$, and whether they are special or not. For the latter characteristics, we ask whether denominators like our $h_x$ (or $h_x^2$), which extends the Thomae formulae to these characteristics as well, might exist.

Next, one may wish to consider the dependence of the Thomae formulae on the actual choice of the function $z$ (rather than just a projection from $X$ to the quotient under the Galois action). Since the denominators $h_x$ are based only on differences between $z$-values, replacing $z$ by $z - \zeta$ for some $\zeta \in \mathbb{C}$ leaves the Thomae constant invariant. On the other hand, multiplying $z$ by some number $u \in \mathbb{C}^*$ multiplies $h_x$ by $u$ raised to the power $\deg h_x$, which changes the constant. This shows, by the way, that all the denominators $h_x$ must have the same degree. One might search for Thomae constants which are independent of the choice of $z$. For example, multiplying all the constants by some global product of differences, which would render the denominators $h_x$ rational functions, should yield an expression which is independent of dilations of $z$ as well. If this is indeed possible, one needs only to consider replacing $z$ by $\frac{1}{z}$ and allowing branching at $\infty$. In addition, assuming that such an invariant constant exists, we observe that some Riemann surfaces may be given several structures of $\mathbb{Z}$-curves. For example, the $Z_n$ curves from Section 3.3 of [FZ] are all hyper-elliptic, hence carry an additional structure of a $\mathbb{Z}_2$ curve. In this case it is natural to ask whether connections between the Thomae constants arising from the different $Z_n$ structures on $X$ can be found.

It may also be of interest to study the dependence on $n$ of the number of non-special divisors of degree $g$ or of $M$-orbits in families of $Z_n$ curves with related equations. More precisely, consider the $Z_n$ curve associated with an equation of the sort

$$w^n = \prod_{i=1}^{p} (z - \lambda_i)^{c_i} \prod_{i=1}^{q} (z - \mu_i)^{n-d_i},$$

where $\sum_{i=1}^{p} c_i = \sum_{i=1}^{q} d_i$ (hence the sum of powers is divisible by $n$), $q \leq p$ (otherwise replace $w$ by $w_{n-1}$), and $n$ is large enough (i.e., $n \to \infty$). For sufficiently large $n$ this yields $t_k = q$ for small $k$ and $t_k = p$ for large $k$ (here $k$ is considered to be in $\mathbb{N}$, and not in $\mathbb{Z}/n\mathbb{Z}$). Note that the non-singular $Z_n$ curves do not describe such a family, since the corresponding number of points depends on $n$. On the other hand, the singular curves from Chapter 5 of [FZ] do lie in such families for every choice of the parameter $m$. We have seen in the second numerical example in Section 7 (or in Section 6.2 of [FZ]) that these numbers are constant (18 divisors for $n \geq 7$ and 6 $M$-orbits for $n \geq 4$). In the last example in Section 7 both numbers vanish for $n \geq 17$. Considerations similar to that example show that $w^n = (z - \lambda)(z - \sigma)^3(z - \tau)^{n-4}$ displays a similar behavior. Furthermore, the choice of $r = 2$, $p = m = 0$, and $q = 1$ in the first example in Section 7 yields the curves of the form $w^n = (z - \lambda_1)(z - \lambda_2)(z - \tau)^{n-2}$. These curves have 4 non-special divisors (for $n \geq 5$) and 2 $M$-orbits which are related via $N$. Note that in all these cases we have $q = 1$, namely $t_k = 1$ for small $k$. On the other hand, the singular curves from Section 3.3 of [FZ] (with $p = q = 2$ and allowing branching at $\infty$)
and the indices \( d_i \) and \( e_i \) being 1) admit \( 2n - 1 \) divisors not containing \( P_0 \) in their support (hence \( 2n - 1 \) orbits of \( M \)). One easily sees that the total number of divisors in this case is \( 4n - 4 \). In the first numerical example of Section 7 (considered in Section 6.1 of [FZ]) we have, for \( n \geq 5 \), \( n + 2 \) characteristics and \( 2n + 5 \) divisors (see Theorem 6.3 of [FZ]). Similar considerations show that by taking \( r = p = q = m = 1 \) in the second family described in Section 7, the \( \mathbb{Z}_n \) curves of the form

\[
w^n = (z - \lambda_1)(z - \lambda_2)(z - \mu_1)^{n-1}(z - \mu_2)^{n-1}(z - \mu_3)^{n-1}
\]

(the \( \mathbb{Z}_n \) curves appearing in Chapter 5 of [FZ] with \( m = 3 \)) the total number of such divisors of degree \( g \) is \( 18n^2 - 45n + 33 \). Of these divisors, \( 3n^2 + 6n - 14 \) do not contain a pre-fixed branch point and correspond to orbits of \( M \) (the latter number indeed becomes \( 10 = \frac{5!}{3!2!} \) for the non-singular \( \mathbb{Z}_n \) curve arising from \( n = 2 \), and is equal to the number 31 from Section 3.2 of [FZ] for \( n = 3 \)). These observations lead us to formulate the following

**Conjecture 8.1.** The number of non-special divisors of degree \( g \) on the \( \mathbb{Z}_n \) curve associated to Equation (14) is described, for large enough \( n \), by a polynomial of degree \( q - 1 \) in \( n \). The same assertion holds for the number of \( M \)-orbits on such a \( \mathbb{Z}_n \) curve.

In fact, taking a closer look into the results of these examples leads to a finer form of Conjecture 8.1:

**Conjecture 8.2.** Let \( X \) be a \( \mathbb{Z}_n \) curve described by Equation (14), and let \( c \) and \( d \) be positive integers. Define \( x_c = |\{i | c_i = c\}| \) and \( y_d = |\{i | d_i = d\}| \). Then \( p = \sum x_c \) and \( q = \sum y_d \) form partitions of \( p \) and \( q \) respectively. The leading coefficients of the two polynomials appearing in Conjecture 8.1 depend only on these partitions of \( p \) and \( q \).

Both Conjectures 8.1 and 8.2 can be formulated in terms of assertions about the numbers of solutions of combinatorial equations (Theorem 1.6 again), where each solution is assigned a multiplicity according to the number of divisors it represents (another combinatorial expression).

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Fachbereich Mathematik, AG 5, Technische Universität Darmstadt, Schloßgartenstrasse 7, D-64289, Darmstadt, Germany
E-mail address: zemel@mathematik.tu-darmstadt.de