THE $C^*$-ALGEBRA OF THE CARTAN MOTION GROUPS

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Abstract. Let $G_0 = K \rtimes \mathfrak{p}$ be the Cartan motion groups. Under some assumption on $G_0$, we describe the $C^*$-algebra $C^*(G_0)$ of $G_0$ in terms of operator fields.

1. Introduction

Let $G$ be a locally compact group. We denote by $\hat{G}$ the unitary dual of $G$. It well-known that $\hat{G}$ equipped with the Fell topology (see [18, 19]). The first representation-theoretic question concerning the group $G$ is the full parametrezation and topological identification of the dual $\hat{G}$. The $C^*$-algebra $C^*(G)$ is the completion of the convolution algebra $L^1(G)$ equipped with the $C^*$-norm $\| \cdot \|_{C^*(G)}$, given by

$$\| f \|_{C^*(G)} := \sup_{\pi \in \hat{G}} \| \pi(f) \|_{\text{op}}.$$  

We denote by $\hat{C^*}(G)$ the unitary dual of the $C^*$-algebra of $G$. Then we have the following bijection

$$\hat{C^*}(G) \simeq \hat{G}.$$  

Furthermore, the $C^*$-algebra $C^*(G)$ of $G$ can be identified with a subalgebra of the large $C^*$-algebra $\ell^\infty(\hat{G})$ of bounded operator fields given by

$$\ell^\infty(\hat{G}) := \left\{ F : \hat{G} \to \bigcup_{\pi \in \hat{G}} B(H_\pi), \pi \mapsto F(\pi) \in B(H_\pi); \| F \|_\infty := \sup_{\pi \in \hat{G}} \| F(\pi) \|_{\text{op}} < \infty \right\}$$

under the Fourier transform $\mathcal{F}$ defined on $C^*(G)$ as follows:

$$\mathcal{F}(f)(\pi) = \pi(f), \; \pi \in \hat{G}, \; f \in C^*(G).$$

Using the fact that $\mathcal{F}$ is an injective homomorphism of $C^*(G)$ into $\ell^\infty(\hat{G})$, then the $C^*$-algebra $C^*(G)$ is isomorphic to a subalgebra $\mathcal{D} := \mathcal{F}(C^*(G))$ of elements in $\ell^\infty(\hat{G})$ verifying some conditions. The elements of $\mathcal{D}$ must naturally fulfil is that of continuity. Then the parametrization and the description of the topology of $\hat{G}$ are required to describe the $C^*$-algebra $C^*(G)$ of $G$.

In this context, we have some works in the literature, for example, J. Ludwig and L. Turowska have described in [16] the $C^*$-algebra of the Heisenberg group and of the thread-like Lie groups in terms of an algebra of operator fields defined over their dual spaces. The description of the $C^*$-algebra of the Euclidean motion group $M_n := SO(n) \rtimes \mathbb{R}^n$, $n \in \mathbb{N}^*$ was established
in [1]. In the present work, we give a similar precise description of the $C^*$-algebra of the Cartan motion groups. Our result is a generalization of analogous results in the case of the Euclidean motion group (see [1]).

The paper is organized as follows. In section 2, we introduce the groups $G_0 = K \ltimes p$, the semi-direct product of the maximal compact connected subgroup $K$ of same connected semisimple Lie group with finite center $G$, acting by adjoint action on $p$ (where $p$ determined by the Cartan decomposition of Lie algebra of the group $G$). We recall the topology of the spectrum of the groups $G_0$ and we determine the convergence in $\widehat{G}_0$. In the last section, we determine the Fourier transform of their group $C^*$-algebras and we describe the elements of the image of the Fourier transform of $C^*(G_0)$ inside the big algebra $\ell^\infty(\widehat{G}_0)$. This is the main result of the paper.

2. The Cartan motion groups $G_0$.

Let $G$ be a connected semisimple Lie group with finite center and $K$ a maximal compact connected subgroup of $G$. Let $g = \mathfrak{t} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$ with $\mathfrak{t} := \text{Lie}(K)$. Then one can form the semidirect product $G_0 := K \ltimes \mathfrak{p}$ with respect to the adjoint action of $K$ on $\mathfrak{p}$. The group $G_0$ is called the Cartan motion group associated to the Riemannian symmetric pair $(G, K)$. The multiplication in this group is given by

$$(k_1, X_1) \cdot (k_2, X_2) = (k_1 k_2, X_1 + \text{Ad}(k_1) X_2), \quad (k_1, k_2) \in K, \quad (X_1, X_2) \in \mathfrak{p}.$$

The group $M_n = SO(n) \ltimes \mathbb{R}^n$ is an example of Cartan motion groups. More precisely, $M_n$ is the Cartan motion group associated to the compact Riemannian symmetric pair $(SO(n + 1), SO(n))$.

Let now $a$ be a maximal abelian subspace of $\mathfrak{p}$. The dimension of the real vector space $a$ is called the rank of the Riemannian symmetric pair $(G, K)$. An important fact worth mentioning here is that every adjoint orbit of $K$ in $\mathfrak{p}$ intersects $a$ (see [9] p. 247). Let $N_K(a)$ and $Z_K(a)$ denote respectively the normalizer and centralizer of $a$ in $K$. The quotient group

$$W := N_K(a)/Z_K(a)$$

is called the Weyl group of the pair $(G, K)$. We shall denote the action of $W$ on $a$ by $H \mapsto s.H$ for $H \in a$ and $s \in W$. Let us take the subspaces $\tilde{a} := i a$, $\tilde{\mathfrak{p}} := i \mathfrak{p}$ and $\tilde{\mathfrak{g}} := \mathfrak{t} \oplus \tilde{\mathfrak{p}}$ of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. An element $H \in a$ is called regular if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$ where $\Sigma$ is the set of all restricted roots associated to the pair $(\mathfrak{g}, a)$. A connected component of the set of regular elements in $a$ is called a Weyl chamber of the pair $(G, K)$. Endow the dual space $\tilde{a}^*$ with a lexicographic ordering and denote by $\Sigma^+$ the set of positive restricted roots. As an example of Weyl chambers, let us set $C^+(a) := i \tilde{C}^+(\tilde{a})$ with

$$\tilde{C}^+(\tilde{a}) = \{ \tilde{H} \in \tilde{a}; \; \alpha(\tilde{H}) > 0, \; \forall \alpha \in \Sigma^+ \}.$$
It is well-known that every $s \in W$ permutes the Weyl chambers and that $W$ acts simply transitively on the set of Weyl chambers. Furthermore, we have the following important result (see [9], p. 322):

**Proposition 2.1.** Let $C \subset a$ be a Weyl chamber. Each orbit of $W$ in $a$ intersects the closure $\overline{C}$ in exactly one point.

We shall briefly review the description of the unitary dual of $G_0$ via Mackey’s little group theory. Let $\varphi$ be a non-zero linear form on $p$. We denote by $\chi_\varphi$ the unitary character of the vector Lie group $p$ given by $\chi_\varphi = e^{i\varphi}$. Let $K_\varphi$ be the stabilizer of $\varphi$ under the coadjoint action of $K$ on $p^*$, and let $\rho$ be an irreducible unitary representation of $K_\varphi$ on some Hilbert space $H_\rho$. The map

$$\rho \otimes \chi_\varphi : (k, X) \mapsto e^{i\varphi(X)} \rho(k)$$

is a representation of the semidirect product $K_\varphi \ltimes p$, which we may induce up so as to obtain a unitary representation of $G_0$. Let $L^2_\rho(K, H_\rho)$ be the subspace of $L^2(K, H_\rho)$ consisting of the maps $f$ which satisfy the covariance condition

$$f(kk_0) = \rho(k_0^{-1}) f(k)$$

for $k_0 \in K_\varphi$ and $k \in K$. The induced representation

$$\pi_{(\rho, \varphi)} := \text{Ind}_{K_\varphi \ltimes p}^{G_0} (\rho \otimes \chi_\varphi)$$

is realized on $H_{\rho, \varphi} := L^2_\rho(K, H_\rho)$ by

$$\pi_{(\rho, \varphi)}(k_0, X) f(k) = e^{i\varphi(\text{Ad}(k^{-1})X)} f(k_0^{-1}k),$$

where $(k_0, X) \in G_0$, $f \in L^2_\rho(K, H_\rho)$ and $k \in K$. Mackey’s theory tells us that the representation $\pi_{(\rho, \varphi)}$ is irreducible and that every infinite dimensional irreducible unitary representation of $G_0$ is equivalent to some $\pi_{(\rho, \varphi)}$. Furthermore, two representations $\pi_{(\rho, \varphi)}$ and $\pi_{(\rho', \varphi')}$ are equivalent if and only if $\varphi$ and $\varphi'$ lie in the same coadjoint orbit of $K$ and the representations $\rho$ and $\rho'$ are equivalent under the identification of the conjugate subgroups $K_\varphi$ and $K_{\varphi'}$. In this way, we obtain all irreducible representations of $G_0$ which are not trivial on the normal subgroup $p$. On the other hand, every irreducible unitary representation $\tau$ of $K$ extends trivially to an irreducible representation, also denoted by $\tau$, of $G_0$ by $\tau(k, X) := \tau(k)$ for $k \in K$ and $X \in p$.

Next, we shall provide a more precise description of the so-called “generic irreducible unitary representations” of $G_0$. Denote again by $\langle , \rangle$ the restriction to $p \times p$ of the $\text{Ad}(K)$-invariant scalar product $\langle , \rangle$ on $g_0$. Let $a$ be a maximal abelian subspace of $p$, and let $M$ be the centralizer of $A = \exp p_G(a)$ in $K$. In general, the compact Lie group $M$ is not connected, and one can prove that $M = M_0 \cdot (M \cap A)$ with $M_0$ being the identity component of $M$. A proof of the following well-known result can be found in [10].
Lemma 2.2. Let \( C \subset a \) be a Weyl chamber. Every adjoint orbit of \( K \) in \( p \) intersects the closure \( C \) in exactly one point.

We conclude that every infinite dimensional unitary representation of \( G_0 \) has the form \( \pi_{(\rho,\varphi_H)} \), where \( H \) is a non-zero vector in \( C^+(a) \) and \( \varphi_H \) is the linear form on \( p \) given by \( \varphi_H(X) = \langle H, X \rangle \). Observe that the isotropy group \( K_{\varphi_H} \) coincides with the centralizer \( Z_K(H) \). Let us fix a regular element \( H \) in \( a \). The subgroups \( K_{\varphi_H} \) and \( M \) of \( K \) are identical. If \( \rho \) is an irreducible representation of \( M \), then the representation \( \pi_{(\rho,\varphi_H)} \) corresponding to the pair \((\rho,\varphi_H)\) is said to be generic. We denote by \( \Gamma_0 \) the set of all equivalence classes of generic irreducible unitary representations of \( G_0 \). Notice that \( \Gamma_0 \) has full Plancherel measure in the unitary dual \( \widehat{G_0} \). Applying Mackey’s analysis and the result of Lemma 2.3, we obtain the bijection

\[
\Gamma_0 \cong \widetilde{M} \times C^+(a).
\]

Note that when the Riemannian symmetric pair \((G,K)\) has rank one, we can find a unit vector \( H_0 \in a \) such that \( C^+(a) = \mathbb{R}_+^* H_0 \). In this case we have the bijections

\[
\Gamma_0 \cong \widetilde{M} \times \mathbb{R}_+^* \quad \text{and} \quad \widehat{G_0} \cong \Gamma_0 \sqcup \Gamma_2
\]

where \( \Gamma_2 := \widetilde{K} \). Now, we define the subset \( \Gamma_1 \subset \widehat{G_0} \) as follows

\[
\Gamma_1 := \left\{ \pi_{(\mu,H)} = Ind_{K_{\varphi_H}}^{K \times p}(\rho \otimes \chi_{\varphi_H}), \rho \in \widetilde{K_{\varphi_H}}, H \in \partial(C^+(a)) \setminus \{0\} \right\}
\]

where \( \partial(C^+(a)) \) is the boundary of \( C^+(a) \). According to Mackey’s theory, we obtain the following parametrization as sets

\[
\widehat{G_0} \cong \Gamma_0 \sqcup \Gamma_1 \sqcup \Gamma_2.
\]

We denote by \( \mathfrak{z} \) the orthogonal complement of \( a \) in \( p \) (\( p = a \oplus \mathfrak{z} \)). Let \( \Lambda : a \rightarrow C \) be a real linear function. Also we denote by \( \Lambda \) the extension of \( \Lambda \) to \( p \) so that \( \mathfrak{z} \subseteq Ker(\Lambda) \), and let \( \rho \in \widetilde{K_{\Lambda}} \). We denote by \( \pi := \pi_{(\rho,\Lambda)} \) the representation of \( G_0 \) induced from

\[
K_{\Lambda} \ltimes p \quad \rightarrow \quad \mathcal{L}(E_{\rho})
\]

\[
(k,X) \quad \mapsto \quad e^{\sqrt{-1\Lambda(X)}}\rho(k).
\]

Now, we describe the Fell topology on \( \widehat{G_0} \). For \( \beta_1, \beta_2 \in a^* \) (the dual vector space of \( a \)), define

\[
|\beta_1 + \sqrt{-1}\beta_2|^2 = B(\beta_1,\beta_1) + B(\beta_2,\beta_2)
\]

where \( B \) is the Cartan-Killing form on \( g_0 \). Let \( \mathcal{F}_\varepsilon \) be the set of all pairs \((\rho,\Lambda)\) where \( \rho \in \widetilde{K_{\Lambda}} \). We take \((\rho,\Lambda) \in \mathcal{F}_\varepsilon \), if \( \varepsilon > 0 \) is sufficiently small then \( |\Lambda - \Lambda'| < \varepsilon \) implies \( K_{\Lambda'} \subseteq K_{\Lambda} \). So the subset

\[
\mathcal{U} := \left\{ (\rho',\Lambda') \in \mathcal{F}_\varepsilon : |\Lambda - \Lambda'| < \varepsilon \quad \text{and} \quad |\rho|_{K_{\Lambda'}} : \rho' > 0 \right\}
\]
([ρ]_{K,\lambda} : \rho') is the multiplicity of ρ' in ρ|_{K,\lambda}) defines a basis for the neighborhoods of (ρ, Λ) in the topology we give \( F_c \) (see [13]). Note that \( W \) acts on \( F_c \) by

\[ w.(\rho, \Lambda) = (w.\rho, w.\Lambda). \]

Let \( F_c/W \) be the quotient space by this action of \( W \), equipped with the quotient topology. Now, let

\[ F := \{(\rho, \Lambda) \in F_c : \Lambda = \sqrt{-1}\beta \text{ where } \beta \text{ is real valued}\}. \]

According to [13], then we have the useful Lemma.

**Lemma 2.3.** The unitary dual \( \hat{G}_0 \) of \( G_0 \) is homeomorphic to \( F/W \).

In the remainder of this paper, we shall assume that the stabilizer \( K_\varphi \) is connected for each \( \varphi \in p^* \). Let \( \rho_\mu \) be an irreducible representation of \( K_\Lambda \) with highest weight \( \mu \). For simplicity, we shall write \( \pi(\mu,H) \) and \( H(\mu,H) \) instead of \( \pi(\rho_\mu,\varphi_H) \) and \( H_{\rho_\mu,\varphi_H} \) respectively. We have:

**Proposition 2.4.** Let \((\pi(\mu^n,H_n))_n\) be a sequence in \( \Gamma_0 \). Then we have:

1. The sequence \((\pi(\mu^n,H_n))_n\) converges to \( \pi(\mu,H) \) in \( \Gamma_0 \) if and only if \((H_n)_n\) converges to \( H \) and \( \mu^n = \mu \) for \( n \) large enough.
2. The sequence \((\pi(\mu^n,H_n))_n\) converges to \( \pi(\mu,H) \) in \( \Gamma_1 \) if and only if \((H_n)_n\) converges to \( H \) and \([\rho_\mu|_H : \rho_{\mu^n}] > 0 \) for \( n \) large enough.
3. The sequence \((\pi(\mu^n,H_n))_n\) converges to \( \tau_\lambda \) in \( \Gamma_2 \) if and only if \((H_n)_n\) converges to \( 0 \) and \([\tau_\lambda|_H : \rho_{\mu^n}] > 0 \) for \( n \) large enough.

**Proof.** By Lemma 2.3 we show that the map

\[ F/W \rightarrow \hat{G}_0 \]

\[ (\rho, \Lambda) \mapsto \pi(\rho,\Lambda) \]

is a homeomorphism (see [13]). Let \( H \) be a non-zero vector in \( C^+(a) \) and \( \Lambda_H \) is the linear form on \( a \) given by

\[ \Lambda_H(X) = \sqrt{-1}(H,X) \quad \forall X \in a. \]

For simplicity, we shall write \( K_H \) instead of \( K_{\lambda_H} \). Now, we assume that \((H_n)_n\) converges to \( H \) and \([\rho_\mu|_{K_{H_n}} : \rho_{\mu^n}] > 0 \) for \( n \) large enough. This is equivalent that the net \((\rho_{\mu^n},\Lambda_{H_n})_n\) converges to \((\rho_{\mu},\Lambda_H)\) in \( F/W \). In view of the continuity of the map \( F/W \ni (\rho,\Lambda) \mapsto \pi(\rho,\Lambda) \in \hat{G}_0 \), we easily see that \((\pi(\mu^n,H_n))_n\) converges to \( \pi(\mu,H) \) in \( \hat{G}_0 \).

Conversely, assume that the net \((\pi(\mu^n,H_n))_n\) converges to \( \pi(\mu,H) \) in \( \hat{G}_0 \). Then the net \((\rho_{\mu^n},\Lambda_{H_n})_n\) converges to \((\rho_{\mu},\Lambda_H)\) in \( F/W \). Recall that the set

\[ \mathcal{V} := \{(\rho',\Lambda') \in F_c : |\Lambda_H - \Lambda'| < \varepsilon \text{ and } [\rho_{\mu}|_{K_{\lambda_H}} : \rho'] > 0 \} \]
defines a basis for the neighborhoods of \((\rho_\mu, \Lambda_H)\). Hence for each \(\varepsilon > 0\), there exists \(n_0 \in I\) such that
\[
\forall n \geq n_0, \ (\rho_\mu^n, \Lambda_{H_n}) \in V.
\]
i.e.; \(\forall n \geq n_0\), we have
\[
|\Lambda_{H_n} - \Lambda_H| < \varepsilon
\]
and
\[
[\rho_\mu|_{K_{H_n}} : \rho_\mu^n] > 0.
\]
Then we obtain the following
\[
H_n \rightarrow H
\]
and
\[
[\rho_\mu|_{K_{H_n}} : \rho_\mu^n] > 0.
\]
for \(n\) large enough. Notice that for \(H, H_n \in C^+(a), \ K_H = K_{H_n} = M\) and we get
\[
[\rho_\mu|_{K_{H_n}} : \rho_\mu^n] > 0 \iff [\rho_\mu|_{M} : \rho_\mu^n] > 0
\]
for \(n\) large enough, which is equivalent to \(\mu^n = \mu\) for \(n\) large enough. This completes the proof of the Proposition. \(\square\)

**Proposition 2.5.** Let \((\pi(\mu^n, H_n))_n\) be a sequence in \(\Gamma_1\). Then we have:

1. The sequence \((\pi(\mu^n, H_n))_n\) converges to \(\pi(\mu, H)\) in \(\Gamma_1\) if and only if \((H_n)_n\) converges to \(H\) and \([\rho_\mu|_{K_{H_n}} : \rho_\mu^n] > 0\) for \(n\) large enough.

2. The sequence \((\pi(\mu^n, H_n))_n\) converges to \(\tau_\lambda\) in \(\Gamma_2\) if and only if \((H_n)_n\) converges to 0 and \([\tau_\lambda|_{K_{H_n}} : \rho_\mu^n] > 0\) for \(n\) large enough.

**Proof.** Applying the same arguments as in the proof of Proposition 2.4. \(\square\)

Of course \(\hat{K}\) has the discrete topology.

Let \(C^*(G_0)\) denote the full \(C^*\)-algebra of \(G_0\). We denote by \(C_0(\hat{K})\) the Banach algebra of all operator fields
\[
F : \hat{K} \rightarrow \bigcup_{\pi \in \hat{K}} \mathcal{B}(\mathcal{H}_\pi),
\]
such that \(F(\pi) \in \mathcal{B}(\mathcal{H}_\pi)\) for each irreducible unitary representation \(\pi\) of \(K\) and such that \(\lim_{\pi \rightarrow \infty} \|F(\pi)\|_{op} = 0\). This algebra is equipped with the norm \(\|F\|_\infty = \sup_{\pi \in \hat{K}} \|F(\pi)\|_{op}, \ F \in C_0(\hat{K})\). Its well-known that the \(C^*\)-algebra \(C^*(K)\) of \(K\) is isomorphic to \(C_0(\hat{K})\) (see, [1]).

In the sequel, we describe the elements of the image of the Fourier transform of \(C^*(G_0)\) inside the big algebra \(\ell^\infty(G_0)\).
Definition 2.6. Let \((\mu, H) \in \Gamma_0 \sqcup \Gamma_1\) and let \(\rho_\mu \in \hat{K}_H\). We define the representation \(\pi_{\mu,0}\) of \(G_0\) by
\[
\pi_{\mu,0} := \text{Ind}_{G_0}^{\hat{K}_H \ltimes p}(\rho_\mu \otimes 1)
\]
and let \(\mathcal{H}_{\mu,0}\) be its Hilbert space. By the Frobenius reciprocity, we obtain
\[
\pi_{\mu,0} = \bigoplus_{\lambda \geq \mu} \tau_\lambda
\]
where \(\lambda \geq \mu\) means that \(\tau_\lambda \in \text{Ind}_{K}^{K_H}(\rho_\mu)\).

3. The \(C^*\)-algebra of the group \(G_0\).

3.1. The Fourier transform. Now, let \(f \in L^1(G_0), h \in K\) and \(\Psi \in \mathcal{H}_{\mu,H}\), then we can give the expression of the operator \(\pi_{(\mu,H)}(f)\) by the following equality
\[
\pi_{(\mu,H)}(f)(\Psi(h)) = \int_{G_0} f(k, X)\pi_{(\mu,H)}(k, X)(\Psi(h))dkdX
\]
\[
= \int_{K} \int_{p} f(k, X)\Psi(k^{-1}h)e^{i(\text{Ad}(h^{-1})X,H)}dkdX
\]
\[
= \int_{K} \hat{f}^2(hk^{-1}, \text{Ad}(h)H)\Psi(k)dk
\]
\[
= \int_{K/K_H} \int_{K_H} \hat{f}^2(hs^{-1}k^{-1}, \text{Ad}(h)H)\rho_\mu(s^{-1})(\Psi(k))dsk.
\]
Then
\[
\pi_{(\mu,H)}(f)(\Psi(h)) = \int_{K/K_H} f_{\mu,H}(h,k)(\Psi(k))dk,
\]
here \(\hat{f}^2\) denotes the partial Fourier transform of \(f\) in the second variable and where
\[
f_{\mu,H} : K \times K \to \mathcal{B}(\mathcal{H}_{\mu,0})
\]
\[
(h,k) \mapsto \int_{K_H} \hat{f}^2(hsk^{-1}, \text{Ad}(h)H)\rho_\mu(s)ds.
\]

Definition 3.1. Let
\[
C_{0,2}(G_0) := \left\{ f \in L^1(G_0) \mid \hat{f}^2 \in C_0(K \times p) \right\}.
\]
This space \(C_{0,2}(G_0)\) is dense in \(L^1(G_0)\) and hence also in \(C^*(G_0)\).

Definition 3.2. For each \(f \in C^*(G_0)\), the Fourier transform \(\mathcal{F}(f)\) of \(f\) is an isometric homomorphism on \(C^*(G_0)\) into \(\ell^\infty(\hat{G}_0)\) which is given by
\[
\mathcal{F}(f)(\mu, H) = \pi_{(\mu,H)}(f) \in \mathcal{B}(\mathcal{H}_{\mu,H}), \ (\mu, H) \in \Gamma_0 \sqcup \Gamma_1,
\]
\[
\mathcal{F}(f)(\lambda) = \tau_\lambda(f) \in \mathcal{B}(\mathcal{H}_\lambda) \forall \lambda \in \Gamma_2.
\]

In the following Proposition, we give a description of elements \(\mathcal{F}(f)\) for each \(f \in C^*(G_0)\).
**Proposition 3.3.** Let $f \in C^*(G_0)$. Then we have:

1. For every $(\mu, H) \in \Gamma_0 \sqcup \Gamma_1$, $\mathcal{F}(\mu, H)$ is a compact operator on $\mathcal{H}_{\mu,H}$.
2. The mapping
   
   \[
   \widehat{G}_0 \longrightarrow \mathcal{B}(\mathcal{H}_{\pi,\gamma})
   \]
   
   $\gamma \mapsto \mathcal{F}(f)(\gamma)$
   
   are norm continuous on the difference sets $\Gamma_i$, $i = 0, 1, 2, \ldots$.
3. $\lim_{|\mu| \to +\infty} \|\mathcal{F}(f)(\mu, H)\|_{op} = 0$.
4. $\lim_{H \to 0} \|\pi(\mu,H)(f) - \pi_{\mu,0}(f)\|_{op} = 0$.

**Proof.** Let $f \in C_{0,2}(G_0)$

1. We have the following bound for the kernel functions $f_{\mu,H}$ where $(\mu, H) \in \Gamma_0 \sqcup \Gamma_1$

   \[
   \|f_{\mu,H}(h,k)\|_{op} \leq \left\| \int_{K_H} \hat{f}^2(hsk^{-1}, Ad(h)H)\rho_{\mu}(s)ds \right\|_{op}
   \]
   
   $\leq \int_{K_H} |\hat{f}^2(hsk^{-1}, Ad(h)H)| \|\rho_{\mu}(s)\|_{op} ds$
   
   $\leq \int_{K_H} |\hat{f}^2(hsk^{-1}, Ad(h)H)| ds$

   \[\text{(3.2)}\]

   Since the dimension of the representation $\rho_{\mu}$ of $K_H$ is finite (denote by $d_{\mu}$), it follows that the norm $\| \cdot \|_{op}$ and $\| \cdot \|_{H.S}$ on the space $\mathcal{H}_{\rho_{\mu}}$ are equivalent and

   \[
   \|f_{\mu,H}(h,k)\|_{H.S} \leq \sqrt{d_{\mu}}\|f_{\mu,H}(h,k)\|_{op}, \ h, k \in K.
   \]

   Then by (3.2) the operator $\pi(\mu,H)(f)$ is Hilbert-Schmidt and its Hilbert-Schmidt norm is given by

   \[
   \|\pi(\mu,H)(f)\|_{H.S}^2 = \int_{K/K_H} \int_{K/K_H} \|f_{\mu,H}(h,k)\|_{H.S}^2 dhdk
   \]

   \[\text{(3.3)}\]

2. Let $(\mu, H_n)_{n \in \mathbb{N}}$ be a sequence converges to $(\mu, H)$ in $\Gamma_0$. Using now 3.1, we obtain

   \[
   \|\pi_{\mu,H_n}(f) - \pi(\mu,H)(f)\|_{H.S}^2
   \]
   
   $= \int_{K/K_H} \int_{K/K_H} \|f_{\mu,H_n}(h,k) - f_{\mu,H}(h,k)\|_{H.S}^2 dhdk$
   
   $\leq d_{\mu} \int_{K/K_H} \int_{K/K_H} \|f_{\mu,H_n}(h,k) - f_{\mu,H}(h,k)\|_{op}^2 dhdk$
   
   $\leq d_{\mu} \int_{K/K_H} \int_{K/K_H} \left( \int_{K} \left| \hat{f}^2(hsk^{-1}, Ad(h)H_n) - \hat{f}^2(hsk^{-1}, Ad(h)H) \right|^2 ds \right) dhdk,$
For \( f \in C_{0,2}(G_0) \) we have
\[
\lim_{n \to \infty} |\hat{f}^2(hsk^{-1}, Ad(h)H_n) - \hat{f}^2(hsk^{-1}, Ad(h)H)| = 0.
\]
Hence
\[
\lim_{n \to \infty} \|\pi(\mu,H_n)(f) - \pi(\mu,H)(f)\|_{H.S} = 0.
\]
Let now \((\mu, H_n)_{n \in \mathbb{N}}\) be a sequence converges to \((\mu, H)\) in \(\Gamma_1\). We have for \(\xi \in L^2(K/K_H, \mu)\)
\[
\|(\pi(\mu,H_n)(f) - \pi(\mu,H)(f))\xi\|_2^2 = \int_K \left|\int_K (f_{\mu,H_n}(h,k) - f_{\mu,H}(h,k))(\xi(k))dk\right|^2_{H_\mu} dk
\]
\[
\leq \sup_{k \in K} \left(\int_K \|f_{\mu,H_n}(h,k) - f_{\mu,H}(h,k)\|_{op} dh\right)^{\frac{1}{2}}
\]
\[
\times \sup_{h \in K} \left(\int_K \|f_{\mu,H_n}(h,k) - f_{\mu,H}(h,k)\|_{op} dk\right)^{\frac{1}{2}} \|\xi\|_2,
\]
since the function \((h,k,s,H) \mapsto |\hat{f}^2(hsk^{-1}, Ad(h)H_n) - \hat{f}^2(hsk^{-1}, Ad(h)H)|\) converges uniformly to 0 as \(n\) tends to \(\infty\), since
\[
\|f_{\mu,H_n}(h,k) - f_{\mu,H}(h,k)\|_{op}
\]
\[
= \|\int_{K_{H_n}} \hat{f}^2(hsk^{-1}, Ad(h)H_n)\rho_{\mu}(s)ds - \int_{K_{H}} \hat{f}^2(hsk^{-1}, Ad(h)H)\rho_{\mu}(s)ds\|_{op}
\]
\[
\leq \|\int_{K_{H_n}} \hat{f}^2(hsk^{-1}, Ad(h)H_n)\rho_{\mu}(s)ds - \int_{K_{H_n}} \hat{f}^2(hsk^{-1}, Ad(h)H)\rho_{\mu}(s)ds\|_{op}
\]
\[
+ \|\int_{K_{H_n}} \hat{f}^2(hsk^{-1}, Ad(h)H)\rho_{\mu}(s)ds - \int_{K_{H}} \hat{f}^2(hsk^{-1}, Ad(h)H)\rho_{\mu}(s)ds\|_{op}
\]
\[
\leq \int_{K_{H}} |\hat{f}^2(hsk^{-1}, Ad(h)H_n) - \hat{f}^2(hsk^{-1}, Ad(h)H)|ds
\]
\[
+ \int_{K} |\hat{f}^2(hsk^{-1}, Ad(h)H)||1_{K_{H_n}}(s) - 1_{K_{H}}(s)|ds
\]
we see therefore that
\[
\lim_{n \to \infty} \|\pi(\mu,H_n)(f) - \pi(\mu,H)(f)\|_{op} = 0
\]
uniformly in \(\mu\).

(3) It remains for us to prove that \(\lim_{|\mu| \to \infty} \|\pi(\mu,H)(f)\|_{H.S} = 0\). This is equivalent to show that
\[
\lim_{|\mu| \to \infty} \|f_{\mu,H}(k,h)\|_{H.S} = 0 \text{ for all } k, h \in K.
\]
Recall that
\[
f_{\mu,H}(h,k) = \int_{K_{H}} \hat{f}^2(hsk^{-1}, Ad(h)H)\rho_{\mu}(s)ds.
\]
Let
\[ \varphi^{h,k}_H(s) = \hat{f}^2(hsk^{-1}, Ad(h)H), \quad s \in K_H. \]
So,
\[ f_{\mu,H}(h,k) = \rho_{\mu}(\varphi^{h,k}_H). \]
Using now the Plancherel formula, we get
\[ \|\varphi^{h,k}_H\|^2_{L^2(K_H)} = \sum_{\mu \in \hat{K}_H} d_{\mu} \|\rho_{\mu}(\varphi^{h,k}_H)\|^2_{H,S}. \]
Therefore
\[ \lim_{|\mu| \to \infty} \|f_{\mu,H}(h,k)\|_{H,S} = \lim_{|\mu| \to \infty} \|\rho_{\mu}(\varphi^{h,k}_H)\|_{H,S} = 0. \]
(4) From (3.1) we have for \( \mu \in \hat{K}_H \) and \( \xi \in L^2(K/K_H, \mu) \)
\[ \|(\pi_{(\mu,H)}(f) - \pi_{\mu,0}(f))\xi\|^2_2 = \int_K \left\| \int_K (f_{\mu,H}(h,k) - f_{\mu,0}(h,k))(\xi(k))dk \right\|^2_{H_{\mu}} dk \]
\[ \leq \sup_{k \in K} \left( \int_K \|f_{\mu,H}(h,k) - f_{\mu,0}(h,k)\|_{op} dh \right)^{\frac{1}{2}} \]
\[ \times \sup_{h \in K} \left( \int_K \|f_{\mu,H}(h,k) - f_{\mu,0}(h,k)\|_{op} dk \right)^{\frac{1}{2}} \|\xi\|_2, \]
where
\[ f_{\mu,0} : K \times K \to B(H_{\mu}) \]
\[ (h,k) \mapsto \int_{K_H} \hat{f}^2(hsk^{-1},0)\rho_{\mu}(s)ds \]
Since the function \( (k,h,s,H) \mapsto |\hat{f}^2(hsk^{-1}, Ad(h)H) - \hat{f}^2(hsk^{-1},0)| \)
converges uniformly to 0 as \( H \) tends to 0 since
\[ \|f_{\mu,H}(h,k) - f_{\mu,0}(h,k)\|_{op} = \left\| \int_{K_H} (\hat{f}^2(hsk^{-1}, Ad(h)H) - \hat{f}^2(hsk^{-1},0))\rho_{\mu}(s)ds \right\|_{op} \]
\[ \leq \int_{K_H} |\hat{f}^2(hsk^{-1}, Ad(h)H) - \hat{f}^2(hsk^{-1},0)| ds \]
we see therefore that
\[ \lim_{H \to 0} \|\pi_{(\mu,H)}(f) - \pi_{\mu,0}(f)\|_{op} = 0 \]
uniformly in \( \mu \).
The proposition follows now from the density of \( C_{0,2}(G_0) \) in \( C^\ast(G_0) \). \( \square \)
3.2. A C*-condition.

Definition 3.4. For an operator field \( F \in \ell^\infty(\hat{G}_0) \), let

\[
F(\mu, 0) := \bigoplus_{\lambda \geq \mu} F(\lambda) \in \mathcal{B}\left(\bigoplus_{\lambda \geq \mu} \mathcal{H}_\mu\right).
\]

We have

\[
\|F(\mu, 0)\|_{\text{op}} = \sup_{\lambda \geq \mu} \|F(\lambda)\|_{\text{op}}.
\]

Definition 3.5. Let \( \mathcal{D} \) be the family consisting of all operator fields \( F \in \ell^\infty(\hat{G}_0) \) satisfying the following conditions:

1. \( F(\mu, H) \) is a compact operator on \( \mathcal{H}_{\mu, H} \) for every \( (\mu, H) \in \Gamma_0 \cup \Gamma_1 \).
2. The mapping \( \hat{G}_0 \to \mathcal{B}(\mathcal{H}_{\mu, H}) : (\mu, H) \mapsto F(\mu, H) \) are norm continuous on the difference sets \( \Gamma_i, i = 0, 1, 2 \).
3. \( \lim_{|\mu| \to \infty} \|F(\mu, H)\|_{\text{op}} = 0 \).
4. \( \lim_{H \to 0} \|F(\mu, H) - F(\mu, 0)\|_{\text{op}} = 0 \) uniformly in \( \mu \in K_H \).
5. \( \lim_{|\lambda| \to \infty} \|F(\lambda)\|_{\text{op}} = 0 \).

Definition 3.6. Let

\[
\mathcal{D}_0 = \{ F \in \mathcal{D} : F(\lambda) = 0 \text{ for all } \lambda \in \hat{K} \}.
\]

Remark 3.7. According to Proposition 4.7 in [1] the unitary dual \( \hat{\mathcal{D}}_0 \) is in bijection with the parameter space \( \Gamma_0 \cup \Gamma_1 \).

Theorem 3.8. \( \mathcal{D} \) is a C*-algebra for the norm \( \| \cdot \|_{\text{op}} \), which is isomorphic to the C*-algebra of Cartan motion groups under the Fourier transform.

Proof. First we show that \( \mathcal{D} \) is a C*-algebra. It is clear that \( \mathcal{D} \) is an involutive sub-algebra of \( \ell^\infty(\hat{G}_0) \). The conditions (1), (2), (3) et (5) are evidently true for every \( F \in \mathcal{D} \). For the condition (4), let \( (F_k)_k \subset \mathcal{D} \) such that \( \lim_{k \to \infty} \|F_k - F\|_\infty = 0 \). We have then \( \lim_{H \to 0} \|F_k(\mu, H) - F_k(\mu, 0)\|_{\text{op}} = 0 \) uniformly in \( \mu \), since for any \( \varepsilon > 0 \) there exists \( k_0 \) such that such that \( \|F - F_k\|_\infty < \varepsilon \) for any \( k \geq k_0 \). Then \( \|F_{k_0}(\mu, H) - F_{k_0}(\mu, 0)\|_{\text{op}} < \varepsilon \) uniformly in \( \mu \). Hence

\[
\begin{align*}
\|F(\mu, H) - F(\mu, 0)\|_{\text{op}} & \leq \|F(\mu, H) - F_{k_0}(\mu, H)\|_{\text{op}} + \|F_{k_0}(\mu, H) - F_{k_0}(\mu, 0)\|_{\text{op}} + \|F_{k_0}(\mu, 0) - F(\mu, 0)\|_{\text{op}} \\
& \leq 3\varepsilon.
\end{align*}
\]

Then \( F \in \mathcal{D} \).

Second, we have the quotient space \( \mathcal{D}/\mathcal{D}_0 \) is isomorphic to \( C^*(K) \), indeed, let

\[
\delta : \mathcal{D} \to C^*(K) \\
F \mapsto (F(\lambda))_{\lambda \in \hat{K}}
\]
We have $\text{ker}(\delta) = D_0$. Since $D$ contains the algebra $\mathcal{F}(C^*(G_0))$ the image of $\delta$ contains the image of the Fourier transform of $C^*(K)$. Then $\delta$ is surjective. In addition $D$ is a type $I$ algebra and $\hat{G}_0 \subset \hat{D}$. Moreover for every irreducible representation $\pi \in \hat{D}$, we have either $\pi|_{D_0} \neq \{0\}$, and then $\pi = \pi_{(\mu,H)}$ since $D_0$ is closed ideal of $D$ and $\hat{D}_0 = \Gamma_0 \cup \Gamma_1$ or $\pi \in \hat{D}/D_0$ and $\pi \in \Gamma_2$. Hence $\pi \in \hat{G}_0$ and $\hat{G}_0 = \hat{D}$ as sets.

Thanks to Stone-Weierstrass’s theorem we have the identity $D = \mathcal{F}(C^*(G_0))$. □

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