Unifying the dynamical effects of quantum and classical noises

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Abstract

We develop a new master equation as a unified description of the effects of both quantum noise (system-bath interaction) and classical noise on a system’s dynamics, using a two-dimensional series expansion method. When quantum and classical noises are both present, their combined effect on a system’s dynamics is not necessarily a simple sum of the two individual effects. Thus previous master equations for open systems and those for classical noise, even when jointly used, may not capture the full physics. Our formalism can determine whether there is interference between quantum and classical noises and will be able to capture and describe such interference if there is any (in a perturbative manner). We find that, interestingly, second-order interference between quantum and classical noises vanishes identically. This work thus also serves to justify simple additive treatments of quantum and classical noises, especially in the weak coupling regime. For a Zeeman-split atom in a stochastic magnetic field interacting with an optical cavity, we use the formalism developed herein to find the overall decoherence rate between the atom’s energy levels.

1 Introduction

Ideally, for a closed quantum system under a deterministic Hamiltonian, its density matrix evolves unitarily according to the von Neumann equation, \[\frac{d}{dt} \rho(t) = -i[H_{\text{deterministic}}(t), \rho(t)].\] (1)
In reality, many quantum systems interact with external quantum degrees of freedom (the “environment”), and such system-environment interactions generally result in non-unitary dynamics at the system’s level. \[6, 7\] This is the scenario of open quantum systems. On the other hand, there are scenarios where no external quantum degree of freedom is formally present, but a (closed) quantum system is subject to a Hamiltonian that is non-deterministic/stochastic across different realizations of the system’s evolution (e.g. across different runs of repeated experiment). This stochasticity is called classical noise. \[5, 7\] For a closed quantum system under a stochastic Hamiltonian, its dynamics is unitary in one particular realization of the system’s evolution (e.g. in a single run of the experiment). However, when we repeat the experiment for multiple times, and if we consider the ensemble average of the system’s statistics over multiple realizations, the system’s average density matrix generally evolves non-unitarily. \[5, 7\]

The study of how system-environment interaction and classical noise affect a system’s dynamics is important. Put most simplistically, quantum coherence is essential to the broad field of quantum information \[1\] and quantum control \[2\]. Two conceptually different sources for the loss of quantum coherence are: (a) decoherence, which generally arises in open quantum systems, resulting from system-environment interaction; \[6, 7\] (b) dephasing, which generally arises under stochastic Hamiltonians, resulting from classical noise. \[5, 7\] Observationally, both may be similarly represented by the decay of some off-diagonal density matrix element(s) in some bases. \[7, 5, 6\] However, the two kinds of “noises” are of different natures - a system can get entangled with the environment in the case of open quantum systems, whereas there is no system-environment entanglement in the case of classical noise. \[7\]

Open quantum systems are extensively studied in the literature. \[6, 3, 4, 14, 15, 16, 17\] There are also studies on quantum systems under classical noise. \[5, 11\] However, there are few studies that consider the effects of both system-environment interaction and classical noise on the system’s dynamics in a unified and systematic way. \(\text{[See 13] [12]}\) for previous works on classical and quantum noises, through “Environment Algebra” and quantum Langevin equations rather than master equations.) In our work, with a two-dimensional series expansion approach, we develop a master equation formalism that takes into account both system-environment interaction (“quantum noise”) and stochastic Hamiltonian (classical noise) and treats their joint effects on a system’s dynamics in a unified and consistent manner.

When both quantum noise and classical noise are present, there may be interference between the two kinds of noises on a system’s dynamics. Most cautiously put, we have no a priori reason to think that their joint effect is merely a simple sum of the two individual effects. If interference exists, then

\(\text{[It may be added that system-environment interaction is sometimes loosely referred to as “quantum noise” in the open quantum system scenario, because observationally it can have similar effects on the system’s statistics like classical noise does. [10]}\)

\[1\]
the master equations for open quantum systems and those for classical noise, even when both are used together, do not describe the full physics. The master equation formalism developed herein will be able to determine whether there is interference between quantum and classical noises on a system’s dynamics (in a perturbative manner). If there is interference (in some perturbative order), our formalism can capture and quantify such interference; if there is no interference (in some perturbative order), our formalism can rule it out. This is a motivation behind this work.

2 Theory

2.1 Derivations

Total unitary dynamics

The total interaction Hamiltonian consists of two terms

\[ H_{\text{int}}^{(j)}(t) = \lambda H_{SE}(t) + \delta H_S^{(j)}(t) \otimes \mathbb{1}_E, \]  

(2)

where \( H_{SE}(t) \) is the system-bath interaction Hamiltonian, \( H_S^{(j)}(t) \otimes \mathbb{1}_E \) is the stochastic Hamiltonian acting on the system, \( \lambda \) and \( \delta \) parametrize the strength of the system-bath interaction and that of the stochastic Hamiltonian respectively, and the index \( j \) denotes the \( j \)-th realization of the stochastic process / experimental run. The first term alone can be lead to system-bath entanglement, which in turn can lead to decoherence in the system’s reduced density matrix. This often goes by the name of “quantum noise” and is the subject of open quantum systems. The second term alone, upon averaging, can lead to dephasing of the system’s density matrix, which is an effect of “classical noise”.

The system-bath total density matrix in the \( j \)-th run of the experiment thus obeys the equation of motion

\[ i \frac{d}{dt} \rho_{\text{total}}^{(j)}(t) = \left[ H_{\text{int}}^{(j)}(t), \rho_{\text{total}}^{(j)}(t) \right] = \left[ \lambda H_{SE}(t) + \delta H_S^{(j)}(t) \otimes \mathbb{1}_E, \rho_{\text{total}}^{(j)}(t) \right]. \]  

(3)

Let \( U^{(j)}(t,0) \) be the unitary evolution operator for the system-bath total density matrix in the in the \( j \)-th run of the experiment such that \( \rho_{\text{total}}^{(j)}(t) = U^{(j)}(t,0) \rho_{\text{total}}(0) U^{(j)\dagger}(t,0) \), then

\[ i \frac{d}{dt} U^{(j)}(t,0) = \left( \lambda H_{SE}(t) + \delta H_S^{(j)}(t) \otimes \mathbb{1}_E \right) U^{(j)}(t,0). \]  

(4)

Two-dimensional series expansion

The first key step in our construction is to suppose that the total unitary operator \( U^{(j)}(t,0) \) can be expanded in a 2-dimensional power series of \( \lambda \) and \( \delta \):

\[ U^{(j)}(t,0) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda^m \delta^n U_{m,n}^{(j)}(t,0). \]  

(5)

\[ ^2 \text{All works herein are within the interaction picture unless otherwise noted.} \]
Plugging Eq. (5) into Eq. (4), we have

\[
i \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda^m \delta^n \frac{d}{dt} U_{m,n}(t, 0) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \lambda^{m+1} \delta^n H_{SE}(t) U_{m,n}(t, 0) + \lambda^m \delta^{n+1} H_S^{(j)}(t) \otimes \mathbb{I}_E U_{m,n}(t, 0) \right) (6)
\]

Comparing like-order terms from the left-hand side and right-hand side of Eq. (6), we have iterative equations for \( U_{M,N}(t, 0) \):

\[
i \frac{d}{dt} U_{0,0}^{(j)}(t, 0) = 0, \quad (M = 0, N = 0) \tag{7}
\]
\[
i \frac{d}{dt} U_{M,0}^{(j)}(t, 0) = H_{SE}(t) U_{M-1,0}^{(j)}(t, 0), \quad (M \geq 1, N = 0) \tag{8}
\]
\[
i \frac{d}{dt} U_{0,N}^{(j)}(t, 0) = H_S^{(j)}(t) \otimes \mathbb{I}_E U_{0,N-1}^{(j)}(t, 0), \quad (M = 0, N \geq 1) \tag{9}
\]
\[
i \frac{d}{dt} U_{M,N}^{(j)}(t, 0) = H_{SE}(t) U_{M-1,N}^{(j)}(t, 0) + \left( H_S^{(j)}(t) \otimes \mathbb{I}_E \right) U_{M,N-1}^{(j)}(t, 0). \quad (M, N \geq 1) \tag{10}
\]

Solving the above iterative equations, we have, for example,

\[
U_{0,0}^{(j)}(t, 0) = I, \tag{11}
\]
\[
U_{1,0}^{(j)}(t, 0) = (-i) \int_{0}^{t} dt' H_{SE}(t'), \tag{12}
\]
\[
U_{2,0}^{(j)}(t, 0) = -\int_{0}^{t} dt' \int_{0}^{t'} dt'' H_{SE}(t') H_{SE}(t''), \tag{13}
\]
\[
U_{0,1}^{(j)}(t, 0) = (-i) \int_{0}^{t} dt' H_S^{(j)}(t') \otimes \mathbb{I}_E, \tag{14}
\]
\[
U_{0,2}^{(j)}(t, 0) = -\int_{0}^{t} dt' \int_{0}^{t'} dt'' \left( H_S^{(j)}(t') H_S^{(j)}(t'') \right) \otimes \mathbb{I}_E, \tag{15}
\]
\[
U_{1,1}^{(j)}(t, 0) = -\int_{0}^{t} dt' \int_{0}^{t'} dt'' \left[ H_{SE}(t') \left( H_S^{(j)}(t'') \otimes \mathbb{I}_E \right) \right] + \left( H_S^{(j)}(t') \otimes \mathbb{I}_E \right) H_{SE}(t''), \tag{16}
\]

Averaged reduced density matrix

Tracing out the environmental degrees of freedom and averaging over the stochastic process, we have the averaged reduced density matrix that describes the measurement statistics

\[
\bar{\rho}_S(t) = \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} Tr_E \left( \rho_{\text{total}}^{(j)}(t) \right). \tag{17}
\]
Thus for initial separability $\rho_{\text{total}}(0) = \rho_S(0) \otimes \rho_{E0}$, we have

$$\bar{\rho}_S(t) = \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} T_{RE} \left( U^{(j)}(t,0) \rho_S(0) \otimes \rho_{E0} U^{(j)\dagger}(t,0) \right)$$

$$= \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} T_{RE} \left( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda^m \delta^m U^{(j)}_{m,n}(t,0) \rho_S(0) \otimes \rho_{E0} \sum_{m'=0}^{\infty} \sum_{n'=0}^{\infty} \lambda^{m'} \delta^{m'} U^{(j)\dagger}_{m',n'}(t,0) \right)$$

$$= \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \lambda^M \delta^N \sum_{m=0}^{M} \sum_{n=0}^{N} \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} T_{RE} \left( U^{(j)}_{M-m,N-n}(t,0) \rho_S(0) \otimes \rho_{E0} U^{(j)\dagger}_{m,n}(t,0) \right) .$$

(18)

With $\rho_t \equiv \bar{\rho}_S(t)$ and $\rho_0 \equiv \rho_S(0)$, the linear mapping from $\rho_0$ to $\rho_t$ can be re-written as

$$\rho_t = (\mathbb{1} + \mathcal{E}_t)(\rho_0) = \mathbb{1}(\rho_0) + \sum_{(M,N) \neq (0,0)} \lambda^M \delta^N \mathcal{E}_{t(M,N)}(\rho_0),$$

(19)

where the $(M,N)$-th order linear map for an arbitrary system density matrix $\rho$ is

$$\mathcal{E}_{t(M,N)}(\rho) \equiv \sum_{m=0}^{M} \sum_{n=0}^{N} \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} T_{RE} \left( U^{(j)}_{M-m,N-n}(t,0) \rho \otimes \rho_{E0} U^{(j)\dagger}_{m,n}(t,0) \right) ,$$

(20)

and

$$\mathcal{E}_t(\rho) \equiv \sum_{(M,N) \neq (0,0)} \lambda^M \delta^N \mathcal{E}_{t(M,N)}(\rho).$$

(21)

The $Y_{Q,t}$ map

The second key step is to introduce a linear map that will be central to our construction

$$Y_{Q,t}(\rho) \equiv \sum_{q=0}^{Q} (-1)^q \mathcal{E}_t^{(q)}(\rho),$$

(22)

where $\mathcal{E}_t^{(q)}(\rho) \equiv \mathcal{E}_t(\mathcal{E}_t(\ldots \mathcal{E}_t(\rho)))$ is a composition of $q$ $\mathcal{E}_t$ maps.

Applying this linear map to the averaged reduced density matrix at time $t$ yields

$$Y_{Q,t}(\rho_t) = \sum_{q=0}^{Q} (-1)^q \mathcal{E}_t^{(q)}((\mathbb{1} + \mathcal{E}_t)(\rho_0))$$
\[
\begin{align*}
= & \quad (\mathbb{1} + \mathcal{E}_t) (\rho_0) - \mathcal{E}_t (\mathbb{1} + \mathcal{E}_t) (\rho_0) + \mathcal{E}_t (\mathcal{E}_t (\mathbb{1} + \mathcal{E}_t) (\rho_0)) - \ldots \\
= & \quad (\mathbb{1} + \mathcal{E}_t) (\rho_0) - \mathcal{E}_t (\rho_0) - \mathcal{E}_t (\mathcal{E}_t (\rho_0)) + \mathcal{E}_t (\mathcal{E}_t (\rho_0)) + \mathcal{E}_t (\mathcal{E}_t (\rho_0)) - \ldots \\
= & \quad \left( \mathbb{1} + (-1)^Q \mathcal{E}_t^{(Q+1)} \right) (\rho_0). 
\end{align*}
\]

Rearranging the terms, we now have the crucial equality in our work:

\[
\rho_0 = Y_{Q,t} (\rho_t) + (-1)^{Q+1} \mathcal{E}_t^{(Q+1)} (\rho_0)
\]

\[
= \sum_{q=0}^{Q} (-1)^q \mathcal{E}_t^{(q)} (\rho_t) + (-1)^{Q+1} \mathcal{E}_t^{(Q+1)} (\rho_0).
\]

What it does is to express the initial state \( \rho_0 \) in terms of the state at time \( t \), \( \rho_t \), that can be neglected to certain perturbative order.

Before proceeding further, let’s examine the order of magnitudes of relevant terms. For the 1st-order term:

\[
\mathcal{E}_t (\rho) = \lambda \mathcal{E}_t^{(1,0)} (\rho) + \delta \mathcal{E}_t^{(0,1)} (\rho) + \ldots
\]

\[
= \mathcal{O}(\lambda) + \mathcal{O}(\delta);
\]

For the 2nd-order term:

\[
\mathcal{E}_t^{(2)} (\rho) = \mathcal{E}_t (\mathcal{E}_t (\rho)) = \mathcal{E}_t \left( \lambda \mathcal{E}_t^{(1,0)} (\rho) + \delta \mathcal{E}_t^{(0,1)} (\rho) + \ldots \right)
\]

\[
= \lambda \mathcal{E}_t^{(1,0)} \left( \lambda \mathcal{E}_t^{(1,0)} (\rho) + \delta \mathcal{E}_t^{(0,1)} (\rho) + \ldots \right) + \delta \mathcal{E}_t^{(0,1)} \left( \lambda \mathcal{E}_t^{(1,0)} (\rho) + \delta \mathcal{E}_t^{(0,1)} (\rho) + \ldots \right) + \ldots
\]

\[
= \mathcal{O}(\lambda^2) + \mathcal{O}(\lambda \delta) + \mathcal{O}(\delta^2);
\]

In general, for the \( Q \)th-order term:

\[
\mathcal{E}_t^{(Q)} (\rho) = \sum_{q=0}^{Q} \mathcal{O} (\lambda^q \delta^{Q-q}).
\]

**Equation of motion**

Differentiating the averaged reduced density matrix with respect to time, we have

\[
\frac{d}{dt} \rho_t = \frac{d}{dt}\left( \mathbb{1} + \mathcal{E}_t \right) (\rho_0)
\]

\[
= \mathcal{E}_t (\rho_0)
\]

\[
= \mathcal{E}_t \left( \sum_{q=0}^{Q} (-1)^q \mathcal{E}_t^{(q)} (\rho_t) + (-1)^{Q+1} \mathcal{E}_t^{(Q+1)} (\rho_0) \right),
\]

\[\text{(28)}\]
where in the last equality we have made use of Eq. (24). Thus we now have

\[
\frac{d}{dt} \rho_t = \sum_{q=0}^{Q} (-1)^q \dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(q)} (\rho_t) \right) + (-1)^{Q+1}\dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(Q+1)} (\rho_0) \right). \tag{29}
\]

Note that no approximation has been made so far and that Eq. (29) is formally exact.

With Eq. (29), we can systematically make approximations, that is, collecting like-order terms in \(\lambda\) and \(\delta\) and truncate the series as needed. For example, suppose we want to consider \(P\)-th-order approximation (i.e. approximations up to \(\sum_{q=0}^{P} \mathcal{O} (\lambda^q \delta^{P-q})\) terms). Because as we have shown in Eq. (27) 

\[
\dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(Q+1)} (\rho_0) \right) \sim \sum_{q=0}^{Q+2} \mathcal{O} (\lambda^q \delta^{Q+2-q}),
\]

we can always choose \(Q \geq P - 1\), so that the residual term \((-1)^{Q+1}\dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(Q+1)} (\rho_0) \right)\) is negligible to our intended approximation.

A formally exact, time-local equation of motion can be formally achieved by taking the \(Q \to \infty\) limit on the right-hand side of Eq. (29). Loosely speaking, as \(\lim_{Q \to \infty} (-1)^{Q+1}\dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(Q+1)} (\rho_0) \right) \sim \lim_{Q \to \infty} \sum_{q=0}^{Q+2} \mathcal{O} (\lambda^q \delta^{Q+2-q}) \to 0\), the residual term can be neglected, and we obtain

\[
\frac{d}{dt} \rho_t = \sum_{q=0}^{\infty} (-1)^q \dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(q)} (\rho_t) \right). \tag{30}
\]

This \(Q \to \infty\) formal treatment and the resulting equation of motion are not necessary in practice, however. See [6] for discussions on this issue.

### 2.2 Equation of motion

#### 2.2.1 Second-order equation of motion

To work out the second-order equation of motion for the average reduced density matrix, we first set \(Q = 1\) in Eq. (29):

\[
\frac{d}{dt} \rho_t = \dot{\mathcal{E}}_t (\rho_t) - \dot{\mathcal{E}}_t (\mathcal{E}_t (\rho_t)) + \dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(2)} (\rho_0) \right), \tag{31}
\]

\(^3\)Note that “\(P\)-th-order approximation” in this context has a slightly different meaning than “\(P\)-th-order approximation” in the case of a 1-dimensional series expansion (e.g. in [6]). In the case of a 2-dimensional series expansion, a “\(P\)-th-order approximation” includes all terms with total power of \(P\), namely \(\lambda^P, \lambda^{P-1} \delta, \lambda^{P-2} \delta^2\), and so on.

\(^4\)As a side note, our formalism would also allow for truncation of the 2-dimensional series to, say \(\mathcal{O} (\lambda^M \delta^N)\), for arbitrary \((M, N)\) of our choice. For example, if the system-bath interaction effect is more significant and the classical noise effect is comparatively less important, we may truncate to the 2-dimensional series to a higher order in \(\lambda^M\) and lower order in \(\delta^N\), that is, for \(M > N\).
so that the residual term \( \dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(2)} (\rho_0) \right) \sim \sum_{g=0}^{3} \mathcal{O} \left( \lambda^g \delta^{3-g} \right) \) is of third order significance and thus negligible in second-order approximation. Thus up to second order we have

\[
\frac{d}{dt} \rho_t = \dot{\mathcal{E}}_t (\rho_t) - \dot{\mathcal{E}}_t (\mathcal{E}_t (\rho_t))
\]

\[
= \lambda \dot{\mathcal{E}}_t (1,0)(\rho_t) + \delta \dot{\mathcal{E}}_t (0,1)(\rho_t) + \lambda^2 \dot{\mathcal{E}}_t (2,0)(\rho_t) + \lambda \delta \dot{\mathcal{E}}_t (1,1)(\rho_t) + \delta^2 \dot{\mathcal{E}}_t (0,2)(\rho_t) + ....
\]

\[
- \lambda \dot{\mathcal{E}}_t (1,0) \left( \lambda \mathcal{E}_t (1,0)(\rho_t) + \delta \mathcal{E}_t (0,1)(\rho_t) + .... \right)
\]

\[
- \delta \dot{\mathcal{E}}_t (0,1) \left( \lambda \mathcal{E}_t (1,0)(\rho_t) + \delta \mathcal{E}_t (0,1)(\rho_t) + .... \right) + ....
\]

(32)

where we have neglected all terms of third or higher orders.

Collecting like-order terms, we obtain the equation of motion for the “average reduced density matrix” \( \rho_t \) of the system (up to second order)

\[
\frac{d}{dt} \rho_t = \lambda \mathcal{L}_t (1,0)(\rho_t) + \delta \mathcal{L}_t (0,1)(\rho_t) + \lambda^2 \mathcal{L}_t (2,0)(\rho_t) + \delta^2 \mathcal{L}_t (0,2)(\rho_t) + \lambda \delta \mathcal{L}_t (1,1)(\rho_t),
\]

(33)

where

\[
\mathcal{L}_t (1,0)(\rho_t) = \dot{\mathcal{E}}_t (1,0)(\rho_t),
\]

(34)

\[
\mathcal{L}_t (0,1)(\rho_t) = \dot{\mathcal{E}}_t (0,1)(\rho_t),
\]

(35)

\[
\mathcal{L}_t (2,0)(\rho_t) = \dot{\mathcal{E}}_t (2,0)(\rho_t) - \dot{\mathcal{E}}_t (1,0) \left( \mathcal{E}_t (1,0)(\rho_t) \right),
\]

(36)

\[
\mathcal{L}_t (0,2)(\rho_t) = \dot{\mathcal{E}}_t (0,2)(\rho_t) - \dot{\mathcal{E}}_t (0,1) \left( \mathcal{E}_t (0,1)(\rho_t) \right),
\]

(37)

\[
\mathcal{L}_t (1,1)(\rho_t) = \dot{\mathcal{E}}_t (1,1)(\rho_t) - \dot{\mathcal{E}}_t (1,0) \left( \mathcal{E}_t (0,1)(\rho_t) \right) - \dot{\mathcal{E}}_t (0,1) \left( \mathcal{E}_t (1,0)(\rho_t) \right).
\]

(38)

with \( \mathcal{E}_t (M,N)(\rho_t) \) defined in Eq.(20).

Even with this abstract form of the second-order equation of motion, we can already see that \( \mathcal{L}_t (2,0)(\rho_t) \) is the familiar decoherence term due to system-bath interaction (i.e. “quantum noise”), \( \mathcal{L}_t (0,2)(\rho_t) \) is the familiar dephasing term due to classical noise, and \( \mathcal{L}_t (1,1)(\rho_t) \) supposedly represents the interference between quantum noise and classical noise on the system’s dynamics, if it does not vanish.

2.2.2 Second-order cross term

The lowest-order contribution of quantum noise to decoherence is \( \mathcal{L}_t (2,0)(\rho_t) \), and the lowest-order contribution of classical noise to dephasing is \( \mathcal{L}_t (0,2)(\rho_t) \), both of which are of second order in nature. To this same order, a cross-term \( \mathcal{L}_t (1,1)(\rho_t) \) as defined by Eq.(25) originates neither from quantum noise alone nor from classical noise alone, but supposedly from both.

To work out the details of \( \mathcal{L}_t (1,1)(\rho_t) \) as in Eq.(25), we first make use of Eq.(20) for relevant values of \( (m,n) \) to work out various individual terms:

\[
\mathcal{E}_t (1,0)(\rho) = \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} \text{Tr}_E \left( U_{1,0}^{(j)}(t,0) \rho \otimes \rho_{0E} + \rho \otimes \rho_{0E} U_{1,0}^{(j)\dagger}(t,0) \right)
\]
\[
\mathcal{L}_{(1,1)}(\rho_t) = \dot{\mathcal{E}}_{(1,1)}(\rho_t) - \dot{\mathcal{E}}_{(1,0)}(\mathcal{E}_{(1,0)}(\rho_t)) - \dot{\mathcal{E}}_{(0,1)}(\mathcal{E}_{(1,0)}(\rho_t)),
\]
which will show if and how quantum and classical noises interfere on the system's dynamics up to second order.
2.2.3 Second-order non-interference

Plugging Eqs. (39-44) into Eq. (38), we can now calculate the cross term $L_{t(1,1)}(\rho_t)$. We will work out the details term by term. The first term $\dot{E}_{t(1,1)}(\rho)$ is readily worked out in Eq. (44); the second term is

$$-\dot{E}_{t(1,0)}(\rho)$$

$$= -(i)Tr_E \left( [H_{SE}(t), \mathcal{E}_{t(0,1)}(\rho) \otimes \rho_{E0}] \right)$$

$$= iTr_E \left( \left[ H_{SE}(t), \left( -i \int_0^t dt' \left[ \frac{H_{SE}(t')}{H_{SE}(t')} \right] \right) \otimes \rho_{E0} \right] \right)$$

$$= \int_0^t dt' Tr_E \left( \left[ H_{SE}(t), \left( H_{SE}(t') \rho \otimes \rho_{E0} - \rho H_{SE}(t') \otimes \rho_{E0} \right) \right] \right)$$

$$= \int_0^t dt' Tr_E \{ H_{SE}(t) \left( \frac{H_{SE}(t')}{H_{SE}(t')} \otimes \mathbb{I}_E \right) (\rho \otimes \rho_{E0}) - H_{SE}(t) (\rho \otimes \rho_{E0}) \left( \frac{H_{SE}(t')}{H_{SE}(t')} \otimes \mathbb{I}_E \right) \}$$

$$= \left( \frac{H_{SE}(t')}{H_{SE}(t')} \otimes \mathbb{I}_E \right) (\rho \otimes \rho_{E0}) H_{SE}(t) + (\rho \otimes \rho_{E0}) \left( \frac{H_{SE}(t')}{H_{SE}(t')} \otimes \mathbb{I}_E \right) H_{SE}(t) \}; \quad (46)$$

and the third term is

$$-\dot{E}_{t(0,1)}(\rho)$$

$$= -(i) \left[ H_{SE}(t), \mathcal{E}_{t(0,1)}(\rho) \right]$$

$$= i \left[ \frac{H_{SE}(t)}{H_{SE}(t)}, \left( -i \int_0^t dt' Tr_E \left( [H_{SE}(t'), \rho \otimes \rho_{E0}] \right) \right) \right]$$

$$= \int_0^t dt' \left[ \frac{H_{SE}(t)}{H_{SE}(t)}, \left( Tr_E \left( H_{SE}(t') (\rho \otimes \rho_{E0}) - (\rho \otimes \rho_{E0}) H_{SE}(t') \right) \right) \right]$$

$$= \int_0^t dt' \left\{ H_{SE}(t) Tr_E \left( H_{SE}(t') (\rho \otimes \rho_{E0}) - (\rho \otimes \rho_{E0}) H_{SE}(t') \right) \right\}$$

$$= \int_0^t dt' Tr_E \left\{ \left( \frac{H_{SE}(t')}{H_{SE}(t)} \right) H_{SE}(t') (\rho \otimes \rho_{E0}) - \left( \frac{H_{SE}(t)}{H_{SE}(t)} \otimes \mathbb{I}_E \right) (\rho \otimes \rho_{E0}) H_{SE}(t') \right\}$$

$$= \left( \frac{H_{SE}(t)}{H_{SE}(t)} \otimes \mathbb{I}_E \right) (\rho \otimes \rho_{E0}) \left( \frac{H_{SE}(t)}{H_{SE}(t)} \otimes \mathbb{I}_E \right) H_{SE}(t') + (\rho \otimes \rho_{E0}) H_{SE}(t') \left( \frac{H_{SE}(t)}{H_{SE}(t)} \otimes \mathbb{I}_E \right). \quad (47)$$

Comparing the three terms Eqs. (44), (46), (47) of $L_{t(1,1)}(\rho_t)$, we can see that they cancel out, that is,

$$L_{t(1,1)}(\rho_t) = \dot{E}_{t(1,1)}(\rho_t) - \dot{E}_{t(1,0)}(\mathcal{E}_{t(0,1)}(\rho_t)) - \dot{E}_{t(0,1)}(\mathcal{E}_{t(1,0)}(\rho_t)) = 0. \quad (48)$$

The second-order cross term is thus shown to vanish identically, that is, free of conditions/assumptions. Physically, this means the effects of quantum noise and classical noise on the system’s average reduced dynamics do not interfere with each other up to second order.
Because both decoherence due to quantum noise and dephasing due to classical noise are primarily second-order effects, we may now say that quantum and classical noises do not interfere at their dominant order. This also implies that interference between quantum and classical noises, if there is any, should be perturbatively less significant than pure decoherence and pure dephasing effects. Thus this result may also be viewed as providing justification for the practice of treating the effects of quantum noise and classical noise in a simple additive manner (in the weak coupling limit where second-order effects dominate).

On the other hand, we don’t have a priori reasons to think that interference between quantum and classical noises must vanish in second order. At least it is not obvious from the definition of the second-order cross term $L_{t(11)}(\rho_t)$ Eq.(38) that it should vanish identically. This second-order non-interference is an interesting finding, the reason of which may be worth further investigation.

2.2.4 Higher-order dynamics

Besides second-order results, the master equation formalism can be used to systematically study a general system’s average reduced dynamics in higher orders. One can use Eqs.(29, 21, 20) to work out higher-order equations of motion mechanically.

In particular, our formalism can be used to work out the details of higher-order cross terms and determine if and how quantum and classical noises interfere in higher orders. For example, it can be shown that an interference term in third order $L_{t(21)}(\rho_t)$ is not identically vanishing, the calculation details of which can be found in Appendix A. This also provides evidence for interference between quantum and classical noises on a system’s dynamics.

3 Example: a Zeeman-splitted atom in stochastic B-field interacting with optical cavity

In the presence of an external magnetic field, an atom can experience the Zeeman effect, where the spacings between the splitted energy levels are linear on the B-field strength. Suppose only two energy levels of the atom are relevant for the purpose of our discussion. Now if the external B-field is stochastic instead of deterministic, then mathematically the two-level atom is subject to a stochastic Hamiltonian (i.e. with classical noise). This noise/stochasticity can result from the fact that experimentalists do not have perfect control over the external B-field. Suppose further that the aforementioned atom is also interacting with an optical cavity. In this case the atom is subject to both classical and quantum noises at the same time, thus the formalism developed herein can be used.
3.1 Problem description

Hamiltonian

The total Hamiltonian in Schrödinger picture is

\[ H_{\text{total}} = \frac{\omega_0}{2} \sigma_z + \frac{a(t)}{2} \sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k g_k \left( \sigma_+ b_k + \sigma_- b_k^\dagger \right), \quad (49) \]

where the first term is the deterministic and time-independent part of the two-level atom’s self-Hamiltonian, \( \omega_0 \) being the energy spacing between the two levels, the second term is the stochastic part of the atom’s self-Hamiltonian, \( a(t) \) being a stochastic process describing the fluctuating energy spacing between the two levels, \( \delta \) the third term is the self-Hamiltonian of the optical cavity, \( \omega_k \) being the frequency of each cavity mode, and the last term is the interaction between the atom and the optical cavity, \( g_k \) being the interaction strength. \[ \delta \]

Switching to the rotating frame generated by the deterministic self-Hamiltonian \( H_0 = \frac{\omega_0}{2} \sigma_z + \sum_k \omega_k b_k^\dagger b_k \), and treating the stochastic self-Hamiltonian and the atom-cavity interaction as a perturbing Hamiltonian \( H_{\text{perturb}} = \frac{a(t)}{2} \sigma_z + \sum_k g_k \left( \sigma_+ b_k + \sigma_- b_k^\dagger \right) \), we may obtain the interaction picture Hamiltonian \[ \delta \]

\[ H_{\text{int}} = e^{itH_0} H_{\text{perturb}} e^{-itH_0} = \frac{a(t)}{2} \sigma_z + \sum_k \left( g_k e^{-i(\omega_k-\omega_0)t} \sigma_+ b_k + g_k e^{i(\omega_k-\omega_0)t} \sigma_- b_k^\dagger \right). \quad (50) \]

Alternatively, to put it into a language conforming to our formalism as in Eq.(2),

\[ H_{\text{int}}^{(j)} = \lambda \sum_k \left( g_k e^{-i(\omega_k-\omega_0)t} \sigma_+ b_k + g_k e^{i(\omega_k-\omega_0)t} \sigma_- b_k^\dagger \right) + \frac{\delta}{2} a^{(j)}(t) \sigma_z \otimes \mathbb{I}_E, \quad (51) \]

where \( \lambda \) parametrizes the strength of the system-bath interaction and \( \delta \) parametrizes the strength of the stochastic self-Hamiltonian, two parameters around which the two-dimensional series can be expanded, and the index \( j \) refers to the \( j \)-th realization of the stochastic process. That is, we make the following identifications:

\[ H_{SE}(t) = \sum_k \left( g_k e^{-i(\omega_k-\omega_0)t} \sigma_+ b_k + g_k e^{i(\omega_k-\omega_0)t} \sigma_- b_k^\dagger \right), \quad (52) \]

\[ H_S^{(j)}(t) = \frac{a^{(j)}(t)}{2} \sigma_z. \quad (53) \]

Initial state of cavity

Suppose the optical cavity is initially in the thermal state, that is, \( \rho_{E0} = \frac{1}{Z} \exp(-\beta H_{\text{cavity}}) \), where \( Z = Tr_E(\exp(-\beta H_{\text{cavity}})) \) is the partition function.
and $\beta = 1/k_BT$ is the inverse temperature. In this example, we have
\[
\rho_{E0} = \prod_k \left( \frac{1}{Z_k} \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k} |m_k\rangle \langle m_k| \right) = \frac{1}{Z} \prod_k \left( \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k} |m_k\rangle \langle m_k| \right),
\]
(54)
where $Z_k = \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k}$ and $Z_k$, $\omega_k$ is the frequency of the $k$-th cavity mode, and $m_k$ is the number of photons in this cavity mode.

Stochastic property of atomic energy spacing

Suppose the stochastic energy spacing between the two atomic levels $a(t)$ is a real-valued Gaussian random process\(^5\) with a constant zero mean,\(^5\)
\[
a(t) = 0.
\]
(55)

3.2 Equation of motion

It can be shown that the equation of motion for this physical example up to second order is
\[
\frac{d}{dt}\rho_{t} = -i [H_{eff}(t), \rho_{t}] - D_R(t) (\sigma_-\rho_{t} + \rho_{t}\sigma_- - 2\sigma+\rho_{t}\sigma_-)
- D'_R(t) (\sigma_+\sigma_- + \rho_{t}\sigma_+\sigma_- - 2\sigma_-\rho_{t}\sigma_+) - 2D_C(t) (\rho_{t} - \sigma_z\rho_{t}\sigma_z),
\]
(56)
where the effective Hamiltonian is $H_{eff}(t) \equiv D_I(t)\sigma_-\sigma_+ - D'_I(t)\sigma_+\sigma_-$ and the prefactors $D_R(t)$, $D_I(t)$, $D'_R(t)$, $D'_I(t)$, and $D_C(t)$ are defined as
\[
D_R(t) \equiv \int_0^t dt' \sum_k |g_k|^2 \bar{N}_k \cos (\omega_{k0}(t - t')), (57)
\]
\[
D_I(t) \equiv \int_0^t dt' \sum_k |g_k|^2 \bar{N}_k \sin (\omega_{k0}(t - t')), (58)
\]
\[
D'_R(t) \equiv \int_0^t dt' \sum_k |g_k|^2 (\bar{N}_k + 1) \cos (\omega_{k0}(t - t')), (59)
\]
\[
D'_I(t) \equiv \int_0^t dt' \sum_k |g_k|^2 (\bar{N}_k + 1) \sin (\omega_{k0}(t - t')), (60)
\]
\[
D_C(t) \equiv \frac{1}{4} \int_0^t dt' a(t) a(t'). \]
(61)

The details of the calculation leading to Eq.(56) can be found in Appendix B.\(^5\)

\(^5\)Note that any non-zero mean can be incorporated into the deterministic part of the self-Hamiltonian and thus into $H_0$. 

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3.3 Decay of coherence

With the equation of motion Eq.(56), one can then study various aspects of the dynamics of a Zeeman splitted atom in the a stochastic B-field interacting with optical cavity. One aspect of particular interest to AMO and quantum information physics is how fast the coherence between atomic energy eigenlevels decays over time [7], that is, the decay rate of off-diagonal element(s) of the atomic density matrix in energy eigenbasis (e.g. $\rho_{01}(t)$).

Total decay rate

To find the decay rate of the off-diagonal element $\rho_{01}(t)$, we sandwich both sides of Eq.(56) with $\langle 0| \ldots |1 \rangle$, with the convention that $|0\rangle$ is spin-up and $|1\rangle$ is spin-down. With $\sigma_+|0\rangle = 0$, $\sigma_+|1\rangle = 2|0\rangle$, $\sigma_-|0\rangle = 2|1\rangle$, and $\sigma_-|1\rangle = 0$, it can be shown from Eqs.(127, 133) that

\begin{align}
\langle 0|L_{t (2,0)}(\rho)|1 \rangle &= 4i (D_I(t) + D'_I(t)) \rho_{01} - 4 (D_R(t) + D'_R(t)) \rho_{01}, \\
\langle 0|L_{t (0,2)}(\rho)|1 \rangle &= -4 D_C(t) \rho_{01},
\end{align}

with which we can proceed to have

\begin{align}
\frac{d}{dt} \rho_{01}(t) &= \langle 0| \frac{d}{dt} \rho_{t}|1 \rangle = \langle 0|L_{t (2,0)}(\rho_t)|1 \rangle + \langle 0|L_{t (0,2)}(\rho_t)|1 \rangle, \\
\Rightarrow \quad \frac{d}{dt} \rho_{01}(t) &= i (4 (D_I(t) + D'_I(t))) \rho_{01}(t) - 4 (D_R(t) + D'_R(t) + D_C(t)) \rho_{01}(t).
\end{align}

We see that the evolution of $\rho_{01}(t)$ is governed by a simple linear ordinary differential equation Eq.(65), that is, the evolution of $\rho_{01}(t)$ is decoupled from the other density matrix elements. The linear ordinary differential equation Eq.(65) is first order in time, which means the rate of change in $\rho_{01}(t)$ is simply given by the right-hand side of the equation. The first term with a pure imaginary prefactor results in a phase shift of $\rho_{01}(t)$, while the second term with a real prefactor results in a decay in the amplitude of $\rho_{01}(t)$. The total decay rate of the coherence $\rho_{01}(t)$ in the presence of both the optical cavity and the stochastic B-field is thus

\begin{equation}
D_{total}(t) = 4 (D_R(t) + D'_R(t) + D_C(t)).
\end{equation}

Decay rate in stochastic B-field alone

Suppose the two-level atom is subject to an external stochastic B-field only, that is, in the absence of the optical cavity. This case of a classical noise scenario is treated systematically in [5]. To find the decay rate of $\rho_{01}(t)$ in this case, we follow the treatment of a single real Gaussian random process in [5]. Quoting the results therein, for a stochastic Hamiltonian of the form

\begin{equation}
H_S(t) = a(t) \sigma_z^2,
\end{equation}

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where \( a(t) \) is a real Gaussian random process, the equation of motion is (up to second order)

\[
\frac{d}{dt} \rho_t = -i \left[ a(t) \left[ \frac{\sigma_z}{2}, \rho_t \right] + \left( a(t) \int_0^t dt' a(t') - \int_0^t dt' a(t) a(t') \right) \left[ \frac{\sigma_z}{2}, \rho_t \right] \right].
\]

(68)

Now that we assume a zero mean for the Gaussian random process in our example, that is, \( a(t) = 0 \), we are left with a simplified equation of motion

\[
\frac{d}{dt} \rho_t = -\int_0^t dt' a(t) a(t') \left[ \frac{\sigma_z}{2}, \rho_t \right] = -2D_C(t) (\rho_t - \sigma_z \rho_t \sigma_z),
\]

(69)

where we have used the same definition Eq.(61) for \( D_C(t) \) as in previous parts of this paper.

Sandwiching both sides of Eq.(69) with \( \langle 0 | . . . | 1 \rangle \), we obtain an ordinary differential equation

\[
\frac{d}{dt} \rho_{01}(t) = -4D_C(t) \rho_{01}(t),
\]

(70)

which shows the decay rate of the coherence \( \rho_{01}(t) \) in the stochastic B-field alone is

\[
D_{B-field}(t) = 4D_C(t).
\]

(71)

Decay rate in optical cavity alone

Suppose the two-level atom is interacting with an optical cavity only, that is, in the absence of the stochastic B-field. This case of a system-bath interaction scenario is treated systematically in [6]. The Hamiltonian for this case is

\[
H_{SE}(t) = \sigma_+ \otimes \left( \sum_k g_k e^{-i\omega_k t} b_k \right) + \sigma_- \otimes \left( \sum_k g_k e^{i\omega_k t} b_k^\dagger \right),
\]

(72)

and the optical cavity is assumed to be initially thermal,

\[
\rho_{E0} = \prod_k \left( \frac{1}{Z_k} \sum_{m_k=0}^{\infty} e^{-m_k \beta \omega_k} |m_k\rangle \langle m_k| \right).
\]

(73)

Using the results in [6] towards Eqs.(72, 73), we can show that the second-order equation of motion for an atom interacting with an optical cavity is

\[
\frac{d}{dt} \rho_t = -i [H_{eff}(t), \rho_t] - D_R(t) (\sigma_- \sigma_+ \rho_t + \rho_t \sigma_- \sigma_+) - 2\sigma_+ \rho_t \sigma_-
\]

\[
- D'_R(t) (\sigma_+ \sigma_- \rho_t + \rho_t \sigma_+ \sigma_-) - 2\sigma_- \rho_t \sigma_+),
\]

(74)

This derivation is similar to one in Appendix B, namely Eqs.(102–128).
where $H_{\text{eff}}(t) \equiv D_I(t)\sigma_- \sigma_+ - D'_I(t)\sigma_+ \sigma_- \equiv D'_R(t), D'_R(t), D'_R(t)$, and $D'_I(t)$ as in previous parts of this paper.

Sandwiching both sides of Eq.(74) with $\langle 0 | \ldots | 1 \rangle$, we obtain an ordinary differential equation for $\rho_{01}(t)$:

$$\frac{d}{dt}\rho_{01}(t) = i\left(4 (D_I(t) + D'_I(t))\rho_{01}(t) - 4 (D_R(t) + D'_R(t))\rho_{01}(t)\right),$$

which shows the decay rate of the coherence $\rho_{01}(t)$ due to atom-cavity interaction alone is

$$D_{\text{cavity}}(t) = 4 (D_R(t) + D'_R(t)).$$

Note that the first term of Eq.(75) with a pure imaginary prefactor only results in a phase shift of $\rho_{01}(t)$ but has no effect on its amplitude.

**Summary**

Comparing Eq.(66) with Eq.(71) and Eq.(76), we see that

$$D_{\text{total}}(t) = D_{\text{cavity}}(t) + D_{B-\text{field}}(t),$$

that is, the total decay rate of the coherence between atomic energy eigenlevels $\rho_{01}(t)$ in the presence of both an optical cavity and a stochastic B-field is a simple sum of the decay rate due to the optical cavity alone and that due to the stochastic B-field alone. In other words, up to second order (i.e. the leading order of decoherence), the optical cavity and the stochastic B-field neither enhance nor undermine each other in terms of decoherence effect. In terms of physical parameters, the total decay rate is

$$D_{\text{total}}(t) = 4 \int_0^t dt' \sum_k |g_k|^2 \bar{N}_k \cos (\omega_k(t - t'))$$

$$+ 4 \int_0^t dt' \sum_k |g_k|^2 (\bar{N}_k + 1) \cos (\omega_k(t - t')) + \int_0^t dt' a(t)a(t').$$

One may further ask if and how the optical cavity and the stochastic B-field would interfere at higher order(s) in terms of the effects on the system’s dynamics. Higher-order equation(s) of motion would be needed for such investigation, an example of which can be found in Appendix A.

**4 Conclusion**

We use a two-dimensional series expansion method to construct a new master equation formalism, which can properly describe the effects of both quantum noise and classical noise on a system’s ensemble-averaged reduced dynamics in
a unified and consistent way. Such a unified treatment is of theoretical importance, because conceptually speaking quantum noise and classical noise are of different natures. Regardless of empirical implications, it is important to have a theory framework that can properly deal with the dynamical effects of both. In particular, this formalism can be used to determine if there is interference between quantum and classical noises on the system’s dynamics and will be able to capture and describe such interference if there is any (in a perturbative manner). Interestingly, we find that second-order interference between quantum and classical noises vanishes identically. This finding may justify simple additive treatments of quantum and classical noises, especially in weak coupling and/or short time regimes where second-order effects dominate. We study the dynamics of a Zeeman-splitted atom in a stochastic B-field interacting with an optical cavity and calculate the decay rate of coherence between the atom’s energy levels, which is (up to second order) a simple sum of the decay rate due to the stochastic B-field alone and that due to the atom-cavity interaction alone. Further details of higher-order dynamics under quantum and classical noises can be worked out systematically using Eqs.(29, 21, 20), and the question of higher-order interference can be further investigated within our formalism. In the future, we can apply this formalism to more realistic experimental setups where the effect of quantum noise and that of classical noise are of comparable (similar) significance and where higher-order contributions matter.

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Appendix A: Higher-order dynamics

A.1 Third-order equation of motion

We will examine higher-order terms in the general master equation to find out about interference. To work out the third-order terms, we first set $Q = 2$ in Eq.(29) so that we have

$$\frac{d}{dt} \rho_t = \dot{\mathcal{E}}_t (\rho_t) - \dot{\mathcal{E}}_t (\mathcal{E}_t (\rho_t)) + \dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(2)} (\rho_t) \right) - \dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(3)} (\rho_0) \right).$$  \hspace{1cm} (79)

Because the last term is of fourth order, that is, $-\dot{\mathcal{E}}_t \left( \mathcal{E}_t^{(3)} (\rho_0) \right) \sim \sum_{q=0}^{4} O (\lambda^q \delta^{4-q})$, it can be neglected for our study of third order effects, and thus we are left with

$$\frac{d}{dt} \rho_t = \dot{\mathcal{E}}_t (\rho_t) - \dot{\mathcal{E}}_t (\mathcal{E}_t (\rho_t)) + \dot{\mathcal{E}}_t \left( \mathcal{E}_t (\mathcal{E}_t (\rho_t)) \right).$$  \hspace{1cm} (80)

Let’s examine the right-hand side of Eq.(80) term by term to work out third-order contributions. For the first term of Eq.(80), it can be shown that the third
Third-order interference

\[ E_t (\rho_t) = 1st \text{ order term} + 2nd \text{ order term} \]
\[ + \lambda^3 \dot{E}_t (2,0) (\rho_t) + \lambda^2 \dot{E}_t (1,2) (\rho_t) + \lambda \delta^2 \dot{E}_t (1,1) (\rho_t) + \delta^3 \dot{E}_t (0,3) (\rho_t) \]
\[ + \text{higher order terms.} \quad (81) \]

For the second term of Eq. (81), it can be shown that the third order contributions are

\[ -\dot{E}_t (E_t (\rho_t)) = 2nd \text{ order term} \]
\[ + \lambda^3 \left( -\dot{E}_t (1,0) (E_t (2,0) (\rho_t)) - \dot{E}_t (2,0) (E_t (1,0) (\rho_t)) \right) \]
\[ + \lambda^2 \delta \left( -\dot{E}_t (1,0) (E_t (1,1) (\rho_t)) - \dot{E}_t (0,1) (E_t (2,0) (\rho_t)) \right) \]
\[ - \dot{E}_t (2,0) (E_t (0,1) (\rho_t)) - \dot{E}_t (1,1) (E_t (1,0) (\rho_t)) \]
\[ + \lambda \delta^2 \left( -\dot{E}_t (0,1) (E_t (0,1) (\rho_t)) - \dot{E}_t (1,0) (E_t (0,2) (\rho_t)) \right) \]
\[ - \dot{E}_t (0,2) (E_t (0,1) (\rho_t)) - \dot{E}_t (1,1) (E_t (0,1) (\rho_t)) \]
\[ + \delta^3 \left( -\dot{E}_t (0,1) (E_t (0,1) (\rho_t)) - \dot{E}_t (0,1) (E_t (0,2) (\rho_t)) \right) \]
\[ + \text{higher order terms.} \quad (82) \]

For the third term of Eq. (81), it can be shown that the third order contributions are

\[ \dot{E}_t (E_t (\rho_t)) = \lambda^3 \dot{E}_t (1,0) (E_t (1,0) (\rho_t)) \]
\[ + \lambda^2 \delta \left( \dot{E}_t (1,0) (E_t (1,0) (\rho_t)) + \dot{E}_t (0,1) (E_t (1,0) (\rho_t)) \right) \]
\[ + \lambda \delta^2 \left( \dot{E}_t (0,1) (E_t (0,1) (\rho_t)) + \dot{E}_t (1,0) (E_t (0,1) (\rho_t)) \right) \]
\[ + \delta^3 \dot{E}_t (0,1) (E_t (0,1) (\rho_t)) \]
\[ + \text{higher order terms.} \quad (83) \]

A.2 Third-order interference \( L_t (2,1) (\rho_t) \)

Collecting all terms with prefactor \( \lambda^2 \delta \) in Eqs. (81, 82, 83), we have the expression for \( L_t (2,1) (\rho_t) \) in the Master equation:

\[ L_t (2,1) (\rho_t) = \dot{E}_t (2,1) (\rho_t) - \dot{E}_t (1,0) (E_t (1,1) (\rho_t)) - \dot{E}_t (0,1) (E_t (2,0) (\rho_t)) \]
\[ - \dot{E}_t (2,0) (E_t (0,1) (\rho_t)) - \dot{E}_t (1,1) (E_t (1,0) (\rho_t)) + \dot{E}_t (1,0) (E_t (1,0) (E_t (0,1) (\rho_t))) \]
\[ + \dot{E}_t (1,0) (E_t (0,1) (E_t (1,0) (\rho_t)) + \dot{E}_t (0,1) (E_t (1,0) (E_t (1,0) (\rho_t))). \quad (84) \]

Recall that in Eq. (48) we have shown that for an arbitrary operator \( \rho \)

\[ \dot{E}_t (1,1) (\rho) - \dot{E}_t (1,0) (E_t (0,1) (\rho)) - \dot{E}_t (0,1) (E_t (1,0) (\rho)) = 0. \quad (85) \]
Let this arbitrary operator be \( \mathcal{E}_t(1,0)(\rho_t) \), in which case we have three terms in Eq.\,(100) cancelling out:

\[
\dot{\mathcal{E}}_t(1,1)(\mathcal{E}_t(1,0)(\rho_t))+\dot{\mathcal{E}}_t(1,0)(\mathcal{E}_t(1,1)(\rho_t))=0.
\]  

(86)

We are thus left with

\[
\mathcal{L}_t(2,1)(\rho_t) = \dot{\mathcal{E}}_t(2,1)(\rho_t) - \dot{\mathcal{E}}_t(2,0)(\mathcal{E}_t(1,1)(\rho_t)) - \dot{\mathcal{E}}_t(0,1)(\mathcal{E}_t(1,0)(\rho_t)) + \dot{\mathcal{E}}_t(1,0)(\mathcal{E}_t(1,0)(\rho_t)).
\]  

(87)

Now we can use definition of \( \mathcal{E}_t(M,N)(\rho_t) \) in Eq.\,(20) to evaluate \( \mathcal{L}_t(2,1)(\rho_t) \).

To facilitate further calculations, let's introduce

\[
\triangle^{(j)}(t) = U_{1,1}^{(j)}(t,0) - U_{1,0}(t,0)U_{0,1}^{(j)}(t,0),
\]  

(88)

noting that we have dropped the superscript \( j \) for \( U_{1,0}^{(j)}(t,0) \) because it does not depend on the stochastic process. It can be shown that

\[
\triangle^{(j)}(t) = -(i) \int_0^t dt' \left[ H_S^{(j)}(t') \otimes I_E, U_{1,0}(t',0) \right] = -\int_0^t dt' \int_0^{t'} dt'' \left[ H_S^{(j)}(t') \otimes I_E, H_{SE}(t'') \right] \Rightarrow \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^R \triangle^{(j)}(t) = -\int_0^t dt' \int_0^{t'} dt'' \left[ H_S(t'') \otimes I_E, H_{SE}(t'') \right] = \overline{\triangle}(t).
\]  

(89)

(90)

With this short-hand notation, it can be shown that the \( \mathcal{L}_t(2,1)(\rho) \) interference term is\(^7\)

\[
\mathcal{L}_t(2,1)(\rho) = -(i) \text{ Tr}_E \left[ \left[ H_{SE}(t), \left[ \overline{\triangle}(t), \rho \otimes \rho_{E0} \right] - \text{ Tr}_E \left( \left[ \overline{\triangle}(t), \rho \otimes \rho_{E0} \right] \otimes \rho_{E0} \right) \right] \right] = i \int_0^t dt' \int_0^{t'} dt'' \text{ Tr}_E \left[ \left[ H_{SE}(t), \left[ H_S(t'') \otimes I_E, H_{SE}(t'') \right], \rho \otimes \rho_{E0} \right] - \text{ Tr}_E \left( \left[ H_S(t'') \otimes I_E, H_{SE}(t'') \right], \rho \otimes \rho_{E0} \right) \otimes \rho_{E0} \right] \right].
\]

(91)

In general, the system-bath interaction can be written as \( H_{SE}(t) = \sum_n S_n(t) \otimes E_n(t) \), which can be plugged into Eq.\,(91) to obtain

\[
\mathcal{L}_t(2,1)(\rho) = i \int_0^t dt' \int_0^{t'} dt'' \sum_m \sum_n \left( \mathcal{C}_{mn}(t,t'') \left[ S_m(t), \left[ H_S(t''), S_n(t'') \right] \rho \right] - \mathcal{C}_{nm}(t'',t) \left[ S_n(t), \rho \left[ H_S(t''), S_n(t'') \right] \right] \right),
\]

(92)

\(^7\)The details of the calculation are mechanical and lengthy, which we will not show here.
where
\[ \mathcal{L}_{jk}(t,t') \equiv \text{Tr}_E \left( E_j(t)E_k(t')\rho_{E0} \right) - \text{Tr}_E \left( E_j(t)\rho_{E0} \right) \text{Tr}_E \left( E_k(t')\rho_{E0} \right). \] (93)

With Eq. (91) or Eq. (92), we see that \( \mathcal{L}_{t(2,1)}(\rho) \) is not identically vanishing. Thus interference between quantum and classical noises exists in third order.

Appendix B: Equation of motion for a Zeeman-split atom subject to stochastic B-field and optical cavity

To work out the equation of motion for the physical example up to second order, we make use of Eqs. (34-38) to work out each term in Eq. (33).

B.1 \( \mathcal{L}_{t(1,0)}(\rho) \)

The first-order term due to atom-cavity interaction alone is

\[ \mathcal{L}_{t(1,0)}(\rho) = \dot{\mathcal{E}}_{t(1,0)}(\rho) = \dot{\mathcal{E}}_{t(1,0)}(\rho) = 0. \] (97)
B.2 $\mathcal{L}_{t(0,1)}(\rho)$

The first-order term due to stochasticity of the external B-field alone is

$$\mathcal{L}_{t(0,1)}(\rho) = \dot{\mathcal{E}}_{t(0,1)}(\rho)$$
$$= (-i) \left[ H_{s}(t), \rho \right]$$
$$= (-i) \frac{a(t)}{2} [\sigma_{z}, \rho],$$

(98)

where $\overline{a(t)} = 0$ for all time. Therefore, the stochastic part of the external B-field does not affect the system’s dynamics in first order,

$$\mathcal{L}_{t(0,1)}(\rho) = \dot{\mathcal{E}}_{t(0,1)}(\rho) = 0. \quad (99)$$

B.3 $\mathcal{L}_{t(2,0)}(\rho)$

The second-order term due to atom-cavity interaction alone is

$$\mathcal{L}_{t(2,0)}(\rho) = \dot{\mathcal{E}}_{t(2,0)}(\rho) - \dot{\mathcal{E}}_{t(1,0)}(\mathcal{E}_{t(1,0)}(\rho))$$

$$= \dot{\mathcal{E}}_{t(2,0)}(\rho)$$
$$= \frac{\partial}{\partial t} \sum_{m=0}^{2} \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} \text{Tr}_{E} \left( U_{2-m,0}(t,0) \rho \otimes \rho_{E0} U_{m,0}^{\dagger}(t,0) \right)$$
$$= \frac{\partial}{\partial t} \text{Tr}_{E} \left( U_{2,0}(t,0) \rho \otimes \rho_{E0} + U_{1,0}(t,0) \rho \otimes \rho_{E0} U_{1,0}^{\dagger}(t,0) + \rho \otimes \rho_{E0} U_{2,0}^{\dagger}(t,0) \right)$$
$$= \text{Tr}_{E} \left( \dot{U}_{2,0}(t,0) \rho \otimes \rho_{E0} + \dot{U}_{1,0}(t,0) \rho \otimes \rho_{E0} U_{1,0}^{\dagger}(t,0) \right.$$ \[\left. + U_{1,0}(t,0) \rho \otimes \rho_{E0} \dot{U}_{1,0}^{\dagger}(t,0) + \rho \otimes \rho_{E0} \dot{U}_{2,0}^{\dagger}(t,0) \right), \quad (100)$$

where in the second equality we have made use of the vanishing of $\dot{\mathcal{E}}_{t(1,0)}(\cdots)$, as is already shown in Eq. (77), and in the fourth equality we drop the stochastic averaging because we are dealing with a deterministic evolution in this case, as is obvious from Eqs. (12-13). Plugging Eqs. (12-13) into Eq. (100), we can carry out further calculation

$$\mathcal{L}_{t(2,0)}(\rho) = \dot{\mathcal{E}}_{t(2,0)}(\rho)$$
$$= \text{Tr}_{E} \left\{ - \int_{0}^{t} dt' H_{SE}(t) H_{SE}(t') \rho \otimes \rho_{E0} - \int_{0}^{t} dt' \rho \otimes \rho_{E0} H_{SE}(t') H_{SE}(t) \right.$$
$$+ (-i) H_{SE}(t) \rho \otimes \rho_{E0}(i) \int_{0}^{t} dt' H_{SE}(t') + (-i) \int_{0}^{t} dt' H_{SE}(t') \rho \otimes \rho_{E0}(i) H_{SE}(t) \}$$
$$= - \int_{0}^{t} dt \text{Tr}_{E} \left\{ H_{SE}(t) H_{SE}(t') \rho \otimes \rho_{E0} + \rho \otimes \rho_{E0} H_{SE}(t') H_{SE}(t) \right.$$
$$- H_{SE}(t) \rho \otimes \rho_{E0} H_{SE}(t') - H_{SE}(t') \rho \otimes \rho_{E0} H_{SE}(t) \} \quad (101)$$

We see that $\mathcal{L}_{t(2,0)}(\rho)$ agrees with the second-order term in the case of mere quantum noise (i.e. without classical noise) as in [6], as it should.
Following the treatment of [6], we use the following notation to facilitate further derivation

\[ H_{SE}(t) = S_1 \otimes E_1(t) + S_2 \otimes E_2(t), \]  
\[ S_1 = \sigma_+ , \]  
\[ S_2 = \sigma_- , \]  
\[ E_1(t) = \sum_k g_k e^{-i\omega_k t} b_k , \]  
\[ E_2(t) = \sum_k g_k e^{i\omega_k t} b_k^\dagger , \]

in which case the second-order term can be re-written as

\[ \mathcal{L}_{1(2,0)}(\rho) = -\sum_{m=1,2} \sum_{n=1,2} \int^t dt' \langle C_{mn}(t,t') [S_m, S_n \rho] - C_{nm}(t',t) [S_m, \rho S_n] \rangle , \]

\[ C_{mn}(t,t') = Tr_E (E_m(t) E_n(t') \rho_{E0}) . \]

We will evaluate the four terms for \( m, n = 1, 2 \) respectively in the following. For the term \( m = n = 1 \), the first prefactor in Eq. (108) is

\[ C_{11}(t,t') = Tr_E (E_1(t) E_1(t') \rho_{E0}) \]

\[ = \sum_k \sum_{k'} g_k g_{k'} e^{-i\omega_k t} e^{-i\omega_{k'} t'} Tr_E (b_k b_{k'} \rho_{E0}) . \]

It is easy to see that the factors \( Tr_E (b_k b_{k'} \rho_{E0}) \) vanish for the thermal state \( \rho_{E0} \). To work it out in detail, for \( k \neq k' \):

\[ Tr_E (b_k b_{k'} \rho_{E0}) = Tr_E \left( b_k b_{k'} \prod_K \left( \frac{1}{Z_K} \sum_{m_K=0}^{\infty} e^{-m_K \beta \omega_K} |m_K\rangle \langle m_K| \right) \right) \]

\[ = \frac{1}{Z_k Z_{k'}} Tr_{E_k} \left( b_k \sum_{m_k=0}^{\infty} e^{-m_k \beta \omega_k} |m_k\rangle \langle m_k| \right) \]

\[ = \frac{1}{Z_k Z_{k'}} \left( \sum_{m_k=0}^{\infty} e^{-m_k \beta \omega_k} \langle m_k| b_k |m_k\rangle \right) \]

\[ = \left( \sum_{m_{k'}=0}^{\infty} e^{-m_{k'} \beta \omega_{k'}} \langle m_{k'}| b_{k'} |m_{k'}\rangle \right) = 0 , \]

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because all the factors $\langle m_k|b_k|m_k \rangle \propto \langle m_k + 1|m_k \rangle = 0$ vanish; for $k = k'$:

$$
Tr_E(b_k b_k^\dagger \rho_{E0}) = \frac{1}{Z_k} Tr_{E_k} \left( b_k b_k \sum_{m_k=0}^{\infty} e^{-m_k \beta \omega_k} \langle m_k | m_k \rangle \right)
= \frac{1}{Z_k} \left( \sum_{m_k=0}^{\infty} e^{-m_k \beta \omega_k} \langle m_k | b_k b_k | m_k \rangle \right)
= 0,
$$

(112)

because all the factors $\langle m_k|b_k b_k|m_k \rangle \propto \langle m_k + 2|m_k \rangle = 0$ vanish. Plugging Eqs. (111,112) back into Eq. (110), we find that the first prefactor in Eq. (110) vanishes, $C_{11}(t,t') = 0$. By the same token, we can show that the second prefactor in Eq. (108) also vanishes, $C_{11}(t',t) = 0$. Therefore, the term for $m = n = 1$ vanishes.

Similarly, it can be shown that the term for $m = n = 2$ vanishes as well, essentially because the prefactors $C_{22}(t,t') = C_{22}(t', t) = 0$ vanish, which in turn is due to the vanishing of the factor $Tr_E(b_k b_k^\dagger \rho_{E0}) = 0$.

Thus we only have to consider the cross terms for $(m = 1, n = 2)$ and $(m = 2, n = 1)$ in Eq. (108), which now reads

$$
\mathcal{L}_{1(2,0)}(\rho) = -\int_0^t dt' \left[ C_{12}(t,t') [S_1, S_2 \rho] - C_{21}(t',t) [S_1, \rho S_2] \right. \\
+ C_{21}(t,t') [S_2, S_1 \rho] - C_{12}(t',t) [S_2, \rho S_1] \bigg]
= -\int_0^t dt' \left[ C_{12}(t,t') (\sigma_+ \sigma_- - \sigma_- \sigma_+) + C_{21}(t',t) (\rho \sigma_+ - \sigma_+ \rho \sigma_-) + C_{21}(t,t') (\rho \sigma_+ - \sigma_+ \rho \sigma_-) \right].
$$

(113)

where in the second equality we have rearrange the order of the terms. The prefactors are to be evaluated as follows. The first prefactor is

$$
C_{12}(t,t') = Tr_E(E_1(t) E_2(t') \rho_{E0})
= \sum_k \sum_{k'} g_k g_{k'} e^{-i \omega_{k'} t} e^{i \omega_k t'} Tr_{E_k} \left( b_k b_k^\dagger \rho_{E0} \right),
$$

(114)

where for $k \neq k'$:

$$
Tr_{E_k} \left( b_k b_k^\dagger \rho_{E0} \right) = \frac{1}{Z_k Z_{k'}} Tr_{E_k} \left( b_k \sum_{m_k=0}^{\infty} e^{-m_k \beta \omega_k} \langle m_k | m_k \rangle \right)
= \frac{1}{Z_k Z_{k'}} \left( \sum_{m_k=0}^{\infty} e^{-m_k \beta \omega_k} \langle m_k | b_k | m_k \rangle \right)
$$
Similarly, the second prefactor is 

\[
\left( \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k} \langle m_k | b_k^\dagger | m_{k'} \rangle \right) = 0, \quad (115)
\]

and for \( k = k' \):

\[
\text{Tr}_E \left( b_k^\dagger b_k \rho_{E0} \right) = \frac{1}{Z_k} \langle t | m_{k'} \rangle \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k} | m_k \rangle \langle m_k | b_k^\dagger b_k | m_{k'} \rangle
\]

\[
= \frac{1}{Z_k} \langle t | m_{k'} \rangle \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k} \langle m_k | b_k b_k^\dagger | m_{k'} \rangle
\]

\[
= \frac{1}{Z_k} \langle t | m_{k'} \rangle \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k} \langle m_k | \left( b_k b_k^\dagger + \hat{1} \right) | m_{k'} \rangle
\]

\[
= \frac{1}{Z_k} \langle t | m_{k'} \rangle \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k} \langle m_k | b_k b_k^\dagger | m_k \rangle + \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k} \langle m_k | \hat{1} | m_{k} \rangle
\]

\[
= \bar{N}_k + 1, \quad (116)
\]

with the average/expected occupation number in the \( k \)-th mode of the bath being

\[
\bar{N}_k \equiv \text{Tr}_E \left( b_k^\dagger b_k \rho_{E0} \right) = \frac{1}{Z_k} \sum_{m_k=0}^{\infty} e^{-m_k\beta\omega_k} \langle m_k | b_k b_k^\dagger | m_k \rangle; \quad (117)
\]

therefore we have

\[
C_{12}(t, t') = \sum_k \sum_{k'} g_k g_{k'} e^{-i\omega_{k0}t'} e^{i\omega_{k0}t} \text{Tr}_E \left( b_k^\dagger b_k \rho_{E0} \right)
\]

\[
= \sum_k |g_k|^2 e^{-i\omega_{k0}(t-t')} \text{Tr}_E \left( b_k^\dagger b_k \rho_{E0} \right)
\]

\[
= \sum_k |g_k|^2 \left( \bar{N}_k + 1 \right) e^{-i\omega_{k0}(t-t')}
\]

\[
= \sum_k |g_k|^2 \left( \bar{N}_k + 1 \right) \left( \cos \left( \omega_{k0}(t-t') - i \sin \left( \omega_{k0}(t-t') \right) \right) \right) \quad (118)
\]

Similarly, the second prefactor is

\[
C_{12}(t', t) = \text{Tr}_E \left( E_1(t') E_2(t) \rho_{E0} \right)
\]

\[
= \sum_k \sum_{k'} g_k g_{k'} e^{-i\omega_{k0}t} e^{i\omega_{k0}t'} \text{Tr}_E \left( b_k^\dagger b_k \rho_{E0} \right)
\]

\[
= \sum_k |g_k|^2 e^{i\omega_{k0}(t-t')} \text{Tr}_E \left( b_k^\dagger b_k \rho_{E0} \right)
\]

\[
= \sum_k |g_k|^2 \left( \bar{N}_k + 1 \right) e^{i\omega_{k0}(t-t')}
\]

\[
= \sum_k |g_k|^2 \left( \bar{N}_k + 1 \right) \left( \cos \left( \omega_{k0}(t-t') + i \sin \left( \omega_{k0}(t-t') \right) \right) \right) \quad (119)
\]
the third prefactor is
\[
C_{21}(t, t') = Tr_E (E_2(t) E_1(t') \rho_{E0})
\]
\[
= \sum_k \sum_{k'} g_k g_{k'} e^{i\omega_{k0} t} e^{-i\omega_{k'0} t'} Tr_E \left( b_k^\dagger b_{k'} \rho_{E0} \right)
\]
\[
= \sum_k |g_k|^2 e^{i\omega_{k0} (t-t')} Tr_E \left( b_k^\dagger b_k \rho_{E0} \right)
\]
\[
= \sum_k |g_k|^2 \bar{N}_k \left( \cos (\omega_{k0} (t-t')) + i \sin (\omega_{k0} (t-t')) \right); \quad (120)
\]

and the fourth prefactor is
\[
C_{21}(t', t) = Tr_E (E_2(t') E_1(t) \rho_{E0})
\]
\[
= \sum_k \sum_{k'} g_k g_{k'} e^{i\omega_{k0} t'} e^{-i\omega_{k'0} t} Tr_E \left( b_k^\dagger b_{k'} \rho_{E0} \right)
\]
\[
= \sum_k |g_k|^2 e^{-i\omega_{k0} (t-t')} Tr_E \left( b_k^\dagger b_k \rho_{E0} \right)
\]
\[
= \sum_k |g_k|^2 \bar{N}_k \left( \cos (\omega_{k0} (t-t')) - i \sin (\omega_{k0} (t-t')) \right). \quad (121)
\]

For convenience, let’s introduce the following definitions:
\[
D_R(t) \equiv \int_0^t dt' \sum_k |g_k|^2 \bar{N}_k \cos (\omega_{k0} (t-t')), \quad (122)
\]
\[
D_I(t) \equiv \int_0^t dt' \sum_k |g_k|^2 \bar{N}_k \sin (\omega_{k0} (t-t')), \quad (123)
\]
\[
D'_R(t) \equiv \int_0^t dt' \sum_k |g_k|^2 (\bar{N}_k + 1) \cos (\omega_{k0} (t-t')), \quad (124)
\]
\[
D'_I(t) \equiv \int_0^t dt' \sum_k |g_k|^2 (\bar{N}_k + 1) \sin (\omega_{k0} (t-t')), \quad (125)
\]

as a result of which Eq. (113) can be re-expressed as
\[
\mathcal{L}_{t, (2,0)}(\rho) = -(D'_R(t) - iD'_I(t)) (\sigma_+ \sigma - \sigma_- \sigma_+ - \sigma_+ \sigma_-)
\]
\[
+ (D'_R(t) + iD'_I(t)) (\rho \sigma_+ \sigma_- - \sigma_- \rho \sigma_+)
\]
\[
+ (D_R(t) + iD_I(t)) (\sigma_- \sigma_+ - \sigma_- \sigma_+)
\]
\[
+ (D_R(t) - iD_I(t)) (\rho \sigma_- \sigma_+ - \sigma_- \rho \sigma_+)
\]
\[
= -D_R(t) (\sigma_+ \sigma + \rho \sigma_- \sigma_+ - 2\sigma_+ \rho \sigma_-)
\]
\[
- D'_R(t) (\sigma_- \sigma_+ + \rho \sigma_- \sigma_+ - 2\sigma_- \rho \sigma_+)
\]
\[
- i (D_I(t) [\sigma_- \sigma_+, \rho] - D'_I(t) [\sigma_+ \sigma_-, \rho]). \quad (126)
\]

Alternatively, put in a compact form, we have
\[
\mathcal{L}_{t, (2,0)}(\rho) = -i [H_{eff}(t), \rho] - D_R(t) (\sigma_+ \rho + \rho \sigma_- \sigma_+ - 2\sigma_+ \rho \sigma_-)
\]
\[
- D'_R(t) (\sigma_- \sigma_+ + \rho \sigma_- \sigma_+ - 2\sigma_- \rho \sigma_+), \quad (127)
\]

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where the effective Hamiltonian is defined as
\[ H_{\text{eff}}(t) \equiv D_t(t)\sigma_+\sigma_- - D'_t(t)\sigma_+\sigma_- \quad (128) \]

**B.4 $\mathcal{L}_{t(0,2)}(\rho)$**

The second-order term due to stochasticity of the external B-field alone is
\[
\mathcal{L}_{t(0,2)}(\rho) = \dot{\mathcal{E}}_{t(0,2)}(\rho) - \dot{\mathcal{E}}_{t(0,1)}(\mathcal{E}_{t(0,1)}(\rho)) \\
= \dot{\mathcal{E}}_{t(0,2)}(\rho) \\
= \frac{\partial}{\partial t} \sum_{n=0}^{2} \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} \text{Tr}_E \left( U_{0,2}^{(j)\dagger}(t,0)\rho \otimes \rho_{E0} U_{0,2}^{(j)}(t,0) \right) \\
= \frac{\partial}{\partial t} \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} \text{Tr}_E \{ U_{0,1}^{(j)}(t,0)\rho \otimes \rho_{E0} U_{0,1}^{(j)\dagger}(t,0) + \rho \otimes \rho_{E0} U_{0,2}^{(j)}(t,0) \} \\
= \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} \text{Tr}_E \{ \dot{U}_{0,2}^{(j)}(t,0)\rho \otimes \rho_{E0} + \rho \otimes \rho_{E0} \dot{U}_{0,2}^{(j)\dagger}(t,0) \} \\
+ \dot{U}_{0,1}^{(j)}(t,0)\rho \otimes \rho_{E0} U_{0,1}^{(j)\dagger}(t,0) + U_{0,1}^{(j)}(t,0)\rho \otimes \rho_{E0} \dot{U}_{0,1}^{(j)\dagger}(t,0) \} \\
= \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^{R} \text{Tr}_E \{ \dot{U}_{0,2}^{(j)}(t,0)\rho \otimes \rho_{E0} + \rho \otimes \rho_{E0} \dot{U}_{0,2}^{(j)\dagger}(t,0) \} \\
+ \dot{U}_{0,1}^{(j)}(t,0)\rho \otimes \rho_{E0} U_{0,1}^{(j)\dagger}(t,0) + U_{0,1}^{(j)}(t,0)\rho \otimes \rho_{E0} \dot{U}_{0,1}^{(j)\dagger}(t,0) \} \\
(129)
\]

where in the second equality we have made use of the vanishing of $\dot{\mathcal{E}}_{t(0,1)}(\cdots)$, as is already shown in Eq. (129).

To facilitate further calculation, let’s first evaluate $U_{0,1}^{(j)}(t,0)$ and $U_{0,2}^{(j)}(t,0)$ in Eqs. (128) with the stochastic part of the Hamiltonian $H_S^{(j)}(t) = \frac{a^{(j)(0)}}{2}\sigma_z$ given in Eq. (53):
\[
U_{0,1}^{(j)}(t,0) = (-i) \int_0^t dt' H_S^{(j)}(t') \otimes \mathbb{I}_E \\
= (-i) \int_0^t dt' \frac{a^{(j)}(t')}{2} \sigma_z \otimes \mathbb{I}_E \\
= \left( -\frac{i}{2} \right) \left( \int_0^t dt' a^{(j)}(t') \right) \sigma_z \otimes \mathbb{I}_E, \quad (130)
\]
\[
U_{0,2}^{(j)}(t,0) = - \int_0^t dt' \int_0^{t'} dt'' \left( H_S^{(j)}(t') H_S^{(j)}(t'') \right) \otimes \mathbb{I}_E \\
= - \int_0^t dt' \int_0^{t'} dt'' \left( \frac{a^{(j)}(t')}{2} \sigma_z \frac{a^{(j)}(t'')}{2} \sigma_z \right) \otimes \mathbb{I}_E \\
= - \frac{1}{4} \left( \int_0^t dt' \int_0^{t'} dt'' a^{(j)}(t') a^{(j)}(t'') \right) \sigma_z^2 \otimes \mathbb{I}_E
\]
= \frac{-1}{4} \left( \int_0^t dt' \int_0^{t'} dt'' a^{(j)}(t') a^{(j)}(t'') \right) I_S \otimes I_E, \quad (131)

where in the last equality we have made use of the property of the Pauli operators \( \sigma_z^2 = I_S \) for two-level systems. Plugging Eqs. (130, 131) into Eq. (129), we have

\[ \mathcal{L}_{t(0,2)}(\rho) = \dot{\mathcal{L}}_{t(0,2)}(\rho) \]

\[ = \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^R Tr_E \left\{ -\frac{1}{4} \left( \int_0^t dt' a^{(j)}(t) a^{(j)}(t') \right) I_S \otimes I_E (\rho \otimes \rho_E) \right. \]

\[ + \frac{-1}{4} \left( \int_0^t dt' a^{(j)}(t) a^{(j)}(t') \right) (\rho \otimes \rho_E) I_S \otimes I_E \]

\[ + \left. \left( -\frac{i}{2} \left( a^{(j)}(t) \right) \sigma_z \otimes I_E (\rho \otimes \rho_E) \left( \frac{i}{2} \left( \int_0^t dt' a^{(j)}(t') \right) \sigma_z \otimes I_E \right) \right) \right\} \]

\[ = \int_0^t dt' \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^R Tr_E \left\{ -\frac{1}{2} \left( a^{(j)}(t) a^{(j)}(t') \right) (\rho \otimes \rho_E) \right. \]

\[ + \frac{1}{2} \left( a^{(j)}(t) a^{(j)}(t') \right) (\sigma_z \rho \sigma_z \otimes I_E) \} \]

\[ = -\frac{1}{2} \int_0^t dt' \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^R \left( a^{(j)}(t) a^{(j)}(t') \right) (\rho - \sigma_z \rho \sigma_z) Tr_E (\rho_E) \]

\[ = -\frac{1}{2} \int_0^t dt' \left( a(t)a(t') \right) \left( \rho - \sigma_z \rho \sigma_z \right), \quad (132) \]

where \( a(t)a(t') = \lim_{R \to \infty} \frac{1}{R} \sum_{j=1}^R \left( a^{(j)}(t) a^{(j)}(t') \right) \).

Put in a more compact form, we have

\[ \mathcal{L}_{t(0,2)}(\rho) = -2D_C(t) (\rho - \sigma_z \rho \sigma_z), \quad (133) \]

where the dephasing rate (due to classical noise) is defined as

\[ D_C(t) = \frac{1}{4} \int_0^t dt' a(t)a(t'). \quad (134) \]

**B.5 Second-order equation of motion**

As has been shown in Eq. (133), the second-order cross term vanishes, that is, \( \mathcal{L}_{t(1,1)}(\rho) = 0 \).

Therefore, with the three terms \( \mathcal{L}_{t(1,0)}(\rho_t), \mathcal{L}_{t(0,1)}(\rho_t) \) and \( \mathcal{L}_{t(1,1)}(\rho_t) \) vanishing in Eq. (133), and the remaining two terms \( \mathcal{L}_{t(2,0)}(\rho_t) \) and \( \mathcal{L}_{t(0,2)}(\rho_t) \) given by Eqs. (127, 133), the equation of motion for this physical example up to second
order is
\[
\frac{d}{dt} \rho_t = -i \left[ H_{\text{eff}}(t), \rho_t \right] - D_R(t) (\sigma_- \sigma_+ + \rho_t \sigma_- \sigma_+ - 2 \sigma_+ \rho_t \sigma_- )
\]
\[-D'_R(t) (\sigma_+ \sigma_- \rho_t + \rho_t \sigma_+ \sigma_- - 2 \sigma_- \rho_t \sigma_+) - 2 D_C(t) (\rho_t - \sigma_z \rho_t \sigma_z),
\]
where the effective Hamiltonian is \(H_{\text{eff}}(t) \equiv D_I(t) \sigma_- \sigma_+ - D'_I(t) \sigma_+ \sigma_-\) and the prefactors \(D_R(t), D_I(t), D'_R(t), D'_I(t), \) and \(D_C(t)\) are as defined in Eqs. (122, 123, 124, 125, 134) respectively.

References

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2nd edition, 2010).
[2] M. Shapiro and P. Brumer, Quantum Control of Molecular Processes (Wiley-VCH, 2nd, revised and enlarged edition, 2012).
[3] H.-P. Breuer, B. Kappler, F. Petruccione, Annals of Physics 291, 36–70 (2001).
[4] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, 2002).
[5] L. Yu and D. F. V. James, arXiv:1111.6686 (2011).
[6] L. Yu and E. J. Heller, arXiv:2004.13130 (2020).
[7] M. Schlosshauer, Decoherence and the Quantum-to-Classical Transition (Springer, 2007).
[8] R. Shankar, Principles of Quantum Mechanics, 2nd edition (Plenum Press, 1994).
[9] H. Walther, B. T. H. Varcoe, B.-G. Englert, T. Becker, Rep. Prog. Phys. 69, 1325–1382 (2006).
[10] O.-P. Saira, V. Bergholm, T. Ojanen, M. Mottonen, Phys. Rev. A 75, 012308 (2007).
[11] A. A. Budini, Phys. Rev. A 64, 052110 (2001).
[12] I. Bardet, arXiv:1511.08683 (2015).
[13] S. Attal and I. Bardet, Ann. Inst. H. Poincare Probab. Statist. 54, no. 4, 2159-2176 (2018).
[14] M. Yamaguchi, T. Yuge, T. Ogawa, Phys. Rev. E 95, 012136 (2017).
[15] J. E. Elenewski, D. Gruss, M. Zwolak, J. Chem. Phys. 147, 151101 (2017).
[16] J. Jeske and J. H. Cole, Phys. Rev. A 87, 052138 (2013).
[17] P. G. Kirton, A. D. Armour, M. Houzet, F. Pistolesi, Phys. Rev. B 86, 081305 (2012).