Exponential stability analysis and stabilization for discrete positive switched linear time-delay systems with ADT

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Abstract. In this paper, the stability analysis and stabilization problem for discrete positive switched linear time delay system with average dwell time switching are investigated. First, by constructing an appropriate multiple linear co-positive type Lyapunov-Krasovskii functional, sufficient condition for the exponential stability are developed by using ADT approach. Then based on the result obtained a linear feedback law is established for the system to make the state exponentially stable.

1. Introduction
Many dynamical systems in real world are positive systems which variables are confined to the positive orthant [1-3]. And if on the same time, a positive system is also a hybrid dynamic system which is composed of a family of subsystems and a switching signal to determine the switching between subsystems, the positive system is called a positive switched system, [4-7]. For the research of positive switched linear systems, stability analysis is a major concern [3, 8]. Researchers have proved that constructing an appropriate linear co-positive Lyapunov-Krasovskii functional is an efficient approach [9, 10]. For general switched systems, there are many switching strategies such as arbitrary switching, [11], dwell time switching [12] and average dwell time (ADT) switching [13,14]. Arbitrary switching requires the existence of a common Lyapunov-Krasovskii function. Therefore, it holds more conservative than other switching laws. Dwell time switching requires that the time interval between any two consecutive switchings is no smaller than a fixed positive constant. It can be shown that it is always possible to maintain stability when all the subsystems are stable and switching is slow enough. Actually, it is common for a system to have a smaller dwell time. For this kind of systems, the researchers define the concept of ADT. It has been well known that the ADT switching characterizes a larger class of stable switching signals than dwell time switching.

On the other hand, time delay as a source of instability and poor performance often appears in many dynamic systems. For example, biological systems, neural network and electrical networks [15-17]. In recent years, more and more researchers begin to pay attention to the stability analysis and stabilization of time delay forward switching systems [11, 15, 18].

In this paper, the stability problems of discrete positive switched linear time-delay systems with ADT switching will be investigated. We first construct an appropriate co-positive type Lyapunov-Krasovskii functional. Sufficient conditions for the exponential stability are proposed by using the average dwell time approach. Furthermore, the desired controller is proposed under which exponential stability of a closed-loop system is obtained.
Notation: Throughout the paper, $A \leq (\geq) 0$ means that all entries of matrix $A$ is non-positive (non-negative); $A < (>) 0$ means that all entries of matrix $A$ is negative (positive); $R^d (R^{+})$ is the set of all real (positive real) numbers; $R^n (R^{+n})$ is the $n$-dimensional matrices; $Z^+$ refers to the set of all non-negative integers. $\Delta x(k) = x(k+1) - x(k)$ and $\|x\| = \sum_{i=1}^{n} |x_i|$, where $x_i$ is the $i^{th}$ element of $x \in R^n$.

2. Problem formulation and preliminaries

Given the following system

$$\begin{align*}
    x(k+1) &= Ax(k) + A_d x(k-d) + Bu(k), k = k_0, k_0 + 1, \ldots \\
    x(s) &= \varphi(s), \\
    s &\in \{-d, -d+1, \ldots, -1, 0\}
\end{align*}$$

(2.1)

where $x(k) \in R^n, u(k) \in R^n (n, p \in N)$ are the state vector and the control input, respectively. $d > 0$ denotes the constant delay. $\varphi(s)$ is the initial condition defined on $\{-d, \ldots, -1, 0\}$ and $\|\varphi\| = \max_{s \in \{-d, -d+1, \ldots, -1, 0\}} \|\varphi(s)\|$.

In system (2.1), let

$$A = \sum_{p=1}^{M} \delta_p (\sigma(k)) A_p, A_d = \sum_{p=1}^{M} \delta_p (\sigma(k)) A_{dp}, B = \sum_{p=1}^{M} \delta_p (\sigma(k)) B_p$$

(2.2)

Then system (2.1) with (2.2) is a discrete positive switched linear system with delays, where $A_p, A_{dp}, B_p, \forall p \in S$ are constant matrices with appropriate dimensions, $p$ denotes the $p^{th}$ subsystem and

$$\delta_p (\sigma(k)) = \begin{cases} 1, \sigma(k) = p \\ 0, \text{otherwise} \end{cases}$$

$\sigma(\cdot)$ is a piecewise constant function of variable, called a switching signal, which takes its values in the finite set $S = \{1, 2, \cdots, M\}$, $M$ is the number of subsystems. Also, for a switching sequence $0 = k_0 < k_1 < \cdots < k_i < k_{i+1} < \cdots, \sigma(k)$ is either autonomous or controlled. When $k_{j} \leq k \leq k_{j+1} - 1$, we say the $\sigma(k)^{th}$ subsystem is active.

**Definition 2.1 ([19])** System (2.1) with (2.2) is said to be positive, if for any switching signal $\sigma(k)$ and any initial condition $\varphi(s), s \in \{-d, -d+1, \cdots, -1, 0\}$, the corresponding trajectory $x(k) > 0$ holds for any $k = 0, 1, 2, \cdots$.

**Lemma 2.1 ([20])** System (2.1) with (2.2) is positive if and only if $A \geq 0, A_d \geq 0, B \geq 0$.

**Definition 2.2 ([6])** The switched system (2.1) with (2.2) is said to be exponentially stable under $\sigma(k)$, if there exist constants $\alpha > 0, 0 < \eta < 1$ such that the solution of system (2.1) with (2.2) satisfies

$$\|x(k)\| \leq \alpha \eta^{k-k_0} \|\varphi\|, \forall k \geq 0, k_0 \geq 0,$$

(2.3)

where $\|\varphi\| = \max_{s \in \{-d, -d+1, \ldots, -1, 0\}} \|\varphi(s)\|$. 

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Definition 2.3 (see [3]) For any switching signal $\sigma(k)$ and any $K_2 \geq K_1 \geq 0$, let $N_{\sigma}(K_1, K_2)$ denote the numbers of the switching of $\sigma(k)$ over the set $\{K_1, K_1 + 1, \ldots, K_2\}$. For any given $K_a \in \mathbb{Z}^+$, $N_0 \geq 0$, if the inequality
\[
N_{\sigma}(K_1, K_2) \leq N_0 + \frac{K_2 - K_1}{K_a}
\] (2.4)
holds, the positive constant $K_a$ is called an average dwell time and $N_0$ is called chattering bound.

As commonly used in the literature, for convenience, we choose $N_0 = 0$ in this paper.

Firstly, we study the asymptotic stability for the system (2.1) with (2.2) and $u(k) = 0$, which is denoted as:
\[
\begin{cases}
  x(k+1) = Ax(k) + A_p x(k-d), k = k_0, k_0 + 1, \ldots \\
  x(s) = \varphi(s), 
\end{cases}
\] (2.5)

3. Stability analysis with ADT

Theorem 3.1 Let $0 < \lambda < 1$, $\mu > 1$ be given constants. If there exist vectors
\[
v_p = (v_{p1}, v_{p2}, \ldots, v_{pn}), v_p = (v_{p1}, v_{p2}, \ldots, v_{pn}), \varphi_p = (\varphi_{p1}, \varphi_{p2}, \ldots, \varphi_{pn}) \in \mathbb{R}^n
\] and $\zeta_p = (\zeta_{p1}, \zeta_{p2}, \ldots, \zeta_{pn}) \in \mathbb{R}^n$, such that for $\forall (p, q) \in S \times S$,
\[
\Pi_p = \text{diag} \{\pi_{p1}, \ldots, \pi_{pn}, \pi_{p1}', \ldots, \pi_{pn}'\} \leq 0
\] (3.1)
\[
\Pi_{pq} = \text{diag} \{\tau_{pq1}, \ldots, \tau_{pqn}, \tau_{pq1}', \ldots, \tau_{pqn}'\} \leq 0
\] (3.2)
\[
\Psi_{pq} = \text{diag} \{\psi_{pq1}, \ldots, \psi_{pqn}, \psi_{pq1}', \ldots, \psi_{pqn}'\} \leq 0
\] (3.3)
where for any $i \in \{1, 2, \ldots, n\}$,
\[
\pi_{pi} = d_{pi}v_p - \lambda v_{pi} + v_{pi} + d_{qi}\varphi_{pi} + \zeta_{pi}, \quad \pi_{pi}' = a_{dpi}v_p - \lambda d_{pi}v_{pi} - \zeta_{pi};
\] (3.4)
\[
\tau_{pq1} = -d_{qi}\zeta_{pi} - \lambda d_{ai}\varphi_{qi}, \quad \tau_{pq1}' = -d_{qi}\zeta_{pi};
\] (3.5)
\[
\psi_{pq1} = v_{pi} + \mu v_{qi} \quad \psi_{pq1}' = \varphi_{pi} - \mu \varphi_{qi}, \quad \varphi_{pi} = \varphi_{pi} - \mu \varphi_{qi};
\] (3.6)
a $a_p(a_{dp})$ represents the $i^{th}$ column vector of matrix $A_p(A_{dp})$. Then system (2.5) is exponentially stable for any switching signal $\sigma(k)$ with average dwell time
\[
K_a \geq K^*_a = -\frac{\ln \mu}{\ln \lambda}.
\] (3.7)

Proof. For any $K > 0$, let $k_0 = 0$ and denote $k_1, \ldots, k_i, k_{i+1}, \ldots, k_{K_a(0,K)}$ the switching times on the interval $[0, K]$. Define the co-positive type Lyapunov-Krasovskii function for system (2.5) as follows:
\[
V_{\sigma(k)}(k, x(k)) = x^T(k) v_{\sigma(k)} + \sum_{\ell = k-1}^{k} \lambda^{-(k-\ell+1)} x^T(\ell) v_{\sigma(k)} + \sum_{\ell = -d}^{k-1} \lambda^{-(k-\ell+1)} x^T(\ell) \varphi_{\sigma(k)}
\]
where $v_p = (v_{p1}, \ldots, v_{pn}), v_p = (v_{p1}, \ldots, v_{pn}), \varphi_p = (\varphi_{p1}, \ldots, \varphi_{pn}) \in \mathbb{R}^n, \forall p \in S$.

Because of (3.3) and (3.6), for $\forall (p, q) \in S \times S$, we have $V_p(k, x(k)) \leq \mu V_q(k, x(k))$.

When $k \in \{k_i, k_i + 1, \ldots, k_{i+1} - 1\}$, assume that the $p^{th}$ subsystem is activated, then
\[ V_p(k+1) = x^T(k+1)v_p + \sum_{\ell=k+1-d}^k \lambda^{-\ell-1} x^T(\ell)u_p + \sum_{\ell=0}^{k} \sum_{\theta=\ell+1-d}^\theta \lambda^{-\ell}(\theta) x^T(\ell) \theta_p \]
\[ = x^T(k)(A^T_{p}v_p - \lambda v_p + \nu_p + d \theta_p) + x^T(k-d)(A^T_{d}v_p - \lambda^d \nu_p) \]
\[ - \sum_{\ell=k-d}^{k-1} \lambda^{-\ell}(k) x^T(\ell) \theta_p + \lambda V_p(k) \quad (3.8) \]

On the other hand, the following two formulas are obvious:
\[ \sum_{\ell=k-d}^{k-1} \Delta x(\ell) = x(k) - x(k-d) \quad (3.9) \]
\[ \sum_{\ell=k-d}^{k-1} \Delta x(\ell) = \sum_{\ell=k-d}^{k-1} ((A-I)x(\ell) + A_d x(\ell-d)) \quad (3.10) \]

From (3.9) and (3.10), the following is obtained for any \( n \)-dimension vector:
\[ \left( x(k) - x(k-d) - \left( \sum_{\ell=k-d}^{k-1} ((A-I)x(\ell) + A_d x(\ell-d)) \right) \right)^T \zeta_p = 0 \quad (3.11) \]

Combining (3.8) and (3.11), we have
\[ V_p(k+1) \leq \lambda V_p(k) + x^T(k)(A^T_{p}v_p - \lambda v_p + \nu_p + d \theta_p + \zeta_p) \]
\[ + x^T(k-d)(A^T_{d}v_p - \lambda^d \nu_p - \zeta_p) \]
\[ - \sum_{\ell=k-d}^{k-1} \left( x(\ell) \right)^T \left( \left( A_q - I \right)^T \zeta_p + \lambda^d \theta_p \right) \]
\[ A^T_{d} \zeta_p \quad (3.12) \]

According to (3.4)-(3.6), it is obtained that, for \( \forall (p, q) \in S \times S \), \( A^T_{p}v_p - \lambda v_p + \nu_p + d \theta_p + \zeta_p \leq 0 \), \( A^T_{d}v_p - \lambda^d \nu_p - \zeta_p \leq 0 \).

Then for \( \forall k \in \{ k, k_i+1, \ldots, k_{i+1} - 1 \} \),
\[ V_p(k+1) \leq \lambda V_p(k) \leq \cdots \leq \lambda^{k-k_i} V_p(k_i) \quad (3.13) \]
\[ V_{\sigma(k)}(K, x(K)) \leq \mu^{N_k} \lambda^{k-k_i} V_{\sigma(0)}(k_0, x(k_0)) \]
\[ \leq \mu \lambda^{k-k_n} V_{\sigma(k_n+1)}(k_{n+1}, x(k_{n+1})) \]
\[ \leq \cdots \]
\[ \leq \mu^{N_n} \lambda^{k-k_0} V_{\sigma(0)}(k_0, x(k_0)) \]
\[ \leq \exp \{ N_0 \ln \mu \} \exp \left( \frac{1}{K_n} \ln \mu + \ln \lambda \right) K \left( V_{\sigma(0)}(0, x(0)) \right) \quad (3.14) \]

Considering the definition of \( V_{\sigma(t)}(t) \), there exist \( \epsilon_1, \epsilon_2, \epsilon_3 \) and \( \epsilon_4 \) such that
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\[
\varepsilon_1 \|x(k)\| \leq \exp \left\{ N_0 \ln \mu + \left( \frac{1}{K_u} \ln \mu + \ln \lambda \right) k \right\} \left( \varepsilon_2 \|x(0)\| + (\varepsilon_3 + \varepsilon_4) \sum_{s=d}^{k-1} \|x(s)\| \right) \]

(3.15)

Denote \( \alpha = \left( \frac{\varepsilon_2}{\varepsilon_1} \right)^2 + \left( \frac{\varepsilon_3 + \varepsilon_4}{\varepsilon_1} \right)^2 \) exp \{ N_0 \ln \mu \}, \( \eta = \exp \left\{ \frac{1}{K_u} \ln \mu + \ln \lambda \right\} \). According to (3.15), the system state satisfies \( \|x(k)\| \leq \alpha \eta^{(k-k_0)} \|\varphi\|, \forall k \geq k_0, \) that is, the underlying system is exponentially stable with switching signal satisfying average dwell time satisfying (3.7).

This completes the proof.

4. Design of feedback controller

In this section, a memory-less state feedback controller is designed to make the corresponding closed-loop system of (2.1) with (2.2) exponentially stable.

Theorem 4.1 Let \( 0 < \lambda < 1 \) and \( \mu > 1 \) be given constants. If there exist vectors \( \nu_p, \nu_q, \chi_p, \chi_q \) defined in Theorem 3.1, such that for \( \forall (p, q) \in S \times S, \forall i \in N = \{1, 2, \ldots, n\} \)

\[
E_p = \text{diag} \{ \bar{E}_p, \ldots, \bar{E}_p, \bar{E}_p \} \leq 0, \\
G_{pq} = \text{diag} \{ \bar{G}_{pq}, \ldots, \bar{G}_{pq}, \bar{G}_{pq} \} \leq 0, \\
F_{pq} = \text{diag} \{ \bar{F}_{pq}, \ldots, \bar{F}_{pq}, \bar{F}_{pq} \} \leq 0,
\]

(4.1) (4.2) (4.3)

where \( \bar{E}_p = \alpha^T v_p + g_{pq} v_p - \lambda v_p + v_p + d \chi_p + \chi_p, \bar{G}_{pq} = -a^T q \chi_p - h_{pq} - \chi_p + \lambda^d \chi_q \) and others are defined in Theorem 3.1, \( g_p = K_p^T B_p^T v_p, h_q = K_q^T B_q^T v_q \). Then system (2.1) with (2.2) is exponentially stable for any switching signal \( \sigma(k) \) with average dwell time \( K_d \geq K_u^* = -\frac{\ln \mu}{\ln \lambda} \). Moreover, the controller is \( u(k) = Kx(k), K = \sum_{p=1}^{M} \delta_p \left( \sigma(k) \right) K_p \).

Proof. Substitute \( u(k) = Kx(k), K = \sum_{p=1}^{M} \delta_p \left( \sigma(k) \right) K_p \) into the system (2.1). Then the corresponding closed-loop system is of the form

\[
x(k+1) = (A + BK)x(k) + A_g x(k-d), k = k_0, k_0 + 1, \ldots \\
x(s) = \varphi(s), s \in \{-d, -d+1, \ldots, -1, 0\}
\]

(4.4)

Replace \( A_p, A_q \) with \( A_p + B_p K_p, A_q + B_q K_q \) in the matrix inequality (3.12) which has occurred in Theorem 3.1. According to Theorem 3.1 the closed-loop system (4.4) is exponentially stable.

This completes the proof.

5. Numerical example

As an illustration, we consider a system in the form (2.1) without less of generality, assume that there two subsystems described by
Choose the discrete time delay \( d = 2 \) and \( \lambda = 0.9 \). Solving the matrix inequalities (4.1) and (4.2) under MATLAB Toolbox, we obtain

\[
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} 0.9535 \\ 0.9535 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0.4693 \\ 0.4589 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0.7099 \\ 0.7214 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0.1709 \\ 0.0611 \end{pmatrix}, \\
\mathbf{v}_5 &= \begin{pmatrix} 0.9063 \\ 0.9063 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} 0.5336 \\ 0.5147 \end{pmatrix}, \quad \mathbf{v}_7 = \begin{pmatrix} 0.7863 \\ 0.7948 \end{pmatrix}, \quad \mathbf{v}_8 = \begin{pmatrix} 0.1891 \\ 0.0673 \end{pmatrix}.
\end{align*}
\]

The state feedback gain matrices are given by

\[
K_1 = (-21.1858, -4.1209), \quad K_2 = (-3.2880, -22.9654).
\]

and according to (4.3), we can get \( \mu = 1.137, K^*_a = 1.218 \).

6. Conclusions
In this paper, the exponential stability and stabilization problem have been investigated for a class of discrete positive switched linear system with time-delay. Firstly, an exponential stability criterion has been obtained by choosing proper co-positive Lyapunov-Krasovskii function. Furthermore, some appropriate feedback controllers have been constructed to ensure the stability of the closed loop systems. Finally, an example is provided to show that the results obtained above are effective, that is, the closed loop system is exponential stable if the average dwell time is no less than a certain constant.

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