Errata to ”An example of Bautin-type bifurcation in a delay differential equation”, JMAA, 329(2007), 777-789

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Abstract

Some errors contained in the author’s previous article ”An example of Bautin-type bifurcation in a delay differential equation”, JMAA, 329(2007), 777-789, are listed and corrected.

In our work [1], we considered the delay differential equation,

\[ \dot{x} = ax(t-r) + x^2(t) + cx(t)x(t-r), \]

and looked for values of the parameters where the conditions for the occurrence of Bautin type bifurcation around the equilibrium point \( x = 0 \) are fulfilled.

For this we first proved that for the linearized around \( x(t) = 0 \) equation, at

\[ a = -1, \quad r = \pi/2, \]

two eigenvalues \( \lambda_{1,2} = \pm i \) exist, while all the other eigenvalues have negative real part. Thus we were entitled to consider the reduction of the problem to the two-dimensional center manifold for these values of the parameters. The reduced problem is a two-dimensional system of differential equations, that can be written as an ODE for a complex valued function

\[ \dot{z} = \pm iz + \sum_{j+k \geq 2} \frac{1}{j!k!} g_{jk}(c) z^j \bar{z}^k. \]

For such problems the Bautin bifurcation was studied in [4] and we followed the method therein for our study. In order to find Bautin bifurcation points we computed the first Lyapunov coefficient and found that this is zero for

\[ c_{1,2} = \frac{18 - 7\pi \pm \sqrt{36 + 212\pi + \pi^2}}{2(3\pi - 2)}. \]

In this note we intend to correct two distinct type of mistakes that, we, unhappily, made in [1].
I. To determine whether the bifurcation point presents a higher order degeneracy or is a proper Bautin bifurcation point, we computed the second Lyapunov coefficient for the reduced on the center manifold problem. For this we needed $w_{21}(0)$, $w_{21}(-r)$, where $w_{21}(\cdot) \in \mathcal{C}([-r,0),\mathbb{R})$ is a coefficient of the series of powers of the function whose graph is the center manifold $w(x,\varpi)(\cdot) = \sum_{j+k \geq 2} \frac{1}{j!k!} w_{jk}(\cdot) z^j \varpi^k$ see [1].

The two algebraic equations that yield $w_{21}(0)$ and $w_{21}(-r)$ proved to be dependent, and at that moment we have chosen arbitrarily $w_{21}(0) = 0$ and we computed $w_{21}(-r)$ from one of the two equations. This is a mistake and we want to correct it here.

By studying more carefully the problem of computing $w_{21}(0)$ and $w_{21}(-r)$, we found out that these can be uniquely determined by using a perturbation technique. This result was published in [2].

The formula obtained there for $w_{21}(0)$, adapted to problem (1), is

$$w_{21}(0) = f_{21}(\Psi_1 + \Psi_2, \rho) - 2g_{11}(\bar{\rho}, w_{20}) - (g_{20} + 2g_{11})(\bar{\rho}, w_{11}) - g_{02}(\bar{\rho}, w_{20}),$$

Hence, by the theory concerning the Bautin bifurcation, in the parameters $a,c$ we found out that these can be uniquely determined by using a perturbation technique. This result was published in [2].

By studying more carefully the problem of computing $w_{21}(0)$ and $w_{21}(-r)$, we found out that these can be uniquely determined by using a perturbation technique. This result was published in [2].

II. We also noticed some other errors in [1], that we correct here:

1. at pg. 8 (784 in JMAA), $w_{20}(0,c)$ should be

$$w_{20}(0,c) = F_{20} \left[ \frac{4(\pi + 4i)}{3(4 + \pi^2)} \frac{1 + 2i}{5} \right] = 2(1 - ic) \left[ \frac{4(\pi + 4i)}{3(4 + \pi^2)} - \frac{1 + 2i}{5} \right];$$

2. at pg. 10 (787 in JMAA), $F_{31}$ should be

$$F_{31} = c_1 \left[ 3w_{21}(-r) + w_{30}(-r) + i w_{30}(0) - 3i w_{21}(0) + w_{20}(0)w_{11}(-r) + 3w_{11}(0)w_{20}(-r) + 6w_{11}(0)w_{20}(0) + 2w_{21}(0) + 2w_{20}(0);$$
3. at pg. 10 (787 in JMAA), $F_{22}$ should be

$$F_{22} = c_1 [2w_{12}(-r)+2w_{21}(-r)+w_{20}(0)w_{02}(-r)+4w_{11}(0)w_{11}(-r)+w_{02}(0)w_{20}(-r)+$$

$$+2iw_{21}(0) - 2iw_{12}(0)] + 2w_{20}(0)w_{02}(0) + 4w_{11}(0)^2 + 4w_{12}(0) + 4w_{21}(0).$$

We apologize to the readers of *Journal of Mathematical Analysis and Applications* for the errors listed and corrected above.

**References**

[1] A. V. Ion, *An example of Bautin-type bifurcation in a delay differential equation*, J. Math. Anal. Appl., **329**(2007), 777-789.

[2] A. V. Ion, *On the computation of the third order terms of the series defining the center manifold for a scalar delay differential equation*, Journal of Dynamics and Differential Equations, **24**, 2(2011), 325-340.

[3] Y. A. Kuznetsov, *Elements of applied bifurcation theory*, Applied Mathematical Sciences, **112**, Springer, New York, 1998.
An example of Bautin-type bifurcation in a delay differential equation

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Abstract
In a previous paper we gave sufficient conditions for a system of delay differential equations to present Bautin-type bifurcation. In the present work we present an example of delay equation that satisfies these conditions.

1 Introduction

In [5] the system of delay differential equations
\[ \dot{x}(t) = A(\alpha)x(t) + B(\alpha)x(t-r) + f(x(t), x(t-r), \alpha), \]
\[ x(s) = \phi(s), \quad s \in [-r, 0], \]
with \( x(t) = (x_1(t),...,x_n(t)) \in \mathbb{R}^n, \alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2, A(\alpha), B(\alpha) n \times n \]
real matrices is considered. Here \( f = (f_1,...,f_n) \) is continuously differentiable on its domain of existence, \( D \subset \mathbb{R}^{2n+2} \). It is also assumed that \( f(0,0,\alpha) = 0 \) and the differential of \( f \) in the first two vectorial variables, calculated at \( (0,0,\alpha) \) is equal to zero. \( \phi \) belongs to the Banach space \( C([-r,0],\mathbb{R}^n) \).

For this system we give in [5] a theorem providing sufficient conditions for the appearance of Bautin-type bifurcation.

Bautin bifurcations are degenerated Hopf bifurcations. As it is known, \[4\], for two-dimensional systems of ODEs depending on a scalar parameter \( \alpha \), Hopf bifurcation around a branch of equilibrium points appears when there is a certain value of the parameter, \( \alpha_0 \), at which:
- a pair of purely imaginary eigenvalues of the linear part exists,
- the real part of the eigenvalues (that is zero at \( \alpha_0 \)) has non-zero derivative at \( \alpha_0 \),
- the first Lyapunov coefficient at \( \alpha_0 \) is non-zero.

The first Lyapunov coefficient is a number defined as follows. The two-dimensional system of real equations is written as a single complex equation
\[ z = \lambda z + g(z,\bar{z},\alpha), \]
and the first Lyapunov coefficient, \( l_1(\alpha) \) is defined in terms of the coefficients (up to the third degree) of the series
\[ g(z,\bar{z},\alpha) = \sum_{j+k\geq 1} \frac{1}{j!k!} g_{jk}(\alpha) z^j \bar{z}^k, \]
\[ ^1 \text{published in Journal of Mathematical Analysis and Application, 329(2007), 777-789} \]
Now, also for an ODEs system, let us assume that the parameter $\alpha$ is bidi-
mensional. When $l_1(\alpha_0) = 0$ and a second Lyapunov coefficient, $l_2(\alpha)$ (that
is defined in terms of the coefficients up to the fifth degree terms of the above
series - see Section 6.2) is non-zero at $\alpha_0$, the Bautin bifurcation takes place [4].
It is characterized by the appearance, for the parameters in a neighborhood of
$\alpha_0$, of two limit cycles (one inside the other).

By using the reduction of the problem (1) to its central manifold, we ex-
tended in the main theorem in [5] the above ideas to the class of systems of
delay differential equations (1).

The problem that arised after this theorem was proved, was whether there
is any delay equation that satisfies its hypotheses, or not.

In this paper we present a scalar differential delay equation that sa-
tisfies the hypotheses of our theorem.

We set below the theoretical frame we used and our result, as a starting
point for the rest of the paper.

2 Theoretical framework

Let us consider the solutions of the equation

$$\det (\lambda I - A(\alpha) - e^{\lambda r} B(\alpha)) = 0,$$

where $I$ is the $n$-dimensional unity matrix. These are the eigenvalues of the
infinitesimal generator of the linearized problem obtained from (1) (see [1], [2]).
Let us consider the hypothesis:

$H1$. There is an open set $U$ in the parameter plane such that for every
$\alpha \in U$, there is a pair of complex conjugated simple eigenvalues
$\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i \omega(\alpha)$, with the property that there is a $\alpha_0 \in U$ such that
$\lambda_{1,2}(\alpha_0) = \pm i \omega_0$, with $\omega_0 > 0$ and there is an $\varepsilon > 0$ such that for every $\alpha \in U$,
$\mu(\alpha) > -\varepsilon$, while all other eigenvalues $\lambda$ have $\text{Re} \lambda < -\varepsilon$.

It is important to assume that, as $\alpha$ varies in $U$, $\mu(\alpha)$ takes both positive
and negative values. This is usually expressed by the hypothesis $d\mu \over d\alpha|_{\alpha_0} \neq 0$,
but in our case it will be covered by hypothesis $H2$ below.

Let $\varphi_1(\alpha), \varphi_2(\alpha) \in C([-r,0], \mathbb{R}^n)$ be the two eigenvectors cor-
responding to $\lambda_1(\alpha)$ respectively $\lambda_2(\alpha)$ (these are simple eigenvalues). Let also
$M_{\{\lambda_{1,2}(\alpha)\}}$ be the space spanned by $\varphi_1(\alpha), \varphi_2(\alpha)$.

For the values of $\alpha \in U$ such that $\mu(\alpha) > 0$ there is a two-dimensional local
invariant manifold, the unstable manifold of the equilibrium point 0 (see [1], [2],
[6]). For $\alpha_0$ there is a two-dimensional local central manifold. In both cases the
manifold is the graph of a differentiable application $w_\alpha$ defined on $M_{\{\lambda_{1,2}(\alpha)\}}$.
We denote the local invariant manifold by $W_{\text{loc}}(\alpha)$.

The restriction of the equation (1) to the invariant manifold for the values
of $\alpha$ mentioned above is

$$\dot{z}(t) = \lambda_1(\alpha) z(t) + \psi_1(\alpha)(0) f((S_\alpha(t) \phi)(0), (S_\alpha(t) \phi)(-r), \alpha),$$

where $\psi_1(\alpha)$ is a certain eigenvector of the adjoint problem (1).

If we take $\phi \in W_{\text{loc}}(\alpha)$, then $S_\alpha(t) \phi \in W_{\text{loc}}(\alpha)$ and thus

(see [6] below).
$$S_\alpha (t) \phi (s) = z(t) \varphi_1 (s) + \bar{z}(t) \bar{\varphi}_1 (s) + w_\alpha (s, z(t), \bar{z}(t)).$$  

(3)

This implies that $f \left( [S_\alpha (t) \phi (s), [S_\alpha (t) \phi (s - r), \alpha) \right]$ is a function of $z, \bar{z}$ and it can be written as a series of powers as

$$f \left( S_\alpha (t) \phi (s), S_\alpha (t) \phi (s - r), \alpha \right) = \sum_{j+k \geq 2} \frac{1}{j!k!} F_{jk} (s, \alpha) z^j \bar{z}^k. \quad (4)$$

Then we can write $\psi_1 (\alpha) (0) f \left( [S_\alpha (t) \phi (0), [S_\alpha (t) \phi (s - r), \alpha) \right]$ also as a function of $z(t), \bar{z}(t), \alpha$ namely

$$\psi_1 (\alpha) (0) f \left( [S_\alpha (t) \phi (0), [S_\alpha (t) \phi (s - r), \alpha) = g (z(t), \bar{z}(t), \alpha). \quad (5)$$

Equation (2) becomes

$$\dot{z}(t) = \lambda_1 (\alpha) z(t) + \sum_{j+k \geq 2} \frac{1}{j!k!} g_{jk} (\alpha) z(t)^j \bar{z}(t)^k. \quad (6)$$

For this equation we can study Bautin bifurcation as in [4]. We consider the first and second Lyapunov coefficients defined in [4], that are functions of $g_{ij}$.

We remind that

$$l_1 (\alpha_0) = \frac{1}{2\omega_0} \text{Re} \left( i g_{20} (\alpha_0) g_{11} (\alpha_0) + \omega_0 g_{21} (\alpha_0) \right), \quad (6)$$

while $l_2 (\alpha_0)$ is a much more complicated expression.

We also define

$$\nu_1 = \frac{\mu (\alpha)}{\omega (\alpha)}, \quad \nu_2 = l_1 (\alpha),$$

and $\nu = (\nu_1, \nu_2)$. Let us consider the following hypothesis:

$H2$. $l_1 (\alpha_0) = 0$, $l_2 (\alpha_0) > 0$, and the map $(\alpha_1, \alpha_2) \rightarrow (\nu_1, \nu_2)$ is regular at $\alpha_0$.

Now we can state the main result of [5].

**Theorem** If $H1$, $H2$ are satisfied for eq. (1), then at $\alpha_0$ a Bautin-type bifurcation takes place.

That is there is a neighbourhood $U_1$ of $\alpha_0$ in the $\alpha$ plane having a subset $V^*$ (with $\alpha_0$ as a limit point) with the property that for every $\alpha \in V^*$, the restriction of problem (1) to the unstable manifold has two limit cycles (one interior to the other).
3 The scalar equation

Let us consider the equation
\[ x' = ax(t - r) + x^2(t) + cx(t)x(t - r), \] (7)
with \( r = \frac{\pi}{2} \). We will study the equation around the equilibrium solution \( x(t) = 0 \).

Define \( \alpha = (a, c) \). The linear part of (7) is
\[ x' = ax(t - r). \] (8)

Let us consider the function
\[ \eta(s) = \begin{cases} -a, & s = -r \\ 0, & s \in (-r, 0] \end{cases}. \] (9)

We observe that, by defining
\[ L\varphi = \int_{-r}^{0} \varphi(s) d\eta(s), \]
and \( x_t(s) = x(t + s) \) for \( s \in [-r, 0] \), equation (8) may be written as
\[ x' = Lx_t. \]

4 The eigenvalues

The characteristic equation is \( \lambda - ae^{-\lambda r} = 0 \), where \( \lambda = \mu + i\omega \) and it is equivalent to the system of two equations
\[ \mu - ae^{-\mu r} \cos \omega r = 0, \]
\[ \omega + ae^{-\mu r} \sin \omega r = 0. \]

These are equivalent with
\[ \omega = \pm \sqrt{a^2e^{-2\mu r} - \mu^2}. \] (10)
\[ \cos \sqrt{a^2e^{-2\mu r} - \mu^2} r = \frac{\mu e^{\mu r}}{a}. \] (11)

We see that at \( a_0 = -1 \), \( r = \frac{\pi}{2} \) the pairs \( \omega = \pm 1 \), \( \mu = 0 \) are solutions of the above equations.

In order to study equation (11) we define \( y = \mu r \), and obtain the new equation
\[ \cos \sqrt{\frac{a^2r^2}{e^{2y}} - y^2} - y^2 = \frac{y}{ar} e^y, \] (12)
that accepts the solution \( y = 0 \) for \( a_0 = -1 \).

Hence \( \lambda_{1,2}(a_0) = \pm i\omega(a_0) = \pm i\omega_0 = \pm i \), \( \omega_0 = 1 \).

Proposition 1. There is an open neighborhood \( V_{-1} \) of \( a_0 = -1 \) such that for every \( a \in V_{-1} \), there is a pair of complex conjugated simple eigenvalues \( \lambda_{1,2}(a) = \mu(a) \pm i\omega(a) \) such that for every \( a \in V_{-1} \), \( \mu(a) > -\frac{1}{3} \), and all other eigenvalues \( \lambda \) have \( \text{Re}\lambda < -\frac{1}{5} \).

Proof. We consider the function \( G(a, y) = \cos \sqrt{\frac{a^2r^2}{e^{2y}} - y^2} - \frac{y}{ar} e^y, \) and we observe that \( G(a_0, 0) = 0 \) and
\[ \frac{\partial G}{\partial y} = \frac{\frac{a^2 y^2}{e^{2y}}}{\sqrt{\frac{a^2 y^2}{e^{2y}} - y^2}} \sin \sqrt{\frac{a^2 y^2}{e^{2y}} - y^2} - \frac{y + 1}{\alpha} e^y, \]

\[ \frac{\partial G}{\partial y}(-1, 0) = \frac{\pi}{2} + \frac{2}{\pi} > 0. \]

The implicit functions theorem implies the existence of: a neighborhood \( W_{-1} \) of \(-1, \) a neighborhood \( W_{0} \) of 0 and an unique function \( y : W_{-1} \to W_{0} \) such that \( G(a, y(a)) = 0 \) for every \( a \in W_{-1} \).

Thus \( \mu(a) = \frac{1}{r} y(a) \), \( \omega(a) = \pm \sqrt{\frac{a^2}{e^{2y}} - \frac{y(a)^2}{r^2}} \) and we can define the eigenvalues

\[ \lambda_{1,2}(a) = \frac{1}{r} y(a) \pm i \sqrt{\frac{a^2}{e^{2y}} - \frac{y(a)^2}{r^2}}. \]

For \( a_0 = -1 \), \( y(a_0) = 0 \), we have \( \sqrt{\frac{a^2 y^2}{e^{2y}} - y^2} = \frac{\pi}{2} \). Let us denote by \( m \) a positive integer such that

\[ \frac{\frac{a^2 y^2}{e^{2y}}}{\sqrt{\frac{a^2 y^2}{e^{2y}} - y^2}} = \left( \frac{\pi}{m} \right)^2 > 0, \quad \epsilon_m < \frac{4}{3}. \]

There is a neighbourhood of \(-1, \) \( W_{m+1} \subset W_{-1} \) such that for \( a \in W_{m+1} \), \( y(a) \in (-\frac{\pi}{m}, \frac{\pi}{m}) \). We shall take \( V_{-1} = (-\frac{2}{\pi}, -\frac{1}{2}) \cap W_{-1} \). This implies

\[ \sqrt{\frac{a^2 y^2}{e^{2y}} - \frac{y(a)^2}{r^2}} \leq \sqrt{\frac{a^2 y^2}{e^{2y}} - y^2} \leq \sqrt{\frac{\frac{a^2 y^2}{e^{2y}}}{\pi^2}} = \frac{2\pi}{\frac{4}{3}} e^{-\frac{\pi}{m}} < \pi, \]

for \( a \in V_{m+1} \).

We consider the equation

\[ \frac{a^2 y^2}{e^{2y}} - y^2 = 0 \]

and denote by \( y_r(a) \) its positive solution. We look for solutions of (11) only at the left of \( y_r(a) \), since only there the expression \( \sqrt{\frac{a^2 y^2}{e^{2y}} - y(a)^2} \) is real.

Since the function \( u \to \frac{\sin u}{u} \) is decreasing on \([0, \pi]\), for \( y \in [0, y_r(a)] \), and \( a < 0 \) we have

\[ \frac{\partial G}{\partial y} > \frac{\frac{a^2 y^2}{e^{2y}} - y^2}{\sqrt{\frac{a^2 y^2}{e^{2y}} - y^2}} \sin \sqrt{\frac{a^2 y^2}{e^{2y}} - y^2} - \frac{y + 1}{\alpha} e^y \geq - \frac{y + 1}{\alpha} e^y > 0. \]

Thus, there are no solutions at the right of \( y(a) \) for \( a \in V_{m+1} \).

Now, for \( y \in (-\frac{\pi}{m}, 0) \)

\[ \sqrt{\frac{a^2 y^2}{e^{2y}} - \frac{y(a)^2}{r^2}} \leq \sqrt{\frac{a^2 y^2}{e^{2y}} - y^2} \leq \frac{2\pi}{\frac{4}{3}} e^{-\frac{\pi}{m}}. \]

We choose \( m = 16 \) and denote the neighborhood \( V_{16} \) by \( V_{-1} \). Then

\[ \sqrt{\frac{a^2 y^2}{e^{2y}} - \frac{y(a)^2}{r^2}} = \frac{\sqrt{16\pi}}{\pi} \geq \frac{\pi}{6}, \quad \frac{\pi}{6} \leq \sqrt{\frac{a^2 y^2}{e^{2y}} - y^2} \leq \frac{2\pi}{3}. \]

and thus
\[
\frac{\partial G}{\partial y} (a, y) = \frac{a e^{2s} + y}{\sqrt{4a^2 + y^2}} \sin \sqrt{\frac{a^2 e^{2s}}{4a^2 + y^2}} - y^2 - \frac{a + 1}{\alpha r} e^y > \\
> \left( \frac{\pi}{4} + y \right) \frac{\pi}{2} - \frac{a + 1}{\alpha r} e^y > \\
\geq \left( \frac{\pi}{4} - \frac{e}{16} \right) \frac{4\pi}{3} + 2 \frac{a + 1}{\alpha r} e^y > 0.
\]

Hence we have no solutions with \( \mu \geq -\frac{e}{16} = -\frac{\pi}{16} \frac{3}{2} = -\frac{1}{8} \) besides \( \mu (a) = \frac{2}{\pi} y (a) \).

To summarize, for each \( a \in V_{-1} \) we have the following:
- There is a pair of eigenvalues of \( 12 \), namely
  \[ \lambda_{1,2} (a) = \frac{2}{\pi} y (a) \pm i \sqrt{\frac{a^2}{\alpha^2 (a)}} - \frac{\mu (a)^2}{\alpha r} \]
  with \( y(a) \) defined above,
- \( \mu (a) > -\frac{1}{8} \),
- all other eigenvalues \( \lambda \) have \( \Re \lambda < -\frac{1}{8} \),
- for \( a_0 = -1 \), \( \lambda_{1,2} (a_0) = \pm i \).

It follows that the hypothesis \( H1 \) is satisfied by our equation.

5 The eigenvectors at \( a_0 \)

The eigenvectors \( 22 \) corresponding to \( \lambda_{1,2} (a_0) \) are \( \varphi_1 (s) = e^{is} \), \( \varphi_2 (s) = e^{-is} \), \( s \in [-r, 0] \), and we denote by \( M^* (\lambda_{1,2} (a_0)) \) the eigenspace spanned by them.

The eigenvectors for the adjoint problem are \( \phi_1 (s) = e^{-is} \), \( \phi_2 (s) = e^{is} \), \( s \in [0, r] \). Let us denote by \( M^* (\lambda_{1,2} (a_0)) \) the space spanned by \( \{\phi_1, \phi_2\} \) in \( C([0, r], \mathbb{R}^n) \).

We define the bilinear form \( (\chi (.), \varphi (.) : M^* (\lambda_{1,2} (a_0)) \times M^* (\lambda_{1,2} (a_0)) \to \mathbb{C}, \)
\( (\chi (.), \varphi (.) ) = \bar{\chi} (0) \varphi (0) - \int_0^\theta \int_{-r}^r \chi (\xi - \theta) \psi (\xi) d\xi d\eta (\theta), \) with \( \eta \) defined by \( 9 \).

Let the numbers \( e_{ij} \) be defined by \( e_{ij} = (\phi_i (.), \varphi_j (.) ) \). We find that the matrix \( E = (e_{ij})_{1 \leq i, j \leq 2} \) is
\[
E = \begin{bmatrix}
\frac{2+\pi i}{2} & 0 \\
0 & -\frac{2-\pi i}{2}
\end{bmatrix}.
\]

The vectors \( \psi_1, \psi_2 \) given by
\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = E^{-1} \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
\]

have the property \( (\psi_1, \varphi_1 ) = \delta_{ij} \).

By \( 13 \) \( \psi_1 (s) = \frac{2}{2+\pi i} e^{is}, \ s \in [0, r], \) and \( \psi_1 (0) = \frac{2 e^{\pi i}}{2+\pi i} \).

6 The Bautin-type bifurcation in the central manifold

We consider \( a_0 = -1 \), when the dynamical system admits a two-dimensional local center manifold, that we denote \( W_{loc} (c) \). Everywhere below, the dependence of \( \alpha = (a, c) \) becomes dependence of \( c \) only.
Obviously, we have for every \( \phi \in C ([r, 0], \mathbb{R}) \),

\[
f ((S_c(t) \phi)(s), (S_c(t) \phi)(s - r), c) = [S_c(t) \phi](s) [S_c(t) \phi](s) + c [S_c(t) \phi](s) [S_c(t) \phi](s - r).
\]

By writing \( w_c(s, z(t), \xi(t)) \) as a series of powers of \( z \) and \( \xi \),

\[
w_c(s, z, \xi) = \sum_{j + k \geq 2} \frac{1}{j!k!} w_{jk}(s, c) z^j \xi^k,
\]

and inserting \( (15) \) and \( (3) \) in \( (14) \), we can obtain the coefficients \( F_{jk} \) of \( (4) \) (here depending on \( s, c \)).

Since

\[
f ((S_c(t) \phi(0), S_c(t) \phi(-r), c) = \left[ z(t) \varphi_1(0) + \xi(t) \bar{\varphi}_1(0) + \frac{1}{2} w_{20}(0) z^2 + w_{11}(0) z \xi + \frac{1}{2} w_{02}(0) \xi^2 + \ldots \right]
\]

and

\[
\left[ z(t) \varphi_1(0) + \xi(t) \bar{\varphi}_1(0) + \frac{1}{2} w_{20}(0) z^2 + w_{11}(0) z \xi + \frac{1}{2} w_{02}(0) \xi^2 + \ldots \right] + \frac{c}{c} \left[ z \varphi_1(-r) + \xi \bar{\varphi}_1(-r) + \frac{1}{2} w_{20}(-r) z^2 + w_{11}(-r) z \xi + \frac{1}{2} w_{02}(-r) \xi^2 + \ldots \right],
\]

we find, by denoting \( F_{jk}(0, c) = F_{jk} \),

\[
F_{20} = 2 (1 - ic), \quad F_{11} = 2, \quad F_{02} = 2 (1 + ic).
\]

By the definition of the function \( g, \) \( (5) \), and of the coefficients \( g_{jk} \),

\[
g_{jk} = \psi_1(0) F_{jk} = \frac{2 - \pi i}{4 + \pi^2} F_{jk}.
\]

Hence, by the above relations, \( g_{20}, g_{11}, g_{02} \) are determined.

Now we look for the second order terms in the series of powers defining \( w_c \), \( (15) \). Differential equations for them are found from the relation \( (5), (5) \)

\[
\frac{\partial}{\partial s} \sum_{j + k \geq 2} \frac{1}{j!k!} w_{jk}(s, c) z^j \xi^k = \sum_{j + k \geq 2} \frac{1}{j!k!} g_{jk}(c) z^j \xi^k \varphi_1(s) + \sum_{j + k \geq 2} \frac{1}{j!k!} \xi \bar{\varphi}_1(s) + \frac{\partial}{\partial t} \sum_{j + k \geq 2} \frac{1}{j!k!} w_{jk}(s, c) z^j \xi^k,
\]

by equating the terms containing the same powers of \( z(t) \) and \( \xi(t) \). The conditions for the determination of the integration constants are obtained from \( (6), (5) \)

\[
\frac{d}{dt} \sum_{j + k \geq 2} \frac{1}{j!k!} w_{jk}(0, c) z^j \xi^k + \sum_{j + k \geq 2} \frac{1}{j!k!} g_{jk}(c) z^j \xi^k \varphi_1(0) + \sum_{j + k \geq 2} \frac{1}{j!k!} \xi \bar{\varphi}_1(0) + \frac{1}{j!k!} g_{jk}(c) z^j \xi^k = - \sum_{j + k \geq 2} \frac{1}{j!k!} g_{jk}(-r, c) z^j \xi^k + \sum_{j + k \geq 2} \frac{1}{j!k!} F_{jk} z^j \xi^k.
\]

Thus, we find for \( w_{20}(s, c) \),
\[ w_{20}' = 2iw_{20}(s,c) + g_{20}(c)e^{is} + \tilde{g}_{02}(c)e^{-is}, \]
\[ 2w_{20}(0,c)i + g_{20}(c) + \tilde{g}_{02}(c) = -w_{20}(-r,c) + 2 - 2ic, \]

and by solving the equation we have
\[ w_{20}(s,c) = w_{20}(0,c)e^{2is} - \frac{i}{2}g_{20}(c) \left( e^{is} - e^{2is} \right) - \frac{i}{\pi^2}g_{02}(c) \left( e^{-is} - e^{2is} \right). \]
\[ w_{20}(0,c) = \frac{2(1+2i)}{15(4+\pi^2)} \left( (4 - 3\pi^2) + 8c + i \left( 8 - 4c + 3\pi^2 \right) \right). \]

For \( w_{11}(s,c) \):
\[ w_{11}' = g_{11}(c)e^{is} + \tilde{g}_{11}(c)e^{-is}, \]
\[ w_{11}(-r) = w_{11}(0) + ig_{11}(c)(i+1) + ig_{11}(c)(i-1), \]

from where
\[ w_{11}(s) = w_{11}(0) - ig_{11}(c)(e^{is} - 1) + ig_{11}(c)(e^{-is} - 1), \]
\[ w_{11}(0) = 2 \frac{e^{i+1} - 1}{4+\pi^2} + g_{11}(c)(1-i) + i\tilde{g}_{11}(c)(1+i). \]

The relation \( w_{02} = \tilde{w}_{20} \) holds true.

### 6.1 The first Lyapunov coefficient

We are now able to calculate \( F_{jk} \) (and thus \( g_{jk} \)) with \( j + k = 3 \).

We find
\[ F_{30} = 3c \left( w_{20}(-r) - w_{20}(0)i \right) + 6w_{20}(0), \]
\[ F_{03} = \tilde{F}_{30}, \]
\[ F_{21} = 2c \left( w_{11}(-r) - iw_{11}(0) \right) + 4w_{11}(0) + c \left( w_{20}(-r) + w_{20}(0)i \right) + 2w_{20}(0), \]
\[ F_{12} = \tilde{F}_{21}, \]

while \( g_{jk} \) are given by (16).

These allow us to calculate the first Lyapunov coefficient, (6):
\[ l_1(c) = \frac{1}{5(4 + \pi^2)} \left[ (8 - 12\pi) c^2 + (72 - 28\pi) c + 144 - 16\pi \right]. \quad (17) \]

We impose the condition
\[ l_1(c) = 0 \iff (8 - 12\pi) c^2 + (72 - 28\pi) c + 144 - 16\pi = 0. \]

The two solutions of this equations are
\[ c_1 = \frac{18 - 7\pi + \sqrt{36 + 212\pi + \pi^2}}{2(3\pi - 2)} \approx 1.52799, \]
\[ c_2 = \frac{18 - 7\pi - \sqrt{36 + 212\pi + \pi^2}}{2(3\pi - 2)} \approx -2.06554. \]

We thus found two values of the parameter \( c \) for which degenerate Hopf bifurcation takes place.
6.2 The second Lyapunov coefficient

The second Lyapunov coefficient at the values $c_1$, $c_2$ of the parameter $c$ has the form

$$12l_2(c_1) = \frac{1}{2\pi} \text{Re} g_{32} +$$

$$+ \frac{1}{4\pi^2} \text{Im} \left[ g_{20}\bar{g}_{31} - g_{11}(4g_{31} + 3\bar{g}_{22}) - \frac{1}{3}g_{02}(g_{40} + \bar{g}_{13}) - g_{30}g_{12} \right] +$$

$$+ \frac{1}{2\pi} \left( \text{Re} \left[ g_{20} \left( \bar{g}_{11}(3g_{12} - \bar{g}_{30}) + g_{02} \left( \bar{g}_{12} - \frac{1}{3}g_{30} \right) + \frac{1}{3}g_{02}g_{30} \right) \right] +$$

$$g_{11} \left( \bar{g}_{02} \left( \frac{4}{3}\bar{g}_{30} + 3g_{12} \right) + \frac{1}{3}g_{02}\bar{g}_{30} - 4g_{11}g_{30} \right) \right] + 3\text{Im} (g_{20}g_{32} \text{Im} g_{21}) +$$

$$+ \frac{1}{2\pi} \left[ \text{Im} (g_{11}\bar{g}_{02} \left( g_{20}^2 - 3\bar{g}_{20}g_{11} - 4\bar{g}_{11}^2 \right) \right] + \text{Im} \left( g_{20}g_{11} \left| 3\text{Re} (g_{20}g_{11}) - 2 \left| g_{02} \right|^2 \right| \right),$$

where $g_{ij}$ are evaluated at $c_i$, $i = 1$ or $i = 2$.

We will calculate first $l_2(c_1)$, and for this, in the sequel, all the $g_{jk}$ will be evaluated at $c_1$.

Hence, we have to calculate $g_{jk}(c_1)$ for $j + k = 4$ and $g_{32}$. In order to do this, we calculate $w_{jk}$, with $j + k = 3$. For $w_{30}$, the equation is

$$w_{30}'(s) = 3iw_{30}(s) + g_{30}e^{is} + \bar{g}_{03}e^{-is} +$$

$$+ 3w_{20}(s)g_{20} + 3w_{11}(s)\bar{g}_{02},$$

with the condition

$$w_{30}(-r) = -3w_{30}(0)i - 3w_{20}(0)g_{20} - 3w_{11}(0)\bar{g}_{02} - g_{30} - \bar{g}_{03} +$$

$$+ 3c_1 \left( w_{20}(-r) - w_{20}(0)i \right) + 6w_{20}(0).$$

The values obtained after solving the above equations, are

$$w_{30}(0) = -327626 - 5.115802i, \quad w_{03}(0) = 5\bar{w}_{30}(0), \quad w_{30}(-r) = -14.190120 - 5.277852i, \quad w_{03}(-r) = \bar{w}_{30}(-r).$$

For $w_{21}$, we have

$$w_{21}'(s) = iw_{21} + g_{21}e^{is} + \bar{g}_{12}e^{-is} + 2w_{20}(s)g_{11} +$$

$$+ w_{11}(s) \left( g_{20} + 2\bar{g}_{11} \right) + w_{02}(s)g_{20}$$

and

$$w_{21}(-r) + iw_{21}(0) = -2w_{20}(0)g_{11} - 2w_{11}(0)\bar{g}_{11} - w_{11}(0)g_{20} \quad (18)$$

$$- w_{02}(0)\bar{g}_{02} - g_{21} - \bar{g}_{12} + F_{21}. $$

In this case, after solving the differential equation above, the second equation for obtaining $w_{21}(-r)$ and $w_{21}(0)$ reads

$$w_{21}(-r) + iw_{21}(0) = F_{21} - F_{21} \frac{8}{4 + \pi^2} + 2g_{11}e^{-ir} \int_0^{-r} w_{20}(\tau)e^{-i\tau} d\tau$$

$$+ e^{-ir} \left( (g_{20} + 2\bar{g}_{11}) \int_0^{-r} w_{11}(\tau)e^{-i\tau} d\tau + \bar{g}_{02} \int_0^{-r} w_{02}(\tau)e^{-i\tau} d\tau \right).$$

The system (18), (19) is not determined. It is to be seen whether it is compatible or not.

We have to compare the right hand sides of (18) and (19). We first notice that
\[ g_{21} + \bar{g}_{12} = \frac{8}{3\pi r} F_{21}. \]

Then, we find by direct calculations that
\[
\begin{align*}
I_1 &= \int_0^{-r} w_{20}(\tau)e^{-ir\tau}d\tau = w_{20}(-r) + iw_{20}(0) - irg_{20} + \bar{g}_{02}, \\
I_2 &= \int_0^{-r} w_{11}(\tau)e^{-ir\tau}d\tau = -w_{11}(-r) - iw_{11}(0) + irg_{11} - \bar{g}_{11}, \\
I_3 &= \int_0^{-r} w_{02}(\tau)e^{-ir\tau}d\tau = \frac{1}{3} [-w_{02}(-r) + iw_{02}(0) + irg_{02} - \bar{g}_{20}],
\end{align*}
\]

and, by using these equalities,
\[
\begin{align*}
-2ig_{11}I_1 + 2w_{20} (0) g_{11} &= 0, \\
-i [g_{20} + 2g_{11}] I_2 + (2g_{11} + g_{20}) w_{11} (0) &= 0, \\
-ig_{02}I_3 + w_{02} (0) \bar{g}_{02} &= 0.
\end{align*}
\]

This leads to the conclusion that the right hand sides of (18) and (19) are equal, and thus, the system in \( w_{21} (-r), w_{21} (0) \) is compatible. We will then take \( w_{21} (0) = 0 \) and it follows that
\[
\begin{align*}
w_{21} (-r) &= -2w_{20} (0) g_{11} - 2w_{11} (0) \bar{g}_{11} - w_{11} (0) g_{20} \\
&\quad - w_{02} (0) \bar{g}_{02} - g_{21} - \bar{g}_{12} + F_{21}.
\end{align*}
\]

Then \( w_{12} (0) = 0 \) and \( w_{12} (-r) = \bar{g}_{21} (-r) \).

We are now able to compute \( g_{jk} \), with \( j + k = 4 \).

Firstly we compute \( F_{4k} \), with \( j + k = 4 \):
\[
\begin{align*}
\frac{1}{24} F_{40} &= \frac{1}{3} w_{30} (0) + \frac{1}{3} w_{20} (0)^2 + c_1 \left( \frac{1}{6} w_{30} (-r) - \frac{1}{6} iw_{30} (0) + \frac{1}{2} w_{20} (0) \frac{1}{2} w_{20} (-r) \right), \\
\frac{1}{6} F_{31} &= c_1 \left( \frac{1}{3} w_{21} (-r) + \frac{1}{2} w_{30} (-r) + \frac{1}{2} iw_{30} (0) + \frac{1}{2} w_{20} (0) w_{11} (-r) \right) \\
&\quad + w_{11} (0) \frac{1}{2} w_{20} (0) + \frac{1}{2} w_{30} (0) + w_{20} (0) w_{11} (0), \\
\frac{1}{12} F_{22} &= c_1 \left( \frac{1}{2} w_{12} (-r) + \frac{1}{2} w_{21} (-r) + \frac{1}{2} w_{30} (0) w_{02} (-r) + w_{11} (0) w_{11} (-r) \right) \\
&\quad + \frac{3}{4} w_{02} (0) w_{20} (-r) + \frac{3}{4} w_{02} (0) w_{02} (0) + w_{11} (0) w_{11} (0),
\end{align*}
\]

We obtain, by using (16):
\[
\begin{align*}
g_{40} &= -70.452908 + 32.020324i, \quad g_{34} = .804019 + 77.383894i, \\
g_{31} &= -13.491939 - 6.450063i, \quad g_{33} = 11.553771 + 9.494531i, \\
g_{22} &= 4.485812 - 7.046298i.
\end{align*}
\]

The only \( g_{jk} \) still to be computed in order to be able to evaluate \( I_2 (c_1) \) is \( g_{32} \). Since
\[
\begin{align*}
F_{32} &= 6w_{22} (0) + 4w_{31} (0) + 6w_{20} (0) w_{12} (0) + 12w_{11} (0) w_{21} (0) + 2w_{02} (0) w_{30} (0) + \\
&\quad + c \left[ 3w_{22} (-r) + 2w_{31} (-r) + 3w_{20} (0) w_{12} (-r) + 6w_{11} (0) w_{21} (-r) \right] + \\
&\quad + w_{02} (0) w_{30} (-r) + w_{30} (0) w_{02} (-r) + 6w_{21} (0) w_{11} (-r) + \\
&\quad + 3w_{12} (0) w_{20} (-r) + 2w_{31} (0) i + 3w_{22} (0) (-i),
\end{align*}
\]
we have to compute \( w_{22}(0), w_{22}(-r), w_{31}(0), w_{31}(-r). \)

The equations for \( w_{22}(s) \) are

\[
\begin{align*}
w_{22} &= g_{22}e^{is} + \tilde{g}_{22}e^{-is} + 2w_{20}(s)g_{12} + 2w_{02}(s)\tilde{g}_{12} + 2w_{11}(s)(g_{21} + \tilde{g}_{21}) + w_{30}(s)g_{02} + w_{03}(s)\tilde{g}_{02} + w_{21}(s)(4g_{11} + \tilde{g}_{20}) + w_{12}(s)(g_{20} + 4\tilde{g}_{11}),
\end{align*}
\]

and

\[
\begin{align*}
w_{22}(-r) &= -(2w_{20}(0)g_{12} + 2w_{11}(0)\tilde{g}_{21} + 2w_{11}(0)g_{21} + +2w_{02}(0)\tilde{g}_{12} + w_{30}(0)g_{02} + 4w_{21}(0)g_{11} + +w_{21}(0)\tilde{g}_{20} + w_{12}(0)g_{20} + 4w_{12}(0)\tilde{g}_{11} + w_{03}(0)\tilde{g}_{02} + g_{22} + \tilde{g}_{22}) + 4w_{12}(0) + 4w_{21}(0) + 2w_{20}(0)w_{02}(0) + 4w_{11}(0)w_{11}(0) + c[2w_{12}(-r) + 2w_{21}(-r) + 2(-i)w_{12}(0) + 2iw_{21}(0)] + +w_{20}(0)w_{02}(-r) + 4w_{11}(0)w_{11}(-r) + w_{02}(0)w_{20}(-r)]
\end{align*}
\]

while those for \( w_{31}(s) \) are

\[
\begin{align*}
w'_{31} &= 2w_{31}i + g_{31}e^{is} + \tilde{g}_{13}e^{-is} + 3w_{20}(s)g_{12} + w_{11}(s)g_{30} + 3w_{11}(s)\tilde{g}_{12} + w_{02}(s)\tilde{g}_{03} + 3w_{30}(s)g_{11} + 3w_{21}(s)g_{20} + 3w_{21}(s)\tilde{g}_{11} + 3w_{12}(s)\tilde{g}_{02},
\end{align*}
\]

and

\[
\begin{align*}
2w_{31}(0)i + w_{31}(-r) &= -(3w_{20}(0)g_{21}(\alpha) + 3w_{11}(0)\tilde{g}_{12}(\alpha) + w_{11}(0)g_{30}(\alpha) + w_{02}(0)\tilde{g}_{03} + 3w_{30}(0)g_{11} + 3w_{21}(0)g_{20} + 3w_{21}(0)\tilde{g}_{11} + 3w_{12}(0)\tilde{g}_{02} + g_{31} + 3\tilde{g}_{13}) + c[3w_{21}(-r) + 3w_{30}(-r) - 3iw_{21}(0) + iw_{30}(0) + +3w_{20}(0)w_{11}(-r) + 3w_{11}(0)w_{20}(-r)] + +[6w_{21}(0) + 2w_{30}(0) + 6w_{20}(0)w_{11}(0)].
\end{align*}
\]

After computations:

\[
\begin{align*}
w_{22}(-r) &= 4.864870928, \\
w_{22}(0) &= -43.85187247, \\
w_{31}(-r) &= -6.41714235 - 18.89415271i, \\
w_{31}(0) &= 17.94690049 + 2.001612024i.
\end{align*}
\]

For \( g_{32} \) we found the value

\[
g_{32} = 28.68605342 + 128.6141166i.
\]

Now we can compute \( l_2(c_1) \). We find:

\[
l_2(c_1) = 13.08553919.
\]

Hence \( l_2(c_1) > 0 \). We are now able to assert and prove

Proposition 2. Hypothesis H2 is satisfied by equation (7).

Proof The only part of Hypothesis H2 that still has to be checked, is that the map \( (\alpha, c) \to (\nu_1, \nu_2) \) is regular at \( (-1, c_1) \), where \( \nu_1 = \frac{\partial \omega}{\partial \omega(a, c)}, \nu_2 = l_1(a, c) \).

We have

\[
\frac{\partial (\nu_1, \nu_2)}{\partial (\alpha, c)} = \begin{pmatrix}
\frac{\partial \omega(a, c)}{\partial a} & 0 \\
\frac{\partial l_1(a, c)}{\partial a} & \frac{\partial l_1(a, c)}{\partial c}
\end{pmatrix}.
\]
We have \( \left( \frac{\mu(a)}{\omega(a)} \right)' \bigg|_{a=-1} = \frac{\mu'(a)\omega(a)-\mu(a)\omega'(a)}{\omega^2(a)} \bigg|_{a=-1} = \frac{\mu'(-1)}{\omega(-1)} = \mu'(-1) \). By taking
the derivative of the equation for \( \mu \), (11), with respect to \( a \), and by evaluating
the result in \( a = -1 \), we find \( \mu'(-1) \neq 0 \).

The form of \( l_1 \), and the fact that the equation \( l_1(-1,c) = 0 \) has two distinct
solutions show that \( \frac{\partial}{\partial c} l_1(-1,c_1) \neq 0 \).

It follows that \( \frac{\partial}{\partial (a,c)} \bigg|_{(-1,c_1)} \neq 0 \) hence the conclusion.\( \square \)

Since all the hypotheses of the Theorem presented in Introduction are satisfied, we may formulate the following result.

**Proposition 3.** The equation (7) presents a Bautin-type bifurcation at
\((a_0,c_0) = (-1,c_1)\).

Following the same path as for \( l_1(c_1) \), we find \( l_2(c_2) < 0 \). In this situation,
as is shown in [4], the two limit cycles (one interior to the other) that should
appear by the Bautin bifurcation for eq. (2), exist for some zone of the quadrant
\( \nu_1 < 0, \nu_2 > 0 \). But \( \nu_1 < 0 \Leftrightarrow \mu(a,c) < 0 \), and we have no theorem to assert
the existence of a bi-dimensional invariant (stable) manifold that is tangent to
\( M_{\lambda_1,2(a)} \). That is why the restriction \( l_2(a) > 0 \) is among the hypotheses of our
Theorem.

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