SUPERCONFORMAL CHANGE OF VARIABLES FOR $N = 1$
NEVEU-SCHWARZ VERTEX OPERATOR SUPeralgebras

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Abstract. Superconformal change of variables formulas for $N = 1$ Neveu-Schwarz vertex operator superalgebras are presented for general invertible superconformal changes of variables. Using the underlying worldsheet supergeometry of propagating superstrings, geometric proofs of the change of variables formulas are given for the case of convergent superconformal changes of variables. More general formal algebraic proofs of the change of variables formulas in the case of formal superconformal changes of variables are then given. Finally, isomorphic families of $N = 1$ NS-VOSAs are derived from the superconformal change of variables formulas.

1. Introduction

In this paper, we prove superconformal change of variables formulas for an $N = 1$ Neveu-Schwarz vertex operator superalgebra ($N = 1$ NS-VOSAs) and for general invertible superconformal changes of variables. Using the correspondence between the worldsheet supergeometry of propagating superstrings in $N = 1$ superconformal field theory and the algebraic structure of $N = 1$ NS-VOSA developed in [B2], [B4] and [B5], we give geometric proofs of the change of variables formulas for the case of convergent superconformal changes of variables. More general formal algebraic proofs of the change of variables formulas in the case of formal superconformal changes of variables are then given. In addition, we introduce isomorphic families of $N = 1$ NS-VOSAs using the superconformal change of variables formulas.

This paper is organized as follows. In Section 2, we give preliminary definitions including the notions of $N = 1$ Neveu-Schwarz Lie superalgebra and superconformal superfunction. We recall the characterization of invertible superconformal functions vanishing at zero given in [B2] and [B4] as exponentials of superderivations which give a representation of the $N = 1$ Neveu-Schwarz algebra. This characterization is crucial for the development of the change of variables formulas. In Section 2.3, we introduce two series of supernumbers, $\Theta^{(1)}_j$ and $\Theta^{(2)}_j$, for $j \in \frac{1}{2} \mathbb{N}$, which are determined by local coordinates vanishing at zero and infinity, respectively, and which will appear in the change of variables formulas.

In Section 3, we recall the notion of $N = 1$ Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra ($N = 1$ NS-VOSA) given in [B2] and [B3] and recall some of the consequences of this notion which we will need later.

In Section 4, we present the change of variables formulas for two different types of changes of variables. First, in Section 4.1, we consider a change of variables by
formal invertible superconformal function vanishing at zero. Second, in Section 4.2, we consider the change of variables associated to a formal invertible superconformal function vanishing at infinity. In both cases, we only state the results; we leave the proofs for Sections 5 and 6. In addition, in Remark 4.3, we indicate how to obtain the change of variables formula for an invertible superconformal change of variables vanishing at a non-zero point from the change of variables for an invertible superconformal function vanishing at zero.

In Section 5, we give geometric proofs of the change of variables formulas in the case of convergent superconformal changes of variables vanishing at zero and infinity. These geometric proofs rely on the worldsheet supergeometry of propagating superstrings in \( N = 1 \) superconformal field theory as developed in [B4], the notion of \( N = 1 \) supergeometric vertex operator superalgebra (\( N = 1 \) SG-VOSA) as introduced in [B5], and the isomorphism between the category of \( N = 1 \) SG-VOSAs and the category of \( N = 1 \) NS-VOSAs as proved in [B5]. In addition to giving alternate proofs to the formal algebraic proofs we will present in Section 6, this section gives the motivation behind the change of variables formulas.

Sections 5.1-5.3 recall the basic structures and results involved in the correspondence between the algebraic and geometric aspects of \( N = 1 \) NS-VOSAs. In Section 5.1, we recall from [B4] the moduli space of superspheres with tubes and the sewing operation and provide two important examples, Examples 5.1 and 5.2, of pairs of superspheres with tubes and the sewing operation. These examples are used in Sections 5.4 and 5.5, respectively, to prove the change of variables formulas. In Section 5.2, we recall the notion of \( N = 1 \) SG-VOSA from [B5]. In Section 5.3, we recall from [B5] how to construct an \( N = 1 \) NS-VOSA from an \( N = 1 \) SG-VOSA, and how to construct an \( N = 1 \) SG-VOSA from an \( N = 1 \) NS-VOSA, and then we recall the Isomorphism Theorem from [B5] which states that the two notions are equivalent because their respective categories are isomorphic. Finally in Sections 5.4 and 5.5, we use Examples 5.1 and 5.2 and the Isomorphism Theorem to prove the change of variables formulas for superconformal changes of variables convergent in a neighborhood of zero and infinity, respectively.

In Section 6, we give proofs of the change of variables formulas for formal algebraic, not necessarily convergent, changes of variables via a formal algebraic argument first employed in [B5] to prove the Isomorphism Theorem. However, in the proof of the Isomorphism Theorem we assumed that all local changes of variables were convergent in some neighborhood. In this section, and in the change of variables formulas presented in Section 4, we do not assume the convergence of the changes of variables and consequently cannot use the Isomorphism Theorem arising from the correspondence between the geometry and algebra as we do in Section 5. However, enough of the algebraic formalism used in the proof of the Isomorphism Theorem in [B5] carries over to the more general setting of formal not necessarily convergent changes of variables, and in Section 6 we retrace that part of the construction used in [B5] in this more general setting to obtain formal algebraic proofs of the change of variables formulas.

In Section 7, we use the results of the change of variables formulas presented in Section 6 to obtain families of isomorphic \( N = 1 \) NS-VOSAs. Finally, in Section 8, we combine the results of Sections 5 and 7 to give the change of variables formulas and family of isomorphic \( N = 1 \) NS-VOSAs for an invertible superconformal change of variables defined on a superannulus.
These results are super-extensions of the change of variables formulas developed by Huang in [H2]. However, there are some typos in [H2] for defining the isomorphic families of VOAs associated to the change of variable formulas in a neighborhood of infinity and in an annulus. We point out those mistakes in Remarks [33] and [34].

2. Preliminaries

2.1. Grassmann algebras and the N = 1 Neveu-Schwarz algebra. Let \( \mathbb{Z}_2 \) denote the integers modulo two. For a \( \mathbb{Z}_2 \)-graded vector space \( V = V^0 \oplus V^1 \), define the \textit{sign function} \( \eta \) on the homogeneous subspaces of \( V \) by \( \eta(v) = i \) for \( v \in V^i \), \( i = 0, 1 \). If \( \eta(v) = 0 \), we say that \( v \) is \textit{even}, and if \( \eta(v) = 1 \), we say that \( v \) is \textit{odd}. A \textit{superalgebra} is an (associative) algebra \( A \) (with identity \( 1 \in A \)), such that: (i) \( A \) is a \( \mathbb{Z}_2 \)-graded algebra; (ii) \( ab = (-1)^{\eta(a)\eta(b)}ba \) for \( a, b \) homogeneous in \( A \).

The exterior algebra over a vector space \( W \), denoted \( \wedge(W) \), has the structure of a superalgebra. Fix \( W_L \) to be an \( L \)-dimensional vector space over \( \mathbb{C} \) for \( L \in \mathbb{N} \) with fixed basis \( \{e_1, ..., e_L\} \) such that \( W_L \subset W_{L+1} \). We denote \( \bigwedge(W_L) \) by \( \bigwedge_L \) and call this the \textit{Grassmann algebra on \( L \) generators}. Note that \( \bigwedge_L \subset \bigwedge_{L+1} \) and taking the direct limit as \( L \to \infty \), we have the \textit{infinite Grassmann algebra} denoted by \( \bigwedge_\infty \).

We use the notation \( \bigwedge_a \) to denote a Grassmann algebra, finite or infinite.

The \( \mathbb{Z}_2 \)-grading of \( \bigwedge_a \) is given explicitly by

\[
\bigwedge_a^0 = \left\{ a \in \bigwedge_a \mid a = \sum_{(i) \in I_a} a_{(i)} \zeta_{i_1} \cdots \zeta_{i_{2n}}, a_{(i)} \in \mathbb{C}, n \in \mathbb{N} \right\}
\]

\[
\bigwedge_a^1 = \left\{ a \in \bigwedge_a \mid a = \sum_{(j) \in J_a} a_{(j)} \zeta_{j_1} \cdots \zeta_{j_{2n+1}}, a_{(j)} \in \mathbb{C}, n \in \mathbb{N} \right\},
\]

where

\[ I_a = \left\{ (i) = (i_1, i_2, ..., i_{2n}) \mid i_1 < i_2 < \cdots < i_{2n}, i_1 \in \{1, 2, ..., \ast\}, n \in \mathbb{N} \right\}, \]

\[ J_a = \left\{ (j) = (j_1, j_2, ..., j_{2n+1}) \mid j_1 < j_2 < \cdots < j_{2n+1}, j_1 \in \{1, 2, ..., \ast\}, j_2 \in \{1, 2, ..., \ast\}, n \in \mathbb{N} \right\}, \]

with \( \{1, 2, ..., \ast\} \) denoting \( \{1, 2, ..., L\} \) if \( \bigwedge_a \) is the finite-dimensional Grassmann algebra \( \bigwedge_L \) and denoting the positive integers \( \mathbb{Z}_+ \) if \( \bigwedge_a = \bigwedge_\infty \). We can also decompose \( \bigwedge_a \) into \textit{body} \( (\bigwedge_a)_B = \{a(0) \in \mathbb{C}\} \), and \textit{soul}

\[(\bigwedge_a)_S = \left\{ a \in \bigwedge_a \mid a = \sum_{(k) \in I_a \cup J_a, k \neq (\emptyset)} a_{(k)} \zeta_{k_1} \cdots \zeta_{k_n}, a_{(k)} \in \mathbb{C} \right\}
\]

subspaces such that \( \bigwedge_a = (\bigwedge_a)_B \oplus (\bigwedge_a)_S \). For \( a \in \bigwedge_a \), we write \( a = a_B + a_S \) for its body and soul decomposition.

A \( \mathbb{Z}_2 \)-graded vector space \( g \) is said to be a \textit{Lie superalgebra} if it has a bilinear operation \( [\cdot, \cdot] \) such that for \( u, v \) homogeneous in \( g \): (i) \( [u, v] \in g^{(\eta(u) + \eta(v)) \mod 2} \); (ii) skew-symmetry holds \( [u, v] = -(-1)^{\eta(u)\eta(v)}[v, u] \); (iii) the Jacobi identity holds \( (-1)^{\eta(u)\eta(v)}[u, [v, w]] + (-1)^{\eta(v)\eta(u)}[[u, v], w] + (-1)^{\eta(u)\eta(v)}[[[u, v], w]] = 0 \).

For any \( \mathbb{Z}_2 \)-graded associative algebra \( A \) and for \( u, v \in A \) of homogeneous sign, we can define \( [u, v] = uv - (-1)^{\eta(u)\eta(v)}vu \), making \( A \) into a Lie superalgebra. The algebra of endomorphisms of \( A \), denoted \( \text{End} A \), has a natural \( \mathbb{Z}_2 \)-grading induced from that of \( A \), and defining \( [X, Y] = XY - (-1)^{\eta(X)\eta(Y)}YX \) for \( X, Y \) homogeneous in \( \text{End} A \), this gives \( \text{End} A \) a Lie superalgebra structure. An element \( D \in (\text{End} A)^1 \), for \( i \in \mathbb{Z}_2 \), is called a \textit{superderivation of sign} \( i \) (denoted \( \eta(D) = i \)) if \( D \) satisfies the super-Leibniz rule \( D(uv) = (Du)v + (-1)^{\eta(D)\eta(u)}uDv \) for \( u, v \in A \) homogeneous.
The Virasoro algebra is the Lie algebra with basis consisting of the central element $d$ (called the \textit{central charge}) and $L_n$, for $n \in \mathbb{Z}$, satisfying commutation relations
\begin{equation}
[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0} d,
\end{equation}
for $m, n \in \mathbb{Z}$. The $N = 1$ Neveu-Schwarz Lie superalgebra is a super-extension of the Virasoro algebra by the odd elements $G_{n+1/2}$, for $n \in \mathbb{Z}$, with supercommutation relations
\begin{align}
\left[G_{m+\frac{1}{2}}, L_n\right] &= (m - n - \frac{1}{2})G_{m+n+\frac{1}{2}} \\
\left[G_{m+\frac{1}{2}}, G_{n-\frac{1}{2}}\right] &= 2L_{m+n} + \frac{1}{3}(m^2 + m)\delta_{m+n,0} d
\end{align}
in addition to (2.1). We denote the $N = 1$ Neveu-Schwarz algebra by $\mathfrak{ns}$.

Let $V = \prod_{k \in (1,2)} V(k)$ be a module for $\mathfrak{ns}$, and let $L(n), G(n - 1/2) \in \text{End } V$ and $c \in \mathbb{C}$ be the representation images of $L_n$, $G_{n-1/2}$ and $d$, respectively, such that $L(0)v = kv$ for $v \in V(k)$. If in addition, $V(k) = 0$ for $k$ sufficiently small, we call $V$ a \textit{positive energy representation} of $\mathfrak{ns}$.

\section*{2.2. Superanalytic and superconformal superfunctions.}

Let $U$ be a subset of $\wedge_*$, and write $U = U^0 \oplus U^1$ for the decomposition of $U$ into even and odd subspaces. Let $z$ be an even variable in $U^0$ and $\theta$ an odd variable in $U^1$. We call $H : U \rightarrow \wedge_*$, mapping $(z, \theta) \mapsto H(z, \theta)$, a $\wedge_*$-superfunction on $U$ in (1,1)-variables or just a superfunction.

Let $z_B$ be a complex variable and $h(z_B)$ a complex analytic function in some open set $U_B \subseteq \mathbb{C}$. For $z$ a variable in $\wedge^0_*$, we define $h(z)$ to be the Taylor expansion about the body of $z = z_B + z_S$. Then $h(z)$ is well defined (i.e., convergent) in the open neighborhood $\{z = z_B + z_S \in \wedge^0_* \mid z_B \in U_B\} = U_B \times (\wedge^0_*)_S \subseteq \wedge^0_*$. Since $h(z)$ is algebraic in each $z(i)$, for $(i) \in I_z$, it follows that $h(z)$ is complex analytic in each of the complex variables $z(i)$.

For $n \in \mathbb{N}$, we introduce the notation $\wedge_{*, n}$ to denote a finite Grassmann algebra $\wedge_{*, n}$ with $L > n$ or an infinite Grassmann algebra. We will use the corresponding index notations for the corresponding indexing sets $I_{*, n}$ and $J_{*, n}$. A \textit{superanalytic $\wedge_{*, n}$-superfunction in (1,1)-variables} $H$ is a superfunction in the form
\begin{align}
H(z, \theta) &= (f(z) + \theta \xi(z), \psi(z) + \theta g(z)) \\
&= \left( \sum_{(i) \in I_{*, -1}} f_{(i)}(z)\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_n} + \theta \sum_{(j) \in J_{*, -1}} \xi_{(j)}(z)\zeta_{j_1}\zeta_{j_2}\cdots\zeta_{j_{2n+1}}, \right. \\
&\left. \sum_{(j) \in J_{*, -1}} \psi_{(j)}(z)\zeta_{j_1}\zeta_{j_2}\cdots\zeta_{j_{2n+1}} + \theta \sum_{(i) \in I_{*, -1}} g_{(i)}(z)\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_n} \right)
\end{align}
where $f_{(i)}(z_B)$, $g_{(i)}(z_B)$, $\xi_{(j)}(z_B)$, and $\psi_{(j)}(z_B)$ are all complex analytic in some non-empty open subset $U_B \subseteq \mathbb{C}$. If each $f_{(i)}(z_B)$, $g_{(i)}(z_B)$, $\xi_{(j)}(z_B)$, and $\psi_{(j)}(z_B)$ is complex analytic in $U_B \subseteq \mathbb{C}$, then $H(z, \theta)$ is well defined (i.e., convergent) for \{$(z, \theta) \in \wedge_{*, n} \mid z_B \in U_B\} = U_B \times (\wedge_{*, n})_S$. Consider the topology on $\wedge_*$ given by the product of the usual topology on $(\wedge_*)_B = \mathbb{C}$ and the trivial topology on $(\wedge_*)_S$. This topology on $\wedge_*$ is called the \textit{DeWitt topology}. The natural domain of any superanalytic $\wedge_{*, n}$-superfunction is an open set in the DeWitt topology on $\wedge_{*, n}$. 
Since $1/a = \sum_{n \in \mathbb{N}} (-1)^n a_B^n / a_B^{n+1}$ is well defined if and only if $a_B \neq 0$, the set of invertible elements in $\mathbb{A}_*$, denoted $\mathbb{A}_*^*$, is given by $\mathbb{A}_*^* = \{ a \in \mathbb{A}_* \mid a_B \neq 0 \}$.

We define the (left) partial derivatives $\partial / \partial z$ and $\partial / \partial \theta$ acting on superfunctions which are superanalytic in some DeWitt open neighborhood $U$ of $(z, \theta) \in \mathbb{A}_{*>0}$ by

$$
\Delta z \left( \frac{\partial}{\partial z} H(z, \theta) \right) + O((\Delta z)^2) = H(z + \Delta z, \theta) - H(z, \theta)
$$

$$
\Delta \theta \left( \frac{\partial}{\partial \theta} H(z, \theta) \right) = H(z, \theta + \Delta \theta) - H(z, \theta)
$$

for all $\Delta z \in \mathbb{A}_{*>0}$ and $\Delta \theta \in \mathbb{A}_{*>0}^1$ such that $z + \Delta z \in U^0 = U_B \times (\mathbb{A}_{*>0})^S$ and $\theta + \Delta \theta \in U^1 = \mathbb{A}_{*>0}^1$. Note that $\partial / \partial z$ and $\partial / \partial \theta$ are endomorphisms of the superalgebra of superanalytic superfunctions, and in fact, are even and odd superderivations, respectively.

If $h(z_B)$ is complex analytic in an open neighborhood of the complex plane, then $h(z_B)$ has a Laurent series expansion in $z_B$, given by $h(z_B) = \sum_{l \in \mathbb{Z}} c_l z_B^l$, for $c_l \in \mathbb{C}$, and we have $h(z) = \sum_{n \in \mathbb{N}} (z_B^n / n!) h^{(n)}(z_B) = \sum_{l \in \mathbb{Z}} c_l (z_B + z_S)^l = \sum_{l \in \mathbb{Z}} c_l z^l$ where $(z_B + z_S)^l$, for $l \in \mathbb{Z}$, is always understood to mean expansion in positive powers of the second variable, in this case $z_S$. Thus if $H$ is a $\mathbb{A}_{*>0}$-superfunction in $(1,1)$-variables which is superanalytic in a (DeWitt) open neighborhood, $H$ can be expanded as

$$
H(z, \theta) = \left( \sum_{l \in \mathbb{Z}} a_l z^l + \theta \sum_{l \in \mathbb{Z}} n_l z^l, \sum_{l \in \mathbb{Z}} m_l z^l + \theta \sum_{l \in \mathbb{Z}} b_l z^l \right)
$$

for $a_l, b_l \in \mathbb{A}_{*>0}^0$ and $m_l, n_l \in \mathbb{A}_{*>0}^1$.

Define $D$ to be the odd superderivation $D = \partial / \partial \theta + \theta \partial / \partial z$ acting on superanalytic superfunctions. Then $D^2 = \partial / \partial z$, and if $H(z, \theta) = (\check{z}, \check{\theta})$ is superanalytic in some DeWitt open subset, then $D$ transforms under $H(z, \theta)$ by $D = (D\check{\theta})\check{D} + (D\check{z} - \check{\theta} D\hat{\theta})\check{D}^2$. We define a superconformal $(1,1)$-superfunction on a DeWitt open subset $U$ of $\mathbb{A}_{*>0}$ to be a superanalytic superfunction $H$ under which $D$ transforms homogeneously of degree one. Thus a superanalytic function $H(z, \theta) = (\check{z}, \check{\theta})$ is superconformal if and only if, in addition to being superanalytic, $H$ satisfies $D\check{z} - \check{\theta} D\hat{\theta} = 0$, for $D\hat{\theta}$ not identically zero, thus transforming $D$ by $D = (D\check{\theta})\check{D}$.

Let $R$ be a superalgebra, let $x$ be an even formal variable, and let $\varphi$ be an odd formal variable. By this we mean that $x$ commutes with all formal variables and all superalgebra elements and $\varphi$ anticommutes with all odd formal variables and all odd superalgebra elements but commutes with even elements. We use the following notational conventions for $V$ a vector space:

$$
V[x] = \{ \sum_{j \in \mathbb{N}} v_j x^j \mid v_j \in V, \text{ all but finitely many } v_j = 0 \}
$$

$$
V[x, x^{-1}] = \{ \sum_{j \in \mathbb{Z}} v_j x^j \mid v_j \in V, \text{ all but finitely many } v_j = 0 \}
$$

$$
V[[x]] = \{ \sum_{j \in \mathbb{N}} v_j x^j \mid v_j \in V \}
$$

$$
V[[x, x^{-1}]] = \{ \sum_{j \in \mathbb{Z}} v_j x^j \mid v_j \in V \}$$
which allows us to define a map \[ \tilde{H} \] coordinates vanishing at zero. Note that if \( R \)

\[ R \text{ power series in } H \]

that \( H \in \) determined by

\[ a \]

this defines a bijection in \( \text{Der}(\mathbb{R}[x, x^{-1}]) \). It is easy to check that \( L_j(x, \varphi) \) and \( G_{j-\frac{1}{2}}(x, \varphi) \) give a representation of the \( N = 1 \) Neveu-Schwarz superalgebra with central charge zero.

For any formal power series \( H \in \mathbb{R}[x, x^{-1}][[\varphi]] \), we say that \( H(x, \varphi) = (\tilde{x}, \tilde{\varphi}) \) is superconformal if \( D\tilde{x} = \tilde{\varphi}D\tilde{\varphi} \) for \( D = \partial/\partial x + \varphi \partial/\partial x \).

Let \( (\mathbb{R}^0)^\infty \) be the set of all sequences \( A = \{a_j\}_{j \in \mathbb{Z}_+} \) of even elements in \( R \), let \( (\mathbb{R}^1)^\infty \) be the set of all sequences \( M = \{M_{j-1/2}\}_{j \in \mathbb{Z}_+} \) of odd elements in \( R \), and let \( \mathbb{R}^\infty = (\mathbb{R}^0)^\infty \oplus (\mathbb{R}^1)^\infty \). Let \( (\mathbb{R}^0)^\infty \) denote the set of invertible elements in \( \mathbb{R}^0 \).

Consider the set of formal superconformal power series \( H \in x \mathbb{R}[x] \oplus \varphi \mathbb{R}[x] \) with invertible even coefficient of \( \varphi \), i.e., of the form

\[ H(x, \varphi) = \left( a_0^\varphi \left( x + \sum_{j \in \mathbb{Z}_+} a_j x^{j+1} + \varphi \sum_{j \in \mathbb{Z}_+} n_j x^j \right) \right) \]

\[ = \left( \sum_{j \in \mathbb{Z}_+} m_{j-\frac{1}{2}} x^j + \varphi \left( 1 + \sum_{j \in \mathbb{Z}_+} b_j x^j \right) \right) \]

with \( a_0^\varphi \in (\mathbb{R}^0)^\infty \), \( (a, m) = \{(a_j, m_{j-\frac{1}{2}})\}_{j \in \mathbb{Z}_+} \in \mathbb{R}^\infty \) and satisfying \( D\tilde{x} = \tilde{\varphi}D\tilde{\varphi} \).

If \( R = \Lambda_\circ \) and \( (x, \varphi) = (z, \theta) \) then these are the invertible superconformal local coordinates vanishing at zero. Note that if \( H \) is given by (2.7), then the condition that \( H \) be superconformal means that \( n_j \) and \( b_j \), for \( j \in \mathbb{Z}_+ \), are completely determined by \( a_0^\varphi \), the \( a_j \)'s and the \( m_{j-\frac{1}{2}} \)'s, for \( j \in \mathbb{Z}_+ \). In [32] and [33], we show that there is a bijection between the set of formal superconformal power series \( H \in x \mathbb{R}[x] \oplus \varphi \mathbb{R}[x] \) with invertible even coefficient of \( \varphi \) and the set of formal power series of the form

\[ \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( A_j L_j(x, \varphi) + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}}(x, \varphi) \right) \right) \cdot (a_0^\varphi x, a_0^\varphi) \]

with \( a_0^\varphi \in (\mathbb{R}^0)^\infty \), and \( (A, M) = \{(A_j, M_{j-\frac{1}{2}})\}_{j \in \mathbb{Z}_+} \in \mathbb{R}^\infty \). As is shown in [34], this defines a bijection

\[ E: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty \]

\[ (A, M) \rightarrow (a, m). \]

which allows us to define a map \( \tilde{E} \) from \( \mathbb{R}^\infty \) to the set of all formal superconformal power series in \( x \mathbb{R}[x] \oplus \varphi \mathbb{R}[x] \) with leading even coefficient of \( \varphi \) equal to one, by
defining
\begin{align}
\varphi \tilde{E}^0(A, M)(x, \varphi) &= \varphi \left( x + \sum_{j \in \mathbb{Z}_+} E_j(A, M) x^{j+1} \right), \\
\varphi \tilde{E}^1(A, M)(x, \varphi) &= \varphi \sum_{j \in \mathbb{Z}_+} E_{j-\frac{1}{2}}(A, M) x^j,
\end{align}
and letting \( \tilde{E}(A, M)(x, \varphi) \) be the unique formal superconformal power series with even coefficient of \( \varphi \) equal to one such that the even and odd components of \( \tilde{E} \) satisfy (2.10) and (2.11), respectively. Similarly, we define a map \( \hat{E} \) from \((R^0)^{\times} \times R^\infty\) to the set of all formal superconformal power series in \( xR[[x]] \oplus \varphi R[[x]] \) with invertible leading even coefficient of \( \varphi \), by defining
\begin{align}
\hat{E}(a_\Box, A, M)(x, \varphi) &= (\tilde{E}^0(a_\Box, A, M)(x, \varphi), \tilde{E}^1(a_\Box, A, M)(x, \varphi)) \\
&= (a_\Box^2 \tilde{E}^0(A, M)(x, \varphi), a_\Box \tilde{E}^1(A, M)(x, \varphi)).
\end{align}
Thus we have the following proposition which is proved in [B4].

**Proposition 2.1.** ([B4]) The map \( \hat{E} \) from \((R^0)^{\times} \times R^\infty\) to the set of all formal superconformal power series \( H(x, \varphi) \in xR[[x]] \oplus \varphi R[[x]] \) of the form
\begin{equation}
\varphi H(x, \varphi) = \varphi \left( a_\Box^2 \left( x + \sum_{j \in \mathbb{Z}_+} a_j x^{j+1} \right), a_\Box \sum_{j \in \mathbb{Z}_+} m_j x^j \right)
\end{equation}
and with even coefficient of \( \varphi \) equal to \( a_\Box \) for \( (a_\Box, a, m) \in (R^0)^{\times} \times R^\infty \), is a bijection.

The map \( \tilde{E} \) from \( R^\infty \) to the set of formal superconformal power series of the form (2.13) with \( a_\Box = 1 \) and even coefficient of \( \varphi \) equal to one is also a bijection.

In particular, we have inverses \( \tilde{E}^{-1} \) and \( \hat{E}^{-1} \).

Note: In [B4], the space \( xR[[x]] \oplus \varphi R[[x]] \) was erroneously denoted \( xR[[x]]|_{\varphi} \).

**Remark 2.2.** Let \( I(x, \varphi) = (1/x, i\varphi/x) \). Then \( I \) is superconformal and vanishing at \((x, \varphi) = (\infty, 0) = \infty \). Note that the composition of two superconformal functions is again superconformal. Thus if \( H \in x^{-1} R[[x^{-1}]]|_{\varphi} \) is superconformal, vanishing at infinity and with even coefficient of \( \varphi x^{-1} \) equal to \( i \), then \( H \circ I^{-1} \) is superconformal, vanishing at zero and with even coefficient of \( \varphi \) equal to one. Then by Proposition 2.1 \( H \circ I^{-1}(x, \varphi) = \tilde{E}(A, M)(x, \varphi) \), and therefore
\begin{align*}
H(x, \varphi) &= \tilde{E}(A, M) \circ I(x, \varphi) \\
&= \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( A_j L_j(x, \varphi) + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}}(x, \varphi) \right) \right) \cdot (x, \varphi) \bigg|_{(x, \varphi) = I(x, \varphi)} \\
&= \exp \left( \sum_{j \in \mathbb{Z}_+} \left( A_j L_{-j}(x, \varphi) + i M_{j-\frac{1}{2}} G_{j+\frac{1}{2}}(x, \varphi) \right) \right) \cdot (x, \varphi).
\end{align*}

2.3. The \( \{\Theta^{(1)}_j, \Theta^{(1)}_{j-1/2}\}_{j \in \mathbb{Z}} \) and \( \{\Theta^{(2)}_j, \Theta^{(2)}_{j-1/2}\}_{j \in \mathbb{Z}} \) series. In this section we recall two series first introduced in [B4] which will play a central role in the change of variables formulas. Though rather technical as introduced here, a geometric motivation for these series will be given in Section 5.
Let $\alpha_0^{1/2}$ and $A_j$, for $j \in \mathbb{Z}_+$, be even formal variables and let $M_{j - \frac{1}{2}}$, for $j \in \mathbb{Z}_+$, be odd formal variables. Let

$$H_{\alpha_0^{1/2}, A, M}^{(1)}(x, \varphi) = \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( A_j L_j(x, \varphi) + M_{j - \frac{1}{2}} G_{j - \frac{1}{2}}(x, \varphi) \right) \right).$$

Let

$$(\tilde{x}, \varphi) = (H_{\alpha_0^{1/2}, A, M}^{(1)})^{-1}(x, \varphi) \in (\alpha_0^{-1} x, \alpha_0^{-\frac{1}{2}} \varphi) + x \mathbb{C}[x, \varphi][\alpha_0^{-\frac{1}{2}}][[A]][[M]].$$

Let $w$ be another even formal variable and define the superconformal shift

$$(2.14) \quad s_{(x, \varphi)}(w, \rho) = (w - x - \rho \varphi, \rho - \varphi).$$

Then $s_{(x, \varphi)} \circ H_{\alpha_0^{1/2}, A, M}^{(1)} \circ s_{(\tilde{x}, \varphi)}^{-1}(\alpha_0^{-1} w, \alpha_0^{-\frac{1}{2}} \rho)$ is in

$$w \mathbb{C}[x, \varphi][\alpha_0^{\frac{1}{2}}][[A]][[M]][[w]] \oplus \rho \mathbb{C}[x, \varphi][\alpha_0^{\frac{1}{2}}][[A]][[M]][[w]],$$

is superconformal in $(w, \rho)$, and the even coefficient of the monomial $\rho$ is an element in $(1 + x \mathbb{C}[x][\alpha_0^{-1/2}][[A]][[M]] \oplus \varphi \mathbb{C}[x][\alpha_0^{-1/2}][[A]][[M]])$.

Let $\Theta_{(1)}^j = \Theta_{(1)}^{(1)}(\alpha_0^{1/2}, A, M, (x, \varphi)) \in \mathbb{C}[x, \varphi][\alpha_0^{1/2}, \alpha_0^{-1/2}][[A]][[M]]$, for $j \in \frac{1}{2} \mathbb{N}$, be defined by

$$(2.15) \quad \left\{ \exp(\Theta_0^{(1)}(\alpha_0^{\frac{1}{2}}, A, M, (x, \varphi))), \left\{ \Theta_j^{(1)}(\alpha_0^{\frac{1}{2}}, A, M, (x, \varphi)) \right\}_{j \in \mathbb{Z}_+} \right\} = \hat{E}^{-1}(s_{(x, \varphi)} \circ H_{\alpha_0^{1/2}, A, M}^{(1)} \circ s_{(\tilde{x}, \varphi)}^{-1}(\alpha_0^{-1} w, \alpha_0^{-\frac{1}{2}} \rho)).$$

In other words, the $\Theta_j^{(1)}$s are determined uniquely by

$$s_{(x, \varphi)} \circ H_{\alpha_0^{1/2}, A, M}^{(1)} \circ s_{(\tilde{x}, \varphi)}^{-1}(\alpha_0^{-1} w, \alpha_0^{-\frac{1}{2}} \rho)$$

$$= \exp \left( \sum_{j \in \mathbb{Z}_+} \left( \Theta_j^{(1)} \left( w^{j+1} \frac{\partial}{\partial w} + \left( \frac{j + 1}{2} \right) \rho w^j \frac{\partial}{\partial \rho} \right) + \Theta_{j - \frac{1}{2}}^{(1)} \left( w^j \left( \frac{\partial}{\partial \rho} - \rho \frac{\partial}{\partial w} \right) \right) \right) \right) \exp \left( \Theta_0^{(1)} \left( 2w \frac{\partial}{\partial w} + \rho \frac{\partial}{\partial \rho} \right) \right).$$

This formal power series in $(w, \rho)$ gives the formal local superconformal coordinate at a puncture of the canonical supersphere obtained from the sewing together of two particular canonical superspheres with punctures as we shall in Example 5.1 in Section 5.1.

The following proposition states results contained in Corollary 3.44 and Proposition 3.45 in [B3]. Corollary 3.44 and Proposition 3.45 are in turn based on Proposition 3.30 of [B3]. However, there is a typo on line six on p.72 in [B3] of the proof of Proposition 3.30 and hence in the statement of Proposition 3.30 and in the
Let \( w, \mathcal{W} \) holds for \( \Theta \).

Line six on p.72 of the proof of Proposition 3.30 in [B4] should read
\[
\exp \left( -\alpha_0 \frac{\partial}{\partial w} x - \alpha_0 \frac{\partial}{\partial w} \phi \frac{\partial}{\partial \rho} \right)
\]
rather than
\[
\exp \left( -(\alpha_0 \partial_x - \alpha_0 \frac{\partial}{\partial w}) x - \alpha_0 \frac{\partial}{\partial w} \phi \frac{\partial}{\partial \rho} \right).
\]

We will need the corrected results from Corollary 3.44 and Proposition 3.45 in [B4], and these results are given in the proposition below. It is the first line of the right-hand side of equation (2.16) below that contains the corrected terms.

\[\text{Proposition 2.3.} \quad \text{The formal series } \Theta_j^{(1)}(t^{-1/2}a_0^{1/2}, A, M, (x, \varphi)), \text{ for } j \in \frac{1}{2} \mathbb{N}, \text{ are } \mathbb{C}[x, \varphi][A][M][a_0^{1/2}][[t^{1/2}]]. \text{ Thus for } a_\square \in (\Lambda_\alpha^{1/2})^\times, \text{ and } (A, M) \in \Lambda_\alpha, \text{ the series } \Theta_j^{(1)}(t^{-1/2}a_\square, A, M, (x, \varphi)), \text{ for } j \in \frac{1}{2} \mathbb{N}, \text{ are well defined and belong to } \\
\Lambda_\alpha[x, \varphi][[t^{1/2}]].
\]

Let \( \tilde{x}(t^{1/2}), \tilde{\varphi}(t^{1/2}) = (H_j^{(1)}(t^{-1/2}a_\square, A, M)^{-1}(x, \varphi), \text{ and let } V \text{ be a positive-energy module for } \mathfrak{ns}. \text{ Then in } \langle \langle \text{End} (\Lambda_\alpha \otimes \mathbb{C} V) \rangle[[t^{1/2}]][[x, x^{-1}]]\varphi \rangle^0, \text{ the following identity holds for } \Theta_j^{(1)}(t^{1/2}) = \Theta_j^{(1)}(t^{-1/2}a_\square, A, M, (x, \varphi)).
\]

\[
(2.16) \quad \exp \left( -\sum_{m=-1}^{\infty} \sum_{j \in \mathbb{Z}_+} \frac{j + 1}{m + 1} t^j a_\square^{-2j} x^j \right)
\]
\[
\left( A_j + 2 \left( \frac{j - m}{j + 1} \right) t^{-\frac{j}{2}} a_\square x^{-1} \varphi M_{j-\frac{j}{2}} \right) L(m) + x^{-1} \left( \left( \frac{j - m}{j + 1} \right) t^{-\frac{j}{2}} a_\square M_{j-\frac{j}{2}} + \varphi \left( \frac{j - m}{2} \right) A_j \right) G(m + \frac{1}{2})
\]
\[
= \exp \left( t^{-1} a_\square \tilde{x}(t^{1/2}) - x - t^{-\frac{j}{2}} a_\square \tilde{\varphi}(t^{1/2}) \varphi \right) L(-1) + (t^{-\frac{j}{2}} a_\square \tilde{\varphi}(t^{1/2}) - \varphi) G(-\frac{1}{2})
\]
\]
\[
\exp \left( -\sum_{j \in \mathbb{Z}_+} \left( \Theta_j^{(1)}(t^{1/2}) L(j) + \Theta_j^{(1)}(t^{1/2}) G(j - \frac{1}{2}) \right) \right) \cdot \exp \left( -2\Theta_0^{(1)}(t^{1/2}) L(0) \right).
\]

Similarly, let \( B_j \), for \( j \in \mathbb{Z}_+ \), be even formal variables and let \( N_{j-\frac{j}{2}} \), for \( j \in \mathbb{Z}_+ \), be odd formal variables. Let
\[
H^{(2)}(x, \varphi) = \exp \left( \sum_{j \in \mathbb{Z}_+} \left( B_j L_{j-\frac{j}{2}}(x, \varphi) + N_{j-\frac{j}{2}} G_{j+\frac{j}{2}}(x, \varphi) \right) \right) \cdot \left( \frac{1}{x^{1/2}} \right)
\]
\[
= \hat{E}(B, -iN) \left( \frac{1}{x^{1/2}} \right)
\]
and let
\[
(\tilde{x}, \tilde{\varphi}) = (H^{(2)}_{B, N})^{-1} \circ I(x, \varphi) \in (x, \varphi) + \mathbb{C}[x^{-1}, \varphi][[B]][N].
\]

Let \( w \) be another even formal variable and \( \rho \) another odd formal variable. Now write \( s_{(x, \varphi)}(w, \rho) = (-x + w - \rho \varphi, \rho - \varphi). \) We will use the convention that we
should expand \((-x + w - \rho \varphi)^j = (-x + w)^j - j \rho \varphi (-x + w)^{j-1}\) in positive powers of the second even variable \(w\), for \(j \in \mathbb{Z}\). Then

\[
s_{(x, \varphi)} \circ I^{-1} \circ H_{B, N}^{(2)} \circ s_{(\tilde{x}, \tilde{\varphi})}^{-1}(w, \rho) \in w \mathbb{C}[x^{-1}, \varphi][[B]][[N]][[w]] \oplus \rho \mathbb{C}[x^{-1}, \varphi][[B]][[N]][[w]],
\]

is superconformal in \((w, \rho)\), and the even coefficient of the monomial \(\rho\) is an element in \((1 + x^{-1} \mathbb{C}[x^{-1}, \varphi][[B]][[N]]\).

Let \(\Theta_j^{(2)} = \Theta_j^{(2)}(B, N, (x, \varphi)) \in \mathbb{C}[x^{-1}, \varphi][[B]][[N]], \) for \(j \in \frac{1}{2} \mathbb{N}\), be defined by

\[
(2.17) \quad \left( \exp(\Theta_0^{(2)}(B, N, (x, \varphi)), \left\{ \Theta_j^{(2)}(B, N, (x, \varphi)), \Theta_{\frac{j}{2}}^{(2)}(B, N, (x, \varphi)) \right\}_{j \in \mathbb{Z}_+} \right)
\]

\[
= \hat{E}^{-1}(s_{(x, \varphi)} \circ I^{-1} \circ H_{B, N}^{(2)} \circ s_{(\tilde{x}, \tilde{\varphi})}^{-1}(w, \rho)).
\]

In other words, the \(\Theta_j^{(2)}\)'s are determined uniquely by

\[
s_{(x, \varphi)} \circ I^{-1} \circ H_{B, N}^{(2)} \circ s_{(\tilde{x}, \tilde{\varphi})}^{-1}(w, \rho)
\]

\[
\exp \left( \sum_{j \in \mathbb{Z}_+} \Theta_j^{(2)}(w^{j+1} \frac{\partial}{\partial w} + (j + \frac{1}{2}) \rho w^j \frac{\partial}{\partial \rho} + \Theta_{j-\frac{1}{2}}^{(2)} w^j \left( \frac{\partial}{\partial \rho} - \rho \frac{\partial}{\partial w} \right)) \right).
\]

This formal power series in \((w, \rho)\) gives the formal local superconformal coordinate at a puncture of the canonical supersphere obtained from the sewing together of two particular canonical superspheres with punctures as we shall see in Example 5.2 in Section 5.1.

The following proposition states results contained in Corollary 3.47 and Proposition 3.48 in [B4]. Corollary 3.47 and Proposition 3.48 are in turn based on Proposition 3.31 of [B4]. However, there is a typo on line five of the proof of Proposition 3.31 (which appears on p.76 of [B4]) and hence in the statement of Proposition 3.31 and in the statements of Corollary 3.47 and Proposition 3.48 which are based on Proposition 3.31. Line five of the proof of Proposition 3.31 on p.76 in [B4] should read

\[
\exp \left( -(\tilde{x} - x - \tilde{\varphi} \varphi) \frac{\partial}{\partial w} - (\tilde{\varphi} - \varphi) \left( \frac{\partial}{\partial \rho} - \rho \frac{\partial}{\partial w} \right) \right).
\]

rather than

\[
\exp \left( -(\tilde{x} - x) \frac{\partial}{\partial w} - (\tilde{\varphi} - \varphi) \left( \frac{\partial}{\partial \rho} - \rho \frac{\partial}{\partial w} \right) \right).
\]

We will need the corrected results from Corollary 3.47 and Proposition 3.48 in [B4], and these results are given in the proposition below. It is the first line of the righthand side of equation (2.18) below that contains the corrected term.

**Proposition 2.4.** ([B4]) The formal series \(\Theta_j^{(2)}(\{t^k B_k, t^{k-1/2} N_{k-1/2}\}_{k \in \mathbb{Z}_+}, (x, \varphi))\), for \(j \in \frac{1}{2} \mathbb{N}\), are in \(\mathbb{C}[x^{-1}, \varphi][[B]][[[t^{1/2}]]]\). Thus for \((B, N) \in \bigwedge_\infty \), the series

\[
\Theta_j^{(2)}(\{t^k B_k, t^{k-\frac{1}{2}} N_{k-\frac{1}{2}}\}_{k \in \mathbb{Z}_+}, (x, \varphi))
\]

are well defined and belong to \(\bigwedge_\infty [x^{-1}, \varphi][[[t^{1/2}]]].\)
Change of variables for $N = 1$ NS-VOSAs

Let $(\hat{x}(t^{1/2}), \hat{\varphi}(t^{1/2})) = (H_{j}^{(2)}(t^{1/2}B_{j}, t_{j}N_{j})_{x})^{-1} \circ I(x, \varphi)$, and let $V$ be a positive-energy module for $\mathfrak{ns}$. Then in $([\text{End} (\Lambda_{\bullet} \otimes_{\mathbb{C}} V)][t^{1/2}][[x, x^{-1}]] \mathbb{C})^{0}$, the following identity holds for $\Theta_{j}^{(2)}(t^{1/2}) = \Theta_{j}^{(2)}((t^{k}B_{k}, t^{k-1/2}N_{k-1/2})_{k} \in \mathbb{Z}_{+}, (x, \varphi))$.

\begin{equation}
\exp \left( \sum_{m=1}^{\infty} \sum_{j \in \mathbb{Z}^{+}} \left( -j + 1 \right) m \right) x^{-j-m} \left( t^{j}B_{j} + 2\varphi t^{j-1/2}N_{j} \right) L(m)
\end{equation}

\begin{equation}
= \exp \left( (\hat{x}(t^{1/2}) - x - \hat{\varphi}(t^{1/2}) \varphi)L(-1) + (\hat{\varphi}(t^{1/2}) - \varphi)G\left( \frac{1}{2} \right) \right).
\end{equation}

3. $N = 1$ Neveu-Schwarz vertex operator superalgebras

In this section, we recall the notion of $N = 1$ Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra and with odd formal variables ($N = 1$ NS-VOSA) given in [B2] and [B3] and recall some of the consequences of this notion which we will need later.

Let $x$, $x_{0}$, $x_{1}$ and $x_{2}$ be even formal variables, and let $\varphi$, $\varphi_{1}$ and $\varphi_{2}$ be odd formal variables. For any formal Laurent series $f(x) \in \Lambda_{\infty}[[x, x^{-1}]]$, we can define

\begin{equation}
f(x + \varphi_{1}\varphi_{2}) = f(x) + \varphi_{1}\varphi_{2}f'(x) \in \Lambda_{\infty}[[x, x^{-1}]] \mathbb{C} \mathbb{C} .
\end{equation}

Recall (cf. [FLM]) the formal $\delta$-function at $x = 1$ given by $\delta(x) = \sum_{n \in \mathbb{Z}} x^{n}$. As developed in [B3], we have the following $\delta$-function of expressions involving three even formal variables and two odd formal variables

\begin{equation}
\delta \left( \frac{x_{1} - x_{2} - \varphi_{1}\varphi_{2}}{x_{0}} \right) = \sum_{n \in \mathbb{Z}} (x_{1} - x_{2} - \varphi_{1}\varphi_{2})^{n} x_{0}^{-n}
\end{equation}

\begin{equation}
= \delta \left( \frac{x_{1} - x_{2}}{x_{0}} \right) - \varphi_{1}\varphi_{2} x_{0}^{-1} \delta' \left( \frac{x_{1} - x_{2}}{x_{0}} \right)
\end{equation}

where $\delta'(x) = d/dx \delta(x) = \sum_{n \in \mathbb{Z}} nx^{n-1}$, and we use the conventions that a function of even and odd variables should be expanded about the even variables and any expression in two even variables (such as $(x_{1} - x_{2})^{n}$, for $n \in \mathbb{Z}$) should be expanded in positive powers of the second variable, (in this case $x_{2}$).

From [B3], we have the following $\delta$-function identity which will be used in the proofs of Lemmas 6.1 and 6.3

\begin{equation}
x_{1}^{-1} \delta \left( \frac{x_{2} + x_{0} + \varphi_{1}\varphi_{2}}{x_{1}} \right) = x_{2}^{-1} \delta \left( \frac{x_{1} - x_{2} - \varphi_{1}\varphi_{2}}{x_{2}} \right).
\end{equation}

Definition 3.1. An $N = 1$ NS-VOSA over $\bigwedge_{\bullet}$, with odd variables is a $\frac{1}{2}$-graded (by weight) $\bigwedge_{\infty}$-module which is also $\mathbb{Z}_{2}$-graded (by sign)

\begin{equation}
V = \bigoplus_{k \in \mathbb{Z}} V_{k} = \bigoplus_{k \in \mathbb{Z}} V_{0}^{k} \oplus \bigoplus_{k \in \mathbb{Z}} V_{1}^{k} = V^{0} \oplus V^{1}
\end{equation}
such that only $\bigwedge_* \subseteq \bigwedge_\infty$ acts nontrivially,

\begin{equation}
\dim V(k) < \infty \quad \text{for} \quad k \in \frac{1}{2}\mathbb{Z},
\end{equation}

\begin{equation}
V(k) = 0 \quad \text{for} \quad k \text{ sufficiently small},
\end{equation}
equipped with a linear map $V \otimes V \rightarrow V[[x, x^{-1}]][\varphi]$, or equivalently,

\begin{equation}
\begin{aligned}
V & \quad \rightarrow \quad \text{(End } V)[[x, x^{-1}]][\varphi] \\
\upsilon & \quad \rightarrow \quad Y(v, (x, \varphi)) = \sum_{n \in \mathbb{Z}} \upsilon_n x^{-n-1} + \varphi \sum_{n \in \mathbb{Z}} \upsilon_{n-\frac{1}{2}} x^{-n-1}
\end{aligned}
\end{equation}

where $\upsilon_n \in (\text{End } V)^0(\upsilon)$ and $\upsilon_{n-\frac{1}{2}} \in (\text{End } V)^{(\eta(\upsilon)+1) \mod 2}$ for $\upsilon$ of homogeneous sign in $V$, $x$ is an even formal variable, and $\varphi$ is an odd formal variable, and where $Y(v, (x, \varphi))$ denotes the \textit{vertex operator associated with $v$}, and equipped also with two distinguished homogeneous vectors $1 \in V_{(0)}^0$ (the \textit{vacuum}) and $\tau \in V_{(3/2)}^1$ (the \textit{Neveu-Schwarz element}). The following conditions are assumed for $u, v \in V$:

\begin{equation}
u_n \upsilon = 0 \quad \text{for} \quad n \in \frac{1}{2}\mathbb{Z} \text{ sufficiently large};\end{equation}

\begin{equation}Y(1, (x, \varphi)) = 1 \quad (1 \text{ on the right being the identity operator});\end{equation}

the \textit{creation property} holds:

\begin{equation}
Y(v, (x, \varphi))1 \in V[[x]][\varphi] \quad \text{and} \quad \lim_{(x, \varphi) \rightarrow 0} Y(v, (x, \varphi))1 = v;
\end{equation}

the \textit{Jacobi identity} holds:

\begin{equation}
x_0^{-1} \delta \left( \frac{x_1 - x_2 - \varphi_1 \varphi_2}{x_0} \right) Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2)) \nonumber\end{equation}

\begin{equation}
-(-1)^{\eta(u) \eta(v)} x_0^{-1} \delta \left( \frac{x_2 - x_1 + \varphi_1 \varphi_2}{-x_0} \right) Y(v, (x_2, \varphi_2)) Y(u, (x_1, \varphi_1)) \nonumber\end{equation}

\begin{equation}
= x_2^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2} \right) Y(Y(u, (x_0, \varphi_1 - \varphi_2))v, (x_2, \varphi_2)),\end{equation}

for $u, v$ of homogeneous sign in $V$; the $N = 1$ Neveu-Schwarz algebra relations hold:

\begin{equation}
[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\text{rank } V),\end{equation}

\begin{equation}
\left[ G(m + \frac{1}{2}), L(n) \right] = (m - \frac{n-1}{2})G(m + n + \frac{1}{2}),\end{equation}

\begin{equation}
\left[ G(m + \frac{1}{2}), G(n - \frac{1}{2}) \right] = 2 L(m + n) + \frac{1}{3}(m^2 + m)\delta_{m+n,0}(\text{rank } V),\end{equation}

for $m, n \in \mathbb{Z}$, where

\begin{equation}
G(n + \frac{1}{2}) = \tau_{n+1}, \quad \text{and} \quad 2L(n) = \tau_{n+\frac{1}{2}} \quad \text{for} \quad n \in \mathbb{Z},\end{equation}

i.e.,

\begin{equation}
Y(\tau, (x, \varphi)) = \sum_{n \in \mathbb{Z}} G(n + \frac{1}{2})x^{-n-\frac{1}{2}-\frac{1}{2}} + 2\varphi \sum_{n \in \mathbb{Z}} L(n)x^{-n-2},\end{equation}

(3.8)
and rank \( V \in \mathbb{C} \);
\[
(3.9) \quad L(0)v = kv \quad \text{for} \quad k \in \frac{1}{2}\mathbb{Z} \quad \text{and} \quad v \in V_{(k)};
\]
and the \( G(-1/2) \)-derivative property holds:
\[
(3.10) \quad \left( \frac{\partial}{\partial \varphi} + \varphi \frac{\partial}{\partial x} \right) Y(v, (x, \varphi)) = Y(G(-\frac{1}{2})v, (x, \varphi)).
\]

The \( N = 1 \) NS-VOSA just defined is denoted by \((V, Y(\cdot, (x, \varphi)), 1, \tau)\), or for simplicity by \( V \).

The following are consequences of the definition of \( N = 1 \) NS-VOSA with odd formal variables which we will need in our proofs of the change of variables formulas (cf. [123]). We have
\[
L(n)1 = G(n + \frac{1}{2})1 = 0, \quad \text{for} \quad n \geq -1
\]
and \( \tau = G(-3/2)1 \). There exists \( \omega = (1/2)G(-1/2)\tau \in V_{(2)} \) such that
\[
(3.11) \quad Y(\omega, (x, \varphi)) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2} - \frac{1}{2} \sum_{n \in \mathbb{Z}} (n + 1)G(n - \frac{1}{2})x^{-n-2},
\]
and \( \omega = L(-2)1 \). The supercommutator formula is given by
\[
(3.12) \quad [Y(u, (x_1, \varphi_1)), Y(v, (x_2, \varphi_2))] = \text{Res}_{x_0} x_0^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2} \right) Y(Y(u, (x_0, \varphi_1 - \varphi_2))v, (x_2, \varphi_2))
\]
where \( \text{Res}_{x_0} \) of a power series in \( x_0 \) is the coefficient of \( x_0^{-1} \). We have
\[
(3.13) \quad x_0^{2L(0)}Y(v, (x, \varphi))x_0^{-2L(0)} = Y(x_0^{2L(0)}v, (x_0^2 x, x_0 \varphi)),
\]
and
\[
(3.14) \quad Y(e^{x_0 L(-1)+\varphi_0 G(-\frac{1}{2})}v, (x, \varphi)) = e^{x_0 \frac{\partial}{\partial x} + \varphi_0 (\frac{\partial}{\partial x} + \varphi \frac{\partial}{\partial \varphi})} Y(v, (x, \varphi)) = Y(v, (x + x_0 + \varphi_0 \varphi, \varphi_0 + \varphi)).
\]

\[
(3.15) \quad \gamma(Y_1(u, (x, \varphi))v) = Y_2(\gamma(u), (x, \varphi))\gamma(v) \quad \text{for} \quad u, v \in V_1,
\]
\( \gamma(1_1) = 1_2 \), and \( \gamma(\tau_1) = \tau_2 \).

**Remark 3.2.** If \((V_1, Y_1(\cdot, (x, \varphi)), 1_1, \tau_1)\) is an \( N = 1 \) NS-VOSA and \( V_2 \) is a \( \Lambda_{\infty} \)-module which is isomorphic to \( V_1 \) for some \( \Lambda_{\infty} \)-module isomorphism \( \gamma : V_1 \to V_2 \), then defining \( Y_2 : V_2 \to (\text{End}V_2)([x, x^{-1}],[\varphi]) \) by
\[
Y_2(u, (x, \varphi))v = \gamma(Y_1(\gamma^{-1}(u), (x, \varphi))\gamma^{-1}(v))
\]
for all $u, v \in V_2$, we have that $(V_2, Y_2(\cdot, (x, \varphi)), \gamma(1), \gamma(\tau))$, with $\frac{1}{2}\mathbb{Z}$-grading and $\mathbb{Z}_2$-grading on $V_2$ induced by $V_1$, is an $N = 1$ NS-VOSA which is isomorphic to $(V_1, Y_1(\cdot, (x, \varphi)), 1, \tau)$.

**Remark 3.3.** Let $(V, Y(\cdot, (x, \varphi)), 1, \tau)$ be an $N = 1$ NS-VOSA. In [13], we also study the notion of $N = 1$ NS-VOSA over a Grassmann algebra without odd formal variables and show that $(V, Y(\cdot, (x, 0)), 1, \tau)$ is such an algebra. Conversely, given an $N = 1$ NS-VOSA without odd formal variables $(V, Y(\cdot, x), 1, \tau)$, we can define $\tilde{Y}(v(\varphi, \varphi)) = Y(v, x) + \varphi Y(G(-1/2)v, x)$, and then $(V, \tilde{Y}(\cdot, (x, \varphi)), 1, \tau)$ is an $N = 1$ NS-VOSA with odd formal variables. Using this correspondence, in [13] we prove that the category of $N = 1$ NS-VOSAs over $\Lambda$ with odd formal variables is isomorphic to the category of $N = 1$ NS-VOSAs over $\Lambda$ without odd formal variables. However, in including the odd formal variables the correspondence with the geometry and the role of the operator $G(-1/2)$, as in the $G(-1/2)$-derivative property, is more explicit. In addition, in a related issue, the choice of the odd components, i.e., the $\varphi$ components of the vertex operators, determines the form of the $G(-1/2)$-derivative property which in turn is directly related to the choice of superconformal operator $D$ for the underlying worldsheet geometry.

4. **Superconformal change of variables formulas for $N = 1$ NS-VOSAs**

Let $(V, Y(\cdot, (x, \varphi)), 1, \tau)$ be an $N = 1$ NS-VOSA. In this section, we present the change of variables formulas for two superconformal changes of variables. First, in Section 4.1, we consider the change of variables $(x, \varphi) \mapsto H(x, \varphi)$ where $H$ is a formal superconformal function vanishing at zero with invertible leading even coefficient of $\varphi$ in its power series expansion about zero. This last condition is equivalent to $H$ being bijective in a neighborhood of zero if $H$ is convergent in a neighborhood of zero, however we do not assume $H$ is convergent in a neighborhood of zero.

Second, in Section 4.2, we consider the change of variables $(x, \varphi) \mapsto H \circ I(x, \varphi)$ where $H^{-1}$ is a formal superconformal function vanishing at infinity with leading even coefficient of $\varphi x^{-1}$ equal to $i$ in its power series expansion about infinity. Thus $H \circ I$ is a formal invertible superconformal function taking infinity to infinity.

The results of this section will be proved in Sections 5 and 6. In Section 5, we give geometric proofs of the change of variables formulas for changes of variables convergent in a neighborhood of zero and infinity, respectively, and in Section 6, we give purely algebraic proofs in the more general case of formal changes of variables not necessarily convergent.

4.1. **Superconformal change of variables at zero.** Let $(V, Y(\cdot, (x, \varphi)), 1, \tau)$ be an $N = 1$ NS-VOSA and let $H$ be a formal invertible superconformal change of variables vanishing at zero. By Proposition 2.1, $H^{-1}$ can be written uniquely as

$$H^{-1}(x, \varphi) = \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( A_j L_j(x, \varphi) + M_{j-2} G_{j-2}(x, \varphi) \right) \right) a^{-2L_0(x, \varphi)}(x, \varphi).$$
for some \( a_0 \in (\Lambda_\infty^0) \times \) and \((A_j, M_{j-1/2}) \in \Lambda_\infty \) for \( j \in \mathbb{Z}_+ \), and by Proposition 3.21 in [B4]

\[(4.2) \quad H(x, \varphi) = a_0^{2L_0(x, \varphi)} \cdot \exp \left( \sum_{j \in \mathbb{Z}_+} \left( A_j L_j(x, \varphi) + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}}(x, \varphi) \right) \right) \cdot (x, \varphi). \]

Let \( t^{1/2} \) be an even formal variable and let \( H_{t^{1/2}}(x, \varphi) \in \Lambda_\infty[[t^{1/2}]][[x, \varphi]] \) be defined by

\[H_{t^{1/2}}(x, \varphi) = (t^{-\frac{1}{2}} a_0)^{2L_0(x, \varphi)} \cdot \exp \left( \sum_{j \in \mathbb{Z}_+} \left( A_j L_j(x, \varphi) + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}}(x, \varphi) \right) \right) \cdot (x, \varphi) \]

so that by Proposition 3.13 in [B4]

\[H_{t^{1/2}}^{-1}(x, \varphi) = \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( A_j L_j(x, \varphi) + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}}(x, \varphi) \right) \right) \cdot \left( t^{-\frac{1}{2}} a_0 \right)^{-2L_0(x, \varphi)} \cdot (x, \varphi). \]

Let \( V \) be a positive energy module for the \( N = 1 \) Neveu-Schwarz algebra. Define

\[\gamma_{H, t^{1/2}} : V \rightarrow V[t^{-\frac{1}{2}}, t^{\frac{1}{2}}] \]

\[v \mapsto \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( A_j L(j) + M_{j-\frac{1}{2}} G(j - \frac{1}{2}) \right) \right) \cdot (t^{-\frac{1}{2}} a_0)^{-2L(0)} \cdot v. \]

Define \( \gamma_H : V \rightarrow V \), by \( \gamma_H = \gamma_{H, t^{1/2}}|_{t^{1/2} = 1} \). Note that \( \gamma_H \) is well-defined and bijective with inverse \( \gamma_H^{-1} = \gamma_{H^{-1}} \).

Define

\[\gamma_{\theta(H, t^{1/2}, (x, \varphi))} : V \rightarrow V[x, \varphi] \cdot [(t^{\frac{1}{2}})] \]

\[v \mapsto (t^{-\frac{1}{2}} a_0)^{-2L(0)} \cdot \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( \Theta_j^{(1)}(t^{\frac{1}{2}}) L(j) \right) + \Theta_j^{(1)}(t^{\frac{1}{2}}) G(j - \frac{1}{2}) \right) \cdot \exp(-\Theta_0^{(1)}(t^{\frac{1}{2}})2L(0)) \cdot v. \]

where the \( \Theta_j^{(1)}(t^{1/2}) = \Theta_j^{(1)}(t^{-1/2}a_0, A, M, (x, \varphi)) \in \Lambda_{\infty}[x, \varphi][[t^{1/2}]], \) for \( j \in \frac{1}{2} \mathbb{Z}_+ \), are defined by [2.16] with \( H_{a_0^{1/2}, A, M}(x, \varphi) = H_{t^{1/2}}^{-1}(x, \varphi) \). That is

\[(4.3) \quad \left\{ \exp(\Theta_0^{(1)}(t^{\frac{1}{2}} a_0, A, M, (x, \varphi))), \Theta_j^{(1)}(t^{\frac{1}{2}} a_0, A, M, (x, \varphi)) \right\}_{j \in \mathbb{Z}_+} \]

\[= \hat{E}^{-1}(s_{(x, \varphi)} \circ H_{t^{1/2}}^{-1} \circ s_{H_{t^{1/2}}(x, \varphi)}(t a_0^{-2} w, t^{\frac{1}{2}} a_0^{-1} \rho)). \]
In other words, the $\Theta^{(1)}_j(t^{1/2})$’s are uniquely determined by

$$s(u,v)H_{t^{1/2}}^{-1}\circ s^{-1}_{H_{t^{1/2}}(x,\varphi)}(t^{2}\partial_x^2 a^{-1}_V)\rho \exp\left(\sum_{j\in \mathbb{Z}_+}\left(\Theta^{(1)}_j(t^2)(w^{j+1}\frac{\partial}{\partial w} + \varphi\omega w^j\frac{\partial}{\partial \varphi}) + \Theta^{(1)}_{j^2}(t^2)w^j\left(\frac{\partial}{\partial \rho} - \rho\frac{\partial}{\partial w}\right)\right)\right).$$

Finally, define

$$\gamma_{\Theta(H,t^{1/2},(x,\varphi))}^{-1} : V \longrightarrow V[x,\varphi][[t^{1/2}]]$$

$$\gamma = \exp(\Theta^{(1)}_0(t^2)2L(0)) \cdot \exp\left(\sum_{j\in \mathbb{Z}_+}\left(\Theta^{(1)}_j(t^2)L(j)\right) + \Theta^{(1)}_{j^2}(t^2)G(j - \frac{1}{2})\right) \cdot (t^{-1}a^{-1}_V)^{2L(0)} \cdot v.$$ 

**Proposition 4.1.** Let $(V, Y(\cdot, (x,\varphi)), 1, \tau)$ be an $N = 1$ NS-VOSA and $H(x,\varphi) \in x\Lambda_{\infty}[x] \oplus \varphi \Lambda_{\infty}[x]$ superconformal and invertible. Then $H$ can be expressed uniquely as $[4,3]$, the map

$$\gamma_{\Theta(H,x,\varphi)} = \gamma_{\Theta(H,t^{1/2},(x,\varphi))}\big|_{t^{1/2}=1} : V \longrightarrow V[[x]][\varphi]$$

is well defined, and in $V[[x^{-1}, x]][\varphi]$, we have the following change of variables formula

$$\gamma_H(Y(u, (x,\varphi))v) = Y(\gamma_{\Theta(H,(x,\varphi))}(u), H(x,\varphi))\gamma_H(v),$$

for $u, v \in V$. In particular, we have that $\gamma_{\Theta(H,(x,\varphi))} = \gamma^{-1}_{\Theta(H,t^{1/2},(x,\varphi))}\big|_{t^{1/2}=1}$ is well-defined, and

$$Y(u, H(x,\varphi))v = \gamma_H(Y(\gamma^{-1}_{\Theta(H,(x,\varphi))}(u), (x,\varphi))\gamma^{-1}_H)(v).$$

Furthermore, if $H$ is convergent in a DeWitt open neighborhood of zero, then for $(z,\theta)$ in the domain of convergence, both sides of $[4,3]$ and of $[4,6]$ exist for $(x,\varphi) = (z,\theta)$ and are equal.

**Remark 4.2.** The odd components $M_{j-1/2}$, for $j \in \mathbb{Z}_+$, and the even soul components $(a_\varphi)_S$ and $(A_j)_S$, for $j \in \mathbb{Z}_+$, of the change of variables related to $H$ act on both the even and odd components of the vertex operator and thus appear even if we restrict ourselves to an $N = 1$ NS-VOSA without odd formal variable components. However, if $V$ is an $N = 1$ NS-VOSA over $\mathbb{C}$ rather than over a Grassmann algebra $\Lambda_\mathbb{C} \neq \mathbb{C}$ then the effect of the soul components of the change of variables related to $H$, i.e., the affects of $(a_\varphi)_S$, $(A_j)_S$ and $M_{j-1/2}$, for $j \in \mathbb{Z}_+$, will be trivial. (Recall that an $N = 1$ NS-VOSA over a Grassmann algebra $\Lambda_\mathbb{C}$ is defined to be a $\Lambda_{\infty}$-module where only the subalgebra $\Lambda_\mathbb{C}$ acts nontrivially.) This trivialization of the soul components for $N = 1$ NS-VOSA over $\mathbb{C}$ occurs regardless of whether the odd variable components of $V$ are included or not. Thus, if $V$ is an $N = 1$ NS-VOSA over $\mathbb{C}$, to realize a superconformal change of coordinates involving soul components, one must consider $\Lambda_\mathbb{C} \otimes_\mathbb{C} V$ with $\Lambda_\mathbb{C} = \Lambda_{\infty}$ or $\Lambda_L$, where $L > 0$. 
Remark 4.3. For invertible superconformal change of variables $F(x, \varphi)$ vanishing at $(z, \theta) \neq 0$, we can write $F(x, \varphi) = H \circ s_{z, \theta}(x, \varphi)$ where $H$ is invertible and vanishing at zero, and we can use the superconformal shift formula given by \[3.15\] which, when combined with the change of variables formulas given in Proposition \[4.3\] above, provide the appropriate change of variables formulas.

Remark 4.4. Replacing the superconformal function $H(x, \varphi)$ by the body portion of $H$, setting all odd variables and the soul portion of supernumbers equal to zero and restricting $V$ to $V^0$ in Proposition \[4.4\] we obtain the change of variables formula for a VOA in the case of an invertible analytic function vanishing at zero as developed in \[4.2\].

4.2. Superconformal change of variables at infinity. Let $(V, Y(\cdot, (x, \varphi)), 1, \tau)$ be an $N = 1$ NS-VOSA. In this section we consider a change of variables $(x, \varphi) \mapsto H \circ I(x, \varphi)$ where $H^{-1}$ is formally superconformal vanishing at infinity with even coefficient of $x^{-1}$ equal to $i$ in its power series expansion about infinity. By Remark \[4.2\] $H^{-1}$ can be written uniquely as

\[
(4.7) \quad H^{-1}(x, \varphi) = \exp \left( \sum_{j \in \mathbb{Z}_+} \left( B_j L_{-j}(x, \varphi) + N_j - \frac{1}{2} G_{j+\frac{1}{2}}(x, \varphi) \right) \right) \cdot \left( \frac{1}{x}, \frac{i \varphi}{x} \right),
\]

for some $(B, N) \in \bigwedge^\infty \mathbb{R}$. Let

\[
H^{-1}_{t/2}(x, \varphi) = \exp \left( \sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^{j-\frac{1}{2}} N_j - \frac{1}{2} G_{j+\frac{1}{2}}(x, \varphi) \right) \right) \cdot \left( \frac{1}{x}, \frac{i \varphi}{x} \right).
\]

Proposition 3.17 in \[4.4\] implies that $t^{-1} \circ H^{-1}_{t/2}(x, \varphi) \in (x, \varphi) + \bigwedge^\infty [t^{1/2}[[x^{-1}]]][\varphi]$ is given by

\[
(4.8) \quad I^{-1} \circ H^{-1}_{t/2}(x, \varphi) = \exp \left( \sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^{j-\frac{1}{2}} N_j - \frac{1}{2} G_{j+\frac{1}{2}}(x, \varphi) \right) \right) \cdot (x, \varphi),
\]

and thus

\[
(4.9) \quad H_{t/2} \circ I(x, \varphi) = \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^{j-\frac{1}{2}} N_j - \frac{1}{2} G_{j+\frac{1}{2}}(x, \varphi) \right) \right) \cdot (x, \varphi).
\]

Let $(V, Y(\cdot, (x, \varphi)), 1, \tau)$ be an $N = 1$ NS-VOSA. Let $V' = \prod_{k \in \frac{1}{2} \mathbb{Z}} V(k)$ be the graded dual space of $V$, and let $L'(j)$ and $G'(j-1/2)$, for $j \in \mathbb{Z}_+$, be the adjoint operators corresponding to $L(-j)$ and $G(-j+1/2)$, respectively. Define

\[
(4.10) \quad \xi_{H_{t/2}, t^{1/2}} : V' \rightarrow V' [t^{1/2}]
\]

\[
\varphi' \mapsto \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( t^j B_j L'(j) + t^{j-\frac{1}{2}} N_j - \frac{1}{2} G'(j - \frac{1}{2}) \right) \right) \cdot \varphi',
\]

and define $\xi_{H_{t/2}} : V' \rightarrow V'$, by $\xi_{H_{t/2}} = |\xi_{H_{t/2}, t^{1/2}}|_{t^{1/2}=1}$. Note that $\xi_{H_{t/2}}$ is well defined and bijective. Let $\tilde{V} = \prod_{k \in \frac{1}{2} \mathbb{Z}} V(k) = (V')^*$ be the algebraic completion of
V. Let
\[ \xi_{H^{t, t^{1/2}}} : V \rightarrow V^* \]
\[ v \mapsto \exp \left( -\sum_{j \in \mathbb{Z}_+} \left( t^j B_j L(-j) + t^{j-\frac{1}{2}} N_{j-\frac{1}{2}} G(-j + \frac{1}{2}) \right) \right) \cdot v \]
be the adjoint operator corresponding to \( \xi \), and let \( \xi^{*}_{H^{t, t^{1/2}}} \) be the adjoint operator corresponding to \( \xi \).

Define
\[ \xi_{\Theta(H^{t, t^{1/2}}, (x, \varphi))} : V \rightarrow V[x^{-1}, \varphi] \]
\[ v \mapsto \exp \left( -\sum_{j \in \mathbb{Z}_+} \left( \Theta_j^{(2)}(t) L(j) \right) \right) \cdot \exp \left( -\Theta_0^{(2)}(t^\frac{1}{2}) 2L(0) \right) \cdot v \]
where the \( \Theta_j^{(2)}(t^{1/2}) = \Theta_j^{(2)} \{ t^k B_k, t^{k-1/2} N_{k-1/2} \}_{k \in \mathbb{Z}_+}, (x, \varphi) \in \Lambda_\infty [x^{-1}, \varphi][[t^{1/2}]] \) are defined by (2.17) with \( H_\Theta^{(2)}(x, \varphi) = H_{t, t^{1/2}}^{-1}(x, \varphi) \).

Thus, \( \Theta_j^{(2)}(t^{1/2})(t) \)'s are uniquely determined by the \( s_{(x, \varphi)} \circ \cdot \circ \circ \circ \circ \cdot \circ s_{(x, \varphi)} \circ \cdot \circ \circ \circ \circ \cdot \circ s_{(x, \varphi)} \circ \cdot \circ \circ \circ \circ \cdot \circ s_{(x, \varphi)} \)’s.

Finally, define
\[ \xi^{-1}_{\Theta(H^{t, t^{1/2}}, (x, \varphi))} : V \rightarrow V[x^{-1}, \varphi] \]
\[ v \mapsto \exp \left( \Theta_0^{(2)}(t^\frac{1}{2}) 2L(0) \right) \cdot \exp \left( \sum_{j \in \mathbb{Z}_+} \left( \Theta_j^{(2)}(t) L(j) \right) + \Theta_j^{(2)}(t^{1/2}) \left( \frac{\partial}{\partial \rho} - \rho \frac{\partial}{\partial w} \right) \right) \cdot \exp \left( \Theta_0^{(2)}(t^\frac{1}{2}) 2L(0) \right) \cdot (w, \rho). \]

**Proposition 4.5.** Let \((V, Y((x, \varphi)), 1, \tau)\) be an \( N = 1 \) NS-VOSA and \( H(x, \varphi) \circ I \in (x, \varphi) + \Lambda_\infty[[x^{-1}]] \) formally superconformal. Then \( H \circ I \) can be expressed uniquely as \( \xi_{\Theta} \) with \( t^{1/2} = 1 \), the map
\[ \xi_{\Theta(H^{t, t^{1/2}}, (x, \varphi))} = \xi_{\Theta(H^{t, t^{1/2}}, (x, \varphi))}|_{t^{1/2}=1} : V \rightarrow V[[x^{-1}]] \]
is well defined, and in $\tilde{V}[[x^{-1}, x]][\phi]$, we have the following change of variables formula

\begin{equation}
Y(u, (x, \varphi))\xi^*_{H\circ I}(v) = \xi^*_{H\circ I}(Y(\xi_{(H\circ I, (x, \varphi)}(u), H \circ I(x, \varphi)))v),
\end{equation}

for $u, v \in V$. In particular, we have that $\xi_{(H\circ I, (x, \varphi)}^{-1}(u) = \xi_{(H\circ I, t^{1/2}, (x, \varphi)}^{-1}(u)$ is well-defined, and

\begin{equation}
Y(u, H \circ I(x, \varphi))v = (\xi^*_{H\circ I})^{-1}(Y(\xi_{(H\circ I, (x, \varphi)}^{-1}(u), (x, \varphi)))\xi^*_{H\circ I}(v)).
\end{equation}

Furthermore, if $H \circ I$ is convergent in a DeWitt open neighborhood of infinity, then for $(x, \varphi)$ in the domain of convergence of $H \circ I$ both sides of (4.12) and of (4.13) exist when $(x, \varphi) = (z, \theta)$ and are equal.

\textbf{Remark 4.6.} Equation (4.12) in Proposition 4.5 above, is the superextension of the change of variables formula given at the top of p. 181 in [H2]. This can be seen by letting Huang’s $f^{-1}(x)$ be the body of $H(x, \varphi)$ (and thus $f^{-1}(1/x)$ is the body of $H \circ I(x, \varphi)$), setting all odd variables and the soul portion of supernumbers equal to zero and restricting $V$ to $V^0$. (Note that Huang’s operator $\xi^*_1/f$ is the body portion of our operator $\xi^*_{H\circ I}$ if $f^{-1}$ is the body of $H$.)

5. Geometric proofs for convergent superconformal changes of variables

In this section, we give geometric proofs of Propositions 4.1 and 4.5 in the case of convergent superconformal changes of variables. These geometric proofs rely on the correspondence between the worldsheet supergeometry of propagating superstrings in $N = 1$ superconformal field theory and the algebraic structure of $N = 1$ NS-VOSA developed in [B4] and [B5]. More than just giving alternate proofs to the formal algebraic proofs we will present in Section 6, this section gives the motivation behind the change of variables formulas.

In Sections 5.1-5.3 we recall some of the basic results and structures of this geometric/algebraic correspondence. For more details, the reader should refer to [B4] and [B5]. In Sections 5.4 and 5.5, we present the actual proofs of Propositions 4.1 and 4.5 respectively.

5.1. The moduli space of $N = 1$ superspheres with tubes and the sewing operation. In [B4], we define the moduli space $SK(n)$ of $N = 1$ superspheres with $n$ incoming ordered tubes and one outgoing tube, for $n \in \mathbb{N}$. Incoming “tubes” represent propagating incoming superstrings and the one outgoing “tube” represents an outgoing propagating superstring in the worldsheet supergeometry of $N = 1$ superconformal field theory. These propagating superstrings sweep out a supersurface or worldsheet in space time which has superconformal coordinate transition functions. Such a supersurface is called a super-Riemann surface. It is shown in [B4] that each incoming or outgoing tube can be represented by an oriented puncture and local superconformal coordinates vanishing at the puncture. Thus the moduli space $SK(n)$ is the set of equivalence classes, under superconformal equivalence, of genus zero super-Riemann surfaces with $n$ incoming ordered punctures, one outgoing puncture, and local superconformal coordinates vanishing at the punctures.
Furthermore the Uniformization Theorem for genus-zero super-Riemann surfaces proved by Crane and Rabin in [CR] states that any compact genus-zero super-Riemann surface (i.e., supersphere) is superconformally equivalent to the super-Riemann sphere denoted $\tilde{S}\hat{C}$. Just as the usual Riemann sphere is identified with the complex plane and a point added at infinity, the super-Riemann sphere $\tilde{S}\hat{C}$ can be thought of as the Grassmann algebra $\bigwedge_{\infty}$ together with a fiber at infinity given by $\infty \times (\bigwedge_{\infty})_S$. Thus $\tilde{S}\hat{C}$ has two coordinate charts $\bigwedge_{\infty} = \tilde{S}\hat{C}\setminus (\infty \times (\bigwedge_{\infty})_S)$ and $\bigwedge_{\infty} \cong \tilde{S}\hat{C}\setminus (0 \times (\bigwedge_{\infty})_S)$ with coordinate transition function from the first coordinate chart to the second given by $I(w, \rho) = (1/w, i\rho/w)$ for $(w, \rho) \in (\bigwedge_{\infty})^\times$; see [BR].

Let

$$\mathcal{H} = \{(A, M) \in \bigwedge_{\infty}^{-} \mid \tilde{E}(A, M)(z, \theta) \text{ is an absolutely convergent power series in some neighborhood of } (z, \theta) = 0\},$$

and for $n \in \mathbb{Z}_+$, let

$$SM^{n-1} = \{(z_1, \theta_1), \ldots, (z_{n-1}, \theta_{n-1}) \mid (z_i, \theta_i) \in \bigwedge_{\infty}^{\times}, (z_i)_B \neq (z_j)_B, \text{ for } i \neq j\}.$$  

Note that for $n = 1$, the set $SM^0$ has exactly one element. It is shown in [BR], that as a set

$$SK(n) = SM^{n-1} \times \mathcal{H} \times \left((\bigwedge_{\infty}^{0})^{\times} \times \mathcal{H}\right)^n$$

for $n \in \mathbb{Z}_+$, and $SK(0) = \{(A, M) \in \mathcal{H} \mid (A_1, M_{1/2}) = (0, 0)\}$. Each element $Q \in SK(n)$ can be thought of as the super-Riemann sphere with $n$ incoming punctures, one outgoing puncture and local superconformal coordinates vanishing at the punctures, where this super-Riemann sphere $Q$ is a canonical representative of the equivalence class of superspheres with tubes superconformally equivalent to $Q$. For example, for $Q \in SK(n)$ given by

$$Q = ((z_1, \theta_1), \ldots, (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a^{(1)}_0, A^{(1)}), (M^{(1)})_{0}, \ldots, (a^{(n)}_0, A^{(n)}), M^{(n)})$$

this is the canonical representative of the equivalence class of superspheres superconformally equivalent to $Q$ where $Q$ is the super-Riemann sphere with outgoing puncture at $(\infty, 0) \in \tilde{S}\hat{C}$ and local coordinate vanishing at infinity given by $\tilde{E}(A^{(0)}), -iM^{(0)})(1/w, i\rho/w)$, the $i$-th incoming puncture at $(z_i, \theta_i) \in \tilde{S}\hat{C}$ and local coordinate vanishing at the puncture given by $\tilde{E}(a^{(i)}_0, A^{(i)}), M^{(i)})(w-z_i-\rho\theta_i, \rho-\theta_i)$, for $1 \leq i \leq n-1$, and last incoming puncture at $(0, 0) \in \tilde{S}\hat{C}$ and local coordinate vanishing at the puncture given by $\tilde{E}(a^{(n)}_0, A^{(n)}), M^{(n)})(w, \rho)$.

In [BR], we define a (partial) sewing operation on $SK = \bigcup_{n \in \mathbb{N}} SK(n)$, denoted by

$$\iota_0 : SK(m) \times SK(n) \rightarrow SK(m+n-1)$$

$$Q_1, Q_2 \rightarrow Q_1 \iota_0 Q_2,$$

where $m \in \mathbb{Z}_+$, $n \in \mathbb{N}$, and $1 \leq i \leq m$. This sewing together of $Q_1$ and $Q_2$ is defined if there exists $r_1 > r_2 > 0$ such that there exists a (DeWitt) open ball of radius $r_2 > 0$ about zero which is contained in the image of the local coordinate map vanishing at the $i$-th puncture of $Q_1$ but whose preimage contains no other punctures, and a (DeWitt) open ball of radius $1/r_1$ about zero which is contained in the image of the local coordinate map vanishing at the outgoing puncture of $Q_2$ but whose preimage contains no other punctures. The sewn sphere is then obtained
by removing the preimage of the ball of radius $r_2$ from $Q_1$ and the preimage of the ball of radius $1/r_1$ from $Q_2$ and identifying the superannuli of the two boundaries using the superconformal inversion $f(w, \rho) = (1/w, i\rho/w)$. See [14] for details.

The resulting sewn supersphere is then in a superconformal equivalence class of superspheres whose canonical representative can be expressed as an element of $SK(m + n - 1)$. Much of the details which we will leave out here, but which are given in [14], concern the determination of the superconformal uniformizing function that maps the resulting sewn supersphere to its canonical representative in $SK(m + n - 1)$.

Below we give two examples of the sewing together of two canonical superspheres and the determination of the resulting canonical supersphere. These examples are the two that we will need for the geometric proofs of Propositions 1.11 and 1.13 in the case of convergent changes of variables.

**Example 5.1.** Denote by $0$ the infinite sequence in $\bigwedge_{\infty}$ that consists of all zeros. Let

$$Q_1 = ((0, (a, A, M)) \in SK(1),$$

i.e., $Q_1$ is the canonical supersphere representative with one outgoing tube at infinity and local coordinate given by $(1/w, i\rho/w)$, and one incoming puncture at zero and local coordinate given by

$$H^{-1}(w, \rho) = \exp \left( - \sum_{j \in \mathbb{Z}_+} \left( A_j L_j(w, \rho) + M_j - \frac{1}{2} G_{j-\frac{1}{2}}(w, \rho) \right) \right) \cdot a - 2L_0(w, \rho) \cdot (w, \rho).$$

In otherwords, the local coordinate vanishing at zero is given by [14] with $(x, \varphi) = (w, \rho)$ where we are assuming that $H^{-1}$ is convergent in a neighborhood of zero, i.e., that $(A, M) \in \mathcal{H}$.

Let

$$Q_2 = ((z, \theta); 0, (1, 0), (1, 0)) \in SK(2),$$

i.e., $Q_2$ is the canonical supersphere representative with one outgoing tube at infinity and local coordinate given by $(1/w, i\rho/w)$, the first incoming puncture at $(z, \theta)$ and local coordinate given by $s_{(z, \theta)}(w, \rho) = (w - z - \rho \theta, \rho - \theta)$, and second incoming puncture at zero and local coordinate $(w, \rho)$. From the definition of sewing on $SK$, $Q_2$ can be sewn to $Q_1$ if there exists a ball of radius $r > 0$ in the image of $H^{-1}$, i.e., in the domain of convergence of $H$, such that $r > |z_B|$. In this case, we can form the sewn supersphere $Q_1 \mathcal{H} Q_2$, and there exists a uniformizing function $F : Q_1 \mathcal{H} Q_2 \rightarrow SK$ which, restricted to $Q_1$, is given by $F_1(w, \rho) = F|_{Q_1}(w, \rho) = (w, \rho)$ and, restricted to $Q_2$, is given by $F_2(w, \rho) = F|_{Q_2}(w, \rho) = H(w, \rho)$.

Thus the resulting canonical supersphere has: outgoing puncture at $F_1(\infty, 0) = (\infty, 0)$ with local coordinate vanishing at the puncture given by $I \circ F_1^{-1}(w, \rho) = I(w, \rho) = (1/w, i\rho/w)$; first incoming puncture at $F_2(z, \theta) = H(z, \theta)$ with local coordinate vanishing at the puncture given by $s_{(z, \theta)} \circ F_2^{-1}(w, \rho) = s_{(z, \theta)} \circ H^{-1}(w, \rho) = s_{(z, \theta)} \circ s_{H(z, \theta)}\circ s_{H(z, \theta)}(w, \rho)$; and last incoming puncture at $F_2(0, 0) = H(0, 0) = (0, 0)$ with local coordinate vanishing at the puncture given by $F_2^{-1}(w, \rho) = H^{-1}(w, \rho)$.
In otherwords, letting \( \Theta_j^{(1)} = \Theta_j^{(1)}(t^{-1/2}a, A, M, (z, \theta)) \big|_{t^{1/2}=1} \in \Lambda_{\infty} \) be defined by (4.3), the resulting canonical supersphere is given by

\[
Q_1 \circlearrowleft_{0} Q_2 = (H(z, \theta); 0, 0, 0, 0) \circlearrowleft_{0} (s_{(z, \theta)} \circ H^{-1} \circ s_{H(z, \theta)}(w, \rho), (a, A, M))
\]

(5.1)

\[
= (H(z, \theta); 0, (a, A, M))
\]

Note in particular that this implies that if \( H(w, \rho) \) is convergent in a neighborhood of zero, i.e., if \( (A, M) \in \mathcal{H} \), and \( (z, \theta) \) is in the radius of convergence, then \( \Theta_j^{(1)}(t^{-1/2}a, A, M, (x, \varphi)) \) is convergent for \( t^{1/2} = 1 \) and \( (x, \varphi) = (z, \theta) \).

**Example 5.2.** Let

\[
Q_1 = ((z, \theta); 0, (1, 0), (1, 0)) \in SK(2).
\]

(which is described in detail in Example 5.1 above) and,

\[
Q_2 = ((B, N), (1, 0)) \in SK(1),
\]

i.e., \( Q_2 \) is the canonical supersphere representative with one outgoing tube at infinity and local coordinate given by

\[
H^{-1}(w, \rho) = \exp\left( \sum_{j \in \mathbb{Z}} \left( B_j L_{-j}(w, \rho) + N_{j-\frac{1}{2}} G_{-j+\frac{1}{2}}(w, \rho) \right) \right) \cdot \left( \frac{1}{w}, i\rho \right)
\]

and one incoming puncture at zero and local coordinate given by \( (w, \rho) \). In otherwords, the local coordinate vanishing at infinity is given by (4.7) with \( (x, \varphi) = (w, \rho) \), where we are assuming that \( H^{-1} \) is convergent in a neighborhood of infinity, i.e., that \( (B, N) \in \mathcal{H} \). from the definition of sewing on \( SK, Q_2 \) can be sewn to the last puncture of \( Q_1 \) if there exists a ball of radius \( 1/r \), for \( r > 0 \), about zero which lies in the image of \( H \) and such that \( 1/|z_B| < 1/r \), i.e., such that \( (z, \theta) \) is in the domain of convergence of \( H \circ I \). In this case, we can form the sewn supersphere \( Q_1 \circlearrowleft_{0} Q_2 \), and there exists a uniformizing function \( F : Q_1 \circlearrowleft_{0} Q_2 \rightarrow \mathcal{S} \mathcal{C} \) which, restricted to \( Q_1 \), is given by \( F_1(w, \rho) = F_{Q_1}(w, \rho) = H \circ I(w, \rho) \) and, restricted to \( Q_2 \), is given by \( F_2(w, \rho) = F_{Q_2}(w, \rho) = (w, \rho) \).

Thus the resulting canonical supersphere has: outgoing puncture at \( F_1(\infty, 0) = H \circ I(\infty, 0) = (\infty, 0) \) with local coordinate vanishing at the puncture given by \( I \circ F_1^{-1}(w, \rho) = H^{-1}(w, \rho) \); first incoming puncture at \( F_1(z, \theta) = H \circ I(z, \theta) \) with local coordinate vanishing at the puncture given by \( s_{(z, \theta)} \circ F_1^{-1}(w, \rho) = s_{(z, \theta)} \circ I^{-1} \circ H^{-1}(w, \rho) \); and last incoming puncture at \( F_2(0, 0) = (0, 0) \) with local coordinate vanishing at the puncture given by \( F_2^{-1}(w, \rho) = (w, \rho) \).

In otherwords, letting \( \Theta_j^{(2)} = \Theta_j^{(2)}(\{t_k B_k, t_k^{-1/2} N_k^{-1/2}\}_{k \in \mathbb{Z}}, (z, \theta)) \big|_{t^{1/2}=1} \in \Lambda_{\infty} \) be defined by (4.10), the resulting canonical supersphere is given by

\[
Q_1 \circlearrowleft_{0} Q_2 = (H \circ I(z, \theta); (B, N), \hat{E}^{-1}(s_{(z, \theta)} \circ I^{-1} \circ H^{-1} \circ s_{H \circ I(z, \theta)}(w, \rho)), (1, 0))
\]

(5.2)

\[
= (H \circ I(z, \theta); (B, N), (e^{\Theta^{(2)}}, \{\Theta_j^{(2)}, \Theta_j^{(2)} \}_{j \in \mathbb{Z}}), (1, 0)).
\]

Note in particular that this implies that if \( H^{-1}(w, \rho) \) is convergent in a neighborhood of infinity, i.e., if \( (B, N) \in \mathcal{H} \), and \( (z, \theta) \) is in the radius of convergence of \( H \circ I \), then \( \Theta_j^{(2)}(\{t_k B_k, t_k^{-1/2} N_k^{-1/2}\}_{k \in \mathbb{Z}}, (z, \theta)) \) is convergent for \( t^{1/2} = 1 \) and \( (x, \varphi) = (z, \theta) \).
5.2. \(N = 1\) supergeometric vertex operator superalgebras. In this section we introduce the notion of \(N = 1\) supergeometric vertex operator superalgebra \((N = 1 \text{ SG-VOSA})\). But before we do this, we will need some further definitions and notations.

Let \(\langle \cdot, \cdot \rangle\) be the natural pairing between \(V'\) and \(\bar{V}\). For \(n \in \mathbb{N}\), let
\[
\mathcal{SF}_V(n) = \text{Hom}_{\Lambda_*}(V^\otimes n, \bar{V}).
\]
For \(m \in \mathbb{Z}_+\), \(n \in \mathbb{N}\), and any positive integer \(i \leq m\), we define the \(t^{1/2}\)-contraction
\[
(f \circ_0 g)_{t^{1/2}} : \mathcal{SF}_V(m) \times \mathcal{SF}_V(n) \to \text{Hom}(V^\otimes (m+n-1), \bar{V}[[t^2, t^{-2}]])
\]
by
\[
(f, g) \mapsto (f \circ_0 g)_{t^{1/2}},
\]
and \(t^{1/2}\)-contraction
\[
(f \circ_0 g)_{t^{1/2}}(v_i \otimes \cdots \otimes v_{m+n-1})
\]
for all \(v_1, \ldots, v_{m+n-1} \in V\), where for any \(k \in \frac{1}{2} \mathbb{Z}\), the map \(P(k) : \bar{V} \to V(k)\) is the canonical projection map.

If for arbitrary \(v' \in V'\), \(v_1, \ldots, v_{m+n-1} \in V\), the formal Laurent series in \(t^{1/2}\)
\[
\langle v', (f \circ_0 g)_{t^{1/2}}(v_1 \otimes \cdots \otimes v_{m+n-1}) \rangle
\]
is absolutely convergent when \(t^{1/2} = 1\), then \((f \circ_0 g)_{1}\) is well defined as an element of \(\mathcal{SF}_V(m+n-1)\), and we define the contraction \((f \circ_0 g)_{t^{1/2}}\) of \(f \) and \(g\) by
\[
f \circ_0 g = (f \circ_0 g)_{t^{1/2} = 1}.
\]

In [3] (resp., in [4]) we define a left action of the symmetric groups \(S_n\) on \(\mathcal{SF}_V(n)\) (resp., on \(\mathcal{SF}_V(n)\)). These actions are needed to define the notion of \(N = 1\) SG-VOSA but will not be used elsewhere in this paper. Therefore we refer the readers to [3] and [4] for details.

A supermeromorphic superfunction on \(SK(n)\), for \(n \in \mathbb{Z}_+\), is a superfunction \(F : SK(n) \to \Lambda_\infty\) of the form
\[
F(Q) = F((z_1, \theta_1), \ldots, (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a^{(1)}_i, A^{(1)}, M^{(1)}), \ldots, (a^{(n)}_i, A^{(n)}, M^{(n)}))
\]
\[
= F_0((z_1, \theta_1), \ldots, (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a^{(1)}_i, A^{(1)}, M^{(1)}), \ldots, (a^{(n)}_i, A^{(n)}, M^{(n)}))
\]
\[
\times \prod_{i=1}^{n-1} z_i^{-s_i} \prod_{1 \leq i < j \leq n-1} (z_i - z_j - \theta_i \theta_j)^{-s_{ij}}
\]
where \(s_i\) and \(s_{ij}\) are nonnegative integers and
\[
F_0((z_1, \theta_1), \ldots, (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a^{(1)}_i, A^{(1)}, M^{(1)}), \ldots, (a^{(n)}_i, A^{(n)}, M^{(n)}))
\]
is a polynomial in the \(z_i^\prime, \theta_i^\prime, A_i^{(i)}\)'s, \(a_i^{(i)}\)'s, \(A_i^{(i)}\)'s, \(M_i^{(i)}\)'s, and \(M_{j-1/2}\)'s. For \(n = 0\) a supermeromorphic superfunction on \(SK(0)\) is a polynomial in the components of elements of \(SK(0)\), i.e., a polynomial in the \(A^{(0)}\)'s, and \(M^{(0)}\)'s, \(M^{(0)}\)'s. For \(F\) of the form [3], we say that \(F\) has a pole of order \(s_{ij}\) at \((z_i, \theta_i) = (z_j, \theta_j)\).
and if \((a, A, M) \in (\Lambda^0_n)^\times \times \mathcal{H}\), the function such that for any \(v\) on \(\mathcal{S}\mathcal{K}_Q\) of rank \(n\) is contained in the following lemma which follows from Proposition 3.33 in [B4].

**Lemma 5.3.** Let \((A, M), (B, N) \in \mathcal{H} \text{ and } a'^{\infty} \in (\Lambda^0_n)^\times\). If either \((A, M) = 0\) or \((B, N) = 0\), then \(\Gamma(a'^{\infty}, A, M, B, N) = 0\).

**Definition 5.4.** An \(N = 1\) SG-VOSA over \(\Lambda_\ast\) is a \(\frac{1}{2}\mathbb{Z}\)-graded (by weight) \(\Lambda_\infty\)-module which is also \(\mathbb{Z}_2\)-graded (by sign)

\[
V = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} V(k) = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} V^0(k) \oplus \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} V^1(k) = V^0 \oplus V^1
\]

such that only the subspace \(\Lambda_n\) of \(\Lambda_\infty\) acts non-trivially on \(V\),

\[
\dim V(k) < \infty \quad \text{for} \quad k \in \frac{1}{2}\mathbb{Z},
\]

and for any \(n \in \mathbb{N}\), a map

\[
\nu_n : \mathcal{S}\mathcal{K}(n) \to \mathcal{S}\mathcal{F}_V(n)
\]

satisfying the following axioms:

1. Positive energy axiom:

\[
V(k) = 0 \quad \text{for } k \text{ sufficiently small.}
\]

2. Grading axiom: Let \(v' \in V', v \in V(k)\), and \(a \in (\Lambda^0_n)^\times\). Then

\[
\langle v', \nu_1(0, (a, 0))(v) \rangle = a^{-2k} \langle v', v \rangle.
\]

3. Supermeromorphicity axiom: For any \(n \in \mathbb{Z}_+\), \(v' \in V'\), and \(v_1, ..., v_n \in V\), the function

\[
Q \mapsto \langle v', \nu_n(Q)(v_1 \otimes \cdots \otimes v_n) \rangle
\]

on \(\mathcal{S}\mathcal{K}(n)\) is a canonical supermeromorphic superfunction (in the sense of [B3]), and if \((z_i, \theta_i)\) and \((z_j, \theta_j)\), for \(i, j \in \{1, ..., n\}, i \neq j\), are the \(i\)-th and \(j\)-th punctures of \(Q \in \mathcal{S}\mathcal{K}(n)\), respectively, then for any \(v_i\) and \(v_j\) in \(V\), there exists \(N(v_i, v_j) \in \mathbb{Z}_+\) such that for any \(v' \in V'\) and \(v_k \in V, k \neq i, j\), the order of the pole \((z_i, \theta_i) = (z_j, \theta_j)\) of \(\langle v', \nu_n(Q)(v_1 \otimes \cdots \otimes v_n) \rangle\) is less than \(N(v_i, v_j)\).

4. Permutation axiom: Let \(\sigma \in S_n\). Then for any \(Q \in \mathcal{S}\mathcal{K}(n)\)

\[
\sigma(\nu_n(Q)) = \nu_n(\sigma(Q)).
\]

5. Sewing axiom: There exists a unique complex number \(c\) (the central charge or rank) such that if \(Q_1 \in \mathcal{S}\mathcal{K}(m)\) and \(Q_2 \in \mathcal{S}\mathcal{K}(n)\) are given by

\[
Q_1 = ((z_1, \theta_1), ..., (z_{m-1}, \theta_{m-1}); (A^{(0)}; M^{(0)}), (a^{(1)}, A^{(1)}, M^{(1)}), ..., (a^{(m)}, A^{(m)}, M^{(m)})),
\]

and

\[
Q_2 = ((z'_1, \theta'_1), ..., (z'_{n-1}, \theta'_{n-1}); (B^{(0)}, N^{(0)}), (b^{(1)}, B^{(1)}, N^{(1)}), ..., (b^{(n)}, B^{(n)}, N^{(n)})),
\]

Finally, the last ingredient we will need to introduce the notion of \(N = 1\) SG-VOSA is the \(\Gamma(a'^{\infty}, A, M, B, N)\) series which was introduced and studied in [B4] and [B5].
and if the \( i \)-th tube of \( Q_1 \) can be sewn with the 0-th tube of \( Q_2 \), then for any \( v' \in V' \) and \( v_1, \ldots, v_{m+n-1} \in V \),

\[
\langle v', (\nu_n(Q_1) i *_0 \nu_n(Q_2))_{1/2} (v_1 \otimes \cdots \otimes v_{m+n-1}) \rangle
\]

is absolutely convergent when \( t^{1/2} = 1 \), and

\[
\nu_{m+n-1}(Q_1 i \infty_0 Q_2) = (\nu_n(Q_1) i *_0 \nu_n(Q_2)) e^{-F(a^{(i)}, A(i), M^{(i)}, B^{(0)}, N^{(0)})c}.
\]

We denote the \( N = 1 \) supergeometric vertex operator superalgebra defined above by

\[
(V, \nu = \{\nu_n\}_{n \in \mathbb{N}})
\]

or just by \((V, \nu)\).

5.3. **The isomorphism between the category of \( N = 1 \) SG-VOSAs and the category of \( N = 1 \) NS-VOSAs.** In this section we recall from [B5] how to construct an \( N = 1 \) NS-VOSA from an \( N = 1 \) SG-VOSA, and how to construct an \( N = 1 \) SG-VOSA from an \( N = 1 \) NS-VOSA. We then recall the Isomorphism Theorem from [B5] which states that the two notions are equivalent by showing that their respective categories are isomorphic. But first we must recall the \( \iota \) function introduced in [B2] and [B3] which maps rational superfunctions to power series expanded about certain variables. This \( \iota \) function is a generalization of the \( \iota \) function of [FILM].

Let \( \Lambda_{\infty}[x_1, x_2, \ldots, x_n]_S \) be the ring of rational functions obtained by inverting (localizing with respect to) the set

\[
S = \left\{ \sum_{i=1}^n a_i x_i \mid a_i \in \Lambda_\infty^0, \text{ not all } (a_i)_B = 0 \right\}.
\]

Recall the map \( \iota_{i_1 \cdots i_n} : \mathbb{F}[x_1, \ldots, x_n]_S \longrightarrow \mathbb{F}[[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]] \) defined in [FILM] where coefficients of elements in \( S \) are restricted to the field \( \mathbb{F} \). We extend this map to \( \Lambda_{\infty}[x_1, x_2, \ldots, x_n]_S[\varphi_1, \varphi_2, \ldots, \varphi_n] = \Lambda_{\infty}[x_1, \varphi_1, x_2, \varphi_2, \ldots, x_n, \varphi_n]_S \) in the obvious way obtaining

\[
\iota_{i_1 \cdots i_n} : \Lambda_{\infty}[x_1, \varphi_1, x_2, \varphi_2, \ldots, x_n, \varphi_n]_S \longrightarrow \Lambda_{\infty}[[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]][\varphi_1, \ldots, \varphi_n].
\]

Let \( \Lambda_{\infty}[x_1, \varphi_1, x_2, \varphi_2, \ldots, x_n, \varphi_n]_{S'} \) be the ring of rational functions obtained by inverting the set

\[
S' = \left\{ \sum_{i<j}^n (a_i x_i + a_{ij} \varphi_i \varphi_j) \mid a_i, a_{ij} \in \Lambda_\infty^0, \text{ not all } (a_i)_B = 0 \right\}.
\]

Since we use the convention that a function of even and odd variables should be expanded about the even variables, we have

\[
\frac{1}{\sum_{i,j=1}^n (a_i x_i + a_{ij} \varphi_i \varphi_j)} = \frac{1}{\sum_{i=1}^n a_i x_i} - \frac{\sum_{i<j}^n a_{ij} \varphi_i \varphi_j}{(\sum_{i=1}^n a_i x_i)^2}.
\]

Thus \( \Lambda_{\infty}[x_1, \varphi_1, x_2, \varphi_2, \ldots, x_n, \varphi_n]_{S'} \subseteq \Lambda_{\infty}[x_1, \varphi_1, x_2, \varphi_2, \ldots, x_n, \varphi_n]_S \) and \( \iota_{i_1 \cdots i_n} \) is well defined on \( \Lambda_{\infty}[x_1, \varphi_1, x_2, \varphi_2, \ldots, x_n, \varphi_n]_{S'} \).

In the case \( n = 2 \), the map \( \iota_2 : \Lambda_{\infty}[x_1, \varphi_1, x_2, \varphi_2]_{S'} \longrightarrow \Lambda_{\infty}[[x_1, x_2]][\varphi_1, \varphi_2] \) is given by first expanding an element of \( \Lambda_{\infty}[x_1, \varphi_1, x_2, \varphi_2]_{S'} \) as a formal series in \( \Lambda_{\infty}[x_1, \varphi_1, x_2, \varphi_2]_S \) and then expanding each term as a series in \( \Lambda_{\infty}[[x_1, x_2]][\varphi_1, \varphi_2] \).
Proposition 5.5. Let \( z \) be the coefficient of \( x \) for negative powers of linear polynomials involving both \( x \) and \( z \), where \( \text{Res}_{\nu} V_{\nu} \) for \( \nu \) containing at most finitely many negative powers of \( 2 \).

Given an \( N = 1 \) supergeometric VOSA, we construct an (algebraic) \( N = 1 \) NS-VOSA. Let \((V, \nu)\) be an \( N = 1 \) SG-VOSA over \( \Lambda_\ast \). We define the vacuum \( 1_\nu \in \bar{V} \) by \( 1_\nu = \nu_0(0) \); an element \( \tau_\nu \in \bar{V} \) by \( \tau_\nu = (-\partial/\partial \nu) \nu_0(0, M(\epsilon, 3/2)) \), where \( M(\epsilon, 3/2) \) is the series of odd supernumbers given by \( \nu \). The vertex operator \( Y_\nu(u, (x, \varphi)) = \sum_{n \in \mathbb{Z}} u_n x^{-n-1} + \varphi \sum_{n \in \mathbb{Z}} u_{n-1/2} x^{-n-1} \) associated with \( u \in V \) by
\[
(5.5) \quad u_n v + \theta u_{n-1} v = \text{Res}_{z} (z^n \nu_2((z, \theta); 0, (1, 0), (1, 0)) (v' \otimes u \otimes v)),
\]
where \( \text{Res}_{z} \) means taking the residue at the singularity \( z = 0 \), i.e., taking the coefficient of \( z^{-1} \).

**Proposition 5.5.** (BG) The elements \( 1_\nu \) and \( \tau_\nu \) of \( \bar{V} \) are in fact in \( V(0) \) and \( V(3/2) \), respectively. If the rank of \((V, \nu)\) is \( c \), then \((V, Y_\nu(\cdot, (x, \varphi)), 1_\nu, \tau_\nu)\) is an \( N = 1 \) NS-VOSA with odd formal variables and with rank \( c \).

Given an \( N = 1 \) NS-VOSA over \( \Lambda_\ast \), with rank \( c \in \mathbb{C}, (V, Y(\cdot, (x, \varphi)), 1, \tau) \), we construct an \( N = 1 \) SG-VOSA by defining a sequence of maps:
\[
\nu^Y_n : SK(n) \to SF_V(n), \quad Q \to \nu^Y_n(Q)
\]
by
\[
\langle \nu^Y_n, \nu^Y_n \rangle (z_1, \theta_1), \ldots, (z_{n-1}, \theta_{n-1}); (A(0), M(0)), (a(1), A(1), M(1)), \ldots, (a(n), A(n), M(n)) \rangle (v_1 \otimes \cdots \otimes v_n)
\]
\[
(5.6) \quad = t_{-n-1}^{-1} e^{-\sum_{j \in \mathbb{Z}_+} \left( A^{(j)}(j') + M^{(j)}(j') \right) G^{(j'-\frac{1}{2})}} v',
\]
\[
Y(\nu \cdot \sum_{j \in \mathbb{Z}_+} \left( A^{(j)}(j)+M^{(j)}(j) \right) G^{(j-\frac{1}{2})}, a(1)-2L(0) \cdot v_1, (x_1, \varphi_1)) \cdots
\]
\[
\left|_{(x, \varphi) = (z, \theta)} \right|
\]
for \( n \in \mathbb{Z}_+, v' \in V', v_1, \ldots, v_n \in V \), and by
\[
\langle v', \nu^Y_0(A(0), M(0)) \rangle = e^{-\sum_{j \in \mathbb{Z}_+} \left( A^{(j)}(j'+M^{(j)}(j)) G^{(j'-\frac{1}{2})} \right) v', 1}.
\]

**Proposition 5.6.** (BG) The pair \((V, \nu^Y)\) is an \( N = 1 \) SG-VOSA over \( \Lambda_\ast \).

Let \( \text{SG}(c, *) \) be the category of \( N = 1 \) SG-VOSAs over \( \Lambda_\ast \) with central charge \( c \), and let \( \text{SV}(\varphi, c, *) \) be the category of \( N = 1 \) NS-VOSAs over \( \Lambda_\ast \) with odd formal variables and central charge \( c \) where \( * \) is \( L \in \mathbb{N} \) or \( \infty \). Let \( 1_{\text{SV}} \) and \( 1_{\text{SG}} \) be the identity functors on \( \text{SV}(\varphi, c, *) \) and \( \text{SG}(c, *) \), respectively.
Theorem 5.7. For any $c \in \mathbb{C}$, the two categories $\mathbf{SV}(\varphi, c, *)$ and $\mathbf{SG}(c, *)$ are isomorphic with isomorphism given by the following functors acting on objects and morphisms
\[
F_{\mathbf{SV}} : \mathbf{SV}(\varphi, c, *) \rightarrow \mathbf{SG}(c, *)
\]
\[
((V, Y(\cdot, (x, \varphi))), 1, \tau) \mapsto ((V, \nu^Y), \gamma)
\]
and
\[
F_{\mathbf{SG}} : \mathbf{SG}(c, *) \rightarrow \mathbf{SV}(\varphi, c, *)
\]
\[
((V, \nu), \gamma) \mapsto ((V, Y_{\nu}(\cdot, (x, \varphi))), 1, \tau_{\nu}), \gamma)
\]

which satisfy $F_{\mathbf{SV}} \circ F_{\mathbf{SG}} = 1_{\mathbf{SG}}$ and $F_{\mathbf{SG}} \circ F_{\mathbf{SV}} = 1_{\mathbf{SV}}$.

5.4. Geometric proof of Proposition 4.1 for superconformal change of coordinates convergent in a neighborhood of zero.

Proof. Assuming that $H(x, \varphi)$ given by (5.2) is convergent and bijective in a neighborhood of zero, we have
\[
Q_1 = (0, (a_{\Box}, A, M)) \in SK(1),
\]
and
\[
Q_2 = ((\varphi, \theta); 0, (1, 0), (1, 0)) \in SK(2).
\]
From Example 5.1 we see that if $(\varphi, \theta)$ is in the radius of convergence of $H$, then $Q_2$ can be sewn to $Q_1$ and
\[
Q_1 1 \infty_0 Q_2 = (H(\varphi, \theta); 0, \hat{E}^{-1}(s_{\varphi, \theta}) \circ H^{-1} \circ s^{-1}_{H(z, \theta)}(w, \rho), (a_{\Box}, A, M))
\]
(5.7)
\[
= (H(z, \theta); 0, (a_{\Box} e^{\Theta(1)}), \{a_{\Box}^2 \Theta(1)^2, a_{\Box}^2 - 1 \Theta(1) \}_{j \in \mathbb{Z}_+}, (a_{\Box}, A, M)).
\]
Therefore from the Isomorphism Theorem 5.7 1 the sewing axiom for an $N = 1$ SG-VOSA, and Lemma 5.3 we have that
\[
\langle v', \gamma_H(Y(u, (x, \varphi))v) \rangle |_{(x, \varphi) = (z, \theta)}
\]
\[
= \frac{1}{\nu_1} v', e^{- \sum_{j \in \mathbb{Z}_+} \left( A_{j} L(j) + M_j \right)} \cdot a_{\Box}^{-2} L(0) \cdot Y(u, (z, \theta)) v)
\]
\[
= \langle v', (Y_1(Q_1) v_1)_{j = 0} v_2(Q_2)(u \otimes v) \rangle
\]
\[
= \langle v', (v'_{\gamma}(Q_1 1 \infty_0 Q_2)(u \otimes v) \rangle
\]
\[
= \langle v', Y(\gamma_{H}(H(u, (x, \varphi))H(v)) |_{(x, \varphi) = (z, \theta)}
\]
(5.8)

for $u, v \in V$, $v' \in V'$, and $(z, \theta)$ in the domain of convergence of $H$. Thus $\gamma_{H}$ is well defined and we have that
\[
\gamma_{H}(Y(u, (x, \varphi))v) = Y(\gamma_{H}(H(u, (x, \varphi))H(v))
\]
as formal power series in $V[1/x, 1/\varphi]$, with both sides convergent and equal if $(x, \varphi) = (z, \theta)$ is in the domain of convergence of $H$. Since $\gamma_{H}(H, t^{1/2}, (x, \varphi)) |_{t^{1/2} = 1}$ is well-defined, so is $\Theta(1)^{(1)} |_{t^{1/2} = 1}$ and thus so is $\gamma_{H}(H, t^{1/2}, (x, \varphi)) |_{t^{1/2} = 1}$. This, the fact that $\gamma_{H}(H, (x, \varphi)) = \gamma_{H}(H, (x, \varphi)) |_{v} = v$ for $v \in V$, and the fact that $\gamma_{H}$ is invertible, give us equation (5.8).
5.5. Geometric proof of Proposition 5.3 for superconformal change of coordinates convergent in a neighborhood of infinity.

Proof. Let

\[ Q_1 = (z, \theta ; 0, (1, 0), (1, 0)) \in SK(2). \]

Assuming that \( H^{-1}(x, \varphi) \) given by \( x, \varphi \in SK(2) \) is convergent and bijective in a neighborhood of infinity, we have

\[ Q_2 = (B, N, (1, 0)) \in SK(1). \]

From Example 5.2, we see that if \( (z, \theta) \) is in the radius of convergence of \( H \circ I \), then \( Q_2 \) can be sewn to \( Q_1 \) and

\[ Q_1 2 \rightarrow \\circ 0 \quad Q_2 \]

\[ = (H \circ I(z, \theta); (B, N), \tilde{E}^{-1}(s_{(z, \theta)} \circ I^{-1} \circ H^{-1} \circ s_{H \circ I(z, \theta)}(w, \rho)), (1, 0)) \]

(5.9) \[ = (H \circ I(z, \theta); (B, N), (e^{\Theta(x, \varphi)} ; \{ \Theta_j^{(2)} \}_j \in \mathbb{Z}_+), (1, 0)). \]

Therefore from the Isomorphism Theorem 5.7, the sewing axiom for an \( N = 1 \)

SG-VOSA, and Lemma 5.8, we have that

\[ \langle v', Y(u, (x, \varphi)) \xi_{H \circ I}(v) \rangle \big|_{(x, \varphi) = (z, \theta)} \]

(5.10) \[ = \sum_{k \in \frac{1}{2} \mathbb{Z}} \langle v', Y(u, (x, \varphi)) P_k(\xi_{H \circ I}(v)) \rangle \big|_{(x, \varphi) = (z, \theta)} \]

\[ = \langle v', \nu_k^2(Q_1) 2 \ast_0 \nu_k^2(Q_2)(u \otimes v) \rangle \]

\[ = \langle v', \nu_k^2(Q_1 2 \rightarrow \circ_0 Q_2)(u \otimes v) \rangle \]

\[ = \langle \xi_{H \circ I}(v'), Y(\xi_{H \circ I}(x, \varphi))_k(u, H \circ I(x, \varphi)v) \rangle \big|_{(x, \varphi) = (z, \theta)} \]

(5.11) \[ = \langle v', \xi_{H \circ I}(Y(\xi_{H \circ I}(x, \varphi))_k(u, H \circ I(x, \varphi)v)) \rangle \big|_{(x, \varphi) = (z, \theta)} \]

for \( u, v \in V, v' \in V' \) and \( (z, \theta) \) in the radius of convergence of \( H \circ I \). In particular, the above equality implies that

\[ \langle v', (\nu_k^2(Q_1) 2 \ast_0 \nu_k^2(Q_2))_{t1/2}(u \otimes v) \rangle \]

is absolutely convergent when \( t1/2 = 1 \), and is equal to

\[ \langle v', Y(u, (x, \varphi)) \xi_{H \circ I}(v) \rangle \big|_{(x, \varphi) = (z, \theta)} \]

\[ = \langle v', \xi_{H \circ I}(Y(\xi_{H \circ I}(x, \varphi))_k(u, H \circ I(x, \varphi)v)) \rangle \big|_{(x, \varphi) = (z, \theta)} \]

implying \( \xi_{\Theta}(H \circ I, (x, \varphi)) = \xi_{\Theta}(H \circ I, t1/2, (x, \varphi)) \big|_{t1/2 = 1} \) is well defined for \( (z, \theta) \) in the radius of convergence of \( H \circ I \). Thus, as formal power series in \( \tilde{V}[x^{-1}, x]]_{(\varphi)} \), we have

(5.12) \[ Y(u, (x, \varphi)) \xi_{H \circ I}(v) = \xi_{H \circ I}(Y(\xi_{\Theta}(H \circ I,(x, \varphi))_k(u, H \circ I(x, \varphi)v)) \]

and both sides are well defined and equal for \( (x, \varphi) = (z, \theta) \) in the radius of convergence of \( H \circ I \). In addition, since \( \xi_{\Theta}(H \circ I, t1/2, (x, \varphi)) \big|_{t1/2 = 1} \) is well-defined, so is \( \Theta_j^{(2)}(t1/2) \big|_{t1/2 = 1} \) and thus so is \( \xi_{\Theta}(H \circ I, t1/2, (x, \varphi)) \big|_{t1/2 = 1} \). This, the fact that \( \xi_{\Theta}(H \circ I, (x, \varphi)) \circ \xi_{\Theta}(H \circ I, (x, \varphi))^{-1}(v) = v \) for \( v \in V \), and the fact that \( \xi_{H \circ I} \) is invertible, give us equation □
6. Formal algebraic proofs for formal superconformal changes of variables

6.1. Formal algebraic proof of Proposition 4.1 for formal superconformal change of coordinates at zero. In this section, we let \( H(x, \varphi) \in \Lambda_\infty[[x]][\varphi] \) be the formal superconformal power series given by (4.2), but this time we do not assume that \( H(x, \varphi) \) converges in a neighborhood of zero.

We will prove Proposition 4.1 by a formal algebraic argument first employed in [B5] to construct an \( N = 1 \) SG-VOSA from an \( N = 1 \) NS-VOSA (i.e., to prove Proposition 6.1 above). This result was then used in [B5] to prove the Isomorphism Theorem 6.1. The difference in the present case is that we are not assuming the convergence of \( H \) and consequently cannot use the Isomorphism Theorem 6.1.2, as we did in Section 5. However, as it turns out, enough of the algebraic formalism used in [B5] carries over to the current more general setting, and we will be retracing that part of the construction used in [B5] in this more general setting.

We first prove the following bracket formula.

**Lemma 6.1.** Let \( (V, Y, (x, \varphi)), 1, \tau \) be an \( N = 1 \) NS-VOSA over a Grassmann algebra. For \( u \in V \), \( t^{1/2} \) a formal even variable, \( a_{\square} \in (\Lambda_\infty^0)^{x} \), and \( (A_j, M_{j-1/2}) \in \Lambda_\infty \), for \( j \in \mathbb{Z}_+ \), we have the following bracket formula in \( (\text{End} V)[[t^{1/2}]][[x^{-1}, x]][\varphi] \)

\[
\sum_{j \in \mathbb{Z}_+} \left( t^j a_{\square}^{-2j} A_j L(j) + t^j a_{\square}^{-2j+1} M_{j-\frac{1}{2}} G(j - \frac{1}{2}) \right) , Y(u, (x, \varphi)) \right] 
\]

(6.1)

\[
= Y\left( \sum_{k=-1}^{\infty} \sum_{j \in \mathbb{Z}_+} \left( j + 1 \right) t^j a_{\square}^{-2j} x^{j-k} \left( A_j + 2 \left( \frac{j-k}{j+1} \right) t^j a_{\square} x^{-1} \varphi M_{j-\frac{1}{2}} \right) L(k) + x^{-1} \left( \frac{j-k}{j+1} t^j a_{\square} M_{j-\frac{1}{2}} + \varphi \frac{j-k}{2} A_j \right) G(k + \frac{1}{2}) \right) , u, (x, \varphi)).
\]

Furthermore, if we apply both sides of (6.1) to \( v \in V \), then we can set \( t^{1/2} = 1 \) and both sides are well-defined elements of \( V((x))[\varphi] \).

**Proof.** From 4.12, we have

\[
[Y(\tau, (x_1, \varphi_1)), Y(u, (x, \varphi))] = \text{Res}_{x_0} x_0^{-1} \left( \frac{x_1 - x_0 - \varphi_1 x}{x} \right) Y(\tau, (x_0, \varphi_1 - \varphi)) u, (x, \varphi)).
\]

Let

\[
l_t(x_1) = \sum_{j \in \mathbb{Z}_+} t^j a_{\square}^{-2j} A_j x_1^{j+1} \quad \text{and} \quad g_t(x_1) = \sum_{j \in \mathbb{Z}_+} t^{j-\frac{1}{2}} a_{\square}^{-2j+1} M_{j-\frac{1}{2}} x_1^{j+1}.
\]

Then

\[
\varphi_1 \text{Res}_{x_1} l_t(x_1) Y(\omega, (x_1, \varphi_1)) = \varphi_1 \sum_{j \in \mathbb{Z}_+} t^j a_{\square}^{-2j} A_j L(j)
\]

\[
\varphi_1 \text{Res}_{x_1} g_t(x_1) Y(\tau, (x_1, \varphi_1)) = \varphi_1 \sum_{j \in \mathbb{Z}_+} t^{j-\frac{1}{2}} a_{\square}^{-2j+1} M_{j-\frac{1}{2}} G(j - \frac{1}{2}),
\]
and by the $G(-1/2)$-derivative property \[3.10\], we have

\[
Y(\omega, (x_1, \varphi_1)) = Y\left(\frac{1}{2}G(-\frac{1}{2})\tau, (x_1, \varphi_1)\right)
\]

\[
= \frac{1}{2} \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) Y(\tau, (x_1, \varphi_1)).
\]

Thus

\[
\varphi_1 \left[ \sum_{j \in \mathbb{Z}_+} \left( t^j a_{-2j} A_j L(j) + t^{j-\frac{1}{2}} a_{-2j+1} M_j - \frac{1}{2} G(j - \frac{1}{2}) \right) , Y(u, (x, \varphi)) \right]
\]

\[
= \varphi_1 \text{Res}_{x_1} \left[ \frac{1}{2} l_i(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_i(x_1) \right] Y(\tau, (x_1, \varphi_1)), Y(u, (x, \varphi)) \]

\[
= \varphi_1 \cdot \left[ \frac{1}{2} l_i(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_i(x_1) \right]
\]

\[
\left[ \text{Res}_{x_0} x^{-1} \delta(x_0 - x_1 - \varphi_1 \varphi) \right] Y(\tau, (x_0, \varphi_1 - \varphi)) u, (x, \varphi) \right)
\]

\[
= Y \left( \varphi_1 \text{Res}_{x_1} \text{Res}_{x_0} \left[ \frac{1}{2} l_i(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_i(x_1) \right]
\]

\[
\left[ x^{-1} \delta(x_0 - x_1 - \varphi_1 \varphi) \right] Y(\tau, (x_0, \varphi_1 - \varphi)) u, (x, \varphi) \right)
\]

However using the $\delta$-function identity \[3.2\], we have

\[
\varphi_1 \text{Res}_{x_1} \text{Res}_{x_0} \left[ \frac{1}{2} l_i(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_i(x_1) \right]
\]

\[
\left[ x^{-1} \delta(x_0 - x_1 - \varphi_1 \varphi) \right] Y(\tau, (x_0, \varphi_1 - \varphi))
\]

\[
= \varphi_1 \text{Res}_{x_1} \text{Res}_{x_0} \left[ \frac{1}{2} l_i(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_i(x_1) \right]
\]

\[
\left[ x^{-1} \delta(x_0 + x_1 + \varphi_1 \varphi) \right] Y(\tau, (x_0, \varphi_1 - \varphi))
\]

\[
= \varphi_1 \text{Res}_{x_1} \text{Res}_{x_0} \left[ \frac{1}{2} l_i(x_1) \left( \frac{\partial}{\partial \varphi_1} x^{-1} \delta(x_0 + x_1 + \varphi_1 \varphi) \right) \right] Y(\tau, (x_0, \varphi_1))
\]

\[
+ \frac{1}{2} l_i(x_1) x^{-1} \delta(x_0 + x_1) \frac{\partial}{\partial \varphi_1} Y(\tau, (x_0, \varphi_1))
\]

\[
+ g_i(x_1) x^{-1} \delta(x_0 + x_1) Y(\tau, (x_0, \varphi_1))
\]

\[
+ g_i(x_1) x^{-1} \delta(x_0 + x_1) Y(\tau, (x_0, \varphi_1))
\]
\[ \varphi_1 \operatorname{Res}_{x_0} \left( \sum_{n \in \mathbb{Z}} n(x + x_0)^{n-1} \varphi x_1^{-n-1} \right) \left( \frac{1}{2} \sum_{j \in \mathbb{Z}^+} t^j a_{\square}^{-2j} A_j x_1^{j+1} \right) \]
\[ \left( \sum_{k \in \mathbb{Z}} G(k + \frac{1}{2}) x_0^{-k-2} \right) + \left( \sum_{n \in \mathbb{Z}} (x + x_0)^n x_1^{-n-1} \right) \left( \frac{1}{2} \sum_{j \in \mathbb{Z}^+} t^j a_{\square}^{-2j} A_j x_1^{j+1} \right) \]
\[ \left( 2 \sum_{k \in \mathbb{Z}} L(k) x_0^{-k-2} \right) + \left( \sum_{n \in \mathbb{Z}} (x + x_0)^n x_1^{-n-1} \right) \left( \sum_{j \in \mathbb{Z}^+} t^j x_1^{j+1} M_j^{-\frac{1}{2}} x_1^j \right) \]
\[ \left( \sum_{k \in \mathbb{Z}} G(k + \frac{1}{2}) x_0^{-k-2} - 2 \varphi L(k) x_0^{-k-2} \right) \]

\[ = \varphi_1 \operatorname{Res}_{x_0} \left( \sum_{l \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \left( \frac{j + 1}{2} \right) x^{j-l} \varphi t^j a_{\square}^{-2j} A_j G(l - \frac{1}{2}) \right) \]
\[ + \left( \frac{j + 1}{2} \right) x^{j-l} t^j a_{\square}^{-2j} A_j L(l - 1) \]
\[ + \left( \frac{j}{2} \right) x^{j-l} t^j a_{\square}^{-2j+1} M_j^{-\frac{1}{2}} \left( G(l - \frac{1}{2}) - 2 \varphi L(l - 1) \right) \]

\[ = \varphi_1 \sum_{k=-1}^{\infty} \sum_{j \in \mathbb{Z}^+} \left( \frac{j + 1}{k + 1} \right) t^j a_{\square}^{-2j} x^{j-k} A_j + 2 \left( \frac{j - k}{k + 1} \right) x^{-k} t^j a_{\square}^{-2j} \varphi M_j^{-\frac{1}{2}} \left( L(k) \right) \]
\[ + \left( \frac{j}{k + 1} \right) t^j a_{\square}^{-2j} x^{j-k+1} \left( \frac{j + 1}{2} \right) \varphi A_j + t^j a_{\square}^{-2j+1} M_j^{-\frac{1}{2}} \left( G(k + \frac{1}{2}) \right) \]

which finishes the proof of (6.1).

Since \( V \) is a positive energy representation for the Neveu-Schwarz algebra and by the truncation condition for \( Y(V, (x, \varphi)) \), when applied to \( v \in V \), each side of (6.1) is in \( V((x)) \{ \varphi \} [t^{1/2}] \) and thus well defined when \( t^{1/2} \)

We have the following immediate corollary to Lemma 6.1.

**Corollary 6.2.**

\[ e^{-\sum_{j \in \mathbb{Z}^+} (t^j a_{\square}^{-2j} A_j L(j) + t^j a_{\square}^{-2j+1} M_j^{-\frac{1}{2}} G(j - \frac{1}{2}))} Y(u, (x, \varphi)) \]
\[ = e^{-\sum_{j \in \mathbb{Z}^+} (t^j a_{\square}^{-2j} A_j L(j) + t^j a_{\square}^{-2j+1} M_j^{-\frac{1}{2}} G(j - \frac{1}{2}))} Y(u, (x, \varphi)). \]
By Corollary 6.2, Proposition 2.3, equation (3.14), and equation (3.13), we have

\[ Y\left(\exp\left(-\sum_{k=1}^{\infty} \sum_{j \in \mathbb{Z}_+} \frac{j+1}{k+1} t^{j-1} a_{j}\right)^{2j} x^{j-k} \right) \]

\[ \left( A_j + 2 \left(\frac{j-k}{j+1}\right) t^{-\frac{j}{2}} a_{j} x^{-1} \varphi M_{j-\frac{1}{2}} \right) L(k) \]

\[ + x^{-1} \left(\left(\frac{j-k}{j+1}\right) t^{-\frac{j}{2}} a_{j} M_{j-\frac{1}{2}} + \varphi \left(\frac{j-k}{2} \right) A_j \right) G(k + \frac{1}{2}) \right) u_{i}(x, \varphi). \]

Furthermore, if we apply both sides of the equation above to \( v \in \mathcal{V} \), we can set \( t^{\frac{1}{2}} = 1 \).

We are now ready to prove Proposition 4.1 in the case of a formal invertible superconformal change of coordinates vanishing at zero.

Proof. Denote

\[ H_{t^{\frac{1}{2}}}(x, \varphi) \]

\[ = (\tilde{x}(t^{\frac{1}{2}}), \tilde{\varphi}(t^{\frac{1}{2}})) \]

\[ = (t^{-\frac{j}{2}} a_{j})^{2L_0(x, \varphi)} \cdot \exp\left(\sum_{j \in \mathbb{Z}_+} \left( A_j L_j(x, \varphi) + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}}(x, \varphi) \right) \right) \cdot (x, \varphi). \]

By Corollary 6.2, Proposition 2.3, equation (3.13), and equation (3.13), we have

\[ \gamma_{H_{t^{\frac{1}{2}}}}(Y(u_{i}, (x, \varphi))v) \]

\[ = e^{-\sum_{j \in \mathbb{Z}_+} \left( A_j L_j + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}} \right) (x, \varphi)} \cdot (t^{-\frac{j}{2}} a_{j})^{-2L(0)} Y(u_{i}, (x, \varphi)) v \]

\[ = e^{-\sum_{j \in \mathbb{Z}_+} \left( A_j L_j + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}} \right) (x, \varphi)} \cdot (t^{-\frac{j}{2}} a_{j})^{-2L(0)} Y(u_{i}, (x, \varphi)) \cdot (t^{-\frac{j}{2}} a_{j})^{2L(0)}. \]

\[ e^{\sum_{j \in \mathbb{Z}_+} \left( A_j L_j + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}} \right) (x, \varphi)} \cdot (t^{-\frac{j}{2}} a_{j})^{-2L(0)}. \]

\[ = (t^{-\frac{j}{2}} a_{j})^{-2L(0)} e^{-\sum_{j \in \mathbb{Z}_+} \left( t^{j} a_{j}^{-2j} A_j L_j + t^{j-\frac{j}{2}} a_{j-1}^{-2j+1} M_{j-\frac{1}{2}} G_{j-\frac{1}{2}} \right) (x, \varphi)} \]

\[ e^{\sum_{j \in \mathbb{Z}_+} \left( t^{j} a_{j}^{-2j} A_j L_j + t^{j-\frac{j}{2}} a_{j-1}^{-2j+1} M_{j-\frac{1}{2}} G_{j-\frac{1}{2}} \right) (x, \varphi)} \]

\[ e^{-\sum_{j \in \mathbb{Z}_+} \left( A_j L_j + M_{j-\frac{1}{2}} G_{j-\frac{1}{2}} \right) (x, \varphi)} \cdot (t^{-\frac{j}{2}} a_{j})^{-2L(0)}. \]
\[
\begin{align*}
\Theta &= (t^{-\frac{1}{2}}a)_{\square}^{-2L(0)} Y \left( \exp \left( -\sum_{k=-1}^{\infty} \sum_{j \in \mathbb{Z}^+} \frac{j+1}{k+1} t^j a_{\square}^{-2j} x^{-k} \right) \right) \\
&= \left( A_j + 2 \left( \frac{j-k}{j+1} \right) t^{-\frac{j}{2}} a_{\square}^{-1} x^{-1} \phi M_j \right) L(k) \\
&+ x^{-1} \left( \left( \frac{j-k}{j+1} \right) t^{-\frac{j}{2}} a_{\square}^{-1} M_j \phi + \phi \left( \frac{j-k}{2} \right) A_j \right) G(k+\frac{1}{2}) \bigg) u, (x, \phi) \\
&= (t^{-\frac{1}{2}}a)_{\square}^{-2L(0)} e^{-\sum_{j \in \mathbb{Z}^+} \left( A_j L(j) + M_j \right) \frac{1}{2} G(j) - \frac{1}{4}} \cdot (t^{-\frac{1}{2}}a)_{\square}^{-2L(0)} , v
\end{align*}
\]

Thus we have
\[
Y \left( (t^{-\frac{1}{2}}a)_{\square}^{-2L(0)} e^{-\sum_{j \in \mathbb{Z}^+} \left( \Theta(j) L(j) + \Theta(j) \phi \right) G(j) - \frac{1}{4}} \cdot (t^{-\frac{1}{2}}a)_{\square}^{-2L(0)} , v \right) = Y \left( (t^{-\frac{1}{2}}a)_{\square}^{-2L(0)} e^{-\sum_{j \in \mathbb{Z}^+} \left( A_j L(j) + M_j \right) \frac{1}{2} G(j) - \frac{1}{4}} \cdot (t^{-\frac{1}{2}}a)_{\square}^{-2L(0)} , v \right)
\]

for \( u, v \in V \). Thus we have
\[
\gamma_{H,t^{1/2}}(Y(u, (x, \phi))) = Y(\gamma_{H,t^{1/2}}(u), H_{t^{1/2}}(x, \phi)) \gamma_{H,t^{1/2}}(v)
\]

By the truncation condition and the fact that \( V \) is a positive energy module for the Neveu-Schwarz algebra, both sides of (6.2) are formal power series in \( V[t^{-1/2}, t^{1/2}]((x)) \), and thus well defined at \( t^{1/2} = 1 \). This proves the first statement of Proposition 4.1 and equation (4.5) for general formal superconformal change of variables \( H \) vanishing at zero. In addition, since \( \gamma_{H,t^{1/2},(x,\phi)} \big|_{t^{1/2}=1} = 1 \) and thus so is \( \Theta^{-1}(t^{1/2}) \big|_{t^{1/2}=1} \) is well-defined, so is \( \Theta^{-1}(t^{1/2}) \big|_{t^{1/2}=1} = 1 \) and thus so is \( \gamma^{-1}_{H,t^{1/2},(x,\phi)} \big|_{t^{1/2}=1} = 1 \). This, the fact that \( \gamma_{H,(x,\phi)} \circ \gamma_{H,(x,\phi)}(v) = v \) for \( v \in V \), and the fact that \( \gamma_{H} \) is invertible, give us equation (4.6).
6.2. Formal algebraic proof of Proposition 4.5 for formal superconformal change of coordinates at infinity. In this section, we let \( H^{-1}(x, \varphi) \in x^{-1} \wedge_\infty[[x^{-1}][\varphi]] \) be the formal superconformal power series given by (4.7), but this time we do not assume that \( H^{-1} \) converges in a neighborhood of infinity.

As in Section 6.1, we will prove Proposition 4.5 by a formal algebraic argument first employed in \[B5\] to construct an \( N=1 \) SG-VOSA from an \( N=1 \) NS-VOSA.

We first prove the following bracket formula.

**Lemma 6.3.** Let \((V, Y((\cdot, (x, \varphi)), 1, \tau))\) be an \( N=1 \) NS-VOSA over a Grassmann algebra. For \( u \in V, \, t^{1/2} \) a formal even variable, and \((B_j, N_j-1/2) \in \wedge_\infty, \) for \( j \in \mathbb{Z}_+ \), we have the following bracket formula in \((\text{End } V)[[t^{1/2}]][[x^{-1}, x]][\varphi] \):

\[
\left[ \sum_{j \in \mathbb{Z}_+} \left( t^j B_j L(-j) + t^{-j/2} N_j - \frac{j}{2} G(-j + \frac{1}{2}) \right), Y(u, (x, \varphi)) \right] = Y\left( \sum_{m=-j \in \mathbb{Z}_+} \frac{(-j+1)}{m+1} x^{-j-m} \left( t^j B_j + 2 \varphi t^{-j/2} N_j - \frac{j}{2} \right) L(m) \right.

\]

\[
+ \left( t^{-j/2} N_j - \frac{j}{2} + \varphi x^{-1} \frac{(-j-m)}{2} t^j B_j \right) G(m + \frac{1}{2}) u, (x, \varphi) \right).
\]

**Proof.** From \(3.12\), we have

\[
[Y(\tau, (x_1, \varphi_1)), Y(u, (x, \varphi))]

= \text{Res}_{x_0} x^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi}{x} \right) Y(\tau, (x_0, \varphi_1 - \varphi)) u, (x, \varphi)).\]

Let

\[
l_t(x_1) = \sum_{j \in \mathbb{Z}_+} t^j B_j x_{1}^{-j+1} \quad \text{and} \quad g_t(x_1) = \sum_{j \in \mathbb{Z}_+} t^{-j/2} N_j x_{1}^{-j+1}.
\]

Then

\[
\varphi_1 \text{Res}_{x_1} l_t(x_1) Y(\omega, (x_1, \varphi_1)) = \varphi_1 \sum_{j \in \mathbb{Z}_+} t^j B_j L(-j),
\]

\[
\varphi_1 \text{Res}_{x_1} g_t(x_1) Y(\tau, (x_1, \varphi_1)) = \varphi_1 \sum_{j \in \mathbb{Z}_+} t^{-j/2} N_j - \frac{j}{2} G(-j + \frac{1}{2}),
\]

and by the \(G(-1/2)\)-derivative property \(3.10\), we have

\[
Y(\omega, (x_1, \varphi_1)) = Y\left( \frac{1}{2} G(-\frac{1}{2}) \tau, (x_1, \varphi_1) \right)

= \frac{1}{2} \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) Y(\tau, (x_1, \varphi_1)).
\]
Thus
\[ \varphi_1 \left[ \sum_{j \in \mathbb{Z}_+} \left( t^j B_j L(-j) + t^{j-\frac{1}{2}} N_{j-\frac{1}{2}} G(-j + \frac{1}{2}) \right), Y(u, (x, \varphi)) \right] \]

\[ = \varphi_1 \text{Res}_{x_0} \left( \left( \frac{1}{2} l_0(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_0(x_1) \right) \right) \]

\[ + \prod_{k=0}^{\infty} \int_{y} \left( \frac{1}{2} l_k(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_k(x_1) \right) \]

\[ x^{-1} \left[ \left( x_{n+1} \cdot \frac{y_{n+1}}{x_1} \right) \right] Y(\tau, (x_0, \varphi_1 - \varphi)) \]

\[ = Y \left( \varphi_1 \text{Res}_{x_0} \left( \frac{1}{2} l_0(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_0(x_1) \right) \right) \]

\[ x^{-1} \left[ \left( x_{n+1} \cdot \frac{y_{n+1}}{x_1} \right) \right] Y(\tau, (x_0, \varphi_1 - \varphi)) \]

\[ = Y \left( \varphi_1 \text{Res}_{x_0} \left( \frac{1}{2} l_0(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_0(x_1) \right) \right) \]

\[ x^{-1} \left[ \left( x_{n+1} \cdot \frac{y_{n+1}}{x_1} \right) \right] Y(\tau, (x_0, \varphi_1 - \varphi)) \]

\[ = \varphi_1 \text{Res}_{x_0} \left( \prod_{k=0}^{\infty} \int_{y} \left( \frac{1}{2} l_k(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_k(x_1) \right) \right) \]

\[ x^{-1} \left[ \left( x_{n+1} \cdot \frac{y_{n+1}}{x_1} \right) \right] Y(\tau, (x_0, \varphi_1 - \varphi)) \]

\[ = \varphi_1 \text{Res}_{x_0} \left( \left( \sum_{n \in \mathbb{Z}} n(x + x_0)^{n-1} \varphi x_1^{n-1} \right) \left( \frac{1}{2} \sum_{j \in \mathbb{Z}_+} t^j B_j x_1^{-j+1} \right) \right) \]

\[ + \prod_{k=0}^{\infty} \int_{y} \left( \frac{1}{2} l_k(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_k(x_1) \right) \]

\[ x^{-1} \left[ \left( x_{n+1} \cdot \frac{y_{n+1}}{x_1} \right) \right] Y(\tau, (x_0, \varphi_1 - \varphi)) \]

\[ = \varphi_1 \text{Res}_{x_0} \left( \left( \sum_{n \in \mathbb{Z}} n(x + x_0)^{n-1} \varphi x_1^{n-1} \right) \left( \frac{1}{2} \sum_{j \in \mathbb{Z}_+} t^j B_j x_1^{-j+1} \right) \right) \]

\[ + \prod_{k=0}^{\infty} \int_{y} \left( \frac{1}{2} l_k(x_1) \left( \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial x_1} \right) + g_k(x_1) \right) \]

\[ x^{-1} \left[ \left( x_{n+1} \cdot \frac{y_{n+1}}{x_1} \right) \right] Y(\tau, (x_0, \varphi_1 - \varphi)) \]
Corollary 6.4. 

We are now ready to prove Proposition 4.5 in the case of a formal invertible superconformal change of variables $H \circ I$ taking infinity to infinity.
Proof. Denote
\[ H_{t^{1/2}} \circ I(x, \varphi) = (\tilde{x}(t^{1/2}), \tilde{\varphi}(t^{1/2})) \]
\[ = \exp \left( -\sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^j - \frac{1}{2} N_j - \frac{1}{2} G_{-j + \frac{1}{2}}(x, \varphi) \right) \right) \cdot (x, \varphi). \]

By Corollary 6.4, Proposition 2.4, and equation (3.14), we have

\[ Y(u, (x, \varphi)) \xi_{H_{t^{1/2}}(t^{1/2})}^*(v) \]
\[ = Y(u, (x, \varphi)) e^{-\sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^j - \frac{1}{2} N_j - \frac{1}{2} G_{-j + \frac{1}{2}}(x, \varphi) \right)} \cdot v \]
\[ = e^{-\sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^j - \frac{1}{2} N_j - \frac{1}{2} G_{-j + \frac{1}{2}}(x, \varphi) \right)} \cdot Y(u, (x, \varphi)) e^{-\sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^j - \frac{1}{2} N_j - \frac{1}{2} G_{-j + \frac{1}{2}}(x, \varphi) \right)} \cdot v \]
\[ = e^{-\sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^j - \frac{1}{2} N_j - \frac{1}{2} G_{-j + \frac{1}{2}}(x, \varphi) \right)} \cdot Y \left( \exp \left( \sum_{m=-1}^{\infty} \sum_{j \in \mathbb{Z}_+} \frac{(-j + 1)}{m + 1} x^{-j-m} \right) \right) \cdot v \]
\[ = e^{-\sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^j - \frac{1}{2} N_j - \frac{1}{2} G_{-j + \frac{1}{2}}(x, \varphi) \right)} \cdot \exp \left( \sum_{j \in \mathbb{Z}_+} \left( \Theta_{j}^{(2)}(t^{1/2}) L(j) + \Theta_{j - \frac{1}{2}}^{(2)}(t^{1/2}) G(j - \frac{1}{2}) \right) \right) \cdot v \]
\[ = e^{-\sum_{j \in \mathbb{Z}_+} \left( t^j B_j L_{-j}(x, \varphi) + t^j - \frac{1}{2} N_j - \frac{1}{2} G_{-j + \frac{1}{2}}(x, \varphi) \right)} \cdot \exp \left( -2 \Theta_{0}^{(2)}(t^{1/2}) L(0) \right) \cdot u, (x, \varphi) \cdot v \]
\[ = \xi_{H_{t^{1/2}}(t^{1/2})}^*(Y(\xi_{\Theta(H_{t^{1/2}},(x,\varphi))}(u), H_{t^{1/2}} \circ I(x, \varphi))(v)), \]

for \( u, v \in V \). Thus we have

(6.4) \[ Y(u, (x, \varphi)) \xi_{H_{t^{1/2}}(t^{1/2})}^*(v) = \xi_{H_{t^{1/2}}(t^{1/2})}^*(Y(\xi_{\Theta(H_{t^{1/2}},(x,\varphi))}(u), H_{t^{1/2}} \circ I(x, \varphi))(v)) \]
as formal power series in $V[[t^{1/2}]]$ of $\xi(x^{-1},x)$. Setting $t^{1/2} = 1$ on the left-hand side of (6.4), we see that both sides of (6.4) are well-defined elements of $V[[x^{-1},x]]$. This proves the first statement of Proposition 4.5 and equation (4.12) for general formal superconformal change of variables $H$. Thus by Remark 3.2, we have the following corollary to Proposition 4.5.

**Corollary 7.1.** Let $H$, $\gamma_H$ and $\gamma_{\Theta(H,(x,\varphi))}$ be as in Section 4.1. Define

$$
V_H = \gamma_H(V),
$$

$$
Y_H(u,(x,\varphi)) = Y(\gamma_{\Theta(H,(x,\varphi))} \circ \gamma_H^{-1}(u), H(x, \varphi)),
$$

$$
1_H = \gamma_H(1) = 1,
$$

$$
\tau_H = \gamma_H(\tau) = a_{\gamma}^{-3}\tau.
$$

Then $(V_H, Y_H(\cdot,(x,\varphi)), 1_H, \tau_H)$ is an $N = 1$ NS-VOSA and is isomorphic to $(V,Y,1,\tau)$.

Similarly for $H^{-1}$ a formally superconformal power series of the form (4.17), we note that since $\xi_{\Theta(H)}$ is invertible, $(\xi_{\Theta(H)})^{-1}$ is equivalent to

$$
(\xi_{\Theta(H)})^{-1}(Y(u,(x,\varphi))v) = Y(\xi_{\Theta(H)}(u), H \circ I(x,\varphi)) \circ \xi_{\Theta(H)}^{-1}(v)) = Y(\xi_{\Theta(H)}(u), H \circ I(x,\varphi)) \circ (\xi_{\Theta(H)})^{-1}(v).
$$

Thus by Remark 3.2, we have the following corollary to Proposition 4.5.

**Corollary 7.2.** Let $H \circ I$, $\xi_{\Theta(H)}$ and $\xi_{\Theta(H,(x,\varphi))}$ be as in Section 4.2. Define

$$
V_{\Theta(H)} = \prod_{n \in \mathbb{Z}} (\xi_{\Theta(H)})^{-1}(V(n)),
$$

$$
Y_{\Theta(H)}(u,(x,\varphi)) = Y(\xi_{\Theta(H)}(u), H \circ I(x,\varphi)),
$$

$$
1_{\Theta(H)} = (\xi_{\Theta(H)})^{-1}(1) = 1 + N_1\tau + B_2\omega + \text{terms of higher weight in } V,
$$

$$
\tau_{\Theta(H)} = (\xi_{\Theta(H)})^{-1}(\tau) = \tau + \text{terms of higher weight in } V.
$$

Then $(V_{\Theta(H)}, Y_{\Theta(H)}(\cdot,(x,\varphi)), 1_{\Theta(H)}, \tau_{\Theta(H)})$ is an $N = 1$ NS-VOSA and is isomorphic to $(V,Y,1,\tau)$.  

7. **Isomorphism classes of $N = 1$ NS-VOSAs arising from superconformal changes of variables**

For $H$ a formally superconformal of the form (4.2), we note that (4.5) is equivalent to

$$
(\xi_{\Theta(H)})^{-1}(Y(u,(x,\varphi))v) = Y(\xi_{\Theta(H)}(u), H(x, \varphi)) \circ \gamma_H^{-1}(v).
$$

This proves the first statement of Proposition 4.5 and equation (4.12) for general formal superconformal change of variables $H$ at infinity. In addition, since $\xi_{\Theta(H)}}(t^{1/2},(x,\varphi)|_{t^{1/2}=1}$ is well defined, $\Theta_j(t^{1/2})|_{t^{1/2}=1}$ must also be well defined, and thus so is $\xi_{\Theta(H)}}(t^{1/2},(x,\varphi)|_{t^{1/2}=1}$. This, the fact that for $v \in V$, we have $\xi_{\Theta(H)}}(x,\varphi)) \circ \xi_{\Theta(H)}^{-1}(v) = v$, and the fact that $\xi_{\Theta(H)}$ is invertible, give us equation (4.15).
Remark 7.3. In [H2], Huang defines families of isomorphic VOAs derived from a change of variables \( f \) where \( f \) is an invertible analytic function vanishing at infinity. However, there is a mistake in his definition of the operator \( Y_{1/f}(u_{1/f}, x) V_{1/f} \) on p. 181. Instead of defining

\[
Y_{1/f}(u_{1/f}, x) V_{1/f} = Y(v_{1/f}(u_{1/f}), 1/f(x)) V_{1/f}
\]

one should define

\[
Y_{1/f}(u_{1/f}, x) V_{1/f} = Y(v_{1/f}(u_{1/f}), 1/f(x)) V_{1/f}
\]

The superextension of this is equivalent to using the equation (4.12) to note that

\[
\xi_{\text{Hor}}^{*}(Y(u, (x, \varphi)) v)
\]

\[
= Y(\xi_{\text{Hor}}(H^{-1}(x, \varphi))^{-1}(u), I^{-1} \circ H^{-1}(x, \varphi)) \xi_{\text{Hor}}^{*}(v)
\]

\[
= Y(\xi_{\text{Hor}}(H^{-1}(x, \varphi))^{-1} \circ \xi_{\text{Hor}}^{*}(u), I^{-1} \circ H^{-1}(x, \varphi)) \xi_{\text{Hor}}^{*}(v).
\]

Thus defining

\[
V_{I^{-1} o H^{-1}} = \prod_{n \in \mathbb{Z}} \xi_{\text{Hor}}^{*}(V(u_n)),
\]

\[
Y_{I^{-1} o H^{-1}}(u, (x, \varphi)) = Y(\xi_{\text{Hor}}(H^{-1}(x, \varphi))^{-1}(u), I^{-1} \circ H^{-1}(x, \varphi)),
\]

\[
\tau_{I^{-1} o H^{-1}} = \xi_{\text{Hor}}^{*}(\tau)
\]

(7.12) gives an \( N = 1 \) NS-VOSA isomorphic to \( (V, Y, 1, \tau) \). One recovers the correct analogous formulas for the non-super case given in [H2] by letting \( f \) be the body component of \( H^{-1} \), setting all odd components and soul portions of supernumbers equal to zero and restricting \( V \) to \( V^0 \). (Recall that Huang’s operator \( \xi_{1/f}^{*} \) is the body portion of our operator \( \xi_{\text{Hor}}^{*} \) if \( f^{-1} \) is the body of \( H \).)

8. Superconformal change of variables in an annulus

Let \( \Sigma_B \) be a closed annulus in the complex plane with Jordan curves \( C_1 \) and \( C_2 \) as its boundary. Let \( \Sigma \) be the superannulus in \( \Lambda_{\infty} \) with body \( \Sigma_B \) and such that \( \Sigma \) is closed under the DeWitt topology, i.e., let

\[
\Sigma = \Sigma_B \times (\Lambda_{\infty})_S.
\]

Let \( H^{-1}(z, \varphi) \) be an invertible orientation-preserving superconformal map defined on \( \Sigma \).

Let \( \Sigma_2 \) be the open subset of \( \Lambda_{\infty} \) in the DeWitt topology whose body is the exterior of \( C_2 \) in the complex plane union \( \{\infty\} \). Assume that \( C_1 \) and \( C_2 \) were chosen such that \( C_1 \subset (\Sigma_2)_B \). Assume that the interior of \( C_2 \) and the interior of the body of the subset \( H^{-1}(C_2 \times (\Lambda_{\infty})_S) \) both contain zero. Let \( \Sigma_1 \) be the open subset of \( \Lambda_{\infty} \) in the DeWitt topology whose body is the interior of the body of \( H^{-1}(C_1 \times (\Lambda_{\infty})_S) \) in the complex plane.
Let 

\[ M = \Sigma_1 \sqcup \Sigma_2 / \{ p = q \text{ if } p \in \Sigma, q \in H^{-1}(\Sigma), \text{ and } H^{-1}(p) = q \}. \]

Then \( M \) is a super-Riemann surface with superconformal coordinate charts given by the two open neighborhoods \( \Sigma_1 \) and \( \Sigma_2 \), and the superconformal coordinate transition \( H^{-1} \). It is clear that topologically, \( M \) is a supersphere. Thus by the Uniformization Theorem for super-Riemann surfaces proved in [CR], \( M \) must be superconformally equivalent to the super-Riemann sphere \( \hat{S} \). Let \( F \) be a superconformal isomorphism from \( M \) to \( \hat{S} \). Then \( F \) restricted to \( \Sigma_1 \) and \( \Sigma_2 \), respectively, gives invertible superconformal functions \( F_1 \) and \( F_2 \), respectively, such that for \( (w, \rho) \) in the interior of \( \Sigma \)

\[ F_2(w, \rho) = F_1 \circ H^{-1}(w, \rho), \]

or equivalently

\[ H^{-1}(w, \rho) = F_1^{-1} \circ F_2(w, \rho). \]

We can assume that

\[ F_1(0, 0) = (0, 0) \]
\[ F_1(\infty, 0) = (\infty, 0) \]
\[ \lim_{w \to \infty} \frac{\partial}{\partial \rho} F_2(w, \rho) = 1, \]

since if equations (8.2) do not hold, we can compose \( F \) with a global superconformal transformation \( T \) from \( \hat{S} \) to \( \hat{S} \), such that equations (8.2) hold for \( T \circ F \); see [11].

Since \( F_1 \) is superconformal and invertible in a neighborhood of zero, and vanishing at zero, \( F_1(x, \varphi) \) can be written in the form of (4.2) and (4.3), and since \( F_2 \) is superconformal satisfying equations (8.3) and (8.4), \( F_2 \) can be written in the form of the right hand side of (4.3) with \( 1^{1/2} = 1 \). Thus by Propositions 1.1 and 1.5 we have

\[ (\xi_{F_2}^*)^{-1} \circ \gamma_{F_2}(Y(u, (x, \varphi)))v = (\xi_{F_2}^*)^{-1}(Y(\gamma_{F_1}(x, \varphi))(u), F_1(x, \varphi)(\gamma_{F_1}(v))) \]
\[ = Y(\gamma_{F_2}(F_1(x, \varphi))(u), F_2^{-1} \circ F_1(x, \varphi)(\xi_{F_2}^*)^{-1} \circ \gamma_{F_1}(v)) \]
\[ = Y(\gamma_{F_2}(F_1(x, \varphi))(u), H(x, \varphi))(\xi_{F_2}^*)^{-1} \circ \gamma_{F_1}(v)) \]
\[ = Y(\gamma_{F_2}(F_1(x, \varphi))(u), \gamma_{F_1}(x, \varphi) \circ \gamma_{F_1}(x, \varphi) \circ F_2^{-1} \circ \gamma_{F_1}(v)) \]

Thus by Remark 8.2, we have the following Corollary to Propositions 1.1 and 1.5.

**Corollary 8.1.** Let \( H^{-1} \) be a superconformal map defined on the superannulus \( \Sigma \) as given above, and let \( F_1 \) and \( F_2 \) be the unique superconformal coordinates defined in neighborhoods of 0 and \( \infty \), respectively, and satisfying (8.2)-(8.4). Let \( (V, Y(x, (x, \varphi)), 1, \tau) \) be an \( N = 1 \) NS-VOSA. We have the following change of
Remark 8.2. We have assumed that the annulus $\Sigma$ and its image under $H^{-1}$ circumscribe zero. However, if it does not, we can use the superfconformal shift should read

$$[B2] \text{K. Barron, The supergeometric interpretation of vertex operator superalgebras}$$

In addition, the isomorphism in Theorem 7.4.8 in [H2] should be $\xi_{F_2} \circ \gamma F_1$.

(8.5) \[(\xi_{F_2}^{-1})^{-1} \circ \gamma F_1 (Y(u, (x, \varphi))v)\]

$$= Y(\xi_{\Theta(F_2^{-1}, F_1(x, \varphi))}^{-1} \circ \gamma_{\Theta(F_1, (x, \varphi))}(u), H(x, \varphi))(\xi_{F_2}^{-1})^{-1} \circ \gamma F_1 (v),$$
i.e.,

(8.6) \[Y(u, H(x, \varphi))v\]

$$= (\xi_{F_2}^{-1})^{-1} \circ \gamma F_1 (Y(\gamma_{\Theta(F_1, (x, \varphi))}^{-1} \circ \xi_{F_2}^{-1} (u), (x, \varphi))\gamma F_1 \circ \xi_{F_2}^{-1} (v)),$$

for $u, v \in V$, and for $(z, \theta)$ in the domain of convergence of $H$, both sides of (8.5) and of (8.6) exist for $(x, \varphi) = (z, \theta)$ and are equal. Furthermore, define

(8.7) \[V_H = (\xi_{F_2}^{-1})^{-1} \circ \gamma F_1 (V),\]

(8.8) \[Y_H(u, (x, \varphi)) = Y(\xi_{\Theta(F_2^{-1}, F_1(x, \varphi))}^{-1} \circ \gamma_{\Theta(F_1, (x, \varphi))} \circ \gamma F_1 \circ \xi_{F_2}^{-1} (u), H(x, \varphi)),\]

(8.9) \[1_H = (\xi_{F_2}^{-1})^{-1} \circ \gamma F_1 (1),\]

(8.10) \[\tau_H = (\xi_{F_2}^{-1})^{-1} \circ \gamma F_1 (\tau).\]

Then $(V_H, Y_H(:, (x, \varphi)), 1_H, \tau_H)$ is an $N = 1$ NS-VOSA and is isomorphic to $(V, Y, 1, \tau)$.

Remark 8.2. We have assumed that the annulus $\Sigma$ and its image under $H^{-1}$ circumscribe zero. However, if it does not, we can use the superfconformal shift change of variables formulas given by (3.13) which, when combined with the change of variables given in Corollary 8.1 above, provide the appropriate change of variables formula.

Remark 8.3. The formula for $Y_{1/f}$ on p. 183 in [H2] is incorrect. The formula should read

\[Y_{1/f}(u_{1/f}, x)v_{1/f}\]

$$= Y(\xi_{(F_2)^{-1} \circ F_1(x)}^{-1} \circ \gamma_{F_1}(u_{1/f}), (F_2)^{-1} \circ F_1(x))v_{1/f}$$

$$= Y(\xi_{f(x)}^{-1} \circ \gamma_{F_1} \circ \xi_{(F_2)^{-1}}^{-1} \circ (u_{1/f}), f(x))v_{1/f}.$$
B5] K. Barron, The notion of $N = 1$ supergeometric vertex operator superalgebra and the isomorphism theorem, Commun. in Contemp. Math., to appear; preprint available at [http://arXiv.org/abs/math.QA/0301273](http://arXiv.org/abs/math.QA/0301273).

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