PATH MODEL FOR QUANTUM LOOP MODULES OF FUNDAMENTAL TYPE

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1. Introduction

1.1. In this paper we construct a combinatorial realisation of a certain class of simple integrable modules with finite dimensional weight spaces over a quantised affine algebra.

The best-known examples of such modules are the highest weight simple integrable modules $V(\lambda)$. These modules are, essentially, combinatorial objects for the following reasons. First of all, they can be defined for an arbitrary quantised Kac-Moody algebra. Next, the formal character of $V(\lambda)$ is given by a universal formula known as Kac-Weyl character formula (cf. [15, Chapter 10]) and determines $V(\lambda)$ uniquely up to an isomorphism. Furthermore, $V(\lambda)$ is a quantum deformation (cf. [23]) of a module over the corresponding Kac-Moody algebra which is also simple and has the same formal character. Finally, after [16, 24], $V(\lambda)$ admits a crystal basis and a global basis.

The properties of a crystal basis, formulated in an abstract way, lead to the notion of a crystal as a set equipped with root operators $\tilde{e}_\alpha$, $\tilde{f}_\alpha$ for each simple root $\alpha$ of the corresponding Kac-Moody algebra and some other operations which will be discussed later. In particular, one associates with $V(\lambda)$ a crystal $B(\lambda)$ which encodes the major properties of the module. For example, one can define, in a natural way, a tensor product of crystals whose properties reflect these of the tensor product of modules for the $V(\lambda)$. Namely, a decomposition of the tensor product of crystals $B(\lambda)$ and $B(\mu)$ yields a decomposition of $V(\lambda) \otimes V(\mu)$.

1.2. The crystals $B(\lambda)$ are known to admit numerous combinatorial realisations. One of the most important, due to its simplicity and universality, is the path model of Littelmann (cf. [21, 22]). In the framework of that model, $B(\lambda)$ is represented as a subset of the set $P$ of piece-wise continuous linear paths in a rational vector subspace a Cartan subalgebra of the Kac-Moody algebra connecting the origin with an integral weight. Then the tensor product of crystals corresponds to the concatenation of paths. The Isomorphism Theorem of Littelmann (cf. [24]) stipulates that any subcrystal of $P$, which is generated over the associative monoid $M$ of root operators by a path which connects the origin with $\lambda$ and lies entirely in the dominant chamber, provides a realisation of $B(\lambda)$. Moreover, any two such realisations for $\lambda$ fixed are isomorphic as crystals. In particular, they are isomorphic
to the subcrystal of $P$ generated over $\mathcal{M}$ by the linear path connecting the origin with $\lambda$.

1.3. The case of affine Lie algebras is somewhat special since they admit, besides the Kac-Moody presentation, an explicit realisation in terms of loop algebras. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of rank $\ell$ over $\mathbb{C}$ and denote by $\hat{\mathfrak{g}}$ the corresponding untwisted affine algebra (cf. 2.2). Two quantum versions of $\hat{\mathfrak{g}}$ are generally considered. They will be denoted by $U_q$ and $\hat{U}_q$ respectively and differ by a choice of the torus (cf. 2.3). The algebra $U_q$ can also be viewed as a subquotient of $\hat{U}_q$.

The algebra $U_q$ admits finite dimensional integrable representations which have been and still are being studied extensively (cf., to name but a few, \[127, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147\]). These modules are parametrised by $\ell$-tuples of polynomials over $\mathbb{C}(q)$ in one variable with constant term 1, known as Drinfel’d polynomials, and are very different, in many respects, from highest weight integrable modules. They are not, in general, determined by their formal character (however, they are determined by their $q$-characters introduced in \[13\]). They do not always admit classical limits and these limits, when exist, are not necessarily simple modules over the corresponding affine Lie algebra and in fact may have a rather complex structure. Finally, it seems that existence of a crystal basis is an exception rather than a rule for this class of modules. The general reason for these discrepancies is that the construction of finite dimensional $U_q$ modules arises from the loop-like (Drinfel’d) presentation of $U_q$ (cf. \[2, 10, 19\]) peculiar to the Kac-Moody algebras of affine type.

1.4. Simple (infinite dimensional) integrable modules with finite dimensional weight spaces were classified in \[3, 6\] for affine Lie algebras and in \[5\] for quantised affine algebras. Namely, such a module is either a highest weight module $V(\lambda)$ (or its graded dual) or a loop module. The modules of the latter class are constructed, in the quantum case, as simple submodules of the loop spaces of finite dimensional simple modules over $U_q$. Namely, let $\pi = (\pi_1, \ldots, \pi_\ell)$, $\pi_i \in \mathbb{C}(q)[u]$ be an $\ell$-tuple of polynomials with constant term 1 and let $V(\pi)$ be the corresponding finite dimensional simple $U_q$-module. Let $m$ be the maximal positive integer such that all the $\pi_i$, $i = 1, \ldots, \ell$ lie in $\mathbb{C}(q)[u^m]$. Then one can show (cf. \[3\]) that the cyclic group $\mathbb{Z}/m\mathbb{Z}$ acts on the loop space $\hat{V}(\pi) := V(\pi) \otimes_{\mathbb{C}(q)} \mathbb{C}(q)[t, t^{-1}]$ and its action commutes with that of $\hat{U}_q$. In particular, simple submodules $\hat{V}(\pi)^{(k)}$, $k = 0, \ldots, m-1$ correspond to distinct irreducible characters of the abelian group $\mathbb{Z}/m\mathbb{Z}$. We say that $\hat{V}(\pi)$ is of fundamental type if $\pi_i(u) = \delta_{i,j}(1-u^m)$ for some $m > 0$ and for some $i \in \{1, \ldots, \ell\}$ fixed. Henceforth we denote such an $\ell$-tuple of polynomials by $\varpi^{(i,m)}$.

It turns out that simple submodules of $\hat{V}(\varpi^{(i,m)})$ are determined by their formal characters up to a twist by an automorphism of $\hat{U}_q$. In the present paper we show that these modules admit a certain analogue of a crystal basis and construct a realisation in the framework of Littelmann’s path model of the crystal associated to that basis in a natural way. The first example of $\mathfrak{g}$ of type $A_\ell$, $m$ arbitrary and $i = 1$, in which case the module $V(\varpi^{(i,1)})$ is isomorphic to the quantum analogue of the natural $(\ell+1)$-dimensional representation of $\mathfrak{g}$ as a module over the quantised enveloping algebra $U_q(\mathfrak{g})$ corresponding to $\mathfrak{g}$, was considered in \[13\]. The case $m = 1$
was later treated, independently, by S. Naito and D. Sagaki (cf. [24]) for \( g \) of all types and for all \( i = 1, \ldots, \ell \). Here we consider all modules of fundamental type for \( g \) of all types which we believe to be the widest class of integrable \( \hat{U}_g \)-modules of level zero with finite dimensional weight spaces which admit a combinatorial realisation inside the path crystal of Littelmann. Our analysis is based on the approach of [14] and on the results of [18] and [25].

1.5. Let us briefly describe the principal results of this paper. It was shown in [18] that \( V(\varpi_{i1}) \) always admits a crystal basis \( B(\varpi_{i1}) \) whose \( m \)th tensor power, for any \( m > 0 \) is indecomposable as a crystal. In order to treat the modules \( V(\varpi_{i,m}) \) for an arbitrary \( m \) one has to introduce the notion of a \( z \)-crystal basis (cf. [5] and Definition 3.1). Roughly speaking, whilst crystal bases are preserved as sets by the root operators of Kashiwara, \( z \)-crystal bases are preserved by these operators up to a multiplication by a power of a complex number \( z \) only. Our first result is the following

**Theorem 1.** The simple module \( \hat{V}(\varpi_{i,m})^{(k)}, k = 0, \ldots, m - 1, m > 0 \), admits a \( z \)-crystal basis \( \hat{B}(\varpi_{i,m})^{(k)} \), where \( z \) is an \( m \)th primitive root of unity.

From the combinatorial point of view multiplication of elements of a basis by roots of unity is not important and one can get rid of it associating a crystal to a \( z \)-crystal basis (cf. 2.2). It turns out that the crystal associated with \( \hat{B}(\varpi_{i,m})^{(k)} \) is indecomposable and these are all indecomposable subcrystals of the affinisation (cf. 2.8) of the finite crystal \( B(\varpi_{i1}) \otimes_m \). That illustrates once again how different loop modules are from highest weight modules. Indeed, the affinisation of \( B(\varpi_{i1}) \otimes_m \) is also isomorphic to the crystal basis of the simple \( \hat{U}_g \)-module \( \hat{V}(\pi) \) where \( \pi = (\pi_1, \ldots, \pi_\ell) \) with \( \pi_j(u) = \delta_{ij}(1 - u)^m \). Thus, the crystal basis of that simple module is a disjoint union of indecomposable crystals.

Let \( \varpi_i, i = 1, \ldots, \ell \) be the fundamental weights of \( g \) extended by zero to weights of \( \widehat{g} \) and let \( \delta \) be the generator of imaginary roots of \( \widehat{g} \) (cf. 2.2). The main result of this paper is the following

**Theorem 2.** The associated crystal of \( \hat{B}(\varpi_{i,m})^{(k)} \) is isomorphic to the subcrystal \( B(m\varpi_i + k\delta) \) of the Littelmann path crystal generated by the linear path connecting the origin with \( m\varpi_i + k\delta \).

Acknowledgements. We are greatly indebted to A. Joseph who taught us all we know about crystals. We are grateful to V. Toledano-Laredo, and the first author thanks B. Leclerc, P. Littelmann and M. Varagnolo, for numerous interesting discussions.

2. Preliminaries and notations

2.1. Let \( C(q) \) be the field of rational functions in \( q \) with complex coefficients, that is, the fraction field of \( C[q] \). Let \( A \subset C(q) \) be the ring \( C[q] \) localized at \( q = 0 \), which identifies with the subring of rational functions in \( q \) regular at \( q = 0 \). Given \( m \geq n \geq 0 \), define

\[
[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_q! = [1]_q \cdots [m]_q, \quad \left[ \frac{m}{n} \right]_q := \frac{[m]_q!}{[n]_q![m-n]_q!}.
\]

All the above are Laurent polynomials in \( q \) over \( Z \).
2.2. Set \( I = \{1, \ldots, \ell\} \) and let \( A = (a_{ij})_{i,j \in I} \) be the Cartan matrix of a finite dimensional simple Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \) of rank \( \ell \). Fix a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) and let \( \{\alpha_i\}_{i \in I} \) (respectively, \( \{\alpha_i^\vee\}_{i \in I} \) ) be a basis of \( \mathfrak{h}^* \) (respectively, of \( \mathfrak{h} \)) such that \( \alpha_i^\vee(\alpha_j) = a_{ij} \). Define the fundamental weights \( \varpi_i \in \mathfrak{h}^* \), \( i \in I \) of \( \mathfrak{g} \) by \( \alpha_i^\vee(\varpi_j) = \delta_{i,j} \), where \( \delta_{i,j} \) is the Kronecker’s symbol, and let \( P_0 \) be the free abelian group generated by the \( \varpi_i \), \( i \in I \). Let \( \theta = \sum_{i \in I} a_i \alpha_i \) be the highest root of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) and denote by \( \theta^\vee = \sum_{i \in I} a_i^\vee \alpha_i^\vee \) the corresponding co-root.

Set \( \tilde{I} = I \cup \{0\} \) and let \( A = (a_{ij})_{i,j \in \tilde{I}} \) be the generalised Cartan matrix of the untwisted affine Lie algebra \( \hat{\mathfrak{g}} \) associated with \( \mathfrak{g} \). As a vector space,

\[
\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C} \subseteq \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \otimes \mathbb{C} \partial,
\]

where \( c \) is the canonical central element and \( \text{ad} \partial = t \frac{d}{dt} \). Then \( \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C} c \oplus \mathbb{C} \partial \) is a Cartan subalgebra of \( \hat{\mathfrak{g}} \). Set \( \alpha_i^\vee := c - \theta^\vee \). Define \( \delta \in \hat{\mathfrak{h}}^* \) by \( \delta(\delta) = 1, \delta_{\mathfrak{h} \oplus \mathbb{C} c} = 0 \) and set \( \alpha_0 = \delta - \theta \). Then \( \{\alpha_i\}_{i \in \tilde{I}} \) (respectively, \( \{\alpha_i^\vee\}_{i \in \tilde{I}} \) ) is a set of simple roots of \( \hat{\mathfrak{g}} \) and \( \alpha_i^\vee(\alpha_j) = a_{ij}, i, j \in \tilde{I} \). Notice that \( \alpha_i^\vee(\delta) = 0 = c(\alpha_i) \) for all \( i \in \tilde{I} \).

Define the fundamental weights \( \Lambda_i \in \hat{\mathfrak{h}}^* \), \( i \in \tilde{I} \) of \( \hat{\mathfrak{g}} \) by conditions \( \alpha_i^\vee(\Lambda_j) = \delta_{i,j}, \partial(\Lambda_i) = \delta_{i,0} \). Let \( P \) be the free abelian group generated by the \( \Lambda_i \), \( i \in \tilde{I} \) and set \( \hat{P} := P \oplus \mathbb{Z}\delta \). Extend the map \( \varpi_i \mapsto \Lambda_i - a_i^\vee \Lambda_0 \) to an embedding of \( P_0 \) into \( P \) and identify \( P_0 \) with its image inside \( P \) which in turn coincides with the set \( \{\lambda \in P : c(\lambda) = 0\} \). Let \( \xi : \hat{P} \to \hat{P}/\mathbb{Z}\delta \) be the canonical projection. Notice that \( \hat{P}/\mathbb{Z}\delta \) identifies with \( \hat{P} \) and that \( \xi(\alpha_0) = -\theta \).

For all \( i \in \tilde{I} \) define an elementary reflection \( s_i \in \text{Aut} \hat{\mathfrak{h}}^* \) by \( s_i \lambda = \lambda - \alpha_i^\vee(\lambda)\alpha_i \) for all \( \lambda \in \hat{\mathfrak{h}}^* \). The Weyl group \( \hat{W} \) of \( \hat{\mathfrak{g}} \) (respectively, the Weyl group \( \hat{W} \) of \( \hat{\mathfrak{g}} \)) identifies with the group generated by the \( s_i : i \in \tilde{I} \) (respectively, \( i \in I \) ). The set of roots of \( \hat{\mathfrak{g}} \) is a disjoint union of the set of real roots \( \cup_{i \in \tilde{I}} \hat{W} \alpha_i \) and imaginary roots \( \mathbb{Z}\delta \setminus \{0\} \). If \( \beta \) is a real root, denote the corresponding co-root by \( \beta^\vee \) and set \( s_\beta \lambda = \lambda - \beta^\vee(\lambda)\beta, \lambda \in \hat{\mathfrak{h}}^* \). Observe that \( s_\lambda = s_\lambda \) as an automorphism of \( P \)

and \( \hat{W} \) identifies with \( W \) when we consider the action of the former group on \( P \).

2.3. Let \( d_i, i \in \tilde{I} \) be positive relatively prime integers such that the matrix \( (d_i a_{ij})_{i,j \in \tilde{I}} \) is symmetric and let \( q_i = q^{d_i}. \) Henceforth, for any symbol \( X \), \( i \in \tilde{I} \), set \( X^{(k)} = X^k/[k]_{q_i} \).

The quantised affine algebra \( \hat{U}_q := \mathcal{U}_q(\hat{\mathfrak{g}}) \) corresponding to \( \hat{\mathfrak{g}} \) is an associative algebra over \( \mathbb{C}(q) \) with generators \( E_i, F_i, K_i^{\pm 1}, i \in \tilde{I}, C^{1/2} \) and \( D^{1/2} \) subjects to the following relations

\[
C^{1/2} \text{ are central and } C = \prod_{i \in I} K_{i}^{\alpha_i} \text{, where } \delta = \sum_{i \in I} a_i \alpha_i
\]

\[
K_i K^{-1}_i = 1, K_i K^{-1}_i = DD^{-1} = D^{-1}D = 1, K_i K_j = K_j K_i, K_i D = DK_i
\]

\[
K_i E_j K^{-1}_i = q_i^{a_{ij}} E_j, K_i F_j K^{-1}_i = q_i^{-a_{ji}} F_j
\]

\[
DE_j D^{-1} = q_{ij} E_j, DF_j D^{-1} = q_{ji}^{-1} F_j
\]

\[
[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}
\]
\[
\sum_{r=0}^{1-a_{ij}} (-1)^r E_i^{(r)} E_j^{(1-a_{ij}-r)} = 0 = \sum_{r=0}^{1-a_{ij}} (-1)^r F_i^{(r)} F_j^{(1-a_{ij}-r)},\quad \text{if } i \neq j.
\]

Let \( U_q^\infty \) be the quotient of \( \widehat{U}_q \) by the two-sided ideal generated by \( C^{\pm1/2} - 1 \). The algebra \( U_q \) is the subalgebra of \( U_q^\infty \) generated by the \( E_i, F_i \) and \( K_i^{\pm1}, i \in \widehat{I} \).

The elements \( E_i, F_i \) and \( K_i^{\pm1}, i \in I \) generate a subalgebra \( U_q^{\text{fin}} \) of \( \widehat{U}_q \) which is isomorphic to the quantised enveloping algebra \( U_q(g) \) of \( g \). Notice also that, for all \( i \in \widehat{I} \) fixed, the elements \( E_i, F_i \) and \( K_i^{\pm1} \) generate a subalgebra of \( \widehat{U}_q \) isomorphic to \( U_q(sl_2) \).

2.4. One can introduce a \( Z \)-grading on \( \widehat{U}_q \) in the following way. We say that \( x \in \widehat{U}_q \) is homogeneous of degree \( k \in Z \) if \( DxD^{-1} = q^k x \). That grading is obviously well-defined since all generators of \( \widehat{U}_q \) are homogeneous and induces a \( Z \)-grading on \( U_q \). Given \( x \in C(q)^\times \), define an automorphism \( \phi_x \) of \( \widehat{U}_q \) by \( \phi_x(x) = x^k x \) if \( x \) is homogeneous of degree \( k \). Evidently, \( \phi_x \) descends to an automorphism of \( U_q \).

Let \( M \) be a \( \widehat{U}_q \) or \( U_q \)-module. Denote by \( \phi_x^\ast M \) the vector space \( M \) with the action of \( \widehat{U}_q \) twisted by the automorphism \( \phi_x \), that is \( x\phi_x^\ast M := \phi_x(x)m \) for all \( x \in \widehat{U}_q \) or \( U_q \), \( m \in M \). Notice that the map \( M \to \phi_x^\ast M \) is trivial as a map of vector spaces or \( U_q^{\text{fin}} \)-modules.

Let \( M \) be a \( U_q \)-module. One can endow the loop space \( \widehat{M} := M \otimes_{C(q)} C(q)[t, t^{-1}] \) of \( M \) with the structure of a \( \widehat{U}_q \)-module by setting

\[
x(m \otimes f(t)) = xm \otimes t^k f(t), \quad D^{\pm1}(m \otimes f(t)) = m \otimes f(q^{\pm1} t), \quad C^{\pm1/2} m = m, \quad \text{for all } m \in M, f \in C(q)[t, t^{-1}] \text{ and for all } x \in \widehat{U}_q \text{ homogeneous of degree } k.
\]

2.5. Let \( M \) be a \( U_q \) (respectively, \( \widehat{U}_q \)) module. We say that \( M \) is a module of type 1 if \( M = \bigoplus_{\nu \in \widehat{P}} M_\nu \) (respectively, \( M = \bigoplus_{\nu \in \widehat{P}} M_\nu \)), where \( M_\nu = \{ m \in M : K_i^{\nu} m = q_{\nu}^{\alpha_i^{\vee}(\nu)} m, \forall i \in \widehat{I} \} \) (respectively, \( M_\nu = \{ m \in M : K_i^{\nu} m = q_{\nu}^{\alpha_i^{\vee}(\nu)} m, \forall i \in \widehat{I}, Dm = q^{\partial(\nu)m} \}) \). The subspaces \( M_\nu \) are called weight subspaces of \( M \) and we call \( M \) admissible if \( \dim M_\nu < \infty \) for all \( \nu \in \widehat{P}_0 \) (respectively, for all \( \nu \in \widehat{P} \)). An element \( \nu \in \widehat{P}_0 \) or \( \widehat{P} \) is a weight of \( M \) if \( M_\nu \neq 0 \).

A module of type 1 is said to be of level \( k \in Z \) if \( C \) acts on \( M \) by \( q^k \) id and is said to be integrable if the generators \( E_i, F_i, i \in \widehat{I} \) act locally nilpotently on \( M \). In other words, \( M \) is a direct sum (possibly infinite), of finite dimensional simple \( U_q(sl_2) \)-modules for all \( i \in \widehat{I} \). Evidently, if \( M \) is a finite dimensional \( U_q \)-module, then \( \widehat{M} \) is an integrable \( \widehat{U}_q \)-module. Moreover, observe that all weights of \( \widehat{M} \) are of the form \( \nu + r \delta \) where \( \nu \in \widehat{P}_0 \) and \( r \in Z \), and that \( \widehat{M}_{\nu + r \delta} \) is spanned by \( m \otimes t^r \) where \( m \in M_\nu \). Thus, \( \widehat{M} \) is admissible.

2.6. It is well-known that \( \widehat{U}_q \) admits a structure of a Hopf algebra. Throughout the rest of this paper we will use the co-multiplication given on generators by the following formulae

\[
\Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i, \quad (2.1)
\]
the elements $K^\pm_1$, $D^\pm_1$ and $C^\pm_1/2$ being group-like. Then one can easily prove by induction on $r$ that
\[
\Delta(E^{(r)}_i) = \sum_{s=0}^r q_i^{-s(r-s)} E^{(s)}_i \otimes E^{(r-s)}_i K_i^{-s},
\]
\[
\Delta(F^{(r)}_i) = \sum_{s=0}^r q_i^{-s(r-s)} E^{(r-s)}_i K_i^s \otimes E^{(s)}_i.
\] (2.2)

Evidently, the above Hopf algebra structure descends to the algebra $U_q$. Henceforth, unless specified otherwise, a tensor product of two $\hat{U}_q$ or $U_q$ modules is assumed to be endowed with a structure of a $\hat{U}_q$ or $U_q$ module with respect to the co-product $\Delta$.

### 2.7. The algebras $\hat{U}_q$ and $U_q$ admit another presentation, known as the Drinfeld’s or loop-like presentation (cf. [2, 10, 19]). Namely, $\hat{U}_q$ is isomorphic to an associative algebra over $C(q)$ generated by the $x_{i,k}$, $h_i$, $K_i^\pm_1$, $i \in I$, $k \in Z$, $r \in Z \setminus \{0\}$, $C^\pm_2$ and $D^\pm_1$ subject to certain relations (see, for example, [2, 3]). Let us only mention that the $x_{i,k}$ and the $h_i$ are homogeneous of degree $k$ and $x_{i,0}$ (respectively, $x_{i,0}$) identifies with $E_i$ (respectively, $F_i$).

For all $i \in I$ and $r \in Z$, define $P_i, r$ by equating the powers of $u$ in the formal power series
\[
\sum_{r \geq 0} P_i, r u^r = \exp \left( - \sum_{k>0} \frac{q_i^{k+1}}{k!} h_i \frac{u^k}{k!} \right).
\]

Then the $P_i, r$, $i \in I$, $r \in Z$ are homogeneous of degree $r$ and generate the same subalgebra of $\hat{U}_q$ as the $h_i, r$.

A $U_q$-module $M$ is called $l$-highest weight with highest weight $(\lambda, \pi, \pm)$, where $\lambda \in P_0$ and $\pi = (\pi^+_1(u), \ldots, \pi^+_\ell(u))$ with $\pi^+_i(u) = \sum_{k \geq 0} \pi^+_i u^k \in C(q)[[u]]$ and $\pi^+_i = 1$, if there exists a non-zero $m \in M_\lambda$ such that $M = U_q m$ and
\[
x_{i,k} m = 0, \quad P_i, k m = \pi_i, k m, \quad \forall i \in I, k \in Z.
\]

Such an $m$ is called an $l$-highest vector. By [3, 19], an $l$-highest weight module $M$ with highest weight $(\lambda, \pi, \pm)$ is simple and finite dimensional provided that $\pi(u) = \pi^+(u) = (\pi_1, \ldots, \pi_\ell)$ is an $l$-tuple of polynomials, $\deg \pi_i = \alpha_i(\lambda)$ and $\deg \pi_i(1) = u^{-1}\partial u^{\alpha_i} \pi_i(1) / (u^{-1}\partial u^{\alpha_i} \pi_i(1))_{u=0}$. Moreover, all finite dimensional simple $U_q$-modules are obtained that way. Henceforth we denote the simple finite dimensional $l$-highest weight module corresponding to an $l$-tuple $\pi$ of polynomials with constant term $1$ by $V(\pi)$. Let $v_\pi$ be the unique, up to a scalar, $l$-highest weight vector of $V(\pi)$.

Let $z \in C^\times$. Since the $P_i, k$ are homogeneous of degree $k$, $P_i, k \phi_i^+(v_\pi) = z^{\pm k} \pi_i, k v_\pi$. It follows that $\phi_i^+ V(\pi)$ is isomorphic to $V(\pi z)$ where $\pi_z(u) = \pi(z u)$.

### 2.8. Let us conclude this section with a brief review of some facts about crystals which we will need later. Throughout the rest of this paper, a crystal is a set $B$ endowed with maps $e_i, f_i : B \rightarrow B \cup \{0\}$, $\varepsilon_i, \varphi_i : B \rightarrow Z$ for all $i \in \hat{I}$ and $\mathrm{wt} : B \rightarrow P$ or $\mathrm{wt} : B \rightarrow \hat{P}$ satisfying the standard axioms (see [17, 1.2] or [20, 5.2]). In particular, $\varphi_i(b) = \varepsilon_i(b) + \alpha_i(\mathrm{wt} b)$ for all $b \in B$ and $e_i, f_i$ for $i$ fixed are quasi-inverses of each other i.e. for all $b, b' \in B$, $e_i b = b'$ if and only if $f_i b' = b$. All crystals we consider are normal, that is $\varepsilon_i(b) = \max\{n : e_i^n b \in B\}$, $\varphi_i(b) = \max\{n : f_i^n b \in B\}$.
is a subcrystal if the trivial embedding is a morphism of crystals that is, if $B$ is an injective morphism of crystals $B \rightarrow B_2$. If $B_1$ is a subset of $B_2$, we say that $B_1$ is a subcrystal if the trivial embedding is a morphism of crystals that is, if $B_1$ is a crystal with respect to the crystal operations on $B_2$ restricted to $B_1$.

Let $\mathcal{M}$ be the associative monoid generated by the operators $e_i, f_i : i \in \hat{I}$. A crystal $B$ is generated by $b \in B$ over $\mathcal{M}$ if $B = M_b := \{xb : x \in \mathcal{M}\} \setminus \{0\}$. We say that a crystal $B$ is indecomposable if it does not admit a non-empty subcrystal different from itself. By say [14, 2.5] a crystal $B$ is indecomposable if and only if $B$ is generated by some $b \in B$ over $\mathcal{M}$. Moreover, if $B = M_b$ for some $b \in B$ then $B = M_{b'}$ for all $b' \in B$.

Given a family of crystals $B_1, \ldots, B_n$ one can introduce a structure of a crystal on the set $B_1 \times \cdots \times B_n$, which is called the tensor product of crystals and denoted by $B_1 \otimes \cdots \otimes B_n$, in the following way (cf. [17, 1.3]). Given $b = b_1 \otimes \cdots \otimes b_n$, $b_i \in B_i$, define the Kashiwara functions $b \mapsto r_i^k(b) : i \in I, k \in \{1, \ldots, n\}$ by

$$r_i^k(b) = e_i(b_k) - \sum_{1 \leq j < k} \alpha_j^\vee(\text{wt } b_j).$$

Then $\varepsilon_i(b)$ is defined to be the maximal value of $r_i^k(b)$ as a function of $k$, wt $b = \text{wt } b_1 + \cdots + \text{wt } b_n$ and $e_i$ (respectively, $f_i$) acts in the lefmost (respectively, rightmost) place where the maximal value of $r_i^k(b)$ is attained. That is known as Kashiwara's tensor product rule. It takes a particularly nice form for $n = 2$ (cf. [13, 3.5]).

Let $B$ be a crystal with wt $: B \rightarrow \mathbb{P}$. Its affinisation $\hat{B} = B \times \mathbb{Z}$ is a crystal with respect to the following operators. Denote the pair $(b, n) \in B \times \mathbb{Z}$ as $b \otimes t^n$. Then $\varepsilon_i(b \otimes t^n) = \varepsilon_i(b)$ and wt $b \otimes t^n = \text{wt } b + n\delta$ in $\hat{P}$. Furthermore, if $e_i b = 0$, set $e_i(b \otimes t^n) = 0$. Otherwise, $e_i(b \otimes t^n) = e_i b \otimes t^{n+\delta_i,0}$. Similarly, if $f_i b = 0$, set $f_i(b \otimes t^n) = 0$. Otherwise, set $f_i(b \otimes t^n) = f_i b \otimes t^{n-\delta_i,0}$. This should be regarded as the crystal analogue of the passage from a $U_q$-module $V$ to a $\hat{U}_q$-module $\hat{V}$ (cf. [24]).

3. General properties of $z$-crystal bases

3.1. Let $M$ be an integrable $U_q$ or $\hat{U}_q$-module of type 1. Fix $i \in \hat{I}$ and let $u$ be a weight vector of $M$ of weight $\nu$. Then $u$ can be written uniquely as

$$u = \sum_{s \geq \max\{0, -\alpha_i^\vee(\nu)\}} F_i^{(s)} u_s,$$

(3.1)

where $u_s \in \ker E_{i} \cap M_{\nu+s\alpha_i}$, and $u_s = 0$ for $s \gg 0$. The crystal operators of Kashiwara are defined as

$$\hat{e}_i u = \sum_{s \geq \max\{1, -\alpha_i^\vee(\nu)\}} F_i^{(s-1)} u_s, \quad \hat{f}_i u = \sum_{s \geq \max\{0, -\alpha_i^\vee(\nu)\}} F_i^{(s+1)} u_s.$$

(3.2)

Observe that, since $M$ is integrable, the operators $\hat{e}_i, \hat{f}_i$ are locally nilpotent.

Definition (cf. [5, 4.8]). Let $z \in \mathcal{C}^\times$. A $z$-crystal basis of $M$ is a pair $(L, B)$, where $L$ is a free $\mathcal{A}$-submodule of $M$ and $B$ is a basis of $\mathcal{C}$-vector space $L/qL$ such that

(i) $M = L \otimes_{\mathcal{A}} \mathcal{C}(q)$,
(ii) $L = \bigoplus_\lambda L_\lambda, \quad B = \bigsqcup_\lambda B_\lambda$, where $L_\lambda = L \cap M_\lambda$ and $B_\lambda = B \cap (L_\lambda/qL_\lambda)$.
(iii) $L$ is preserved by the operators $\hat{e}_i$, $\hat{f}_i$ for all $i \in \hat{I}$. In particular, $\hat{e}_i$, $\hat{f}_i$ act on $L/qL$.
(iv) $\hat{e}_i B, \hat{f}_i B \subset z^{\mathbb{Z} \delta_i, 0} B \cup \{0\}$, for all $i \in \hat{I}$.
(v) For all $b, b' \in B$, $i \in \hat{I}$, $\hat{e}_i b = z^{\mathbb{Z} \delta_i, 0} b'$ if and only if $\hat{f}_i b' = z^{-\mathbb{Z} \delta_i, 0} b$.

For $z = 1$ the above definition reduces to Kashiwara’s definition of crystal bases (cf. for example \cite{16}).

Let $(L, B)$ be a $z$-crystal basis of an integrable $U_q$ or $\hat{U}_q$-module $M$. Given $b \in B$, let $\varepsilon_i(b)$ (respectively, $\varphi_i(b)$) be the minimal non-negative integer $n$ such that $\hat{e}_i^{n+1} b$ (respectively, $\hat{f}_i^{n+1} b$) equals zero modulo $qL$. These are well-defined since $\hat{e}_i$, $\hat{f}_i$ are locally nilpotent. Furthermore, if $b \in B\lambda$, set $wt b = \lambda$.

We will need the following simple modification of \cite{24} Lemma 20.1.2 (cf. \cite{5} Lemma 4.8).

**Lemma.** Fix $i \in \hat{I}$. Let $u \in L_\lambda$ and write $u = \sum_{s \geq \max\{0, -\alpha_i^\vee(u)\}} F_i^{(s)} u_s$ as in \eqref{8}. Then

(i) $F_i^{(r)} u_s \in L$ for all $r, s \geq 0$.
(ii) If $u \pmod{qL} \in B$ then there exists $s_0$ such that $u_s \in qL$, $s \neq s_0$, $u_{s_0} \pmod{qL}$ $\in \mathbb{Z} \delta_i, 0 B$ and $u = F_i^{(s_0)} u_{s_0}$.

**Proof.** The proof is an obvious modification of that of Lemma 20.1.2 in \cite{24}. \hfill $\square$

3.2. Let $(L, B)$ be a $z$-crystal basis of an integrable $U_q$ or $\hat{U}_q$-module $M$. It follows immediately from Definition 3.1 that the set $\hat{B} = \bigcup_{r \in \mathbb{Z}} z^r B$ is a normal crystal. Define an equivalence relation on $\hat{B}$ by setting $b \sim_z b'$ if and only if $b = z^r b'$ for some $r \in \mathbb{Z}$. Then $\hat{B}/\sim_z$ identifies with $B$ as a set.

**Lemma.** The set $\hat{B}/\sim_z$ is a normal crystal with respect to the operators $\hat{e}_i, \hat{f}_i, \varepsilon_i, \varphi_i$, $i \in \hat{I}$ and $wt$.

**Proof.** Immediate. \hfill $\square$

We call $\hat{B}/\sim_z$ the crystal associated with $B$.

3.3. The following proposition justifies the definition of $z$-crystal bases.

**Proposition.** Let $M$ be a finite-dimensional $U_q$-module and assume that $M' = \phi^*_u M$ is not isomorphic to $M$. Suppose that $M$ admits a crystal basis $(L, B)$. Then $(L', B')$ where $L' = \phi^*_u L$, $B' = \phi^*_u B \subset L'/qL'$ is a $z$-crystal basis of $M'$. Moreover, if $b_1, b_2 \in B$ such that $\hat{e}_0 b_1 = b_2$ and $b'_1, b'_2$ are their images in $B'$ then $\hat{e}_0 b'_1 = z b'_2$.

**Proof.** Since $M \cong M'$ as $U_q^{fin}$-module, it is sufficient to verify (iii)-(v) for $i = 0$.

Let $u \in L_\lambda$ be a weight vector and let $u_s$, $s \geq 0$ be as in \eqref{8}. Set $u' = \phi^*_u u$. Then

$$u' = \sum_{s \geq \max\{0, -\alpha_0^\vee(u)\}} z^s F_0^{(s)} \phi^*_u u_s = \sum_{s \geq \max\{0, -\alpha_0^\vee(u)\}} F_0^{(s)} u'_s,$$

where $u'_s = z^s \phi^*_u u_s$. This provides the unique decomposition of the form \eqref{8} for $u' \in L'$. Then

$$\phi^*_u (\hat{e}_0 u) = \sum_{s \geq \max\{1, -\alpha_0^\vee(u)\}} z^{s-1} F_0^{(s-1)} \phi^*_u u_s = z^{-1} \sum_{s \geq \max\{1, -\alpha_0^\vee(u)\}} F_0^{(s-1)} u'_s = z^{-1} \hat{e}_0 u'.$$
Since \( \tilde{e}_0u \in L \), it follows that \( \tilde{e}_0u' = z\phi_*^z(\tilde{e}_0u) \in L' \). Similarly, \( \tilde{f}_0u' = z^{-1}\phi_*^z(\tilde{f}_0u) \in L' \). Since \( u_s \in L \) for all \( s \geq 0 \) by Lemma 3.1(i), it follows that \( u'_s \in L' \) for all \( s \geq 0 \).

Furthermore, suppose that \( b = u \pmod{qL} \in B \). Then by Lemma 3.1(ii) there exists \( s_0 \) such that \( u_s \in qL, s \neq s_0, u_{s_0} \pmod{qL} \in z^2B \) and \( b = F^0_{s_0}u_{s_0} \pmod{qL} \). Moreover, \( \tilde{e}_0b = F^0_{s_0}u_{s_0} \pmod{qL}, \tilde{f}_0b = F^0_{s_0+1}u_{s_0} \pmod{qL} \).

Let \( b' = \phi_*^z b \pmod{qL'} \in B' \). It follows immediately that \( \tilde{e}_0b' = z\phi_*^z(\tilde{e}_0u) \pmod{qL'} \in z^2B' \cup \{0\} \), \( \tilde{f}_0b' = z^{-1}\phi_*^z(\tilde{f}_0u) \pmod{qL'} \in z^2B' \cup \{0\} \).

Finally, suppose that \( \tilde{e}_0b = b_1 \in B \) and let \( b'_1 = \phi_*^z b_1 \). Then, as above, \( \tilde{e}_0b' = zb'_1 \). On the other hand, \( b'_1 = z^{s_0-1}F^{s_0-1}(u_{s_0}) \pmod{qL'} \), whence \( \tilde{f}_0b'_1 = z^{s_0-1}F^{s_0-1}(u_{s_0}) = z^{-1}b'_1 \).

**Remark.** Similarly, one can prove that if \( (L, B) \) is a \( z \)-crystal basis of \( M \) and \( \phi_*^z M \) is not isomorphic to \( M \) then \( (\phi_*^z L, \phi_*^z B) \) is a \( z \)-crystal basis of \( \phi_*^z M \). Moreover, if \( \tilde{e}_0b = z^kB_1 \) for some \( b, b_1 \in B \) and \( b', b'_1 \) denote their images in \( \phi_*^z B \) then \( \tilde{e}_0b' = z^{k+1}b'_1 \).

**3.4.** The following Lemma is rather standard (cf. [24 Corollary 17.4.2]). We deem it necessary to present its proof here since the argument in [24] is based on the use of Kashiwara’s bilinear form and cannot be modified for \( z \)-crystal bases.

**Lemma.** Let \( M_i, i = 1, 2 \) be finite dimensional \( U_q(\mathfrak{sl}_2) \) modules of type 1 and fix \( v_i, i = 1, 2 \) such that \( Kv_i = q^iv_i, t_i \geq 0 \) and \( Ev_i = 0 \). Let \( \mathcal{L}_i \), be the \( A \)-module generated by the \( F^{(s)} v_i, 0 \leq s \leq t_i \). Then

\[
(\text{i}) \quad \mathcal{L} := \mathcal{L}_1 \otimes_A \mathcal{L}_2 \text{ is preserved by the operators } \tilde{e}, \tilde{f} \text{ acting on the module } M_1 \otimes M_2.
\]

\[
(\text{ii}) \quad \text{There exist unique, up to multiplication by an element of } 1 + qA, u_r \in \ker E \cap \mathcal{L}, \quad 0 \leq r \leq \min\{t_1, t_2\} \text{ such that } Ku_r = q^{1+t_2-2r}ur, \quad \mathcal{L} \text{ is a direct sum of } A \text{-modules generated by } F^{(s)} u_r, 0 \leq b \leq t_1 + t_2 - r + 1 \text{ and for all } 0 \leq s_i \leq t_i, \quad i = 1, 2, \text{ there exists a unique } s, 0 \leq s \leq \min\{t_1, t_2\} \text{ such that } F^{(s)} v_1 \otimes F^{(s)} v_2 = F^{(s+s_2-s)} u_s \pmod{q\mathcal{L}}.
\]

\[
(\text{iii}) \quad \text{For all } 0 \leq s_i \leq t_i,
\]

\[
\tilde{e}(F^{(s_1)} v_1 \otimes F^{(s_2)} v_2) = F^{(s_1-1)} v_1 \otimes F^{(s_2)} v_2 \pmod{q\mathcal{L}}, \quad t_1 \geq s_1 + s_2
\]

\[
\tilde{e}(F^{(s_1)} v_1 \otimes F^{(s_2)} v_2) = F^{(s_1)} v_1 \otimes F^{(s_2-1)} v_2 \pmod{q\mathcal{L}}, \quad t_1 < s_1 + s_2
\]

\[
\tilde{f}(F^{(s_1)} v_1 \otimes F^{(s_2)} v_2) = F^{(s_1+1)} v_1 \otimes F^{(s_2)} v_2 \pmod{q\mathcal{L}}, \quad t_1 > s_1 + s_2
\]

\[
\tilde{f}(F^{(s_1)} v_1 \otimes F^{(s_2)} v_2) = F^{(s_1)} v_1 \otimes F^{(s_2+1)} v_2 \pmod{q\mathcal{L}}, \quad t_1 \leq s_1 + s_2
\]

where \( F^{(s)} v_i = 0 \) if \( s < 0 \).

**Proof.** Let \( V(n), n \geq 0 \) denote the unique \((n + 1)\)-dimensional simple \( U_q(\mathfrak{sl}_2) \) module. It is sufficient to prove the Lemma for \( M \cong V(t_i) \). The argument is by induction on \( t_1 \) and is rather standard.

1° Suppose first that \( t_1 = 0 \). Then \( F^{(s)} (v_1 \otimes v_2) = v_1 \otimes F^{(s)} v_2 \) and \( E(v_1 \otimes v_2) = 0 \). The proposition is then trivial.

2° Suppose that \( t_1 = 1 \) and set \( u_0 = v_1 \otimes v_1, \)

\[
u_1 = v_1 \otimes Fv_2 - q^{(2t_1)} v_1 \otimes v_2 = v_1 \otimes Fv_2 - \frac{q - q^{2t_1}}{1 - q^2} F v_1 \otimes v_2.
\]
Then $u_0, u_1$ generate $\ker E \cap \mathcal{L}$ and $u_1 = v_1 \otimes F v_2$ (mod $q \mathcal{L}$). Furthermore,
\begin{equation}
F^{(b)} u_0 = q^b v_1 \otimes F^{(b)} v_2 + F v_1 \otimes F^{(b-1)} v_2
\end{equation}
\begin{equation}
F^{(b)} u_1 = \frac{1 - q^{2(b+1)}}{1 - q^2} v_1 \otimes F^{(b+1)} v_2 + q \frac{d^{t_2} - q^b}{1 - q^2} F v_1 \otimes F^{(b)} v_2,
\end{equation}
where we used \eqref{2.2}. Since $F^{(b)} v_2 = 0$ if $b > t_2$, it follows immediately that $F^{(b)} u_0, F^{(b)} u_1 \in \mathcal{L}$ for all $b \geq 0$. Moreover, by the above formulae, $v_1 \otimes F^{(0)} v_2 = F^{(b-1)} u_1$ (mod $q \mathcal{L}$) whilst $F v_1 \otimes F^{(b-1)} v_2 = F^{(b)} u_0$ (mod $q \mathcal{L}$), $b > 0$. It follows that the matrix of $F^{(b)} u_0, F^{(b-1)} u_1$ in the basis of $F v_1 \otimes F^{(b-1)} v_2$, $v_1 \otimes F^{(b)} v_2$ is diagonal and its diagonal entries equal 1 (mod $q \mathcal{A}$). Therefore, that matrix is invertible over $\mathcal{A}$ and so the $F^{(b)} u_0, F^{(b-1)} u_1$ and $F v_1 \otimes F^{(b-1)} v_2$, $v_1 \otimes F^{(b)} v_2$ generate the same $\mathcal{A}$-module which completes the proof of (ii).

Since $E u_0 = 0$, it follows that $\tilde{e} (v_1 \otimes v_2) = F^{(b-1)} u_0 = F v_1 \otimes v_2$ (mod $q \mathcal{L}$), which agrees with the formulae in (iii).

Suppose now that $b > 0$. By the above, $F^{(v)} v_1 \otimes F^{(b-s)} v_2 = x_s F^{(b)} u_0 + y_s F^{(b-1)} u_1$, where $x_s \in \delta_{s,1} + q \mathcal{A}$, $y_s \in \delta_{s,0} + q \mathcal{A}$. Then, by definition of Kashiwara’s operators,
\begin{equation}
\tilde{e} (F^{(s)} v_1 \otimes F^{(b-s)} v_2) = x_s F^{(b-1)} u_0 + y_s F^{(b-2)} u_1,
\end{equation}
In particular, $\tilde{e}$ preserves $\mathcal{L}$. If $s = 0$ then the above expression equals $F^{(b-2)} u_1$ (mod $q \mathcal{L}$) $= v_1 \otimes F^{(b-1)} v_2$ (mod $q \mathcal{L}$) provided that $b \geq 2$ (and so $s_1 + s_2 = b > t_1$), which agrees with the formulae in (iii). If $b = 1$ (that is, $s_1 + s_2 = t_1$), $\tilde{e} (v_1 \otimes F^{(b)} v_2) = x_{s_1} u_0 = 0$ (mod $q \mathcal{L}$) as expected. Similarly, if $s = 1$, we get
\begin{equation}
\tilde{e} (F v_1 \otimes F^{(b-1)} v_2) = F^{(b-1)} u_0 \quad \text{(mod $q \mathcal{L}$)} = F v_1 \otimes F^{(b-2)} v_2 \quad \text{(mod $q \mathcal{L}$),}
\end{equation}
as desired. The formulae for the action of $\tilde{f}$ are proved similarly.

3? Suppose that (i)-(iii) are proved for all $t_1 \leq t$, $t > 0$. It is well-known (cf., for example, \cite{20} 4.3) that $V(t+1)$ can be realised as a simple submodule of $V(1) \otimes V(t)$ generated by the tensor product of the corresponding highest weight vectors. Thus, we can write $v_1$ from the assertion of the Lemma as $v'_1 \otimes v'_1$, where $E v'_1 = E v'_1 = 0$, $F^2 v'_1 = 0$, $F^{t_1} v'_1 = 0$. Let $\mathcal{L}'$ be the $\mathcal{A}$-module generated by the $F^{(s)} v'_1 \otimes F^{(s)} v'_2$ and denote by $u'_r$, $0 \leq r \leq \min \{ t, t_2 \}$ the elements of $\ker E \cap \mathcal{L}'$ satisfying $K u'_r = q^{t_1} - 2 r u'_r$ given by the induction hypothesis. Let $\mathcal{L}'' = \mathcal{A} u'_1 + \mathcal{A} F u'_1$. It follows from \cite{24} that the $\mathcal{A}$-module $\mathcal{L}$ generated by the $F^{(s)} v'_1 \otimes F^{(s)} v'_2$ is an $\mathcal{A}$-submodule of $\mathcal{L}'' \otimes \mathcal{A} \mathcal{L}'$.

By the induction hypothesis, $\mathcal{L}' = \bigoplus_s \mathcal{A} F^{(b)} u'_r$, $0 \leq r \leq \min \{ t, t_2 \}$. Applying the second part of the proof to $\mathcal{L}'' \otimes \mathcal{A} \bigoplus_s \mathcal{A} F^{(b)} u'_r$, $0 \leq r \leq \min \{ t, t_2 \}$, we conclude that $\tilde{e}$, $\tilde{f}$ preserve $\mathcal{L}'' \otimes \mathcal{A} \mathcal{L}'$. Since $\mathcal{L}$ is contained in the intersection of $\mathcal{L}'' \otimes \mathcal{A} \mathcal{L}'$ with a submodule of $V(1) \otimes V(t) \otimes V(t_2)$, it follows that $\tilde{e}$, $\tilde{f}$ preserve $\mathcal{L}$.

The next step is to prove the formulae in (iii). Since $\tilde{e}$, $\tilde{f}$ preserve $\mathcal{L}$, we can do all the computations modulo $q \mathcal{L}$.

Consider first $v_1 \otimes F^{(s_2)} v_2 = v''_1 \otimes v'_1 \otimes F^{(s_2)} v_2$, $s_2 \leq t_2$. By the induction hypothesis, $v''_1 \otimes F^{(s_2)} v_2 = F^{(s_2 - s')} u'_r$ (mod $q \mathcal{L}'$) for some $0 \leq s \leq \min \{ t, t_2 \}$. Suppose first that $s_2 \leq t$. Then $\tilde{e} (v''_1 \otimes F^{(s_2)} v_2) = 0$ (mod $q \mathcal{L}'$) $= F^{(s_2 - s')} u'_r$ by the induction hypothesis, whence $s_2 = s$. It follows that $\tilde{e} (v_1 \otimes F^{(s_2)} v_2) = \tilde{e} (v''_1 \otimes u'_r)$ (mod $q \mathcal{L}'$) $= 0$, as desired. Suppose that $s_2 = t + k$, $k > 0$. Then $v''_1 \otimes F^{(s_2)} v_2 = \tilde{f}^k (v''_1 \otimes F^{(t)} v_2)$ (mod $q \mathcal{L}'$) $= F^{(t)} u'_1$ (mod $q \mathcal{L}'$) by the induction hypothesis. Then $\tilde{e} (v_1 \otimes F^{(s_2)} v_2) = \tilde{e} (v''_1 \otimes F^{(k)} u'_1)$ (mod $q \mathcal{L}'$). By the first part of the proof, the latter expression equals
zero if \( k = 1 \) (that is, \( s_2 = t + 1 \)) and \( v''_i \otimes F^{(k-1)}u'_i = v_1 \otimes F^{(s_2-1)}v_2 \) (mod \( q\mathcal{L} \)) if \( k > 1 \) (that is, \( s_2 > t + 1 \)). Both agree with the formulae in (iii).

It follows from (iii) that \( F^{(s_1)}v_1 = F^{(s_1-1)}v'_1 \) (mod \( q\mathcal{L}\)). Thus, \( F^{(s_1)}v_1 = F^{(s_1-1)}v'_1 \) (mod \( q\mathcal{L}\)).

Finally, assume that \( s_2 - t \). Then \( F^{(s_1)}v''_1 \otimes u''_s = F^{(s_1-1)}v'_1 \otimes v''_s \) (mod \( q\mathcal{L}\)). Similarly, if \( s_2 > t \), \( F^{(s_1)}v_1 \otimes F^{(s_2)}v_2 = F^{(s_1-1)}v'_1 \otimes F^{(s_2)}v_2 \) (mod \( q\mathcal{L}\)) by the second part of the proof. Thus, \( F^{(s_1)}v_1 \otimes F^{(s_2)}v_2 = v_1 \otimes F^{(s_2)}v_2 \) (mod \( q\mathcal{L}\)) as desired.

Consider now \( F^{(s_1)}v_1 \otimes F^{(s_2)}v_2 \) with \( 0 < s_1 \leq t + 1 \). Using the induction hypothesis, we get

\[
\tilde{e}(F^{(s_1)}v_1 \otimes F^{(s_2)}v_2) = \tilde{e}(F^{(s_1-1)}v'_1 \otimes F^{(s_2)}v_2) \quad \text{(mod } \mathcal{L} \text{)}
\]

for some \( s, 0 \leq s \leq \min\{t, t_2\} \). Suppose first that \( s_1 + s_2 - s \leq 1 \). Then, by the second part of the proof,

\[
\tilde{e}(F^{(s_1+1)}v'_1 \otimes F^{(s_2-s-1)}u'_s) = v''_i \otimes F^{(s_1+s_2-s-1)}u'_s \quad \text{(mod } \mathcal{L} \text{)}.
\]

Yet \( s_1 + s_2 - s \geq 1 \), hence \( F^{(s_1-1)}v'_1 \otimes F^{(s_2)}v_2 \) (mod \( \mathcal{L} \)). In particular,

\[
\tilde{e}(F^{(s_1-1)}v'_1 \otimes F^{(s_2)}v_2) = 0 \quad \text{(mod } \mathcal{L} \text{)}.
\]

Suppose that \( s_1 + s_2 \leq t + 1 \).

We conclude that \( s_1 = t + 1 \). Thus

\[
\tilde{e}(F^{(s_1)}v_1 \otimes F^{(s_2)}v_2) = v''_i \otimes v'_i \otimes F^{(s_2)}v_2 \quad \text{(mod } \mathcal{L} \text{)} = v_1 \otimes F^{(s_2)}v_2 \quad \text{(mod } \mathcal{L} \text{)},
\]

which agrees with the formulae in (iii). On the other hand, if \( s_1 + s_2 > t + 1 \), then, by the induction hypothesis, \( \tilde{e}(F^{(s_1-1)}v'_1 \otimes F^{(s_2)}v_2) = F^{(s_1-1)}v'_1 \otimes F^{(s_2-1)}v_2 \), whence \( s_2 = 0 \) and \( s_2 > t + 1 \) which is a contradiction.

Finally, assume that \( s_1 + s_2 - s \geq 1 \). Then, by the second part of the proof,

\[
\tilde{e}(F^{(s_1+1)}v'_1 \otimes F^{(s_2-s-1)}u'_s) = v''_i \otimes F^{(s_1+s_2-s-2)}u'_s \quad \text{(mod } \mathcal{L} \text{)}.
\]

Yet, by the induction hypothesis, \( F^{(s_1+s_2-s-2)}u'_s = \tilde{e}(F^{(s_1)}v'_1 \otimes F^{(s_2)}v_2) \) (mod \( \mathcal{L} \)). The latter expression equals \( \mathcal{L} \), by the induction hypothesis, \( F^{(s_1-2)}v'_1 \otimes F^{(s_2)}v_2 \) if \( s_1 + s_2 - 1 \leq t \) and \( F^{(s_1)}v'_1 \otimes F^{(s_2-1)}v_2 \) otherwise.

Thus,

\[
\tilde{e}(F^{(s_1)}v_1 \otimes F^{(s_2)}v_2) = v''_i \otimes (s_1-2+k)v'_i \otimes F^{(s_2-k)}v_2 \quad \text{(mod } \mathcal{L} \text{)}
\]

where \( k \) equals zero if \( s_1 + s_2 \leq t + 1 \) and 1 otherwise. That proves the first two formulae in (iii). In order to prove the last two formulae, observe that, since \( s_1 + s_2 - s \geq 1 \), \( \tilde{e}(F^{(s_1)}v''_i \otimes F^{(s_1+s_2-s-1)}u'_s) = v''_i \otimes F^{(s_1+s_2-s)}u'_s \) (mod \( \mathcal{L} \)). It remains to apply the induction hypothesis.

The last step is to prove (ii). Set, for \( 0 \leq r \leq \min\{t + 1, t_2\} \),

\[
u_r = \sum_{a=0}^r c_{r,a} F^{(a)}v_1 \otimes F^{(r-a)}v_2,
\]

where \( c_{r,0} = 1 \) and, for \( 1 \leq a \leq r \),

\[
c_{r,a} = (-1)^a \prod_{j=1}^a q^{j-2(r-j)} \frac{(t_2-r+j)q}{(t-j+2)} + (-1)^a q^{a(t-r+2)} \prod_{j=1}^a \frac{1 - q^{2(t_2-r+j)}}{1 - q^{2(t-j+2)}}.
\]
Then $Eu_r = 0$ and $Ku_r = q^{t+1+2r}u_r$. Evidently, $u_r \in \mathcal{L}$ and $u_r = v_1 \otimes F(v_2)$ (mod $q\mathcal{L}$). We claim that, for all $0 \leq s_1 \leq t + 1$, $0 \leq s_2 \leq t_2$ there exist a unique $0 \leq s \leq \min\{t + 1, t_2\}$ such that $F^{(s_1)}v_1 \otimes F^{(s_2)}v_2 = \tilde{f}^{s_1+s_2-s}u_s$ (mod $q\mathcal{L}$). Evidently, (ii) follows immediately from the claim.

In order to prove the claim, observe first that $v_1 \otimes F^{(s_2)}v_2 = u_s$ (mod $q\mathcal{L}$), $0 \leq \min\{t + 1, t_2\}$. If $t_2 \leq t + 1$ that gives $v_1 \otimes F^{(s_2)}v_2$ for all $0 \leq s_2 \leq t_2$. Otherwise, by (iii), $\tilde{f}^{k}(v_1 \otimes F^{(t+1)}v_2) = v_1 \otimes F^{(t+k+1)}v_2$ (mod $q\mathcal{L}$). Thus, $v_1 \otimes F^{(s_2)}v_2 = \tilde{f}^{s_2-t-1}u_{t+1}$, $s_2 > t + 1$. Consider further $F^{(s_1)}v_1 \otimes F^{(s_2)}v_2$, $s_1 > 0$.

We use induction on $s_1$. If $s_1 + s_2 < t + 1$ then we have, by (iii), $F^{(s_1)}v_1 \otimes F^{(s_2)}v_2 = \tilde{f}F^{(s_1)}v_1 \otimes F^{(s_2)}v_2$ (mod $q\mathcal{L}$) = $\tilde{f}^{s_1+s_2-s}u_s$ (mod $q\mathcal{L}$), where $s$ is such that $F^{(s_1)}v_1 \otimes F^{(s_2)}v_2 = \tilde{f}^{s_1+s_2-s}u_s$ (mod $q\mathcal{L}$). Finally, suppose that $s_1 + s_2 = t+1+k$, $k \geq 0$. We may assume that $s_1 < t+1$ for otherwise $F^{(s_1)}v_1 = 0$. Set $l = t+1-s_1 > 0$. Then $s_2 = k+l \geq l$ and $F^{(s_1)}v_1 \otimes F^{(l)}v_2 = \tilde{f}^{s_1+l-s_1}u_s$ (mod $q\mathcal{L}$) by the induction hypothesis. Using (iii) repeatedly we conclude that $F^{(s_1)}v_1 \otimes F^{(s_2)}v_2 = \tilde{f}^{s_2-t-l+2}F^{(s_1)}v_1 \otimes F^{(l)}v_2$ (mod $q\mathcal{L}$) = $\tilde{f}^{s_1+s_2-s-1}u_s$ (mod $q\mathcal{L}$), which completes the proof of the claim.

3.5. Let $M_i, i = 1, 2$ be finite dimensional $U_q\mathcal{L}$-modules or admissible integrable $\tilde{U}_q\mathcal{L}$-modules. Suppose that $M_1$ admits a crystal basis $(L_1, B_1)$ and that $M_2$ admits a $z$-crystal basis $(L_2, B_2)$ for some $z \in \mathbb{C}^\times$.

Proposition. The pair $(L, B)$, where $L = L_1 \otimes A L_2$ and $B = \{b_1 \otimes b_2 : b_i \in B_i\}$, is a $z$-crystal basis of $M_1 \otimes M_2$. Moreover, for all $b_i \in B_i$, $i = 1, 2$

\[
\begin{align*}
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{e}_i(b_1) \otimes b_2, & \tilde{e}_i(b_1) \geq \varepsilon_i(b_2) \\
\tilde{e}_i(b_1) \otimes \tilde{e}_i(b_2), & \tilde{e}_i(b_1) < \varepsilon_i(b_2) 
\end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{f}_i(b_1) \otimes b_2, & \tilde{f}_i(b_1) > \varepsilon_i(b_2) \\
\tilde{f}_i(b_1) \otimes \tilde{f}_i(b_2), & \tilde{f}_i(b_1) \leq \varepsilon_i(b_2) 
\end{cases}
\end{align*}
\]

Proof. The proof is essentially the same as that of [24] Theorem 20.2.2). We only have to verify the properties of a $z$-crystal basis for $i = 0$. Set

\[G_i^1 := \{v \in L_i : K_0v = q_0^iv, \; E_0v = 0, \; v \in B_i \pmod{qL_i}\}.\]

Then, by Lemma Lemma 3.3 $F_0^{(s_1)}v_1 \in B_1$ (mod $qL_1$) for all $v \in G_i^1$ and $0 \leq s_1 \leq t_1$ and all elements of $B_1$ are obtained that way. Similarly, for all $v_2 \in G_i^2$ and $0 \leq s_2 \leq t_2$ there exists $r = r(v_2, s_2) \in \mathbb{Z}$ such that $z^rF_0^{(s_2)}v_2 \in B_2$ (mod $qL_2$) and all elements of $B_2$ are obtained that way. Since the weight spaces of $M_i, i = 1, 2$ are finite-dimensional, it follows by Nakayama’s Lemma that the $\mathcal{A}$-module $L_i$ is generated over $\mathcal{A}$ by the $F_0^{(s_1)}v_1, v_i \in G_i^1$, $0 \leq s_i \leq t_i$. Therefore, $L$ is generated over $\mathcal{A}$ by the $F_0^{(s_1)}v_1 \otimes F_0^{(s_2)}v_2, v_i \in G_i^1$, $0 \leq s_i \leq t_i, i = 1, 2$. Using Lemma 3.3 (iii) we conclude that $\tilde{e}_0, \tilde{f}_0$ map the generators of the $\mathcal{A}$-module $L$ into $L$ and hence act on $L$. The rest of the properties of a $z$-crystal basis and Kashiwara’s tensor product rule follows readily from Lemma 3.3 (iii).

3.6. Let $V$ be a finite-dimensional simple $U_q\mathcal{L}$-module and assume that $V$ admits a $z$-crystal basis $(L, B)$ for some $z \in \mathbb{C}^\times$. Let $\tilde{V}$ be as in [24].
Lemma. Set $\hat{L} = L \otimes_{\mathcal{A}} \mathcal{A}[t, t^{-1}]$, $\hat{B} = \{ b \otimes t^r : b \in B, r \in \mathbb{Z} \}$. Then $(\hat{L}, \hat{B})$ is a $z$-crystal basis of the $\hat{U}_q$-module $\hat{V}$. Moreover, for all $b \in B$, $r \in \mathbb{Z}$,

$$\hat{e}_i(b \otimes t^r) = (\hat{e}_i b) \otimes t^{r+\delta_i,0} \pmod{q \hat{L}}, \quad \hat{f}_i(b \otimes t^r) = (\hat{f}_i b) \otimes t^{r-\delta_i,0} \pmod{q \hat{L}}.$$  

In other words, the associated crystal of $\hat{B}$ is the affinisation of the associated crystal of $B$ in the sense of the definition given in 2.8.

Proof. Take $u \in L$ of weight $\lambda$ and write $u = \sum_{s \geq \max\{0, -\alpha_i^\vee(\lambda)\}} F_i^{(s)} u_s$ as in (3.1). Evidently,

$$u \otimes t^r = \sum_{s \geq \max\{0, -\alpha_i^\vee(\lambda)\}} F_i^{(s)} (u_s \otimes t^{r+\delta_i,0}),$$

which is the decomposition (3.1) for $u \otimes t^r$. Then by the definition of Kashiwara’s operators,

$$\hat{e}_i(u \otimes t^r) = \sum_{s \geq \max\{1, -\alpha_i^\vee(\lambda)\}} F_i^{(s-1)} (u_s \otimes t^{r+\delta_i,0}) = \left( \sum_{s \geq \max\{1, -\alpha_i^\vee(\lambda)\}} F_i^{(s-1)} u_s \right) \otimes t^{r+\delta_i,0},$$

$$\hat{f}_i(u \otimes t^r) = \sum_{s \geq \max\{0, -\alpha_i^\vee(\lambda)\}} F_i^{(s+1)} (u_s \otimes t^{r+\delta_i,0}) = \left( \sum_{s \geq \max\{0, -\alpha_i^\vee(\lambda)\}} F_i^{(s+1)} u_s \right) \otimes t^{r-\delta_i,0}.$$

The assertion follows immediately from the above formulae and the properties of a $z$-crystal basis.

\[ \square \]

4. Quantum loop modules and their $z$-crystal bases

4.1. Let $\pi^0$ be an $\ell$-tuple of polynomials over $\mathbb{C}(q)$ with constant term 1 and suppose that $\pi^0(zu) \neq \pi^0(u)$, $z \in \mathbb{C}^\times$ as a set of polynomials. Given $\ell$-tuples of polynomials $\pi = (\pi_i)_{i \in I}$, $\pi^\ell = (\pi^0_i)_{i \in I}$ set $\pi \pi^\ell = (\pi_i \pi^0_i)_{i \in I}$.

Retain the notations of 2.4 and suppose that the finite dimensional $U_q$-module $V(\pi^0)$ admits a crystal basis $(L(\pi^0), B(\pi^0))$. Fix $m \in \mathbb{N}$ which does not exceed the multiplicative order of $z$ and set $\pi = \pi^0_0 \cdots \pi^0_{z^{m-1}}$. Then $V(\pi)$ is isomorphic to $V(\pi^0) \otimes V(\pi^0_2) \otimes \cdots \otimes V(\pi^0_{z^{m-1}})$ by [41]. Furthermore, set $L(\pi) = L(\pi^0) \otimes_{\mathcal{A}} \phi^*_z L(\pi^0) \otimes \cdots \otimes_{\mathcal{A}} \phi^*_z L(\pi^0)$ and define $B(\pi)$ accordingly. Since $\phi^*_z$ is the identity map on the level of vector spaces, $B(\pi)$ identifies with $B(\pi^0)^{\otimes m} = \{ b_1 \otimes \cdots \otimes b_m : b_i \in B(\pi^0) \}$. 

Proposition. The pair $(L(\pi), B(\pi))$ is a $z$-crystal basis of $V(\pi)$. Moreover, for all $b_1, \ldots, b_m \in B(\pi^0)$,

$$\hat{e}_i(b_1 \otimes \cdots \otimes b_m) = z^{r-1} b_1 \otimes \cdots \otimes b_{r-1} \otimes \hat{e}_i b_r \otimes b_{r+1} \otimes \cdots \otimes b_m,$$

$$\hat{f}_i(b_1 \otimes \cdots \otimes b_m) = z^{-s+1} b_1 \otimes \cdots \otimes b_{s-1} \otimes \hat{e}_i b_s \otimes b_{s+1} \otimes \cdots \otimes b_m,$$

where $r$ and $s$ are determined by Kashiwara’s tensor product rule. In particular, the associated crystal of $B(\pi)$ is isomorphic to $B(\pi^0)^{\otimes m}$. 

\[ \square \]
Proof. The proof is by induction on \( m \), the induction base being trivial. Recall that \( V(\pi_0^m) = \phi^m_0 V(\pi_0) \). Set \( V_k = V(\pi_0) \otimes V(\pi_0^k) \otimes \cdots \otimes V(\pi_0^{k-1}) \), \( k > 0 \) and define \( L_k, B_k \). Suppose that \( (L_k, B_k) \) is a \( z \)-crystal basis for \( V_k \). Then \( V_{k+1} \cong V_1 \otimes \phi^*_0 V_k \) and \( (\phi^*_0 L_k, \phi^*_0 B_k) \) is a \( z \)-crystal basis of \( V_k \) by Remark 3.3. Then \( (L_1 \otimes \phi^*_0 L_k, B_1 \otimes \phi^*_0 B_k) = (L_{k+1}, B_{k+1}) \) is a \( z \)-crystal basis of \( V_k \) by Proposition 3.3. The formulae follow immediately from these in Proposition 3.3. \( \square \)

4.2. Let \( \zeta \) be an \( m \)th primitive root of unity. Let \( \pi^0 \) be a tuple of polynomials such that \( \pi^0(\zeta u) \neq \pi^0(u) \) as a set of polynomials. Fix an \( l \)-highest weight vector \( v_{\pi^0} \) in \( V(\pi^0) \) and write \( v_{\pi^0} = \phi^m_0 v_{\pi^0} \). Let \( V(\pi) = V(\pi^0) \otimes V(\pi_0^k) \otimes \cdots \otimes V(\pi_0^{m-1}) \) and set \( v_\pi = v_{\pi^0} \otimes v_{\pi_0^k} \otimes \cdots \otimes v_{\pi_0^{m-1}} \). By [3], \( V(\pi) \) is a simple \( U_\pi \)-module and there exists a unique isomorphism of \( U_\pi \)-modules

\[
\tau : V(\pi) \rightarrow V(\pi_0^{m-1}) \otimes V(\pi_0^k) \otimes \cdots \otimes V(\pi_0^{m-2})
\]

which maps \( v_\pi \) to the corresponding permuted tensor product of the \( v_{\pi^0} \). Define \( \eta : V(\pi) \rightarrow V(\pi) \) by \( \eta := (\phi^m_0)^m \circ \tau \). Then, for all \( x \in U_\pi \) homogeneous of degree \( k \) and for all \( v \in V(\pi) \) we have \( \eta(xv) = \zeta^{-k} x\eta(v) \) (cf. [3] Lemma 2.6)). In particular, since \( \eta(v_\pi) = v_\pi \), we conclude that

\[
V(\pi) = \bigoplus_{k=0}^{m-1} V(\pi)^{(k)}, \quad \text{where} \quad V(\pi)^{(k)} := \{ v \in V(\pi) : \eta(v) = \zeta^k v \}.
\]

Define \( \hat{\eta} : \hat{V}(\pi) \rightarrow \hat{V}(\pi) \) by \( \hat{\eta}(v \otimes t^r) = \zeta^r \eta(v) \otimes t^r \). Then \( \hat{\eta} \in \text{End}_{U_\pi} \hat{V}(\pi) \) (cf. [3] Lemma 2.7)). Moreover, by [3] Lemma 2.8], \( \hat{V}(\pi) \) is a direct sum of simple \( \hat{U}_\pi \)-submodules \( \hat{V}(\pi)^{(r)} \), \( r = 0, \ldots, m - 1 \) which are in turn the eigenspaces of \( \hat{\eta} \) corresponding to the eigenvalues \( \zeta^r \). Observe also that \( \hat{V}(\pi)^{(r)} \) is spanned by \( v \otimes t^r \), where \( v \in V(\pi)^{(k)} \), \( k = r - s \) (mod \( m \)). By [3] Theorem 5], all simple integrable admissible \( \hat{U}_\pi \)-modules of level zero are obtained that way.

4.3. Following [3, 4.3], set, for all \( v \in V(\pi), r, s \in \mathbb{Z} \)

\[
\Pi_s(v) := \frac{1}{m} \sum_{j=0}^{m-1} \zeta^{-js} \eta^j(v), \quad \Pi_s(v \otimes t^r) := \Pi_{s-r}(v) \otimes t^r.
\]

By [3] Lemma 4.3], \( \Pi_s \) (respectively, \( \hat{\Pi}_s \)) is an orthogonal projector onto \( V(\pi)^{(s)} \) (respectively, onto \( \hat{V}(\pi)^{(s)} \)). Moreover, if \( x \in U_\pi \) is homogeneous of degree \( k \), then

\[
\Pi_s(xv) = \frac{1}{m} \sum_{j=0}^{m-1} \zeta^{-j(s+k)} x\eta^j(v) = x \Pi_{s+k}(v).
\]

The map \( \hat{\Pi}_s \) is obviously a homomorphism of \( \hat{U}_\pi \)-modules.

In the reminder of this section we will prove that \( \hat{V}(\pi)^{(r)} \) admits a \( \zeta \)-crystal basis provided that \( V(\pi_0^0) \) admits a crystal basis.

4.4. Suppose that \( V(\pi_0^0) \) is a “good” \( U_\pi \)-module (we refer the reader to [18, Sect. 8] for the precise definition). In particular, \( V(\pi_0^0) \) admits a crystal basis \( (L(\pi_0^0), B(\pi_0^0)) \) and \( B(\pi_0^0) \) is indecomposable as a crystal for all \( m > 0 \). It is proved in [18, Proposition 5.15] that the module \( V(\pi_0^m) \) corresponding to \( \pi_0^m = \pi_{i;1} \) is good.
Let $z_1, z_2 \in \mathbb{C}^\times$. Let $\tau_{z_1, z_2}$ be the isomorphism $V(\pi_0) \otimes V(\pi_0) \rightarrow V(\pi_0) \otimes V(\pi_0)$ normalized so that it preserves the tensor product of highest weight vectors. By [18, Proposition 9.3], $\tau_{z_1, z_2}$ maps $\phi^*_1 L(\pi^0) \otimes_A \phi^*_r L(\pi^0)$ into $\phi^*_1 L(\pi^0) \otimes \phi^*_r L(\pi^0)$. Moreover, there is a unique map $\chi : B(\pi^0)^{\otimes 2} \rightarrow \mathbb{Z}$ such that

$$\tau_{z_1, z_2}(b_1 \otimes b_2) = (z_1/z_2)^{\phi_1(b_1 \otimes b_2)} b_1 \otimes b_2 \pmod{q(\phi^*_1 L(\pi^0) \otimes_A \phi^*_r L(\pi^0))}.$$

and $\chi(b_{\pi^0} \otimes b_{\pi^0}) = 0$ where $b_{\pi^0} \in B(\pi^0)$ is the $l$-highest weight vector.

**Lemma.** Let $b_1, b_2 \in B(\pi^0)$ and suppose that $\hat{f}_i(b_1 \otimes b_2) \neq 0$. Then

$$\chi(\hat{f}_i(b_1 \otimes b_2)) = \begin{cases} \chi(b_1 \otimes b_2) + \delta_i,0, & \varphi_i(b_1) > \varepsilon_i(b_2) \\ \chi(b_1 \otimes b_2) - \delta_i,0, & \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Similarly, if $\hat{e}_i(b_1 \otimes b_2) \neq 0$, then

$$\chi(\hat{e}_i(b_1 \otimes b_2)) = \begin{cases} \chi(b_1 \otimes b_2) - \delta_i,0, & \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ \chi(b_1 \otimes b_2) + \delta_i,0, & \varphi_i(b_1) < \varepsilon_i(b_2). \end{cases}$$

**Proof.** Observe that $\hat{f}_i$ commutes with $\tau_{z_1, z_2}$. Indeed, given $u \in V(\pi_{z_1}) \otimes V(\pi_{z_2})$, write, as in [14], $u = \sum_s f(s) u_s$. Since $\tau_{z_1, z_2}$ is an isomorphism of $U_q$-modules, $\tau_{z_1, z_2}(\hat{f}_i u) = \sum_s f(s) \tau_{z_1, z_2}(u_s)$. On the other hand, $\tau_{z_1, z_2}$ commutes with $E_i$, $K_i^{-1}$, hence $\tau_{z_1, z_2}(u_s)$ has the same weight as $u_s$ and is annihilated by $E_i$. It follows that $\tau_{z_1, z_2}(u) = \sum_s f(s) \tau_{z_1, z_2}(u_s)$ is the unique decomposition of the form [14]. Therefore, $\hat{f}_i \tau_{z_1, z_2}(u) = \sum_s f(s) \tau_{z_1, z_2}(u_s) = \tau_{z_1, z_2}(\hat{f}_i u)$.

It is sufficient to prove the formula for $\chi(\hat{f}_i(b_1 \otimes b_2))$ since the formula for $\chi(\hat{e}_i(b_1 \otimes b_2))$ follows from that one by the properties of crystals. Suppose that $\varphi_i(b_1) > \varepsilon_i(b_2)$, the other case being similar. Then $\hat{f}_i(b_1 \otimes b_2) = z_1^{-\delta_i,0} \hat{f}_i b_1 \otimes b_2$ by Lemma 3.11 and Proposition 3.8. Therefore,

$$\tau_{z_1, z_2}(\hat{f}_i(b_1 \otimes b_2)) = z_1^{-\delta_i,0} (z_1/z_2)^{\phi_1(b_1 \otimes b_2)} \hat{f}_i b_1 \otimes b_2.$$

On the other hand,

$$\hat{f}_i(\tau_{z_1, z_2}(b_1 \otimes b_2)) = (z_1/z_2)^{\phi_1(b_1 \otimes b_2)} z_2^{-\delta_i,0} \hat{f}_i b_1 \otimes b_2.$$

Since $\hat{f}_i$ commutes with $\tau_{z_1, z_2}$ it follows that $\chi(\hat{f}_i b_1 \otimes b_2) = \chi(b_1 \otimes b_2) + \delta_i,0$. \hfill \square

The map $\chi : B(\pi^0)^{\otimes 2} \rightarrow \mathbb{Z}$ is called the energy function.

**4.5.** Retain the notations of 4.2. Using the isomorphism $\tau_{z_1, z_2}$, we can write $\tau$ as

$$\tau = \tau^{(0)} \circ \cdots \circ \tau^{(m-2)},$$

where

$$\tau^{(k)} := \text{id}^{\otimes k} \otimes \tau^{c_{m-k-1}} \otimes \text{id}^{\otimes m-k-2}.$$ Take some $b_1, \ldots, b_m \in B(\pi^0)$ and consider $b = b_1 \otimes \cdots \otimes b_m \in B(\pi)$. Then

$$\tau(b) = \zeta^{\text{Maj}_b} b,$$

where $\chi : B(\pi^0)^{\otimes 2} \rightarrow \mathbb{Z}$ is the energy function and

$$\text{Maj}_b(b) = \sum_{r=1}^{m-1} r \chi(b_r \otimes b_{r+1})$$
is the generalised major index of MacMahon. Indeed, in the case of \( \mathfrak{g} \) of type \( A_t \) and \( \pi^0 = \varpi_{1,1} \), there exists a total order on \( B(\pi^0) \) such that, for all \( b, b' \in B(\pi^0)\), 
\[
\chi(b \otimes b') = 0 \quad \text{if} \quad b \geq b' \quad \text{whilst} \quad \chi(b \otimes b') = 1 \quad \text{if} \quad b < b' \quad (\text{cf. [14]}).\]
Thus, in that case \( \text{Maj}_\chi(b) \) is just the usual major index of MacMahon for a word in a monoid over a completely ordered alphabet.

**Lemma.** Let \( V(\pi)^{(k)} \) be the eigenspace of \( \eta \) corresponding to the eigenvalue \( \zeta^k \).

Set \( L(\pi)^{(k)} := L(\pi) \cap V(\pi)^{(k)} \), \( B(\pi)^{(k)} := \{ b \in B(\pi) : \text{Maj}_\chi(b) = k \pmod{m} \} \).

(i) \( L(\pi)^{(k)} \) is a free \( \mathcal{A} \)-module, \( V(\pi)^{(k)} = L(\pi)^{(k)} \otimes \mathcal{A} \mathbf{C}(q) \) and \( B(\pi)^{(k)} \) is a basis of the \( \mathbf{C} \)-vector space \( L(\pi)^{(k)}/qL(\pi)^{(k)} \).

(ii) Let \( u \in L(\pi)^{(k)} \) and write \( u = \sum_s F_i^{(s)} u_s \) as in (4.1). Then \( u_s \in L(\pi)^{(k-s\delta_i,0)} \), \( \tilde{e}_i u \in L(\pi)^{(k-\delta_i,0)} \) and \( \hat{f}_i u \in L(\pi)^{(k+\delta_i,0)} \).

(iii) Suppose that \( b \in B(\pi)^{(k)} \). Then
\[
\tilde{e}_i b \in \zeta^{\delta_i,0} B(\pi)^{(k-\delta_i,0)} \cup \{0\}, \quad \hat{f}_i b \in \zeta^{\delta_i,0} B(\pi)^{(k+\delta_i,0)} \cup \{0\}.
\]

**Proof.** Take \( u \in L(\pi) \) such that \( u = b \pmod{qL(\pi)} \). Since \( L(\pi) \) is a free module and \( B(\pi) \) is a basis of \( L(\pi)/qL(\pi) \), such \( u \) generate \( L(\pi) \) as an \( \mathcal{A} \)-module by Nakayama’s Lemma. Then, since \( \eta \) maps \( L(\pi) \) into itself,
\[
\Pi_s(u) = \frac{1}{m} \sum_{r=0}^{m-1} \zeta^{r(Maj_\chi(b)-s)} b \pmod{qL(\pi)}.
\]
It follows that \( \Pi_s(u) = b \pmod{qL(\pi)} \) if \( s = \text{Maj}_\chi(b) \pmod{m} \) whilst \( \Pi_s(u) = 0 \pmod{qL(\pi)} \) otherwise.

Since \( \Pi_k \) is an orthogonal projector onto \( V(\pi)^{(k)} \) and maps \( L(\pi) \) into itself, it follows that \( L(\pi)^{(k)} = \Pi_k(L(\pi)) \). Then \( B(\pi)^{(k)} \) is a basis of \( L(\pi)^{(k)}/qL(\pi)^{(k)} \). Indeed, elements of \( B(\pi)^{(k)} \) are contained in \( L(\pi)^{(k)}/qL(\pi)^{(k)} \) by the above and are linearly independent, whence \( \dim_{\mathbf{C}} L(\pi)^{(k)}/qL(\pi)^{(k)} \geq \#B(\pi)^{(k)} \). Yet, \( \#B(\pi)^{(k)} = \sum_{k=0}^{m-1} \#B(\pi)^{(k)} \leq \sum_{k=0}^{m-1} \dim_{\mathbf{C}} L(\pi)^{(k)}/qL(\pi)^{(k)} = \dim_{\mathbf{C}} L(\pi)/qL(\pi) = \#B(\pi) \).
It follows that \( \dim_{\mathbf{C}} L(\pi)^{(k)}/qL(\pi)^{(k)} = \#B(\pi)^{(k)} \). Then \( L(\pi)^{(k)} \) is generated by the \( \Pi_k(u) \), \( u = b \pmod{qL(\pi)} \), with \( \text{Maj}_\chi(b) = k \) by Nakayama’s Lemma.

For the second part, suppose that \( u \in L(\pi)^{(k)} \). Then by (4.1),
\[
u = \Pi_k(u) = \sum_s \Pi_k(F_i^{(s)} u_s) = \sum_s F_i^{(s)} \Pi_k-\delta_i,0(u_s).
\]
Since \( K_i \) commutes with the \( \Pi_r \) and \( E_i \Pi_r(u_s) = \Pi_{r-\delta_i,0}(E_i u_r) = 0 \) if it follows that \( \Pi_{k-s\delta_i,0}(u_s) \) is of the same weight as \( u_s \) and is annihilated by \( E_i \). Then \( u_s = \Pi_{k-s\delta_i,0}(u_s) \) by the uniqueness of the decomposition (3.1). Furthermore,
\[
\Pi_r(\tilde{e}_i u) = \sum_s \Pi_r(F_i^{(s-1)} u_s) = \sum_s F_i^{(s-1)} \Pi_{r-(s-1)\delta_i,0}(u_s).
\]
It remains to observe that \( \Pi_{r-(s-1)\delta_i,0}(u_s) = 0 \) unless \( r = k-\delta_i,0 \). The proof for \( \hat{f}_i \) is similar.

The last part follows immediately from (i), (ii) and the properties of the \( z \)-crystal basis. However, we prefer to present a direct proof since it involves a property of \( \text{Maj}_\chi \) which we will need later. Evidently, it is enough to prove the statement for \( \hat{f}_i \). Write \( b = b_1 \otimes \cdots \otimes b_m, b_i \in B(\pi^0) \) and suppose that \( \hat{f}_i b \neq 0 \). Then \( \hat{f}_i b = \)
\[ \zeta^{-s+1}b_1 \otimes \cdots \otimes \tilde{f}_i b_s \otimes \cdots \otimes b_m \] for some \( 1 \leq s \leq m \). Suppose first that \( 1 \leq s < m \). Then

\[
\text{Maj}_\chi(\tilde{f}_i b) - \text{Maj}_\chi(b) = (s - 1)(\chi(b_{s-1} \otimes \tilde{f}_i b_s) - \chi(b_{s-1} \otimes b_s)) + s(\chi(\tilde{f}_i b_s \otimes b_{s+1}) - \chi(b_s \otimes b_{s+1})) = -(s - 1)\delta_{i,0} + s\delta_{i,0} = \delta_{i,0},
\]

where we used Lemma 4.4. Finally, if \( s = m \), then

\[
\text{Maj}_\chi(\tilde{f}_i b) - \text{Maj}_\chi(b) = (m - 1)(\chi(b_{m-1} \otimes \tilde{f}_i b_m) - \chi(b_{m-1} \otimes b_m)) = -(m - 1)\delta_{i,0} = \delta_{i,0} \pmod{m}. \]

\[ \Box \]

\[ \textbf{4.6.} \quad \text{Retain the notations of } 4.1, 4.2 \text{ and } 4.5. \]

\[ \textbf{Theorem.} \quad \text{Suppose that } V(\pi^0) \text{ is a good module and let } (L(\pi^0), B(\pi^0)) \text{ be its crystal basis. Set } \pi = \pi^0 \pi_1 \cdots \pi_{m-1}, \text{ where } \zeta \text{ is an } m \text{th primitive root of unity, and define } L(\pi), B(\pi) \text{ as in } 4.1. \text{ The simple submodule } \hat{V}(\pi)^{(k)}, k = 0, \ldots, m - 1 \text{ of } \hat{V}(\pi) \text{ admits a } \zeta \text{-crystal base } (\hat{L}(\pi)^{(k)}, \hat{B}(\pi)^{(k)}), \text{ where }
\]

\[
\hat{L}(\pi)^{(k)} = \hat{n}_k \hat{L}(\pi) = \bigoplus_{r \in \mathbb{Z}, 0 \leq r \leq m-1, r+s=k \pmod{m}} L(\pi)^{(s)} \otimes t',
\]

\[
\hat{B}(\pi)^{(k)} = \{ b \otimes t' : b \in B(\pi)^{(s)}, r \in \mathbb{Z}, r + s = k \pmod{m} \}
\]

\[ \text{Proof.} \quad \text{This follows immediately from Lemma } 3.6 \text{ and Lemma } 4.6. \quad \Box \]

\[ \text{Our Theorem } 4 \text{ is a particular case of the above statement since, as shown in } [13], \text{ the module } V(\pi^0) \text{ with } \pi^0 = \varpi_{i;1} \text{ satisfies all the required conditions and the corresponding } \pi \text{ obviously coincides with } \varpi_{i;m}. \]

\section{5. Path model for } \( z \)-\text{crystal bases of quantum loop modules}

In the present section we will construct a combinatorial model, in the framework of Littelmann’s path crystal, of \( z \)-crystal bases of simple components of quantum loop modules of fundamental type. The necessary facts about Littelmann’s path crystal will be reviewed as the need arises. Throughout this section we identify \( P \) with \( \hat{P} / \mathbb{Z} \delta \).

\[ \textbf{5.1.} \quad \text{Given } a, b \in \mathbb{Q}, a < b, \text{ set } [a, b] := \{ x \in \mathbb{Q} | a \leq x \leq b \}. \text{ Let } \mathbb{P} \text{ (respectively, } \hat{\mathbb{P}}) \text{ be the set of piece-wise linear continuous paths in } P \otimes_{\mathbb{Z}} \mathbb{Q} \text{ (respectively, in } \hat{P} \otimes_{\mathbb{Z}} \mathbb{Q}) \text{ starting at zero and terminating at an element of } P \text{ (respectively, } \hat{P}). \text{ In other words, } \pi \in \mathbb{P} \text{ (respectively, } \hat{\mathbb{P}}) \text{ is a piece-wise linear continuous map of } [0, 1] \text{ into } P \otimes_{\mathbb{Z}} \mathbb{Q} \text{ (respectively, into } \hat{P} \otimes_{\mathbb{Z}} \mathbb{Q}) \text{ such that } \pi(0) = 0 \text{ and } \pi(1) \in P \text{ (respectively, } \hat{P}). \text{ We consider two paths as identical if they coincide up to a continuous piece-wise linear non-decreasing reparametrisation.}
\]

After Littelmann (cf. [21, 22]) one can introduce a structure of a normal crystal on \( \mathbb{P} \) or on \( \hat{\mathbb{P}} \) in the following way. Given \( \pi \in \mathbb{P} \) or \( \hat{\mathbb{P}} \) and \( i \in \hat{I} \), set \( h^n_i(\tau) = -\alpha^*_i(\pi(\tau)), \tau \in [0, 1] \). Let \( \varepsilon_i(\pi) \) be the maximal integral value attained by \( h^n_i \) on \([0, 1] \). Furthermore, set \( c^n_i(\pi) = \min\{\tau \in [0, 1] : h^n_i(\tau) = \varepsilon_i(\pi)\} \). If \( \varepsilon_i(\pi) = 0 \)
Finally, set \( e_i \pi = 0 \). Otherwise, set \( e_i^\pi(\pi) = \max\{\tau \in [0, e_i^\pi(\pi)] : h_i^\pi(\tau) = e_i(\pi) - 1\} \) and define

\[
(e_i^\pi)(\pi) = \begin{cases} 
\pi(\tau), & \tau \in [0, e_i^\pi(\pi))] \\
(\pi(e_i^\pi(\pi))) + s_i(\pi(\tau) - \pi(e_i^\pi(\pi))), & \tau \in [e_i^\pi(\pi), e_i^\pi(\pi)] \\
\pi(\tau) + \alpha_i, & \tau \in [e_i^\pi(\pi), 1],
\end{cases}
\]

where \( s_i \) acts point-wise. Similarly, in order to define \( f_i \), let \( f_i^\pi(\pi) = \max\{\tau \in [0, 1] : h_i^\pi(\tau) = e_i(\pi)\} \). If \( f_i^\pi(\pi) = 1 \), set \( f_i \pi = 0 \). Otherwise, set \( f_i^\pi(\pi) = \min\{\tau \in [f_i^\pi(\pi), 1] : h_i^\pi(\tau) = e_i(\pi) - 1\} \) and define

\[
(f_i^\pi)(\pi) = \begin{cases} 
\pi(\tau), & \tau \in [0, f_i^\pi(\pi))] \\
(\pi(f_i^\pi(\pi))) + s_i(\pi(\tau) - \pi(f_i^\pi(\pi))), & \tau \in [f_i^\pi(\pi), f_i^\pi(\pi)] \\
\pi(\tau) - \alpha_i, & \tau \in [f_i^\pi(\pi), 1],
\end{cases}
\]

Finally, \( \wt \pi \) is defined as the endpoint \( \pi(1) \) of \( \pi \).

**Remark.** As in [14], we use the definition of crystal operations on \( \mathcal{P} \) given in [20] 6.4.4 which differs by the sign of \( h_i^\pi \) from the definition in [21] 1.2. That choice is more convenient for us since it makes the comparison with Kashiwara’s tensor product easier.

### 5.2.
Following [22] Theorem 8.1, one can introduce an action of the Weyl group \( \hat{W} \) on \( \mathcal{P} \) and \( \hat{P} \). Namely, given \( \pi \in \mathcal{P} \) or \( \pi \in \hat{P} \), set

\[
s_i \pi = \begin{cases} 
f_i^{\alpha_i^\pi(\pi)}(\pi), & \alpha_i^\pi(\pi(1)) \geq 0, \\
e_i^{\alpha_i^\pi(\pi)}(\pi), & \alpha_i^\pi(\pi(1)) \leq 0.
\end{cases}
\]

Given \( \lambda \in \mathcal{P} \) or \( \hat{P} \), denote by \( \pi_{\lambda} \) the linear path \( \tau \mapsto \pi_\lambda \). One can easily see from the definitions in [21] that \( \epsilon_i(\pi_{\lambda}) = \max\{0, -\alpha_i^\lambda(\lambda)\} \) and \( \varphi_i(\pi_{\lambda}) = \max\{0, \alpha_i^\lambda(\lambda)\} \).

**Lemma.** For all \( \lambda \in \mathcal{P} \) or \( \hat{P} \), \( s_i \pi_{\lambda} = \pi_{s_i \lambda} \). In particular, if \( B \) is a subcrystal of \( \mathcal{P} \) or \( \hat{P} \) and \( \pi_{\lambda} \in B \) for some \( \lambda \in \mathcal{P} \) or \( \hat{P} \) then \( \pi_{w \lambda} \in B \) for all \( w \in \hat{W} \).

**Proof.** The second assertion is an immediate corollary of the first one which in turn follows from the formulae

\[
f_i^n \pi_{\lambda} = \begin{cases} 
s_i \lambda \tau, & \tau \in [0, \frac{n}{\alpha_i^\lambda(\lambda)}] \\
\lambda \tau - n \alpha_i, & \tau \in [\frac{n}{\alpha_i^\lambda(\lambda)}, 1],
\end{cases}
\]

\[
e_i^n \pi_{\lambda} = \begin{cases} 
\lambda \tau, & \tau \in \left[0, 1 - \frac{n}{\alpha_i^\lambda(\lambda)}\right] \\
\tau \lambda + (\alpha_i^\lambda(\lambda) - n) \alpha_i, & \tau \in \left[1 - \frac{n}{\alpha_i^\lambda(\lambda)}, 1\right],
\end{cases}
\]

These can be deduced easily from the formulae in [5.1] by induction on \( n \).

### 5.3.
Given \( \lambda \in \mathcal{P} \) or \( \hat{P} \) and \( \mu \), \( \nu \in \hat{W} \lambda \), write, following [22], \( \nu \geq \mu \) if there exist a sequence \( \{\nu_0 = \nu, \nu_1, \ldots, \nu_s = \mu\} \), \( \nu_i \in \mathcal{P} \) or \( \hat{P} \) and positive real roots \( \beta_1, \ldots, \beta_s \) of \( \hat{g} \) such that

\[
\nu_i = s_{\beta_i}(\nu_{i-1}), \quad \beta_i^\nu(\nu_{i-1}) < 0, \quad i = 1, \ldots, s.
\]

If \( \nu \geq \mu \), let \( \dist(\nu, \mu) \) be the maximal length of such a sequence.
Let \( \nu = \{ \nu_1, \ldots, \nu_r \} \) be a sequence of elements of \( \hat{W} \lambda \) and \( a = \{ a_0 = 0 < a_1 < \cdots < a_r = 1 \} \) be a sequence of rational numbers. Denote by \( \pi_{\nu,a} \) the piece-wise linear path
\[
\pi_{\nu,a}(\tau) = \sum_{i=1}^{j-1} (a_i - a_{i-1}) \nu_i + (t - a_{j-1}) \nu_j, \quad \tau \in [a_{j-1}, a_j].
\]
(5.1)

In other words, it is a concatenation of straight lines joining \( \lambda_{j-1} \) and \( \lambda_j, j = 0, \ldots, r \), where \( \lambda_j = \sum_{i=1}^{j} (a_i - a_{i-1}) \nu_i \).

**Definition**. Fix \( \lambda \in P \) or \( \hat{P} \). A path of the form \( \pi_{\nu,a} \), where \( \nu = \{ \nu_1 \geq \cdots \geq \nu_r \} \), \( \nu_i \in \hat{W} \lambda \) and \( a = \{ a_0 = 0 < a_1 < \cdots < a_r = 1 \} \) is called a *Lakshmibai-Seshadri (LS) path of class \( \lambda \)* if, for all \( 1 \leq i \leq r-1 \), either \( \nu_i = \nu_{i+1} \) or there exists a sequence \( \lambda_{0,i} = \nu_i > \lambda_{1,i} > \cdots > \lambda_{s,i} = \nu_{i+1} \), \( \lambda_{j,i} \in \hat{W} \lambda \) such that
\[
\lambda_{j,i} = s_{\beta_{j,i}}(\lambda_{j-1,i}), \quad a_i \beta_{j,i}(\lambda_{j-1,i}) \in \mathbb{N}, \quad \text{dist}(\lambda_{j-1,i}, \lambda_{j,i}) = 1,
\]
for some real positive roots \( \beta_{j,i} \).

It is known (cf. [22] Lemma 4.5i) that an LS-path \( \pi = \pi_{\nu,a} \) of class \( \lambda \) is an element of \( P \) or \( \hat{P} \) and has the integrality property, that is, the maximal value attained by the function \( h^i_n \) on \([0,1] \) is an integer for all \( i \in \mathbb{N} \). Moreover, by [22] Lemma 4.5j all local maxima of \( h^i_n \) are integers.

**5.4.** Given a collection \( \pi_1, \ldots, \pi_k \) of paths in \( P \) or \( \hat{P} \), define their concatenation
\[
(\pi_1 \oplus \cdots \oplus \pi_k)(\tau) = \sum_{1 \leq s < j} \pi_s(1) + \pi_1((\tau - \sigma_j-1)/(\sigma_j - \sigma_j-1)), \quad \tau \in [\sigma_j-1, \sigma_j]
\]
for some \( 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{k-1} < \sigma_k = 1, \sigma_j \in \mathbb{Q} \). This definition does not depend on the \( \sigma_j \), up to a reparametrisation. By [22] 2.6, the concatenation of paths satisfies Kashiwara’s tensor product rule.

Let \( \pi \in P \) or \( \hat{P} \) be an LS-path and define the path \( n\pi \) by \( (n\pi)(\tau) = n\pi(\tau), \forall \tau \in [0,1] \). Evidently, \( \epsilon_i(n\pi) = n\epsilon_i(\pi), \varphi_i(n\pi) = n\varphi_i(\pi) \), \( i \in \mathbb{N} \) and \( wt n\pi = wt \pi \). Let \( S_n \) be the map \( \pi \mapsto n\pi \). Then by [22] Lemma 2.4, \( S_n(\epsilon_i) = \epsilon_i^n S_n(\pi) \) and \( S_n(f_j) = f_j^n S_n(\pi) \), \( j \in \mathbb{N} \). Observe that, for a linear path \( \pi_\lambda \), \( S_n(\pi_\lambda) = \pi_{n\lambda} = \pi_\lambda \).

**5.5.** Fix \( \lambda \in P \) (respectively, \( \lambda \in \hat{P} \)) and let \( B(\lambda, P) \) (respectively, \( B(\lambda, \hat{P}) \)) be the subcrystal of \( P \) (respectively, \( \hat{P} \)) generated over the monoid \( M \) (cf. [23]) by the linear path \( \pi_\lambda \). Henceforth we write \( B(\lambda) \) for \( B(\lambda, P) \). Then by [22] Corollary 2 of Proposition 4.7 all elements of \( B(\lambda) \) or \( B(\lambda, \hat{P}) \) are LS-paths of class \( \lambda \). Suppose further that \( B(\lambda) \) is a finite set. Then there exists \( N \in \mathbb{N}^+ \) and \( a = \{ 0 = a_0 < a_1 < \cdots < a_N = 1 \} \) such that every element of \( B(\lambda) \) can be represented as \( \pi_{\nu,a} \) for some sequence of weights \( \nu = \{ \nu_1 \geq \cdots \geq \nu_N \} \). Observe that \( \pi_{\nu',a'}(\tau) = \pi_{\nu,a}(\tau) \), for all \( \tau \in [0,1] \), where
\[
a_j' = a_j, \quad \nu_j' = \nu_j, \quad j = 0, \ldots, r
\]
\[
a_{j+1} = x, \quad \nu_{j+1} = \nu_{j+1},
\]
\[
a_j' = a_{j-1}, \quad \nu_j' = \nu_{j-1}, \quad j = r + 2, \ldots, N + 1
\]
for any \( 0 \leq r < N \) and for any rational \( x, a_r < x < a_{r+1} \). Therefore we may assume, without loss of generality, that \( a_j = j/N \) and in that case we omit \( a \).
Lemma. (i) $S_N(\pi_{\nu}) = \pi_{\nu_1} \otimes \cdots \otimes \pi_{\nu_N}$. In particular, $S_N$ can be viewed as an injective map $B(\lambda) \to B(\lambda)^{\otimes N}$.
(ii) If $e_i \pi_{\nu} \neq 0$ then $e_i \pi_{\nu} = \pi_{\nu'}$ with
\[ \nu' = \left\{ \nu_1, \ldots, \nu_k, s_i(\nu_{k+1}), \ldots, s_i(\nu_l), \nu_{l+1}, \ldots, \nu_N \right\} \]
where $k = Ne_i^j(\pi_{\nu})$ and $l = Ne_i^s(\pi_{\nu})$. Similarly, if $f_i \pi_{\nu} \neq 0$ then $f_i \pi_{\nu} = \pi_{\nu''}$ with
\[ \nu'' = \left\{ \nu_1, \ldots, \nu_r, s_i(\nu_{r+1}), \ldots, s_i(\nu_s), \nu_{s+1}, \ldots, \nu_N \right\}, \]
where $r = Nf_i^j(\pi_{\nu})$ and $s = Nf_i^s(\pi_{\nu})$.
(iii) Let $k$, $l$, $r$ and $s$ be as above. Then
\[ \sum_{j=l+1}^{s} \alpha_i'(\nu_j) = -N, \quad \sum_{j=k+1}^{l} \alpha_i'(\nu_j) = N. \]

Proof. Let $\pi = \pi_{\nu}$. The first part follows immediately from (ii) and \ref{thm:crystal} with $\sigma_j = j/N$, $j = 0, \ldots, N$. In order to prove (iii) observe that, by the choice of $a$, there exist $0 \leq k < l \leq N$ such that $k/N = e_i^j(\pi)$ whilst $l/N = e_i^s(\pi)$. Then
\[ 1 = h_n^l(e_i^j(\pi)) - h_n^j(e_i^s(\pi)) = -\frac{1}{N} \sum_{r=k+1}^{l} \alpha_i'(\nu_r), \]
by \ref{thm:crystal}. The second formula in (iii) is proved in the same way. Furthermore, for $\tau \in [(j-1)/N, j/N]$ one has by \ref{thm:crystal}
\[ (e_i \pi)(\tau) = \frac{1}{N} \sum_{r=1}^{j-1} \nu_r + (\tau - (j-1)/N)\nu_j, \quad 1 \leq j \leq k \]
\[ (e_i \pi)(\tau) = \frac{1}{N} \sum_{r=1}^{j-1} s_i\nu_r + (\tau - (j-1)/N)s_i\nu_j + \pi(k/N) - s_i\pi(k/N) \]
\[ = \frac{1}{N} \sum_{r=1}^{k} \nu_r + \frac{1}{N} \sum_{r=k+1}^{j} s_i\nu_r + (\tau - (j-1)/N)s_i\nu_j, \quad k \leq j \leq l \]
\[ (e_i \pi)(\tau) = \frac{1}{N} \sum_{r=1}^{j-1} \nu_r + (\tau - (j-1)/N)\nu_j + \alpha_i, \quad l \leq j \leq N. \]

It is now obvious that $(e_i \pi)(\tau) = \pi_{\nu'}(\tau)$, $0 \leq \tau \leq l/N$. Finally, observe that $\alpha_i = -\frac{1}{N} \sum_{r=k+1}^{l} \alpha_i'(\nu_r)\alpha_i = \frac{1}{N} \sum_{r=1}^{l} (s_i\nu_r - \nu_r)$ by (iii). Thus, we can write for $\tau \in [(j-1)/N, j/N]$ and $l \leq j \leq N$
\[ (e_i \pi)(\tau) = \frac{1}{N} \left( \sum_{1 \leq r \leq k, \ l \leq l \leq N} \nu_r + \sum_{1 \leq r \leq k, \ l \leq l \leq N} s_i\nu_r \right) + (\tau - (j-1)/N)\nu_j = \pi_{\nu'}(\tau). \]

The second formula in (ii) for the action of $f_i$ is proved in a similar way. \qed

5.6. Let $\xi : \hat{\mathbb{P}} \to \mathbb{P}$ be the canonical projection. Define the map $\Xi : \hat{\mathbb{P}} \to \mathbb{P}$ by $(\Xi \pi)(\tau) = \xi(\pi(\tau))$, for all $\tau \in [0, 1]$.

Lemma. The map $\Xi$ is a morphism of crystals and $\Xi(B(\lambda, \hat{\mathbb{P}})) = B(\xi(\lambda))$. 

Proof. Since $\alpha^y_i(\delta) = 0$ for all $i \in \hat{I}$, we conclude that $h_{-\pi_i}(\tau) = h_{\pi_i}(\tau)$ for all $i \in \hat{I}$ and for all $\tau \in [0,1]$. For the same reason, one has $\xi(s_i(\pi(\tau))) = s_i((\xi\pi)(\tau))$. It is now obvious that $\Xi$ commutes with the operators defined in 5.1. In order to prove the second assertion, observe that $\Xi\pi = \pi\xi(\lambda)$. □

5.7. It was shown in [25, Theorem 1.1] that $B(\pi, \hat{P})$ is isomorphic to the crystal basis $\hat{B}(\pi, 1)$ of the simple integrable $\hat{U_q}$-module $\hat{V}(\pi, 1)$. Thus, for all $\pi \in B(\pi, \hat{P})$ there exists $x \in M$ such that $\pi = x\pi_{\infty}$. Following [25, 5.3], one introduces a translation operator $z$ on $B(\pi, \hat{P})$ by $zx = x\pi_{\infty} + \delta$. It follows that $(z\pi)(\tau) = \pi(\tau) + \tau\delta$ for all $\tau \in [0,1]$. By [25, Proposition 5.8], $B(\pi, \hat{P})/\sim$, where $\sim$ is an equivalence relation on $B(\pi, \hat{P})$ defined by $\pi \sim \pi'$ if and only if $\pi = z^k\pi''$ for some $k \in \mathbb{Z}$, is a crystal isomorphic to the crystal basis $B(\pi, 1)$ of the finite dimensional simple $U_q$-module $V(\pi, 1)$. Evidently, $B(\pi, \hat{P})/\sim$ is isomorphic to $\Xi(B(\pi, 1)) = B(\pi) \subset \hat{P}$ in the notations of [25].

Thus, $B(\pi, 1)$ is isomorphic to $B(\pi, 1)$ as a crystal. In particular, $B(\pi, 1)$ is finite, $B(\pi) \hat{\otimes}^m$ is indecomposable as a crystal for all $m > 0$ (hence $B(\pi) \hat{\otimes}^m = B(m\pi)$) and there exists a unique map $\chi : B(\pi) \hat{\otimes}^2 \to \mathbb{Z}$ satisfying $\chi(\pi_{\infty} \otimes \pi_{\infty}) = 0$ and the properties listed in Lemma 5.2.

We will now construct an injective map $\psi$ from $\hat{B}(\pi) \hat{\otimes}^m$ into $\hat{P}$. Given an arbitrary collection of weights $\lambda = \{\lambda_0, \ldots, \lambda_K\}$, where $\lambda_0 = 0$ and $\lambda_j \in P \otimes \mathbb{Q}$ or $\hat{P} \otimes \mathbb{Q}$, and a collection of rational numbers $a = \{a_0 = 0 < a_1 < \cdots < a_K = 1\}$ denote by $p_{\lambda, a}$ the path

$$ p_{\lambda, a}(\tau) = \lambda_j - (\lambda_j - \lambda_{j-1}) \left( \frac{\tau - a_{j-1}}{a_j - a_{j-1}} \right), \quad \tau \in [a_{j-1}, a_j]. $$

In other words, $p_{\lambda, a}$ is a concatenation of straight lines joining $\lambda_j$ with $\lambda_j$, $j = 1, \ldots, K$. As before, we omit $a$ if $a_j = j/K$.

Let $b = b_1 \otimes \cdots \otimes b_m$ be an element of $B(\pi) \hat{\otimes}^m$ and choose $N \in \mathbb{N}^+$ as in 5.2. Actually, it is sufficient to take as $N$ the least common multiple of coefficients of $a_i^y$ in all co-roots of $\mathfrak{g}$. Then $b_k = \mu_{\nu_k}$ with $\nu_k = \{w_1^k \pi_{\infty} \geq \cdots \geq w_N^k \pi_{\infty}\}$ for some $w_j^k \in \hat{W}$. Set $\nu_{rN+s} = w_{r+s} \pi_{\infty}$, $r = 0, \ldots, m-1$, $s = 1, \ldots, N$. The element $b$ then corresponds to the linear path $p_{\lambda} \in \hat{P}$ with $\lambda = \{\lambda_0, 0, \lambda_1, \ldots, \lambda_N\}$ where $\lambda_j = \frac{1}{N} \sum_{k=1}^N \nu_k$. On the other hand, using the map $5.2$ and the associativity of the tensor product of crystals, we associate with $b$ the element $T_N(b) := \pi_1 \otimes \cdots \otimes \pi_N \in B(\pi) \hat{\otimes}^{Nm}$, where $\pi_r := \pi_{\nu_r}$. Set $\text{Maj}_\lambda(T_N(b)) = \sum_{r=1}^{Nm-1} r\chi(\pi_r \otimes \pi_{r+1})$. This expression is defined since $\nu_r \in \hat{W}\pi_{\infty}$ for all $r = 1, \ldots, Nm$ and so $\pi_r \in B(\pi_{\infty})$ by Lemma 5.2.

For all $n \in \mathbb{Z}$ we associate with $b \otimes t^n \in B(\pi) \hat{\otimes}^m$ a path $p_{\lambda(n)} \in \hat{P}$, where $\lambda(n) = \{\lambda_0 = 0, \lambda_1, \ldots, \lambda_N\}, \lambda_j = \lambda_j + \kappa_j(b \otimes t^n)\delta$ and

$$ \kappa_j(b \otimes t^n) = \frac{j}{Nm} \left( \frac{1}{N} \text{Maj}_\lambda(T_N(b)) + n \right) - \frac{1}{N} \sum_{s=1}^{j-1} s\chi(\pi_s \otimes \pi_{s+1}) - \frac{j}{N} \sum_{s=j}^{Nm-1} \chi(\pi_s \otimes \pi_{s+1}). $$

Proposition. The map $\psi : B(\pi) \hat{\otimes}^m \to \hat{P}$ sending $p_{\lambda} \otimes t^n$ to $p_{\lambda(n)}$ is an injective morphism of crystals.
Proof. The injectivity is obvious. Let \( b = p_\lambda \). By the definition of affinisation, wt \( b \otimes t^n = wt b + n\delta = \lambda_N m + n\delta \). On the other hand, wt \( \psi (b \otimes t^n) = \lambda_N m + \kappa_{N m} (b \otimes t^n) \delta \) and

\[
\kappa_{N m} (b \otimes t^n) = \frac{1}{N} \text{Maj}_N(T_N(b)) + n - \frac{1}{N} \sum_{s=1}^{N-1} s\chi(\pi_s \otimes \pi_{s+1}) = n,
\]

hence \( wt b \otimes t^n = wt \psi (b \otimes t^n) \). Furthermore, observe that by construction \( \Xi \mu_{\lambda(n)} = p_\lambda \). Then it follows from Lemma 5.6 that \( \varepsilon_j \) and hence \( \varphi_j \) commute with \( \psi \) for all \( j \in \hat{I} \).

Since both \( B(\overline{\nu_j}) \otimes m \) and \( \hat{\Pi} \) are normal, it remains to prove that \( \psi (e_j (b \otimes t^n)) = \psi (e_j (b \otimes t^{n+\delta_j, 0})) = e_j \psi (b \otimes t^n) \) provided that \( e_j b \neq 0 \).

By the choice of \( N \), there exist \( 0 \leq k < l \leq N m \) such that \( e_j (b) = k/N m \) and \( e_j (b) = l/N m \). As above, we conclude immediately that \( e_j (\psi (b \otimes t^n)) = k/N m \) and \( e_j (\psi (b \otimes t^n)) = l/N m \). Set \( p_\mu = e_j b \) and \( p_\lambda = e_j \psi (b \otimes t^n) = e_j \psi (b \otimes t^n) \). We aim to prove that \( \hat{\mu} (n + \delta_j, 0) = \hat{\lambda}' \). Using the formulae from 5.1 and the definition of \( \psi \) we obtain

\[
\hat{\mu}_s = \begin{cases} 
\lambda_s + K_s \delta, & 0 \leq s \leq k \\
\lambda_s + \alpha_j (\lambda_k - \lambda_s) \xi (\alpha_j) + K_s \delta, & k \leq s \leq l \\
\lambda_s + \xi (\alpha_j) + K_s \delta, & l \leq s \leq N m,
\end{cases}
\]

where \( K_s := \kappa (e_j b \otimes t^{n+\delta_j, 0}) \). On the other hand,

\[
\hat{\lambda}' = \begin{cases} 
\lambda_s + K_s \delta, & 0 \leq s \leq k \\
\lambda_s + \alpha_j (\lambda_k - \lambda_s) \alpha_j + K_s \delta, & k \leq s \leq l \\
\lambda_s + \alpha_j + K_s \delta, & l \leq s \leq N m,
\end{cases}
\]

where \( K_s := \kappa (b \otimes t^n) \). Since \( \alpha_j = \delta_j, 0 + \xi (\alpha_j) \), \( j \in \hat{I} \), the above formulae imply that \( \hat{\mu} (n + \delta_j, 0) = \hat{\lambda}' \) if and only if

\[
\begin{align*}
K_s' &= K_s, & 0 \leq s \leq k \\
K_s' &= K_s + \delta_j, 0, & l \leq s \leq N m.
\end{align*}
\]

Write \( T_N (e_j b) = \pi_1' \otimes \cdots \otimes \pi_{N m}' \). By Lemma 5.3, \( \pi_r' = \pi_r, 1 \leq r \leq k \) or \( l < r \leq N m \) whilst \( \pi_r' = s_j \pi_r = \pi_{s_j \pi_r}, r = k + 1, \ldots, l \). Then

\[
K_s' - K_s = \frac{s}{N m} \left( \frac{1}{N} (\text{Maj}_N (T_N (e_j b)) - \text{Maj}_N (T_N (b))) + \delta_j, 0 \right) - \frac{1}{N} \sum_{r=1}^{s-1} r (\chi (\pi_r \otimes \pi_{r+1}') - \chi (\pi_r \otimes \pi_{r+1})) - \frac{s}{N} \sum_{r=s}^{N m - 1} (\chi (\pi_r \otimes \pi_{r+1}') - \chi (\pi_r \otimes \pi_{r+1}))
\]

Evidently, \( \chi (\pi_r \otimes \pi_{r+1}') = \chi (\pi_r \otimes \pi_{r+1}) \) for all \( 1 \leq r < k \) and for all \( l < r < N m \). The crucial point in our argument is the following.
Claim. One has
\[
\chi(\pi_k \otimes \pi_{k+1}) = \chi(\pi_k \otimes s_j \pi_{k+1}) = \chi(\pi_k \otimes \pi_{k+1}) - \alpha_j^\vee(\nu_{k+1})\delta_{j,0} = 1 \quad (5.6)
\]
\[
\chi(\pi_r \otimes \pi_{r+1}^\prime) = \chi(s_j \pi_r \otimes s_j \pi_{r+1}) = \chi(\pi_r \otimes \pi_{r+1}) + \alpha_j^\vee(\nu_r - \nu_{r+1})\delta_{j,0}, \quad k < r < l \quad (5.7)
\]
\[
\chi(\pi_r^\prime \otimes \pi_{r+1}^\prime) = \chi(s_j \pi_r \otimes \pi_{r+1}) + \alpha_j^\vee(\nu_r)\delta_{j,0} = 1 \quad (5.8)
\]

Before we establish the claim, let us prove that \(5.2\) - \(5.4\) follow from \(5.6\) and \(5.8\). Observe first that for \(j \neq 0\), the above formulae imply that \(\chi(\pi_r \otimes \pi_{r+1}^\prime) = \chi(\pi_r \otimes \pi_{r+1})\) for all \(1 \leq r < Nm\), so in that case there is nothing to prove. Furthermore, suppose that \(j = 0\) and \(1 \leq l < Nm\). Then

\[
\text{Maj}_\chi(T_N(e_0 b)) - \text{Maj}_\chi(T_N(b)) = \sum_{r=k+1}^{l} r\alpha_0^\vee(\nu_r) - \sum_{r=k}^{l-1} r\alpha_0^\vee(\nu_{r+1}) = -N \quad (5.9)
\]

by the choice of \(k\) and \(l\), whence by \(5.6\) - \(5.8\)

\[
K_s' - K_s = \frac{1}{N} \min\{s, l\} \sum_{r=k}^{l} r(\chi(\pi_r \otimes \pi_{r+1}) - \chi(\pi_r^\prime \otimes \pi_{r+1}^\prime)) + \frac{s}{N} \sum_{r=\max\{s, k\}}^{l} (\chi(\pi_r \otimes \pi_{r+1}) - \chi(\pi_r^\prime \otimes \pi_{r+1}^\prime)) \quad (5.10)
\]

For \(s = 1, \ldots, k\) the first sum is empty whilst the second sum reduces to

\[
\sum_{r=k}^{l-1} a_0^\vee(\nu_{r+1}) - \sum_{r=k+1}^{l} a_0^\vee(\nu_r) = 0.
\]

Thus, \(K_s' = K_s\), \(s = 1, \ldots, k\). Furthermore, for \(s = k+1, \ldots, l-1\) we get

\[
K_s' - K_s = \frac{1}{N} \left( \sum_{r=k}^{s-1} r\alpha_0^\vee(\nu_{r+1}) - \sum_{r=k+1}^{s} r\alpha_0^\vee(\nu_r) \right) + \frac{s}{N} \left( \sum_{r=s}^{l-1} a_0^\vee(\nu_{r+1}) - \sum_{r=s}^{l} a_0^\vee(\nu_r) \right) = \frac{1}{N} \sum_{r=k+1}^{s} a_0^\vee(\nu_r) = \alpha_0^\vee(\lambda_k - \lambda_s).
\]

Finally, for \(s = l, \ldots, Nm - 1\) the second sum in \(5.10\) vanishes and so

\[
K_s' - K_s = \frac{1}{N} \left( \sum_{r=k}^{l-1} r\alpha_0^\vee(\nu_{r+1}) - \sum_{r=k+1}^{l} r\alpha_0^\vee(\nu_r) \right) = -\frac{1}{N} \sum_{r=k+1}^{l} a_0^\vee(\nu_r) = 1
\]

by \(5.7\).

Similarly, for \(l = Nm\) we get

\[
\text{Maj}_\chi(T_N(e_0 b)) - \text{Maj}_\chi(T_N(b)) = \sum_{r=k+1}^{l} a_0^\vee(\nu_r) - l\alpha_0^\vee(\nu_l) = -N - Nm\alpha_0^\vee(\nu_l),
\]
whence

\[ K'_s - K_s = -\frac{s}{N} \alpha_0^\vee (\nu_{Nm}) + \frac{1}{N} \sum_{r=k}^{s-1} r(\chi(\pi_r \otimes \pi_{r+1}) - \chi(\pi'_r \otimes \pi'_{r+1})) + \frac{s}{N} \sum_{r=\max\{s,k\}}^{Nm-1} \left( \chi(\pi_r \otimes \pi_{r+1}) - \chi(\pi'_r \otimes \pi'_{r+1}) \right). \]

For \( s = 1, \ldots, k \) the first sum in (5.11) is empty and so

\[ K'_s - K_s = -\frac{s}{N} \alpha_0^\vee (\nu_{Nm}) + \frac{s}{N} \sum_{r=k}^{Nm-1} \alpha_0^\vee (\nu_r) = 0. \]

Finally, for \( s = k + 1, \ldots, Nm \),

\[ K'_s - K_s = -\frac{1}{N} \sum_{r=k+1}^{s} \alpha_0^\vee (\nu_r) = \alpha_0^\vee (\lambda_k - \lambda_s), \]

as required.

It remains to prove the claim. By Kashiwara’s tensor product rule, \( e_j b = b_1 \otimes \cdots \otimes e_j b_p \otimes \cdots \otimes b_m \) for some \( 1 \leq p \leq m \). Since \( b_p \) is an LS-path, the function \( h^j_{bp} \) is strictly increasing on the interval \( [e^j_-(b_p), e^j_+(b_p)] = [k'/N, l'/N] \), where \( k' = k - (p-1)N, \) \( l' = l - (p-1)N \) and \( 0 \leq k' < l' < N \) by [22] Proposition 4.7(b) (recall that the definition of crystal operators on \( \hat{B} \) we use differs by the sign of \( h^j \) from that of \( \hat{B} \)). Therefore, \( \alpha_j^\vee (\nu_r) < 0, \) \( r = k + 1, \ldots, l \). It follows from 5.2 that \( 0 = \varphi_j(\pi_r) < \varepsilon_j(\pi_r) = -\alpha_j^\vee (\nu_r) \). Then, by Kashiwara’s tensor product rule

\[ e^{-\alpha_j^\vee (\nu_{r+1})}(\pi_r \otimes \pi_{r+1}) = \pi_r \otimes e^{-\alpha_j^\vee (\nu_{r+1})} = \pi_r \otimes s_j \pi_{r+1}, \quad k < r < l, \]

where we used 5.2. Therefore, \( \chi(\pi_r \otimes s_j \pi_{r+1}) = \chi(\pi_r \otimes \pi_{r+1}) - \delta_{j,0} \alpha_j^\vee (\nu_{r+1}) \) by Lemma 4.4. Furthermore, since \( e_j(\pi_{r+1}) = \varepsilon_j(\pi_{r+1}) + \alpha_j^\vee (\nu_{r+1}) = 0, \)

\[ e_j^{-\alpha_j^\vee (\nu_r)}(\pi_r \otimes s_j \pi_{r+1}) = e_j^{-\alpha_j^\vee (\nu_r)} \pi_r \otimes s_j \pi_{r+1} = s_j \pi_r \otimes s_j \pi_{r+1}. \]

Then \( \chi(s_j \pi_r \otimes s_j \pi_{r+1}) = \chi(\pi_r \otimes s_j \pi_{r+1}) + \delta_{j,0} \alpha_j^\vee (\nu_r) \) by Lemma 4.4, whence 5.7.

Suppose that \( k > 0 \). Since all local maxima of \( h^j_\tau(\tau) \) are integers, it follows by the choice of \( k \) and \( l \) that \( h^j_\tau(\tau) \leq h^j_\tau(k/Nm) \) for \( \tau \leq k/Nm \). Then \( \alpha_j^\vee (\nu_k) \leq 0, \)
\( \text{whence } 0 = \varphi_j(\pi_k) < \varepsilon_j(\pi_k) = -\alpha_j^\vee (\nu_k + 1). \) Using Kashiwara’s tensor product rule, we obtain

\[ e_j^{-\alpha_j^\vee (\nu_{k+1})}(\pi_k \otimes \pi_{k+1}) = \pi_k \otimes e_j^{-\alpha_j^\vee (\nu_{k+1})} = \pi_k \otimes s_j \pi_{k+1}, \]

and 5.6 follows by Lemma 4.4. Finally, suppose that \( l < Nm \). Since \( h^j_\tau \) reaches its local maximum at \( l/Nm \), we conclude that \( \alpha_j^\vee (\nu_{l+1}) \geq 0, \)
\( \text{whence } \varphi_j(\pi_l) = 0 = \varepsilon_j(\pi_{l+1}). \) Therefore,

\[ e_j^{-\alpha_j^\vee (\nu_l)}(\pi_l \otimes \pi_{l+1}) = e_j^{-\alpha_j^\vee (\nu_l)} = s_j \pi_l \otimes \pi_{l+1}, \]

which yields 5.6 by Lemma 4.4. \( \square \)

**Corollary.** The associated crystal of a \( \zeta \)-crystal basis \( \hat{B}(\varpi_{i:\lambda}) \) of the integrable \( \hat{U}_q \)-module \( \hat{V}(\varpi_{i:\lambda}) \) is isomorphic to \( \hat{V}(\varpi_{i:\lambda}) \odot m \).
5.8. Let us compute $\psi(\pi_{m,0} \otimes t^n)$, $n \in \mathbb{Z}$. Recall that $\chi(\pi_{m,0} \otimes \pi_{m,0}) = 0$. Since, evidently, $T_N(\pi_{m,0} \otimes \pi_{m,0}) = \pi_{m,Nm}$ we conclude that $\operatorname{Maj}_x(T_N(\pi_{m,0} \otimes \pi_{m,0})) = 0$ and so $\kappa_\pi(\pi_{m,0} \otimes t^n) = sn/Nm$. Thus, $\psi(\pi_{m,0} \otimes t^n) = \pi_{m,n+t}$. In particular, $B(m\pi_1 + n\delta, \hat{\mathbb{P}})$, $n \in \mathbb{Z}$ is an indecomposable subcrystal of $\psi(B(\pi_1)^{\otimes m})$.

**Proposition.** The image of $B(\pi_1)^{\otimes m}$ in $\hat{\mathbb{P}}$ under $\psi$ is a disjoint union of indecomposable crystals $B(m\pi_1 + n\delta, \hat{\mathbb{P}})$, $n = 0, \ldots, m - 1$.

**Proof.** By [14, Theorem 1.1], a crystal basis of $\hat{V}(\pi_1)$ is isomorphic to $B(\pi_1, \hat{\mathbb{P}})$ which is indecomposable, being generated by the linear path $\pi_1$, $\in \hat{\mathbb{P}}$. On the other hand, $B(\pi_1)$ is isomorphic to the affinisation $B(\pi_1, \hat{\mathbb{P}})$ of the crystal basis $B(\pi_1)$ of $V(\pi_1)$ which in turn is isomorphic to the affinisation of $B(\pi_1) \subset \mathbb{P}$. We conclude that $B(\pi_1)$ is indecomposable as a crystal. Thus there exists a monomial $x \in \mathcal{M}$ such that $x(\pi_{m,0} \otimes 1) = \pi_{m,0} \otimes t$.

Set $\deg x_j = \delta_{j,0}$ and $\deg f_j = -\delta_{j,0}$. That defines a grading on $\mathcal{M}$. Since $x(x(\pi_{m,0} \otimes 1) = x(\pi_{m,0} \otimes t^{\deg x})$ by the definition of affinisation, it follows that $\deg x = 1$ and $x(\pi_{m,0}) = \pi_{m,0}$. Write $x = x_{j_1} \cdots x_{j_t}$, where $x_{j_r}$ is either $e_{j_r}$ or $f_{j_r}$ and $j_r \in \hat{\mathbb{I}}$ and set $x^{(m)} = x_{j_1} \cdots x_{j_t}$. Evidently, $\deg x^{(m)} = m$. We claim that $x^{(m)}(\pi_{m,0} \otimes \pi_{m,0}) = \pi_{m,0} \otimes \pi_{m,0}$. Indeed, $\pi_{m,0} = \pi_{m,0}(\pi_{m,0})$ and so $x^{(m)}(\pi_{m,0}) = \pi_{m,0}(\pi_{m,0})$. It remains to use induction on $k$.

Since $\deg x^{(m)} = m$ it follows that $x^{(m)}(\pi_{m,0} \otimes t^k) = \pi_{m,0} \otimes t^{k+m}$ for all $k \in \mathbb{Z}$. On the other hand, let $y = x_{j_1} \cdots x_{j_k}$, where $x_{j_r} = e_{j_r}$ or $f_{j_r}$, and vice versa. Evidently, $\deg y = -1$. Since $\pi_{m,0} = \pi_{m,0}$ and $e_i$, $f_i$ are pseudo-inverses of each other, it follows that $y^{(m)}(\pi_{m,0} \otimes t^k) = \pi_{m,0} \otimes t^{k-m}$ for all $k \in \mathbb{Z}$. Therefore, $\pi_{m,0}(\pi_{m,0}) = \pi_{m,0}(\pi_{m,0})$ if $r = s$ (mod $m$).

Since $B(\pi_1)^{\otimes m}$ is indecomposable, for all $b \in B(\pi_1)^{\otimes m}$ there exists a monomial $x \in \mathcal{M}$ such that $b = x^{\pi_{m,0} \otimes m}$. Then for all $k \in \mathbb{Z}$, $b \otimes t^k = x^{(\pi_{m,0} \otimes t^{\deg x})}$.

It follows that $\psi(b \otimes t^k) \in B(m\pi_1 + n\delta, \hat{\mathbb{P}})$ for some $n \in \mathbb{Z}$, that is $\psi(B(\pi_1)^{\otimes m}) = \bigcup_{n=0}^{m-1} B(m\pi_1 + n\delta, \hat{\mathbb{P}})$.

Since the crystals $B(m\pi_1 + n\delta, \hat{\mathbb{P}})$ are indecomposable, it remains to prove that $B(m\pi_1 + r\delta, \hat{\mathbb{P}}) \neq B(m\pi_1 + s\delta, \hat{\mathbb{P}})$, $r \neq s$ (mod $m$). For, observe that by the proof of Lemma [14, Lemma 2] $\operatorname{Maj}_x(xb) = \operatorname{Maj}_x(b) - \deg x$ (mod $m$) provided that $xb \neq 0$, $x \in \mathcal{M}$. Therefore, the set $C_s := \{ b \otimes t^k : b \in B(\pi_1)^{\otimes m}, k \in \mathbb{Z}, \operatorname{Maj}_x(b) + s = (mod m) \}$ is a subcrystal of $B(\pi_1)^{\otimes m}$ and $B(\pi_1)^{\otimes m} = \bigsqcup_{s=0}^{m-1} C_s$. Moreover, since $\operatorname{Maj}_x(\pi_{m,0}) = 0$, $\pi_{m,0} \otimes t^r \in C_s$ if and only if $r = s$ (mod $m$). Thus, $\psi(C_s)$ contains an element of $B(m\pi_1 + r\delta, \hat{\mathbb{P}})$ if and only if $r = s$ (mod $m$). It follows that $\psi(C_s) = B(m\pi_1 + s\delta)$.

**Corollary** (Theorem 2). The associated crystal of $\zeta$-crystal basis $B(\pi_1)^{(k)}$ of the simple $U_q$-module $V(\pi_1)^{(k)}$ is isomorphic to $B(m\pi_1 + k\delta, \hat{\mathbb{P}})$ and hence indecomposable.

**Proof.** By Theorem 2 B(\pi_1)^{(k)} = \{ b \otimes t^m : b \in B(\pi_1), n \in \mathbb{Z}, \operatorname{Maj}_x(b) + n = k \ (mod m) \}$. Since the associated crystal of $B(\pi_1)^{(m)}$ is isomorphic to $B(\pi_1)^{(m)}$
by \textsuperscript{25} and Proposition \textsuperscript{14} we conclude that $\hat{B}(\varpi_{i,m})^{(k)}$ is isomorphic to $C_k$ in the notation of the proof of the above proposition. □

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