Jackson’s type estimate of nearly coconvex approximation

G.A. Dzyubenko

Yu.A. Mitropolsky International Mathematical Center of NASU, Tereschenkivska str. 3, 01601 Kyiv-4, Ukraine
dzyuben@gmail.com

Abstract

Suppose that a continuous on the real axis $2\pi$-periodic function $f$ changes its convexity at $2s$, $s \in \mathbb{N}$, points $y_i$ on each period: $-\pi \leq y_{2s} < y_{2s-1} < \ldots < y_1 < \pi$, and for the rest $i \in \mathbb{Z}$, the points $y_i$ are defined periodically. In the paper, for each $n \geq N$, a trigonometric polynomial $P_n$ of order $cn$ is found such that:

$$P_n \text{ has the same convexity as } f, \text{ everywhere except, perhaps, the small neighborhoods of the } y_i : (y_i - \pi/n, y_i + \pi/n)$$

and

$$\|f - P_n\| \leq c(s) \omega_4(f, \pi/n),$$

where $N$ is a constant depending only on $\min_{i=1,\ldots,2s} \{y_i - y_{i+1}\}$, $c$ and $c(s)$ are constants depending only on $s$, $\omega_4(f, \cdot)$ is the modulus of continuity of the 4-th order of the function $f$, and $\| \cdot \|$ is the max-norm.

1 Introduction and the main theorem

By $C$ we denote the space of continuous $2\pi$-periodic functions $f : \mathbb{R} \to \mathbb{R}$ with the uniform norm

$$\|f\| = \max_{x \in \mathbb{R}} |f(x)|,$$

and by $\mathbb{T}_n$, $n \in \mathbb{N}$, denote the space of trigonometric polynomials

$$P_n(x) = a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx), \quad a_j \in \mathbb{R}, b_j \in \mathbb{R},$$

of degree $\leq n$. Recall the classical Jackson-Zygmund-Akhiezer-Stechkin estimate (obtained by Jackson for $k = 1$, Zygmund and Akhiezer for $k = 2$, and Stechkin for $k \geq 3$, $k \in \mathbb{N}$): if a function $f \in C$, then for each $n \in \mathbb{N}$ there is a polynomial $P_n \in \mathbb{T}_n$ such that

$$\|f - P_n\| \leq c(k) \omega_k (f, \pi/n),$$

where $c(k)$ is a constant depending only on $k$, and $\omega_k (f, \cdot)$ is the modulus of continuity of order $k$ of the function $f$. For details, see, for example, [2].
In 1968 Lorentz and Zeller [8] for \( k = 1 \) obtained a bell-shaped analogue of the inequality (1.1), i.e., when bell-shaped (even and nonincreasing on \([0, \pi]\)) \( 2\pi \)-periodic functions are approximated by bell-shaped polynomials.

In papers [10] and [14] two coconvex analogues of the inequality (1.1) were proved for \( k = 2 \) and \( k = 3 \), respectively. Moreover, in [15] arguments from the papers [11], [12] of Shvedov and [1] of DeVore, Leviatan and Shevchuk were used to show that for \( k > 3 \) there is no coconvex analogue of the inequality (1.1).

Nevertheless, as we know from the coconvex approximation on a closed interval (by algebraic polynomials, see, for details [5]) if some relaxation of the condition of coconvexity for the approximating polynomials is allowed, then an extra order of the approximation can be achieved, and, as it seems, no more than one extra order, though the corresponding counterexample is not constructed yet.

So, in the paper in Theorem 1 we prove a trigonometric analogue of the algebraic result [5]. To write it we give necessary notations.

Suppose that on \([-\pi, \pi]\) there are \( 2s \), \( s \in \mathbb{N} \), fixed points \( y_i : \)

\[-\pi \leq y_{2s} < y_{2s-1} < \cdots < y_1 < \pi,\]

while for other indices \( i \in \mathbb{Z} \), the points \( y_i \) are defined periodically by the equality

\[ y_i = y_{i+2s} + 2\pi \quad \text{(i.e., } y_0 = y_{2s} + 2\pi, ..., y_{2s+1} = y_1 - 2\pi, \text{...)}\]

Denote \( Y := \{y_i\}_{i \in \mathbb{Z}} \). By \( \Delta^{(2)}(Y) \) we denote the set of all functions \( f \in C \) which are convex on \([y_1, y_0]\), concave on \([y_2, y_1]\), convex on \([y_3, y_2]\), and so on. The functions in \( \Delta^{(2)}(Y) \) are coconvex with one another. Note, if a function \( f \) is twice differentiable, then \( f \in \Delta^{(2)}(Y) \) if and only if

\[ f''(x)\Pi(x) \geq 0, \quad x \in \mathbb{R}, \]

where

\[ \Pi(x) := \Pi(x, Y) := \prod_{i=1}^{2s} \sin \frac{x - y_i}{2} \quad (\Pi(x) > 0, \quad x \in (y_1, y_0)). \]

**Theorem 1** If a function \( f \in \Delta^{(2)}(Y) \), then there exists a constant \( N(Y) \) depending only on \( \min_{i=1, \ldots, 2s} \{y_i - y_{i+1}\} \) such that for each \( n \geq N(Y) \) there is a polynomial \( P_n \in \mathbb{T}_cn \) satisfying

\[ P_n''(x)\Pi(x) \geq 0, \quad x \in \mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} (y_i - \pi/n, y_i + \pi/n), \quad (1.2) \]

and

\[ \|f - P_n\| \leq c(s) \omega_4(f, \pi/n), \quad (1.3) \]

where \( c \) and \( c(s) \) are constants depending only on \( s \).

The following Theorem 2 is a simple corollary of Theorem 1 and Whitney’s inequality [13]

\[ \|f - f(0)\| \leq 3\omega_4(f, 4\pi). \]
Theorem 2  If a function $f \in \Delta^{(2)}(Y)$, then for each $n \in \mathbb{N}$ there is a polynomial $P_n \in T_n$ such that

\begin{equation}
 P'_n(x)\Pi(x) \geq 0, \quad x \in \mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} (y_i - c/n, y_i + c/n),
\end{equation}

\begin{equation}
 \|f - P_n\| \leq C(Y) \omega_4(f, \pi/n),
\end{equation}

where $c$ is a constant depending only on $s$, and $C(Y)$ is a constant depending only on $\min_{i=1,...,2s} \{y_i - y_{i+1}\}$.

Remark 1  We believe that $\omega_4$ in (1.3) and (1.5) cannot be replaced by $\omega_k$ with $k > 4$. Also we believe that the constants $N(Y)$ and $C(Y)$ in Theorems 1 and 2 cannot be replaced by constants independent of $\min_{i=1,...,2s} \{y_i - y_{i+1}\}$ (and depending, say, on $s$). These both assumptions are not made further in the paper. Also, we do not pay attention to the constant $c$ in the both theorems, i.e., we did not try to replace it by an absolute constant or/and by a smallest possible one.

2 Auxiliary facts I

For each $n \in \mathbb{N}$ denote

$$h := h_n := \frac{\pi}{n}, \quad x_j := x_{j,n} := -j h, \quad I_j := I_{j,n} := [x_j, x_{j-1}], \quad j \in \mathbb{Z}.$$ 

Let $m \in \{1, 2, 3, 10, 20, 30\}$. For a fixed $Y = \{y_i\}_{i \in \mathbb{Z}}$ and a fixed $n$ denote

$$O_{i,m} := O_i(Y, n, m) := (x_{j+m+1}, x_{j-m}) \quad \text{if} \quad y_i \in [x_j, x_{j-1}) =: [x_{j,i}, x_{j,i-1}).$$

Set

$$O_m := O(Y, n, m) := \bigcup_{i \in \mathbb{Z}} O_{i,m}.$$ 

We will write

$$j \in H(Y, n, m) \quad \text{if} \quad x_j \subset \mathbb{R} \setminus O_m.$$ 

Let

$$H_m := \{j : j \in H(Y, n, m), \, |j| \leq n\}.$$ 

Choose $N(Y) := N(Y, 30) \in \mathbb{N}$ sufficiently large so that

\begin{equation}
 O_{i,30} \cap O_{i-1,30} = \emptyset
\end{equation}

for all $n \geq N(Y)$ and all $i = 1, ..., 2s$ (thus, $N(Y)$ depends only on $\min_{i=1,...,2s} \{y_i - y_{i+1}\}$).

In what follows $n > N(Y)$.

Denote

$$\chi(x, a) := \begin{cases} 
0, & \text{if} \quad x \leq a, \quad a \in \mathbb{R}, \\
1, & \text{if} \quad x > a,
\end{cases} \quad \chi_j(x) := \chi(x, x_j), \quad (x - x_j)_+ := (x - x_j)\chi_j(x),$$

$$\Gamma_j(x) := \Gamma_{j,n}(x) := \min \left\{ 1, \frac{1}{n \sin \frac{x - (x_j + h/2)}{2}} \right\}, \quad j \in \mathbb{Z}, \, n \in \mathbb{N}.$$
and note that

\[(2.2) \quad \left\| \sum_{j=1-n}^{n} \Gamma_{j}^{2} \right\| < 6,\]

for details, see [9].

For each \( j \in \mathbb{Z} \) and \( b \in \mathbb{N} \) we set the positive polynomial \( J_{j} \in \mathbb{T}_{(n-1)b}, \ n \in \mathbb{N}, \)

\[(2.3) \quad J_{j}(x) := J_{j,n}(x) := \left( \frac{\sin \frac{n(x-x_{j})}{2}}{\sin \frac{x-x_{j}}{2}} \right)^{2b} + \left( \frac{\sin \frac{n(x-x_{j-1})}{2}}{\sin \frac{x-x_{j-1}}{2}} \right)^{2b},\]

(i.e., the sum of two "adjacent" kernels of Jackson type).

For each \( j \in H_{10} \) denote

\[(2.4) \quad t_{j}(x) := t_{j,n}(x, b, Y) := \frac{\int_{x_{j-\pi}}^{x_{j+\pi}} J_{j}(u) \Pi(u) du}{\int_{x_{j-\pi}}^{x_{j+\pi}} J_{j}(u) \Pi(u) du}.\]

In what follows \( c_{i} = c_{i}(b) = c_{i}(s, b), \ i = 1, \ldots, 8, \) stand for positive constants which may depend only on \( s \) and \( b. \)

**Lemma 3** [7] Lemma 1]. *If \( j \in H_{10} \) and \( b \geq s + 2, \) then*

\[(2.5) \quad t'_{j}(x) \Pi(x) \Pi(x_{j}) \geq 0, \ x \in \mathbb{R},\]

\[(2.6) \quad |\chi_{j}(x) - t_{j}(x)| \leq c_{1} (\Gamma_{j}(x))^{2b-2s-1}, \ x \in [x_{j} - \pi, x_{j} + \pi],\]

\[(2.7) \quad |t'_{j}(x)| \leq c_{2} \frac{1}{h} (\Gamma_{j}(x))^{2b-2s}, \ x \in \mathbb{R},\]

\[(2.8) \quad |t'_{j}(x)| \geq c_{3} \frac{1}{h} (\Gamma_{j}(x))^{2b+2s}, \ x \in \mathbb{R} \setminus O_{10},\]

\[(2.9) \quad |t'_{j}(x)| \geq c_{3} \frac{1}{h} (\Gamma_{j}(x))^{2b+2s} \frac{|x - y_{i}|}{|x_{j} - y_{i}|}, \ x \in O_{i,10}, \ i \in \mathbb{Z}.\]

Note that Lemma 3 is proved by using the inequalities

\[(2.10) \quad \frac{1}{c_{4}h} \Gamma_{j}^{2b}(x) \left| \frac{\Pi(x)}{\Pi(x_{j})} \right| \leq |t'_{j}(x)| \leq c_{4} \frac{1}{h} \Gamma_{j}^{2b}(x) \left| \frac{\Pi(x)}{\Pi(x_{j})} \right|, \]

\[\left| \frac{\Pi(x)}{\Pi(x_{j})} \right| \leq 2^{2s} \Gamma_{j}^{-2s}(x), \ x \in \mathbb{R}, \ j \in H_{m}, \ m \geq 10,\]

\[(2.11) \quad \left| \int_{x}^{x_{j+\pi}} \Gamma_{j}^{b}(u) du \right| \leq c_{5} h \Gamma_{j}^{b-1}(x), \ b \in \mathbb{N}, \ x \in [x_{j}, x_{j} + 2\pi],\]

\[\left| \int_{x}^{x_{j-\pi}} \Gamma_{j}^{b}(u) du \right| \leq c_{5} h \Gamma_{j}^{b-1}(x), \ b \in \mathbb{N}, \ x \in [x_{j} - 2\pi, x_{j}],\]

for details, see [9].
For each $j \in H_{20}$ set the function

$$
(2.12) \quad \tau_j(x) := \tau_{j,n}(x, b, t_j) := \alpha \int_{x_{j-\pi}}^{x} t_{j+10}(u) du + (1 - \alpha) \int_{x_{j-\pi}}^{x} t_{j-10}(u) du,
$$

where the number $\alpha \in [0, 1]$ is chosen from the condition

$$
\tau_j(x_j + \pi) = \pi
$$

(note that the inequalities $0 \leq \alpha \leq 1$ follow from the estimate (2.6) and the choice of the indices $j \pm 10$ if $b \geq s + 2$, for details, see [10, p. 923]).

Note that the functions $t_j$ and $\tau_j$ can be expressed on $\mathbb{R}$ as

$$
(2.13) \quad t_j(x) = \frac{1}{2\pi} x + \tilde{R}_j(x), \quad j \in H_{10},
$$

$$
(2.14) \quad \tau_j(x) = \frac{1}{4\pi} x^2 + \frac{\pi - x_j}{2\pi} x + \tilde{R}_j(x), \quad j \in H_{20},
$$

where $\tilde{R}_j$ and $\tilde{R}_j$ are polynomials from $\mathbb{T}_{\alpha,n}$ (see similar cases in [3] and [10], respectively).

In what follows $c > 0$ denote different absolute constants or constants depending only on $s$. They can be different even if they are in the same line.

Denote two functions $\tilde{t}_j$ and $\tau_j$:

$$
\tilde{t}_j(x) := \tilde{t}_{j,n}(x, b) := \tilde{t}_j(x) + \sum_{i=1}^{2s} \frac{x_j(y_i) - \tilde{t}_j(y_i)}{\tilde{t}_j(y_i)} \hat{t}_j(x), \quad j \in H_{10},
$$

where $\tilde{t}_j(x) := t_{j,n}(x, \bar{b}, \emptyset)$ is the function defined by (2.4) with $\Pi(x) := 1$ and $\bar{b} = b + 3$, and

$$
\tilde{t}_j(x) := \tilde{t}_{j,n}(x, \bar{b}, \emptyset) := \tilde{t}_j(x) + \sum_{i=1}^{2s} \frac{y_i - x_j}{\tilde{t}_j(y_i)} \hat{t}_j(x), \quad j \in H_{20},
$$

where $\tilde{t}_j(x) := t_{j,n}(x, \bar{b}, \emptyset)$ is the function defined by (2.4) with $\Pi(x) := 1$ and $\bar{b} = b + 3$, and

$$
\hat{t}_j(x) := (\tilde{t}_{j+10}(x) - \tilde{t}_{j-10}(x)) \frac{\Pi(x, Y_i)}{\Pi(x_j, Y_i)},
$$

is the polynomial, where $j_i$ is an index $j$ such that $y_i \in [x_j, x_{j-1})$, $i = 1, \ldots, 2s$, $\tilde{t}_j(x) := t_{j,n}(x, \bar{b}, \tilde{Y}_i)$ is the function (2.4) with $\tilde{Y}_i := \{y_i - \pi \nu\}_{\nu \in \mathbb{Z}}$, and

$$
Y_i := (Y \setminus \{y_i + 2\pi \nu\}_{\nu \in \mathbb{Z}}) \cup \{y_i^* + 2\pi \nu\}_{\nu \in \mathbb{Z}},
$$

where $y_i^*$ is the left endpoint of the interval $O_{i,20}$, if $i$ is odd, and $-\pi$ the right one, if $i$ is even; and

$$
\tilde{t}_j(x) := \tilde{t}_{j,n}(x, b) := \tau_{j,n}(x, b, \tilde{t}_j) + \sum_{i=1}^{2s} \frac{y_i - x_j}{\tilde{t}_j(y_i)} \hat{t}_j(x), \quad j \in H_{20}.
$$

Note that the following Lemma 4 can be proved with the arguments similar to [6, Lemma 5.3].

**Lemma 4** [3, Lemmas 4 and 5]. For each $j \in H_{10}$ and $b \geq 3s + 2$ the function $\tilde{t}_j$ satisfies the relations (2.6), (2.13), and in addition,

$$
(2.15) \quad \left| x_j(x) - \tilde{t}_j(x) \right| \leq c_7 (\Gamma_j(x))^{2b - 2s - 1} \left| \frac{x - y_i}{x_j - y_i} \right|, \quad x \in O_{i,10}, \ i = 1, \ldots, 2s,
$$
(in particular, $\chi_j(y_i) - \tilde{\tau}_j(y_i) = 0$). For each $j \in H_2$ and $b \geq 3s + 2$ the function $\tilde{\tau}_j$ satisfies the relation (2.14), and in addition,

\begin{equation}
|(x - x_j)_+ - \tilde{\tau}_j(x)| \leq c_8 h (\Gamma_j(x))^{2(b-s-1)}, \quad x \in [x_j - \pi, x_j + \pi],
\end{equation}

\begin{equation}
(2.16)
\end{equation}

\begin{equation}
|(x - x_j)_+ - \tilde{\tau}_j(x)| \leq c_8 h (\Gamma_j(x))^{2(b-s-1)} \left| \frac{x - y_i}{x_j - y_i} \right|, \quad x \in O_{i,10}, \; i = 1, \ldots, 2s,
\end{equation}

\begin{equation}
(2.17)
\end{equation}

(in particular, $(y_i - x_j)_+ - \tilde{\tau}_j(y_i) = 0$).

3 Auxiliary facts II

Since we prove Theorem 1 using an intermediate approximation by a spline, i.e., the inequality $||f - S + S - P_n|| \leq ||f - S|| + ||S - P_n||$, we describe the $S$ in this section. Without special references we will use Whitney inequality [13]

$$|f(x) - L_3(x; a; f)| \leq \omega_4 \left(f, (b - a)/4, [a, b]\right), \quad x \in [a, b],$$

where $L_3$ is Lagrange polynomial interpolating $f$ at $a, b - \frac{b - a}{3}, b - \frac{b - a}{3}$ and $b$. Fix $j \in \mathbb{Z}$. Let

$$\Psi_3(x, x_j) := (x - x_j)_+ (x - x_j - 1)(x - x_j - 2), \; d_j := x_j - 1,$$

$$a_\nu := a_{j, \nu} := x_j \vee x_j - 1 \vee x_j - 2, \; \tilde{h}_\nu := -h \vee 0 \vee h, \; \tilde{h}_\nu := 2h^2 \vee -h^2 \vee 2h^2,$$

if $\nu = 1 \vee 2 \vee 3$ respectively. In the following $\nu \in \{1, 2, 3\}$ only.

Introduce three functions $\Psi_{j, \nu} \in \mathbb{C}$ coinciding with $\Psi_3(x, x_j)$ almost everywhere

$$\Psi_{j, \nu}(x) := \Psi_3(x, x_j) \chi(x, a_\nu) = (x - a_\nu)^2_+ + 3\tilde{h}_\nu(x - a_\nu)^2_+ + \tilde{h}_\nu(x - a_\nu)_+.$$

That is,

\begin{equation}
\Psi_{j, \nu}(x) = \Psi_3(x, x_j), \; x \in \mathbb{R} \setminus [x_j, a_\nu]; \quad |\Psi_3(x, x_j) - \Psi_{j, \nu}(x)| \leq c h^3, \; x \in [x_j, a_\nu],
\end{equation}

\begin{equation}
(3.1)
\end{equation}

\begin{equation}
\Psi_{j, \nu}(x) = \int_{d_j - \pi}^x \left( 6 \int_{d_j - \pi}^t \left( (u - a_\nu)_+ + \tilde{h}_\nu \chi(u, a_\nu) \right) du + \tilde{h}_\nu \chi(t, a_\nu) \right) dt,
\end{equation}

\begin{equation}
(3.2)
\end{equation}

and for $\nu_1, \; \nu_2 \in \{1, 2, 3\}$, we have

\begin{equation}
\frac{\Psi'_{j, \nu_1}(x) - \Psi'_{j-1, \nu_2}(x)}{3h} = 6(x - d_j) - 6(x - d_j - 1) = 2, \; x \in (\max\{a_{\nu_1}, a_{\nu_2}\}, \infty).
\end{equation}

\begin{equation}
(3.3)
\end{equation}

Without loss of generality suppose that $y_1 = x_{30}$ (i.e., points $Y$ are far from $-\pi$ and $\pi$), also recall that $H_3 \subset H_2 \subset H_1$.

Construction of the nearly coconvex cubic spline

Denote two divided differences of $f$

$$F_j := [x_j, x_{j-1}, x_{j-2}; f], \; j = 2, 1, \ldots, n,$$

$$\Phi_j := [x_{j+1}, x_j, x_{j-1}, x_{j-2}, x_{j-3}; f], \; j = 3, 2, \ldots, n - 1.$$
Remark, $\Phi_j 4h = \frac{F_{j+1} - F_j}{3h} - \frac{F_j - F_{j-1}}{3h}$.

Introduce new functions $\Psi_j(x)$, $j = 3 - n, ..., n - 1$. For each $j \in H_2$, put

$$\Psi_j(x) := \Psi_{j,2}(x) \text{ if } \Phi_j \Pi(x_j) \leq 0,$$

otherwise put

$$\Psi_j(x) := \begin{cases} 
\Psi_{j,1}(x) & \text{if } |F_{j+1}| > |F_j| \geq |F_{j-1}|
\Psi_{j,3}(x) & \text{if } |F_{j+1}| \leq |F_j| < |F_{j-1}|,
\alpha_j \Psi_{j,1}(x) + (1 - \alpha_j) \Psi_{j,3}(x) & \text{if } |F_{j+1}| > |F_j| < |F_{j-1}|,
\end{cases}$$

where $\alpha_j := \frac{F_{j+1}}{F_{j+1} + F_{j-1}} \in (0, 1)$. For other $j = 3 - n, ..., n - 1$, such that $j \not\in H_2$ (i.e., $j : x_j \in O_{i,2}$, $i = 1, ..., 2s$) put

$$\Psi_j(x) := \begin{cases} 
\Psi_{j,2}(x) & \text{if } \Phi_j \Pi(x_j, \bar{Y}_i) \leq 0, 
\Psi_{j,1}(x) & \text{otherwise},
\end{cases}$$

Set

$$\Psi_n(x) := \Psi_3(x, x_n), \quad \Psi_{2-n}(x) := 0.$$

Remark 2 For the both "strange" cases in (d.4) it is sufficient to take simply $\Psi_j(x) = \Psi_{j,2}(x)$ to have the nearly coconvexity of the spline below with $f$ however the setting (d.4) is more convenient to verify the nearly copositivity of $P_n^a$ feather.

Show that the cubic spline

$$S(x) = L_3(x; x_n; f) + 4h \sum_{j=3-n}^{n-1} \Phi_j \Psi_j(x),$$

or equivalently,

$$S(x) = L_1(x; x_n; f) + F_n \left( (x - x_n)(x - x_{n-1}) - \frac{\Psi_n(x) - \Psi_{n-1}(x)}{3h} \right) + \sum_{j=3-n}^{n-1} F_j A_j(x) + F_2 \frac{\Psi_3(x)}{3h},$$

where

$$A_j(x) := \bar{A}_j(x) - A_j(x) := \frac{\Psi_{j+1}(x) - \Psi_j(x)}{3h} - \frac{\Psi_j(x) - \Psi_{j-1}(x)}{3h},$$

(having been continued periodically) is nearly coconvex with $f$, i.e.,

$$S''(x) \Pi(x) \geq 0, \quad x \in I_j, \ j \in H_3,$$

and satisfies the inequality

$$\|f - S\| = \|f - S\|_{[-\pi, \pi]} \leq c \omega_4(f, h)$$

(it is convenient to look at the sums in (3.4) and (3.5) starting from the last addend, for the details of such a kind of representations, see [3 Proposition 1]).
With the help of (3.4) and (3.5) verify (3.6). Represent the set \([-\pi, \pi] \cap (\bigcup_{j \in H_3} I_j)\), as a union of nonintersecting intervals \([a_\mu, b_\mu], \mu = 1, \ldots, 2s + 1\), \(b_{\mu+1} < a_\mu\). Let \(j = j(\mu)\) and \(\overline{j} = \overline{j}(\mu)\) denote the indexes \(j\) such that \(x_j = a_\mu\) and \(x_{\overline{j}} = b_\mu\), respectively. For each \(\mu = 1, \ldots, 2s + 1\), set

\[
G_\mu := (d_{j+1}, d_{\overline{j}}), \quad G := \bigcup_{\mu = 1}^{2s+1} G_\mu.
\]

Without loss of any generality verify (3.6) only for one interval \(G_\mu\), i.e., fix \(\mu\), and let it be such that \(\Pi(x) > 0, x \in G_\mu\). For a conveniens let \(n > j\) and \(\overline{j} > 3 - n\), the cases \(n = j\) and \(\overline{j} = 3 - n\) are proved analogously with respecting (d.5).

Let

\[
\Pi_\mu := \{j + 1, \ldots, \overline{j}\}.
\]

Note, \(\Pi_\mu \subset H_3\). It follows from (3.1), (d.0)-(d.3) that the function \(S'\), at the points \(a_\nu\) defined separately for each \(\Psi_j\) with \(j \in \overline{H}_\mu\), satisfies the inequality

\[
S'(a_\nu) - S'(a_\nu) \leq S'(a_\nu).
\]

Note, \(F_j \geq 0\) for \(j \in \{j + 2, \ldots, \overline{j} - 1\} =: \overline{H}_\mu \subset H_1\). Therefore, in particular, it follows from the inequalities \(F_{j+1} \leq F_j \geq F_{j-1}\) that

\[
\Phi_j \Pi(x_j) \leq 0, \quad j \in \overline{H}_\mu.
\]

Taking this into account, remark that in \(\overline{H}_\mu\) there is not any \(j\) for which, in according with the definitions (d.0)-(d.4), the following settings where made

\[
\Psi_j = \Psi_{j,3} \quad \text{and} \quad \Psi_{j-1} = \Psi_{j-1,1},
\]

as well as the settings like

\[
\Psi_{j+1} = \Psi_{j+1,3} \quad \text{and} \quad \Psi_j = \alpha_j \Psi_{j,1} + (1 - \alpha_j) \Psi_{j,3} \quad \text{and} \quad \Psi_{j-1} = \Psi_{j-1,1}.
\]

By the other words,

\[
a_\nu \text{ (defined for } \Psi_j) \leq a_\nu \text{ (defined for } \Psi_{j-1}).
\]

From this and (3.3) note,

\[
A_j''(x) = 0, \quad x \notin (a_{j+1}, a_{j-1}],
\]

where \(a_j := a_1\) and \(a_3 := a_3\) if (d.3) otherwise \(a_j = a_{j+1}\). Denote \(a_\nu\) defined for \(\Psi_j\) by (d.0)-(d.2) or (d.4) (if \(a_{j+1} = a_{j-1}\) then \((a_{j+1}, a_{j-1}] := \emptyset\)).

Using the equality \(A_{j+1} = A_j\), extract from (3.5) four addends involving the function \(\Psi_j\)

\[
- F_{j+1} A_j(x) + F_j A_j(x) - F_j A_j(x) + F_{j-1} A_j(x).
\]

Taking into account (3.9)-(3.12), fix \(j \in \overline{H}_\mu\), and show that

\[
S''(x) \geq 0, \quad \begin{cases} x \in (a_1, a_3) & \text{if (d.0)}, \\ x \in (a_1, a_2) & \text{if (d.1)}, \\ x \in (a_2, a_3) & \text{if (d.2)}, \\ x \in (a_1, a_3) & \text{if (d.3)}, \end{cases}
\]

\[c.0\]
\[c.1\]
\[c.2\]
\[c.3\]
Only these three points \(a_1\), \(a_2\) and \(a_3\) will take part in the sentences below.

We start from the case (c.1). Describe it on \((a_1,a_2)\). The function \(\Psi_{j+1}\) can be stated by (d.0) or (d.4) or (d.1) only, whereas \(\Psi_{j-1}\) is any of the four by (d.0)-(d.4). Anyway, \(\Psi''_{j+1} = 6(x-a_1)\), \(\Psi''_j = 6(x-a_2)\) and \(\Psi''_{j-1} = 0\). Hence, by virtue of (17), write

\[
F_{j+1} 2 - F_{j+1} 2 + \frac{6(x-a_2)}{x_{j-3} - x_j} + F_{j-1} \frac{6(x-a_2)}{x_{j-3} - x_j} \geq 0, \quad x \in (a_1,a_2),
\]

since \(F_j \geq F_{j-1}\).

In the case (c.2) \(\Psi_{j+1}\) is any of the four potential settings, whereas \(\Psi_{j-1}\) is defined by (d.0) or (d.4) or (d.2) only, but always

\[
\overline{A}''_j(x) = 2 + \frac{6(x-a_2)}{x_{j-2} - x_{j+1}} \quad \text{and} \quad \underline{A}''_j(x) = 0 \quad \text{for} \quad x \in (a_2,a_3],
\]

where we used (17) in the first equality. Thus,

\[
F_{j+1} 2 - F_{j+1} 2 + \frac{6(x-a_2)}{x_{j-2} - x_{j+1}} + F_j \left(2 + \frac{6(x-a_2)}{x_{j-2} - x_{j+1}}\right) \geq 0, \quad x \in (a_2,a_3],
\]

since \(F_{j+1} \leq F_j\).

To see (c.3) note that \(\Psi_{j+1}\) and \(\Psi_{j-1}\) are defined by (d.0) or (d.4) or (d.1) and by (d.0) or (d.4) or (d.2), respectively. Anyway, \(\Psi''_{j+1}(x) = 6(x-a_1)\), \(\Psi''_j(x) = \alpha_j 6(x-a_2)\) and \(\Psi''_{j-1}(x) = 0\) for \(x \in (a_1,a_3]\). Write

\[
F_{j+1} 2 - F_{j+1} \frac{6(x-a_1) - \alpha_j 6(x-a_2)}{3h} + F_j \frac{6(x-a_1) - \alpha_j 6(x-a_2)}{3h} - F_j \frac{\alpha_j 6(x-a_2)}{3h} + F_{j-1} \frac{\alpha_j 6(x-a_2)}{3h}
\]

\[
= F_j \left(2 - (1-\alpha_j) \frac{6(x-a_1) - 6(x-a_2) - 6(x-a_1)}{3h} - \alpha_j \frac{6(x-a_2) - 6(x-a_3) + 6(x-a_3)}{3h}\right)
\]

\[
= F_j \left(2 - (1-\alpha_j) 2 + (1-\alpha_j) \frac{2(x-a_1)}{h} - \alpha_j 2 - \alpha_j \frac{2(x-a_3)}{h}\right) \geq 0, \quad x \in (a_1,a_3].
\]

For the last case (c.0) note that \(\Psi_{j+1}\) can both be any of the four potential settings but it’s sufficient to verify this case only when \(\Psi_{j+1} = \Psi_{j+1,2}\) and \(\Psi_{j-1} = \Psi_{j-1,2}\) because for the other settings the positivity of \(S''\) on \((a_1,a_2) \cup (a_2,a_3]\) is guaranteed by just considered three cases, namely, on \((a_1,a_2] - \text{by (c.2) or (c.3)},\) and on \((a_2,a_3] - \text{by (c.1) or (c.3)}.\) So, for \(x \in (a_1,a_3]\) we have

\[
\overline{A}'_j(x) = \frac{6(x-a_1) - 6(x-a_2)}{3h} \quad \text{and} \quad \underline{A}_j''(x) = \frac{6(x-a_2)}{3h},
\]

\]
that together with \((17)\) yield
\[
F_{j+1} - F_{j+1} A''(x) + F_j A''(x) - F_j A''(x) + F_{j-1} A''(x) \geq 0.
\]
The inequalities \((c.0)-(c.3)\) are proved.

Finally, since the intervals in \((c.0)-(c.3)\) cover all \(G_\mu\) if \(j\) runs through \(H_\mu\), then
\[
(3.13) \quad S''(x) = \sum_{j \in H_\mu} F_j A''(x) \geq 0, \quad x \in G_\mu,
\]
that together with \((3.8)\) leads to \((3.6)\).

To prove \((3.7)\) we need the estimate
\[
(3.14) \quad |\Phi_j| \leq c \frac{\omega_4(f, h)}{h^4},
\]
see, for example, in \([2]\), \((3.1)\) and the technical spline
\[
s(x) = L_3(x; x_n; t) + 4h \sum_{j=3-n}^{n-1} \Phi_j \Psi_3(x, x_j),
\]
that interpolates \(f\) without restrictions by cubic parabolas in each \(x_j\), see, \([4]\). Now let \(x \in [x_{j^*+1}, x_{j^*-3}]\), then it follows from \((3.4)\) that
\[
|f(x) - S(x)| = |f(x) - s(x) + s(x) + S(x)|
\leq c \omega_4(f, h) + 4h \sum_{j=3-n}^{n-1} |\Phi_j| |\Psi_3(x, x_j) - \Psi_j(x)|
\leq c \omega_4(f, h) + \sum_{j=\min\{n-1, j^*-3\}}^{\max\{3-n, j^*-3\}} |\Phi_j| 4h |\Psi_3(x, x_j) - \Psi_j(x)|
\leq c \omega_4(f, h)
\]
and therefore \((3.7)\) is correct.

4 Proof of Theorem 1

Denote the numbers
\[
b_1 := s + 2, \quad b_2 := 3(s + 1), \quad c_9 := \max \left\{ \frac{6(2\pi)^2 b_2 \max\{c_1(b_2), c_7(b_2)\} + c_8(b_2) + 2}{3c_3(b_1)}, 2 \right\},
\]
\[
n_1 := 2[c_9 + 1] n, \quad h_1 := h_{n_1},
\]
\[
c_{10} := \max \left\{ c_5(b_2) \left( \frac{c_8(b_2)}{2c_9} + c_1(b_2) \right), 10 \right\}, \quad n_2 := 2[c_{10} + 1] n_1, \quad h_2 := h_{n_2},
\]
where \([\cdot]\) stands for the integer part.

Fix \(j = 3-n, \ldots, n-1\). For each point \(a_\nu, \nu = 1, 2, 3\), let \(j_\nu\) denotes the index such that \(x_{j_\nu} := x_{j_\nu, n_1} = a_\nu\), and let \(j^{*}_\nu\) denotes the index such that \(x_{j^{*}_\nu} := x_{j^{*}_\nu, n_2} = x_{j_\nu} (= x_{j_\nu, n_1})\).
Let \( j \in H_3 \). For each \( j, \nu = 1, 2, 3 \), we take
\[
\bar{\tau}_{j,\nu}(x) = \bar{\tau}_{j,\nu,n_2}(x, b_2), \quad \bar{t}_{j,\nu}(x) = \bar{t}_{j,\nu,n_2}(x, b_2),
\]
and put
\[
\varphi_{j,\nu}(x) := 6 \int_{d_j-\pi}^{d_j+\pi} \left( \bar{\tau}_{j,\nu}(u) + \bar{h}_\nu \left( \alpha_\nu \bar{t}_{(j,\nu+1),\nu}(u) + (1 - \alpha_\nu) \bar{t}_{(j,\nu-1),\nu}(u) \right) \right) du, \quad \nu = 1, 3,
\]
\[
\varphi_{j,2}(x) := 6 \int_{d_j-\pi}^{d_j+\pi} \left( \bar{\tau}_{j,2}(u) - \frac{1}{12} h^2 \left( \alpha_2 \bar{t}_{(j,2+5),\nu}(u) + (1 - \alpha_2) \bar{t}_{(j,2-5),\nu}(u) \right) \right) du,
\]
where \( \alpha_\nu \in [0, 1], \nu = 1, 2, 3 \), can be chosen such that
\[
(4.1) \quad \varphi_{j,\nu}(d_j + \pi) = 3(\pi + h)(\pi - h), \quad \nu = 1, 3, \quad \varphi_{j,2}(d_j + \pi) = 3\pi^2 - \pi h^2 / 2.
\]
Indeed, for example, using (2.16), (2.3) for \( \bar{t}_{j,\nu} \), and (2.11), we, for a fixed \( j, \nu = 3 \) and \( \alpha_3 = 1 \), have the estimate
\[
\varphi_{j,3}(d_j + \pi) = 6 \int_{d_j-\pi}^{d_j+\pi} \left( \bar{\tau}_{j,3}(u) - (u - a_3)_+ + h \left( \bar{t}_{(j,3+1),\nu}(u) - \chi(u, x_{j,3+1}) \right) \right) du + 6 \int_{d_j-\pi}^{d_j+\pi} ((u - a_3)_+ + h \chi(u, a_3)) du \geq 3(\pi^2 - h^2) + 6h h_1
\]
\[
- 6 \int_{d_j-\pi}^{d_j+\pi} \left( \bar{\tau}_{j,3}(u) - (u - a_3)_+ + h \left( \bar{t}_{(j,3+1),\nu}(u) - \chi(u, x_{j,3+1}) \right) \right) du \geq 3(\pi^2 - h^2)
\]
\[
+ 6h h_1 - 6c_8(b_2)h_2 \int_{d_j-\pi}^{d_j+\pi} \Gamma_{(j,3+1),n_2}^{2(b_2-1)}(u) du - 6c_1(b_2)h \int_{d_j-\pi}^{d_j+\pi} \Gamma_{(j,3+1),n_2}(u) du \geq 3(\pi^2 - h^2) + 6h h_1 - 6c_5(b_2) \left( c_8(b_2)h_2^2 + c_1(b_2)h h_2 \right) > 3(\pi^2 - h^2),
\]
whereas for \( \alpha_3 = 0 \) we (again due to \( h_1 >> h_2 \)) analogously have the opposite inequality
\[
\varphi_{j,3}(d_j + \pi) < 3(\pi^2 - h^2). \quad \text{So, (4.1) is proved for } \nu = 3, \text{ and for } \nu = 1, 2 \text{ it can be proved by analogy.}
\]
Now, we take
\[
t_{j,\nu}(x) = t_{j,\nu,n_2}(x, b_2, Y), \quad t_{j,\nu}(x) = t_{j,\nu,n_1}(x, b_1, Y),
\]
and put
\[
\psi_{j,\nu}(x) := \int_{d_j-\pi}^{d_j+\pi} \left[ \varphi_{j,\nu}(u) + \hat{h}_\nu \left( \beta_\nu t_{(j,\nu+1),\nu}(u) + t_{j,\nu}(u) + (1 - \beta_\nu) t_{(j,\nu-1),\nu}(u) \right) \right] du,
\]
where \( \hat{h}_\nu := h^2 \vee -h^2 / 4 \vee h^2, \nu = 1, 2, 3, \) respectively.
Lemma 5 If a fixed $j$ belongs to $H_2$, then $\beta_\nu \in [0, 1], \ \nu = 1, 2, 3,$ can be chosen such that

\[(4.2) \quad \psi_{j,\nu}(d_j + \pi) = (\pi + h)\pi(\pi - h),\]

and then three functions $\psi_{j,\nu}$ satisfy the inequalities

\[(4.3) \quad (\psi_{j,\nu}''(x) - \Psi_{j,\nu}''(x)) \Pi(x) \Pi(x_j) \geq 0, \quad \nu = 1, 3, \quad x \in [-\pi, \pi],\]

\[(4.4) \quad |\Psi_{j,\nu}(x) - \psi_{j,\nu}(x)| \leq ch_j^3 \Gamma_{j,n}^6(x), \quad \nu = 1, 2, 3, \quad x \in [-\pi, \pi].\]

In addition,

\[(4.5) \quad \psi_{j,\nu}(x) = \frac{1}{8\pi} x^4 + \frac{\pi - d_j}{2\pi} x^3 + \frac{5d_j^2 - 6d_j\pi - h^2}{4\pi} x^2 + \frac{(\pi - d_j)(5d_j^2 - 2\pi^2 - h^2)}{2\pi} x + Q_{j(\nu)}(x), \quad \nu = 1, 2, 3,\]

where $Q_{j(\nu)} \in \mathbb{T}_{cn}.$

Proof. The relations [(4.2)–(4.4)] can be proved with the arguments similar to proving [(4.1), or [1]] Lemma 5, or [3] Lemma 6, using the choice of $n_1$ and $n_2,$ and the inequalities $\Gamma_{j_1,\nu_1,\nu_2}(x) < \Gamma_{j_2,\nu_1,\nu_2}(x) < 2\pi \Gamma_{j_1,\nu_1,\nu_2}(x) < 2\pi \Gamma_{j_1,\nu_1,\nu_2}(x), \quad x \in \mathbb{R}.$ We will calculate here the presentation [(4.5)] only, with $\nu = 1,$ for definiteness. By [(2.13) and (2.14)] write

\[
\tilde{r}_{j_1}(x) = \frac{1}{2\pi} x + \hat{R}_{j_1}(x), \quad \tilde{\tau}_{j_1}(x) = \frac{1}{4\pi} x^2 + \frac{\pi - x_j}{2\pi} x + \hat{R}_{j_1}(x),
\]

\[
\hat{r}_{j_1}(x) := \hat{R}_{j_1}(x) - \hat{R}_{j_1,0}, \quad \tilde{r}_{j_1}(x) := \tilde{R}_{j_1}(x) - \hat{R}_{j_1,0},
\]

where $\hat{R}_{j_1,0}$ and $\tilde{R}_{j_1,0}$ are free terms of polynomials $\hat{R}_{j_1}, \tilde{R}_{j_1} \in \mathbb{T}_{cn},$ respectively. Then

\[
\varphi_{j,1}(x) = \left(\frac{1}{2\pi} x^3 + \frac{3(\pi - x_j)}{2\pi} x^2 + 6\hat{R}_{j_1,0} x\right) - \left(...(d_j - \pi)\right)
\]

\[-6h\left(\frac{1}{4\pi} x^2 + \left(\alpha_1 \hat{R}_{(j_1 + 1)^*,0} + (1 - \alpha_1) \hat{R}_{(j_1 - 1)^*,0}\right) x\right) + 6h\left(...(d_j - \pi)\right)
\]

\[+6 \int_{d_j - \pi}^{x} \left(\tilde{r}_{j_1}(u) - h(\alpha_1 \hat{r}_{(j_1 + 1)^*,0} + (1 - \alpha_1) \hat{r}_{(j_1 - 1)^*,0}(u))\right) du
\]

\[= \frac{1}{2\pi} x^3 + \frac{3(\pi - d_j)}{2\pi} x^2 + 6Ax \left(\frac{1}{2\pi} (d_j - \pi)^3 + \frac{3(\pi - d_j)}{2\pi} (d_j - \pi)^2 + 6A(d_j - \pi)^2\right) + q_{j_1}(x),\]

where

\[A := \hat{R}_{j_1,0} - h\left(\alpha_1 \hat{R}_{(j_1 + 1)^*,0} + (1 - \alpha_1) \hat{R}_{(j_1 - 1)^*,0}\right),\]

and $q_{j_1} \in \mathbb{T}_{cn}$ does not have a free term. Taking this and [(4.1)] we derive the value of $A$

\[3(\pi^2 - h^2) = \frac{1}{2\pi} ((d_j + \pi)^3 - (d_j - \pi)^3) + \frac{3(\pi - d_j)}{2\pi} ((d_j + \pi)^2 - (d_j - \pi)^2) + 12\pi A\]
and so,

\[ \varphi_{j,1}(x) = \frac{1}{2\pi} x^3 + \frac{3(\pi - d_j)}{2\pi} x^2 + \frac{5d_j^2 - 6d_j\pi - 3h^2}{2\pi} x + \frac{(\pi - d_j)(3d_j^2 - 2d_j\pi - 2\pi^2 - 3h^2)}{2\pi} + q_{j,1}(x). \]

Having this, (2.13) and (4.2) we get (4.5) analogously. Lemma 5 is proved.

\[
\Rightarrow A = \frac{5d_j^2 - 6d_j\pi - 3h^2}{12\pi},
\]

Construction of the nearly coconvex polynomial

For each \( j = 3 - n, n - 1 \) introduce the polynomial \( \psi_j(x) \in \mathbb{T}_{cn} \). If \( j \in H_2 \) then set

\[ \psi_j(x) := \psi_{j,2}(x) \quad \text{if} \quad \Phi_j \Pi(x_j) \leq 0, \]

otherwise set

\[
\psi_j(x) := \begin{cases} 
\psi_{j,1}(x) & \text{if} \quad |F_{j+1}| > |F_j| \geq |F_{j-1}|, \\
\psi_{j,3}(x) & \text{if} \quad |F_{j+1}| \leq |F_j| < |F_{j-1}|, \\
\alpha_j \psi_{j,1}(x) + (1 - \alpha_j) \psi_{j,3}(x) & \text{if} \quad |F_{j+1}| > |F_j| < |F_{j-1}|. 
\end{cases}
\]

If \( j \notin H_2 \) (i.e., \( j : x_j \in O_i, 2, i = 1, \ldots, 2s \)) then let

\[ \psi_j(x) := \begin{cases} 
\psi_{j,2}(x) \quad \text{if} \quad \Phi_j \Pi(x_j, \tilde{Y}_i) \leq 0, \\
\psi_{j,1}(x) \quad \text{otherwise}. 
\end{cases} \]

Now, put

\[ P_n(x) = L_3(x; x_n; f) + 4h \sum_{j=3-n}^{n-1} \Phi_j \psi_j(x). \]

The fact that \( P_n \) is a polynomial from \( \mathbb{T}_{cn} \) can be directly verified arithmetically like in [14], or [3], using (4.5), i.e., all the arithmetical terms in (4.5), having been evaluated in the sum (4) together with the corresponding divided differences, including the \( L_3 \), are equal 0.

Verify (1.2). Remark that Lemma 5 will be used in two senses: in an “ordinary” one for \( j \in H_2 \) = \( H(n, Y, 2) \), and for \( j \notin H_2 \) in the sense that \( j \in H(n, \tilde{Y}, 2) \). So, (4.3), (3.2), (3.4), (3.5) and (3.13) imply

\[
P''_n(x) \Pi(x) = \left( L''_3(x; x_n; f) + 4h \sum_{j=3-n}^{n-1} \Phi_j \left( \psi''_j(x) - \Psi''_j(x) \right) + 4h \sum_{j=3-n}^{n-1} \Phi_j \Psi''_j(x) \right) \Pi(x)
\]

\[
= 4h \sum_{j \in H_2} \frac{1}{\Pi^2(x_j)} \Phi_j \Pi(x_j) \left( \psi''_j(x) - \Psi''_j(x) \right) \Pi(x) \Pi(x_j)
\]

\[
+ 4h \sum_{i=1}^{2s} \sum_{j: x_j \in O_i, 2} \frac{1}{\Pi^2(x_j, \tilde{Y}_i)} \Phi_j \Pi(x_j, \tilde{Y}_i) \left( \psi''_{j,2\nu_1}(x) - \Psi''_{j,2\nu_1}(x) \right) \Pi(x, Y) \Pi(x_j, \tilde{Y}_i)
\]
\( + \left( F_n \left( 2 - \Psi''_n(x) - \Psi''_{n-1}(x) \right) \right) + \sum_{j=3-n}^{n-1} F_j A''_j(x) + F_2 \frac{\Psi''_n(x)}{3h} \) \Pi(x) =: A(x) + B(x) + C(x),

\[
A(x) \geq 0, \quad x \in \mathbb{R},
\]
\[
B(x) \geq 0, \quad x \in \mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} (x_{i+5}, y_i),
\]
\[
C(x) \geq 0, \quad x \in G \text{ on all periods},
\]

that leads to (1.2). To prove (1.3) we use (3.7), (3.14), (4.4) and (2.2). Namely,

\[
\|f - P_n\| = \|f - S + S - P_n\| = \left\| f - S + \sum_{j=3-n}^{n-1} \Phi_j 4h \left( \Psi_j(\cdot) - \psi_j(\cdot) \right) \right\|_{[-\pi, \pi]} \leq c \omega_4(f, h) + c \left\| \sum_{j=3-n}^{n-1} \omega_4(f, h) \Gamma_j^6(\cdot) \right\|_{[-\pi, \pi]} \leq c \omega_4(f, h).
\]

Theorem 1 is proved. \(\square\)

References

[1] R.A. DeVore, D. Leviatan, I.A. Shevchuk, Approximation of monotone functions: a counter example, Curves and surfaces with applications in CAGD (Chamonix-Mont-Blanc, 1996) 95-102. Vanderbilt Univ. Press, Nashville, TN, 1997.
[2] V.K. Dzyadyk, I.A. Shevchuk, Theory of Uniform Approximation of Functions by Polynomials, Walter de Gruyter, Berlin and New York, 2008.
[3] G. A. Dzyubenko, Nearly comonotone approximation of periodic functions, to appear.
[4] G. A. Dzyubenko, J. Gilewicz, Nearly coconvex pointwise approximation, East J. Approxim. 6 (2000), 357-383.
[5] G. A. Dzyubenko, J. Gilewicz, Nearly coconvex pointwise approximation by cubic splines and polynomials, East J. Approxim. 12 (2006), No. 4, 417-439.
[6] G. A. Dzyubenko, J. Gilewicz, I. A. Shevchuk, Piecewise monotone pointwise approximation, Constr. Approx., 14 (1998), 311-348.
[7] G.A. Dzyubenko, M.G. Pleshakov, Comonotone Approximation of Periodic Functions, Mat. Zametki 83 (2008), no. 2, 199-209; Engl. transl. in Math. Notes 83 (2008), 180-189.
[8] G.G. Lorentz, K.L. Zeller, Degree of Approximation by Monotone Polynomials I, J. Approx. Theory 1 (1968), 501-504.
[9] M.G. Pleshakov, Comonotone Jackson’s Inequality, J. Approx. Theory 99 (1999), 409-421.
[10] P.A. Popov, An analogue of Jeckson inequality for coconvex approximation of periodic functions, Ukr. Matem. Zhurn., 53 (2001), 919-928; Engl. transl. in Ukrainian Math. J. 53 (2001), 1093-1105.
[11] A. S. Shvedov, Comonotone approximation of functions by polynomials, Dokl. Akad. Nauk SSSR 250 (1980), no. 1, 39-42 (Russian).
[12] A. S. Shvedov, Orders of coapproximations of functions by algebraic polynomials, Mat. Zametki 29 (1981), no. 1, 117-130; Engl. transl. in Math. Notes 29 (1981).
[13] H. Whitney, On Functions with Bounded n-th Differences, J. Math. Pures Appl. 6(9) (1957), 36, 67-95.
[14] V.D. Zalizko, Coconvex approximation of periodic functions, Ukr. Matem. Zhurn., 59 (2007), No. 1, 29-42; Engl. transl. in Ukrainian Math. J. 59 (2003), 28-44.
[15] V.D. Zalizko, A counterexample for coconvex approximation of periodic functions, in Candidate’s Dissertation. Kyiv: Inst. of Math. of NASU, 1997.