Convergence of vertex-reinforced jump processes to an extension of the supersymmetric hyperbolic nonlinear sigma model

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Abstract

In this paper, we define an extension of the supersymmetric hyperbolic nonlinear sigma model introduced by Zirnbauer. We show that it arises as a weak joint limit of a time-changed version introduced by Sabot and Tarrès of the vertex-reinforced jump process. It describes the asymptotics of rescaled crossing numbers, rescaled fluctuations of local times, asymptotic local times on a logarithmic scale, endpoints of paths, and last exit trees.

1 Introduction and results

1.1 Extension of the susy hyperbolic nonlinear sigma model

The supersymmetric hyperbolic nonlinear sigma model This model, called $H^{2/2}$ model for short, was introduced by Zirnbauer in [Zir91]. Concerning its original motivation, Zirnbauer writes that it may serve as a toy model for studying diffusion and localization in disordered one-electron systems. The $H^{2/2}$ model is a statistical mechanics type model defined over a finite undirected graph $G = (V, E)$. Any undirected edge $\{i, j\} \in E$ is given a weight $W_{ij} = W_{ji} > 0$. In its original form, which is not used in this paper, the “spin variables” at any vertex take their value in a supermanifold $H^{2/2}$ having the hyperbolic plane $H^2$ as its base manifold. Written in so-called “horospherical coordinates”, the model associates to any vertex $i \in V$ two kinds of “spin variables”: two real-valued variables $s_i$ and $u_i$ and two Grassmann (anticommuting) variables $\psi_i$ and $\bar{\psi}_i$.

In the description with Grassmann variables, the model has useful supersymmetries as is shown in the paper [DSZ10] by Disertori, Spencer, and Zirnbauer; note that $u_i$ is called $t_i$ in that paper.

However, in the current paper we use an equivalent purely probabilistic description of the $H^{2/2}$ model where the Grassmann variables $\psi_i$ and $\bar{\psi}_i$ are replaced by a discrete

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variable \( T' \) taking values in the set \( \mathcal{T} \) of undirected spanning trees of \( G \). Any undirected spanning tree is viewed as a set of undirected edges. The tree variant of the \( H^{2|2} \) model has the disadvantage that the supersymmetries become hidden, but the advantage that it is phrased solely in probabilistic terms. It is defined as follows. Given a fixed reference point \( i_0 \in V \), the vectors \( s = (s_i)_{i \in V} \) and \( u = (u_i)_{i \in V} \) take values in the set
\[
\Omega_{i_0} = \{ u \in \mathbb{R}^V : u_{i_0} = 0 \}. \tag{1.1}
\]

**Definition 1.1 (Tree version of the \( H^{2|2} \) model)** The tree version of the supersymmetric hyperbolic nonlinear sigma model is the following probability measure on \( \Omega_{i_0}^2 \times \mathcal{T} \):
\[
\mu^{\text{susy}}_{i_0}(ds \, du \, dT') = \exp \left( \sum_{\{i,j\} \in E} W_{ij} \left( 1 - \cosh(u_i - u_j) - \frac{1}{2} e^{u_i + u_j} (s_i - s_j)^2 \right) \right) \cdot \prod_{\{i,j\} \in T'} W_{ij} e^{u_i + u_j} \prod_{i \in V \setminus \{i_0\}} e^{-u_i} ds_i \, du_i \cdot dT', \tag{1.2}
\]
where \( ds_i \) and \( du_i \) denote the Lebesgue measure on \( \mathbb{R} \), and \( dT' \) means the counting measure on \( \mathcal{T} \).

It is a non-trivial fact that \( \mu^{\text{susy}}_{i_0} \) is a probability measure, i.e.
\[
\mu^{\text{susy}}_{i_0}(\Omega_{i_0}^2 \times \mathcal{T}) = 1. \tag{1.3}
\]

For the version of the \( H^{2|2} \) model with Grassmann variables the corresponding fact is stated in formula (5.1) and proved in Proposition 2 in appendix C of [DSZ10]; its proof heavily uses the supersymmetry of the model. Note that the reference point \( i_0 \) is not mentioned explicitly in [DSZ10], but the pinning strengths \( \varepsilon_i \) in that paper play the role of the weights \( W_{i_0 i} \) connecting any vertex \( i \in V \setminus \{i_0\} \) to the reference vertex \( i_0 \), while the coupling constants \( \beta J_{ij} \) play the role of all other weights \( W_{ij}, \{i, j\} \in E \) with \( i, j \neq i_0 \).

There are at least two other proofs of the normalization of \( \mu^{\text{susy}}_{i_0} \) that do not use supersymmetry. In the paper [ST15] of Sabot and Tarrès, the normalization comes from the fact that the marginal in \( u \) of the measure is interpreted as a probability distribution for a random variable associated to asymptotic behavior of the vertex-reinforced jump process. In a more recent paper [STZ15a], Sabot, Tarrès, and Zeng give another proof of the normalization using an interpretation of \( u \) through the Green’s function of random Schrödinger operators.

In the present paper, we deal also with the finite time behavior of the jump process, which makes other variables such as \( s \) and \( T' \) appear naturally.

The link between the \( H^{2|2} \) model in horospherical coordinates and its tree version, used in the current paper, is given by the matrix tree theorem stated in formula (2.17) in [DSZ10]. More precisely, the sum \( \sum_{T' \in \mathcal{T}} \prod_{\{i,j\} \in T'} W_{ij} e^{u_i + u_j} \) arises from the matrix tree theorem as the same determinant that occurs when integrating out the Grassmann variables \( \bar{\psi}_i \) and \( \psi_i \). A variant of this argument concerning only the marginal of \( u \) with respect to \( \mu^{\text{susy}}_{i_0} \) is also described in [DSZ10]; see formula (1.4) in that paper for the statement.
Aim of this paper  The main goal of the present paper is to give an interpretation of all random variables $s, u, T'$, jointly distributed according to $\mu_{i_0}^{\text{susy}}$, in terms of limits of vertex-reinforced jump processes. For linearly edge-reinforced random walks, which are processes in discrete time, a similar asymptotic analysis was given in [KR00]; see also Theorem 3.2 in [MR06]. However, due to continuous time, the analysis in the current paper requires additional considerations, in particular, when dealing with local times and their fluctuations. In the present setup, even more random variables than only $s, u, T'$ occur naturally: $v = (v_i)_{i \in V} \in \mathbb{R}^V$, $i_1 \in V$, $i'_1 \in V$, a second spanning tree $T'$ playing a similar role as $T'$, and two vectors $\kappa$ and $\kappa'$. More precisely, one may view $\kappa$ and $\kappa'$ as currents flowing through the edges of the graph. They take values in the space $H$ of sourceless currents, defined as follows: Let $E' = \{(i, j) : \{i, j\} \in E\}$ denote the set of directed edges, where each undirected edge in $E$ is replaced by two directed edges with opposite directions. Let $H$ denote the linear subspace of $\mathbb{R}^E$ consisting of all $\kappa = (\kappa_{ij})_{(i, j) \in E'}$ that satisfy the homogeneous Kirchhoff rules given by

$$
\sum_{j \in V : \{i, j\} \in E} (\kappa_{ij} - \kappa_{ji}) = 0 \quad \text{for all } i \in V. \quad (1.4)
$$

We endow $H$ with the Lebesgue measure $d\kappa_H$ defined as follows: Take any directed reference spanning tree $\vec{T}_0$ of $G$. Note that the restriction

$$
\iota : H \to \mathbb{R}^{E' \setminus \vec{T}_0} \quad (1.5)
$$

of the restriction map $\mathbb{R}^E \to \mathbb{R}^{E' \setminus \vec{T}_0}$ is an isomorphism. Let $d\kappa_H$ be the image under $\iota^{-1}$ of the Lebesgue measure on $\mathbb{R}^{E' \setminus \vec{T}_0}$. Note that $d\kappa_H$ does not depend on the choice of $\vec{T}_0$. For $(i, j) \in E$, we abbreviate

$$
\omega_{ij} = \frac{W_{ij}}{2} e^{v_i + v_j}, \quad \omega'_{ij} = \frac{W_{ij}}{2} e^{u_i + u_j}. \quad (1.6)
$$

Using this, we define an extended version of the supersymmetric hyperbolic nonlinear sigma model that involves not only the original variables $s, u, T'$, but also the new variables $\kappa, \kappa', v, u, i_1, i'_1$, and $T$.

Definition 1.2 (An extended version of the $H^{2/2}$ model) We define the function $\rho_{i_0}^{\text{big}} : (\mathbb{R}^E)^2 \times \Omega_{i_0}^3 \times V^2 \times T' \to \mathbb{R}^+$ by

$$
\rho_{i_0}^{\text{big}} = \rho_{i_0}^{\text{big}}(\kappa, \kappa', s, u, i_1, i'_1, T, T') \quad (1.7)
$$

$$
= \frac{4^{|V| - 1}}{(2\pi)^{2|E|}} \exp \left( \sum_{(i, j) \in E} W_{ij} \left( 1 - \cosh(u_i - u_j) - \frac{1}{2} e^{u_i + u_j} (s_i - s_j)^2 \right) \right) \cdot \prod_{(i, j) \in T} \omega'_{ij} \cdot \prod_{(i, j) \in E} \frac{1}{(\omega'_{ij})^2} \cdot \exp \left( - \sum_{(i, j) \in E} \frac{\kappa_{ij}^2 + (\kappa'_{ij})^2}{2\omega_{ij}^2} \right) \sum_{i \in V} e^{2v_i} \sum_{j \in V} e^{2u_j} \prod_{i \in V \setminus \{i_0\}} e^{-u_i}. \quad 3
$$
Furthermore, we define the measure
\[
\mu_{i_0}^{\text{big}}(d\kappa d\kappa' ds du dv d\iota_1 d\iota_1' dT dT') = \rho_{i_0}^{\text{big}} d\kappa d\kappa' \prod_{i \in V \setminus \{i_0\}} 1_{\{u_i = v_i\}} d\iota_i d\iota_i' dT dT' \tag{1.8}
\]
on \mathcal{H}^2 \times \Omega_{i_0}^3 \times V^2 \times \mathcal{T}^2$, where in the last expression $1_{\{u_i = v_i\}} d\iota_i$ denotes the Lebesgue measure on the diagonal of $\mathbb{R}^2$, $d\iota_i, d\iota_i'$ denote the counting measure on $V$ and $dT, dT'$ mean the counting measure on $\mathcal{T}$.

The reader might wonder why we introduce a seemingly redundant variable $v = u$. The reason becomes clear below when we describe the asymptotics of vertex-reinforced jump processes; cf. the considerations following (1.35) and Theorem 1.7.

The following theorem explains the link between the extended $H_{1/2}^2$ model and the tree version of the $H_{1/2}^2$ model.

**Theorem 1.3 (The $H_{1/2}^2$ model as a marginal of its extended version)**
The marginal of $(s, u, i_1, i_1', T, T')$ with respect to $\mu_{i_0}^{\text{big}}$ equals
\[
\mathcal{L}_{\mu_{i_0}^{\text{big}}}(s, u, i_1, i_1', T, T') = \frac{1}{\pi |V| - 1} \exp \left( \sum_{\{i,j\} \in E} W_{ij} \left( 1 - \cosh(u_i - u_j) - \frac{1}{2} e^{u_i + u_j} (s_i - s_j)^2 \right) \right) \prod_{\{i,j\} \in T'} \omega_{ij}'
\cdot \frac{\prod_{\{i,j\} \in T} \omega_{ij}}{\prod_{\{i,j\} \in S} \omega_{ij}^S} \left( \sum_{j \in V} e^{2u_j} \right)^2 \prod_{i \in V \setminus \{i_0\}} e^{-u_i} ds_i d\iota_i \cdot d\iota_i' dT dT' \tag{1.9}
\]
Furthermore, the marginal of $(s, u, T')$ with respect to $\mu_{i_0}^{\text{big}}$ equals the tree version of the non-linear supersymmetric sigma model:
\[
\mathcal{L}_{\mu_{i_0}^{\text{big}}}(s, u, T') = \mu_{i_0}^{\text{susy}}. \tag{1.10}
\]

In particular, $\mu_{i_0}^{\text{big}}$ is a probability measure.

In the interpretation of the extended $H_{1/2}^2$ model in terms of vertex-reinforced jump processes explained in the next subsection, two different time scales $\sigma \ll \sigma'$ play a role. Then, the list of variables $\kappa, \kappa', s, v, u, i_1, i_1', T, T'$ splits into two groups: $\kappa, v, i_1$, and $T$ involve only the first time scale $\sigma$, whereas the remaining variables $\kappa', s, u, i_1'$, and $T'$ involve both time scales. This is why we are also interested in the following marginal of the extended $H_{1/2}^2$ model.
Theorem 1.4 (Single-time-scale marginal) The marginal of \((\kappa, v, i_1, T)\) with respect to \(\mu_{i_0}^{\text{big}}\) equals the following probability measure on \(H \times \Omega_{i_0} \times V \times T\):

\[
L_{\mu_{i_0}^{\text{big}}} (\kappa, v, i_1, T) = \frac{1}{\pi |E|} \exp \left( \sum_{(i,j) \in E} W_{ij} (1 - \cosh (v_i - v_j)) \right) \cdot \prod_{\{i,j\} \in E \setminus T} \frac{1}{W_{ij} e^{v_i + v_j}} \cdot \exp \left( - \sum_{(i,j) \in \tilde{E}} \frac{\kappa_{ij}^2}{W_{ij} e^{v_i + v_j}} \right) \prod_{i \in V \setminus \{i_0\}} e^{2v_i} \prod_{i \in V \setminus \{i_0\}} e^{-v_i} d\kappa_H \prod_{i \in V \setminus \{i_0\}} dv_i \cdot di_1 dT. \tag{1.11}
\]

1.2 Vertex-reinforced jump processes

Consider again a finite undirected graph \(G\) with edge weights \(W = (W_{ij})_{i,j \in V}\), where we set \(W_{ij} = 0\) for \(i,j \in V\) with \(\{i,j\} \notin E\) to simplify notation. The vertex-reinforced jump process (VRJP) is a stochastic jump process \(Y = (Y_t)_{t \geq 0}\) in continuous time with càdlàg paths, taking values in the vertex set \(V\) of \(G\). The process starts in \(Y_0 = i_0 \in V\). Let \(P_{i_0}\) denote the underlying probability measure. The jump probabilities are defined in terms of the local times with offset 1 given by

\[
L_i(t) = 1 + \int_0^t 1_{\{Y_\tau = i\}} d\tau, \quad i \in V, \tag{1.12}
\]

at times \(t \geq 0\) of \(Y\). In other words, the local time \(L_i(t)\) is 1 plus the time the process \(Y\) spends in vertex \(i\) up to time \(t\). Given two different vertices \(i,j \in V\), a time \(t \geq 0\), and another (small) time \(\Delta t > 0\), on the event \(\{Y_t = i\}\) and conditionally on the past \(\mathcal{F}_t = \sigma(Y_\tau : 0 \leq \tau \leq t)\) up to time \(t\), its jump probability is given by

\[
P_{i_0} (Y_{t + \Delta t} = j | \mathcal{F}_t, Y_t = i) = W_{ij} L_j(t) \Delta t + o(\Delta t) \quad \text{as } \Delta t \downarrow 0. \tag{1.13}
\]

In other words, the process has the jump rates \(W_{ij} L_j(t)\).

In the following we consider the time-changed version of the vertex-reinforced jump process \(Z = (Z_\sigma)_{\sigma \geq 0}\) on \(G\) which was introduced in [ST15]. Let us review its definition. The time change is defined by

\[
D(t) = \sum_{i \in V} (L_i(t)^2 - 1). \tag{1.14}
\]

The time-changed version \(Z\) is defined by

\[
Z_\sigma = Y_{D^{-1}(\sigma)}, \quad \sigma \geq 0. \tag{1.15}
\]

The local time \(l(\sigma) = (l_i(\sigma))_{i \in V}\) (without offset) of the process \(Z\) is defined by

\[
l_i(\sigma) = \int_0^\sigma 1_{\{Z_\zeta = i\}} d\zeta = L_i(D^{-1}(\sigma))^2 - 1, \quad i \in V. \tag{1.16}
\]
The jump rates of the process $Y$ specified in (1.13) are transformed by the time-change as follows. Given two different vertices $i,j \in V$, a time $\sigma \geq 0$, and another (small) time $\Delta \sigma > 0$, on the event $\{Z_\sigma = i\}$ and conditionally on the past $G_\sigma = \sigma(Z_\zeta : 0 \leq \zeta \leq \sigma)$ up to time $\sigma$, its jump probability is given by

$$P_{i_0}(Z_{\sigma+\Delta\sigma} = j | G_\sigma, Z_\sigma = i) = \frac{W_{ij}}{2} \sqrt{\frac{1 + l_j(\sigma)}{1 + l_i(\sigma)}} \Delta \sigma + o(\Delta \sigma) \quad \text{as } \Delta \sigma \downarrow 0. \quad (1.17)$$

In other words, the process $Z$ has the jump rates $\frac{W_{ij}}{2} \sqrt{\frac{1 + l_j(\sigma)}{1 + l_i(\sigma)}} \Delta \sigma + o(\Delta \sigma)$.

**History of the model**  The vertex-reinforced jump process was initially proposed by Werner and introduced by Davis and Volkov in [DV02] on the integers and studied in [DV04] on trees. Further analysis on regular and Galton-Watson trees was conducted by Collevecchio in [Col06] and [Col09] and by Basdevant and Singh in [BS12].

Tarrés [Tar11] and Sabot and Tarrés in [ST15] showed that the VRJP is related to the linearly edge-reinforced random walk and to the $u$-marginal of the supersymmetric hyperbolic sigma model $H^{2|2}$ defined above; cf. formula (1.18), below. Using this second link and the results from [DSZ10] and [DST10], they proved in the same paper recurrence of VRJP on any graph of bounded degree for strong reinforcement, i.e. $W_{ij}$ small for all edges $\{i,j\}$, and transience on $\mathbb{Z}^d$, $d \geq 3$, for small constant reinforcement, i.e. $W_{ij}$ large and constant for all edges. Disertori, Sabot, and Tarrés [DST15] give a generalization to non-constant reinforcement. An alternative proof of recurrence of VRJP under the same conditions, not using the connection with the $H^{2|2}$ model, was given by Angel, Crawford, and Kozma in [ACK14]. In [DMR14], Disertori, Merkl, and Rolles show recurrence of VRJP on two-sided infinite strips with translationally invariant $W_{ij}$.

Further links of VRJP with a random Schrödinger operator and with Ray-Knight second generalized theorem were investigated by Sabot, Tarrés, and Zeng in [STZ15b], [SZ15], and [ST16].

**Current setup**  The components $u$ in the $H^{2|2}$ model have the following interpretation in terms of VRJP: Proposition 1 and Theorem 2 in [ST15] imply that $l_i(\sigma)/l_{i_0}(\sigma)$, $i \in V$, converge jointly $P_{i_0}$-almost surely to a limit having the law

$$\mathcal{L}_{P_{i_0}} \left( \lim_{\sigma \to \infty} \left( \frac{l_i(\sigma)}{l_{i_0}(\sigma)} \right)_{i \in V} \right) = \mathcal{L}_{\mu_{i_0}^{\text{susy}}} \left( \left( e^{2u_i} \right)_{i \in V} \right) \quad (1.18)$$

with the tree version $\mu_{i_0}^{\text{susy}}$ of the $H^{2|2}$ model given in Definition 1.11. One of the goals of the current paper is to show that all the components $\kappa, \kappa', s, u, v, i_1, i_1', T$, and $T'$ in the extended $H^{2|2}$ model have an interpretation in terms of VRJP as well. We shall show below that the components $s_i$ can be interpreted in terms of fluctuations of $l_i(\sigma)/l_{i_0}(\sigma)$ around its asymptotic value $\lim_{\sigma' \to \infty} l_i(\sigma')/l_{i_0}(\sigma')$. For this analysis, it is natural to consider two different time scales $1 \ll \sigma \ll \sigma'$. First, we consider only two fixed times $\sigma, \sigma' > 0$. 


Local times} We set \( l'(\sigma, \sigma') = (l_i'(\sigma, \sigma'))_{i \in V} \) with
\[
l_i'(\sigma, \sigma') = l_i(\sigma + \sigma') - l_i(\sigma).
\]
This is the local time the process \((Z_\sigma)_{\sigma \geq 0}\) spends in vertex \(i\) during the time interval \([\sigma, \sigma + \sigma']\). Let
\[
\mathcal{L}_\sigma = \left\{ l \in (0, \infty)^V : \sum_{i \in V} l_i = \sigma \right\}, \quad \mathcal{L}_{\sigma, \sigma'} = \mathcal{L}_\sigma \times \mathcal{L}_{\sigma'}.
\]
In particular, \((l(\sigma), l'(\sigma, \sigma')) \in \mathcal{L}_{\sigma, \sigma'}\).

**Last exit trees** We shall show below that the component \(T'\) in the \(H^{2\|}\) model has an interpretation in terms of last exit trees of the process \(Z\). These last exit trees are defined as follows. Given a time interval \([\sigma_1, \sigma_2]\), let \(V_{[\sigma_1, \sigma_2]} = \{Z_t : t \in [\sigma_1, \sigma_2]\}\) denote the set of all vertices visited between times \(\sigma_1\) and \(\sigma_2\). For \(i \in V_{[\sigma_1, \sigma_2]} \setminus \{Z_{\sigma_2}\}\), let \(e_i^{\text{last exit}}(\sigma_1, \sigma_2)\) denote the directed edge of the form \((i, j)\) which the process \(Z\) has crossed when it left vertex \(i\) for the last time during the time interval \([\sigma_1, \sigma_2]\). Let
\[
\overline{T}^{\text{last exit}}(\sigma_1, \sigma_2) = \bigcup_{i \in V_{[\sigma_1, \sigma_2]} \setminus \{Z_{\sigma_2}\}} \{e_i^{\text{last exit}}(\sigma_1, \sigma_2)\}
\]
be the collection of directed edges taken by the process \(Z\) for the last departures from all vertices visited in the time interval \([\sigma_1, \sigma_2]\) except the endpoint. Sometimes, we need also the undirected version of \(\overline{T}^{\text{last exit}}(\sigma_1, \sigma_2)\); it is denoted by \(T^{\text{last exit}}(\sigma_1, \sigma_2)\). If the process \(Z\) has visited all vertices between times \(\sigma_1\) and \(\sigma_2\), then \(\overline{T}^{\text{last exit}}(\sigma_1, \sigma_2)\) is a spanning tree of \(G\), directed towards the endpoint \(Z_{\sigma_2}\). For \(i_1 \in V\) let \(\overline{T}_{i_1}\) denote the set of spanning trees of \(G\) which are directed towards \(i_1\).

**Edge crossings and currents with sources** We define \(k(\sigma) = (k_{ij}(\sigma))_{(i,j) \in E}\) by
\[
k_{ij}(\sigma) = |\{t \leq \sigma : Z_t = i, Z_t = j\}|,
\]
which denotes the number of crossings from \(i\) to \(j\) up to time \(\sigma\). Similarly, for the time interval of length \(\sigma'\) starting at \(\sigma\), we define \(k'(\sigma, \sigma') = (k'_{ij}(\sigma, \sigma'))_{(i,j) \in E}\) by
\[
k'_{ij}(\sigma, \sigma') = k_{ij}(\sigma + \sigma') - k_{ij}(\sigma).
\]
In other words, \(k'_{ij}(\sigma, \sigma')\) equals the number of crossings from \(i\) to \(j\) in the time interval \([\sigma, \sigma + \sigma']\). We denote by \(\delta_i(j) = 1_{\{i=j\}}\) Kronecker’s delta. For \(i_0, i_1 \in V\), let \(\mathcal{K}_{i_0, i_1}\) denote the set of all \(k \in \mathbb{Z}^E\) such that the inhomogeneous Kirchhoff rules
\[
\sum_{j \in V} (k_{ij} - k_{ji}) = \delta_{i_0}(i) - \delta_{i_1}(i), \quad i \in V,
\]
\[
(1.19)
\]
\[
(1.20)
\]
\[
(1.21)
\]
\[
(1.22)
\]
\[
(1.23)
\]
\[
(1.24)
\]
hold. For an additionally given \( i'_1 \in V \), let

\[
K_{i_0,i_1,i'_1} = K_{i_0,i_1} \times K_{i_1,i'_1} \quad \text{and} \quad K_{i_0,i_1,i'_1}^+ = K_{i_0,i_1,i'_1} \cap (\mathbb{N}^E \times \mathbb{N}^E).
\]

(1.25)

Thus, \( K_{i_0,i_1,i'_1}^+ \) is obtained by a restriction to strictly positive integers. One can imagine \( k \) as a current flowing through the graph with a source of size 1 at \( i_0 \) and a sink of size 1 at \( i_1 \). Similarly, the current \( k' \) has a source at \( i_1 \) and a sink at \( i'_1 \). In particular, \((k(\sigma), k'(\sigma, \sigma')) \in K_{i_0,z_\sigma,z_{\sigma+\sigma'}}\).

The following definition introduces some events which are useful to study the joint law of the random variable \( \xi_{\sigma,\sigma'} \).

**Definition 1.5 (Events concerning local times and last exit trees)** Let \( i_0, i_1, i'_1 \in V \), \((k, k') \in K_{i_0,i_1,i'_1}^+\), \( \sigma, \sigma' > 0 \), \( A \subseteq \mathcal{L}_{\sigma,\sigma'} \) be measurable, \( \vec{T} \in \vec{T}_{i_1} \), and \( \vec{T}' \in \vec{T}_{i'_1} \). Let \( T \) and \( T' \) denote the undirected version of \( \vec{T} \) and \( \vec{T}' \), respectively. In this setup, we define the following events

\[
K_{k,\sigma,k',\sigma'} = \{k(\sigma) = k, k'(\sigma, \sigma') = k'\},
\]

(1.27)

\[
L_{\sigma,\sigma'}(A) = \{l(\sigma), l'(\sigma, \sigma') \in A\},
\]

(1.28)

\[
E_{i_1,\vec{T},\sigma,i'_1,\vec{T}',\sigma'} = E_{i_1,T,\sigma,i'_1,T',\sigma'}
= \{Z_{\sigma} = i_1, Z_{\sigma+\sigma'} = i'_1, T^{\text{last exit}}(0, \sigma) = T, T^{\text{last exit}}(\sigma, \sigma + \sigma') = T'\}.
\]

(1.29)

The following theorem describes explicitly the distribution of the random variable \( \xi_{\sigma,\sigma'} \).

**Theorem 1.6 (Joint density of edge crossings, local times, and last exit trees)** In the setup of Definition 1.5, the following holds

\[
P_{i_0}(K_{k,\sigma,k',\sigma'} \cap L_{\sigma,\sigma'}(A) \cap E_{i_1,\vec{T},\sigma,i'_1,\vec{T}',\sigma'})
= \int_{A} \exp \left( \sum_{\{i,j\} \in E} W_{ij} \left( 1 - \sqrt{1 + l_i + l'_i \sqrt{1 + l_j + l'_j}} \right) \right) \prod_{i \in V \setminus \{i_0\}} \frac{1}{\sqrt{1 + l_i + l'_i}}
\cdot \mathcal{P}(k,l,\vec{T}) \mathcal{P}(k',l',\vec{T}') \prod_{i \in V \setminus \{i_0\}} dl_i dl'_i,
\]

(1.30)

where we abbreviate

\[
\mathcal{P}(k,l,\vec{T}) = \prod_{(i,j) \in E} \left( \frac{W_{ij} l_i}{2} \right)^{k_{ij}} \frac{1}{k_{ij}!} \prod_{(i,j) \in \vec{T}} \frac{k_{ij}}{l_i}.
\]

(1.31)

It follows from Theorem 1.6 that conditionally on \( l(\sigma) \) and \( l'(\sigma, \sigma') \), the random variables \( k(\sigma) \) and \( k'(\sigma, \sigma') \) follow an oriented random current model, i.e. a product of Poisson distributions conditioned on Kirchhoff’s rule.
1.3 The extended $H^{2|2}$ model as a limit of VRJP

Recall that the motivation for taking two different time scales $1 \ll \sigma \ll \sigma'$ was to study fluctuations of local times. In that view, it is natural to take the limit $\sigma' \to \infty$ first and only second the limit $\sigma \to \infty$. More generally, it turns out that we can also take $\sigma$ and $\sigma'$ simultaneously to infinity, as long as $\min\{\sigma, \sigma'^{-2}\} \to \infty$.

By a slight abuse of notation, we will abbreviate henceforth

$$k = k(\sigma), \quad l = l(\sigma), \quad \text{and} \quad k' = k(\sigma, \sigma'), \quad l' = l(\sigma, \sigma').$$

(1.32)

Rescaling of local times and their fluctuations

The considerations around (1.18) motivate us to study the cross-ratio

$$\frac{l_i/l_{i_0}}{l'_i/l'_{i_0}}.$$  \quad (1.33)

In order not to divide by 0, given $\sigma, \sigma' > 0$, we consider the event

$$A_{\sigma, \sigma'} = \{l_i > 0, \; l'_i > 0 \text{ for all } i \in V\}.$$  \quad (1.34)

On this event, motivated by (1.18), we introduce new variables $v_i = v_i(\sigma)$ and $u_i = u_i(\sigma, \sigma')$ for $i \in V$ by

$$l_i = l_{i_0}e^{2v_i}, \quad l'_i = l'_{i_0}e^{2u_i}.$$ \quad (1.35)

Although $v_i$ and $u_i$ are certainly different random variables, formula (1.18) shows us that they coincide $P_{i_0}$-almost surely asymptotically in the limit as $\sigma' \gg \sigma \to \infty$. Not very unexpectedly for fluctuations, the right scale for the logarithm of the cross-ratio (1.33) turns out to be $\sqrt{l_{i_0}}$, i.e. roughly the square root of the smaller time scale. This motivates us to define

$$s_i = s_i(\sigma, \sigma') = -\frac{1}{2} \sqrt{l_{i_0}} \log \frac{l_i/l_{i_0}}{l'_i/l'_{i_0}} = \sqrt{l_{i_0}}(u_i - v_i), \quad i \in V.$$ \quad (1.36)

Note that $s_{i_0} = u_{i_0} = v_{i_0} = 0$. In other words, $(s_i)_{i \in V}$, $(u_i)_{i \in V}$, and $(v_i)_{i \in V}$ belong to the space $\Omega_{i_0}$ defined in (1.1). An interpretation of $s_i$ is most easily described in the special case of taking the limit $\sigma' \to \infty$ first and only then $\sigma \to \infty$. In this case, $\lim_{\sigma' \to \infty} s_i(\sigma, \sigma')$ describes the fluctuations of $l_i(\sigma)/l_{i_0}(\sigma)$ around $\lim_{\sigma' \to \infty} l_i(\sigma')/l_{i_0}(\sigma')$ on the appropriate scale.

Rescaling of crossing numbers

It turns out that the random variables $k_{ij}(\sigma)$ are centered roughly around $\frac{1}{2}W_{ij}\sqrt{l_{i0}l_{j0}}$, with fluctuations on the scale $\sqrt{l_{i0}}$. This motivates
us to introduce, again on the event $A_{\sigma, \sigma'}$, the rescaled crossing numbers $\kappa_{ij} = \kappa_{ij}(\sigma)$ and $\kappa'_{ij} = \kappa'_{ij}(\sigma, \sigma')$ for $(i, j) \in \mathcal{E}$ by

\[
\begin{align*}
\kappa_{ij} &= \frac{k_{ij}}{l_{i0}'^{\frac{1}{2}}} = \frac{k_{ij}}{l_{i0}^{\frac{1}{2}}} - \frac{W_{ij} e^{v_i + v_j}}{2} \sqrt{l_{i0}^{\frac{1}{2}}}, \quad (1.37) \\
\kappa'_{ij} &= \frac{k'_{ij}}{l_{i0}'^{\frac{1}{2}}} = \frac{k'_{ij}}{l_{i0}^{\frac{1}{2}}} - \frac{W_{ij} e^{u_i + u_j}}{2} \sqrt{l_{i0}^{\frac{1}{2}}}.
\end{align*}
\]

The rescaling with the factors $l_{i0}^{-1/2}$ and $(l_{i0}')^{-1/2}$, which $P_{i0}$-a.s. converge to 0 as $\sigma' \gg \sigma \to \infty$, makes the sources $\pm 1$ of the currents $k$ and $k'$ in the vertices $i_0$, $Z_\sigma$, and $Z_{\sigma+\sigma'}$ asymptotically negligible. This explains intuitively why the homogeneous Kirchhoff rules (1.4) rather than the inhomogeneous Kirchhoff rules (1.24) apply asymptotically to $\kappa$ and $\kappa'$.

For a truncation parameter $M > 0$, we consider the event

\[
B_{\sigma, \sigma'}(M) = \{ |\kappa_{ij}|, |\kappa'_{ij}|, |s_i|, |u_i|, |v_i| \leq M \text{ for all } i, j \in V \} \cap A_{\sigma, \sigma'}.
\]

**Notation for error terms** We write $f(\sigma) = O_M(g(\sigma))$ as $\sigma \to \infty$ if there exists a constant $c(M) > 0$ depending on the parameter $M$ such that $|f(\sigma)| \leq c(M)|g(\sigma)|$ for all $\sigma$ large enough. If we use $O$ with more than one subscript, the constant may depend on all subscripts.

The following theorem connects the function $\rho_{i0}^{\text{big}}$, which was used as a density in the Definition 1.2 of the extended version of the $H^{2|2}$ model, to the asymptotics of VRJP.

**Theorem 1.7 (Limiting joint density)** Consider the setup of Definition 1.2. For $M > 0$, on the events $B_{\sigma, \sigma'}(M)$, one has the following in the limit as $\min\{\sigma, \sigma'\sigma^{-2}\} \to \infty$:

\[
\begin{align*}
&\quad P_{i0}(K_{k, \sigma, k', \sigma'} \cap L_{\sigma, \sigma'}(A) \cap E_{i_0, T, \sigma, i_1', T', \sigma'}) \\
&= \left(1 + O_{M,W,G}(\sigma^{-1/2} + \frac{\sigma^2}{\sigma'})\right) \int_A \rho_{i0}^{\text{big}} \Lambda_{\sigma, \sigma', i_0}(dl dl') \quad (1.40)
\end{align*}
\]

with the following measure on $\mathcal{L}_{\sigma, \sigma'}$

\[
\Lambda_{\sigma, \sigma', i_0}(dl dl') = \frac{4^{1-|V|}}{l_{i0}'(l_{i0}')^{\frac{1}{2}}} \frac{\sigma \sigma'}{l_{i0}' l_{i0}'} \prod_{i \in V \setminus \{i_0\}} l_{i0}' l_{i1}' dl_i dl_i' \quad (1.41)
\]

and the function $\rho_{i0}^{\text{big}} = \rho_{i0}^{\text{big}}(\kappa, \kappa', s, v, u, i_1, i_1', T, T')$ defined in (1.7).

The random variable $\xi_{\sigma, \sigma'}$ defined in (1.26) is only interesting on the event $\{\xi_{\sigma, \sigma'} \in \mathcal{O}_{\sigma, \sigma', i_0}\}$ with

\[
\begin{align*}
&\quad \mathcal{O}_{\sigma, \sigma', i_0} = \bigcup_{i_1, i_1' \in V} \mathcal{K}_{i_0, i_1, i_1'} \times \mathcal{L}_{\sigma, \sigma'} \times \{i_1\} \times \{i_1'\} \times T_{i_1} \times T_{i_1'}.
\end{align*}
\]
One has \( \xi_{\sigma, \sigma'} \notin O_{\sigma, \sigma', i_0} \) if some \( l_i \) or \( l'_i \) equals 0 or if \( T^{\text{last exit}}(0, \sigma) \) or \( T^{\text{last exit}}(\sigma, \sigma + \sigma') \) is not spanning. Furthermore, we consider the map

\[
F_{\sigma, \sigma', i_0} : O_{\sigma, \sigma', i_0} \to (\mathbb{R}^2)^2 \times \Omega^3_{i_0} \times V^2 \times T^2,
\]

\[
F_{\sigma, \sigma', i_0}(k, k', l, l', i_1, i'_1, T, T') = (\kappa, \kappa', s, v, u, i_1, i'_1, T, T')
\] (1.43)
defined by the equations (1.35), (1.36), (1.37), and (1.38).

The following main theorem shows that the extended \( H^{2|2} \) model describes the asymptotics of the time-changed version \( Z \) of the vertex-reinforced jump process. To be more precise, it occurs as the joint limit of the rescaled crossing numbers, the rescaled fluctuations of local times, the asymptotic local times on a logarithmic scale, the endpoints of paths, and last exit trees as follows. Let \( E_{i_0} \) denote the expectation operator with respect to \( P_{i_0} \).

**Theorem 1.8 (Weak convergence to the extended \( H^{2|2} \) model)** The joint sub-probability distribution of

\[
(\kappa(\sigma), \kappa'(\sigma, \sigma'), s(\sigma, \sigma'), v(\sigma), u(\sigma, \sigma'), Z_{\sigma}, Z_{\sigma + \sigma'}, T^{\text{last exit}}(0, \sigma), T^{\text{last exit}}(\sigma, \sigma + \sigma'))
\] (1.44)

with respect to \( P_{i_0}(\cdot \cap \{\xi_{\sigma, \sigma'} \in O_{\sigma, \sigma', i_0}\}) \) converges weakly as \( \min\{\sigma, \sigma'\sigma^{-2}\} \to \infty \) to \( \mu_{i_0}^{\text{big}} \). In other words, for any bounded continuous test function \( f : (\mathbb{R}^2)^2 \times \Omega^3_{i_0} \times V^2 \times T^2 \to \mathbb{R} \), one has

\[
\lim_{\min\{\sigma, \sigma'\sigma^{-2}\} \to \infty} E_{i_0}[f(F_{\sigma, \sigma', i_0}(\xi_{\sigma, \sigma'}))], \xi_{\sigma, \sigma'} \in O_{\sigma, \sigma', i_0} = \int f \, d\mu_{i_0}^{\text{big}}.
\] (1.45)

In particular,

\[
P_{i_0}(\xi_{\sigma, \sigma'} \in O_{\sigma, \sigma', i_0}) \to 1 \quad \text{as} \quad \min\{\sigma, \sigma'\sigma^{-2}\} \to \infty.
\] (1.46)

In order to phrase a variant of this theorem involving only a single time scale \( \sigma \), we introduce the following reduced versions of the random vector \( \xi_{\sigma, \sigma'} \) and the set \( O_{\sigma, \sigma', i_0} \), respectively.

\[
\xi_{\sigma} = (k(\sigma), l(\sigma), Z_{\sigma}, T^{\text{last exit}}(0, \sigma)),
\] (1.47)

\[
Q_{\sigma, i_0} = \bigcup_{i_1 \in V} K_{i_0, i_1} \times L_{\sigma} \times \{i_1\} \times T_{i_1}.
\] (1.48)

**Corollary 1.9 (Weak convergence of single-time marginals)** The joint sub-probability distribution of

\[
(\kappa(\sigma), v(\sigma), Z_{\sigma}, T^{\text{last exit}}(0, \sigma))
\] (1.49)

with respect to \( P_{i_0}(\cdot \cap \{\xi_{\sigma} \in Q_{\sigma, i_0}\}) \) converges weakly as \( \sigma \to \infty \) to the law \( \mathcal{L}_{\mu_{i_0}^{\text{big}}}(\kappa, v, i_1, T) \) described in Theorem 1.4.
How this paper is organized In the rest of the paper, we prove the theorems stated so far. Section 2 contains a proof of Theorem 1.6. Using path counting arguments and calculating volume factors, we determine precisely the density of the random variable $ξ_{σ,σ'}$ for fixed times $σ,σ'$. In Section 3 we derive the asymptotics of this density, appropriately rescaled, in the limit $\min\{σ,σ'σ^{-2}\} → \infty$. This yields a proof of Theorem 1.7. The proof of Theorem 1.8, i.e. of weak convergence to the extended $H^{2|2}$ model, consists of two pieces. The first step shows vague convergence rather than weak convergence. This involves a continuum limit and is done in Section 4. The second step deduces weak convergence from vague convergence. The key ingredients here are on the one hand the normalization (1.3) of the $H^{2|2}$ model $µ^\text{susy}$ and on the other hand the fact, stated in Theorem 1.3, that the extended $H^{2|2}$ model $µ^\text{big}$ has $µ^\text{susy}$ as a marginal. It would be possible to prove Corollary 1.9 which deals with single-time-scale marginals, directly without introduction of a second time scale $σ'$. However, in order to avoid duplication of arguments, we deduce it as an immediate consequence of the more general Theorem 1.8. All this, including the explicit description of the limiting marginal in Theorem 1.4, is proven in Section 5.

2 Joint density of local times and last exit trees

In this section, the time horizons $σ$ and $σ'$ are kept fixed. Using combinatorial arguments, we derive the density of the distribution of the random variable $ξ_{σ,σ'}$.

Proof of Theorem 1.6. For $0 < σ_1 < σ_2$, let discrete($Z_{[σ_1, σ_2]}$) denote the path in discrete time obtained from $(Z_σ)_{σ∈[σ_1, σ_2]}$ by taking only the values immediately before the jumps. For $i_0, i_1 ∈ V, k ∈ K_{i_0,i_1}^+$, and $\vec{T} ∈ \vec{T}_{i_1}$ let $Π_{i_0,i_1}(k,\vec{T})$ denote the set of finite paths in discrete time which start in $i_0$, end in $i_1$, cross every $(i, j) ∈ \vec{E}$ precisely $k_{ij}$ times and have last exit tree $\vec{T}$. Let in addition $i'_1 ∈ V, k' ∈ K_{i_1,i'_1}^+$, and $\vec{T}' ∈ \vec{T}_{i'_1}$. Given $π ∈ Π_{i_0,i_1}(k,\vec{T})$ and $π' ∈ Π_{i_1,i'_1}(k',\vec{T}')$, we consider the event

$$C_{σ,σ'}(π, π') = \{\text{discrete}(Z_{[0,σ]} = π, \text{discrete}(Z_{[σ,σ+σ']} = π')\}. \quad (2.1)$$

Recall the definition (1.28) of $L_{σ,σ'}(A)$. Using the proof of Theorem 3 in ST16 (first displayed formula on page 569, see also formula (2) in Section 3.3 of Zen16), we obtain the following for any measurable $A ⊆ L_{σ,σ'}$ with appropriate volume factors $V(k, l, i_1)$ and $V(k', l', i'_1)$ specified in (2.7) and (2.8) below:

$$P_{i_0}(C_{σ,σ'}(π, π') ∩ L_{σ,σ'}(A)) = \int_A \exp\left(\sum_{(i,j)∈E} W_{ij} \left(1 - \sqrt{1 + l_i + l'_{i'}} \sqrt{1 + l_j + l'_{j'}}\right)\right) \prod_{i∈ V\setminus\{i'_1\}} \frac{1}{\sqrt{1 + l_i + l'_{i'}}} \prod_{(i,j)∈E} (W_{ij}/2)^{k_{ij}+k'_{ij}} \cdot V(k, l, i_1) V(k', l', i'_1) \prod_{i∈ V\setminus\{i_0\}} dl_i dl'_{i}. \quad (2.2)$$

Note that the right hand side in the last equation depends only on the choice of $k, k', i_0, i_1,$ and $i'_1$, but neither on the choice of $\vec{T}$ and $\vec{T}'$ nor on the choice of $π ∈ Π_{i_0,i_1}(k,\vec{T})$ and
\[ \pi' \in \Pi_{i, i'}(k, T) \text{.} \] Consequently,

\[
P_{i_0}(K_{k,\sigma,k',\sigma'} \cap L_{\sigma,\sigma'}(A) \cap E_{i_1, i', i_1'}(T, \sigma_k, \sigma'))
= |\Pi_{i_0, i_1}(k, T)| \cdot |\Pi_{i_1, i_1'}(k', T')| \int_A \exp \left( \sum_{(i,j)} W_{ij} \left( 1 - \sqrt{1 + l_i + l'_i \sqrt{1 + l_j + l'_j}} \right) \right)
\]

\[
\cdot \prod_{i \in V \setminus \{i_1\}} \frac{1}{1 + l_i + l'_i} \prod_{(i,j) \in E} \left( \frac{W_{ij}}{2} \right)^{k_j + k'_j} \cdot \mathcal{V}(k, l, i_1) \mathcal{V}(k', l', i'_1) \prod_{i \in V \setminus \{i_0\}} dl_i dl'_i. \tag{2.3}
\]

The volume factors \( \mathcal{V}(k, l, i_1) \) and \( \mathcal{V}(k', l', i'_1) \) consist of a product of contributions from each vertex. We determine them as follows. For \( i \in V \), we set

\[
k_i = \sum_{j \in V : (i,j) \in E} k_{ij}, \quad k'_i = \sum_{j \in V : (i,j) \in E} k'_{ij}. \tag{2.4}
\]

Then, \( k_i \) equals the number of departures from vertex \( i \). Given the directed edge crossings \( k \), to have for all vertices \( i \in V \) local time \( l_i \) at vertex \( i \) at time \( \sigma \) we need jump times

\[
0 = t_0^{(i)} < t_1^{(i)} < \cdots < t_{k_i}^{(i)} \leq l_i,
\]

where for all \( i \in V \setminus \{i_1\} \) we have moreover \( t_k^{(i)} = l_i \). For \( i \neq i_1 \) these are \( k_i - 1 \) jumps in the time interval \( (0, l_i) \). Integrating over \( t_1^{(i)}, \ldots, t_{k_i}^{(i)} \) gives the volume factor contribution from vertex \( i \neq i_1 \)

\[
\mathcal{V}(k_i - 1, l_i) = \lambda^{k_i - 1} \left( \left\{ (t_1^{(i)}, \ldots, t_{k_i - 1}^{(i)}) \in (0, l_i)^{k_i - 1} : t_1^{(i)} < \cdots < t_{k_i - 1}^{(i)} \right\} \right) = \frac{l_{k_i - 1}^{k_i - 1}}{(k_i - 1)!}; \tag{2.5}
\]

here \( \lambda^{k_i - 1} \) denotes the Lebesgue measure on \( \mathbb{R}^{k_i - 1} \). For \( i = i_1 \), given \( l_{i_1} \), there is one degree of freedom more. Integrating over the jump times \( t_1^{(i_1)}, \ldots, t_{k_{i_1}}^{(i_1)} \) gives the volume factor

\[
\mathcal{V}(k_{i_1}, l_{i_1}) = \lambda^{k_{i_1}} \left( \left\{ (t_1^{(i_1)}, \ldots, t_{k_{i_1}}^{(i_1)}) \in (0, l_{i_1})^{k_{i_1}} : t_1^{(i_1)} < \cdots < t_{k_{i_1}}^{(i_1)} \right\} \right) = \frac{l_{k_{i_1}}^{k_{i_1}}}{k_{i_1}!}. \tag{2.6}
\]

Altogether this yields the volume factor

\[
\mathcal{V}(k, l, i_1) = \mathcal{V}(k_{i_1}, l_{i_1}) \prod_{i \in V \setminus \{i_1\}} \mathcal{V}(k_i - 1, l_i) = \frac{l_{k_{i_1}}^{k_{i_1}}}{k_{i_1}!} \prod_{i \in V \setminus \{i_1\}} \frac{l_{k_i - 1}^{k_i - 1}}{(k_i - 1)!}. \tag{2.7}
\]

Similarly, integration over the jump times between times \( \sigma \) and \( \sigma + \sigma' \) yields the volume factor

\[
\mathcal{V}(k', l', i'_1) = \mathcal{V}(k'_{i_1}, l'_{i_1}) \prod_{i \in V \setminus \{i'_1\}} \mathcal{V}(k'_i - 1, l'_i) = \frac{(l'_{k'_{i_1}})^{k'_{i_1}}}{(k'_{i_1})!} \prod_{i \in V \setminus \{i'_1\}} \frac{(l'_i)^{k'_i - 1}}{(k'_i - 1)!}. \tag{2.8}
\]
To determine the cardinality of the set of paths \( \Pi_{i_0,i_1}(k, \vec{T}) \), we use the calculations leading to Lemma 6 of [KR00]. They give

\[
|\Pi_{i_0,i_1}(k, \vec{T})| = \frac{\prod_{i \in V} k_i!}{\prod_{(i,j) \in E} k_{ij}!} \times \frac{\prod_{(i,j) \in T} k_{ij}!}{\prod_{i \in V \setminus \{i_1\}} (k_i - 1)!} \times \prod_{(i,j) \in E} \frac{k_{ij}}{k_{ij}'}.
\]

Using the above notation, we rewrite the rescaling of the crossing numbers

\[
|\Pi_{i_0,i_1}(k', \vec{T}')| = k_{i_1}'! \prod_{i \in V \setminus \{i_1'\}} (k_i' - 1)! \prod_{(i,j) \in E} k_{ij}'.
\]

Combining the contributions from (2.9) and (2.7) and using that \( \vec{T} \) is a spanning tree directed towards \( i_1 \), we obtain

\[
|\Pi_{i_0,i_1}(k, \vec{T})| \gamma(k, l, i_1) = \frac{\prod_{(i,j) \in T} k_{ij}}{\prod_{(i,j) \in E} k_{ij}'} = \prod_{(i,j) \in E} \frac{k_{ij}}{k_{ij}'}.
\]

By the same argument, the combination of (2.10) and (2.8) yields

\[
|\Pi_{i_1,i_1'}(k', \vec{T}')| \gamma(k', l', i_1') = \prod_{(i,j) \in E} \frac{k_{ij}'}{k_{ij}''}.
\]

The claim follows from (2.3), (2.11), and (2.12).

### 3 Asymptotics

In this section, we use Taylor arguments and Stirling’s formula to asymptotically describe the density of the random variable \( \xi_{\sigma,\sigma'} \) but rewritten in terms of rescaled variables. Recall the definition (1.6) of \( \omega_{ij} \) and \( \omega_{ij}' \).

**Scales of the variables** Using the above notation, we rewrite the rescaling of the crossing numbers \( k \) and \( k' \) given in (1.37) and (1.38) as follows

\[
k_{ij} = l_{io} \omega_{ij} + \sqrt{l_{io} \kappa_{ij}}, \quad k_{ij}' = l_{io}' \omega_{ij}' + \sqrt{l_{io}' \kappa_{ij}'},
\]

Using the definition (1.35) of \( v \) and \( u \) we deduce

\[
\frac{\sigma}{l_{io}} = \sum_{i \in V} e^{2v_i} \quad \text{and} \quad \frac{\sigma'}{l_{io}'} = \sum_{i \in V} e^{2u_i}.
\]

In particular, \( \sigma \) and \( l_{io} \) live on the same scale when all \( v_i \) are bounded. A similar statement holds for \( \sigma', l_{io}' \), and \( u_i \). Consequently, on the event \( B_{\sigma,\sigma'}(M) \) defined in (1.39), all \( l_i \) and all \( k_{ij} \) have the same order of magnitude as \( \sigma \) and all \( l_i' \) and all \( k_{ij}' \) have the same order of magnitude as \( \sigma' \). By the definition (1.30) of \( s_i \), one has \( v_i = u_i - l_{io}^{-1/2} s_i \). Hence, for any given \( M > 0 \), on the event \( B_{\sigma,\sigma'}(M) \), in the limit as \( \sigma \to \infty \), one has

\[
e^{v_i} = \exp\left(1 + O_M(\sigma^{-1/2})\right), \quad \omega_{ij} = \omega_{ij}'(1 + O_M(\sigma^{-1/2})).
\]
Lemma 3.1 (Asymptotics of the combinatorial factors) Consider the setup of Definition 1.1 and recall the notation (1.31). Given \(M > 0\), on the event \(B_{\sigma,\sigma'}(M)\), one has the following as \(\sigma \to \infty\):

\[
\mathcal{P}(k, l, \bar{T}) = \frac{1}{(2\pi l_{io})^{|E|}} \exp \left( l_{io} \sum_{(i,j) \in E} \omega_{ij} - v_{i1} - \sum_{(i,j) \in E} \frac{k_{ij}^2}{2\omega_{ij}} \right)
\cdot \prod_{i \in V \setminus \{i_1\}} e^{-2v_{i}} \prod_{(i,j) \in E \setminus \bar{T}} \frac{1}{\omega_{ij}} \prod_{(i,j) \in E} \omega_{ij}' \cdot (1 + O_{M,W,G}(\sigma^{-1/2})). \tag{3.4}
\]

Similarly, on the same event, one has the following as \(\sigma' \to \infty\):

\[
\mathcal{P}(k', l', \bar{T}') = \frac{1}{(2\pi l'_{io})^{|E|}} \exp \left( l'_{io} \sum_{(i,j) \in E} \omega_{ij}' + u_{i1} - u_{i1}' - \sum_{(i,j) \in E} \frac{(k_{ij}')^2}{2\omega_{ij}'} \right)
\cdot \prod_{i \in V \setminus \{i_1'\}} e^{-2u_{i}} \prod_{(i,j) \in E \setminus \bar{T}'} \frac{1}{\omega_{ij}'} \prod_{(i,j) \in E} \omega_{ij}' \cdot (1 + O_{M,W,G}(\sigma'^{-1/2})). \tag{3.5}
\]

Proof. To have a uniform notation for the proof, we set

\[
\nabla v = v_{i0} - v_{i1} = -v_{i1}, \quad \nabla u = u_{i1} - u_{i1}'. \tag{3.6}
\]

In the whole proof, we work only on the events \(B_{\sigma,\sigma'}(M)\). Firstly, we prove formula (3.23) below, which will imply the first claim (3.4), and similarly the second claim (3.5) will follow with

\[
\sigma, k_{ij}, l_{i}, \kappa_{ij}, v_{i}, \nabla v, \omega_{ij}, \bar{T} \text{ replaced by } \sigma', k'_{ij}, l'_{i}, \kappa'_{ij}, u_{i}, \nabla u, \omega'_{ij}, \bar{T}', \text{ respectively.} \tag{3.7}
\]

In particular, \(l_{io}\) is replaced by \(l'_{io}\), the limit \(\sigma \to \infty\) is replaced by \(\sigma' \to \infty\), but the events \(B_{\sigma,\sigma'}(M)\) remain unchanged.

For the proof of formula (3.4), all Landau symbols \(O\) are understood in the limit as \(\sigma \to \infty\). Note that on \(B_{\sigma,\sigma'}(M)\), we have \(k_{ij} \to \infty\) for any \((i, j) \in \bar{E}\) as \(\sigma \to \infty\). Hence, by Stirling’s formula,

\[
k_{ij}! = \sqrt{2\pi e^{-k_{ij} k_{ij}^{k_{ij} + \frac{1}{2}}}} (1 + O_{M}(\sigma^{-1})). \tag{3.8}
\]

For \((i, j) \in \bar{T}\), the \(k_{ij}\)-dependent part in the definition (1.31) of \(\mathcal{P}(k, l, \bar{T})\) is given by

\[
k_{ij} \left( \frac{W_{ij} l_{i}}{2} \right)^{k_{ij}} \frac{1}{k_{ij}!} = \sqrt{\frac{k_{ij}}{2\pi}} \left( \frac{W_{ij} l_{i} e^{W_{ij} l_{i}}}{2k_{ij}} \right)^{k_{ij}} (1 + O_{M}(\sigma^{-1})). \tag{3.9}
\]

Using

\[
k_{ij} = l_{io} (\omega_{ij} + l_{io}^{-1/2} \kappa_{ij}) = l_{io} \omega_{ij} (1 + O_{M,W}(\sigma^{-1/2})), \tag{3.10}
\]

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and the symmetry $\omega_{ij} = \omega_{ji}$, we deduce
\[
\prod_{(i,j) \in E} \sqrt{\frac{k_{ij}}{2\pi}} = \left( \frac{l_{io}}{2\pi} \right)^{|E|} \prod_{(i,j) \in E} \omega_{ij} \cdot (1 + O_{M,W,G}(\sigma^{-1/2})). \tag{3.11}
\]
One has
\[
\frac{W_{ij}l_{io}}{2k_{ij}}e^{1+2v_i} = \frac{W_{ij}l_{io}e^{1+2v_i}}{2(l_{io}\omega_{ij} + l_{io}^{1/2}\kappa_{ij})} = \frac{\omega_{ij}e^{1+v_i-v_j}}{\omega_{ij} + l_{io}^{-1/2}\kappa_{ij}}. \tag{3.12}
\]
Consequently, we obtain
\[
\log \left[ \frac{W_{ij}l_{io}}{2k_{ij}} \right] = l_{io}(\omega_{ij} + l_{io}^{-1/2}\kappa_{ij}) (\log \omega_{ij} + 1 + v_i - v_j) - l_{io}(\omega_{ij} + l_{io}^{-1/2}\kappa_{ij}) \log (\omega_{ij} + l_{io}^{-1/2}\kappa_{ij}). \tag{3.13}
\]
Using the Taylor expansion $x \log x = x_0 \log x_0 + (1 + \log x_0)(x - x_0) + \frac{(x-x_0)^2}{2x_0} + O((x-x_0)^3)$ as $x \to x_0$ at $x_0 = \omega_{ij}$ for the second term, we deduce
\[
\log \left[ \frac{W_{ij}l_{io}}{2k_{ij}} \right] = l_{io}(\omega_{ij} + l_{io}^{-1/2}\kappa_{ij}) (\log \omega_{ij} + 1 + v_i - v_j) - l_{io}(\omega_{ij} + l_{io}^{-1/2}\kappa_{ij}) \log (\omega_{ij} + l_{io}^{-1/2}\kappa_{ij}) + \frac{\kappa_{ij}^2}{2\omega_{ij}l_{io}} + O_M\left(\sigma^{-3/2}\right).
\]
Since $\omega_{ij} = \omega_{ji}$ and with $(i,j) \in \bar{E}$ there is $(j,i) \in \bar{E}$ as well, we have
\[
\sum_{(i,j) \in \bar{E}} \omega_{ij}(v_i - v_j) = 0. \tag{3.15}
\]
Note that
\[
\sum_{(i,j) \in \bar{E}} \kappa_{ij}v_j = \sum_{(j,i) \in \bar{E}} \kappa_{ji}v_i = \sum_{(i,j) \in \bar{E}} \kappa_{ji}v_i. \tag{3.16}
\]
Using this, Kirchhoff’s rule \[1.24\] for $k_{ij}$, and the definition \[3.6\] of $\nabla v$, we deduce
\[
\sqrt{l_{io}} \sum_{(i,j) \in \bar{E}} \kappa_{ij}(v_i - v_j) = \sqrt{l_{io}} \sum_{(i,j) \in \bar{E}} (\kappa_{ij} - \kappa_{ji})v_i = \sum_{(i,j) \in \bar{E}} (k_{ij} - k_{ji})v_i = \sum_{i \in V} v_i \sum_{j \in V; \{i,j\} \in \bar{E}} (k_{ij} - k_{ji}) = \sum_{i \in V} v_i (\delta_{io}(i) - \delta_{i}(i)) = \nabla v. \tag{3.17}
\]
We remark that \( k' \in K_{i_1,i_1}' \) satisfies the Kirchhoff rules
\[
\sum_{j \in V; \{i,j\} \in E} (k'_{ij} - k'_{ji}) = \delta_{i_1}(i) - \delta_{i_1}(i), \quad i \in V. \tag{3.18}
\]

Hence, the equation analogous to (3.17) for the proof of (3.5) reads as follows:
\[
\sqrt{l_{i_0}} \sum_{(i,j) \in \bar{E}} k'_{ij} (u_i - u_j) = \sum_{i \in V} u_i (\delta_{i_1}(i) - \delta_{i_1}(i)) = \nabla u. \tag{3.19}
\]

Combining (3.15) and (3.17) with (3.14) yields
\[
\prod_{(i,j) \in \bar{E}} \left( \frac{W_{ij} l_i e^{k_{ij}}}{2k_{ij}} \right)^k_{ij} = \exp \left( l_{i_0} \sum_{(i,j) \in \bar{E}} \omega_{ij} + \nabla v - \sum_{(i,j) \in \bar{E}} \frac{k_{ij}^2}{2\omega_{ij}} + O_{M,G} (\sigma^{-1/2}) \right). \tag{3.20}
\]

Inserting (3.11) and (3.20) into (3.9) yields the following for the left-hand side in the claim (3.4):
\[
\mathcal{P}(k,l,\bar{T}) = \exp \left( l_{i_0} \sum_{(i,j) \in \bar{E}} \omega_{ij} + \nabla v - \sum_{(i,j) \in \bar{E}} \frac{k_{ij}^2}{2\omega_{ij}} \right) \cdot \left( \frac{l_{i_0}}{2\pi} \right)^{|\bar{E}|} \prod_{(i,j) \in \bar{E}} \omega_{ij} \prod_{(i,j) \in \bar{T}} \frac{1}{k_{ij}} \prod_{(i,j) \in \bar{E} \setminus \bar{T}} \frac{1}{k_{ij}} \cdot (1 + O_{M,W,G}(\sigma^{-1/2})). \tag{3.21}
\]

Using (3.10) and the fact that \( \bar{T} \) is a spanning tree directed towards \( i_1 \), we obtain
\[
\prod_{(i,j) \in \bar{T}} \frac{1}{l_i} \prod_{(i,j) \in \bar{E} \setminus \bar{T}} \frac{1}{k_{ij}} = \frac{l_{i_0}^{|\bar{E}|}}{2\pi} \prod_{i \in V \setminus \{i_1\}} e^{-2v_i} \prod_{(i,j) \in \bar{E} \setminus \bar{T}} \frac{1}{\omega_{ij}} \cdot (1 + O_{M,W,G}(\sigma^{-1/2})). \tag{3.22}
\]

Combining this with (3.21) yields
\[
\mathcal{P}(k,l,\bar{T}) = \frac{1}{(2\pi l_{i_0})^{|\bar{E}|}} \exp \left( l_{i_0} \sum_{(i,j) \in \bar{E}} \omega_{ij} + \nabla v - \sum_{(i,j) \in \bar{E}} \frac{k_{ij}^2}{2\omega_{ij}} \right) \cdot \prod_{i \in V \setminus \{i_1\}} e^{-2v_i} \prod_{(i,j) \in \bar{T}} \frac{1}{\omega_{ij}} \prod_{(i,j) \in \bar{E}} \omega_{ij} \cdot (1 + O_{M,W,G}(\sigma^{-1/2})). \tag{3.23}
\]

Precisely the same argument with the replacements in (3.7) applied proves (3.5).

It remains to prove (3.4). We substitute the definition (3.6) of \( \nabla v = -v_{i_1} \) into (3.23). The result is already almost the claim (3.4), but still some \( \omega_{ij}'s \) need to be replaced by \( \omega_{ij}'s \). The second identity in (3.3) allows us to do these replacements. Note that the first occurrence of \( \omega_{ij} \) in (3.23), i.e. in \( l_{i_0} \sum_{(i,j) \in \bar{E}} \omega_{ij} \), is kept without replacement \( \omega_{ij} \neq \omega_{ij}' \), as it is scaled with \( l_{i_0} \). Summarizing, we obtain claim (3.4). \( \blacksquare \)

Next, we study the asymptotic behavior of the middle line in (1.30).
Lemma 3.2 (Asymptotics of the density of a path) For $M > 0$, on the events $B_{\sigma, \sigma'}(M)$, one has the following in the limit as $\min\{\sigma, \sigma'\sigma^{-2}\} \to \infty$:

$$
\exp \left( \sum_{\{i,j\} \in E} W_{ij} \left( 1 - \sqrt{1 + l_i + l'_i \sqrt{1 + l_j + l'_j}} \right) \right) \prod_{i \in V \setminus \{i'_1\}} \frac{1}{\sqrt{1 + l_i + l'_i}} = (l'_{i_0})^{\frac{1}{2l_i-1}} \exp \left( \sum_{\{i,j\} \in E} W_{ij} \left( 1 - \cosh(u_i - u_j) - \frac{1}{2} e^{u_i + u_j} (s_i - s_j)^2 \right) \right) 
\cdot \prod_{(i,j) \in E} \exp(-l_{i_0} \omega_{ij} - l'_{i_0} \omega'_{ij}) \prod_{i \in V \setminus \{i'_1\}} e^{-u_i} \cdot \left( 1 + O_{M,W,G} \left( \sigma^{-1/2} + \frac{\sigma^2}{\sigma'} \right) \right).
\label{eq:3.28}
$$

Proof. During the proof, we work on the events $B_{\sigma, \sigma'}(M)$ in the limit $\min\{\sigma, \sigma'\sigma^{-2}\} \to \infty$. Note that this implies $\sigma' \gg \sigma \to \infty$. Furthermore, all $\sigma' l'_i$ and $\sigma l_i$ are bounded from above and below by $M$-dependent positive constants. We use the following Taylor expansion:

$$
\sqrt{1 + l_i + l'_i} = \sqrt{l'_i} + \frac{1 + l_i}{2\sqrt{l'_i}} + O \left( \frac{(1 + l_i)^2}{(l'_i)^{3/2}} \right) = \sqrt{l'_i} + \frac{1 + l_i}{2\sqrt{l'_i}} + O_M \left( \sigma^2 (\sigma')^{-3/2} \right).
\label{eq:3.27}
$$

Hence, we obtain

$$
\sqrt{1 + l_i + l'_i} \sqrt{1 + l_j + l'_j} = \sqrt{l'_i l'_j} + \frac{1}{2} \sqrt{l'_i} \sqrt{l'_j} (1 + l_j) + \frac{1}{2} \sqrt{l'_i} \sqrt{l'_j} (1 + l_i) + O_M \left( \frac{\sigma^2}{\sigma'} \right).
\label{eq:3.26}
$$

Note that

$$
\prod_{(i,j) \in E} \exp(l'_{i_0} \omega'_{ij}) = \prod_{\{i,j\} \in E} \exp(2l'_{i_0} \omega'_{ij}) = \exp \left( \sum_{\{i,j\} \in E} W_{ij} \sqrt{l'_i l'_j} \right).
\label{eq:3.27}
$$

Consequently, we deduce

$$
\exp \left( \sum_{\{i,j\} \in E} W_{ij} \left( 1 - \sqrt{1 + l_i + l'_i \sqrt{1 + l_j + l'_j}} \right) \right) \prod_{(i,j) \in E} \exp(l'_{i_0} \omega'_{ij})
= \exp \left\{ \sum_{\{i,j\} \in E} W_{ij} \left[ 1 - \frac{1}{2} \left( \sqrt{\frac{l'_i}{l'_j}} + \sqrt{\frac{l'_j}{l'_i}} + \sqrt{\frac{l'_i}{l'_j}} l_j + \sqrt{\frac{l'_j}{l'_i}} l_i \right) \right] \right\} \left( 1 + O_{M,W,G} \left( \frac{\sigma^2}{\sigma'} \right) \right).
\label{eq:3.28}
$$

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Inserting the definition (1.36) of $s_i$ yields

\[
\frac{1}{2} \left( \sqrt{\frac{l_i'}{l_j'}} + \sqrt{\frac{l_j'}{l_i'}} \right) = \frac{1}{2} \left( e^{u_i-u_j} + e^{u_j-u_i} \right) = \cosh(u_i-u_j), \tag{3.29}
\]

\[
\frac{1}{2} \left( \sqrt{\frac{l_i'}{l_j'}} + \sqrt{\frac{l_j'}{l_i'}} \right) = \frac{1}{2} \left( e^{u_i-u_j+2v_j} + e^{u_j-u_i+2v_i} \right) = \frac{1}{2} l_{i0} e^{v_i+v_j} \left( e^{(u_i-v_i)-(u_j-v_j)} + e^{(u_j-v_j)-(u_i-v_i)} \right) = l_{i0} e^{v_i+v_j} \cosh((u_i-v_i)-(u_j-v_j)). \tag{3.30}
\]

Next, we insert the definition (1.36) of $s_i$ and replace $\cosh$ by its Taylor expansion $\cosh x = 1 + \frac{1}{2} x^2 + O(x^3)$, $x \to 0$. We obtain

\[
\frac{1}{2} \left( \sqrt{\frac{l_i'}{l_j'}} + \sqrt{\frac{l_j'}{l_i'}} \right) = l_{i0} e^{v_i+v_j} \cosh \left( \frac{s_i-s_j}{\sqrt{l_{i0}}} \right) = l_{i0} e^{v_i+v_j} \left( 1 + \frac{(s_i-s_j)^2}{2l_{i0}} + O \left( \frac{(s_i-s_j)^3}{l_{i0}^{3/2}} \right) \right) = l_{i0} e^{v_i+v_j} + \frac{1}{2} e^{v_i+v_j}(s_i-s_j)^2 + O_M (\sigma^{-1/2}). \tag{3.31}
\]

In analogy to (3.27), we deduce

\[
\prod_{(i,j) \in E} \exp(l_{i0} \omega_{ij}) = \exp \left( \sum_{(i,j) \in E} W_{ij} \sqrt{l_i l_j} \right) = \exp \left( \sum_{\{i,j\} \in E} W_{ij} l_{i0} e^{v_i+v_j} \right). \tag{3.32}
\]

Inserting (3.29), (3.31), and the last equation into (3.28), we obtain

\[
\exp \left( \sum_{(i,j) \in E} W_{ij} \left( 1 - \sqrt{1 + l_i + l_i'} \sqrt{1 + l_j + l_j'} \right) \prod_{(i,j) \in E} \exp(l_{i0} \omega_{ij} + l_{i0}' \omega_{ij}') \right)
\]
\[
= \exp \left( \sum_{(i,j) \in E} W_{ij} \left( 1 - \cosh(u_i-u_j) - \frac{1}{2} e^{v_i+v_j}(s_i-s_j)^2 \right) \right)
\]
\[
\cdot \left( 1 + O_{M,W,G} \left( \sigma^{-1/2} + \frac{\sigma^2}{\sigma'} \right) \right). \tag{3.33}
\]

By (3.3), it follows

\[
e^{v_i+v_j}(s_i-s_j)^2 = e^{u_i+u_j}(s_i-s_j)^2 + O_M (\sigma^{-1/2}). \tag{3.34}
\]
Furthermore, using \( \sqrt{1 + l_i + l'_i} = \sqrt{l'_i(1 + O_M(\sigma/\sigma'))} \), we calculate
\[
\prod_{i \in V \setminus \{i'_i\}} \frac{1}{\sqrt{1 + l_i + l'_i}} = \left(1 + O_{M,G} \left(\frac{\sigma'}{\sigma'}\right)\right) \prod_{i \in V \setminus \{i'_i\}} e^{-u_i}. \tag{3.35}
\]

Combining these facts with \( \text{(3.33)} \) completes the proof of the lemma. \( \blacksquare \)

The last two lemmas are combined in the following proof.

**Proof of Theorem 1.7.** Substituting formula \( \text{(3.24)} \) from Lemma 3.2 and formulas \( \text{(3.4)} \) and \( \text{(3.5)} \) from Lemma 3.1 into the assertion \( \text{(1.30)} \) of Theorem 1.6 yields
\[
P_{i_0}(K_{k,\sigma,k',\sigma'} \cap L_{\sigma,\sigma'}(A) \cap E_{i_1,T,\sigma,i'_1,T',\sigma'})
= \int_A \left(\prod_{i \in V \setminus \{i'_i\}} \frac{1!}{\sqrt{l'_i}}\right) \exp \left( \sum_{i,j \in E} W_{ij} \left(1 - \cosh(u_i - u_j) - \frac{1}{2} e^{u_i + u_j} (s_i - s_j)^2 \right) \right)
\cdot \prod_{(i,j) \in E} \exp(-l_{i_0} \omega_{ij} - l'_{i_0} \omega'_{ij}) \prod_{i \in V \setminus \{i'_i\}} e^{-u_i} \cdot \left(1 + O_{M,W,G} \left(\frac{\sigma'}{\sigma'}\right)\right) 
\cdot \prod_{i \in V \setminus \{i'_i\}} e^{-v_i} \prod_{(i,j) \in E \setminus T} \frac{1}{\omega'_{ij}} \prod_{i,j \in E} \omega'_{ij} \cdot (1 + O_{M,W,G}(\sigma^{-1/2}))
\cdot \prod_{i \in V \setminus \{i'_i\}} e^{-u_i} \prod_{(i,j) \in E \setminus T} \frac{1}{\omega'_{ij}} \prod_{i,j \in E} \omega'_{ij} \cdot (1 + O_{M,W,G}(\sigma^{-1/2})) \prod_{i \in V \setminus \{i'_i\}} dl_i dl'_i. \tag{3.36}
\]

The error term \( 1 + O_{M,W,G} \left(\frac{\sigma^{-1/2} + \sigma^2}{\sigma'}\right) \) dominates all other error terms in this formula. The symmetry \( \omega_{ij} = \omega_{ji} \) yields the following formula, which connects products indexed by directed edges with products indexed by undirected edges.
\[
\prod_{(i,j) \in E \setminus T} \frac{1}{\omega'_{ij}} \prod_{i,j \in E} \omega'_{ij} = \prod_{i,j \in E} \omega'_{ij} \prod_{(i,j) \in E \setminus T} \frac{1}{\omega'_{ij}}. \tag{3.37}
\]

Collecting the factors \( e^{-u_i} \) and \( e^{-v_i} \) and using \( u_{i_0} = v_{i_0} = 0 \) and \( \text{(3.3)} \) gives
\[
\prod_{i \in V \setminus \{i'_i\}} e^{-u_i} \cdot \exp(-v_i) \prod_{i \in V \setminus \{i'_i\}} e^{-2v_i} \cdot \exp(v_i - u_i) \prod_{i \in V \setminus \{i'_i\}} e^{-2u_i}
= e^{u_i + v_i + 2u_i} \prod_{i \in V \setminus \{i'_i\}} e^{-3u_i - 2v_i} = e^{2v_i + 2u_i} \prod_{i \in V \setminus \{i'_i\}} e^{-3u_i - 2v_i} (1 + O_M(\sigma^{-1/2})). \tag{3.38}
\]

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Substituting (3.37) and (3.38) into (3.36) and simplifying the remaining terms yields

\[
P_{\ell_0}(K_{k,\sigma,k',\sigma'} \cap L_{\sigma,\sigma'}(A) \cap E_{i_1,T_{\sigma},i',T_{\sigma'}}) = \left( 1 + O_{M,W,G} \left( \sigma^{-1/2} + \frac{\sigma^2}{\sigma'} \right) \right)
\]

\[
\cdot \int_A \frac{|E|}{(2\pi)^{|E|}} \exp \left( \sum_{(i,j) \in E} W_{ij} \left( 1 - \cosh(u_i - u_j) - \frac{1}{2} e^{u_i + u_j}(s_i - s_j)^2 \right) \right)
\]

\[
\cdot \prod_{\{i,j\} \in T} \omega'_{ij} \prod_{\{i,j\} \in T'} \omega_{ij} \prod_{(i,j) \in E} \frac{1}{(\omega_{ij})^2} \cdot \exp \left( - \sum_{(i,j) \in E} \kappa^2 + (\kappa_{ij})^2 \right)
\]

\[
e^{2u_1 + 2u'_1} \prod_{i \in V \setminus \{i_0\}} e^{-3u_i - 2u_i} dl_idl_i'.
\]  

(3.39)

Using the definition of \( \rho_{\ell_0}^{\text{big}} \) and the relations (1.35) and (3.2), claim (1.40) follows. □

4 Continuum limit and vague convergence

The main result in this section, stated in Corollary 4.3 below, deals with a vague convergence of the random vector in (1.44). This requires two ingredients. First, we need to calculate the Jacobian of the transformation \((s,u) \mapsto (l,l')\). Second, we deal with convergence of a Riemann sum indexed by \((\kappa(\sigma),\kappa'(\sigma,\sigma'))\) to an integral. These indices live on a lattice and still fulfill inhomogeneous Kirchhoff rules. On the other hand, the limiting integral is taken over the linear space \(H\) described by homogeneous Kirchhoff rules (1.41). These two ingredients are treated in the following two lemmas. Recall the definition (1.41) of the measure \(\Lambda_{\sigma,\sigma',\ell_0}\).

Lemma 4.1 (Jacobian of the variable transformation) The following formula describes the Jacobian determinant for the transformation \((s,u) \mapsto (l,l')\):

\[
(l_i l'_i)^{|E| + \frac{1}{2} |V|} \Lambda_{\sigma,\sigma',\ell_0}(dl dl') = (4 \sqrt{l_{i_0} l'_{i_0}})^{1 - |V|} \frac{\sigma \sigma'}{l_{i_0} l'_{i_0}} \prod_{i \in V \setminus \{i_0\}} \frac{l_i l'_i}{l_{i_0} l'_{i_0}} dl_i dl'_i = \prod_{i \in V \setminus \{i_0\}} ds_i du_i
\]

(4.1)

Proof. In this proof, we use the abbreviations

\[
z_i = e^{2u_i} = \frac{l_i}{l_{i_0}} = \frac{l_i}{\sigma - \sum_{j \in V \setminus \{i_0\}} l_j},
\]

(4.2)

\[
z'_i = e^{2u'_i} = \frac{l'_i}{l'_{i_0}} = \frac{l'_i}{\sigma' - \sum_{j \in V \setminus \{i_0\}} l'_j},
\]

(4.3)

for \(i \in V, i \neq i_0\); see (1.35). We obtain the Jacobi matrix elements

\[
\frac{\partial z_i}{\partial l_j} = \frac{1}{l_{i_0}} (\delta_{ij} + l_i) = \frac{1}{l_{i_0}} (\delta_{ij} + z_i),
\]

(4.4)
which are the matrix elements of a perturbation of $l_{i_0}^{-1} \text{Id}$ by a rank 1 matrix. The corresponding Jacobi determinant equals

$$2^{|V|-1} \prod_{i \in V \setminus \{i_0\}} z_i \cdot \det \left( \frac{\partial v_i}{\partial l_j} \right)_{i,j \in V \setminus \{i_0\}} = \det \left( \frac{\partial z_i}{\partial l_j} \right)_{i,j \in V \setminus \{i_0\}} = l_{i_0}^{-|V|} \sigma.$$  \hspace{1cm} (4.5)

The same argument with the primed variables gives us

$$2^{|V|-1} \prod_{i \in V \setminus \{i_0\}} z'_i \cdot \det \left( \frac{\partial u_i}{\partial l'_j} \right)_{i,j \in V \setminus \{i_0\}} = (l'_{i_0})^{-|V|} \sigma'. \hspace{1cm} (4.6)$$

Combining (4.5) and (4.6), we deduce

$$\prod_{i \in V \setminus \{i_0\}} dl_i dl'_i = (4l_{i_0} l'_{i_0})^{|V|-1} l_{i_0} l'_{i_0} \prod_{i \in V \setminus \{i_0\}} z_i z'_i dv_i du_i. \hspace{1cm} (4.7)$$

Using the relation (1.36) between $s$ and $u,v$ the claim follows. \hfill \blacksquare

Let $i_0, i_1, i'_1 \in V$ and $\sigma, \sigma' > 0$. Recall the definitions of $\mathcal{K}_{i_0,i_1,i'_1}$ and $\mathcal{K}^+_{i_0,i_1,i'_1}$ stated in (1.25) and before, and the definitions (1.20) of $\mathcal{L}_{\sigma,\sigma'}$ and (1.1) of $\Omega_{i_0}$. We consider the measure

$$\lambda^+_{\sigma,\sigma',i_0,i_1,i'_1} = \sum_{(k,k')} \delta_{k,k'} \Lambda_{\sigma,\sigma',i_0}(dl \, dl') \hspace{1cm} (4.8)$$

defined on $\mathcal{K}_{i_0,i_1,i'_1} \times \mathcal{L}_{\sigma,\sigma'}$. Let $\lambda_{\sigma,\sigma',i_0,i_1,i'_1}$ be defined as $\lambda^+_{\sigma,\sigma',i_0,i_1,i'_1}$ with the only difference that the summation over $\mathcal{K}^+_{i_0,i_1,i'_1}$ is replaced by $\mathcal{K}_{i_0,i_1,i'_1}$. We introduce the following variant of the map $F_{\sigma,\sigma',i_0}$, cf. (1.43):

$$F_{\sigma,\sigma',i_0,i_1,i'_1} : \mathcal{K}_{i_0,i_1,i'_1} \times \mathcal{L}_{\sigma,\sigma'} \rightarrow (\mathbb{R}^E)^2 \times \Omega_{i_0}^3, \hspace{1cm} (k,k',l,l') \mapsto (k, k', s, v, u) \hspace{1cm} (4.9)$$

using again the equations (1.35), (1.36), (1.37), and (1.38).

**Lemma 4.2 (Vague convergence of the reference measure)**

The image measure $F_{\sigma,\sigma',i_0,i_1,i'_1} \left[ \lambda^+_{\sigma,\sigma',i_0,i_1,i'_1} \right]$ converges vaguely as $\sigma, \sigma' \rightarrow \infty$ to

$$d\kappa_H d\kappa'_H \prod_{i \in V \setminus \{i_0\}} 1_{\{u_i = v_i\}} \, ds_i \, du_i. \hspace{1cm} (4.10)$$
as its image under the restriction map \( R \), corresponding \((\kappa \sigma, \sigma)\) holds for \( \sigma, \sigma \) implies \((\kappa \kappa, \kappa)\). Given a test function \( \sigma, \sigma \) such that for any \((k, k', l, l') \in \mathcal{F}^{-1}[\supp f] \), all components of \( f \) and \( l/\sigma \) and \( l'/\sigma' \) are bounded away from 0, say by \( \varepsilon > 0 \), and bounded above by 1. For \( (i, j) \in E \), the facts \( \omega_{ij}, \omega'_{ij} \geq \frac{W_{ij}}{2}e^{-2M} > 0 \) and \( |\kappa_{ij}|, |\kappa'_{ij}| \leq M \) together with \((4.11)\) imply \( k_{ij}, k'_{ij} > 0 \) for \( \sigma, \sigma' \) large enough. This proves \((4.12)\). Summarizing, vague convergence of the image measure \( F^{-1}_{\sigma, \sigma', i_0, i_1, i'_1}[\lambda_{\sigma, \sigma', i_0, i_1, i'_1}] \) as \( \sigma, \sigma' \to \infty \) implies vague convergence of \( F^{-1}_{\sigma, \sigma', i_0, i_1, i'_1}[\lambda_{\sigma, \sigma', i_0, i_1, i'_1}] \) to the same limit.

Fix a path from \( i_0 \) to \( i_1 \) and another one from \( i_1 \) to \( i'_1 \). Let \( \pi = (\pi_{ij})_{(i,j) \in E} \) be the corresponding edge crossing numbers. Then \( (k - \pi, k' - \pi') \in \mathcal{H}^2 \) implies \((\kappa - l_{i_0}^{-1/2} \pi, \kappa' - (l'_{i_0})^{-1/2} \pi') \in \mathcal{H}^2 \). Let \( \Gamma \subset \mathcal{H} \) denote the lattice which has \( \mathbb{Z}^{\mathcal{E} \setminus \mathcal{T}_0} \) as its image under the restriction map \( \mathbb{R}^E \to \mathbb{R}^{\mathcal{E} \setminus \mathcal{T}_0} \). When \( (k, k') \) runs over \( \mathcal{K}_{i_0, i_1, i'_1} \), the corresponding \( (\kappa - l_{i_0}^{-1/2} \pi, \kappa' - (l'_{i_0})^{-1/2} \pi') \) runs over the lattice \( l_{i_0}^{-1/2} \Gamma \times (l'_{i_0})^{-1/2} \Gamma \). In other words, for any \((l, l') \in \mathcal{L}_{\sigma, \sigma'} \), one has

\[
\sum_{(k, k') \in \mathcal{K}_{i_0, i_1, i'_1}} f(F^{-1}_{\sigma, \sigma', i_0, i_1, i'_1}(k, k', l, l')) = \sum_{\kappa \in l_{i_0}^{-1/2} \Gamma} \sum_{\kappa' \in (l'_{i_0})^{-1/2} \Gamma} f\left(\kappa + \frac{\pi}{\sqrt{l_{i_0}}}, \kappa' + \frac{\pi'}{\sqrt{l'_{i_0}}}, s, v, u\right),
\]

where according to \((1.35)\) and \((1.36)\)

\[
v_i = \frac{1}{2} \log \frac{l_i}{l'_{i_0}}, \quad u_i = \frac{1}{2} \log \frac{l'_i}{l'_{i_0}}, \quad s_i = \frac{\sqrt{l_{i_0}}}{2} \left(\log \frac{l'_i}{l'_{i_0}} - \log l_i\right).
\]
Integrating first (4.13) over \( l \) and \( l' \) with appropriate weights and using Lemma 4.1 in the second equality, we obtain

\[
\int_{K_{i_0,i_1,i_1'}} f \circ F_{\sigma,\sigma',i_0,i_1,i_1'} d\lambda_{\sigma,\sigma',i_0,i_1,i_1'} (4.15)
\]

\[
= \int_{L_{\sigma,\sigma'}} \sum_{\tilde{\kappa} \in (l''_{i_0})^{-1/2}\Gamma} f \left( \tilde{\kappa} + \frac{\pi}{\sqrt{l_{i_0}}}, \tilde{\kappa}' + \frac{\pi'}{\sqrt{l''_{i_0}}}, s, v, u \right) \Lambda_{\sigma,\sigma',i_0} (dl \, dl') \tag{4.16}
\]

with the substitution

\[
v_i = u_i - \frac{s_i}{\sqrt{l_{i_0}}}, \quad l_{i_0} = \frac{\sigma}{\sum_{i \in V} e^{2u_i - 2s_i/\sqrt{l_{i_0}}}}, \quad l''_{i_0} = \frac{\sigma'}{\sum_{i \in V} e^{2u_i}}. \tag{4.17}
\]

Note that \(|u_i - v_i| \leq M/\sqrt{l_{i_0}}\) holds, whenever the integrand in (4.15) is non-zero. We interpret the Riemann sum in (4.16) as an integral over functions which are constant on boxes associated to \(l^{-1/2}\Gamma \times (l''_{i_0})^{-1/2}\Gamma\). These boxes have volume \((l_{i_0} l''_{i_0})^{-1}\). Using the dominated convergence theorem to perform the limit \(\sigma, \sigma' \to \infty\), the claim (4.14) follows.

We endow the set \(O_{\sigma,\sigma',i_0}\) defined in (1.42) with the measure \(\lambda^+_{\sigma,\sigma',i_0}\) which is characterized as follows. When we restrict \(\lambda^+_{\sigma,\sigma',i_0}\) to \(K_{i_0,i_1,i_1'} \times L_{\sigma,\sigma'} \times \{i_1\} \times \{i_1'\} \times \{T\} \times \{T'\}\) for any \(i, i_1, i_1' \in V, T \in T_i, \text{ and } T' \in T_{i_1'}\) and project it down to \(K_{i_0,i_1,i_1'} \times L_{\sigma,\sigma'}\), it becomes \(\lambda^+_{\sigma,\sigma',i_0,i_1,i_1'}\). Recall the definition (1.44) of the map \(F_{\sigma,\sigma',i_0}\) and the definition (1.26) of the random variable \(\xi_{\sigma,\sigma'}\). Theorem 1.7 and the last lemma are combined in the following corollary.

**Corollary 4.3 (Vague convergence to the extended \(H^{2|2}\) model)** The joint sub-probability distribution of

\[
(\kappa(\sigma), \kappa'(\sigma, \sigma'), s(\sigma, \sigma'), v(\sigma), u(\sigma, \sigma'), Z_{\sigma}, Z_{\sigma + \sigma'}, T^{\text{last exit}}(0, \sigma), T^{\text{last exit}}(\sigma, \sigma + \sigma')) \tag{4.18}
\]

with respect to \(P_{i_0}(\cdot \cap \{\xi_{\sigma,\sigma'} \in O_{\sigma,\sigma',i_0}\})\) converges vaguely as \(\min\{\sigma, \sigma' - 2\} \to \infty\) to \(\mu_{\text{big}}^{i_0}\). In other words, for any continuous compactly supported test function \(f : (\mathbb{R}^2)^3 \times \Omega_{i_0}^3 \times V^2 \times T^2 \to \mathbb{R}\), one has

\[
\lim_{\min\{\sigma, \sigma' - 2\} \to \infty} E_{i_0} [f(F_{\sigma,\sigma',i_0}(\xi_{\sigma,\sigma'})), \xi_{\sigma,\sigma'} \in O_{\sigma,\sigma',i_0}] = \int_{\mathcal{H}^2 \times \Omega_{i_0}^3 \times V^2 \times T^2} f \, d\mu_{\text{big}}^{i_0}. \tag{4.19}
\]
Proof. Because $f$ is compactly supported, we can choose a constant $M > 0$ such that for any $(\kappa, \kappa', s, v, u, i_1, i_1', T, T') \in \text{supp} f$ all the components of $\kappa, \kappa', s, v, u$ are bounded in absolute value by $M$. Theorem 1.7 yields in the limit as $\min\{\sigma, \sigma'\sigma'^{-2}\} \to \infty$

$$E_{i_0} [f(F_{\sigma,\sigma'}(\xi_{\sigma,\sigma'}))], \xi_{\sigma,\sigma'} \in O_{\sigma,\sigma',i_0} = \left(1 + O_{M,W,G} \left(\sigma^{-1/2} + \sigma'\right)\right) \int_{O_{\sigma,\sigma',i_0}} f_{i_0} \bigg|_{\lambda^{+}_{\sigma,\sigma',i_0}} d(\lambda^{+}_{\sigma,\sigma',i_0}) \tag{4.19}$$

Note that the density $f_{i_0}$ is continuous. Hence, Lemma 4.2 implies that the last integral converges as $\min\{\sigma, \sigma'\sigma'^{-2}\} \to \infty$ to the following integral.

$$\int_{H^2 \times \Omega \times S \times \mathcal{T}} (f_{i_0} (\kappa, \kappa', s, u, u, i_1, i_1', T, T')) d\kappa d\kappa' ds_1 du_1 di_1' dT dT'$$

$$= \int_{H^2 \times \Omega \times S \times \mathcal{T}} f d\mu_{i_0} \bigg|_{i_0} \tag{4.20},$$

where we used the definition (1.8) of $\mu_{i_0}$ in the last step. This proves the claim. ■

5 Marginals and weak convergence

In this section, we calculate marginals of the extended $H^{2|2}$ measure $\mu_{i_0}$ by integrating out the current vectors $\kappa$ and $\kappa'$ and summing over the endpoints $i_1$ and $i_1'$ of paths and summing over the spanning tree $T$. The integral over $\kappa$ and $\kappa'$ is a Gaussian integral over $H \times H$. It is performed in the following lemma.

Lemma 5.1 (Gaussian integral over currents) The following Gaussian integral formulas hold:

$$\int_{\mathcal{H}} \exp \left( - \sum_{(i,j) \in E} \frac{(k_{ij}')^2}{2\omega_{ij}'} \right) dk_{ij}' = 2^{|E| - |V| + 1} \pi^{\frac{|E| - |V| - 1}{2}} \frac{\prod_{(i,j) \in E} \omega_{ij}'}{\sqrt{\sum_{s \in \mathcal{T}} \prod_{(i,j) \in S} \omega_{ij}'}}, \tag{5.1}$$

$$\int_{\mathcal{H}^2} \exp \left( - \sum_{(i,j) \in E} \frac{(\kappa_{ij}')^2}{2\omega_{ij}'} \right) dk_{ij}' dk_{ij} = 4^{|E| - |V| + 1} \pi^{2|E| - |V| + 1} \frac{\prod_{(i,j) \in E} (\omega_{ij}')^2}{\sum_{s \in \mathcal{T}} \prod_{(i,j) \in S} \omega_{ij}'}. \tag{5.2}$$

Proof. We endow every undirected edge in $E$ with a counting direction. For $\kappa \in \mathcal{H}$ and $i, j \in V$ such that $\{i, j\} \in E$, we introduce the following variables

$$I_{ij} = \frac{1}{\sqrt{2}} (\kappa_{ij} - \kappa_{ji}), \quad J_{ij} = \frac{1}{\sqrt{2}} (\kappa_{ij} + \kappa_{ji}). \tag{5.3}$$

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Note that $I$ is antisymmetric ($I_{ij} = -I_{ji}$) and $J$ is symmetric ($J_{ij} = J_{ji}$). Recall that $\tilde{T}_0$ denotes a directed reference spanning tree. Let $T_0$ be its undirected version. Recall that the restriction map $\iota : \mathcal{H} \to \mathbb{R}^{E \setminus T_0}$ defined in (1.3) is an isomorphism. In other words, the components $\kappa_{\alpha \beta}, (\alpha, \beta) \in \tilde{E} \setminus \tilde{T}_0$, of $\kappa \in \mathcal{H}$ can be chosen arbitrarily while all other $\kappa_{ij}, (i, j) \in \tilde{T}_0$, are determined by the first. We define now a linear map $L : \mathbb{R}^{E \setminus T_0} \to \mathbb{R}^{E \setminus T_0} \times \mathbb{R}^E$. Given $\tilde{\kappa} \in \mathbb{R}^{E \setminus T_0}$, we set $\kappa = \iota^{-1}(\tilde{\kappa})$ and

$$L(\tilde{\kappa}) = ( (I_{ij}(\kappa))_{\{i,j\} \in E \setminus T_0}, (J_{ij}(\kappa))_{\{i,j\} \in E} ),$$

(5.4)

where the vertices $i$ and $j$ in $I_{ij}$ are ordered with respect to the counting direction of the edge $\{i, j\}$, in order to have no ambiguity with the sign of $I_{ij}$. We claim that the determinant of $L$ equals $\pm 2^{\frac{|\mathcal{H}| - 1}{2}}$. In other words, this yields the change of measure

$$\iota[d\kappa_{\mathcal{H}}] = \prod_{\{i,j\} \in E \setminus T_0} d\kappa_{ij} = 2^{-\frac{|\mathcal{H}| - 1}{2}} \prod_{\{i,j\} \in E \setminus T_0} dI_{ij} \prod_{\{i,j\} \in E} dJ_{ij}.$$

(5.5)

Indeed, the matrix associated to $L$ is

$$
\begin{pmatrix}
\frac{\partial J_{ij}}{\partial \kappa_{\alpha \beta}}_{\{i,j\} \in T_0, (\alpha, \beta) \in \tilde{T}_0} \\
\frac{\partial J_{ij}}{\partial \kappa_{\alpha \beta}}_{\{i,j\} \in \mathcal{H} \setminus T_0, (\alpha, \beta) \in \tilde{E} \setminus \tilde{T}_0}
\end{pmatrix},
$$

(5.6)

which can be written as follows, by an appropriate choice of order on the second index:

$$
\begin{pmatrix}
\frac{\partial J_{ij}}{\partial \kappa_{\alpha \beta}}_{\{i,j\} \in T_0, (\beta, \alpha) \in \tilde{T}_0} & \frac{\partial J_{ij}}{\partial \kappa_{\alpha \beta}}_{\{i,j\} \in \mathcal{H} \setminus T_0, (\alpha, \beta) \in \tilde{E} \setminus \tilde{T}_0} \\
\frac{\partial J_{ij}}{\partial \kappa_{\alpha \beta}}_{\{i,j\} \in \mathcal{H} \setminus T_0, (\alpha, \beta) \in \tilde{T}_0} & \frac{\partial J_{ij}}{\partial \kappa_{\alpha \beta}}_{\{i,j\} \in \mathcal{H} \setminus T_0, (\alpha, \beta) \in \tilde{E} \setminus \tilde{T}_0}
\end{pmatrix}.
$$

(5.7)

We order the indices $(\alpha, \beta)$ with $(\alpha, \beta) \in E \setminus T_0$ in the second block column successively by groups of two associated to each nonoriented edge $(\alpha, \beta) \in E \setminus T_0$, taking first the oriented edge corresponding to the arbitrary counting direction. We claim that the Jacobian matrix above takes the following block triangular form:

$$
\begin{pmatrix}
\sqrt{2} \text{id}_{|T_0| \times |T_0|} & (\ast)|T_0| \times 2|E \setminus T_0| \\
(0)_{2|E \setminus T_0| \times |T_0|} & \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \text{id}_{|E \setminus T_0| \times |E \setminus T_0|}
\end{pmatrix},
$$

(5.8)

In order to see why the first block column takes the claimed form, let $(\beta, \alpha) \in \tilde{T}_0$ and take $\tilde{\kappa} = (\delta_\beta(i) \delta_\alpha(j))_{\{i,j\} \in E \setminus T_0} \in \mathbb{R}^{\tilde{E} \setminus \tilde{T}_0}$. Then, $\kappa = \iota^{-1}(\tilde{\kappa})$ is given by $\kappa_{\alpha \beta} = \kappa_{\beta \alpha} = 1$ and
$\kappa_{ij} = 0$ otherwise. This implies $I_{ij}(\kappa) = 0$ for all $\{i, j\} \in E$, $J_{\alpha\beta}(\kappa) = \sqrt{2}$, and $J_{ij}(\kappa) = 0$ otherwise. This explains the blocks $\sqrt{2} \text{id}$ and 0. The expression for the lower right block in the matrix (5.8) follows from the definition (5.3) using that $\kappa_{\alpha\beta}$ with $(\alpha, \beta) \in \bar{E} \setminus \bar{T}_0$ are linearly independent variables. We conclude

$$| \det L | = \sqrt{2} | T_0 | \left[ \det \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right]^{| E \setminus T_0 |} = 2^{\frac{| V \setminus T_0 |}{2}} 1^{1 | E \setminus T_0 |} = 2^{\frac{| V \setminus T_0 |}{2}},$$

(5.9)
in other words (5.5) holds.

Note that for all $\{i, j\} \in E$,

$$I_{ij}^2 + J_{ij}^2 = \frac{1}{2} ( (\kappa_{ij} - \kappa_{ji})^2 + (\kappa_{ij} + \kappa_{ji})^2 ) = \kappa_{ij}^2 + \kappa_{ji}^2.$$

(5.10)

Consequently, we obtain

$$\int_{\mathcal{H}} \exp \left( - \sum_{(i, j) \in E} \frac{\kappa_{ij}^2}{2 \omega_{ij}^2} \right) d\kappa_{\mathcal{H}} = 2^{-\frac{| V \setminus T_0 |}{2}} \int_{\mathbb{R}^{E \setminus T_0}} \prod_{(i, j) \in E} \exp \left( - \frac{I_{ij}^2}{2 \omega_{ij}^2} \right) \prod_{(i, j) \in E \setminus T_0} dI_{ij} \prod_{(i, j) \in E} \int_{\mathbb{R}} \exp \left( - \frac{J_{ij}^2}{2 \omega_{ij}^2} \right) dJ_{ij}
$$

$$= 2^{-\frac{| E \setminus T_0 |}{2}} \frac{\pi^{| E \setminus T_0 |}}{2} \int_{\mathbb{R}^{E \setminus T_0}} \prod_{(i, j) \in E} \exp \left( - \frac{I_{ij}^2}{2 \omega_{ij}^2} \right) \prod_{(i, j) \in E \setminus T_0} dI_{ij} \prod_{(i, j) \in E} \sqrt{\omega_{ij}^2}.$$

(5.11)

For $e \in E \setminus T_0$, let $c_e$ be the unique oriented cycle in $T_0 \cup \{e\}$ containing the edge $e$ in its counting direction. We define the matrix $B = (B_{ef})_{e, f \in E \setminus T_0}$ by

$$B_{ee} = \sum_{g \in c_e} \frac{1}{\omega_g^2}, \quad B_{ef} = \sum_{g \in c_e \cap c_f} \pm \frac{1}{\omega_g^2} \quad \text{for } e \neq f,$$

(5.12)

where the signs in the last sum are chosen to be +1 if the edge $g$ has in $c_e$ and $c_f$ the same direction and −1 otherwise. As was shown on page 68 of [MR06], one has

$$\sum_{(i, j) \in E} \frac{I_{ij}^2}{2 \omega_{ij}^2} = \frac{1}{2} I^T B I$$

(5.13)

with the restriction $I = (I_{ij})_{(i, j) \in E \setminus T_0}$ to $E \setminus T_0$. We abbreviate $dI = \prod_{(i, j) \in E \setminus T_0} dI_{ij}$. We can apply the calculation on page 20 of [KR00] to obtain

$$\int_{\mathbb{R}^{E \setminus T_0}} \prod_{(i, j) \in E} \exp \left( - \frac{I_{ij}^2}{2 \omega_{ij}^2} \right) \prod_{(i, j) \in E \setminus T_0} dI_{ij} = \int_{\mathbb{R}^{E \setminus T_0}} e^{-\frac{1}{2} I^T B I} dI$$

$$= (2\pi)^{\frac{| E \setminus T_0 |}{2} + \frac{| V \setminus T_0 |}{2}} \frac{\prod_{(i, j) \in E} \sqrt{\omega_{ij}^2}}{\sqrt{\sum_{S \in T} \prod_{(i, j) \in S} \omega_{ij}^2}}.$$

(5.14)
We conclude the first claim (5.1):

\[
\int_{H} \exp \left( - \sum_{(i,j) \in E} \frac{\kappa_{ij}^2}{2 \omega'_{ij}} \right) d\kappa_H = 2^{\frac{|E|-|V|+1}{2}} \frac{\sqrt{\omega'}_{ij}}{2} \frac{\prod_{\{i,j\} \in E} \sqrt{\omega'}_{ij}}{\prod_{\{i,j\} \in E} \sqrt{\omega'}_{ij}} \frac{\prod_{\{i,j\} \in E} \sqrt{\omega'}_{ij}}{\prod_{\{i,j\} \in E} \sqrt{\omega'}_{ij}}
\]

\[
= 2^{\frac{|E|-|V|+1}{2}} \frac{\prod_{\{i,j\} \in E} \sqrt{\omega'}_{ij}}{\prod_{\{i,j\} \in E} \sqrt{\omega'}_{ij}} \frac{\prod_{\{i,j\} \in E} \sqrt{\omega'}_{ij}}{\prod_{\{i,j\} \in E} \sqrt{\omega'}_{ij}}.
\]

(5.15)

Taking the square of this equation, the second claim (5.2) follows also. ■

The main theorems follow now easily by combining the previous results:

**Proof of Theorem 1.3.** Combining Lemma 5.1 with the definition (1.8) of \( \mu_{\text{big}}^{\text{susy}} \) and the definition (1.7) of \( \rho_{\text{big}}^{\text{susy}} \), claim (1.9) follows. Note that the following holds:

\[
\prod_{\{i,j\} \in T} \omega'_{ij} = 2^{-|V|-1} \prod_{\{i,j\} \in T} W_{ij} e^{u_i + u_j}
\]

(5.16)

Summing over \( i_1, i'_1 \in V \) and \( T \in T \), claim (1.10) follows. Using this and the fact that \( \mu_{\text{big}}^{\text{susy}} \) is a probability measure, cf. (1.3), imply that \( \mu_{\text{big}} \) is a probability measure as well. ■

**Proof of Theorem 1.4.** To integrate the \( s \)-dependent part in \( \rho_{\text{big}}^{\text{susy}} \), we calculate the following Gaussian integral:

\[
\int_{\Omega_1} \exp \left( - \frac{1}{2} \sum_{\{i,j\} \in E} W_{ij} e^{u_i + u_j} (s_i - s_j)^2 \right) \prod_{i \in V \setminus \{i_0\}} ds_i
\]

\[
= \int_{\Omega_1} \exp \left( - \sum_{\{i,j\} \in E} \omega'_{ij} (s_i - s_j)^2 \right) \prod_{i \in V \setminus \{i_0\}} ds_i = \frac{\pi^{|V|-1}}{\sqrt{\prod_{\{i,j\} \in S} \omega'_{ij}}}.
\]

(5.17)

Note that \( \mu_{\text{big}}^{\text{susy}} - \text{almost surely} \) \( u = v \) and hence \( \omega'_{ij} = \omega_{ij} \). Using formula (5.1) from Lemma 5.1 for the integration over \( \kappa' \), we obtain

\[
\int_{H \times \Omega_1 \times V \times T} \rho_{\text{big}}^{\text{susy}}(\kappa, \kappa', s, v, u, i_1, i'_1, T, T') d\kappa_H \prod_{i \in V \setminus \{i_0\}} 1_{u_i = v_i} \cdot d\kappa'_{H'} \prod_{i \in V \setminus \{i_0\}} 1_{u'_i = v'_i}
\]

\[
= \frac{2^{|V|-1}}{2^{|E|}} \exp \left( \sum_{\{i,j\} \in E} W_{ij} (1 - \cosh(v_i - v_j)) \right) \prod_{\{i,j\} \in T} \omega_{ij} \cdot \prod_{\{i,j\} \in T'} \omega_{ij}
\]

\[
\cdot \exp \left( - \sum_{\{i,j\} \in E} \frac{\kappa_{ij}^2}{2 \omega_{ij}} \right) \sum_{i \in V} e^{2v_i} \prod_{i \in V \setminus \{i_0\}} e^{-v_i}.
\]

(5.18)

Using the definition (1.6) of \( \omega_{ij} \), the claim follows. ■
Finally, we prove the weak convergence results.

**Proof of Theorem 1.8.** By Theorem 1.3 the measure $\mu_{i_0}^{\text{big}}$ is a probability measure. Because vague convergence of sub-probability measures to a probability measure implies weak convergence, Corollary 4.3 yields the claimed weak convergence. For the constant test function $f = 1$, the last claim (1.46) follows.

**Proof of Corollary 1.9.**

As a consequence of Theorem 1.8, the reduced vector $(\kappa(\sigma), v(\sigma), Z_{\text{last exit}}(0, \sigma))$ converges weakly to the marginal $\mathcal{L}_{\mu_{i_0}^{\text{big}}}^{\text{big}}(\kappa, v, i_1, T)$ as $\min\{\sigma, \sigma'\sigma^{-2}\} \to \infty$ with respect to the sub-probability measure $P_{i_0}(\cdot \cap \{\xi_{\sigma,\sigma'} \in \mathcal{O}_{\sigma,\sigma',i_0}\})$. Because of

$$\{\xi_{\sigma,\sigma'} \in \mathcal{O}_{\sigma,\sigma',i_0}\} \subseteq \{\xi_{\sigma} \in \mathcal{Q}_{\sigma,i_0}\}$$

we have the same weak convergence also with respect to the sub-probability measure $P_{i_0}(\cdot \cap \{\xi_{\sigma} \in \mathcal{Q}_{\sigma,i_0}\})$, again as $\min\{\sigma, \sigma'\sigma^{-2}\} \to \infty$. However, the second time scale $\sigma'$ does not play any role in the last statement anymore. Hence, we may replace the limit $\min\{\sigma, \sigma'\sigma^{-2}\} \to \infty$ by the single-time limit $\sigma \to \infty$.

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