GROUND STATES FOR ASYMPTOTICALLY PERIODIC FRACTIONAL KIRCHHOFF EQUATION WITH CRITICAL SOBOLEV EXPONENT

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Abstract. In this paper, we study the following fractional Kirchhoff equation with critical nonlinearity

\[
(a + b \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx)(-\Delta)^s u + V(x)u = K(x)|u|^{2^*_s - 2} u + \lambda g(x, u), \quad \text{in } \mathbb{R}^3,
\]

where \(a, b > 0, \lambda > 0, \) \((-\Delta)^s\) is the fractional Laplace operator with \(s \in (\frac{3}{4}, 1)\) and \(2^*_s = \frac{6}{3 - 2s}\), \(V, K\) and \(g\) are asymptotically periodic in \(x\). The existence of a positive ground state solution is obtained by variational method.

1. Introduction. This paper deals with the existence of positive ground state solution to the following asymptotically periodic fractional Kirchhoff equation involving critical Sobolev exponent

\[
(a + b \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx)(-\Delta)^s u + V(x)u = K(x)|u|^{2^*_s - 2} u + \lambda g(x, u), \quad \text{in } \mathbb{R}^3,
\]

where \(a, b > 0, \lambda > 0, \) \((-\Delta)^s\) is the fractional Laplace operator defined as (see [12, Lemma 3.2])

\[
(-\Delta)^s u = C_{3,s} P.V. \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3 + 2s}} dy = -\frac{C_{3,s}}{2} \int_{\mathbb{R}^3} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{3 + 2s}} dy.
\]

Here, \(C_{3,s}\) is a positive depending on \(3\) and \(s\).

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When \( s = 1 \), problem \((1.1)\) reduces to the following classical Kirchhoff equation
\[
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + V(x) u = f(x, u), \; \mathbb{R}^3, 
\]
which is related to the stationary analogue of the Kirchhoff equation
\[
u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, u) 
\]
Equation \((1.4)\) was proposed by Kirchhoff [23] as a result of extending the well-know D’Alembert wave equation. Equation \((1.3)\) is called to be nonlocal due to the presence of the term \( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \), which makes the research of such problem very difficult. It should be pointed out that nonlocal problem \((1.3)\) also appears in many fields such as biological systems [3]. Problem \((1.3)\) received a great deal of attention after Lions [27] introduced a functional analysis approach, see, for example, [1, 15, 16, 24, 39] and the references therein. Recently, Fiscella and Valdinoci [17] firstly gave a very detailed introduction of the following generalized fractional Kirchhoff equations in physical background and their applications
\[
\begin{align*}
\{ & M \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y) \, dxdy \right) \mathcal{L}_K u = \lambda f(x, u) + |u|^{2^*_s - 2} u \text{ in } \Omega, \\
& u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, 
\end{align*}
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( M \) is a Kirchhoff function (which contains the case \( M(t) = a + bt \)). The nonlocal operator \( \mathcal{L}_K \) is defined by:
\[
\mathcal{L}_K(x) := \frac{1}{2} \int_{\mathbb{R}^N} \left( u(x+y) + u(x-y) - 2u(x) \right) K(y) \, dy, \; x \in \mathbb{R}^N,
\]
where \( K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty) \) is a measurable function and satisfies following assumptions:
there exists \( \theta > 0 \) and \( s \in (0, 1) \) such that
\[
\theta |x|^{-(N+2s)} \leq K(x) \leq \theta^{-1} |x|^{-(N+2s)}, \forall x \in \mathbb{R}^N \setminus \{0\}.
\]
They showed the existence of non-negative solutions for equation \((1.5)\) under suitable conditions of \( M \) and \( f \). We also refer to [19, 20, 33, 35, 34, 36] and the references therein for more results of nonlocal operator \( \mathcal{L}_K \) (in which covers the special case \( (-\Delta)^s \) if \( K(y) = |y|^{-(N+2s)} \)). The general analysis methods for elliptic PDEs involving classical Laplacian operator \(-\Delta\) can not be directly applied to problem \((1.1)\) or \((1.5)\) since fractional operator \((-\Delta)^s\) is nonlocal. In [7], Caffarelli and Silvestre developed a crucial tool for transferring the nonlocal equation \((1.1)\) into a local problem. The additional details on the extension method from [7] can be found in [6, Chapter4.2]. Zhang, Xu and Zhang [42] obtained the existence of ground states and infinitely many geometrically distinct solutions for equation \((1.1)\) with \( b = 0 \) when \( V \) and \( f \) are asymptotically periodic and periodic in \( x \), respectively. A similar result for the critical case \( f(x, u) = |u|^{2^*_s} + \lambda g(x, u) \) was established in [25]. For further details about problem \((1.1)\) with \( b = 0 \), we refer to [5, 11, 14] and the references therein. In [4], Autuori, Fiscella and Pucci studied problem \((1.1)\) involving a generalized Kirchhoff function, and they obtained the the existence and the asymptotic behavior of nonnegative solutions. Pucci, Xiang and Zhang [31] investigated a nonhomogeneous fractional \( p\)-Laplacian equation of Schrödinger-Kirchhoff type. We also make reference to [8] and the references therein.
for more results of fractional Kirchhoff type involving generalized Kirchhoff function, in which contains the so-called degenerate case (when $b = 0$). Zhang et al. [43] used the $s$-harmonic extension technique in [7] and established the Pohožăev identity of (1.1), they obtained some results on the existence of ground state solutions for $s \in [\frac{1}{2}, 1)$ and proved the non-existence result for $s \in (0, \frac{1}{2})$. Here we refer to [2, 9, 10, 26, 29, 30, 32] and the references therein for more results of the fractional Kirchhoff type equations involving variational methods. We also refer to [13] for more results about nonlinear and nonlocal problems in the whole space.

Inspired by the works formulated above, in the present paper, we obtain the following conditions:

(V) $V \in L^{\infty}(\mathbb{R}^3)$, $V_0 := \inf_{x \in \mathbb{R}^3} V(x) > 0$, and there exists a function $V_p \in L^{\infty}(\mathbb{R}^3)$, 1-periodic in $x_i$, $1 \leq i \leq 3$, such that $V - V_p \in F$, and $0 < V(x) \leq V_p(x)$, $\forall x \in \mathbb{R}^3$, where

$$F := \left\{ h \in L^{\infty}(\mathbb{R}^3) : \text{for every } \varepsilon \geq 0, \text{ meas}\{x \in \mathbb{R}^3 : |h(x)| \geq \varepsilon \} < +\infty \right\}.$$  

(K) $K \in L^{\infty}(\mathbb{R}^3)$, $K(x) > 0$ and $K(x) - K(x_0) = O(|x - x_0|^\alpha)$ as $x \to x_0$, where $\alpha > 0$ and $K(x_0) = \max_{x \in \mathbb{R}^3} K(x) > 0$, and there exists a function $K_p \in L^{\infty}(\mathbb{R}^3)$, 1-periodic in $x_i$, $1 \leq i \leq 3$, such that $K - K_p \in F$, $0 < K_p(x) \leq K(x)$, for all $x \in \mathbb{R}^3$.

(g1) $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $g(x, t) = o(|t|)$ uniformly in $x$ as $|t| \to 0$ and $g(x, t) = 0$ for $t \leq 0$.

(g2) $|g(x, t)| \leq c_1 (1 + |t|^{q-1})$ form some $c_1 > 0$ and $2 < q < 2_s^* = \frac{6}{3 - 2s}$.

(g3) $tg(x, t) - 4G(x, t) \geq g(x, \theta t) \theta t - 4G(x, \theta t)$, $\forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}$ and $\forall \theta \in [0, 1]$, where $G(x, t) = \int_0^t g(x, \tau) d\tau$.

(g4) There exists a function $g_p \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, 1-periodic in $x_i$, $1 \leq i \leq 3$, such that

(i) $|g(x, t)| \geq |g_p(x, t)|, \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}$.

(ii) $|g(x, t) - g_p(x, t)| \leq |h(x)|(1 + |t|^{q-1}), \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}, h \in F$.

(iii) $tg_p(x, t) - 4G_p(x, \theta t) \geq g_p(x, \theta t) \theta t - 4G_p(x, \theta t), \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}$ and $\forall \theta \in [0, 1]$, where $G_p(x, t) = \int_0^1 g_p(x, \tau) d\tau$.

Under the hypothesis (V) and (K) we show that (1.1) admits a ground state solution (a solution whose energy is minimal among the set of nontrivial solution of (1.1)). Our main result is the following.

**Theorem 1.1.**

(1) Assume (V), (K) and (g1) - (g4) hold. Let $\alpha \geq 3 - 2s$ and $g$ satisfies

(g5) There exist $p \in (2, 2_1^*)$ such that $|G(x, t)| \geq C_0 |t|^p$ for some $C_0 > 0$.

(i) If $p \in (\frac{4s}{3-2s}, 2_1^*)$, then problem (1.1) has a positive ground state solution for any $\lambda > 0$;

(ii) If $p \in (2, \frac{4s}{3-2s})$, then problem (1.1) possesses a positive ground state solution provided that $\lambda > 0$ is sufficiently large.

(2) Assume (V1), (V2), (K) and (g1) - (g4) hold. Let $0 < \alpha \leq 3 - 2s$ and $g$ satisfies
Preliminary results.

of positive ground state solutions for asymptotically periodic problem (1.1).

liminary Lemmas which will be used later. In section 3, we prove that the existence endowed with the natural norm $D^{s,2}$.

This paper is organized as follows. In section 2 we collect some necessary preliminary Lemmas which will be used later. In section 3, we prove that the existence of positive ground state solutions for asymptotically periodic problem (1.1).

2. Preliminary results. In this section, we firstly define the homogeneous fractional Sobolev space $D^{s,2}(\mathbb{R}^3)$ as follows

$$D^{s,2} := \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x) - u(y)|}{|x-y|^{3+2s}} \in L^2(\mathbb{R}^3) \times \mathbb{R}^3) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the Gagliardo semi-norm

$$\|u\|_{D^{s,2}} := [u]_{s,2} = \left( \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} dxdy \right)^{\frac{1}{2}},$$

for any $s \in (0,1)$. The fractional Sobolev space $H^s(\mathbb{R}^3)$ is defined by

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \frac{u(x) - u(y)}{|x-y|^{\frac{3}{2}+s}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\},$$

equipped with the natural norm

$$\|u\|^2_{H^s(\mathbb{R}^3)} := \left( \int_{\mathbb{R}^3} |u|^2 dx + \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} dxdy \right)^{\frac{1}{2}}.$$
From [12, Proposition 3.4 and Proposition 3.6], we know that
\[ \|(-\Delta)^{\frac{\alpha}{2}} u\|^2 = \frac{1}{2} C_{3,s} |u|_{s,2}^2. \]

It is well known that the embedding \( H^s(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3) \) is continuous for any \( t \in [2, 2^*_s] \), and there exists a best constants \( S_s > 0 \) such that
\[ S_s = \inf_{u \in D^{1,2}_{s}(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{(\int_{\mathbb{R}^3} |u|^{2^*_s} dx)^{\frac{2}{2^*_s}}}, \]  
(2.1)
which is called the fractional Sobolev critical exponent. Moreover, the embedding \( H^s(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3) \) is locally compact for any \( t \in [2, 2^*_s] \) (see, e.g., [12, Theorem 6.2]).

We consider the Hilbert space \( H^s(\mathbb{R}^3) \) endowed with one of the following norms:
\[ \|u_n\| = \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} V(x) u^2(x) dx \right)^{\frac{1}{2}}, \]
\[ \|u_n\|_p = \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} V_p(x) u^2(x) dx \right)^{\frac{1}{2}}, \]
From the hypotheses (V), the norms \( \| \cdot \| \) and \( \| \cdot \|_p \) are equivalent to the standard norm in \( H^s(\mathbb{R}^3) \). \( H^s(\mathbb{R}^3) \) is a Hilbert space equipped with the inner product
\[ (u, v) = a \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} V(x) uv dx, \]
and the norm
\[ \|u\|^2 = (u, u) = a \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} V(x) u^2 dx. \]
The energy functional associated with (1.1), \( I : H^s(\mathbb{R}^3) \rightarrow \mathbb{R} \) is defined as
\[ I(u) = \frac{1}{2} \|u\|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^2 - \frac{1}{2^*_s} \int_{\mathbb{R}^3} K(x)|u|^{2^*_s} dx - \lambda \int_{\mathbb{R}^3} G(x, u) dx. \]
Obviously, \( I \) is well-defined in \( u \in H^s(\mathbb{R}^3) \) and \( I \in C^1(H^s(\mathbb{R}^3), \mathbb{R}) \). Moreover,
\[ \langle I'(u), v \rangle = (a + b \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} V(x) uv dx
- \int_{\mathbb{R}^3} K(x)|u|^{2^*_s - 2} uv dx - \lambda \int_{\mathbb{R}^3} g(x, u) v dx, \]
for any \( u, v \in H^s(\mathbb{R}^3) \). Therefore, a critical point of \( I \) is a weak solution of (1.1).

Let us define by \( \mathcal{N} \) the Nehari manifold [41] associated to \( I \), given by
\[ \mathcal{N} := \left\{ u \in H^s(\mathbb{R}^3) \setminus \{0\} \mid \langle I'(u), u \rangle = 0 \right\}. \]
The least energy on \( \mathcal{N} \) is defined by
\[ c := \inf_{u \in \mathcal{N}} I(u). \]  
(2.2)

We firstly recall the following vanishing Lemma [14].

**Lemma 2.1.** Let \( \{u_n\} \) be a bounded sequence in \( H^s(\mathbb{R}^3) \) and it satisfies
\[ \lim_{n \to +\infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |u_n(x)|^2 dx = 0, \]
where \( R > 0 \). Then \( u_n \to 0 \) in \( L^t(\mathbb{R}^3) \) for every \( 2 < t < 2^*_s \).
We need the following results to overcome the non-differentiability of $N$.

**Lemma 2.2.** If the assumptions (V), (K) and $(g_1) - (g_4)$ hold, then the following statements hold:

(i) For any $u \in H^s(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $h(t_u) = \max_{t \geq 0} h(t)$, $h'(t) > 0$, $\forall t \in (0, t_u)$ and $h'(t) < 0$, $\forall t \in (t_u, \infty)$. Moreover, $tu \in N$ if and only if $t = t_u$, here $h(t) := I(tu)$.

(ii) There is $\tau > 0$ such that $t_u > \tau$, $\forall u \in S_1$; and for each compact subset $W \subset S_1$, there exists $C_W > 0$ such that $C_W \geq t_u$, $\forall u \in W$.

**Proof.** (i) By $(g_1) - (g_4)$, for each $\delta > 0$, there is $C_\delta > 0$ such that

$$|g(x, t)| \leq \delta |t| + C_\delta |t|^{q-1}$$

and

$$|G(x, t)| \leq \delta |t|^2 + C_\delta |t|^q, \forall t \in \mathbb{R}.$$  \hfill (2.3)

By $(2.3)$, for $t > 0$ small,

$$h(t) = I(tu) = \frac{1}{2} t^2||u||^2 + \frac{b}{4} t^4 \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{4}} u|^2 dx \right)$$

$$- \frac{t^{2s}}{2s} \int_{\mathbb{R}^3} K(x)|u|^{2s} dx - \lambda \int_{\mathbb{R}^3} G(x, tu) dx$$

$$\geq \frac{1}{2} t^2||u||^2 - \delta \lambda C t^2 ||u||^2 - \lambda C_\delta t^q ||u||^q - Ct^{2s}||u||^{2s} > 0$$

and

$$h'(t) = t||u||^2 + bt^3 \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{4}} u|^2 dx \right)$$

$$- t^{2s-1} \int_{\mathbb{R}^3} K(x)|u|^{2s} dx - \lambda \int_{\mathbb{R}^3} g(x, tu) u dx$$

$$\geq t||u||^2 - \delta \lambda C t ||u||^2 - \lambda C_\delta t^{q-1} ||u||^q - Ct^{2s-1}||u||^{2s} \geq 0.$$  

Moreover,

$$h(t) \leq \frac{1}{2} t^2||u||^2 + \frac{1}{2} t^4||u||^4 - \frac{t^{2s}}{2s} \int_{\mathbb{R}^3} K(x)|u|^{2s} dx \to -\infty$$

as $t \to \infty$. Hence $h$ has a positive maximum, that is, there exists a $t_u > 0$ such that $h'(t_u) = 0$ and $h'(t) > 0$ for $0 < t < t_u$. We claim that $h'(t) \neq 0$, $\forall t > t_u$. Suppose by contradiction that there exists $t_u < t_1 < +\infty$ such that $h'(t_1) = 0$ and $h(t_u) \geq h(t_1)$. It follows from $(g_3)$ that

$$h(t_1) = h(t_1) - \frac{t_1}{4} h'(t_1)$$

$$= \frac{t_1^2}{4}||u||^2 + \frac{1}{4} \int_{\mathbb{R}^3} \left( g(x, t_1 u) t_1 u - 4G(x, t_1 u) \right) dx + \left( \frac{1}{4} - \frac{1}{2s} \right) \frac{t_1^{2s}}{2s} \int_{\mathbb{R}^3} K(x)|u|^{2s} dx$$

$$\geq \frac{t_1^2}{4}||u||^2 + \frac{1}{4} \int_{\mathbb{R}^3} \left( g(x, t_1 u) t_1 u - 4G(x, t_2 u) \right) dx + \left( \frac{1}{4} - \frac{1}{2s} \right) \frac{t_1^{2s}}{2s} \int_{\mathbb{R}^3} K(x)|u|^{2s} dx$$

$$= h(t_u) - \frac{t_u}{4} h'(t_u) = h(t_u),$$

which is contradiction. So $h'(t) \neq 0$ for all $t > t_u$. Moreover, it is easy to see that $tu \in N$ if and only if $t = t_u$ by $g'(t) = t^{-1} \langle I'(tu), tu \rangle$. This completes the proof of Lemma 2.2(i).
(ii) Since \( u \in S_1 \) and \( t_n u \in N \), we obtain
\[
t_n^2 ||u||^2 \leq t_n^2 ||u||^2 + \beta t^4 \left( \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} u|^2 dx \right)^2
\]
\[
= t_n^2 \int_{\mathbb{R}^3} K(x)||u||^{2^*} dx + \lambda \int_{\mathbb{R}^3} g(x, t_n u) t_n u dx
\]
\[
\leq \lambda C_\delta t_n^q + \lambda C_\delta t_n^q + C t_n^{2^*},
\]
this implies that \( t_n \geq \tau > 0 \). If there is \( \{w_n\} \subset W \) with \( t_n := t_{w_n} \to +\infty \), then there exists a \( w \in W \) such that \( w_n \rightharpoonup w \) in \( H^s(\mathbb{R}^3) \) since \( W \) is compact. Direct computations yield
\[
I(t_n w_n) \leq \frac{1}{2} t_n^2 ||w_n||^2 + \frac{b t^4}{4} \left( \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} w_n|^2 dx \right)^2 - \frac{t_n^{2^*}}{2^*} \int_{\mathbb{R}^3} K(x)||w_n||^{2^*} dx
\]
\[
= t_n \left\{ \frac{1}{t_n^2} + \frac{b}{4} \left( \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} w_n|^2 dx \right)^2 - \frac{t_n^{2^*}}{2^*} \int_{\mathbb{R}^3} K(x)||w_n||^{2^*} dx \right\} \to -\infty.
\]
On the other hand, by \( v_n := t_n w_n \in N \) and condition \((g_1)\), we deduce that
\[
I(t_n w_n) = I(v_n) = I(v_n) - \frac{1}{4} (P'(v_n), v_n)
\]
\[
= \frac{1}{4} ||v_n||^2 + \frac{1}{4} \int_{\mathbb{R}^3} \left( g(x, v_n) v_n - 4G(x, v_n) \right) dx + \left( \frac{1}{4} - \frac{1}{2^*} \right) \int_{\mathbb{R}^3} K(x)||v_n||^{2^*} dx \geq 0,
\]
which yields a contradiction. Thus, the conclusion (ii) holds. \( \square \)

**Lemma 2.3.** There exists a \( \rho > 0 \) such that \( c := \inf_{u \in N} I(u) \geq \inf_{u \in S_\rho} I(u) > 0 \), where \( S_\rho := \{ u \in H^s(\mathbb{R}^3) : ||u|| = \rho \} \).

**Proof.** For \( u \in H^s(\mathbb{R}^3) \backslash \{0\} \), by (2.3) and Sobolev embedding inequality, we have
\[
I(u) \geq \frac{1}{2} ||u||^2 - \lambda C_\delta ||u||^2 - \lambda C_\delta ||u||^q - C ||u||^{2^*}.
\]
Hence, we can choose some \( \rho \in (0, 1) \) such that \( I(u) \geq \alpha > 0 \) with \( ||u|| = \rho \). Lemma 2.2(i) implies that \( I(u) = \max_{t > 0} I(tu), \forall u \in N \). We may take a \( l > 0 \) such that \( lu \in S_\rho \). Then
\[
I(u) \geq I(\alpha u) = \inf_{u \in S_\rho} I(u) \geq \alpha > 0 \text{ and } c := \inf_{u \in N} I(u) \geq \inf_{u \in S_\rho} I(u) > 0.
\]
\( \square \)

**Lemma 2.4.** The functional \( I \) is coercive on \( N \).

**Proof.** If this result does not hold, then there exist a sequence \( \{u_n\} \subset N \) and a \( d > 0 \) such that \( ||u_n|| \to +\infty \) and \( I(u_n) \leq d \). Let \( v_n = \frac{u_n}{||u_n||} \), going to a subsequence if necessary, we may assume that \( v_n \rightharpoonup v \) in \( H^s(\mathbb{R}^3) \), \( v_n \to v \) in \( L_{\text{loc}}^q(\mathbb{R}^3) \) for \( q \in [2, 2^*) \), \( v_n(x) \to v(x) \) a.e. in \( \mathbb{R}^3 \). Since \( v_n \neq 0 \), there exists a point \( y \in \mathbb{R}^3 \) such that
\[
\delta := \inf_{B_1(z)} ||v_n||^2 dx > 0.
\]
Set \( \xi(z) := \inf_{B_1(z)} ||v_n||^2 dx \). It is easy to see that \( \xi(z) \) is continuous function on \( \mathbb{R}^3 \). There exists a \( R > 0 \) such that \( \int_{\mathbb{R}^3 \backslash B_R(0)} ||v_n||^2 dx < \delta \). Then,
\[
\xi(z) = \inf_{B_1(z)} ||v_n||^2 dx < \delta, \; z \in \mathbb{R}^3 \backslash B_{R+1}(0).
\]
Hence
\[ \sup_{x \in \mathbb{R}^3} \xi(z) = \sup_{B_{R+1}(0)} \xi(z). \]

It follows from the continuity of \( \xi \) and the compactness of \( B_{R+1}(0) \) that there exists \( y_n \in B_{R+1}(0) \) such that \( \xi(y_n) = \sup_{B_{R+1}(0)} \xi(z) \), and then
\[ \int_{B_{1}(y_n)} |v_n|^2 dx = \sup_{x \in \mathbb{R}^3} \int_{B_1(z)} |v_n|^2 dx. \]

We claim that \( \limsup_{n \to +\infty} \int_{B_1(y_n)} |v_n|^2 dx > 0 \). Otherwise, \( \lim_{n \to +\infty} \int_{B_1(y_n)} |v_n|^2 dx = 0 \).

By Lemma 3.4 in [40] we have and \( v_n \to 0 \) in \( L^{2^*_n}(\mathbb{R}^3) \). In view of the interpolation inequality, we get that
\[ ||v_n||_{L^t(\mathbb{R}^3)} \leq ||v_n||^{1-\theta}_{L^2(\mathbb{R}^3)} ||v_n||^{\theta}_{L^{2^*_n}(\mathbb{R}^3)}, \]
where \( t \in (2, 2^*_n) \) and \( \theta = \frac{3(t-2)}{2t} \). Then we have \( v_n \to 0 \) in \( L^t(\mathbb{R}^3) \) for \( t \in (2, 2^*_n] \).

Hence, by (2.3), we have,
\[ d \geq I(u_n) = \max_{t > 0} I(tu_n) \geq I\left( \frac{t}{||u_n||} u_n \right) = I(tv_n) \]
\[ \geq \frac{t^2}{2} ||v_n||^2 - \lambda \delta t^2 \int_{\mathbb{R}^3} |v_n|^2 dx - \lambda C_\delta t^q \int_{\mathbb{R}^3} |v_n|^q dx - \frac{t^{2^*_n}}{2^*_n} \int_{\mathbb{R}^3} K(x)|u|^2^* dx - \frac{t^2}{4} \]
as \( n \to +\infty \), which is a contradiction. Thus,
\[ 0 \leq \frac{I(u_n)}{||u_n||^2} \leq -\frac{1}{2^*_n} \int_{\mathbb{R}^3} K(x)|v_n|^{2^*_n} dx + o(1) \leq -\frac{1}{2^*_n} \int_{B_1(y_n)} K(x)|v_n|^{2^*_n} dx + o(1) < 0 \]
for large \( n \), which yields a contradiction. \( \square \)

Let the mapping \( m : S_1 \to N \) and the functional \( \Phi : S_1 \to \mathbb{R} \) defined by \( m(u) = tu_n \) and \( \Phi(u) = I(m(u)) \), respectively.

By Lemma 2.2-2.4, we may get the following Lemmas 2.5-2.6 (see, e.g., [37]).

**Lemma 2.5.** Let \((V), (K)\) and \((g_1)-(g_4)\) hold, then \( m \) is a homeomorphism between \( S_1 \) and \( N \), and the inverse of \( m \) is given by \( m^{-1}(u) = \frac{u}{||u||} \).

**Lemma 2.6.** Let \((V), (K)\) and \((g_1)-(g_4)\) hold, then the following properties hold:
(i) \( \{m(w_n)\} \) is a (PS) sequence of \( I \) if \( \{w_n\} \) is a (PS) sequence of \( \Phi \); \( \{m^{-1}(u_n)\} \) is a (PS) sequence of \( \Phi \) if \( \{u_n\} \subset N \) is a bounded (PS) sequence of \( I \).
(ii) \( w \in S_1 \) is a critical point of \( \Phi \) if and only if \( m(w) \) is a nontrivial point of \( I \).
Moreover, the corresponding values of \( \Phi \) and \( I \) coincide and \( \inf_{S_1} \Phi = \inf_{N} I \).

(iii) A minimizer of \( I \) on \( N \) is a ground state of equation (1.1).

**Lemma 2.7.** The following inequality holds
\[ c < \frac{a}{2} S_4 \hat{T}^2 + \frac{b}{4} S^2 \hat{T}^4 - \frac{||K||_{L^\infty}}{2^*_n} \hat{T}^{2^*_n}, \]
where \( \hat{T} > 0 \) independent of \( \lambda \).

**Proof.** We define
\[ u_\varepsilon(x) = \psi(x) U_\varepsilon(x), \quad x \in \mathbb{R}^3, \]
where \( U_\varepsilon(x) = \varepsilon^{-\frac{3-2^*}{2^*}} u^*(\frac{x}{\varepsilon}), u^*(x) = \frac{\tilde{u}(x/\sqrt{\kappa})}{||\tilde{u}\||_{L^2}} \), \( \kappa \in \mathbb{R} \setminus \{0\}, \mu_0 > 0 \) and \( x_0 \in \mathbb{R}^3 \) are fixed constants, \( \tilde{u}(x) = \kappa (\mu_0 + |x-x_0|^2)^{-\frac{3}{2}} \) (see [36], Section 4), and \( \psi \in C^\infty(\mathbb{R}^3) \).
such that $0 \leq \psi \leq 1$ in $\mathbb{R}^3$, $\psi \equiv 1$ in $B_{r/2}$ and $\psi \equiv 0$ in $\mathbb{R}^3 \setminus B_r$. From Proposition 21 and Proposition 22 in [36] we know that

$$A_\varepsilon := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon(x)|^2 dx \leq S_*^2 + O(\varepsilon^{3-2s}), \quad (2.4)$$

$$B_\varepsilon := \int_{\mathbb{R}^3} |u_\varepsilon(x)|^2 dx = O(\varepsilon^{3-2s}), \quad (2.5)$$

$$C_\varepsilon := \int_{\mathbb{R}^3} |u_\varepsilon(x)|^4 dx = \begin{cases} O(\varepsilon^{3-\frac{3}{2s}}), & t > \frac{3}{3-2s}, \\ O(\varepsilon^{3-\frac{3}{2s}}|\log \varepsilon|), & t = \frac{3}{3-2s}, \\ O(\varepsilon^{3-2s}), & t < \frac{3}{3-2s}, \end{cases} \quad (2.6)$$

and

$$D_\varepsilon := \int_{\mathbb{R}^3} |u_\varepsilon(x)|^{2s} dx = S_*^2 + O(\varepsilon^3), \quad (2.7)$$

We may assume without loss of generality that $K(0) = ||K||_\infty$. Since $K(x) - K(0) = O(|x|^\alpha)$ as $x \to 0$, we can chose $\rho$ small enough such that $K(0) - K(x) \leq C|x|^\alpha$, $\forall x \in B_\rho(0)$. Therefore, we deduce that

$$E_\varepsilon = \int_{\mathbb{R}^3} (K(0) - K(x)) |u_\varepsilon|^{2s} dx$$

$$= \varepsilon^{-3} \int_{B_\varepsilon(0)} |\tilde{u}|^{2s} dx + \int_{\mathbb{R}^3 \setminus B_\varepsilon(0)} |\tilde{u}|^{2s} dx$$

$$\leq \varepsilon^{-3} C \int_{B_\varepsilon(0)} |x|^\alpha dx + \varepsilon^{-3} C \int_{\mathbb{R}^3 \setminus B_\varepsilon(0)} \frac{1}{(\mu^2 + \frac{x^2}{\varepsilon S_*^{1+2s}})^3} dx$$

$$\leq \varepsilon^\alpha C \int_{\mathbb{R}^3 \setminus B_{\varepsilon S_*^{1+2s}}} \frac{|x|^\alpha}{(\mu^2 + x^2)^3} dx + C \int_{\mathbb{R}^3 \setminus B_{\varepsilon S_*^{1+2s}}} \frac{1}{(\mu^2 + x^2)^3} dx$$

$$\leq \varepsilon^\alpha C \int_0^{\varepsilon S_*^{1+2s}} \frac{r^{\alpha + 2}}{(\mu^2 + r^2)^3} dr + C \int_{\varepsilon S_*^{1+2s}}^{\infty} \frac{r^2}{(\mu^2 + r^2)^3} dr$$

$$\leq \varepsilon^\alpha C \int_0^{\varepsilon S_*^{1+2s}} r^{\alpha - 4} dr + C \int_{\varepsilon S_*^{1+2s}}^{\infty} r^{-4} dr$$

$$\leq \begin{cases} O(\varepsilon^\alpha), & \alpha < 3, \\ O(\varepsilon^\alpha |\log \varepsilon|), & \alpha = 3, \\ O(\varepsilon^3), & \alpha > 3. \end{cases} \quad (2.8)$$

By Lemma 2.2 there exists $t_\varepsilon := t_{u_\varepsilon} > 0$ such that $I(t_\varepsilon u_\varepsilon) = \max_{t \geq 0} I(t u_\varepsilon)$. Hence, we have

$$t_\varepsilon^2 \|u_\varepsilon\|^2 + \varepsilon^4 \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx \right)^2 = t_\varepsilon^{2s} \int_{\mathbb{R}^3} K(x)|u_\varepsilon|^{2s} dx + \lambda \int_{\mathbb{R}^3} g(x, t_\varepsilon u_\varepsilon) t_\varepsilon u_\varepsilon dx.$$

By $(g_1)$ we have that

$$t_\varepsilon^2 \|u_\varepsilon\|^2 + \varepsilon^4 \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx \right)^2 \geq t_\varepsilon^{2s} \int_{\mathbb{R}^3} K(x)|u_\varepsilon|^{2s} dx,$$

which implies that $|t_\varepsilon| < C_1$, where $C_1 > 0$ is independent of $\varepsilon > 0$ small.
On the other hand, we may assume that there is a positive constant \( C_2 > 0 \)
such that \( t_\varepsilon > C_2 \) for \( \varepsilon > 0 \) small. Otherwise, we may assume
that there exists a sequence \( \varepsilon_n \to 0 \) as \( n \to \infty \) such that \( t_\varepsilon_n \to 0 \)
as \( n \to \infty \). Hence,
\[
0 < c \leq \max_{t \geq 0} \mathcal{I}(tu_\varepsilon) = \mathcal{I}(t_\varepsilon u_\varepsilon) \to 0,
\]
which is a contradiction. Therefore, \( C_2 \leq t_\varepsilon_n \leq C_1 \).

Define
\[
J(t) := \frac{aA_\varepsilon}{2} t^2 + \frac{bA_\varepsilon^2}{4} t^4 - \frac{||K||_\infty}{2^*} D_\varepsilon t^{2^*},
\]
then
\[
c \leq \sup_{t > 0} J(t) + B_\varepsilon + E_\varepsilon - \lambda C_\varepsilon.
\]
By (2.5) (2.7) and (2.8), we have that \( \sup_{t > 0} J(t) \geq \frac{c}{3} \) uniformly for \( \varepsilon > 0 \) small.
As above, there are \( C_3, C_4 > 0 \) (independent of \( t > 0 \)) such that
\[
\sup_{t \in [C_3, C_4]} J(t) \geq 0\text{ small}
\]
and (2.9) we deduce that \( \alpha > 3 - 2s \), by (2.4)-(2.8) and (2.9) we deduce that
\[
I(t_\varepsilon u_\varepsilon) = \frac{a}{2} t_\varepsilon^2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{3}{2}} u_\varepsilon(x)|^2 dx + \frac{b}{4} t_\varepsilon^4 \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{3}{2}} u_\varepsilon(x)|^2 dx \right)^2
\]
\[
+ \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^3} |u_\varepsilon(x)|^2 dx - \frac{t_\varepsilon^2}{2^*} \int_{\mathbb{R}^3} K(x) |u_\varepsilon|^2 dx + \lambda \int_{\mathbb{R}^3} G(x, t_\varepsilon u_\varepsilon) dx
\]
\[
\leq K(S_{\varepsilon}^{3-2s}) + O(\varepsilon^{3-2s}) - C \int_{\mathbb{R}^3} (K(0) - K(0)) |u_\varepsilon|^{2^*} dx - \lambda C \int_{\mathbb{R}^3} |u_\varepsilon|^p dx. \tag{2.10}
\]

It follows from (2.8) that
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} (K(0) - K(x)) |u_\varepsilon|^{2^*} dx \leq \begin{cases}
0 & 3 - 2s \leq \alpha < 3, \\
\lim_{\varepsilon \to 0} \frac{O(\varepsilon^\alpha)}{\varepsilon^{3-2s}} = 0 & \alpha = 3, \\
\lim_{\varepsilon \to 0} \frac{O(\varepsilon^\alpha \log \varepsilon)}{\varepsilon^{3-2s}} = 0 & \alpha > 3.
\end{cases}
\]

Moreover, by \( \frac{3}{4} < s < 1 \), we deduce that
\[
\lim_{\varepsilon \to 0} \lambda \int_{\mathbb{R}^3} \left| u_\varepsilon(x) \right|^p dx = \begin{cases} 
\lim_{\varepsilon \to 0} \lambda \frac{O\left(\varepsilon^{3-\frac{2s}{2}}\right)}{\varepsilon^{\frac{3}{2}} + p} = +\infty, & \frac{4s}{3-2s} \leq p < \frac{6}{3-2s}, \\
\lim_{\varepsilon \to 0} \lambda \frac{O\left(\varepsilon^{3-\frac{2s}{2}}\right)}{\varepsilon^{\frac{3}{2}} + p} = \frac{4s}{3-2s} < p < \frac{6}{3-2s}, \\
\lim_{\varepsilon \to 0} \lambda \frac{O\left(\varepsilon^{3-\frac{2s}{2}}\right)}{\varepsilon^{\frac{3}{2}} + p} = \frac{3}{3-2s}, & p = \frac{3}{3-2s}, \\
\lim_{\varepsilon \to 0} \lambda \frac{O\left(\varepsilon^{3-\frac{2s}{2}}\right)}{\varepsilon^{\frac{3}{2}} + p} = 2 < p < \frac{3}{3-2s}.
\end{cases}
\]

We can choose \( \lambda \) large enough such that the above three limit equal to +\( \infty \), for instance, \( \lambda = \varepsilon^{-2s} \). Therefore,

\[
\varepsilon \leq I(t_\varepsilon u_\varepsilon) < K(S_s^{\frac{3}{3-2s}} T).
\]

If \( 0 < \alpha < 3 - 2s \), we may choose \( \varepsilon \) so small that \( B_\varepsilon(0) \subset \Omega \), then, by \((g_5)\) and definition of \( U_\varepsilon(x) \) we have

\[
t_\varepsilon U_\varepsilon(x) > \frac{t_\varepsilon \kappa S_s^{\frac{3}{3-2s}}}{\varepsilon^{\frac{3}{2}} + \| \tilde{u} \|_{2^*}^2 \left( S_s^2 \mu^2 + x^2 \right)^{\frac{3-2s}{2}}} \geq C \varepsilon^{\frac{2a-3}{2}} \to +\infty
\]
as \( \varepsilon \to 0^+ \), which implies that for any \( M > 0 \) there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \),

\[
\int_{\mathbb{R}^3} G(x, t_\varepsilon u_\varepsilon) dx \geq \int_{B_\varepsilon(0)} G(x, t_\varepsilon U_\varepsilon(x)) dx \geq CM \int_{B_\varepsilon(0)} \varepsilon^{a-3} dx \geq CM \varepsilon^a. \tag{2.11}
\]

Therefore,

\[
I(t_\varepsilon u_\varepsilon) = \frac{a}{2} t_\varepsilon^2 \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} u_\varepsilon(x)|^2 dx + \frac{b}{4} t_\varepsilon^2 \left( \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} u_\varepsilon|^2 dx \right)^2 + \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^3} |u_\varepsilon(x)|^2 dx - \frac{t_\varepsilon^2}{2^*} \int_{\mathbb{R}^3} K(x) |u_\varepsilon|^2 dx - \lambda \int_{\mathbb{R}^3} G(x, t_\varepsilon u_\varepsilon) dx
\]
\[
\leq K(S_s^{\frac{3}{3-2s}} T) + O(\varepsilon^{3-2s}) - C \left( K(x) - K(0) \right) |u_\varepsilon|^2 dx - \lambda M \varepsilon^a
\]
\[
\leq K(S_s^{\frac{3}{3-2s}} T) + O(\varepsilon^{3-2s}) + O(\varepsilon^a) - \lambda CM \varepsilon^a. \tag{2.12}
\]

Since \( 3 - 2s > \alpha = \frac{2a-3}{2} m + 3 \), by \((2.8)\) and \((2.12)\) we obtain

\[
I(t_\varepsilon u_\varepsilon) \leq K(S_s^{\frac{3}{3-2s}} T) + O(\varepsilon^a) - \lambda M \varepsilon^a.
\]

Hence, if \( M \) is large enough, then \( I(t_\varepsilon u_\varepsilon) < K(S_s^{\frac{3}{3-2s}} T) \) for \( \varepsilon \) small enough. Thus we completed the proof of Lemma \(2.7\). \( \square \)

3. **Proof of Theorem 1.1.** In this section we first study the following periodic problem associated to \((1.1)\)

\[
(a + b \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} u|^2 dx) (-\Delta)^\frac{s}{2} u + V_p(x) u = K_p(x)|u|^{2^* - 2} u + \lambda g_p(x, u), \text{ in } \mathbb{R}^3. \tag{3.1}
\]

The weak solutions of \((3.1)\) are the critical points of the \( C^1 \)–functional \( I_p : H^s(\mathbb{R}^3) \to \mathbb{R} \) given by

\[
I_p(u) = \frac{1}{2} |u|^2_p + b \left( \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} u|^2 dx \right)^2 - \frac{1}{2^*} \int_{\mathbb{R}^3} K_p(x)|u|^{2^*} dx - \lambda \int_{\mathbb{R}^3} G_p(x, u) dx.
\]
Moreover, for any \(u, v \in H^s(\mathbb{R}^3)\), we get
\[
\langle I_p'(u), v \rangle = \left( a + b \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \right) \int_{\mathbb{R}^3} (-\Delta) \frac{v}{2} u (-\Delta) \frac{v}{2} v \, dx + \int_{\mathbb{R}^3} V_p(x) u v \, dx \\
- \int_{\mathbb{R}^3} K_p(x)|u|^{2^* - 2} u v \, dx - \lambda \int_{\mathbb{R}^3} g_p(x, u) v \, dx.
\]

The Nehari manifold associated to \(I_p\) is defined by
\[
\mathcal{N}_p := \left\{ u \in H^s(\mathbb{R}^3) \setminus \{0\} : \langle I_p'(u), u \rangle = 0 \right\}.
\]

We define the number \(c_p\) by \(c_p := \inf_{u \in \mathcal{N}_p} I_p(u)\), which is the ground energy corresponding to (3.1).

**Lemma 3.1.** Let \((V), (K)\) and \((g_1) - (g_4)\) hold, then \(I(u) \leq I_p(u)\) for all \(u \in H^s(\mathbb{R}^3)\) and \(c \leq c_p\).

**Lemma 3.2.** Assume that \(\{u_n\} \subset H^s(\mathbb{R}^3)\) is bounded and \(\varphi_n(x) := \varphi(x - x_n)\), where \(\varphi \in H^s(\mathbb{R}^3)\) and \(x_n \in \mathbb{R}^3\). If \(|x_n| \to \infty\), then
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} \left( K(x) - K_p(x) \right) |u_n|^{2^* - 2} u_n \varphi_n \, dx = 0.
\]

**Proof.** For any \(\varepsilon > 0\) and \(R > 0\), set \(D_\varepsilon(R) := \{ x \in \mathbb{R}^3 : |h| \geq \varepsilon, |x| \geq R \}\). By \((K)\) there exists \(R_1 > 0\) such that \(|D_\varepsilon(R)| < \varepsilon\). Notice that
\[
K := \int_{\mathbb{R}^3} \left( K(x) - K_p(x) \right) |u_n|^{2^* - 2} u_n \varphi_n \, dx \\
\leq \int_{\mathbb{R}^3} \left| K(x) - K_p(x) \right| |u_n|^{2^* - 1} |\varphi(x - x_n)| \, dx \\
= \left\{ \int_{D_\varepsilon(R_1)} + \int_{B_{R_1}(0)} + \int_{\mathbb{R}^3 \setminus (D_\varepsilon(R_1) \cup B_{R_1}(0))} \right\} \left| K(x) - K_p(x) \right| |u_n|^{2^* - 1} |\varphi(x - x_n)| \, dx \\
:= K_1 + K_2 + K_3.
\]

Given \(\varepsilon > 0\), since \(\varphi \in L^{2^*_n}(\mathbb{R}^3)\), there exists \(\varepsilon \in (0, \varepsilon)\) such that, for every measurable \(A \subset \mathbb{R}^3\) satisfying \(|A| \leq \varepsilon\), \(\int_A |\varphi|^{2^*_n} \leq \varepsilon\). Thus,
\[
K_1 \leq 2||K||_{\infty} \left( \int_{\mathbb{R}^3} |u_n|^{2^*_n} \, dx \right)^{\frac{2^*_n - 1}{2^*_n}} \left( \int_{D_\varepsilon(R_1)} |\varphi_n|^{2^*_n} \, dx \right)^{\frac{1}{2^*_n}} \leq C \varepsilon^\frac{1}{2^*_n}
\]

Since \(\varphi \in L^{2^*_n}(\mathbb{R}^3)\) and \(|x_n| \to \infty\), we have
\[
K_2 \leq 2||K||_{\infty} \left( \int_{\mathbb{R}^3} |u_n|^{2^*_n} \, dx \right)^{\frac{2^*_n - 1}{2^*_n}} \left( \int_{B_{R_1}(0)} |\varphi(x - x_n)|^{2^*_n} \, dx \right)^{\frac{1}{2^*_n}} \\
\leq C \left( \int_{B_{R_1}(-x_n)} |\varphi(x)|^{2^*_n} \, dx \right)^{\frac{1}{2^*_n}} \leq C \varepsilon \text{ as } n \to \infty.
\]

Moreover,
\[
K_3 \leq \varepsilon \left( \int_{\mathbb{R}^3} |u_n|^{2^*_n} \, dx \right)^{\frac{2^*_n - 1}{2^*_n}} \left( \int_{\mathbb{R}^3} |\varphi_n|^{2^*_n} \, dx \right)^{\frac{1}{2^*_n}} \leq C \varepsilon.
\]

Therefore, \(K \leq C(\varepsilon + \varepsilon^\frac{1}{2^*_n})\) for large \(n\), which implies that Lemma 3.2 holds. \(\square\)
The following result is important for proving Theorem 1.1, one also consult [42] or [25].

**Lemma 3.3.** If \( \{u_n\} \subset H^s(\mathbb{R}^3) \) satisfies \( u_n \rightharpoonup 0 \) and \( \varphi_n \in H^s(\mathbb{R}^3) \) is bounded, then

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} \left( V(x) - V_p(x) \right) u_n \varphi_n \, dx = 0, \tag{3.2}
\]

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} \left( g(x, u_n) - g_p(x, u_n) \right) \varphi_n \, dx = 0, \tag{3.3}
\]

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} \left( G(x, u_n) - G_p(x, u_n) \right) \, dx = 0. \tag{3.4}
\]

**Proof of Theorem 1.1.** If \( \{w_n\} \subset S_1 \) is a minimizing sequence with \( \Phi(w_n) \to \inf_{S_1} \Phi \).

We may suppose \( \Phi'(w_n) \to 0 \) by using the Ekeland variational principle. Then, by Lemma 2.6 we obtain that \( I'(u_n) \to 0 \) and \( I(u_n) = \Phi(w_n) \to c \), where \( u_n = m(w_n) \in N \). Moreover, by Lemma 2.6, \( u_n \) is bounded in \( H^s(\mathbb{R}^3) \). Then, up to a subsequence, \( u_n \rightharpoonup u \) in \( H^s(\mathbb{R}^3) \), \( u_n \to u \) in \( L^{p_q}_{loc}(\mathbb{R}^3) \) for \( q \in [2, 2^*_s) \), \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^3 \).

Next, we verify that \( I'(u) = 0 \). If \( u = 0 \), we get \( I'(u) = 0 \). If \( u \neq 0 \), we claim that

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u_n|^2 \, dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx. \tag{3.5}
\]

Indeed, we may suppose that

\[
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx \leq \lim_{n \to +\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u_n|^2 \, dx = A > 0.
\]

By the fact that \( I'(u_n) \to 0 \), \( u \) is a solution of the following equation

\[
(a + bA)(-\Delta)^s u(x) + V(x)u = K(x)|u|^{2^*_s - 2}u + \lambda g(x, u), \text{ in } \mathbb{R}^3.
\]

Then

\[
\left( a + b \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx \right) \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx + \int_{\mathbb{R}^3} V(x)u^2 \, dx \\
\leq \left( a + bA \right) \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx + \int_{\mathbb{R}^3} V(x)u^2 \, dx \\
= \int_{\mathbb{R}^3} K(x)|u|^{2^*_s} \, dx + \lambda \int_{\mathbb{R}^3} G(x, u) \, dx. \tag{3.6}
\]

Set

\[
h_1(t) := \langle I'(tu), tu \rangle \text{ for } t \geq 0.
\]

If \( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx = A \), the (3.5) holds. If \( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx < A \), by (3.6) we have that \( h_1(1) < 0 \). From (2.3) we have

\[
h_1(t) \geq t^2\|u\|^2 - \delta \lambda C t^2 \int_{\mathbb{R}^3} |u|^2 \, dx - \lambda C \delta^t \int_{\mathbb{R}^3} |u|^q \, dx - t^2 \int_{\mathbb{R}^3} K(x)|u|^{2^*_s} \, dx,
\]
which yields that \( h_1(t) > 0 \) for \( t > 0 \) small. Thus, there exists a \( t_0 \in (0, 1) \) such that \( h_1(t_0) = (I'(t_0u), t_0u) = 0 \), that is, \( t_0u \in \mathcal{N} \). By (2.3) we have

\[
c \leq I(t_0u) = I(t_0u) - \frac{1}{4} \langle I'(t_0u), t_0u \rangle
\]

\[
= \frac{t_0^2}{4} \|u\|^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \left( g(x, t_0u)t_0u - 4G(x, t_0u) \right) dx + \left( \frac{1}{4} - \frac{1}{2s} \right) t_0^2 \int_{\mathbb{R}^3} K(x)|u|^{2^*} dx
\]

\[
< \frac{1}{4} \|u\|^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \left( g(x, u)u - 4G(x, u) \right) dx + \left( \frac{1}{4} - \frac{1}{2s} \right) \int_{\mathbb{R}^3} K(x)|u|^{2^*} dx
\]

\[
\leq \liminf_{n \to +\infty} \left\{ I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle \right\} = c,
\]

(3.7)

this contradiction implies that (3.5) holds true and hence \( I'(u) = 0 \).

We next continue our arguments by distinguishing the following two cases: \( u \neq 0 \) and \( u = 0 \).

**Case 1** \( u \neq 0 \). In this case, \( u \in \mathcal{N} \) and \( I(u) \geq c \). By (2.3) we have

\[
c = \lim_{n \to +\infty} \left\{ I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle \right\}
\]

\[
\geq \liminf_{n \to +\infty} \left\{ \frac{1}{4} \|u_n\|^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \left( \frac{1}{4} g(x, u_n)u_n - G(x, u_n) \right) dx + \left( \frac{1}{4} - \frac{1}{2s} \right) \int_{\mathbb{R}^3} K(x)|u_n|^{2^*} dx \right\}
\]

\[
\geq \frac{1}{4} \|u\|^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \left( \frac{1}{4} g(x, u)u - G(x, u) \right) dx + \left( \frac{1}{4} - \frac{1}{2s} \right) \int_{\mathbb{R}^3} K(x)|u|^{2^*} dx
\]

\[
= I(u) - \frac{1}{4} \langle I'(u), u \rangle
\]

\[
= I(u),
\]

which implies that \( I(u) \leq c \). Therefore, \( I(u) = c \).

**Case 2** \( u = 0 \). That is \( \{u_n\} \) is vanishing or non-vanishing. If \( \{u_n\} \) is vanishing, i.e.,

\[
\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |u_n(x)|^2 dx = 0,
\]

it follows from Lemma 2.1 that \( u_n \to 0 \) in \( L^t(\mathbb{R}^3) \), \( 2 < t < 2^*_s \). By (2.3) we get

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} g(x, u_n)u_n dx = 0 \text{ and } \lim_{n \to +\infty} \int_{\mathbb{R}^3} G(x, u_n) dx = 0.
\]

Thus,

\[
a \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} u_n|^2 dx + b \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} u_n|^2 dx \right) = \int_{\mathbb{R}^3} K(x)|u_n|^{2^*} dx + o(1). \quad (3.9)
\]

By (3.9) we have

\[
a S_\alpha \left( \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx \right)^{\frac{2^*_s}{2^*_s - 2}} + b S_\alpha \left( \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx \right)^{\frac{2^*_s}{2^*_s - 2}}
\]

\[
\leq a \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} u_n|^2 dx + b \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} u_n|^2 dx \right)^2
\]

\[
\leq ||K||_\infty \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx.
\]
Denote \( \int_{\mathbb{R}^3} |u_n|^2^* \, dx = L^2 + o_n(1) \). By (3.10) we obtain that
\[
aS_s L^2 + bS_s^2 L^4 - ||K||_{\infty} L^2 \leq 0,
\]
which implied \( L > \hat{T} \). Therefore,
\[
c + o_n(1)
= I(u_n) - \frac 14 (I'(u_n), u_n)
= \frac a 4 \int_{\mathbb{R}^3} |(-\Delta)^{\frac 12} u_n|^2 \, dx + \frac 14 \int_{\mathbb{R}^3} V(x) u_n^2 \, dx + \lambda \int_{\mathbb{R}^3} \left( \frac 14 g(x, u_n) u_n - G(x, u_n) \right) \, dx
+ \left( \frac 14 - \frac 1{2^{s_1}} \right) \int_{\mathbb{R}^3} K(x) |u_n|^2 \, dx
\geq \frac a 4 \int_{\mathbb{R}^3} |(-\Delta)^{\frac 12} u_n|^2 \, dx + \left( \frac 14 - \frac 1{2^{s_1}} \right) \left\{ a \int_{\mathbb{R}^3} |(-\Delta)^{\frac 12} u_n|^2 \, dx
+ b \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac 12} u_n|^2 \, dx \right)^2 \right\}
\geq \frac a 4 S_s \left( \int_{\mathbb{R}^3} |u_n|^2^* \, dx \right)^{\frac 1{2^*}} + \left( \frac 14 - \frac 1{2^{s_1}} \right) \left\{ a S_s \left( \int_{\mathbb{R}^3} |u_n|^2^* \, dx \right)^{\frac 1{2^*}}
+ bS_s^2 \left( \int_{\mathbb{R}^3} |u_n|^2^* \, dx \right)^{\frac 1{2^*}} \right\}
= \frac a 4 S_s L^2 + \left( \frac 14 - \frac 1{2^{s_1}} \right) \left( a S_s L^2 + bS_s^2 L^4 \right)
\geq \frac a 4 S_s \hat{T}^2 + \left( \frac 14 - \frac 1{2^{s_1}} \right) \left( a S_s \hat{T}^2 + bS_s^2 \hat{T}^4 \right)
= \frac a 2 S_s \hat{T}^2 + \frac b 4 S_s^2 \hat{T}^4 + \frac {||K||_{\infty}}{2^*} \hat{T}^2^*,
\]
which contradicts with Lemma 2.7. Therefore, \( \{u_n\} \) is non-vanishing. Then there exist \( x_n \in \mathbb{R}^3 \) and \( \delta_0 > 0 \) such that
\[
\int_{B_{\delta}(x_n)} |u_n(x)|^2 \, dx \geq \delta_0.
\]
Without loss of generality, we assume that \( x_n \in \mathbb{Z}^N \) with \( |x_n| \to \infty \). Denote \( \tilde{u}_n \) by \( \tilde{u}_n(\cdot) := u_n(\cdot + x_n) \), up to a subsequence, \( \tilde{u}_n \to \tilde{u} \) in \( H^s(\mathbb{R}^3) \), \( \tilde{u}_n \to \tilde{u} \) in \( L^q_{loc}(\mathbb{R}^3) \) for \( q \in [2, 2^*_s) \) and \( \tilde{u}_n(x) \to \tilde{u}(x) \) a.e. in \( \mathbb{R}^3 \). By using (3.11) we have \( \tilde{u} \neq 0 \).

We first claim that
\[
I_p'(\tilde{u}) = 0.
\]
Indeed, for all \( \psi \in H^s(\mathbb{R}^3) \), let \( \psi_n(\cdot) := \psi(\cdot - x_n) \). By Lemma 3.2, replacing \( \varphi_n \) by \( \psi_n \), we have
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} \left( V(x) - V_p(x) \right) u_n \psi_n \, dx = 0,
\]
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} \left( K(x) - K_p(x) \right) |u_n|^{2^* - 2} u_n \varphi_n \, dx = 0
\]
and
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} \left( g(x, u_n) - g_p(x, u_n) \right) \psi_n \, dx = 0.
\]
Therefore, we have
\[ \langle I'(u_n), \psi_n \rangle - \langle I'_p(u_n), \psi_n \rangle \to 0. \]
By the fact that \( I'(u_n) \to 0 \) and \( ||\psi_n|| = ||\psi|| \), we have \( \langle I'(u_n), \psi_n \rangle \to 0 \). Thus, \( \lim_{n \to +\infty} \langle I'_p(u_n), \psi_n \rangle = 0 \). Moreover, by \((V), (K)\) and \((g_4)\) we have \( \langle I'_p(\tilde{u}_n), \psi \rangle = \langle I'_p(u_n), \psi_n \rangle \). Then \( \langle I'_p(\tilde{u}_n), \psi \rangle \to 0 \) and \( I'_p(\tilde{u}_n) \to 0 \) in \( H^s(\mathbb{R}^3) \).
We next verify that
\[ I_p(\tilde{u}) \leq c. \] (3.13)
If we replace \( \varphi_n \) by \( u_n \) in Lemma 3.2, then
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^3} (V(x) - V_p(x))|u_n|^2 \, dx = 0, \]
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^3} (g(x, u_n) - g_p(x, u_n))u_n \, dx = 0 \]
and
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^3} (G(x, u_n) - G_p(x, u_n)) \, dx = 0. \]
Combining the above equations with \((K)\), we obtain that
\[ c = \lim_{n \to -\infty} \left\{ I(u_n) - \frac{1}{4}\langle I'(u_n), u_n \rangle \right\} \]
\[ = \lim_{n \to -\infty} \left\{ \frac{1}{4}|u_n|^2 + \lambda \int_{\mathbb{R}^3} \left( \frac{1}{4}g(x, u_n)u_n - G(x, u_n) \right) \, dx \right. \]
\[ \left. + \left( \frac{1}{4} - \frac{1}{2s} \right) \int_{\mathbb{R}^3} K(x)|u_n|^2 \, dx \right\} \]
\[ \geq \lim_{n \to -\infty} \left\{ \frac{1}{4}|u_n|^2 + \lambda \int_{\mathbb{R}^3} \left( \frac{1}{4}g_p(x, u_n)u_n - G_p(x, u_n) \right) \, dx \right. \]
\[ \left. + \left( \frac{1}{4} - \frac{1}{2s} \right) \int_{\mathbb{R}^3} K_p(x)|u_n|^2 \, dx \right\} \]
\[ = \lim_{n \to -\infty} \left\{ \frac{1}{4}|\tilde{u}_n|^2 + \lambda \int_{\mathbb{R}^3} \left( \frac{1}{4}g_p(x, \tilde{u}_n)\tilde{u}_n - G_p(x, \tilde{u}_n) \right) \, dx \right. \]
\[ \left. + \left( \frac{1}{4} - \frac{1}{2s} \right) \int_{\mathbb{R}^3} K_p(x)|\tilde{u}_n|^2 \, dx \right\} \]
\[ \geq ||\tilde{u}||^2 + \lambda \int_{\mathbb{R}^3} \left( \frac{1}{4}g_p(x, \tilde{u})\tilde{u} - G_p(x, \tilde{u}) \right) \, dx + \left( \frac{1}{4} - \frac{1}{2s} \right) \int_{\mathbb{R}^3} K_p(x)|\tilde{u}|^2 \, dx \]
\[ = I_p(\tilde{u}) - \frac{1}{4}\langle I'_p(\tilde{u}), \tilde{u} \rangle = I_p(\tilde{u}). \]
Thus, we have \( I_p(\tilde{u}) \leq c \).
Since \( \tilde{u}_n \rightharpoonup \tilde{u} \) in \( H^s(\mathbb{R}^3) \), we may assume that
\[ \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{u}|^2 \, dx \leq \lim_{n \to +\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{u}_n|^2 \, dx = \tilde{A} > 0. \]
It follows from \( I'_p(\tilde{u}_n) \to 0 \) that \( \tilde{u} \) is a solution of the following problem
\[ (a + b\tilde{A})(-\Delta)^s u(x) + V_p(x)u = K_p(x)|u|^{2^*_s - 2}u + \lambda g_p(x, u), \text{ in } \mathbb{R}^3. \]
Then
\[ (a + b \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} \tilde{u}|^2 dx) \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} \tilde{u}|^2 dx + \int_{\mathbb{R}^3} V_p(x) \tilde{u}^2 dx \]
\[ \leq (a + b \bar{A}) \int_{\mathbb{R}^3} |(-\Delta)^\frac{s}{2} \tilde{u}|^2 dx + \int_{\mathbb{R}^3} V_p(x) \tilde{u}^2 dx \]
\[ = \int_{\mathbb{R}^3} K_p(x)|\tilde{u}|^2 dx + \lambda \int_{\mathbb{R}^3} G_p(x, \tilde{u}) dx. \]

Set
\[ h_2(t) := \langle I'_p(t\tilde{u}), t\tilde{u} \rangle \text{ for } t \geq 0. \]

Then inequality (3.14) implies that \( h_2(1) \leq 0. \)

Suppose that \( h_2(1) = 0 \), that is, \( \langle I'_p(\tilde{u}), \tilde{u} \rangle = 0 \). Otherwise, \( h_2(1) < 0 \). By (2.3) we have
\[ h_2(t) \geq t^2||\tilde{u}||_p^2 - \delta \lambda C t^2 \int_{\mathbb{R}^3} |\tilde{u}|^2 dx - \lambda C \delta t^q \int_{\mathbb{R}^3} |\tilde{u}|^q dx - t^2 \int_{\mathbb{R}^3} K(x)|\tilde{u}|^2 dx, \]

which implies that \( h_2(t) > 0 \) for \( t > 0 \) small. Thus, there exists a \( t_\ast \in (0, 1) \) such that \( h_2(t_\ast) = \langle I'(t_\ast u), t_\ast u \rangle = 0 \), that is, \( t_\ast u \in \mathcal{N}_p \). Hence, by (g3) we have
\[
\begin{align*}
  c_p & \leq I_p(t_\ast u) = I_p(t_\ast u) - \frac{1}{4} \langle I'_p(t_\ast u), t_\ast u \rangle \\
  &= \frac{t^2}{4}||\tilde{u}||_p^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \left( g_p(x, t_\ast \tilde{u}) t_\ast \tilde{u} - 4G_p(x, t_\ast \tilde{u}) \right) dx \\
  & \quad + \left( \frac{1}{4} - \frac{1}{2s} \right) t_\ast^2 \int_{\mathbb{R}^3} K_p(x)|\tilde{u}|^2 dx \\
  & < \frac{1}{4}||\tilde{u}||_p^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \left( g_p(x, \tilde{u}) \tilde{u} - 4G_p(x, \tilde{u}) \right) dx + \left( \frac{1}{4} - \frac{1}{2s} \right) \int_{\mathbb{R}^3} K_p(x)|\tilde{u}|^2 dx \\
  & = \lim_{n \to +\infty} \left\{ I_p(\tilde{u}_n) - \frac{1}{4} \langle I'_p(\tilde{u}_n), \tilde{u}_n \rangle \right\} \\
  & = c, \end{align*}
\]
which is a contradiction. Thus, the claim holds and \( \langle I'_p(\tilde{u}), \tilde{u} \rangle = 0 \).

Similarly to Lemma 2.2, it is easy to see that max \( I_p(t\tilde{u}) = I_p(\tilde{u}) \). By Lemma 2.2 there exists \( t_\ast > 0 \) such that \( t_\ast \tilde{u} \in \mathcal{N} \). It follows from Lemma 3.1 that
\[ I(t_\ast \tilde{u}) \leq I_p(t_\ast \tilde{u}) = \max_{t \geq 0} I_p(t\tilde{u}) = I_p(\tilde{u}). \]

By (3.13) we have \( I(t_\ast \tilde{u}) \leq c \). So, we get \( I(t_\ast u) \geq c \) by \( t_\ast \tilde{u} \in \mathcal{N} \). Then \( I(t_\ast \tilde{u}) = c \).

In a word, the infimum \( c \) is attained, and then the corresponding minimizer is a ground state of (1.1).

Next, we will prove ground state solution of (1.1) is positive. Indeed, all the analysis above can be repeated word by word, replacing \( I(u) \) by \( I^+(u) \), the functional \( I^+(u) \) defined by
\[ I^+(u) = \frac{1}{2}||u||^2 + \frac{b}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{1}{2s} \int_{\mathbb{R}^3} K(x)|u^+|^2 dx - \lambda \int_{\mathbb{R}^3} G(x, u^+) dx, \]
where \( u^+ = \max\{u, 0\} \) and \( u^- = \min\{u, 0\} \) denote the positive part and negative part of a function \( u \), respectively. In this way we obtain a ground state solution \( u \).
of the following equation
\[
(a + b \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx)(-\Delta)^{s} u + V(x)u = K(x)|u^+|^{2^*_s - 1} + \lambda g(x, u^+), \text{ in } \mathbb{R}^3.
\]
By using \(u^-\) as a test function in (3.16) we obtain
\[
0 = (a + b \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} u^- dx + \int_{\mathbb{R}^3} V(x)|u^-|^2 dx
\geq a \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u^-|^2 dx + \int_{\mathbb{R}^3} V(x)|u^-|^2 dx \geq 0,
\]
which yields that \(u^- = 0\) and hence \(u \geq 0\). Furthermore, if \(u(x_0) = 0\) for some \(x_0 \in \mathbb{R}^3\), then \((-\Delta)^{s} u(x_0) = 0\). It follows from (1.2) and \(u(x_0) = 0\) that
\[
(-\Delta)^{s} u(x_0) = -C_{3,s} \frac{1}{2} \int_{\mathbb{R}^3} \frac{u(x_0 + y) + u(x_0 - y)}{|y|^{3+2s}} dy = 0,
\]
which implies \(u \equiv 0\). Therefore, \(u\) is positive. \(\square\)

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