MOMENTS AND ONE LEVEL DENSITY OF QUADRATIC HECKE L-FUNCTIONS OF $\mathbb{Q}(\omega)$

PENG GAO AND LIANGYI ZHAO

Abstract. In this paper, we evaluate explicitly certain quadratic Hecke Gauss sums of $\mathbb{Q}(\omega), \omega = \exp\left(\frac{2\pi i}{36}\right)$. As applications, we study the moments of central values of quadratic Hecke $L$-functions of $\mathbb{Q}(\omega)$, and establish quantitative non-vanishing result for the $L$-values. We also establish an one level density result for the low-lying zeros of quadratic Hecke $L$-functions of $\mathbb{Q}(\omega)$.

Mathematics Subject Classification (2010): 11L05, 11L40, 11M06, 11M26, 11M41, 11M50, 11R16

Keywords: Hecke $L$-functions, low-lying zeros, one level density, quadratic Hecke characters

1. Introduction

Gauss sums and their averages play crucial roles in the study of $L$-functions, objects that are of important arithmetic interest. Although it is relatively easy to determine the absolute values of the Gauss sums, the determinations of their arguments are considerably more difficult. The famous result of Gauss on the sign of the value of quadratic Gauss sum in $\mathbb{Q}$ gives the first known result of this kind. For cubic and quartic Gauss sums, the exact values are determined by C. R. Matthews [30, 31]. A general formula for Hecke Gauss sums in quadratic number fields is given by H. Boylan and N. Skoruppa [3].

In this article, we shall evaluate explicitly certain quadratic Hecke Gauss sums in $K = \mathbb{Q}(\omega)$ with $\omega = \exp\left(\frac{2\pi i}{36}\right)$ and apply the results to two problems arising from the study of the Hecke $L$-functions of $K$.

First, we study the moments of quadratic Hecke $L$-functions of $K$ at the central point. M. Jutila [24] obtained asymptotic formulas for the first and second moments of $L(1/2, \chi)$ of quadratic Dirichlet characters and the error terms in his results were subsequently improved in [15, 36, 38]. W. Luo [28] studied the moments of $L(1/2, \chi)$ for cubic Hecke characters on $\mathbb{Q}(\omega)$ while S. Baier and M. P. Young [11] obtained similar results for cubic Dirichlet characters. In [11,12], the authors studied the moments of $L(1/2, \chi)$ for quadratic and quartic Hecke characters on $\mathbb{Q}(\omega)$. See [5,7,8,16,28] also for moments of Hecke $L$-functions associated with various families of characters of a fixed order.

Let $W$ be a smooth Schwarz class function compactly supported in $(1,2)$ and that $0 \leq W(t) \leq 1$ for all $t$. Our result in this direction is:

Theorem 1.1. For $y \to \infty$ and any $\varepsilon > 0$, we have

\begin{equation}
\sum_{c \equiv 1 \pmod{36}}^* L\left(\frac{1}{2}, \chi_c\right) W\left(\frac{N(c)}{y}\right) = \frac{(3 + \sqrt{3})\pi^2 A}{19440 \zeta(\omega)(2)} \hat{W}(0) y \log y + C \hat{W}(0) y + O(y^{(3+\theta)/4}),
\end{equation}

and

\begin{equation}
\sum_{c \equiv 1 \pmod{36}}^* \left| L\left(\frac{1}{2}, \chi_c\right) \right|^2 W\left(\frac{N(c)}{y}\right) \ll \varepsilon y^{1+\varepsilon},
\end{equation}

where $\chi_c = \left(\frac{c}{\omega}\right)$ is the quadratic residue symbol in $\mathbb{Q}(\omega)$, $\zeta(\omega)(s)$ is the Dedekind zeta function of $\mathbb{Q}(\omega)$, $\theta = 131/416$, \(A = \prod_{\pi \text{ prime in } K \pmod{6}=1} \left(1 - \frac{1}{(N(\pi) + 1)N(\pi)}\right), \hat{W}(0) = \int_1^2 W(x) \, dx, \)

\(C\) is a constant and $\sum^*$ denotes summation over square-free elements of $\mathbb{Z}[\omega]$ congruent to $1 \pmod{36}$.

From Theorem 1.1 we readily deduce, via a standard argument (see [28]), the following

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\(C\) is a constant and $\sum^*$ denotes summation over square-free elements of $\mathbb{Z}[\omega]$ congruent to $1 \pmod{36}$.

From Theorem 1.1 we readily deduce, via a standard argument (see [28]), the following
Corollary 1.2. For $y \to \infty$ and any $\varepsilon > 0$, we have

$$\# \left\{ c \in \mathbb{Z}[\omega] : c \equiv 1 \pmod{36}, \ N(c) \leq y, \ L \left( \frac{1}{2}, \chi_c \right) \neq 0 \right\} \gg y^{1-\varepsilon}.$$  

Next, we study one level density of low-lying zeros of quadratic family of Hecke $L$-functions of $K$. Let

$$C(X) = \{ c \in \mathbb{Z}[\omega] : (c, 2) = 1, \ c \text{ squarefree}, \ X \leq N(c) \leq 2X \}.$$ 

We shall define in Section 2.1 the primitive quadratic Kronecker symbol $\chi(-8c)$ for $c \in C(X)$. We denote the non-trivial zeroes of the Hecke $L$-function $L(s, \chi(-8c))$ by $\frac{1}{2} + i\gamma_{\chi(-8c), j}$. Without assuming the generalized Riemann hypothesis (GRH), we order them as

$$\ldots \leq \Re\gamma_{\chi(-8c), -2} \leq \Re\gamma_{\chi(-8c), -1} < 0 \leq \Re\gamma_{\chi(-8c), 1} \leq \Re\gamma_{\chi(-8c), 2} \leq \ldots .$$

We set

$$\hat{\gamma}_{\chi(-8c), j} = \frac{\gamma_{\chi(-8c), j}}{2\pi} \log X$$

and define for an even Schwartz class function $\phi$,

$$S(\chi(-8c), \phi) = \sum_{j} \phi(\hat{\gamma}_{\chi(-8c), j}).$$

We further let $\Phi_X(t)$ be a non-negative smooth function supported on $(1, 2)$, satisfying $\Phi_X(t) = 1$ for $t \in (1 + 1/U, 2 - 1/U)$ with $U = \log \log X$ and such that $\Phi_X^{(j)}(t) \ll U^j$ for all integers $j \geq 0$. Our result is

Theorem 1.3. Suppose that GRH is true. Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ has compact support in $(-2, 2)$, then

$$\lim_{X \to +\infty} \frac{1}{\# \mathbb{P}(X)} \sum_{(c, 2) = 1}^* S(\chi(-8c), \phi) \Phi_X \left( \frac{N(c)}{X} \right) = \int_{\mathbb{R}} \phi(x) W_{USp}(x) dx, \quad W_{USp}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x}.$$  

Here the “*” on the sum over $c$ means that the sum is restricted to square-free elements $c$ of $\mathbb{Z}[\omega]$.

The left-hand side expression of (1.4) is known as the one level density of low-lying zeros of the quadratic family of Hecke $L$-functions in $\mathbb{Q}(\omega)$. Given a natural family of $L$-functions, the density conjecture of N. Katz and P. Sarnak [25, 26] states that the distribution of zeros (the $n$-level densities) near the central point of a family of $L$-functions is the same as that of eigenvalues near $1$ of a corresponding classical compact group. The kernel of the integral $W_{USp}$ in (1.4) is the same function which occurs on the random matrix theory side, when studying the eigenvalues of unitary symplectic matrices. This shows that the family of quadratic Hecke $L$-functions of $\mathbb{Q}(\omega)$ is a symplectic family.

Various families of quadratic $L$-functions are among the most important families studied for the $n$-level densities. For the family of quadratic Dirichlet $L$-functions, A. E. Özlük and C. Snyder [31] computed the one level density for test functions $\phi(x)$ whose Fourier transforms $\hat{\phi}(u)$ is supported in $|u| < 2$ while assuming GRH. Their work is extended to all $n$-level densities without assuming GRH by M. O. Rubinstein [33] for test functions $\phi(x_1, \ldots, x_n)$ whose Fourier transforms $\hat{\phi}(u_1, \ldots, u_n)$ is supported in $\sum_{i=1}^n |u_i| < 1$. Yet assuming GRH, the first-named author [9] computed the $n$-level densities for $\phi(u_1, \ldots, u_n)$ being supported in $\sum_{i=1}^n |u_i| < 2$. It is shown in [9, 24] that all the results obtained above agree with random matrix theory and the family of quadratic Dirichlet $L$-functions is a symplectic family. In [13], the authors computed the one level density of quadratic Hecke $L$-functions of $\mathbb{Q}(i)$ and showed that this family is also a symplectic family. This together with the result given in Theorem 1.3 provides two other families of quadratic $L$-functions that are symplectic.

Our approaches towards establishing Theorems 1.1 and 1.3 are analogue to those in [12, 13]. In particular, we apply a two dimensional Poisson summation over $\mathbb{Z}[\omega]$ in the study of one level density result in the proof of Theorem 1.3.

1.4. Notations. The following notations and conventions are used throughout the paper.

- $\Phi(t)$ for $\Phi_X(t)$.
- $e(z) = \exp(2\pi iz) = e^{2\pi iz}$.
- $f = O(g)$ or $f \ll g$ means $|f| \leq cg$ for some unspecified positive constant $c$.
- $f = o(g)$ means $\lim_{x \to \infty} f(x)/g(x) = 0$.
- $\mu[\omega]$ denotes the Möbius function on $\mathbb{Z}[\omega]$.
- $\zeta_{\mathbb{Q}(\omega)}(s)$ denotes the Dedekind zeta function of $\mathbb{Q}(\omega)$.
- $\chi_{[-1,1]}$ denotes the characteristic function of $[-1,1]$.  

In this section, we collect the information needed in the proof of our results.

2. Preliminaries

2.1. Quadratic symbol and Kronecker symbol. It is well-known that $K = \mathbb{Q}(\omega)$ has class number 1, and the symbol $(\frac{\omega}{n})$ is the quadratic residue symbol in the ring of integers $\mathcal{O}_K = \mathbb{Z}[\omega]$. For a prime $\pi \in \mathbb{Z}[\omega], \pi \neq 2$, the quadratic character is defined for $a \in \mathbb{Z}[\omega], (a, \pi) = 1$ by $(\frac{a}{\pi}) \equiv a^{(N(\pi)-1)/2} \pmod{\pi}$, with $(\frac{a}{\pi}) \in \{\pm 1\}$. When $\pi|a$, we define $(\frac{a}{\pi}) = 0$. Then the quadratic character can be extended to any composite $n$ with $(N(n), 2) = 1$ multiplicatively. We further define $(\frac{\omega}{n}) = 1$ when $n$ is a unit in $\mathbb{Z}[\omega]$.

Note that in $\mathbb{Z}[\omega]$, every ideal co-prime to 3 has a unique generator congruent to 1 (mod 3). Such a generator is called primary. Observe that a non-unit $n = a + b\omega$ in $\mathbb{Z}[\omega]$ is congruent to 1 (mod 3) if and only if $a \equiv 1 \pmod{3}$, and $b \equiv 0 \pmod{3}$ (see the discussions before [22, Proposition 9.3.5]).

We shall say that any $n \in \mathbb{Z}[\omega]$ is cubic $E$-primary if $n^3 = a + b\omega$ with $a, b \in \mathbb{Z}$ such that $6|b$ and $a + b \equiv 1 \pmod{4}$. Any cubic $E$-primary number is thus co-prime to 2.

It follows from [27, Lemma 7.9] that any $n = a + b\omega \in \mathbb{Z}[\omega]$ is cubic $E$-primary if and only if

\begin{equation}
(2.1) \quad a + b \equiv 1 \pmod{4}, \quad \text{if} \quad 2|b,
\quad b \equiv 1 \pmod{4}, \quad \text{if} \quad 2|a,
\quad a \equiv 3 \pmod{4}, \quad \text{if} \quad 2 \nmid ab.
\end{equation}

Furthermore, the following quadratic reciprocity law holds for two cubic $E$-primary, co-prime numbers $n, m \in \mathbb{Z}[\omega]:$

\begin{equation}
(2.2) \quad \left(\frac{n}{m}\right) = \left(\frac{m}{n}\right) (-1)^{(N(n)-1)/2((N(m)-1)/2)}.
\end{equation}

Let $c \in \mathbb{Z}[\omega]$, we say that $c$ is $E$-primary if we can write it as $c = -(1 - \omega)^r c'$ with $r \geq 0, r \in \mathbb{Z}, (c', 6) = 1, c'$ is cubic $E$-primary and either $c'$ or $-c'$ is primary. Note that our definition of $E$-primary here is the same as that defined in [27, Section 7.3] when $(c, 6) = 1$. One checks easily that every ideal co-prime to 2 in $\mathbb{Z}[\omega]$ has a unique $E$-primary generator.

One also has the following supplementary laws for $n = a + b\omega, (n, 6) = 1$ being $E$-primary (see [27, Theorem 7.10]),

\begin{equation}
(2.3) \quad \left(\frac{-1}{n}\right) = (-1)^{(N(n)-1)/2}, \quad \left(\frac{1 - \omega}{n}\right) = \left(\frac{a}{3}\right)_n \quad \text{and} \quad \left(\frac{2}{n}\right) = \left(\frac{2}{N(n)}\right)_n,
\end{equation}

where $(\cdot)_n$ denotes the Jacobi symbol in $\mathbb{Z}$. One checks that the last expression above holds in fact for all $E$-primary numbers.

For any element $c \in \mathbb{Z}[\omega], (c, 2) = 1$, we can define a quadratic Dirichlet character $\chi^{(-8c)} (\mod 8c)$ such that for any $n \in (\mathbb{Z}[\omega]/(8c\mathbb{Z}[\omega]))^*$,

\[\chi^{(-8c)}(n) = \left(\frac{-8c}{n}\right)\].

One deduces from (2.3) and the quadratic reciprocity that $\chi^{(-8c)}(n) = 1$ when $n \equiv 1 \pmod{8c}$. It follows from this that $\chi^{(-8c)}(n)$ is well-defined. As $\chi^{(-8c)}(n)$ is clearly multiplicative and of order 2 and is trivial on units, it can be regarded as a quadratic Hecke character (mod 8c) of trivial infinite type. We denote $\chi^{(-8c)}$ for this Hecke character as well and we call it the Kronecker symbol. Furthermore, when $c$ is square-free, $\chi^{(-8c)}$ is non-principal and primitive. To see this, we write $c = \omega_{c_1} \omega_1 \cdots \omega_k$ with $\omega_c$ a unit and $\omega_k$ being $E$-primary primes. Suppose $\chi^{(-8c)}$ is induced by some $\chi$ modulo $c'$ with $\omega_j \nmid c'$, then by the Chinese Remainder Theorem, there exists an $n$ such that $n \equiv 1 \pmod{8c/\omega_j}$ and $(\frac{\omega}{n}) \neq 1$. It follows that $\chi(n) = 1$ but $\chi^{(-8c)}(n) \neq 1$, a contradiction. Thus, $\chi^{(-8c)}$ can only be possibly induced by some $\chi$ modulo 4c. By the Chinese Remainder Theorem, there exists an $n$ such that $n \equiv 1 \pmod{c}$ and $n \equiv 1 + 4\omega \pmod{8}$. As this $n \equiv 1 \pmod{4}$, it follows that $n \equiv 1 \pmod{4c}$, hence $\chi(n) = 1$ but $\chi^{(-8c)}(n) = (\frac{\omega}{n}) = -1 \neq 1$ (note that $(\frac{\omega}{n}) = 1$ when $u$ is a unit in $\mathbb{Z}[\omega]$) and this implies that $\chi^{(-8c)}$ is primitive. This also shows that $\chi^{(-8c)}$ is non-principal.
2.2. Quadratic Gauss sums. For a non-unit \( n \in \mathbb{Z}[\omega] \), \((n, 2) = 1\), the quadratic Gauss sum \( g(n) \) is defined by

\[
g(n) = \sum_{x \mod n} \left( \frac{x}{n} \right) \overline{e} \left( \frac{x}{n} \right),
\]

where \( \overline{e}(z) = e \left( \frac{-z}{\sqrt{3}} \right) \).

It follows from the definition that \( g(1) = 1 \). The following well-known relation (see [5]) now holds for all \( n \):

\[
|g(n)| = \begin{cases} \sqrt{N(n)} & \text{if } n \text{ is square-free}, \\ 0 & \text{otherwise}. \end{cases}
\]

The following properties of \( g(n) \) can be easily derived from definition:

\[
(2.4) \quad g(n_1 n_2) = \left( \frac{n_2}{n_1} \right) g(n_1) g(n_2), \quad (n_1, n_2) = 1.
\]

In what follows we compute the value of \( g(c) \) where \( c \equiv 1 \pmod{36} \) is square-free in \( \mathbb{Z}[\omega] \). We first evaluate the Gauss sum at each \( E \)-primary prime \( \varpi \). We have the following

**Lemma 2.3.** Let \( \varpi \) be an \( E \)-primary prime in \( \mathbb{Z}[\omega] \). Then

\[
g(\varpi) = \begin{cases} N(\varpi)^{1/2} & \text{if } N(\varpi) \equiv 1 \pmod{4}, \\ -iN(\varpi)^{1/2} & \text{if } N(\varpi) \equiv -1 \pmod{4}. \end{cases}
\]

**Proof.** Note that \((-1 - \omega)\) is an \( E \)-primary prime lying above the rational prime 3 and direct computation shows that \( g(-1 - \omega)) = -\sqrt{3}i \). Now we consider the case when \( N(\varpi) = p \) is a prime \( \equiv 1 \pmod{3} \) in \( \mathbb{Z} \). It follows that \( \varpi \) is also a prime in \( \mathbb{Z}[\omega] \) such that \((\varpi, \overline{\varpi}) = 1\). Here we use \( \overline{n} \) to denote the complex conjugate of \( n \), for any \( n \in \mathbb{C} \). We consider the Gauss sum associated to the quadratic Dirichlet character \( \chi_p = \left( \frac{p}{\omega} \right) \zeta \):

\[
\tau(\chi_p) = \sum_{1 \leq x \leq p} \left( \frac{x}{p} \right) \zeta e \left( \frac{x}{N(\varpi)} \right) = \sum_{1 \leq x \leq N(\varpi)} \left( \frac{x}{\varpi} \right) e \left( \frac{x}{N(\varpi)} \right).
\]

Now write \( x = y\varpi + \overline{\varpi}y \), where \( y \) varies over a set of representatives in \( \mathbb{Z}[\omega] \pmod{\varpi} \), then it is easy to see that as \( y \) varies \( \pmod{\varpi} \), \( x \) varies \( \pmod{N(\varpi)} \) in \( \mathbb{Z} \). We deduce that

\[
\tau(\chi_p) = \sum_{y \mod \varpi} \left( \frac{y\varpi}{\varpi} \right) e \left( \frac{y}{\varpi} \right) = \left( \frac{\varpi}{\varpi} \right) \sum_{y \mod \varpi} \left( \frac{y}{\varpi} \right) e \left( \frac{y}{\varpi} \right).
\]

Observe that

\[
g(\varpi) = \sum_{x \mod \varpi} \left( \frac{x}{\varpi} \right) e \left( \frac{1}{\sqrt{3}i} \left( \frac{x}{\varpi} - \frac{\varpi}{\varpi} \right) \right) = \left( \frac{\sqrt{3}i}{\omega} \right) \sum_{x \mod \varpi} \left( \frac{x}{\varpi} \right) e \left( \frac{x}{\varpi} + \frac{\varpi}{\varpi} \right) = \left( \frac{\varpi(1 - \omega)}{\omega} \right) \tau(\chi_p).
\]

Next, we shall show that

\[
\left( \frac{\omega(1 - \omega)\varpi}{\varpi^3} \right) = \left( \frac{\omega(1 - \omega)\varpi}{\varpi^3} \right).
\]

For \( \varpi \) being \( E \)-primary, our discussion above implies that we can write \( \varpi^3 = a + b\omega \) with \( 6|b, a + b \equiv 1 \pmod{4} \) and it follows that \( \varpi^3 = a + b\omega^2 \). Hence

\[
\left( \frac{\omega(1 - \omega)\varpi}{\varpi^3} \right) = \left( \frac{(1 - \omega)(b + a\omega)}{a + b\omega} \right) = \left( \frac{(1 - \omega)(b + a\omega - a - b\omega)}{a + b\omega} \right) = \left( \frac{(1 - \omega)(b - a)}{a + b\omega} \right) = \left( \frac{b - a}{a + b\omega} \right).
\]

Note that when \( a + b\omega \) satisfies \( 6|b, a + b \equiv 1 \pmod{4} \), \( a - b \) is also \( E \)-primary. Thus, it follows from (2.2) that

\[
\left( \frac{b - a}{a + b\omega} \right) = \left( \frac{-1}{a + b\omega} \right) \left( \frac{a - b}{a + b\omega} \right) = \left( \frac{-1}{a + b\omega} \right) \left( \frac{-1}{a + b\omega} \right) \left( \frac{1}{a + b\omega} \right) = \left( \frac{a + b\omega}{a - b} \right) \left( \frac{1 + \omega}{a - b} \right).
\]

Note that \( N(a - b) = (a - b)^2 \equiv 1 \pmod{4} \). We then conclude that

\[
\left( \frac{b - a}{a + b\omega} \right) \left( \frac{a + b\omega}{a - b} \right) = \left( \frac{-1}{a + b\omega} \right) \left( \frac{a + b\omega}{a - b} \right) = \left( \frac{-1}{a + b\omega} \right) \left( \frac{a + b\omega}{a - b} \right) = \left( \frac{-1}{a + b\omega} \right) \left( \frac{a + b\omega}{a - b} \right) = \left( \frac{-1}{a + b\omega} \right) \left( \frac{a + b\omega}{a - b} \right).
\]
Using the relation \(1 + \omega + \omega^2 = 0\), we see that

\[
\frac{b-a}{a+b\omega} = \frac{-1}{a+b\omega} \frac{-a}{a-b} \frac{\omega^2}{a-b} = \left( \frac{-1}{a+b\omega} \right) \left( \frac{-a}{a-b} \right).
\]

We note that for two co-prime \(a, b \in \mathbb{Z}\) (see [27, p. 219]), we have

\[
\left( \frac{a}{b} \right) = 1.
\]

We then conclude from the above discussions that we have

\[
g(\omega) = \left( \frac{-1}{a+b\omega} \right) \tau(\chi_p) = \left( \frac{-1}{\omega} \right) \tau(\chi_p) = (-1)^{(N(\omega)-1)/2} \tau(\chi_p).
\]

As it follows from [41, Chap. 2] that \(\tau(\chi_p) = p^{1/2} / N(\omega)^{1/2}\) when \(p \equiv 1 \mod 4\) and \(\tau(\chi_p) = ip^{1/2} = iN(\omega)^{1/2}\) when \(p \equiv 3 \mod 4\), this completes the proof for the case when \(N(\omega)\) is a rational prime \(\equiv 1 \mod 3\).

Next, let \(p \equiv 2 \mod 3, p \neq 2\) be a prime in \(\mathbb{Z}\), then \(p\) is also a prime in \(\mathbb{Z}[\omega]\), we now compute \(g(p)\). As in [41, Chap. 2] (note that in this case, we still have \(\sum_{x \mod p} \tilde{e}(x/p) = 0\), we have

\[
g(p) = \sum_{x \mod p} \tilde{e}(\frac{x^2}{p}).
\]

We now write \(x = a + b\omega\) with \(a, b \mod p\) in \(\mathbb{Z}\) to see that

\[
g(p) = \sum_{x \mod p} \tilde{e}(\frac{x^2}{p}) = \sum_{a=1}^{p} \sum_{b=1}^{p} \tilde{e}(\frac{2ab - b^2}{p}) = p = N(p)^{1/2}.
\]

As \(N(p) = p^2 \equiv 1 \mod 4\), this completes the proof of the lemma.

Now for a fixed square-free \(c \neq 1, c \equiv 1 \mod 36\), \(c\) is both primary and \(E\)-primary. By writing \(c\) as products of primary primes and adjusting by a possible factor of \(-1\), we can write \(c = \omega_1 \cdots \omega_k \) or \(-\omega_1 \cdots \omega_k\) with \(\omega_i\) being distinct \(E\)-primary primes. Then \(c^3 = \omega_1^3 \cdots \omega_k^3\) or \(-\omega_1^3 \cdots \omega_k^3\). We write \(c^3 = a + b\omega\) and note that \(a, b\) satisfy \(2.1\) by definition. The same consideration for \((\omega_1 \cdots \omega_k)^3\) enables us to conclude that \(c = \omega_1 \cdots \omega_k\) as \(c\) and \(c^3\) differ only by a possible factor of \(-1\). As \(N(c) \equiv 1 \mod 4\), we conclude that there must be an even number of \(\omega_j\) in the decomposition of \(c\) such that \(N(\omega_j) \equiv -1 \mod 4\). We may assume that \(\omega_1, \cdots, \omega_{2k_0}\) are such primes. It follows from \([2.4]\) and \([2.2]\) and Lemma 2.3 that

\[
g(c) = g(\omega_1 \cdots \omega_{2k_0}) \prod_{j=2k_0+1}^{k} g(\omega_j) = g(\omega_1 \cdots \omega_{2k_0}) N\left( \prod_{j=2k_0+1}^{k} \omega_j \right)^{1/2}.
\]

Using Lemma 2.3 and induction on \(k_0\) shows that

\[
g(\omega_1 \cdots \omega_{2k_0}) = N\left( \prod_{j=1}^{2k_0} \omega_j \right)^{1/2}.
\]

We then conclude that

\[
g(c) = N(c)^{1/2}.
\]

Now, we define more generally, for any \(n, k \in \mathbb{Z}[\omega], (n, 2) = 1\),

\[
g(k, n) = \sum_{x \mod n} \left( \frac{x}{n} \right) \tilde{e}(\frac{kx}{n}) \quad \text{and} \quad G(k, n) = \left( \frac{1-i}{2} + \left( \frac{-1}{n} \right) \frac{1+i}{2} \right) g(k, n).
\]

Note that

\[
(2.5) \quad g(k, n) = \left( \frac{1+i}{2} + \left( \frac{-1}{n} \right) \frac{1-i}{2} \right) G(k, n).
\]

The notion of \(G(k, n)\) is motivated by the work of K. Soundararajan [36, Section 2.2], to compensate for the fact that \(g(k, n)\) is not multiplicative with respect to \(n\). The following lemma shows that \(G(k, n)\) is indeed multiplicative with respect to \(n\) and allows us to evaluate \(G(k, n)\) for \(n\) being \(E\)-primary explicitly. We omit the proof here as it is similar to that of [36, Lemma 2.3] and [13, Lemma 2.4], using the quadratic reciprocity [22].
Lemma 2.4. (i) We have

\[ G(rs, n) = \left( \frac{r}{n} \right) G(r, n), \quad (s, n) = 1, \]
\[ G(k, mn) = G(k, m)G(k, n), \quad m, n \text{ E-primary and } (m, n) = 1. \]

(ii) Let \( \varpi \) be an E-primary prime in \( \mathbb{Z}[\omega] \). Suppose \( \varpi^h \) is the largest power of \( \varpi \) dividing \( k \). If \( k = 0 \) then set \( h = \infty \).
Then for \( l \geq 1 \),

\[ G(k, \varpi^l) = \begin{cases} 
0 & \text{if } l \leq h \quad \text{is odd}, \\
\varphi(\varpi^l) = \#(\mathbb{Z}[\omega]/(\varpi^l))^* & \text{if } l \leq h \quad \text{is even}, \\
-N(\varpi)^{l-1} & \text{if } l = h + 1 \quad \text{is even}, \\
\left( \frac{-k}{\varpi^h} \right) N(\varpi)^{l-1/2} & \text{if } l = h + 1 \quad \text{is odd}, \\
0 & \text{if } l \geq h + 2.
\end{cases} \]

2.5. The approximate functional equation. Let \( \chi \) be a primitive Hecke character, the Hecke \( L \)-function associated with \( \chi \) is defined for \( \Re(s) > 1 \) by

\[ L(s, \chi) = \sum_{0 \neq A \subseteq \mathcal{O}_K} \chi(A)(N(A))^{-s}, \]

where \( A \) runs over all non-zero integral ideals in \( K \) and \( N(A) \) is the norm of \( A \). As shown by E. Hecke, \( L(s, \chi) \) admits analytic continuation to an entire function and satisfies a functional equation. We refer the reader to \([11, 12, 18, 28]\) for a more detailed discussion of these Hecke characters and \( L \)-functions.

Let \( \chi \) be a primitive Hecke character (mod \( m \)) of trivial infinite type. Let \( G(s) \) be any even function which is holomorphic and bounded in the strip \(-4 < \Re(s) < 4\) satisfying \( G(0) = 1 \). We have the following expression for \( L(1/2 + it, \chi) \) for \( t \in \mathbb{R} \) (see \([12\text{ Section 2.3}]\)):

\[ L \left( \frac{1}{2} + it, \chi \right) = \sum_{0 \neq A \subseteq \mathcal{O}_K} \frac{\chi(A)}{N(A)^{1/2+it}} V_t \left( \frac{2\pi N(A)}{x} \right) \]
\[ + \frac{W(\chi)}{N(m)^{1/2}} \left( \frac{(2\pi)^2}{|D_k|N(m)} \right)^{it} \frac{\Gamma(1/2-it)}{\Gamma(1/2+it)} \sum_{0 \neq A \subseteq \mathcal{O}_K} \frac{\chi(A)}{N(A)^{1/2-it}} V_t^{-1} \left( \frac{2\pi N(A)x}{|D_k|N(m)} \right), \]

where

\[ V_t(\xi) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(s+1/2+it)}{\Gamma(1/2+it)} G(s) \frac{\xi^{-s}}{s} ds. \]

We write \( V \) for \( V_0 \) and note that for a suitable \( G(s) \) (for example \( G(s) = e^{s^2} \)), we have for any \( c > 0 \) (see \([23\ Proposition 5.4]\)):

\[ V_t(\xi) \ll \left( 1 + \frac{\xi}{1+|t|} \right)^{-c}. \]

On the other hand, when \( G(s) = 1 \), we have (see \([36\ Lemma 2.1]\)) for the \( j \)-th derivative of \( V(\xi) \),

\[ V^{(j)}(\xi) = 1 + O(\xi^{1/2-c}) \quad \text{for } 0 < \xi < 1 \quad \text{and} \quad V^{(j)}(\xi) = O(e^{-\xi}) \quad \text{for } \xi > 0, \ j \geq 0. \]

When \( c \in \mathcal{O}_K \) which is square-free and congruent to 1 (mod 36), it follows from \([23]\) that

\[ \left( \frac{-1}{c} \right) = \left( \frac{1-\omega}{c} \right) = \left( \frac{2}{c} \right) = 1. \]

This shows that \( \chi_c = (\frac{\cdot}{c}) \) is trivial on units, it can be regarded as a primitive character of the ray class group \( h(c) \).
We recall here that for any \( c \), the ray class group \( h(c) \) is defined to be \( I(c)/P(c) \), where \( I(c) = \{ A \in I : (A, (c)) = 1 \} \) and \( P(c) = \{ (a) \in P : a \equiv 1 \pmod{c} \} \) with \( I \) and \( P \) denoting the group of fractional ideals in \( K \) and the subgroup of principal ideals, respectively. The functional equation \([23\ Theorem 3.8]\) for \( L(s, \chi_c) \) becomes

\[ \Lambda(s, \chi_c) = g(c)(N(c))^{-1/2}\Lambda(1-s, \overline{\chi}_c), \]

where \( g(c) \) is the Gauss sum defined in Section 2.1 and

\[ \Lambda(s, \chi_c) = (|D_K|N(c))^{s/2}(2\pi)^{-s}\Gamma(s)L(s, \chi_c), \]
where $D_K = -3$ is the discriminant of $K$.

We now derive readily from (2.7) by setting $x = (|D_K|N(c))^{1/2}$ the following expression for $L(1/2, \chi_c)$:

$$L \left( \frac{1}{2}, \chi_c \right) = 2 \sum_{0 \neq \mathbf{A} \subseteq O_K} \frac{\chi_c(\mathbf{A})}{N(\mathbf{A})^{1/2}} V \left( \frac{2\pi N(\mathbf{A})}{(3N(c))^{1/2}} \right).$$

2.6. The large sieve with quadratic symbols. The large sieve inequality for quadratic Hecke characters will be an important ingredient of this paper. The study of the large sieve inequality for characters of a fixed order is of independent interest. We refer the reader to [1, 10, 17, 19, 20].

**Lemma 2.7.** [33, Theorem 1] Let $M, N$ be positive integers, and let $(a_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of complex numbers, where $n$ runs over $\mathbb{Z}[\omega]$. Then we have

$$\sum_{\substack{m \in \mathbb{Z}[\omega] \\ N(m) \leq M}} \sum_{\substack{n \in \mathbb{Z}[\omega] \\ N(n) \leq N}} |a_n|^2 \ll \varepsilon (M + N)(MN)^{\varepsilon} \sum_{N(n) \leq N} |a_n|^2,$$

for any $\varepsilon > 0$, where the asterisks indicate that $m$ and $n$ run over $E$-primary square-free elements of $\mathbb{Z}[\omega]$ and $(\frac{\omega}{m})$ is the quadratic residue symbol.

2.8. The Explicit Formula. Our approach in this paper relies on the following explicit formula, which essentially converts a sum over zeros of an $L$-function to a sum over primes. As it is similarly to that of [13, Lemma 2.7], we omit its proof here.

**Lemma 2.9.** Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ is compactly supported. For any square-free $c \in \mathbb{Z}[\omega], (c, 2) = 1$, $X \leq N(c) \leq 2X$, we have

$$S(\chi^{(-8c)}, \phi) = \int_{-\infty}^{\infty} \phi(t) dt - \frac{1}{2} \int_{-\infty}^{\infty} \hat{\phi}(u) du - 2S(\chi^{(-8c)}, X, \hat{\phi}) + O \left( \frac{\log \log 3X}{\log X} \right),$$

with the implicit constant depending on $\phi$, where

$$S(\chi^{(-8c)}, X, \hat{\phi}) = \frac{1}{\log X} \sum_{\omega \in E_{primary}} \log N(\omega) \chi^{(-8c)}(\omega) \hat{\phi} \left( \frac{\log N(\omega)}{\log X} \right).$$

We shall also need the following

**Lemma 2.10.** Suppose that GRH is true. For any non-principal Hecke character $\chi$ of trivial infinite type with modulus $n$, we have for $x \geq 1$,

$$S(x, \chi) = \sum_{N(\omega) \leq x, \omega \in E_{primary}} \chi(\omega) \log N(\omega) \ll \min \left\{ x, x^{1/2} \log^3(x) \log N(n) \right\}.$$

2.11. Poisson Summation. The proof of Theorem 3 requires the following Poisson summation formula.

**Lemma 2.12.** Let $n \in \mathbb{Z}[\omega], (n, 2) = 1$ and $(\frac{\omega}{n})$ be the quadratic symbol $\pmod{n}$. For any Schwartz class function $\Psi$, we have for all $a > 0$,

$$\sum_{\substack{m \in \mathbb{Z}[\omega] \\ (m, 2) = 1}} \left( \frac{m}{n} \right) \Psi \left( \frac{aN(m)}{X} \right) = \frac{X}{N(n)} \left( \frac{2}{n} \right) \sum_{k \in \mathbb{Z}[\omega]} \left( 1 + (-1)^N(k) \right) \frac{1}{4} G(k, n) \bar{\Psi} \left( \sqrt{\frac{N(k)X}{4N(n)}} \right),$$

where

$$\bar{\Psi}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(N(x + y\omega)) \bar{e}(-t(x + y\omega)) dx dy, \quad t \geq 0.$$

**Proof.** For any $a > 0$, we have the following result from [13, Lemma 2.6]:

$$\sum_{\substack{m \in \mathbb{Z}[\omega] \\ (m, 2) = 1}} \left( \frac{m}{n} \right) \Psi \left( \frac{aN(m)}{X} \right) = \frac{X}{aN(n)} \sum_{k \in \mathbb{Z}[\omega]} g(k, n) \bar{\Psi} \left( \sqrt{\frac{N(k)X}{aN(n)}} \right).$$
We apply (2.3) to write \( g(k, n) \) in terms of \( G(k, n) \), using that \( G(k, n) = \left( \frac{-1}{n} \right) G(-k, n) \) (by (2.6)), and recombining the \( k \) and \( -k \) terms to see that

\[
(2.11) \quad \sum_{m \in \mathbb{Z}[\omega]} \left( \frac{m}{n} \right) \Psi \left( \frac{aN(m)}{X} \right) = \frac{X}{aN(n)} \sum_{k \in \mathbb{Z}[\omega]} G(k, n) \widetilde{\Psi} \left( \sqrt{\frac{N(k)X}{aN(n)}} \right).
\]

It follows from this that

\[
\sum_{m \in \mathbb{Z}[\omega]} \left( \frac{m}{n} \right) \Psi \left( \frac{N(m)}{X} \right) = \sum_{m} \left( \frac{m}{n} \right) \Psi \left( \frac{N(m)}{X} \right) - \left( \frac{2}{n} \right) \sum_{m} \left( \frac{m}{n} \right) \Psi \left( \frac{4N(m)}{X} \right)
\]

\[
= \frac{X}{N(n)} \sum_{k \in \mathbb{Z}[\omega]} G(k, n) \widetilde{\Psi} \left( \sqrt{\frac{N(k)X}{N(n)}} \right) - \left( \frac{2}{n} \right) \frac{X}{4N(n)} \sum_{k \in \mathbb{Z}[\omega]} G(k, n) \widetilde{\Psi} \left( \sqrt{\frac{N(k)X}{4N(n)}} \right).
\]

Using (2.6), we have

\[ G(2k, n) = \left( \frac{2}{n} \right) G(k, n). \]

We can rewrite the first sum in the last expression of (2.12) as

\[
\sum_{k \in \mathbb{Z}[\omega]} G(k, n) \widetilde{\Psi} \left( \sqrt{\frac{N(k)X}{N(n)}} \right) = \left( \frac{2}{n} \right) \sum_{k \in \mathbb{Z}[\omega]} G(2k, n) \widetilde{\Psi} \left( \sqrt{\frac{N(2k)X}{4N(n)}} \right) = \left( \frac{2}{n} \right) \sum_{k \in \mathbb{Z}[\omega]} G(k, n) \widetilde{\Psi} \left( \sqrt{\frac{N(k)X}{4N(n)}} \right)
\]

\[
= \left( \frac{2}{n} \right) \sum_{k \in \mathbb{Z}[\omega]} \frac{1 + (-1)^{N(k)}}{2} \frac{G(k, n) \widetilde{\Psi} \left( \sqrt{\frac{N(k)X}{4N(n)}} \right)}{G(2k, n)}. \]

Substituting this back to last expression in (2.12), we get the desired result. \( \square \)

Suppose that \( \Psi(t) \) is a non-negative smooth function supported on \((1, 2)\), satisfying \( \Psi(t) = 1 \) for \( t \in (1 + 1/U, 2 - 1/U) \) and \( \Psi^{(j)}(t) \ll_j U^j \) for all integers \( j \geq 0 \). The following estimations for \( \widetilde{\Psi} \) and its derivatives are from \([13, \text{Section 2.8}]\) (beware of the differences between the support of the functions): we have \( \widetilde{\Psi}(t) \in \mathbb{R} \) for any \( t \geq 0 \) and

\[
(2.13) \quad \widetilde{\Psi}^{(j)}(t) \ll_j \min\{1, U^{j-1}t^{-j}\}
\]

for all integers \( \mu \geq 0, j \geq 1 \) and all \( t > 0 \).

We also have

\[
(2.14) \quad \widetilde{\Psi}(0) = \frac{2\pi}{\sqrt{3}} + O \left( \frac{1}{U} \right).
\]

As \( \Phi(t) \) satisfies the assumptions on \( \Psi(t) \), the estimation (2.13) is also valid for \( \Phi(t) \). So in the sequel, we shall use these estimations for \( \Phi(t) \) without further justification.

### 3. Proof of Theorem 1.1

Note first that (1.2) is an easy consequence of the following bound:

\[
(3.1) \quad \sum_{\substack{(a, 2)=1 \atop a \equiv 1 \mod 3 \atop N(a) \leq M}} \left| L \left( \frac{1}{2} + it, \psi\chi^{(a)} \right) \right|^2 \ll (M(1 + |t|))^{1+\epsilon},
\]

where \( \psi \) is any ray class character of a fixed modulus. For \((a, 2) = 1, a \equiv 1 \mod 3\), \( \chi^{(a)} \) is a Hecke character (mod 36a) such that for any ideal \((c)\) co-prime to 6, with \( c \) being the unique primary generator of \((c)\), \( \chi^{(a)}((c)) \) is defined as \( \chi^{(a)}((c)) = \left( \frac{a}{c} \right) \). One checks easily that \( \chi^{(a)} \) is a Hecke character (mod 36a) of trivial infinite type. By an abuse of notation, we shall also write \( \chi^{(a)}(c) \) for \( \chi^{(a)}((c)) \). As the proof of the above bound is similar to the arguments in \([12, \text{Section 3.3}]\), using Lemma (2.7) instead, we omit the proof here.
We are therefore left to prove (1.1). We have, using (2.10) with \(G(s) = 1\), that
\[
M = \sum_{c \equiv 1 \mod 36}^* L \left( \frac{1}{2}, \chi_c \right) W \left( \frac{N(c)}{y} \right) = 2 \sum_{c \equiv 1 \mod 36}^* \sum_{0 \neq A \subseteq \mathcal{O}_K} \frac{\chi_c(A)}{N(A)^{1/2}} V \left( \frac{2\pi N(A)}{(3N(c))^{1/2}} \right) W \left( \frac{N(c)}{y} \right).
\]

Since any integral non-zero ideal \(A\) in \(\mathbb{Z}[\omega]\) has a unique generator \(2^r(1 - \omega)^s a\), with \(r_1, r_2 \in \mathbb{Z}, r_1, r_2 \geq 0, a \in \mathbb{Z}[\omega]\), \((a, 2) = 1, a \equiv 1 \pmod{3}\), it follows from (1.9) and the definition of \(\chi(a)\) that \(\chi_c(A) = \chi_c(a)\).

The above discussions allow us to recast \(M\) as
\[
M = 2 \sum_{(r_1, r_2) \geq 0} \sum_{c \equiv 1 \mod 3 \atop a \equiv 1 \mod 3} \frac{1}{2^{r_1} 3^{r_2} N(a)^{1/2}} M(r,a),
\]
where
\[
M(r,a) = \sum_{c \equiv 1 \mod 36}^* \chi^{(a)}(c)V \left( \frac{2^{2r_1+1}3^{2r_2-1/2}N(a)}{y^{1/2}} \frac{y^{1/2}}{N(c)^{1/2}} \right) W \left( \frac{N(c)}{y} \right).
\]

Now we use M"obius inversion to detect the condition that \(c\) is square-free, getting
\[
M(r,a) = \sum_{(l,2) = 1} \mu_{[\omega]}(l) \chi^{(a)}(l^2) M(l,r,a),
\]
with
\[
M(l,r,a) = \sum_{c \equiv 1 \mod 3 \atop cl^2 \equiv 1 \mod 36}^* \chi^{(a)}(c)V \left( \frac{2^{2r_1+1}3^{2r_2-1/2}N(a)}{y^{1/2}} \frac{y^{1/2}}{N(cl^2)^{1/2}} \right) W \left( \frac{N(cl^2)}{y} \right).
\]

By Mellin inversion, we have
\[
V \left( \frac{2^{2r_1+1}3^{2r_2-1/2}N(a)}{y^{1/2}} \frac{y^{1/2}}{N(cl^2)^{1/2}} \right) W \left( \frac{N(cl^2)}{y} \right) = \frac{1}{2\pi i} \int_{(2)} \left( \frac{y}{N(cl^2)} \right)^s \tilde{f}(s) \, ds,
\]
where
\[
\tilde{f}(s) = \int_0^\infty V \left( \frac{2^{2r_1+1}3^{2r_2-1/2}N(a)}{(xy)^{1/2}} \right) W(x)x^{s-1} \, dx.
\]

Integration by parts and using (2.8) shows \(\tilde{f}(s)\) is a function satisfying the bound for all \(\Re(s) > 0\), and any integer \(E > 0\),
\[
\tilde{f}(s) \ll (1 + |s|)^{-E} \left( 1 + \frac{2^{2r_1+1}3^{2r_2-1/2}N(a)}{y^{1/2}} \right)^{-E}.
\]

With this notation, we have
\[
M(l,r,a) = \frac{1}{2\pi i} \int_{(2)} \tilde{f}(s) \left( \frac{y}{N(l^2)} \right)^s \sum_{c \equiv 1 \mod 3 \atop cl^2 \equiv 1 \mod 36} \chi^{(a)}(c) \frac{N(c)^s}{N(c)} \, ds.
\]

We now use the ray class characters to detect the condition that \(cl^2 \equiv 1 \pmod{36}\), getting
\[
M(l,r,a) = \frac{1}{\#h_{(36)}} \sum_{\psi \mod 36} \psi(l^2) \int_{(2)} \tilde{f}(s) \left( \frac{y}{N(l^2)} \right)^s L(s, \psi \chi^{(a)}) \, ds,
\]
where \(\psi\) runs over all ray class characters \(\pmod{36}\), \(\#h_{(36)} = 108\) and
\[
L(s, \psi \chi^{(a)}) = \sum_{A \neq 0} \psi(A) \chi^{(a)}(A) N(A)^s.
\]

We estimate \(M\) by shifting the contour to the half line. When \(\psi \chi^{(a)}\) is principal, the Hecke \(L\)-function has a pole at \(s = 1\). We set \(M_0\) to be the contribution to \(M\) of these residues, and \(M_1\) to be the remainder.

Using arguments similar to those in [12] Section 3.3 and [3.1], we have that
\[
M_1 \ll y^{3/4 + \varepsilon}.
\]
It now remains to determine $M_0$. Note that $\psi \chi^{(a)}$ is principal if and only if both $\psi$ and $\chi^{(a)}$ are principal. Hence $a$ must be a square. We denote $\psi_0$ for the principal ray class character $\pmod{36}$. Then we have
\[
L \left( s, \psi_0 \chi^{(a^2)} \right) = \zeta_{Q(\omega)}(s) \prod_{(\pi)|6a} \left( 1 - N(\pi)^{-s} \right).
\]

Let $c_0 = \frac{\sqrt{a}}{3}$, the residue of $\zeta_{Q(\omega)}(s)$ at $s = 1$. Then we have
\[
M_0 = \frac{2y}{\#h_{(36)}} \sum_{r_1, r_2 \geq 0 \atop (a, 2) = 1} \frac{1}{2r_1 3^{r_2/2} N(a)} \hat{f}(1) \text{Res}_{s=1} L \left( s, \psi_0 \chi^{(a^2)} \right) \sum_{l \equiv 1 \pmod{3}} \frac{\mu_{[\omega]}(l) \chi^{(a^2)}(l^2)}{N(l^2)}
\]
\[
= \frac{2c_0 y}{\#h_{(36)} \zeta_{Q(\omega)}(2)} \sum_{r_1, r_2 \geq 0 \atop (a, 2) = 1} \frac{1}{2r_1 3^{r_2/2} N(a)} \hat{f}(1) \prod_{\pi \equiv 0 \pmod{6a}} \left( 1 - N(\pi)^{-1} \right) \prod_{(\pi)|6a} \left( 1 - N(\pi)^{-2} \right)^{-1}
\]
\[
= \frac{2c_0 y}{\#h_{(36)} \zeta_{Q(\omega)}(2)} \sum_{r_1, r_2 \geq 0 \atop (a, 2) = 1} \frac{1}{2r_1 3^{r_2/2} N(a)} \hat{f}(1) \prod_{(\pi)|6a} \left( 1 + N(\pi)^{-1} \right)^{-1}.
\]

Note that
\[
\prod_{(\pi)|6a} \left( 1 + N(\pi)^{-1} \right)^{-1} = \sum_{(d)|6a} \frac{\mu_{[\omega]}(d)}{\sigma(d)},
\]
where $\sigma(d)$ denotes the sum of the norms of the integral ideal divisors of $d$.

Applying this, we see that
\[
(3.2) \quad \sum_{(a, 2) = 1 \atop N(a) \leq x} \frac{1}{N(a)} \prod_{(\pi)|6a} \left( 1 + N(\pi)^{-1} \right)^{-1} = \sum_{(d)|12x} \frac{\mu_{[\omega]}(d)}{\sigma(d)} \sum_{(a, 2) = 1 \atop N(a) \leq x} \frac{1}{N(a)}.
\]

We note the following result from counting the lattice points inside an ellipse (we can take $\theta = 131/416$, see [21])
\[
(3.3) \quad \sum_{N(a) \leq x} 1 = \frac{2\pi}{\sqrt{3}} x + O \left( x^\theta \right).
\]

This implies that
\[
\sum_{(n, 2) = 1 \atop n \equiv 1 \pmod{3} \atop N(a) \leq x} 1 = \frac{\pi}{6\sqrt{3}} x + O \left( x^\theta \right).
\]

Now partial summation yields
\[
\sum_{(a, 2) = 1 \atop a \equiv 1 \pmod{3} \atop N(a) \leq x} \frac{1}{N(a)} = \frac{\pi}{6 \sqrt{3}} \log x + C_0 + O(x^\theta),
\]
where $C_0$ is a constant. We then deduce from the above formula and (3.2) that
\[
\sum_{(a, 2) = 1 \atop N(a) \leq x} \frac{1}{N(a)} \prod_{(\pi)|6a} \left( 1 + N(\pi)^{-1} \right)^{-1} = \frac{\sqrt{3} \pi}{30} A \log x + C_1 + O(x^\theta),
\]
where $A$, $C_1$ are constants with $A$ defined in (1.3).
We then deduce that
\[
\sum_{\substack{a=1 \mod 3 \\ (a,2)=1}} \frac{1}{N(a)} \int (1) \prod_{(\pi)(6a)} (1 + N(\pi))^{-1} = \int W(x) \sum_{\substack{a=1 \mod 3 \\ (a,2)=1}} \frac{1}{N(a)} \prod_{(\pi)(6a)} (1 + N(\pi))^{-1} V \left( \frac{\pi^{2r_1 + 1} r_2 - 1/2 N(a)^2}{(xy)^{1/2}} \right) dx.
\]

Applying partial summation and (2.8), we get
\[
\sum_{\substack{a=1 \mod 3 \\ (a,2)=1}} \frac{1}{N(a)} \prod_{(\pi)(6a)} (1 + N(\pi))^{-1} V \left( \frac{\pi^{2r_1 + 1} r_2 - 1/2 N(a)^2}{(xy)^{1/2}} \right) = \left\{ \begin{array}{l}
\frac{\sqrt{3\pi}}{30} A \log_{\pi^{1/2}} \left( \frac{(xy)^{1/4}}{r_{1/2} r_1 + 1/2 + 2r_2 - 1/4} \right) + C_2 + O \left( \frac{(xy)^{1/4}}{(xy)^{1/4}} \right)^{1-\theta} \quad (xy)^{1/2} > \pi^{2r_1 + 1} r_2 - 1/2, \\
O \left( \frac{(xy)^{1/4}}{(xy)^{1/4}} \right)^{-1} \quad \pi^{2r_1 + 1} r_2 - 1/2 \geq (xy)^{1/2}, \end{array} \right.
\]

for some constant $C_2$.

We then conclude that by a straightforward calculation (we may assume that $y$ is large),
\[
M_0 = \frac{(1 + \sqrt{3})c_0\pi A}{20 \# h(36) \zeta(2)} \tilde{W}(0)y \log y + C\tilde{W}(0)y + O(y^{3+\theta}/4),
\]
where $C$ is the same constant $C$ appearing in $|L|$.

Combining the results for $M_0$ and $M_1$, we complete the proof of Theorem 1.1

4. Proof of Theorem 1.3

4.1. Evaluation of $C(X)$. We have
\[
\sum_{N(c) \leq X \atop (c,2)=1} \mu^2(c) = \sum_{N(c) \leq X \atop (c,2)=1} \mu(c) = \sum_{N(c) \leq X \atop (c,2)=1} \mu(c) = \sum_{N(c) \leq X \atop (c,2)=1} \mu(c) = \sum_{N(c) \leq X \atop (c,2)=1} 1,
\]

where the “$*$” on the sum over $c$ means that the sum is restricted to square-free elements $c$ of $\mathbb{Z}[\omega]$.

Similar to the arguments in 3.3 Section 3.1, we deduce from (3.3) that as $X \to \infty$,
\[
\sum_{(c,2)=1} \Phi \left( \frac{N(c)}{X} \right) \sim \# C(X) \sim 8\sqrt{3\pi} \frac{X}{15\zeta(2)}.
\]

Just as shown in 3.3 Section 3.1 and using Lemma 2.9 we see that in order to establish Theorem 1.3 it suffices to show that for any Schwartz function $\hat{\phi}$ with $\hat{\phi}$ supported in $(-2 + \varepsilon, 2 - \varepsilon)$ for any $0 < \varepsilon < 1$,
\[
\lim_{X \to \infty} S(X, Y; \hat{\phi}, \Phi) = \frac{-2\sqrt{3\pi}}{15\zeta(2)} \int_{-\infty}^{\infty} \left( 1 - \chi(-1,1)(t) \right) \hat{\phi}(t) dt,
\]

where
\[
S(X, Y; \hat{\phi}, \Phi) = \sum_{(c,2)=1} \prod_{N(\pi)\leq Y} \chi(\pi) \log N(\pi) \frac{\log N(\pi)}{X} \Phi \left( \frac{N(c)}{X} \right) \hat{\phi}(t).
\]

Here we set $Y = X^{2-\varepsilon}$ and write the condition $N(\pi) \leq Y$ explicitly throughout this section.
4.2. Expressions $S_M(X,Y;\hat{\phi},\Phi)$ and $S_R(X,Y;\hat{\phi},\Phi)$. Let $Z = \log^5 X$ and write $\mu_{[\omega]}(c) = M_Z(c) + R_Z(c)$ where

$$M_Z(c) = \sum_{\substack{l \mid c \leq Z \wedge \varphi(l)}} \mu_{[\omega]}(l)$$ \quad and \quad $$R_Z(c) = \sum_{\substack{l \mid c \leq Z \wedge \varphi(l)}} \mu_{[\omega]}(l).$$

Define

$$S_M(X,Y;\hat{\phi},\Phi) = \sum_{\substack{(c,2) = 1}} M_Z(c) \sum_{\substack{\varphi(l) \leq \sqrt{N(l)} \leq Y \wedge l \equiv \varphi(l) \mod N(l)}} \frac{\log N(\varphi)}{\sqrt{N(l)}} \left( -\frac{8c}{\omega} \right) \hat{\phi} \left( \frac{\log N(\varphi)}{\log X} \right) \Phi \left( \frac{N(c)}{X} \right),$$

and

$$S_R(X,Y;\hat{\phi},\Phi) = \sum_{\substack{(c,2) = 1}} R_Z(c) \sum_{\substack{\varphi(l) \leq \sqrt{N(l)} \leq Y \wedge l \equiv \varphi(l) \mod N(l)}} \frac{\log N(\varphi)}{\sqrt{N(l)}} \left( -\frac{8c}{\omega} \right) \hat{\phi} \left( \frac{\log N(\varphi)}{\log X} \right) \Phi \left( \frac{N(c)}{X} \right),$$

so that $S(X,Y;\hat{\phi},\Phi) = S_M(X,Y;\hat{\phi},\Phi) + S_R(X,Y;\hat{\phi},\Phi)$.

Using standard techniques (see [13] Section 3.3), we have that

$$S_R(X,Y;\hat{\phi},\Phi) = o(X \log X).$$

We now give another expression for $S_M(X,Y;\hat{\phi},\Phi)$ using Possion summation. We write it as

$$S_M(X,Y;\hat{\phi},\Phi) = \sum_{\varphi \text{-primary}} \frac{\log N(\varphi)}{\sqrt{N(l)}} \left( -\frac{8c}{\omega} \right) \hat{\phi} \left( \frac{\log N(\varphi)}{\log X} \right) \sum_{\substack{l \mid c \leq Z \wedge \varphi(l)}} \mu_{[\omega]}(l) \frac{(\omega)^2}{(c)(\omega)} \Phi \left( \frac{N(l^2\varphi)}{4N(l^2\varphi)} \right).$$

Applying Possion summation, Lemma 2.12 we obtain that

$$\sum_{c \in Z,\varphi(l) \leq Y \wedge l \equiv \varphi(l) \mod N(l)} \frac{(\omega)^2}{(c)(\omega)} \Phi \left( \frac{N(c^2)}{X} \right) = \frac{X}{N(l^2\varphi)} \left( \frac{2}{\omega} \right) \sum_{k \in Z,\varphi(l)} \left( \frac{1}{2} - \frac{1}{4} \right) G(k,\varphi) \Phi \left( \sqrt{\frac{N(k)X}{4N(l^2\varphi)}} \right).$$

We can now recast $S_M(X,Y;\hat{\phi},\Phi)$ as

$$S_M(X,Y;\hat{\phi},\Phi) = X \sum_{\substack{l \mid c \leq Z \wedge \varphi(l) \equiv \varphi(l) \mod N(l) \wedge (l,2) = 1}} \frac{\mu_{[\omega]}(l)}{N(l^2)} \sum_{k \in Z,\varphi(l)} \left( \frac{1}{2} - \frac{1}{4} \right) G(k,\varphi) \Phi \left( \sqrt{\frac{N(k)X}{4N(l^2\varphi)}} \right),$$

as one deduces easily from Lemma 2.3 that $G(0,\varphi) = 0$ and

$$G(k,\varphi) = \left( \frac{-k}{\omega} \right) N(\varphi)^{1/2}, \quad k \neq 0.$$

4.3. Estimation of $S_M(X,Y;\hat{\phi},\Phi)$. We show in what follows that the terms $k = \square (k$ is a square), $k \neq 0$ in $S_M(X,Y;\hat{\phi},\Phi)$ contribute a second main term. Before we proceed, we need the following result:

Lemma 4.4. For $y > 0$, \[ \sum_{k \in Z, k \neq 0} \left( \frac{1}{2} - \frac{1}{4} \right) \Phi \left( \frac{N(k)}{y} \right) = -\frac{3}{4} \Phi(0) + O \left( \frac{y^2}{y^{1/2}} \right). \]
Proof. Note that
\[
\sum_{k \in \mathbb{Z}[\omega], \ k \neq 0} \left( \frac{1 + (-1)^N(k)}{2} - \frac{1}{4} \right) \Phi \left( \frac{N(k)}{y} \right) = \sum_{k \in \mathbb{Z}[\omega]} \left( \frac{1 + (-1)^N(k)}{2} - \frac{1}{4} \right) \Phi \left( \frac{N(k)}{y} \right) - \frac{3}{4} \Phi(0)
\]
and
\[
\sum_{k \in \mathbb{Z}[\omega]} \left( \frac{1 + (-1)^N(k)}{2} - \frac{1}{4} \right) \bar{\Phi} \left( \frac{N(k)}{y} \right) = \sum_{k \in \mathbb{Z}[\omega]} \bar{\Phi} \left( \frac{4N(k)}{y} \right) - \frac{1}{4} \sum_{k \in \mathbb{Z}[\omega]} \bar{\Phi} \left( \frac{N(k)}{y} \right).
\]
By taking \( n = 1 \) in (2.11), we immediately obtain that
\[
\sum_{k \in \mathbb{Z}[\omega]} \Phi \left( \frac{4N(k)}{y} \right) = \frac{y}{4} \sum_{j \in \mathbb{Z}[\omega]} \phi \left( \frac{N(j)y}{4} \right) \quad \text{and} \quad \sum_{k \in \mathbb{Z}[\omega]} \bar{\Phi} \left( \frac{N(k)}{y} \right) = y \sum_{j \in \mathbb{Z}[\omega]} \phi \left( \sqrt{N(j)y} \right),
\]
where
\[
\Phi(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{\Phi}(N(u + v\omega))e(-t(u + v\omega)) \, du \, dv, \quad t \geq 0.
\]
It follows that
\[
\sum_{k \in \mathbb{Z}[\omega]} \left( \frac{1 + (-1)^N(k)}{2} - \frac{1}{4} \right) \Phi \left( \frac{N(k)}{y} \right) = \frac{y}{4} \sum_{j \in \mathbb{Z}[\omega]} \phi \left( \sqrt{N(j)y} \right).
\]
We have, when \( t > 0 \), using (14, (2.6)) and via integration by parts
\[
\Phi(t) \ll \frac{U^2}{t^3}.
\]
The lemma follows immediately from this bound and (14).

By a change of variables \( k \rightarrow k^2 \) and noting that \( k^2 = k_1^2 \) if and only if \( k = \pm k_1 \), we see that the terms \( k = \Box, k \neq 0 \) in \( S_M(X, Y; \phi, \Phi) \) contribute
\[
S_{\Box,1}(X, Y; \phi, \Phi) = \frac{X}{2} \sum_{\substack{(l) \in \mathbb{Z}[\omega] \leq Z, (l,2) = 1 \atop \chi(l) = 1 \atop \chi \equiv E-\text{primary}}} \frac{\mu_{\omega}(l)}{N(l^2)} \sum_{k \in \mathbb{Z}[\omega], \ k \neq 0} \frac{\log N(\chi)}{N(\chi)} \phi \left( \frac{\log N(\chi)}{\log X} \right) \sum_{k \in \mathbb{Z}[\omega], \ k \neq 0} \left( \frac{1 + (-1)^N(k)}{2} - \frac{1}{4} \right) \Phi \left( \sqrt{N(k)X} \right)
\]
\[
= S_{\Box} - S_{\Box}',
\]
where
\[
S_{\Box} = \frac{X}{2} \sum_{\substack{(l) \in \mathbb{Z}[\omega] \leq Z, (l,2) = 1 \atop \chi(l) = 1 \atop \chi \equiv E-\text{primary}}} \frac{\mu_{\omega}(l)}{N(l^2)} \sum_{k \in \mathbb{Z}[\omega], \ k \neq 0} \frac{\log N(\chi)}{N(\chi)} \phi \left( \frac{\log N(\chi)}{\log X} \right) \sum_{k \in \mathbb{Z}[\omega], \ k \neq 0} \left( \frac{1 + (-1)^N(k)}{2} - \frac{1}{4} \right) \Phi \left( \sqrt{N(k) \frac{X}{4N(l^2\chi)}} \right)
\]
and
\[
S_{\Box}' = \frac{X}{2} \sum_{\substack{(l) \in \mathbb{Z}[\omega] \leq Z, (l,2) = 1 \atop \chi(l) = 1 \atop \chi \equiv E-\text{primary}}} \frac{\mu_{\omega}(l)}{N(l^2)} \sum_{k \in \mathbb{Z}[\omega], \ k \neq 0} \frac{\log N(\chi)}{N(\chi)} \phi \left( \frac{\log N(\chi)}{\log X} \right) \sum_{k \in \mathbb{Z}[\omega], \ k \neq 0} \left( \frac{1 + (-1)^N(k)}{2} - \frac{1}{4} \right) \Phi \left( N(k) \sqrt{\frac{X}{4N(l^2\chi)}} \right).
\]
We further rewrite \( S_{\Box} = S_{\Box,1} + S_{\Box,2} \) where
\[
S_{\Box,1} = \frac{X}{2} \sum_{\substack{(l) \in \mathbb{Z}[\omega] \leq Z, (l,2) = 1 \atop \chi(l) = 1 \atop \chi \equiv E-\text{primary}}} \frac{\mu_{\omega}(l)}{N(l^2)} \sum_{k \in \mathbb{Z}[\omega], \ k \neq 0} \frac{\log N(\chi)}{N(\chi)} \phi \left( \frac{\log N(\chi)}{\log X} \right) \sum_{k \in \mathbb{Z}[\omega], \ k \neq 0} \left( \frac{1 + (-1)^N(k)}{2} - \frac{1}{4} \right) \Phi \left( N(k) \sqrt{\frac{X}{4N(l^2\chi)}} \right),
\]
and
and
\[ S_{\square,2} = \frac{X}{2} \sum_{N(l) \leq Z \atop (l,2) = 1} \frac{\mu_\omega(l)}{N(l)^2} \sum_{\chi \sim \text{E-prime}} \frac{\log N(\chi)}{N(\chi)} \phi \left( \frac{\log N(\chi)}{\log X} \right) \sum_{k \in \mathbb{Z}[\omega] \atop k \neq 0} \left( \frac{1 + (-1)^{N(k)}}{2} \right) - \frac{1}{4} \phi \left( \sqrt{\frac{X}{4N(l^2 \omega)}} \right). \]

We now apply Lemma 4.4 to see that \( S_{\square,1} = S_{\square,M} + S_{\square,R} \), where
\[ S_{\square,M} = -\bar{\Phi}(0) \frac{3X}{8} \sum_{N(l) \leq Z \atop (l,2) = 1} \frac{\mu_\omega(l)}{N(l)^2} \int_{X/N(l^2)}^\infty \phi \left( \frac{\log u}{\log X} \right) \log u \, du + o(X \log X) \]
and
\[ S_{\square,R} \ll X^{1+1/4}U^2 \sum_{N(l) \leq Z} \frac{1}{N(l)^{1+1/4}} \sum_{\text{E-prime} \atop N(\chi) \geq X/N(l^2)} \frac{\log N(\chi)}{N(\chi)} \phi \left( \frac{\log N(\chi)}{\log X} \right). \]

Applying (2.13) and A consequence of the prime ideal theorem [32, Theorem 8.9] is that for \( x \geq 1 \),
\[ \sum_{N(\chi) \leq x \atop \chi \sim \text{E-prime}} \frac{\log N(\chi)}{N(\chi)} = \log x + O(\log \log 3x). \]

Applying (2.13) and the above formula, we see that
\[ S_{\square}' + S_{\square,2} + S_{\square,M} + S_{\square,R} = o(X \log X). \]

Moreover, we have
\[ S_{\square,M} = -\bar{\Phi}(0) \frac{3X}{8} \sum_{N(l) \leq Z \atop (l,2) = 1} \frac{\mu_\omega(l)}{N(l)^2} \int_{X/N(l^2)}^\infty \phi \left( \frac{\log u}{\log X} \right) \log u \, du + o(X \log X) \]
\[ = -\bar{\Phi}(0) \frac{3X \log X}{8} \sum_{N(l) \leq Z \atop (l,2) = 1} \frac{\mu_\omega(l)}{N(l)^2} \int_{1}^{\infty} \phi(t) \log t \, dt + o(X \log X) \]
\[ = -\bar{\Phi}(0) \frac{3X \log X}{15 \zeta_Q(2)} \int_{-\infty}^{\infty} (1 - \chi_{[-1,1]}(t)) \phi(t) \log t \, dt + o(X \log X) \]
\[ = -2 \sqrt{3} \pi X \log X \int_{-\infty}^{\infty} (1 - \chi_{[-1,1]}(t)) \phi(t) \log t \, dt + o(X \log X), \]
where the last equality follows from (2.14).

Gathering the estimations for \( S_{\square}', S_{\square,2}, S_{\square,M} \) and \( S_{\square,R} \), we obtain that
\[ S_{M,\square}(X, Y; \hat{\phi}, \Phi) = -\frac{2 \sqrt{3} \pi X \log X}{15 \zeta_Q(2)} \int_{-\infty}^{\infty} (1 - \chi_{[-1,1]}(t)) \phi(t) \log t \, dt + o(X \log X). \]

Now, the sums in \( S_{M}(X, Y; \hat{\phi}, \Phi) \) corresponding to the contribution of \( k \neq 0, \square \) can be written as \( XR \), where
\[ R = \sum_{N(l) \leq Z \atop (l,2) = 1} \frac{\mu_\omega(l)}{N(l)^2} \sum_{k \in \mathbb{Z}[\omega] \atop k \neq 0, \square} \frac{1 + (-1)^{N(k)}}{2} \phi \left( \frac{\log N(\chi)}{\log X} \right) \Phi \left( \sqrt{\frac{N(k)X}{4N(l^2 \omega)}} \right). \]

We define \( \chi^{(k^2/\omega)} \) to be \( \left( \frac{k^2}{\omega} \right) \). Similar to our discussions in Section 2.1 when \( k \) is not a square, \( \chi^{(k^2/\omega)} \) can be regarded as a non-principle Hecke character modulo \( 8k^2 \) of trivial infinite type.
Similar to the estimations done in section 3.6 of [13], we deduce via Lemma 2.10 that the contribution of $k \neq 0$, \Box is
\begin{equation}
XR \ll \log^4 X \log(XZ) \sqrt{Y} U^3. \tag{4.4}
\end{equation}

4.5. Conclusion. We now combine the bounds (4.2), (4.3) (recall that $Z = \log^5 X, U = \log \log X$) to obtain that
\begin{equation}
S(X, Y; \hat{\phi}, \Phi) = -\frac{2\sqrt{3\pi X \log X}}{15 \zeta(2)} \int_{-\infty}^{\infty} \left(1 - \chi_{-1,1}(t)\right) \hat{\phi}(t) dt + o(X \log X),
\end{equation}
which implies (4.1) and this completes the proof of Theorem 1.3.

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