ON INTERVAL BASED GENERALIZATIONS OF ABSOLUTE CONTINUITY FOR FUNCTIONS ON $\mathbb{R}^n$

MICHAEL DYMOND, BEATA RANDRIANANTOANINA, AND HUAQIANG XU

Abstract. We study notions of absolute continuity for functions defined on $\mathbb{R}^n$ similar to the notion of $\alpha$-absolute continuity in the sense of Bongiorno. We confirm a conjecture of Malý that 1-absolutely continuous functions do not need to be differentiable a.e., and we show several other pathological examples of functions in this class. We establish containment relations of the class $1$-$AC_{W_{1,2}^n}$ which consists of all functions in $1$-$AC$ which are in the Sobolev space $W_{1,2}^n$, are differentiable a.e. and satisfy the Luzin (N) property, with previously studied classes of absolutely continuous functions.

1. Introduction

The classical Vitali’s definition says that when $\Omega \subseteq \mathbb{R}$, a function $f : \Omega \rightarrow \mathbb{R}$ is absolutely continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ so that for every finite collection of disjoint intervals $\{[a_i, b_i]\}_{i=1}^k \subset \Omega$ we (below $L^n$ denotes the Lebesgue measure on $\mathbb{R}^n$)

$$\sum_{i=1}^k L^n[a_i, b_i] < \delta \Rightarrow \sum_{i=1}^k |f(a_i) - f(b_i)| < \varepsilon.$$  

(1.1)

The study of the space of absolutely continuous functions on $[0, 1]$ and their generalizations to domains in $\mathbb{R}^n$ is connected to the problem of finding regular subclasses of Sobolev spaces which goes back to Cesari and Calderón [11] [9]. On the other hand, the Banach space of generalized absolutely continuous functions on $[0, 1]$ is closely related to the famous James space, and it is an example of a separable space not containing $\ell_1$ but with a non-separable dual [22] [21]. Moreover it has a very rich subspace structure [1] [2], but several questions about the Banach space structure of this space remain open, see [1].

There are several natural ways of generalizing the definition of absolute continuity for functions of several variables (cf. [12] [20] [27] [33] [1]).

One approach is to replace the intervals in the antecedent of (1.1) by balls in $\mathbb{R}^n$ and differences in the conclusion of (1.1) by oscillations of $f$.
on the images of balls from (1.1). This approach goes back to Banach, Vitali and Tonelli [5, 32, 31] (cf. [20]). More recently Malý [25] suggested another fruitful approach which is to replace the intervals in the antecedent of (1.1) by balls of a selected norm in $\mathbb{R}^n$ and replace sums in the conclusion of (1.1) by sums of oscillations raised to the power equal to the dimension of the domain space. This generalized notion gives functions in the Sobolev space $W^{1,n}_{loc}(\Omega)$, when $\Omega \subset \mathbb{R}^n$, and it has been extensively studied by Malý, Csörnyei, Hencl and Bongiorno [13, 17, 18, 19, 8]. Csörnyei [13] proved that this notion does depend on the shape of the balls substituted for intervals in (1.1). Thus the incomparable classes $Q$-$AC$ and $B$-$AC$ are defined where cubes (i.e. balls in the $\ell_\infty^n$-norm) or Euclidean balls are used, respectively. Hencl [17] introduced a shape-independent class $AC_H$ (see Definition 2.2) which contains both classes $Q$-$AC$ and $B$-$AC$ and so that $AC_H$ is contained in the Sobolev space $W^{1,n}_{loc}$ and that all functions in $AC_H$ are differentiable a.e. and satisfy the Luzin (N) property and the change of variable formula.

Bongiorno [6] introduced another generalization of Vitali’s classical definition for functions of several variables, which is simultaneously similar to Arzelà’s notion of bounded variation for functions on $\mathbb{R}^2$, cf. [12], and to Malý’s definition [25].

**Definition 1.1.** (Bongiorno [6]) Let $0 < \alpha < 1$. A function $f : \Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is open, is said to be $\alpha$-absolutely continuous (denoted $\alpha$-$AC^{(n)}(\Omega, \mathbb{R}^l)$ or $\alpha$-$AC$) if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for any finite collection of disjoint $\alpha$-regular intervals $\{[a_i, b_i] \subset \Omega\}_{i=1}^k$ we have

$$\sum_{i=1}^k \mathcal{L}^n([a_i, b_i]) < \delta \Rightarrow \sum_{i=1}^k |f(a_i) - f(b_i)|^n < \varepsilon.$$  

Here, for $a \in \mathbb{R}^l$, $|a|$ denotes the Euclidean norm of $a$, and we say that an interval $[a, b] \stackrel{\text{def}}{=} \{x = (x_\nu)_{\nu=1}^n \in \mathbb{R}^n : a_\nu \leq x_\nu \leq b_\nu, \nu = 1, \ldots, n\}$ is $\alpha$-regular if

$$\frac{\mathcal{L}^n([a, b])}{(\max_\nu |a_\nu - b_\nu|)^n} \geq \alpha.$$  

Bongiorno [6] showed that for all $0 < \alpha < 1$,

$$Q$-$AC^{(n)}(\Omega, \mathbb{R}^l) \subsetneq \alpha$-$AC^{(n)}(\Omega, \mathbb{R}^l) \subsetneq AC_H^{(n)}(\Omega, \mathbb{R}^l).$$

In 2012, Malý [23] asked us about the properties of absolutely continuous functions in a sense similar to Definition [1.1] but without restriction to $\alpha$-regular intervals for a specified $0 < \alpha < 1$. This question led us to the following definitions:
Definition 1.2. We say that a function \( f : \Omega \to \mathbb{R}^l \) (\( \Omega \subset \mathbb{R}^n \) open) is 0-absolutely continuous, denoted 0-\( AC^n(\Omega, \mathbb{R}^l) \) or 0-\( AC \), (resp. strongly 0-absolutely continuous, denoted strong-0-\( AC^n(\Omega, \mathbb{R}^l) \) or strong-0-\( AC \)) if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that for any finite collection of disjoint arbitrary intervals \( \{[a_i, b_i] \subset \Omega\}_{i=1}^k \) we have

\[
\sum_{i=1}^k (\max_{\nu} |a_{i,\nu} - b_{i,\nu}|)^n < \delta \Rightarrow \sum_{i=1}^k |f(a_i) - f(b_i)|^n < \varepsilon,
\]

(1.3) respectively,

\[
\sum_{i=1}^k \mathcal{L}^n([a_i, b_i]) < \delta \Rightarrow \sum_{i=1}^k |f(a_i) - f(b_i)|^n < \varepsilon.
\]

(1.4)

Note that the antecedent of implication (1.3) is equivalent to the antecedent of (1.2), since intervals in (1.2) are \( \alpha \)-regular for a fixed \( \alpha \). The antecedent of (1.4) is much weaker since there is no assumption of \( \alpha \)-regularity of intervals.

We show that, when \( n \geq 2 \), the condition (1.4) characterizes constant functions, and (1.3) characterizes Lipschitz functions (Theorem 3.1).

The main goal of this paper is to study an analog of Bongiorno’s notion for \( \alpha = 1 \).

Definition 1.3. We say that a function \( f : \Omega \to \mathbb{R}^l \) (\( \Omega \subset \mathbb{R}^n \) open) is 1-absolutely continuous, denoted 1-\( AC^n(\Omega, \mathbb{R}^l) \) or 1-\( AC \), if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that for any finite collection of disjoint 1-regular intervals \( \{[a_i, b_i] \subset \Omega\}_{i=1}^k \) we have

\[
\sum_{i=1}^k \mathcal{L}^n([a_i, b_i]) < \delta \Rightarrow \sum_{i=1}^k |f(a_i) - f(b_i)|^n < \varepsilon.
\]

We show that the class 1-\( AC \) is not contained in \( AC_H \) or even in the Sobolev space \( W^{1,n}_{\text{loc}}(\Omega) \). We show that, similarly as the Sobolev space \( W^{1,n}(\Omega) \), when \( \Omega \subset \mathbb{R}^n \) and \( n > 1 \), cf. [15, 24], the class 1-\( AC \) contains functions with pathological properties such as:

(i) compactly supported, but unbounded,
(ii) bounded but discontinuous,
(iii) continuous but nowhere differentiable,
(iv) differentiable but without the Luzin (N) property,
(v) differentiable, with the Luzin (N) property but not in the class \( W^{1,n}_{\text{loc}} \).

However we prove that every function in 1-\( AC \) has a directional derivative in the direction \((1, \ldots, 1)\) at a.e. point of the domain (Theorem 5.9).
Moreover the class $1-\text{AC}$ is useful for the study of the Bongiorno’s classes $\alpha-\text{AC}$. Namely, in [28] it is proved that

**Theorem 1.4.** ([28, Theorem 3.2]) For all $0 < \alpha < 1$,

$$\alpha-\text{AC} = 1-\text{AC} \cap \text{AC}_H.$$ 

We finish the paper by showing where the class $1-\text{AC}_{\text{WDN}}$ which consists of all functions in $1-\text{AC}$ which are in the Sobolev space $W^{1,n}_{\text{loc}}$, are differentiable a.e. and satisfy the Luzin (N) property, fits in the hierarchy of previously studied classes. Namely we prove that (Theorem 5.10):

(1.5) $$\mathcal{Q}-\text{AC} \subsetneq \alpha-\text{AC} = 1-\text{AC}_{\text{WDN}} \cap \text{AC}_H \subseteq 1-\text{AC}_{\text{WDN}}$$

$$\subseteq 1-\text{AC}_{\text{WDN}} \cup \text{AC}_H \subseteq \text{lin span}(1-\text{AC}_{\text{WDN}} \cup \text{AC}_H) \subseteq 1-\text{AC}_{\text{HWDN}},$$

where $1-\text{AC}_{\text{HWDN}}$ denotes the set of functions in $1-\text{AC}_H$ (see Definition 4.1 and Remark 4.3) which are in the Sobolev space $W^{1,2}_{\text{loc}}$, are differentiable a.e. and satisfy the Luzin (N) property. We pose a few related open questions in Remark 5.11.

On the other hand we observe that a small adjustment of the function constructed by Csörgő in [13, Theorem 2] (see (5.8) below) shows that

(1.6) $$1-\text{AC}_{\text{WDN}} \setminus B-\text{AC} \neq \emptyset, \quad \text{and} \quad B-\text{AC} \setminus 1-\text{AC}_{\text{WDN}} \neq \emptyset.$$ 

2. Preliminaries

Let $C_0(\mathbb{R}^n, \mathbb{R}^l)$ denote the set of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}^l$ with compact support. For $f \in C_0(\mathbb{R}^n, \mathbb{R}^l)$, and a measurable set $A \subset \mathbb{R}^n$, let $\text{osc}(f, A)$ denote the oscillation of $f$ on $A$, i.e.

$$\text{osc}(f, A) = \text{diam}f(A).$$

Let $K_0 \subset \mathbb{R}^n$ be a fixed symmetric closed convex set with non-empty interior, and let $\mathcal{K}$ denote the set of all balls of $\mathbb{R}^n$ in the norm defined by set $K_0$, i.e.,

$$\mathcal{K} = \{a + rK_0 : a \in \mathbb{R}^n, r > 0\}.$$

**Definition 2.1.** (Csörgő [13]) We say that a function $f \in C_0(\mathbb{R}^n, \mathbb{R}^l)$ is absolutely continuous with respect to $\mathcal{K}$ (denoted $f \in \mathcal{K}-\text{AC}$) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every finite collection of disjoint sets $\{K_i\}_{i=1}^k \subset \mathcal{K}$,

$$\sum_{i=1}^k \mathcal{L}^n(K_i) < \delta \Rightarrow \sum_{i=1}^k \text{osc}^n(f, K_i) < \varepsilon.$$
Malý [25] considered functions absolutely continuous with respect to the family \( \mathcal{B} \) of Euclidean balls in \( \mathbb{R}^n \) and showed that all functions in \( \mathcal{B} - AC \) are differentiable a.e. and satisfy the change of variables formula, similarly as functions in \( \mathcal{Q} - AC \), where \( \mathcal{Q} \) denotes the family of cubes, i.e. balls in the \( \ell_\infty^n \)-norm. Csörnyei [13] and Hencl and Malý [19] showed that the classes \( \mathcal{B} - AC \) and \( \mathcal{Q} - AC \) are incomparable.

In 2002, Hencl [17] introduced the following shape-independent class of absolutely continuous functions which contains both classes \( \mathcal{Q} - AC \) and \( \mathcal{B} - AC \).

**Definition 2.2.** (Hencl [17]) We say that a function \( f : \Omega \to \mathbb{R}^l \) (\( \Omega \subset \mathbb{R}^n \) open) is in \( AC_H^{(n)}(\Omega, \mathbb{R}^l) \) (briefly \( AC_H \)) if there exists \( \lambda \in (0,1) \) (equivalently, for all \( \lambda \in (0,1) \)) so that for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) so that for any finite collection of disjoint closed balls \( \{B(x_i, r_i) \subset \Omega\}_{i=1}^k \)

\[
\sum_{i=1}^k \mathcal{L}^n(B(x_i, r_i)) < \delta \Rightarrow \sum_{i=1}^k osc^n(f, B(x_i, \lambda r_i)) < \varepsilon.
\]

Hencl [17] proved that \( AC_H \subset W^{1,n}_{loc} \) and that all functions in \( AC_H \) are differentiable a.e. and satisfy the Luzin (N) property and the change of variables formula.

### 3. Classes 0-AC and strong-0-AC

The main result of this section is the following.

**Theorem 3.1.** Let \( n \geq 2 \) and \( f : \Omega \to \mathbb{R}^l \) be a function from an open connected set \( \Omega \subset \mathbb{R}^n \). Then:

(a) \( f \in \text{strong-0-AC} \) if and only if \( f \) is constant on \( \Omega \),

(b) \( f \in \text{0-AC} \) if and only if \( f \) is Lipschitz.

**Proof of (a).** We show that every strong-0-AC\((n) (\Omega, \mathbb{R}^l) \) function \( f \) is constant. It is clear that if \( f \in \text{strong-0-AC}^{(n)}(\Omega, \mathbb{R}^l) \) then \( f \) is continuous.

Let \( a \in \Omega \). We claim that the set \( \Omega_a \stackrel{\text{def}}{=} \{ b \in \Omega : f(b) = f(a) \} \) is equal to \( \Omega \). Since \( f \) is continuous, \( \Omega_a \) is closed in \( \Omega \). We will show that \( \Omega_a \) is open.

Fix \( \varepsilon > 0 \). Let \( r > 0 \) be such that \( B(a, r) \subset \Omega \), and let \( b \in B(a, r) \). For each \( j = 0, 1, \ldots, n-1 \), let

\[
(c^{(j)})_i = \begin{cases} b_i, & \text{if } i \leq j, \\ a_i, & \text{if } i > j. \end{cases}
\]
Then $c^{(0)} = a$, $c^{(n-1)} = b$, and for each $j \leq n-1$, $c^{(j)} \in B(a, r)$, and $c^{(j)}$ and $c^{(j+1)}$ differ only on the $(j+1)$-th coordinate. Since $f$ is continuous, there exists $\delta_1 > 0$, such that for all $j \leq n-1$ and all $x$ with $|x - c^{(j)}| < \delta_1$ we have

$$\tag{3.1} |f(c^{(j)}) - f(x)| < \frac{\varepsilon}{2n}.$$ 

For any $t > 0$, let

$$(x^{(j+1)})(t)_i = \begin{cases} (c^{(j+1)})_{j+1}, & \text{if } i = j + 1, \\ (c^{(j+1)})_i + t \text{sgn}((c^{(j+1)})_{j+1} - (c^{(j)})_{j+1}), & \text{if } i \neq j + 1. \end{cases}$$

Then

$$\tag{3.2} |x^{(j+1)}(t) - c^{(j+1)}| = \sqrt{n - 1} t,$$

and the Lebesgue measure of the interval with endpoints $x^{(j+1)}$ and $c^{(j)}$ is equal to $t^{n-1}|a_{j+1} - b_{j+1}|$. By (1.4), there exists $\delta_2 > 0$, such that if $t^{n-1}|a_{j+1} - b_{j+1}| \leq t^{n-1}r < \delta_2$, then

$$\tag{3.3} |f(c^{(j)}) - f(x^{(j+1)}(t))|^n < \left(\frac{\varepsilon}{2n}\right)^n.$$ 

Now let $\delta = \min(\delta_1/\sqrt{n - 1}, n^{-1}\sqrt{\delta_2/r})$ and $0 < t < \delta$. Then, by (3.1), (3.2) and (3.3), we have

$$|f(c^{(j)}) - f(c^{(j+1)})| \leq |f(c^{(j)}) - f(x^{(j+1)}(t))| + |f(x^{(j+1)}(t)) - f(c^{(j+1)})|$$

$$< \frac{\varepsilon}{2n} + \frac{\varepsilon}{2n} = \frac{\varepsilon}{n}.$$

Thus

$$|f(a) - f(b)| \leq \sum_{j=0}^{n-1} |f(c^{(j)}) - f(c^{(j+1)})| < n\frac{\varepsilon}{n} = \varepsilon.$$ 

Since $\varepsilon$ is arbitrary, we get $f(b) = f(a)$, and thus $B(a, r) \subseteq \Omega_a$. \hfill \Box

**Proof of (b).** Since the Euclidean norm and the $\ell_1$-norm $\|\cdot\|_1$ are equivalent on $\mathbb{R}^n$, we can without loss of generality replace $|\cdot|$ with $\|\cdot\|_1$ in the definition of 0-absolute continuity, which we will do to simplify some computations.

It is not difficult to see that Lipschitz functions are 0-absolutely continuous. Indeed, suppose that $f : \Omega \rightarrow \mathbb{R}^l$ ($\Omega \subset \mathbb{R}^n$ open) is Lipschitz with constant $M > 0$. Then for any $\varepsilon > 0$ and for any finite collection of non-overlapping arbitrary intervals $\{[a_i, b_i] \subset \Omega\}$ with

$$\sum_{i=1}^{k} \left(\max_{\nu} |a_{i,\nu} - b_{i,\nu}|\right)^n < \frac{\varepsilon}{(Mn)^n},$$

Since $\varepsilon$ is arbitrary, we get $f(b) = f(a)$, and thus $B(a, r) \subseteq \Omega_a$. \hfill \Box
we have
\[
\sum_{i=1}^{k} \| f(a_i) - f(b_i) \|^n < \sum_{i=1}^{k} M^n \| a_i - b_i \|^n \\
\leq \sum_{i=1}^{k} M^n \cdot (n \max_{\nu} |a_{i,\nu} - b_{i,\nu}|)^n \\
< (Mn)^n \cdot \frac{\varepsilon}{(Mn)^n} = \varepsilon,
\]
which completes the proof.

For the other direction, we note that all functions in $0-AC$ are continuous, and let $f : \Omega \to \mathbb{R}$ be with compact support and not Lipschitz. Then for every $m \in \mathbb{N}$, there exists $x_m, y_m$ so that
\[
(3.4) \quad \| f(x_m) - f(y_m) \|_1 \geq m \| x_m - y_m \|_1.
\]

For each $j = 0, 1, \ldots, n$, let
\[
(z^{(j)}_m)_i = \begin{cases} (x_m)_i, i \leq j, \\ (y_m)_i, i > j. \end{cases}
\]

Then $z^{(0)}_m = y_m$, $z^{(n)}_m = x_m$, and
\[
\| x_m - y_m \|_1 = \sum_{j=1}^{n} \| z^{(j)}_m - z^{(j-1)}_m \|_1.
\]

If for every $j = 1, \ldots, n$
\[
\| f(z^{(j)}_m) - f(z^{(j-1)}_m) \|_1 < m \| z^{(j)}_m - z^{(j-1)}_m \|_1,
\]
then we will have
\[
\| f(x_m) - f(y_m) \|_1 \leq \sum_{j=1}^{n} \| f(z^{(j)}_m) - f(z^{(j-1)}_m) \|_1 \\
< m \cdot \sum_{j=1}^{n} \| z^{(j)}_m - z^{(j-1)}_m \|_1 = m \cdot \| x_m - y_m \|_1.
\]

which contradicts (3.4). Thus there exists $j_m \in 1, \ldots, n$ so that
\[
(3.5) \quad \| f(z^{(j)}_m) - f(z^{(j-1)}_m) \|_1 \geq m \cdot \| z^{(j)}_m - z^{(j-1)}_m \|_1
\]

Note that all coordinates of $z^{(j_m)}_m$ and $z^{(j_m-1)}_m$ coincide, with the exception of the $j_m$-th coordinate.

Since $f$ is a continuous function with compact support, $f$ is bounded and uniformly continuous and thus (3.5) implies that
\[
\lim_{m \to \infty} \| f(z^{(j_m)}_m) - f(z^{(j_m-1)}_m) \|_1 = 0
\]
and
\[
\lim_{m \to \infty} \left\| z^{(j_m)}_m - z^{(j_m-1)}_m \right\|_1 = 0.
\]

We claim that \( f \not\in 0-\text{AC} \). Indeed, let \( \varepsilon = \frac{1}{3^{m+1}} \), \( \delta > 0 \) and \( m \in \mathbb{N} \) so that \( \frac{1}{m} < \delta \). Then, by (3.5), we have
\[
\| z^{(j_m)}_m - z^{(j_m-1)}_m \|_1 < \delta.
\]
Denote \( \gamma = \| f(z^{(j_m)}_m) - f(z^{(j_m-1)}_m) \|_1 \), and choose \( \mu \in \mathbb{N} \) so that
\[
(3.6) \quad \frac{1}{3} \gamma^{-n} \leq \mu \leq \gamma^{-n}.
\]
Then
\[
(3.7) \quad \mu \cdot \| z^{(j_m)}_m - z^{(j_m-1)}_m \|_1 < \delta.
\]
Since \( f \) is uniformly continuous, there exists \( 0 < \eta < \| z^{(j_m)}_m - z^{(j_m-1)}_m \|_1 \) so that for all \( x, y \)
\[
(3.8) \quad \| x - y \|_1 < \eta \Rightarrow \| f(x) - f(y) \|_1 < \frac{1}{3} \gamma.
\]
For \( i = 1, \ldots, \mu \), we define
\[
a_i = z^{(j_m)}_m + i \left( \frac{\eta}{n(\mu + 1)} \right) \cdot \left( \sum_{\nu \neq j_m} e_\nu \right)
\]
\[
b_i = z^{(j_m-1)}_m + (i + 1) \left( \frac{\eta}{n(\mu + 1)} \right) \cdot \left( \sum_{\nu \neq j_m} e_\nu \right).
\]
Then for every \( i \) we have
\[
\max_\nu |a_{i,\nu} - b_{i,\nu}| = \| z^{(j_m)}_m - z^{(j_m-1)}_m \|_1,
\]
\[
\| a_i - z^{(j_m)}_m \|_1 = i \left( \frac{\eta}{n(\mu + 1)} \right) \cdot (n - 1) < \eta,
\]
\[
\| b_i - z^{(j_m-1)}_m \|_1 = (i + 1) \left( \frac{\eta}{n(\mu + 1)} \right) \cdot (n - 1) < \eta.
\]
Thus, by (3.8) we get
\[
\| f(a_i) - f(b_i) \|_1 \geq \| f(z^{(j_m)}_m) - f(z^{(j_m-1)}_m) \|_1 - \frac{2}{3} \gamma
\]
\[
= \frac{1}{3} \| f(z^{(j_m)}_m) - f(z^{(j_m-1)}_m) \|_1
\]
Hence, by (3.6) we get
\[
\sum_{i=1}^\mu \| f(a_i) - f(b_i) \|_1 \geq \mu \cdot \frac{1}{3^n} \gamma^n \geq \frac{1}{3} \cdot \frac{1}{3^n} \geq \frac{1}{3^{n+1}}.
\]
On the other hand, by (3.7) we have
\[
\sum_{i=1}^{\mu} (\max_{\nu} |a_{i,\nu} - b_{i,\nu}|)^n = \sum_{i=1}^{\mu} \|z_i^{(m)} - z_i^{(m-1)}\|_1^n < \delta
\]
which ends the proof that \( f \notin 0\text{-}AC \). \( \square \)

4. The Hencl Type Extension of the Class 1-AC

We give an analog of Definition 2.2, and we prove that, similarly as for other classes of absolutely continuous functions, the classes 1-AC\(\lambda\) do not depend on \( \lambda \) when \( 0 < \lambda < 1 \).

Following [7], we will use the following notation. Given interval \([x, y]\), we denote \( |f([x, y])| = |f(y) - f(x)| \), and given \( 0 < \lambda < 1 \), we denote by \( \lambda[x, y] \) the interval with center \((x + y)/2\) and sides of length \( \lambda(y_\nu - x_\nu) \), \( \nu = 1, \ldots, n \).

Definition 4.1. (cf. [7]) Let \( \alpha \in (0, 1] \) and \( \lambda \in (0, 1) \). A function \( f : \Omega \to \mathbb{R}^l \) (\( \Omega \subset \mathbb{R}^n \) open) is said to be in \( \alpha\text{-AC}_\lambda^{(n)}(\Omega, \mathbb{R}^l) \) (briefly \( \alpha\text{-AC}_\lambda \)) if for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any finite collection of disjoint \( \alpha \)-regular intervals \( \{[a_i, b_i] \subset \Omega\}_{i=1}^{k} \) we have

\[
\sum_{i=1}^{k} L^n([a_i, b_i]) < \delta \Rightarrow \sum_{i=1}^{k} |f(\lambda[a_i, b_i])|^n < \varepsilon.
\]

Bongiorno [7] proved that for all \( \alpha < 1 \), the class \( \alpha\text{-AC}_\lambda \) is independent of \( \lambda \). We prove the same result for 1-AC\(\lambda\).

Theorem 4.2. Let \( 0 < \lambda_1 < \lambda_2 < 1 \). Then

\[
1\text{-AC}_{\lambda_1}^{(n)}(\Omega, \mathbb{R}^l) = 1\text{-AC}_{\lambda_2}^{(n)}(\Omega, \mathbb{R}^l).
\]

Proof. It is easy to see that \( 1\text{-AC}_{\lambda_2} \subset 1\text{-AC}_{\lambda_1} \). For the other direction, suppose \( f \in 1\text{-AC}_{\lambda_1} \). Fix \( p \in \mathbb{N} \) such that \( p > \frac{2\lambda_2}{(1-\lambda_2)\lambda_1} \). For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for each finite family of non-overlapping 1-regular intervals \( \{[\overline{a_i}, \overline{b_i}] \subset \Omega\} \), we have

\[
\sum_{i} L^n([\overline{a_i}, \overline{b_i}]) < \delta \Rightarrow \sum_{i} |f(\lambda_1[\overline{a_i}, \overline{b_i}])|^n < \frac{\varepsilon}{p^n}.
\]

Let \( \{[\overline{a_i}, \overline{b_i}]\} \) be a finite family of non-overlapping 1-regular intervals in \( \Omega \) with \( \sum_{i} L^n([\overline{a_i}, \overline{b_i}]) < \delta \). Let \( c_i, d_i \) be such that \( [c_i, d_i] = \lambda_2[\overline{a_i}, \overline{b_i}] \).
Then
\[ |f(\lambda_2[a_i, b_i])| = |f([c_i, d_i])| \leq \sum_{j=0}^{p-1} \left| f\left( \left[ c_i + \frac{(d_i - c_i)j}{p}, c_i + \frac{(d_i - c_i)(j + 1)}{p} \right] \right) \right| \]

Hence there exists \( j_0 \in \{0, \ldots, p-1\} \) such that for
\[ [\overline{a_i}, \overline{b_i}] = \frac{1}{p} \left[ c_i + \frac{(d_i - c_i)j_0}{p}, c_i + \frac{(d_i - c_i)(j_0 + 1)}{p} \right], \]
we have
\[ |f(\lambda_2[a_i, b_i])| \leq p|f(\lambda_1[\overline{a_i}, \overline{b_i}])|. \]

Since \( p > \frac{2\lambda_2}{(1-\lambda_2)\lambda_1} \), we obtain \([\overline{a_i}, \overline{b_i}] \subseteq [a_i, b_i]\) and hence the intervals \([\overline{a_i}, \overline{b_i}]\) are pairwise disjoint. Since \( \sum_i L^\alpha([\overline{a_i}, \overline{b_i}]) \leq \sum_i L^\alpha([a_i, b_i]) < \delta \), by (4.2) and (4.1), we get
\[ \sum_i |f(\lambda_2[a_i, b_i])|^n \leq p^n \sum_i |f(\lambda_1[\overline{a_i}, \overline{b_i}])|^n < p^n \varepsilon = \varepsilon. \]

\[ \square \]

Remark 4.3. By Theorem 4.2 in analogy with Definition 2.2 and [7] we will use the notation \( 1-\text{AC}^H \) and \( 1-\text{AC}^{(n)}_H(\Omega, \mathbb{R}^l) \) instead of \( 1-\text{AC}_\lambda \) and \( 1-\text{AC}_{\lambda}^{(n)}(\Omega, \mathbb{R}^l) \).

Corollary 4.4.
\[ \text{AC}_{H}^{(n)}(\Omega, \mathbb{R}^l) \subset 1-\text{AC}_{H}^{(n)}(\Omega, \mathbb{R}^l). \]

Proof. Note that it follows from the definition of \( \alpha \)-regularity of intervals that if \( \alpha < \beta \) and \( f \in \alpha-\text{AC}_\lambda \) then \( f \in \beta-\text{AC}_\lambda \). In particular, for all \( \alpha < 1 \), \( \alpha-\text{AC}_\lambda \subset 1-\text{AC}_\lambda \). Bongiorno [7] proved that for all \( \alpha < 1 \), \( \alpha-\text{AC}_H = \text{AC}_H \). Thus \( \text{AC}_H \subset 1-\text{AC}_H \).

\[ \square \]

5. Class 1-AC

To simplify notation, the results of this section, except Theorem 5.9, are stated for functions defined on subsets of \( \mathbb{R}^2 \) with range in \( \mathbb{R} \). However they can be easily generalized to functions from \( \Omega \subseteq \mathbb{R}^n \) to \( \mathbb{R}^l \) for any \( n \geq 2, l \in \mathbb{N} \).

We start from a structural result which will allow us to give examples of functions in 1-AC.
Theorem 5.1. Let \( d > 0 \) and let \( S_d \) denote the square with vertices \((\pm 2d, 0), (0, \pm 2d)\). Let \( h : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) be nonzero measurable functions with support contained in \([-d, d]\). We consider \( \mathbb{R}^2 \) with the basis \( \{x_1 = (-1, 1), x_2 = (1, 1)\} \) and define \( f : \mathbb{R}^2 \to \mathbb{R} \) with support contained in \( S_d \) by
\[
f(sx_1 + tx_2) = h(s)g(t).
\]
(Alternatively, in the standard basis of \( \mathbb{R}^2 \), \( f(x, y) = h(y - x^2)g(y + x^2) \).)

(a) If \( h \) is bounded and \( g \) is Lipschitz, then \( f \) is in \( 1-AC^2(\mathbb{R}^2, \mathbb{R}) \).

(b) If \( f \in 1-AC^2_H(\mathbb{R}^2, \mathbb{R}) \), then \( g \) is Lipschitz.

Proof of (a). Let \( M \in \mathbb{R} \) be a bound of \( h(s) \) and \( L \in \mathbb{R} \) be a Lipschitz constant for \( g(t) \). Then for all \( s, t, 1, t_2 \in [-d, d] \) we have
\[
|f(sx_1 + t_1x_2) - f(sx_1 + t_2x_2)| = |h(s)| \cdot |g(t_1) - g(t_2)| \leq ML|t_1 - t_2|.
\]

Let \( \varepsilon > 0 \). Put \( \delta = \frac{\varepsilon}{M^2L^2} \). Let \( \{[a_i, b_i] \subset S_d\}_{i=1}^k \) be any finite collection of disjoint 1-regular intervals with \( \sum_{i=1}^k \mathcal{L}^2([a_i, b_i]) < \delta \). By 1-regularity of intervals \( [a_i, b_i] \), there exist \( s_i, t_{i,1}, t_{i,2} \in [-d, d] \) so that \( a_i = s_i x_1 + t_{i,1}x_2, b_i = s_i x_1 + t_{i,2}x_2 \) and \( \mathcal{L}^2([a_i, b_i]) = |t_{i,1} - t_{i,2}|^2 \). Thus, by (5.1), we have
\[
\sum_{i=1}^k |f(a_i) - f(b_i)|^2 \leq \sum_{i=1}^k (ML|t_{i,1} - t_{i,2}|)^2
\]
\[
= M^2L^2 \sum_{i=1}^k \mathcal{L}^2([a_i, b_i]) < M^2L^2 \cdot \delta = \varepsilon.
\]

So \( f \in 1-AC^2(\mathbb{R}^2, \mathbb{R}) \). \( \square \)
Proof of (b). Let \( I \subseteq [-d, d] \) be a set of positive measure so that there exists \( c > 0 \) such that \( |h(s)| \geq \frac{1}{c} \) for all \( s \in I \). Let \( \varepsilon = 1 \) and \( \lambda \in (0, 1) \). Since \( f \in 1-AC_H \), there exists \( \delta > 0 \) so that for all finite families of disjoint 1-regular intervals \( \{[a_i, b_i]\} \) we have

\[
\sum_i L^n([a_i, b_i]) < \delta \Rightarrow \sum_i |f(\lambda [a_i, b_i])|^2 < 1.
\]

Without loss of generality \( \sqrt{\delta} < L^1(I) \). Let \( t, t' \in [-d, d] \) be such that \( t < t' \) and \( |t' - t| < \lambda \sqrt{\delta} \), and let \( k \in \mathbb{N} \) be such that \( \lambda \sqrt{\delta} k + 1 \leq |t - t'| \leq \lambda \sqrt{\delta} k \).

Let \( \{s_i\}_{i=1}^k \subseteq I \) be such that \( |s_i - s_j| > \sqrt{\frac{k}{k}} \) for all \( i \neq j \).

Put

\[
a_i = s_i x_1 + \left( \frac{t + t'}{2} - \frac{1}{\lambda} \frac{|t - t'|}{2} \right) x_2, \]
\[
b_i = s_i x_1 + \left( \frac{t + t'}{2} + \frac{1}{\lambda} \frac{|t - t'|}{2} \right) x_2.
\]

Then \( \{[a_i, b_i]\}_{i=1}^k \) is a family of disjoint 1-regular intervals with \( L^2[a_i, b_i] = \left( \frac{1}{\lambda} |t - t'| \right)^2 < \frac{\delta}{k} \), and \( \lambda [a_i, b_i] = [s_i x_1 + t x_2, s_i x_1 + t' x_2] \), for all \( i \). Thus, by (5.2), we have

\[
1 > \sum_{i=1}^k |f(s_i x_1 + t x_2) - f(s_i x_1 + t' x_2)|^2 \\
= \sum_{i=1}^k |h(s_i)|^2 |g(t) - g(t')|^2 \\
\geq k \cdot \frac{1}{c^2} |g(t) - g(t')|^2.
\]

By (5.3) we get

\[
|g(t) - g(t')|^2 \leq \frac{c}{k} \leq \frac{c^2}{\delta^2 \lambda^2} |t - t'|^2.
\]

Thus \( g \) is continuous, and therefore bounded on \([-d, d]\), and \( g \) is Lipschitz with Lipschitz constant \( L = \max(\sqrt{\frac{c^2}{\delta^2 \lambda^2}}, \frac{2M}{\lambda \sqrt{\delta}}) \), where \( M = \max\{|g(t)| : t \in [-d, d]\} \).

As a consequence of Theorem 5.1 we obtain that, on the one hand, not every differentiable function belongs to \( 1-AC^2_H(\mathbb{R}^2, \mathbb{R}) \) and that
$W_{loc}^{1,2}(\mathbb{R}^2) \not\subset 1-AC_R^2(\mathbb{R}^2, \mathbb{R})$, and on the other hand $1-AC^2(\mathbb{R}^2, \mathbb{R})$ contains examples of several types of functions with pathological properties.

**Corollary 5.2.** (a) There exists a function differentiable everywhere which does not belong to $1-AC_H$, and (b) $W_{loc}^{1,2}(\mathbb{R}^2) \not\subset 1-AC_H^2(\mathbb{R}^2, \mathbb{R})$.

**Proof.** It is enough to take a function like in Theorem 5.1 with $h$ constant and, for (a), $g$ differentiable but not Lipschitz, e.g. $g(t) = \sqrt{t}$ on $[-\frac{d}{2}, \frac{d}{2}]$, for (b), $g$ in $W^{1,2}(\mathbb{R})$ but not Lipschitz. \hfill \Box

**Corollary 5.3.** There exists a function in $1-AC$ which is bounded and discontinuous everywhere on its support.

**Proof.** It is enough to take a function like in Theorem 5.1 with $g$ constant and $h$ bounded but discontinuous everywhere. \hfill \Box

**Corollary 5.4.** There exists a function in $1-AC$ which is supported on a compact set and unbounded.

**Proof.** An example is provided by a function like in Theorem 5.1 with $g$ constant and $h$ unbounded on $[-d, d]$ and $g(t) = 1$ for $t \in [-d, d]$.

The sum of $f$ and any function in $1-AC$ which is not constant on segments with slope 1 gives an example of an unbounded function in $1-AC$ which is not constant on segments with slope 1. \hfill \Box

**Corollary 5.5.** There exists a function in $1-AC$ which is differentiable everywhere but is not in the Sobolev space $W_{loc}^{1,2}(\mathbb{R}^2)$.

**Proof.** An example is provided by a function like in Theorem 5.1 with $g$ constant and $h \in W^{1,1}(-1, 1) \setminus W^{1,2}(-1, 1)$. \hfill \Box

**Corollary 5.6.** There exists a continuous function $f \in 1-AC^2(\mathbb{R}^2, \mathbb{R})$ such that for every $p \in \text{supp} f$ and every direction $v \neq (1, 1)$ the directional derivative $D_v f(p)$ does not exist. In particular $f$ is not differentiable anywhere on its support.

**Proof.** For the proof we will use the Takagi function $T$ which is continuous on $[0, 1]$ but is nowhere differentiable (29, cf. 30, p. 36) and which is defined by

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \text{dist}(2^k x, \mathbb{Z}) = \sum_{k=0}^{\infty} \frac{1}{2^k} \inf_{m \in \mathbb{Z}} |2^k x - m|.$$ 

Let $g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ be defined by $g(t) = 1 - 2|t|$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(sx_1 + tx_2) = T(s + \frac{1}{2}) g(t).$$
Since \( T(s) \) is bounded and \( g \) is Lipschitz, by Theorem 5.1(a), \( f \in 1-AC \).

We claim that \( f \) is not differentiable anywhere. Indeed, for any point \( p = s_0x_1 + t_0x_2 \) \( \in \) supp \( f \), for any direction \( v = v_1x_1 + v_2x_2 \) other than \( x_2 = (1, 1) \), we have

\[
\lim_{h \to 0} \frac{f(p + hv) - f(p)}{h} = \lim_{h \to 0} \frac{T(s_0 + \frac{1}{2} + hv_1)g(t_0 + hv_2) - T(s_0 + \frac{1}{2})g(t_0)}{h}.
\]

Since \( g(t_0 + hv_2) = 1 - 2(|t_0| + hv_2) \) for \( |h| < \frac{|t_0|}{\nu_2} \), \( g(t_0) = 1 - 2|t_0| \), and \( T(s) \) is continuous, we get

\[
\lim_{h \to 0} \frac{(1 - 2(|t_0| + hv_2))T(s_0 + \frac{1}{2} + hv_1) - (1 - 2|t_0|)T(s_0 + \frac{1}{2})}{h}
\]

\[
= \lim_{h \to 0} (1 - 2|t_0|) \left[ \frac{T(s_0 + \frac{1}{2} + hv_1) - T(s_0 + \frac{1}{2})}{h} \right] \pm \lim_{h \to 0} 2v_2T(s_0 + \frac{1}{2} + hv_1)
\]

\[
= \lim_{h \to 0} (1 - 2|t_0|) \left[ \frac{T(s_0 + \frac{1}{2} + hv_1) - T(s_0 + \frac{1}{2})}{h} \right] \pm 2v_2T(s_0 + \frac{1}{2}).
\]

But, since \( v_1 \neq 0 \) and the Takagi function \( T \) is nowhere differentiable,

\[
\lim_{h \to 0} \frac{T(s_0 + \frac{1}{2} + hv_1) - T(s_0 + \frac{1}{2})}{h}
\]

does not exist anywhere. Therefore \( D_x f \) does not exist anywhere. \( \Box \)

Next we study the Luzin (N) property for differentiable functions in \( 1-AC(\mathbb{R}^n, \mathbb{R}^l) \) where \( n > 1 \). Recall that a function \( f : \mathbb{R}^n \to \mathbb{R}^l \) is said to have the Luzin (N) property if \( \mathcal{H}^n(f(E)) = 0 \) whenever \( E \subseteq \mathbb{R}^n \) and \( \mathcal{L}^n(E) = 0 \), where \( l \geq n \) and \( \mathcal{H}^n \) denotes the \( n \)-dimensional Hausdorff measure on \( \mathbb{R}^l \), see e.g. [33].

It is known that all absolutely continuous functions on \( \mathbb{R} \) and all functions in \( AC_H \) satisfy the Luzin (N) condition. However, this property is not guaranteed for differentiable functions in \( 1-AC(\mathbb{R}^n, \mathbb{R}^l) \) where \( n > 1 \).

**Theorem 5.7.** Suppose \( n > 1 \) and \( l \geq 1 \) are integers. Then there exists a differentiable function \( f \) in \( 1-AC(\mathbb{R}^n, \mathbb{R}^l) \) and a set \( U \subseteq \mathbb{R}^n \) with \( \mathcal{L}^n(U) = 0 \) and \( \mathcal{L}^l(f(U)) > 0 \).

**Remark 5.8.** Note that when \( l \geq n \), any subset of \( \mathbb{R}^l \) with positive \( l \)-dimensional Lebesgue measure, has positive \( n \)-dimensional Hausdorff measure. In fact, if \( l > n \), the \( n \)-dimensional Hausdorff measure of such a set is necessarily infinite. Hence, Theorem 5.7 shows that functions in the class \( 1-AC_D(\mathbb{R}^n, \mathbb{R}^l) \) may be severely expanding; when \( l \geq n \), the conclusion of Theorem 5.7 is stronger than the assertion that \( f \) does not have the Luzin (N) property.
Proof of Theorem 5.7. Let $\varphi : \mathbb{R} \to [0, 1]$ be an extension of the Cantor function (see [10], [14]) on $[0, 1]$ such that $\varphi$ is constant on $\mathbb{R} \setminus [0, 1]$. Let $C$ denote the standard ternary Cantor set. Recall that $\varphi(C) = [0, 1]$ and $\varphi$ is constant on each connected component of $[0, 1] \setminus C$. Hence, $\varphi$ is differentiable with derivative zero almost everywhere.

Let $p : [0, 1] \to \mathbb{R}^{l-1}$ denote a space filling curve with $L^{l-1}(p([0, 1])) > 0$, (see [3]). Note that the function $p \circ \varphi : [0, 1] \to \mathbb{R}^{l-1}$ is differentiable with derivative zero almost everywhere.

Let $x_1, \ldots, x_n$ of $\mathbb{R}^n$ be an orthonormal basis of $\mathbb{R}^n$ so that $x_n = (1/\sqrt{n})(e_1 + \ldots + e_n)$. Define a set $U \subseteq \mathbb{R}^n$ by

$$U = \{t_1x_1 + \ldots + t_nx_n : t_1 \in C, t_2, \ldots, t_n \in \mathbb{R}\}.$$ 

We note that $U$ has Lebesgue measure zero, since it is an isomorphic image of $C \times \mathbb{R}^{n-1}$. Let $f : \mathbb{R}^n \to \mathbb{R}^l$ be the function defined by

$$f(t_1x_1 + \ldots + t_nx_n) = \begin{cases} \varphi(t_1) & \text{if } l = 1, \\ (p \circ \varphi(t_1), t_2) & \text{if } l > 1. \end{cases}$$

Then

$$f(U) = \begin{cases} [0, 1] & \text{if } l = 1, \\ p([0, 1]) \times \mathbb{R} & \text{if } l > 1. \end{cases}$$

Hence, $L^l(f(U)) > 0$, and $f$ does not have the Luzin (N) property. Further, note that $f$ is differentiable almost everywhere.

It only remains to verify that $f \in 1\text{-}AC(\mathbb{R}^n, \mathbb{R}^l)$. One can check that each component of the function $f$ has the form given by the generalization of Theorem 5.1 part (a) (for $\mathbb{R}^n$ rather than $\mathbb{R}^2$). By an adaptation of the proof of Theorem 5.1 part (a), we get that $f \in 1\text{-}AC(\mathbb{R}^n, \mathbb{R}^l)$.

The next theorem contains a positive result about properties of functions in $1\text{-}AC_H$.

Theorem 5.9. Every function $f \in 1\text{-}AC_H^{(n)}(\mathbb{R}^n, \mathbb{R}^l)$ is differentiable a.e. in the direction $e_1 + e_2 + \cdots + e_n$.

Proof. Fix $f \in 1\text{-}AC_H^{(n)}(\mathbb{R}^n, \mathbb{R}^l)$. If $n = 1$ then $f = (f_1, \ldots, f_l)$ where each $f_i : \mathbb{R} \to \mathbb{R}$ is a function in $AC_H$. Thus each $f_i$ is differentiable almost everywhere, and the theorem follows.

Let $n > 1$. Let $x_n = e_1 + e_2 + \cdots + e_n$ and choose vectors $x_1, \ldots, x_{n-1} \in \mathbb{R}^n$ so that $x_1, \ldots, x_n$ is an orthonormal basis. In what follows we will identify $\mathbb{R}^{n-1}$ with the $n - 1$ dimensional subspace of $\mathbb{R}^n$ spanned by $x_1, \ldots, x_{n-1}$ via the correspondence $s \leftrightarrow s_1x_1 + \ldots + s_{n-1}x_{n-1}$. For
We will show that the set
\[ E := \{ \mathbf{u} \in \mathbb{R}^n : \text{Lip}(f_{\mathbf{u}}, 0) = \infty \} \]
has \( n \)-dimensional Lebesgue measure zero.

We claim that this suffices: Indeed, if \( E \) has measure zero then, using Fubini’s Theorem, we get that for almost every \( s \in \mathbb{R}^{n-1} \), \( \text{Lip}(f_{s+tx_n}, 0) < \infty \) for almost every \( t \in \mathbb{R} \). Observe that \( \text{Lip}(f_{s}, t) = \text{Lip}(f_{s+tx_n}, 0) \) for all \( s \in \mathbb{R}^{n-1} \) and \( t \in \mathbb{R} \). It follows that \( \text{Lip}(f_{s}, t) < \infty \) for almost every \( s \in \mathbb{R}^{n-1} \) and almost every \( t \in \mathbb{R} \). Now, applying the Stepanov Theorem [4], we conclude that for almost every \( s \in \mathbb{R}^{n-1} \), \( f_{s} \) is differentiable almost everywhere (in \( \mathbb{R} \)). Clearly \( f \) is differentiable at \( s+tx_n \) in the direction \( x_n \) if and only if \( f_{s} \) is differentiable at \( t \). Hence \( f \) is differentiable in the direction \( x_n \) almost everywhere.

We now prove that \( E \) has measure zero. Note that \( E \) is measurable. Fix \( \varepsilon > 0 \) and choose \( \delta > 0 \) such that
\[
\sum_{k=1}^{N} \mathcal{L}^n([a_k, b_k]) < \delta \Rightarrow \sum_{k=1}^{N} \left| f\left(\frac{a_k + 3b_k}{4}\right) - f\left(\frac{3a_k + b_k}{4}\right) \right|^n < \varepsilon, 
\]
whenever \( \{[a_k, b_k]\}_{k=1}^{N} \) is a finite collection of pairwise disjoint, 1-regular intervals in \( \mathbb{R}^n \). Let \( \{A_m\}_{m=1}^{\infty} \) be a countable collection of closed intervals with pairwise disjoint, non-empty interiors such that \( \mathbb{R}^n = \bigcup_{m=1}^{\infty} A_m \) and \( \mathcal{L}^n(A_m) < \delta \) for all \( m \). For each \( m \in \mathbb{N} \), we define a collection of intervals \( \mathcal{V}_m \) by
\[
\mathcal{V}_m = \left\{ [a, b] \subseteq A_m : \frac{\left| f\left(\frac{a+3b}{4}\right) - f\left(\frac{3a+b}{4}\right) \right|}{\|b-a\|^{2}} \geq m \right\}.
\]
Note that \( \mathcal{V}_m \) is a Vitali cover of \( E \cap \text{Int}(A_m) \). Hence, by the Vitali Covering Theorem, we can find a collection \( \{I_k^{(m)} = [a_k^{(m)}, b_k^{(m)}]\}_{k=1}^{\infty} \) of pairwise disjoint intervals from \( \mathcal{V}_m \) such that
\[
\mathcal{L}^n \left( (E \cap \text{Int}(A_m)) \setminus \bigcup_{k=1}^{\infty} I_k^{(m)} \right) = 0.
\]

Choose an integer \( K_m \geq 1 \) so that \( \sum_{k=1}^{K_m} \mathcal{L}^n(I_k^{(m)}) > \mathcal{L}^n(E \cap A_m) - \frac{\varepsilon}{2^m} \). Since the intervals \( I_k^{(m)} \) are pairwise disjoint and contained in \( A_m \) we
have that $\sum_{k=1}^{K_m} \mathcal{L}^n(I_k^{(m)}) < \mathcal{L}^n(A_m) < \delta$. Thus, using (5.5) and (5.6), we get

$$
\varepsilon > \sum_{k=1}^{K_m} \left| f\left(\frac{a_k^{(m)} + 3b_k^{(m)}}{4}\right) - f\left(\frac{3a_k^{(m)} + b_k^{(m)}}{4}\right)\right|^n
$$

$$
\geq 2^{-n_m} \sum_{k=1}^{K_m} \left| b_k^{(m)} - a_k^{(m)} \right|^n
$$

$$
\geq 2^{-n_m} \sum_{k=1}^{K_m} \mathcal{L}^n(I_k^{(m)}).
$$

Hence,

$$
\sum_{k=1}^{K_m} \mathcal{L}^n(I_k^{(m)}) \leq \frac{2^n \varepsilon}{m^n}.
$$

We now deduce that

$$
\sum_{m=1}^{\infty} \sum_{k=1}^{K_m} \mathcal{L}^n(I_k^{(m)}) \leq 2^n \varepsilon \sum_{m=1}^{\infty} \frac{1}{m^n} < 2^{n+1} \varepsilon,
$$

whilst

$$
\sum_{m=1}^{\infty} \sum_{k=1}^{K_m} \mathcal{L}^n(I_k^{(m)}) \geq \sum_{m=1}^{\infty} \left( \mathcal{L}^n(E \cap A_m) - \frac{\varepsilon}{2^m} \right) \geq \mathcal{L}^n(E) - \varepsilon.
$$

Thus $\mathcal{L}^n(E) \leq (2^{n+1} + 1)\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\mathcal{L}^n(E) = 0$. □

In view of the presented above pathological examples of functions in $1-AC$, it makes sense to consider the class $1-AC_{\text{WDN}}$ which consists of all functions in $1-AC$ which are in the Sobolev space $W^{1,n}_{\text{loc}}$, are differentiable a.e. and satisfy the Luzin (N) property.

Our final result shows where the class $1-AC_{\text{WDN}}$ fits in the hierarchy of previously studied classes.

**Theorem 5.10.** The following holds

(5.7)

$$
\mathcal{Q} - AC \subsetneq \alpha - AC = 1 - AC_{\text{WDN}} \cap AC_H \subsetneq 1 - AC_{\text{WDN}} \subsetneq 1 - AC_{\text{HWDN}},
$$

and

(5.8)

$$
1 - AC_{\text{WDN}} \setminus \mathcal{B} - AC \neq \emptyset, \quad \text{and} \quad \mathcal{B} - AC \setminus 1 - AC_{\text{WDN}} \neq \emptyset.
$$

where $1 - AC_{\text{HWDN}}$ denotes the set of functions in $1 - AC_H$ which are in the Sobolev space $W^{1,2}_{\text{loc}}$, are differentiable a.e. and satisfy the Luzin (N) property.
Remark 5.11. Bongiorno [8] introduced another class of absolute continuity denoted $AC^A_n(\Omega, \mathbb{R}^l)$, or simply $AC^A$, so that $AC^A \subset AC^A_n(\Omega, \mathbb{R}^l)$ and all functions in $AC^A_n(\Omega, \mathbb{R}^l)$ are differentiable a.e. and satisfy the Luzin (N) property, but $AC^A_n(\Omega, \mathbb{R}^l) \subsetneq W_{loc}^{1,n}(\Omega, \mathbb{R}^l)$. It would be interesting to determine what are the classes $1-AC^A \cap AC^A_n$, $1-AC^A \cap AC^A_H$, and $1-AC^A \cap AC^HWDN$. It also would be interesting to determine what are the relations between $AC^A \cap W_{loc}^{1,n}(\Omega, \mathbb{R}^l)$, $1-AC^A_HWDN \cap AC^A$, and the linear span of $1-AC^AWDN \cup AC^A_H$.

Proof of Theorem 5.10. The first containment of (5.7) is due to Bongiorno [6]. The next equality follows from Theorem 1.4, since all functions in $AC^A_H$ are in the Sobolev space $W_{loc}^{1,2}(\Omega, \mathbb{R}^l)$, are differentiable a.e. and satisfy the Luzin (N) property. All following containments are clear.

The proof that

\[(5.9) \quad 1-AC^AWDN \setminus AC^A_H \neq \emptyset\]

is technical and we postpone it till the end.

The Bongiorno’s example [6, Example 4.(1)] shows that

\[(5.10) \quad AC^A_H \setminus 1-AC^AWDN \neq \emptyset.\]

Since both $AC^A_H$ and $1-AC^AWDN$ are linearly closed it follows from (5.10) that $\text{lin span}(1-AC^AWDN \cup AC^A_H) \setminus (1-AC^AWDN \cup AC^A_H) \neq \emptyset$.

The first part of (5.8) follows from (5.9) since $BAC \subset AC^A_H$.

The other part of (5.8) follows from a small adjustment of the function constructed by Csörnyei in [13, Theorem 2]. Indeed, let $f$ be the function defined in [13, Theorem 2]. We rotate $f$ clockwise by $90^\circ$ to obtain the function $g$. Csörnyei showed that $f \notin Q-AC^2(\Omega, \mathbb{R})$. By a similar argument, using the same notation, since the right-upper corner of each $Q_{mk} = [a_{mk}, b_{mk}]$ is a 1-regular interval for $m \in \mathbb{N}$ and $k = 1, 2, \ldots, r_m$, and since the collection $\{Q_{mk}\}$ forms a pairwise disjoint system of 1-regular intervals, we have for the new function $g$

\[|g(b_{mk}) - g(a_{mk})| = \omega_m.\]

Thus,

\[\sum_{m=1}^{\infty} \sum_{k=1}^{r_m} |g(b_{mk}) - g(a_{mk})|^2 = \sum_{m=1}^{\infty} \sum_{k=1}^{r_m} \omega_m^2 = \sum_{m=1}^{\infty} r_m \omega_m^2 = \sum_{m=1}^{\infty} \frac{1}{4m} = \infty.\]

Therefore, $g \notin 1-AC^2(\Omega, \mathbb{R}^l)$. However, by the same argument as in [13, Theorem 2], $g$ is in $BAC^2(\Omega, \mathbb{R})$.

We now prove (5.9). The construction is an adjustment of [13, Theorem 2] and [6, Example 4.(2)]. Since the construction is very technical we provide all details for the convenience of the reader.
We define \( f \) as the sum of an absolutely convergent series of non-negative continuous functions \( f_m \) so that the support of each \( f_m \) is covered by the union of pairwise disjoint squares

\[
Q_m, Q_{m1}, Q_{m2}, \ldots, Q_{mr_m},
\]

with

\[
\mathcal{L}^2 \left( \bigcup_{k=1}^{r_{m+1}} Q_{(m+1)k} \right) < \frac{1}{8} \mathcal{L}^2 \left( \bigcup_{k=1}^{r_m} Q_{mk} \right) < \left( \frac{1}{8} \right)^m \mathcal{L}^2(Q_{11}),
\]

and such that

\[
\max f_m = \omega_m = \frac{1}{2^m m!}.
\]

The square \( Q_{11} \) is arbitrarily chosen in \( \Omega \). Assume that, for a given \( m \), the functions \( f_1, f_2, \ldots, f_{m-1} \) and the squares \( Q_{hk}, 1 \leq h \leq m, 1 \leq k \leq r_m \), have been defined. We define \( f_m \) and the squares \( Q_{(m+1)j} \), for \( j = 1, \ldots, r_{m+1} \) as follows: for a fixed \( k \in 1, \ldots, r_m \), we put a horizontal and a vertical line through the midpoint \( O = (o_1, o_2) \) of the square \( Q_{mk} \). Denote by \( 2d_1 \) the length of the side of \( Q_{mk} \) and by

\[
A_1 = (o_1 + d_1, o_2 + d_1), \quad A_2 = (o_1 + d_1, o_2 - d_1),
\]

\[
A_3 = (o_1 - d_1, o_2 - d_1), \quad A_4 = (o_1 - d_1, o_2 + d_1),
\]

the vertices of the square \( Q_{mk} \).

Let \( d = \frac{1}{2} d_1 \). We construct a smaller square \( A_1 B_1 C_1 D_1 \) in the upper right corner of the square \( A_1 A_2 A_3 A_4 \), where

\[
B_1 = (o_1 + d_1, o_2 + d), \quad C_1 = (o_1 + d, o_2 + d), \quad D_1 = (o_1 + d_1, o_2 + d).
\]

The length of side \( A_1 B_1 = d_1 - d = d \). Note that the interval \( T_{mk} = [O, B_1] \subset Q_{mk} \) is \( \frac{1}{2} \)-regular since

\[
\frac{\mathcal{L}^2(T_{mk})}{d_1^2} = \frac{\mathcal{L}^2([O, B_1])}{d_1^2} = \frac{d \cdot d_1}{d_1^2} = \frac{1}{2}.
\]

Similarly we construct a square \( A_2 B_3 C_3 D_3 \) in the lower left corner of the square \( A_1 A_2 A_3 A_4 \), where

\[
B_3 = (o_1 - d_1, o_2 - d), \quad C_3 = (o_1 - d, o_2 - d), \quad D_3 = (o_1 - d, o_2 - d_1).
\]

Look at the square \( A_1 B_1 C_1 D_1 \) with the side length equal to \( d \), cf. Figure 1. For every interval

\[
[a_i, b_i] = \begin{cases} 
\left[ \frac{d}{2^{i-1}}, d \right], & i = 1, 2, \ldots, m - 1, \\
\left[ 0, \frac{d}{2^{m-1}} \right], & i = m,
\end{cases}
\]
Figure 1. $Q_{mk}$ when $m = 2$

we put $2(m+1)$ small disjoint squares $Q_{(m+1)j}$ inside each strip

$$M_i = \left\{ (x, y) \in \mathbb{R}^2, \rho(x, y) \in \left[ \frac{2a_i + b_i}{3}, \frac{a_i + 2b_i}{3} \right] \cap \Box A_1 B_1 C_1 D_1 \right\},$$

where $\rho(x, y) = |x - o_1 - d| + |y - o_2 - d|$ (as shaded in Figure 1); and

we also put $2(m + 1)$ small disjoint squares $Q_{(m+1)j}$ inside each strip

$$M'_i = \left\{ (x, y) \in \mathbb{R}^2, \rho'(x, y) \in \left[ \frac{2a_i + b_i}{3}, \frac{a_i + 2b_i}{3} \right] \cap \Box A_3 B_3 C_3 D_3 \right\},$$

where $\rho'(x, y) = |x - o_1 + d| + |y - o_2 + d|$.

Thus all squares $Q_{(m+1)j}$ are contained in the union of of squares $A_1 B_1 C_1 D_1$ and $A_3 B_3 C_3 D_3$, whose union has measure equal to $\frac{1}{8}$ of the measure of square $Q_{mk}$, so that (5.11) is satisfied.

We require that the distribution of the squares inside the strips is the following (we will describe the situation for the strip $M_i$, and the strip $M'_i$ will be symmetric):
Moreover, on the intervals \( [a_{i-1} + 2b_i, a_i + 2b_i] \) we define \( f_m \) by
\[
\tilde{f}_m(\rho) = \begin{cases} 
\frac{f_m(a_i) + f_m(b_i)}{2}, & \text{if } \rho \in \left[ a_{i-1} + 2b_i, a_i + 2b_i \right]; \\
\text{linear}, & \text{if } \rho \in \left[ a_i, \frac{2a_i + b_i}{3} \right] \cup \left[ \frac{a_i + 2b_i}{3}, b_i \right].
\end{cases}
\]

We perform the same construction in the square \( A_3B_3C_3D_3 \).

On the polygon \( B_1C_1D_1B_2C_2D_2 \) the function \( f_m \) is constant along the lines with slope 1. Outside the polygon \( B_1A_1D_1B_3A_3D_3 \) the function \( f_m \) is 0. We define the function
\[
f = \sum_{m=1}^{\infty} f_m.
\]

For every \( x \in \Omega \), there exists the \( m_x \in \mathbb{N} \) such that for all \( m \geq m_x \), \( f_m(x) = 0 \). Thus \( f \) is well defined.
Claim 1: \( f \) is differentiable almost everywhere.

By the Rademacher-Stepanov theorem (see e.g. [16, Theorem 3.1.9], a short proof in [26]), it is enough to prove that \( \text{Lip}(f, x) < \infty \) for a.e. \( x \in \Omega \), where \( \text{Lip}(f, x) \) was defined in (5.4).

For every \( x \in \Omega \), there exists the smallest \( m_x \in \mathbb{N} \) such that for all \( m \geq m_x \), \( f_m(x) = 0 \) and without loss of generality \( m_x > 1 \). Moreover for every \( x \) there exists a neighborhood \( B(x, r) \) such that for any \( y \in B(x, r) \) and any \( m \neq m_x \), we have \( f_m(x) = f_m(y) \). Thus \( \text{Lip}(f, x) = \text{Lip}(f_{m_x}, x) \). Since functions \( f_m \) are Lipschitz for every \( m \), we conclude that \( \text{Lip}(f_{m_x}, x) \), and thus also \( \text{Lip}(f, x) \), is finite.

Claim 2: \( f \in W^{1,2}(\mathbb{R}^2) \) and \( f \) satisfies the Luzin (N) property.

Since each function \( f_m \) is Lipschitz, \( f_m \in W^{1,2}(\mathbb{R}^2) \), and thus \( f \) as a weak limit of finite sums of \( f_m \)'s also belongs to \( W^{1,2}(\mathbb{R}^2) \).

Similarly, each \( f_m \) satisfies the Luzin (N) property, and thus \( f \) as a countable sum of \( f_m \)'s also satisfies the Luzin (N) property.

Claim 3: \( f \notin \frac{1}{2}\text{AC}^2(\Omega, \mathbb{R}) \).

By (5.13) for each \( m \in \mathbb{N} \), and \( k = 1, 2, \ldots, r_m \), the disjoint intervals \( T_{mk} \subseteq Q_{mk} \) are \( \frac{1}{2} \)-regular, and by (5.11), for every \( \delta > 0 \), there exists \( m_0 \in \mathbb{N} \) so that
\[
\mathcal{L}^2 \left( \bigcup_{m=m_0}^{\infty} \bigcup_{k=1}^{r_m} T_{mk} \right) < \frac{1}{7} \cdot \frac{1}{8^{m_0-2}} \mathcal{L}^2(Q_{11}) < \delta.
\]
Since, for each \( m \in \mathbb{N} \), and \( k = 1, 2, \ldots, r_m \), \( |f(T_{mk})| = \omega_m \), we have
\[
\sum_{m=m_0}^{\infty} \sum_{k=1}^{r_m} |f(T_{mk})|^2 = \sum_{m=m_0}^{\infty} \sum_{k=1}^{r_m} \omega_m^2 = \sum_{m=m_0}^{\infty} r_m \omega_m^2 = \sum_{m=m_0}^{\infty} \frac{1}{4m} = \infty.
\]
Therefore, \( f \notin \frac{1}{2}\text{AC}^2(\Omega, \mathbb{R}) \).

Claim 4: \( f \in 1\text{-AC}^2(\Omega, \mathbb{R}) \setminus \text{AC}^2_H(\Omega, \mathbb{R}) \).

By Theorem [13] and Claim 3, it is enough to show that \( f \) is in \( 1\text{-AC}^2(\Omega, \mathbb{R}) \).

To see this, first note that, by an adaptation of [13 Lemma 3], for every 1-regular interval \( I = [a, b] \), there exists an index \( m = m(I) \), so that,
\[
|f(a) - f(b)|^2 \leq 16|f_{m(I)}(a) - f_{m(I)}(b)|^2.
\]
Let
\[
\mathcal{D}_1 = \left\{ I = [a, b] : I \text{ is 1-regular and } |f_{m(I)}(a) - f_{m(I)}(b)| \leq \frac{9 \omega_m(I)}{m(I)} \right\},
\]
\[ \mathcal{D}_2 = \left\{ I = [a, b] : I \text{ is 1-regular and } |f_{m(I)}(a) - f_{m(I)}(b)| > \frac{\omega_{m(I)}}{m(I)} \right\}. \]

We will prove that there exist two measures \( \mu_1 \) and \( \mu_2 \), absolutely continuous with respect to the Lebesgue measure, such that
\[
|f_{m(I)}(a) - f_{m(I)}(b)|^2 \leq \mu_1([a, b])
\]
for each \([a, b] \in \mathcal{D}_1\), and
\[
|f_{m(I)}(a) - f_{m(I)}(b)|^2 \leq \mu_2([a, b])
\]
for each \([a, b] \in \mathcal{D}_2\).

If such measures exist, then the absolute continuity of \( \mu_1 \) and \( \mu_2 \) implies that for all \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that for each finite collection of non-overlapping 1-regular intervals \([a_j, b_j]\) with \( \mathcal{L}^2(\bigcup_j [a_j, b_j]) < \delta \),
\[
\mu_1(\bigcup_j [a_j, b_j]) < \frac{\varepsilon}{32}, \quad \mu_2(\bigcup_j [a_j, b_j]) < \frac{\varepsilon}{32}.
\]
Hence, we obtain
\[
\sum_j |f(a_j) - f(b_j)|^2 \leq 16 \sum_j |f_{m([a_j, b_j])}(a_j) - f_{m([a_j, b_j])}(b_j)|^2
\]
\[
\leq 16 \sum_j \mu_1([a_j, b_j]) + 16 \sum_j \mu_2([a_j, b_j])
\]
\[
= 16 \mu_1(\bigcup_j [a_j, b_j]) + 16 \mu_2(\bigcup_j [a_j, b_j]) < \varepsilon,
\]
which proves that \( f \in 1-AC^2(\Omega, \mathbb{R}) \).

Thus, to complete the proof, it is enough to prove the existence of measures \( \mu_1 \) and \( \mu_2 \).

**Existence of the measure \( \mu_1 \)**

For a fixed 1-regular interval \( I = [a, b] \in \mathcal{D}_1 \), let \( m = m(I) \) be such that \( |f(a) - f(b)| \leq 4|f_{m(I)}(a) - f_{m(I)}(b)| \). If \( |f_m(a) - f_m(b)| = 0 \), then we can remove this interval \([a, b]\) without affecting our results. If \( |f_m(a) - f_m(b)| > 0 \), then let \( I' = [a', b'] \) be the smallest 1-regular sub-interval of \( I \) such that \( |f_m(a') - f_m(b')| = |f_m(a) - f_m(b)| \). Then \( I' \subset Q_{mk} \) for some \( k \), and there are three cases:

\( \circ \) Both points \( a' \) and \( b' \) are in \( \triangle B_1C_1D_1 \).

\( \circ \) Both points \( a' \) and \( b' \) are in \( \triangle B_3C_3D_3 \).

\( \circ \) Point \( b' \) is in \( \triangle B_3C_3D_3 \), and \( a' \) is in \( \triangle B_1C_1D_1 \).
In case (c), let \( F_1 \) be the intersection point of line \( a'b' \) and the side \( B_1C_1 \) (or \( C_1D_1 \)), and \( I^* = [F_1, b'] \subseteq I' \). If \( |f_m(F_1) - f_m(b')| > \frac{9}{m} \), then we put this interval \( I^* \) into set \( D_2 \). If \( |f_m(F_1) - f_m(b')| \leq \frac{9}{m} \), then we set \( I' = I^* \). Therefore, combining these cases, without loss of generality, we can assume that \( I' \subseteq \square A_1B_1C_1D_1 \).

For \( 1 \leq i \leq m - 9 \), we set

\[
S_i = \left\{ x \in \mathbb{R}^2 : \rho(x) \in \bigcup_{j=i}^{i+9} [a_j, b_j] \right\}.
\]

Since \( |f_m(a') - f_m(b')| \leq \frac{9}{m} \), we have \( a', b' \in S_i \) for some integer \( 1 \leq i \leq m - 9 \). Now notice that \( f_m \) is Lipschitz on \( \bigcup_{j=i}^{i+9} [a_j, b_j] \) with Lipschitz constant

\[
K = \frac{\omega_m/2m}{\frac{1}{3} \cdot \sqrt{2} \cdot (b_{i+9} - a_{i+9})} \leq \frac{\omega_m/2m}{\frac{1}{3} \cdot \sqrt{2} \cdot (d/2^{i+9} - d/2^{i+9})} = 3 \cdot 2^{9-1} \cdot \sqrt{2} \cdot \frac{\omega_m}{m} \cdot \frac{1}{d/2^i}.
\]

Therefore

\[
|f_m(a') - f_m(b')|^2 \leq 9 \cdot 2^{17} \cdot \frac{\omega_m^2}{m^2} \cdot \frac{(\text{diam} I')^2}{(d/2^i)^2}.
\]

Since \( (\text{diam} I')^2 \leq 2\mathcal{L}^2(I') \) and \( \mathcal{L}^2(S_i) < 8 \cdot (d/2^i)^2 \), we have

\[
|f_m(b') - f_m(a')|^2 \leq 9 \cdot 2^{21} \cdot \frac{\omega_m^2}{m^2} \cdot \frac{\mathcal{L}^2(I')}{\mathcal{L}^2(S_i)} = \int_{I'} 9 \cdot 2^{21} \cdot \frac{\omega_m^2}{m^2} \cdot \frac{1}{\mathcal{L}^2(S_i)}.
\]

Thus if we set \( \mu_1 = \int g \), where

\[
g(x) = 9 \cdot 2^{21} \cdot \sum_{m=1}^{\infty} \sum_{k=1}^{r_m} \sum_{i=1}^{m-9} \left( \frac{\omega_m^2}{m^2} \cdot \frac{\chi_{S_i}(x)}{\mathcal{L}^2(S_i)} \right) \in L^1(\mathbb{R}^2),
\]

then \( \mu_1 \) satisfies (5.14) for each interval \( [a, b] \in \mathcal{D} \) (cf. [13] p. 154 for details).

**Existence of the measure \( \mu_2 \)**

Let \( \mu_2 \) be an absolutely continuous measure for which

\[
\mu_2(Q_{mk}) = \frac{4}{m \cdot r_m}, \quad m \in \mathbb{N}, k = 1, \ldots, r_m.
\]
This measure exists because
\[
\sum_{j=1}^{r_m+1/r_m} \mu_2(Q_{(m+1)j}) = \frac{4}{(m+1) \cdot r_m} < \frac{4}{m \cdot r_m} = \mu_2(Q_{mk}),
\]
and
\[
\sum_{j=1}^{r_m} \mu_2(Q_{mk}) = \frac{4}{m} \to 0.
\]

As before, for a fixed 1-regular interval \(I = [a, b] \in \mathcal{D}_2\), let \(m = m(I)\) be such that \(|f(a) - f(b)| \leq 4|f_{m(I)}(a) - f_{m(I)}(b)|\). Let \(I' = [a', b']\) be the smallest 1-regular sub-interval of \(I\) such that \(|f_m(a') - f_m(b')| = |f_m(a) - f_m(b)|\). Then \(I' \subset Q_{mk}\) for some \(k\), and by the argument above, without loss of generality, we can assume that \(I' \subset A_1B_1C_1D_1\).

Let
\[
\beta = \frac{|f(a) - f(b)| \cdot m}{\omega_m}.
\]

Since \(I \in \mathcal{D}_2\), we have \(\beta \geq 9\). For simplicity, we can assume that \(C_1 = (0, 0)\). Let \(j\) be the smallest integer with \(\rho(b') \geq \frac{d}{2^j}\), and \(i\) be the biggest integer with \(\rho(a') \geq \frac{d}{2^j}\). We denote \(a' = (a_1, a_2), b' = (b_1, b_2)\), and, for \(1 \leq v \leq i\), we set
\[
S_v = \left\{ x \in \mathbb{R}^n : \rho(x) \in \left[ \frac{d}{2j+v}, \frac{d}{2j+v-1} \right] \right\}.
\]

Let \(a_{1t}\) and \(a_{2t}\) be the points on the lines \(y = 2x\) and \(y = x/2\), respectively, so that
\[
\rho(a_{1t}) = \rho(a_{2t}) = \frac{t}{3} \cdot \frac{d}{2j+i}, \text{ with } 4 \leq t \leq 5.
\]

Let \(b_{1t}, b_{2t}\) and \(c_{1t}, c_{2t}\) be the images of the orthogonal projections of \(a_{1t}, a_{2t}\) onto the horizontal line and onto the vertical line through \(C_1\), respectively. Finally let \(d_t = (\frac{t}{3} \cdot \frac{d}{2j+i}, 0)\), \(f_t = (0, \frac{t}{3} \cdot \frac{d}{2j+i})\). Since
\[
|C_1 - c_{1t}| = |a_{1t} - b_{1t}| = 2|a_{1t} - c_{1t}| = 2|a_{1t} - c_{1t}|,
\]
\[
|b_{1t} - d_t| = |a_{1t} - b_{1t}|, \text{ and } |a_{1t} - c_{1t}| = |f_t - c_{1t}|,
\]
we have
\[
\text{(5.16)} \quad \frac{t}{3} \cdot \frac{d}{2j+v} = |C_1 - d_t| = 3|C_1 - b_{1t}|,
\]
and
\[
\text{(5.17)} \quad \frac{t}{3} \cdot \frac{d}{2j+v} = |C_1 - f_t| = 3|a_{1t} - c_{1t}|.
\]
Moreover, since
\[
|a_{2t} - c_{2t}| = |C_1 - b_{2t}| = 2|a_{2t} - b_{2t}| = 2|C_1 - c_{2t}|,
\]

\[ |a_{2t} - b_{2t}| = |b_{2t} - d_t|, \text{ and } |c_{2t} - f_t| = |a_{2t} - c_{2t}|, \]

we have
\[ t \cdot \frac{d}{2^{j+v}} = |C_1 - d_t| = 3|a_{2t} - b_{2t}|, \]

and
\[ t \cdot \frac{d}{2^{j+v}} = |C_1 - f_t| = 3|C_1 - c_{2t}|. \]

Therefore, if \( v \leq i - 1 \), by (5.16),
\[ |a_{1t} - b_{1t}| = 2 \cdot |C_1 - b_{1t}| = \frac{t \cdot d}{3^2 \cdot 2^{j+v-1}} > \frac{4 \cdot d}{3^2 \cdot 2^{j+v-1}} > \frac{d}{\frac{2^{j+v}}{2^{j+i}}} \geq \frac{d}{\frac{2^{j+i}}{2^{j+i}}} > a_1, \]

and, by (5.19),
\[ |a_{2t} - c_{2t}| = 2 \cdot |C_1 - c_{2t}| = \frac{t \cdot d}{3^2 \cdot 2^{j+v}} > \frac{4 \cdot d}{3^2 \cdot 2^{j+v}} > \frac{d}{\frac{2^{j+v+2}}{2^{j+i}}} \geq \frac{d}{\frac{2^{j+i}}{2^{j+i}}} > a_1. \]

Moreover, if \( v \leq i - 2 \), by (5.18)
\[ |a_{2t} - b_{2t}| = \frac{1}{3} \cdot |C_1 - d_t| = \frac{t \cdot d}{3^2 \cdot 2^{j+v}} > \frac{4 \cdot d}{3^2 \cdot 2^{j+v}} > \frac{d}{\frac{2^{j+v+2}}{2^{j+i}}} \geq \frac{d}{\frac{2^{j+i}}{2^{j+i}}} > a_2, \]

and, by (5.17),
\[ |a_{1t} - c_{1t}| = \frac{1}{3} \cdot |C_1 - f_t| = \frac{t \cdot d}{3^2 \cdot 2^{j+v+1}} > \frac{4 \cdot d}{3^2 \cdot 2^{j+v+1}} > \frac{d}{\frac{2^{j+v+2}}{2^{j+i}}} \geq \frac{d}{\frac{2^{j+i}}{2^{j+i}}} > a_1. \]

Thus, to see that the points \( a_{1t}, a_{2t} \) belong to the interval \( I' \), it suffices to show that
\[ |a_{1t} - b_{1t}| < b_2 \text{ and } |a_{2t} - b_{2t}| < b_2; \]
\[ |a_{1t} - c_{1t}| < b_1 \text{ and } |a_{2t} - c_{2t}| < b_1. \]

First of all, we have
\[ b_1 > \frac{d}{2^{j+3}} \text{ and } b_2 > \frac{d}{2^{j+3}}. \]

Indeed, if \( b_1 \leq \frac{d}{2^{j+1}} \), then \( b_2 > \frac{d}{2^{j+1}} \). Thus
\[ b_2 - a_2 > \frac{d}{2^{j+1}} - \frac{d}{2^{j+1}} > \frac{d}{2^{j+3}} \geq b_1 > b_1 - a_1, \]

which contradicts 1-regularity of \([a', b']\). Similarly \( b_2 \leq \frac{d}{2^{j+1}} \).
Therefore, if \( v \geq 4 \), we have

\[
|a_{1t} - b_{1t}| < \frac{5}{3^2} \cdot \frac{d}{2^{j+v-1}} < \frac{d}{2^{j+v-1}} < \frac{d}{2^{j+3}} \leq b_2;
\]

\[
|a_{2t} - b_{2t}| < \frac{5}{3^2} \cdot \frac{d}{2^{j+v}} < \frac{d}{2^{j+v}} < \frac{d}{2^{j+3}} \leq b_2;
\]

\[
|a_{1t} - c_{1t}| < \frac{5}{3^2} \cdot \frac{d}{2^{j+v}} < \frac{d}{2^{j+v}} < \frac{d}{2^{j+3}} \leq b_1;
\]

\[
|a_{1t} - c_{1t}| < \frac{5}{3^2} \cdot \frac{d}{2^{j+v-1}} < \frac{d}{2^{j+v-1}} \leq d_{2^j} + 3 < b_1.
\]

In conclusion, we have proved that, if \( 4 \leq v \leq i - 2 \), then the interval \( I' \) contains the points \( a_{1t}, a_{2t} \) for all \( 4 \leq t \leq 5 \). Thus \( I' \) contains the intervals \( [a_{14}, a_{15}] \) and \( [a_{24}, a_{25}] \). Therefore, \( I' \) covers at least \( (i-5)m/2 \) of the \( r_{m+1} \) squares \( Q_{(m+1)h}, 1 \leq h \leq r_{m+1} \). Thus

\[
\mu_2([a', b']) \geq (i - 5)m \cdot \frac{4}{(m + 1)r_{m+1}} = (i - 5) \cdot \frac{4m}{(m + 1)^2} \cdot \omega_m^2,
\]

and, since \( \beta \leq m \) and \( \beta \leq i + 2 \), we have

\[
|f(a') - f(b')|^2 = \frac{\beta^2}{m^2} \cdot \omega_m^2 \leq \frac{\beta^2}{m^2} \cdot \frac{(m + 1)^2}{4m(i - 5)} \cdot \mu_2([a', b'])
\]

\[
< \frac{(i + 2)(m + 1)^2}{4(i - 5)m^2} \cdot \mu_2([a', b']).
\]

Moreover, since \( m \geq 9 \), we have \( 5m^2 - 22m + 7 > 0 \). Since \( i \geq 9 \), we have \( i(3m^2 - 2m + 1) > 22m^2 + 4m + 2 \). Thus, \( (i + 2)(m + 1)^2 < 2(i - 5)m^2 \), and hence

\[
|f(a') - f(b')|^2 < \mu_2([a', b']).
\]

\( \square \)

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**School of Mathematics, University of Birmingham, Birmingham, B15 2TT, UK**

*E-mail address*: dymondm@maths.bham.ac.uk

**Department of Mathematics, Miami University, Oxford, OH 45056, USA**

*E-mail address*: randrib@miamioh.edu

**Department of Mathematics, Miami University, Oxford, OH 45056, USA**

*E-mail address*: xuhuaqiang1990@gmail.com