A NOTE ON THE $R_\infty$ PROPERTY FOR GROUPS

$F\text{Alt}(X) \leq G \leq \text{Sym}(X)$

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ABSTRACT. Given a set $X$, the group $\text{Sym}(X)$ consists of all maps from $X$ to $X$, and $\text{FSym}(X)$ is the subgroup of maps with finite support i.e. those that move only finitely many points in $X$. We describe the automorphism structure of groups $\text{FSym}(X) \leq G \leq \text{Sym}(X)$ and use this to state some conditions on $G$ for it to have the $R_\infty$ property. Our main results are that if $G$ is infinite, torsion, and $\text{FSym}(X) \leq G \leq \text{Sym}(X)$, then it has the $R_\infty$ property. Also, if $G$ is infinite and residually finite, then there is a set $X$ such that $G$ acts faithfully on $X$ and, using this action, $\langle G, \text{FSym}(X) \rangle$ has the $R_\infty$ property. Finally we have a result for the Houghton groups, which are a family of groups we denote $H_n$, where $n \in \mathbb{N}$. We show that, given any $n \in \mathbb{N}$, any group commensurable to $H_n$ has the $R_\infty$ property.

1. INTRODUCTION

The notion of twisted conjugacy and its relationship to fixed point theory has attracted significant attention. For any group $G$ and any $\phi \in \text{Aut}(G)$, we say that two elements $a, b \in G$ are $\phi$-twisted conjugate (denoted $a \sim_\phi b$) if there exists an $x \in G$ such that

$$(x^{-1})\phi ax = b.$$ (1)

Notice that when $\phi = \text{id}_G$ this becomes the equation for conjugacy. Now, given any $\phi \in \text{Aut}(G)$, define the Reidemeister number of $\phi$, denoted $R(\phi)$, to be the number of $\phi$-twisted conjugacy classes in $G$. Thus $R(\text{id}_G)$ records the number of conjugacy classes of $G$ and deciding whether this is infinite has been studied for some time (e.g. [HNN49] where an infinite group with $R(\text{id}_G)$ finite was constructed). We say that $G$ has the $R_\infty$ property if $R(\phi) = \infty$ for every $\phi \in \text{Aut}(G)$.

Notation. For a non-empty set $X$, let $\text{Sym}(X)$ denote the group of all permutations of $X$. Furthermore, let $\text{FSym}(X)$ denote the group of all permutations of $X$ with finite support, and let $F\text{Alt}(X)$ denote the group of all even permutations of $X$ with finite support.

A first example one may consider for the $R_\infty$ property is $\mathbb{Z}$. Although this has infinitely many conjugacy classes, the only non-trivial automorphism has Reidemeister number 2. Similarly, for any $m \in \mathbb{N} := \{1, 2, \ldots\}$, the automorphism $\psi$ of $\mathbb{Z}^m$ which sends $a$ to $a^{-1}$ for all $a \in \mathbb{Z}^m$ has Reidemeister number $2^m$. In [ILS17]
and \[GP16\] however, the family of Houghton groups, which (for any \(n \in \mathbb{N}\)) are denoted \(H_n\), act on \(\{1, \ldots, n\} \times \mathbb{N} =: X_n\), and which lie in the short exact sequence
\[
1 \rightarrow \text{FSym}(X_n) \rightarrow H_n \rightarrow \mathbb{Z}^{n-1} \rightarrow 1
\]
were shown to have the \(R_\infty\) property. In this note we start with a simpler, more general proof of their theorem, and then develop this in various directions.

**Definition 1.1.** A group \(G\) fully contains \(\text{FAlt}(X)\) if \(\text{FAlt}(X) \leq G \leq \text{Sym}(X)\). Since we only wish to investigate infinite groups, we will always consider \(X\) to be infinite. We do not, however, place any other cardinality assumptions on \(X\).

Note that any Houghton group \(H_n\) fully contains \(\text{FAlt}(X_n)\), but let us justify that this is a large class of groups, using a construction from \[HO16\]. For any infinite group \(G\), we have that \(G \leq \text{Sym}(X)\) for some \(X\) (with the possibility that \(X = G\) since \(G\) can always be embedded into \(\text{Sym}(G)\) using a regular representation of \(G\)). Then \(\langle G, \text{FAlt}(X) \rangle\) fully contains \(\text{FAlt}(X)\).

**Conjecture 1.2.** Let \(G\) be an infinite group that acts faithfully on a set \(X\). Then \(\langle G, \text{FAlt}(X) \rangle\) and \(\langle G, \text{FSym}(X) \rangle\) both have the \(R_\infty\) property.

We make some progress with this conjecture. We first confirm it for the case where \(G\) is torsion i.e. we show that, for any infinite set \(X\) and any embedding of \(\Psi : G \rightarrow \text{Sym}(X)\), \(\langle (G)\Psi, \text{FAlt}(X) \rangle\) has the \(R_\infty\) property for any torsion group \(G\). We then use this work to show that if \(G\) is an infinite residually finite group, then there is a set \(X\) on which \(G\) acts faithfully and, using this action, \(\langle G, \text{FAlt}(X) \rangle\) has the \(R_\infty\) property. We end by showing that, if is commensurable to a Houghton group \(H_n\) (where \(n \in \mathbb{N}\)) then \(G\) has the \(R_\infty\) property.

Let us now describe these results more precisely, and better indicate the path that the paper takes. We start by describing the automorphism group for groups fully containing \(\text{FAlt}(X)\), so to approach twisted conjugacy.

**Definition.** A group \(G\) is monolithic if it has a non-trivial normal subgroup that is contained in every non-trivial normal subgroup of \(G\) i.e. if it has a minimal non-trivial normal subgroup.

Let \(N_{\text{Sym}(X)}(G) := \{ \rho \in \text{Sym}(X) \mid \rho^{-1}g\rho \in G \}\), the normaliser of \(G\) over \(\text{Sym}(X)\).

**Proposition 1.** \([\text{Lem. 2.4, Prop. 2.5, Rem. 2.6}]\) Let \(G\) fully contain \(\text{FAlt}(X)\), where \(X\) is infinite. Then \(\text{FAlt}(X)\) is characteristic in \(G\), \(\text{Aut}(G) \cong N_{\text{Sym}(X)}(G)\), and \(G\) is monolithic. Moreover, since elements of \(\text{Aut}(G)\) preserve the cycle type of elements of \(G\), if \(\text{FSym}(X) \leq G\), then \(\text{FSym}(X)\) is characteristic in \(G\).

The following well known lemma implies that if \(G\) is a group with the \(R_\infty\) property and \(G\) acts faithfully on a set \(X\), then \(\langle G, \text{FAlt}(X) \rangle\) and \(\langle G, \text{FSym}(X) \rangle\) have the \(R_\infty\) property (since Proposition 1 states that \(\text{FAlt}(X)\) and \(\text{FSym}(X)\) are characteristic in \(G\)).

**Lemma 1.3.** \([\text{MS14}, \text{Lem 2.1}]\) For any short exact sequence of groups
\[
1 \rightarrow D \rightarrow E \rightarrow F \rightarrow 1
\]
if \(D\) is characteristic in \(E\) and \(F\) has the \(R_\infty\) property, then \(E\) has the \(R_\infty\) property.

We then work with arguments using cycle type (using that the conjugacy classes of \(\text{Sym}(X)\) are well known: each consists of all elements of the same cycle type).
Definition. Let $g \in \text{Sym}(X)$. Then an orbit of $g$ is $\{xg^d \mid d \in \mathbb{Z}\}$ where $x \in X$. Also, $g$ has an infinite orbit if there is a $y \in X$ such that $\{yg^d \mid d \in \mathbb{Z}\}$ is infinite.

Proposition 3.4. Let $G$ fully contain $\text{FAlt}(X)$, where $X$ is an infinite set. If for every $\rho \in N_{\text{Sym}(X)}(G)$ there is an $s \in \mathbb{N}$ such that $\rho$ has finitely many orbits of size $s$, then $G$ has the $R_\infty$ property.

From the structure of $\text{Aut}(H_n)$, where $H_n$ denotes the $n^{\text{th}}$ Houghton group, Proposition 3.4 immediately yields that, for any $n \geq 2$, $H_n$ has the $R_\infty$ property.

Corollary 3.6. Let $G$ fully contain $\text{FAlt}(X)$, where $X$ is an infinite set. If for every $g \in G$, $g$ does not have an infinite orbit, then $G$ has the $R_\infty$ property.

Clearly torsion groups satisfy Corollary 3.6.

Corollary 3.7. Let $G$ be an infinite torsion group which fully contains $\text{FAlt}(X)$. Then $G$ has the $R_\infty$ property.

This means that any torsion group $T$ can be embedded into an infinite torsion group (of any cardinality greater than or equal to $|T|$) which has the $R_\infty$ property. It is in fact easy to construct an uncountable family of such groups.

Corollary 3.11. There exist uncountably many countable torsion groups which have the $R_\infty$ property.

This result can be strengthened by using an already known family of countable, finitely generated, torsion groups.

Corollary 3.12. There exist uncountably many finitely generated torsion groups which have the $R_\infty$ property.

Residually finite groups are exactly those who have a faithful action on their finite quotients. This action therefore only has finite orbits, meaning that Corollary 3.6 applies.

Corollary 3.15. Let $G$ be an infinite residually finite group, and $X$ be the union of the finite quotients of $G$. Then, using this action, $(G, \text{FAlt}(X))$ has the $R_\infty$ property.

A few conventions will be used throughout this note:

i) we shall always work with right actions;
ii) unless specified, $X$ will refer to an infinite set;
iii) we shall always consider elements from $\text{Sym}(X)$ to be written in disjoint cycle notation;
iv) for all of the results in this note, the same proofs can be used if $\text{FAlt}$ is replaced with $\text{FSym}$.

Remark. Let $g \in \text{Sym}(X)$. We shall say ‘a cycle of $g$’ to refer, for some $x \in X$, to an orbit $\{xg^d \mid d \in \mathbb{Z}\}$. If there is an $x \in X$ such that this set is infinite, then this is an infinite cycle of $g$ and $g$ contains an infinite cycle. If there is an $x \in X$ such that this set has cardinality $r$, then this is an $r$-cycle of $g$ and $g$ contains an $r$-cycle. If, for some $s \in \mathbb{N}$, there are only finitely many $x \in X$ such that $|\{xg^d \mid d \in \mathbb{Z}\}| = s$, then we shall say that $g$ has finitely many $s$-cycles. Similarly $g$ may have finitely many infinite cycles.
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2. Preliminary observations

The groups $\text{FAlt}(X)$, $\text{FSym}(X)$, and $\text{Sym}(X)$ often arise when considering permutation groups (see, for example, [Cam99] and [DM96]). Note that any countable group can be considered as a subgroup of $\text{Sym}(X)$ where $X$ is countable (for example set $X := G$ and use the regular representation of $G$). Let us start by recalling some elementary observations about $\text{FAlt}$.

**Lemma 2.1.** If $G$ fully contains $\text{FAlt}(X)$ where $X$ is infinite, then $G$ is centreless.

**Proposition 2.2.** For any infinite set $X$, $\text{FAlt}(X)$ is generated by $S$, where $S$ is the set of all 3-cycles with support in $X$.

**Proof.** Using $S$ we can produce any element which is a product of two 2-cycles (for example choose $(a_1 a_2)(a_2 b_1)$ and $(a_2 b_1)(b_1 b_2)$ whose product is $(a_1 a_2)(b_1 b_2)$). Now, given an element $\sigma \in \text{FAlt}(X)$, write $\sigma$ as a product of 2-cycles. By definition this product will consist of an even number of 2-cycles. Now, each pair of 2-cycles will either be: trivial; a 3-cycle; or a product of two 2-cycles. □

For any infinite set $X$ we therefore have that $|X| = |S| = |\text{FAlt}(X)|$. Moreover $\text{FAlt}(X)$ is an index 2 subgroup of $\text{FSym}(X)$, and so for any infinite set $X$ we also have that $|X| = |\text{FSym}(X)|$.

**Lemma 2.3.** For any infinite set $X$, $\text{FAlt}(X)$ is simple.

**Proof.** Assume that $1 \neq \sigma \in N$, a non-trivial normal subgroup of $\text{FAlt}(X)$. Then $\sigma \in A_n$ where $n \geq 5$. But $A_n \cap N \leq A_n$, and (since $N$ is non-trivial and $A_n$ is simple for $n \geq 5$) we have that $N \cap A_n = A_n$. Thus $N$ contains a 3-cycle and so $N = \text{FAlt}(X)$ by the previous lemma. □

Note that no infinite simple group can be residually finite, and so if $G$ fully contains $\text{FAlt}(X)$, then $G$ is not residually finite. Also, given any infinite set $X$, any group $G$ fully containing $\text{FAlt}(X)$ will have $\text{FAlt}(X)$ as a normal subgroup. Thus, unless $G = \text{FAlt}(X)$, $G$ will not be simple.

**Notation.** Let $G \leq \text{Sym}(X)$. For any given $\rho \in N_{\text{Sym}(X)}(G)$, let $\phi_\rho$ denote the automorphism of $G$ induced by conjugation by $\rho$ i.e., $\phi_\rho(g) := \rho^{-1}g\rho$ for all $g \in G$.

The three groups $\text{FAlt}(X), \text{FSym}(X), \text{Sym}(X)$ have the property that

\begin{equation}
N_{\text{Sym}(X)}(G) \to \text{Aut}(G), \rho \mapsto \phi_\rho \text{ is an isomorphism.}
\end{equation}

This means that $\text{Aut}(\text{FAlt}(X)) \cong N_{\text{Sym}(X)}(\text{FAlt}(X)) = \text{Sym}(X) \cong \text{Aut}(\text{FSym}(X))$ and that $\text{FAlt}(X)$ is characteristic in $\text{FSym}(X)$ which is characteristic in $\text{Sym}(X)$.

Our first aim is to show that any group $G$ fully containing $\text{FAlt}(X)$ satisfies \textbf{(2)}. We do this by showing that $\text{FAlt}(X)$ is characteristic in such a $G$ and then apply the following lemma.
Lemma 2.4. Let $G \leqslant \text{Sym}(X)$ and $\text{FAlt}(X)$ be a characteristic subgroup of $G$. Then $N_{\text{Sym}(X)}(G) \cong \Psi \text{ Aut}(G)$ where $\Psi : \rho \mapsto \phi_{\rho}$.

Proof. Running the proof of [GP16] Cor. 3.3] using 3-cycles rather than 2-cycles yields the result. □

For any group $G$ satisfying (2), we may use the following reformulation of twisted conjugacy, which has been used extensively by many authors working with the $R_{\infty}$ property. Recall that $\phi_{\rho}$ denotes the automorphism induced by conjugation by $\rho \in \text{Sym}(X)$.

Thus,

$$(x^{-1})\phi_{\rho}ax = b \Rightarrow \rho^{-1}(x^{-1})\rho ax = b \Rightarrow x^{-1}\rho ax = \rho b.$$  

We may then show that $R(\phi_{\rho}) = \infty$ by finding a set of elements $\{a_{k} \in G \mid k \in \mathbb{N}\}$ such that

$$\rho a_{i} \sim_{G} \rho a_{j} \iff i = j.$$  

This is because, if such a set of elements exist, then each $a_{k}$ lies in a distinct $\phi_{\rho}$-twisted conjugacy class, and so $R(\phi_{\rho}) = \infty$. Thus showing, for each $\rho \in N_{\text{Sym}(X)}(G)$, that there exists a set of elements $\{a_{k} \in G \mid k \in \mathbb{N}\}$ where (4) holds is sufficient to show that $G$ has the $R_{\infty}$ property.

Proposition 2.5. If $G$ fully contains $\text{FAlt}(X)$, then $\text{FAlt}(X)$ is a unique minimal normal subgroup of $G$. Moreover $\text{FAlt}(X)$ is a characteristic subgroup of $G$.

Proof. We first show that $\text{FAlt}(X)$ is a unique minimal normal subgroup of $G$, known as the monolithic property. Clearly $\text{FAlt}(X)$ is normal in $G$, since it is normal in $\text{Sym}(X)$ (conjugation in $\text{Sym}(X)$ preserves cycle type).

Consider $N \leqslant G$. We have $N \cap \text{FAlt}(X) \leqslant \text{FAlt}(X)$, and since $\text{FAlt}(X)$ is simple, $N \cap \text{FAlt}(X)$ must either be trivial or $\text{FAlt}(X)$. Let $g \in N \setminus \{1\}$. This must either: be in $\text{FSym}(X)$; contain infinitely many finite cycles; or contain an infinite cycle.

We now show that there exists a $\sigma \in \text{FAlt}(X)$ such that $\sigma^{-1}g\sigma g^{-1} \in \text{FAlt}(X) \setminus \{1\}$. Since $N$ is normal in $G$, $g$ and $\sigma^{-1}g\sigma$ are in $N$ and so this will prove the claim. For the case where $g \in \text{FSym}(X)$, choose $\sigma$ so that $\sigma^{-1}g\sigma$ and $g$ have disjoint supports. For the case where $g$ contains infinitely many finite cycles, pick 4 distinct cycles (each of length greater than 1) of $g$ and points $b_{1}, b_{2}, b_{3}, b_{4}$: one from each cycle. A suitable $\sigma$ is then $(b_{1} b_{2})(b_{3} b_{4})$. Finally, assume that $g$ contains an infinite cycle. Let $x_{0} \in X$ lie in some infinite cycle of $g$, and for every $i \in \mathbb{Z}$ let $x_{i} := x_{0}g^{i}$. Let $a := (\ldots x_{-3} x_{-2} x_{-1} x_{0} x_{1} x_{2} x_{3} \ldots)$ and let $\mu := (x_{-1} x_{0} x_{1})$. Straightforward computation shows that $\mu^{-1}a\mu^{-1} = (x_{-2} x_{-1} x_{1})$. Moreover, since $\mu$ commutes with $ga^{-1}$, we have that $\mu^{-1}g\mu^{-1} = (x_{-2} x_{-1} x_{1})$. Thus $\mu$ is a suitable candidate for $\sigma$ in this case.

Now, let $\phi \in \text{Aut}(G)$ and consider $\text{FAlt}(X) \cap (\text{FAlt}(X))\phi$. As above, this must be trivial or $\text{FAlt}(X)$. If it were trivial, this would contradict the uniqueness of $\text{FAlt}(X)$ as a minimal, non-trivial, normal subgroup in $G$, and hence $\text{FAlt}(X)$ is characteristic in $G$. □

Remark 2.6. We may use Lemma 2.4 and Proposition 2.5 to prove that all automorphisms of $\text{Sym}(X)$ are inner. Also, consider if $\text{FSym}(X) \leqslant G \leqslant \text{Sym}(X)$. Then, for all $\rho \in N_{\text{Sym}(X)}(G)$ and all $g \in \text{FSym}(X)$, we have that $(g)\phi_{\rho}$ has the same cycle type as $g$. Thus $\text{FSym}(X)$ is characteristic in $G$. 

A NOTE ON THE $R_{\infty}$ PROPERTY FOR GROUPS $\text{FAlt}(X) \leqslant G \leqslant \text{Sym}(X)$
We are now ready to produce conditions on the cycle type of elements in $G$ and in $N_{\text{Sym}(X)}(G)$ for automorphisms to have infinite Reidemeister number. In order to do this we will use the condition equivalent to showing that $R(\phi_\rho) = \infty$ (labelled $\square$ above) and well known facts about $\text{Sym}(X)$ regarding cycle type.

3. Results using facts about conjugacy in $\text{Sym}$

**Lemma 3.1.** Let $Y$ be an infinite set and $X$ be an infinite subset of $Y$. If $\text{FAlt}(X)$ is a subgroup of $G \leq \text{Sym}(Y)$, then $R(\text{id}_G) = \infty$.

*Proof.* We produce an infinite family of elements in $G$ which all lie in distinct conjugacy classes. We have the equation $g^{-1}ag = b$. Conjugation by elements of $G$ cannot change the cycle type of elements of $\text{Sym}(X) \leq \text{Sym}(Y)$. Thus choosing $a_k$ to be a cycle of length $2k + 1$ (or any infinite family of elements of $\text{FAlt}(X)$ with distinct cycle types) proves the claim. \hfill $\square$

The following is well known.

**Lemma 3.2.** Let $G$ be any group. Then, for any $\psi \in \text{Aut}(G)$ and $\phi \in \text{Inn}(G)$, we have that $R(\psi \phi) = R(\psi)$.

**Lemma 3.3.** Let $G$ be a group with subgroup $\text{FAlt}(X)$, where $X$ is an infinite set, and with $\text{Aut}(G) = \text{Inn}(G)$. Then $G$ has the $R_\infty$ property.

*Proof.* Let $\phi \in \text{Aut}(G)$. By assumption $\text{Aut}(G) = \text{Inn}(G)$. Therefore, by the previous lemma, $R(\phi) = R(\text{id}_G)$. Now $R(\text{id}_G) = \infty$ by Lemma 3.1 \hfill $\square$

Lemma 3.3 implies that, for any infinite set $X$, $\text{Sym}(X)$ has the $R_\infty$ property.

**Notation.** For any $g \in \text{Sym}(X)$ and $x \in X$, let $O_x(g) := \{xg^d : d \in \mathbb{Z}\}$. Also, let $\eta_r(g) := |\{x \in X : |O_x(g)| = r\}|/r$, the number of $r$-cycles in $g$. We shall use $\eta_1(g)$ to denote the number of fixed points of $g$ and $\eta_\infty(g)$ to denote the number of distinct infinite orbits induced by $g$. If any of these values is infinite then, since our arguments will be unaffected by the size of this infinity, we shall write $\eta_r(g) = \infty$.

From the previous section, for any group fully containing $\text{FAlt}(X)$ we have that the map $\Psi : N_{\text{Sym}(X)}(G) \to \text{Aut}(G), \phi \mapsto \phi_\rho$, is an isomorphism. We may therefore consider elements of $\text{Aut}(G)$ as elements of $\text{Sym}(X)$.

**Proposition 3.4.** Let $G$ fully contain $\text{FAlt}(X)$, where $X$ is an infinite set, and let $\rho \in N_{\text{Sym}(X)}(G)$. If there is an $r \in \mathbb{N}$ such that $\eta_r(\rho) < \infty$, then $R(\phi_\rho) = \infty$.

*Proof.* We shall work with the reformulation of twisted conjugacy in $\square$ above and argue for any $\rho \in N_{\text{Sym}(X)}(G)$ using three cases. Let $s \in \mathbb{N}$ be the smallest number such that $\eta_s(\rho)$ is finite.

**Case A:** $s = 1$ and $\eta_\infty(\rho) > 0$. As with the proof of Proposition 2.5, let $x_0$ lie in an infinite cycle of $\rho$ and, for each $i \in \mathbb{Z}$, let $x_i := x_0g^i$. For each $k \in \mathbb{N}$, let

$$a_k := \prod_{i=0}^{k-1} (x_{2i} \ x_{2i+1}).$$

The set of elements lying in disjoint $\phi_\rho$-twisted conjugacy classes is then given by $\{a_k \mid k \in \mathbb{N}\} \subset \text{FAlt}(X)$. This is because $\eta_1(\rho a_k)$ is finite for all $k \in \mathbb{N}$, and is strictly increasing as a function of $k$. Thus, if $i \neq j$, the elements $\rho a_i$ and $\rho a_j$ have a different number of fixed points and hence are not conjugate in $G \leq \text{Sym}(X)$.
Case B: \( s = 1 \) and \( \eta_\infty(\rho) = 0 \). Since \( \rho \) has finitely many fixed points and no infinite cycles, \( \rho \) contains infinitely many finite cycles. Thus \( \rho \) has infinitely many odd length cycles or infinitely many even length cycles. First assume that \( \rho \) has infinitely many odd length cycles and index a countable infinite subset of these by the natural numbers. Let \( \rho = \rho' \prod_{i \in \mathbb{N}} \rho_i \), where each \( \rho_i \) is a finite cycle of odd length and \( \rho' \in \text{Sym}(X) \) has cycles with disjoint support from all of the \( \rho_i \)'s. Now, for any \( m \in \mathbb{N} \), \( \rho(\rho_m)^{-1} \) has more fixed points than \( \rho \). Defining
\[
a_k := \prod_{i=1}^{k} \rho_i^{-1} \in \text{FAlt}(X)
\]
means that \( i < j \Rightarrow \eta_i(\rho a_i) < \eta_i(\rho a_j) \) and so \( \{a_k \mid k \in \mathbb{N}\} \) provides our infinite family of elements which are pairwise not \( \phi_\rho \)-twisted conjugate. Similarly, if \( \rho \) has infinitely many even length cycles, complete the same construction with \( \rho = \rho' \prod_{i \in \mathbb{N}} \rho_i \) where each \( \rho_i \) is a finite cycle of even length and \( \rho' \in \text{Sym}(X) \) has cycles with disjoint support from all of the \( \rho_i \)'s.

Case C: \( s > 1 \). All we shall use is that \( \rho \) has infinitely many fixed points. For any \( k \in \mathbb{N} \), let \( a_k \) consist of \( 2k \) \( s \)-cycles such that \( \text{supp}(a_k) \subseteq X \setminus \text{supp}(\rho) \). We then have, for all \( k \in \mathbb{N} \): that \( a_k \in \text{FAlt}(X) \); that \( \eta_k(\rho a_k) \) is finite; and that \( \eta_k(\rho a_k) \) is strictly increasing as a function of \( k \).

**Proposition 3.5.** Let \( a, b \in \text{Sym}(X) \), \( \text{supp}(b) \subseteq \text{supp}(a) \), and \( g \in \text{Sym}(X) \) satisfy \( g^{-1}ag = b \). Then \( \eta_\infty(g) > 0 \).

**Proof.** We assume, for a contradiction, that \( \eta_\infty(g) = 0 \). Since \( g^{-1}ag = b \), \( g \) must restrict to a bijection from \( \text{supp}(a) \) to \( \text{supp}(b) \) i.e.
\[
(\text{supp}(a) \cup \text{supp}(b)) \setminus (\text{supp}(a) \cap \text{supp}(b)) \subseteq \text{supp}(g)
\]
which from our hypotheses implies that
\[
\text{supp}(a) \setminus \text{supp}(b) \subseteq \text{supp}(g)
\]
where \( \text{supp}(a) \setminus \text{supp}(b) \neq \emptyset \) since \( \text{supp}(b) \neq \text{supp}(a) \). Thus \( g \) sends some \( n \in \text{supp}(a) \setminus \text{supp}(b) \) to some \( m \in \text{supp}(b) \). Now, since all of the cycles in \( g \) are finite, there is a \( k \in \mathbb{N} \) such that \( (n)g^k = n \). Therefore \( g \) sends a point in \( \text{supp}(b) \) to a point in \( X \setminus \text{supp}(b) \). This would mean that \( \text{supp}(g^{-1}ag) \cap (X \setminus \text{supp}(b)) \neq \emptyset \) and that \( g^{-1}ag \) and \( b \) have different supports, a contradiction. \( \square \)

**Corollary 3.6.** Let \( G \) be a group fully containing \( \text{FAlt}(X) \). If \( \eta_\infty(g) = 0 \) for all \( g \in G \), then \( G \) has the \( R_\infty \) property.

**Proof.** By Proposition 3.4 if \( \phi_\rho \in \text{Aut}(G) \) has \( \eta_s(\rho) < \infty \) for some \( s \in \mathbb{N} \), then \( R(\phi_\rho) = \infty \). We may therefore assume that \( \eta_r(\rho) = \infty \) for all \( r \in \mathbb{N} \). This implies that \( X \setminus \text{supp}(\rho) \) is an infinite set.

Our aim is to show that there is an infinite set of elements in \( G \) which are not \( \phi_\rho \)-twisted conjugate. Let \( b_0 := 1 \), the identity element of \( G \). For each \( k \in \mathbb{N} \), let \( b_k := b_1 b_1^{-1} \) where \( \eta_2(b_k) = 2 \), \( |\text{supp}(b_k')| = 4 \), \( \text{supp}(b_k') \subseteq X \setminus \text{supp}(\rho) \), and \( \text{supp}(b_k') \cap \text{supp}(b_{k-1}) = \emptyset \). Thus, for each \( k \in \mathbb{N} \), \( b_k \in \text{FAlt}(X) \) and \( \eta_2(b_k) = 2k \).

If \( i < j \), then \( \text{supp}(b_i) \subseteq \text{supp}(b_j) \) and so \( \text{supp}(\rho b_i) \subseteq \text{supp}(\rho b_j) \). Since \( \eta_\infty(g) = 0 \) for all \( g \in G \), Proposition 3.5 implies that not two elements in \( \{\rho b_k \mid k \in \mathbb{N}\} \) are conjugate in \( G \) i.e. \( R(\phi_\rho) = \infty \). \( \square \)
Remark 3.9. A consequence of this proof is that if all elements of \( |G| = \infty \), then the elements of \( \sigma g \) have only finite orbits. Consider an element \( \sigma g \) where \( \sigma \in \text{FSym}(X) \) and \( g \in G \). It suffices to show that \( \sigma g \) is torsion. Let \( k := \max \{ r \in \mathbb{N} : \text{supp}(g_r) \cap \text{supp}(\sigma) \neq \emptyset \} \) and let \( F := \bigcup_{1 \leq i \leq k} \text{supp}(g^{-i}\sigma^i) \).

Now \(|F| < \infty \) (implying that \( \text{Sym}(F) \) is a finite group) and \( \sigma g \) restricts to a bijection on \( F \). Also \( g|(X \setminus F) \) must be torsion since otherwise \( g \) cannot be torsion.

Remark 3.10. Let \( G \) be a torsion group. For every \( \alpha \geq |G| \), there exists a torsion group \( H_\alpha \) with the \( R_\infty \) property and contains an isomorphic copy of \( G \).

Proof. Let \( G \) be torsion, \( G \) denote the right regular representation of \( G \), and let \( \alpha \geq G \). Then there is a set \( Y_\alpha \) such that \( |Y_\alpha| = \alpha \). Also \( \hat{G} \leq \text{Sym}(G) \hookrightarrow \text{Sym}(G \cup Y_\alpha) \) via the natural inclusion of the set \( G \) into the set \( G \cup Y_\alpha \). Let \( G_\alpha \) denote the image of \( \hat{G} \) in \( \text{Sym}(G \cup Y_\alpha) \), using the restriction of this map. Now \( H_\alpha := \langle G_\alpha, \text{FAlt}(G \cup Y_\alpha) \rangle \) has cardinality \( \alpha \). Moreover it is torsion by Lemma 3.8 and so has the \( R_\infty \) property by Corollary 3.7.

There are also groups which are not torsion and have no infinite cycles. Consider an element \( \rho \in \text{Sym}(X) \) with \( \eta_\rho(\rho) \) non-zero for infinitely many \( r \in \mathbb{N} \). Then \( \rho \) has infinite order, but need not contain an infinite cycle. Therefore \( \rho \) generates an infinite cyclic group, but \( \langle \rho, \text{FSym}(X) \rangle \) is not finitely generated. This is an interesting example since \( \text{FSym}(\mathbb{Z}) \times \mathbb{Z} \), which also consists of the group \( \text{FSym} \) together with a single element of infinite order, is 2-generated (being the second Houghton group \( H_2 \)). For another example, consider \( G = \prod_{i \in \mathbb{N}} C_2 \). This can be seen as a subgroup \( G_1 \) of \( \text{Sym}(\{1, 2\} \times \mathbb{N}) \) where the \( i \)th \( C_2 \) transposes the points \( (1, i) \) and \( (2, i) \) and fixes all other points of \( \{1, 2\} \times \mathbb{N} \). Now \( \prod_{i \in \mathbb{N}} C_2 \cong \bigoplus_{i \in \mathbb{N}} C_2 \) (both are vector spaces of rank \( |\mathbb{R}| \) over \( \mathbb{F}_2 \)) and so \( G \) can also be seen as a subgroup \( G_2 \) of \( \text{Sym}(\{1, 2\} \times \mathbb{R}) \) with generators \( g_i \) (for each \( i \in \mathbb{R} \)) that transpose the points \( (1, i) \) and \( (2, i) \) and fix all other points of \( \{1, 2\} \times \mathbb{R} \). But \( \langle G_1, \text{FAlt}(\{1, 2\} \times \mathbb{N}) \rangle \neq \langle G_2, \text{FAlt}(\{1, 2\} \times \mathbb{R}) \rangle \), since \( \langle G_1, \text{FAlt}(\{1, 2\} \times \mathbb{N}) \rangle \leq \text{Sym}(\{1, 2\} \times \mathbb{N}) \) and \( \text{FAlt}(\{1, 2\} \times \mathbb{R}) \) does not embed into \( \text{Sym}(\{1, 2\} \times \mathbb{N}) \) by \cite{BH15}.

Corollary 3.11. There exist uncountably many countable torsion groups which have the \( R_\infty \) property.

Proof. We will work within \( \text{Sym}(\mathbb{N} \times \mathbb{N}) \). For each \( n \geq 2 \), define \( \phi^{(n)} : C_n \hookrightarrow \text{Sym}(\mathbb{N} \times \mathbb{N}), (1 \ldots n) \mapsto \rho_n \) where \( \text{supp}(\rho_n) = \{ (m, n) | m \in \mathbb{N} \} \) and \( (m, n)\rho_n := \begin{cases} (m - n + 1, n) & \text{if } m \equiv 0 \mod n \\ (m + 1, n) & \text{otherwise} \end{cases} \).
i.e. \( \rho_n \) consists of \( n \)-cycles ‘all the way along’ the \( n \)-th copy of \( \mathbb{N} \).

Let \( \mathbb{P} \) denote the set of all prime numbers. Then, for any subset \( S \subseteq \mathbb{P} \), let \( G_S := \bigoplus_{p \in S} C_p \). Note that there are uncountably many choices for \( S \). Also,
\[
\bigoplus_{p \in S} C_p \hookrightarrow \text{Sym}(\mathbb{N} \times \mathbb{N})
\]
by using the maps \( \phi^{(n)} \) defined above. For any \( S \subseteq \mathbb{P} \), let \( \hat{G}_S := \langle G_S, \text{FAlt}(\mathbb{N} \times \mathbb{N}) \rangle \), which fully contains \( \text{FAlt}(\mathbb{N} \times \mathbb{N}) \) and, by Lemma 3.8, is torsion. Hence Corollary 3.7 applies to \( \hat{G}_S \) and it has the \( R_\infty \) property. Our final aim is therefore to show that if \( S \neq S' \), then \( G_S \) and \( G_{S'} \) are not isomorphic. By Proposition 2.5, \( G_S \) and \( G_{S'} \) each have \( \text{FAlt}(\mathbb{N} \times \mathbb{N}) \) as a unique minimal normal subgroup. Since \( G_S \) and \( G_{S'} \) contain no non-trivial elements of finite support,
\[
\hat{G}_S/\text{FAlt}(\mathbb{N} \times \mathbb{N}) \cong G_S \quad \text{and} \quad \hat{G}_{S'}/\text{FAlt}(\mathbb{N} \times \mathbb{N}) \cong G_{S'}.
\]
Hence if \( \hat{G}_S \) and \( \hat{G}_{S'} \) are isomorphic, then \( G_S \) and \( G_{S'} \) are isomorphic. But since \( S \neq S' \), there is a \( p \in \mathbb{P} \) in one set that is not in the other. Without loss of generality let \( p \in S \setminus S' \). By construction, \( G_S \) has \( p \)-torsion but \( G_{S'} \) does not. Hence \( \hat{G}_S \not\cong \hat{G}_{S'} \).

**Corollary 3.12.** There exist uncountably many finitely generated torsion groups which have the \( R_\infty \) property.

**Proof.** In [Ols82] the Tarski monsters, an uncountable family of finitely generated infinite \( p \)-groups, are described. Let \( M_1 \) and \( M_2 \) be non-isomorphic Tarski monsters. For any group \( G \), let \( \hat{G} \) denote the right regular representation of \( G \) and let \( \hat{G} := \langle \hat{G}, \text{FSym}(G) \rangle \). By Lemma 3.8, \( \hat{M}_1 \) and \( \hat{M}_2 \) are torsion. By [HO16, Prop 5.10], \( \hat{M}_1 \) and \( \hat{M}_2 \) are finitely generated. Moreover \( \hat{M}_1 \not\cong \hat{M}_2 \) since they are both monolithic (by Proposition 2.5) but if we quotient by this unique minimal normal subgroup then we obtain non-isomorphic groups. \( \Box \)

There are many equivalent definitions of the following.

**Definition 3.13.** A group \( G \) is residually finite if for each non-trivial element \( g \in G \) there exists a finite group \( F_g \) and a homomorphism \( \phi_g : G \to F_g \) such that \( (g)\phi_g \neq 1 \).

It is the following well known reformulation that shall be of use to us.

**Lemma 3.14.** A group \( G \) is residually finite if and only if it can be embedded inside the direct product of a family of finite groups. Moreover the family comprises of the finite quotients of \( G \).

**Corollary 3.15.** Let \( G \) be an infinite residually finite group, and \( X \) be the union of the finite quotients of \( G \). Then, using this action, \( \langle G, \text{FAlt}(X) \rangle \) has the \( R_\infty \) property.

**Proof.** Since \( G \) is residually finite, it can be embedded inside the direct product of a family of finite groups (which are those groups appearing as finite quotients of \( G \)). Therefore any element \( g \in G \) has only finite orbits, and by Remark 3.9, any element in \( \langle G, \text{FSym}(X) \rangle \) also only has finite orbits. Hence Corollary 3.6 applies, and \( \langle G, \text{FSym}(X) \rangle \) has the \( R_\infty \) property. \( \Box \)
4. The $R_\infty$ Property and Commensurable Groups

This final section involves results for commensurable groups.

**Notation.** Let $N \leq_f G$ denote that $N$ is normal and finite index in $G$.

**Definition 4.1.** Let $G$ and $H$ be groups. We say that $G$ is commensurable to $H$ if and only if there exist $N_G \cong N_H$ with $N_G \leq_f G$ and $N_H \leq_f H$.

We will work towards Theorem 4.6 which applies to the Houghton groups, a family of groups $H_n$ indexed over $\mathbb{N}$ where, for each $n \in \mathbb{N}$, $H_n$ acts on a set $X_n$ and $\text{FSym}(X_n) \leq H_n \leq \text{Sym}(X_n)$. Each group $H_n$ therefore fully contains $\text{FAlt}(X_n)$. These were first introduced in [Hou78], but we rely heavily on [Cox17] where an introduction to these groups can be found and a description, for all $n \geq 2$, of the structure of the automorphism group for all finite index subgroups of $H_n$ is given. We start with three well known results.

**Lemma 4.2.** If $H \leq_f G$, then $\exists N \leq_f H$ which is normal in $G$.

*Proof.* Let $H$ have index $n$ in $G$ and let $N := \bigcap_{g \in G} (g^{-1}Hg)$. Then $G$ acts on $H \setminus G$ by right multiplication, and so there is a homomorphism $\phi : G \to S_n$. Now $h \in \ker(\phi)$ if, $Hgh = Hg$ for all $g \in G$ $\Leftrightarrow ghg^{-1} = H$ for all $g \in G$ $\Leftrightarrow h \in g^{-1}Hg$ for all $g \in G$.

Hence $\ker(\phi) = N$ and $N$ is normal. Moreover $G/\ker(\phi) \cong \text{Im}(\phi) \leq S_n$, and so $N$ has index $m$ in $G$ where $m \leq n!$ and $m$ divides $n!$. $\square$

**Lemma 4.3.** If $H \leq_f G$ and $G$ is finitely generated, then $\exists K \leq_f H$ which is characteristic in $G$.

*Proof.* Suppose $H \leq_n G$. We first show that there exist only finitely many subgroups of $G$ of a given index. As in the previous lemma, right multiplication by $G$ on $H \setminus G$ gives a homomorphism $\phi_H : G \to S_n$. Note that $\text{Stab}(H) = H$ since $g \in \text{Stab}(H) \Leftrightarrow Hg = H$. Thus, by choosing $1 \in \mathbb{Z}_n$ to correspond to the coset $H$ in $H \setminus G$, the preimage of $\text{Stab}(1)$ in $S_n$ is $H$. Hence $H = H' \Leftrightarrow \phi_H = \phi_{H'}$.

But $G$ finitely generated $\Rightarrow \exists$ only finitely many homomorphisms $G \to S_n$ (there are $(n!)^{|S|}$ maps from $S$ to $S_n$) and so there can only be finitely many index $n$ subgroups. Now let

$$K := \bigcap_{\phi \in \text{Aut}(G)} (H)\phi$$

and note that, for any $\phi \in \text{Aut}(G)$, $(H)\phi \leq_n G$. But there are only finitely many possible images for $H$ in (4), and so (since the intersection of finitely many subgroups of finite index is of finite index) $K$ is finite index in $G$. Finally, $K$ is characteristic in $G$ since the image of $K$ under $\psi \in \text{Aut}(G)$ is contained within

$$\bigcap_{\phi \in \text{Aut}(G)} ((H)\phi \psi)$$

which is equal to $K$. $\square$
Lemma 4.4. [MS14, Lem 2.2(ii)] Let $D$ be a group with the $R_\infty$ property and

$$1 \rightarrow D \rightarrow E \rightarrow F \rightarrow 1$$

be a short exact sequence of groups. If $D$ is characteristic in $E$ and $F$ is any finite group, then $E$ has the $R_\infty$ property.

Combining the previous two results provides an easier condition to check in order to show that all commensurable groups have the $R_\infty$ property.

Lemma 4.5. Let $G$ be a finitely generated group. If $G$ and all finite index subgroups of $G$ have the $R_\infty$ property, then all groups commensurable to $G$ have the $R_\infty$ property.

Proof. Let $H$ be commensurable to $G$. Then $\exists\ N \trianglelefteq_f G, H$. By Lemma 4.3 there exists a group $U$ which is characteristic in $H$ and such that $U \trianglelefteq_f G, H$. From our assumption that all finite index subgroups of $G$ have the $R_\infty$ property, $U$ has the $R_\infty$ property. Hence, by Lemma 4.4, $H$ has the $R_\infty$ property. □

Our final aim is the following. Although it was done independently, our argument has similarities to [GP16, First proof of Thm. 3.8]. Their argument produces elements of different orders, whilst we produce elements of different cycle types. The flexibility that this affords allows our arguments to generalise from $H_n$ to certain subgroups $U_p \leq H_n$.

Theorem 4.6. Let $n \in \mathbb{N}$. If $G$ is any group commensurable to $H_n$, the $n^{th}$ Houghton group, then $G$ has the $R_\infty$ property.

Proof. We first work with $\text{FAlt}$. If $G$ is commensurable to $\text{FAlt}(X)$, then there exists $N \leq_f \text{FAlt}(X), G$. Now, since $\text{FAlt}(X)$ is simple and infinite, $N = \text{FAlt}(X)$. Hence we have the short exact sequence

$$1 \rightarrow \text{FAlt}(X) \rightarrow G \rightarrow F \rightarrow 1$$

where $F$ is some finite group. Let $\phi \in \text{Aut}(G)$ and consider $\text{FAlt}(X) \cap (\text{FAlt}(X))\phi$. This has finite index in $\text{FAlt}(X)$. Using Lemma 4.2 and that $\text{FAlt}(X)$ is simple, we have $(\text{FAlt}(X))\phi = \text{FAlt}(X)$ i.e. that $\text{FAlt}(X)$ is characteristic in $G$. Since $\text{FAlt}(X)$ is torsion, Corollary 3.7 states that it has the $R_\infty$ property. Hence Lemma 4.3 applies to $G$ implying that $G$ has the $R_\infty$ property.

We now work with $n \geq 2$. From Lemma 4.5 it is sufficient to show that, for any $n \geq 2$, all finite index subgroups of $H_n$ have the $R_\infty$ property.

Fix an $n \geq 2$. There are a family of finite index, characteristic subgroups of $H_n$ defined in [BCM16] and denoted $U_p$ where $p \in \mathbb{N}$. They showed that, for any $U \leq_f H_n$, there exists a $p \in \mathbb{N}$ such that $U_p \leq_f U$. This was strengthened in [Cox17, Prop. 5.12] by showing that, for any $U \leq_f H_n$, there exists an $m \in \mathbb{N}$ such that $U_m \leq_f U$ and

$$\text{Aut}(U) \Psi \cong N_{\text{Sym}(X_n)}(U) \leq N_{\text{Sym}(X_n)}(U_m) \cong \Psi \text{Aut}(U_m)$$

where $\Psi : N_{\text{Sym}(X_n)}(G) \rightarrow \text{Aut}(G)$ is defined by $(g)\Psi = \phi_g$. Furthermore, by [Cox17, Lem. 5.9], there is a monomorphism $\mu : N_{\text{Sym}(X_n)}(U_m) \rightarrow N_{\text{Sym}(H_{nm})}(H_{nm})$ and, for any $k \geq 2$, $N_{\text{Sym}(X_n)}(H_k) = H_k \times S_k$. Importantly, this monomorphism preserves cycle type. We shall apply Proposition 3.3 to show that any group with automorphism group contained within $N_{\text{Sym}(H_k)}(H_k)$ for some $k \geq 2$ has the $R_\infty$ property.
Fix a $k \geq 2$. Notice that for all $r \in \mathbb{N} \setminus \{1\}$ and for all $g \in H_k$, $\eta_r(g)$ is finite. Given a $\rho \in H_k \rtimes S_k$, which is isomorphic to $\text{Aut}(H_k)$ via the map $\rho \mapsto \phi_\rho$, we have that $\eta_r(\rho)$ is infinite if and only if $\rho$ induces a cyclic permutation of $r$ branches of $X_k$. Thus, for all $\rho \in N\text{Sym}(X_k)(H_k)$ and all $g \in H_k$, $\eta_r(\rho)(g)$ is infinite if and only if $\rho$ induces a cyclic permutation of $r$ branches of $X_k$. Hence, for all $\rho \in N\text{Sym}(X_k)(H_k)$ and all $r > k$, we have that $\eta_r(\rho)$ is finite.

Now, for any $U \leq H_n$, there exists an $m \in \mathbb{N}$ such that $N\text{Sym}(X_n)(U) \leq N\text{Sym}(X_n)(U_m)$. Consider if $\rho \in N\text{Sym}(X_n)(U_m)$. Using the above homomorphism $\mu : N\text{Sym}(X_n)(U_m) \to N\text{Sym}(X_{nm})(H_{nm})$, we have that $\eta_r(\rho)\mu$ is finite for all $r > nm$. Since $\mu$ preserves cycle type, $\eta_r(\rho)$ is also finite for all $r > nm$. Hence, by Proposition [5.3], $R(\phi_\rho) = \infty$ and so all automorphisms of $U$ have infinite Reidemeister number. Thus all finite index subgroups of $H_n$ have the $R_\infty$ property and so Lemma [4.5] yields the result. □

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