On Holographic Thermalization and Dethermalization of Quark-Gluon Plasma

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ABSTRACT: We estimate the ratio of the thermalization time over the freeze-out time using a holographic AdS-Vaidya model for quark-gluon plasma formed in the heavy ion collisions. In the model the process of thermalization is described as formation of the black hole in AdS space while dethermalization process, related with the freeze-out, as the black hole evaporation due to the Hawking radiation that is modeled by the Vaidya metric with a negative mass. In this model the thermalization takes place only at small scales and absent in the infrared region. At small scales the system tends towards a state of the thermal equilibrium only for the short time after which the processes of dethermalization starts. In this simple model the dethermalization time has a low bound about 7 fm/c which is consistent with experimental data.

KEYWORDS: Holography, thermalization, AdS/CFT.
Contents

1. Introduction 2
2. Holographic Thermalization with AdS-Vadiya metric 3
3. The Hawking Radiation and Dethermalization 4
4. Bounds on Thermalization and Dethermalization Times 7
5. Conclusion 10
A. Hawking radiation and nonequal time correlators. 11
1. Introduction

In the process of collision of two ions one observes a rapid local thermalization of the system of partons with further expansion and cooling of system which leads to hadronization and multiple production of particles. Experiments indicate to a very short thermalization time, $\tau_{\text{therm}} \sim 1 \text{ fm/c}$ for the quark-gluon plasma (QGP) formed in heavy ion collisions, while the freeze-out time is of order $20 \text{ fm/c}$ [1, 2, 3, 4, 5, 6].

In more details, the time schedule of the process is the following. Up to a time $\sim 0.02 \text{ fm/c}$ ”hard” processes take place and they are responsible for ”hard” particles, which can be observed at detectors. Up to a time $\sim 0.2 \text{ fm/c}$ ”semi-hard” processes take place and they produce the most of the ”multiplicity” in the final state. Then at the thermalization time of order $\tau_{\text{therm}} \sim 1 \text{ fm/c}$ the system reaches a local thermal equilibrium state, called QGP. After that the evolution of QGP is described by equations of hydrodynamics and after the time of order $\tau_{\text{hadr}} \sim 10 \text{ fm/c}$, when due to the separation of the colliding ions the temperature becomes lower than the deconfinement temperature, a hot hadron gas is formed. Upon the further expansion and cooling, around the freeze-out time, we refer to this time as the dethermalization time $\tau_{\text{det}} \sim 20 \text{ fm/c}$, the density of the hadron gas become sufficiently low and the system decays into free hadrons, which can be observed at detectors. Therefore in experiments on heavy ion collisions there is the following hierarchy of time scales:

$$\tau_{\text{therm}} < \tau_{\text{hydro}} < \tau_{\text{hadr}} < \tau_{\text{det}}. \quad (1.1)$$

There are many attempts to describe the process of heavy ion collisions and QGP formation in the QCD framework. One of difficulties is that one has to compute the time depended correlation functions in the strong coupling regime since the QGP is a strong coupling system [2]. In the recent years a powerful approach to these problems is pursued which is based on a holographic duality between the strong coupling quantum field in $d$-dimensional Minkowski space and classical gravity in $d + 1$-dimensional anti-de Sitter space (AdS) [7, 8, 9]. In particular, there is a considerable progress in the holographic description of equilibrium QGP [10].

Holographic thermalization describes the thermalization process as a process of formation of a black hole in AdS. There are several scenarios to produce a black hole in AdS [11]-[31]. One of them uses the AdS-Vaidya metric [32, 33, 28, 26, 27, 29]. In this paper we consider an influence of the Hawking radiation on the thermalization process. We describe the evaporation of black hole due to the Hawking radiation by the Vaidya metric with negative mass. We use a special form of the AdS-Vaidya metric, see eq. (4.1) below, to describe thermalization and subsequent dethermalization. We show that the evaporation of the black hole in AdS leads to an interesting phenomena in the $d$-dimensional Minkowski space-time – thermalization is possible only at small distances and impossible in the infrared region.
The Hawking radiation from a Vaidya black hole was considered in [34] where it was shown that allowance for the nonstationarity reduces to allowing for the dependence of the temperature of the Hawking radiation on the retarded time. Note that the Hawking radiation in holographic approach was also discussed in [35].

The paper is organized as follows. In Sect. 2 we remind the basic formulas related with the AdS-Vaidya metric which are used for the holographic thermalization [26]-[29]. Sect.3 discusses the AdS-Vaidya metric with negative mass to model the Hawking radiation. We show that thermalization for the two-point correlation functions holds only at small distances. In Sect. 4 we discuss attempts to fit the time hierarchy (1.1) obtained in Pb+Pb collisions at LHC and in Au+Au collisions at RIHC, by a special form of the AdS-Vaidya metric. In Appendix we present an explicit formula for a non-equal time two points correlation function for the de-thermalization process. This formula is a counterpart of non-equal time two points correlation functions for the thermalization process found in [26]-[29].

2. Holographic Thermalization with AdS-Vaidya metric

Holographic duality prescribes that the vacuum correlation functions in quantum field theory in the strong coupling regime in 4-dimensional Minkowski space can be computed by using the action functional for corresponding fields in \((d+1)\)-dimensional AdS [7, 8, 9]. Thermal states also admit a holographic description by using a black hole AdS metric [9, 38]. The holographic description of equilibrium QGP was tested in numerical works, see the review paper [10] and refs therein.

Hypothesis on holographic thermalization asserts that the thermalization process can be described in the dual framework as the process of formation of a black hole in AdS. The process of formation of a black hole can be initiated by a perturbation of the initial AdS metric. Formations of black holes in gravitational collapse or in the collision of ultra-relativistic particles are traditional difficult problems in general relativity, see for example [37] and refs therein. In AdS there are specific features related with the compression of AdS metric. In the context of holographic description of heavy-ions collisions formation of black holes in collision of gravitational waves in AdS has been considered in numerous works [11, 14, 13, 15, 17, 16, 18, 19]. Gravitational collapse occurring as a result of the weak perturbation in AdS is considered in [20], where it is shown that the system evolves to the state of thermal equilibrium almost instantly. This result stimulates application of the AdS-Vaidya metric for the thin shell in the holographic description of thermalization [32, 33, 28, 26, 27, 24].

The \((d+1)\)-dimensional infalling matter shell in AdS in Poincaré coordinates is described by the Vaidya metric

\[
ds^2 = \frac{1}{z^d} \left[ - (1 - m(v)z^d) \, dv^2 - 2 \, dv \, dz + dx^2 \right] ,
\]  

(2.1)
where \( v \) is the null coordinate, \( x = (x_1, \ldots, x_{d-1}) \) are the spatial coordinates on the boundary \( z = 0 \) and we have set the AdS radius equal to 1. We take \( m(v) \) in the form

\[
m(v) = M \theta(v),
\]

where \( M \) is a constant and \( \theta(v) \) is the Heaviside function.

For \( m(v) = M \) the change of variables

\[
dv = dt - \frac{dz}{1 - M z^d}
\]

brings (2.1) to the standard metric of the black hole in AdS in the Poincare coordinates

\[
ds^2 = \frac{1}{z^2} \left[- (1 - M z^d) dt^2 + \frac{dz^2}{1 - M z^d} + dx^2 \right].
\]

For \( v < 0 \) the metric (2.1) with \( m(v) \) in the form (2.2) is just the AdS metric.

The holographic prescription permits to calculate an equal-time two-point function of operators \( O(x, t) \) with large conformal dimension \( \Delta \) in dual theory of gravity by using the geodesic approximation [36]:

\[
\langle O(x, t) O(0, t) \rangle \sim \exp[-\Delta \mathcal{L}(x, t)].
\]

Here \( \mathcal{L}(x, t) \) is the length of the geodesic that begins and ends at the boundary points \((0, t)\) and \((x, t)\). The vacuum correlation functions correspond to the computation of geodesics in AdS, thermal ones in AdS with a black hole (\( BH AdS \)) while correlation functions describing the process of thermalization correspond to geodesics in the AdS-Vaidya (2.1) with (2.2). These geodesics are studied in papers [33, 28, 26, 27, 29]. These considerations give an estimation of the thermalization time for two equal time points at the boundary. In particular, for \( d = 2 \) the thermalization time is equal to the half of the space distance between these points. For \( d > 2 \) the thermalization time decreases. The Vaidya Reissner-Nordstrom AdS metric increases the thermalization time in all dimensions [30, 31].

3. The Hawking Radiation and Dethermalization

In this section we consider the Hawking radiation in the holographic approach by using the AdS-Vaidya metric and show that the Hawking radiation prevents to thermalization at large distances. In other words, the Hawking radiation in AdS produces the dethermalization in dual QGP in the infrared region. To describe the Hawking radiation we use the AdS-Vaidya metric with negative energy (with negative mass function), compare with [34], where the Hawking radiation has been studied for the Minkowski Vaidya metric and see [29] where the AdS-Vaidya with negative mass
has been considered. The process of the black hole Hawking radiation in this model corresponds to

\[ m(v) = M - M\theta(v), \]  

(3.1)

and is presented in Fig. 1.

**Figure 1:** Cartoon of the black hole evaporation: A. The geodesic connected points ±l/2 is totally in the black hole region. B. A partial evaporation of the black hole from the point of view of two points ±l/2 correlator, i.e. the geodesic is partially in the black hole region. C. The total evaporation of the black hole from the point of view of this correlator, i.e. the geodesic totally abandons the black hole region. D. The geodesic is totally in the empty region in all subsequent momenta of time.

The evolution of the 2-point correlation function (3.3) corresponding to the black hole evaporation process can be written explicitly in the case \( d = 2 \), see (3.2), (3.3) below. These formulas are similar to the evolution formulas describing the black hole formation (26). The formulas are obtained by combining relations between the change of the value of the affine parameter and the value of the change of the coordinate \( x \) at the different parts of the geodesic. These relations have the form

\[ l = l_{AdS,1} + l_{AdS,2} + l_{BHAdS,1} + l_{BHAdS,2} \]

\[ = \frac{4}{p_x r_c} \left( r_c - \sqrt{r_c^2 - p_x^2} \right) - 2p_x^2 \ln \left( \frac{r_c^2 (2 - r_c^2 G_+ + 2 \sqrt{F(r_c)})}{r_c^2 (2 - r_c^2 G_+ + 2 \sqrt{F(r_c)})} \right) \]  

(3.2)

and

\[ L_{ren} = \delta L_{ren} + L_{AdS,1} + L_{AdS,2} + L_{BHAdS,2} + L_{BHAdS,2} \]

\[ = -4 \log(r_c + \sqrt{r_c^2 - p_x^2}) + \ln \left( \frac{-(p_x^2 + 1 - E_B^2) + 2r_c^2 + 2 \sqrt{D(r_c)}}{-(p_x^2 + 1 - E_B^2) + 2r_c^2 + 2 \sqrt{D(r_c)}} \right) \]  

(3.3)

Here \( ren \) means the renormalized length (compare with the action renormalization considered in [34]) and the following notations are used

\[ G_+ = p_x^2 + 1 - E_B^2, \quad G_- = -p_x^2 + 1 + E_B^2 \]  

(3.4)

\[ D(r) = r^4 + (E_B^2 - 1 - p_x^2)r^2 + p_x^2 \]  

(3.5)

\[ F(r) = p_x^2 r^4 - (p_x^2 + 1 - E_B^2 p_x^2)r^2 + 1 \]  

(3.6)
The above formulas correspond to the case when in the empty AdS space the geodesic has zero energy (the case of nonzero energy corresponds to non-equal time correlator), and the energy in the black space (in BHAdS space) is defined from the refraction condition

\[
E_B = -\frac{1}{2r_c^2} \sqrt{r_c^2 - p_x^2}.
\]  

(3.7)

Here \( r_c \) is the coordinate of the crossing point, \( p_x \) is the angular momentum that is the integral of motion and has no a junction under crossing the geodesic (as opposed to the energy, that has according to (3.7) a junction under a crossing of the shell), \( r_t \) is the turning point of the geodesic. There is a relation

\[
I_{t+}^2 = \frac{(1 + p_x^2 - E_B^2) \pm \sqrt{(1 + p_x^2 - E_B^2)^2 - 4p_x^2}}{2}
\]  

(3.8)

Let us clarify the meaning of (3.2), (3.3), (3.7) and (3.8). Here \( r_c \) is a free parameter that specifies the position of the shell, \( p_x \) is a parameter that does not change in shell moving, \( E_B \) is given by (3.7), and \( r_t \) given by formula (3.8). Therefore, under \( r_c \) and \( p_x \) fixed, formulas (3.2) and (3.3) give the relation between renormalized geodesic length \( L_{ren} \) and the distance \( l \).

The process of the black hole creation and subsequent radiation within our model is presented in Fig. 2. The metric of this process is given by the formula (2.1) with \( m(v) \):

\[
m(v) = M\theta(v) - M_1\theta(v - v_1),
\]  

(3.9)

The case of the total dethermalization corresponds to \( M = M_1 \) and will be considered in what follows. If the distance \( l \) is much less then \( v_1 \), then at the given moment of time the geodesic cross the shell no more than 2 times. This case is schematically presented in Fig. 2.

\[\text{Figure 2: Cartoon of the black formation and the subsequent black hole evaporation: transition from A to B shows the black hole creation; C shows the black hole evaporation and D shows the total black hole evaporation.}\]

As we can see from the figures, for \( 0 < t < v_1 \) the length of the geodesic connected the points \((\pm l/2, t_0, z_0)\) is given by formula from (2.4). We assume that here \( t_{therm} < v_1 \). Then the length of the geodesic does not change in the interval...
$t_{\text{therm}} < t < v_1$, Fig 2B. Starting from $v_1 = t$ the geodesic crosses the shell with the negative mass (the magenta line in Fig. 4, C) at 2 points, and the length of the geodesic is given by formulas (3.2) and (3.3). As it can be is seen from Fig. 2D, starting from $t \geq v_1 + t_{\text{det}}$ the geodesic related the points $(\pm \ell/2, t_0, z_0)$ occurs totally outside the black hole and the geodesic length will be given by formula $\delta \mathcal{L} = 2 \ln(\ell/2)$.

For large $\ell$ the geodesic cross the shell at four points, see Fig. 3, and the length of the geodesic is given by a more complicated formula, but in any case at large times the length does not change and is given by $\delta \mathcal{L} = 2 \ln(\ell/2)$, i.e. the thermalization does not occur.

**Figure 3:** Thermalization process does not occur for a long distance correlations: B. and C. show that the black hole does not have enough time to form, since the evaporation process has already started; A, D as in Fig. 2

### 4. Bounds on Thermalization and Dethermalization Times

In this section we establish bounds on thermalization and dethermalization times in the AdS-Vadiya holographic model.

Let us consider the AdS-Vadiya metric (2.1) with

$$m(v) = M(\theta(v) - \theta(v - v_1)), \quad (4.1)$$

where $M > 0$ and $v_1$ is larger than the thermalization time, $v_1 > t_{\text{therm}}$.

Let us suppose that the vector $\vec{\ell}$, characterizing the equal times two points on the boundary, at which the geodesic states and ends, has only one nonzero component, $l_1 \equiv \ell$, and denote $J_1 = J$. The thermalization time for the correlation function at for these two points is given by

$$\tau_{\text{therm}} = \int_J^\infty \frac{dr}{r^2(1 - \frac{M}{r^2})}, \quad (4.2)$$

here $J$ is the first component of the conserved angular momentum related with $\ell$

$$\ell = 2J \int_J^\infty \frac{dr}{r^2 \sqrt{(r^2 - J^2)(1 - \frac{M}{r^2})}}. \quad (4.3)$$
From these formulas we get that

\[ \frac{\tau_{\text{ther}}}{\ell} = F(m^2, d), \quad (4.4) \]

where \( m^2 = M/J^d \rho^d \) and \( F(m^2, d) \) is given by

\[ F(m^2, d) = \frac{\int_1^{\infty} \frac{dp}{\rho^2(1 - \frac{m^2}{\rho^d})}}{2 \int_1^{\infty} \frac{dp}{\rho^2 \sqrt{\rho^2 - 1}(1 - \frac{m^2}{\rho^d})}}. \quad (4.5) \]

Now, in the 4-dimensional Minkowski space, i.e. for \( d = 4 \) for the function \( F(m^2, 4) \) as a function of \( m \), \( 0 < m < 1 \), we have the bound

\[ 0.39 \leq F(m^2, 4) \leq 0.5, \quad (4.6) \]

see Fig.4. Therefore we obtain the bound for the thermalization time (d=4)

\[ 0.39 \leq \frac{\tau_{\text{ther}}}{\ell} \leq 0.5, \quad (4.7) \]

The similar bounds take place for other \( d > 2 \) [26, 28, 29].

The dethermalization time, under assumption that \( v_1 > \tau_{\text{th}} \) and for the same points on the boundary, is defined by the formulas (4.2) and (4.3) with \( M = 0 \). Since \( F(0, d) = 1/2 \) we have

\[ \frac{\tau_{\text{det}}}{\ell} = \frac{1}{2} \quad (4.8) \]

Note, that (4.8) does not depend on the space time dimension \( d \).

From (4.7), (4.8) we obtain the following relation between thermalization and dethermalization times for observables at the same distance:

\[ 0.78 < \frac{\tau_{\text{ther}}}{\tau_{\text{det}}} < 1 \quad (4.9) \]

We see that this ratio is universal and does not depend on the distance between two points till the distance is less then \( v_1 \).

Therefore, the minimal ratio of thermalization time to dethermalization time, that can be realized in the \( d = 4 \) AdS-Vadiya model is 0.78. Increasing \( d \) one gets a possibility to decrease this ratio, see 4.B. As it has been noted in [30, 31] involving the nonzero chemical potential one increases the ratio of \( \tau_{\text{therm}} \) to \( \ell \), and therefore, in our model, this increases \( \tau_{\text{therm}}/\tau_{\text{der}} \). One can also try to add by hands an effective locking potential, for example, the quadratic one. This corresponds to a change \( (1 - m^2/\rho^d) \to (1 - m^2/\rho^d + q\rho^2) \) in (4.3). This locking potential decreases the ratio \( F(m, q, 4) = \tau_{\text{therm}}/\tau_{\text{der}} \), see Fig.4.C.

It is known that the experimental data on heavy ion collisions determine the thermalization time as \( \tau_{\text{ther}} \sim 1 \text{ fm/c} \) and the dethermalization time as \( \tau_{\text{det}} \sim 10 - 20 \).
Figure 4: A. The plot of $F(m^2, 4)$ as function of $m$. B. The plot shows the dependence of the below bound of $F(m^2, d)$ on dimension $d$. $d=2$ corresponds to the red line, $d=4,6,8$ to blue, green and magenta lines, respectively. C. The plot of $F(m^2, q, 4)$ as function of $q$ and $m^2 = 0.99$.

fm/c, so $\tau_{\text{ther}}/\tau_{\text{det}} \sim 0.1 - 0.05$ . It seems at the first sight that it is difficult to explain this data using our model. This is in fact so, if one thinks that thermalization and the dethermalization have to be happen at the same scale, but it is not so if the scales of thermalization and the dethermalization are different. For the thermalization time, the relevant length scale according [26] can be taken about $\ell \sim 0.6 \text{ fm}$, that is the thermal scale $l \sim h/T$ for the temperature value $T \sim 300 - 400 \text{ MeV}$ at heavy ion collider energies, and one obtains the estimate $\tau_{\text{therm}} \sim 0.3 \text{ fm/c}$, which is smaller then the experimental data.

One can fit better the experimental data using the scale $l \sim 2 \text{ fm}$. One of possible explanations of this scale is a classical estimation of the distance between the nucleons inside the nucleus. One gets this estimation by taking into account that the radius of the nucleus of Pb is about $r_{\text{Pb}} \approx 7 \text{ fm}$, and in the sphere with this radius one can pack 208 ($A=208$ for Pb) balls with radius

$$r_n = \frac{3}{208} \sqrt[3]{\frac{\eta_K}{208}} r_{\text{Pb}} \approx 1.07 \text{ fm}. \quad (4.10)$$

Here $\eta_K$ is the Kepler number $\eta_K = \pi/\sqrt{18} \approx 0.74$. From this consideration it seems natural to take $l = 2r_n \sim 2 \text{ fm}$ as a typical scale of the thermalization. In this case one has $\tau_{\text{therm}} \sim 1 \text{ fm/c}$.

For an estimation of the dethermalization time, one can take as the typical scale the size of the nucleus, i.e. $l_{\text{det}} \sim 2r_{\text{Pb}} \sim 14 \text{ fm}$. Then one gets the dethermalization time $\tau_{\text{det}} \sim 7 \text{ fm/c}$. Note that this estimation gets a low bound, since in the model we have the free parameter $v_1$. For the ratio one gets

$$\frac{\tau_{\text{ther}}}{\tau_{\text{det}}} = \frac{\tau_{\text{ther}}}{0.5 \cdot l_{\text{ther}}} \cdot \frac{l_{\text{ther}}}{l_{\text{det}}} = 0.39 \cdot \frac{2}{14} \approx 0.056, \quad (4.11)$$

which is in agreement with the experimental data.
5. Conclusion

In this paper we have used the holographic approach to study the processes of thermalization and dethermalization in strongly coupled field theories relevant to the QGP formation in the heavy ion collisions.

We show that the evaporation of black hole due to the Hawking radiation, which is modeled by the AdS-Vaidya metric with negative energy, leads to an interesting phenomena in the dual theory – thermalization is possible only at small distances and impossible in the infrared region.

We have shown that for the ratio of the thermalization to the dethermalization time, considering at the the same length scale, has the universal bound and does not depend on the scale. The assumption that thermalization and dethermalization take place at the same scale does not fit well enough with experimental data. However it is more reasonable to consider thermalization and dethermalization at different length scales, since thermalization is local, while dethermalization is more suitable to relate with largest space scale of the system under consideration. It is obtained that the ratio of thermalization time to dethermalization time is equal to 0.05 that is consistent with experimental data.

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A. Hawking radiation and nonequal time correlators.

Considerations in Sect.3 concern to the case $E = 0$ in the AdS space. Here we consider geodesic with nonzero energy in the AdS space. One of such geodesic is presented in Fig.5. This figure corresponds to the case when the shell has no time to reach the horizon.

![Figure 5: A projection of the three dimensional picture in coordinates $t, r, x$ of evolution of the shell describing the process for black hole formation. The picture contains a geodesic, related two point on the boundary with different times and with non-zero energy in the empty space. The green arrows show the time direction. The blue arrows indicate the time coordinates. The down thick line shows the shell position at the moment $t^\text{AdS} = t_1$, and the up thick line shows it at the moment $t^\text{AdS} = t_2$. A part of the geodesic with nonzero energy in the empty space is depicted by the dashed line and a part in the black hole region by the thick dashed line. geodesic abandon the plane $t = \text{const}$. The tick green segments show the time junctions under transition from $AdS$ to $BHAdS$.](image)

In this case we have

$$\ell = l_\text{AdS,1} + l_\text{AdS,2} + l_\text{BHAdS,1} + l_\text{BHAdS,2}$$

$$= \frac{2p_x}{(p_x^2 - E_1^2)} \left(1 - \sqrt{1 + \frac{E_1^2 - p_x^2}{r_1^2}}\right) + \frac{2p_x}{(p_x^2 - E_2^2)} \left(1 - \sqrt{1 + \frac{E_2^2 - p_x^2}{r_2^2}}\right) \quad (A.1)$$

$$- p_x^2 \ln \left(\frac{(2 - r_1^2 G_+ + 2 \sqrt{F(r_1)})(2 - r_2^2 G_+ + 2 \sqrt{F(r_2)})}{p_x^2 r_1^2 r_2^2}\right) \quad (A.2)$$

$$+ 2p_x^2 \ln \left(\frac{(2 - r_1^2 G_+ + 2 \sqrt{F(r_1)})}{p_x^2 r_1^2}\right) \quad (A.3)$$
\[ L_{AdS,1} + L_{AdS,2} + L_{BHAdS,2} + L_{BHAdS,2} + \text{reg.} = -2 \ln \left( r_1 + \sqrt{r_1^2 - (p_x^2 - E_1^2)} \right) \left( r_2 + \sqrt{r_2^2 - (p_x^2 - E_2^2)} \right) \]
\[ + \frac{1}{2} \ln \left( -\frac{1}{2} G_+ + r_1^2 + \sqrt{D(r_1)} \right) \left( -\frac{1}{2} G_+ + r_2^2 + \sqrt{D(r_2)} \right) \]
\[ - 2 \ln \left( -\frac{1}{2} G_+ + r_t^2 + \sqrt{D(r_t)} \right) \]

and

\[ T_{AdS,1} + \Delta t_1 + T_{BHAdS,2} + \Delta t_2 + T_{AdS,2} = \frac{2E_1}{(p_x^2 - E_1^2)} \left( 1 - \sqrt{1 + \frac{E_1^2 - p_x^2}{r_t^2}} \right) - \frac{1}{2} \ln \frac{|r_1 - 1|}{r_1 + 1} - \frac{1}{r_1} \]
\[ + \frac{2E_2}{(p_x^2 - E_2^2)} \left( 1 - \sqrt{1 + \frac{E_2^2 - p_x^2}{r_t^2}} \right) + \frac{1}{r_2} + \frac{1}{2} \ln \frac{|r_2 - 1|}{r_2 + 1} \]
\[ - \frac{E}{2E_B} \arctanh \left( \frac{1}{2} \frac{G_r^2 - G_+}{\sqrt{D(r_t)}} \right) - \frac{E}{2E_B} \arctanh \left( \frac{1}{2} \frac{G_r^2 - G_+}{\sqrt{D(r_t)}} \right) \]
\[ - \frac{E}{E_B} \arctanh \left( \frac{1}{2} \frac{G_r^2 - G_+}{\sqrt{D(r_t)}} \right) \]

\[ E_B = E_1 - \frac{E_1}{2r_1^2} - \frac{1}{2r_1^2} \sqrt{r_1^2 + E_1^2 - p_x^2} \]  
\[ E_B = E_2 - \frac{E_2}{2r_2^2} - \frac{1}{2r_2^2} \sqrt{r_2^2 + E_2^2 - p_x^2} \]

The turning point is now in BHAdS and it is satisfied

\[ r_t^2 = \frac{(1 + p_x^2 - E_B^2) \pm \sqrt{(1 + p_x^2 - E_B^2)^2 - 4p_x^2}}{2} \]

Here we use

\[ \Delta t_1 = \Delta_{AdS \rightarrow BHAdS} t(r_1) \equiv [t_{BHAdS} - t_{AdS}] = -\frac{1}{2} \ln \frac{|r_1 - 1|}{r_1 + 1} - \frac{1}{r_1} \]  
\[ \Delta t_2 = \Delta_{BHAdS \rightarrow AdS} t(r_1) \equiv [t_{AdS} - t_{BHAdS}] = \frac{1}{r_2} + \frac{1}{2} \ln \frac{|r_2 - 1|}{r_2 + 1} \]

It is clear that if \( r_1 = r_2 \), then \( \delta t = \Delta t_1 + \Delta t_2 = 0 \).
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