Marginally Stable Topologically Non-Trivial Solitons in the Gross-Neveu Model

Joshua Feinberg\textsuperscript{1,2,3}

\textsuperscript{1}Department of Physics, University of Haifa at Oranim, Tivon 36006, Israel.
\textsuperscript{2}Department of Physics, Technion, Haifa 32000, Israel.
\textsuperscript{3}Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan.

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We show that a kink and a topologically trivial soliton in the Gross-Neveu model form, in the large-$N$ limit, a marginally stable static configuration, which is bound at threshold. The energy of the resulting composite system does not depend on the separation of its solitonic constituents, which serves as a modulus governing the profile of the compound soliton. Thus, in the large-$N$ limit, a kink and a non-topological soliton exert no force on each other.

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The problem of finding the particle spectrum (e.g., bound states) of quantum field theory is a major objective of nonperturbative studies thereof. This issue may be addressed quantitatively in model field theories in 1 + 1 space-time dimensions such as the Gross-Neveu (GN) model\cite{1}, in the large $N$ limit. The GN model (and other similar models) are particularly appealing, since they exhibit, among other things, asymptotic freedom and dynamical mass generation, like more realistic four dimensional models.

One version of writing the action of the 1 + 1 dimensional GN model is

$$S = \int d^2x \sum_{a=1}^{N} \bar{\psi}_a (i\gamma^\mu \partial_\mu - \sigma) \psi_a - \frac{1}{2g^2} \sigma^2 ,$$  

(1)

where the $\psi_a (a = 1, \ldots, N)$ are $N$ flavors of massless Dirac fermions, with Yukawa coupling to the scalar auxiliary field $\sigma(x)$. This action is evidently symmetric under the simultaneous transformations $\sigma \rightarrow -\sigma$ and $\psi \rightarrow \gamma_5 \psi$, which generate the so-called discrete (or $\mathbb{Z}_2$) chiral symmetry of the GN model. The GN action has also flavor symmetry $O(2N)$, which can be seen by breaking the $N$ Dirac spinors into $2N$ Majorana spinors. Related to this is the fact that the model is also invariant under charge-conjugation $\mathbb{Z}_2$. Thus, focusing on bound states of the Dirac equation

$$[i\gamma^\mu \partial_\mu - \sigma(x)] \psi = 0$$  

(2)

associated with the GN action, if $\psi_b(x,t) = e^{-i\omega_b t} u_b(x)$ (with $0 \leq \omega_b^2 < m^2$) is a bound state solution of (2), so is its charge conjugate spinor $\bar{\psi}_b(x,t) = e^{i\omega_b t} \bar{u}_b(x)$, which has bound state frequencies in pairs: $\pm \omega_b$. If, however, (2) has a bound state at $\omega_b = 0$, it is of course unpaired, i.e., self-charge-conjugate.

Performing functional integration over the Grassmannian variables in the GN action leads to the partition function $Z = \int D\sigma \exp \{iS_{eff} [\sigma] \}$ where the bare effective action is

$$S_{eff} [\sigma] = -\frac{1}{2g^2} \int d^2x \sigma^2 - iN \text{Tr} \log (i\sigma - \sigma)$$  

(3)

and the trace is taken over both functional and Dirac indices.

The theory \cite{3} has been studied in the limit $N \rightarrow \infty$ with $Ng^2$ held fixed\cite{4}. In this limit the partition function $Z$ is governed by saddle points of (3) and the small fluctuations around them. In this Letter, as in \cite{2}, we will consider only the leading term in the $1/N$ expansion, and thus will not compute the effect of the fluctuations around the saddle points. The most general saddle point condition reads

$$\delta S_{eff} \over \delta \sigma(x,t) = -\frac{\sigma(x,t)}{g^2} + iN \text{Tr} \left[ (x,t) \frac{1}{i\sigma - \sigma} (x,t) \right] = 0 .$$  

(4)

In particular, the non-perturbative vacuum of the GN model is governed by the simplest saddle point of the path integral associated with it, where the composite scalar operator $\bar{\psi} \psi$ develops a space-time independent expectation value, signaling the dynamical breakdown of the discrete chiral symmetry by the non-perturbative vacuum. Thus, the fermions acquire mass $m$ dynamically.

Associated with this breakdown of the discrete symmetry is a topological soliton, the so-called Callan-Coleman-Gross-Zee (CCGZ) kink \cite{2,3,4}, $\sigma(x) = m \tanh(mx)$, with mass $M_{kink} = \frac{Nm}{\pi}$ ($m$ is the dynamically generated fermion mass). It is topology which insures the stability of these kinks: they are the lightest topologically nontrivial solitons in the GN model. The Dirac equation (2) in the kink background has a single self-charge-conjugate bound state at $\omega_b = 0$, which can populate at most $N$ valence fermions. Thus, it gives rise to a multiplet of $2^N$ degenerate states, with mass equal to $M_{kink}$, which can be identified as the (reducible) spinor representation of $O(2N)$\cite{5,6}. The expectation value of the fermion number operator $N_F$ in a state in which the kink traps $n$ valence fermions is $n - \frac{N}{2}$, due to fluctuations of the fermion field in the topologically nontrivial background. Thus, in the kink multiplet $-\frac{N}{2} \leq N_F \leq \frac{N}{2}$.

The GN model bears also non-topological solitons,
which were discovered by Dashen, Hasslacher and Neveu (DHN) (2) (after the work of 3). Henceforth, we shall refer to them as “DHN solitons”. These non-topological solitons are stabilized dynamically, by trapping fermions and releasing binding energy. In (2), DHN used inverse scattering analysis 7 to find static soliton solutions to the large-N saddle point equations of the GN model. (DHN also found in (2) oscillatory, time dependent solutions of the saddle point equations, which we will not discuss in this Letter.) The remarkable discovery DHN made was that all the physically admissible static, space-dependent solutions of (4), i.e., the static bag configurations in the GN model (the CCGZ kink being a non-trivial example of which) were reflectionless. That is, the static $\sigma(x)$’s that solve the saddle point equations of the GN model (subjected to the obvious boundary condition $\sigma(\pm\infty) = \pm m$) are such that the reflection coefficient of the Dirac equation (2) associated with the GN action, vanishes identically.

The Dirac equation (2) in a DHN soliton background has a pair of charge conjugate bound states at $\pm\omega_b$. The $O(2N)$ flavor symmetry mixes particles and antiparticles. At the level of the Dirac equation (2) this means that we have to consider the pair $\pm\omega_b$ of bound state eigenfrequencies together in the following way: Due to Pauli’s principle, we can populate each of the bound states $\pm\omega_b$ with up to $N$ (non-interacting) fermions. Then all the multi-particle states in which the negative frequency state is populated by $N - h$ fermions (i.e., has $h$ holes) and the positive frequency state contains $p$ fermions, with $h + p = n$ fixed are degenerate in energy, and thus form a $C_{2N}^p = n!(2N)!/(n!(2N - n)!)$ dimensional irreducible $O(2N)$ multiplet, namely, an antisymmetric tensor of rank $n$ (2). Superficially, $0 \leq n \leq 2N$, in accordance with Pauli’s principle. However, for dynamical reasons, as explained in Section 3 of (3), only solitons with $0 < n < N$ are realized. The expectation value of the fermion number operator $N_F$ in a state in which the DHN soliton traps $p$ particles and $h$ holes is simply $N_F = p - h = 2n$, i.e., purely the naive valence contribution. (There is no fractional contribution due to the trivial topology.) Thus, in the DHN soliton multiplet $-n \leq N_F \leq n$. DHN found that in this case $\omega_b = m \cos \left(\frac{\pi n}{2N}\right)$, the mass of such a soliton is $M_n = \frac{2N\pi}{\omega_b} \sin \left(\frac{\pi n}{2N}\right)$ and its profile is $\sigma(x) = \sigma(\infty) + \kappa \tanh [\kappa(x - x_0)] - \kappa \tanh \left[\kappa(x - x_0) + \frac{1}{2} \log \left(\frac{m + \kappa}{m - \kappa}\right)\right]$, where $\sigma(\infty) = \sigma(-\infty) = \pm m$, $\kappa = m \sin \left(\frac{\pi n}{2N}\right) = \sqrt{m^2 - \omega_b^2}$, and $x_0$ is a translational collective mode. Note that both $M_n$ and $M_{kink}$ are of order $N \sim \frac{1}{\sqrt{\epsilon}}$, as typical of soliton masses in weakly interacting QFT. The binding energy $B_n = nm - M_n$ of the DHN soliton (as well as the binding energy per-fermion, $B_n/n$) increase with the number $n$ of trapped valence fermions (this is the so-called “mattress effect” in the physics of fermion bags). Thus, a DHN soliton is stable against decaying into a bunch of non-interacting fundamental fermions. It is also stable against decaying into lighter DHN bags, because any such presumed process can be shown to violate either energy or fermion number conservation. Thus, DHN solitons are stable. Note that $M_{n=N} = \frac{2N\pi}{\omega_b} = 2M_{\text{kink}}$, and more over, that as $n \rightarrow N$ (i.e., $\kappa \rightarrow m$ and $\omega_b \rightarrow 0$), the profile $\sigma(x)$ tends to $\sigma(\infty) + m \tanh (mx) - m \tanh [m(x + R)]$, $R \rightarrow \infty$. Thus, the configuration at $n = N$ is that of infinitely separated CCGZ kink and anti-kink bound at threshold. This singular behavior occurs because a DHN soliton, being a topologically trivial configuration, cannot support a normalizable zero mode, as explained below.

The soliton solutions (both topological and non-topological) in the GN model serve as concrete calculable examples of fermion-bag 2 10 formation. Furthermore, these “multi-quark” bound states of the GN model are analogous to baryons in QCD in the limit of large number of colors 11. Since the work of DHN, these fermion bags were discussed in the literature several other times, using alternative methods 12. For a recent review on these and related matters, see 13.

In this Letter, we show that the spectrum of the GN model contains, in the large-N limit, a composite, marginally stable topological soliton, which may be interpreted as a kink and a non-topological soliton bound at threshold. Furthermore, the energy of this system does not depend on the intersoliton distance, which thus serves as a modulus controlling the shape of the corresponding static solution of the large-N saddle point equation.

It is a general feature of the Dirac equation (2) in the background of a static topologically non-trivial $\sigma(x)$ configurations, that the spectrum contains an unpaired bound state at $\omega = 0$. For example, for kink boundary conditions ($\sigma(\infty) = -\sigma(-\infty) = m$), such a normalizable zero-mode is given by $u_0 \exp -\int^x \sigma(y) dy$, where $i\gamma^1 u_0 = -u_0$.

To make our point, we have to find a topologically non-trivial static solution of (4), which in a certain limit appears as well separated kink and a DHN soliton. Thus, we must find a reflectionless $\sigma(x)$ configuration, such that (2) has a bound state at $\omega = 0$, as required by topology, and a pair of charge conjugate bound states at some $\pm\omega_b \neq 0$, with $\omega_b$ considered a free parameter. To this end, we apply the machinery of (2) (see also 14) and find the most general such reflectionless background as
\[
\sigma(x) = m + \frac{2\kappa}{1 + \frac{m + \kappa}{m - \kappa} e^{2\kappa(x - y_0)}} - 2(m + \kappa) \left( 1 + \frac{m + \kappa}{m - \kappa} \right) \frac{1}{2} e^{2m(x - x_0)} + \frac{m + \kappa}{m - \kappa} m e^{2\kappa(x - y_0)}
\]

\[
= \kappa \tanh [\kappa(x - y_0 + R)]
\]

\[
+ \omega_b \frac{\sinh [m(x - x_0) + \kappa(x - y_0) + 2\kappa R] + \sinh [m(x - x_0) - \kappa(x - y_0)]}{e^{-\kappa R} \cosh [m(x - x_0) + \kappa(x - y_0) + 2\kappa R] + e^{\kappa R} \cosh [m(x - x_0) - \kappa(x - y_0)]}
\]

where \( \kappa = \sqrt{m^2 - \omega_b^2} \), \( \kappa R = \frac{1}{2} \log \frac{m + \omega_b}{m - \omega_b} \), and where \( x_0 \) and \( y_0 \) are arbitrary real parameters. The latter two quantities arise in the inverse scattering formalism as arbitrary parameters, independent of the parameter \( \omega_b \), which determine the coefficients in front of the asymptotic exponentially decaying normalized bound state wave functions. Thus, the normalized bound state at \( \omega = 0 \) behaves asymptotically as \( \sqrt{2m} \exp -m(x - x_0) \), and the normalized bound states at \( \pm \omega_b \) behave asymptotically as \( \sqrt{2\kappa} \exp -\kappa(x - y_0) \), as \( x \to \infty \). Here, of course, \( x_0 \) and \( y_0 \) are translational collective coordinates of the soliton \( \sigma(x) \). Note that \( \sigma(x) \) in (4) satisfies kink boundary conditions: \( \sigma(\infty) = m \) and \( \sigma(-\infty) = m + 2\kappa - 2(\kappa + m) = -m \). Similarly, \( -\sigma(x) \) is the desired extremal configuration with boundary conditions of an anti-kink.

The effective action \( S_{eff} \) in (3), evaluated at the background (4), is an ordinary function \( S_{eff}(\omega_b, y_0 - x_0) \) of the parameters which determine the shape of (4), where we have invoked translational invariance of (3). (With no loss of generality, we can always set one of these collective coordinates, say \( x_0 \), to zero.) Minus the value of \( S_{eff}(\omega_b, y_0 - x_0) \) per unit time is the rest energy, or mass \( M(\omega_b, y_0 - x_0) \), of the static inhomogeneous condensate (5). We still have to extremize \( M(\omega_b, y_0 - x_0) \) with respect to \( \omega_b \). As in the case of the DHN soliton, the extremal value is determined by the number \( n \) of valence particles and antiparticles which are trapped in the bound states \( \pm \omega_b \). Following the same technique used by DHN in (1) to calculate the mass of the DHN soliton, we find that the extremal value is again \( \omega_b = m \cosh (\frac{\pi n}{2N}) \), and that the mass of (4), which we will refer to as the “heavier topological soliton” (HTS), is

\[
M_{HTS,n} = \frac{Nm}{\pi} + 2Nm \frac{\sin (\frac{\pi n}{2N})}{\pi}
\]

(6)

Thus, \( M_{HTS,n} \) coincides with the sum of masses of a CCGZ kink and a DHN soliton trapping \( n \) valence fermions. Clearly, the \( O(2N) \) quantum numbers of the HTS are those of the direct product of the \( 2^N \) dimensional spinorial representation and the antisymmetric tensor representation of rank \( n \). More details of the construction of (5) and the associated extremum condition on \( \omega_b \) will be given elsewhere (6).

Note that \( M_{HTS,n} \) is independent of the remaining collective coordinate \( y_0 \). By varying \( y_0 \) (while keeping \( \omega_b \) fixed at its extremal value), we can modify the shape of \( \sigma(x) \) in (4) without affecting the mass of the soliton. The translational collective coordinate \( y_0 \) is thus a flat direction of the energy functional, or a modulus. In the following we will show that the modulus \( y_0 \) is essentially the separation between a CCGZ kink and a DHN soliton which we interpret as the loosely bound constituents of the HTS. Thus, the fact that \( \partial M_{HTS,n}/\partial y_0 = 0 \), means that these solitons exert no force on each other, whatever their separation is.

This is a somewhat surprising result, since one would normally expect soliton-soliton interactions to be of the order \( \frac{1}{N} \sim N \) in a weakly interacting field theory, which is consistent, of course, with what one should expect from general counting rules. Indeed, drawing further the analogy between the solitons discussed in this Letter and baryons in QCD with large \( N_{color} \), the HTS would correspond to a dibaryon. From the general counting rules (6), the baryon-baryon interaction is expected to be of order \( N \). Yet, due to dynamical reasons which elude us at this point, the solitonic constituents of the HTS avoid these general considerations and do not exert force on each other.

It is straightforward to obtain the asymptotic behavior of (5) for large \( |y_0| \) (and \( x_0 = 0 \)). In the limit \( y_0 \to -\infty \), the shape of (4) is that of a CCGZ kink, with a little “DHN bump”, centered on its left wing at \( x_{bump} = y_0 + (3/4\kappa) \log \frac{m + \omega_b}{m - \omega_b} \approx y_0 \), of width \( 1/2\kappa \) and maximum value of \( m - 2\omega_b \). Its shape is given approximately by \( \sigma(x_{bump} + z) \approx -m + 2\kappa^2/(m + \omega_b \cosh 2\kappa z) \). In the other limit, \( y_0 \to +\infty \), (4) has the shape of a CCGZ kink, with a little “DHN dip”, centered on its right wing at \( x_{dip} = y_0 + (1/4\kappa) \log \frac{m + \omega_b}{m - \omega_b} \approx y_0 \), of width \( 1/2\kappa \) and minimum value of \( m - 2\kappa (m + \kappa - \omega_b)/(m + \kappa + \omega_b) \). Its shape is given approximately by \( \sigma(x_{dip} + z) \approx
$m - 2\kappa(m + \kappa)/(m + \kappa + \omega_b \exp 2\kappa z) + 2\kappa\omega_b/(\omega_b + (m + \kappa) \exp 2\kappa z)$. For $y_0$ in the range such that $\kappa/|y_0| \sim 1$, the “DHN disturbance” and the kink partly overlap. These statements are demonstrated in the figures, for the case $m = 2\kappa = 1$.

![FIG. 1: The soliton (5) for $m = 2\kappa = 1; 2\kappa y_0 = -20$](image1.png)

![FIG. 2: The soliton (5) for $m = 2\kappa = 1; 2\kappa y_0 = -1$](image2.png)

The limit $n \to N$ is of some interest. Strictly speaking, there is no HTS with $n = N$, since at $n = N$ the pair of bound states at $\pm \omega_b$ of the Dirac equation would coincide with the bound state at $\omega = 0$, which already exists due to non-trivial topology. Clearly, such a degeneracy cannot occur in the spectrum in one spatial dimension. (For more details see Sections 3.1.2 and B.3.2 in [8].) However, it is possible to study HTS’s with $n$ arbitrarily close to $N$. In this case, $\kappa \to m$, with $R \to \infty$ in [5]. Thus, for $|x|, |x_0|, |y_0| \ll R$, [5] tends in this limit to

$$\sigma(x) = \frac{1 - e^{-2m(x-y_0)} - e^{-2m(x-x_0)}}{1 + e^{-2m(x-y_0)} + e^{-2m(x-x_0)}}$$

(7)

(where for clarity of presentation we have reinstated the parameter $x_0$ into the expression for $\sigma(x)$). In the asymptotic region $1 \ll m |x_0 - y_0| (\ll m R)$ [7] simplifies further, and appears as a kink $m \tanh[m(x-x_{\text{max}})]$, located at $x_{\text{max}} = \max\{x_0, y_0\}$. This clearly has mass $M_{kink} = \frac{N m}{\pi}$, but according to [6], $M_{HTS,n\geq N}$ should tend to $\frac{3N m}{\pi} = 3M_{kink}$. The extra mass $2M_{kink}$ corresponds, of course, to the kink-anti-kink pair which recedes to spatial infinity.

Finally, we must settle the important issue of stability of the HTS. Due to conservation of the topological charge $q = (\sigma(\infty) - \sigma(-\infty))/2m$, the final static configuration will obey the same kink boundary conditions as [5]. The HTS is too light to decay into a configuration of CCGZ kink-kink-antikink (see [6]). Thus, it can only decay into a CCGZ kink plus a bunch of lighter DHN bags and/or free fermions. The binding energy $B_n = nm + M_{kink} - M_{HTS,n}$ of the HTS coincides with that of a DHN bag with the same quantum number $n$. Thus, similarly to the latter, the HTS is stable against evaporation into a CCGZ kink and a cloud of non-interacting fermions. Decay of the HTS (with $O(2N)$ quantum number $n$) into a CCGZ kink and a DHN bag (of quantum number $n'$) is almost totally forbidden: Energy conservation obviously requires $0 < n' \leq n$. Fermion number conservation, on the other hand, requires (see [8] for more details) $n \leq n' < N$, i.e., the complimentary set of the possible range $0, N$ for $n'$. The two conservation laws are compatible only at $n' = n$. Thus, the mass of the decay products equals the mass of the parent HTS, and the allowed channel in this case has no phase space. Finally, one can show, using elementary considerations as above, that energy conservation and fermion number con-
servation strictly forbid decay of the HTS into a lighter HTS and a DHN soliton, or into a CCGZ kink plus any number of DHN solitons, or into any final state containing the time dependent solitons discovered by DHN in [2]. Since the only allowed channel has no phase space, the HTS is marginally stable. This must be a manifestation of the fact that the translational collective coordinate $y_0$ is a flat direction of the energy functional.

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[1] D.J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).
[2] R.F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D 12, 2443 (1975).
[3] C.G. Callan, S. Coleman, D.J. Gross and A. Zee, unpublished; This work is described by D.J. Gross in Methods in Field Theory, R. Balian and J. Zinn-Justin (Eds.), Les-Houches session XXVIII 1975 (North Holland, Amsterdam, 1976).
[4] J. Feinberg, Phys. Rev. D 51, 4503 (1995).
[5] E. Witten Nucl. Phys. B142, 285 (1978).
[6] R. Jackiw and C. Rebbi, Phys. Rev. D 12, 3398 (1976).
[7] L.D. Faddeev, J. Sov. Math. 5 (1976) 334. This paper is reprinted in L.D. Faddeev, 40 Years in Mathematical Physics (World Scientific, Singapore, 1995). S. Novikov, S.V. Manakov, L.P. Pitaevsky and V.E. Zakharov, Theory of Solitons - The Inverse Scattering Method (Consultants Bureau, New York, 1984) (Contemporary Soviet Mathematics).
[8] J. Feinberg, All about the Static Fermion Bags in the Gross-Neveu Model [hep-th/0305240].
[9] T. D. Lee and G. Wick, Phys. Rev. D 9, 2291 (1974); R. Friedberg, T.D. Lee and R. Sirlin, Phys. Rev. D 13, 2739 (1976); R. Friedberg and T.D. Lee, Phys. Rev. D 15, 1694 (1976), ibid. 16, 1096 (1977); A. Chodos, R. Jaffe, K. Johnson, C. Thorn, and V. Weisskopf, Phys. Rev. D 9, 3471 (1974).
[10] W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein and T. M. Yan, Phys. Rev. D 11, 1094 (1974); M. Creutz, Phys. Rev. D 10, 1749 (1974).
[11] E. Witten, Nucl. Phys. B160, 57 (1979).
[12] See e.g., A. Klein, Phys. Rev. D 14, 558 (1976); R. Pausch, M. Thies and V. L. Dolman, Z. Phys. A 338, 441 (1991).
[13] V. Schönh and M. Thies, 2d Model Field Theories at Finite Temperature and Density, in At the Frontiers of Particle Physics - Handbook of QCD (vol. 3), M. Shifman (Ed.), (World Scientific, Singapore, 2001.) [hep-th/0008175].
[14] H. B. Thacker, C. Quigg and J. L. Rosner, Phys. Rev. D 18, 274 (1978).