LARGE AMPLITUDE STATIONARY SOLUTIONS OF THE MORROW MODEL OF GAS IONIZATION

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Abstract. We consider the steady states of a gas between two parallel plates that is ionized by a strong electric field so as to create a plasma. We use global bifurcation theory to prove that there is a curve $X$ of such states with the following property. The curve begins at the sparking voltage and either the particle density becomes unbounded or the curve ends at the anti-sparking voltage.

1. The model. This paper is concerned with a model for the ionization of a gas such as air due to a strong applied electric field. The high voltage thereby creates a plasma, which may possess very hot electrical arcs. A century ago Townsend experimented with a pair of parallel plates to which he applied a strong voltage that produces cascades of free electrons and ions. This phenomenon is called the Townsend discharge or avalanche. The collision of gas particles within the plasma is sometimes called the $\alpha$-mechanism. For more details, we refer the reader to [17].

Many models have been proposed to describe this phenomenon (see [1, 8, 9, 10, 12, 13, 14, 15]). In 1985 Morrow [15] was perhaps the first to provide a model of its detailed mechanism. The model consists of continuity equations for the electrons and ions coupled to the Poisson equation for the electrostatic potential. For simplicity in this paper we consider only electrons and positive ions. We do not consider certain much smaller mechanisms such as ‘attachment’ and ‘recombination’, which were denoted by $\eta$ and $\beta$ in equations (1)-(3) in Morrow’s paper. Thus the model...
in this paper is as follows:

$$\begin{align}
\partial_t \rho_i + \partial_x (\rho_i u_i) &= a \exp\left(-b|\partial_x \Phi|^{-1}\right) \rho_e |v_e|, \\
\partial_t \rho_e + \partial_x (\rho_e u_e) &= a \exp\left(-b|\partial_x \Phi|^{-1}\right) \rho_e |v_e|, \\
\partial^2_x \Phi &= \rho_i - \rho_e,
\end{align}$$

where $L$ is the distance between the planar parallel plates, $\rho_i$ is the density of positive ions, $\rho_e$ is the electron density, and $-\Phi$ is the electrostatic potential. Moreover, $k_i$, $k_e$, $a$, and $b$ are positive constants. The ion and electron velocities $u_i$ and $u_e$ are assumed to obey the constitutive velocity relations (1d), which are due to the fact that the ions are much heavier than the electrons.

The right sides of (1a) and (1b) come from the $\alpha$-mechanism. They express the number of ion-electron pairs generated per unit volume by the impacts of the electrons. Specifically, the coefficient $a = a \exp\left(-b|\partial_x \Phi|^{-1}\right)$ is the first Townsend ionization coefficient, which can be found in equation (A1) of Morrow’s paper.

The interesting article [8] of Degond and Lucquin-Desreux derives the model from the general Euler-Maxwell system by scaling assumptions, in particular by assuming a very small mass ratio between the electrons and ions. In an appropriate limit the Morrow model is obtained at the end of their paper in equations (160) and (163), which we have specialized to assume constant temperature and no neutral particles. We are also ignoring the $\gamma$-mechanism, which refers to the secondary emission of electrons caused by the impacts of the ions with the cathode.

Now let us consider the structure of the model (1). Substituting the constitutive velocity relations (1d) into the continuity equations (1a) and (1b), we observe that the system is of hyperbolic-parabolic-elliptic type. We may consider the initial-boundary value problem for (1) by prescribing the initial and boundary data

$$\begin{align}
(\rho_i, \rho_e)(0, x) &= (\rho_{i0}, \rho_{e0})(x), \\
\rho_{i0}(x) &\geq 0, \\
\rho_{e0}(x) &\geq 0, \\
\rho_i(t, 0) &= \rho_i(t, 0) = \Phi(t, 0) = 0, \\
\rho_e(t, L) &= 0, \\
\Phi(t, L) &= V_e > 0.
\end{align}$$

The boundaries $x = 0$ and $x = L$ correspond to the anode and cathode, respectively, since $-\Phi$ is the electrostatic potential. The boundary condition (1f) means that, at each instant, electrons are absorbed by the anode and ions are repelled from the anode. Due to the assumed lack of a $\gamma$-mechanism, $\rho_e$ is assumed to vanish at the cathode $x = L$. Of course the non-negativity of the mass densities $\rho_{i0}$ and $\rho_{e0}$ is a natural condition.

Suzuki and Tani in [18] gave the first mathematical analysis of this model. Typical shapes of the cathode and anode in physical and numerical experiments are a sphere or a plate. Therefore they proved the time-local solvability of an initial boundary value problem over domains with a pair of boundaries that are plates or spheres. In another paper [19] they did a deeper analysis of problem (1). They proved that there exists a certain threshold of voltage at which the trivial solution (with $\rho_i = \rho_e = 0$) goes from stable to unstable. This fact means that gas discharge can occur and continue for a voltage greater than the threshold. The remarkable point is that gas discharge can occur even if $\gamma$-mechanism is not taken into account, in contrast to Townsend’s theory which required the $\gamma$-mechanism for gas discharge to occur.
2. Stationary solutions. In this paper we consider the stationary problem. First of all, there are the trivial solutions $\rho_i \equiv 0, \rho_e \equiv 0, \partial_x \Phi \equiv \text{constant}$. Unless the electric field is strong enough, avalanche does not occur. The critical threshold value of the voltage is called the sparking voltage $V^*_c$. The ionization coefficient $a$ in (1a) and (1b) must be large enough, depending on $b$ and $L$, in order to reach this threshold. In that case it was proven in [19] that $(0, 0, V^*_c)$ is a bifurcation point. The local bifurcation theorem is stated below.

Our goal in this paper is to extend the local bifurcation curve to a global one, thereby obtaining a one-parameter family of stationary solutions of large amplitude. In Section 3 we apply a functional-analytical global bifurcation theorem to construct a global curve $\mathcal{K}$ of stationary solutions $(\rho_i, \rho_e, V)$. This global curve includes solutions with positive densities as well as solutions with negative “densities”. In Section 4 we restrict our attention to positive solutions.

Ultimately we prove our main result, namely, that there is a curve $\mathcal{K}_{\text{pos}} \subset \mathcal{K}$ along which $\rho_i > 0$ and $\rho_e > 0$ in $(0, L)$ and either $\rho_i + \rho_e$ is unbounded in $L^\infty(0, L]$ or $\mathcal{K}_{\text{pos}} \text{ “ends” at a point } (0, 0, V^*_c)$, where $V^*_c > V^*_e$ is another critical voltage defined below. See Theorem 4.5 for a precise statement.

In Figure 4 in Appendix A, we present the voltage–current curve in a typical laboratory experiment. The sparking voltage in Figure 4 is denoted by $V_S$, and the current is denoted by $I := -\rho_e u_e + \rho_i u_i$. Notice that the current appears to be unbounded.

For mathematical convenience, we rewrite initial–boundary value problem (1) by using the new unknown function

$$R_e := \rho_e e \frac{V_c}{L}$$

and the explicit functions

$$h(x) := a \exp \left(\frac{-b}{|x|}\right)|x|, \quad g(V_c) := h \left(\frac{V_c}{L}\right) - \frac{V_c^2}{4L^2},$$

which will play an essential role in our analysis. We also decompose the electrostatic potential as

$$\Phi = V(x) + \frac{V_c}{L} x,$$

so that $\partial^2_x V = \rho_i - e^{-\frac{V_c}{L} x}R_e$ with the boundary conditions $V(0) = V(L) = 0$. Occasionally we will denote it by

$$V = [V_c, \rho_i, R_e].$$

As a result, we have the following system for stationary solutions:

$$k_i \partial_x \left\{ \left( \partial_x V + \frac{V_c}{L} \right) \rho_i \right\} = k_c h \left( \partial_x V + \frac{V_c}{L} \right) e^{-\frac{V_c}{L} x} R_e,$$

$$-k_e \partial^2_x R_e - k_e g(V_c) R_e = k_e f_e[V_c, R_e, V],$$

$$\partial^2_x V = \rho_i - e^{-\frac{V_c}{L} x} R_e,$$

where the nonlinear term $f_e$ is defined as

$$f_e[V_c, R_e, V] := \left( \partial_x V \right) \left( \partial_x R_e \right) - \frac{V_c}{2L} R_e \partial_x V + R_e \partial^2_x V - \left\{ h \left( \frac{V_c}{L} \right) - h \left( \partial_x V + \frac{V_c}{L} \right) \right\} R_e.$$
The graph of $g$ was drawn in [19] depending on the physical parameters $a$, $b$, and $L$. There are two cases, illustrated in Figures 1 and 2. As mentioned above, here and in [19] only the case in Figure 1 is considered, in order that $g(V_c)$ can be sufficiently large. Then the *sparking voltage* $V_c^* > 0$ for the Degond–Lucquin-Desreux–Morrow model is uniquely defined by

$$g(V_c^*) = \frac{\pi^2}{L^2}, \quad g'(V_c^*) > 0.$$  \hfill (6)

We also define the *anti-sparking voltage* $V_c^\# > 0$ by

$$g(V_c^\#) = \frac{\pi^2}{L^2}, \quad g'(V_c^\#) < 0.$$  \hfill (7)

Note that $\pi^2/L^2$ is the lowest eigenvalue of $-\partial_x^2$ on $[0, L]$.

Local bifurcation was proven in [19]. That is, there is a unique non-trivial stationary solution curve in a neighborhood of the point $(\rho_i, R_c, V_c) = (0, 0, V_c^*)$ where the voltage $V_c$ is regarded as the bifurcation parameter. The precise result is summarized as follows. (In [19] the theorem was stated in terms of $R_i = e^{-Lx/V_c} \rho_i$ instead of $\rho_i$, but for the analysis of global bifurcation it is more convenient to adopt $\rho_i$.). Let $I$ be the interval $(0, L)$ and denote $H^1_{0\partial}(I) = \{ v \in H^1(I) : v(0) = 0 \}$. We use $s$ as a parameter along the curve.
Theorem 2.1 (Local Bifurcation). There exists a value $s_0 > 0$, a voltage $V_c \in C^2([-s_0, s_0]; \mathbb{R})$, and a pair of densities $(\rho_i, R_e) \in C^2([-s_0, s_0]; H^1_0(I) \times (H^1_0(I) \cap H^2(I)))$ with the following properties: $V_c(0) = V_c^*$, $(\rho_i, R_e)(0) = (0, 0)$, and for all $s \in [-s_0, s_0]$ they solve the stationary problem (4). Furthermore, the solutions have the form $(\rho_i, R_e)(s) = s(\varphi_i(V_c^*, \varphi_c), \varphi_c) + o(s)$ for $s \in [-s_0, s_0]$, where

$$
\varphi_i(x) := \frac{k_e}{k_i} e^{-\frac{sL}{L}} \int_0^x e^{-\frac{V_c^*}{L}} y \varphi_c(y) dy, \quad \varphi_c(x) := \sin \frac{\pi x}{L}.
$$

The following additional properties of the local curve of solutions were proven in [19]. However, these properties are not used in the rest of the present article.

Corollary 1. (a) $\frac{d\varphi_i}{ds}(0) \leq 0$ holds if and only if

$$
-2Lg'(V_c^*) \int_0^L \varphi_i^2 \partial_x^2 (V[V_c^*, \varphi_i, \varphi_c]) dx - \int_0^L \varphi_i^2 \partial_x^2 (V[V_c^*, \varphi_i, \varphi_c]) dx \leq 0. \quad (8)
$$

(b) If $\frac{d\varphi_i}{ds}(0)$ in Theorem 2.1 is nonzero, then there exists $s_1 > 0$ such that the solutions $(\rho_i(s), R_e(s))$ satisfy

$$
\frac{d\varphi_i}{ds}(0) \rho_i(s, x) > 0, \quad \frac{d\varphi_i}{ds}(0) R_e(s, x) > 0 \quad \text{for} \quad s \in [-s_1, s_1] \backslash \{0\}, \quad x \in I. \quad (9)
$$

(c) The positive non-trivial solution is linearly stable if $\frac{d\varphi_i}{ds}(0) > 0$ and is linearly unstable if $\frac{d\varphi_i}{ds}(0) < 0$.

The main purpose of this article is to prove that there exist many more stationary solutions, including ones of large amplitude. This is accomplished by a global bifurcation technique. We introduce some notation for the stationary system as follows. We write the system (4) as

$$
\mathcal{F}_j(\lambda, \rho_i, R_e, V) = 0 \quad \text{for} \quad j = 1, 2, 3, \quad (10)
$$

where we denote

$$
\lambda := \frac{V_c}{L},
$$

$$
\mathcal{F}_1(\lambda, \rho_i, R_e, V) := k_e \partial_x \left\{ (\partial_x V + \lambda) \rho_i \right\} - k_e h (\partial_x V + \lambda) e^{-\frac{\varphi_i}{L}} R_e,
$$

$$
\mathcal{F}_2(\lambda, \rho_i, R_e, V) := - \partial_x^2 R_e - (\partial_x V) \partial_x R_c
$$

$$
+ \left\{ \frac{\lambda}{2} (\partial_x V) - (\partial_x^2 V) + \frac{\lambda^2}{4} - h (\partial_x V + \lambda) \right\} R_c,
$$

$$
\mathcal{F}_3(\lambda, \rho_i, R_e, V) := \partial_x^2 V - \rho_i + e^{-\frac{\varphi_i}{L}} R_e.
$$

3. Global bifurcation. In this section, we apply a functional-analytic global bifurcation theorem to the stationary problem (10). The theory of global bifurcation goes back to Rabinowitz [16, 11] using topological degree. A different version using analytic continuation goes back to Dancer [7, 4]. The specific version that is most convenient to use here is Theorem 6 in [5], which is the following:

Theorem 3.1 ([5]). Let $X$ and $Y$ be Banach spaces, $\mathcal{O}$ be an open subset of $\mathbb{R} \times X$ and $\mathcal{F} : \mathcal{O} \to Y$ be a real-analytic function. Suppose that

(H1) $(\lambda, 0) \in \mathcal{O}$ and $\mathcal{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$;

(H2) for some $\lambda^* \in \mathbb{R}$, $N(\partial_0 \mathcal{F}(\lambda^*, 0))$ and $Y \backslash R(\partial_0 \mathcal{F}(\lambda^*, 0))$ are one-dimensional, with the null space generated by $u^*$, which satisfies the transversality condition

$$
\partial_{\lambda, u}^2 \mathcal{F}(\lambda^*, 0)(1, u^*) \notin R(\partial_0 \mathcal{F}(\lambda^*, 0)).
$$
where $\partial_u$ and $\partial^2_{\lambda,u}$ mean Fréchet derivatives for $(\lambda, u) \in \mathcal{O}$, and $N(L)$ and $R(L)$ denote the null space and range of a linear operator $L$ between two Banach spaces;

(H3) $\partial_u \mathcal{F}(\lambda, u)$ is a Fredholm operator of index zero for any $(\lambda, u) \in \mathcal{O}$ that satisfies the equation $\mathcal{F}(\lambda, u) = 0$;

(H4) for some sequence $\{O_j\}_{j \in \mathbb{N}}$ of bounded closed subsets of $\mathcal{O}$ with $\mathcal{O} = \cup_{j \in \mathbb{N}} O_j$, the set $\{(\lambda, u) \in \mathcal{O}; \mathcal{F}(\lambda, u) = 0\} \cap O_j$ is compact for each $j \in \mathbb{N}$.

Then there exists in $\mathcal{O}$ a continuous curve $\mathcal{K} = \{(\lambda(s), u(s)); s \in \mathbb{R}\}$ of $\mathcal{F}(\lambda, u) = 0$ such that:

(C1) $(\lambda(0), u(0)) = (\lambda^*, 0)$;

(C2) $u(s) = su^* + o(s)$ in $X$ as $s \to 0$;

(C3) there exists a neighborhood $\mathcal{W}$ of $(\lambda^*, 0)$ and $\varepsilon > 0$ sufficiently small such that

$$\{(\lambda, u) \in \mathcal{W}; u \neq 0 \text{ and } \mathcal{F}(\lambda, u) = 0\} = \{(\lambda(s), u(s)); 0 < |s| < \varepsilon\};$$

(C4) $\mathcal{K}$ has a real-analytic reparametrization locally around each of its points;

(C5) one of the following two alternatives occurs:

(I) for every $j \in \mathbb{N}$, there exists $s_j > 0$ such that $(\lambda(s), u(s)) \notin O_j$ for all $s \in \mathbb{R}$ with $|s| > s_j$;

(II) there exists $T > 0$ such that $(\lambda(s), u(s)) = (\lambda(s + T), u(s + T))$ for all $s \in \mathbb{R}$.

Moreover, such a curve of solutions of $\mathcal{F}(\lambda, u) = 0$ having the properties (C1)-(C5) is unique (up to reparametrization).

Hypothesis (H2) is the local bifurcation condition while (H3) and (H4) are the global ones. (C1) – (C3) are local conclusions, (C4) is a statement of regularity, and (C5) is the global conclusion which states that either the curve reaches the boundary of the set $O_j$ or the curve is periodic (that is, forming a closed loop).

To apply the theorem to our situation, we define the two spaces

$$X : \rho_i \in \{f \in C^1([0, L]); f(0) = 0\}, \quad R_e \in \{f \in C^2([0, L]); f(0) = f(L) = 0\},$$

$$V \in \{f \in C^3([0, L]); f(0) = f(L) = 0\};$$

$$Y : \mathcal{F}_1 \in C^0([0, L]), \quad \mathcal{F}_2 \in C^0([0, L]), \quad \mathcal{F}_3 \in C^1([0, L])$$

and the sets

$$\mathcal{O} := \{(\lambda, \rho_i, R_e, V) \in (0, \infty) \times X; \partial_x V + \lambda > 0\} = \bigcup_{j \in \mathbb{N}} \mathcal{O}_j,$$

where

$$\mathcal{O}_j := \{(\lambda, \rho_i, R_e, V) \in (0, \infty) \times X; \lambda + \|\rho_i, R_e, V\|_X \leq j, \lambda \geq \frac{1}{2}, \partial_x V + \lambda \geq \frac{1}{2}\}.$$ 

Note that $\mathcal{O}$ is an open set and each $\mathcal{O}_j$ is a closed bounded subset of $\mathcal{O}$. Furthermore, the $\mathcal{F}_j$ are real-analytic operators because they are polynomials in $(\lambda, \rho_i, R_e, V)$ and their $x$-derivatives, except for the factor $h(\partial_x V + \lambda)$. However, $\partial_x V + \lambda > 0$ in $\mathcal{O}$ and the function $s \to h(s)$ is analytic for $s > 0$. Hypothesis (H1) is obvious. The local bifurcation condition (H2) is easily checked in exactly the same way as in [19]. The conditions (H3) and (H4) are validated in the following two lemmas.

**Lemma 3.2.** For any $(\lambda, \rho_i^0, R_e^0, V^0) \in \mathcal{O}$, the Fréchet derivative $L^0 = \partial(\rho_i, R_e, V)\mathcal{F}(\lambda, \rho_i^0, R_e^0, V^0)$ is a linear Fredholm operator of index zero.
Proof. For any fixed choice of \((\lambda, \rho_i^0, R_0^0, V^0)\), we know that \(\inf_x \partial_x V^0 + \lambda > 0\). The operator \(L^0 = (L_1, L_2, L_3)\) has the form

\[
\begin{align*}
L_1 &= L_1(S_i, S_e, W) = \partial_x (\{\partial_x V^0 + \lambda\} S_i) + b_1 \partial_x^2 W + b_2 S_i + b_3 S_e + b_4 \partial_x W, \quad (11) \\
L_2 &= L_2(S_i, S_e, W) = -\partial_x^2 S_e + a_1 \partial_x S_e + b_5 S_e + b_6 \partial_x^2 W + b_7 \partial_x W, \quad (12) \\
L_3 &= L_3(S_i, S_e, W) = -\partial_x^2 W + a_2 S_i + a_3 S_e, \quad (13)
\end{align*}
\]

where the coefficients \(a_1, a_2\) and \(a_3\) belong to \(C^1([0, L])\) and the coefficients \(b_1, \ldots, b_7\) belong to \(C^0([0, L])\).

Let us first show that the linear operator \(L^0\) has a finite-dimensional nullspace and a closed range. By [20, Theorem 12.12] or [3, Exercise 6.9.1], it is equivalent to prove that \(L^0\) satisfies the estimate

\[
C\|\!\|\!(S_i, S_e, W)\|\!\|_X \leq \|L^0(S_i, S_e, W)\|_Y + \|\!(S_i, S_e, W)\|_Y
\]

for all \((S_i, S_e, W) \in X\) and for some constant \(C\) depending only on \((\lambda, \rho_i^0, R_0^0, V^0)\). Indeed, keeping in mind that \(\partial_x V^0 + \lambda \geq 1/j\), we see from (11) and (13) that

\[
\|\partial_x S_i\|_{C^0} = (\partial_x V^0 + \lambda)^{-1} \left( \{\partial_x (\partial_x V^0 + \lambda)\} S_i + b_1 \partial_x^2 W + b_2 S_i + b_3 S_e + b_4 \partial_x W - L_1 \right) \|_{C^0} \\
\leq C(\|S_i\|_{C^0} + \|S_e\|_{C^0} + \|W\|_{C^2} + \|L_1\|_{C^0}) \\
\leq C\|L^0(S_i, S_e, W)\|_Y + C\|\!(S_i, S_e, W)\|_Y.
\]

By writing (12) as

\[
\partial_x^2 S_e - a_1 \partial_x S_e = -L_2 + b_5 S_e + b_6 (-L_3 + a_2 S_i + a_3 S_e) + a_7 \partial_x W;
\]

and \(a_1 \partial_x S_e = \partial_x (a_1 S_e) - (\partial_x a_1) S_e\), we see that

\[
\|\partial_x^2 S_i\|_{C^0} \leq \|L^0(S_i, S_e, W)\|_Y + \|\!(S_i, S_e, W)\|_Y.
\]

Finally, (13) leads to

\[
\|\partial_x^2 W\|_{C^1} = \|\!\|\!(a_2 S_i + a_3 S_e - L_3)\|\!\|_{C^1} \leq C\|\!\|\!\|L^0(S_i, S_e, W)\|\!\!\!\|_Y + C\|\!(S_i, S_e, W)\|_Y.
\]

Combining (15)–(17), we find the estimate (14).

Because \(L^0\) has a finite-dimensional nullspace and a closed range, it is called a semi-Fredholm operator. By the proof of Theorem 2.1, we know that at the bifurcation point the nullspace of \(\partial_{(\rho_i, R_e, V)}\mathcal{F}(V_e^*/L, 0, 0, 0)\) has dimension one and the codimension of its range is also one, so that its index is zero. Since \(O\) is connected and the index is a topological invariant [2, Theorem 4.51, p166], \(L^0\) also has index zero. In particular, this implies that the codimension of \(L^0\) is also finite.

**Lemma 3.3.** For each \(j \in \mathbb{N}\), the set \(K_j = \{\!(\lambda, \rho_i, R_e, V) \in O_j; \mathcal{F}(\lambda, \rho_i, R_e, V) = 0\}\) is compact in \(\mathbb{R} \times X\).

**Proof.** Let \(\{(\lambda_n, \rho_{in}, R_{en}, V_n)\}\) be any sequence in \(K_j\). It suffices to show that it has a convergent subsequence whose limit also belongs to \(K_j\). By the assumed bound \(|\lambda_n| + \|\!(\rho_{in}, R_{en}, V_n)\|_X \leq j\), there exists a subsequence, still denoted by \(\{(\lambda_n, \rho_{in}, R_{en}, V_n)\}\), and \((\lambda, \rho_i, R_e, V)\) such that

\[
\begin{align*}
\lambda_n &\rightarrow \lambda \quad \text{in} \quad \mathbb{R}, \\
\rho_{in} &\rightarrow \rho_i \quad \text{in} \quad C^0([0, L]), \\
R_{en} &\rightarrow R_e \quad \text{in} \quad C^1([0, L]), \\
V_n &\rightarrow V \quad \text{in} \quad C^2([0, L]).
\end{align*}
\]
Furthermore, 
\[ \partial_x V + \lambda \geq \frac{1}{T}. \]
Since \( \mathcal{O}_j \) is closed in \( X \), it remains to show that
\[ \mathcal{F}_j(\lambda, \rho_i, R_e, V) = 0 \quad \text{for } j = 1, 2, 3, \]
\[ \rho_{in} \to \rho_i \text{ in } C^1([0, L]), \quad R_{en} \to R_e \text{ in } C^2([0, L]), \quad V_n \to V \text{ in } C^3([0, L]). \]
Now the first equation \( \mathcal{F}_1(\lambda_n, \rho_{in}, R_{en}, V_n) = 0 \) with \( \rho_{in}(0) = 0 \) is equivalent to
\[ \rho_{in}(x) = \frac{k_e}{k_i}(\partial_x V_n(x) + \lambda_n)^{-1} \int_0^x h(\partial_x V_n(y) + \lambda_n)e^{-\frac{3}{4}y}R_{en}(y) \, dy. \]
Taking the limit and using (18), we see that
\[ \rho_i(x) = \frac{k_e}{k_i}(\partial_x V(x) + \lambda)^{-1} \int_0^x h(\partial_x V(y) + \lambda)e^{-\frac{3}{4}y}R_e(y) \, dy, \]
where the right hand side converges in \( C^1([0, L]) \). Hence, we see that \( \mathcal{F}_1(\lambda, \rho_i, R_e, V) = 0 \) and \( \rho_{in} \to \rho_i \) in \( C^1([0, L]) \).
Taking the limit using (18) in the third equation \( \mathcal{F}_3(\lambda_n, \rho_{in}, R_{en}, V_n) = 0 \) immediately leads to
\[ \partial_x^2 V = \rho_i - e^{-\frac{3}{4}x}R_e. \]
Hence \( \mathcal{F}_3(\lambda, \rho_i, R_e, V) = 0 \) and \( V_n \to V \) in \( C^3([0, L]) \).
The second equation \( \mathcal{F}_2(\lambda_n, \rho_{in}, R_{en}, V_n) = 0 \) can be written as
\[ \partial_x \{ \partial_x R_{en} - (\partial_x V_n)R_{en} \} = \left( \frac{\lambda}{2} + \frac{\lambda^2}{3} - h(\partial_x V_n + \lambda_n) \right) R_{en}. \]
Because the right side converges in \( C^1([0, L]) \), we see that \( \{ \partial_x R_{en} - (\partial_x V_n)R_{en} \} \) converges in \( C^2([0, L]) \). But \( (\partial_x V_n)R_{en} \) converges in \( C^1([0, L]) \). Hence \( \partial_x R_{en} \) converges in \( C^1([0, L]) \), which means that \( R_{en} \) converges in \( C^2([0, L]) \). \hfill \( \Box \)

As we have checked all conditions in Theorem 3.1, the following conclusion is valid.

**Theorem 3.4.** There exists in \( \mathcal{O} \) a continuous curve \( \mathcal{K} = \{(\lambda(s), \rho_i(s), R_e(s), V(s)) \}; \) \( s \in \mathbb{R} \) of stationary solutions to problem (10) such that

(\( C1 \)) \( (\lambda(0), \rho_i(0), R_e(0), V(0)) = (V_c^*/L, 0, 0, 0) \), where \( V_c^* \) is defined in (6);

(\( C2 \)) \( (\rho_i(s), R_e(s), V(s)) = \sigma(\varphi_i[V_c^*, \varphi_i], \varphi_i, V[V_c^*, \varphi_i[V_c^*, \varphi_i]], \sigma(s)) \in \) the space \( X \) as \( s \to 0 \), where \( \varphi_i, \varphi_i[V_c^*, \varphi_i], \varphi_i[V_c^*, \varphi_i], \varphi_i[V_c^*, \varphi_i] \) are defined in (3) and Theorem 2.1;

(\( C3 \)) there exists a neighborhood \( \mathcal{W} \) of \( (V_c^*/L, 0, 0, 0) \) and \( \varepsilon < 1 \) such that
\[ \{(\lambda, \rho_i, R_e, V) \in \mathcal{W} \}; (\rho_i, R_e, V) \neq (0, 0, 0, 0), \mathcal{F}(\lambda, \rho_i, R_e, V) = 0 \} \]
\[ = \{(\lambda(s), \rho_i(s), R_e(s), V(s)) \}; 0 < |s| < \varepsilon \}; \]

(\( C4 \)) \( \mathcal{K} \) has a real-analytic reparametrization locally around each of its points;

(\( C5 \)) at least one of the following five alternatives occurs:

(\( a \)) \( \lim_{s \to \infty} \lambda(s) = 0; \)

(\( b \)) \( \lim_{s \to \infty} \left( \inf_{x \in I} \partial_x V(x, s) + \lambda(s) \right) = 0; \)

(\( c \)) \( \lim_{s \to \infty} \lambda(s) = \infty; \)

(\( d \)) \( \lim_{s \to \infty} (\|\rho_i\|_{C^1} + \|R_e\|_{C^2} + \|V\|_{C^3})(s) = \infty; \)

(\( e \)) there exists \( T > 0 \) such that
\[ (\lambda(s), \rho_i(s), R_e(s), V(s)) = (\lambda(s + T), \rho_i(s + T), R_e(s + T), V(s + T)) \]
for all \( s \in \mathbb{R} \).
Moreover, such a curve of solutions to problem (10) having the properties (C1)-(C5) is unique (up to reparametrization).

Conditions (C1)-(C3) are an expression of the local bifurcation, while (C4)-(C5) are assertions about the global curve $\mathcal{K}$. Alternatives (c) and (d) assert that $\mathcal{K}$ may be unbounded. Alternative (e) asserts that $\mathcal{K}$ may form a closed curve (a 'loop').

4. Positive densities. Of course, we should keep in mind that for the physical problem $\rho_i$ and $R_e$ are densities of particles and so they should be non-negative. In this section we investigate the part of the curve $\mathcal{K}$ that corresponds to such densities.

A simple observation is the following proposition, which states that either $\rho_i$ and $R_e$ remain positive or the curve of positive solutions forms a half-loop going from $V_c^*$ to $V_c^\#$. Here $V_c^*$ and $V_c^\#$ are defined in (6) and (7). The bifurcation diagram of the half-loop case is qualitatively sketched in Figure 3.

![Figure 3. alternative (ii)](image)

**Proposition 1.** For the global bifurcation curve $(\lambda(s), \rho_i(s), R_e(s), V(s))$ in Theorem 3.4, one of the following two alternatives occurs:

(i) $\rho_i(s, x) > 0$ and $R_e(s, x) > 0$ for all $0 < s < \infty$ and $x \in I = (0, L)$.

(ii) there exists a finite parameter value $T^\# > 0$ such that

1. $\rho_i(s, x) > 0$ and $R_e(s, x) > 0$ for all $s \in (0, T^\#)$ and $x \in I$;
2. $(\lambda(T^\#), \rho_i(T^\#), R_e(T^\#), V(T^\#)) = (V_c^\#/L, 0, 0, 0)$;
3. $(\rho_i(s), R_e(s)) = (T^\# - s)(\varphi_i[V_c^\#, \varphi_e], \varphi_e) + o(|s - T^\#|)$ as $s \to T^\#$, where $\varphi_i[\cdot, \cdot]$ and $\varphi_e$ are defined in Theorem 2.1;
4. $\rho_i(s, x) < 0$ and $R_e(s, x) < 0$ for $0 < s - T^\# \ll 1$ and $x \in I$.

**Proof.** We define

$$s_* := \inf\{s > 0 : R_e(s, x_0) = 0 \text{ for some } x_0 \in I\}.$$  

We shall show that $s_* = T^\#$ satisfies (ii). By Corollary 1, $s_* > 0$. If $s_* = \infty$, then alternative (i) occurs. Indeed, $(\partial_x V + \lambda)$ is positive owing to $(\lambda(s), \rho_i(s), R_e(s), V(s)) \in \emptyset$ and then the following formula also yields $\rho_i > 0$.

$$\rho_i = \frac{k_e}{k_i} (\partial_x V + \lambda)^{-1} \int_0^x h(\partial_x V + \lambda) e^{-\frac{1}{2} y} R_e \, dy.$$  

(19)
If \( s_* < \infty \), the function \( R_e(s_*, \cdot) \) takes the value zero, which is its minimum, at some point \( x_0 \in I \). In case \( x_0 \in I \), \( \partial_x R_e(s_*, x_0) = 0 \) obviously holds. In case \( x_0 \in \{0, L\} \), there exists a sequence \( \{(s_n, x_n)\}_{n \in \mathbb{N}} \) such that \( R_e(s_n, x_n) = 0 \), \( s_n \searrow s_* \), and \( x_n \to x_0 \). Applying Rolle's theorem to the interval between \( x_0 \) and \( x_n \), we see that \( \partial_x R_e(s_*, x_0) = 0 \). Thus, solving \( F_2(\lambda, \rho_i, R_e, V) = 0 \) with \( R_e(s_*, x_0) = \partial_x R_e(s_*, x_0) = 0 \), we see by uniqueness that \( R_e = 0 \). Owing to this and (19), we have \( \rho_i \equiv 0 \) and thus \( V \equiv 0 \). Hence \( (\rho_i, R_e, V)(s_*) = (0, 0, 0) \) is the trivial solution.

Following the nodal argument of [6], the linearized operator of \( F_2 \) around \((0, 0, 0)\) is \(-\partial_x^2 - g(V_0)I\), which has the eigenfunction \( \sin(n\pi x/L) \) for \( g(V_0) = n^2\pi^2/L^2 \), for all \( n \in \mathbb{N} \). If \( n \geq 2 \), this eigenfunction changes its sign in \((0, L)\). However, the bifurcation curve \((\lambda(s), \rho_i(s), R_e(s), V(s))\) connects at \( s = s_* \) to the point \((g^{-1}(n^2\pi^2/L^2), 0, 0, 0)\). So if \( n \geq 2 \), the function \( R_e(s) \) would change its sign as \( s \) approaches \( s_* \), which would contradict the definition of \( s_* \). Therefore \( n = 1 \) so that \( g(V_0^*) = \pi^2/L^2 = g(V_0^*) \) and \( \lambda(s_*) = V_0^*/L \). Thus alternative (ii) occurs.

We are interested in the positive solutions, since \( \rho_i \) and \( R_e e^{-V_0 x/L} \) are the densities of the ions and electrons, respectively. Let us investigate in detail the case that the global positivity alternative (i) in Proposition 1 occurs. More precisely, the next three lemmas show that if any one of the alternatives (a), (b) or (c) in Theorem 3.4 occurs, then alternative (d) also occurs.

**Lemma 4.1.** Assume alternative (i) in Proposition 1. If \( \lim_{s \to \infty} \lambda(s) = 0 \), then \( \sup_{s > 0} ||V(s)||_{C^2} \) is unbounded.

**Proof.** Suppose that \( \sup_{s > 0} ||V(s)||_{C^2} \) is bounded. Because \( \lim_{s \to \infty} \lambda(s) = 0 \) and \((\partial_x V + \lambda)(s, x) > 0\), there exists a sequence \( \{s_n\}_{n \in \mathbb{N}} \) and a function \( V^*(\cdot) \) such that

\[
\begin{aligned}
\lambda(s_n) &\to 0 \quad \text{in} \quad \mathbb{R}, \\
V(s_n, \cdot) &\to V^*(\cdot) \quad \text{in} \quad C^1([0, L]), \\
V^*(0) &= V^*(L) = 0, \\
\partial_x V^* &\geq 0.
\end{aligned}
\tag{20}
\]

The boundary condition (21) means that \( \int_0^L \partial_x V^*(x) \, dx = 0 \). This together with (22) implies \( \partial_x V^* \equiv 0 \). Using (21) again, we have \( V^* \equiv 0 \).

It follows that for suitably large \( n \) the expressions \( |\lambda(s_n)| \), \( ||V(s_n)||_{C^2} \), and \( ||h(\partial_x V(s_n) + \lambda(s_n))||_{C^0} \) are arbitrarily small. We multiply \( F_2(\lambda(s_n), \rho_i(s_n), R_e(s_n), V(s_n)) = 0 \) by \( R_e(s_n) \) and integrate by parts over \([0, L] \). Then using Poincaré's inequality and taking \( n \) suitably large, we obtain

\[
\begin{aligned}
\int_0^L (\partial_x R_e)^2(s_n) \, dx \\
&= - \int_0^L \partial_x V(s_n) R_e(s_n) \partial_x R_e(s_n) \, dx \\
&\quad - \int_0^L \left\{ \frac{\lambda(s_n)}{2} \partial_x V(s_n) + \frac{\lambda^2(s_n)}{4} - h(\partial_x V(s_n) + \lambda(s_n)) \right\} R_e^2(s_n) \, dx \\
&\leq \frac{1}{2} \int_0^L (\partial_x R_e)^2(s_n) \, dx
\end{aligned}
\]

since \( \partial_x^2 V(s_n) \) is bounded. Hence \( \partial_x R_e(s_n) \equiv 0 \). Since \( R_e \) vanishes at the endpoints, we conclude that \( R_e(s_n) \equiv 0 \), which contradicts the assumed positivity.
Lemma 4.2. Assume alternative (i) in Proposition 1. If \( \lim_{s \to \infty} \lambda(s) = \infty \), then \( \sup_{s > 0} ||V(s)||_{C^2} \) is unbounded.

Proof. Suppose that \( \sup_{s > 0} ||V(s)||_{C^2} \) is bounded. We take a subsequence as above. For suitably large \( n \), the expressions \( \frac{1}{\lambda(s_n)} \) and \( \|(h(\partial_x V(s_n) + \lambda(s_n))/\lambda^2(s_n))\|_{C^0} \) are arbitrarily small. Write \( s = s_n \) for brevity. Multiplying \( F_2(\lambda(s), \rho_i(s), R_e(s), V(s)) = 0 \) by \( R_e(s)/\lambda^2(s) \) and then integrating by parts over \([0, L] \), we obtain

\[
\int_0^L \frac{\partial_x R_e(s)^2}{\lambda^2(s)} dx + \frac{1}{4} R_e^2(s) dx
\]

\[
= \int_0^L \left\{ \frac{\partial_x^2 V(s)}{2\lambda^2(s)} - \frac{\partial_x V(s)}{2\lambda(s)} + \frac{h(\partial_x V(s) + \lambda(s))}{\lambda^2(s)} \right\} R_e^2(s) dx
\]

\[
\leq \frac{1}{8} \int_0^L R_e^2(s) dx.
\]

Once again this leads to \( R_e(s) \equiv 0 \), which contradicts the assumed positivity. \( \square \)

Lemma 4.3. Assume alternative (i) in Proposition 1. If \( \lim_{s \to \infty} \{\inf_{x \in I} \partial_x V(s, x) + \lambda(s)\} = 0 \), then \( \sup_{s > 0} \{||\rho_i(s)||_{C^0} + ||R_e(s)||_{C^2} + ||V(s)||_{C^2}\} \) is unbounded.

Proof. On the contrary, suppose that \( \sup_{s > 0} \{||\rho_i(s)||_{C^0} + ||R_e(s)||_{C^2} + ||V(s)||_{C^2}\} \) is bounded. We see from Lemma 4.2 and \( \lim_{s \to \infty} \{\inf_{x \in I} \partial_x V(s, x) + \lambda(s)\} = 0 \) that there exists a sequence \( \{s_n\} \in \mathbb{N} \) and \((\lambda^*, \rho_i^*, R_e^*, V^*)\) with \( \lambda^* < \infty \) such that

\[
\begin{align*}
\lambda(s_n) & \rightarrow \lambda^* \quad \text{in } \mathbb{R}, \\
\rho_i(s_n) & \rightarrow \rho_i^* \quad \text{in } L^\infty(0, L) \quad \text{weakly-star,} \\
R_e(s_n) & \rightarrow R_e^* \quad \text{in } C^1([0, L]), \\
\partial_x^2 R_e(s_n) & \rightarrow \partial_x^2 R_e^* \quad \text{in } L^\infty(0, L) \quad \text{weakly-star,} \\
V(s_n) & \rightarrow V^* \quad \text{in } C^2([0, L]), \\
\partial_x^2 V(s_n) & \rightarrow \partial_x^2 V^* \quad \text{in } L^\infty(0, L) \quad \text{weakly-star,} \\
R_e^*(0) = R_e^*(L) = V^*(0) = V^*(L) = 0,
\end{align*}
\]

(23)

\[
\rho_i^* \geq 0, \quad R_e^* \geq 0,
\]

(24)

\[
\inf_{x \in [0, L]} (\partial_x V^* + \lambda^*)(x) = 0.
\]

(26)

We shall show that

\[
F_j(\lambda^*, \rho_i^*, R_e^*, V^*) = 0 \quad \text{for a.e. } x \text{ and } j = 1, 2, 3.
\]

The equation \( F_1(\lambda(s_n), \rho_i(s_n), R_e(s_n), V(s_n)) = 0 \) with \( \rho_i(s_n, 0) = 0 \) is equivalent to

\[
(\partial_x V(s_n) + \lambda(s_n))\rho_i(s_n) = \frac{k_i}{k_i} \int_0^x h(\partial_x V(s_n) + \lambda(s_n))e^{-\frac{\lambda(s)}{k_i}} R_e(s_n) dy.
\]

Multiplying by a test function \( \varphi \in C^0([0, L]) \) and integrating over \([0, L] \), we obtain

\[
\int_0^L (\partial_x V(s_n) + \lambda(s_n))\rho_i(s_n)\varphi dx = \int_0^L \frac{k_i}{k_i} \left( \int_0^x h(\partial_x V(s_n) + \lambda(s_n))e^{-\frac{\lambda(s)}{k_i}} R_e(s_n) dy \right) \varphi dx.
\]

(27)
We divide our proof into two cases. 

This means that

where

taking the limit

for any \( \phi \in C^0([0, L]) \). This immediately gives

\[
(\partial_x V^* + \lambda^*) \rho^*_i \varphi dx = \frac{k_2}{k_i} \int_0^x h(\partial_x V^* + \lambda^*) e^{-\frac{1}{2} y R_e^* dy} \varphi dx
\]

which is equivalent to \( \mathcal{F}_1(\lambda^*, \rho^*_i, R_e^*, V^*) = 0 \ a.e. \)

We can write \( \mathcal{F}_2(\lambda(s_n), \rho_i(s_n), R_e(s_n), V(s_n)) = 0 \) and \( R_e(s_n, 0) = R_e(s_n, L) = 0 \) weakly as

\[
\int_0^L \partial_x R(e(s_n)) \partial_x \varphi dx + \frac{(\lambda(s_n))^2}{4} \int_0^L R_e(s_n) \varphi dx = - \int_0^L G_{2n} \varphi dx
\]

for any \( \varphi \in H^1_0(0, L) \), where

\[
G_{2n} := - \partial_x V(s_n) \partial_x R_e(s_n)
\]

\[
+ \left\{ \frac{\lambda(s_n)}{2} \partial_x V(s_n) - \partial_x^2 V(s_n) - h(\partial_x V(s_n) + \lambda(s_n)) \right\} R_e(s_n).
\]

Noting that

\[
\int_0^L (\partial_x^2 V(s_n) R_e(s_n) - (\partial_x^2 V^*) R_e^*) \varphi dx
\]

\[
\leq \int_0^L \partial_x^2 V(s_n) (R_e(s_n) - R_e^*) \varphi dx + \int_0^L (\partial_x^2 V(s_n) - \partial_x^2 V^*) R_e^* \varphi dx,
\]

taking the limit \( n \to \infty \) in the weak form, and using (23), we have

\[
\int_0^L (\partial_x R_e^*) (\partial_x \varphi) dx + \frac{\lambda^2}{4} \int_0^L R_e^* \varphi dx = - \int_0^L G_2^* \varphi dx \quad \text{for any } \varphi \in H^1_0(0, L),
\]

where

\[
G_2^* := - (\partial_x V^*) \partial_x R_e^* + \left\{ \frac{\lambda}{2} \partial_x V^* - \partial_x^2 V^* - h(\partial_x V^* + \lambda^*) \right\} R_e^* \in L^2(0, L).
\]

This means that \( R_e \in H^1_0(0, L) \) is a weak solution to \( \mathcal{F}_2 = 0 \). Furthermore, the standard theory of elliptic equations ensures that \( R_e \in H^1_0(0, L) \cap H^2(0, L) \) is also a strong solution to \( \mathcal{F}_2(\lambda^*, \rho^*_i, R_e^*, V^*) = 0 \). Similarly we can show \( \mathcal{F}_3(\lambda^*, \rho^*_i, R_e^*, V^*) = 0 \).

We now set

\[
x_* := \inf \{x \in [0, L] ; (\partial_x V^* + \lambda^*)(x) = 0\}.
\]

We divide our proof into two cases \( x_* = 0 \) and \( x_* > 0 \).
We first consider the case $x_0 > 0$. The equation (28), which holds for a sequence $x_\nu \to x_0$, yields the inequality

$$0 = (\partial_x V^* + \lambda^*)\|\rho_0\|_{L^\infty(I)} \geq \frac{k_c}{k_i} \int_0^{x_0} h(\partial_x V^* + \lambda^*) e^{-\frac{\lambda^*}{2} y} R^*_e dy.$$  

Together with the positivity (25) this implies that $(h(\partial_x V^* + \lambda^*) e^{-\frac{\lambda^*}{2} y} R^*_e)(x) = 0$ for $x \in [0, x_0]$. From the definition of $x_0$, we see that

$$(\partial_x V^* + \lambda^*)(x) > 0 \quad \text{for} \quad x \in [0, x_0),$$

so that $h(\partial_x V^* + \lambda^*) > 0$ on $[0, x_0]$. Therefore, $R^*_e(x) \equiv 0$ in $[0, x_0]$. Hence from (28) and (29), $\rho_0^* = 0$ a.e. in $[0, x_0]$. Now from the equation $\mathcal{F}_3(\lambda^*, \rho_0^*, R^*_e, V^*) = 0$ we see that $\partial_x V^*$ is a constant in $(0, x_0)$. Thus $\partial_x V^* + \lambda^* = 0$ in $[0, x_0]$. This contradicts the definition of $x_0$.

Now consider the other case $x_0 = 0$. We first suppose that there exists $y_0 > 0$ such that $(\partial_x V^* + \lambda^*)(y_0) > 0$. Let us set

$$y^* := \sup\{x < y_0; (\partial_x V^* + \lambda^*)(x) = 0\}.$$  

Note that $y^* \in [0, y_0)$ and $(\partial_x V^* + \lambda^*)(y^*) = 0$. On the other hand, integrating $\mathcal{F}_1(\lambda^*, \rho_0^*, R^*_e, V^*) = 0$ a.e. over $[y^*, y]$ for any $y \in [y^*, y_0]$ and using $\mathcal{F}_3(\lambda^*, \rho_0^*, R^*_e, V^*) = 0$, we have

$$(\partial_x V^* + \lambda^*)(\partial_x^2 V^* + e^{-\frac{\lambda^*}{2} y} R^*_e)(y) \leq \int_{y^*}^y \frac{k_c}{k_i} h(\partial_x V^* + \lambda^*) e^{-\frac{\lambda^*}{2} z} R^*_e dz$$

for a.e. $y \in [y^*, y_0]$. By (25) and (26), the left hand side is estimated from below as

$$(\partial_x V^* + \lambda^*)(\partial_x^2 V^* + e^{-\frac{\lambda^*}{2} y} R^*_e) \geq (\partial_x V^* + \lambda^*)\partial_x^2 V^* = \frac{1}{2} \partial_x \left\{ (\partial_x V^* + \lambda^*)^2 \right\} \quad \text{a.e.}$$

since $\partial_x V^*$ is absolutely continuous. The integrand on the right hand side of (30) is estimated from above by $Ce^{-b(\partial_x V^* + \lambda^*)^{-1}}$, due to the behavior of $h$; see (2). Consequently, substituting these expressions into (30), integrating the result over $[y^*, x]$, and using $(\partial_x V^* + \lambda^*)(y^*) = 0$, we have

$$(\partial_x V^* + \lambda^*)^2(x) \leq C \int_{y^*}^x \int_{y^*}^y e^{-b(\partial_x V^*(z) + \lambda^*)^{-1}} dz dy$$

for $x \in [y^*, y_0]$. Now let us define $x_n$ by

$$x_n := \inf\left\{ x \leq y_0; \partial_x V^*(x) + \lambda^* = \frac{1}{n} \right\}.$$  

Notice that $y^* < x_n$ and $(\partial_x V^* + \lambda^*)(x) \leq 1/n$ for any $x \in [y^*, x_n]$, since the continuous function $(\partial_x V^* + \lambda^*)$ vanishes at $x = y^*$. Then we evaluate (31) at

$$x = x_n$$

to obtain

$$\frac{1}{n^2} \leq C \int_{y^*}^{x_n} \int_{y^*}^y e^{-b(\partial_x V^*(z) + \lambda^*)^{-1}} dz dy \leq Ce^{-bn}.$$  

For suitably large $n$, this clearly does not hold. So once again we have a contradiction.

The remaining case is that $x_0 = 0$ and $(\partial_x V^* + \lambda^*) \equiv 0$. In this case, $\partial_x^2 V^* \equiv 0$ and so the equation $\mathcal{F}_2(\lambda^*, \rho_0^*, R^*_e, V^*) = 0$ yields $\partial_x^2 (e^{-\lambda^* x^2} R^*_e) = 0$. Solving this with (24) yields $R^*_e \equiv 0$. Then from $\mathcal{F}_3(\lambda^*, \rho_0^*, R^*_e, V^*) = 0$ we obtain $\rho_0^* \equiv 0$. Solving $\mathcal{F}_2(\lambda^*, \rho_0^*, R^*_e, V^*) = 0$ with (24) and $\rho_0^* \equiv R^*_e \equiv 0$, we also have $V^* \equiv 0$.  


Consequently $\lambda^* = 0$ holds and $\lim_{s \to 0} \lambda(s) = 0$. This contradicts Lemma 4.1, since $\sup_{s > 0} ||V(s)||_{C^2}$ is bounded.

Let us reduce Condition (d) in Theorem 3.4 to a simpler condition. We write the result directly in terms of the ion density $\rho_i$ and the electron density $\rho_e = R_e e^{-\lambda x/2}$.

**Lemma 4.4.** Assume the global positivity alternative (i) in Proposition 1. If we assume that the maximum of the densities $\sup_{s > 0} \{||\rho_i(s)||_{C^0} + ||\rho_e(s)||_{C^0}\}$ is bounded, then $\sup_{s > 0} \{||\rho_i(s)||_{C^1} + ||R_e(s)||_{C^2} + ||V(s)||_{C^0}\}$ is also bounded.

**Proof.** It is clear from $\mathcal{F}_3 = 0$ together with the definition $\rho_e = R_e e^{-\lambda x/2}$, that

$$\sup_{s > 0} ||V(s)||_{C^2} \leq C \sup_{s > 0} \{||\rho_i(s)||_{C^0} + ||\rho_e(s)||_{C^0}\} < +\infty. \quad (32)$$

Then from Lemma 4.2 we have

$$\sup_{s > 0} |\lambda(s)| < +\infty. \quad (33)$$

From (33) and the boundness of $||\rho_e(s)||_{C^0}$, we also have

$$\sup_{s > 0} ||R_e(s)||_{C^0} < +\infty. \quad (34)$$

Applying a standard estimate of one-dimensional elliptic equations to $\mathcal{F}_2 = 0$ with (32)–(34), we also deduce that $\sup_{s > 0} ||R_e(s)||_{C^2} < +\infty$. Now Lemma 4.3 implies that $\lim_{s \to 0} \{\inf_x (\partial_x V + \lambda)(s, x)\} \neq 0$. Together with (19), this leads to $\sup_{s > 0} ||\rho_i(s)||_{C^1} < +\infty$. Finally $\sup_{s > 0} ||\partial_2^2 V(s)||_{C^0} < +\infty$ follows from $\mathcal{F}_3(\lambda(s), \rho_i(s), R_e(s), V(s)) = 0$. \qed

We conclude with the following main result, which asserts that either $\rho_i$ and $\rho_e$ are positive along all of $\mathcal{K}$ with $0 < s < +\infty$ and $\rho_i + \rho_e$ is unbounded or else there is a half-loop of positive solutions going from $V^*_c$ to $V^-_c$.

**Theorem 4.5.** One of the following two alternatives occurs:

(A) Both $\rho_i(s, x)$ and $\rho_e(s, x) = (R_e e^{-\lambda x/2})(s, x)$ are positive for any $s \in (0, \infty)$ and $x \in I$. Furthermore, $\lim_{s \to +\infty} \{||\rho_i(s)||_{C^0} + ||\rho_e(s)||_{C^0}\} = \infty$.

(B) there exists $T^* > 0$ such that the following conditions hold:

1. Both $\rho_i(s, x)$ and $\rho_e(s, x)$ are positive for any $s \in (0, T^*)$ and $x \in I$;
2. $(\lambda(T^*), \rho_i(T^*), \rho_e(T^*)) = (V^*_c / L, 0, 0, 0)$;
3. $\rho_i(s, x) < 0$ and $\rho_e(s, x) < 0$ for $0 < s - T^* \ll 1$ and $x \in I$.

**Proof.** Suppose that (B), which is the same as the second alternative (ii) in Proposition 1, does not hold. Then the first alternative (i) in Proposition 1 must hold. Now in Theorem 3.4 there are five alternatives. Alternative (c) cannot happen because $\rho_i$ and $R_e$ are negative on part of the loop. Lemmas 4.1 - 4.3 assert that any one of (a) or (b) or (c) implies that $\sup_{s > 0} \{||\rho_i(s)||_{C^0} + ||R_e(s)||_{C^2} + ||V(s)||_{C^0}\}$ is unbounded. Then Lemma 4.4 implies that $\sup_{s > 0} \{||\rho_i(s)||_{C^0} + ||\rho_e(s)||_{C^0}\}$ must also be unbounded. This means that (A) holds. \qed

**Appendix A. Voltage–current curve.** Figure 4 shows some experimental data together with some comments that appear in [21], where the sparking voltage is denoted as $V_S$. The current appears to be unbounded, which is consistent with Alternative (A) of Theorem 4.5.
The Voltage-Current Characteristics

Let's now consider a typical laboratory discharge, taking place in a glass tube with metallic electrodes. The nature of the electrodes has little effect on the characteristics of the discharge. Commonly-used materials are carbon, platinum, iron, nickel or tungsten. The voltage source $E$ is connected in series with a current-limiting resistance $R$, so that the voltage between anode and cathode is $V = E - IR$. This relation is expressed by the load lines in the diagram, for values of $R$ equal to $R_1 > R_2 > R_3$. The irregular curve is the V-I characteristic of this device, distorted to show the various regions conveniently. Point $A$ is a stable point of operation for $R = R_1$. This can be seen as follows: suppose the current $I$ to be slightly reduced for some reason. Then $V$ becomes greater, according to the load line, while the voltage between anode and cathode becomes smaller. The difference in voltage acts to increase the current, restoring it to the value before the disturbance. If the current is slightly increased, we find a voltage deficit, which reduces the current, again bringing the operating point back to the original place. This will always happen if the V-I curve is more steeply inclined than the load line. At point $A$, the current is no more than a microampere, the discharge is dark, and is not self-sustained. We are in the Townsend region.

Now imagine $R$ reduced steadily from $R_1$ to $R_2$. Point $A$ moves up the curve until the sparking potential is reached. Now the voltage is sharply reduced, and the operating point is $B$, which is stable. The discharge is now self-sustaining as a glow discharge, and cathode heating is not sufficient to cause transition to an arc. If $R$ is further decreased, towards $R_3$, the voltage across the discharge increases until point $B'$ is reached. Although $B'$ is stable with respect to small fluctuations, cathode heating may be enough to increase the electron supply and lower the discharge voltage. This change is cooperative, and the discharge quickly moves to point $C$, where $V$ is lower and $I$ is greater. This is the arc, and operating point $C$ is stable. However, if $R$ is further reduced, the current will increase without bound until something melts. The regions where the discharge type changes are shown cross-hatched, to show that the actual values may not be clearly defined. This characteristic tells a lot about the circuit behavior of discharges, but it does not say much about the dynamic relations, only about the stable operating points.

Figure 4. voltage–current curve

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