Multi-player Multi-armed Bandits for Optimal Assignment in Heterogeneous Networks

Harshvardhan Tibrewal†, Sravan Patchala‡, Manjesh K. Hanawal and Sumit J. Darak

Abstract—We consider an ad hoc network where multiple users access the same set of channels. The channel characteristics are unknown and could be different for each user (heterogeneous). No controller is available to coordinate channel selections by the users, and if multiple users select the same channel, they collide and none of them receive any rate (or reward). For such a completely decentralized network we develop algorithms that aim to achieve optimal network throughput. Due to lack of any direct communication between the users, we allow each user to exchange information by transmitting in a specific pattern and sense such transmissions from others. However, such transmissions and sensing for information exchange do not add to network throughput. For the wideband sensing and narrowband sensing scenarios, we first develop explore-and-commit algorithms that converge to near-optimal allocation with high probability in a small number of rounds. Building on this, we develop an algorithm that gives logarithmic regret, even when the number of users changes with time. We validate our claims through extensive experiments and show that our algorithms perform significantly better than the state-of-the-art CSM-MAB, DE⁢TLS and DE⁢T⁢S⁢T⁢S⁢ algorithms.

Index Terms—Multi-player Multi-armed Bandits, Optimal regret, Pure exploration, Distributed learning

I. INTRODUCTION

Cognitive cellular networks are one of the key components of the next generation wireless networks. It promises seamless and high-speed connectivity by combining the features of both cellular and ad hoc networks [1,2], which is a much-desired requirement in all mission-critical and Internet of Things (IoT) applications. In the ad hoc component of such networks, a central controller may not always exist which makes the coordination among the users challenging. Further, users may not know characteristics (mean rewards) of the available channels, and these characteristics could be statistically different across users due to their geographical separations. Thus for effective utilization of network resources, users not only need to learn the channel characteristics experienced by them but also that experienced by the others. This work[3] develops distributed learning algorithms for such networks that achieve optimal network performance using signaling schemes.

A. Sensing and Signaling

Ad hoc networks are usually dynamic in nature and users may not know how many others are present in the network. If multiple users select the same channel, a collision occurs and all the users involved in a collision will get zero reward, thus degrading the total network throughput. The users in ad hoc networks are often equipped with sensing mechanisms which can be used to learn about the presence of other users and avoid collisions. Further, users can leverage sensing capabilities to exchange information through signaling without a central coordinator. A signal can be a specific pattern of transmission by a user which the other users can sense to gather information.

The amount of signaling required to achieve coordination depends on how effectively users can sense the channels. A network where users can sense all channels simultaneously (wideband sensing) requires fewer signals than a network where users can sense on only one channel at a time (narrowband sensing). However, wideband sensing is significantly costlier than narrowband sensing, as it requires multiple RF chains, higher speed ADCs and computationally intensive digital baseband processing algorithms [3]. Though using advanced sensing and computational power at the users’ end is at the heart of ad hoc and cognitive radio networks (CRNs), a distributed algorithm should not require too advanced capabilities to keep the cost of user equipment low. We thus focus on distributed algorithms that use minimal sensing capabilities to achieve optimal network performance. We first consider wideband sensing capability and then relax it to the more challenging narrowband case.

B. Multi-Player Multi-Armed Setting

The standard Multi-Armed Bandit (MAB) setting and its extensions involving multiple players are widely used to model performance of learning policies in CRNs [4,5], where channels are cast as arms of bandits, rates received on them as rewards and the users as players. The standard MAB deals with a single user in the network where the goal is to learn a policy that minimizes the regret [6], where regret of a given policy is defined as the difference between best expected cumulative reward (obtained with full knowledge of mean rewards) and that obtained by the given policy. In the multi-player setup, the regret is defined as the difference between the best expected cumulative network reward (sum over all players) with full knowledge of mean rewards and that obtained from the given policy. In the multi-player case, unlike the single-player case, users will not receive reward in each slot – in slots where collisions are incurred or signals for coordination are sent, no reward is collected and thus adds to regret.

We refer our setting to be static when the users in the system do not enter or leave throughout the algorithm. In realistic scenarios, the number of users can change with time. This is referred to as dynamic setting. We first develop an explore-and-commit algorithm that reaches near-optimal allocation quickly and then build on it to develop an algorithm that has sub-linear (logarithmic) regret. With minimal modifications we extend our algorithms to handle the dynamic setting.

C. Optimal Assignment and Performance Metric

Achieving optimal network performance i.e. maximizing sum of mean rewards obtained for all users, in our setup, is
equivalent to solving a maximum weight-matching problem on a weighted bipartite graph (or an assignment problem). Here, the users and channels correspond to the two sets of nodes that are to be paired and the mean rewards correspond to the edge weights. If mean rewards seen by all the users are known (referred to as full information), the users can apply any maximum weight matching algorithm, e.g., Hungarian method \([7, 8]\), to obtain an optimal assignment and occupy the channels as per the same.

Achieving the optimal assignment in a finite time is an impossible task in our setting as it requires all users to not only learn their channel characterization accurately but also exchange this information with all other users without any errors. We thus focus on developing a Probably Approximately Correct (PAC) algorithm that guarantees near-optimal assignment with high confidence. Our PAC metric for multi-player MAB is a natural generalization of PAC metric used in pure exploration setting of the standard MABs \([9, 10]\). This algorithm leverages the fact that if each user learns the mean rewards of channels experienced by herself and that by others within an \(\epsilon > 0\) accuracy, then the optimal assignment computed by them from their estimates gives a network reward within a \(2\epsilon N\) neighborhood of the optimal value (see Thm. \([1]\)). Thus, once all users gather ‘sufficiently’ many samples of channel rates and send ‘sufficiently’ many signals to indicate their observations, the network can reach a near-optimal assignment. We develop efficient signaling schemes that allow users to exchange their estimates so that all of them have ‘near full information’ of network rewards. We then develop algorithms to achieve logarithmic regret by repeating exploration, signaling and exploitation phases. Our contributions can be summarized as follows:

- We motivate our algorithm design from the max-weight matching method \([7]\). Our approach revolves around efficient exchange of information required for all users to learn the optimal assignment in a distributed fashion.
- We develop explore-and-commit algorithms for both wideband sensing (WS) and narrowband sensing (NS) capabilities, achieving near-optimal allocation with high probability in finite rounds.
- For NS, we develop Explore-Signal-Exploit-Repeat (ESER) algorithm and a variant (named mESER) that guarantees near-logarithmic and logarithmic regret, respectively under the assumption that the number of optimal assignment is known. To the best of our knowledge, mESER is the only algorithm that guarantees logarithm regret without requiring any problem-specific information (sub-optimality gap) and the time horizon.
- With minor changes in the signaling protocol, we extend the mESER to handle the the dynamic case. Unlike many algorithms in the literature, our algorithm does not require to restart when there is a change in number of users.
- We validate the performance of our algorithms via numerical experiments and show that performance-gains of our algorithm are significant, compared to the state-of-the-art algorithms, and the gains improve with the increase in the number of users in the network.

**Organization of paper:** In Section \([1]\) we describe the model and setup the performance metrics. Section \([3]\) bounds degradation in network reward due to lack of full knowledge of rate matrix. In Section \([5]\) we consider the static case and develop an explore-and-commit algorithm that guarantees near-optimal assignment with high confidence. In section \([5]\) we develop Explore-Signal-Exploit-Repeat (ESER) and show that expected regret achieved by its variant, named mESER, is almost logarithmic in \(T\). In Section \([7]\) we describe how mESER can be made to improve performance when the channel gains are homogenous. We provide a discussion on algorithm initialization in section \([7]\). We illustrate how our algorithm scales in the dynamic scenario in section \([8]\). Numerical experiments in Section \([9]\) validate our claims. Conclusions and future directions are given in Section \([10]\).

### D. Related work

Several works have addressed the problem of learning and coordination in multi-user ad hoc networks. In this subsection, we focus on literature employing the multi-armed bandit (MAB) based approach for channel selection, as it outperforms other approaches employing random or sequential-hopping-based channel selection \([11]\).

The \(\rho^{\text{rand}}\) \([4]\) approach employs MAB based upper-confidence bound (UCB) algorithm for channel characterization and selection. The UCB is combined with the rank-based-randomization approach to orthogonalize users in the best \(N\) channels, where \(N\) denotes the number of users in the network. Subsequent algorithms in \([12, 13]\) are based on \(\rho^{\text{rand}}\) and they offer further improvement in performance by reducing the number of collisions among users. However, major drawbacks of these algorithms are that they need prior knowledge of \(N\) and require that all channels are identical for all users (homogeneous channels). Recently, various algorithms \([14–21]\) have been proposed to improve the performance of CRNs. Among them, algorithms in \([14–16]\) work only for homogeneous channels while algorithms in \([17, 21]\) apply to heterogeneous channels and are closer to our work.

The heterogeneous channels are considered in \([17, 21]\). The \(\text{de}^2\) and \(\text{de}^3\)-TS algorithms in \([17, 19]\) employ distributed Bertsekas auction algorithm where players in a static network agree upon an allocation via exchange of bids. Their method finds a near-optimal allocation within finite number of rounds. Both the algorithms assume narrowband sensing and achieve near-logarithmic regret bounds. The CSM-MAB algorithm in \([18]\) aims to achieve a stable allocation, assuming wideband sensing capability. CSM-MAB guarantees convergence to stable allocation in finite time with high probability. The GYRO algorithm in \([20]\) assumes a centralized system, where users communicate their channel estimates to the controller who finds an allocation with low computational complexity. The authors in \([21]\) consider a static setting where users do not change and propose a fully-distributed algorithm with \(\log^2(T)\) regret bound.

Our work differs from the rest of literature as we consider the more challenging setting with unknown number of users that can enter and leave channel arbitrarily and see different gains across the channels (heterogeneous channels). We set our goal to achieve optimal allocation in the network and develop a distributed algorithm motivated by the Hungarian algorithm \([7]\), that achieves \(\log(T)\) regret bound.

Part of this work is accepted for presentation in INFOCOM 2019 \([22]\). The conference version gives expected regret bounds that hold with high confidence but does not study the dynamic setting. Moreover, the improved algorithm for the homogeneous case only appears in this work.

**II. Model and Setup**

Let \(N\) and \(K\) denote the number of users/players and number of channels/arms, respectively. We assume \(N \leq K\) and denote
Let $\pi : [N] \rightarrow [K]$ denote an assignment such that no two players are assigned the same arm. We denote set of all assignment functions as $\Pi$. If all the players know the mean reward matrix $M = \{\mu_{nk}\}$, then the optimal policy is to use an assignment given by

$$\pi^* = \arg \max_{\pi \in \Pi} \sum_{i \in [N]} \mu_{i, \pi(i)}.$$ 

To find the optimal solution $\pi^*$ in a distributed fashion requires each player to sample arms prohibitively large number of times and also require many signaling rounds to exchange information with other players, making it infeasible for all practical purposes. We thus focus on learning ‘approximate’ optimal solution with high probability, defined as follows:

**Definition 1:** For a given $\epsilon > 0$ and $\delta \in (0, 1)$, a policy $\pi$ is said to be $(\epsilon, \delta)$-optimal on a mean reward matrix $M$ if there exists a positive integer $T := T(M)$ such that

$$\Pr\left\{ \sum_{n \in [N]} \mu_{n, \pi^*_n} \geq \sum_{n \in [N]} \mu_{n, \pi(n)} - \epsilon \right\} \geq 1 - \delta \quad \forall \ t \geq T. \quad (1)$$

The above definition can be viewed as generalization of probably-approximately-correct performance guarantee in the pure exploration MAB problems with single player \cite{9,10} to the multi-player case.

We define expected regret of a policy $\alpha$ over a period $T$ as

$$R(T, \alpha) = T \sum_{n \in [N]} \mu_{n, \pi^*_n} - \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{n \in [N]} X^n_{\pi(n)} \right]. \quad (2)$$

We aim to develop policies that give $(\epsilon, \delta)$-optimal performance within a short time period $T$. Also, for a given $T$, we develop an algorithm that has sub-linear regret.

**Sensing model:** Any policy needs to incorporate an appropriate signaling mechanism so that the players can exchange information with each other. In our setup, information exchange happens through sensing. We first assume that each player can sense all the arms simultaneously in each round as in \cite{18}. This model is applicable in some restricted scenarios where players are equipped with sophisticated RF chain that can do wideband sensing. This setting also helps us better explain the main ideas in our algorithm. We then relax this requirement and allow players to sense only on one arm in each round. This model is applicable in more scenarios where players only need to do narrowband sensing. We refer to the first and second scenarios as ‘wideband sensing’ and ‘narrowband sensing’ models respectively.

## III. Optimal Allocation with Perturbed Matrix

Given full knowledge of the mean reward matrix $M$ to all the players, each player can find an optimal assignment using well known bipartite matching methods like Hungarian algorithm \cite{17,18} and play the arm as specified by the optimal assignment. However, in our setting each player can only estimate her mean rewards and signal the same to other players. Hence the players can only have an estimate of $M$, which at best can be guaranteed to be within a small interval around the true values. An optimal assignment obtained using the estimated matrix can be sub-optimal with respect to the true reward matrix. The following result bounds the loss incurred by playing this sub-optimal assignment. Denote by $f(M, \pi) := \sum_{i \in [N]} \mu_{i, \pi(i)}$, the network reward by applying assignment $\pi$ on a reward matrix $M$. 

\begin{figure}[h] 
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{A network with 2 users, 3 channels and rate matrix $M$.}
\end{figure}
Theorem 1: Let $M = \{\mu_{n,k}\}$ and $\hat{M} = \{\hat{\mu}_{n,k}\}$ be any mean reward matrices such that $|\mu_{n,k} - \hat{\mu}_{n,k}| \leq \epsilon \forall n \in [N], k \in [K]$ for some $\epsilon > 0$. Let $\pi^{*}$ and $\hat{\pi}^{*}$ denote the optimal allocations corresponding $M$ and $\hat{M}$ respectively. We have

$$f(M, \pi^{*}) - f(M, \hat{\pi}^{*}) \leq 2N\epsilon.$$  

Proof: We prove the result from the following inequalities:

$$-N\epsilon \leq f(M, \hat{\pi}^{*}) - f(\hat{M}, \hat{\pi}^{*}) \leq N\epsilon \tag{3}$$

and

$$-N\epsilon \leq f(\hat{M}, \hat{\pi}^{*}) - f(M, \pi^{*}) \leq N\epsilon. \tag{4}$$

The first inequality is obvious, since for any $\pi \in \Pi$, $f(M, \pi) - f(\hat{M}, \pi) = f(M - \hat{M}, \pi)$. Using the fact that each element in $M$ is far from $\hat{M}$ by at most $\epsilon$ and using $\pi = \hat{\pi}^{*}$, (3) follows. Note that adding or subtracting the same constant to all elements of $M$, does not change the optimal assignment.

The theorem statement thus follows from (3) and (4). □

This result suggests a method to achieve a network reward that is $\epsilon$-close to the optimal. Specifically, if all the players estimate each entry of $M$ within $\epsilon/2N$ accuracy and play an arm specified by an optimal allocation computed on the estimated matrix, then the network reward is $\epsilon$-optimal. The goal of estimating the elements of $M$ within $\epsilon/2N$ accuracy can be achieved in two steps. In the first step, each user estimates her mean rewards within $\epsilon/4N$ accuracy. In the second step, the players signal their estimated mean rewards within $\epsilon/4N$ accuracy to other players in a round-robin fashion. Then each player will have an estimated mean reward matrix that is within the $\epsilon/2N$ accuracy of the underlying true reward matrix. However, the issue with this approach is if the optimal assignment on the estimated matrix is not unique, then the optimal assignment obtained by each player could be different and multiple players may select the same arm resulting in collisions. We first assume that the optimal assignment computed on the estimated matrix is unique and then justify why this assumption holds, in Subsection IV-C.

We next develop distributed learning algorithms that exploit these facts to obtain optimal assignments.

IV. EXPLORE AND COMMIT ALGORITHMS

In this section, we develop PAC algorithms. In these algorithms, players explore the arms initially and exchange their estimates with all users via signaling. After this, the players compute the optimal allocation using the estimated mean reward matrix and play an arm assigned by it thereafter. The challenges are to avoid collisions among the players and learn mean rewards as quickly as possible. Also, we need efficient signaling schemes for players to exchange information.

A. Wideband sensing

In this subsection, we develop an algorithm named Distributed Optimal Assignment with Wideband Sensing (DOA-WS). The inputs to the algorithm are $K, T_{r}, T_{b}$ and the algorithm guarantees an $(\epsilon, \delta)$-optimal assignment within a finite number of rounds, where $\epsilon$ and $\delta$ are dependent on $K, T_{r}, T_{b}$ and $T_{b}$. The algorithm consists of three phases namely 1) Random Hopping (RH) 2) Sequential Hopping (SH) and 3) Bernoulli Signaling (BS). These phases run sequentially to estimate the matrix $M$. In the end, all players apply the Hungarian method and play the arm given by the optimal assignment thereafter. All subroutines are written in a distributed fashion. The pseudo-code of DOA-WS is given in Algorithm 1. The algorithm is run by each player separately. We suppress index $n$ to avoid cluttering in the presentation.

Algorithm 1 DOA-WS

1: Input: $K, T_{r}, T_{b}$
2: Initial Orthogonal Assignment: $(N, k, n) = RH(K, T_{r})$
3: Exploration: $\hat{\mu}_{n} = SH(K, k, T_{r})$
4: Signaling: $\hat{M} = BS(N, \hat{\mu}_{n}, T_{b})$
5: Find optimal assignment $\hat{\pi}^{*}$ on $\hat{M}$
6: Exploitation: Play $\hat{\pi}^{*}(k)$ till end

The RH phase ensures that all players are on different arms (orthogonalized). During each round of this phase, each player selects an arm uniformly at random until they experience a collision-free round. Once it happens, they continue playing that arm till the end of the phase. The length of the phase is set such that all players are orthogonalized with high probability.

Phase 1: Random Hopping (RH)

1: Input: $K, T_{r}$
2: Initialize: Set $Lock = 0$
3: for $t = 1 \ldots T_{r}$ do
4: \hspace{1em} if $Lock == 1$ then
5: \hspace{2em} Select the same arm, $\alpha_{t} = \alpha_{t-1}$
6: \hspace{1em} else
7: \hspace{2em} Randomly select a channel, $\alpha_{t} \sim U([K])$
8: \hspace{2em} Set $Lock = 1$ if no collision is observed
9: \hspace{1em} end if
10: end for
11: Set $k = \alpha_{t}$. Transmit on $k$ and sense all other arms
12: $A = \{i \in [K]:$ transmission is sensed on arm $i \neq k\}$
13: Set $N = |A| + 1$ and $n = \{|j \in A: j < k\} + 1$
14: Return $N, n$ and index of current arm $k$.

The SH phase ensures that all the players learn mean rewards they see on the arms. In this phase, all the players select the arms sequentially i.e. arm $(k+1) \mod K$ is played in the next round after playing arm $k$ in the current round. Since an orthogonal allocation is maintained in each round, no collisions occur in this phase. The length of the phase is set such that the mean rewards are estimated with high accuracy.

In the BS phase, each player signals her observation of mean rewards to others and also learns reward seen by others, through their signals. This is achieved as follows: The BS

2The Hungarian algorithm gives an assignment that minimizes the sum reward. Since we want the sum reward to be maximized, the input given to this algorithm is the negative of the rate matrix, thus ensuring the sum reward is maximized. Henceforth, it is assumed implicitly that the negative of the rate matrix is provided to the algorithm.
Phase 2: Sequential Hopping (SH)

1: Input: $K, T_s, k, n$
2: Set $a_{T_s+K} = k$, $r_i = 0$ $\forall i \in [K]$
3: for $t = T_s + K + 1 \ldots K T_s + K$ do
4: Play arm $a_t = (a_{t-1} + 1)$ mod $K$
5: Observe reward $X_{a_t}$ and update $r_{a_t} = r_{a_t} + X_{a_t}$
6: end for
7: Estimate $\hat{\mu}_{n,k} = r_k / T_s$ $\forall k \in [K]$
8: Return $\hat{\mu}_n = [\hat{\mu}_{n,k}]$.

Phase 3: Bernoulli Signaling (BS)

1: Input: $K, T_b, \hat{\mu}_n$
2: Set $a_{i,j} = 0$ $\forall i,j \in [K]$
3: for $j = 1, 2 \ldots K$ do
4: for $T_b$, number of rounds do
5: Transmit with probability $\hat{\mu}_{n,j}$ on current arm
6: $A = \{ k \in [K] : \text{transmission is sensed on arm } k \}$
7: Update $a_{i,j} = a_{i,j} + 1$ for all $i \in A$
8: end for
9: end for
10: Estimate $\hat{\mu}_{i,j} = a_{i,j} / T_b$ $\forall i,j \in [K]$
11: Return $M = [\hat{\mu}_{i,j}]$.

At the end of the BS phase, each player will have the same estimate of the rate matrix $M$. Each player then finds the optimal assignment on $\hat{M}$ and plays it thereafter. We state the following lemmas before providing the guarantees of DOA-WS:

Lemma 1: Let $\delta \in (0, 1)$. If RH phase is run for $T_r = \frac{\log(\delta/3K)}{\log(1-1/4K)}$ number of rounds then all the players will be orthogonalized with probability at least $1 - \delta/3$.

Lemma 2: For any given $\delta \in (0, 1)$ and $\epsilon > 0$ set $T_s = \frac{8N\epsilon^2}{\delta^2} \log \left( \frac{6NK}{\delta} \right)$. Then estimated mean rewards at the end of the SH phase are such that $|\mu_{n,k} - \hat{\mu}_{n,k}| \leq \epsilon/4N$ for all $n \in [N], k \in [K]$ with probability at least $1 - \delta/3$.

Lemma 3: For any given $\delta \in (0, 1)$ and $\epsilon > 0$ set $T_b = \frac{8N^2\epsilon^2}{\delta} \log \left( \frac{6NK}{\delta} \right)$. Let $\mu_n$ denotes the mean reward vector that a player $n \in [N]$ signals. Then estimated mean rewards matrix $\hat{M} = [\hat{\mu}_{i,j}]$ at the end of the BS phase will be such that $|\mu_{n,k} - \hat{\mu}_{n,k}| \leq \epsilon/4N$ for all $n \in [N], k \in [K]$ with probability at least $1 - \delta/3$.

Proof for Lemma 1 is similar to [10] [Lemma 1]. The proofs for Lemmas 2 and 3 are straightforward applications of the Hoeffding’s concentration inequalities [23] and have been omitted due to space constraints.

Theorem 2: For a given $\epsilon > 0, \delta \in (0, 1)$ let $T_r, T_s$ and $T_b$ be set as in the Lemmas [4], [2] and [3] respectively. Then the DOA-WS policy is $(\epsilon, \delta)$-optimal after $T_r + KT_s + KT_b$ number of rounds for any mean reward matrix $M$.

Proof of Theorem [2]: If (A) denotes all players orthogonalized in RH phase, (B) denotes all players estimate their mean reward within $\epsilon/4N$ accuracy and (C) denotes all players signal their estimates to others within $\epsilon/4N$ accuracy, the estimated matrix $\hat{M}$ at each player will be such that $|\hat{M} - M| \leq \epsilon/2N$ after $T := T_r + K(T_s + T_b)$ rounds, where $I$ is an all-ones $N \times K$ matrix. Theorem [1] then implies that if all players play the optimal assignment obtained from $\hat{M}$, DOA-WS policy will be $\epsilon$-optimal for all $t > T$. Hence we have

$\Pr \{ \text{DOA-WS is } \epsilon \text{-optimal for } t > T \} \geq \Pr \{ \text{A,B,C holds} \}
= 1 - \Pr \{ \text{any of A,B,C doesn’t hold} \} > 1 - 3\delta/3 = 1 - \delta,$

where the inequality follows from Lemma [1], [2] and [3].

Remark 1: Note that mean reward corresponding to player $n$ in $\hat{M}$ is $\hat{\mu}_n$ and not $\mu_n$, i.e., the values that players signaled to other players in the BS phase and not what she estimated in the SH phase. This is done to ensure that all players have the same $\hat{M}$ so that they find the same optimal assignment.

Remark 2: The performance in the BS phase can be improved by listening only on the arms on which players are transmitting. In this case, we can set $T_b = \frac{8N^2\epsilon^2}{\delta} \log \left( \frac{6NK}{\delta} \right)$ and still achieve the same result. We skip the details.

B. Packetized vs Bernoulli Signaling

In the BS phase of DOA-WS, players exchange information with high accuracy using Bernoulli distributions. This can be achieved in multiple other ways. One possibility is that each player encodes her estimates and signals this using a certain accuracy. Theorem 2 will hold for packetized signaling as well, by setting $T_b = \frac{8N^2\epsilon^2}{\delta} \log \left( \frac{6NK}{\delta} \right)$.

Packetized signaling requires less number of time slots compared to Bernoulli signaling for information exchange. However, it is more prone to errors. An error in a single bit, especially if it is a leading bit, can cause significant distortion in the information exchanged by packetized signaling, whereas Bernoulli signaling is more robust to such distortions. The choice of a signaling scheme depends on sensing accuracy of players and how robustly signaling can be done.

C. Handling Multiple Optimal Assignments

In the design of DOA-WS, we assumed that the optimal allocation computed from $M$ will be unique. However, there could exist multiple optimal assignments due to which the players may compute different optimal allocations. In such a case, it may happen that two players are assigned to the same arm leading to continuous collisions.

When multiple allocations exist, if the ties are always broken in a deterministic way, for example, in favor of a player with the smaller index, the algorithm will always return an optimal assignment that is same across all players. This is
true because, given a matrix, the Hungarian algorithm is a deterministic algorithm when provided with a deterministic method of breaking ties (see scipy implementation [24]).

At the end of a signaling phase, each player will have the same reward matrix and hence evaluate the same assignment.

D. Narrowband Sensing

In this subsection, we study the case of narrowband sensing where players can listen or transmit on one arm at a time. With narrowband sensing, more time slots are required to exchange information. We present Distributed Optimal Allocation for Narrowband sensing (DOA-NS) algorithm, which is a modification of the DOA-WS algorithm [1]. All the phases in DOA-NS are same as that in DOA-WS except for the signaling phase and estimation of the number of users (end of RH phase).

Estimating number of users can be completed in $K$ time slots in DOA-NS as follows: After $T_r$ rounds of RH phase, random hopping stops and every player notes her current channel (referred to as a reserved channel). Subsequently, in round $(T_r + k)$, $k \in [K]$, the player whose reserved arm is the $k^{th}$ channel transmits while all others listen on it. At the end of $(T_r + K)$ rounds, all players count the number of transmissions sensed in the last $K$ rounds. This gives the number of players in the network, utilizing $T_r + K$ rounds.

The BS phase in DOA-NS, referred to as BS-NS, is executed as follows: For the first $K T_b$ rounds, player 1 signals on channel 1 while all other listen on it. Player 1 transmits with probability $\bar{\mu}_1$ for $T_b$ rounds, with probability $\bar{\mu}_2$ in the next $T_b$ rounds, and continuing like this, she transmits with probability $\bar{\mu}_K$ in the last $T_b$ rounds. Then, the process is repeated by player 2 for the next $K T_b$ rounds using her estimates of channel rates as probabilities, while all other players listen on channel 2. This phase ends after the $N^{th}$ player completes signaling her rates. In total, BS-NS takes $N K T_b$ rounds and everyone gets an estimate of the rate matrix.

After the BS-NS phase, players use the Hungarian Algorithm to find optimal allocation and play the arm as per the assignment. In total, DOA-NS takes $(T_r + K) + K T_b + N K T_b$ time slots to give an $(\epsilon, \delta)$-optimal policy. Note that the length of the signaling phase increases by a factor $N$ compared to that in the DOA-WS algorithm. The proof for this is similar to that of the Theorem [1] and is omitted.

V. ALGORITHM FOR REGRET MINIMIZATION

In this section, we focus on algorithms that minimize regret. The expected regret is shown at least logarithm in $T$, i.e., lower bounded as $\Omega(\log T)$ in [25]. We first develop an algorithm, named Explore-Signal-Exploit-Repeat (ESER), that has near-logarithmic regret and give a modification that achieves the optimal regret. We restrict to the more challenging narrowband sensing scenario.

The pseudo code of ESER is given in Algorithm 2 which is run by each player independently. To avoid clutter in notation, we drop the player’s index. The algorithm runs in epochs of growing lengths and takes as inputs $K, T_r, \{T_b(i)\}$ and $\{b(i)\}$, where the latter two quantities are positive sequences.

The system is initialized with an orthogonal allocation using the RH subroutine. Next, the algorithm proceeds in epochs. Each epoch consists of exploration, signaling and exploitation phases and corresponds to one run of the DOA-NS algorithm. Firstly, the subroutine SH is called, where players explore arms to estimate their rewards using all the samples collected so far. The updated estimates are communicated through a signaling protocol. Any signaling mechanism, packetized or Bernoulli can be used. Every player now applies Hungarian method on the estimated rate matrix to find the optimal assignment and plays the arm she gets, for double the time than in the previous epoch. The end of the exploitation phases marks the end of the epoch and the process is repeated.

Algorithm 2 Explore-Signal-Exploit Repeat (ESER)

1: Input: $K, T_r, T_b(i), b(i), \forall i \geq 1$
2: Initialization: $l = 1, M = 0$
3: Initial Orthogonal Allocation: $(N, k, n) = RH(K, T_r)$
4: while $(t < T)$ do
5: Exploration phase: $\bar{\mu}_n = SH(K, k, T_b(i))$
6: $\hat{\mu}_n = ((l - 1) * \bar{\mu}_n + \mu_n) / l$
7: Signaling: $M = BS-NS(N, \hat{\mu}_n, T_b(i))$
8: Find optimal assignment $\pi^*$ on $M$
9: Exploitation phase: Play $\pi^*(n)$ for $2^l$ rounds
10: $l = l + 1 ; t = t + K T_b(i) + N K T_b(i) + 2^l$
end while

A. Regret Analysis

The regret bound for ESER is obtained by bounding the regret from the i) Sequential Hopping, ii) Signaling and iii) Exploitation phases separately. The following notations are helpful to state results. Let $\pi_1(M)$ and $\pi_2(M)$ denote the optimal and next-best assignments on matrix $M$, i.e.

$$\pi_1(M) := \{ \pi : \max_{\pi \in \Pi \leq \{\pi_1\}} \sum_{n \in [N]} \mu_n \pi(n) \}$$

and $\pi_2(M) := \{ \pi : \max_{\pi \in \Pi \leq \{\pi_1\}} \sum_{n \in [N]} \mu_n \pi(n) \}$

$$\Delta_{\text{max}} := \max_{\pi \in \Pi} \{ \sum_{n \in [N]} \mu_n \pi(n) - \sum_{n \in [N]} \mu_n \pi(n) \}$$

$$\Delta_{\text{min}} := \{ \sum_{n \in [N]} \mu_n \pi(n) - \sum_{n \in [N]} \mu_n \pi(n) \}$$

The quantity $\Delta_{\text{max}}$ corresponds to the maximum regret incurred in a round and the quantity $\Delta_{\text{min}}$ corresponds to the difference in rewards between the best and the next-best allocations. Notice that when the optimal allocation is not unique, $\Delta_{\text{min}} = 0$. In the following we assume that there exists a unique optimal allocation, i.e., $\Delta_{\text{min}} > 0$. We use shorthand $R(T) := R(T, ESER)$.

Theorem 3: Assume the players are orthogonalized at the end of RH phase. Using packetized communication, the regret bound for ESER algorithm is as follows

i) If $\Delta_{\text{min}}$ is known, set $\epsilon < \Delta_{\text{min}}$ and $T_s = 8N^2/\epsilon^2$, then

$$R(T) \leq (KT_s \Delta_{\text{max}} + N^2 K T_b) \log(T) + C_1 N K \Delta_{\text{max}} \tag{5}$$

$$\leq O(N^2 K \log(T))$$

ii) If $\Delta_{\text{min}}$ is unknown, choose $\beta \in (0, 1)$ and set $\epsilon(l) = \log^{1-\beta/2}(t_l)$ and $T_s(l) = 8N^2 \log^{1-\beta}(t_l)$, where $t_l$ denotes the start of the $l^{th}$ exploration phase. Then

$$R(T) \leq 8 N^2 K \Delta_{\text{max}} \log^{1-\beta}(T) + N^2 K \log^{1-\beta/2}(T) + \frac{N^2 K \log^2(4N \log(T)) + C_2 + 3 N K \Delta_{\text{max}}}{N^2 K \Delta_{\text{max}} \log^{1-\beta}(T)}, \tag{6}$$

where $C_1 = 6/(\epsilon^2)$, $C_2 = 2^{\beta/2} (1 + \Delta_{\text{max}})$, $C_3 = 3 r^\nu (1 - r)$ with $r = 2e^{-1}$ and $l^\nu = (\Delta_{\text{min}}/\log(2))$.

A proof is given in the Appendix. If we use Bernoulli Signaling instead of packetized signaling, the bound in (5) worsens to $O(N^2 K \log^2(T))$ and that in (6) worsens to $O(N^2 K \log^{2+\beta}(T))$. Hence, ESER achieves near-logarithmic regret with packetized signaling.
B. Modified ESER (mESER) algorithm

In this subsection, we propose a methodology to achieve logarithmic regret from ESER when the optimal allocation is unique. In the Modified ESER (mESER) given in Algorithm 4, we estimate \( \Delta_{\text{min}} \) at the end of the \( l^{th} \) epoch using \( M = \{\mu_n,k\} \), which is an estimate of \( M \). Specifically, each player computes

\[
\hat{\Delta}_{\text{min}}(l) := \left\{ \sum_{n \in \mathcal{N}} \mu_{n,k}(n) - \sum_{n \in \mathcal{N}} \hat{\mu}_{n,k}(n) \right\},
\]

where \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) are computed on \( \hat{M} \), which is a ‘good’ estimate of \( \Delta_{\text{min}} \) and its accuracy improves with epochs. This estimate of \( \Delta_{\text{min}} \) will help us set \( \epsilon(l) \) appropriately to achieve logarithmic regret as we will see later in this subsection. The following result is key to improve performance of ESER and achieve logarithmic regret bounds. The modification to ESER is as follows: Once the condition \( \hat{\Delta}_{\text{min}}(l) > \epsilon(l) \) holds for some \( l \), we update \( \epsilon(l+1) \) to any values less than \( \hat{\Delta}_{\text{min}}(l) - \epsilon(l) \) and maintain this value for all later epochs.

**Lemma 4:** For any \( \epsilon \geq 0 \) and \( \Delta_{\text{min}} > 0 \), if \( |M - \hat{M}| \leq \epsilon l / 2N \), we have

\[
- \epsilon \leq \Delta_{\text{min}} - \hat{\Delta}_{\text{min}} \leq \epsilon,
\]

where \( \hat{\Delta}_{\text{min}} \) is computed from \( \hat{M} \).

Proof is given in the Appendix.

Algorithm 3 Modified ESER (mESER)

1. **Input:** \( K, T_r, T_b(l), T_b(l) \forall l \geq 1 \)
2. **Initialization:** \( l = 1, M = 0, \text{lock} = \text{False} \)
3. **Initial Orthogonal Allocation:** \( (N, k, n) = RH(K, T_r) \)
4. **while** \( (t < T) \) **do**
5. **Exploration phase:** \( \tilde{\mu}_n = SH(K, k, T_b(l)) \)
6. **Initialization:** \( \tilde{\mu}_n = ((l-1) * \mu_n + \mu_n) / l \)
7. **Signaling:** \( M(l) = BS-NS(N, \tilde{\mu}_m, T_b(l)) \)
8. **Exploration phase:** Find \( \tilde{\pi}'(n) \) and play for \( 2l \) rounds
9. **Calculate** \( \hat{\Delta}_{\text{min}}(l) \) from \( \hat{M}(l) \)
10. **if** \( \text{lock} = \text{False} \) **then**
11. **if** \( \hat{\Delta}_{\text{min}}(l) > \epsilon(l) \) **then**
12. Fix \( \epsilon(l+1) = \epsilon_0 \) and \( T_b(l+1) = 8N^2 / \epsilon_0^2 \)
13. **else** \( \epsilon(l+1) = \log^{-\beta/2}(T_l) \);
14. **end if**
15. **else** \( \epsilon(l+1) = \epsilon(l) \)
16. **end if**
17. \( l = l + 1 ; t = t + KT_b(l) + NKT_b(l) + 2l \)
18. **end while**

**Theorem 4:** Assume the players are orthogonalized at the end of RH phase. The regret of mESER is, denoted \( R(T) := R(T, mESER) \) upper bounded as

\[
R(T) \leq \Delta_{\text{max}}N^2KC_1 + N^2KC_2 + C_3 + C_4NK\Delta_{\text{max}} + \Delta_{\text{max}}T_0K + N^2KC_1 \log(T) = O(N^2K \log(T))
\]

where \( T_0, T_1, C_1, C_2, C_3, C_4 \) are constants independent of \( T \).

**Proof Outline:** Setting \( T_b(l) = 8N^2log^{\beta/2}(T_l) \) and \( \epsilon(l) = log^{-\beta/2}(T_l) \) for \( \beta \in (0,1) \), there exists \( l_1 \) such that \( \hat{\Delta}_{\text{min}}(l_1) > \epsilon(l_1) \). This is because of the decreasing nature of \( \epsilon(l) = log^{-\beta/2}(T_l) \) and the bounded nature of \( \hat{\Delta}_{\text{min}}(l_1) \in [\Delta_{\text{min}} - \epsilon(l_1), \Delta_{\text{min}} + \epsilon(l_1)] \) (as shown in Lemma 4).

After \( l_1 \) epochs of Modified ESER algorithm, we set \( \epsilon(l) = \epsilon_0 \) where \( \epsilon_0 < \hat{\Delta}_{\text{min}}(l_1) - \epsilon(l_1) \). For all further epochs \( l > l_1 \) we set \( T_b(l) = T_0 = 8N^2 / \epsilon_0^2 \).

Now, \( \forall l > l_1, \epsilon(l) < \{\hat{\Delta}_{\text{min}}(l_1) - \epsilon(l_1)\} \in [\Delta_{\text{min}} - 2\epsilon(l_1), \Delta_{\text{min}}] \leq \Delta_{\text{min}} \) (from Lemma 4) \( \Rightarrow \epsilon(l) < \Delta_{\text{min}} \) and thus from here on now, we will not be incurring any regret with probability \( 1 - \delta(l) \).

Since \( l_1 \) is finite, the regret incurred in the initial \( l_1 \) epochs is constant (independent of \( T \)). But after \( l_1 \) epochs, the regret equations are similar to the case when \( \Delta_{\text{min}} \) was known in the ESER algorithm. Consequently, we obtain logarithmic regret even when \( \Delta_{\text{min}} \) is unknown. The proof is continued in the appendix.

**Remark 3:** We assume each player has enough computational resources to calculate best and second-best assignments [26] in every epoch. mESER has logarithmic regret bound and outperforms other algorithms in literature (see Fig. 3 in Sec. IX). More importantly, it does not require any problem specific information (\( \Delta_{\text{min}} \)).

**Remark 4:** If we allow players to simultaneously sense and transmit on different channels in the narrowband sensing scenario, the regret bound for packetized signaling changes from \( O(N^2) \) to \( O(N\Delta_{\text{max}}) \).

**Remark 5:** If the optimal allocation is not unique, then the mESER will not give logarithmic regret because the stopping criterion is never met. However if we know the number of optimal allocation a priori, then the algorithm mESER can be adjusted to still achieve logarithmic regret. The modification is as follows: let \( p \) denote the number of optimal allocations. Compute \( \hat{\Delta}_{\text{min}} \) in every epoch as the difference in the sum rewards between the \( p^{th} \) best and the \((p + 1)^{th} \) best allocation in mESER will ensure that the regret remains logarithmic. This is because the stopping criterion will be met under this calculation and the set of optimal allocations will be distinguished from the sub-optimal ones.

VI. HOMOGENEOUS NETWORK

The mESER algorithm is developed for the worst case scenario where the mean rewards on the channels can be potentially different for all the users. The performance of mESER can be significantly improved if the mean rewards for all the users is the same across all the channels, i.e., rewards are homogeneous across the users. This could happen, for example, when all the users are nearby in a geographical location and experience similar ambient conditions. We derive the algorithm in homogeneous setting by simplifying our mESER. In mESER, each user learned an \( N \times K \) matrix of mean rewards, whereas now, this collapses to just one row of \( K \) elements. Further, the optimal allocation is now just the top \( N \) channels, irrespective of the order in which they are distributed among users. To manage these changes effectively, we appoint a leader, who can simply be the user on the first non-empty channel at the beginning of the algorithm.

Modified mESER algorithm for the homogeneous case is given in Algorithm 4.

The modifications are as follows:

- After the indexing, a leader is chosen (by default user with index 1)
- During the exploration phase only the leader explores the arms while others remain silent.
- The leader communicates channel rankings and \( \hat{\Delta}_{\text{min}} \) to others via signaling.

In the exploitation phase, the users play top \( N \) channels as indicated by the leader. Each user picks a channel based on his index ranking. We consider any permutations in the assignment.
SIC-MMAB is different from ours as they just focus on the homogeneous case and its regret is also upper bounded to the SIC-MMAB algorithm in [27]. SIC-MMAB works as follows.

We note that our algorithm for the homogeneous case is similar to [27].

Algorithm 4 mESER (Homogeneous)

1: Input: $K, T_r, T_b(l), T_b(l) \forall l \geq 1$
2: Initialization: $l = 1, M = 0, lock = False$
3: Initial Orthogonal Allocation: $(N, k, n) = RH(K, T_r)$
4: Select Leader
5: while $(t < T)$ do
6:  Exploration phase: Leader explores each channel
7:  and updates based on channel reward feedback
8:  Signaling: Signal channel ranking and $\Delta_{min}$
9:  Exploitation phase: Find $\hat{n}(n)$ and play for $2^l$ rounds
10:  if $lock == False$ then
11:  \hspace{1em} if $\Delta_{min}(l) > \epsilon(l)$ then
12: \hspace{2em} Fix $\epsilon(l + 1) = \epsilon_l$ and $T_s(l + 1) = T_0$
13: \hspace{2em} for some $\epsilon_0 < \Delta_{min}(l) - \epsilon(l)$ : Set $lock = True$
14:  \hspace{1em} else $\epsilon(l + 1) = \log^{\frac{1}{2l}(t_l)}$
15:  \hspace{1em} end if
16:  else $\epsilon(l + 1) = \epsilon(l)$
17:  end if
18:  $l = l + 1$ ; $t = t + KT_s(l) + KT_b(l) + 2^l$
19: end while

VII. ORTHOGONALIZATION IN RANDOM HOPPING

The regret bound in Theorem 4 \[4\] is conditional as it holds provided that the Random hopping phase results in an orthogonal allocation. In this section we obtain regret bound that holds irrespective of whether or not the random hopping phase results in orthogonal allocations. We first consider the simple case when time horizon is known and then modify the mESER algorithm to make it an anytime algorithm that do not require to know $T$.

A. Time $T$ is known

Running the random hopping phase for a duration of $T_r = \frac{\log(\delta_k/K)}{\log(1-1/\sqrt{K})}$ (from Lemma 1) at the beginning of the mESER algorithm, the player orthogonalize with probability $1 - \delta_R$ and the regret bound of Thm. 5 \[5\] holds. If the the player do not orthogonlaze, the regret will be $NT_r$. Hence the expected regret of mESER is as follows:

\[ R(T) \leq (1 - \delta_R)R(T|O) + \delta_R R(T|NO) + NT_r \]

where $R(T|O)$ denotes the regret of mESER when players orthogonalize at the end of RH and $R(T|NO)$ denotes the regret when they do not. If $T$ is known apriori, then we can set $\delta_R = \frac{1}{T}$ and get

\[ R(T) \leq (1 - \delta_R)R(T|O) + \delta_R NT + NT_r \]

\[ = (1 - \frac{1}{T})R(T|O) + \frac{NT}{T} + NT_r = O(N^2K\log(T)) \]

which is of the same order as that in Thm 4 \[4\].

B. Time $T$ is unknown (anytime algorithm)

In this section we discuss a version of mESER that achieves logarithmic regret without the knowledge of $T$. In this version of mESER, RH phase is repeated in each epoch for a fixed number still the players find a collision free transmission on a channel and locks on it, and once locked, she continues to transmit on her locked channel in the RH phase of the subsequent channels. Only the players who do not get locked on a channel continue RH phase in the next epoch. We refer to this version of mESER as mESER1. In mESER1, the probability that the players are not orthogonalized in an epoch decreases with the epoch number. This ensures $O(N^2K\log(T))$ regret as shown in the following theorem.

Theorem 6: Let the length of the RH phase in mESER1 is $T_r := -\log^2(1 - \frac{1}{4K})$. Then, the regret of mESER1 is $O(N^2K\log(T))$

Proof: In mESER1, at the end of the $l^{th}$ epoch, a total of $\Omega_l^l$ rounds of random hopping would have taken place if the player has not yet locked on a channel. Then the probability that the players are not orthogonalized is at most $\delta_R(l) = Ke^{-\frac{l}{2}}$ (from Lemma 1). The expected regret incurred in the $l^{th}$ epoch is bounded as follows:

\[ R(l) \leq (1 - \delta_R(l))R_m(l) + N\delta_R(l)^2 + NT_r \leq R_m(l) + NT_r + NK e^{-\frac{l}{2}}, \]

where $R_m(l)$ denotes the regret incurred by the mESER in the $l$ epoch if the players are orthogonalized. The total regret
incurred by mESER, denoted \( R(T) := R(T, mESER) \), is then given by

\[
R(T) = \sum_{l=1}^{T} R(I) \leq \sum_{l=1}^{T} R_{mESER}(l) + NT, l_0 + NK \sum_{l=1}^{T} e^{-l} T \leq R(T, mESER) + NT, \log(T) + NK(2/e) \log(T) = O(N^2K \log(T))
\]

Remark 6: The users who are still performing random hopping have yet to learn their own rates. As soon as a user is ‘locked’ in the system, she learns her rates via exploration, and then communicates rates with others. This adjustment of a user in the system is similar to an arrival of a ‘new user’ in a dynamic setting. This is described in detail in VIII-B.

VIII. DYNAMIC mESER

In a real network users may join and leave the network continuously. In this section we propose an approach to extend the mESER algorithm to such dynamic scenarios. In this section, our analysis is focused on a heterogeneous system with users having wideband sensing capabilities.

The analysis we present holds in general, but for the ease of understanding we assume users arrive or depart uniformly at random, one at a time. But as time increases, the duration of exploitation phase increases exponentially. Thus the probability of arrivals or departures happening in an exploitation phase is significantly more than that happening in other phases. Therefore we consider arrivals and departures happening only in the exploitation phase.

With wideband sensing capabilities detecting an arrival or departure is straightforward. During the exploitation phase, all users keep track of the number of other channels on which transmissions occur, which would precisely be equal to \( N - 1 \) (without counting her own channel). An increase or decrease in the number of such channels uniquely signals an arrival or a departure, respectively, to all the users. The channel on which the change has occurred is also uniquely determined by all the users.

We proceed to show how arrivals and departures of users would perturb the system and how the algorithm could be modified to adapt to the situations.

A. Departures

When a subset of users leave, all the remaining users know it and they simply recompute the optimal assignment and take over the new channel as per the new assignment. The re-allocation of the users is done immediately after users leave because existing users might get better rewards on the channels which were previously in use.

Lemma 6: The mESER algorithm will adapt to departures and converge to optimal assignment with \( O(\log(T)) \) regret.

Proof: Since the all current users know when a subset of users leave the network, each of them re-index themselves and calculate \( \Delta_{\min} \) after the departures. Based on the condition between \( \epsilon(l) \) and \( \Delta_{\min} \) from line (10) in Algorithm the algorithm proceeds and adjusts to this perturbation. The system might require additional epochs to settle to optimal assignment, but the order of regret is still \( O(\log(T)) \).

B. Arrivals

In case of an arrival during the exploitation phase, we modify the mESER algorithm to accommodate the learning phase for the new user. When a new user joins, the existing users and the new user start selecting the channels sequentially as per the channel index and allow the new user to learn her rates on all the channels. When the players select the channels sequentially no collisions occur in the network. To ensure everyone in the system has estimates of the rates with the same precision (same \( \epsilon \), one of the old users signals to the new user the number of exploration rounds required and the number of epochs covered so far. This signalling requires only a constant number of rounds.

The new user now explores for the number of rounds specified, during which the remaining old users stay silent and incur full regret (in the worst case scenario). We could have them exploiting other channels, but for simplicity, assume otherwise. Once the new user finishes the required duration of exploration, everyone in the system now has the estimates within the same precision. They all undergo the Signalling phase in which the new rate matrix is communicated, followed by exploitation and then normal execution of the algorithm.

Lemma 7: The above stated modification to mESER algorithm will adapt to an arrival and converge to the optimal assignment with \( O(\log(T)) \) regret.

Proof: When the new user joins, the total regret which the system incurs is proportional to the total time required for the new user to learn her rates with the same precision. Since the user arrives at a finite time instant, the amount of time for which he would be required to do exploration, and thus the amount of time for which the system incurs full regret, would remain finite. This regret also does not scale with the total time \( T \) of the algorithm. Thus in the overall regret for the system, the addition of a new user would simply add a constant regret which is dependent on the arrival instant of the user and does not scale with the total time \( T \) of the algorithm. Thus the regret remains \( O(\log(T)) \) if the mentioned modifications are made to the mESER algorithm.

C. Constraints on the effective arrival rate

As seen in the previous two subsections, a single arrival or departure, results in no change in the order of the regret in the mESER algorithm. But if the number of arrivals occur at certain rates, then the regret from each of the arrivals, despite being a constant, could add up to be more in order than that of mESER. Specifically, if the effective arrivals (taking the departures into account) is lesser than \( O(\log(T)) \) till round \( T \), then the overall order of the regret incurred in the dynamic scenario still remains \( O(\log(T)) \). This is summarized in the following theorem.

Theorem 7: Let the total number of arrival and departure over a period \( T \) is at most \( O(\log(T)) \). Then the regret of mESER algorithm adapted to the dynamic case is at most \( O(\log(T)) \).

Proof follows from the discussion above.

For the narrowband case, modifications to the algorithm can be constructed in a similar fashion as done in the earlier sections. All the results hold for the stricter narrowband sensing scenario as well.

IX. EXPERIMENTAL RESULTS AND OBSERVATIONS

To demonstrate the effectiveness of our algorithms, we present simulations and comparisons with state-of-the-art algorithms in literature. For all simulations, we consider that
each player draws Bernoulli rewards from each arm. For each player-arm pair, the mean reward is set uniformly at random from \([0, 1]\). Our setup can be viewed as communication over a binary symmetric channel with unknown transmission rates. All performance curves are obtained by averaging 50 simulation runs. We assumed packetized signaling in all the runs. For a fair comparison, we compare our algorithms against algorithms that use similar sensing capabilities.

A. Reward comparison in Wideband sensing setting

We compare the performance of DOA-WS with the CSM-MAB algorithm \([13]\) that also assumes wideband sensing. CSM-MAB aims to achieve stable allocation in the network by allowing players to swap channels with each other. In CSM-MAB, time slots are divided into frames of size \(2K\). In each frame, an initiator (a player selected randomly) requests other players occupying channels better than her current channel, to swap channels with her. The requests are sent as per a preference list of the initiator. The frame structure is such that the initiator proposes a swap by signaling on the channel she likes to occupy, and the player occupying that channel signals back if she accepts the swap. CSM-MAB guarantees convergence to a stable allocation in finite time with high probability.

We run DOA-WS and CSM-MAB for \(T = 10^6\) rounds on the same problem-instance each time. Average network reward obtained are shown in Fig (2) where we fix number of channels as \(K = 12\) and vary the number of users as \(N = \{8, 10, 12\}\). Similar observations were made when we fixed the number of users and varied the number of channels, but we skip those figures due to space constraints. The DOA-WS performs better than the CSM-MAB in all cases, and the improvement is significant, particularly for higher values of \(N\).

The performance of DOA-NS, which permits sensing of only one channel at a time, is also shown in Fig (3) Even though DOA-NS has restricted capabilities, it performs better than CSM-MAB eventually. The better performance of DOA-WS and DOA-NS can be attributed to the following observations:

Optimal assignment vs Stable allocation: An optimal allocation is also a stable allocation. Hence DOA-WS and DOA-NS aim to reach the "best" stable allocation among all possible allocations. In CSM-MAB, players use UCB indices to rank channels and always try to move to their best channels. It may happen that the players reach some stable allocation that is far from the optimal in the initial stages and after that no player can switch to her better channels, resulting in lower network throughput.

Faster convergence: A player can move to better channel in CSM-MAB only when she is the initiator. Since the initiator is chosen in a probabilistic fashion, multiple time slots pass where allocations will not improve, keeping the network in a sub-optimal "state" for a longer duration. Also, even after reaching a stable allocation, there can be collisions in the network as some players that become initiators propose swaps in each frame. However, this never happens in DOA-WS (or NS) as no collisions occur after the random hopping phase.

Higher number of empty channels: With the increase in the number of empty channels, we see in Fig (2) that the performance gain DOA-WS has over CSM-MAB reduces. Intuitively, this is because the initiators will have higher degrees of freedom with more empty channels thus increasing their chances of reaching a better stable allocation.

B. Regret Comparison in Narrowband Sensing

We compare the performance of the mESER and ESER algorithms against \(d^3, d^3-\text{TS}\) algorithms given in \([19]\). All four algorithms assume a narrowband sensing scenario. \(d^3\) and \(d^3-\text{TS}\) alternate between exploration and exploitation phases with an auction mechanism in between the phases. In the exploration phase, each player samples the channels for a fixed number of rounds \((\gamma)\) in a round-robin fashion (only one player explores at a time). After every exploration phase, players use Bertsekas auction mechanism to find an \(\epsilon\)-optimal allocation. Players then exploit the allocation for the exponentially large number of rounds (growing over phases). We set \(\gamma = 100\) and \(\gamma = 400\) for \(d^3\) and \(d^3-\text{TS}\) respectively, and \(\epsilon = 0.001\) as suggested in \([19]\) [Sec. VI]. We set \(T_s = 100\) in ESER and mESER. We use packetized signaling for all four algorithms and the number of bits is set according to \(\epsilon\) accuracy. All the algorithms are run for \(T = 10^6\) rounds.

Fig (3) demonstrates cumulative regret plots of \(d^3\), \(d^3-\text{TS}\), ESER and mESER for a fixed number of channels \(K = 12\) and varying number of users \(N = \{6, 10, 12\}\). As seen, mESER and ESER perform significantly better than both \(d^3\) and \(d^3-\text{TS}\). Also, from Thm. \([4]\) we know that mESER achieves logarithmic regret while ESER (Thm. \([3]\)), \(d^3\) and \(d^3-\text{TS}\) \([19]\) all incur near-logarithmic regret. This performance improvement is also visible in Fig (3). When \(N\) increases, the performance of our algorithms improve significantly which can be attributed to the following two reasons:

Simultaneous exploration: Recall that in the SH phase all players are on different channels and sample arms simultaneously without colliding with each other. Whereas in both \(d^3\) and \(d^3-\text{TS}\), the players take turns to explore the arms which makes learning slow.

Bipartite Matching The number of iterations required by the Bertsekas auction algorithm is of order \(O(N^2/\epsilon)\) and hence increases fast with the number of players. Further, as the number of players increases, the number of "losing bids" increases. Every player corresponding to a "losing bid" does not have any allocation at the end of the auction iteration and thus incurs heavy regret for that round. Thus, for an \(N\) user system, the time complexity for communication through auctioning is \(O(N^3/\epsilon)\) whereas our algorithms perform signaling in \(O(N^2)\).

C. Reward comparison in the Dynamic Setting

We look at the performance of the mESER algorithm in the dynamic setting, where the number of players change with time. We compare the average reward of our algorithm with D-CSM-MAB algorithm of \([28]\), which also considers users varying over time. We run both the algorithms for \(2 \times 10^5\) rounds for \(K = 10\) channels and varying number of users. As shown in Fig. (4) the arrivals and departures are indicated as green and red vertical lines respectively. We expect that during departures, our algorithm immediately adjusts to this perturbation and can recompute optimal system reward. This is observed near the red vertical lines in the figure. Further, we observe that our algorithm incurs some regret during arrivals but quickly converges to the optimal allocation and maintains higher average reward compared to the D-CSM-MAB. This transition time comes from the initial exploration performed by the newly arrived user in the system. Since this time is of the order-logarithmic in the time elapsed, it ensures that the regret remains of order \(-\log(T)\).
Fig. 2: Average reward comparison. We set $K = 12$ and $N = \{8, 10, 12\}$ in plots (a),(b),(c) respectively.

Fig. 3: Regret Comparison (with 95% confidence intervals). We set $K = 12$ and $N = \{6, 10, 12\}$ in (a), (b), (c), respectively.

Fig. 4: Average reward comparison in dynamic case. We set $K = 10$ and $N$ varies between 7 to 9.

X. Conclusion

We studied an ad hoc network with multiple players with each player experiencing different and unknown rates on different channels. There is no central coordinator to facilitate communication between the players and the goal is to achieve the highest possible throughput in the network in a distributed fashion. We provide explore-and-commit algorithms for both wideband and narrowband scenarios that guarantee network throughput that converges close to optimal with high confidence in finite time. We then extended the algorithm to achieve logarithmic regret, even in cases where the number of users was changing with time. To the best of our knowledge, ours is the only algorithm that guarantees logarithmic regret in the multi-player setting without requiring to know any problem specific details (sub-optimality gap). We showed that our algorithms perform significantly better than the state-of-the-art CSM-MAB, $dE^3$ and $dE^3$-TS algorithms.

In our work, we primarily focused on the distributed algorithms. It is interesting to establish the lower bounds and to know what is the best one can achieve. Though our algorithm mESER has a $\log(T)$ dependence on $T$, its dependence on $N$ is $O(N^2)$. It is interesting to see if this can be improved.

Appendix

A. Proof of Lemma 4

Let $\hat{M}(l)$ be the estimated matrix after learning and signaling in the $l^{th}$ epoch such that $|M - \hat{M}| \leq \epsilon I$. Let $\hat{\pi}_1(l)$ and $\hat{\pi}_2(l)$ be the best and second-best allocations on $M(l)$ respectively as calculated in V-A. Let $\pi_1$ and $\pi_2$ be similar quantities in case of the matrix $M$. Also, we define $\hat{\Delta}_{\min}(l) = f(M(l), \hat{\pi}_1(l)) - f(M(l), \hat{\pi}_2(l))$.

Case 1: If $\pi_1 = \hat{\pi}_1$, $f(M, \hat{\pi}_2) \leq f(M + \epsilon I, \hat{\pi}_2) \leq f(M + \epsilon I, \pi_2)$ where the second inequality holds because $\pi_1 = \hat{\pi}_1 \neq \hat{\pi}_2$ and thus $\hat{\pi}_2$ cannot give better reward on $M$ than $\pi_2$.

Case 2: If $\pi_1 \neq \hat{\pi}_1$, $f(M, \hat{\pi}_2) \leq f(M, \hat{\pi}_1) \leq f(M + \epsilon I, \hat{\pi}_1) \leq f(M + \epsilon I, \pi_2)$ where the last inequality holds because $\pi_1 \neq \hat{\pi}_1$ and thus $\hat{\pi}_1$ cannot give better reward than $\pi_2$ on $M$.

$\Rightarrow$ $f(M + \epsilon I, \pi_2) = f(M, \pi_2) + N \epsilon$, $f(M, \hat{\pi}_2) - f(M, \pi_2) \leq N \epsilon$.

A similar comparison of $\hat{M}$ with $M - \epsilon I$ instead of $M + \epsilon I$ proves that $f(M, \pi_2) - f(M, \hat{\pi}_2) \leq N \epsilon$. 

We thus show \(-2N\epsilon \leq \Delta_{\text{min}} - \hat{\Delta}_{\text{min}} \leq 2N\epsilon\) from Eqn. 4 and the above result. Replacing \(\epsilon\) by \(\epsilon/2N\) ends the proof. \(\blacksquare\)

\[R^o(T) \leq \sum_{t=1}^{l,b} \Delta_{\text{max}} T_s K = \Delta_{\text{max}} KT_s \log(T)\]

In the signalling phase, set \(T_b = \lceil \log_2(4N/\epsilon) \rceil\)

\[R^s(T) \leq \sum_{t=1}^{l,b} T_b K N^2 = N^2 KT_s \log(T)\]

In the exploitation phase, since \(\epsilon < \Delta_{\text{min}}\), the expected regret would be \(R^e(T) \leq 2\Delta_{\text{max}}\epsilon\).

\[R^e(T) = \Delta_{\text{max}} \sum_{t=1}^{l,b} \epsilon 2^l \Pr\{\hat{M}(l) - M > \epsilon/2N\} \leq \Delta_{\text{max}} \sum_{t=1}^{l,b} 2^l * 3NKe^{-\epsilon^2 lT_s/8N^2} \leq 3NK\Delta_{\text{max}} \sum_{t=1}^{\infty} 2^{-l} = C_1 NK\Delta_{\text{max}}\]

\[\text{Proof of ii): In the exploitation phase, every epoch has } T_s(l) = 8N^2 \log^8(\beta(l)) \text{ time slots till the } l^{th} \text{ epoch after which it is a constant equal to } T_0, \text{ giving a total regret of}\]

\[R^o(T) \leq \sum_{t=1}^{l-1} \Delta_{\text{max}} T_s(l) K + \sum_{t=l+1}^{l+b} \Delta_{\text{max}} T_s(l) K \leq \Delta_{\text{max}} K \log(l_T) T_s(l) + \Delta_{\text{max}} T_0 K \log(T) \leq \Delta_{\text{max}} KC_1 + \Delta_{\text{max}} T_0 K \log(T)\]

For the packetized signaling, \(T_b(l) = \lceil \log_2(4N/(\epsilon/l)) \rceil\) hence \(T_b(l) \leq \log_2(4N) + \log^8(\beta(l))\). Also, \(\forall l > l_1, T_b(l) = T_1 = \lceil \log_2(4N/(\epsilon_0) \rceil\). Thus, regret in the signalling phase is

\[R^s(T) \leq \sum_{t=1}^{l,b} T_b K N^2 = N^2 KT_s \log(T)\]

We now look at the regret in the exploitation phase. Over the \(l^{th}\) epoch, the expected regret would be \(R^e(l) \leq 2\epsilon (1/\Delta(l) + \Delta_{\text{max}}\delta(l))\). After \(l_1\) epochs, we set \(\epsilon(l) = \epsilon_0 < \Delta_{\text{min}}\). Thus the overall regret \(R^e(T) = \sum_{l=1}^{l_b} R^e(l)\) would be:

\[R^e(T) \leq \sum_{l=1}^{l_1} 2\epsilon(l) + \Delta_{\text{max}} \sum_{l=1}^{l_1} 2\delta(l) = R^e_1(T) + R^e_2(T)\]

\[R^e_1(T) = \sum_{l=1}^{l_1} 2\epsilon(l) + \Delta_{\text{max}} \sum_{l=1}^{l_1} 2\delta(l) = 2^{l+1} + \Delta_{\text{max}} = C_3\]

\[R^e_2(T) = \sum_{l=1}^{l_1} 2\epsilon(l) + \Delta_{\text{max}} \sum_{l=1}^{l_1} 2\delta(l) = \Delta_{\text{max}} \sum_{l=1}^{l_1} 2\epsilon(l) + \Delta_{\text{max}} \sum_{l=1}^{l_1} 2\delta(l) \leq \Delta_{\text{max}} \sum_{l=1}^{l_1} 2\epsilon(l) + \Delta_{\text{max}} \sum_{l=1}^{l_1} 2\delta(l) \leq 3NK\Delta_{\text{max}} \sum_{l=1}^{l_1} 2\epsilon(1-l_1) = C_4 NK\Delta_{\text{max}}\]

\[\text{where } C_4 = 3r^l/(1-r) \text{ and } r = 2e^{-1}. \]

\[\square\]

\[D. \text{ Proof of Theorem 5}\]

We assume the algorithm is orthogonalized by the random hopping phase. For users to explore arms with precision \(\epsilon\) and error probability \(\delta\), the number of samples needed are \(T_s\). This is given by,

\[\Pr(\tau=1,...,K|\mu_{ij} - \mu_{ij} > \epsilon) \leq \sum_{i=1}^{j=K} \Pr(\mu_{ij} - \mu_{ij} > \epsilon) < K * 2e^{-2\epsilon^2 T}\]

\[K * 2e^{-2\epsilon^2 T/2} < \delta \implies T_s > \frac{1}{2e^2} \log e \frac{2K}{\delta}\]

The exploitation phase hence requires \(KT_s\) slots for estimation of mean rewards of channels. On relaxing assumptions in Thm 1 for the homogeneous setting, we require a precision estimate of \(2\epsilon\) instead of \(2N\epsilon\) to be less than \(\Delta_{\text{min}}\). The reason is outlined in Lemma 5.

Let’s assume that after \(l_1\) epochs, the mESER algorithm finds an \(\epsilon < \Delta_{\text{min}}/2\). Use \(T_s(l) = 1/2\epsilon(l)^2\) and \(T_0 = 1/2\epsilon_0^2\). Then,

\[R^o(T) \leq \sum_{l=1}^{l-1} \Delta_{\text{max}} T_s(l) K + \sum_{l=1}^{l+b} \Delta_{\text{max}} T_s(l) K \leq \Delta_{\text{max}} K \log(T) \leq \Delta_{\text{max}} KT_0 K \log(T)\]

\[R^s(T) \leq \sum_{l=1}^{l,b} T_b K N^2 = N^2 KT_s \log(T)\]
For the packetized signaling, $T_p(l) = \lceil \log_2(1/(e(l)) \rceil$ hence $T_p(l) \leq \log_2(1/\epsilon(l))$. Also, $\forall l > 1$, $T_p(l) = T_1 = \lceil \log_2(1/\epsilon_0) \rceil$.

Thus, regret in the signaling phase is

$$R^S(T) \leq \sum_{l=1}^{T_p(l)} T_0(l) K = K \sum_{l=1}^{T_p(l)} T_0(l) \leq K \sum_{l=1}^{T_p(l)} T_0(l) + K \sum_{l=1}^{T_p(l)} T_0(l) = K \log(\epsilon_0^{-1} + 1) + KT_1 \log(T) = K C_2 + KT_1 \log(T)$$

We now look at the regret in the exploitation phase. Over the $l$th epoch, the expected regret would be $R^E(l) \leq (\epsilon(l) - \delta(l)) \Delta_{max} \beta(l)$. After $l_1$ epochs, we set $\epsilon(l) = \epsilon_0 < \Delta_{min}$. Thus the overall regret $R^E(T) = \sum_{l=1}^{T_p(l)} R^E(l)$ would be:

$$R^E(T) \leq \sum_{l=1}^{T_p(l)} 2^{\epsilon(l) + \delta(l)} \Delta_{max} \beta(l) \leq 2^{\epsilon_0^l} (1 + \Delta_{max}) = C_3$$

$$R^E_1(T) = \sum_{l=1}^{T_p} 2^{\epsilon(l)} \leq 2^{\epsilon_0^l (1 + \Delta_{max})} = C_3$$

$$R^E_2(T) = \Delta_{max} \sum_{l=1}^{T_p(l)} 2^{\epsilon(l)} = \Delta_{max} \sum_{l=1}^{T_p(l)} 2^{\epsilon_0^l} \leq \Delta_{max} \sum_{l=1}^{T_p(l)} 2^{\epsilon_0^l}$$

$$\leq 2K \Delta_{max} \sum_{l=1}^{T_p(l)} (2^{\epsilon_0^l}) = C_4 K \Delta_{max}$$

where $C_4 = \epsilon_0^l / (1 - r)$ and $r = 2 \epsilon_0^{-1}$.

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