Statistical Measures of Complexity for Strongly Interacting Systems

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Abstract

In recent studies, new measures of complexity for nonlinear systems have been proposed based on probabilistic grounds, as the LMC measure (Phys. Lett. A \textbf{209} (1995) 321) or the SDL measure (Phys. Rev. E \textbf{59} (1999) 2). All these measures share an intuitive consideration: complexity seems to emerge in nature close to instability points, as for example the phase transition points characteristic of critical phenomena. Here we discuss these measures and their reliability for detecting complexity close to critical points in complex systems composed of many interacting units. Both a two-dimensional spatially extended problem (the 2D Ising model) and a \infty-dimensional (random graph) model (random Boolean networks) are analysed. It is shown that the LMC and the SDL measures can be easily generalized to extended systems but fails to detect real complexity.

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1 Introduction

The quantitative characterization of complexity in nature has received considerable attention over the last decade [1-4]. Although there is no universal agreement about the definition of what complexity is, increasing evidence suggests that it often emerges close to marginal instability points of some kind. Critical phase transitions are a particularly well-studied example [5]. In all these systems, some generic properties like self-similar structures, $1/f$ noise and maximum information transfer seem to emerge in a rather spontaneous way.

In order to quantify the degree of complexity in a system, many measures have been proposed [1,6-9]. Comparative classifications have been made based on general characteristics like the type of partitions in phase space, statistical structure, etc [10,11]. In spite of the large number of proposed measures of complexity, it seems to be a matter of fact that these measures have to share some stringent conditions.

2 The complexity measure LMC

Recently, López-Ruiz, Mancini and Calbet (LMC) presented a new measure based on a simple set of assumptions [12]. In short, let us assume that an arbitrary dynamical system has $N$ accessible states which belong to a set $\Sigma_\mu = \{x_i(\mu) ; i = 1, \ldots, N\}$ and have an associated probability distribution

$$\Pi_\mu = \{p_i(\mu) = P[x = x_i(\mu)]; i = 1, \ldots, N\}$$ (1)

Here $\mu$ stands for a given parameter which allows to describe the transition from the ordered (low $\mu$) to the disordered (high $\mu$) regimes. As usual, $\sum_{i=1}^{N} p_i(\mu) = 1$ and $p_i(\mu) > 0 \forall i = 1, \ldots, N$. The LMC measure is based on a combination of two different quantities: (a) the Boltzmann entropy,

$$H_\mu(P_\mu) = -\sum_{j=1}^{N} p_j(\mu) \log p_j(\mu)$$ (2)

where $P_\mu \equiv (p_1(\mu), \ldots, p_N(\mu))$ and (b) the so-called disequilibrium $D_\mu$, defined as:

$$D_\mu(P_\mu) = \sum_{j=1}^{N} \left( p_j(\mu) - \frac{1}{N} \right)^2$$ (3)

Using (2) and (3) the LMC measure of complexity is defined by the functional:

$$C_\mu(P_\mu) = H_\mu(P_\mu)D_\mu(P_\mu)$$ (4)

The basic idea involves the interplay of two different tendencies: the increase of entropy as the system becomes more and more disordered and the decrease in $D_\mu$ as the system
approaches equiprobability/disorder. The previous definition can be easily generalized to the continuum, i.e.:

\[ C_\mu = H_\mu D_\mu = -\left( \int p_\mu(x) \log p_\mu(x) \, dx \right) \left( \int p_\mu^2(x) \, dx \right) \]  

This measure gives small values for highly homogeneous (ordered) or heterogeneous (disordered) states but it must reach a maximum at some intermediate value \( \mu^* \). The application of this measure to some well known nonlinear dynamical systems described by iterative maps gave consistent results. For example, the study of the logistic map \( x_{n+1} = \mu x_n (1 - x_n) \) showed that \( C_\mu \) undergoes a rapid increase for \( (\mu - \mu_c) \to 0 \) close to the critical point defining the onset of the period-three window.

![Figure 1: Entropy (open circles) and disequilibrium (open squares) for a single spin from a 40 x 40 Ising model. In all figures concerning this model, \( \tau = 2 \times 10^4 \) transients were discarded and averages were performed over \( T = 10^4 \) time steps (Glauber dynamics). We can see that both quantities reach their extreme values near \( T = T_c (= 2.27) \). Inset: 1-spin LMC complexity (eq. 12) computed from the previous sets of values. A maximum is obtained at \( T = T_c \) with a sharp decay to zero for \( T > T_c \). But there is no guarantee that the maximum of \( C_\mu \) will occur at the appropriate \( \mu^* \). In fact, the LMC measure can be written as a combination of various \( H_\beta \)-Renyi entropies.
of a probability measure $\eta$. This seems reasonable, but there are too many possible ansatizes which would do the same. These problems have been recently discussed [13] in relation with the LMC measure. In this context, Feldman and Crutchfield [13] have shown that the LMC measure is neither an intensive nor an extensive thermodynamic variable. Another problem in the previous definition is the specific form of $D_\mu$. The equiprobability is a consequence of an underlying assumption of weak or no interaction. Such distribution can be reached from a variational argument. Specifically, the variation of the functional:

$$
\delta \left[ H_\mu - \alpha \left( \sum_{j=1}^{N} p_j - 1 \right) \right] = 0
$$

leads to $p_j = 1/N$ and so a maximum $H_\mu = \log N$. But if other constraints play a role (as it occurs in far-from-equilibrium systems) then disorder does imply equiprobability.

Let us first show that LMClike measures can be easily extended to complex, spatially extended systems. The presence of correlations/interactions in a given system can be mea-
sured through the spatial correlation function [14]. For a given sequence \( S = s_1, s_2, \ldots, s_N \) when \( s_i \in \Sigma = \{ a_\alpha \} \) (with \( i = 1, \ldots, k \)) it is given by

\[
\Gamma(\delta) = \sum_\alpha \sum_\beta a_\alpha a_\beta P_{\alpha\beta}(\delta) - \left( \sum_\alpha P_\alpha \right)^2
\]  

(7)

Two-point correlation functions are used in statistical physics as a quantitative characterization of phase transitions. In the thermodynamic limit \( \Gamma(\delta) \) is expected to diverge at criticality or reach a maximum if the system size is finite. For one of these systems, both temporal and spatial patterns contain information about the existence of correlations arising from the local interactions.

As a standard example, let us consider the 2D Ising model. In this paper all of our results for the Ising model are obtained with a \( L = 40 \) lattice and using Glauber dynamics. This example allows us to consider a binary \( (N = 2) \) set of states and so the relation between \( H_\mu \) and \( C_\mu \) is guaranteed to be univocal [12]. Here \( \mu \) is the temperature \( T \) and the set of possible states for each unit is simply \( \Sigma_T = \{ \uparrow, \downarrow \} \), where the arrows stands for spin up and down, respectively. The mutual information \( M(S) \) for two \( \delta \)-neighbors spins clearly fulfil the requeriment of being low at the ordered and chaotic regimes and maximum at the transition point. We have:

\[
M(\delta) = \sum_{i \in \{\uparrow, \downarrow\}} \sum_{j \in \{\uparrow, \downarrow\}} P_{ij}(\delta) \log \left( \frac{P_{ij}(\delta)}{P_i(\delta)P_j(\delta)} \right)^2
\]  

(8)

Now at \( T < T_c \) most of the spins are in the same direction (say up) so \( P_{\uparrow, \uparrow} \approx 1 \) and \( P_i = P_j = P(\uparrow) \approx 1 \) and so \( M(\delta) \approx 0 \). Here \( \delta_{ab} \) is the Dirac delta. When \( T > T_c \) we have \( P(i,j) \approx P_i P_j \) and so \( M(\delta) \to 0 \) as \( T \) grows. Information transfer becomes maximum at \( T_c \) [4] as a consequence of long-range correlations. In fact it can be shown that -for binary sequences- a formal relation between the correlation function and the mutual information can be derived [14]:

\[
M(\delta) \approx \frac{1}{2} \left( \frac{\Gamma(\delta)}{P_0(\delta)P_1(\delta)} \right)^2
\]  

(9)

which leads to the conclusion that \( M(\delta) \) decay to zero at a faster rate than \( \Gamma(\delta) \). So if \( \Gamma(\delta) \approx \delta^{-\beta} \) then \( M(\delta) \approx \delta^{-2\beta} \). Beyond these specific relations, any properly defined complexity measure should give equivalent results.

Clearly, if only a single spin is used, we can write the previous definitions for the disequilibrium and entropy as

\[
D_T(P(\uparrow)) = \left( P(\uparrow) - \frac{1}{2} \right)^2 + \left( (1 - P(\uparrow)) - \frac{1}{2} \right)^2
\]  

(10)

\[
H_T(P(\uparrow)) = -P(\uparrow) \log P(\uparrow) - ((1 - P(\uparrow)) \log(1 - P(\uparrow))
\]  

(11)
and the LMC measure will be given by

\[ C_T(P(\uparrow)) = D_T(P(\uparrow))H_T(P(\uparrow)) \]  \hspace{1cm} (12)

Our results are shown in figure (1). We can see that \( H_T \) grows fast when approaching \( T_c = 2.27 \) from \( T < T_c \). This is an expected result as far as \( \langle M(T = T_c) \rangle = 0 \) and so on the average \( P(\uparrow) = 1/2 \). On the other hand if we take \( D_T \) we can see that it drops to zero at \( T_c \), as expected. The LMC measure is shown in the inset. We see a fast growth of \( C_T \) with a sharp decay at the critical point. So this characterization gives nearly null complexity for \( T > T_c \). This result, however, does not correspond with the analysis of the model by means of standard techniques. As an example, we show in figure 2 the average fluctuation of the magnetization for the Ising model. It shows a maximum at \( T_c \) and a decay for \( T > T_c \). Such decay is not as sharp as the one given from the LMC measure, and it measures in fact the existence of correlations for \( T > T_c \) which are not taken into account by the 1-spin LMC quantity.

3 LMC extended and distance to independence

The LMC measure can be extended by considering the statistics of interacting units. The reason of this choice is clear: long range correlations are created through interactions between nearest spins, as defined by the Ising Hamiltonian \( H_{\text{Ising}} = -\sum J S_i S_j \), being \( J \) the coupling constant. A first possibility is to use the statistics of \( K \)-spin blocks. The joint entropy for a \( K \)-block spin system is defined as

\[ H_T^{(K)} = - \sum_{i_1 \in \{\uparrow, \downarrow\}} \sum_{i_2 \in \{\uparrow, \downarrow\}} ... \sum_{i_K \in \{\uparrow, \downarrow\}} p(i_1, i_2, ..., i_K) \log p(i_1, i_2, ..., i_K) \]  \hspace{1cm} (13)

and the corresponding disequilibrium will be given by

\[ D_T^{(K)} = - \sum_{i_1 \in \{\uparrow, \downarrow\}} \sum_{i_2 \in \{\uparrow, \downarrow\}} ... \sum_{i_K \in \{\uparrow, \downarrow\}} \left( p(i_1, i_2, ..., i_K) - \frac{1}{2^K} \right)^2 \]  \hspace{1cm} (14)

For \( K = 2 \), two given spins (here we take two nearest neighbors) are considered and a set of joint probability measures can be used \( p_{ij} = P[s_a = i; s_b = j] \), i.e., the joint probability of having the \( a \)-spin in state \( i \) and the \( b \)-spin in state \( j \). Again, we have a normalization condition \( \sum p_{ij} = 1 \). Because \( p_{ij} \) introduces correlations in a very general way, we can redefine the disequilibrium in terms of these values as:

\[ D_T^{ij} = \sum_{i \in \{\uparrow, \downarrow\}} \sum_{j \in \{\uparrow, \downarrow\}} \left( p_{ij} - \frac{1}{4} \right)^2 \]  \hspace{1cm} (15)
and the joint entropy as:

\[ H_T = - \sum_{i \in \{\uparrow, \downarrow\}} \sum_{j \in \{\uparrow, \downarrow\}} p_{ij} \log p_{ij} \]  

These are simple extensions of the previous LMC approach but now the interaction (and so the intrinsic correlations) between parts of the system are taken into account. In figure 3 the previous quantities are shown. We can clearly observe that both quantities behave smoothly close to \( T_c \). This behavior leads to an LMC-complexity that shows again a maximum at \( T_c \) but with a slower decay for \( T > T_c \). This is consistent with our explicit consideration of correlations into the joint probabilities.

![Graph showing joint entropy and disequilibrium for the 2D Ising model.](image)

Figure 3: Joint entropy (16) (open squares) and disequilibrium (15) (filled triangles) for the 2D Ising model. Inset: corresponding LMC complexity (product of Joint entropy and disequilibrium). Now a maximum at \( T_c \) is also obtained but correlations on both sides are detected (compare with figure 1, inset)

An important drawback of these measures, as pointed before, is the definition of the disequilibrium based on the distance to equal probabilities. We should remind that for the 2D-Ising model the entropy raises sharply close to \( T_c \) towards its maximum value, But this is not the case in most problems. A simple generalization can be obtained by taking into account a much more natural measure, i.e. the distance to independence defined,
for a $K$-block as:

$$D_T^{(K)} = - \sum_{i_1 \in \{\uparrow, \downarrow\}} \sum_{i_2 \in \{\uparrow, \downarrow\}} \cdots \sum_{i_K \in \{\uparrow, \downarrow\}} \left( p(i_1, i_2, \ldots, i_K) - \prod_{j=1}^{K} p(i_j) \right)^2$$

(17)

which, for $K = 2$ simply reads

$$D_T^{ij} = \sum_{i \in \{\uparrow, \downarrow\}} \sum_{j \in \{\uparrow, \downarrow\}} (p_{ij} - p_{i} p_{j})^2$$

(18)

Now, $D_T^{ij}$ is a measure of how far are the subsystems from being independent (i.e., from the identity $p_{ij} = p_{i} p_{j}$). The distance to independence $D_T^i$, does not behave as $D_T^{E}$. It can be easily shown that the distance to independence acts similarly to the joint information (8) at both extremes (where total disorder leads to $p_{ij} = p_{i} p_{j}$ or when a complete homogeneous distribution leads to $p_{ii} = p_{i} p_{i}$ and zero for the other cases $j \neq i$). In fact:

$$M = \sum_{i,j} p_{ij} \log \frac{p_{ij}}{p_{i} p_{j}} = - \sum_{i,j} p_{ij} \log \left( 1 + \frac{p_{i} p_{j} - p_{ij}}{p_{i} p_{j}} \right)$$

(19)

approximating to second order the logarithm, we have:

$$M \approx - \sum_{i,j} \left( p_{ij} \left( \frac{p_{i} p_{j} - p_{ij}}{p_{i} p_{j}} \right) - \frac{1}{2} \left( \frac{p_{i} p_{j} - p_{ij}}{p_{i} p_{j}} \right)^2 \right) = \frac{1}{2} \sum_{i,j} \frac{(p_{ij} - p_{i} p_{j})^2}{p_{ij}}$$

(20)

i.e, the distance of the independence is a non-weighted approximation to the joint information.

The behavior of this measure for the 2D Ising model is shown in the inset of figure 2. We see that $D_T^i$ behaves as the fluctuations in the magnetization. This measure is able to detect the onset of complex fluctuations (and the underlying correlations at many scales) and decays for $T > T_c$ in the same way as the average fluctuation. So in fact $D_T^i$ itself seems to be a simple and consistent measure of complexity very close to the underlying physics. As we will see below, this measure is also consistent in other cases.

### 4 Testing LMC and SDL with Random Boolean Networks

The previous results might suggest that the generalized LMC complexity is an appropriate measure of correlations. This intuition, however, is not supported from the analysis of other complex systems. As an example, let us consider the dynamics of a random Boolean network (RBN) [15]. This model is defined by a set of $N$ discrete maps:

$$S_i(t + 1) = \Lambda_i(S_{i_1}, S_{i_2}, \ldots, S_{i_K})$$

(21)
Figure 4: Distance to independence, $D_p^I$, as a function of the bias $p$, for RBN’s of $K = 3$ and $N = 250$ spins. $p$ has been varied in intervals of size $\Delta p = 10^{-2}$. After a transient of length $\tau = 250$ is discarded, we measure over 250 time steps. Each point is the average over 400 samples. The theoretical analysis gives a critical point at $p_c \approx 0.79$ (derived from (20)) which is revealed by the distance to independence (21). The snapshots show particular examples of the dynamics for $p = 0.6$ (chaotic regime), $p = 0.79$ (transition point) and $p = 0.9$ (ordered regime). Inset: the corresponding LMC complexities for the single-spin (open triangles) and two-spin (black circles) measures under identities compute conditions. A maximum at $p \approx 0.69$ is obtained.

where $S_i \in \Sigma = \{0, 1\}; i = 1, 2, ..., N$ and $\Lambda_i(S_{i_1}, S_{i_2}, ..., S_{i_K})$ is a Boolean function randomly chosen from the set $\mathcal{F}_K$ of all the Boolean functions with $K$ variables. So each spin in (21) receives exactly $K$ inputs from a set of randomly chosen neighbors. This is no longer a finite-dimensional extended system. Here the dynamics takes place on a random directed graph. In fact, the statistical properties of this model can be understood in terms of damage spreading on a Bethe lattice [16].

It is well known that RBNs show a phase transition at some critical points. Specifically, if $p = P[\Lambda_i(S_{i_1}, S_{i_2}, ..., S_{i_K}) = 1]$ (i. e. $p$ is the bias in the sampling of Boolean functions) a phase transition curve in the $(K, p)$ plane is shown to exist [16-19] and is given by

$$K = \frac{1}{2p(1-p)}$$

The analysis of this system shows that at the critical point percolation of damage takes place for the first time i. e. a single flip of a binary unit can generate an avalanche of
changes through the whole system and appropriate order parameters for the (second order) phase transition can be defined.

As argued in the previous example, maximum structure and correlation should be expected at the critical point, and so maximum complexity. In figure 4 we show the behavior of $D_I^p$ as a function of the bias $p$ for a $K = 3$ RBN. We can see a maximum at $p_c = 0.79$, as predicted by the critical condition given by (22). So the distance to the independence shows a maximum at the critical point, as expected from a properly defined complexity measure. Here $D_I^p$ has been computed by averaging

$$D_I^p = \left\langle \sum_{j \in \{1,2,...,K\}} (p_{ij} - p_i p_j)^2 \right\rangle_N$$

where $\langle ... \rangle_N$ stands for average over all the units. We can similarly define the corresponding single-spin and two-spin LMC measures for the RBN and the results are also shown in the inset of figure 4. Although both LMC measures have a maximum at a given $0.5 < p^* < 1.0$, they fail to detect the critical point. Here $p^* \approx 0.69p_c$. This is understandable as far as the $p$ parameter strongly influences (a priori) the statistical distribution of state frequencies.

In [20], Shiner, Davison and Landsberg introduce a new measure of complexity (hereafter SDL measure):

$$C_{\alpha\beta} = H^\alpha (1 - H)^\beta$$

where $H$ is the normalized Boltzmann entropy. The SDL measure also tries to satisfy the criteria of maximum complexity between order and disorder as in LMC (in fact the LMC measure can be viewed as an approximation to the SDL as shown in [20]). $H$ is interpreted as a measure of disorder and $1 - H$ as a measure of order. The parametrization with $\alpha$ and $\beta$ allow to fit ad hoc according to the specific cases considered.

The SDL measure has been criticized in [21]: the SDL measure give the same value for systems structurally different but with identical $H$. This problem is severe in RBN’s. The entropy $H$ in RBN’s depends, exclusively on the bias $p$, thus nets with different connectivities (and then with different structural dynamic) but identical bias show identical $H$ and then identical $C$ curves. Since the maxima of complexity depend too on $K$, the SDL measure will fail: we can always fit $\alpha$ and $\beta$ in different ways for each connectivity $K$ in such a way that we can recover the maximum. The value of the normalized entropy $H$ that maximize the SDL measure is $\alpha/(\alpha + \beta)$ [20]. But it is easy to see from (22) and fixed $K$, that:

$$H = -p_c \log p_c - (1 - p_c) \log(1 - p_c) = \alpha/(\alpha + \beta)$$

allows an infinite number solutions of values for $\alpha$ and $\beta$, which is far from satisfactory. Similarly, we can define $H$ in blocks, but the problem persists.
5 DISCUSSION

In summary, we have analysed the validity of the LMC approach as an effective measure of complexity for systems composed by many units in interaction. Two different (although standard) problems have been considered. For the 2D Ising model, where a phase transition is known to occur at the Curie temperature, the LMC measures showed a maximum close to $T = T_c$. A measure based on the statistics from a single spin failed, however, to detect correlations for $T > T_c$ although such correlations exist. The reason was the lack of information about interactions between nearest spins and the fact that $P(\uparrow) \to 1/2$ as the critical temperature is approached from below. This problem was solved by considering an extension of the LMC measure to a joint probability distribution. Such extension was able to show a maximum at $T_c$ and well-defined correlations. A simple extension of the disequilibrium function was proposed -the distance to independence $D^I$- as an alternative measure of complexity. This quantity (which is in fact a second-order mutual Renyi information) was shown to consistently detect the critical point for the 2D Ising model and RBNs with a lower computational effort than LMC complexity. A direct extension of the LMC measure to the random Boolean network model was shown to fail in detecting maximum complexity at the critical point. The reason for this result is clear is we consider that both the entropy and the disequilibrium change smoothly with the $p$ parameter. Such parameter strongly influences the statistics of the numbers of units in one of the two states, and in so doing it definitely influences the values of both functions. Although correlations can be in principle detected, the relevance of $p$ in defining a priori the values of entropies and disequilibrium is very important. These results confirm the intuition that the LMC and SML measures (and similar quantities) will fail to detect real complexity.

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