Propagation of dipole solitons in inhomogeneous highly dispersive optical fiber media

Houria Triki and Vladimir I. Kruglov

1Radiation Physics Laboratory, Department of Physics, Faculty of Sciences, Badji Mokhtar University, P. O. Box 12, 23000 Annaba, Algeria
2Centre for Engineering Quantum Systems, School of Mathematics and Physics, The University of Queensland, Brisbane, Queensland 4072, Australia

We consider the ultrashort light pulse propagation through an inhomogeneous monomodal optical fiber exhibiting higher-order dispersive effects. Wave propagation is governed by a generalized nonlinear Schrödinger equation with varying second-, third-, and fourth-order dispersions, cubic nonlinearity, and linear gain or loss. We construct a new type of exact self-similar soliton solutions that takes the structure of dipole via a similarity transformation connected to the related constant-coefficients one. The conditions on the optical fiber parameters for the existence of these self-similar structures are also given. The results show that the contribution of all orders of dispersion is an important feature to form this kind of self-similar dipole pulse shape. The dynamic behaviors of the self-similar dipole solitons in a periodic distributed amplification system are analyzed. The significance of the obtained self-similar pulses is also discussed.

PACS numbers: 05.45.Yv, 42.65.Tg

I. INTRODUCTION

Light pulses in optical fibers are referred to generically as solitons and are usually described within the framework of the cubic nonlinear Schrödinger equation (NLSE) that includes only basic effects on waves such as group velocity dispersion (GVD) and self-phase modulation (SPM) [1-5]. Depending on the anomalous dispersion or the normal dispersion, the NLSE model allows for either bright or dark solitons, respectively [4]. The formation of these solitons result from the exact balance between the GVD and SPM of the material. Due to the robust and stable nature of solitons, such wave packets have been successfully used as the information carriers (optical bits) to transmit digital signals over long propagation distances. However, many applications in various areas such as ultrahigh-bit-rate optical communication systems, infrared time-resolved spectroscopy, ultrafast physical processes, and optical sampling systems require ultrashort (femtosecond) pulses [3-5], that leads to the appearance of different higher-order effects in the optical material. For instance, the third-order dispersion plays a significant role in propagation if short pulses whose widths are nearly 50 femtosecond have to be injected [6,7]. The fourth-order dispersion is also important when pulses are shorter than 10 femtosecond [6,7]. In such a situation, the wave dynamics can be described by the higher-order nonlinear Schrödinger equation incorporating the contribution of various physical phenomena on short-pulse propagation and generation. One notes that higher-order effects introduce several new important phenomena into the system dynamics that are absent in nonlinear media of the Kerr type.

In any material, however, there exists always some nonuniformities. Among the many factors that cause nonuniformities in optical fiber systems [3,10], we note (i) the variation in the lattice parameters of the fiber media, (ii) the imperfection of manufacture, and (iii) the fluctuation of the fiber diameters. These nonuniformities may significantly influence various effects in the nonlinear medium such as dispersion, phase modulation, gain (or loss), and others [9]. From the theoretical standpoint, the description of the optical pulse propagation in inhomogeneous fibers is generally based on the generalized NLSE with distributed coefficients [i.e., allowing nonlinearity, dispersion, and gain profiles to change with the distance along the direction of propagation] [11,12]. Such generalized model possesses a rich variety of exact self-similar solutions that are characterized by a linear chirp [11,12]. These self-similar pulses (also called ‘similaritons’) have attracted considerable interest in recent years because of their extensive applications in photonics and fiber-optic telecommunications [13]. Importantly, these similaritons can maintain the overall shape while allowing their parameters such as amplitudes and widths to change with the modulation of system parameters [15].

Recently, propagation of self-similar waves has drawn much interest and many important results have been presented, which is an essential prerequisite for understanding the dynamical processes and mechanism of the complicated phenomena in different inhomogeneous media [16-29]. For example, Dai et al. [16] investigated the dynamic behaviors of spatial similaritons in inhomogeneous nonlinear cubic-quintic media. They also discussed controllable optical rogue waves in the femtosecond regime [17]. In Ref. [18], the authors constructed explicit chirped and chirp-free self-similar solitary wave and cnoidal wave solutions of the generalized cubic-quintic NLSE by applying the similarity transformation method. Very recently, Pal et al. [19] found self-similar wave solutions of the quadratic-cubic NLSE describing wave propagation through a tapered graded-index waveguide. Choudhuri et al. [20] derived the exact self-similar localized pulse solutions for the NLSE with distributed cubic-quintic nonlinearities. Triki et al. [21]
investigated the propagation of self-similar optical solitons on a continuous-wave background in a quadratic-cubic non-centrosymmetric waveguide. Liu et al. [22] constructed a variety of spatiotemporal self-similar wave solutions for the (3 +1)-dimensional variable-coefficients NLSE with cubic-quintic nonlinearities. Serkin et al. [13] discovered solitary nonlinear Bloch waves of the bright and dark types in dispersion managed fiber systems and soliton lasers.

However, most investigations on the propagation of self-similar solitons in inhomogeneous fiber systems have been focused on bright, dark, and kink type self-similar solitary waves or solitons, as well as rogue and cnoidal waves. But many novel localized structures including for example dipole solitons, vortex solitons, and soliton trains have been demonstrated experimentally and theoretically in both one- and two-dimensional nonlinear media [22, 23]. To our knowledge, no exact self-similar ‘dipole’ soliton solutions have been previously reported within the framework of the variable-coefficient NLSE models. Moreover, the control of self-similar localized pulses under the combined influence of GVD, TOD, FOD, and Kerr nonlinearity management has not been widespread. In this paper, we demonstrate the existence of self-similar pulses that takes a dipole structure in inhomogeneous highly dispersive optical fibers and investigate their propagation dynamics for different parameters.

Our results are presented as follows. Section II presents the method used for obtaining traveling wave solutions of the extended NLSE that describes the propagation of extremely short pulses inside a highly dispersive optical fiber medium. In Sec. III, we derive analytical bright and dipole soliton solutions of the model and their characteristics. In Sec. IV, the variation of fiber dispersions, nonlinearity, and gain or loss is considered and a similarity transformation to reduce the generalized nonlinear Schrödinger equation with varying coefficients to the related constant-coefficients one is presented. Self-similar dipole structures of the generalized NLSE and their dynamical behaviors in a periodic distributed amplification system are reported in Sec. V. Conclusions and future research directions are addressed in Sec. VI.

II. TRAVELING WAVES

In this section, we reduce the extended NLSE to an ordinary differential equation. The extended NLSE is derived for the assumptions of slowly varying envelope, instantaneous nonlinear response, and no higher order nonlinearities [30]. This nonlinear equation has the next form for the optical pulse envelope $E(z, \tau)$,

$$iE_z = \alpha E_{\tau\tau} + i\sigma E_{\tau\tau\tau} - \epsilon E_{\tau\tau\tau\tau} - \gamma |E|^2 E,$$  \hspace{1cm} (1)

where $z$ is the longitudinal coordinate, $\tau = t - \beta_1 z$ is the retarded time, and $\alpha = \beta_2/2$, $\sigma = \beta_3/6$, $\epsilon = \beta_4/24$, and $\gamma$ is the nonlinear parameter. The parameters $\beta_k = (d^k \beta/d\omega^k)_{\omega=\omega_0}$ are the k-order dispersion of the optical fiber and $\beta(\omega)$ is the propagation constant depending on the optical frequency.

This equation has been intensively studied for its importance from various view points [30]-[34]. In particular, the modulational instability phenomena of Eq. (1) have been analyzed in the region of the minimum group-velocity dispersion in [30]. In a very recent work, Kruglov and Harvey [31] presented an exact solitary wave solution having the functional form of $\text{sech}^2$ for the NLSE (1) including second- , third- , and fourth-order dispersion effects. Moreover, Karpman et al. [32, 33] have studied the time behavior of the amplitudes, velocities, and other parameters of radiating solitons. Shagalov [34] has investigated the effect of the third- and fourth-order dispersions on the modulational instability. Roy et al. [35] have also studied the role of TOD and FOD in the radiation emitted by fundamental soliton pulses in the form of dispersive waves within the framework of the dimensionless form of the NLSE (1).

We consider the solution of the generalized NLSE in the form,

$$E(z, \tau) = u(x) \exp[i(\kappa z - \delta \tau + \theta)],$$  \hspace{1cm} (2)

where $u(x)$ is a real function depending on the variable $x = \tau - qz$, and $q = v^{-1}$ is the inverse velocity. Also, $\kappa$ and $\delta$ are the respective real parameters describing the wave number and frequency shift, while $\theta$ represents the phase of the pulse at $z = 0$.

Equations (1) and (2) lead to the next system of the ordinary differential equations,

$$\left(\sigma + 4\epsilon \delta\right) \frac{d^3 u}{dx^3} + (q - 2\alpha \delta - 3\sigma \delta^2 - 4\epsilon \delta^3) \frac{du}{dx} = 0,$$  \hspace{1cm} (3)

$$\epsilon \frac{d^4 u}{dx^4} - (\alpha + 3\sigma \delta + 6\epsilon \delta^2) \frac{d^2 u}{dx^2} + \gamma u^3 - (\kappa - \alpha \delta^2 - \sigma \delta^3 - \epsilon \delta^4) u = 0,$$  \hspace{1cm} (4)

In the general case the system of Eqs. (3) and (4) is overdetermined because we have two differential equations for the function $u(x)$. However, if some constraints for the parameters in Eq. (3) are fulfilled the system of Eqs. (3) and
has non-trivial solutions. We refer the solution of the extended NLSE where the function \( E(z, \tau) \) is given by Eq. (2) with \( u(x) \neq \) constant as non-plain wave or traveling wave solution.

The system of Eqs. (3) and (4) with \( \epsilon \neq 0 \) yields the non-plain wave solutions if and only if the next relations are satisfied:

\[
q = 2\alpha\delta + 3\sigma\delta^2 + 4\epsilon\delta^3, \quad \delta = -\frac{\sigma}{4\epsilon}.
\]

The system of Eqs. (3) and (4) with \( \epsilon = 0 \) has the non-plain wave solutions only when the parameter \( \sigma = 0 \). Note that Eq. (3) is satisfied for an arbitrary function \( u(x) \) according to the conditions in Eq. (5) with \( \epsilon \neq 0 \). The relations in Eq. (5) lead to the next expression for the velocity \( v = 1/q \) defined in the retarded frame,

\[
v = \frac{8\epsilon^2}{\sigma(\sigma^2 - 4\alpha\epsilon)}.
\]

The relations in Eq. (5) reduce the system of Eqs. (3) and (4) to the ordinary nonlinear differential equation,

\[
\epsilon d^4 u dx^4 + b d^2 u dx^2 - cu + \gamma u^3 = 0,
\]

where the parameters \( b \) and \( c \) are

\[
b = \frac{3\sigma^2}{8\epsilon} - \alpha, \quad c = \kappa + \frac{\sigma^2}{16\epsilon^2} \left( \frac{3\sigma^2}{16\epsilon} - \alpha \right).
\]

By analytically solving Eq. (7), we obtain the soliton structures that can propagate in the highly dispersive fiber medium. However, it would be very difficult to find the closed form solutions of such ordinary differential equation in which two even-order derivative terms coexist. Obtaining solutions in analytic form is of great interest since these are useful for instance to compare experimental results with theory. In following, families of soliton solutions having the functional form of ‘sech\(^2\)(.)’ and ‘sech(.)th(.)’ are derived in presence of all physical parameters.

### III. DIPOLE SOLITON IN HIGHLY DISPERSIVE OPTICAL FIBER

In this section, we consider the traveling wave solutions of Eq. (7) in the form,

\[
u(x) = \frac{F_N(x)}{G_M(x)} = \frac{\sum_{n=-N}^{N} A_n \exp[-nw(x-\eta)]}{\sum_{n=-M}^{M} B_n \exp[-nw(x-\eta)]}.
\]

The quartic dark soliton solution of this form is given by

\[
u(x) = A + B \text{ th}^2[w(x-\eta)] = D - \frac{B}{\text{ch}^2[w(x-\eta)]},
\]

where \( D = A + B \) and \( D \neq 0 \). In the case when \( D = 0 \) we have the sech\(^2\) solitary wave.

Substituting the function (10) into Eq. (7) and setting the coefficients of independent terms equal to zero, we obtain the following equations:

\[
cD = \gamma D^3, \quad c = 16\epsilon w^4 + 4bw^2 + 3\gamma D^2,
\]

\[
40\epsilon w^4 + 2bw^2 + \gamma DB = 0, \quad 120\epsilon w^4 + \gamma B^2 = 0.
\]

We now discuss solutions to these parametric equations for two cases: (1) for \( D = 0 \), and (2) for \( D \neq 0 \). In the case (1) with \( D = 0 \) and \( E_0 = -B \) we have

\[
w = \frac{1}{4} \sqrt{\frac{8\alpha - 3\sigma^2}{10\epsilon}}, \quad E_0 = \pm \sqrt{\frac{-3}{10\gamma\epsilon} \left( \frac{3\sigma^2}{8\epsilon} - \alpha \right)}.
\]
Further, we get from Eqs. (11) and (12) a condition on the parameter \( c \) as

\[
c = -\frac{4}{25\epsilon} \left( \frac{3\sigma^2}{8} - \alpha \right)^2.
\] (14)

Incorporating these results into Eq. (10), we obtain the following soliton solution to the extended NLSE (1) [31]:

\[
E(z, \tau) = E_0 \text{sech}^2(w\xi) \exp[i(\kappa z - \delta \tau + \theta)],
\] (15)

where \( \xi = \tau - v^{-1}z - \eta \), with \( \eta \) being the position of the pulse at \( z = 0 \). The wave number \( \kappa \) follows from Eqs. (8) and (14) as

\[
\kappa = -\frac{4}{25\epsilon^3} \left( \frac{3\sigma^2}{8} - \alpha \right)^2 - \frac{\sigma^2}{16\epsilon^3} \left( \frac{3\sigma^2}{16} - \alpha \right).
\] (16)

The velocity \( v \) and frequency shift \( \delta \) in this soliton solution are given by Eqs. (5) and (6).

Physically, Eq. (15) describes a bright stationary pulse with amplitude \( E_0 \) and inverse temporal width \( w \) depending on all order of dispersion as well as nonlinearity. It follows from Eq. (13) that this \( \text{sech}^2 \) solitary wave exists when the next two conditions are satisfied:

\[
\gamma \epsilon < 0, \ 8\alpha \epsilon > 3\sigma^2.
\]

In the case (2) with \( D \neq 0 \), Eqs. (11) and (12) lead to the complex values for parameters \( B \) and \( D \). However the function \( u(x) \) in Eq. (7) is real which contradicts to such complex parameters. Thus the extended NLSE given by Eq. (1) does not have a quartic dark soliton solution.

We have also found that Eq. (7) admits an exact dipole soliton solution of the form,

\[
u(x) = E_0 \text{sh}[w(x - \eta)]/\text{ch}[w(x - \eta)],
\] (17)

Inserting this solution into Eq. (7) and equating the coefficients of independent terms, one obtains

\[
c = \epsilon w^4 + bw^2, \quad 120\epsilon w^4 = \gamma E_0^2,
\] (18)

\[
60\epsilon w^4 + 6bw^2 - \gamma E_0^2 = 0.
\] (19)

These equations yield the dipole soliton solution as

\[
E(z, \tau) = E_0 \text{sech}(w\xi) \text{th}(w\xi) \exp[i(\kappa z - \delta \tau + \theta)],
\] (20)

where \( \xi = \tau - v^{-1}z - \eta \). Thus we have the following relations for the pulse inverse width \( w \) and amplitude \( E_0 \),

\[
w = \frac{1}{4} \sqrt{\frac{3\sigma^2 - 8\alpha \epsilon}{5\epsilon^2}}, \quad E_0 = \pm \sqrt{\frac{6}{5\gamma \epsilon}} \left( \frac{3\sigma^2}{8\epsilon} - \alpha \right).
\] (21)

Eqs. (18) and (19) yield the parameter \( c = 11b^2/100\epsilon \). Hence it follows from Eq. (8) that the wave number \( \kappa \) in the dipole soliton solution is given by

\[
\kappa = \frac{11}{100\epsilon^3} \left( \frac{3\sigma^2}{8} - \alpha \epsilon \right)^2 - \frac{\sigma^2}{16\epsilon^3} \left( \frac{3\sigma^2}{16} - \alpha \right).
\] (22)

The velocity \( v \) and frequency shift \( \delta \) in this dipole soliton solution are given by Eqs. (5) and (6). It follows from Eq. (21) that the dipole soliton solution exists when the next two conditions are satisfied: \( \gamma \epsilon > 0, \ 3\alpha^2 > 8\epsilon \).

The corresponding energy \( \mathcal{E} \) of the dipole solitons is given by

\[
\mathcal{E} = \int_{-\infty}^{+\infty} |E(z, \tau)|^2 d\tau = \frac{(3\sigma^2 - 8\alpha \epsilon)^{3/2}}{4\gamma \epsilon \sqrt{5\epsilon^2}}.
\] (23)

Note that the energy of the pulse \( \mathcal{E} \) is the integral of motion \( (d\mathcal{E}/dz = 0) \) of the NLSE (1) for any pulses satisfying the boundary condition: \( E(z, \tau) \to 0 \) for \( \tau \to \pm \infty \).
IV. TRANSFORMATION TO GENERALIZED NLSE WITH VARIABLE COEFFICIENTS

We consider in this section the variations of fiber dispersion, nonlinearity, and gain or loss. For our purpose, the dynamics of pulses is described by the following generalized NLSE with distributed coefficients:

\[ iU_s = D(s)U_{tt} + iP(s)U_{ttt} - Q(s)U_{tttt} - R(s)|U|^2U + i\Gamma(s)U. \]

(24)

where \(D(s), P(s)\) and \(Q(s)\) are the variable GVD, TOD, and FOD coefficients, respectively. The function \(R(s)\) stands for the varying Kerr nonlinearity coefficient, while \(\Gamma(s)\) denotes the amplification \((\Gamma(s) > 0)\) or absorption coefficient \((\Gamma(s) < 0)\).

In the simplest case, when all the coefficients are constants and \(\Gamma(s) = 0\), then Eq. (24) can be transformed into the constant-coefficient NLSE. It is of interest to control optical solitons in communication systems when all orders of dispersion, nonlinearity, and gain or loss are varied as described by the NLSE (24). In following, we first search for exact self-similar soliton solutions of the variable-coefficient NLSE (24) by employing the similarity transformation method and then discuss their propagation behaviors in a specified soliton control system.

We first construct the transformation

\[ U(s, t) = A(s)E[z(s), \tau(s, t)]e^{i\phi(s, t)}, \]

(25)

where \(A(s)\) is the amplitude, \(E(z, \tau)\) is the optical pulse envelope, \(z = z(s)\) and \(\tau = \tau(s, t)\) are two unknown functions to be determined, and \(\phi(s, t)\) is the phase function.
FIG. 2: Evolution of the dipole self-similar intensity wave profile $|U(s,t)|^2$ as computed from Eq. (44) when (a) $\Gamma_0 = -0.02$, (b) $\Gamma_0 = 0.02$. The other parameters are the same as in Fig. 1.

Upon substituting Eq. (25) into Eq. (24) leads to (1), but now we must have the following set of equations:

$A_s - \Gamma A - DA\phi_{ttt} + 3PA\phi_t\phi_{tt} + QA\phi_{ttt} - 6QA\phi_t^2\phi_{tt} = 0$,  \hspace{1cm} (26)

$\tau_s - 2D\tau_{tt} + 3P\tau_t\phi_t^2 - P\tau_{ttt} + 4Q\tau_{ttt}\phi_t + 6Q\tau_{ttt}\phi_{tt} - 4Q\tau_t\phi_t^3 + 4Q\tau_t\phi_{ttt} = 0$,  \hspace{1cm} (27)

$\phi_s - D\phi_t^2 - P\phi_{ttt} + P\phi_t^3 + 4Q\phi_t\phi_{ttt} - Q\phi_t^4 + 3Q\phi_{ttt}^2 = 0$,  \hspace{1cm} (28)

$(3P\phi_t - D)\tau_{tt} + 3P\tau_t\phi_t + Q\tau_{tttt} - 12Q\tau_t\phi_t\phi_{tt} - 6Q\tau_{ttt}\phi_t^2 = 0$,  \hspace{1cm} (29)

$-3P\tau_t\phi_t + 12Q\tau_t\tau_{tt}\phi_t + 6Q\tau_t^2\phi_{tt} = 0$,  \hspace{1cm} (30)

$RA^2 = \gamma z_s$,  \hspace{1cm} (31)

$(D - 3P\phi_t)\tau_t^2 - 4Q\tau_t\tau_{tt} + 6Q\tau_t^2\phi_t^2 - 3Q\tau_{tt}^2 = \alpha z_s$,  \hspace{1cm} (32)

$-P\tau_t^3 + \sigma z_s + 4Q\tau_t^3\phi_t = 0$,  \hspace{1cm} (33)

$Q\tau_t^4 = \epsilon z_s$,  \hspace{1cm} (34)

$\tau_{tt} = 0$,  \hspace{1cm} (35)

Solving these equations self-consistently allows us to find the following parameters that characterize the self-similar pulse:

$A(s) = A_0 \exp \left[ \int_0^s \Gamma(\zeta) d\zeta \right]$,  \hspace{1cm} (36)

$\tau(s,t) = k \left[ t + p \left( \frac{4p^2 + 2\alpha k^2}{\epsilon} + 3p\sigma k \right) \int_0^s Q(\zeta) d\zeta \right] + t_0$,  \hspace{1cm} (37)

$z(s) = \frac{k^4}{\epsilon} \int_0^s Q(\zeta) d\zeta$,  \hspace{1cm} (38)

$\phi(s,t) = p \left[ t + p \left( \frac{3p^2 + \alpha k^2}{\epsilon} + 2p\sigma k \right) \int_0^s Q(\zeta) d\zeta \right] + \phi_0$,  \hspace{1cm} (39)

where $k$ and $p$ are parameters relative to pulse width and phase shift, respectively, $\tau(s,t)$ is the mapping variable, and $z(s)$ represents the effective propagation distance. Here the subscript 0 denotes the initial values of the corresponding
FIG. 3: Evolution of the dipole self-similar intensity wave profile $|U(s,t)|^2$ as computed from Eq. (44) when $\Gamma(s) = \sin(s)$. The other parameters are the same as in Fig. 1 except $A_0 = 0.1$.

parameters at distance $s = 0$. Furthermore, the constraint conditions on the inhomogeneous fiber parameters are given as

$$D(s) = \left(6p^2 + \frac{\alpha k^2 + 3p\sigma k}{\epsilon}\right)Q(s),$$  \hspace{1cm} (40)

$$R(s) = \frac{\gamma k^4}{\epsilon A_0^2} \exp \left[-2 \int_0^s \Gamma(\zeta)d\zeta\right] Q(s),$$  \hspace{1cm} (41)

$$P(s) = \left(4p + \frac{\sigma k}{\epsilon}\right)Q(s).$$  \hspace{1cm} (42)

Equations (40) and (41) show that the effective propagation distance and phase are strongly dependent on the FOD parameter $Q(s)$. The latter influences the GVD coefficient $D(s)$, Kerr nonlinear coefficient $R(s)$, and TOD coefficient $P(s)$ as seen from the preceding constraints. Hence, one can control the dynamics of propagating self-similar dipole solitons in the fiber medium by selecting the profile of this parameter suitably.

Thus, the general form of self-similar solutions of the generalized NLSE (24) is of the form

$$U(s,t) = A_0 E \left\{ \frac{k^4}{\epsilon} \int_0^s Q(\zeta)d\zeta, k \left[ t + p \left(4p^2 + \frac{2\alpha k^2 + 3p\sigma k}{\epsilon}\right) \int_0^s Q(\zeta)d\zeta \right] + t_0 \right\} $$

$$\times \exp \left[ \int_0^s \Gamma(\zeta)d\zeta + i\phi(s,t) \right].$$  \hspace{1cm} (43)

where the phase function $\phi(s,t)$ is given by Eq. (39) and $E(z, \tau)$ are the exact solutions of Eq. (1).

Therefore exact self-similar solutions to Eq. (24) can be constructed by using the exact solutions of Eq. (1) via the transformation (43). One thus needs to use the closed form solutions of the constant-coefficient NLSE (1) presented above.

V. SELF-SIMILAR DIPOLE SOLITON SOLUTIONS

Making use of the exact solution given in Eq. (20) of the extended constant-coefficient NLSE (1), the transformations in Eq. (43), and Eqs. (36)-(39), we can construct the self-similar solutions of the generalized NLSE with varying
coefficients (24). The self-similar dipole soliton solution of Eq. (24) is then given by

\[ U(s, t) = A_0 E_0 \exp \left[ \int_0^s \Gamma(\zeta) d\zeta \right] \text{sech}(w\xi) \text{th}(w\xi) \exp \left[ i \Phi(s, t) \right], \] (44)

where the traveling coordinate \( \xi \) is given by

\[ \xi(s, t) = kt - \eta + \left\{ kp \left( 4p^2 + 2\alpha k^2 + 3pek \right) - \frac{k^4}{v^2} \right\} \int_0^s Q(\zeta) d\zeta + t_0, \] (45)

and the phase of the field \( \Phi \) has the form

\[ \Phi(z, t) = \kappa z - \delta \tau + \theta + \phi(s, t), \] (46)

where \( \tau \) and \( z \) are given by Eqs. (37) and (38) respectively, while the phase \( \phi(s, t) \) is given by Eq. (39).

From the results obtained above, we see that the contribution of all orders of dispersion is necessary for the existence of self-similar dipole soliton solutions for the generalized NLSE with distributed coefficients (24). This is markedly different from the dipole structures of many constant coefficient NLSE models describing femtosecond pulse dynamics in homogeneous fibers [24]-[26], which exist only when both GVD, TOD and FOD are compensated. It should be noted that the simultaneous compensation of various order of dispersion is generally more difficult in optical systems [24]. Therefore, our results could be of importance in applications of dipole type solitons in optical fiber systems exhibiting dispersive effects up to the fourth order.

To examine the dynamical evolution of the obtained self-similar structure in the optical fiber medium, it is worthwhile to consider a specific soliton control system. Here, we focus on studying the propagation of self-similar waves through a periodically distributed amplification system similar to that of Ref. [17]. Especially, we suppose that the FOD management takes the form of a cosinelike space-dependent rapidly varying function as [17]: \( Q(s) = d_4 \cos(gz) \), while the gain function is given by \( \Gamma(s) = \Gamma_0 \). Here \( d_4 \) and \( g \) are the parameters to describe FOD and \( \Gamma_0 \) represents the constant net gain or loss. From the practical point of view, the propagation with periodic dispersion is of great importance as it has application in enhancing the signal to noise ratio and reducing Gordon-Hauss time jitter and is also helpful in suppressing the phase matched condition for four-wave mixing [36, 37]. Then according to Eq. (38), the effective propagation distance can be obtained as \( z(s) = \frac{d_4 k^4}{eg} \sin(gz) \), implying that \( z \) varies periodically with
the propagation distance $s$. Furthermore, the amplitude can be calculated from Eq. (36) as: $A(s) = A_0 \exp(\Gamma_0 s)$. This means that the amplitude of the self-similar pulse will undergo increase ($\Gamma_0 > 0$) and decrease ($\Gamma_0 < 0$) along the propagation distance, while it remains a constant when the gain (loss) vanishes ($\Gamma_0 = 0$). As concerns the other parameters, they can be obtained exactly through Eqs. (10), (11) and (12).

Consider first the most interesting case when the optical fiber medium does not subject to the effect of the gain or loss effect [i.e., $\Gamma_0 = 0$]. The evolution of the self-similar dipole soliton solution (44) calculated with the framework of the generalized NLSE (24) is shown in Figs. 1(a) and 1(b) with the parametric values: $\alpha = -1, \gamma = 2, \sigma = 1$, and $\epsilon = \frac{1}{2}$. Also we take $k = p = 1$, $A_0 = 0.3098$, $d_4 = \frac{1}{5}$, $\eta = 0$, $g = 1$, and $t_0 = 0$. From these figures, one can clearly see that the self-similar structure display a snakelike behavior along the propagation distance due to the presence of periodic distributed dispersion parameter $Q(s)$. For such oscillatory trajectory, the self-similar pulse keep no change in propagating along optical medium although its position oscillate periodically (which is called “Snakelike” in Ref. [28]).

When the self-similar dipole soliton is subjected to the action of a constant gain or loss, that is, $\Gamma(s) = \Gamma_0$, its intensity decreases when $\Gamma_0 < 0$ and increase when $\Gamma_0 > 0$, and the time shift and the group velocity of the soliton pulse are changing while the soliton keeps its shape in propagation along the fiber [Figs. 2(a)-(b)]. One readily concludes that the gain parameter affects only the evolution of soliton peak and has no influence on the width or shape of the pulse.

Another interesting behavior appears when the gain or loss function is chosen to vary periodically with the propagation distance as $\Gamma(s) = \sin(s)$. This spatial profile of gain (loss) was first used in studying soliton management in inhomogeneous pure Kerr media [29]. The corresponding intensity profiles of self-similar dipole soliton are shown in Figs. 3(a)-(b) for the same values of parameters as those in Fig. 1 except $A_0 = 0.1$. As can be seen from this figure, in the presence of periodic gain, the dipole solitons emerge periodically in the inhomogeneous fiber system.

Let us now investigate the propagation dynamics of self-similar dipole pulses in a distributed fiber system whose FOD and gain or loss parameters are distributed according to [29]: $Q(s) = \tanh(s)$ and $\Gamma(s) = \sin(s)$. Figures 4(a)-(b) show the nonlinear evolution of the self-similar solution (44) for the same values of parameters as those in Fig. 1 except $A_0 = 0.1$. We observe an interesting periodic occurrence of dipole solitons appearing for this choice of dispersion and gain or loss management, as can be seen from Fig. 4.

VI. CONCLUSION

In this paper, we have investigated the variable-coefficient nonlinear Schrödinger equation incorporating, at the highest order, a fourth-order dispersion, which governs the femtosecond optical pulse propagation in an inhomogeneous highly dispersive fiber media. We have first constructed the relation between this generalized wave equation and the related constant-coefficients one via a similarity transformation. Then based on the obtained transformation, we have derived the exact self-similar dipole soliton solutions of the considered model. Conditions on the varying optical fiber parameters for the existence of these self-similar structures are also presented. It is found that the existence of these self-similar dipole solitons in an inhomogeneous highly dispersive optical fiber media crucially depends, indeed, on all orders of dispersion. We have further discussed the propagation dynamics of self-similar waves in dispersion changing periodically fiber system. It is observed that the self-similar wave structure and dynamical behavior can be controlled by choosing appropriate parameters of fourth-order dispersion and gain or loss.

An issue of prime importance is the stability of self-similar solitons with respect to perturbations. This is because only stable (or weakly unstable) solitary waves are promising for experimental observations and practical applications [20,40]. It should be noted that the stability analysis can be achieved by numerical simulations and the linear stability theory of the solutions with perturbations initially implanted. For the present generalized nonlinear Schrödinger equation model with varying coefficients, we have found that the obtained self-similar dipole structures essentially exist due to a balance among all order of dispersion and Kerr nonlinearity effect. The stability aspects of such privileged localized light pulses typically require detailed individual analysis based on such balance aspects. Detailed stability analysis are now under investigation.

[1] A. Hasegawa and F. Tappert, Appl. Phys. Lett. 23, 142 (1973).
[2] A. Hasegawa and F. Tappert, Appl. Phys. Lett. 23, 171 (1973).
[3] G.P. Agrawal, Applications of Nonlinear Fiber Optics (Academic, San Diego, 2001).
[4] R.Y. Hao, L. Li, Z.H. Li, R. C. Yang, G.S. Zhou, Opt. Commun. 245, 383 (2005).
[5] Aka, A. Goyal, R. Gupta and C. N. Kumar, and T. S. Raju, Phys. Rev. A 84, 063830 (2011).
[6] S. L. Palacios, J. M. Fernández-Díaz, Opt. Commun. 178, 457 (2000).
[7] S. L. Palacios, J. Opt. A: Pure Appl. Opt. 5, 180 (2003).
[8] L. Wang, J. H. Zhang, C. Liu, M. Li, and F. H. Qi, Phys. Rev. E 93, 062217 (2016).
[9] L. Li, Z. Li, S. Li and G. Zhou, Opt. Commun. 234, 169 (2004).
[10] A. Mahalingam, K. Porsezian, M. S. Mani Rajan and A. Uthayakumar, J. Phys. A: Math. Theor. 42, 165101 (2009).
[11] V. I. Kruglov, A. C. Peacock, and J. D. Harvey, Phys. Rev. Lett. 90, 113902 (2003).
[12] V. I. Kruglov, A. C. Peacock, and J. D. Harvey, Phys. Rev. E 71, 056619 (2005).
[13] V. N. Serkin and A. Hasegawa, Phys. Rev. Lett. 85, 4502 (2000).
[14] J. F. Zhang, L. Wu, L. Li, D. Mihalache, and B. A. Malomed, Phys. Rev. A 81, 023836 (2010).
[15] L. Wu, J. F. Zhang, L. Li, C. Finot, and K. Porsezian, Phys. Rev. A 78, 053807 (2008).
[16] C. Q. Dai, J. F. Ye, X. F. Chen, Opt. Commun. 285, 3988 (2012).
[17] C. Q. Dai, G. Q. Zhou, and J. F. Zhang, Phys. Rev. E 85, 016603 (2012).
[18] C. Q. Dai, Y. Y. Wang, C. Yan, Opt. Commun. 283, 1489 (2010).
[19] R. Pal, S. Loomba, C. N. Kumar, Annals of Physics 387, 213 (2017).
[20] A. Choudhuri, H. Triki, and K. Porsezian, Phys. Rev. A 94, 063814 (2016).
[21] H. Triki, C. Bensalem, A. Biswas, S. Khan, Q. Zhou, S. Adesanya, S. P. Moshokoa, M. Belic, Opt. Commun. 437, 392 (2019).
[22] X. B. Liu, X. F. Zhang, B. Li, Opt. Commun. 285, 779 (2012).
[23] J. K. Yang, I. Makasyuk, A. Bezryadina, and Z. Chen, Opt. Lett. 29, 1662 (2004).
[24] A. Choudhuri, K. Porsezian, Opt. Commun. 285, 364 (2012).
[25] F. Azzouzi, H. Triki, Ph. Grelu, Appl. Math. Modell. 39, 1300 (2015).
[26] H. Triki, F. Azzouzi, Ph. Grelu, Opt. Commun. 309, 71 (2013).
[27] D. N. Neshev, T. J. Alexander, E. A. Ostrovskaya, Y. S. Kivshar, I. Martin, H. Makasyuk, and Z. G. Chen, Phys. Rev. Lett. 92, 123903 (2004).
[28] X. Wang, Z. Chen, J. Wang, and J. Yang, Phys. Rev. Lett. 99, 243901 (2007).
[29] V. N. Serkin, M. Matsumoto, T. L. Belyaeva, Opt. Commun. 196, 159 (2001).
[30] S. B. Cavalcanti, J. C. Cressoni, H. R. da Cruz, and A. S. Gouveia-Neto, Phys. Rev. A 43, 6162 (1991).
[31] V. I. Kruglov and J. D. Harvey, Phys. Rev. A 98, 063811 (2018).
[32] V.I. Karpman, Phys. Lett. A 244, 397 (1998).
[33] V.I. Karpman and A.G. Shagalov, Phys. Lett. A 254, 319 (1999).
[34] A.G. Shagalov, Phys. Lett. A 239, 41 (1998).
[35] S. Roy, S. K. Bhadra, G. P. Agrawal, Opt. Commun. 282, 3798 (2009).
[36] S. Konar, M. Mishra, and S. Jana, Fiber Integr. Opt. 24, 537 (2005).
[37] S. Loomba and H. Kaur, Phys. Rev. E 88, 062903 (2013).
[38] Z.-Y. Yang, L.-C. Zhao, T. Zhang,Y.-H. Li, and R.-H. Yue, Phys. Rev. A 81, 043826 (2010).
[39] J. F. Zhang, C. Q. Dai, Q. Yang, J. M. Zhu, Opt. Commun. 252, 408 (2005).
[40] Z. Shi, J. Wang, Z. Chen, and J. Yang, Phys. Rev. A 78, 063812 (2008).