Noise tailoring for quantum circuits via unitary 2\(t\)-design

Linxi Zhang\(^{1,2}\), Yan Yu\(^1\), Changhua Zhu\(^1\) & Changxing Pei\(^1\)

Because of environmental variations and imperfect operations, real-world quantum computers produce different coherent errors that are difficult to estimate. Here, we propose a method whereby the twirled noise over a unitary 2\(t\)-design (a set of unitary matrices that approximate the entire unitary group) for quantum circuits can be tailored into stochastic noise. Then, we prove that local random circuits for twirling separable noisy channel over the Clifford group can be used to construct a unitary 2\(t\)-design, which is easy to implement in experiments. Moreover, we prove that our method is robust to gate-dependent and gate-independent noise. The stochastic noise can be both estimated by average fidelity and directly obtained by randomized benchmarking via unitary 2\(t\)-designs. Obtaining such tailored noise is an important guarantee for achieving fault-tolerant quantum computation.

Preliminaries

We begin by considering a \(d^n\)-dimensional \(t\)-tensor product completely positive trace-preserving (CPTP) channel \(\Lambda^{(t)}\) for characterizing the noise, i.e.,

\[^{1}\) State Key Laboratory of Integrated Services Networks, Xidian University, Xi’an, 710071, China. \(^{2}\) Science and Technology on Communication Networks Laboratory, Shijiazhuang, 050081, China. Correspondence and requests for materials should be addressed to Y.Y. (email: yuyanxidian@126.com)
models of the noisy channel $\Lambda^\otimes t$, (a) A specific model of $\Lambda^\otimes t$ can be considered as $t$ copies of the $d^n$-dimensional noisy channels $\Lambda_i$ for $j \in \{1, \ldots, t\}$. Here, $t$ is mainly used to indicate which type of unitary 2t-design to use for twirling the noisy channel.

$$\Lambda^\otimes t (\rho^\otimes t) = \sum_p A_p^\otimes t \rho^\otimes t (A_p^\otimes t)^\dagger,$$

(1)

with $A_p^\otimes t \in \mathbb{L}(\mathbb{C}^{d^n})$ being a linear operator for any $d^n$-dimensional input density operator $\rho^\otimes t = |\psi\rangle^\otimes t \langle \psi|^\otimes t$ shown in Fig. 1, where $\mathbb{L}(\mathbb{C}^{d^n})$ is with respect to the space of $d^n$-dimensional complex linear operators. For Fig. 1(a), the $t$-tensor product CPTP channel can be composed of $t$ copies of the $d^n$-dimensional CPTP noisy channel $\Lambda_j$ where we define $\Lambda_j (\rho) = \sum_{p=1}^{d^n} A_{p,j}^{d^n} \rho A_{p,j}^{d^n \dagger}$ with any $d^n$-dimensional input state $\rho$. The above model can be used to characterize the corresponding properties of $t$ experiments with coherent measurements. For Fig. 1(b), it is a more general model whereby the $t$-tensor product CPTP channel can also be composed of $t$ different $d^n$-dimensional CPTP noisy channels $\Lambda_j$ for $j \in \{1, \ldots, t\}$, where $\Lambda_j (\rho) = \sum_{p=1}^{d^n} A_{p,j}^{d^n} \rho A_{p,j}^{d^n \dagger}$ and $A_p^\otimes t = \otimes_{j=1}^{t} A_{p,j}^n = A_{p,1} \otimes \cdots \otimes A_{p,t}$.

We define the superoperator representation of the noisy channel $\Lambda^\otimes t$ based on ref.26, i.e.,

$$\hat{\Lambda}^\otimes t = \sum_{m,p=1}^{d^n} A_p^\otimes t \otimes (A_m^{d^n})^\dagger = \left( \sum_{p=1}^{d^n} \otimes_{j=1}^{t} A_{p,j}^n \right) \otimes \left( \sum_{m=1}^{d^n} \otimes_{i=1}^{t} A_{m,i}^n \right)^\dagger,$$

(2)

where $^\dagger$ denotes complex conjugation.

The advantage of such representation is that the cascade of any two noisy channels $\Lambda_1$ and $\Lambda_2$ corresponds to matrices multiplication, i.e.,

$$\Lambda_1 \circ \Lambda_2 \rightarrow \hat{\Lambda}_1 \cdot \hat{\Lambda}_2.$$

(3)

**Definition of the unitary $t$-design.** Then, we introduce a definition of the unitary $t$-design based on ref.27-29.

In this paper, we use unitary 2t-design for twirling of a noisy channel according to the following definition, as defined below.

Let $t$ be a natural number and $\mathbb{L}(d^n)$ be the set of unitary operators in a $d^n$-dimensional Hilbert space. A finite subset $|U|_{D=1}^D \subset \mathbb{L}(d^n)$ is called a unitary $t$-design if

$$\frac{1}{D} \sum_{i=1}^{D} U_i^\otimes t \otimes (U_i^\otimes t)^\dagger = \int_{\mathbb{L}(d^n)} U^\otimes t \otimes (U^\otimes t)^\dagger \mu_{\text{Haar}} (dU),$$

(4)

and the corresponding superoperator representation can be written as

$$\frac{1}{D} \sum_{i=1}^{D} \hat{U}_i^\otimes t = \int_{\mathbb{L}(d^n)} \hat{U}^\otimes t \mu_{\text{Haar}} (dU),$$

(5)

where $\mu_{\text{Haar}} (\cdot)$ is a uniform distribution and the integrals over $\mathbb{L}(d^n)$ are the unitarily invariant Haar measure. Let $p_{(t,t)}(U)$ be any polynomial that is homogeneous of degree $t$ in the matrix elements of $U \in \mathbb{L}(d^n)$ and homogeneous of degree $t$ in their complex conjugate elements of $U^\dagger$. Therefore, a unitary $t$-design $|U|_{D=1}^D$ can also be written as

$$\frac{1}{D} \sum_{i=1}^{D} p_{(t,t)}(U_i) = \int_{\mathbb{L}(d^n)} p_{(t,t)}(U) \mu_{\text{Haar}} (dU).$$

(6)
Finally, we introduce the method of local random circuits, which is an efficient scheme for constructing unitary $t$-design. In each step of the walk, an index $i$ is chosen uniformly at random from the set $\{1, \ldots, n\}$. A two-qubit unitary operator $U_{i,i+1}$ drawn from a set of Haar measures $\mathcal{U}(d^2)$ is applied to the two neighboring qubits $i$ and $i+1$ (because of the finite number of qubits, we suppose the $(n+1)$-th qubit as being equal to the 1-st qubit).

The operator $H_{n,t}$ is a $d^{2n}$-dimensional quantum local Hamiltonian composed of $n$ local subsystem operators $H_{i,i+1}$ such that

$$H_{n,t} = \sum_{i=1}^{n} H_{i,i+1}$$

with local terms $H_{i,i+1} = I_d - P_{i,i+1}$. Here, $P_{i,i+1}$ is the projector of two neighbors $i, i+1$, on $\mathcal{U}(d^2)$ such that

$$P_{i,i+1} = \int_{\mathcal{U}(d^2)} (U_{i,i+1})^\dagger U_{i,i+1} U_{i,i+1}^\dagger U_{i,i+1}^\dagger U_{i,i+1} \mu_{\text{Haar}}(dU)$$

with $U^{\mathcal{U}(d^2)} = U \otimes (U^*)^\dagger$.

The properties of local random circuits are as follows:

- (Periodic boundary conditions) The $(n+1)$-th subsystem is identified with the first;
- (Zero ground-state energy) $\lambda_{\text{min}}(H_{n,t}) = 0$, with $\lambda_{\text{min}}(H_{n,t})$ being the minimum eigenvalue of $H_{n,t}$;
- (Frustration-freeness) Every state $|\psi\rangle$ in the groundstate manifold, composed of all eigenvectors with eigenvalue zero, is such that

$$H_{i,i+1} |\psi\rangle = 0, (i = 1, 2, \ldots, n).$$

For a physical example of local random circuits, we assume $U \in \mathcal{U}(4)$ and that the matrix elements of each $U$ must be algebraic. We introduce a physical construction of parallel local random circuits on $n$ qubits shown in Fig. 2. At each step, we perform with probability $1/2$ either the ‘even’ unitary operation $U_{1,2} \otimes U_{3,4} \otimes \cdots \otimes U_{n-2,n} \otimes U_{n-1,n}$ or the ‘odd’ $U_{2,3} \otimes U_{4,5} \otimes \cdots \otimes U_{n-2,n-1} \otimes U_{n-1,n}$, where each $U_{i,i+1}$ is uniformly randomly sampled from $U$. Starting in an ‘even’ configuration, applying instead an ‘odd’ operation can be accomplished by a shift operation, defined over the $n$ input and two ancilla qubits 0 and $n+1$, such that

$$U_k = S_{n,n+1} \prod_{i=0}^{n-2} S_{i,i+1},$$

where $S_{i,i+1} \in \mathcal{U}(4)$ is the swap operation between qubits $i$ and $i+1$. Iterating the circuit in Fig. 2 therefore produces a local random circuit.

**Results**

We propose a method using the unitary $2t$-design for twirling of the noisy channel. The analysis for noisy channels can also be used for the analysis of the noise characterizing quantum gates. Therefore, we call this method noise tailoring for quantum circuits.

Here, we divide the quantum circuit into $K$ rounds of quantum gates. We propose a method to tailor the noise of each round into stochastic noise via unitary $2t$-designs. For the near term, it is a general method for characterizing the noise in a quantum circuit without quantum error correction. For the long term, many quantum circuits are being packaged for specific functions and given corresponding parameters. We need to estimate the
noise between such devices. The method via unitary 2t-designs for twirling of the noise is an efficient scheme for estimating the stochastic noise for large-scale quantum circuits.

We begin by proposing the method for twirling the noisy channel via a unitary 2t-design. We define a unitary 2t-design for the twirling of a noisy channel shown in Fig. 3. If $\Lambda \otimes 1$ is a unitary 2t-design, the twirled t-tensor product noisy channel via a unitary 2t-design is given by

$$
\Lambda_{\otimes t}^{\otimes t} \otimes (U)_{\otimes t}^{\otimes t}
$$

where

$$
\rho_0 \Lambda = \sum \rho_0 \Lambda \otimes 1
$$

and

$$
A_p \otimes 1 = \otimes_{j=1}^p A_p \Lambda_j
$$

with a fixed input density operator $\rho_0 \Lambda = \langle \psi_0 | \langle \psi_0 | \Lambda_0 \rangle \Lambda_0$. We propose a method to tailor noise using unitary 2t-designs, where the unitary operators shown in Fig. 4(b) should ideally be uniformly selected from the unitary 2t-design for each round of the target gate. Consequently, uniformly averaging over the unitary 2t-design for twirling the noise in each round reduces the noise in the k-th round to the tailored noise, i.e.

---

**Figure 3.** Twirling of noise model via unitary 2t-designs. (a) A general model for twirling of a $d^m$-dimensional noisy channel $\Lambda^{\otimes t}$ via a unitary 2t-design. (b) The corresponding simplified model. The shaded square represents the t-tensor product complex conjugate transpose of the corresponding unitary operator.

---

**Figure 4.** (a) Model of an original circuit that is arranged into cycles wherein each cycle consists of a round of the specified circuit and a round of the corresponding noise. (b) Model of a circuit wherein the gates are chosen from the unitary 2t-design for the twirling of the noise inserted at both ends of the corresponding noise of the gates.

---

**Compiled quantum circuits via unitary 2t-designs.** In Fig. 4(a), an original circuit is composed of $K$ rounds of the quantum circuit and the corresponding noise. Here, $C_{\otimes t}$ is with respect to the k-th t-tensor product noise channel and $A_{k}^{\otimes t}$ is the corresponding noise. Following the method of twirled noise using unitary 2t-designs, we can tailor the noise of each round of the quantum circuits into the stochastic noise. Compared with the division of easy and hard gates, it is a more flexible method.

We now specify how to compile the target circuit in the above form to tailor the noise into an effective stochastic noise. The k-th round of the corresponding noise can be written as

$$
\Lambda_{k}^{\otimes t} (\rho_0^{\otimes t}) = \sum_{p=1}^{d^m} A_{k,p}^{\otimes t} \rho_0^{\otimes t} (A_{k,p}^{\otimes t})^\dagger,
$$

where $A_{k,p}^{\otimes t} \in L(C_{\otimes t})$ is a linear operator for a fixed input $d^m$-dimensional input density operator $\rho_0^{\otimes t} = \langle \psi_0 | \langle \psi_0 | \Lambda_0 \rangle \Lambda_0$.

We propose a method to tailor noise using unitary 2t-designs, where the unitary operators shown in Fig. 4(b) should ideally be uniformly selected from the unitary 2t-design for each round of the target gate. Consequently, uniformly averaging over the unitary 2t-design for twirling the noise in each round reduces the noise in the k-th round to the tailored noise, i.e.
\[ T_k^{2^t}(\rho_0^{2^t}) = E_{U_k^t}[U_k^t(f_k^{2^t}(U_k^t)^\dagger(U_k^t)^{2^t})U_k^t] \]
\[ = \frac{1}{D} \sum_{d=1}^{D} (U_k^t)^{2^t} f_k^{2^t}(U_k^t)^{2^t} \rho_0^{2^t}(U_k^t)^{2^t} dU_k^t \]
\[ = \int_{\hat{\mathcal{H}}(d^t)} (U_k^t)^{2^t} f_k^{2^t}(U_k^t)^{2^t} \rho_0^{2^t}(U_k^t)^{2^t} \mu_{\text{Haar}}(dU_k^t) \]  
(12)

with a \(d^t\)-dimensional fixed input density operator \(\rho_0^{2^t}\). Because the unitary operators uniformly selected from the unitary \(2^t\)-design are independent of the fixed input density operator \(\rho_0^{2^t}\), the expected noise over \(D\) experiments is exactly the tailored noise. The superoperator representation of the tailored noise can be written as

\[ \hat{T}_k^{2^t} = E_{U_k^t}[U_k^t(f_k^{2^t}(U_k^t)^\dagger(U_k^t)^{2^t})U_k^t] \]
\[ = \frac{1}{D} \sum_{d=1}^{D} (\hat{U}_k^t)^{2^t} f_k^{2^t}(\hat{U}_k^t)^{2^t} U_k^t \]
\[ = \int_{\hat{\mathcal{H}}(d^t)} (\hat{U}_k^t)^{2^t} f_k^{2^t}(\hat{U}_k^t)^{2^t} \mu_{\text{Haar}}(dU_k^t), \]  
(13)

where \(\hat{U}_k^t = U_k^{2^t} \otimes (U_k^{2^t})^\dagger\). From the above equality, the tailored noise is not implemented in any given \(K\) rounds of quantum gates. Instead, it is the average over unitary \(2^t\)-designs. This method can be performed in conjunction with a classical computer or with fast control. Moreover, this fast control is exactly equivalent to the control required in quantum error correction and thus does not impose an additional experimental burden.

**Robustness to gate-independent and gate-dependent noise.** Our first approach is that the method of unitary \(2^t\)-design for twirled noisy channel is applied to tailor the noise of each round of the quantum gates.

**Theorem 1** Uniformly sampling the unitary operators from a unitary \(2^t\)-design independently in each round tailors the noise of quantum gates into stochastic noise when the noise is gate independent.

Theorem 1 establishes that the noise in each round can be exactly tailored into stochastic noise. In ref.12, the authors use a Pauli operator randomly selected from the Pauli group to tailor the noise of each round of the corresponding quantum gate, which is a method using the unitary 2-design for twirling noise. However, the above method can only be applied to quantum circuits with a single-qubit gate. Here, we propose a more general method to tailor noise using the unitary \(2^t\)-design. We can use the construction of the unitary \(2^t\)-design to tailor any independent noise for large-scale quantum circuits. Moreover, the error of the estimation caused by pseudo-randomness is avoided.

Our second approach is to give a specific construction of the unitary \(2^t\)-design.

**Theorem 2** For any completely positive separable noisy channel, an exact unitary \(2^t\)-design for the twirling of channels can be constructed by local random circuits with a uniform Haar measure over the Clifford group.

A unitary \(2^t\)-design is highly similar to a spherical \(2^t\)-design independently in each round tailors noise. Therefore, choosing a reasonable construction for unitary \(2^t\)-design to characterize noise is the key to solving this problem.

Theorem 2 shows a specific construction for unitary \(2^t\)-design. Local random circuits are easier to implement in experiments and avoid errors caused by constructing a specific set of unitary operators. Roughly speaking, randomized benchmarking is a special case for estimating the average fidelity using local random circuits over the Clifford group for twirling noise.

Our third approach is to give the relationship between the gate-dependent and gate-independent noise.

**Theorem 3** Let \(\hat{C}_{GD}^{2^t}\) and \(\hat{C}_{GI}^{2^t}\) be the superoperator representation of two tailored circuits with \(K\) rounds of gate-dependent and gate-independent noise, respectively. Then,

\[ \left\| \hat{C}_{GD}^{2^t} - \hat{C}_{GI}^{2^t} \right\|_1 \leq E_{K=1}^{K} \left\| \hat{f}_k^{2^t}(C_k^{2^t}) - \hat{f}_k^{2^t}(\hat{C}_k^{2^t}) \right\|_1, \]  
(14)

where \(E_{K=1}^{K}\) is with respect to expectations from \(K\)-th to first round.

Note that the diamond norm of a superoperator \(\Delta\) is defined as in ref.34:

\[ \left\| \Delta \right\|_1 = \sup_{\|L\|_1 \leq 1} \left\| \Delta \otimes L \right\|_1 \]  
(15)

Here, the \(p \rightarrow q\) induced Schatten norm is

\[ \left\| \Delta(X) \right\|_q = \sup_{X=0} \left\| \Delta(X) \right\|_q. \]  

The diamond norm is generally used as the quantity to prove the fault-tolerance thresholds.

Theorem 3 establishes the robustness to gate-dependent and gate-independent noise in each round of the quantum circuit. In near-term applications without quantum error correction, the above theorem can be applied
to characterize the noise for the entire quantum circuit. In the long term for applications with packaged quantum devices, the above theorem can be an important basis for quantum error correction. However, rigorously determining the bound would require estimating the gate-dependent noise, which is currently an open problem. In particular, if each round of a gate in a quantum circuit is an element in the set of local random circuits, the gate-dependent noise can also be tailored into the stochastic noise. For a large-scale quantum circuit that satisfies the above conditions, it can be estimated directly by the stochastic noise model.

**Numerical Simulations**

Tailoring experimental noise into stochastic noise using unitary 2-designs provides several dramatic advantages, which we now illustrate via numerical simulations. In our simulation, we assume $d = 2$, $n = 2$, and $t = 2$ for the four-qubit quantum circuits as a physical example. Because an arbitrary two-qubit unitary gate can be decomposed in terms of three CNOT (controlled-NOT) gates and corresponding single-qubit unitary gates36–39 shown in Fig. 5, our simulations are all of four-qubit circuits with single-qubit unitary operations and CNOT gates shown in Fig. 6(a). Such circuits are universal for quantum computation.

For our simulation, we add gate-dependent noise to each gate shown in Fig. 6(b), that is, we perturb one of the eigenvectors of each gate by $e^{i \phi}$. For single-qubit gates the choice of eigenvector is irrelevant, while for the CNOT gate, we add the phase to the $|11\rangle$ state. We then apply unitary 4-designs composed of local random circuits to tailor the noise shown in Fig. 6(c).

We quantifiy the total noise in a noisy quantum circuit $C_{\text{noisy}}$, of an ideal circuit $C_{\text{ideal}}$ by the variational distance

$$
\epsilon = \frac{1}{2} \sum_{j} |\Pr(j|C_{\text{noisy}}) - \Pr(j|C_{\text{ideal}})|
$$

between the probabilities for ideal computational basis measurements after applying $C_{\text{noisy}}$ and $C_{\text{ideal}}$ to a system initialized in the $|0\rangle^\otimes n$ state. We do not maximize over states and measurements; rather, our results indicate the effect of noise under practical choices of preparations and measurements.

We perform two sets of numerical simulations to illustrate the properties. Figure 7 shows that our technique introduces an improvement as the disturbance of the noise $\delta$ decreases. For the original circuits, each data point is the variational distance of $20$ cycles of $2$-tensor product of two-qubit unitary operations, each composed of three CNOT gates and eight randomly selected single-qubit unitary gates. For the tailored circuits, each data point is the variational distance between $\Pr(j|C_{\text{ideal}})$ and the probability $\Pr(j|C_{\text{noisy}})$ averaged over $1000$ randomizations of the unitary 4-designs.

We take four tests, each as shown in Fig. 7, to obtain the properties of the mean-square errors about the original and tailored circuits. Table 1 shows that our technique make the change of total noise more stable. Therefore, tailored noise via unitary 2-designs composed of local random circuits is an efficient method to characterize noise of quantum circuits.

**Conclusion**

We have shown that arbitrary Markovian noise processes can be tailored into stochastic noise using unitary 2-design for twirling noise. This technique can effectively estimate the coherent noise error for large-scale quantum circuits. Then, we proved that local random circuits over the Clifford group for twirled separable noisy channel can construct an exact unitary 2-design. This method can be performed efficiently on a classical computer or with fast control with no additional experimental burden. Furthermore, our method of tailored noise is robust against gate-dependent errors. In particular, the gate-dependent noise in all but the final round can be tailored into stochastic noise.

A significant open problem is the construction of the unitary 2-design. A unitary 2-design is highly similar to a spherical 2-design in terms of the Jamiolkowski isomorphism and the frame potential. We need to continue to study the use of spherical 2-design to find a simpler construction method for unitary 2-design.

**Methods**

**Proof of Theorem 1.** The average fidelity estimation is an efficient method for partially characterizing noise. The average fidelity of the $k$-th $d^n$-dimensional noise $\Lambda_k^{d^n}$, $k \in \{1, \ldots, K - 1\}$ can be written as

![Figure 5](https://example.com/figure5.png)

Figure 5. An arbitrary 2-qubit unitary gate $U \in \mathcal{U}(4)$ can be decomposed by canonical decomposition, where $H = h_x \sigma_x \otimes \sigma_x + h_y \sigma_y \otimes \sigma_y + h_z \sigma_z \otimes \sigma_z + i/4 \sigma_x \sigma_y \sigma_z \geq |h_x| \geq |h_y| \geq |h_z|$. It can also be decomposed in terms of three CNOT gates and eight single-qubit unitary gates.
Figure 6. Example of a 4-qubit gate-dependent noisy circuit. (a) The 4-qubit quantum circuit in the simulation is composed of \( K \) cycles of corresponding circuits. For the \( k \)-th cycle, there are 2-tensor product circuits, each composed of 3 CNOT gates and 8 single-qubit unitary gates. In the simulation, we let \( K = 20 \). (b) A part of the original circuit which represents the part in the dotted box of the circuit with corresponding gate-dependent noise. In the simulation, we perturb one of the eigenvectors of each gate by \( e^{\delta} \) for the single-qubit gates. For the CNOT gates, we add the phase to the corresponding \( |11\rangle \) state. (c) A part of the tailored circuit which represents the part in the dotted box in the circuit with twirled noise via unitary 4-designs. In the simulation, the unitary 4-designs are composed of local random circuits \( U_{\text{local}}^{i} \) for \( i \in \{1, \ldots, 7\} \).

Figure 7. Plots of the error \( \epsilon \) with respect to computational basis measurement outcomes in four-qubit original (blue hollow circles) and tailored (green diamonds) circuits. Each data point corresponds to an independent random circuit with 20 cycles, where we perturb one of the eigenvectors of each single-qubit gate by \( e^{\delta} \) and add the phase to the \( |11\rangle \) state of the CNOT gate. The data points for the tailored noise correspond to an average over 1000 independent randomizations of the unitary 4-designs. The blue line and green line are the linear fit of the corresponding error points about the original and tailored circuits, respectively.

| Mean-square error (10^{-5}) | test 1 | test 2 | test 3 | test 4 |
|----------------------------|-------|-------|-------|-------|
| Original circuits          | 8.37  | 8.35  | 9.10  | 7.44  |
| Tailored circuits          | 1.86  | 1.56  | 2.42  | 1.12  |

Table 1. Main-square error of the original and tailored circuits.
with the $d^n$-dimensional input state $|\psi_0^{\otimes t}\rangle$. We can also use $k$-th unitary operators uniformly selected from the unitary 2t-design $\{U_{k,i}\}_1^D$, for twirling the noisy channel to estimate the average fidelity, i.e.,

$$F(\Lambda_k^{\otimes t}) = \mathbb{E}_{(\psi_0^{\otimes t})}(\Lambda_k^{\otimes t} | \psi_0^{\otimes t} \rangle \langle \psi_0^{\otimes t}|)$$

$$= \int \left| \langle \psi_0^{\otimes t} | \Lambda_k^{\otimes t} | \psi_0^{\otimes t}\rangle \right|^2 \mathcal{N}_{\text{Haar}}(d\psi)$$

(17)

with the $d^n$-dimensional input state $|\psi_0^{\otimes t}\rangle$. The Eq. (17) is equal to Eq. (18). It can be recognized in the experiment as the average fidelity by uniformly averaging over all unitary operators with a fixed input state instead of averaging over all input states.

From Schur's lemma, the $k$-th tailored noise can be expressed as a depolarizing channel, i.e.,

$$T_k^{\otimes t}(\rho_0^{\otimes t}) = \rho_0^{\otimes t} + (1 - p_k) \text{Tr}(\rho_0^{\otimes t}) \frac{I_{d^n}}{d^n}$$

(19)

with the strength parameter

$$p_k = \frac{\text{Tr}(\tilde{T}_k^{\otimes t}) - 1}{d^{2n} - 1}.$$  

(20)

**Proof of Theorem 2.** We consider a completely positive noisy channel composed of a linear mapping expressed as $\Lambda^{\otimes t}(\rho_0^{\otimes t}) = A^{\otimes t} \rho_0^{\otimes t} B^{\otimes t} [A^{\otimes t}, B^{\otimes t} \in \mathcal{L}(C(d^n))]$ and local random circuits $[M_{n,t}, M_{n,t}^\dagger]$ at both ends for any $d^n$-dimensional input state $\rho_0^{\otimes t}$. Our approach is to prove that local random circuits with a uniform distribution over the Clifford group are exact unitary 2t-designs for the twirling of separable noisy channels. This can be achieved by proving that $\Lambda^{\otimes t}$ is a normal subgroup of $\text{CU}(d^n)$.

We first define the superoperator representation of local random circuits as

$$\hat{M}_{n,t} = \frac{1}{n} \sum_{i=1}^n \hat{I}_{d^{2n-1}} \otimes \hat{T}_{i,t+1} \otimes \hat{I}_{d^{2n-1}},$$

(21)

with

$$\hat{T}_{i,t+1} = \int_{|d^2|} (U_{i,t+1})^{\otimes t} \mathcal{N}_{\text{Haar}}(dU).$$

(22)

In theory, we can prove local random circuits with uniform distribution over the Clifford group to construct an exact unitary 2t-design for the twirling of separable noisy channels as follows. In practice, we generally build parallel random unitary operators using the even and odd tensor products which we will not introduce in this paper (an introduction is given in ref.30).

For a $d^n$-dimensional quantum channel $\Lambda^{\otimes t}$, we define

$$\text{Sum}_{d^n}(\rho^{\otimes t}) = \frac{1}{|\mathcal{C}(d^n)|} \sum_{U_{i,t+1} \in \mathcal{C}(d^n)} M_{n,t}^{\dagger} \Lambda^{\otimes t}(M_{n,t}^{\otimes t} \rho_0^{\otimes t} M_{n,t}^\dagger) M_{n,t}$$

$$= \frac{1}{|\mathcal{C}(d^n)|} \sum_{U_{i,t+1} \in \mathcal{C}(d^n)} \left| 1 - \frac{1}{n} \right| \Lambda^{\otimes t}(\rho^{\otimes t})$$

$$+ \frac{1}{n} \sum_{i=1}^n \left( U_{i,t+1}^{\dagger} \Lambda_{i,t+1}^{\otimes t} (U_{i,t+1}^{\otimes t} \rho_0^{\otimes t} (U_{i,t+1}^{\otimes t})^\dagger) U_{i,t+1}^{\dagger} \right)$$

(23)

and
Note that $\sum_{i=1}^{n}$ represents the sum elements of a $d^n$-dimensional matrix and $\Lambda_{i,i+1}$ is with respect to the $d^n$-dimensional matrix of noise between two neighboring qubits $i$ and $i+1$. Our approach is only to prove that Eqs (23) and (24) are equal. Now, we need to prove that

$$\sum_{i=1}^{n} (U_{i,i+1}^{(\otimes)} \Lambda_{i,i+1}^{(\otimes)} (U_{i,i+1}^{(\otimes)})^\dagger) U_{i,i+1}^{(\otimes)} = \int_{(d^2)^1} (U_{i,i+1}^{(\otimes)} \Lambda_{i,i+1}^{(\otimes)} (U_{i,i+1}^{(\otimes)})^\dagger) U_{i,i+1}^{(\otimes)} \mu_{\text{Haar}}(dU).$$

(25)

Following the Eqs (19) and (20), we obtain

$$\text{Int}_{d^n} (\rho^{\otimes}) = \frac{\text{Tr}(U_{i,i+1}^{(\otimes)} \Lambda_{i,i+1}^{(\otimes)} (U_{i,i+1}^{(\otimes)})^\dagger) - 1}{d^n - 1} \rho_{i,i+1}^{\otimes} + \frac{d^n - \text{Tr}(U_{i,i+1}^{(\otimes)} \Lambda_{i,i+1}^{(\otimes)} (U_{i,i+1}^{(\otimes)})^\dagger)}{d^n - 1} \text{Tr}(\rho_{i,i+1}^{\otimes}) \|_{d^n}$$

$$= \frac{d^n - \text{Tr}(A_{i,i+1}^{(\otimes)} \rho_{i,i+1}^{\otimes})}{d^n - 1} \text{Tr}(\rho_{i,i+1}^{\otimes}) \|_{d^n}$$

$$+ \frac{d^n - \text{Tr}(A_{i,i+1}^{(\otimes)} \rho_{i,i+1}^{\otimes})}{d^n - 1} \text{Tr}(\rho_{i,i+1}^{\otimes}) \|_{d^n}$$

(26)

with $\text{Tr}(A_{i,i+1}^{(\otimes)} (\rho_{i,i+1}^{\otimes})) = \text{Tr}(A_{i,i+1}^{(\otimes)} B_{i,i+1}^{(\otimes)}) = d^2$.

We denote the elements of the Pauli group $\mathcal{P}(d^2)$ as $[P_{i,j}]_{i=1}^{d^2}$, consisting of all $2$-fold $t$-tensor products of the one-qubit Pauli operators $[I_d, X, Y, Z]$, where $P_1$ is the $2$-fold $t$-tensor product of $I_d$. We can define $A_{i,i+1}^{(\otimes)} = \sum_{a,b} \alpha_a \beta_b P_{a,i}^{\otimes} P_{b,i+1}^{\otimes}$ and $\rho_{i,i+1}^{\otimes} = \sum_{a,b} \beta_a \alpha_b P_{a,i}^{\otimes} P_{b,i+1}^{\otimes}$. Then we have $\text{Tr}(A_{i,i+1}^{(\otimes)} (\rho_{i,i+1}^{\otimes})) = d^2 \alpha_1$ and $\text{Tr}(B_{i,i+1}^{(\otimes)} (\rho_{i,i+1}^{\otimes})) = d^2 \beta_1$. The expression of the Pauli-twirled superoperator $\Lambda_{i,i+1}^{(\otimes)}$ is given by

$$\Lambda_{i,i+1}^{(\otimes)} (\rho^{\otimes}) = \frac{1}{d^4} \sum_{j=1}^{d^4} \sum_{a,b=1}^{d^2} \alpha_a \beta_b \sum_{j=1}^{d^2} \omega^{(a,b)j} P_{a,i}^{\otimes} P_{b,i+1}^{\otimes}$$

$$= \sum_{a,b=1}^{d^2} \alpha_a \beta_b P_{a,i}^{\otimes} P_{b,i+1}^{\otimes}$$

(27)

where $\sum_{j=1}^{d^2} \omega^{(a,b)j} = d^4 \delta_{a,b}$.

From the above equalities, the $\text{SL}(d^n)$-twirl yields

$$\frac{1}{|\text{SL}(d^n)|} \sum_{U_{i,i+1}^{(\otimes)} \in (d^2)^1} U_{i,i+1}^{(\otimes)} \Lambda_{i,i+1}^{(\otimes)} (U_{i,i+1}^{(\otimes)})^\dagger U_{i,i+1}^{(\otimes)}$$

$$= \frac{\text{Tr}(\rho_{i,i+1}^{\otimes})}{|\mathcal{P}(d^2)|} \sum_{U_{i,i+1}^{(\otimes)} \in (d^2)^1} \sum_{j=1}^{d^4} \alpha_j \beta_j \left( \sum_{a,b=1}^{d^2} \omega^{(a,b)j} P_{a,i}^{\otimes} P_{b,i+1}^{\otimes} \right)$$

$$= \left( \sum_{j=1}^{d^4} \alpha_j \beta_j \right) \rho_{i,i+1}^{\otimes} + \frac{d^4 - \sum_{j=1}^{d^4} \alpha_j \beta_j}{d^4 - 1} \text{Tr}(\rho_{i,i+1}^{\otimes}) \|_{d^n}$$

(28)

where $\text{Tr}(\rho_{i,i+1}^{\otimes}) \|_{d^n} = \frac{1}{d^n} \sum_{j=1}^{d^4} \rho_{i,i+1}^{\otimes} P_{j}^{\otimes}$.

Now, we have the conclusion

$$\text{Sum}_{d^n} (\rho^{\otimes}) = \text{Int}_{d^n} (\rho^{\otimes}).$$

(29)
Proof of Theorem 3. We use the superoperator representation to represent the two tailored circuits \( \hat{\mathcal{C}}_{GD}^{\otimes t} \) and \( \hat{\mathcal{C}}_{GL}^{\otimes t} \) with \( K \) rounds of gate-dependent and gate-independent noise. Let

\[
\begin{align*}
\hat{\mathcal{A}}_{GD, t}^{\otimes t} &= \hat{A}_k^{\otimes t} \hat{A}_k^{\dagger} (C_k^{\otimes t}) \\
\hat{\mathcal{B}}_{GL, t}^{\otimes t} &= \hat{A}_k^{\otimes t} \hat{A}_k^{\dagger} 
\end{align*}
\]

Then, the tailored circuits under gate-dependent noise is

\[
\begin{align*}
\hat{\mathcal{C}}_{GD}^{\otimes t} &= \mathbb{E}_{K, t} \hat{\mathcal{A}}_{K, 1}^{\otimes t} = \frac{1}{D^t} \prod_{k=1}^{K} \hat{C}_k^{\otimes t} \hat{U}_k^{\otimes t} \hat{A}_k^{\otimes t} (C_k^{\otimes t}) \hat{U}_k^{\otimes t} \\
&= \int \mathcal{D}(\hat{U}) \prod_{k=1}^{K} \hat{C}_k^{\otimes t} \hat{U}_k^{\otimes t} \hat{A}_k^{\otimes t} (C_k^{\otimes t}) \hat{U}_k^{\otimes t} \prod_{k=1}^{K} \mu_{\text{Haar}}^k (d\hat{U}_k) \\
&= \int \mathcal{D}(\hat{U}) \prod_{k=1}^{K} \hat{C}_k^{\otimes t} \hat{U}_k^{\otimes t} \hat{A}_k^{\otimes t} (C_k^{\otimes t}) \hat{U}_k^{\otimes t} \prod_{k=1}^{K} \mu_{\text{Haar}}^k (d\hat{U}_k),
\end{align*}
\]

where \( \mu_{\text{Haar}}^k \) is with respect to the expectation over all \( K \) rounds of the tailored noise. Similarly, we can obtain the tailored circuits under gate-independent noise using unitary 2t-designs \( \{U_k^{\otimes t}\}_{i=1}^{D^t} \), i.e.,

\[
\begin{align*}
\hat{\mathcal{C}}_{GL}^{\otimes t} &= \mathbb{E}_{K, t} \hat{\mathcal{B}}_{K, 1}^{\otimes t} = \frac{1}{D^t} \prod_{k=1}^{K} \hat{C}_k^{\otimes t} \hat{U}_k^{\otimes t} \hat{A}_k^{\otimes t} (C_k^{\otimes t}) \hat{U}_k^{\otimes t} \\
&= \int \mathcal{D}(\hat{U}) \prod_{k=1}^{K} \hat{C}_k^{\otimes t} \hat{U}_k^{\otimes t} \hat{A}_k^{\otimes t} (C_k^{\otimes t}) \hat{U}_k^{\otimes t} \prod_{k=1}^{K} \mu_{\text{Haar}}^k (d\hat{U}_k),
\end{align*}
\]

If we assume \( \hat{\mathcal{A}}_{K+1, 1}^{\otimes t} = \hat{B}_{0, 1}^{\otimes t} = \mathbb{I}_{D^{2t}} \), the diamond norm of the difference between the two tailored circuits \( \hat{\mathcal{C}}_{GD}^{\otimes t} \) and \( \hat{\mathcal{C}}_{GL}^{\otimes t} \) can be written as

\[
\begin{align*}
\| \hat{\mathcal{C}}_{GD}^{\otimes t} - \hat{\mathcal{C}}_{GL}^{\otimes t} \|_D &= \left\| \mathbb{E}_{K, t} \sum_{k=1}^{K} \hat{\mathcal{A}}_{K, k+1}^{\otimes t} (\hat{\mathcal{A}}_k^{\otimes t} - \hat{B}_k^{\otimes t}) \hat{B}_{k+1}^{\otimes t} \right\|_D \\
&\leq \mathbb{E}_{K, 1} \left\| \sum_{i=1}^{K} \hat{\mathcal{A}}_{K, i+1}^{\otimes t} (\hat{\mathcal{A}}_k^{\otimes t} - \hat{B}_k^{\otimes t}) \hat{B}_{k+1}^{\otimes t} \right\|_D \\
&\leq \left\| \sum_{i=1}^{K} \hat{\mathcal{A}}_{K, i+1}^{\otimes t} (\hat{C}_k^{\otimes t} - \hat{A}_k^{\otimes t}) \right\|_D,
\end{align*}
\]

where the normalization \( \left\| \hat{A}_k^{\otimes t} \right\|_D = \left\| \hat{B}_k^{\otimes t} \right\|_D = 1 \) which holds for all quantum channels.

References
1. Lidar, D. & Brunner, T. Quantum Error correction. (Cambridge University Press).
2. Aliferis, P., G., D. & Preskill, J. Quantum accuracy threshold for concatenated distance-3 codes. Quantum Info. Comput. 6, 97–165 (2006).
3. Fortescue, B., N., S. & Byrd, M. Fault-tolerance against loss for photonic QEC. arXiv:1405.1766v1 [quant-ph] (2014).
4. Varnava, M., B., D. & Rudolph, T. Loss tolerance in one-way quantum computation via counterfactual error correction. Phys. Rev. Lett. 97, 120501 (2006).
5. Whiteside, A. C. & Fowler, A. G. Upper bound for loss in practical topological-cluster-state quantum computing. Phys. Rev. A 90, 052316 (2014).
6. Blume-Kohout, R. et al. Demonstration of qubit operations below a rigorous fault tolerance threshold with gate set tomography. Nat. Commun. 8 (2017).
7. Chuang, I. L. & Nielsen, M. A. Prescription for experimental determination of the dynamics of a quantum black box. J. Mod. Opt. 44, 2455–2467 (1997).
8. Poyatos, J. F., C., J. I. & Zoller, P. Complete experimental characterization of a quantum process: the two-bit quantum gate. Phys. Rev. Lett. 78, 390–393 (1997).
9. D’Ariano, G. M., P., M. A. & Sacchi, M. F. Quantum tomography. Advances in Imaging and Electron Physics 128, 205–308 (2003).
10. Merkel, S. T. et al. Self-consistent quantum process tomography. Phys. Rev. A 87, 062113 (2013).
11. Knill, E. Fault-tolerant postselected quantum computation: Threshold analysis. arXiv:0404104 [quant-ph] (2004).
12. Wallman, J. J. & Emerson, J. Noise tailoring for scalable quantum computation via randomized compiling. Phys. Rev. A 94, 052325 (2016).
13. Brandao, F. G. S. L. & Horodecki, M. Exponential quantum speed-ups are generic. Q. Inf. Comp 13, 0901 (2013).
14. Brandao, F. G. S. L., Harrow, A. W. & Horodecki, M. Local random quantum circuits are approximate polynomial-designs. Communications in Mathematical Physics 346, 397–434 (2016).
15. Wallman, J., J., B. & Emerson, J. Robust characterization of loss rates. Phys. Rev. Lett. 115, 060501 (2015).
16. Magesan, E., G., J. M. & Emerson, J. Characterizing quantum gates via randomized benchmarking. Phys. Rev. A 85, 042311 (2012).
17. Wallman, J., G., C. H. R. & Flammini, S. T. Estimating the coherence of noise. New J. Phys. 17, 113020 (2015).
18. Knill, E. et al. Randomized benchmarking of quantum gates. Phys. Rev. A 77, 012307 (2007).
19. Magesan, E., G., J. M. & Emerson, J. Robust randomized benchmarking of quantum processes. Phys. Rev. Lett. 106, 180504 (2010).
20. Gaebler, J. P. et al. Randomized benchmarking of multi-qubit gates. Phys. Rev. Lett. 108, 260503 (2012).
21. Magesan, E. et al. Efficient measurement of quantum gate error by interleaved randomized benchmarking. Phys. Rev. Lett. 109, 080505 (2012).
22. Wallman, J. J. & Flammia, S. T. Randomized benchmarking with confidence. New J. Phys. 16, 043021 (2014).
23. Wallman, J. J., B., M. & Emerson, J. Characterization of leakage errors via randomized benchmarking. New J. Phys. 18, 043021 (2016).
24. Helsen, J., J. J., F. S. T. & Wehner, S. Multi-qubit randomized benchmarking using few sample. arXiv:1701.04299v1 [quant-ph] (2017).
25. Proctor, T., R., K., Y., K., S., M. & Blume-Kohout, R. What randomized benchmarking actually measures. Phys. Rev. Lett. 119, 130502 (2017).
26. Emerson, J., A., R. & Zyczkowski, K. Scalable noise estimation with random unitary operators. J. Opt. B: Quantum Semiclass. Opt. 7, 347–352 (2005).
27. Roy, A. & Scott, A. J. Unitary designs and codes. Designs Codes and Cryptography 53, 13–31 (2009).
28. Dankert, C., C., R., E., J. & Livine, E. Exact and approximate unitary 2-designs: Constructions and applications. Phys. Rev. A 80, 012304 (2009).
29. Gross, D., A., K. & Eisert, J. Evenly distributed unitaries: on the structure of unitary designs. Journal of Mathematical Physics 48, 2171 (2007).
30. Turner, P. S. & Markham, D. Derandomizing quantum circuits with measurement based unitary designs. Phys. Rev. Lett. 116, 200501 (2016).
31. Bannai, E. & Bannai, E. A survey on spherical designs and algebraic combinatorics on spheres. European Journal of Combinatorics 30, 1392–1425 (2009).
32. Delarte, P. G., J. M. & Seidel, J. J. Spherical codes and designs. Geometriae Dedicata 6, 68–93 (1991).
33. Benedetto, J. J. & Fickus, M. Finite normalized tight frames. Advances in Computational Mathematics 18, 257–385 (2003).
34. Kueng, R., L., D. M., D., A. C. & Flammia, S. T. Comparing experiments to the fault-tolerance threshold. Phys. Rev. Lett. 117, 170502 (2016).
35. Aharonov, D. & Ben-Or, M. Fault-tolerant quantum computation with constant error rate. SIAM Journal on Computing 38, 1207–1282 (2008).
36. Vidal, G. & Dawson, C. M. A universal quantum circuit for two-qubit transformations with three cnot gates. Physical Review A 69, 010301–1–010301–4 (2003).
37. Kraus, B. & Cirac, J. I. Optimal creation of entanglement using a two-qubit gate. Physical Review A 63, 21–22 (2001).
38. Childs, A. M., Hanglebroe, H. L. & Nielsen, M. A. Lower bounds on the complexity of simulating quantum gates. Physical Review A 68, 162–162 (2003).
39. Zhang, J., Vala, J., Sastry, S. & Whaley, K. B. Geometric theory of nonlocal two-qubit operations. Physical Review A 67, 645–648 (2002).
40. Goodman, R. & Wallach, N. R. Symmetry, representations, and invariants. (Springer).
41. Chau, H. F. Unconditionally secure key distribution in higher dimensions by depolarization. IEEE Transactions on Information Theory 51, 1451–1468 (2005).

Acknowledgements

This research work was supported by the National Natural Science Foundation of China (Grant Nos 61372076, 61701375), the 111 Project (No. B08038), the Key Research and Development Plan of Shannxi Province (No. BBD24017290001) and Foundation of Science and Technology on Communication Networks Laboratory (KX172600031).

Author Contributions

L.Z. conceived of and designed the study and wrote the initial draft of the manuscript; Y.Y. contributed to the interpretation of the results and assisted in the preparation of the manuscript; C.Z. and C.P. critically reviewed the manuscript; and all authors contributed to the manuscript preparation.

Additional Information

Competing Interests: The authors declare no competing interests.

Publisher’s note: Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this license, visit http://creativecommons.org/licenses/by/4.0/.

© The Author(s) 2019