Abstract. Jacobi matrices probably are the most classical object in spectral theory, while CMV matrices are a comparably fresh one, although they are related to a very classical topic, namely to orthogonal polynomials on the unit circle (in the same way as Jacobi matrices are related to orthogonal polynomials on the real axis). We will discuss the third member of this family. Our matrices are generated by the orthonormal systems of functions related to the so-called Strong Moment Problem. For this reason we call them SMP matrices. For instance, one can describe the spectral sets of periodic SMP matrices. Similarly to the case of their counterparts, the description is given by means of conformal mappings on hyperbolic, in this case, comb domains. One can represent functional models associated with periodic and almost periodic SMP matrices. We are especially enthusiastic about the role, which such matrices can play in the Killip-Simon-problem related to Jacobi matrices with the essential spectrum on two arbitrary intervals. The parametric description of SMP matrices of the Killip-Simon-class with their essential spectrum on two arbitrary intervals is the main result of this paper.

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1. Introduction

Jacobi matrices probably are the most classical object in spectral theory, while CMV matrices are a comparably fresh one [7, 8], although they are related to a very classical topic, namely to orhtogonal polynomials on the unit circle (in the same way as Jacobi matrices are related to orthogonal polynomials on the real axis). We will discuss the third member of this family. Our matrices are generated by the orthonormal systems of functions related to the so-called Strong Moment Problem (see, for instance, [6] and references therein). For this reason we call them SMP matrices.

In their breakthrough paper [4] R. Killip and B. Simon completely described spectral properties of Jacobi matrices $J_+\nu$ which are Hilbert-Schmidt-class pertubations of matrices with constant coefficients. That is, let

$$J_+e_n = a_n e_{n-1} + b_n e_n + a_{n+1} e_{n+1}, \quad a_0 = 0,$$

where $\{e_n\}_{n\geq 0}$ is the standard basis in $\ell^2_+$. Then

$$\sum_n ((a_n - 1)^2 + b_n^2) < \infty$$

if and only if the spectrum of $J_+$ is the union of the interval $[-2, 2]$ and an at most countable system of points $X = \{x_k\}$ accumulating to the ends of the interval only and

$$\int_{-2}^2 |\log \sigma'(x)| \sqrt{4 - x^2} \, dx + \sum_{x_k \in X} \sqrt{x_k^2 - 4} < \infty,$$

where $\sigma$ denotes the spectral measure of $J_+$.

In [3] this result was generalized to the case of pertubations of (non-degenerated) periodic matrices. The main idea behind this is a reduction of the initial problem to the problem related to pertubations of Jacobi matrices with constant matrix-block coefficients.

Recall that the spectrum $E$ of a $p$-periodic Jacobi matrix $J_0$ is of the form $E = T^{-1}([-2, 2])$, where $T(z)$ is a polynomial. In the non-degenerated case $E$ is of degree $p$ and $E$ consists of exactly $p$ intervals. Let $J_0(E)$ be the collection of all periodic matrices with given spectrum $E$. Then the so-called Magic Formula holds true:

$$T(J_0) = S^p + S^{-p}$$

for every $J_0 \in J_0(E)$. Now, the Killip-Simon-class $KS_J(E)$ can be fully characterized as the collection of $J$’s for which $T(J) - (S^p + S^{-p})$ belongs to the Hilbert-Schmidt-class. Thus, $T(J)$ is a block-Jacobi matrix whose block-coefficients are close to the $p \times p$ block-matrix $S^p + S^{-p}$ with constant coefficients. After that, the class $KS_J(E)$ was completely characterized in spectral terms as well as w.r.t. coefficient sequences (by means of approaching of the shifts of $J$ to the isospectral set $J_0(E)$).

In [2] the following problem was posed: “Is there an extention of the Damanik-Killip-Simon theorem to the general $E$ case?”. Note that, in the two-interval case, the Damanik-Killip-Simon method can only be used if $E$ is the union of two intervals of equal length. We are able to study the case of two arbitrary intervals, but for SMP matrices. We propose a magic formula
for periodic SMP matrices $A_0$ with the given spectrum $E$, i.e., (compare to (1.1))

$$V(A_0) = S^2 + S^{-2},$$

where $V(z)$ denotes a special rational function (see Theorem 3.3.2 for more details). Subsequently, we will define the Killip-Simon-class $\text{KSSMP}(E)$ for SMP matrices $A$ by $V(A) - (S^2 + S^{-2})$ being a Hilbert-Schmidt operator. Thus, the theorem on perturbations of Jacobi matrices with block-matrix coefficients can be used perfectly well in our case. In particular, we can get spectral properties of $A$ in the same way as in [3]. As the main result of this paper we give a parametric description of the coefficients of the elements from $\text{KSSMP}(E)$, see Theorem 3.6.1.

Furthermore, we present a similar explicit parametrization for Jacobi matrices with their essential spectrum on two symmetric intervals. We begin by dealing with this as an introductory problem. The structure of the paper is given in the table of contents.

2. Jacobi matrices

2.1. An introduction to Jacobi matrices. In this section we consider a real measure $\sigma$, whose support we assume to be compact. By applying the Gram-Schmidt orthogonalization procedure to the family of polynomials $\{x^n\}_{n \geq 0}$ w.r.t. $\sigma$, we obtain orthonormal polynomials $P_n = \alpha^{(n)}_nx^n + \cdots + \alpha^{(n)}_0$, where $\alpha^{(n)}_0 > 0$. It is a well-known fact that these polynomials obey the following three-term recurrence

$$zP_n = a_{n+1}P_{n+1} + b_nP_n + a_nP_{n-1},$$

with $a_n$ positive and where we set $P_{-1} \equiv 0$.

The corresponding three-diagonal Jacobi matrix we denote by $J_+$, that is

$$J_+ = S_+A_+ + B_+ + A_+S_+^*,$$

where $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 0}$ are two bounded real sequences such that $a_n > 0$. Furthermore, $A_+ = \text{diag}\{a_n\}$ and $B_+ = \text{diag}\{b_n\}$ and $S_+$ denotes the shift operator on the positive half axis. Recall that any Jacobi matrix, which defines a bounded self-adjoint operator in $\ell^2_+$, can be obtained in this way.

In the next section we will consider periodic Jacobi matrices. Such matrices possess a natural extension to the negative half axis. That is, we will also consider two-sided Jacobi matrices.

Given two bounded real sequences $\{a_n\}_{n \in \mathbb{Z}}$, $a_n > 0$, and $\{b_n\}_{n \in \mathbb{Z}}$ we define a two-sided Jacobi matrix by

$$J = S\mathcal{A} + \mathcal{B} + \mathcal{A}S^{-1},$$

where $\mathcal{A}$ (resp. $\mathcal{B}$) is a two-sided diagonal matrix, whose entries are the elements of the sequence $\{a_n\}$ (resp. $\{b_n\}$) and where $S$ denotes the two-sided shift operator acting on the entire $\ell^2$.

2.2. The isospectral torus of period-2 Jacobi matrices. In this section we are going to discuss two-periodic Jacobi matrices whose essential spectrum is composed of two disjoint intervals of equal length. To this end
we introduce $\tilde{E}$ as the union of these intervals. To any such $\tilde{E}$ there corresponds a quadratic polynomial $\tilde{T}(z)$ such that $\tilde{E} = \tilde{T}^{-1}([-2, 2])$. Through a linear change of variables, however, we may as well consider

$$T(z) = z^2 - \lambda, \quad \lambda > 2$$

and $E = T^{-1}([-2, 2])$.

Also, if we have an arbitrary two-sided Jacobi matrix of period two

$$\tilde{J} = \begin{pmatrix} \ldots & \ldots & \ldots \\ \ldots & a_0 & b_0 & a_1 \\ \ldots & a_1 & b_1 & a_0 \\ \ldots & a_0 & b_0 & a_1 \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

where $a_i > 0$ and $b_i \in \mathbb{R}$, $i \in \{0, 1\}$, then there exist constants $\tilde{a}$ and $\tilde{b}$ such that

$$J_0 := \tilde{a}\tilde{J} + \tilde{b} = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots \\ \ldots & a & b & \frac{1}{a} & -b & a \\ \ldots & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & -b & a \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

for some $a > 0$ and $b \in \mathbb{R}$.

In what follows, let us denote by $J_0(E)$ the set of all period-2 Jacobi matrices $J_0$ of the above form such that the spectrum of $J_0$ is given by $E$, or, equivalently, such that $T(J_0) = S^2 + S^{-2}$.

**Theorem 2.2.1.** For $\lambda > 2$ we have that

$$J_0 \in J_0(E) \iff u^2 + b^2 = r^2,$$

where $u = a - a^{-1}$ and $r = \sqrt{\lambda - 2}$.

**Proof.** From $J_0^2 - \lambda = S^2 + S^{-2}$ we immediately obtain

$$0 = a^2 + \frac{1}{a^2} + b^2 - \lambda = \left( a - \frac{1}{a} \right)^2 + b^2 - (\lambda - 2).$$

Hence, the isospectral equation describes a circle in the $(u, b)$-plane.

### 2.3. The Killip-Simon-class for Jacobi matrices.

**Definition 2.3.1.** A two-sided matrix $A$ is in the *Hilbert-Schmidt-class* ($A \in \text{HS}$) if $\text{tr}(A^*A) < \infty$, where, as usual for positive matrices, the trace $\text{tr}(\cdot)$ can be defined as a series of diagonal elements. Moreover, we say that a Jacobi matrix $J$ is in the *Killip-Simon-class* ($J \in \text{KS}_J(E)$) if $T(J) - (S^2 + S^{-2})$ is of the Hilbert-Schmidt class.
This class was originally discovered and studied in [3]. For recent progress in the understanding of finite gap Jacobi matrices and their perturbations see [2].

In what follows, we give an explicit parametric description of Jacobi matrices which belong to the Killip-Simon-class. To this end, let from now on $\lambda > 2$ and let $J$ be an arbitrary, two-sided Jacobi matrix. The following result immediately follows from the above definition.

**Lemma 2.3.2.** $J$ is in the Killip-Simon-class if and only if the following conditions are fulfilled:

1. $a_n a_{n+1} - 1 \in \ell^2$  
2. $a_{n+1} (b_n + b_{n+1}) \in \ell^2$.  
3. $a_n^2 + a_{n+1}^2 + b_n^2 - \lambda \in \ell^2$.

**Remark 2.3.3.** Note that, due to condition (J3), we have that

$$\frac{4}{\lambda + 1} < a_n^2 < \lambda + 1$$

and as $a_n > 0$ we get that $\{\log a_n\}$ is uniformly bounded.

**Lemma 2.3.4.** The set of conditions (J1), (J2), (J3) is equivalent to

1. $u_n + u_{n+1} \in \ell^2$,  
2. $b_n + b_{n+1} \in \ell^2$,  
3. $u_n^2 + b_n^2 - r^2 \in \ell^2$,

where $u_n = a_n - a_{n-1}$ and $r = \sqrt{\lambda - 2}$.

**Proof.** For (J1) we obtain

$$u_n + u_{n+1} = a_n - \frac{1}{a_n} + a_{n+1} - \frac{1}{a_{n+1}} = (a_n a_{n+1} - 1) \left( \frac{1}{a_n} + \frac{1}{a_{n+1}} \right).$$

Due to Remark 2.3.3 the last factor is uniformly bounded and hence this is equivalent to (J1). Also because of the uniform boundedness of $\{\log a_n\}$ we see that (J2) $\iff$ (J2'). For the proof of the last equivalence we first observe that

$$a_n^2 + a_{n+1}^2 = \frac{1}{a_n^2} (a_n^2 a_{n+1}^2 - 1) + a_n^2 + \frac{1}{a_n^2}.$$

Note that $a_n^2 a_{n+1}^2 - 1 \in \ell^2$ iff $a_n a_{n+1}^2 - 1 \in \ell^2$, which is (J1). Thus,

$$a_n^2 + a_{n+1}^2 + b_n^2 - \lambda \in \ell^2 \iff \left( a_n - \frac{1}{a_n} \right)^2 + b_n^2 - (\lambda - 2) \in \ell^2.$$

We now rewrite our parameters $u_n$ and $q_n$ by means of polar coordinates, i.e.

$$u_n = r_n \cos \phi_n \quad \text{and} \quad b_n = r_n \sin \phi_n.$$

This allows us to state the main theorem of this section.
Theorem 2.3.5. Let \( \lambda > 2 \) and let \( J \) be a two-sided Jacobi matrix. Then, \( J \in KS_J(E) \) if and only if there exist two \( \ell^2 \)-sequences \( \{\alpha_n\}, \{\beta_n\} \) such that

\[
r_n = \sqrt{\lambda - 2} + \alpha_n \quad \text{and} \quad \phi_n = \pi n + \sum_{k=0}^{n} \beta_k,
\]

where

\[
a_n - \frac{1}{a_n} = r_n \cos \phi_n \quad \text{and} \quad b_n = r_n \sin \phi_n.
\]

Proof. Obviously,

\[
[J3^2] \Leftrightarrow r_n^2 - r^2 \in \ell^2 \Leftrightarrow r_n - r = \alpha_n \in \ell^2,
\]

Considering this in \([J1']\) and in \([J2']\) we get that

\[
\cos \phi_n + \cos \phi_{n+1} \in \ell^2 \quad \text{and} \quad \sin \phi_n + \sin \phi_{n+1} \in \ell^2.
\]

Rewriting the cosines and sines by means of exponential functions and adding the above conditions yields

\[
[J1'], [J2'] \Leftrightarrow e^{i\phi_{n+1}} + e^{i\phi_n} \in \ell^2 \Leftrightarrow \cos \left( \frac{\phi_{n+1} - \phi_n}{2} \right) \in \ell^2
\]

\[
\Leftrightarrow \sin \left( \frac{\phi_{n+1} - \phi_n}{2} - \frac{\pi}{2} \right) \in \ell^2
\]

\[
\Leftrightarrow \phi_{n+1} - \phi_n - \frac{\pi}{2} = \beta_n \in \ell^2.
\]

\[ \square \]

3. SMP matrices

3.1. The origin and construction of one-sided SMP matrices. Suppose we are given a compactly supported real measure \( \sigma \) whose support does not contain zero and consider the linearly independent family of functions

\[
1, \frac{-1}{z}, z, \left( \frac{-1}{z} \right)^2, z^2, \ldots
\]

Subsequently, we orthogonalize these functions with respect to \( d\sigma \), giving

\[
\tilde{\Psi}_0 = 1, \quad \tilde{\Psi}_{2n-1} = \left( \frac{-1}{z} \right)^n + \alpha_1 z^{n-1} + \cdots, \quad \tilde{\Psi}_{2n} = z^n + \beta_1 \left( \frac{-1}{z} \right)^n + \cdots,
\]

and normalize them by setting

\[
(3.1) \quad \Psi_n = \frac{\tilde{\Psi}_n}{\|\tilde{\Psi}_n\|}.
\]

We denote this family by \( \Psi = \{\Psi_n\}_{n\in\mathbb{N}_0} \). Note that \( \Psi \) forms a complete orthonormal system in \( L^2(d\sigma) \). This system of orthonormal rational functions is also considered in [6 (2.2) and (2.3)] and for more general orthogonal systems of rational functions see [1].
Lemma 3.1.1. Using the same notation as in the above paragraphs we have
\[
\begin{align*}
  z\Psi_m &= r_m \Psi_{m-2} + p_m \Psi_{m-1} + q_m \Psi_m + p_{m+1} \Psi_{m+1} + r_{m+2} \Psi_{m+2}, \\
  \frac{1}{z} \Psi_m &= \rho_m \Psi_{m-2} + \sigma_m \Psi_{m-1} + \pi_m \Psi_m + \pi_{m+1} \Psi_{m+1} + \rho_{m+2} \Psi_{m+2},
\end{align*}
\]
where \(p_m, q_m, r_m, \pi_m, \sigma_m\) and \(\rho_m\) denote some coefficient sequences. Moreover, \(r_{2n} > 0, r_{2n+1} = 0\) and \(\rho_{2n} = 0, \rho_{2n+1} < 0\).

We now pass from square integrable functions w.r.t. \(d\sigma\) to the one-sided sequence space \(\ell^2_+\). To this end we define \(A_+\) as the matrix of the multiplication operator by \(z\) and \(A_+^{-1}\) is the matrix of the multiplication operator by \(z^{-1}\). We refer to \(A_+\) as one-sided SMP matrix. As a direct consequence of Lemma 3.1.1 we obtain the following theorem.

Theorem 3.1.2. We have that
\[
A_+ = S^2_+ \mathcal{R}_+ + S_+ \mathcal{P}_+ + \mathcal{Q}_+ + \mathcal{P}S^+ + \mathcal{R}_+(S^+_2)^2,
\]
where, for \(n \geq 0\), \(\mathcal{P}_+ = \text{diag}\{p_n\}\), \(\mathcal{Q}_+ = \text{diag}\{q_n\}\) as well as \(\mathcal{R}_+ = \text{diag}\{r_n\}\) are one-sided diagonal matrices and where \(r_{2n} > 0, r_{2n+1} = 0\). Moreover,
\[
A_+^{-1} = S^2_+ \mathcal{P}_+ + S_+ \mathcal{P}_+ \mathcal{S}_+ + \mathcal{S}_+ + \mathcal{S}_+ \mathcal{S}_+ + \mathcal{P}_+(S^+_2)^2,
\]
where \(\mathcal{P}_+ = \text{diag}\{\rho_n\}\), \(\mathcal{S}_+ = \text{diag}\{\sigma_n\}\) and \(\mathcal{S}_+ = \text{diag}\{\pi_n\}\) and where \(\rho_{2n+1} < 0, \rho_{2n} = 0\).

Remark 3.1.3. Note that \(A_+\) possesses the same structure as its shifted inverse equipped with a negative sign. Also, since \(A_+A_+^{-1} = I\) there exist several algebraic relations between the entries of \(A_+\) and \(A_+^{-1}\).

3.2. The introduction of two-sided SMP matrices. In what follows we will consider periodic SMP matrices. These were introduced and partially studied in [5]. Periodic SMP matrices can be naturally extended to the negative half axis. Similarly to the case of Jacobi matrices, such an extension appears to be very useful.

Following the result of the previous theorem we make the following definition:

Definition 3.2.1. We say that a two-sided real self-adjoint pentadiagonal matrix \(A\) is SMP-structured if its entries are uniformly bounded and, additionally, if all even entries on the most outer diagonals vanish and all the odd ones are positive. This means that \(A\) can be written in the following way
\[
A = S^2 \mathcal{R} + S \mathcal{P} + \mathcal{Q} + \mathcal{P}S^{-1} + \mathcal{R}S^{-2},
\]
where \(\mathcal{P} = \text{diag}\{p_n\}\), \(\mathcal{Q} = \text{diag}\{q_n\}\), \(\mathcal{R} = \text{diag}\{r_n\}\) are real diagonal matrices with \(r_{2n+1} > 0, r_{2n} = 0\) and where the sequences \(\{p_n\}\), \(\{q_n\}\) as well as \(\{r_n\}\) are uniformly bounded. Furthermore, we demand that \(\{r_{2n+1}\}\) is uniformly bounded from zero, i.e. there exists an \(\eta > 0\) such that \(r_{2n+1} \geq \eta\) for all \(n \in \mathbb{Z}\).

We now call \(A\) an SMP matrix, or \(A \in \text{SMP}\), if it is invertible and both \(A\) and \(A^T := -SA^{-1}S^{-1}\) are SMP-structured.
Similar to Jacobi matrices we can represent SMP matrices as a two-dimensional perturbation of a block-diagonal matrix
\[
A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} + a_0 (\tilde{e}_0 \langle \cdot, e_{-1} \rangle + e_{-1} \langle \cdot, \tilde{e}_0 \rangle),
\]
where \(a_0 = \sqrt{p_0^2 + r_1^2}\) and \(\tilde{e}_0 = \frac{1}{a_0} (p_0 e_0 + r_1 e_1)\). In this notation the matrix \(A_-\) corresponds to a one-sided SMP matrix which was defined in Section 3.1.

**Notation.** For simplicity, in what follows, we write the inverse of an SMP matrix \(A\) as
\[
A^{-1} = S^2 P + S \Sigma + \Pi \Sigma^{-1} + P S^{-2},
\]
where \(P = \text{diag}\{\rho_n\}, \Sigma = \text{diag}\{\sigma_n\}\) and \(\Pi = \text{diag}\{\pi_n\}\). Hence, considering the above definition, we have that \(\rho_{2n+1} = 0\) and \(\rho_{2n} < 0\).

**Theorem 3.2.2.** Let \(r_{2n} = 0, r_{2n+1} > 0\) and let the sequences \(\{\log r_{2n+1}\},\) \(\{\rho_n\},\) \(\{q_{2n+1}\}\) be uniformly bounded. Then, \(A\) is an SMP matrix if and only if there exists an \(\varepsilon > 0\) such that
\[
q_{2n-1} p_{2n-2} p_{2n+1} - p_{2n-2} p_{2n} r_{2n+1} - p_{2n-1} p_{2n+1} r_{2n-1} \leq -\varepsilon
\]
and the coefficient sequences satisfy
\[
q_{2n} = \frac{p_{2n} p_{2n+1}}{r_{2n+1}}.
\]
Moreover, in this case, \(A^{-1}\) exists and its elements are given by
\[
\rho_{2n+1} = 0,\quad \rho_{2n} = \frac{r_{2n-1} r_{2n+1}}{p_{2n-1} p_{2n+1}},
\]
\[
\pi_{2n+1} = \frac{p_{2n} p_{2n+1}}{r_{2n+1} r_{2n+3}},\quad \pi_{2n} = \frac{p_{2n-2} p_{2n}}{r_{2n-1}},
\]
\[
\sigma_{2n+1} = \frac{p_{2n} p_{2n+1}}{r_{2n+1} r_{2n+3}} \quad \text{and} \quad \sigma_{2n} = \frac{p_{2n} (\pi_{2n+1} r_{2n+1} + q_{2n-1} \pi_{2n} + p_{2n-1} \rho_{2n})}{p_{2n}^2 + p_{2n+1}^2}.
\]

**Proof.** First, assume that \(A \in \text{SMP}\). From \(AA^{-1} = I\) we can derive the following identities
\[
S^4 \left[(S^{-2} R S^2) P\right] = 0, \quad [R (S^{-2} P S^2)] S^{-4} = 0,
\]
\[
S^3 \left[(S^{-1} R S) \Pi + (S^{-2} P S^2) \Pi P\right] = 0, \quad [P (S^{-1} P S) + R (S^{-2} \Pi S^2)] S^{-3} = 0,
\]
\[
S^2 \left[R \Sigma + (S^{-1} P S) \Pi + (S^{-2} Q S^2) P\right] = 0, \quad [Q P + P (S^{-1} \Pi S) + R (S^{-2} \Sigma S)] S^{-2} = 0,
\]
\[
S \left[S R \Pi S^{-1} + P \Sigma + (S^{-1} Q S) \Pi + (S^{-1} P S) P\right] = 0, \quad [S P S^{-1} + Q \Pi + P (S^{-1} \Sigma S) + R (S^{-1} \Pi S)] S^{-1} = 0,
\]
\[
S^2 \left[R P S^{-2} + S \Pi S^{-1} + Q \Sigma + P \Pi + R P\right] = I,
\]
where 0 denotes the zero operator.
From the first equation we immediately see that $\rho_{2n+1} = 0$. Considering the fact that, by definition, $r_{2n+1}$ is uniformly greater than zero, we obtain

$$\pi_{2n+2} = -\frac{p_{2n+2}}{r_{2n+1}}$$

from the third equation. Inserting this into the equation for the second upper diagonal yields

$$q_{2n} = -\frac{p_{2n+1}\pi_{2n+2}}{\rho_{2n+2}} = \frac{p_{2n}p_{2n+1}}{r_{2n+1}},$$

where we used that $\rho_{2n} < 0$ uniformly. We may now continue by solving the above system for the entries of $A^{-1}$ resulting in the identities from the claim.

Since the inverse of an invertible matrix is unique, these are the coefficient sequences of $A^{-1}$ indeed. We once again refer to the fact that both $r_{2n+1}$ and $-\rho_{2n}$ are uniformly bounded from above and from zero. Together with the expression which we obtained for $\rho_{2n}$ this implies (3.3).

For the proof of the if-part, we first of all verify that all the sequences involved are uniformly bounded. For $q_{2n}$ this, again, follows from the fact that so is $\log r_{2n+1}$. Hence, it also does for $\pi_{2n}$, $\pi_{2n+1}$ and $\sigma_{2n+1}$. In addition to that, we see from (3.3) that $p_{2n}^2 + p_{2n+1}^2$, too, is uniformly greater than zero. Indeed, suppose that w.l.o.g. $p_{2n}$ and $p_{2n+1}$ converge to zero. Then the left handside of (3.3) tends to zero as $n \to \infty$, which clearly leads to a contradiction. Consequently, $\sigma_{2n}$ is uniformly bounded too.

Secondly, note that the identities from the claim fulfill all the equations from (3.4). Thus, $A$ is invertible and, additionally, the coefficient sequences of its inverse satisfy the conditions from Definition 3.2.1 and hence $A \in \text{SMP}$. \hfill $\square$

3.3. The isospectral torus of period-2 SMP matrices. Suppose we are given two disjoint intervals, whose union we denote by $\tilde{E}$. Then we can always find a function

$$\tilde{V}(z) = az + b + \frac{\varrho}{c - z}, \quad a > 0, \quad \varrho > 0,$$

such that

$$\tilde{E} = \tilde{V}^{-1}([-2, 2]) = \left\{ z : \tilde{V}(z) \in [-2, 2] \right\}.$$

By a linear change of variables we can reduce $\tilde{V}$ such that it suffices to consider

(3.5) \hspace{1cm} V(z) := az + b - \frac{1}{z},

where $a > 0$ and $b \in \mathbb{R}$. Thus, in what follows $E$ is assumed to be the preimage of $V$ of $[-2, 2]$, i.e.

$$E := \left\{ z : az + b - \frac{1}{z} \in [-2, 2] \right\}.$$

$E$ constitutes the union of two arbitrary intervals about zero, see Fig. 1, whose end points are given by
\[
b_0 = -\sqrt{\left(\frac{b + 2}{2a}\right)^2 + \frac{1}{a} - \frac{b + 2}{2a}}, \quad a_1 = -\sqrt{\left(\frac{b - 2}{2a}\right)^2 + \frac{1}{a} - \frac{b - 2}{2a}}, \tag{3.6}
\]
\[
b_1 = \sqrt{\left(\frac{b + 2}{2a}\right)^2 + \frac{1}{a} - \frac{b + 2}{2a}}, \quad a_0 = \sqrt{\left(\frac{b - 2}{2a}\right)^2 + \frac{1}{a} - \frac{b - 2}{2a}}.
\]

**Figure 1.** Exemplary plot of \(E\) for \(a = 2.5\) and \(b = -3\).

**Definition 3.3.1.** By \(A_0(E)\) we denote the set of all SMP matrices of period two with their spectrum on \(E\).

**Theorem 3.3.2.** Let \(a > 0, b \in \mathbb{R}\) and let \(V\) and \(E\) be given as above. Then the following three statements are equivalent:

(i) \(A_0 \in A_0(E)\),

(ii) \(V(A_0) = S^2 + S^{-2}\),

(iii) \(r_1 = r = \frac{1}{a}, \quad q_0 + q_1 = -\frac{b}{a}\) and

\[
p_2^2 + p_1^2 + a^2 p_0^2 p_1^2 + b p_0 p_1 = \frac{1}{a}, \tag{3.7}
\]

Proof. From the identity \(V(A_0) = S^2 + S^{-2}\) we can derive the following system of equations:

(i) \(a r = 1\)

(ii) \(-\rho_0 = -\rho = 1\)

(iii) \(a p_0 - \pi_0 = 0\)

(iv) \(a p_1 - \pi_1 = 0\)

(v) \(a q_0 + b - \sigma_0 = 0\)

(vi) \(a q_1 + b - \sigma_1 = 0\).

The first two items obviously imply that \(r = a^{-1}\) and \(\rho = -1\). Also, we immediately see that, due to Theorem 3.2.2 (iii) and (iv) are automatically fulfilled. We use (vi) to find that

\[q_1 = \frac{1}{a} (\sigma_1 - b) = -a p_0 p_1 - \frac{b}{a},\]
which, in turn, renders \((v)\) obsolete. The only equation that we have not yet fully exploited is the second one. We have

\[
p_0^2 r + p_1^2 r - q_1 p_0 p_1 = r^2 \iff p_0^2 + p_1^2 + a^2 p_0^2 + b p_0 p_1 = \frac{1}{a}
\]

\(\square\)

Let us rewrite the isospectral equation \((3.7)\) in the following way

\[
(p_0 + p_1)^2 + \left( a p_0 p_1 + \frac{b - 2}{2a} \right)^2 = \frac{1}{a} + \left( \frac{b - 2}{2a} \right)^2.
\]

If we use the identities from \((3.6)\) and put \(u = p_0 + p_1\) and \(v = a p_0 p_1\), then we obtain

\[
u^2 + \left( v - \frac{a_0 + a_1}{2} \right)^2 = \left( \frac{a_0 - a_1}{2} \right)^2.
\]

Note that the critical points of the mapping \((p_0, p_1) \mapsto (u, v)\) correspond to \(p_0 = p_1 = p\) on this curve, see Fig. 3. From

\[a^2 p^4 + (b + 2)p^2 - \frac{1}{a} = 0\]

we see that these are given by

\[
\pm u_c = \pm 2p = \pm \left( \frac{4}{a} \left( \sqrt{\left( \frac{b + 2}{2a} \right)^2 + \frac{1}{a} - \frac{b + 2}{2a}} \right) \right)^{\frac{1}{2}} = \pm \sqrt{b_1 \left( a_0 + a_1 - b_1 - b_0 \right)}
\]

and

\[
v_c = ap^2 = \sqrt{\left( \frac{b + 2}{2a} \right)^2 + \frac{1}{a} - \frac{b + 2}{2a}} = b_1.
\]

Thus, what is obtained in the \((u, v)\)-plane, as a matter of fact, is a part of a circle centered at \((0, c)\), \(c = (a_1 + a_0)/2\), with radius \((a_0 - a_1)/2\) and end points \((\pm u_c, v_c)\). Figure 2 shows an exemplary plot of the isospectral equation in the \((u, v)\)-plane.

### 3.4. The Killip-Simon-class for SMP matrices.

**Definition 3.4.1.** We say that an SMP matrix \(A\) is in the **Killip-Simon-class**, or, in short, \(A \in \text{KS}_{\text{SMP}}(E)\) if and only if

\[
V(A) - (S^2 + S^{-2}) \in \text{HS}, \tag{3.8}
\]

where \(V\) is defined in \((3.5)\).

The following lemma describes when an SMP matrix belongs to the Killip-Simon-class in terms of the parameters \(p_{2n}, p_{2n+1}, q_{2n+1}, r_{2n+1}\).
Figure 2. Exemplary plot of the isospectral equation in the 
$(u,v)$-plane for $a = 2.5$ and $b = -3$.

Lemma 3.4.2. Let $A$ be an SMP matrix. Then, $A \in \text{KS}_{\text{SMP}}(E)$ if and only if the coefficient sequences satisfy the following conditions:

(C1) \quad $r_{2n+1} - \frac{1}{a} \in \ell^2$.
(C2) \quad $p_n - p_{n-2} \in \ell^2$.
(C3) \quad $q_{2n+1} + b + ap_{2n}p_{2n+1} \in \ell^2$.
(C4) \quad $p_{2n} + p_{2n+1}^2 + a^2 p_{2n}^2 p_{2n+1}^2 + bp_{2n}p_{2n+1} - \frac{1}{a} \in \ell^2$.

Proof. From (3.8) we immediately get the conditions

(c1) \quad $r_{2n+1} - \frac{1}{a} \in \ell^2$,
(c2) \quad $p_{2n} + 1 \in \ell^2$,
(c3) \quad $ap_{2n} - \sigma_{2n} \in \ell^2$,
(c4) \quad $ap_{2n+1} - \sigma_{2n+1} \in \ell^2$,
(c5) \quad $aq_{2n} + b - \sigma_{2n} \in \ell^2$ and
(c6) \quad $aq_{2n+1} + b - \sigma_{2n+1} \in \ell^2$,

so (C1) is already obtained. Furthermore, it follows from Theorem 3.2.2 together with the fact that the sequences \{$\log r_{2n+1}$\} and \{$p_n$\} are uniformly bounded that

(c3) \quad $ap_{2n}r_{2n-1} + p_{2n-2}p_{2n} \in \ell^2$ \quad $\iff$ \quad $p_{2n} - p_{2n-2} \in \ell^2$.

In the same way we can deduce $p_{2n+1} - p_{2n-1} \in \ell^2$ from (c4) and thus obtain (C2).

For the simplification of (c6) we once again use the fact that $r_{2n+1} > 0$ uniformly and that \{$p_n$\} is a uniformly bounded sequence as well as the
identities given in Theorem 3.2.2:

\[(c6) \iff aq_{2n+1}^2 + b - \frac{p2n+3p2n+2}{r_{2n+3}} \in \ell^2\]

\[(c1) \& (c2) \iff \frac{b}{a} + ap_{2n+3} \in \ell^2\]

\[(c2) \iff \frac{b}{a} + ap_{2n+1} \in \ell^2\]

\[\iff (C3).\]

For more or less the same reasons we can reduce (c5) to

\[(c5) \iff \begin{aligned}
    ap_{2n+1}^2 + b &+ \frac{1}{p_{2n+1}^2} \times \\
    \left( \frac{1}{a} (p_{2n} (p_{2n+3} - p_{2n-1}) + p_{2n+1} (p_{2n-2} - p_{2n+2})) + \\
    q_{2n-1}p_{2n-2}p_{2n} + q_{2n+1}p_{2n+1}p_{2n+3} \right) \in \ell^2 \\
    ap_{2n+1} + \frac{q_{2n-1}p_{2n}^2 + q_{2n+1}p_{2n+1}^2}{p_{2n}^2 + p_{2n+1}^2} \in \ell^2 \\
    \end{aligned}\]

\[\iff (C3),(C2) \not\implies 0 \in \ell^2,\]

so this condition turns out to be obsolete.

Finally, we draw our attention to (c2). By Theorem 3.2.2 we have that the denominator of \(\rho_{2n}\) is uniformly smaller than zero and hence

\[(c2) \iff \begin{aligned}
    \frac{1}{a} - \frac{q_{2n-1}p_{2n}^2}{p_{2n}^2 + a^2 p_{2n+1}^2 + b p_{2n+1}^2} - \frac{1}{a} \in \ell^2.
    \end{aligned}\]

\[\square\]

3.5. The Damanik-Killip-Simon-condition.

**Definition 3.5.1.** For two sequences \(a = \{a_n\}_{n \in \mathbb{N}_0}\) and \(a' = \{a'_n\}_{n \in \mathbb{N}_0}\) we define their distance as

\[\text{dist}^2(a, a') = \sum_{n=0}^{\infty} 2^{-n} (a_n - a'_n)^2.\]

Let us denote by \(\mathcal{C}\) the set of constant sequences, i.e., \(a' \in \mathcal{C}\) if \(a'_n = a'_0\) for all \(n \geq 0\). We define

\[\text{dist}^2(a, \mathcal{C}) = \inf_{a' \in \mathcal{C}} \text{dist}^2(a, a').\]

We have the following lemma:

**Lemma 3.5.2.** Let \(a = \{a_n\}_{n \in \mathbb{N}_0}\) and \(b = \{b_n\}_{n \in \mathbb{N}_0}\) be two real one-sided sequences. If

\[(3.9) \sum_{k \geq 0} \text{dist}^2 \left( (S_+^k)^k a, \mathcal{C} \right) < \infty \quad \text{and} \quad \sum_{k \geq 0} \text{dist}^2 \left( (S_+^k)^k a, \mathcal{C} \right) < \infty,\]
Proof. We assume (3.9). Let
\[ \sum_{k \geq 0} \text{dist}^2 \left( (S_+^*)^k (a + b), C \right) < \infty. \]

In case that, additionally, \( a, b \in \ell_+^\infty \) we have that
\[ \sum_{k \geq 0} \text{dist}^2 \left( (S_+^*)^k (a \cdot b), C \right) < \infty. \]

Moreover, if \( a_n = \sum_{k=0}^n \alpha_j \) and \( \alpha \in \ell_+^2 \) then
\[ \sum_{k \geq 0} \text{dist}^2 \left( (S_+^*)^k a, C \right) < \infty. \]

Proof. We assume (3.9). Let \( c^{(k)} = (c_k, c_k, \ldots) \in C \) and \( d^{(k)} = (d_k, d_k, \ldots) \in C \) such that
\[
\text{dist}^2 \left( (S_+^*)^k a, c^{(k)} \right) \leq \text{dist}^2 \left( (S_+^*)^k a, C \right) + 2^{-k},
\]
\[
\text{dist}^2 \left( (S_+^*)^k b, d^{(k)} \right) \leq \text{dist}^2 \left( (S_+^*)^k b, C \right) + 2^{-k}.
\]

It follows that
\[
\text{dist}^2 \left( (S_+^*)^k (a + b), C \right) \leq \text{dist}^2 \left( (S_+^*)^k (a + b), c^{(k)} + d^{(k)} \right)
= \sum_{n \geq 0} 2^{-n} (a_{n+k} + b_{n+k} - c_k - d_k)^2
\leq \sum_{n \geq 0} 2^{-n+1} \left( (a_{n+k} - c_k)^2 + (b_{n+k} - d_k)^2 \right)
= 2 \text{dist}^2 \left( (S_+^*)^k a, c^{(k)} \right) + 2 \text{dist}^2 \left( (S_+^*)^k b, d^{(k)} \right)
\leq 2 \left( 2^{-k+1} + \text{dist}^2 \left( (S_+^*)^k a, C \right) + \text{dist}^2 \left( (S_+^*)^k b, C \right) \right),
\]
which proves (3.10).

Let additionally \( b \in \ell_+^\infty \). Then \( d = \{d_1, d_2, \ldots \} \in \ell_+^\infty \) and
\[
\text{dist}^2 \left( (S_+^*)^k (a \cdot b), C \right) \leq \text{dist}^2 \left( (S_+^*)^k (a \cdot b), c^{(k)} \cdot d^{(k)} \right)
= \sum_{n \geq 0} 2^{-n} (a_{n+k} (b_{n+k} - d_k) + d_k (a_{n+k} - c_k))^2
\leq \sum_{n \geq 0} 2^{-n+1} \left( \|a\|_\infty^2 (b_{n+k} - d_k)^2 + \|d\|_\infty^2 (a_{n+k} - c_k)^2 \right)
= 2 \|d\|_\infty^2 \text{dist}^2 \left( (S_+^*)^k a, c^{(k)} \right) + \|a\|_\infty^2 \text{dist}^2 \left( (S_+^*)^k b, d^{(k)} \right)
\leq 2 \|d\|_\infty^2 \left( \text{dist}^2 \left( (S_+^*)^k a, c^{(k)} \right) + 2^{-k} \right) + 2 \|a\|_\infty^2 \left( \text{dist}^2 \left( (S_+^*)^k b, d^{(k)} \right) + 2^{-k} \right)
\]
and hence (3.11) follows.
Finally, let \( a_n = \sum_{j=0}^{n} \alpha_j \) and \( \alpha \in \ell^2_+ \). In order to estimate the infimum we choose \((a_k, a_k, \ldots)\) as a constant sequence for each \( k \) and thus obtain

\[
\sum_{k \geq 0} \text{dist}^2 \left( (S_k^+)^k a, C \right) \leq \sum_{k \geq 0} \sum_{n \geq 0} 2^{-n} \left( \sum_{j=1}^{n} \alpha_{j+k} \right)^2 \leq \sum_{n \geq 0} 2^{-n} \sum_{j=1}^{n} \sum_{k \geq 0} \alpha_{j+k}^2 \leq \sum_{n \geq 0} 2^{-n} n^2 \sum_{k \geq 0} \alpha_k^2 < \infty.
\]

\( \square \)

**Definition 3.5.3.** Let \( \{p_n, q_n, r_n\}_{n \geq 0} \) (resp. \( \{p'_n, q'_n, r'_n\}_{n \geq 0} \)) be a set of sequences defining a one-sided matrix \( A_+ \) (resp. \( A'_+ \)). Then we define their distance as

\[
\text{dist}^2(A_+, A'_+) = \text{dist}^2(p_n, p'_n) + \text{dist}^2(q_n, q'_n) + \text{dist}^2(r_n, r'_n).
\]

Furthermore, for a subset \( S \) of the SMP class we set

\[
\text{dist}^2(A_+, S) = \inf_{A' \in S} \text{dist}^2(A_+, A'_+).
\]

Considering the above lemma as well as Theorem 3.2.2 and Lemma 3.4.2 we have the following theorem.

**Theorem 3.5.4.** Let \( A \in \text{KS}_{\text{SMP}}(E) \). Then

\[
\sum_{k=-1}^{\infty} \text{dist}^2 \left( (S^{-2k}A S^{2k})^+ a, C \right) < \infty.
\]

### 3.6. The parametrization theorem for SMP matrices of the Killip-Simon-class.

We set

\[
p_{2n} = R_n \cos \phi_n, \quad p_{2n+1} = R_n \sin \phi_n
\]

and obtain the following result.

**Theorem 3.6.1.** \( A \in \text{KS}_{\text{SMP}}(E) \) if and only if there exist sequences \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \in \ell^2 \) such that

\[
(\text{KS1}) \quad \phi_n = \phi_0 + \sum_{k=1}^{n} \alpha_k,
\]

\[
(\text{KS2a}) \quad p_{2n} = (R(\phi_n) + \beta_n) \cos \phi_n,
\]

\[
(\text{KS2b}) \quad p_{2n+1} = (R(\phi_n) + \beta_n) \sin \phi_n,
\]

\[
(\text{KS3}) \quad q_{2n+1} = -\frac{1}{2} R^2(\phi_n) \sin(2\phi_n) - \frac{b}{a} + \gamma_n
\]

\[
(\text{KS4}) \quad r_{2n+1} = \frac{1}{a} + \delta_n,
\]

where

\[
R^2(\phi) := \frac{1}{a} \frac{4}{\sqrt{(2 + b \sin(2\phi))^2 + 4a \sin^2(2\phi) + 2 + 2b \sin(2\phi)}}.
\]
Proof. Obviously, \([\text{KS4}]\) is just a reformulation of \([\text{C1}]\). Let us continue with \([\text{C4}]\). There exists a sequence \(\{\beta_n^\prime\} \in \ell^2\) such that
\[
R^2_n + a^2 R^4_n \frac{\sin(2\phi_n)}{4} + b R^2_n \frac{\sin(2\phi_n)}{2} - \frac{1}{a} = \beta_n^\prime.
\]
We solve this equation for \(R^2_n\) and obtain
\[
R^2_n = \frac{- (2 + b \sin(2\phi_n)) \pm \sqrt{(2 + b \sin(2\phi_n))^2 + 4a(1 + a\beta_n^\prime) \sin^2(2\phi_n)}}{a^2 \sin^2(2\phi_n)}.
\]
Note that \(R^2_n\) has to be non-negative. Hence, we have to choose the solution with the plus sign. Further simplifications yield
\[
R^2_n = \frac{1}{a} \frac{4(1 + a\beta_n^\prime)}{\sqrt{(2 + b \sin(2\phi_n))^2 + 4a(1 + a\beta_n^\prime) \sin^2(2\phi_n) + 2 + b \sin(2\phi_n)}}.
\]
Moreover, the function
\[
g(x) := \sqrt{(2 + bx)^2 + 4ax^2 + 2 + bx}, \quad x \in \mathbb{R}
\]
attains its minimum at
\[
x_0 = -\frac{4b}{4a + b^2} \quad \text{and} \quad g(x_0) = \frac{16a}{4a + b^2} > 0.
\]
Hence, \(R^2_n - R^2(\phi_n) \in \ell^2\). Thus, we immediately obtain \([\text{KS2a}], [\text{KS2b}]\) and \([\text{KS3}]\).

Now, consider \([\text{C2}]\), i.e.
\[
R_n \cos \phi_n - R_{n-1} \cos \phi_{n-1} \in \ell^2 \quad \text{and} \quad R_n \sin \phi_n - R_{n-1} \sin \phi_{n-1} \in \ell^2
\]
Rewriting the cosines and sines by means of the exponential function and adding these two conditions gives
\[
[\text{C2}] \iff e^{i\phi_n} - e^{i\phi_{n-1}} \in \ell^2 \iff \sin \left(\frac{\phi_n - \phi_{n-1}}{2}\right) \in \ell^2 \iff \phi_n - \phi_{n-1} \in \ell^2,
\]
which is exactly \([\text{KST}]\). \(\square\)

**Remark 3.6.2.** Let \(g\) be defined as in the above proof. For \(x \in [-1, 1]\) we set
\[
f(x) := \frac{1}{2a} g(x) = \sqrt{\left(\frac{2 + bx}{2a}\right)^2 + \frac{x^2}{a} + \frac{2 + bx}{2a}}.
\]
Note that, for \(b < 0\), we have that \(f(x) \leq f(-1) = a_0\) and, for \(b > 0\), \(f(x) \leq f(1) = -b_0\) (for \(b = 0\) these two values coincide). Therefore, the radius of the incircle of the isospectral curve is given by
\[
r_{in} = \sqrt{\frac{2}{a^2 \max\{a_0, -b_0\}}}.
\]

Figure 3 shows a plot of the isospectral curve \((R(\phi) \cos(\phi), R(\phi) \sin(\phi))\) in the \((p_0, p_1)\)-plane.
Figure 3. The isospectral curve in the \((p_0, p_1)\)-plane for \(a = 2.5\) and \(b = -3\).

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