Pattern of fermion masses from high-scale evolution

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Dynamical equations for fermion masses are derived using high scale universal mass generation and consequent mass evolution due to \textit{SU}(3), \textit{SU}(2), and \textit{U}(1) gauge interaction. Assuming mass generation at the GUT scale $M = 10^{14}$ GeV, one obtains hierarchy and a large spread in fermion masses with roughly correct values of $m_\nu, m_\tau, m_t, m_b$ in the third generation. The smallness of neutrino mass, $\nu_3 \sim 10^{-12} m_t$, naturally arises in the solution.

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1. The problem of fermion masses is being studied for many years (see [1] for reviews and references). The most striking points are large spread and hierarchy of masses both in vertical (inside one generation) and horizontal directions (i.e. from one generation to another), and also the extreme smallness of neutrino mass.

   In the Standard Model (SM) scenario fermion mass generation is related to the Yukawa Higgs constants. When one tries to understand fundamental dynamics behind Higgs field and express all effects in terms of fields at high scale and known gauge fields, one realizes that visible masses at our scale ($\sim 1$ GeV) are due to several sources.

   First of all, resulting fermion masses are to be created in the original chiral symmetry breaking (CSB) process (possibly at high scale), and then they are evolved by all known gauge interactions, and finally (or originally) mixed and shifted by general CKM mechanism.

   In recent publications [2–4] the author has argued, that the original mass generation process can be associated with CSB due to topological charges in the electroweak (EW) vacuum. This process is similar to CSB in the instanton gas, which was studied in different approaches in QCD [5, 6]; in what follows we shall use the formalism of [7, 8].

   General setting of the problem is given in [4] in the framework of the Pati-Salam $G(2, 2, 4)$ group [9], but for present paper the details of $SO(10)$ group, which is splitted down to $G(2, 2, 4)$, are not important, and one can use the \textit{SU}(2) instanton as the basic element of $SO(2n)$ or $SU(n)$ group [10]. The interesting feature of the \textit{SU}(2) instanton (or any local topcharge) is that it produces integral equation with the kernel of the same structure as in the fermion self-energy equations [11, 12], (see [13] for review and earlier references).

   The main emphasis of [11–13] is on the possible new type of evolution, given by a linearized equation, while for the present analysis the first solution and the nonlinear regime are relevant.

2. We start with general equation describing the process of CSB and mass generation at some high scale $M$ and the consequent mass evolution. For the Euclidean momentum-dependent fermion mass $\mu_i(p)$, one can write

$$\mu_i(p) = \int_0^M b_i(q) \mu_i(p_1) d^4p_1 \frac{d^4q}{q^2(p_1 + \mu_i(p_1))}, \quad q \equiv p - p_1. \quad (1)$$

Here $i$ refers to fermions within the highest generation $i = (\nu, \tau, t, b)$. Taking into account, that the mass of each fermion is generated with the interaction constant $b_i^{(0)}(q)$ and is subject to evolution due to gauge fields of group $SU(3)_c \times U(1)_{em}$, one can write

$$b_i(q) = b_i^{(0)} + \sum_{n=1,3} \rho_n(q) \frac{\alpha_n(q)}{\pi} \quad (2)$$

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1 Strictly speaking, both CSB and mass generation mechanisms are not derived in [5–8] in a gauge invariant formalism, since the instanton gas with net zero topcharge is introduced in a certain gauge. A gauge invariant mass generation mechanism (outside of elementary Higgs model and confining phase of QCD) is not known to the author.

2 The main emphasis of [11, 13] is on the possible new type of evolution, given by a linearized equation, while for the present analysis the first solution and the nonlinear regime are relevant.
where $\nu_1^{(i)}$ is the weight of charge $n$ for fermion $i$. Here we shall consider only the third generation to avoid confinement complications at low scale, hence $\nu_1^{(i)} = (0, 1, \frac{1}{3}, \frac{1}{3})$ and $\nu_3^{(i)} = (0, 0, 1, 1)$ for $(\nu, \tau, t, b)$, while $\nu_2^{(i)}$ is calculated from the $Z_0$ exchanges and will be neglected in the first approximation. The constant $b_1^{(0)}$ is proportional to topcharge density and depends on $i$ in the $SU(2)$ broken vacuum [14].

For $\alpha_n(q^2)$ one can use the one-loop evolution,

$$\frac{\alpha_n(q^2)}{\pi} = \frac{c_n}{1 + \omega_n \ln q^2/M^2}, \quad \omega_n = \frac{\beta_0^{(n)}}{4\pi} \alpha_n(M^2),$$

(3)

where $c_n = \frac{1}{4} \alpha_n(M^2)$, $\beta_0^{(n)} = \frac{11n - 2n - 4}{4n}$, $n = 3$, and $\beta_0^{(1)} = -\frac{\pi}{2}$. For $\alpha_3(q)$ we implicitly introduce IR freezing at small $q^2$, (see [14] for review and references), which contributes less than 10% for the third generation.

Integrating in [14] over angles and introducing $s \equiv p^2/M^2$ and $\kappa(s) \equiv \mu(p^2/M^2)/M$, one obtains integral equation

$$\kappa_i(s) = \int_0^\infty \frac{b_i(s)\kappa_i(s_1)s_1 ds_1}{\sigma(s_1 + \kappa_i^2(s_1))}, \quad \sigma = \max(s, s_1),$$

(4)

where

$$b_i(s) = b_i^{(0)} + \sum_{n=1}^3 \nu_n^{(i)} c_n(s), \quad i = \nu, \tau, t, b$$

(5)

differential equation [11–13], (up to small terms $O(\omega_n)$).

$$s^2 \kappa^\prime_i(s) + 2s \kappa_i(s) + \frac{s b_i(s)\kappa_i(s)}{s + \kappa_i^2(t)} = 0.$$  

(6)

For a constant $b_i \equiv b$ and for $s \gg \kappa_i^2$ one finds two solutions $\kappa_i(s) = s^\delta$, $\delta_1 = -\frac{1}{2} + \sqrt{1 - 4b} \approx -b$, $\delta_2 \approx -1 + b$. In what follows only the first solution will be appropriate in the integral equation (14) for small $s$. For $s \lesssim \kappa_i^2(0)$ one has a solution of nonlinear equation (10)

$$\kappa_i(s) \approx \sqrt{s \kappa_i^2(0) - b_i s}.$$  

(7)

Now taking into account the evolution of $b_i(s)$, given by (5), (3), the solution of the linearized Eq. (6) acquires the form

$$\kappa_i(s) = \kappa_i(s_0) \left(\frac{s}{s_0}\right)^{-b_i(0)} \prod_{n=1}^{1,2,3} \left(1 + \frac{\omega_n \ln s_0}{1 + \omega_n \ln s_0}\right)^{-a_n^{(i)}}, \quad s \geq s_0$$  

(8)

with $a_n^{(i)} = \frac{\nu_n^{(i)} c_n}{\omega_n}$. We are now matching two solutions and their derivatives, Eq. (7) for $s \leq s_0$, and Eq. (8) for $s \geq s_0$, which yields $s_0 \approx 2(\kappa_i(0))^2$, $\kappa_i^{(0)} \equiv \kappa_i(0)$. To express $\kappa_i^{(0)}$ through $M$ (which is the only mass parameter of our problem), one can insert the matched solution into Eq. (14) at $s = s_0$, which yields

$$b_i^{(0)} \ln s_0 + \sum_n a_n^{(i)} \ln(1 + \omega_n \ln s_0) = f_i,$$  

(9)

where $f_i \equiv \ln(b_i(s_0)0.45)$, $s_0 = 2(\kappa_i(0))^2$.

In the simplest approximation, when $\ln(1 + \omega_n \ln s_0) \approx \omega_n \ln s_0$ the solution, for fermion mass $\mu_1^{(0)} \equiv \mu_i(p^2 = 2(\mu_i^{(0)})^2)$ is

$$2 \left(\frac{\mu^{(0)}}{M}\right)^2 = \exp \left(\frac{f_i}{b_i^{(0)} + \sum_n a_n^{(i)} \alpha_i(M)}\right).$$  

(10)

In general case one has instead

$$2 \left(\frac{\mu^{(0)}}{M}\right)^2 = \exp \left(\frac{f_i - \sum_n a_n^{(i)} \ln(1 + \omega_n \ln s_0)}{b_i^{(0)}}\right).$$  

(11)
Both forms, (10) and (11) demonstrate an extreme sensitivity of the resulting fermion mass to the parameter $b_i^{(0)}$. The coefficients $\nu_{n_i}^{(i)}$, which define both the spread and the hierarchy of fermion masses $\mu_i^{(0)}$, are discussed below.

3. To make numerical predictions, we start with the mass $M = 10^{14}$ GeV, which is around the point where all constants $\alpha_n(M)$ are nearly intersecting, as it happens in $SU(5)$ and $SO(10)$ groups \[13\], so we take $\alpha_\mu(M) = 1/43, M = 10^{14}$ GeV, and $\alpha_\tau(M) \simeq 0.01$. Variations of $M$ in the range $10^{14} \pm 10^{16}$ GeV with unequal $\alpha_n(M)$ do not change results qualitatively, if the appropriate change in $b_i^{(0)}$ is made. For simplicity we also neglect contribution of $\alpha_\tau$, which can be compensated by a small change $0(10^{-4})$ in $b_i^{(0)}$. As a first approximation we consider an unbroken $SU(2)$ at high scale with a common $b_i^{(0)}$. We have chosen $b_i^{(0)}$ in the interval $[0.03; 0.05]$; results for the masses $\mu_i$ are shown in Table 1 and compared with experimental values for the third generation. Note, that due to very high sensitivity of Eq. (11) to entering numbers, the accuracy of results in Table 1 is low and the entries are rather indicative of orders of magnitude.

The first thing is to check, whether (10), (11) predict correct hierarchy within the generation ($\mu_t > \mu_d > \mu_b > \mu_\tau$), when one keeps $b_i^{(0)}$ constant, i.e. whether the hierarchy is due to $SU(3)_c \times U(1)_{em}$ evolution. Looking at Table 1, one indeed can see that the mass hierarchy is kept correct for all values of $b_i^{(0)}$. From Eq. (10), one can understand why the hierarchy is natural in our approach: what enters in the denominator of the exponent is the weighted sum of $(2)$ symmetry group violation. A similar analysis for $SU_\nu, \tau, t, b$ for ($\alpha_n$ constants $1$ change results qualitatively, if the appropriate change in $b_2^{(0)}$ is made. The coefficients $\Delta_i$, which define both the spread and the hierarchy of fermion masses $\mu_i^{(0)}$, are nearly intersecting, as it happens in $SU(3)_c \times U(1)_{em}$ evolution. For simplicity we also neglect contribution of $\alpha_\tau$, which can be compensated by a small change $0(10^{-4})$ in $b_i^{(0)}$.

Thus two main properties of fermion spectra: the hierarchy and the large mass spread are qualitatively reproduced by the simplest variant of our model with a common $b_i^{(0)}$.

However, the absolute values and mass ratios are in many cases far from experiment. Especially the ratios $\mu_\nu/\mu_\tau$ and $\mu_\tau/\mu_b$ are nine orders of magnitude off the experimental values. To improve agreement we shall take into account the $SU(2)$ splitting, discovered in \[4\], which splits $b_i^{(0)}$ from $b_2^{(0)}$, and $b_\nu^{(0)}$ from $b_\tau^{(0)}$.

Results of this analysis with the central value $b_i^{(0)} = 0.045$ are positive in the sense, that indeed all four masses of the third generation are well reproduced, when $b_i^{(0)} = b_i^{(0)} + \Delta b_i$, where $\Delta b_i \simeq (-0.009; +0.01; +0.004; -0.004)$ for ($\nu, \tau, t, b$). It is remarkable, that the structure $\Delta b \simeq (-2\varepsilon; +2\varepsilon; \varepsilon; -\varepsilon), \varepsilon \ll b_i^{(0)}$ may be indicative of a certain $SU(2)$ symmetry group violation. A similar analysis for $M = 10^{10}$ GeV allows to reproduce experimental masses with $b_1 = b_i^{(0)} + \Delta b_i, b_i^{(0)} = 0.07, \Delta b_i = (-0.028; 0; 0; -0.007)$.

Till now we have considered only the heaviest, third generation. Lower generations can be also included, they correspond to the splitted quadruplets centered around $b_i^{(0)} = 0.036$ and $b_2^{(0)} = 0.03$ for the second and first generations respectively. This calculation, however, does not explain the generation mechanism, but rather readdresses it to the high scale dynamics, where $\mu_i$ and $b_i$ in Eq. (11) must become matrices in generation indices, which naturally provide both masses and mixing coefficients. This point is now under investigation.

It is also interesting, that the same equation (11) would result if the high-scale mass generation is given without top charge by mechanism of gauge field exchanges, in which case $b_i^{(0)}$ is the squared charge $\nu_n \alpha_\nu / n$ for this new interaction, and $M$ is the scale, where new interaction splits off from the original GUT.

Summarizing, we have derived dynamical equation for fermion masses, where the $SU(3) \times U(1)$ evolution and the (10-20)% spread in $b_i^{(0)}$ naturally explains the hierarchy and the spread of masses within one generation, and especially the smallness of neutrino masses through the model parameter – the topological mass generation constant $b_i^{(0)}$. With a small variation of this parameter the exact masses of the third generation are reproduced. Of course, our discussion above is oversimplified. We have ignored processes like breaking of $SU(4)_c$ and neglected $SU(2)_R$ and the $U(1)_Y$ contribution, stressing only qualitative mechanism. The explicit type of gauge interaction yielding constant $b_i^{(0)}$, was not specified in the paper, and it is an open question, whether it would pass precise EW tests. The author is grateful for financial support to the grant RFBR no. 09-02-00620a.

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TABLE I: Fermion masses (in GeV/c^2) of the third generation for different the topological constant $b^{(0)}$ in comparison with experimental values

| $b^{(0)}$ | 0.03   | 0.035  | 0.0442 | 0.05   | experiment |
|----------|--------|--------|--------|--------|------------|
| $\mu_{\nu}$ | $0.6 \cdot 10^{-17}$ | $0.16 \cdot 10^{-12}$ | $0.5 \cdot 10^{-5}$ | $3.1 \cdot 10^{-3}$ | $5 \cdot 10^{-11}$ |
| $\mu_{\tau}$ | $0.17 \cdot 10^{-14}$ | $2 \cdot 10^{-10}$ | $1.1 \cdot 10^{-4}$ | $0.034$ | $1.78$ |
| $\mu_t$   | 0.009  | 0.5    | 24     | 214    | 174        |
| $\mu_b$   | 0.0025 | 0.18   | 12     | 110    | 4.2        |

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