QUANTITATIVE ERGODIC THEOREMS FOR ACTIONS OF GROUPS OF POLYNOMIAL GROWTH

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Abstract. We strengthen the maximal ergodic theorem for actions of groups of polynomial growth to a form involving jump quantity, which is the sharpest result among the family of variational or maximal ergodic theorems. As a consequence, we deduce in this setting the quantitative ergodic theorem, in particular, the upcrossing inequalities with exponential decay. The ideas or techniques involve probability theory, non-doubling Calderón-Zygmund theory, almost orthogonality argument and some delicate geometric argument involving the balls and the cubes on the group equipped with a not necessarily doubling measure.

1. Introduction

1.1. Background and main results. In the past few decades, a great deal of significant results related to the pointwise ergodic theorems for group actions have been established. The earliest pointwise ergodic theorems, to our knowledge, was obtained by Birkhoff [6], where he established pointwise ergodic theorems for one-parameter flow (such as a translation group on \( \mathbb{R} \) or \( \mathbb{Z} \)). Wiener [57] extended Birkhoff’s result to the case of several commuting flows. Even further, these pointwise ergodic results were generalized by Calderón [11] for an increasing family of compact symmetric neighborhoods of the identity satisfying doubling condition, which is abundant on non-commutative groups with polynomial volume growth. Calderón’s works [11, 12] motivate further research on pointwise ergodic theorems, such as [5, 13, 17, 21, 25, 27, 53, 54]. In particular, Breuillard [9] (see also Tessera [56]) showed that the sequence of balls with respect to any fixed word metric on groups of polynomial growth satisfy the doubling condition and are asymptotically invariant, and thus established the corresponding pointwise ergodic theorem; actually these results apply to more general metrics such as the periodic pseudodistances as defined in [9] (and recalled in Section 7). This settled a long-standing problem in ergodic theory since Calderón’s classical paper [11] in 1953. Lindenstrauss [43] established the pointwise ergodic theorem for tempered Følner sequences; this result resolves the problem of the existence of a Følner sequence which satisfies the pointwise ergodic theorem on an arbitrary amenable group. For more details we refer the reader to the survey works [2, 49].

In this paper, we aim at establishing the quantitative pointwise ergodic theorems for actions of polynomial growth groups in terms of the following jump quantity. Given a sequence of measurable functions \( \{\alpha_r(x) : r > 0\} \) and \( \lambda > 0 \), the \( \lambda \)-jump function of the sequence \( \alpha = \{\alpha_r(x) : r \in \mathcal{I}\} \) is defined by

\[
\mathcal{N}_\lambda(\alpha)(x) = \sup\{N \mid \exists r_0 < r_1 < \cdots < r_N, r_i \in \mathcal{I} : \min_{0 \leq i \leq N} |\alpha_{r_i}(x) - \alpha_{r_{i-1}}(x)| > \lambda\}.
\]

where \( \mathcal{I} \) is a subset of \((0, \infty)\) and the supremum is taken over all finite increasing sequences in \( \mathcal{I} \).

Key words: pointwise ergodic theorems, groups of polynomial growth, variational inequalities, jump inequalities, upcrossing inequalities, exponential estimates.
Let $G$ be a locally compact group equipped with a measure $m$, and let $d$ be a metric on $G$. For $r > 0$ and $h \in G$, we define the ball $B(h, r) = \{g \in G : d(g, h) \leq r\}$, and we will write it simply $B_r$ when $x = e$ (the identity in $G$). Let $r_0 > 0$. Let $\epsilon \in (0, 1]$, we say that $(G, d, m)$ satisfies the $(\epsilon, r_0)$-annular decay property if there exists a constant $K > 0$ such that for all $h \in G$, $r \in (r_0, \infty)$ and $s \in (0, r]$, 

\begin{equation}
(1.1) \quad m(B(h, r + s)) - m(B(h, r)) \leq K \left(\frac{\epsilon}{r}\right)^s m(B(h, r)).
\end{equation}

Let $D_0 > 0$, we call that $(G, d)$ satisfies the $(D_0, 4r_0)$-geometrically doubling property if for every $r \in (0, 4r_0]$ and every ball $B(h, r)$, there are at most $D_0$ balls $B(h_i, r/2)$ such that 

\begin{equation}
(1.2) \quad B(h, r) \subseteq \bigcup_{1 \leq i \leq D_0} B(h_i, r/2).
\end{equation}

Let $p \in [1, \infty)$ and $f \in L^p(G, m)$, we consider the following averages 

\begin{equation}
(1.3) \quad A_r f(h) = \frac{1}{m(B(h, r))} \int_{B(h, r)} f(g) dm(g).
\end{equation}

One of the main result of this paper is the following theorem.

**Theorem 1.1.** Assume that $(G, d, m)$ satisfies (1.1) and (1.2). Let $A_r' = \{A_r' : r \geq r_0\}$ be the sequence of averaging operators given by (1.3). Then the following assertions hold true.

(i) For any $p \in [1, \infty)$, $\lambda \sqrt{N_{\lambda}(A')} \text{ is of strong type } (p, p) \text{ uniformly in } \lambda > 0$, that is, there exists a constant $c_p > 0$ such that 

$$
\sup_{\lambda > 0} \|\lambda \sqrt{N_{\lambda}(A')} f\|_{L^p(G, m)} \leq c_p \|f\|_{L^p(G, m)}, \forall f \in L^p(G, m).
$$

(ii) For $p = 1$, $\lambda \sqrt{N_{\lambda}(A')} \text{ is of weak type } (1, 1) \text{ uniformly in } \lambda > 0$, that is, there exists a constant $c > 0$ such that for any $\gamma > 0$, 

$$
\sup_{\lambda > 0} m(\{g \in G : \lambda \sqrt{N_{\lambda}(A')} f(g) > \gamma\}) \leq \frac{c}{\gamma} \|f\|_{L^1(G, m)}, \forall f \in L^1(G, m).
$$

Conditions (1.1) and (1.2) are related to the geometric structure of group $G$; there exist lots of examples such as the ones introduced in [49, 56], see Section 7 for more details. Moreover, as a consequence of Theorem 1.1, we obtain the quantitative ergodic theorems for actions of groups of polynomial growth.

Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space and $T$ an action of $G$ on the associated $L^p$-spaces $L^p(X, \mu)$, under some additional assumptions recalled in later sections. In particular, if $T$ is induced by a $\mu$-preserving measurable transformation $\tau$ on $X$, then $T$ extends to an isometric action on $L^p(X, \mu)$ for all $1 \leq p \leq \infty$, given by $T_\gamma f(x) = f(\tau_\gamma^{-1} x)$. Given an action $T$ and a ball $B_r$, the associated averaging operator is given by 

\begin{equation}
(1.4) \quad A_r f(x) = \frac{1}{m(B_r)} \int_{B_r} T_\gamma f(x) dm(g).
\end{equation}

**Theorem 1.2.** Assume that $G$ is of polynomial growth with a symmetric compact generating set $V$ and $d$ is the associated word metric on $G$, that is, 

$$
d(g, h) = \min\{n \in \mathbb{N}, \ g^{-1} h \in V^n\}.
$$

Let $m$ be a Haar measure. Let $A_r = \{A_r : r \in \mathbb{N}\}$ the corresponding sequence of averaging operators with respect to an action $T$. 

(i) If $T$ is an action induced by a measure-preserving measurable transformation, then $\frac{1}{\lambda}N_{\lambda}(A)$ is of weak type $(1, 1)$ and of strong type $(p, p)$ for all $1 < p < \infty$ uniformly in $\lambda > 0$.

(ii) If $T$ is a strongly continuous regular action of $G$ on $L^p(X, \mu)$ $(1 < p < \infty)$, then $\frac{1}{\lambda}N_{\lambda}(A)$ is of strong type $(p, p)$ uniformly in $\lambda > 0$.

The notions of regular action will be recalled in Subsection 6.2. If we take $G$ to be the integer group $\mathbb{Z}$ and $d$ to be the usual word metric, then we recover the usual ergodic average $A_n = \frac{1}{n} \sum_{k=-n}^{n} T^k$ for an automorphism $T$, as is treated in [8, 30]. Moreover, from the definition of jump quantity, Theorems 1.2 imply that the underlying sequence of functions $A_n f$ converges almost everywhere as $r \to \infty$, that is the pointwise ergodic theorem.

The jump quantity appeared first implicitly in [52], and then explicitly in [30], whose study was motivated by the research of variational inequality arising from probability theory [40]. For $q \in (0, \infty]$, the $q$-variation (semi-)norm $V_q$ of a sequence $\{\alpha_r(x) : r \in (0, \infty)\}$ of complex-valued functions is defined by

$$V_q(\alpha_r(x) : r \in \mathcal{I}) = \sup_{0 < r_0 < \cdots < r_J} \left( \sum_{j=0}^{J} |\alpha_{r_{j+1}}(x) - \alpha_{r_{j}}(x)|^q \right)^{1/q}.$$ 

The $\infty$-variation norm is nothing but equivalent to the maximal norm; moreover any $q$-variation norm dominates the maximal norm

$$\sup_{r \in \mathcal{I}} |\alpha_r(x)| \leq |\alpha_{r_1}(x)| + V_q(\alpha_r(x) : r \in \mathcal{I}), \forall r_1 > 0.$$ 

As the jump quantity, finite $q$-variation norm with $q < \infty$ also deduces immediately the pointwise convergence of the underlying sequence of operators without density argument. This idea was firstly exploited by Bourgain [8] to study the pointwise ergodic theorem for dynamical systems where the density argument is not available. Bourgain’s work [8] inspired many studies on variational inequalities and ergodic theory, see [30, 31, 37, 38, 46, 58] and the references therein.

On the other hand, by Chebychev’s inequality, the $q$-variation norm dominates the jump quantity

$$\sup_{\lambda > 0} \lambda(\mathcal{N}_{\lambda}(\alpha_r(x) : r \in \mathcal{I}))^{1/q} \leq V_q(\alpha_r(x) : r \in \mathcal{I}).$$

However, it is shown in [32, 41] that the 2-variational inequality (and thus $q < 2$) does not hold true in general. Conversely, for any family of linear operators $T = \{T_t\}_{t \in \mathcal{I}}$, the mapping properties of $\lambda \sqrt{\mathcal{N}_{\lambda}(T)}$ imply the corresponding $q$-variational inequalities for all $q > 2$ by interpolation, see e.g. [8, 33, 47]. Thus for all $q > 2$, the $q$-variational version of Theorem 1.1 and 1.2, in particular the maximal ergodic theorem, holds true.

Furthermore, restricted to bounded function, from Theorem 1.2, we may deduce a kind of exponential decay estimates by using Vitali covering lemma and geometric argument.

**Theorem 1.3.** Assume that $G$ is a group of polynomial growth with a symmetric finite generating set and $d$ is the resulting word metric. Let $T$ be an action induced by a measure-preserving measurable transformation and $A_r = \{A_r : r \in \mathbb{N}\}$ the sequence of averaging operators with respect to action $T$ given by (1.4). Let $\lambda > 0$. Then for every $p \in [1, \infty)$, there are two constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 \in (0, 1)$ depending on $\lambda$, $p$ and group $G$ such that for any $f \in L^p(X, \mu)$ with $\|f\|_{L^p(X, \mu)} \leq 1$,

$$\mu\{x \in X : \mathcal{N}_{\lambda}(A)(x) > n\} \leq \tilde{c}_1\tilde{c}_2^p \|f\|_{L^p(X, \mu)}^p.$$ 

This result immediately yields the upcrossing inequalities with exponential decay. Recall that for two real numbers $a$ and $b$ with $b > a$, the upcrossings of a family of real-valued functions $\alpha = \{\alpha_r(x) : r \in \mathcal{I}\}$, $\mathcal{N}_{a,b}(\alpha)(x)$ is defined by

$$\sup\{N \in \mathbb{N} \in \exists s_1 < r_1 < \cdots < s_N < r_1, r_i, s_i \in \mathcal{I} \text{ such that } \alpha_{s_i}(x) < a \text{ and } \alpha_{r_i}(x) > b\}.$$
Taking $\lambda = b - a$, it is easy to see that $\mathcal{N}_{a,b}(\alpha)(x) \leq 2\mathcal{N}_{\lambda/2}(\alpha)(x)$. Thus we obtain

**Corollary 1.4.** Let $a$ and $b$ be two real numbers with $b > a$. Then for every $p \in [1, \infty)$, there are two constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 \in (0, 1)$ depending on $\lambda$, $p$ and group $G$ such that for any real-valued function $f \in L^p(X, \mu)$ with $\|f\|_{L^\infty(X, \mu)} \leq 1$,

$$\mu(\{x \in X : \mathcal{N}_{a,b}(\mathbf{A}f)(x) > n\}) \leq \tilde{c}_1 \tilde{c}_2^p \|f\|_{L^p(X, \mu)}^p. \quad (1.5)$$

A variant of (1.5) for non-negative functions has been recently proved by Moriakov [48], as a generalization of Kalikow and Weiss’s exponential estimate for $G = \mathbb{Z}$ [34]. Note that, our method here is completely different from the one used by Moriakov that is based on a generalization of Vitali covering theorem. On the other hand, Kalikow and Weiss’s estimate was motivated by Bishop’s fundamental result [7] and the similar estimates in the martingale setting [19, 20]. Recently the upcrossing inequalities with exponential decay have been extended to stationary process and find many applications to ergodic theory and information theory, see e.g. [26].

1.2. Methods and more results. The theorems rely on several key results obtained in this paper.

The first step to show Theorem 1.1 is now standard, that is, to control the jump quantity by the ‘dyadic’ jump and the short variation operator, see e.g. [33]. More precisely, let $f \in L^p(G, m)$, we dominate the jump quantity $\lambda \sqrt{\mathcal{N}_\lambda(\mathbf{A}f)}$ in the following way,

$$\lambda \sqrt{\mathcal{N}_\lambda(\mathbf{A}f)} \leq 2\lambda \sqrt{\mathcal{N}_{\lambda/6}(\mathbf{A}f') : n > n_r}; + 16 \left( \sum_{n \geq n_r} V_2(A'_f : r \in [\delta^n, \delta^{n+1})^2 \right)^{1/2},$$

where $\delta > 1$ is a constant depending on $G$ that will be determined in Proposition 2.2 and $n_r$ is the unique integer such that $\delta^{n_r} < r_0 \leq \delta^{n_r+1}$. Moreover, compared with the resulting martingale $\mathbb{E}f = (\mathbb{E}_n f : n \in \mathbb{N})$ (see Definition 2.3), the dyadic jump is controlled by

$$96\sqrt{2} \left( \sum_{n > n_r} |A'_{\mathbb{E}_n} f - \mathbb{E}_n f|^2 \right)^{1/2} + 2\sqrt{2} \lambda \sqrt{\mathcal{N}_{\lambda/24}(\mathbb{E}_n f : n > n_r)}.$$

Thus we obtain the following pointwise estimate,

$$\lambda \sqrt{\mathcal{N}_\lambda(\mathbf{A}f)} \leq 96\sqrt{2} \left( \sum_{n > n_r} |A'_{\mathbb{E}_n} f - \mathbb{E}_n f|^2 \right)^{1/2}$$

$$+ 16 \left( \sum_{n \geq n_r} V_2(A'_f : r \in [\delta^n, \delta^{n+1})^2 \right)^{1/2} + 2\sqrt{2} \lambda \sqrt{\mathcal{N}_{\lambda/24}(\mathbb{E}_n f : n > n_r)}.$$  \quad (1.6)

This inequality implies that in order to bound $\lambda \sqrt{\mathcal{N}_\lambda(\mathbf{A}f)}$, it suffices to estimate the three parts on the right hand, respectively. Note that the boundedness of the jump operator $\lambda \sqrt{\mathcal{N}_\lambda(\mathbf{E}_f)}$ was proved in [30, 52], see Lemma 2.4; and so for our purposes we only need to estimate the first two terms on the right hand side of inequality (1.6). For abbreviation, we denote

$$S(f) := \left( \sum_{n > n_r} |A'_{\mathbb{E}_n} f - \mathbb{E}_n f|^2 \right)^{1/2}$$

and

$$SV(f) := \left( \sum_{n \geq n_r} V_2(A'_f : r \in [\delta^n, \delta^{n+1})^2 \right)^{1/2}.$$
For these two operators, we will show more results including \((L^\infty, \text{BMO})\) estimate which is also necessary to obtain the result for \(2 < p < \infty\) (see Section 5 for the definition of BMO). In what follows, \(L^\infty\) denotes the compactly supported \(L^\infty\) functions.

**Theorem 1.5.** Assume that \((G, d, m)\) satisfies conditions (1.1) and (1.2), then

(i) for every \(p \in (1, \infty)\), there exists a constant \(c_p > 0\) such that for every \(f \in L^p(G, m)\),

\[
\|S(f)\|_{L^p(G, m)} \leq c_p \|f\|_{L^p(G, m)};
\]

(ii) there exists a constant \(c > 0\) such that for every \(f \in L^1(G, m)\),

\[
m\{g \in G : S(f)(g) > \gamma\} \leq \frac{c}{\gamma} \|f\|_{L^1(G, m)}, \quad \forall \gamma > 0,
\]

and for every \(f \in L^\infty(G, m)\),

\[
\|S(f)\|_{BMO} \leq c \|f\|_{L^\infty(G, m)}.
\]

The same results hold still true for the short variation operator \(SV\).

**Theorem 1.6.** Assume that \((G, d, m)\) satisfies conditions (1.1) and (1.2), then

(i) for every \(p \in (1, \infty)\), there exists a constant \(c_p > 0\) such that for every \(f \in L^p(G, m)\),

\[
\|SV(f)\|_{L^p(G, m)} \leq c_p \|f\|_{L^p(G, m)};
\]

(ii) there exists a constant \(c > 0\) such that for every \(f \in L^1(G, m)\),

\[
m\{g \in G : SV(f)(g) > \gamma\} \leq \frac{c}{\gamma} \|f\|_{L^1(G, m)}, \quad \forall \gamma > 0,
\]

and for every \(f \in L^\infty(G, m)\),

\[
\|SV(f)\|_{BMO} \leq c \|f\|_{L^\infty(G, m)}.
\]

The above two theorems, parallel to Theorem 2.3 and 2.4 of [28], are not a surprise (cf. [28]). However, note that under conditions (1.1) and (1.2), \((G, d, m)\) is not necessarily a doubling metric measure space; this induces several new difficulties in showing Theorems 1.5 and 1.6, such as, that the ‘dyadic cubes’ constructed in Proposition 2.2 may not admit the small boundary property (cf. [14, Theorem 11]) and that the standard Calderón-Zygmund decomposition for homogeneous space is not enough etc.. For these reasons, we have to pay great attention to the geometric argument involving the cubes and the balls and explore the non-doubling Calderón-Zygmund theory etc..

To deduce Theorem 1.2 from Theorem 1.1, there needs two transference principles: the first one is for actions induced by measure-preserving measurable transforms, while another one is for regular actions. We refer the reader to (6.7) for the definition of regular actions and the related constant \(\|\cdot\|_{r}\). A group \(G\) is called amenable if it admits a Folner sequence \((F_n)_{n \in \mathbb{N}}\), that is, for every \(g \in G\),

\[
\lim_{n \to \infty} \frac{m((F_n g) \triangle F_n)}{m(F_n)} = 0,
\]

where \(\triangle\) denotes the usual symmetric difference of two sets.

**Theorem 1.7.** Let \(G\) be an amenable group equipped with invariant metric \(d\) and a right Haar measure \(m\), and \(T\) an action on \(L^p(X, \mu)\). Let \(A_r = \{A_r : r \in \mathcal{I}\}\) and \(A = \{A_r : r \in \mathcal{I}\}\) be two sequences of averaging operators given by (1.3) and (1.4), respectively.
(i) Let \( p \in [1, \infty) \). If \( T \) is an action induced by a measure-preserving measurable transformation and \( \lambda_1 \sqrt{N_1(A)} \) is of weak (resp. strong) type \((p, p)\) uniformly in \( \lambda > 0 \), then \( \lambda_1 \sqrt{N_1(A)} \) is of weak (resp. strong) type \((p, p)\) uniformly in \( \lambda > 0 \), and moreover \( \sup_{\lambda > 0} \| \lambda_1 \sqrt{N_1(A)} \|_{L^p \rightarrow L^{p'}} \leq \sup_{\lambda > 0} \| \lambda_1 \sqrt{N_1(A)} \|_{L^p \rightarrow L^{p'}} \leq \sup_{\lambda > 0} \| \lambda_1 \sqrt{N_1(A)} \|_{L^p \rightarrow L^{p'}} \).

(ii) Let \( p \in (1, \infty) \). If \( T \) is a strongly continuous regular action of \( G \) on \( L^p(X) \) and \( \lambda_1 \sqrt{N_1(A)} \) is of strong type \((p, p)\) uniformly in \( \lambda > 0 \), then \( \lambda_1 \sqrt{N_1(A)} \) is of strong type \((p, p)\) uniformly in \( \lambda > 0 \), and moreover there exists a constant \( c_p > 0 \) such that \( \sup_{\lambda > 0} \| \lambda_1 \sqrt{N_1(A)} \|_{L^p \rightarrow L^{p'}} \leq c_p \sup_{h \in G} \| T_h \|_p^2 \sup_{\lambda > 0} \| \lambda_1 \sqrt{N_1(A)} \|_{L^p \rightarrow L^{p'}} \).

It is a little bit surprising that Theorem 1.7(i) holds true due to the fact that \( \lambda_1 \sqrt{N_1(A)} \) is a priori not a norm; while to prove Theorem 1.7(ii), in addition to the use of the fact that \( \sup_{\lambda > 0} \| \lambda_1 \sqrt{N_1(A)} \|_{p} \) is equivalent to a norm (cf. [47]) for \( p > 1 \), the argument is subtle since the appearance of the supremum over \( \lambda \) outside \( L^p \) norm.

An outline of this paper is as follows. In Section 2, we first recall some necessary preliminaries concerning the definition of ‘dyadic cubes’, which was constructed by Hytönen and Kairema [29] in metric space; and then present some technical lemmas, which are essential in showing Theorems 1.5 and 1.6. The last two theorems will be proved in Section 3-5. In Section 6, we prove the transfer principles, namely Theorem 1.7. In Section 7, we discuss the \((\epsilon, r_0)\)-annular decay property, providing examples and formulating problems; in particular, the balls with respect to a word metric over groups of polynomial growth satisfies conditions (1.1), (1.2) and (1.13), and thus we obtain Theorem 1.2. Finally, in Section 8, we give a proof of Theorem 1.3.

Throughout this paper, we always denote \( C \) by a positive constant respectively that may vary from line to line, while \( c_p \) denote positive constant possibly depending on the subscripts.

2. Preliminaries and Some Technical Lemmas

In this section, we first recall the ‘dyadic cubes’ constructed on the measure space \((G, d, m)\) satisfying conditions (1.1) and (1.2) which might not be a measure doubling metric space, and the resulting martingale inequalities will play a key role in the probabilistic approach to jump or variational inequalities. We then collect several technical lemmas which involve or rely on the estimates of the boundaries of ‘dyadic cubes’ or balls or certain configuration among them: these estimates are subtle, and thus we will set and fix in the whole paper several constants such as \( k_1, n_0, c_0, C_0, C_1, L_0, L_1, \delta \) which depend on conditions (1.1) and (1.2) and the construction.

These preliminary results or technical lemmas, collected in such a way, will facilitate much the presentation of the proof of Theorems 1.5 and 1.6.

We will exploit the system of ‘dyadic cubes’ constructed in the setting of geometrically doubling metric measure spaces, which means that each ball of radius \( r > 0 \) can be covered by fixed finitely many balls of radius \( r/2 \) (cf. [29, Theorem 2.2]). It is not difficult to see that the measure space \((G, d, m)\) satisfying conditions (1.1) and (1.2) is geometrically doubling, but which might not be measure doubling. Indeed, by a simple computation, the \((\epsilon, r_0)\)-annular decay property—condition (1.1)—implies the measure doubling condition for large balls, that is, for every \( x \in G \) and \( r_0 < r \leq R < \infty \),

\[
\frac{m(B(x, R))}{m(B(x, r))} \leq (K + 1) \left( \frac{R}{r} \right)^\epsilon.
\]

This yields the geometrically doubling condition for large balls, which, combined with condition (1.2), deduces the geometrically doubling condition property of the space \((G, d, m)\), namely, each ball of radius \( r > 0 \) in \( G \) can be covered by no more than \( D = \max\{D_0, 9^\epsilon(K + 1) + 1\} \)
balls of radius \( r/2 \). Moreover, from the geometrically doubling condition, one can easily deduce the following property.

**Proposition 2.1.** Let \((G, d, m)\) satisfy the conditions of Theorem 1.1. Let \(0 < r \leq R\), any ball \(B(x, R)\) can be covered by no more than \(D^{\log_2(R/r) + 1}\) balls of radius \(r\).

For more information about geometrically doubling property we refer the reader to [16].

**Proposition 2.2.** [29, Theorem 2.2] Let \((G, d, m)\) satisfy the conditions of Theorem 1.1. Fix constants \(0 < c_0 < C_0 < \infty\) and \(\delta > 1\) such that

\[
18C_0\delta^{-1} \leq c_0.
\]

Let \(I_k (k \in \mathbb{Z})\) be an index set and \(\{z^k_\alpha \in G : \alpha \in I_k, k \in \mathbb{Z}\}\) be a collection of points with the properties that

\[
d(z^k_\alpha, z^k_\beta) \geq c_0\delta^k (\alpha \neq \beta), \quad \min_{\alpha} d(x, z^k_\alpha) < C_0\delta^k, \quad \forall \ x \in G, \ k \in \mathbb{Z}.
\]

Then there exist a family of sets \(\{Q^k_\alpha\}_{\alpha \in I_k}\) associating with \(\{z^k_\alpha\}_{\alpha \in I_k}\), and constants \(a_0 := c_0/3\) and \(C_1 := 2C_0\) such that

(i) \(\forall \ k \in \mathbb{Z}, U_{\alpha \in I_k} Q^k_\alpha = G\).

(ii) If \(k \leq l\) then either \(Q^k_\alpha \subset Q^l_\beta\) or \(Q^k_\alpha \cap Q^l_\beta = \emptyset\).

(iii) For each \((k, \alpha)\) and each \(k < n\) there is a unique \(\beta\) such that \(Q^k_\alpha \subset Q^n_\beta\), and for \(n = k + 1\), we call such \(Q^k_\beta\) the parent of \(Q^k_\alpha\).

(iv) \(B(z^k_\alpha, a_0\delta^k) \subseteq Q^k_\alpha \subseteq B(z^k_\alpha, C_1\delta^k)\).

We remark that the geometrically doubling property ensures that the minimum of the second inequality in (2.2) is attained.

For \(k \in \mathbb{Z}\), let \(\mathcal{F}_k\) be the \(\sigma\)-algebra generated by the ‘dyadic cubes’ \(\{Q^k_\alpha : \alpha \in I_k\}\). We recall the following notions associated to the (reverse) martingale theory.

**Definition 2.3.** Let \(f : G \to \mathbb{C}\) be a locally integrable function and \(k \in \mathbb{Z}\), the conditional expectation of \(f\) with respect to \(\mathcal{F}_k\) is defined by

\[
E_k f(x) = \sum_{\alpha \in I_k} \frac{1}{m(Q^k_\alpha)} \int_{Q^k_\alpha} f(y) dm(y) \mathbf{1}_{Q^k_\alpha}(x);
\]

the resulting martingale difference operator \(D_k\) is defined as

\[
D_k f = E_{k-1} f - E_k f.
\]

We check at once \(E_k \circ E_j = E_{\max(j, k)}\) and that for \(f \in L^2\), \(f = \sum_{k \in \mathbb{Z}} D_k f\) and

\[
\|\left(\sum_{k \in \mathbb{Z}} |D_k f|^2\right)^{1/2}\|_{L^2} = \|f\|_{L^2}.
\]

Denote \(E = \{E_k : k \in \mathbb{Z}\}\). We remark that the strong type \((p, p)\) with \(1 < p < \infty\) and the weak type \((1, 1)\) estimates for operator \(\lambda \sqrt{N_\lambda(E)}\) were given implicitly in [52] and explicitly in [30]. We state the results as follows.

**Lemma 2.4.** Let \(E = \{E_k : k \in \mathbb{Z}\}\) be defined as above.

(i) When \(p \in (1, \infty)\), there is a constant \(c_p > 0\) such that for all \(f \in L^p(G, m)\),

\[
\sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(E)f}\|_{L^p(G, m)} \leq c_p \|f\|_{L^p(G, m)}.
\]

(ii) For \(p = 1\), there is a constant \(c > 0\) such that for every \(\gamma > 0\) and \(f \in L^1(G, m)\),

\[
\sup_{\lambda > 0} m\{x \in G : \lambda \sqrt{N_\lambda(E)f}(x) > \gamma\} \leq \frac{c}{\gamma} \|f\|_{L^1(G, m)}.
\]
Since \((G, d, m)\) might not be a measure doubling metric space, the small boundary property of the ‘dyadic cubes’ constructed from (2.2) (see e.g. [14]) does not hold in general. However, from the \((\epsilon, r_0)\) annular decay property, we do have some boundary property—Lemma 2.6, which will be enough for our purpose in the present paper.

Set
\[
L_0 = \lfloor \log_2(12/c_0) \rfloor + 1, \quad L_1 = \lfloor \log_2(36r_0/c_0) \rfloor + 1.
\]

**Lemma 2.5.** Let \(k, L \in \mathbb{Z}\) satisfy \(L_0 < L < k + L_0 - L_1\) and \(\alpha \in I_k\). Then we have
\[
m(\{x \in Q^k_\alpha : d(x, G \setminus Q^k_\alpha) \leq \delta^{k-L} \}) \leq \frac{(K + 1)^2}{(L - L_0 + 1)} \left( \frac{72C_0}{c}\right)^{2\epsilon} m(Q^k_\alpha).
\]

**Proof.** For a fixed point \(x \in Q^k_\alpha\) with \(d(x, G \setminus Q^k_\alpha) \leq \delta^{-L}\delta^k\), we claim that there exists a chain \(Q^k_{\sigma_{k-L+L_0}} \subset \cdots \subset Q^k_{\sigma_{k-1}} \subset Q^k_{\sigma_0} = Q^k_\alpha\) such that \(x \in Q^k_{\sigma_{k-L+L_0}}\) and
\[
d(z_{j', j}, z_{j', j}) \geq c_0\delta^j/12, \quad \forall k - L + L_0 \leq j < i \leq k.
\]
Indeed, let \(x \in Q^k_\alpha\), by Proposition 2.2(i)-(iii), for any \(n \leq k\), there exists a ‘dyadic cube’ \(Q^k_n\) such that \(x \in Q^k_n \subseteq Q^k_\alpha\). Hence there exists a chain \(Q^k_{\sigma_{k-L+L_0}} \subset \cdots \subset Q^k_{\sigma_{k-1}} \subset Q^k_{\sigma_0} = Q^k_\alpha\) with \(x \in Q^k_{\sigma_{k-L+L_0}}\). Now we verify (2.4). First, by Proposition 2.2(iv), for every \(k - L + L_0 \leq j \leq k\), we have \(x \in B(z_{j', j}, C_1\delta^j)\). We also have \(B(z_{j', j}, a_0\delta^j) \subset Q^k_{\sigma_{j', j}} \subset Q^k_\alpha\). If (2.4) were not true, then
\[
a_0\delta^j \leq d(z_{j', j}, G \setminus Q^k_\alpha) \leq d(z_{j', j}^1, z_{j', j}) + d(z_{j', j}^1, x) + d(x, G \setminus Q^k_\alpha) < \frac{c_0}{12} \delta^j + C_1\delta^j + \delta^{-L}\delta^k
\]
\[
\leq \frac{a_0}{4} \delta^j + \frac{a_0}{3} \delta^{j+1} + \delta^{-L_0}\delta^j \leq \frac{a_0}{4} \delta^j + \frac{a_0}{3} \delta^j + \frac{a_0}{4} \delta^j < a_0\delta^j,
\]
which leads to a contradiction and the claim is proved.

For every ‘dyadic cube’ \(Q^k_n\), we denote it briefly by \((m, \beta)\). Let \(E = \{x \in Q^k_n : d(x, G \setminus Q^k_\alpha) \leq \delta^{-L}\delta^k\}\). For each \(x \in E\), there exists a chain of pair \((i, \beta(i, x))\) with the properties proved in the first paragraph. Set \(S_i = \bigcup_{E \in E} \{z^i_{\beta(i, x)}\}\) for \(k - L + L_0 \leq i \leq k\). In the following, we abbreviate \(z_{\beta(i, x)}^i\) to \(z_{i(x)}\). We have the following observation: for \(z_{i(x)} \neq z_{j(y)}\),
\[
(2.5) \quad B(z_{i(x)}, a_0\delta^j/36) \cap B(z_{j(y)}, a_0\delta^j/36) = \emptyset, \quad \forall z_{i(x)} \in S_i, \ z_{j(y)} \in S_j, \ k - L + L_0 \leq i, j \leq k.
\]
For \(i = j\), by (2.2), the above assertion is trivially right. For \(i \neq j\), without loss of generality, we assume \(i > j\). Note that by the definition of \(S_j\), for each \(z_{j(y)} \in S_j\), there exists a point \(\zeta \in E\) such that \(d(z_{j(y)}, G \setminus Q^k_\alpha) \leq d(\zeta(i, j), G \setminus Q^k_\alpha) \leq C_1\delta^j + \delta^{-L}\delta^k\). It follows that if \(B(z_{i(x)}, c_0\delta^j/36) \cap B(z_{j(y)}, c_0\delta^j/36) \neq \emptyset\), then there exists a point \(z \in B(z_{i(x)}, c_0\delta^j/36) \cap B(z_{j(y)}, c_0\delta^j/36)\) such that
\[
a_0\delta^j \leq d(z_{i(x)}, G \setminus Q^k_\alpha) \leq d(z_{i(x)}, z_{j(y)}) + d(z_{j(y)}, G \setminus Q^k_\alpha)
\]
\[
\leq \frac{c_0}{36} \delta^j + \frac{c_0}{36} \delta^j + C_1\delta^j + \delta^{-L}\delta^k \leq \frac{c_0}{18} \delta^j + C_1\delta^j + \delta^{-L}\delta^k \leq \frac{a_0}{6} \delta^j + \frac{a_0}{3} \delta^{j+1} + \delta^{-L_0}\delta^j
\]
\[
\leq \frac{a_0}{6} \delta^j + \frac{a_0}{3} \delta^j + \frac{a_0}{4} \delta^j < a_0\delta^j,
\]
where we used \(c_0 = c_0/3, \ C_1 = 2C_0, \ 3C_1 \leq a_0\delta^j\) and \(\delta^{-L_0} \leq \delta^{-\log_2(12/c_0)} = c_0/12\) again. This leads to a contradiction and so (2.5) holds.
We now prove the desired result. In the following, we write $z_i$ for $z_i(x)$. Setting $G_i = \cup_{z_i \in S_i} B(z_i, c_0 \delta^i/36)$, for any $k - L + L_0 \leq i \leq k$, we have
\[
m(E) \leq \sum_{z_{k-L+L_0} \in S_{k-L+L_0}} m(B(z_{k-L+L_0}, C_1 \delta^{k-L+L_0}))
\leq (K+1) \left( \frac{36C_1}{c_0} \right)^{c} \sum_{z_{k-L+L_0} \in S_{k-L+L_0}} m(B(z_{k-L+L_0}, c_0 \delta^{k-L+L_0}/36))
= (K+1) \left( \frac{36C_1}{c_0} \right)^{c} \sum_{S_i \subseteq S_k} \sum_{w_i \in S_i} m(B(z_{k-L+L_0}, c_0 \delta^{k-L+L_0}/36))
\leq (K+1) \left( \frac{36C_1}{c_0} \right)^{c} \sum_{w_i \in S_i} m(B(w_i, C_1 \delta^i))
\leq (K+1)^2 \left( \frac{36C_1}{c_0} \right)^{2c} \sum_{w_i \in S_i} m(B(w_i, c_0 \delta^i/36)) = (K+1)^2 \left( \frac{72C_0}{c_0} \right)^{2c} m(G_i),
\]
where $z_{k-L+L_0} \leq w_i$ in the third line means that the inclusion of the corresponding ‘dyadic cubes’ $Q_{\beta(k-L-L_0)} \subset Q_{\beta(i)}$, and we used (2.1) in the second and last inequalities since $\delta^i \geq \delta^{k-L+L_0} \geq \delta^{L_1} > 36c_0/c_0$. The equality in the third line follows from (2.5), as does the equality in the last line. From the above inequalities and the disjointness of the sets $G_i$, we obtain
\[
m(E) \leq \frac{(K+1)^2}{(L-L_0+1)} \left( \frac{72C_0}{c_0} \right)^{2c} \sum_{i=k-L+L_0}^k m(G_i) \leq \frac{(K+1)^2}{(L-L_0+1)} \left( \frac{72C_0}{c_0} \right)^{2c} m(Q_{a_0}^k),
\]
and the lemma follows. \qed

Set
\[
L_2 = \lfloor \log_2(4C_0 + 1) \rfloor + 1, \quad L_3 = \lfloor 2(K+1)^2 \left( \frac{72C_0}{c_0} \right)^{2c} \rfloor + L_0 + L_2,
\]
\[
\eta = (\log_2 2)/L_3, \quad C_2 = 4(K+1)^2(72C_0/c_0)^{2c}, \quad C'_2 = (K+1)(3C_1/\alpha_0)^{c_0}.
\]

**Lemma 2.6.** Under the assumption of Lemma 2.5, we have
\[
m(\{ x \in Q_{a_0}^k : d(x, G \setminus Q_{a_0}^k) \leq \delta^{k-L} \}) \leq C_2 \delta^{-L_0} m(Q_{a_0}^k),
\]
\[
m(\{ x \in G \setminus Q_{a_0}^k : d(x, Q_{a_0}^k) \leq \delta^{k-L} \}) \leq C_2 \delta^{-L_0} m(Q_{a_0}^k).
\]

**Proof.** Let us focus on the first inequality. Let $\ell \in \mathbb{N}$, set
\[
E_-(Q_{a_0}^k) = \{ Q_{a_0}^{k-\ell} \subset Q_{a_0}^k : d(Q_{a_0}^{k-\ell}, G \setminus Q_{a_0}^k) \leq (C_1 + 1) \delta^{k-\ell} \},
\]
where $d(Q_{a_0}^{k-\ell}, G \setminus Q_{a_0}^k) = \inf_{x \in Q_{a_0}^{k-\ell}} d(x, G \setminus Q_{a_0}^k)$. We denote by
\[
e_-(Q_{a_0}^k) = \{ x : x \in Q_{a_0}^{k-\ell} \text{ with } Q_{a_0}^{k-\ell} \text{ in } E_-(Q_{a_0}^k) \}
\]
the underlying point set. We proceed to show that
\[
(2.6) \quad \{ x \in Q_{a_0}^k : d(x, G \setminus Q_{a_0}^k) \leq \delta^{k-\ell} \} \subseteq e_-(Q_{a_0}^k) \subseteq \{ x \in Q_{a_0}^k : d(x, G \setminus Q_{a_0}^k) \leq \delta^{k-\ell} + L_2 \}.
\]
Fix $x \in Q_{a_0}^k$ such that $d(x, G \setminus Q_{a_0}^k) \leq \delta^{k-\ell}$. There exists ‘dyadic cube’ $Q_{a_0}^{k-\ell}$ such that $x \in Q_{a_0}^{k-\ell} \subset Q_{a_0}^k$, and then
\[
d(Q_{a_0}^{k-\ell}, G \setminus Q_{a_0}^k) \leq C_1 \delta^{k-\ell} + d(x, G \setminus Q_{a_0}^k) \leq (C_1 + 1) \delta^{k-\ell}.
\]
On the other hand, fix a point \( x \in e^{-t}(Q^k_\alpha) \), then there exists \( Q^k_{\beta, t} \in E_{-t}(Q^k_\alpha) \) such that
\[
dx, G \setminus Q^k_\alpha \leq C_1 \delta^{k-t} + d(Q^k_{\beta, t}, G \setminus Q^k_\alpha) \leq (2C_1 + 1)\delta^{k-t} \leq \delta^{k-L+L_2},
\]
since \( C_1 = 2C_0 \), and (2.6) is proved.

To achieve our goal, we split \( L \) into two cases: \( L_0 < L \leq 2L_3 \) and \( L > 2L_3 \). We first prove the case \( L > 2L_3 \). Set \( M_0 = [L/L_3] \); here and below, \([t]\) denotes the integer part of a real number \( t \).

Let \( F_1(Q^k_\alpha) = E_{-L_3}(Q^k_\alpha) \) and
\[
F_n(Q^k_\alpha) = \bigcup_{Q^k_{\beta, (n-1)L_3} \in F_{n-1}(Q^k_\alpha)} E_{-L_3}(Q^k_{\beta, (n-1)L_3}),
\]
for \( 2 \leq n \leq M_0 \). Let \( f_n(Q^k_\alpha) \) be the underlying point set. It is easy to check that
(2.7)
\[
e^{-nL_3}(Q^k_\alpha) \subset f_n(Q^k_\alpha).
\]
Moreover, for each \( Q^k_{\beta, (n-1)L_3} \in F_{n-1}(Q^k_\alpha) \) and \( 1 \leq n \leq M_0 \), plugging (2.6) into Lemma 2.5, we obtain
\[
m(e_{-L_3}(Q^k_{\beta, (n-1)L_3})) \leq m(Q^k_{\beta, (n-1)L_3}) \leq \delta^{k-nL_3+L_2}
\leq \frac{(K + 1)^2}{L_3 - L_2 - L_0 + 1} \left( \frac{72C_0}{c_0} \right)^{2e} m(Q^k_{\beta, (n-1)L_3}) \leq \frac{1}{2} m(Q^k_{\beta, (n-1)L_3}),
\]
since \( nL_3 \leq L < k + L_0 - L_1 \). Then
\[
m(f_n(Q^k_\alpha)) \leq \frac{1}{2} m(f_{n-1}(Q^k_\alpha)),
\]
and iterating the above inequality we have \( m(f_{M_0}(Q^k_\alpha)) \leq 2^{-M_0} m(Q^k_\alpha) \). From this inequality and (2.7), one has
\[
m(\{x \in Q^k_\alpha : d(x, G \setminus Q^k_\alpha) \leq \delta^{k-L}\}) \leq m(e_{-L}(Q^k_\alpha)) \leq m(e_{-M_0L_3}(Q^k_\alpha)) \leq m(f_{M_0}(Q^k_\alpha)) \leq 2^{-M_0} m(Q^k_\alpha) \leq 2^{-L/L_3+1} m(Q^k_\alpha) = 2\delta^{-L/\alpha} m(Q^k_\alpha),
\]
where the second inequality in the first line follows from the definition of \( e_{-L}(Q^k_\alpha) \).

For the case \( L_0 < L \leq 2L_3 \), using Lemma 2.5 again we have
\[
m(\{x \in Q^k_\alpha : d(x, G \setminus Q^k_\alpha) \leq \delta^{k-L}\}) \leq (K + 1)^2 \left( \frac{72C_0}{c_0} \right)^{2e} m(Q^k_\alpha) \leq \delta^{-L_k} \delta_{2\eta L_3}(K + 1)^2 \left( \frac{72C_0}{c_0} \right)^{2e} m(Q^k_\alpha) = 4(K + 1)^2 \left( \frac{72C_0}{c_0} \right)^{2e} \delta^{-L_k} m(Q^k_\alpha).
\]

We now prove the second inequality. Let \( \widetilde{E} = \{x \in G \setminus Q^k_\alpha : d(x, Q^k_\beta) \leq \delta^{k-L}\} \). Define the set \( \widetilde{T} = \{\beta : Q^k_\beta \cap \widetilde{E} \neq \emptyset, \beta \in I_k\} \) and \( e(Q^k_\beta) = \{x \in Q^k_\beta : d(x, G \setminus Q^k_\beta) \leq \delta^{k-L}\} \). Then by Proposition 2.2(i), we obtain \( \widetilde{E} = \bigcup_{\beta \in \widetilde{T}} e(Q^k_\beta) \).

Fix \( \beta \in \widetilde{T} \). There exists a point \( y_0 \in Q^k_\beta \cap \widetilde{E} \). By Proposition 2.2(iv), for every \( y \in Q^k_\beta \), we have
\[
d(y, z^k_\alpha) \leq d(y, y_0) + d(y_0, z^k_\alpha) \leq C_1 \delta^k + \delta^{k-L} + C_1 \delta^k \leq 3C_1 \delta^k.
\]
Hence $Q^k_\beta \subseteq B(z^k_\alpha, 3C_1\delta^k)$. By Proposition 2.2(ii), the 'dyadic cubes' $Q^k_\beta$ are disjoint, hence $\cup_{\beta \in \mathcal{I}} Q^k_\beta \subseteq B(z^k_\alpha, 3C_1\delta^k)$. Then by the first inequality and (2.1), we obtain

$$m(\bar{E}) \leq m(\cup_{\beta \in \mathcal{I}} e(Q^k_\beta)) \leq \sum_{\beta \in \mathcal{I}} m(e(Q^k_\beta)) \leq \sum_{\beta \in \mathcal{I}} C_2 \delta^{-L\eta} m(Q^k_\beta) \leq C_2 \delta^{-L\eta} m(B(z^k_\alpha, 3C_1\delta^k)) \leq C_2(K + 1)(3C_1/a_0)^r m(B(z^k_\alpha, a_0\delta^k)) \leq C_2(K + 1)(3C_1/a_0)^r m(Q^k_{\alpha}) \leq C_2(K + 1)(3C_1/a_0)^r m(Q^k_{\alpha}),$$

which completes the proof. \qed

Given a ball $B_{\delta^n}$ and a 'dyadic cube' $Q^k_{\alpha}$, define

$$\mathcal{H}(B_{\delta^n}, Q^k_{\alpha}) = \{ x \in Q^k_{\alpha} : B(x, \delta^n) \cap (Q^k_{\alpha})^c \neq \emptyset \}.$$

Set

$$n_0 = \max\{L_1 - L_0, 0\}.$$

**Lemma 2.7.** Let $n > n_0$ and $k > L_0$. Then for every 'dyadic cube' $Q^k_{\alpha}$, we have

$$m(\mathcal{H}(B_{\delta^n}, Q^k_{\alpha})) \leq C_2 \delta^{-kn} m(Q^k_{\alpha}).$$

**Proof.** We check at once that for every $x \in \mathcal{H}(B_{\delta^n}, Q^k_{\alpha})$, the distance $d(x, (Q^k_{\alpha})^c)$ is not bigger than $\delta^n$, and so

$$\mathcal{H}(B_{\delta^n}, Q^k_{\alpha}) \subseteq \{ x \in Q^k_{\alpha} : d(x, G \setminus Q^k_{\alpha}) \leq \delta^n \}.$$

Note that $k > L_0$ and $n > L_1 - L_0$, then by Lemma 2.6, we obtain

$$m(\mathcal{H}(B_{\delta^n}, Q^k_{\alpha})) \leq m(\{ x \in Q^k_{\alpha} : d(x, G \setminus Q^k_{\alpha}) \leq \delta^n \}) \leq C_2 \delta^{-kn} m(Q^k_{\alpha}),$$

which is the desired conclusion. \qed

**Remark 2.8.** Set $\tilde{\mathcal{H}}(B_{\delta^n}, Q^k_{\alpha}) = \{ x \in G \setminus Q^k_{\alpha} : B(x, \delta^n) \cap (Q^k_{\alpha})^c \neq \emptyset \}$. Similar to Lemma 2.7, by Lemma 2.6, we have

$$m(\tilde{\mathcal{H}}(B_{\delta^n}, Q^k_{\alpha})) \leq C_2 \delta^{-kn} m(Q^k_{\alpha}).$$

Set

$$K'_\epsilon = (2^\epsilon + 1)K + 2^\epsilon.$$

**Lemma 2.9.** Let $r \geq 2r_0$, then for every $s \in (0, r]$, we have

$$m(B(x, r + s)) - m(B(x, r - s)) \leq K'_\epsilon \left( \frac{s}{r} \right) \epsilon \ m(B(x, r)).$$

**Proof.** We only need to estimate the part $m(B(x, r)) - m(B(x, r - s))$, since by the $(\epsilon, r_0)$-annular decay property—(1.1), we have $m(B(x, r + s)) - m(B(x, r)) \leq K(s/r)^\epsilon m(B(x, r))$. In the following we split the $s$ into two cases $s \in (0, r/2)$ and $s \in [r/2, r]$. For $s \in (0, r/2)$, since $r \geq 2r_0$, then $r - s > r/2 \geq r_0$, applying the $(\epsilon, r_0)$-annular decay property again, we have

$$m(B(x, r)) - m(B(x, r - s)) \leq K \left( \frac{s}{r - s} \right)^\epsilon \ m(B(x, r - s)) \leq K \left( \frac{r}{r - s} \right)^\epsilon \left( \frac{s}{r} \right)^\epsilon \ m(B(x, r)) \leq K \left( \frac{r}{r - s} \right)^\epsilon \left( \frac{s}{r} \right)^\epsilon \ m(B(x, r)).$$
For $s \in [r/2, r)$, since $1/2 \leq s/r < 1$, so $2^s (s/r)^r \geq 1$ and $m(B(x, r)) - m(B(x, r - s)) \leq 2^s (s/r)^r m(B(x, r))$. Combining the two cases, we obtain

$$m(B(x, r + s)) - m(B(x, r - s)) \leq \left((2^s + 1)K + 2^s\right) m(B(x, r)),$$

and the lemma follows. \qed

Given a ball $B(x, r)$ and an integer $n$, we define

$$\mathcal{I}(B(x, r), n) = \cup_{\alpha} \{Q^n_{\alpha} \cap B(x, r) : Q^n_{\alpha} \cap \partial B(x, r) \neq \emptyset\}.$$

Set

$$n_1 = \min\{n \in \mathbb{N} : \delta^n \geq 2n_0\}, k_1 = \max\{k \in \mathbb{Z} : C_1 \delta^k \leq 1\}.$$

Unless otherwise stated, we assume that $n > n_1$, $k < k_1$ in the following three lemmas.

**Lemma 2.10.** For any $x \in G$, we have

$$\sup_{r \in [\delta^n, \delta^{n+1}]} \frac{m(\mathcal{I}(B(x, r), n + k))}{m(B(x, r))} \leq K_{\varepsilon} C_1 \delta^k.$$

**Proof.** Note that for $k < k_1$ and $r \in [\delta^n, \delta^{n+1}]$, $\mathcal{I}(B(x, r), n + k)$ is contained in the annulus $B(x, r + C_1 \delta^{n+k}) \setminus B(x, r - C_1 \delta^{n+k})$. For every $n > n_1$ and $r \in [\delta^n, \delta^{n+1}]$, Lemma 2.9 yields

$$m(\mathcal{I}(B(x, r), n + k)) \leq m\left(B(x, r + C_1 \delta^{n+k}) \setminus B(x, r - C_1 \delta^{n+k})\right) \leq K_{\varepsilon} \left(C_1 \delta^{n+k} \frac{r}{\delta^{n+k}}\right)^{\varepsilon} m(B(x, r)) \leq K_{\varepsilon} C_1 \delta^k m(B(x, r)),$$

which is the desired conclusion. \qed

To state the next technical lemmas, we need the following estimate (cf. [1, Theorem 3.5]). Recall that $A_r^G$ is the averaging operator given by (1.3).

**Proposition 2.11.** Let $r > 0$, then for every $p \in [1, \infty]$, we have $\|A_r^G\|_{L^p(G,m)} \rightarrow L^p(G,m) \leq D^{1/p}$.

With $\mathcal{I}(B(x, r), n + k)$ being defined as above, we define

$$M_{n+k}f(x) = \sup_{r \in [\delta^n, \delta^{n+1}]} \frac{1}{m(B(x, r))} \int_{\mathcal{I}(B(x, r), n + k)} f(y)dm(y).$$

**Lemma 2.12.** Let $p \in [1, \infty]$ and $p'$ be its conjugate index. There exists a constant $D_p = \left((K + 1)(\delta + 1)^r\right)^{1/p'} D^{1/p}(K_{\varepsilon} C_1)^{1/p'}$ such that for all $f \in L^p(G,m)$,

$$\|M_{n+k}f\|_{L^{p'}(G,m)} \leq D_p \delta^{k/p'} \|f\|_{L^p(G,m)}.$$

**Proof.** For $p = \infty$, using Lemma 2.10, the conclusion is obvious. For $p \in [1, \infty)$, fix $x \in G$, note that $\mathcal{I}(B(x, r), n + k) \subseteq B(x, r + C_1 \delta^{n+k})$. Then using the H"older inequality and Lemma 2.10, we obtain

$$M_{n+k}f(x) \leq \sup_{r \in [\delta^n, \delta^{n+1}]} \frac{m(\mathcal{I}(B(x, r), n + k))^{1/p'}}{m(B(x, r))} \left(\int_{\mathcal{I}(B(x, r), n + k)} |f(y)|^p dm(y)\right)^{1/p} \leq (K_{\varepsilon} C_1)^{1/p'} \delta^{k/p'} \left(\frac{1}{m(B(x, \delta^n))} \int_{B(x, \delta^{n+1} + C_1 \delta^{n+k})} |f(y)|^p dm(y)\right)^{1/p},$$
and by inequality (2.1), the above inequality yields
\[ M_{n+k} f(x) \leq (K C_1)\delta^{k/p'} \left( \frac{(K + 1)(\delta + 1)}{m(B(x, \delta^{n+1} + C_1 \delta^{n+k}))} \right) \int_{B(x, \delta^{n+1} + C_1 \delta^{n+k})} |f(y)|^p dm(y) \]^{1/p}

Then applying Proposition 2.11, we have
\[ \|M_{n+k} f\|_{L^p} \leq \left( (K + 1)(\delta + 1) \right)^{1/p} D^{1/p} (K C_1)^{1/p'} \delta^{k/p'} \|f\|_{L^p}. \]
and the lemma follows.

Given an annulus \( B(x, r) \setminus B(x, s) \) and integers \( k, n \), define
\[ \mathcal{I}(B(x, r) \setminus B(x, s), n) = \bigcup_{i} \left( Q^n_i \cap (B(x, r) \setminus B(x, s)) : Q^n_i \cap \partial(B(x, r) \setminus B(x, s)) \neq \emptyset \right), \]
and
\[ M_{n+k}^S f(x) = \sup_{\delta^n \leq r_0 < \ldots < r_{\delta^{n+1}}} \left( \frac{1}{m(B(x, r_i))} \int_{\mathcal{I}(B(x, r_i) \setminus B(x, r_{i-1}), n+k)} |f(y)|^2 dm(y) \right)^{1/2}. \]

**Lemma 2.13.** There exists a constant \( C_3 = (2(K + 1)D\delta C_1 K_r)^{1/2} \) such that for all \( f \in L^2(G, m), \)
\[ \|M_{n+k}^S f\|_{L^2(G, m)} \leq C_3 \delta^{k/2} \|f\|_{L^2(G, m)}. \]

**Proof.** Let \( x \in G \). Fixing \( \delta^n \leq r_{i-1} < r_i \leq \delta^{n+1} \), by the Cauchy-Schwarz inequality and Lemma 2.10, we have
\[
\frac{1}{m(B(x, r_i))} \int_{\mathcal{I}(B(x, r_i) \setminus B(x, r_{i-1}), n+k)} |f(y)|^2 dm(y) \]
\[
\leq \frac{m(\mathcal{I}(B(x, r_i), B(x, r_{i-1}), n+k))}{m(B(x, r_i))^2} \int_{\mathcal{I}(B(x, r_i) \setminus B(x, r_{i-1}), n+k)} |f(y)|^2 dm(y) \]
\[
\leq \frac{2C_1 K_r \delta^{k}}{m(B(x, \delta^n))} \int_{\mathcal{I}(B(x, r_i) \setminus B(x, r_{i-1}), n+k)} |f(y)|^2 dm(y),
\]
where we used the fact that \( \mathcal{I}(B(x, r_i) \setminus B(x, r_{i-1}), n+k) \subseteq \mathcal{I}(B(x, r_i), n+k) \cup \mathcal{I}(B(x, r_{i-1}), n+k) \) in the last inequality. From this, inequality (2.1) and the observation that \( \mathcal{I}(B(x, r_i) \setminus B(x, r_{i-1}), n+k) \subseteq B(x, \delta^{n+1}) \), it follows that
\[ M_{n+k}^S f(x) \leq \left( \frac{2C_1 K_r \delta^{k}}{m(B(x, \delta^{n+1}))} \right) \int_{B(x, \delta^{n+1})} |f(y)|^2 dm(y) \]
\[
\leq \left( \frac{2(K + 1)\delta^{k} C_1 K_r \delta^{k}}{m(B(x, \delta^{n+1}))} \right)^{1/2} \int_{B(x, \delta^{n+1})} |f(y)|^2 dm(y) \]
\[
\leq \left( \frac{2(K + 1)\delta^{k} C_1 K_r \delta^{k}}{m(B(x, \delta^{n+1}))} \right)^{1/2}. \]

Then using Proposition 2.11, we conclude
\[ \|M_{n+k}^S f\|_{L^2} \leq (2(K + 1)D\delta C_1 K_r)^{1/2} \delta^{k/2} \|f\|_{L^p}. \]

Let \( f \) be a locally integrable function on \( G \). We define
\[ S_n(f) = |A_{n} f - \mathbb{E}_n f|, \quad S_{V_n}(f) = V_2(A_{n} f : r \in [\delta^n, \delta^{n+1}] ). \]
Note that
\[
\frac{1}{m(B_r)} \int_{B_r} f(xy)dm(y) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y)dm(y). \]
Fixing $\delta^n \leq r_{i-1} < r_i < \delta^{n+1}$, we have
\[
\left| \frac{1}{m(B(x, r_i))} \int_{B(x, r_i)} f(y) dm(y) - \frac{1}{m(B(x, r_{i-1}))} \int_{B(x, r_{i-1})} f(y) dm(y) \right| \\
\leq \left| \frac{1}{m(B(x, r_i))} \int_{B(x, r_i) \setminus B(x, r_{i-1})} f(y) dm(y) \right| \\
+ \left| \left( \frac{1}{m(B(x, r_i))} - \frac{1}{m(B(x, r_{i-1}))} \right) \int_{B(x, r_{i-1})} f(y) dm(y) \right|.
\]

From the above observations and the triangle inequality of $\ell^2$-norm, we obtain
\[(2.8)\quad SV_n(f)(x) \leq SV_I(f)(x) + SV_{II}(f)(x),\]
where
\[
SV_I(f)(x) = \left( \sup_{\delta^n \leq r_0 \cdots \leq r_j \leq \delta^{n+1}} \sum_{i=1}^{j} \left| \frac{1}{m(B(x, r_i))} - \frac{1}{m(B(x, r_{i-1}))} \right| \int_{B(x, r_{i-1})} f(y) dm(y) \right)^{1/2}
\]
and
\[
SV_{II}(f)(x) = \left( \sup_{\delta^n \leq r_0 \cdots \leq r_j \leq \delta^{n+1}} \sum_{i=1}^{j} \left| \frac{1}{m(B(x, r_i))} - \frac{1}{m(B(x, r_{i-1}))} \right| \int_{B(x, r_{i-1})} f(y) dm(y) \right)^{1/2}.
\]

Moreover, since the $\ell^1$ norm is not less than the $\ell^2$ norm, then
\[(2.9)\quad SV_n(f)(x) \leq \sup_{\delta^n \leq r_0 \cdots \leq r_j \leq \delta^{n+1}} \sum_{i=1}^{j} \left| \frac{1}{m(B(x, r_i))} - \frac{1}{m(B(x, r_{i-1}))} \right| \int_{B(x, r_{i-1})} f(y) dm(y) \\
+ \sup_{\delta^n \leq r_0 \cdots \leq r_j \leq \delta^{n+1}} \sum_{i=1}^{j} \left| \frac{1}{m(B(x, r_i))} - \frac{1}{m(B(x, r_{i-1}))} \right| \int_{B(x, r_{i-1})} f(y) dm(y) \\
\leq \frac{1}{m(B(x, \delta^n))} \int_{B(x, \delta^{n+1}) \setminus B(x, \delta^n)} |f(y)| dm(y) \\
+ \left( \frac{1}{m(B(x, \delta^n))} - \frac{1}{m(B(x, \delta^{n+1}))} \right) \int_{B(x, \delta^{n+1})} |f(y)| dm(y) \\
\leq \frac{2}{m(B(x, \delta^n))} \int_{B(x, \delta^{n+1})} |f(y)| dm(y).
\]

By (2.1) and Proposition 2.11, the above discussions imply

**Lemma 2.14.** Let $p \in [1, \infty]$. There exists a constant $C_4 = 2(K+1)D^{1/p} \delta^s$ such that
\[
\sup_{n \geq n_0} \|S_n\|_{L^p(G,m) \to L^p(G,m)} \leq D^{1/p} + 1, \quad \sup_{n \geq n_0} \|SV_n\|_{L^p(G,m) \to L^p(G,m)} \leq C_4.
\]

Finally, we state the following version of the almost orthogonality principle, see e.g. [31] for a proof.

**Lemma 2.15.** Let $\{T_n\}_{n \in \mathbb{Z}}$ be a sequence of sub-linear operators from $L^2$ to $L^2$ on some $\sigma$-finite measure space. Let $\{u_n\}_{n \in \mathbb{Z}}$ and $\{v_n\}_{n \in \mathbb{Z}}$ be two sequences of $L^2$ functions. Let $N$ be a positive integer number. Assume that for $n \geq N$, there exists a sequence of positive constant $\{a(j)\}_{j \in \mathbb{Z}}$ with $w = \sum_{k \in \mathbb{Z}} a(k) < \infty$ such that
\[(2.10)\quad \|T_n(u_{k+n})\|_{L^2} \leq a(k)\|v_{k+n}\|_{L^2},\]
then
$$\sum_{n \geq N} \| T_n \left( \sum_{j \leq m} u_{k+n} \right) \|^2 \leq w^2 \sum_{n \in \mathbb{Z}} \| v_n \|^2.$$  

Furthermore, if $T_n := \max_{0<n \leq N} \| T_n \|^2 < \infty$, $T_n$ is strongly continuous for every $n \in \mathbb{N}$, $f = \sum_{n \in \mathbb{Z}} u_n$ and $\sum_{n \in \mathbb{Z}} \| v_n \|^2 \leq C \| f \|^2$, then we have
$$\sum_{n \in \mathbb{N}} \| T_n f \|^2 \leq (Cw^2 + NT_n^2) \| f \|^2.$$  

3. Strong type (2, 2) estimates

In this section, we prove that the square function $S(f)$ and the short variation operator $SV(f)$ are of strong type $(2, 2)$. We begin with the square function $S(f)$.

Proof of (1.7) in the case $p = 2$. Fix $f \in L^2(G, m)$. Recall that
$$S(f) = \left( \sum_{n > n_{\alpha}} |A_{\alpha}^n f - E_n f|^2 \right)^{1/2}.$$  

Set $n_2 = \max\{n_0, n_1, n_{\alpha}\}$. By Lemma 2.14, we first have
$$\left\| \left( \sum_{n_{\alpha} < n \leq n_2} |A_{\alpha}^n f - E_n f|^2 \right)^{1/2} \right\|_{L^2} \leq 2(n_2 - n_{\alpha}) \| f \|_{L^2}.$$  

Note that $f = \sum_{n \in \mathbb{N}} D_n f$. Set $T_n = A_{\alpha}^n - E_n$, $u_n = v_n = D_n f$, $N = n_2$ in Lemma 2.15; it suffices for our purposes to prove that for every $n > n_2$ and $k \in \mathbb{Z}$, there exists a sequence $a(k)$ of positive numbers with $w = \sum_{k \in \mathbb{Z}} a(k) < \infty$ such that
$$\| A_{\alpha}^n D_n + k f - E_n D_{n+k} f \|_{L^2} \leq a(k) \| D_{n+k} f \|_{L^2}. \quad (3.1)$$

In order to achieve our goal, we first set
$$k_2 = \max\{L_0 + 1, |k_1|, \min\{k \in \mathbb{N} : a_0 \delta^{k-1} > 1\}\},$$
and then divide $k$ into three cases $-k_2 \leq k \leq k_2$, $k > k_2$ and $k < -k_2$.

Case $-k_2 \leq k \leq k_2$. Applying Lemma 2.14 for function $D_n + k f$, we obtain
$$\| A_{\alpha}^n D_n + k f - E_n D_{n+k} f \|_{L^2} \leq (D^{1/2} + 1) \| D_{n+k} f \|_{L^2}.$$  

Case $k > k_2$. Note that $E_n D_{n+k} f = D_{n+k} f$, we write
$$\| A_{\alpha}^n D_n + k f - E_n D_{n+k} f \|^2_{L^2(G, m)} = \sum_{\alpha} \int_{Q_{\alpha}^{n+k-1}} |A_{\alpha}^n D_{n+k} f(x) - D_{n+k} f(x)|^2 dm(x).$$  

Fix a ‘dyadic cube’ $Q_{\alpha}^{n+k-1}$. Note that $D_{n+k} f(x)$ is a constant valued function on $Q_{\alpha}^{n+k-1}$, and so it follows that on $Q_{\alpha}^{n+k-1}$, $A_{\alpha}^n D_{n+k} f(x) - D_{n+k} f(x) \neq 0$ only if $x$ belongs to the set $\mathcal{H}(B_{\alpha}, Q_{\alpha}^{n+k-1})$. Applying Lemma 2.7, we have
$$m(\mathcal{H}(B_{\alpha}, Q_{\alpha}^{n+k-1})) \leq C_2 \delta^{(1-k)\eta} m(Q_{\alpha}^{n+k-1}). \quad (3.3)$$

Let $x \in \mathcal{H}(B_{\alpha}, Q_{\alpha}^{n+k-1})$ and define the set $I_x = \{ \beta : B(x, \delta^n) \cap Q_{\alpha}^{n+k-1} \neq \emptyset \}$. Setting $I_{\alpha} = \bigcup_{x \in \mathcal{H}(B_{\alpha}, Q_{\alpha}^{n+k-1})} I_x$. Fix $\beta \in I_{\alpha}$, there exist a point $x \in \mathcal{H}(B_{\alpha}, Q_{\alpha}^{n+k-1})$ and $y_0 \in Q_{\alpha}^{n+k-1}$ such that $y_0 \in B(x, \delta^n) \cap Q_{\alpha}^{n+k-1}$. Then for every $y \in Q_{\alpha}^{n+k-1}$, by Proposition 2.2(iv), we have
$$d(y, z_{\alpha}^{n+k-1}) \leq d(y, y_0) + d(y_0, x) + d(x, z_{\alpha}^{n+k-1}) \leq 2C_1 \delta^{n+k-1} + \delta^n.$$
It follows that for every $\beta \in I_\alpha$, $Q_\beta^{n+k-1} \subseteq B(z_\alpha^{n+k-1}, 2C_1\delta^{n+k-1} + \delta^n)$. By Proposition 2.2(ii), the ‘dyadic cubes’ $Q_\beta$ are disjoint, hence $\bigcup_{\beta \in I_\alpha} Q_\beta^{n+k-1} \subseteq B(z_\alpha^n, 2C_1\delta^{n+k-1} + \delta^n)$. On the other hand, by Proposition 2.1, $B(z_\alpha^{n+k-1}, 2C_1\delta^{n+k-1} + \delta^n)$ is covered by at most $D \log_2((2C_1+1)/a_0)+1$ balls of radius $a_0\delta^{n+k-1}$. Since by Proposition 2.2(iv), we know that each $Q_\beta^{n+k-1}$ contains a ball $B(z_\alpha^{n+k-1}, a_0\delta^{n+k-1})$, hence

\[(3.4)\quad \# \{I_\alpha\} \leq D \log_2((2C_1+1)/a_0)+1,\]

Here and in what follows, $\# \{A\}$ stands for the number of the set $A$. Set

$$m_\alpha = \max_{\beta \in I_\alpha} |\mathbb{D}_{n+k} f(z_\beta^{n+k-1})|,$$

where $z_\beta^{n+k-1}$ is the point associated with the corresponding ‘dyadic cube’ $Q_\beta^{n+k-1}$. Since $\mathbb{D}_{n+k} f(x)$ is a constant-value function on $Q_\beta^{n+k-1}$. It follows that

\[(3.5)\quad m_\alpha^2 \leq \sum_{\beta \in I_\alpha} \frac{1}{m(Q_\beta^{n+k-1})} \int_{Q_\beta^{n+k-1}} |\mathbb{D}_{n+k} f(x)|^2 dm(x).\]

By the above inequalities, Proposition 2.2(iv), (3.3) and (2.1), we conclude

\[
\sum_\alpha \int_{Q_\alpha^{n+k-1}} |A_\alpha' \mathbb{D}_{n+k} f(x) - \mathbb{D}_{n+k} f(x)|^2 dm(x) \\
= \sum_\alpha \int_{H(B_{\alpha^n}, Q_\alpha^{n+k-1})} |A_\alpha' \mathbb{D}_{n+k} f(x) - \mathbb{D}_{n+k} f(x)|^2 dm(x) \\
\leq 2 \sum_\alpha m_\alpha^2 m(H(B_{\alpha^n}, Q_\alpha^{n+k-1})) \\
\leq 2C_2 \delta^{(1-k)\eta} \sum_\alpha m_\alpha^2 m(Q_\alpha^{n+k-1}) \\
\leq 2C_2 \delta^{(1-k)\eta} \sum_\alpha \sum_{\beta \in I_\alpha} \frac{m(Q_\beta^{n+k-1})}{m(Q_\alpha^{n+k-1})} \int_{Q_\beta^{n+k-1}} |\mathbb{D}_{n+k} f(x)|^2 dm(x) \\
\leq 2C_2 \delta^{(1-k)\eta} \sum_\alpha \sum_{\beta \in I_\alpha} \frac{m(B(z_\alpha^{n+k-1}, 2C_1\delta^{n+k-1} + \delta^n))}{m(B(z_\beta^{n+k-1}, a_0\delta^{n+k-1}))} \int_{Q_\beta^{n+k-1}} |\mathbb{D}_{n+k} f(x)|^2 dm(x) \\
\leq 2C_2 (K + 1)((2C_1 + 1)/a_0)^\eta D \log_2((2C_1+1)/a_0)+1 \delta^{(1-k)\eta} \int_G |\mathbb{D}_{n+k} f(x)|^2 dm(x).\]

**Case** $k < -k_2$. Since $E_n \mathbb{D}_{n+k} f = 0$ and $\int_{Q_\alpha^{n+k-1}} \mathbb{D}_{n+k} f = 0$ for every $Q_\alpha^{k+n} \in \mathcal{F}_{k+n}$, thus for any $x \in G$,

\[
|A_\alpha' \mathbb{D}_{n+k} f(x)| = \left| \frac{1}{m(B(x, \delta^n))} \sum_\alpha \int_{Q_\alpha^{n+k-1} \cap B(x, \delta^n)} \mathbb{D}_{n+k} f(y) dm(y) \right| \\
= \left| \frac{1}{m(B(x, \delta^n))} \int_{B(x, \delta^n), n+k} \mathbb{D}_{n+k} f(y) dm(y) \right| \\
\leq M_{n+k} \mathbb{D}_{n+k} f(x).
\]

Form this and Lemma 2.12, we obtain

\[
\|A_\alpha' \mathbb{D}_{n+k} f\|_{L^2} \leq \|M_{n+k} \mathbb{D}_{n+k} f\|_{L^2} \leq D_2 \delta^{k_2/2} \|\mathbb{D}_{n+k} f\|_{L^2}.
\]
Therefore, for every \( n > n_2 \), we determine
\[
a(k) = \begin{cases} 
    2C(2)(1)((2C_1 + 1)/\alpha)D^{\log_2((2C_1 + 1)/\alpha)+1}\delta^{\alpha} & k > k_2, \\
    D^{1/2} + 1 & -k_2 \leq k \leq k_2, \\
    D_2\delta^{k/2} & k < -k_2,
\end{cases}
\]
and \( \sum_{k \in \mathbb{Z}} a(k) < \infty \), which completes the proof. \( \square \)

The proof of strong type \((2,2)\) estimate for \( SV(f) = \left( \sum_{n \geq n_{\alpha}} |SV_n(f)|^2 \right)^{1/2} \) is similar in spirit to that of the square function \( S(f) \).

**Proof of (1.10) in the case \( p = 2 \).** First by Lemma 2.14, we have
\[
\left\| \left( \sum_{n \geq n_{\alpha}} |SV_n(f)|^2 \right)^{1/2} \right\|_{L^2} \leq 2(n_2 - n_{\alpha} + 1)C_4\|f\|_{L^2}.
\]
Set \( T_n = SV_n, u_n = v_n = D_n, N = n_2 \) in Lemma 2.15. For every \( n > n_2 \), we divide \( k \) into three cases \( -k_2 \leq k \leq k_2, k > k_2 \) and \( k < -k_2 \), where \( k_2 \) is defined in (3.2).

**Case** \(-k_2 \leq k \leq k_2\). For function \( D_{n+k}f \), using Lemma 2.14 again, we have
\[
\|V_2(A_0' D_{n+k}f : r \in [\alpha^n, \alpha^{n+1}] )\|_{L^2} \leq C_4 \|D_{n+k}f\|_{L^2}.
\]

**Case** \( k > k_2 \). We write
\[
\|V_2(A_0' D_{n+k}f : r \in [\alpha^n, \alpha^{n+1}] )\|_{L^2(G,m)}^2 = \sum_{\alpha} \int_{Q^{n+k-1}_{\alpha}} |V_2(A_0' D_{n+k}f(x) : r \in [\alpha^n, \alpha^{n+1}] )|^2 dm(x).
\]
Fix a ‘dyadic’ cube \( Q^{n+k-1}_{\alpha} \). Recall that \( D_{n+k}f(x) \) is a constant valued function on \( Q^{n+k-1}_{\alpha} \), it follows that for every \( \delta^n \leq r_i < r_{i+1} < \delta^{n+1} \) and \( x \in Q^{n+k-1}_{\alpha} \), \( |A_0' D_{n+k}f(x) - A_0' D_{n+k}f(x)| \neq 0 \) only if there exists at least one ball \( B(x, r_i) \) or \( B(x, r_{i+1}) \) intersecting with \( Q^{n+k-1}_{\alpha} \). So \( V_2(A_0' D_{n+k}f : r \in [\alpha^n, \alpha^{n+1}] ) \) is supported on \( \mathcal{H}(B^{n+1}_{\alpha^n}, Q^{n+k-1}_{\alpha}) \). By Lemma 2.7, we have
\[
m(\mathcal{H}(B^{n+1}_{\alpha^n}, Q^{n+k-1}_{\alpha})) \leq C_2(2-\delta^{k/2})m(\alpha^n).
\]
On the other hand, by (2.8), we know that \( V_2(A_0' D_{n+k}f(x) : r \in [\alpha^n, \alpha^{n+1}] ) \) is controlled by the sum of \( SV_1(D_{n+k}f(x) \) and \( SV_1(D_{n+k}f(x) \), where
\[
SV_1(D_{n+k}f(x)) = \left( \sup_{\delta^n \leq r_0 < \cdots < r_j < \delta^{n+1}} \sum_{i=1}^j \frac{1}{m(B(x, r_i))} \left| \int_{B(x,r_i) \setminus B(x,r_{i-1})} D_{n+k}f(y) dm(y) \right|^2 \right)^{1/2}
\]
and
\[
SV_1(D_{n+k}f(x)) = \left( \sup_{\delta^n \leq r_0 < \cdots < r_j < \delta^{n+1}} \sum_{i=1}^j \left( \frac{1}{m(B(x, r_i-1))} - \frac{1}{m(B(x, r_{i-1}))} \right) \int_{B(x,r_{i-1})} D_{n+k}f(y) dm(y) \right)^{1/2}.
\]
Let \( x \in \mathcal{H}(B^{n+1}_{\alpha^n}, Q^{n+k-1}_{\alpha}) \) and set \( I_x = \{ \beta : B(x, \delta^{n+1}) \cap Q^{n+k-1}_{\beta} \neq \emptyset \} \). We define the set \( I_\alpha = \cup_{x \in \mathcal{H}(B^{n+1}_{\alpha^n}, Q^{n+k-1}_{\alpha})} I_x \). Similar to (3.4), we have \#\{I_\alpha\} \leq D^{\log_2((2C_1 + 1)/\alpha)+1}. Write
\[
m_\alpha = \max_{\beta \in I_\alpha} |D_{n+k}f(\alpha^n)|.
\]
It follows from (2.1) that

$$SV_I(\mathbb{D}_{n+k})^2(x)$$

$$\leq \frac{m(B(x, \delta^{n+1}) \setminus B(x, \delta^n))}{m(B(x, \delta^n))} \sup_{\delta^n \leq r_0 < \cdots < r_j < \delta^{n+1}} \sum_{i=1}^{j} \int_{B(x, r_i) \setminus B(x, r_{i-1})} |\mathbb{D}_{n+k} f(y)|^2 dm(y)$$

$$\leq \left( \frac{m(B(x, \delta^{n+1}) \setminus B(x, \delta^n))}{m(B(x, \delta^n))} \right)^2 m_\alpha^2 \leq (K + 1)^2 \delta^{2e} m_\alpha^2$$

and

$$SV_{II}(\mathbb{D}_{n+k})^2(x)$$

$$\leq m_\alpha^2 m(B(x, \delta^{n+1}))^2 \left( \sup_{\delta^n \leq r_0 < \cdots < r_j < \delta^{n+1}} \sum_{i=1}^{j} \frac{1}{m(B(x, r_{i-1}))} - \frac{1}{m(B(x, r_i))} \right)^2$$

$$\leq m_\alpha^2 m(B(x, \delta^{n+1}))^2 \left( \frac{1}{m(B(x, \delta^n))} - \frac{1}{m(B(x, \delta^{n+1}))} \right)^2 \leq (K + 1)^2 \delta^{2e} m_\alpha^2.$$

Combining the above two inequalities with (3.5) and (3.6), we have

$$\int_{Q_n^{k+1}} |V_2(A_{i}, \mathbb{D}_{n+k} f(x)) : r \in [\delta^n, \delta^{n+1})|^2 dm(x)$$

$$= \int_{\mathcal{H}(B_{\delta^{n+1}}, Q_n^{k+1})} |V_2(A_{i}, \mathbb{D}_{n+k} f(x)) : r \in [\delta^n, \delta^{n+1})|^2 dm(x)$$

$$\leq 2 \int_{\mathcal{H}(B_{\delta^{n+1}}, Q_n^{k+1})} SV_I(\mathbb{D}_{n+k})^2(x) + SV_{II}(\mathbb{D}_{n+k})^2(x) dm(x)$$

$$\leq 4(K + 1)^2 \delta^{2e} m(\mathcal{H}(B_{\delta^{n+1}}, Q_n^{k+1})) m_\alpha^2$$

$$\leq 4C_2(K + 1)^2 \delta^{2e} \delta^{(2-k)\eta} \sum_{\beta \in I_n} \frac{m(Q_n^{k+1})}{m(Q_{\beta}^{k+1})} \int_{Q_n^{k+1}} |\mathbb{D}_{n+k} f(x)|^2 dm(x)$$

$$\leq 4C_2(K + 1)^2 ((2C_1 + 1)/a_0)^{x} \delta^{2e} \delta^{(2-k)\eta} \sum_{\beta \in I_n} \int_{Q_n^{k+1}} |\mathbb{D}_{n+k} f(x)|^2 dm(x),$$

and summing over all $\alpha$ shows,

$$\|V_2(A_{i}, \mathbb{D}_{n+k} f : r \in [\delta^n, \delta^{n+1})\|_{L^2}$$

$$\leq \left( 4C_2(K + 1)^2 ((2C_1 + 1)/a_0)^{x} \delta^{2e} \delta^{(2-k)\eta} \right)^{1/2} \|\mathbb{D}_{n+k} f\|_{L^2}.$$
so $SV_I(\mathbb{D}_{n-k} f)(x) = M_{n-k}^\infty(\mathbb{D}_{n-k} f)(x)$. Moreover, using the estimate (2.1) and the fact that the $\ell^1$-norm is greater than the $\ell^2$-norm, we obtain

$$SV_I(\mathbb{D}_{n-k} f)(x)$$

$$\leq \sup_{\delta^n \leq r_0 < \ldots < r_j < \delta^{n+1}} \sum_{i=1}^J \left| \frac{1}{m(B(x,r_i))} - \frac{1}{m(B(x,r_{i-1}))} \right| \int_{B(x,r_{i-1})} \mathbb{D}_{n-k} f(y) dm(y)$$

$$\leq (K + 1)\delta^\epsilon \sup_{\delta^n \leq r_0 < \ldots < r_j < \delta^{n+1}} \sum_{i=1}^J \frac{m(B(x,\delta^{n+1}))}{m(B(x,\delta^n))} - \frac{m(B(x,\delta^{n+1}))}{m(B(x,\delta^n))} \mathbb{D}_{n-k}(\mathbb{D}_{n-k} f)(x)$$

$$\leq (K + 1)^2 \delta^{2\epsilon} M_{n-k}(\mathbb{D}_{n-k} f)(x).$$

Using Lemma 2.13 and Lemma 2.12, respectively, we have

$$\|V_2(A'_{\mathbb{D}_{n-k} f} : r \in [\delta^n, \delta^{n+1}])\|_{L^2} \leq \|SV_I(\mathbb{D}_{n-k} f)\|_{L^2} + \|SV_I(\mathbb{D}_{n-k} f)\|_{L^2}$$

$$\leq \|M_{n-k}^\infty(\mathbb{D}_{n-k} f)\|_{L^2} + (K + 1)^2 \delta^{2\epsilon} \|M_{n-k}(\mathbb{D}_{n-k} f)\|_{L^2}$$

$$\leq (C_3 + (K + 1)^2 \delta^{2\epsilon} D_2) \delta^{k/2} \|\mathbb{D}_{n-k} f\|_{L^2}.$$

Hence, for every $n > n_2$, we determine

$$a(k) = \begin{cases} 
\left(4C_2(K + 1)^2((2C_1 + 1)/a_0)^4 \log_2((2C_1 + 1)/a_0) + 1/2\delta(\epsilon+\eta)\right)^{1/2} \delta^{-kn/2}, & k > k_2, \\
(3 + (K + 1)^2 \delta^{2\epsilon} D_2) \delta^{k/2}, & k < -k_2,
\end{cases}
$$

and $\sum_k a(k) < \infty$, which completes the proof. \qed

4. Weak type $(1,1)$ estimates

In this section, we prove that the square function $f \to S(f)$ and the short variation operator $f \to SV(f)$ are of weak type $(1,1)$.

Under the conditions (1.1) and (1.2), the space $(G, d, m)$ might not be a doubling measure space and the usual Calderón-Zygmund decomposition does not work any more. We need the following version of Calderón-Zygmund decomposition, which is motivated by Gundy’s decomposition from martingale theory. This decomposition was constructed in [44] for non-doubling measure on $\mathbb{R}^d$, and construction works without alterations for the space $(G, d, m)$.

Let $f : G \to \mathbb{C}$ be a locally integrable function. The ‘dyadic’ maximal function is defined by

$$M_df(x) = \sup_{k \in \mathbb{Z}} \mathbb{E}_k(\lfloor f \rfloor)(x).$$

Given a ‘dyadic cube’ $Q^k_\alpha$, let $\tilde{Q}^k_\alpha = \{y \in G : d(y, z^k_\alpha) \leq 3C_1 \delta^{k+1}\}$ and $\hat{Q}^k_\alpha$ be its parent. We denote $z^k_\alpha$ the corresponding point of $Q^k_\alpha$. Set $\langle f \rangle_{Q^k_\alpha} = \frac{1}{m(\tilde{Q}^k_\alpha)} \int_{\tilde{Q}^k_\alpha} f$.\]

**Lemma 4.1.** Let $f \in L^1(G, m)$, and $\gamma > 0$, let

$$\{ x \in G : M_df(x) > \gamma \} = \cup_{k \in \mathbb{Z}} \Omega_k,$$

where

$$\Omega_k = \{ x \in G : \mathbb{E}_k(\lfloor f \rfloor)(x) > \gamma, \mathbb{E}_j(\lfloor f \rfloor)(x) \leq \gamma, j > k \}.$$

Then we decompose

$$\Omega = \cup_{k \in \mathbb{Z}} \Omega_k = \cup_{k \in \mathbb{Z}} (\cup_{\alpha \in \Lambda_k} Q^k_\alpha).$$
into a disjoint union of maximal ‘dyadic cubes’ $Q_k^-$, where $\{\Lambda_k\}_k$ stands for the sequence of the corresponding index set. Let

$$g(x) = f \mathbf{1}_{\Omega} + \sum_{k}^{\sum_{\alpha \in \Lambda_k}} \langle f \rangle_{Q_k^+} \mathbf{1}_{Q_k^+}(x) + \sum_{k}^{\sum_{\alpha \in \Lambda_k}} \langle f \rangle_{Q_k^-} - \langle f \rangle_{Q_k^+} \frac{m(Q_k^+)}{m(Q_k^-)} \mathbf{1}_{Q_k^-}(x),$$

$$b(x) = \sum_{k} b_k = \sum_{k}^{\sum_{\alpha \in \Lambda_k}} b_k^\alpha(x) = \sum_{k}^{\sum_{\alpha \in \Lambda_k}} (f(x) - \langle f \rangle_{Q_k^-}) \mathbf{1}_{Q_k^-}(x),$$

$$\xi(x) = \sum_{k} \xi_k = \sum_{k}^{\sum_{\alpha \in \Lambda_k}} \xi_k^\alpha(x) = \sum_{k}^{\sum_{\alpha \in \Lambda_k}} (\langle f \rangle_{Q_k^+} - \langle f \rangle_{Q_k^-}) \left( \mathbf{1}_{Q_k^+}(x) - \frac{m(Q_k^+)}{m(Q_k^-)} \mathbf{1}_{Q_k^-}(x) \right).$$

Then

(i) $f = g + b + \xi$;
(ii) let $p \in [1, \infty)$ and $m = [p] + 1$, the function $g$ satisfies

$$\|g\|_p \leq 3 \cdot 2^p (m!)^{\frac{1}{m-1}} \gamma^{p-1} \|f\|_{L^1};$$

(iii) the function $b$ satisfies

$$\int_G b_k^\alpha = 0, \quad \|b\|_{L^1} = \sum_{k}^{\sum_{\alpha \in \Lambda_k}} \|b_k^\alpha\|_{L^1} \leq 2\|f\|_{L^1};$$

(iv) the function $\xi$ satisfies

$$\int_G \xi_k^\alpha = 0, \quad \|\xi\|_{L^1} = \sum_{k}^{\sum_{\alpha \in \Lambda_k}} \|\xi_k^\alpha\|_{L^1} \leq 4\|f\|_{L^1}.$$

With the above decomposition, the almost orthogonality principle which was exploited well in [30] or [28], doesn’t seem to work. Thereby, we have to provide another method to achieve our goal. A new input is the observation that the operators $S$ and $SV$ are essentially dyadic in the small scale—small cubes versus large balls.

We first deal with the operator $S$.

Proof of (1.8). Fix $f \in L^1(G, m)$ and $\gamma > 0$. Keeping the notations in Lemma 4.1, we get $f = g + b + \xi$. We first have

$$m\{x \in G : S(f)(x) > \gamma\} \leq m\{x \in G : S(g)(x) > \gamma/3\}$$

$$+ m\{x \in G : S(b)(x) > \gamma/3\} + m\{x \in G : S(\xi)(x) > \gamma/3\}.$$

By the $L^2$-boundedness of the operator $S$ and Lemma 4.1(ii), the first term on the right side is controlled by

$$m\{x \in G : S(g)(x) > \gamma/3\} \leq \frac{9}{\gamma^2} \int_G |S(g)(x)|^2 dm(x) \leq \frac{9\gamma^2}{\gamma^2} \int_G |g(x)|^2 dm(x)$$

$$\leq \frac{108\sqrt{6\gamma^2}}{\gamma} \int_G |f(x)| dm(x).$$

It remains to handle the other two terms, we set

$$k_3 = \min\{k : a_0 \delta^k > r_0\}, \quad \tilde{\Omega} = \left( \cup_{k < k_3} \Omega_k \right) \cup \left( \cup_{k \geq k_3} \tilde{\Omega}_k \right).$$
where $\tilde{\Omega}_k = \cup_{\alpha \in \Lambda_k} \tilde{Q}^k_{\alpha}$. By (2.1) and the fact that $M_d$ is of weak type $(1, 1)$, $m(\tilde{\Omega})$ is controlled by

\[
m(\tilde{\Omega}) = \sum_{k < k_3} m(\Omega_k) + \sum_{k \geq k_3} m(\tilde{\Omega}_k) \leq \sum_{k < k_3} \sum_{\alpha \in \Lambda_k} m(Q^k_{\alpha}) + \sum_{k \geq k_3} \sum_{\alpha \in \Lambda_k} m(\tilde{Q}^k_{\alpha}) \leq \sum_{k < k_3} \sum_{\alpha \in \Lambda_k} m(Q^k_{\alpha}) + \sum_{k \geq k_3} \sum_{\alpha \in \Lambda_k} m(\tilde{Q}^k_{\alpha}) \leq (K + 1)(3C_1\delta/a_0)\sum_{k < k_3} \sum_{\alpha \in \Lambda_k} m(Q^k_{\alpha}) \leq (K + 1)(3C_1\delta/a_0)^\gamma \|f\|_{L^1}.
\]

We now focus on the term $m\{(x \in G \setminus \tilde{\Omega} : S(\xi)(x) > \gamma/3)\}$. Recall that $n_2 = \max\{n_0, n_1\}$, we decompose

\[
m\{(x \in G \setminus \tilde{\Omega} : S(\xi)(x) > \gamma/3)\} \leq m\{(x \in G \setminus \tilde{\Omega} : \sum_{n_0 < n \leq n_2} |A_{\delta^n}\xi(x) - \mathbb{E}_{\mathcal{R}_n}\xi(x)|^2)^{1/2} > \gamma/6\} + m\{(x \in G \setminus \tilde{\Omega} : \sum_{n > n_2} |A_{\delta^n}\xi(x) - \mathbb{E}_{\mathcal{R}_n}\xi(x)|^2)^{1/2} > \gamma/6\}.
\]

For the first part of the right side of the above inequality, using Lemma 2.14 and Lemma 4.1(iv), we have

\[
m\{(x \in G \setminus \tilde{\Omega} : \sum_{n_0 < n \leq n_2} |A_{\delta^n}\xi(x) - \mathbb{E}_{\mathcal{R}_n}\xi(x)|^2)^{1/2} > \gamma/6\} \leq \sum_{n_0 < n \leq n_2} \frac{6}{\gamma} \int_{G \setminus \tilde{\Omega}} |A_{\delta^n}\xi(x) - \mathbb{E}_{\mathcal{R}_n}\xi(x)| dm(x) \leq \frac{12(n_2 - n_0)}{\gamma} \|\xi\|_{L^1(G, m)} \leq \frac{48(n_2 - n_0)}{\gamma} \|f\|_{L^1(G, m)}.
\]

For the second part, we first have

\[
m\{(x \in G \setminus \tilde{\Omega} : \sum_{n > n_2} |A_{\delta^n}\xi(x) - \mathbb{E}_{\mathcal{R}_n}\xi(x)|^2)^{1/2} > \gamma/6\} \leq \frac{6}{\gamma} \int_{G \setminus \tilde{\Omega}} \left( \sum_{n > n_2} |A_{\delta^n}\xi(x) - \mathbb{E}_{\mathcal{R}_n}\xi(x)|^2 \right)^{1/2} dm(x) \leq \frac{6}{\gamma} \int_{G \setminus \tilde{\Omega}} \sum_{n > n_2} |A_{\delta^n}\xi(x) - \mathbb{E}_{\mathcal{R}_n}\xi(x)| dm(x) \leq \frac{6}{\gamma} \sum_{n > n_2} \sum_{\kappa \in \mathbb{Z}} \sum_{\alpha \in \Lambda \land \kappa} |A_{\delta^n}\xi_{\alpha}^{n+k}(x) - \mathbb{E}_{\mathcal{R}_n}\xi_{\alpha}^{n+k}(x)| dm(x).
\]

We now deal with integral term $\int_{G \setminus \tilde{\Omega}} |A_{\delta^n}\xi_{\alpha}^{n+k}(x) - \mathbb{E}_{\mathcal{R}_n}\xi_{\alpha}^{n+k}(x)| dm(x)$. Let

\[k_4 = \max\{k : k < 0 \land C_1\delta^{k+1} 1\}, \quad k_5 = \max\{|k_4|, k_3\}.
\]

We split the $k$ into three cases: $-k_5 \leq k \leq k_5$, $k > k_5$, and $k < -k_5$. We will prove that

\[
\int_{G \setminus \tilde{\Omega}} |A_{\delta^n}\xi_{\alpha}^{n+k}(x) - \mathbb{E}_{\mathcal{R}_n}\xi_{\alpha}^{n+k}(x)| dm(x) = a(k)\|\xi_{\alpha}^{n+k}\|_{L^1(G, m)}.
\]
where

\[ a(k) = \begin{cases} 
(D + 1), & -k_3 \leq k \leq k_3; \\
0, & k > k_3; \\
6^*(K + 1)^2C^n D K_\varepsilon \delta^{sk}, & k < -k_3.
\end{cases} \]

Assume this result momentarily. Combining (4.1), (4.2) with Lemma 4.1(iv), one can see immediately that

\[
m \left( \{ x \in G \setminus \hat{\Omega} : \left( \sum_{n \geq n_2} |A_{n}^* \xi(x) - \mathbb{E}_n \xi(x)|^2 \right)^{1/2} > \gamma / 6 \} \right) \leq \frac{6}{\gamma} \sum_{n > n_2} \sum_{k \in \mathbb{Z} \cap \Lambda_{n+k}} a(k) \| \xi_{n+k}^\alpha \|_{L^1(G,m)}
\]

\[
\leq \frac{6}{\gamma} \left( \sum_{k \in \mathbb{Z}} a(k) \right) \left( \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \Lambda} \| \xi_n^\alpha \|_{L^1(G,m)} \right) \leq \frac{C_\delta \alpha}{\gamma} \| f \|_{L^1(G,m)}.
\]

We now prove (4.2).

**Case** \(-k_3 \leq k \leq k_3\). Using Lemma 2.14 for function \( \xi_{n+k}^\alpha \), we have

\[
\int_{G \setminus \hat{\Omega}} |A_{n}^* \xi_{n+k}^\alpha(x) - \mathbb{E}_n \xi_{n+k}^\alpha(x)| dm(x) \leq \int_G |A_{n}^* \xi_{n+k}^\alpha(x) - \mathbb{E}_n \xi_{n+k}^\alpha(x)| dm(x)
\]

\[
\leq (D + 1) \| \xi_{n+k}^\alpha \|_{L^1(G,m)}.
\]

**Case** \(k > k_3\). Note that \( \xi_{n+k}^\alpha \) is supported on \( \hat{Q}_{n+k}^\alpha \). Recall that \( \hat{Q}_{n+k}^\alpha = \{ y \in G : d(y, z_{n+k}^\alpha) \leq 3C_1 \delta_{n+k+1} \} \). Let \( y \in \hat{Q}_{n+k}^\alpha \), we have \( d(y, z_{n+k}^\alpha) \leq d(y, z_{n+k+1}^\alpha) + d(z_{n+k}^\alpha, z_{n+k+1}^\alpha) \leq 2C_1 \delta_{n+k+1} \). This gives \( \hat{Q}_{n+k}^\alpha \subseteq Q_{n+k}^\alpha \).

Fix \( x \in G \setminus \hat{\Omega} \). There exists an unique ‘dyadic cube’ \( Q_{n+k}^\alpha \) containing \( x \). We claim that

\[
B(x, \delta^n) \cap \hat{Q}_{n+k}^\alpha = \emptyset, \quad Q_{n+k}^\alpha \cap \hat{Q}_{n+k}^\alpha = \emptyset.
\]

Since

\[
A_{n}^* \xi_{n+k}^\alpha(x) = \frac{1}{m(B(x, r))} \int_{B(x, r) \cap \hat{Q}_{n+k}^\alpha} \xi_{n+k}^\alpha(y) dm(y)
\]

and

\[
\mathbb{E}_n \xi_{n+k}^\alpha(x) = \frac{1}{m(Q_{\beta}^\alpha)} \int_{Q_{\beta}^\alpha \cap \hat{Q}_{n+k}^\alpha} \xi_{n+k}^\alpha(y) dm(y),
\]

it follows from the claim that \( A_{n}^* \xi_{n+k}^\alpha = \mathbb{E}_n \xi_{n+k}^\alpha = 0 \).

We now prove the claim. If \( Q_{\beta}^\alpha \cap \hat{Q}_{n+k}^\alpha \neq \emptyset \), by Proposition 2.2(ii), it follows that \( x \in Q_{\beta}^\alpha \subseteq \hat{Q}_{n+k}^\alpha \subseteq Q_{n+k}^\alpha \); a contradiction with \( x \in G \setminus \hat{\Omega} \). On the other hand, if there exists a point \( z \in B(x, \delta^n) \cap Q_{n+k}^\alpha \), then \( d(z, z_{n+k+1}^\alpha) \leq d(x, z) + d(z, z_{n+k+1}^\alpha) \leq \delta^n + C_1 \delta_{n+k+1} < 2C_1 \delta_{n+k+1} \), and so it follows that

\[
d(x, z_{n+k}^\alpha) \leq d(x, z_{n+k+1}^\alpha) + d(z_{n+k}^\alpha, z_{n+k+1}^\alpha) < 3C_1 \delta_{n+k+1},
\]

contrary to \( x \in G \setminus \hat{\Omega} \) and (4.3) is proved. This gives the conclusion.

**Case** \(k < -k_3\). We also have

\[
\mathbb{E}_n \xi_{n+k}^\alpha(x) = 0, \forall x \in G \setminus \hat{\Omega}.
\]

Indeed, let \( x \in Q_{\beta}^\alpha \), if \( Q_{\beta}^\alpha \cap \hat{Q}_{n+k}^\alpha \neq \emptyset \), then by Proposition 2.2(ii), it follows that \( \hat{Q}_{n+k}^\alpha \subseteq Q_{\beta}^\alpha \)

and

\[
\mathbb{E}_n \xi_{n+k}^\alpha(x) = \int_{\hat{Q}_{n+k}^\alpha} \xi_{n+k}^\alpha(y) dm(y) = 0.
\]
Given a ball $B(x, r)$ and ‘dyadic cube’ $Q_0^k$, we define the set $\mathcal{I}(B(x, r), Q_0^k) = \{Q_0^k \cap B(x, r) : Q_0^k \cap \partial B(x, r) \neq \emptyset\}$. Since $\varepsilon_{\alpha}^{n+k}$ is supported on $\tilde{Q}_{\alpha}^{n+k}$ and $\int_{\tilde{Q}_{\alpha}^{n+k}} \varepsilon_{\alpha}^{n+k} = 0$, it follows that

$$|A^{\delta_{\alpha}} \varepsilon_{\alpha}^{n+k}(x)| = \left| \frac{1}{m(B(x, \delta_{\alpha}))} \int_{\mathcal{I}(B(x, \delta_{\alpha}), \tilde{Q}_{\alpha}^{n+k})} \varepsilon_{\alpha}^{n+k}(y) dm(y) \right| \leq \frac{1}{m(B(x, \delta_{\alpha}))} \int_{\delta_{\alpha} - C_1 \delta_{\alpha}^{n+k+1} \leq d(x, y) \leq \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}} |\varepsilon_{\alpha}^{n+k}(y)| dm(y).$$

Then by the above estimate we obtain

$$\int_G |A^{\delta_{\alpha}} \varepsilon_{\alpha}^{n+k}(x)| dm(x) \leq \int_G \frac{1}{m(B(x, \delta_{\alpha}))} \int_{\delta_{\alpha} - C_1 \delta_{\alpha}^{n+k+1} \leq d(x, y) \leq \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}} |\varepsilon_{\alpha}^{n+k}(y)| dm(y) dm(x) \leq \int_G \{\varepsilon_{\alpha}^{n+k}(y)\} \cdot \int_G \frac{1}{m(B(x, \delta_{\alpha}))} \frac{1}{m(B(y, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}))} dm(x) dm(y)$$

Set $A(y) = B(y, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}) \setminus B(y, \delta_{\alpha} - C_1 \delta_{\alpha}^{n+k+1})$. Fix $y \in G$. By (2.1), we have

$$\frac{1}{m(B(x, \delta_{\alpha}))} \leq 2(K + 1) \frac{1}{m(B(x, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}))}. \frac{1}{m(B(y, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}))}$$

By the same argument as in the proof of [1, Theorem 3.5], we have the following properties. For any $0 < \varepsilon < 1$, there exist $M$-points $\{u_i : 1 \leq i \leq M\}$ with $M \leq D$ in $B(y, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1})$ such that $B(y, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}) \setminus \bigcup_{i=1}^{M} B(u_i, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}) = \emptyset$. Fix $x \in B(y, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1})$, and let $j$ be the first index such that

$$x \in B(u_j, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}), \quad m(B(u_j, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1})) \leq (1 + \varepsilon) m(B(x, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1})).$$

By the above discussions, we first have

$$\int_G \frac{1}{m(B(x, \delta_{\alpha}))} dm(x) \leq (K + 1) \int_G \frac{1}{m(B(u_j, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}))} \frac{1}{m(B(x, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}))} dm(x) \leq (K + 1) \int_G \frac{1}{m(B(u_j, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}))} dm(x) \leq (K + 1) \sum_{i=1}^{M} \frac{1 + \varepsilon}{m(B(u_i, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}))} dm(x)$$

Note that each $u_i \in B(y, \delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1})$, then $B(y, \delta_{\alpha}) \subseteq B(u_i, 2\delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1})$. It follows from (2.1) that

$$\frac{m(B(y, \delta_{\alpha}))}{m(B(u_i, \delta_{\alpha}))} \leq \frac{m(B(u_i, 2\delta_{\alpha} + C_1 \delta_{\alpha}^{n+k+1}))}{m(B(u_i, \delta_{\alpha}))} \leq 3(K + 1).$$
Moreover, by Lemma 2.9, we have $m(A(y))/m(B(y,\delta^n)) \leq K_n C_1^k \delta^k$. Combining these two estimates with (4.5), we conclude

\begin{equation}
\int_G \frac{1}{m(B(x,\delta^n))} \, dm(x) \leq (1+\varepsilon)6'(K+1)^2C_1^k DK_1 \delta^k.
\end{equation}

Combining (4.6) with (4.4), and then letting $\varepsilon \to 0$, we obtain $\|A_n^\alpha \xi_n^{\alpha+k}\|_{L^1} \leq 6'(K+1)^2C_1^k DK_1 \delta^k \|\xi_n^{\alpha+k}\|_{L^1}$. Together with the above three cases for $k$, (4.2) is proved.

The estimate of $m(\{x \in G \setminus \hat{\Omega} : S(b)(x) > \gamma/3\})$ is similar, and will only be indicated briefly. Fix $n > n_2$. We first prove that

\begin{equation}
E_n b(x) = 0, \forall x \in G \setminus \hat{\Omega}.
\end{equation}

Note that $b = \sum_k \sum_{\alpha \in \Lambda_k} b_\alpha^k$, by the linearity of operator $E_n$, we only need to prove that for each $b_\alpha^k$, $E_n b_\alpha^k = 0$.

Fix $x \in G \setminus \hat{\Omega}$. Let $Q_\alpha^n \supseteq x$. Since $b_\alpha^k$ is supported on $Q_\alpha^n$, it follows that $E_n b_\alpha^k(x) = \int_{Q_\alpha^n} b_\alpha^k(y) dm(y)$. If $Q_\alpha^n \cap Q_\beta^n \neq \emptyset$, we split $k$ into two cases: $k > n$ and $k < n$. For $k > n$, by Proposition 2.2(ii), it follows that $x \in Q_\beta^n \subseteq Q_\alpha^n$, this leads to a contradiction since $Q_\alpha^n \subseteq \hat{\Omega}$. For $k < n$, using Proposition 2.2(ii) again, we have $Q_\alpha^n \subseteq Q_\beta^n$, it follows immediately that

$$E_n b_\alpha^k(x) = \frac{1}{m(Q_\beta^n)} \int_{Q_\alpha^n} b_\alpha^k(y) dm(y) = 0.$$ 

This gives (4.7). Similar to (4.2), we can also establish the following inequality

\begin{equation}
\int_{G \setminus \hat{\Omega}} |A_{\alpha^n} b_\alpha^{n+k}(x)| dm(x) \leq \begin{cases} D \|b_\alpha^{n+k}\|_{L^1}, & -k_5 \leq k \leq k_5; \\ 0, & k > k_5; \\ 6'(K+1)^2C_1^k DK_1 \delta^k \|b_\alpha^{n+k}\|_{L^1}, & k < -k_5. \end{cases}
\end{equation}

Note that for $-k_5 \leq k \leq k_5$, the estimate holds by Proposition 2.11. For $k > k_5$, the estimate holds true by (4.3) and the fact that $b_\alpha^{n+k}$ is supported on $Q_\alpha^{n+k}$. For $k < -k_5$, we first have

$$|A_{\alpha^n} b_\alpha^{n+k}(x)| = \frac{1}{m(B(x,\delta^n))} \int_{B(x,\delta^n)Q_\alpha^{n+k}} b_\alpha^{n+k}(y) dm(y) \leq \frac{1}{m(B(x,\delta^n))} \int_{\delta^n - C_1 \delta^{n+k} \leq d(x,y) \leq \delta^n + C_1 \delta^{n+k}} |b_\alpha^{n+k}(y)| dm(y).$$

By the same estimates as (4.4) and (4.6), the above inequality yields

$$\int_{G \setminus \hat{\Omega}} |A_{\alpha^n} b_\alpha^{n+k}(x)| dm(x) \leq \frac{1}{m(B(x,\delta^n))} \int_G \int_{\delta^n - C_1 \delta^{n+k} \leq d(x,y) \leq \delta^n + C_1 \delta^{n+k}} |b_\alpha^{n+k}(y)| dm(y) dm(x) \leq 6'(K+1)^2C_1^k DK_1 \delta^k \|b_\alpha^{n+k}\|_{L^1(G,m)};$$

together with the above two cases, this proves (4.8), and the proof is complete.

The proof of weak type (1,1) estimate for operator $SV$ is similar to $S$. Let us explain it briefly.

\textit{Proof of (1.11).} Fix $f \in L^1(G, m)$. Decompose $f = g + b + \xi$. The desired estimate for $g$ is true by the fact that $SV$ is of strong type $(2,2)$. In what follows, we only state the proof for $\xi$ since the proof for $b$ is similar.
By similar arguments as in the previous proof, we mainly need to prove the following inequality for $n > n_2$,

$$\int_{G \setminus \Omega} V_2(A'_n \xi_{n,k}^\alpha) : r \in [\delta^n, \delta^{n+1}])dm(x) \leq a(k) \|\xi_{n,k}^\alpha\|_{L^1(G,m)},$$

where

$$a(k) = \begin{cases} C_4, & -k_5 \leq k \leq k_5; \\ 0, & k > k_5; \\ 2 \cdot 3^r(\delta + 1)^r(K + 1)^2C_4^rDK_5\delta^k, & k < -k_5. \end{cases}$$

We now focus on the above inequality. Fix $n > n_2$. Note that for $-k_5 \leq k \leq k_5$, by Lemma 2.14, the estimate is true. For $k > k_5$, similar to (4.3), we can prove that for every $x \in G \setminus \Omega$ and $r \in [\delta^n, \delta^{n+1}]$, $B(x,r) \cap \tilde{Q}_n^{\alpha,k} = \emptyset$, hence $V_2(A'_n \xi_{n,k}^\alpha) : r \in [\delta^n, \delta^{n+1}) = 0$. It remains to prove the case $k < -k_5$.

Given an annular $B(x,R) \setminus B(x,r)$ and ‘dyadic cube’ $Q_n^{\alpha}$, we define the set $I(B(x,R) \setminus B(x,r),Q_n^{\alpha}) = \{Q_n^{\alpha} \cap (B(x,R) \setminus B(x,r)) : \tilde{Q}_n^{\alpha,k} \subset \partial(B(x,R) \setminus B(x,r)) \neq \emptyset\}$. Since $\int_{\tilde{Q}_n^{\alpha,k}} \xi_{n,k}^\alpha = 0$, it follows that

$$V_2(A'_n \xi_{n,k}^\alpha) : r \in [\delta^n, \delta^{n+1})$$

$$\leq \sup_{\delta^n \leq r_0 < \cdots < r_j < \delta^{n+1}} \sum_{i=1}^j \frac{1}{m(B(x,r_i))} \int_{I(B(x,r_i),B(x,r_{i-1}),\tilde{Q}_n^{\alpha,k})} \xi_{n,k}^\alpha(y)dm(y)$$

$$+ \sup_{\delta^n \leq r_0 < \cdots < r_j < \delta^{n+1}} \sum_{i=1}^j \left( \frac{1}{m(B(x,r_i))} - \frac{1}{m(B(x,r_{i-1}))} \right) \int_{I(B(x,r_i),\tilde{Q}_n^{\alpha,k})} \xi_{n,k}^\alpha(y)dm(y)$$

$$\leq \frac{2}{m(B(x,\delta^n))} \int_{\delta^{n+1} - C_1 \delta^n \leq d(x,y) \leq \delta^{n+1} + C_1 \delta^n} |\xi_{n,k}^\alpha(y)|dm(y).$$

Using the same argument as (4.4) and (4.6), the above inequality yields

$$\int_{G \setminus \Omega} V_2(A'_n \xi_{n,k}^\alpha) : r \in [\delta^n, \delta^{n+1})dm(x) \leq 2 \cdot 3^r(\delta + 1)^r(K + 1)^2C_4^rDK_5\delta^k\|b_{n,k}^\alpha\|_{L^1(G,m)},$$

which completes the proof.  \(\square\)

5. \((L^\infty, \text{BMO})\) estimates

In this section, we prove that both operator $S$ and operator $SV$ map $L^\infty_c$ to dyadic BMO space.

Given a locally integrable function $f \in L^1_{\text{loc}}(G,m)$, the dyadic sharp maximal function is defined by

$$M^d f(x) = \sup_{(a,k) \in Q_n^{\alpha}} \inf_c \frac{1}{m(Q_n^{\alpha})} \int_{Q_n^{\alpha}} |f(y) - c|dm(y),$$

and then the dyadic BMO space is defined as $BMO = \{f \in L^1_{\text{loc}}(G,m) : \|M^d f\|_{L^\infty} < \infty\}$ with the norm $\|f\|_{BMO} = \|M^d f\|_{L^\infty}$.

We first handle operator $S$, which was defined as

$$S(f) = \left( \sum_{n>n_0} |A_{n} f - E_n f|^2 \right)^{1/2}.$$

As in dealing with the weak type $(1, 1)$ estimate, we also need the non-doubling analysis.
By the triangle inequality, it follows that
\[ G \]

We now focus on the first term of the right hand side. By the Cauchy-Schwarz inequality, (\ref{eq:Cauchy-Schwarz}) and the fact that the operator \( S \) is of strong type \((2,2)\), we have
\[
\frac{1}{m(Q)_{\beta}} \int_{Q} |S(f)(y)|dm(y) \leq \frac{1}{m(Q)_{\beta}} \int_{Q} |S(f)(y) - S(f_2)(y)|dm(y) \\
+ \frac{1}{m(Q)_{\beta}} \int_{Q} |S(f_2)(y) - c|dm(y) \\
\leq \frac{1}{m(Q)_{\beta}} \int_{Q} |S(f_1)(y)|dm(y) + \frac{1}{m(Q)_{\beta}} \int_{Q} |S(f_2)(y) - c|dm(y).
\]

We now focus on the first term of the right hand side. By the Cauchy-Schwarz inequality, (\ref{eq:Cauchy-Schwarz}), (\ref{eq:Strong-Type}) and the fact that the operator \( S \) is of strong type \((2,2)\), we have
\[
\frac{1}{m(Q)_{\beta}} \int_{Q} |S(f_1)(y)|dm(y) \leq \left( \frac{1}{m(Q)_{\beta}} \int_{Q} |S(f_1)(y)|^2dm(y) \right)^{1/2} \\
\leq \left( \frac{c_2}{m(Q)_{\beta}} \int_{Q} |f(y)|^2dm(y) \right)^{1/2} \\
\leq c_2(3C_11/\alpha)\|f\|_{L^\infty}.
\]

It remains to handle the term \( \frac{1}{m(Q)_{\beta}} \int_{Q} |S(f_2)(y) - c|dm(y) \). Taking \( c = S(f_2)(z_{\beta}^k) \), we first claim that
\[
E_n f_2(x) - E_n f_2(z_{\beta}^k) = 0, \quad \forall \ x \in Q_{\beta}^k.
\]

Indeed, let \( Q_{\alpha}^n \ni x \), if \( n \leq k \), by Proposition \ref{prop:Strong-Type}(ii), then \( Q_{\alpha}^n \subseteq Q_{\beta}^k \). Since \( f_2 \) is supported on \( G \setminus Q_{\beta}^k \), it follows that \( E_n f_2(x) = E_n f_2(z_{\beta}^k) = 0 \). If \( n > k \), using Proposition \ref{prop:Strong-Type}(ii) again, we have \( Q_{\beta}^k \subseteq Q_{\alpha}^n \) and \( x, z_{\beta}^k \in Q_{\alpha}^n \). It follows that \( E_n f_2(x) = E_n f_2(z_{\beta}^k) \), then the claim is proved.

On the other hand, let \( n_3 = k_3 + [\log_\delta(2C_1/\alpha_0)] + 2 \). By the triangle inequality, we first have
\[
\left( \sum_{n > n_3} |A_{\alpha} \ast f_2(x) - A_{\alpha} \ast f_2(z_{\beta}^k)|^2 \right)^{1/2} \\
\leq \left( \sum_{n_3 < n \leq n_3} |A_{\alpha} \ast f_2(x) - A_{\alpha} \ast f_2(z_{\beta}^k)|^2 \right)^{1/2} \\
+ \left( \sum_{n > n_3} |A_{\alpha} \ast f_2(x) - A_{\alpha} \ast f_2(z_{\beta}^k)|^2 \right)^{1/2} \\
\leq 2(n_3 - n_3)\|f\|_{L^\infty} + \left( \sum_{n > n_3} |A_{\alpha} \ast f_2(x) - A_{\alpha} \ast f_2(z_{\beta}^k)|^2 \right)^{1/2}.
\]

Let \( d(Q_{\beta}^k) \) be the diameter of \( Q_{\beta}^k \). Set \( n_4 = \min\{n : \delta^n > d(Q_{\beta}^k)\} \). We now consider two cases for \( n_3: n_3 \geq n_4 \) and \( n_3 < n_4 \).
Combining the above inequality with condition (5.2), the proof will be completed if we proved the following inequality
\[
\left( \sum_{n > n_3} |A_{\delta^n} f_2(x) - A_{\delta^n} f_2(z_{\beta^n})|^2 \right)^{1/2} \leq \frac{2(K + 2^\epsilon K(K + 1))}{1 - \delta^{-\epsilon}} \|f\|_{L^\infty}, \quad \forall \, x \in Q_{\beta^n}^k.
\]

We now focus on the above inequality. By the triangle inequality, we have
\[
|A_{\delta^n} f_2(x) - A_{\delta^n} f_2(z_{\beta^n})| = \left| \frac{1}{m(B(x, \delta^n))} \int_{B(x, \delta^n)} f_2(y) dm(y) - \frac{1}{m(B(z_{\beta^n}, \delta^n))} \int_{B(z_{\beta^n}, \delta^n)} f_2(y) dm(y) \right| \\
\leq \left| \frac{1}{m(B(x, \delta^n))} \int_{B(x, \delta^n)} f_2(y) dm(y) - \frac{1}{m(B(x, \delta^n))} \int_{B(z_{\beta^n}, \delta^n)} f_2(y) dm(y) \right| \\
+ \left| \frac{1}{m(B(x, \delta^n))} \int_{B(z_{\beta^n}, \delta^n)} f_2(y) dm(y) - \frac{1}{m(B(z_{\beta^n}, \delta^n))} \int_{B(z_{\beta^n}, \delta^n)} f_2(y) dm(y) \right| \\
\leq \frac{m(B(x, \delta^n) \Delta B(z_{\beta^n}, \delta^n))}{m(B(x, \delta^n))} \|f\|_{L^\infty} + \frac{m(B(z_{\beta^n}, \delta^n))}{m(B(x, \delta^n))} - 1 \|f\|_{L^\infty} \\
\leq 2 \frac{m(B(x, \delta^n) \Delta B(z_{\beta^n}, \delta^n)) |\|f\|_{L^\infty}}{m(B(x, \delta^n))}.
\]

Note that
\[
B(x, \delta^n) \Delta B(z_{\beta^n}, \delta^n) \subseteq \left( B(x, d(x, z_{\beta^n}) + \delta^n) \setminus B(x, \delta^n) \right) \cup \left( B(z_{\beta^n}, d(x, z_{\beta^n}) + \delta^n) \setminus B(z_{\beta^n}, \delta^n) \right).
\]

For \( n > n_3 \), we have \( \delta^n > r_0 \). Note that \( x \in Q_{\beta^n}^k \), then by Proposition 2.2(iv), we have \( B(z_{\beta^n}, \delta^n) \subseteq B(x, \delta^n + d(Q_{\delta^n}^k)) \). It follows from (2.1) that
\[
\frac{m(B(z_{\beta^n}, \delta^n))}{m(B(x, \delta^n))} \leq \frac{m(B(x, \delta^n + d(Q_{\delta^n}^k)))}{m(B(x, \delta^n))} \leq 2^\epsilon(K + 1).
\]

Combining the above inequality with condition (1.1), we have
\[
\frac{m(B(x, \delta^n) \Delta B(z_{\beta^n}, \delta^n))}{m(B(x, \delta^n))} \leq \frac{m\left( B(x, d(x, z_{\beta^n}) + \delta^n) \setminus B(x, \delta^n) \right)}{m(B(x, \delta^n))} + \frac{m\left( B(z_{\beta^n}, d(x, z_{\beta^n}) + \delta^n) \setminus B(z_{\beta^n}, \delta^n) \right)}{m(B(z_{\beta^n}, \delta^n))} \frac{m(B(z_{\beta^n}, \delta^n))}{m(B(x, \delta^n))} \\
\leq (K + 2^\epsilon K(K + 1)) \left( \frac{d(x, z_{\beta^n})}{\delta^n} \right)^\epsilon.
\]

Therefore,
\[
|A_{\delta^n} f_2(x) - A_{\delta^n} f_2(z_{\beta^n})| \leq 2(K + 2^\epsilon K(K + 1)) \left( \frac{d(Q_{\delta^n}^k)}{\delta^n} \right)^\epsilon \|f\|_{L^\infty}.
\]

By the above inequality, we have
\[
\left( \sum_{n > n_3} |A_{\delta^n} f_2(x) - A_{\delta^n} f_2(z_{\beta^n})|^2 \right)^{1/2} \leq 2(K + 2^\epsilon K(K + 1)) \|f\|_{L^\infty} \sum_{n, \delta^n > d(Q_{\delta^n}^k)} \left( \frac{d(Q_{\delta^n}^k)}{\delta^n} \right)^\epsilon \\
\leq 2 \frac{(K + 2^\epsilon K(K + 1))}{1 - \delta^{-\epsilon}} \|f\|_{L^\infty}.
\]

**Case** \( n_3 \geq n_4 \). Note that \( \delta^{n_3} > d(Q_{\delta^n}^k) \). By (5.2), the proof will be completed if we proved the following inequality
\[
\left( \sum_{n > n_3} |A_{\delta^n} f_2(x) - A_{\delta^n} f_2(z_{\beta^n})|^2 \right)^{1/2} \leq 2(K + 2^\epsilon K(K + 1)) \|f\|_{L^\infty}, \quad \forall \, x \in Q_{\beta^n}^k.
\]
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**Case** \( n_3 < n_4 \). Note that \( d(Q_{\beta}^k) \geq \delta^{n_3-1} \). It follows from Proposition 2.2(iv) that

\[
a_0 \delta^k \geq (a_0/2C_1)\delta^{n_3-1} > \delta^{k_2} \geq r_0,
\]
and \( f_2 = f 1_{G \setminus Q_{\beta}^k} \). We claim that for \( n_3 < n \leq n_4 \),

\[
A_{n^3} f_2(x) = 0, \forall x \in Q_{\beta}^k.
\]

Indeed, fix \( x \in Q_{\beta}^k \) and let \( y \in B(x, \delta^n) \), then \( d(y, z_{\beta}^k) \leq d(y, x) + d(z_{\beta}^k, x) \leq \delta^n + C_1 \delta^k \leq d(Q_{\beta}^k) + C_1 \delta^k \leq 3C_1 \delta^k \). From this, we have \( B(x, \delta^n) \subset \tilde{Q}_{\beta}^k \). Then the claim follows from the fact that \( f_2 \) is supported on \( G \setminus \tilde{Q}_{\beta}^k \).

Combining (5.4) with a similar argument as in the previous proof, we also have

\[
\left( \sum_{n > n_3} |A_{n^3} f_2(x) - A_{n^3} f_2(z_{\beta}^k)|^2 \right)^{1/2} \leq \frac{2(K + 2^K K(K + 1))}{1 - \delta^{-x}} ||f||_{L^\infty}, \forall x \in Q_{\beta}^k,
\]
and the proof is complete. \( \square \)

The proof of \((L_c^\infty, BMO)\) boundedness of \( SV \) is similar to \( S \). Let us give a sketch of the proof.

**Proof of** (1.12). Let \( f \) be a \( L_c^\infty \) function. Recall that the short variation operator is defined by

\[
SV(f) = (\sum_{n > n_3} V_2(A_r f : r \in [\delta^n, \delta^{n+1})^2)^{1/2}.
\]
Fix a ‘dyadic cube’ \( Q_{\beta}^k \) and let = \( f 1_{Q_{\beta}^k} + f 1_{G \setminus Q_{\beta}^k} \). We only state the proof for case \( n_3 \geq n_4 \), namely \( \delta^{n_3} > d(Q_{\beta}^k) \), since the proof for case \( n_3 < n_4 \) is similar. By an argument similar to that in the proof for operator \( S \), it is sufficient to prove that there exists a constant \( C_{\epsilon, \delta, K} > 0 \) such that for all \( x \in Q_{\beta}^k \),

\[
\left( \sum_{n > n_3} V_2(A_r f_2(x) - A_r f_2(z_{\beta}^k) : r \in [\delta^n, \delta^{n+1})^2)^{1/2} \leq C_{\epsilon, \delta, K} ||f||_{L^\infty}.
\]

Fix \( n > n_3 \) and \( x \in Q_{\beta}^k \). By the definition of short variation operator, there exists a sequence \( \{r_i\} \subset [\delta^n, \delta^{n+1}) \) such that

\[
V_2(A_r f_2(x) - A_r f_2(z_{\beta}^k) : r \in [\delta^n, \delta^{n+1})^2)
\]

\[
\leq 2 \left( \sum_{i} |A_{r_i} f_2(x) - A_{r_i} f_2(z_{\beta}^k) - (A_{r_i} f_2(x) - A_{r_i} f_2(z_{\beta}^k))|^2 \right)^{1/2}.
\]

We split such sequence \( \{r_i\} \) into two cases: \( J_1 = \{i : r_i - r_{i-1} \leq d(Q_{\beta}^k) / (\delta^{(i-1)n}) \} \) and \( J_2 = \{i : r_i - r_{i-1} > d(Q_{\beta}^k) / (\delta^{(i-1)n}) \} \). Therefore, the right hand side of the above inequality is controlled by the sum of

\[
SV_{J_1}(f_2)(x) = \left( \sum_{i, i \in J_1} |A_{r_i} f_2(x) - A_{r_i} f_2(z_{\beta}^k) - (A_{r_i} f_2(x) - A_{r_i} f_2(z_{\beta}^k))|^2 \right)^{1/2},
\]

and

\[
SV_{J_2}(f_2)(x) = \left( \sum_{i, i \in J_2} |A_{r_i} f_2(x) - A_{r_i} f_2(z_{\beta}^k) - (A_{r_i} f_2(x) - A_{r_i} f_2(z_{\beta}^k))|^2 \right)^{1/2}.
\]
We now focus on $SV_{T_n}(f_2)(x)$. By the triangle inequality, we have

$$SV_{T_n}(f_2)(x) \leq \left( \sum_{i \in J_1} |A'_{r_i}f_2(x) - A'_{r_i-1}f_2(x)|^2 \right)^{1/2} + \left( \sum_{i \in J_1} |A'_{r_i}f_2(z_{j_i}) - A'_{r_i-1}f_2(z_{j_i})|^2 \right)^{1/2}.$$ 

Note that for any $z \in Q^k$, we have

$$|A'_{r_i}f_2(z) - A'_{r_i-1}f_2(z)| \leq \frac{1}{m(B(z, r_i))} \int_{B(z, r_i) \setminus B(z, r_i-1)} |f_2(y)| dm(y)$$

$$+ \left( \frac{1}{m(B(z, r_i))} - \frac{1}{m(B(z, r_i-1))} \right) \int_{B(z, r_i-1)} |f_2(y)| dm(y)$$

$$\leq \frac{m(B(z, r_i)) - m(B(z, r_i-1))}{m(B(z, r_i))} \|f_2\|_{L^\infty} + \left( \frac{m(B(z, r_i-1))}{m(B(z, r_i))} - \frac{m(B(z, r_i-1))}{m(B(z, r_i))} \right) \|f_2\|_{L^\infty}$$

$$\leq 2 \left( \frac{m(B(z, r_i)) - m(B(z, r_i-1))}{m(B(z, r_i))} \right) \|f_2\|_{L^\infty}$$

$$\leq 2 \|f\|_{L^\infty} \int_{m(B(z, r_i-1))}^{m(B(z, r_i))} \frac{1}{u} du,$$

by condition (1.1), and the integral term of the above inequality is controlled by

$$\int_{m(B(z, r_i-1))}^{m(B(z, r_i))} \frac{1}{u} du \leq \left( \frac{m(B(z, r_i)) - m(B(z, r_i-1))}{m(B(z, r_i))} \right)^{1/2} \left( \int_{m(B(z, r_i-1))}^{m(B(z, r_i))} \frac{1}{u^2} du \right)^{1/2}$$

$$\leq K \left( \frac{r_i - r_{i-1}}{r_{i-1}} \right)^{\epsilon/2} \left( \int_{m(B(z, r_i-1))}^{m(B(z, r_i))} \frac{m(B(z, \delta^{n+1}))}{u^2} du \right)^{1/2}.$$ 

Then using (2.1), the above inequality yields

$$\left( \sum_{i \in J_1} |A'_{r_i}f_2(x) - A'_{r_i-1}f_2(x)|^2 \right)^{1/2}$$

$$\leq 2K \|f\|_{L^\infty} \left( \sum_{i \in J_1} \left( \frac{r_i - r_{i-1}}{r_{i-1}} \right)^{\epsilon} \int_{m(B(x, r_{i-1}))}^{m(B(x, r_i))} \frac{m(B(x, \delta^{n+1}))}{u^2} du \right)^{1/2}$$

$$\leq 2K \|f\|_{L^\infty} \left( \frac{d(Q^k)}{\delta^n} \right)^{\epsilon/2} \left( \int_{m(B(x, \delta^n))}^{m(B(x, \delta^{n+1}))} \frac{m(B(x, \delta^{n+1}))}{u^2} du \right)^{1/2}$$

$$\leq 2K ((K + 1)\delta^n)^{1/2} \|f\|_{L^\infty} \left( \frac{d(Q^k)}{\delta^n} \right)^{\epsilon/2}.$$ 

By a similar argument, we have

$$\left( \sum_{i \in J_1} |A'_{r_i}f_2(z_{j_i}) - A'_{r_i-1}f_2(z_{j_i})|^2 \right)^{1/2} \leq 2K ((K + 1)\delta^n)^{1/2} \|f\|_{L^\infty} \left( \frac{d(Q^k)}{\delta^n} \right)^{\epsilon/2}.$$
Therefore,
\[ SV_{I_n}(f_2)(x) \leq 4K ((K + 1)\delta)^{1/2} \|f\|_{L^\infty} \left( \frac{d(Q^k_{\beta})}{\delta^n} \right)^{\varepsilon/2}. \]

For the part \( SV_{II_n}(f_2)(x) \), using the triangle inequality, we first have
\[ SV_{II_n}(f_2)(x) \leq \left( \sum_{t_1 \in I_2} |A'_{t_1}f_2(x) - A_{t_1}f_2(z^k_{\beta})|^2 \right)^{1/2} \]
\[ + \left( \sum_{t \in J_2} |A'_{t_1 \cdots t}f_2(x) - A_{t_1 \cdots t_1}f_2(z^k_{\beta})|^2 \right)^{1/2}. \]

Note that \( \#(J_2) \leq (\delta - 1)\delta^n / d(Q^k_{\beta})^2 \). Similar to (5.3), for any \( z \in Q^k_{\beta} \) and \( r \in [\delta^n, \delta^{n+1}] \), we have
\[ |A'_{r}f_2(z) - A'_{r}f_2(z^k_{\beta})| \leq 2(K + 2'K(K + 1)) \left( \frac{d(Q^k_{\beta})}{\delta^n} \right)^{\varepsilon} \|f\|_{L^\infty}. \]

Combining the above inequalities, we have
\[ SV_{II_n}(f_2)(x) \leq 4(K + 2'K(K + 1)) \|f\|_{L^\infty} \left( (\delta - 1) \left( \frac{\delta^n}{d(Q^k_{\beta})} \right)^{\varepsilon} \left( \frac{d(Q^k_{\beta})}{\delta^n} \right) \right)^{1/2} \]
\[ = 4(K + 2'K(K + 1))(\delta - 1)^{1/2} \|f\|_{L^\infty} \left( \frac{d(Q^k_{\beta})}{\delta^n} \right)^{\varepsilon/2}. \]

Finally, together the estimates of \( SV_{I_n}(f_2)(x) \) and \( SV_{II_n}(f_2)(x) \) with (5.5), we have
\[ \left( \sum_{n>n_3} V_3(A'_{r}f_2(x) - A'_{r}f_2(z^k_{\beta}) : r \in [\delta^n, \delta^{n+1}]) \right)^{1/2} \]
\[ \leq 8(K + 2'K(K + 1))(\delta - 1)^{1/2} \|f\|_{L^\infty} \left( \sum_{n, \delta^n > d(Q^k_{\beta})} \left( \frac{d(Q^k_{\beta})}{\delta^n} \right)^{\varepsilon} \right)^{1/2} \]
\[ + 8K ((K + 1)\delta)^{1/2} \|f\|_{L^\infty} \left( \sum_{n, \delta^n > d(Q^k_{\beta})} \left( \frac{d(Q^k_{\beta})}{\delta^n} \right)^{\varepsilon/2} \right)^{1/2} \]
\[ \leq 8(K + 2'K(K + 1))(\delta - 1)^{1/2} \|f\|_{L^\infty} + 8K \left( \frac{(K + 1)\delta}{1 - \delta^{-\varepsilon}} \right)^{1/2} \|f\|_{L^\infty}, \]

and the proof is complete. \( \square \)

6. Transference principles

In this section, we establish the transference principles for the jump operator. Recall that a sequence of compact sets \( \{F_n\}_{n \in \mathbb{N}} \) with positive measures in a locally compact group \( G \) is called a Folner sequence if for every \( g \in G \)
\[ \lim_{i} \frac{m((F_n \triangle F_i) \cap F_n)}{m(F_n)} = 0, \]
or equivalently for all compact set \( K \) in \( G \),
\[ \lim_{n} \frac{m(F_n \triangle K)}{m(F_n)} = 1. \]
A group $G$ is called amenable if it admits such a Følner sequence. It is well known that if $G$ is a group with polynomial volume growth, then it is amenable (cf. [24]), and in particular the family of balls $\{B_r\}_{r>0}$ generated by any word metric on $G$ is a Følner sequence (cf. [9, 49, 56]). For more information about Følner sequences and amenable groups we refer the reader to [50].

6.1. Weak type inequalities.

Proof of Theorem 1.7(i). We only give the proof of transference principle for weak type inequalities, since the strong type one can be done verbatim. Let $p \in [1, \infty)$. Given a $f \in L^p(X, \mu)$. Let $T$ be an action induced by a $\mu$-preserving measurable transformation $\tau$, that is for all $g \in G$, $T_g f(x) = f(\tau_g x)$. Fix $x \in X$ and a compact set $A$, define

$$F_A(g) = \begin{cases} T_g f(x), & g \in A; \\ 0, & g \notin A. \end{cases}$$

Let $N$ be an integer large enough and $K$ a compact set such that for $r \leq N$ we have $B_r \subseteq K$. Clearly, $F_{AK}(g) = T_g f(x)1_{AK}(g)$. Moreover, if $h \in A$ and $k \in K$, we have

$$T_h T_k f(x) = T_{hk} f(x) = F_{AK}(hk).$$

It follows that for all $h \in A$ and $r \in (0, N] \cap \mathbb{Z}$, we have

$$T_h A_r f(x) = \frac{1}{m(B_r)} \int_{B_r} T_g f(x)1_{AK}(h g) dm(g) = \frac{1}{m(B_r)} \int_{B_r} F_{AK}(h g) dm(g).$$

Let $A_N f = \{A_r f : r \in (0, N] \cap \mathbb{Z}\}$ and $A'_N f = \{A'_r f : r \in (0, N] \cap \mathbb{Z}\}$. For all $h \in G$, set $T_h A_N f = \{T_h A_r f : r \in (0, N] \cap \mathbb{Z}\}$; here and subsequently, for a sequence of measurable functions $\alpha = \{\alpha_r : r \in \mathbb{Z}\}$, $T\alpha$ stands for $\{T\alpha_r : r \in \mathbb{Z}\}$. From (6.2), we have

$$\lambda \sqrt{N_A(T_h A_N f)(x)} = \lambda \sqrt{N_A(A'_N F_{AK})(h)}.$$  

Fix $\gamma > 0$, define the set

$$D(\gamma) = \{(h, x) \in A \times X : \lambda \sqrt{N_A(T_h A_N f)(x)} > \gamma\}.$$  

Fix $h \in A$ and define the set

$$D^h(\gamma) = \{x \in X : \lambda \sqrt{N_A(T_h A_N f)(x)} > \gamma\}.$$  

Since $T_h(1_{D^h(\gamma)})(x) = 1_{D^h(\gamma)}(x)$, it follows that

$$\int_X 1_{D(\gamma)}(x) d\mu(x) = \int_X T_h(1_{D^h(\gamma)})(x) d\mu(x) = \int_X 1_{D^h(\gamma)}(x) d\mu(x).$$  

On the other hand, fix $x \in X$ and define the set

$$D_x(\gamma) = \{h \in A : \lambda \sqrt{N_A(T_h A_N f)(x)} > \gamma\}.$$  

By (6.3), one can see that

$$D_x(\gamma) = \{h \in A : \lambda \sqrt{N_A(A'_N F_{AK})(h)} > \gamma\}.$$  

Moreover, using the assumption that the jump operator is of weak type $(p, p)$, we have

$$m(D_x(\gamma)) \leq \frac{\|\lambda \sqrt{N_A(A'_N)}\|_{L^p(G, \mu)}^p}{\gamma^p} \frac{\|F_{AK}\|_{L^p(G, \mu)}^p}{\gamma^p}.$$  


It follows from the above inequality that
\[
\int_X \int_A \mathbb{1}_{D_\varepsilon(\gamma)}(h) dm(h) d\mu(x) \leq \frac{\|\lambda \sqrt{N_\lambda(A_N)}\|^p_{L^p \to L^{p,\infty}}}{\gamma^p} \int_X \int_G |F_{AK}(h)|^p dm(h) d\mu(x) \\
= \frac{\|\lambda \sqrt{N_\lambda(A_N)}\|^p_{L^p \to L^{p,\infty}}}{\gamma^p} m(AK) \int_X |f(x)|^p d\mu(x).
\]

By the Fubini theorem, we can see that
\[
\int_X \int_A \mathbb{1}_{D_\varepsilon(\gamma)}(h) dm(h) d\mu(x) = \int_{G \times X} \mathbb{1}_{D_\varepsilon(\gamma)}(h, x) dm(h) d\mu(x) \\
= \int_A \int_X \mathbb{1}_{D_\varepsilon(\gamma)}(x) dm(x) d\mu(h).
\]

Using (6.4), we have
\[
\int_A \int_X \mathbb{1}_{D_\varepsilon(\gamma)}(x) dm(x) d\mu(h) = m(A) \int_X \mathbb{1}_{D_\varepsilon(\gamma)}(x) d\mu(x) \\
= m(A) \mu(\{x \in X : \lambda \sqrt{N_\lambda(A_N f)(x)} > \gamma\}).
\]

Together (6.5), (6.6) with the above inequality, we conclude
\[
\mu(\{x \in X : \lambda \sqrt{N_\lambda(A_N f)(x)} > \gamma\}) \leq \frac{\|\lambda \sqrt{N_\lambda(A_N)}\|^p_{L^p \to L^{p,\infty}}}{\gamma^p} \frac{m(AK)}{m(A)} \int_X |f(x)|^p d\mu(x).
\]

Since \(G\) is an amenable group, by (6.1), for any \(\varepsilon > 0\) we can choose the above subset \(A\) such that \(m(AK)/m(A) \leq (1 + \varepsilon)\). By the arbitrariness of \(\varepsilon\) and the monotone convergence theorem, letting \(N \to \infty\), we have
\[
\mu(\{x \in X : \lambda \sqrt{N_\lambda(A f)(x)} > \gamma\}) \leq \frac{\|\lambda \sqrt{N_\lambda(A)}\|^p_{L^p \to L^{p,\infty}}}{\gamma^p} \int_X |f(x)|^p d\mu(x).
\]

which is the desired conclusion. \(\square\)

6.2. **Strong type inequalities.** In this subsection, we assume that the action \(T\) is a strongly continuous regular action of \(G\) on \(L^p(X, \mu)\). Before stating the result to be proved, we give some notations and lemmas. The following lemma was proved in [47].

**Lemma 6.1.** Let \(\alpha = \{\alpha_r(x), r \in \mathcal{I}\}\) be a sequence of measurable functions on measurable space \((X, \mu)\). For every \(p \in (1, \infty)\) and \(\theta \in (0, 1)\), there exists a positive constant \(c_{p, \theta}\) such that
\[
c_{p, \theta} \sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(\alpha)}\|_{L^p} \leq [L^\infty(X; V_\infty), L^{\theta p}(X; V_{2\theta})]_{\theta, \infty}(\alpha) \leq c_{p, \theta} \sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(\alpha)}\|_{L^p}.
\]

Moreover, if \(\max\{1/p, 1/2\} < \theta < 1\), then the vector-valued interpolation space \([L^\infty(X; V_\infty), L^{\theta p}(X; V_{2\theta})]_{\theta, \infty}\) admits an equivalent norm; in particular, if \(p > 1\), \(\sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(\cdot)}\|_{L^p}\) admits an equivalent norm.

See [47] for the definition of vector-valued interpolation spaces and more details about jump quasi-norms.

An operator \(T : L^p(X, \mu) \to L^p(X, \mu)\) is called regular if there exists a constant \(C > 0\) such that
\[
\|\sup_{k \geq 1} |T(f_k)|\|_{L^p} \leq C \|\sup_{k \geq 1} |f_k|\|_{L^p},
\]
for any finite sequence \(\{f_k : k \geq 1\}\) in \(L^p(X, \mu)\). Let us denote by \(\|T\|\), the smallest \(C\) for which this holds. Let \(\mathfrak{B}\) be a Banach space. If \(T\) is a regular operator on \(L^p(X, \mu)\), then the
tensor product operator $T \otimes id_{\mathcal{B}} : L^p(X, \mu) \otimes \mathcal{B} \to L^p(X, \mu) \otimes \mathcal{B}$ extends to a bounded operator $\widetilde{T} \otimes id_{\mathcal{B}}$ from the Bochner space $L^p(X; \mathcal{B})$ to $L^p(X; \mathcal{B})$, and

$$\|\widetilde{T} \otimes id_{\mathcal{B}}\|_{L^p(X; \mathcal{B}) \to L^p(X; \mathcal{B})} \leq \|T\|_r.$$  \hfill (6.8)

For more information on regular operators we refer the reader to [51]. A group action $T$ of $G$ on $L^p(X, \mu)$ is called regular if for any $g \in G$, $T_g$ is regular and $\sup_{g \in G}\|T_g\|_r < \infty$.

Moreover, together Lemma 6.2 with (6.8), one can obtain the following lemma.

**Lemma 6.2.** Fix $p \in (1, \infty)$. Let $T : L^p(X, \mu) \to L^p(X, \mu)$ be a regular operator. Given a sequence of measurable functions $\alpha = \{\alpha_r(x) : r \in I\}$ in $L^p(X, \mu)$, there exists a constant $c_p > 0$ such that

$$\sup_{\lambda > 0}\|\lambda \sqrt{\mathcal{N}_\lambda(T\alpha)}\|_{L^p} \leq c_p\|T\|_r \sup_{\lambda > 0}\|\lambda \sqrt{\mathcal{N}_\lambda(\alpha)}\|_{L^p}.$$  

In what follows, we state the strong type $(p, p)$ transference principle for strongly continuous regular group actions.

**Proof of Theorem 1.7(ii).** Let $p \in (1, \infty)$ and $f \in L^p(X, \mu)$. Fix $x \in X$ and a compact set $A$, define $F_A(y) = T_g f(x)\chi_A(y)$. Let $N$ be an integer large enough and $K$ a compact set such that for every $r \leq N$ we have $B_r \subseteq K$. Clearly, $F_A K(h) = T_h f(x)\chi_A K(h)$.

Keeping the notations introduced in the proof of weak type inequalities and using (6.3), we have

$$\int_A |\lambda \sqrt{\mathcal{N}_\lambda(T_h A_N f)}(x)|^p dm(h) = \int_A |\lambda \sqrt{\mathcal{N}_\lambda(A_N f)}(h)|^p dm(h) \leq \int_G |\lambda \sqrt{\mathcal{N}_\lambda(A_N f)}(h)|^p dm(h).$$

Using the strong type $(p, p)$ jump inequality which is assumed for the translation action, we obtain

$$\int_A |\lambda \sqrt{\mathcal{N}_\lambda(T_h A_N f)}(x)|^p dm(h) \leq \|\lambda \sqrt{\mathcal{N}_\lambda(A_N f)}\|_{L^p \to L^p}^p \int_G |F_A K(h)|^p dm(h)$$

$$= \|\lambda \sqrt{\mathcal{N}_\lambda(A_N f)}\|_{L^p \to L^p}^p \int_{AK} |T_h f(x)|^p dm(h).$$

Moreover, integrating both sides of the above inequality over $X$ and using the Fubini theorem, we have

$$\int_X \int_A |\lambda \sqrt{\mathcal{N}_\lambda(T_h A_N f)}(x)|^p dm(h) d\mu(x)$$

$$\leq \|\lambda \sqrt{\mathcal{N}_\lambda(A_N f)}\|_{L^p \to L^p}^p \int_{AK} \int_X |T_h f(x)|^p d\mu(x) dm(h)$$

$$= \|\lambda \sqrt{\mathcal{N}_\lambda(A_N f)}\|_{L^p \to L^p}^p \sup_{h \in G} \|T_h\|_p m(AK) \int_X |f(x)|^p d\mu(x).$$  \hfill (6.9)

On the other hand, by the assumption that $T$ is a strongly continuous regular action of $G$ on $L^p(X, \mu)$ and Lemma 6.2, we have

$$\sup_{\lambda > 0}\|\lambda \sqrt{\mathcal{N}_\lambda(A_N f)}\|_{L^p} = \inf_{h \in G} \sup_{\lambda > 0}\|\lambda \sqrt{\mathcal{N}_\lambda(T_h^{-1} T_h A_N f)}\|_{L^p}$$

$$\leq c_p \sup_{h \in G}\|T_h\|_r \inf_{h \in A \lambda > 0} \sup_{\lambda > 0}\|\lambda \sqrt{\mathcal{N}_\lambda(T_h A_N f)}\|_{L^p}.$$
By the above inequality, we have
\[
\sup_{\lambda > 0} \int_X |\lambda \sqrt{N_\lambda (A_N f)(x)}|^p d\mu(x) = \frac{1}{m(A)} \int_A \sup_{\lambda > 0} \int_X |\lambda \sqrt{N_\lambda (A_N f)(x)}|^p d\mu(x) dm(h) \\
\leq c_p \sup_{h \in G} ||T_h||^p \frac{1}{m(A)} \int_A \inf_{\lambda > 0} \sup_{h \in A} \int_X |\lambda \sqrt{N_\lambda (T_h A_N f)(x)}|^p d\mu(x) dm(h) \\
= c_p \sup_{h \in G} ||T_h||^p \sup_{\lambda > 0} \frac{1}{m(A)} \int_A \inf_{h \in A} \int_X |\lambda \sqrt{N_\lambda (T_h A_N f)(x)}|^p d\mu(x) dm(h) \\
\leq c_p \sup_{h \in G} ||T_h||^p \sup_{\lambda > 0} \frac{1}{m(A)} \int_A \int_X |\lambda \sqrt{N_\lambda (T_h A_N f)(x)}|^p d\mu(x) dm(h).
\]
Using the Fubini theorem and (6.9), the above inequality yields
\[
\sup_{\lambda > 0} \int_X |\lambda \sqrt{N_\lambda (A_N f)(x)}|^p d\mu(x) \\
\leq c_p \sup_{\lambda > 0} ||\lambda \sqrt{N_\lambda (A_N')||}^p ||_p \sup_{h \in G} ||T_h||^p \frac{m(A)}{m(A)} \int_X |f(x)|^p d\mu(x).
\]
By (6.1) and a similar argument as in the proof of weak type inequalities, letting \(N \to \infty\), we have
\[
\sup_{\lambda > 0} ||\lambda \sqrt{N_\lambda (A f)||}^p \leq c_p \sup_{\lambda > 0} ||\lambda \sqrt{N_\lambda (A')}||^p \sup_{h \in G} ||T_h||^p ||f||_p,
\]
which is the desired conclusion. 

7. Annular decay property

In this section, we discuss the annular decay property. We first recall the \((\epsilon, 1)\)-annular decay property of word metrics and verify that this property is stable under \((1, C)\)-quasi isometry (recalled below), and thus obtain the quantitative ergodic theorems, including Theorem 1.2, on the polynomial growth group equipped with a metric that is \((1, C)\)-quasi isometric to a word metric. We then check that all the known examples of periodic metric, which was introduced by Breuillard [9], satisfy some \((\epsilon, r_0)\)-annular decay property. At the moment of writing, we do not know how to verify this property for all periodic metrics.

Let \(G\) be a polynomial growth group with a symmetric compact generating set \(V\). Recall that the word metric \(d\) is defined by
\[
\forall x, y \in G, \quad d(x, y) = \inf\{n \in \mathbb{N}, x^{-1} y \in V^n\}.
\]
It is clear that \(d\) is a (left-) invariant metric on \(G\). Let \(r > 0\) and \(B_r\) be the ball generated by word metric \(d\) of radius \(r\) in \(G\). It is well known that there exist two constants \(C_V > 0\) and \(D_G > 0\) such that for every \(r \in (0, \infty)\),
\[
C_V^{-1} r^{D_G} \leq m(B_r) \leq C_V r^{D_G}.
\]
Form the above inequality, it is easy to check that \((G, d, m)\) satisfies the measure doubling condition, it follows that such \((G, d, m)\) satisfies condition (1.2). By an argument same as in the proof of [56, Theorem 4], we have the following proposition.

**Proposition 7.1.** Let \(G\) be a polynomial growth group with a symmetric compact generating set \(V\) and \(\{B_r\}_{r > 0}\) be the balls given by the corresponding word metric. Then there exist two constants \(\theta = \log_2 (1 + \frac{1}{c_T^{10 \log C}})\) and \(c_V = (1 + \frac{1}{c_T^{10 \log C}})^3\) such that for all \(r \in [1, \infty)\) and \(s \in (0, r]\),
\[
m(B_{r+s}, B_r) \leq c_V \left(\frac{s}{r}\right)^\theta m(B_r).
\]
Remark 7.2. In terms of our terminology, the above ball annular decay property is just the \((\epsilon, 1)\)-annular decay property (1.1) of \((G, d, m)\). Note that all the groups of polynomial growth are amenable (cf. [24]), thus Theorem 1.2 follows from Theorem 1.1 by the transference principles.

In fact, in [56, Theorem 4], the \((\epsilon, 1)\)-annular decay property is established for every metric measure space satisfying the measure doubling condition and Property (M). Recall that a metric space \(\mathcal{X}, d, \mu\) is said to satisfy Property (M) if there exists a constant \(C > 0\) such that the Hausdorff distance between any pair of balls with same center and any radii between \(r\) and \(r + 1\) is less than \(C\). In other words, for all \(x \in \mathcal{X}\), \(r > 0\) and \(y \in B(x, r + 1)\), we have \(d(y, B(x, r)) \leq C\).

Property (M) is equivalent to the property: there exists a constant \(C < \infty\) such that for all \(r > 0\), \(s \geq 1\), \(y \in B(x, r + s)\)

\begin{equation}
\tag{7.3}
d(y, B(x, r)) \leq Cs;
\end{equation}

and is also equivalent to that the metric space \(\mathcal{X}, d, \mu\) admits monotone geodesics, see e.g. [56, Proposition 2] for the relevant definitions and proofs. Moreover, Property (M) is invariant under Hausdorff equivalence but unstable under quasi-isometry in the sense of [56, Page 50], where one can find the relevant counterexamples.

We prove that Property (M) is invariant under the \((1, C)\)-quasi-isometry. Two metrics \(d_1\) and \(d_2\) on \(\mathcal{X}\) are called \((1, C)\)-quasi-isometric if there exists a constant \(C > 0\) such that for any \(x, y \in \mathcal{X}\), \(|d_1(x, y) - d_2(x, y)| \leq C\).

Proposition 7.3. Let \(d_1\) and \(d_2\) be two metrics on \(\mathcal{X}\). Assume that \(d_1\) is \((1, C_1)\)-quasi-isometric to \(d_2\) and \((\mathcal{X}, d_1)\) satisfies property (M), then \((\mathcal{X}, d_2)\) satisfies property (M).

Proof. Fix a point \(x \in \mathcal{X}\), \(s \geq 1\) and \(r > 0\). We denote by \(B_1(x, r)\) and \(B_2(x, r)\) the balls generated by \(d_1\) and \(d_2\), respectively. Since \(d_1\) is \((1, C_1)\)-quasi-isometric to \(d_2\), it follows that \(B_2(x, r) \subseteq B_1(x, r + C_X)\). Let \(z \in B_2(x, r + s)\). We split \(r\) into two cases: \(r > C_X\) and \(r \leq C_X\).

For \(r > C_X\), we have \(B_1(x, r - C_X) \subseteq B_2(x, r)\) and it follows that

\[d_2(z, B_2(x, r)) \leq d_1(z, B_1(x, r - C_X)) + C_X.\]

Combining (7.3) for \(d_1\) with the fact that \(s \geq 1\) and \(z \in B_1(x, r + s + C_X)\), the above inequality yields

\[d_2(z, B_2(x, r)) \leq C(s + 2C_X) + C_X \leq (C + 2CC_X + C_X)s.\]

For \(r \leq C_X\), it is easy to check that \(d_2(z, B_2(x, r)) \leq r + s \leq (C_X + 1)s\). Combining this case with the case \(r > C_X\), we prove that \((\mathcal{X}, d_2)\) satisfies (7.3), which completes the proof. \(\square\)

Remark 7.4. By the above observations, we can see that any left-invariant metric, defined on a polynomial volume growth group \(G\) which is \((1, C)\)-quasi-isometric to a word metric, satisfies the \((\epsilon, 1)\)-annular decay property. Thus Theorem 1.2 holds for any metric that is \((1, C)\)-quasi-isometric to a word metric.

Motivated by the notion—Property (M), we can introduce Property \((M_{r_0})\), namely there exists a positive constant \(C < \infty\) such that the Hausdorff distance between any pair of balls with same center and any radii belonging to \([r, r + 1]\) with \(r > r_0\) is less than \(C\). Similar to [56, Theorem 4], one can show that for a doubling metric measure space \((\mathcal{X}, d, \mu)\) with Property \((M_{r_0})\), then there exists \(\theta > 0\) and a constant \(C > 0\) such that for all \(x \in \mathcal{X}\), \(r \in [r_0, \infty)\) and \(s \in (0, r]\), we have \(\mu(B_{x+s} \setminus B_r) \leq C(s/r)^\theta \mu(B_r)\). As shown in Proposition 7.3, Property \((M_{r_0})\) is also stable under \((1, C)\)-quasi-isometry.

If a metric measure space satisfies \((\epsilon, r_0)\)-annular decay property for all \(r_0 > 0\), then we call it satisfy the \(\epsilon\)-annular decay property. This terminology, to our best knowledge, was introduced by
Buckley [10, (1.1)] on metric space. A slight variant on manifold was introduced by Colding and Minicozzi [15], which they called the $\epsilon$-volume regularity property. In recent years the $\epsilon$-annular decay property has been widely exploited in harmonic analysis, see [3, 4, 35, 36, 42, 59] for more details.

The following example tells us that there are numerous metric measure spaces only satisfying the $(\epsilon, r_0)$-annular decay property for some $r_0$ but not all $r_0 > 0$.

**Example 7.5.** Fix a positive integer $r_0$. Let $\mathcal{X} = \mathbb{Z}$ endowed with the counting measure $\mu$. The metric $d$ is given by

$$d(x, y) = \begin{cases} 0, & x = y, \\ r_0/2, & 0 < |x - y| \leq r_0/2, \\ \max\{|x - y|, r_0\}, & |x - y| > r_0/2. \end{cases}$$

One can check that for any $k \in (-\infty, -1) \cap \mathbb{Z}$, $\mu(B(0, r_0) \setminus B(0, r_0 - 2^k)) = r_0$, it follows that this metric space does not satisfy $\epsilon$-annular decay property but satisfies the $(\epsilon, r_0)$-annular decay property.

In [10], Buckley proved that the metric space $(\mathcal{X}, d, \mu)$ has the $\epsilon$-annular decay property when it satisfies the measure doubling condition and the $(\alpha, \beta)$-chain ball property. Lin, Nakai and Yang [42] established the $\epsilon$-annular decay property for the metric space $(\mathcal{X}, d, \mu)$ if $(\mathcal{X}, d, \mu)$ satisfies the measure doubling condition and the weak geodesic property (i.e., (7.3) holds for all $s > 0$). Actually, Lin et al proved that the weak geodesic property is equivalent to the $(\alpha, \beta)$-chain ball property, and is also equivalent to the monotone geodesic property. A typical class of metric spaces having the $(\alpha, \beta)$-chain ball property (or the weak geodesic property) is the length spaces. The following proposition was established in [10, Corollary 2.2].

**Proposition 7.6.** If $(\mathcal{X}, d, \mu)$ is a length space (a metric space in which the distance between any two points is the infimum of the lengths of all curves joining the two points) and satisfies the measure doubling condition. Then $(\mathcal{X}, d, \mu)$ satisfies the $\epsilon$-annular decay property.

Basic examples of length spaces include the graph which is defined in [56, Page 51], the homogeneous groups endowed with homogeneous metrics (such as $\mathbb{R}^n$ and the Heisenberg group $\mathbb{H}^n$), the Riemannian metric spaces, the Finsler metric spaces, the subRiemannian metric and the subFinsler metric spaces (also called the Carnot-Carathéodory metric spaces). For more examples we refer the reader to the book [23].

In addition to length spaces, the RD-spaces with condition $(H_0)$, $\alpha \in (0, 1]$ satisfy also the $\epsilon$-annular decay property, see e.g. [42, Example 4.1(iii)] for the assertion and the related notions.

Finally, we focus on the periodic metric spaces. Recall that a pseudodistance $d$ on locally compact group $G$ is called a periodic metric if it satisfies the following properties:

(i) $d$ is invariant under left translations by a closed co-compact subgroup $H$, meaning that for all $x, y \in G$ and all $h \in H$, $d(hx, hy) = d(x, y)$;

(ii) $d$ is locally bounded and proper;

(iii) $d$ is asymptotically geodesic.

For more information about the periodic metrics we refer the reader to [9, Section 4]. Moreover, in [9] the following result is proved. There exist two constants $d_G > 0$ and $c_d > 0$ such that

$$\lim_{r \to \infty} \frac{m(B_r)}{r^{d_G}} = c_d,$$

where $B_r$ is a ball given by the periodic metric of radius $r$ in a polynomial growth group $G$. From the above identity, we can see that equipped with a left-invariant periodic metric $d$, $(G, d, m)$...
satisfies the doubling measure condition and (1.13). However condition (1.1) seems quite inaccessible from the above estimate. But all the periodic metrics provided in [9, Section 4] satisfy condition (1.1).

Example 7.7. (i) Let $G$ be a polynomial volume growth group with a compact symmetric generating set. The corresponding word metric is a periodic metric.

(ii) Let $G$ be a simply connected nilpotent Lie group. Let $\Gamma$ be a finitely generated torsion free nilpotent group which is embedded as a co-compact discrete subgroup of $G$. We denote by $V$ the generating set of $\Gamma$ and $d_V$ the word metric on $\Gamma$. The metric $d$ on $G$ which is defined by $d(x, y) = d_V(h_x h_y)$ is a periodic metric, where $x \in h_x F$ and $y \in h_y F$ and $F$ is some fixed fundamental domain for $\Gamma$ in $G$.

(iii) Let $G/\Gamma$ be a nilmanifold with universal cover $G$ and fundamental group $\Gamma$. Let $d$ be a Riemannian metric on $G/\Gamma$. The Riemannian metric $d$ on $G$ which is extended by Riemannian metric $d$ is a periodic metric.

(iv) Let $G$ be a connected Lie group. The left invariant Carnot-Carathéodory metric or left invariant Riemannian metric on $G$ is a periodic metric.

Note that Examples (i) and (ii) are the spaces endowed with word metrics, then by Proposition 7.1, such spaces satisfy the $(\epsilon, 1)$-annular decay property; Examples (iii) and (iv) are actually length spaces, and so by Proposition 7.6, such spaces satisfy the $\epsilon$-annular decay property.

Based on the above examples, one can not expect the $\epsilon$-annular decay property for all periodic metrics, but the following conjecture is still expected; if this were the case, we would be able to obtain the quantitative ergodic theorems for all periodic metrics on polynomial growth groups.

Question 7.8. Let $G$ be a polynomial growth group endowed with a left-invariant periodic metric $d$ and a Haar measure $m$. Then there exists one $r_0 > 0$ such that $(G, d, m)$ satisfies the $(\epsilon, r_0)$-annular decay property.

Combining Proposition 7.3 with the proof of Corollary 1.6 in [9], we can see that in order to prove the $(\epsilon, r_0)$-annular decay property for polynomial growth group $G$ endowed with left-invariant periodic metric, the question is left to prove the case when $G$ is a simply connected solvable Lie group of polynomial growth.

8. Exponential decay estimates

In this section, we establish the jump estimates with exponential decay, namely Theorem 1.3. Let $G$ be a group of polynomial growth with a symmetric finite generating set $V$ in this section, and $A' = \{A_r : r \in \mathbb{N}\}$ be the sequence of averaging operators given by (1.3). It is certainly interesting to establish Theorem 1.3 in the more general setting. However, we do not know how to prove it.

We start with several lemmas. The first one is a transference principle, which can be established verbatim using the arguments as in the proof of Theorem 1.7(i). We omit the details.

Lemma 8.1. Let $p \in [1, \infty)$. Let $T$ be an action induced by a $\mu$-preserving measurable transformation $\tau$ on $X$. Let $A = \{A_r : r \in \mathbb{N}\}$ be the sequence of averaging operators given by (1.4). If there exist two constants $C_\lambda > 0$ and $c_\lambda \in (0, 1)$ such that for every $n \in \mathbb{N}$ and $F \in L^p(G, m)$

$$m\{g \in G : N_\lambda(A'F)(g) > n\} \leq C\lambda c_\lambda\|F\|_{L^p(G, m)}^p,$$

then for every $n \in \mathbb{N}$ and $f \in L^p(X, \mu)$,

$$\mu\{x \in X : N_\lambda(Af)(x) > n\} \leq C\lambda c_\lambda\|f\|_{L^p(X, \mu)}^p.$$

With the above transference principle, it suffices to show Theorem 1.3 when $X = G$. The second lemma that we need is a trivial jump estimate following from Theorem 1.2.
Lemma 8.2. For every $p \in [1, \infty)$, there exists a constant $c_p > 0$ such that for all $F \in L^p(G, m)$,

$$m\{g \in G : \mathcal{N}_\lambda(A^f)(g) > n\} \leq \frac{c_p}{(\lambda \sqrt{n})^p} \|F\|_{L^p(G, m)}^p.$$ 

To present the next two lemmas, we fix $\lambda > 0$ and $F \in L^p(G, m)$ with $\|F\|_{L^\infty} \leq 1$. For $q \in \mathbb{N}$, define

$$\mathcal{F}_q = \mathcal{F}_q(\lambda, F) = \{x \in G : \mathcal{N}_\lambda(A^f)(x) > q\},$$

and

$$\mathcal{F}_q'(F, \lambda) = \{(x, r_0) : x \in G, \exists 1 \leq r_0 < r_1 < \cdots < r_q \text{ such that } \min_{1 \leq i \leq q} |A'_{r_i}F(x) - A'_{r_{i-1}}F(x)| > \lambda\}.$$ 

Let $\mathcal{G}_q' = \mathcal{G}_q'(F, \lambda) = \{B(x, r_0) : (x, r_0) \in \mathcal{F}_q'(F, \lambda)\}$ and $\mathcal{G}_q = \mathcal{G}_q(F, \lambda) = \cup_{B \in \mathcal{G}_q'} B$. It is clear that $\mathcal{F}_q \subseteq \mathcal{G}_q$.

Set

$$C_{V, \lambda} = \min\{1/(8c_{V}), 1/\lambda\},$$

$$\Phi(q) = 2^p \cdot 3 D_{\alpha} c_p C_{V, \lambda}^{-\alpha} \lambda^{-\frac{\alpha q}{q} - \frac{p}{q}}.$$ 

where the constants $D_{\alpha}$, $C_{V, \lambda}$, $c_{V, \lambda}$ and $\theta$ were given in Section 7.

Lemma 8.3. For any $q \in \mathbb{N}$, one has

$$m(\mathcal{G}_q(F, \lambda)) \leq \Phi(q) \|F\|_{L^p(G, m)}^p.$$ 

Proof. We first prove the following inequality

$$(8.1) \quad m(B(x, r)) \leq C_V^2 (C_{V, \lambda})^{D_{\alpha}/\alpha} m(B(x, (C_{V, \lambda})^{1/\alpha} r)), \quad \forall \ r \in \mathbb{N}.$$ 

Indeed, if $(C_{V, \lambda})^{1/\alpha} r < 1$, since $m$ is a counting measure, then $m(B(x, (C_{V, \lambda})^{1/\alpha} r)) = 1$. By (7.1), we have $m(B(x, r)) \leq C_V (C_{V, \lambda})^{\frac{\alpha}{D_{\alpha}}} m(B(x, (C_{V, \lambda})^{1/\alpha} r))$. If $(C_{V, \lambda})^{1/\alpha} r \geq 1$, using (7.1) again, we have $m(B(x, r)) \leq C_V^2 (C_{V, \lambda})^{-\frac{\alpha q}{q}} m(B(x, (C_{V, \lambda})^{1/\alpha} r))$, and so (8.1) is proved.

Applying the Vitali covering lemma, we can select a subset

$$\{B_{j_1}, B_{j_2}, \cdots, B_{j_q}, \cdots\} \subseteq \mathcal{G}_q'$$

of pairwise disjoint balls satisfying

$$(8.2) \quad \mathcal{G}_q \subseteq \cup_{i \leq q} 3B_{j_i}.$$ 

For each ball $B_{j_i} = B(x_{j_i}, r_{j_i})$ selected, by the definition of $\mathcal{G}_q'$, there exists a sequence $1 \leq r_{j_i} < r_{j_i+1} < \cdots < r_{j_i+q}$ such that

$$|A'_{r_{j_i+k}}F(x_{j_i}) - A'_{r_{j_i+k-1}}F(x_{j_i})| > \lambda, \quad \forall \ 1 \leq k \leq q.$$ 

Now we fix such ball $B(x_{j_i}, r_{j_i})$ and sequence $1 \leq r_{j_i} < r_{j_i+1} < \cdots < r_{j_i+q}$. We claim that for all $y \in B(x_{j_i}, (C_{V, \lambda})^{1/\alpha} r_{j_i})$ and $1 \leq k \leq q$,

$$|A'_{r_{j_i+k}}F(y) - A'_{r_{j_i+k-1}}F(y)| > \lambda/2,$$

namely $B(x_{j_i}, (C_{V, \lambda})^{1/\alpha} r_{j_i}) \in \mathcal{F}_q'(\frac{\lambda}{2}, F)$.
Assume this claim momentarily. We have

\[
m(\cup_i B_{j_i}) = \sum_i m(B_{j_i}) \leq C_V^2 (C_{V,\lambda})^{-\frac{p\alpha}{\alpha}} \sum_i m(B(x_{j_i}, (C_{V,\lambda})^{1/\theta} r_{j_i}))
\]

\[
\leq C_V^2 (C_{V,\lambda})^{-\frac{p\alpha}{\alpha}} m(\bigcup_i B(x_{j_i}, (C_{V,\lambda})^{1/\theta} r_{j_i}))
\]

\[
\leq C_V^2 (C_{V,\lambda})^{-\frac{p\alpha}{\alpha}} \left( F_q \left( \frac{\lambda}{2} F \right) \right) \leq C_V^2 (C_{V,\lambda})^{-\frac{p\alpha}{\alpha}} \frac{2^p c_p}{\lambda \sqrt{q}} \|F\|_p^p,
\]

where the equality follows from the disjointness of the balls \(B_{j_i}\), the first inequality follows from (8.1), the second inequality follows from the fact \(C_{V,\lambda} \leq 1\) and the disjointness of the balls \(B_{j_i}\), the third inequality follows from the claim and the last inequality follows from Lemma 8.2.

Moreover, combining the above inequality with (8.2), we have

\[
m(G_q) \leq m(\cup_i 3B_{j_i}) \leq 3^D C_V^2 \sum_i m(B_{j_i}) \leq 2^p \cdot 3^D c_p C_V^4 C_{V,\lambda}^{-\frac{p\alpha}{\alpha}} \lambda^{-\frac{p\alpha}{\alpha}} q^{-\frac{p}{2}} \|F\|_p^p,
\]

and the conclusion is proved.

We now prove the claim. Fix \(y \in B(x_{j_i}, (C_{V,\lambda})^{1/\theta} r_{j_i})\). By a simple geometric argument, we can check at once that for every \(0 \leq k \leq q\), \(B(x_{j_i}, r_{j_i+k}) \Delta B(y, r_{j_i+k})\) is contained in

\[
\left( B(x_{j_i}, r_{j_i+k} + (C_{V,\lambda})^{1/\theta} r_{j_i}) \setminus B(x_{j_i}, r_{j_i+k}) \right) \cup \left( B(y, r_{j_i+k} + (C_{V,\lambda})^{1/\theta} r_0) \setminus B(y, r_{j_i+k}) \right).
\]

Using the inequality (7.2) and the fact that the measure \(m\) is invariant under the translation, the above inequality implies

\[
m\left(B(x_{j_i}, r_{j_i+k}) \Delta B(y, r_{j_i+k})\right) \leq m\left(B(x_{j_i}, r_{j_i+k} + (C_{V,\lambda})^{1/\theta} r_{j_i}) \setminus B(x_{j_i}, r_{j_i+k})\right) + m\left(B(y, r_{j_i+k} + (C_{V,\lambda})^{1/\theta} r_0) \setminus B(y, r_{j_i+k})\right)
\]

\[
\leq 2 v \left( \frac{(C_{V,\lambda})^{1/\theta} r_{j_i}}{r_{j_i+k}} \right) \lambda m(B_{r_{j_i+k}})
\]

\[
\leq \frac{\lambda}{4} m(B_{r_{j_i+k}}).
\]

Combining the above inequality with \(\|F\|_\infty \leq 1\), one has

\[
\left| \int_{B(y, r_{j_i+k})} F(z) dm(z) - \int_{B(x_{j_i}, r_{j_i+k})} F(z) dm(z) \right| = \left| \int_{B(x_{j_i}, r_{j_i+k}) \Delta B(y, r_{j_i+k})} F(z) dm(z) \right|
\]

\[
\leq \frac{\lambda}{4} m(B_{r_{j_i+k}}).
\]

It follows that

\[
|A'_{r_{j_i+k}} F(x_{j_i}) - A'_{r_{j_i+k}} F(y)| \leq \frac{\lambda}{4} \quad \forall \ 0 \leq k \leq q.
\]

Using the triangle inequality, for every \(1 \leq k \leq q\), we have

\[
|A'_{r_{j_i+k}} F(x_{j_i}) - A'_{r_{j_i+k-1}} F(x_{j_i})| \leq |A'_{r_{j_i+k}} F(y) - A'_{r_{j_i+k-1}} F(y)| + |A'_{r_{j_i+k}} F(x_{j_i}) - A'_{r_{j_i+k}} F(y)|
\]

\[
+ |A'_{r_{j_i+k-1}} F(x_{j_i}) - A'_{r_{j_i+k-1}} F(y)|
\]

\[
\leq |A'_{r_{j_i+k}} F(y) - A'_{r_{j_i+k-1}} F(y)| + \lambda/2.
\]

By the above inequality, we have \(|A'_{r_{j_i+k}} F(y) - A'_{r_{j_i+k-1}} F(y)| > \lambda/2\), and the claim is proved. \(\square\)
Lemma 8.4. For positive integers \( p \) and \( q \), one has
\[
m(G_{(p+1)q}(F, \lambda)) \leq 3^{D_G} C_G^2 \Phi(q)m(G_{pq}(F, \lambda)).
\]

The proof of this lemma is inspired by the proof of [30, Inequality (5.7)].

Proof. For every ball \( B = B(x, r) \in G'_{(p+1)q} \) by the definition of \( G'_{(p+1)q} \), there exists a sequence 
\( r = r_0 < r_1 < \cdots < r_{(p+1)q} \) such that
\[
|A'_{r_k} F(x) - A'_{r_{k-1}} F(x)| > \lambda, \quad \forall \ 1 \leq k \leq (p+1)q.
\]
So we have \( B(x, r_q) \in G'_{pq} \). Write \( B = B(x, r_q) \). Set \( B = G'_{(p+1)q} \)
and 
\[
B' = \{ B' : B' \in G'_{pq} \text{ satisfies } B' = \tilde{B} \text{ for some } \tilde{B} \in B \}.
\]
Note that \( B' \subseteq G'_{pq} \), then by the definition of \( G_{(p+1)q} \), the proof is finished if we show
\[
\tag{8.4}
m(\cup_{B \in B} B) \leq 3^{D_G} C_G^2 \Phi(q)m(\cup_{B' \in B'} B').
\]
We now focus on the above inequality. Before proving this estimate, we introduce some new notations. We assume that \( B_1 \) is the maximal size ball of \( B' \). We set \( B_1 = B, B'_1 = B' \) and define the sets
\[
I_1 = \{ B | B \in B_1, \tilde{B} \cap B_1 \neq \emptyset \},
\]
\[
I'_1 = \{ B' | B' \in B'_1, B' \cap B_1 \neq \emptyset \}.
\]
We first prove the following estimate
\[
\tag{8.5}
m(\cup_{B \in I_1} B) \leq \Phi(q)m(3B_1),
\]
where \( 3B_1 \) denotes the ball with the same center as \( B_1 \) and its radius is 3 times that of \( B_1 \).

Fix \( B = B(x, r) \in I_1 \). Since \( B(x, r) \in G'_{(p+1)q} \), then there exists \( r = r_0 < r_1 < \cdots < r_{(p+1)q} \) such that for all \( 1 \leq k \leq (p+1)q \),
\[
|A'_{r_k} F(x) - A'_{r_{k-1}} F(x)| > \lambda.
\]
Since \( B_1 \) is maximal size ball of \( B' \) and \( B_1 \cap B(x, r_q) \neq \emptyset \), so for all \( 0 \leq k \leq q \), \( B(x, r_k) \subseteq 3B_1 \).

It follows that
\[
|A'_{r_k} (F 1_{3B_1})(x) - A'_{r_{k-1}} (F 1_{3B_1})(x)| > \lambda, \quad \forall \ 1 \leq k \leq q.
\]
Hence \( B \in G'_q(F 1_{3B_1}, \lambda) \). Combining this observation with Lemma 8.3 and (7.1), we have
\[
m(\cup_{B \in I_1} B) \leq m(\cup_{B \in G'_q(F 1_{3B_1}, \lambda)} B) = m(G_q(F 1_{3B_1}, \lambda)) \leq \Phi(q) \| F 1_{3B_1} \|^p_{L^p(G, m)}
\]
\[
\leq \Phi(q)m(3B_1) \leq 3^{D_G} C_G^2 \Phi(q)m(B_1).
\]
So (8.5) is proved.

Let \( B_2 = B_1 \setminus I_1 \) and \( B'_2 = B'_1 \setminus I'_1 \). Select \( B_2 \in B'_2 \) is a maximal size ball of \( B'_2 \). Note that
\( B_1 \in I'_1 \), then \( B_2 \) is disjoint from \( B_1 \). We define the sets
\[
I_2 = \{ B | B \in B_2, \tilde{B} \cap B_2 \neq \emptyset \},
\]
\[
I'_2 = \{ B' | B' \in B'_2, B' \cap B_2 \neq \emptyset \}.
\]
By similar discussions of (8.5), we have
\[
m(\cup_{B \in I_2} B) \leq \Phi(q)m(3B_2) \leq 3^{D_G} C_G^2 \Phi(q)m(B_2).
\]
Repeating the above process, then we can select the pairwise disjoint balls \( B_1, B_2, \cdots \) which belongs to \( B' \) and the sets \( I_1, I_2, \cdots, I'_1, I'_2, \cdots \) with the properties
\[
\cup_i I_i = B, \quad \cup_i I'_i = B', \quad m(\cup_{B \in I_i} B) \leq 3^{D_G} C_G^2 \Phi(q)m(B_i).
\]
Summing $i$ for the latter inequality and using property that the balls $\{B_i\}$ are pairwise disjoint, we have

$$m(\cup_{B \in B} B) = m(\cup_{i} \cup_{B \in B_i} B) \leq \sum_i m(\cup_{B \in B_i} B) \leq 3^{D_G} C_V^2 \Phi(q) \sum_i m(B_i)$$

$$= 3^{D_G} C_V^2 \Phi(q) m(\cup_{i} B_i) \leq 3^{D_G} C_V^2 m(\cup_{B \in B'} B').$$

We obtained (8.4), and the lemma follows. \qed

**Proof of Theorem 1.3.** By Lemma 8.1, it suffices for our purpose to show that for any $\lambda > 0$, there exist two constants $\tilde{c}_1$ and $\tilde{c}_2$ in $(0, 1)$ such that for any $n \in \mathbb{N}$ and $F \in L^p(G, m)$ with $\|F\|_{L^\infty} \leq 1$,

$$m\{g \in G : \mathcal{N}_S(A^k F)(g) > n\} \leq \tilde{c}_1 \tilde{c}_2^n \|F\|_{L^p(G, m)}^p.$$ 

From now on, fix one $\lambda > 0$ and one $F \in L^p(G, m)$ with $\|F\|_{L^\infty} \leq 1$. First, by Lemma 8.3, we set $q_0 = \min\{q \in \mathbb{N} : \Phi(q) \leq \frac{1}{2}\}$. For each $n > 0$, compared with $q_0$, we divide the $n$ into two cases: $n \geq q_0$ and $1 \leq n < q_0$.

We first consider the case $1 \leq n < q_0$. By Lemma 8.2, we can set $\tilde{c}_1 = 2c_r/\lambda^p$, $\tilde{c}_2 = (1/2)^{1/q_0}$. It remains to consider the case $n \geq q_0$. Write $n = s q_0 + r$ with $0 \leq r < q_0$. Using Lemma 8.4, we have

$$m(\mathcal{F}_n) \leq m(\mathcal{F}_{s q_0}) \leq m(\mathcal{G}_{s q_0}) \leq \left(\frac{1}{2}\right)^{s-1} m(\mathcal{G}_{q_0}).$$

On the other hand, using Lemma 8.3 again, we have

$$m(\mathcal{G}_{q_0}) \leq \Phi(q_0) \|F\|_{L^p}^p \leq \frac{1}{2}\|F\|_{L^p}^p.$$ 

Note that $s = (n-r)/q_0$, by the above discussions we have

$$m(\mathcal{F}_n) \leq \left(\frac{1}{2}\right)^{-r/q_0} \left(\frac{1}{2}\right)^{n/q_0} \|F\|_{L^p}^p,$$

and so we can set $\tilde{c}_1 = 2$ and $\tilde{c}_2 = (1/2)^{1/q_0}$ in this case.

Finally, we can take

$$\tilde{c}_1 = \max\{2, 2c_r/\lambda^p\}, \quad \tilde{c}_2 = (1/2)^{1/q_0},$$

and the proof is complete. \qed

**References**

[1] J. M. Aldaz, *Boundedness of averaging operators on geometrically doubling metric spaces*, Ann. Acad. Sci. Fenn. Math. 44 (2019), no. 1, 497-503.

[2] C. Anantharaman, J.-P. Anker, M. Babillot, A. Bonami, B. Demange, S. Grellier, F. Havard, P. Jaming, E. Lesigne, P. Maheux, J.-P. Otal, B. Schapira and J.-P. Schreiber, *Théorèmes ergodiques pour les actions de groupes*, Monographies de L’Enseignement Mathématique, Vol. 41, L’Enseignement Mathématique, Geneva, 2010.

[3] Á. Arroyo, J. G. Lorente, *On a class of singular measures satisfying a strong annular decay condition*, Proc. Amer. Math. Soc. 147 (2019), no. 10, 4409-4423.

[4] P. Auscher, E. Routin, *Local Tb theorems and Hardy inequalities*, J. Geom. Anal. 23 (2013), no. 1, 303-374.

[5] T. Bewley, *Extension of the Birkhoff and von Neumann ergodic theorems to semigroup actions*, Ann. Inst. H. Poincaré Sect. B (N.S.) 7 (1971), 283-291.

[6] G. D. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. USA 17 (1931), 656-660.

[7] E. Bishop, *Foundations of constructive analysis*, McGraw-Hill, NY, 1967.

[8] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets*, Publ. Math. IHES. 69 (1989), 5-41.

[9] E. Breuillard, *Geometry of locally compact groups of polynomial growth and shape of large balls*, Groups Geom. Dyn. 8 (2014), no. 3, 669-732.
[10] S. M. Buckley, *Is the maximal function of a Lipschitz function continuous?*, Ann. Acad. Sci. Fenn. Math. 24 (1999), no. 2, 519-528.

[11] A. P. Calderón, *A general ergodic theorem*, Ann. of Math. (2) 58 (1953), 182-191.

[12] A. P. Calderón, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. U.S.A. 59 (1968), 349-353.

[13] J. Chatard, *Applications des propriétés de moyenne d’un groupe localement compact à la théorie ergodique*, Ann. Inst. H. Poincaré Sect. B (N.S.) 6 (1970), 307-326.

[14] M. Christ, *A T(b) theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. 60/61 (1990), no. 2, 601-628.

[15] T. H. Colding, W. P. Minicozzi II, *Liouville theorems for harmonic sections and applications*, Comm. Pure Appl. Math. 51 (1998), no. 2, 113-138.

[16] R. Coifman, G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Étude de certaines intégrandes singulières, Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971.

[17] R. Coifman, G. Weiss, *Transference Methods in Analysis*, CBMS Regional Conferences in Mathematics, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 31. American Mathematical Society, Providence, R.I., 1976.

[18] G. David, J.-L. Journé, S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoamericana 1 (1985) 1-56.

[19] J. L. Doob, *Stochastic Processes*, John Wiley & Sons, Inc., New York; Chapman & Hall, Limited, London, 1953.

[20] Y. Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Sc. Math. France 101 (1973), 353-379.

[21] C. Herz, *The theory of p-spaces with an application to convolution operators*, Trans. Amer. Math. Soc. 154 (1971), 69-82.

[22] G. Hong, B. Liao, S. Wang, *Noncommutative maximal ergodic inequalities associated with doubling conditions*, to appear in Duke. Math. J.

[23] G. Hong, T. Ma, *Vector valued q-variation for differential operators and semigroups I*, Math. Z. 286 (2017), no. 1-2, 89-120.

[24] T. Hytönen, A. Kairema, *Systems of dyadic cubes in a doubling metric space*, Colloq. Math. 126 (2012), no. 1, 1-33.

[25] R. L. Jones, A. Seeger, J. Wright, *Strong variational and jump inequalities in harmonic analysis*, Trans. Amer. Math. Soc. 360, 6711-6742 (2008).

[26] S. Kalikow, B. Weiss, *Fluctuations of ergodic averages. Proceedings of the Conference on Probability, Ergodic Theory, and Analysis (Evanston, IL, 1997)*, Illinois J. Math. 43 (1999), no. 3, 480-488.

[27] J. Kinnunen, P. Shukla, *The structure of reverse Hölder classes on metric measure spaces*, Nonlinear Anal. 95 (2014), 666-675.

[28] J. Kinnunen, P. Shukla, *The distance of $L^\infty$ from BMO on metric measure spaces*, Adv. Pure Appl. Math. 5 (2014), no. 2, 117-129.

[29] B. Krause, *Polynomial Ergodic Averages Converge Rapidly: Variations on a Theorem of Bourgain*, (preprint). Available at arXiv:1402.1803, 2014.
B. Krause, P. Zorin-Kranich, Weighted and vector-valued variational estimates for ergodic averages, Ergodic Theory Dynam. Systems 38 (2018), no. 1, 244-256.

D. Lépingle, La variation d’ordre p des semi-martingales, Z. Wahrsch. Verw. Gebiete 36 (1976), no. 4, 295-316.

A. Lewko, M. Lewko, Estimates for the square variation of partial sums of Fourier series and their rearrangements, J. Funct. Anal. 262 (2012), no. 6, 2561-2607.

H. Lin, E. Nakai, D. Yang, Boundedness of Lusin-area and $g_λ^*$ functions on localized BMO spaces over doubling metric measure spaces, Bull. Sci. Math. 135 (2011), no. 1, 59-88.

E. Lindenstrauss, Pointwise theorems for amenable groups, Invent. Math. 146 (2001), no. 2, 259-295.

L. D. Lópe-Sánchez, J. M. Martell, J. Parcet, Dyadic harmonic analysis beyond doubling measures, Adv. Math. 267 (2014), 44-93.

V. Losert, On the structure of groups with polynomial growth, Math. Z. 195 (1986), 109-118.

M. Mirek, E. M. Stein, B. Trojan, $ℓ^p(ℤ)$-estimates for discrete operators of Radon type: variational estimates, Invent. Math. 209 (2017), no. 3, 665-748.

M. Mirek, E. M. Stein, P. Zorin-Kranich, Jump inequalities via real interpolation, Math. Ann. 376 (2020), no. 1-2, 797-819.

N. Moriakov, Fluctuations of ergodic averages for actions of groups of polynomial growth, Studia Math. 240 (2018), no. 3, 255-273.

A. Nevo, Pointwise ergodic theorems for actions of groups, Handbook of dynamical systems. Vol. 1B, 871-982, Elsevier B. V., Amsterdam, 2006.

A. L. T. Paterson, Amenability, Mathematical Surveys and Monographs, 29. American Mathematical Society, Providence, RI, 1988.

G. Pisier, Complex interpolation and regular operators between Banach spaces, Arch. Math. (Basel) 62 (1994), no. 3, 261-269.

G. Pisier, Q. Xu, The strong $p$-variation of martingale and orthogonal series, Probab. Theory Related fields 77 (1988), no. 4, 497-514.

A. Tempelman, Ergodic theorems for general dynamical systems, (Russian) Dokl. Akad. Nauk SSSR 176 (1967), 790-793.

A. Tempelman, Ergodic theorems for general dynamical systems, (Russian) Trudy Moskov. Mat. Obšč. 26 (1972), 95-132.

A. Tempelman, Pointwise ergodic theorems for bounded Lamperti representations of amenable groups, Proc. Amer. Math. Soc. 143 (2015), no. 11, 4989-5004.

R. Tessera, Volume of spheres in doubling metric measured spaces and groups of polynomial growth, Bull. Soc. Math. France 135 (2007), no. 1, 47-64.

N. Wiener, The ergodic theorem, Duke Math. J. 5 (1939), no. 1, 1-18.

P. Zorin-Kranich, Variation estimates for averages along primes and polynomials, J. Funct. Anal. 268 (2015), no. 1, 210-238.

P. Zorin-Kranich, Variational truncations of singular integrals on spaces of homogeneous type, (preprint). Available at arXiv:2004.04541, 2020.

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