The wavefront set of spherical Arthur representations

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April 14, 2022

Abstract

We calculate the wavefront set of spherical Arthur representations of split \( p \)-adic reductive groups - the spherical representations expected to occur as local factors of automorphic representations. We do this by developing new finer invariants for irreducible admissible representations of \( p \)-adic groups and showing that for spherical Arthur representations they can be computed in terms of equivariant perverse sheaves on the nilpotent cone of the Langlands dual group.

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1 Introduction

Let $k$ be a $p$-adic field with finite residue field $f$ of sufficiently large characteristic, algebraic closure $\overline{k}$ and let $G(k)$ be the $k$-points of a connected reductive group $G$ defined over $k$ and split over an unramified extension of $k$. For an admissible irreducible representation $(\pi, X)$ of $G(k)$, the wavefront set of $X$, denoted $WF(X)$, is a harmonic analytic invariant of $X$ of fundamental importance. Roughly speaking it measures the direction of the singularities of the character distribution $\Theta_X$ of $X$ near the identity. More precisely: the Harish-Chandra–Howe local character expansion dictates that there exists an open neighbourhood $V$ of the identity and coefficients $c_\mathcal{O}(X) \in \mathbb{C}$, one for each $\mathcal{O}$ in the collection, $\mathcal{N}_o(k)$, of $k$-rational nilpotent orbits of $g$, such that

$$\Theta_\pi(f) = \sum_{\mathcal{O} \in \mathcal{N}_o(k)} c_\mathcal{O}(X)\hat{\mu}_\mathcal{O}(f \circ \exp), \quad \forall f \in C^\infty_c(V)$$

(1)

where $\hat{\mu}_\mathcal{O}$ denotes the Fourier transform of the nilpotent orbital integral associated to $\mathcal{O} \in \mathcal{N}_o(k)$. The ($p$-adic) wavefront set is the set (not necessarily a singleton)

$$WF(X) := \max\{\mathcal{O} : c_\mathcal{O}(X) \neq 0\} \subseteq \mathcal{N}_o(k)$$

where the maximum is taken with respect to the closure ordering on the $k$-rational nilpotent orbits.
Much can be said about $X$ from the wavefront set. When $WF(X)$ consists of regular nilpotent elements, a famed result of Rodier [Rod75] states that $X$ admits a Whittaker model. If an orbit in $WF(X)$ meets a Levi subgroup of $G(k)$ then work of Moeglin and Waldspurger [MW87] shows that $X$ cannot be supercuspidal. From an analytic point of view the wavefront set controls the asymptotic growth of the space of vectors fixed by the Moy–Prasad filtration subgroups as you go further along the filtration [BM97, Section 5.1]. Finally for representations of Arthur type (by which we mean lie in an Arthur packet), the wavefront set of $X$, along with two other invariants should (according to a conjecture by Kawanaka stated in the introduction of [Mœg96]) uniquely determine $X$.

This distinguishes the wavefront set as a particularly powerful invariant to study. However, its determination in practice is notoriously difficult to compute. Moeglin and Waldspurger [MW87] have calculated the wavefront set for irreducible smooth representations of $GL_n$ and for irreducible subquotients of the regular principal series for split classical groups, but little is known in general. A slightly coarser invariant which one might hope to have more control over is the geometric wavefront set $\overline{k}WF(X)$. This is defined to be

$$
\overline{k}WF(X) := \max\{N_o(\overline{k}/k)(\mathcal{O}) : c_0(X) \neq 0\} \subseteq N_o(\overline{k})
$$

where $N_o(\overline{k})$ denotes that set of $\overline{k}$-rational nilpotent orbits and $N_o(\overline{k}/k)(\mathcal{O})$ denotes the (unique) $\overline{k}$-rational nilpotent orbit of $\mathfrak{g}$ that $\mathcal{O}$ lies in. Much more is known about the geometric wavefront set. For classical groups, Moeglin [Mœg96] showed that it must always be a special orbit, and in [Wal18] and [Wal20], Waldspurger computed $\overline{k}WF(X)$ for anti-tempered and tempered unipotent representations of the pure inner twists of the split form of $SO(2n+1)$. Moreover, in analogy with real reductive groups and finite groups of Lie type, it is expected that there is a single nilpotent orbit in the geometric wavefront set and that all the nilpotent orbits in $WF(X)$ lie in it. In this sense, $\overline{k}WF(X)$ is a good first approximation for $WF(X)$.

In the first part of this paper we shall concern ourselves with studying the wavefront set over an intermediate field. This will result in two new invariants which provide richer information than the geometric wavefront set. Let $K$ be the maximal unramified extension of $k$ in $\overline{k}$ and let $\mathfrak{f}$ be its residue field. Note that $\mathfrak{f}$ is naturally an algebraic closure of $\mathfrak{o}$. Define the unramified wavefront set to be

$$
\overline{k}WF(X) := \max\{N_o(K/k)(\mathcal{O}) : c_0(X) \neq 0\} \subseteq N_o(K)
$$

where $N_o(K)$ denotes the set of $K$-rational nilpotent orbits (which we henceforth refer to as unramified nilpotent orbits) and $N_o(K/k)(\mathcal{O})$ denotes the $K$-rational nilpotent orbit of $\mathfrak{g}$ that $\mathcal{O}$ lies in. The inspiration for this modification comes from the work of Barbasch and Moy [BM97] where they relate the coefficients $c_0(X)$ to representations of the reductive quotients of the parahoric subgroups of $G(k)$. The introduction of the field $K$ is inevitable if one wants to interpret geometric results from the reductive quotients back on the $p$-adic side, and much trouble is taken in [BM97] to then obtain results over $k$. What we show in this paper is that one should accept the field $K$ as
a fact of life - and once one does this many statements take a more natural form and a lot of new structure becomes apparent.

The first main result of this paper - which we now state - illustrates this well. Let $\mathcal{B}$ denote the Bruhat–Tits building of $G(k)$. For each face $c$ of $\mathcal{B}$ recall that we have the following short exact sequence

$$1 \to U_c(\mathfrak{o}) \to P_c(\mathfrak{o}) \to L_c(F_q) \to 1.$$ 

The group $L_c(F_q)$ - the reductive quotient of the parahoric subgroup $P_c(\mathfrak{o})$ - is a finite group of Lie type, and the space of fixed vectors $X^{U_c(\mathfrak{o})}$ is a finite dimensional representation of $L_c(F_q)$. As we alluded to earlier, the wavefront set is an invariant which also makes sense for representations of finite groups of Lie type and we write

$$\mathfrak{F}WF(X^{U_c(\mathfrak{o})})$$

for the (geometric) wavefront set of the representation $X^{U_c(\mathfrak{o})}$ of $L_c(F_q)$. This is a collection of nilpotent orbits of $L_c$ over $\mathfrak{F} = \mathfrak{f}$ (see Section 2.3.2 for precise details).

Inspired by the work of [BM97], [DeB02b] and [Wal96] we introduce a lifting map $L_c$ from the partially ordered set of $\mathfrak{F}$-rational nilpotent orbits of $L_c$ to the partially ordered set of $K$-rational nilpotent orbits of $G(k)$. The wavefront sets of the representations of the reductive quotients are then related to the wavefront set of $X$ by the following theorem.

**Theorem 1.1.** Let $(\pi, X)$ be a depth-0 representation of $G(k)$. Then

$$K\mathfrak{F}WF(X) = \max_{c \subseteq \mathcal{B}} \mathcal{L}_c(\mathfrak{F}WF(X^{U_c(\mathfrak{o})})).$$

(2)

In fact one can restrict $c$ to range over the faces (or vertices) of any fixed chamber of $\mathcal{B}$.

This easily yields the following corollary.

**Corollary 1.2.** Let $(\pi, X)$ be a depth-0 representation of $G(k)$. Then

$$\mathfrak{F}WF(X) = \max_{c_{c_0} \subseteq c_0} \mathcal{N}_c(\mathfrak{k}/K)(\mathcal{L}_c(\mathfrak{F}WF(X^{U_c(\mathfrak{o})})))$$

(3)

where $c_0$ is any chamber of $\mathcal{B}$.

This corollary cleanly repackages the main idea of [BM97, Section 5.1] (indeed it makes precise [BM97, Proposition 5.2]) and in a sense this alone would be a satisfactory conclusion to our foray into unramified territory. However from a philosophical perspective Theorem 1.1 suggests that the unramified wavefront set is natural in its own right and warrants further investigation.

The next section of this paper investigates the unramified wavefront set closer. The main practical obstruction to doing this is the partially ordered set $\mathcal{N}_c(K)$ for which very little is known. The main result of this section is a concrete parameterisation of this set. Let $G$ denote the complex reductive group with the same absolute root dataum as $G$ and let $\mathcal{N}_{c,c}$ denote the set of all pairs $(\mathcal{O}, C)$ where $\mathcal{O}$ is a complex nilpotent orbit of $G$ and $C$ is a conjugacy class of $A(\mathcal{O})$ - the $G$-equivariant fundamental group of $\mathcal{O}$. 
Theorem 1.3. There is a canonical bijection

\[ \theta : N_o(K) \sim N_{o,c}. \] (4)

The set \( N_{o,c} \) might look familiar to readers for its resemblance to the parameters arising in the Springer correspondence. In that setting one considers the set of pairs \((O, \rho)\) where \( O \) is a complex nilpotent orbit of \( G \) and \( \rho \) is an irreducible representation of \( A(O) \). The similarity is of course not precise, but certainly suggests some tantalising connections (the paper [CMBO21] explores this more concretely). More than simply being suggestive however, the set \( N_{o,c} \) is well studied in its own right. It naturally arises (non-canonically) as the parameterising set for nilpotent orbits of finite groups of Lie type and in Sommers’ work generalising the Bala–Carter theorem for nilpotent orbits. It is also the domain for a powerful extension of the incredibly important Barbasch–Lusztig–Spaltenstein–Vogan duality map \( d : N_o \rightarrow N_o^\vee \) going from complex nilpotent orbits of \( G \) to complex nilpotent orbits of the complex Langlands dual group \( G^\vee \) of \( G \). This extension \( d_S : N_{o,c} \rightarrow N_o^\vee \), discovered by Sommers, but also apparent in earlier work by Lusztig, extends the map \( d \) in the sense that

\[ d_S(O, 1) = d(O) \]

where 1 denotes the trivial conjugacy class, and is notable because in contrast to \( d \), the map \( d_S \) is surjective. Using the bijection \( \theta \) from Theorem 1.3 one can of course interpret the map \( d_S \) as a map

\[ d_S : N_o(K) \rightarrow N_o^\vee \]

purely between nilpotent orbits. This raises the very natural - and pertinent - question: is \( d_S \) an order reversing map? From a purely philisophical perspective the answer ought to be yes - duality maps should be order reversing. Indeed we provide some evidence to support this belief in Lemma 3.24. However the partial order on \( N_o(K) \) is difficult to study and the bijection \( \theta \) is not so well suited to give easy answers to this question. The best we can do in this paper is leave this as a conjecture.

Conjecture 1.4. The map \( d_S : N_o(K) \rightarrow N_o^\vee \) is order reversing.

But let us set aside this issue for the moment. The work of Achar in [Ach03] strongly suggests that there is merit in declaring \( d_S \) to be order-reversing. What we mean by this is we introduce a new order on \( N_o(K) \) where \( d_S \) is order reversing by design. For \( O_1, O_2 \in N_o(K) \) define

\[ O_1 \leq_A O_2 \quad \text{if} \quad N_o(\bar{k}/K)(O_1) \leq N_o(\bar{k}/K)(O_2) \quad \text{and} \quad d_S(O_1) \geq d_S(O_2). \]

This is the finest pre-order on \( N_o(K) \) that makes \( N_o(\bar{k}/K) \) order preserving and \( d_S \) order reversing. It is not a partial order because there are orbits \( O_1, O_2 \in N_o(K) \) such that \( O_1 \leq_A O_2 \) and \( O_2 \leq_A O_1 \) but \( O_1 \neq O_2 \). However, if we define \( O_1 \sim O_2 \)
when \( \mathcal{O}_1 \leq_A \mathcal{O}_2 \) and \( \mathcal{O}_2 \leq_A \mathcal{O}_1 \) then \( \leq_A \) descends to a partial order on \( \mathcal{N}_o(K) / \sim_A \), and there is a canonical bijection

\[
\tilde{\theta} : \mathcal{N}_o(K) / \sim_A \sim \mathcal{N}_{o,e}
\]

where \( \mathcal{N}_{o,e} \) is the set of all pairs \((\mathcal{O}, \bar{C})\) of complex nilpotent orbits of \( G \) and conjugacy classes \( \bar{C} \) of \( \bar{A}(\mathcal{O}) \) - Lusztig’s canonical quotient of \( A(\mathcal{O}) \). The crucial takeaway is that the partial order \( \leq_A \) on \( \mathcal{N}_o(k)/\sim_A \) is compatible with the saturation map \( \mathcal{N}_o(\bar{k}/K) \) and is considerably easier to compute than the closure ordering on \( \mathcal{N}_o(K) \). Using this new partial order we define the canonical unramified wavefront set to be

\[
K\mathrm{WF}(X) := \max_{c \leq_B} [\mathcal{L}_c(\bar{\theta}(\mathrm{WF}(X_{U,c(o)})))] \quad (\leq \mathcal{N}_o(K)/\sim_A)
\]

where \([\bullet] : \mathcal{N}_o(K) \to \mathcal{N}_o(K)/\sim_A\) is the natural quotient map. The compatibility between \( \leq_A \) and \( \mathcal{N}_o(\bar{k}/K) \) ensures that an analogue of Corollary \([12]\) holds for \( K\mathrm{WF}(X) \), but \( K\mathrm{WF}(X) \) also has the added benefit that it is frequently (conjecturally always) a singleton. When \( K\mathrm{WF}(X) \) is a singleton, if we view \( K\mathrm{WF}(X) \) as an element of \( \mathcal{N}_{o,e}(\bar{k}/K) \) (via \( \tilde{\theta} \)) and \( \bar{K}\mathrm{WF}(X) \) as an element of \( \mathcal{N}_o(\bar{k}) \) (under the natural isomorphism between \( \mathcal{N}_o(\bar{k}) \) and \( \mathcal{N}_o(k) \)), then \( K\mathrm{WF}(X) \) takes the form

\[
K\mathrm{WF}(X) = (\bar{K}\mathrm{WF}(X), \bar{C})
\]

for some \( \bar{C} \in \bar{A}(\bar{K}\mathrm{WF}(X)) \). For those familiar with representations of real reductive groups, this is reminiscent of the associated cycle of a representation, but of course differs crucially in that we are dealing with conjugacy classes rather than irreducible representations, and with \( \bar{A}(\bar{K}\mathrm{WF}(X)) \) instead of \( A(\bar{K}\mathrm{WF}(X)) \).

Let us now briefly digress to explain the terminology used. In the language of partial orders, Conjecture \([12]\) is equivalent to \([\bullet] : \mathcal{N}_o(K) \to \mathcal{N}_o(K)/\sim_A \) being a homomorphism. Under the assumption that this is true then

\[
K\mathrm{WF}(X) = \max\{[\mathcal{N}_o(K/k)(\mathcal{O})] \colon c_0(X) \neq 0\}
\]

for depth-0 representations and so indeed \( K\mathrm{WF}(X) \) is a ‘wavefront set’. If we further assume that all the orbits of \( K\mathrm{WF}(X) \) lie in a single geometric orbit, then \( K\mathrm{WF}(X) \) simply picks out the \( \sim_A \) classes of elements in \( K\mathrm{WF}(X) \) that minimise \( d_S \). Conjecturally there is a unique such class - a ‘canonical’ such class with respect to this property if you will. Perhaps ‘distinguished’ would have been a better modifier, but that adjective already has an important meaning in the context of nilpotent orbits.

The third and final section of this paper is dedicated to developing the tools needed to compute \( K\mathrm{WF}(X) \) for irreducible representations in the principal block of \( \mathrm{Rep}(G(k)) \) when \( G \) is split over \( k \). Recall that the principal block of \( \mathrm{Rep}(G(k)) \), which we denote \( \mathrm{Rep}_1(G(k)) \), consists of those representations that are generated by their Iwahori fixed vectors and is equivalent to the category of finite dimensional modules of the Iwahori–Hecke algebra \( \mathcal{H}_I \) of \( G(k) \). Moreover, by the theory of unrefined minimal \( K \)-types, the representations \( X_{U,c(o)}^{\mathfrak{u}_c(o)} \) are sums of principal series unipotent representations. The wavefront sets of such representations have a particularly simple
expression connected to the Hecke algebra of $L_c(F_q)$. We use the compatibility of these Hecke algebras with the Iwahori–Hecke algebra and a simple deformation argument to obtain an explicit algorithm for computing $K\ WF(X)$.

Finally, we use the tools developed in this section to compute $K\ WF(X)$ and $\bar{k}\ WF(X)$ for the spherical Arthur representations of a split adjoint group over $k$. For $G$ a reductive group defined and split over a number field $F$, these are expected to be the spherical representations arising as local factors of irreducible subrepresentations of $L^2_{\text{disc}}(G(F)\backslash G(A_F))$ (see [Jac84], [Lan76], [Kim96], [Mil13], [Meg91], [MW89], for proofs in various special cases. See [MHO17] for a uniform proof that all the spherical Arthur representations arise in this way). Note that knowledge of the geometric wavefront set of spherical Arthur representations provides valuable structural insight for automorphic representations. In particular the geometric wavefront set of the local factors bound the Fourier coefficients of the automorphic form (see [GGS20]). We now state our results for the non-archimedean spherical Arthur representations in terms of their Arthur parameters. Let $G$ be defined and split over the $p$-adic field $k$. Let $W_k$ denote the Weil group of $k$, and

$$WD_k = W_k \times \SL(2, \mathbb{C})$$

the Weil–Deligne group of $k$. Let $(\pi, X)$ be the spherical Arthur representation of $G(k)$ lying in the Arthur packet $\psi : WD_k \times \SL(2, \mathbb{C}) \to G^\vee$ that is trivial on $WD_k$. Within this packet, $X$ is the representation corresponding to the trivial representation of $A_\psi$ - the component group of the centraliser of the image of $\psi$. Let $\psi_0 = \psi \mid_{1 \times \SL(2, \mathbb{C})}$ and

$$n = d(\psi_0) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$  \hfill (5)

The nilpotent orbit $O^\vee := G^\vee.n$ completely determines the representation $X$ among the spherical Arthur representations and so we refer to $X$ as the spherical Arthur representation with parameter $O^\vee$. The final ingredient that we need in order to state the result is a refinement $d_A$ due to Achar [Ach03] of the duality map $d_S$. Suppose $G$ is the complex reductive group with dual $G^\vee$. Let $N^\vee_{o,c}, N^\vee_{o,\bar{c}}$ be the corresponding sets for $G^\vee$. The duality $d_A$ is a map

$$d_A : N^\vee_{o,c} \to N^\vee_{o,\bar{c}}$$

satisfying certain properties. In particular,

$$d_A(O^\vee, 1) = (d(O^\vee), C'),$$

for some class $C'$ which is the trivial class when $O^\vee$ is special in the sense of Lusztig.

**Theorem 1.5.** Let $X$ be the spherical Arthur representation with parameter $O^\vee \in N^\vee_o$. Then $K\ WF(X)$ is a singleton and

$$\tilde{\theta}(K\ WF(X)) = d_A(O^\vee, 1), \quad \bar{k}\ WF(X) = d(O^\vee).$$ \hfill (6)
1.1 Acknowledgements

I would like to thank Dan Ciubotaru and Kevin McGerty for their invaluable guidance and many helpful conversations. I would also like to thank Lucas Mason-Brown and Beth Romano for their careful proof reading of this paper, Maarten Solleveld for helpful comments and corrections on an earlier draft of this paper, Marcelo De Martino for teaching me about the Bruhat-Tits building, and Jonas Antor and Ruben La for many enlightening discussions on perverse sheaves.

2 The Wavefront Set

2.1 Setup and Notation

2.1.1 Basic Notation

Let $k$ be a nonarchimedean local field of characteristic 0 with residue field $\mathbb{F}_q$ of sufficiently large characteristic, ring of integers $\mathfrak{o} \subset k$, and valuation $\text{val}_k$. Let $p \subset \mathfrak{o}$ be the maximal ideal of $\mathfrak{o}$, fix an algebraic closure $\bar{k}$ of $k$ and let $K \subset \bar{k}$ be the maximal unramified extension of $k$ in $\bar{k}$. Let $\mathcal{O}$ be the ring of integers of $K$ and $\mathfrak{f}$ be the residue field for $K$. Note that $\mathfrak{f}$ is an algebraic closure for $\mathbb{F}_q$. Let $\chi : k \to \mathbb{C}^\times$ be an additive character of $k$ that is trivial on $p$ and non-trivial on $\mathfrak{o}$. $\chi$ descends to a character of $\mathbb{F}_q$ and we will refer to the resulting character also as $\chi$.

Let $G$ be a connected reductive algebraic group defined over $k$, that splits over an unramified extension of $k$. Let $G_K$ denote the base change of $G$ along $\text{Spec}(K/ \to \text{Spec}(k))$.

Note that $G_K$ is a split group. Let $T_K \subset G_K$ be a maximal $K$-split torus. For any field extension $F$ of $k$, we write $G(F)$, $\mathfrak{g}(F)$ etc. for the $F$ rational points.

Write $X^*(T_K, \bar{k})$ (resp. $X_*(T_K, \bar{k})$) for the lattice of algebraic characters (resp. co-characters) of $T_K(\bar{k})$, $\langle , \rangle$ for the canonical pairing between $X^*(T_K, \bar{k}), X_*(T_K, \bar{k})$ and write $\Phi(T_K, \bar{k})$ (resp. $\Phi^\vee(T_K, \bar{k})$) for the set of roots (resp. co-roots). Write

$$\mathcal{R} = (X^*(T_K, \bar{k}), \Phi(T_K, \bar{k}), X_*(T_K, \bar{k}), \Phi^\vee(T_K, \bar{k}), \langle , \rangle)$$

for the absolute root datum of $G$, and let $W$ the associated (finite) Weyl group.

Let $G_Z$ be the connected reductive algebraic group corresponding to the root datum $\mathcal{R}$ defined (and split) over $\mathbb{Z}$. Let $G^\vee_Z$ be the Langlands dual group of $G$, i.e. the connected reductive algebraic group corresponding to the root datum

$$\mathcal{R}^\vee = (X_*(T_K, \bar{k}), \Phi^\vee(T_K, \bar{k}), X^*(T_K, \bar{k}), \Phi(T_K, \bar{k}), \langle , \rangle)$$

defined (and split) over $\mathbb{Z}$. Set $T^\vee = X^*(T_K, \bar{k}) \otimes_\mathbb{Z} \mathbb{C}^\times$, regarded as a maximal torus in $G^\vee := G^\vee_Z(\mathbb{C})$ with Lie algebra $t^\vee = X^*(T_K, \bar{k}) \otimes_\mathbb{Z} \mathbb{C}$, a Cartan subalgebra of the Lie algebra $\mathfrak{g}^\vee$ of $G^\vee$. Define

\begin{align*}
T^\vee_R &= X^*(T_K, \bar{k}) \otimes_\mathbb{Z} \mathbb{R}_{>0} \\
\mathfrak{t}^\vee_\mathbb{R} &= X^*(T_K, \bar{k}) \otimes_\mathbb{Z} \mathbb{R} \\
T^\vee_c &= X^*(T_K, \bar{k}) \otimes_\mathbb{Z} S^1.
\end{align*}
There is a polar decomposition $T^\vee = T_o^\vee T_g^\vee$.

Let $\text{Field}_k$ denote the category of field extensions of $k$. Let $N$ be the functor from $\text{Field}_k$ to Set which takes a field extension $F$ of $k$ to the set of nilpotent elements of $\mathfrak{g}(F)$. By nilpotence in this context we mean the unstable points (in the sense of GIT) with respect to the adjoint action of $G(F)$, see [Deb02b, Section 2]. For $F$ algebraically closed this coincides with all the usual notions of nilpotence. Let $\mathcal{N}_o$ be the functor which takes $F$ to the set of orbits in $\mathcal{N}(F)$ under the adjoint action of $G(F)$. We briefly remark how $\mathcal{N}_o$ behaves on morphisms. Given field extensions $F_1, F_2 \in \text{Field}_k$ and a morphism $F_1 \to F_2$ we have natural inclusion maps

$$\mathfrak{g}(F_1) \to \mathfrak{g}(F_2) \text{ and } G(F_1) \to G(F_2).$$

Thus given a $G(F_1)$ orbit $\mathcal{O} \subset \mathcal{N}(F_1)$ we can form the orbit

$$G(F_2).\mathcal{O} \subset \mathcal{N}(F_2).$$

We define $\mathcal{N}_o(F_1 \to F_2)(\mathcal{O})$ to be this orbit. When we wish to emphasise the group we are working with we include it as a superscript e.g. $\mathcal{N}_o^{G_\mathbb{C}}$.

When $F$ is algebraically closed, we view $\mathcal{N}_o(F)$ as a partially ordered set with respect to the closure ordering in the Zariski topology. When $F$ is $k$ or $K$, we view $\mathcal{N}_o(F)$ as a pre-ordered set with respect to the closure ordering in the topology induced by the topology on $F$. When $F = k$ it is well known that the pre-order is a partial order [cite DeBacker]. When $F = K$ we will show in [3.8] that the pre-order is a partial order. For brevity we will write $\mathcal{N}(F'/F)$ (resp. $\mathcal{N}_o(F'/F)$) for $\mathcal{N}(F \to F')$ (resp. $\mathcal{N}_o(F \to F')$) where $F \to F'$ is a morphism of fields.

Recall the following classical result.

**Lemma 2.1** ([Pom77], [Pom80]). Let $F \in \text{Field}_k$ be algebraically closed with good characteristic for $G$. Then there is canonical isomorphism of partially ordered sets $\Lambda_F : \mathcal{N}_o(F) \cong \mathcal{N}_o^{G_\mathbb{C}}(\mathbb{C})$.

When $F$ is algebraically closed let $\mathcal{N}_{o,sp}(F)$ denote the set of special orbits in the sense of Lusztig.

### 2.1.2 Buildings, Parahorics and Associated Notation

Let $\mathcal{B}(G, k)$ (resp. $\mathcal{B}(G, K)$) denote the (enlarged) Bruhat–Tits building for $G(k)$ (resp. $G(K)$). We identify $\mathcal{B}(G, k)$ with the Gal($K/k$)-fixed points of $\mathcal{B}(G, K)$. We use the notation $c \subseteq \mathcal{B}(G)$ to indicate that $c$ is a face of $\mathcal{B}$. Given a maximal $k$-split torus $T$, write $\mathcal{A}(T, k)$ for the corresponding apartment in $\mathcal{B}(G, k)$. For an apartment $\mathcal{A}$ of $\mathcal{B}(G, k)$ and $\Omega \subseteq \mathcal{A}$ we write $\mathcal{A}(\Omega, \mathcal{A})$ for the smallest affine subspace of $\mathcal{A}$ containing $\Omega$. We write $\Phi(T, k)$ (resp. $\Psi(T, k)$) for the set of roots (resp. affine roots) of $T(k)$ on $G(k)$. For $\psi \in \Psi(T, k)$ write $\psi \in \Phi(T, k)$ for the gradient of $\psi$, and $W = W(T, k)$ for the Weyl group of $G(k)$ with respect to $T(k)$. For a face $c \subseteq \mathcal{B}(G)$ there is a reductive subgroup $P_c^+$ of $G$ defined over $\mathfrak{o}$ such that $P_c^+(\mathfrak{o})$ is the stabiliser of $c$ in $G(k)$. There is an exact sequence

$$1 \to U_c(\mathfrak{o}) \to P_c^+(\mathfrak{o}) \to L_c^+(\mathbb{F}_q) \to 1,$$

(8)
where $U_c(o)$ is the pro-unipotent radical of $P_c^+(o)$ and $L_c^+$ is the special fibre of $P_c^+$. Let $L_c$ denote the identity component of $L_c^+$, and let $P_c$ be the reductive subgroup of $P_c^+$ defined over $o$ such that $P_c(o)$ is the inverse image of $L_c(F_q)$ in $P_c^+(o)$. The groups $P_c$ are called parahoric subgroups of $G(k)$. We have analogous short exact sequences

$$1 \to U_c(o) \to P_c(o) \to L_c(F_q) \to 1,$$

and one on the level of the Lie algebra

$$0 \to u_c(o) \to p_c(o) \to l_c(F_q) \to 0.$$  

When $c$ is a chamber in the building, then we call $P_c$ an Iwahori subgroup of $G$. Let

$$U = \bigcup_{c \subseteq B(G(k))} U_c(o) \subseteq g(k), \quad U = \bigcup_{c \subseteq B(G(k))} U_c(o) \subseteq G(k).$$

These are the topologically nilpotent and topologically unipotent elements of $g(k)$ and $G(k)$ respectively.

### 2.1.3 Fourier Transforms

By [AR00, Proposition 4.1], for $p$ sufficiently large (see reference for a precise bound for $p$), there exists a symmetric, non-degenerate $G(k)$-invariant bilinear form

$$B : g(k) \times g(k) \to k$$

such that for every face $c$ of $B(G(k))$ we have

$$p_c(o) = \{ X \in g(k) : B(X, Y) \in p, \forall Y \in u_c(o) \}.$$ 

Such a bilinear form naturally descends for each face $c$ of $B(G(k))$ to a symmetric, non-degenerate $L_c(F_q)$-invariant bilinear form

$$B_c : l_c(F_q) \times l_c(F_q) \to F_q.$$ 

Fix a Haar measure $\mu_{g(k)}$ on $g(k)$. For a function $f \in C_c^\infty(g(k))$ we define the Fourier transform of $f$ to be

$$\hat{f}(X) := FT_{g(k)}(f)(X) := \int_{g(k)} \chi(B(X, Y)) f(Y) d\mu_{g(k)}(Y).$$  \hfill (11)

Let $c$ be a face of $B(G(k))$ and $h : l_c(F_q) \to \mathbb{C}$ a function. We define the Fourier transform of $h$ to be

$$\hat{h}(x) := FT_{l_c(F_q)}(h)(x) := \sum_{y \in l_c(F_q)} \chi(B_c(x, y)) h(y).$$  \hfill (12)

We define $\tilde{h} : g(k) \to \mathbb{C}$ to be the function given by

$$\tilde{h}(X) = \begin{cases} 
  h(X + u_c(o)) & \text{if } X \in p_c(o) \\
  0 & \text{otherwise.}
\end{cases}$$  \hfill (13)

We say $\tilde{h}$ is inflated form $h$ and we have that

$$FT_{g(k)}(\tilde{h})(X) = \mu_{g(k)}(u_c(o)) \cdot FT_{l_c(F_q)}(h)(X).$$
2.1.4 The Harish-Chandra–Howe Local Character Expansion

Let \( \exp: \mathfrak{u} \to U \) be the exponential map defined in [BM97, Lemma 3.2] and [Wal95, Section 3.3] (see references for bounds on \( p \)). The map \( \exp: \mathfrak{u} \to U \) has the property that for every face \( c \) of \( B(\mathbf{G}(k)) \), \( \exp(\mathfrak{u}_c(\mathfrak{o})) = U_c(\mathfrak{o}) \) and \( \exp \) descends to the exponential from the nilpotent elements of \( L_c(\mathbb{F}_q) \) to the unipotent elements of \( L_c(\mathbb{F}_q) \). For a function \( f \in C^\infty_c(U) \) let \( f \circ \exp \) denote the function in \( C^\infty_c(\mathfrak{g}(k)) \) given by

\[
 f \circ \exp(X) = \begin{cases} 
 f(\exp(X)) & \text{if } X \in \mathfrak{u} \\
 0 & \text{if } X \notin \mathfrak{u}.
\end{cases}
\] (14)

For \( \mathcal{O} \in \mathcal{N}_o(k) \) let \( \mu_\mathcal{O} \) denote the corresponding nilpotent orbital integral. We have the following result due to DeBacker (building on work by Waldspurger [Wal95]).

**Theorem 2.2.** ([Deb02a, Theorem 3.5.2]) Let \((\pi, X)\) be a depth 0 admissible representation of \( \mathbf{G}(k) \). Then there exists \( c_\mathcal{O}(X) \in \mathbb{C} \) for each \( \mathcal{O} \in \mathcal{N}_o(k) \) such that for \( f \in C^\infty_c(U) \) we have

\[
 \Theta_\pi(f) = \sum_{\mathcal{O} \in \mathcal{N}_o(k)} c_\mathcal{O}(X) \hat{\mu}_\mathcal{O}(f \circ \exp). \] (15)

We remark that the local character expansion, and in particular the coefficients \( c_\mathcal{O}(X) \), always exists for admissible smooth representations. The point of this theorem is that for depth 0 representations the expansion is valid for functions supported on \( U \).

2.1.5 The Wavefront Set of Representations of p-adic Groups

Let \((\pi, X)\) be an admissible representation of \( \mathbf{G} \). The \( (p\text{-adic}) \) wavefront set is

\[
 \text{WF}(X) := \max_{\mathcal{O}: \mathcal{O}_c(X) \neq 0} \mathcal{O},
\]

the unramified wavefront set is

\[
 \text{KW}(X) := \max_{\mathcal{O}: \mathcal{O}_c(\mathfrak{g}_k) \neq 0} \mathcal{N}_o(K/k)(\mathcal{O}),
\]

and the geometric wavefront set is

\[
 \text{GW}(X) := \max_{\mathcal{O}: \mathcal{O}_c(\mathfrak{g}_k) \neq 0} \mathcal{N}_o(\mathfrak{k}/k)(\mathcal{O}).
\]

**Remark 2.3.** In analogy with real groups and finite groups of Lie type it is expected that \( \text{KW}(X) \) consists of a single nilpotent orbit - \( \mathcal{O} \) say. Moreover it is expected that for all \( \mathcal{O}' \in \text{WF}(X) \), \( \mathcal{N}_o(\mathfrak{k}/k)(\mathcal{O}') = \mathcal{O} \) (this is a strictly stronger condition than \( \text{KW}(X) \) being a singleton).
2.2 Lifting Nilpotent Orbits and Closure Relations

2.2.1 The Lifting Map

Let $h$ be the Coxeter number of the absolute Weyl group for $G$. In this section we assume that $p > 3(h - 1)$ and that $p$ satisfies the conditions of section 2.1.4 so that we can apply the results of [DeB02b] to $\mathfrak{g}(k)$. Let $A$ be an apartment of $B(G(k))$. For faces $c_1, c_2$ in $A$ with $A(c_1, A) = A(c_2, A)$ the projection maps

\[ P_{c_1}(o) \cap P_{c_2}(o) \rightarrow L_{c_1}(\mathbb{F}_q), \quad P_{c_1}(o) \cap P_{c_2}(o) \rightarrow L_{c_2}(\mathbb{F}_q) \]

are both surjective with kernel $U_{c_1}(o) \cap U_{c_2}(o)$ and so there is an isomorphism

\[ i_{c_2,c_1} : L_{c_1}(\mathbb{F}_q) \rightarrow L_{c_2}(\mathbb{F}_q). \]

We similarly obtain an isomorphism

\[ j_{c_2,c_1} : L_{c_1}(\mathbb{F}_q) \rightarrow L_{c_2}(\mathbb{F}_q) \]

which is compatible with $i_{c_2,c_1}$ in the following sense:

\[ j_{c_2,c_1}(Ad(h)x) = Ad(i_{c_2,c_1}(h))j_{c_2,c_1}(x) \quad (16) \]

for all $h \in L_{c_1}(\mathbb{F}_q), x \in L_{c_1}(\mathbb{F}_q)$. Let

\[ I^k = \{(c, x) : c \subseteq B(G(k)), x \in \mathcal{N}_c^L(\mathbb{F}_q)\}. \]

Let $I^k_d$ denote the set of pairs $(c, x) \in I^k$ where $x$ is a distinguished nilpotent element of $L_c(\mathbb{F}_q)$. For $(c, x) \in I^k$ let $C(c, x)$ denote the preimage of $x$ in $p_c(o)$. For $(c_1, x_1), (c_2, x_2) \in I^k$ we define $(c_1, x_1) \sim_k (c_2, x_2)$ if there exists an $H \in G(k)$ and an apartment $A$ such that

\[ A(c_2, A) = A(Hc_1, A), \text{ and } x_2 = j_{c_2,Hc_1}(Hx_1). \]

Given an $(c, x) \in I^k$ one can attach to it, as in [BM97] and [DeB02b], a well defined nilpotent orbit $L_c(x) \in \mathcal{N}_c(o(k))$ called its lift. It has the following two useful equivalent characterisations (due to Debacker in [DeB02b] Lemma 5.3.3):

1. If $x$ is included into an $s_{t_2}$-triple $x, h, y \in L_c(\mathbb{F}_q)$, and $X, H, Y \in p_c(o)$ is an $s_{t_2}$-triple such that their images in $L_c(\mathbb{F}_q)$ are $x, h, y$ respectively (such an $s_{t_2}$-triple always exists), then $L_c(x) = G(k)X$. 

2. $L_c(x)$ is the unique minimal element of \{ $\mathcal{O} \in \mathcal{N}_c(o(k)) : \mathcal{O} \cap C(c, x) \neq \emptyset$ \}.

Let $\mathcal{O} \in \mathcal{N}_c(o(k))$. The nilpotent orbit $L_c(x)$ is independent of the choice of $x \in \mathcal{O}$; we write $L_c(\mathcal{O})$ for the resulting nilpotent orbit. Define

\[ I^k_0 = \{(c, \mathcal{O}) : c \subseteq B(G(k)), \mathcal{O} \in \mathcal{N}_c^L(\mathbb{F}_q)\} \]

and define $I^k_{o,d}$ to be the subset of $I^k$ consisting of pairs $(c, \mathcal{O})$ where $\mathcal{O}$ is a distinguished nilpotent orbit of $L_c(\mathbb{F}_q)$. For $\mathcal{O} \in \mathcal{N}_c(o(k))$ write $I^k(\mathcal{O})$ for the set $\{(c, x) \in I^k : L_c(x) = \mathcal{O}\}$. Analogously define $I^k_0(\mathcal{O}), I^k_d(\mathcal{O}), I^k_{o,d}(\mathcal{O})$. We have the following result due to Barbasch and Moy [BM97] and Debacker [DeB02b] classifying the nilpotent orbits of $\mathfrak{g}(k)$.
Theorem 2.4. The map $I^k_d \to \mathcal{N}_c(k)$, $(c,x) \mapsto \mathcal{L}_c(x)$ descends to a bijective correspondence between $I^k_d/\sim_k$ and $\mathcal{N}_c(k)$.

Note that for all the results in this section we are using the results from [DeB02b] with $r = 0$.

We can similarly define $I^K$, $I^K_d$, $I^K_o$, $I^K_{o,d}$, $I^K(\emptyset)$, $I^K_d(\emptyset)$, $I^K_o(\emptyset)$, $I^K_{o,d}(\emptyset)$, $\sim_K$, $\mathcal{C}$, and $\mathcal{L}_c$ for $G(K)$ and the results in this section hold verbatim for these objects too.

Let $I$ be a face of $B(G,K)$ and let $X,H,Y$ be an $\mathfrak{sl}_2$-triple contained in $\mathfrak{p}_c(\mathfrak{D})$. Then

$$U_c(\mathfrak{D})(X + c_{u_c(\mathfrak{D})}(Y)) = X + u_c(\mathfrak{D})$$

(17)

where $c_{u_c(\mathfrak{D})}(Y)$ denotes the centraliser of $Y$ in $u_c(\mathfrak{D})$.

Proof. Since we are only looking at $G(K) = G_K(K)$, and $G_K$ is split, we may as well assume that $G$ is also split over $k$. Since $G(K)$ acts transitively on the apartments of $B(G,K)$ we may also assume that $c \in B(G,k)$. There is nothing to prove for the $\subseteq$ direction. For the $\supseteq$ direction let $Z \in X + u_c(\mathfrak{D})$. Since $Z, X, Y \in \mathfrak{g}(K)$ there is a finite unramified extension $F$ of $k$ such that $Z, X, Y \in \mathfrak{g}(F)$. Let $\mathfrak{o}_F$ be the ring of integers for $F$. Then since $\mathfrak{D} \cap F = \mathfrak{o}_F$ we have that $Z \in X + u_c(\mathfrak{o}_F)$. Since $F$ is complete we can apply [DeB02b, Lemma 5.2.1] to $G(F)$ and so

$$U_c(\mathfrak{o}_F)(X + c_{u_c(\mathfrak{o}_F)}(Y)) = X + u_c(\mathfrak{o}_F)$$

Thus

$$Z \in U_c(\mathfrak{o}_F)(X + c_{u_c(\mathfrak{o}_F)}(Y)) \subset U_c(\mathfrak{D})(X + c_{u_c(\mathfrak{D})}(Y))$$

as required.

2.2.2 Closure relations

The set $I^K_o$ comes with additional structure that $I^K_o$ does not. For $(c_1, \mathcal{O}_1), (c_2, \mathcal{O}_2) \in I^K_o$ define

$$(c_1, \mathcal{O}_1) \leq (c_2, \mathcal{O}_2) \text{ if } c_1 = c_2 \text{ and } \mathcal{O}_1 \leq \mathcal{O}_2.$$ 

The following result is implied by the proofs in [BM97, Section 3.14].

Proposition 2.6. Let $(c_1, \mathcal{O}_1), (c_2, \mathcal{O}_2) \in I^K_o$ and suppose $(c_1, \mathcal{O}_1) \leq (c_2, \mathcal{O}_2)$. Then $\mathcal{L}_{c_1}(\mathcal{O}_1) \leq \mathcal{L}_{c_2}(\mathcal{O}_2)$.

In other words, the map $\mathcal{L} : I^K_o \to \mathcal{N}_c(K)$, $(c, \mathcal{O}) \mapsto \mathcal{L}_c(\mathcal{O})$ is non-decreasing. In section 3.2.3 we prove that $\mathcal{L}$ is in fact strictly increasing. Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{N}_c(K)$ and suppose $\mathcal{O}_1 \leq \mathcal{O}_2$.

1. We say $\mathcal{O}_1 \leq \mathcal{O}_2$ is a lifted closure relation if there exists a face $c$ of $B(G,K)$ and $(c, \mathcal{O}_1^i) \in I^K_o(\mathcal{O}_i)$ for $i = 1, 2$ such that $(c, \mathcal{O}_1^i) \leq (c, \mathcal{O}_2^i)$ (cf. proposition 2.6).
2. We say $\mathcal{O}_1 \leq \mathcal{O}_2$ is a degenerate closure relation if there exists $(c,x) \in I^K(\mathcal{O}_1)$ such that $\mathcal{O}_2 \cap \mathcal{C}(c,x) \neq \emptyset$ (cf. property 2 of section 2.2.1).

The following proposition shows that every closure relation in $\mathcal{N}_o(K)$ can be broken down into a lifted closure relation and a degenerate closure relation.

**Theorem 2.7.** Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{N}_o(K)$, and suppose $\mathcal{O}_1 \leq \mathcal{O}_2$. Then there exists a $\mathcal{O}_{1,5} \in \mathcal{N}_o(K)$ such that $\mathcal{O}_1 \leq \mathcal{O}_{1,5} \leq \mathcal{O}_2$ where $\mathcal{O}_1 \leq \mathcal{O}_{1,5}$ is a lifted closure relation and $\mathcal{O}_{1,5} \leq \mathcal{O}_2$ is a degenerate closure relation.

**Proof.** Let $(c, \mathcal{O}_1') \in I^K(\mathcal{O}_1)$. Let $X \in \mathcal{O}_1 \cap p_c(\mathcal{O})$ and let $x$ denote the image of $X$ in $l_c(\overline{F}_q)$. Since $p_c(\mathcal{O})$ is open in $g(K)$, $p_c(\mathcal{O}) \cap \mathcal{O}_2 \neq \emptyset$. Let $\mathcal{O}'$ be the image of $\mathcal{O}_2 \cap p_c(\mathcal{O})$ in $l_c(\overline{F}_q)$. We claim that $\mathcal{O}_1' \subseteq \mathcal{O}'$. Let $U$ be an open subset of $l_c(\overline{F}_q)$ containing $x$. Let $\overline{U}$ be the preimage in $p_c(\mathcal{O})$. Since $u_c(\mathcal{O})$ is open, the projection map $p_c(\mathcal{O}) \rightarrow l_c(\overline{F}_q)$ is continuous and so $\overline{U}$ is open and contains $X$. Thus $\overline{U} \cap \mathcal{O}_2 \neq \emptyset$ and so $U \cap \mathcal{O}' \neq \emptyset$. This proves the claim. Write $\mathcal{O}' = \cup_i \mathcal{O}'(i)$ as a union of $l_c(\overline{F}_q)$ nilpotent orbits. Then since $\mathcal{O}_1 \subseteq \overline{U}$ there exists an $i$ such that $\mathcal{O}_1 \leq \mathcal{O}'(i)$. Let $\mathcal{O}_1' = \mathcal{O}'(i)$. By construction $\mathcal{O}_1' := l_c(\mathcal{O}_1')$ has the required properties. \hfill \Box

For $\mathcal{O} \in \mathcal{N}_o(k)$ we say $X, H, Y$ is an $\mathfrak{s}l_2$-triple for $\mathcal{O}$ if they are an $\mathfrak{s}l_2$-triple and $X \in \mathcal{O}$. We now show that $\mathcal{N}_o(K/k) : \mathcal{N}_o(k) \rightarrow \mathcal{N}_o(K)$ and $\mathcal{N}_o(k/k) : \mathcal{N}_o(k) \rightarrow \mathcal{N}_o(k)$ are strictly increasing.

**Lemma 2.8.** Let $\mathcal{O} \in \mathcal{N}_o(k)$ and let $X, H, Y$ be an $\mathfrak{s}l_2$-triple for $\mathcal{O}$. Let $s = X + c_{g(k)}(Y)$ (a Slodowy slice for $\mathcal{O}$). Then

1. $\mathcal{O} \cap s = \{X\}$,
2. if $\mathcal{O}' \in \mathcal{N}_o(k)$ and $\mathcal{O}' \cap s \neq \emptyset$ then $\mathcal{O}' \cap s \neq \emptyset$.

**Proof.** We have the decomposition

$$g(k) = [g(k), X] \oplus c_{g(k)}(Y) \quad (18)$$

and $[g(k), X]$ is the tangent space of $\mathcal{O}$ at $X$. Thus

$$T_X(s \cap \mathcal{O}) \hookrightarrow T_Xs \cap T_X\mathcal{O} = 0$$

and so $s \cap \mathcal{O}$ is discrete. However, if $X'$ is in $s \cap \mathcal{O}$ let $\lambda_{X'} : G_m \rightarrow G$ be a 1-parameter $k$ subgroup such that

$$\lambda_{X'}(t).X' = t^{-2}X'$$

and write $X' = X + Z$ where $Z \in c_{g(k)}(Y)$. Let $\lambda$ be the 1-parameter $k$ subgroup attached to $H$. Write $g(k)(i)$ for the set of $W \in g(k)$ such that $\lambda(t).W = t^iW$ and let

$$g(k)_{\leq 0} = \bigoplus_{i \leq 0} g(k)(i).$$
Since \( c_{g(k)}(Y) \subseteq g(k)(\leq 0) \), write \( Z = \sum_{i \leq 0} Z_i \) where \( Z_i \in g(k)(i) \). Then

\[
\lambda(t^{-1})\lambda_X(t^{-1})X' = X + \sum_{i \leq 0} t^{2-i}Z_i \in \mathcal{O} \cap s
\]  

for all \( t \in k \) and \( \to X \) as \( t \to 0 \). Since \( s \cap \mathcal{O} \) is discrete this means that \( X' = X \). This proves 1.

Let \( \text{Ad} : G(k) \times s \to g(k) \) be the restriction of the adjoint map. Ad is smooth with differential \( T_{1,X} \) which is onto by equation 18. All the varieties in question are smooth and so there exists a Zariski open (and hence open in the topology induced by \( k \)) subset \( V \) in \( G(k).s \) containing \( X \). We have that \( \overline{V} \cap s \neq \emptyset \) and so \( \overline{V} \cap G(k).s \neq \emptyset \). Since \( s \cap \overline{V} \) is closed and non-empty the argument for part 1. shows that \( X \in s \cap \overline{V} \). It follows that \( X \in \overline{V} \cap V \). But \( V \) is open and so \( \mathcal{O}' \cap V \neq \emptyset \). It follows that \( \mathcal{O}' \cap G(k).s \neq \emptyset \) and so \( \mathcal{O}' \cap s \neq \emptyset \) as required. \( \blacksquare \)

**Corollary 2.9.** Let \( \mathcal{O}, \mathcal{O}' \in N_o(k) \). Let \( X, H, Y \) be an \( sl_2 \)-triple for \( \mathcal{O} \) and \( s = X + c_{g(k)}(Y) \). Then

1. \( \mathcal{O} = \mathcal{O}' \) iff \( \mathcal{O}' \cap s \) is a singleton,
2. \( \mathcal{O} < \mathcal{O}' \) iff \( \mathcal{O}' \cap s \) has more than one element,

**Proof.** By Lemma 2.8.1, if \( \mathcal{O} = \mathcal{O}' \) then \( \mathcal{O}' \cap s = \{X\} \). If \( \mathcal{O}' \cap s \) is a singleton then it is closed and the same argument as Lemma 2.8.1 gives that \( \mathcal{O}' \cap s = \{X\} \) and so \( \mathcal{O} = \mathcal{O}' \). If \( \mathcal{O} < \mathcal{O}' \) then \( X \in \overline{V} \cap s \). By Lemma 2.8.2, \( \mathcal{O}' \cap s \neq \emptyset \). It cannot consist of a single element since by part 1 this would imply \( \mathcal{O} = \mathcal{O}' \). Thus \( \mathcal{O}' \cap s \) consists of more than one element. If \( \mathcal{O}' \cap s \) consists of more than one element then \( \overline{V} \cap s \neq \emptyset \) and so contains \( X \). Thus \( \mathcal{O} \leq \mathcal{O}' \). But \( \mathcal{O} \neq \mathcal{O}' \) since \( \mathcal{O}' \cap s \) is not a singleton and so \( \mathcal{O} < \mathcal{O}' \). \( \blacksquare \)

Analogous results to Lemma 2.8 and corollary 2.9 hold for nilpotent orbits of \( g(\bar{k}) \), though different proof methods must be used (see [BDT20] Lemma 5.10) for details.

**Theorem 2.10.** Let \( \mathcal{O}, \mathcal{O}' \in N_o(k) \).

1. If \( \mathcal{O} < \mathcal{O}' \) then \( N_o(K/k)(\mathcal{O}) < N_o(K/k)(\mathcal{O}') \).
2. If \( \mathcal{O} < \mathcal{O}' \) then \( N_o(\bar{k}/k)(\mathcal{O}) < N_o(\bar{k}/k)(\mathcal{O}') \).

**Proof.** Clearly 2 implies 1 so it suffices to show that \( \mathcal{O} < \mathcal{O}' \implies N_o(\bar{k}/k)(\mathcal{O}) < N_o(\bar{k}/k)(\mathcal{O}') \). Let \( X, H, Y \) be an \( sl_2 \)-triple for \( \mathcal{O} \) and let \( s = X + c_{g(\bar{k})}(Y) \). If \( \mathcal{O} < \mathcal{O}' \) then \( \mathcal{O} \cap s \) has more than one element. Let \( s = X + c_{\bar{g}}(Y) \supseteq s \). Then \( N_o(\bar{k}/k)(\mathcal{O}) \cap s \) contains \( \mathcal{O}' \cap s \) and so also has more than one element. Thus \( N_o(\bar{k}/k)(\mathcal{O}) < N_o(\bar{k}/k)(\mathcal{O}') \). \( \blacksquare \)
2.3 The Wavefont Set of Representations of Finite Groups of Lie Type

2.3.1 Generalised Gelfand–Graev Representations

For this section and the next only let \( G \) be a connected reductive group defined over \( \mathbb{F}_q \). Fix an algebraic closure \( \overline{\mathbb{F}}_q \) of \( \mathbb{F}_q \) and let \( F : \mathbb{G}(\overline{\mathbb{F}}_q) \to \mathbb{G}(\mathbb{F}_q) \) be the associated geometric Frobenius (so that \( \mathbb{G}(\overline{\mathbb{F}}_q)^F = \mathbb{G}(\mathbb{F}_q) \)). Let \( h \) be the coxeter number of the absolute Weyl group for \( G \) and suppose \( p > 3(h-1) \). Then we can operate with \( \mathfrak{g}(\mathbb{F}_q) \) as if we were in characteristic 0 [BM97, Section 2.1]. In particular the nilpotent cone \( \mathcal{N}(\mathbb{F}_q) \) of \( \mathfrak{g}(\mathbb{F}_q) \) may identified with the unipotent cone of \( \mathbb{G}(\mathbb{F}_q) \) via the exponential map \( \exp : \mathcal{N}(\mathbb{F}_q) \to \mathcal{U}(\mathbb{F}_q) \). Let \( \mathcal{B} : \mathfrak{g}(\mathbb{F}_q) \times \mathfrak{g}(\mathbb{F}_q) \to \mathbb{F}_q \) be a symmetric, non-degenerate \( G \)-invariant bilinear form and \( \chi : \mathbb{F}_q \to \mathbb{C}^\times \) be a non-trivial character. Recall that for a function \( f : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C} \) the fourier transform of \( f \) is \( \hat{f}(x) = \sum_{x \in \mathfrak{g}(\mathbb{F}_q)} \chi(\mathcal{B}(x,y))f(y) \).

For a nilpotent element \( n \in \mathcal{N}(\mathbb{F}_q) \) we may associate to it a representation \( \Gamma_n \) of \( \mathbb{G}(\mathbb{F}_q) \) called the associated Generalised Gelfand–Graev Representation or GGGR for short (see [Lus92, Section 2] for details on its construction). Write \( \gamma_n \) for the character of \( \Gamma_n \). \( \gamma_n \) has the following key properties (due to Kawanaka [Kaw87], [Kaw82])

1. \( \gamma_n \) only depends on the \( \text{Ad}(\mathbb{G}(\mathbb{F}_q)) \)-orbit of \( n \). If \( \bigcirc = \mathbb{G}(\mathbb{F}_q).n \) write \( \Gamma_{\bigcirc} \) (resp. \( \gamma_{\bigcirc} \)) for the resulting representation (resp. character),

2. the support of \( \gamma_n \) is contained in the closure of \( \mathbb{G}(\mathbb{F}_q).\exp(n) \).

Let \( n = e, h, f \) be an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g}(\mathbb{F}_q) \) and \( \Sigma = -f + c_{\mathfrak{g}(\mathbb{F}_q)}(e) \). Let \( r(n) = \frac{1}{2}(\dim \mathfrak{g}(\mathbb{F}_q) - \dim c_{\mathfrak{g}(\mathbb{F}_q)}(n)) \). We have the following result due to Lusztig about the fourier transform of \( \gamma_n \circ \exp \) (which we will also refer to as \( \gamma_n \)).

**Proposition 2.11.** [Lus92, Proposition 2.5, Proposition 6.13] Let \( n, n' \in \mathcal{N}(\mathbb{F}_q) \).

1. \( \hat{\gamma}_n(y) = q^{r(n)}\mathbb{N}\{g \in \mathbb{G}(\mathbb{F}_q) : g \cdot y \in \Sigma \} \) for all \( y \in \mathfrak{g}(\mathbb{F}_q) \),

2. if \( \hat{\gamma}_n(n') \neq 0 \), then \( n \) must lie in the closure of \( \mathbb{G}(\mathbb{F}_q).n' \),

3. if \( n' \in \mathbb{G}(\mathbb{F}_q).n \) and \( \hat{\gamma}_n(n') \neq 0 \), then \( n' \in \mathbb{G}(\mathbb{F}_q).n \),

4. \( \hat{\gamma}_n(n) = q^{r(n)}\mathbb{N}_\mathbb{G}(\mathbb{F}_q)(n) \).

2.3.2 The Kawanaka Wavefront Set

The Kawanaka wavefront set is the analagous notion of the \( p \)-adic wavefront set for representations of finite groups of Lie type. In the finite groups of Lie type setting it was introduced by Kawanaka and so we refer to it as the Kawanaka wavefront set. Let \( (\rho, V) \) be an irreducible representation of \( \mathbb{G}(\mathbb{F}_q) \) and \( \chi_V \) be the character afforded by \( V \). The Kawanaka wavefront set \( \overline{\text{WF}}(V) \) of \( V \) is defined to be a nilpotent orbit \( \bigcirc \in \mathcal{N}_{\rho}(\mathbb{F}_q) \) satisfying
1. There exists an $O' \in \mathcal{N}_a(F_q)$ such that $\langle \gamma_{O'}, \chi_V \rangle \neq 0$ and $\mathcal{N}_a(F_q/F_q)(O') = \emptyset$.

2. If $O' \in \mathcal{N}_a(F_q)$ and $\langle \gamma_{O'}, \chi_V \rangle \neq 0$ then $\mathcal{N}_a(F_q/F_q)(O') \leq O$.

It is not clear a priori that such an orbit exists, but if it does it is clear that it is unique.

The existence of the Kawanaka wavefront set has a somewhat long and complicated history. Originally conjectured to always exist by Kawanaka, he proved that this is indeed the case for adjoint groups of type $A_n$, or of exceptional type [Kaw87]. He also gave a conjectural description of $\mathbb{F}_q\operatorname{WF}(V)$ in terms of Lusztig’s classification of the irreducible representations of $G(F_q)$. In particular, let $G^*$ be the dual group of $G$ defined over $\mathbb{F}_q$ with corresponding Frobenius $F'$ and suppose $V$ corresponds to the $F'$-stable special $G^*(\mathbb{F}_q)$-conjugacy class $C$ (in the sense of [Lus92, Section 13.2]). Pick an element $g$ of $C^F$ and let $g = su$ be its Jordan decomposition. We can attach to the Weyl group $W(s)$ of $C_{G^*}(F)$ an irreducible (special) representation $E$ via the Springer correspondence applied to $u$ and the trivial local system. The representation $E' = j_{W(s)}^w E$ obtained via truncated induction then corresponds to an $F$-stable nilpotent orbit $O$ and the trivial local system with respect to the Springer correspondence on $g(\mathbb{F}_q)$. The orbit $O$ was Kawanaka’s candidate for $\mathbb{F}_q\operatorname{WF}(V)$. This conjecture was partially proved by Lusztig in his paper [Lus92]. In this paper Lusztig attached to $V$ an $F$-stable unipotent class $C$ of $G(\mathbb{F}_q)$ called the unipotent support of $V$ which is the unipotent class of $G(\mathbb{F}_q)$ of maximal dimension satisfying

$$\sum_{g \in C^F} \chi_V(g) \neq 0.$$  

(20)

He then showed that the (log of the) unipotent support of the Alvis–Curtis dual of $V$ is the unique nilpotent orbit of maximal dimension satisfying condition 1. above. This essentially settled the existence claim, modulo the slight weakening of condition 2. above. This however was fixed in later work by Achar and Aubert in [AA07] (and Taylor [Tay16] with weakened conditions on the characteristic) - finally settling the matter fully.

In chapter 4 of this thesis we will need to know the Kawanaka wavefront set for the case when $G$ is split and for irreducible constituents of $\text{Ind}^G_{B(\mathbb{F}_q)} \mathbb{1}$ - the unipotent principal series representations - where $B$ is an $F$-stable Borel subgroup of $G$. We record a precise formula for $\mathbb{F}_q\operatorname{WF}$ for this case.

First recall that for unipotent principal series representations we have a $q \to 1$ operation arising from Lusztig’s isomorphism

$$\mathbb{C}[W] \to \mathcal{H}(B(\mathbb{F}_q) \backslash G(\mathbb{F}_q)/B(\mathbb{F}_q))$$  

(21)

that gives a bijection between the irreducible constituents of $\text{Ind}^G_{B(\mathbb{F}_q)} \mathbb{1}$ and irreducible $W$-representations. For $V$ a constituent of $\text{Ind}^G_{B(\mathbb{F}_q)} \mathbb{1}$ we write $V_{q \to 1}$ for the corresponding irreducible representation of $W$. Second, recall Lusztig’s partition of $\text{Irr}(W)$ into families so that each family contains a unique special representation. Each special
representation of \( W \) corresponds via the Springer correspondence to a special nilpotent orbit \( O \) and the trivial local system. For an irreducible representation \( E \) of \( W \) we write \( \mathcal{O}^s(E) \) for the special nilpotent orbit corresponding to the special representation in the same family as \( E \otimes \text{sgn} \). Unravelling the above recipe for the Kawanaka wavefront set we get that for unipotent principal series representations \( V \) of \( G(\mathbb{F}_q) \) it is given by

\[
\mathbb{F}_q \text{WF}(V) = \mathcal{O}^s(V_q \rightarrow 1).
\] (22)

We also record the following proposition.

**Proposition 2.12.** Let \( (\rho, V) \) be an irreducible representation of \( G(\mathbb{F}_q) \) and \( \rho^* \) denote its contragredient. Then \( \mathbb{F}_q \text{WF}(V) = \mathbb{F}_q \text{WF}(V^*) \).

**Proof.** The unipotent support of a representation and its contragredient are trivially the same (see equation (21)). The result then follows from the fact that Alvis–Curtis duality commutes with taking contragredients. \( \blacksquare \)

Finally, we make the following convenient definitions. When \( \rho \) is not necessarily irreducible, the Kawanaka wavefront set of \( \rho \) is the collection of maximal orbits among the wavefront sets of the irreducible constituents. An element of the Kawanaka wavefront set is called a Kawanaka wavefront-set nilpotent.

### 2.4 Relating the \( \mathbb{Q}_p \) and \( \mathbb{F}_p \) Wavefront Sets

#### 2.4.1 Inflated Generalised Gelfand–Graev Representations

Let \( (c, O) \in I^k_o \). Define the function \( f_{c,O} = \tilde{\gamma}_O \) where \( \gamma_O \) is the character of the GGGR of \( L_c(\mathbb{F}_q) \) attached to the orbit \( O \). We will also write \( f_{c,O} \) for \( f_{c,O} \circ \exp \). The following result is essentially due to Barbasch and Moy in [BM97], but is a sharper result than in loc. cit.

**Theorem 2.13.** Let \( (c, O) \in I^k_o \). Then

1. \( f_{c,O} \) is supported on the topologically unipotent elements \( u \),
2. \( \hat{\mu}_{O'}(f_{c,O}) = 0 \) unless \( \mathcal{L}_c(O) \subseteq O' \).
3. Suppose \( O' \subseteq N_o(k) \) is such that \( N_o(\overline{k}/k)(O') = N_o(\overline{k}/k)(\mathcal{L}_c(O)) \). Then
   (a) If \( \mathcal{L}_c(O) \neq O' \), then \( \hat{\mu}_{O'}(f_{c,O}) = 0 \).
   (b) If \( \mathcal{L}_c(O) = O' \), then \( \hat{\mu}_{O'}(f_{c,O}) \neq 0 \).
4. For any irreducible smooth admissible representation \( (\pi, X) \) of \( G(k) \), we have

\[
\Theta_X(f_{c,O}) = \langle \Gamma_{c,O}, \hat{X}^{u_c(O)} \rangle.
\]

Here \( \hat{X} \) denotes the contragredient (i.e. the smooth dual) of \( X \). The proof for part 2 in [BM97] however only shows that \( \mathcal{L}_c(O) \) lies in the closure of \( N_o(\overline{k}/k)(O') \cap g(k) \) in \( g(k) \). We now give a complete proof of part 2 using ideas from [Wal01].
Proof. Let \( x, h, y \) be an \( \mathfrak{sl}_2 \)-triple for \( \mathbb{O} \). Let \( X, H, Y \) denote a lift of \( x, h, y \) to an \( \mathfrak{sl}_2 \)-triple of \( \mathfrak{g}(k) \). We proceed by first showing that

\[
\text{supp}(\hat{f}_{c,0}) = \{ h(-Y + Z) : h \in \mathfrak{p}_c(0), Z \in Z_{\mathfrak{p}_c(0)}(X) \}.
\] (23)

We have that \( \hat{f}_{c,0}(W) \neq 0 \) iff \( W \in \mathfrak{p}_c(0) \) and \( \hat{\gamma}_0(w) \neq 0 \) where \( w \) is the image of \( W \) in \( I_c(F_q) \). By [Lus92, Equation 2.4 (a)], \( \hat{\gamma}_0(w) \neq 0 \) iff \( w \in \mathfrak{L}_c(F_q)(-y + Z_{I_c(F_q)}(e)) \). By the proof of [Wal01, Lemma IX.3], we know that the image of \( Z_{\mathfrak{p}_c(0)}(X) \) in \( I_c(F_q) \) is \( Z_{\mathfrak{L}_c(F_q)}(x) \). Thus the support consists of those \( W \) in \( \mathfrak{P}_c(0)(-Y + Z_{\mathfrak{p}_c(0)}(X)) + \mathfrak{u}_c(0) = \mathfrak{P}_c(0)(-Y + Z_{\mathfrak{p}_c(0)}(X) + \mathfrak{u}_c(0)) \). But from the same proof in [Wal01] we also know that

\[-Y + Z + \mathfrak{u}_c(0) = \{ h(-Y + Z + Z') : Z' \in Z\mathfrak{u}_c(0), h \in \mathfrak{u}_c(0) \} \] (24)

Thus \(-Y + Z + \mathfrak{u}_c(0) = \mathfrak{U}_c(0)(-Y + Z\mathfrak{p}_c(0)) \) and so equation (23) holds.

Now let \( \mathbb{O}' \) be a nilpotent orbit with \( \hat{\mu}_{\mathbb{O}'}(f_{c,0}) \neq 0 \). Then \( \text{supp}(\hat{f}_{c,0}) \cap \mathbb{O}' \neq \emptyset \). Thus there is a \( \mathbb{X}' \in \mathbb{O}' \), \( h \in \mathfrak{p}_c(0) \) and \( Z \in Z_{\mathfrak{p}_c(0)}(X) \) such that \( X' = h(-Y + Z) \).

Now let \( \mathbb{X}' \in \mathbb{O}' \), \( h \in \mathfrak{p}_c(0) \) and \( Z \in Z_{\mathfrak{p}_c(0)}(X) \) such that \( X' = h(-Y + Z) \).

Since \( h^{-1}X' \in \mathbb{O} \) we can assume \( h = 1 \) and so \( X' = -Y + Z \). Let \( \lambda_{X'} : G_m \to G \) be a 1-parameter \( k \)-subgroup so that \( \text{Ad}(\lambda_{X'}(t))X' = t^2X' \), and \( \lambda \) be the 1-parameter subgroup determined by \( H \). Since \( Z_{\mathfrak{g}(k)}(X) \subseteq \mathfrak{g}(k)(\geq 0) \), write \( Z = \sum_{i \geq 0} Z_i \) where \( Z_i \in \mathfrak{g}(k)(i) \).

Then

\[
\lambda(t)\lambda_{X'}(t)X' = -Y + \sum_{i \geq 0} t^{2i}Z_i \to -Y
\] (25)

as \( t \to 0 \). Thus \( \mathbb{O} = \mathfrak{g}(k)(-Y) \leq \mathbb{O}' \).

Note that Theorem 2.10 together with part 2 of this theorem immediately imply part 3 (a) of this theorem.

**Proposition 2.14.** Suppose \((\pi, X)\) has depth 0. Let

\[
\Xi(X) = \{ \mathbb{O} \in \mathcal{N}_c(k) : \text{there exists } (c, \mathbb{O}') \in I_c^k(\mathbb{O}) \text{ such that } \Theta_X(f_{c,\mathbb{O}'}) \neq 0 \}.
\]

Write \( \Xi^{\max}(X) \) for the maximal orbits of \( \Xi(X) \). Then

1. \( \text{WF}(X) = \Xi^{\max}(X) \),

2. if \( \mathbb{O} \in \Xi^{\max}(X) \) then \( \Theta_X(f_{c,\mathbb{O}'}) \neq 0 \) for all \( (c, \mathbb{O}') \in I_c^k(\mathbb{O}) \).

**Proof.** Let \( \mathbb{O} \in \text{WF}(X) \) and \((c, \mathbb{O}') \in I_c^k(\mathbb{O}) \). Since \( X \) has depth 0 and the inflated GGGRs have support in \( \mathbb{U} \) we have

\[
\Theta_X(f_{c,\mathbb{O}'}) = \sum_{\mathbb{O}'' \in \mathcal{N}_c(k)} c_{\mathbb{O}''} \hat{\mu}_{\mathbb{O}''}(f_{c,\mathbb{O}'}).
\] (26)

Then by Theorem 2.13 part 2 we have that

\[
\Theta_X(f_{c,\mathbb{O}'}) = \sum_{\mathbb{O} \leq \mathbb{O}''} c_{\mathbb{O}''} \hat{\mu}_{\mathbb{O}''}(f_{c,\mathbb{O}'})
\] (27)
But if \( \varnothing < \varnothing' \) then \( c_{\varnothing'} = 0 \) and so \( \Theta_X(f_{c,\varnothing'}) = c_{\varnothing'}(X) \mu_{\varnothing}(f_{c,\varnothing'}) \neq 0 \). Thus \( \WF(X) \subseteq \Xi(X) \).

Now suppose \( \varnothing \in \WF(X) \) and \( \varnothing_1 \) is a nilpotent orbit with \( \varnothing < \varnothing_1 \). Let \( (c_1, \varnothing'_1) \in I^k_\varnothing(\varnothing_1) \). Then \( \Theta_X(f_{c_1,\varnothing_1}) = \sum_{\varnothing_2 \leq \varnothing_1} c_{\varnothing_2} \mu_{\varnothing_2}(f_{c_1,\varnothing'_1}) \). But if \( \varnothing_1 \leq \varnothing_2 \) then \( \varnothing < \varnothing_2 \) and so \( c_{\varnothing_2}(X) = 0 \) and so \( \Theta_X(f_{c_1,\varnothing'_1}) = 0 \). Thus we get that \( \WF(X) \subseteq \Xi^{\max}(X) \).

Finally, suppose \( \varnothing \) is in \( \Xi^{\max}(X) \) and \( (c, \varnothing') \in I^k_\varnothing(\varnothing) \). \( \varnothing \) must be \( \leq \varnothing_1 \) for some \( \varnothing_1 \in \WF(X) \) (since otherwise \( c_{\varnothing_2}(X) = 0 \) for all \( \varnothing \leq \varnothing_2 \) and so \( \Theta_X(f_{c,\varnothing'}) = 0 \). But \( \WF(X) \subseteq \Xi(X) \) and so by maximality we must have \( \varnothing = \varnothing_1 \) and so \( \Xi^{\max}(X) \subseteq \WF(X) \). This establishes 1.

To establish 2., note by the first part we have that \( \Theta_X(f_{c,\varnothing'}) \neq 0 \) for all \( \varnothing \in \WF(X) \) and \( (c, \varnothing') \in I^k_\varnothing(\varnothing) \). Then use the fact that \( \Xi_{\varnothing}(K/k) \) consists of the maximal orbits of \( \WF(\varnothing) \) and \( \varnothing \) is equal to \( \Xi^{\max}(X) \). Thus we get that \( \Xi^{\max}(X) = \WF(X) \).

\[ \text{Proposition 2.15.} \]

1. \( KWF(X) \) consists of the maximal orbits of \( \{ N_\varnothing(K/k)(\varnothing) : \varnothing \in \WF(X) \} \).

2. \( \hat{K}WF(X) \) consists of the maximal orbits of \( \{ N_\varnothing(k/K)(\varnothing) : \varnothing \in KWF(X) \} \).

\[ \text{Proof.} \] This follows from \( N_\varnothing(K/k) \) and \( N_\varnothing(k/K) \) being non-decreasing.

\[ \text{Corollary 2.16.} \] Suppose \( (\pi, X) \) has depth 0. Then \( KWF(X) \) is equal to the set of maximal orbits of \( N_\varnothing(K/k)(\Xi(X)) \).

### 2.4.2 Local Wavefront Sets

We now define local wavefront sets \( \WF_c \) and \( KWF_c \) for \( c \) a face of \( F(G(k)) \). Let \( \varnothing_1, \ldots, \varnothing_k \in N_\varnothing^L_\varnothing(\F_q) \) be the Kawanaka wavefront-set nilpotents of \( W_c := V^{U(\varnothing)} \), a representation of \( L_c(\F_q) \). Let \( \varnothing'_1, \ldots, \varnothing'_l \in N_\varnothing^L_\varnothing(\F_q) \) be the nilpotent orbits in \( \bigcup_i \varnothing_i \) such that \( \langle \Gamma_{\varnothing'_i}, W_c \rangle \neq 0 \). Define

\[ \WF_c(X) = \{ \mathcal{L}_c(\varnothing'_i) : 1 \leq i \leq l \} \quad \text{and} \quad KWF_c(X) = \{ \mathcal{L}_c(\varnothing_i) : 1 \leq i \leq k \}. \]

It is clear that

\[ KWF_c(X) = \{ N_\varnothing(K/k)(\varnothing) : \varnothing \in \WF_c(X) \}. \]

Let \( \mathcal{C} \) denote a collection of faces in \( F(G(k)) \) such that for all \( \varnothing \in N_\varnothing(k) \) there exists a \( (c, \varnothing') \in I^k_\varnothing(\varnothing) \) such that \( c \in \mathcal{C} \). Examples of \( \mathcal{C} \) that satisfy this property are:

1. the faces of a fixed chamber \( c_0 \);
2. the vertices of a fixed chamber \( c_0 \);
3. a choice of vertex from each \( G(k) \) orbit of the vertices of a fixed chamber \( c_0 \).

\[ \text{Lemma 2.17.} \] Let \( (\pi, X) \) be depth 0. Then

\[ KWF(X) = \max_{c \in \mathcal{C}} KWF_c(X). \]
Proof. Let $K\Xi(X) = \mathcal{N}_o(K/k)(\Xi(X))$. We will show that $K\text{WF}_c(X) \subseteq K\Xi(X)$ and that if $\mathcal{O}$ is a maximal element of $K\Xi(X)$ then $\mathcal{O} \in K\text{WF}_c(X)$ for some $c \in \mathcal{C}$. Corollary 2.16 then implies the result.

The first part is straightforward. Let $c$ be any face of $B(G(k))$ and $\mathcal{O} \in K\text{WF}_c(X)$. Write $\mathcal{O}$ as $\mathcal{N}_o(K/k)(O_1)$ where $O_1$ is in $\text{WF}_c(X)$. By definition of $\text{WF}_c(X)$, $O_1$ is in $\Xi(X)$. Thus we have $K\text{WF}_c(X) \subseteq K\Xi(X)$.

For the second part let $O$ be a maximal element of $K\Xi(X)$. Write $O = \mathcal{N}_o(K/k)(O_1)$ where $O_1 \in \text{WF}_c(X)$. By definition of $\text{WF}_c(X)$, $O_1$ is in $\Xi(X)$. Thus we have $K\text{WF}_c(X) \subseteq K\Xi(X)$.

Corollary 2.16 then implies the result.

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Corollary 2.16 then implies the result.
Proof. By Proposition 2.15 it suffices to show the first equality. By Lemma 2.17 it suffices to show that $KWF_c(X) = KWF_c(\bar{X})$ for any face $c$ of $B(G(k))$. This last equality follows from the previous corollary.

In the following example we use the local wavefront set operators to show that the geometric wavefront set of an anti-spherical representation is the regular nilpotent orbit.

Example 2.20. Let $(\pi, X)$ be a spherical representation of $G(k)$. Let $A$ be an apartment of $B(G(k))$ and $c$ be the origin of $A$ (i.e. the common vanishing point of all the roots $\Phi(T, k) \subseteq \Psi(T, k)$). By definition, $V^{\text{un},(o)}$ contains the trivial $L_c(\mathbb{F}_q)$-representation. Let $AZ(X)$ denote the Aubert–Zelevinsky dual of $X$ [Anb95]. Then $AZ(V)^{U_c(o)}$ contains the Steinberg representation. By Equation 22, the Kawanaka wavefront set of the Steinberg is $O_{\text{reg}}$ - the regular nilpotent orbit of $L_c(\mathbb{F}_q)$. This is the unique maximal nilpotent orbit of $L_c(\mathbb{F}_q)$ and so the Kawanaka wavefront set of $V^{U_c(o)}$ is $O_{\text{reg}}$. Let $\mathcal{O} = L_c(\mathcal{O}_{\text{reg}})$. Then $KWF_c(AZ(X)) = \mathcal{O}$. By Theorem 3.5, $\mathcal{N}_c(k/K)(\mathcal{O})$ is the regular nilpotent orbit of $g$ and so $kWF(AZ(X))$ must also be the regular nilpotent orbit.

3 Unramified Nilpotent Orbits

3.1 Notation

Let $G = G_Z(\mathbb{C})$ and $\mathcal{N}_o := \mathcal{N}_o^{G_Z(\mathbb{C})}$. Let $T_Z$ be the split torus of $G_Z$ under which the identification of root data of $G_K$ and $G_Z$ is taken with respect to $T_K$. The extended affine Weyl group naturally acts on $A$ by affine transformations. For a $c \subseteq A$ let $W_c$ denote the subgroup of $W$ generated by the reflections through hyperplanes containing $c$. The torus $T_K$ is in fact defined over $\mathbb{Q}$, is a subgroup of $T_c$ for each $c \subseteq A$, and the special fibre of $T_K$, denoted $\bar{T}_K$, is a $\mathbb{F}_q$-split torus of $L_c(\mathbb{F}_q)$. Write $\Phi_c(\bar{T}_K, \mathbb{F}_q)$ for the root system of $L_c(\mathbb{F}_q)$ with respect to $\bar{T}_K$. Then $\Phi_c(\bar{T}_K, \mathbb{F}_q)$ naturally identifies with the set of $\psi \in \Psi(T, k)$ that vanish on $A(c, A)$, and the Weyl group of $L_c$ with respect to $T_K$ is naturally isomorphic to $W_c$. Let $B_K$ be a Borel of $G_K$ containing $T_K$ and let $\Delta := \Delta(T_K, B_K) \subseteq \Phi(T, K)$ be the simple roots determined by $B_K$ and let $\tilde{\Delta}$ be the simple roots of $\Psi$ corresponding to $\Delta$. There is a unique labelling function $\text{Type}$ of the faces of $B(G, K)$ in terms of the set $\{J \subsetneq \tilde{\Delta}\}$ [Gar97 Section 5.2]. Let $E = X^* \otimes \mathbb{Z} \mathbb{R}, V = X_* \otimes \mathbb{Z} \mathbb{R}$ and define the
Proposition 3.2. There exists an $G$-invariant characterisation of pseudo-Levi subgroups. Let $\mathfrak{X}_\alpha$ denote the normaliser of $\mathfrak{X}$ in $G$. For a pseudo-Levi subgroup $\mathfrak{L}_{\mathfrak{X}}$ of $G$, the following convenient characterisation.

3.2 Parameterising Unramified Nilpotent Orbits

3.2.1 Affine Bala–Carter Theory

Let

$$I^K_{d,A} = \{(c, x) \in I^K_d : c \subseteq A\} \quad \text{and} \quad I^K_{o,d,A} = \{(c, \mathbb{O}) \in I^K_{o,d} : c \subseteq A\}.$$ 

Since $G(K)$ acts transitively on ordered pairs of apartments and chambers the inclusion map $I^K_{o,d,A} \to I^K_{o,d}$ descends to a bijection on $\sim_K$ equivalence classes. With the apartment $\mathfrak{A}$ now fixed we write $\mathfrak{A}(c)$ for $\mathfrak{A}(c, A)$ for faces $c$ in $\mathfrak{A}$. Let $\mathcal{N}$ denote the stabiliser of $\mathfrak{A}$ in $G(K)$. For $(c_1, \mathbb{O}_1), (c_2, \mathbb{O}_2) \in I^K_{o,d,A}$ declare $(c_1, \mathbb{O}_1) \sim_\mathcal{N} (c_2, \mathbb{O}_2)$ if there exists an $n \in \mathcal{N}$ such that $\mathfrak{A}(c_1) = \mathfrak{A}(nc_2)$ and $\mathbb{O}_1 = j_{c_1, nc_2}(n, \mathbb{O}_2)$.

Proposition 3.2. The map

$$I^K_{d,A} \to I^K_{o,d,A}, \quad (c, x) \mapsto (c, L_c(\overline{\mathbb{F}}_q)x)$$

descends to bijection between $I^K_{d,A}/\sim_K$ and $I^K_{o,d,A}/\sim_\mathcal{N}$.

Proof. The map $I^K_{d,A} \to I^K_{o,d,A}$ is clearly surjective. Suppose there is a $h \in G(K)$ such that $\mathfrak{A}(c_1) = \mathfrak{A}(hc_2)$ and $x_1 = j_{c_1, hc_2}(h, x_2)$. Then we can write $h = nh_0$ where $n \in \mathcal{N}, h_0 \in \mathfrak{P}_{c_2}(\mathfrak{O})$. Thus

$$L_{c_1}(\overline{\mathbb{F}}_q)x_1 = L_{c_1}(\overline{\mathbb{F}}_q)j_{c_1, hc_2}(h, x_2) = L_{c_1}(\overline{\mathbb{F}}_q)j_{c_1, nc_2}(nh_0, x_2) = j_{c_1, nc_2}(L_{c_2}(\overline{\mathbb{F}}_q), (nh_0, x_2)) \quad (29)$$

$$L_{c_2}(\overline{\mathbb{F}}_q)(h_0, x_2) = L_{c_2}(\overline{\mathbb{F}}_q)x_2 = j_{c_1, nc_2}(n(L_{c_2}(\overline{\mathbb{F}}_q), (h_0, x_2))). \quad (30)$$

But since $h_0 \in \mathfrak{P}_{c_2}(\mathfrak{O})$, we have that $L_{c_2}(\overline{\mathbb{F}}_q)(h_0, x_2) = L_{c_2}(\overline{\mathbb{F}}_q)x_2$ and so

$$(c_1, L_{c_1}(\overline{\mathbb{F}}_q)x_1) \sim_\mathcal{N} (c_2, L_{c_2}(\overline{\mathbb{F}}_q)x_2).$$

Thus $I^K_{d,A} \to I^K_{o,d,A}/\sim_\mathcal{N}$ descends to a well defined map $I^K_{d,A}/\sim_K \to I^K_{o,d,A}/\sim_\mathcal{N}$. Now suppose there is an $n \in \mathcal{N}$ such that $\mathfrak{A}(c_1) = \mathfrak{A}(nc_2)$ and $L_{c_1}(\overline{\mathbb{F}}_q)x_1 = j_{c_1, nc_2}(n(L_{c_2}(\overline{\mathbb{F}}_q)x_2))$. 23
Then there exists an \( h'_0 \in L_{c_2}(\overline{F}_q) \) such that \( x_1 = j_{c_1,nc_2}(nh_0x_2) \). Let \( h_0 \in P_{c_2}(\Omega) \) be a lift of \( h'_0 \) and \( h = nh_0 \). Then \( A(c_1) = A(hc_2) \) and

\[
x_1 = j_{c_1,nc_2}(nh_0x_2) = j_{c_1,hc_2}(hx_2).
\]

Thus \( I^K_{o,d,A}/ \sim_K \rightarrow I^K_{o,d,\hat{A}}/ \sim_{\hat{A}} \) is injective and hence a bijection as required. \( \blacksquare \)

Note that there is a surjection \( \pi : \mathcal{N} \rightarrow \tilde{W} \) given by quotienting by the elements of \( \mathcal{N} \) that act trivially on \( A \). Since additionally

\[
\ker \pi \subseteq \bigcap_{c \subseteq A} P_c(\Omega),
\]

if \( n_1, n_2 \in \mathcal{N} \) and \( \pi(n_1) = \pi(n_2) \) then for any facet \( c \) of \( A \) and nilpotent orbit \( \mathcal{O} \in \mathcal{N}_{c}^L(\overline{F}_q) \), we have \( n_1c = n_2c \) and \( n_1 \mathcal{O} = n_2 \mathcal{O} \). Thus for \( w \in \tilde{W} \), \( c \subseteq A \) and \( \mathcal{O} \in \mathcal{N}_{c}^L(\overline{F}_q) \), defining \( w.\mathcal{O} \) to be \( n.\mathcal{O} \) where \( n \) is any element of \( \mathcal{N} \) such that \( \pi(n) = w \), is well defined. Now define a relation \( \sim_A \) on \( I_{o,d,A} \) by declaring \( (c_1, \mathcal{O}_1) \sim_A (c_2, \mathcal{O}_2) \) if there exists \( w \in \tilde{W} \) such that \( A(c_1) = A(wc_2) \) and \( \mathcal{O}_1 = i_{c_1,wc_2}(w.\mathcal{O}_2) \). Then clearly \( \sim_{\mathcal{N}} \) and \( \sim_A \) are the same equivalence relation on \( I^K_{o,d,A} \).

Let \( ABC(\tilde{\Delta}) \) be the set of pairs \((J, J')\) where \( J \) is a proper subset of \( \tilde{\Delta} \) and \( J' \subseteq J \) is a distinguished subset of \( J \) in the sense of Bala–Carter (we can think of \( J \) as the simple roots of a crystallographic root system). Given \( c \subseteq B(G, K) \), \( \text{Type}(c) \) is a set of simple roots for \( \Phi_c(\mathcal{T}_K, \overline{F}_q) \), and so given a distinguished \( \mathcal{O} \in \mathcal{N}_{c}^L(\overline{F}_q) \), let \( BC_c(\mathcal{O}) \) be the corresponding distinguished subset of \( \text{Type}(c) \) perscribed by the Bala–Carter classification of distinguished nilpotent orbits [blah]. Let \( c_0 \) denote the chamber of \( A \) cut out by the simple roots \( \tilde{\Delta} \). Let

\[
I^K_{o,d,c_0} = \{ (c, \mathcal{O}) \in I^K_{o,d,A} : c \subseteq c_0 \}.
\]

Since \( \tilde{W} \) acts transitively on the chambers in \( A \), the inclusion map \( I^K_{o,d,c_0} \rightarrow I^K_{o,d,A} \) descends to a bijection. There is a natural bijection between \( I^K_{o,d,c_0} \) and \( ABC(\tilde{\Delta}) \) given by

\[
ABC : (c, \mathcal{O}) \mapsto (\text{Type}(c), BC_c(\mathcal{O})).
\]

Write \( I^K_{o,d,c_0} : ABC(\tilde{\Delta}) \rightarrow I^K_{o,d,c_0} \) for the inverse map. For \((J_1, J'_1), (J_2, J'_2) \in ABC(\tilde{\Delta}) \) define

\[
(J_1, J'_1) \sim_{\tilde{W}} (J_2, J'_2) \text{ if there exists } w \in \tilde{W} : J_2 = w.J_1, J'_2 = w.J'_1.
\]

**Proposition 3.3.** Let \((c_1, \mathcal{O}_1), (c_2, \mathcal{O}_2) \in I^K_{o,d,c_0} \). Then

\[
(c_1, \mathcal{O}_1) \sim_A (c_2, \mathcal{O}_2) \text{ if and only if } ABC(c_1, \mathcal{O}_1) \sim_{\tilde{W}} ABC(c_2, \mathcal{O}_2).
\]
Proof. Let \((J_i, J'_i) = ABC(c_i, \mathbb{O}_i)\) for \(i = 1, 2\). \((\Rightarrow)\) Suppose there is a \(w \in \tilde{W}\) such that \(\mathcal{A}(c_1) = \mathcal{A}(wc_2)\) and \(\mathbb{O}_1 = i_{c_1, wc_2}(w, \mathbb{O}_2)\). Note that \(J_i\) is a root basis for \(\psi_{\mathcal{A}(c_i)}\). Thus since \(\mathcal{A}(c_1) = \mathcal{A}(wc_2) = w \mathcal{A}(c_2)\), we have that \(J_1\) and \(wJ_2\) are both root bases for \(\psi_{\mathcal{A}(c_1)}\). Thus there exists \(w_0 \in W_{c_1}\) such that \(J_1 = w_0wJ_2\). Now, the Bala–Carter data for \(i_{c_1, wc_2}(w, \mathbb{O}_2)\) with respect to \(wJ_2\) is \(wJ'_2\). With respect to the root basis \(J_1\) it is thus \(w_0wJ'_2\). Thus \((J_1, J'_1) \sim_{\tilde{W}} (J_2, J'_2)\). \((\Leftarrow)\) Suppose there is some \(w \in \tilde{W}\) such that \((J_1, J'_1) = w.(J_2, J'_2)\). Then \(J_1 = wJ_2\) implies that \(\mathcal{A}(c_1) = \mathcal{A}(wc_2)\). Moreover, the Bala–Carter data of \(i_{c_1, wc_2}(w, \mathbb{O}_2)\) is \(wJ'_2\) with respect to \(wJ_2 = J_1\). But \(wJ'_2 = J'_1\) and so \(\mathbb{O}_1\) and \(i_{c_1, wc_2}(w, \mathbb{O}_2)\) have the same Bala–Carter data and so must be equal. \(\blacksquare\)

**Theorem 3.4.** The map

\[
\mathcal{L} : ABC(\tilde{\Delta}) \to N_o(K), \quad (J, J') \mapsto \mathcal{L} \circ I^K_{o,d,c}(J, J')
\]

descends to a bijection \(ABC(\tilde{\Delta})/\sim_{\tilde{W}} \to N_o(K)\).

**Proof.** This follows immediately from the previous Proposition. \(\blacksquare\)

Recall that regular Bala–Carter theory states that there is a map from the set \(BC(\Delta)\) of pairs \((J, J')\) where \(J \subseteq \Delta\) and \(J'\) is distinguished in \(J\), to \(N'_o(\tilde{k})\) that descends to a bijection \(BC(\Delta)/\sim_W \to N'_o(\tilde{k})\) (where \(\sim_W\) is the obvious analogue of \(\sim_{\tilde{W}}\)). So Theorem 3.4 is an affine version of the combinatorial Bala–Carter Theorem in a very literal sense (\(ABC\) stands for Affine Bala–Carter). In Proposition 3.21 we give a precise statement about how these two parameterisations relate.

### 3.2.2 Properties of Lifting of Nilpotent Orbits

Let \(x_0\) be the origin of \(\mathcal{A}\) and let \(L = L_{x_0}, 1 = 1_{x_0}\). Note that \(L(\mathbb{F}_q) = G_{\mathbb{Z}}(\mathbb{F}_q)\) since \(G_K\) is split. Let \(\mathfrak{t}_K\) be the Lie algebra of \(T_K\). There is a natural identification between the character lattices of \(T_K\) and \(\mathfrak{t}_K\). For \(\alpha \in \Phi\) let \(\bar{\alpha}\) denote the image under this identification. Let \(\Phi = \{\bar{\alpha} : \alpha \in \Phi\}\). Let \(\mathfrak{p}_\Phi\) be the maximal ideal of \(\mathfrak{g}\). Recall that a choice of uniformiser of \(K\) induces an isomorphism of additive groups between \(\mathfrak{p}_i^{j}/\mathfrak{p}_i^{j+1} \to \mathfrak{p}_j^{j}/\mathfrak{p}_j^{j+1}\) for any \(i, j \in \mathbb{Z}\). For any facet \(c\) of \(\mathcal{A}\) this in turn induces an isomorphism from \(L_c(\mathbb{F}_q)\) onto the pseudo-levi of \(L(\mathbb{F}_q)\) corresponding to \(T_K(\mathbb{F}_q)\) and \(\hat{\Phi}_c\). For a uniformiser \(\tilde{\omega}\) let \(i_{c, \tilde{\omega}} : L_c \to L\) denote the corresponding homomorphism and \(j_{c, \tilde{\omega}} : l_c \to l\) the associated morphism of lie algebras. One important property of this map is that the following diagram commutes

\[
\begin{array}{ccc}
\hat{T}_K & \xrightarrow{j_{c, \tilde{\omega}}} & l_c \\
\downarrow & & \downarrow \\
l_c & \xrightarrow{i_{c, \tilde{\omega}}} & l.
\end{array}
\]

Moreover, since \((l_c)_a\) maps to \(l_a\) the resulting map of nilpotent orbits does not depend on the choice of uniformiser for \(K\) (this follows from Bala–Carter theory). Thus we
obtain a canonical map \( j_{c,o} : \mathcal{N}_o^L(\mathbb{F}_q) \to \mathcal{N}_o^R(\mathbb{F}_q) \). Note since \( L \) and \( G_Z \) have the same root data, by lemma 2.1 there is an order preserving isomorphism

\[
\Lambda^L : \mathcal{N}_o^L(\mathbb{F}_q) \to \mathcal{N}_o.
\]

Let \( \Lambda^k : \mathcal{N}_o(\bar{k}) \to \mathcal{N}_o \) be the isomorphism from lemma 2.1 applied to \( F = \bar{k} \).

**Theorem 3.5.** Let \((c, \mathcal{O}) \in I_{o,A}^K \). Then

\[
\Lambda^k \circ \mathcal{N}_o(\bar{k}/K) \circ \mathcal{L}_c(\mathcal{O}) = \Lambda^L \circ j_{c,o}(\mathcal{O}).
\]

**Proof.** Let \( x, h, y \) be an \( sl_2 \)-triple for \( \mathcal{O} \) and let \( X, H, Y \) be a lift to an \( sl_2 \)-triple for \( \mathcal{L}_c(\mathcal{O}) \). We have that \( \alpha(H) \in \mathbb{Z} \) for all \( \alpha \in \Phi \). There exists a \( w \in W \) such that \( \alpha(wH) \geq 0 \) for all \( \alpha \in \Delta \). In particular then \( \alpha(wH) \in \{0,1,2\} \) for all \( \alpha \in \Delta \). We also have \( j_{c,\alpha}(\mathcal{O}) = w.j_{c,\alpha}(\mathcal{O}) = w\mathcal{L}_c(\alpha,\mathcal{O}) \) and \( w\mathcal{L}_c(\mathcal{O}) = \mathcal{L}_c(\mathcal{O}) \). Thus by replacing \((c, \mathcal{O}) \) with \((wc, w\mathcal{O}) \) we can assume that \( \alpha(H) \in \{0,1,2\} \) for all \( \alpha \in \Delta \). Then \( j_{c,\alpha}(x), j_{c,\alpha}(h), j_{c,\alpha}(y) \) is an \( sl_2 \)-triple for \( j_{c,\alpha}(\mathcal{O}) \). But since \( \bar{\alpha}(j_{c,\alpha}(h)) = \bar{\alpha}(h) \) equals the image of \( \alpha(H) \) in \( \mathbb{F}_q \) for all \( \alpha \in \Delta \), \( j_{c,\alpha}(\mathcal{O}) \) and \( \mathcal{N}_o(\bar{k}/K)(\mathcal{L}_c(\mathcal{O})) \) have the same weighted Dynkin diagram with respect to \( \Delta \) and \( \Delta \) respectively and so \( \Lambda^k \circ \mathcal{N}_o(\bar{k}/K) \circ \mathcal{L}_c(\mathcal{O}) = \Lambda^L \circ j_{c,o}(\mathcal{O}). \)

**Corollary 3.6.** Let \((c, \mathcal{O}_1), (c, \mathcal{O}_2) \in I_{o}^K \) and suppose that \( \mathcal{O}_1 \subset \mathcal{O}_2 \). Then

\[
\mathcal{N}_o(\bar{k}/K)(\mathcal{L}_c(\mathcal{O}_1)) < \mathcal{N}_o(\bar{k}/K)(\mathcal{L}_c(\mathcal{O}_2)).
\]

**Proof.** By [BDT20, Theorem 5.5], if \( \mathcal{O}_1 < \mathcal{O}_2 \), then \( j_{c,o}(\mathcal{O}_1) < j_{c,o}(\mathcal{O}_2) \). Thus

\[
\mathcal{N}_o(\bar{k}/K)(\mathcal{O}_1) = \Lambda^L \circ j_{c,o}(\mathcal{O}_1) < \Lambda^L \circ j_{c,o}(\mathcal{O}_2) = \mathcal{N}_o(\bar{k}/K)(\mathcal{O}_2).
\]

**Corollary 3.7.** The map \( \mathcal{L} : I_{o}^K \to \mathcal{N}_o(K) \) is strictly increasing.

**Proof.** Suppose that \((c, \mathcal{O}_1' \subset (c, \mathcal{O}_2') \). Let \( \mathcal{O}_i = \mathcal{L}_c(\mathcal{O}_i) \) for \( i = 1, 2 \). Since \( \mathcal{L} \) is non-decreasing we have that \( \mathcal{O}_1 \subset \mathcal{O}_2 \). By corollary 3.6 we have that \( \mathcal{N}_o(\bar{k}/K)(\mathcal{O}_1) \neq \mathcal{N}_o(\bar{k}/K)(\mathcal{O}_2) \). Therefore \( \mathcal{O}_1 \neq \mathcal{O}_2 \) and so \( \mathcal{O}_1 \subset \mathcal{O}_2 \) as required.

**Corollary 3.8.** The closure ordering on \( \mathcal{N}_o(K) \) is a partial order.

**Proof.** Let \( \mathcal{O}_1, \mathcal{O}_2 \in \mathcal{N}_o(K) \) and suppose \( \mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_1 \). By Theorem 2.7 there exists \( \mathcal{O}_{1,5} \in \mathcal{N}_o(K) \) with \( \mathcal{O}_1 \subset \mathcal{O}_{1,5} \subset \mathcal{O}_2 \) such that \( \mathcal{O}_1 \subset \mathcal{O}_{1,5} \subset \mathcal{O}_2 \) is degenerate. Since \( \mathcal{N}_o(\bar{k}/K)(\mathcal{O}_1) = \mathcal{N}_o(\bar{k}/K)(\mathcal{O}_2) \), by corollary 3.7 we must have \( \mathcal{O}_1 = \mathcal{O}_{1,5} \). Let \((c, \mathcal{O}_{1,5}) \in I_{o}^K(\mathcal{O}_{1,5}) \) be such that \( \mathcal{C}(c, \mathcal{O}_{1,5}) \cap \mathcal{O}_2 \neq \emptyset \) and let \( X, H, Y \in \mathcal{P}_c(\mathcal{O}) \) be an \( sl_2 \)-triple for \( \mathcal{O}_{1,5} \). By lemma 2.7 we have that

\[
X + u_c(\mathcal{O}) = u_c(\mathcal{O}) \cdot (X + c_{u_c(\mathcal{O})}(Y)).
\]

It follows that

\[
X + c_{u_c(\mathcal{O})}(Y) \cap \mathcal{O}_2 \neq \emptyset.
\]

However,

\[
X + c_{u_c(\mathcal{O})}(Y) \subseteq X + c_{g(\bar{k})}(Y)
\]

and since \( \mathcal{N}_o(\bar{k}/K)(\mathcal{O}_1) = \mathcal{N}_o(\bar{k}/K)(\mathcal{O}_2) \) we have that

\[
X + c_{g(\bar{k})}(Y) \cap \mathcal{N}_o(\bar{k}/K)(\mathcal{O}_2) = X.
\]

Thus \( X + c_{u_c(\mathcal{O})}(Y) \cap \mathcal{O}_2 = X \) and so \( \mathcal{O}_1 = \mathcal{O}_2 \).
3.2.3 Sommers and McNinch’s Parameterisation of $N_{o,c}$

For a finite group $G_0$ write $C(G_0)$ for the set of conjugacy classes of $G_0$. For $\mathcal{O} \in N_o$ let $A(\mathcal{O})$ denote the $G$-equivariant fundamental group of $\mathcal{O}$. Let

$$N_{o,c} = \{(\mathcal{O}, C) : \mathcal{O} \in N_o, C \in C(A(\mathcal{O}))\}.$$ 

Let $\mathcal{F}$ denote the set of pairs $(L,t\mathbb{Z}^o)$ such that $L$ is a pseudo- Levi of $G$ and $t\mathbb{Z}^o$ is an element of $Z/\mathbb{Z}^o$ where $Z$ is the center of $L$ and $L = C_G^o(t\mathbb{Z}^o)$. Let $I_d^c$ denote the set of all triples $(L,t\mathbb{Z}^o,x)$ such that $(L,t\mathbb{Z}^o) \in \mathcal{F}$, and $x$ is a distinguished nilpotent element of $I$, the Lie algebra of $L$. McNinch and Sommers prove the following result.

**Theorem 3.9.** (MS03) The map $(L,t\mathbb{Z}^o,x) \mapsto (x,tC_G^o(x))$ yields a bijection between $I_d^c/G$ and $N_{o,c}$.

Note that every semisimple element can be conjugated to lie in $T$. Thus if we define $I^c_{d,T}$ to be the subset of $I_d^c$ consisting of triples $(L,t\mathbb{Z}^o,x)$ such that $T \subseteq L$, then the map in Theorem 3.9 descends to a bijection between $I^c_{d,T}/G$ and $N_{o,c}$. Define $\mathcal{F}_T = \{(L,t\mathbb{Z}^o) \in \mathcal{F} : T \subseteq L\}$. We will additionally find it convenient to work with orbits rather than elements. Define $I^c_{o,d,T}$ to be the set of all triples $(L,t\mathbb{Z}^o,\mathcal{O})$ where $(L,t\mathbb{Z}^o) \in \mathcal{F}_T$, and $\mathcal{O}$ is a distinguished nilpotent orbit of $I$.

**Proposition 3.10.** The map $(L,t\mathbb{Z}^o,x) \mapsto (L,t\mathbb{Z}^o,L,x)$ induces a bijection between $I^c_{d,T}/G$ and $I^c_{o,d,T}/N$.

**Proof.** The induced map $\phi : I^c_{d,T} \to I^c_{o,d,T}/N$ is clearly a surjection. Suppose $g \in G$ is such that $g(L_1,t_1\mathbb{Z}_1^o,x_1) = (L_2,t_2\mathbb{Z}_2^o,x_2)$. Then $T, gT$ are both maximal tori of $L_2$ and so there is a $l \in L_2$ so that $lgT = T$ and so $lg = n$ for some $n \in N$. Clearly $nL_1 = L_2$ and $n(t_1\mathbb{Z}_1^o) = t_2\mathbb{Z}_2^o$. Also $n(L_1x_1) = L_2nx_1 = L_2gx_1 = L_2x_2$. Thus $\phi$ factors through $\mathcal{F}_T/G$. Now suppose there exists a $n \in N$ such that $n(L_1,t_1\mathbb{Z}_1^o,L_1x_1) = (L_2,t_2\mathbb{Z}_2^o,L_2x_2)$. Then $nL_1 = L_2$ and $nL_1x_1 = L_2x_2$. Thus $L_2nx_1 = L_2x_2$ and so there exists an $l \in L_2$ such that $lnx_1 = x_2$. Clearly $ln(t_1\mathbb{Z}_1^o) = t_2\mathbb{Z}_2^o$ since $t_2\mathbb{Z}_2^o$ lies in the center of $L_2$. Thus $ln(L_1,t_1\mathbb{Z}_1^o,x_1) = (L_2,t_2\mathbb{Z}_2^o,x_2)$. Thus $\phi$ descends to a bijection as required. 

Note that $N$ stabilises $I^c_{o,d,T}$ and $T$ acts trivially $I^c_{o,d,T}$. Thus $W$ acts on $I^c_{o,d,T}$ and $I^c_{o,d,T}/W = I^c_{o,d,T}/N$. We thus have canonical bijections

$$I^c_{d,T}/G \sim I^c_{o,d,T}/W \sim N_{o,c}.$$ 

(33)

3.2.4 From Faces of the Apartment to Pseudo-Levis

Recall the following standard results about closed subgroups of $T$ and $T$. 

27
Proposition 3.11. There is a $W$-equivariant bijective correspondence

$$\{\text{closed subgroups } H \leq T\} \leftrightarrow \{\mathbb{Z}\text{-submodules } M \leq X^*\}$$  \hspace{1cm} (34)

$$H \rightarrow X^*_H = \{\chi \in X^* : \chi(H) = 1\}$$  \hspace{1cm} (35)

$$\{t \in T : \chi(t) = 1 \ \forall \chi \in M\} \leftrightarrow M.$$  \hspace{1cm} (36)

Proposition 3.12. There is a $W$-equivariant bijective correspondence

$$\{\text{closed subgroups } \mathbb{H} \leq \mathbb{T}\} \leftrightarrow \{\mathbb{Z}\text{-submodules } M \leq \hat{T}\}$$  \hspace{1cm} (37)

$$\mathbb{H} \rightarrow X^*_\mathbb{H} = \{\chi \in X^* : \chi(\mathbb{H}) = 0 + \mathbb{Z}\}$$  \hspace{1cm} (38)

$$\{t \in \mathbb{T} : \chi(t) = 1 \ \forall \chi \in M\} \leftrightarrow M.$$  \hspace{1cm} (39)

Moreover, for every closed subgroup $\mathbb{H} \leq \mathbb{T}$ the connected component of $\mathbb{H}$ containing the identity, $\mathbb{H}^0$, coincides with the annihilator of the torsion elements of the Pontryagin dual of $\mathbb{H}$.

For an affine subspace $A \subseteq V$, call $A$ admissible if it is equal to the vanishing set of a subset of the affine roots $\Psi$.

Proposition 3.13. For every admissible $A \subseteq V$, there is a pseudo-Levi $L$ with $\Phi_L = \Phi_A$.

Proof. This follows easily from [Ste68, Remark 5.2 (b)].

Write $\mathcal{A}$ for the set of admissible affine subspaces of $V$. Cocharacters pair integrally with the roots $\Phi$ and so $X_\ast$ acts on the collection of admissible affine subspaces by translation. For $A \in \mathcal{A}$ write $[A]$ for the orbit of $A$ in $\mathcal{A}$ under the action of $X_\ast$. For $A \in \mathcal{A}$ write $L_A$ for the pseudo-Levi containing $T$ with $\Phi_{L_A} = \Phi_A$. For $v \in X_\ast$ and $w \in W$ we have $L_{A+v} = L_A$ and $L_{wA} = w.L_A$ (where $w.L_A$ denotes $wL_Aw^{-1}$).

Remark 3.14. For a closed subgroup $H \leq T$, $X^*_H = M X^*_H \cap X^*$. Moreover, for $tH^0 \in H/H^0$ we have $X^*_H \subseteq X^*_tH^0 \subseteq X^*_H$ and $\Phi_H \subseteq \Phi_tH^0 \subseteq \Phi_H$. Identical statements hold for closed subgroups $\mathbb{H} \leq \mathbb{T}$.

Lemma 3.15. Let $\mathbb{H} \leq \mathbb{T}$ be a closed subgroup, $\pi : V \rightarrow \mathbb{T}$ be the projection map and $z + \mathbb{H}^0 \in \mathbb{H}/\mathbb{H}^0$. Then

1. there is an affine subspace $A \subseteq V$ such that $\pi^{-1}(z+\mathbb{H}^0) = A + X_\ast$, $X^*_A = X^*_{z+\mathbb{H}^0}$ and $\dim A = \text{rk } \mathbb{H}^0$.

2. if $B$ is any other affine subspace with $\pi^{-1}(z+\mathbb{H}^0) = B + X_\ast$ then $B \in [A]$, $X^*_B = X^*_{z+\mathbb{H}^0}$ and $\dim B = \text{rk } \mathbb{H}^0$.

Proof. Let $N$ be annihilator of $\mathbb{H}^0$ in $X^\ast$. By Proposition 3.12 $X^\ast/N$ is a free $\mathbb{Z}$-module and so the short exact sequence

$$0 \rightarrow N \rightarrow X^\ast \rightarrow X^\ast/N \rightarrow 0$$  \hspace{1cm} (40)
splits. We can thus find a basis \( \chi_1, \ldots, \chi_n \) of \( X^* \) such that \( \chi_1, \ldots, \chi_k \) is a basis for \( N \) and \( \chi_{k+1}, \ldots, \chi_n \) project onto a basis for \( X^*/N \). Let \( \gamma_1, \ldots, \gamma_n \) be the dual basis in \( X_* \). Then \( \mathbb{H}^o \) is evidently equal to the image of \( W = \sum_{i=k+1}^n \mathbb{R}\gamma_i = \bigcap_{i=1}^k \ker \chi_i \) under \( \pi : V \to \mathbb{T} \). Let \( w \in V \) be such that \( \pi(w) \in v + \mathbb{H}^o \). Then \( A = w + W \) maps onto \( z + \mathbb{H}^o \) and so \( \pi^{-1}(z + \mathbb{H}^o) = A + X_* \). It is clear that \( \chi \in X^* \) takes integer values on \( A \) if and only if we obtain the desired result.

**Theorem 3.16.** Let \( L \) be a pseudo-Levi with center \( Z \). There is a canonical bijection

\[
\frac{Z/Z^o}{\{A \in \mathcal{A} : \Phi_Z \subseteq \Phi_A \subseteq \Phi_{Z^o}\}} \leftrightarrow \{A \in \mathcal{A} : \Phi_A = \Phi_L\}/X_*.
\]

(41)

with the property that if \([A]\) is the image of \( tZ^o \) then \( \Phi_A = \Phi_{C_{\mathcal{O}}(tZ^o)} \). In particular, there is a bijection

\[
\{tZ^o \in Z/Z^o : C_{\mathcal{O}}(tZ^o) = L\} \leftrightarrow \{A \in \mathcal{A} : \Phi_A = \Phi_L\}/X_*.
\]

(42)

**Proof.** Fix a pseudo-Levi \( L \) and let \( Z \) denote its center. The annihilator of \( Z \) in \( X^* \) is \( Z\Phi_L \). Write \( N \) for the annihilator for \( Z^o \) in \( X^* \). Recalling the identification between \( X^* \) and \( \mathbb{T} \), let \( \mathbb{H} \) denote the annihilator of \( Z\Phi_L \) in \( \mathbb{T} \). Then both the character group of \( Z/Z^o \) and the Pontryagin dual of \( \mathbb{H}/\mathbb{H}^o \) naturally identify with \( N/Z\Phi_L = \text{tor}(X^*/Z\Phi_L) \) - the torsion subgroup of \( X^*/Z\Phi_L \). But \( Z/Z^o \) and \( \mathbb{H}/\mathbb{H}^o \) are both finite groups and since the Pontryagin dual and character group (which we denote by \( X^* \)) coincide for finite groups we obtain a canonical isomorphism \( f_L \) given by the composition

\[
\frac{Z/Z^o}{\rightarrow X^*(\text{tor}(X^*/Z\Phi_L)) \rightarrow (\text{tor}(X^*/Z\Phi_L))^\wedge \rightarrow \mathbb{H}/\mathbb{H}^o.}
\]

(43)

**Claim 3.16.1.** The isomorphism \( f_L \) has the property that for all \( tZ^o \in Z/Z^o \), \( X^*_{tZ^o} = X^*_{f_L(tZ^o)} \).

**Proof.** Suppose \( \chi \in X^* \) vanishes on \( tZ^o \). Then since \( \chi \) is multiplicative, it must vanish on \( Z^o \) too and hence lie in \( N \). Its image in \( N/Z\Phi_L = \text{tor}(X^*/Z\Phi_L) = X^*(Z/Z^o) \) - the character group of \( Z/Z^o \) - must thus lie in \( X^*(Z/Z^o)_{tZ^o} \). Conversely any lift of an element \( \tilde{\chi} \in X^*(Z/Z^o)_{tZ^o} \) to \( X^* \) clearly lies in \( X^*_{tZ^o} \). Thus \( X^*_{tZ^o} \) is equal to all possible lifts of elements in \( X^*(Z/Z^o)_{tZ^o} \). We may similarly characterise \( X^*_{f_L(tZ^o)} \) and so we obtain the desired result.

Now fix a \( tZ^o \), let \( v + \mathbb{H}^o = f_L(tZ^o) \) and let \( \pi : V \to \mathbb{T} \) denote the projection map. By Lemma 3.15 we may assign to \( z + \mathbb{H}^o \) a well defined class \([A] \) with the property that for any \( B \in [A] \) we have \( \pi^{-1}(z + \mathbb{H}^o) = B + X_* \), \( \dim B = \text{rk} \mathbb{H}^o \) and \( X^*_B = X^*_{z + \mathbb{H}^o} \). Define \( \lambda_L(tZ^o) \) to be \([A]\)
Claim 3.16.2. \([A]\) is an admissible class with \(\Phi_Z \subseteq \Phi_A \subseteq \Phi_{Z^o}\).

Proof. The conditions on \(\Phi_A\) are evident from the fact that \(\Phi_A = \Phi_{z+H^o}\) and remark 3.14. It remains to show that \(A\) is admissible. We know that \(\Phi_L \subseteq X^*_z + H^o = X^*_A\) and so \(A\) lies in the vanishing set of \(\{\alpha - \alpha(A): \alpha \in \Phi_L\} \subseteq \Psi\). It thus suffices to check that for all \(w \in \mathbb{Z}\) there exists an \(\lambda\) where the forwards map is given by \(\Phi(\lambda)\) = \(\Phi_{C_G(tZ^o)}\).

Thus \(\lambda_L\) gives us a well defined forwards map for equation 41. Moreover \(\Phi_A = \Phi_{tZ^o} = \Phi_{C_G(tZ^o)}\) and so \(\lambda_L\) has the advertised property. It thus remains to show that the map has an inverse.

Let \([A]\) be an admissible class with \(\Phi_Z \subseteq \Phi_A \subseteq \Phi_{Z^o}\). Then \(\pi(A) \subseteq \mathbb{H}\) and is connected so \(\pi(A) \subseteq z + \mathbb{H}^o\) for some \(z \in \mathbb{H}\). But also
\[
\Phi_Z \subseteq \Phi_A \subseteq \Phi_{Z^o} \subseteq X^*_Z = X^*_A \subseteq \mathbb{R} \subseteq X^*_Z = \mathbb{R} \Phi_Z
\]
and so \(\mathbb{R} \Phi_A = \Phi Z\). Thus, since \(A\) is admissible, \(\dim A = \dim V - \dim \mathbb{R} \Phi A = \dim V - \dim \mathbb{R} \Phi Z = \text{rk} \mathbb{H}^o\). We must therefore have that \(\pi(A) = z + \mathbb{H}^o\).

Applying \(f_L^{-1}\) we obtain a backwards map which is clearly inverse to \(\lambda_L\).

Proposition 3.17. There is a canonical \(W\)-equivariant bijection
\[
\mathcal{F}_T \leftrightarrow \mathcal{A}/X^*_x
\]
where the forwards map is given by \(\lambda : (L, tZ^o) \mapsto \lambda_L(tZ^o)\).

Proof. We show first that \(\lambda\) is injective. Let \((L_1, t_1Z^o_1), (L_2, t_2Z^o_2) \in \mathcal{F}_T\) and write \([A_1] = \lambda(L_1, t_1Z^o_1)\). Suppose we have \([A_1] = [A_2]\). Then \(\Phi_{L_1} = \Phi_{t_1Z^o_1} = \Phi_{A_1} = \Phi_{A_2} = \Phi_{t_2Z^o_2} = \Phi_{L_2}\)
and so \(L_1 = L_2 =: L\). We then get that \(t_1Z^o_1 = t_2Z^o_2\) from the fact that \(\lambda_L\) is a bijection. Now let \([A]\) be an admissible class. By Proposition 3.13 there exists a pseudo-Levi \(L\) with \(\Phi_L = \Phi_A\). Let \(Z\) denote the center of \(L\). Then by Theorem 3.16 there exists an \(tZ^o\) such that \(\lambda(L, tZ^o) = [A]\) and so \(\lambda\) is surjective. It remains to show that \(\lambda\) is \(W\)-equivariant. Since the projection map \(V \to \mathbb{H}\) is \(W\) equivariant, it suffices to check that for all \(w \in W\), the outer square of the following diagram commutes.
But it is clear that all the inner squares commute and so the outer square must do too.

An important consequence of this proposition is the following corollary.

**Corollary 3.18.** Let
\[ \mathcal{L} : \{ c \subseteq A \} \rightarrow \mathcal{F}_T, \quad c \mapsto \lambda^{-1}(A(c) + X_\ast). \] (48)

Then \( \mathcal{L} \) is \( W \)-equivariant surjection and \( c_1, c_2 \) lie in the same fibre iff \( A(c_1) + X_\ast = A(c_2) + X_\ast \). Moreover, if \( \mathcal{L}(c) = (L, tZ^*_L) \) then \( L \) is the complex reductive group with the same root datum as \( L_{c}(\overline{F}_q) \).

For \( c \subseteq A \) and \( (L, tZ^*_L) = \mathcal{L}(c) \) write \( L_c \) for \( L \). Since \( L_c(\mathbb{F}_q) \) and \( L_c \) have the same root datum, by lemma \( 2.1 \), there is an isomorphism of partial orders
\[ \Lambda^*_c : \mathcal{N}^L_{o,c}(\mathbb{F}_q) \rightarrow \mathcal{N}^L_{o,c}(C). \] (49)

3.2.5 A Parameterisation of Unramified Nilpotent Orbits

**Theorem 3.19.** The map
\[ I^K_{o,d,A} \rightarrow I^C_{o,d,T}, \quad (c, \mathcal{O}) \mapsto (\mathcal{L}(c), \Lambda^*_c(\mathcal{O})) \]
induces a bijection
\[ I^K_{o,d,A}/\sim_A \rightarrow I^C_{o,d,T}/W. \] (50)

**Proof.** The map \( I^K_{o,d,A} \rightarrow I^C_{o,d,T} \) is clearly a surjection. Let
\[ (c_1, \mathcal{O}_1), (c_2, \mathcal{O}_2) \in I^K_{o,d,A} \]
and suppose that \( (c_1, \mathcal{O}_1) \sim_A (c_2, \mathcal{O}_2) \). Then there exists a \( w \in \tilde{W} \) such that
\[ A(c_1) = A(wc_2) \text{ and } \mathcal{O}_1 = j_{c_1,wc_2}(w, \mathcal{O}_2). \]
By corollary \( 3.18 \) we have that \( \mathcal{L}(c_1) = \tilde{w} \cdot \mathcal{L}(c_2). \) Moreover
\[ \Lambda^*_c(\mathcal{O}_1) = \Lambda^*_c(j_{c_1,wc_2}(w, \mathcal{O}_2)) = \Lambda^*_c(w, \mathcal{O}_2) = \tilde{w} \Lambda^*_c(\mathcal{O}_2). \]
Thus
\[ (\mathcal{L}(c_1), \Lambda^*_c(\mathcal{O}_1)) = \tilde{w} \cdot (\mathcal{L}(c_2), \Lambda^*_c(\mathcal{O}_2)) \]
and so \( I^K_{o,d,A} \rightarrow I^C_{o,d,T}/W \) descends to a map
\[ I^K_{o,d,A}/\sim_A \rightarrow I^C_{o,d,T}/W. \]
To see that this map is injective suppose that \((c_1, \mathcal{O}_1), (c_2, \mathcal{O}_2) \in I^{K}_{o,d,a}\) are such that there is a \(w_0 \in W\) with
\[
(\Sigma(c_1), \Lambda_{\chi}^{\mathcal{O}_1}(\mathcal{O}_1)) = w_0.(\Sigma(c_2), \Lambda_{\chi}^{\mathcal{O}_2}(\mathcal{O}_2)).
\]
Since \(\Sigma(c_1) = w_0 \Sigma(c_2) = \Sigma(w_0c_2)\) we have that
\[
\mathcal{A}(c_1) + X_\ast = \mathcal{A}(w_0c_2) + X_\ast
\]
and so there is a \(w \in \tilde{W}\) such that \(\tilde{w} = w_0\), and \(\mathcal{A}(c_1) = A(wc_2)\). Moreover
\[
\Lambda_{\chi}^{\mathcal{O}_1}(\mathcal{O}_1) = \tilde{w}\Lambda_{\chi}^{\mathcal{O}_2}(\mathcal{O}_2) = \Lambda_{\chi}^{\mathcal{O}_2}(w;\mathcal{O}_2) = \Lambda_{\chi}^{\mathcal{O}_2}(j_{c_1,wc_2}(w;\mathcal{O}_2))
\]
and so \(\mathcal{O}_1 = j_{c_1,wc_2}(w;\mathcal{O}_2)\). Thus \((c_1, \mathcal{O}_1) \sim_{\mathcal{A}} (c_2, \mathcal{O}_2)\) as required. \(\blacksquare\)

Let \(pr_1 : \mathcal{N}_{o,c} \rightarrow \mathcal{N}_o\) be the projection onto the first factor. Let \(\theta : \mathcal{N}_o(K) \rightarrow \mathcal{N}_{o,c}\) be the composition

\[
\xymatrix{ 
\mathcal{N}_o(K) \ar[r]_{\sim} & I^K_d/\sim_K & \ar[l]^-{\sim} I^K_{o,d,a}/\sim_{A} \ar[r]_{\sim} & I^C_{o,d,T}/W \ar[r]_{\sim} & \mathcal{N}_{o,c}.
}
\]

**Theorem 3.20.** The map \(\theta : \mathcal{N}_o(K) \rightarrow \mathcal{N}_{o,c}\) is a bijection and for all \(\mathcal{O} \in \mathcal{N}_o(K)\)
\[
\Lambda^{\mathcal{O}}(\mathcal{N}_o(k/K)(\mathcal{O})) = pr_1(\theta(\mathcal{O})).
\]

**Proof.** Since all the maps involved are bijections, \(\theta\) is also a bijection. Let \(\mathcal{O} \in \mathcal{N}_o(K)\) and \((c, \mathcal{O}') \in I^K_{o,d,a}(\mathcal{O})\). By theorem 3.15
\[
\Lambda^{\mathcal{O}} \circ \mathcal{N}_o(k/K)(\mathcal{O}) = \Lambda_{\chi}^{\mathcal{O}} \circ j_{c,o}(\mathcal{O}').
\]
By equation 49
\[
\Lambda_{\chi}^{\mathcal{O}} \circ j_{c,o}(\mathcal{O}') = G.\Lambda_{\chi}^{\mathcal{O}}(\mathcal{O}').
\]
But
\[
G.\Lambda_{\chi}^{\mathcal{O}}(\mathcal{O}') = pr_1(\theta(\mathcal{O})).
\]
as required. \(\blacksquare\)

**Proposition 3.21.** Let \(\mathcal{O} \in \mathcal{N}_o\) and let \([(J, J')]_W \in \mathbf{BC}(\Delta)\) be the Bala–Carter parameter for \(\mathcal{O}\) with respect to \(\Delta\). Then the affine Bala–Carter parameter for \(\theta^{-1}(\mathcal{O}, 1)\) is \([(J, J')]_W\).

**Proof.** For \(I \Subset \tilde{\Delta}\) let \(c(I)\) denote the face of \(c_0\) with Type equal to \(I\). Let \(\mathcal{O} \in \mathcal{N}_o\) and let \([(J, J')]_W \in \mathbf{BC}(\Delta)\) be the Bala–Carter parameter for \(\mathcal{O}\). Then \(\mathcal{O}\) intersects non-trivially with \(I_{c(J)}\) and the distinguished orbit \(\mathcal{O}_{J'}\) of \(I_{c(J)}\) parameterised by \(J'\) lies in this intersection. Since \(L_{c(J)}\) is a Levi it is equal to \(C^o_G(Z^\mathcal{O}_{J'})\) where \(Z_J\) is the center of \(L_{c(J)}\). Thus \((L_{c(J)}, Z^\mathcal{O}_{J'}, \mathcal{O}_{J'}) \in I^C_{o,d,T}\) and its image in \(\mathcal{N}_{o,c}\) is \((\mathcal{O}, 1)\). We also have that \(\lambda(L_{c(J)}, Z^\mathcal{O}_{J'}) = [A]\) where \(A\) is the vanishing set of \(J\), and \((\Lambda_{\chi}^{\mathcal{O}})^{-1}(\mathcal{O}_{J'})\) is the nilpotent orbit of \(I_{c(J)}\) corresponding to \(J'\). It follows that \(\theta^{-1}(\mathcal{O}, 1)\) has affine Bala–Carter parameter \([(J, J')]_W\) as required. \(\blacksquare\)
This result implies that the diagram below commutes.

\[
\begin{array}{c}
\text{BC}(\Delta) \sim_{W} \xrightarrow{i} \text{ABC}(\tilde{\Delta}) \sim_{W} \xrightarrow{\sim} \text{BC}(\Delta) \sim_{W} \\
\mathcal{N}_{\alpha}(\bar{k}) \xrightarrow{\oplus \theta^{-1}(\Lambda^{\pm}(O),1)} \mathcal{N}_{\alpha}(K) \xrightarrow{\sim} \mathcal{N}_{\alpha}(K) \xrightarrow{\text{id}} \mathcal{N}_{\alpha}(\bar{k}).
\end{array}
\]

(52)

Here the map \(i\) is the map induced by the inclusion map

\[
\text{BC}(\Delta) \rightarrow \text{ABC}(\tilde{\Delta}).
\]

(53)

### 3.2.6 Example: \(G_2\)

Consider the case when \(G\) is \(K\)-split semisimple of type \(G_2\) (there is only one isogeny type). Let \(\{\alpha_1, \alpha_2\}\) denote the simple roots where \(\alpha_2\) is the short root and let \(\alpha_0\) be 1 minus the highest root. The extended Dynkin diagram of \(G_2\) is \(\bullet - \bullet\). The set \(\text{ABC}(\tilde{\Delta})\) consists of the pairs

\[
(0, 0), (\{\alpha_0\}, 0), (\{\alpha_1\}, 0), (\{\alpha_2\}, 0), (\{\alpha_1, \alpha_2\}, 0),
\]

\[
(\{\alpha_1, \alpha_2\}, \{\alpha_2\}), (\{\alpha_0, \alpha_2\}, 0), (\{\alpha_0, 0, \alpha_1\}, 0).
\]

(54)

Under the equivalence relation \(\sim_{W}\), all equivalence classes are singletons except \((\{\alpha_0\}, 0) \sim_{W} (\{\alpha_1\}, 0)\) so \(\mathcal{N}_{\alpha}(K)\) has size 7. We also obtain 7 nilpotent orbits using the parameterisation in terms of \(\mathcal{N}_{\alpha,c}\). The only nilpotent orbit with non-trivial \(G_{\mathbb{C}}\)-equivariant fundamental group is \(G_2(\alpha_1)\). In this case \(A(G_2(\alpha_1)) = S_3\) which has 3 conjugacy classes which we denote by representatives \(1, (12), (123)\). Using theorem 3.20, the unramified orbits can be parameterised by

\[
(0, 1), (A_1, 1), (\bar{A}_1, 1), (G_2(\alpha_1), 1), (G_2(\alpha_1), (12)), (G_2(\alpha_1), (123)), (G_2, 1) \in \mathcal{N}_{\alpha,c}.
\]

(55)

We now demonstrate how to match up the two parameterisations. We already have from Proposition 3.21 the following matchings

\[
(0, 0) \leftrightarrow (1, 1),
\]

\[
(\{\alpha_0\}, 0) \leftrightarrow (A_1, 1)
\]

\[
(\{\alpha_2\}, 0) \leftrightarrow (\bar{A}_1, 1)
\]

\[
(\{\alpha_1, \alpha_2\}, \{\alpha_1\}) \leftrightarrow (G_2(\alpha_1), 1)
\]

\[
(\{\alpha_1, \alpha_2\}, 0) \leftrightarrow (G_2, 1)
\]

(56)

It remains to determine how to match up \((G_2(\alpha_1), (12))\) and \((G_2(\alpha_1), (123))\) together with \((\{\alpha_0, \alpha_1\}, 0)\) and \((\{\alpha_0, \alpha_2\}, 0)\). For this we must look at the map \(\lambda\). Figure 3.2.6 is a diagram of an apartment of \(G\) together with the coroot lattice (in grey), the coroots of the simple roots and minus the highest root (in blue), and a fundamental domain for the topological torus \(V/X_\ast\) (in red).
Figure 1: Apartment with coroots and fundamental domain for the torus $\mathbb{T}$.

Figure 2: Vanishing set of $\{\alpha_0, \alpha_1\}$ lifted to the apartment.

Figure 3: Vanishing set of $\{\alpha_0, \alpha_2\}$ lifted to the apartment.
Figures 2 and 3 respectively show the vanishing sets of \( \{\alpha_0, \alpha_1\} \) and \( \{\alpha_0, \alpha_2\} \) in \( \mathbb{T} \) lifted to the apartment. Having the same color indicates having the same image in \( \mathbb{T} \). Note that the yellow and blue lattices are \( W \) conjugate, and the red, cyan and green lattices are also \( W \) conjugate. Thus respectively they give rise to the same element of \( \mathcal{N}_{\alpha,c} \). It suffices thus to focus on the blue and green lattices respectively. Clearly the blue lattice gives rise to a conjugacy class of order 3, while the green one gives rise to a conjugacy class of order 2. The remaining matchings are thus

\[
\begin{align*}
\{\alpha_0, \alpha_1\}, \emptyset &\leftrightarrow (G_2(a_1), (123)) , \\
\{\alpha_0, \alpha_2\}, \emptyset &\leftrightarrow (G_2(a_1), (12)).
\end{align*}
\]

(57)

### 3.3 Achar’s Partial Order

#### 3.3.1 A Duality Map for Unramified Nilpotent Orbits

Write \( \mathcal{N}_o^{\vee} \) for \( \mathcal{N}_o^{G^{\vee}}(\mathbb{C}) \). Let \( c \subseteq A \) and let \( E \) be a special representation of \( \tilde{W}_c \). Recall from [Lus92] that the representation \( j_{W_c}^W E \) obtained through truncated induction corresponds under the Springer correspondence for \( \mathfrak{g}^{\vee}(\mathbb{C}) \) to a nilpotent orbit \( \mathfrak{O} \in \mathcal{N}_o^{\vee} \) and the trivial local system. This gives us a map \( \text{Ind}_c^{W} \) from \( \mathcal{N}_{o,sp}(\overline{F}_q) \) to \( \mathcal{N}_o^{\vee} \). Let \( d_{\ell} \) denote Lusztig–Spaltenstein duality on \( I_c \). Define the map

\[
d_S : I_{o,A}^K \to \mathcal{N}_o^{\vee}, \quad (c, \mathcal{O}) \mapsto \text{Ind}_c^{W} \circ d_{\ell}(\mathcal{O}).
\]

(58)

This is our candidate for a duality map on the unramified orbits. A priori it is not obvious that it factors through \( \mathcal{L} : I_{o,A}^K \to \mathcal{N}_o(K) \), but as we will see in this next theorem - this is indeed the case.

**Theorem 3.22.** The map \( d_S : I_{o,A}^K \to \mathcal{N}_o^{\vee} \) factors through the lifting map \( \mathcal{L} : I_{o,A}^K \to \mathcal{N}_o(K) \).

**Proof.** Let \( c' \subseteq c \) be faces of \( A \). Then \( I_c(\overline{F}_q) \) is the reductive quotient of the parabolic \( p_c(\mathcal{O})/\mathcal{U}_c(\mathcal{O}) \) in \( I_{c'}(\overline{F}_q) \). For \( \mathcal{O} \in \mathcal{N}_o^{L_c^{\vee}}(\overline{F}_q) \), write \( \text{Ind}_{c'}^c \mathcal{O} \) for the Lusztig–Spaltenstein induction of \( \mathcal{O} \) from \( I_{c'}(\overline{F}_q) \) to \( I_c(\overline{F}_q) \). Let \( \text{Sat}_{c'}^c \) denote the map from \( \mathcal{N}_{o}^{L_c^{\vee}}(\overline{F}_q) \) to \( \mathcal{N}_{o}^{L_c^{\vee}}(\overline{F}_q) \) obtained by \( L_c(\overline{F}_q) \)-saturation (note that although \( I_c(\overline{F}_q) \) is naturally only a subquotient of \( I_{c'}(\overline{F}_q) \), it is standard to identifying it with a Levi factor of \( p_c(\mathcal{O})/\mathcal{U}_c(\mathcal{O}) \) and the saturation map is independent of this choice of Levi factor).

Recall that

\[
d_{\ell} \circ \text{Sat}_{c'}^c \mathcal{O} = \text{Ind}_{c'}^c \circ d_{\ell}(\mathcal{O}).
\]

Then since \( \text{Ind}_c^{W} \circ \text{Ind}_{c'}^c \mathcal{O} = \text{Ind}_c^{W} \mathcal{O} \) and \( \mathcal{L}_c(\mathcal{O}) = \mathcal{L}_{c'}(\text{Sat}_{c'}^c \mathcal{O}) \) it suffices to show that \( d_S \) restricted to \( I_{o,d,A}^K \) factors through \( \mathcal{L} \) i.e. that it is constant on \( \sim_A \) equivalence classes. Thus suppose we have \( (c_1, \mathcal{O}_1), (c_2, \mathcal{O}_2) \in I_{o,d,A}^K \) such that \( (c_1, \mathcal{O}_1) \sim_A (c_2, \mathcal{O}_2) \). For \( i = 1, 2 \) let \( (L_i, t_i Z_i) = \mathfrak{L}(c_i) \). Then \( L_i \) has Weyl group \( W_{c_i} \) and \( (L_1, t_1 Z_1, \mathcal{N}_{c_1}(\mathcal{O}_1)) \) and \( (L_2, t_2 Z_2, \mathcal{N}_{c_2}(\mathcal{O}_2)) \) are \( W \)-conjugate by theorem 3.19. In particular under the isomorphism in equation 3.33 they give the same element of \( \mathcal{N}_{o,c} \). The fact that \( d_S((c_1, \mathcal{O}_1)) = d_S((c_2, \mathcal{O}_2)) \) is then exactly [Som01, Proposition 7].

35
The map $d_S$ thus descends to a map $\mathcal{N}_c(K) \to \mathcal{N}_c'$. Call this map also $d_S$. One important consequence of the proof just given is that $d_S$ then coincides exactly with Sommers’ duality map under the identification between $\mathcal{N}_c(K)$ and $\mathcal{N}_{o,c}$ (hence the subscript $S$ in $d_S$). This raises some intriguing questions. First of all, there are now two natural pre/partial order structures on $\mathcal{N}_c(K)$. There is the one induced by the closure relations, but also a second one, first defined on $\mathcal{N}_{o,c}$ by Achar in [Ach03], which we can transport to $\mathcal{N}_c(K)$ via the identification between $\mathcal{N}_c(K)$ and $\mathcal{N}_{o,c}$. This second pre-order, which we denote by $\leq_A$ can be rewritten using our new machinery as follows: $\mathcal{O}_1 \leq_A \mathcal{O}_2$ iff $\mathcal{N}_c(\bar{k}/K)(\mathcal{O}_1) \leq \mathcal{N}_c(\bar{k}/K)(\mathcal{O}_2)$ and $d_S(\mathcal{O}_1) \geq d_S(\mathcal{O}_2)$. Now let

$$\mathcal{N}_{o,c} = \{ (\mathcal{O}, C) : \mathcal{O} \in \mathcal{N}_c(C), C \in C(\bar{A}(\mathcal{O})) \}$$

where $\bar{A}(\mathcal{O})$ is Lusztig’s canonical quotient of $A(\mathcal{O})$ as defined in [Som01] (so that it exists for non-adjoint groups and non-special orbits too). For a conjugacy class $C$ of $A(\mathcal{O})$, write $\bar{C}$ for its image in $\bar{A}(\mathcal{O})$. Write $\Omega : \mathcal{N}_{o,c} \to \mathcal{N}_{o,c}$ for the map induced by $C \mapsto \bar{C}$. Achar showed in [Ach03] that the $\leq_A$-equivalence classes (which we denote by $\sim_A$) are precisely the fibres of $\Omega$ and so $\leq_A$ descends to a partial order on $\mathcal{N}_{o,c}$ (also denoted $\leq_A$). Thus $\theta$ descends to a bijection $\tilde{\theta} : \mathcal{N}_0(K) / \sim_A \to \mathcal{N}_{o,c}$. Note that the closure ordering $\leq$ on $\mathcal{N}_c(K)$ and $\leq_A$ are therefore in general not the same since the former is a partial order, but the latter is often not.

**Lemma 3.23.** Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{N}_c(K)$. If $\mathcal{O}_1 \sim_A \mathcal{O}_2$ then

$$\mathcal{N}_c(\bar{k}/K)(\mathcal{O}_1) = \mathcal{N}_c(\bar{k}/K)(\mathcal{O}_2).$$

**Proof.** Obvious. ■

The saturation map $\mathcal{N}_c(\bar{k}/K)$ is constant on $\sim_A$ classes.

**Lemma 3.24.** Let $(c, \mathcal{O}_1), (c, \mathcal{O}_2) \in I^K_{o,A}$ and let $\mathcal{O}_i = \mathcal{L}_c(\mathcal{O}_i)$ for $i = 1, 2$. If $\mathcal{O}_1 \leq \mathcal{O}_2$ then $d_S(\mathcal{O}_2) \leq d_S(\mathcal{O}_1)$. In particular $\mathcal{O}_1 \leq_A \mathcal{O}_2$.

**Proof.** If $\mathcal{O}_1 \leq \mathcal{O}_2$ then $d_L(\mathcal{O}_2) \leq d_L(\mathcal{O}_1)$. But Ind$_L^G$ is order preserving by [Spa82, Theorem 12.5] and so $d_S(\mathcal{O}_2) \leq d_S(\mathcal{O}_1)$. ■

### 3.3.2 The Canonical Unramified Wavefront Set

Let $(\pi, X)$ be a depth-0 representation of $G(k)$. Define the canonical unramified wavefront set of $(\pi, X)$ to be

$$KWF(X) := \max_{c \leq B(G,k)} \{ \mathcal{L}_c(\tilde{\mathcal{W}}F(X_{U_c})) \} \quad (\subseteq \mathcal{N}_o(K) / \sim_A).$$

The canonical unramified wavefront set has many of the nice properties that $KWF(\pi)$ has.

**Lemma 3.25.** Let $(\pi, X)$ be a depth 0 representation of $G(k)$. Then

$$KWF(X) = \max[KWF(X)].$$
Proof. The proof is exactly the same as Lemma 2.17 except we need to be careful for the second part which we now provide the details for. Let $O$ be a maximal element of $K\Xi(\pi)$ with respect to $\leq_A$. Write $O = N_o(K/k)(O_1)$ where $O_1 \in \Xi(\pi)$. Let $(c, O'_1) \in I^*_o(O_1)$. Let 

\[ O'_2 = N_o^{\Lambda}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right)(O'_1). \]

Let $O'_3$ be a wavefront set nilpotent for $V_{U,c}(\pi)$ such that $O'_2 \leq O'_3$. Let $O_3 = L_c(O'_3)$. $O'_3$ is an element of $K\Xi(\pi)$. By Lemma 3.24, $O'_2 \leq O'_3$ implies that $O \leq_A O_3$. By maximality of $O$ in $K\Xi(\pi)$ we get that 

\[ O \sim_A O_3. \]

In particular 

\[ N_o(\overline{k}/K)(O) = N_o(\overline{k}/K)(O_3). \]

Thus by corollary 3.6 we have that $O'_2 = O'_3$. It follows that $O \in K\WF_c(\pi)$.

Theorem 3.26. Let $(\pi, X)$ be a depth 0 representation of $G(k)$. Then 

\[ \overline{k}\WF(X) = \max N(\overline{k}/K)(K\WF(X)). \]

Proof. By Proposition 2.15 and corollary 2.16, $\overline{k}\WF(X)$ consists of the maximal elements of $N_o(\overline{k}/K)(K\Xi(X))$. Since $O_1 \leq_A O_2$ implies that $\overline{k}O_1 \leq \overline{k}O_2$, the results follows.

3.3.3 Achar’s Duality Map

Let $N_o^{\vee}, N_o^{\vee}$ be the analogous objects of $N_{o,c}$ and $N_{o,c}$ for $G^\vee$. In [Ach03], Achar introduces duality maps 

\[ N_{o,c} \xrightarrow{d_A} N_{o,c}^{\vee}. \]  

which extend $d_S$ in the sense that the following diagram commutes

\[ N_o(K) \xrightarrow{d_S} N_o^{\vee} \quad \xrightarrow{p_{1}^{\vee}} \quad N_{o,c} \xrightarrow{d_A} N_{o,c}^{\vee}. \]

Here $p_{1}^{\vee} : N_{o,c} \rightarrow N_{o,c}^{\vee}$ denotes the projection onto the first factor.

Note that since $d_S$ factors through $\Omega$, write $d_S$ for the resulting map $N_{o,c} \rightarrow N_{o,c}^{\vee}$.

Let $d : N_{o,c}^{\vee} \rightarrow N_{o}$ denote Barbasch–Vogan duality [BV85] and $d_{LS} : N_{o} \rightarrow N_{o}$ denote Lusztig–Spaltenstein duality [Spa82]. We also write $d$ for the map going from $N_{o} \rightarrow N_{o}^{\vee}$.

Theorem 3.27. 1. Let $O \in N_o(K)$. Then $\Lambda^k \circ N_o(\overline{k}/K)(O) \leq d \circ d_S(O)$. 

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2. Let \( \mathcal{O}^\vee \in \mathcal{N}_o^\vee \). Then
\[
\{ \theta(\mathcal{O}) \in \mathcal{N}_o(k): d_S(\mathcal{O}) = \mathcal{O}^\vee, \mathcal{N}_o(k/K)(\mathcal{O}) = d(\mathcal{O}^\vee) \} = \Omega^{-1}(d_A(\mathcal{O}^\vee, 1)).
\]
In particular, this set is non-empty.

3. Let \( \mathcal{O} \in \mathcal{N}_o(k) \) and \( \mathcal{O}^\vee \in \mathcal{N}_o^\vee \). If \( d_S(\mathcal{O}) \geq \mathcal{O}^\vee \) then \( \bar{\theta}(\mathcal{O}) \leq d_A(\mathcal{O}^\vee, 1) \).

Proof. In [Ach03, Proposition 2.3] Achar proves that \( d_S(\mathcal{O}', C) \leq d_S(\mathcal{O}', 1) \) for all \( \mathcal{O}' \in \mathcal{N}_o \) and \( (\mathcal{O}', C) \in \mathcal{N}_o_c \). But \( d_S(\mathcal{O}', 1) = d(\mathcal{O}') \). Thus \( d \circ d(\mathcal{O}') \leq d \circ d_S(\mathcal{O}', C) \). Since \( \mathcal{O}' \leq d_L \circ d_L(\mathcal{O}') = d \circ d(\mathcal{O}') \), the first part follows.

The second part is an immediate consequence of [Som01, Remark 14] and [Ach03, Proposition 2.8].

For the third part, note that by part 1 of this theorem, we have that
\[
\Lambda^k \circ \mathcal{N}_o(k/K)(\mathcal{O}) \leq d(\mathcal{O}^\vee).
\]
By assumption, we also have
\[
d_S(\mathcal{O}) \geq d_S(d_A(\mathcal{O}^\vee, 1)) = \mathcal{O}^\vee.
\]
Since \( p_1(d_A(\mathcal{O}^\vee, 1)) = d(\mathcal{O}^\vee) \), it follows by definition that \( \bar{\theta}(\mathcal{O}) \leq d_A(\mathcal{O}^\vee, 1) \).  

4 The Wavefront Set for the Principal Block

4.1 Notation

If \( H \) is a complex group and \( x \) is an element of \( H \) or its Lie algebra \( \mathfrak{h} \), we write \( H(x) \) for the centralizer of \( x \) in \( H \), and \( A_H(x) \) for the group of connected components of \( H(x) \). If \( S \) is a subset of \( H \) or \( \mathfrak{h} \) (or indeed, of \( H \cup \mathfrak{h} \)), we can similarly define \( H(S) \) and \( A_H(S) \). We will sometimes write \( A(x) \), \( A(S) \) when the group \( H \) is implicit.

Write \( B^\vee \) for the flag variety of \( G^\vee \), i.e. the variety of Borel subgroups \( B^\vee \subset G^\vee \). Note that \( B^\vee \) has a natural left \( G^\vee \)-action. For \( g \in G^\vee \), write
\[
B_g^\vee = \{ B^\vee \in B^\vee \mid g \in B^\vee \}.
\]
(this coincides with the subvariety of Borels fixed by \( s \)). Similarly, for \( x \in \mathfrak{g}^\vee \), write
\[
B_x^\vee = \{ B^\vee \in B^\vee \mid x \in \mathfrak{b}^\vee \}.
\]
If \( S \) is a subset of \( G^\vee \) or \( \mathfrak{g}^\vee \) (or indeed of \( G^\vee \cup \mathfrak{g}^\vee \)), write
\[
B_S^\vee = \bigcap_{x \in S} B_x^\vee.
\]
The singular cohomology group \( H^i(B_S^\vee, \mathbb{C}) = H^i(B_S^\vee) \) carries an action of \( A(S) = A_{G^\vee}(S) \). For an irreducible representation \( \rho \in \text{Irr}(A(S)) \), let
\[
H^i(B_S^\vee)\rho := \text{Hom}_{A(S)}(\rho, H^i(B_S^\vee)).
\]
We will often consider the subset
\[
\text{Irr}(A(S))_0 := \{ \rho \in \text{Irr}(A(S)) \mid H^{\top}(B_S^\vee)\rho \neq 0 \}.
\]
4.2 Local Wavefront Sets for the Principal Block

4.2.1 Representations with Unipotent Cuspidal Support

Definition 4.1. Let $X$ be an irreducible smooth $G(k)$-representation. We say that $X$ has unipotent cuspidal support if there is a parahoric subgroup $P_c \subset G$ such that $X^{U_c(o)}$ contains an irreducible Deligne–Lusztig cuspidal unipotent representation of $L_c(F_q)$. Write $\Pi^{Lus}(G(k))$ for the set of all such representations.

Recall that an irreducible $G(k)$-representation $V$ is Iwahorii spherical if $V^{I(o)} \neq 0$ for an Iwahori subgroup $I$ of $G$. We note that all such representations have unipotent cuspidal support, corresponding to the case $P_c = I$ and the trivial representation of $T(F_q)$.

We will now recall the classification of irreducible representations of unipotent cuspidal support. Write $L(G(k))$ for the set of $G^\vee$-orbits (under conjugation) of triples $(s, n, \rho)$ such that

- $s \in G^\vee$ is semisimple,
- $n \in g^\vee$ such that $\text{Ad}(s)n = qn$,
- $\rho \in \text{Irr}(A_{G^\vee}(s, n))$ such that $\rho|_{Z(G^\vee)}$ is a multiple of the identity.

Without loss of generality, we may assume that $s \in T^\vee$. Note that $n \in g^\vee$ is necessarily nilpotent. The group $G^\vee(s)$ acts with finitely many orbits on the $q$-eigenspace of $\text{Ad}(s)$

$$g_q^\vee = \{ x \in g^\vee \mid \text{Ad}(s)x = qx \}$$

In particular, there is a unique open $G^\vee(s)$-orbit in $g_q^\vee$.

Fix an $\mathfrak{sl}(2)$-triple $\{ n^-, h, n \} \subset g^\vee$ with $h \in t_q^\vee$ and set

$$s_0 := sq^{-\frac{1}{2}}.$$ 

Then $\text{Ad}(s_0)n = n$.

Theorem 4.2 (Local Langlands correspondence, [KL87, Lus95a, Lus02a]). Suppose that $G$ is adjoint and $k$-split. There is a natural bijection

$$L(G(k)) \overset{\sim}{\to} \Pi^{Lus}(G(k)), \quad (s, n, \rho) \mapsto X(s, n, \rho),$$

such that

1. $X(s, n, \rho)$ is tempered if and only if $s_0 \in T_c^\vee$ and $G^\vee(s)n = g_q^\vee$,
2. $X(s, n, \rho)$ is square integrable (modulo the center) if and only if it is tempered and $A_{G^\vee}(s, n)$ contains no nontrivial torus.
3. $X(s, n, \rho)^{I(o)} \neq 0$ if and only if $\rho \in \text{Irr}(A(s, n))_0$, i.e., $\rho$ is such that

$$H^{\text{top}}(B^\vee_{s, n})^\rho \neq 0.$$
For each parameter \((s, n, \rho) \in \mathbf{L}(\mathbf{G}(k))\), there is an associated *standard representation* \(Y(s, n, \rho) \in \mathfrak{C}(\mathbf{G}(k))\). For Iwahori spherical representations, the relevant results are [KLS7] Theorems 7.12, 8.2, 8.3], see also [CG10]. For the more general setting of representations with unipotent cuspidal support, see [Lus95b §10] and [Lus02b Theorems 1.15, 1.21, 1.22] (in the equivalent setting of geometric graded Hecke algebras) and [Lus02a Theorems 10.4, 10.5]. See [AMS21 §2.3] for a discussion of the compatibility between these classifications.

### 4.2.2 The Borel–Casselman Equivalence

In this chapter we assume that \(\mathbf{G}\) is \(k\)-split. Let \(\mathbf{I}\) be an Iwahori subgroup of \(\mathbf{G}\). Recall the Iwahori–Hecke algebra associated to \(\mathbf{G}(k)\)

\[
\mathcal{H}_I = \{ f \in C_c^\infty(\mathbf{G}(k)) \mid f(i_1 g i_2) = f(g), \ i_1, i_2 \in \mathbf{I}(\mathfrak{o}) \}.
\]

Multiplication in \(\mathcal{H}_I\) is given by convolution with respect to a fixed Haar measure of \(\mathbf{G}(k)\). Let \(\text{Rep}_I(\mathbf{G}(k))\) denote the Iwahori category, i.e. the full subcategory of \(\text{Rep}(\mathbf{G}(k))\) consisting of representations \(X\) such that \(X\) is generated by \(X^I(\mathfrak{o})\). The simple objects in this category are the (irreducible) Iwahori spherical representations. By the Borel–Casselman Theorem [Bor76 Corollary 4.11], there is an exact equivalence of categories

\[
m_I : \text{Rep}_I(\mathbf{G}(k)) \rightarrow \text{Mod}(\mathcal{H}_I), \quad m_I(X) = X^I(\mathfrak{o}).
\]

This equivalence induces a group isomorphism

\[
m_I : R_I(\mathbf{G}(k)) \simeq R(\mathcal{H}_I),
\]

where \(R_I(\mathbf{G}(k))\) (resp. \(R(\mathcal{H}_I))\) is the Grothendieck group of \(\text{Rep}_I(\mathbf{G}(k))\) (resp. \(\text{Mod}(\mathcal{H}_I))\).

### 4.2.3 Deformations of Modules

Suppose \(\mathbf{P}_c\) is a parahoric subgroup containing \(\mathbf{I}\) with pro-unipotent radical \(\mathbf{U}_c\) and reductive quotient \(\mathbf{L}_c\). The finite Hecke algebra \(\mathcal{H}_c\) of \(\mathbf{L}_c(\mathbb{F}_q)\) embeds as a subalgebra of \(\mathcal{H}_I\). For \(X \in \text{Rep}_I(\mathbf{G}(k))\), the Moy–Prasad theory of unrefined minimal \(K\)-types [MP94] implies that the finite dimensional \(\mathbf{L}_c(\mathbb{F}_q)\)-representation \(X^{U_c(\mathfrak{o})}\) is a sum of principal series unipotent representations and so corresponds to an \(\mathcal{H}_c\)-module with underlying vector space

\[
(X^{U_c(\mathfrak{o})})^I(\mathfrak{o})/U_c(\mathfrak{o}) = X^I(\mathfrak{o}).
\]

The \(\mathcal{H}_c\)-module structure obtained in this manner coincides naturally with that of \(\text{Res}_{\mathcal{H}_c}^{\mathcal{H}_I} m_I(X)\).

Let

\[
\mathcal{J}_c : \mathcal{H}_c \rightarrow \mathbb{C}[W_c]
\]
be the isomorphism introduced by Lusztig in [Lus81]. Given any \( \mathcal{H}_c \)-module \( M \) we can use the isomorphism \( \mathcal{Z} \) to obtain a \( W_c \)-representation which we denote by \( M_{q+1} \). Define

\[
X|_{W_c} := (\text{Res}_{\mathcal{H}_c}^{{H}} m_1(X))_{q+1}.
\]  

(66)

We will need to recall some structural facts about the Iwahori–Hecke algebra. Let \( \mathcal{X} := X_*(T, \widehat{k}) = X^*(T^\vee, \widehat{k}) \) and consider the (extended) affine Weyl group \( \widetilde{W} := W \ltimes \mathcal{X} \). Let

\[
S := \{ s_\alpha \mid \alpha \in \Delta \} \subset W
\]
denote the set of simple reflections in \( W \). For each \( x \in \mathcal{X} \), write \( t_x \in \widetilde{W} \) for the corresponding translation. If \( W \) is irreducible, let \( \alpha_0 \) be the highest root and set \( s_0 = s_{\alpha_0} t_{-\alpha_0} \). \( S^a = S \cup \{ s_0 \} \). If \( W \) is a product, define \( S^a \) by adjoining to \( S \) the reflections \( s_0 \), one for each irreducible factor of \( W \). Consider the length function \( \ell : \widetilde{W} \to \mathbb{Z}_{\geq 0} \).

For each \( w \in \widetilde{W} \), choose a representative \( \bar{w} \) in the normalizer \( N_{\mathcal{G}(k)}(I(o)) \). Recall the affine Bruhat decomposition

\[
\mathcal{G}(k) = \bigsqcup_{w \in \widetilde{W}} I(o) \bar{w} I(o),
\]

For each \( w \in \widetilde{W} \), write \( T_w \in \mathcal{H}_I \) for the characteristic function of \( I(o) \bar{w} I(o) \subset \mathcal{G}(k) \). Then \( \{ T_w \mid w \in \widetilde{W} \} \) forms a \( \mathbb{C} \)-basis for \( \mathcal{H}_I \).

The relations on the basis elements \( \{ T_w \mid w \in \widetilde{W} \} \) were computed in [IM65, Section 3]:

\[
T_w \circ T_{w'} = T_{ww'}, \quad \text{if } \ell(ww') = \ell(w) + \ell(w'),
\]

\[
T_s^2 = (q - 1)T_s + q, \quad s \in S^a.
\]  

(67)

Let \( R \) be the ring \( \mathbb{C}[v, v^{-1}] \) and for \( a \in \mathbb{C}^* \) let \( \mathbb{C}_a \) be the \( R \)-module \( R/(v - a) \). Let \( \mathcal{H}_{I,v} \) denote the Hecke algebra with base ring \( R \) instead of \( \mathbb{C} \) and where \( q \) is replaced with \( v^2 \) in the relations \( \text{[67]} \). By specializing \( v \) to \( \sqrt{q} \), 1, we obtain

\[
\mathcal{H}_{I,v} \otimes_R \mathbb{C}_{\sqrt{q}} \cong \mathcal{H}_I, \quad \mathcal{H}_{I,v} \otimes_R \mathbb{C}_1 \cong \mathbb{C}[\widetilde{W}].
\]  

(68)

Suppose \( Y = Y(s, n, \rho) \) is a standard Iwahori spherical representation, see Section 4.2.1 and let \( M = m_1(Y) \). By [KL87, Section 5.12] there is a \( \mathcal{H}_{I,v} \) module \( M_v \), free over \( R \), such that

\[
M_v \otimes_R \mathbb{C}_{\sqrt{q}} \cong M
\]
as \( \mathcal{H}_I \)-modules. We can thus construct the \( \widetilde{W} \)-representation

\[
Y_{q+1} := M_v \otimes_R \mathbb{C}_1.
\]
Let $R(\tilde{W})$ be the Grothendieck group of $\text{Rep}(\tilde{W})$. Since the standard modules form a $\mathbb{Z}$-basis for $R_1(G(k))$, the Grothendieck group of $\text{Rep}_1(G(k))$, there is a unique homomorphism

$$(\bullet)_{q^{-1}} : R_1(G(k)) \to R(\tilde{W})$$

extending $Y \mapsto Y_{q^{-1}}$. Moreover, since

$$\text{Res}_{\tilde{W}_c}^W Y_{q^{-1}} = Y|_{W_c}$$

for the Iwahori spherical standard modules we have that

$$\text{Res}_{\tilde{W}_c}^W X_{q^{-1}} = X|_{W_c}$$

for all $X \in R_1(G(k))$.

### 4.2.4 Reduction to Weyl Groups

Recall the definition of $O^*$ in Section 2.3.2.

**Theorem 4.3.** Suppose $X \in \text{Rep}_1(G(k))$. Let $c \subseteq c_0$. Then

$$\mathbb{F}_q \text{WF}(X^{U_c(s)}) = \max \{O^*(E) : [\text{Res}_{\tilde{W}_c}^W (X^{I(s)}) : E > 0] \}$$

and $K \text{WF}_c(\pi)$ is the lift of these orbits.

**Proof.** This is just putting together Equation (70) and the results in Section 2.3.2. $\Box$

### 4.3 The Wavefront Set of Spherical Arthur Representations

In this section we will prove the following theorem.

**Theorem 4.4.** Let $(\pi, X)$ be the spherical Arthur representation with parameter $O^\vee \in \mathcal{N}_c^\vee$. Then

$$K \text{WF}(X) = d_A(O^\vee, 1), \quad \tilde{k} \text{WF}(X) = d(O^\vee).$$

Let $n \in O^\vee$. Our strategy is to apply Theorem 4.3. The conditions apply since $X$ is Iwahori spherical (since it is spherical). The first step is thus to get a grasp on the $\tilde{W}$ structure of $(X^{I(s)})_{q^{-1}}$. Note that since $X$ is spherical, $AZ(X)$ is equal to its standard module. With respect to Lusztig’s parameterisation of representations with cuspidal unipotent support, $X$ corresponds to $[(q^{1/2}h^\vee, 0, \text{triv})]$, where $h^\vee$ is a neutral element for $O^\vee$, and $AZ(X)$ corresponds to $[(\tau, n, \text{triv})]$ where

$$\tau = \psi_0 \left( \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \right).$$

and $\psi_0 : SL_2 \to G^\vee$ is an $SL_2$ homomorphism such that

$$d\psi_0 \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = n.$$
In particular, as shown in [Rec00], we have that \((AZ(X)^{H(o)})_{q \to 1} = \text{sgn} \otimes H^*(\mathcal{B}_n)_{\text{triv}}\) where \(\mathcal{B}_n\) denotes the space of Borel subgroups whose Lie algebra contains \(n\), and \(X_s\) acts trivially on the total cohomology space. It follows that \((X^{H(o)})_{q \to 1} = H^*(\mathcal{B}_n)_{\text{triv}}\) and that the action of \(\tilde{W}\) factors through \(\cdot : \tilde{W} \to W\). It thus suffices to understand \(H^*(\mathcal{B}_n)_{\text{triv}}\) as a \(W\)-module and to do this we will use the theory of perverse sheaves.

We adopt the conventions of [Sho88], except that we will work with perverse sheaves over \(\mathbb{C}\) rather than \(l\)-adic sheaves, and we will ignore shifts since we are exclusively interested in the total cohomology. For a proper subset \(J \subsetneq \Delta\), we will write \(c(J)\) for the face of \(c_0\) with type equal to \(J\), and we write \(W_J\) in place of \(W_{c(J)}\) for notational convenience.

Let \(\pi : \tilde{g}^\vee \to g^\vee\) be the Grothendieck resolution and \(\square\) be the constant sheaf on \(\tilde{g}^\vee\). Let \(N^\vee\) be the nilpotent cone on \(g^\vee\). Then \(\pi_*\square = IC(g^\vee, L)\) where \(L\) is the local system on the regular semisimple elements \(g^\vee\). This map is an etale covering with Galois group \(W^\vee\) and \(L\) is a \(G^\vee\)-equivariant sheaf of \(W^\vee\) representations. We can thus decompose it as \(L = \bigoplus_{E \in \text{Irr}(W^\vee)} E \otimes L_E\). Thus \(\pi_*\square = \bigoplus_{E \in \text{Irr}(W^\vee)} E \otimes IC(g^\vee, L_E)\) and we have that \(IC(g^\vee, L_E)|_{N^\vee} = IC(O_E^\vee, E_E)\) where the map \(E \mapsto (O_E^\vee, E_E)\) is the Springer correspondence [Sho88]. Thus \(\pi_*\square|_{N^\vee} = \bigoplus_{E \in \text{Irr}(W^\vee)} IC(O_E^\vee, E_E)\).

**Lemma 4.5.** Let \(J \subsetneq \Delta\) and \(E \in \text{Irr}(W_J)\). View \(W_J\) as a subgroup of \(W^\vee\) via

\[
W_J \to W \xrightarrow{\cdot} W^\vee.
\]  

The first map is the projection map \(\cdot : \tilde{W} \to W\). Define \(K_{J,E}\) to be the perverse sheaf

\[
K_{J,E} := \bigoplus_{E' \in \text{Irr}(W^\vee)} [E' : \text{Ind}^{W^\vee}_{W_J} E] IC(O_E^\vee, E_E).
\]

Then the support of \(K_{J,E}\) lies in \(d_S(c(J), O'(E))\).

**Proof.** By [AA07] Proposition 4.3, \([\text{Ind}^{W^\vee}_{W_J} E : E'] > 0\) implies that \(O_E^\vee \leq \text{Ind}^{W^\vee}_{W_J} O''\) where \(O''\) is the Springer support of the unique special representation in the same family as \(E\). But \(O'' = d_{LS}(O'(E))\) and so in fact \(O_E^\vee \leq d_S(c(J), O'(E))\). Therefore \(\text{supp}(K_{J,E})\) is contained in the closure of \(d_S(c(J), O'(E))\) as required. \(\square\)

**Theorem 4.6.** Let \(n \in N^\vee\) and \(O_n^\vee = G^\vee.n\).

1. For all \(J \subsetneq \Delta\) and \(E \in \text{Irr}(W_J)\) such that \(E\) is an irreducible constituent of \(\text{Res}^{W_J}_{W} H^*(\mathcal{B}_n)_{\text{triv}}\), we have that \(O_n^\vee \leq d_S(c(J), O'(E))\).

2. Let \(O \in \tilde{O}^{-1}(d_A(O_n^\vee, 1))\) and \((c(J), O') \in I_{\alpha_d,c_0}^K(O).\) Let \(E\) be the special representation of \(W_J\) corresponding to \(d_{O,J}(O')\). Then

\[
[E : \text{Res}^{W_J}_{W} H^*(\mathcal{B}_n)_{\text{triv}}] = 1.
\]
Proof. 1. Let $J \subseteq \Delta$ and $E \in \text{Irr}(W_J)$. We have that

$$[E : \text{Res}_{W_J}^W H^*(B_n)] = [E : \text{Res}_{W_J}^{W'} \bigoplus_{E' \in \text{Irr}(W')} E' \otimes H_n^* \text{IC}(\overline{O}_{E'}, E_{E'})]$$

$$= \sum_{E' \in \text{Irr}(W')} [E' : \text{Ind}_{W_J}^{W'} E] \dim H_n^* \text{IC}(\overline{O}_{E'}, E_{E'})$$

$$= \dim H_n^* K_{J,E}.$$ (78)

Thus if $[E : \text{Res}_{W_J}^W H^*(B_n)] > 0$ then $n \in \text{supp} K_{J,E} \subseteq \overline{d_S(c(J), O'(E))}$. But $[E : \text{Res}_{W_J}^W H^*(B_n)] > 0$ implies that $[E : \text{Res}_{W_J}^W H^*(B_n)] > 0$ and so indeed $
 \leq d_S(c(J), O'(E))$. 

2. Let $\overline{O}, c, O', E$ be as in the statement of the theorem. Since $O'$ is distinguished, it is special and so $O'(E) = O'$ and $\overline{O} = \overline{\text{c}(c(J), O'(E))}$. Let $E' = j_{W_J}^{W'} E$. Then $[\text{Ind}_{W_J}^{W'} E : E'] = 1$, $O'_{E'}, = d_S(c(J), O'(E)) = O_{n}'$, and $E_{E'}$ is the trivial local system. Since $[E' : \text{Ind}_{W_J}^{W'} E] > 0$ implies that $O_{E'} \leq d_S(c(J), O'(E)) = \overline{O}_{n}'$ [AA07 Proposition 4.3], we have that

$$K_{J,E} \mid_{\overline{O}_{n}'} = \bigoplus_{E' \in \text{Irr}(W') : O'_{E'} = O_{n}'} [E' : \text{Ind}_{W_J}^{W'} E] E_{E'}.$$ (81)

By a similar calculation as above we have that $[E : \text{Res}_{W_J}^W H^*(B_n)] = [\text{triv} : K_{J,E} \mid_{\overline{O}_{n}'}]$. Thus $[E : \text{Res}_{W_J}^W H^*(B_n)] = [E' : \text{Ind}_{W_J}^{W'} E] = 1$ as required. \hfill ■

We can now give a proof of Theorem 4.3.

Proof. Let $J \subseteq \Delta$. By Theorem 4.3 we know that

$$K_{W, c(J)}(\pi) = \{ \overline{O} (c, O'(E)) : E \in \text{Irr}(W_J), [E : \text{Res}_{W_J}^W V^{1_{c(J)}}] > 0 \}. $$ (82)

By part 1 of Theorem 4.6 for any $\overline{O} \in K_{W, c(J)}(\pi)$ we have $d_S(\overline{O}) \geq \overline{O}'. \: \text{By Theorem 3.27 it follows that } \overline{\delta}(\overline{O}) \leq d_A(\overline{O}', 1). \: \text{But by part 2 of Theorem 4.6 we have that for any } \overline{O} \in \overline{\delta}^{-1}(d_A(\overline{O}', 1)) \text{ and } (c(J), O') \in \text{Irr}(\overline{O}'). \: \text{Since } k \overline{O} = d_{BV}(\overline{O}') \text{ for any } \overline{O} \in \overline{\delta}^{-1}(d_A(\overline{O}', 1)) \text{ it follows that } \overline{\delta}(\overline{O}) = d_{BV}(\overline{O}')$. \hfill ■

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