PARTITIONS INTO A SMALL NUMBER OF PART SIZES

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Abstract. We study $\nu_k(n)$, the number of partitions of $n$ into $k$ part sizes, and find numerous arithmetic progressions where $\nu_2$ and $\nu_3$ take on values divisible by 2 and 4. We show $\nu_2(An + B) \equiv 0 \pmod{4}$ for $(B,A) = (30,36)$, $(42,72)$, $(70,196)$, $(114,252)$, and likely many other progressions. We show $\nu_3(36n + 30) \equiv 0 \pmod{2}$ and offer evidence for similar close relations in other arithmetic progressions. We end with open questions of interest.

1. Introduction

Denote the number of partitions of $n$ in which exactly $k$ sizes of part appear by $\nu_k(n)$. For instance, $\nu_2(5) = 5$, counting $4 + 1$, $3 + 2$, $3 + 1 + 1$, $2 + 2 + 1$, and $2 + 1 + 1 + 1$. This easily stated function has been studied by Major P. A. MacMahon [2], George Andrews [1], and more recently Tani and Bouroubi [9], the latter specifically interested in $\nu_2$. The author in a recent paper [5] stated several theorems concerning $\nu_2$ and ventured further conjectures regarding $\nu_2$ and $\nu_3$, which it is the purpose of this paper to prove and expand. Despite attention from these authors, results of the kind found in other areas of partition theory, such as congruences in arithmetic progressions, have not been forthcoming; here we provide several, with a proof strategy easily adaptable to future possible candidates.

Besides the inherent interest of a simply stated problem, data on $\nu_k(n)$ could be useful for the study of overpartitions. An overpartition of $n$ is a partition of $n$ in which the last appearance of a given size of summand is either overlined or not. The overpartitions of 3 are

$$3, \overline{3}, 2 + 1, \overline{2} + 1, 2 + \overline{2}, \overline{1} + 1 + 1, 1 + 1 + 1 + 1.$$

Often attributed originally to Major MacMahon, overpartitions have seen a surge of interest in recent years since the 2004 publication of a paper by Corteel and Lovejoy [2], placing them in the context of more recent work in partition theory.

Denote the number of overpartitions of $n$ by $\overline{\nu}(n)$. Then it is clear that

$$\overline{\nu}(n) = 2\nu_1(n) + 4\nu_2(n) + 8\nu_3(n) + \ldots .$$

Thus data about $\nu_k(n)$ can inform or be informed by results on overpartitions. An example by Byungchan Kim [4] is the theorem that $\overline{\nu}(n) \equiv 0 \pmod{8}$ if $n$ is neither a square nor twice a square; this is equivalent to the claim that for such numbers, $\frac{1}{2}\nu_1(n)$ and $\nu_2(n)$ are simultaneously both even or both odd.

Our main theorems include several on $\nu_2(An + B) \pmod{4}$ and $\nu_3 \pmod{2}$:

Theorem 1. $\nu_2(n) \equiv 0 \pmod{4}$ if $n \equiv 30 \pmod{36}, 42 \pmod{72}, 70 \pmod{196}$, or $114 \pmod{252}$. 

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Theorem 2. $\nu_3(36j + 30) \equiv 0 \pmod{2}$.

There are many other candidate progressions. The proof techniques we give should be easily adaptable.

In the next section we give much of the background information necessary to verify the results in this paper, including useful formulas from MacMahon and Andrews for $\nu$, and several facts concerning modular forms which are central to methodology provided by Jeremy Rouse on MathOverflow [8] answering a question in [5].

In Section 3 we prove our theorems on $\nu_2$ by isolating as much as possible common to all such progressions using MacMahon and Andrews’ results, reducing the proof in each individual case to a short catalog and calculation, which we then perform using an expansion of Rouse’ proof. In Section 4 we prove the theorem on $\nu_3$ and discuss challenges and possible avenues of attack in proceeding further. The last section gives a number of open questions which we think are of general interest.

2. Background Theorems

Since partitions into exactly one size of part have Ferrers diagrams which are just rectangles of area $n$, $\nu_1(n)$ is just $d(n)$, the divisor function, which is perfectly understood. If the factorization of $n$ into primes is $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots$, then

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1)\cdots.$$ 

We are thus more interested in $\nu_2$ and $\nu_3$.

MacMahon and Andrews gave generating functions for $\nu_k$ and, along with Karl Dilcher independently [3], all derived the identities

(1) $\nu_2(n) = \frac{1}{2} \left( \sum_{k=1}^{n-1} d(k)d(n-k) - \sigma_1(n) + d(n) \right)$

and

(2) $\nu_3(n) = \frac{1}{3} d(n) - \frac{1}{2} \sigma_1(n) + \frac{1}{6} \sigma_2(n) - \frac{1}{2} \sum_{k=1}^{n-1} d(k)\sigma_1(n-k)$

$$+ \frac{1}{2} \sum_{k=1}^{n-1} d(k)d(n-k) + \frac{1}{6} \sum_{k=1}^{n-2} \sum_{j=1}^{n-k-1} d(k)d(j)d(n-k-j)$$

where $\sigma_k(n) = \sum_{d|n} d^k$. (Dilcher’s identity is different in form but closely related.)

Using these ideas, the author showed in [5] that

Theorem 3. If $n \equiv 2 \pmod{4}$, or $n$ has two or more primes appearing to odd order in its prime factorization, then $\nu_2(n) \equiv 0 \pmod{2}$.

Together with Rouse, it was further shown that

Theorem 4. $\nu_2(16j + 14) \equiv 0 \pmod{4}$.

Since related theorems are among our main results, we sketch the proof strategy for Theorem [4].
One observes that for \( n = 16j + 14 \), \( \sigma_1(n) \equiv 0 \pmod{8} \) and that \( d(n) \equiv d \left( \frac{n}{2} \right)^2 \pmod{8} \), so these can be removed from equation (1) and it remains to show that

\[
\sum_{k=1}^{\frac{n}{2}} d(k)d(n-k) \equiv 0 \pmod{4}.
\]

There are no odd terms, since \( n \) is not the sum of two squares (observe quadratic residues \( \pmod{16} \)), and therefore we wish to show that there are an even number of terms that are not multiples of 4. The only terms that are not multiples of 4 are those in which \( k \) or \( n-k \) is square, and the other term is 2 mod 4, i.e. \( n-k \) or \( k \) respectively is \( py^2 \) for \( p \) a prime, with \( s_p(y) \equiv 0 \pmod{2} \) where \( s_p(y) \) is the power of \( p \) in the prime factorization of \( y \).

Thus the theorem reduces to showing that there are an even number of representations of \( n \) in the form \( n = x^2 + py^2 \) with the appropriate conditions on the prime \( p \), since for each such pair \( k \) will be the smaller of the two terms \( x^2 \) or \( py^2 \). In order to analyze the parity of the number of such representations, we avail ourselves of the congruences

\[
F(q) := \sum_{n=0}^{\infty} \sigma_1(2n+1)q^{2n+1} \equiv \sum_{n=1}^{\infty} q^{(2n+1)^2} \pmod{2}
\]

and

\[
G(q) := \frac{1}{2} \sum_{n=0}^{\infty} \sigma_1(8n+5)q^{8n+5} \equiv \sum_{p \equiv 5 \pmod{8}, \ p \geq 13} q^{py^2} \pmod{2}.
\]

This is advantageous since \( F(q) \) and \( G(q) \) are modular forms. We refer the interested reader to any textbook on modular forms for a more detailed study of their properties; we here summarize the properties we need.

- A modular form is said to be of weight \( k \) for \( \Gamma_0(N) \), a certain subgroup of the modular group on the upper half-plane. Its level is the minimum possible \( N \). Such a form is also of weight \( k \) for any \( \Gamma_0(cN) \), \( c \in \mathbb{N} \).
- Modular forms of a given weight for \( \Gamma_0(N) \) form a vector space over \( \mathbb{C} \).
- The substitutions \( q \rightarrow q^c \) for \( c \in \mathbb{N} \) send forms of weight \( k \) for \( \Gamma_0(N) \) to forms of weight \( k \) for \( \Gamma_0(cN) \).
- The product of two modular forms for \( \Gamma_0(N) \) of weights \( k \) and \( \ell \) is a modular form for \( \Gamma_0(N) \) of weight \( k + \ell \).
- For a form \( f(q) \) of weight \( k \) and level \( N \), if all coefficients of \( q^i \) in \( f \) for \( i \) below the Sturm bound \( \frac{1}{12}N \prod_{p|N} \left( \frac{N}{p} \right) \) (the product is over all primes dividing \( N \)) are divisible by a given prime, then all coefficients of \( f \) are so divisible.
- If \( f(q) = \sum_{n=0}^{\infty} a(n)q^n \) is a modular form of weight \( k \) and level \( N \), then for \( m|N \), \( g(q) = \sum_{n=0}^{\infty} a(mn)q^n \) is also a modular form of weight \( k \) and level \( N \), and if \( \chi \) is a primitive Dirichlet character \( \pmod{M} \), then \( g(q) = \sum_{n=0}^{\infty} a(n)\chi(n)q^n \) is a modular form of weight \( k \) and level \( NM^2 \).

The last property allows us to dissect modular forms as needed for our proofs, for it is clear that by selecting characters that cancel properly when the forms are added, we may construct from the form \( f(q) = \sum_{n=0}^{\infty} a(n)q^n \) a form \( g(q) = \sum_{n=0}^{\infty} b(n)q^n \).
\[ \sum_{n=0}^{\infty} a(An + B)q^{An+B} \] of the same weight and higher level. Our proofs will require the facts that \( F(q) \) (defined above) is of weight 2 and level 4, and \( H(q) := \sum_{n=0}^{\infty} \sigma_1(3n + 1)q^{3n+1} \) is of weight 2 and level 9.

3. Partitions into 2 sizes of part

We begin by proving a conjecture advanced in the previous paper.

**Theorem 5.** \( \nu_2(36j + 30) \equiv 0 \pmod{4} \).

**Proof.** Set \( n = 36j + 30 \). We again observe that \( \sigma_1(n) \equiv 0 \pmod{8} \) (since \( 3 \mid n \) and at least one prime \( 6f + 5 \) appears to odd order in its factorization) and that \( d(n) \equiv d(\frac{n}{2})^2 \pmod{8} \), and so again it suffices to show

\[ \sum_{k=1}^{n-2} d(k)d(n-k) \equiv 0 \pmod{4}. \]

By the same argument as before, we wish to show that there exist an even number of representations \( n = x^2 + py^2, \quad s_p(y) \equiv 0 \pmod{2} \). There are now six possible residue classes for \( x \), with several possible values \( \pmod{36} \) of \( p \) and \( y \) for each. We summarize these in the following table.

| \( x^2 \pmod{36} \) | \( py^2 \pmod{36} \) | Possible (\( p, y^2 \)) \pmod{36} |
|-----------------|-----------------|----------------------------------|
| 1               | 29              | \{29, 1\}, \{17, 25\}, \{5, 13\} |
| 13              | 17              | \{17, 1\}, \{5, 25\}, \{29, 13\} |
| 25              | 5               | \{5, 1\}, \{29, 25\}, \{17, 13\} |
| 4               | 26              | \{2, 13\}                        |
| 16              | 14              | \{2, 25\}                        |
| 28              | 2               | \{2, 1\}                         |

We construct the following modular forms (all \( q \)-series congruences are \( \pmod{2} \)):

\[
F_{x,1}(q) := \sum_{j=0}^{\infty} \sigma_1(36j + 1)q^{36j+1} = \sum_{j=0}^{\infty} q^{(18j+1)^2} + q^{(18j+17)^2}
\]

\[
F_{x,25}(q) := \sum_{j=0}^{\infty} \sigma_1(36j + 25)q^{36j+25} = \sum_{j=0}^{\infty} q^{(18j+5)^2} + q^{(18j+13)^2}
\]

\[
F_{x,13}(q) := \sum_{j=0}^{\infty} \sigma_1(36j + 13)q^{36j+13} = \sum_{j=0}^{\infty} q^{(18j+7)^2} + q^{(18j+11)^2}
\]
\[ G_{y,29}(q) := \frac{1}{2} \sum_{j=0}^{\infty} \sigma_1(36j + 29)q^{36j+29} \equiv \sum_{\substack{p \equiv 29 \pmod{36} \\ y \equiv \pm 1 \pmod{18} \atop sp(y) \text{ even}}} q^{py^2} + \sum_{\substack{p \equiv 17 \pmod{36} \\ y \equiv \pm 5 \pmod{18} \atop sp(y) \text{ even}}} q^{py^2} + \sum_{\substack{p \equiv 5 \pmod{36} \\ y \equiv \pm 7 \pmod{18} \atop sp(y) \text{ even}}} q^{py^2} \]

\[ G_{y,5}(q) := \frac{1}{2} \sum_{j=0}^{\infty} \sigma_1(36j + 5)q^{36j+5} \equiv \sum_{\substack{p \equiv 5 \pmod{36} \\ y \equiv \pm 1 \pmod{18} \atop sp(y) \text{ even}}} q^{py^2} + \sum_{\substack{p \equiv 29 \pmod{36} \\ y \equiv \pm 5 \pmod{18} \atop sp(y) \text{ even}}} q^{py^2} + \sum_{\substack{p \equiv 17 \pmod{36} \\ y \equiv \pm 7 \pmod{18} \atop sp(y) \text{ even}}} q^{py^2} \]

\[ G_{y,17}(q) := \frac{1}{2} \sum_{j=0}^{\infty} \sigma_1(36j + 17)q^{36j+17} \equiv \sum_{\substack{p \equiv 5 \pmod{36} \\ y \equiv \pm 1 \pmod{18} \atop sp(y) \text{ even}}} q^{py^2} + \sum_{\substack{p \equiv 29 \pmod{36} \\ y \equiv \pm 5 \pmod{18} \atop sp(y) \text{ even}}} q^{py^2} + \sum_{\substack{p \equiv 17 \pmod{36} \\ y \equiv \pm 7 \pmod{18} \atop sp(y) \text{ even}}} q^{py^2} \]

\[ F_{x,4}(q) := \sum_{j=0}^{\infty} \sigma_1(9j + 1)(q^4)^{9j+1} \equiv \sum_{j=0}^{\infty} \sigma_1(36j + 4)q^{36j+4} \equiv \sum_{j=0}^{\infty} q^{(18j+2)^2} + q^{(18j+16)^2} \]

\[ F_{x,16}(q) := \sum_{j=0}^{\infty} \sigma_1(9j + 4)(q^4)^{9j+4} \equiv \sum_{j=0}^{\infty} \sigma_1(36j + 16)q^{36j+16} \equiv \sum_{j=0}^{\infty} q^{(18j+4)^2} + q^{(18j+14)^2} \]

\[ F_{x,28}(q) := \sum_{j=0}^{\infty} \sigma_1(9j + 7)(q^4)^{9j+7} \equiv \sum_{j=0}^{\infty} \sigma_1(36j + 28)q^{36j+28} \equiv \sum_{j=0}^{\infty} q^{(18j+8)^2} + q^{(18j+10)^2} \]

\[ G_{y,26}(q) := \sum_{j=0}^{\infty} \sigma_1(18j + 13)(q^2)^{18j+13} \equiv \sum_{j=0}^{\infty} \sigma_1(36j + 26)q^{36j+26} \equiv \sum_{j=0}^{\infty} (q^2)^{(18j+7)^2} + (q^2)^{(18j+11)^2} \]

\[ G_{y,14}(q) := \sum_{j=0}^{\infty} \sigma_1(18j + 7)(q^2)^{18j+7} \equiv \sum_{j=0}^{\infty} \sigma_1(36j + 14)q^{36j+14} \equiv \sum_{j=0}^{\infty} (q^2)^{(18j+5)^2} + (q^2)^{(18j+13)^2} \]

\[ G_{y,2}(q) := \sum_{j=0}^{\infty} \sigma_1(18j + 1)(q^2)^{18j+1} \equiv \sum_{j=0}^{\infty} \sigma_1(36j + 2)q^{36j+2} \equiv \sum_{j=0}^{\infty} (q^2)^{(18j+1)^2} + (q^2)^{(18j+17)^2} \]

These modular forms have the following weights and levels: \( F_{x,1}, F_{x,25}, F_{x,13}, G_{y,29}, G_{y,5}, \) and \( G_{y,17} \) are all dissections of \( F(q) \) by characters mod 36, and so they all of weight 2 for \( \Gamma_0(36^2 \cdot 4) = \Gamma_0(5184) \). \( F_{x,4}, F_{x,16}, \) and \( F_{x,28} \) are dissections of \( H(q) \) by characters mod 9, thereafter magnified by the substitution \( q \to q^4 \), so they of weight 2 for \( \Gamma_0(4 \cdot 9^3) = \Gamma_0(2916) \). \( G_{y,26}, G_{y,14} \) and \( G_{y,2} \) are all dissections of \( H(q) \) by characters mod 18, thereafter magnified by the substitution \( q \to q^2 \), so they are modular forms of weight 2 for \( \Gamma_0(9 \cdot 18^2 \cdot 2) = \Gamma_0(5832) \).

The product of any two of these modular forms of weight 2 for \( \Gamma_0(N_1) \) and \( \Gamma_0(N_2) \) is a modular form of weight 4 for \( \Gamma_0(lcm(N_1, N_2)) \). For the odd \( F \) and \( G \), we have \( F_{x,i}G_{y,j} \) of weight 4 for \( \Gamma_0(5184) \). The even cases \( F_{x,2i}G_{y,2j} \) are of weight 4 for \( \Gamma_0(5832) \). Finally, the sum of all these is a modular form of weight 4 for \( \Gamma_0(lcm(5184, 5832)) = \Gamma_0(46656) = \Gamma_0(6^6) \).

Define \( S = \{ 1, 25, 13, 4, 16, 28 \} \) and set

\[ R(q) = \sum_{n=0}^{\infty} r(n)q^n = \sum_{i \in S} F_{x,i}(q)G_{y,30-i}(q). \]
Then \( r(n) \) is 0 for terms other than \( n = 36j + 30 \), and for these terms is of the same parity as the number of representations of \( n \) of the form sought. The Sturm bound for \( R(q) \) is \( \frac{q}{12} 46656 \left( \frac{3}{2} \right) = 31104 \).

It is a straightforward calculation to construct all these forms in Mathematica or another symbolic computation package, expand the series to the Sturm bound, and check that all coefficients up to \( q^{31104} \) are even. Hence, all coefficients are even, and so the number of representations of \( 36j + 30 \) of the form required is also even. Thus, \( \sum_{k=1}^{n-2} d(k)d(n-k) \equiv 0 \pmod{4} \), and the theorem holds.

For progressions \( n = Aj + B \) with \( A \) even in which \( \sigma_1(n) \equiv 0 \pmod{8} \) and \( d(n) \equiv d\left( \frac{n}{2} \right)^2 \pmod{8} \), the sum reduces the same way to showing

\[
\sum_{k=1}^{n-2} d(k)d(n-k) \equiv 0 \pmod{4}.
\]

If the candidate progression is among those which can never contain sums of two squares, then we again reduce the question to analyzing parity of the number of representations of \( n \) of the form \( x^2 + py^2 \), \( s_p(y) \equiv 0 \pmod{2} \), which if \( Aj + B \) is a suitable progression we can analyze exactly as before. (We note that computation has not yet suggested a candidate progression in which these conditions do not hold. It would be reasonable to conjecture that the conditions are necessary.)

The following progressions can be analyzed in such a fashion. We omit the repetitive details; the necessary parity checks can be easily verified by a symbolic computation package on a laptop computer.

**Theorem 6.** The following progressions \( Aj + B \) all satisfy \( \nu_2(Aj + B) \equiv 0 \pmod{4} \):

\[
\begin{align*}
n &= 72j + 42, \\
n &= 196j + 70, \\
n &= 252j + 114.
\end{align*}
\]

These are not exhaustive even of candidates of small moduli.

### 4. Partitions into 3 sizes of part

We observed in the introduction that the result of Kim, that \( p(n) \equiv 0 \pmod{8} \) for \( n \neq k^2, 2k^2 \), is equivalent to the claim that for such \( n \), \( \frac{1}{5} \nu_1(n) \) and \( \nu_2(n) \) are simultaneously both even or both odd. Relations between \( \nu_k \) for different \( k \) appear to be a more common phenomenon, one which it would be interesting to explore. For instance, we have the following theorem concerning \( \nu_3 \):

**Theorem 7.** \( \nu_3(36n + 30) \equiv 0 \pmod{2} \).

**Proof.** By Corollary 9 of [4], \( p(36n + 30) \equiv 0 \pmod{16} \), since \( 36n + 30 \) cannot be the sum of two squares. (Alternatively, one may extract the \( q^{12n+10} \) terms from equation 4.29 of [10]. Reduce mod 16 instead of 32, and employ other identities found in that paper to find that the \( q^{12n+10} \) terms vanish mod 16.)

Since \( p(m) = 2\nu_1(m) + 4\nu_2(m) + 8\nu_3(m) + 16\nu_4(m) + \ldots \), and we have already shown that \( \nu_1(36n + 30) \equiv 0 \pmod{8} \) and \( \nu_2(36n + 30) \equiv 0 \pmod{4} \), it must hold that \( \nu_3(36n + 30) \equiv 0 \pmod{2} \).

**Remark:** Naturally, it would be of considerable interest to see this theorem proved directly from analysis of the parity of the terms involved. This would thus prove the overpartition statement rather than the other way around.
Numerical experimentation suggests that $\nu_3(An + B) \equiv 0 \pmod{2}$ for each of the congruences listed for $\nu_2$ mod 4 in the previous chapter, and for many of the other candidates. That is, when $\nu_2(An + B) \equiv 0 \pmod{4}$, it frequently appears to be the case that $\nu_3(An + B) \equiv 0 \pmod{2}$. (The relation does not hold for all candidates. Of course, it may be possible that the absence of one or the other suggests that a candidate congruence is spurious.) There is good reason to think that such relations exist, as we illustrate below.

Suppose that one wishes to show directly that $\nu_3(n) = \nu_3(36j+30) \equiv 0 \pmod{2}$. Then we wish to analyze the parity of the terms in equation (2). We have already shown that $\nu_2(n) \equiv 0 \pmod{4}$, and we know $d(n) \equiv 0 \pmod{8}$, $\sigma_1(n) \equiv 0 \pmod{8}$, and $\sum_{k=1}^{n-1} d(k)d(n - k) \equiv 0 \pmod{8}$. (In any other arithmetic progression, if any three of these are true, all four are, because we may subtract the other terms in equation (1) from $\nu_2(n)$.)

We may then subtract these terms from equation (2) for $\nu_3(n)$ to obtain

$$
\nu_3(n) = \frac{1}{3}d(n) - \frac{1}{2}\sigma_1(n) + \frac{1}{6}\sigma_2(n) - \frac{1}{2}\sum_{k=1}^{n-1} d(k)\sigma_1(n - k)
$$

$$
+ \frac{1}{2}\sum_{k=1}^{n-1} d(k)d(n - k) + \frac{1}{6}\sum_{k=1}^{n-2} \sum_{\ell=1}^{n-k-1} d(k)d(\ell)d(n - k - \ell)
$$

$$
\equiv -\frac{1}{6}d(n) + \frac{1}{6}\sigma_2(n) - \frac{1}{2}\sum_{k=1}^{n-1} d(k)\sigma_1(n - k)
$$

$$
+ \frac{1}{6}\sum_{k=1}^{n-2} \sum_{\ell=1}^{n-k-1} d(k)d(\ell)d(n - k - \ell) \pmod{2}.
$$

It is not difficult to show that $d(36j + 30) \equiv -\sigma_2(36j + 30) \pmod{12}$ (both functions being multiplicative, one simply observes the values mod 6 of each factor) and hence we can write

$$
\nu_3(n) \equiv -\frac{1}{3}d(n) - \frac{1}{2}\sum_{k=1}^{n-1} d(k)\sigma_1(n - k)
$$

$$
+ \frac{1}{6}\sum_{k=1}^{n-2} \sum_{\ell=1}^{n-k-1} d(k)d(\ell)d(n - k - \ell) \pmod{2}.
$$

We now note that many terms in the final sum can be cast out modulo 2. If exactly one of $k$, $\ell$, or $n - k - \ell$ is a nonsquare and the other two terms are not equal, we can group the six permutations of the three entries, the product of which are even, and discard them. If exactly one entry is a square, we can again do so – we may have only three permutations, but the product is divisible by 4.

If all three are non-squares, the only term we cannot permute and cast out is when $k = \ell = 12j + 10$, which may not have $d(12j + 10) \equiv 0 \pmod{3}$. But $d(12j + 10) = \frac{1}{2}d(36j + 30)$. If we add one-sixth of this to $-\frac{1}{3}d(n)$, we obtain $-\frac{1}{4}d(n)$, which we know to be an even integer.

If all three are squares, then they cannot be the same square (10 is not a quadratic residue mod 12) and thus they have 3 or 6 permutations; however, we may not be
able to cast out such terms. For instance, $30 = 25 + 4 + 1$ and the six permutations thereof, and this is the only such representation of 30. We are also left with terms in which exactly one entry is a non-square and the other two are equal squares.

We may multiply by 3 and take the representative of these in which $k = \ell$ are the squares. Thus, we end up interested in representations of $n$ by three distinct squares, or twice a square and a non-square.

Now observe that in $\frac{1}{2} \sum_{k=1}^{n-1} d(k)\sigma_1(n-k), \sigma_1(n-k) \equiv d(n-k) \pmod{2}$ unless $n - k$ is twice a square, and we already know that $\frac{1}{2} \sum_{k=1}^{n-1} d(k)d(n-k) \equiv 0 \pmod{4}$. Although we must further analyze $\sigma_1 \pmod{4}$, it seems highly plausible that summands from our two terms would cancel mod 2, giving a direct proof of the congruence. The search for such a proof seems like a natural question of interest.

5. Open questions

A number of open questions on this topic present themselves.

(1) Treat the set of possible progressions in a more unified fashion, probably through the theory of eigenforms.

(2) Numerical experimentation to date has found no progressions $An + B$ in which $\nu_2(An + B) \equiv 0 \pmod{N}$ for any $N$ other than 2 or 4; and none for $\nu(3)$ other than $N = 2$. If different moduli occur, they may have large progression modulus $A$. Do these occur, and if so, how can they be efficiently found, or, are they forbidden?

(3) Experimentation has yielded no progressions with nontrivial modulus for $\nu_k$ with $k > 3$. It is plausible that these never occur, since from the formulas in Andrews [1] these values involve sums concerning $d(k)\sigma_2(n-k)$, and $\sigma_2(j)$ is not part of the same framework of modular forms and their symmetries as $\sigma_1$. (When it appeared in $\nu_3$ it was a single term and hence more tractable.) Again, can these occur, and if so, why, or are they forbidden?

(4) Elaborate on the relationships between $\nu_1$ and $\nu_2$, and between $\nu_2$ and $\nu_3$. State conditions necessary and/or sufficient for simultaneous congruences.

(5) Complete the combinatorial proof for $\nu_3(36j + 30)$ and generalize to other progressions.

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