SPECTRAL PROPERTIES OF MATRIX-VALUED DISCRETE DIRAC SYSTEM

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ABSTRACT. In this paper, we find a polynomial-type Jost solution of a self-adjoint matrix-valued discrete Dirac system. Then we investigate analytical properties and asymptotic behavior of this Jost solution. Using the Weyl compact perturbation theorem, we prove that matrix-valued discrete Dirac system has continuous spectrum filling the segment $[-2, 2]$. Finally, we examine the properties of the eigenvalues of this Dirac system and we prove that it has a finite number of simple real eigenvalues.

1. Introduction

Consider the boundary value problem (BVP) consisting of the Sturm–Liouville equation

\[
\begin{cases}
- y'' + q(x)y = \lambda^2 y, & 0 \leq x < \infty \\
y(0) = 0,
\end{cases}
\]

and the boundary condition

where $q$ is a real-valued function and $\lambda$ is a spectral parameter. The bounded solution of (1.1) satisfying the condition

\[
\lim_{x \to \infty} y(x, \lambda)e^{-i\lambda x} = 1, \quad \lambda \in \mathbb{C}_+ := \{ \lambda \in \mathbb{C} : \text{Im} \lambda \geq 0 \}
\]

will be denoted by $e(., \lambda)$. The solution $e(., \lambda)$ is called the Jost solution of (1.1). In \cite{19}, the author presented a condition depending on the function $q$ that guaranteed $e(., \lambda)$ has an integral representation as

\[
e(x, \lambda) = e^{i\lambda x} + \int_x^{\infty} K(x, t)e^{i\lambda t} \, dt < \infty,
\]

where the function $K$ is defined in terms of $q$. Moreover, the author showed that $e(., \lambda)$ is analytic with respect to $\lambda$ in $\mathbb{C}_+ := \{ \lambda \in \mathbb{C} : \text{Im} \lambda > 0 \}$.

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continuous in \( \overline{\mathbb{C}}_+ \) and satisfies
\[
\ell(x, \lambda) = e^{i\lambda x} [1 + o(1)], \quad \lambda \in \overline{\mathbb{C}}_+, \quad x \to \infty.
\]
The function \( e(\lambda) := e(0, \lambda) \) is called Jost function of the BVP (1.1). The functions \( e(., \lambda) \) and \( e(\lambda) \) play an important role in the solutions of direct and inverse problems of the quantum scattering theory [6, 13, 19, 20]. The Jost solutions are especially useful in the study of the spectral analysis of differential and difference operators [?, 11, 18, 21, 22]. Therefore Jost solutions of Dirac systems, Schrödinger and discrete Sturm–Liouville equations have been obtained in [9, 11, 12].

Discrete boundary value problems have been intensively studied in the last decade. The modeling of certain linear and nonlinear problems from economics, optimal control theory and other areas of study has led to the rapid development of the theory of difference equations. Also, the spectral analysis of differential and difference operators [15, 18, 21, 22] has been treated by various authors in connection with the classical moment problem [15, 18, 21, 22]. The spectral theory of the difference equations has also been applied to the solution of classes of nonlinear discrete Korteweg-de Vries equations and Toda lattices [5]. Let us introduce the Hilbert space \( \ell_2(N, \mathbb{C}^{2m}) \) consisting of all vector sequences \( y = \{y_n\}, y_n = (y_n^{(1)}, y_n^{(2)}) \), where \( y_n^{(i)} \in \mathbb{C}^m, i = 1, 2, n \in \mathbb{N} \) and \( \mathbb{C}^m \) is \( m \)-dimensional \( (m < \infty) \) Euclidean space. In \( \ell_2(N, \mathbb{C}^{2m}) \), the norm and inner product are defined by
\[
\|y\|_{\ell_2}^2 := \sum_{n=1}^{\infty} \left( \|y_n^{(1)}\|_{\mathbb{C}^m}^2 + \|y_n^{(2)}\|_{\mathbb{C}^m}^2 \right) < \infty,
\]
\[
\langle y, z \rangle_{\ell_2} := \sum_{n=1}^{\infty} \left[ \langle y_n^{(1)}, z_n^{(1)} \rangle_{\mathbb{C}^m} + \langle y_n^{(2)}, z_n^{(2)} \rangle_{\mathbb{C}^m} \right],
\]
where \( \|\cdot\|_{\mathbb{C}^m} \) and \( \langle \cdot, \cdot \rangle_{\mathbb{C}^m} \) denote the norm and inner product in \( \mathbb{C}^m \), respectively. Now consider the matrix-valued discrete Dirac system
\[
\begin{align*}
A_n y_{n+1}^{(2)} + B_n y_{n}^{(2)} + P_n y_{n}^{(1)} &= \lambda y_{n}^{(1)}, \\
A_{n-1} y_{n-1}^{(1)} + B_n y_{n}^{(1)} + Q_n y_{n}^{(2)} &= \lambda y_{n}^{(2)},
\end{align*}
\]
(1.2)
with the boundary condition
\[
y_0^{(1)} = 0,
\]
(1.3)
where \( A_n, n \in N \cup \{0\} \) and \( B_n, P_n, n \in \mathbb{N} \) are linear operators (matrices) acting in \( \mathbb{C}^m \). Throughout the paper, we will assume that \( \det A_n \neq 0, A_n = A_n^* \ (n \in \mathbb{N} \cup \{0\}), \det B_n \neq 0, B_n = B_n^*, Q_n = Q_n^* \) and \( P_n = P_n^* \ (n \in \mathbb{N}) \), where \( \ast \) denotes the adjoint operator. Let \( L \) denote the operator generated in \( \ell_2(N, \mathbb{C}^{2m}) \) by the BVP (1.2)–(1.3). The
operator $L$ is self-adjoint, i.e., $L = L^*$. In the following, we will assume that the matrix sequences $\{A_n\}, \{B_n\}, \{P_n\}$, and $\{Q_n\}$ ($n \in \mathbb{N}$), satisfy

\begin{equation}
\sum_{n=1}^{\infty} \left( \| I - A_n \| + \| I + B_n \| + \| P_n \| + \| Q_n \| \right) < \infty,
\end{equation}

where $\| . \|$ and $I$ denote the matrix norm and identity matrix in $\mathbb{C}^m$, respectively. The setup of this paper is as follows: In Section 2, we find a polynomial-type Jost solution of (1.2), investigate analytical properties and asymptotic behavior of this Jost solution. In Section 3, we show that $\sigma_c(L) = [-2, 2]$, where $\sigma_c(L)$ denotes the continuous spectrum of $L$. Also, we prove that under the condition (1.4), the operator $L$ has a finite number of simple real eigenvalues. To the best of our knowledge, this paper is the first one that focuses on matrix-valued discrete Dirac system including a polynomial-type Jost solution.

2. JOST SOLUTION OF (1.2)

Assume $P_n \equiv Q_n \equiv 0$, $B_n \equiv -I$ for all $n \in \mathbb{N}$, and $A_n \equiv I$ for all $n \in \mathbb{N} \cup \{0\}$ in (1.2). Then we get

\begin{equation}
\begin{cases}
y_{n+1}^{(2)} - y_n^{(2)} = \left[ -iz - (iz)^{-1} \right] y_n^{(1)} \\
y_{n-1}^{(1)} - y_n^{(1)} = \left[ -iz - (iz)^{-1} \right] y_n^{(2)}
\end{cases}
\end{equation}

for $\lambda = -iz - (iz)^{-1}$. It is clear that

\begin{equation}
e_n(z) = \begin{pmatrix} e_n^{(1)}(z) \\
e_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} z \\
-i \end{pmatrix} z^{2n}, \quad n \in \mathbb{N},
\end{equation}

is a solution of (2.1). Now, we will find the solution $\begin{pmatrix} F_n(z) \\
G_n(z) \end{pmatrix}, n \in \mathbb{N} \cup \{0\}$ of (1.2) for $\lambda = -iz - (iz)^{-1}$, satisfying the condition

\begin{equation}
\begin{pmatrix} F_n(z) \\
G_n(z) \end{pmatrix} = [I + o(1)] e_n(z), \quad |z| = 1, \quad n \to \infty.
\end{equation}

The solution $\begin{pmatrix} F_n(z) \\
G_n(z) \end{pmatrix}, n \in \mathbb{N}$ is called the Jost solution of (1.2) for $\lambda = -iz - (iz)^{-1}$.

**Theorem 2.1.** Assume (1.4). Then for $\lambda = -iz - (iz)^{-1}$ and $|z| = 1$, (1.2) has the solution $\begin{pmatrix} F_n(z) \\
G_n(z) \end{pmatrix}, n \in \mathbb{N}$ having the representation

\begin{equation}
\begin{pmatrix} F_n(z) \\
G_n(z) \end{pmatrix} = T_n \left( I + \sum_{m=1}^{\infty} K_{nm} z^{2m} \right) \begin{pmatrix} z \\
-i \end{pmatrix} z^{2n}, \quad n \in \mathbb{N},
\end{equation}

where $T_n$ is a unitary matrix.
(2.4) \[ F_0(z) = T_0^{11} z + T_0^{11} \sum_{m=1}^{\infty} K_{0m}^{11} z^{2m+1} - iT_0^{11} \sum_{m=1}^{\infty} K_{0m}^{12} z^{2m}, \]

where \( K_{nm} = \left( \begin{array}{cc} K_{nm}^{11} & K_{nm}^{12} \\ K_{nm}^{21} & K_{nm}^{22} \end{array} \right), \) \( T_n = \left( \begin{array}{cc} T_n^{11} \\ T_n^{21} \\ T_n^{22} \end{array} \right) \) and these are expressed in terms of \( \{A_n\}, \{B_n\}, \{P_n\} \) and \( \{Q_n\}. \)

Proof. Substituting \((F_n(z))\) defined by (2.3) into (1.2) and taking \( \lambda = -iz - (iz)^{-1}, |z| = 1 \), we get

\[
T_n^{12} = 0, \quad T_n^{22} = \left( \prod_{p=n}^{\infty} (-1)^{n-p} A_p B_p \right)^{-1}, \]

\[
T_n^{11} = -B_n \left( \prod_{p=n}^{\infty} (-1)^{n-p} A_p B_p \right)^{-1}, \]

\[
T_n^{21} = -Q_n T_n^{22} - A_{n-1} T_{n-1}^{11} \left( \sum_{p=n}^{\infty} (T_p^{11})^{-1} B_p Q_p T_p^{22} - \sum_{p=n}^{\infty} (T_p^{11})^{-1} P_p T_p^{11} \right), \]

\[
K_{n1}^{12} = \sum_{p=n+1}^{\infty} (T_p^{11})^{-1} B_p Q_p T_p^{22} - \sum_{p=n+1}^{\infty} (T_p^{11})^{-1} P_p T_p^{11}, \]

\[
K_{n1}^{11} = \sum_{p=n+1}^{\infty} \left[ -I + (T_p^{11})^{-1} (B_p^2 T_p^{11} + A_p T_p^{21}) + B_p Q_p T_p^{21} + B_p T_p^{22} + P_p T_p^{11} K_{p1}^{12} \right], \]

\[
K_{n1}^{22} = (T_n^{22})^{-1} [B_n T_n^{11} + Q_n T_n^{21} + T_n^{22} + A_{n-1} T_{n-1}^{11} K_{n-1,1}^{11} - T_n^{21} K_{n1}^{12}], \]

\[
K_{n1}^{21} = \sum_{p=n+1}^{\infty} (T_p^{22})^{-1} T_p^{21} (K_{p1}^{11} - I) + \sum_{p=n+1}^{\infty} \left[ (T_p^{22})^{-1} (B_p T_p^{11} + Q_p T_p^{21}) K_{p1}^{12} \right]
\]

\[+ \sum_{p=n+1}^{\infty} (T_p^{22})^{-1} Q_p T_p^{22} K_{p1}^{12} + \sum_{p=n+1}^{\infty} (T_p^{22})^{-1} A_{p-1} T_{p-1}^{11} K_{p2,1}^{12} \]

\[+ \sum_{p=n+1}^{\infty} (T_p^{22})^{-1} A_p^2 T_p^{21} + \sum_{p=n+1}^{\infty} (T_p^{22})^{-1} \left[ A_p P_p T_p^{11} + A_p B_p T_p^{21} \right] K_{p1}^{11}, \]

\[
K_{n2}^{12} = \sum_{p=n+1}^{\infty} \left[ (T_p^{11})^{-1} B_p Q_p (T_p^{21} K_{p1}^{12} + T_p^{22} K_{p1}^{12}) \right] - \sum_{p=n+1}^{\infty} K_{p1}^{12}
\]

\[+ \sum_{p=n+1}^{\infty} (T_p^{11})^{-1} \left[ B_p^2 T_p^{11} K_{p1}^{12} - P_p T_p^{11} K_{p1}^{12} - A_p T_p^{21} - B_p T_p^{21} \right], \]
\[ K_{n2}^{11} = \sum_{p=n+1}^{\infty} \left( T_p^{11} \right)^{-1} \left[ B_p T_{p_1}^{11} K_{p1}^{11} + P_p T_{p_1}^{11} K_{p1}^{12} + B_p T_{p_1}^{22} K_{p1}^{22} + B_p T_{p_1}^{21} K_{p1}^{12} \right] \\
+ \sum_{p=n+1}^{\infty} \left( T_p^{11} \right)^{-1} \left[ B_p Q_p T_{p_1}^{21} K_{p1}^{11} + B_p Q_p T_{p_1}^{22} K_{p1}^{21} \right] - \sum_{p=n+1}^{\infty} K_{p1}^{11} \\
+ \sum_{p=n+2}^{\infty} \left( T_{p-1}^{11} \right)^{-1} \left[ A_{p-1} T_{p_1}^{22} K_{p1}^{21} + A_{p-1} T_{p_1}^{21} K_{p1}^{11} \right], \\

K_{n2}^{22} = -\sum_{p=n+1}^{\infty} \left( T_p^{22} \right)^{-1} \left[ B_p T_{p_1}^{11} K_{p1}^{11} - T_p^{21} K_{p2}^{12} + Q_p T_{p_1}^{21} K_{p1}^{12} + Q_p T_{p_1}^{22} K_{p1}^{21} + T_p^{21} K_{p1}^{12} \right] \\
+ \sum_{p=n+1}^{\infty} \left( T_p^{22} \right)^{-1} \left[ A_{p-1} T_{p_1}^{12} K_{p1}^{11} + A_{p-1} B_{p-1} T_{p_1}^{21} K_{p1}^{12} \right] \\
+ \sum_{p=n+1}^{\infty} \left( T_p^{22} \right)^{-1} \left[ -A_{p-1} T_{p_1}^{11} K_{p1}^{11} \right] - \sum_{p=n+1}^{\infty} K_{p1}^{22} \\
+ \sum_{p=n+1}^{\infty} \left( T_p^{22} \right)^{-1} \left[ A_{p-1} T_{p_1}^{22} K_{p1}^{12} + A_{p-1} T_{p_1}^{21} K_{p1}^{12} \right], \\

K_{n2}^{21} = \sum_{p=n}^{\infty} \left( T_{p+1}^{22} \right)^{-1} \left[ A_p T_{p_1}^{11} K_{p2}^{12} + A_p P_p T_{p_1}^{11} K_{p2}^{11} + A_p B_p T_{p_1}^{21} K_{p2}^{12} \right] \\
+ \sum_{p=n+1}^{\infty} \left( T_p^{22} \right)^{-1} \left[ A_{p-1} T_{p_1}^{21} K_{p1}^{11} + A_{p-1} T_{p_1}^{22} K_{p1}^{12} + B_p T_{p_1}^{11} K_{p2}^{12} \right] - \sum_{p=n+1}^{\infty} K_{p1}^{21} \\
+ \sum_{p=n+1}^{\infty} \left( T_p^{22} \right)^{-1} \left[ Q_p T_{p_1}^{21} K_{p2}^{12} + Q_p T_{p_1}^{22} K_{p2}^{22} + T_p^{21} K_{p2}^{12} - T_p^{21} K_{p1}^{11} \right], \\

where n \in \mathbb{N}. Furthermore, for m \geq 3 and n \in \mathbb{N}, we obtain that \\

\[ K_{nm}^{12} = -\sum_{p=n+1}^{\infty} \left( T_p^{11} \right)^{-1} \left[ P_p T_{p_1}^{11} K_{p,m-1}^{11} - B_p T_{p}^{11} K_{p,m-1}^{12} + B_p T_{p_1}^{21} K_{p,m-2}^{11} \right] \\
+ \sum_{p=n+1}^{\infty} \left( K_{p,m-2}^{21} - K_{p,m-1}^{12} \right) + \sum_{p=n+1}^{\infty} \left( T_p^{11} \right)^{-1} \left[ B_p Q_p T_{p}^{21} K_{p,m-2}^{11} \right] \\
- \sum_{p=n+1}^{\infty} \left( T_p^{11} \right)^{-1} \left[ A_p T_{p+1}^{21} K_{p+1,m-2}^{11} + A_p T_{p+1}^{22} K_{p+1,m-2}^{21} \right] \]
By the condition (1.4), the infinite products and the series in the definition of 

\[ K_{nm}^{11} = + \sum_{p=n+1}^{\infty} (T_p^{11})^{-1} \left[ P_p T_n^{11} K_{nm}^{11} + B_p Q_p T_p^{21} K_{p,m-1}^{11} \right] 
- \sum_{p=n+1}^{\infty} K_{p,m-1}^{11} + \sum_{p=n+1}^{\infty} (T_p^{11})^{-1} \left[ B_p Q_p T_p^{22} K_{p,m-1}^{21} \right] + \sum_{p=n+1}^{\infty} (T_p^{11})^{-1} \left[ A_p T_p^{22} K_{p,1,m-1} + A_p T_p^{21} K_{p,1,m-1} \right] + \sum_{p=n+1}^{\infty} (T_p^{11})^{-1} \left[ B_p T_p^{21} K_{p,m-1}^{12} + B_p T_p^{22} K_{p,m-1}^{22} \right], \]

\[ K_{nm}^{22} = K_{nm-1}^{22} + (T_n^{22})^{-1} \left[ A_{n-1} T_{n-1}^{11} K_{n-1,m}^{11} - T_n^{21} K_{nm}^{21} + B_n T_n^{11} K_{nm}^{11} \right] + (T_n^{22})^{-1} \left[ Q_n T_n^{21} K_{n,m-1}^{11} + Q_n T_n^{22} K_{n,m-1}^{21} + T_n^{21} K_{n,m-1}^{12} \right], \]

\[ K_{nm}^{21} = K_{nm-1}^{21} + (T_n^{22})^{-1} \left[ -A_{n-1} T_{n-1}^{11} K_{n-1,m+1}^{12} - B_n T_n^{11} K_{nm}^{12} \right] + (T_n^{22})^{-1} \left[ -Q_n T_n^{21} K_{nm}^{12} + T_n^{21} K_{n,m-1}^{11} - Q_n T_n^{22} K_{nm}^{22} + T_n^{21} K_{nm}^{11} \right]. \]

By the condition (1.4), the infinite products and the series in the definition of \( T_{ij}^{ij} \) and \( K_{nm}^{ij} \) \((i, j = 1, 2)\) are absolutely convergent. Therefore, \( T_{ij}^{ij} \) and \( K_{nm}^{ij} \) \((i, j = 1, 2)\) can uniquely be defined by \( \{A_n\}, \{B_n\}, \{P_n\} \) and \( \{Q_n\} \), \( n \in \mathbb{N} \), \( i.e. \), the system (1.2) for \( \lambda = -iz - (iz)^{-1} \) has the solution \( (F_n(z)) \) given by (2.3). \( \square \)

**Theorem 2.2.** If the condition (1.4) holds, then (2.5)

\[ \| K_{nm}^{ij} \| \leq C \sum_{p=n+1}^{\infty} \left( \| I - A_p \| + \| I + B_p \| + \| Q_p \| + \| P_p \| \right), \quad i, j = 1, 2, \]

where \( \left\lfloor \frac{m}{2} \right\rfloor \) is the integer part of \( \frac{m}{2} \) and \( C > 0 \) is a constant.
Proof. We will use the method of induction to prove the theorem. For $m = 1$, we get that

\[
\left\| K^{12}_{n1} \right\| = \left\| \sum_{p=n+1}^{\infty} (T^{11}_{p})^{-1} \left[ B_p Q_p T^{22}_{p} - P_p T^{11}_{p} \right] \right\| \leq A \sum_{p=n+1}^{\infty} \left\| B_p Q_p T^{22}_{p} - P_p T^{11}_{p} \right\|
\]

\[
\leq A' \sum_{p=n+1}^{\infty} \left\| Q_p \right\| + \sum_{p=n+1}^{\infty} \left\| P_p \right\| \leq C \sum_{p=n+1}^{\infty} \left( \left\| Q_p \right\| + \left\| P_p \right\| \right)
\]

\[
= C \sum_{p=n+\left\lceil \frac{n}{2} \right\rceil}^{\infty} \left( \left\| Q_p \right\| + \left\| P_p \right\| \right),
\]

where $A = \left\| (T^{11}_{p})^{-1} \right\|$, $A' = A \left\| B_p \right\| \left\| T^{22}_{p} \right\|$, $C = \max \left\{ 1, A' \right\}$. Similar to this inequality, we can get (2.5) for $K^{11}_{n1}$, $K^{22}_{n1}$ and $K^{21}_{n1}$. Now, if we suppose that (2.5) is correct for $m = k$, then we can write

\[
\left\| K^{12}_{n,k+1} \right\| \leq \left\| \sum_{p=n+1}^{\infty} K^{21}_{p,k} - \sum_{p=n+1}^{\infty} K^{12}_{p,k} \right\|
\]

\[
+ \left\| \sum_{p=n+1}^{\infty} (T^{11}_{p})^{-1} \left\{ -P_p T^{11}_{p} K^{11}_{p,k} + B_p^2 T^{11}_{p} K^{12}_{p,k} - B_p T^{22}_{p} K^{11}_{p,k-1} \right\} \right\|
\]

\[
+ \left\| \sum_{p=n+1}^{\infty} (T^{11}_{p})^{-1} \left\{ -A_p T^{21}_{p+1,k} K^{11}_{p+1,k-1} - A_p T^{22}_{p+1} K^{21}_{p+1,k-1} \right\} \right\|
\]

\[
+ \left\| \sum_{p=n+1}^{\infty} (T^{11}_{p})^{-1} \left\{ B_p Q_p A_{p-1} T^{11}_{p-1} K^{11}_{p-1,k} + B_p Q_p B_p T^{11}_{p} K^{11}_{p,k-1} \right\} \right\|
\]

\[
+ \left\| \sum_{p=n+1}^{\infty} (T^{11}_{p})^{-1} \left\{ B_p Q_p^2 T^{21}_{p} K^{11}_{p,k-1} + B_p Q_p T^{22}_{p} K^{21}_{p,k-1} \right\} \right\|
\]

\[
+ \left\| \sum_{p=n+1}^{\infty} (T^{11}_{p})^{-1} \left\{ B_p Q_p T^{21}_{p} K^{12}_{p,k-1} + B_p Q_p T^{22}_{p} K^{22}_{p,k-1} \right\} \right\|
\]

If we use $T^{22}_{p+1} = A_p T^{11}_{p}$ for last inequality, we find

\[
\left\| K^{12}_{n,k+1} \right\| \leq \sum_{p=n+1}^{\infty} \left\| K^{21}_{p,k-1} - K^{21}_{p+1,k-1} \right\| + \sum_{p=n+1}^{\infty} \left\| (T^{11}_{p})^{-1} (I - A_p^2) T^{11}_{p} K^{21}_{p+1,k-1} \right\|.
\]
\[ + \sum_{p=n+1}^{\infty} \left\| (T_p^{11})^{-1} (-B_p - I) T_p^{21} K_{p,k-1}^{11} \right\| \\
+ \sum_{p=n+1}^{\infty} \left\| (T_p^{11})^{-1} (I - A_p) T_{p+1}^{21} K_{p+1,k-1}^{21} \right\| \\
+ \sum_{p=n+1}^{\infty} \left\| (T_p^{11})^{-1} (T_{p+1}^{21} K_{p,k-1}^{11} - T_{p+1}^{21} K_{p+1,k-1}^{11}) \right\| \\
+ C'' \sum_{p=n+1}^{\infty} \| B_p + I \| \sum_{s=p+\left\lfloor \frac{k}{2} \right\rfloor}^{\infty} \| N_s \| + \sum_{p=n+1}^{\infty} \| P_p \| \sum_{s=p+\left\lfloor \frac{k}{2} \right\rfloor}^{\infty} \| N_s \| \\
+ B \sum_{p=n+1}^{\infty} \| Q_p \| \sum_{s=p+\left\lceil \frac{k-2}{2} \right\rceil}^{\infty} \| N_s \| + D \sum_{p=n+1}^{\infty} \| Q_p \| \sum_{s=p+\left\lceil \frac{k-1}{2} \right\rceil}^{\infty} \| N_s \| , \]

where \( \| N_s \| = \| I - A_s \| + \| I + B_s \| + \| Q_s \| + \| P_s \| \) and \( C'', B, D \) are constants. It follows from that

\[ \| K_{n+1,k+1}^{12} \| \leq \| K_{n+1,k-1}^{21} \| + D' \sum_{p=n+1}^{\infty} \| I - A_p \| \sum_{s=p+\left\lfloor \frac{k}{2} \right\lfloor}^{\infty} \| N_s \| \\
+ D'' \sum_{p=n+1}^{\infty} \| B_p + I \| \sum_{s=p+\left\lfloor \frac{k-2}{2} \right\rfloor}^{\infty} \| N_s \| \\
+ D'' \sum_{p=n+1}^{\infty} \| I - A_p \| \sum_{s=p+\left\lfloor \frac{k-1}{2} \right\rfloor}^{\infty} \| N_s \| + T \| K_{n+1,k-1}^{11} \| \\
+ C'' \sum_{p=n+1}^{\infty} \| B_p + I \| \sum_{s=p+\left\lfloor \frac{k-1}{2} \right\rceil}^{\infty} \| N_s \| \\
+ \| P_p \| \sum_{s=p+\left\lfloor \frac{k}{2} \right\lceil}^{\infty} \| N_s \| + B \sum_{p=n+1}^{\infty} \| Q_p \| \sum_{s=p+\left\lfloor \frac{k-2}{2} \right\rceil}^{\infty} \| N_s \| \\
+ D \sum_{p=n+1}^{\infty} \| Q_p \| \sum_{s=p+\left\lceil \frac{k-1}{2} \right\rceil}^{\infty} \| N_s \| , \]
where $D', D'', D'''$ and $T$ are also constants. Using last inequality, we obtain
\[
\|K_{n,k+1}^{12}\| \leq C \sum_{p=n+1}^{\infty} \|N_p\| + TC \sum_{p=n+1}^{\infty} \|N_p\|
\]
\[
+ \left(D' + D''\right) \sum_{p=n+1}^{\infty} \|I - A_p\| \sum_{s=p+\left[\frac{k+1}{2}\right]}^{\infty} \|N_s\|
\]
\[
+ \max\left\{D'', D\right\} \sum_{p=n+1}^{\infty} (\|B_p + I\| + \|Q_p\|) \sum_{s=p+\left[\frac{k+2}{2}\right]}^{\infty} \|N_s\|
\]
\[
+ \max\left\{C'', 1\right\} \sum_{p=n+1}^{\infty} (\|B_p + I\| + \|P_p\|) \sum_{s=p+\left[\frac{k}{2}\right]}^{\infty} \|N_s\|
\]
\[
+ B \sum_{p=n+1}^{\infty} \|N_p\| \sum_{s=p+\left[\frac{k+2}{2}\right]}^{\infty} \|N_s\|
\]
and
\[
\|K_{n,k+1}^{12}\| \leq Z \sum_{p=n+\left[\frac{k+1}{2}\right]}^{\infty} \|N_p\| + Y \sum_{p=n+1}^{\infty} \|N_p\| \sum_{s=p+\left[\frac{k+2}{2}\right]}^{\infty} \|N_s\|
\]
\[
\leq Z \sum_{p=n+\left[\frac{k+1}{2}\right]}^{\infty} \|N_p\| + Y \left\{ \sum_{p=n+1}^{\infty} \|N_p\| \sum_{s=p+\left[\frac{k}{2}\right]}^{\infty} \|N_s\| \right\}
\]
\[
\leq Z \sum_{p=n+\left[\frac{k+1}{2}\right]}^{\infty} \|N_p\| + Y' \sum_{p=n+\left[\frac{k}{2}\right]}^{\infty} \|N_p\|
\]
\[
\leq 2Z \sum_{p=n+\left[\frac{k+1}{2}\right]}^{\infty} \|N_p\| + Y' \sum_{p=n+\left[\frac{k+1}{2}\right]}^{\infty} \|N_p\| \leq G \sum_{p=n+\left[\frac{k+1}{2}\right]}^{\infty} \|N_p\|,
\]
where $C + TC = Z$, $Y = D' + D'' + B + \max\{C'', 1\} + \max\{D'', D\}$, $Y' = Y \sum_{p=n+1}^{\infty} \|N_p\|$ and $2Z + Y' = G$. Similar to $K_{n,k+1}^{12}$, we can easily obtain (2.5) for $K_{n,k+1}^{11}$, $K_{n,k+1}^{21}$ and $K_{n,k+1}^{22}$.

It follows from (2.3) and (2.5) that $\left(F_n(z)\right)_{n \in \mathbb{N} \cup \{0\}}$ has analytic continuation from $D_0 := \{z \in \mathbb{C} : |z| = 1\}$ to $\{z \in \mathbb{C} : |z| < 1\} \setminus \{0\}$. 

\[\square\]
Theorem 2.3. Assume \((1.4)\). Then the Jost solution satisfies
\[
\begin{pmatrix} F_n(z) \\ G_n(z) \end{pmatrix} = [I + o(1)] \begin{pmatrix} z \\ -i \end{pmatrix} z^{2n}, \ n \to \infty
\]
for \(z \in D := \{z \in \mathbb{C} : |z| \leq 1\} \setminus \{0\}\).

Proof. It follows from \((2.3)\) that
\[
\begin{pmatrix} F_n(z) \\ G_n(z) \end{pmatrix} = \begin{pmatrix} T_n^{11} & T_n^{12} \\ T_n^{21} & T_n^{22} \end{pmatrix} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{m=1}^{\infty} \begin{pmatrix} K_{nm}^{11} & K_{nm}^{12} \\ K_{nm}^{21} & K_{nm}^{22} \end{pmatrix} z^{2m} \right] \begin{pmatrix} z^{2n+1} \\ -iz \end{pmatrix},
\]
then using \((1.4), (2.5)\) and the definition of \(T_{ij}n\) for \(i, j = 1, 2\), we get
\[
(2.7) \quad \begin{pmatrix} T_n^{11} & T_n^{12} \\ T_n^{21} & T_n^{22} \end{pmatrix} \to I, \ n \to \infty,
\]
and
\[
(2.8) \quad \sum_{m=1}^{\infty} \begin{pmatrix} K_{nm}^{11} & K_{nm}^{12} \\ K_{nm}^{21} & K_{nm}^{22} \end{pmatrix} z^{2m} = o(1), \ z \in D, \ n \to \infty.
\]
From \((2.3), (2.7), (2.8)\), we find \((2.6)\). \[\square\]

3. Continuous and discrete spectrum of \(L\)

Theorem 3.1. Under the condition \((1.4)\), \(\sigma_c(L) = [-2, 2]\).

Proof. Let \(L_0\) denote the operator generated in \(\ell_2(\mathbb{N}, \mathbb{C}^{2m})\) by the difference expression
\[
(l_1y)_n := \begin{cases} y^{(2)}_{n+1} - y^{(2)}_n \\ y^{(1)}_{n+1} - y^{(1)}_n \end{cases}
\]
with the boundary condition \(y^{(1)}_0 = 0\). We also define the operator \(J\) in \(\ell_2(\mathbb{N}, \mathbb{C}^{2m})\) by
\[
J \begin{pmatrix} y^{(1)}_n \\ y^{(2)}_n \end{pmatrix} := \begin{pmatrix} P_n & 0 \\ 0 & Q_n \end{pmatrix} \begin{pmatrix} y^{(1)}_n \\ y^{(2)}_n \end{pmatrix} + \begin{pmatrix} I + B_n & 0 \\ 0 & I + B_n \end{pmatrix} \begin{pmatrix} y^{(2)}_n \\ y^{(1)}_n \end{pmatrix} + \begin{pmatrix} A_n - I & 0 \\ 0 & A_{n-1} - I \end{pmatrix} \begin{pmatrix} y^{(2)}_{n+1} \\ y^{(1)}_{n+1} \end{pmatrix} + \begin{pmatrix} A_n - I & 0 \\ 0 & A_{n-1} - I \end{pmatrix} \begin{pmatrix} y^{(2)}_n \\ y^{(1)}_n \end{pmatrix} + \begin{pmatrix} A_n - I & 0 \\ 0 & A_{n-1} - I \end{pmatrix} \begin{pmatrix} y^{(2)}_n \\ y^{(1)}_n \end{pmatrix} + \begin{pmatrix} A_n - I & 0 \\ 0 & A_{n-1} - I \end{pmatrix} \begin{pmatrix} y^{(2)}_n \\ y^{(1)}_n \end{pmatrix}.
\]
It is clear that \(L_0 = L_0^*\) and \(L = L_0 + J\). Moreover, we can easily prove that
\[
\sigma(L_0) = \sigma_c(L_0) = [-2, 2],
\]
and using (1.4), we get that the operator $J$ is compact in $\ell_2(\mathbb{N}, \mathbb{C}^{2m})$ [16]. By the Weyl theorem [17, p. 13] of a compact perturbation, we obtain
\[
\sigma_c(L) = \sigma_c(L_0) = [-2, 2].
\]
This completes the proof.

Since the operator $L$ is self-adjoint, the eigenvalues of $L$ is real. From the definition of the eigenvalues, we can write
\[
\sigma_d(L) = \{ \lambda \in \mathbb{R} : \lambda = -iz - (iz)^{-1}, iz \in (-1, 0) \cup (0, 1), \det F_0(z) = 0 \},
\]
where $\sigma_d(L)$ denotes the set of all eigenvalues of $L$.

**Definition 3.2.** The multiplicity of a zero of the function $\det F_0(z)$ is called the multiplicity of the corresponding eigenvalue of $L$.

**Theorem 3.3.** Assume (1.4). Then the operator $L$ has a finite number of simple real eigenvalues.

**Proof.** To prove the theorem, we have to show that the function $\det F_0(z)$ has a finite number of simple zeros. Let $z_0$ be one of the zeros of $\det F_0(z)$. Hence $\det F_0(z_0) = 0$, there is a non-zero vector $u$ such that $F_0(z_0)u = 0$ [7]. As we know, $(F_{n(z)}^\ast)_{n=0}^\ast$ is the Jost solution of (1.2) for $\lambda = -iz - (iz)^{-1}$, i.e.,
\[
\begin{align*}
\{ A_nG_{n+1}(z) + B_nG_n(z) + P_nF_n(z) = [-iz - (iz)^{-1}] F_n(z) \\
A_{n-1}F_{n-1}(z) + B_nF_n(z) + Q_nG_n(z) = [-iz - (iz)^{-1}] G_n(z).
\end{align*}
\]
Differentiating (3.2) with respect to $z$, we have
\[
\begin{align*}
A_n \frac{d}{dz} G_{n+1}(z) + B_n \frac{d}{dz} G_n(z) + P_n \frac{d}{dz} F_n(z) &= [-iz - (iz)^{-1}] \frac{d}{dz} F_n(z) - i (1 - z^{-2}) F_n(z) \\
A_{n-1} \frac{d}{dz} F_{n-1}(z) + B_n \frac{d}{dz} F_n(z) + Q_n \frac{d}{dz} G_n(z) &= [-iz - (iz)^{-1}] \frac{d}{dz} G_n(z) - i (1 - z^{-2}) G_n(z).
\end{align*}
\]
Using (3.2) and (3.3), we obtain
\[
\begin{align*}
\left( \frac{d}{dz} F_n(z) \right)^\ast A_nG_{n+1}(z) + \left( \frac{d}{dz} F_n(z) \right)^\ast B_nG_n(z) \\
- \left( \frac{d}{dz} G_{n+1}(z) \right)^\ast A_nF_n(z) - \left( \frac{d}{dz} G_n(z) \right)^\ast B_nF_n(z) &= [-iz - (iz)^{-1}] \left( \frac{d}{dz} F_n(z) \right)^\ast F_n(z)
\end{align*}
\]
From (3.4) and (3.5), we get

\[ -[iz - (iz)^{-1}] \left( \frac{d}{dz} F_n(z) \right)^* F_n(z) + i (1 - z^{-2}) F_n^*(z) F_n(z) \]

and

\[ (3.5) \]

\[ \left( \frac{d}{dz} G_n(z) \right)^* A_{n-1} F_{n-1}(z) + \left( \frac{d}{dz} G_n(z) \right)^* B_n F_n(z) \]

\[ - \left( \frac{d}{dz} F_{n-1}(z) \right)^* A_{n-1} G_n(z) - \left( \frac{d}{dz} F_n(z) \right)^* B_n G_n(z) \]

\[ = [-iz - (iz)^{-1}] \left( \frac{d}{dz} G_n(z) \right)^* G_n(z) \]

\[ - [-iz - (iz)^{-1}] \left( \frac{d}{dz} G_n(z) \right)^* G_n(z) + i (1 - z^{-2}) G_n^*(z) G_n(z). \]

From (3.4) and (3.5), we get

\[ (3.6) \]

\[ \left( \frac{d}{dz} G_1(z) \right)^* A_0 F_0(z) - \left( \frac{d}{dz} F_0(z) \right)^* A_0 G_1(z) \]

\[ = [-iz - (iz)^{-1}] \sum_{n=1}^{\infty} \left[ \left( \frac{d}{dz} F_n(z) \right)^* F_n(z) + \left( \frac{d}{dz} G_n(z) \right)^* G_n(z) \right] \]

\[ - \left( iz - (iz)^{-1} \right) \sum_{n=1}^{\infty} \left[ \left( \frac{d}{dz} F_n(z) \right)^* F_n(z) + \left( \frac{d}{dz} G_n(z) \right)^* G_n(z) \right] \]

\[ + i(1 - z^{-2}) \sum_{n=1}^{\infty} [F_n^*(z) F_n(z) + G_n^*(z) G_n(z)]. \]

If we write (3.6) for \( z = z_0 \), we obtain

\[ (3.7) \]

\[ \left( \frac{d}{dz} G_1(z_0) \right)^* A_0 F_0(z_0) - \left( \frac{d}{dz} F_0(z_0) \right)^* A_0 G_1(z_0) \]

\[ = -i \left( 1 - \frac{1}{z_0^2} \right) \sum_{n=1}^{\infty} [F_n^*(z_0) F_n(z_0) + G_n^*(z_0) G_n(z_0)] \]

using \( iz_0 \in (-1, 0) \cup (0, 1) \). Then if we multiply (3.7) with the vector \( u \) on the right side \( (u \in \ell_2(\mathbb{N}, \mathbb{C}^{2m}), u \neq 0) \), we get

\[ \left\langle A_0 G_1(z_0) u, \frac{d}{dz} F_0(z_0) u \right\rangle = \left( i - \frac{i}{(iz_0)^2} \right) \left\{ \sum_{n=1}^{\infty} \|F_n(z_0) u\|^2 + \sum_{n=1}^{\infty} \|G_n(z_0) u\|^2 \right\}. \]
Since \( iz_0 \neq 0 \) and \( iz_0 \neq 1 \), we can write \( i - \frac{i}{(iz_0)^2} \neq 0 \). Also we can write \( \| F_n(z_0) u \| \neq 0 \) and \( \| G_n(z_0) u \| \neq 0 \) for all \( n \in \mathbb{N} \), so
\[
\left( A_0 G_1(z_0) u, \frac{d}{dz} F_0(z_0) u \right) \neq 0.
\]
This shows that \( \frac{d}{dz} F_0(z_0) u \neq 0 \), that is, all zeros of \( \det F_0(z) \) are simple. To complete the proof of theorem, we have to show that the function \( \det F_0(z) \) has a finite number of zeros. Let we take the function
\[
M(z) = z^{-1} \left( T_{011}^{11} \right)^{-1} F_0(z) = I + A(z),
\]
where \( A(z) = \sum_{m=1}^{\infty} K_{0m}^{11} z^{2m} - i \sum_{m=1}^{\infty} K_{0m}^{12} z^{2m-1} \). Since \( A(z) \) is matrix-valued analytic function on \( D \), the function \( M \) has inverse on the boundary of \( D \) [4, Theorem 5.1], i.e., the limit points of the set of zeros of
\[
\text{det} \ F_0(z) = 0
\]
is empty. Therefore, the set of zeros of (3.8) in \( D \) is finite, i.e., the operator \( L \) has a finite number of eigenvalues. \( \square \)

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