Distributionally Robust Variance Minimization: Tight Variance Bounds over $f$-Divergence Neighborhoods

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Abstract Distributionally robust optimization (DRO) is a widely used framework for optimizing objective functionals in the presence of both randomness and model-form uncertainty. A key step in the practical solution of many DRO problems is a tractable reformulation of the optimization over the chosen model ambiguity set, which is generally infinite dimensional. Previous works have solved this problem in the case where the objective functional is an expected value. In this paper we study objective functionals that are the sum of an expected value and a variance penalty term. We prove that the corresponding variance-penalized DRO problem over an $f$-divergence neighborhood can be reformulated as a finite-dimensional convex optimization problem. This result also provides tight uncertainty quantification bounds on the variance.

Keywords distributionally robust optimization · uncertainty quantification · variance minimization · $f$-divergence

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1 Introduction

Optimization problems that depend on incompletely known parameter values or involve systems with inherently noisy dynamics are often naturally phrased as stochastic programming (SP) problems of the general form

\[(SP) \quad \min_{x \in X} H[P, x]. \tag{1}\]

The objective functional, $H[P, x]$, depends on the underlying probability measure, $P$, which models the inherent randomness and/or parameter uncertainty,
and on the control variable, \( x \). An important and much studied case of (1) is the minimization of an expected value, \( H[P, x] = E_P[\rho_x] \), for some \( x \)-dependent random variable, \( \rho_x \), often thought of as a cost. In this paper we consider objective functionals of the form, \( H[P, x] = E_P[\rho_x] + \text{Var}_P[\phi_x] \), consisting of an expected cost and a variance penalty. Such objective functionals arise in resource allocation [22], stochastic control [25], Markov decision processes [16], and portfolio optimization [36,33], where the variance penalty enforces a certain risk aversion.

In practice, the model \( P \) is often learned from data and so is itself uncertain, i.e., there is model-form uncertainty. This motivates generalizing (1) to the following distributionally robust optimization (DRO) problem:

\[
\begin{align*}
\text{(DRO)} & \quad \min_{x \in X} \sup_{Q \in \mathcal{U}} H[Q, x],
\end{align*}
\]

i.e., minimizing the worst-case ‘cost’ over the neighborhood of models (ambiguity set) \( \mathcal{U} \). The ambiguity set encodes the degree and form of uncertainty regarding the ‘true’ model. A key step in the practical solution of many DRO problems is a tractable reformulation of the inner optimization over \( Q \), which is often infinite dimensional. Finite dimensional reformulations are known when the objective functional is an expected value and for various types of ambiguity sets, including moment constraints [19][20][34], Kullback–Leibler or \( f \)-divergence neighborhoods [31][20][4][21], and Wasserstein neighborhoods [17][27][7]. Such reformulations are also needed to solve problems with distributionally robust chance constraints [5][20][21][35].

Considerably less is known about DRO for more general objective functionals, beyond expected values. In this paper we study the following variance penalized DRO (VP-DRO) problem:

\[
\begin{align*}
\text{(VP-DRO)} & \quad \min_{x \in X} \sup_{Q \in \mathcal{U}} \{E_Q[\rho_x] + \text{Var}_Q[\phi_x]\}.
\end{align*}
\]

In [31] VP-DRO was studied for moment constraints, resulting in an inherently finite dimensional minimax problem; see also [31]. Here we consider \( f \)-divergence ambiguity sets, \( \mathcal{U}(P) = \{Q : D_f(Q, P) \leq \eta\} \), where \( P \) is a given baseline model. In this case, the optimization over \( Q \) in (3) is infinite dimensional and presents a considerable challenge on its own. Our focus for the remainder of the paper will thus be on the inner maximization problem in (3). Specifically, we show it can be rewritten as the following finite dimensional convex optimization problem:

\[
\begin{align*}
\sup_{Q, D_f(Q, P) \leq \eta} \{E_Q[\rho] + \text{Var}_Q[\phi]\} & = \inf_{\lambda > 0, \beta \in \mathbb{R}, \nu \in \mathbb{R}} \left\{ \nu^2/4 + \beta + \eta \lambda + \lambda E_P[f^*((\rho + \phi^2 - \nu \phi - \beta)/\lambda)] \right\},
\end{align*}
\]

where \( f^* \) denotes the Legendre transform of \( f \). Full details and assumptions can be found in Theorem 1 below and the proof is given in Section 3.
In addition to its importance for DRO, note that Eq. (4) also constitutes an uncertainty quantification (UQ) bound over the $f$-divergence model neighborhood. UQ bounds for expected values have been heavily studied \cite{11,15,8,9,18,6,2,14}. The formula (4) extends these works to provide a tight UQ bound on the variance.

1.1 Background on $f$-Divergences

Before proceeding to the main theorem, we first provide some required background on $f$-divergences. For $-\infty \leq a < b \leq \infty$ we let $F_1(a, b)$ denote the set of convex functions $f : (a, b) \to \mathbb{R}$ with $f(1) = 0$. Such functions are continuous and extend to convex, lower semicontinuous functions $f : \mathbb{R} \to (-\infty, \infty]$ by defining $f(a) = \lim_{t \searrow a} f(t)$ and $f(b) = \lim_{t \nearrow b} f(t)$ (when either $a$ and/or $b$ is finite) and $f|_{[a, b]} = \infty$ (see Appendix B for further details on $F_1(a, b)$).

Functions $f \in F_1(a, b)$ are appropriate for defining $f$-divergences as follows \cite{10,28}: Let $\mathcal{P}(\Omega)$ denote the set of probability measures on a measurable space $(\Omega, \mathcal{M})$. For $P, Q \in \mathcal{P}(\Omega)$ and $f \in F_1(a, b)$ the $f$-divergence of $Q$ with respect to $P$ is defined by

$$D_f(Q, P) = \begin{cases} E_P[f(dQ/dP)], & Q \ll P \\ \infty, & Q \not\ll P \end{cases}. \quad (5)$$

We will also use the following variational characterization of $f$-divergences \cite{10,28}:

$$D_f(Q, P) = \sup_{\phi \in \mathcal{M}_b(\Omega)} \{E_Q[\phi] - E_P[f^*(\phi)]\}, \quad (6)$$

where $\mathcal{M}_b(\Omega)$ denotes the set of bounded measurable real-valued functions on $\Omega$ and $f^*$ is the Legendre transform of $f$.

2 Tight Variance Bounds

The main result in this paper is the following tight bound on an expected value with variance penalty over an $f$-divergence neighborhood:

**Theorem 1** Suppose:

i. $f \in F_1(a, b)$ with $a \geq 0$.

ii. $P \in \mathcal{P}(\Omega)$.

iii. $\phi : \Omega \to \mathbb{R}$, $\phi \in L^1(P)$, and there exists $c_+, c_- > 0$, $\nu_+, \nu_- \in \mathbb{R}$ such that $E_P[[f^*(\pm c_+ \phi - \nu_\pm)]^+] < \infty$.

iv. $\rho : \Omega \to \mathbb{R}$ is measurable and if $Q \in \mathcal{P}(\Omega)$ with $D_f(Q, P) < \infty$ then $E_Q[\rho_-] < \infty$. 


Then $\phi \in L^1(Q)$ for all $Q \in \mathcal{P}(\Omega)$ that satisfy $D_f(Q, P) < \infty$ and for all $\eta > 0$ we have

$$\sup_{Q : D_f(Q, P) \leq \eta} \{ E_Q[\rho] + \text{Var}_Q[\phi] \} \leq \eta \{ E_Q[\rho] + \text{Var}_Q[\phi] \}$$

(7)

$$= \inf_{\lambda > 0, \beta \in \mathbb{R}, \nu \in \mathbb{R}} \left\{ \frac{\nu^2}{4} / \lambda + \beta + \eta \lambda + \lambda E_P[f^\ast((\rho + \phi^2 - \nu \phi - \beta)/\lambda)] \right\},$$

where the map $(0, \infty) \times \mathbb{R} \times \mathbb{R} \to (-\infty, \infty]$,

$$(\lambda, \beta, \nu) \to \frac{\nu^2}{4} / \lambda + \beta + \eta \lambda + \lambda E_P[f^\ast((\rho + \phi^2 - \nu \phi - \beta)/\lambda)]$$

(8)

is convex.

Remark 1 We use $g^\pm$ to denote the positive and negative parts of a (extended) real-valued function $g$, so that $g^\pm \geq 0$ and $g = g^+ - g^-$. The above assumptions imply $E_P[f^\ast((\rho + \phi^2 - \nu \phi - \beta)/\lambda)]$ exists in $(-\infty, \infty]$ for all $\lambda > 0$, $\beta \in \mathbb{R}$, $\nu \in \mathbb{R}$. Assumption (iv) is required to ensure that $E_Q[\rho] + E_Q[\phi^2] \neq -\infty + \infty$. Often $\rho$ is a non-negative cost function and so this assumption is trivial. Otherwise, Lemma 1 below (applied to $-\rho$) provides a concrete condition that ensures condition (iv) holds.

The proof of Theorem 1 which follows from the solution of a more general convex optimization problem, will be given in Section 3. In some cases, the optimization over various parameters on the right hand side of Eq. (7) can be evaluated explicitly. Below are two such examples.

2.1 Tight Variance Bounds: Relative Entropy

The relative entropy (i.e., Kullback–Leibler divergence) is the $f$ divergence corresponding to $f(t) = t \log(t)$, with Legendre transform $f^\ast(y) = e^y - 1$ (we write $R(Q\|P)$ for $D_f(Q, P)$). Assuming the conditions of Theorem 1 hold, we can evaluate the infimum over $\beta$ in (7) to find

$$\sup_{Q : R(Q\|P) \leq \eta} \{ E_Q[\rho] + \text{Var}_Q[\phi] \} \leq \eta \{ E_Q[\rho] + \text{Var}_Q[\phi] \}$$

(9)

$$= \inf_{\lambda > 0, \nu \in \mathbb{R}} \left\{ \frac{\nu^2}{4} / \lambda + \eta \lambda + \lambda \inf_{\beta \in \mathbb{R}} \left\{ \frac{\beta}{\lambda} + e^{-\beta/\lambda} E_P[\exp((\rho + \phi^2 - \nu \phi)/\lambda)] \right\} \right\}$$

$$= \inf_{\lambda > 0, \nu \in \mathbb{R}} \left\{ \frac{\nu^2}{4} / \lambda + \eta \lambda + \lambda \log \left( E_P \left[ e^{(\rho + \phi^2 - \nu \phi)/\lambda} \right] \right) \right\},$$

where the optimum occurs at $\beta_\lambda = \lambda (\log E_P[\exp((\rho + \phi^2 - \nu \phi)/\lambda)] + 1)$. 
2.2 Tight Variance Bounds: \( \alpha \)-Divergences

The family of \( \alpha \)-divergences is defined via the convex function
\[
f_\alpha(t) = \begin{cases} 
|y|^{-\alpha/(1-\alpha)}(1-\alpha)^{-\alpha/(1-\alpha)} - \frac{1}{\alpha(1-\alpha)}, & y < 0 \\
\infty, & y \geq 0 
\end{cases}
\]

For \( \alpha \in (0, 1) \) the \( \alpha \)-divergence has the upper bound
\[
D_{f_\alpha} \leq \frac{1}{\alpha(1-\alpha)}
\]
and so we assume \( 0 < \eta < \frac{1}{\alpha(1-\alpha)} \). In this case, and assuming the conditions of Theorem 1 hold, we can evaluate the infimum over \( \lambda \) in (7) to find
\[
\sup_{Q : D_{f_\alpha}(Q, P) \leq \eta} \{ \mathbb{E}_Q[\rho] + \text{Var}_Q[\phi] \} 
\]
(11)
\[
= \inf_{\beta \in \mathbb{R}, \nu \in \mathbb{R}, C_{\beta, \nu} \neq \infty} \left\{ \frac{\nu^2}{4} + \beta + \inf_{\lambda > 0} \left\{ \eta \lambda + \lambda \left( C_{\beta, \nu} \frac{\nu}{\lambda} - \frac{1}{\alpha(1-\alpha)} \right) \right\} \right\}
\]
(12)
\[
= \inf_{\beta \in \mathbb{R}, \nu \in \mathbb{R}, C_{\beta, \nu} \neq \infty} \left\{ \frac{\nu^2}{4} + \beta - \alpha \left( \frac{1}{C_{\beta, \nu}} \right)^{\frac{\nu}{\lambda}} \left( \frac{1}{\alpha(1-\alpha)} - \eta \right)^\frac{\nu}{\lambda} \right\}
\]
where \( C_{\beta, \nu} \in (0, \infty) \) is given by
\[
C_{\beta, \nu} = \begin{cases} 
\mathbb{E}_P[\rho + \phi^2 - \nu\phi - \beta], & P(\rho + \phi^2 - \nu\phi - \beta \geq 0) = 0 \\
\infty, & \text{otherwise.}
\end{cases}
\]

2.3 Formal Solution of the Optimization Problem

The objective functionals, both in the general case (8) and in either of the simplifications (9) or (11), are convex and are relatively straightforward to minimize via (stochastic) gradient descent. However, we can gain some insight by formally solving the optimization problem on the right hand side of Eq. (7) (see Section 3 for the rigorous derivations): Fix \( \eta > 0 \) and suppose the optimum in (7) is achieved at \( (\lambda_\eta, \beta_\eta, \nu_\eta) \). Differentiating with respect to \( \beta \) at the optimizer we find
\[
\mathbb{E}_P[(f^*')'(\Psi_\eta)] = 1, \quad \Psi_\eta = (\rho + \phi^2 - \nu_\eta\phi - \beta_\eta)/\lambda_\eta.
\]
(13)
\( f^* \) is nondecreasing (see Appendix B), hence \( (f^*)' \geq 0 \) and \( dQ_\eta \equiv (f^*)'(\Psi_\eta)dP \) is a probability measure. Next, by differentiating with respect to \( \lambda \) we obtain
\[
\eta = \mathbb{E}_P[(f^*')'(\Psi_\eta)\Psi_\eta - f^*(\Psi_\eta)].
\]
(14)
Differentiating with respect to $\nu$ we obtain
\[ E_{Q_\eta}[\phi] = \nu_\eta/2. \] (15)

Putting these together we can compute
\[ E_{Q_\eta}[\rho] + \operatorname{Var}_{Q_\eta}[\phi] = \nu_\eta^2/4 + \beta_\eta + \lambda_\eta E_{Q_\eta}[\Psi_\eta] = \nu_\eta^2/4 + \beta_\eta + \lambda_\eta(\eta + EP[f^*(\Psi_\eta)]), \] (16)

which equals the right hand side of (7). Therefore the probability measure that achieves the optimum on the left hand side of (7) is the tilted measure
\[ dQ_\eta = (f^*)'(\rho + \phi^2 - \nu_\eta\phi - \beta_\eta)/\lambda_\eta) dP. \] (17)

Below we present a simple example where some computations can be done explicitly, illustrating the above formal calculations.

**Example:** Suppose $\rho = 0$, $\phi$ is distributed as $N(0, \sigma^2)$ under $P$, and consider the KL-divergence ambiguity sets. Then $f^*(y) = e^{y-1}$ and the optimizing measure has the form
\[ dQ_\eta = e^{-\beta_\eta/\lambda_\eta} \exp((\phi^2 - \nu_\eta\phi)/\lambda_\eta) dP. \] (18)

In particular, $\phi$ is still normally distributed under $Q_\eta$. Also note that $\beta_\eta$ simply sets the normalization constant; this is the reason we were able to explicitly optimize over $\beta$ in (9). From (9) we have
\[ \sup_{Q: R(Q\|P) \leq \eta} \operatorname{Var}_Q[\phi] = \frac{1}{2} \log(1 - 2\sigma^2/\lambda_\eta) + \frac{\sigma^2}{\lambda - 2\sigma^2} = \eta. \] (20)

Substituting into (18) we see that $\phi$ is distributed as $N(0, \tilde{\sigma}_\lambda^2)$ under $Q_\eta$. To double check that $Q_\eta$ is the optimizer, note that the left hand side of Eq. (20) equals $R(N(0, \tilde{\sigma}_\lambda^2)||N(0, \sigma^2))$ and substituting Eq. (20) into Eq. (14) we see that
\[ \sup_{Q: R(Q\|P) \leq \eta} \operatorname{Var}_Q[\phi] = \operatorname{Var}_{Q_\eta}[\phi]. \] (21)
3 Proof of Theorem 1

We now work towards the proof of Theorem 1. We will require a number of intermediate results, the first being a useful condition that ensures certain expectations exist.

Lemma 1 Let \( f \in \mathcal{F}_1(a, b) \) and \( P \in \mathcal{P}(\Omega) \). Suppose \( \phi : \Omega \to \mathbb{R} \) is measurable and \( E_P[|f^*(c_0\phi - \nu_0)|^+] < \infty \) for some \( \nu_0 \in \mathbb{R} \) and \( c_0 > 0 \). Then for all \( Q \in \mathcal{P}(\Omega) \) with \( D_f(Q, P) < \infty \) we have \( E_Q[\phi^+] < \infty \).

Proof Fix \( b \in \mathbb{R} \) for which \( f^*(b) \) is finite and define
\[
\phi_n^+ = \phi 1_{0 \leq \phi < n} + (b + \nu_0)/c_0 1_{\phi \geq |n, n)\cdot
\] (22)
Hence \( c_0\phi_n^+ - \nu_0 \in \mathcal{M}_b(\Omega) \) and the variational formula (3) gives
\[
D_f(Q, P) \geq E_Q[c_0\phi_n^+ - \nu_0] - E_P[f^*(c_0\phi_n^+ - \nu_0)],
\] (23)
where \( E_P[f^*(c_0\phi_n^+ - \nu_0)] \) is defined in \((-\infty, \infty)\). Hence
\[
E_Q[c_0\phi_n^+] - D_f(Q, P) \leq \nu_0 + E_P[f^*(c_0\phi_n^+ - \nu_0)].
\] (24)

We can bound
\[
f^*(c_0\phi_n^+ - \nu_0) = f^*(c_0\phi - \nu_0)1_{0 \leq \phi < n} + f^*(b)1_{\phi \geq |n, n)}
\] (25)
and so
\[
E_P[f^*(c_0\phi_n^+ - \nu_0)] \leq E_P[f^*(c_0\phi - \nu_0)] + |f^*(b)|.
\] (26)
Combined with Eq. (24), this implies
\[
E_Q[c_0\phi_n^+] - D_f(Q, P) \leq \nu_0 + E_P[f^*(c_0\phi - \nu_0)] + |f^*(b)| < \infty
\] (27)
for all \( n \). \( \phi_n^+ \) are uniformly bounded below, therefore Fatou’s Lemma implies
\[
E_Q[\liminf_n \phi_n^+] \leq \liminf_n E_Q[\phi_n^+]
\] (28)
\[
\leq c_0^{-1}(\nu_0 + E_P[f^*(c_0\phi - \nu_0)] + |f^*(b)| + D_f(Q, P)).
\]
We have the pointwise limit \( \phi_n^+ \to \phi^+ + (b + \nu_0)/c_0 1_{\phi < 0} \), hence
\[
E_Q[\phi^+] + (b + \nu_0)/c_0Q(\phi < 0) = E_Q[\liminf_n \phi_n^+]
\] (29)
\[
\leq c_0^{-1}(\nu_0 + E_P[f^*(c_0\phi - \nu_0)] + |f^*(b)| + D_f(Q, P)) < \infty.
\]
This proves the claim.

The following lemma is an intermediate step towards Eq. (1). It show how to express certain suprema over \( f \)-divergence neighborhoods in terms of finite dimensional maximin problems.
Lemma 2 Suppose:

i. \( f \in \mathcal{F}_1(a,b) \) with \( a \geq 0 \).

ii. \( P \in \mathcal{P}(\Omega) \).

iii. \( \phi_i : \Omega \rightarrow \mathbb{R} \), \( i = 1, ..., k \), \( \phi_i \in L^1(P) \), and there exists \( c_i^+, c_i^- > 0 \), \( \nu_i^+, \nu_i^- \in \mathbb{R} \) such that \( E_P[|f^*(\pm c_i^+ \phi_i - \nu_i^+)|^+] < \infty \) for \( i = 1, ..., k \).

iv. \( \psi : \Omega \rightarrow \mathbb{R} \) is measurable and \( \psi^\pm \in L^1(P) \).

v. \( g : \mathbb{R}^k \rightarrow \mathbb{R} \) is convex.

Then

1. \( \phi_i \in L^1(Q) \), \( i = 1, ..., k \), for all \( Q \in \mathcal{P}(\Omega) \) that satisfy \( D_f(Q,P) < \infty \).

2. For all \( \eta > 0 \) there exists \( M_{i,\eta}^\pm \in \mathbb{R} \), \( i = 1, ..., k \) such that

\[
E_Q[\phi_i] \in [-M_{i,\eta}^-, M_{i,\eta}^+] \tag{30}
\]

for all \( i = 1, ..., k \) and all \( Q \in \mathcal{P}(\Omega) \) that satisfy \( D_f(Q,P) \leq \eta \).

3. For all \( \eta > 0 \) and all \( C \subset \mathbb{R}^k \) with \( \prod_{i=1}^k [-M_{i,\eta}^-, M_{i,\eta}^+] \subset C \) we have

\[
\sup_{Q : D_f(Q,P) \leq \eta} \{ E_Q[\psi] - g(E_Q[\phi_1], ..., E_Q[\phi_k]) \}
= \sup_{z \in C, \nu \in \mathbb{R}^k} \inf_{\lambda > 0, \beta \in \mathbb{R}} \{ \beta + \eta \lambda + \lambda E_P[f^*(\psi - \nu \cdot \phi - \beta)/\lambda]) \}
\]

and \( E_P[f^*(\psi - \nu \cdot \phi - \beta)/\lambda]) \) exists in \( (-\infty, \infty) \) for all \( \lambda > 0, \beta \in \mathbb{R}, \nu \in \mathbb{R}^k \).

Remark 2 In Eq. (31), and in the following, we define \( \infty - \infty \equiv \infty \) so that expectations are defined for all measurable functions; such a term is only possible in Eq. (31) on the left hand side and, under appropriate assumptions, can be ruled out entirely via Lemma 1.

Proof

1. This is a direct consequence of Lemma 1.

2. Let \( \eta > 0 \). The tight bound on expected values over \( f \)-divergence neighborhoods from [1][2][3], which we recall in Theorem 3 in Appendix 3 implies

\[
E_Q[\pm \phi_i] \leq \frac{\eta}{c_i^+} + \frac{1}{c_i^-} (\nu_i^+ + E_P[f^*(\pm c_i^+ \phi_i - \nu_i^+)]) \equiv M_{i,\eta}^\pm \tag{32}
\]

for all \( Q \in \mathcal{P}(\Omega) \) with \( D_f(Q,P) \leq \eta \). Assumption (iii) implies \( M_{i,\eta}^\pm \in \mathbb{R} \). The lower bound \( f^*(y) \geq y \) (see Appendix 3) together with \( \phi_i \in L^1(P) \) then implies \( M_{i,\eta}^\pm \in \mathbb{R} \).

3. Fix \( \eta > 0 \), let \( M_{i,\eta}^\pm \) be as in part (2) and fix \( C \subset \mathbb{R}^k \) with \( \prod_{i=1}^k [-M_{i,\eta}^-, M_{i,\eta}^+] \subset C \). Using \( f^*(y) \geq y \), for \( \lambda > 0, \beta \in \mathbb{R}, \nu \in \mathbb{R}^k \) we find

\[
f^*(\psi - \nu \cdot \phi - \beta)/\lambda) \leq \frac{1}{\lambda}(\psi^- + (\nu \cdot \phi + \beta)^+) \in L^1(P). \tag{33}
\]
Hence $E_P[f^*((\psi - \nu \cdot \phi - \beta) / \lambda)]$ exists in $(-\infty, \infty]$. The following formula will be key during the remainder of the derivation: For any $\nu \in \mathbb{R}^k$ we have $(\psi - \nu \cdot \phi)^- \in L^1(P)$ and so Theorem 3 implies

$$
\sup_{Q: D_f(Q,P) \leq \eta} E_Q[\psi - \nu \cdot \phi] = \inf_{\lambda > 0, \beta \in \mathbb{R}} \left\{ \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta) / \lambda)] \right\},
$$

where $-\infty - \infty \equiv \infty$.

To prove the claimed equality (31) we consider two cases. First, suppose there exists $\nu_0 \in \mathbb{R}^k$ such that $E_P[f^*((\psi - \nu_0 \cdot \phi - \beta) / \lambda)] = \infty$ for all $\beta \in \mathbb{R}, \lambda > 0$: Taking $\nu = \nu_0$ in Eq. (34) we see that

$$
\sup_{Q: D_f(Q,P) \leq \eta} E_Q[\psi - \nu_0 \cdot \phi] = \infty.
$$

For any other $\nu$ we have

$$
E_Q[(\nu - \nu_0) \cdot \phi] \geq -\sum_i |(\nu - \nu_0)^i| \max\{M^-_{i,\eta}, M^+_{i,\eta}\} \equiv -C
$$

and hence

$$
\sup_{Q: D_f(Q,P) \leq \eta} E_Q[\psi - \nu \cdot \phi] = \infty
$$

for all $\nu \in \mathbb{R}^k$. Eq. (35) then implies

$$
\inf_{\lambda > 0, \beta \in \mathbb{R}} \left\{ \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta) / \lambda)] \right\} = \infty
$$

for all $\nu \in \mathbb{R}^k$ and so we see that the right hand side of Eq. (31) equals $+\infty$.

Eq. (39) together with continuity of $g$ imply that there exists $D \in \mathbb{R}$ such that $g(E_Q[\phi]) \leq D$ for all $Q \in \mathcal{P}(\Omega)$ with $D_f(Q,P) \leq \eta$. Therefore

$$
\sup_{Q: D_f(Q,P) \leq \eta} \{ E_Q[\psi] - g(E_Q[\phi]) \} \geq -D + \sup_{Q: D_f(Q,P) \leq \eta} E_Q[\psi].
$$

Eq. (34) for $\nu = 0$ then implies that the left hand side of Eq. (31) also equals $+\infty$.

Now consider the alternative case, where for all $\nu \in \mathbb{R}^k$ there exists $\beta \in \mathbb{R}, \lambda > 0$ such that $E_P[f^*((\psi - \nu \cdot \phi - \beta) / \lambda)] < \infty$. We will show that the optimization problem $\inf_{Q \in X} F(Q,H(Q))$ can be written as an iterated optimization over level sets of $H$. To do this we will use the convex optimization result given in Lemma 4 of Appendix B. This result relies on a variant of the Slater conditions, which we now verify: Let $V$ be the vector space consisting of all finite, real linear combinations of measures in $\{Q \in \mathcal{P}(\Omega) : D_f(Q,P) < \eta\}$ and let

$$
X = \{Q : D_f(Q,P) \leq \eta, E_Q[\psi^-] < \infty\}.
$$
Then \( P \in X \subset V \), \( X \) is convex, and \( \psi \in L^1(Q) \) for all \( Q \in X \) (Theorem 3) implies that \( E_Q[\psi^+] < \infty \) for all \( Q \) with \( D_Q(Q, P) \leq \eta \). Define the convex function \( F : X \times \mathbb{R}^k \to \mathbb{R} \) by \( F(Q, z) = g(z) - E_Q[\psi] \) and the linear function \( H : V \to \mathbb{R}^k \), \( H(\mu) = (\int \phi_1 d\mu, \ldots, \int \phi_k d\mu) \). For all \( \nu \in \mathbb{R}^k \), \( z \in \mathbb{R}^k \) we can use Eq. (34) to compute

\[
\inf_{Q \in X} \{ F(Q, z) + \nu \cdot (H(Q) - z) \} = g(z) - \nu \cdot z - \sup_{Q : D_Q(Q, P) \leq \eta} \{ E_Q[\psi - \nu \cdot \phi] \}
\]

\[
= g(z) - \nu \cdot z - \inf_{\lambda > 0, \beta \in \mathbb{R}} \{ \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \}.
\]

In the case currently under consideration, there exists \( \lambda > 0, \beta \) such that
\[
E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)^+] < \infty \quad \text{and so} \quad \inf_{Q \in X} \{ F(Q, z) + \nu \cdot (H(Q) - z) \} > -\infty.
\]

With this we have shown that all of the hypotheses of Lemma 4 from Appendix B hold and hence we obtain the following: For all \( K \subset \mathbb{R}^k \) with \( H(X) \subset K \) we have

\[
\inf_{Q \in X} F(Q, H(Q)) = \inf_{z \in K} \sup_{\nu \in \mathbb{R}^k} \inf_{Q \in X} \{ F(Q, z) + \nu \cdot (H(Q) - z) \},
\]

i.e.,

\[
\sup_{Q : D_Q(Q, P) \leq \eta} \{ E_Q[\psi] - g(E_Q[\phi_1], \ldots, E_Q[\phi_k]) \}
\]

\[
= \sup_{z \in K} \inf_{\nu \in \mathbb{R}^k} \{ \nu \cdot z - g(z) + \inf_{\lambda > 0, \beta \in \mathbb{R}} \{ \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \} \}.
\]

From part (2) we see that \( H(X) \subset \prod_{i=1}^k [-M_{i,\eta}, M_{i,\eta}] \subset C \) and so we can take \( K = C \), thus completing the proof.

Given further assumptions on \( g \) we can exchange the order of the minimization and maximization in Eq. (41) and evaluate the supremum over \( z \), thereby expressing the result as a finite dimensional convex minimization problem.

**Theorem 2** Suppose:

i. \( f \in F_1(a, b) \) with \( a \geq 0 \).

ii. \( P \in \mathcal{P}(\Omega) \).

iii. \( \phi_i : \Omega \to \mathbb{R}, \ i = 1, \ldots, k, \ \phi_i \in L^1(P) \), and there exists \( c_i^+, c_i^- > 0, \nu_i^+, \nu_i^- \in \mathbb{R} \) such that \( E_P[f^*((c_i^+ \phi_i - c_i^- \phi_i)^+) < \infty \) for \( i = 1, \ldots, k \).

iv. \( \psi : \Omega \to \mathbb{R} \) is measurable and \( \psi^{-} \in L^1(P) \).

v. \( g : \mathbb{R}^k \to \mathbb{R} \) is convex, \( C^1 \), and

\[
\lim_{z \to \infty} g(z)/\|z\| = \lim_{z \to \infty} \|\nabla g(z)\| = \lim_{\nu \to \infty} g^*(\nu)/\|\nu\| = \infty.
\]

Let \( \eta > 0 \). Then
1. \[
\sup_{Q \in \mathcal{P}(Q,P) \leq \eta} \{ E_Q[\psi] - g(E_Q[\phi_1], ..., E_Q[\phi_k]) \} = \inf_{\lambda > 0, \beta \in \mathbb{R}, \nu \in \mathbb{R}^k} \{ g^*(\nu) + \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \}. \tag{45}
\]

2. The map \((0, \infty) \times \mathbb{R} \times \mathbb{R}^k \rightarrow (-\infty, \infty), \)
\[
(\lambda, \beta, \nu) \rightarrow g^*(\nu) + \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \tag{46}
\]
is convex.

**Proof** First we collect some useful facts regarding \(g^*\): We have assumed \(g\) is \(C^1\) and \(\lim_{z \rightarrow \infty} g(z)/||z|| = \infty\), therefore for all \(\nu \in \mathbb{R}^k\) we have \(g^*(\nu) = \nu \cdot z^*_\nu - g(z^*_\nu)\) for some \(z^*_\nu \in \mathbb{R}^k\) with \(\nabla g(z^*_\nu) = \nu\). In particular, \(g^*\) is a real valued convex function on \(\mathbb{R}^k\) and hence is continuous.

By convexity of \(f^*\) and of the perspective of a convex function we see that the map \((0, \infty) \times \mathbb{R} \times \mathbb{R}^k \rightarrow (-\infty, \infty), \)
\[
(\lambda, \beta, \nu) \rightarrow \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \tag{47}
\]
is convex. Together with the convexity of \(g^*\), we therefore conclude that \(\text{46}\) is convex.

To prove (1), first use Lemma 2 to conclude the following: For all \(Q \in \mathcal{P}(\Omega)\) with \(D_f(Q,P) < \infty\) we have \(\phi_i \in L^1(Q), i = 1, ..., k\), and for all \(C \subset \mathbb{R}^k\) with \(\prod_{i=1}^k [-M_{i,\eta}, M_{i,\eta}] \subset C\) we have
\[
\sup_{Q \in \mathcal{P}(Q,P) \leq \eta} \{ E_Q[\psi] - g(E_Q[\phi_1], ..., E_Q[\phi_k]) \} = \sup_{z \in C} \inf_{\nu \in \mathbb{R}^k} \{ \nu \cdot z - g(z) + \inf_{\lambda > 0, \beta \in \mathbb{R}} \{ \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \} \}. \tag{48}
\]
The claimed equality, Eq. (45), will follow if we can show that
\[
\sup_{z \in C} \inf_{\nu \in \mathbb{R}^k} \{ \nu \cdot z - g(z) + \inf_{\lambda > 0, \beta \in \mathbb{R}} \{ \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \} \}
\]
\[
= \inf_{\lambda > 0, \beta \in \mathbb{R}, \nu \in \mathbb{R}^k} \{ g^*(\nu) + \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \} \tag{49}
\]
for some set \(C\) containing \(\prod_{i=1}^k [-M_{i,\eta}, M_{i,\eta}]\).

Eq. (49) is trivial if \(E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] = \infty\) for all \(\nu \in \mathbb{R}^k, \lambda > 0, \beta \in \mathbb{R}\), so suppose not. First we rewrite the left hand side of Eq. (49) in a more convenient form: The map
\[
(\lambda, \beta, \nu) \rightarrow \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \tag{50}
\]
is convex on \((0, \infty) \times \mathbb{R} \times \mathbb{R}^k\), hence we can conclude that
\[
h : \nu \in \mathbb{R}^k \rightarrow \inf_{\lambda > 0, \beta \in \mathbb{R}} \{ \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] \} \tag{51}
\]
is also convex, provided $h > -\infty$ (see Proposition 2.22 in [30]); the latter follows by using $f^*(y) \geq y$ to compute

$$\inf_{\lambda > 0, \beta \in \mathbb{R}} \{ \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)]\} \geq E_P[\psi] - E_P[\nu \cdot \phi] > -\infty. \quad (52)$$

Therefore Lemma 3 in Appendix B implies that the infimum can be restricted to the relative interior of the domain of $h$:

$$\sup_{z \in C} \inf_{\nu \in \mathbb{R}^k} \nu \cdot z - g(z) + \inf_{\lambda > 0, \beta \in \mathbb{R}} \{ \beta + \eta \lambda + \lambda E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)]\}$$

$$= \sup_{z \in C} \inf_{\nu \in \text{ri}(\text{dom } h)} \nu \cdot z - g(z) + h(\nu). \quad (53)$$

We now proceed to show that the required equality holds for $C = [-R, R]^k$ when $R$ is sufficiently large. We restrict to $R \geq R_0 \equiv \max\{M_{\infty}^{-}\} \geq \prod\{M_{\infty}^{-}, M_{\infty}^{+}\} \subset [-R, R]^k$. We will now show that the supremum over $z$ and infimum over $\nu$ in Eq. (53) can be commuted: The map $(\nu, z) \rightarrow \nu \cdot z - g(z) + h(\nu)$ is continuous on $\text{ri}(\text{dom } h) \times [-R, R]^k$, concave in $z$, convex in $\nu$. The domain of $z$ is compact, hence Sion’s minimax theorem (see [32,23]) implies

$$\sup_{z \in [-R, R]^k} \inf_{\nu \in \text{ri}(\text{dom } h)} \nu \cdot z - g(z) + h(\nu)$$

$$= \inf_{\nu \in \text{ri}(\text{dom } h)} \{ \sup_{z \in [-R, R]^k} \nu \cdot z - g(z) + h(\nu) \} \leq \inf_{\nu \in \text{ri}(\text{dom } h)} \{ g^*(\nu) + h(\nu) \}. \quad (54)$$

Next we show that equality holds in the last line of (54) when $R$ is sufficiently large: We are in the case where $E_P[f^*((\psi - \nu \cdot \phi - \beta)/\lambda)] < \infty$ for some $\nu_0 \in \mathbb{R}^k$, $\lambda_0 > 0$, $\beta_0 \in \mathbb{R}$ and so

$$\inf_{\nu \in \text{ri}(\text{dom } h)} g^*(\nu) + h(\nu) \leq g^*(\nu_0) + h(\nu_0) < \infty. \quad (55)$$

Convexity of $h$ implies that it has an affine lower bound. Hence there exists $D \geq 0$ such that $h(\nu) \geq -D/\|\nu\|_1 + d$ for all $\nu$. Fix $\hat{R} > \max\{R_0, D\}$ and choose $C > 0$ such that

$$\hat{R}(1 - D/\hat{R}) C + d > \inf_{\nu \in \text{ri}(\text{dom } h)} g^*(\nu) + h(\nu) + \max_{w : w_i \in \{\pm \hat{R}\}, i = 1, \ldots, k} g(w). \quad (56)$$

Note that this is possible because of Eq. (56). Finally, $\lim_{z \to \infty} \|\nabla g(z)\| = \infty$ and so we can choose $R > \hat{R}$ such that $\|\nabla g(z)\|_1 > C$ for all $z \not\in [-R, R]^k$.

To prove equality in Eq. (54), let $\nu \in \text{ri}(\text{dom } h)$ and consider the following two cases.
a. Suppose \( \{ z : \nabla g(z) = \nu \} \subset [-R, R]^k \): We know that \( g^*(\nu) = \nu \cdot z^*_\nu - g(z^*_\nu) \) \( \nu \in [-R, R]^k \) and so
\[
\begin{align*}
  h(\nu) + \sup_{z \in [-R, R]^k} \{ \nu \cdot z - g(z) \} & \geq h(\nu) + \nu \cdot z^*_\nu - g(z^*_\nu) \\
\geq h(\nu) + g^*(\nu) & \geq \inf_{\nu \in \text{ri}(\text{dom } h)} \{ g^*(\nu) + h(\nu) \}. 
\end{align*}
\]

b. Suppose there exists \( z_0 \not\in [-R, R]^k \) with \( \nabla g(z_0) = \nu \): Let \( w_i = \text{sgn}(\nu_i) \tilde{R} \) so that \( w \in [-R, R]^k \), \( \nu \cdot w = \tilde{R} \| \nu \|_1 \) and
\[
\begin{align*}
  h(\nu) + \sup_{z \in [-R, R]^k} \{ \nu \cdot z - g(z) \} & \geq h(\nu) + \tilde{R} \| \nu \|_1 - g(w) \\
\geq \tilde{R}(1 - D/\tilde{R}) & > 0.
\end{align*}
\]
The definitions of \( R \) and \( \tilde{R} \) imply \( \| \nabla g(z_0) \|_1 > C \) and \( \tilde{R}(1 - D/\tilde{R}) > 0 \), hence
\[
\begin{align*}
  h(\nu) + \sup_{z \in [-R, R]^k} \{ \nu \cdot z - g(z) \} & \geq \tilde{R}(1 - D/\tilde{R})C + d - g(w) \\
\geq \inf_{\nu \in \text{ri}(\text{dom } h)} \{ g^*(\nu) + h(\nu) \} + \max_{w: w_i \in \{-R, R\}, i = 1, \ldots, k} \{ g(w) \} - g(w) \\
\geq \inf_{\nu \in \text{ri}(\text{dom } h)} \{ g^*(\nu) + h(\nu) \}. 
\end{align*}
\]
Combining these two cases, we see that
\[
\inf_{\nu \in \text{ri}(\text{dom } h)} \{ \sup_{z \in [-R, R]^k} \{ \nu \cdot z - g(z) \} + h(\nu) \} \geq \inf_{\nu \in \text{ri}(\text{dom } h)} \{ g^*(\nu) + h(\nu) \}. 
\]
Therefore we have equality in Eq. (54). Applying Lemma 3 to the continuous function \( g^* \) and the convex function \( h \) and then using the formula (51) for \( h \) we arrive at
\[
\begin{align*}
  \sup_{z \in [-R, R]^k} \inf_{\nu \in \text{ri}(\text{dom } h)} \{ \nu \cdot z - g(z) + h(\nu) \} & = \inf_{\nu \in R^k} \{ g^*(\nu) + h(\nu) \} \\
= \inf_{\lambda > 0, \beta \in R, \psi \in R^k} \{ g^*(\nu) + \beta + \eta \lambda + \lambda E_P[\psi - \nu \cdot \phi - \beta]/\lambda] \}. 
\end{align*}
\]
Combining (53) with (61) we arrive at (49) and so the result is proven.

**Corollary 1** Applying Theorem 3 to \( \phi, \psi \equiv \rho + \phi^2, g(z) = z^2 \), and \( g^*(\nu) = \nu^2/4 \) we obtain the tight variance bound stated in Theorem 4.

### A Tight Bounds on Expected Values

A key ingredient in the proof of Theorem 3 is the following tight bound on expected values over \( f \)-divergence neighborhoods which was proven for bounded integrands, \( \phi \), in Theorem 1 of [14] and was extended to unbounded integrands in [19].

**Theorem 3** Suppose:
i. \( f \in \mathcal{F}_1(a, b) \) with \( a \geq 0 \).

ii. \( P \in \mathcal{P}(\Omega) \).

iii. \( \phi : \Omega \to \mathbb{R} \) is measurable and \( \phi^- \in L^1(P) \).

Then for all \( \eta > 0 \) we have

\[
\sup_{Q, D_f(\Omega, P) \leq \eta} E_Q[\phi] = \inf_{\lambda > 0, \beta \in \mathbb{R}} \{ (\beta + \eta \lambda) + \lambda E_P[f^*((\phi - \beta)/\lambda)] \},
\]

where we use the convention \( -\infty \equiv \infty \) to extend the definition of \( E_Q[\phi] \) to all \( Q \).

**Remark 3** Note that \( f^* \) is nondecreasing and hence \( (-\infty, \infty) \) to extend the definition of \( E_Q[\phi] \) to all \( Q \).

\[ f^*((\phi - \beta)/\lambda) \geq (\phi - \beta)/\lambda \in L^1(P) \]

for all \( \beta \in \mathbb{R}, \lambda > 0 \), and hence \( E_P[f^*((\phi - \beta)/\lambda)] \) in Eq. (29) exists in \((-\infty, \infty)\).

## B Convex Functions and Optimization

Here we recall several key results from convex analysis and optimization that will be needed above, as will as fix our notation; for much more detail on these subjects see, e.g., [29,26, 30]. We will also prove a pair of lemmas that are key tools in the proof of Theorem 1 above.

We denote the relative interior of a convex set \( C \subset \mathbb{R}^n \) by \( ri(C) \), the affine hull by \( aff(C) \), and the domain of a convex function \( f : C \to (-\infty, \infty) \) by \( \text{dom} f \equiv \{ f < \infty \} \) (we do not allow convex functions to take the value \(-\infty \) in this work). Recall that a convex function on \( \mathbb{R}^n \) is continuous on the relative interior of its domain (see Theorem 10.1 in [29]). The Legendre transform of \( f \), defined by

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \{ yx - f(x) \}, \]

is a convex lower semicontinuous function on \( \mathbb{R}^n \), provided that \( f \) is not identically \( \infty \). We will be especially concerned with the following classes of convex functions, which are used to define \( f \)-divergences: For \(-\infty \leq a < 1 < b \leq \infty \) we let \( \mathcal{F}_i(a, b) \) be the set of convex functions \( f : (a, b) \to \mathbb{R} \) with \( f(1) = 0 \). Such functions are continuous and extend to convex, lower semicontinuous functions, \( f : \mathbb{R} \to (-\infty, \infty) \), by defining \( f(a) = \lim_{x \to a^+} f(x) \) and \( f(b) = \lim_{x \to b^-} f(x) \) (where appropriate; if \( a \) and/or \( b \) is finite then the corresponding limit is guaranteed to exist in \((-\infty, \infty)\)) and \( f[|a, b|] = \infty \). The Legendre transform of this extension can be computed via

\[ f^*(y) = \sup_{x \in (a, b)} \{ yx - f(x) \}. \]

Note that \( f(1) = 0 \) implies that \( f^*(y) \geq y \) for all \( y \in \mathbb{R} \). From Eq. (30) we also see that if \( a \geq 0 \) then \( f^* \) is nondecreasing and hence \( (-\infty, d] \subset \text{dom} f^* \) for some \( d \in \mathbb{R} \).

The following lemma shows that to minimize the sum of a continuous and convex function, it suffices to minimize over the relative interior of the domain.

**Lemma 3** Let \( g : \mathbb{R}^n \to \mathbb{R} \) be continuous and \( f : \mathbb{R}^n \to (-\infty, \infty] \) be convex. Then

\[ \inf_{\mathbb{R}^n} \{ g + f \} = \inf_{ri(\text{dom} f)} \{ g + f \}. \]

**Proof** The result is trivial if \( \text{dom} f = \emptyset \) so suppose not. Then \( ri(\text{dom} f) \neq \emptyset \) and we can fix \( x_0 \in ri(\text{dom} f) \). For \( x \in \text{dom} f \) we have \( (1 - t)x + tx_0 \in ri(\text{dom} f) \) for all \( t \in (0, 1) \) (see Theorem 6.1 in [29]) therefore, using convexity of \( f \), we find

\[
\inf_{ri(\text{dom} f)} \{ g + f \} \leq g((1 - t)x + tx_0) + f((1 - t)x + tx_0) \\
\leq g((1 - t)x + tx_0) + (1 - t)f(x) + tf(x_0)
\]

where \( \lambda \geq 1 \), \( 0 \leq \lambda \leq 1 \), and \( \beta \in \mathbb{R} \) are parameters such that \( \lambda + \beta \leq 1 \) and \( \beta \leq 0 \).
for all $t \in (0, 1)$. By taking $t \searrow 0$ we arrive at
\[ \inf_{\nu(\text{dom } f)} \{ g + f \} \leq g(x) + f(x) \tag{68} \]
for all $x \in \text{dom } f$. Eq. (68) trivially holds when $x \notin \text{dom } f$ and hence we obtain
\[ \inf_{\nu(\text{dom } f)} \{ g + f \} \leq \inf_{\mathbb{R}^n} \{ g + f \}. \tag{69} \]

The reverse inequality is trivial.

The following lemma is one of our key technical tools. It splits a convex minimization problem into an iterated optimization of Lagrangians over the level sets of a given linear function and relies on a variant of the Slater conditions. This result is new, to the best of the author’s knowledge.

**Lemma 4** Let $V$ be a real vector space and suppose:

i. $X \subset V$, $C \subset \mathbb{R}^k$ are nonempty convex subsets,

ii. $f : X \times C \rightarrow \mathbb{R}$ is jointly convex,

iii. $h : V \rightarrow \mathbb{R}^k$ is linear with $h(X) \subset C$,

iv. $\inf_{x \in X} \{ f(x, z) + \nu \cdot (h(x) - z) \} > -\infty$ for all $\nu \in \mathbb{R}^k$, $z \in C$.

Then for all $K \subset \mathbb{R}^k$ with $h(X) \subset K \subset C$ we have
\[ \inf_{x \in X} f(x, h(x)) = \inf_{x \in K} \sup_{\nu \in \mathbb{R}^k} \inf_{x \in X} \{ f(x, z) + \nu \cdot (h(x) - z) \}. \tag{70} \]

**Proof** We will need several properties of the function $F : C \rightarrow (-\infty, \infty]$ defined by
\[ F(z) = \sup_{\nu \in \mathbb{R}^k} \inf_{x \in X} \{ f(x, z) + \nu \cdot (h(x) - z) \}. \tag{71} \]

The function $(z, x) \rightarrow f(x, z) + \nu \cdot (h(x) - z)$ is convex and so $z \rightarrow \inf_{x \in X} \{ f(x, z) + \nu \cdot (h(x) - z) \}$ is convex for all $\nu$ (see Proposition 2.22 in [30] and note that it never equals $-\infty$ by assumption (iv)). This holds for all $\nu$ and so $F$ is convex (see Proposition 2.9 in [31]). We extend $F$ by $F|_{C^c} \equiv \infty$; this extension is also convex.

Next we show that $F$ is finite on $h(X)$: Let $z_0 \in h(X)$. Then there exists $x_0 \in X$ with $h(x_0) = z_0$ and for any $\nu$ we have
\[ \inf_{x \in X} \{ f(x, z_0) + \nu \cdot (h(x) - z_0) \} \leq f(x_0, z_0) + \nu \cdot (h(x_0) - z_0) = f(x_0, z_0). \tag{72} \]

Hence
\[ F(z_0) = \sup_{\nu \in \mathbb{R}^k} \inf_{x \in X} \{ f(x, z_0) + \nu \cdot (h(x) - z_0) \} \leq f(x_0, z_0) < \infty. \tag{73} \]

Finally we show that $F(z) = \infty$ for all $z \in \overline{h(X)} \cap C$: Let $z_0 \in \overline{h(X)} \cap C$. The separating hyperplane theorem implies that there exists $v \in \mathbb{R}^k$, $v \neq 0$, and $c \in (0, \infty)$ such that $v \cdot (z_0 - w) \geq c$ for all $w \in h(X)$. Letting $\nu = -tv$, $t > 0$, we have
\[ F(z_0) \geq \inf_{x \in X} \{ f(x, z_0) - tv \cdot (h(x) - z_0) \} = \inf_{x \in X} \{ f(x, z_0) + tv \cdot (z_0 - h(x)) \} \geq tv + \inf_{x \in X} f(x, z_0). \tag{74} \]

From assumption (iv) we see that $\inf_{x \in X} f(x, z_0) > -\infty$ and so by taking $t \rightarrow \infty$ we find $F(z_0) = \infty$.

Now we prove the claimed equality (70). If $h(X) = \{ z_0 \}$ for some $z_0$ then
\[ \inf_{x \in X} f(x, h(x)) = \inf_{x \in X} f(x, z_0) \tag{75} \]
and
\[
\inf_{x \in X} \sup_{\nu \in \mathbb{R}^k} \inf_{z \in X} \{ f(x, z) + \nu \cdot (h(x) - z) \} = \inf_{x \in X} \inf_{\nu \in \mathbb{R}^k} \sup_{z \in X} \{ \nu \cdot (z_0 - z) \} \quad (76)
\]
\[
= \inf_{z \in X} \inf_{x \in X} f(x, z) + \infty I_{x \neq z_0} = \inf_{z \in X} f(x, z_0).
\]

This proves the claim in this case.

Now suppose \( h(X)^o \neq \emptyset \). As a first step, we have
\[
\inf_{x \in X} f(x, h(x)) = \inf_{z \in h(X)^o} \inf_{x \in X} f(x, z) \quad (77)
\]
Fix \( z_0 \in h(X)^o \). \( h(X) \) is convex, hence for any \( \tilde{z} \in h(X) \) and any \( t \in [0, 1] \) we have \((1 - t)z_0 + t\tilde{z} \in h(X)^o \) (see Theorem 6.1 in [29]). Take \( x_0, \tilde{x} \in X \) with \( h(x_0) = z_0 \) and \( h(\tilde{x}) = \tilde{z} \). Then \((1 - t)x_0 + t\tilde{x} \in X \), \( h((1 - t)x_0 + t\tilde{x}) = (1 - t)z_0 + t\tilde{z} \) and
\[
\inf_{z \in h(X)^o} \inf_{x \in X} f(x, z) \leq f((1 - t)x_0 + t\tilde{x}, (1 - t)z_0 + t\tilde{z})
\]
\[
\leq tf(\tilde{x}, \tilde{z}) + (1 - t)f(x_0, z_0)
\]
for all \( t \in [0, 1] \). Taking \( t \to 1 \) we find
\[
\inf_{z \in h(X)^o} \inf_{x \in X} f(x, h(x)) \leq f(\tilde{x}, \tilde{z}) \quad (79)
\]
for all \( \tilde{x} \in X \) with \( h(\tilde{x}) = \tilde{z} \) and hence
\[
\inf_{x \in X} f(x, h(x)) = \inf_{z \in h(X)^o} \inf_{x \in X} f(x, z) = \inf_{z \in h(X)^o} \inf_{x \in X} f(x, h(x)) = \inf_{z \in h(X)^o} \inf_{x \in X} f(x, z). \quad (80)
\]
For \( z \in h(X)^o \) the Slater conditions hold for the convex function \( f(\cdot, z) \) and affine constraint \( h(\cdot) - z \) (see Theorem 8.3.1 and Problem 8.7 in [29]) and so we have strong duality:
\[
\inf_{x \in X} f(x, h(x)) = \sup_{\nu \in \mathbb{R}^k} \inf_{x \in X} \{ f(x, z) + \nu \cdot (h(x) - z) \}. \quad (81)
\]
Therefore \( \inf_{x \in X} f(x, h(x)) = \inf_{z \in h(X)^o} F(z) \). The properties of \( F \) proven above imply
\[
h(X) \subseteq \text{dom} \, F \subseteq h(X) \quad (82)
\]
and hence \( \langle \text{dom} \, F \rangle^o = h(X)^o \) (see Theorem 6.3 in [29]). Therefore Lemma 3 implies
\[
\inf_{h(X)^o} F = \inf_{\langle \text{dom} \, F \rangle^o} F = \inf_{R^k} F \quad \text{and so for any } K with h(X) \subseteq K \subseteq C we have}
\[
\inf_{x \in X} f(x, h(x)) = \inf_{z \in K} F(z) \quad (83)
\]
as claimed.

Finally when \( h(X) \) contains more than one element and \( h(X)^o = \emptyset \) we will transform the problem into an equivalent one with that fits under the previously proven case. Intuitively, in this case \( h(X) \) lies in a hyperplane and has nonempty relative interior. Hence by using an affine transformation we can push it down to a lower dimensional space where it has nonempty interior: There exists \( m \in \mathbb{Z}^+ \) and affine maps \( \Phi : \mathbb{R}^m \to \text{aff}(h(X)) \), \( \Psi : \mathbb{R}^k \to \mathbb{R}^m \) such that \( \Phi^{-1} = \Psi \big|_{\text{aff}(h(X))} \) and \( \Psi(h(X))^o \neq \emptyset \). Write \( \Phi(\cdot) = A(\cdot) + a \) and \( \Psi(\cdot) = B(\cdot) + b \) with \( A, B \) linear and \( a, b \) constant. The above properties imply that \( \Phi \circ \Psi \circ h|_X = h|_X \) and \( (Bh(X))^o \neq \emptyset \). Define
1. \( \tilde{C} = Bh(X) \), a nonempty convex subset of \( \mathbb{R}^m \),
2. \( f : X \times \tilde{C} \to \mathbb{R}, f(x, z) = f(x, a + A(b + z)), \) a convex map,
3. \( \tilde{h} : V \to \mathbb{R}^m, \tilde{h} = Bh, \) a linear map with \( h(X) = \tilde{C} \).
For any $\nu \in \mathbb{R}^m$, $\tilde{z} \in \tilde{C}$ we have $\tilde{z} = Bz$ for some $z \in h(X) \subset C$, hence
\[
\inf_{x \in X} \left\{ \bar{f}(x, \tilde{z}) + \nu \cdot \langle \tilde{h}(x) - \tilde{z} \rangle \right\} = \inf_{x \in X} \left\{ f(x, a + A(b + Bz)) + \nu \cdot B(h(x) - z) \right\} = \inf_{x \in X} \left\{ f(x, z) + (B^T \nu) \cdot \langle h(x) - z \rangle \right\} > -\infty
\]
by assumption (iv). In addition, $\tilde{h}(X)^o = \tilde{C}^o = (Bh(X))^o \neq \emptyset$. Therefore $\bar{f}, \tilde{h}$ satisfy the assumptions (i)-(iv) of this lemma and this system falls under the previously proven case. Hence
\[
\inf_{x \in X} f(x, h(z)) = \inf_{x \in X} \bar{f}(x, \tilde{h}(x)) = \inf_{\tilde{z} \in \tilde{C}, x \in X} \sup_{\tilde{\nu} \in \mathbb{R}^m} \left\{ \tilde{f}(x, \tilde{z}) + \tilde{\nu} \cdot \langle \tilde{h}(x) - \tilde{z} \rangle \right\}
\]
where the last inequality is due to weak duality.

Therefore we have proven
\[
\inf_{x \in X} f(x, h(x)) = \inf_{x \in h(X)} \sup_{\tilde{\nu} \in \mathbb{R}^k} \inf_{\tilde{z} \in \tilde{C}} \left\{ f(x, z) + \tilde{\nu} \cdot \langle h(x) - z \rangle \right\} = \inf_{x \in h(X)} F(z).
\]
From Lemma 3 we find $\inf_{(\text{dom } F) \cap h(X)} F = \inf_{h(X)} F$. From Eq. (82) we see that $\text{dom } F \cap \text{ri}(h(X)) = \text{ri}(h(X)) \neq \emptyset$ and so dom $F$ is not contained in the relative boundary of $h(X)$. Therefore Corollary 6.5.2 in [29] implies $\text{ri}(\text{dom } F) \subset \text{ri}(h(X))$ and hence for any $K \subset \mathbb{R}^k$ with $h(X) \subset K \subset C$ we have
\[
\inf_{x \in X} f(x, h(x)) = \inf_{x \in \text{ri}(h(X))} \sup_{\tilde{\nu} \in \mathbb{R}^k} \inf_{\tilde{z} \in \tilde{C}} \left\{ f(x, z) + \tilde{\nu} \cdot \langle h(x) - z \rangle \right\}
\]
which proves the result in this final case.

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