A cohomological proof for the integrability of strict Lie 2-algebras

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We prove a series of van Est type theorems relating the cohomologies of strict Lie 2-groups and strict Lie 2-algebras, and use them to prove the integrability of Lie 2-algebras anew.

1 Introduction

Associated with any Lie algebra \( g \), there is a canonical extension

\[
0 \rightarrow \mathfrak{z}(g) \rightarrow g \xrightarrow{\text{ad}} \text{ad}(g) \rightarrow 0
\]

of the linear Lie subalgebra \( \text{ad}(g) \leq \mathfrak{gl}(g) \) by the center \( \mathfrak{z}(g) \). Such extensions are classified by \( H^2_{CE}(\text{ad}(g), \mathfrak{z}(g)) \), the Chevalley-Eilenberg cohomology of \( \text{ad}(g) \) with values on \( \mathfrak{z}(g) \). Recall the following theorem due to van Est:

**Theorem 1.1.** [15] Let \( G \) be a Lie group with Lie algebra \( g \) and a representation on \( V \). If \( G \) is \( k \)-connected, then the map that differentiates group cochains into Chevalley-Eilenberg cochains induces isomorphisms

\[
\Phi : H^n_G(G, V) \rightarrow H^n_{CE}(g, V)
\]

for all \( n \leq k \) and is injective when \( n = k + 1 \).

Observe that, being linear, \( \text{ad}(g) \) is integrable and recall that one can always pick a \( 2 \)-connected integration, say \( G \). Thus, Theorem 1.1 implies there exists a unique cohomology class \( [\int \omega_g] \in H^2_{Gp}(G, \mathfrak{z}(g)) \), whose image under (2) is \( [\omega_g] \in H^2_{CE}(\text{ad}(g), \mathfrak{z}(g)) \), the cohomology class that corresponds to the extension (1). Since \( H^2_{Gp}(G, V) \) classifies extensions of \( G \) by \( V \), there is a unique extension

\[
1 \rightarrow \mathfrak{z}(g) \rightarrow G \rightarrow G \rightarrow 1
\]

of the linear Lie subalgebra \( \text{ad}(g) \leq \mathfrak{gl}(g) \) by the center \( \mathfrak{z}(g) \), that \( G \) is a Lie group integrating \( g \).

The purpose of this article is to adapt this strategy -that we henceforth refer to as the **van Est strategy**-[16 11] to prove the integrability of strict Lie 2-algebras.

A strict Lie 2-group is a groupoid

\[
\mathcal{G} \times_H \mathcal{G} \xrightarrow{m} \mathcal{G} \xrightarrow{s} H \xrightarrow{u} \mathcal{G},
\]

in which the spaces of objects, arrows and composable arrows are Lie groups, and all of whose structural morphisms are Lie group homomorphisms. Differentiating the whole structure, one gets a (strict) Lie 2-algebra

\[
\mathfrak{g}_1 \times_H \mathfrak{g}_1 \xrightarrow{m} \mathfrak{g}_1 \xrightarrow{s} \mathfrak{h} \xrightarrow{u} \mathfrak{g}_1.
\]
In [12], it is proven using the path method that all finite-dimensional Lie 2-algebras are the infinitesimal counterpart of a Lie 2-group. In the sequel, we present a cohomological proof of this fact. Such approach still works in infinite dimensions and was historically used to construct the first example of a non-integrable Lie algebra [17]; thus, it bears the potential to improve our current understanding of the Lie theory of other categorified objects (see, e.g., 8, 14, 16).

Let us list the necessary ingredients for the van Est strategy to run:

1) The canonically associated adjoint extension 8.
2) Global and infinitesimal cohomology theories that classify extensions.
3) A van Est map and theorem.
4) That linear Lie algebras be integrable to 2-connected Lie groups.

Lie 2-algebras have a canonically associated adjoint representation (see Example 2.1 below). In 1, 2, complexes whose second cohomology classify respectively extensions of Lie 2-algebras and extensions of Lie 2-groups are introduced. Each of these is the total complex of a triple complex of sorts. In order to describe them, let us fix notation.

First, recall that the categories of Lie 2-algebras and Lie 2-groups are respectively equivalent to the categories of linear invertible self-functors of a 2-vector space and of Lie groups 4, 6, 8.

**Definition 1.2.** A crossed module of Lie algebras is a Lie algebra morphism \( g \rightarrow \mathfrak{h} \) together with a Lie algebra action by derivations \( \mathcal{L} : \mathfrak{h} \rightarrow \mathfrak{gl}(g) \) satisfying

\[
\mu(L_y x) = [y, \mu(x)], \quad \mathcal{L}_{\mu(x_0) x_1} = [x_0, x_1]
\]

for all \( y \in \mathfrak{h} \) and \( x, x_0, x_1 \in g \). Following the convention in the literature, we refer to these equations respectively as equivariance and Peiffer. □

**Definition 1.3.** A crossed module of Lie groups is a Lie group homomorphism \( G \rightarrow H \) together with a right action of \( H \) on \( G \) by Lie group automorphisms satisfying

\[ i(g^h) = h^{-1}i(g)h, \]

\[ g_1^{-1}g_2 = g_2^{-1}g_1g_2, \]

for all \( h \in H \) and \( g, g_1, g_2 \in G \), where we write \( g^h \) for \( h \) acting on \( g \). Following the convention in the literature, we refer to these equations respectively as equivariance and Peiffer. □

Representations of both Lie 2-algebras and Lie 2-groups take values on so-called 2-vector spaces. These are (flat) abelian objects in either category, which, in crossed module presentation, correspond simply to 2-term complexes of vector spaces \( W \rightarrow \phi V \). The category of linear invertible self-functors of a 2-vector space and homomorphic natural transformations \( GL(\phi) \) has got the structure of a Lie 2-group whose Lie 2-algebra \( \mathfrak{gl}(\phi) \) is the category of linear functors and linear natural transformations (see Subsection 2.1.2 for details). Respectively, representations are by definition maps of Lie 2-groups to \( GL(\phi) \) and maps of Lie 2-algebras to \( \mathfrak{gl}(\phi) \).

Lastly, we assume the following unconventional notation for the spaces of p-composable arrows:

\[ G_p := \{ (\gamma_1, ..., \gamma_p) \in G^p : s(\gamma_k) = t(\gamma_{k+1}), \quad \forall k \}, \quad \text{and} \quad g_p := \{ (\xi_1, ..., \xi_p) \in g^p : s(\xi_k) = t(\xi_{k+1}), \quad \forall k \}. \]

We are ready to define the three-dimensional lattices of vector spaces underlying the definition of the complexes of Lie 2-algebra and Lie 2-group cochains taking values on the 2-vector space \( W \rightarrow \phi V \). For a Lie 2-algebra 5 with associated crossed module \( g \rightarrow \mathfrak{h} \), set

\[ C^p_q(g_1, \phi) := \bigwedge^q g_p \otimes \bigwedge^r g^\ast \otimes W \quad \text{for} \; r > 0, \quad \text{and} \quad C^0_q(g_1, \phi) := \bigwedge^q g_p \otimes V, \]

where \( g_0 = \mathfrak{h} \). For a Lie 2-group 4 with associated crossed module \( G \rightarrow H \), set

\[ C^p_q(G, \phi) := C^p_q(G_p \times G^r, W) \quad \text{for} \; r > 0, \quad \text{and} \quad C^0_q(G, \phi) := C^q(G_p^0, V), \]

where \( G_0 = H \), \( G_1 = G \) and \( C(X, A) \) is the vector space of \( A \)-valued smooth functions.

These lattices come together with a three-dimensional grid of maps that is a complex in each direction; in the Lie 2-algebra case, the grid is built out of Chevalley-Eilenberg complexes, while in the Lie 2-group case, the
grid is built out of groupoid cochain complexes (see Subsection 2.1.3 for details). We refrain from calling either grid a triple complex because not all differentials commute with one another. In each case, two of the building differentials commute only up to homotopy (or up to isomorphism when \( r = 0 \)). In [1, 2], it is explained how adding the homotopies to the total differential \( \nabla \) makes up for this defect, ultimately turning

\[
C^p_V(g_1, \phi) = \bigoplus_{p+q+r=n} C^p_{\mathcal{V}}(g_1, \phi) \quad \text{and} \quad C^n_V(G, \phi) = \bigoplus_{p+q+r=n} C^p_{\mathcal{V}}(G, \phi)
\]

into actual complexes.

The fundamental property of the complexes (9), as it was mentioned, is that their second cohomology classify extensions. In fact, the equivalence happens at the level of cocycles, i.e., a 2-cocycle in either \( C^2_{\mathcal{V}}(g_1, \phi) \) or \( C^2_{\mathcal{V}}(G, \phi) \) univocally defines an extension by the 2-vector space \( W \xrightarrow{\phi} V \), and two such extensions are isomorphic if and only if the cocycles are cohomologous. In so, the map that linearizes a Lie 2-group extension induces a map

\[
\Phi : C^2_{\mathcal{V}}(G, \phi) \longrightarrow C^2_{\mathcal{V}}(g_1, \phi)
\]

whenever \( g_1 \) is the Lie 2-algebra of \( G \). Due to its nature, we are bound to call \( \Phi \) the van Est map. This map can be proved to be assembled from groupoid van Est maps (see Section 3). Recall that the van Est theorem (Theorem 1.5) admits an extension to Lie groupoids.

**Theorem 1.4.** [1, 2] Let \( G \longrightarrow M \) be a Lie groupoid with Lie algebroid \( A \) and a representation on the vector bundle \( E \). If the source fibres of \( G \) are \( k \)-connected, the map that differentiates groupoid cochains into algebroid cochains induces isomorphisms

\[
\Phi : H^n_{Gpd}(G, E) \longrightarrow H^n_{C, E}(A, E)
\]

for all \( n \leq k \) and it is injective for \( n = k + 1 \). \( \Box \)

We refer to Theorem 1.4 as the Crainic-van Est theorem in the sequel. The Crainic-van Est theorem can be rephrased as a vanishing result for the cohomology of the mapping cone of \( \Phi \) (see Proposition 2.4).

**Theorem 1.5.** If \( G \) is source \( k \)-connected, then

\[
H^n(\Phi) = (0), \quad \text{for all } n \leq k,
\]

where \( H^n(\Phi) \) is the cohomology of the mapping cone of \( \Phi \). \( \Box \)

One can combine the fact that \( \Phi \) is assembled from groupoid van Est maps and Theorem 1.5 to prove a van Est type theorem. To see how, let us momentarily take \( W = (0) \). In this case, the three-dimensional grids (7) and (8) collapse to honest double complexes

\[
C(H^2, V) \xrightarrow{\delta} C(G^2, V) \xrightarrow{\delta} C(G_2^2) \xrightarrow{\delta} \ldots \quad \text{and} \quad \Lambda^2 h^* \otimes V \xrightarrow{\delta} \Lambda^2 g_1^* \otimes V \xrightarrow{\delta} \Lambda^2 g_2^* \otimes V \xrightarrow{\delta} \ldots
\]

\[
C(H, V) \xrightarrow{\delta} C(G, V) \xrightarrow{\delta} C(G_2) \xrightarrow{\delta} \ldots \quad \Phi \quad h^* \otimes V \xrightarrow{\delta} g_1^* \otimes V \xrightarrow{\delta} g_2^* \otimes V \xrightarrow{\delta} \ldots
\]

\[
V \xrightarrow{\delta} V \xrightarrow{\delta} V \xrightarrow{\delta} \ldots
\]

where \( \Phi \) is defined columnwise by \( \Phi_p : C^p_{Gpd}(G_p, V) \longrightarrow C^p_{C, E}(g_p, V) \), the usual van Est map for \( G_p \). Now, \( \Phi \) is a map of double complexes if and only if

\[
C(\Phi_0) \longrightarrow C(\Phi_1) \longrightarrow C(\Phi_2) \longrightarrow \ldots
\]
is a double complex, and the first page of the spectral sequence of its filtration by columns is

\[ E_1^{p,q} : \]

\[ H^2(\Phi_0) \longrightarrow H^2(\Phi_1) \longrightarrow H^2(\Phi_2) \longrightarrow \ldots \]

\[ H_0(\Phi_0) \longrightarrow H_0(\Phi_1) \longrightarrow H_0(\Phi_2) \longrightarrow \ldots \]

Therefore, if we suppose that \( H_4 \) is 2-connected and \( G_\) is 1-connected, \( \Phi \) induces isomorphisms between the total cohomologies in all degrees less than or equal to 2.

The general case is not much more complicated. Indeed, when \( W \neq (0) \), the restriction

\[ \Phi_p : C_*^p(G, \phi) \longrightarrow C_*^p(g_1, \phi) \]

is a map of honest double complexes for which one can apply the above reasoning. The cohomology of \( \Phi \) coincides with the total cohomology of a double complex of sorts each of whose columns is the total complex of \( \Phi_p \) (see (61) below). In spite of not being an actual double complex, this object can be filtrated by columns and the first page of its spectral sequence essentially coincides with (12), ultimately, allowing us to prove:

**Theorem 1.6.** Let \( G \) be a Lie 2-group with associated crossed module \( G \longrightarrow H \), Lie 2-algebra \( g_1 \) and a representation on the 2-vector space \( W \longrightarrow V \). If \( H \) and \( G \) are both 2-connected, the map that extends the linearization of extensions induces isomorphisms

\[ \Phi : H^n_v(G, \phi) \longrightarrow H^n_v(g_1, \phi) \]

for all \( n \leq 2 \) and it is injective for \( n = 3 \).

We can now run van Est strategy as the final ingredient, that linear Lie 2-algebras are integrable is proved in [13], and one can choose such an integration \( G \longrightarrow H \) with both \( H \) and \( G \) 1-connected. Since 1-connected Lie groups are automatically 2-connected, we conclude that there is a unique Lie 2-group extension integrating the canonical adjoint extension of a Lie 2-algebra, thus implying its integrability.

The body of the article is dedicated to formalize how to apply the van Est strategy to the case of strict Lie 2-algebras. In Section 2, we convene notation and state some basic facts. In Section 3 we define the van Est map and prove it is a map of complexes. In Section 4 we prove a van Est type theorem realizing the van Est map as a composition of groupoid van Est maps.

2 Preliminaries

In this section, we settle notation by reviewing the necessary notions to formally define the complexes of Lie 2-algebra and Lie 2-group cochains with 2-vector space coefficients, and recall the elements of homological algebra of which we make use below.

2.1 Lie 2-algebras, Lie 2-groups and their cohomology

2.1.1 The equivalence with the categories of crossed modules

In the sequel, we make no distinction between a Lie 2-group or a Lie 2-algebra and their corresponding crossed modules. For future reference, we make explicit the equivalence at the level of objects.

Let \( G \) be a Lie 2-group \[ 4. \] In order to make clear the difference between the group operation and the groupoid operation in \( G \), we assume the following convention:

\[ \gamma_1 \times \gamma_2, \quad \gamma_3 \ast \gamma_4 \]
stand respectively for the group multiplication and the groupoid multiplication whenever \((\gamma_1, \gamma_2) \in G^2\) and \((\gamma_3, \gamma_4) \in G \times G\). This notation intends to reflect that we think of the group multiplication as being “vertical”, and the groupoid multiplication as being “horizontal”.

Since the source map is a surjective submersion and the unit map is a canonical splitting thereof, letting \(G\) be the Lie subgroup \(\ker s \leq G\), \(G \cong G \times H\). The associated crossed module map is given by \(G \overset{\text{hterm.}}{\longrightarrow} H\).

Thinking of the underlying 2-vector space of a Lie 2-algebra \(\mathcal{G}\) as an abelian Lie 2-group, this construction yields a canonical isomorphism \(g_1 \cong g \oplus h\), where \(g := \ker s\) and the crossed module map \(\mathcal{G} \overset{\text{hterm.}}{\longrightarrow} h\).

In the Lie 2-group case, the right action is given by conjugation by units in the group \(G\):

\[
g^h := u(h)^{-1} x u(h),
\]

for \(h \in H\) and \(g \in G\). We stress that the \(-1\) power stands for the inverse of the group multiplication\(\mathcal{G}\). In the Lie 2-algebra case, the action by derivations is given by

\[
\mathcal{L}_y x := [u(y), x]_1,
\]

for \(y \in h\) and \(x \in g\). Here, \([\cdot, \cdot]_1\) stands for the Lie bracket of \(g_1\).

Conversely, given a crossed module \(G \overset{\iota}{\longrightarrow} H\) as in Definition \([\ref{crossed_module}3]\) the space of arrows of its associated Lie 2-group \(G\) is defined to be the product \(G \times H\) together with the structural maps

\[
s(g, h) = h, \quad t(g, h) = h \iota (g), \quad \iota (g, h) = (g^{-1}, h \iota (g)), \quad u(h) = (1, h),
\]

\[
(g', h \iota (g)) \circlearrowleft (g, h) := (gg', h)
\]

for \(h \in H\) and \(g, g' \in G\). Thinking of the underlying 2-term complex of vector spaces of a crossed module \(g \overset{\mu}{\longrightarrow} h\) as in Definition \([\ref{crossed_module}2]\) as an abelian crossed module of Lie groups, this construction yields a 2-vector space \(g_1 := g \oplus h\) with structural maps given by the same formulae that we transcribe using additive notation

\[
\hat{s}(x, y) = y, \quad \hat{t}(x, y) = y + \mu(x), \quad \hat{i}(x, y) = (-x, y + \mu(x)), \quad \hat{u}(y) = (0, y),
\]

\[
(x', y + \mu(x)) \circlearrowleft (x, y) = \hat{m}(x', y + \mu(x); x, y) := (x + x', y)
\]

for \(y \in h\) and \(x, x' \in g\). In the Lie group case, \(G\) is endowed with the group structure of the semi-direct product \(G \ltimes H\) with respect to the \(H\)-action. Explicitly, the product is given by

\[
(g_1, h_1) \times (g_2, h_2) = (g_1^1 g_2, h_1^1 h_2),
\]

for \((g_1, h_1), (g_2, h_2) \in G \ltimes H\). In the Lie algebra case, \(g_1\) is endowed with the bracket of the semi-direct sum \(g \oplus \mathcal{G}\), explicitly given by

\[
[[x_0, y_0], (x_1, y_1)]_{\mathcal{L}} := ([x_0, x_1] + \mathcal{L}_{y_0, x_1} - \mathcal{L}_{y_0, x_0}, [y_0, y_1]),
\]

for \((x_0, y_0), (x_1, y_1) \in g \oplus h\).

### 2.1.2 The general linear Lie 2-group and the linear Lie 2-algebra

The General Linear Lie 2-group \(GL(\phi)\) \([\ref{general_linear_group}11 \ [\ref{linear_algebra}13]\) is the Lie 2-group which plays the rôle of space of automorphisms of the 2-vector space \(W \overset{\phi}{\longrightarrow} V\). \(GL(\phi)\) is by definition the crossed module

\[
\Delta : GL(\phi)_1 \longrightarrow GL(\phi)_0 : A \longrightarrow (I + A \phi, I + \phi A),
\]

where

\[
GL(\phi)_1 = \{A \in Hom(V, W) : (I + A \phi, I + \phi A) \in GL(W) \times GL(V)\},
\]

\[
GL(\phi)_0 = \{(F, f) \in GL(W) \times GL(V) : \phi \circ F = f \circ \phi\},
\]
and the right action given by
\[ A^{(F, f)} := F^{-1}Af, \]
for \((F, f) \in GL(\phi)_0, \ A \in GL(\phi)_1.\) (22)

The group structure on the Whitehead group \(GL(\phi)_1\) is given by
\[ A_1 \odot A_2 := A_1 + A_2 + A_1 \phi A_2, \]
\[ A_1, A_2 \in GL(\phi)_1, \]
whose identity element is the 0 map, and where the inverse of an element \(A\) is given by either \(-A(I + \phi A)^{-1} = -(I + A\phi)^{-1}A\).

The Lie 2-algebra of \(GL(\phi)\) is \(\mathfrak{gl}(\phi)\), explicitly given by (13)
\[ \Delta' : \mathfrak{gl}(\phi)_1 \longrightarrow \mathfrak{gl}(\phi)_0 : A \longrightarrow (A\phi, \phi A), \]
(24)

where
\[ \mathfrak{gl}(\phi)_1 = \text{Hom}(V, W), \quad \mathfrak{gl}(\phi)_0 = \{(F, f) \in \mathfrak{gl}(W) \oplus \mathfrak{gl}(V) : \phi \circ F = f \circ \phi\}, \]
and the action is given by
\[ L_{(F, f)}^\phi A := FA - Af, \quad \text{for } (F, f) \in \mathfrak{gl}(\phi)_0, \ A \in \mathfrak{gl}(\phi)_1. \] (25)

The Lie bracket on \(\mathfrak{gl}(\phi)_1\) is given by
\[ [A_1, A_2]_\phi := A_1 \phi A_2 - A_2 \phi A_1, \quad A_1, A_2 \in \mathfrak{gl}(\phi)_1. \] (26)

Regarding the maps (21) and (24) as diagonal homomorphisms to \(GL(W \oplus V)\) and \(\mathfrak{gl}(W \oplus V)\), one can deduce the formula for the exponential \(\exp_{GL(\phi)_1} : \mathfrak{gl}(\phi)_1 \longrightarrow GL(\phi)_1\),
\[ \exp_{GL(\phi)_1}(A) = A \sum_{n=0}^{\infty} \frac{(\phi A)^n}{(n+1)!} = \sum_{n=0}^{\infty} \frac{(A\phi)^n}{(n+1)!} A. \] (27)

A representation of a Lie 2-group \(G\) with crossed module \(G \overset{i}{\longrightarrow} H\) on \(W \overset{\phi}{\longrightarrow} V\) is a morphism of Lie 2-groups
\[ \rho : G \longrightarrow GL(\phi). \] (28)

Explicitly, \(\rho\) consists of Lie group representations of \(H\) on \(W\) and on \(V,\)
\[ \rho_0 : H \longrightarrow GL(\phi)_0 \leq GL(W) \times GL(V) : h \longrightarrow (\rho_0^h(h), \rho_0^h(h)), \] (29)

intertwining \(\phi, \) i.e., such that \(\phi \circ \rho_0^h(h) = \rho_0^h(h) \circ \phi\) for all \(h \in H;\) and a Lie group homomorphism
\[ \rho_1 : G \longrightarrow GL(\phi)_1, \] (30)

so that
\[ \rho_1(g_0g_1) = \rho_1(g_0) + \rho_1(g_1) + \rho_1(g_0) \circ \phi \circ \rho_1(g_1), \quad \text{for all } g_0, g_1 \in G. \] (31)

Due to the compatibility with the crossed module structure, the following relations hold for all \(g \in G\) and \(h \in H:\)
\[ \rho_0^h(i(g)) = I + \phi \circ \rho_1(g), \quad \rho_0^h(i(g)) = I + \rho_1(g) \circ \phi, \quad \rho_1(g^h) = \rho_0^h(h)^{-1} \rho_1(g) \rho_0^h(h). \] (32)

A representation of a Lie 2-algebra \(\mathfrak{g}_1\) with crossed module \(\mathfrak{g} \overset{\mu}{\longrightarrow} \mathfrak{h}\) on \(W \overset{\phi}{\longrightarrow} V\) is a morphism of Lie 2-algebras
\[ \rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(\phi). \] (33)
Explicitly, \( \rho \) consists of Lie algebra representations of \( \mathfrak{h} \) on \( W \) and on \( V \),

\[
\rho_0 : \mathfrak{h} \rightarrow \mathfrak{gl}(\phi)_0 \leq \mathfrak{gl}(W) \oplus \mathfrak{gl}(V) : y \rightarrow (\rho_0^1(y), \rho_0^0(y)),
\]

(34)

intertwining \( \phi \), i.e., such that \( \phi \circ \rho_0^0(y) = \rho_0^0(y) \circ \phi \) for all \( y \in \mathfrak{h} \); and a Lie algebra homomorphism

\[
\rho_1 : \mathfrak{g} \rightarrow \mathfrak{gl}(\phi)_1.
\]

(35)

Due to the compatibility with the crossed module structure, the following relations hold for all \( x \in \mathfrak{g} \) and \( y \in \mathfrak{h} \):

\[
\rho_0^0(\mu(x)) = \phi \circ \rho_1(x), \quad \rho_0^1(\mu(x)) = \rho_1(x) \circ \phi, \quad \rho_1(\mathcal{L}_y x) = \rho_0^1(y) \rho_1(x) - \rho_1(x) \rho_0^0(y),
\]

(36)

where \( \mathcal{L} \) is the action of \( \mathfrak{h} \) on \( \mathfrak{g} \).

Clearly, the differential of a Lie 2-group representation \( \rho \) yields a representation \( \dot{\rho} \) of its Lie 2-algebra.

**Example 2.1.** [The adjoint representation] Define the adjoint representation of a Lie 2-algebra on itself by:

\[
\begin{align*}
\text{ad}_1 : & \quad \mathfrak{g} \rightarrow \mathfrak{gl}(\mu) \\
\text{ad}_0^1 : & \quad \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g}) \\
\text{ad}_0^0 : & \quad \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{h})
\end{align*}
\]

\[
\begin{align*}
\text{ad}_1(x)(u) := & \ -\mathcal{L}_u x, \quad x \in \mathfrak{g}, u \in \mathfrak{h} \\
\text{ad}_0^1(y)(v) := & \ \mathcal{L}_y v, \quad y \in \mathfrak{h}, v \in \mathfrak{g} \\
\text{ad}_0^0(y)(u) := & \ [y, u], \quad y, u \in \mathfrak{h}.
\end{align*}
\]

\[
\Box
\]

### 2.1.3 The cochain complexes with coefficients

We define the differentials \( \nabla \) for the complexes in \((33)\). As it was briefly stated, both differentials are defined as a graded sum of the differentials in a three-dimensional grid whose vertices are the spaces in \((7)\) and \((8)\) together with difference maps that account for the fact that not all these commute.

Let \( \mathcal{G} \) be a Lie 2-group \((41)\) with associated crossed module \( G \rightarrow H \). The face maps of the simplicial structure on the nerve of \( \mathcal{G} \) are given by

\[
\partial_k(\gamma_0, ..., \gamma_p) = \begin{cases} 
(\gamma_1, ..., \gamma_p) & \text{if } k = 0 \\
(\gamma_0, ..., \gamma_{k-1} \triangleright \triangleright \gamma_k, \gamma_{k+1}, ..., \gamma_p) & \text{if } 0 < k \leq p \\
(\gamma_0, ..., \gamma_{p-1}) & \text{if } k = p + 1,
\end{cases}
\]

(37)

for a given element \((\gamma_0, ..., \gamma_p) \in \mathcal{G}_{p+1} \). Under the canonical isomorphism \( \mathcal{G} \cong G \times H \) (see Subsection 2.1.1), the space of \( p \)-composable arrows \( \mathcal{G}_p \) corresponds to \( G^p \times H \); hereafter, we consider this isomorphism to be fixed and treat it as an equality when necessary. For each coordinate \( \gamma_j \) of \( \gamma = (\gamma_1, ..., \gamma_p) \in \mathcal{G}_p \leq \mathcal{G}^p \), there is a corresponding \((g_j, h_j) \in G \times H \). According to Eq. \((17)\), the defining relation for \( \mathcal{G}_p \) then reads \( h_j = h_{j+1}(g_{j+1}) \), thus making the map \( \gamma \mapsto (g_1, ..., g_p; h_p) \) an isomorphism with inverse \((g_1, ..., g_p; h) \mapsto (g_1, h(g_1), ..., g_{p-1}, h(g_{p-1})h(g_p)) \). Under this isomorphism, the face maps become

\[
\begin{align*}
\partial_k(g_0, ..., g_p; h) = & \ \begin{cases} 
(g_1, ..., g_p; h) & \text{if } k = 0 \\
(g_0, ..., g_{k-2}, g_kg_{k-1}, g_{k+1}, ..., g_p; h) & \text{if } 0 < k \leq p \\
(g_0, ..., g_{p-1}; h(g_p)) & \text{if } k = p + 1.
\end{cases}
\end{align*}
\]

(38)

Let \( t_p : \mathcal{G}_p \rightarrow H : (\gamma_1, ..., \gamma_p) \mapsto t(\gamma_1) \) be the map that returns the final target of a \( p \)-tuple of composable arrows. We remark that \( t_p \) is a composition of face maps and hence a group homomorphism, and rewrite it as

\[
t_p : G^p \times H \rightarrow H : (g_1, ..., g_p; h) \mapsto h(s(\prod_{j=0}^{p-1} g_{p-j})).
\]

(39)

If \( E \) is a vector bundle over \( H \) and \( \mathcal{G}_s \times H E \rightarrow E : (\gamma, e) \mapsto \Delta_{\gamma} e \) is a left action along the projection, the differential of the complex of Lie groupoid cochains with values on \( E \), \( \partial : C^\bullet(\mathcal{G}; E) \rightarrow C^{\bullet+1}(\mathcal{G}; E) \), where \( C^p(\mathcal{G}; E) := \Gamma(t_p^* E) \), is defined by

\[
(\partial \varphi)(\gamma_0, ..., \gamma_p) = \Delta_{\gamma_0} \partial_0^\varphi(\gamma_0, ..., \gamma_p) + \sum_{k=1}^{p+1} (-1)^k \partial_k^\varphi(\gamma_0, ..., \gamma_p)
\]

(40)
for \( \varphi \in C^p(\mathcal{G}; E) \) and \( (\gamma_0, ..., \gamma_p) \in \mathcal{G}_{p+1} \). Analogously, when \( E \) is a right representation, one uses the initial source map to define \( C^p(\mathcal{G}; E) := \Gamma(s^*E) \), and, keeping the notation for the action, the differential for \( \varphi \in C^p(\mathcal{G}; E) \) and \( (\gamma_0, ..., \gamma_p) \in \mathcal{G}_{p+1} \) is defined to be

\[
(\partial \varphi)(\gamma_0, ..., \gamma_p) = \sum_{k=0}^{p} (-1)^k \partial_k^* \varphi(\gamma_0, ..., \gamma_p) + (-1)^{p+1} \Delta_{\gamma_p} \partial_{p+1}^* \varphi(\gamma_0, ..., \gamma_p).
\]

The differential of the Chevalley-Eilenberg complex of a Lie algebra \( \mathfrak{g} \) with values in a representation \( \rho \) on the vector space \( V \),

\[
\delta_{CE} : \Lambda^* \mathfrak{g}^* \otimes V \longrightarrow \Lambda^{*+1} \mathfrak{g}^* \otimes V,
\]

is defined by

\[
(\delta_{CE}(X))(\gamma) = \sum_{j=0}^{q} (-1)^j \rho(x_j) \omega(X(j)) + \sum_{m<n} (-1)^{m+n} \omega([x_m, x_n], X(m, n))
\]

for \( \omega \in \Lambda^q \mathfrak{g}^* \otimes V \) and \( X = (x_0, ..., x_q) \in \mathfrak{g}^q \). We adopt the convention that, for \( 0 \leq a_1 < ... < a_k \leq q \),

\[
X(a_1, ..., a_k) := (x_0, ..., x_{a_1-1}, x_{a_1+1}, ..., x_{a_k-1}, x_{a_k+1}, ..., x_q),
\]

as opposed to the usual \( \cdot \) notation.

To define the complex of Lie 2-algebra cochains with values on \( W \longrightarrow \phi \longrightarrow V \), fix a Lie 2-algebra \( [\mathfrak{g}, \mathfrak{h}] \) with associated crossed module \( \mathfrak{g} \longrightarrow \mathfrak{h} \) whose action we write \( \mathcal{L} \) and fix a representation \( \rho \) \( \mathfrak{h} \). We think of \( \rho \) as a triple \( (\rho_0^0, \rho_0^1; \rho_1) \), where \( \rho_0 = (\rho_0^0, \rho_0^1) \) is the map \( \mathfrak{h} \mathfrak{g} \) and \( \rho_1 \) is the map \( \mathfrak{g} \mathfrak{h} \). Let \( C^p_q(\mathfrak{g}_1, \phi) \) be given by \( (\cdot) \).

**The p-direction** - For constant \( r \), when \( q = 0 \), one has got the trivial complexes

\[
C^0_r(\mathfrak{g}_1, \phi) \longrightarrow C^1_r(\mathfrak{g}_1, \phi) \longrightarrow \cdots
\]

for \( q > 0 \), we set the complexes

\[
\partial : \Lambda^q \mathfrak{g}^* \otimes V \longrightarrow \Lambda^q \mathfrak{g}^*_{q+1} \otimes V \quad \text{and} \quad \partial : \Lambda^q \mathfrak{g}^* \otimes \Lambda^r \mathfrak{g}^* \otimes W \longrightarrow \Lambda^q \mathfrak{g}^*_{q+1} \otimes \Lambda^r \mathfrak{g}^* \otimes W
\]

to be the subcomplexes of multilinear alternating groupoid cochains of the complex of \( \mathfrak{g}^*_1 \longrightarrow \mathfrak{h}^q \) with values in the trivial representation on \( \mathfrak{h}^q \times V \longrightarrow \mathfrak{h}^q \) and \( \mathfrak{h}^q \times (\Lambda^r \mathfrak{g}^* \otimes W) \longrightarrow \mathfrak{h}^q \) respectively.

**The q-direction** - For constant \( p \) and \( r = 0 \), we set the complex

\[
\delta : \Lambda^q \mathfrak{g}^*_p \otimes V \longrightarrow \Lambda^{q+1} \mathfrak{g}^*_p \otimes V
\]

to be the Chevalley-Eilenberg complex of \( \mathfrak{g}_p \) with values in the pull-back representation \( \rho_p := \mathcal{L}_p^r \rho_0^0 \) on \( V \). For constant \( r > 0 \), we set the complex

\[
\delta : \Lambda^q \mathfrak{g}^*_p \otimes \Lambda^r \mathfrak{g}^* \otimes W \longrightarrow \Lambda^{q+1} \mathfrak{g}^*_p \otimes \Lambda^r \mathfrak{g}^* \otimes W
\]

to be the Chevalley-Eilenberg complex of \( \mathfrak{g}_p \) with values in the pull-back representation \( \rho_p^{(r)} := \mathcal{L}_p^r \rho^{(r)} \) on \( \Lambda^r \mathfrak{g}^* \otimes W \), where \( \rho^{(r)} : \mathfrak{h} \longrightarrow \mathfrak{g}(\Lambda^r \mathfrak{g}^* \otimes W) \) is the dual representation given for \( \omega \in \Lambda^r \mathfrak{g}^* \otimes W \), \( y \in \mathfrak{h} \) and \( z_1, ..., z_r \in \mathfrak{g} \) by

\[
\rho^{(r)}(y)\omega(z_1, ..., z_r) := \rho_0^1(y)\omega(z_1, ..., z_r) - \sum_{k=1}^{r} \omega(z_1, ..., \mathcal{L}_y z_k, ..., z_r).
\]

**The r-direction** - For constant \( p \) and \( q \), we set the complex

\[
\delta_{(1)} : \Lambda^q \mathfrak{g}^*_p \otimes \Lambda^r \mathfrak{g}^* \otimes W \longrightarrow \Lambda^q \mathfrak{g}^*_p \otimes \Lambda^{r+1} \mathfrak{g}^* \otimes W
\]
to be the Chevalley-Eilenberg complex of \( g \) with values in \( \rho_{(1)} : g \rightarrow \mathfrak{gl}(\bigwedge^q g_p^* \otimes W) \) given for \( \omega \in \bigwedge^q g_p^* \otimes W, \Xi \in g_p^* \) and \( z \in g \) by

\[
\rho_{(1)}(z)\omega(\Xi) := \rho_0^1(\mu(z))\omega(\Xi).
\]

(45)

Since the 0th degree is \( \bigwedge^q g_p^* \otimes V \) instead of \( \bigwedge^q g_p^* \otimes W \), we specify the first differential to be

\[
\delta' : \bigwedge^q g_p^* \otimes V \rightarrow \bigwedge^q g_p^* \otimes g^* \otimes W : \delta' \omega(\Xi; z) = \rho_1(z)\omega(\Xi),
\]

(46)

where \( \Xi \in g_p^* \) and \( z \in g \).

**Difference maps** - The 6th difference map

\[
\Delta_k : C^{p,q}_{r-k}(g_1, \phi) \rightarrow C^{p+1,q+k}_{r-k}(g_1, \phi)
\]

is defined for \( \omega \in C^{p,q}_{r-k}(g_1, \phi), Z \in g^{r-k} \) and \( \Xi = (\xi_0, ..., \xi_{q+k-1}) \in g^{q+k+1}_p \) by

\[
(\Delta_k \omega)(\Xi; Z) := \sum_{a_1 < ... < a_k} (-1)^{a_1 + ... + a_k} \omega(\hat{\delta}_0 \Xi (a_1, ..., a_k); x^0_{a_1}, ..., x^0_{a_k}, Z).
\]

(47)

Here, we used the notation convention (42), and identified each \( \xi_j \) with \( j \in \{0, ..., q + k - 1\} \) with \( (x^0_j, ..., x^p_j; g_j) \).

In the special case \( k = r \), the map \( \Delta_r \) is essentially defined by Eq. (17), but composed with \( \phi \), so that it takes values in the right vector space. We sometimes drop the subindex of the first difference map and write \( \Delta \) instead of \( \Delta_1 \).

The three dimensional grid knit by \( \partial, \delta \) and \( \delta_{(1)} \) falls short of defining a triple complex because \( \partial \) and \( \delta \) do not commute. Since all difference maps are homogeneous of degree +1 with respect to the diagonal grading, it makes sense to use them as differentials:

\[
\nabla := \partial + (-1)^{p+q}(\delta + \delta_{(1)}) + \sum_{k=1}^r \Delta_k.
\]

(48)

The higher difference maps in Eq. (43) make up for the non-commuting differentials, so that \( \nabla \) squares to zero [1]. Thus defined, the complex \( (C^{p,q}_{r}(g_1, \phi), \nabla) \) verifies the following property.

**Theorem 2.2.** \( H^*_\nabla(g_1, \phi) \) is in one-to-one correspondence with isomorphism classes of extensions of the Lie 2-algebra \( g_1 \) by the 2-vector space \( W \rightarrow V \).

To define the complex of Lie 2-group cochains with values on \( W \rightarrow V \), fix a Lie 2-group \( G \) with associated crossed module \( G \rightarrow H \) for whose action we use exponential notation and fix a representation \( \rho \) (28). We think of \( \rho \) as a triple \( (\rho_0^1, \rho_0^0, \rho_1) \), where \( \rho_0 = (\rho_0^1, \rho_0^0) \) is the map (20) and \( \rho_1 \) is the map (20). Let \( C^{p,q}_{r}(G, \phi) \) be given by (38).

**The p-direction** - For constant \( r \), when \( q = 0 \), one has got the trivial complexes

\[
C^{p,0}_{r}(G, \phi) \xrightarrow{\partial=0} C^{1,0}_{r}(G, \phi) \xrightarrow{\partial=1d} C^{2,0}_{r}(G, \phi) \xrightarrow{\partial=0} C^{3,0}_{r}(G, \phi) \rightarrow \cdots
\]

When \( q > 0 \) and \( r = 0 \), we set the complex

\[
\partial : C(G^1_p, V) \rightarrow C(G^2_{p+1}, V)
\]

to be the cochain complex of the product groupoid \( G^0 \rightarrow H^q \) with respect to the trivial representation on the vector bundle \( H^q \times V \rightarrow H^q \). For any other value of \( r \), we set the complex

\[
\partial : C(G^2_p \times G^r, W) \rightarrow C(G^3_{p+1} \times G^r, W)
\]

to be the cochain complex of the product groupoid \( G^q \times G^r \rightarrow H^q \times G^r \) with respect to the left representation on the trivial bundle \( H^q \times G^r \rightarrow H^q \times G^r \) given for \( \gamma_k = (g_k, h_k) \in G \) and \( f \in G \) by

\[
(\gamma_1, ..., \gamma_q; f) \cdot (h_1, ..., h_q; f, w) := (h_1 i(g_1), ..., h_q i(g_q); f, \rho_0^1(i(pr_G(\gamma_1 \Xi \cdots \Xi \gamma_q)))^{-1} w).
\]

(49)
The \( q \)-direction - When \( r = 0 \), we set the complex

\[
\delta : C(G^r_p, V) \longrightarrow C(G^{r+1}_p, V)
\]

to be the group complex of \( G_p \) with values in the pull-back of the representation \( \rho_0^1 \) along the final target map \( t_p \); when \( r \neq 0 \), we set the complex

\[
\delta : C(G^r_p \times G^r, W) \longrightarrow C(G^{r+1}_p \times G^r, W)
\]

to be the cochain complex of the (right!) transformation groupoid \( G_p \times G^r \longrightarrow G^r \) with respect to the right representation

\[
(g_1, \ldots, g_r; w) \cdot (\gamma; g_1, \ldots, g_r) := (g_1^{t_0(\gamma)}, \ldots, g_r^{t_0(\gamma)}; \rho_0^1(t_0(\gamma))^{-1}w)
\]

(50)
on the trivial vector bundle \( G^r \times W \longrightarrow G^r \), where \( g_1, \ldots, g_r \in G_p, g \in G_p \) and \( w \in W \). When writing the groupoid differential, we use the shorthand \( \rho_0^1(\gamma; g)w \) instead of the lengthier Eq. (50).

The \( r \)-direction - When \( q = 0 \), we set the complex

\[
\delta_{(1)} : C(G^*, W) \longrightarrow C(G^{*+1}, W)
\]

to be the group complex of \( G \) with values in the pull-back of the representation \( \rho_0^1 \) along the crossed module homomorphism \( i \), but for the 0th degree; when \( q \neq 0 \), we set the complex

\[
\delta_{(1)} : C(G^q_p \times G^*, W) \longrightarrow C(G^q_p \times G^{*+1}, W),
\]

(51)
again except for the 0th degree, to be the cochain complex of the Lie group bundle \( G^q_p \times G \longrightarrow G^q_p \) with respect to the left representation

\[
(\gamma_1, \ldots, \gamma_q; g) \cdot (\gamma_1, \ldots, \gamma_q; w) := (\gamma_1, \ldots, \gamma_q; \rho_0^1(i(\gamma_1^{t_1(\gamma_2)}\cdots t_q(\gamma_q)))w)
\]

(52)
on the trivial vector bundle \( G^q_p \times W \rightarrow G^q_p \), where \( \gamma_1, \ldots, \gamma_q \in G_p, g \in G_p \) and \( w \in W \). Though right and left representations of a Lie group bundle coincide, we emphasize that Eq. (52) be taken as a left representation.

The missing maps \( \delta' : V \longrightarrow C(G, W) \) and \( \delta' : C(G^q_p, V) \longrightarrow C(G^q_p \times G, W) \) are defined respectively for \( v \in V, g \in G, \omega \in C(G^q_p, V) \) and \( \gamma_1, \ldots, \gamma_q \in G_p \) by

\[
(\delta'(v)(g)) := \rho_1^1(v)(g) \quad \text{and} \quad (\delta' \omega)(\gamma_1, \ldots, \gamma_q; g) = \rho_0^1(t_0(\gamma_1)\cdots t_0(\gamma_q))^{-1}\rho_1^1(g)\omega(\gamma_1, \ldots, \gamma_q).
\]

(53)

**Difference maps** - In order to define the first difference maps, we introduce the following notation. We think of an element \( \bar{\gamma} \in G^q_p \) as having components

\[
\bar{\gamma} = \begin{pmatrix}
\gamma_1 \\
\vdots \\
\gamma_q
\end{pmatrix} = \begin{pmatrix}
\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1p} \\
\gamma_{21} & \gamma_{22} & \cdots & \gamma_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{q1} & \gamma_{q2} & \cdots & \gamma_{qp}
\end{pmatrix} = \begin{pmatrix}
g_{11} & g_{12} & \cdots & g_{1p} & h_1 \\
g_{21} & g_{22} & \cdots & g_{2p} & h_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
g_{q1} & g_{q2} & \cdots & g_{qp} & h_q
\end{pmatrix},
\]

(54)
where the last equality is a notation abuse corresponding to the row-wise isomorphism \( G_p \cong G^r \times H \). Here, for each value of \( a \) and \( b \), \( (g_{ab}, h_{ab}) \) is the image of \( \gamma_{ab} \) under the canonical isomorphism \( G \cong G \times H \) (see Subsection 2.1.1).

We proceed to define the difference maps

\[
\Delta_{a,b} : C_{r+1-a,b}(G, \phi) \longrightarrow C_{r+1-a,b}(G, \phi)
\]

for \( (a, b) \in \{(1, 1), (r, 1), (1, r)\} \). If \( a + b = r + 1 \), \( \omega \in C(G^q_p \times G^{a+b-1}_p, W) \) and \( \bar{\gamma} \in G^{q+b}_p \), set

\[
(\Delta_{a,b} \omega)(\bar{\gamma}) := \rho_0^1(t_0(\delta_0^a \gamma_1)\cdots t_0(\delta_0^b \gamma_{q+b})) \circ \phi(\omega(\delta_0^a \delta_0^b \bar{\gamma}; c_{a,b}(\bar{\gamma}))),
\]

(55)
where \( c_{a,b} : G_{p+a}^{q+b} \rightarrow G^r \) are respectively
\[
c_{1,1}(\gamma) := g_{11}, \quad c_{r,1}(\gamma) := (g_{1r}, g_{1r-1}, ..., g_{11}), \quad \text{and} \quad c_{1,r}(\gamma) := (g_{11}^{b_1}, ..., g_{11}^{b_r}, g_{21}^{b_1}, ..., g_{21}^{b_r}). \quad (56)
\]

When \( r > 1 \),
\[
\Delta_{1,1} : C(G_{p}^{q} \times G^r, W) \longrightarrow C(G_{p+1}^{q+1} \times G^{r-1}, W)
\]
is defined for \( \omega \in C(G_{p}^{q} \times G^r, W), f = (f_1, ..., f_{r-1}) \in G^{r-1} \) and \( \gamma \in G_{p+1}^{q+1} \) as in Eq. (55) by
\[
(\Delta_{1,1}(\omega) ; \bar{f}) = \rho_0(i(pr_G(\gamma_{21}z \cdots z\gamma_{(q+1)})))^{-1} \rho_0(i(g_{11}^{h_1}, ..., g_{11}^{h_{(r+1)}}))^{-1} \omega(\partial_0 \delta_0 \gamma; (\bar{f}^{h_{11}}; g_{11})) + \\
+ \sum_{n=1}^{r-1} (-1)^{n+1} \omega(\partial_0 \delta_0 \gamma; c_{2n-1}(\bar{f}; \gamma_{11})) - \omega(\partial_0 \delta_0 \gamma; c_{2n}(\bar{f}; \gamma_{11})),
\]
where \( (\bar{f})^{h_{11}} : = (f_{11}^{h_{11}}, ..., f_{r-1}^{h_{11}}) \) and \( e_{2n-1}, e_{2n} : G^{r-1} \times G^r \rightarrow G^r \) are respectively given by
\[
e_{2n-1}(\bar{f}; \gamma_{11}) := (f_{11}^{h_{11} i(g_{11})}, ..., f_{r-2}^{h_{11} i(g_{11})}, f_{r-1}^{h_{11} i(g_{11})})^{-1} f_{r-1}^{h_{11} i(g_{11})}, \quad (57)
\]
\[
e_{2n}(\bar{f}; \gamma_{11}) := (f_{11}^{h_{11} i(g_{11})}, ..., f_{r-2}^{h_{11} i(g_{11})}, f_{r-1}^{h_{11} i(g_{11})})^{-1} f_{r-1}^{h_{11} i(g_{11})}, \quad (58)
\]
for \( 0 < n < r \). We often drop the subindex of the difference map \( \Delta_{1,1} \) and write \( \Delta \).

In the Lie 2-group case, \( \partial \) and \( \delta \) do not commute either and stand in the way of yielding a triple complex. Nonetheless, the difference maps are homogeneous of degree +1 with respect to the diagonal grading, and make up for the non-commuting differentials \( \delta \) by setting:
\[
\nabla := (-1)^p(\delta_{(1)} + \sum_{a+b>0} (-1)^{(a+1)(r+b+1)} \Delta_{a,b}), \quad (59)
\]
where \( \Delta_{0,0} := \partial, \Delta_{0,1} := \delta, \) and \( \Delta_{a,0} = \Delta_{0,b} = 0 \) whenever \( a, b > 1 \). Thus defined, the complex \( (C_\nabla(G, \phi), \nabla) \) verifies the following property.

**Theorem 2.3.** \( \cong H^2_{\nabla}(G, \phi) \) is in one-to-one correspondence with isomorphism classes of split extensions of the Lie 2-group \( G \) by the 2-vector space \( W \rightarrow V \).

**Remark 1.** As of the writing of this paper, a general formula for \( \Delta_{a,b} \) is still unavailable. In \[2\], there are formulas for several families of difference maps and, in particular, the complex of Lie 2-group cochains with values on a 2-vector is defined up until degree 5. Due to the scope of our application, we just included the necessary maps to define the complex up to degree 2.

### 2.2 A brief excursus on homological algebra

As it was stated in the Introduction, we make use of the van Est theorem written in terms of the mapping cone. Though we could not find Proposition 2.3 as stated in the literature, it follows from standard techniques that can be found in [18].

Given a map \( \Phi : (A^\bullet, d_A) \rightarrow (B^\bullet, d_B) \), the mapping cone of \( \Phi \) is defined to be
\[
C(\Phi) := (A[1] \oplus B, d_\Phi),
\]
where \( d_\Phi = \begin{pmatrix} -d_A & 0 \\ \Phi & d_B \end{pmatrix} \). \( C(\Phi) \) is complex if and only if \( \Phi \) is a map of complexes. We write \( H(\Phi) \) for the cohomology of the mapping cone of \( \Phi \).

**Proposition 2.4.** Let \( \Phi : A^\bullet \rightarrow B^\bullet \) be a map of complexes. The following are equivalent:

i) \( H^n(\Phi) = (0) \) for \( n \leq k \).
ii) The induced map in cohomology

\[ \Phi^n : H^n(A) \longrightarrow H^n(B), \]

is an isomorphism for \( n \leq k \), and it is injective for \( n = k + 1 \).

**Proof.** Clearly, \( C(\Phi) \) fits in an exact sequence

\[ 0 \longrightarrow B \xrightarrow{j} C(\Phi) \xrightarrow{\pi} A[1] \longrightarrow 0, \] (60)

whose associated long exact sequence in cohomology is

\[ 0 \longrightarrow H^{-1}(B) \xrightarrow{0} H^{-1}(\Phi) \xrightarrow{\pi^{-1}} H^{-1}(A[1]) \longrightarrow H^0(B) \xrightarrow{j^n} H^0(\Phi) \xrightarrow{\pi^n} \cdots \]

\[ \cdots \xrightarrow{\pi^0} H^0(A[1]) \longrightarrow H^1(B) \xrightarrow{j^1} H^1(\Phi) \xrightarrow{\pi^1} H^1(A[1]) \longrightarrow \cdots \]

After observing that \( H^n(A[1]) = H^{n+1}(A) \), and that the connecting homomorphism is \( \Phi^* \), the proof is straightforward.

Proposition 2.4 holds for the complexes defined by Eq.’s (18) and (59); however, it will be relevant to us that the cone of \( \Phi \) can be endowed with a triple grading inherited from \( \Phi \). First, suppose \( \Phi : A^{\bullet \bullet} \longrightarrow B^{\bullet \bullet} \) is a map between double complexes, we define the **mapping cone double** \( C^{p,q}(\Phi) = A^{p,q+1} \oplus B^{p,q} \) by setting the \( p \)-th column to be the mapping cone of \( \Phi|_{A_{p\bullet}} \). The product of the horizontal differentials defines a map of complexes between columns, and thus a double complex, if and only if \( \Phi \) is a map of double complexes. \( C^{p,q}(\Phi) \) fits in an exact sequence of double complexes analogous to (60); therefore, the conclusion of Proposition 2.4 holds for its total cohomology \( H_{tot}(\Phi) \) and the total cohomologies of \( A \) and \( B \).

Next, consider a map \( \Phi : C(\Phi) \longrightarrow C(\Phi) \) that respects the triple grading of the complexes \( \Phi \). We define the **mapping cone triple** \( C^{p,q}(\Phi) = C^{p,q+1}(\Phi) \oplus C^{p,q}(\Phi_1, \phi) \) by setting, for constant \( p \), the \( p \)-th page to be the mapping cone double of the map \( \Phi_p \) \( \text{[13]} \) and formally summing the remaining differentials and difference maps. The total differential \( \nabla_\phi \) squares to zero if and only if \( \Phi \) is a map of complexes. \( C^{p,q}(\Phi) \) fits in an exact sequence analogous to (60); therefore, the conclusion of Proposition 2.4 holds for its total cohomology \( H_{\nabla}(\Phi) \) and the cohomologies of \( \Phi \) and \( \Phi_1 \).

We point out that the cohomology of the mapping cone triple of \( \Phi \) tautologically coincides with the total cohomology of

\[ \cdots \]
\[ C^1_{tot}(\Phi_0) \longrightarrow C^2_{tot}(\Phi_1) \longrightarrow C^3_{tot}(\Phi_2) \longrightarrow \cdots \]
\[ C^1_{tot}(\Phi_0) \longrightarrow C^2_{tot}(\Phi_1) \longrightarrow C^3_{tot}(\Phi_2) \longrightarrow \cdots \]
\[ C^1_{tot}(\Phi_0) \longrightarrow C^2_{tot}(\Phi_1) \longrightarrow C^3_{tot}(\Phi_2) \longrightarrow \cdots \]
\[ \cdots \]

each of whose columns is the total complex of the mapping cone double of \( \Phi_p \) \( \text{[13]} \) and where the maps \( C^q_{tot}(\Phi_p) \longrightarrow C^{q+1-a}_{tot}(\Phi_{p+a}) \) - exemplified respectively for \( a \in \{1, 2, 3\} \) by the horizontal, dashed and pointed arrows in \( \text{[61]} \) - correspond to the sum \( \sum_{b=0}^a \Delta_{a,b} \). One may filter \( \text{[61]} \) by columns giving rise to a spectral sequence whose first page is \( E_1^{p,q} = H_{tot}(\Phi_p) \).

We close this section with the following lemma, which is much used below. It follows from a simple application of spectral sequences (see, e.g., \( \text{[18]} \)).
Lemma 2.5. Let \((E^p,q, d_r)\) be the spectral sequence of a double graded object \(C^{*, *}\) that can be filtered by columns. If there is a page for which \(E^p,q\) is zero for all \((p, q)\) satisfying \(p + q \leq k\), then

\[ H^n_{\text{tot}}(C) = (0) \quad \text{for} \ n \leq k. \]

\[ \square \]

3 The 2-van Est Map

In this section, we define the van Est map and prove that it defines a map of complexes. Throughout, let \(G\) be a Lie 2-group with Lie 2-algebra \(g\) and let \(\rho\) be a representation on \(W \overset{\phi}{\longrightarrow} V\). Define the van Est map

\[
\Phi : C^p,q(G, \rho) \longrightarrow C^p,q(g_1, \rho), \quad (\Phi \omega)(\xi_1, ..., \xi_q; z_1, ..., z_r) := \sum_{\sigma \in S_q} \sum_{\phi \in S_r} |\sigma||\rho| \tilde{R}_{\sigma(\xi)} \tilde{R}_{\phi(\omega)},
\]

where \(\Xi = (\xi_1, ..., \xi_q) \in g_0^q\), \(Z = (z_1, ..., z_r) \in g_r\), \(|\cdot|\) stands for the sign of the permutation, and

\[
\tilde{R}_{\sigma(\xi)} \tilde{R}_{\phi(\omega)}(\gamma) := \frac{d}{d\tau_r} \cdot \cdot \cdot \frac{d}{d\tau_1} \cdot \cdot \cdot \frac{d}{d\tau_1} \omega(\gamma; \exp(\tau_1 z_{\phi(1)}), ..., \exp(\tau_r z_{\phi(r)})), \quad \text{for} \ \gamma \in G_0^q;
\]

\[
\tilde{R}_{\sigma(\xi)} \tilde{R}_{\phi(\omega)} \tilde{R}_{\sigma(\xi)}(\omega) = \frac{d}{d\lambda_q} \cdot \cdot \cdot \frac{d}{d\lambda_0} (\tilde{R}_\rho(\omega)(\exp(\lambda_1 \xi_{\sigma(1)}), ..., \exp(\lambda_q \xi_{\sigma(q)}))).
\]

Regarding each of these \(\tilde{R}\) operators as compositions of independent \(R_x\) operators that lower degree by differentiating a single entry in the direction of the right-invariant vector field of \(\bullet\) and evaluating at the identity, it is clear that \(\Phi \omega \in C^p,q(g_1, \rho)\) and \(\Phi\) is thus well-defined. Observe that, in the page \(r = 0\), the formula

\[
\Phi = \sum_{\rho \in \rho} \left( \omega_0 \rho \right) \left( \exp \left( \sum_{\rho \in \rho} \rho \right) \right),
\]

and, clearly, \(\Phi = \Phi \omega\). Hence, \(\Phi \omega\) is a 2-cocycle, in particular, so are \(\Phi \omega\) and \(\Phi \omega\).

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We show that, under the correspondences of Theorems 2.2 and 2.3, \(\Phi\) is the map that takes in a Lie 2-group extension (or an isomorphism class thereof) and returns its Lie 2-algebra. Let \(G = (\varphi, \omega_0, \alpha, \omega_1) \in C^1 \times (G, \phi) \oplus C^0 \times (\exp(\lambda_1 \xi_{\sigma(1)}), ..., \exp(\lambda_q \xi_{\sigma(q)}))).

In the course of proving Theorem 2.3, one learns that \(\omega\) is a 2-cocycle if and only if

\[
G_{\rho^x_1} \times \omega_1 ? \overset{\Phi}{\longrightarrow} G_{\rho^x_0} \times \omega_0 V : (g, w) \longrightarrow (i(g), \phi(w) + \varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix})
\]

with action

\[
(g, w)^{(h, v)} = (g^h, \rho^v_1(h)^{-1}(w + \rho_1(g)v) + \alpha; g) \quad \text{for} \ (g, w) \in G \times W, (h, v) \in H \times V,
\]

is a crossed module. Here, we use the isomorphism of Subsection 2.1.1 to cast \(\varphi\) as a function of \(G \times H\). Also, \(X_\rho \times \omega_1 A\) stands for the twisted semi-direct product with respect to the representation \(\rho\) of \(X\) on \(A\) defined by \(\omega \in C^2_X(\rho, \omega_1)\), which is a Lie group if and only if \(\omega\) is a Lie group 2-cocycle.

If \(\omega\) is a 2-cocycle, in particular, so are \(\omega_0\) and \(\omega_1\). Hence, \(\Phi \omega_0\) and \(\Phi \omega_1\) are Lie algebra 2-cocycles respectively defining the Lie algebras of \(H^0 \times \omega_0\) \(V\) and \(G^0 \times \omega_1\) \(W\) as the twisted semi-direct sums \(h^0 \oplus \Phi \omega_0\) \(V\) and \(\rho^x_1 \oplus \Phi \omega_1\) \(W\). Differentiating \(\Phi\) at the identity yields

\[
\Phi \omega_0 \oplus \Phi \omega_1 \omega \longrightarrow \Phi \omega_0 \oplus \Phi \omega_1 \omega : (x, w) \longrightarrow (\mu(x), \phi(w) + d(1,1)\varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix})
\]

and, clearly, \(d(1,1)\varphi = \Phi \varphi\). As for the action, each \((h, v) \in H \times V\) defines a Lie group automorphism

\[
(\_)^{(h, v)} : G^0 \times \omega_1 W \longrightarrow G^0 \times \omega_1 W.
\]

Differentiating \(\Phi\) at the identity yields a Lie algebra automorphism, for which we use the same notation

\[
(-)^{(h, v)} : \rho^x_1 \omega_1 \longrightarrow \rho^x_1 \omega_1.
\]
One way of regarding the category of vector spaces as a subcategory of 2-vector spaces is realizing a vector
3.1 we deal with the case where the representation takes values on an honest vector space; then, we move on to the
complex of which is \( (0) \rightarrow \) 

\[ L_{(y, v)}(x, w) = \left( L_x y, \rho_0(y) w - \rho_1(x) v + \frac{d}{d \lambda} \bigg|_{\lambda=0} \alpha(\exp(\lambda y); \exp(\tau x)) \right) \]

for \((y, v) \in h \oplus V \) and \((x, w) \in g \oplus W \). Together with this action, \( (67) \) is naturally seen as an extension of \( g_1 \) by
\( W \) whose classifying 2-cocycle under Theorem 2.2 is \( (\Phi \phi, \Phi \Theta, \Phi \Omega) \in C^2_T(g_1, \phi) \).

In the remainder of this section, we prove that \( \Phi \) commutes with all differentials and difference maps defining
\( \nabla \), thus defining a map of complexes. To illustrate the van Est strategy, we consider two cases separately: First, we deal with the case where the representation takes values on an honest vector space; then, we move on to the
general case.

### 3.1 Vector space coefficients

One way of regarding the category of vector spaces as a subcategory of 2-vector spaces is realizing a vector
space \( V \) as the groupoid that has \( s \rightarrow \) 

\[ (\partial \xi, \partial \varphi, \partial \phi) \]

the left and right squares commute because they lie in their corresponding double complexes. Due to the
observation that \( (62) \) coincides with the classic Van Est map, the front and back squares commute as well. We are
left to prove the following lemma.

**Lemma 3.1.** In \( (67) \), \( \Phi \circ \partial = \partial \circ \Phi \). \( \square \)

**Proof.** Each face map \( \partial_k \) in a Lie 2-group is a homomorphism whose derivative is the face map \( \hat{\partial}_k \) in its Lie
2-algebra; therefore, if \( \xi \in g_{p+1} \),

\[ \partial_k(\exp_{\varphi_{p+1}}(\xi)) = \exp_{\varphi_{p}}(\hat{\partial}_k \xi). \]

Let \( \omega \in C(G^q, V) \) and \( \Xi = (\xi_1, \ldots, \xi_q) \in g_{p+1}^q \). Then

\[ \overrightarrow{R}_{\Xi}(\partial \omega) \]

\[ = \sum_{k=0}^{p+1} (-1)^k \omega(\exp_{\varphi_{p+1}}(\lambda_k \xi_1), \ldots, \exp_{\varphi_{p+1}}(\lambda_q \xi_q)) \]

[61x266]V

\[ = \sum_{k=0}^{p+1} (-1)^k \omega(\exp_{\varphi_{p}}(\lambda_k \hat{\partial}_1 \xi_1), \ldots, \exp_{\varphi_{p}}(\lambda_q \hat{\partial}_q \xi_q)) = \sum_{k=0}^{p+1} (-1)^k \overrightarrow{R}_{\partial_k \Xi} \omega, \]
and
\[ \Phi(\partial \omega)(\Xi) = \sum_{\sigma \in S_q} |\sigma| R_{\sigma}(\partial \omega) = \sum_{\sigma \in S_q} |\sigma| (\sum_{k=0}^{p+1} (-1)^k R_{\partial,\sigma}(\omega) = \sum_{k=0}^{p+1} (-1)^k (\Phi(\partial \omega)(\partial_k \Xi) = \partial(\Phi \omega)(\Xi). \]

In this case, we have the following van Est type theorem.

**Theorem 3.2.** Let \( G \) be a Lie 2-group with associated crossed module \( G \rightarrow H \), Lie 2-algebra \( g_1 \) and a representation on the 2-vector space \( V \). If \( H \) is \( k \)-connected, \( G \) is \((k-1)\)-connected and \( \Phi \) is the van Est map \( \Phi \), then the total cohomology of its mapping cone double vanishes:
\[ H^n_{\text{tot}}(\Phi) = 0, \quad \text{for all degrees } n \leq k. \]

**Proof.** The first page of the spectral sequence of the filtration by columns of \( C^{•, •}(\Phi) \) is \( E_1 \). Recall that \( G_\mu \cong G^p \times H \); hence, using the Künneth formula and the connectedness hypotheses for \( H \) and \( G \), it follows that \( G_\mu \) is \((k-1)\)-connected. Theorem \( \ref{thm:van_est} \) implies all columns in \( E_1^{p,q} \) vanish below \( k \), and so the result follows from Lemma \( \ref{lemma:van_est} \).

Rephrasing with Proposition \( \ref{prop:van_est} \)

**Corollary 3.3.** Under the hypotheses of Theorem \( \ref{thm:van_est} \) the van Est map \( \Phi \) induces isomorphisms
\[ \Phi : H^n_C(G, V) \cong H^n_V(g_1, V), \]
for \( n \leq k \), and it is injective for \( n = k + 1 \).

As an application, we prove the following partial integrability result.

**Theorem 3.4.** Let \( g_1 \) be a Lie 2-algebra with associated crossed module \( g \rightarrow \mathfrak{h} \). If
\[ g \cap c(\hat{u}(\mathfrak{h})) = 0, \]
where \( c(\hat{u}(\mathfrak{h})) \) is the centralizer of \( \hat{u}(\mathfrak{h}) \) in \( g_1 \), then \( g_1 \) is integrable.

**Proof.** We are to use the van Est strategy. Consider the exact sequence
\[ 0 \rightarrow \ker(\text{ad}_1)^C \rightarrow g \rightarrow g_1 \rightarrow 0 \]
\[ 0 \rightarrow \ker(\text{ad}_0)^C \rightarrow \mathfrak{h} \rightarrow \mathfrak{h}_0 \rightarrow 0, \]
associated to the adjoint representation of Example \( \ref{ex:adjoint} \). If \( x \in g \), from Eq. \( \ref{eq:intersection} \), there exists a \( y_x \in \mathfrak{h} \) such that \( [x, \hat{u}(y_x)]_1 = \text{ad}_1(x)(y_x) \neq 0 \). Consequently, \( \text{ad}_1(x) \neq 0 \in \text{Hom}(\mathfrak{h}, g) \) for every \( x \in g \), and \( \ker(\text{ad}_1) = 0 \). Let \( [\omega] \in H^2_C(\mathfrak{g}_1, \ker(\text{ad}_0)) \) be the class corresponding to \( \omega \) under Theorem \( \ref{thm:van_est} \).

The image of any linear functor between 2-vector spaces yields a Lie subgroupoid, and Lie 2-subalgebras of \( g\mathfrak{l}(\phi) \) can be integrated using exponentials \( \ref{ex:exponential} \); hence, \( \text{ad}_1(g) \rightarrow \text{ad}_0(\mathfrak{h}) \) is integrable to a Lie 2-group \( G \) with associated crossed module \( G \rightarrow H \). Picking \( G \) and \( H \) 1-connected, we may use Corollary \( \ref{cor:van_est} \) to conclude
\[ [\omega] = \Phi[\int \omega], \quad \text{for a unique } [\omega] \in H^2_C(G, \ker(\text{ad}_0)). \]

The extension of \( G \) by \( \ker(\text{ad}_0) \) that corresponds to \( [\omega] \) under Theorem \( \ref{thm:van_est} \) integrates \( g_1 \).

**Remark 2.** One could cast Eq. \( \ref{eq:intersection} \) in terms of the isotropy Lie algebras of the action of \( \mathfrak{h} \) on \( g \) by asking equivalently that \( \dim\{y \in \mathfrak{h} : L_y x = 0\} > 0 \) for all \( x \in g \).
3.2 The general case

We devote the remainder of this section to prove that, when values are taken on a 2-vector space $W \xrightarrow{\phi} V$ with $W \neq (0)$, the van Est map $\Phi$ still defines a map of complexes. In particular, we show that $\Phi$ commutes with all differentials in Subsection 2.1.3 and all difference maps necessary to define the complexes (9) up to degree 2. The proofs we present boil down to long and unfortunately unenlightening computations.

For notational convenience, we adopt the following shorthands. For an index set $I = \{1, \ldots, n\}$,

$$
\frac{d^I}{d\tau^I} \bigg|_{\tau = 0} := \frac{d}{d\tau_1} \bigg|_{\tau_2 = 0} \cdots \frac{d}{d\tau_n} \bigg|_{\tau_n = 0}.
$$

On the other hand, for any Lie algebra $\mathfrak{g}$, if $X = (x_1, \ldots, x_n) \in \mathfrak{g}^n$, we define

$$
\exp(\tau I \cdot X) := (\exp[\tau_1 x_1], \ldots, \exp[\tau_n x_n]) \in G^n,
$$

where $G$ is a Lie group integrating $\mathfrak{g}$.

We also use the following partitions of the symmetric group $S_n$. First,

$$
S_n = \bigcup_{a=1}^n S_{n-1}(m|a),
$$

where $S_{n-1}(m|a)$ is the set of permutations that fix the $m$th element to be $a$; in symbols, $S_{n-1}(m|a) := \{\sigma \in S_n : \sigma(m) = a\}$. Each element $\sigma \in S_{n-1}(m|a)$ can be factored as $\sigma = \sigma' \circ \sigma^m_a$ where,

$$
\sigma^m_a(j) := \begin{cases} a & \text{if } j = m \\ j - 1 & \text{if } m < j \leq a \\ j + 1 & \text{if } a \leq j < m \\ j & \text{otherwise.} \end{cases}
$$

The residual permutation $\sigma'$ leaves the $m$th element alone and shifts the remaining $n - 1$ elements; thus, one can regard $\sigma'$ as belonging to the permutation group $S_{n-1}$, incidentally justifying the notation. Since $\sigma^m_a$ is a composition of $|m - a|$ transpositions, $|\sigma| = |\sigma'||\sigma^m_a| = (-1)^{m-a}|\sigma'|$.

Iterating this process, for any $k < n$ and $a_0 < \ldots < a_{k-1}$, one can partition the symmetric group as

$$
S_n = \bigcup_{a_0 < \ldots < a_{k-1} \in S_k} \bigcup_{a \in S_k} S_{n-k}(m|a_{(0)} \ldots a_{(k-1)}),
$$

where $S_{n-k}(m|a_0 \ldots a_{k-1}) := \{\sigma \in S_n : \sigma(m + j) = a_j, 0 \leq j < k\}$. Each element $\sigma \in S_{n-k}(m|a_{(0)} \ldots a_{(k-1)})$ can be factored as $\sigma = \sigma^{(r)} \circ \sigma^m_{a_0 \ldots a_{k-1}} \circ \varphi$, where $\varphi$ is interpreted to act only on $\{a_0, \ldots, a_{k-1}\}$ and $\sigma^m_{a_0 \ldots a_{k-1}} = \sigma_{a_0 \ldots a_{k-1}} \circ \cdots \circ \sigma^m_1 \circ \sigma^m_a$. Again, the residual permutation $\sigma^{(r)}$ leaves fixed the $k$ elements following $m$ and shifts the remaining $n - k$ elements, thus allowing it to be regarded as belonging to $S_{n-k}$. The sign is computed to be

$$
|\sigma| = |\sigma^{(r)}||\sigma^m_{a_0 \ldots a_{k-1}}||\varphi| = (-1)^{km + (k-1)(k-2)/2 - (a_0 + \ldots + a_{k-1})}|\sigma^{(r)}||\varphi|.
$$

The r-direction - The following results prove that the van Est map $\Phi$ commutes with the differentials in the $r$-direction.

**Lemma 3.5.** Let $\omega \in C^p_0(G, \phi)$, then

$$
\Phi(\delta' \omega) = \delta'(\Phi \omega) \in C^p_1(\mathfrak{g}, \phi).
$$

**Proof.** For $q = 0$, $\omega = v \in V$ and $z \in \mathfrak{g}$,

$$
\Phi(\delta' v)(z) = \frac{d}{d\tau} \bigg|_{\tau=0} (\delta'(\exp(\tau z))) = \frac{d}{d\tau} \bigg|_{\tau=0} \rho_1(\exp(\tau z))v = \dot{\rho}_1(z)v = (\delta' v)(z),
$$

and $\Phi$ is defined to be the identity when $(q, r) = (0, 0)$.
where $t_p$ is a composition of face maps and so its derivative $\dot{t}_p$; therefore, it is a group homomorphism and, if $\xi \in \mathfrak{g}_p$,

$$t_p(\exp_{\mathfrak{g}_p}(\xi)) = \exp_H(\dot{t}_p(\xi)).$$

(71)

For $q > 0$, let $\Xi = (\xi_1, ..., \xi_q) \in \mathfrak{g}^q_p$, then setting $h_j := \prod_{k=j}^q \exp(\lambda_k \dot{t}_p(\xi_k)) = \exp(\lambda_j \dot{t}_p(\xi_j)) ... \exp(\lambda_q \dot{t}_p(\xi_q)) \in H$, we compute

$$\overline{R}_\Xi \epsilon \dot{R}_\epsilon (\delta' \omega) = \frac{d\epsilon}{d\lambda} \bigg|_{\lambda = 0} \frac{d\delta'}{d\tau} \bigg|_{\tau = 0} \left( \delta' (\exp(\lambda_1 \cdot \epsilon); \exp(\tau_x)) = \frac{d\epsilon}{d\lambda} \bigg|_{\lambda = 0} \rho_0(\dot{h}_1)^{-1} \dot{\rho}_1(z) \omega (\exp(\lambda_1 \cdot \epsilon)) \right) = \rho_0(\dot{h}_2)^{-1} \dot{\rho}_1(z) \epsilon (\exp(\lambda_2 \xi_2), ..., \exp(\lambda_q \xi_q));$$

inductively implying $\overline{R}_\Xi \epsilon \dot{R}_\epsilon (\delta' \omega) = \dot{\rho}_1(z)(\overline{R}_\Xi \omega)$. Thus,

$$\Phi(\delta' \omega)(\Xi; z) = \sum_{\sigma \in S_q} \sigma |\overline{R}_\sigma(\Xi) \epsilon \dot{R}_\epsilon (\delta' \omega) = \sum_{\sigma \in S_q} \sigma |\dot{\rho}_1(z)(\overline{R}_\sigma(\Xi) \omega) = \dot{\rho}_1(z)((\Phi(\omega))(\Xi)) = \delta'(\Phi(\omega; \Xi; z)).$$

Lemma 3.6. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $V$ be a vector space. If $\omega \in C(G^p, V)$, $X = (x_0, ..., x_p) \in \mathfrak{g}^{p+1}$, and $\partial_k$ is the $k$th face map of $\mathfrak{g}$ with $0 < k < p + 1$, then for all $m < n$,

$$\sum_{\sigma \in S_{p-1}(k-1) \cap S_{q-1}(k-1) \cap S_{n-1}} |\sigma| \overline{R}_{(X^k_m) | \epsilon \dot{R}_\epsilon (\delta' \omega) = (-1)^{m+n} \sum_{\sigma' \in S_{p-1}} |\sigma'| \overline{R}_{(X^k_m) | \epsilon \dot{R}_\epsilon (\delta' \omega),}$$

where, we interpret $S_{p-1}$ as the space of bijections between the set of coordinates of $X(k-1, k)$ and the set of coordinates of $X(m, n)$, and $q(X^{(k)}_{nm}) := (q(x_0), ..., q(x_{k-2}), [x_m, x_n], q(x_{k+1}), ..., q(x_p))$.

Proof. Let $\sigma \in S_{p-1}(k-1 \cap mn)$, then $\sigma = \sigma'' \circ \sigma^{-1}$ and there exists a unique $\sigma \in S_{p-1}(k-1 \cap mn)$ given by $\sigma = \sigma'' \circ \sigma^{-1}$. By definition, if $\varphi \in C(G, V)$,

$$R(\epsilon x_n \varphi) = \frac{d\epsilon}{d\tau} \bigg|_{\tau = 0} \frac{d\varphi(\exp(\tau_1 x_n) \exp(\tau_2 x_m)) - \varphi(\exp(\tau_1 x_n) \exp(\tau_2 x_n))}{\tau_1 = 0}$$

hence, regarding $\sigma''$ as belonging to $S_{p-1}$,

$$|\sigma'| \overline{R}_{(X^k_m) | \epsilon \dot{R}_\epsilon (\delta' \omega) + |\sigma'| \overline{R}_{(X^k_m) | \epsilon \dot{R}_\epsilon (\delta' \omega) = (-1)^{m+n}|\sigma''| \overline{R}_{\sigma''(X^k_m) \omega).}$$

Proposition 3.7. Let $r > 0$ and $\omega \in C^p_q(G, \phi)$, then

$$\Phi(\delta(1) \omega) = \delta(1)(\Phi(\omega)) \in C^p_q(\mathfrak{g}_1, \phi).$$

Proof. Let $\Xi = (\xi_1, ..., \xi_q) \in \mathfrak{g}^q_p$ and $Z = (z_0, ..., z_r) \in \mathfrak{g}^{r+1}$, $\overline{R}_\Xi \overline{R}_\epsilon (\delta(1) \omega)$ has got three types of terms corresponding respectively to the dual face maps $\delta^*_{\epsilon}$ for $1 \leq k \leq r$ and $\delta^*_{r+1}$ in $\delta(1) \omega$:

$I := \frac{d\epsilon}{d\lambda} \bigg|_{\lambda = 0} \frac{d\epsilon}{d\tau} \bigg|_{\tau = 0} \rho_0(i(\exp(\tau_0 z_0)) \Pi_{\tau_1 \epsilon}(\exp(\lambda_1 \xi_1))) \omega(\exp(\lambda_1 \cdot \Xi); \exp(\tau_1 z_1), ..., \exp(\tau_r z_r))$

$II := \frac{d\epsilon}{d\lambda} \bigg|_{\lambda = 0} \frac{d\epsilon}{d\tau} \bigg|_{\tau = 0} \omega(\exp(\lambda_1 \cdot \Xi); \exp(\tau_0 z_0), ..., \exp(\tau_k z_k) \exp(\tau_{k+1} z_{k+1}), ..., \exp(\tau_r z_r))$

$III := \frac{d\epsilon}{d\lambda} \bigg|_{\lambda = 0} \frac{d\epsilon}{d\tau} \bigg|_{\tau = 0} \omega(\exp(\lambda_1 \cdot \Xi); \exp(\tau_0 z_0), ..., \exp(\tau_{r-1} z_{r-1})).$

Type III terms are constant with respect to $\tau_r$ and thus vanish.

For any fixed $0 \leq k \leq r$, partition $S_{r+1} = \bigcup_{m < n} (S_{r+1}(k-1 \cap mn) \cup S_{r-1}(k-1 \cap mn))$ and use Lemma 3.6 to conclude

$$\sum_{\rho \in S_{r+1}} |\rho| \overline{R}_{\epsilon \rho(\Xi) \epsilon \dot{R}_\epsilon (\delta(1) \omega) = \sum_{r \in S_{r-1}} (-1)^{m+n} \sum_{\rho' \in S_{r-1}} |\rho'| \overline{R}_{\rho'(\rho(\Xi) \epsilon \dot{R}_\epsilon (\delta(1) \omega).}$$
For each pair \( m < n \), using \( S_r = \bigcup_{k=1}^r S_{r-1}(0|k) \),
\[
\sum_{q \in S_r} |q| \tilde{R}_{q,z}([z_\alpha,z_\beta], Z(m,n)) \omega = \sum_{k=1}^r \sum_{q' \in S_{r-1}} (1) \cdot |q'| \tilde{R}_{q',(z_{k+1}^n)} \omega;
\]
thus, summing all type II terms yields
\[
\sum_{\sigma \in S_q} \sum_{y \in S_{q+1}} \sum_{k=1}^r \sum_{j=1}^r (-1)^k |\sigma||\bar{q}| \tilde{R}_{\sigma(\Xi)} \tilde{R}_{q(z)}(\delta_k \omega) = \sum_{m<n} (1)^{m+n}(\Phi \omega)(\Xi;[z_m,z_n], Z(m,n)).
\]
Concludingly, recall that for \( y \in \mathfrak{g} \) and \( x \in \mathfrak{g} \),
\[
\frac{d}{d\lambda} \bigg|_{\lambda=0} \frac{d}{dt} \bigg|_{t=0} \rho_0^1(i(exp(\tau x)exp(\lambda y))) = \frac{d}{d\lambda} \bigg|_{\lambda=0} \rho_0^1(\mu(exp(\lambda y))) = -\rho_0^1(\mu(\mathcal{L}_y x));
\]
hence, setting \( h_j := \prod_{k=1}^r \exp(\lambda_k t_p(\xi_k)) = \exp(\lambda_j t_p(\xi_j))...\exp(\lambda_q t_p(\xi_q)) \in H \), we compute
\[
\frac{d}{d\lambda} \bigg|_{\lambda=0} \rho_0^1(\mu(z_0 \lambda^h)) \tilde{R}_{\Xi} \tilde{R}_{Z(\lambda)}(\omega)(\exp(\lambda t \cdot \Xi)) = \rho_0^1(\mu(\exp(0)h)) R_{\xi}(\tilde{R}_{\Xi} \tilde{R}_{Z(0)}(\omega)(\exp(\lambda t \xi_\lambda),...)) + \rho_0^1(\mu(\exp(0)h)) R_{\xi}(\tilde{R}_{\Xi} \tilde{R}_{Z(0)}(\omega)(\exp(\lambda t \xi_\lambda),...)) \notin \rho_0^1(\mu(\exp(0)h)) R_{\xi}(\tilde{R}_{\Xi} \tilde{R}_{Z(0)}(\omega)) \exp(\lambda t \xi_\lambda),...)
\]
and inductively, \( I = \rho_0^1(\mu(z_0 \lambda^h)) \tilde{R}_{\Xi} \tilde{R}_{Z(\lambda)}(\omega). \) Using the partition of \( S_{r+1} \) by \( S_r(0|k) \)'s and summing type I terms yields
\[
\sum_{\sigma \in S_q} \sum_{y \in S_{q+1}} \sum_{k=1}^r \sum_{j=1}^r (-1)^k |\sigma||\bar{q}| \tilde{R}_{\sigma(\Xi)} \tilde{R}_{q(z)}(\delta_k \omega) = \sum_{m<n} (1)^{m+n}(\Phi \omega)(\Xi;[z_m,z_n], Z(k)),
\]
and the result follows.

**The q-direction** - The following results prove that the van Est map \( \Phi \) commutes with the differentials in the q-direction. Since the differentials in the r-direction do commute with differentials in the q-direction, we prove that for constant p, \( \Phi \) yields a map of double complexes that we refer to as p-pages.

**Lemma 3.8.** Let \( T : V \times \ldots \times V \rightarrow W \) be an r-multilinear map. If \( H_\lambda \) is a differentiable path of automorphisms of \( V \) with \( H_0 = Id_V \), then
\[
\frac{d}{d\lambda} \bigg|_{\lambda=0} T(H_\lambda(v_1), \ldots, H_\lambda(v_r)) = \sum_{k=1}^r T(v_1, \ldots, v_{k-1}, \frac{d}{d\lambda} \bigg|_{\lambda=0} H_\lambda(v_k), v_{k+1}, \ldots, v_r)
\]

**Proof.** Let \( \{e_i\}_{i=1}^n \) be a basis for \( V \). In these coordinates, \( R_{a_1 \ldots a_r} := R(e_{a_1}, \ldots, e_{a_r}) \) and \( H_\lambda(e_a) = H_\lambda^b(\lambda)e_b. \) Since \( H_0 = Id_V, H_\lambda^b(0) = \delta_0^b \), on basic elements,
\[
\frac{d}{d\lambda} \bigg|_{\lambda=0} R(H_\lambda(e_{a_1}), \ldots, H_\lambda(e_{a_r})) = \frac{d}{d\lambda} \bigg|_{\lambda=0} R(H_\lambda^b(\lambda)e_{a_1}, \ldots, H_\lambda^b(\lambda)e_{a_r}) = \frac{d}{d\lambda} \bigg|_{\lambda=0} H_\lambda^b(\lambda) R_{a_1 \ldots a_r} = \sum_{k=1}^r H_\lambda^b(\lambda) R_{a_1 \ldots a_k} \tilde{H}_{a_k}^b(0)
\]
as desired.
Proposition 3.9. Let $r > 0$ and $\omega \in C^p_r(G, \phi)$, then

$$\Phi(\delta \omega) = \delta(\Phi \omega) \in C^p_{r+1}(g_1, \phi).$$

\[\square\]

Proof. Let $\Xi = (\zeta_0, \ldots, \zeta_q) \in g^+_p$ and $Z = (z_1, \ldots, z_r) \in g^r \backslash g$. The result follows.

For constant $q$, the following results prove that the van Est map $\Phi$ (62) commutes with the differentials in the $p$-direction. Since the differentials in the $r$-direction do commute with differentials in the $p$-direction, we prove that for constant $q$, $\Phi$ yields a map of double complexes that we refer to as $q$-pages.

Theorem 3.10. For constant $p$, $\Phi$ restricts to a map $\Phi_p$ (13) of double complexes.\[\square\]
Proposition 3.11. Let \( r > 0 \) and \( \omega \in C^{p,q}_r(G, \phi) \), then
\[
\Phi(\partial \omega) = \partial(\Phi \omega) \in C^{p+1,q}_r(g_1, \phi).
\]

\[\square\]

Proof. Let \( \Xi = (\xi_1, \ldots, \xi_q) \in g^q_{p+1} \) and \( Z \in g^r \). Writting \( \exp_{g_{p+1}}(\xi) = (\xi^1, \ldots, \xi^q) \in g_{p+1} \leq g^{p+1} \), for \( \xi \in g_{p+1} \), define
\[
g_j := pr_G(\lambda^j_1 \xi^1 \lambda^{j+1}_1 \xi^{j+1} \ldots \lambda_{q-j} \xi_{q-j} \lambda^j_1) \in G, \quad \text{for } 1 \leq j \leq q.
\]

Using Eq. (68),
\[
\overrightarrow{R}_Z \overrightarrow{R}_Z(\partial \omega) = \frac{d}{d\lambda_1} \Bigg|_{\lambda_1 = 0} \left[ \rho_0^{-1}(\overrightarrow{R}_Z \omega)(\partial_0 \exp(\lambda_1 \Xi)) + \sum_{j=1}^{p+1} (-1)^j (\overrightarrow{R}_Z \omega)(\partial_j \exp(\lambda_1 \Xi)) \right]
\]
\[
= \sum_{j=1}^{p+1} (-1)^j \overrightarrow{R}_{\partial_j \Xi} \overrightarrow{R}_Z \omega + \frac{d}{d\lambda_1} \Bigg|_{\lambda_1 = 0} \left[ \rho_0^{-1}(\overrightarrow{R}_Z \omega)(\partial_0 \exp(\lambda_1 \Xi)) \right]
\]
\[
+ \left( \frac{d}{d\lambda_1} \Bigg|_{\lambda_1 = 0} \left[ \rho_0^{-1}(\overrightarrow{R}_Z \omega)(\partial_0 \exp(\lambda_1 \Xi)) \right] \right) (\overrightarrow{R}_Z \omega)(\exp(0), \exp(\lambda_2 \partial_0 \xi_2), \ldots, \exp(\lambda_q \partial_0 \xi_q)).
\]

Now,
\[
\frac{d}{d\lambda_1} \Bigg|_{\lambda_1 = 0} \rho_0^{-1}(\overrightarrow{R}_Z \omega)(\exp(\lambda_0 \partial_0 \Xi)) = \rho_0^{-1}(\overrightarrow{R}_Z \omega)(\exp(\lambda_0 \partial_0 \xi_0), \ldots, \exp(\lambda_q \partial_0 \xi_q)) +
\]
\[
+ \left( \frac{d}{d\lambda_1} \Bigg|_{\lambda_1 = 0} \left[ \rho_0^{-1}(\overrightarrow{R}_Z \omega)(\partial_0 \exp(\lambda_1 \Xi)) \right] \right) (\overrightarrow{R}_Z \omega)(\exp(0), \exp(\lambda_2 \partial_0 \xi_2), \ldots, \exp(\lambda_q \partial_0 \xi_q));
\]
thus, inductively, \( \overrightarrow{R}_Z \overrightarrow{R}_Z(\partial \omega) = \sum_{j=0}^{p+1} (-1)^j \overrightarrow{R}_{\partial_j \Xi} \overrightarrow{R}_Z \omega \), and as a consequence,
\[
\Phi(\partial \omega)(\Xi; Z) = \sum_{\sigma \in S_q} \sum_{\varrho \in S_r} \sum_{j=0}^{p+1} (-1)^j |\sigma| |\varrho| \overrightarrow{R}_{\varrho(\partial_j \Xi)} \overrightarrow{R}_{\varrho(Z)} \omega = \sum_{j=0}^{p+1} (-1)^j \Phi(\omega)(\partial_j \Xi; Z) = \partial(\Phi(\omega)(\Xi; Z)).
\]

\[\square\]

Theorem 3.12. For constant \( q \), \( \Phi \) restricts to a map of double complexes.

Difference maps - We conclude this section by proving that the van Est map \( \Phi \) commutes with the difference maps. We restrict our attention to the difference maps necessary to prove that \( \Phi \) defines a map of complexes up to degree 2.

Under the isomorphisms of Subsection 2.1.1, \( \xi = (\xi_1, \ldots, \xi_q) \in g^q_p \) corresponds to a unique element \( (x_1, \ldots, x_p; y) \in g^p \oplus h \), where each individual \( \xi_m \in g_1 \) corresponds to \( (x_m, y_m) = (x_m, y + \mu(x_{m+1} + \ldots + x_p)) \in g \oplus h \). In the upcoming computations, we often need to consider \( \xi \) as the sum
\[
\xi = (X^k_0, 0) + (0, \partial_0 \xi) = (x_1, \ldots, x_r, 0, \ldots; 0) + (0, \ldots, 0, x_{r+1}, \ldots, x_p; y),
\]
where \( X^k_0 := (x_1, \ldots, x_r) \in G^r \). Here, we abused notation and wrote an \( = \) sign to mean the image under the isomorphism of Subsection 2.1.1. Since the inclusion of \( G \) in \( g \) is a group homomorphism, and the full nerve of the Lie 2-group lies in the category of Lie groups,
\[
\exp_{g_p}(0, \partial_0 \xi) = (1, \exp_{g_{p-r}}(\partial_0 \xi)) = (1, \partial_0 \xi \exp_{g_{p-r}}(\xi))
\]
\[
\exp_{g_p}(0, 0, x_0, 0, 0; y_0) = (1, \ldots, 1, \exp_{G}(x_1), 1, \ldots, 1; 1) \in G^p \times H \cong g_p.
\]

In the latter equation, if \( x \in g \) is in the \( k \)th position, so is \( \exp_{G}(x) \in G \). Note that, for \( 1 < r \leq p \), none of the inclusions of \( G^r \) in \( g_p \) is a Lie group homomorphism; hence, there is no relation analogous to (75) when there is more than one non-zero entry.

Proposition 3.13. Let \( r > 0 \) and \( \omega \in C^{p,q}_r(G, \phi) \), then
\[
\Phi(\Delta \omega) = \Delta(\Phi \omega) \in C^{p+1,q+1}_{r-1}(g_1, \phi).
\]

\[\square\]
Proof. If \( r = 1 \), let \( \Xi = (\xi_0, ..., \xi_t)^T \in g^{q+1}_{p+1} \). If \( \xi_j \in g_{p+1} \) corresponds to \( (x_j^0, ..., x_j^p; y_j) \in g^{p+1}_{q+1} \otimes h \), using the convention of Eq. (74), write \( \Xi = \Xi_1 + \Xi_2 \), where

\[
\Xi_1 = \begin{pmatrix} x_0^0 \xi_0(0) \\ 0 \end{pmatrix}, \quad \Xi_2 = \begin{pmatrix} 0 \\ \tilde{\partial}_0 \xi_0(0) \end{pmatrix}.
\]

(76)

Since \( \mathcal{R}_\ast(\Delta \omega) \in \otimes g^{q+1}_{p+1} \otimes V, \mathcal{R}(\Xi(\Delta \omega)) = \mathcal{R}(\Xi_1(\Delta \omega)) + \mathcal{R}(\Xi_2(\Delta \omega)) \). Now, from Eq. (75), \( c_{11}(\Xi_2) = 1 \); hence, \( R(\Xi_2(\Delta \omega)) = 0 \). On the other hand, letting \( I' = \{ 1, ..., q \} \),

\[
\begin{aligned}
\mathcal{R}(\Xi(\Delta \omega)) &= \left. \frac{d^q}{d\lambda^q} \right|_{\lambda = 0} \phi(\mathcal{R}(\Delta \omega) |_{\lambda = 0} - \nabla(\phi \mathcal{R}(\Delta \omega) |_{\lambda = 0} 
\end{aligned}
\]

inductively implying \( \mathcal{R}(\Xi(\Delta \omega)) = \phi(\mathcal{R}(\Xi_{\lambda = 0} R(\xi \omega)) \). Using the partition \( S_{q+1} = \bigcup_{j=0}^{q} S_q(0, j) \), one concludes

\[
\Phi(\Delta \omega)(\Xi) = \sum_{j=0}^{q} \sum_{\sigma \in S_q(0,j)} (-1)^j |\sigma'| \phi(\mathcal{R}_{\Sigma(\Xi(\Xi_j) \Sigma_j)} R(\xi_{\omega})) = \sum_{j=0}^{q} (-1)^j (\Phi(\omega))((\tilde{\partial}_0 \Xi(j)); x_j^0) = \Delta(\Phi(\omega))(\Xi).
\]

If \( r > 1 \), let \( Z = (z_1, ..., z_{r-1}) \in g^{r-1} \) and \( \Xi \) as before. This time round, \( \mathcal{R}_\ast \mathcal{R}(\Delta \omega) \in \otimes g^{q+1}_{p+1} \otimes W \); thus, using Eq. (70), \( \mathcal{R}_\ast \mathcal{R}(\Delta \omega) = \mathcal{R}(\Xi_1(\Delta \omega)) + \mathcal{R}(\Xi_2(\Delta \omega)) \). The fact that \( c_{11}(\Xi_2) = 1 \) remains; hence, \( \mathcal{R}(\Xi_2(\Delta \omega)) = 0 \). Now, the terms of type \( c_{2n} \) are constant with respect to \( \tau_{r-1} \) and thus vanish. Writing \( \gamma(\lambda_0) \) for \((exp(\partial \lambda_0)^0_{r-1}) \), we compute

\[
\begin{aligned}
\frac{d}{d\lambda_0} &\left|_{\lambda_0=0} \right. \frac{d}{d\tau_j} \bigg|_{\tau_j=0} \frac{d}{d\tau_j} \bigg|_{\tau_j=0} \Phi(\Delta \omega) |_{\lambda_0=0} R(\xi_{\omega}) \gamma(\lambda_0)
\end{aligned}
\]

Ultimately, putting \( \xi'(Z)_{x_j^0} = (z_{\xi'}(1), ..., z_{\xi'}(j-1), x_j^0, z_{\xi'}(j), ..., z_{\xi'}(r-1)) \), and using successively the partitions \( S_{q+1} = \bigcup_{j=0}^{q+1} S_q(0, j) \) and \( S_r = \bigcup_{n=0}^{r-1} S_{r-1}(j, n) \), we compute

\[
\begin{aligned}
\Phi(\Delta \omega)(\Xi; Z) &= \sum_{j=0}^{q} \sum_{\sigma' \in S_q(0, j)} \sum_{\xi' \in \Sigma_{r-1}} (-1)^j |\sigma'| \phi(\mathcal{R}_{\Sigma(\Xi(\Xi_j) \Sigma_j)} \mathcal{R}_{\xi_{\omega}(Z)} \mathcal{R}(\xi_{\omega}))
\end{aligned}
\]

which yields \( \Delta(\Phi(\omega))(\Xi; Z) \) as desired.

\[\square\]

**Proposition 3.14.** Let \( r > 1 \) and \( \omega \in C^p_{p,q} (\mathcal{G}, \phi) \), then

\[
\Phi(\Delta_{r, 1\omega}) = 0 \in C^p_{0, r, q+1}(\mathcal{G}_1, \phi).
\]

\[\square\]
Proposition 3.15. Let \( \Xi = (\xi_0, \ldots, \xi_r)^T \in \mathfrak{g}_{p+r}^0 \oplus \mathfrak{h} \) and \((x_0^0, \ldots, x_r^0; y_0) \in \mathfrak{g}_{p+r}^0 \oplus \mathfrak{h}\) be the image of \( \xi_0 \in \mathfrak{g}_{p+r} \) under the isomorphism of Subsection 2.11. Using the convention of Eq. (74), write \( \Xi = \Xi_0 + \cdots + \Xi_r \), where

\[
\Xi_0 = \begin{pmatrix} 0 & \partial_0^0 \xi_0 \end{pmatrix}, \quad \Xi_k = \begin{pmatrix} 0 \cdots 0 \ x_k^0 \ 0 \cdots 0 \end{pmatrix}, \quad \text{for } 1 \leq k \leq r.
\]

Since \( \overrightarrow{\mathcal{R}}_s (\Delta_r, \omega) \in \mathfrak{g}_{p+r}^0 \otimes V \), \( \overrightarrow{\mathcal{R}}_s (\Delta_r, \omega) = \overrightarrow{\mathcal{R}}_{\Xi_0} (\Delta_r, \omega) + \cdots + \overrightarrow{\mathcal{R}}_{\Xi_r} (\Delta_r, \omega) \). It then follows from Eq. (74) that

\[
\overrightarrow{\mathcal{R}}_{\Xi_0} (\Delta_r, \omega) = \frac{d^T}{d\lambda^T} \bigg|_{\lambda=0} \rho_0^0 (t_p (\partial_0^0 \exp(\lambda_0 \xi_0)) \cdots t_p (\partial_0^0 \exp(\lambda_r \xi_0))) \circ \phi (\omega (\partial_0^0 \exp(\lambda_r \xi_0)) ; 1, \ldots, 1) = 0,
\]

and, for \( 1 \leq k \leq r \),

\[
\overrightarrow{\mathcal{R}}_{\Xi_k} (\Delta_r, \omega) = \frac{d^T}{d\lambda^T} \bigg|_{\lambda=0} \rho_0^0 (t_p (\partial_0^0 \exp(\lambda_1 \xi_1)) \cdots t_p (\partial_0^0 \exp(\lambda_r \xi_0))) \circ \phi (\omega (\partial_0^0 \exp(\lambda_r \xi_0)) ; 1, \ldots, exp(\lambda_0 \xi_0), \ldots, 1) = 0.
\]

\[ \square \]

Proposition 3.15. Let \( r > 1 \) and \( \omega \in C^{p,q} (\mathcal{G}, \phi) \), then

\[
\Phi (\Delta_r, \omega) = (-1)^{\frac{(r+1)}{2}} \Delta_r (\Phi \omega) \in C^{p+1,q+r}_0 (g_1, \phi).
\]

\[ \square \]

Proof. Let \( \Xi = (\xi_1, \ldots, \xi_{r+1})^T \in \mathfrak{g}_{p+r}^0 \oplus \mathfrak{h} \) and \((x_1^0, \ldots, x_r^0; y_1) \in \mathfrak{g}_{p+r}^0 \oplus \mathfrak{h}\) be the image of \( \xi_1 \in \mathfrak{g}_{p+1} \) under the isomorphism of Subsection 2.11. For \( \xi \in \mathfrak{g}_{p+1} \), write \((g_0^0, \ldots, g_r^0; \hat{x}_1) \in \mathfrak{g}^{p+1} \times H \) for the image of exp\( \mathfrak{g}_{p+1} \) (\( \xi \)) under the isomorphism of Subsection 2.11.

Making \( \Xi = \Xi_1 + \Xi_2 \) in the manner of Eq. (70), \( \overrightarrow{\mathcal{R}}_s (\Delta_r, \omega) = \overrightarrow{\mathcal{R}}_{\Xi_1} (\Delta_r, \omega) + \overrightarrow{\mathcal{R}}_{\Xi_2} (\Delta_r, \omega) \) as \( \overrightarrow{\mathcal{R}}_s (\Delta_r, \omega) \in \mathfrak{g}_{p+1} \oplus \otimes V \). Given that

\[
\omega (\partial_0^0 \exp(\lambda_1 \Xi_2); 1^{h_{\lambda_2} \xi_2} \cdots h_{\lambda_{r-1} \xi_{r-1}}; (g_0^0 h_{\lambda_2 \xi_2}^{1^{h_{\lambda_2} \xi_2}} \cdots h_{\lambda_{r-1} \xi_{r-1}}; (g_0^0 h_{\lambda_{r-1} \xi_{r-1}}^{1^{h_{\lambda_{r-1} \xi_{r-1}}}} \cdots h_{\lambda_r}^{1^{h_{\lambda_r}}}, g_0^0) = 0,
\]

\( \overrightarrow{\mathcal{R}}_{\Xi_2} (\Delta_r, \omega) = 0 \); therefore, inductively using Eq. (70) on \( (1, \ldots, k) \), one ultimately concludes \( \overrightarrow{\mathcal{R}}_s (\Delta_r, \omega) = \overrightarrow{\mathcal{R}}_{\Xi_0} (\Delta_r, \omega) \), where

\[
\Xi (r) := \begin{pmatrix} x_1^0 & 0 & \cdots & 0 \\ \vdots \\ x_r^0 & 0 \\ (\Xi_1, \ldots, r) \end{pmatrix},
\]

and, as in the proof of Proposition 5.13

\[
\overrightarrow{\mathcal{R}}_{\Xi_0} (\Delta_r, \omega) = \phi (\overrightarrow{\mathcal{R}}_{\hat{x}_1 (\Xi_1, \ldots, r)} X_{x_1, \ldots, x_r} \omega), \quad \text{where } X_{x_1, \ldots, x_r} = (x_1^0, \ldots, x_r^0) \in \mathfrak{g}^r.
\]

Partitioning \( S_{q+r} = \bigcup_{a_1 < \cdots < a_r} \bigcup_{a \in S_r} S_q (1 \sigma (a_1) \cdots a_{q(r)}) \),

\[
\Phi (\Delta_r, \omega) (\Xi) = \sum_{a_1 < \cdots < a_r} \sum_{a \in S_r} \sum_{\sigma \in \sigma (a_1) \cdots a_{q(r)}} (-1)^{a_1 + \cdots + a_r + \frac{(q(1))}{2}} |\sigma (r)| |\sigma (a) (a_1) \cdots a_{q(r)} | \Phi (\overrightarrow{\mathcal{R}}_{\hat{x}_0 (\Xi_1, \ldots, r)} X_{x_0, \ldots, x_r} \omega)
\]

\[
= \sum_{a_1 < \cdots < a_r} (-1)^{a_1 + \cdots + a_r + \frac{(q(r))}{2}} \phi (\Phi (\omega) (\partial_0 (\xi_1, \ldots, \xi_k)); X_{x_0, \ldots, x_r} (a_1, \ldots, a_k)) = (-1)^{\frac{(q(r)+1)}{2}} \Delta_r (\Phi (\omega) (\Xi)).
\]

\[ \square \]

All the previous results add up to the following theorem.

Theorem 3.16. The van Est map \( \Phi \) induces a map of complexes

\[
\Phi : C^r (\mathcal{G}, \phi) \rightarrow C^r (g_1, \phi)
\]

between the complexes \( \mathfrak{g}^r \) for \( n \leq 2 \).

\[ \square \]

Remark 3. Continuing Remark \( \# \), Theorem 3.14 extends to \( n \leq 5 \) for the difference maps made explicit in \( \alpha \).

We abstain from presenting a proof because it lies outside of our application.

\[ \square \]
4 A Collection of van Est Type Theorems

In this section, we prove a van Est type theorem relating the cohomology of Lie 2-groups and Lie 2-algebras:

**Theorem 4.1.** Let $G$ be a Lie 2-group with associated crossed module $\xymatrix{\mathfrak{g}_1 \ar[r] & H \ar[r] & \mathfrak{g}_2}$ and a representation on the 2-vector space $W \xrightarrow{\phi} V$. If $H$ and $G$ are both $k$-connected and the van Est map $\Phi$ induces a map of complexes between the complexes $(\bullet)$ for $n \leq k + 1$, then

$$\Phi : H^k_\bullet(G, \phi) \longrightarrow H^k_\bullet(\mathfrak{g}_1, \phi),$$

is an isomorphism for $n \leq k$ and it is injective for $n = k + 1$.

**Theorem 4.1** follows from the vanishing of the cohomology of the mapping cone of $\Phi$, which in turn, using the spectral sequence of the filtration of $(\bullet)$ by columns, follows from van Est type theorems that ensure the vanishing of its columns below the diagonal. Throughout, fix a Lie 2-group $G$ with associated crossed module

$$\xymatrix{\mathfrak{g}_1 \ar[r] & H \ar[r] & \mathfrak{g}_2}$$

in the sense of Ehresmann [7], where the top groupoid is a Lie group bundle over $G$ and the left groupoid is the right action groupoid of $H$ over $G$. Furthermore, for $r > 0$, the $p$-pages of $C^p_r(G, \phi)$ can be thought of as naturally associated double complexes induced by a map of double Lie groupoids

$$\xymatrix{H \times G \ar[r]^{\rho_H} & GL(W) \times GL(W)}$$

where $W$ is a vector space and the double Lie groupoid to the right is the one given by the right action by conjugation of $GL(W)$ on itself (cf. [33]).

Associated to a double Lie groupoid, there are two LA-groupoids [9], which are roughly given by passing the Lie functor in the vertical and the horizontal directions. In the case of $H \times G$, these are respectively given by

$$\xymatrix{\mathfrak{h} \times G \ar[r] & \mathfrak{h} \ar[r] & H \times \mathfrak{g}}$$

Bear in mind that the notation $\times$ stands alternatively for the transformation groupoid associated to an action of a Lie group and the action Lie algebroid associated to the action of a Lie algebra. Correspondingly, there are two morphisms of LA-groupoids which correspond to the differentiation of the map $(\bullet)$ in each direction. The subsequent results say there are double complexes naturally associated to each of the latter derivatives.

**Proposition 4.2.** Let $\mathfrak{h} \times G \longrightarrow \mathfrak{gl}(W) \times GL(W) : (y; g) \longrightarrow (\rho_\mathfrak{h}(y), \rho_G(g))$ be the differentiation of the map $(\bullet)$ in the vertical direction. Then, for every $q \geq 0$, there is a representation $\rho^q_\mathfrak{h}$ of the Lie group bundle

$$\mathfrak{h}^q \times G \longrightarrow \mathfrak{h}^q \ar[r] & \mathfrak{h}^q, \quad \text{on } \mathfrak{h}^q \times W \longrightarrow \mathfrak{h}^q,$$

and for every $r \geq 0$, there is a representation $\rho^r_G$ of the action Lie algebroid

$$\mathfrak{h} \times G^r \longrightarrow G^r \ar[r] & G^r \ar[r] & G^r \quad \text{on } G^r \times W \longrightarrow G^r.$$
such that the grid

\[
\begin{array}{c}
\bigwedge^3 h^* \otimes W & \xrightarrow{\partial} & C(G, \bigwedge^3 h^* \otimes W) & \xrightarrow{\partial} & C(G^2, \bigwedge^3 h^* \otimes W) & \cdots \\
\bigwedge^2 h^* \otimes W & \xrightarrow{\partial} & C(G, \bigwedge^2 h^* \otimes W) & \xrightarrow{\partial} & C(G^2, \bigwedge^2 h^* \otimes W) & \cdots \\
h^* \otimes W & \xrightarrow{\partial} & C(G, h^* \otimes W) & \xrightarrow{\partial} & C(G^2, h^* \otimes W) & \cdots \\
W & \xrightarrow{\partial} & C(G, W) & \xrightarrow{\partial} & C(G^2, W) & \cdots \\
\end{array}
\]

whose rows are the subcomplexes of alternating \(q\)-multilinear Lie algebroid cochains with values in \(\rho_G^q\) and whose columns are Lie algebroid complexes with values on \(\rho_b^q\), is a double complex.

**Proof.** Since (77) is a map of double groupoids, \(\rho_b \times \rho_G\) is a map of LA-groupoids and its restrictions to the base Lie group and to the side Lie algebra give respectively representations \(\rho_G\) of \(G\) and \(\rho_b\) of \(h\), both on \(W\). These are the representations for the group bundle over a point \(G\) and for the trivial action Lie algebroid \(h \times \ast \longrightarrow h\). For each \(q > 0\), define \(\rho_G^q := pr_G^q \rho_G\), which is a representation of the group bundle \(\gamma\) as the projection onto \(G\) is a Lie groupoid homomorphism. Analogously, for each \(r > 0\), define \(\rho_b^q := \hat{t}_r^q \rho_b\), where \(\hat{t}_r : (h \times G)_r \cong h \times G^r \longrightarrow h\) is the final target map of the LA-groupoid and hence a Lie algebroid map.

Since for any group bundle, \((h^q \times G)_r \cong h^q \times G^r\), \((h^q \times G^r) \cong C(h^q \times G^r, W) \cong C(h^q \times G^r, W)\), one can make sense of the subspace of alternating \(q\)-multilinear in the \(h\)-coordinates \(r\)-cochains of (79) with values on \(\rho_G^q\),

\[
C^r_{\text{lin}}(h^q \times G, W) := \{ \omega \in C^r(h^q \times G; h^q \times W) : \omega(-; \tilde{g}) \in \bigwedge^q h^* \otimes W, \forall \tilde{g} \in G^r \}.
\]

\(C^r_{\text{lin}}(h^q \times G, W)\) is a subcomplex of \(C^r(h^q \times G; h^q \times W)\). Indeed, for \(Y \in h^q\) and \(\tilde{g} = (g_0, \ldots, g_r) \in G^{r+1}\), \(\partial_k(Y; \tilde{g}) = (Y; \delta_k \tilde{g})\), where \(\delta_k\) is the \(k\)th face map in the nerve of the Lie groupoid \(G\). Thus, for \(\omega \in C^r_{\text{lin}}(h^q \times G, W)\),

\[
\partial \omega(Y; \tilde{g}) = \rho_G^q(Y; g_0) \omega(Y; \delta_0 \tilde{g}) + \sum_{k=1}^{r+1} (-1)^k \omega(Y; \delta_k \tilde{g}),
\]

but \(\rho_G^q(Y; g_0) = \rho_G(g_0)\); hence, \(\partial \omega(-; \tilde{g})\) is a linear combination of alternating \(q\)-multilinear maps. Since by definition

\[
C^r_{\text{lin}}(h^q \times G, W) = C(G^r, \bigwedge^q h^* \otimes W) = \Gamma \left( \bigwedge^q (h \times G^r)^* \otimes (G^r \times W) \right),
\]

for fixed \((q, r)\), the spaces of \(q\)-multilinear \(r\)-cochains of the Lie groupoid (80) with values on \(\rho_G^q\) and of \(q\)-cochains of the Lie algebroid \(\gamma\) with values on \(\rho_b^q\) coincide.

We are left to prove that the generic square

\[
\begin{array}{c}
C(G^r, \bigwedge^q h^* \otimes W) & \xrightarrow{\partial} & C(G^{r+1}, \bigwedge^q h^* \otimes W) \\
\downarrow \delta & & \downarrow \delta \\
C(G^r, \bigwedge^q h^* \otimes W) & \xrightarrow{\partial} & C(G^{r+1}, \bigwedge^q h^* \otimes W)
\end{array}
\]

commutes. Given that the brackets of all the Lie algebroids involved are completely determined by the bracket of \(h\), we restrict to prove the commutativity of (82) for constant sections. Let \(\omega \in C(G^r, \bigwedge^q h^* \otimes W)\),
while on the other hand, ρ̄^r+1(Y; y_0)ω(Y(j); δ_0ḡ) = ω^a ρ̄_b(y_j)ρ̄_G(g_0)e_a + \left( \frac{d}{dλ} \right)_{λ=0} ω^a(Y(j); (δ_0ḡ)^{exp(λy_j)}) \rhō_G(g_0) e_a + ω^a(Y(j); δ_0ḡ) \left( \frac{d}{dλ} \right)_{λ=0} ρ̄_G(g_0)^{exp(λy_j)} e_a

= ω^a ρ̄_b(y_j)ρ̄_G(g_0)e_a + \omega^a ρ̄_G(g_0) \left( \frac{d}{dλ} \right)_{λ=0} ω^a(Y(j); (δ_0ḡ)^{exp(λy_j)}) e_a + ω^a ρ̄_b(y_j) e_a + ω^a ρ̄_G(g_0) e_a,

ultimately implying the commutativity of the square (82).

We aim at approximating the cohomology of the map Φ_p of Theorem 8.10 by assembling van Est maps from C_2^*(G, φ) to the complexes of Proposition 4.2. Since, each p-page is induced by the double groupoid map

G_p × G \rightarrow GL(W) × GL(W) : (γ; g) \mapsto (ρ̄^p(λ_p(γ))^{-1}, ρ̄_0(i(g))),

replacing the first column of maps by (83), we introduce a first column replacement for the associated double complex of Proposition 4.2.
Lemma 4.3. Let $\omega \in \bigwedge^q g_p^{*} \otimes V$, $\Xi \in g_p^{n}$ and $g \in G$. If $\partial' \omega(\Xi; g) := \rho_1(g)\omega(\Xi)$, then, using the notation of Proposition 4.2,

$$\bigwedge^q g_p^{*} \otimes V \xrightarrow{\partial'} C_{lin}^1(g_p^{*} \times G, W) \xrightarrow{\delta} C_{lin}^2(g_p^{*} \times G, W)$$

is a complex.

Proof. First, notice that $\partial'$ is well-defined. Then, letting $g_0, g_1 \in G$, if follows from Eq.'s (31) and (32) that

$$\partial(\partial' \omega)(\Xi; g_0, g_1) = \rho_1^0(\Xi; g_0)\rho_1(g_1)\omega(\Xi) - \rho_1(g_0g_1)\omega(\Xi) + \rho_1(g_0)\omega(\Xi) = \rho_1^0(i(g_0))\rho_1(g_1)\omega(\Xi) - (I + \rho_1(g_0) \circ \delta)\rho_1(g_1)\omega(\Xi) = 0.$$

Proposition 4.4. For each $p \geq 0$, define $C_{LA}(g_p \ltimes G, \phi)$ to be

$$
\begin{array}{c}
\bigwedge^3 g_p^{*} \otimes V & \xrightarrow{\sigma} & C(G, \bigwedge^3 g_p^{*} \otimes W) & \xrightarrow{\partial} & C(G^2, \bigwedge^3 g_p^{*} \otimes W) & \ldots \\
\bigwedge^2 g_p^{*} \otimes V & \xrightarrow{\sigma} & C(G, \bigwedge^2 g_p^{*} \otimes W) & \xrightarrow{\partial} & C(G^2, \bigwedge^2 g_p^{*} \otimes W) & \ldots \\
g_p^{*} \otimes V & \xrightarrow{\sigma} & C(G, g_p^{*} \otimes W) & \xrightarrow{\partial} & C(G^2, g_p^{*} \otimes W) & \ldots \\
V & \xrightarrow{\delta'} & C(G, W) & \xrightarrow{\delta(1)} & C(G^2, W) & \ldots,
\end{array}
$$

where the maps in the first column are given by Lemma 4.3 and the rest by Proposition 4.2. Then $C_{LA}(g_p \ltimes G, \phi)$ is a double complex, the vertical LA-double complex of $C_{LA}^*(G, \phi)$.

Proof. The proof reduces to show that, in the first column, all squares commute. For a constant section $\xi \in \Gamma(g_p \ltimes G)$ and $g \in G$, one has

$$
\rho_{\theta_p}^1(\xi) \rho_1(g) = \rho_{\theta_p}(\xi) \rho_1(g) + \frac{d}{d\lambda} \rho_1(g^{\theta}(\exp(\lambda \xi)))
$$

$$
= \rho_1^0(\tilde{t}(\xi)) \rho_1(g) + \frac{d}{d\lambda} \rho_1^0(\tilde{t}(\exp(\lambda \xi)))^{-1} \rho_1(g) \rho_1^0(\tilde{t}(\exp(\lambda \xi)))
$$

$$
= \rho_1^0(\tilde{t}(\xi)) \rho_1(g) + \rho_1^0(\tilde{t}(\xi)) \rho_1(g) + \rho_1(g) \rho_1^0(\tilde{t}(\xi)) = \rho_1(g) \rho_1^0(\tilde{t}(\xi)).
$$

If $q = 0$, let $v \in V$, then $\delta(\partial' \omega)(\xi; g) := \rho^1_{\theta_p}(\xi) \rho_1(g) v = \rho_1(g) \rho^0_{\theta_p}(\tilde{t}(\xi)) v = \partial'(\delta \omega)(\xi; g)$. If $q > 0$, let $\omega \in \bigwedge^q g_p^{*} \otimes V$ and $\Xi = (\xi_0, \ldots, \xi_q) \in g_p^{q+1}$, then

$$
\delta(\partial' \omega)(\Xi; g) = \sum_{j=0}^q (-1)^j \rho_{\theta_p}^1(\xi_j) \rho_1(g) \omega(\Xi(j)) + \sum_{m<n} (-1)^{m+n} \rho_1(g) \omega([\xi_m, \xi_n], \Xi(m, n))
$$

$$
= \rho_1(g) \left( \sum_{j=0}^q (-1)^j \rho^0_{\theta_p}(\tilde{t}(\xi_j)) \omega(\Xi(j)) + \sum_{m<n} (-1)^{m+n} \omega([\xi_m, \xi_n], \Xi(m, n)) \right) = \partial'(\delta \omega)(\Xi; g).
$$

\[\square\]
Let
\[ \Phi_V : C^\bullet_p(G, \phi) \longrightarrow C^*_L A(G_p \ltimes G, \phi) \]
be defined by assembling column-wise van Est maps
\[ \Phi^0_V : C(G^*_p, V) \longrightarrow \bigwedge^q g^*_p \otimes V \quad \text{and} \quad \Phi_V : C(G^*_p \rtimes G, W) \longrightarrow C(G^r, \bigwedge^\bullet g^*_p \otimes W). \]

To describe \( \Phi_V \) explicitly consider a \( q \)-cochain \( \omega \in C(G^*_p \rtimes G, W) \), \( q \) sections \( \xi_1, \ldots, \xi_q \in I(G^*_p \rtimes G^r) \) and let \( \{ u_a \} \) be a basis for \( g_p \). Then, writing \( (\xi_j) g = \xi^a_j (g) u_a \) for \( g \in G^r \), the right-invariant vector field \( \xi_j \in \mathfrak{X}(G_p \rtimes G^r) \) associated to \( \xi_j \) is given by
\[
(\xi_j)(\gamma; g) := \frac{d}{d\lambda_a} \bigg|_{\lambda_a=0} \xi^a_j ((g)^{t_p}(\gamma)) \exp(g_a) (\lambda_a u_a); (g)^{t_p}(\gamma) \triangleright (\gamma; g) \\
= \xi^a_j ((g)^{t_p}(\gamma)) \frac{d}{d\lambda_a} \bigg|_{\lambda_a=0} \gamma \exp(g_a) (\lambda_a u_a); (g)^{t_p}(\gamma) = \xi^a_j ((g)^{t_p}(\gamma)) d\gamma u_a \in T_\gamma G_p \leq T_\gamma G_p \otimes T_g G^r.
\]

Consequently, if \( \gamma_2, \ldots, \gamma_q \in G_p, \ (R_{\xi_q}\omega)(\gamma_2, \ldots, \gamma_q; g) = \xi^a_j ((g)^{t_p}(\gamma_2) \ldots (g)^{t_p}(\gamma_q)) \frac{d}{d\lambda_a} \bigg|_{\lambda_a=0} \omega((\exp(g_a) (\lambda_a u_a)); \gamma_2, \ldots, \gamma_q; g) \)
and
\[
R_{\xi_q-1} (R_{\xi_q}\omega)(\gamma_3, \ldots, \gamma_q; g) = \xi^b_j ((g)^{t_p}(\gamma_3) \ldots (g)^{t_p}(\gamma_q)) \frac{d}{d\lambda_b} \bigg|_{\lambda_b=0} \big( R_{\xi_q}\omega \big)((\exp(g_b) (\lambda_b u_b), \gamma_3, \ldots, \gamma_q; g) \\
= \xi^b_j ((g)^{t_p}(\gamma_3) \ldots (g)^{t_p}(\gamma_q)) \frac{d}{d\lambda_b} \bigg|_{\lambda_b=0} \omega((\exp(g_b) (\lambda_b u_b)); \gamma_3, \ldots, \gamma_q; g) \\
= \xi^b_j ((g)^{t_p}(\gamma_3) \ldots (g)^{t_p}(\gamma_q)) \frac{d}{d\lambda_b} \bigg|_{\lambda_b=0} \omega((\exp(g_b) (\lambda_b u_b)), \gamma_3, \ldots, \gamma_q; g).
\]

Inductively, for \( \Xi = (\xi_1, \ldots, \xi_q) \), \( \overrightarrow{R}_\Xi = (\xi^1 \xi^2 \ldots \xi^q)((g) \frac{d}{d\lambda_t} \bigg|_{\lambda_t=0} \omega((\exp(\lambda_t \cdot \sigma(\Xi)); g)) \)
and
\[
(\Phi^0_{V'})(\Xi; g) = \sum_{\sigma \in S_q} |\sigma| \frac{d}{d\lambda_t} \bigg|_{\lambda_t=0} \omega((\exp(\lambda_t \cdot \sigma(\Xi)); g)).
\]

**Proposition 4.5.** The map \( \Phi_V \) \( \{ \Xi \} \) is a map of double complexes.

**Proof.** Since by definition \( \Phi_V \) defines a map of complexes when restricted to columns, we prove that it is compatible with the horizontal differentials in \( \{ \Xi \} \). For \( r = 0 \), let \( \omega \in C(G^*_p, V) \), \( \Xi = (\xi_1, \ldots, \xi_q) \in g^*_p \) and \( g \in G \), then
\[
\overrightarrow{R}_\Xi \vec{d}^r \omega(g) = \frac{d}{d\lambda_t} \bigg|_{\lambda_t=0} \rho^0_0 (t_p(\exp(g \lambda_1 \xi_1) \ldots \exp(g \lambda_q \xi_q)))^{-1} \rho_1 (g) \omega(\exp(\lambda_1 \cdot \Xi));
\]
inductively yielding \( \overrightarrow{R}_\Xi \vec{d}^r \omega(g) = \rho_1 (g) \overrightarrow{R}_\Xi \omega \). Taking the alternating sum over \( S_q \), \( \Phi^0_{V'} \vec{d}^r \omega = \vec{d}^r \Phi^0_V \omega \). For \( r > 0 \), let \( \omega \in C(G^*_p \times G^r, W) \), \( g = (g_0, \ldots, g_r) \in G^{r+1} \) and \( \Xi \) as above, then
\[
\overrightarrow{R}_\Xi (\delta(1) \omega)((g)) = \frac{d}{d\lambda_t} \bigg|_{\lambda_t=0} \rho^0_0 (t_p(\exp(\lambda_1 \xi_1) \ldots t_p(\exp(\lambda_1 \xi_1))) \omega(\exp(\lambda_1 \cdot \Xi) \delta \omega(g)) + \sum_{k=1}^{r+1} (-1)^k \omega(\exp(\lambda_1 \cdot \Xi) \delta_k g).
\]
Inductively, \( \frac{d}{d\lambda_t} \bigg|_{\lambda_t=0} \rho^0_0 (t_p(\exp(\lambda_1 \xi_1) \ldots t_p(\exp(\lambda_1 \xi_1))) \omega(\exp(\lambda_1 \cdot \Xi) \delta \omega(g)) = \rho^0_0 (i(g_0)) \overrightarrow{R}_\Xi \omega(\delta \omega(g)); \) hence, taking the alternating sum over \( S_q \) and recalling \( \rho^0_0 (i(g_0)) = \frac{d}{d\lambda_t} (i(g_0)) \)
\[
\Phi^0_{V'} (\delta(1) \omega)(\Xi; g) = \rho^0_0 (i(g_0)) \Phi_{V'} \omega(\Xi; \delta \omega(g)) + \sum_{k=1}^{r+1} (-1)^k \Phi^0_{V'} \omega(\Xi; \delta_k \omega(g)) = \overrightarrow{\partial} (\Phi_{V'} \omega)(\Xi; g).
\]
\[
\Phi_{V'}^{k+1}(\delta(1) \omega)(\Xi; g) = \rho^0_0 (i(g_0)) \Phi_{V'} \omega(\Xi; \delta \omega(g)) + \sum_{k=1}^{r+1} (-1)^k \Phi_{V'} \omega(\Xi; \delta_k \omega(g)) = \overrightarrow{\partial} (\Phi_{V'} \omega)(\Xi; g).
\]

**Theorem 4.6.** If \( G_p \) is \( k \)-connected, \( H^q_{tot}(\Phi_V) = (0) \) for all degrees \( n \leq k \).
Proposition 4.7. $\Phi$ follows from Lemma 2.5.

Since $\Phi$ is defined column-wise by the van Est maps $\Phi_i$, the $r$th column of the mapping cone double coincides with the mapping cone of $\Phi_i$. Invoking Theorem 1.5, the $r$th column of $E_1^{p,q}$ vanishes below $k$; indeed, $G_p$ is the $s$-slice of $G \times G$ and $G$ and is $k$-connected by hypothesis. Given that $E_1^{p,q} \Rightarrow H_{tot}^{p+q}(\Phi_V)$, the result follows from Lemma 2.5. \hfill \Box

4.2 Second approximation and the main theorem

Observe that the $q$th row of $C_{LA}(G_p \times G, \phi)$ coincides with the cochain complex of the Lie group $G$ with values in the representation

$$\rho_{(q)} : G \longrightarrow GL(\Lambda^q G_p \otimes W), \quad \rho_{(q)}(g)\omega = \rho_q^i(i(g))\omega \quad \text{for} \quad \omega \in \Lambda^q G_p \otimes W$$

except in degree 0. Note that assembling row-wise van Est map extended by the identity in degree 0 defines a map

$$\Phi_{row} : C_{LA}(G_p \times G, \phi) \longrightarrow C_{\bullet}^q(G_1, \phi) \quad (87)$$

that casually lands in the $p$-page of the grid of the Lie 2-algebra. Let $\Phi_{row}^q : C(G^r, \Lambda^q G_p \otimes W) \longrightarrow \Lambda^q G_p \otimes \Lambda^\cdot g \otimes W$ be the van Est map defined by the van Est map for $r > 0$ and the identity of $\Lambda^q G_p \otimes V$ for $r = 0$. $\Phi_{row}^q$ is explicitly given by

$$(\Phi_{row}^q \omega)(\Xi, Z) = \sum_{\xi \in S_r} |\xi| \left. \frac{d^\cdot}{d\tau_j} \right|_{\tau_j=0} \omega(\hat{\Xi}; \exp(\tau_j \cdot g(Z_{\tau}))) \quad (88)$$

for $\omega \in C(G^r, \Lambda^q G_p \otimes W)$, $\Xi \in G_p^q$ and $Z = (z_1, \ldots, z_r) \in g^r$.

Proposition 4.7. $\Phi_{row}$ is a map of double complexes. \hfill \Box

Proof. Since $\Phi_{row}$ defines a map of complexes when restricted to rows and to the first column, we are left to prove that it is compatible with the vertical differentials in (84). Let $\omega \in C(G^r, \Lambda^q G_p \otimes W)$, $\Xi = (\xi_0, \ldots, \xi_q) \in G_p^q+1$ and $Z = (z_1, \ldots, z_r) \in g^r$, then

$$\overrightarrow{R}\hat{\delta}_{\Xi} \omega(\Xi) = \left. \frac{d^\cdot}{d\tau_j} \right|_{\tau_j=0} \sum_{j=0}^{q} \left(-1\right)^{j} \rho_{\hat{\xi}_{p_j}}(\xi_j)\omega(\Xi(j); \exp(\tau_j \cdot Z)) + \sum_{m<n} \omega(\xi_m, \xi_n, \Xi(m, n); \exp(\tau_j \cdot Z)).$$

Let $\{v_a\}$ be a basis for $W$ and $\omega(\Xi(j); \exp(\tau_j \cdot Z)) = \omega^a(\Xi(j); \exp(\tau_j \cdot Z))v_a$. By definition

$$\rho_{\hat{\xi}_{p_j}}(\xi_j)\omega(\Xi(j); \exp(\tau \cdot X)) = \omega^a(\Xi(j); \exp(\tau_j \cdot Z))\rho_{\hat{\xi}_{p_j}}(\xi_j)v_a + \left. \frac{d^\cdot}{d\lambda} \right|_{\lambda=0} \omega^a(\Xi(j); \exp(\tau_j \cdot Z))^\cdot_{p_j}(\exp(\lambda \xi_j))v_a;$$
Lemma 4.8. A van Est type theorem for $\Phi$ cannot use the Crainic-van Est Theorem directly as we replaced the space of 0-cochains. In the sequel, we prove by induction that the result follows from a simple induction.

Lemma 4.10. Let $\delta'$ be the map of Lemma 4.3 and $\delta'$ the map of Eq. (46). For constant $p, q \geq 0$, if $G$ is connected, $\ker \delta' = \ker \delta'$.

Proof. (1) If $\omega \in \ker \delta$, $(\delta' \omega)(\Xi; g) = 0$ for all $(\Xi; g) \in g_0^p \times G$. Then, for $z \in g$,

\[
(\delta' \omega)(\Xi; z) = \Phi_{\delta'}(\delta' \omega)(\gamma; z) = \frac{d}{dr} \bigg|_{r=0} (\delta' \omega)(\Xi; \exp_G(\gamma z)) = 0.
\]

(2) Conversely, if $\omega \in \ker \delta$, $(\delta' \omega)(\Xi; x) = \hat{\rho}_1(z) \omega(\Xi) = 0$ for all $(\Xi; z) \in g_0^p \times g$. Being connected, $G$ is generated by $\exp_G(U) \subset G$ for some neighborhood of the identity $U$. Therefore, for all $g \in G$, there exist $z_1, \ldots, z_n \in g$ such that $g = \exp_G(z_1) \ldots \exp_G(z_n)$. Since $\rho_1$ is a Lie group homomorphism, it follows from Eq. (27) that

\[
\rho_1(\exp_G(z)) \omega(\Xi) = \exp_G(\phi)(\hat{\rho}_1(z)) \omega(\Xi) = \sum_{n=0}^{\infty} \frac{(\hat{\rho}_1(z) \phi)^n}{(n+1)!} \hat{\rho}_1(z) \omega(\Xi) = 0
\]

for all $z \in g$. Now, $\delta' \omega(\Xi; g) = \rho_1(g) \omega(\Xi) = \rho_1(\exp_G(z_1) \ldots \exp_G(z_n)) \omega(\Xi)$ and it follows from Eq. (31) that

\[
\rho_1(\exp_G(z_1) \ldots \exp_G(z_n)) = \rho_1(\exp(z_1) \ldots \exp(z_{n-1}+1)) + \rho_1(\exp(z_1) \ldots \exp(z_n))
\]

hence, the result follows from a simple induction.

As $\Phi_{\delta'}$ restricted to the first column of $[S]$ is the identity and there are no cochains of negative degree, Lemma 4.3 is a van Est type theorem in degree 0 and a consequence of several pieces of Lie theory. In contrast, the following lemma is stated as a general result of homological algebra and implies naturally that if $G$ is 1-connected, $\Phi_{\delta'}$ induces isomorphism in degree 1.

Definition 4.9. If $(C^g_1, d_{C_1})$ and $(C^g_2, d_{C_2})$ are equal complexes except in degree zero, and there is a map $\phi : C^g_1 \rightarrow C^g_2$ such that $d_{C_1} = d_{C_2} \circ \phi$, then they are called $\phi$-related.

Lemma 4.10. Let $(C^g_1, d_{C_1})$, $(C^g_2, d_{C_2})$ be $\phi$-related complexes, and let $(D^g_1, d_{D_1})$, $(D^g_2, d_{D_2})$ be $\phi_D$-related complexes. Let $\Phi_1 : C^g_1 \rightarrow D^g_1$ and $\Phi_2 : C^g_2 \rightarrow D^g_2$ be maps of complexes that coincide except in degree zero, where

\[
\begin{array}{ccc}
C^0_1 & \xrightarrow{\Phi_1} & D^0_1 \\
\phi_C & \downarrow & \downarrow \phi_D \\
C^1_1 & \xrightarrow{\Phi_1 = \Phi_2} & D^1_1 = D^1_2 \\
\downarrow & & \\
C^2_2 & \xrightarrow{\phi_2} & D^2_2
\end{array}
\]

If $\Phi_1$ induces an isomorphisms $H^1(C_1) \cong H^1(D_1)$, then $\Phi_2$ induces an isomorphism $H^1(C_2) \cong H^1(D_2)$.
Proof. Let $Z^1_{B_k} := \ker(d_{X_k} : X^1_k \longrightarrow X^2_k)$ and $B^1_{X_k} := d_{X_k}(X^0_k)$, for $X \in \{C, D\}$ and $k \in \{1, 2\}$, and consider the maps of exact sequences

$$0 \longrightarrow B^1_{C_k} \longrightarrow Z^1_{C_k} \longrightarrow H^1(C_k) \longrightarrow 0$$

whose associated long exact sequence is

$$0 \longrightarrow \ker(\Phi_k|_{B^1_{C_k}}) \longrightarrow \ker \Phi_k \longrightarrow \ker[\Phi_k] \longrightarrow \coker (\Phi_k|_{B^1_{C_k}}) \longrightarrow \coker \Phi_k \longrightarrow \coker [\Phi_k] \longrightarrow 0.$$ 

By hypothesis, $\ker[\Phi_1]$ and $\coker [\Phi_1]$ are trivial thus implying $\ker (\Phi_1|_{B^1_{C_1}}) \cong \ker \Phi_1$ and $\ker(\Phi_1|_{B^1_{C_1}}) = \ker \Phi_1$, which we interpret as $\ker \Phi_1 \subseteq B^1_{C_1}$. Since for every element $x \in C^1_1$, $d_{C_1}(x) = d_{C_2}(\phi_C(x))$, we have got that $B^1_{C_1} \subseteq B^1_{C_2}$; therefore, $\ker(\Phi_1|_{B^1_{C_2}}) = \ker \Phi_1 \cap B^1_{C_2} = \ker \Phi_1$. As a consequence, $\ker[\Phi_2]$ vanishes, so the induced map in cohomology is injective and we are left with the short exact sequence

$$0 \longrightarrow \coker (\Phi_2|_{B^1_{C_2}}) \longrightarrow \coker \Phi_2 \longrightarrow \coker [\Phi_2] \longrightarrow 0.$$ 

Now, $d_{D_1} = d_{D_2} \circ \phi_D$ implies $B^1_{D_1} \subseteq B^1_{D_2}$, so there is a map of exact sequences

$$0 \longrightarrow \coker (\Phi_1|_{B^1_{C_1}}) \longrightarrow \coker \Phi_1 \longrightarrow 0 \longrightarrow 0$$

where, for $y \in D^1_1$, $\alpha(d_{D_1}(y) + \Phi(B^1_{C_1})) := d_{D_2}(\phi_D(y)) + \Phi(B^1_{C_2})$. The long exact sequence of (89) tells us that $\alpha$ is an isomorphism and $\ker[\Phi_2]$ is trivial, so the induced map in cohomology is surjective.

Remark 4. In the proof of Lemma 4.10 the inclusions $B^1_{X_1} \subseteq B^1_{X_2}$ also give rise to exact sequences

$$0 \longrightarrow B^1_{X_1} \longrightarrow Z^1_{C_1} \longrightarrow H^1(X_1) \longrightarrow 0$$

out of whose long exact sequences one reads

$$H^1(X_1) \cong H^1(X_2) \oplus B^1_{X_2}/B^1_{X_1}.$$ 

What the proof of Lemma 4.10 ultimately says is that the isomorphism $H^1(C) \cong H^1(D)$ is diagonal with respect to the direct sum decompositions of Eq. (90).

Proposition 4.11. For constant $q \geq 0$, if $G$ is $k$-connected, $H^n(\Phi^q_{\text{tor}}) = (0)$ for all degrees $n \leq k$.

Proof. Since $G$ is the $s$-fibre of the group bundle $g^q \times G \longrightarrow g^q$ and is $k$-connected by hypothesis, Theorem 4.3 implies the result for $1 < n \leq k$. That there is an isomorphism in degree $0$ follows from Lemma 4.8. As for degree 1, the result follows from Lemma 4.10 after noticing that letting

$$\phi^q_\omega : \Lambda^q g^q \otimes W \longrightarrow \Lambda^q g^q \otimes V \quad (\phi^q_\omega(\Xi) := \phi(\omega(\Xi))) \quad \text{for } \Xi \in g^q,$$

the $q$th rows of (81) and (84) are $\phi^q_\omega$-related and the same holds for the Chevalley-Eilenberg complex of $g$ with values in (45) and the $q$th row of the $p$-page $C^p_\bullet(\frak{g}_1, \phi)$.
Theorem 4.12. If $G$ is $k$-connected, $H^n_{\text{tot}}(\Phi_{\text{row}}) = (0)$ for all degrees $n \leq k$. □

Proof. We compute the cohomology of the (transposed) mapping cone double of $\Phi_{\text{row}}$ using the spectral sequence of its filtration by rows, whose first page is

$$E^p_q : H^p(\Phi^r_{\text{row}}) \rightarrow H^1(\Phi^r_{\text{row}}) \rightarrow H^2(\Phi^r_{\text{row}}) \rightarrow \cdots$$

Since $\Phi_{\text{row}}$ is defined row-wise by van Est maps, the $q$th row of the transposed mapping cone double coincides with the mapping cone of $\Phi^q_{\text{row}}$. Invoking Proposition 4.11, the $q$th row of $E^p_q$ vanishes below $k$; indeed, $G$ is $k$-connected by hypothesis. Given that $E^p_q \Rightarrow H^{p+q}_{\text{tot}}(\Phi_{\text{row}})$, the result follows from Lemma 2.5.

Remark 5. One can prove results analog to those in Subsection 4.1 for the horizontal differential of (77) yielding as a first approximation a map to the double complex associated to the other LA-groupoid in (78). In that case, the ideas needed to prove that its cohomology vanishes parallel those of Lemmas 4.8 and 4.10. We opted for the present approach because, in the second approximation, one would need a van Est theory adapted to the subcomplex of multilinear cochains.

We are ready to prove that the cohomology of the restriction $\Phi^p$ (13) of the van Est map $\Phi$ (62) to the $p$-pages vanishes.

Theorem 4.13. If $H$ and $G$ are both $k$-connected, $H^n_{\text{tot}}(\Phi_p) = (0)$ for all degrees $n \leq k$. □

Proof. As in the proof of Theorem 4.2 it follows from the Künneth formula and the $k$-connectedness of $H$ and $G$, that $G_p$ is $k$-connected as well. Theorem 4.6 and Proposition 2.4 imply that $\Phi_V$ induces isomorphism in cohomology for $n \leq k$, and it is injective for $n = k + 1$. Analogously, as $G$ is $k$-connected, Theorem 4.12 and Proposition 2.4 imply that the same holds for $\Phi_{\text{row}}$. The result follows from Proposition 2.4 after noticing that (cf. Eq.’s (86) and (88)), for constant $p$,

$$C(G^r_p \times G^r, W) \xrightarrow{\Phi_p} \Lambda^q g^*_p \otimes \Lambda^r g^* \otimes W,$$

As announced in the introduction, we can now prove the main Theorem.
Proof. (of Theorem 4.11) We compute the cohomology of the mapping cone triple of $\Phi$ using the spectral sequence of the filtration by columns of (61), whose first page is (schematically)

\[ E_1^{p,q} : \]

\[ H_{\text{tot}}^0(\Phi_0) \rightarrow H_{\text{tot}}^0(\Phi_1) \rightarrow H_{\text{tot}}^0(\Phi_2) \rightarrow H_{\text{tot}}^0(\Phi_3) \rightarrow \ldots \]

\[ H_{\text{tot}}^1(\Phi_0) \rightarrow H_{\text{tot}}^1(\Phi_1) \rightarrow H_{\text{tot}}^1(\Phi_2) \rightarrow H_{\text{tot}}^1(\Phi_3) \rightarrow \ldots \]

\[ H_{\text{tot}}^2(\Phi_0) \rightarrow H_{\text{tot}}^2(\Phi_1) \rightarrow H_{\text{tot}}^2(\Phi_2) \rightarrow H_{\text{tot}}^2(\Phi_3) \rightarrow \ldots \]

\[ H_{\text{tot}}^3(\Phi_0) \rightarrow H_{\text{tot}}^3(\Phi_1) \rightarrow H_{\text{tot}}^3(\Phi_2) \rightarrow H_{\text{tot}}^3(\Phi_3) \rightarrow \ldots \]

By definition, the $p$th column in (91) is given by the cohomology of the mapping cone double of $\Phi_p$. Invoking Theorem 4.13, the $p$th column of $E_1^{p,q}$ vanishes below $k$; indeed, both $H$ and $G$ are $k$-connected by hypothesis. Given that $E_1^{p,q} \Rightarrow H_{\text{tot}}^{p+q}(\Phi)$, the result follows from Lemma 2.5 and Proposition 2.4.

And having all the ingredients to run van Est’s strategy:

**Corollary 4.14.** Every finite-dimensional Lie 2-algebra is integrable.

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