Vortex Origin of Tricritical Point in Ginzburg-Landau Theory

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Motivated by recent experimental progress in the critical regime of high-$T_c$ superconductors we show how the tricritical point in a superconductor can be derived from the Ginzburg-Landau theory as a consequence of vortex fluctuations. Our derivation explains why usual renormalization group arguments always produce a first-order transition, in contrast to experimental evidence and Monte Carlo simulations.

1. The critical regime of old-fashioned superconductors can be described extremely well by the Ginzburg-Landau theory [1] at the mean-field level [2,3]. The reason is the smallness of the Ginzburg temperature interval $\Delta T_G$ around the mean-field critical temperature $T_{c\,MF}$ where fluctuation become important [4]. This made discussions of the order of the superconductive phase transition started by Halperin, Lubensky, and Ma in 1972 [5] rather academic.

The situation has changed with the advent of modern high-$T_c$ superconductors temperature superconductors. In these the Ginzburg temperature interval is large enough to study field fluctuations and critical behavior. Several experiments have found a critical point of the XY universality class [6]. In addition, there seems to be recent evidence for an additional critical behavior associated with the so-called charged fixed point [7]. In view of future experiments, it is important to understand precisely the nature of critical fluctuations.

The Ginzburg-Landau theory [1] describes a superconductor with the help of an energy density

$$\mathcal{H}(\psi, \nabla \psi, \mathbf{A}, \nabla \mathbf{A}) = \frac{1}{2} \left\{ \left[ (\nabla - ie \mathbf{A}) \psi \right]^2 + \tau |\psi|^2 + \frac{g}{2} |\psi|^4 \right\} + \frac{1}{2} (\nabla \times \mathbf{A})^2, \quad (1)$$

where $\psi(x)$, $\mathbf{A}(x)$ are pair field and vector potential, respectively, and $e$ is the charge of the Cooper pairs. The parameter $\tau \equiv T / T_{c\,MF} - 1$ is a reduced temperature measuring the distance from the characteristic temperature $T_{c\,MF}$ at which the $\tau$-term changes sign. The theory needs gauge fixing, which is usually done by setting

$$\psi(x) = \rho(x) e^{i\theta(x)},$$

rewriting the covariant derivative of $\psi$ as

$$D\psi = [i(\nabla \theta - e \mathbf{A}) \rho + \nabla \rho] e^{i\theta}, \quad (2)$$

and eliminating the phase variable $\theta(x)$ by a local gauge transformation $\mathbf{A} \rightarrow \mathbf{A} + \nabla \theta / e$. This brings $\mathcal{H}(\psi, \nabla \psi, \mathbf{A}, \nabla \mathbf{A})$ to the form

$$\mathcal{H}_1 = \frac{1}{2} (\nabla \rho)^2 + V(\rho) + \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{\rho^2 \xi^2}{2} \mathbf{A}^2, \quad (3)$$

where $V(\rho)$ is the potential of the $\rho$-field:

$$V(\rho) = \frac{\tau}{2} \rho^2 + \frac{2}{4} \rho^4. \quad (4)$$

The last term in (3) is the famous Meissner-Higgs mass $m_A = pe$ [2,3] of the vector potential $\mathbf{A}$, whose analog in the gauge theory of electroweak interactions is the famous Meissner-Higgs mass term in (3).

At the mean-field level, the energy density (3) describes a second order phase transition. It takes place if $\tau$ drops below zero where the pair field $\psi(x)$ acquires the nonzero expectation value $\langle \psi(x) \rangle = \rho_0 = \sqrt{-\tau / g}$, the order parameter of the system. The $\rho$-fluctuations around this value have a coherence length $\xi = 1/\sqrt{2\tau}$. The Meissner-Higgs mass term in (3) gives rise to a finite penetration depth of the magnetic field $\lambda = 1/m_A = 1/\rho_0 e$.

The ratio of the two lengths scales $\kappa \equiv \lambda / \sqrt{2\xi}$, which for historic reasons carries a factor $\sqrt{2}$, is the Ginzburg parameter whose mean field value is $\kappa_{MF} \equiv g/e^2$. Type I superconductors have small values of $\kappa$, type II superconductors have large values. At the mean-field level, the dividing line is lies at $\kappa = 1/\sqrt{2}$.

In high-$T_c$ superconductors, field fluctuations become important. These can be taken into account by calculating the partition function and field correlation functions from the functional integral

$$Z = \int D\rho \, D\mathbf{A} \, e^{-\int d^2 x \, \mathcal{H}} \quad (5)$$

(in natural units with $k_B T = 1$). So far, all approximations to $Z$ found since the initial work [5] have had notorious difficulties in accounting for the order of the superconductive phase transition. In [5], simple renormalization group arguments [8] in $4 - \epsilon$ dimensions suggested that the transition should be of first order. The technical signal for this is the nonexistence of an infrared-stable fixed point in the renormalization group flow of the coupling constants $e$ and $g$ as a function of the renormalization scale. Due to the smallness of the Ginzburg interval $\Delta T_G$, the first order was never verified experimentally. Since then, there has been much work [9] trying to find an infrared-stable fixed point in higher loop orders or by different resummations of the divergent perturbations expansions.

The simplest argument suggesting a first-order nature of the transition arises at the mean-field level of the pair
field $\rho$ as follows: The fluctuations of the vector potential are Gaussian and can be integrated out in (5). Assuming $\rho$ to be smooth, this may be done in the Thomas-Fermi approximation [10], leading to an additional cubic term in the potential (4), which becomes

$$V(\rho) = \frac{\tau}{2} \rho^2 + \frac{g}{4} \rho^4 - \frac{c}{3} \rho^3, \quad c \equiv \frac{\epsilon^3}{2\pi}. \quad (6)$$

The cubic term generates, for $\tau < c^2/4g$, a second minimum at

$$\tilde{\rho}_0 = \frac{c}{2g} \left( 1 + \sqrt{1 - \frac{4\tau g}{c^2}} \right). \quad (7)$$

If $\tau$ decreases below

$$\tau_1 = \frac{2c^2}{9g}. \quad (8)$$

the new minimum lies lower than the one at the origin (see Fig. 1), so that the order parameter jumps from zero to

$$\rho_1 = \frac{2c}{3g} \quad (9)$$
in a phase transition. At this point, the coherence length of the $\rho$-fluctuations $\xi = 1/\sqrt{\tau + 3g\rho^2 - 2c\rho}$ has the finite value (the same as the fluctuations around $\rho = 0$)

$$\xi_1 = \frac{3}{c} \sqrt{\frac{\tau}{2}}. \quad (10)$$

The phase transition is therefore of first-order.

![FIG. 1. Potential for the order parameter $\rho$ with cubic term. A new minimum develops around $\rho_1$ causing a first-order transition for $\tau = \tau_1$.](image)

This conclusion is reliable only if the jump of $\rho_0$ is sufficiently large. For small jumps, the mean-field discussion of the energy density (6) cannot be trusted. The place where the transition becomes second order has, so far, never been explained satisfactorily within the Ginzburg-Landau theory. An explanation has been found only with the help of a dual disorder field theory derived from the Ginzburg-Landau theory in Ref. [11,12]. This theory is constructed in such a way that its Feynman diagrams are direct pictures of the vortex lines of the superconductor. The dual disorder field theory shows that there is a first-order transition only if the Ginzburg parameter $\kappa \equiv \lambda / \sqrt{2} \xi$ is smaller than the tricritical value $\kappa_{\text{tric}} \approx 0.8 / \sqrt{2}$. This point is close to the mean-field value $\kappa = 1 / \sqrt{2}$ where the superconductor changes from type II to type I, and the average short-range repulsion between vortex lines changes into an attraction, so that the quartic term in the dual field theory becomes negative [11].

In contrast to the Ginzburg-Landau theory, the vector potential of the disorder field theory is massive from the outset, so that its fluctuations do not generate a cubic term. Instead, they change the sign of the quartic term, making the transition first-order for $\kappa < \kappa_{\text{tric}}$, while leaving it second-order for $\kappa > \kappa_{\text{tric}}$.

The purpose of this note is to show how the tricritical point can be derived from the original Ginzburg-Landau theory by a proper inclusion of fluctuation corrections. The mistake in the above argument lies in the neglect of vortex fluctuations. In fact, the transformation of the covariant derivative $D\psi$ to the $\rho$-$\theta$ expression in Eq. (2) is false. Since $\theta(x)$ and $\theta(x) + 2\pi$ are physically indistinguishable — the complex field $\psi(x)$ is the same for both — the correct substitution is

$$D\psi = [i(\nabla \theta - 2\pi \theta^2 - cA)\rho + \nabla \rho]e^{i\delta}, \quad (11)$$

The cyclic nature of the scalar field $\theta(x)$ requires the presence of a vector field $\theta^v(x)$ called vortex gauge field. This field is a sum of $\delta$-functions on Volterra surfaces across which $\theta(x)$ has jumps by $2\pi$. The boundary lines of the surfaces are vortex lines. They are found from the vortex gauge field $\theta^v(x)$ by forming the curl

$$\nabla \times \theta^v(x) = j^v(x), \quad (12)$$

where $j^v(x)$ is the vortex density, a sum over $\delta$-functions along the vortex lines $\delta(L;x) \equiv \int V dx \delta(x - \overline{x})$. Vortex gauge transformations are deformations of the surfaces at fixed boundary lines which add to $\theta^v(x)$ pure gradients of the form $\nabla \delta(V;x)$, where $\delta(V;x) \equiv \int V d^3x \delta(x - \overline{x})$ are $\delta$-functions on the volumes $V$ over which the surfaces have swept. The theory of these fields has been developed in the textbook [12] and the Cambridge lectures [13]. Being a gauge field, $\theta^v(x)$ may be modified by a further gradient of a smooth function to make it purely transverse, $\nabla \cdot \theta^v_T(x) = 0$, as indicated by the subscript $T$. Since the vortex gauge field is not a gradient, it cannot be absorbed into the vector potential by a gauge transformation. Hence it survives in the last term in Eq. (3), and the correct partition function is

$$Z \approx \int D\theta^v \int D\rho \rho D\mathbf{A} \exp \left[ -\frac{1}{2}(\nabla \rho)^2 - \frac{\tau}{2} \rho^2 - \frac{g}{4} \rho^4 - \frac{\tau}{2}(\nabla \times \mathbf{A})^2 - \rho^2 \epsilon^2 (A - 2\pi \theta^v_T/\epsilon)^2 \right]. \quad (13)$$

The symbol $\int D\theta^v$ does not denote an ordinary functional integral. It is defined as a sum over all numbers.
and shapes of Volterra surfaces $S$ in $\theta_{T}^\mu$, across which the phase jumps by $2\pi$ [13].

The important observation is now that the partial partition function contained in (13)

$$Z_1[\rho] = \int D\theta_{T}^\mu D\mathbf{A} \exp \left\{ -\frac{1}{2} \int d^3x (\nabla \times \mathbf{A})^2 - \frac{\rho^2}{2} \int d^3x |e\mathbf{A} - 2\pi\theta_{T}^\mu|^2 \right\}$$

(14)

can give rise to a second-order transition of the XY-model type if the Ginzburg parameter $\kappa$ is sufficiently large. To see this we integrate out the $\mathbf{A}$-field and obtain

$$Z_1[\rho] = \exp \left[ \frac{4\pi^2\rho^2}{2} \int d^3x \left( \frac{\theta_{T}^\mu - \nabla^2}{-\nabla^2 + \rho^2 e^2 \theta_{T}^\mu} \right) \right].$$

(15)

The second integral can be simplified to

$$\frac{4\pi^2\rho^2}{2} \int d^3x \left( \frac{\theta_{T}^\mu - \nabla^2}{-\nabla^2 + \rho^2 e^2 \theta_{T}^\mu} \right).

(16)

Integrating this by parts, and replacing $\nabla, \theta_{T}^\mu, \nabla, \theta_{T}^\mu$ by $(\nabla \times \theta_{T}^\mu)^2 = j^\nu$, the partition function (15) without the prefactor takes the form

$$Z_2[\rho] = \int D\theta_{T}^\mu \exp \left[ -\frac{4\pi^2\rho^2}{2} \int d^3x \left( j^\nu - \frac{1}{-\nabla^2 + \rho^2 e^2 j^\nu} \right) \right].

(17)

This is the partition function of the grand-canonical ensemble of closed fluctuating vortex lines. The interaction between them has a finite range equal to the penetration depth $\lambda = 1/\rho e$.

It is well-known how to compute pair and magnetic fields of the Ginzburg-Landau theory for a single straight vortex line from the extrema of the energy density [2]. In an external magnetic field, there exist triangular and various other regular arrays of vortex lattices and various phase transitions. In the core of each vortex line, the pair field $\rho$ goes to zero over a distance $\xi$. If we want to sum over grand-canonical ensemble of fluctuating vortex lines of any shape in the partition function (13), the space dependence of $\rho$ causes complications. These can be avoided by an approximation, in which the system is placed on a simple-cubic lattice of spacing $a = \alpha \xi$, with $\alpha$ of the order of unity, and a fixed value $\rho = \rho_0$ given by Eq. (7). Thus we replace the partial partition function (17) approximately by

$$Z_2[\rho_0] = \sum_{\{\nabla f = 0\}} \exp \left[ -\frac{4\pi^2\rho_0^2}{2} \sum \mathbf{l}(\mathbf{x}) v_{\rho_0 e}(\mathbf{x} - \mathbf{x'}) \mathbf{l}(\mathbf{x'}) \right].

(18)

The sum runs over the discrete versions of the vortex density in (12). These are integer-valued vectors $\mathbf{l}(\mathbf{x}) = (l_1(\mathbf{x}), l_2(\mathbf{x}), l_3(\mathbf{x}))$ which satisfy $\nabla \cdot \mathbf{l}(\mathbf{x}) = 0$, where $\nabla$ denotes the lattice derivative. This condition restricts the sum over all $\mathbf{l}(\mathbf{x})$-configurations in (18) to all non-selfbacktracking integer-valued closed loops. The function

$$v_m(\mathbf{x}) = \prod_{i=1}^3 \int d^3(\mathbf{k}_i) \exp \left\{ i(\mathbf{k}_1 x_1 + \mathbf{k}_2 x_2 + \mathbf{k}_3 x_3) \right\}

= \int d\mathbf{s} e^{-i (\mathbf{m} + \mathbf{x}) \cdot \mathbf{s}} I_{x_1}(2s) I_{x_2}(2s) I_{x_3}(2s).

(19)

is the lattice Yukawa potential [14].

The lattice partition function (18) is known to have a second-order phase transition in the universality class of the XY-model. This can be seen by a comparison with the Villain approximation [15] to the XY model, whose partition function is a lattice version of

$$Z_V[\rho] = \int D\theta \int D\theta_{T}^\mu \exp \left[ -\frac{b}{2} \int d^3x (\nabla \theta - \theta_{T}^\mu)^2 \right].$$

After integrating out $\theta(\mathbf{x})$, this becomes

$$Z_V[\rho] = \text{Det}^{-1/2}(\nabla^2) \int D\theta_{T}^\mu \exp \left[ -\frac{b}{2} \int d^3x (\theta_{T}^\mu)^2 \right].

(20)

and we can replace $\theta_{T}^\mu$ by $\nabla \times \theta_{T}^\mu (\nabla^2)^{-1}(\nabla \times \theta_{T}^\mu)^2 = j^\nu (\nabla^2)^{-1}j^\nu$. By taking this expression to a simple-cubic lattice we obtain the partition function (18), but with $\rho_0^2 a$ replaced by $\beta_V \equiv ba$, and the Yukawa potential $v_{\rho_0 e}(\mathbf{x})$ replaced by the Coulomb potential $v_0(\mathbf{x})$.

The partition function (18) has the same transition at roughly the same place as its local approximation

$$Z_2[\rho_0] \approx \sum_{\{\nabla f = 0\}} \exp \left[ -\frac{4\pi^2\rho_0^2}{2} v_{\rho_0 e}(\mathbf{0}) \sum \mathbf{l}(\mathbf{x}) \right].

(21)

A similar approximation holds for the Villain model with $v_0(\mathbf{x})$ instead of $v_{\rho_0 e}(\mathbf{x})$ and $\rho_0^2 a$ replaced by $\beta_V \equiv ba$.

The Villain model is known to undergo a second-order phase transition of the XY-model type at $\beta_V = r/3$ with $r \approx 1$, where the vortex lines become infinitely long [17]. Thus we conclude that also the partition function (21) has a second-order phase transition of the XY-model type at $\rho_0^2 v_{\rho_0 e}(\mathbf{0}) = v_0(\mathbf{0})/3$. The potential (19) at the origin has the hopping expansion [16]

$$v_m(\mathbf{0}) = \sum_{n=0,2,4} \frac{H_n}{(a^2 m^2 + 6)^{n+1}}, \quad H_0 = 1, H_2 = 6, \ldots,

(22)

To lowest order, this yields the ratio $v_m(\mathbf{0})/v_0(\mathbf{0}) \approx 1/(m^2/6 + 1)$. A more accurate fit to the ratio $v_m(\mathbf{0})/v_0(\mathbf{0})$ which is good up to $m^2 \approx 10$ (thus comprising all interesting $\kappa$-values since $m^2$ is of the order of $3/\kappa^2$) is $1/(\sigma m^2/6 + 1)$ with $\sigma \approx 1.38$. Hence the transition takes place at
\[
\frac{\rho_{0}^{3} \alpha}{(\sigma \alpha^{2} \rho_{0} e^{2}/6 + 1)} \approx \frac{r}{3} \quad \text{or} \quad \rho_{0} \approx \frac{1}{\sqrt{3} \alpha} \sqrt{\frac{r}{1 - \sigma \alpha e^{2}/18}}.
\]

(23)

This transition will persist until \(\rho_{0}\) reaches the value \(\rho_{1} = 2e/3g\) of Eq. (9). From there on, the transition will no longer be of the XY-model type but occur discontinuously as a first-order transition.

Replacing in (23) \(a\) by \(\alpha \zeta_{1}\) of Eq. (10), and \(\rho_{0}\) by \(\rho_{1}\), we find the equation for the mean-field Ginzburg parameter \(\kappa_{\text{MF}} = \sqrt{g/\alpha^{2}}:\)

\[
\kappa_{\text{MF}}^{3} + \alpha^{2} \sigma \kappa_{\text{MF}}^{3} - \frac{2 \alpha}{\pi r} = 0.
\]

(24)

Inserting \(\sigma \approx 1.38\) and choosing \(\alpha \approx r \approx 1\), the solution of this yields the tricritical value

\[
\kappa_{\text{MF}}^{\text{tric}} \approx 0.81/\sqrt{2}.
\]

(25)

In spite of the roughness of the approximations, this result is very close to the value \(0.8/\sqrt{2}\) derived from the dual theory in [11]. The approximation has three uncertainties. First, the identification of the effective lattice spacing \(a = \alpha \zeta\) with \(\alpha \approx 1\); second the associated neglect of the \(x\)-dependence of \(\rho\) and its fluctuations, and third the localization of the critical point of the XY-model type transition in Eq. (23) setting \(g \approx 1\). By modifying slightly the parameters \(q\) and \(r\) we can, of course, obtain complete agreement with the previous result.

Our goal has been achieved: We have shown the existence of a tricritical point in a superconductor directly within the fluctuating Ginzburg-Landau theory, by taking the vortex fluctuations into account. For \(\kappa > 0.76/\sqrt{2}\) these give rise to an XY-model type second-order transition before the cubic term becomes relevant. The cubic term causes a discontinuous transition only for \(\kappa < 0.76/\sqrt{2}\).

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[16] See Eq. (6.122) in Part I of Ref. [12] (ibid/b1/gifs/v1-168s.html).
[17] See the figure on p. 501 of Ref. [12] (ibid/b1/gifs/v1-501s.html).