Submanifolds with harmonic mean curvature vector field in contact 3-manifolds

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Abstract

Biharmonic or polyharmonic curves and surfaces in 3-dimensional contact manifolds are investigated.

AMS Mathematics Subject Classification: 2000 53C42 53D10

Keywords and Phrases: Biharmonicity, polyharmonicity, Sasaki manifolds

Introduction

This paper concerns curves and surfaces in 3-dimensional contact manifolds, whose mean curvature vector field is in the kernel of certain elliptic differential operators.

First we study submanifolds whose mean curvature vector field is in the kernel of Laplacian (submanifolds with harmonic mean curvature vector fields).

The study of such submanifolds is inspired by a conjecture of Bang-yen Chen [14]:

Harmonicity of the mean curvature vector field implies harmonicity of the immersion {?}

The harmonicity equation \( \Delta \mathbf{H} = 0 \) for the mean curvature vector field \( \mathbf{H} \) of an immersed submanifold \( x : M^m \to E^n \) in Euclidean \( n \)-space is equivalent to the biharmonicity of the immersion: \( \Delta \Delta x = 0 \), since \( \Delta x = -m \mathbf{H} \).

A submanifold \( x : M \to E^n \) is said to be a biharmonic submanifold if \( \Delta \mathbf{H} = 0 \).

In 1985, Chen proved the nonexistence of proper biharmonic surfaces in Euclidean 3-space. The conjecture by Chen is still open.

Some partial and positive answers have been obtained by several authors [16-19, 25-27].

The biharmonicity equation is regarded as a special case of the following condition:

\[ \Delta \mathbf{H} = \lambda \mathbf{H}, \quad \lambda \in \mathbb{R}. \]

Namely the mean curvature vector field is an eigenfunction of the Laplacian.
The study of Euclidean submanifolds with $\Delta H = \lambda H$ was initiated by Chen in 1988 (See [14]). It is known that submanifolds in $E^n$ satisfying $\Delta H = \lambda H$ are either biharmonic ($\lambda = 0$), of 1-type or null 2-type. In particular all surfaces in $E^3$ with $\Delta H = \lambda H$ are of constant mean curvature. Moreover a surface in $E^3$ satisfies $\Delta H = \lambda H$ if and only if it is minimal, an open portion of a totally umbilical sphere or an open portion of a circular cylinder.

F. Defever [17] showed that hypersurfaces satisfying $\Delta H = \lambda H$ are of constant mean curvature. Note that Chen [12], [13] studied spacelike submanifolds with $\Delta H = \lambda H$ in Minkowski space, hyperbolic space or de Sitter space. M. Barros and O. J. Garay showed that Hopf cylinders in $S^3$ with $\Delta H = \lambda H$ are Hopf cylinders over circles in the 2-sphere $S^2$. A. Ferrández, P. Lucas and M. A. Meroño [24] studied such submanifolds in anti de Sitter 3-space $H^3_1$.

In non-constant curvature ambient spaces, results on biharmonic submanifolds are very few.

Recently, T. Sasahara [37–38] studied Legendre surfaces in the Sasakian space form $R^{5}(-3)$ satisfying $\Delta H = \lambda H$. Moreover Sasahara introduced the notion of “$\varphi$-position vector field” and “$\varphi$-mean curvature vector field” for submanifolds in Sasakian space form $R^{2n+1}(-3)$. Sasahara investigated submanifolds in $R^{2n+1}(-3)$ whose $\varphi$-mean curvature vector field $\mathbb{H}_\varphi$ satisfies $\Delta \mathbb{H}_\varphi = \lambda \mathbb{H}_\varphi$. In particular he classified curves and surfaces in $R^{3}(-3)$ with $\Delta \mathbb{H}_\varphi = \lambda \mathbb{H}_\varphi$. Since both $R^{2n+1}(-3)$ and $S^{2n+1}$ are typical examples of Sasakian space form, it seems to be interesting to study biharmonic submanifolds in general Sasakian space forms.

Based on these observations, in the first part of this paper, we shall study harmonicity of mean curvature vector fields of curves and surfaces in 3-dimensional Sasakian space forms. Several results for 3-dimensional sphere $S^3$ due to Spanish research group (Barros, Garay Ferrández, Lucas and Meroño) will be generalised to 3-dimensional Sasakian space forms.

Next, in the second part, we shall study another “biharmonicity” suggested by J. Eells and J. H. Sampson [23]. A smooth map $\phi : M \to N$ between Riemannian manifolds is said to be a biharmonic map (or polyharmonic map of order 2) if its bitension field $T_2(\phi)$ vanishes. In [9], “biharmonic” curves and surfaces in $S^3$ are classified. We shall classify Legendre curves and Hopf cylinders in 3-dimensional Sasakian space forms, which are biharmonic in this sense.

In particular we shall show the existence of non-minimal biharmonic Hopf cylinders in Sasakian space forms of holomorphic sectional curvature greater than 1 (Berger spheres).

The author would like to thank Dr. Cezar Dumitru Oniciuc (University “AL. I. Cuza”) and Dr. Tooru Sasahara (Hokkaido University) for their useful comments.
Part I

1 Preliminaries

1.1 Contact manifolds

We begin by recalling fundamental ingredients of contact Riemannian geometry from [7].

Let $M$ be a $(2n + 1)$-manifold. A one form $\eta$ is called a contact form on $M$ if $(d\eta)^n \wedge \eta \neq 0$. A $(2n + 1)$-manifold $M$ together with a contact form is called a contact manifold. The contact distribution $D$ of $(M, \eta)$ is defined by

$$D = \{X \in TM \mid \eta(X) = 0\}.$$ 

On a contact manifold $(M, \eta)$, there exists a unique vector field $\xi$ such that

$$\eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0.$$

This vector field $\xi$ is called the Reeb vector field or characteristic vector field of $(M, \eta)$.

Moreover there exists an endomorphism field $\varphi$ and a Riemannian metric $g$ on $M$ such that

1. $\varphi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$,
2. $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$, $g(\xi, \cdot) = \eta$,
3. $d\eta(X, Y) = 2g(X, \varphi Y)$

for all vector fields $X, Y$ on $M$. On an almost contact manifold $(M, \eta; \xi, \varphi)$, there exists a Riemannian metric $g$ satisfying (2). Such a metric $g$ is called a compatible metric of $M$. A contact manifold $(M, \eta)$ together with structure tensors $(\xi, \varphi, g)$ is called a contact Riemannian manifold.

Proposition 1.1 Let $(M, \eta, \xi, \varphi, g)$ be a contact Riemannian manifold. Then $M$ is a Killing vector field if and only if

$$\nabla_X \xi = -\varphi X, \quad X \in \mathfrak{X}(M).$$

Here $\nabla$ is the Levi-Civita connection of $(M, g)$.

Definition 1.1 A contact Riemannian manifold $(M, \eta, \xi, \varphi, g)$ is said to be a Sasaki manifold if

$$\nabla_{X\varphi}Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in \mathfrak{X}(M).$$
Note that on a Sasaki manifold, $\xi$ is a Killing vector field.

Let $(M, \eta; \xi, \varphi, g)$ be a contact Riemannian manifold. A tangent plane at a point of $M$ is said to be a holomorphic plane if it is invariant under $\varphi$. The sectional curvature of a holomorphic plane is called holomorphic sectional curvature. If the sectional curvature function of $M$ is constant on all holomorphic planes in $TM$, then $M$ is said to be of constant holomorphic sectional curvature. Complete and connected Sasaki manifolds of constant holomorphic sectional curvature are called Sasaki space forms. Let us denote by $R$ the Riemannian curvature tensor of the metric $g$ which is defined by

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \quad X, Y \in X(M).$$

When $(M, \eta; \xi, \varphi, g)$ is a Sasakian space form of constant holomorphic sectional curvature $c$, then $R$ is described by the following formula:

$$R(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(Z,X)Y\} + \frac{c-1}{4}\{\eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X \\
+ g(Z,X)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\
- g(Y,\varphi Z)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\}.$$ 

Note that even if the holomorphic sectional curvature is negative, a Sasakian space form is not negatively curved. In fact, the sectional curvature of plane sections containing $\xi$ is 1 on any Sasaki manifold.

It is known that every 3-dimensional Sasakian space form is realised as a Lie group together with a left invariant Sasaki structure. More precisely the following is known (cf. [6]):

**Proposition 1.2** Simply connected 3-dimensional Sasakian space form of constant holomorphic sectional curvature is isomorphic to

1. special unitary group $SU(2)$;
2. Heisenberg group $\mathbb{R}^3(-3)$;
3. the universal covering group of the special linear group $SL_2\mathbb{R}$

together with canonical left invariant Sasaki structure. In particular simply connected Sasakian space form of constant holomorphic sectional curvature 1 is the $SU(2)$ with biinvariant metric of constant curvature 1 (hence isometric to the unit 3-sphere $S^3$).
1.2 Boothby-Wang fibration

Let \((M^{2n+1}, \eta, \xi, \varphi, g)\) be a contact Riemannian manifold. Then \(M\) is said to be regular if \(\xi\) generates a one-parameter group \(K\) of isometries on \(M\), such that the action of \(K\) on \(M\) is simply transitive. Note that if \(M\) is regular, then both \(\varphi\) and \(\eta\) are automatically \(K\)-invariant, i.e., \(\mathcal{L}_\xi \varphi = 0\) and \(\mathcal{L}_\xi \eta = 0\). The Killing vector field \(\xi\) induces a regular one-dimensional Riemannian foliation on \(M\). We denote by \(\overline{M} := M/K\) the orbit space (the space of all leaves) of a regular contact Riemannian manifold \(M\) under the \(K\)-action.

Let \(\bar{X}_p\) be a tangent vector of the orbit space \(\overline{M}\) at \(\bar{p} = \pi(p)\). Then there exists a tangent vector \(\bar{X}_p^*\) of \(\bar{M}\) at \(p\) which is orthogonal to \(\xi\) such that \(\pi^* \bar{X}_p = \bar{X}_p^*\). The tangent vector \(\bar{X}_p^*\) is called the horizontal lift of \(\bar{X}_p\) to \(\bar{M}\) at \(p\). The horizontal lift operation \(* : \bar{X}_p \mapsto \bar{X}_p^*\) is naturally extended to vector fields.

The contact structure on \(M\) induces an almost Hermitian structure on the orbit space \(\overline{M}\):

\[
\mathcal{J}\bar{X} = \pi^*(\varphi \bar{X}^*), \quad \bar{X} \in \mathfrak{X}(\bar{M}).
\]

Let us denote by \(\bar{\nabla}\) the Levi-Civita connection of \(\bar{M}\). Then, by using the fundamental equations for Riemannian submersions due to O’Neill [33], we have the following results.

**Proposition 1.3** ([32]) Let \(M\) be a regular contact Riemannian manifold. Then for any \(\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})\):

\[
\bar{\nabla}_{\bar{X}} \bar{Y}^* = (\bar{\nabla}_{\bar{X}} \bar{Y})^* - g(\bar{X}^*, \varphi \bar{Y}^*)\xi.
\]

**Proposition 1.4** ([32]) Sasakian space forms are regular Sasaki manifolds. The orbit space of a Sasakian space form of constant holomorphic sectional curvature \(c\) is a complex space form of constant holomorphic sectional curvature \(c + 3\).

W. M. Boothby and H. C. Wang [8] proved that if \(M\) is a compact regular contact manifold, then the natural projection \(\pi : M \to \bar{M}\) defines a principal circle bundle over a symplectic manifold \(\bar{M}\) and the symplectic form \(\Omega\) of \(\bar{M}\) determines an integral cocycle. Furthermore the contact form \(\eta\) gives a connection form of this circle bundle and satisfies \(\pi^* \Omega = d\eta\). The fibering \(\pi : M \to \bar{M}\) is called the Boothby-Wang fibering of a regular compact contact manifold \(M\). Based on this result, we call the fibering \(\pi : M \to \bar{M}\) of a regular contact Riemannian manifold \(M\), the “Boothby-Wang fibering” of \(M\) even if \(M\) is noncompact.

The unit sphere \(S^{2n+1}\) is a typical example of regular compact Sasaki manifold. For \(S^{2n+1}\), the Boothby-Wang fibering coincides with the Hopf fibering \(S^{2n+1} \to \mathbb{C}P^n\).

In 3-dimensional case, the Boothby-Wang fibering of Sasakian space forms have the following matrix group models [6]:

\[
\begin{align*}
\pi : \text{SU}(2) & \to S^2(\mathbb{C}) = \text{SU}(2)/U(1), \\
\pi : \mathbb{R}^3(-3) & \to \mathbb{C} = \mathbb{R}^3(-3)/\mathbb{R}, \\
\pi : \text{SL}_2\mathbb{R} & \to H^2(\mathbb{C}) = \text{SL}_2\mathbb{R}/\text{SO}(2).
\end{align*}
\]
Here $S^2(c)$ and $H^2(c)$ are sphere and hyperbolic space of curvature $c$, respectively.

### 1.3 Hopf cylinders

Now we shall restrict our attention to 3-dimensional regular contact Riemannian manifold $M$.

Let $\tilde{\gamma}$ be a curve parameterized by arc length in $\mathbb{M}$ with curvature $\tilde{\kappa}$. Taking the inverse image $S_{\tilde{\gamma}} := \pi^{-1}\{\tilde{\gamma}\}$ of $\tilde{\gamma}$ in $M^3$.

Here we compute the fundamental quantities of $S_{\tilde{\gamma}}$.

Let us denote by $\tilde{\mathbf{P}} = (\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)$ the Frenet frame field of $\tilde{\gamma}$. By using the complex structure $J$ of $\mathbb{M}^2$, $\tilde{\mathbf{p}}_2$ is given by $\tilde{\mathbf{p}}_2 = J\tilde{\mathbf{p}}_1$.

Then the Frenet-Serret formula of $\tilde{\gamma}$ is given by

$$\tilde{\nabla}_{\tilde{\gamma}'} P = P \begin{pmatrix} 0 & -\tilde{\kappa} \\ \tilde{\kappa} & 0 \end{pmatrix}.$$  

Here the function $\tilde{\kappa}$ is the (signed) curvature of $\tilde{\gamma}$.

Let $t = (\tilde{\mathbf{p}}_1)^*$ the horizontal lift of $\tilde{\mathbf{p}}_1$ with respect to the Boothby-Wang fibering. Then $(t, \xi)$ gives an orthonormal frame field of $S$. We choose a unit normal vector field $\mathbf{n}$ by $\mathbf{n} = (\tilde{\mathbf{p}}_2)^*$. Since $\tilde{\mathbf{p}}_2$ is defined by $\tilde{\mathbf{p}}_2 = J\tilde{\mathbf{p}}_1$, $\mathbf{n} = \varphi t$. In fact,

$$(\tilde{\mathbf{p}}_2)^* = (J\tilde{\mathbf{p}}_1)^* = \varphi(\tilde{\mathbf{p}}_1)^* = \varphi t.$$  

Let us denote by $\nabla^S$ the Levi-Civita connection of $S$. The second fundamental form $II$ derived from $\mathbf{n}$ is defined by the Gauss formula:

$$\nabla_XY = \nabla^S_XY + II(X,Y)\mathbf{n}, \quad X, Y \in \mathfrak{X}(S).$$  

By using (7),

$$\nabla_t t = (\nabla_{\tilde{\mathbf{p}}_1}\tilde{\mathbf{p}}_1)^* - g(t, \varphi t)\xi = (\tilde{\kappa} \circ \pi)\mathbf{n}.$$  

Hence $\nabla^S_t t = 0$. Since $\xi$ is Killing, we have $\nabla^S_t \xi = \nabla^S_\xi \xi = 0$. Thus $S_{\tilde{\gamma}}$ is flat. The second fundamental form $II$ is described as

$$II(t, t) = \tilde{\kappa} \circ \pi, \quad II(t, \xi) = -1, \quad II(\xi, \xi) = 0.$$  

The mean curvature is $H = (\tilde{\kappa} \circ \pi)/2$ and the mean curvature vector field $\mathbb{H}$ is $\mathbb{H} = H \mathbf{n}$.

In case $M = S^3$, $S_{\tilde{\gamma}}$ is called the Hopf cylinder. In particular if $\tilde{\gamma}$ is closed, then $S_{\tilde{\gamma}}$ is a flat torus in $S^3$ and called the Hopf torus over $\tilde{\gamma}$ (H. B. Lawson, cf. [31], [35]). The Hopf torus over a geodesic in $S^2(4)$ coincides with the Clifford minimal torus. We call the flat surface $S_{\tilde{\gamma}}$ in a regular contact Riemannian manifold $M$ a Hopf cylinder over the curve $\tilde{\gamma}$ in $\mathbb{M}$.  

6
1.4 Curves in Riemannian 3-manifolds

Let \((M, g)\) be a Riemannian manifold and \(\gamma = \gamma(s) : I \to M\) a curve parametrised by the arclength parameter in \(M\). We regard \(\gamma\) as a 1-dimensional Riemannian manifold with respect to the metric induced by \(g\).

We recall the following definition (cf. [2]).

**Definition 1.2** If \(\gamma(s)\) is a unit speed curve in a Riemannian 3-manifold \((M^3, g)\), we say that \(\gamma\) is a Frenet curve if there exists an orthonomal frame field \(P = (p_1, p_2, p_3)\) along \(\gamma\) and two nonnegative functions \(\kappa\) and \(\tau\) such that \(P\) satisfies the following Frenet-Serret formula:

\[
\nabla_{\gamma'} P = P \begin{pmatrix}
0 & -\kappa & 0 \\
\kappa & 0 & -\tau \\
0 & \tau & 0
\end{pmatrix}, \quad p_1 = \gamma'(s).
\]

The functions \(\kappa\) and \(\tau\) are called the curvature and torsion of \(\gamma\) respectively.

Geodesics can be regarded as Frenet curves with \(\kappa = 0\). A curve with constant curvature and zero torsion is called a (Riemannian) circle. A helix is a curve whose curvature and torsion are constants. Riemannian circles are regarded as degenerate helices. Helices, which are not circles, are frequently called proper helices.

Note that, in general ambient space \((M^3, g)\), geodesics may have non-vanishing torsion. In fact, as we shall see later, Legendre geodesics in a Sasakian 3-manifold have constant torsion 1.

The Frenet-Serret formula of \(\gamma\) implies that the mean curvature vector field \(\mathbb{H}\) of a Frenet curve \(\gamma\) is given by

\[
\mathbb{H} = \nabla_{\gamma'} \gamma' = \kappa p_2.
\]

Let us denote by \(\Delta\) the Laplace operator acting on the space \(\Gamma(\gamma^*TM)\) of all smooth sections of the vector bundle:

\[
\gamma^*TM := \bigcup_{s \in I} T_{\gamma(s)}M
\]

over \(I\). Then \(\Delta\) is given explicitly by

\[
\Delta = -\nabla_{\gamma'} \nabla_{\gamma'}.
\]

**Lemma 1.1** The mean curvature vector field \(\mathbb{H}\) of a Frenet curve \(\gamma\) is harmonic in \(\gamma^*TM\) \((\Delta \mathbb{H} = 0)\) if and only if

\[
\nabla_{\gamma'} \nabla_{\gamma'} \nabla_{\gamma'} \gamma' = 0.
\]

When \(M\) is the Euclidean space \(\mathbb{E}^m\), a curve \(\gamma\) satisfies \(\Delta \mathbb{H} = 0\) if and only if \(\gamma\) is biharmonic, i.e., \(\Delta \Delta \gamma = 0\) since \(\Delta \gamma = -\mathbb{H}\).

The following general result is essentially obtained in [2].
Theorem 1.1 Let \( \gamma \) be a Frenet curve in a Riemannian 3-manifold \((M, g)\). Then \( \gamma \) satisfies \( \Delta H = \lambda H \) in \( \gamma^*TM \) if and only if \( \gamma \) is a geodesic (\( \lambda = 0 \)) or a helix satisfying \( \lambda = \kappa^2 + \tau^2 \).

Proof. Let \( I \) be an open interval and \( \gamma = \gamma(s) : I \to M \) be a curve parametrised by the arclength parameter \( s \) with Frenet frame field \( P = (p_1, p_2, p_3) \). Direct computation shows that

\[
\nabla_{\gamma'}H = -\kappa^2 p_1 + \kappa'p_2 + \kappa\tau p_3. \tag{9}
\]

Let us compute the Laplacian of \( H \):

\[
-\Delta H = \nabla_{\gamma'} \nabla_{\gamma'} H = -3\kappa\kappa' p_1 + (\kappa'' - \kappa^3 - \kappa\tau^2)p_2 + (2\kappa'\tau + \kappa\tau')p_3.
\]

Hence \( \Delta H = \lambda H \) if and only if

\[
\kappa\tau' = 0, \quad \kappa^3 + \kappa\tau^2 = \lambda \kappa.
\]

These formulae imply that \( \gamma \) is a geodesic or a helix satisfying \( \lambda = \kappa^2 + \tau^2 \).

Conversely every geodesic satisfies \( \Delta H = 0 \). Helices satisfy \( \Delta H = \lambda H \) with \( \lambda = \kappa^2 + \tau^2 \).

Corollary 1.1 ([19]) Let \( \gamma \) be a curve in Euclidean 3-space \( \mathbb{E}^3 \). Then \( \gamma \) is biharmonic if and only if \( \gamma \) is a straight line.

On the contrary, in indefinite semi-Euclidean space, there exist nongeodesic biharmonic curves. Chen and Ishikawa [15] classified biharmonic spacelike curves in \( \mathbb{E}^m_{\nu} \). (See also [28]).

1.5 Curves with normal-harmonic mean curvature

The results in the preceding subsection say that to characterise curves which are non geodesics we need to use another differential operator for our purpose.

In this subsection we use the normal Laplacian.

Let \( \gamma : I \to M \) be a Frenet curve in an oriented Riemannian 3-manifold \( M \) parametrised by the arclength. Denote by \( P = (p_1, p_2, p_3) \) the Frenet frame field of \( \gamma \) as before. Then the normal bundle \( T^\perp \gamma \) of the curve \( \gamma \) is given by

\[
T^\perp \gamma = \bigcup_{s \in I} T^\perp_{s}\gamma, \quad T^\perp_{s}\gamma = \mathbb{R} p_2(s) \oplus \mathbb{R} p_3(s).
\]

The normal connection \( \nabla^\perp \) is a connection of \( T^\perp \gamma \) defined by

\[
\nabla^\perp_{\gamma'}X = \text{normal component of } \nabla_{\gamma'}X
\]

for any section \( X \) of the normal bundle \( T^\perp \gamma \).

By using the Frenet frame field, \( \nabla^\perp \) can be represented as

\[
\nabla^\perp_{\gamma'}X = \nabla_{\gamma'}X - g(\nabla_{\gamma'}X, p_1)p_1.
\]
Let us denote by $\Delta^\perp$ the Laplace operator acting on the space $\Gamma(T^\perp\gamma)$ of all smooth sections of the normal bundle $T^\perp\gamma$. The operator $\Delta^\perp$ is called the normal Laplacian of $\gamma$ in $M$. The normal Laplacian $\Delta^\perp$ is given by

$$\Delta^\perp X = -\nabla^\perp_{\gamma'} \nabla^\perp_{\gamma'} X, \quad X \in \Gamma(T^\perp\gamma).$$

Now we compute $\Delta^\perp H$. From (9), we have

$$\nabla^\perp_{\gamma'} H = \kappa' p^2 + \kappa \tau p^3.$$  

From this equation, we get

$$-\Delta^\perp H = (\kappa'' - \kappa \tau^2) p^2 + (2\kappa' \tau + \kappa \tau') p^3.$$ 

**Theorem 1.2** (cf. [24]) A curve $\gamma$ satisfies $\Delta^\perp H = \lambda H$ if and only if

$$\kappa'' - \kappa \tau^2 = -\lambda \kappa, \quad 2\kappa' \tau + \kappa \tau' = 0.$$ 

**Corollary 1.2** A curve $\gamma$ satisfies $\Delta^\perp H = 0$ if and only if

$$\kappa'' - \kappa \tau^2 = 0, \quad 2\kappa' \tau + \kappa \tau' = 0.$$ 

We shall apply these general results for curves in Sasakian 3-manifolds in the next section. Note that Barros and Garay classified curves, which satisfy $\Delta^\perp H = \lambda H$ in space forms [4],[5].

**2 Curves and surfaces in 3-dimensional Sasaki manifolds**

**2.1 Curves in 3-dimensional Sasaki manifolds**

Now let $M^3 = (M, \eta, \xi, \varphi, g)$ be a contact Riemannian 3-manifold with an associated metric $g$. A curve $\gamma = \gamma(s) : I \to M$ parametrised by the arclength parameter is said to be a Legendre curve if $\gamma$ is tangent to the contact distribution $D$ of $M$. It is obvious that $\gamma$ is Legendre if and only if $\eta(\gamma') = 0$.

Let $\gamma$ be a Legendre curve in $M^3$. Then we can take a Frenet frame field $P = (p_1, p_2, p_3)$ so that $p_1 = \gamma'$ and $p_3 = \xi$. (See [2]).

Now we assume that $M$ is a Sasaki manifold. Then by definition, the Frenet-Serret formula of $\gamma$ is given explicitly by

$$\nabla_{\gamma'} P = P \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. $$
Namely every Legendre curve has constant torsion 1 \cite{2}.

Now we investigate curves with harmonic or normal-harmonic mean curvature vector field in Sasakian 3-manifolds.

The following two results are direct consequence of Theorem 1.1 and Theorem 1.2 respectively.

**Corollary 2.1** Let $\gamma$ be a Legendre curve in 3-dimensional Sasaki manifold. Then $\gamma$ satisfies $\Delta H = \lambda H$ in $\gamma^*TM$ if and only if $\gamma$ is a Legendre geodesic ($\lambda = 0$) or a Legendre helix satisfying $\lambda = \kappa^2 + 1$ ($\lambda \neq 0$).

**Remark 2.1** Sasaki manifolds together with compatible Lorentz metric are called Sasakian spacetimes \cite{20,40}. On Sasakian spacetimes, the Reeb vector fields are timelike. Every 3-dimensional Sasakian spacetime contains proper biharmonic Legendre curves. In fact, in a 3-dimensional Sasakian spacetime biharmonic Legendre curves are Legendre geodesics or Legendre helices with curvature 1. (cf. 28).

**Proposition 2.1** Let $\gamma$ be a Legendre curve in a Sasakian 3-manifold. Then $\Delta^\bot H = \lambda H$ if and only if $\gamma$ is a Legendre geodesic ($\lambda = 0$) or a Legendre helix with constant nonzero curvature ($\lambda \neq 0$). In the latter case, $\lambda = 1$.

### 2.2 Biharmonic Hopf cylinders

In this section we study harmonicity and normal-harmonicity of the mean curvature of Hopf cylinders.

Let $M^3$ be a regular Sasaki manifold with Boothby-Wang fibration $\pi : M \to \bar{M}$.

Take a curve $\bar{\gamma} = \bar{\gamma}(s)$ parametrised by the arclength $s$ in the base space form $\bar{M}$. Let us denote by $\bar{S} = \bar{S}_{\bar{\gamma}}$ the Hopf cylinder of $\bar{\gamma}$. (See Section 1.3)

Let $t = (\bar{p}_1)^*$ be the horizontal lift of $\bar{p}_1$ with respect to the Boothby-Wang fibering. Then $(t, \xi)$ gives an orthonormal frame field of $M$. The unit normal vector field $n$ is the horizontal lift of $\bar{p}_2$. Note that $n = \varphi t$.

The mean curvature vector field $\bar{H}$ of $\bar{S}$ is $\bar{H} = H n = (\kappa \circ \pi)n/2$.

Now we study harmonicity and normal-harmonicity of $\bar{H}$. Denote by $\iota$ the inclusion map of $S$ into $\bar{M}$. Then the Laplace operator $\Delta$ acting on the space $\Gamma(\iota^*TM)$ and the normal Laplacian $\Delta^\bot$ of $S$ are given by

$$
\Delta = -(\nabla_t \nabla_t + \nabla_\xi \nabla_\xi), \quad \Delta^\bot = -\left(\nabla^\bot_t \nabla^\bot_t + \nabla^\bot_\xi \nabla^\bot_\xi\right),
$$

respectively. Direct computation shows that

\[\nabla_\xi \bar{H} = -2H^2 t + H' n + H \xi, \quad \nabla^\bot_t \bar{H} = H' n, \quad \nabla_\xi \bar{H} = H t, \quad \nabla^\bot_\xi \bar{H} = 0,\]

\[\nabla_\xi \nabla_\xi \bar{H} = -H n.\]
Thus we get
\[ -\Delta \mathbb{H} = -6HH'\mathbf{t} + (H'' - 4H^3 - 2H)\mathbf{n} + 2H'\xi, \]
\[ -\Delta \mathbb{H} = H''\mathbf{n}. \]

**Theorem 2.1** A Hopf cylinder \( S_{\gamma} \) in a 3-dimensional regular Sasaki manifold satisfies \( \Delta \mathbb{H} = \lambda \mathbb{H} \) in \( \tau^*\mathcal{M} \) if and only if \( \gamma \) is a geodesic (\( \lambda = 0 \)) or a Riemannian circle (\( \lambda \neq 0 \)). In case that \( \lambda \neq 0 \), the eigenvalue \( \lambda \) is \( \lambda = 4H^2 + 2 > 2 \).

**Remark 2.2** Every Hopf cylinder in a 3-dimensional regular Sasaki manifold is anti invariant. Sasahara showed that an anti invariant surface in \( \mathbb{R}^3(-3) \) satisfies \( \Delta \mathbb{H} = \lambda \mathbb{H}, \lambda \neq 0 \) if and only if it is a Hopf cylinder over a circle with \( \lambda > 2 \). See Proposition 11 in [37].

**Lemma 2.1** A Hopf cylinder \( S_{\gamma} \) satisfies \( \Delta \mathbb{H} = \lambda \mathbb{H} \) if and only if \( \gamma \) is defined by one of the following natural equations:

1. \( \bar{\kappa}(s) = as + b, \ a, b \in \mathbb{R}, \ \lambda = 0; \)
2. \( \bar{\kappa}(s) = a\cos(\sqrt{\lambda}s) + b\sin(\sqrt{\lambda}s), \ \lambda > 0; \)
3. \( \bar{\kappa}(s) = a\exp(\sqrt{-\lambda}s) + b\exp(-\sqrt{-\lambda}s), \ \lambda < 0. \)

*Proof.* The Hopf cylinder \( S_{\gamma} \) satisfies \( \Delta \mathbb{H} = \lambda \mathbb{H} \) if and only if \( \gamma \) satisfies
\[ \bar{\kappa}'' + \lambda \bar{\kappa} = 0. \]
Thus the result follows. \( \Box \)

**Theorem 2.2** A Hopf cylinder \( S_{\gamma} \) satisfies \( \Delta \mathbb{H} = 0 \) if and only if \( \gamma \) is one of the following:

1. a geodesic;
2. a Riemannian circle or;
3. a Riemannian clothoid (Cornu spiral).

Here a Riemannian clothoid is a curve in \( \bar{M}^2 \) whose curvature is a linear function of the arclength.

**Remark 2.3** On curves in Riemannian 2-space forms, the following result is obtained [24].

**Theorem 2.3** Let \( \tilde{\gamma} \) be a curve in Riemannian 2-manifold \( \bar{M}^2 \). To avoid the confusion, let us denote by \( \Delta \mathbb{H} \) and \( \mathbb{H}_{\gamma} \) the normal Laplacian of \( \tilde{\gamma} \) and the mean curvature vector in \( \bar{M}^2 \) respectively. Then \( \Delta \mathbb{H}_{\gamma} = \lambda \mathbb{H}_{\gamma} \) if and only if
(1) \( \tilde{\gamma} \) is a geodesic, Riemannian circle or a Riemannian clothoid;

(2) \( \tilde{\kappa}(s) = a\cos(\sqrt{\lambda} s) + b\sin(\sqrt{\lambda} s), \lambda > 0; \)

(3) \( \tilde{\kappa}(s) = a\exp(\sqrt{-\lambda} s) + b\exp(-\sqrt{-\lambda} s), \lambda < 0. \)

**Corollary 2.2** Let \( M \) be a 3-dimensional regular Sasaki manifold with Boothby-Wang fibering \( \pi : M \to \bar{M} \). Let \( \tilde{\gamma} \) be a curve in \( \bar{M} \). Then the Hopf cylinder \( S = S_{\tilde{\gamma}} \) satisfies \( \Delta^\perp \mathbb{H} = \lambda \mathbb{H} \) if and only if \( \tilde{\gamma} \) satisfies \( \Delta^\perp \mathbb{H}_{\tilde{\gamma}} = \lambda \mathbb{H}_{\tilde{\gamma}}. \)

Theorem 2.2 is a generalisation of a result obtained by Barros and Garay [3]. In fact, if we choose \( M^3 = S^3 \) then we obtain the following.

**Theorem 2.4** (3) A Hopf cylinder \( S_{\tilde{\gamma}} \) in the unit 3-sphere \( S^3 \) satisfies \( \Delta^\perp \mathbb{H} = 0 \) if and only if \( \gamma \) is one of the following:

(1) a geodesic,
(2) a Riemannian circle or
(3) a Riemannian clothoid.

Here a Riemannian clothoid is a curve in the 2-sphere \( S^2(1/2) \) of radius \( 1/2 \) whose curvature is a linear function of the arclength.

Riemannian clothoids are called “Cornu spirals” in [3].

**Part II**

3 Polyharmonic maps

Let \( (M^n, g) \) and \( (N^n, h) \) be Riemannian manifolds and \( \phi : M \to N \) a smooth map. The tension field \( T(\phi) \) is a section of the vector bundle \( \phi^*(TN) \) defined by

\[
T(\phi) := \text{tr}(\nabla d\phi).
\]

A smooth map \( \phi \) is said to be a harmonic map if its tension field vanishes. It is well known that \( \phi \) is harmonic if and only if \( \phi \) is a critical point of the energy:

\[
E(\phi) = \int \frac{1}{2} |d\phi|^2 dv_g
\]

over every compact supported region of \( M \).

Now let \( \phi \) be a harmonic map. Then the Hessian \( \mathcal{H}_\phi \) of the energy is given by the following second variation formula:

\[
\mathcal{H}_\phi(V, W) = \int h(\mathcal{J}_\phi(V), W) dv_g, \quad V, W \in \Gamma(\phi^*TN).
\]
Here the operator $J_\phi$ is the Jacobi operator of the harmonic map $\phi$ defined by

$$J_\phi(V) := \bar{\Delta}_\phi V - R_\phi(V), \quad V \in \Gamma(\phi^*TN),$$

$$\bar{\Delta}_\phi := -\sum_{i=1}^m (\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla^\phi e_i}), \quad R_\phi(V) = \sum_{i=1}^m R^N(V, d\phi(e_i))d\phi(e_i).$$

Here $\nabla^\phi$, $R^N$ and $\{e_i\}$ denote the induced connection of $\phi^*TN$, curvature tensor of $N$ and a local orthonormal frame field of $M$, respectively.

For general theory of harmonic maps and their Jacobi operators, we refer to [21] and [42].

J. Eells and J. H. Sampson suggested to study polyharmonic maps (See [23] and [21], p. 77 (8.7)). Let $\phi : M \to N$ be a smooth map as before. Then $\phi$ is said to be a polyharmonic map of order $k$ if it is an extremal of the functional:

$$E_k(\phi) = \int |(d + d^*)^k \phi|^2 dv_g.$$  

Here $d^*$ is the codifferential operator. In particular, if $k = 2$, we have

$$E_2(\phi) = \int |\mathcal{T}(\phi)|^2 dv_g.$$  

The Euler-Lagrange equation of the functional $E_2$ was computed by Caddeo and Oproiu (See [9], p. 867) and G. Y. Jiang [29]–[30], independently. The Euler-Lagrange equation of $E_2$ is

$$\mathcal{T}_2(\phi) := -J_\phi(\mathcal{T}(\phi)) = 0.$$  

**Remark 3.1** Let $\phi : M \to N$ be an isometric immersion. Then its tension field is $m\mathbb{H}$. Thus the functional $E_2$ is given by

$$E_2(\phi) = m^2 \int |\mathbb{H}|^2 dv_g.$$  

In case that $M$ is 2-dimensional, $E_2(\phi)$ the total mean curvature of $M$ up to constant multiple. See [11], Section 5.3.

In particular, if $N = \mathbb{E}^n$ and $\phi$ an isometric immersion, then

$$\mathcal{T}_2(\phi) = -\Delta_M \Delta_M \phi,$$

since $\Delta_M \phi = m\mathbb{H}$. Here $\Delta_M$ is the Laplacian of $(M, g)$. Thus the polyharmonicity (of order 2) for an isometric immersion into Euclidean space is equivalent to the biharmonicity in the sense of Chen. On this reason, polyharmonic maps of order 2 are frequently called biharmonic maps (or 2-harmonic maps) [9], [29], [30], [34].

Obviously, the notion of $p$-harmonic map in the sense of [22], p. 397 is different from that of polyharmonic map of order $p$. 

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Hereafter we call polyharmonic maps of order 2 by the name “polyharmonic maps” in short.

Caddeo, Montaldo and Oniciuc classified polyharmonic curves in 3-dimensional Riemannian space forms. More precisely they showed the following two results.

**Theorem 3.1** ([9]) Let $N$ be a 3-dimensional Riemannian space form of non-positive curvature. Then all the polyharmonic curves are geodesics.

Next for the study of polyharmonic curves in positively curved space forms, we may assume that $N^3$ is the unit 3-sphere.

**Theorem 3.2** ([9]) Let $\gamma : I \rightarrow S^3$ be a polyharmonic curve parametrised by the arclength. Then $0 \leq \kappa \leq 1$ and $\gamma$ is one of the following:

1. $\kappa$ is a geodesic ($\kappa = 0$).
2. If $k = 1$ then $\gamma$ is a Riemannian circle of curvature 1;
3. If $0 < \kappa < 1$ then $\gamma$ is a geodesic of the Clifford minimal torus of $S^3$.

The preceding theorem implies the following result:

**Corollary 3.1** Let $\gamma : I \rightarrow S^3$ be a Legendre curve parametrised by the arclength. Then $\gamma$ is polyharmonic if and only if $\gamma$ is a Legendre geodesic.

In fact, curves in the latter two classes can not be Legendre. (Recall that every Legendre curve has constant torsion 1).

Now we study polyharmonic Legendre curves in contact Riemannian 3-manifolds.

Let $M^3$ be a contact Riemannian 3-manifold and $\gamma : I \rightarrow M$ a Frenet curve framed by $(p_1, p_2, p_3)$. Then direct computation shows that

$$T_2(\gamma) = -3k\kappa'p_1 + (\kappa'' - \kappa^3 - \kappa\tau^2)p_2 + (2\kappa'\tau + \kappa\tau')p_3 + \kappa R(p_2, p_1)p_1.$$

Now assume that $M$ is a Sasakian space form of constant holomorphic sectional curvature $c$ then

$$R(p_2, p_1)p_1 = \frac{c+3}{4}p_2 + \frac{c-1}{4}\{\eta(p_2)\eta(p_1)p_1 - \eta(p_1)^2p_2 - \eta(p_2)\xi + 3g(p_2, \varphi p_1)\varphi p_1\}.$$

In particular, if $\gamma$ is Legendre, then $R(p_2, p_1)p_1 = c p_2$. Thus a Legendre curve $\gamma$ in $M$ is polyharmonic if and only if

$$\kappa = \text{constant}, \quad \kappa^3 - (c-1)\kappa = 0, \quad \tau = 1.$$
If we look for nongeodesic polyharmonic Legendre curves, we obtain
\[ \kappa = \text{constant}, \quad \kappa^2 = c - 1, \quad \tau = 1. \]

Thus we obtain the following result which is a generalisation of Corollary 3.1.

**Theorem 3.3** Let \( M^3(c) \) be a Sasakian space form of constant holomorphic sectional curvature \( c \) and \( \gamma : I \rightarrow M \) a polyharmonic Legendre curve parametrised by the arclength.

1. **If** \( c \leq 1 \), **then** \( \gamma \) **is a Legendre geodesic**;
2. **If** \( c > 1 \), **then** \( \gamma \) **is a Legendre geodesic or a Legendre helix of curvature** \( \sqrt{c - 1} \).

Let \( \phi : M \rightarrow N \) be an isometric immersion. Then \( \phi \) is a critical point of the volume functional if and only if \( \phi \) is minimal. The *Jacobi operator* \( J \) of a minimal immersion \( \phi \) (with respect to the volume functional) is appeared in the second variation formula of the volume and given by

\[ J V = \Delta^\perp V - S V + R(V), \quad V \in \Gamma(T^\perp M). \]

Here the operators \( S \) and \( R \) are defined by

\[ h(SV, W) = \text{tr}(A_V \circ A_W), \quad R(V) = \sum_{i=1}^{m} \left( R^N(d\phi(e_i), V)d\phi(e_i) \right)^\perp. \]

Here \( A_V \) denotes the Weingarten operator with respect to \( V \).

Arroyo, Barros and Garay studied submanifolds in \( S^3 \) whose mean curvature vector fields are eigen-section of the Jacobi operator with respect to the volume functional \([1], [4], [5]\). Such study for surfaces in 5-dimensional Sasakian space forms can be found in \([37]\).

It seems to be interesting to study similar problems for submanifolds in space forms or Sasakian space forms with respect to the energy functional.

In \([9]\), all the polyharmonic surfaces in \( S^3 \) are classified. More precisely, the only non-minimal polyharmonic surfaces are totally umbilical 2-spheres.

Based on this result, we would like to propose the following problem:

*Are there non-minimal and non totally umbilical polyharmonic submanifolds in homogeneous Riemannian manifolds?*

To close this paper, we study polyharmonic Hopf cylinders in 3-dimensional Sasakian space forms. Moreover we show the existence of non-minimal and non totally umbilical polyharmonic surfaces in Sasakian space forms.

First we recall the following result which is a consequence of the main result in \([9]\):
**Proposition 3.1** There are no non minimal polyharmonic Hopf cylinders in the unit 3-sphere $S^3$.

Now we generalise this result to Sasakian space forms.

Let $S = S_\gamma$ be a Hopf cylinder and $\iota : S \subset M^3(c)$ its inclusion map into a Sasakian space form $M^3(c)$. Then the bitension field $T_2(\iota)$ is given by

$$T_2(\iota) = -J_\iota(\mathcal{J}(\iota)) = -2J_\iota(\mathbb{H}).$$

We use the orthonormal frame field \{t, \xi\} as before. Then since $S$ is flat, we have

$$\bar{\Delta}_\mathcal{H} = \Delta \mathbb{H}, \quad \mathcal{R}(\mathbb{H}) = H(R(n, t)t + R(n, \xi)\xi).$$

Using the curvature formula of Sasakian space form, we get

$$\mathcal{R}(\mathbb{H}) = (c + 1)H n.$$

Hence

$$\mathcal{J}_\iota(\mathbb{H}) = 6HH't - (H'' - 4H^3 + (c - 1)H)n - 2H'\xi.$$

Thus $\mathcal{J}_\iota(\mathbb{H}) = \lambda \mathbb{H}$ if and only if

$$H' = 0, \quad 4H^3 = (c - 1 + \lambda)H$$

and hence $H = 0$ or $\lambda = 4H^2 + 1 - c, \ H \neq 0$.

**Theorem 3.4** Let $S$ be a Hopf cylinder in a Sasakian space form $M^3(c)$. Then $S$ satisfies $\mathcal{J}_\iota(\mathbb{H}) = \lambda \mathbb{H}$ if and only if the base curve of $S$ is a Riemannian circle or a geodesic. In case that the base curve is not a geodesic, then $\lambda = 4H^2 + 1 - c$.

**Corollary 3.2** Let $\iota : S_\gamma \to M^3(c)$ be a polyharmonic Hopf cylinder in a Sasakian space form.

1. If $c \leq 1$ then $\bar{\gamma}$ is a geodesic;
2. If $c > 1$ then $\bar{\gamma}$ is a geodesic or a Riemannian circle of curvature $\bar{\kappa} = \sqrt{c - 1}$.

In particular, there exist nonminimal polyharmonic Hopf cylinders in Sasakian space forms of holomorphic sectional curvature greater than 1.

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