Towards noncommutative supersymmetric quantum cosmology

W. Guzmán\textsuperscript{1} \textsuperscript{*}, M. Sabido\textsuperscript{1} \textsuperscript{†}, and J. Socorro\textsuperscript{1,2} \textsuperscript{‡}

\textsuperscript{1} Instituto de Física de la Universidad de Guanajuato, A.P. E-143, C.P. 37150, León, Guanajuato, México
\textsuperscript{2} Facultad de Ciencias de la Universidad Autónoma del Estado de México, Instituto Literario No. 100, Toluca, C.P. 50000, Edo de Mex, México

Using the factorization approach of quantum mechanics, we obtain a family of iso-spectral scalar potentials for noncommutative quantum cosmology. The family we build is based on a scattering Wheeler-DeWitt solution for the potential \( V(\phi) = V_0 e^{-\lambda \phi} \). We analyze the effects of noncommutativity on the iso-potentials and the possible relationship between noncommutativity and dark energy.

PACS numbers: 02.30.Jr; 04.60.Kz; 12.60.Jv; 98.80.Qc.

The first venture into quantum cosmology was engaged in the 1970’s, through canonical quantization of minisuperspace models \cite{1} in which the gravitational and matter variables have been reduced to a finite number of degrees of freedom. The interest in the field was rekindled in the 80’s by Hawking \cite{2}, emphasizing the path integral approach, revising the interest in minisuperspace quantization. Recently, loop quantum cosmology has given new life to minisuperspace models, and as in the original proposal, the objective is to get some insight into the complete quantum theory of gravity by studying some simplified models.

Supersymmetry, solves many problems, such as the hierarchy problem, or the dark matter conundrum. If one believes that supersymmetry is a fundamental property of nature, then supergravity should be the fundamental theory of gravity. This line of reasoning applied to cosmology gave birth to what is known as supersymmetric quantum cosmology (SUSY-QC) \cite{3}. Several approaches to SUSY-QC were developed during the 90’s \cite{4}, all of them with their respective advantages and problems, yet one of the simplest approaches is the use of susy quantum mechanics to QC.

Since the beginning of this century, there has been a lot of interest in the old idea of noncommutative spacetime \cite{5}. This renewed interest is a consequence of the developments in string theory, where a noncommutative gauge theory action \cite{6,7} appears in a natural way. The current formulations of gravity in noncommutative spacetime \cite{6,7} are highly nonlinear. Directly solving the cosmological equations from noncommutative gravity is a daunting task. In the last few years there have been several attempts to study the possible effects of noncommutativity in the cosmological scenario \cite{10,11}. In particular, in \cite{12}, the authors, analyzed the effects of noncommutativity in quantum cosmology by deforming the minisuperspace in a similar way as in noncommutative quantum mechanics \cite{13,14}. This is achieved by introducing the Moyal product of functions in the Wheeler-DeWitt (WDW) equation, cunningly avoiding the difficulties of analyzing noncommutative cosmology.

The goal of this short paper is to use the methods of SUSY-QM, which can be considered an equivalent formulation of the Darboux transformation method \cite{13}, that is well-known in mathematics, and apply them to non-commutative quantum cosmology (NCQC). This is done by introducing noncommutativity in the minisuperspace of the quantum cosmological model, and by using SUSY-QM we get the iso-spectral wave functions as well as the iso-potentials for noncommutative quantum cosmology. This ideas have recently been applied in SUSY-QM \cite{16}.

With this in mind, we start with a flat, homogeneous and isotropic universe, the Friedmann-Robertson-Walker (FRW) metric

\[
ds^2 = -N^2(t) dt^2 + e^{2\alpha(t)} \left[ dr^2 + r^2 d\Omega^2 \right],
\]

where \( a(t) = e^{\alpha(t)} \) is the scale factor, \( N(t) \) is the lapse function. From the Einstein-Hilbert action, coupled to a scalar field \( \phi \) with scalar potential \( V(\phi) = V_0 e^{-\lambda \phi} \),

\[
S = \int dx^4 \sqrt{-g} \left[ R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right],
\]

we can write the canonical Hamiltonian by means of the Legendre transformation and arrive to the FRW Hamiltonian

\[
\mathcal{H} = \frac{N}{12} e^{-3\alpha} \left[ \Pi_\phi^2 - 6\Pi_\phi^2 - 12e^{6\alpha} V(\phi) \right],
\]

where we have used the units \( 8\pi G = 1 \). In order to simplify the calculations from now on we will be working in the gauge \( N = 12 e^{3\alpha} \), this will simplify the noncommutative formulation and because of the reparametrization invariance of the theory, the physical implications are independent of the chosen gauge. Also in order to simplify the calculations we make the canonical transformation

\[\text{arXiv:0812.4999v1 [hep-th]} [30 Dec 2008]

\\textsuperscript{*}Electronic address: wguzman@fisica.ugto.mx
\textsuperscript{†}Electronic address: msabido@fisica.ugto.mx
\textsuperscript{‡}Electronic address: socorro@fisica.ugto.mx
\[ x = -6a + \lambda \phi, \quad \Pi_x = \frac{1}{\lambda^2 - 6} (\Pi_\alpha + \lambda \Pi_\phi), \quad (4) \]
\[ y = -\sqrt{\lambda} \phi + \sqrt{6} \phi, \quad \Pi_y = \frac{1}{\sqrt{6(6 - \lambda^2)}} (\lambda \Pi_\alpha + 6 \Pi_\phi). \]

From this point forward our analysis will be done in this new set of minisuperspace variables, the advantages of this selection has been analyzed in \[17\]. The classical Hamiltonian has the simple form \[ H = -\beta \Pi_\alpha^2 + \beta \Pi_\phi^2 - 12V_0 e^{-xy} \approx 0, \] where \( \beta \) is defined as \( \beta = 6(\lambda^2 - 6). \) The usual canonical quantization is done by the usual identifications \( \Pi_{\phi} = -i d_{\phi}, \) and we arrive to the WDW equation
\[
\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2} + \frac{2V_0}{6 - \lambda^2} e^{-xy} \Psi = 0, \quad (5)
\]
in this formalism \( \Psi \) is called the wave function of the universe.

The proposal to introduce the noncommutative minisuperspace deformation is achieved by introducing the following commutation relation between the minisuperspace variables
\[
[x, y] = i\theta, \quad (6)
\]
this can be seen as an effective noncommutativity that could arise from a fundamental noncommutative theory of gravity. For example, if we start with the Lagrangian derived in \[49\], the noncommutative fields are a consequence of noncommutativity among the coordinates and then the minisuperspace variables would inherit some effective noncommutativity. This we assume to be encoded in \[6\], otherwise we would have a very complicated Hamiltonian for the higher order Lagrangian.

This effective noncommutativity can be formulated in terms of product of functions of the mini-superspace variables, with the Moyal star product of functions. The noncommutative WDW (NCWDW) equation is obtained by replacing the products of functions by star products. We can show \[14\] that the effects of the Moyal star product are reflected only in a shift in the potential \( V(x, y) \) \* \( \Psi(x, y) = V(x + \frac{2}{\theta} \Pi y, y - \frac{2}{\theta} \Pi x) \Psi(x, y). \) Taking this in to account, we arrive to
\[
\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \gamma e^{-\left(x-\frac{\theta}{\gamma}y\right)} \right) \Psi = 0, \quad (7)
\]
with \( \gamma \equiv \frac{2V_0}{6 - \lambda^2}. \)

Using the the ansatz \( \Psi(x, y) = e^{\pm i\eta y} \Psi_x(x) \) and the property \( e^{i\theta x} e^{i\eta y} = e^{i\eta y} e^{i\theta x} \) we arrive to
\[
\frac{d^2 \Psi_x}{d x^2} + \left( \gamma e^{-\left(x+\frac{\theta}{\gamma}y\right)} + \eta^2 \right) \Psi_x = 0, \quad (8)
\]
equation over we employ the susy isospectral method, who solution for \( \lambda^2 < 6 \) become
\[
\Psi(x, y) = e^{\pm i\eta y} \left[ j_{2\eta} \left( 2\sqrt{\gamma} e^{-\left(\frac{x}{2} + \frac{\theta}{\gamma} y\right)} \right) + J_{-2\eta} \left( 2\sqrt{\gamma} e^{-\left(\frac{x}{2} - \frac{\theta}{\gamma} y\right)} \right) \right], \quad (9)
\]
this of course in the minisuperspace variables \( x \) and \( y, \) which reduces to the corresponding commutative solution for \( \theta = 0. \) When writing down the noncommutative WDW equation, one expects additional ordering ambiguities to arise, for the example we are working, this can be ignored due to simple form of the potential and the classical canonical transformation we are using.

As already mentioned, the goal of this letter is to apply the factorization approach of supersymmetric quantum mechanics \[15\], to noncommutative quantum cosmology (for a full review of these techniques see \[18\]). For this we start with the WDW equation (or the Schroedinger equation if we are working in quantum mechanics), and proceed to find first order differential operators that factorize the hamiltonian. The factorization technique is based on the fact that once we have a solution to the original WDW equation a super-potential function can be constructed, and a whole family of potentials can be found, with the particularity that all have the same energy spectrum \[19\]. These potentials are known as the iso-potentials. The family of iso-potentials can be parametrized and in the limit when the parameter goes to infinity we return to the original problem. Due to simple modification on the noncommutative WDW equation with respect to its commutative counterpart factorizing the noncommutative WDW equation Eq.\[8\] immediately yields the iso-potentials and the iso-wavefunctions.

The total WDW isospectral wave function has the generic form \[19\]
\[
\Psi_{iso}(x, y; \tau) = \frac{g(\tau) \Psi(x, y)}{I + \tau}, \quad (10)
\]
where \( g(\tau) = \sqrt{\tau(\tau + 1)}, \) \( \tau \) is a continuous free parameter time-like and the function \( I \) become
\[
I(x) = \int_0^x [\Psi_x(t)]^2 dt, \quad (11)
\]
where the effects of the mini-superspace noncommutativity are incorporated on the Bessel functions. The iso-potentials are constructed from the noncommutative wave functions, so they will also be influenced by the noncommutative parameter \( \theta, \) and although a closed expression for the noncommutative iso-potentials is difficult to construct, some general features can be seen in figures \[11\] and \[2\].

We can see from Fig\[11\] the effects of noncommutativity on the iso-potentials. As the value of \( \theta \) is increased the number of local minima and maxima on the potential also increases. This is reminiscent to the effects of the noncommutative parameter on the probability density in NC-QC, where new maxima and minima appear as the value of \( \theta \) is increased; this can be interpreted in the framework of QC as new less probable universes to which our current universe can tunnel \[12\]. Another interesting effect of noncommutativity seems to be related to the value of the cosmological constant. The value of the minimum of the potential, is related to the value of \( \Lambda, \)
The slow roll condition is encoded in the parameter $\epsilon = \frac{1}{2} (\frac{\dot{V}(\phi)}{V(\phi)})^2 << 1$ (in units of $M_p = 1$), when $\epsilon \geq 1$ the slow roll condition is broken and the universe no longer accelerates. In particular for the exponential potential $V(\phi) = e^{-\lambda \phi}$ the slow roll parameter is $\epsilon = \frac{1}{2} \lambda^2$, this means that once we fix $\lambda$ to have a non-accelerating universe there is no mechanism to start an inflationary epoch.

In order to simplify the analysis we write the slow roll parameter in terms of the potential as a function of $x$ instead of the scalar field $\phi$: taking the form $\epsilon = \frac{1}{2} \lambda^2 (V''(x)/V(x) + b)^2$, where $b = 6(\lambda^2 - 6)^{-1}$, for the original exponential potential $V'(x)/(V(x)) = -\lambda^2 b$, so again, once we fix the value of $\lambda$ to have inflation there is no way to end it. For the iso-spectral potentials the approximate relation holds $V_{iso}(\phi) \approx V_{iso}(x)$, where $V_{iso}(x)$ is the iso-spectral potential, and again we may write the slow roll parameter in a similar manner, now in contrast with the original potential, once we fix the value of $\lambda = \sqrt{2}$ for a non-inflating epoch, as we increase the value of $x$ the ratio $V_{iso}(x)/V_{iso}(x)$ can be as small as we want, this can be seen from the plots of the iso-potentials (Fig.1). In particular near the critical points of $V(x)$, $V_{iso}(x)$ can be small enough, in order to satisfy the slow roll condition, and give an accelerating epoch for the universe.

In this letter we have applied the factorization technique in order to find noncommutative iso-potentials to the noncommutative WDW equation, the scenario we have used corresponds to FRW cosmological model coupled to a scalar field. We speculate on the possible relationship between dark energy and the noncommutative parameter, of course more realistic models need to be constructed, but the possible relationship between noncommutativity and dark energy is very attractive. Another possible area to which these ideas can be applied is in the very early universe, particularly in connection to inflation where noncommutativity might play a relevant role [17], the reason being that the iso-spectral method gives a complete family of iso-potentials that might give better agreement to the observational data, research in this line of reasoning in being done and will be reported elsewhere.

Finally we believe that this technique can be applied to other areas of physics where the factorization approach of QM is used, the procedure gives a new parameter that can be introduced in a straightforward way and that it might be used to have a better phenomenological agreement.

Acknowledgments

This work was partially supported by CONACYT grants 47641 and 62253. DINPO 38/07 and PROMEP grants UGTO-CA-3, UGTO-PTC-085 and CONCYTEG grant 07-16-K662-062 A01.
[1] For reviews, see M. P. Ryan, Hamiltonian Cosmology (Speinger, Berlin, 1972); M. MacCallum, in General Relativity: An Einstein Centenary Survey, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[2] S. Hawking, Pontif. Acad. Sci. Varia 48, 563 (1982); J. B. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960 (1983).

[3] A. Macias, O. Obregon and M. Ryan, Class. Quantum Grav. 4, 1477 (1987).

[4] J. Bene and R. Graham, Phys. Rev. D 49, 799 (1994); O. Obregon, J. Socorro and J. Benitez, Phys. Rev. D 47, 4471 (1993); V.I. Tkach, J.J. Rosales and O. Obregon, Class. Quantum Grav. 13, 2349 (1996); P.D. D'Eath, Supersymmetry quantum cosmology (Cambridge University Press, Cambridge, England, 1996); J. Socorro and E.R. Medina, Phys. Rev. D 61, 087702 (2000).

[5] H. Snyder, Phys. Rev. 71, 38 (1947).

[6] N. Seiberg and E. Witten, JHEP 9909:032 (1999).

[7] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73, 977 (2001) [arXiv:hep-th/0106048].

[8] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, Class. Quant. Grav. 23 (2006) 1883.

[9] H. Garcia-Compean, O. Obregon, C. Ramirez and M. Sabido, Phys. Rev. D 68, 044015 (2003); H. Garcia-Compean, O. Obregon, C. Ramirez and M. Sabido, Phys. Rev. D 68, 045010 (2003); A.H. Chamseddine, J. Math. Phys. 44, 2534 (2003); J.W. Moffat, Phys. Lett. B 491, 345 (2000); P. Aschieri, C. Blohm, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, Class. Quant. Grav. 22, 3511 (2005); L. Alvarez-Gaume, F. Meyer and M. A. Vazquez-Mozo, Nucl. Phys. B 753, 92 (2006).

[10] J. M. Romero and J. A. Santiago, Mod. Phys. Lett. A 20 (2005) 781 [arXiv:hep-th/0310266].

[11] R. Brandenberger and P. M. Ho, Phys. Rev. D 66 (2002) 023517; Q. G. Huang and M. Li, Nucl. Phys. B 713 (2005) 219; Q. G. Huang and M. Li, JHEP 0306 (2003) 04; H. Kim, S. S. Lee, H. W. Lee and Y. S. Myung, Phys. Rev. D 70 (2004) 043521; H. Kim, G. S. Lee and Y. S. Myung, Mod. Phys. Lett. A 20 (2005) 271; D. J. Liu and X. Z. Li, Phys. Rev. D 70 (2004) 123504.

[12] H. Garcia-Compean, O. Obregon and C. Ramirez, Phys. Rev. Lett. 88, 161301 (2002).

[13] J. Gamboa, M. Loewe and J. C. Rojas, Phys. Rev. D 64, 067901.

[14] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. 86, 2716.

[15] G. Darboux, C.R. Acad. Sci. (Paris) 94, 1456 (1882).

[16] A. Das, H. Falomir, J. Gamboa and F. Mendez, arXiv:0809.1405 [hep-th].

[17] W. Guzman, M. Sabido and J. Socorro, Phys. Rev. D 76 (2007) 087302.

[18] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).

[19] H. Rosu and J. Socorro, Il Nuovo Cimento B 113, 683-689 (1998), gr-qc/9606030.

[20] E. Mena, O. Obregon and M. Sabido, arXiv:0802.3393 [hep-th].

[21] J. Socorro, M.A. Reyes and F.A. Gelbert, Physics Lett. A 313, 338 (2003)