Rigidity of area-minimizing two-spheres in three-manifolds

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We give a sharp upper bound for the area of a minimal two-sphere in a three-manifold $(M,g)$ with positive scalar curvature. If equality holds, we show that the universal cover of $(M,g)$ is isometric to a cylinder.

1. Introduction

A classical result in differential geometry due to Toponogov [14] states that every simple closed geodesic $\gamma$ on a two-dimensional surface $(\Sigma, g)$ satisfies

$$\text{length}(\gamma)^2 \inf_{\Sigma} K \leq 4\pi^2,$$

where $K$ denotes the Gaussian curvature of $\Sigma$. Moreover, equality holds if and only if $(\Sigma, g)$ is isometric to the standard sphere $S^2$ up to scaling (see also [8] for a different proof).

We next consider a three-manifold $(M,g)$ with positive scalar curvature. By a theorem of Schoen and Yau [13], any area-minimizing surface in $M$ is homeomorphic to either $S^2$ or $\mathbb{RP}^2$. The case of area-minimizing projective planes was studied in [3]. In particular, if $\Sigma$ is an embedded $\mathbb{RP}^2$ of minimal area, then the area of $\Sigma$ can be estimated from above by

$$\text{area}(\Sigma, g) \inf_{M} R \leq 12\pi$$

(cf.[3, Theorem 1]). Moreover, equality holds if and only if $(M, g)$ is isometric to $\mathbb{RP}^3$ up to scaling. A survey of related rigidity results involving scalar curvature can be found in [4].

In this paper, we consider the case of area-minimizing two-spheres. We shall assume throughout that $(M,g)$ is a compact three-manifold with $\pi_2(M) \neq 0$. We denote by $\mathcal{F}$ the set of all smooth maps $f : S^2 \to M$ which represent a non-trivial element of $\pi_2(M)$. We define

$$\mathcal{A}(M, g) = \inf \{ \text{area}(S^2, f^* g) : f \in \mathcal{F} \}. $$

We now state the main result of this paper:
Theorem 1.1. We have
\begin{equation}
\mathcal{A}(M, g) \inf_M R \leq 8\pi,
\end{equation}
where $R$ denotes the scalar curvature of $(M, g)$. Moreover, if equality holds, then the universal cover of $(M, g)$ is isometric to the standard cylinder $S^2 \times \mathbb{R}$ up to scaling.

Inequality (1.2) follows directly from the formula for the second variation of area. We now describe the proof of the rigidity statement. By scaling, we may assume that $\mathcal{A}(M, g) = 4\pi$ and $\inf_M R = 2$. It follows from results of Meeks and Yau [12] that the infimum in (1.1) is attained by a smooth immersion $f \in \mathcal{F}$ (see also [9]). Using the implicit function theorem, we construct a one-parameter family of immersed two-spheres with constant mean curvature. Using the formula for the second variation of area, we are able to show that these surfaces all have area $\mathcal{A}(M, g) = 4\pi$. Consequently, these spheres are all round and totally geodesic. This allows us to construct a local isometry from the cylinder $S^2 \times \mathbb{R}$ into $M$. The use of constant mean curvature surfaces is motivated in part by work of Bray [1] and Huisken and Yau [11] (see also [2, 10]).

We note that Cai and Galloway [5] have obtained a similar rigidity theorem for minimal tori in three-manifolds of nonnegative scalar curvature. The proof in [5] uses a different argument based on a deformation of the metric to strictly positive scalar curvature. The arguments in this paper can be adapted to give an alternative proof of Theorem 1 in [5]. See also [7] for related work in this direction.

2. Proof of (1.2)

Let us consider a smooth immersion $f : S^2 \to M$. Since $f$ has trivial normal bundle, there exists a globally defined unit normal field $\nu$. In other words, for each point $x \in S^2$, $\nu(x) \in T_{f(x)} M$ is a unit vector which is orthogonal to the image of $df_x : T_x S^2 \to T_{f(x)} M$. The following result is a consequence of the Gauss–Bonnet theorem.

Proposition 2.1. For any immersion $f : S^2 \to M$, we have
\[
\int_{S^2} (R - 2\, \text{Ric}(\nu, \nu) - |\II|^2) \, d\mu_{f^* g} \leq 8\pi,
\]
where $\II$ denotes the second fundamental form of $f$. 
Proof. By the Gauss equations, we have
\[ R - 2 \text{Ric}(\nu, \nu) - |II|^2 = 2K - H^2, \]
where \( H \) and \( K \) denote the mean curvature and the Gaussian curvature, respectively. This implies
\[ \int_{S^2} (R - 2 \text{Ric}(\nu, \nu) - |II|^2) \, d\mu_{f^*g} \leq 2 \int_{S^2} K \, d\mu_{f^*g} = 8\pi \]
by the Gauss–Bonnet theorem. \( \Box \)

We next consider a map \( f \in \mathcal{F} \) which attains the infimum in (1.1). The existence of a minimizer is guaranteed by the following result.

**Proposition 2.2.** There exists a smooth map \( f \in \mathcal{F} \) such that area \((S^2, f^*g) = \mathcal{A}(M, g)\). Moreover, the map \( f \) is an immersion.

Proposition 2.2 is an immediate consequence of Theorem 7 in [12] (see also [9, Theorem 4.2]). In fact, Meeks and Yau show that either \( f \) is an embedding or a two-to-one covering map whose image is an embedded \( \mathbb{RP}^2 \). We will not use this stronger statement here.

Let \( f \in \mathcal{F} \) be a smooth immersion with area \((S^2, f^*g) = \mathcal{A}(M, g)\). Using the formula for the second variation of area, we obtain
\[ \int_{S^2} (\text{Ric}(\nu, \nu) + |II|^2) \, u^2 \, d\mu_{f^*g} \leq \int_{S^2} |\nabla u|^2_{f^*g} \, d\mu_{f^*g} \]
for every smooth function \( u : S^2 \to \mathbb{R} \). Choosing \( u = 1 \) gives
\[ \int_{S^2} (\text{Ric}(\nu, \nu) + |II|^2) \, d\mu_{f^*g} \leq 0. \]

Using Proposition 2.1, we obtain
\[ \text{area}(S^2, f^*g) \inf_M R \leq \int_{S^2} (R + |II|^2) \, d\mu_{f^*g} \]
\[ \leq 8\pi + 2 \int_{S^2} (\text{Ric}(\nu, \nu) + |II|^2) \, d\mu_{f^*g} \]
\[ \leq 8\pi. \]
(2.1)

This completes the proof of (1.2).
3. The case of equality

In this section, we analyze the case of equality. Suppose that
\[ \mathcal{A}(M,g) \inf_M R = 8\pi. \]

After rescaling the metric if necessary, we may assume that \( \mathcal{A}(M,g) = 4\pi \) and \( \inf_M R = 2 \). By Proposition 2.2, we can find a smooth immersion \( f \in \mathcal{F} \) such that \( \text{area}(S^2, f^* g) = 4\pi \).

**Proposition 3.1.** Let \( f \in \mathcal{F} \) be a smooth immersion such that area \( (S^2, f^* g) = 4\pi \). Then the surface \( \Sigma = f(S^2) \) is totally geodesic. Moreover, we have \( R = 2 \) and \( \text{Ric}(\nu, \nu) = 0 \) at each point on \( \Sigma \).

**Proof.** By assumption, we have area \( (S^2, f^* g) = 4\pi \) and \( \inf_M R = 2 \). Consequently, the inequalities in (2.1) are all equalities. In particular, we have
\begin{align*}
(3.1) & \quad \int_{S^2} (R + |\nabla|^2) \, d\mu f^* g = 8\pi \\
(3.2) & \quad \int_{S^2} (\text{Ric}(\nu, \nu) + |\nabla|^2) \, d\mu f^* g = 0.
\end{align*}

It follows from (3.2) that the constant functions lie in the nullspace of the Jacobi operator \( L = \Delta f^* g + \text{Ric}(\nu, \nu) + |\nabla|^2 \). This implies
\[ \text{Ric}(\nu, \nu) + |\nabla|^2 = 0 \]
at each point on \( \Sigma \). Moreover, since area \( (S^2, f^* g) = 4\pi \) and \( \inf_M R = 2 \), the identity (3.1) implies that \( R = 2 \) and \( |\nabla|^2 = 0 \) at each point on \( \Sigma \). This completes the proof. \( \square \)

**Proposition 3.2.** Let \( f \in \mathcal{F} \) be a smooth immersion such that area \( (S^2, f^* g) = 4\pi \). Then there exists a positive real number \( \delta_1 \) and a smooth map \( w : S^2 \times (-\delta_1, \delta_1) \to \mathbb{R} \) with the following properties:

- For each point \( x \in S^2 \), we have \( w(x,0) = 0 \) and \( \frac{\partial}{\partial t} w(x,t) \Big|_{t=0} = 1 \).
- For each \( t \in (-\delta_1, \delta_1) \), we have \( \int_{S^2} (w(\cdot, t) - t) \, d\mu f^* g = 0 \).
For each \( t \in (-\delta_1, \delta_1) \), the surface
\[
\Sigma_t = \{ \exp_{f(x)}(w(x, t)\nu(x)) : x \in S^2 \}
\]
has constant mean curvature.

Proof. The Jacobi operator associated with the minimal immersion \( f : S^2 \to M \) is given by \( L = \Delta_{f^*g} + \text{Ric}(\nu, \nu) + |\text{II}|^2 \). Using Proposition 3.1, we conclude that \( L = \Delta_{f^*g} \). Hence, the assertion follows from the implicit function theorem. 

For each \( t \in (-\delta_1, \delta_1) \), we define a map \( f_t : S^2 \to M \) by \( f_t(x) = \exp_{f(x)}(w(x, t)\nu(x)) \). Clearly, \( f_0(x) = f(x) \) for all \( x \in S^2 \). To fix notation, we denote by \( \nu_t(x) \in T_{f_t(x)}M \) the unit normal vector to the surface \( \Sigma_t = f_t(S^2) \) at the point \( f_t(x) \). We assume that \( \nu_t \) depends smoothly on \( x \) and \( t \), and \( \nu_0(x) = \nu(x) \) for all \( x \in S^2 \). Moreover, we denote by \( \text{II}_t \) the second fundamental form of \( f_t \).

Lemma 3.3. There exists a positive real number \( \delta_2 < \delta_1 \) with the following property: if \( t \in (-\delta_2, \delta_2) \) and \( u : S^2 \to \mathbb{R} \) is a smooth function satisfying \( \int_{S^2} u d\mu_{f_t^*g} = 0 \), then
\[
\int_{S^2} |\nabla u|_{f_t^*g}^2 d\mu_{f_t^*g} - \int_{S^2} (\text{Ric}(\nu_t, \nu_t) + |\text{II}_t|^2) u^2 d\mu_{f_t^*g} \geq 0.
\]

Proof. We can find a uniform constant \( c > 0 \) such that
\[
\int_{S^2} |\nabla u|_{f_t^*g}^2 d\mu_{f_t^*g} \geq c \int_{S^2} u^2 d\mu_{f_t^*g}
\]
for each \( t \in (-\delta_1, \delta_1) \) and every smooth function \( u : S^2 \to \mathbb{R} \) satisfying \( \int_{S^2} u d\mu_{f_t^*g} = 0 \). Moreover, it follows from Proposition 3.1 that
\[
\sup_{S^2}(\text{Ric}(\nu_t, \nu_t) + |\text{II}_t|^2) \to 0
\]
as \( t \to 0 \). Putting these facts together, the assertion follows. 

Lemma 3.4. For each \( t \in (-\delta_1, \delta_1) \), we have
\[
\int_{S^2} (\text{Ric}(\nu_t, \nu_t) + |\text{II}_t|^2) d\mu_{f_t^*g} \geq 0.
\]
Proof. Since \( f \) minimizes area in its homotopy class, we have
\[
\text{area}(S^2, f^* g) \geq \text{area}(S^2, f^* g) = 4\pi.
\]
Moreover, we have \( \inf_M R = 2 \). Applying Proposition 2.1 to the map \( f_t : S^2 \to M \), we obtain
\[
8\pi \leq \text{area}(S^2, f_t^* g) \leq 8\pi + 2 \int_{S^2} (\text{Ric}(\nu_t, \nu_t) + |II_t|^2) \, d\mu_{f_t^* g}.
\]
From this, the assertion follows. \( \square \)

By assumption, the surface \( \Sigma_t \) has constant mean curvature. The mean curvature vector of \( \Sigma_t \) can be written in the form \( -H(t) \nu_t \), where \( H(t) \) is a smooth function of \( t \). For each \( t \in (-\delta_1, \delta_1) \), the lapse function \( \rho_t : S^2 \to \mathbb{R} \) is defined by
\[
\rho_t(x) = \left\langle \nu_t(x), \frac{\partial}{\partial t} f_t(x) \right\rangle.
\]
Clearly, \( \rho_0(x) = 1 \) for all \( x \in S^2 \). By continuity, we can find a positive real number \( \delta_3 < \delta_2 \) such that \( \rho_t(x) > 0 \) for all \( x \in S^2 \) and all \( t \in (-\delta_3, \delta_3) \). The lapse function \( \rho_t : S^2 \to \mathbb{R} \) satisfies the Jacobi equation
\[
\Delta_{f_t^* g} \rho_t + (\text{Ric}(\nu_t, \nu_t) + |II_t|^2) \rho_t = -H'(t)
\]
(cf.[10, equation (1.2)]).

Proposition 3.5. We have \( \text{area}(S^2, f_t^* g) = 4\pi \) for all \( t \in (-\delta_3, \delta_3) \).

Proof. Let \( \bar{\rho}_t \) denote the mean value of the lapse function \( \rho_t : S^2 \to \mathbb{R} \) with respect to the induced metric \( f_t^* g \); that is,
\[
\bar{\rho}_t = \frac{1}{\text{area}(S^2, f_t^* g)} \int_{S^2} \rho_t \, d\mu_{f_t^* g}.
\]
It follows from Lemma 3.3 that
\[
\int_{S^2} |\nabla \rho_t|_{f_t^* g}^2 \, d\mu_{f_t^* g} - \int_{S^2} (\text{Ric}(\nu_t, \nu_t) + |II_t|^2) (\rho_t - \bar{\rho}_t)^2 \, d\mu_{f_t^* g} \geq 0
\]
Rigidity of area-minimizing two-spheres in three-manifolds

for all $t \in (-\delta_2, \delta_2)$. Moreover, Lemma 3.4 implies that

$$\bar{\rho}_t^2 \int_{S^2} (\text{Ric}(\nu_t, \nu_t) + \|II_t\|^2) \, d\mu \geq 0$$

for all $t \in (-\delta_1, \delta_1)$. Adding both inequalities yields

(3.5) $$\int_{S^2} |\nabla \rho_t|^2 f^*_tg \, d\mu_{f^*_tg} + \int_{S^2} (\text{Ric}(\nu_t, \nu_t) + \|II_t\|^2) \rho_t (2 \bar{\rho}_t - \rho_t) \, d\mu_{f^*_tg} \geq 0$$

for all $t \in (-\delta_2, \delta_2)$.

In the next step, we multiply Equation (3.4) by $2 \bar{\rho}_t - \rho_t$ and integrate. This gives

(3.6) $$\int_{S^2} |\nabla \rho_t|^2 f^*_tg \, d\mu_{f^*_tg} + \int_{S^2} (\text{Ric}(\nu_t, \nu_t) + \|II_t\|^2) \rho_t (2 \bar{\rho}_t - \rho_t) \, d\mu_{f^*_tg} = -H'(t) \int_{S^2} (2 \bar{\rho}_t - \rho_t) \, d\mu_{f^*_tg} = -H'(t) \int_{S^2} \rho_t \, d\mu_{f^*_tg}.$$

Putting these facts together, we obtain

(3.7) $$H'(t) \int_{S^2} \rho_t \, d\mu_{f^*_tg} \leq 0$$

for each $t \in (-\delta_2, \delta_2)$. Therefore, we have $H'(t) \leq 0$ for all $t \in (-\delta_3, \delta_3)$. Since $H(0) = 0$, it follows that $H(t) \geq 0$ for all $t \in (-\delta_3, 0]$ and $H(t) \leq 0$ for all $t \in [0, \delta_3)$. Using the identity

$$\frac{d}{dt} \text{area}(S^2, f^*_tg) = \int_{S^2} \left\langle H(t) \nu_t, \frac{\partial}{\partial t} f_t \right\rangle \, d\mu_{f^*_tg} = H(t) \int_{S^2} \rho_t \, d\mu_{f^*_tg},$$

we obtain

$$\text{area}(S^2, f^*_tg) \leq \text{area}(S^2, f^*g) = 4\pi$$

for all $t \in (-\delta_3, \delta_3)$. Since $f$ minimizes area in its homotopy class, we conclude that $\text{area}(S^2, f^*_tg) = 4\pi$ for all $t \in (-\delta_3, \delta_3)$. □

**Proposition 3.6.** For each $t \in (-\delta_3, \delta_3)$, the surface $\Sigma_t$ is totally geodesic, and we have $R = 2$ and $\text{Ric}(\nu_t, \nu_t) = 0$ at each point on $\Sigma_t$. Moreover, the lapse function $\rho_t : S^2 \to \mathbb{R}$ is constant.

**Proof.** By Proposition 3.5, we have $\text{area}(S^2, f^*_tg) = 4\pi$. Hence, it follows from Proposition 3.1 that $\Sigma_t$ is totally geodesic, and $R = 2$ and $\text{Ric}(\nu_t, \nu_t) = 0$ at each point on $\Sigma_t$. Substituting this into (3.4), we obtain $\Delta_{f^*_tg} \rho_t = 0$. Therefore, the function $\rho_t : S^2 \to \mathbb{R}$ is constant, as claimed. □
**Corollary 3.7.** The normal vector field $\nu_t$ is a parallel vector field near $\Sigma$. In particular, each point on $\Sigma$ has a neighborhood which is isometric to a Riemannian product.

**Proof.** By Proposition 3.6, the lapse function $\rho_t : S^2 \to \mathbb{R}$ is constant. This implies

$$\left\langle D_{\frac{\partial f_t}{\partial t}} \nu_t, \frac{\partial f_t}{\partial t} \right\rangle - \left\langle D_{\frac{\partial f_t}{\partial x_i}} \nu_t, \frac{\partial f_t}{\partial x_i} \right\rangle = \frac{\partial}{\partial x_i} \left\langle \nu_t, \frac{\partial f_t}{\partial t} \right\rangle - \frac{\partial}{\partial t} \left\langle \nu_t, \frac{\partial f_t}{\partial x_i} \right\rangle = \frac{\partial}{\partial x_i} \rho_t(x) = 0$$

for each point $x \in S^2$. Moreover, we have

$$\left\langle D_{\frac{\partial f_t}{\partial x_i}} \nu_t, \frac{\partial f_t}{\partial x_j} \right\rangle - \left\langle D_{\frac{\partial f_t}{\partial x_j}} \nu_t, \frac{\partial f_t}{\partial x_i} \right\rangle = \frac{\partial}{\partial x_i} \left\langle \nu_t, \frac{\partial f_t}{\partial x_j} \right\rangle - \frac{\partial}{\partial x_j} \left\langle \nu_t, \frac{\partial f_t}{\partial x_i} \right\rangle = 0$$

for all $x \in S^2$. Putting these facts together, we obtain

$$\left\langle D_{\frac{\partial f_t}{\partial x_i}} \nu_t, V \right\rangle - \left\langle D_{\nu_t} \frac{\partial f_t}{\partial x_i} \right\rangle = 0$$

for each point $x \in S^2$ and all vectors $V \in T_{f(x)} M$. In particular, we have

$$\left\langle D_{\frac{\partial f_t}{\partial x_i}} \nu_t, \nu_t \right\rangle - \left\langle D_{\nu_t} \frac{\partial f_t}{\partial x_i} \right\rangle = 0$$

for each point $x \in S^2$. Since the vector field $\nu_t$ has unit length, we conclude that $D_{\nu_t} \nu_t = 0$. On the other hand, it follows from Proposition 3.6 that the surfaces $\Sigma_t$ are totally geodesic. This implies $D_{\frac{\partial f_t}{\partial x_i}} \nu_t = 0$ for each point $x \in S^2$. Thus, we conclude that the normal vector field $\nu_t$ is parallel. This completes the proof of Corollary 3.7.

We now consider the product $S^2 \times \mathbb{R}$, where $S^2$ is equipped with the induced metric $f^*g$. We define a map $\Phi : S^2 \times \mathbb{R} \to M$ by $\Phi(x, t) = \text{exp}_{f(x)}(t \nu(x))$. It follows from Corollary 3.7 that the restriction $\Phi|_{S^2 \times (-\delta, \delta)}$ is a local isometry if $\delta > 0$ is sufficiently small.

**Proposition 3.8.** The map $\Phi : S^2 \times \mathbb{R} \to M$ is a local isometry.

**Proof.** We first show that $\Phi|_{S^2 \times [0, \infty)}$ is a local isometry. Suppose this is false. Let $\tau$ be the largest positive real number with the property that $\Phi|_{S^2 \times [0, \tau]}$
is a local isometry. We now define a map \( \tilde{f} : S^2 \to M \) by \( \tilde{f}(x) = \Phi(x, \tau) \). Clearly, \( \tilde{f} \) is homotopic to \( f \); consequently, \( \tilde{f} \) represents a non-trivial element of \( \pi_2(M) \). Moreover, we have \( \text{area}(S^2, \tilde{f}^*g) = \text{area}(S^2, f^*g) = 4\pi \). Therefore, \( \tilde{f} \) has minimal area among all maps in \( \mathcal{F} \). By Corollary 3.7, each point on the surface \( \tilde{\Sigma} = \tilde{f}(S^2) \) has a neighborhood which is isometric to a Riemannian product. Hence, if \( \delta > 0 \) is sufficiently small, then the map \( \Phi|_{S^2 \times [0, \tau + \delta]} \) is a local isometry. This contradicts the maximality of \( \tau \).

Therefore, the restriction \( \Phi|_{S^2 \times [0, \infty)} \) is a local isometry. An analogous argument shows that \( \Phi|_{S^2 \times (-\infty, 0]} \) is a local isometry. This completes the proof of Proposition 3.8.

Since \( \Phi : S^2 \times \mathbb{R} \to M \) is a local isometry, it follows that \( \Phi \) is a covering map (cf.[6, Section 1.11]). Consequently, the universal cover of \( (M, g) \) is isometric to \( S^2 \times \mathbb{R} \), equipped with the standard metric. This completes the proof of Theorem 1.1.

Acknowledgments

The authors would like to thank Fernando Marques for discussions, and Willie Wong for comments on an earlier version of this paper.

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**Received February 24, 2010**