A local action for fermionic unconstrained higher spin gauge fields in (A)dS space

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ABSTRACT: The local free action principle for bosonic unconstrained higher spin gauge fields in d-dimensional (A)dS_d space has been already established by Segal. In this paper, we will present a similar action principle à la Segal for fermions. Moreover, we will demonstrate how the Fronsdal and the Fang-Fronsdal equations in d-dimensional (A)dS_d space are related, respectively, to the Euler-Lagrange equations of the bosonic and fermionic actions mentioned above.

KEYWORDS: Higher spin, (Anti-)de-Sitter space
1 Introduction

From the massless limit of the Singh-Hagen formulation [1, 2], the local constrained Lagrangian formulation describing free bosonic (fermionic) massless higher spin fields was established by Fronsdal (Fang and Fronsdal) in flat and Anti-de-Sitter (AdS) spacetimes [3, 4] ([5, 6]) in metric-like approach (see [7] for a recent review)\(^1\). Along with the constrained formulations, unconstrained ones\(^2\), in both the BRST and geometric approaches, have been already studied in Minkowski and (A)dS spacetimes [16]-[45]\(^3\). An unconstrained Lagrangian formulation should be useful to make links between the higher spin theories and the BRST form of the string theory [23, 47], as well as it is thought that it might be helpful for studying a possible Lagrangian formulation for the Vasiliev equations,

\(^1\) For frame-like approach see the references of [8]-[10] in Minkowski space, and [11]-[14] in AdS space (see also [15] for massless mixed-symmetry fermionic fields).

\(^2\) The constrained formulation includes some conditions on the gauge fields and parameters, while in the unconstrained one there is no conditions on the gauge fields and parameters.

\(^3\) We notice that in particular the formulation proposed in [42] does not appear to be fully unconstrained (the gauge parameter is still subject to a transversality condition), however, one can parameterize the gauge symmetry of the corresponding Maxwell-like Lagrangians in terms of fully unconstrained gauge parameters so as to obtain a fully unconstrained system [46] (we thank Dario Francia for comments and pointing out this issue).
describing interacting massless higher spin fields [48] - [50]. We note that one of the main open problems in classical field theory is constructing a Lagrangian formalism to describe interacting higher spin fields (see e.g. [52] - [55] for reviews).

In 2001, Segal suggested a simple model for the massless noninteracting bosonic higher spin fields in $d$-dimensional (A)dS$_d$ spacetime, in terms of unconstrained gauge fields and parameters [56]. In fact, he constructed a generating formulation describing an infinite sum of the actions for massless integer-spin fields, $s = 0, 1, \ldots, \infty$, such that every spin enters only one time. The obtained action includes an integral localized on the “constraint surface” $p^2 - 1 = 0$ in the cotangent bundle of AdS space. He was shown that the action can be thought as a decomposition of an infinite sum over all Fronsdal actions in (A)dS$_d$ spacetime. It should be emphasized that the Segal’s unconstrained formulation is devoid of any auxiliary fields, differently from what has been obtained e.g. in [34] where there exist indeed two auxiliary fields in the Lagrangian.

Later on, in 2014, Schuster and Toro in their researches on constructing a free action principle for continuous spin particle (CSP) presented an unconstrained formulation for bosonic higher spin gauge fields in flat space [57]. This action principle was obtained from a limit of the CSP theory when the continuous spin parameter vanishes, and is a reformulation of the Segal’s 4-dimensional action with vanishing cosmological constant ($\Lambda = 0$). They also recovered this action as a sum of the Fronsdal actions in another approach. Indeed, they integrated out the auxiliary space dependence of the action to reproduce the Schwinger-Fronsdal tensor actions, using a field decomposition and partially gauge fixing.

Afterwards, in 2015, we found an unconstrained formulation for fermionic continuous spin particle in four-dimensional flat spacetime [58]. On the other hand, to our knowledge, a fermionic action à la Segal has not been already discussed in the literature. Therefore, to a great extent, the present work was motivated by the previous studies on the bosonic [57] and fermionic [58] continuous spin gauge theories, which are formulated in an unconstrained formalism. In fact, we realize that the unconstrained formalism of the bosonic [56] and fermionic (present work) higher spin gauge theories in 4-dimensional flat spacetime are the higher spin limit of [57] and [58], when the continuous spin parameter goes to zero.

In the present work, we obtain a local and covariant action principle for free fermionic higher spin gauge fields in $d$-dimensional (A)dS$_d$ spacetime as
\[ \mathcal{A} = \int d^d x \ d^d \eta \ e^{\Psi(x,\eta)} \delta'(\eta^2 + 1) \ (\gamma \cdot \eta + i) \left[ \gamma \cdot D - (\gamma \cdot \eta - i) \ (\partial_\eta \cdot D) \right. \\
+ \left. \frac{i \sqrt{\Lambda}}{2} \left( 2 N_\eta d - 4 + (\gamma \cdot \eta)(\gamma \cdot \partial_\eta) - 3 i (\gamma \cdot \partial_\eta) \right) \right]\Psi(x,\eta), \] 

where \( \eta^a \) is a \( d \)-dimensional auxiliary Lorentz vector localized to the unit hyperboloid \( \eta^2 = -1 \), \( \delta' \) is the derivative of the Dirac delta function with respect to its argument \( \delta'(a) = \frac{d}{da} \delta(a) \), and \( \gamma^a \) are gamma matrices in \( d \) dimensions. In addition, we define the Dirac adjoint as \( \Psi^\dagger = \Psi \gamma^0 \), \( \partial_a \eta := \partial / \partial \eta^a \), \( N_\eta := \eta \cdot \partial_\eta \), and \( e := \det e^a_{\mu} \), where \( e^a_{\mu} \) stands for vielbein of (A)dS space. We also denote \( D_a := \epsilon^\mu_a D_\mu \) where \( D_\mu \) stands for fermionic Lorentz covariant derivative, defined in the Appendix A, and

\[ \Lambda = \begin{cases} 
+ 1, & \text{for dS space;} \\
0, & \text{for flat space;} \\
- 1, & \text{for AdS space,} 
\end{cases} \] 

such that the (A)dS radius is set to be one. The gauge field \( \Psi \) is unconstrained and introduces as the generating function

\[ \Psi(x,\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \eta^{a_1} \ldots \eta^{a_n} \Psi_{a_1 \ldots a_n}(x), \] 

where \( \Psi_{a_1 \ldots a_n}(x) \) are totally symmetric spinor-tensor fields of all half-integer spin fields \( s = n + \frac{1}{2} \), in such a way that the spinor index is left implicit. Note that in the infinite tower of spins (1.3), every spin state interns only once, and the spin states are not mixed under the Lorentz boost \( ^7 \).

The action (1.1) is invariant under the gauge transformations

\[ \delta_{\xi_1} \Psi(x,\eta) = \left[ (\gamma \cdot D)(\gamma \cdot \eta - i) - (\eta^2 + 1)(\partial_\eta \cdot D) \right. \\
+ \left. \frac{i \sqrt{\Lambda}}{2} \left( 2(\gamma \cdot \eta) + (\gamma \cdot \eta + i)^2 (\gamma \cdot \partial_\eta) - (\gamma \cdot \eta + i)(2 N_\eta d) \right) \right] \xi_1(x,\eta), \] 

\[ \delta_{\xi_2} \Psi(x,\eta) = (\eta^2 + 1)(\gamma \cdot \eta + i) \xi_2(x,\eta), \] 

where \( \xi_1 \) and \( \xi_2 \) are the unconstrained gauge transformation parameters. Varying the action (1.1) with respect to the gauge field \( \Psi \) yields the following equation of motion

\[ \delta'(\eta^2 + 1)(\gamma \cdot \eta + i) \left[ \gamma \cdot D - (\gamma \cdot \eta - i)(\partial_\eta \cdot D) \right. \\
+ \left. \frac{i \sqrt{\Lambda}}{2} \left( 2 N_\eta d - 4 + (\gamma \cdot \eta)(\gamma \cdot \partial_\eta) - 3 i (\gamma \cdot \partial_\eta) \right) \right] \Psi(x,\eta) = 0. \]

We will show later how this equation of motion can be related to the Fang-Fronsdal equation.

\(^7 \) We note that in the context of the continuous spin gauge theory, the gauge field has a form of the one in (1.3), but with the difference that the spin states are mixed under the Lorentz boost.
The structure of this paper is as follows. In Sec. 2 we will discuss the constrained Fronsdal(-like) formulation in order to construct an unconstrained formalism, leading then to the Segal action [56]. In Sec. 3 we shall elaborate on the action (1.1), by studying constrained Fang-Fronsdal(-like) formulation, and obtaining an unconstrained formalism to discover the fermionic action presented in (1.1). Conclusions and further directions will be presented in Sec. 4. We present our conventions in the Appendix A. Useful relations are given in the Appendix B. In order to be self-contained, a short discussion on the Segal action [56] will be presented in the Appendix C.

2 The Segal action

In this section, we aim to construct the Segal action [56] from the Fronsdal one [4]. In fact, we will make a relationship between the Fronsdal equation and the obtained Euler-Lagrange equation of the Segal action. To this end, we first briefly review the Fronsdal formulation in \( d \)-dimensional (A)dS space, in which the gauge field and parameter obey traceless constraints. Then by applying a field redefinition, we construct a Fronsdal-like formulation in such a way that the gauge field and parameter satisfy shifted traceless conditions. Finally, we perform a Fourier transformation on the auxiliary space, solve the shifted traceless conditions using distributions, and construct an unconstrained formulation for the equation of motion. We will find the obtained equation of motion is nothing but the Euler-Lagrange equation of the Segal action.

2.1 Fronsdal formulation

The action describing an arbitrary massless spin-\( s \) field in the Minkowski [3] and (A)dS [4] space-times were first found by Fronsdal in the metric-like approach. In \( d \)-dimensional (A)dS space-time, the free action for a given spin \( s \) reads

\[
\mathcal{I}_s = \frac{s!}{2} \int d^d x \ e \varphi_s(x, \partial \omega) \left[ 1 - \frac{1}{4} \omega^2 (\partial \omega \cdot \partial \omega) \right] \mathcal{F}_s \varphi_s(x, \omega) \bigg|_{\omega=0},
\]

where

\[
\mathcal{F}_s = - \Box_{(A)dS} + (\omega \cdot \nabla)(\partial \omega \cdot \nabla) - \frac{1}{2} (\omega \cdot \nabla)^2 (\partial \omega \cdot \partial \omega) - \Lambda \left[ s^2 + s(d-6) - 2(d-3) + \omega^2 (\partial \omega \cdot \partial \omega) \right],
\]

is the Fronsdal operator. The operator \( \Box_{(A)dS} \) denotes the D’Alembert operator of (A)dS space, and \( \nabla_a := e^\mu_a \nabla_\mu \) while \( \nabla_\mu \) stands for the Lorentz covariant derivative (see the Appendix A). The gauge field \( \varphi_s(x, \omega) \) in (2.1) is double-traceless

\[
(\partial \omega \cdot \partial \omega)^2 \varphi_s(x, \omega) = 0, \quad \varphi_s(x, \omega) = \frac{1}{s!} \omega^{a_1} \ldots \omega^{a_s} \varphi_{a_1 \ldots a_s}(x),
\]

describing a totally symmetric double-traceless tensor field \( \varphi_{a_1 \ldots a_s}(x) \) of any integer spin \( s \), where \( \omega^a \) is a \( d \)-dimensional auxiliary vector and \( \partial_a^\omega := \partial / \partial \omega_a \). The action (2.1) is

\[
\text{The operator } \mathcal{F}_s \text{ in (2.2) was first found in [62]. See also the presented formulation in [63].}
\]
invariant under the gauge transformation

\[ \delta_\epsilon \varphi_s(x, \omega) = (\omega \cdot \nabla) \epsilon_s(x, \omega) , \]  

(2.4)

where \( \epsilon_s \) is the gauge transformation parameter subject to the traceless condition

\[ (\partial_\omega \cdot \partial_\omega) \epsilon_s(x, \omega) = 0 , \quad \epsilon_s(x, \omega) = \frac{1}{(s-1)!} \omega^{a_1} \cdots \omega^{a_{s-1}} \epsilon_{a_1 \cdots a_{s-1}}(x) , \]

(2.5)

for the rank-\((s-1)\) symmetric and traceless gauge parameter \( \epsilon_{a_1 \cdots a_{s-1}(x)} \). We note that the generating functions in (2.3) and (2.5) satisfy, respectively, the following homogeneity conditions:

\[ (N_\omega - s) \varphi_s(x, \omega) = 0 , \quad (N_\omega - s + 1) \epsilon_s(x, \omega) = 0 , \]

(2.6)

where \( N_\omega := \omega \cdot \partial_\omega \). We end up here the constrained Fronsdal formulation and present a Fronsdal-like system in next subsection.

2.2 Fronsdal-like formulation

By the Fronsdal-like formulation for a massless higher spin field we mean a formulation, in which the gauge field and parameter are redefined objects by the operators \( P_\Phi \) and \( P_\epsilon \)

\[ \Phi_s(x, \omega) = P_\Phi \varphi_s(x, \omega) , \quad P_\Phi = \sum_{n=0}^{\infty} \omega^{2n} \frac{1}{2^{2n} n! (N_\omega + \frac{d}{2} - 1)_n} , \]

(2.7)

\[ \epsilon_s(x, \omega) = P_\epsilon \epsilon_s(x, \omega) , \quad P_\epsilon = \sum_{n=0}^{\infty} \omega^{2n} \frac{1}{2^{2n} n! (N_\omega + \frac{d}{2})_n} , \]

(2.8)

so that the new gauge field \( \Phi_s \) and parameter \( \epsilon_s \) satisfy, respectively, the shifted traceless conditions

\[ (\partial_\omega \cdot \partial_\omega - 1)^2 \Phi_s(x, \omega) = 0 , \quad (\partial_\omega \cdot \partial_\omega - 1) \epsilon_s(x, \omega) = 0 . \]

(2.9)

Indeed, the operators \( P_\Phi \) and \( P_\epsilon \) play a role to convert the shifted traceless conditions (2.9) to the traceless ones (2.3), (2.5); and \((a)_n\) in the denominators denotes the rising Pochhammer symbol (B.1) (for specific details of the calculation, see the appendices of [64]).

Considering a similar generating function, as before, for the redefined gauge field \( \Phi_s \) and parameter \( \epsilon_s \), we can find the “Fronsdal-like equation” \(^9\)

\[ \left[ - \square(A)_{dS} + (\omega \cdot \nabla)(\partial_\omega \cdot \nabla) - \frac{1}{2} (\omega \cdot \nabla)^2 (\partial_\omega \cdot \partial_\omega - 1) - \Lambda \left( s^2 + s(d-6) - 2(d-3) + \omega^2 (\partial_\omega \cdot \partial_\omega) - 2 \omega^2 \right) \right] \Phi_s(x, \omega) = 0 , \]

(2.10)

\(^9\) Note that the Fronsdal-like formulation, describing a single bosonic continuous spin particle (CSP), was first discussed in [65]. That formulation satisfies similar conditions as (2.9) for the bosonic CSP gauge field and the parameter. In this sense, we called here our formulation the “Fronsdal-like formulation”, however we note that it actually describes the bosonic higher spin gauge theory.
which is invariant under the gauge transformation
\[ \delta \varepsilon \Phi_s(x, \omega) = (\omega \cdot \nabla) \varepsilon_s(x, \omega). \] (2.11)

To illustrate that the obtained Fronsdal-like equation (2.10) will reproduce the Fronsdal equation, one can first use the homogeneity condition \((N_\omega - s) \Phi_s = 0\) to rewrite the equation (2.10) in terms of \(N_\omega\). Then, by plugging (2.7) into (2.10), and applying the relations (B.9)-(B.12), it is straightforward to demonstrate that the Fronsdal-like equation (2.10) will precisely recover the Fronsdal equation,

\[ F(s) \varphi_s(x, \omega) = 0, \]

up to terms of order \(O(\omega^4)\) vanishing due to the double-traceless condition \(\varphi(x, \partial_\omega)(\omega^2)^2 = 0\) at the level of the action (2.1).

The relations (2.9) - (2.11) are formulated for an arbitrary integer massless spin \(s\) field. However, we might be interested in a formulation where its gauge field \(\Phi(x, \omega)\) decomposes into an infinite tower of all integer spins \((s = 0, 1, 2, \ldots, \infty)\), in which every spin state interns only once and the spin states of the symmetric tensor fields \(\Phi_{a_1 \ldots a_s}(x)\) are not mixed under the Lorentz boost\(^{10}\). Therefore, to construct such a formulation, we first use the homogeneity condition of the gauge field (2.6) into (2.10), and then take into account an infinite sum over all integer spins in the relations (2.9) - (2.11). The obtained result gives us the traceless conditions
\[ (\partial_\omega \cdot \partial_\omega - 1)^2 \Phi(x, \omega) = 0, \quad \quad \quad \quad (\partial_\omega \cdot \partial_\omega - 1) \varepsilon(x, \omega) = 0, \] (2.13)
the Fronsdal-like equation
\[ \left[ - \Box_{(A)dS} + (\omega \cdot \nabla)(\partial_\omega \cdot \nabla) - \frac{1}{2} (\omega \cdot \nabla)^2 (\partial_\omega \cdot \partial_\omega - 1) \right. \]
\[ \quad \quad \quad \left. - \Lambda \left( N_\omega^2 + N_\omega(d-6) - 2(d-3) + \omega^2 (\partial_\omega \cdot \partial_\omega) - 2\omega^2 \right) \right] \Phi(x, \omega) = 0, \] (2.14)
and the gauge transformation
\[ \delta \varepsilon \Phi(x, \omega) = (\omega \cdot \nabla) \varepsilon(x, \omega), \] (2.15)
in terms of the gauge field \(\Phi\) (2.12) and the gauge parameter \(\varepsilon\)\(^{11}\).

In flat spacetime, it would be useful for our future purpose to take into account the equation of motion (2.14) in the momentum space and then perform a suitable gauge

\(^{10}\) We note that this is in contrast to the bosonic continuous spin gauge field, where the gauge field has a similar decomposition as (2.12), but actually the spin states of the symmetric tensor fields \(\Phi_{a_1 \ldots a_s}(x)\) are mixed under the Lorentz boost such that the degree of mixing is determined by a continuous spin parameter.

\(^{11}\) The gauge parameter \(\varepsilon\) has a similar decomposition as the gauge field \(\Phi\), if we substitute \(\Phi\) with \(\varepsilon\), and \(s\) with \(s - 1\) into (2.12).
fixing (see the procedure applied in [65]). Consequently the massless bosonic higher spin equations in terms of the gauge-invariant distribution, $\Phi(p, \omega) = \delta(p \cdot \omega) \Phi(p, \omega)$, become:

\begin{align*}
p^2 \Phi(p, \omega) &= 0, \\
(p \cdot \omega) \Phi(p, \omega) &= 0, \\
(p \cdot \partial_\omega) \Phi(p, \omega) &= 0, \\
(\partial_\omega \cdot \partial_\omega - 1) \Phi(p, \omega) &= 0.
\end{align*}

We notice that these equations in their Fourier-transformed auxiliary space are precisely the Wigner equations [66] with a vanishing continuous spin parameter.

We note also that, as an Euler-Lagrange equation, the equation (2.14) (or (2.10)) cannot be directly obtained from an action principle. Strictly speaking, if we consider the Fronsdal-like equation (2.14) as a form of $F \Phi(x, \omega) = 0$, then the operator

$$K = \left[1 - \frac{1}{4} (\omega^2 - 1)(\partial_\omega \cdot \partial_\omega - 1)\right] F,$$

would not be a Hermitian operator. In a sense, this is similar to what happens in [65] for the Fronsdal-like equation. However, we know that one can treat in two ways. One way is redefining the gauge field in the equation (2.14), which comes it back again to the Fronsdal equation, derived from the Fronsdal action (2.1). Another way is performing the Fourier transformation on the auxiliary space $\omega$ in the equation (2.14) and working in $\eta$-space, as well as solving the (double-)traceless condition using the Dirac delta distributions. In [61], this procedure has been explained in detail for CSP, and here, in next subsection, we will shortly discuss it for higher spin and demonstrate how it leads to the Segal action.

### 2.3 Unconstrained formulation

The Segal action is an unconstrained formulation for describing the bosonic higher spin gauge field in $d$-dimensional (A)dS$_d$ space [56], which is formulated in $\eta$-space. Thus we will write the Fronsdal-like formulation in $\eta$-space, by performing the Fourier transformation on the auxiliary space $\omega$

$$\tilde{\Phi}(x, \eta) = \int \frac{d^4 \omega}{(2\pi)^2} \exp(-i \eta \cdot \omega) \Phi(x, \omega).$$

In $\eta$-space, the shifted double-traceless condition (2.13) becomes $(\eta^2 + 1)^2 \tilde{\Phi}(x, \eta) = 0$, which can be generally solved by the Dirac delta distribution

$$\tilde{\Phi}(x, \eta) = \delta'((\eta^2 + 1) \Phi(x, \eta),$$

where $\Phi(x, \eta)$ is now an arbitrary unconstrained function and $\delta'(a) = \frac{d}{da} \delta(a)$. We then take into account the Fronsdal-like equation (2.14) in its Fourier-transformed auxiliary space, which is

$$\left[-\Box_{(A)dS} - (\partial_\eta \cdot \nabla)(\eta \cdot \nabla) - \frac{1}{2} (\partial_\eta \cdot \nabla)^2 (\eta^2 + 1) \right.$$

$$\left. - \Lambda \left( (N_\eta + d)^2 - (N_\eta + d)(d - 6) - 2(d - 3) + (\partial_\eta \cdot \partial_\eta) \eta^2 + 2 (\partial_\eta \cdot \partial_\eta) \right)\right] \tilde{\Phi}(x, \eta) = 0,$$
with \( N_\eta := \eta \cdot \partial_\eta \) and \( \partial^a_\eta := \partial/\partial \eta^a \). Afterwards, by plugging (2.18) into (2.19), and applying the identities (B.6) and (B.8), we will conveniently arrive at the equation of motion

\[
\delta'(\eta^2 + 1) \left[ - \Box_{(A)dS} + (\eta \cdot \nabla)(\partial_\eta \cdot \nabla) - \frac{1}{2} (\eta^2 + 1)(\partial_\eta \cdot \nabla)^2 \right. \\
- \Lambda \left( N^2_\eta + N_\eta (d-6) - 2(d-3) + \eta^2 (\partial_\eta \cdot \partial_\eta) + 2 (\partial_\eta \cdot \partial_\eta) \right) \Phi(x,\eta) = 0.
\]

The obtained equation of motion (2.20) can be derived from the action

\[
\mathcal{I} = \frac{1}{2} \int d^d x d^d \eta \ e \ \Phi(x,\eta) \ \delta'(\eta^2 + 1) \left[ - \Box_{(A)dS} + (\eta \cdot \nabla)(\partial_\eta \cdot \nabla) - \frac{1}{2} (\eta^2 + 1)(\partial_\eta \cdot \nabla)^2 \right. \\
- \Lambda \left( N^2_\eta + N_\eta (d-6) - 2(d-3) + \eta^2 (\partial_\eta \cdot \partial_\eta) + 2 (\partial_\eta \cdot \partial_\eta) \right) \Phi(x,\eta) .
\]

This action is precisely the Segal action presented in [56], describing the bosonic higher spin gauge field in \( d \)-dimensional (A)dS space (see the Appendix C for more detail on the Segal action). Note that, as Segal mentioned, we find the action (2.21) is equal to a sum of the Fronsdal actions (2.1) up to some coefficients

\[
\mathcal{I} = \sum_{s=0}^{\infty} \alpha_s \mathcal{I}_s . \tag{2.22}
\]

It is notable that in 4-dimensional flat space \( \Lambda = 0 \), the authors of [57] also proposed the action (2.21) for presenting their massless bosonic higher spin formulation. They directly solved the integral over \( \eta \)-space in the action (2.21) and illustrated the outcome (3.23) in another fashion.

With a similar procedure as what was done in this subsection to obtain the Segal action, we can find the invariance of the action (2.21) under the gauge transformations (see the method in [61])

\[
\delta_\varepsilon \Phi(x,\eta) = \left[ \eta \cdot \nabla - \frac{1}{2} (\eta^2 + 1)(\partial_\eta \cdot \nabla) \right] \varepsilon(x,\eta), \tag{2.23}
\]

\[
\delta_\chi \Phi(x,\eta) = (\eta^2 + 1)^2 \chi(x,\eta), \tag{2.24}
\]

where the gauge parameters \( \varepsilon \) and \( \chi \) are two unconstrained arbitrary functions.

3 The fermionic action

In this section, we will construct the fermionic action presented in (1.1), by following the steps which led to the Segal action. For this purpose, we will demonstrate a relationship between the Fang-Fronsdal equation [6] and the obtained equation of motion in (1.6). Following the previous section, we first review the Fang-Fronsdal formulation. Then, using a field redefinition, we shall construct the Fang-Fronsdal-like formulation. By performing a Fourier transformation and solving the gamma trace conditions, we will arrive at the equation of motion (1.6), which can be directly derived from the fermionic action (1.1).
3.1 Fang-Fronsdal formulation

The action describing an arbitrary massless half-integer spin field \( s = n + \frac{1}{2} \) in \( d \)-dimensional (A)dS\(_d\) space-time was first proposed by Fang and Fronsdal [6] in metric-like approach\(^{12}\). The free action is given by

\[
\mathcal{A}_n = \int d^d x \ e^{-\psi_n(x,\partial \omega)} \left[ 1 - \frac{1}{2} (\gamma \cdot \omega)(\gamma \cdot \partial \omega) - \frac{1}{4} \omega^2 (\partial \omega \cdot \partial \omega) \right] \mathcal{F}(n) \psi_n(x,\omega) \bigg|_{\omega=0} ,
\]

where

\[
\mathcal{F}(n) = i \gamma \cdot D - i (\omega \cdot D)(\gamma \cdot \partial \omega) - \frac{1}{2} \sqrt{\Lambda} \left[ 2n + d - 4 + (\gamma \cdot \omega)(\gamma \cdot \partial \omega) \right]
\]

is the Fang-Fronsdal operator (see the Appendix A for conventions). The action (3.1) is invariant under the gauge transformation

\[
\delta \zeta \psi_n(x,\omega) = \left( \omega \cdot D + \frac{i \sqrt{\Lambda}}{2} \gamma \cdot \omega \right) \zeta_n(x,\omega),
\]

where the spinor gauge field \( \psi_n \) and parameter \( \zeta_n \), using an auxiliary vector \( \omega^a \), are introduced as the generating functions

\[
\psi_n(x,\omega) = \frac{1}{n!} \omega^{a_1} \ldots \omega^{a_n} \psi_{a_1 \ldots a_n}(x),
\]

\[
\zeta_n(x,\omega) = \frac{1}{(n-1)!} \omega^{a_1} \ldots \omega^{a_{n-1}} \zeta_{a_1 \ldots a_{n-1}}(x),
\]

obeying the gamma traceless conditions

\[
(\gamma \cdot \partial \omega)^3 \psi_n(x,\omega) = 0, \quad (\gamma \cdot \partial \omega) \zeta_n(x,\omega) = 0,
\]

and the homogeneity ones

\[
(N_\omega - n) \psi_n(x,\omega) = 0, \quad (N_\omega - n + 1) \zeta_n(x,\omega) = 0.
\]

Note the spinor indices are left implicit, and \( \psi_{a_1 \ldots a_n} \) in (3.4) denotes totally symmetric spinor-tensor field of half-integer spin, while \( \zeta_{a_1 \ldots a_{n-1}} \) in (3.5) stands for the relevant spinor gauge parameter. We also note, using the homogeneity condition on the spinor gauge field (3.7), one can be shown that the action (3.1) is precisely equivalent to the Metsaev action [68], in the limit of massless fields.

3.2 Fang-Fronsdal-like formulation

Similar to the bosonic case, we introduce the Fang-Fronsdal-like formulation in terms of the redefined spinor gauge field and parameter

\[
\Psi_n(x,\omega) = P_\Psi \psi_n(x,\omega), \quad P_\Psi = \sum_{k=0}^\infty \left[ (\gamma \cdot \omega)^{2k} + 2k(\gamma \cdot \omega)^{2k-1} \right] \frac{1}{2^{2k} k!(N_\omega + d/2 - 1)_k},
\]

\[
\zeta_n(x,\omega) = P_\zeta \zeta_n(x,\omega), \quad P_\zeta = \sum_{k=0}^\infty \left[ (\gamma \cdot \omega)^{2k} + 2k(\gamma \cdot \omega)^{2k-1} \right] \frac{1}{2^{2k} k!(N_\omega + d/2)_k},
\]

\(^{12}\) To clarify the equivalence of metric- and frame-like formulations of higher spin fermions see e.g. [67].
such that the new spinor gauge field $\Psi_n$ and parameter $\xi_n$ satisfy, respectively, the shifted gamma traceless conditions

$$(\gamma \cdot \partial_\omega - 1)(\partial_\omega \cdot \partial_\omega - 1) \Psi_n(x, \omega) = 0, \quad (\gamma \cdot \partial_\omega - 1) \xi_n(x, \omega) = 0.$$ (3.10)

The “Fang-Fronsdal-like equation” can be then find as

$$\left[i\gamma \cdot D - i(\omega \cdot D)(\gamma \cdot \partial_\omega - 1)
- \frac{1}{2} \sqrt{\Lambda} \left[2n + d - 4 + (\gamma \cdot \omega)(\gamma \cdot \partial_\omega) - 3(\gamma \cdot \omega)\right]\right]\Psi_n(x, \omega) = 0,$$

which is invariant under the gauge transformation

$$\delta_\xi \Psi_n(x, \omega) = \left(\omega \cdot D + \frac{i \sqrt{\Lambda}}{2} \gamma \cdot \omega\right) \xi_n(x, \omega).$$ (3.12)

To demonstrate the obtained Fang-Fronsdal-like equation (3.11) is equivalent to the Fang-Fronsdal one $F_n \psi_n = 0$, we can first use the homogeneity condition $(N_\omega - n) \Psi_n(x, \omega) = 0$ within (3.11). Then by plugging (3.8) into (3.11), and applying the relations (B.13) - (B.16), the Fang-Fronsdal equation, $F_n \psi_n = 0$, will be conveniently reproduced (up to terms of order $O(\omega^3)$ vanishing at the level of the action, due to the triple gamma-trace condition on the spinor gauge field $\psi_n(x, \partial_\omega)(\gamma \cdot \omega)^3 = 0$).

If we are interested in a formulation in terms of the spinor gauge field

$$\Psi(x, \omega) = \sum_{n=0}^{\infty} \Psi_n(x, \omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \omega^{a_1} \ldots \omega^{a_n} \Psi_{a_1 \ldots a_n}(x),$$ (3.13)

comprising an infinite tower of all half-integer spins, and consider a similar decomposition for the spinor gauge parameter $\xi$, the shifted gamma traceless conditions read

$$(\gamma \cdot \partial_\omega - 1)(\partial_\omega \cdot \partial_\omega - 1) \Psi(x, \omega) = 0, \quad (\gamma \cdot \partial_\omega - 1) \xi(x, \omega) = 0.$$ (3.14)

The Fang-Fronsdal-like equation, therefore, becomes

$$\left[i\gamma \cdot D - i(\omega \cdot D)(\gamma \cdot \partial_\omega - 1)
- \frac{1}{2} \sqrt{\Lambda} \left[2N_\omega + d - 4 + (\gamma \cdot \omega)(\gamma \cdot \partial_\omega) - 3(\gamma \cdot \omega)\right]\right]\Psi(x, \omega) = 0,$$

with the gauge symmetry

$$\delta_\xi \Psi(x, \omega) = \left(\omega \cdot D + \frac{i \sqrt{\Lambda}}{2} \gamma \cdot \omega\right) \xi(x, \omega).$$ (3.16)

Note again that the Fang-Fronsdal-like formulation, describing a single fermionic continuous spin particle (CSP), was first discussed in [65]. That formulation satisfies similar conditions as (3.10) for the fermionic CSP gauge field and parameter. In this sense, we called here our formulation the “Fang-Fronsdal-like formulation”, however it actually describes the fermionic higher spin gauge theory.
Again as the previous section, considering the flat space limit of the equation of motion (3.15) in the momentum space, one can find the massless fermionic higher spin equations

\[
\begin{align*}
(\gamma \cdot p) \Psi(p, \omega) &= 0, \\
(p \cdot \omega) \Psi(p, \omega) &= 0, \\
(p \cdot \partial_\omega) \Psi(p, \omega) &= 0, \\
(\gamma \cdot \partial_\omega - 1) \Psi(p, \omega) &= 0,
\end{align*}
\]

(3.17) in terms of the gauge-invariant distribution \( \Psi = \delta(p \cdot \omega) \Psi \). We note again that these equations are the massless higher spin limit of the Wigner equations \([66]\), if we replace the fourth equation by \((\partial_\omega \cdot \partial_\omega - 1) \Psi(p, \omega) = 0\) (for details see the explanations in \([65]\)).

Similar to what was already discussed about the bosonic case, the Fang-Fronsdal-like equation (3.15) (or (3.11)) cannot be directly acquired from an action principle. Nevertheless, one can follow the method in the previous section to find an equation of motion, which can be derived from the action (1.1).

### 3.3 Unconstrained formulation

In \(\eta\)-space, the triple-gamma traceless condition (3.14) on the spinor gauge field becomes \((\gamma \cdot \eta + i)(\eta^2 + 1) \bar{\Psi}(x, \eta) = 0\), which can be generally solved by

\[
\bar{\Psi}(x, \eta) = \delta'(\eta^2 + 1)(\gamma \cdot \eta - i) \Psi(x, \eta),
\]

(3.18)

where \(\Psi\) is an unconstrained arbitrary function. We then take into account the Fang-Fronsdal-like equation (3.15) in its Fourier transformed auxiliary space, which is

\[
\begin{align*}
&\left[ i \gamma \cdot D + i (\partial_\eta \cdot D)(\gamma \cdot \eta + i) \\
&+ \frac{1}{2} \sqrt{\Lambda} \left[ 2N_\eta + d + 4 + (\gamma \cdot \partial_\eta)(\gamma \cdot \eta) + 3 i (\gamma \cdot \partial_\eta) \right] \right] \bar{\Psi}(x, \eta) = 0.
\end{align*}
\]

(3.19)

Plugging (3.18) into (3.19), and applying the identities (B.7) and (B.8), we will arrive at the equation of motion

\[
\hat{K} \Psi(x, \eta) = \delta'(\eta^2 + 1) (\gamma \cdot \eta + i) \left[ \gamma \cdot D - (\gamma \cdot \eta - i)(\partial_\eta \cdot D) \\
+ \frac{i \sqrt{\Lambda}}{2} \left( 2N_\eta + d - 4 + (\gamma \cdot \eta)(\gamma \cdot \partial_\eta) - 3 i (\gamma \cdot \partial_\eta) \right) \right] \Psi(x, \eta) = 0.
\]

(3.20)

This equation of motion is precisely the one in (1.6), which obtained from the action (1.1). Therefore, at the level of the equation of motions, we illustrated how the Fang-Fronsdal equation, \(\mathcal{F}(\alpha) \psi_\alpha = 0\), can be related to the Euler-Lagrange equation of (1.6). Practically, to find the action (1.1), we indeed introduced the Hermitian conjugation

\[
(\partial_x)^\dagger \equiv - \partial_x, \quad (\partial_\eta)^\dagger \equiv - \partial_\eta, \quad \eta^\dagger \equiv \eta,
\]

(3.21)
such that $\hat{\mathcal{K}}^\dagger = \gamma^0 \hat{\mathcal{K}} \gamma^0$. Using this fact, we were be able to write the fermionic action (1.1) à la Segal as
\[
A = \int d^d x \, d^d \eta \, e^{\Psi(x, \eta) \hat{\mathcal{K}} \Psi(x, \eta)}
\]
\[
= \int d^d x \, d^d \eta \, e^{\Psi(x, \eta) \delta'(\eta^2 + 1) \left( \gamma \cdot D - (\gamma \cdot \eta - i) (\partial_\eta \cdot D) + \frac{i \sqrt{\Lambda}}{2} \left( 2N_\eta + d - 4 + (\gamma \cdot \eta)(\gamma \cdot \partial_\eta) - 3 i (\gamma \cdot \partial_\eta) \right) \right) \Psi(x, \eta)}.
\]

We note that, similar to the bosonic case, we find the action (1.1) is equal to a sum of the Fang-Fronsdal actions (3.1) with some coefficients
\[
A = \sum_{s=0}^{\infty} \beta_s A_s.
\]

A similar procedure can be easily done to obtain the gauge symmetries (1.4), (1.5). For instance, in order to obviously see the invariance of the action (1.1), and consequently the equation of motion (1.6), under the gauge symmetry (1.5), we can use the identities (B.7) and (B.8), to simplify the equation of motion (1.6), after some calculations, to the following form
\[
\left[ \gamma \cdot D + (\partial_\eta \cdot D) (\gamma \cdot \eta + i) \right] - \frac{i \sqrt{\Lambda}}{2} \left( 2N_\eta + d + 4 + (\gamma \cdot \partial_\eta)(\gamma \cdot \eta) + 3 i (\gamma \cdot \partial_\eta) \right) \delta'(\eta^2 + 1) (\gamma \cdot \eta - i) \Psi(x, \eta) = 0.
\]

Then, at a glance, this equation would be clearly invariant under the $\xi_2$ symmetry (1.5), by applying the Dirac delta function’s property: $a^2 \delta'(a) = 0$.

4 Conclusions and future directions

In this work, we presented a local and covariant action principle (1.1), devoid of any auxiliary field, to describe free fermionic higher spin gauge fields in $d$-dimensional (A)dS$_d$ spacetime. The action is invariant under the gauge symmetries (1.4), (1.5) where both the gauge field and the parameter were unconstrained.

We also applied a field redefinition on the Fronsdal equation to obtain a Fronsdal-like formulation for the bosonic higher spin gauge fields, in which the gauge field and parameter were traceless-like, instead of being traceless. Then by solving the traceless-like conditions in terms of distributions, we rewrote the Fronsdal-like formulation on its Fourier-transformed auxiliary space which was led to the Euler-Lagrange equation of the Segal action.

Finally, a similar procedure was applied on the Fang-Fronsdal equation, so that the Fang-Fronsdal-like formulation was found and the origin of the fermionic action (1.1) was clarified.
It should be emphasized that, indeed, we made a relationship between the Fronsdal equation and the Euler-Lagrange equation of (2.20), as well as a connection between the Fang-Fronsdal equation and the Euler-Lagrange equation of (1.6). However making a connection at the level of the actions is still an open problem; i.e. link the Segal action (2.21) to the Fronsdal one (2.1) (or link the fermionic action (1.1) to the Fang-Fronsdal one (3.1)).

In 4-dimensional flat space, the authors of [57] were directly shown that solving the $\eta$-dependent part of the Segal action will lead to a direct sum of all Fronsdal actions\textsuperscript{14}. However, It would be interesting to investigate and illustrate explicitly that their applied fashion will work for the fermionic action (1.1) as well; i.e. do the integral on the auxiliary space in the action (1.1), and reproduce a direct sum of all Fang-Fronsdal actions. Extending the manner to the higher spin theories in $d$-dimensional de-Sitter and anti-de-Sitter backgrounds would be attractive too.

The fermionic action presented here, together with the bosonic action of Segal, can be applied to construct supersymmetric higher spin theories in this approach, which its formulation presumably seems to be simpler than other existing theories. Moreover, it would be interesting to generalize the bosonic and fermionic formulations, à la Segal, to the partially-massless, mixed-symmetry and massive fields.

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A Conventions

Our conventions are as follows. $x^a$ and $\eta^a$ (or its Fourier transformed $\omega^a$) denote respectively coordinates and auxiliary coordinates in $d$-dimensional flat space-time, where the Latin (flat) indices take values: $a = 0, 1, \ldots, d - 1$. Derivatives with respect to $x^a$ and $\eta^a$ are defined as: $\partial_a := \partial/\partial x^a$, $\partial_\eta^a := \partial/\partial \eta^a$. We use the mostly minus signature for the flat metric tensor $\eta^{ab}$, and define the operators $N_\eta := \eta \cdot \partial_\eta$ and $N_\omega := \omega \cdot \partial_\omega$.

The \textbf{bosonic} covariant derivative $\nabla_a$ is given by

$$\nabla_a := e_a^\mu \nabla_\mu, \quad \nabla_\mu := \partial/\partial x^\mu + \frac{1}{2} \omega_\mu^{ab} M_{ab}, \quad M^{ab} := \eta^a \partial_\eta^b - \eta^b \partial_\eta^a,$$

where $e_a^\mu$ is inverse vielbein of (A)dS$_d$ space, $\nabla_\mu$ stands for the Lorentz covariant derivative, $\omega_\mu^{ab}$ is the Lorentz connection of (A)dS$_d$ space, and $M^{ab}$ denotes the spin operator of the

\textsuperscript{14}We note that the integral on the auxiliary space in the Segal action can not be solved in the Lorentzian signature (see appendices of [61] for more detail).
Lorentz algebra, while the Greek (curved) indices take values: \( \mu = 0, 1, \ldots, d - 1 \). The D’Alembert operator of \((A)dS_d\) space \( \Box_{(A)dS} \) is defined by

\[
\Box_{(A)dS} := \nabla^a \nabla_a + \epsilon_a^\mu \omega^b_{\mu b} \nabla_b . \tag{A.2}
\]

Flat and curved indices of the covariant totally symmetric tensor fields of \((A)dS_d\) space are related to each other as: \( \Phi_{a_1 \ldots a_s}(x) = e^b_{a_1} \ldots e^b_{a_s} \Phi_{\mu_1 \ldots \mu_s}(x) \).

The **fermionic** covariant derivative \( D_a \) is given by

\[
D_a := e^a_\mu D_\mu , \quad D_\mu := \partial / \partial x^\mu + \frac{1}{2} \omega^{ab}_\mu (M_{ab} + \gamma_{ab}) , \quad \gamma^{ab} := \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a) , \quad \tag{A.3}
\]

where \( \gamma^a \) are the \( d \)-dimensional Dirac gamma matrices satisfying the Clifford algebra

\[
\lbrace \gamma^a , \gamma^b \rbrace = 2 \eta^{ab} , \quad \tag{B.1}
\]

\[
(\gamma^a)^\dagger = \gamma^0 \gamma^a \gamma^0 , \quad (\gamma^0)^\dagger = + \gamma^0 , \quad (\gamma^i)^\dagger = - \gamma^i , \quad (i = 1, \ldots, d - 1) . \tag{A.4}
\]

We choose the mostly minus signature for the metric, however, it would be useful to stress the bosonic and fermionic formulations in the mostly plus signature for the metric as well. To this end, e.g. for the bosonic action (2.21), we have to apply the following replacements

\[
X \rightarrow - X , \quad Y \rightarrow Y , \quad \tag{A.5}
\]

where \( X \) denotes: \( \Lambda , \eta^2 , (\partial_\eta \cdot \partial_\eta) , (\partial_\eta \cdot \nabla) , \Box_{(A)dS} \); while \( Y \) stands for: \( N_\eta , (\eta \cdot \nabla) \). For the fermionic action (1.1), the substitutions should be taken into account as

\[
\mathcal{X} \rightarrow i \mathcal{X} , \quad (\gamma \cdot \eta) \rightarrow - i (\gamma \cdot \eta) , \quad \mathcal{Y} \rightarrow - \mathcal{Y} , \quad \tag{A.6}
\]

where \( \mathcal{X} \) denotes: \( \gamma^a , \overline{\Psi} , (\gamma \cdot \partial_\eta) , (\gamma \cdot D) \); while \( \mathcal{Y} \) stands for: \( \Lambda , \eta^2 , (\partial_\eta \cdot D) \).

## B Useful relations

The **rising Pochhammer symbol** \((a)_n\) is defined as

\[
(a)_n := a (a + 1) (a + 2) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)} , \quad n \in \mathbb{N} \quad \text{and} \quad a \in \mathbb{R} . \tag{B.1}
\]

The following useful relations can be conveniently derived (see the Appendix of [68] for more detail)

\[
[D^a , \eta^2] = 0 , \quad [\partial^2_\eta , D^a] = 0 , \quad [D^a , \gamma \cdot \eta] = 0 , \tag{B.2}
\]

\[
[\partial^2_\eta , \eta \cdot D] = 2 \partial_\eta \cdot D , \quad [\gamma \cdot \partial_\eta , \eta \cdot D] = \gamma \cdot D , \quad \{ \gamma \cdot D , \gamma \cdot \eta \} = 2 \eta \cdot D , \tag{B.3}
\]

\[
[\gamma \cdot D , \eta \cdot D] = \Lambda \left( \gamma \cdot \eta \left[ N_\eta + \frac{d - 1}{2} \right] - \eta^2 (\gamma \cdot \partial_\eta) \right) \tag{B.4}
\]

\[
[\partial_\eta \cdot D , \gamma \cdot \eta] = \Lambda \left( \left[ N_\eta + \frac{d - 1}{2} \right] \gamma \cdot \partial_\eta - (\gamma \cdot \eta) \partial^2_\eta \right) \tag{B.5}
\]
The identities
\[ (\partial_\gamma \cdot \partial_\eta) \delta'(\eta^2 + 1) = \delta'(\eta^2 + 1) (\partial_\gamma \cdot \partial_\eta) + 4 \delta''(\eta^2 + 1) (h - 3) - 4 \delta'''(\eta^2 + 1), \tag{B.6} \]
\[ (\gamma \cdot \partial_\eta) \delta'(\eta^2 + 1) = 2 (\gamma \cdot \eta) \delta''(\eta^2 + 1) + \delta'(\eta^2 + 1) (\gamma \cdot \partial_\eta), \tag{B.7} \]
\[ N_\gamma \delta'(\eta^2 + 1) = \delta'(\eta^2 + 1) (N_\gamma - 4) - 2 \delta''(\eta^2 + 1), \tag{B.8} \]
can be easily obtained, where \( \delta''(a) \) (or \( \delta'''(a) \)) is the derivative of \( \delta'(a) \) (or \( \delta''(a) \)) with respect to its argument \( a \).

The quantities \( \partial_\omega^a, \partial_\omega^2, \omega^a \) and \( N_\omega \) on the bosonic operator \( \mathbf{P}_\Phi \), introduced in (2.7), act as (for more detail, see the Appendices in [64])
\[ \partial_\omega^a \mathbf{P}_\Phi = \mathbf{P}_\Phi \left[ \partial_\omega^a - \omega^a \frac{1}{(2N + d)(2N + d - 2)} \right], \tag{B.9} \]
\[ \partial_\omega^2 \mathbf{P}_\Phi = \mathbf{P}_\Phi \left[ (\partial_\omega \cdot \partial_\omega) - \omega^2 \frac{2}{(2N + d)(2N + d - 2)} (\partial_\omega \cdot \partial_\omega) \right. \]
\[ \left. + \frac{2N + d}{(2N + d - 2)} - \omega^2 \frac{2}{(2N + d)(2N + d - 2)^2} + \mathcal{O}(\omega^4) \right], \tag{B.10} \]
\[ \omega^a \mathbf{P}_\Phi = \mathbf{P}_\Phi \left[ \omega^a + \omega^2 \omega^a \frac{1}{(2N + d)(2N + d - 2)} + \mathcal{O}(\omega^4) \right], \tag{B.11} \]
\[ N_\omega \mathbf{P}_\Phi = \mathbf{P}_\Phi \left[ N_\omega + \omega^2 \frac{1}{(2N + d - 2)} + \mathcal{O}(\omega^4) \right], \tag{B.12} \]
where the terms containing \( \mathcal{O}(\omega^4) \) will be eliminated at the level of the action, due to the double-traceless condition on the gauge field \( \Phi(x, \partial_\omega) (\omega^2)^2 = 0 \). On the other hand, the quantities \( \gamma \cdot \mathbf{D}, \omega \cdot \mathbf{D}, \gamma \cdot \partial_\omega \) and \( N_\omega \) on the fermionic operator \( \mathbf{P}_\Psi \), given by (3.8), act as
\[ (\gamma \cdot \mathbf{D}) \mathbf{P}_\Psi = \mathbf{P}_\Psi \left[ (\gamma \cdot \mathbf{D}) + (\omega \cdot \mathbf{D}) - (\gamma \cdot \omega)(\gamma \cdot \mathbf{D}) \frac{2}{(2N + d - 2)} \right. \]
\[ \left. - (\gamma \cdot \omega)(\omega \cdot \mathbf{D}) \frac{2}{(2N + d)(2N + d - 2)} + \omega^2 (\gamma \cdot \mathbf{D}) \frac{2}{(2N + d)(2N + d - 2)} + \mathcal{O}(\omega^3) \right], \tag{B.13} \]
\[ (\omega \cdot \mathbf{D}) \mathbf{P}_\Psi = \mathbf{P}_\Psi \left[ (\omega \cdot \mathbf{D}) + (\gamma \cdot \omega)(\omega \cdot \mathbf{D}) \frac{2}{(2N + d)(2N + d - 2)} + \mathcal{O}(\omega^3) \right], \tag{B.14} \]
\[ (\gamma \cdot \partial_\omega) \mathbf{P}_\Psi = \mathbf{P}_\Psi \left[ (\gamma \cdot \partial_\omega) - (\gamma \cdot \omega) \frac{2}{(2N + d - 2)^2} \right. \]
\[ + \omega^2 \frac{1}{(2N + d)^2} (\gamma \cdot \partial_\omega) + \frac{2N + d}{(2N + d - 2)} \right. \]
\[ \left. - (\gamma \cdot \omega) \frac{2(2N + d - 1)}{(2N + d)(2N + d - 2)} (\gamma \cdot \partial_\omega) + \mathcal{O}(\omega^3) \right]. \tag{B.15} \]

\footnote{We note, in this paper, the metric signature is the mostly minus while, in [64], the one is the mostly plus.}
\[ N_\omega \mathbf{P}_\psi = \mathbf{P}_\psi \left[ N_\omega + (\gamma \cdot \omega) \frac{1}{2N + d - 2} + \omega^2 \frac{2N + d - 1}{(2N + d)(2N + d - 2)} + \mathcal{O}(\omega^3) \right], \tag{B.16} \]

where the terms containing \( \mathcal{O}(\omega^3) \) will be vanished, at the level of the action, because of the triple gamma-trace condition on the spinor gauge field \( \bar{\Psi}(x, \partial_\omega) (\gamma \cdot \omega)^3 = 0 \).

## C The Segal action in (A)dS

In this Appendix, we review briefly the Segal action [56] in a more convenient form to compare with our results in Sec. 2.

The invariant action of the bosonic higher spin gauge fields on AdS\(_d\) spacetime, in the mostly negative signature\(^{16}\), is given by [56]

\[
S = \frac{1}{2} \int d^d x \, d^d p \, \sqrt{-g} \, \bar{h} \, \delta'(p^2 + 1) \left\{ -AB + 2BA + V_{11} - \frac{1}{2} \left( p^2 + 1 \right) \left( A^2 - V_{21} \right) \right\} \tilde{h},
\tag{C.1}
\]

where \( p^\mu \) is an auxiliary \( d \)-dimensional vector, \( g = \det(g_{\mu \nu}) \), \( \delta'(a) = \frac{d}{da} \delta(a) \) and the unconstrained gauge field \( \tilde{h} \) is considered as the generating function

\[
\tilde{h} = \tilde{h}(x, p) = \sum_{s=0}^{\infty} \frac{1}{s!} p^{\mu_1} \cdots p^{\mu_s} \, h_{\mu_1 \cdots \mu_s}(x), \tag{C.2}
\]

with \( h_{\mu_1 \cdots \mu_s}(x) \) corresponding to totally symmetric tensor fields of all integer spins (one row Young tableaux) in any dimension \( d \). The operators \( A, B, V_{11} \) and \( V_{21} \) in the action (C.1) are given by\(^{17}\)

\[
\begin{align*}
A &= \partial_\mu \cdot \nabla, \\
V_{11} &= 2 \Lambda \left( 2p \cdot \partial_\mu + d - 3 \right), \\
B &= p \cdot \nabla, \\
V_{21} &= -4 \Lambda \left( \partial_\mu \partial_\nu - \partial_\nu \partial_\mu \right),
\end{align*} \tag{C.3}
\]

where \( \partial_\mu := \frac{\partial}{\partial p^\mu} \), \( \Lambda \) is the constant scalar curvature defined in (1.2), and \( \nabla_\mu \) is the bosonic “covariant derivative”

\[
\nabla_\mu = \partial_\mu + \Gamma^\alpha_{\mu \nu}(x) p_\alpha \partial_\nu, \quad \partial_\mu := \frac{\partial}{\partial x^\mu}, \tag{C.5}
\]

with the Riemannian connection \( \Gamma_{\mu \nu}^\alpha(x) \) corresponding to the metric \( g_{\mu \nu}(x) \), so that

\[
\left[ \nabla_\mu, \nabla_\nu \right] f(x, p) = p_\alpha R_{\mu \nu \alpha \beta}(x) \partial_\beta f(x, p) = \Lambda \left( p_\mu \partial_\nu - p_\nu \partial_\mu \right) \partial_\mu f(x, p). \tag{C.6}
\]

Using the latter, and \( [\partial_\mu, \partial_\nu] = \delta_{\mu \nu} \), it is straightforward to demonstrate the following commutator

\[
[A, B] = \square_{\text{AdS}} + \Lambda \left( N^2 + N(d - 2) - p^2 \partial_\mu \partial_\mu \right), \tag{C.7}
\]

where \( N = p \cdot \partial_\mu \) and

\[
\square_{\text{AdS}} = \nabla_\mu \nabla^\mu + 2 \Gamma^\alpha_{\mu \nu} p_\alpha \partial_\mu \nabla_\nu. \tag{C.8}
\]

Then, we can rewrite the action (C.1) by substituting (C.3), (C.4) and (C.7) in (C.1) as

\[
S = \frac{1}{2} \int d^d x \, d^d p \, \sqrt{-g} \, \bar{h} \, \delta'(p^2 + 1) \left\{ -\Box_{\text{AdS}} + (p \cdot \nabla) (\partial_\mu \nabla^\mu) - \frac{1}{2} (p^2 + 1) (\partial_\mu \nabla^\mu)^2 \\
- \Lambda \left( N^2 + N(d - 6) - 2(d - 3) + p^2 (\partial_\mu \partial_\nu) + 2 (\partial_\mu \partial_\nu) \right) \right\} \tilde{h}. \tag{C.9}
\]

\(^{16}\)Note that in [56], the action is written in the mostly plus signature for the metric.

\(^{17}\)There is a typo in the operator \( V_{11} \) introduced in [56]. The corrected one is given here in (C.3).
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