Neutrosophic variational inequalities with applications in decision-making

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Abstract
In this paper, we introduced some new concepts of a neutrosophic set such as neutrosophic convex set, strongly neutrosophic convex set, neutrosophic convex function, strongly neutrosophic convex function, the minimum and maximum of a function \( f \) with respect to neutrosophic set, min, and max neutrosophic variational inequality, neutrosophic general convex set, neutrosophic general convex function, and min, max neutrosophic general variational inequality. We introduced some basic results on these new concepts. Moreover, we discussed the application of the neutrosophic set in optimization theory. We developed an algorithm using neutrosophic min and max variational inequalities and identified the maximum and minimum profit of the company.

Keywords Neutrosophic set · Neutrosophic convex set · Neutrosophic convex function · Max and min neutrosophic variational inequality

1 Introduction

Zadeh (1965) suggested the theory of fuzzy sets (FSs) to solve various forms of uncertainties. This theory has now been successfully implemented in different fields (Pedrycz 1990; Zadeh 1975). A single value \( \mu_A(x) \in [0, 1] \) is used by traditional FSs to describe the degree of membership of the fuzzy set \( A \), which is specified on a universal scale, and they are unable to manage those instances where it is difficult to describe \( \mu_A \) by specific value one. Atanassov introduced intuitionistic fuzzy sets (IFSs) (Atanassov 1999), which are an extension of Zadeh’s FSs, to address the lack of knowledge of non-membership degrees. Moreover, vague sets were described by Gau and Buehrer (1993). IFSs have been commonly used to solve multi-criteria decision-making (MCDM) problems (Xu 2011; Pei and Zheng 2012; Zeng and Su 2011), medical diagnosis (Shirin et al 2014), neural networks (Sotirov et al 2009), market prediction (Joshi and Kumar 2012), and color region extraction (Chaira 2011).

The membership degree, non-membership degree, and degree of hesitation are taken into account simultaneously by the IFSs. They are thus more flexible and realistic than traditional FSs when discussing fuzziness and ambiguity. Moreover, the membership degree, non-membership degree, and hesitation degree of an element in IFSs may not be a specific number in some real cases. Thus, they were extended to interval-valued intuitionistic fuzzy sets (Atanassov 1999). Moreover, Torra (2010) introduced hesitant fuzzy sets in order to deal with situations where people are hesitant when expressing their preferences regarding objects in the decision-making process. Arqub et al (2016) presented a new method for solving fuzzy differential equations based on the reproducing kernel theory under strongly generalized differentiability. They investigated the analytic and approximate solutions of second-order, two-point fuzzy boundary value problems based on the reproducing kernel theory under the assumption of strongly generalized differentiability (Arqub et al 2017). Arqub (2017) proposed the reproducing kernel Hilbert space method to obtain the exact and the numeri-
cal solutions of fuzzy Fredholm–Volterra integrodifferential equations. Moreover, Arqub and Al-Smadi (2020) proposed a new definition of fuzzy fractional derivative, so-called fuzzy conformable.

Although the theory of FSs has been developed and generalized, in various real-life problems, it does not deal with all uncertainty. For example, it is not possible to deal with certain kinds of uncertainty, such as indeterminate and inconsistent information. For example, when an expert is asked for his or her opinion about a certain statement, he or she may say that the possibility that the statement is true is 0.5, that it is false is 0.6 and the degree that he or she is not sure is 0.2 (Wang et al 2005b). This problem is beyond the reach of FSs and IFSs, so it needs some new theories.

Smarandache (1999) suggested neutrosophic (NS) sets and neutrosophic logic. An NS is a set where each element of the universe has the degrees of truth, indeterminacy, and falsity and it lies in $[0, 1]^3$, the non-standard unit interval. This is simply an extension to the standard interval $[0, 1]$ of the IFSs. Moreover, the uncertainty presented here, i.e., the indeterminacy element, is independent of the values of true and falsity, while the incorporated uncertainty depends on the degree of belonging and non-belonging to IFSs. However, in practical situations, NSs are difficult to apply without a particular description. In various areas of knowledge, this theory is being used. See recent examples as Crespo Berti (2020) in modeling real-life problems; Hatip (2020), and Saqlain et al (2020) who developed extensions of it. A new framework for dealing with imprecision is provided by Neutrosophic Theory. It is well known that statistical concepts and methods can be expanded using a neutrosophic point of view, see Smarandache (2013), Smarandache (2014), Schweizer (2020), Almeida et al (2020).

Single-valued neutrosophic sets (SVNSs), which are a variant of NSs, were proposed in Majumdar and Samanta (2014). Moreover, the information energy of SVNSs, their coefficient of correlation and correlation, and the process of decision-making used by them were proposed in Ye (2013). In addition, Ye (2014a) introduced the simplified neutrosophic sets (SNSs), which can be represented by three real numbers in the real unit interval $[0, 1]$, and proposed a method of MCDM using SNS aggregation operators. Moreover, Majumdar and Samanta (2014) introduced a measure of SVNS entropy. The definition of interval neutrosophic sets (INSs) was proposed by Wang et al (2005a). In addition, Ye (2014b) proposed similarity measures between SVNSs and INSs based on the relationship between measures of similarity and distances.

There are different concepts developed from NS, neutrosophic probability, and neutrosophic statistics. Each of these is interactive. The sets derived from NS are intuitionistic set, paraconsistent set, paradoxist set, trivialist set, nihilist set, dialetheist set, and fallibilist set. Tautological probability and statistics, intuitionistic probability and statistics, dialetheist probability and statistics, fallibilist probability and statistics, paraconsistent probability and statistics, trivialist probability and statistics, and nihilist probability and statistics are derived from neutrosophic probability and statistics. Nabeeh et al (2019) proposed an approach that would facilitate a personal selection process by incorporating the process of neutrosophic analytical hierarchy to demonstrate the ideal solution between various options similar to an ideal solution for order preference technique (TOPSIS). Abdel-Basset et al (2019b) developed a new kind of technique for neutrosophy called neutrosophic numbers of type 2. They suggested a novel T2NN-TOPSIS process, combining type 2 neutrosophic number and TOPSIS, which is very useful in group decision-making. A multi-criteria group decision-making method of the analytical network process method and the VIKOR method was investigated in a neutrosophical setting dealing with high-order imprecision and incomplete information (Abdel-Baset et al 2019b). M. A. Baset introduced a new technique for estimating the GDM selection process for smart medical devices in a vague decision-making environment. Neutrosophic with the TOPSIS strategy is used in decision-making processes to deal with incomplete information, vagueness, and ambiguity, taking into account the decision-making criteria in the information obtained by decision-makers in Abdel-Basset et al (2019a). They proposed a robust ranking method with NS to manage the performance of the supply chain management (GSCM) and methods that have been commonly used to promote environmental sustainability and achieve competitive advantages. The principle of the N.S. has been used to handle imprecision, linguistic imprecision, ambiguous details, and incomplete information (Abdel-Baset et al 2019a). Moreover, Abdel-Basset et al (2018) et al. used NS for evaluation techniques and decision-making to identify and analyze factors influencing the selection of suppliers for the supply chain management. Bera and Mahapatra (2018a) et al. characterized a neutrosophic norm for a soft linear space known as a neutrosophic soft linear space. They also explore the notion of neutrosophic soft (Ns) prime ideal over a ring. They introduced the idea of N’s completely prime ideals, N’s fully semi-primary ideals, and N’s prime K-ideals Bera and Mahapatra (2018b). Moreover, Bera and Mahapatra (2018c) established the concept of connectedness and compactness in N’s topological space along with its various characteristics. Shah and Hussain (2016) et al. studied the P-OR, P-intersection and P-union, and P-AND of neutrosophic cubic sets and their associated properties. Shah Shah and Hussain (2016) et al. discussed neutrosophic soft graphs. They proposed a connection between the neutrosophic soft sets and the graphs.

In decision-making problems, the use of optimization approaches is ubiquitous. The purpose of this article is
twofold. The first half aims to present the theoretical foundations of neutrosophic in optimization such as neutrosophic variational inequalities and neutrosophic convex function, and the second half aims to present these theoretical foundations and key techniques in convex optimization, decision-making, and the principle of the neutrosophic variational inequalities in a coherent manner. The purpose of these innovative concepts is to provide a new approach with useful mathematical tools to address the fundamental problem of decision-making (e.g., maximization and minimization of the problem). The generality of the neutrosophic variational inequalities system is given special importance, illustrating how many interesting optimization decision-making problems can be formulated as a problem of neutrosophic variational inequalities. These applied contexts provide solid evidence of the wide applications of the neutrosophic variational inequality approach to model and research decision-making problems. This article will stimulate the interest in neutrosophic variational inequality and its application in optimization.

In this paper, we introduce some new concepts of a neutrosophic set such as neutrosophic convex set, strongly neutrosophic convex set, neutrosophic convex function, strongly neutrosophic convex function, the minimum and maximum of a function $f$ with respect to neutrosophic set, min, and max neutrosophic variational inequality, neutrosophic general convex set, neutrosophic general convex function, and min, max neutrosophic general variational inequality. We study some basic results on these new concepts. Moreover, we discuss the application of the neutrosophic set in optimization theory. We propose a method using neutrosophic min and max variational inequality and identify the maximum and minimum profit of the company.

## 2 Preliminaries

We will define here some new concepts on the neutrosophic set and also discuss particular examples of these new concepts.

**Definition 1** Smarandache (1999) Let $X$ be a space of points, and let $x \in X$. A neutrosophic set $N$ in $X$ is characterized by a truth membership function $T_N$, an indeterminacy membership function $I_N$, and a falsity membership function $F_N$. $T_N(x)$, $I_N(x)$, and $F_N(x)$ are real standard or non-standard subsets of $[0^-, 1^+]$, and $F_N$, $T_N(x)$, $I_N(x)$, $F_N(x) : X \rightarrow [0^-, 1^+]$. The neutrosophic set $N$ can be represented as:

$$N = \{(x, T_N(x), I_N(x), F_N(x)) : x \in X\}.$$

There is no restriction on the sum of $T_N(x)$, $I_N(x)$, and $F_N(x)$, so

$$0^- \leq T_N(x), I_N(x), F_N(x) \leq 3^+.$$

**Definition 2** Ali and Smarandache (2017) Let $N = (T_N(x)$, $I_N(x)$, $F_N(x))$ be a neutrosophic set. Then, the complement of a neutrosophic set $N$ is denoted by $N^c$ and is defined by

$$T_{N^c}(x) = F_N(x), I_{N^c}(x) = 1 - I_N(x), F_{N^c}(x) = T_N(x) ; \forall x \in X.$$

**Definition 3** Ali and Smarandache (2017) Let $N_1$ and $N_2$ be two neutrosophic sets in a universe of discourse $X$. Then, the union of $N_1$ and $N_2$ is denoted by $N_1 \cup N_2$, which is defined by

$$N_1 \cup N_2 = \{(x, T_{N_1}(x) \lor T_{N_2}(x), I_{N_1}(x) \land I_{N_2}(x), F_{N_1}(x) \lor F_{N_2}(x)) : x \in X\},$$

for all $x \in X$, and $\lor$, $\land$ represent the max and min operators, respectively.

**Definition 4** Ali and Smarandache (2017) Let $N_1$ and $N_2$ be two neutrosophic sets in a universe of discourse $X$. Then, the intersection of $N_1$ and $N_2$ is denoted by $N_1 \cap N_2$, which is defined by

$$N_1 \cap N_2 = \{(x, T_{N_1}(x) \land T_{N_2}(x), I_{N_1}(x) \lor I_{N_2}(x), F_{N_1}(x) \lor F_{N_2}(x)) : x \in X\},$$

for all $x \in X$, and $\lor$, $\land$ represent the max and min operators, respectively.

**Definition 5** Let $N$ be a neutrosophic set on $X$, and $\mu_N = (T_N(x)$, $I_N(x)$, $F_N(x))$ denotes their membership function. Then, $N$ is said to be convex if

$$N((1 - t)x + ty) \geq \min(N(x), N(y)).$$

Or

$$\mu_N((1 - t)x + ty) \geq \min(\mu_N(x), \mu_N(y)),$$

for all $x, y \in \mathbb{R}^n$, and $t \in [0, 1]$.

**Note:** Every neutrosophic convex set is a neutrosophic set, but the converse is not true.

**Example 1** Let $t = 0.5$, and $\mu_N(0.5x + 0.5y) = (0.1, 0.5, 0.9)$, $\mu_N(x) = (0.4, 0.5, 0.6)$, $\mu_N(y) = (0.6, 0.5, 0.3)$ be the membership functions of $N$ under $0.5x + 0.5y$, $x$ and $y$, respectively. Now, we have to show that $N$ is not a neutrosophic convex set, as

$$\mu_N((1 - t)x + ty) = (0.1, 0.5, 0.9), \quad \text{for } t = 0.5, \quad (1)$$
and
\[ \min(\mu_N(x), \mu_N(y)) = \langle 0.4, 0.5, 0.6 \rangle. \] (2)

From (1) and (2), we have
\[ \mu_N((1-t)x + ty) \geq \min(\mu_N(x), \mu_N(y)). \] (3)

Hence, \( N \) is not a neutrosophic convex set.

**Definition 6** Let \( N \) be a neutrosophic set on \( X \), and \( \mu_N = (T_N(x), I_N(x), F_N(x)) \) denotes their membership function. Then, \( N \) is said to be strongly convex if
\[ N((1-t)x + ty) > \min(N(x), N(y)). \]

Or
\[ \mu_N((1-t)x + ty) > \min(\mu_N(x), \mu_N(y)), \]

for all \( x \neq y, x, y \in \mathbb{R}^n \), and \( t \in [0, 1] \).

**Note:** The strongly neutrosophic convex set is the neutrosophic convex set, but the converse is not true.

**Definition 7** Let \( N \) be a neutrosophic convex set on \( X = \mathbb{R}^n \) and is characterized by a membership function \( \mu_N = (T_N(x), I_N(x), F_N(x)) \) for all \( x \in X \). Let \( f : \tau \rightarrow \tau \) be a function, where \( \tau = \{N(x) = (T_N(x), I_N(x), F_N(x)) : x \in X\} \) denotes the collection of neutrosophic convex sets. Then, the function \( f \) on neutrosophic convex sets \( N \) is said to be a neutrosophic convex function if the following condition holds:
\[ f(N((1-t)x + ty)) \geq \min(f(N(x)), f(N(y))). \] (4)

for all \( x, y \in \mathbb{R}^n \), and all \( t \in [0, 1] \).

The inequality (4) can be written as:
\[ \min(f(N(x)), f(N(y)) \leq f(N((1-t)x + ty)). \]

Note that the neutrosophic convex function is more significant in optimization theory. They are used in models for optimization problems (maximization and minimization problems).

**Example 2** The identity function on the neutrosophic convex set \( N \) is a neutrosophic convex function.

**Definition 8** A neutrosophic set \( N_i(x) \in \tau \), is called a minimum of \( f \), if \( f(N_i(x)) \leq f(N_j(x)) \) for all \( N_i(x) \in \tau \).

**Definition 9** Let \( N \) be a neutrosophic convex set on \( X = \mathbb{R}^n \) and is characterized by a membership function \( \mu_N = (T_N(x), I_N(x), F_N(x)) \) for all \( x \in X \). Let \( f : \tau \rightarrow \tau \) be a neutrosophic convex function, where \( \tau = \{N_i(x) = (T_N(x), I_N(x), F_N(x)) : x \in X\} \) denotes the collection of neutrosophic convex sets. Then, the inequality
\[ \{f(N_i(x)), f(N_i(x)) \cap f(N_j(x))\} \leq f(N_i(x)) \circ f(N_j(x)); i \neq j, \]

\[ \forall N_i(x), N_j(y) \in \tau \]

is called neutrosophic min variational inequality.

**Example 3** Let \( N_1 = \left[ \left(\frac{0.4 \cdot 0.7 \cdot 1}{x} + \frac{0.5 \cdot 0.8 \cdot 0.2}{y} + \frac{0.2 \cdot 0.5 \cdot 1}{z} \right) \right] \) and \( N_2 = \left[ \left(\frac{0.6 \cdot 0.9 \cdot 1}{x} + \frac{0.3 \cdot 0.8 \cdot 0.6}{y} + \frac{0.3 \cdot 0.7 \cdot 0.9}{z} \right) \right] \) be two neutrosophic sets and \( f \) be a function defined by
\[ \frac{0.4, 0.7, 1}{x} \rightarrow \frac{0.2, 0.5, 1}{y}, \frac{0.5, 0.8, 0.2}{z} \]
\[ \frac{0.3, 0.6, 0.2}{x} \rightarrow \frac{0.2, 0.5, 1}{y}, \frac{0.3, 0.6, 0.2}{z} \]
\[ \frac{0.6, 0.9, 1}{x} \rightarrow \frac{0.4, 0.7, 1}{y}, \frac{0.7, 0.9, 0}{z} \]
\[ \frac{0.5, 0.7, 0}{x} \rightarrow \frac{0.3, 0.7, 0}{y}, \frac{0.1, 0.5, 0}{z} \].

We have
\[ f(N_1) = \left[ \left(\frac{0.2 \cdot 0.5 \cdot 1}{x} + \frac{0.3 \cdot 0.6 \cdot 0.2}{y} + \frac{0.3 \cdot 0.7 \cdot 0.9}{z} \right) \right] \]
\[ f(N_2) = \left[ \left(\frac{0.4 \cdot 0.7 \cdot 1}{x} + \frac{0.5 \cdot 0.7 \cdot 0.9}{y} + \frac{0.1 \cdot 0.5 \cdot 0.9}{z} \right) \right] \]
\[ \{f(N_1), f(N_1) \cap f(N_2)\} \]
\[ = \left[ \left(\frac{0.04, 0.35, 1}{x} + \frac{0.09, 0.42, 0}{y} + \frac{0.15, 0.15, 1}{z} \right) \right]. \] (6)

Now,
\[ f(N_1) \circ f(N_2) = \left[ \left(\frac{0.08, 0.35, 1}{x} + \frac{0.15, 0.42, 0}{y} + \frac{0.01, 0.15, 0.9}{z} \right) \right]. \] (7)

From (6) and (7), we have
\[ \{f(N_1), f(N_1) \cap f(N_2)\} \leq f(N_1) \circ f(N_2). \]

**Definition 10** A neutrosophic set \( N_i(x) \in \tau = \{N_i(x) = (T_N(x), I_N(x), F_N(x)) : x \in X\} \) is called a maximum of \( f \), if \( f(N_i(x)) \geq f(N_j(x)) \) for all \( N_j(x) \in \tau \).
Definition 11 Let $N$ be a neutrosophic convex set on $X = \mathbb{R}^n$, and $\mu_N = (T_N(x), I_N(x), F_N(x))$ denotes their membership function. Let $f : \tau \rightarrow \tau$ be a function, where $\tau = \{N(x) = \{T_N(x), I_N(x), F_N(x) : x \in X\}$ denotes the collection of neutrosophic convex sets. Then, the inequality

$$\{f(N_i(x)), f(N_j(x)) \cup f(N_j(x))\}$$

$$\geq f(N_i(x)) \circ f(N_j(x)); i \neq j,$$

$$\forall N(x), N(y) \in \tau$$

is called neutrosophic max variational inequality.

Example 4 Let $N_1 = \left[\frac{[0.4,0.7,1]}{x} + \frac{[0.5,0.8,0.2]}{y} + \frac{[0.2,0.5,1]}{z}\right]$ and $N_2 = \left[\frac{[0.6,0.9,1]}{x} + \frac{[0.3,0.8,0.6]}{y} + \frac{[0.3,0.7,0.9]}{z}\right]$ be two neutrosophic sets and $f$ be a function defined by

$$f\left[\frac{[0.4,0.7,1]}{x}\right] = \frac{[0.2,0.5,1]}{y}$$

$$f\left[\frac{[0.3,0.6,0.2]}{y}\right] = \frac{[0.2,0.5,1]}{x}$$

$$f\left[\frac{[0.6,0.9,1]}{y}\right] = \frac{[0.4,0.7,1]}{x}$$

Then,

$$\{f(N_1), f(N_1) \cup f(N_2)\}$$

$$= \left\{\frac{[0.7,0.8,0.2]}{x} + \frac{[0.6,0.7,0.4]}{y} + \frac{[1.0,6,0.5]}{z}\right\},$$

$$\cup \left\{\frac{[0.4,0.7,1]}{y} + \frac{[0.5,0.6,0.9]}{x} + \frac{[1.0,5,0.6]}{z}\right\}$$

For all $g(x), g(y) \in \mathbb{R}^n$, and $t \in [0, 1]$. From (8) and (9), we have

$$(f(N_1), f(N_1) \cup f(N_2)) \geq f(N_1) \circ f(N_2).$$

2.1 Generalized convex sets and convex functions

In the problems, if the domain set may not be a convex set, in those situations, the non-convex set can be made a convex set with respect to an arbitrary function. These sets are called general convex sets, and the function defined on the general convex set is called general convex function.

Definition 12 Let $N$ be a neutrosophic set on $X$, and $\mu_N = (T_N(g(x)), I_N(g(x)), F_N(g(x)))$ denotes their membership function. Then, $N$ is said to be general convex if

$$N((1-t)g(x) + tg(y)) \geq \min(N(g(x)), N(g(y))).$$

Or

$$\mu_N((1-t)g(x) + tg(y)) \geq \min(\mu_N(g(x)), \mu_N(g(y))).$$

for all $g(x), g(y) \in \mathbb{R}^n$, and $t \in [0, 1]$.

Definition 13 Let $N$ be a neutrosophic general convex set on $X = \mathbb{R}^n$, and $\mu_N = (T_N(g(x)), I_N(g(x)), F_N(g(x)))$ denotes their membership function. Let $f : \tau \rightarrow \tau$ be a function, where $\tau = \{N_i(g(x)) = \{T_N(x), I_N(x), F_N(x) : x \in X\}$ denotes the collection of neutrosophic general convex sets. Then, the function $f$ on neutrosophic general convex sets $N$ is said to be a neutrosophic general convex function if

$$f(N((1-t)g(x) + tg(y))) \geq \min(f(N(g(x)), f(N(g(y))))$$

for all $g(x), g(y) \in \mathbb{R}^n$, and $t \in [0, 1]$.

Case 1 $f g = I$ in inequality (10), then the neutrosophic general convex function is the neutrosophic convex function.

Definition 14 Let $N$ be a neutrosophic general convex set on $X = \mathbb{R}^n$, and $\mu_N = (T_N(g(x)), I_N(g(x)), F_N(g(x)))$ denotes their membership function. Let $f : \tau \rightarrow \tau$ be a neutrosophic general convex function, where

$$\tau = \{N_i(g(x))$$

$$= \{T_N(g(x)), I_N(g(x), F_N(g(x))) : g(x) \in X\},$$

denotes the collection of neutrosophic general convex sets. Then, the inequality

$$\{f(N_i(g(x))), f(N_i(g(x))) \cup f(N_j(g(x)))\}$$

$$\leq f(N_i(g(x))) \circ f(N_j(g(x))); i \neq j,$$

$$\forall N_i(g(x), N_j(g(x)) \in \tau$$
is called neutrosophic min general variational inequality.

**Definition 15** Let \( N \) be a neutrosophic general convex set on \( X \), and \( \mu_N = (T_N(g(x)), I_N(g(x)), F_N(g(x))) \), denotes their membership function. Let \( f : \tau \to \tau \) be a neutrosophic general convex function, where

\[
\tau = \{ N_i(g(x)) = [T_N(g(x)), I_N(g(x)), F_N(g(x))]: g(x) \in X \},
\]

denotes the collection of neutrosophic general convex sets. Then, the inequality

\[
\{ f(N_i(g(x))), f(N_i(g(x))) \cup f(N_j(g(x))) \} \geq f(N_i(g(x))) \cup f(N_j(g(x)));
\]

\[
\forall N_i(g(x)), N_j(g(y)) \in \tau
\]

is called neutrosophic max general variational inequality.

### 3 Main results

**Proposition 2** Let \( \tau = \tau = \{ N_i(x) = [T_N(g(x)), I_N(g(x)), F_N(g(x))]: g(x) \in X \} \) be a collection of neutrosophic convex sets and \( N_i(x) \in \tau \) be a minimum of the neutrosophic convex function \( f \) on \( \tau \). Then, \( N_i(x) \) satisfies the neutrosophic min variational inequality.

**Proof** Let \( N_i(x) \in \tau \) be the minimum of \( f \). Then,

\[
f(N_i(x)) \leq f(N_j(x)); \forall N_j(x) \in \tau.
\]  

(11)

Also, from inequality (11), we have

\[
(f(N_i(x)) \cap f(N_j(x))) \leq f(N_j(x)); \forall N_j(x) \in \tau.
\]  

(12)

The inequality (12) can be written as:

\[
f(N_i(x)) \cap f(N_j(x)) \leq f(N_i(x)) \cap f(N_j(x)); \forall N_i(x), N_j(y) \in \tau.
\]

Thus, \( N_i(x) \in \tau \) satisfies the neutrosophic min variational inequality.

**Proposition 3** Let \( \tau = \tau = \{ N_i(x) = [T_N(g(x)), I_N(g(x)), F_N(g(x))]: g(x) \in X \} \) be a collection of neutrosophic convex sets and \( N_i(x) \in \tau \) be a maximum of the neutrosophic convex function \( f \) on \( \tau \). Then, \( N_i(x) \) satisfies the neutrosophic max variational inequality.

**Proof** Let \( N_i(x) \in \tau \) be the maximum of \( f \). Then,

\[
f(N_i(x)) \geq f(N_j(x)); \forall N_j(x) \in \tau.
\]  

(13)

Also, from inequality (13), we have

\[
f(N_i(x)) \cup f(N_j(x)) \geq f(N_j(x)); \forall N_j(x) \in \tau.
\]  

(14)

The inequality (14) can be written as:

\[
f(N_i(x)) \circ (f(N_i(x)) \cup f(N_j(x)))
\]

\[
\geq f(N_i(x)) \circ f(N_j(y))
\]

\[
\{ f(N_i(x)), f(N_j(x)) \} \cap f(N_j(x))
\]

\[
\geq f(N_i(x)) \circ f(N_j(x));
\]

\[
\forall N_i(x), N_j(x) \in \tau.
\]

Thus, \( N_i(x) \in \tau \) satisfies the neutrosophic max variational inequality.

**Proposition 4** Let \( \tau = \{ N_i(g(x)) = [T_N(g(x)), I_N(g(x)), F_N(g(x))]: g(x) \in X \} \) be a collection of neutrosophic convex sets and \( N_i(g(x)) \in \tau \) be a minimum of the neutrosophic convex function \( f \) on \( \tau \). Then, \( N_i(g(x)) \) satisfies the neutrosophic general variational inequality.

**Proof** Let \( N_i(g(x)) \in \tau \) be the minimum of \( f \). Then,

\[
f(N_i(g(x))) \leq f(N_j(g(x)));
\]

\[
\forall N_j(g(x)) \in \tau.
\]  

(15)

Also, from inequality (15), we have

\[
f(N_i(g(x))) \cap f(N_j(g(x)))
\]

\[
\leq f(N_j(g(x))); \forall N_j(g(x)) \in \tau.
\]  

(16)

The inequality (16) can be written as:

\[
f(N_i(g(x))) \circ (f(N_i(g(x))) \cap f(N_j(g(x)))
\]

\[
\leq f(N_i(g(x))) \circ f(N_j(g(x)));
\]

\[
\forall N_i(g(x)), N_j(g(x)) \in \tau.
\]

Thus, \( N_i(g(x)) \in \tau \) satisfies the neutrosophic general min variational inequality.

**Proposition 5** Let \( \tau = \{ N_i(g(x)) = [T_N(g(x)), I_N(g(x)), F_N(g(x))]: g(x) \in X \} \) be a collection of neutrosophic convex sets and \( N_i(g(x)) \in \tau \) be a maximum of the neutrosophic convex function \( f \) on \( \tau \). Then, \( N_i(g(x)) \) satisfies the neutrosophic max general variational inequality.

**Proof** Let \( N_i(g(x)) \in \tau \) be the maximum of \( f \). Then,

\[
f(N_i(g(x))) \geq f(N_j(g(x)));
\]

\[
\forall N_j(g(x)) \in \tau.
\]  

(17)

Also, from inequality (17), we have
Proposition 6 For any two neutrosophic convex sets $N_1$ and $N_2$, $N_1 \cup N_2$ is also a neutrosophic convex set.

**Proof** Since $N_1$ is a neutrosophic convex set, we have

$$\mu_{N_1}((1-t)x + ty) \leq \min(\mu_{N_1}(x), \mu_{N_1}(y)).$$

Also, $N_2$ is a neutrosophic convex set, then

$$\mu_{N_2}((1-t)x + ty) \leq \min(\mu_{N_2}(x), \mu_{N_2}(y)).$$

Now,

$$\mu_{N_1 \cup N_2}((1-t)x + ty) \leq \min(\mu_{N_1 \cup N_2}(x), \mu_{N_1 \cup N_2}(y)).$$

Thus, $N_1 \cup N_2$ is a neutrosophic convex set. □

Proposition 7 If $N_i(x) \in \tau = \{N_i(x) = \{T_{N_i}(x), I_{N_i}(x), F_{N_i}(x)\} : x \in X\}$ be a minimum of the neutrosophic convex function $f$. Then,

$$\{f(N_i(x)), f(N_i(x)) \cap f(N_j(x))\} \leq \{f(N_i(x)), f(N_i(x)) \cup f(N_j(x))\}; \forall N_j(x) \in \tau.$$

**Proof** Assume that $N_i(x) \in \tau$ be a minimum of $f$. Then,

$$\{f(N_i(x)), f(N_i(x)) \cap f(N_j(x))\} \leq \{f(N_i(x)), f(N_i(x)) \cup f(N_j(x))\}; \forall N_j(x) \in \tau.$$

As $N_i(x) \in \tau$, we have

$$f(N_i(x)) \circ f(N_j(x)) \leq \{f(N_i(x)), f(N_i(x)) \cup f(N_j(x))\}. \quad (19)$$

From (19) and (20), we have

$$\{f(N_i(x)), f(N_i(x)) \cap f(N_j(x))\} \leq \{f(N_i(x)), f(N_i(x)) \cup f(N_j(x))\}; \forall N_j(x) \in \tau. \quad \Box$$

Proposition 8 If $N_i(x) \in \tau = \{N_i(x) = \{T_{N_i}(x), I_{N_i}(x), F_{N_i}(x)\} : x \in X\}$ be a maximum of the neutrosophic convex function $f$. Then,

$$\{f(N_i(x)), f(N_i(x)) \cup f(N_j(x))\} \leq \{f(N_i(x)), f(N_j(x)) \cap f(N_i(x))\}; \forall N_j(x) \in \tau.$$

**Proof** Assume that $N_i(x) \in \tau$ be a maximum of $f$. Then,

$$\{f(N_i(x)), f(N_i(x)) \cup f(N_j(x))\} \leq f(N_i(x)) \circ f(N_j(x)). \quad (21)$$

As $N_i(x) \in \tau$, we have

$$\{f(N_i(x)), f(N_i(x)) \cap f(N_j(x))\} \leq f(N_i(x)) \circ f(N_j(x)). \quad (22)$$

From (21) and (22), we have

$$\{f(N_i(x)), f(N_i(x)) \cap f(N_j(x))\} \leq \{f(N_i(x)), f(N_i(x)) \cup f(N_j(x))\}; \forall N_j(x) \in \tau. \quad \Box$$

Theorem 1 Let $f : \tau \to \tau'$ be a mapping and "\sim" be a relation defined in the following way: "f($N_1$), f($N_2$) \in \tau', f($N_1$) \sim f($N_2$) if the min variational inequality holds. Show that the relation "\sim" is an order relation.

**Proof** To prove the relation "\sim" is an order relation, we have to show the following.

(i). The relation "\sim" is reflexive; that is, $f(N_1) \sim f(N_1)$.

(ii). The relation "\sim" is antisymmetric; that is, if $f(N_1) \sim f(N_2)$ and $f(N_2) \sim f(N_1)$, then $f(N_1) = f(N_2)$.

(iii). The relation "\sim" is transitive; that is, if $f(N_1) \sim f(N_2)$ and $f(N_2) \sim f(N_3)$, then $f(N_1) \sim f(N_3)$.

(i). **Reflexive**

The relation "\sim" is reflexive, since, for any $f(N_i) \in \tau'$, we have

$$f(N_i(x)) \circ f(N_i(x)) \leq f(N_i(x)) \circ f(N_i(x)).$$

Hence $f(N_i) \sim f(N_i)$.

Thus, the relation "\sim" is reflexive.

(ii). **Antisymmetric**

Assume that $f(N_1) \sim f(N_2)$ and $f(N_2) \sim f(N_1)$, then

$$f(N_1(x)) \circ f(N_2(x)) \leq f(N_1(x)) \circ f(N_2(x)). \quad (23)$$

The inequality (23) implies that

$$f(N_1(x)) \leq f(N_2(x)). \quad (24)$$
Also,
\[
\langle f(N_2(x)), f(N_2(x)) \cap f(N_1(x)) \rangle \leq f(N_2(x)) \circ f(N_1(x)).
\] (25)

The inequality (25) implies that
\[
f(N_2(x)) \leq f(N_1(x)).
\] (26)

From (24) and (26), we have \(f(N_1) = f(N_2)\). Thus, the relation “\(\sim\)” is antisymmetric. 

(iii). Transitive

Assume that \(f(N_1) \sim f(N_2)\) and \(f(N_2) \sim f(N_3)\), then
\[
\langle f(N_1(x)), f(N_1(x)) \cap f(N_2(x)) \rangle \leq f(N_1(x)) \circ f(N_2(x)).
\] (27)

The inequality (27) implies that
\[
f(N_1(x)) \leq f(N_2(x)).
\] (28)

Also,
\[
\langle f(N_2(x)), f(N_2(x)) \cap f(N_3(x)) \rangle \leq f(N_2(x)) \circ f(N_3(x)).
\] (29)

The inequality (29) implies that
\[
f(N_2(x)) \leq f(N_3(x)).
\] (30)

From (28) and (30), we have \(f(N_1) \leq f(N_3)\). Hence, \(f(N_1) \sim f(N_3)\).

Thus, the relation “\(\sim\)” is transitive and consequently the relation “\(\sim\)” is an order relation. \(\square\)

**Theorem 2** Let \(f : \tau \rightarrow \tau'\) be a mapping and “\(\sim\)” be a relation defined in the following way “\(f(N_1), f(N_2) \in \tau'\), \(f(N_1) \sim f(N_2)\) if the max variational inequality holds. Show that the relation “\(\sim\)” is an order relation.

**Proof** To prove the relation “\(\sim\)” is an order relation, we have to show the following.

(i). The relation “\(\sim\)” is reflexive; that is, \(f(N_1) \sim f(N_1)\).

(ii). The relation “\(\sim\)” is antisymmetric; that is, if \(f(N_1) \sim f(N_2)\) and \(f(N_2) \sim f(N_1)\), then \(f(N_1) = f(N_2)\).

(iii). The relation “\(\sim\)” is transitive; that is, if \(f(N_1) \sim f(N_2)\) and \(f(N_2) \sim f(N_3)\), then \(f(N_1) \sim f(N_3)\).

(i). Reflexive

The relation “\(\sim\)” is reflexive, since, for any \(f(N_1) \in \tau'\), we have
\[
\langle f(N_1(x)), f(N_1(x)) \cup f(N_1(x)) \rangle \geq f(N_1(x)) \circ f(N_1(x)).
\]

Hence, \(f(N_1) \sim f(N_1)\).

Thus, the relation “\(\sim\)” is reflexive.

(ii). Antisymmetric

Assume that \(f(N_1) \sim f(N_2)\) and \(f(N_2) \sim f(N_1)\), then
\[
\langle f(N_1(x)), f(N_1(x)) \cup f(N_2(x)) \rangle \geq f(N_1(x)) \circ f(N_2(x)).
\] (31)

The inequality (31) implies that
\[
f(N_1(x)) \geq f(N_2(x)).
\] (32)

Also,
\[
\langle f(N_2(x)), f(N_2(x)) \cup f(N_1(x)) \rangle \geq f(N_2(x)) \circ f(N_1(x)).
\] (33)

The inequality (33) implies that
\[
f(N_2(x)) \geq f(N_1(x)).
\] (34)

From (32) and (34), we have \(f(N_1) = f(N_2)\). Thus, the relation “\(\sim\)” is antisymmetric.

(iii). Transitive

Assume that \(f(N_1) \sim f(N_2)\) and \(f(N_2) \sim f(N_3)\), then
\[
\langle f(N_1(x)), f(N_1(x)) \cup f(N_2(x)) \rangle \geq f(N_1(x)) \circ f(N_2(x)).
\] (35)

The inequality (35) implies that
\[
f(N_1(x)) \geq f(N_2(x)).
\] (36)

Also,
\[
\langle f(N_2(x)), f(N_2(x)) \cup f(N_3(x)) \rangle \geq f(N_2(x)) \circ f(N_3(x)).
\] (37)

The inequality (37) implies that
\[
f(N_2(x)) \geq f(N_3(x)).
\] (38)

From (36) and (38), we have \(f(N_1) \geq f(N_3)\). Hence, \(f(N_1) \sim f(N_3)\).

Thus, the relation “\(\sim\)” is transitive and consequently the relation “\(\sim\)” is an order relation. \(\square\)

### 4 Applications

We are going to discuss a real-life application of newly defined neurotrophic max and neurotrophic min variational Inequalities. In fact, we will discuss that how our novel concepts have real-life applications. Specifically, the neurotrophic max and neurotrophic min variational inequality explains how to get the maximum and minimum profit of the company.
We will discuss the algorithm by using the neutrosophic max variational inequality and neutrosophic min variational inequality. In this algorithm, we will discuss how the trucking company gets maximum profit and minimum profit.

Algorithm
Suppose ABC Trucking is a company that operates 20 trucks for transport and logistics. When they are full and on the track, trucks make the most money for the company. ABC Trucking has the following vector entities or groups:

(i). Truck Company (truck type, age, engine size).
(ii). Income \((\text{Euro}_1, \text{Euro}_2, \text{Euro}_3)\).

A neutrosophic set \(N_1, N_2, \text{and} N_3\) in \(X = \text{truck type}, Y = \text{age}, Z = \text{engine size}\) is characterized by a truth membership function \(T_{N_1}, T_{N_2}, T_{N_3}\), an indeterminacy membership function \(I_{N_1}, I_{N_2}, I_{N_3}\), and a falsity membership function \(F_{N_1}, F_{N_2}, F_{N_3}\); \(T_{N_1}, T_{N_2}, T_{N_3}, I_{N_1}, I_{N_2}, I_{N_3}, \text{and} F_{N_1}, F_{N_2}, F_{N_3}\) are real standard or non-standard subsets of \([0, 1]^{+}\). A neutrosophic set \(N'_1, N'_2, \text{and} N'_3\) in \(X' = \text{Euro}, Y' = \text{Dollar}, Z = \text{Riyal}\) is characterized by a truth membership function \(T'_{N'_1}, T'_{N'_2}, T'_{N'_3}\), an indeterminacy membership function \(I'_{N'_1}, I'_2, I'_{N'_3}\), and a falsity membership function \(F'_{N'_1}, F'_2, F'_{N'_3}\); \(T'_{N'_1}, T'_{N'_2}, T'_{N'_3}, I'_{N'_1}, I'_{N'_2}, I'_{N'_3}, \text{and} F'_{N'_1}, F'_2, F'_{N'_3}\) are real standard or non-standard subsets of \([0, 1]^{+}\).

The trucking company needs to optimize the use of its trucks and workers for the highest possible profits. To find the probability of the maximum or minimum profit of the trucking company, we define a relation \(f : \tau \to \tau'\) by

\[
\begin{align*}
(\text{truck type} = a, T_{N_1}(a), I_{N_1}(a), F_{N_1}(a)) & \rightarrow ((\text{Euro}_1), T_{N'_1}(x), I_{N'_1}(x), F_{N'_1}(x)), \\
(\text{age} = b, T_{N_2}(b), I_{N_2}(b), F_{N_2}(b)) & \rightarrow ((\text{Euro}_2), T_{N'_2}(x), I_{N'_2}(x), F_{N'_2}(x)), \\
(\text{engine size} = c, T_{N_3}(c), I_{N_3}(c), F_{N_3}(c)) & \rightarrow ((\text{Euro}_3), T_{N'_3}(x), I_{N'_3}(x), F_{N'_3}(x)).
\end{align*}
\]

Now, if the relation \(f\) satisfies the max variational inequality, that is,

\[
\{f(N_i(x)), f(N_i(x)) \cup f(N_j(y))\}
\geq f(N_i(x)) \circ f(N_j(y)); \forall N_i(x) \neq N_j(y) \in \tau.
\] (39)

Taking the left-hand side of the inequality (39), we have

\[
\{f(N_i(x)), f(N_i(x)) \cup f(N_j(y))\} = N' = \tau, T_{N'_i}(z), I_{N'_i}(z), F_{N'_i}(z).
\] (40)

which gives the maximum profit with a neutrosophic set \(N'\) characterized by a truth membership function \(T_{N'_i}\), an indeterminacy membership function \(I_{N'_i}\), and a falsity membership function \(F_{N'_i}\).

If the relation \(f\) satisfies the min variational inequality, that is,

\[
\{f(N_i(x)), f(N_i(x)) \cap f(N_j(y))\}
\leq f(N_i(x)) \circ f(N_j(y)); \forall N_i(x) \neq N_j(y) \in \tau.
\] (41)

Taking the left-hand side of the inequality (41), we have

\[
\{f(N_i(x)), f(N_i(x)) \cap f(N_j(y))\} = N'' = (z, T_{N''}(z), I_{N''}(z), F_{N''}(z)),
\] (42)

which gives the minimum profit characterized by a neutrosophic set \(N''\) with a truth membership function \(T_{N''}\), an indeterminacy membership function \(I_{N''}\), and a falsity membership function \(F_{N''}\).

Example 5 Suppose ABC Trucking is a company that operates 20 trucks for transport and logistics. When they are full and on the track, trucks make the most money for the company. ABC Trucking has the following vector entities or groups:

(i) Truck Company \((X = \text{truck type}, Y = \text{age}, Z = \text{engine size})\).
(ii) Income \((X' = \text{Euro}_1, Y' = \text{Euro}_2, Z' = \text{Euro}_3)\).

The neutrosophic sets \(N_1, N_2, \text{and} N_3\) in \(X, Y, \text{and} Z\) are:

\[
N_1 = \left(\frac{0.4, 0.5, 0.8}{x}\right),
\]

\[
N_2 = \left(\frac{0.2, 0.7, 0.3}{x}\right), \quad N_3 = \left(\frac{0.5, 0.4, 1}{x}\right).
\]

Let \(f\) be a function defined by

\[
\begin{align*}
\left(\frac{0.4, 0.5, 0.8}{x}\right) & \xrightarrow{f} \left(\frac{0.8, 0.6, 0.2}{x}\right), \\
\left(\frac{0.2, 0.7, 0.3}{x}\right) & \xrightarrow{f} \left(\frac{0.7, 0.5, 0.8}{x}\right), \\
\left(\frac{0.5, 0.4, 1}{x}\right) & \xrightarrow{f} \left(\frac{0.5, 0.3, 1}{x}\right).
\end{align*}
\]

Now, neutrosophic max variational inequality is:

\[
\{f(N_1(x)), f(N_1(x)) \cup f(N_j(x))\}
\geq f(N_1(x)) \circ f(N_j(x)); \quad j = 2, 3.
\] (43)

Taking the left side of the inequality (43), we have
\[
\begin{align*}
\langle f(N_1(x)), f(N_1(x) \cup f(N_2(x)) \\ = \left\{ \frac{0.8, 0.6, 0.2}{x}, \frac{0.8, 0.6, 0.2}{x} \right\} \cup \left\{ \frac{0.7, 0.5, 0.8}{x} \right\} \\
= \left\{ \frac{0.8, 0.6, 0.2}{x}, \frac{0.8, 0.5, 0.2}{x} \right\} \\
\langle f(N_1(x)), f(N_1(x) \cup f(N_2(x)) \\ = \left\{ \frac{0.64, 0.30, 0.04}{x} \right\}.
\end{align*}
\]

Also,
\[
\begin{align*}
\langle f(N_1(x)), f(N_1(x) \cup f(N_3(y)) \\ = \left\{ \frac{0.8, 0.6, 0.2}{x}, \frac{0.8, 0.6, 0.2}{x} \right\} \cup \left\{ \frac{0.5, 0.3, 1}{x} \right\} \\
= \left\{ \frac{0.8, 0.6, 0.2}{x}, \frac{0.8, 0.3, 0.2}{x} \right\} \\
\langle f(N_1(x)), f(N_1(x) \cup f(N_3(y)) \\ = \left\{ \frac{0.64, 0.18, 0.04}{x} \right\}.
\end{align*}
\]

Now, we have two neutrosophic values with respect to \( f(N_1(x)) \), that is,
\[
\begin{align*}
\left\{ \frac{0.64, 0.30, 0.04}{x}, \frac{0.64, 0.18, 0.04}{x} \right\}.
\end{align*}
\]

The max value of \( \left\{ \frac{0.64, 0.30, 0.04}{x}, \frac{0.64, 0.18, 0.04}{x} \right\} \) is:
\[
\begin{align*}
\max \left\{ \frac{0.64, 0.30, 0.04}{x}, \frac{0.64, 0.18, 0.04}{x} \right\} \\
= \left\{ \frac{0.64, 0.30, 0.04}{x} \right\}.
\end{align*}
\]

Thus, the maximum profit with a neutrosophic value \( N' \) is characterized by a truth membership function \( T_{N'} = 0.40 \), an indeterminacy membership function \( I_{N'} = 0.30 \), and a falsity membership function \( F_{N'} = 0.2 \).

Also, the neutrosophic min variational inequality is:
\[
\begin{align*}
\langle f(N_3(x)), f(N_3(x) \cap f(N_j(x)) \\ \leq f(N_3(x)) \circ f(N_j(x)); \quad j = 1, 2.
\end{align*}
\]

Taking the left side of the inequality (47), we have
\[
\begin{align*}
\langle f(N_3(x)), f(N_3(x) \cap f(N_1(x)) \\ = \left\{ \frac{0.5, 0.3, 1}{x}, \frac{0.5, 0.3, 1}{x} \right\} \cap \left\{ \frac{0.8, 0.6, 0.2}{x} \right\} \\
= \left\{ \frac{0.5, 0.3, 1}{x}, \frac{0.5, 0.6, 1}{x} \right\} \\
= \left\{ \frac{0.40, 0.36, 0.2}{x} \right\}.
\end{align*}
\]

Also,
\[
\begin{align*}
\langle f(N_3(x)), f(N_3(x) \cap f(N_2(x)) \\ = \left\{ \frac{0.5, 0.3, 1}{x}, \frac{0.5, 0.3, 1}{x} \right\} \cap \left\{ \frac{0.7, 0.5, 0.8}{x} \right\} \\
= \left\{ \frac{0.5, 0.3, 1}{x}, \frac{0.5, 0.5, 1}{x} \right\} \\
\langle f(N_3(x)), f(N_3(x) \cap f(N_2(x)) \\ = \left\{ \frac{0.40, 0.36, 0.2}{x} \right\}.
\end{align*}
\]

Now, we have two neutrosophic values with respect to \( f(N_3(x)) \), that is,
\[
\begin{align*}
\left\{ \frac{0.40, 0.36, 0.2}{x}, \frac{0.40, 0.30, 0.2}{x} \right\}.
\end{align*}
\]

The min value of \( \left\{ \frac{0.40, 0.36, 0.2}{x}, \frac{0.40, 0.30, 0.2}{x} \right\} \) is:
\[
\begin{align*}
\min \left\{ \frac{0.40, 0.36, 0.2}{x}, \frac{0.40, 0.30, 0.2}{x} \right\} \\
= \left\{ \frac{0.40, 0.30, 0.2}{x} \right\}.
\end{align*}
\]

The comparison of the two neutrosophic values, one is \( N' \) and another one is \( N'' \), can be made by calculating their min and max values. The min value of \( N' \) is \( 0.40 \), and the min value of \( N'' \) is \( 0.30 \). Also, the max value of \( N' \) is \( 0.40 \), and the max value of \( N'' \) is \( 0.30 \).

5 Comparison

The neutrosophic set has many applications in many fields of science. Here, we discussed the application of neutrosophic set in decision-making problems. In this practical application, one of the main issues is that how to choose a suitable model. We examined this idea in-depth and used the neutrosophic max and min variational inequalities.

In the real world, fuzziness is a common phenomenon and is unavoidable in many realistic fields. In 1965, Zadeh (1965) suggested the idea of fuzzy sets and developed the theory of fuzzy sets. It is used in many fields, including fuzzy control, fuzzy optimization, fuzzy analysis of data, fuzzy time series, etc. Here are some interesting references: (Buckley 1988, 1989) used possibility distribution, Herrera et al (1993) used fuzzified constraints and objective functions, transformed a fuzzy linear optimization problem to a classical one by using the structural properties of fuzzy numbers. With the development of computer science and evolutionary computation theory, evolutionary computation methods came into play in fuzzy optimization problems. Razavi HAJIAGHA et al...
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Author Contributions All authors contributed equally.

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6 Conclusion
In this paper, we have introduced some new concepts of a neutrosophic set such as neutrosophic convex set, strongly neutrosophic convex set, neutrosophic convex function, strongly neutrosophic convex function, the minimum and maximum of a function $f$ with respect to neutrosophic set, min, and max neutrosophic variational inequality, neutrosophic general convex set, neutrosophic general convex function, and min, max neutrosophic general variational inequality. We have discussed some basic results on these new concepts. We proposed the application of the neutrosophic set in optimization theory. This work and further study of neutrosophic max and min variational inequalities will give a new direction of application in the field of optimization. Moreover, neutrosophic differential equation is important in applied science and engineering for modeling of uncertainty.

Declarations

Conflict of interests The authors declare that there is no conflict of interest regarding the publication of this article.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References
Abdel-Baset M, Chang V, Gamal A (2019) Evaluation of the green supply chain management practices: a novel neutrosophic approach. Comput Ind 108:210–220
Abdel-Baset M, Chang V, Gamal A et al (2019) An integrated neutrosophic anp and vikor method for achieving sustainable supplier selection: A case study in importing field. Comput Ind 106:94–110
Abdel-Basset M, Manogaran G, Gamal A et al (2018) A hybrid approach of neutrosophic sets and dematel method for developing supplier selection criteria. Des Autom Embed Syst 22(3):257–278
Abdel-Basset M, Manogaran G, Gamal A et al (2019) A group decision making framework based on neutrosophic topsis approach for effective decision making under type-2 neutrosophic number. Appl Soft Comput 77:438–452
Ali M, Smarandache F (2017) Complex neutrosophic set. Neural Comput Appl 28(7):1817–1834
Almeida GC, Morillo JC, Palacios MC (2020) Discernimiento e inferencia de la reinserción social en ecuador, basada en conjuntos de números de 2-tuplas. Investigación Operacional 41(5):637–647
Arqub OA, Al-Smadi M (2020) Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions. Soft Comput 1–22
Arqub OA (2017) Adaptation of reproducing kernel algorithm for solving fuzzy fredholm-volterra integrodifferential equations. Neural Comput Appl 28(7):1591–1610
Arqub OA, Mohammed AS, Momani S et al (2016) Numerical solutions of fuzzy differential equations using reproducing kernel hilbert space method. Soft Comput 20(8):3283–3302
Arqub OA, Al-Smadi M, Momani S et al (2017) Application of reproducing kernel algorithm for solving second-order, two-point fuzzy boundary value problems. Soft Comput 21(23):7191–7206
Atanassov KT (1999) Interval valued intuitionistic fuzzy sets. In Intuitionistic fuzzy sets. Springer, pp 139–177
Atanassov K (2016) Intuitionistic fuzzy sets. Int J Bioautom 20(1)
Bera T, Mahapatra NK (2018) On neutrosophic soft topological space. Neutrosophic Sets Syst 19(1):3–15
Bera T, Mahapatra NK (2018) Neutrosophic soft normed linear spaces, vol 23. Neutrosophic Sets Syst
Bera T, Mahapatra NK (2018) On neutrosophic soft prime ideal. Infinite Study
Berti LAC (2020) Application of the neutrosophic system to tax havens with a criminal approach. Int J Neutrosoph Sci 5(2):91–106
Buckley J (1988) Possibilistic linear programming with triangular fuzzy numbers. Fuzzy Sets Syst 26(1):135–138
Buckley J (1989) Solving possibilistic linear programming problems. Fuzzy Sets Syst 31(3):329–341
Chaira T (2011) A novel intuitionistic fuzzy c means clustering algorithm and its application to medical images. Appl Soft Comput 11(2):1711–1717
Chakraborty D, Jana DK, Roy TK (2014) A new approach to solve intuitionistic fuzzy optimization problem using possibility, necessity, and credibility measures. Int J Eng Math 2014:1–12
Gau WL, Buehrer DJ (1993) Vague sets. IEEE Trans Systems Man Cybern 23(2):610–614
Hatip A (2020) The special neutrosophic functions. Int J Neutrosoph Sci 4(2):104–116
Herrera F, Kovacs M, Verdegay J (1993) Optimality for fuzzified mathematical programming problems: a parametric approach. Fuzzy Sets Syst 54(3):279–285
Joshi BP, Kumar S (2012) Fuzzy time series model based on intuitionistic fuzzy sets for empirical research in stock market. Int J Appl Evolut Comput (IJAECE) 3(4):71–84
Majumdar P, Samanta SK (2014) On similarity and entropy of neutrosophic sets. J Intell Fuzzy Syst 26(3):1245–1252
Nabeeh NA, Smarandache F, Abdel-Basset M et al (2019) An integrated neutrosophic-topsis approach and its application to personnel selection: a new trend in brain processing and analysis. IEEE Access 7:29,734–29,744
Pedrycz W (1990) Fuzzy sets in pattern recognition: methodology and methods. Pattern Recognit 23(1–2):121–146
Pei Z, Zheng L (2012) A novel approach to multi-attribute decision making based on intuitionistic fuzzy sets. Expert Syst Appl 39(3):2560–2566
Razavi HAJIAGHA SH, Mahdiraji HA, Zavadskas EK, et al (2014) Maximizing and minimizing sets in solving fuzzy linear programming. Econ Comput Econ Cybern Stud Res 48(2)
Saqlain M, Hamza A, Farooq S (2020) Linear and non-linear octagonal neutrosophic numbers: its representation, α-cut and applications. Int J Neutrosoph Sci 3(1):29–43
Schweizer P (2020) The natural bases of neutrosophy. Int J Neutrosoph Sci 9:100–109
Shah N, Hussain A (2016) Neutrosophic soft graphs. Neutrosoph Sets Syst 11:31–44
Shirin S et al (2014) Application of fuzzy optimization problem in fuzzy environment. Dhaka Univ J Sci 62(2):119–125
Smarandache F (1999) A unifying field in logics, neutrosophy: Neutrosophic probability, set and logic
Smarandache F (2013) Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability. Infinite Study
Smarandache F (2014) Introduction to neutrosophic statistics: infinite study. Romania-Educational Publisher, Columbus
Sotirov S, Sotirova E, Orozova D (2009) Neural network for defining intuitionistic fuzzy sets in e-learning. Notes Intuit Fuzzy Sets 15(2):33–36
Torra V (2010) Hesitant fuzzy sets. Int J Intell Syst 25(6):529–539
Wang H, Smarandache F, Sunderraman R, et al (2005) Interval neutrosophic sets and logic: theory and applications in computing: theory and applications in computing, vol 5. Infinite Study
Wang H, Smarandache F, Zhang Y, et al (2005) Single valued neutrosophic sets, vol 5. Multispace Multistruct
Xu Z (2011) Intuitionistic fuzzy multiattribute decision making: an interactive method. IEEE Trans Fuzzy Syst 20(3):514–525
Ye J (2013) Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. Int J General Syst 42(4):386–394
Ye J (2014) A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. J Intell Fuzzy Syst 26(5):2459–2466
Ye J (2014) Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making. J Intell Fuzzy Syst 26(1):165–172
Zadeh LA (1965) Fuzzy sets, information and control, 8: 338-353. MathSciNet zbMATH
Zadeh LA (1975) Fuzzy logic and approximate reasoning. Synthese 30(3):407–428
Zeng S, Su W (2011) Intuitionistic fuzzy ordered weighted distance operator. Knowl-Based Syst 24(8):1224–1232

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