PRICING OPTIONS ON INVESTMENT PROJECT EXPANSIONS UNDER COMMODITY PRICE UNCERTAINTY

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Abstract. In this work we develop PDE-based mathematical models for valuing real options on investment project expansions when the underlying commodity price follows a geometric Brownian motion. The models developed are of a similar form as the Black-Scholes model for pricing conventional European call options. However, unlike the Black-Scholes’ model, the payoff conditions of the current models are determined by a PDE system. An upwind finite difference scheme is used for solving the models. Numerical experiments have been performed using two examples of pricing project expansion options in the mining industry to demonstrate that our models are able to produce financially meaningful numerical results for the two non-trivial test problems.

1. Introduction. Investments are always accompanied by risks as well as opportunities due to uncertainties in our fast-changing economic environments and conditions. It is important for investors to have flexible plans to cope with the uncertainties in the form of options such as an option to expand a project at a certain time. Although the traditional Discounted Cash Flow (DCF) model has been widely adopted in the management of enterprise practices, it has been pointed out in [23, 15, 9] that DCF has many limitations in practice. More specifically, DCF is unable to capture the values of flexibilities or options that are embedded in a project. On the other hand, such an option can be used as an instrument to represent value components resulted from the interaction of flexibility and uncertainty. The aforementioned type of option is called a ‘real option’ which is first introduced by Stewart C. Myers. In [25] the author presents an extensive classification of real options which include Option to Defer, Staged Investment Option, Option to Alter Operating Scale, Option to Abandon, Option to Switch, Growth Option, and Interacting Option. In [24] the author analyzed the value of a multinational natural resource project with real options. Costa Lima and Suslick [8] pointed out that real option theory is a useful method for the evaluation of mining projects. The value of a real option is a function of time and the underlying asset price. How to price such an option accurately has been a challenge in financial engineering. Valuation of real options has been studied by various authors such as Moyen [22], Moel and Tufano [21], Abdel Sabour and Poulin [1], Brennan and Schwartz [4, 5].

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also studied the problem of how to estimate the value of a copper mining project with a high degree of uncertainty in output prices. In [10] the authors propose a partial differential equation (PDE) approach to value a project with an expansion option. However, unlike pricing conventional financial options, studies on how to price investment extension options using a PDE approach are very limited.

In this work, we present a systematic study on a PDE approach to pricing ‘options to expand’. More specifically, we develop PDE models for valuing expansion and compound expansion options embedded in a development project. These models are of the form of Black-Scholes model for pricing conventional European options with a vanilla payoff function, but unlike the Black-Scholes model, the payoff condition of such a real option is defined by another PDE system. We also propose a finite difference scheme to solve the developed models. In the rest of this paper, we will demonstrate this approach using a commodity-based project, though the idea can easily be used for other types of projects. The rest of this paper is organized as follows.

In Section 2, we will first derive a mathematical model for valuing an expansion option of a project whose underlying commodity price follows a geometric Brownian motion, based on the idea for valuing a resource project in [12]. This model is of the form of a European call option. We will then extend the model to pricing expansion-on-expansion, or call-on-call (CoC), compound options [11]. In Section 3, we will propose a finite difference scheme for numerically solving the models. In Section 4 we will present some numerical experimental results using two non-trivial examples of iron-ore projects and the numerical results show that our models and numerical methods produce financially meaningful results.

2. The pricing models. In this section we will present mathematical or PDE models for valuing project expansion options. For clarity, we will use an investment project in a nature resource (e.g., iron ore) industry to demonstrate our approach. The idea can easily be used for other types of project expansion options as well.

2.1. The models for pricing expansion options. Consider an investment project in a natural resource industry whose underlying commodity price follows a geometric Brownian motion in time $t$. The value of the project is a function of $P$ and $t$. In this work we assume that $P$ satisfies the following stochastic differential equation proposed in [7]

$$dP = P(r - \delta)dt + P\sigma dz$$  \hspace{1cm} (2.1) $$

for $t \geq 0$, where $z$ is a Wiener process or geometric Brownian motion, $\sigma$ is the instantaneous standard deviation, or volatility, of $P$, $r$ is the risk free interest rate, and $\delta$ is the mean convenience yield on holding one unit of the output. We assume that $\sigma$, $r$, and $\delta$ are non-negative constants.

Now, we consider the project in the following two different cases.

C0. The project does not have any expansion options.

C1. There is an embedded production expansion option in the project that is exercisable at a fixed time $T > 0$ (expiry date) and requires the amount $K > 0$ (strike price) for the expansion.

We denote the value functions of the project in the above two cases as $V_0(P,t)$ and $V_1(P,t)$ respectively and expect $V_1(P,t) \geq V_0(P,t)$ for all $t \geq 0$ and $P \geq 0$ if the investors can make the right decision on whether or not exercising the option. This is because Case C1 has the right, not an obligation, to increase the production rate
at $T$ and the option is exercised only when this decision adds value to the project. Whether the option is exercised or not at $T$ depends on the commodity price $P$ at maturity which is used to define the payoff of the option at $t = T$. When the payoff function is determined, the value of the option, denoted as $W(P, t)$, for $t < T$ is the difference between $V_1$ and $V_0$, i.e., $W(P, t) = V_1(P, t) - V_0(P, t)$ for $t < T$ and $P \geq 0$.

To determine $W(P, t)$ we let $q_k(t)$, $k = 0, 1$, be the the production rates associated with Cases C0 and C1 respectively. Then the after-tax cash flow rates for C0 and C1 are respectively given by (see, for example, [12])

$$
D_k(P, t) := q_k(t)[P(t)(1 - R) - C(t)](1 - B), \quad k = 0, 1,
$$

(2.2)

where $C(t)$ is the average cash cost rate of production per unit output, $R$ is the rate of state royalties and $B$ the company income tax rate.

Using a Taylor expansion and (2.1), and omitting terms of orders higher than $dt$, we have the following instantaneous change $dV_k$ in the value of the project

$$
dV_k = \frac{\partial V_k}{\partial P} dP + \frac{\partial V_k}{\partial t} dt + \frac{1}{2} P^2 \sigma^2 \frac{\partial^2 V_k}{\partial P^2} dt, \quad k = 0, 1.
$$

(2.3)

Following the $\Delta$-hedging strategy in [2, 3], we construct a risk-free portfolio $\Pi_k$ consisting of a long position in the value of the project and a short position in $\frac{\partial V_k}{\partial P}$ units of the commodity at unit price $P$, i.e., $\Pi_k = V_k - \frac{\partial V_k}{\partial P} P$ for $k = 0, 1$. Then, when $dt$ is sufficiently small, using the result in [12] we have that the return of the portfolio $\Pi_k$ in time period $dt$ is a combination of the changes in the values of the project and the short position, given respectively by $dV_k$ and $\frac{\partial V_k}{\partial P} dP$, the convenience yield of the short position given by $\delta \frac{\partial V_k}{\partial P} P dt$, and the cash flow in the period given by $D_k(P, t) dt$, where $D_k$ is defined in (2.2), i.e.,

$$
d\Pi_k = dV_k - \frac{\partial V_k}{\partial P} dP - \delta \frac{\partial V_k}{\partial P} P dt + D_k(P, t) dt
$$

(2.4)

for $k = 0, 1$. Replacing $dV_k$ in (2.4) by the one defined in (2.3) and using the fact that $\Pi_k$ is risk-less, i.e., $d\Pi_k = r \Pi_k dt$, we have the following PDE.

$$
\mathcal{L} V_k := \frac{\partial V_k}{\partial t} + (r - \delta) P \frac{\partial V_k}{\partial P} + \frac{1}{2} P^2 \sigma^2 \frac{\partial^2 V_k}{\partial P^2} - r V_k = -D_k(P, t)
$$

(2.5)

for $k = 0, 1$. Eq.(2.5) is satisfied by the value functions of the project in both Cases C0 and C1. Therefore, taking both sides of the equation for $V_0$ away from the corresponding sides of that for $V_1$, we have the following PDE for the value $W$ of the expansion option:

$$
\mathcal{L} W = D_0(P, t) - D_1(P, t),
$$

(2.6)

for $P > 0$ and $t \in (0, T]$. In fact, for an expansion option, we usually have $D_0(P, t) - D_1(P, t) = 0$ for $t \in ]0, T]$ since the change of operating rate starts from $t = T$. However, we still leave the above equation in its general form.

In computation, (2.6) needs to be solved on a finite domain $(P, t) \in (0, P_{max}) \times [0, T]$, where $P_{max}$ is a positive constant satisfying $P_{max} \gg C(t)$ for any $t$ in the life-time of the project. We now determine the boundary and payoff conditions for (2.6) at $P = 0$, $P = P_{max}$ and $t = T$ in order for (2.6) to determine the correct value of the expansion option. When $P = 0$, it is clear that exercising the option will add negative value to the investment and thus the option is of no value, i.e.,

$$
W(0, t) = 0, \quad t \in [0, T).
$$

(2.7)
To determine the value (or payoff) of the option at \( t = T \), we note that when \( V_1(P,T) > V_0(P,T) + K \), the value \( W(P,T) \) of the option is non-zero. Otherwise, \( W(P,T) = 0 \), as in this case, exercising the option does not add any value to the investment project. Thus, the terminal condition for \( W \) is defined as the following Vanilla payoff function.

\[
W(P,T) = [V_1(P,T) - V_0(P,T) - K]^+, \quad P \in (0, P_{\text{max}}).
\] (2.8)

For the boundary condition of \( W \) at \( P = P_{\text{max}} \) we define

\[
W(P_{\text{max}}, t) = V_1(P_{\text{max}}, t) - V_0(P_{\text{max}}, t) - e^{-r(T-t)} K
\] (2.9)

for \( t \in [0, T) \). From (2.7)–(2.9) we see that the payoff and boundary conditions satisfy the compatibility condition that \( W(P,t) \) is continuous at the points \((0,T)\) and \((P_{\text{max}},T)\). Note that from (2.8) and (2.9) we see that \( V_1(P,T) - V_0(P,T) \) and \( V_1(P_{\text{max}},t) - V_0(P_{\text{max}},t) \) are needed to determine the payoff and boundary conditions. These functions can be determined using a different approach. We now discuss it.

To estimate the value of the expanding option for the investment project, we need to calculate the lifetime of the project. Let \( Q \) denote the reserve of the resource and \( T_k^* \) be life time of the project when the production rate is \( q_k \). Then these quantities should satisfy

\[
Q = \int_0^{T_k^*} q_k(t) dt, \quad k = 0, 1.
\] (2.10)

Since we expect the life-span of a production project is much longer than that of a real option, it is reasonable to assume that the expiry date of the option satisfies \( T < \min\{T_0^*, T_1^*\} \). For simplicity, we let \( T^* = \max\{T_0^*, T_1^*\} \) and use \( T^* \) as the life time of the project for both Cases C0 and C1. This requires that if the reserve is exhausted before \( T^* \), we simply set the corresponding production rate in (2.2) to zero. For this expansion option, we have \( T^* = T_0^* \),

\[
q_0(t) = q_1(t), \quad t \in [0, T), \quad q_0(t) < q_1(t), \quad t \in [T, T_1^*), \quad q_1(t) = 0, \quad t \in [T_1^*, T^*].
\] (2.11)

When \( P_{\text{max}} \) is sufficiently large and fixed, we may use the ‘Net Present Value’ to estimate the value \( V_k(P_{\text{max}}, t) \) for \( t \geq T \). Since the cash flow of the project is \( D_k(P_{\text{max}}, t) \), the rate of change at \( \tau > t \) in the present value of the resource project at time \( t \) is

\[ dV_k = D_k(P_{\text{max}}, t)e^{-r(\tau-t)} d\tau, \quad \tau \geq t. \]

This expression is also true if \( P_{\text{max}} \) is replaced with 0. Therefore, the present values of the project at \( t \) when \( P = 0 \) and \( P = P_{\text{max}} \) are

\[
V_k(P, t) = \int_t^{T_0^*} D_k(P, \tau)e^{-r(\tau-t)} d\tau
\] (2.12)

for \( P = 0, P_{\text{max}} \), where \( D_k \) is defined in (2.2) for \( k = 0, 1 \). Using (2.12) we are able to define the boundary condition (2.9).

It now remains to determine \( V_1(P,T) - V_0(P,T) \) in (2.8). Note that when \( t = T^* \), the project has zero value for both of the two cases. Using (2.12), we pose the
following problem for the \( \hat{W} \):

\[
\begin{aligned}
\mathcal{L} \hat{W} &= -(D_1(P,t) - D_0(P,t)), \quad (P,t) \in (0, P_{\text{max}}) \times [T, T^*), \\
\hat{W}(0,t) &= \int_{t}^{T^*} [D_1(0,\tau) - D_0(0,\tau)] e^{-r(\tau-t)} d\tau, \quad t \in [T, T^*), \\
\hat{W}(P_{\text{max}},t) &= \int_{t}^{T^*} [D_1(P_{\text{max}},\tau) - D_0(P_{\text{max}},\tau)] e^{-r(\tau-t)} d\tau, \quad t \in [T, T^*), \\
\hat{W}(P,T^*) &= 0, \quad P \in (0, P_{\text{max}}).
\end{aligned}
\]

(2.13)

Solving (2.13), we can determine the payoff condition for (2.6) by replacing \( V_1(P,T) - V_0(P,T) \) with \( \hat{W}(P,T) \) in (2.8).

2.2. The pricing model for CoC compound options. We now extend the above pricing model to Call-on-Call (CoC) compound expansion options [11]. In this discussion, we consider the following case.

C2 The investment project has a production expansion option with a given expansion rate and strike price \( K_1 > 0 \) exercisable at time \( T_1 > 0 \). This option has an embedded option of a further production expansion, exercisable at time \( T_2 > T_1 \), with another given operating rate and strike price \( K_2 > 0 \).

In this case, we have three different cash flow rates \( D_k(P,t) \), \( k = 0, 1, 2 \), corresponding to three production rates \( q_0(t) \), \( q_1(t) \) and \( q_2(t) \) respectively. The life-spans \( T_k^* \) corresponding to \( q_k \) for \( k = 0, 1, 2 \) can be determined by (2.10) and we define \( T^* = \max\{T_0^*, T_1^*, T_2^*\} \). Since we are only concerned with production expansions at the fixed time points \( T_1 \) and \( T_2 \), we have \( T^* = T_0^* \geq T_1^* \geq T_2^* \). More specifically, we assume that the rates satisfy

\[
\begin{align*}
q_0(t) &= q_1(t) = q_2(t), \quad t \in [0, T_1), \\
q_0(t) &< q_1(t) = q_2(t), \quad t \in [T_1, T_2), \\
q_0(t) &< q_1(t) < q_2(t), \quad t \in [T_2, T_2^*], \\
q_1(t) &= 0, \quad t \in [T_1^*, T^*], \\
q_2(t) &= 0, \quad t \in [T_2, T^*].
\end{align*}
\]

Let \( W_p \) denote the value of the option exercisable at \( T_2 \). Then, following the deduction of the model for pricing normal options in the previous subsection, we define the following problem to determine \( W_p \).

\[
\begin{aligned}
\mathcal{L} W_p &= D_0(P,t) - D_2(P,t), \quad (P,t) \in (0, P_{\text{max}}) \times [T_1, T_2), \\
W_p(0,t) &= 0, \\
W_p(P_{\text{max}},t) &= V_2(P_{\text{max}},t) - V_0(P_{\text{max}},t) - e^{-r(T_2-t)} K_2, \quad t \in [T_1, T_2), \\
W_p(P,T_2) &= [V_2(P,T_2) - V_0(P,T_2) - K_2]^+, \quad P \in (0, P_{\text{max}}).
\end{aligned}
\]

(2.14)

The solution to the above problem defines the value of the compound option in \([T_1, T_2]^*\). However, the boundary and terminal conditions in (2.14) require the differences \( V_2(P_{\text{max}},t) - V_0(P_{\text{max}},t) \) for \( t \in [T_1, T_2) \) and \( V_2(P,T_2) - V_0(P,T_2) \) for \( P \in (0, P_{\text{max}}) \). The former can be calculated using (2.12) and the latter by solving (2.13) with \( D_1 \) and \( T \) replaced with \( D_2 \) and \( T_2 \) respectively.

To determine the value \( W_c \) of the compound option exercisable at \( T_1 \), we solve the following problem

\[
\begin{aligned}
\mathcal{L} W_c &= D_0(P,t) - D_1(P,t), \quad (P,t) \in (0, P_{\text{max}}) \times [0, T_1), \\
W_c(0,t) &= 0, \\
W_c(P_{\text{max}},t) &= V_2(P_{\text{max}},t) - V_0(P_{\text{max}},t) - e^{-r(T_1-t)} K_1, \quad t \in [0, T_1), \\
W_c(P,T_1) &= [W_p(P,T_1) - K_1]^+, \quad P \in (0, P_{\text{max}}).
\end{aligned}
\]

(2.15)
where \( W_p(P, t) \) used in the payoff and boundary conditions of (2.15) is the solution to (2.14).

3. Discretization. In this section we present a discretization scheme for (2.6) along with the boundary and terminal conditions (2.7)–(2.12). Various finite difference and finite volume based discretization schemes have been developed for solving Black-Scholes equations such as those in [4, 30, 27, 29, 6, 20]. In this work we will use a finite difference scheme with an upwind technique for (2.6) based on that used in [28, 18, 19, 16]. The discretization scheme developed for (2.6) is also applicable to (2.13) as well as the problems governing the CoC option. We divide this discussion into two parts – discretization of (2.6) and that of (2.7)–(2.12).

3.1. Discretization of (2.6). For any positive integers \( M \) and \( N \), We partition \((0, P_{\text{max}}) \times (0, T)\) into \( M \times N \) rectangles with nodes \((P_i, t_j)\) for \( i = 0, 1, \ldots, M \) and \( j = 0, 1, \ldots, N \) satisfying

\[
0 = P_0 < P_1 < \ldots < P_M = P_{\text{max}},
\]

\[
T = t_0 > t_1 > \ldots > t_N = 0.
\]

Set \( h_i = P_{i+1} - P_i > 0 \) for \( i = 0, 1, \ldots, M - 1 \) and \( \Delta t_j = t_{j+1} - t_j < 0 \) for \( j = 0, 1, \ldots, N - 1 \). On this mesh, we define the following finite difference equation approximating (2.6).

\[
\frac{W_i^{j+1} - W_i^j}{\Delta t_j} + \frac{P_i^2 \sigma^2}{h_i + h_{i+1}} \left( \frac{W_i^{j+1} - W_{i+1}^{j+1}}{h_{i+1}} - \frac{W_i^{j+1} - W_{i-1}^{j+1}}{h_i} \right) + P_i(r - \delta) \left( \frac{\text{sgn}(r - \delta) + 1}{2} \frac{W_{i+1}^{j+1} - W_i^{j+1}}{h_i} - \frac{\text{sgn}(r - \delta) - 1}{2} \frac{W_i^{j+1} - W_{i-1}^{j+1}}{h_{i-1}} \right) - rW_i^{j+1} = f_i^{j+1} \tag{3.1}
\]

for \( i = 1, 2, \ldots, M - 1 \) and \( j = 0, 1, \ldots, N - 1 \), where \( \text{sgn}(\cdot) \) denotes the sign function, \( W_i^j \) denotes an approximation to \( W(P_i, t_j) \) and \( f_i^{j+1} = D_{0i}(P_i, t_{j+1}) - D_i(P_i, t_{j+1}) \). In (3.1), we used the backward difference to approximate the term \( \frac{\partial W}{\partial t} \), the central difference to approximate \( \frac{\partial^2 W}{\partial x^2} \), and the upwind difference for \( \frac{\partial W}{\partial x} \). Upwind techniques have often been used for solving convection-diffusion problems (see, for example, [14]).

Re-arranging (3.1), we have

\[
\alpha_i^j W_{i-1}^{j+1} + \beta_i^j W_i^{j+1} + \gamma_i^j W_{i+1}^{j+1} = \frac{W_i^j}{\Delta t_j} - f_i^{j+1} \tag{3.2}
\]

for \( i = 1, 2, \ldots, M - 1 \) and \( j = 0, 1, \ldots, N - 1 \), where

\[
\alpha_i^j = -\frac{P_i^2 \sigma^2}{h_i + h_{i-1}} - \frac{\text{sgn}(r - \delta) - 1}{2h_{i-1}} P_i(r - \delta) \tag{3.3}
\]

\[
\beta_i^j = -\frac{1}{\Delta t_j} + \frac{\sigma^2 P_i^2}{h_i} + \left( \frac{\text{sgn}(r - \delta) - 1}{2h_{i-1}} + \frac{\text{sgn}(r - \delta) + 1}{2h_i} \right) P_i(r - \delta) + r \tag{3.4}
\]

\[
\gamma_i^j = -\frac{P_i^2 \sigma^2}{h_i(h_i + h_{i-1})} - \frac{\text{sgn}(r - \delta) + 1}{2h_i} (r - \delta) P_i \tag{3.5}
\]
Using (2.7)–(2.9) we define boundary and terminal conditions for (3.2) as follows.
\[ W_0^i = 0, \quad W_M^i = V_i(P_{\text{max}}, t_j) - V_0(P_{\text{max}}, t_j) - K e^{-r(T-t_j)}, \quad W_i^0 = [\hat{W}(P, T) - K]^+ \]

(3.6)

for \( i = 1, 2, ..., M-1 \) and \( j = 1, 2, ..., N \), where \( \hat{W} \) denotes the solution to (2.13). In practice, the values of \( V_i(P, t) - V_0(P, t) \) and \( \hat{W}(P, T) \) at the boundary and terminal mesh points need to be approximated. We will discuss this in the next subsection.

We write (3.2) along with (3.6) as the following matrix form:
\[
A^j W^{j+1} = -\frac{1}{\Delta t_j} W^j + b^{j+1},
\]

(3.7)

where
\[
A^j = \begin{bmatrix}
\beta_1^j & \gamma_1^j & 0 & 0 & \cdots & 0 & 0 & 0 \\
\alpha_2^j & \beta_2^j & \gamma_2^j & 0 & \cdots & 0 & 0 & 0 \\
0 & \alpha_3^j & \beta_3^j & \gamma_3^j & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \beta_{M-3}^j & \gamma_{M-3}^j & 0 \\
0 & 0 & 0 & 0 & \cdots & \alpha_{M-2}^j & \beta_{M-2}^j & \gamma_{M-2}^j \\
0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{M-1}^j & \beta_{M-1}^j
\end{bmatrix}
\]

(3.8)

\[
b^{j+1} = \begin{bmatrix}
-f_1^{j+1} - \alpha_1^j W_0^{j+1} \\
-f_2^{j+1} \\
\vdots \\
-f_{M-2}^{j+1} \\
-f_{M-1}^{j+1} - \gamma_{M-1}^j W_M^{j+1}
\end{bmatrix}, \quad W^m = \begin{bmatrix}
W_1^m \\
W_2^m \\
\vdots \\
W_{M-2}^m \\
W_{M-1}^m
\end{bmatrix}
\]

for \( m = j \) and \( j + 1 \).

The solution to the above linear system, along with the boundary and payoff conditions in (3.6) defines an approximation to \( W(P, t) \) at the mesh nodes. In the following theorem, we prove that \( A^j \) in (3.7) is an \( M \)-matrix.

**Theorem 3.1.** The matrix \( A^j \) is an \( M \)-matrix for \( j = 0, 1, ..., N-1 \).

**Proof.** To prove \( A^j \) is an \( M \)-matrix, it suffices to show that it is irreducible and strictly diagonally dominant with positive diagonal and non-positive off-diagonal entries.

It is obvious that \( A^j \) is irreducible, as otherwise (3.7) can be solved as two or more independent sub-problems. Note that \( A^j \) is a tri-diagonal matrix as defined in (3.8), where the non-zero entries are given in (3.3)–(3.5). Since \( r \geq 0 \) and \( \Delta t_j < 0 \), we have from (3.3)–(3.5) that \( \alpha_i^j < 0, \beta_i^j > 0 \) and \( \gamma_i^j < 0 \) for all \( i = 1, 2, ..., M-1 \). Also,
\[
\beta_i^j = |\alpha_i^j| + |\gamma_i^j| + r - \frac{1}{\Delta t_j} > |\alpha_i^j| + |\gamma_i^j|,
\]
since \( r \geq 0 \) and \( \Delta t_j < 0 \). Therefore \( A^j \) is strictly diagonally dominant. By [26] we have that \( A^j \) is an \( M \)-matrix. \( \square \)

**Remark 3.1.** We comment that in fact, both \( \alpha_i^j \) and \( \gamma_i^j \) can be simply denoted as \( \alpha_i \) and \( \gamma_i \), because from (3.3) and (3.5) we see both of them are independent of \( j \). However, we will still keep the subscript \( j \) as the above scheme can also be used for problems in which any of \( \sigma, r \) and \( \delta \) is time-dependent.
Remark 3.2. The upwind finite difference method has been used for solving various types of differential equations (see, for example, [14, 28, 19, 13, 16, 17]). In particular it is used for a nonlinear Black-Scholes equation arising in pricing a conventional option in [16] in which the authors also prove the convergence of the method by showing that it is consistent, monotone, stable. For brevity, we will omit the discussion on the convergence of the above method, but refer readers to the aforementioned papers.

3.2. Discretization of the boundary and payoff conditions. The boundary and payoff conditions in (3.6) depend on the values of $V_0$ and $V_1$ at the mesh points on the boundaries of the solution domain. These values need to be calculated numerically using (2.10)–(2.13). Before considering approximations of (2.10)–(2.13), we first propose the following algorithm to determine $T^*_k$.

1. Let the reserve of the resource $Q \gg 0$ and $q_k(t)$ be given. Choose $\Delta t > 0$ sufficiently small. Set $j = 0$ and $Q_0 = 0$.
2. Evaluate $Q_{j+1} = Q_j + q_k((j + 1/2)\Delta t)\Delta t$.
3. If $Q_{j+1} \geq Q$, then set $T^*_k = (j + 1)\Delta t$ and stop. Otherwise, set $j = j + 1$ and go to Step 2.

In the above algorithm, we evaluate the integral in (2.10) using the mid-point quadrature rule until the total reserve $Q$ is reached approximately. From the previous discussion we see that $T^* = \max\{T_0^*, T_1^*\} = T_0^*$ and $q_1$ is defined in (2.11).

To determine the payoff condition $W_0^0, i = 1, 2, ..., M - 1$ on (3.6), we need to solve (2.13) numerically to find approximations to $\hat{W}(P_t, T)$ for $i = 1, ..., M - 1$. To achieve this, we further divide $(T, T^*)$ into $N^*$ subintervals with the nodes $T^* = \tau_0 > \tau_1 > \cdots > \tau_{N^*} = T$ and solve (2.13) on the mesh with nodes $(P_i, \tau_j)$ for $i = 0, 1, ..., M$ and $j = 0, 1, ..., N^*$ using the discretization scheme (3.7) with the terminal and boundary conditions

$$
\hat{W}_i^{N^*} = 0, \quad W_0^j = \hat{V}_{i,0}^j - \hat{V}_0^j, \quad \hat{W}_M^j = \hat{V}_{1,M}^j - \hat{V}_0^j,
$$

for $i = 0, 1, ..., M$ and $j = 1, 2, ..., N^*$, where $\hat{V}_{k,i}$ is the following approximation to $V_k(P_t, \tau_j)$ defined in (2.12):

$$
\hat{V}_{k,i}^j = \sum_{l=1}^{j} (\tau_{l-1} - \tau_l)D_k (P_t, \tau_{l-1}) e^{-(\tau_{l-1} - \tau_l)}
$$

for $i = 0$ and $M$ with $\tau_{l-1} = (\tau_{l-1} + \tau_l)/2$. The above scheme is based on the composite mid-point quadrature rule for (2.12). It is easy to verify that the above expression can also be rewritten as the following recursive relation:

$$
\hat{V}_{k,i}^0 = 0, \quad \hat{V}_{k,i}^j = \hat{V}_{k,i}^{j-1} e^{-(\tau_{j-1} - \tau_l)} + (\tau_{j-1} - \tau_l)D_k (P_t, \tau_{j-1}) e^{-(\tau_{j-1} - \tau_l)}
$$

(3.9)

for $j = 1, 2, ..., N^*$, $i = 0, M$ and $k = 0, 1$.

To determine the boundary condition $W_0^j$ for $j = 1, 2, ..., N$ in (3.6), we may use a formula similar to (3.9) which approximates (2.12). However, note that $D_0(P_t, t) = D_1(P_t, t)$ for $t \in [0, T]$. We may simply use the following formula to determine the boundary condition of $W_M^j$:

$$
W_M^j = \left(\hat{W}_M^{N^*} - K\right) e^{-(T - \tau_j)}, \quad j = 0, 1, ..., N,
$$

where $\hat{W}_M^{N^*} = \hat{V}_{1,M}^{N^*} - \hat{V}_0^{N^*}$ with $\hat{V}_0^{N^*}$ and $\hat{V}_{1,M}^{N^*}$ determined by (3.9).
4. Numerical experiments. We now solve two non-trivial iron-ore project expansion test problems using our models and numerical method developed in this paper to demonstrate that our methods are able to produce numerical solutions which are financially meaningful.

**Test 1: Option to expand**

This test problem is to price the value of an option to expand an investment project in the iron ore industry. All data for the option and project are given in Table 4.1.

To price the option, we choose \( P_{\text{max}} = 100 \) (USD) and using the algorithm in Subsection 3.2 we find \( T^* \approx 75 \) (years). Now, the problems (2.13) and (2.6)–(2.9) are solved on the uniform meshes with mesh sizes \( h = 1 \) and \( \Delta t = 0.02 \). The numerical solution for \( \kappa = 2 \) in Table 4.1 is depicted in Figure 4.1 in which the option price is in million USD. From the figure we see that this option as a function of \((P, t)\) behaves like a conventional option. To further investigate the behaviours of the option, we solve the problem on the meshes for various values of expansion rate \( \kappa \) and volatility \( \sigma \) and plot the numerical results at \( t = 0 \) in Figure 4.2. From Figure 4.2(a) we see that the value of the option is an increasing function of \( \kappa \), while Figure 4.2(b) demonstrates that, for a fixed \( \kappa \), the value of the option increases when \( \sigma \) increases from 0.35 to 0.7. However, as \( \sigma \) decreases from 0.35 to 0.1, the value of the option increases (respectively decreases) when \( P \) is larger (respectively smaller) than some critical values. This may represent the fact that the commodity price has more chance to decrease (respectively increase) for \( \sigma = 0.35, 0.55 \) and 0.7 than for \( \sigma = 0.2 \) and 0.1 when \( P \) is larger (respectively smaller) than some critical values. Thus, the option values when \( \sigma = 0.1 \) and 0.2 are higher (respectively lower) than those of the other cases when \( P \) is greater (respectively smaller) than some critical values.

**Test 2: Compound expansion option**

As described in Case C2 in Subsection 2.2, this compound expansion option consists of two sequential options – one exercisable at \( T_1 \) with strike price \( K_1 \) and one at \( T_2 > T_1 \) with strike price \( K_2 \). The latter option is available only if the former is exercised. In addition to the market and project constants and functions defined in Table 4.1, other required parameters and functions are listed in Table 4.2.

To solve this problem numerically, we choose \( P_{\text{max}} = 100 \) (USD). Note that the numerical solution of the problem is divided into three sequential sub-problems which are (2.13) for determining the payoff condition at \( t = T_2 \), (2.14) for \( W_p \) and (2.15) for \( W_c \). The sub-regions corresponding to the three sub-problems are partitioned into uniform meshes with mesh sizes \((h, \Delta t) = (1, 0.02)\).

The computed value \( W_c \) of the compound option for \( t \in (0, T_1) \) is displayed in Figure 4.3(a). For comparison, we also compute the value \( W \) of the normal option with maturity \( T = 2 \), \( \kappa = 4 \) and strike price \( K = 3 \times 10^4 \) (million USD) and plot the
result in Figure 4.3(b). Both of the expansion options require the same amount of capital. However, in the compound option case, the investment is in two stages. The difference $W_c - W$ in the region $(0, P_{\text{max}}) \times (0, 2)$ is depicted in Figure 4.4(a) from which we see that the compound option is more valuable than the normal one when $P$ is around the average cost function $C(t)$ and cheaper than the normal option when $P$ is large. To further demonstrate this, we plot the values of the two options and their difference at $t = 0$ in Figure 4.4(b). From the figure we see that $W_c \approx W$ when $P < 20$, $W_c > W$ when $P$ is roughly between 20 to 60 and $W_c < W$ when $P > 60$. This observation is meaningful as the compound option has the flexibility for investors to avoid risks by not to exercise the 2nd component of option at $T_2$ when $P$ is in a certain range near $C_0$ (in this case $P \in (20, 60)$) in which the return of the investment is very uncertain due to commodity price changes. Therefore, $W_c$ is more valuable than $W$ in a range. However, when $P$ is large, the normal option
is more valuable than the compound one which is financially true as in this case investor should increase the production rate as soon as possible.

![Figure 4.3. Computed values of compound and normal options.](image)

![Figure 4.4. Computed option values and their differences for Test 2.](image)

Finally, we comment that since the exact solutions to the above test problems are unknown, we are unable to calculate the computational errors from our methods. However, demonstrated above, the numerical results produced by our approach are financially very meaningful.

5. **Conclusion.** In this work we have proposed new PDE-based models to price real and compound real options to expand investment projects when the underlying asset price follows geometric Brownian motion. An upwind finite difference method has been proposed to numerically solve the pricing models. We have also
proved that the system matrix from the finite difference method is an $M$-matrix and thus numerical solutions are non-negative. Numerical experiments have been performed to demonstrate that the models and numerical methods produce financially meaningful results. Through our case studies we have found that the value of an expansion option of a project is affected notably by the volatility and the expansion rate. Also, a compound option on a project expansion is more valuable than the corresponding normal one when the commodity price is not far from the average unit operating cost.

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