A COLIMIT OF TRACES OF REFLECTION GROUPS

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Abstract. Li-Nadler proposed a conjecture about traces of Hecke categories, which implies the semistable part of the Betti Geometric Langlands Conjecture of Ben-Zvi-Nadler in genus 1. We prove a Weyl group analogue of this conjecture. Our theorem holds in the natural generality of reflection groups in Euclidean or hyperbolic space.

1. Introduction

Let $W$ be a reflection group in Euclidean or hyperbolic space. For $I$ a facet, denote by $W_I$ the subgroup fixing $I$. For $C$ a chamber, denote by $\mathcal{F}_C$ be the category (or poset) of faces in $\overline{C}$ ($:= \text{the closure of } C$). We view $W$ (with discrete topology) as an algebra object in $\mathcal{S} := \text{the } \infty\text{-category of topological spaces}$, and denote its trace by $\text{Tr}(W) \in \mathcal{S}$. Our main theorem is:

**Theorem 1.1.** The natural map

$$
\colim_{I \in \mathcal{F}_C} \text{Tr}(W_I) \longrightarrow \text{Tr}(W)
$$

is fully-faithful.

The proof uses Lurie’s $\infty$-categorical Seifert-Van Kampen theorem and basic properties of reflection groups.

When $W = W_{\text{aff}}$ the affine Weyl group of a simply-connected reductive group $G$, Theorem 1.1 confirms a Weyl group analogue of the following conjecture in [LN15]. Let $G$ be a simply-connected reductive algebraic group, $LG$ the loop group of $G$. Let $C$ be an affine alcove. For each facet $I$ of $C$, denote by $G_I$ the Levi of the parahoric subgroup of $LG$ corresponding to $I$. Let $\mathcal{H}_I$ be the Hecke category of $G_I$, and $\mathcal{H}_{\text{aff}}$ be the affine Hecke category.

**Conjecture 1.2 ([LN15, Claim 1.12]).** The natural map of $\infty$-categories

$$
\colim_{I \in \mathcal{F}_C} \text{Tr}(\mathcal{H}_I) \longrightarrow \text{Tr}(\mathcal{H}_{\text{aff}})
$$

is fully-faithful.

This conjecture comes from the consideration of Geometric Langlands. Roughly speaking, the Betti Geometric Langlands Conjecture [BZN16] predicts the equivalence of two ($\infty$)-categories: the automorphic category $A_g$ and the spectral category $B_g$. As explained in [LN15], the above conjecture implies that for genus $g = 1$, one can embed the semistable automorphic category $A_{1,ss} \subset A_1$ fully-faithfully into $B_1$, and hence implies this part of Geometric Langlands. Note that $W_I$ is the Weyl group of $G_I$, Weyl groups are specializations of Hecke algebra, and Hecke algebras are decategorifications of Hecke categories. Hence Theorem 1.1 confirms an easier analogue of Conjecture 1.2.

**Remark 1.3.** (1) For a topological group $G$ acting on a topological space $X$, we denote by $X/G$ the topological space $X \times_G EG$, where $EG$ is a contractible space with free $G$ action. It is not hard to see that $\text{Tr}(G) \simeq G/G$, for the adjoint action of $G$ (Proposition 2.3). Denote $\bullet$ a single point. It is known that $\colim_{I \in \mathcal{F}_C} \frac{\bullet}{W_I} \simeq \bullet/W$ (see e.g. [Li18]). This equivalence sits inside Theorem 1.1 via the commutative diagram

$$
\begin{array}{ccc}
\colim_{I \in \mathcal{F}_C} \frac{\bullet}{W_I} & \sim & \frac{\bullet}{W} \\
\downarrow & & \downarrow \\
\colim_{I \in \mathcal{F}_C} \frac{W_I}{W_I} & \longrightarrow & \frac{W}{W}
\end{array}
$$
where the vertical maps take $\bullet$ to 1. A similar statement of the top equivalence for the Bruhat-Tits building was used to prove that the representation category of a $p$-adic group has global dimension $\leq \dim(C)$ (see e.g., Bernstein’s lectures on Representation of $p$-adic groups). It may be interesting to see the meaning of bottom arrow in $p$-adic representation theory.

(2) We can get the map in Theorem 1.1 by applying $\text{Map}(S^1, -)$ to the equivalence $\text{colim} \bullet / W_l \simeq \bullet / W$. However, this resulting map is no longer an equivalence in general. This reflects the fact that $\text{Map}(S^1, -)$ does not preserve colimits. I.e., the loop space are not calculated locally. To see a concrete example when the surjectivity fails: take $W = W_{\text{aff}}$, then $\pi_0(LHS)$ is finite but $\pi_0(RHS) \supset \{\text{dominant coweights}\}$, which is infinite.

**Example 1.4.** We gives some examples of Theorem 1.1. Denote by $\mathcal{S}_n$ the symmetric group on $n$ letters,

(1) $W$ is the Weyl group of a reductive algebraic group $G$. Then $\mathcal{P}^p_C$ has a final object $O$ the origin, and $W_O = W$. Hence Theorem holds trivially since the LHS is also $\text{T}r(W)$.

(2) $W$ is the affine Weyl group of $SL_2$. $\mathcal{P}^p_C$ is the category $\bullet \leftarrow \bullet \rightarrow \bullet$, $\mathcal{S}_2 / \mathcal{S}_2 \simeq \bullet / \mathcal{S}_2 \amalg \mathcal{S}_2 \bullet / \mathcal{S}_2$

$LHS = \text{colim}$

$\mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2$

$\simeq \text{colim}$

$\bullet / \mathcal{S}_2 \amalg \bullet / \mathcal{S}_2 \amalg \bullet / \mathcal{S}_2 \amalg \bullet / \mathcal{S}_2 \amalg \bullet / \mathcal{S}_2$

$\simeq \bullet / W \amalg (\bullet / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2) \amalg (\bullet / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2) \amalg (\bullet / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2) \amalg (\bullet / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2) \rightarrow W / W$

The image is the full subgroupoid consist of objects $1, s_1, s_0$, for $s_1, s_0$ two simple reflections in $W$. The map being fully-faithful reflects the fact that $s_1$ and $s_0$ are not conjugate in $W$, and the centralizer of each is $\mathcal{S}_2$.

(3) $W$ is the affine Weyl group of $SL_3$. Note that $\mathcal{S}_3 / \mathcal{S}_3 \simeq \bullet / \mathcal{S}_3 \amalg \bullet / \mathcal{S}_2 \amalg \bullet / (\mathcal{Z} / 3)$.

$LHS = \text{colim}$

$\mathcal{S}_3 / \mathcal{S}_3 \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2$

$\simeq \text{colim}$

$\bullet / \mathcal{S}_3 \amalg \bullet / \mathcal{S}_3 \amalg \bullet / \mathcal{S}_3 \amalg \bullet / \mathcal{S}_3 \amalg \bullet / \mathcal{S}_3$

$\simeq (\bullet / \mathcal{S}_2 \times \partial C) \amalg (\bullet / (\mathcal{Z} / 3)) \amalg (\bullet / (\mathcal{Z} / 3))$.

The second factor $\bullet / \mathcal{S}_2 \times \partial C$ can be identified as the full subgroupoid in $W / W$ consists of reflections: let $s_I \in W$ be the reflection corresponding to a face $I$ of $C$. The centralizer $C_W(s_I) \simeq < s_I > \times X_*(\mathcal{G}_I) \simeq \mathcal{S}_2 \times \mathcal{Z}$. Hence the subgroupoid at $s_I$ is equivalent to $\bullet / \mathcal{S}_2 \amalg \bullet / G \simeq \bullet / \mathcal{S}_2 \times S^1 \simeq \bullet / \mathcal{S}_2 \times \partial C$. Also note that all reflections are conjugate in this case.

(4) Let $W$ be the triangle group $(2, 3, \infty)$. It is a reflection group in hyperbolic plane.

$LHS = \text{colim}$

$\mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_2 / \mathcal{S}_2$

$= \text{colim} \{ (\mathcal{S}_2 / \mathcal{S}_2) / (\mathcal{S}_2 / \mathcal{S}_2) \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_3 / \mathcal{S}_3 \}$

$= (\mathcal{S}_2 / \mathcal{S}_2) / (\mathcal{S}_2 / \mathcal{S}_2) \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_3 / \mathcal{S}_3 \}$

$= (\mathcal{S}_2 / \mathcal{S}_2) / (\mathcal{S}_2 / \mathcal{S}_2) \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_3 / \mathcal{S}_3 \}$

$= (\mathcal{S}_2 / \mathcal{S}_2) / (\mathcal{S}_2 / \mathcal{S}_2) \amalg \mathcal{S}_2 / \mathcal{S}_2 \amalg \mathcal{S}_3 / \mathcal{S}_3 \}$

$= \bullet / (\mathcal{Z} / 3)$

We see that in this case, $C_{W_I}(w) = C_W(w)$, for any $w \in W_I$. 

2. Preliminaries

2.1. Discrete groups generated by reflections. References for this section are [Bou02, V] and [Vin88]. Let $X$ be an Euclidean space $\mathbb{E}^n$ or hyperbolic space $\mathbb{H}^n$. Let $\mathcal{H}$ be a collection of hyperplanes in $X$. Let $W$ be the group generated by the orthogonal reflection along the hyperplanes $H \in \mathcal{H}$. Assume that:

1. For any $w \in W$ and $H \in \mathcal{H}$, we have $w(H) \in \mathcal{H}$.
2. $W$ provided with discrete topology, acts properly on $X$.

Given two points $x$ and $y$ of $X$, denote by $R(x, y)$ the equivalence relation:

For any hyperplane $H \in \mathcal{H}$, either $x \in H$ and $y \notin H$ or $x$ and $y$ are strictly on the same side of $H$.

**Definition 2.1.** (1) A facet of $X$ is an equivalence class of the equivalence relation defined above.

2. For two facets $I, J$, denote $I \leq J$ if $I \subset \overline{J}$. Then $\leq$ defines a partial order on the set of facets.

3. A chamber of $X$ is a facet $C$ that is maximal along this partial order.

4. For any facet $J$, denote $\mathcal{T}_J$ be the category corresponding to the poset $\{I | I \leq J\}$.

5. The star of $I$ is $X_I := \bigcup_{\{I \leq J\}} J \subset X$; and $W_I := \{w \in W : \text{w}|_I = \text{id}\}$.

**Proposition 2.2.** (1) A facet is a polytope.

2. For any chamber $C$, the closure $\overline{C}$ of $C$ is a fundamental domain for the action of $W$ on $X$, i.e., every orbit of $W$ in $X$ meets $\overline{C}$ in exactly one point.

3. For $I$ a facet, the group $W_I$ is generated by the reflections fixing $I$.

4. $W_I$ acts on $X_I$ with fundamental domain $X_I \cap \overline{C}$.

**Proof.** (1) by definition since each facet is a intersection of hyperplanes and half spaces. (2) See [Bou02, V.3.3 Theorem 2]. (3) See [Bou02, V.3.3 Prop 2]. (4) Let $J \geq I$ be a facet in $X_I$, then $w(J) \geq w(I) = I$, hence $w(J) \subset X_I$, and therefore $W_I$ is a reflection group, hence by (2), $X_I \cap \overline{C}$ is a fundamental domain.

2.2. Traces of algebras. Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category such that all colimit exist. Let $A \in \text{Alg}(\mathcal{C})$ be an algebra object in $\mathcal{C}$. The trace (or Hochschild homology) of $A$ is by definition $\text{Tr}(A) := A \otimes_{A \otimes A^{op}} A \in \mathcal{C}$. We view $\mathcal{S}$ as a symmetric monoidal $\infty$-category, where $\otimes$ is given by the Cartesian product.

**Proposition 2.3.** Let $G$ be a topological group. View $G \in \text{Alg}(\mathcal{S})$, then $\text{Tr}(G) \simeq G/G := G \times_G EG$, where $G$ acts on $G$ by conjugation.

**Proof.** We have an isomorphism $G \simeq G \otimes G^{op} \otimes_G \bullet$ as $G \otimes G^{op}$ modules, where $G$ maps to $G \otimes G^{op}$ diagonally. Then $G \otimes_{G \otimes G^{op}} G \simeq G \otimes_{G \otimes G^{op}} (G \otimes G^{op} \otimes G^{op}) \simeq G \otimes_G \bullet \simeq G/G$, where the action of $G$ on $G$ is the conjugation.

2.3. Topological groupoid and open descent. We denote a topological groupoid $\mathcal{G}$ to be the data consist of a discrete group $G$ acting properly discontinuously on a topological space $Y$. And we use the notation $\mathcal{G} = [Y/G]$. Let $G' = [Y'/G']$ be another topological groupoid. A morphism $F : \mathcal{G} \to G'$ consists of the data $(f, \varphi)$, where $f : Y \to Y'$ continuous maps and $\varphi : G \to G'$ injective homomorphisms, such that $f(a \cdot y) = \varphi(a) \cdot f(z)$, for all $a \in G, y \in Y$. We denote TopGrpd the category of topological groupoids. A morphism $F$ is an open embedding if the induced map $Y \times_G G' \to Y'$ is an open embedding. We denote $\underline{Y}$ the underline set of $Y$, and by $\underline{\mathcal{G}} := \underline{Y}/G \in \mathcal{S}$ the underline $\infty$-groupoid of points (recall that $G$ is assumed to be discrete). And define $\mathcal{G}_h := \underline{Y}/G \in \mathcal{S}$ the homotopy type of $\mathcal{G}$.

**Proposition 2.4.** Let $F : \mathcal{G} \to G'$, assume that $f$ is open embeddings and the induced map $\underline{F} : \underline{\mathcal{G}} \to \underline{G'}$ is fully-faithful. Then $F$ is an open embedding.

**Proof.** The base changed map $Y \times_G G' \to Y'$ is a local homeomorphism since $G$ and $G'$ are discrete. The underline set map $\underline{Y} \times_G G' \to \underline{Y}'$ is fully-faithful (i.e injective), because fully-faithful map between groupoid are stable under base change. These imply the map $Y \times_G G' \to Y'$ is an open embedding.

For $\mathcal{I}$ a category, we denote $\mathcal{I}^+$ the category by adding one final object $*$ to $\mathcal{I}$. We say a functor $K : \mathcal{I}^+ \to \mathcal{J}$ is a colimit diagram if the induced map colim$K|_\mathcal{I} \to K(*)$ is an isomorphism.
Proposition 2.5 (∞-categorical Seifert-van Kampen theorem for topological groupoids). Let $K : \mathcal{P} \to \text{TopGrpd}$ be a functor, assume that all arrows in $\mathcal{P}$ go to open embeddings, and the induced functor $\underline{K} : \mathcal{P} \to \mathcal{S}$ is a colimit diagram. Then the induced functor $K_h : \mathcal{P} \to \mathcal{S}$ is a colimit diagram.

Proof. By base change, one can assume that $K$ takes value in $\text{Top}$ the category of topological spaces. Then this is the ∞-categorical Seifert-van Kampen theorem [Lur12, Theorem A.3.1]. Note that the condition $(\ast)$ loc. cit. is equivalent to the condition on $\underline{K}$. 

Remark 2.6 (Topological groupoids as topological stacks). Denote $\text{Top}$ the category of topological space with continuous map. One can define a topological stack as a functor $X : \text{Top} \to \mathcal{S}$, satisfying certain descent and representibility conditions. Then Yoneda embedding gives $\iota : X \to \text{TopStack}$. One can define embedding $\iota' : \text{TopGrpd} \to \text{TopStack}$, via $[Y/G] \to \text{colim}_{w \in W} \mathcal{S}(\underline{G} \times \bullet \times Y)$. In this case, $\mathcal{G} = \iota'(\mathcal{G})(\ast)$ and $\mathcal{G}_h$ is also the homotopy type of $\iota'(\mathcal{G})$. Proposition 2.5 is most naturally presented in the context of topological stacks (with local homeomorphisms), but we shall not use this generality.

3. Proof of main theorem

For any $w \in W$, let $X^w$ be the fixed locus of $w$. For $I$ a facet, put $X^w_I := X_I \cap X^w$. Let $W^f := \{ w \in W : w(I) = I, \text{for some facet } I \}$. We define a functor $K : \mathcal{P}^{op} \to \text{TopGrpd}$, by $I \mapsto \left(\coprod_{w \in W^f} X^w_I\right)/W_I$, and $\ast \mapsto \left[\left(\coprod_{w \in W^f} X^w\right)/W\right]$.

Lemma 3.1. $K(I) = \coprod_{\{J \geq I\}} (J \times W_J)/W_J$, and $K(\ast) = \coprod_{\{J \geq I\}} (J \times W_J)/W_J$.

Proof. As sets, we have $X^w_I = \coprod_{\{J \geq I\}} (J \times W_J)$, and $\coprod_{w \in W_I} X^w_I = \coprod_{w \in W_I} \coprod_{\{J \geq I\}} (J \times W_J) = \coprod_{\{J \geq I\}} \left(\coprod_{w} (J \times W_J)\right) = \coprod_{\{J \geq I\}} (J \times W_J)/W_J$. Hence $K(I) = \coprod_{\{J \geq I\}} (J \times W_J)/W_J = \coprod_{\{J \geq I\}} (J \times W_J)/W_J$, where the last equality is by Proposition 2.2 (2). The second statement follows from similar argument.

Lemma 3.2. $(1)$ For any $I' \to I$ in $\mathcal{P}^{op}$, $K(I') \to K(I)$ is fully-faithful.

$(2)$ $K$ is a colimit diagram.

Proof. $(1)$ One check that under the identification in Lemma 3.1, the map $K(I') \to K(I)$ is induced by the inclusion of indexing sets $\{J | C \geq J \geq I'\} \to \{J | C \geq J \geq I\}$.

$(2)$ For any $J \leq C$, define $K_J : \mathcal{P}^{op} \to \mathcal{S}$ by

$$(3.3) \quad K_J(I) := \begin{cases} (J \times W_J)/W_J, & \text{if } I \leq J \\ \emptyset, & \text{otherwise} \end{cases}$$

We see that $\text{colim}_{\mathcal{P}^{op}} K_J \simeq |\mathcal{P}| \times (J \times W_J)/W_J \simeq (J \times W_J)/W_J \simeq K_J(\ast)$. The second equivalence follows from the fact that the geometric realization $|\mathcal{P}| \simeq J$ is contractible. Hence $K_J$ is a colimit diagram, and $K \simeq \coprod_{J \leq C} K_J$ is also a colimit diagram.

Proof of Theorem 1.1. By Proposition 2.3, it is equivalent to show the natural map $\text{colim}_{I \in \mathcal{P}} W_I/W_I \to W/W$ is fully-faithful. We claim the functor $K$ satisfies the assumption of Proposition 2.5. We first show that all arrows in $\mathcal{P}^{op}$ goes to open embeddings. For any $I' \geq I$, the natural map $\coprod_{w \in W_I} X^w_I \to \coprod_{w \in W_{I'}} X^w_I$ is an open embedding. Hence by Proposition 2.4, and Lemma 3.2 (1), $K(I') \to K(I)$ is an open embedding. And $\underline{K}$ is a colimit diagram by Lemma 3.2 (2). Hence we conclude that $K_h$ is a colimit diagram by Proposition 2.5. Now we have a commutative diagram in $\mathcal{S}$:

$$\begin{array}{ccc}
\text{colim}_{I \in \mathcal{P}} (\coprod_{w \in W_I} X^w_I)/W_I & \simeq & (\coprod_{w \in W_I} X^w_I)/W \\
\downarrow & & \downarrow \\
\text{colim}_{I \in \mathcal{P}} W_I/W_I & \xrightarrow{p} & W_I/W \\
\downarrow & \overset{q}{\nearrow} & \downarrow i \\
W/W & & 
\end{array}$$
In the top square, the top horizontal arrow is an equivalence from the definition of $K_h$ being a colimit diagram. The two vertical arrows are given by $X^w_I (\text{resp. } X^w) \mapsto \{w\}$, hence they are equivalences since $X^w_I$ and $X^w$ are contractible. We conclude that $p$ is an equivalence. Now $i$ is fully-faithful by definition, hence $q$ is fully-faithfully.

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\begin{thebibliography}{99}
\bibitem[Bou02]{Bou02} Nicolas Bourbaki. Lie groups and Lie algebras. chapters 4–6. translated from the 1968 french original by andrew pressley. elements of mathematics, 2002.
\bibitem[BZN16]{BZN16} David Ben-Zvi and David Nadler. Betti geometric langlands. \textit{arXiv preprint arXiv:1606.08523}, 2016.
\bibitem[Li18]{Li18} Penghui Li. Derived categories of character sheaves. \textit{arXiv preprint arXiv:1803.04289}, 2018.
\bibitem[LN15]{LN15} Penghui Li and David Nadler. Uniformization of semistable bundles on elliptic curves. \textit{arXiv preprint arXiv:1510.08762}, 2015.
\bibitem[Lur12]{Lur12} Jacob Lurie. Higher algebra, 2012.
\bibitem[Vin88]{Vin88} Ernest B Vinberg. Geometry. ii, encyclopaedia of mathematical sciences, 29, 1988.
\end{thebibliography}

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