Linear irreversible heat engines based on local equilibrium assumptions

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Abstract

We formulate an endoreversible finite-time Carnot cycle model based on the assumptions of local equilibrium and constant energy flux, where the efficiency and the power are expressed in terms of the thermodynamic variables of the working substance. By analyzing the entropy production rate caused by the heat transfer in each isothermal process during the cycle, and using the endoreversible condition applied to the linear response regime, we identify the thermodynamic flux and force of the present system and obtain a linear relation that connects them. We calculate the efficiency at maximum power in the linear response regime by using the linear relation, which agrees with the Curzon–Ahlborn (CA) efficiency known as the upper bound in this regime. This reason is also elucidated by rewriting our model into the form of the Onsager relations, where our model turns out to satisfy the tight-coupling condition leading to the CA efficiency.

1. Introduction

The physics of heat engines originates from the Carnot’s great discovery of the fundamental upper bound of the thermodynamic efficiency $\eta$ of heat engines working between two heat reservoirs with temperatures $T_h$ and $T_c$ ($T_h > T_c$) [1–3]:

$$\eta \leq 1 - \frac{T_c}{T_h} \equiv \eta_C \text{ (Carnot efficiency)},$$

where the equality holds for an infinitely slow process (quasistatic limit) with zero dissipation realized in, e.g., the Carnot cycle. Indeed, this discovery may be regarded as the origin of thermodynamics itself. However, the quasistatic limit is an ideal case, and the thermodynamic processes observed in daily life occur at finite rates. Remembering that we always demand power for our use of electric devices, which may originally be generated from power plants converting heat flux into electric power, we require the physics of powerful heat engines free from the limitation of the equilibrium thermodynamics. This deep understanding of powerful heat engines is becoming more important due to the worldwide energy crisis and climate change.

The physics of heat engines maximizing the power rather than the efficiency was developed in a classical paper by Curzon and Ahlborn [4]. They showed that, under an assumption of the endoreversible condition and the Fourier law of heat conduction between the working substance and the heat reservoir, the efficiency at maximum power $\eta^*$ of a finite-time Carnot cycle is given by the following Curzon–Ahlborn (CA) efficiency:

$$\eta^* = 1 - \sqrt{\frac{T_c}{T_h}} \equiv \eta_{CA} \approx \frac{\Delta T_R}{2 T^R} \approx \frac{\Delta T_R^2}{8 T^R} + \cdots,$$

where we have defined the temperature difference between the heat reservoirs as $\Delta T_R \equiv T_h - T_c$ and the averaged temperature between the heat reservoirs as $T^R \equiv \frac{T_h + T_c}{2}$. We note that the same formula was derived...
more previously in, e.g. [5, 6] in the 1950s, and it is claimed that its origin even goes back to a paper from the 1920s according to a recent work [7]. Because this CA efficiency displays a similar simplicity to the Carnot efficiency, it led to the development of a new discipline of finite-time thermodynamics that aims to account for the efficiency of actual power plants and thermal devices [8–14]. The key to the derivation of the CA efficiency is the phenomenological assumption of the endoreversible condition (as named by Rubin [8] later), which means that the irreversibility occurs only by the heat transfer process between the working substance and the heat reservoir, and that the state of the working substance is internally reversible whose entropy change along the cycle is expressed by a Clausius-like equality (see equation (15)). Under this condition, the efficiency of the finite-time Carnot cycle is given by the Carnot efficiency using the ratio of the temperatures of the working substance (see equation (14)), and the efficiency at maximum power $\eta^*$ is expressed by using the square root of the temperature ratio $T_2/T_1$ as in equation (2) [4].

Despite its importance, even until recently, there has been no argument showing whether the CA efficiency is universal as $\eta^*$ from the viewpoint of fundamental physics. The role of the CA efficiency has become increasingly important after van den Broeck [15] proved that the CA efficiency is the upper bound of $\eta^*$ in the linear response regime by using the Onsager relations of the linear irreversible thermodynamics framework [16–18]:

$$\eta^* \leq \frac{\Delta T^R}{2T^R} = \eta_{CA} + O\left(\Delta T^{R^2}\right),$$

where the bound is realized under the tight-coupling (no heat-leakage) condition [15]. Because the Carnot efficiency equation (1) can be approximated as $\eta_{C} \approx \frac{\Delta T^R}{2T^R}$ in the linear response regime, it is also equivalent to say that $\eta^*$ in the linear response regime is bounded from above by one half of the Carnot efficiency as $\eta^* \leq \frac{\eta_{C}}{2}$. As reviewed in [19–22], since the paper by van den Broeck [15], various studies on finite-time heat engines have been conducted, which include linear response [23–30], nonlinear response [31–40], stochastic [41–43], quantum [44–49], thermoelectric [50–52], photoelectric [53], molecular dynamics [54–57], and experimental [58] studies.

In [15], van den Broeck established a view that the process of heat energy conversion into work in the linear response regime is ruled by a cross effect based on the Onsager relations:

$$J_1 = L_{11}X_1 + L_{12}X_2,$$

$$J_2 = L_{21}X_1 + L_{22}X_2,$$

where $X_1$ is an ‘external’ thermodynamic force and $X_2$ is a ‘thermal’ thermodynamic force that is proportional to $\Delta T^R$, $J_1$ and $J_2$ are their conjugate thermodynamic fluxes, and $L_{ij}$’s are the Onsager coefficients with reciprocity $L_{ij} = L_{ji}$ (see section 2.3 for details). While this viewpoint is familiar in steady-state heat energy conversion, such as in thermoelectric devices usually analyzed with the Onsager relations [1, 22, 49, 50], an identical formulation has also been established even for cyclic heat engines such as a finite-time Carnot cycle [27, 28].

Despite these successes, these theories of finite-time heat engines may still be abstract compared to the theory of the quasistatic heat engine. One reason could be that a general state of a finite-time heat engine cannot be drawn on a thermodynamic plane, thus a clear picture is lacking, unlike the quasistatic cycle with well-defined thermodynamic variables of the working substance such as the pressure, temperature, and so on. Even in a finite-time cycle, however, it may still be possible to assume that the state of the working substance is specified by a unique combination of the thermodynamic variables at any instant along the cycle, and hence the working substance and the heat reservoirs are in a local equilibrium, but not in a global equilibrium similar to the quasistatic cycle. According to this local equilibrium assumption, we can draw the finite-time cycle on the thermodynamic plane as well as the quasistatic cycle, and can also assume that the fundamental thermodynamic relation holds between the thermodynamic variables of the working substance during a finite-time cycle. In fact, Rubin introduced this local equilibrium thermodynamic description to the endoreversible cycle very previously [8] (see also [36]), where the endoreversible condition is expressed by the entropy change of the working substance during one cycle. However, this local equilibrium assumption has not fully been taken into account in the recent literatures, and hence we are naturally motivated to elucidate how these endoreversible heat engine models based on the local equilibrium assumption are unified with the more recent linear irreversible thermodynamic description using the Onsager relations equations (4) and (5).

In the present study, we formulate an endoreversible finite-time Carnot cycle model based on the assumptions of local equilibrium and constant energy flux. In our framework, the power and the efficiency can be expressed in terms of the thermodynamic variables of the working substance. From the analysis of the entropy production rate caused by the heat transfer in each isothermal process during the cycle, we identify the thermodynamic flux and force in each isothermal process, where the flux and force are assumed to be related by the Fourier law. We also find that, due to the endoreversible condition applied to the linear response regime, these thermodynamic forces in the isothermal processes are not independent, and we can identify the reduced thermodynamic force, whose conjugate thermodynamic flux turns out to be an averaged heat flux. We then
obtain a linear relation that connects these thermodynamic flux and force. From the calculation of the efficiency at maximum power by using this linear relation, we obtain the CA efficiency as \( \eta^* \) of the present model. Then, suitably changing the variables, we elucidate that the linear relation in our framework can be rewritten into the form of the Onsager relations, from which we can directly confirm the tight-coupling condition leading to the CA efficiency in the framework of \([15]\). Therefore our work establishes a precise connection between the finite-time thermodynamic approach to heat engines by Curzon and Ahlborn \([4]\) and the linear irreversible thermodynamic approach based on the Onsager relations \([15]\) via the local equilibrium assumption that gives us a more intuitive picture of the finite-time heat engines.

The present paper is organized as follows. In section 2.1, we introduce our model based on the assumptions of local equilibrium and constant energy flux. In section 2.2, by analyzing the entropy production rate of our heat engine with the aid of the endoreversible condition applied to the linear response regime, we naturally identify the thermodynamic flux and force and a linear relation that connects them. We then calculate the efficiency at maximum power in the linear response regime by using this linear relation. In section 2.3, we elucidate the relationship between the linear relation obtained in section 2.2 and the Onsager relations equations (4) and (5), explicitly showing that our model surely satisfies the tight-coupling (no heat-leakage) condition leading to the CA efficiency. In section 3, we discuss a few aspects related to our formulation in section 2, and summarize our study.

2. Model and results

2.1. Local equilibrium thermodynamic formulation of the endoreversible finite-time Carnot cycle

Our heat engine model consists of the working substance, the hot heat reservoir with temperature \( T^h \), and the cold heat reservoir with temperature \( T^c \). We assume that the working substance is always in a local equilibrium state specified by a unique combination of the well-defined thermodynamic variables, and the heat reservoirs are also in a local equilibrium state. Denoting the internal energy and the entropy of the heat reservoir by \( U^h \) and \( S^h \) (\( i = h, c \)), respectively, the temperature is defined by \( \frac{1}{T^i} \equiv \frac{\partial S^i}{\partial U^i} \). We also denote the internal energy and entropy of the working substance by \( U \) and \( S \), respectively. Hereafter, we use the suffix \( i \) to denote the thermodynamic variable of the working substance when it contacts with the heat reservoir with the temperature \( T^i \). We then obtain the first law of thermodynamics (energy-conservation law) by using these thermodynamic variables as \([8]\)

\[
-\frac{dU^i}{dt} = \frac{dU}{dt} + P \frac{dV}{dt},
\]

where we define \( \frac{1}{\tilde{U}^i} \equiv \frac{\partial S^i}{\partial U^i} \) and \( \frac{P}{\tilde{V}} \equiv \frac{\partial S}{\partial P} \) through the fundamental thermodynamic relation

\[
dU_i \left( S_i, V_i \right) = \frac{\partial U_i}{\partial S_i} dS_i + \frac{\partial U_i}{\partial V_i} dV_i = T_i dS_i - P_i dV_i,
\]

with \( T_i, P_i, \) and \( V_i \), being the temperature, pressure, and volume of the working substance, respectively. Equation (6) states that the heat flux from the heat reservoir \(- \frac{dU^i}{dt}\), which is the internal-energy change rate of the heat reservoir, is decomposed into the internal-energy change rate of the working substance \( \frac{dU}{dt} \) and the instantaneous power output \( P \frac{dV}{dt} \).

Our heat engine experiences a thermodynamic cycle that consists of (I) an isothermal expansion process in contact with the hot heat reservoir with the temperature \( T^h \), (II) an adiabatic expansion process, (III) an isothermal compression process in contact with the cold heat reservoir with the temperature \( T^c \), and (IV) an adiabatic compression process (see figure 1(a)). We assume that we can take the durations of the adiabatic processes to be sufficiently short compared to the ones of the isothermal processes in a finite-time cycle, and that the thermodynamic states of the working substance move along the quasistatic adiabatic curves in the thermodynamic plane, while the thermodynamic states of the working substance during the isothermal processes do not agree with the quasistatic isothermal curves. This assumption is quite natural because relaxation to the equilibrium state during the adiabatic process is occurred by internal processes inside the working substance itself, which is much faster than the speed of relaxation to the (global) equilibrium during the isothermal process determined by interactions with the heat reservoir. From this, our heat engine can be regarded to run one cycle in the cycle time \( t_{\text{cycle}} \equiv t_h + t_c \), where we denote by \( t_i \) the duration of the isothermal process in contact with the heat reservoir at \( T^i \). Additionally, in our formulation, we also assume that the energy flux corresponding to each term in equation (6) is constant at any instant along each isothermal process, and the temperature \( T_i \) does not change during the isothermal process, where these assumptions are also adopted in the original CA model \([4]\) (see section 3 for a discussion on this assumption). Using equation (7), we can rewrite
equation (6) as
\[-\frac{dU^R}{dt} = T_i \frac{dS_i}{dt}.\]  

The heat from the heat reservoir during the isothermal process \(Q_i\) is calculated by using equation (8) as
\[Q_h \equiv -\int_0^{t_i} \frac{dU^R}{dt} dt = T_h \Delta S_h,\]
\[Q_c \equiv -\int_0^{t_c} \frac{dU^R}{dt} dt = T_c \Delta S_c,\]

where we defined the entropy change of the working substance during the isothermal process as
\[\Delta S_h \equiv \int_0^{t_h} \frac{dS_h}{dt} dt, \quad \Delta S_c \equiv \int_0^{t_c} \frac{dS_c}{dt} dt.\]  

Because we require that the cycle is closed after one cycle and we also assume that the adiabatic processes are regarded as quasistatic processes, the following relations should hold:
\[\Delta S_h = -\Delta S_c \equiv \Delta S.\]  

We note that the endoreversibility condition [4, 8]
\[\frac{Q_h}{T_h} + \frac{Q_c}{T_c} = 0\]  

automatically holds from equations (9), (10) and (12). This condition manifests that the entropy production is occurred only by the heat transfer between the heat reservoir and the working substance, and the internal state of the working substance is, as assumed above, in a local equilibrium state, which implies that the entropy change of the working substance along the cycle is expressed by the Clausius-like equality using the temperature of the working substance. Equation (13) also implies that the efficiency of this type of the heat engines is given by the ‘endoreversible Carnot efficiency’ using the temperatures of the working substance as
\[\eta \equiv \frac{W}{Q_h} = 1 - \frac{T_c}{T_h}.\]

Although this differs from the usual Carnot efficiency equation (1) that uses the temperatures of the heat reservoirs, it is remarkable that the efficiency is still expressed by the thermodynamic variables of the working substance. In this way, the endoreversibility condition equation (13) largely constrains the behavior of the heat engines. The power output \(W\) of the heat engine is also expressed by using the entropy change as
\[ W \equiv \frac{W}{t_{\text{cyc}}} = \frac{Q_h + Q_c}{t_{\text{cyc}}} = \frac{\Delta T \Delta S}{t_{\text{cyc}}}, \]  

(15)

where \( \Delta T \equiv T_h - T_c \) is the temperature difference of the working substance between the isothermal processes and we used the first law of thermodynamics \( W = Q_h + Q_c \) for one cycle. Hereafter we denote by the dot the quantity divided by the cycle period. Then the rightmost expression in equation (15) using the ‘endoreversible work’ \( W = \Delta T \Delta S \) written by the thermodynamic variables of the working substance like the quasistatic work \( W = \Delta T \Delta S \) may be regarded as ‘endoreversible power.’ The power equation (15) and the efficiency equation (14), which are defined by using the thermodynamic variables of the working substance in this way, are a characteristic of our local equilibrium description of the endoreversible heat engine model. Especially, the ‘endoreversible power’ expression has never been used in a context of determining the efficiency at maximum power so far. We note that these expressions also give us an intuitive picture underlying heat energy conversion in a finite-time Carnot cycle, because the efficiency and the work output are still given by the quasistatic-like expressions.

### 2.2. Efficiency at maximum power in the linear response regime

In this subsection, we consider the efficiency at maximum power of our heat engine in the linear response regime \( \Delta T^R \to 0 \), based on the local equilibrium assumption introduced in section 2.1.

Our analysis using the linear irreversible thermodynamics begins from the entropy production rate. Because we assume that the heat reservoirs and the working substance are always in a local equilibrium state with the well-defined entropies, the entropy production rate \( \frac{d\sigma}{dt} \) of the total system (the working substance and the heat reservoirs) at any instant along the isothermal process is expressed by the sum of the entropy change rates of these partial systems:

\[ \frac{ds}{dt} \equiv \frac{dS_h^R}{dt} + \frac{dS_i}{dt} = \frac{\partial S^R}{\partial U^R_i} \frac{dU^R_i}{dt} + \left( \frac{\partial S_i}{\partial U_i} \frac{dU_i}{dt} + \frac{\partial S_i}{\partial V_i} \frac{dV_i}{dt} \right) = \left( \frac{1}{T_i} - \frac{1}{T_c} \right) \frac{dU^R_i}{dt}, \]  

(16)

where we used equation (6). Then, the entropy production rate for one cycle \( \dot{\sigma} \) is written as

\[ \dot{\sigma} = \frac{1}{t_{\text{cyc}}} \int_0^{t_{\text{cyc}}} ds(t) \frac{dt}{dt} = \frac{1}{t_{\text{cyc}}} \int_0^{t_{\text{cyc}}} \frac{dU^R_{ih}}{dt} \frac{dt}{dt} + \frac{1}{t_{\text{cyc}}} \int_{t_h}^{t_{\text{cyc}}} \left( \frac{1}{T_i} - \frac{1}{T_c} \right) \frac{dU^R_i}{dt} \frac{dt}{dt} = \sum_1^n \left( \frac{1}{T_i^c} - \frac{1}{T_i^h} \right) \Delta U^R_i \frac{1}{t_{\text{cyc}}} \equiv \sum_1^n J_{Q_i} X_{T_i}, \]  

(17)

where we defined the internal energy change of the heat reservoir during the isothermal process as

\[ \Delta U^R_i = \int_0^{t_h} \frac{dU^R_{ih}}{dt} dt, \Delta U^R_i \equiv \int_{t_h}^{t_{\text{cyc}}} \frac{dU^R_i}{dt} dt. \]  

(18)

From equation (17), we naturally define the thermodynamic force as the (inverse) temperature difference between the working substance and the heat reservoir, and the conjugate thermodynamic flux as the heat flux from the heat reservoir [36] (see figure 1(b)):

\[ J_{Q_i} \equiv -\frac{\Delta U^R_i}{t_{\text{cyc}}} = \frac{Q_i}{t_{\text{cyc}}}, X_{T_i} \equiv -\left( \frac{1}{T_i^c} - \frac{1}{T_i^h} \right). \]  

(19)

Because the energy flux and the temperature of the working substance is assumed to be constant along each isothermal process, we obtain

\[ \frac{\Delta U^R_i}{t_i} = -\frac{dU^R_i}{dt} = T_i \frac{dS_i}{dt}, \]  

(20)

by using equation (8). Then the thermodynamic flux \( J_{Q_i} \) can also be expressed by using the time derivative of the thermodynamic variable of the working substance as

\[ J_{Q_i} = a_i \frac{dS_i}{dt}, \]  

(21)

where we denote by \( a_i \) the ratio of the duration of each isothermal process \( t_i \) to the cycle time \( t_{\text{cyc}} \). To proceed further, we need a relation that connects \( J_{Q_i} \) and \( X_{T_i} \), in addition to the local equilibrium thermodynamic formulation mentioned in section 2.1. Because we are adopting the local equilibrium thermodynamic assumption, it is also quite natural to assume that the heat flows in proportion to the temperature difference (the Fourier law) in the same way as the original CA model [4]:
where we denote by \(\kappa\), the thermal conductance between the heat reservoir and the working substance. Using equation (22), we then obtain the following relationship between \(J_Q\) and \(X_T\):

\[
J_Q = \frac{Q_i}{T_{cyc}} = a_i \kappa_i \left( T_R^T - T_i \right) \eta_i,
\]

(23)

In the following, we consider the linear response regime \(\Delta T^R \rightarrow 0\). By substituting equation (22) with \(a_i \kappa_i \) instead of \(\eta_i\) into the endoreversible Carnot efficiency condition equation (13), and expanding that with respect to \(\Delta T\) and \(\Delta T^R\), we can write equation (13) as

\[
\left( a_i \kappa_h + a_e \kappa_e \right) \left( \frac{T_R^T}{T} - 1 \right) + \left( a_h \kappa_h - a_e \kappa_e \right) \frac{T_R^T}{2T} \left( -\frac{\Delta T}{T} + \frac{\Delta T^R}{T^R} \right) + O \left( \Delta T^R \Delta T, \Delta T^2 \right) = 0,
\]

(24)

where \(T \equiv \frac{T_h + T_c}{2}\) is the averaged temperature of the working substance between the isothermal processes. Then, by substituting \(T = c^{(0)} + c^{(1)} \Delta T^R + c^{(1)} \Delta T\) into equation (24), we obtain the expansion coefficients order by order as follows:

\[
c^{(0)} = T^R,
\]

(25)

\[
c^{(1)} = \frac{a_h \kappa_h - a_e \kappa_e}{2 \left( a_h \kappa_h + a_e \kappa_e \right)},
\]

(26)

\[
c^{(1)} = -\frac{a_h \kappa_h - a_e \kappa_e}{2 \left( a_h \kappa_h + a_e \kappa_e \right)}.
\]

(27)

Then, \(T_h = T + \frac{\Delta T}{2}\) and \(T_c = T - \frac{\Delta T}{2}\) are expressed as

\[
T_h = T^R + \frac{a_h \kappa_h - a_e \kappa_e}{2 \left( a_h \kappa_h + a_e \kappa_e \right)} \Delta T^R + \left( \frac{1}{2} - \frac{a_h \kappa_h - a_e \kappa_e}{2 \left( a_h \kappa_h + a_e \kappa_e \right)} \right) \Delta T,
\]

(28)

\[
T_c = T^R + \frac{a_h \kappa_h - a_e \kappa_e}{2 \left( a_h \kappa_h + a_e \kappa_e \right)} \Delta T^R + \left( \frac{1}{2} - \frac{a_h \kappa_h - a_e \kappa_e}{2 \left( a_h \kappa_h + a_e \kappa_e \right)} \right) \Delta T.
\]

(29)

By using equations (25)–(27), we also find that \(X_T\)'s are expressed in terms of \(\Delta T\) and \(\Delta T^R\) as follows:

\[
X_T \equiv \frac{T - T_{cyc}}{T^R} \frac{T_R^T - T_{cyc}}{T^R} \frac{a_h \kappa_h + a_e \kappa_e}{a_h \kappa_h - a_e \kappa_e} \frac{\Delta T^R - \Delta T}{T^R} = \frac{a_e \kappa_e}{a_h \kappa_h + a_e \kappa_e} X_T, \tag{30}
\]

\[
X_T \equiv \frac{T - T_{cyc}}{T^R} \frac{T_R^T - T_{cyc}}{T^R} \frac{a_h \kappa_h + a_e \kappa_e}{a_h \kappa_h - a_e \kappa_e} \frac{\Delta T^R - \Delta T}{T^R} = -\frac{a_h \kappa_h}{a_h \kappa_h + a_e \kappa_e} X_T, \tag{31}
\]

where we defined the ‘reduced thermodynamic force’ \(X_T\) as

\[
X_T \equiv \frac{\Delta T^R - \Delta T}{T^R}.
\]

(32)

From equations (30) and (31), we find that \(X_T\) are proportional with each other in the linear response regime. By using equations (30) and (31) and approximating \(J_Q\) in equation (23) as \(J_Q \approx a_i \kappa_i T^R X_T\), we can simplify the entropy production rate equation (17) by using \(X_T\) up to the quadratic order of \(X_T\) as

\[
\dot{s} = \frac{1}{T_{cyc}} \oint \delta s = \sum_i J_Q_i X_{T_i} \approx \sum_i a_i \kappa_i T^R X_{T_i}^2 = \frac{a_h \kappa_h a_e \kappa_e}{a_h \kappa_h + a_e \kappa_e} T^R X_T^2 \equiv J_Q X_T, \tag{33}
\]

where \(J_Q\) is the averaged heat flux \([39, 59, 60]\) defined as

\[
J_Q \approx \frac{J_{Q-h} - J_{Q-c}}{2} = \frac{a_h \kappa_h a_e \kappa_e}{a_h \kappa_h + a_e \kappa_e} T^R X_T, \tag{34}
\]

Therefore the description of the present heat engine model is reduced to this linear relation equation (34). In the linear response regime, the endoreversible Carnot efficiency equation (14) and the endoreversible power equation (15) are also approximated by using \(X_T\) as

\[
\eta \approx \frac{\Delta T}{T^R} = \frac{\Delta T^R}{T^R} - T^R X_T, \tag{35}
\]

\[
W = \frac{J_Q \Delta T}{T^R} = \frac{a_h \kappa_h a_e \kappa_e}{a_h \kappa_h + a_e \kappa_e} T^R X_T \left( \Delta T^R - T^R X_T \right), \tag{36}
\]
respectively, where \( W \) is a quadratic function of the reduced thermodynamic force \( X_f \). Because \( X_f \) is proportional to the temperature difference between the working substance and the heat reservoir as in equations (30) and (31), it is natural that we can control the power by changing \( X_f \); the quasistatic limit is realized under no temperature difference between them as \( X_f^0 = 0 \) \( (\Delta T^R = \Delta T^F) \). Then the heat flux \( J_0 \) also vanishes from the linear Fourier law equation (23) via equations (30) and (31), and we obtain no power \( W = 0 \) from equation (36) as in the usual Carnot cycle. As \( X_f \) increases from \( X_f^0 \), the heat flux \( J_0 \) becomes finite. Since the power \( W \) is a quadratic function of \( X_f \), it can take a maximum value at a certain point, which is determined by \( \frac{\partial W}{\partial X_f} = 0 \):

\[
X_f^* = \frac{\Delta T^R}{2T^R \lambda} \left( \Delta T^* = \frac{\Delta T^R}{2} \right).
\]

In the quasistatic limit \( X_f^0 = 0 \), we attain the Carnot efficiency \( \eta^C \equiv \frac{\Delta T^R}{2T^R} \) as the maximum efficiency from equation (35). At the maximum power, from equations (35) and (37), the efficiency \( \eta^* \) is given by

\[
\eta^* = \frac{\Delta T^R}{2T^R}.
\]

This is the CA efficiency up to first order of \( \Delta T^R \) that corresponds to the equality in equation (3). The maximum power \( W^* \) is also given as

\[
W^* = \frac{a_2 \kappa_0 \kappa_x}{a_1 \kappa_0 \kappa_x + a_2 \kappa_x} \frac{\Delta T^R}{4T^R},
\]

which depends on the thermal conductivity [4]. Here we note that we obtain the CA efficiency in equation (38) without using the Onsager relations equations (4) and (5), but using the linear relation between the thermodynamic flux and force equation (34) with the aid of the endoreversible expression of the efficiency and power equations (35) and (36). In the next section 2.3, we consider this connection.

### 2.3 Formulation of the endoreversible finite-time Carnot cycle model using Onsager relations

As we have shown in section 2.2, the efficiency at maximum power \( \eta^* \) in equation (38), which is based on the local equilibrium thermodynamic formulation using the linear relation equation (34), is the upper bound in equation (3), while the inequality in equation (3) comes from the formulation based on the Onsager relations [15]. Therefore, we elucidate the relationship between these formulations in this section.

First, we briefly review the derivation of the inequality for the efficiency at maximum power in equation (3) [15]. Denoting an external force and its conjugate variable by \( F \) and \( x \), respectively, we can generally express the power of the heat engine \( W \) as \( W = -F \dot{x} \). Then the entropy production rate of the total system is decomposed into the sum of the entropy increase rate of each heat reservoir because the state of the working substance should return to the original state after one cycle:

\[
\dot{\sigma} = -\frac{Q_h}{T_h} - \frac{Q_f}{T_f} = \left( 1 - \frac{1}{T^R} \right) \dot{Q}_h - \frac{W}{T_f} \equiv \frac{\Delta T^R}{T_f} Q_h = J_1 X_1 + J_2 X_2.
\]

Here, the thermodynamic fluxes \( J_1 \equiv \dot{x} \) and \( J_2 \equiv \dot{Q}_h \), and their conjugate thermodynamic forces \( X_1 \equiv \frac{\Delta T^R}{T_f} \) and \( X_2 \equiv \frac{\Delta T^R}{T_f} \) are related through the Onsager relations equations (4) and (5). We note that, in this case, we do not necessarily assume that the working substance along the cycle is expressed in terms of the well-defined thermodynamic variables, in contrast to our formulation in section 2.2. Using these thermodynamic fluxes and forces, the power and the efficiency are given as

\[
W = -J_1 X_1 T^R, \quad \eta = \frac{W}{Q_h} = -\frac{J_1 X_1 T^R}{J_2}.
\]

With these expressions as well as the Onsager relations equations (4) and (5), we find that the maximum power is realized at \( X_1^* = -\frac{L_{12} X_2}{2L_{11}} \) from \( \frac{\partial W}{\partial X_1} = 0 \). Its efficiency \( \eta^* \) is given as

\[
\eta^* = \frac{q^2}{2} \frac{\Delta T^R}{2T^R},
\]

which is a monotonically increasing function of \(|q|\), where the coupling strength \( q \) is defined by

\[
q \equiv \frac{L_{12}}{\sqrt{L_{11}L_{22}}},
\]

From the non-negativity of the entropy production rate \( \dot{\sigma} \equiv J_1 X_1 + J_2 X_2 \), the Onsager coefficients \( L_{ij} \)'s should satisfy \( L_{11} \geq 0, L_{22} \geq 0 \), and \( L_{11} L_{22} - L_{12}^2 \geq 0 \), and they impose the following constraint on \( q \):

\[
q \equiv \frac{L_{12}}{\sqrt{L_{11}L_{22}}}.
\]
where the equality is known as the tight-coupling (no heat-leakage) condition [15, 18]. Under this tight-coupling condition, $\eta^*$ in equation (43) attains the upper bound given by the CA efficiency $\eta_{CA}$ as in equation (3). An essential point of the derivation of the formula equation (43) is that the non-zero cross-coefficient $L_{12}$ plays an important role in $\eta^*$ in equation (43), which is clear from the definition equation (44).

Returning to our original problem, from equation (23), we formally obtain the following ‘Onsager coefficients’ under our choice of the thermodynamic forces $F_i$ and forces $X_j$:

$$L_{ij} = \begin{pmatrix} a_i \kappa_h T^R & 0 \\ 0 & a_i \kappa_c T^R \end{pmatrix}, \quad (46)$$

where there are no nondiagonal elements. This contrasts to the formulation using equations (4) and (5) where the cross-terms play an important role in the heat energy conversion into work [15]. Equation (46) is natural if the entropy production originating from the heat transfer between the working substance and the heat reservoir in each isothermal process is independent of each other. However, as seen from equations (30) and (31), $X_T$’s are not independent of each other, unlike $X_i$’s in equation (40), but $X_T$’s and $X_i$’s should be related with each other by a variable change. Moreover, how the external force $F$ in the definition of $X_i = F/T^R$ against the heat engine [15] is related to the thermodynamic variable of the working substance is not obvious in our local equilibrium description of the finite-time Carnot cycle where the efficiency and the power is expressed by them as in equations (14) and (15). To elucidate these points, we restate our expression of the entropy production rate equation (17) with $X_T$’s using independent thermodynamic forces as (see figure 1(b))

$$\dot{\sigma} = \frac{1}{T_{cy}} \oint ds = \sum_i \frac{1}{T_i} \frac{1}{T_c} \Delta S_i^R \equiv - \frac{1}{T_c} \sum_i \frac{Q_{hi}}{T_c} + \frac{\Delta S}{T_c} \equiv J_{Q_i} Y_{T^R} + J_s Y_T. \quad (47)$$

We can make this restatement by using the endoreversibility condition equation (13), the first law of the thermodynamics $W = Q_h + Q_c$, for one cycle, and equation (15), where we defined the heat flux from the hot heat reservoir as a new thermodynamic flux and its conjugate new thermodynamic force as

$$J_{Q_h} = \frac{Q_h}{T_{cy}} = a_i \kappa_h \frac{dS_h}{dt}, \quad Y_{T^R} = \frac{1}{T_c} \frac{dS_h}{dt} - \frac{1}{T_c} \frac{dS_c}{dt}. \quad (48)$$

In addition, we defined the entropy flux as another new thermodynamic flux and its conjugate new thermodynamic force as

$$J_s = \frac{\Delta S}{T_{cy}} = a_i \kappa_h \frac{dS_h}{dt}, \quad Y_T = \frac{\Delta T}{T_c}. \quad (49)$$

In this way, all the thermodynamic fluxes and forces are expressed in terms of the combination of the thermodynamic variables of the working substance and the heat reservoirs owing to the local equilibrium assumption. From equation (49), in particular, the new thermodynamic force $Y_T$ is proportional to the temperature difference of the working substance between the isothermal processes (figure 1(b)), which has never been proposed so far. The external force $F$ in the linear irreversible thermodynamics framework in equation (41) by which the power is maximized is abstract in the case of the finite-time Carnot cycle, where the heat flux from the hot heat reservoir is just transferred to the cold heat reservoir without generating any work. Then the power must take a maximum with an intermediate $\Delta T$ between these two zeros.

In the linear response regime, these new thermodynamic fluxes and forces are approximated as

$$J_{Q_h} \approx a_i T^R \frac{dS_h}{dt}, \quad Y_{T^R} \approx \frac{\Delta T}{T^R}, \quad J_s = \frac{\Delta S}{T_{cy}} = a_i \kappa_h \frac{dS_h}{dt}, \quad Y_T \approx - \frac{\Delta T}{T_c}. \quad (50)$$

We note that $Y_{T^R} (Y_{T^R})$ corresponds to $X_1 (X_2)$, and $J_s (J_{Q_h})$ does to $J_1 (J_2)$ in the definition of the general formulation in equation (40), respectively. From equation (50), we immediately notice that $J_{Q_h}$ and $J_s$ are in proportion to each other:

$$J_{Q_h} = T^R J_s. \quad (51)$$

In fact, the proportionality between the two thermodynamic fluxes in equation (51) indirectly implies the tight-coupling condition of this system $|q| = 1$, because we can easily show from equations (4) and (5) the relation

$$J_2 = \frac{L_{12}}{L_{11}} J_1 + L_{22} (1 - q^2) X_2$$

between the two thermodynamic fluxes. However, to understand this relationship more directly and precisely, we express the present system by the following Onsager relations using the new thermodynamic fluxes and forces:
\[ J_\text{S} = L_{TT} Y_T + L_{TT^a} Y_{T^a}, \]
\[ J_{Qe} = L_{T^{T^a} T} Y_T + L_{T^{T^a} T^a} Y_{T^a}. \]

To obtain the new Onsager coefficients from the previous coefficients in equation (46), we relate the thermodynamic forces \( X_T \) (\( i = h, c \)) and \( Y_m \) (\( m = T, T^R \)) by using equations (30), (31), and (50) as follows:
\[
\begin{pmatrix} X_T \\ X_{T^a} \end{pmatrix} = \begin{pmatrix} X_T \\ X_{T^a} \end{pmatrix} = \begin{pmatrix} \frac{1}{a_h k_h + a_c k_c} \left( - a_h k_c \right) \\ \frac{1}{a_h k_h + a_c k_c} \left( - a_h k_h \right) \end{pmatrix} \begin{pmatrix} a_t k_c \\ a_t k_h \end{pmatrix} T^R \begin{pmatrix} \frac{1}{a_h k_h + a_c k_c} \left( - a_h k_h - a_h k_h T^R \right) \end{pmatrix} Y_T. \tag{54}
\]

Rewriting equation (54) as \( X_T \equiv F_{im} Y_m \) in Einstein notation, we obtain the new Onsager matrix
\[
\dot{L}_{mn} = - \frac{a_h k_h a_t k_c}{a_h k_h + a_c k_c} \begin{pmatrix} T^R & T^R \end{pmatrix} \tag{55}
\]
from the relation \( \dot{L}_{mn} = F_{im} L_{ji} F_{jm} \) that conserves the entropy production rate as \( \dot{\sigma} = L_{ij} X_i Y_j = \dot{L}_{mn} Y_m Y_n \).

Alternatively, we can directly obtain the Onsager coefficients \( L_{T^{T^a} T} \) and \( L_{T^{T^a} T^a} \) from the expression of
\[
J_{Qe} = a_h k_h T^{T^a} X_{T^a} = \frac{a_h k_h a_t k_c}{a_h k_h + a_c k_c} T^{T^a} X_T = - \frac{a_h k_h a_t k_c}{a_h k_h + a_c k_c} T^R Y_T + \frac{a_h k_h a_t k_c}{a_h k_h + a_c k_c} T^{T^a} Y_{T^a}, \tag{56}
\]
which is obtained using equations (30) and (54), and we can also obtain \( L_{TT} \) and \( L_{T^a T^a} \) from equation (51):
\[
J_\text{S} = \dot{J}_0 = a_h k_h T^{T^a} Y_{T^a} = \frac{a_h k_h a_t k_c}{a_h k_h + a_c k_c} T^{T^a} X_T = - \frac{a_h k_h a_t k_c}{a_h k_h + a_c k_c} T^R Y_T + \frac{a_h k_h a_t k_c}{a_h k_h + a_c k_c} T^{T^a} Y_{T^a}. \tag{57}
\]

From equation (55), it is straightforward to confirm that the Onsager reciprocity and the tight-coupling (no heat-leakage) condition are fulfilled:
\[
\dot{L}_{TT} = \dot{L}_{T^a T^a}, \quad q = \frac{\dot{L}_{TT}}{\sqrt{\dot{L}_{T^a T^a} L_{TT}}} = 1. \tag{58}
\]

This implies that our model attains the upper bound (the CA efficiency) in equation (3) corresponding to \(|q| = 1 \) in equation (43), from the viewpoint of the linear irreversible thermodynamics framework. For completeness, we explicitly confirm that the maximization of the power by \( Y_T \) leads to the CA efficiency by using the Onsager relations along with the general formulation introduced in the beginning of this section. Let us compare the two ways of maximization of the power by \( X_T \) as in section 2.2 and by \( Y_T \). As we have seen in equation (54), the reduced thermodynamic force \( Y_T \) introduced in equation (32) is linearly connected by \( Y_T \) and \( Y_{T^a} \). By substituting equation (34) into equation (36), we can express the power \( W \) in equation (36) by using \( Y_T \) and \( Y_{T^a} \) instead of using \( X_T \) as
\[
W = - \frac{a_h k_h a_t k_c}{a_h k_h + a_c k_c} T^{T^a} Y_T^2 - \frac{a_h k_h a_t k_c}{a_h k_h + a_c k_c} T^{T^a} Y_{T^a} Y_T, \tag{59}
\]
which can also be obtained from the expression \( W = -J_\text{S} Y_T T^R \) in equations (41) and (52). We find that the quasistatic limit \( X_T^0 = 0 \) corresponds to \( Y_T^0 = -T^R Y_{T^a} \) from \( W = 0 \) by using equation (59). We also find that the maximum power point \( X_T^0 = \frac{\Delta T^R}{T^R} \) in equation (37) also corresponds to \( Y_T^0 = -\frac{T^R}{T} Y_{T^a} \) from \( \frac{\partial W}{\partial Y_T} = 0 \) by using equation (59), from which we can obtain the efficiency \( \eta^* = -J_\text{S} Y_T T^R J_{Qe}^t \) by using equations (52) and (53) as
\[
\eta^* = \frac{\Delta T^R}{2 T^R} = \eta_{CA} + O \left( \Delta T^{R^2} \right). \tag{60}
\]

Therefore, for our endoreversible heat engine model based on the local equilibrium assumption, we conclude that the efficiency at maximum power attains the upper bound in equation (3) from a viewpoint of the linear irreversible thermodynamics framework using the Onsager relations [15]. We note that the above derivation is quite general because it does not rely on any particular working substance or thermal conductance.

3. Discussion and summary

We discuss a few aspects related to our formulation in section 2.

First, we note that the endoreversible power as determined by the product of the heat flux and the endoreversible Carnot efficiency in equation (36) in section 2.2 should also be determined by the first law of the thermodynamics \( W = J_\text{S} + J_{Qe} \) for one cycle. To this end, we need to consider \( J_{Qe} \) with higher-order corrections of \( \Delta T \) and \( \Delta T^R \), while we have considered only the lowest order in section 2.2. We utilize the expressions \( J_\text{S} = \dot{T}_f J_S \) and \( J_{Qe} = -\dot{T}_f J_S \), which are obtained from equations (9), (10), (12), (19), and (49). Then, by using equations (28), (29), and (57), we directly obtain the explicit form of \( J_\text{S} \) with the nonlinear terms as...
from which we can confirm \( W = J_{Qh} + J_Q \) by comparing it with equation (36).

Second, we point out that an extension of our formulation introduced in section 2.1 to the general case where the energy flux may not be constant along the cycle may be an interesting challenge. This would be more clarified by considering under what conditions the assumption of the constant energy flux strictly holds in a specific example: let us consider the following first law of thermodynamics equation (6) for the three-dimensional ideal gas during an isothermal process:

\[
\frac{3}{2} Nk_B \frac{dT_i}{dt} = \kappa_i (T_i^R - T_i) = \frac{Nk_B T_i \, dV_i}{V_i} \frac{dV_i}{dt},
\]

where we used the internal energy \( U_i = \frac{3}{2} Nk_B T_i \) and the equation of state \( P_i = \frac{Nk_B T_i}{V_i} \) for the ideal gas, and we denote by \( N \) and \( k_B \) the particle number and the Boltzmann constant, respectively.

For each term in equation (63) to be constant, \( T_i = \text{const} \) solution to equation (63) can be realized only under a specific protocol \( V_i (t) \) that satisfies \( \frac{dV_i}{dt} = \text{const} \). Therefore even in this simplest case of the ideal gas, our assumption of the constant energy flux introduced in section 2.1 (as also assumed in the original CA model) seems too restricted. In addition, although we assumed that the thermal conductance \( \kappa_i \) is a constant in the present model (as also assumed in the original CA model), it can also depend on the thermodynamic variables such as \( T_i \) and \( V_i \) \([27, 41]\), and hence on time through them in general. Since actual heat engines may consist of complicated combinations of these factors, it is difficult to specify generic conditions for arbitrary working substances and thermal conductances under which the assumption of the constant energy flux holds in a strict manner. Instead, for more natural description of the endoreversible finite-time Carnot cycle under more generic conditions, we are motivated to extend our formulation introduced in section 2.1 to non-constant energy flux cases that include the constant energy flux case as a special case. We are also in progress toward this goal from the linear irreversible thermodynamics point of view, which will be presented elsewhere \([61]\).

However, we stress that our present formulation of the constant energy flux case would serve as the ‘zeroth approximation’ or a starting point for analyzing the performance of such actual heat engines. Indeed, this is also supported by the fact that observed data of the efficiency of actual heat engines are distributed around the CA efficiency \([33]\), which implies that they may be regarded as more or less variants of the constant energy flux case.

In the present study, we formulated an endoreversible finite-time Carnot cycle model based on the assumptions of local equilibrium and constant energy flux. In our framework, the power and the efficiency are expressed in terms of the thermodynamic variables of the working substance. From the analysis of the entropy production caused by the heat transfer in each isothermal process, we identified the thermodynamic flux and force in each isothermal process, which are related by the Fourier law. By applying the endoreversible condition to the linear response regime, we found that those thermodynamic forces are not independent, and identified the reduced thermodynamic force and its conjugate thermodynamic flux, which are connected by the linear relation different from the Onsager relations. We calculated the efficiency at maximum power by using this linear relation, and obtained the CA efficiency. We also elucidated that by suitable change of the variables, the linear relation between the thermodynamic flux and force in our framework can be rewritten into the form of the Onsager relations, where the novel thermodynamic force that is proportional to the temperature difference of the working substance between the isothermal processes is introduced. Then we directly confirmed that our model satisfies the tight-coupling condition that ensures the CA efficiency as is the upper bound in the linear irreversible thermodynamics framework. We stress that our framework is quite universal because it only assumes that the working substance is in a local equilibrium state specified by a unique combination of thermodynamic variables at any instant along the cycle. We expect that our study unifies recent development of the theories of heat engines based on the universal nonequilibrium thermodynamics framework and the more phenomenological finite-time thermodynamics approach that was developed for application to real power plants and heat devices.

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