A cellular automaton model of pulsar glitches

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ABSTRACT

A cellular automaton model of pulsar glitches is described, based on the superfluid vortex unpinning paradigm. Recent analyses of pulsar glitch data suggest that glitches result from scale-invariant avalanches, which are consistent with a self-organized critical system (SOCS). A cellular automaton provides a computationally efficient means of modelling the collective behaviour of up to $10^{16}$ vortices in the pulsar interior, whilst ensuring that the dominant aspects of the microphysics are not lost. The automaton generates avalanche distributions that are qualitatively consistent with a SOCS and with glitch data. The probability density functions of glitch sizes and durations are power laws, and the probability density function of waiting times between successive glitches is Poissonian, consistent with statistically independent events. The output of the model depends on the physical and computational parameters used. The fitted power-law exponents $a$ and $b$ (the size and duration distributions, respectively) decrease as the strength of the vortex pinning increases. Similarly, the exponents increase as the fraction of vortices that are pinned decreases. For the physical and computational parameters considered, one finds $-4.3 \leq a \leq -2.0$, $-5.5 \leq b \leq -2.2$, and mean glitching rates in the range $0.0023 \leq \lambda \leq 0.13$ in units of inverse time.

Key words: stars: interior – stars: neutron – pulsars: general.

1 INTRODUCTION

It has recently been shown that glitches in individual pulsars obey avalanche statistics (Melatos, Peralta & Wyithe 2008). Such statistical behaviour, in conjunction with the wide dynamical range of glitches, suggests that the underlying physics is that of a self-organized critical system (SOCS) (Pines & Alpar 1985; Bak 1996; Jensen 1998; Melatos et al. 2008). A SOCS is characterized by transitions between metastable states via scale-invariant avalanches (Jensen 1998). The fractional increase in spin frequency spans seven decades across the entire glitch population ($10^{-11} \leq \delta \nu/\nu \leq 10^{-4}$), and up to four decades in any individual pulsar. A natural way to model a scale invariant, avalanching system is to construct a cellular automaton that is driven slowly to a threshold of instability (Field, Witt & Nori 1995). Such models have been explored in detail in the context of earthquakes (Sornette, Sornette & Vanneste 1991), granular assemblies (Bak, Tang & Wiesenfeld 1988; Wiesenfeld, Tang & Bak 1989; Bak 1996; Frette et al. 1996; Pruessner & Jensen 2003), solar flares (Lu & Hamilton 1991) and superconducting vortices (Jensen 1998a; Field et al. 1995; Bassler & Paczuski 1998; Linder et al. 2004).

Theories of pulsar glitches centre around the mass unpinning model first proposed by Anderson & Itoh (1975) and extended by many others (Alpar et al. 1984; Pines & Alpar 1985; Alpar, Cheng & Pines 1989; Link & Cutler 2002). The neutron superfluid in the stellar interior is threaded by many ($\sim 10^{16}$) vortices, approximately 1 per cent of which are pinned to the stellar crust at grain boundaries and/or nuclear lattice sites (Alpar et al. 1989). As the pulsar crust spins down electromagnetically, a lag builds up between the velocity of the pinned vortex lines (corotating with the crust) and the superfluid. When the transverse Magnus force (directly proportional to the lag) surpasses a threshold value (equal to the strength of the pinning force), a catastrophic unpinning of vortices occurs, transferring angular momentum to the crust. In order for this mechanism to generate glitches on the scale observed, it requires up to $10^{12}$ vortices to unpin simultaneously, exhibiting a high level of collective, non-local behaviour.

The mass unpinning model spawned much activity in quantifying the microphysics that would lead to such macroscopic behaviour (Pines & Alpar 1985; Jones 1997, 1998a,b; De Blasio & Elgarò 1999; Elgarøy & De Blasio 2001; Donati & Pizzochero 2003; Avogadro et al. 2007). In particular, the strength of the pinning force has recently been discussed in detail using a self-consistent, semi-analytical model for the vortex–nucleus interaction (Avogadro et al. 2007), yielding the unexpected result that the strongest pinning occurs in regions of the crust that are of high and low density, rather than intermediate density (Donati & Pizzochero 2006). An interest has also been taken in the evolution and morphology of the neutron star crust, with particular attention paid to impurities and defects, which determine the density of pinning centres available to the vortices, and their relative strengths (De Blasio & Lazzari 1998). Link

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Self-organized behavioural patterns. A system is described as a SOCS based on two underlying behavioural patterns. In Section 6, the model's validity and a summary of our findings are contained in the model's behaviour. In Section 5, the statistical behaviour of the model is explored with respect to driver parameters, internal physical parameters, and experimental data. Combined with local thresholds, this ensures that the system evolves quasi-statically through a history-dependent sequence of metastable states.

Regardless of the model one chooses to examine, there is a common need to synthesize the microphysics with the global, collective dynamics in order to accurately describe the observational data. In order to produce a realistic model of the global dynamics, some subtle aspects of the microphysics need to be sacrificed. Cellular automaton models of complex systems aim to employ only the dominant aspects of the microphysics to generate replicable and efficient automaton rules. Such an approach was attempted by Morley & Schmidt (1996) using a cellular automaton platelet model of mass accretion on to the stellar crust. Although the microphysical details differ from those treated in this paper, the underlying philosophy of using a simple model to describe large-scale behaviour is identical to ours. We are similarly encouraged by the marked parallels between superfluid vortices and those that populate hard, type II superconductors. The agreement between cellular automaton models of flux creep and vortex avalanches in superconductors and experimental data is excellent (Jensen 1990; Field et al. 1995; Bassler & Paczuski 1998; Nicodemi & Jensen 2001a,b; Linder et al. 2004), in particular with respect to the measurement of a power law over several decades in the avalanche (cf. glitch) size.

A general, first principles theory of self-organized criticality does not yet exist for any SOCS, let alone for pulsar glitches. In this paper, we look for empirical agreement between the output of our cellular automaton and the gross qualitative features in pulsar glitch data. A detailed quantitative comparison between a first principles theory and data is not possible at this stage.

In this paper, we present a cellular automaton model of pulsar glitches based on the mass unpinning model. After describing the key features of a SOCS, we revisit the observational pulsar glitch data in Section 2. Section 3 reviews the vortex unpinning paradigm and recent advances in the understanding of neutron star structure which are relevant to the paradigm. Section 4 describes in detail the cellular automaton model and justifies physically the automaton rules. In Section 5, the statistical behaviour of the model is explored with respect to driver parameters, internal physical parameters and variations in the automaton rules. Further discussion of the model's validity and a summary of our findings are contained in Section 6.

2 SELF-ORGANIZED CRITICAL SYSTEMS

2.1 General properties

A system is described as a SOCS based on two underlying behavioural patterns. Self-organized implies the ability to develop structures and patterns in the absence of manipulation by an external agent. Critical implies that, like in phase transitions near the critical temperature, a small local perturbation can propagate throughout the entire system (Jensen 1998).

For a system to be in a self-organized critical (SOC) state, it must first satisfy the following three conditions (Jensen 1998; Melatos et al. 2008).

(i) It is composed of many discrete, mutually interacting elements, whose motions are dominated by local (e.g. nearest-neighbour) rather than global (e.g. mean-field) forces.

(ii) Each element moves when the local force exceeds a threshold. In this way, stress accumulates sustainedly at certain random locations (metastable stress reservoirs) while relaxing quickly elsewhere.

(iii) An external force drives the system slowly, in the sense that elements adjust to local forces rapidly compared to the driver time-scale. Combined with local thresholds, this ensures that the system evolves quasi-statically through a history-dependent sequence of metastable states.

If the above conditions are met, the following properties are generically observed (Jensen 1998; Melatos et al. 2008).

(iv) Transitions from one metastable state to the next occur via avalanches: spatially connected chains of local equilibration events, in which one element relaxes and redistributes some local stress to its neighbours, which in turn can exceed their thresholds and relax (knock-on effect).

(v) Avalanches have no preferred scale: they can involve a few (commonly) or all (rarely) of the elements in the system. Their sizes and lifetimes follow power-law distributions, whose exponents are related. Numerical values of the exponents depend on the spatial dimensionality of the system, the symmetry and strength of the local forces (Field et al. 1995), and the level of conservation (Vespignani & Zapperi 1998; Pruessner & Jensen 2002).

(vi) Over the long term, the system tends to a critical state, which is stationary on average but fluctuates instantaneously.

A driving force is not essential to the operation of a SOCS. Pruessner & Jensen (2002) assert that an external driving force compensates for a lack of conservation in the microscopic dynamics of a SOCS, and hence is superfluous if the system is conservative on microscopic scales. We claim that our model is locally non-conservative and hence a driving force is essential to maintain criticality (see Section 4.7).

Avalanches result from the relaxation of isolated metastable stress reservoirs. The storage capacity of these reservoirs is scale invariant. Each relaxation event is statistically independent from other relaxation events in size and when it occurs. Two or more simultaneous relaxation events at different locations cannot be distinguished by their global effect (e.g. spin-up of a neutron star), which is the cumulative result of multiple relaxations. Likewise, relaxation of two or more smaller reservoirs causes an avalanche which is indistinguishable from that produced by the relaxation of a single large reservoir. Statistical independence implies that the waiting times between avalanches follow Poisson statistics, and this is confirmed by models based on cellular automata.

Alpar, Kızıloğlu & van Paradijs (1995) and Alpar et al. (1996) discussed the formation of depleted and capacitive regions of vortices in a neutron star interior, generated by steep gradients in the vortex pinning potential. Wherever the pinning potential is greater, the vortex density is also greater, resulting in a stronger local inter-vortex interaction (Magnus force). When the Magnus force exceeds the pinning threshold, vortices are redistributed locally into neighbouring regions, relaxing the capacitive region (stress reservoir). This qualitative picture bears the hallmark of a SOCS.

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Melatos et al. (2008) compiled all of the available glitch data for pulsars, analysing in detail the nine pulsars that have been observed to glitch at least five times. We consider here the subset that has glitched more than six times ($N_g \geq 6$). Of the eight pulsars satisfying this criterion, six are consistent with a power-law probability density function in the fractional glitch size $\delta \nu / \nu$, where $\nu$ is the spin frequency of the pulsar, implying a cumulative probability of the form

$$P(\delta \nu) = \frac{(\delta \nu)^{1-a} - (\delta \nu)^{1-a}_{\min}}{(\delta \nu)^{1-a}_{\max} - (\delta \nu)^{1-a}_{\min}},$$

(1)

where $(\delta \nu / \nu)_{\max}$ is the maximum observed glitch size and $(\delta \nu / \nu)_{\min}$ is the minimum observed glitch size. Melatos et al. (2008) found that $a$ takes a different value for each pulsar; when the data from all nine pulsars with $N_g \geq 6$ are considered, equation (1) with a single value of $a$ is not a good description of the aggregate $\delta \nu / \nu$ distribution.

Two pulsars also exhibit quasi-periodic behaviour, which is also a natural manifestation of avalanche dynamics; when the global forces (driven externally) overwhelm the nearest-neighbour interactions, quasi-periodicity in the waiting time distribution is expected (Jensen 1998).

For each of the six pulsars analysed here, we find all values of the exponent $a$, for which the data are not inconsistent with the analytic distribution at the 1σ confidence level according to the Kolmogorov–Smirnov (KS) test. The range of $a$ is presented in Table 1. Table 1 also gives the exponent that minimizes the KS statistic, along with the smallest glitch $(\delta \nu / \nu)_{\min}$, for each pulsar (Melatos et al. 2008). However, it should be noted that all values of $a$ within the 1σ range (bounded by $a_{\min}$ and $a_{\max}$) are equally likely; the exponent that minimizes the KS statistic is not a best fit. To appreciate why, suppose that the true underlying distribution of $\delta \nu / \nu$ is known for some pulsar. Each time the pulsar glitches it samples this underlying $\delta \nu / \nu$ distribution. The glitch data for this pulsar represent one possible realization of the distribution containing $N_g$ glitches. There is no reason to attach greater significance to the $a$ value that minimizes the KS statistic with respect to the particular realization than to any other in the 1σ range.

Melatos et al. (2008) also investigated the distribution of waiting times $\Delta t$, between successive glitches. Based on the statistical independence of each avalanche and the assumption that the system is driven at a constant rate, it is expected that the probability density function for $\Delta t$ is Poissonian (Jensen 1998), implying a cumulative probability distribution of the form

$$P(\Delta t) = \frac{1}{N_g} \sum_{\Delta t_{\min}} \frac{\exp(-\lambda \Delta t_{\min}) - \exp(-\lambda \Delta t)}{\exp(-\lambda \Delta t_{\min}) - \exp(-\lambda \Delta t_{\max})},$$

(2)

The sum in equation (2) is taken over the minimum observable waiting time $\Delta t_{\min}$, which is set by the gap between data spans in which a glitch is localized. $\Delta t_{\max}$ is a function of the observing schedule and hence of epoch; effectively, therefore, it takes a different value for each glitch (Melatos et al. 2008). $\Delta t_{\max}$ is the total interval over which a pulsar is observed.

Fig. 1 shows cumulative histograms of waiting times $\Delta t$, for the same six pulsars listed in Table 1, plotted against $\lambda \Delta t$, where $\lambda$ is the Poisson rate for each pulsar in units of yr$^{-1}$. Once again, $\lambda$ is chosen to minimize the KS statistic for definiteness, but any value in the range $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$ would do just as well at the 1σ confidence level. The observational data are overlaid with the ideal cumulative Poissonian $1 - e^{-\lambda \Delta t}$ for comparison, taking $\Delta t_{\min} = 0$ and $\Delta t_{\max} \rightarrow \infty$; that is, we do not correct for data gaps and a finite total observation for simplicity, unlike in Melatos et al. (2008). The agreement with the universal Poissonian is good for these six objects, with $0.35 \leq \lambda \leq 5.6$ yr$^{-1}$. An observational confirmation that pulsar glitch waiting times obey Poisson statistics upholds the prediction that glitches arise from the relaxation of metastable stress reservoirs isolated by relaxed zones.

Alpar & Baykal (2006) also modelled the waiting time distribution as a Poissonian in their efforts to explain anomalous braking indices $(\dot{\nu} / \nu^2 \gg 3)$. In the context of the vortex creep model (see e.g. Alpar et al. 1984), values for $\lambda$ are derived for glitching and non-glitching pulsars. Alpar & Baykal (2006) devoted particular attention to explaining anomalous positive $\dot{\nu}$ by postulating the occurrence of unobserved glitches, based on a smaller data set than Melatos et al. (2008), containing different objects. As the glitch data base continues to expand (both as a result of longer observation periods in the future, and reinpection of existing data sets), it will be of great interest to compare the values of $\lambda$ inferred by Melatos et al. (2008) to those calculated by Alpar & Baykal (2006).

The observed distribution of glitch durations cannot be discussed at this time, as very few glitches have been resolved temporally (McCulloch et al. 1990).

### Table 1. Parameters of the glitch size and waiting time distributions for pulsars with $N_g \geq 6$ (Melatos et al. 2008). $a$ and $\lambda$ are chosen to minimize the KS statistic.

| PSR J   | $\lambda_{\min}$ | $\lambda_{\max}$ | $\lambda$ | $a_{\min}$ | $a_{\max}$ | $10^6 (\delta \nu/\nu)_{\min}$ |
|---------|------------------|------------------|----------|-----------|-----------|-----------------------------|
| 1825–0935 | 0.09             | 0.13             | 1.1      | 0.3       | 1.2       | 0.7                         |
| 0534+2200 | 0.09             | 0.13             | 1.1      | 0.3       | 1.2       | 0.7                         |
| 1801–2304 | 0.55             | 0.88             | 0.8      | 0.57      | 1.1       | 4                           |
| 1341–6220 | 1.8              | 5.6              | 1.4      | 1.2       | 2.7       | 1.33                        |
| 0631+1036 | 0.95             | 1.8              | 1.2      | 0.98      | 1.3       | 0.7                         |
| 1740–3015 | 1.5              | 2.5              | 1.1      | 0.98      | 1.3       | 0.7                         |

### Figure 1. Cumulative distributions of glitch waiting times $\Delta t$ for six pulsars with $N_g \geq 6$. The horizontal scale of glitch waiting times $\Delta t$ is different for each pulsar; $\lambda$ is the mean glitch rate that minimizes the KS statistic for that pulsar as listed in Table 1. The solid curve represents the ideal, Poissonian, cumulative distribution, $P(\Delta t) = 1 - e^{-\lambda \Delta t}$. The waiting time distribution in an ideal SOCS follows this Poissonian.

### 3 VORTEX UNPINNING IN A PULSAR

The mass unpinning paradigm of pulsar glitches assumes that the inner crust is a dense nuclear lattice permeated by a superfluid...
that is threaded by many quantized vortices. The vortices interact repulsively via the Magnus force\(^1\) (per unit length of the vortex line) given by

\[
F_M = \rho_s \kappa \times (\mathbf{v}_L - \mathbf{v}_s),
\]

where \(\rho_s\) is the superfluid density, \(\kappa\) is a vector pointing along the vortex, whose modulus \(|\kappa| = h/2m_n\) is the circulation around a single vortex (\(m_n\) is the neutron mass), \(\mathbf{v}_L\) is the vortex line velocity and \(\mathbf{v}_s\) is the local bulk superfluid velocity. If allowed to move freely, vortices organize into a rectilinear array spaced evenly according to Feynman’s rule (vortex area density \(n = 4\pi v/\kappa\)). In keeping with condition (i) in Section 2.1, each vortex is considered a discrete element of a SOCS. For the purpose of practical modelling, given that we are dealing with \(\sim 10^{16}\) vortices in a neutron star, we group vortices into bundles (in internal equilibrium) and regard the bundles as discrete elements. The velocity field generated by each vortex is inversely proportional to the distance from the vortex line, so local interactions dominate global forces.

Condition (ii) in Section 2.1 exactly describes the pinning of vortices to the pulsar crust, whether the pinning centres are nuclear lattice sites, defects (monovacancies or impurities De Blasio & Lazzari 1998), or grain boundaries. Condition (ii) does not rule out pinning of neutron and proton vortices in the neutron star’s core. Vortices are unpinned by the bulk superfluid flow when the Magnus force exceeds the pinning force. Jones (1997) suggested that vortex pinning is too weak to occur in the pulsar crust, and that strong magnetic interactions between neutron and proton vortices are a feasible alternative. In the event that strong pinning is provided by monovacancies alone, Jones (1998a,b) calculated the monovacancy unpinned vortices (remembering that only \(\sim 10^{16}\) vortices in a neutron star, we group vortices into bundles (in internal equilibrium) and regard the bundles as discrete elements. The velocity field generated by each vortex is inversely proportional to the distance from the vortex line, so local interactions dominate global forces.

The external driver fulfilling condition (iii) in Section 2.1 is the accumulating Magnus force generated by the build up of rotational lag between the vortex lines and the superfluid flow due to the electromagnetic spin-down of the star. The effect of this slow build up is two-fold. First, the vortices tend to slowly move radially outward, as the system strives to maintain the Feynman vortex density (Alpar et al. 1984; Jones 1998a). Secondly, abrupt adjustments occur when locally the pinning force threshold is exceeded. Once a vortex is unpinned, it readjusts rapidly, either by returning to the ‘soup’ of unpinned vortices (remembering that only \(~1\) per cent of vortices is pinned at any one time), or by repinning at a nearby lattice or defect site. Most mesoscopic vortex models assume that the inertia of vortex lines is negligible: once unpinned, vortices move immediately with the bulk superfluid flow, regardless of the direction of the unpinning force (Donnelly 1991). Field et al. (1995) pointed out that, at least in the context of flux creep in superconductors, such an assumption leads to ‘stick-slip’ dynamics that are more likely to resemble those of an ideal sandpile (the quintessential SOCS) because inertial effects do not dominate the effect of the threshold. The time-scale of the readjustment is much shorter than the driving time-scale of the stellar spin-down.

Throughout this paper, we assume that superfluid vortices are straight and parallel to the axis of rotation. Jones (1998b) argued that the high, but finite, tension in vortex lines justifies this assumption, as the displacements caused by the pinning centres are much smaller than the nuclear separation. Contrarily, it is the finite tension in vortices that allows them to be drawn into the potential well of the pinning centres and pin at all (Jones 1998b). Link & Cutler (2002) suggested pinning centres in the nuclear lattice effectively span approximately 30 lattice sites, due to finite tension, where the internuclear spacing is \(\sim 10^{-14}\) m. Relative to the coarse graining scale of a practical cellular automaton (linear cell dimension up to \(\approx 2 \times 10^3\) m), this deviation from straight vortices is clearly insignificant. However, global hydrodynamic simulations suggest that the array breaks up into a vortex tangle above a critical rotational lag, driven by the Donnelly–Glazebrook instability, with a net polarization parallel to the rotation axis (see e.g. Peralta et al. 2005, 2006a,b).

### 4 Cellular Automaton

The cellular automaton model presented in this paper takes the basic physical features of the inner crust-superfluid system and interprets them as simple rules. Much of the microphysics that has emerged from detailed studies of pinning strengths, crustal structure and turbulent flow (see references in Section 1) has not been taken into account directly when constructing the automaton rules, but it is relevant for understanding how the system can be ‘scaled up’ (renormalized) to the necessary coarse grain. In terms of reproducing the large scale, collective dynamics of the SOCS, these simple cellular automaton rules have many advantages. In particular, the simple rules allow for a large range of system sizes and parameter space to be modelled for long enough to obtain stationary statistics, without prohibitive computational cost. Table 2 summarizes the nomenclature used in this and subsequent sections.

#### 4.1 Grid

The simulation grid contains \(N_{\text{grid}}^2\) cells, covering a circular ‘star’ of radius \(R\), representing a projection on to the equatorial plane. The grid points are arranged in a rectilinear (square) array in order to easily identify nearest neighbours and to ensure that all cells are equal in size. In reality, the equilibrium configuration of superfluid vortices is a triangular Abrikosov lattice. For the purposes of this model, we assume that the two configurations are indistinguishable when vortices are bundled in large groups. Vortex positions are restricted to the coordinates of the centre of each grid cell. As mentioned in Section 3, there is an inherent assumption that the vortices are straight and parallel to the axis of rotation, and that the

Table 2. Default physical and computational parameters used to analyse the model.

| Parameter                  | Symbol | Default value |
|----------------------------|--------|---------------|
| Grid dimension             | \(N_{\text{grid}}\) | 100           |
| Stellar radius             | \(R\)  | 10 km         |
| Initial spin frequency     | \(v_0\) | 30.25 Hz      |
| Current spin frequency     | \(v\)  | 13.90 Hz      |
| Spin-down rate             | \(\dot{v}\) | 0 Hz s\(^{-1}\) |
| Total vortices in simulation | \(N_v\) | \(4\pi v^2 R^2/\kappa\) |
| Vortices per bundle        | \(N_{\text{vort}}\) | \(N_v/N_{\text{grid}}^2\) |
| Bundle factor (average bundles per cell) | \(B\) | 1             |
| Pinning threshold          | \(F_{\text{pin}}\) | \(10^4\) m s\(^{-1}\) |
| Pinned fraction            | \(\epsilon\) | 0.01          |

\(1\) The Magnus force is in fact a fictitious force, similar in nature to the Coriolis or centrifugal forces. A free vortex, in the absence of external forces, moves with the local superfluid flow. In order for a vortex to move with respect to the local flow, a force must be applied such that the total force per unit length of vortex line, \(\rho_s \kappa \times (\mathbf{v}_L - \mathbf{v}_s)\), is sufficient to sustain the motion of the line \(\gamma_{\text{L}}\) with respect to the local flow.
vortex dynamics are independent of the vortex length. Indeed, the intervortex force is calculated as a force per unit length.

The number of vortices that occupy the grid is calculated using the Feynman relation:

\[ \oint v_\phi \cdot dl = \kappa N_v, \]  

(4)

where the integral is taken around a closed path (e.g. a circle of radius equal to the stellar radius) and \( N_v \) is the total number of vortices enclosed by the path. In the limit of an infinite vortex distribution, the equilibrium distribution is a rectilinear array. Although we have a finite array of vortices, we assume that they form a rectilinear array, such that the total number of vortices is

\[ N_v(R) = \frac{4\pi^2 v R^2}{\kappa}. \]  

(5)

Once we know how many vortices should occupy the system as a whole, we bundle them such that, on average, there is one bundle per cell. Partly, this is done to reduce the number of discrete elements and keep the computation practical. However, it is also done because uncertainties in pulsar timing limit the smallest glitch we can observe (corresponding to one bundle). For any given pulsar, we model the observed glitch range: the minimum glitch size \( G_{\text{min}} \) occurs when one cell unpins, and the maximum \( G_{\text{max}} \) occurs when all cells unpin (what constitutes a glitch is discussed in Section 4.5).

This restriction ensures that our model does not produce glitches outside the observationally imposed range. This is in contrast with the model of Morley & Schmidt (1996) in which the minimum glitch size is set by timing noise and the maximum by the total available energy. Alternatively, we can have an average of \( B > 1 \) bundles per cell. In this scenario, the minimum glitch size is less than the minimum observed glitch size, and hence we can disregard all glitches resulting from the unpinning of fewer than \( B \) vortex bundles. We choose our grid size such that \n
\[ N^2_{\text{grid}} = \frac{G_{\text{max}}}{G_{\text{min}}}. \]  

(6)

Pulsar timing data suggest, via equation (6), a range of grid sizes varying from \( N_{\text{grid}} = 10 \) to a maximum of \( N_{\text{grid}} = 100^2 \). The number of vortices per bundle is then

\[ N_{\text{ Bun}} = N_v / (B N^2_{\text{grid}}). \]  

(7)

In an astrophysical context, however, we expect that glitches smaller than the observable minimum do take place, and with equal or greater frequency. If the system is in a self-organized critical state, we expect glitches to occur at all scales, within the bounds of the system. To resolve smaller \( \Delta v/v \) in the model, we must fractionate into smaller bundles. In order to make comparisons with the observational data, we preclude a priori these smaller glitches from occurring in our model. If the system is truly scale invariant, this effective window function (coarse-grain scale) should not interfere with its overall statistical behaviour. In the flux creep model for superconductors developed by Jensen (1990), each cell holds at most one particle. In our model, each cell holds as many bundles as the avalanche history dictates. Bassler & Paczuski (1998) take a similar approach, allowing many point pins in an extended cell.

\[ N^2_{\text{grid}} \] is the total number of cells in the square grid. However, only cells that lie within the stellar radius are actually available.

4.2 Initial conditions

Initially, the vortex bundles are laid out at random\(^3\) such that the mean occupancy of a cell is \( B \) bundles. Whether or not it is realistic to have a vortex distribution which is inhomogeneous on the scale of the grid cells (\( \sim 2 \times 10^3 \) m) depends upon the distribution of the pinning centres. This point is discussed in more detail in Section 6. From the glitch data analysed by Melatos et al. (2008), we hypothesize that our system is in a SOC state at any given epoch, with an inhomogeneous configuration of pinned vortices, and take this to be the initial state of the automaton. We do not model directly how this inhomogeneous state arises from an initially homogeneous state because the latter configuration is inaccessible to observations; it is disrupted by avalanches almost immediately after the neutron star is born. Given that the random initial conditions of the model do not necessarily define a critical state of the system, we allow the simulation to complete many (\( \sim 10^5 \)) time-steps before considering the output, to ensure that criticality has been established.

4.3 Pinning threshold

The strength with which vortices are pinned to the stellar crust, \( F_{\text{thresh}} \), plays an essential role in the mass unpinning model of pulsar glitches. Recent calculations imply that pinning to nuclear lattice sites is the strongest in regions of low (~4.9 \( \times 10^{13} \) kg m\(^{-3} \)) and high density (~1.6 \( \times 10^{17} \) kg m\(^{-3} \)), with a maximum energy of \( E_p \sim 3 \) MeV (Avogadro et al. 2007). We take the pinning force to be the same everywhere on the grid, with value \( E_p/\xi \), where \( \xi \sim 10^{-14} \) m is the superfluid coherence length (Elgarøy & De Blasio 2001); i.e. \( F_{\text{thresh}} \sim 5 \times 10^{15} \) J m\(^{-3} \). For the purposes of the model, we convert this to a threshold on the velocity lag, \( \Delta v_{\text{pin}} = (v_1 - v_2) \sim 10^4 \) m s\(^{-1} \).

Vortex pinning at lattice defects is also possible. Link & Cutler (2002) argued that if the pinned (coupled) fluid makes up 1 per cent of the total moment of inertia, then pinning must occur throughout the crust, suggesting that the crust should be treated as an amorphous solid with random pinning sites. The randomness ensures that the contributions to the force of attraction towards the nuclear sites on either side of the vortex line do not cancel (Jones 1998a). Another possibility is to adopt the random pinning potential used to model flux creep in superconductors (Bassler & Paczuski 1998).

Vortices unpin when the relative velocity of the superfluid and the vortex lines is large enough to produce a Magnus force \( F_M \) greater than the pinning threshold \( F_{\text{thresh}} \). If the Magnus force does not exceed \( F_{\text{thresh}} \), the pinning force adjusts to balance the Magnus force exactly. Given that the vortices are assumed to be inertialless, the excess force beyond the pinning threshold is of no relevance, as once the vortices unpin they flow with the ambient superfluid:

\[ F_{\text{pin}} = \begin{cases} F_M & \text{if } F_M < F_{\text{thresh}}, \\ 0 & \text{otherwise.} \end{cases} \]  

(8)

In our model, the superfluid velocity at a given point is the sum of three components: the contribution from all pinned vortices that lie on or within the star-centred circle passing through the point (these vortices are assumed to be in a regular Feynman array on average); the contribution from all of the unpinned vortices lying within the above circle (also assumed to be in a regular Feynman
unpinned superfluid. Over short time-scales, we include thermally ramps up as the lag builds up between the pinned vortices and the

\[ F_{NN} \]

are calculated using equation (4), assuming that the velocity field is

The first two contributions, hereafter named the mean-field terms, are calculated using equation (4), assuming that the velocity field is exactly tangential to the closed path (a good approximation which becomes exact in the thermodynamic limit \( N_v \to \infty \)).

By assuming that the vortices are arranged in a regular array, we can evaluate equation (4) with tangential \( v_s \), without knowing the location of each vortex (the validity of this approach is verified in Section 5.1). The Magnus force is then calculated from equation (3), with \( v_L \) equal to the velocity of the stellar crust and \( v_{s,NC} \) given by equation (9). Initially, the mean Magnus force is very small. As the star spins down, however, the pinned contribution does not diminish, while the unpinned contribution does, and the mean \( \Delta v \) increases.

We emphasize that the automaton represents a locally averaged model where each cell covers many pinning sites. Most of these sites are unoccupied because, for pinning at lattice defects (cf. seismic faults), the total number of vortices is much less than the number of pinning sites.

### 4.4 Driver

In this paper, we consider two separate driving forces. Over long time-scales (>\( \delta \nu/\dot{\nu} \)), the subject of Section 5.7, the Magnus force ramps up as the lag builds up between the pinned vortices and the unpinned superfluid. Over short time-scales, we include thermally activated unpinning of the vortex bundles in random cells. In both scenarios (short and long time-scales), small variations in \( \Delta \nu = v_L - v_s \) due to the eight nearest-neighbour cells trigger avalanches in the system. By treating the nearest-neighbour interactions with greater precision than the longer range contributions, we are emphasizing the property outlined in condition (i) of Section 2.

### 4.5 Operational definition of glitches

In the automaton model, a glitch corresponds to the unpinning of vortex bundles. Its size is determined by the total number of bundles (and hence vortices) unpinned. The associated fractional spin-up of the pulsar is then given by the fraction of the total moment of inertia carried by the recently unpinned fluid. For example, if each vortex bundle represents the unpinning of fluid carrying \( 10^{-3} \) of the moment of inertia of the system, then an avalanche of 10 bundles results in a glitch of size \( \delta \nu/\dot{\nu} = 10^{-6} \).

In practice, the angular momentum transferred to the crust by vortex unpinning depends on the product of the number of vortices unpinned (equivalently, the reduction in superfluid rotation rate) and the moment of inertia of the region through which they move before repinning, or until the avalanche stops, reaching a new quasi-equilibrium state. The reason for this is found in the Onsager–Feynman relation (4). The superfluid velocity at some distance from the rotation axis is proportional to the number of vortices enclosed, so the further the unpinned vortices travel radially, the more the superfluid decelerates, and hence the more angular momentum is transferred to the crust. The definition of glitch size given in this paper is therefore an approximation, valid only when three conditions are met: (i) all vortices travel the same radial distance during an iteration of the automaton; (ii) avalanches peter out after traveling one or two grid cells radially; and (iii) there are no steep radial gradients of vortex density (trivially satisfied by our uniform automaton, but not necessarily realistic). We show in Section 5.4 that these conditions are fulfilled by our automaton primarily because the unpinning activity is restricted to a narrow annulus. However, a more sophisticated automaton which incorporates radial gradients is needed to address this issue properly.

To facilitate future comparisons between the model and data, we clarify that the automaton describes spin-up events where the jump in spin frequency is unresolved by current timing experiments; the
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4.7 Vortex motion

The rules for moving vortices to an adjacent cell following an unpinning event rely on the (incorrect) assumption that $|\nu|_i$ is the same in the vicinity of all supercritical cells, such that the time taken to travel from one pinning site to the next is the same throughout the grid. That is, the discrete nature of the automaton means that the small time-step sets the characteristic intercell travel time. This approximation is justified in two ways in the model. First, as long as the time to move one cell is much shorter than the driver time-scale, it is reasonable to set the small time-step using the slowest vortices. If, as we claim in Section 5.1, most of the avalanche activity takes place in a thin annulus, this approach is accurate, because $|\nu|_i$ is roughly uniform over the annulus. Secondly, only half the vortices in a supercritical cell are redistributed to its neighbours. The travel time for vortices at the far side of the supercritical (departure) cell (relative to the destination cell) is greater than the travel time for vortices starting near the border. Crudely, if vortices are evenly spaced throughout the departure cell, about half of them have sufficient time to reach the destination cell during a small time-step. As a corollary, the model contains a dissipative mechanism, which ensures that avalanches eventually terminate. This feature means that our system is non-conservative. Following Pruessner & Jensen (2002), we require an external driving force to maintain the system in a critical state.

5 STATISTICS

5.1 General features

Three quantities, in particular, are used to characterize our cellular automaton model of pulsar glitches: the distribution of glitch sizes $\delta \nu/\nu$, durations $t$ and waiting times $\Delta t$. The glitch duration is defined as the number of consecutive time-steps at which there exist supercritical cells.

The hypothesis that the automaton results in avalanche dynamics, driven slowly by electromagnetic spin-down or random thermal unpinning, is now tested. The model is allowed to run for $10^5$ big time-steps. In Sections 5.2–5.6, we consider time periods over which $\nu$ does not decrease significantly. In Section 5.7, we consider longer time periods over which spin-down becomes important. We characterize the model by exploring its response to changes in the controlling parameters, both physical and computational. The physical parameters are the spin frequency $\nu$, the pinning threshold $\Delta \nu_{\text{pin}}$ and the fraction of vortices that are pinned $\epsilon$. The computational parameters are the numbers of bundles $N_{\text{num}} \equiv BN_{\text{grid}}^2$ and grid cells $N_{\text{grid}}^2$.

Fig. 3 shows the raw output of the model in the form of a time-series of glitch sizes for a standard set of parameters (defined in Table 2). Unless otherwise specified, these are the parameters used to generate the model output. This particular set of parameters is chosen to balance the need for sufficient avalanches to generate good statistics versus computational efficiency. The vertical axis gives the size in terms of the fraction of the total superfluid moment of inertia that unpins during the avalanche. The inset zooms in on a segment of the time-series spanning $10^3$ big time-steps. Although the simulation runs for a total of $10^5$ time-steps (without spin-down), the first $10^4$ time-steps are discarded to allow the system to establish stationary statistics. Jensen (1998) asserted that fractal dynamics necessarily arise in a SOCS. Although we do not endeavour to quantify this mathematically, we are able to demonstrate, by considering time-series such as the one shown in Fig. 3, that our system
Section 5.7). Even more encouragingly, we find (for the parameter ranges studied in this paper, we find $2.0 \leq a \leq 4.3$ and $2.2 \leq b \leq 5.5$) that for $10^5$ avalanches, the correlation coefficient is $\epsilon = 0.01$. The main panel displays a time-series of avalanche sizes beginning after $10^4$ time-steps elapse, in order to ensure that a self-organized critical state has time to emerge from the initially random conditions. The inset zooms in on $5 \times 10^3$ big time-steps. Bottom panel: probability density functions of $\delta v/v$ (left-hand panel), durations $t$ (centre) and $\Delta t$ (right-hand panel) plotted as histograms on a log–log (left-hand and centre panels) and log-linear (right-hand panel) scale, using $10^2$ logarithmically (left-hand and centre panels) and linearly (right-hand panel) spaced bins. The same parameters are used as for the top panel. The size and duration distributions are fitted with power laws of the form $p(\delta v/v) = (\delta v/v)^{-a}$ and $p(t) \propto e^{1.2t}$, respectively. The waiting time distribution is fitted with an exponential of the form $p(\Delta t) = e^{-0.01\Delta t}$, where $\lambda$ is in units of the reciprocal of one big time-step. All fitted functions are plotted as dashed lines.

Figure 3. Top panel: automaton output for $9 \times 10^4$ big time-steps. The radius of the star is $R = 10$ km, the unpinning threshold is $|\Delta v_{\text{pin}}| = 1.0 \times 10^6$ ms$^{-1}$, the neutron superfluid density is $\rho_s = 10^{16}$ kg m$^{-3}$, the initial spin frequency is $v_0 = 30.25$ Hz and the current spin frequency is $\nu = 13.9$ Hz. The avalanche size is given as the fraction of the total fluid moment of inertia that unpins during the avalanche. The fraction of pinned vortices is taken to be $\epsilon = 0.01$. The main panel displays a time-series of avalanche sizes beginning after $10^4$ time-steps elapse, in order to ensure that a self-organized critical state has time to emerge from the initially random conditions. The inset zooms in on $5 \times 10^3$ time-steps. Bottom panel: probability density functions of $\delta v/v$ (left-hand panel), durations $t$ (centre) and $\Delta t$ (right-hand panel) plotted as histograms on a log–log (left-hand and centre panels) and log-linear (right-hand panel) scale, using $10^2$ logarithmically (left-hand and centre panels) and linearly (right-hand panel) spaced bins. The same parameters are used as for the top panel. The size and duration distributions are fitted with power laws of the form $p(\delta v/v) = (\delta v/v)^{-a}$ and $p(t) \propto e^{1.2t}$, respectively. The waiting time distribution is fitted with an exponential of the form $p(\Delta t) = e^{-0.01\Delta t}$, where $\lambda$ is in units of the reciprocal of one big time-step. All fitted functions are plotted as dashed lines.

produces self-similar dynamics; qualitatively, the time-series looks the same whether we plot $10^5$ or $10^4$ big time-steps.

The lower panel of Fig. 3 shows the probability density functions of glitch sizes $\delta v/v$, durations $t$ and waiting times $\Delta t$, plotted as histograms, for the standard set of parameters defined in the caption. Both $\delta v/v$ and $t$ obey power-law statistics, with probability density functions $p(\delta v/v) \propto (\delta v/v)^{-a}$ and $p(t) \propto e^{-b}$, respectively. This is a characteristic feature of a SOCS (see Section 2.1). For the standard parameters in Fig. 3, we obtain $a = 2.9$ and $b = 4.8$. More generally, for the parameter ranges studied in this paper, we find $2.0 \leq a \leq 4.3$ and $2.2 \leq b \leq 5.5$ (see Sections 5.2–5.6). In a SOCS, $a$ and $b$ are related by (Jensen 1998):

$$b = 1 + (a - 1)\frac{\gamma_2}{\gamma_3},$$

such that the size of an avalanche scales with its linear extent $L$ and $(\delta v/v) \propto L^{\gamma_2}$ and the duration scales as $t \propto L^{\gamma_3}$. The exponents in Fig. 3 imply $\gamma_2/\gamma_3 \approx 2.0$, which is exactly what one expects for compact, two-dimensional avalanches with $\gamma_2 = 2$ and $\gamma_3 = 1$ (see Section 5.7). Even more encouragingly, we find $(b - 1)/(a - 1) = \gamma_2/\gamma_3 \approx 2.0$ in many of our numerical experiments where criticality applies, even though $a$ and $b$ individually cover wide ranges.

Based on the data analysed in Melatos et al. (2008), as well as the likeness between our model and other cellular automata that display self-organized critical behaviour, we expect the distribution of waiting times to be exponential, in keeping with a Poisson process (statistically independent avalanches). The bottom right-hand panel in Fig. 3 confirms this expectation: the waiting time probability density function is $p(\Delta t) \propto e^{-0.01\Delta t}$, with $\lambda = 6.8 \times 10^{-3}$ in units of an inverse big time-step. To further verify the property of statistical independence, we calculate the linear Pearson correlation coefficient relating the size of an avalanche and the size of the avalanche immediately preceding it, as plotted in Fig. 4. We find that for $10^5$ avalanches, the correlation coefficient is $-0.012$, implying that there is almost no correlation between an avalanche and its predecessor.

Contrary to the general idea that a SOCS tends naturally to its critical state, we are forced to fine tune the parameters to elicit avalanches. The system is particularly sensitive to $\epsilon$ and $|\Delta v_{\text{pin}}|$. Similarly, the exponents of the best power-law fits to the glitch size and duration distributions (see Sections 5.2 and 5.6) depend
on the same tuned parameters. This is also observed to be the case experimentally when superconducting vortices respond to a change in the ramp rate of the magnetic field (Field et al. 1995).

In cellular automaton models of forest fires, the number of trees between two ignitions needs to be tuned according to system size, to avoid a cut-off or bump in the size distribution (Pruessner & Jensen 2002).

Before analysing the results produced by the cellular automaton, we assess the accuracy of the mean-field approximation outlined in Section 4.3. Fig. 5 shows the velocity imbalance across the grid (as both a vector field and along a linear cut) generated using the mean-field approximation for a random distribution of vortex bundles (with a mean occupancy of one bundle per cell). The result is compared with an exact, ‘N-body’ algorithm, in which \( \Delta v \) at any cell is computed by summing the \( v_i \) produced at that cell by every bundle on the grid. The exact algorithm is not used elsewhere in this paper due to computational cost. Discrepancies between the mean field and exact calculations are greater when a line is taken proportional to the square root of its contents. The fitted exponent is estimated using a least-squares algorithm with the fitted exponent indicated on each plot. By the central limit theorem, the uncertainty in each bin is inversely proportional to the square root of its contents. The fitted exponent is shown as a function of \( N_{\text{grid}} \) in Fig. 7, with errorbars taken from the least-squares algorithm. There is a trend in both \( a \) and \( b \) to increase with increasing \( N_{\text{grid}} \), in keeping with the increasing dynamic range with \( N_{\text{grid}} \). When the simulation is run for \( 2 \times 10^5 \) time-steps, the dynamic range does, as expected, increase; however, it still falls short of \( G_{\text{max}}/G_{\text{min}} \). The number of bundles available to unpin is restricted to a small fraction (\( \sim 15\% \)) of \( BN_{\text{grid}}^2 \) that lie in the vicinity of the critical circle. To better match the desired dynamic range, we should select the grid size such that \( G_{\text{max}}/G_{\text{min}} \) cells lie in the active zone near the critical circle.

The glitch sizes generated by the model should be treated as indicative only. Evidently, several of the pulsars analysed by Melatos et al. (2008) exhibit glitches several orders of magnitude smaller than those produced by the automaton. It is easy to elicit such small glitches from the automaton by adjusting \( F_{\text{pin}} \) or \( v - v_0 \) so that the active circle lies closer to the rotation axis, thereby enclosing fewer vortices. We have not explored these possibilities here as there is too much freedom in the choice of parameters at present.

(See Section 5.5 for further discussion.) If pinning is strongest in two or more annuli within the crust, the model predicts that glitches fall into two or more size brackets, depending on where the annuli are located. It should be noted, however, that Melatos et al. (2008) found no evidence for a bimodal distribution when analysing the latest statistics from individual pulsars, whose size distributions are consistent with unbroken power laws.

### 5.2 Dynamic range

In Section 4.1, we motivate the choice of grid size by referring to the observed dynamic range of pulsar glitches. In any individual pulsar the maximum dynamic range of \( \Delta v/v \) is \( 10^4 \) (for PSR J1704–3015), and the minimum is \( 10^1 \) (for PSR J0537–6910). Hence, from equation (6), we run our simulation on grids ranging from \( N_{\text{grid}} = 10 \) to 100 (the physical size remains unchanged at \( R = 20 \) km). By construction, the range of avalanches outputted by the model should not exceed \( G_{\text{max}}/G_{\text{min}} \) (i.e. \( 10^4 \) for the \( 10 \times 10 \) grid and \( 10^5 \) for \( N_{\text{grid}} = 100 \) grid). Fig. 6 confirms this expectation. The top panels are histograms of the avalanches produced by systems with \( N_{\text{grid}} = 100, N_{\text{grid}} = 60 \) and \( N_{\text{grid}} = 10 \) over 10\(^5\) time-steps. The dynamic range is greatest for the largest grid (\( 10^{-6.1} \leq \Delta v/v < 10^{-4.2} \)) and smallest for the smallest grid (\( 10^{-4.2} \leq \Delta v/v < 10^{-3.2} \)). It is not expected that these dynamic ranges span the full range \( G_{\text{min}} \leq \Delta v/v \leq G_{\text{max}} \), as many of the grid cells lie inactive, well within the critical circle discussed in Section 5.1. It is, however, encouraging that the larger grid sizes do in fact produce a larger dynamic range.

Each of the histograms in Fig. 6 is overlaid with a power-law best fit of the form \( p(\Delta v/v) \propto (\Delta v/v)^{-a} \), estimated using a least-squares algorithm, with the fitted exponent indicated on each plot. The central limit theorem, the uncertainty in each bin is inversely proportional to the square root of its contents. The fitted exponent is shown as a function of \( N_{\text{grid}} \) in Fig. 7, with errorbars taken from the least-squares algorithm. There is a trend in both \( a \) and \( b \) to increase with increasing \( N_{\text{grid}} \), in keeping with the increasing dynamic range with \( N_{\text{grid}} \). When the simulation is run for \( 2 \times 10^5 \) time-steps, the dynamic range does, as expected, increase; however, it still falls short of \( G_{\text{max}}/G_{\text{min}} \). The number of bundles available to unpin is restricted to a small fraction (\( \sim 15\% \)) of \( BN_{\text{grid}}^2 \) that lie in the vicinity of the critical circle. To better match the desired dynamic range, we should select the grid size such that \( G_{\text{max}}/G_{\text{min}} \) cells lie in the active zone near the critical circle.

The glitch sizes generated by the model should be treated as indicative only. Evidently, several of the pulsars analysed by Melatos et al. (2008) exhibit glitches several orders of magnitude smaller than those produced by the automaton. It is easy to elicit such small glitches from the automaton by adjusting \( F_{\text{pin}} \) or \( v - v_0 \) so that the active circle lies closer to the rotation axis, thereby enclosing fewer vortices. We have not explored these possibilities here as there is too much freedom in the choice of parameters at present.

(See Section 5.5 for further discussion.) If pinning is strongest in two or more annuli within the crust, the model predicts that glitches fall into two or more size brackets, depending on where the annuli are located. It should be noted, however, that Melatos et al. (2008) found no evidence for a bimodal distribution when analysing the latest statistics from individual pulsars, whose size distributions are consistent with unbroken power laws.

### 5.3 Vortex bundling

An underlying assumption of the model is that the scale on which the pinning centres and vortices is coarse grained does not alter the behaviour of the output qualitatively. In this section, we explore the impact of maintaining the grid size whilst varying the mean number of bundles per cell \( B \). The simulation is designed such that the smallest (largest) avalanche occurs when one (every) bundle unpins. In a scale-invariant system, avalanches should occur below

\[ \frac{\Delta v}{v} = \frac{1}{10^4} \]

\[ \frac{\Delta v}{v} = \frac{1}{10^3} \]

5 This is one of two pulsars that glitch quasi-periodically; the other is Vela.
Figure 5. Left-hand panel: a vector plot of the velocity imbalance $\Delta v = v_L - v_s$, generated using the mean-field approximation described in equation (9). The inset zooms in on the central $10 \times 10$ grid cells. The solid black circle shows the position of the critical radius, where the mean-field contribution to $\Delta v$ is approximately equal to $\Delta v_{\text{pin}}$. Right-hand panel: profiles of $\Delta v$ along a straight-line cut through the grid, as a function of distance from the origin, as calculated by the mean-field (black) and exact ($N$-body) (grey) algorithms. The solid and dashed curves correspond to the vertical and diagonal cuts drawn in the left-hand panel. The two vertical dashed lines show the position of the critical circle. The horizontal dashed line marks $\Delta v_{\text{pin}}$.

Figure 6. Probability density functions $p(\delta v/\nu)$ of avalanche sizes $\delta v/\nu$, plotted as histograms on a log–log scale, using $10^2$ logarithmically spaced bins, for $10^5$ big time-steps. The histograms are fitted with a power law of the form $p(\delta v/\nu) \propto (\delta v/\nu)^{-a}$ (dashed lines). Top panel: grids with $N_{\text{grid}} = 100$ (left-hand panel), 60 (centre panel) and 10 (right-hand panel). All panels have bundle factor $B = 1$. Middle panel: bundle factors $B = 1$ (left-hand panel), $10^3$ (centre panel) and $10^5$ (right-hand panel). All panels have $N_{\text{grid}} = 100$. Bottom panel: $B = 10^5$ and $N_{\text{grid}} = 100$ (left-hand panel), 60 (centre panel) and 20 (right-hand panel). All panels have $B = 10^5$. The vertical grey line denotes the crossover scale. Physical parameters are $|\Delta v_{\text{pin}}| = 10^4 \text{ m s}^{-1}$, $\rho_s = 10^{16} \text{ kg m}^{-3}$, $\nu_0 = 30.25 \text{ Hz}$, $\nu = 13.9 \text{ Hz}$, $\epsilon = 0.01$, $R = 20 \text{ km}$.

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Figure 7. Fitted power-law exponents for the distribution of glitch sizes (a) and durations (b). The errorbars represent the 1σ uncertainty returned by the least-squares fitting algorithm. Top left-hand panel: $10 \leq N_{\text{grid}} \leq 100$. Top right-hand panel: $9.5 \times 10^4 \leq |\Delta \nu_{\text{pin}}| \leq 10.4 \times 10^4$ ms$^{-1}$. Bottom left-hand panel: $0.0092 \leq \epsilon \leq 0.0101$. Bottom right-hand panel: $12.73 \leq \nu \leq 15.91$ Hz. Automaton parameters: $N_{\text{grid}} = 100, B = 1, 10^5$ big time-steps. Physical parameters: $\rho_s = 10^{16}$ kg m$^{-3}, v_0 = 30.25$ Hz, $R = 20$ km.

Figure 8. Probability density functions $p(t)$ of avalanche durations $t$, for bundle factors $B = 1$ (left-hand panel), $10^3$ (centre panel) and $10^5$ (right-hand panel), plotted as histograms on a log–log scale, using $10^2$ logarithmically spaced bins. The histograms are fitted with a power law of the form $p(t) \propto t^{-b}$ (dashed lines). Automaton parameters: $N_{\text{grid}} = 100, 10^5$ big time-steps. Physical parameters are as $|\Delta \nu_{\text{pin}}| = 10^4$ ms$^{-1}, \rho_s = 10^{16}$ kg m$^{-3}, v_0 = 30.25$ Hz, $\nu = 13.9$ Hz, $\epsilon = 0.01, R = 20$ km.

and above the observational limits. For this reason, we probe the effect of increasing the number of vortex bundles (equivalent to decreasing the number of vortices per bundle, for fixed $\nu$) on the distribution of avalanches.

The avalanche size distributions for different values of the bundle factor $B$ are shown in the middle panels of Fig. 6. The histograms are overlaid with a power-law best fit. For $B = 5 \times 10^2$ and $B = 1 \times 10^4$, there is an obvious ‘bump’ in the glitch size distribution, whose peak occurs at the same value of $\delta \nu / \nu$ for different $B[\log_{10}(\delta \nu / \nu)_{\text{bump}} = -4]$. The bump is also present in the duration distribution (see Fig. 8), and there is a clear turnover in the slope of the waiting time distribution (see Fig. 9), which may be interpreted as a crossover from power-law statistics to exponential decay. Importantly, crossover is evidence that the system is no longer in a SOC state, as there is a preferred scale for avalanches.

In SOCS generally, there is a system-size-dependent crossover scale above which the power-law distribution of glitch sizes gives way to exponential decay (see Section 2.1). In a genuine critical state, the crossover scale should be an increasing function of the system size (Jensen 1998). In the examples discussed by Jensen (1998), the system size corresponds to the number of particles in the system (e.g. rice or sand grains), not the physical extent of the
system. That the bump does not appear for small $B$ and becomes increasingly prominent for large $B$ suggests a system-size effect. On the other hand, the crossover scale, defined by the bump, does not increase with system size (i.e. $B$) in Fig. 6. Nor does system size explain why the bump is a local maximum in $p(\delta \nu/\nu)$ instead of a turnover. In their experiments on superconductors, Field et al. (1995) saw a crossover to steeper decay in the avalanche size distribution for large values of the magnetic field ramp rate, but not a local maximum. If instead we interpret the system size to be the number of cells, not bundles, then the bump becomes more prominent for larger systems (e.g. with $B = 1 \times 10^5$) as in Fig. 6. Despite the bump not being present for smaller systems ($N_{\text{grid}} \leq 40$), there is a value of $\delta \nu/\nu$, for each $N_{\text{grid}}$, above which $p(\delta \nu/\nu)$ falls off faster than a power law, which is the expected behaviour beyond the crossover scale defined by Jensen (1998) (see Fig. 6). Bumps like the ones in Fig. 6 also arise in SOCS when one changes the morphology of the grains (e.g. rice versus sand) (Frette et al. 1996; Jensen 1998; Pruessner & Jensen 2003).

The violation of scale invariance is at least partly due to a mismatch in how the pinning centres (cells) and vortices (bundles) are renormalized. The most scale-invariant results, closest in spirit to a SOCS, are obtained for $B = 1$ (one bundle per cell, on average). By increasing $B$ without changing $N_{\text{grid}}$, we are renormalizing the vortices on a smaller coarse grain, whilst maintaining the coarseness of the grid, which is not a justifiable physical scenario. Pruessner & Jensen (2002) claimed that inappropriately chosen parameters produce a bump in the size distribution, which further suggests that $B \gg 1$ alters the fundamental statistical properties of the driven system. The implications for the microphysics of pinning in pulsars are explored in Section 6.

In summary, bundling in the model introduces a preferred scale into the system. (This scale greatly exceeds the vortex quantum $\kappa$, which is too small to be resolved by a practical simulation.) When the average number of bundles per cell is kept near one, the system exhibits scale-invariant behaviour, as demonstrated by the power laws in glitch size. In this regime, the grid cells and bundles are well matched. If, however, the number of cells is significantly outnumbered by the vortex bundles, the scale invariance of the system is broken. In the absence of computational limitations, one should represent each pinning site and each vortex uniquely, eliminating the need for bundling. On the other hand, bundling confers a distinct advantage: it circumvents the subtleties of vortex–vortex interactions and reconnections by renormalizing these effects into effective vortex–vortex forces on a coarsened grid.

### 5.4 Pinning threshold

The pinning threshold $F_{\text{pin}}$ determines the minimum lag between the superfluid and pinned vortices necessary to unpin vortex bundles. The pinning threshold can be re-expressed in terms of a minimum $|\Delta \nu_{\text{pin}}|$ between the pinned vortices (comoving with the crust) and the unpinned superfluid.

As $|\Delta \nu_{\text{pin}}|$ increases, the number of avalanches occurring within a fixed time ($10^5$ big time-steps) decreases, from 33 549 for $|\Delta \nu_{\text{pin}}| = 9.5 \times 10^3$ ms$^{-1}$ to 2227 for $|\Delta \nu_{\text{pin}}| = 10.4 \times 10^3$ ms$^{-1}$, as shown in Fig. 10. There are two reasons for this trend. First, greater imbalance is needed in the nearest-neighbour cells to overcome the pinning force and trigger an avalanche. The greater is the required imbalance, the less probable it becomes. Second is the issue of fine tuning. For a given $|\Delta \nu_{\text{pin}}|$, there is a critical circle where the mean-field contribution to the velocity imbalance matches the pinning threshold almost exactly. Well outside the circle, every cell is supercritical and the vortices never pin. Near the circle, however, we find a mixture of sub and supercritical cells because the
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Figure 10. Probability density functions $p(\delta\nu/\nu)$ of avalanche sizes $\delta\nu/\nu$, plotted as histograms on a log–log scale, using $10^2$ logarithmically spaced bins. The histograms are fitted with a power law of the form $p(\delta\nu/\nu) \propto (\delta\nu/\nu)^{-a}$ (dashed lines). Top panel: current spin frequency $\nu = 15.9$ (left-hand panel), 14.7 (middle panel) and 13.1 Hz (right-hand panel). Bottom panel: pinning threshold $|\Delta v_{\text{pin}}| = 9.5 \times 10^3$ (left-hand panel), $9.9 \times 10^3$ (middle panel) and $10.4 \times 10^3$ ms$^{-1}$ (right-hand panel). Automaton parameters: $N_{\text{grid}} = 100, 10^5$ big time-steps, $B = 100$. Physical parameters are as $\rho_s = 10^{16}$ kg m$^{-3}$, $\nu_0 = 30.25$ Hz, $\epsilon = 0.01$, $R = 20$ km.

Figure 11. Images of the velocity imbalance $|\Delta v|$ for $|\Delta v_{\text{pin}}| = 10^3$ m s$^{-1}$ (upper panel) and $9 \times 10^3$ m s$^{-1}$ (lower panel). White cells are supercritical; the darker the cell, the smaller the $|\Delta v|$. Nearest-neighbour imbalance can act to reduce $|\Delta v|$ at a given cell, if it opposes the mean-field contribution. As $|\Delta v_{\text{pin}}|$ increases, the critical circle moves towards the origin, and hence the number of cells in its vicinity decreases. Fig. 11 depicts images of the grid after $10^5$ big time-steps. The colours encode the value of $|\Delta v|$ on each cell. Supercritical cells are white; all other cells are subcritical. The pinning threshold in the upper panel of Fig. 11 is $|\Delta v_{\text{pin}}| = 10^4$ m s$^{-1}$. We observe a smattering of supercritical cells near the critical circle. Inside the circle, all cells are subcritical. In the lower panel of Fig. 11, with $\Delta v_{\text{pin}} = 9 \times 10^3$ m s$^{-1}$, we observe an annulus in which most cells are supercritical (where the vortices never pin). Just inside the annulus, there is, like in the upper panel, a mixture of sub and supercritical cells. It is in these mixed regions where we claim that self-organized criticality controls the unpinning dynamics. These results are evidence that the unpinning activity is restricted to a narrow annulus, and hence unaffected by the use of a full circle as the simulation grid, as discussed in Section 4.5.

A power law describes the size distribution well for all three values of $|\Delta v_{\text{pin}}|$ in Fig. 10. The best-fitting power law and its exponent are shown in each panel of Fig. 10. We find that $a$ decreases as...
\(|\Delta v|\) increases, however there is a slight upturn in both \(a\) and \(b\) for the largest values of \(|\Delta v|\) considered. For larger values of \(|\Delta v|\), avalanches involving many cells (and hence many vortex bundles) are less likely to occur, both because the active annulus shrinks, and because it is relatively unlikely that the nearest-neighbour imbalance sends a cell supercritical. Hence, \(a\) decreases because the dynamic range decreases; the area under the histogram equals the number of avalanches.

Fig. 5 demonstrates that, for a finite, square vortex array, the profile of \(\Delta v\) along a linear cut flattens out in the region where \(r \approx R\), compared to smaller radii. Hence if \(|\Delta v|\) approaches the maximum \(\Delta v\) on the grid, more cells may participate in an avalanche, in comparison to when the critical circle lies further in. This may account for the upturn in \(a\) versus \(|\Delta v|\) seen in Fig. 7.

5.5 Pinned fraction

The pinned fraction \(\epsilon\) impacts on the dynamics in two main ways: it changes the size of each vortex bundle and alters the relative contribution to the velocity imbalance from each term in equation (9). Increasing the size of the vortex bundles increases the minimum avalanche size, whilst maintaining the maximum avalanche size. The larger bundles also enhance the role of the nearest-neighbour cells. As more vortices pin and unpin, the first and third terms in equation (9) decrease with respect to the contribution from the unpinned vortices. Given that the unpinned vortices are homogeneously distributed, they are the foremost contributors to the mean-field critical circle discussed in Section 5.4. Decreasing their relative contribution reduces the mean-field effect, emphasizing the nearest neighbour, stochastic dynamics of the model. The lower left-hand panel of Fig. 7 supports this argument: the \(1\sigma\) error in the fitted \(a\) and \(b\) values decreases with increasing \(\epsilon\). The lower left-hand panel of Fig. 7 also shows that both \(a\) and \(b\) increase with increasing \(\epsilon\). For \(\epsilon \geq 0.0097\), the \(a\) distribution flattens out, whereas the \(b\) distribution continues to increase.

For \(\epsilon \leq 0.009\), no avalanches are observed; the model is badly tuned. Moreover, for \(\epsilon \gtrsim 0.010\), the model slows down computationally, as a large number of cells regularly pin and unpin. This computational limitation arises because our algorithm does not excise regions in which vortices are permanently unpinned (see the lower panel of Fig. 11), instead calculating \(\Delta v\) unnecessarily in these regions at each small time-step. We intend to address this flaw in future work.

5.6 Spin frequency

Two values of the spin frequency enter the model: at birth (\(v_0\)) and today (\(v\)). As a first approximation, the total number of pinned vortices is taken to be unchanged over the lifetime of the star and hence is dictated by \(v_0\) as in equation (5). The pinned portion of the superfluid does not spin-down with the stellar crust, whereas the unpinned superfluid does; its circulation is dictated by \(v\). In the current model, we hold \(v\) and \(v_0\) constant (cf. Section 5.7), with \(v_0 = 30.25\) Hz, and \(12.73 \leq v \leq 15.91\) Hz.

By decreasing \(v\), we increase \(|\Delta v|\) and hence the force imbalance at every cell. The impact is similar to decreasing \(|\Delta v|\) because the critical circle shrinks. The two right-hand panels of Fig. 7 display a similar trend for increasing \(|\Delta v|\) and \(v\). The main difference is that \(v\) also determines the relative size of the first two terms in equation (9), given that the number of pinned vortices remains constant while the number of unpinned vortices decreases with \(v\). These effects increase the range of avalanche sizes as \(v\) decreases as shown in Fig. 10. For large values of \(v\), the dynamic range goes to zero, as the critical circle has moved outside the star. For small \(v\), a turnover emerges in the size distribution for small \(\delta v/v\), which is also present in the size distributions for small values of \(|\Delta v|\) in Fig. 10. This may happen because the critical circle lies well within the star, so there is a sizable region outside the critical circle where vortices are permanently unpinned. The automaton does not recognize that permanently unpinned vortices do not, in fact, participate in avalanches in real physical systems, so the output is biased towards large avalanches.

5.7 Long-term activity

To model the long-term glitch behaviour of a pulsar, we must allow \(v\) to mimic the electromagnetic spin-down of the stellar crust. Observed spin-down rates differ by several orders of magnitude; in this paper, we take \(\dot{\nu} = -7 \times 10^{-13}\) Hz s\(^{-1}\) by way of example. The observations of pulsar glitches span a period of approximately 40 yr (~1970 to present), which equates to a total spin-down of \(\sim 0.1\) Hz. In our model, each big time-step must therefore span \(\sim 10^6\) s, in order to represent a total observing time of 40 yr. Such a large time-step determines the scale on which we can resolve individual avalanches. Observationally, glitches can be resolved down to a 40 s window (McCulloch et al. 1990; Dodson et al. 2002).

Spin-down causes the critical circle to shrink towards the axis of the star, because the number of pinned vortices is fixed, whilst the number of unpinned vortices decreases with \(v\). The rate of spin-down thus determines how quickly the glitch behaviour changes as the critical circle moves through the strata of the star. Fig. 12 shows the size distribution (\(\delta v/v\)) for time-steps \(\Delta t_0 = 10, 10^2\) and \(5 \times 10^3\) s. Fig. 13 shows the fitted power-law exponents for the sizes and durations, for \(10 \leq \Delta t_0 \lesssim 5 \times 10^3\). Within the uncertainty of the fitting algorithm, \(a\) and \(b\) are the same for all values of \(\Delta t_0\) tested. For the largest \(\Delta t_0\), the time elapsed after \(10^3\) big time-steps is \(5 \times 10^3\) s, which equates to \(\Delta v = -0.007\) Hz. This moves the critical circle towards the origin by a negligible \(\sim 0.007\) per cent.

The long-term fate of a pulsar, with respect to its glitch behaviour, depends on the location of its pinning zones. If pinning can occur throughout the pulsar, glitches continue to occur indefinitely, as the critical circle moves inwards. As time passes, however, the glitches become smaller, on average, as the number of cells in the vicinity of the critical circle diminishes.\(^6\) If, as claimed by Donati & Pizzochero (2003), pinning occurs at intermediate densities, no glitches occur once the critical circle shrinks inside the radius at which these densities occur. Alternatively, if pinning occurs at several strata (high and low densities e.g. Avogadro et al. 2007), glitches occur when the critical circle lies in the vicinity of the pinning zones, which could mean that episodes of glitching are punctuated by ‘quiet’ periods.

In the models of post-glitch relaxation developed by Alpar et al. (1996), the superfluid spins down when vortices move radially outward. The outward motion is driven by the distribution of capacitive and depleted zones; outward motion is energetically favourable due to a radial bias in vortex–vortex scattering. In our automaton, spin-down is an input; it does not arise self-consistently from vortex motion. By reducing the number of unpinned vortices with time

\(^6\) Interestingly, there is no guarantee that glitches emanating from different critical circles are, in fact, drawn from the same statistical distribution (e.g. value of \(\alpha\)). If so, we do not expect a pure power law in \(\delta v/v\) over long times.

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A cellular automaton model of pulsar glitches

6 DISCUSSION

In this paper, we present a cellular automaton model for glitches in neutron stars, based on the vortex unpinning paradigm (Anderson & Itoh 1975; Alpar et al. 1984). Despite the gross idealizations in such a discrete model, the resulting statistical behaviour agrees with recent analyses of radio timing data (Wang et al. 2000; Wong et al. 2001; Melatos et al. 2008). Of particular interest are the power-law distributions of glitch sizes and durations produced by the model, providing further evidence that glitches are a scale-invariant phenomenon. For this reason, as observations improve in resolution and lengthen in duration, we expect to discover smaller and larger glitches than have been seen so far. Moreover, the exponential waiting time distribution generated by the model supports the idea that the each vortex avalanche is a statistically independent event. The values of the power-law exponents and Poisson glitching rates depend on the particular parameter set. Our fits return size and duration exponents in the ranges $2.0 \leq a \leq 4.3$ and $2.2 \leq b \leq 5.5$, respectively, and mean glitching rates in the range $0.0023 \leq \lambda \leq 0.13$. Although we do not have a first principles theory against which to compare our results, we claim that our cellular automaton is consistent with a SOCS.

The operation of the cellular automaton relies heavily on the assumption that vortices are inhomogeneously distributed. Alpar et al. (1996) suggested that crust cracking creates capacitive regions where many vortices pin simultaneously. These regions do not permit a vortex current, so pinned vortices in these regions do not participate in outward vortex creep. Crust cracking is a non-uniform process, suggesting that the regions themselves are inhomogeneously distributed. Our model conforms qualitatively with this picture. Of particular interest is the claim by Alpar et al. (1995) that the neutron star crust is divided coarsely into depleted and capacitive regions. If verified, this claim suggests that the level of coarse graining necessary to render our cellular automaton computationally tractable matches the level of inhomogeneity which is realistically present in a neutron star.

Of course, the opposite viewpoint is also viable: the crust forms a large, defect-free crystal, allowing the vortices to occupy a contiguous sequence of pinning positions and preventing the type of inhomogeneity represented by our initial conditions (Jones 1998a). Importantly, such a scenario challenges any model based on collective unpinning of numerous vortices, not just our cellular automaton. De Blasio & Lazzari (1998) concluded that the crust is relatively free of microcrystalline structures and vacancies, with impurities, or point defects, the most likely disruptions to the crystal lattice (faults and dislocations were not considered). Alternatively, if pinning occurs mostly at grain boundaries and cracks, as originally postulated by Anderson & Itoh (1975), then inhomogeneity on macroscopic scales is possible.

Irrespective of the density and distribution of pinning sites in reality, the model assumes for simplicity that vortex pinning occurs throughout the pulsar. Several recent studies have shown that

Figure 12. Probability density functions $p(\delta v/v)$ of avalanche sizes $\delta v/v$, for time-steps lasting $\Delta t_{\text{av}} = 10$ (left-hand panel), $10^3$ (centre panel) and $5 \times 10^5$ (right-hand panel), plotted as histograms on a log–log scale, using $10^2$ logarithmically spaced bins. The histograms are fitted with a power law of the form $p(\delta v/v) \propto (\delta v/v)^{-\alpha}$ (dashed lines). Automaton parameters: $N_{\text{grid}} = 100, 10^3$ big time-steps. Physical parameters are as $|\Delta v_{\text{pin}}| = 10^6$ m s$^{-1}$, $\rho_s = 10^{16}$ kg m$^{-3}$, $v_0 = 30.25$ Hz, $v = 13.9$ Hz, $\epsilon = 0.01$, $R = 20$ km.

Figure 13. Fitted power-law exponents for the distribution of glitch sizes ($a$) and durations ($b$), for time-steps in the range $10 \leq \Delta t_{\text{av}} \leq 5 \times 10^5$. The errorbars represent the 1σ uncertainty returned by the least-squares fitting algorithm. Automaton parameters: $N_{\text{grid}} = 100, 10^3$ big time-steps. Physical parameters are as $|\Delta v_{\text{pin}}| = 10^6$ m s$^{-1}$, $\rho_s = 10^{16}$ kg m$^{-3}$, $v_0 = 30.25$ Hz, $v = 13.9$ Hz, $\epsilon = 0.01$, $R = 20$ km.

(e.g. Section 5.7), we attempt to recreate the scenario in which outward radial motion eventually makes vortices exit the pinning zones in the star. The automaton rules do inadvertently allow for outward radial motion thanks to the rectangular nature of the grid: azimuthal motion (i.e. in the direction of rigid body rotation) is never truly possible, as the vortices can only ever move horizontally, vertically or diagonally. To incorporate outward motion and hence spin-down self-consistently, the automaton rules should add a radial component to the bulk superfluid flow when computing $v_L$. (This does not remove the artificial radial motion arising from the rectangular grid. In this context, an annular grid topology is preferable.)
pinning forces sufficient to oppose the Magnus force develop only in specific zones of the crust, e.g. regions of high and low, rather than intermediate, density (Avogadro et al. 2007). Restricting the simulation to these optimal pinning zones obviates the need to allow for a range of pinning strengths if the zones are thin layers. A model in which pinning does not occur everywhere in the star leads to a different computational topology, e.g. an annulus, or several nested annuli. However, we choose to be conservative in this introductory paper by not imposing a grid topology a priori, in order to see if a preferred topology emerges by itself. In fact, it does; we find that a narrow active annulus, situated at the critical radius, participates in the avalanches. However, it is important to experiment with other topologies in future work, because the active annulus we find in this paper may still be an artefact. If the active region lies near the rotation axis, an automaton executed in an annulus is free of the shortcomings introduced by the ‘permanently unpinned’ regions mentioned in Section 5.5. To use the full circular grid in a meaningful way, we must include pinning of proton and neutron vortices in the core of the neutron star, together with pinning of neutron vortices in the inner crust. In this case, the additional physics of entrainment (Link & Epstein 1996; Ruderman, Zhu & Chan 1998; Sedrakian & Sedrakian 1995; Andersson, Comer & Prio 2004) and induced electric currents implies a more detailed set of automaton rules, outside the scope of the model presented here. It should also be noted that if the superfluid vortices are tangled (Peralta et al. 2006b,a), rather than forming a regular Abrikosov array, these two-dimensional annular and circular topologies are clearly too restrictive.

The locations of pinning zones also determine the long-term fate of a pulsar as it spins down. If pinning occurs everywhere, spin-down increases the proportion of vortices with $|\Delta \nu| > |\Delta \nu_{\text{pin}}|$ which are always supercritical (outside the critical circle) and do not participate in avalanche dynamics as they never pin. If pinning is restricted to one or more annuli, the glitch behaviour of the pulsar changes significantly when the critical circle moves into and out of the pinning zones.

Clearly, our cellular automaton does not include the response of the pulsar to glitches (i.e. the spin-up of the pulsar crust and the subsequent ‘relaxation’). Nor do we account for the fine time-scale on which the unpinned superfluid couples to the crust, governed by the Hall–Vinen–Bekarevich–Khalatnikov equations (Peralta et al. 2005, 2006b). We make the approximation that this time-scale is restricted to one or more annuli, the glitch behaviour of the pulsar changes significantly when the critical circle moves into and out of the pinning zones.

To elicit avalanches from our model, we require fine tuning in both the physical and computational parameters, such that $|\Delta \nu_{\text{max}}| \approx |\Delta \nu_{\text{pin}}|$. This condition ensures that there are enough vortex bundles that switch between becoming sub and supercritical as time passes. We achieve this by choosing $\nu, \epsilon$ and $\Delta \nu_{\text{pin}}$ to place the critical circle near the surface of the star. We emphasize that for a larger (smaller) star, we would resize the critical circle proportionally, for computational rather than physical reasons. Fine tuning is also required in the ratio of vortex bundles to grid cells $B$. For $B \gg 1$, the automaton output is no longer consistent with a SOCS.

In conclusion, we present an empirical cellular automaton model of pulsar glitches based on the mass vortex unpinning paradigm. We find that for certain physical and computational parameters the model produces dynamics that are consistent with a SOCS and with radio timing data from pulsars. There exists no general, first principles theory of SOC, let alone of pulsar glitches, so many of our results are empirical. In particular, there is no way known at present to predict theoretically the size and duration distribution exponents, and the mean rate of the waiting time distribution. We do demonstrate that the basic physical principles governing inter vortex interactions can produce the type of collective behaviour necessary to explain pulsar glitches within the mass unpinning paradigm, especially the puzzle of how so many vortices can unpin in sympathy during a glitch, and why their number varies so much from glitch to glitch.

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