THE LOVÁSZ-CHERKASSKY THEOREM FOR
LOCALLY FINITE GRAPHS WITH ENDS
– NOTE –

RAPHAEL W. JACOBS, ATTIKA JOÓ, PAUL KNAPPE, JAN KURKOFKA, AND RUBEN MELCHER

Abstract. Lovász and Cherkassky discovered independently that, if $G$ is a finite graph and $T \subseteq V(G)$ such that the degree $d_G(v)$ is even for every vertex $v \in V(G) \setminus T$, then the maximum number of edge-disjoint paths which are internally disjoint from $T$ and connect distinct vertices of $T$ is equal to $\frac{1}{2} \sum_{t \in T} \lambda_G(t, T \setminus \{t\})$ (where $\lambda_G(t, T \setminus \{t\})$ is the size of a smallest cut that separates $t$ and $T \setminus \{t\}$). From another perspective, this means that for every vertex $t \in T$, in any optimal path-system there are $\lambda_G(t, T \setminus \{t\})$ many paths between $t$ and $T \setminus \{t\}$. We extend the theorem of Lovász and Cherkassky based on this reformulation to all locally-finite infinite graphs and their ends. In our generalisation, $T$ may contain not just vertices but ends as well, and paths are one-way (two-way) infinite when they establish a vertex-end (end-end) connection.

1. Introduction

A non-trivial path $P$ is a $T$-path for a set $T$ of vertices if $P$ has its endvertices but no inner vertex in $T$. For disjoint vertex sets $X$ and $Y$ in a graph $G$, we write $\lambda_G(X, Y)$ for the size of a smallest cut in $G$ that separates $X$ and $Y$.

Now let $T$ be any set of vertices in a finite graph $G$. In a set $P$ of edge-disjoint $T$-paths, there are at most $\lambda_G(t, T \setminus \{t\})$ many paths that link a vertex $t \in T$ to $T \setminus \{t\}$. It follows that $|P| \leq \frac{1}{2} \sum_{t \in T} \lambda_G(t, T \setminus \{t\})$. The question about the sharpness of this upper bound can be formulated in the following structural way. For every vertex $t \in T$, let $P_t$ be a set of $\lambda_G(t, T \setminus \{t\})$ many edge-disjoint $t$–($T \setminus \{t\}$) paths.

Question. Can we choose the paths in the sets $P_t$ for each vertex $t \in T$ in such a way that the union of all the sets $P_t$ is an edge-disjoint path-system?

Clearly, the answer is no: the three leaves of a star $K_{1,3}$ form a set $T$ where each set $P_t$ must consist of a single path, but the union $\bigcup_{t \in T} P_t$ always contains two distinct paths that share an edge, no matter how we choose the paths in each set $P_t$. Lovász and Cherkassky independently showed that, perhaps surprisingly, the answer is yes under the additional assumption that the graph $G$ is inner-Eulerian for $T$ in that every vertex of $G$ which is not in $T$ has even degree in $G$.

Theorem 1.1 (Lovász-Cherkassky Theorem [4,15]). Let $G$ be any finite graph, and let $T \subseteq V(G)$ such that $G$ is inner-Eulerian for $T$. Then the maximum number of pairwise edge-disjoint $T$-paths

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in $G$ is equal to

$$\frac{1}{2} \sum_{t \in T} \lambda_G(t, T \setminus \{t\}).$$

Diestel asked whether Theorem 1.1 extends to all locally-finite infinite graphs and their ends [9]. In this note, we answer his question in the affirmative. For this, we employ the Freudenthal compactification in the spirit of [6, 7, 13], as customary in the study of locally-finite infinite graphs [2, 3, 5, 10–12]. For infinite graphs and their ends, we follow and assume familiarity with the terminology in [8, §8], in particular in §8.6.

We allow a graph to have parallel edges, but we do not allow any loops; if a graph has no parallel edges, we call it simple. The degree of a vertex $v$ in a graph $G$ is the number $d_G(v) \in \mathbb{N} \cup \{\infty\}$ of edges of $G$ incident with $v$. If all the vertices of $G$ have finite degree, then we say that $G$ is locally finite. Note that in a locally finite graph, there can be only finitely many parallel edges between any two vertices. We write $\hat{V}(G)$ for the union of the vertex set $V(G)$ of $G$ and the set $\Omega(G)$ of all ends of $G$. An arc $A$ in the end compactification $|G|$ of a locally-finite connected graph $G$ is a $T$-arc for a set $T \subseteq \hat{V}(G)$ if $A$ has its endpoints but no inner points in $T$. An arc $A$ is an $X$–$Y$ arc between two sets $X$ and $Y$ if $A$ intersects $X$ precisely in one endpoint and $Y$ precisely in the other. We call an arc graphic if it is defined by a finite graph-theoretic path, a ray, or a double ray with its tails in distinct ends. Two arcs in $|G|$ are edge-disjoint if they do not meet in inner points of edges.

A subset $X \subseteq \hat{V}(G)$ lives in a subgraph $C \subseteq G$ or a vertex set $C \subseteq V(G)$ if all the vertices of $X$ lie in $C$ and all the rays of ends in $X$ have tails in $C$ or $G[C]$, respectively. A finite cut $F$ of a graph $G$ is an $X$–$Y$ cut for two sets $X, Y \subseteq \hat{V}(G)$ if $X$ and $Y$ live in distinct sides of $F$. If $Y \subseteq \hat{V}(G)$ is a set of vertices and ends of a locally finite graph $G$, and $x \in \hat{V}(G)$ is not contained in the closure of $Y$ in $|G|$, then $G$ admits a (finite) $x$–$Y$ cut, and we denote the least size of such a cut by $\lambda_G(x, Y)$.

Our main result reads as follows:

**Theorem 1.** Let $G$ be any locally-finite graph, and let $T \subseteq \hat{V}(G)$ be discrete in $|G|$. If every finite cut of $G$ such that $T$ lives on one of its sides is even, then $|G|$ contains a set $A$ of pairwise edge-disjoint graphic $T$-arcs such that for every $t \in T$, the number of $t$–$(T \setminus \{t\})$ arcs in $A$ is equal to $\lambda_G(t, T \setminus \{t\})$.

On first sight, one might wonder why the notion of ‘inner-Eulerian’ from the premise of Theorem 1.1 has been replaced in the premise of Theorem 1 with a new condition on the vertex sets $X$. In short, in finite graphs ‘inner-Eulerian’ implies the new premise, so Theorem 1 is more general; see Appendix A for a discussion of the necessity of the new condition for end-compactifications of locally finite graphs.

The assumption that $T$ is a discrete subset of $|G|$ naturally arises here as it is equivalent to asking that there exists a $t$–$(T \setminus \{t\})$ cut in $G$ for each end $t \in T$, which precisely ensures that $\lambda_G(t, T \setminus \{t\})$ is defined for all $t \in T$. We remark that the assumption that $T$ is discrete is
also motivated by the work of Bruhn, Diestel, and Stein [2], which is a generalisation of the Erdős-Menger theorem by Aharoni and Berger [1] from infinite graphs to infinite graphs and their ends, under a similar assumption on the ends which implies discreteness in our setting. (Diestel discusses this assumption in detail in [5, §3].)

We conclude the introduction with an example that discusses why a natural weakening of the discreteness-assumption in Theorem 1 cannot be made.

\[ \mu_G(t, T \setminus \{t\}) = \begin{cases} 4 & \text{if } t \in T \cap V(G), \\ 1 & \text{if } t \in T \cap \Omega(G). \end{cases} \]

But any \( t-(T \setminus \{t\}) \) arc for some \( t \in T \cap \Omega(G) \) does already preclude the existence of four \( t'-(T \setminus \{t'\}) \) arcs for some of the \( t' \in T \cap V(G) \), and hence there is no desired arc-system for \( T \).

1.1. The Lovász-Cherkassky Theorem for infinite graphs without ends. The second author has extended Theorem 1.1 to infinite – not necessarily locally finite – graphs where the set \( T \) is a countably infinite set of vertices, but does not contain ends. A cut \( F \) is said to lie on a set \( P \) of edge-disjoint paths in \( G \) if \( F \) consists of a choice of exactly one edge from each path in \( P \). For a vertex set \( X \subseteq V(G) \), we denote by \( d_G(X) \) the number of edges of \( G \) between \( X \) and its complement \( V(G) \setminus X \). Note that this notation is consistent with the above definition of the degree \( d_G(v) \) of a vertex \( v \) of \( G \) in that \( d_G(v) = d_G(\{v\}) \) for every vertex \( v \in G \).

**Theorem 1.3** ([14, Theorem 1.3]). Let \( G \) be any graph, and let \( T \subseteq V(G) \) be a countable vertex set such that there is no \( X \subseteq V(G) \setminus T \) for which \( d_G(X) \) is an odd natural number. Then \( G \) contains a set \( P \) of edge-disjoint \( T \)-paths such that for each vertex \( t \in T \), the graph \( G \) contains a \( t-(T \setminus \{t\}) \) cut on the set of \( t-(T \setminus \{t\}) \) paths in \( P \).

We will use Theorem 1.3 in the proof of Theorem 1.

Theorem 1.3 compares to Theorem 1 as follows. On the one hand, Theorem 1.3 can be applied in the setting of countable graphs that are not locally finite, where Theorem 1 cannot be applied. In the setting of locally finite graphs, on the other hand, Theorem 1 is more general than Theorem 1.3, as it allows \( T \) to consist of vertices and ends alike. Indeed, the locally-finite version

**Figure 1.** \( T \) consists of the black vertices and ends
of Theorem 1.3 can easily be re-obtained from Theorem 1 since every set of vertices of a locally finite graph $G$ is discrete in $|G|$ and does not contain any other vertices in its closure.

2. Proof of the main result

We need one auxiliary result for the proof Theorem 1, and to state this lemma we make the following definition. If $S$ is a set of vertices of a graph $G$ and $\omega$ is an end of $G$, then by an $S$–$\omega$ ray we mean a ray which has precisely its first vertex in $S$ and belongs to $\omega$.

Lemma 2.1 ([3, Lemma 10]). Let $G$ be a locally finite connected graph, let $\omega$ be an end of $G$, and let $S$ be a finite set of vertices in $G$. Then the maximum number of edge-disjoint $S$–$\omega$ rays is equal to the minimum size of a cut that separates $S$ and $\omega$.

Originally, Lemma 2.1 has been proved only for simple graphs. However, it extends to graphs with parallel edges: Given a graph with parallel edges, just subdivide each edge once, apply the original result, and then suppress all subdividing vertices.

Proof of Theorem 1. Without loss of generality, we may assume that $G$ is connected.

Let us first show that $T$ is countable. Since $G$ is locally finite, $|G|$ is second-countable, meaning that the topology on $|G|$ has some countable base $\mathcal{U}$. Recall that by assumption, $T$ is discrete in
\begin{equation}
|G| \text{. Therefore, we find for each } t \in T \text{ a basic neighbourhood } U_t \in \mathcal{U} \text{ so that } U_t \text{ and } T \setminus \{t\} \text{ are disjoint; in particular, } U_t \neq U_{t'} \text{ for distinct } t, t' \in T \text{. Then } T \text{ is countable because } \mathcal{U} \supseteq \{U_t : t \in T\} \text{ is countable.}
\end{equation}

Since \( T \cap \Omega(G) \) is countable, we may fix an enumeration \( (\omega_n : n < \kappa) \) of \( T \cap \Omega(G) \) where \( \kappa := |T \cap \Omega(G)| \leq N_0 \). Next, we recursively find for each \( n < \kappa \) an \( \omega_n-(T \setminus \{\omega_n\}) \) cut \( F_n \) of \( G \) such that the component \( C_n \) of \( G - F_n \) in which \( \omega_n \) lives is disjoint from the component \( C_m \) of \( G - F_m \) in which \( \omega_m \) lives for all \( m < \kappa \) other than \( n \) (see also Figure 2 for a visualisation of the whole proof).

Given any \( n < \kappa \), assume that we have already found suitable finite cuts \( F_i \) for all \( i < n \). Let \( G_n \) be the graph obtained from \( G \) by contracting each component \( C_i \) for \( i < n \) to a single vertex \( v_i \), keeping all the parallel edges that may arise. Since all cuts \( F_i \) for \( i < n \) are finite, the contraction minor \( G_n \) is again locally finite. Let

\[ T_n := (T \setminus \{\omega_i : i < n\}) \cup \{v_i : i < n\} \subseteq \hat{V}(G_n). \]

Note that \( |G_n| = |G| / \{\overline{C_i} : i < n\} \), since all the \( F_i \) are finite.

Since \( T \) is discrete in \( |G| \) and the components \( C_i \) for \( i < n \) are disjoint, \( T_n \) is discrete in \( |G_n| \). Therefore, the end \( \omega_n \) is not contained in \( T_n \setminus \{\omega_n\} \) where the closure is taken in \( |G_n| \). Thus, there is a smallest \( \omega_n-(T_n \setminus \{\omega_n\}) \) cut \( F_n^* \) in \( G_n \). This finite cut \( F_n^* \) in \( G_n \) defines a finite cut \( F_n \) of \( G \) of the same size.

Finally, we observe that the component \( C_n \subseteq G_n \) of \( G_n - F_n^* \) in which \( \omega_n \) lives does not contain any \( v_i \) with \( i < n \), since \( F_n^* \) separates \( \omega_n \) and \( T_n \setminus \{\omega_n\} \supseteq \{v_i : i < n\} \). Thus, \( C_n \) is also a component of \( G - F_n \) and disjoint from each previous \( C_i \). Altogether, this shows that the cut \( F_n \) is as desired.

Next, we simultaneously contract each component \( C_n \) for \( n < \kappa \) to a single vertex \( v_n \), again keeping all the parallel edges that may arise, and obtain a contraction minor \( G_\kappa \) of \( G \). As before, this contraction minor \( G_\kappa \) is locally finite since all the cuts \( F_n \) are finite. Let

\[ T_\kappa := (T \setminus \Omega(G)) \cup \{v_n : n < \kappa\}, \]

and note that \( T_\kappa \subseteq V(G_\kappa) \). Moreover, \( T_\kappa \) is countable because it has the same size as \( T \).

We show that there is no vertex set \( X \subseteq V(G) \) whose closure in \( |G| \) is disjoint from \( T \) and for which \( d_G(X) \) is an odd natural number: Let \( X \) be a set of vertices of \( G \) such that its closure in \( |G| \) is disjoint from \( T \) and \( d_G(X) \) is finite. It follows from the Jumping Arc Lemma [8, Proposition 8.6.3 (i)] that \( T \) lives in \( \hat{V}(G) \setminus X \). Hence, \( d_G(X) = |E_G(X, V(G) \setminus X)| \) is even by assumption.

This yields that there is no vertex set \( X \subseteq V(G_\kappa) \) whose closure in \( |G_\kappa| \) is disjoint from \( T_\kappa \) and for which \( d_{G_\kappa}(X) \) is an odd natural number. Since \( T_\kappa \) contains no ends, it further follows that there is no vertex set \( X \subseteq V(G_\kappa) \setminus T_\kappa \) for which \( d_{G_\kappa}(X) \) is an odd natural number. Therefore, we can apply Theorem 1.3 in \( G_\kappa \) to \( T_\kappa \) to obtain a set \( \mathcal{P} \) of edge-disjoint \( T_\kappa \)-paths in \( G_\kappa \) with the following property: For every vertex \( t' \in T_\kappa \), there is a cut \( F_{t'}^* \) of \( G \) on the set \( t'-(T_\kappa \setminus \{t'\}) \) paths in \( \mathcal{P} \). Note that all cuts \( F_{t'}^* \) are finite, because \( G_\kappa \) is locally finite and the paths in \( \mathcal{P} \) are
edge-disjoint. It remains to translate the set $\mathcal{P}$ of edge-disjoint $T_n$-paths in $G_\kappa$ into the desired set $A$ of pairwise edge-disjoint graphic $T$-arcs in $|G|$.

For every $n < \kappa$, the finite $\omega_n-(T \setminus \{\omega_n\})$ cut $F_n$ has smallest size. Let $S_n$ be the set of those endvertices of $F_n$ in $G - C_n$. By definition, $F_n$ is a minimal $S_n-\omega_n$ cut in $G$. So we can apply Lemma 2.1 to $S_n$ and $\omega_n$ to find a set $R_n$ of $|F_n|$ many edge-disjoint $S_n-\omega_n$ rays. Note that the rays in $R_n$ use only vertices of $V(C_n) \cup S_n$ since $F_n$ is a cut. In particular, there is for each edge in $F_n$ precisely one ray in $R_n$ which starts in this edge.

Now every $T_\kappa$-path $P \in \mathcal{P}$ in $G_\kappa$ uniquely defines a graphic $T$-arc in $|G|$ as follows: If the first or last edge $e$ of $P$ runs between a contraction vertex $v_n$ and another vertex $u$, then we replace it with the unique $u-\omega_n$ ray in $R_n$ which traverses the inner points of $e$ and add the end $\omega_n$ to it. Here we allow one exception: if $P$ consists of just one edge $e$ between two vertices $v_n$ and $v_m$ in $T_\kappa$, then we replace $e$ with the double ray which arises from the two rays in $R_n$ and $R_m$ that start in $e$, and add the ends $\omega_n$ and $\omega_m$ to it. In either case, let us write $A(P)$ for the graphic arc defined by $P$ in this way. We claim that $A := \{A(P) \mid P \in \mathcal{P}\}$ is the desired set of graphic $T$-arcs.

The arcs in $A$ are edge-disjoint because the paths in $\mathcal{P}$ are edge-disjoint, the components $C_n$ are disjoint, and the rays in each set $R_n$ are edge-disjoint. It remains to show that for each $t \in T$, the number of arcs in $A$ that link $t$ to $T \setminus \{t\}$ is equal to $\lambda_G(t, T \setminus \{t\})$. Given any $t \in T$, let $t' \in T_\kappa$ be equal to $t$ if $t$ is a vertex, and let $t' := v_n$ if $t$ is an end $\omega_n$. The finite $t'-(T_\kappa \setminus \{t'\})$ cut $F'_t$ of $G_\kappa$ witnesses that the set of $t'-(T_\kappa \setminus \{t'\})$ paths in $\mathcal{P}$ form a maximal sized edge-disjoint $t'-(T_\kappa \setminus \{t'\})$ path system in $G_\kappa$. Since the $t-(T \setminus \{t\})$ arc system defined by $A$ arises from this path system by replacing each path $P$ with the arc $A(P)$, the number of arcs in this system is equal to the size of the finite cut $F'_t$. As the $t'-(T_\kappa \setminus \{t'\})$ cut $F'_t$ of $G_\kappa$ induces a $t-(T \setminus \{t\})$ cut of $G$ of the same size, we have $\lambda_G(t, T \setminus \{t\}) \leq |F'_t|$, which completes the proof. 

\appendix
\section{A Degree Condition for Vertices and Ends}

We recall that an end $\omega$ of a locally finite graph $G$ is \textit{even}, or \textit{has even degree}, if there is a finite vertex set $S \subseteq V(G)$ such that for every finite set $S' \supseteq S$ of vertices, the maximum number of edge-disjoint $S'-\omega$ rays is even; otherwise, $\omega$ is \textit{odd} or \textit{has odd degree}. We refer to [3, §3] for a discussion of the parity of ends. With the parity of ends at hand, it seems natural to extend the notion of \textit{inner-Eulerian} from finite to locally-finite infinite graphs and their ends, as follows. A locally finite graph $G$ is \textit{inner-Eulerian} for a set $T \subseteq \hat{V}(G)$ if every vertex and every end in $\hat{V}(G) \setminus T$ have even degree in $G$.

This raises the following three questions:

1. Is it necessary to consider ends in this generalisation of ‘inner-Eulerian’?
2. Why does the premise of Theorem 1 not use this notion of ‘inner-Eulerian’?
3. How does the the notion of ‘inner-Eulerian’ compare to the premise of Theorem 1?

The first question is answered by the following example.
Example A.1. Theorem 1 becomes wrong if we replace its premise on the vertex sets $X$ with an ‘inner-Eulerian’ condition that ignores end degrees. Indeed, let us consider the tree $G$ in Figure 3, and let $T$ consist of its leaves. All the non-leaves of $G$ have even degree, but the unique end of $G$ has degree one. Any set $A$ of pairwise edge-disjoint $T$-arcs has size at most one, so $A$ contains no $t$–$(T \setminus \{t\})$ arc for at least one $t \in T$, even though $\lambda_G(t, T \setminus \{t\}) = 1$.

We answer the second question twofold: On the one hand, with the premise of Theorem 1 it is clear that Theorem 1 implies Theorem 1.3 for locally-finite graphs. On the other hand, the premise of $G$ being inner-Eulerian for $T \subseteq \hat{V}(G)$ is more specific than the premise on the finite cuts in Theorem 1, see Lemma A.3 below. The latter also answers the third question.

Lemma A.2 (Infinite Handshaking Lemma [3, Proposition 15]). The number of odd vertices and ends in a locally finite graph is even or infinite.

Lemma A.3. If $G$ is a locally finite graph which is inner-Eulerian for $T \subseteq \hat{V}(G)$, then every finite cut of $G$ such that $T$ lives on one of its sides is even.

Proof. Assume for a contradiction that there is an odd finite cut $E_G(A, B)$ such that $T$ lives in $A$. Consider the graph $H$ that arises from $G$ by contracting $G - B$ to a single vertex $v$, keeping parallel edges. Then $v$ has odd degree in $H$, but no other vertex or end of $H$ has odd degree, which contradicts Lemma A.2.

The converse of Lemma A.3 fails if $G$ is a ray and $T = V(G)$. Hence Theorem 1 is more general than its corollary below (which follows with Lemma A.3):

Corollary A.4. Let $G$ be any locally-finite graph, and let $T \subseteq \hat{V}(G)$ be discrete in $|G|$ such that $G$ is inner-Eulerian for $T$. Then $|G|$ contains a set $A$ of pairwise edge-disjoint graphic $T$-arcs such that for every $t \in T$, the number of $t$–$(T \setminus \{t\})$ arcs in $A$ is equal to $\lambda_G(t, T \setminus \{t\})$.

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Attila Joó: Universität Hamburg, Department of Mathematics, Bundesstrasse 55 (Geomatikum), 20146 Hamburg, Germany

Email address: attila.joo@uni-hamburg.de

Raphael W. Jacobs, Paul Knappe, Jan Kurkofka, Ruben Melcher: Universität Hamburg, Department of Mathematics, Bundesstrasse 55 (Geomatikum), 20146 Hamburg, Germany

Email address: raphael.jacobs@uni-hamburg.de, paul.knappe@uni-hamburg.de

Email address: j.lastname@bham.ac.uk, ruben.melcher@uni-hamburg.de