Scalars from Gauge Fields

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Abstract

In an Euclidean SU(2) ⊗ U(1) gauge theory without fermions, we identify scalar-field variables, functionals of the gauge fields and coming in different representations of isospin, which (i) are of mass dimension one in $d = 4$, (ii) couple to their parent gauge fields through suitable gauge-covariant derivatives, and (iii) can be endowed with a hypercharge despite their parents having none. They can be interpreted as projections of the gauge vectors onto an orthonormal basis that is defined by the fields themselves. We inquire as to whether these scalars can perform the usual tasks, normally fulfilled by external scalar fields, of spontaneous symmetry breaking and mass generation through vacuum expectation values. The gauge Lagrangian, expressed in terms of these scalars, automatically has quartic and cubic terms; no extra coupling constant for quartic scalar self-interactions is needed. VEV formation takes place in one of four scalar fields populating the classical potential-energy minimum. There are nine massive Higgs particles, a neutral triplet at a mass of $m_Z \sqrt{2}$, and three conjugate pairs of charged ones at $m_W \sqrt{2}$. Seven quasi-Goldstone scalars remain massless. This results in a qualitatively correct pattern of heavy-vector masses and mixing, with the analog of the mixing angle determined by theory. Higgs-type hypercharge and charge assignments emerge naturally.

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1. In search of “intrinsic” scalars

The Lorentz-scalar Higgs fields providing for spontaneous symmetry breaking and mass generation in present-day electroweak theory are introduced in addition to the gauge-vector fields. In this article we refer to those as scalars external to the gauge-vector system. It is a valid question whether such fields could, alternatively, arise from the dynamics of the gauge fields themselves. The latter, after all, possess self-interactions that could in principle stabilize scalar combinations with distinguishable experimental signals. Clearly, such intrinsic scalars, if indeed they are to play their assigned roles in electroweak theory, would from the outset have to meet nontrivial restrictions. To make mass formation along the known lines feasible, they would, in addition to being true Lorentz scalars, have to couple to their parent gauge fields through something like gauge-covariant derivatives. For this, in turn, they would need to be, like ordinary scalar fields, of mass dimension one in \( d = 4 \) dimensions (or \( 1 - \epsilon \) in \( d = 4 - 2\epsilon \) dimensions) in order for these interactions to fit into, or be derivable from, the known renormalizable gauge-field action. They would, moreover, have to carry the extra property of hypercharge, since it is only through this quantum number that the abelian U(1) field can couple dynamically. It is not clear a priori whether all these properties can be realized simultaneously. The question is not merely one of economy: it is known that the quartic self-interaction of the “external” scalar fields does not allow for asymptotic freedom, and it is still true that “there is today a widespread feeling that interacting quantum fields that are not asymptotically free .... are not mathematically consistent”[1]. In the following we hope to elucidate some sides of the above questions, not least because their study may reveal aspects of gauge theory itself that do not seem to have attracted attention previously.

In what follows, section 2 initiates, and section 3 completes, the identification of “intrinsic” scalar fields with desirable properties as functionals of the gauge-vector fields. This is done for the simplest case of a pure U(2) gauge theory, without either charged-vs.-neutrals mixing or coupling to fermions. Without the introduction of hypercharge, this theory is partly trivial, since in the absence of both fermions and external scalars, and without neutral-sector mixing, the abelian field decouples. It will however provide a simple workspace in which to try things out.

Section 4 leads, in several steps, to the identification of these scalars with projections of the gauge vectors onto an orthonormal basis defined by the vector fields themselves. Section 5 discusses the rewriting of the gauge action in terms of “intrinsic” scalars, for the case where the latter are not yet endowed with hypercharge. Still for the same case, section 6 describes the modification of the usual concept of a covariant derivative that becomes necessary in building gauge or BRS-invariant terms for the action. Section 7 suggests a way of conferring a hypercharge onto the scalars, despite the fact that their parent gauge fields carry none, and of adapting the covariant-derivative concept to this situation. On this basis, section 8 then discusses a “mixed” formulation of the action, where both types of fields, and their relationship as enforced through suitable delta functionals, make their appearance. Section 9 analyzes the modified, gauge-covariant derivative in more depth and discusses its transformation to an isospin representation making electric charge diagonal, such that electrically neutral scalars, capable of formation of vacuum expectation values, can be identified. Along the way,
Higgs-type charge and hypercharge assignments emerge naturally.

In this framework, section 10 finally takes up the issue of mass formation, both for the heavy vector bosons and for the scalars themselves. It will be seen that a qualitatively correct, and even semi-quantitative, pattern of vector masses and mixings results without further assumptions. In addition there are a set of nine massive Higgs-type scalars, both charged and neutral, at a fixed mass ratio to their vector counterparts. In closing, section 11 offers an enumeration of questions not yet taken up or unresolved.

That one and the same action functional, when written in different variables, should describe different excitations of the same underlying field system is not, of course, a new phenomenon. A well-known example is Coleman’s rewriting of the massive Thirring model in terms of sine-Gordon solitons, where the solitonic excitation appears as a kind of coherent state built from the ”particle” excitations. The analogy is not as far-fetched as it would seem: we shall see in sect. 3 that the “intrinsic” scalars contain factors that can indeed be read as coherent superpositions of gauge-vector excitations.

2. Introducing the Scalar Fields

The many composite scalar fields that may be formed from the set of four vector fields of an SU(2) ⊗ U(1) theory clearly are not the answer to our needs, as they all are of higher mass dimension. Although we will begin with a simple field of this type – one that is easily identifiable in the standard gauge Lagrangian – we will then have to perform a process of “extraction of roots”, in a sense to be made precise below, to arrive at fields of mass dimension one.

We denote, as usual, by $A^a_\mu (x)$ the three SU(2) fields with isospin indices $a = 1, 2, 3$, and by $B_\mu (x)$ the U(1) field in a four-dimensional Euclidean $x$ space, but we shall soon be in need of a uniform notation for all four fields and thus write

$$A^C_\mu (x), C = 1, 2, 3, 4, \quad \text{with} \quad A_4^4 (x) := B_\mu (x),$$

(2.1)

with a capital index running over the four directions of an extended isospace (technically, the adjoint-representation space of the group U(2)), while lower-case indices will continue to take their three values, and appropriate summation conventions apply to both. (Thus, for example, $\delta^A^c \delta^B^c + \delta^A_4 \delta^B_4 = \delta^{AB}$). Then in the standard (Euclidean) gauge-field action

$$S_E [A, \bar{c}, c] = S_2 [A] + S_3 [A] + S_4 [A] + S_{GFG} [A, c, \bar{c}],$$

(2.2)

where $S_{GFG}$ contains gauge-fixing-and-ghosts terms, and where $S_{(2,3,4)}$ denote the bilinear, trilinear, and quadrilinear portions of the classical action, the bilinear term might be written as

$$S_2 [A] = \frac{1}{4} \int d^4x \left( \partial_\mu A^C_\nu - \partial_\nu A^C_\mu \right)^2 = \frac{1}{2} \int d^4x A^C_\mu \left[ \delta_{\mu \nu} (-\partial_\lambda \partial_{\lambda}) + \partial_\mu \partial_\nu \right] A^C_\nu,$$

(2.3)

whereas the $S_{3,4}$ terms would involve lower-case isospin indices only. We begin by inspecting

$$S_4 [A] = \frac{1}{4} g_2^2 \epsilon^{abc} \epsilon^{ade} \int d^4x A^d_\mu (x) A^e_\nu (x) A^b_\mu (x) A^c_\nu (x).$$

(2.4)
The observation of this being a contraction of a Lorentz tensor \( \epsilon^{abc} A^a_{\mu} A^b_{\mu} \) with itself, could motivate a definition of this tensor as a composite new field, and (since a glance at its gauge behavior would quickly suggest its completion to the non-abelian field-strength tensor \( G^a_{\mu \nu} \)) this would naturally lead to the introduction of the \( G^a_{\mu \nu} \) as new, tensorial field variables, as envisaged in the field-strength formulation of Halpern[2, 3]. But we might pair off the Lorentz indices the other way around and introduce the Lorentz, or rather Euclidean, scalar composite fields

\[
\Xi^{ab}(x) := A^a_{\mu}(x) A^b_{\mu}(x),
\]  

which form a symmetric tensor in isospace, of mass dimension two, and evidently positive semi-definite in the Euclidean. As a collection of the scalar products between three vectors, this isotensor resembles what in linear algebra is called a Gramian matrix, except that the number of vectors does not fit the spatial dimension. At this point, in order to avoid a rather repetitious later discussion, we anticipate that we will be forced to consider the extension of (2.5) to a four-dimensional matrix in extended isospace,

\[
X^{AB}(x) := A^A_{\mu}(x) A^B_{\mu}(x),
\]  

in spite of the fact that only its \( 3 \times 3 \) submatrix \( \Xi \) appears in (the non-abelian part of) the action. This is now a genuine Gramian matrix, and it is well known[5] that this matrix, at a certain Euclidean point \( x \), encodes the linear-dependence properties of the set of four vectors at that point: it is nonsingular if and only if those vectors are linearly independent. In that case, it is then also positive definite. We will follow [2] in assuming that this is in fact the generic case, i.e. that a vanishing determinant of \( X \) will at most occur along lower-dimensional submanifolds of the \( E^4 \) space that make zero contribution to the action integral. For our purposes, \( X \) is then a generically positive definite \( 4 \times 4 \) matrix field.

The process we referred to as extracting a root now consists in invoking the simple fact[5] that a symmetric and positive semi-definite matrix like \( X \) can always be decomposed as

\[
X(x) = Q(x) \cdot Q^T(x),
\]  

with \( T \) denoting a matrix transpose. If \( X \) is nonsingular, so is \( Q \), but otherwise \( Q \) is a general \( 4 \times 4 \) matrix field, of mass dimension one in \( d = 4 \), and with 16 independent component fields. Note that this is now a matrix factorization in the extended isospace that in contrast to eq. (2.6) no more involves the Lorentz indices. Clearly the choice of \( Q \) is not unique, as a given \( Q \) can always be transformed, without changing \( X \), into \( Q \cdot O \) with \( O \) an orthogonal \( 4 \times 4 \) matrix. Thus in order to parameterize \( Q \), we may start with the simplest possibility, \( Q = X^\frac{1}{2} \), where \( X^\frac{1}{2} \) is, at each space point \( x \), the positive semi-definite matrix square root of \( X \) and is, like \( X \) itself, a symmetric matrix. It thus provides ten independent component fields. Then the general \( Q \) can be represented as

\[
Q(x) = X^\frac{1}{2}(x) \cdot \exp[\Omega(x)],
\]  

where the orthogonal-matrix field is generated by

\[
\Omega^{AB}(x) = -\Omega^{BA}(x),
\]
an antisymmetric and dimensionless $4 \times 4$ matrix field, which provides the remaining six independent component fields. Thus $\exp [\Omega(x)]$ is an element of $\text{SO}(4)$, the real orthogonal group in four dimensions, but it is to be noted that the rotation here takes place, not in the Euclidean space, but rather in the four-dimensional extended isospace referred to by the upper-case indices. Eq. 2.8 represents what is known as the right polar decomposition of $Q$, a decomposition of type modulus $\times$ phase. The determination of $\exp [\Omega(x)]$ by further requirements will be the subject of the next section.

Since $X$ has a distinguished upper-left $3 \times 3$ submatrix (2.5) that enters the quadrilinear action $S_4$, it seems natural to adopt three-plus-one partitionings for both $X$ and its “root” $Q$, writing

$$Q = \begin{pmatrix} \Psi & \chi \\ \eta^T & \psi \end{pmatrix}. \quad (2.10)$$

Here $\chi^a = Q^{a4}$ ($a = 1, 2, 3$) is a column 3-vector, $(\eta^T)^a = Q^{4a}$ ($a = 1, 2, 3$) a collection of 3 singlets, and $\psi = Q^{44}$ a singlet in isospace, but these designations are to be taken with a grain of salt – they do not, in these cases, imply homogeneous transformation properties under local-gauge and BRS transformations. Here, since all our scalars are defined through the gauge fields, they will inherit in some form the inhomogeneous transformation laws of the latter. (We collect the gauge/BRS variations in appendix A; they will typically feature, in addition to an homogeneous piece, an inhomogeneous term involving derivatives $\partial_\mu \theta^A(x)$ of the gauge functions $\theta^A$).

The $3 \times 3$ matrix $\Psi$ carries a reducible representation of isospin that naively would be expected to reduce according to the dimension count $9 = 1 + 3 + 5$, but a glance at eq. (A5) shows that in this case we rather have a set of three (global) isotriplets:

$$9 = 3 + 3 + 3, \quad (2.11)$$

The $\chi$ entry provides a further (global) isotriplet. Similarly, the last line of (2.10) comprises four (global) isosinglets, so the sixteen-dimensional representation $Q$ gets reduced according to

$$16 = 4 \times 3 + 4 \times 1 \quad (2.12)$$

We reemphasize that because of the special gauge/BRS behavior of our scalars, we may at most speak of global isomultiplets, i.e. multiplets under spatially constant but not under local gauge transformations.

Using (2.10) and its transpose, the $X$ of eq. (2.7) now gets partitioned as

$$X = \begin{pmatrix} \Psi \cdot \Psi^T + \chi \cdot \chi^T & \Psi \cdot \eta + \psi \cdot \chi \\ (\Psi \cdot \eta + \psi \cdot \chi)^T & \eta^T \cdot \eta + \psi^2 \end{pmatrix}. \quad (2.13)$$

(We employ the usual notation where $\Psi \cdot \Psi^T$ is a product matrix while $\eta^T \cdot \eta$ is a scalar product of vectors). The point here is that the upper-left $3 \times 3$ matrix $\Xi$ does not appear simply as $\Psi \cdot \Psi^T$, as it would if we had “extracted a root” from $\Xi$ alone, but has an additional separable term $\chi \chi^T$ that still preserves its (generically) positive definite character. In terms of components, then,

$$\Xi^{ab}(x) = \Psi^{ac}(x) \Psi^{bc}(x) + \chi^a(x) \chi^b(x). \quad (2.14)$$
For later use, we note that a similar decomposition may also be given for the complete four-dimensional matrix $X$: from eq. (2.13) we have

$$X = P \cdot P^T + q \cdot q^T,$$

(2.15)

where $P$ is the $4 \times 4$ matrix with three-plus-one partitioning

$$P = \begin{pmatrix}
\Psi & 0 \\
\eta^T & 0
\end{pmatrix},$$

(2.16)

while $q$ denotes the 4-isovector

$$q = \begin{pmatrix}
\chi^1 \\
\chi^2 \\
\chi^3 \\
\psi
\end{pmatrix}.$$

(2.17)

The difference from eq. (2.14) is that obviously $\det P = 0$ and therefore

$$\det(P \cdot P^T) = 0.$$

(2.18)

More precisely, $P$ and $P P^T$ are generically of rank three.

With these scalar-isomatrix fields, the quadrilinear part (2.4) of the action can now be rewritten in either of two equivalent forms: the first,

$$S_4[Q, A] = \frac{1}{4} g_s^2 \int d^4 x \, A_\mu^b(x) \left\{ \epsilon^{abc} \epsilon^{ade} \Xi^{ce}(x) \right\} A_\mu^d(x)$$

(2.19)

is, by eq. (2.14), bilinear in our scalar fields. It is thus a scalar-gauge interaction of the “seagull” type. The second,

$$S_4[Q] = \frac{1}{4} g_s^2 \int d^4 x \, \left\{ \epsilon^{abc} \epsilon^{ade} \Xi^{bd}(x) \Xi^{ce}(x) \right\}$$

(2.20)

is a quadrilinear self-interaction of the scalar fields. Recall that these two kinds of interactions appear separately in the standard-model Higgs action, whereas here they are equivalent forms of the same term.

The foregoing construction of the matrix field $Q$ makes essential use of the theory operating in Euclidean space, since it is only there that the scalar products in $X$ are positive definite. It may therefore not be redundant to recall one of the basic tenets of Euclidean field theory: analytic continuation to the Minkowskian domain is to be performed for the final, c-number correlation functions (Schwinger functions) generated, not in the equations of motion, or the path integrals, of the theory, whose treatment must be performed entirely in the Euclidean.
3. The trilinear coupling

We have rewritten the four-fields term $S_4$ as an interaction between the new scalars $Q$ and their “parent” gauge fields. We now wish to do the same for the trilinear term $S_3$. This will turn out to also determine partially, though not completely, the orthogonal-matrix factor $\exp(\Omega)$ of eq. (2.8). Of course, the gauge-invariant way of generating both types of coupling is through (scalar contractions of) gauge-covariant derivatives, but here we defer the introduction of a suitable concept of covariant derivative to section 6, where it will be facilitated by the use of an orthonormal vector system attached, in a sense, to the gauge fields. The results of the present section will in fact lead directly to that very convenient tool.

For physical reasons, we expect the trilinear term to assume the form of a coupling of the gauge field to scalar-field currents. We inspect the $S_3$ term of (2.2), which again features only lower-case isospin indices:

$$S_3[A] = \frac{1}{2} g_2 \int d^4x \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right) \left( \epsilon^{abc} A^b_\mu(x) A^c_\nu(x) \right).$$  \hspace{1cm} (3.1)

With some index reshuffling, this can be rewritten as

$$S_3[A] = \frac{1}{2} g_2 \epsilon^{abc} \int d^4x \left[ A^b_\lambda(x) \leftrightarrow A^c_\mu(x) \right] \cdot A^a_\mu(x),$$  \hspace{1cm} (3.2)

where the usual notation, $A \leftrightarrow B = A(\partial B) - (\partial A)B$, has been employed. The desired structure will emerge if we can choose our matrix-of-scalars $Q$ such that

$$A^B_\lambda(x) \leftrightarrow A^C_\mu(x) = \left[ Q \leftrightarrow \partial_\mu Q^T \right]^{BC},$$  \hspace{1cm} (3.3)

(of which, again, only the $3 \times 3$ submatrix $(B,C) \rightarrow (b,c)$ enters into (3.2)), but at first sight this seems an unlikely prospect: the freedom we have in $Q$ at this point consists in the six independent components of $\Omega$, eq. (2.9), but while the two sides of (3.3) are antisymmetric in their U(2) indices, the relation is to hold for each of the four values of $\mu$ and thus implies $6 \times 4 = 24$ conditions. Obviously, this can work only if substantial constraints apply among the four $\mu$ values. It is all the more remarkable, and perhaps a piece of insight into the structure of gauge theory, that such constraints can be identified and are in fact fulfilled.

The key to bringing eq. (3.3), which as a condition on $\Omega$ is rather unwieldy, to a transparent form making the necessary constraints evident, is the introduction of a quadruple of vector fields obtained by orthonormalizing the four gauge fields. This is possible by means of the Gram matrix $X$ of (2.6): $X$ being generically nonsingular, so is its matrix root $X^{1/2}$, and we can define

$$N^A_\mu(x) := \left[ X(x)^{-1/2} \right]^{AB} A^B_\mu(x) = A^B_\mu(x) \left[ X(x)^{-1/2} \right]^{BA},$$  \hspace{1cm} (3.4)

the second equation holding due to the symmetry of $X^{-1/2}$. It is immediate to see that these form an orthonormal system:

$$N^A_\mu(x) N^B_\mu(x) = \left[ X(x)^{-1/2} \right]^{AC} \left[ A^C_\mu(x) A^D_\mu(x) \right] \left[ X(x)^{-1/2} \right]^{DB}$$

$$= \delta^{AB}.$$  \hspace{1cm} (3.5)
Moreover, in 4-dimensional Euclidean space they are a complete set, which translates as

\[ N^A_\mu (x) N^A_\nu (x) = \delta_{\mu \nu} . \]  

This orthonormalization process is known as Löwdin orthogonalization \[7\] and was first developed for applications in quantum chemistry. In our context, the basis constructed is different at each point \( x \) in \( \mathbb{E}^4 \). Thus we have a bundle of orthonormal frames, attached to all points of space. The only change such a basis can undergo while maintaining its orthonormality is a rigid rotation, so \( N \) bases at different points of \( \mathbb{E}^4 \) differ at most by an \( \text{SO}(4) \) rotation. Definition (3.4) may be turned around to give an expansion of the \( A(x) \) field in terms of the orthonormal system at \( x \):

\[ A^B_\mu (x) = \left[ X(x)^{1/2} \right]^{BC} N^C_\mu (x) . \]  

Using this expansion on the l.h.s. of eq. (3.3), we have

\[ A^B_\lambda (x) \overset{\rightarrow}{\partial}_\mu A^C_\lambda (x) = \left[ X(x)^{1/2} \right]^{BD} \left[ N^D_\lambda \overset{\rightarrow}{\partial}_\mu N^E_\lambda \right] \left[ X(x)^{1/2} \right]^{EC} + \left\{ N^D_\lambda N^E_\lambda \right\} \left[ \left( X(x)^{1/2} \right)^{BD} \overset{\rightarrow}{\partial}_\mu \left( X(x)^{1/2} \right)^{EC} \right] . \]  

On the other hand the r.h.s. of eq. (3.3), upon using the polar decomposition (2.8) of \( Q \), becomes

\[ \left[ Q \overset{\rightarrow}{\partial}_\mu Q^T \right]^{BC} = \left[ X(x)^{1/2} \right]^{BD} \left[ e^\Omega \overset{\rightarrow}{\partial}_\mu e^{-\Omega} \right]^{DE} \left[ X(x)^{1/2} \right]^{EC} + \left\{ e^\Omega e^{-\Omega} \right\} \left[ \left( X(x)^{1/2} \right)^{BD} \overset{\rightarrow}{\partial}_\mu \left( X(x)^{1/2} \right)^{EC} \right] . \]  

Orthonormality turns the curly bracket in eq. (3.8) into a \( \delta^{DE} \), so the second lines of both sides drop out from the condition. Note that this simplification, which is crucial to the entire subsequent development, would not happen if we were to scale both sides of eq. (3.3) by different numerical factors. The result, as desired, is purely a condition on \( e^\Omega \):

\[ \left[ e^\Omega \overset{\rightarrow}{\partial}_\mu e^{-\Omega} \right]^{BC} = N^B_\lambda \overset{\rightarrow}{\partial}_\mu N^C_\lambda . \]  

From the trivial relations

\[ \partial_\mu \left( e^\Omega e^{-\Omega} \right) = 0 , \quad \partial_\mu \left( N^A_\mu N^B_\mu \right) = 0 , \]  

it follows that on both sides of (3.10), the second (minus) term of the \( \overset{\rightarrow}{\partial}_\mu \) construct equals the first – in other words, both sides are antisymmetric matrices in the \( \text{U}(2) \) space, and are thus structurally compatible. We may then simplify condition (3.10) to read

\[ \partial_\mu e^\Omega = \mathcal{M}_\mu e^\Omega , \]  

with the \( 4 \times 4 \) matrices \( \mathcal{M}_\mu \) given by

\[ \left( \mathcal{M}_\mu \right)^{AB} (x) = \frac{1}{2} \left[ -N^A_\lambda (x) \overset{\rightarrow}{\partial}_\mu N^B_\lambda (x) \right] \]  

and

\[ = -N^A_\lambda \left( \partial_\mu N^B_\lambda \right) = \left( \partial_\mu N^A_\lambda \right) N^B_\lambda . \]
The latter two forms are again valid because of (3.11).

It is now evident which constraints must be met for eq. (3.3) to be solvable by a suitable choice of \( \Omega \) in (2.8): these constraints are identical with the compatibility conditions for the system of four partial differential equations (3.12):

\[
\frac{\partial}{\partial \nu} (M_{\mu} e^{\Omega}) = \frac{\partial}{\partial \mu} (M_{\nu} e^{\Omega}).
\]  

(3.15)

If these are fulfilled, the four right-hand sides of (3.12) are in fact first partial derivatives of a single antisymmetric \( 4 \times 4 \) matrix quantity, with six independent components, so the freedom in \( \Omega \) is just sufficient for building that quantity. Upon working out these conditions\[6\], one finds

\[
\frac{\partial}{\partial \mu} M_{\nu} - \frac{\partial}{\partial \nu} M_{\mu} - [M_{\mu}, M_{\nu}] = 0,
\]  

(3.16)

the brackets denoting a matrix commutator. This means that \(-M_{\mu}\), when viewed as a matrix-valued gauge potential, produces zero field strength. (It must then be a “pure gauge”, which is exactly what eq. (3.12) is saying). Now compute

\[
(\frac{\partial}{\partial \mu} M_{\nu} - \frac{\partial}{\partial \nu} M_{\mu})^{AB} = -\left[\frac{\partial}{\partial \mu} N_{A}^{\lambda}\right] \left[\frac{\partial}{\partial \nu} N_{B}^{\lambda}\right] + \left[\frac{\partial}{\partial \nu} N_{A}^{\lambda}\right] \left[\frac{\partial}{\partial \mu} N_{B}^{\lambda}\right] - \left[\frac{\partial}{\partial \nu} N_{A}^{\lambda}\right] \left[\frac{\partial}{\partial \mu} N_{B}^{\lambda}\right],
\]  

(3.17)

\[
[M_{\mu}, M_{\nu}]^{AB} = \{N_{\kappa}^{C} N_{\lambda}^{C}\} \cdot \left[\left(\frac{\partial}{\partial \mu} N_{\kappa}^{A}\right) \left(\frac{\partial}{\partial \nu} N_{\lambda}^{B}\right) - \left(\frac{\partial}{\partial \nu} N_{\kappa}^{A}\right) \left(\frac{\partial}{\partial \mu} N_{\lambda}^{B}\right)\right].
\]  

(3.18)

This time it is completeness of the \( N \)'s, eq. (3.6), that turns the curly bracket in (3.18) into \( \delta_{\kappa \lambda} \) and, remarkably, makes the two quantities cancel. It now becomes obvious why we had to work with the four-gauge-vectors formulation, rather than with the non-abelian fields alone: had we proceeded from the 3-dimensional \( \Xi \) of eq. (2.5), we would have obtained, from the analog of eq. (3.4), a set of three orthonormal vectors that would have been incomplete in four Euclidean dimensions. It is interesting that in the present context, the additional U(1) is not simply an empirical fact but is forced upon us by the formalism.

An explicit form of the solution of system (3.12) will not be absolutely necessary in the following, but is convenient to have. Since \( M_{\mu} \) is both \( x \)-dependent and a noncommuting quantity, such a solution can in any case be exhibited only in a formal sense, as a path-ordered exponential:

\[
e^{\Omega}[A, C; x) = O[A, C; x) e^{\Omega_{0}[A]} ,
\]  

(3.19)

where

\[
O[A, C; x) = \mathcal{P} \left\{ \exp \left[ -\frac{1}{2} \int_{C(x_{0}, x)} ds_{\mu} \left( N_{\lambda}(s) \hat{\partial}_{\mu} N_{\lambda}(s) \right) \right] \right\} ,
\]  

(3.20)

so that \( O[A, C; x_{0}) = 1_{4} \). Here we employ the obvious matrix notation \( N_{\lambda} \hat{\partial}_{\mu} N_{\lambda} \), where

\[
( N_{\lambda} \hat{\partial}_{\mu} N_{\lambda})^{AB} = N_{\lambda}^{A} \hat{\partial}_{\mu} N_{\lambda}^{B}.
\]  

(3.21)

Eqs. (3.19) do, however, illustrate a number of qualitative features. First, note the presence of a matrix-valued “integration constant”, \( e^{\Omega_{0}} \), with \( \Omega_{0}[A] = \Omega[A; x = x_{0}] \), a spatially constant SO(4) isoration, which by convention we have written as a right-hand factor. Thus, we still have not
pinned down the matrix field $Q$ uniquely, but are left with a *residual freedom* encoded in one constant orthogonal matrix.

Second, since the integrand $ds_\mu \cdot M_\mu$ is a Lorentz (or Euclidean) scalar, $Q$ continues to be defined in terms of scalars alone.

Third, upon expanding the path-ordered exponential one obtains a series of multiple integrals with increasing numbers of the bilinear (3.21), which may be viewed as a kind of coherent-state expansion, as mentioned in section 1.

Fourth, since $O[A; C; x]$ involves a path $C$, which we always take from a conventional starting point $x_0$ to $x$, it is *prima vista* a nonlocal functional of the original gauge fields $A$, in apparent contrast to the $X^\frac{1}{2}$ factor in (2.7), which is built purely from the fields at its own argument $x$. Since the $M_\mu$ in the integral is not a pure gradient, this path dependence seems to be unavoidable. The (provisional) functional notation in eq. (3.19) refers to this feature.

However, such a conclusion would not do justice to the nonabelian nature of the expression, as encoded in the path ordering $P$. To examine this question, we first parameterize $C$ as

$$C(0, x) = \{ s_\mu(x, t) : \mu = 1, 2, 3, 4; \ t = 0 \ldots 1, \ s(x, 0) = x_0, s(x, 1) = x \} .$$ (3.22)

The path ordering $P$ now turns into a time ordering $T$ with respect to the path parameter $t$. The tool of choice is then a quantity that interpolates $O[A; C; x]$ in this curve parameter$^{[11]}$, $U[C; x, t \leftarrow 0) = T \left\{ \exp \left[ \int_0^t dt' \frac{\partial s_\mu(x, t')}{\partial t'} M_\mu (s(x, t')) \right] \right\}.$ (3.23)

$U$ is a unitary-matrix function obeying the differential equation

$$\frac{\partial}{\partial t} U[C; x, t \leftarrow 0) = \frac{\partial s_\mu(x, t)}{\partial t} M_\mu (s(x, t)) U[C; x, t \leftarrow 0),$$ (3.24)

and the boundary conditions,

$$U[C; x, 0 \leftarrow 0) = 1_4; \quad U[C; x, 1 \leftarrow 0) = O[C; x)$$ (3.25)

where in the last term we omitted the functional dependence on $A$. (Notation: $1_n$ and $\text{tr}_n$ are the unit matrix and trace, respectively, in $n$ isodimensions, for $n = 2, 3, 4$). The infinitesimal change in $U$,

$$\delta_C U := U[C + \delta C; x, t \leftarrow 0) - U[C; x, t \leftarrow 0),$$ (3.26)

produced by an infinitesimal change in the path,$$
C \rightarrow C + \delta C : \ s_\mu(x, t) \rightarrow s_\mu + \delta s_\mu(x, t); \quad \delta s_\mu(x, 0) = \delta s_\mu(x, 1) = 0,$$ (3.27)

is given by a standard formula for $T$-ordered functionals$^{[9]}$,

$$\delta_C U = \int_0^1 U[s; x, t \leftarrow \tau) \left\{ \frac{\delta}{\delta s_\nu(\tau)} \left[ \frac{\partial s_\mu}{\partial \tau} M_\mu (s(\tau)) \right] \right\} U^{-1}[s; x, t \leftarrow \tau) \delta s_\nu(\tau) d\tau,$$

$$\cdot U[s; x, t \leftarrow 0),$$ (3.28)
where
\[ U^{-1} [s; x, t \leftarrow \tau] = U [s; x, \tau \leftarrow t) . \] (3.29)

By spelling out the bracketed functional derivative, applying partial integration, using the differential equation (3.24) as adapted to the variable \( \tau \), and setting \( t = 1 \), one finally obtains a result derived via a different route in [10]:
\[
\{ \deltaCO [C; x) \} OT [C; x) = - \int_0^1 d\tau U [s; x, 1 \leftarrow \tau) \left\{ \frac{\partial}{\partial s_\mu} M_\nu - \frac{\partial}{\partial s_\nu} M_\mu - [M_\mu, M_\nu] \right\}_{s(\tau)}
\times U^{-1} [s; x, 1 \leftarrow \tau) \cdot \frac{\partial s_\mu(\tau)}{\partial \tau} \delta s_\nu(x, \tau) .
\] (3.30)

A glance at eq. (3.16) then shows that \( \deltaCO = 0 \). Contrary to appearances, the matrix \( O \) is therefore not a path functional; it continues to depend functionally on the gauge fields, which we may call its parents, but otherwise is just a local function of \( x \).

In the following, when we wish to refer to a completely specified quantity, we will therefore choose, without loss of generality, the straight path \( s_\mu(x, t) = t \cdot (x - x_0)_\mu \), in which case
\[
O[A; x) = T \left\{ \exp \left[ - \frac{1}{2} (x - x_0)_\mu \int_0^1 dt (N_\chi(s) \rightarrow s_\mu N_\chi(s))_{s=t(x-x_0)} \right] \right\} .
\] (3.31)

This orthogonal-matrix function obeys the initial condition \( O[A; x = x_0) = 1 \).

With the possibility of eq. (3.3) now assured, we may finally rewrite (3.2) in the form envisaged, that of a set of scalar-field currents coupling to the SU(2) gauge fields:
\[
S_3[A] = \frac{1}{2} g_2 \epsilon^{abc} \int d^4x [Q \rightarrow s_\mu Q^T]^{bc} \cdot A_\mu^a (x) .
\] (3.32)

In terms of the component fields of eq. (2.10), the \( Q \) current gets partitioned as
\[
Q \rightarrow s_\mu Q^T = \begin{pmatrix}
\Psi \rightarrow s_\mu \Psi^T & \chi \rightarrow s_\mu \chi^T & \Psi \rightarrow s_\mu \eta + \psi \rightarrow s_\mu \chi \\
\Psi \rightarrow s_\mu \eta + \psi \rightarrow s_\mu \chi^T & 0
\end{pmatrix}
\] (3.33)

The last entry obtains because \( \eta^a \rightarrow s_\mu \eta^a = \psi \rightarrow s_\mu \psi = 0 \). One sees that the upper-left submatrix that enters into (3.32),
\[
[Q(x) \rightarrow s_\mu Q^T(x)]^{bc} = [\Psi(x) \rightarrow s_\mu \Psi^T(x)]^{bc} + \chi^b(x) \rightarrow s_\mu \chi^c(x) ,
\] (3.34)
again involves not only the \( 3 \times 3 \) matrix \( \Psi \) but has an additional, separable \( \chi \) term.

4. Constant orthonormal frame

This section is a technical interlude which does, however, lead up to something conceptually significant: the interpretation, in eq. (4.18) below, of the scalars \( Q^{AB} \) as projections of the gauge-field vectors onto a constant orthonormal basis that is defined by the fields themselves.
Eq. (3.3) has further consequences that are best formulated by introducing the modified Löwdin basis vectors
\[ \tilde{N}_A^\lambda := \left( e^{-\Omega} \right)^{AB} N_B^\lambda = N_B^\lambda \left( e^\Omega \right)^{BA}. \] (4.1)
Since these are obtained from the \( N' \)s by an orthogonal transformation in extended isospace, they too form an orthonormal and complete system:
\[ \tilde{N}_\mu^A \tilde{N}_\mu^B = \delta^{AB}, \] (4.2)
\[ \tilde{N}_\mu^C \tilde{N}_\nu^C = \delta_{\mu \nu}. \] (4.3)
At this point they seem to depend on \( x \), like the \( N \) vectors. By eqs. (3.4) and (2.8), they can be expressed as
\[ \tilde{N}_\mu^A = \left( Q(x)^{-1} \right)^{AB} A_B^\mu(x) = A_B^\mu(x) \left( Q^T(x)^{-1} \right)^{BA}. \] (4.4)
Using orthonormality, this can be turned around to give an instructive representation of the \( Q \)-matrix elements:
\[ Q^{AB} = A_B^\lambda \tilde{N}_\lambda^A. \] (4.5)
While once again presenting them as Lorentz scalars, this formula identifies the \( Q^{AB} \) as the sixteen projections of the four gauge fields onto the four orthonormal \( \tilde{N} \) vectors.

The salient property of the latter emerges when relation (3.3), now established, is combined with the derivative of (2.7),
\[ A_B^\lambda \left( \partial_\mu A_\lambda^C \right) + \left( \partial_\mu A_B^\lambda \right) A_C^\lambda = \left[ Q \left( \partial_\mu Q^T \right) + \left( \partial_\mu Q \right) Q^T \right]^{BC}. \] (4.6)
By adding and subtracting these one finds the relation
\[ \left[ \left( \partial_\mu Q \right) Q^T \right]^{BC} = \left( \partial_\mu A_B^\lambda \right) A_C^\lambda, \] (4.7)
and its transpose. Or, by combining this with eq. (4.4),
\[ \left( \partial_\mu Q \right)^{BC} = \left( \partial_\mu A_B^\lambda \right) \tilde{N}_\lambda^C. \] (4.8)
Upon comparing this with the partial derivatives of eq. (4.5), we conclude that
\[ A_B^\lambda \left( \partial_\mu \tilde{N}_\lambda^C \right) = 0 \quad (\text{all } \mu, B, C, \lambda) \] (4.9)
Thus each partial derivative of an \( \tilde{N} \) vector is orthogonal to all gauge vectors. The latter being generically linearly independent, we conclude that, generically, the \( \tilde{N} \)’s are spatially constant:
\[ \partial_\mu \tilde{N}_\lambda^A = 0 \quad (\text{all } \mu, \lambda, A) \]. (4.10)
This somewhat unexpected property shows that the transformation of eq. (4.1) has undone the \( x \)-dependent 4-rotation by which the original Löwdin frames at various points differed. It is clear that the \( x \)-independence property simplifies calculations in these frames significantly.
Of course, we cannot exclude that the $\tilde{N}$’s, while spatially constant in a given gauge-field configuration
\[
A := \left\{ A^C_\mu(x) \mid \text{all } x \text{ in } E^4; \mu, C = 1,...,4 \right\},
\] (4.11)
may still be different for different field configurations. But such a variation is strongly restricted by condition (4.10) and orthonormality. Since $\tilde{N}$’s must be constant separately in each of two different configurations $A, A'$, they can differ between these only by a constant rotation, which because of the perfect symmetry between eqs. (4.2) and (4.3) may be taken to be an $SO(4)$ element on either Euclidean or extended isospace. After selecting some reference configuration $A_0$, we then have for any $A$,
\[
\tilde{N}_\lambda^B[A] = \left( e^{-\Gamma_0[A]} \right)^{BC} \tilde{N}_\lambda^C[A_0],
\] (4.12)
with $-\Gamma_0[A]$ a constant, antisymmetric generator matrix typical of configuration $A$. Now recall that $\tilde{N}$, through the $e^{-\Omega}$ factor in its definition (4.1), contains just a matching freedom in the form of the integration-constant matrix in eq. (3.19). Writing (4.12) with (4.1) as
\[
e^{\Gamma_0[A]}\tilde{N}_\lambda[A] = \left( e^{\Gamma_0[A]} e^{-\Omega_0[A]} \right) \left( O^T[A; x] N_\lambda[A; x] \right) = \tilde{N}_\lambda[A_0],
\] (4.13)
and choosing the “integration constant” as
\[
\Omega_0[A] = \Gamma_0[A],
\] (4.14)
we see that the once-more-modified L"owdin orthonormal system,
\[
n_\mu^B := \left( O^T[A; x] \right)^{BC} N_\mu^C[A; x] = \left( e^{\Gamma_0} \right)^{BD} \tilde{N}_\mu^D,
\] (4.15)
is always equal to $\tilde{N}_\mu^B[A_0]$ for the reference configuration, and contrary to appearances is therefore independent of both $x$ and of the gauge-field configuration $A$:
\[
\partial_\lambda n_\mu^A = 0; \quad \delta n_\mu^A = 0 \quad \text{under } A \rightarrow A + \delta A.
\] (4.16)
Since the latter change may arise from a gauge or BRS transformation, the $n$ system is, in particular, a gauge and BRS invariant.

A slight modification of the $Q$ matrix,
\[
Q'(x) = Q(x) e^{-\Gamma_0} = X^{\frac{3}{2}}(x) O(x),
\] (4.17)
which is still of the general form of eq. (2.8), then has matrix elements
\[
(Q')^{A B} = A^A_\lambda n_\lambda^B,
\] (4.18)
which are the scalar projections of the $A$ vectors onto the $n$ vectors – a pleasantly simple, geometric interpretation.

We emphasize that for our task of reformulating the gauge action, it is not necessary to actually construct quantities such as $\Gamma_0[A]$ or $\tilde{N}_\lambda[A_0]$ explicitly; it suffices to know that a BRS-invariant basis
\[ n^B_\lambda \] exists and that the \( Q' \) matrix obeying eqs. (4.17) and (4.18) continues to fulfill relations (2.7) and (3.3). In the main text we therefore drop the prime, referring to it again as \( Q \).

Note that there is a difference, both conceptual and practical, between these projections and the components of \( A \) in the arbitrary and fixed coordinate system tacitly understood when writing them as \( A^C_\mu \). These are not scalars but vector components. The orthonormal \( n \)-frame vectors, while spatially constant and independent of field configuration, do change their components in the arbitrary external basis in the same way as the \( A \) vectors do when that basis is changed, in such a way as to keep the contraction over Lorentz indices in eq. (4.18) unchanged. In this sense the \( Q \) elements are indeed scalar fields.

5. Action in terms of \( Q \)'s

We seek to rewrite the generating functional of Euclidean correlation functions,

\[
G[J,K] = \frac{Z[J,K]}{Z[0,0]} \quad (5.1)
\]

\[
Z[J,K] = \int \mathcal{D}[A] \mathcal{D}[c,\bar{c}] \ e^{-S_E[A,\bar{c},c] + (J,A) + (K,Q[A])} \quad (5.2)
\]

in a way that facilitates a study of the dynamics of either the scalar \( Q \) fields among themselves, or of their coupling to their parent gauge fields. In eq. (5.2), since we are not interested here in ghost amplitudes, we have omitted sources for the ghost fields, but we do carry sources \( K = \{K_{AB}\} \) for easier generation of correlations of the \( Q \) scalars, which at this point are viewed as functionals \( Q[A] \) of the \( A \)'s. Like the gauge-field sources \( J \), they are of mass dimension three. Standard scalar-product notation such as \((J,A) = \int d^4x J_\mu^C(x) A_\mu^C(x)\) has been employed. Here and in section 8, we examine two ways of reformulating \( Z \) using the scalar fields.

First, we might consider a complete reformulation in terms of scalars, which formally turns gauge theory into a theory of scalar fields interacting through tri- and quadrilinear couplings. For this, one uses the inversion of eq. (4.18) via the completeness relation for the \( n \) basis,

\[
A^B_\mu = Q^{BC} n^C_\mu. \quad (5.3)
\]

and in particular, for \( B = b = 1, 2, 3 \),

\[
A^b_\mu = \Psi^{bc} n^C_\mu + \chi^b n^4_\mu. \quad (5.4)
\]

Thanks to the configuration independence of the \( n \) basis, this relation is effectively linear, and the functional Jacobian is a constant that drops out of the ratio (5.1). The \( J \)-source term may now be omitted from eq. (5.2), and the \([A]\) argument from the \( Q \)-source term. The rewriting uses, in addition to eqs. (2.20) and (3.3), the two pieces of the integrand in the bilinear term (2.3),

\[
\frac{1}{2} \left( A^C_\mu, \delta_{\mu \nu} (-\partial_\lambda \partial_\lambda) A^C_\nu \right) = \frac{1}{2} \text{tr}_4 \left( \partial_\lambda Q, \partial_\lambda Q_T^T \right), \quad (5.5)
\]

\[
\frac{1}{2} \left( A^C_\mu, (\partial_\mu \partial_\nu) A^C_\nu \right) = -\frac{1}{2} \text{tr}_4 \left( \partial_\mu Q, (n_\mu \cdot n_\nu) \partial_\nu Q_T^T \right). \quad (5.6)
\]
In the latter equation, we note the occurrence of the invariant, orthonormal-basis vectors $n^C_{\mu}$, introduced in eq. (4.15), in the tensorial combination
\[
\left( n_\mu \cdot n_\nu^T \right)^{BC} := n^B_{\mu} n^C_{\nu} . \tag{5.7}
\]
Eq. (5.5) shows that the “diagonal” piece alone of $S_2$ already yields the complete, normalized kinetic term for the scalars:
\[
\frac{1}{2} \text{tr}_4 \left( \partial_\lambda Q, \partial_\lambda Q^T \right) = \frac{1}{2} \text{tr}_3 \left( \partial_\lambda \Psi, \partial_\lambda \Psi^T \right) + \frac{1}{2} \left[ \left( \partial_\lambda \chi^a, \partial_\lambda \chi^a \right) + \left( \partial_\lambda \eta^b, \partial_\lambda \eta^b \right) \right] + \frac{1}{2} \left( \partial_\lambda \psi, \partial_\lambda \psi \right) . \tag{5.8}
\]
On the other hand, eq. (5.6) for the “longitudinal” piece of $S_2$ illustrates the fact that in this formulation, uncontracted $n$-basis vectors remain in some, though not all, terms of the action. This feature will reoccur, in particular, in the trilinear terms such as $S_3$ and the fermion-gauge and ghost-gauge interactions. For these, the formulation is intrinsically awkward, since it tends to obscure the vector nature of these couplings. The explicit $n$ vectors present an artificial element, conceptually different from but practically somewhat similar to the constant external vectors that one introduces in the axial gauges for vector gauge theories. The piece (5.6) with its $n$-tensor (5.7) is best treated together with the gauge-fixing term in $S_{GFG}$.

By contrast, the “all-$Q$’s” formulation is quite suitable for an inspection of the $S_4$ term, in the form of eq. (2.20), in which all $n$’s drop out by contraction. Upon applying, as usual, the SU(2)-specific relation
\[
\epsilon^{abc} \epsilon^{ade} = \delta^{db} \delta^{ce} - \delta^{dc} \delta^{eb} , \tag{5.9}
\]
that term assumes the form,
\[
S_4 = \frac{1}{4} g_2^2 \int d^4 x \left\{ \left[ \text{tr}_3 \Xi(x) \right]^2 - \text{tr}_3 \left[ \Xi(x)^2 \right] \right\} . \tag{5.10}
\]
Note that the coupling constant here is the usual $g_2$ of the non-abelian sector; in the present context there is no need to introduce an extra constant (usually denoted $\lambda$) for the quartic scalar term. For a positive-definite matrix such as $\Xi$, the integrand in curly brackets is nonnegative,
\[
\left[ \text{tr}_3 \Xi(x) \right]^2 \geq \text{tr}_3 \left[ \Xi(x)^2 \right] , \tag{5.11}
\]
(proof: write the difference in terms of the eigenvalues of $\Xi$) and the minimum value of zero is attained, apart from the trivial case $\Xi = 0$, when $\Xi$ becomes proportional to a projector onto a one-dimensional subspace (facts familiar from the properties of the density matrix). A glance at (2.14) shows that there is a natural candidate for a term proportional to a projector – the $\chi \chi^T$ matrix. The minimum of zero is thus assumed when
\[
\Xi(x) = \chi(x) \chi^T(x) , \quad \text{i.e.} \quad \Psi(x) = 0 , \quad (5.12)
\]
Recall that in the standard Higgs action, a nontrivial minimum of the quartic potential at a finite value of the scalar field is ensured through the addition, by hand, of a “wrong-sign mass term”, $-\mu^2 \phi^\dagger \phi$, which is really just a device for creating a minimum at a finite and constant value $\phi^\dagger \phi = \frac{1}{2} v^2$, where
\( v \propto \mu \). It is interesting to observe that at least for the U(2) gauge group, an essentially similar structure, with a minimum at finite field values, is naturally present in the \( S_4 \) term. Also, the term quartic in \( \chi \) cancels in the integrand of eq. (5.10), so that no mass term can form for the \( \chi_s \), and the difference (5.11) may be rewritten

\[
\frac{1}{4} \{ [\text{tr}_3 \Xi(x)]^2 - \text{tr}_3 [\Xi(x)^2] \} = \text{tr}_3 \left\{ \frac{1}{2} \Psi^T M^2[\chi] \Psi + \frac{1}{4} \left( \Psi^T f_a \Psi \right)^4 \right\} 
\]

(5.13)

with \( f_a \) the adjoint SU(2) generators of eq. (6.3) below. Remember \( \Psi \) is real. What is different here is that the minimum field configuration is not a constant but rather an \( x \)-dependent quantity, proportional to the projector

\[
\tilde{\Pi}(x) \ : \ \left[ \tilde{\Pi}(x) \right]^{ab} = \frac{\chi^a(x) \chi^b(x)}{|\chi(x)|^2} ; \quad \chi(x) = [\chi^c(x) \chi^c(x)]^{1/2} 
\]

(5.15)

onto the \( \chi \) direction in isospace, locally at \( x \). Thus in general it minimizes only the \( S_4 \) term of the action, whereas the standard \( \phi = \frac{1}{\sqrt{2}} v \) also makes derivative terms vanish, and thus pushes the entire Higgs-field action into its minimum. We may however select a constant one among the minimum configurations (5.12), based on the vacuum expectation values of our scalar fields, which are expected to settle in the classical potential minimum but are \( x \)-independent. Thus, with a hat denoting quantum-field operators,

\[
\hat{\chi}^a(x) = \nu u^a + \hat{\chi}^a(x) ; \quad \nu u^a := \langle 0 | \hat{\chi}^a(x) | 0 \rangle = \langle 0 | \hat{\chi}^a(0) | 0 \rangle ; \quad \langle 0 | \hat{\chi}^a(x) | 0 \rangle = 0 .
\]

(5.16)

Here \( u^a \) is a constant unit isovector, \( u^a u^a = 1 \), and the field \( \hat{\chi}^a(x) \), an isovector under global gauging, sweeps the non-constant remainder of the configurations (5.12).

On the other hand, the field degrees of freedom describing deviations from the minimum configuration, \( \Xi - \Xi_{\text{min}} = \Psi \cdot \Psi^T \), being given by the elements of the \( \Psi \) matrix, are nine in number, rather than the four of the usual Higgs scenario. The “mass matrix” of eq. (5.14) now takes the form,

\[
\left\{ M^2[\chi] \right\}^{ab} = [\nu u^c + \hat{\chi}^c(x)]^2 \left[ 1_3 - \tilde{\Pi}(x) \right]^{ab} ,
\]

(5.17)

proportional to a projector onto a two-dimensional subspace. The \( S_4 \) portion of the action therefore splits off a mass term

\[
\frac{1}{2} \nu^2 g_2^2 \text{tr}_3 \left( \Psi^T , \Psi' \right) ; \quad \Psi' := \left( 1_3 - \tilde{\Pi} \right) \Psi ,
\]

(5.18)

with a mass of \( m_{\text{scalar}}^2 = (\nu g_2)^2 \), for only \( 3 \times 2 = 6 \) of the nine \( \Psi \) fields. The other three, contained in \( \Psi'' = \tilde{\Pi} \Psi \), remain massless, as did the three \( \chi_s \). It is these six fields that one might envisage transforming away by a gauge choice, although at this stage, with only three SU(2) gauge parameters available, it is not clear how this could work. The question will, however, come up again in the setting of section 10.
The complete “all-Q’s” action, upon putting in the $S_2$ pieces from eqs. (5.5) and (5.6) and the $S_3$ term from eqs. (5.32) and (5.34), comes out as

$$
S_E \left[ Q, \bar{c}, c \right] - (K, Q) = \frac{1}{2} \text{tr}_3 \left[ (\partial_\lambda \Psi^T, \partial_\lambda \Psi) + (\nu g_2)^2 \left( \Psi^{T'}, \Psi' \right) \right] 
+ \frac{1}{2} \left[ (\partial_\lambda \chi^{a\alpha}, \partial_\lambda \chi^{a\alpha}) + (\partial_\lambda \eta^b, \partial_\lambda \eta^b) + (\partial_\lambda \psi, \partial_\lambda \psi) \right] 
+ \frac{1}{2} g_2 \epsilon^{abc} \left( \sum_{\mu} \partial_\mu \Psi^T \right) \frac{b c}{\chi^n \partial_\mu \chi^{n c}} + 2 \frac{\nu g_2}{\chi^n \partial_\mu \chi^{n c}} + \chi^a n^d \chi^a n^{4 d} 
+ g_2^2 \text{tr}_3 \left\{ \left( \Psi^{T'}, [\nu u^c \chi^{i c} + \frac{1}{2} \chi^{i c} \chi^{i c}] \Psi' \right) + \frac{1}{4} \left( \left[ \Psi^T f_a \Psi \right]^\dagger \left[ \Psi^T f_a \Psi \right] \right) \right\} 
+ \left\{ \frac{1}{2} \text{tr}_4 \left( \partial_\mu \Psi, (n_\mu \cdot n^T_\nu) \partial_\nu \Psi^T \right) + S_{GF} \left[ Q, c, \bar{c} \right] \right\} - (K, Q). \tag{5.19}
$$

To limit the proliferation of terms, we have refrained from introducing the splitting $\Psi = \Psi' + \Psi''$ of the $\Psi$ field in the 3rd and 4th lines of eq. (5.19), as well as in the first term.

Although we will be led below to the introduction of a minimal complex extension of our present real $Q$ fields, this discussion of the $S_4$ term will be seen to remain valid, provided we replace the transpose $\Psi^T$ everywhere by the hermitean conjugate, $\Psi^\dagger$. More important will be the generalization of $S_4$ obtained in sect. 7 below, which aims at bringing all four vector fields into play.

6. A covariant derivative

Conceptually more interesting would be a mixed $A$-and-$Q$ formulation, describing vector and scalar field variables on the same level. For this it is obviously necessary to be able to formulate the interaction of the scalars with their “parent” fields in terms of (a scalar contraction of) gauge-covariant derivatives. This, in particular, is the only known, gauge-invariant way of generating the seagull-type terms that can provide for vector-mass generation through vacuum expectations of scalars.

Now the very concept of a covariant derivative, as commonly understood, tacitly presupposes homogeneous gauge/BRS transformation properties in the fields acted upon, as illustrated by the standard Higgs doublet. We already emphasized that the scalars considered here exhibit gauge changes (appendix A) that are at least partly inhomogeneous, but in building action functionals we require a modified form of covariant derivative for $Q$ scalars which still transforms homogeneously so its contraction with itself will form an invariant. It is interesting that such a construct, gauge-transforming homogeneously in spite of the fact that the scalars acted upon do not, is in fact feasible, and again possesses a relatively simple interpretation. We approach this construct heuristically, discussing in this section the pure SU(2) case, and extending it in the following section to the U(2) context through the introduction of hypercharge.

We start from the usual form of a covariant derivative for an U(2) local gauge group,

$$
D_\mu := 1_4 \partial_\mu - ig_2 A^d_\mu F^c_d - ig_1 B_\mu Y, \tag{6.1}
$$

(the couplings $g_2$, $g_1$ being often denoted alternatively as $g$, $g'$), where the $4 \times 4$ matrices $F^c$ have
three-plus-one partitions,

\[ F_c = \begin{pmatrix} f_c & 0 \\ 0 & 0 \end{pmatrix}. \]  

(6.2)

Here \( f^c \) are the hermitean SU(2) generators in the adjoint representation,

\[ (f_c)^{ab} = -i \epsilon_{cab} , \]  

(6.3)

while the hermitean, \( 4 \times 4 \) hypercharge matrix \( Y \) should commute with the three \( F^c \)'s,

\[ [F_c, Y] = 0. \]  

(6.4)

For the moment we do not specify \( Y \) further; its form as adapted to our context – eq. (9.27) below – will emerge only after we have effected the transformation to a charge-diagonal basis for our scalars. In this section, since our construction so far has been entirely in terms of vector gauge fields carrying zero hypercharge, we start by setting \( g_1 = 0 \), which amounts to omitting the \( B_\mu Y \) term from definition (6.1) altogether, and to considering

\[ \tilde{D}_\mu := 1_4 \partial_\mu - ig_2 A^{d}_\mu F_d \]  

(6.5)

instead. We straightforwardly evaluate, using the properties of the matrices (6.2), the infinitesimal gauge variation of \( \tilde{D}_\mu Q \) under the nonhomogeneous gauge variation of \( Q \), eqs. (A9/A10). The result is

\[ \delta_\theta \left[ \left( \tilde{D}_\mu Q(x) \right)^{A,B} \right] \]  

\[ = \left[ \left( i g_2 \delta \theta^d(x) F_d \right)^{A,C} \left( \tilde{D}_\mu Q(x) \right)^{C,B} \right] + \tilde{D}_\mu^{A,C} \left[ \left( \partial_\nu \delta \theta^C(x) \right) n^B_\nu \right] , \]  

(6.6)

On the r.h.s. of eq. (6.6), the first term gives the infinitesimal form of an homogeneous transformation law, while the second collects the contributions arising from the inhomogeneous parts (containing derivatives of the gauge functions \( \delta \theta^C(x) \)). We recast the latter term in a form with the order of the two derivatives interchanged. This leads to

\[ \tilde{D}_\mu^{A,C} \left[ \left( \partial_\nu \delta \theta^C(x) \right) n^B_\nu \right] = \delta_\theta \left[ \partial_\nu \left( A^{A}_\mu n^B_\nu \right) \right] + \left( i g_2 \delta \theta^d(x) F_d \right)^{A,C} \left[ -\partial_\nu \left( A^{A}_\mu n^B_\nu \right) \right] . \]  

(6.7)

Using this result in eq. (6.6), bringing its first term to the l.h.s. of that formula, and applying eq. (5.3), we conclude that the modified covariant derivative for \( Q \) fields without hypercharge,

\[ (\tilde{\nabla}_\mu Q)^{A,B} := -i g_2 \delta \theta^d(x) F_d \]  

\[ \left( n_\mu \cdot n^C_\nu \right) \]  

(6.8)

\[ = \partial_\mu Q^{A,B} - \left( \partial_\nu Q^{A,C} \right) \left( n_\mu \cdot n^C_\nu \right) , \]  

(6.9)

obeys a purely homogeneous gauge-transformation law,

\[ \delta_\theta \left[ \left( \tilde{\nabla}_\mu Q(x) \right)^{A,B} \right] \]  

\[ = \left( i g_2 \delta \theta^d(x) F_d \right)^{A,C} \left( \tilde{\nabla}_\mu Q(x) \right)^{C,B} , \]  

(6.10)
despite the nonhomogeneous law for \( Q \) itself. In eq. (6.9), one observes again the occurrence of the tensorial combination (5.7) of \( n \)-basis vectors.

In this no-hypercharge case, the homogeneous-transformation property of \( \tilde{\nabla} Q \) can also be seen in a different way, namely by formally rewriting it entirely in terms of \( A \) fields and \( n \) vectors, using again eq. (4.18). One finds for \( A = a = 1, 2, 3, 4 \),

\[
\left( \tilde{\nabla}_\mu Q \right)^{ab} = G^{a}_{\mu \nu} n^b_\nu, \quad \left( \tilde{\nabla}_\mu Q \right)^{a4} = G^{a}_{\mu \nu} n^4_\nu, \quad (6.11)
\]

where \( G^{a}_{\mu \nu} \) are the components of the nonabelian field strength, while for \( A = 4 \),

\[
\left( \tilde{\nabla}_\mu Q \right)^{4b} = F_{\mu \nu} n^b_\nu, \quad \left( \tilde{\nabla}_\mu Q \right)^{44} = F_{\mu \nu} n^4_\nu, \quad (6.12)
\]

with \( F_{\mu \nu} \) the abelian field strength. Just as the \( Q \) fields themselves can be viewed as scalar projections of the gauge-vector fields, the components of \( \tilde{\nabla} Q \) can thus be interpreted as projections, with respect to the second of their tensorial indices, of the field strengths onto the same \( n \)-vector system. Since the field strengths transform homogeneously (\( F_{\mu \nu} \) being a trivial case) while the \( n \) system is invariant, we again conclude that \( \tilde{\nabla} Q \), too, transforms homogeneously.

By using completeness of that system in the form

\[
n^b_\nu n^b_\lambda + n^4_\nu n^4_\lambda = \delta_{\nu \lambda},
\]

we then obtain the gauge-invariant, four-isospace contraction of two such modified derivatives in the form

\[
\frac{1}{4} \text{tr}_4 \left[ \left( \tilde{\nabla}_\mu Q(x) \right) \cdot \left( \tilde{\nabla}_\mu Q(x) \right)^\dagger \right] = \frac{1}{4} \left( G^{a}_{\mu \nu} G^{a}_{\mu \nu} + F_{\mu \nu} F_{\mu \nu} \right) = S_{gauge}[A]. \quad (6.14)
\]

In a sense, this contraction is therefore just a fancy rewriting of the classical gauge action – a rewriting, though, that captures the interaction between two different excitations, vector and scalar, of the same field system.

It is instructive to digress briefly on how far one can go with this no-hypercharge scenario in the direction of vector-mass formation. For this purpose one would declare

\[
S_{\text{scalar–gauge}} = \frac{1}{4} \int d^4 x \text{tr}_4 \left[ \left( \tilde{\nabla}_\mu Q(x) \right) \cdot \left( \tilde{\nabla}_\mu Q(x) \right)^\dagger \right] \quad (6.15)
\]

as the interaction term in a mixed \( A \)-and-\( Q \) formulation of the problem, as detailed in sect. 8 below. Upon writing this out, one finds, first, a “kinetic” term bilinear in the \( Q \) fields, which after partial integrations reproduces the integral of the sum of the two terms encountered already in eqs. (5.5) and (5.6). Second, the trilinear, two-\( Q \)'s-one-\( A \) pieces, after some rewriting, turn into the \( S_3 \) action term in the “mixed” form of eq. (3.32). Our focus at this moment is on, third, the quadrilinear piece, which turns into the \( S_4 \) action term, again in its “mixed” form as given in eq. (2.19), and which can be written,

\[
S_4 [Q, A] = \frac{1}{4} g_2^2 \int d^4 x A^b_\mu(x) \left\{ \chi(x)^2 1_3 - \chi(x) \chi^T(x) \right\}
+ \left[ \text{tr}_3 \left( \Psi(x) \Psi^T(x) \right) \cdot 1_3 - \Psi(x) \Psi^T(x) \right]^{b c} A^c_\mu(x). \quad (6.16)
\]
Upon application of eq. (5.16), this splits off a bare-mass term

\[ \frac{1}{2} \left( \frac{1}{2} \nu^2 g_2^2 \right) \left( A'^T, A' \right) ; \quad A' := \left( 1_3 - \Pi \right) A, \tag{6.17} \]

which, since \( 1_3 - \Pi \) projects on two isodimensions, confers a bare mass of \( m_{\text{gauge}}^2 = \frac{1}{2} (\nu g_2)^2 \) on only two of the gauge fields. In this it is phenomenologically deficient at this stage, and it is not hard to check that the deficiency will not be cured by the introduction of vector-field mixing. But it is interesting that upon comparing with (5.18), eq. (6.17) yields a value of

\[ m_{\text{scalar}} : m_{\text{gauge}} = \sqrt{2} \tag{6.18} \]

for the tree-level mass ratio, for which the standard scenario offers no clue. This would hardly qualify as an accurate pre- or postdiction, but in view of the utter simplicity of the reasoning advanced here in its favor, its cost-to-benefit ratio may nevertheless be deemed acceptable.

The occurrence of the projector \( 1_3 - \Pi \), as in eq. (5.17), is of course welcome, as it allows for the appearance of a massless photon. But at the same time, the result shows clearly that without hypercharge we have one interacting vector field too few in the mass game, and this forces the next point upon us.

### 7. Hypercharge ?

In order to introduce hypercharge, as a means of bringing in one more vector field capable of mass formation, it is of course not sufficient to simply restore the \( g_1 B_\mu Y \) term in eq. (6.1). A gauge-covariant construct can result only if the \( Q \) fields acted upon are first endowed with hypercharge, in the sense of a homogeneous transformation behavior under a local \( U(1) \) group, and this is not trivial – we have built the \( Q \)'s wholly from gauge fields, which carry no hypercharge. Can we confer upon our scalar fields a quantum number that their “parents” do not have?

The answer offered in the following is tentative, and it has drawbacks that we shall discuss. It does, however, establish the required transformation behavior, allows for a gauge-covariant extension of eq. (6.9), and will presently be seen to go quite some way in the vector-mixing and mass-formation problems. In the following we proceed heuristically, arguing that the natural way of embarking on this problem is to try to further exploit the freedom in \( Q \) as encountered in choosing the \( \exp(\Omega(x)) \) “phase” factor in eq. (2.8). Instead of taking the phase matrix purely real and orthogonal, we now allow for a minimal complex extension, writing

\[ \bar{Q}(x) = e^{ig_4 h(x)} F_4 Q(x), \tag{7.1} \]

where \( F_4 \) is an hermitean \( 4 \times 4 \) matrix that will turn out to be related, though not identical, to hypercharge. For the time being, we require of \( F_4 \) only that it commute with the other \( F \)'s:

\[ [F_4, F_a] = 0 \quad (a = 1, 2, 3). \tag{7.2} \]
Moreover, \( h \) denotes the integral
\[
h(x) = \int d^4 y \, K_\mu(x - y) \, B_\mu(y)
\] (7.3)
with a weight function given by
\[
K_\mu(x - y) = \frac{x - y}{2\pi^2 \left[ (x - y)^2 \right]^2}.
\] (7.4)
As a consequence of the relations
\[
K_\mu(x - y) = \frac{1}{4\pi^2 (x - y)^2} \left[ \frac{1}{4\pi^2} \frac{1}{(x - y)^2} \right] = \delta^4(x - y),
\] (7.5)
h\( (x) \) has two simple infinitesimal properties:
\[
\partial_\mu h(x) = B_\mu(x),
\] (7.6)
\[
\delta_\theta h(x) = \delta\theta^4(x).
\] (7.7)
The \( \bar{Q} \) fields are now complex. The use of complex “dressing factors” as in eq. (7.1) is again not new; similar factors have been used as far back as the 1930’s by Dirac to define physical-field variables for charged fermions. It must be emphasized that the hypercharge coupling \( g_4 \) as introduced through eq. (7.1) bears no relation to the constant \( g_1 \) appearing in the traditional eq. (6.1); it is one feature in which the \( \bar{Q} \) scalars depart from their vectorial ancestry. Also, we should remember that eq. (7.1) does not define \( g_4 \) and \( F_4 \) separately but only their product; some kind of normalization of \( F_4 \) will be necessary to calibrate \( g_4 \).

We now explore the consequences of postulating the \( \bar{Q} \)'s, rather than the \( Q \)'s, to be the physical scalar fields (apart perhaps from a linear mixing that parallels the usual mixing of gauge vectors into mass and charge eigenstates). From eq. (7.7) one infers that the \( \bar{Q} \)'s now have acquired the sought-after, homogeneous terms in their gauge responses to local U(1) transformations with parameter \( \theta^4(x) \) (see appendix A):
\[
\delta_\theta \bar{Q}(x) = e^{ig_4 h(x) F_4} \left[ \delta_\theta Q(x) + ig_4 F_4 \delta\theta^4(x) Q(x) \right] .
\] (7.8)
From eqs. (7.6) and (7.2) it follows that
\[
\left[ 1_\mu \partial_\mu - ig_4 B_\mu(x) F_4 \right] \bar{Q}(x) = e^{ig_4 h(x) F_4} \left[ \partial_\mu Q(x) \right] .
\] (7.9)
Using these relations, it is now straightforward to establish that the quantity
\[
\nabla_\mu \bar{Q} := \bar{D}_\mu \bar{Q} - \left( \partial_\nu - ig_4 B_\nu F_4 \right) \left( \bar{Q} n_\mu \cdot n_\nu^T \right)
\] (7.10)
relates to the \( \nabla \) of (6.9) by the simple formula
\[
\nabla_\mu \bar{Q} = e^{ig_4 h(x) F_4} \left( \nabla_\mu Q \right)
\] (7.11)
and therefore possesses the homogeneous U(2) transformation law
\[
\delta_\theta \left[ \nabla_\mu \bar{Q}(x) \right] = i \left[ g_2 \delta\theta^d(x) F^d + g_4 \delta\theta^4(x) F_4 \right] \left[ \nabla_\mu \bar{Q}(x) \right] .
\] (7.12)
Thus $\nabla_\mu$ is our \textit{new covariant derivative with hypercharge}, designed to act on complex $\bar{Q}$ rather than real $Q$ fields. It can be written more explicitly as

$$\left(\nabla_\mu \bar{Q}\right)^{AB} = \left(\nabla_\mu\right)^{AB,CD} \bar{Q}^{CD}, \quad (7.13)$$

where $\nabla_\mu$, in addition to being an Euclidean four-vector, now emerges as a fourth-rank tensor over the four-dimensional isospace:

$$\left(\nabla_\mu\right)^{AB,CD} := \delta^{AC} \left[ \delta^{DB} \partial_\mu - \left( n_\mu \cdot n_\nu \right)^{DB} \partial_\nu \right] - i \left\{ g_2 \left( A_\mu^e F^e \right)^{AC} \delta^{DB} + g_4 F_4^{AC} \left[ B_\mu \delta^{DB} - \left( n_\mu \cdot n_\nu \right)^{DB} B_\nu \right]\right\}. \quad (7.14)$$

The first line here generates in eq. (7.13) the part linear in fields and containing differential operators. The second line produces terms bilinear in fields; these comprise, as expected, the terms familiar from the “old” eq. (6.1), but also a new, unfamiliar term

$$\left. + i g_4 F_4^{AC} \left( n_\mu \cdot n_\nu \right)^{DB} B_\nu \bar{Q}^{CD}. \right. \quad (7.15)$$

This can be written, using the $A - Q$ relations of eqs. (4.18/5.3), in various equivalent forms, among which we choose, heuristically, by arguing that a covariant derivative in Euclidean direction $\mu$ should provide coupling to vector fields $A_\mu$, not $A_\nu$. Thus the form

$$- i g_4 A_\mu^D \left[ \left( -e^{ig_4 F_4} \right)^{AD} \left( e^{-ig_4 F_4} \right)^{4C} \right] \bar{Q}^{CB}. \quad (7.16)$$

is indicated for expression (7.15) – incidentally, it is also the only form having no uncontracted $n$ vectors.

To deal with the square bracket, we need to say more about the matrix $F_4$. If $F_4$ were restricted to its upper-left $3 \times 3$ block, requirement (7.2) would force it to be a multiple of $1_3$, since the three $F_a$’s generate an irreducible representation (Schur’s lemma). Again arguing heuristically, we then note that the simplest extension of this to four isodimensions, respecting (7.2), is to still keep $F_4$ diagonal but to insert a nonzero $(4, 4)$ element. Thus we try

$$F_4 = \text{diag} \{ \sigma_2, \sigma_2, \sigma_2, \sigma_4 \}. \quad (7.17)$$

This turns the square bracket of expression (7.16) into

$$\delta^{AD} \delta^{4C} \sigma(D) e^{\sigma_4 h(x)[\sigma(D) - \sigma_4]}, \quad (7.18)$$

where $(D)$ is not summed over, and has values of 2 for $D = 1, 2, 3$, and 4 for $D = 4$. The exponent, if nonzero, would produce x-dependent generator matrices in eq. (7.16) through the function $h(x)$, which in a covariant derivative is excluded. We are forced to assume

$$\sigma_2 = \sigma_4 = \sigma, \quad \text{i.e.} \quad F_4 = \sigma 1_4; \quad e^{ig_4 h(x)F_4} = \left( e^{ig_4 h(x)} \right) \cdot 1_4. \quad (7.19)$$

Thus $F_4$ commutes with everything not only in the three-dimensional but also in the four-dimensional isospace; the dressing factor reduces to a number-valued, x-dependent complex phase. This allows
us to get rid of a minor awkward feature of definition (7.1) as combined with eq. (2.8): so far, the complex “phase” matrix stands to the left, the real one to the right of the $X^{\frac{1}{2}}$ “modulus” matrix, but now we may write

$$\bar{Q}(x) = X^{\frac{1}{2}}(x) \cdot \exp \left( \bar{\Omega}(x) \right), \quad (7.20)$$

with a single, unitary-matrix factor,

$$\exp \left( \bar{\Omega}(x) \right) = \mathcal{P} \left\{ \exp \left[ \int_{C(x_0,x)} M_{\mu}(s) ds_{\mu} + i g_4 \sigma \int d^4 y K_{\mu}(x-y) B_{\mu}(y) \ 1_4 \right] \right\}. \quad (7.21)$$

The $M_{\mu}$ portion of the exponent is real and antisymmetric, the $i \ 1_4$ portion imaginary and symmetric. With this, our basic factorization of eq. (2.7) generalizes simply to

$$X(x) = \bar{Q}(x) \cdot \bar{Q}^\dagger(x). \quad (7.22)$$

Returning now to eq. (7.14) and introducing into it the simplified form of the term (7.16), we obtain

$$\left( \nabla_{\mu} \right)^{AB,CD} := \delta^{AC} \left[ \delta^{DB} \partial_{\mu} - \left( n_{\mu} \cdot n_\nu \right)^B D \partial_{\nu} \right]$$

$$- i \tilde{g} A_\mu^E(x) \left( \bar{I}_E \right)^{AC} \delta^{DB}, \quad (7.23)$$

featuring the four (now duly constant) $4 \times 4$ matrices

$$\left( \bar{I}_E \right)^{AC} := c_g \delta_{E,e} \left( F_e \right)^{AC} + s_g \sigma \left[ \delta^4_E \delta^{AC} - \delta^A_E \delta^4_C \right] \quad (7.24)$$

for $E = 1 \ldots 4$. Here we have parameterized the couplings $g_2, g_4$ in terms of a common coupling $\tilde{g}$ and an angle $\vartheta_g$ by putting

$$g_2 = \tilde{g} c_g ; \quad c_g := \cos \vartheta_g, \quad (7.25)$$

$$g_4 = \tilde{g} s_g ; \quad s_g := \sin \vartheta_g, \quad (7.26)$$

$$\tilde{g} := \sqrt{g_2^2 + g_4^2}. \quad (7.27)$$

In more detail, the matrices (7.24) are,

$$\left( \bar{I}_d \right)^{AC} = c_g \left( F_d \right)^{AC} - s_g \sigma \left( \Theta_d \right)^{AC} \quad (D = d = 1 \ldots 3), \quad (7.28)$$

$$\left( \bar{I}_4 \right)^{AC} = s_g \left( F_4 \right)^{AC} - s_g \sigma \left( \Theta_4 \right)^{AC} = s_g \sigma \cdot \text{diag} \{ 1, 1, 1, 0 \}. \quad (7.29)$$

We have introduced $4 \times 4$ matrices $\Theta_D$, first encountered in (7.18), with elements,

$$\left( \Theta_D \right)^{AC} = \delta^A_D \delta^4_C, \quad (D = 1 \ldots 4), \quad (7.30)$$

that is, with a single, fourth-column entry. It is these that will turn our generator algebra into a truly four-dimensional one.

Note that the matrix $\bar{I}_4$ that has replaced $F_4$, and now governs the coupling of our scalars to the $A_\mu^4 = B_\mu$ field, is diagonal. (Up to a factor it is, in fact, the projector onto the 3-dimensional isospace).
By contrast, the $\tilde{I}_{1,2,3}$ matrices, through their $\Theta_d$ terms, are non-hermitean. Together, these four matrices no more obey the U(2) commutation relations – in this sense, the covariance property of eq. (7.12) is no more manifest. (This comes as no surprise, since in writing eq. (7.23) we have torn apart the natural unit of eq. (7.9) into its two pieces, in order to separate “kinetic” and “interaction” parts). To be more precise, we calculate the commutation relations

$$\begin{align*}
[F_a, F_b] &= i \epsilon_{abc} F_c, \\
[F_a, \Theta_b] &= i \epsilon_{abc} \Theta_c, \\
[\Theta_a, \Theta_b] &= 0.
\end{align*}$$

(7.31)
(7.32)
(7.33)

These will be recognized as defining the Lie algebra of the group of rigid motions in three-dimensional Euclidean space, variously denoted as E(3) or ISO(3), the semi-direct product of the 3-dimensional rotation and translation groups. (Here, of course, we are talking of isospace). The effect of the novel two-fields term (7.15), in the form (7.16), has been to enlarge the U(2) group behind eq. (6.1) to this isospace Euclidean group, with the $\Theta_a$ matrices acting as generators of translations in isospace. It is well known that finite-dimensional representations of this non-compact group are not unitary, and their generators therefore not hermitean, which may help remove some of the exotism of the non-hermiticity of the $\Theta_a$’s, and lend some after-the-fact plausibility to our heuristic guessing that led to their appearance in (7.18).

A second, isomorphic E(3) group is generated by the $F_c$ together with the hermitean adjoints of the $\Theta_a$’s:

$$\begin{align*}
\left( \Theta_D^\dagger \right)^{AC} &= \delta^{A4} \delta_D^C, \quad (D = 1 \ldots 4),
\end{align*}$$

(7.34)

which fulfill the same commutation relations. Finally, the matrix $\Theta_4$, which is hermitean and represents the projector onto the fourth isodimension, obeys

$$\begin{align*}
[\Theta_4, \Theta_a] &= -\Theta_a; \quad \left[ \Theta_4, \Theta_b^\dagger \right] = \Theta_b^\dagger,
\end{align*}$$

(7.35)

while commuting, trivially, with the $F_a$.

With the expression resulting from (7.23),

$$\begin{align*}
\left( \nabla_\mu \tilde{Q} \right)^{AB} &= \left[ \delta^{DB} \delta_{\mu\nu} - \left( n_\mu \cdot n_\nu^\dagger \right)^{DB} \right] \partial_\nu \tilde{Q}^{DA}(x) \\
&\quad - i \tilde{g} \left[ A_\mu^E(x) \left( \tilde{I}_E \right)^{AC} \right] \tilde{Q}^{CB},
\end{align*}$$

(7.36)

the contraction-of-covariant-derivatives action with hypercharge,

$$S_{\text{scalar–gauge}} = \frac{1}{4} \int d^4x \text{tr}_4 \left[ \left( \nabla_\mu \tilde{Q}(x) \right) \cdot \left( \nabla_\mu \tilde{Q}(x) \right)^\dagger \right],$$

(7.37)

finally has terms with zero, one, and two vector fields,

$$S_{\text{scalar–gauge}} \left[ \tilde{Q}, A \right] = C_2 \left[ \tilde{Q} \right] + C_3 \left[ \tilde{Q}, A \right] + C_4 \left[ \tilde{Q}, A \right],$$

(7.38)
where $C_2$ now is a kinetic term for the $\bar{Q}$ scalars analogous to (the integral of) the sum of terms (5.5) and (5.6),

$$C_2 [\bar{Q}] = \frac{1}{2} \int d^4x \text{tr}_4 \left[ \left( \partial_\mu \bar{Q}(x) \right) \cdot \left( \partial_\nu \bar{Q}^\dagger(x) \right) - \left( \partial_\mu \bar{Q}(x) \right) \left( n_\mu \cdot n_\nu^T \right) \left( \partial_\nu \bar{Q}^\dagger(x) \right) \right],$$  (7.39)

and $C_3$ is a trilinear interaction of the vector fields with a “current” of the $\bar{Q}$ scalars,

$$C_3 [\bar{Q}, A] = \tilde{g} \int d^4x A_\mu^C(x) \left( -\frac{i}{4} \right) \text{tr}_4 \left\{ + \tilde{I}_C \left[ \bar{Q}(x) \left( \delta_{\mu \nu} 1_4 - n_\nu \cdot n_\mu^T \right) \left( \partial_\nu \bar{Q}^\dagger(x) \right) \right] \right. 
\left. - \left( \tilde{I}_C \right)^\dagger \left[ \left( \partial_\nu \bar{Q}(x) \right) \left( \delta_{\mu \nu} 1_4 - n_\nu \cdot n_\mu^T \right) \bar{Q}^\dagger(x) \right] \right\},$$  (7.40)

a generalization of eq. (3.32) to four isodimensions. Finally, $C_4$ denotes the seagull-type, four-fields term

$$C_4^{(A)} [\bar{Q}, A] = \tilde{g}^2 \int d^4x A_\mu^C(x) \left\{ \frac{1}{4} \text{tr}_4 \left[ \left( \tilde{I}_C \right)^\dagger \bar{I}_D Q(x) \bar{Q}^\dagger(x) \right] \right\} A_\mu^D(x),$$  (7.41)

which generalizes eqs. (2.19) (6.16) to four isodimensions. Like the latter, it has an alternative and equivalent form as a quadrilinear self-interaction of scalars that mirrors eq. (2.20):

$$C_4^{(Q)} [\bar{Q}] = \tilde{g}^2 \int d^4x \left( \bar{Q}(x) \bar{Q}^\dagger(x) \right)^C_D \left\{ \frac{1}{4} \text{tr}_4 \left[ \left( \tilde{I}_C \right)^\dagger \bar{I}_D Q(x) \bar{Q}^\dagger(x) \right] \right\}. $$  (7.42)

Eqs. (7.41) (7.42) will be the starting point for section 9 below.

It is clear from the combination of eqs. (6.14) and (7.11) that the action of eq. (7.37) is still a fancy rewriting of the classical gauge action. Since the latter “knows nothing” about the coupling $g_4$, it follows that (7.37), contrary to appearances, is in fact independent of $g_4$. As we shall see in section 9, this invariance will be broken in the scalar sector by imposing the condition that the scalar fields carry a truly four-dimensional representation of the extended isospin – this condition will be met at a nonzero value of $g_4 \sigma/g_2$ (eq. (9.7) below). In the world of “intrinsic” scalars, once they have been endowed with hypercharge in the way attempted here, there is therefore the possibility of “spontaneous creation of a coupling”.

8. Mixed-variables action

**Enforcing the $A – Q$ Relation.** We now need to address the fact that use of an action like (7.37) (7.38) makes sense only in a *mixed A-and-Q formulation* of the problem – which, incidentally, would also lend itself best to a comparison with the standard Higgs scenario. Such a formulation involves rewriting the path integral (5.2) as an integral over both vector and scalar fields, while imposing the relation between these through delta functionals. In this section we discuss this rewriting on the simplest level of the original, real $A$ and $Q$ fields, leaving the modifications for complex fields such as $W^\pm$ – eqs. (10.20) (10.21) below – or $\bar{Q}$ to the reader. On this level, we have to enforce the relation (4.18) between the two kinds of variables by inserting

$$1 = \int \mathcal{D}[Q] \delta \left[ Q - A_\lambda n_\lambda \right]$$  (8.1)
into a double path integral:

\[
Z \left[ J, K \right] = \int \mathcal{D}[Q] \mathcal{D}[A] \mathcal{D}[c, \bar{c}] \delta \left[ Q - A_\lambda n_\lambda \right] \exp \left\{ -S_{\text{mixed}}[A, Q, \bar{c}, c] + (J, A) + (K, Q) \right\} . \tag{8.2}
\]

Here \( S_{\text{mixed}} \) stands for a suitable form of the gauge action, to be detailed below, which involves both vector and scalar fields. Exponentiation of the delta functional, in order to convert it into additional action terms, can be achieved by various techniques. One might insert

\[
\delta \left[ Q - A_\lambda n_\lambda \right] = \int \mathcal{D}[R] \exp \left\{ i \left( R, [Q - A_\lambda n_\lambda] \right) \right\} , \tag{8.3}
\]

as is done in the field-strength formulation of ref. [3]. This is technically unobjectionable, but as an action term the imaginary exponent seems somewhat out of place in an Euclidean theory (except when, as in ref. [3], it is immediately absorbed by a Gaussian integration). Alternatively, one could utilize the limit representation,

\[
\delta \left[ Q - A_\lambda n_\lambda \right] = \lim_{\beta \to 0} \text{const.} \exp \left\{ \int d^4x \frac{(Q(x) - A_\lambda(x)n_\lambda)^2}{2\beta^2} \right\} , \tag{8.4}
\]

which, before the limit is executed, would resemble the method of Kondo[4] for introducing new functional variables defined “on the Gaussian average”. The (infinite) constant again drops out of the ratio (5.1).

There is yet a third method, more specifically attuned to the context of a gauge theory, on which we concentrate in the following. It consists in appealing to a functional theorem that underlies the De Witt - Fadde’ev-Popov quantization procedure. In the formulation of [1], and with minimally adapted notation, the theorem runs as follows:

**Theorem:** Let \( \phi_n(x) \) be a set of gauge and matter fields and \( \mathcal{D}\phi = \prod_{n,x} d\phi_n(x) \) their functional volume element. Let \( \mathcal{G}[\phi] \) be a functional of the \( \phi_n \) satisfying the gauge-invariance condition

\[
\mathcal{G}[\phi_\theta] \mathcal{D}[\phi_\theta] = \mathcal{G}[\phi] \mathcal{D}[\phi] , \tag{8.5}
\]

where \( \phi_\theta \) are the \( \phi \) fields locally gauge-transformed with gauge parameters \( \theta^A(x) \). Moreover, let \( f^A[\phi, x] \) be a set of gauge-noninvariant functionals of \( \phi \) and functions of \( x \), and \( \mathcal{B}[f[\phi]] \), in turn, a functional of the \( f^A \). Finally, let \( \mathcal{F} \) denote the continuous matrix of functional derivatives,

\[
\mathcal{F}_{A,x; B,y} = \left. \frac{\delta f^A[\phi_\theta; x]}{\theta \mathcal{B}(y)} \right|_{\theta=0} , \tag{8.6}
\]

Then the functional integral

\[
\mathcal{J} = \int \mathcal{D}[\phi] \mathcal{G}[\phi] \mathcal{B}[f[\phi]] \text{Det} \mathcal{F}[\phi] \tag{8.7}
\]

is actually independent (within broad limits) of the noninvariant functionals \( f^A[\phi, x] \), depends on the choice of the functional \( \mathcal{B}[f] \) only through an irrelevant constant factor, and in fact equals

\[
\mathcal{J} = \frac{C}{\Omega} \int \mathcal{D}[\phi] \mathcal{G}[\phi] , \tag{8.8}
\]

26
where $C = \int \mathcal{D}[f] \mathcal{B}[f]$ is the irrelevant constant, and where

$$
\Omega = \int \mathcal{D}[\theta] \rho[\theta] \quad \left( \mathcal{D}[\theta] = \prod_{x \in E^4, A} d\theta^A(x) \right)
$$

is the volume of the gauge group (in other words, $\rho[\theta]$ is the invariant or Haar measure on the space of group parameters).

This theorem is usually applied for the purpose of changing between gauge-fixing schemes, but it can also be used as a method of enforcing relations between functional variables. In the present context, it evidently cannot be applied to the $\mathcal{D}[\theta] \mathcal{D}[A, c, \bar{c}]$ integration of eq. (8.2) as a whole, since the obvious choice for the “initial” functional $f$,

$$
(f^{(0)})^C_{\mu}[Q, A; x] = A^C_{\mu} - Q^C D_n^D \tag{8.10}
$$

would, in the presence of the “initial” $B$ functional

$$
\mathcal{B}^{(0)} \left[ f^{(0)} \right] = \delta \left[ f^{(0)} \right] = \prod_{\mu=1}^4 \left\{ \prod_{C=1}^4 \prod_{x \in E^4} \delta \left[ (f^{(0)})^C_{\mu}[Q, A; x] \right] \right\} = \prod_{\mu=1}^4 \mathcal{B}^{(0)}_{\mu} \left[ f^{(0)} \right], \quad (8.11)
$$

be effectively gauge invariant under combined gauge transformation of both the $A$ and $Q$ fields (the kernel $F$ of eq. (8.6) would vanish). It can, however, be applied to the “inner” $A$ integration at any fixed configuration $Q$ of the “outer” integration of (8.2), where reference to the fixed $Q$ breaks invariance under gauging of $A$ alone. We also require a slight modification of the above theorem, since the $f^{(0)}$ of (8.10) now also carries a spacetime index $\mu$. Here we observe that the gauge transformations on $A$ fields do not mix the spacetime indices, and that the $\mathcal{B}^{(0)}$ functional of eq. (8.11), as well as the functional volume element $\mathcal{D}[A] = \prod_{\mu=1}^4 \mathcal{D}[A_{\mu}]$, have a product structure with respect to $\mu$. Therefore the theorem can be applied consecutively to each of the four nested path integrations on $A_{\mu}$. For this, however, it is necessary at each step to first multiply and divide by the corresponding functional determinant $\text{Det} \mathcal{F}^{(0)}_{\mu}$; the inverse determinant, which in the presence of the $\delta$ functionals effectively depends only on $Q$, can be shifted into the “outer” $Q$ integration. The theorem then permits, for one $\mu$ at a time, the replacements

$$
(f^{(0)})^C_{\mu} \rightarrow f^C_{\mu}, \quad \mathcal{B}^{(0)}_{\mu} \left[ f^{(0)} \right] \rightarrow \mathcal{B}_{\mu} \left[ f_{\mu} \right], \quad \text{Det} \mathcal{F}^{(0)}_{\mu} \rightarrow \text{Det} \mathcal{F}_{\mu} \quad (8.12)
$$

inside the $A$ integral, with wide freedom in the choice of the “final” $f_{\mu}$ and $\mathcal{B}_{\mu}$ functionals.

For the functional-derivative kernel $\mathcal{F}^{(0)}_{\mu} = \delta f^{(0)} / \delta \theta^C$, using the notation of eq. (A12) of appendix A and the presence of the delta functional $\mathcal{B}^{(0)}_{\mu}$, we obtain

$$
(\mathcal{F}^{(0)}_{\mu})_{B,x;C,y} = \frac{\delta (f^{(0)})^B_{\mu}[A_{\theta}; x]}{\delta \theta^C(y)} \bigg|_{\theta=0, A_{\mu} \rightarrow Qn_{\mu}}
$$

$$
= \left\{ \frac{\partial}{\partial x_{\mu}} \delta^4(x - y) \right\} \delta^{BC} - g_2 \delta^4(x - y) \epsilon^{BC d} Q^d E(y)n^E_{\mu} \right\}. \quad (8.13)
$$
The inverse determinants of these kernels, having accumulated outside the \(A\) integration, form a product-inverse determinant that can be represented by a path integral over bosonic ghost fields:

\[
\left[ \operatorname{Det} \mathcal{F}(0) \right]^{-1} = \prod_{\mu=1}^{4} \left[ \operatorname{Det} (\mathcal{F}_{\mu}^{(0)})^{-1} \right] \\
= \text{const.} \int \mathcal{D}[b, \bar{b}] \exp \left[ -\sum_{\mu=1}^{4} \left( \bar{b}_{\mu}^{A}, (\mathcal{F}_{\mu}^{(0)})_{AB} b^{A}_{\mu} \right) \right], \quad (8.14)
\]

with volume element

\[
\mathcal{D}[b, \bar{b}] := \prod_{\mu=1}^{4} \prod_{A=1}^{4} \left\{ \mathcal{D}[b_{\mu}]\mathcal{D}[\bar{b}_{\mu}] \right\}. \quad (8.15)
\]

Since \(\mathcal{F}^{(0)}\) has unit mass dimension, the \(b\)'s and \(\bar{b}\)'s are bosonic variables with the unconventional mass dimension of \(3/2\).

For the “final” functionals of the process, which should provide for the desired exponentiation of the delta functionals, we follow closely the gauge-fixing examples by choosing

\[
B[f] = \prod_{\mu=1}^{4} B_{\mu} [f_{\mu}[A]] = \exp \left[ -\frac{1}{2\zeta} \left( f_{\mu}^{C}[A], f_{\mu}^{C}[A] \right) \right]; \quad (8.16)
\]

\[
f_{\mu}^{C}[A] = \partial_{\mu} g^{C}[A]; \quad g^{C}[A] = A_{\nu}^{C} n_{\nu}^{4} - Q^{C} 4, \quad (8.17)
\]

with \(\zeta\) a dimensionless, real parameter. Then \(B[f]\) is accompanied by the product of the determinants of the four functional-derivative kernels

\[
(F_{\mu})_{B,x;C,y} = \left. \frac{\delta f_{\mu}^{B}[A_{\theta};x]}{\delta \theta^{C}(y)} \right|_{\theta=0} = \left\{ \left[ \frac{\partial^{2}}{\partial x_{\mu} \partial x_{\nu}} \delta^{4}(x-y) \right] \delta^{BC} - g_{2} \left[ \frac{\partial}{\partial x_{\mu}} \delta^{4}(x-y) \right] \epsilon^{BCd} A_{\nu}^{d}(y) \right\} n_{\nu}^{4}. \quad (8.18)
\]

Since this can be written as

\[
(F_{\mu})_{B,x;C,y} = \int d^{4}z \left[ \frac{\partial}{\partial x_{\mu}} \delta^{4}(x-z) n_{\nu}^{4} \right] \\
\times \left\{ \left[ \frac{\partial}{\partial z_{\nu}} \delta^{4}(z-y) \right] \delta^{BC} - g_{2} \delta^{4}(z-y) \epsilon^{BCd} A_{\nu}^{d}(y) \right\}, \quad (8.19)
\]

and since the determinant of the square-bracketed kernel in the first line is a field-independent factor that cancels in the ratio \((5.1)\), we may as well replace

\[
(F_{\mu})_{B,x;C,y} \rightarrow (F_{\mu}^{(\text{red})})_{B,x;C,y}[A]
\]

\[
= \left\{ \left[ \frac{\partial}{\partial x_{\mu}} \delta^{4}(x-y) \right] \delta^{BC} - g_{2} \delta^{4}(x-y) \epsilon^{BCd} A_{\nu}^{d}(y) \right\}, \quad (8.20)
\]

a first-order differential kernel almost identical to \(F_{\mu}^{(0)}\), but depending on \(A\) rather than \(Q\). (Remember the delta functional, at this stage, is no more available). Then

\[
\operatorname{Det} F^{(\text{red})} = \prod_{\mu=1}^{4} \operatorname{Det} F_{\mu}^{(\text{red})}. \quad (8.21)
\]
can be represented as an integral over additional Grassmannian ghost fields, but one may of course also opt for a purely bosonic-ghosts representation of the quotient kernel \( \left\{ (\mathcal{F}^{(0)}[Q]) (\mathcal{F}^{(\text{red})}[A])^{-1} \right\}^{-1} \). We see no necessity to spell these alternatives out in detail. The point of this exercise has been to show that the task of enforcing the \( Q - A \) relation in the double path integral can be shifted onto a purely kinetic addition, eqs. (8.16)/(8.17), to the \( A - Q \) action, at the expense of a substantial increase in the number of ghost-field integrations.

The still unbroken invariance of the combined \( A - Q \) path integrand under simultaneous gauge transformation of both \( A \)'s and \( Q \)'s should, at least for a perturbative treatment, be dealt with by standard schemes of gauge fixing, such as those implemented by the term \( S_{\text{GFG}} \); and the concomitant Grassmann ghosts \( \bar{c}, c \) and as a self-interaction of the scalars, and since with the rescaling of \( \tilde{\sigma} \) standard schemes of gauge fixing, such as those implemented by the term \( T \).

To this should be added the exponent of expression (8.16),

\[
\frac{1}{2\zeta} \left( f^C_\mu [A], f^C_\mu [A] \right) = \frac{1}{2\zeta} \left( (A_\mu n_\mu^4 - \phi_C), (-\partial_\lambda \partial_\lambda) (A_\nu n_\nu^4 - \phi_C) \right),
\]

We see no necessity to spell these alternatives out in detail. The point of this exercise has been to show that the task of enforcing the \( Q - A \) relation in the double path integral can be shifted onto a purely kinetic addition, eqs. (8.16)/(8.17), to the \( A - Q \) action, at the expense of a substantial increase in the number of ghost-field integrations.

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Each of the latter two terms carries the unrescaled coupling \( \tilde{\sigma} \) whereas the other terms, such as \( C_3 \) or the pure-\( A \) part of the action, appear with rescaled couplings. (That the total is still gauge invariant is then no more immediately visible). The schematic structure of the action is

\[
S_{\text{mixed}}[A, \bar{Q}, \bar{c}, c, \bar{c}, c] = S_{\text{gauge}}[A, \bar{c}, c; g_2 \sqrt{2}] + S_{\text{GFG}}[A, \bar{c}, c; g_2 \sqrt{2}]
\]

\[
+ C_2[\bar{Q}] + C_3[\bar{Q}, A; \tilde{\sigma} \sqrt{2}] + C_4^{(A)}[\bar{Q}, A; \tilde{\sigma}] + C_4^{(Q)}[\bar{Q}; \tilde{\sigma}].
\]

To this should be added the exponent of expression (8.16),

\[
\frac{1}{2\zeta} \left( f^C_\mu [A], f^C_\mu [A] \right) = \frac{1}{2\zeta} \left( (A_\mu n_\mu^4 - \phi_C), (-\partial_\lambda \partial_\lambda) (A_\nu n_\nu^4 - \phi_C) \right),
\]
as well as the exponents and extra ghost integrations representing the functional determinants \([\text{Det} F(0)]^{-1} \text{Det} F\), as discussed above. The fields entering here are really the primed ones of eq. (8.23), but since they are only path-integration variables, we may as well drop the primes.

9. Charge-diagonal representation

In order to decide which of the scalar fields are eligible for VEV formation, we must first of all select those with zero electric charge, i.e. linear combinations of the Q-isomatrix components that belong to eigenvalue zero of the electric-charge matrix (in units of the positron charge \(e\)),

\[
\frac{1}{e} E = T_3 + Y ,
\]

where \(T_3\) and \(Y\) denote, respectively, the diagonalized forms of the third component of isospin and of hypercharge. But at this moment, it is not yet clear what the isospin and hypercharge matrices are in a truly four-dimensional representation of \(U(2)\). The \(F_a\) matrices of eq. (6.2), being just trivial extensions of the three-dimensional generators, are not satisfactory. The \(\tilde{I}_A\) matrices of eq. (7.24) do not qualify, as the first three lack hermiticity, and the fourth does not commute with them. We have once more to rely on heuristic procedure.

For a four-dimensional representation of the isospin generators, the simplest trial choice that suggests itself is to take the hermitean parts of the \(\tilde{I}_a\). It is not the least among the little surprises offered by this study that this seemingly simplistic concept actually works. Since the calculations involved are all standard matrix algebra, we proceed straight to a statement of the results.

After a further rescaling of the coupling according to

\[
\tilde{g} \rightarrow 2c_g \tilde{g} = 2g_2 ,
\]

which restores the original, nonabelian coupling \(g_2\) but with a factor of 2, and a corresponding rescaling of the \(\tilde{I}_A\) matrices,

\[
I_C := \frac{1}{2c_g} \tilde{I}_C ;
\]

\[
(I_C)^{AB} = \frac{1}{2} \delta_C e (F_e)^{AB} + \frac{g_2 \sigma}{2c_g} \left[ \delta^A_C \delta^{AB} - \delta^A_C \delta^{4B} \right] ,
\]

so that \(\tilde{g} \tilde{I} = 2g_2 I\), the hermitean parts of the first three matrices,

\[
R_a := \frac{1}{2} \left( I_a + I_a^\dagger \right) = \frac{1}{2} F_a - \frac{g_2 \sigma}{4c_g} \left( \Theta_a + \Theta_a^\dagger \right) ,
\]

together fulfill the SU(2) commutation relations,

\[
[R_a , R_b] = i \epsilon_{abc} R_c ,
\]

provided the parameters \(\theta_g\) and \(\sigma\) occurring in them obey the condition

\[
\frac{g_2 \sigma}{2c_g} = 1 , \quad \text{or} \quad g_4 \sigma = 2g_2 .
\]
This relation simplifies the $I_C$ matrices, as well as the $R_a$ themselves:

\[
(I_C)^{AB} = \frac{1}{2} \delta_{C,C'} (F_c)^{AB} + \left[ \delta_C^4 \delta^{AB} - \delta_C^A \delta^{4B} \right], \tag{9.8}
\]

\[
R_a = \frac{1}{2} \left[ F_a - \left( \Theta_a + \Theta_a^\dagger \right) \right]. \tag{9.9}
\]

The latter, in addition to being hermitean, are traceless and normalized,

\[
tr_4 R_a = 0; \quad \| R_a \| = 1, \tag{9.10}
\]

where the usual matrix norm has been employed:

\[
\| M \|^2 = (M, M); \quad (M_1, M_2) := tr_4 \left( M_1^\dagger M_2 \right). \tag{9.11}
\]

It is remarkable that condition (9.7) – that is, the condition for the $R_a$ matrices to form a four-dimensional set of generators for the isospin-$SU(2)$ group – has fixed the $I_C$ and their hermitean parts $R_c$ absolutely, with no reference anymore to $c_g, s_g, \text{or } s_g \sigma$. The conclusions we draw in the next section from the (rescaled) action terms (7.41) and (7.42), concerning VEV’s, masses, and mixing, will therefore not depend on either of these quantities.

It is then possible, but neither necessary nor of much consequence, to force the rescaling factor $2c_g$ to be unity, in which case

\[
s_g = \frac{\sqrt{3}}{2}; \quad c_g = \frac{1}{2}, \tag{9.12}
\]

and the scale $\sigma$ of the $F_4$ matrix is then fixed, by condition (9.7), at $\sigma = 2/\sqrt{3}$. But it would just as well be possible to work the other way around and normalize $F_4$ by setting

\[
\sigma = 1; \quad \text{i.e. } F_4 = 1_4, \tag{9.13}
\]

which, again by (9.7), would entail $s_g/c_g = 2$, or

\[
s_g = \frac{2}{\sqrt{5}}; \quad c_g = \frac{1}{\sqrt{5}}. \tag{9.14}
\]

It is obvious that neither of these choices have anything to do with the usual electroweak mixing angle.

To extend the four-dimensional representation to the full $U(2)$ group, we would like to find a hypercharge-generator matrix $R_4$ to match the $R_a$, but since this can be done properly only after diagonalizing $R_3$, we defer this point for a moment; the result in eq. (9.29) below will be seen to have all the required properties,

\[
[R_4, R_a] = 0; \quad tr_4 R_4 = 0; \quad \| R_4 \| = 1, \tag{9.15}
\]

for a consistent and uniformly normalized set of $U(2)$ generators.

The “leftovers” $L_A$ in the decomposition

\[
I_C = R_C + L_C \quad (C = 1 \ldots 4) \tag{9.16}
\]
are, for $C = c = 1 \ldots 3$, the antihermitean parts

$$L_c := \frac{1}{2} (I_c - I_c^\dagger) = -\frac{1}{2} (\Theta_c - \Theta_c^\dagger) = -L_c^\dagger,$$  \hspace{1cm} (9.17)

which are real and antisymmetric. For $C = 4$, the determination of $L_4$ will once more have to wait for a moment, until the correct choice of $R_4$ has been identified. The matrices (9.17) do not generate a group, as is evident from their commutation relations written in the form

$$[-2L_a, -2L_b] = -i \epsilon_{abc} F_c .$$  \hspace{1cm} (9.18)

Taken together with the relations

$$[F_c, -2L_d] = i \epsilon_{cde} (-2L_e) ,$$  \hspace{1cm} (9.19)

they will be recognized as those of the boost transformations in the homogeneous Lorentz group, but associated with the “old” rotation generators $F_c$, rather than with the “new” $R_a$. Since there is no obvious physical meaning to the notion of a boost in isospace, these observations remain purely formal.

The four-dimensional $U(2)$ representation generated by the $R_A$ is reducible. The reducing transformation

$$T_A = U R_A U^\dagger , \quad (A = 1 \ldots 4) ,$$  \hspace{1cm} (9.20)

which converts all $R_A$’s to a $2 \oplus 2$ block-diagonal form, while making $T_3$ diagonal, is effected by the unitary matrix

$$U = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 & i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & i & 0 & 0 \end{pmatrix} ,$$  \hspace{1cm} (9.21)

with inverse,

$$U^\dagger = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} .$$  \hspace{1cm} (9.22)

(No danger of confusion with the matrix function (9.23). The first three $T$’s then assume the $2 \oplus 2$-block form

$$T_a = \begin{pmatrix} \frac{1}{2} \tau_a & 0 \\ 0 & \frac{1}{2} \tau_a \end{pmatrix} (a = 1 \ldots 3) ,$$  \hspace{1cm} (9.23)

where $\tau_a$ are the Pauli matrices, so that in particular

$$T_3 = \text{diag} \left\{ \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\}$$  \hspace{1cm} (9.24)

is diagonal. In this new representation, the hypercharge and electric-charge matrices, too, should then be diagonal. From the irreducibility of the fundamental representation generated by the $\tau_a$, it is clear
that for the hypercharge matrix $T_4 = Y$ to commute with the $T_a$’s requires it to be a multiple of unity in each block:

$$
T_4 = Y = \begin{pmatrix}
  c_+ & 1 & 0 \\
  0 & c_- & 1 \\
\end{pmatrix} (a = 1 \ldots 3). 
$$

(9.25)

For more insight we turn to the transformation of the scalar fields. We would expect the isomatrix $Q(x)$ to transform into $\Phi(x) = U Q(x) U^\dagger$, which is given in eq. (B17) of the appendix, but since only the product $\Phi \Phi^\dagger = (U Q)(U Q)^\dagger$ appears in the $C_4$ action terms, and $\Phi \partial \Phi^\dagger$ in $C_3$, it is sufficient in practice to consider the simpler quantity $U Q(x)$. This also seems more consistent with the fact that the fourth column of $Q$, being the four-isovector $q$ of (2.17) as appearing in the decomposition (2.15), would naturally transform into

$$
\phi(x) := U q(x) = \begin{pmatrix}
  \phi_1 = \frac{1}{\sqrt{2}} ( - \chi_1 + i \chi_2 ) \\
  \phi_2 = \frac{1}{\sqrt{2}} ( \chi_3 + \psi ) \\
  \phi_3 = \frac{1}{\sqrt{2}} ( \chi_3 - \psi ) \\
  \phi_4 = \frac{1}{\sqrt{2}} ( + \chi_1 + i \chi_2 )
\end{pmatrix}. 
$$

(9.26)

This says it all, for every column of $U Q$. (Moreover, the fields in the remainder matrix $U P(x)$ will be easier to interpret physically in section 10 below). The electric charges here are self-suggestive: $\phi_{2,3}$ are real scalar fields and should therefore be electrically neutral. That is, the 2nd and 3rd eigenvalues in the diagonal matrix (9.1) should vanish. Taking this together with eq. (9.24) we conclude that $c_{\pm} = \pm \frac{1}{2}$. This in turn gives $\pm 1$ for the 1st and 4th charge eigenvalues, which fits the interpretation of $\phi_{1,4}$ as a charge- conjugate pair of fields with integer charges $\pm 1$. It is remarkable how naturally these assignments, which are essentially those of the standard Higgs sector, flow from the reduction of the $R$ representation. We now have

$$
Y = T_4 = \text{diag} \left\{ + \frac{1}{2}, + \frac{1}{2}, - \frac{1}{2}, - \frac{1}{2} \right\},
$$

(9.27)

$$
\frac{1}{e} E = \text{diag} \left\{ + 1, 0, 0, -1 \right\}. 
$$

(9.28)

The four $T_A$’s again form an hermitean, traceless, and uniformly normalized set of generators for what we now recognize as the direct sum of two fundamental representations of $U(2)$, distinguished only by their $Y$ eigenvalues of $\pm \frac{1}{2}$.

We may now go back and determine the $R_4$ hypercharge generator of the $R$ representation by applying the $U$ transform backwards:

$$
R_4 = U^\dagger T_4 U = \begin{pmatrix}
  \frac{1}{2} \tau_2 & 0 \\
  0 & \frac{1}{2} \tau_1
\end{pmatrix}. 
$$

(9.29)

Its commutativity with $R_3$ is obvious, since for $a = 3$ has the block-diagonal form

$$
R_3 = \begin{pmatrix}
  \frac{1}{2} \tau_2 & 0 \\
  0 & - \frac{1}{2} \tau_1
\end{pmatrix},
$$

(9.30)

but it also extends, less obviously, to $R_{1,2}$. (From the fact that $R_3 + R_4$, the not-yet-diagonal precursor of the electric charge (9.1), has a vanishing lower-right block, we might have already concluded that
there must be two vanishing scalar-charge eigenvalues). The corresponding matrix $L_4$ follows from formulas (9.16) and (9.8):

$$L_4 = \begin{pmatrix}
1_2 & \frac{1}{2} \tau_2 & 0 \\
0 & \pi_+ & \frac{1}{2} \tau_1 \\
0 & \pi_+ & \frac{1}{2} \tau_1
\end{pmatrix} = L_4^\dagger, \tag{9.31}
$$

where $\pi_+ = \frac{1}{2} (1_2 + \tau_3)$ is the projector onto the upper element of a 2-isospinor. Like $R_4$, the matrix is block diagonal; its matrix norm is $\| L_4 \| = 2$. We do not dwell on its various commutation relations, since these do not lead to a closed algebra until the level of the general linear group in four dimensions, $GL(4)$ – a framework too wide to be of interest in our context.

While the precise group-theoretic status of the $L_A$ matrices remains obscure, their contribution will still be important when extracting masses from the $C_4$ action terms, since there it is not only the $R_C$ parts but the complete $I_C$ matrices that come into play, albeit in the hermitean bilinear combination of eq. (10.11) below.

10. VEVs, Masses, and Mixing

The question of mass formation, for both the vector and scalar fields, now poses itself anew, since eqs. (7.41) and (7.42), in contrast to eq. (6.16), now indeed involve all four vector fields. It turns out that in dealing with this question, it is important not to impose a mixing scheme for the neutral vector fields from the outside[13]. Once the possibilities of formation of vacuum expectation values (VEV’s) have been located by the charge-and-hypercharge diagonalization of the preceding section, the present formalism will produce such a scheme, and indeed a qualitatively correct picture of vector masses, all by itself.

VEV formation. In sect. 5 we pinpointed, on the basis of minimality properties, the separable $\chi^a \chi^b$ term of eq. (2.14) as the place where formation of vacuum-expectation values should occur. In the present, four-fields situation, the natural extension is to rely instead on the decomposition of eq. (2.15), generalized, as in eq. (7.22), to read

$$\bar{Q}(x) \cdot \bar{Q}^\dagger(x) = \bar{P} \cdot \bar{P}^\dagger + \bar{q} \cdot \bar{q}^\dagger, \tag{10.1}$$

with obvious definitions. In the same vein,

$$\bar{\phi}(x) = \exp[i g_4 \sigma h(x)] \phi(x) \tag{10.2}$$

represents the “hypercharged”, complex generalization of the four-isovector (9.26). To find out which scalars can develop nonzero VEV’s, we should examine the eigenvectors – in the present case, the elements of the matrix $\bar{\Phi}$, and more specifically of the four-isovector $\bar{\phi}$ – of electric charge. Since the complex dressing factor commutes, and the $C_4$ term in either form only involves the real and symmetric combination

$$\bar{Q} \bar{Q}^\dagger = Q Q^T = X, \tag{10.3}$$
we may immediately take over the observations made on eq. (9.26): it is the component fields \( \phi_2 \) and \( \phi_3 \) that are electrically neutral and therefore qualify for VEV formation. We combine this with the plausible postulate that VEV’s do not mix different eigenvalues of hypercharge – in other words, that they respect a kind of superselection rule with respect to hypercharge. We first pursue the consequences of choosing \( \phi_2 \),

\[
\langle 0 \mid \hat{\phi}_2(x) \mid 0 \rangle = \frac{1}{\sqrt{2}} v,
\]

so that a four-dimensional extension of eq. (5.16) holds:

\[
\phi(x) = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \end{pmatrix} = \frac{1}{\sqrt{2}} v + \phi'(x),
\]

where \( \phi_4(x) = \phi'^1(x) \). This, incidentally, is also the isospin-hypercharge assignment of the VEV-forming neutral scalar in the standard Higgs scenario. (The other possibility will be commented upon below). The extraction of mass-squared terms from the \( C_4 \) action integrals will then proceed by performing the substitutions

\[
\left( \Phi(x) \cdot \Phi^\dagger(x) \right)^{AB} \rightarrow \Phi_A(x) \cdot \Phi_B^\dagger(x) \rightarrow \delta^{A2} \delta^{B2} \frac{1}{2} v^2
\]

on one of the \( QQ^T \) factors. We remark in passing that, strictly speaking, it is only the bilinear correspondence of \( \Phi \Phi^\dagger \) rather than the linear one in \( \phi \phi^\dagger \), that is necessary in order to exhibit the VEV contributions to the action. In other words, we only need to assume that the two-point Schwinger function of \( \phi_2 \) develop a constant condensate \( \frac{1}{2} v^2 \), but not necessarily that this constant stems from its disconnected term.

**Vector masses.** Start from the form (7.41) of the seagull term and relate the \( \bar{Q} \bar{Q}^\dagger \) to \( \Phi \Phi^\dagger \) by writing

\[
\left( \bar{Q} \bar{Q}^\dagger \right)^{AB} = \left( U^\dagger \right)^{AC} \left( \Phi(x) \cdot \Phi^\dagger(x) \right)^{CD} U^{DB} ,
\]

then perform the substitution (10.7), and read off the necessary \( U^\dagger \) and \( U \) matrix elements from eqs. (9.21, 9.22). You end up with the replacement

\[
\left( \bar{Q} \bar{Q}^\dagger \right)^{AB} \rightarrow \frac{1}{4} v^2 \left( \delta^{A3} + \delta^{A4} \right) \left( \delta^{B3} + \delta^{B4} \right) .
\]

As for the matrix products \( I_C^\dagger I_D \) figuring in both terms of the \( C_4 \) action term (8.24), we observe that in both cases they are contracted with a quantity symmetric under exchange of indices \( C \) and \( D \). We may therefore symmetrize those products as \( \frac{1}{2} \left( I_C^\dagger I_D + I_D^\dagger I_C \right) \). The latter matrices are hermitean. Their imaginary parts are therefore antisymmetric, and in a trace \( tr_4 \) with the symmetric product (10.3) they make no contribution. Overall, the substitution

\[
I_C^\dagger I_D \rightarrow M_C^2 ;
\]

\[
M_C^2 = \Re \left\{ \frac{1}{2} \left( I_C^\dagger I_D + I_D^\dagger I_C \right) \right\}
\]
is permitted. Combining this with (10.9), we find that the “seagull” action (7.11) splits off a mass term for the vector fields,

\[ S_{\text{vmass}}[A] = \frac{1}{2} \int d^4x \left( m^2 \right)_{C,D} A_C^\mu(x) A_D^\mu(x) , \]  

(10.12)

where the mass-squared matrix \( m^2 \) is real and symmetric:

\[ \left( m^2 \right)_{C,D} = \frac{1}{2} g^2 v^2 \sum_{A,B=3}^4 \left( M_C^2 D^B \right)^{AB} . \]  

(10.13)

Evaluating the matrices (10.11) from eq. (9.8), we have for their matrix elements,

\[ \left( M_C^2 D^B \right)^{AB} = \frac{1}{4} \left[ \delta^{AB} \delta^{CD} - \frac{1}{2} \left( \delta^{AC} \delta^{BD} + \delta^{AD} \delta^{CB} \right) \right] + \frac{3}{4} \left( \delta^{AB} \delta^{C4} \delta^{D4} + \delta^{CA} \delta^{B4} \delta^{D4} \right) . \]  

(10.14)

Using these in eq. (10.13), we obtain a vector mass-squared matrix with block-diagonal structure,

\[ m^2 = \frac{1}{2} g^2 v^2 \cdot \begin{pmatrix} 5 & 12 & 0 \\ 5 & -1 & 0 \\ 0 & 1 & N_0^2 \end{pmatrix} , \]  

(10.15)

where the matrix \( N_0^2 \) in the \( A^3 - A^4 \) subspace is given by

\[ N_0^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} . \]  

(10.16)

The latter matrix, visibly degenerate, has eigenvalues 2 and 0. The corresponding eigenvectors give the \( A^3 - A^4 \) combinations that diagonalize the quadratic form of eq. (10.12),

\[ Z_\mu(x) = \frac{1}{\sqrt{2}} \left[ A^3_\mu(x) - i A^4_\mu(x) \right] , \]  

(10.17)

\[ A_\mu(x) = \frac{1}{\sqrt{2}} \left[ A^3_\mu(x) + i A^4_\mu(x) \right] , \]  

(10.18)

the latter being massless. The fully diagonalized form of \( m^2 \) thus reads,

\[ (m^2)_{\text{diag}} = g^2 v^2 \cdot \text{diag} \left\{ \frac{5}{8}, \frac{5}{8}, 1, 0 \right\} . \]  

(10.19)

On the other hand, in the mass-degenerate \( A^1 - A^2 \) subspace, where any linear combination of these two fields is an eigenvector, the argument follows the familiar pattern: the charge superselection rule singles out the charge-conjugate pair

\[ W^+(x) = \frac{1}{\sqrt{2}} \left[ A^1_\mu(x) - i A^2_\mu(x) \right] , \]  

(10.20)

\[ W^-(x) = \frac{1}{\sqrt{2}} \left[ A^1_\mu(x) + i A^2_\mu(x) \right] , \]  

(10.21)
as the physical ones. What is noteworthy here is that once the modified gauge-covariant derivative of eq. (7.23) has been established, the mechanics of the $I_A$ matrices, when triggered by the VEV formation (10.1) but with no further input, produces all by itself a qualitatively correct picture of vector masses: a massive, charge-conjugate pair, a neutral vector with slightly larger mass, and a massless photon.

The heavy-vector masses and their ratio are,

$$m_W = g_2 v \sqrt{\frac{5}{8}}, \quad m_Z = g_2 v; \quad m_W / m_Z = \sqrt{\frac{5}{8}} \approx 0.79.$$ (10.22)

The empirical value is $\approx 0.87^{14}$. The comments made above for eq. (6.18) apply again: there is no accurate fit with experiment, but given the facts that these are tree-level values, and that they have been obtained without adjustable parameters, the result (10.22) would seem to be tolerable.

By comparison, we recall that in the standard Higgs scenario the counterparts to eqs. (10.15/10.16) and eqs. (10.17/10.18) are parameterized by the electroweak mixing angle, $\theta_W$, which is defined in analogy to eqs. (7.25/7.26), but for the couplings $g_2, g_1$ of the ordinary covariant derivative (6.1).

It is a new constant of nature, to be adjusted to experiment, and it appears (at tree level) in two roles: first, it gives the $m_W / m_Z$ ratio as $c_W = \cos(\theta_W)$, from which it is determined empirically as $c_W \approx 0.87, \theta_W \approx 28.7$ degrees. Second, it appears in a degenerate hermitean matrix analogous to (10.16), and consequently produces for the mixing scheme analogous to eqs. (10.17 /, 10.18) an $SO(2)$ orthogonal matrix parameterized by this very angle. In the present formalism, these two roles are decoupled: the mass ratio (again at tree level) is as in (10.22), an algebraic number, while the orthogonal mixing matrix has an angle of 45 degrees.

It remains to comment on the alternative to eq. (10.4), where $\hat{\phi}_3(x)$ rather than $\hat{\phi}_2(x)$ is allowed to develop a vacuum expectation. One finds quickly that mass-squared eigenvalues remain the same, and that the only change to eqs. (10.17/10.18) is an exchange of the plus and minus signs, which amounts to redefining $-A^4$ as $A^4$. Since $A^4$ appears only bilinearly in the original action, this clearly changes nothing.

The end result of this section is then the desired form of the vector-mass term (10.12) in the action,

$$S_{\text{vmass}}[WZ] = \frac{1}{2} \int d^4x \left[ 2 m_W^2 W^+_{\mu}(x) W^-_{\mu}(x) + m_Z^2 Z_{\mu}(x) Z_{\mu}(x) \right],$$ (10.23)

with $m_W$ and $m_Z$ as in eq. (10.22).

**Scalar masses.** The discussion of sect. 5 on mass formation for the scalars, based purely on the $\Xi$ fields, also needs to be reconsidered in the new context created by the introduction of hypercharge. A convenient starting point is eq. (7.32), with both the symmetrization (10.10) and the transform of eq. (10.8) applied:

$$\frac{1}{2} C_4^{(Q)} \left[ Q; g \sqrt{2} \right] = g^2 \sqrt{2} \int d^4x \frac{1}{4} \left( M^2_{C,D} \right)^{AB} \left[ U^\dagger \left( \Phi(x) \cdot \Phi^\dagger(x) \right) U \right]^{C,D} \left[ U^\dagger \left( \Phi(x) \cdot \Phi^\dagger(x) \right) U \right]^{B,A}. \quad (10.24)$$
Upon introducing the $U$ transform of the decomposition (10.1) into (10.24), and taking eq. (9.2) into account, we have

$$\frac{1}{2} C_4^{(Q)} \left[ \bar{Q} \gamma \sqrt{2} \right] = (2g_2)^2 \int d^4 x \frac{1}{4} \left( M_{CD}^2 \right)^{AB}$$

$$\left\{ (U^\dagger)^{C,E} \left[ \left( \Pi \cdot \Pi^\dagger \right)^{E,F} + \phi_E \cdot \phi_F^\dagger \right] U^{F,D} \right\} \left\{ (U^\dagger)^{B,G} \left[ \left( \Pi \cdot \Pi^\dagger \right)^{G,H} + \phi_G \cdot \phi_H^\dagger \right] U^{H,A} \right\},$$

(10.25)

where $\Pi$ is the $U$ transform of the matrix $P$ of (2.16):

$$\Pi(x) = U P(x) U^\dagger.$$  

(10.26)

We now put to work three useful properties of the symmetrized 16-plet (10.11) of matrices:

- The contraction of $M^2$ with one and the same four-isovector in all four indices vanishes. This is exemplified by the four-$\phi$s term of (10.11): by eq. (9.26),

$$\left( M_{CD}^2 \right)^{AB} \left\{ (U^\dagger)^{C,E} \left[ \phi_E(x) \cdot \phi_F^\dagger(x) \right] U^{F,D} \right\} \left\{ (U^\dagger)^{B,G} \left[ \phi_G(x) \cdot \phi_H^\dagger(x) \right] U^{H,A} \right\} = \left( M_{CD}^2 \right)^{AB} q^C q^D q^B q^A = 0.$$  

(10.27)

This implies, in particular, that there is no $\phi$ mass term, nor are there cubic or quartic $\phi$ self-interactions. The $\phi$ fields remain massless; they figure as quasi-Goldstone bosons. (The property, actually, is already true for contraction with one four-isovector in only three of the four indices).

- Besides being symmetric, by construction, both in their index pair $(A,B)$ and in the matrix-enumeration pair $(C,D)$, the sixteen matrices possess the remarkable additional symmetry

$$\left( M_{CD}^2 \right)^{AB} = \left( M_{AB}^2 \right)^{C,D},$$

(10.28)

such that the ensemble may be viewed as a single supermatrix, with indices consisting of symmetric pairs of ordinary matrix indices, and moreover symmetric under the exchange of its two index pairs:

$$\hat{M}_{AB,CD}^2 = \hat{M}_{C,D,AB}^2.$$  

(10.29)

Since each of those pairs has ten different values, the supermatrix $\hat{M}^2$ is ten-dimensional, with $\frac{1}{2} \cdot 10 \cdot (10 + 1) = 55$ different matrix elements. (It has some formal similarity with a curvature tensor, except that the latter is antisymmetric within each of its index pairs). The above statement is proven by inspection of eq. (10.14).

Due to this additional symmetry, the two mixed $\Pi - \phi$ terms in eq. (10.25) are equal. In the final form of (10.24),

$$\frac{1}{2} C_4^{(Q)} \left[ \bar{Q} \gamma \sqrt{2} \right] = g_2^2 \int d^4 x \left( M_{CD}^2 \right)^{AB}$$

$$\left\{ (U^\dagger)^{C,E} \left[ 2 \phi_E \cdot \phi_F^\dagger + \left( \Pi \cdot \Pi^\dagger \right)^{E,F} \right] U^{F,D} \right\} \left\{ (U^\dagger)^{B,G} \left[ \left( \Pi \cdot \Pi^\dagger \right)^{G,H} \right] U^{H,A} \right\},$$

(10.30)
the mixed term therefore appears with a factor of 2, whose origin is the same as for the 2 in the mixed $\Psi - \chi$ term of eq. (5.13), and which will again be crucial for the scalars-to-vectors mass ratio. In addition, there is of course a four-$\Pi$ self-interaction term, which we do not discuss, except for pointing out that its coupling constant is just the old $g^2$ of the nonabelian sector.

- The supermatrix $\hat{M}^2$ is positive semi-definite on the subspace of positive semi-definite, symmetric matrices. To check this, we consider its mean value in an ordinary matrix $X = X^\dagger$ of this type:

$$\left( X, \hat{M}^2 X \right) = \sum_{A,B,C,D} X_{AB} \hat{M}^2_{AB,C,D} X_{CD} \, .$$

Its evaluation gives

$$\left( X, \hat{M}^2 X \right) = \frac{1}{4} \left[ (\text{tr}_4 X)^2 - \text{tr}_4 \left( X^2 \right) \right] + \frac{3}{2} \left[ (\text{tr}_4 X) X_{4,4} - \left( X^2 \right)_{4,4} \right] \, .$$

In the first line we recognize the four-dimensional analog of (5.11), which is positive semi-definite for the same reasons. But the second-line term is also positive semi-definite: using the unitary matrix $V$ which carries $X$ to its diagonal form $X_{\text{diag}} = V X V^\dagger$, with (by assumption) nonnegative eigenvalues $\lambda^A, A = 1 \ldots 4$, it becomes

$$\frac{3}{2} \sum_{A,B} |V^{4,A}|^2 \lambda^A \lambda^B \left( 1 - \delta^{AB} \right) \, ,$$

from which, since $|V|^2$’s are between zero and unity, one concludes

$$0 \leq \frac{3}{2} \left[ \ldots \right] \leq \frac{3}{2} \left[ (\text{tr}_4 X)^2 - \text{tr}_4 \left( X^2 \right) \right] \, .$$

Thus the integrand of $C_4^{(Q)}$ has a minimum of zero at $\Phi \Phi^\dagger = \phi \phi^\dagger$, i.e. at $\Pi = 0$, a situation largely analogous to the one in three isodimensions at the end of sect. 5. We can expect massive Higgs-type particles in those $\Pi$ directions where there is a stable minimum, but this time with more fields in the game.

We may now retrace the steps which in the vector case led from eq. (7.41) to the isolation of the mass term (10.23), after the replacement of eq. (10.7). We find that the mixed $\phi - \Pi$ part of expression (10.30) splits off a scalar-mass term,

$$S_{\text{smass}} [\Pi] = \frac{1}{2} \int d^4x \left( \mu^2 \right)_{AB} \left[ \Pi(x) \Pi^\dagger(x) \right]^{BA} \, ,$$

where the scalar mass-squared matrix $\mu^2$ is given by

$$\left( \mu^2 \right)_{AB} = 2 \left[ \hat{M}^2_{CD,EF} \frac{1}{2} g^2 v^2 \left( \delta^{C3} + \delta^{C4} \right) \left( \delta^{D3} + \delta^{D4} \right) \right] \left( U^\dagger \right)^F B U^{AE} \, .$$

In the square bracket, one recognizes the element $(m^2)_{EF}$ of the vector mass-squared matrix (10.13). We therefore have the relation

$$\mu^2 = 2 U m^2 U^\dagger \, .$$
between the two kinds of (squared) mass matrices. Upon carrying out the $U$ transform one finds that \textit{en passant} it diagonalizes the matrix

$$
\mu^2 = g_2^2 v^2 \text{diag} \left\{ \frac{5}{4}, 0, 2, \frac{5}{4} \right\}
= \text{diag} \left\{ (m_W \sqrt{2})^2, 0, (m_Z \sqrt{2})^2, (m_W \sqrt{2})^2 \right\}.
$$

(10.38)

The eigenvalues, apart from a different order of appearance, are twice those for the squared vector masses, and $\sqrt{2}$ times those for the vector masses themselves. Thus $T_3, Y = T_4$, and $\mu^2$ are all diagonal in the same $U(2)$ representation, as one would expect in an ordinary Goldstone environment.

Since by eq. (10.26) we have $\Pi \Pi^\dagger = (UP)(UP)^\dagger$, we may regroup the twelve $\Pi$ scalars as

$$
H^{Ab}(x) = [UP(x)]^{Ab}, \quad (b = 1 \ldots 3),
$$

(10.39)

for along with $P$, $UP$ also has vanishing fourth column. Then,

$$
\left[ \Pi(x) \Pi^\dagger(x) \right]^{AB} = H^{Ac}(x) \left( H^\dagger \right)^cB(x).
$$

(10.40)

By eq. (10.38), only $B = A$ elements appear in the action mass term (10.35), which therefore reads,

$$
S_{\text{mass}}[\Pi] = \frac{1}{2} \int d^4x \sum_{c=1}^3 \left\{ (m_W \sqrt{2})^2 \left[ |H^{1c}(x)|^2 + |H^{4c}(x)|^2 \right] + (m_Z \sqrt{2})^2 |H^{3c}(x)|^2 \right\}.
$$

(10.41)

A glance at eq. (9.28) shows that $H^{1c}(x)$ and $H^{4c}(x)$ are charge conjugates, with electric charges $\pm 1$, while $H^{3c}(x)$ is electrically neutral. Thus there are a mass-degenerate sextuplet of charged Higgs-type scalars at a mass of $\mu^{(\pm)} = m_W \sqrt{2}$, and a degenerate triplet of neutral Higgs scalars at $\mu^{(0)} = m_Z \sqrt{2}$.

Similarly to what we saw at the end of section 5, three other neutral $H$ fields, $H^{2c}(x)$, remain massless, as did all the $\phi$ or $\phi^\dagger$ fields, although the former are not in the classical valley of zero potential energy defined by $\Pi = 0$. So there are now a total of seven massless, candidate Goldstone fields.

11. Loose Ends

We have seen that “intrinsic” scalar fields with several desirable properties can be identified in an $SU(2) \otimes U(1)$ gauge theory, and that one can go quite some distance in replaying with them the standard scenario of spontaneous symmetry breaking and mass generation. Nevertheless, the article, despite its length, has left several unresolved questions. It is useful to enunciate these in closing.

- We have argued, in section 4, for the orthonormal vector system $n^A_{\mu}$ being both $x$-independent and independent of gauge-field configuration. A question that suggests itself, but to which at present we have no clue, is whether that system can then be chosen arbitrarily, or whether it retains some vestige of its gauge-theoretic construction that sets it apart. It would be helpful here to have some explicit examples of gauge fields for which that construction can be carried out explicitly.
• The true interrelationship of the seven massless fields, candidates for Goldstone bosons, is arguably not fully understood. The $\phi(x)$ doublet-plus-doublet of fields populate the region $\Pi = 0$ of field space in which the quartic scalar potential $C_4^{(Q)}$ assumes its minimum of zero, while remaining themselves massless. Now the Higgs-type scalars of eq. (10.39) also include a massless triplet – what precisely is the difference? It may help to reflect on the fact that any projector in isospace onto the direction of a single isovector may become a zero minimum, namely if all fields other than this isovector vanish. If we complement definition (10.39) by

$$H^{A4}(x) = [UQ(x)]^{A4} = \phi_A(x),$$

(11.1)

then the projector onto the direction of the isovector $H^{2B}$, $B = 1 \ldots 4$ would also give a minimum of zero for $C_4^{(Q)}$, but in the region where all fields $H^{AB}(x)$ with $A \neq 2$, including $\phi_A(x)$ with $A \neq 2$, would vanish. Thus the two minima are in general incompatible – in the density-matrix analogy invoked in section 5, they correspond to projectors onto nonorthogonal pure states. The field configuration where only $H^{24}(x) = \phi_2(x)$ is nonvanishing “sits at the crossroads”; it is the only one belonging to both classical minima, and in this sense particularly susceptible to VEV formation. This may cast some light on the special role of $\phi_2$ in our treatment, but it does not fully elucidate the respective roles of the two remaining massless triplets in each minimum.

• We have not touched upon the practically important problem of finding a unitarity gauge, in which the massless scalars would be gotten rid of at the expense of a loss of manifest renormalizability. For this problem, the observation in section 7 of the presence of a six-parameter group of motions in isospace may be of relevance, although in ways that have not been demonstrated. Nor have we exhibited manifestly renormalizable gauges of the $R_\xi$ type that would be welcome for calculations.
A Gauge and BRS variations

Gauge variations of $Q$ scalars. Under infinitesimal local gauge transformations characterized by gauge functions $\delta \theta^a(x)$, $a = 1, 2, 3$, and $\delta \theta^4(x)$, the $A^a$ and $B$ gauge fields undergo the well-known changes

$$\delta A^a_{\mu}(x) = \left[ \delta^{ab} \partial_\mu - g_2 \epsilon^{abc} A^c_\mu(x) \right] \delta \theta^b(x), \quad (A1)$$

$$\delta B_\mu(x) = \left[ \partial_\mu \delta \theta^4(x) \right]. \quad (A2)$$

These formulas remain valid for BRS transformations when replacing the infinitesimal gauge functions according to

$$\delta \theta^a(x) \rightarrow \lambda c^a(x); \quad \delta \theta^4(x) \rightarrow \lambda c^4(x), \quad (A3)$$

with $\lambda$ a Grassmann-valued, constant parameter ($\lambda^2 = 0$). By virtue of the “projection” relation (4.18) (with prime dropped) and the property (4.16) of the $n^A$ vector system, it becomes straightforward to derive from these the gauge/BRS variations of the $Q^{AB}$ scalar fields. We merely list the results:

$$\delta Q^{AB} = \left( \delta A^A_\mu \right) n^B_\mu. \quad (A4)$$

with special cases,

$$\delta \Psi^{ab} = \left\{ \left[ \delta^{ad} \partial_\mu - g_2 \epsilon^{adc} A^c_\mu(x) \right] \delta \theta^d(x) \right\} n^b_\mu, \quad (A5)$$

$$\delta \chi^a = \left\{ \left[ \delta^{ad} \partial_\mu - g_2 \epsilon^{adc} A^c_\mu(x) \right] \delta \theta^d(x) \right\} n^4_\mu, \quad (A6)$$

and moreover,

$$\delta \eta^b = \left[ \partial_\mu \delta \theta^4(x) \right] n^b_\mu, \quad (A7)$$

$$\delta \psi = \left[ \partial_\mu \delta \theta^4(x) \right] n^4_\mu. \quad (A8)$$

When eqs. (A3) and (A6) are rewritten as

$$\delta \Psi^{ab} = i g_2 \left[ \delta \theta^c(x) f_c \right]^{ad} \Psi^{db} + \partial_\mu \delta \theta^a(x) n^b_\mu, \quad (A9)$$

$$\delta \chi^a = i g_2 \left[ \delta \theta^c(x) f_c \right]^{ad} \chi^d + \partial_\mu \delta \theta^a(x) n^4_\mu, \quad (A10)$$

in terms of the adjoint-generator matrices $f_c$ of $f_R$, one recognizes the first terms on the r. h. sides as infinitesimal versions of homogeneous gauge transformations $\Psi \rightarrow \tilde{U}(x) \Psi$, $\chi \rightarrow \tilde{U}(x) \chi$ with matrix $\tilde{U}(x) = \exp [ i g_2 \delta \theta^c(x) f_c ]$. The additional inhomogeneous or “abelian” terms, involving $\partial_\mu \delta \theta^a(x)$, that our scalars inherit from their parent gauge fields are what sets them apart from the usual Higgs scalars. In eqs. (A7) and (A8), only these abelian terms appear.

Gauge variations of $\bar{Q}$ scalars. From the definition of $\bar{Q}$, eq. (7.1), extension of the above to the complex $\bar{Q}$ scalars is simple:

$$\delta_\theta \bar{Q} = \exp(i g_4 h(x) F_4) \left\{ \delta_\theta \bar{Q} + i g_4 d_4(\delta \theta^4(x) \bar{Q}) \right\}. \quad (A11)$$
The results can be collected in a compact form by writing
\[ \delta_\theta \hat{Q}^{AB} = \left[ e^{[i2g_2h(x)]} \partial_\mu \delta \theta^A(x) \right] n^B_\mu + i \left[ g_2 \delta \theta^d(x) F_d + 2g_2 \delta \theta^4(x) \right] A^C \hat{Q}^{CB}. \] (A12)

It is understood that for \( d = 1, 2, 3 \),
\[ (F_d)^{AC} = 0, \quad \text{if } A \text{ or } C = 4. \] (A13)

**B Explicit Matrix Representations**

The matrices \( \tilde{I}_E \) of eq. (7.24), defining the interaction part (second line of eq. (7.23)) of the covariant derivative \( \nabla_\mu \hat{Q} \), are listed below:

\[
\tilde{I}_1 = \begin{pmatrix} 0 & 0 & 0 & -s_g \sigma \\ 0 & 0 & -ic_g & 0 \\ 0 & +ic_g & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{I}_2 = \begin{pmatrix} 0 & 0 & +ic_g & 0 \\ 0 & 0 & 0 & -s_g \sigma \\ -ic_g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad (B1)
\]
\[
\tilde{I}_3 = \begin{pmatrix} 0 & -ic_g & 0 & 0 \\ +ic_g & 0 & 0 & 0 \\ 0 & 0 & 0 & -s_g \sigma \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{I}_4 = s_g \sigma \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (B2)
\]

For the first three of these, the hermitean parts, candidates for a set of four-dimensional isospin-SU(2) generators, are

\[
\tilde{R}_1 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} s_g \sigma \\ 0 & 0 & -ic_g & 0 \\ 0 & +ic_g & 0 & 0 \\ -\frac{1}{2} s_g \sigma & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{R}_2 = \begin{pmatrix} 0 & 0 & +ic_g & 0 \\ 0 & 0 & 0 & -\frac{1}{2} s_g \sigma \\ -ic_g & 0 & 0 & 0 \\ 0 & -\frac{1}{2} s_g \sigma & 0 & 0 \end{pmatrix}; \quad (B3)
\]
\[
\tilde{R}_3 = \begin{pmatrix} 0 & -ic_g & 0 & 0 \\ +ic_g & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} s_g \sigma \\ 0 & 0 & -\frac{1}{2} s_g \sigma & 0 \end{pmatrix} = \begin{pmatrix} c_g \tau_2 & 0 \\ 0 & -\frac{1}{2} s_g \sigma \tau_1 \end{pmatrix}. \quad (B4)
\]

The last term uses the \( 2 \oplus 2 \) block-diagonal form. From these, one may evaluate the commutator

\[
[\tilde{R}_1, \tilde{R}_2] = \begin{pmatrix} 0 & c_g^2 + \left( \frac{1}{2} s_g \sigma \right)^2 & 0 & 0 \\ c_g^2 - \left( \frac{1}{2} s_g \sigma \right)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -ic_g s_g \sigma \\ 0 & 0 & -ic_g s_g \sigma & 0 \end{pmatrix}. \quad (B5)
\]
This has the same pattern of nonzero entries as \( i \tilde{R}_3 \). It is therefore proportional to \( i \tilde{R}_3 \) if the two ratios of corresponding elements coincide:

\[
\frac{c_g + \left( \frac{1}{2} s_g \sigma \right)^2}{c_g} = k; \quad \frac{-i c_g s_g \sigma}{-\frac{1}{2} s_g \sigma} = k; \quad (B6)
\]

with \( k \) a constant. These conditions give \( k = 2 c_g \) and \( s_g \sigma = 2 c_g \), and therefore the rescaling of eqs. (9.2, 9.4) and the condition (9.6) for SU(2) generators. The rescaled, absolutely fixed \( R \) matrices then read,

\[
R_1 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -i & 0 \\
0 & +i & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}; \quad R_2 = \frac{1}{2} \begin{pmatrix}
0 & 0 & +i & 0 \\
0 & 0 & 0 & -1 \\
-i & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}; \quad (B7)
\]

\[
R_3 = \frac{1}{2} \begin{pmatrix}
0 & -i & 0 & 0 \\
+i & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix} \tau_2 & 0 \\
0 & -\tau_1 \end{pmatrix}; \quad (B8)
\]

\[
R_4 = \frac{1}{2} \begin{pmatrix}
0 & -i & 0 & 0 \\
+i & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix} \tau_2 & 0 \\
0 & \tau_1 \end{pmatrix}; \quad (B9)
\]

We next list explicit representations of the \( L_A \) matrices of eqs. (9.17) and (9.31):

\[
L_1 = -\frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}; \quad L_2 = -\frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}; \quad (B10)
\]

\[
L_3 = -\frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & 0 \\
0 & i \tau_2 \end{pmatrix}; \quad (B11)
\]

\[
L_4 = \begin{pmatrix}
1 & i \frac{\tau_1}{2} & 0 & 0 \\
-i \frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 1 & -i \frac{\tau_1}{2} \\
0 & 0 & -i \frac{1}{2} & 0
\end{pmatrix} = \begin{pmatrix} \mathbf{1}_2 - \frac{1}{2} \tau_2 & 0 \\
0 & \frac{1}{2}( \mathbf{1}_2 + \tau_3 - \tau_1) \end{pmatrix}; \quad (B12)
\]

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For the $T_A$ matrices in the charge-diagonal representation, the forms of eqs. (9.23/9.27) are sufficiently explicit. We therefore list explicit forms only for the transforms $S_A$ of the $L_A$'s:

$$S_1 = \frac{1}{4} \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} = -S_1^T; \quad S_2 = \frac{1}{4} \begin{pmatrix} 0 & -i & +i & 0 \\ -i & 0 & 0 & -i \\ i & 0 & 0 & i \\ 0 & -i & +i & 0 \end{pmatrix} = -S_2^T; \quad (B13)$$

$$S_3 = \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 & \tau_1 - i \tau_2 \\ -\tau_1 - i \tau_2 & 0 \end{pmatrix} = -S_3^T; \quad (B14)$$

$$S_4 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1_2 + \tau_3 & \tau_1 - i \tau_2 \\ \tau_1 + i \tau_2 & 5_2 - \tau_3 \end{pmatrix}. \quad (B15)$$

One verifies that $T_4 + S_4$ gives the $U$ transform of $I_4$,

$$U I_4 U^† = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (B16)$$

Finally we write the explicit form of the transformed matrix $\Phi = U Q U^†$ mentioned in section 9:

$$\Phi(x) = U Q(x) U^† = \begin{pmatrix} \Phi_{11} & -\Phi_{12}^* & -\Phi_{13}^* & \Phi_{14}^* \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & -\Phi_{24}^* \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & -\Phi_{34}^* \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44}^* \end{pmatrix}. \quad (B17)$$

Here the sixteen real fields of eq. (2.10) appear recombined into six complex and four real amplitudes:

$$\Phi_{11} = \frac{1}{2} \left[ (\Psi_{11} + \Psi_{22}) + i (\Psi_{12} + \Psi_{21}) \right], \quad (B18)$$

$$\Phi_{12} = \frac{1}{2} \left[ (\Psi_{11} + \Psi_{22}) - i (\Psi_{12} + \Psi_{21}) \right], \quad (B19)$$

$$\Phi_{21} = \frac{1}{2} \left[ - (\Psi_{31} + \eta_1) - i (\Psi_{32} + \eta_2) \right], \quad (B20)$$

$$\Phi_{31} = \frac{1}{2} \left[ - (\Psi_{31} - \eta_1) - i (\Psi_{32} - \eta_2) \right], \quad (B21)$$

$$\Phi_{41} = \frac{1}{2} \left[ (\Psi_{13} + \chi_1) + i (\Psi_{23} + \chi_2) \right], \quad (B22)$$

$$\Phi_{43} = \frac{1}{2} \left[ (\Psi_{13} - \chi_1) + i (\Psi_{23} - \chi_2) \right], \quad (B23)$$


\[ \Phi_{22} = \frac{1}{2} \left[ (\Psi_{33} + \eta_3) + (\chi_3 + \psi) \right], \quad (B24) \]
\[ \Phi_{23} = \frac{1}{2} \left[ (\Psi_{33} + \eta_3) - (\chi_3 + \psi) \right], \quad (B25) \]
\[ \Phi_{32} = \frac{1}{2} \left[ (\Psi_{33} - \eta_3) + (\chi_3 - \psi) \right], \quad (B26) \]
\[ \Phi_{33} = \frac{1}{2} \left[ (\Psi_{33} - \eta_3) - (\chi_3 - \psi) \right]. \quad (B27) \]

The corresponding matrix elements for the Matrix \( \Pi(x) \) of eq. (10.26) are obtained for \( \chi_a = 0 \) and \( \psi = 0 \). One then notes that the two middle columns of this matrix become equal, which checks with our earlier observation that \( \det P = 0 \).
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[9] This follows by straightforward adaptation of the formulas given in H. M. Friedman, *Green’s Functions and Ordered Exponentials*, Cambridge (UK) 2002, Cambridge University Press, pp. 15/16. There are probably earlier references which I have been unable to locate. The end result in eq. (3.30) below has been derived via a different route in [10].

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[13] Version 1 of this article, in its section 9, imposed the standard neutral mixing, parameterized by the electroweak mixing angle $\theta_W$, from the outset, and consequently was forced into a complicated yet unsatisfactory construction for vector-mass formation, which the present version shows to be unnecessary.

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