Variety of cosmic acceleration models from massive $F(R)$ bigravity.

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We study the accelerating cosmology in massive $F(R)$ bigravity via the reconstruction scheme. The consistent solution of the FRW equations is presented: it includes Big and Little Rip, quintessence, de Sitter and decelerating universes described by the physical $g$ metric while the corresponding solution of the universe described by the reference $f$ metric is also found. It is demonstrated that in general the cosmological singularities of $g$ metric are not always manifested as cosmological singularities of the reference $f$ metric. We give two consistent ways to describe the Big and Little Rip, quintessence, de Sitter and decelerating universes. In one of the consistent solutions, the two metrics $g$ and $f$ coincide with each other, which may indicate the connection with the convenient single metric background formulation. For another solution, where two metrics $g$ and $f$ do not coincide with each other, there always appears super-luminal mode.

PACS numbers: 95.36.+x, 98.80.Cq

I. INTRODUCTION

Massive gravity history has started from the work [1] (for recent review, see [2]). It also has been known that massive gravity eventually contains the Boulware-Deser ghost [3] and vDVZ discontinuity [4] in the limit of $m \to 0$. The attempts to screen this discontinuity are based on the Vainstein mechanism [5] as it was shown, for instance, in Ref. [6] on the example of the DGP model [7].

It has been realized recently that non-linear massive gravity [8, 9] (with non-dynamical background metric) may be extended to the ghost-free construction with dynamical metric [10]. The most general proof of absence of ghost in massive gravity has been given in [11]. Especially in case of the minimal model, which we treat in this paper in section III, the qualitative difference with the case of usual $F(R)$ gravity is caused by constraints which are provided by scalar equations and do not coincide with Bianchi identities.

The present work is devoted to the study of accelerating cosmology in massive $F(R)$ bigravity and its relation with observable universe. After brief introduction to massive $F(R)$ bigravity in second section, we develop the consistent reconstruction scheme where extra ghost is not generated in the section III. The qualitative difference with the case of usual $F(R)$ gravity is caused by constraints which are provided by scalar equations and do not coincide with Bianchi identities.

We also study the relation between background cosmologies induced by two metrics. We introduce four kinds of metrics, $g_{\mu \nu}$, $g^J_{\mu \nu}$, $f_{\mu \nu}$, and $f^J_{\mu \nu}$. The physical observable metric $g^J_{\mu \nu}$ is the metric in the Jordan frame. The metric $g_{\mu \nu}$ corresponds to the metric in the Einstein frame and it is known as cosmological singularities of $g$ metric are not always manifested as cosmological singularities of the reference $f$ metric. We give two consistent ways to describe the Big and Little Rip, quintessence, de Sitter and decelerating universes. In one of the consistent solutions, the two metrics $g$ and $f$ coincide with each other, which may indicate the connection with the convenient single metric background formulation. For another solution, where two metrics $g$ and $f$ do not coincide with each other, there always appears super-luminal mode.

Section IV is devoted to the search of accelerating cosmologies. Variety of accelerating cosmologies which include Big Rip, de Sitter, quintessence, and Little Rip universes are constructed for the space-time described by the metric...
$g^J_{\mu\nu}$ in the situation when the Einstein frame metric $g_{\mu\nu}$ is fixed. These cosmologies are in good correspondence with observational data which can be shown in the analogy with Ref. [20]. It is demonstrated that in general, cosmological singularity in physical universe given by the metric $g^J_{\mu\nu}$ is manifested in the universe given by the reference metric $f_{\mu\nu}$ or $f^J_{\mu\nu}$ and vice-versa (the corresponding observation for $R$ massive gravity with matter is presented in Ref. [20]). However, we show that there are models where cosmological singularity does not occur in the universe described by the metric $g^J_{\mu\nu}$ although it occurs in the universe described by the metric $f_{\mu\nu}$ or $f^J_{\mu\nu}$. It is presented the example where the universe given by the metric $g^J_{\mu\nu}$ accelerates while the universe given by the metric $f_{\mu\nu}$ or $f^J_{\mu\nu}$ decelerates. In this sense, the space-time given by the metric $f_{\mu\nu}$ or $f^J_{\mu\nu}$ (i.e., massive graviton effect) plays a role of dark energy in our universe, where the metric is given by $g^J_{\mu\nu}$. We also propose the qualitative scenario of dissolution of the background metric $f^J_{\mu\nu}$ by the physical metric. The explicit examples of Big Rip, quintessence, de Sitter, Little Rip or decelerating universes as solutions of FRW equations in $F(R)$ bigravity when $f$ metric coincides with $g$ metric are found by using reconstruction scheme. General arguments for future singularity occurrence in the universe given by the metric $f^J_{\mu\nu}$ but its avoidance in the physical universe given by $g^J_{\mu\nu}$ are presented. Section V is devoted to formulation of perturbation theory. It turns out that for the study of stability of cosmological solutions under discussion one should investigate the eigenvalues of eight by eight matrix.

Recently, the super-luminal mode in the massive gravity has been discussed (see [31], [32] and references therein). In section VI, we also observed that the massless particle in the space-time given by the metric $f_{\mu\nu}$ or $f^J_{\mu\nu}$ can be super-luminal. Some physical consequences of this effect are briefly mentioned. Finally, some summary and outlook are given in Discussion section.

II. $F(R)$ BIGRAVITY

In this section, we review the construction of ghost-free $F(R)$ bigravity, following Ref. [19] (for recent review of convenient $F(R)$ gravity, see [21], [22]). The consistent model of bimetric gravity, which includes two metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$, was proposed in Ref. [10]. It contains the massless spin-two field, corresponding to graviton, and massive spin-two field. The gravity model which only contains the massive spin-two field is called massive gravity but we consider the model including both of massless and massive spin two field, which is called bigravity. It has been shown that the Boulware-Deser ghost [3] does not appear in such a theory.

The starting action is given by

$$S_{bh} = M^2_g \int d^4x \sqrt{-det g} R^{(g)} + M^2_f \int d^4x \sqrt{-det f} R^{(f)} + 2m^2M^2_{\text{eff}} \int d^4x \sqrt{-det g} \sum_{n=0}^4 \beta_n e_n \left( \sqrt{g^{-1} f} \right). \tag{1}$$

Here $R^{(g)}$ is the scalar curvature for $g_{\mu\nu}$ and $R^{(f)}$ is the scalar curvature for $f_{\mu\nu}$. $M_{\text{eff}}$ is defined by

$$\frac{1}{M^2_{\text{eff}}} = \frac{1}{M^2_g} + \frac{1}{M^2_f}. \tag{2}$$

Furthermore, tensor $\sqrt{g^{-1} f}$ is defined by the square root of $g^{\mu\nu} f_{\rho\nu}$, that is, $\left( \sqrt{g^{-1} f} \right)_{\mu}^\rho \left( \sqrt{g^{-1} f} \right)^\rho_\nu = g^{\mu\nu} f_{\rho\nu}$. For general tensor $X^\mu_{\nu}$, $e_n(X)$'s are defined by

$$e_0(X) = 1, \quad e_1(X) = [X], \quad e_2(X) = \frac{1}{2}([X]^2 - [X^2]), \quad e_3(X) = \frac{1}{6}([X]^3 - 3[X][X^2] + 2[X^3]), \quad e_4(X) = \frac{1}{24}([X]^4 - 6[X]^2[X^2] + 3[X]^2 + 8[X][X^3] - 6[X^4]),$$

$$e_k(X) = 0 \quad \text{for } k > 4. \tag{3}$$

Here $[X]$ expresses the trace of arbitrary tensor $X^\mu_{\nu}$: $[X] = X^\mu_{\mu}$.

In order to construct the consistent $F(R)$ bigravity, we add the following terms to the action (1):

$$S_{\varphi} = -M^2_g \int d^4x \sqrt{-det g} \left\{ \frac{3}{2} g^{\mu\nu} \partial_\mu \varphi \partial_{\nu} \varphi + V(\varphi) \right\} + \int d^4x \mathcal{L}_{\text{matter}} \left( e^{\varphi} g_{\mu\nu}, \Phi_i \right), \tag{4}$$

$$S_{\xi} = -M^2_f \int d^4x \sqrt{-det f} \left\{ \frac{3}{2} f^{\mu\nu} \partial_{(\mu} \xi \partial_{\nu)} \xi + U(\xi) \right\}. \tag{5}$$
By the conformal transformations $g_{\mu\nu} \to e^{-\varphi}g_{\mu\nu}^3$ and $f_{\mu\nu} \to e^{-\xi}f_{\mu\nu}^3$, the total action $S_F = S_{bi} + S_\varphi + S_\xi$ is transformed as

$$
S_F = M_F^2 \int d^4x \sqrt{-\det f} \left\{ e^{-\xi}R^{(f)} - e^{-2\xi}U(\xi) \right\}
+ 2m^2M_{\text{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^4 \beta_n e^{(\frac{n}{2} - 2)\varphi - \frac{n}{2}\xi} e_n \left( \sqrt{g^{(1-f)}} \right)
+ M_g^2 \int d^4x \sqrt{-\det g} \left\{ e^{-\xi}R^{(g)} - e^{-2\xi}V(\varphi) \right\} + \int d^4x L_{\text{matter}} \left( g_{\mu\nu}^3, \Phi_1 \right).
$$

(6)

The kinetic terms for $\varphi$ and $\xi$ vanish. By the variations with respect to $\varphi$ and $\xi$ as in the case of convenient $F(R)$ gravity \cite{22}, we obtain

$$
0 = 2m^2M_{\text{eff}}^2 \sum_{n=0}^4 \beta_n \left( \frac{n}{2} - 2 \right) e^{(\frac{n}{2} - 2)\varphi - \frac{n}{2}\xi} e_n \left( \sqrt{g^{(1-f)}} \right) + M_g^2 \left\{ e^{-\varphi}R^{(g)} + 2e^{-2\xi}V(\varphi) + e^{-2\xi}V'(\varphi) \right\},
$$

(7)

$$
0 = -2m^2M_{\text{eff}}^2 \sum_{n=0}^4 \beta_n \left( \frac{n}{2} - 2 \right) e^{(\frac{n}{2} - 2)\varphi - \frac{n}{2}\xi} e_n \left( \sqrt{g^{(1-f)}} \right) + M_g^2 \left\{ e^{-\xi}R^{(f)} + 2e^{-2\xi}U(\xi) + e^{-2\xi}U'(\xi) \right\}.
$$

(8)

The Eqs. (7) and (8) can be solved algebraically with respect to $\varphi$ and $\xi$ as $\varphi = \varphi \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right)$ and $\xi = \xi \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right)$. Substituting above $\varphi$ and $\xi$ into (6), one gets $F(R)$ bigravity:

$$
S_F = M_F^2 \int d^4x \sqrt{-\det f} F^{(f)} \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right)
+ 2m^2M_{\text{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^4 \beta_n e^{(\frac{n}{2} - 2)\varphi} \left( R^{(g)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) e_n \left( \sqrt{g^{(1-f)}} \right)
+ M_g^2 \int d^4x \sqrt{-\det g} F^{(g)} \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) + \int d^4x L_{\text{matter}} \left( g_{\mu\nu}^3, \Phi_1 \right),
$$

(9)

$$
F^{(g)} \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) \equiv \left\{ e^{-\varphi} \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) R^{(g)}
- e^{-2\varphi} \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) V \left( \varphi \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) \right) \right\},
$$

(10)

$$
F^{(f)} \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) \equiv \left\{ e^{-\xi} \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) R^{(f)}
- e^{-2\xi} \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) U \left( \xi \left( R^{(g)}, R^{(f)}, e_n \left( \sqrt{g^{(1-f)}} \right) \right) \right) \right\}.
$$

Note that it is difficult to solve Eqs. (7) and (8) with respect to $\varphi$ and $\xi$ explicitly. Therefore, it might be easier to define the model in terms of the auxiliary scalars $\varphi$ and $\xi$ as in (6).

### III. COSMOLOGICAL RECONSTRUCTION

Let us consider the cosmological reconstruction program following Ref. \cite{19} but in slightly extended form. For simplicity, we start from the minimal case

$$
S_{bi} = M_g^2 \int d^4x \sqrt{-\det g} R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f} R^{(f)}
+ 2m^2M_{\text{eff}}^2 \int d^4x \sqrt{-\det g} \left( 3 - \text{tr} \sqrt{g^{-1}f} + \text{det} \sqrt{g^{-1}f} \right).
$$

(11)
In order to evaluate $\delta \sqrt{g^{-1}f}$, two matrices $M$ and $N$, which satisfy the relation $M^2 = N$ are taken. Since $\delta MM + M\delta M = \delta N$, one finds

$$\text{tr}\, \delta M = \frac{1}{2} \text{tr}\, (M^{-1}\delta N).$$

(12)

For a while, we consider the Einstein frame action \((\text{11})\) with \((\text{4})\) and \((\text{5})\) but matter contribution is neglected. Then by the variation over $g_{\mu\nu}$, we obtain

$$0 = M_f^2 \left( \frac{1}{2} f_{\mu\nu} R^{(g)} - R_{\mu\nu}^{(g)} \right) + m^2 M_{\text{eff}}^2 \left\{ g_{\mu\nu} \left[ 3 - \sqrt{g^{-1}f} \right] + \frac{1}{2} f_{\mu\nu} \left( \sqrt{g^{-1}f} \right)^{-1} + \frac{1}{2} f_{\mu\rho} \left( \sqrt{g^{-1}f} \right)^{-1} \right\}$$

$$+ M_f^2 \left[ \frac{1}{2} \left( \frac{3}{2} g^{\rho\sigma} \partial_{\rho} \varphi \partial_{\sigma} \varphi + V(\varphi) \right) g_{\mu\nu} - \frac{3}{2} \partial_{\mu} \varphi \partial_{\nu} \varphi \right].$$

(13)

On the other hand, by the variation over $f_{\mu\nu}$, we get

$$0 = M_f^2 \left( \frac{1}{2} f_{\mu\nu} R^{(f)} - R_{\mu\nu}^{(f)} \right)$$

$$+ m^2 M_{\text{eff}}^2 \sqrt{\det (f^{-1}g)} \left\{ -\frac{1}{2} f_{\mu\rho} \left( \sqrt{g^{-1}f} \right)^{\rho}_{\nu} - \frac{1}{2} f_{\nu\rho} \left( \sqrt{g^{-1}f} \right)^{\rho}_{\mu} + \det \left( \sqrt{g^{-1}f} \right) f_{\mu\nu} \right\}$$

$$+ M_f^2 \left[ \frac{1}{2} \left( \frac{3}{2} f^{\rho\sigma} \partial_{\rho} \xi \partial_{\sigma} \xi + U(\xi) \right) f_{\mu\nu} - \frac{3}{2} \partial_{\mu} \xi \partial_{\nu} \xi \right].$$

(14)

We should note that $\sqrt{g} \sqrt{g^{-1}f} \neq \sqrt{f}$ in general. The variations of the scalar fields $\varphi$ and $\xi$ are given by

$$0 = -3 \Box g \varphi + V'(\varphi), \quad 0 = -3 \Box f \xi + U'(\xi).$$

(15)

Here $\Box_g (\Box_f)$ is the d’Alembertian with respect to the metric $g (f)$. By multiplying the covariant derivative $\nabla^g_{\mu}$ with respect to the metric $g$ with Eq. \((\text{13})\) and using the Bianchi identity $0 = \nabla^g_{\mu} \left( \frac{1}{2} g_{\mu\nu} R^{(g)} - R_{\mu\nu}^{(g)} \right)$ and Eq. \((\text{15})\), we obtain

$$0 = -g_{\mu\nu} \nabla^g_{\mu} \left( \sqrt{g^{-1}f} \right) + \frac{1}{2} g^\mu_{\nu} \left\{ f_{\mu\rho} \left( \sqrt{g^{-1}f} \right)^{-1}_{\rho} + f_{\nu\rho} \left( \sqrt{g^{-1}f} \right)^{-1}_{\rho} \right\}.$$ 

(16)

Similarly by using the covariant derivative $\nabla^{f}_{\mu}$ with respect to the metric $f$, from \((\text{14})\), we obtain

$$0 = \nabla^{f}_{\mu} \left[ \sqrt{\det (f^{-1}g)} \left\{ -\frac{1}{2} \left( \sqrt{g^{-1}f} \right)^{-1}_{\nu} g^{\rho\sigma} - \frac{1}{2} \left( \sqrt{g^{-1}f} \right)^{-1}_{\mu} g^{\rho\sigma} + \det \left( \sqrt{g^{-1}f} \right) f_{\mu\nu} \right\} \right].$$

(17)

In case of the Einstein gravity, the conservation law of the energy-momentum tensor depends from the Einstein equation. It can be derived from the Bianchi identity. In case of bigravity, however, the conservation laws of the energy-momentum tensor of the scalar fields are derived from the scalar field equations. These conservation laws are independent of the Einstein equation. The Bianchi identities give equations \((\text{16})\) and \((\text{17})\) independent of the Einstein equation.

We now assume the FRW universes for the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ and use the conformal time $t$ for the universe with metric $g_{\mu\nu}$.

$$ds_g^2 = \sum_{\mu, \nu = 0}^3 g_{\mu\nu} dx^\mu dx^\nu = a(t)^2 \left( -dt^2 + \sum_{i=1}^3 (dx^i)^2 \right), \quad ds_f^2 = \sum_{\mu, \nu = 0}^3 f_{\mu\nu} dx^\mu dx^\nu = -c(t)^2 dt^2 + b(t)^2 \sum_{i=1}^3 (dx^i)^2.$$ 

(18)

Then \((t, t)\) component of \((\text{16})\) gives

$$0 = -3 M_g^2 H^2 - 3 M^2 M_{\text{eff}}^2 \left( a^2 - ab \right) + \left( \frac{3}{4} \varphi^2 + \frac{1}{2} V(\varphi) a(t)^2 \right) M_g^2.$$ 

(19)

1 In Ref. \((\text{14})\), we have used the cosmological time instead of the conformal time. The use of the conformal time simplifies the formulation.
and \((i, j)\) components give

\[
0 = M_g^2 \left( 2\dot{H} + H^2 \right) + m^2 M_{\text{eff}}^2 \left( 3a^2 - 2ab - ac \right) + \left( \frac{3}{4} \varphi^2 - \frac{1}{2} V(\varphi) a(t)^2 \right) M_g^2. \tag{20}
\]

Here \(H = \dot{a}/a\). On the other hand, \((t, t)\) component of \(\ref{eq:14}\) gives

\[
0 = -3M_f^2 K^2 + m^2 M_{\text{eff}}^2 c^2 \left( 1 - \frac{a^3}{b^2} \right) + \left( \frac{3}{4} \xi^2 - \frac{1}{2} U(\xi) c(t)^2 \right) M_f^2, \tag{21}
\]

and \((i, j)\) components give

\[
0 = M_f^2 \left( 2\dot{K} + 3K^2 - 2LK \right) + m^2 M_{\text{eff}}^2 \left( \frac{a^3 c}{b^2} - c^2 \right) + \left( \frac{3}{4} \xi^2 - \frac{1}{2} U(\xi) c(t)^2 \right) M_f^2. \tag{22}
\]

Here \(K = \dot{b}/b\) and \(L = \dot{c}/c\). Both of Eq. \(\ref{eq:16}\) and Eq. \(\ref{eq:17}\) give the identical equation:

\[
cH = bK \text{ or } \frac{c\dot{a}}{a} = \dot{b}. \tag{23}
\]

If \(\dot{a} \neq 0\), we obtain \(c = \dot{a}/a\). On the other hand, if \(\dot{a} = 0\), we find \(\dot{b} = 0\), that is, \(a\) and \(b\) are constant and \(c\) can be arbitrary. We should note that the scheme of the reconstruction in Ref. \(\ref{ref:19}\), where \(a(t) = c(t) = 1\) is not, unfortunately, consistent with Eq. \(\ref{eq:23}\) in general.

We now redefine scalars as \(\varphi = \varphi(\eta)\) and \(\xi = \xi(\zeta)\) and identify \(\eta\) and \(\zeta\) with the conformal time \(t\), \(\eta = \zeta = t\). Hence, one gets

\[
\omega(t)M_g^2 = -4M_g^2 \left( \dot{H} - H^2 \right) - 2m^2 M_{\text{eff}}^2 (ab - ac), \tag{24}
\]

\[
\dot{V}(t)a(t)^2 M_g^2 = M_g^2 \left( 2\dot{H} + 4H^2 \right) + m^2 M_{\text{eff}}^2 (6a^2 - 5ab - ac), \tag{25}
\]

\[
\sigma(t)M_f^2 = -4M_f^2 \left( \dot{K} - LK \right) - 2m^2 M_{\text{eff}}^2 \left( -\frac{c}{b} + 1 \right) \frac{a^3c}{b^2}, \tag{26}
\]

\[
\ddot{U}(t)c(t)^2 M_f^2 = M_f^2 \left( 2\dot{K} + 6K^2 - 2LK \right) + m^2 M_{\text{eff}}^2 \left( \frac{a^3c}{b^2} - 2c^2 + \frac{a^3c^2}{b^3} \right). \tag{27}
\]

Here

\[
\omega(\eta) = 3\varphi'(\eta)^2, \quad \dot{V}(\eta) = V(\varphi(\eta)), \quad \sigma(\zeta) = 3\xi'(\zeta)^2, \quad \ddot{U}(\zeta) = U(\xi(\zeta)). \tag{28}
\]

Therefore for arbitrary \(a(t), b(t),\) and \(c(t)\) if we choose \(\omega(t), \dot{V}(t), \sigma(t),\) and \(\ddot{U}(t)\) to satisfy Eqs. \(\ref{eq:24} \text{--} \ref{eq:27}\), the cosmological model with given \(a(t), b(t)\) and \(c(t)\) evolution can be reconstructed.

IV. ACCELERATING COSMOLOGICAL MODELS

Let us construct some examples of cosmological models which describe Big Rip universe \(\text{\cite{24, 25}}\), quintessence universe, de Sitter universe, decelerating universe, and Little Rip universe \(\text{\cite{26, 28}}\). The physical metric, where the scalar does not directly couple with matter, is given by multiplying the scalar field to the metric in the Einstein frame in \(\text{\cite{4}}\) or \(\text{\cite{11}}\):

\[
g^J_{\mu\nu} = e^\varphi g_{\mu\nu}. \tag{29}
\]

In the bigravity model, there appears another (unphysical) metric tensor \(f_{\mu\nu}\), besides \(g_{\mu\nu}\). In our model, since the matter only couples with \(g_{\mu\nu}\), the physical metric could be given by \(g^J_{\mu\nu}\) in \(\text{\cite{29}}\). In principle, however, there could be a matter coupled with \(f_{\mu\nu}\). Then one may consider two space-times, one is described by \(g_{\mu\nu}\) (or \(g^J_{\mu\nu}\)) and another by \(f_{\mu\nu}\). Although the matter in the space-time described by \(f_{\mu\nu}\) could not directly couple with the matter in the space-time described by \(g_{\mu\nu}\), the matters can interact with each other via the propagation of massless and massive gravitons. Furthermore, as clear from Eqs. \(\text{\cite{13}}\) and \(\text{\cite{14}}\), the metric \(f_{\mu\nu}\) affects the geometry of the space-time described by \(g_{\mu\nu}\), and vice-versa.
In this section, we show that in general, there occurs future singularity in the space-time described by \( g_{\mu\nu} \) when the singularity occurs in the space-time described by \( f_{\mu\nu} \). We also find some examples where the future singularity does not appear in universe described by \( g_{\mu\nu} \) (more exactly by \( g^J_{\mu\nu} \)) even if future singularity appears in universe described by \( f_{\mu\nu} \). This might be a counterexample for the observation of Ref. \[29\] where \( R \)-bigravity with matter was considered. Although the physical metric \( g^J_{\mu\nu} \) in \( g \) universe is given by (29), it is not so clear what could be a physical metric in \( f \) universe since it depends on the coupling with matter. Anyway we may consider both of \( f_{\mu\nu} \) in the Einstein frame and the metric

\[
f^J_{\mu\nu} = e^\xi f_{\mu\nu},
\]

in the Jordan frame.

**A. Conformal description of accelerating universe**

In our formulation, it is convenient to use the conformal time description. Hence, let us describe how the known cosmologies can be expressed by using the conformal time. Especially we present the explicit expressions of the de Sitter, phantom, quintessence, decelerating, and also Little Rip universes.

The conformally flat FRW universe metric is given by

\[
ds^2 = \tilde{a}(t)^2 \left(-dt^2 + \sum_{i=1}^{3} (dx^i)^2\right). \tag{31}
\]

Eq. (29) with (31) shows

\[
e^{\varphi(t)} a(t)^2 = \tilde{a}(t)^2, \tag{32}
\]

that is,

\[
\varphi = -2 \ln a(t) + 2 \ln \tilde{a}(t). \tag{33}
\]

Using (28), we find

\[
\omega(t) = 12 \left(H - \tilde{H}\right)^2. \tag{34}
\]

Here \( \tilde{H} \equiv \frac{1}{a} \frac{da}{dt} \).

In Eq. (31), when \( \tilde{a}(t)^2 = \frac{t^2}{e^2} \), the metric (31) corresponds to the de Sitter universe which may describe inflation or dark energy in the model under consideration. On the other hand if \( \tilde{a}(t)^2 = \frac{t^2}{e^n} \) with \( n \neq 1 \), by redefining the time coordinate as

\[
d\tilde{t} = \pm \frac{t^n}{t^{n-1}}dt, \tag{35}
\]

that is,

\[
\tilde{t} = \pm \frac{t^n}{n-1} t^{1-n}, \tag{36}
\]

the metric (31) can be rewritten as

\[
ds^2 = -d\tilde{t}^2 + \left(\pm(n-1)\frac{\tilde{t}}{l}\right)^{-\frac{2}{1-n}} \sum_{i=1}^{3} (dx^i)^2. \tag{37}
\]

Eq. (37) shows that if \( 0 < n < 1 \), the metric corresponds to the phantom universe, if \( n > 1 \) to the quintessence universe, and if \( n < 0 \) to decelerating universe. In case of the phantom universe \( 0 < n < 1 \), one should choose + sign in \( \pm \) of (35) or (36) and shift \( \tilde{t} \) in (37) as \( \tilde{t} \rightarrow \tilde{t} - t_0 \). Then \( \tilde{t} = t_0 \) corresponds to the Big Rip and the present time is \( \tilde{t} < t_0 \) and \( t \rightarrow \infty \) corresponds to the infinite past (\( \tilde{t} \rightarrow -\infty \)). In case of the quintessence universe \( n > 1 \), we may again choose + sign in \( \pm \) of (35) or (36). Then \( t \rightarrow 0 \) corresponds to \( t \rightarrow +\infty \) and \( t \rightarrow +\infty \) to \( \tilde{t} \rightarrow 0 \), which may correspond to the Big Bang. In case of the decelerating universe \( n < 0 \), we may choose − sign in \( \pm \) of (35) or (36).
Then $t \to 0$ corresponds to $\tilde{t} \to +\infty$ and $t \to +\infty$ to $\tilde{t} \to 0$, which may correspond to the Big Bang, again. We should also note that in case of the de Sitter universe ($n = 1$), $t \to 0$ corresponds to $\tilde{t} \to +\infty$ and $t \to \pm \infty$ to $\tilde{t} \to -\infty$.

One may also consider the Little Rip universe, where there is no future singularity but the Hubble rate in terms of the cosmological time $\tilde{t}$ becomes infinite when $\tilde{t}$ goes to infinity. As the universe expands, the relative acceleration between two points separated by a comoving distance $l$ is given by $l \left( \frac{1}{\tilde{a}} \right) \left( \frac{d^2\tilde{a}}{dt^2} \right)$, where $a$ is the scale factor. An observer at comoving distance $l$ away from a mass $m$ will measure an inertial force on the mass of

$$F_{\text{iner}} = \frac{ml}{\tilde{a}} \frac{d^2\tilde{a}}{dt^2}.$$  \hspace{1cm} (38)

Let us assume the two particles are bound by a constant force $F_0$. If $F_{\text{iner}}$ is positive and greater than $F_0$, the two particles become unbound. This is the “rip” produced by the accelerating expansion. It leads to finite-time disintegration of bound objects much before the singularity.

An example is given by

$$\frac{1}{\tilde{a}} \frac{d\tilde{a}}{dt} = g_0 \lambda e^{\lambda \tilde{t}} - \lambda.$$  \hspace{1cm} (39)

The last term $-\lambda$ is added for the convenience in the explicit calculation but this term can be neglected for large $\tilde{t}$ compared with the first term. By the results of the Supernova Cosmology Project \cite{30}, the parameter $\lambda$ in (39) is bounded as

$$2.37 \times 10^{-3} \text{Gyr}^{-1} < \lambda < 8.37 \times 10^{-3} \text{Gyr}^{-1}.$$  \hspace{1cm} (40)

Eq. (39) gives

$$t = \frac{e^{-g_0e^{\lambda \tilde{t}}}}{\lambda g_0}, \quad \tilde{a} = e^{g_0e^{\lambda \tilde{t}} - \lambda \tilde{t}} = -\frac{1}{\lambda t \ln(\lambda g_0 t)}.$$  \hspace{1cm} (41)

By using (38), we find the inertial force is given by,

$$F_{\text{iner}} = m l \left\{ g_0 \lambda^2 e^{\lambda \tilde{t}} + \left( g_0 \lambda e^{\lambda \tilde{t}} - \lambda \right)^2 \right\},$$  \hspace{1cm} (42)

which goes to infinity at infinite $\tilde{t}$.

Combining (39) and (41), one finds

$$\ddot{H} = \frac{1}{\tilde{a}} \frac{d\tilde{a}}{dt} = -\frac{1}{\lambda t} \left( 1 + \frac{1}{\ln(\lambda g_0 t)} \right).$$  \hspace{1cm} (43)

Thus we get an explicit example of $\ddot{a}$ and $H$ corresponding to the realistic Little Rip universe.

If the space-time described by the metric $g^1_{\mu\nu}$ describes the universe where we live, the functions $c(t)$ and $b(t)$ are not directly related with the expansion of our universe since the functions $c(t)$ and $b(t)$ correspond to the degrees of freedom in the Einstein frame metric $f_{\mu\nu}$.

Therefore one may choose $c(t)$ and $b(t)$ in the consistent way convenient for the calculation. This does not mean $c(t)$ and $b(t)$ are not relevant for the physics besides the expansion of our universe since the matter in the universe given by the metric $f_{\mu\nu}$ or $f^1_{\mu\nu}$, if any, weakly interacts with the matter in our universe via the massless and massive gravitons. In the following, we consider two choices of $c(t)$ and $b(t)$, that is, $a(t) = c(t) = 1$ case and $a(t) = c(t) = b(t)$ case. Note that the last case probably helps to simplify the formal description of the theory: indeed reference metric seems to be dissolved by physical one in such situation.

**B. Dark energy universe with $a(t) = b(t) = 1$**

In this section, making the choice $a(t) = b(t) = 1$, we explicitly construct Big Rip (phantom), quintessence, de Sitter, decelerating or Little Rip universes. We should note that the choice $a(t) = b(t) = 1$ satisfies the constraint \cite{23}.

When $a(t) = b(t) = 1$, the Einstein frame metric $g_{\mu\nu}$ expresses the flat Minkowski space although the metric we observe is given by $g^1_{\mu\nu}$. Eqs. (23), (24), (25), and (26) with (41) are simplified as follows,

$$\omega(t)^2 M_g^2 = 12 M_g^2 \ddot{H}^2 = m^2 M_{\text{eff}}^2 \left( e - 1 \right).$$  \hspace{1cm} (44)
\[ \tilde{V}(t)M_g^2 = m^2 M_{\text{eff}}^2 (1 - c) = -6M_g^2 \tilde{H}^2, \quad (45) \]
\[ \sigma(t)M_g^2 = 2m^2 M_{\text{eff}}^2 (c - 1) = 12M_g^2 \tilde{H}^2, \quad (46) \]
\[ \tilde{U}(t)M_f^2 = m^2 M_{\text{eff}}^2 (1 - c) = -6M_g^2 \tilde{H}^2 \left(1 + \frac{6\tilde{H}^2}{m^2 M_{\text{eff}}^2} \right). \quad (47) \]

Eq. (44) can be solved with respect to \( c \) as
\[ c = 1 + \frac{6\tilde{H}^2}{m^2 M_{\text{eff}}^2}. \quad (48) \]

We should note that both of \( \omega(t) \) and \( \sigma(t) \) are positive, there does not appear ghost in the theory.

1. **Construction of the models describing Big Rip, quintessence, de Sitter and decelerating universes**

As shown in Subsection IV A, Big Rip, quintessence, de Sitter and decelerating universes are described by the scale factor \( \tilde{a}(t)^2 = \frac{l^2}{n^2 t^2} \) in terms of the conformal time \( t \). Let us construct the models with the scale factor \( \tilde{a}(t)^2 = \frac{l^2}{n^2 t^2} \), that is \( \tilde{H} = \frac{n}{t} \). Studying the properties of such models, we show that there does not appear future singularity in the space-time described by \( f_{\mu\nu} \) although a future singularity appears in the space-time described by \( g_{\mu\nu} \).

By using (44), (45), (46), and (47), we find
\[ \omega(t)^2 M_g^2 = \frac{12n^2 M_g^2}{t^2}, \quad \tilde{V}(t)M_g^2 = -\frac{6n^2 M_g^2}{t^2}, \quad \sigma(t)M_f^2 = \frac{12n^2 M_g^2}{t^2}, \quad \tilde{U}(t)M_f^2 = -\frac{6n^2 M_g^2}{t^2} \left(1 + \frac{6n^2}{m^2 M_{\text{eff}}^2 t^2} \right). \quad (49) \]

Eq. (49) and \( \sigma(t) \) in (28) indicate
\[ e^\xi = \frac{n^2}{t^2}. \quad (50) \]

Then by using (48), ‘physical’ metric \( f_{\mu\nu} \) in (30) is given by
\[ \left(ds_f^2\right)^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu} dx^\mu dx^\nu = e^\xi ds_f^2 = \frac{n^2}{t^2} \left( - \left(1 + \frac{6n^2}{m^2 M_{\text{eff}}^2 t^2} \right)^2 dt^2 + (dx^i)^2 \right). \quad (51) \]

When \( t \sim 0 \), by defining
\[ \tilde{t} \sim \frac{\alpha}{2t^2}, \quad \alpha \equiv \frac{6n^3}{m^2 M_{\text{eff}}^2 t^2}, \quad (52) \]
we find the metric (53)
\[ \left(ds_f^2\right)^2 \sim -d\tilde{t}^2 + \frac{2n^2 \tilde{t}}{\alpha} (dx^i)^2. \quad (53) \]

Because Eq. (52) shows that \( t \to 0 \) corresponds to \( \tilde{t} \to +\infty \), there does not occur singularity in the metric \( \left(ds_f^2\right)^2 \) because the scale factor \( \tilde{a} \) which is proportional to \( \tilde{t} \) corresponds to the universe filled with radiation.

In summary, we presented the model where there does not occur cosmological singularity in the universe described by \( f_{\mu\nu} \) but there occurs finite-time future singularity in the universe described by \( g_{\mu\nu} \).

2. **Little Rip universe**

Let us discuss Little Rip universe which is realistic description of current universe. It may be consistent with observational bounds for LCDM as it was demonstrated earlier. In this section we show that the Little Rip cosmology
in the space-time described by the metric $g_{\mu\nu}^J$ corresponds to the radiation dominated universe in the space-time described by the metric $f_{\mu\nu}^J$. By substituting \( \text{[41]} \) and \( \text{[43]} \) into \( \text{[44]} \), \( \text{[45]} \), \( \text{[46]} \), and \( \text{[47]} \), we find

\[
\begin{align*}
\omega(t)^2 M_g^2 &= \frac{12 M_g^2}{t^2} \left( 1 + \frac{1}{\ln (\lambda g_0 t)} \right)^2, \\
\sigma(t) M_f^2 &= \frac{12 M_g^2}{t^2} \left( 1 + \frac{1}{\ln (\lambda g_0 t)} \right)^2, \\
\tilde{V}(t) M_g^2 &= -\frac{6 M_g^2}{t^2} \left( 1 + \frac{1}{\ln (\lambda g_0 t)} \right)^2, \\
\tilde{U}(t) M_f^2 &= -\frac{6 M_g^2}{t^2} \left( 1 + \frac{1}{\ln (\lambda g_0 t)} \right)^2 \left\{ 1 + \frac{6}{m^2 M_{\text{eff}}^2} \left( 1 + \frac{1}{\ln (\lambda g_0 t)} \right)^2 \right\}.
\end{align*}
\]

Then the metric $f_{\mu\nu}^J$ in \( \text{[50]} \) is given by

\[
(ds_f^I)^2 = \frac{1}{(\lambda t \ln (\lambda g_0 t))^2} \left[ -\left\{ 1 + \frac{6}{m^2 M_{\text{eff}}^2} \left( 1 + \frac{1}{\ln (\lambda g_0 t)} \right)^2 \right\} dt^2 + (dx^i)^2 \right].
\]

When $t$ is small, the metric behaves as,

\[
(ds_f^I)^2 \sim \frac{1}{(\lambda t \ln (\lambda g_0 t))^2} \left[-\left( \frac{6}{m^2 M_{\text{eff}}^2} t^2 \right)^2 dt^2 + (dx^i)^2 \right].
\]

We now define a new variable $\tilde{t}$ by

\[
\tilde{t} \equiv \tilde{\alpha} \int \frac{dt}{t^3 \ln (\lambda g_0 t)} = -\frac{2\tilde{\alpha}}{t^2 \ln \left( \frac{\lambda g_0 t}{\lambda g_0} \right)} \sum_{n=0}^{\infty} \frac{n!}{n \ln \left( \lambda g_0 t \right)^n}, \quad \tilde{\alpha} \equiv \frac{6}{\lambda m^2 M_{\text{eff}}^2}.
\]

Then when $t \to 0+$, we have $\tilde{t} \to +\infty$ and

\[
(\lambda g_0 t)^2 \sim \frac{2\tilde{\alpha}}{\tilde{t} \ln \left( \frac{\lambda g_0 t}{\lambda g_0} \right)}.
\]

Then the metric in \( \text{[56]} \) behaves as

\[
(ds_f^I)^2 \sim -d\tilde{t}^2 + \frac{g_{\mu\nu}^J}{\alpha} (dx^i)^2.
\]

The asymptotic behavior of such the universe is identical with the universe filled by radiation.

**C. Dark energy universe with $a(t) = b(t) = c(t)$**

As we observed above, general bigravity formally describes two gravities which are related via some coupling term. Generally speaking, this is rather formal presentation as second universe described by reference metric is kind of effective description of exotic matter (massive graviton). Nevertheless, it may be useful to clarify the role of reference metric in the better way. To do this, we propose that in the course of the evolution the second universe metric may become equal to the physical universe metric (of course, perturbation theories are different). In other words, $f$ metric is dissolved by $g$ metric. After that the future background evolution is conveniently described by the single metric object.

Let us choose $a(t) = c(t) = b(t)$, which satisfy the condition \( \text{[28]} \), and therefore $H = K = L$. From \( \text{[24]} \) and \( \text{[26]} \), we find $\omega(t) = \sigma(t)$ and therefore $\varphi(t) = \xi(t)$, and also $V(t) = U(t)$ from \( \text{[25]} \) and \( \text{[24]} \), which tells, not only $g_{\mu\nu} = f_{\mu\nu}$, but $g_{\mu\nu}^J = f_{\mu\nu}^J$ from \( \text{[29]} \) and \( \text{[30]} \). Hence if there is any singularity in the space-time described by $f_{\mu\nu}$ or $f_{\mu\nu}^J$, there appears an identical singularity in the universe described by $g_{\mu\nu}$ or $g_{\mu\nu}^J$. Note that, for the choice $a(t) = c(t) = b(t)$ in this Subsection, there does appear the ghost as it will be shown below.

By choosing $a(t) = c(t) = b(t)$, Eqs. \( \text{[24]} \), \( \text{[25]} \), \( \text{[26]} \), and \( \text{[27]} \) are simplified as

\[
3 \left( H - \dot{H} \right)^2 = -\dot{H} + H^2,
\]

\[
\dot{\rho} = \frac{3}{8} \rho H (H - \dot{H})^2,
\]

\[
\rho = \frac{3}{8} \dot{H} (H - \dot{H})^2.
\]
\[ \tilde{V}(t)a(t)^2 = \left( 2\dot{H} + 4H^2 \right), \quad (61) \]
\[ \sigma(t) = 4 \left( -\dot{H} + H^2 \right), \quad (62) \]
\[ \tilde{U}(t)a(t)^2 = \left( 2\dot{H} + 4H^2 \right). \quad (63) \]

Here, (34) is used. Comparing (60) with (62), we find
\[ \sigma(t) = \omega(t) = 4 \left( \dot{H} - \tilde{\dot{H}} \right)^2 \quad (64) \]

Since \( \sigma(t) \) is positive, there does not appear ghost. In order to determine \( a \), however, one should solve the differential equation (60). In case it has no solution, we cannot reconstruct such a model.

1. Construction of Big Rip, quintessence, de Sitter and decelerating universes

Let us consider the construction of the models, which describe Big Rip, quintessence, de Sitter and decelerating universes, where the scale factor is given by \( \tilde{a}(t)^2 = \frac{l^2}{t^2} \).

In case \( \tilde{a}(t)^2 = \frac{l^2}{t^2} \), the solution of (60) is given by
\[ H = \frac{h_0}{t}, \quad h_0 = -2 + 3n \pm \sqrt{(2 - 3n)^2 + n^2}. \quad (65) \]
We should note that \( h_0 \) is always real. From Eq. (65) it follows
\[ a = a_0 t^{h_0}. \quad (66) \]
Here \( a_0 \) is a constant. Then from Eqs. (60), (61), (62), and (63), we find
\[ \omega(\eta) = \frac{12 (h_0 + n)^2}{\eta^2}, \quad \sigma(\zeta) = \frac{12 (h_0 + n)^2}{\zeta^2}, \quad \tilde{V}(\eta) = \frac{4h_0^2}{a_0^2 \eta^2 - 2h_0}, \quad \tilde{U}(\zeta) = \frac{4h_0^2}{a_0^2 \zeta^2 + 2h_0}. \quad (67) \]
Thus, we have constructed the models, which describe Big Rip, quintessence, de Sitter and decelerating universes. Moreover, \( f \) metric seems to be dissolved by \( g \) metric for such the background evolution. Note that we can present the dynamical solution with similar properties. Starting from \( f \) universe where \( b(t) = c(t) \) one can reconstruct the evolution where these parameters just approach to \( a(t) \) so that they happen to coincide from some specific moment (for instance, pre-inflationary stage). Such a moment will correspond to dynamical background dissolution of \( f \) metric (the perturbation theories are still different). We will not present here the corresponding results because they are quite complicated.

2. Little Rip universe

The next example is Little Rip universe unifying again the description of both of cosmologies given by \( g_{\mu\nu} \) and \( f_{\mu\nu} \). When \( t \sim 0 \), Eq. (43) indicates \( \dot{H} \sim -\frac{1}{t^2} \). Then the solution of Eq. (60) is given by
\[ H = -\frac{1 \pm \sqrt{7}}{t} + \mathcal{O} \left( \frac{1}{t (\ln (\lambda g_0 t))} \right). \quad (68) \]
Then, one gets
\[ \omega(t) = \sigma(t) \sim \frac{12 (2 \pm \sqrt{2})^2}{t^2}, \quad \tilde{V} \sim \tilde{U} \sim \frac{10 \pm 4 \sqrt{2}}{t^2}. \quad (69) \]
Thus, the background evolution of both of the universes described by \( g_{\mu\nu} \) and \( f_{\mu\nu} \) happens to coincide as Little Rip cosmology.
D. General arguments for future singularity occurrence

In Subsection [V.B.1] we consider the case that the finite future or past singularity at \( t = 0 \) in the space-time described by the metric \( g^J_{\mu\nu} \) is mapped into the infinite future or past in the space and therefore the singularity is removed in the space-time described by the metric \( f^J_{\mu\nu} \). This requires \( \int dt e^{\frac{\phi}{2m}} c(t) \) to diverge.

In this subsection, we may consider the cases that there occurs singularity in the finite future or past universe described by \( f^J_{\mu\nu} \) but there does not occur finite future or past cosmological singularity in the universe described by \( g^J_{\mu\nu} \).

- Even if \( b(t) \) and \( c(t) \) in the metric \( J_{\mu
u} \) are not singular, Eq. (23) tells \( a(t) \) is not singular. Then from (24) and (28), we find that \( \varphi(t) \) and therefore \( g_{\mu\nu} \) and \( g^J_{\mu\nu} \) is not singular.

- Even if \( b(t) \) and \( c(t) \) have a singularity, if \( b(t) - c(t) \) is non-singular, Eq. (23) shows \( a(t) \) is not singular. Then from (24) and (28), we find that \( \varphi(t) \) and therefore \( g_{\mu\nu} \) and \( g^J_{\mu\nu} \) is not singular although (25) indicates that \( V(t) \) is singular.

- Let assume \( b(t) - c(t) \) has a singularity at \( t = 0 \). Then (23), (24) and (28) demonstrate that \( a(t) \) and/or \( \varphi(t) \) are singular at \( t = 0 \). If \( \hat{a}(t) \) (22) behaves as

\[
\hat{a}(t) \sim \frac{1}{t}, \tag{70}
\]

the metric \( g^J_{\mu\nu} \) describes (asymptotic) de Sitter space and therefore \( g \) universe is not singular. Using (24), (70), and Eq. (24) can be written as

\[
12 \left( \frac{1}{t^2} + \frac{2H(t)}{t} + H(t)^2 \right) = -4 \left( \dot{H} - H^2 \right) - \frac{2m^2M_g^2}{M_g^2} \frac{b(t) - c(t)}{a(t)}. \tag{71}
\]

Then one can consider the following cases:

- When \( a(t) \) and therefore \( H(t) \) are regular at \( t = 0 \), if

\[
b(t) - c(t) \sim - \frac{6M_g^2}{m^2M_{\text{eff}}^2a(0)} \left( \frac{1}{t^2} + \frac{2H(0)}{t} \right), \tag{72}
\]

the universe described by \( g^J_{\mu\nu} \) is not singular even if the universe described by \( f^J_{\mu\nu} \) could be singular.

- Let assume that when \( t \to 0 \), \( a(t) \) behaves as \( a(t) \sim a_0t^{h_0} \) with constant \( a_0 \) and \( h_0 \). Then Eq. (71) demonstrates if \( b(t) - c(t) \) behaves as

\[
b(t) - c(t) \sim \frac{2 \left( 3 + 5h_0 + 2h_0^2 \right) M_g^2}{m^2M_{\text{eff}}^2a_0^2t^2 - h_0}, \tag{73}
\]

the space-time described by \( g^J_{\mu\nu} \) is not singular.

It looks that all possible cases where the space-time described by \( f_{\mu\nu} \) or \( f^J_{\mu\nu} \) has a singularity but there does not occur any cosmological singularity in \( g \) universe are presented in this subsection.

V. STABILITY OF BACKGROUND SOLUTIONS

In the previous sections, we have obtained several solutions describing the FRW universe. We now discuss the (in)stability of the obtained background solution by considering the fluctuation from the background like \( a \to a + \delta a, b \to b + \delta b, c \to c + \delta c, \varphi \to \varphi + \delta \varphi, \xi \to \xi + \delta \xi \).

By substituting Eq. (19) into Eq. (20) and using the constraint (23), we find

\[
\ddot{a} - \frac{\dot{a}^2}{a} + \frac{m^2M_{\text{eff}}^2}{2M_g^2} a^2 \left( b - \frac{\dot{a}h}{\dot{a}} \right) + \frac{1}{6} \left( 4\varphi^2a - Va^3 \right) = 0. \tag{74}
\]
By considering the perturbation from the background, we obtain the linear perturbation equation from Eq. (74):

\[
\delta \ddot{a} + A_1 \delta \dot{a} + A_2 \delta a + A_3 \delta \dot{b} + A_4 \delta b + A_5 \delta \dot{\varphi} + A_6 \delta \varphi = 0 ,
\]

(75)

where

\[
A_1 = \frac{3bm^2M_{\text{eff}}^2}{2a^2M_g^2} - \frac{2\dot{a}}{a} , \quad A_2 = \frac{a^2V}{2} + \frac{2\dot{\varphi}^2}{3} - \frac{3a^2\dot{b}m^2M_{\text{eff}}^2}{2aM_g^2} + \frac{abm^2M_{\text{eff}}^2}{M_g^2} + \frac{\dot{a}^2}{a^2} , \\
A_3 = -\frac{a^3m^2M_{\text{eff}}^2}{2aM_g^2} , \quad A_4 = \frac{a^2m^2M_{\text{eff}}^2}{2M_g^2} , \quad A_5 = \frac{4a\dot{\varphi}}{3} , \quad A_6 = -\frac{a^2V'}{6}.
\]

(76)

On the other hand, using (21), (22), and (23), we obtain

\[
\dot{b}^2 \left[ -3 \frac{M_g^2}{b^2} + m^2 M_{\text{eff}}^2 \left( \frac{a^2}{b^2} \right) \left( 1 - \frac{a^3}{b^3} \right) + \frac{1}{2} U \frac{a^2}{b^2} M_f^2 \right] + \frac{3}{4} \dot{\xi}^2 M_f^2 = 0 .
\]

(77)

Then the linear perturbation equation is given by

\[
\delta \ddot{b} + B_1 \delta \dot{b} + B_2 \delta a + B_3 \delta \dot{a} + B_4 \delta \dot{\xi} + B_5 \delta \xi = 0 .
\]

(78)

where

\[
B_1 = \frac{3M_f^2 \left( \frac{6M_f^2}{a^2} + \frac{3a^4m^2M_{\text{eff}}^2}{a^2b^2} \right) \dot{\xi}^2}{8b \left[ \frac{a^2M_f^2U}{2a^2} - \frac{3M_f^2}{a^2} + \frac{a^2 \left( 1 - \frac{a^3}{b^3} \right) m^2M_{\text{eff}}^2}{a^2b^2} \right]^2} , \quad B_2 = \frac{3M_f^2 \dot{\xi}^2 \left( -\frac{a^2M_f^2U}{a^2} + \frac{3a^2 \left( 1 - \frac{a^3}{b^3} \right) m^2M_{\text{eff}}^2}{a^2b^2} \right)}{8b \left[ \frac{a^2M_f^2U}{2a^2} - \frac{3M_f^2}{a^2} + \frac{a^2 \left( 1 - \frac{a^3}{b^3} \right) m^2M_{\text{eff}}^2}{a^2b^2} \right]^2} , \\
B_3 = \frac{3M_f^2 \dot{\xi}^2 \left[ \frac{a^2M_f^2U}{a^2} + \frac{2a \left( 1 - \frac{a^3}{b^3} \right) m^2M_{\text{eff}}^2}{a^2b^2} - \frac{3a^4m^2M_{\text{eff}}^2}{a^2b^2} \right]}{8b \left[ \frac{a^2M_f^2U}{2a^2} - \frac{3M_f^2}{a^2} + \frac{a^2 \left( 1 - \frac{a^3}{b^3} \right) m^2M_{\text{eff}}^2}{a^2b^2} \right]^2} , \quad B_4 = \frac{3M_f^2 \dot{\xi}^2 \left( \frac{a^2M_f^2U}{a^2} + \frac{2a \left( 1 - \frac{a^3}{b^3} \right) m^2M_{\text{eff}}^2}{a^2b^2} \right)}{4b \left[ \frac{a^2M_f^2U}{2a^2} - \frac{3M_f^2}{a^2} + \frac{a^2 \left( 1 - \frac{a^3}{b^3} \right) m^2M_{\text{eff}}^2}{a^2b^2} \right]^2} , \\
B_5 = \frac{-3a^2M_f^2U \dot{\xi}^2}{16\dot{a}^2b \left[ \frac{a^2M_f^2U}{2a^2} - \frac{3M_f^2}{a^2} + \frac{a^2 \left( 1 - \frac{a^3}{b^3} \right) m^2M_{\text{eff}}^2}{a^2b^2} \right]^2} .
\]

(79)

The equation of motion for the scaler field \( \varphi \) in (15) has the following form:

\[
\ddot{\varphi} + 2\frac{\dot{a}}{a} \dot{\varphi} + \frac{1}{3} V' a^2 = 0 .
\]

(80)

From this equation, the perturbed equation for \( \varphi \) is

\[
\delta \ddot{\varphi} + C_1 \delta \dot{\varphi} + C_2 \delta \varphi + C_3 \delta \dot{a} + C_4 \delta a = 0 ,
\]

(81)

where

\[
C_1 = 2\frac{\dot{a}}{a} , \quad C_2 = \frac{1}{3} V'' a^2 , \quad C_3 = \frac{2\dot{\varphi}}{a} , \quad C_4 = -2\frac{\ddot{a}}{a^2} + \frac{2}{3} V' a .
\]

(82)

The equation of motion for the scaler field \( \xi \) in (14) has the following form:

\[
\ddot{\xi} + \left( -\frac{\dot{c}}{c} + 3\frac{\dot{b}}{b} \right) \dot{\xi} + \frac{1}{3} U' c^2 = 0 .
\]

(83)

Using the constraint (23), Eq. (83) can be rewritten as

\[
\ddot{\xi} + \left( -\frac{\dot{a}}{a} + \frac{\dot{a}}{a} + 3\frac{\dot{b}}{b} \right) \dot{\xi} + \frac{1}{3} U' a^2 \dot{b}^2 = 0 .
\]

(84)
One gets the following perturbation equation:

$$\begin{align*}
\delta \dot{\xi} + \frac{\dot{\xi}}{a} \dot{a} - \frac{\dot{\xi}}{b} \dot{b} &= \left( -\frac{\dot{\xi}}{a^2} + \frac{\ddot{a}}{a^2} - \frac{\ddot{b}}{b^2} + \frac{2}{3} \frac{U'^2 b^2}{a^2} \right) \delta \dot{a} + \left( 3 \frac{\dot{\xi}}{b} + \frac{\ddot{b}}{b^2} \xi + \frac{2}{3} U'' a^2 b^2 \right) \delta b + \left( \frac{\dot{a}}{a^2} \xi + \frac{2}{3} U' \frac{ab^2}{a^2} \right) \delta a \\
-3 \frac{\dot{b}}{b^2} \xi \delta b &= \left( -\frac{\dot{a}}{a} + \frac{\ddot{a}}{a} + \frac{3 \dot{b}}{b} - \frac{\ddot{b}}{b} \right) \delta \xi + \frac{1}{3} U'' a^2 b^2 \frac{\delta a}{\dot{a}} \delta \xi = 0.
\end{align*}$$

(85)

Thus, we presented all the equations which are necessary for the study of the (in)stability. The investigation of the (in)stability is, however, still tedious as it requests complicated numerical work. This is because there are too many degrees of freedom. We have $a(t)$, $b(t)$, and $c(t)$ for the metric ansatz and two scalar fields $\varphi$ and $\xi$. Although we have deleted $c(t)$ by using (23), there are four degrees of freedom. We need also to include their derivatives with respect to time. Then totally we have eight degrees of freedom and in order to investigate the stability, we need to find the eigenvalues of eight by eight matrix. Preliminary study of de Sitter space via these equations indicates to its stability. We will investigate the (in)stability for several accelerating cosmological models in a future work.

VI. SUPER-LUMINAL MODE IN BIGRAVITY

In this subsection, we would like to stress that in $F(R)$ bigravity, there may appear super-luminal mode, that is, there can be a signal whose speed is larger than the speed of light.

The Eqs. (18) show that the speed $v_g$ of the massless particle which propagates in the universe described by $g^J_{\mu \nu}$ or $g_{\mu \nu}$ is given by $v_g^2 = (dx/dt)^2 = 1$ as usually in the special relativity. Note that the light speed in the universe described by $g^J_{\mu \nu}$ is identical with the light speed in the universe described by $g_{\mu \nu}$. The speed $v_f$ of the massless particle which propagates in the universe described by $f^J_{\mu \nu}$ or $f_{\mu \nu}$, however, is given by $v_f^2 = (dx/dt)^2 = c(t)^2/b(t)^2$. The light speed in the universe described by $f^J_{\mu \nu}$ is identical with the light speed in the universe described by $f_{\mu \nu}$, again. Therefore if $c(t)/b(t) > 1$, the speed $v_f$ is greater than the speed of light, which propagates in $g$ universe.

When we consider the cosmology with $a(t) = b(t) = 1$ in Subsection VI B, Eq. (48) shows that $c(t) > 1$ except of $\dot{H} = 0$. Furthermore, because $b(t) = 1$, $v_f$ is given by

$$v_f = 1 + \frac{6\ddot{H}^2}{m^2 M^2_{\text{eff}}} > 1.$$  

(86)

Therefore, in general, $v_f$ is greater than the speed of light in the universe described by $g^J_{\mu \nu}$ or $g_{\mu \nu}$. There might be no direct interaction between the matter in the universe described by $g^J_{\mu \nu}$ and the matter in the universe described by $f^J_{\mu \nu}$ but the two kinds of matter can interact via massless and massive graviton. This shows that if there exists any massless particle propagating in the universe described by $f^J_{\mu \nu}$ or $f_{\mu \nu}$, the signal can propagate even in the universe described by $g^J_{\mu \nu}$ or $g_{\mu \nu}$. The super-luminal mode can appear because $g_{\mu \nu} \neq f_{\mu \nu}$ or $g^J_{\mu \nu} \neq f^J_{\mu \nu}$. On the other hand, in the cosmology with $a(t) = b(t) = c(t)$, because $g_{\mu \nu} = f_{\mu \nu}$ and $g^J_{\mu \nu} = f^J_{\mu \nu}$, there does not appear super-luminal mode. It is interesting that the search of super-luminal particles at some era may serve as kind of observational probe for the existence of the universe described by $f^J_{\mu \nu}$ or $f_{\mu \nu}$ (or better to say as indication to massive gravity manifestation).

VII. DISCUSSION

In summary, we studied massive $F(R)$ bigravity in the conventional description with two metrics. The variety of cosmic acceleration cosmologies is found as explicit solution of FRW equations. In particular, Big and Little Rip, de Sitter, quintessence and decelerating universes are constructed for Jordan frame physical metric $g$ when Einstein frame metric $g$ is fixed and corresponding solution for reference metric $f$ is also presented. The relation between properties of $g$ and $f$ cosmologies is investigated in detail. For instance, it is demonstrated that, in general, the physical $g$ cosmological singularity is manifested as metric $f$ cosmological singularity. However, there are examples where cosmological singularity of physical $g$ universe does not occur in the universe described by reference metric $f$ and vice-versa. Furthermore, the structure of singularity may qualitatively change: what looks like Big Rip in one space-time manifests as Little Rip in other universe. The solution of FRW equations where two metrics just happen to coincide is presented for Big and Little Rip, quintessence, de Sitter and decelerating universes. In this case, the background evolution is described via single metric which looks quite convenient even keeping in mind that second
metric is just the effective description of exotic matter (the perturbation theory for two metrics is also different). Perturbation theory for cosmological solutions under discussions is also developed.

We also observed that the massless particle in the space-time given by the metric $f_{\mu\nu}$ or $f_{\mu\nu}^3$ can be super-luminal. Then if there appears any indication that there exists a signal whose speed is greater than the speed of light, it may be the indication for another space-time existence (massive graviton effect).

It could be interesting to consider how we can observe the manifestation of the space-time described by the metric $f_{\mu\nu}$ or $f_{\mu\nu}^3$. As we mentioned, we can observe the matter via massless and massive gravitons. Therefore not only dark energy but also dark matter might be a matter in the space-time given by the metric $f_{\mu\nu}$ or $f_{\mu\nu}^3$.

Acknowledgments.

We are grateful to G. Gabadadze, N. Kaloper, M. Sami, A. Starobinsky, J. Soda and K. Bamba for useful discussions. The work by SN is supported in part by Global COE Program of Nagoya University (G07) provided by the Ministry of Education, Culture, Sports, Science & Technology and by the JSPS Grant-in-Aid for Scientific Research (S) # 22224003 and (C) # 23540296. The work by SDO is supported in part by MICINN (Spain), project FIS2010-15640 and by AGAUR (Generalitat de Catalunya), contract 2009SGR-994.
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