SOME RESIDUALLY SOLVABLE ONE-RELATOR GROUPS

DELARAM KAHROBAEI, ANDREW F. DOUGLAS, AND KATALIN BENCSÁTH

Abstract. This communication records some observations made in the course of studying one-relator groups from the point of view of residual solvability. As a contribution to classification efforts we single out some relator types that render the corresponding one-relator groups residually solvable.

1. Introduction

It is well known that free groups are residually nilpotent and, consequently, residually solvable. There is a sizable amount of literature devoted to investigations when (residual, virtual) properties of free groups are inherited by one-relator groups. The purpose of this communication is to offer a collection of facts and examples gathered in the course of attempts to characterize the residually solvable one-relator groups in terms of the (form of) the single defining relator. We prove a number of sufficiency results for certain cases when the relator is a commutator, and then raise some questions. We start with reviewing some of the literature that motivated our interest in the topic.

G. Baumslag in [2] showed that positive one-relator groups, which is to say that the relator has only positive exponents, are residually solvable. In the same paper he provides a specific example to show that not all one-relator groups are residually solvable. A free-by-cyclic group is necessarily residually solvable. It is worth mentioning that the Baumslag-Solitar groups $B_{1,n}$ are solvable, but not polycyclic. On the other hand $B_{m,n}$ are free-by-solvable by a result of Peter Kropholler [15] who showed their second derived subgroup to be free, therefore residually solvable. However, there are non-Hopfian Baumslag-Solitar’s groups amongst them, and those are certainly not residually nilpotent. In [8] Baumslag, Fine, Miller and Troeger established that many one-relator groups, in particular cyclically pinched one-relator groups are either free-by-cyclic or virtually free-by-cyclic. These point to another large class of residually solvable one-relator groups. In view of the recent result of M. Sapir and I.Spakulova in [17] that with probability tending to 1, a one-relator group with at least 3 generators and the relator of length $n$ is residually finite, even virtually residually (finite $p$)-group and coherent for all sufficiently large $p$. In addition in [16] M. Sapir focuses on residual properties of one-relator groups with at least 3 generators.

Accordingly our focus is mainly on two generator one-relator groups. Our two main results concern the situation where $G$ is a one-relator group whose relator is a commutator. We provide certain sufficient conditions where $G$ is residually solvable but also give examples to show that in general almost anything can occur.

Clearly the attempts to find criteria for residual solvability are influenced by outcomes of recent and older studies on (fully) residually freeness of one-relator groups, in particular.
Examples of such one-relator groups are surface groups, which are known to be fully residually free \[6\] and therefore are residually solvable. Also in \[4\], B. Baumslag shows residual freeness of the one-relator groups of the type \[\langle a_1, \cdots, a_k; a_1^{w_1} \cdots a_k^{w_k} = 1 \rangle\] where \(k > 3\), and \(w_i\)s in the ambient free group on \(a_1, \cdots, a_k\) satisfy certain conditions, implies residual solvability of such one-relator groups.

2. Preliminaries

For convenience, we start with a list of some the definitions, facts, and theorems that form our starting base.

**Theorem 2.1.** (Von Dyck) Suppose \(G = \langle X; R \rangle\) and \(D = \langle X; R \cup S \rangle\), with presentation maps \(\gamma\) and \(\mu\) respectively. Then \(x\mu \mapsto x\gamma\) \((x \in X)\) defines a homomorphism of \(G\) onto \(D\).

**Theorem 2.2.** (Freiheitssatz) \[14\] Let \(G\) be a one-relator group, i.e., \(G = \langle x_1, \cdots, x_q; r = 1 \rangle\). Suppose that the relator \(r\) is cyclically reduced, i.e., the first and the last letters in \(r\) are not (formal) inverses of each other. If each of \(x_1, \cdots, x_q\) actually appears in \(r\), then any proper subset of \(\{x_1, \cdots, x_q\}\) is a free basis for a free subgroup of \(G\).

W. Magnus’ method of structure analysis \[14\] for groups with a single defining relation has the following immediate consequence:

**Lemma 2.3.** Let \(G = \langle b, x, \cdots, c; r = 1 \rangle\) be a one-relator group. Suppose that \(b\) occurs in \(r\) with exponent sum zero and that upon re-expressing \(r\) in terms of the conjugates \(b^i x b^{-i} = x_i, \cdots, b^i c b^{-i} = c_i, (i \in \mathbb{Z})\) and renaming \(r\) as \(r_0\), \(\mu\) and \(\nu\) are respectively the minimum and maximum subscripts of \(x\) occurring in \(r_0\). If \(\mu < \nu\) and if both \(x_\mu\) and \(x_\nu\) occur only once in \(r_0\) then \(N = gp_G(x, \cdots, c)\) is free. If \(G\) is a two-generator group with generators \(b\) and \(x\), then \(N\) is free of rank \(\nu - \mu + 1\).

**Definition 2.4.** A group \(G\) is residually solvable if for each \(w \in G\) \((w \neq 1)\), there exists a solvable group \(S_w\) and an epimorphism \(\phi : G \rightarrow S_w\) such that \(w\phi \neq 1\).

**Theorem 2.5.** (Kahrobaei \[12, 13\]) Any generalized free product of two finitely generated torsion-free nilpotent groups, amalgamating a cyclic subgroup is an extension of a residually solvable group by a solvable group, therefore is residually solvable.

**Theorem 2.6.** (Kahrobaei \[12, 13\]) Any generalized free product of an arbitrary number of finitely generated nilpotent groups of bounded class, amalgamating a subgroup central in each of the factors, is an extension of a free group by a nilpotent group, therefore is residually solvable.

**Theorem 2.7.** (Kahrobaei \[12, 13\]) The generalized free product of a finitely generated torsion-free abelian group and a nilpotent group is (residually solvable)-by-abelian-by-(finite abelian), consequently residually solvable.

Note that the groups in all three of these theorems above satisfy the conditions of K. Gruenberg’s portent observation \[10\] that we record here as

**Lemma 2.8.** Suppose \(P\) is any group, \(K \triangleleft P\) with \(P/K\) solvable and \(K\) residually solvable. Then \(P\) is residually solvable.
3. The single relator is a commutator

We first recall a result from [3] for a particular class of non-positive one-relator groups. Let $G$ be a group that can be presented in the form,

$G = \langle t, a, ..., c; uw^{-1} = 1 \rangle$,

where $u$ and $w$ are positive words in the given generators and each generator occurs with exponent sum zero in $uw^{-1}$. Then $G$ is residually solvable. In fact, $G$ is free-by-cyclic.

Now consider the group,

$H = \langle t, a, ..., c; [u, w] = 1 \rangle$.

If $u$ and $w$ are positive, $H$ can be recognized as one of the groups in the preceding class (1). Hence $H$ is free-by-cyclic and therefore residually solvable. However, known examples show that residual solvability for $H$ is not guaranteed once the requirement that both $u$ or $w$ be positive is relaxed:

**Example 3.1.** [3] If $G = \langle a, b, ..., c; [u, v] = 1 \rangle$ where $u = a$, $v = [a, b][w, w^b]$, and $[a, b]^{-1}[a, b]^a$, then $G$ is not residually solvable. For it follows from Magnus’ solution of the word problem that $w \neq 1$ [14]. Furthermore since $[u, v] = 1$ we find that $[a, b]^a[w, w^b]^a = [a, b][w, w^b]$, so that $w = [a, b]^{-1}[a, b]^a = [w, w^b([w, w^b]^a)^{-1}$.

Thus $w$ lies in every term of the derived series of $G$.

In contrast, the next example is a residually solvable one-relator group.

**Example 3.2.** The group $G = \langle a, b; [a, [a, b]] \rangle$ is free-by-cyclic.

**Proof.** We expand and re-express the relator,

$r = [a, [a, b]] = a^{-1}[a, b]^{-1}a[a, b] = a^{-1}b^{-1}a^{-1}bab^{-1}ab$.

Observe that in $r_0 = b_1^{-1}b_2b_1^{-1}b_0$ and $\mu = 2$, $\nu = 0$, and $b_0$ and $b_2$ both occur only once and we can invoke lemma 2.3. Therefore $G$, as a cyclic extension of the free group $N = gp_G(b)$ is residually solvable by (c.f. 2.8). \hfill \Box

4. Connection between Generalized Free Products and One-Relator Groups

Over the years since W. Magnus developed his treatment of one-relator groups the increased interest in them yielded many new results. Karrass-Solitar in 1971 showed that a subgroup of a one-relator group is either solvable or contains a free subgroup of rank two. Baumslag-Shalen showed that every one-relator group with at least four generators can be decomposed into a generalized free product of two groups where the amalgamated subgroup is proper in one factor and of infinite index in the other. Fine-Howie-Rosenberger [7] and Culler-Morgan [9] showed that any one-relator group with torsion and at least three generators can be decomposed, in a non-trivial way, as an amalgamated free product.
These results made it seem a reasonable preassumption that a closer look at the residual solvability of generalized free products of two groups could help in establishing residual solvability of further one-relator groups. The following result confirms that assumption.

**Theorem 4.1.** The group $G = \langle a, b; [a, w] \rangle$, where $w = [a, b]^n$ $(n \in \mathbb{N})$, is residually solvable.

**Proof.** Put $N = gp_G(b)$, the normal closure of $b$ in $G$. Using the Magnus break-down, we consider:

(4)  
$N_0 = \langle b_0, b_1, b_2; (b_1b_0)^n = (b_2^{-1}b_1)^n \rangle$.

Now let

$x_0 = b_1^{-1}b_0$, $x_1 = b_2^{-1}b_1$, $y = b_1$.

Tietze transformations confirm that

(5)  
$N_0 = \langle x_0x_1y; (x_0)^n = (x_1)^n = \langle x_0, x_1; (x_0)^n = (x_1)^n \rangle \ast \langle y \rangle \rangle$.

Next let $K = \langle x_0, x_1; (x_0)^n = (x_1)^n \rangle$. Clearly

(6)  
$K = \{ \langle x_0 \rangle \ast \langle x_1 \rangle; \langle x_0^n \rangle = \langle x_1^n \rangle \} \}

Since each factor of $K$ is abelian, by theorem 2.5 $K$ is residually solvable. The free factor of $N_0$, $\langle y \rangle$ is also residually solvable. Therefore $N_0$ is residually solvable, and it follows for every $i \in \mathbb{N}$ that $N_i$ is residually solvable. If we put $N_{i,j} = gp(N_i, N_{i+1}, \ldots, N_j)$, the proceeding approach gives

$N_{i,j} = \langle x_i \rangle \ast \langle x_i^n \rangle = \langle x_i^{n+1} \rangle \ast \cdots \ast \langle x_j \rangle \ast \langle x_j^n \rangle = \langle x_j^{n+1} \rangle \ast \langle y \rangle \rangle$.

Therefore, by Theorem 2.6, $N_{i,j}$ is residually solvable. A task that remains for completing the proof is to show that the ascending union $N = \bigcup_{r<s>0} N_{r,s}$ is residually solvable, which will be taken care of by the following proposition 4.2. Granted that, the residual solvability of $G$ follows with the use of corollary 2.8.

**Proposition 4.2.** $N = \bigcup_{r<s>0} N_{r,s}$ is residually solvable.

**Proof.** We will retain notation from the proof of Theorem 4.1 and start with the assumption that $N_{i,j}$ is residually solvable for all $i, j \in \mathbb{N}$ $(i \leq j)$. For the derived series of $N = \bigcup_{r<s>0} N_{r,s}$, we have

$\delta_i N = \delta_i(\bigcup_{r<s>0} N_{r,s})$.

Every element in $g \in \delta_i N$ is a finite product of commutators of elements from a (finite) subset of the $N_{r,s}$ groups. So $g \in \delta_i N_{r,s}$ for suitably small value of $r < 0$ and suitably large value of $s > 0$. Thus $\delta_i N = \bigcup_{r<s>0} N_{r,s}$. Now, if $j, k$ are a pair of fixed integers the infinite union above can be rewritten as

$\delta_i N = \bigcup_{r<s>0} \delta_i N_{j+r,k+s}$,

so that,

$\delta_i N \cap N_{j,k} = (\bigcup_{r<s>0} \delta_i N_{j+r,k+s}) \cap N_{j,k}$.

Equivalently,

$\delta_i N \cap N_{j,k} = \bigcup_{r<s>0} (\delta_i N_{j+r,k+s} \cap N_{j,k})$.

Further, each term in the union an be written as

$\delta_i N_{j+r,k+s} \cap N_{j,k} = (\delta_i N_{j,k+s} \cap N_{j,k}) \cap (\delta_i N_{j+r,k} \cap N_{j,k})$. 
And because \( s < 0 \) and \( r > 0 \), an argument fashioned after that in [2] (p. 175, Lemma 4.3) yields, that after conjugations by suitable powers
\[
\delta_i \cap N_{j,k} = \delta_i \cap N_{j,k},
\]
and
\[
\delta_i \cap N_{j,r,k} = \delta_i \cap N_{j,k}.
\]
So each term in the union can be re-expressed as
\[
\delta_i \cap N_{j,r,k} \cap N_{j,k} = \delta_i \cap N_{j,k},
\]
and
\[
\delta_i \cap N_{j+r,k} \cap N_{j,k} = \delta_i \cap N_{j,k}.
\]
Notice that this expression is independent of \( r \) and \( s \). Thus, we get
\[
\delta_i \cap N_{j,k} = \delta_i \cap N_{j,k}([2][pg. 175, line 16]).
\]
We claim that
\[
\delta_i \cap N_{j,k} = \delta_i \cap N_{j,k}.
\]
implies that \( N \) is residually solvable. To see this, let \( a \) be a non-trivial element of
\[
N = \bigcup_{r<0, s>0} N_{r,s}.
\]
Then, there is an integer \( j = j(a) \in \mathbb{N} \) such that \( a \in N_{j,j} \). By our (inductive) hypothesis at the outset \( N_{j,j} \) is residually solvable. Consequently, there exists an integer \( i \in \mathbb{N} \) such that \( a \notin \delta_i N_{j,j} \). Then, since \( \delta_i N \cup N_{j,j} = \delta_i N_{j,j} \), we see that \( a \notin \delta_i N \cap N_{j,j} \). But \( a \in N_{j,j} \). So it must be the case that \( a \notin \delta_i N \). Thus we have found a normal subgroup \( \delta_i N \lhd N \) with the property that \( a \notin \delta_i N \) and \( N/\delta_i N \) is solvable. Hence \( N \) is residually solvable. \( \square \)

5. THE RELATOR IS A BASIC COMMUTATOR

The tools of the Magnus theory were of good use for proving residual solvability through gaining information about the structure of the two-generator one-relator groups where the relator is a particular type of basic commutator.

We begin with recalling P.Hall’s [11] definition of the basic commutators (in terms of the free group \( F \) on \( \{x_1, \ldots, x_q\} \)) and their linear ordering (in terms of their weights).

**Definition 5.1.** Basic Commutators.

1. The basic commutators of weight one with their linear order are \( x_1 < x_2 < \cdots < x_q \); for their weights we write \( wt(x_i) = 1 \).
2. Having defined the basic commutators of weight less than \( n \), the basic commutators with weight \( n \) are of the form \( c_n = [c_i, c_j] \) where \( c_i \) and \( c_j \) are all the basic commutators satisfying \( wt(c_i) + wt(c_j) = n \), \( c_i > c_j \), and such that if \( c_i = [c_s, c_t] \), then \( c_j \geq c_t \).

In the following, for positive integers \( k \) we will use the notation \( s_1 = x \), and \( s_{k+1} = [s_k, y] \).

**Theorem 5.2.** The group \( G = \langle x, y; r = [s_k, y] \rangle \) is free-by-cyclic, therefore residually solvable.

**Proof.** Following the Magnus theory, we put \( x_i = y^{-i}xy^i \). Using induction on the weight of the commutator and the relationship
\[
[s_k, y] = s_k^{-1}(s_k)^y, (k > 0)
\]
we see that the minimum index and maximum index in \( r \) are 0 and \( k \), respectively, and both of \( x_0 \) and \( x_k \) only occur once in \( r \). By Lemma [2.3] it follows, similarly to previous cases, that \( G \) is free-by-cyclic. \( \square \)
6. Open Problems

(1) Is it algorithmically decidable whether a one-relator group is residually solvable?

(2) Are one-relator groups generically residually solvable? In other word, are they in most cases residually solvable? In [5], a conjecture of I.Kapovich is quoted and it states that many one-relator groups are finitely generated free-by-cyclic.

(3) Do there exist residually finite one-relator groups that are not residually solvable? (This is recasting a question in [1] in this context.)

(4) Certain basic commutators make were shown in this paper to produce residually solvable one-relator groups. Would all basic commutators have that property? If not can the techniques described here be extended to more one relator groups where the relator is a basic commutator?

(5) Find further examples of non-positive one relator groups that fail to be residually solvable.

(6) Find examples of residually solvable one-relator groups that are not free-by-cyclic.

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Doctoral Program in Computer Science Department, CUNY Graduate Center, City University of New York, 365 Fifth Ave., USA 10016
E-mail address: dkahrobaei@gc.cuny.edu

Department of Mathematics, New York City College of Technology, CUNY, 300 Jay Street, Brooklyn, New York, USA 11201
E-mail address: adouglas@citytech.cuny.edu

Department of Mathematics and Computer Science, Manhattan College, Riverdale, New York 10471
E-mail address: katalin.bencsath@manhattan.edu