NONVANISHING OF CARTAN CR CURVATURE ON BOUNDARIES OF GRAUERT TUBES AROUND HYPERBOLIC SURFACES

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ABSTRACT. We show that the boundaries of thin strongly pseudoconvex Grauert tubes, with respect to the Guillemin-Stenzel Kähler metric canonically associated with the Poincaré metric on closed hyperbolic real-analytic surfaces, has nowhere vanishing Cartan CR-curvature. This result provides a wealth of examples of compact 3-dimensional Levi nondegenerate CR manifolds having no CR-umbilical point.

We provide two proofs utilizing two recent formulas for determining the Cartan CR-curvature of any local $C^6$-smooth hypersurfaces in $\mathbb{C}^2$. One was obtained in 2012 by the second named author joint with Sabzevari, and it is an expanded explicit formula, valid for locally graphed hypersurfaces, containing millions of terms. The other formula, which we published in 2018 when studying Webster’s ellipsoidal hypersurfaces, is not expanded, but more suitable for calculations with a hypersurface in $\mathbb{C}^2$ that is represented as the zero locus of some implicit — but ‘simple’ in some sense, e.g. quadratic — defining function.

We also discuss Grauert tubes constructed with respect to extrinsic metrics depending on embeddings in complex surfaces, together with a certain combinatorics of product metrics.

1. Introduction

The equivalence problem for local real-analytic hypersurfaces with respect to local biholomorphisms in $\mathbb{C}^2$ was first studied by Poincaré [23], and was later solved by Cartan [4] with the introduction of the so-called method of equivalence. The theory was later developed in $\mathbb{C}^{n+1}$ by Chern and Moser [5], and resulted in the set up of invariant CR-curvatures, called Cartan curvatures in complex dimension 2, and Hachtroudi-Chern curvatures when $n \geq 3$.

For a long time, little was known about these curvatures due to their high computational complexity. Nonetheless, Webster [25], and later Huang and Ji [16] were able to investigate the case of real ellipsoidal hypersurfaces. In
recent years, new variants and explicit formulas (see [7,20,21,10]) made it possible to determine the vanishing locus of the Cartan curvatures for new classes of 3-dimensional CR manifolds. For instance, we were able to find a whole explicit curve of points of vanishing Cartan curvature on general ellipsoids in \( \mathbb{C}^2 \) in [10].

In their landmark paper [5, p. 247], Chern and Moser raised the following

**Problem 1.1.** Are there compact strictly pseudoconvex hypersurfaces \( M^3 \subset \mathbb{C}^2 \) without CR-umbilical points? Are there such manifolds diffeomorphic to the sphere \( S^3 \subset \mathbb{C}^2 \).

It is well known that a standard 2-torus in \( \mathbb{R}^3 \) has no Riemannian-umbilic point. Similarly, it is not difficult to verify ([7]) that the boundaries of thin Grauert tubes around the flat 2-dimensional torus \( T^2 = S^1 \times S^1 \subset \mathbb{C}^2 \) have empty CR-umbilical locus. Thus, a topological restriction like \( M^3 \cong S^3 \) must be assumed.

In this paper, we are interested in the question of whether a similar phenomenon holds for higher genus surfaces. Let therefore \( S \) be a closed compact real-analytic \( (C^\omega) \) surface of genus \( \geq 2 \) which is hyperbolic in the sense that its universal cover is the unit disc \( \mathbb{D} \subset \mathbb{C} \). As a special case of a theorem of Bruhat and Whitney [3] in dimension 2, \( S \) admits an extrinsic complexification, namely there exists a complex manifold \( M^c \) of complex dimension 2, together with an analytic totally real embedding of \( S \) into \( M^c \). Moreover, the work [15] of Guillemin and Stenzel provides a canonical Kähler potential \( \rho \) defined in a small neighborhood of \( S \) in \( M^c \) (see Section 2 below). In particular, for each \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \ll 1 \), the set \( \Omega_\varepsilon := \rho^{-1}([0,\varepsilon)) \), called the Grauert tube of radius \( \varepsilon \) around \( S \), has strongly pseudoconvex \( C^\omega \) boundary \( M_\varepsilon := \rho^{-1}(\varepsilon) \) contained in the complex surface \( M^c \), to which Cartan’s method of equivalence applies. Our main result is the following.

**Theorem 1.2.** There exists \( 0 < \varepsilon_0 \ll 1 \) such that for every \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \), the real and imaginary parts of the primary complex Cartan curvature vanish nowhere on the boundary of \( M_\varepsilon \).

Equivalently:

**Corollary 1.3.** The boundaries of these \( M_\varepsilon \) have no CR-umbilical point.

So far, our construction of the Grauert tubes \( \Omega_\varepsilon \) take a complete intrinsic point of view, since the Guillemin-Stenzel potential is obtained only from a given intrinsic metric on the surface \( S \). It is then natural to look at the
Grauert tubes from an extrinsic point of view, that is we consider the surface $S$ as being totally really embedded in a given (local) complex surface equipped with a given metric. Already in the case of a torus embedded in the standard $\mathbb{C}^2$, the extrinsic construction will provide several new examples of compact hypersurfaces without CR-umbilical points (see Example 7.4). Further constructions in this vein are provided in Section 7.

This paper is organized as follows. In Section 2 we recall the construction of the canonical Kähler potential of Guillemin and Stenzel in [15], and we find an explicit formula for the potential in the case of hyperbolic surfaces. Section 3 discusses two standard examples of complexification of the round sphere and the flat torus. In Section 4 we work out the defining function for the Grauert tube around the Poincaré upper half-plane. The formula then will be used in Section 5 to calculate the Cartan curvatures on the boundaries of Grauert tubes of hyperbolic surfaces by explicit expressions given in [20] and [10], and to show that the Cartan curvatures do not vanish for small enough radii. Section 6 explains in details how nonvanishing of the Cartan curvature on the boundary of Grauert tubes around hyperbolic surfaces can be deduced from the calculations in Section 5. Finally, in Section 7 we discuss some extrinsic constructions of Grauert tubes based on product metrics.

2. The Canonical Kähler Potential on Grauert Tubes

For any compact real-analytic ($\mathcal{C}^\omega$) manifold $M$ of dimension $n \geq 1$, Bruhat and Whitney showed in [3] that there exists an $n$-dimensional complex manifold $M^c$, and a real-analytic embedding $M \hookrightarrow M^c$ which is totally real, i.e. such that the real tangent spaces to $M$ contain no complex lines in the complex tangent spaces to $M^c$. The $\mathcal{C}^\omega$ changes of charts $\mathbb{R}^n \ni x \mapsto x' = \varphi(x) \in \mathbb{R}^n$ for $M$, where $x = (x_1, \ldots, x_n)$, become $\mathbb{C}^n \ni z \mapsto z' = \varphi(z) \in \mathbb{C}^n$, where $z = x + \sqrt{-1}y \in \mathbb{C}^n$, and where $\varphi(z)$ means substituting $z$ for $x$ in the punctual convergent power series of $\varphi$, giving the complex manifold structure of $M^c$.

The Taylor coefficients of such $\mathcal{C}^\omega$ diffeomorphisms $\varphi = \overline{\varphi}$ are real, the complex conjugation $z \mapsto \overline{z}$ transfers coherently as $\overline{z'} = \varphi(\overline{z})$, which shows that $M = \text{Fix}(\varphi)$ is the set of fixed points of the antiholomorphic involution $\varphi : M^c \rightarrow M^c$ obtained from $z \mapsto \overline{z}$ in any chart.

Also by substituting $z$ for $x$ in power series, every $\mathcal{C}^\omega$ function $f : M \rightarrow \mathbb{R}$ extends uniquely as a holomorphic function $f^c : U^c \rightarrow \mathbb{C}$ with $f^c|_M = f$, in some open neighborhood $U^c$ of $M$ in $M^c : M \subset U^c \subset M^c$, and $f^c|_M \equiv 0$ if and only if $f^c \equiv 0$ in some subneighborhood $V^c : M \subset V^c \subset U^c$. 
According to Grauert \cite{14}, there exists a \(C^\infty\) strictly plurisubharmonic function \(\rho: U^c \to [0, 1)\) defined in some open neighborhood \(U^c\) of \(M\) in \(M^c\), with \(\rho \circ \sigma = \rho, M = \rho^{-1}(0), d\rho|_M \equiv 0\), and such that \(\rho\) has no critical point in \(V^c \setminus M\), for some subneighborhood \(V^c: M \subset V^c \subset U^c\). Hence for all small enough \(\varepsilon: 0 < \varepsilon \ll \varepsilon_0 \ll 1\), the domain \(\Omega_{\varepsilon} = \{\rho < \varepsilon\}\), a tubular neighborhood of \(M\) in \(M^c\), has \(C^\infty\) strictly pseudoconvex boundary \(M_{\varepsilon} = \{\rho = \varepsilon\}\), and is called the Grauert tube of radius \(\varepsilon\) around \(M\).

When the manifold \(M\) is equipped with some \(C^\omega\) Riemannian metric \(g\), Guillemin and Stenzel gave in \cite{15} a very elegant construction of such a strictly plurisubharmonic function

\[ \rho = \rho_g: M^c \to [0, 1) \]

uniquely associated to \(g\) that will be called the canonical Kähler potential on \(M^c\). Their construction can be summarized as follows.

Embed \(M \hookrightarrow M \times M\) by \(x \mapsto (x, x)\) and let \(W\) be an open neighborhood of \(M\) in \(M \times M\). If \(W\) is thin enough, for any pair \((x, u) \in W\), the local uniqueness and distance minimizing properties of geodesics with respect to \(g\) guarantees that \(\text{dist}_g(x, u)\) is the \(g\)-length of the geodesic from \(x\) to \(u\), and an inspection of the \(g\)-length formula convinces that the (symmetric) squared distance function:

\[ f(x, u) := \left(\text{dist}_g(x, u)\right)^2 \quad (x, u \in W) \]

is \(C^\omega\), hence can be complexified.

Since in local coordinates, we will denote \(x = (x_1, \ldots, x_n)\) and \(u = (u_1, \ldots, u_n)\) in \(\mathbb{R}^n\) and introduce \(z := x + \sqrt{-1} y\) with \(w := u + \sqrt{-1} v\) in \(\mathbb{C}^n\), let us denote a pair of points in the global abstract product similarly as \((z, w) \in M^c \times M^c\), and let us abbreviate \(\sigma: M^c \to M^c\) as \(z \mapsto z\). Also, let us use the embedding:

\[ M^c \ni z \mapsto (z, \overline{z}) \in M^c \times M^c, \]

compatible with \(x \mapsto (x, x)\) which makes \(M^c\) totally real in \(M^c \times M^c\), and let \(W^c\) be a thin open neighborhood of \(M^c\) in \(M^c \times M^c\) invariant under the conjugation \((z, w) \mapsto (\overline{w}, \overline{z})\) and satisfying \(W = W^c \cap (M \times M)\).
Then \( f(x, u) \) complexifies as \( f^c(z, w) \) defined and holomorphic for \((z, w) \in W^c\), with \( f^c \mid_M \equiv f \) and enjoys the symmetry \( f^c(w, z) = f^c(z, w) \).

Furthermore, the reality condition \( f(x, u) = f(x, u) \) of \( f \) yields via complexification:

\[
\overline{f^c(z, w)} \equiv f^c(\overline{z}, \overline{w}),
\]

hence putting \( w := \overline{z} \), and using the symmetry, we see the reality:

\[
\overline{f^c(z, \overline{z})} \equiv f^c(\overline{z}, z) \equiv f^c(z, \overline{z}).
\]

**Proposition 2.1.** ([15], p. 565) The real-valued function \( f^c(z, \overline{z}) \) is equal to 0 on \( M \hookrightarrow M^c \times M^c \) and takes values < 0 outside \( M \).

So in \( W^c \setminus \{ f^c = 0 \} \), the square root \( \sqrt{f^c} \) is 2 : 1-valued, and the canonical Kähler potential \( \rho = \rho_g \) is defined to be:

\[
(2.2)
\rho := -f^c,
\]

so that \( \sqrt{\rho} \) is well defined in \( \mathbb{R}_+ \).

Finally, a consequence of Gauss’ orthogonality lemma ([15], p. 564) which provides the annihilation:

\[
0 \equiv \det \left( \frac{\partial^2 \sqrt{f}}{\partial x_i \partial y_j}(x, y) \right) \quad (\forall (x, y) \in W \setminus M),
\]

yields via complexification the Monge-Ampère equation:

\[
0 \equiv \det \left( \frac{\partial^2 \sqrt{\rho}}{\partial z_i \partial w_j}(z, w) \right) \quad (\forall (z, w) \in W^c \setminus M^c).
\]
In [15], Guillemin and Stenzel established the uniqueness of the Kähler metric \( \omega := \sqrt{-1} \partial \bar{\partial} \rho_g \) on \( M \) satisfying this and restricting to \( g = \omega \bigr|_M \) on \( M \).

Of particular interest to us is the computational fact that \( \rho = \rho_g \) has explicit, workable expressions once \( g \) is given, especially in the case of surfaces.

### 3. Two Examples: Round Sphere and Flat Torus

**Example 3.1.** [15] Section 4] Consider \( M := \mathbb{S}^2 \) to be the 2-dimensional sphere:

\[
\mathbb{S}^2 := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \},
\]
equipped with the standard round metric, whence the squared geodesic distance between two points \( x, y \in \mathbb{S}^2 \) is:

\[
f(x, y) = \left( 2 \arcsin \left( \frac{1}{2} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} \right) \right)^2.
\]

The Bruhat-Whitney complexification of \( \mathbb{S}^2 \) can be represented extrinsically as:

\[
(\mathbb{S}^2)^c := \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1 \},
\]
and on it, we have the useful relation:

\[
(\text{Im} z_1)^2 + (\text{Im} z_2)^2 + (\text{Im} z_3)^2 = \left( z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3 - 1 \right)/2.
\]

The complexification of \( f \) is:

\[
f^c(z, w) = \left( 2 \arcsin \left( \frac{1}{2} \sqrt{(z_1 - w_1)^2 + (z_2 - w_2)^2 + (z_3 - w_3)^2} \right) \right)^2,
\]
hence letting \( w := \bar{z} \) and using the two identities:

\[
\arcsin (\sqrt{-1} t) = \sqrt{-1} \arcsinh(t), \quad 2 \arcsinh t = \text{arccosh} \left( 1 + 2 t^2 \right),
\]
we get:

\[
f^c(z, \bar{z}) = \left( 2 \arcsin \left( \pm \sqrt{-1} \sqrt{(\text{Im} z_1)^2 + (\text{Im} z_2)^2 + (\text{Im} z_3)^2} \right) \right)^2
\]
\[= \left( \pm 2 \sqrt{-1} \arcsinh \left( \sqrt{(z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3 - 1)/2} \right) \right)^2
\]
\[= - \left( \text{arccosh} \left( z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3 \right) \right)^2,
\]
whence, coming back to the definition (2.2) of \( \rho := - f^c \), we obtain:

\[
\rho(z, \bar{z}) = \left( \text{arccosh} \left( z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3 \right) \right)^2.
\]
Example 3.3. [7] Section 3] Consider $M := T^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$ to be the flat torus. Its complexification is $M^c := \mathbb{C}^2/(2\pi\mathbb{Z}^2)$. The geodesic distance between two close points on $T^2$ is computed along straight lines within the flat universal cover $(\mathbb{R}^2, d_{\text{Eucl}})$. So, in a fundamental domain for $T^2$ on $\mathbb{R}^2$, the squared distance and its complexification are

\[ f((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2, \]

\[ f^c((z_1, z_2), (w_1, w_2)) = (z_1 - w_1)^2 + (z_2 - w_2)^2, \]

hence letting $(w_1, w_2) = (\overline{z}_1, \overline{z}_2)^c$, we get by the definition (2.2) of $\rho := -f^c$:

\[ \rho(z, \overline{z}) = 4 (\text{Im } z_1)^2 + 4 (\text{Im } z_2)^2. \]

4. Semi-global Grauert Tube Around Poincaré’s Upper Half-Plane

For our purpose, we need to find the Kähler potential $\rho$ locally on the Bruhat-Whitney complexification of any compact $C^\omega$ surface $S$ of genus $\geq 2$. When $S$ is viewed as a Riemann surface, the uniformization theorem ([11] Chap. 27) states that its universal cover is the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$, and that:

\[ S \cong \mathbb{H}/\pi_1(S). \]

We will then transfer geometric objects from $\mathbb{H}$ to $S$.

But in this section, our calculations will be done entirely in $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R} : x_2 > 0 \}$, viewed as a real $C^\omega$ surface equipped with the Poincaré metric $ds^2 = \frac{dx_1^2 + dx_2^2}{x_2^2}$. Since the squared Poincaré distance between two points $(x_1, x_2)$ and $(y_1, y_2)$ of $\mathbb{H}$, with $x_2, y_2 > 0$, is:

\[ f((x_1, x_2), (y_1, y_2)) = \left( \text{arccosh} \left( 1 + \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{2x_2y_2} \right) \right)^2, \]

it comes by complexification

\[ f^c((z_1, z_2), (\overline{z}_1, \overline{z}_2)) = \left( \text{arccosh} \left( 1 - 2 \frac{(\text{Im } z_1)^2 + (\text{Im } z_2)^2}{(\text{Re } z_2)^2 + (\text{Im } z_2)^2} \right) \right)^2, \]

with $z_1 = \text{Re } z_1 + \sqrt{-1} \text{Im } z_1$ and $z_2 = \text{Re } z_2 + \sqrt{-1} \text{Im } z_2$, provided that certain inequalities are satisfied by $\text{Im } z_1$ and $\text{Im } z_2$ for this formula to be meaningful. Here, the complexification of $\mathbb{H}$ reads as:

\[ \mathbb{H}^c := \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 > 0 \}. \]

Lemma 4.2. The domain of definition of $f^c$ in $\mathbb{H}^c$ contains:

\[ \{(\text{Im } z_1)^2 < (\text{Re } z_2)^2\}. \]
Proof. Indeed, the argument $1 - 2Q$ of arccosh in (4.1) is real and $\leq 1$. But with $s = \sigma + \sqrt{-1}t$, for $\cosh s$ to be real $\leq 1$, since its imaginary part:

$$2 \Im(\cosh s) = 2 \Im(e^{\sigma+it} + e^{-\sigma-it}) = (e^\sigma - e^{-\sigma}) \sin t,$$

vanishes if and only if $t \equiv 0 \text{ mod } \pi$, and since $\cosh \sigma > 1$ whenever $\sigma \in \mathbb{R}\{0\}$, necessarily $s = \sqrt{-1}t \in \sqrt{-1}\mathbb{R}$, hence:

$$\arccosh(1 - 2Q) =: \sqrt{-1}T \in \sqrt{-1}\mathbb{R}$$

for some $T \in \mathbb{R}$, whence:

$$1 - 2Q = \cosh(\sqrt{-1}T) = \cos T \quad (T \in \mathbb{R}).$$

Then $-1 \leq \cos T \leq 1$ forces:

$$-1 \leq 1 - 2 \frac{(\Im z_1)^2 + (\Im z_2)^2}{(\Re z_2)^2 + (\Im z_2)^2} \leq 1,$$

the first inequality being equivalent to $(\Im z_1)^2 \leq (\Re z_2)^2$, while the second holds trivially. □

For later convenience, let us rewrite the local complex coordinates as $z_1 = u + \sqrt{-1}v$ and $z_2 = x + \sqrt{-1}y$. Furthermore, let us restrict our considerations to the subdomain of the above domain $\{v^2 \leq x^2\}$ defined by:

$$0 \leq 1 - 2 \frac{y^2 + v^2}{x^2 + y^2} \leq 1 \iff 2y^2 + v^2 \leq x^2,$$

which guarantees that $\arccosh\left(1 - 2 \frac{y^2 + v^2}{x^2 + v^2}\right)$ is single valued in $[0, \frac{\pi}{2}]$.

Drawing $\mathbb{H} = \{x > 0\}$ as a single right half-axis in order to keep two directions for the $y$- and $v$-axes, this domain $\{2y^2 + v^2 < x^2\}$ looks like a "security cone" which will contain all subsequent Grauert tubes $\Omega_{\varepsilon}$.

Then by the relation:

$$\arccosh(t) = \sqrt{-1}\arccos t \quad (0 \leq t \leq 1),$$

we get from (4.1) in this subdomain $\{2y^2 + v^2 \leq x^2\}$ of $\mathbb{H}^c$:

$$f_c = -\left(\arccos\left(1 - 2 \frac{y^2 + v^2}{x^2 + v^2}\right)\right)^2,$$
hence coming back to (2.2):

\[(4.3) \quad \rho = \left( \arccos \left( 1 - 2 \frac{y^2 + v^2}{x^2 + y^2} \right) \right)^2. \]

**Lemma 4.4.** For every \(0 < \varepsilon < \left( \frac{\pi}{2} \right)^2\), the Grauert tube around \(\mathbb{H}\) in \(\mathbb{H}^c\) for the canonical Kähler potential associated with the Poincaré metric on \(\mathbb{H}\):

\[\Omega_{\varepsilon} := \{(u + \sqrt{-1} v, x + \sqrt{-1} y) \in \mathbb{H}^c: \sqrt{\rho}(u, v, x, y) < \sqrt{\varepsilon}\},\]

has \(C^\omega\) strongly pseudoconvex boundary \(\partial \Omega_{\varepsilon} = \{\rho = \varepsilon\}\) of equation:

\[2v^2 - \left(1 - \cos \sqrt{\varepsilon}\right) x^2 + \left(1 + \cos \sqrt{\varepsilon}\right) y^2 = 0.\]

**Proof.** Since the function \(\arccos\) is a decreasing \(C^\omega\) diffeomorphism \([0, 1) \rightarrow (0, \frac{\pi}{2}]\), we have:

\[\arccos \left( 1 - 2 \frac{y^2 + v^2}{x^2 + y^2} \right) < \sqrt{\varepsilon} \iff 1 - 2 \frac{y^2 + v^2}{x^2 + y^2} > \cos \sqrt{\varepsilon} \iff 2v^2 - \left(1 - \cos \sqrt{\varepsilon}\right) x^2 + \left(1 + \cos \sqrt{\varepsilon}\right) y^2 < 0.\]

Since \(x > 0\), the term \(2x dx\) in the differential \(dr_{\varepsilon}\) guarantees that \(\partial \Omega_{\varepsilon} = \{r_{\varepsilon} = 0\}\) is geometrically smooth at every point.

Furthermore, with \(w := u + \sqrt{-1} v\) and \(z := x + \sqrt{-1} y\), dropping pluriharmonic terms:

\[r_{\varepsilon} \equiv w\overline{w} - \left(1 - \cos \sqrt{\varepsilon}\right) \frac{x^2}{2} + \left(1 + \cos \sqrt{\varepsilon}\right) \frac{y^2}{2},\]

we see that \(r_{\varepsilon}\) is strictly plurisubharmonic, whence \(\Omega_{\varepsilon} = \{r_{\varepsilon} < 0\}\) is strongly pseudoconvex. \(\square\)

In particular, the result holds for thin tubes corresponding to \(0 < \varepsilon \ll \left( \frac{\pi}{2} \right)^2\).
5. Calculation of the Complex Cartan Curvature of $\partial \Omega_\varepsilon \subset \mathbb{H}^c$

In [7], the authors proved the non-existence of CR-umbilical points on the boundaries of Grauert tubes around flat tori by showing the nonvanishing of a certain invariant determinant introduced in [6], which vanishes exactly when the Cartan curvatures vanish. In this paper, we shall use an explicit expression of Cartan curvatures obtained before by the second named author and Sabzevari in [20, 21] for locally graphed hypersurfaces, and alternatively a formula in [10] for hypersurfaces given as zero locus of implicit functions.

For a $C^6$-smooth Levi-nondegenerate real 3-dimensional hypersurface $M \subset \mathbb{C}^2$ represented in complex coordinates $z = x + \sqrt{-1} y$, $w = u + \sqrt{-1} v$ by a local graphing function:

$$v = \varphi(x, y, u),$$

the Cartan essential curvatures of $M$ are two real invariants $\Delta_1, \Delta_4$ expressed in [20, Theorem 1.1] by following a Tanaka approach, explicitly in terms of $J^6_{x,y,u}\varphi$, both containing more than 1,500,000 terms when expanded.

An equivalent approach [21] closer to Cartan’s [4] can be summarized as follows. Local generators of $T^{1,0} M$ and $T^{0,1} M$ are:

$$\mathcal{L} := \frac{\partial}{\partial z} - \frac{\varphi_z}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u} \quad \text{and} \quad \overline{\mathcal{L}} := \frac{\partial}{\partial \overline{z}} - \frac{\varphi_{\overline{z}}}{-\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u},$$

and their commutator:

$$\mathcal{T} := \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}] = \ell \frac{\partial}{\partial u},$$

incorporates the real coefficient, so-called Levi factor:

$$\ell := 2 \frac{\varphi_{\overline{z}}(1 + \varphi_u^2) - \sqrt{-1} \varphi_{\overline{z}}\varphi_{2u} + \sqrt{-1} \varphi_{\overline{z}}\varphi_{u}}{(1 + \varphi_u^2)^2},$$

which is nowhere vanishing if and only if $M$ is Levi nondegenerate.

Abbreviating the coefficients of $\mathcal{L}$ and $\overline{\mathcal{L}}$ as:

$$A := -\frac{\varphi_z}{\sqrt{-1} + \varphi_u} \quad \text{and} \quad \overline{A} := -\frac{\varphi_{\overline{z}}}{-\sqrt{-1} + \varphi_u},$$

then in terms of the following key function (the expansion of which is 1 page long):

$$\mathcal{P} := \frac{\ell \overline{\mathcal{A}}_u + \overline{A} \ell_u}{\ell},$$
the (single) essential Cartan complex invariant expresses in non-expanded form as:

\[
J := \frac{1}{6} \frac{1}{c^3} \left( -2 \mathcal{L} \left( \mathcal{L} \left( \mathcal{P} \right) \right) + 3 \mathcal{L} \left( \mathcal{L} \left( \mathcal{P} \right) \right) - 7 \mathcal{P} \mathcal{L} \left( \mathcal{P} \right) + 4 \mathcal{P} \mathcal{L} \left( \mathcal{P} \right) - \mathcal{L} \left( \mathcal{P} \right) 2 \mathcal{P} \mathcal{L} \left( \mathcal{P} \right) \right),
\]

and a comparison with [20] done at the end of [21] shows that it also expresses as:

\[
J = \frac{4}{c^3} \left( \Delta_1 + \sqrt{-1} \Delta_4 \right),
\]

where the quantity \( c \in \mathbb{C} \backslash \{0\} \) is a group parameter of a certain initial \( G \)-structure, and it has the following signification.

Suppose there really is a local biholomorphic equivalence \( h: \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) which transfers \( M \) into \( M' := h(M) \), so that in some appropriate target coordinates \( z' = x' + \sqrt{-1} y', w' = u' + \sqrt{-1} v' \), the (localized) image is also graphed as:

\[
v' = \varphi' \left( x', y', u' \right).
\]

Compute similarly \( \mathcal{L}', \mathcal{P}', \ell', \mathcal{P}', \mathcal{I}' \), but extract parts independent of group parameters:

\[
J = \frac{1}{c^3} J_* \quad \text{and} \quad J' = \frac{1}{c'^3} J'_*.
\]

Because the differential \( h_*: T\mathbb{C}^2 \rightarrow T\mathbb{C}^2 \) leaves invariant complex tangents, whence \( h_* (\mathcal{T}^{1,0} M) = \mathcal{T}^{1,0} M' \), there is a nowhere vanishing function \( c': M' \rightarrow \mathbb{C} \backslash \{0\} \) such that:

\[
h_* (\mathcal{L}) = c' \mathcal{L}'.
\]

At a basic level, it is an easy exercise ([19, p. 44]) to express the invariancy of the levi factors \( \ell \) and \( \ell' \) through the biholomorphism \( h \) as:

\[
\ell = c' c \ell',
\]

and at a higher level, a standard feature of Cartan’s method of equivalence then shows that:

\[
J_* = c' c^3 J',
\]

which justifies, since \( c' \neq 0 \) vanishes nowhere, the invariancy, under changes of holomorphic coordinates, of the following

**Définition 5.2.** A point \( p \in M \) at which \( J(p) = 0 \) is called a CR-umbilical point.
In continuation with Lemma 4.4 above, we are now ready to state and to establish the main proposition. Inside the complexification of Poincaré’s upper half-plane:
\[ H^c = \{(u + \sqrt{-1} v, x + \sqrt{-1} y) \in \mathbb{C}^2 : x > 0\}, \]
consider for every \(0 < \varepsilon < \left(\frac{\pi}{2}\right)^2\) the hypersurface:
\[ M_\varepsilon := \partial \Omega_\varepsilon = \{(u + \sqrt{-1} v, x + \sqrt{-1} y) \in H^c : v^2 - \frac{\cos \sqrt{\varepsilon}}{1 + \cos \sqrt{\varepsilon}} x^2 + \frac{1 + \cos \sqrt{\varepsilon}}{2} y^2 = 0\} . \]

**Proposition 5.3.** All hypersurfaces \(M_\varepsilon \subset H^c\) with \(0 < \varepsilon < \left(\frac{\pi}{2}\right)^2\) have no CR-umbilical point.

**Proof.** The plain global linear biholomorphism of \(H^c\):
\[ w' := w, \quad z' := z \sqrt{\frac{1 + \cos \sqrt{\varepsilon}}{2}}, \]
transforms \(M_\varepsilon\) into:
\[ M'_\varepsilon := \left\{(u' + \sqrt{-1} v', x' + \sqrt{-1} y') \in H^c : v'^2 - \frac{\cos \sqrt{\varepsilon}}{1 + \cos \sqrt{\varepsilon}} x'^2 + y'^2 = 0\right\}, \]
and it is appropriate to set — mind the change \(\varepsilon \mapsto \epsilon\) —:
\[ \epsilon := \sqrt{\frac{1 - \cos \sqrt{\varepsilon}}{1 + \cos \sqrt{\varepsilon}}}, \]
so that the equation of \(M'_\varepsilon := M'_\epsilon\) becomes a bit simpler (dropping the primes):
\[ v^2 - \epsilon^2 x^2 + y^2 = 0. \]

Since this fractional map \(\varepsilon \mapsto \epsilon(\varepsilon)\) has derivative:
\[ \frac{d}{d\varepsilon} \sqrt{\frac{1 - \cos \sqrt{\varepsilon}}{1 + \cos \sqrt{\varepsilon}}} = \frac{1}{2\sqrt{\varepsilon}} \frac{\sin(\sqrt{\varepsilon})}{\sqrt{\frac{1 - \cos \sqrt{\varepsilon}}{1 + \cos \sqrt{\varepsilon}} (1 + \cos \sqrt{\varepsilon})^2}} \]
everywhere positive, it is a \(C^\omega\) diffeomorphism \((0, \frac{\pi^2}{4}) \rightarrow (0, 1)\), so that the new \(\epsilon\) varies plainly in the open unit real segment:
\[ 0 < \epsilon < 1. \]

Reminding that \(x > 0\), this new equation:
\[ y^2 + v^2 = \epsilon^2 x^2, \]
shows that, a bit similarly as for the flat torus in Example 3.3, either \(v \neq 0\) or \(y \neq 0\) at any point.

Suppose therefore firstly that \(v \neq 0\). For the \(C^\omega\) graph:
\[ v = \sqrt{\epsilon^2 x^2 - y^2}, \]
a direct calculation of $I$ from the formula (5.1), by hand or with help of a computer, provides a compact, serendipitous expression:

$$I = -\frac{9}{16} \frac{1 - \epsilon^4}{(\epsilon^2 x^2 - y^2)^2} \left(\frac{x + \sqrt{-1} y}{x - \sqrt{-1} y}\right)^2,$$

which visibly vanishes nowhere since $x > 0$ whence $(x + \sqrt{-1} y)^4 \neq 0$.

Suppose secondly that $y \neq 0$. Since only points with $v = 0$ are not already examined, assume $v = 0$. For the $C^\infty$ graph:

$$y = \sqrt{\epsilon^2 x^2 - v^2},$$

at points with $v = 0$, another direct calculation of the invariant $I$ from (5.1) also provides a compact, nowhere vanishing expression:

$$I = \frac{9}{16} \frac{(1 - \epsilon^2)}{(\epsilon + \sqrt{-1})^2 \epsilon^4 x^4},$$

and this completes the proof of inexistence of CR-umbilical points on $M_\epsilon$.

\[ \square \]

Second proof of Proposition 5.3 The formula (5.1), explicit as it is, usually gives long and complicated expression for the combined complex-valued Cartan invariant $J$. This reality is due to the iterated process of taking roots, derivatives, quotients, etc. when the graphing function of the hypersurfaces under consideration is not simple, including taking roots for example (see, for example, the formulas given in [21] and [9]). There are instances where the hypersurfaces actually have much simpler representation by mean of implicit functions. An example is the case of general ellipsoidal hypersurfaces in $\mathbb{C}^2$ considered in [10], where a direct calculation from the formula (5.1) for a graphing function of the ellipsoids gives a very complicated expression for $J$, while an alternative formula (cf. [10, Corollary 12]) applied to simple implicit defining functions of the ellipsoids allows one to see a whole curve of CR-umbilical points. As the implicit defining function of $M_\epsilon$ is also very simple, we shall use the formulation in [10] to verify the nonvanishing of the Cartan curvature of $M_\epsilon$ once again.

Let us recall the necessary formulas from [10]. For a Levi nondegenerate analytic hypersurface $M$ in $\mathbb{C}^2$ given by an implicit defining function:

$$0 = F(z, w, \bar{z}, \bar{w}),$$
we set
\[ L := -F_w \frac{\partial}{\partial z} + F_z \frac{\partial}{\partial w}, \]
\[ \bar{L} := -F_w \frac{\partial}{\partial \bar{z}} + F_z \frac{\partial}{\partial \bar{w}}. \]
\[ h(F) := F_z F_z F_{ww} - 2F_z F_w F_{zw} + F_w F_w F_{zz}, \]
\[ l(F) := F_z F_z F_{w\bar{w}} - F_z F_w F_{z\bar{w}} - F_w F_z F_{z\bar{w}} + F_w F_w F_{\bar{z}\bar{z}}. \]

**Theorem 5.4.** (10) On the domain \( \{ F_w \neq 0 \} \), the Cartan invariant \( \mathcal{I} \) of \( M \) vanishes exactly on the zero locus of

\[ I_{[w]} := 12 (F_w)^9 \left( \sum_{i=1}^{7} I_i \right), \]

where
\[ I_1 = \left( \frac{l(F)}{F_w} \right)^3 \cdot T^4 \left( \frac{h(F)}{F_w} \right), \]
\[ I_2 = -6 \left( \frac{l(F)}{F_w} \right)^2 \cdot T^3 \left( \frac{l(F)}{F_w} \right) \cdot T \left( \frac{h(F)}{F_w} \right), \]
\[ I_3 = -4 \left( \frac{l(F)}{F_w} \right)^2 \cdot T^2 \left( \frac{l(F)}{F_w} \right) \cdot T \left( \frac{h(F)}{F_w} \right), \]
\[ I_4 = - \left( \frac{l(F)}{F_w} \right)^2 \cdot T^3 \left( \frac{l(F)}{F_w} \right) \cdot T \left( \frac{h(F)}{F_w} \right), \]
\[ I_5 = 15 \left( \frac{l(F)}{F_w} \right) \cdot \left[ T \left( \frac{l(F)}{F_w} \right) \right]^2 \cdot T^2 \left( \frac{h(F)}{F_w} \right), \]
\[ I_6 = 10 \left( \frac{l(F)}{F_w} \right) \cdot T \left( \frac{l(F)}{F_w} \right) \cdot T^2 \left( \frac{l(F)}{F_w} \right) \cdot T \left( \frac{h(F)}{F_w} \right), \]
\[ I_7 = -15 \left[ T \left( \frac{l(F)}{F_w} \right) \right]^3 \cdot T \left( \frac{h(F)}{F_w} \right). \]

With this formula (5.5) for checking the nonvanishing of the Cartan curvature at hand, we now return to our hypersurface \( M_z \). We again take advantage of the elementary biholomorphic transformation as above, and consider the equivalent model \( M'_z \) whose defining function writes \( v^2 - \epsilon^2 x^2 + y^2 = 0 \), with \( 0 < \epsilon < 1 \). Switching the notation for coordinates in order to reach \( F_w \neq 0 \), namely using instead:

\[ z = u + \sqrt{-1} v \quad \text{and} \quad w = x + \sqrt{-1} y, \]

we can then rewrite:
\[ v^2 - \epsilon^2 x^2 + y^2 = \left( \frac{z - \bar{z}}{2\sqrt{-1}} \right)^2 - \epsilon^2 \left( \frac{w + \bar{w}}{2} \right)^2 + \left( \frac{w - \bar{w}}{2\sqrt{-1}} \right)^2 \]
\[ = \frac{-1}{4} \left[ (z - \bar{z})^2 + (1 + \epsilon^2)(w^2 + \bar{w}^2) - 2(1 - \epsilon^2) w\bar{w} \right] \]
\[ =: \frac{-1}{4} F(z, w, \bar{z}, \bar{w}), \]
so that \( M'_z = \{ F = 0 \} \), and then as wanted we have the nowhere vanishing:

\[ F_w = (1 + \epsilon^2) 2w - 2(1 - \epsilon^2) \bar{w} = 4 (\epsilon^2 x + \sqrt{-1} y) \neq 0. \]
on $M'$ thanks to our constant assumption $x > 0$. Thus, the vanishing locus of $\mathcal{I}_{[w]}$ is exactly the set of CR-umbilical points of $M'$ in this case.

Now, direct calculation from the formula (5.5), by hand or preferably on a computer, and keeping in mind that on $M'$ we always have $v^2 = \epsilon^2 x^2 - y^2$, gives us:

$$\mathcal{I}_{[w]} = \frac{27}{64} \epsilon^8 (1 - \epsilon^4) w^2 w^6$$

$$= \frac{27}{64} \epsilon^8 (1 - \epsilon^4)(x - \sqrt{-1}y)^2(x + \sqrt{-1}y)^6.$$

It is then evident that $\mathcal{I}_{[w]}$ is everywhere nonzero on $M'$ because $x > 0$. This completes our second justification of the inexistence of CR-umbilical points on $M_\epsilon \sim M'_\epsilon$.

**Proof. 6. Transfer to Hyperbolic Genus $g \geq 2$ Compact Surfaces**

Now, let $S$ be a closed compact oriented $C^\infty$ surface of genus $g \geq 2$, considered as a Riemann surface. The Poincaré-Köbe uniformization theorem provides a holomorphic covering:

$$\tau: \ \mathbb{H} \longrightarrow S \cong \mathbb{H} / \pi_1(S).$$

The Poincaré metric $ds^2_\mathbb{H} = \lambda (dx_1^2 + dx_2^2)$ with $\lambda := \frac{1}{x^2}$ on $\mathbb{H}$ has constant Gaussian curvature:

$$-\frac{1}{2} \lambda \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) (\log \lambda) = -1,$$

and is furthermore kept invariant by all elements of the group $\text{Aut} \mathbb{H} \cong PSL(2, \mathbb{R})$ of holomorphic automorphisms of $\mathbb{H}$:

$$\left( \text{Aut} \mathbb{H} \right)^* (ds^2_\mathbb{H}) = ds^2_\mathbb{H},$$

which acts transitively (and isometrically) on the homogeneous space $\mathbb{H}$.

Furthermore, the group of all covering automorphisms of $\mathbb{H} \to S$ happens to be a discrete subgroup:

$$\text{Aut} \left( \mathbb{H} \to S \right) \subset PSL(2, \mathbb{R}) = \text{Aut} \mathbb{H}.$$

Consequently (and as is well known), $ds^2_\mathbb{H}$ descends by push-forward, independently of preimage points, as a metric on $S$:

$$ds^2_S := \tau_* (ds^2_\mathbb{H}),$$

having the same curvature $-1$.

Next, forget the holomorphic structure on $S$, consider now $S$ as a $C^\infty$ real surface equipped with this $C^\infty$ metric $ds^2_S$, and denote the Bruhat-Whitney complexification of $S$ by $S^c$. Then Section 2 gives by complexification a
unique strictly plurisubharmonic $\mathcal{C}^\omega$ Kähler potential $\rho: S^c \to \mathbb{R}_+$ whose sublevel sets:

$$\Delta_\varepsilon := \{ \rho < \varepsilon \} \subset S^c,$$

for all small enough $0 < \varepsilon \leq \varepsilon_0 \ll 1$, are strongly pseudoconvex domains bounded by the $\mathcal{C}^\omega$ hypersurfaces:

$$\partial \Delta_\varepsilon = \{ \rho = \varepsilon \}.$$ 

Here, $\varepsilon_0$ might well be quite small, depending on the convergence radii of the real-analytic objects that are complexified.

**Lemma 6.1.** Shrinking $\varepsilon_0 > 0$ if necessary, $M_\varepsilon$ has no CR-umbilical point for all $0 < \varepsilon \leq \varepsilon_0$.

**Proof.** The uniformizing map, viewed as a $\mathcal{C}^\omega$ map $\tau: \mathbb{H} \to S$, also complexifies to become a holomorphic map:

$$\mathbb{H}^c \supset V^c \xrightarrow{\tau^c} U^c \subset S^c,$$

where $V^c$ is some open neighborhood of $\mathbb{H}$ in $\mathbb{H}^c$: $\mathbb{H} \subset V^c \subset \mathbb{H}^c$, possibly narrowing much as one reaches $\partial \mathbb{H} = \{ x_2 = 0 \}$, and where $U^c$ is also an open neighborhood of $S$ in $S^c$: $S \subset U^c \subset S^c$.

Since $\tau: \mathbb{H} \to S$ is a covering map, hence a local $\mathcal{C}^\omega$ diffeomorphism, each point $p \in S$ has a small open neighborhood $p \in U_p \subset S$ on which there exist $\mathcal{C}^\omega$-diffeomorphic inverses of $\tau$, namely maps:

$$\chi_p: U_p \xrightarrow{\sim} \chi_p(U_p) =: V_{\chi_p}(p) \subset \mathbb{H},$$

that are uniquely defined as soon as a central point $\chi_p(p) \in \tau^{-1}(p) \subset \mathbb{H}$ has been chosen in the fiber to fix a level. Shrinking $U_p$ if necessary, the complexification $\chi^c_p$ of $\chi_p(p)$ is also locally biholomorphic at $p$.

By compactness of $S \subset S^c$, there exists a finite open cover $U^c_1, \ldots, U^c_K \subset S^c$ of $S$:

$$S \subset U^c_1 \cup \cdots \cup U^c_K \subset U^c \quad (K \geq 1),$$

together with biholomorphic inverses of the complexification $\tau^c: V^c \to U^c$:

$$\chi_k^c: U_k^c \xrightarrow{\sim} \chi_k^c(U_k^c) =: V_k^c \subset \mathbb{H}^c \quad (1 \leq k \leq K).$$

If necessary, shrink $\varepsilon_0 > 0$ so that, for all $0 < \varepsilon \leq \varepsilon_0$:

$$\Delta_\varepsilon \subset \Delta_{\varepsilon_0} \subset U^c_1 \cup \cdots \cup U^c_K.$$

Now, take any point $q \in \partial \Delta_\varepsilon$. How to convince oneself that the Cartan CR-curvatures of the strongly pseudoconvex hypersurface $\partial \Delta_\varepsilon$ is nonzero at $q$?

This is very simple. For sure, $q \in U^c_k$ for some $1 \leq k \leq K$. Remind also the tube $\Omega_\varepsilon \subset \mathbb{H}^c$. Then because the metric on $S$ is the push-forward
of Poincaré’s metric on $\mathbb{H}$, the tubes $\Omega_\varepsilon$ and $\Delta_\varepsilon$ correspond to each other, namely $\chi_\varepsilon^k$ sends $\Delta_\varepsilon \cap U_k^\varepsilon$ biholomorphically onto $\Omega_\varepsilon \cap V_k^\varepsilon$ with:

$$\chi_\varepsilon^k(q) \in \partial \Omega_\varepsilon,$$

and since the nonvanishing of Cartan CR-curvatures is a biholomorphically invariant property, Proposition 5.3 offers what was wanted. $\square$

With some basic knowledge on Fuchsian groups, we can also provide a Variation on the proof of Lemma 6.1 As already seen, the quotient map:

$$\tau: \mathbb{H} \longrightarrow S \cong \mathbb{H}/\pi_1(S)$$

is locally isometric. Abbreviate:

$$G := \text{Aut} \left( \mathbb{H} \frac{\tau}{\tau(S)} \cong \pi_1(S) \right).$$

Définition 6.2. A fundamental domain for $S$ is an open subset $D \subset \mathbb{H}$ whose $G$-translates cover:

$$\mathbb{H} = \bigcup_{g \in G} g(D),$$

being mutually disjoint:

$$\emptyset = D \cap g(D) \quad (\forall g \in G \setminus \{\text{Id}\}),$$

and which has the further property of being locally finite in the sense that each compact subset $K \subset \mathbb{H}$ meets only finitely many $G$-images of $D$.

Theorem 6.3. ([1, Chap. 9]) Relatively compact fundamental domains $D \subset \mathbb{H}$ having piecewise $C^\omega$ boundary consisting of $4g$ geodesic segments always exist on the universal cover $\tau: \mathbb{H} \longrightarrow S$ of any genus $g \geq 2$ compact Riemann surface. $\square$

Then in place of a (rough) finite Borel-Lebesgue covering $S \subset U_1 \cup \cdots \cup U_k$ as used in the first proof, we can employ a geometrically more meaningful covering. For such a fundamental domain $D \subset \mathbb{H}$ of $S$, there is an atlas of $S$ consisting of $4g + 1$ open charts:

- $V_0 := D$ itself;
- slightly thickened thin neighborhoods $V_1, \ldots, V_{4g}$ of the $4g$ sides of $D$.

Further, one can arrange that the restrictions:

$$\tau: V_i \longrightarrow \tau(V_i) =: U_i \subset S \quad (i = 0, 1, \ldots, 4g)$$

are $C^\omega$ diffeomorphisms. Complexifying their inverses $\chi_i: U_i \sim \rightarrow V_i$ as:

$$\chi_i^\varepsilon: U_i^\varepsilon \sim \rightarrow V_i^\varepsilon$$

we can now reason similarly as in the first proof, and this concludes. $\square$
Remark 6.4. We observe the following interesting facts about the (non)vanishing of the essential curvatures $\Delta_1$ and $\Delta_4$ on the boundaries of Grauert tubes of small radii around closed surfaces $S$.

1. If $S$ is the 2-sphere with the standard round metric, both $\Delta_1$ and $\Delta_4$ vanish identically.

2. If $S$ is a 2-dimensional flat torus, we leave as an exercise to the reader to verify that $\Delta_1$ never vanishes, while $\Delta_4$ vanishes identically.

3. If $S$ is a closed genus $g \geq 2$ hyperbolic surface, then both $\Delta_1$ and $\Delta_4$ vanish nowhere.

7. Grauert Tubes with Respect to Extrinsic Metrics

In Section 2, Grauert tubes are constructed with respect to metrics obtained from given intrinsic Riemannian metrics on surfaces. In this section, we look at constructions of Grauert tubes around surfaces from an extrinsic point of view. More precisely, let us consider a totally real embedding of a surface $S$ into a complex manifold $X$ of complex dimension 2. We will identify the surface $S$ with its image under the embedding, so that $S$ is viewed as a submanifold of $X$. A given Riemannian metric $d_X$ on $X$ always induces an extrinsic metric on $S$ and the Grauert tubes $\Omega_\varepsilon$ around $S$ also can be defined with respect to $d_X$ as $\Omega_\varepsilon := \{ x \in X : d_X(x, S) < \varepsilon \}$ for small enough positive $\varepsilon$.

Recall that for a real $n$-dimensional submanifold $M$ of a complex $n$-dimensional manifold $X$, a point $p$ of $M$ is called a complex point if the tangent vector space of $M$ at $p$ contains at least one complex line with respect to the complex structure $J \in \text{End}(TX)$ on the tangent bundle of $X$, that is $T_pM \cap J(T_pM) \neq \{0\}$. An embedding of $M$ into $X$ is called a totally real embedding if $M$ does not contain any complex point.

It is known that every affine $n$-dimensional totally real vector subspace $V \subset \mathbb{C}^n$ is affinely holomorphically equivalent to $\mathbb{R}^n \subset \mathbb{C}^n$. It is also known that every $C^\omega$ real $n$-dimensional submanifold $M \subset \mathbb{C}^n$ is locally holomorphically equivalent to $\mathbb{R}^n \subset \mathbb{C}^n$, namely at any point $p \in M$, there is an open neighborhood $p \in U \subset \mathbb{C}^n$ and a biholomorphism $h: U \sim h(U) =: V$ with $h(p) = 0$ such that $h(M \cap U) = \mathbb{R}^n \cap V$. Hence an alternative description of maximally real $C^\omega$ submanifolds $M \subset \mathbb{C}^n$ is as follows.

Définition 7.1. A real $n$-dimensional $C^\omega$ submanifold $M$ of a complex $n$-dimensional manifold $X$ is totally real if there exists a family indexed by
\( \alpha \in A \) of biholomorphisms:

\[
\varphi_\alpha : \quad U_\alpha \xrightarrow{\sim} \varphi_\alpha (U_\alpha) =: V_\alpha \subset X
\]

with \( U_\alpha \subset \mathbb{C}^n \) open, with \( V_\alpha \subset X \) open, with \( X = \bigcup_\alpha V_\alpha \), such that:

- if \( \varphi_\alpha(0) \notin M \), then \( \varphi_\alpha(U_\alpha) \cap M = \emptyset \);
- if \( \varphi_\alpha(0) \in M \), then the restriction:

\[
\varphi_\alpha \big|_{\mathbb{R}^n \cap U_\alpha} : \quad \mathbb{R}^n \cap U_\alpha \xrightarrow{\sim} M \cap V_\alpha,
\]

is a \( \mathcal{C}^\infty \) real diffeomorphism.

**Example 7.2.** By looking at the standard complex atlas of the complex projective space \( \mathbb{C}\mathbb{P}^n \), it is clear that \( \mathbb{R}\mathbb{P}^n \) is totally real in \( \mathbb{C}\mathbb{P}^n \). On \( \mathbb{R}\mathbb{P}^n \), there is a canonical round metric induced from the round metric on its double cover \( S^n \). The Guillemin-Stenzel metric associated to this round metric on \( \mathbb{C}\mathbb{P}^n \) is nothing but the Fubini-Study metric on \( \mathbb{C}\mathbb{P}^n \). The complexified manifold \( (S^n)^c \) is a double cover of \( \mathbb{C}\mathbb{P}^n \), which is a real \( 2n \)-dimensional submanifold in the \( S^1 \)-fibration \( S^{2n+1} \) of \( \mathbb{C}\mathbb{P}^n \).

**Example 7.3.** Of particular interest for us here is the fact that a product of two totally real submanifolds is also totally real, which is evident from either definition.

**Example 7.4.** Let us look at Example 3.3 once again, this time from an extrinsic point of view. Consider a 2-dimensional real vector subspace \( V \) of \( \mathbb{C}^2 \) which passes through the origin, with coordinates \( (z, w) \in \mathbb{C}^2 \). The intersections of \( V \) with the \( z \)-axis and \( w \)-axis are two real line. Therefore \( V \) can be written in exactly one of the following three forms.

**Case 1:** \( V = \{ y = \alpha x, v = \beta u \} \), where \( \alpha, \beta \) are real. The Grauert tube \( \Omega_\varepsilon(V) \) of radius \( \varepsilon \) around \( V \) with respect to the standard distance in \( \mathbb{C}^2 \) is given by:

\[
\left\{ (x + \sqrt{-1} y, u + \sqrt{-1} v) \in \mathbb{C}^2 : \frac{(\alpha x - y)^2}{(\alpha^2 + 1)^2} + \frac{(\beta u - v)^2}{(\beta^2 + 1)^2} < \varepsilon^2 \right\}.
\]

In order to obtain a compact hypersurface, we take the quotient \( \tilde{\Omega}_\varepsilon(V) \) of \( \Omega_\varepsilon(V) \) by the translations by \( 2\pi \) on each real coordinates of \( V \). Then \( \tilde{\Omega}_\varepsilon(V) \) can be embedded into \( \mathbb{C}^2 \) as:

\[
\left\{ (z, w) \in \mathbb{C}^2 : \left( \log |e^{i \sqrt{-1} z} \right)^2 + \left( \log |e^{i \sqrt{-1} z} \right)^2 < \varepsilon^2 \right\}.
\]

Any point on the boundary of \( \tilde{\Omega}_\varepsilon(V) \) admits the same local defining function as its preimage on the boundary of \( \Omega_\varepsilon(V) \). Solving the local defining
function for the variable \( v \) gives the graph:

\[
v = \beta u - (\beta^2 + 1) \sqrt{\varepsilon^2 - \frac{(\alpha x - y)^2}{(\alpha^2 + 1)^2}}.
\]

A direct calculation of the Cartan invariant using the formula (5.1) provides:

\[
\tilde{J}_* = -9 \frac{(\alpha + \sqrt{-1})^9 (-\alpha + \sqrt{-1})^{11} (\beta + \sqrt{-1})^{16} (-\beta + \sqrt{-1})^{16} \varepsilon^8}{(-\alpha x + y + \varepsilon + a^2 \varepsilon)^9 (\alpha x - y + \varepsilon + a^2 \varepsilon)^8},
\]

and this result is nowhere vanishing. So the boundary of \( \tilde{\Omega}_\varepsilon(V) \) also does not contain any CR-umbilical point.

Case 2: \( V = \{x = 0, v = \beta u\} \), where \( \beta \) is again real. The Grauert tube of radius \( \varepsilon \) around \( V \) with respect to the standard distance in \( \mathbb{C}^2 \) is now given by:

\[
\Omega_\varepsilon(V) = \left\{(x + \sqrt{-1} y, u + \sqrt{-1} v) \in \mathbb{C}^2 : x^2 + \frac{(\beta u - v)^2}{(\beta^2 + 1)^2} < \varepsilon^2\right\}.
\]

A point on the boundary of \( \tilde{\Omega}_\varepsilon(V) \) or of \( \Omega_\varepsilon(V) \) admits the local graphing function:

\[
v = \beta u - (\beta^2 + 1) \sqrt{\varepsilon^2 - x^2},
\]

of which the (relative) Cartan curvature can be computed from the formula (5.1) to be:

\[
\tilde{J}_* = 9 \frac{(\beta^2 + 1)^{16} \varepsilon^8}{(x^2 - \varepsilon^2)^8}.
\]

Thus, the (relative) invariant \( \tilde{J}_* \) is also nowhere vanishing on the boundary.

Note that \( \tilde{\Omega}_\varepsilon(V) \) can be embedded into \( \mathbb{C}^2 \) as:

\[
\left\{(z, w) \in \mathbb{C}^2 : \left(\log |e^z|\right)^2 + \left(\log |e^{\sqrt{-1} w}|\right)^2 < \varepsilon^2\right\}.
\]

Case 3: \( V = \{x = 0 = u\} \). A point on \( \tilde{\Omega}_\varepsilon(V) \) which can be embedded into \( \mathbb{C}^2 \) as:

\[
\left\{(z, w) \in \mathbb{C}^2 : \left(\log |e^z|\right)^2 + \left(\log |e^w|\right)^2 < \varepsilon^2\right\},
\]

now admits the local defining function

\[
x^2 + u^2 = \varepsilon^2.
\]

In this case, we do not obtain a local graphing function of the form \( v = \phi(x, y, u) \), but a simple calculation using the alternative formula (5.5) for the implicit defining function \( F(z, w, \bar{z}, \bar{w}) = (\frac{z + \bar{z}}{2})^2 + (\frac{w + \bar{w}}{2})^2 - \varepsilon^2 \) shows that the relative invariant \( \tilde{J}_* \) is proportional to:

\[
\frac{27 (x^2 + u^2)^4}{64} = \frac{27 \varepsilon^8}{64}.
\]
So it is evident that the boundary of $\tilde{\Omega}_z(V)$ also does not contain any CR-umbilical point.

For two given Riemannian manifolds $(X,d_X),(Y,d_Y)$, the distance $d_{X \times Y}$ with respect to the product metric on $X \times Y$ is:

$$d_{X \times Y}((x_1,y_1),(x_2,y_2)) = d_X^2(x_1,x_2) + d_Y^2(y_1,y_2),$$

assuming that $X,Y$ are uniquely geodesic, i.e. there exists a unique geodesic between any two points.

Our next examples of Grauert tubes in $\mathbb{C} \times \mathbb{C}$ will be constructed with respect to products of two extrinsic metrics on $\mathbb{C} \supset \mathbb{R}$. For the two possible component metrics on $\mathbb{C}$, we will consider the three standard ones: flat, elliptic and hyperbolic.

- Flat metric on $\mathbb{C}$. Denote by $d_{\text{Flat}}$ the flat Pythagorean metric on $\mathbb{C} \ni x + \sqrt{-1}y$. Consider the totally real line $V_{\text{Flat}} = \{y = 0\}$ in $U_{\text{Flat}} = \mathbb{C}$. The flat distance from any point $z \in U_{\text{Flat}}$ to $V_{\text{Flat}}$ is:

$$d_{\text{Flat}}(z, V_{\text{Flat}}) = |\text{Im}(z)| = |y|.$$  

- Elliptic metric on $\mathbb{CP}^1$. For the elliptic metric $d_{\text{Ell}}$, we look at the local chart $U_0 = \{[1 : z] : z \in \mathbb{C}\}$ of $\mathbb{CP}^1$. Since $(\mathbb{CP}^1,d_{\text{Ell}})$ is not uniquely geodesic, we consider a small neighborhood $U_{\text{Ell}} = \{[1 : z] : |z| < \delta\}$ of $[1 : 0]$ in $U_0$, which is uniquely geodesic for small positive $\delta$ thanks to the fact that the injective radius of $(\mathbb{CP}^1,d_{\text{Ell}})$ is positive. Then $V_{\text{Ell}} = \{[1 : \text{Re}(z)] : \text{Re}(z) < \delta\}$ is totally real in $U_{\text{Ell}}$.

**Lemma 7.7.** The elliptic distance from any point $(x,y) \approx [1 : (x + \sqrt{-1}y)]$ of $U_{\text{Ell}}$ to $V_{\text{Ell}}$ is given by:

$$d_{\text{Ell}}((x,y), V_{\text{Ell}}) = \arccos\left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2+y^2}}\right).$$

**Proof.** A point $[1 : (x + \sqrt{-1}y)]$ of $\mathbb{CP}^1$ corresponds to the point $(\frac{x}{\sqrt{1+x^2+y^2}}, \frac{y}{\sqrt{1+x^2+y^2}}, \frac{1}{\sqrt{1+x^2+y^2}})$ of $S^2$ embedded in $\mathbb{R}^3$, and a point $[1 : \alpha]$ of $V_{\text{Ell}}$ corresponds to $(\frac{1}{\sqrt{1+\alpha^2}}, \frac{\alpha}{\sqrt{1+\alpha^2}}, 0)$.

Now $d_{\text{Ell}}((x,y), V_{\text{Ell}})$ is exactly the spherical distance between $P = (\frac{1}{\sqrt{1+x^2+y^2}}, \frac{x}{\sqrt{1+x^2+y^2}}, \frac{y}{\sqrt{1+x^2+y^2}})$ and the arc $\{Q_\alpha = (\frac{1}{\sqrt{1+\alpha^2}}, \frac{\alpha}{\sqrt{1+\alpha^2}}, 0) : \alpha > 0\}$, that is:

$$\cos d_{\text{Ell}}((x,y), V_{\text{Ell}}) = \max_{\alpha>0} \frac{(P, Q_\alpha)}{|P| |Q_\alpha|}.$$
Using the Cauchy-Schwartz inequality, we have:
\[
\langle P, Q_\alpha \rangle = \frac{1 + \alpha x}{\sqrt{1 + \alpha^2 \sqrt{1 + x^2 + y^2}}} \leq \frac{\sqrt{1 + x^2}}{\sqrt{1 + x^2 + y^2}},
\]
where the maximum is attained at \( \alpha = x \).

- Hyperbolic metric on \( \mathbb{H} \). For the hyperbolic metric \( d_{\text{Hyp}} \), we may consider a small open neighborhood \( U \) of 0 in the Poincaré disc, and the totally real interval \( U \cap \{ \text{Im}(z) = 0 \} \) in \( U \), but it is more convenient to work with the corresponding domain \( U_{\text{Hyp}} = \{ z = x + \sqrt{-1} y \} \) of \( U \) on the upper-half plane model, which is an open neighborhood of \( \sqrt{-1} \). The corresponding totally real interval in \( U_{\text{Hyp}} \) is \( V_{\text{Hyp}} = U_{\text{Hyp}} \cap \{ \text{Re}(z) = 0 \} \).

**Lemma 7.9.** The hyperbolic distance from any point \((x, y) \approx z = x + \sqrt{-1} y\) in \( U_{\text{Hyp}} \) to \( V_{\text{Hyp}} \) is given by:

\[(7.10) \quad d_{\text{Hyp}}((x, y), V_{\text{Hyp}}) = \text{arccosh} \left( \frac{\sqrt{x^2 + y^2}}{y} \right).\]

**Proof.** Recall that for a hyperbolic triangle on the upper-half plane with angles \( A, B, C \) and opposite sides of lengths \( a, b, c \), the rule of sine reads:

\[
\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}.
\]

Thus, given the angle \( A \) and the side \( a \), the side \( b \) is of maximal length when \( B = \frac{\pi}{2} \) because the function sinh is monotone and because:

\[
\sinh b = \sinh B \frac{\sinh a}{\sinh A} \leq \frac{\sinh a}{\sinh A}.
\]

It follows that to find the hyperbolic distance from a given point \( z = x + \sqrt{-1} y \) to the line \( V_{\text{Hyp}} \), we look at the geodesic line passing through \( z \) and orthogonal to \( V_{\text{Hyp}} \), which is the half-circle on the upper-half plane model with centre at 0 and of radius \( |z| = \sqrt{x^2 + y^2} \). This geodesic line intersects \( V_{\text{Hyp}} \) at the point \((0, \sqrt{x^2 + y^2}) \approx 0 + \sqrt{-1} \sqrt{x^2 + y^2} \). Thus, we
have:
\[ d_{\text{Hyp}}((x, y), V_{\text{Hyp}}) = d_{\text{Hyp}}((x, y), (0, \sqrt{x^2 + y^2})) = \arccosh\left(1 + \frac{(x - 0)^2 + (y - \sqrt{x^2 + y^2})^2}{2y\sqrt{x^2 + y^2}}\right) = \arccosh\left(\frac{\sqrt{x^2 + y^2}}{y}\right). \]

We are now in position to give some non-trivial examples of Grauert tubes with respect to extrinsic metrics.

**Proposition 7.11.** The Grauert tubes of radius \(\varepsilon\) with respect to the product metric \(d_1 \times d_2\) around the totally real submanifold \(V_1 \times V_2\) in \(U_1 \times U_2\) admit local defining functions:
\[
\rho(x, y, u, v) := \left[d_1((x, y), V_1)\right]^2 + \left[d_2((u, v), V_2)\right]^2 < \varepsilon^2,
\]
where \((U_i, V_i, d_i)\) for \(i = 1, 2\) is one of the three models: \((U_{\text{Flat}}, V_{\text{Flat}}, d_{\text{Flat}})\), \((U_{\text{Ell}}, V_{\text{Ell}}, d_{\text{Ell}})\), \((U_{\text{Hyp}}, V_{\text{Hyp}}, d_{\text{Hyp}})\).

In particular, we obtain six examples of Grauert tubes with respect to the corresponding extrinsic product metrics.

**Remark 7.12.** Notice here that our examples are of local nature, and not compact. When both \(d_1\) and \(d_2\) are flat metrics, one recovers the local graphing function of the flat torus as in Example 3.3, since:
\[
\left[d_{\text{Flat}}((x, y), V_{\text{Flat}})\right]^2 + \left[d_{\text{Flat}}((u, v), V_{\text{Flat}})\right]^2 = y^2 + v^2.
\]
However, the remaining five examples are very different from those obtained from intrinsic metrics in Example 3.1, Example 3.3 and Lemma 4.4. Thus, the Grauert tubes around the same totally real manifolds with respect to intrinsic and extrinsic metrics look very different.

**Lemma 7.13.** In terms of:
\[
H := \sqrt{\varepsilon^2 - \left[\arccosh\frac{\sqrt{x^2 + y^2}}{y}\right]^2}
\]
and of:
\[
E := \sqrt{\varepsilon^2 - \left[\arccos\frac{\sqrt{1 + x^2}}{\sqrt{1 + x^2 + y^2}}\right]^2},
\]
the local defining functions for the boundaries of the Grauert tubes of radius \(\varepsilon\) with respect to the product metrics are given by Table 1.
TABLE 1.

| Product metrics | Defining functions |
|------------------|---------------------|
| $d_{\text{Flat}} \oplus d_{\text{Flat}}$ | $v = \sqrt{\varepsilon^2 - y^2}$ |
| $d_{\text{Ell}} \oplus d_{\text{Flat}}$ | $v = (\varepsilon^2 - \arcsin \frac{y}{\sqrt{1+x^2+y^2}})^{1/2}$ |
| $d_{\text{Hyp}} \oplus d_{\text{Flat}}$ | $v = (\varepsilon^2 - \arcsinh \frac{y}{y})^{1/2}$ |
| $d_{\text{Hyp}} \oplus d_{\text{Hyp}}$ | $v = \frac{u}{\sinh H}$ |
| $d_{\text{Ell}} \oplus d_{\text{Hyp}}$ | $v = \frac{u}{\sinh E}$ |
| $d_{\text{Ell}} \oplus d_{\text{Ell}}$ | $v = \frac{1+\sqrt{1-4(1+u^2)(\sin E)^2}}{2 \sin E}$ |

**Proof.** We only treat the case of the product between the hyperbolic and flat metrics, in which the local graphing function is given by:

(7.14) $\rho(x, y, u, v) = \left[\arccosh \left(\frac{\sqrt{x^2 + y^2}}{y}\right)\right]^2 + v^2 < \varepsilon^2,$

while the calculations for the other cases can be done in a similar way.

The defining function for the boundary of the Grauert tube is obtained by solving the equation $\rho = \varepsilon^2$ for the variable $v$ as follows:

$\rho = \varepsilon^2 \implies \arccosh \left(\frac{\sqrt{x^2 + y^2}}{y}\right) = \sqrt{\varepsilon^2 - v^2}$

$\implies \frac{\sqrt{x^2 + y^2}}{y} = \cosh \left(\sqrt{\varepsilon^2 - v^2}\right)$

$\implies 1 + \frac{x^2}{y^2} = \left[\cosh \left(\sqrt{\varepsilon^2 - v^2}\right)\right]^2 = \left[\sinh \left(\sqrt{\varepsilon^2 - v^2}\right)\right]^2 + 1$

$\implies \frac{x}{y} = \sinh \left(\sqrt{\varepsilon^2 - v^2}\right)$.

So, the defining function belongs to the rigid case with the graph:

$v = \sqrt{\varepsilon^2 - \arcsinh \left(\frac{z}{y}\right)}$. \hspace{1cm} \square$

Unfortunately, except for the case of $d_{\text{Flat}} \oplus d_{\text{Flat}}$, the expressions of the Cartan invariant obtained by calculations with either formula (5.1) or (5.5), though explicit, are overwhelmingly complicated, and so do not allows us to see the CR-umbilical locii.
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