Constructing a quadrilateral inside another one

1 The quadrilateral ratio problem

The description of Project 54 in 101 Project Ideas for the Geometer’s Sketchpad [Key] reads (in part):

On the Units panel of Preferences, set Scalar Precision to hundredths. Construct a generic quadrilateral and the midpoints of the sides. Connect each vertex to the midpoint of an opposite side in consecutive order to form an inner quadrilateral. Measure the areas of the inner and original quadrilateral and calculate the ratio of these areas. What conjecture are you tempted to make? Change Scalar Precision to thousandths and drag until you find counterexamples.

A figure similar to the following accompanies the project.
Let $r$ be the ratio of the quadrilateral areas,

$$r = \frac{\text{area}(EFGH)}{\text{area}(ABCD)}.$$  \hspace{2cm} (1)

The tempting conjecture is that $r = 1/5$. In Theorem 1 we show that this is true in case the original quadrilateral is a parallelogram. However, the conjecture is false in general. Instead, the ratio can be any real number in the interval $(1/6, 1/5]$. This is our Corollary 3. (Since $1/6 < 0.17 < 1/5$, it is possible to find counterexamples even with the Sketchpad Scalar Precision set to hundredths; however, it takes very industrious dragging to find them.)

Suppose now that instead of a quadrilateral we had a triangle. Of course, joining each vertex to the opposite midpoint would not yield an inner triangle, since the three lines are medians, which are concurrent in a point. To look for an analogous result for a triangle, we can look for points which are not midpoints, but rather divide each side a ratio $\rho$ of the distance from one point to the next, $0 < \rho < 1$. For definiteness, we assume that “next point” in this definition is based on movement in a counterclockwise direction. We call these points $\rho$-points. It turns out that for a given $\rho$, the ratio of the area of the inner triangle to the area of the outer triangle is constant independent of the initial triangle and is given by $\frac{(2\rho-1)^2}{\rho^2-\rho+1}$. Note that when $\rho = 1/2$, this reduces to 0, which provides a convoluted proof that the medians of a triangle are concurrent. When $\rho = 1/3$, the area ratio is $1/7$, which the Nobel Prize-winning physicist Richard Feynman once proved, though he was probably not the first to do so. The result for general $\rho$ is known, and a proof is given in [DeV], along with the Feynman story.
Inspired by this result, we will study the quadrilateral question for $\rho$-points. Let $ABCD$ be a convex quadrilateral and let $N_1, N_2, N_3,$ and $N_4$ be chosen so that $N_1$ is the $\rho$-point of $BC$, $N_2$ is the $\rho$-point of $CD$, $N_3$ is the $\rho$-point of $DA$, and $N_4$ is the $\rho$-point of $AB$. For fixed $\rho$ ($0 < \rho < 1$) connect each vertex of $ABCD$ to the $\rho$-point of the next side. (A to $N_1$, B to $N_2$, C to $N_3$, and D to $N_4$.) The intersections of the four line segments form the vertices of a convex quadrilateral $EFGH$.

Define the area ratio

$$r(\rho, ABCD) = \frac{\text{Area}(EFGH)}{\text{Area}(ABCD)}.$$ 

Theorem 2 below states that as $ABCD$ varies, the values of $r(\rho, ABCD)$ fill the interval

$$(m, M] := \left( \frac{(1-\rho)^3}{\rho^2 - \rho + 1}, \frac{(1-\rho)^2}{\rho^2 + 1} \right)$$

and that it is possible to give an explicit characterization of the set of convex quadrilaterals with maximal ratio $M$. The fact that $M - m$ has a maximum value of about .034 and is usually much smaller explains the near constancy of $r(\rho, ABCD)$ as $ABCD$ varies. Here are the graphs of $M$ and $m$. 
A more delicate look at the graph of $M - m = \rho^3(\rho - 1)^2 / (\rho^2 + 1)(\rho^2 - \rho + 1)$ shows that as “constant” as $r$ is in the original $\rho = 1/2$ case, it is even “more constant” when $\rho$ is close to the endpoints 0 and 1. (Actually the maximum value of $M - m$ of about .034 is achieved at the unique real zero of $\rho^5 - \rho^4 + 6\rho^3 - 6\rho^2 + 7\rho - 3$ which is about .55.)

The characterization proved in Theorem 2 below shows that not only do parallelograms have maximal ratio $M(\rho)$ for every $\rho$, but also they are the only quadrilaterals that have maximal ratio $M(\rho)$ for more than one $\rho$.

2 The midpoint case for parallelograms

**Theorem 1** If each vertex of a parallelogram is joined to the midpoint of an opposite side in clockwise order to form an inner quadrilateral, then the area of the inner quadrilateral is one fifth the area of the original parallelogram.

**Proof.** In this picture,
$ABCD$ is a parallelogram, and each $M_i$ is a midpoint of the line segment it lies on. Cut apart the figure along all lines. Then by rotating clockwise $180^\circ$ about point $M_3$, the reader can verify that we get $AHGG'$ (where $G'$ is the image of $G$ under the rotation) congruent to $EFGH$. Similarly, each of the triangles $ABE$, $BCF$, and $CDG$ may be dissected and rearranged to form a parallelogram, each congruent to $EFGH$. Thus, the pieces of $ABCD$ can be rearranged into five congruent parallelograms, one of which is $EFGH$, which therefore has area $1/5$ the area of $ABCD$.

This result is a special case of Corollary 4 below, but is included because of the elegant and elementary nature of its proof.

3 The filling of $(m, M]$ and the characterization

**Theorem 2** Let $A, B, C, D$ be (counterclockwise) successive vertices of a convex quadrilateral. Define $EFGH$ as the inner quadrilateral formed by joining vertices to $\rho$-points as described in Section 1. Construct point $P$ so that $ABCP$ is a parallelogram. Locate (as in the following figure) $C'$ and $C''$ on $BC$ so that $C'$ is a distance $\rho BC$ from $C$ and $C''$ is a distance $(1/\rho)BC$ from $C$ and let $S = \overrightarrow{C'P} \cup \overrightarrow{C''P}$. Then the ratio $r$ defined by equation (7) is maximal exactly when $D$ is on $S^* = S \cap \text{int}(\angle ABC) \cap \text{ext}(\triangle ABC)$. Furthermore the set of possible ratios is

$$\frac{(1 - \rho)^3}{\rho^2 - \rho + 1} \cdot \frac{(1 - \rho)^2}{\rho^2 + 1}.$$

In the figure below, $S^*$ is indicated by the thickened portions of the rays composing $S$. 


Proof. Fix $\rho$ and apply an transformation that maps $A, B, C, D$ successively to $(0, 1), (0, 0), (1, 0), (x, y)$. Since an affine transformation preserves both linear length ratios and area ratios, it is enough to prove the theorem after the transformation has been applied. Observe that $P$ has become $(1, 1)$, and the image of $S$ has become a pair of perpendicular rays through $(1, 1)$ with slopes $\rho$ and $-1/\rho$. Here is the situation.

The line from $(0, 0)$ to $(x, y)$ divides the outer quadrilateral into two triangles, one of area $x/2$ and the other of area $y/2$, so that its area is $(x + y)/2$. To find the area of the inner quadrilateral, we first determine $\{r_1, s_1, \ldots, s_4\}$ in terms of $x, y$ and $\rho$.
by equating slopes. For example, the equations

\[
\begin{align*}
\frac{s_1 - 0}{r_1 - 0} &= \frac{\rho y - 0}{1 + \rho (x - 1) - 0} \\
\frac{s_1 - 0}{r_1 - \rho} &= \frac{0 - \rho}{0 - \rho}
\end{align*}
\]

can easily be solved for \( r_1 \) and \( s_1 \). The area of the interior quadrilateral is

\[
\frac{(r_1 s_2 - r_2 s_1) + (r_2 s_3 - r_3 s_2) + (r_3 s_4 - r_4 s_3) + (r_5 s_1 - r_1 s_5)}{2}
\]

This is the \( n = 4 \) case of a well-known formula for the area of an \( n \)-gon\[Bra\], which can be proved by first proving the formula for triangles and then using induction, or by using Green’s Theorem. Some computer algebra produces this formidable and seemingly intractable formula for \( r(x, y) \).

\[
(r - 1)^2 \left( \begin{array}{c}
\rho^4 y^4 - \rho^3 y^4 - 3 \rho^5 x y^3 + 2 \rho^4 x y^3 + \rho^3 x y^3 - 2 \rho^2 x y^3 \\
+2 \rho^5 y^3 - 6 \rho^4 y^3 + 3 \rho^3 y^3 + 2 \rho^2 y^3 - \rho y^3 + \rho^3 x^2 y^2 \\
- \rho^5 x^2 y^2 - 6 \rho^4 x^2 y^2 + 4 \rho^3 x^2 y^2 - \rho^2 x^2 y^2 - \rho x^2 y^2 \\
-3 \rho^5 x y^2 + 10 \rho^4 x y^2 + 3 \rho^3 x y^2 - 13 \rho^3 x y^2 + 5 \\
* \rho^4 x y^2 + \rho x y^2 - \rho x^3 y - \rho^6 x^2 y - 7 \rho^5 x^2 y + 6 \rho^4 x^2 y + 7 \\
* \rho^4 y^2 - 7 \rho^2 y^2 + 2 \rho y^2 + 2 \rho^5 x^3 y - 2 \rho^4 x^3 y - 3 \rho^3 x^3 y \\
* x^3 y + 2 \rho^2 x^3 y - \rho x^3 y - 3 \rho^6 x^2 y - 5 \rho^5 x^2 y + 15 \rho^4 \\
* x^2 y - \rho^3 x^2 y - 7 \rho^2 x^2 y + 4 \rho x^2 y - x^2 y + 5 \rho^6 x y \\
-3 \rho^5 x y - 21 \rho^4 x y + 18 \rho^3 x y - \rho^2 x y - 3 \rho x y + x \\
* y - 2 \rho^6 y + 5 \rho^5 y + 4 \rho^4 y - 12 \rho^3 y + 6 \rho^2 y - \rho y \\
+ \rho^4 x^4 - \rho^3 x^4 - 3 \rho^5 x^3 - 2 \rho^4 x^3 + 5 \rho^3 x^3 - 2 \rho^2 x^3 \\
+ \rho^2 x^2 + 8 \rho^5 x^2 - 6 \rho^4 x^2 - 5 \rho^3 x^2 + 5 \rho^2 x^2 - \rho x^2 \\
-2 \rho^6 x - 5 \rho^5 x + 12 \rho^4 x - 4 \rho^3 x - 2 \rho^2 x + \rho x + \rho^6 \\
-4 \rho^4 + 4 \rho^3 - \rho^2
\end{array} \right) \left( \begin{array}{c}
(y + \rho^2 x - \rho x + x + \rho - 1)(\rho y + x + \rho^2 - \rho) \\
* (\rho^2 y + \rho x - \rho + 1) \\
* (\rho^2 y - \rho y + y + \rho^2 x - \rho^2 + \rho)
\end{array} \right)
\]

Convexity means that \((x, y)\) is constrained to the open “northeast corner” of the first quadrant bounded by \( Y \cup T \cup X, Y = \{(0, y) : y \geq 1\}, T = \{(x, 1 - x) : 0 \leq x \leq 1\},\ X = \{(x, 0) : x \geq 1\}.\) Restricting \( r \) to \( Y \), we get a formula for \( r(y) = r(0, y) \). Taking the derivative unexpectedly gives this simple, completely factored formula:

\[
r'(y) = \frac{(r - 1)^2 \rho^5 (y - \frac{\rho y + 1}{\rho}) (y - (r - 1))}{(\rho^2 y - (r - 1))^2 ((\rho^2 + 1 - \rho) y - \rho (r - 1))^2}.
\]
The only solution to \( r'(y) = 0 \) with \( y \in Y \) has \( y = \frac{\rho + 1}{\rho} \). It quickly follows that on \( Y \), \( r \) attains a maximum value of \( M \) at \( \left(0, \frac{\rho + 1}{\rho}\right) \) and is minimized by \( m \) at the endpoints \((0, 1)\) and \((0, \infty)\). (By this we mean that \( \lim_{y \to \infty} r(0, y) = m \).) Similarly on \( T \), the derivative of \( r(x) = r(x, 1 - x) \) has the following fully factored form

\[
r'(x) = \frac{\partial}{\partial x} r(x, 1 - x) = \frac{(\rho + 1)(\rho - 1)^2 \rho^3 ((\rho - 1)x - \rho)(x - \frac{\rho}{\rho + 1})}{((\rho - 1)x - \rho^3)^2 (\rho (\rho - 1)x - (\rho^2 + (1 - \rho)))^2},
\]

so that \( r \) has minimum value \( m \) at the endpoints \((0, 1)\) and \((1, 0)\) and maximum value \( M \) at \( \left(\frac{\rho}{\rho + 1}, \frac{1}{\rho + 1}\right) \); while on \( X \),

\[
r'(x) = \frac{\partial}{\partial x} r(x, 0) = \frac{(\rho - 1)^2 \rho^3 (\rho^2 + 1)(x - (\rho + 1))(x - (1 - \rho))}{(x + \rho (\rho - 1))^2 ((\rho^2 + 1 - \rho)x + \rho - 1)^2},
\]

so that \( r \) has minimum value \( m \) at the endpoints \((1, 0)\) and \((\infty, 0)\) and maximum value \( M \) at \((\rho + 1, 0)\). Motivated by these results we now sweep the region of permissible values of \((x, y)\) by line segments with \( y \)-intercept \( \eta \) and slope \(-1/\rho\). From

\[
r'(x) = \frac{\partial}{\partial x} r \left( x, -\frac{1}{\rho}x + \eta \right) = \frac{(\rho - 1)^2 \rho^6 (\rho^2 + 1)^2 \left( \eta - \frac{\rho + 1}{\rho} \right)^2 \left( x - \frac{\rho^2 + \eta \rho - \rho}{\rho^2 + 1} \right)}{\left( \left( \rho^3 + 3\rho^2 - 3\rho^2 - \eta^3 + 3\rho - 1 \right) x \right)}
\]

\[
\left( \left( \rho^3 - \rho^2 - \rho + 1 \right) \left( \left( \rho^3 - \rho^2 + \rho - 1 \right) x + \rho^2 + \eta \rho - \rho \right)^2 \right)
\]

it is clear that \( \eta = \frac{\rho + 1}{\rho} \) produces one arm of the image of \( S \). Finally for all other \( \eta \), \( r \) has mound-shaped behavior with minimum values on the coordinate axes and reaches a maximum of \( M \) where \( y = -\frac{1}{\rho} x + \eta \) intersects the other arm.

Recall that we have defined \( \rho \)-points in terms of counterclockwise orientation. Although Theorem 2 is true for clockwise orientation, we stress that the value of \( r \) depends, in general, on the orientation. In fact, clockwise and counterclockwise orientations always give different values of \( r \) unless \( D \) lies on the diagonal \( BP \).

Setting \( \rho = 1/2 \) in Theorem 2 yields this corollary.

**Corollary 3** Let \( A, B, C, D \) be (counterclockwise) successive vertices of a convex quadrilateral. Define \( EFGH \) as the inner quadrilateral formed by joining vertices to midpoints as described in Section 11. Construct point \( P \) so that \( ABCP \) is a parallelogram. Locate \( C' \) and \( C'' \) on \( BC \) so that \( C' \) is a distance \((1/2) BC \) from \( C \).
and $C''$ is a distance $2BC$ from $C$ and let $S = C'\overrightarrow{P} \cup C''\overrightarrow{P}$. Then the ratio $r$ defined by equation (1) is maximal exactly when $D$ is on $S^* = S \cap \text{int}(\angle ABC) \cap \text{ext}(\triangle ABC)$. Furthermore the set of possible ratios is 

$$\left[\frac{1}{5}, \frac{1}{6}\right].$$

Another corollary of Theorem 2 is the following generalization of Theorem 1 from midpoints to $\rho$-points.

**Corollary 4** If each vertex of a parallelogram is joined to the $\rho$-point of an opposite side in counterclockwise order to form an inner quadrilateral, then the area of the inner quadrilateral is $$\frac{(1-\rho^2)}{\rho^2+1}$$ times the area of the original parallelogram.

A nice geometry exercise is to prove this corollary avoiding the calculus part of the proof of Theorem 2. Hint: Performing the affine transformation we may assume that the original quadrilateral is the unit square. Use slope considerations to see that the interior quadrilateral is actually a rectangle. Use length considerations to see that it is a square of side length $\sqrt{\frac{(1-\rho^2)}{\rho^2+1}}$.

**References**

[Bra] B. Braden, The Surveyor’s Area Formula, *College Math. Journal* 17 (1986), 326–337.

[CW] R.J. Cook and G. V. Wood, Note 88.46: Feynman’s Triangle, *The Math. Gazette* 88 (2004), 299–302.

[DeV] M. De Villiers, *Feedback*, Feynman’s Triangle, *The Math. Gazette* 89 (2005), 107. See also [http://mysite.mweb.co.za/residents/profm_d/feynman.pdf](http://mysite.mweb.co.za/residents/profm_d/feynman.pdf).

[Key] Key Curriculum Press, 101 Project Ideas for Geometer’s Sketchpad, Version 4, Key Curriculum Press, Emeryville CA, 2007.