The logarithmic Cauchy quotient mean

Martin Himmel\textsuperscript{a} and Janusz Matkowski\textsuperscript{b}

\textsuperscript{a}Faculty of Mathematics and Computer Science, Institute of Applied Analysis, Technical University Mountain Academy Freiberg, Freiberg, Germany; \textsuperscript{b}Institute of Mathematics, University of Zielona Góra, Góra, Poland

**ABSTRACT**

Motivated by recent results on beta-type functions, a new family of means, which are of logarithmic Cauchy quotient type, are determined and characterized.

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**1. Introduction**

The relationship between the Euler Gamma function and the Beta function inspired to introduce the beta-type function \[1\]. Here we propose the \(k\)-variable logarithmic Cauchy quotients, the logarithmic counterpart of beta-type functions, as follows. Given a positive integer \(k \geq 2\), and a function \(f : I \to (0, +\infty)\) (or \(f : I \to (-\infty, 0)\)) where \(I \subset (0, +\infty)\) is an interval that is closed under multiplication, we define the \(k\)-variable logarithmic Cauchy quotient \(L_{f,k} : I^k \to (0, +\infty)\) by

\[
L_{f,k}(x_1, \ldots, x_k) = \frac{f(x_1) + \cdots + f(x_k)}{f(x_1 \cdots x_k)},
\]

and we refer to \(f\) as its generator (Section 2, Definition 2.1). Similarly to the case of beta-type functions \[1\], we give conditions under which \(L_{f,k}\) is a premean or a mean (see Lemma 4.1, Theorems 5.1–6.2).

In Section 3, assuming that \(1 \in I\), we prove that two \(k\)-variable logarithmic Cauchy quotients coincide if and only if their generators are proportional (Theorem 3.1).

In Section 4, applying the theory of iterative functional equations \[3\], we determine the general solution of the functional equation

\[
f(x) = \frac{x}{k} f(x^k),
\]

that is the reflexivity condition of \(L_{f,k}\) (Lemma 4.1). Based on this lemma, in Section 5, we prove Theorem 5.1, our main result, which says that \(L_{f,k}\) is a \(k\)-variable mean in \((1, +\infty)\).
iff there is \( c \neq 0 \) such that
\[
f(x) = c \frac{\log x}{\sqrt[1-k]{x}},
\]
for all \( x \in (1, +\infty) \), or, equivalently, that \( L_{f,k} = \mathcal{L}_k \), where \( \mathcal{L}_k \) is a new \( k \)-variable mean, called the \( k \)-variable logarithmic Cauchy quotient mean (Definition 5.1), and defined by
\[
\mathcal{L}_k(x_1, \ldots, x_k) := \sum_{i=1}^{k} \frac{\log x_i}{\sum_{l=1}^{k} \log x_l} G_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k),
\]
for all \( x_1, \ldots, x_k \in (1, +\infty) \), where \( G_{k-1} : (0, +\infty)^{k-1} \to (0, +\infty) \) is the \((k-1)\)-variable geometric mean,
\[
G_{k-1}(x_1, \ldots, x_{k-1}) = k^{-1} \sqrt[k-1]{x_1 \cdots x_{k-1}}.
\]
Moreover, some properties of \( \mathcal{L}_k \) are discussed, the results for the interval \((0,1)\) is formulated, as well as the corresponding extension of the logarithmic Cauchy quotient mean on the interval \((0, +\infty)\) (denoted by \( \mathcal{L}_k \)) is proposed.

We end our paper with two characterizations of the mean \( \mathcal{L}_k \). In Section 6, applying a variant of the Krull theorem on difference equations [2] given in Kuczma [3], we show that \( L_{f,k} = \mathcal{L}_k \) iff \( L_{f,k} \) is reflexive in \((1, +\infty)\) and the function \( \log f \circ \exp \circ \exp \) is convex. In Section 7, assuming that \( f : (1, +\infty) \to (0, +\infty) \) is extendable to a function of the class \( C^2 \) in \([1, +\infty)\), we prove that \( L_{f,k} \) is a premean in \((1, +\infty)\) iff it coincides with the mean \( \mathcal{L}_k \).

### 2. Some basic notions

Throughout this paper \( I \subset \mathbb{R} \) stands for an interval.

Let \( k \in \mathbb{N}, k \geq 2 \). A function \( M : I^k \to \mathbb{R} \) is called a \( k \)-variable mean in \( I \), if
\[
\min(x_1, \ldots, x_k) \leq M(x_1, \ldots, x_k) \leq \max(x_1, \ldots, x_k), \quad x_1, \ldots, x_k \in I;
\]
and it is called strict, if these inequalities are strict for all nonconstant \( k \)-tuples \( (x_1, \ldots, x_k) \in I^k \).

Let us note the following easy to verify properties of means.

**Remark 2.1:** If \( M \) is a \( k \)-variable mean in an interval \( I \), then

1. for every subinterval \( J \subset I \), \( M \) restricted to \( J^k \) is a mean in \( J \), and \( M(J^k) = J \), in particular, \( M : I^k \to I \);
2. \( M \) is reflexive, i.e.
\[
M(x, \ldots, x) = x, \quad x \in I.
\]

A function \( M : I^k \to \mathbb{R} \) is called \( k \)-variable premean in \( I \), if it reflexive and \( M(I^k) = I \) (see [7], also [8, p. 29]).

**Remark 2.2:** If a reflexive function \( M : I^k \to \mathbb{R} \) is (strictly) increasing in each variable, then it is a (strict) \( k \)-variable mean in \( I \).
Let us introduce some notion playing here a significant role.

**Definition 2.1:** Let \( k \in \mathbb{N}, k \geq 2 \), be fixed, and let \( I \subset (0, +\infty) \) be an interval that is closed under multiplication. For a function \( f : I \rightarrow (0, +\infty) \) (or \( f : I \rightarrow (-\infty, 0) \)), the function \( L_{f,k} : I^k \rightarrow (0, +\infty) \) defined by (1) is called \( k \)-variable logarithmic Cauchy quotient, and \( f \) is called a generator of \( L_{f,k} \).

**Remark 2.3:** An open interval \( I \subset \mathbb{R} \) is closed under multiplication iff \( I = (p, +\infty) \) for some \( p \in [1, +\infty) \); or \( I = (0, p) \) for some \( p \in (0, 1] \), or \( I = \mathbb{R} \).

From the definitions of the logarithmic Cauchy quotient \( L_{f,k} \) and the reflexivity we obtain

**Remark 2.4:** Under the assumptions of this definition, the logarithmic Cauchy quotient \( L_{f,k} : I^k \rightarrow (0, +\infty) \) of a generator \( f : I \rightarrow (0, +\infty) \) is reflexive (or it is a mean or a premean) if its generator \( f \) satisfies the iterative functional equation (2).

### 3. Equality of two logarithmic Cauchy quotients and a functional equation

**Remark 3.1:** Let \( k \in \mathbb{N}, k \geq 2 \), and interval \( I \subset (0, +\infty) \) satisfy conditions of Definition 2.1 and let \( f, g : I \rightarrow (0, +\infty) \). Then \( L_{g,k} = L_{f,k} \) iff the functions \( f \) and \( g \) satisfy the functional equation

\[
\frac{g(x_1 \cdots x_k)}{f(x_1 \cdots x_k)} = \frac{g(x_1) + \cdots + g(x_k)}{f(x_1) + \cdots + f(x_k)}, \quad x_1, \ldots, x_k \in I. \tag{5}
\]

Moreover, if \( 1 \in I \), then \( f \) and \( g \) satisfy this equation if, and only if, \( g = cf \) for some \( c > 0 \).

**Proof:** The first fact is an immediate consequence of Definition 2.1. To show the remaining one, assume that \( f \) and \( g \) satisfy this equation. Putting \( x_1 = x \) and \( x_2 = x_3 = \cdots = x_k = 1 \) gives

\[
\frac{g(x)}{f(x)} = \frac{g(x) + (k - 1)g(1)}{f(x) + (k - 1)f(1)}, \quad x \in I,
\]

whence

\[
(k - 1)f(1)g(x) = (k - 1)g(1)f(x), \quad x \in I.
\]

Since \( f \) and \( g \) are positive functions, it follows that \( f(1) \neq 0 \) and \( g(1) \neq 0 \). Setting \( c := g(1)/f(1) \) we hence get \( g = cf \). The converse implication is obvious.

In the sequel we have to exclude 1 from the interval \( I \), as we are mainly interested in the case when \( f(1) = 0 = g(1) \). It turns out that in this case the above functional equation is not trivial. We prove

**Lemma 3.1:** Let \( k \in \mathbb{N}, k \geq 2 \), be fixed. If the functions \( f, g : (1, +\infty) \rightarrow (0, +\infty) \) (or \( f, g : (0, 1) \rightarrow (0, +\infty) \)) satisfy Equation (5) with \( I = (1, +\infty) \) (or \( I = (0, 1) \)), and

\[
c := \lim_{x \to 1} \frac{g(x)}{f(x)}
\]

exists, then \( g = cf \).
Proof: Put $h := g/f$. Setting $x_1 = x_2 = \cdots = x_k = x$ in (5), we get $h(x^k) = h(x)$ for all $x \in (1, +\infty)$, or equivalently,

$$h(x) = h\left(x^{1/k}\right), \quad x \in (1, +\infty),$$

whence, by induction,

$$h(x) = h\left(x^{1/k^n}\right), \quad n \in \mathbb{N}, x \in (1, +\infty).$$

Letting $n \to +\infty$ we hence get $h(x) = c$ for all $x \in (1, +\infty)$. ■

From this lemma we obtain

**Theorem 3.1:** Let $k \in \mathbb{N}, k \geq 2$, be fixed. Assume that $f, g : (1, +\infty) \to (0, +\infty)$ (or $f, g : (0, 1) \to (0, +\infty)$) are such that the limit $\lim_{x \to 1} g(x)/f(x)$ exists. Then $L_{f,k} = L_{g,k}$ if and only if $g = cf$ for some $c > 0$.

### 4. Reflexivity of the logarithmic Cauchy quotient

Applying the theory of the iterative functional equations (see [3, p. 46, Theorem 2.1]) one gets

**Lemma 4.1:** Fix an integer $k \geq 2$ and $p \in (1, +\infty)$. Then

(i) a function $f : [p, +\infty) \to (0, +\infty)$ satisfies Equation (2) for all $x \in [p, +\infty)$ if and only if

$$f(x) = k^nx^{\frac{k^n-1}{k-1}}f_0\left(x^{\frac{1}{k^n}}\right)$$

for all $x \in [p^{k^n}, p^{k^n+1})$ and $n \in \mathbb{N}_0$, where $f_0 := f\mid_{[p,p^k)}$; moreover, $f$ is continuous if and only if so is $f_0$ and

$$\lim_{x \to p^k-} f_0(x) = \frac{k}{p}f_0(p).$$

(ii) a function $f : (p, +\infty) \to (0, +\infty)$ satisfies Equation (2) for all $x \in (p, +\infty)$ if and only if condition (6) holds for all $x \in [p^{k^n}, p^{k^n+1})$ and $n \in \mathbb{N}_0$, where $f_0 := f\mid_{(p,p^k)}$; moreover, $f$ is continuous if and only if so is $f_0$ and (7) holds true.

(iii) a function $f : (1, +\infty) \to (0, +\infty)$ satisfies Equation (2) for all $x \in (1, +\infty)$ if and only if condition (6) holds for all $x \in [p^{k^n}, p^{k^n+1})$ and $n \in \mathbb{Z}$, where $f_0 := f\mid_{(p,p^k)}$; moreover, $f$ is continuous if and only if so is $f_0$ and (7) holds true.
5. Means of the logarithmic Cauchy quotient type

Definition 5.1: The function \( L_k : (1, +\infty)^k \to (1, +\infty) \), given by

\[
L_k (x_1, \ldots, x_k) := \sum_{i=1}^{k} \frac{\log x_i}{\sum_{l=1}^{k} \log x_l} \left( \prod_{j=1, j \neq i}^{k} x_j \right)^{1/(k-1)},
\]

that is,

\[
L_k (x_1, \ldots, x_k) = \sum_{i=1}^{k} \frac{\log x_i}{\sum_{l=1}^{k} \log x_l} G_{k-1} (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)
\]

for all \( x_1, \ldots, x_k \in (1, +\infty) \), where \( G_{k-1} \) is the \((k - 1)\)-variable symmetric geometric mean in \((1, +\infty)\), is called \( k \)-variable logarithmic Cauchy quotient mean in \((1, +\infty)\).

We also use some elementary fact on the Jensen equation of two or more variables.

Lemma 5.1: Let \( C \) be a convex set of a linear space. A function \( f : C \to \mathbb{R} \) is a Jensen function of \( k \) variables for some \( k \in \mathbb{N}, k \geq 2 \), i.e., it satisfies the equality

\[
f \left( \frac{x_1 + \cdots + x_k}{k} \right) = \frac{f(x_1) + \cdots + f(x_k)}{k}, \quad x_1, \ldots, x_k \in C,
\]

if and only if it is a Jensen function of two variables, i.e.,

\[
f \left( \frac{x + y}{2} \right) = \frac{f(x) + f(y)}{2}, \quad x, y \in C.
\]

Proof: Indeed, for arbitrary \( x, y \in C \), using (9), we have

\[
f \left( \frac{x + y}{2} \right) = f \left( \frac{x + y + \sum_{i=1}^{k-2} \frac{x+y}{2}}{k} \right) = \frac{f(x) + f(y) + \sum_{i=1}^{k-2} f \left( \frac{x+y}{2} \right)}{k},
\]

whence \( f((x+y)/2) = (f(x) + f(y))/2 \), so \( f \) is a Jensen function of two variables.

If \( f \) is a Jensen function of two variables, then (see Kuczma [4, p. 126], Lemma 3.1, where the Jensen convexity is considered), by induction, for every \( n \in \mathbb{N} \), we get

\[
f \left( \frac{x_1 + \cdots + x_{2^n}}{2^n} \right) = \frac{f(x_1) + \cdots + f(x_{2^n})}{2^n}, \quad x_1, \ldots, x_{2^n} \in C.
\]

Let \( x_1, \ldots, x_k \in C \) be arbitrarily fixed. Choosing \( n \) such that \( k \leq 2^n \) and setting here

\[
x_{k+1} = x_{k+2} = \cdots = x_{2^n} := \frac{x_1 + \cdots + x_k}{k},
\]
we get

\[
\begin{align*}
f\left( \frac{x_1 + \cdots + x_k}{k} \right) &= f\left( \frac{x_1 + \cdots + x_k + (2^n - k) \frac{x_1 + \cdots + x_k}{k}}{2^n} \right) \\
&= f\left( \frac{x_1 + \cdots + x_k + \sum_{j=k+1}^{2^n} \frac{x_1 + \cdots + x_k}{k}}{2^n} \right) \\
&= \frac{f(x_1) + \cdots + f(x_k) + \sum_{j=k+1}^{2^n} f\left( \frac{x_1 + \cdots + x_k}{k} \right)}{2^n} \\
&= \frac{f(x_1) + \cdots + f(x_k) + (2^n - k) f\left( \frac{x_1 + \cdots + x_k}{k} \right)}{2^n},
\end{align*}
\]

whence

\[
kf\left( \frac{x_1 + \cdots + x_k}{k} \right) = f(x_1) + \cdots + f(x_k),
\]

which shows that \( f \) is a Jensen function of \( k \) variables. \( \blacksquare \)

The main result of this paper reads as follows.

**Theorem 5.1:** Fix an integer \( k \geq 2 \) and a function \( f : (1, +\infty) \to (0, +\infty) \) [\( f : (0, 1) \to (0, +\infty) \)]. The following statements are pairwise equivalent:

(i) the logarithmic Cauchy quotient function \( L_{f,k} : (1, +\infty)^k \to (0, +\infty) \) [\( L_{f,k} : (0, 1)^k \to (0, +\infty) \)] is a \( k \)-variable mean in \( (1, +\infty) \) [in \( (0, 1) \)];

(ii) there is a positive [negative] \( c \) such that equality (3) holds for all \( x \in (1, +\infty) \) [for all \( x \in (0, 1) \)];

(iii) the equality

\[
L_{f,k} = \mathcal{L}_k
\]

holds in \( (1, +\infty)^k \) [in \( (0, 1)^k \)].

**Proof:** To prove the implication (i)\( \Rightarrow \) (ii), assume that \( L_{f,k} \) is a mean in \( (1, +\infty) \). Fix arbitrarily \( p > 1 \) and put \( f_0 := f|_{[p^k, p^{k+1})} \). It follows from Remarks 2.1(ii) and 2.4 that, the function \( f : (1, +\infty) \to (0, +\infty) \) satisfies (2). Thus, by part (iii) of Lemma 4.1, for every \( n \in \mathbb{Z} \),

\[
f(x) = k^{-n}x^{(k^n-1)/(k-1)}f_0\left(x^{k^n}\right), \quad x \in \left[p^{k-n}, p^{k-n+1}\right).
\]

Hence, for all \( x_1, \ldots, x_k \in [p^{k-n}, p^{k-n+1}) \), we have

\[
x_1 \cdot \cdots \cdot x_k \in \left[p^{k-(n-1)}, p^{k-(n-2)}\right),
\]
Choosing \( y_1, \ldots, y_k \in [p, p^k) \) arbitrarily, we have, for every \( n \in \mathbb{Z} \),

\[
x_j = y_j^{k-n} = \left[p^{k-n}, p^{k-1+n}\right] \quad \text{for} \ j = 1, \ldots, k.
\]

Setting these numbers into the above inequalities, and, assuming that

\[
y_1 = \min (y_1, \ldots, y_k) \quad \text{and} \quad y_k = \max (y_1, \ldots, y_k),
\]

(which can be done without any loss of generality), we get

\[
y_1^{k-n} \leq \frac{1}{k} \sum_{j=1}^{k} y_j^{(1-k^{-n})(k-1)} f_0(y_j) \left( \prod_{j=1}^{k} y_j \right)^{k-1} \leq y_k^{k-n}, \quad y_2, \ldots, y_{k-1} \in [p, p^k).
\]

Letting here \( n \to +\infty \), we obtain

\[
1 \leq \frac{1}{k} \sum_{j=1}^{k} y_j^{1/(k-1)} f_0(y_j) \left( \prod_{j=1}^{k} y_j \right)^{k-1} \leq 1, \quad y_1, \ldots, y_k \in [p, p^k),
\]

whence,

\[
\frac{1}{k} \sum_{j=1}^{k} y_j^{1/(k-1)} f_0(y_j) = \left( \prod_{j=1}^{k} y_j \right)^{1/(k-1)} f_0 \left( \prod_{j=1}^{k} y_j \right)^{k-1}, \quad y_1, \ldots, y_k \in [p, p^k).
\]

Defining \( g : [p, p^k) \to (0, +\infty) \) by

\[
g(y) := y^{1/(k-1)} f_0(y), \quad y \in [p, p^k),
\]
we can write this equality as follows

\[ g \left( \prod_{j=1}^{k} y_j \right)^{k-1} = \frac{1}{k} \sum_{j=1}^{k} g(y_j), \quad y_1, \ldots, y_k \in [p, p^k). \]

Since, for arbitrary \( s_j \in [\log p, \log p^k], j = 1, \ldots, k, \) we have

\[ y_j = e^{s_j} \in [p, p^k), \quad j = 1, \ldots, k, \]

we hence get

\[ g \left( e^{(1/k)(s_1 + \cdots + s_k)} \right) = \frac{1}{k} \left[ g(e^{s_1}) + \cdots + g(e^{s_k}) \right], \quad s_1, \ldots, s_k \in [\log p, \log p^k). \]

Thus, the function

\[ h := g \circ \exp \]

satisfies the Jensen functional equation

\[ h \left( \frac{1}{k} (s_1 + \cdots + s_k) \right) = \frac{1}{k} [h(s_1) + \cdots + h(s_k)], \quad s_1, \ldots, s_k \in [\log p, \log p^k). \]

By [4, p. 315], Theorem 3.1, and Lemma 5.1 there exists an additive function \( a : \mathbb{R} \to \mathbb{R} \) and \( b \in \mathbb{R} \) such that

\[ h(s) = a(s) + b, \quad s \in \left[ \log a, \log a^k \right). \]

From the definitions of the functions \( h, g \) and \( f_0 \), we obtain

\[ g(y) = a(\log y) + b, \quad y \in [p, p^k), \]

and, using the \( \mathbb{Q} \)-homogeneity of the additive function \( a \),

\[ f_0(y) = \frac{a(\log y) + b}{y^{1/(k-1)}}, \quad y \in [p, p^k) \]

Hence, by Lemma 4.1(iii), we have, for every \( n \in \mathbb{Z} \),

\[ f(x) = \frac{1}{x^{1/(k-1)}} \left( a(\log x) + \frac{b}{k^n} \right), \quad x \in \left[ p^{kn}, p^{kn+1} \right). \]

Setting this into Equation (2), we get

\[ \frac{1}{x^{1/(k-1)}} \left( a(\log x) + \frac{b}{k^n} \right) = \frac{1}{x^{1/(k-1)}} \left( a(\log x) + \frac{b}{k^{n+1}} \right), \]

and thus

\[ b = 0. \]
Since $f$ is assumed to be positive, the function $a$ must be continuous, i.e. there is $c > 0$ such that
\[ a(x) = cx, \quad x \in \mathbb{R}. \]
Consequently, for every $n \in \mathbb{Z}$,
\[ f(x) = \frac{c}{x^{1/(k-1)}} \log x, \quad x \in \left[p^k, p^{k+1}\right). \]
This proves the implication (i) $\implies$ (ii).
Assume (ii) holds. Then, by Definition 2.1, we get, for all $x_1, \ldots, x_k \in (1, +\infty)$,
\[
L_{f,k}(x_1, \ldots, x_k) = \frac{c}{x_1^{1/(k-1)}} \log x_1 + \cdots + \frac{c}{x_k^{1/(k-1)}} \log x_k
\]
\[= \left(x_1 \cdots x_k\right)^{1/(k-1)} \frac{\log x_1}{x_1^{1/(k-1)}} + \cdots + \frac{\log x_k}{x_k^{1/(k-1)}}
\]
\[= \frac{\sum_{i=1}^{k} \left(\prod_{j=1, j \neq i}^{k} x_j^{1/(k-1)}\right) \log x_i}{\sum_{i=1}^{k} \log x_i} = \frac{\sum_{i=1}^{k} \log x_i}{\sum_{i=1}^{k} \log x_i} \log x_i \left(\prod_{j=1, j \neq i}^{k} x_j\right)^{1/(k-1)}
\]
\[= \sum_{i=1}^{k} \log x_i - G_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)
\]
\[= \mathcal{L}_k(x_1, \ldots, x_k),
\]
where $\mathcal{L}_k : (1, +\infty)^k \to (0, +\infty)$ is defined by formula (4), and $G_{k-1}$ the $(k - 1)$-variable geometric mean,
\[G_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) = \left(\prod_{j=1, j \neq i}^{k} x_j\right)^{1/(k-1)}, \quad i = 1, \ldots, k.
\]
For arbitrary $x_1, \ldots, x_k \in (1, +\infty)$ put $x_{\min} := \min(x_1, \ldots, x_k)$ and $x_{\max} := (x_1, \ldots, x_k)$. Since
\[
x_{\min} = \sum_{i=1}^{k} \frac{\log x_i}{\sum_{l=1}^{k} \log x_l} \leq \sum_{i=1}^{k} \frac{\log x_i}{\sum_{l=1}^{k} \log x_l} G_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)
\]
\[\leq \sum_{i=1}^{k} \frac{\log x_i}{\sum_{l=1}^{k} \log x_l} = x_{\max}
\]
we have $x_{\min} \leq \mathcal{L}_k(x_1, \ldots, x_k) \leq x_{\max}$ (and these inequalities are strict if the $k$-tuple $(x_1, \ldots, x_k)$ is not constant) which shows that $\mathcal{L}_k$ is a $k$-variable mean in $(1, +\infty)$. Thus (ii) $\implies$ (iii).

The implication (iii)$\Rightarrow$(i) is obvious. This completes the proof. \qed
In the context of Theorem 5.1 the natural question arises if it is possible to extend the mean $\mathcal{L}_k$ onto $(0, +\infty)^k$. An answer gives the following

**Remark 5.1:** The function $\mathcal{L}_k : (0, +\infty)^k \rightarrow (0, +\infty)$ defined by

$$
\mathcal{L}_k(x_1, \ldots, x_k) := \begin{cases} 
\frac{\sum_{i=1}^k G_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \log x_i}{\sum_{i=1}^k \log x_i} & \text{if } (x_1, \ldots, x_k) \in (0, 1)^k \cup (1, +\infty)^k \\
1 & \text{if } (x_1, \ldots, x_k) \notin (0, 1)^k \cup (1, +\infty)^k
\end{cases}
$$

is a $k$-variable mean in $(0, +\infty)$, and it is the only increasing extension of the means $\mathcal{L}_k : (1, +\infty)^k \rightarrow (1, +\infty)$ and $\mathcal{L}_k : (0, 1)^k \rightarrow (0, 1)$.

**Proof:** By Theorem 5.1, the restriction $\mathcal{L}_k|_{(0,1)^k}$ is a mean in $(0, 1)$, and $\mathcal{L}_k|_{(1,\infty)^k}$ is a mean in $(1, \infty)$. If $(x_1, \ldots, x_k) \notin ((0, 1)^k \cup (1, +\infty)^k)$ then,

$$\min(x_1, \ldots, x_k) \leq 1 \leq \max(x_1, \ldots, x_k),$$

and, clearly, the number 1 is the only possible value for an increasing mean at such a point $(x_1, \ldots, x_k)$.  

To get an involutory counterpart of $\mathcal{L}_k$, which could be denoted by $\mathcal{L}_k^{\text{inv}}$, consider the following

**Remark 5.2:** Let $k \in \mathbb{N}$, $k \geq 2$. A function $M : (1, +\infty)^k \rightarrow (1, +\infty)$ [resp., $M : (0, 1)^k \rightarrow (0, 1)$] is a $k$-variable mean in $(1, +\infty)$ [resp. in $(0, 1)$] iff the function $M^{\text{inv}} : (0, 1)^k \rightarrow (0, 1)$ [resp. $M^{\text{inv}} : (1, +\infty)^k \rightarrow (1, +\infty)$] defined by

$$
M^{\text{inv}}(x_1, \ldots, x_k) := \frac{1}{M\left(\frac{1}{x_1}, \ldots, \frac{1}{x_k}\right)}
$$

is a $k$-variable mean in $(0, 1)$ [resp. in $(1, +\infty)$].

It easy to verify

**Remark 5.3:** The mean $\mathcal{L}_k^{\text{inv}} : (0, 1)^k \rightarrow (0, 1)$, the involutory conjugate mean to $\mathcal{L}_k$, is of the form

$$
\mathcal{L}_k^{\text{inv}}(x_1, \ldots, x_k) = \frac{\sum_{i=1}^k \left(\frac{x_i \log x_i}{\sum_{j=1}^k \log x_j}\right) G_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)}{\sum_{i=1}^k x_i \log x_i}, \quad x_1, \ldots, x_k \in (0, 1).
$$

Let us note some properties of the mean $\mathcal{L}_k$ in

**Proposition 5.1:**

(i) $\mathcal{L}_k$ is a symmetric and strict mean, but is neither homogeneous nor translative.
(ii) \( \mathcal{L}_2 \) is the Beekenbach-Gini mean of generator \( \log \), i.e.

\[
\mathcal{L}_2(x, y) = \frac{y \log x + x \log y}{\log x + \log y}, \quad x, y \in (1, +\infty);
\]

and its involutory conjugate mean

\[
\mathcal{L}_2^{\text{inv}}(x, y) = xy \frac{\log x + \log y}{x \log x + y \log y}, \quad x, y \in (0, 1);
\]

(iii) the bivariable geometric mean \( \mathcal{G} \) is invariant with respect to the mean-type mapping \( (\mathcal{L}_2^{\text{inv}}, \mathcal{L}_2) \), i.e. \( \mathcal{G} \circ (\mathcal{L}_2^{\text{inv}}, \mathcal{L}_2) = \mathcal{G} \), and the sequence \( ((\mathcal{L}_2^{\text{inv}}, \mathcal{L}_2)^n : n \in \mathbb{N}) \) of iterates of \( (\mathcal{L}_2^{\text{inv}}, \mathcal{L}_2) \) converges uniformly on compact subsets of \((1, +\infty)^2\) to \((\mathcal{G}, \mathcal{G})\) (see Theorem 1 in [6]).

**Example 5.1**: Indeed, for \( k = 2 \), we have

\[
\mathcal{L}_2(2, 3) = \frac{3 \log 2 + 2 \log 3}{\log 2 + \log 3} = \frac{\log 72}{\log 6},
\]

\[
\mathcal{L}_2(2t, 3t) = \frac{3t \log 2t + 2t \log 3t}{\log 2t + \log 3t},
\]

and

\[
\mathcal{L}_2(2 + t, 3 + t) = \frac{(3 + t) \log (2 + t) + (2 + t) \log (3 + t)}{\log (2 + t) + \log (3 + t)}.
\]

Setting \( t = 2 \), we get

\[
2\mathcal{L}_2(2, 3) = \frac{\log 144}{\log 6} \neq \mathcal{L}_2(4, 6) = \frac{\log 5308416}{\log 24}, \quad \text{and} \quad 2 + \mathcal{L}_2(2, 3) = \frac{\log 5184}{6} \neq \mathcal{L}_2(4, 5) = \frac{\log 640000}{\log 20}. \]

Thus \( \mathcal{L}_2 \) is neither homogeneous nor translative. A similar argument gives (i) of Proposition 5.1.

### 6. A characterization of \( \mathcal{L}_k \) with the aid of reflexivity of \( L_{f, k} \) and a special type of convexity of its generator

Applying a generalized version of the Krull theorem on linear difference equations ([2]) given in Kuczma [3, p. 114, Theorem 5.11]), we give the following characterization of the logarithmic Cauchy quotient mean \( \mathcal{L}_k \).

**Theorem 6.1**: Let \( k \in \mathbb{N}, k \geq 2 \), be fixed, and assume that \( f : (1, +\infty) \to (0, +\infty) \) is differentiable and such that the function \( \log \circ f \circ \exp \circ \exp \) is convex. Then the following conditions are pairwise equivalent:

(i) the function \( L_{f, k} \) is reflexive in \((1, +\infty)\);
(ii) there is \( c > 0 \) such that \( f \) is given by (3) for all \( x \in (1, +\infty) \);
(iii) \( L_{f, k} = \mathcal{L}_k \).
Proof: Assume (i). By Definition 2.1 and Remark 2.4, the function $f$ satisfies the iterative functional equation:

$$f(x) = \frac{x}{k} f(x^k), \quad x \in (1, +\infty).$$

Taking log on both sides gives us

$$\log f(x) = \log f(x^k) + \log x - \log k, \quad x \in (1, +\infty).$$

Putting $t = \log x$ here we come to the equivalent equality

$$\log f(e^t) = \log f(e^{kt}) + t - \log k, \quad t \in (0, +\infty).$$

Setting $g : (0, +\infty) \to \mathbb{R}$, defined by

$$g = \log \circ f \circ \exp,$$

we can write this equation in the form

$$g(t) = g(kt) + t - \log k, \quad t \in (0, +\infty),$$

that is

$$g(e^{\log t}) = g(e^{\log t + \log k}) + e^{\log t} - \log k, \quad t \in (0, +\infty).$$

Setting $\tau = \log t$ we get

$$g(e^\tau) = g(e^{\tau + \log k}) + e^\tau - \log k, \quad \tau \in \mathbb{R},$$

and, consequently, the function $h : \mathbb{R} \to \mathbb{R}$, defined by

$$h := g \circ \exp = \log \circ f \circ \exp \circ \exp,$$

satisfies the functional equation

$$h(\tau + \log k) = h(\tau) + \log k - e^\tau, \quad \tau \in \mathbb{R}.$$

Differentiating both sides with respect to $\tau$, we obtain

$$h'(\tau + \log k) = h'(\tau) - e^\tau, \quad \tau \in \mathbb{R}.$$

Put

$$F(\tau) := -e^\tau, \quad \tau \in \mathbb{R}.$$

Note that $F$ is concave, and

$$\lim_{\tau \to -\infty} [F(\tau + \log k) - F(\tau)] = \lim_{\tau \to -\infty} \left[ -e^{\tau + \log k} - (-e^\tau) \right]$$

$$= \lim_{\tau \to -\infty} \left[ e^\tau (-k + 1) \right] = 0.$$
Therefore, in view of the theorem of Krull ([3, p. 114, Theorem 5.11]), there exists exactly one, up to an additive constant, convex solution \( h' : \mathbb{R} \to \mathbb{R} \) of the functional equation

\[
h'(\tau + \log k) = h'(\tau) + F(\tau), \quad \tau \in \mathbb{R}.
\]

It is easy to verify that, if \( f \) is given by formula (3) in part (ii), then \( h = \log \circ f \circ \exp \circ \exp \) satisfies this equation, as

\[
\log \circ f \circ \exp \circ \exp(\tau) = \log c + \tau - \frac{1}{k-1}e^\tau, \quad \tau \in \mathbb{R}.
\]

Since \( (\log \circ f \circ \exp \circ \exp)' \) is decreasing, the function \( \log \circ f \circ \exp \circ \exp \) is concave. Indeed, we have

\[
(\log \circ f \circ \exp \circ \exp(\tau))'' = -\frac{1}{k-1}e^\tau, \quad \tau \in \mathbb{R}, \quad (10)
\]

implying the concavity of the function \( \log \circ f \circ \exp \circ \exp \). Thus we have shown (ii). Since logarithmic Cauchy quotients for a given generator \( f \) are uniquely determined, the implication (ii) \( \Rightarrow \) (iii) follows. The remaining implication is due to part (ii) of Remark 2.1. This finishes the proof.\[\blacksquare\]

Weakening the assumption on \( L_{f,k} \) while adding some regularity assumption on the generator \( f \), and making use of the idea applied in [5], one gets the following characterization of the logarithmic Cauchy mean.

**Theorem 6.2:** Let \( k \in \mathbb{N}, k \geq 2 \) be fixed. Assume that \( f : (1, +\infty) \to (0, +\infty) \) is such that, for some \( c > 0 \), the function

\[
(0, +\infty) \ni x \mapsto \frac{f(x) - c (x - 1)}{(x - 1)^2}
\]

is bounded in a right vicinity of 1.

Then the following conditions are pairwise equivalent

(i) the function \( L_{f,k} \) is reflexive in \((1, +\infty)\);
(ii) there is \( c > 0 \) such that \( f \) satisfies (3) for all \( x \in (1, +\infty) \);
(iii) \( L_{f,k} = L_k \).

**Proof:** From (11) we have

\[
f(x) = c (x - 1) + \varphi(x) (x - 1)^2, \quad x \in (1, +\infty),
\]

where the function \( \varphi : (1, +\infty) \to \mathbb{R} \) defined by

\[
\varphi(x) := \frac{f(x) - c (x - 1)}{(x - 1)^2}, \quad x \in (1, +\infty),
\]

is bounded in an interval \((1, 1 + r)\), for some \( r > 0 \).
Assume (i). In view of Remark 2.4, the generator \( f \) of \( L_{f,k} \) satisfies the functional equation (2), that is equivalent to the functional equation
\[
f(x) = \frac{k}{x^{1/k}} f \left( x^{1/k} \right), \quad x \in (1, +\infty). \tag{13}
\]
Taking into account (12), we conclude that \( \phi \) satisfies the functional equation
\[
c(x - 1) + \phi(x) (x - 1)^2 = \frac{k}{x^{1/k}} \left[ c \left( x^{1/k} - 1 \right) + \left( x^{1/k} - 1 \right)^2 \phi \left( x^{1/k} \right) \right], \quad x \in (1, +\infty),
\]
which can be written in the form
\[
\phi(x) = \frac{c \left( 1 + k - x - kx^{-1/k} \right)}{(x - 1)^2} + kx^{-1/k} \left( \frac{x^{1/k} - 1}{x - 1} \right)^2 \phi \left( x^{1/k} \right), \quad x \in (1, +\infty); \quad i = 1, 2. \tag{14}
\]
and, moreover, \( \phi \) is bounded in an interval \((1, 1 + r)\).

Assume that the functions \( \phi_1, \phi_2 : (1, +\infty) \to \mathbb{R} \) are bounded in \((1, 1 + r)\) for some \( r > 0 \), and satisfy Equation (14), that is
\[
\phi_i(x) = \frac{c \left( 1 + k - x - kx^{-1/k} \right)}{(x - 1)^2} + kx^{-1/k} \left( \frac{x^{1/k} - 1}{x - 1} \right)^2 \phi_i \left( x^{1/k} \right), \quad x \in (1, +\infty); \quad i = 1, 2.
\]
Hence, putting
\[
\psi := |\phi_1 - \phi_2| \quad \text{and} \quad \alpha(x) := x^{1/k} \quad \text{for} \quad x \in (1, +\infty),
\]
we see that \( \psi \) is nonnegative and bounded solution of the functional equation
\[
\psi(x) = kx^{-1/k} \left( \frac{x^{1/k} - 1}{x - 1} \right)^2 \psi \left( \alpha(x) \right), \quad x \in (1, +\infty). \tag{15}
\]
Note that
\[
\frac{x^{1/k} - 1}{x - 1} = \frac{x^{1/k} - 1}{(x^{1/k})^k - 1} = \frac{x^{1/k} - 1}{(x^{1/k})^k - 1} \left( (x^{1/k})^{k-1} + (x^{1/k})^{k-2} + \cdots + x^{1/k} + 1 \right)
\]
\[
= \frac{1}{(x^{1/k})^{k-1} + (x^{1/k})^{k-2} + \cdots + x^{1/k} + 1},
\]
so, for all \( x \in (1, +\infty) \), we have
\[
kx^{-1/k} \left( \frac{x^{1/k} - 1}{x - 1} \right)^2 = \frac{kx^{-1/k}}{(x^{1/k})^{k-1} + (x^{1/k})^{k-2} + \cdots + x^{1/k} + 1}^2.
\]
Hence

\[
\lim_{x \to 1} k^{x^{-1/k}} \left( \frac{x^{1/k} - 1}{x - 1} \right)^2 = \lim_{x \to 1} \frac{k^{x^{-1/k}}}{\left( (x^{1/k})^{k-1} + (x^{1/k})^{k-2} + \cdots + x^{1/k} + 1 \right)^2} = \frac{k}{k^2} = \frac{1}{k},
\]

and, as \( k \geq 2 \), there is \( r > 0 \) such that

\[
k^{x^{-1/k}} \left( \frac{x^{1/k} - 1}{x - 1} \right)^2 \leq \frac{1}{2}, \quad x \in (1, 1 + r).
\]

Since \( \alpha((1, 1 + r)) \subset (1, 1 + r) \), in view of (15),

\[
0 \leq \psi(x) \leq \frac{1}{2} \psi(\alpha(x)), \quad x \in (1, 1 + r),
\]

the boundedness of \( \psi \) implies that

\[
\psi(x) = 0, \quad x \in (1, 1 + r).
\]

Now, from (15), taking into account that

\[
\lim_{n \to \infty} \alpha^n(x) = 1, \quad x \in (1, +\infty),
\]

we conclude that \( \psi(x) = 0 \) for every \( x \in (0, +\infty) \) which shows that \( \varphi_1 = \varphi_2 \). This proves that there is at most one solution of equation (13) satisfying condition (11).

Now the implication (i) \( \Rightarrow \) (ii) follows from the fact that the function

\[
(1, +\infty) \ni x \mapsto \frac{c \log x}{k - \sqrt{x}},
\]

is a solution of the reflexivity equation (13) and satisfies condition (11).

The remaining implications are obvious. \( \blacksquare \)

Since twice continuously differentiable functions satisfy condition (11), the following result is an immediate consequence of the above result.

**Corollary 6.1:** Let \( k \in \mathbb{N}, \ k \geq 2 \) be fixed. Assume that \( f : (1, +\infty) \to (0, +\infty) \) is of the class \( C^2 \) and the function

\[
(1, +\infty) \ni x \mapsto f(x)
\]

has an extension that is of the class \( C^2 \) in the interval \([1, +\infty)\).

Then the following conditions are pairwise equivalent:

(i) the function \( L_{f,k} \) is a premean in \((1, +\infty)\);
(ii) there is \( c > 0 \) such that \( f \) satisfies (3) for all \( x \in (1, +\infty) \);
(iii) \( L_{f,k} = \mathcal{L}_k \).
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