Sensitivity of Resonant Axion Haloscopes to Quantum Electromagnetodynamics

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Recently, interactions between putative axions and magnetic monopoles have been revisited by two of the authors.[1] It is shown that significant modifications to conventional axion electrodynamics arise due to these interactions, so that the axion–photon coupling parameter space is expanded from one parameter \( g_{aYY} \) to three (\( g_{aYY} \), \( g_{aAB} \), \( g_{aBB} \)). Poynting theorem is implemented to determine how to exhibit sensitivity to \( g_{aAB} \) and \( g_{aBB} \) using resonant haloscopes, allowing new techniques to search for axions and a possible indirect way to determine if magnetically charged matter exists.

1. Introduction

Axions are putative particles, thought to exist because they solve the strong CP problem,[2–10] as well as being prime candidates for cold dark matter.[11–18] In this work, we implement Poynting theorem[19] to calculate the generalized sensitivity of resonant haloscopes from axion-modified electrodynamics, which includes the usual two photon anomaly coupling parameter, \( g_{aYY} \), as well as two other electromagnetic anomaly coupling parameters, \( g_{aAB} \) and \( g_{aBB} \) if magnetic monopoles exist, as suggested by implementing quantum electromagnetodynamics (QEMD).[1,20,21] We conceptualize resonant experiments with either background DC or AC magnetic and electric fields and show various configurations that can put limits on the axion coupling parameters, while also defining the equivalent experimental form factors.

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2. Axion-Modified Electrodynamics from QEMD

The generalized axion electrodynamics, expanded to include all possible couplings between axion and electromagnetic field, was shown to be given by (in SI units),

\[
\vec{\nabla} \cdot \vec{E}_1 = g_{aYY} c \vec{B}_0 \cdot \vec{v}_a - g_{aAB} \vec{E}_0
\]

\[
\vec{\nabla} \times \vec{B}_1 = \frac{\vec{E}_0}{\mu_0} = \epsilon_0 \partial_0 \vec{E}_1 + \vec{f}_1 + g_{aYY} \epsilon_0 \left( -\vec{v}_a \times \vec{E}_0 - \partial_0 a \vec{B}_0 \right)
\]

\[
\vec{\nabla} \cdot \vec{B}_1 = -\frac{g_{aAB}}{c} \vec{E}_0 \cdot \vec{v}_a + g_{aAB} \vec{B}_0 \cdot \vec{v}_a,
\]

\[
\vec{\nabla} \times \vec{E}_1 = -\partial_0 \vec{B}_1 + \frac{g_{aBB}}{c} \left( c^2 \vec{v}_a \times \vec{B}_0 - \partial_0 a \vec{E}_0 \right)
\]

\[
\vec{\nabla} \times \vec{B}_0 = \mu_0 \epsilon_0 \partial_0 \vec{E}_0 + \mu_0 \vec{f}_0, \quad \vec{\nabla} \cdot \vec{B}_0 = 0, \quad \vec{\nabla} \times \vec{E}_0 = -\partial_0 \vec{B}_0,
\]

while \( \vec{B}_1 \) and \( \vec{E}_1 \) are the axion-generated fields of the first order in axion couplings to photons, which will also generate the associated free current and charge densities, \( \vec{f}_1 \) and \( \rho_0 \), respectively, within the haloscope detector.

For cold dark matter, it is usual to assume \( \vec{v}_a = 0 \), and in this limit, (1)–(4) become,

\[
\vec{\nabla} \cdot \vec{E}_1 = \epsilon_0^{-1} \rho_0
\]

\[
\vec{\nabla} \times \vec{B}_1 = \frac{\epsilon_0 \partial_0 \vec{E}_1 + \vec{f}_1 + (g_{aAB} \epsilon_0 \vec{E}_0 - g_{aYY} \epsilon_0 \vec{B}_0) \partial_0 a}{\mu_0}
\]
\[ \vec{V} \cdot \vec{B}_1 = 0 \]  
(8)

\[ \vec{V} \times \vec{E}_1 = -\partial \vec{B}_1 + (g_{\alpha\beta\gamma} \vec{B}_0 - \frac{g_{\alpha\beta\gamma}}{c} \vec{E}_0)\partial_\alpha \]  
(9)

For this set of equations, we notice the axion electric displacement current is generalized to,

\[ J_{e\alpha} = (g_{\alpha\beta\gamma} \epsilon_0 \vec{E}_0 - g_{\alpha\beta\gamma} \epsilon_0 \vec{B}_0)\partial_\alpha \]  
(10)

and an axion magnetic displacement current exists

\[ J_{m\alpha} = (\frac{g_{\alpha\beta\gamma}}{c} \vec{E}_0 - g_{\alpha\beta\gamma} \vec{B}_0)\partial_\alpha \]  
(11)

Thus, in QEMD, there are three extra axion current terms due to \(g_{\alpha\beta\gamma}\) and \(g_{\alpha\beta\gamma}\) compared to standard axion electrodynamics.

We may also use the method of writing the modifications as effective polarizations and magnetizations,\(^{[1,22-25]}\) by implementing the following vector identities, \(\vec{V} \cdot \vec{x} = \vec{V} \cdot (a \vec{x}) - a(\vec{V} \cdot \vec{x})\) and \(\vec{V} \times \vec{x} = \vec{V} \times (a \vec{x}) - a(\vec{V} \times \vec{x})\), to Equations (1)–(4). In this case, Gauss’ and Ampere’s laws become

\[ \vec{V} \cdot (\epsilon_0 \vec{E}_1 + \frac{1}{c} \vec{B}_1) = g_{\alpha\beta\gamma} \epsilon_0 \rho_{\alpha\beta\gamma} + \rho_{\alpha\beta\gamma} \]  
(12)

\[ \vec{V} \times (\frac{1}{c} \vec{B}_1 - \hat{M}_{\alpha\beta\gamma}) - \partial_\alpha (\epsilon_0 \vec{E}_1 + \frac{1}{c} \vec{B}_1) = \vec{J}_{\alpha\beta\gamma} + g_{\alpha\beta\gamma} \vec{J}_{\alpha\beta\gamma} \]  
(13)

where we define the effective polarization and magnetization as

\[ \frac{1}{\epsilon_0} \vec{P}_1 = g_{\alpha\beta\gamma} \epsilon_0 \vec{E}_0 - g_{\alpha\beta\gamma} \epsilon_0 \vec{B}_0, \quad \text{and} \]  
(14)

\[ \mu_0 \hat{M}_{\alpha\beta\gamma} = -g_{\alpha\beta\gamma} \frac{1}{c} \vec{E}_0 - g_{\alpha\beta\gamma} \epsilon_0 \vec{B}_0 \]

Applying the same vector identities to the magnetic Gauss’ and Faraday law, we obtain,

\[ \vec{V} \cdot (\vec{B}_1 - \mu_0 \hat{M}_{\alpha\beta\gamma}) = g_{\alpha\beta\gamma} \mu_0 \rho_{\alpha\beta\gamma} \]  
(15)

\[ \vec{V} \times (\vec{E}_1 + \frac{1}{\epsilon_0} \vec{P}_1) + \partial_\alpha (\vec{B}_1 - \mu_0 \hat{M}_{\alpha\beta\gamma}) = -g_{\alpha\beta\gamma} \epsilon_0 \mu_0 \vec{J}_{\alpha\beta\gamma} \]  
(16)

with the following definitions of effective polarization and magnetization:

\[ \frac{1}{\epsilon_0} \vec{P}_1 = -g_{\alpha\beta\gamma} \epsilon_0 \vec{E}_0 - g_{\alpha\beta\gamma} \epsilon_0 \vec{B}_0, \quad \text{and} \]  
(17)

\[ \mu_0 \hat{M}_{\alpha\beta\gamma} = g_{\alpha\beta\gamma} \frac{1}{c} \vec{E}_0 + g_{\alpha\beta\gamma} \epsilon_0 \vec{B}_0 \]

We have reversed the symbols and adopted an opposite sign convention for \(\vec{P}_1\) and \(\hat{M}_{\alpha\beta\gamma}\) compared to ref. [1]; this means we keep the vector \(\vec{P}\) as an electrical polarization and vector \(\vec{M}\) as a magnetic polarization (or magnetization); then, the subscript refers to whether it is induced from an effective axion like electric (e) or magnetic (m) charge. The opposite sign convention is used to keep them consistent with how the auxiliary fields (\(\vec{D}\) and \(\vec{H}\)) are defined in matter, which then may be generalized to

\[ \vec{D}_1 = \epsilon_0 \vec{E}_1 + \vec{P}_1 + \vec{P}_{m1} = \epsilon_0 \vec{E}_1 - (g_{\alpha\beta\gamma} + g_{\alpha\beta\gamma}) \epsilon_0 \vec{B}_0, \quad \text{and} \]  
(18)

\[ \vec{H}_1 = \frac{1}{\mu_0} \vec{B}_1 - \hat{M}_{e1} - \hat{M}_{m1} = \frac{1}{\mu_0} \vec{B}_1 + (g_{\alpha\beta\gamma} + g_{\alpha\beta\gamma}) \epsilon_0 \vec{B}_0 \]

Note in this representation of axion-modified electrodynamics, both the electric field and magnetic flux densities may have both vector and scalar potentials as dictated by two potential theory.\(^{[20,21,26-29]}\) One can also write these electrodynamical equations in terms of the auxiliary fields. Thus, assuming \(\vec{V}_\alpha = 0\) and combining (18) with (6)–(9), we may write the axion-modified electrodynamics as

\[ \vec{V} \cdot \vec{D}_1 = \rho_{\alpha\beta\gamma} \]  
(19)

\[ \vec{V} \times \vec{H}_1 = \epsilon_0 \vec{E}_1 + \frac{1}{c} \vec{B}_1 + (g_{\alpha\beta\gamma} \epsilon_0 \vec{E}_0 - g_{\alpha\beta\gamma} \epsilon_0 \vec{B}_0)\partial_\alpha \]  
(20)

\[ \vec{V} \cdot \vec{H}_1 = (g_{\alpha\beta\gamma} + g_{\alpha\beta\gamma}) \epsilon_0 \rho_{\alpha\beta\gamma} \]  
(21)

\[ \frac{1}{\epsilon_0} \vec{V} \times \vec{D}_1 = -\partial_\alpha \vec{B}_1 + (g_{\alpha\beta\gamma} \vec{B}_0 - \frac{g_{\alpha\beta\gamma}}{c} \vec{E}_0)\partial_\alpha \]  
(22)

another form of the modified axion electrodynamics.

To calculate the sensitivity of experiments to \((g_{\alpha\beta\gamma} + g_{\alpha\beta\gamma})\), one can use Poynting theorem, with different choices of vectors considering the electric and magnetic fields and auxiliary fields.\(^{[13]}\) However, similar to what has been shown before,\(^{[13]}\) for resonant and radiative systems, the choice of Poynting vector to calculate the sensitivity gives the same first order solution; so in the following analysis, we use the simplest form given by Equations (6)–(9) (this is straightforward to show and not included here).

3. Phasor Form and Complex Poynting Theorem

For harmonic solutions, the axion pseudoscalar \(a(t)\) may be written as, \(a(t) = \frac{1}{2}(\bar{a}e^{-j\omega t} + a^* e^{j\omega t}) = Re(\bar{a}e^{-j\omega t})\), and thus, in phasor form, \(A = \bar{a}e^{-j\omega t}\) and \(A^* = a^* e^{j\omega t}\). In contrast, the electric and magnetic fields as well as the electric current are represented as vector-phasors. For example, we set \(\vec{E}_1(\vec{r}, t) = \frac{1}{2}(\bar{E}_1(\vec{r})e^{-j\omega t} + \bar{E}_1(\vec{r})e^{j\omega t}) = Re[\bar{E}_1(\vec{r})e^{-j\omega t}]\); so we define the vector phasor (bold) and its complex conjugate by, \(\vec{E}_1(\vec{r}, t) = \bar{E}_1(\vec{r})e^{j\omega t}\) and \(\vec{E}_1(\vec{r}, t) = \bar{E}_1(\vec{r})e^{-j\omega t}\), respectively. Following these definitions, the axion-modified Ampere’s law in (2), in phasor form becomes

\[ \frac{1}{\mu_0} \vec{V} \times \vec{B}_1 = J_{\alpha\beta\gamma} + j\omega \epsilon_0 \vec{E}_1 + j\omega g_{\alpha\beta\gamma} \epsilon_0 \vec{B}_0 - j\omega g_{\alpha\beta\gamma} \epsilon_0 \vec{E}_0 \]  
(23)
while Faraday’s law in (4) becomes
\[
\nabla \times \mathbf{E}_1 = j \omega_1 \mathbf{B}_1 + \frac{\alpha_0 g_{aaBB}}{c} \hat{\mathbf{a}} \mathbf{E}_0 - j \omega_0 g_{BBAA} \hat{\mathbf{a}} \mathbf{B}_0
\]
\[
\mathbf{E}_1^* = j \omega_1^* \mathbf{B}_1^* - \frac{\alpha_0 g_{aaBB}}{c} \hat{\mathbf{a}}^* \mathbf{E}_0^* + j \omega_0 g_{BBAA} \hat{\mathbf{a}}^* \mathbf{B}_0^*
\]

To implement Poynting theorem to calculate the sensitivity of a resonant system, we need to calculate the real power flow at resonance as the reactive power is zero at the resonant frequency. The complex Poynting vector and its complex conjugate are defined by
\[
\mathbf{S}_1 = \frac{1}{2 \mu_0} \mathbf{E}_1 \times \mathbf{B}_1 \quad \text{and} \quad \mathbf{S}_1^* = \frac{1}{2 \mu_0} \mathbf{E}_1^* \times \mathbf{B}_1
\]
respectively, where \(\mathbf{S}_1\) is the complex power density of the harmonic electromagnetic wave or oscillation, with the real part equal to the time averaged power density and the imaginary term equal to the reactive power, which may be inductive (magnetic energy dominates) or capacitive (electrical energy dominates). For resonant systems as analyzed in this paper, the inductive and capacitive imaginary terms cancel as verified in ref. [19]; so we only need to consider the real term. However, for reactive systems, it is the reactive power that dominates, but these systems are not considered in this paper. Unambiguously, we may calculate the real part of the Poynting vector by
\[
\text{Re} \left( \mathbf{S}_1 \right) = \frac{1}{2} (\mathbf{S}_1 + \mathbf{S}_1^*)
\]
Taking the divergence of Equation (26), we find
\[
\nabla \cdot \text{Re} \left( \mathbf{S}_1 \right) = \frac{1}{2} \nabla \cdot (\mathbf{S}_1 + \mathbf{S}_1^*)
\]
The next step is to calculate \(\nabla \cdot \mathbf{S}_1\) and \(\nabla \cdot \mathbf{S}_1^*\)
\[
\nabla \cdot \mathbf{S}_1 = \frac{1}{2} \nabla \cdot (\mathbf{E}_1 \times \frac{1}{\mu_0} \mathbf{B}_1) = \frac{1}{2 \mu_0 \mathbf{B}_1^*} \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \frac{1}{2 \mu_0} \nabla \times \mathbf{B}_1
\]
and
\[
\nabla \cdot \mathbf{S}_1^* = \frac{1}{2} \nabla \cdot (\mathbf{E}_1^* \times \frac{1}{\mu_0} \mathbf{B}_1^*) = \frac{1}{2 \mu_0} \mathbf{B}_1 \cdot \nabla \times \mathbf{E}_1^* - \mathbf{E}_1^* \cdot \frac{1}{2 \mu_0} \nabla \times \mathbf{B}_1
\]
Substituting Equations (23) and (24) into Equations (28) and (29) leads to,
\[
\nabla \cdot \mathbf{S}_1 = \frac{j \omega_1}{2} (\mathbf{c} \cdot \mathbf{B}_1 - \mathbf{E}_1 \cdot \mathbf{E}_1^*) - \frac{1}{2} \mathbf{E}_1 \cdot \mathbf{J}_e^*
\]
\[
- \frac{j \omega_0 g_{BBAA}}{2} (\mathbf{c} \cdot \mathbf{B}_1^* \cdot \hat{\mathbf{a}} \mathbf{B}_0 + \mathbf{E}_1^* \cdot \hat{\mathbf{a}}^* \mathbf{E}_0^*)
\]
\[
+ \frac{j \omega_0 g_{CCAA}}{2} (g_{BBAA} \mathbf{B}_1^* \cdot \hat{\mathbf{a}} \mathbf{E}_0 + g_{CCAA} \mathbf{E}_1 \cdot \hat{\mathbf{a}}^* \mathbf{B}_0^*)
\]
\[
\nabla \cdot \mathbf{S}_1^* = \frac{j \omega_1^*}{2} (\mathbf{c} \cdot \mathbf{B}_1^* - \mathbf{E}_1^* \cdot \mathbf{E}_1) - \frac{1}{2} \mathbf{E}_1^* \cdot \mathbf{J}_e^*
\]
\[
+ \frac{j \omega_0 g_{BBAA}}{2} (\mathbf{c} \cdot \mathbf{B}_1 \cdot \hat{\mathbf{a}}^* \mathbf{B}_0^* + \mathbf{E}_1 \cdot \hat{\mathbf{a}}^* \mathbf{E}_0^*)
\]
\[
- \frac{j \omega_0 g_{CCAA}}{2} (g_{BBAA} \mathbf{B}_1 \cdot \hat{\mathbf{a}} \mathbf{E}_0^* + g_{CCAA} \mathbf{E}_1^* \cdot \hat{\mathbf{a}}^* \mathbf{B}_0^*)
\]
Now by substituting (30) and (31) into (27), we find
\[
\nabla \cdot \text{Re} \left( \mathbf{S}_1 \right) = -\frac{1}{4} (\mathbf{E}_1 \cdot \mathbf{J}_e^* - \mathbf{E}_1^* \cdot \mathbf{J}_e)
\]
\[
+ \frac{j \omega_0 g_{BBAA}}{4} (\mathbf{B}_1 \cdot \hat{\mathbf{a}}^* \mathbf{B}_0^* - \mathbf{B}_1^* \cdot \hat{\mathbf{a}} \mathbf{B}_0)
\]
\[
+ \frac{j \omega_0 g_{CCAA}}{4} (\mathbf{E}_1^* \cdot \hat{\mathbf{a}} \mathbf{E}_0 - \mathbf{E}_1 \cdot \hat{\mathbf{a}}^* \mathbf{E}_0^*)
\]
\[
+ \frac{j \omega_0 g_{BBAA}}{4} (\mathbf{B}_1^* \cdot \hat{\mathbf{a}} \mathbf{E}_0 - \mathbf{B}_1 \cdot \hat{\mathbf{a}}^* \mathbf{E}_0^*)
\]
\[
+ \frac{j \omega_0 g_{CCAA}}{4} (\mathbf{E}_1 \cdot \hat{\mathbf{a}}^* \mathbf{B}_0^* - \mathbf{E}_1^* \cdot \hat{\mathbf{a}} \mathbf{B}_0^*)
\]
Then, applying the divergence theorem, we obtain
\[
\oint \text{Re} \left( \mathbf{S}_1 \right) \cdot \hat{\mathbf{n}} ds = \int \left( -\frac{1}{4} (\mathbf{E}_1 \cdot \mathbf{J}_e^* - \mathbf{E}_1^* \cdot \mathbf{J}_e)
\right.
\]
\[
+ \frac{j \omega_0 g_{BBAA}}{4} (\mathbf{B}_1 \cdot \hat{\mathbf{a}}^* \mathbf{B}_0^* - \mathbf{B}_1^* \cdot \hat{\mathbf{a}} \mathbf{B}_0)
\]
\[
+ \frac{j \omega_0 g_{CCAA}}{4} (\mathbf{E}_1^* \cdot \hat{\mathbf{a}} \mathbf{E}_0 - \mathbf{E}_1 \cdot \hat{\mathbf{a}}^* \mathbf{E}_0^*)
\]
\[
+ \frac{j \omega_0 g_{BBAA}}{4} (\mathbf{B}_1^* \cdot \hat{\mathbf{a}} \mathbf{E}_0 - \mathbf{B}_1 \cdot \hat{\mathbf{a}}^* \mathbf{E}_0^*)
\]
\[
+ \frac{j \omega_0 g_{CCAA}}{4} (\mathbf{E}_1 \cdot \hat{\mathbf{a}}^* \mathbf{B}_0^* - \mathbf{E}_1^* \cdot \hat{\mathbf{a}} \mathbf{B}_0^*) \right) dV
\]
This equation may be used to calculate the expected power in the photon–axion energy conversion in a resonant system.

### 4. Sensitivity of Axion Resonant Haloscopes under DC Magnetic Fields

In this section, we calculate the sensitivity of conventional Sikivie-type resonant axion haloscope. If we assume that the background degree of freedom is only excited by a DC magnetic field, then background field equations (5) become
\[
\mathbf{\nabla} \cdot \mathbf{B}_0 = \mu_0 \mathbf{J}_e, \quad \mathbf{\nabla} \cdot \mathbf{B}_0 = 0, \quad \mathbf{E}_0 = 0
\]
Since the phase of the photon leaving the cavity is not observable, we may arbitrarily set the axion phase, setting \(a_0 = \hat{\mathbf{a}} = \hat{\mathbf{a}}^* = 0\)
\[ \sqrt{2}(a_0), \] (33) becomes,
\[ \oint \text{Re} \left( \mathbf{S}_1 \right) \cdot \hat{n} \, ds = \int \left( -\frac{1}{4} \left[ \mathbf{E}_1 \cdot \mathbf{J}_1^* + \mathbf{E}_1^* \cdot \mathbf{J}_1 \right] + \frac{j\omega_0 \varepsilon_0(a_0)}{2\sqrt{2}} \mathbf{B}_0 \cdot \mathbf{E}_1 \cdot \mathbf{J}_1 \right) \, dV \] (35)

The first thing we notice is that it is quite clear that a haloscope with a DC magnetic field is not sensitive to \( g_{\text{aBB}} \).

In the following, we assume a resonant cavity of volume, \( V_1 \), with resonant modes of stored electromagnetic energy, \( U_1 \), given by,
\[ U_1 = \frac{\varepsilon_0}{4} \int \mathbf{E}_1 \cdot \mathbf{E}_1^* \, dV + \frac{1}{4\mu_0} \int \mathbf{B}_1 \cdot \mathbf{B}_1^* \, dV \] (36)
where the first term is the electric stored energy and the second term is the magnetic. The electric and magnetic energy must be equal; thus, we may also write \( U_1 = \frac{1}{2} \int \mathbf{E}_1 \cdot \mathbf{E}_1^* \, dV \).

The dissipative components of \( \mathbf{E}_1 \) and \( \mathbf{B}_1 \) fields are attenuated in the cavity walls in a similar way and are effectively in phase, and we may write the fields as complex with respect to a loss tangent (\( \tan \delta_1 \sim \frac{1}{\omega_1} \)), so \( \mathbf{B}_1 \approx (1 - j \frac{1}{\omega_1}) \text{Re}(\mathbf{B}_1) \) and \( \mathbf{E}_1 \approx (1 - j \frac{1}{\omega_1}) \text{Re}(\mathbf{E}_1) \), where the real terms relate to the oscillating fields over the volume and the imaginary terms relate to dissipative fields, and thus \( \mathbf{E}_1 - \mathbf{E}_1^* = -j \frac{1}{\omega_1} \text{Re}(\mathbf{E}_1) \).

Furthermore, on resonance and in steady state, we may assume \( \oint \text{Re} \left( \mathbf{S}_1 \right) \cdot \hat{n} \, ds = 0 \), meaning there is no external energy inputted at the resonant mode frequency; so the axion generated signal power, \( P_{\text{d}} \), in the resonant mode is equal to the dissipated power, \( P_{\text{d}} \), so \( P_{\text{d}} \) in Equation (35) may be identified as,
\[ P_{\text{d}} = P_{\text{d}} = \frac{\alpha_1 U_1}{Q_1} \]
\[ = g_{0\text{AB}} \frac{\omega_0 \varepsilon_0(a_0)}{\sqrt{2} Q_1} \int \mathbf{B}_0 \cdot \text{Re}(\mathbf{E}_1) \, dV \] (38)

Combining (36) with (38), we may obtain
\[ \sqrt{U_1} = \frac{\alpha_0 \sqrt{\varepsilon_0(a_0)}}{\alpha_1} \times \left[ \frac{g_{0\text{AB}}}{\sqrt{2} Q_1} \int \mathbf{B}_0 \cdot \text{Re}(\mathbf{E}_1) \, dV + \frac{g_{0\text{AB}}}{\sqrt{2} Q_1} \int \mathbf{B}_0 \cdot \text{Re}(\mathbf{B}_1) \, dV \right] \] (39)

Now defining the following form factors of the haloscope
\[ C_{1\text{aBB}} = \frac{\left( \int \mathbf{B}_0 \cdot \text{Re}(\mathbf{E}_1) \, dV \right)^2}{B_0^2 V_1 / \int \mathbf{E}_1 \cdot \mathbf{E}_1^* \, dV} \]
and
\[ C_{1\text{aAB}} = \frac{\left( \int \mathbf{B}_0 \cdot \text{Re}(\mathbf{B}_1) \, dV \right)^2}{B_0^2 V_1 / \int \mathbf{B}_1 \cdot \mathbf{B}_1^* \, dV} \]

and setting \( \langle a_0 \rangle^2 = \frac{\rho_a B_0^2}{\omega_0} \), where \( \rho_a \) is the axion dark matter density, the signal gained at the output of the haloscope becomes
\[ P_1 = \sqrt{\omega_0 Q_1 U_1} \]
\[ = \left( g_{0\text{AB}} C_{1\text{aAB}} + \left| \frac{g_{0\text{AB}}}{\sqrt{2} Q_1} \right| \right) \mu_0 \varepsilon_0 c V_1 \]
\[ = \left( g_{0\text{AB}} C_{1\text{aAB}} + \left| \frac{g_{0\text{AB}}}{\sqrt{2} Q_1} \right| \right) \mu_0 \varepsilon_0 c V_1 \] (40)

Considering the modes in a cylindrical cavity, with the z-axis aligned with a DC magnetic field, so \( \mathbf{B}_0 = B_0 \mathbf{z} \), then \( \mathbf{B}_0 \cdot \mathbf{z} \, dV = 0 \) for all modes if \( B_0 \) is constant, and the calculation is consistent with what has been derived previously[18,34,35] equivalent to the well-known sensitivity equation of a Sikivie-type axion haloscope, which to first order is not sensitive to \( g_{\text{aAB}} \) or \( g_{\text{aBB}} \) and is only sensitive to \( g_{0\text{BB}} \).

To gain sensitivity to \( g_{\text{aAB}} \) with a DC haloscope, one can apply a nonuniform DC magnetic field with other vector components or a DC electric field (see next section), in a similar way to scalar-field dark matter experiments recently proposed in ref.[56], which set limits on the dilaton scalar coupling parameter, \( g_{\phi\gamma\gamma} \). It turns out that the limits set on \( g_{\phi\gamma\gamma} \) in ref.[56] are equivalent to limits on the axion–photon parameter, \( g_{\text{aAB}} \); so in effect, this paper also sets limits on \( g_{\text{aAB}} \).

5. Sensitivity of Axion Resonant Haloscopes under DC Electric Fields

If we assume that the background degree of freedom is only excited by a DC electric field, then background field equations (5) become
\[ \nabla \times \mathbf{E}_0 = 0, \mathbf{E}_0 \cdot \mathbf{E}_0 = \rho_{\text{a}} B_0 = 0 \] (42)
Since the phase of the photon leaving the cavity is not an observable, we may arbitrarily set the axion phase, setting $\hat{a} = \hat{a}^* = -a_0 = -\sqrt{2}(a_0)$ (33) becomes,

$$\oint \text{Re} \left( S_1 \right) \cdot \hat{n} \, ds = \int \left( -\frac{1}{4} (E_1 \cdot J_1^* - E_1^* \cdot J_1) + \frac{j \omega a \varepsilon_0 (a_0) g_{aAB} \tilde{E}_0 \cdot (E_1 - E_1^*)}{2\sqrt{2}} + \frac{j \omega a \varepsilon_0 (a_0) g_{aBB} \tilde{E}_0 \cdot (B_1 - B_1^*)} \right) \, dV$$

(43)

Now defining the following form factors of the haloscope

$$C_{1ABm} = \frac{(\int \tilde{E}_0 \cdot \text{Re}(E_1) \, dV)^2}{E_0^2 V_1 / \int E_1 \cdot E_1^* \, dV} \quad \text{and}$$

$$C_{1BB} = \frac{(\int \tilde{E}_0 \cdot \text{Re}(B_1) \, dV)^2}{E_0^2 V_1 / \int B_1 \cdot B_1^* \, dV}$$

the signal gained at the output of the haloscope becomes

$$\sqrt{P_1} = \sqrt{\omega_0 Q_1 U_1} = (g_{aBB} \sqrt{C_{1BB}} + g_{aAB} \sqrt{C_{1aABm}}) (a_0) E_0 \sqrt{\omega_0 Q_1 \varepsilon_0 V_1}$$

(45)

$$= (g_{aBB} \sqrt{C_{1BB}} + g_{aAB} \sqrt{C_{1aABm}}) E_0 \sqrt{\frac{\rho_0 \varepsilon_0 c^2 V_1}{\omega_0}}$$

Thus, by applying a DC electric field to a cavity resonator, the experiment becomes sensitive to $g_{aBB}$ and $g_{aAB}$, where the latter is sensitive to TM$_{0,0,0}$ modes in a cylindrical cavity resonator if an electric field is applied along the cylinder axis. However, to attain sensitivity to $g_{aBB}$, a more complicated DC electric field is required. It is a much harder experiment to supply a large DC electric field across a high-Q tunable cavity, even though the QCD axion is supposed to have a larger coupling to $g_{aBB}$ than $g_{aAB}$.[31] Thus, the resonant cavity technique might not be the optimum way to make use of this larger coupling, as it is much easier to configure an experiment with a large DC magnetic field in comparison to a large DC electric field.

6. Sensitivity of Upconversion Resonant Haloscopes

The upconversion technique makes use of two resonant modes in the same resonant cavity, a background mode, which we label with subscript 0, and a readout mode, which we label with subscript 1. This technique upconverts the axion to the carrier frequency of the readout mode and allows the use of high-Q cavities instead of large magnetic fields and allows search for axions in the frequency range $\omega_0 << \omega_1$. This technique was first proposed in ref. [57, 58] and experimentally demonstrated in ref. [59] and showed that a putative dark matter axion background would perturb the frequency (or phase) and amplitude (or power) of the readout mode. The former we call the “frequency technique” and the latter the “power technique.” The power technique was also proposed in ref. [60–63] and later performed in ref. [64]. In this work, we calculate the sensitivity of this experiment to the three axion coupling parameters ($g_{aTT}$, $g_{aAB}$, $g_{aBB}$).

6.1. Power Technique

Here, we use the Poynting vector equation (33) to derive the sensitivity of the power technique, where the background field will mix with the axion to generate power at the readout mode frequency. Since the phase of the photon leaving the cavity is not our observable, we may arbitrarily set the axion phase, setting $a_0 = \hat{a} = \hat{a}^* = \sqrt{2}(a_0)$; Equation (33) becomes,

$$\oint \text{Re} \left( S_1 \right) \cdot \hat{n} \, ds = \int \left( -\frac{1}{4} (E_1 \cdot J_1^* - E_1^* \cdot J_1) + \frac{j \omega a \varepsilon_0 (a_0) g_{aBB} \tilde{E}_0 \cdot (B_1^* \cdot E_0 - B_1 \cdot E_0^*)}{4} \right) \, dV$$

(46)

where we can identify the power generated by the axion as

$$P_{a1} = \frac{j \omega a \varepsilon_0 (a_0) g_{aBB} \sqrt{2} (a_0)}{4}$$

$$\times \left( B_1^* \cdot E_0 - B_1 \cdot E_0^* \right) \, dV$$

(47)

Note, the sensitivity coefficients to $g_{aAB}$ drop out, as in the lossless limit they are identically zero.

Ignoring losses in the background fields, we individually consider them real for both $E_0$ and $B_0$.

$$P_{a1} = \frac{\omega a \varepsilon_0 (a_0)}{\sqrt{2} Q_1}$$

$$\times \int \left( g_{aTT} \text{Re}(E_1) \cdot \text{Re}(B_0) - g_{aBB} \text{Re}(B_1) \cdot \text{Re}(E_0) \right) \, dV$$

(48)

Equating $P_{a1}$ in (48) to $P_{d} = \frac{n_{a1}}{Q_1}$, we can write the stored energy as

$$U_1 = \frac{\omega a \varepsilon_0 (a_0)}{\sqrt{2} \omega_1} \left( g_{aTT} \int \text{Re}(E_1) \cdot \text{Re}(B_0) \, dV - g_{aBB} \int \text{Re}(E_1) \cdot \text{Re}(B_1) \, dV \right)$$

(49)
Since \( U_1 = \frac{1}{2} \int E_1 \cdot E_1^* dV \), we obtain

\[
\sqrt{U_1} = \frac{\omega_a e_0 (a_0)}{\sqrt{2} \omega_1} \times \left( g_{agg} \frac{\sqrt{2}}{\sqrt{2}} \int E_1 \cdot E_1^* dV - g_{aBB} \frac{\sqrt{2}}{\sqrt{2}} \int E_1 \cdot E_1^* dV \right)
\]

(50)

where the overlap functions are defined by \cite{57}

\[
\xi_{10} = \int e_1 \cdot b_0 dV \quad \text{and} \quad \xi_{01} = \int e_0 \cdot b_1 dV
\]

(52)

Then, defining the unit vectors, so \( cB_0 = E_{00}b_0 \) and \( E_0 = E_{00}e_0 \), and \( cB_1 = E_{01}b_1 \) and \( E_1 = E_{01}e_1 \), then (50) becomes,

\[
\sqrt{U_1} = \frac{\omega_a \sqrt{E_{00}} (a_0)}{\omega_1} \times \left( \frac{1}{\sqrt{2}} \int e_1 \cdot b_0 dV - \frac{1}{\sqrt{2}} \int b_1 \cdot e_0 dV \right)
\]

(51)

\[
= (a_0) \frac{\omega_a}{\omega_1} \frac{2P_{\text{inc}}}{\ell^2} \left( g_{agg} \xi_{10} - g_{aBB} \xi_{01} \right)
\]

(53)

where the square of the overlap functions are analogous to the form factors in the previous sections.

Here, \( E_{00} = \sqrt{2P_{\text{inc}}} \) where \( P_{\text{inc}} \) is the circulating power of the background mode over the cavity volume \( V \), which is related to the incident power, \( P_{\text{inc}} \), by,

\[
P_{\text{inc}} = \frac{4\beta_0 q_{\ell_0}}{(\beta_0 + 1)^2} P_{\text{inc}}
\]

where \( \beta_0 \) is the background mode coupling to the cavity and \( q_{\ell_0} \) the mode loaded quality factor. Now, we can determine the square root power in the coupling circuit of the readout mode to be

\[
\sqrt{P_{\text{out}}} = \frac{\sqrt{\omega_1 U_1 \sqrt{P_{\text{inc}}}}}{\sqrt{1 + \beta_1 (1 + 4 Q_{\ell_1}^2 \frac{\delta\omega_a}{\omega_1})}}
\]

(54)

where we define \( \delta\omega_a = \omega_1 + \omega_a - \omega_0 \), so when \( \delta\omega_a = 0 \) then \( \omega_a = \omega_0 \) and the axion induced power is upconverted to the frequency, \( \omega_1 \). Thus, \( \delta\omega_a \) defines the detuning of the induced power with respect to the readout mode frequency. Combining (51)–(54), we obtain

\[
\sqrt{P_{\text{out}}} = K_{agg} (a_0) + K_{aBB} (a_0)
\]

(55)

where

\[
K_{agg} = \frac{\xi_{10} 2 \sqrt{2} \omega_0 \sqrt{P_{\text{inc}}}}{\omega_1 \sqrt{1 + \beta_1} (\beta_0 + 1)} \left( 1 + 4 Q_{\ell_1}^2 \frac{\delta\omega_a}{\omega_1} \right)^{\frac{1}{2}}
\]

(56)

\[
K_{aBB} = - \frac{\xi_{01} 2 \sqrt{2} \omega_0 \sqrt{P_{\text{inc}}}}{\omega_1 \sqrt{1 + \beta_1} (\beta_0 + 1)} \left( 1 + 4 Q_{\ell_1}^2 \frac{\delta\omega_a}{\omega_1} \right)^{\frac{1}{2}}
\]

which are the axion gain coefficients in units of square root power. Equation (55) can be used to calculate the sensitivity for the power technique and is sensitive to the effective monopole coupling term \( g_{aBB} \) unlike the Sikivie-type detectors that utilizes a DC \( B \) field.

To calculate the signal to noise ratio (SNR) for virialized axion dark matter from the galactic halo, we take into account that it presents as a narrow band noise source with a line width of a part in \( 10^6 \). For SI units, we may relate the axion amplitude to the background dark matter density in the galactic halo, \( \rho_a \), by \( (a_0) = \frac{\sqrt{2\pi}}{\omega_1} \). For \( i = \alpha \) or \( i = \beta \), then limits on the axion couplings can be undertaken independently by calculating,

\[
SNR_{\alpha} = \frac{g_{\alpha} |K_{P_{\alpha}}|}{\alpha_\ell V_{\ell} \ell^{3}} \left( \frac{1}{\Delta f_{\ell}} \right)^{\frac{1}{2}}
\]

(57)

where, \( P_{\alpha} \) (Watts/Hz) is the noise power competing with the axion signal and \( \Delta f_{\ell} \) is the axion bandwidth in Hz, where \( \Delta f_{\ell} = \frac{T_{\text{temp}}}{\ell^2} \) for virialized dark matter. This assumes the measurement time, \( t \) is greater than the axion coherence time so that \( t > \Delta f_{\ell}^{-1} \). For measurement times of \( t < \frac{10^4}{\Delta f_{\ell}} \), we substitute \( \left( \frac{10^4}{\Delta f_{\ell}} \right)^{\frac{1}{2}} \rightarrow t^{\frac{1}{2}} \). The noise power in such experiments is dominated by thermal noise in the readout mode resonator of effective temperature, \( T_{\text{r}} \) and the noise temperature of the first amplifier, \( T_{\text{ramp}} \) after the readout mode, and is given by \cite{65,66}

\[
P_N \sim \frac{4\beta_1}{(\beta_0 + 1)^2} \left( 1 + 4 Q_{\ell_1}^2 \left( \frac{\delta\omega_a}{\omega_1} \right)^2 \right) \frac{k_B T_1}{2} + \frac{k_B T_{\text{ramp}}}{2}
\]

(58)

In the case \( \beta_1 \sim 1 \) and \( \delta\omega_a \sim 0 \), then \( P_N \sim \frac{k_B T_{\text{ramp}} + T_{\text{ramp}}}{2} \), and assuming \( \beta_0 \sim 1 \), the signal to noise ratios become

\[
SNR_{\alpha} \sim \frac{g_{\alpha} |\xi_{10}|}{\alpha_\ell V_{\ell} \ell^{3}} \sqrt{\frac{2 \omega_0 Q_{\ell_1} P_{\text{inc}} P_{\text{ramp}}}{\alpha_\ell \alpha_\ell k_B T_{\text{ramp}} (T_{\text{ramp}} + T_{\text{ramp}})}} \left( \frac{10^4}{\ell} \right)^{\frac{1}{2}}
\]

(59)

6.2. Resonant Haloscope Frequency Shift from Perturbation Analysis

As pointed out previously, there is no first order frequency shift for DC Sikivie haloscopes as they are only second order sensitive
to frequency shifts. However, when two AC modes are excited, the situation is different as the DC and AC halosopes belong to different classes of detectors. Since virtual photons or static fields carry no phase, the DC haloscope belongs to the class of phase insensitive systems. In contrast, the AC scheme relies on a pump signal carrying relative phases to the readout signal and axion field, and since this occurs in a resonant cavity, phase shifts are converted to frequency shifts. This is analogous to existing amplifiers that can be grouped into DC (phase insensitive) amplifiers, where energy is drawn from a static power supply and parametric (phase sensitive) amplifiers, where energy comes from oscillating fields.

To calculate frequency shifts, here we adapt the perturbation theory technique to axion-modified electrodynamics as opposed to the quantum optics technique used in the past,[57,58] which gives the same result. We consider the perturbed readout resonator mode fields, \((E_1', B_1')\) and frequency, \(\alpha_1'\), due to the mixing of the axion and the background pump mode, where the unperturbed modes and frequency are given by Ampere's law and Faraday's law in standard electrodynamics:

\[
\frac{1}{\mu_0} \nabla \times B_1 = J_{d1} - j\omega_1 e_0 E_1, \quad \nabla \times E_1 = j\omega_1 B_1
\]

and the perturbed Ampere's law and Faraday's law are derived from (23) and (24), and written as

\[
\frac{1}{\mu_0} \nabla \times B_1' = J_{d1}' - j\omega_1 e_0 E_1', \quad \nabla \times E_1' = j\omega_1 B_1'
\]

Substituting (60), (61) and (62) into (63), one obtains,

\[
\frac{1}{\mu_0} \int_V \nabla \cdot (E_1' \times B_1' + E_1' \times B_1^*')dV
\]

\[
= \frac{1}{\mu_0} \int_V (B_1' \cdot \nabla \times E_1' - E_1' \cdot \nabla \times B_1' + B_1' \cdot \nabla \times E_1' - E_1' \cdot \nabla \times B_1')dV
\]

where \(\delta \omega_1 = \alpha_1' - \omega_1\). Now to apply perturbation theory, we set all unperturbed fields and currents to approximately their unperturbed values, and given that 

\[
\frac{1}{\mu_0} \int_V \nabla \cdot (E_1' \times B_1 + E_1 \times B_1^*)dV = -\int_V (E_1' \cdot J_{d1} + E_1 \cdot J_{d1}^*)dV
\]

and

\[
\delta \omega_1 \approx \frac{\omega_1 e_0}{4\alpha_1 U_1} \left( \int_V (g_{\alpha}E_1' \cdot \vec{a}B_0 - g_{\alpha AB}B_1' \cdot \hat{a}E_0 - g_{\alpha AB}(c^2B_1' \cdot \vec{a}B_0 - E_1' \cdot \hat{a}E_0)dV \right)
\]

This equation may be used to calculate resonant cavity frequency shifts based on upconversion.

### 6.3. Frequency Technique

In this section, we use perturbation analysis via Equation (65) to derive the sensitivity of the frequency technique, where the background field mixes with the axion to perturb the frequency of the readout mode. Since the integrals must be real, then in terms of unit vectors, we obtain

\[
\frac{\delta \omega_1}{\alpha_1} = \frac{\omega_1 e_0}{2E_0 \alpha_1} \frac{1}{\sqrt{\frac{2P_e}{m_{c1} c^2}}} \int \frac{g_{\alpha}cE_1' \cdot \vec{a}b_0 - g_{\alpha AB}B_1' \cdot \hat{a}e_0}{P_{inc}} dV
\]

Now given that \(E_{01} = \sqrt{\frac{2P_e}{m_{c1} c^2}}\) and \(P_{inc} = \frac{4\bar{\rho}_{Q, 1}^2}{\rho_{1+1}^2}\), and by considering the sidebands transferred to the coupling circuit, taking the root mean square average of both sides, then (66) may be written as

\[
\langle \frac{\delta \omega_1}{\alpha_1} \rangle = \mathcal{K}_{\omega_{\alpha}} g_{\alpha} \langle a_0 \rangle + \mathcal{K}_{\omega_{\alpha AB}} g_{\alpha AB} \langle a_0 \rangle
\]

where

\[
\mathcal{K}_{\omega_{\alpha}} = \frac{1}{2} \frac{\alpha_1}{\sqrt{\frac{2\rho_{000}}{m_{c1} c^2}}} \frac{\sqrt{\frac{P_{inc}}{P_{inc}}}}{\sqrt{\frac{P_{inc}}{P_{inc}}}} \sim P_{inc} \epsilon_{01}^{10}
\]

\[
\mathcal{K}_{\omega_{\alpha AB}} = -\frac{1}{2} \frac{\alpha_1}{\sqrt{\frac{2\rho_{000}}{m_{c1} c^2}}} \frac{\sqrt{\frac{P_{inc}}{P_{inc}}}}{\sqrt{\frac{P_{inc}}{P_{inc}}}} \sim P_{inc} \epsilon_{01}^{10}
\]
and depend on the same normalized overlap functions as in the power technique.

To calculate the SNR for virialized axion dark matter from the galactic halo, for \( i = \alpha a \) or \( i = \alpha a a a \), then limits on the axion couplings can be undertaken independently by calculating,

\[
SNR_{i} = \frac{\left| K_{\omega} \right|}{\omega_{i} S_{i}} \sqrt{P_{\text{inc}} c^{3}} \left( \frac{1}{\Delta f_{b}} \right)^{2} \tag{69}
\]

where, \( S_{i} = 1/\text{Hz} \) is the fractional frequency fluctuations competing with the axion signal in the readout oscillator.

The lowest noise oscillators are frequency stabilized by a phase detection scheme, which in principle is limited by the effective readout system noise temperature \( T_{RS} \) of the internal phase detector (and includes the resonator and amplifier noise temperature), which is close to ambient temperature for a well-designed system,\(^{[67]}\) and in such a case, the oscillator noise will be

\[
\sqrt{S_{i}} = \sqrt{\frac{\kappa_{i} T_{RS}}{2Q_{i} \sqrt{P_{\text{inc}}^{2} + P_{\text{inc}}^{2}}}} \left( 1 + \frac{\beta_{i}}{2 \beta_{i}} \sqrt{\frac{2Q_{i} \delta v_{\omega}}{\omega_{i}}} \right)^{2} + 1 \tag{70}
\]

where \( P_{\text{inc}} \) is power incident on the input port to the readout mode. In this configuration, usual operating conditions will require \( \beta_{i} \sim 1, \beta_{i} \sim 1 \) and \( \delta v_{\omega} \sim 0 \); then, the SNRs become

\[
SNR_{\text{SNR}} \sim g_{\alpha a} \sqrt{\frac{\delta v_{\omega}}{\omega_{i} \delta v_{\omega}^{2}}} \sqrt{\frac{2Q_{i} \delta v_{\omega}^{2} P_{\text{inc}}^{2} c^{3}}{\omega_{i} \delta v_{\omega}^{2} \kappa_{i} T_{RS}}} \left( \frac{10^{4} t}{f_{a}} \right)^{2} \tag{71}
\]

and

\[
SNR_{\text{SNR}} \sim g_{a a a a} \sqrt{\frac{\delta v_{\omega}}{\omega_{i} \delta v_{\omega}^{2}}} \sqrt{\frac{2Q_{i} \delta v_{\omega}^{2} P_{\text{inc}}^{2} c^{3}}{\omega_{i} \delta v_{\omega}^{2} \kappa_{i} T_{RS}}} \left( \frac{10^{4} t}{f_{a}} \right)^{2} \tag{72}
\]

Note, the power and frequency techniques derive the same SNR, which depends on the system noise temperature. Thus, the inherent sensitivity of both techniques is the same; which one is better will depend on which can be configured better experimentally to be less influenced by the relevant noise sources and external systems.

### 6.4. Sensitivity to Ultra-Light Axions

Another upconversion technique worth mentioning is the use of the anyon cavity resonator, which uniquely allows the detection of ultra-light axion dark matter due to the non-zero normalized helicity of the cavity mode, given by

\[
\mathcal{H}_{i} = \frac{2 \Im \left[ \int B_{i}(\hat{r}) \cdot E_{i}^{*}(\hat{r}) \, dV \right]}{\sqrt{\int E_{i}(\hat{r}) \cdot E_{i}(\hat{r}) \, dV / \int B_{i}(\hat{r}) \cdot B_{i}(\hat{r}) \, dV}} \tag{73}
\]

so the single mode may act as its own background field.\(^{[68,69]}\) This technique has been detailed in ref.\(^{[68]}\) and included the QEMD terms to show that this technique was sensitive to the sum of \( g_{\alpha a} \) and \( g_{a a a a} \), where the helicity is equivalent to the overlap functions defined for the two-mode upconversion detectors. In this case, both \( g_{\alpha a} \) and \( g_{a a a a} \) have the same overlap function.

### 7. Discussion

In this paper, we have applied Poisson's theorem to axion-modified electrodynamics. In QEMD, there are three parameters to put limits on \( g_{\alpha a} \), \( g_{a a a a} \), and \( g_{a a a a a} \). In principle maybe related if the axion is a QCD axion\(^{[1]}\) and is a generalization of the two-photon chiral anomaly if magnetic charge exists. We have shown if the background electromagnetic field is a DC magnetic field; then we can only simply configure the experiment to be directly sensitive to \( g_{\alpha a} \). Nevertheless, with a more complicated DC background magnetic field, sensitivity to both \( g_{a a a a} \) and \( g_{a a a a a} \) may be obtained. Conversely, if the background electromagnetic field is a DC electric field, then we can only simply configure the experiment to be directly sensitive to \( g_{a a a a a} \). Nevertheless, with a more complicated DC background electric field, sensitivity to both \( g_{a a a a a} \) and \( g_{a a a a a} \) may be obtained. However, if the background field is an oscillating electromagnetic field, the experiment can be sensitive to \( g_{\alpha a} \) and \( g_{a a a a} \) at the same time, as we have shown for the upconversion experiments.

Note, the upconversion experiments search mass ranges less than a \( \mu eV \), which is a much lower mass range when compared to the DC Sikivie haloscope.\(^{[9]}\) In particular, the anyon cavity haloscope is uniquely sensitive to ultra-light axions.\(^{[68]}\) Other ways to search for axions in the low mass range include reactive axion haloscopes, such as those based on capacitors\(^{[19,56]}\) or inductors\(^{[70]}\); however, they were not considered in this work.

The caveat is that we have treated all the axion photon couplings to the axion field, \( a \), as independent, and then for each of the couplings, the terms \( \alpha \mathcal{V} \mathcal{A} \) may be obtained. However, it was shown in ref.\(^{[1]}\) that \( \mathcal{V} a \) multiplied by \( g_{a a a a} \) can be of the same order of magnitude as \( \delta a \) multiplied by \( g_{a a a a a} \), which could lead to extra sensitivity than to what is calculated in this paper.

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### Conflict of Interest

The authors declare no conflict of interest.

### Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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axion dark matter, axion haloscopes, magnetic charged matter
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