Low-lying zeros of $L$-functions for Maass forms over imaginary quadratic fields

Sheng-Chi Liu and Zhi Qi

Abstract. We study the 1- or 2-level density of families of $L$-functions for Hecke–Maass forms over an imaginary quadratic field $F$. For test functions whose Fourier transform is supported in $\left( -\frac{1}{2}, \frac{1}{2} \right)$, we prove that the 1-level density for Hecke–Maass forms over $F$ of square-free level $q$, as $N(q)$ tends to infinity, agrees with that of the orthogonal random matrix ensembles. For Hecke–Maass forms over $F$ of full level, we prove similar statements for the 1- and 2-level densities, as the Laplace eigenvalues tend to infinity.

1. Introduction

Katz and Sarnak [KS2, KS1] predict that the limiting behavior of the low-lying zeros (zeros near the central point $s = \frac{1}{2}$) of a family of $L$-functions agrees with that of the eigenvalues near 1 of the ensemble of random matrices associated to the family. There is now a vast literature on verifying this conjectural agreement (the Katz–Sarnak heuristics) for various families of $L$-functions. We refer the reader to the pioneer work of Iwaniec–Luo–Sarnak [ILS], and also the survey articles [BFMTB, MMR] for an extensive set of references.

The purpose of this paper is to verify the Katz–Sarnak heuristics (for suitably restricted test functions) for families of Hecke–Maass forms over an imaginary quadratic field. Our work is in parallel with [AAI, AM], where they consider Maass forms over $\mathbb{Q}$. Moreover, we shall give a simple proof of (a variant of) the result in [AM] and, in addition, consider the families of even and odd Maass forms for $\text{SL}_2(\mathbb{Z})$.

1.1. The Katz–Sarnak density conjecture. The main statistic we study in this paper is the 1- or 2-level density.

Definition 1.1 (1- and 2-level densities). Define the 1-level density of an $L$-function $L(s, f)$ with non-trivial zeros $\frac{1}{2} + i\gamma_f$ (note that $\gamma_f \in \mathbb{R}$ if the generalized Riemann hypothesis holds for $L(s, f)$) by

$$D_1(f, \phi, R) = \sum_{\gamma_f} \phi \left( \frac{\log R}{2\pi} \gamma_f \right).$$

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where \( \phi : \mathbb{R} \to \mathbb{C} \) is an even Schwartz function such that its Fourier transform

\[
\hat{\phi}(y) = \int_{-\infty}^{\infty} \phi(x)e(-xy)dx,
\]

has compact support (as usual, \( e(x) = e^{2\pi ix} \)), and \( R \) is a scaling parameter. Similarly, the 2-level density is defined by

\[
D_2(f, \phi_1, \phi_2, R) = \sum_{j_1 \neq j_2} \phi_1 \left( \frac{\log R}{2\pi} \gamma_f^{(j_1)} \right) \phi_2 \left( \frac{\log R}{2\pi} \gamma_f^{(j_2)} \right),
\]

where the zeros \( \frac{1}{2} + iy_f^{(j)} \) are labeled such that \( \gamma_f^{(j)} = -\gamma_f^{(-j)} \). The \( n \)-level density may be defined in the same way.

**Remark 1.2.** The definitions in (1.1) and (1.3) make sense (even if \( \gamma_f \) are not real) irrespective of the Riemann hypothesis for \( L(s, f) \) because \( \phi \) is entire. Unless otherwise stated, we shall not assume any generalized Riemann hypothesis in this paper (the Riemann hypothesis for the Riemann \( \zeta \)-function or Dirichlet L-functions will be assumed only in the two theorems in Appendix A).

To be precise, let \( \mathcal{F} = \bigcup Q \mathcal{F}(Q) \) be a family of automorphic forms, ordered by their conductors \( Q \). The Katz–Sarnak density conjecture states that as the conductor \( Q \to \infty \), the averaged \( n \)-level density \( D_n(\mathcal{F}(Q), \phi) \) (its definition will be made precise in our setting as in (1.7), (1.8)) with extra weights and averages) converges to the limiting \( n \)-level density for the associated symmetry group \( G(\mathcal{F}) \):

\[
\lim_{Q \to \infty} D_n(\mathcal{F}(Q), \phi) = \int \cdots \int \phi(x_1, \ldots, x_n) W_n(G(\mathcal{F}))(x_1, \ldots, x_n) dx_1 \cdots dx_n,
\]

where \( \phi(x_1, \ldots, x_n) = \phi_1(x_1) \cdots \phi_n(x_n) \),

| \( G \) | \( W_n(G) \) |
|---|---|
| \( U \) | \( \det (K_0(x_j, x_k))_{j,k \leq n} \) |
| \( \text{Sp} \) | \( \det (K_{-1}(x_j, x_k))_{j,k \leq n} \) |
| \( \text{SO(even)} \) | \( \det (K_1(x_j, x_k))_{j,k \leq n} \) |
| \( \text{SO(odd)} \) | \( \det (K_{-1}(x_j, x_k))_{j,k \leq n} \) |
| & | \( + \sum_{\nu=1}^{n} \delta_0(x_\nu) \det (K_{-1}(x_j, x_\nu))_{j,k \neq \nu} \) |

and

\[
W_n(\text{O}) = \frac{1}{2} \left( W_n(\text{SO(even)}) + W_n(\text{SO(odd)}) \right).
\]

with

\[
K_\varepsilon(x, y) = \frac{\sin (\pi (x - y))}{\pi (x - y)} + \varepsilon \frac{\sin (\pi (x + y))}{\pi (x + y)}.
\]

It is often convenient to look at the Fourier transform side; for the 1-level densities, we have

\[
\hat{\hat{W}}_1(\text{O})(y) = \delta_0(y) + \frac{1}{2}.
\]
\[ \hat{W}_1(\text{SO(even)})(y) = \delta_0(y) + \frac{1}{2} \eta(y), \]
\[ \hat{W}_1(\text{SO(odd)})(y) = \delta_0(y) - \frac{1}{2} \eta(y) + 1, \]
\[ \hat{W}_1(U)(y) = \delta_0(y), \]
\[ \hat{W}_1(\text{Sp})(y) = \delta_0(y) - \frac{1}{2} \eta(y), \]
where \( \eta(y) = 1, \frac{1}{2}, 0 \) for \( |y| < 1, y = \pm 1, |y| > 1 \), and \( \delta_0(y) \) is the Dirac delta distribution at \( y = 0 \).

The different classical compact groups have distinguishable 1-level densities when the support of \( \hat{\theta} \) exceeds \([-1, 1]\). However, for support in \((1, 1)\) the three orthogonal flavors are mutually indistinguishable (though they are different from unitary and symplectic). In cases that it is infeasible to extend the support beyond \([-1, 1]\), one could study the 2-level density, which Miller [Mi1, Mi2] showed distinguishes the three orthogonal ensembles for arbitrarily small support.

**Setup.** Let \( F \) be an imaginary quadratic field, and let \( \mathcal{O} \) be its ring of integers. Assume for simplicity that the class number \( h_F = 1 \).

Let \( \Gamma_0(q) \) be the Hecke congruence subgroup of \( \text{GL}_2(\mathcal{O}) \) with square-free level \( q \). Let \( H^*(q) \) be the collection of primitive Hecke–Maass newforms of level \( q \). For each \( f \in H^*(q) \), we denote by \( 1 + 4\sigma_2^2 \) the Laplacian eigenvalue of \( f \), and by \( \omega_f^1 \) the spectral weight of \( f \) (as defined in (2.17)). We use the notation \( \text{Avg}_q(A; w) \) to denote the average of \( A \) over \( H^*(q) \) weighted by \( w \). That is,

\[ \text{Avg}_q(A; w) = \frac{\sum_{f \in H^*(q)} w(f)A(f)}{\sum_{f \in H^*(q)} w(f)}. \]

### 1.2. Main results

We start with describing the space of weight functions. The space \( \mathcal{H}(S, N) \) below is introduced in [B1] Definition 5.1 and [V1] Definition 4.

**Definition 1.3 (Space of weight functions).** Let \( S > \frac{3}{2} \) and \( N > 6 \). We set \( \mathcal{H}(S, N) \) to be the space of functions \( h : \mathbb{R} \to \mathbb{C} \) which extend to an even holomorphic function on the strip \( \{ t + i\sigma : |\sigma| \leq S \} \) such that, for \( |\sigma| < S \), we have uniformly

\[ h(t + i\sigma) \ll e^{-\pi|\sigma|/(|t| + 1)^{1-N}}. \]

Let \( \mathcal{H}^+(S, N) \) be the space of non-negative valued functions \( h(t) \neq 0 \) in \( \mathcal{H}(S, N) \).

For \( 1 < T < T' \) and \( h(t) \in \mathcal{H}^+(S, N) \), define

\[ h_{T,M}(t) = h((t - T)/M) + h((t + T)/M), \]

which also lies in the space \( \mathcal{H}^+(S, N) \).

### 1.2.1. Low-lying zeros in the level aspect

For \( h(t) \in \mathcal{H}^+(S, N) \) as in Definition 1.3, we let \( R = N(q) \) and define

\[ \mathcal{D}_1(H^*(q), \phi; h) = \text{Avg}_q(\mathcal{D}_1(f, \phi, N(q)); \omega_f^1(h(t))). \]

\[ ^1 \text{In the literature, many authors like to consider congruence subgroups of } \text{SL}_2(\mathcal{O}). \text{ The translation between } \text{SL}_2 \text{ and } \text{GL}_2 \text{ however is usually straightforward (see for example [V1]). The main reason for working on } \text{GL}_2(\mathcal{O}) \text{ is to make the definitions of Hecke operators and } L\text{-functions valid over ideals.} \]
Theorem 1.4. Fix $h \in \mathcal{H}^+(S, N)$. Let $\phi$ be an even Schwartz function with the support of $\hat{\phi}$ in $(-\frac{1}{2}, \frac{1}{2})$. Then

$$\lim_{N(s) \to \infty} D_1(H^*(q), \phi; h) = \int_{-\infty}^{\infty} \phi(x) W_1(O)(x) dx.$$ 

Remark 1.5. It is possible to remove the spectral weights $\omega^*_T$ in the definition (1.7), and this is done in [LLS]. For this, one needs to assume the Riemann hypothesis for $L(s, f)$ and $L(s, \text{Sym}^2 f)$.

1.2.2. Low-lying zeros in the $t$-aspect. Next we investigate the case where $q$ is fixed. For ease of exposition we take $q = 1$. For $h_{T, M}(t)$ as in (1.6), we let $R = T^4$ and define

$$D_1(H^*(1), \phi; h_{T, M}) = \text{Avg}_{(1)} \left( D_1(f, \phi, T^4); \omega^*_T h_{T, M}(t_f) \right).$$

A typical choice of the weight function used by Xiaoqing Li in [LL1] for the subconvexity problem in the $t$-aspect is the following

$$h_{T, M}(t) = e^{-(t-T)^2/M^2}.$$ 

The class of weight functions $h_{T, M}(t)$ in Definition 1.3 serves very well the purpose of localizing to the conductors near $T$ and works far better than the class of $h_T(t)$ used by Alpoge and Miller [AM].

In the real case, by the arguments in [LL1], one may easily deduce that the corresponding (real) Bessel integral $H_{T, M}^+(x)$ is negligibly small for any $1 < x \ll TM^{1-\varepsilon}$ as long as $M = T^\mu$ for some $0 < \mu < 1$. We may therefore avoid the complicated analysis in [AM] which occupies 9 pages. See Appendix A for a simple proof of their result (on the 1-level density for the family of $\text{SL}_2$ Hecke–Maass forms over $\mathbb{Q}$).

When one distinguishes the even and odd Maass forms, there also arises the Bessel integral $H_{T, M}^{-}(x)$. However, it behaves quite differently—$H_{T, M}^{-}(x)$ has an essential support on a neighborhood of $T/2\pi$ with length $M^{1+\varepsilon}$. Under the Riemann hypothesis for classical Dirichlet $L$-functions, it turns out that $H_{T, M}^{-}(x)$ would contribute a main term to the asymptotics. See Appendix A for a more detailed discussion.

In the complex case, however, the analogous analysis can only show that the (complex) Bessel integral $H_{T, M}^{-}(x + iy)$ is negligibly small for $1 < |x| \ll TM^{1-\varepsilon}$ but $|y| \ll T$. Indeed, there is a dramatic change of behavior of $H_{T, M}^{-}(x + iy)$ in the transition range $|y| \approx T$. When $|y| \gg T$, $H_{T, M}^{-}(x + iy)$ is essentially supported on a very narrow sector-like region enclosing the $y$-axis. As a consequence, the angular (or horizontal) derivative $\partial H_{T, M}^{-}(xe^{i\theta})/\partial \theta$ could be very large on this supportive region. See Lemma 5.3 and Remark 5.6.

Theorem 1.6. Let $T, M > 1$ be such that $M = T^\mu$ with $0 < \mu < 1$. Fix $h \in \mathcal{H}^+(S, N)$ and define $h_{T, M}$ by (1.6) in Definition 1.3. Let $\phi$ be an even Schwartz function with the support of $\hat{\phi}$ in $(-1, 1)$. Then

$$\lim_{T \to \infty} D_1(H^*(1), \phi; h_{T, M}) = \int_{-\infty}^{\infty} \phi(x) W_1(\text{SO(even)})(x) dx.$$ 

Even if we assume the Riemann hypothesis for Hecke-character $L$-functions over $F$, it is unlikely that the support of $\hat{\phi}$ can be extended beyond the segment $[-1, 1]$. This will be explained in Remark 5.4. Suffice it to say, the arguments that worked in the real case are invalidated by the largeness of $\partial H_{T, M}^{-}(xe^{i\theta})/\partial \theta$. Therefore we must resort to the averaged 2-level density

$$D_2(H^*(1), \phi_1, \phi_2; h_{T, M}) = \text{Avg}_{(1)} \left( D_2(f, \phi_1, T^4); \omega^*_T h_{T, M}(t_f) \right).$$
According to [Mi12] Theorem 3.2, for an even function \( \hat{\phi}_1(y_1) \hat{\phi}_2(y_2) \) supported in \(|y_1| + |y_2| < 1\),

\[
\int_{-\infty}^{\infty} \hat{\phi}_1(y_1) \hat{\phi}_2(y_2) \hat{W}_2(\text{SO(even)})(y_1, y_2) dy_1 dy_2 = \left( \hat{\phi}_1(0) + \frac{1}{2} \hat{\phi}_1(0) \right) \left( \hat{\phi}_2(0) + \frac{1}{2} \hat{\phi}_2(0) \right)
\]

(1.10)

+2 \int_{-\infty}^{\infty} \left| y \right| \left| \hat{\phi}_1(y) \hat{\phi}_2(y) \right| dy - 2 \hat{\phi}_1(0) \hat{\phi}_2(0),

and for arbitrarily small support, the three 2-level densities for the groups \( \text{O}, \text{SO(even)} \) and \( \text{SO(odd)} \) differ. Thus the following theorem implies that the symmetry type in the \( t_f \)-aspect is \( \text{SO(even)} \).

**Theorem 1.7.** Let \( T, M \) and \( h_{T,M} \) be as in Theorem [1.6]. Let \( \phi_1 \) and \( \phi_2 \) be even Schwartz functions with the supports of \( \hat{\phi}_1 \) and \( \hat{\phi}_2 \) in \((-\frac{1}{2}, \frac{1}{2})\). Then

\[
\lim_{T \to \infty} D_2(H^*(1), \phi_1, \phi_2; h_{T,M}) = \int \phi_1(x_1) \phi_2(x_2) W_2(\text{SO(even)})(x_1, x_2) dx_1 dx_2.
\]

**Notation.** By \( B \ll D \) or \( B = O(D) \) we mean that \( |B| \leq cD \) for some constant \( c > 0 \), and by \( B \approx D \) we mean that \( B \ll D \ll D \). We write \( B \ll_{P, Q, ...} D \) or \( B = O_{P, Q, ...}(D) \) if the implied constant \( c \) depends on \( P, Q, ... \). Throughout this article \( T > 1 \) will be a large parameter, and we say \( B \) is negligibly small if \( B = O(T^{-A}) \) for any \( A > 0 \). We adopt the usual \( \epsilon \)-convention of analytic number theory; the value of \( \epsilon \) may differ from one occurrence to another.

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## 2. Preliminaries

Throughout this paper, fix an imaginary quadratic field \( F \) of class number \( h_F = 1 \). Let \( \mathcal{O} \) be the ring of integers, \( \mathfrak{b} \) be the different ideal, and \( \mathcal{O}^\times \) be the unit group. Let \( d_F \) be the discriminant and \( w_F = |\mathcal{O}^\times| \). Let \( N \) and \( \text{Tr} \) be the norm and the trace map on \( F \).

In general, we shall use Gothic letters \( a, m, n, p, q, v, ... \) to denote nonzero fractional (mostly integral) ideals of \( F \). As convention, let \( p \) always stand for a prime ideal. For a nonzero integral ideal \( n \), let \( N(n) = |\mathcal{O}/n| \) be the norm of \( n \).

For \( n_1, n_2 \in \mathcal{O} \) and \( c \in \mathfrak{b} \setminus \{0\} \), define the Kloosterman sum

\[
S(n_1, n_2; c) = \sum_{a \pmod{c\mathfrak{b}^{-1}}} e \left( \text{Tr} \left( \frac{n_1 a + n_2 \overline{a}}{c} \right) \right),
\]

where \( \sum^* \) means that \( a \) runs over representatives of \( (\mathcal{O}/c\mathfrak{b}^{-1})^\times \) and \( a\overline{a} \equiv 1 \pmod{c\mathfrak{b}^{-1}} \). We have Weil’s bound

\[
S(n_1, n_2; c) \ll N(n_1, n_2, c\mathfrak{b}^{-1})^{1/2} N(c)^{1/2} \epsilon,
\]

where the brackets \( (\cdot, \cdot, \cdot) \) denote greatest common divisor.

### 2.1. Automorphic forms on \( \mathbb{H}^3 \)

In this section, we recollect some basic notions in the theory of automorphic forms on \( \mathbb{H}^3 \). We refer the reader to [EGM] or [LG] for further details.

#### 2.1.1. The three-dimensional hyperbolic space

We let

\[
\mathbb{H}^3 = \{ w = z + jr = x + iy + jr : x, y, r \text{ real, } r > 0 \}
\]
denote the three-dimensional hyperbolic space, with the action of \( GL_2(\mathbb{C}) \) or \( PGL_2(\mathbb{C}) \) (= \( PSL_2(\mathbb{C}) \)) given by
\[
z(g \cdot w) = \frac{(az + b)(cz + d) + adr^2}{|cz + d|^2 + |c|^2r^2}, \quad r(g \cdot w) = \frac{r|\det g|}{|cz + d|^2 + |c|^2r^2}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]
while the action of \( GL_2(\mathbb{C}) \) on the boundary \( \partial \mathbb{H}^3 = \mathbb{C} \cup \{ \infty \} \) is by the Möbius transform. \( \mathbb{H}^3 \) is equipped with the \( GL_2(\mathbb{C}) \)-invariant hyperbolic metric \((dx^2 + dy^2 + dr^2)/r^2 \) and hyperbolic measure \( dx\,dy\,dr/r^3 \). The associated hyperbolic Laplace–Beltrami operator is given by \( \Delta = r^2 (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial r^2) - r^2/\partial r \).

2.1.2. Hecke congruence groups. For a nonzero integral ideal \( q \), define the Hecke congruence group \( \Gamma_0(q) \) of \( GL_2(\mathbb{C}) \) as follows,
\[
\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}) : c \equiv 0 \pmod{q} \right\}.
\]
In this paper, we assume that \( q \) is square-free.

2.1.3. Maass cusp forms. Let \( \mathcal{B}(q) \) be an orthogonal basis of Maass cusp forms in the \( L^2 \)-cuspidal spectrum of the Laplace–Beltrami operator \( \Delta \) on \( \Gamma_0(q) \backslash \mathbb{H}^3 \). For \( f \in \mathcal{B}(q) \), let \( \|f\| \) denote the \( L^2 \)-norm of \( f \) on \( \Gamma_0(q) \backslash \mathbb{H}^3 \),
\[
\|f\|^2 = \int_{\Gamma_0(q) \backslash \mathbb{H}^3} |f(w)|^2 \frac{dx\,dy\,dr}{r^3}.
\]
For \( f \) with Laplacian eigenvalue \( 1 + 4r_f^2 \), we have the Fourier expansion (cf. [EGM], Theorem 3.3.1)
\[
f(z, r) = \sum_{m \in \mathbb{Z} \backslash \{0\}} \rho_f(m)rK_{2r_f}(4\pi|m|r)e(\text{Tr}(mz)).
\]
Since \( \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(q) \) for any \( \epsilon \in \mathbb{C}^\times \), \( \rho_f(m) \) only depends on the ideal \( m = (m) \), and we may therefore set \( \rho_f(m) = \rho_f(m) \). According to the Kim–Sarnak bound in [BB], we know that \( r_f \) is either real or purely imaginary with \( |\text{Im}(r_f)| \leq \frac{7}{8\pi} \).

2.1.4. Hecke–Maass newforms and their \( L \)-functions. For a nonzero integral ideal \( n = (n) \), we define the Hecke operator \( T_n \) by
\[
T_nf(w) = \frac{1}{w_f} \sum_{\substack{\mathfrak{a} \mid n \in \mathbb{Z} \backslash \{0\} \sum_{b \bmod{d}} f(\begin{pmatrix} a & b \\ c & d \end{pmatrix} w).}
\]
Hecke operators commute with each other as well as the Laplacian operator.

Let \( H^* (q) \) be the set of primitive newforms for \( \Gamma_0(q) \) which are eigenfunctions of all the \( T_n \). A form \( f \) is called primitive if \( \rho_f(b^{-1}) = 1 \). Let \( \lambda_f(n) \) denote the Hecke eigenvalue of \( T_n \) for \( f \), then
\[
\rho_f(b^{-1}n) = \lambda_f(n).
\]
For a newform \( f \) in \( H^* (q) \), we have the Hecke relation
\[
\lambda_f(n_1)\lambda_f(n_2) = \sum_{\substack{a|n_1, a|n_2 \\( (a, n) = 1 \)}} \lambda_f(n_1n_2a^{-2}),
\]
and
\[
\lambda_f(p)^2 = \frac{1}{N(p)}, \quad p \mid q.
\]
Moreover, for $p \nmid q$, if we let $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$ and $\alpha_f(p)\beta_f(p) = 1$, then we have the Kim–Sarnak bound in [BB]:

\[(2.5) \quad |\alpha_f(p)|, |\beta_f(p)| \leq N(p)^{-\frac{1}{2}}, \quad p \nmid q.\]

For a newform $f \in \mathcal{H}^*(q)$, its associated $L$-function $L(s, f)$ is defined by

\[L(s, f) = \sum_{n \in \mathbb{O}} \frac{\lambda_f(n)}{N(n)^s}, \quad \text{Re}(s) > 1.\]

Its Euler product is

\[L(s, f) = \prod_{p \mid q} (1 - \lambda_f(p)N(p)^{-s})^{-1} \prod_{p \nmid q} (1 - \lambda_f(p)N(p)^{-s} + N(p)^{-2s})^{-1}.\]

The $L$-function $L(s, f)$ analytically continues to an entire function of the complex plane. The completed $L$-function

\[\Lambda(s, f) = (2\pi)^{-2s}\Gamma(s - it_f)\Gamma(s + it_f)|d_F|^{-\frac{1}{2}}N(q)^{s/2}L(s, f)\]

satisfies the functional equation:

\[\Lambda(s, f) = \epsilon_f \Lambda(1 - s, f),\]

where $\epsilon_f = \mu(q)\lambda_f(q) \sqrt{N(q)} = \pm 1$. Note that the local factor of $L(s, f)$ at $p \nmid q$ factors further as

\[(1 - \alpha_f(p)N(p)^{-s})^{-1} (1 - \beta_f(p)N(p)^{-s})^{-1}.\]

2.2. The spectral Kuznetsov trace formula for $\Gamma_0(q) \backslash \mathbb{H}^3$. For $t$ real, define the Bessel function (cf. [QH] §15.31)

\[(2.6) \quad J_a(z) = \frac{2\pi^2}{\sin(2\pi it)} (J_{-2\theta}(4\pi z)J_{-2\theta}(4\pi \zeta) - J_{2\theta}(4\pi z)J_{2\theta}(4\pi \zeta)),\]

in which $J_a(z)$ is the Bessel function of the first kind. This is a well-defined even function in the sense that the expression on the right of (2.6) is independent on the choice of the argument of $z$ modulo $\pi$. It is understood that in the non-generic case when $t = 0$ the right-hand side should be replaced by its limit.

Let $h(t)$ be a weight function in the space $\mathcal{H}(S, N)$ defined as in Definition 1.3. For nonzero integral ideals $n_1, n_2$, define

\[(2.7) \quad \Delta_0(n_1, n_2; h) = \sum_{f \in \mathcal{H}(S)} \omega_f h(t_f)\rho_F\left(b^{-1}n_1\right)\rho_F\left(b^{-1}n_2\right),\]

\[(2.8) \quad \Xi_0(n_1, n_2; h) = \frac{4}{\pi \omega_F \sqrt{|d_F|}} \sum_{v \mid q} \int_{-\infty}^{\infty} \omega_0(t)h(t)\eta\left(\frac{t_1}{2} + it\right) \eta\left(\frac{t_2}{2} - it\right) \, dt,\]

in which

\[(2.9) \quad \omega_f = \frac{f_f}{\sinh(2\pi it_f)} \left[|f_f|\right], \quad \omega(v) = \frac{8\pi |\xi_{F, v}(1 + 2it)|^2}{|d_F|N(vq)|\xi_{F, v}(1 + 2it)|^2}, \quad v \mid q,\]

\[(2.10) \quad \eta(n, s) = \sum_{a \mid n} N(na^{-2})^{-\frac{s}{2}}, \quad \xi_F(s) = \sum_{n \in \mathbb{O}} \frac{1}{N(n)^s}, \quad \xi_{F, v}(s) = \prod_{v \mid q} \frac{1}{1 - N(v)^{-s}}.\]

\[^2\]According to [LG], the Kuznetsov trace formula is valid for an enlarged space of weight functions. Thus our results should still be valid if [IS] is replaced by the Schwartz condition.
The spectral Kuznetsov trace formula for $\Gamma_0(q)$ follows from [LG] Theorem 11.3.3 [see also [BM] and [Ven]] and computations for the Eisenstein series similar to those in [CL §3] or [Q2, §3.1].

**Proposition 2.1** (Kuznetsov trace formula for $\Gamma_0(q)$). We have

$$\Delta_\phi(n_1, n_2; h) + \Xi_\phi(n_1, n_2; h) = \frac{8}{\pi^2} \sqrt{|d_F|} \delta_{n_1, n_2} H$$

$$+ \frac{4}{\pi^2 |d_F|} \sum_{c \in \mathfrak{O}^\times / \mathfrak{O}^\times c} \sum_{c \in \mathfrak{O} \setminus \{0\}} S(en_1, n_2; c) \frac{H\left(\sqrt{|en_1n_2| / c}\right)}{N(c\mathfrak{d}^{-1})},$$

where

$$H = \int_{-\infty}^{\infty} h(t) r dt, \quad H(z) = \int_{-\infty}^{\infty} h(t) J_n(z) r dt,$$

$\delta_{n_1, n_2}$ is the Kronecker $\delta$-symbol, $\mathfrak{O}^\times = \{e^2 : e \in \mathfrak{O}^\times \}$, $n_1 = (n_1)$, $n_2 = (n_2)$, and $S(en_1, n_2; c)$ is the Kloosterman sum defined by (2.1).

Next we express $\Delta_\phi(n_1, n_2; h)$ in terms of newforms.

For $\psi = \psi_\omega$ and primitive newform $f \in H^*(\psi)$, let $S(w; f)$ denote the linear space spanned by the forms $f_\phi(z, r) = f(z, |d|r)$, with $d \in \psi \setminus \{0\}$. The space of cusp forms for $\Gamma_0(q)$ decomposes into the orthogonal sum of $S(w; f)$. By the calculations in [ILS §2], one may construct an orthonormal basis of $S(w; f)$ in terms of $f_\phi(z)$. Using this collection of bases as our $B(q)$, the sum $\Delta_\phi(n_1, n_2; h)$ in (2.7) can be arranged into a sum over the primitive newforms in $H^*(\psi)$ for all $\psi | q$. To be precise, following [ILS §2] (see also [Q2, §3.2.2]), for $(n_1, n_2, q) = (1)$, we may derive the formula

$$\Delta_\phi(n_1, n_2; h) = \frac{1}{N(q)} \sum_{\psi | q} \sum_{f \in H^*(\psi)} \omega_f^* h(t_f) \lambda_f(n_1) \lambda_f(n_2),$$

where

$$\omega_f^* = \frac{64\pi^2 Z_\phi(1, f)}{|d_F|^2 \zeta_F(2) Z(1, f)}, \quad f \in H^*(\psi), \ \psi | q,$$

with

$$Z(s, f) = \sum_{n \in \mathfrak{O}} \lambda_f(n^2) N(n)^s, \quad Z_\phi(s, f) = \sum_{n \in \mathfrak{O}} \lambda_f(n^2) N(n)^s.$$

Here $\zeta_F(2)$ arises from $\text{Vol}(\text{PSL}_2(\psi)) / \mathbb{H}^3 = |d_F|^{3/2} \zeta_F(2) / 4\pi^2$ as in [EGM §7.1, Theorem 1.1] (note that $\text{PSL}_2(\psi)$ is of index 2 in $\text{PGL}_2(\psi)$).

For square-free $q$, we introduce

$$\Delta_\phi^*(n_1, n_2; h) = \sum_{f \in H^*(\phi)} \omega_f^* h(t_f) \lambda_f(n_1) \lambda_f(n_2),$$

$$\Xi_\phi^*(n_1, n_2; h) = \int_{-\infty}^{\infty} \omega_f^* (t) h(t) \eta(n_1, \frac{1}{2} + it) \eta(n_2, \frac{1}{2} - it) dt,$$

with

$$\omega_f^* = \frac{64\pi^2 Z_\phi(1, f)}{|d_F|^2 \zeta_F(2) Z(1, f)}, \quad \omega_f^* = \frac{32|\zeta_F(1 + 2it)|^2}{w_F |d_F|^{3/2} N(q) \zeta_F(1 + 2it)}/, \quad f \in H^*(\phi).$$

The constants in the formula are adopted from (16) in [Ven Proposition 1].
Then (2.12) may be further written as
\[ \Delta_q(n_1, n_2; h) = \frac{1}{N(q)} \sum_{a=p^n} \sum_{l \mid |w^\infty} \frac{1}{N(l)} \Delta^*_q(n_1^2, n_2; h), \]
for \((n_1 n_2, q) = (1)\), and similarly (2.13) as
\[ \Xi_q(n_1, n_2; h) = \frac{1}{N(q)} \sum_{a=p^n} \sum_{l \mid |w^\infty} \frac{1}{N(l)} \Xi^*_q(n_1^2, n_2; h). \]

From Möbius inversion, we arrive at the following lemma.

**Lemma 2.2.** For \((n_1 n_2, q) = (1)\), we have
\[
\Delta^*_q(n_1, n_2; h) = \sum_{a=p^n} \mu(w)N(v) \sum_{l \mid |w^\infty} \frac{1}{N(l)} \Delta_q(n_1^2, n_2; h),
\]
\[
\Xi^*_q(n_1, n_2; h) = \sum_{a=p^n} \mu(w)N(v) \sum_{l \mid |w^\infty} \frac{1}{N(l)} \Xi_q(n_1^2, n_2; h),
\]
where \(\mu\) is the Möbius function for \(F\).

In view of (2.16) and (2.17), \(\Xi^*_q(n_1, n_2; h)\) may be easily estimated using the lower bound \(|\zeta_F(1 + 2it)| \gg F/1/\log(1 + 3)|t|^3\).

**Lemma 2.3.** We have
\[
\Xi^*_q(n_1, n_2; h) \ll_F \frac{\tau(n_1)\tau(n_2)K}{\varphi(q)},
\]
where, as usual, \(\tau(n)\) is the divisor function for \(F\), \(\varphi(q) = N(q)/\zeta_F,1(1)\) is Euler’s totient function for \(F\), and
\[
K = \int_{-\infty}^{\infty} h(t) \log^2(|t| + 3) dt.
\]

For \((n, q) = (1)\), define
\[
\Delta_q(n; h) = \sum_{f \in H^\infty(q)} \omega_f h(f) \lambda_f(n).
\]

**Corollary 2.4.** Let \(p \nmid q\). For \(n = (1)\), \(p\) or \(p^2\), we have
\[
\Delta_q(n; h) = \varphi(q)\Delta(h) \cdot \delta_{\infty,(1)} + K \mathcal{B}_q(n; h) + O_F \left( \frac{K}{\varphi(q)} \right),
\]
where
\[
\Delta(h) = \frac{8H}{\pi^2 \sqrt{|d_F|}},
\]
\[
\mathcal{B}_q(n; h) = \sum_{a=p^n} \mu(w)N(v) \sum_{l \mid |w^\infty} \frac{KB_q(n^2; h)}{N(l)},
\]
with
\[
KB_q(n; h) = \frac{4}{\pi^2 |d_F|} \sum_{\epsilon \in \mathbb{C}^*} \sum_{\epsilon \in \mathbb{C}^*} \frac{S(\epsilon n, 1; c)}{N(c)} H \left( \frac{\sqrt{\epsilon n}}{c} \right),
\]
for \(n = (n)\); here \(H, H(z)\) and \(K\) are defined as in (2.11) and (2.18).

\[\text{(2.18)}\]

\[\text{(2.19)}\]

\[\text{(2.20)}\]

\[\text{(2.21)}\]

\[\text{(2.22)}\]

\[\text{(2.12)}\]

\[\text{(2.13)}\]

\[\text{(2.16)}\]

\[\text{(2.17)}\]

\[\text{(2.11)}\]
By Poisson’s integral representation for

\[ \Delta_{(1)}(n_1, n_2; h) = \Delta(h) \cdot \delta_{n_1, n_2} + KB_{(1)}(n_1, n_2; h) + O_F(K), \]

where \( K \) and \( \Delta(h) \) are defined as in (2.18) and (2.21), and

\[ KB_{(1)}(n_1, n_2; h) = \frac{4}{4 \pi^2 |d_F|} \sum_{e \in \Theta_2/\Theta_2} \sum_{e \in B(0)} S \left( e n_1, n_2; c_1 \right) \frac{H \left( \sqrt{en_1 n_2} \right)}{N(c)}, \]

with \( n_1 = (n_1) \) and \( n_2 = (n_2) \).

2.3. Stationary phase. The following lemma is an improvement of [BKY] Lemma 8.1 (cf. [AHLQ] Lemma A.1). It will be used to show that certain exponential integrals are negligibly small in the absence of stationary phase.

**Lemma 2.6.** Let \( w(x) \) be a smooth function supported on \([a, b]\) and \( f(x) \) be a real smooth function on \([a, b]\). Suppose that there are parameters \( Q, U, X, Y, R > 0 \) such that

\[ f^{(i)}(x) \ll_i Y/Q^i, \quad w^{(j)}(x) \ll_j X/U^j, \]

for \( i \geq 2 \) and \( j \geq 0 \), and

\[ |f'(x)| \geq R. \]

Then for any \( A \geq 0 \) we have

\[ \int_a^b e(f(x))w(x)dx \ll_A (b-a)X \left( \frac{Y}{R^2 Q^2} + \frac{1}{RU} + \frac{1}{RU} \right)^A. \]

3. Analysis of Bessel integrals

Let \( h(t), h_{T, M}(t) \in \mathcal{H}^+(S, N) \) be as in Definition 1.3. Let \( H(z) \), respectively \( H_{T, M}(z) \), be the Bessel integral as defined in (2.11) for \( h(t) \), respectively for \( h_{T, M}(t) \). Set \( M = T^\mu \) for \( 0 < \mu < 1 \).

Before reading this section, we refer the reader to Appendix A.2 and A.4 for the analysis of real Bessel integrals \( H_{T, M}(x) \) and \( H_{T, M}(x) \).

3.1. The case \(|z| \leq 1\).

**Lemma 3.1.** We have \( H(z) \ll_k |z|^2 \) for \(|z| \leq 1\).

**Proof.** By the definitions in (2.6) and (2.11), we have

\[ H(z) = -4\pi^2 \int_{-i\infty}^{i\infty} h(t) J_{2\mu} \left( 4\pi |z| \right) J_{2\mu} \left( 4\pi \right) \sin(2\pi t) \pi^2 t dt. \]

Note that the Plancherel measure \( \pi^2 t dt \) vanishes at \( t = 0 \). Shifting the line of integration to \( \text{Im}(t) = -\frac{1}{2} \) and crossing only the pole at \( t = -\frac{1}{2} i \), we obtain

\[ H(z) = -\pi^2 \left( \frac{1}{2} i \right) J_{2\mu} \left( 4\pi \right) J_{2\mu} \left( 4\pi \right) \sin(2\pi t) \left( t - \frac{1}{2} i \right)^2 \pi^2 t dt. \]

By Poisson’s integral representation for \( J_\nu(z) \) (cf. [Wat] 3.3 (6) or [GR] 8.411.4),

\[ J_\nu(z) = \frac{\left( \frac{1}{2} i \right) z^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi e^{\nu \cos \theta} \sin^{2\nu} \theta d\theta, \quad \text{Re}(\nu) > -\frac{1}{2}, \]

we have

\[ H(z) \ll_k |z|^2. \]
we infer that
\[ |J_v(z)| \leq \frac{|z^v|}{\Gamma(v + \frac{1}{2})}, \quad |z| \leq 4\pi. \]

Hence, Stirling’s formula yields the estimates
\[ J_1(4\pi z)J_1(4\pi \overline{z}) \leq |z|^2, \quad \frac{J_{2i\theta} + \frac{1}{2} (4\pi z) J_{2i\theta} + \frac{1}{2} (4\pi \overline{z})}{\cosh(2\pi r)} \leq \left( \frac{|z|}{|z| + 1} \right)^3, \]
for \(|z| \leq 1\). Consequently, in view of (1.5), we have
\[ H(z) \ll |z|^2, \]
for \(|z| \leq 1\). Q.E.D.

**Lemma 3.2.** Let \( M = T^\mu \) with \( 0 < \mu < 1 \). We have \( H_{T, M}(z) \ll M |z|^2 / T \) for \(|z| \leq 1\).

**Proof.** We modify the proof of Lemma 3.1 by replacing \( h(t) \) by \( h_{T, M}(t) \). An estimation by (1.5) and (1.6) yields
\[ H_{T, M}(z) \ll e^{-T/M} |z|^2 + \frac{M |z|^3}{T} \ll \frac{M |z|^2}{T}, \]
for \(|z| \leq 1\). Q.E.D.

By shifting the integral contour further, one may prove that \( H_{T, M}(z) \ll M |z|^2 / T^{4\delta - 2} \) as long as \( \delta < S \).

**3.2. The case \(|z| > 1\).**

**Lemma 3.3.** We have \( H(z) \ll 1/|z| \) for \(|z| > 1\).

**Proof.** This follows immediately from (cf. [Qi2] Lemma 4.1)
\[ J_{n}(z) \ll \frac{r^2 + 1}{|z|}, \quad |z| > 1. \]
Q.E.D.

**Lemma 3.4.** Let \( M = T^\mu \) with \( 0 < \mu < 1 \). Let \(|z| > 1\). We have \( H_{T, M}(z) = H_{T, M}^{\mu}(z) + O_A(T^{-A}) \) for any \( A > 0 \), with
\[ H_{T, M}(xe^{i\theta}) = 4MT^2 \int_0^{2\pi} \int_{-M/M}^{M/M} \hat{k}(-2Mr/\pi)e(2f_T(r, \omega; x, \theta)) \, dr \, d\omega, \]
where the phase function is
\[ f_T(r, \omega; x, \theta) = Tr/\pi + 2x(\cos r \omega \cos \theta - \sin r \sin \omega \sin \theta). \]
and \( k(t) = (Mt/T + 1)^2h(t) \) is a Schwartz function.

**Proof.** First, let us recall the following integral representation in [Qi1] §6.8.3) (see also [BM2] Theorem 12.1),
\[ J_{n}(xe^{i\theta}) = 4\pi \int_0^{\infty} J_0(4\pi x |ye^{i\theta} + 1/ye^{i\theta}|) y^{a-1} \, dy. \]
On letting \( y = e^r \), we infer that
\[ J_{n}(xe^{i\theta}) = 4\pi \int_{-\infty}^{\infty} J_0(8\pi x |\cosh (r + i\theta)|) e(2tr/\pi) \, dr. \]
So
\[ H_{T, M}(xe^{i\theta}) = 8\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{r}h((t - T)/M)J_0(8\pi x |\cosh (r + i\theta)|) e(2tr/\pi) \, dr \, dt. \]
This double integral is absolutely convergent as \( J_0(8\pi x |\cosh (r + i\theta)|) \ll 1/\sqrt{x}e^{r/2} \) for \( x > 1 \) and \( r > 1 \). On changing the variable from \( t \) to \( Mt + T \), this integral turns into
\[
8\pi MT^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(t)e(2(Mt + T)r/\pi)J_0(8\pi x |\cosh (r + i\theta)|) \, dt \, dr,
\]
with
\[
k(t) = (Mt/T + 1)^2 h(t).
\]
By (1.3), we have a similar estimate
\[
k(t + i\epsilon) \ll e^{-\pi |t|} (|t| + 1)^{2-N}.
\]
It suffices to know that \( k(t) \) is a Schwartz function (this is because the derivatives of \( k(t) \) also satisfy the above estimate by Cauchy’s integral formula). By interchanging the order of integration, we find that the integral above is equal to
\[
8\pi MT^2 \int_{-\infty}^{\infty} \hat{k}(-2Mr/\pi) e(2Tr/\pi)J_0(8\pi x |\cosh (r + i\theta)|) \, dr,
\]
where \( \hat{k}(t) \) is the Fourier transform of \( k(t) \). Now \( \hat{k}(r) \) is also Schwartz. Hence, the integral may be effectively truncated at \( |r| = M^2/M \), and we need to consider
\[
8\pi MT^2 \int_{-M^2/M}^{M^2/M} \hat{k}(-2Mr/\pi) e(2Tr/\pi)J_0(8\pi x |\cosh (r + i\theta)|) \, dr.
\]
Finally, we apply the Bessel integral representation of \( J_0(2\pi x) \) (cf. [Wat §2.2]),
\[
J_0(2\pi x) = \frac{1}{2\pi} \int_{0}^{2\pi} e(x \cos \omega) \, d\omega,
\]
so that the integral above turns into \( H^{1}_{T,M}(xe^{\theta}) \) defined by (3.2) and (3.3).

Q.E.D.

**Lemma 3.5.** Let \( M = T^\mu \) with \( 0 < \mu < 1 \). Suppose that \( |z| > 1 \). Fix a constant \( 0 < c < 1 \).

1. When \( |\text{Im}(z)| \leq cT/2\pi \) and \( |\text{Re}(z)| \leq TM^{1-\epsilon} \), we have \( H_{T,M}(z) = O_A(T^{-\Lambda}) \).

2. Suppose that \( \mu > 1/2 \). When \( cT/2\pi < |\text{Im}(z)| \leq M^{2-\epsilon} \) and \( |\text{Re}(z)| \leq M^{2+\epsilon} \), we have
\[
H_{T,M}(z) = O_{T,\Lambda} \ll MT^{2+\epsilon}/|z|,
\]
and \( H_{T,M}(z) = O_A(T^{-\Lambda}) \) if
\[
|\text{Re}(z)| \ll |z|^2/M^{1-\epsilon}T.
\]

**Proof.** For notational simplicity, we shall work in the Cartesian coordinates, \( z = x + iy \). In view of Lemma [4.3] we need to consider
\[
H_{T,M}^2(x + iy) = 4MT^2 \int_{0}^{2\pi} \int_{-M^2/M}^{M^2/M} \tilde{k}(-2Mr/\pi)e(2f_T(r, \omega, x, y)) \, dr \, d\omega,
\]
where
\[
f_T(r, \omega, x, y) = Tr/\pi + 2x \cosh r \cos \omega - 2y \sinh r \sin \omega,
\]
for which
\[
\frac{\partial}{\partial r}f_T(r, \omega, x, y) = T/\pi + 2x \sinh r \cos \omega - 2y \cosh r \sin \omega,
\]
\[
\frac{\partial}{\partial \omega}f_T(r, \omega, x, y) = -2x \cosh r \sin \omega - 2y \sinh r \cos \omega.
\]

1. In the first case that \( |y| \leq cT/2\pi (c < 1) \) and \( |x| \leq TM^{1-\epsilon} \), note that \( (\partial/\partial r)f_T(r, \omega, x, y) \geq T/\pi - 2|x| \sinh r - 2|y| \cosh r \),
so it is clear that \((\partial / \partial r)f_T(r, \omega; x, y) \gg T\) for \(|r| \leq M^\varepsilon / M\). Along with the assumption
\(M = T^\eta \leq T^{1-\varepsilon} \ (0 < \mu < 1)\), Lemma 2.6 with \(Y = TM^{1-\varepsilon}, Q = 1, U = 1/M\) and \(R = T\), implies that the integral \(H_T(x + i y)\) in (3.7) is negligibly small (for applying Lemma 2.6 it would be more rigorous if we had used a smooth truncation at \(|r| = M^\varepsilon / M\).

(2). In the second case that \(cT / 2\pi < |y| \leq M^{2+\varepsilon} \) and \(|x| \leq M^{2+\varepsilon}\), Lemma 2.6 with \(Y = M^{2+\varepsilon}, Q = 1, U = 1/M\) and \(R = M^{1+\varepsilon}\), would again imply that the \(r\)-integral in (3.7) is negligibly small unless

\[(3.9)\]
\[|T/\pi - 2y \sin \omega| \leq M^{1+\varepsilon}.
\]

Note here that \(\cosh r = 1 + O(M^\varepsilon / M^2)\) and that \(|y\sin\omega|\leq O(M^\varepsilon)\). We may therefore (smoothly) truncate the \(\omega\)-integral at \(|\sin\omega - T / 2\pi y| = M^{1+\varepsilon}|y|\), and we infer that

\[(3.10)\]
\[H_{T,M}(x + iy) \ll \frac{MT^{2+\varepsilon}}{|y|}.
\]

For \(\omega\) satisfying (3.9) and \(|r| \leq M^\varepsilon / M\), we have

\[|((\partial / \partial \omega)f_T(r, \omega; x, y)| \gg T|y|/|y| - 2|y||\sinh r|,
\]

and hence \(|((\partial / \partial \omega)f_T(r, \omega; x, y)| \gg T|x|/|y|\) provided that \(|x| \gg |y|^2 / M^{1-\varepsilon}T\). Applying Lemma 2.6 with \(Y = |x|, Q = 1, U = |y|/M^{1+\varepsilon}\) and \(R = T|x|/|y|\), we deduce that the integral \(H_{T,M}(x + iy)\) in (3.7) is negligibly small unless

\[(3.11)\]
\[|x| \ll |y|^2 / M^{1-\varepsilon}T.
\]

Finally, note that \(|x| \ll |y|^2 / M^{1-\varepsilon}T\) amounts to \(|x| \ll |z|^2 / M^{1-\varepsilon}T\), and in this case \(|y| = |z|\) so that (3.5) follows from (3.10).

Q.E.D.

**Remark 3.6.** From the uniform asymptotic formulae in [Bal, Dun] (cf. also [Olv1, Olv2]) for \(K_v(vz)\) and \(K_v(e^{\pm \pi v} vz)\) \((v > 0)\), one may derive the uniform asymptotic formula

\[J_v(Z/2\pi) \sim \frac{2\pi}{\sqrt{|t^2 + Z^2|}} \cos(4\pi \text{Im} \xi(Z/it)/\pi)
\]

for \(|\text{Re}(Z/it)| > \varepsilon' > \pi / 2\), where

\[\xi(Z/it) = i \log \left(\frac{t + \sqrt{t^2 + Z^2}}{Z}\right) - \frac{\pi}{2} - i \sqrt{1 + \frac{Z^2}{t^2}}.
\]

It may be shown that the same assertions in Lemma 3.5(2) are valid for \(|\text{Im}(z)| > cT / 2\pi\) with \(c > \pi / 2\) (note the condition \(c < 1\) in Lemma 3.5). There is no room for any improvement (though the \(\varepsilon\) in (3.5) is removable). This is because there is no oscillation in the Bessel function \(J_v(Z/2\pi)\) when \(|\text{Re}(Z)|\) is small; for example, when \(Z = 2\pi iy\) \((y\ \text{real and } \ |y| > t/4)\), we would have

\[J_v(iy) \sim \frac{2\pi}{\sqrt{4\pi^2 y^2 - t^2}}.
\]

By the proof of Lemma 3.5 one may also show that

\[(3.12)\]
\[\frac{\partial^2 H_{T,M}(x e^{\theta})}{\partial \theta^2} \ll MT^{2+\varepsilon}, \quad \frac{\partial^2 H_{T,M}(x e^{\theta})}{\partial x \partial \theta} \ll \frac{T^{1+\varepsilon}}{x},
\]

and that they are negligibly small unless \(|\cos \theta| \ll x / M^{1+\varepsilon}T\). For reasons similar to the above, these assertions can not be improved in an essential way. Note that one loses \(x = |z|\) when taking the \(\theta\)-derivative and the bound \(MT^{2+\varepsilon}\) is on the order of \(H_{T,M} (\approx MT^2)\).

For the proof of Theorem 1.6 we only need Lemma 3.5(1), indeed, its simple corollary as below.
Corollary 3.7. We have $H_{T,M}(z) = O(T^{-A})$ for $1 < |z| \ll T$.

3.3. Estimates for $\widetilde{KB}_\alpha(n;h)$ and $KB_{(1)}(n_1,n_2;h_{T,M})$. Recall the definitions of the Kloosterman–Bessel terms $KB_\alpha(n;h)$, $KB_\alpha(n;h)$ and $KB_{(1)}(n_1,n_2;h)$ in \((2.21), (2.22)\) and \((2.24)\).

Lemma 3.8. We have
\begin{equation}
(3.13) \quad \widetilde{KB}_\alpha(n;h) \ll_{h,F,c} N(n)^{\frac{1}{2}-\epsilon}/N(q)^{\frac{1}{2}-\epsilon},
\end{equation}
and, for $N(n_1n_2) \ll T^4$,
\begin{equation}
(3.14) \quad KB_{(1)}(n_1,n_2;h_{T,M}) \ll_{h,F} M N(n_1n_2)^{\frac{1}{2}+\epsilon}/T.
\end{equation}

Proof. We first prove \((3.13)\) by appealing to the Weil bound \((2.2)\) for Kloosterman sums and the estimates for $H(z)$ in Lemma 3.1 and 3.3. For $N(n) \ll N(qb)^2$, we have
\[
KB_\alpha(n;h) \ll \sum_{c \in \mathcal{B}_0} \frac{N(c)^{\frac{1}{2}+\epsilon} N(n)^{\frac{1}{2}}}{N(c)} \ll \frac{N(n)^{\frac{1}{2}}}{N(q)^{\frac{1}{2}-\epsilon}}.
\]
For $N(n) > N(qb)^2$, we have
\[
KB_\alpha(n;h) \ll \sum_{c \in \mathcal{B}_0} \frac{N(c)^{\frac{1}{2}+\epsilon} N(n)^{\frac{1}{2}}}{N(c)} + \sum_{c \in \mathcal{B}_0} \frac{N(c)^{\frac{1}{2}+\epsilon} N(c)^{\frac{1}{2}}}{N(c)} N(n)^{\frac{1}{2}} \ll \frac{N(n)^{\frac{1}{2}+\epsilon}}{N(q)}.
\]
Hence uniformly
\[
KB_\alpha(n;h) \ll_{h,F,c} N(n)^{\frac{1}{2}-\epsilon}/N(q)^{\frac{1}{2}-\epsilon}.
\]
Then
\[
\widetilde{KB}_\alpha(n;h) = \sum_{q \equiv \sigma \mod{p}} \mu(q) N(\nu) \sum_{l \equiv \omega \mod{\infty}} \frac{KB_\alpha(l;n;h)}{N(l)} \ll \frac{N(n)^{\frac{1}{2}-\epsilon}}{N(q)^{\frac{1}{2}-\epsilon}} \sum_{q \equiv \sigma \mod{p}} N(\nu) \sum_{l \equiv \omega \mod{\infty}} \frac{1}{N(l)^\epsilon} \ll \frac{N(n)^{\frac{1}{2}-\epsilon}}{N(q)^{\frac{1}{2}-\epsilon}}.
\]
To prove \((3.14)\), we use Lemma 3.2 and Corollary 3.7. For $N(n_1n_2) \ll T^4$, we have
\[
KB_{(1)}(n_1,n_2;h_{T,M}) \ll \frac{M}{T} \sum_{c \in \mathcal{B}_0} \frac{N(n_1,n_2,c;e^{-1})^\frac{1}{2+\epsilon} N(c)^{\frac{1}{2}+\epsilon} N(n_1n_2)^{\frac{1}{2}}}{N(c)} \ll \frac{MN(n_1n_2)^{\frac{1}{2}+\epsilon}}{T}.
\]
Q.E.D.

Corollary 3.9. Let $h(t), h_{T,M}(t) \in \mathcal{H}^+(S,N)$ be as in Definition 1.3. Then
\begin{equation}
(3.15) \quad \Delta_\alpha(1;h) = \sum_{f \in \mathcal{H}^+} \omega_f^* h(f) = \varphi(a),
\end{equation}
and, when $a = (1)$,
\begin{equation}
(3.16) \quad \Delta_{(1)}(1;h_{T,M}) = \sum_{f \in \mathcal{H}^+(1)} \omega_f^{*} h_{T,M}(f) \sim \frac{16MT^2}{\pi^2 \sqrt{|d_F|}} \int_{-\infty}^{\infty} h(t)dt \approx MT^2,
\end{equation}
with the implied constants depending only on $h$ and $F$. 

Proof. Let \( n = (1) \) in (2.20). Then (3.15) and (3.16) are obvious in view of the estimates for the Kloosterman–Bessel term as in (3.13) and (5.14). Note that
\[
H_{T,M} = \int_{-\infty}^{\infty} h_{T,M}(t)^2 dt \sim 2MT^2 \int_{-\infty}^{\infty} h(t) dt,
\]
\[
K_{T,M} = \int_{-\infty}^{\infty} h_{T,M}(t) \log^2(|t| + 3) dt \sim 2M \log^2 T \int_{-\infty}^{\infty} h(t) dt.
\]
Q.E.D.

4. The explicit formula

Recall the notation in §2.1.4. Let \( f \in H^*(q) \). Define its 1-level density \( D_1(f, \phi, R) \) as in Definition 1.1 (see also Remark 1.2).

Following from [ILS] §4, we have the explicit formula
\[
D_1(f, \phi, R) = \frac{\hat{\phi}(0)}{\log R} \left( \log N(q) + 2 \log |df| - 4 \log 2\pi \right)
\]
\[
+ \frac{2}{\log R} \int_{-\infty}^{\infty} \Gamma^\prime \left( \frac{1}{2} \pm it_f + \frac{2\pi i x}{\log R} \right) \phi(x) dx
\]
\[
- 2 \sum_{\nu=1}^{\infty} \sum_p \hat{\phi} \left( \frac{\nu \log N(p) \lambda_f(p^\nu) \log N(p)}{\log R} \right) \frac{\log N(p)}{N(p)^{\nu/2} \log R},
\]

where \( \alpha_f(p^\nu) = \alpha_f(p^\nu) + \beta_f(p^\nu) \) for \( p \mid q \) and \( \alpha_f(p^\nu) = \lambda_f(p) \) for \( p \nmid q \).

Lemma 4.1. For \( f \in H^*(q) \), we have
\[
D_1(f, \phi, R) = \frac{\hat{\phi}(0)}{\log R} \left( \log N(q) + 2 \log \left( \frac{1}{4} + t_f^2 \right) \right) + \frac{1}{2} \phi(0)
\]
\[
- P_1(f, \phi, R) - P_2(f, \phi, R) + O \left( \frac{\log \log 3N(q) \log R}{\log R} \right).
\]

where
\[
P_\nu(f, \phi, R) = 2 \sum_{p \mid q} \hat{\phi} \left( \frac{\nu \log N(p) \lambda_f(p^\nu) \log N(p)}{\log R} \right) \frac{\log N(p)}{N(p)^{\nu/2} \log R}, \quad \nu = 1, 2.
\]

Proof. First, we have (cf. [ILS] (4.14), (4.15))
\[
\sum_{\pm} \frac{2}{\log R} \int_{-\infty}^{\infty} \Gamma^\prime \left( \frac{1}{2} \pm it_f + \frac{2\pi i x}{\log R} \right) \phi(x) dx = \frac{2}{\log R} \log \left( \frac{1}{4} + t_f^2 \right) \phi(0) + O \left( \frac{1}{\log R} \right).
\]

For \( p \mid q \), we have \( |\alpha_f(p^\nu)| = N(p)^{-\nu} \) by (2.3) and (2.4). For \( p \nmid q \), in view of (2.5), we have \( |\alpha_f(q)|, |\beta_f(q)| \leq N(p)^{\frac{1}{4}} \) and hence \( |\alpha_f(p^\nu)| \leq 2N(p)^{\frac{1}{4}} \). Consequently, by trivial estimations,
\[
\sum_p \sum_{\nu=3}^{\infty} \hat{\phi} \left( \frac{\nu \log N(p)}{\log R} \right) \frac{\alpha_f(p^\nu) \log N(p)}{N(p)^{\nu/2} \log R} \leq \frac{1}{\log R},
\]
and
\[
\sum_{p \nmid q} \sum_{\nu=1,2} \hat{\phi} \left( \frac{\nu \log N(p)}{\log R} \right) \frac{\alpha_f(p^\nu) \log N(p)}{N(p)^{\nu/2} \log R} \leq \frac{\log \log 3N(q) \log R}{\log R}.
\]
Moreover, the Landau Prime Ideal Theorem for \( F \) implies
\[
\sum_{p \leq q} \left( \frac{2 \log N(p)}{\log R} \right) \frac{2 \log N(p)}{N(p) \log R} = \frac{1}{2} \phi(0) + O \left( \frac{1}{\log R} \right).
\]
Finally, \( \alpha_f(p) + \beta_f(p) = \lambda_f(p) \) and \( \alpha_f^2(p) + \beta_f^2(p) = \lambda_f(p^2) - 1 \). Q.E.D.

5. Proof of the theorems

5.1. Proof of Theorem 1.4. In view of Lemma 4.1 and Corollary 2.4 we have
\[
D_1(H^*(\varphi), \phi; h) = \hat{\phi}(0) + \frac{1}{2} \phi(0) - (P_1 + P_2)(H^*(\varphi), \phi; h) + O \left( \frac{\log \log 3N(a)}{\log N(a)} \right),
\]
with
\[
P_v(H^*(\varphi), \phi; h) = \text{Avg}_{\varphi} \left( P_v(f, \phi, N(a)); \omega_v h(t_f) \right)
= \sum_{p \leq q} \hat{\phi} \left( \frac{4 \log N(p)}{4 \log T} \right) \frac{2 \log N(p)}{N(p) \log T} \frac{\log N(p)}{2N(p)^{3/2}} \frac{\log N(p)}{h(v) + O(1/\varphi(a))}.
\]
Also recall from (3.15) that \( \Delta_v(1; h) = \varphi(a) \).

Suppose that \( \phi \) is supported on \([-v, v] \). We only consider the Kloosterman–Bessel contribution in the case \( \nu = 1 \). The other cases follow entirely analogously (in fact, we obtain better bounds). By (3.13),
\[
\sum_{p \leq q} \hat{\phi} \left( \frac{4 \log N(p)}{4 \log T} \right) \frac{2 \log N(p)}{N(p) \log T} \frac{\log N(p)}{h(v) + O(1/\varphi(a))} \ll \frac{1}{N(a)^{\frac{1}{2}+\epsilon}} \sum_{q \leq N(a)^{\frac{1}{2}}} \frac{1}{N(p)^{\epsilon}}
\ll N(a)^{-\frac{1}{2}+\epsilon}.
\]
Therefore this sum is \( o(\varphi(a)) \) as long as \( \nu < \frac{1}{2} \), so that the Kloosterman–Bessel term in \( P_1(H^*(\varphi), \phi; h) \) would have no contribution to the main term of \( D_1(H^*(\varphi), \phi; h) \).

5.2. Proof of Theorem 1.6. By Lemma 4.1 and Corollary 2.5 we have
\[
\lim_{T \to \infty} D_1(H^*(1), \phi; h_T, M) = \hat{\phi}(0) + \frac{1}{2} \phi(0) - \lim_{T \to \infty} (P_1 + P_2)(H^*(1), \phi; h_T, M),
\]
with
\[
P_v(H^*(1), \phi; h_T, M) = \text{Avg}_{\varphi} \left( P_v(f, \phi, T^4) ; \omega_v h_T, M(t_f) \right)
= \sum_{p \leq q} \hat{\phi} \left( \frac{4 \log N(p)}{4 \log T} \right) \frac{2 \log N(p)}{2N(p)^{3/2}} \frac{\log N(p)}{h(v)} + O \left( \frac{M \log^2 T}{h(v)} \right).
\]
For the term \( \hat{\phi}(0) \), we have used here
\[
\text{Avg}_{\varphi} \left( \frac{4 \log N(p)}{4 \log T} ; \omega_v h_T, M(t_f) \right) \sim \log^2 T,
\]
as
\[
\int_{-\infty}^{\infty} h_T, M(t) \log \left( \frac{1}{2} + \frac{1}{4} t_f^2 \right) \omega_v h_T, M(t_f) dt \sim 2MT^2 \log T \int_{-\infty}^{\infty} h(t) dt.
\]
Suppose that \( \phi \) is supported on \([-v, v] \) for \( \nu < 1 \). Since \( T^{4\nu} \ll T^4 \), the estimate (3.14) in Lemma 3.8 yields
\[
\sum_{p \leq q} \hat{\phi} \left( \frac{4 \log N(p)}{4 \log T} \right) \frac{2 \log N(p)}{2N(p)^{3/2}} \frac{\log N(p)}{h(v)} + O \left( \frac{M \log^2 T}{h(v)} \right) \ll \frac{M}{T} \sum_{N(p) \leq T^{4\nu}} \frac{\log N(p)}{N(p)^{\frac{3}{2}+\epsilon}} \ll MT^{3\nu-1+\epsilon},
\]
which is $o(MT^2)$ as desired. Recall here that the total mass $\Delta_{(1)}(1; h_{T,M})$ is on the order of $MT^2$.

**Remark 5.1.** When we extend the support of $\tilde{\phi}$ beyond the segment $[-1, 1]$, new terms in $\mathcal{D}_3(\phi; h_{T,M})$ are expected to contribute to the asymptotics. For this, we need to consider

$$
\sum_{b(\phi)} \frac{1}{N(c)} \sum_p S(p, 1; c) \frac{\log N(p)}{\sqrt{N(p)}} H_{T,M}^1 \left( \frac{\sqrt{p}}{c} \right) \tilde{\phi} \left( \frac{\log N(p)}{4 \log T} \right),
$$

(5.1)

where the $p$-sum is over prime integers in $\mathcal{O}$, and $H_{T,M}^1(z)$ is defined in Lemma 3.4. Assuming the Riemann hypothesis for Hecke-character $L$-functions over $F$, we have

$$
\sum_{|p| \leq \theta} S(p, 1; c) \frac{\log N(p)}{\sqrt{N(p)}} = \frac{u_F \chi \mu(c^{-1})^2}{\pi} \varphi(c^{-1}) + O(\log N(x(c))^{1/2} + \phi(0)).
$$

(5.2)

It follows that the $p$-sum in (5.1) is equal to

$$
\pi \log T \int_0^\infty e^{2\pi x} \left\{ \chi \theta \frac{\mu(c^{-1})^2}{\varphi(c^{-1})} + O(\log N(x(c))^{1/2} + \phi(0)) \right\} x d\theta dx.
$$

The main term of this integral will eventually contribute the desired

$$
- \frac{8H_{T,M}}{\pi^2} \left( \int_{-\infty}^{\infty} \phi(x) \sin 2\pi x \frac{dx}{2\pi x} - \frac{1}{2} \phi(0) \right).
$$

The error term is bounded by

$$
N(c)\langle N(c)T \rangle \int_0^{\sqrt{N(c)} \hat{\theta}} \int_0^{2\pi} \left| \frac{\partial^2 H_{T,M}^1(xe^{\theta})}{\partial x \partial \theta} \right| + \left| \frac{\partial H_{T,M}^1(xe^{\theta})}{\partial \theta} \right| d\theta dx.
$$

However, the estimates (5.12) in Remark 3.6 for these derivatives could only be used to bound the error-term contribution by $T^{2\epsilon + 1/2}/M$. Unfortunately, this is too large if $\epsilon > 1$.

**5.3. Proof of Theorem 1.4** Since the sign of functional equation of $L(s, f)$ for $f$ in the family $H^*(1)$ is always $\epsilon_f = 1$, the vanishing order of $L(s, f)$ is even at $s = \frac{1}{2}$. Thus by definition (see Definition 1.1), we have

$$
D_2(f, \phi_1, \phi_2, R) = D_1(f, \phi_1, R)D_1(f, \phi_2, R) - 2D_1(f, \phi_1\phi_2, R).
$$

(5.3)

Before applying Lemma 4.1 to (5.4)

$$
\frac{\partial}{\partial t} D_2(F^*(1), \phi_1, \phi_2; h_{T,M}) = \text{Avg}_{(1)}(D_1(f, \phi_1, T^k)D_1(f, \phi_2, T^k; \omega_f h_{T,M}(t_f)),
$$

we note (see (5.2)) that

$$
\text{Avg}_{(1)}(P_\nu(f, \phi_\mu, T^k; \omega_f h_{T,M}(t_f))) = o(1), \quad \nu, \mu = 1, 2,
$$

for the support of $\tilde{\phi}_p$ contained in $(-1, 1)$. We now apply Lemma 4.1 and Corollary 2.5 to (5.4). Some simple calculations show that

$$
\lim_{t \to \infty} \frac{\partial}{\partial t} D_2(F^*(1), \phi_1, \phi_2; h_{T,M}) = \left( \tilde{\phi}_1(0) + \frac{1}{2} \phi_1(0) \right) \left( \tilde{\phi}_2(0) + \frac{1}{2} \phi_2(0) \right)
$$

$$
+ \lim_{t \to \infty} \left( P_1 + P_2 \right)(H^*(1), \phi_1, \phi_2)
$$

$$
+ \lim_{t \to \infty} \left( P_{11} + \cdots + P_{22} \right)(H^*(1), \phi_1, \phi_2; h_{T,M}),
$$

where $P_k$ denotes the $k$-th derivative of $L(s, f)$, and $\text{Avg}_{(1)}(\cdot)$ denotes the average over $f$ in the family $H^*(1)$. The terms on the right-hand side are easy to evaluate, and we obtain

$$
\lim_{t \to \infty} \frac{\partial}{\partial t} D_2(F^*(1), \phi_1, \phi_2; h_{T,M}) = \left( \tilde{\phi}_1(0) + \frac{1}{2} \phi_1(0) \right) \left( \tilde{\phi}_2(0) + \frac{1}{2} \phi_2(0) \right)
$$

$$
+ \sum_{k=1}^2 P_k \left( H^*(1), \phi_1, \phi_2 \right) + \cdots + P_{22} \left( H^*(1), \phi_1, \phi_2; h_{T,M} \right).
$$

(5.5)

The terms on the right-hand side can be evaluated using the same methods as in the proof of Theorem 1.4. We omit the details here, and refer the reader to [1].
forms, respectively. For each $f$

\[ p(A.2) \]

and define the averaged 1-level density (see Definition 1.1):

\[ \log N(p_1) \log N(p_2) \]

where

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let \( \nu \) be the spectral weight of \( \nu \)

Finally, by Theorem 1.6,

\[ \nu \phi \] for the

Suppose that the supports of \( \hat{\phi}_1 \) and \( \hat{\phi}_2 \) are both contained in \((-\frac{1}{2}, \frac{1}{2})\). Similar to the estimation in \( \S 5.2 \) one may prove that the off-diagonal \( \nu_{\text{off}} \) are all \( o(1) \). For the diagonal \( \nu_{\text{diag}} \), the Prime Ideal Theorem for \( F \) implies that

\[ (\nu_{\text{diag}} + \nu_{\text{diag}})(H^*(1), \phi_1, \phi_2) \sim 2 \int_{-\infty}^{\infty} |y|\hat{\phi}_1(y)\hat{\phi}_2(y)dy. \]

Finally, by Theorem 1.6

\[ \text{Avg}_{1}(D_1(f, \phi_1, \phi_2, T^4; \omega f_{\text{Hecke}}(y, T))) = \hat{\phi}_1 \hat{\phi}_2(0) + \frac{1}{2} \phi_1(0) \phi_2(0). \]

The proof is complete by combining the forgoing results.

**Appendix A. Low-lying zeros of \( L \)-functions for \( SL_2(\mathbb{Z}) \)-Maass forms**

In this appendix, we give a simple proof of (a variant of) the result in [AM] for the class of weight functions \( h_{T, M}(t) \) as in Definition 1.3. Moreover, we verify the Katz–Sarnak heuristic for even and odd Hecke–Maass forms. Our analysis of the Bessel integrals is essentially due to Xiaoqing Li [L1].

Let \( H^*(1) \) be the collection of (orthogonal) primitive Maass–Hecke cusp forms for \( SL_2(\mathbb{Z}) \). We shall also consider separately the subsets \( H^+(1) \) and \( H^-(1) \) of even and odd forms, respectively. For each \( f \in H^*(1) \), let \( \frac{1}{2} + i \gamma \) be the Laplacian eigenvalue of \( f \), let

\[ (A.1) \]

be the spectral weight of \( f \) ( \( ||f|| \) is the \( L^2 \)-norm of \( f \) with respect to the Petersson inner product), and let \( \lambda_f(n) \) be the Hecke eigenvalues of \( f \). For \( h_{T, M}(t) \) as in Definition 1.3 define the averaged 1-level density (see Definition 1.1):

\[ (A.2) \]

where \( \sigma = \ast, +, - \).

The following theorem is a variant of [AM] Theorem 1.3.

**Theorem A.1.** Let \( T, M > 1 \) be such that \( M = T^\mu \) with \( 0 < \mu < 1 \). Fix \( h \in \mathcal{H}^+(S, N) \) and define \( h_{T, M} \) by (1.6) in Definition 1.3. Let \( \phi \) be an even Schwartz function with the support of \( \hat{\phi} \) in \((-1 - \mu, 1 + \mu)\). Assume the Riemann hypothesis (for the Riemann zeta function). Then

\[ \lim_{T \to \infty} D_1(H^*(1), \phi; h_{T, M}) = \int_{-\infty}^{\infty} \phi(x)W_1(0)(x)dx. \]
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Remark A.2. When considering the Eisenstein terms, the trivial bound is insufficient if the support of \( \phi \) is extended beyond \([-1, 1]\). However, Alpoge and Miller [AM] claim with no proof in their (29) a bound that seems too strong to be true and it leads to an estimate \( T \log T \) for the total Eisenstein contribution in their (71) (saving a whole \( T \) over the main term). An easy way to rectify this is to assume the Riemann hypothesis and use it to deduce a nontrivial bound for the Eisenstein terms as in (A.19) (half as strong as their claimed (29)). It seems to require some efforts to work without the Riemann hypothesis.

For \( f \in H^\pm(1) \), its Fourier coefficients satisfy
\[
\rho_f(\pm n) = \pm \lambda_f(n), \quad n > 0,
\]
and \( \epsilon_f = \pm 1 \).

A new result towards the Katz–Sarnak conjecture for the family \( H^\pm(1) \) is as follows.

Theorem A.3. Let \( \mu, T, M, h_{T, M} \) and \( \phi \) be as in Theorem A.1. Assume the Riemann hypothesis for Dirichlet L-functions (including the Riemann zeta function). When \( \mu > \frac{1}{3} \), we have
\[
\lim_{T \to \infty} D_1(H^+(1), \phi; h_{T, M}) = \int_{-\infty}^{\infty} \phi(x) W_1(O(\text{even}))(x)dx,
\]
\[
\lim_{T \to \infty} D_1(H^-(1), \phi; h_{T, M}) = \int_{-\infty}^{\infty} \phi(x) W_1(O(\text{odd}))(x)dx.
\]

A.1. The Kuznetsov trace formula for \( \text{SL}_2(\mathbb{Z}) \). Let \( h(t) \in \mathcal{H}(S, N) \) be as in Definition 1.3. For \( n > 0 \), define
\[
\Delta^*_n(h; \phi) = \sum_{f \in \mathcal{H}^+(1)} \omega_f(t_f) \rho_f(n) \rho_f(\pm 1),
\]
\[
\Xi(n; h) = \frac{1}{\pi} \int_{-\infty}^{\infty} \omega(t) h(t) \eta(n, \frac{t}{2} + it) dt,
\]
with
\[
\omega(t) = \frac{1}{\zeta(1 + 2it)} , \quad \eta(n, s) = \sum_{\alpha \mid n} (n/\alpha^2)^{-s}.\]
The Kuznetsov trace formula for \( \text{SL}_2(\mathbb{Z}) \) is as follows (cf. [CL] §3 or [Kuz]):
\[
\Delta^*_n(n; h) + \Xi(n; h) = 2\delta_{n, \pm 1} H + 2K B^\pm(n; h),
\]
where
\[
K B^\pm(n; h) = \sum_{c=1}^{\infty} \frac{\sqrt{n}}{c} H^\pm \left( \frac{\sqrt{n}}{c} \right),
\]
\[
H = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t) \tanh(\pi t) dt,
\]
\[
H^+(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} h(t) J_{2d}(4\pi x) \frac{tdt}{\cosh(\pi t)},
\]
\[
H^-(x) = \frac{2}{\pi^2} \int_{-\infty}^{\infty} h(t) K_{2d}(4\pi x) \sinh(\pi t) dt,
\]
\( J_d(x) \) and \( K_d(x) \) are Bessel functions, \( \delta_{n, \pm 1} \) is the Kronecker \( \delta \)-symbol, and \( S(n, \pm 1; c) \) is the Kloosterman sum.
By (A.3), one may restrict the spectral sum in (A.7) to even or odd forms in $H^\pm(1)$. Precisely, for $n > 0$, define

$$
\Delta^\pm(n; h) = \sum_{f \in H^\pm(1)} \omega_f h(t_f) \lambda_f(n),
$$

then

$$
\Delta^\pm(n; h) + \delta_{\pm, \mp} \Xi(n; h) = \delta_{n,1} H + KB^+(n; h) \mp KB^-(n; h).
$$

### A.2. Analysis of the Bessel integral $H^s_{T,M}(x)$

Let $h_{T,M}(t) \in C^\infty(S, N)$ be as in Definition 1.3. Let $H^s_{T,M}(x)$ be the Bessel integral defined in (A.10) for $h_{T,M}(t)$. Set $M = T^\mu$ for $0 < \mu < 1$.

**Lemma A.4.** We have $H^s_{T,M}(x) \ll h M^x T^2$ for $x \leq 1$.

**Proof.** Let $3 < 2\delta < \min\{5, 2S\}$. By contour shift,

$$
\pi i h_{T,M}(x) = \int_{-\infty}^\infty h_{T,M}(t - \delta i) J_s(4\pi x) \frac{t - \delta i}{\cos(\pi(t + \delta))} dt.
$$

By Poisson’s integral representation for $J_s(x)$ (see (3.1)), along with Stirling’s formula, we infer that

$$
J_3(4\pi x) \ll x, \quad J_s(4\pi x) \ll x^3, \quad \frac{J_{2n+2\delta}(4\pi x)}{\cos(\pi(t + \delta))} \ll \left( \frac{x}{|t| + 1} \right)^{2\delta},
$$

for $x \leq 1$. Consequently (1.5) and (1.6) yield

$$
H^s_{T,M}(x) \ll e^{-\delta M} (x + x^3) + \frac{M x^{2\delta}}{T^{2\delta - 1}} \ll \frac{M x}{T^2},
$$

for $x \leq 1$.

**Q.E.D.**

**Lemma A.5.** When $1 < x \leq T M^{1-\varepsilon}$, we have $H^s_{T,M}(x) = O_x(T^{-A})$ for any $A > 0$.

**Proof.** First of all, by [Wat 6.21 (12)], we may derive that

$$
J_{2n}(4\pi x) - J_{-2n}(4\pi x) = \frac{2}{\pi i} \sinh(\pi t) \int_{-\infty}^{\infty} e(tr/\pi) \cos(4\pi x \cosh r) dr.
$$

This is a real analogue of (3.4), but the integral here is not absolutely convergent. When $x > 1$, it follows from partial integration that only a negligibly small error will be lost if the integral above is truncated at $|r| = T^\varepsilon$. Therefore, up to a negligible error, $H^s_{T,M}(x)$ is equal to

$$
\frac{2}{\pi^2} \int_{-\infty}^{\infty} \int_{-T^\varepsilon}^{T^\varepsilon} t \tanh(\pi t) h((t - T)/M) e(tr/\pi) \cos(4\pi x \cosh r) dr dt.
$$

By changing the variable from $t$ to $Mt + T$, this integral turns into

$$
\frac{2MT}{\pi^2} \int_{-\infty}^{\infty} \int_{-T}^{T} k(t) e((Mt + T)r/\pi) \cos(4\pi x \cosh r) dr dt,
$$

with

$$
k(t) = (Mt/T + 1) \tanh(\pi(Mt + T)) h(t).
$$

By (1.5), it is easy to verify that

$$
k(t + i\sigma) \ll e^{-\pi|x|(|t| + 1)^{1-N}}.
$$
and hence \( k(t) \) is a Schwartz function by Cauchy’s integral formula. After changing the order of integration, we see that the integral above is equal to
\[
\frac{2MT}{\pi^2} \int_{-T}^{T} \hat{k}(Mr/\pi) e(Tr/\pi) \cos(4\pi x \cosh r) dr.
\]
Since \( \hat{k}(r) \) is Schwartz, this integral may be effectively truncated at \( |r| = M^{\varepsilon}/M \), and we need to consider
\[
(A.16) \quad H_{T,M}^{\pm}(x) = \frac{2MT}{\pi^2} \int_{-M^{\varepsilon}/M}^{M^{\varepsilon}/M} \hat{k}(Mr/\pi) e(Tr/\pi) \cos(4\pi x \cosh r) dr.
\]
We now apply Lemma \( 2.6 \) with the phase function \( f_{T,\pm}(r) = Tr/\pi \pm 2x \cosh r \). Observe that
\[
|f_{T,\pm}'(r)| = |T/\pi \pm 2x \sinh r| \geq T/\pi - 2x/M^{1-\varepsilon} \geq T
\]
when \( x \ll M^{1-\varepsilon} \). Lemma \( 2.6 \) implies that the integral is negligibly small for \( x \ll M^{1-\varepsilon} \) (for applying Lemma \( 2.6 \) it would be more rigorous if we had used a smooth truncation at \( |r| = M^{\varepsilon}/M \)).

Q.E.D.

Applying Lemma \( A.4 \) and \( A.5 \) along with the Weil bound for Kloosterman sums \( S(n,1;c) \ll c^{1+\varepsilon} \), the Kloosterman–Bessel term \( KB^+(n,h_{T,M}) \) as in \( (A.8) \) may be estimated as follows,
\[
(A.17) \quad KB^+(n,h_{T,M}) \ll \frac{M}{T^2} \sum_{c \gg \sqrt{n}} \frac{c^{1+\varepsilon} n^{\varepsilon}}{c} + \frac{1}{T^3} \ll \frac{Mn^{1+\varepsilon}}{T^2},
\]
for any \( n \ll T^2 M^{-2-\varepsilon} \). Moreover, in view of the lower bound \( |\zeta(1+2it)| \gg 1/\log(|t|+3) \) (cf. [13] (3.11.10)), the Eisenstein contribution \( \Xi(n,h_{T,M}) \) has the following bound,
\[
(A.18) \quad \Xi(n,h_{T,M}) = \frac{1}{n} \int_{-\infty}^{\infty} \frac{h_{T,M}(t) \eta(n,\frac{1}{2}+it)}{\zeta(1+2it)^2} dt = O(M \log^2 T), \quad n = 1, p^2.
\]
The trivial bound \( O(M \log^2 T) \) above also holds for \( \Xi(p,h_{T,M}) \), but this is not sufficient for our purpose. In order to get a better bound, we assume the Riemann hypothesis, under which we have
\[
(A.19) \quad \Xi(p,h_{T,M}) = O_{\delta,\varepsilon} \left( \frac{MT^\varepsilon}{p^{\delta}} \right),
\]
for any fixed \( \delta < \frac{1}{2} \) and \( \varepsilon > 0 \). To prove this, we write
\[
\Xi(p,h_{T,M}) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{h_{T,M}(t)p^{it}}{\zeta(1+2it)} dt,
\]
and shift the integral contour to \( \text{Im}(t) = \delta < \frac{1}{2} \). We have uniformly
\[
\frac{1}{\zeta(\sigma+it)} = \begin{cases} 
O_{\delta,\varepsilon}(|t|+3)^\delta, & \text{if } \frac{1}{2} < 1-2\delta \leq \sigma \leq 1, \\
O(\log(|t|+3)), & \text{if } \sigma \geq 1,
\end{cases}
\]
where the first bound is valid under the Riemann hypothesis (cf. [13] (3.11.8), (14.2.6)). It follows that
\[
\Xi(p,h_{T,M}) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{h_{T,M}(t+i\delta)p^{it}}{\zeta(1+2it)} dt = O_{\delta,\varepsilon} \left( \frac{MT^\varepsilon}{p^{\delta}} \right),
\]
as desired.
As a consequence of (A.17) and (A.18) (with \( n = 1 \)), in view of (A.7), the total mass of the average
(A.20) \[ \Delta^*_n(1; h_{T,M}) = \sum_{f \in H_n^*(1)} \omega_f h_{T,M}(t_f) \sim \frac{2MT}{\pi^2} \int_{-\infty}^{\infty} h(t) dt = MT. \]

A.3. Proof of Theorem A.1. By standard arguments as in [AM] using the explicit formula and the Kuznetsov trace formula (A.7), one is reduced to bounding certain sums involving the Eisenstein terms \( \Xi \) and the Kloosterman–Bessel terms \( \text{KB}^+ \) by quantities which are of order strictly lower than \( MT \). Two typical sums are
\[ E_1(\phi; h_{T,M}) = \sum_p \phi \left( \frac{\log p}{2 \log T} \right) \frac{\log p}{\sqrt{p \log T}} \Xi(p; h_{T,M}), \]
and
\[ B^*_1(\phi; h_{T,M}) = \sum_p \phi \left( \frac{\log p}{2 \log T} \right) \frac{\log p}{\sqrt{p \log T}} \text{KB}^+(p; h_{T,M}). \]

Suppose that \( \hat{\phi} \) is supported on \([-v, v]\) for \( v < 1 + \mu \).

For the first sum \( E_1(\phi; h_{T,M}) \), it follows from (A.19) that
\[ \sum_p \phi \left( \frac{\log p}{2 \log T} \right) \frac{\log p}{\sqrt{p \log T}} \Xi(p; h_{T,M}) \ll MT^\epsilon \sum_{p \leq T^\delta} \frac{\log p}{p^{\frac{1}{2} + \epsilon}} \ll MT^{(1-2\delta)v + \epsilon}, \]
and \( MT^{(1-2\delta)v + \epsilon} = o(MT) \) if we choose \( \delta \) so that \( 1 - 1/v < 2\delta < \frac{1}{2} \).

As for the second sum \( B^*_1(\phi; h_{T,M}) \), note that \( T^{-2\epsilon} \leq T^2 M^{2-\epsilon} \). Hence (A.17) yields
\[ \sum_p \phi \left( \frac{\log p}{2 \log T} \right) \frac{\log p}{\sqrt{p \log T}} \text{KB}^+(p; h_{T,M}) \ll \frac{M}{T^2} \sum_{p \leq T^\delta} \frac{\log p}{p^{\frac{3}{2} - \epsilon}} \ll MT^{2+2-\epsilon}, \]
which is \( o(MT) \) as desired.

A.4. Analysis of the Bessel integral \( H_{T,M}(x) \). Let \( H_{T,M}(x) \) be the Bessel integral defined in (A.11) for \( h_{T,M}(t) \). Let \( M = T^\mu \) with \( 0 < \mu < 1 \).

**Lemma A.6.** We have \( H_{T,M}(x) \ll h_M x/T^2 \) for \( x \leq 1 \).

**Proof.** Recall the connection formula ([Wat 3.7 (6))]:
\[ 2K_v(z) = \pi \frac{I_{\nu}(z) - I_{-\nu}(z)}{\sin(\pi \nu)}, \]
where \( I_v(z) \) is the \( I \)-Bessel function. So (A.11) may be rewritten as
\[ H^-(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} h(t) I_{2\nu}(4\pi x) \frac{tdt}{\cosh(\pi t)}. \]

Note that \( I_v(z) = e^{-\frac{1}{2} \pi^2} J_v(iz) \) ([Wat 3.7]). We may then follow the same line of arguments in the proof of Lemma A.3. Q.E.D.

**Lemma A.7.** For \( x > 1 \), we have \( H_{T,M}(x) = H_{T,M}^{-\frac{1}{2}}(x) + O(T^{-A}) \) with
(A.21) \[ H_{T,M}^{-\frac{1}{2}}(x) = \frac{2MT}{\pi^2} \int_{-M^\delta/M}^{M^\delta/M} \tilde{k}(-Mr/\pi) e(Tr/\pi) \cos(4\pi x \sinh r) dr, \]
where \( k(t) \) is a Schwartz function defined by (A.15).
Proof. Starting with the formula ([Wat] 6.23 (13))

\[ K_{2\mu}(4\pi x) = \frac{1}{2 \cosh(\pi t)} \int_{-\infty}^{\infty} e(i\pi x \sinh r) dr, \]

we may prove this lemma by the arguments that led us to (A.16). Q.E.D.

Define \( H_{T, M}^{-\pm}(x) = dH_{T, M}^{-\pm}(x)/dx \). By (A.21), we have

\[ H_{T, M}^{-\pm}(x) = -\frac{8MT}{\pi} \int_{-M/\pi}^{M/\pi} \hat{k} (-Mr/\pi \sinh r \cdot e(Tr/\pi) \sin(4\pi x \sinh r) dr. \]

**Lemma A.8.** For \( x > 1 \), we have

\[ H_{T, M}^{-\pm}(x) \ll T^{1+\epsilon}, \quad H_{T, M}^{-\pm}(x) \ll T^{1+\epsilon}/M. \]

Moreover, if \( \mu > \frac{1}{4} \), then \( H_{T, M}^{-\pm}(x) \), \( H_{T, M}^{-\pm}(x) = O_{A}(T^{-A}) \) unless \( T - 2\pi x \leq M^{1+\epsilon} \).

Proof. Trivial estimation yields (A.24). As for the second statement, we apply Lemma 2.6 with the phase function \( f_{T, \pm}(r) = Tr/\pi \pm 2x \sinh r \). Observe that

\[ f_{T, +}'(r) = T/\pi + 2x \cosh r > T/\pi, \]
\[ f_{T, -}'(r) = T/\pi - 2x \cosh r = T/\pi - 2x + O(xM/\pi M^2), \]

and hence \( |f_{T, \pm}'(r)| \geq M^{1+\epsilon} \) provided that \( T < M^3 \) and \( |T - 2\pi x| > M^{1+\epsilon} \). Q.E.D.

Similar to (A.17), we may deduce from Lemma A.6 and A.7 that

\[ KB^{-}(n, h_{T, M}) \ll \frac{Mn^{1+\epsilon}}{T^{2}}, \]

if \( n \ll T^{-1} \). Consequently, by (A.13) and the estimates in (A.17), (A.18) and (A.25), we infer that

\[ \Delta^{\pm}(1; h_{T, M}) = \sum_{f \in \mathbb{Z}^{(1)}} \omega_{f} h_{T, M}(t_{f}) \sim \frac{H_{T, M}}{2\pi^{2}} = MT. \]

**A.5. Proof of Theorem A.3** In view of the Kuznetsov trace formula for \( H^{\pm}(1) \) as in (A.13), we now also need to consider the contribution from the Kloosterman–Bessel terms \( KB^{-} \). To be precise, we are left to consider

\[ \mathcal{B}_{1}^{1}(\phi; h_{T, M}) = \sum_{p} \phi \left( \frac{\log p}{2 \log T} \right) \frac{\log p}{\sqrt{p \log T}} \phi(p; h_{T, M}), \]

\[ \mathcal{B}_{2}^{1}(\phi; h_{T, M}) = \sum_{p} \phi \left( \frac{\log p}{\log T} \right) \frac{\log p}{p \log T} \phi(p; h_{T, M}). \]

Suppose that \( \hat{\phi} \) is supported on \([-v, v]\) for \( v < 1 + \mu \). Set \( P = T^{2v} \).

By (A.25), it is readily seen that \( \mathcal{B}_{2}^{1}(\phi; h_{T, M}) = O(M^{1+\epsilon}/T^{2}) \) and hence only contributes an error term.

Since the support of \( \hat{\phi} \) is extended beyond the segment \([-1, 1]\), the sum \( \mathcal{B}_{2}^{1}(\phi; h_{T, M}) \) would contribute to the asymptotics. For Theorem A.3 sufficient is to prove that it has the following asymptotic.

**Lemma A.9.** Assume the Riemann hypothesis for the classical Dirichlet L-functions. Then, for \( M = T^{\mu} \) with \( \frac{1}{4} < \mu < 1 \), we have

\[ \mathcal{B}_{1}^{1}(\phi; h_{T, M}) = -\frac{H_{T, M}}{2\pi^{2}} \left( \int_{-\infty}^{\infty} \phi(x) \sin(2\pi x) 2\pi x - \frac{1}{2} \phi(0) + O \left( \frac{1}{\log T} \right) \right), \]
where $H_{T,M}$ is the integral defined in (A.9) for $h_{T,M}$.

Proof. By Lemma [A.6, A.7] and [A.8] we write

$$(A.30) \quad B^{-1}_1(\phi; h_{T,M}) = \sum_{c \leq \sqrt{T}/T} \frac{Q^\phi_{T,M}(c; \phi)}{c} + O\left(\frac{MP^\phi}{T^2}\right),$$

with

$$(A.31) \quad Q^\phi_{T,M}(c; \phi) = \sum_p S(p, -1; c) \frac{\log p}{\sqrt{p} \log T} \frac{H^{-1}_{T,M}}{c} \left(\frac{\sqrt{T}}{c}\right) \phi \left(\frac{\log p}{2 \log T}\right).$$

Under the Riemann hypothesis for classical Dirichlet $L$-functions (including the Riemann zeta function), it follows from Lemma 6.1 in [ILS] (see the paragraph above (6.5) in [ILS]) that

$$(A.32) \quad \sum_{p \leq x} S(p, -1; c) \frac{\log p}{\sqrt{p}} = 2\mu(c)^2 \frac{1}{\varphi(c)} \sqrt{x} + O(c(x)^{\epsilon}).$$

By (A.31) and (A.32), we get

$$Q^\phi_{T,M}(c; \phi) = -\frac{1}{\log T} \int_0^\infty \left\{ \frac{2\mu(c)^2}{\varphi(c)} \sqrt{x} + O(c(x)^\epsilon) \right\} d\hat{H}^\phi_{T,M} \left(\frac{\sqrt{x}}{c}\right) \phi \left(\frac{\log x}{2 \log T}\right).$$

The main term is equal to

$$\frac{2\mu(c)^2}{\log T \varphi(c)} \int_0^\infty \hat{H}^\phi_{T,M}(x) \phi \left(\frac{\log (cx)}{\log T}\right) dx.$$

The error term is bounded by

$$c(cP)^\epsilon \int_0^{\sqrt{T}/c} \left( \left| \hat{H}^\phi_{T,m}(x) \right| + \left| \hat{H}^{-1}_{T,M}(x)^{\epsilon} \right| \right) dx = 2c(cP)^\epsilon \int_0^{\sqrt{T}/c} \left( \left| \hat{H}^\phi_{T,M}(x) \right| + \left| \hat{H}^{-1}_{T,M}(x)^{\epsilon} \right| \right) dx,$$

where $\hat{H}^\phi_{T,M}(x)$ and $\hat{H}^{-1}_{T,M}(x)$ are defined by (A.21) and (A.23). In view of Lemma [A.8], one easily sees that it is bounded by $T^{1+\epsilon} (\hat{H}^\phi_{T,M}(x)$ and $\hat{H}^{-1}_{T,M}(x)$ are essentially supported near $T/2\pi$), and that it only contributes an $O(T^{1+\epsilon}) = o(MT)$ to $B^{-1}_1(\phi; h_{T,M})$ as in (A.30). We now change the $\hat{H}^{-1}_{T,M}(x)$ in the main term to $\hat{H}^\phi_{T,M}(x)$ and then complete the $c$-sum. By doing so, in view of Lemma [A.6, A.7] we produce some extra terms which can be absorbed into the error term in (A.30). By the definition of $\hat{H}^\phi_{T,M}(x)$ as in (A.11), we write

$$(A.33) \quad B^{-1}_1(\phi; h_{T,M}) = \int_{-\infty}^\infty h_{T,M}(t) K_T(t; \phi) \sinh(\pi t) dt + O\left(\frac{MP^\phi}{T^2}\right),$$

with

$$(A.34) \quad K_T(t; \phi) = \frac{4}{\pi^2} \sum_{c=1}^\infty \frac{\mu(c)^2}{\varphi(c)} \int_0^\infty K_{2\mu}(4\pi x) \phi \left(\frac{\log (cx)}{\log T}\right) dx \log T.$$

Note that we need $|t|$ to be very close to $T$. We now follow the arguments in §7 of [ILS]. By the definition (12) and the formula ([Wat] 13.21 (8))

$$\int_0^\epsilon K_\nu(x)x^{\nu-1} dx = 2^{\nu-\epsilon} \pi^\frac{\nu}{2} \Gamma \left(\frac{\mu - \nu}{2}\right) \Gamma \left(\frac{\mu + \nu}{2}\right), \quad \Re(\mu) > |\Re(\nu)|,$$
Moreover, by Stirling’s formula, we find that the integral in (A.34) is equal to
\[
\frac{1}{4} \int_{-\infty}^{\infty} \phi(2y \log T) \frac{(2\pi/c)^{4iy}}{\cosh(\pi \sqrt{2} / y)} \Gamma \left( \frac{1}{4} + it - 2\pi iy \right) \Gamma \left( \frac{1}{4} + it + 2\pi iy \right) dy.
\]
Next we interchange the integration over \( y \) with the summation over \( c \). For the convergence we introduce \( \varepsilon > 0 \) so that
\[
K_T(t, \phi) = \frac{1}{\pi^2} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \phi(2y \log T) \frac{\chi(\varepsilon + 4\pi iy)(2\pi)^{4iy}}{\cosh(\pi \sqrt{2} / y)} \Gamma \left( \frac{1}{4} + it - 2\pi iy \right) \Gamma \left( \frac{1}{4} + it + 2\pi iy \right) dy,
\]
where
\[
\chi(s) = \sum_{c=1}^{\infty} \mu(c)^2 \frac{\varphi(c)}{c^s}.
\]
We need \( \chi(s) \) for \( s \ll (\log T)^{-1} \). By comparing their Euler products, we find that the Laurent expansions of \( \chi(s) \) and \( \zeta(1 + s) \) near \( s = 0 \) have the same leading term \( s^{-1} \). Thus
\[
\chi(s) = s^{-1} + O(1).
\]
Moreover, by Stirling’s formula
\[
\Gamma \left( \frac{1}{2} + it + 2\pi iy \right) = \Gamma \left( \frac{1}{2} + it - 2\pi iy \right) \frac{1 + O \left( \frac{|t|}{\log T} \right)}{\pi^2 \cosh(\pi \sqrt{2} / y)}.
\]
Hence we deduce that
\[
K_T(t, \phi) = \frac{1}{\pi^2} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \phi(2y \log T) \frac{\left| \frac{|t|}{2\pi} \right|^{4iy}}{\pi^2 \cosh(\pi \sqrt{2} / y)} \frac{dy}{\varepsilon - 4\pi iy} + O \left( \frac{1}{\log T} \right).
\]
The last integral becomes
\[
- \int_{-\infty}^{\infty} \phi(2y \log T) \sin(4\pi y \log(|t|/2\pi)) \frac{dy}{4\pi y + ie} + \int_{-\infty}^{\infty} \phi(2y \log T) \cos(4\pi y \log(|t|/2\pi)) \frac{dy}{(4\pi y)^2 + 1}.
\]
Now we can take the limit as \( \varepsilon \to 0 \), getting
\[
\frac{1}{2} \int_{-\infty}^{\infty} \phi(y) \sin \frac{2\pi y}{2\pi} dy + \frac{1}{2} \phi(0) + O \left( \frac{1}{\log T} \right),
\]
for \( |t| \) very close to \( T \). Thus
\[
K_T(t, \phi) = - \frac{1}{2\pi^2 \cosh(\pi \sqrt{2} / y)} \left\{ \int_{-\infty}^{\infty} \phi(y) \sin \frac{2\pi y}{2\pi} dy - \frac{1}{2} \phi(0) + O \left( \frac{1}{\log T} \right) \right\}.
\]
We arrive at (A.29) by introducing this into (A.33). Q.E.D.

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LOW- LYING ZEROS OVER IMAGINARY QUADRATIC FIELDS

DEPARTMENT OF MATHEMATICS AND STATISTICS, WASHINGTON STATE UNIVERSITY, PULLMAN, WA 99164-3113, USA
E-mail address: scliu@math.wsu.edu

SCHOOL OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, 310027, CHINA
E-mail address: zhi.qi@zju.edu.cn