Mathematical and Stochastic Growth Models

V. Pavankumari¹, Venkataramana Musala², Akasam Srinivasulu³, M. Bhupathi Naidu⁴

¹Research Scholar, Department of Statistics, Sri Venkateshwara University, Tirupati, Andhra Pradesh, India
²Assistant Professor, Department of Mathematics, Malla Reddy Engineering College(A), Hyderabad, Telangana, India
³Research Scholar, Department of Statistics, Sri Venkateshwara University, Tirupati, Andhra Pradesh, India
⁴Professor &DDE, Sri Venkateshwara University, Tirupati, Andhra Pradesh, India

Abstract: Many statistical and mathematical models of growth are developed in the literature and effectively applied to various conditions in the existent world that involve many research problems in the different fields of applied statistics. Nevertheless, still, there is an equally large number of conditions, which have not yet been mathematically or statistically modeled, due to the complex situations or formed models are mathematically or statistically inflexible. The present study is based on mathematical and stochastic growth models. The specification of both the growth models is depicted. A detailed study of newly modified growth models is mentioned. This research will give substantial information on growth models, such as proposed modified exponential growth models and their specifications clearly motioned which gives scope for future research.

I. INTRODUCTION

Growth modeling is the heart of different fields of applied statistics like Demometrics, Biometrics, Econometrics, Business and Industrial Statistics, Time series, and forecasting. The methodology of the Growth model is widely used in the modeling of different research problems in these fields. For many years, a great deal of research has been directed in almost all applied fields of statistics to either the mathematical growth model or stochastic growth model and creating the functional relationships among various characteristics by fitting different linear or nonlinear models of growth.

The conventional mathematical growth model process is in the given statements:
(a) Selection of mathematical or Formulation model of growth to the existent world problem (b) Findingsolutions and Solving for the chosen mathematical model of growth (c) Observing or analyzing the validity or goodness of formulated mathematical model of growth. The conventional stochastic or statistical growth of the model continues along with the following statements: (a) Formulation or Specification of Statistical model of growth. (b) Evaluation of limitations of the identified statistical model of growth. (c) Testing of hypothesis and constructing confidence intervals concerning the factors of the statistical model of growth. (d) Estimating or forecasting validity of the predicted statistical model of growth. (e) Using the predictable statistical model of growth for making and controlling policy decisions.

II. REVIEW OF LITERATURE

It is observed that during the past decade the make use of Logistic Growth and Exponential Growth modeling have been explored from their original acceptance in epidemiologic research. Bai, D.S et.al(1993) the study emphasized on Estimation of Constant Stress Partially Accelerated Life Test for Fre’chet Distribution with Type-I Censoring Modern reliability engineering accelerated life tests (ALT) and partially accelerated life tests (PALT) in determining the failed manners of the items at routine conditions by using the information of the data generated from the experiment. Results explained in statistical properties of the parameters indicated that the constant stress PALT plan works well. Bakshi, G. and Chen, Z.(1994) in this study explored on Baby Boom, Population Aging and Capital Market as part of business terminology and illustrated with results have demographic results influence with capital market and over so ever uniqueness shown in the social and statistical controls. Ananda, M.M. et.al (1996), there is inter relation of our with titled on Adaptive Bayes Estimators of the Gompertz survival model but exploration variates have taken dimension advances with shape and statistical adoptions. Tahir, M. (2003) the author enumerates with Estimation of the Failure Rate in a Partially Accelerated Life Test: A Sequential Approach and concluded with the stages, the units are put on the test under normal stress up to time t, where t is determined as a stopping time that minimizes the expected loss plus cost of running the test. In the second stage, the stress is raised to a higher level for those units that did not fail by time t and held constant until they all fail. The accumulated data are then used to estimate with the Bayes estimator. Feng, Y and Tong, X. (2019) the study modeled with a new cellular automata framework of urban growth modeling by incorporating statistical and heuristic methods and concluded with effectiveness calibrated elsewhere to simulate the dynamic urban growth and assess the resulting natural and socioeconomic impacts. Our Study focused with new attentive approach with existing new era concepts and found that there is invites us to do some form of research gap in mathematical and growth model.
These growth models are now normally explored in numerous fields. They are not limited only to Biomedical Research, Business, and Finance, Criminology, Ecology, Engineering, Forestry, Demography, and others. At the same time, there has been an identical amount of endeavor in research on all Stochastic aspects of the Exponential growth models too.

The word stochastic is being derived from a Greek word that means ‘random’ or ‘chance’. Generally, a deterministic model foresees a single outcome from the given set of circumstances and a static model predicts a set of expected outputs weighed by their possibilities or probabilities.

Scientific modeling generally would have three components:
1) A natural occurrence under the study
2) A logical system for reasoning the simplification of the phenomenon and

This paradigm is used to analyze the logical system by connecting the elements of the natural system together. The development of the models will benefit a wide range of research and management. The eminent statisticians France and Thornley (1984) highlighted the various ways in which mathematical models assist to the advancement of research and management. They are:

- Mathematical hypotheses would allow for a quantitative description and explanation of physical and biological phenomena.
- In order to construct a mathematical completeness model, it should give a conceptual framework that may help identify places where knowledge is lacking and where new ideas and experimental techniques can be inspired.
- One of the good approaches to provide a procedure by which research knowledge is made available in an easy-to-use form by the farm manager is to employees.
- An agro-economic model could often study and highlight the techniques of economic gain provided by researchers, so encouraging the adoption of enhanced production methods.
- Modeling development may lead to less adhoc experimentation, as developing models can sometimes make it easier to construct tests to answer specific questions or discriminate between mechanisms.
- Several components inside a system may give a model or a method of bringing information about the pieces together to provide a cohesive perspective of the behavior of the entire system. mathematical model.
- Modeling could assist in providing strategic and tactical support for an upcoming research project by motivating scientists and fostering teamwork.
- Any model can give a powerful way of summing up data as well as a tool for interpolation and cautious extrapolation.
- As data grows more accurate and expensive, finding a model that can exploit all available data becomes more difficult.
- The effective model’s predictive control can be applied in a variety of ways, including prioritizing research and development (R&D), management, and planning.

III. OBJECTIVES OF THE PRESENT STUDY

The central aim of this research work focused on various Mathematical and Stochastic aspects of growth curves.
1) To discuss various models of Stochastic Linear Growth models
2) To review the different Mathematical and Stochastic aspects of Growth models.

IV. METHODS

A. Stochastic Linear Growth Models
1) Population Growth Models: If \( x(t) \) be the size of the population at time \( t \) and let \( b \) and \( d \) be the birth and death rates, i.e the number of individuals born or dying per individuals per unit time, then in time interval \( (t, t+\Delta t) \), the number of births and deaths would be \( bx\Delta t + 0(\Delta t) \) and \( dx\Delta t + 0(\Delta t) \) where \( \Delta t \) is an infinitesimal Which approaches zero as \( \Delta t \) approaches zero, so that

\[
x(t + \Delta t) - x(t) = (bx(t) - dx(t)\Delta t + o(\Delta t) \quad (1)
\]

So that by diving by \( \Delta t \) and proceeding to the limit as \( \Delta t \to 0 \), we get

\[
\frac{dy}{dx} = (b - d)x = ax \quad (say) \quad (2)
\]

integrating (2), we get

\[
x(t) = x(0)e^{at} \quad (3)
\]
So that the population grows exponentially if \( a > 0 \) decays exponentially if \( a < 0 \) and remains constant if \( a = 0 \)

(i) If \( a > 0 \), the population will become double its present size at the time \( T \) where
\[
2x(0) = x(0)e^{at} \quad \text{Or} \quad e^{at} = 2
\]
Or
\[
T = \frac{1}{a} \ln 2
\]

This is considered as the doubling period of the population and it may be noted that this doubling period is independent of \( x(0) \). It depends only on \( a \) and is such that the greater the value of \( a \) (i.e. greater the difference between birth and death rates), the smaller is the doubling period

(ii) If \( a < 0 \), the population will become half its present size in time \( T^t \) when
\[
\frac{1}{2}x(0) = x(0)e^{at^t} \quad \text{Or} \quad e^{at^t} = \frac{1}{2}
\]
Or
\[
T^t = \frac{1}{a} \ln 2
\]

It may also be noted that \( T^t \) is also independent of \( x(0) \) and since \( a < 0 \), \( T^t > 0 \) may be called the half-life period of the population and it decreases as the excess of death rate over birth rate increases.

B. Computation of Linear Growth Rate and its test of Significance

1) Linear Growth Rate: Linear growth rate (LGR) in a study variable \( y \), in order to get the total change in a time variable \( t \) which is defined as the ratio of relative change in \( y \) for the fixed change in \( t \), that is multiplied by 100.

\[
i.e. \ LGR = \left( \frac{\text{relative change in } y}{\text{absolute change in } t} \right) 100
\]

Small changes in \( Y \) and \( t \) LGR may be approximated symbolically as given below

\[
LGR = \left( \frac{\Delta y}{\Delta t} \right) 100
\]

\[
LGR = \left( \frac{y_2 - y_1}{t_2 - t_1} \right) 100
\]

Or

\[
LGR = \left( \frac{y_2 - y_1}{t_2 - t_1} \right) 100
\]

\[
\therefore \ LGR = \left( \frac{y_2 - y_1}{t_2 - t_1} \right) 100
\]

Where \( y = \frac{y_2 + y_1}{2} \)

\( y_1 \) and \( y_2 \) is considered as the values of \( y \) for the time periods of \( t_1 \) and \( t_2 \) in the above equation.

If a linear relationship exists between a study variable "\( y \)" and a time variable "\( t \)" as

\[
y_i = \alpha + \beta t_i, \quad i = 1, 2, 3, \ldots, n
\]
If the time variable \( t \) is being coded in the place of \( t \), the linear model can be written as

\[
y_i = \alpha + \beta x_i, \quad i = 1, 2, 3, \ldots, n
\]

Or simply, \( y = \alpha + \beta x \). By adding an error term \( \varepsilon \), the statistical linear regression model is given by

\[
y = \alpha + \beta x + \varepsilon
\]

Wherever, \( y \): Study variable (dependent variable)

\( x \): Coded time variable (independent variable)

And if \( \alpha, \beta \) are the parameters of the linear model. The Least Squares Estimation of \( \alpha \) and \( \beta \) are given below likewise

\[
\hat{\beta} = \frac{\sum xy}{\sum x^2}
\]

And

\[
\hat{\alpha} = y - \hat{\beta}x
\]

Here \( x = \frac{\sum x}{n}, \quad y = \frac{\sum y}{n} \) and \( n \) is the observation. The estimate linear model is shown by

\[
\hat{y} = \hat{\alpha} + \hat{\beta}x. \quad \text{This estimate model is being used for the prediction of analysis.}
\]

An estimate of LGR is given below now as:

\[
LGR = \frac{\hat{\beta}}{100}
\]

C. Different Mathematical and Stochastical aspects of Growth Models

A method of fitting the proposed modified exponential growth model:

Consider the growth model for the modified Exponential Growth Curve can be considered as

\[
y_t = \alpha + \beta y^t
\]

Where \( y_t \) is the value of the study variable at the time period \( t \) and \( \alpha, \beta, y \) are unknown parameters

1) Method for three Selected Points

One may take three ordinates \( y_1, y_2, \) and \( y_3 \) to three equidistant values of \( \quad t = t_1, t_2, \) and \( t_3 \) respectively such that

\[
t_2 - t_1 = t_3 - t_2
\]

Substituting values of \( t_1, t_2, \) and \( t_3 \) in one may get

\[
y_1 = \alpha + \beta y^{t_1}
\]

\[
y_2 = \alpha + \beta y^{t_2}
\]

\[
y_3 = \alpha + \beta y^{t_3}
\]

\[
y_2 - y_1 = \beta y^{t_1}(y^{t_2-t_1} - 1)
\]

\[
y_3 - y_2 = \beta y^{t_2}(y^{t_3-t_2} - 1)
\]
\[
\frac{y_3 - y_2}{y_2 - y_1} = e^{t_2 - t_1} \tag{21}
\]

\[
y = \left(\frac{y_3 - y_2}{y_2 - y_1}\right)^{\frac{1}{t_2 - t_1}} \tag{22}
\]

Substituting for \(y\) in (19) we get

\[
\beta = \frac{(y_2 - y_1)^2}{(y_3 - 2y_2 + y_1)} \left(\frac{y_2 - y_1}{y_3 - y_2}\right)^{\frac{t_1}{t_2 - t_1}} \tag{23}
\]

Substituting \(\beta\) and \(y\) in (16) we get

\[
\alpha = y_1 - \beta y^{t_1}
\]

\[
\alpha = \frac{y_1 y_3 - y_2^2}{(y_3 - 2y_2 + y_1)}
\]

Substitution for \(\alpha, \beta\) and \(y\) from (16), (17), and (18) get the equation of the Modified Exponential Curve fitted to the given series data

\(y_1, y_2,\) and \(y_3\) being ordinates of the freehand curve corresponding to the three selected points \(t_1, t_2\) and \(t_3\)

2) **Method of Partial Sums**

The given time series data are split up into three equal parts each containing (Say) \(n\) consecutive values of \(y_t\) corresponding to

\(t = 1, 2, 3, \ldots, n; \quad t = n + 1, n + 2, \ldots, 2n;\)

\(t = 2n + 1, 2n + 2, \ldots, 3n.\) Let \(S_1, S_2,\) and \(S_3\) represent the partial sums of the three parts respectively so that

\[
S_1 = \sum_{t=1}^{n} y_t \quad S_2 = \sum_{t=n+1}^{2n} y_t \quad S_3 = \sum_{t=2n+1}^{3n} y_t
\]

Substituting for \(y_t\), one may get

\[
S_1 = \sum_{t=1}^{n} (\alpha + \beta y^t) = n\alpha + \beta (y + y^2 + \ldots + y^n) = n\alpha + \beta y \left(\frac{y^n - 1}{y - 1}\right) \tag{25}
\]

Similarly, we shall get

\[
S_2 = n\alpha + \beta y^{n+1} \left(\frac{y^n - 1}{y - 1}\right) \tag{26}
\]

And

\[
S_3 = n\alpha + \beta y^{2n+1} \left(\frac{y^n - 1}{y - 1}\right) \tag{27}
\]

Substituting (24) from (25), and (26) from (27) one may get

\[
S_2 - S_1 = \beta y \left[\frac{(y^n - 1)^2}{y - 1}\right] \tag{28}
\]

\[
S_3 - S_2 = \beta y^{n+1} \left[\frac{(y^n - 1)^2}{y - 1}\right] \tag{29}
\]

Dividing (28) with (29), we have

\[
\frac{S_3 - S_2}{S_2 - S_1} = y^n
\]
\[ y = \left( \frac{S_3-S_2}{S_2-S_1} \right)^{\frac{1}{n}} \]  

(30)

Substituting for \( y^n \) in (28), we get

\[ S_2 - S_1 = \frac{\beta y}{y-1} \left( \frac{S_3 - S_2}{S_2 - S_1} - 1 \right)^2 \]

Finally substituting the values of \( \beta \) and \( y \) in

\[ \alpha = \frac{1}{n} \left[ S_1 - \frac{\beta y}{y-1} (y^n - 1) \right] \]

\[ = \frac{1}{n} \left[ S_1 - \frac{(S_2-S_3)^3}{(S_2-2S_2+S_1)^2} (y^n - 1) \right] \]  

[From (31)]

\[ \frac{1}{n} \left[ S_1 S_2 - S_2^2 \right] \frac{S_2-S_1}{S_2-2S_2+S_1} \]

(32)

3) *Fitting of gompertz curve*

\[ y_t = \alpha \beta^t \]  

(33)

Where \( y_t \) is time series value at time \( t \) and \( \alpha, \beta, y \) are its parameters

\[ \log y_t = \log \alpha + \gamma t \log \beta \]

\[ Y_t = A + B y^t \]  

(34)

Where \( Y_t = \log y_t, A = \log \alpha, B = \log \beta \)

Now (34) is a modified exponential curve and the constants \( A, B, \gamma \) can be estimated by the method of three selected points or by the method of partial sums

**V. CONCLUSIONS**

In the modelling of growth, the growth model methodology is extensively employed. Many domains of study, including economics, business, management, demography, biology, industry, and others, use models to explain growth behavior through time. In the field of population biology, growth happens in plants, animals, organisms, and so on. The sort of model required in a given situation is determined by the type of growth experienced. In general, growth models are mechanical rather than experimental in nature. In the current research work, an effort has been made to develop the models to depict the mathematical and Stochastic aspects of growth and draw from some new Stochastic growth models by using logistic and Exponential growth models. A Logistic growth model has been specified as a number of families of generalized linear models. The concepts of odds ratio and model Deviance is defined and a method is described to estimate the odds ratio. The maximum method of estimation is discussed to estimate the parameters of the multiple logistic growth models. The validity of the integrity of fit of the logistic growth model is tested by using the model deviance. An Exponential Growth model is specified by using the poisson probability model for count data. Identity link function and log link function are measured under the maximum likelihood estimation of parameters of the Exponential Growth model. Residuals are attained from logistic and Exponential Growth models, to test for the competence of the models; and to be used in verifying the assumptions and in the correctness of the selected link function. The studied Conclusions evidenced with holds of the title.
REFERENCES

[1] Bai, D.S., Chung, S.W. and Chun Y.R. (1993), “Optimal Design of partially Accelerated Life Tests for the Exponential Distribution under Type-I Censoring”. IEEE Trans. On Reliability vol. 41 (3), 400-406.

[2] Bakshi, G. and Chen, Z. “Baby Boom, Population Aging and Capital Market”, Journal of business 67(1994):165-202.

[3] Ananda, M.M., Dalpatakud, R.J. and Singh A.K.(1996), “Adaptive Bayes Estimators of the Gompertz survival model” Applied Mathematics and computation, Vol. 75, Issue 2, 167-177.

[4] El-Saidi, M.A., Dimitrov B. and Chukova S. (1996). Some moment properties and limit theorems of the reversed generalized logistic distribution with application. Comma. Statist. Theory meth,25, 609 – 630.

[5] G.Anadurai, S. Rajesh Babu, V.R.Srinivasmoorthy(2000).” Development of mathematical models(Logistic, Gompertz and Richards models) describing the growth pattern of Pseudomonas putida(NICM2174)”. Bioprocess Engineering,2000

[6] Tahir, M. (2003), “Estimation of the Failure Rate in a Partially Accelerated Life Test: A Sequential Approach “, Stochastic Analysis and applications vol. 21, No. 4, PP 909-915.

[7] Jian-Jun Cheng,(2007) “The Images of Restrictions Maps and Dirac Cohomology”, Communications in Algebra.

[8] Jacopo Iannacci.2013.Com pact Modelling of RF-MEMS Devices”, Wiley

[9] Mar tha L. Abell, James P. Braselton,(2014) “Applications of Systems of Ordinary Differential Equations”, Elsevier BV.

[10] Mehmet Korkmaz,(2018). “ N partial Sum approaches to estimate the parameters of some growth models”, AIP Publishing.

[11] Feng, Y., & Tong, X. (2019). A new cellular automata framework of urban growth modeling by incorporating statistical and heuristic methods. International Journal of Geographical Information Science, 34(1), 74–97.
