The Steklov problem on triangle-tiling graphs in hyperbolic space

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Abstract

We introduce a graph $\Gamma$ which is roughly isometric to the hyperbolic plane and we study the Steklov eigenvalues of a subgraph with boundary $(\Omega, B)$ of $\Gamma$. For $(\Omega_l, B_l)_{l \geq 1}$ a sequence of subgraphs of $\Gamma$ such that $|\Omega_l| \to \infty$, we prove that for each $k \in \mathbb{N}$, the $k^{th}$ eigenvalue tends to 0 proportionally to $1/|B_l|$. The idea of this proof consists in finding a bounded domain $(N, \Sigma)$ of the hyperbolic plane which is roughly isometric to $(\Omega, B)$, giving an upper bound for the Steklov eigenvalues of $(N, \Sigma)$ and transferring this bound to $(\Omega, B)$ via a process called discretization.

1 Introduction

Let $(M, g)$ be a smooth connected compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary $\partial M$. The Steklov problem on $(M, g)$ consists in finding all $\sigma \in \mathbb{R}$ such that there exists a non-zero harmonic function $f : M \to \mathbb{R}$ satisfying $\frac{\partial f}{\partial \nu} = \sigma f$ on $\partial M$, where $\frac{\partial}{\partial \nu}$ denotes the outward-pointing normal derivative on $\partial M$.

Such a $\sigma$ is called a Steklov eigenvalue of $M$ and a corresponding $f$ is called a Steklov eigenfunction. The (ordered) set of eigenvalues is called the Steklov spectrum of $(M, g)$.

It is well known that the Steklov spectrum of $M$ forms a discrete sequence

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \nearrow \infty,$$

where each eigenvalue is repeated with multiplicity.

There exists a discrete equivalent to the Steklov problem, which is called the discrete Steklov problem and which is defined on graphs with boundary. Let us begin by defining it.

Definition 1. A graph with boundary is a triplet $(\bar{\Omega}, E', B)$, where $(\bar{\Omega}, E')$ is a simple connected undirected graph and $B \subset \bar{\Omega}$ is a non-empty set of vertices, called the boundary. The set $B^c$ is called the interior of the graph.

In this paper, all graphs will always be simple connected and undirected.

For $v, w \in \bar{\Omega}$, we write $v \sim w$ when $v$ is adjacent to $w$. For $A \subset \bar{\Omega}$, we write $|A|$ the cardinal of $A$, which is the number of vertices contained in $A$. For the purpose of this article, all graphs with boundary are finite. We denote by $\mathbb{R}^{\bar{\Omega}}$ the space of all function $u : \bar{\Omega} \to \mathbb{R}$, which is isomorphic to the Euclidean space of dimension $|\bar{\Omega}|$. Similarly, we denote by $\mathbb{R}^B$ the space of functions $u : B \to \mathbb{R}$, which is the Euclidean space of dimension $|B|$.

We can now introduce the discrete Laplacian operator $\Delta : \mathbb{R}^{\bar{\Omega}} \to \mathbb{R}^{\bar{\Omega}}$, defined by

$$\Delta u : \bar{\Omega} \to \mathbb{R},$$

$$v \mapsto \Delta u(v) = \sum_{w \sim v} (u(v) - u(w)).$$
The normal derivative \( \frac{\partial}{\partial \nu} : \mathbb{R}^\Omega \rightarrow \mathbb{R}^B \) is defined by
\[
\frac{\partial u}{\partial \nu} : B \rightarrow \mathbb{R}
\]
\[v \mapsto \frac{\partial u}{\partial \nu}(v) = \sum_{w \sim v} (u(v) - u(w)).\]

**Definition 2.** The discrete Steklov problem on a finite graph with boundary \((\bar{\Omega}, E', B)\) consists in finding all \(\sigma \in \mathbb{R}\) such that there exists a non-zero harmonic function \(u \in \mathbb{R}^\bar{\Omega}\) such that \(\frac{\partial u}{\partial \nu}(v) = \sigma u(v)\) for all \(v \in B\).

Such a \(\sigma\) is called a Steklov eigenvalue and a corresponding \(u\) is called a Steklov eigenfunction of \((\bar{\Omega}, E', B)\). As said in [12], the Steklov spectrum of a graph with boundary \((\bar{\Omega}, E', B)\) forms a sequence as follows:

\[0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_{|B|-1}.\]

This problem has recently received a particular attention, one can cite for instance [5, 8, 12, 13]. An investigation has been made by Colbois, Girouard and Raveendran in [4], allowing us to understand some spectral links between the Steklov problem on a manifold and the discrete Steklov problem of a graph associated to this manifold. These links will be very useful in this paper. The main problem that we will have to face is to place ourselves in the hypotheses of Theorem 3 of [4], in order to use it to our advantage.

Among other things, a question that has been studied by some authors is that of providing an upper bound for the first - and then for the \(k\)th - eigenvalue of some particular graphs with boundary. These particular graphs that have been studied are those called subgraphs of an (infinite) host graph. A subgraph of a host graph can be interpreted as the discrete equivalent of a bounded domain in a manifold. Let us define what it is exactly.

**Definition 3.** Let \(\Gamma = (V, E)\) be a graph and let \(\Omega \subset V\) be a finite subset of vertices connected for \(\Gamma\), i.e for each \(v, w \in \Omega\), there exist \(l \in \mathbb{N}\) and \(v_0 = v, v_1, \ldots, v_l = w \in \Omega\) satisfying \(\{v_i, v_{i+1}\} \in E\) for all \(i = 0, \ldots, l-1\). The subgraph \((\bar{\Omega}, E', B)\) of \(\Gamma\) associated with \(\Omega\) is the graph with boundary defined as follows:

- \(B = \{w \in V \setminus \Omega : \exists v \in \Omega \text{ such that } \{v, w\} \in E\}\);
- \(\bar{\Omega} = \Omega \cup B\);
- \(E' = \{\{v, w\} \in E : v \in \Omega, w \in \bar{\Omega}\}\).

Such a graph with boundary is denoted \((\Omega, B)\), with \(\Omega\) the interior and \(B\) the boundary of the subgraph. We refer to \(\Gamma\) as the host graph of \((\Omega, B)\).

Some interesting results have recently been discovered, providing us with bounds for the eigenvalues, depending on the host graph \(\Gamma\). We recall here some of these results.

**Theorem 4** (Han, Hua, 2019). Let \(\mathbb{Z}^d\) be the integer lattice of dimension \(d\). Let \((\Omega, B)\) be a subgraph of \(\mathbb{Z}^d\). Then we have
\[
\sum_{l=1}^d \frac{1}{\sigma_l(\Omega, B)} \geq C' \cdot |\Omega|^{\frac{1}{d}} - C'' |\Omega|,
\]
where \(C' = (64d^3\omega_d^{\frac{1}{d}})^{-1}, C'' = \frac{1}{32d}\) and \(\omega_d\) is the volume of the unit ball in \(\mathbb{R}^d\).
This theorem - and its proof - can be found in [5]. Another investigation gives some control over the spectrum of a subgraph of a Cayley graph. We remind that, given a finitely generated group $G$ and a finite generating subset $S$ of $G$, one can define a graph, called Cayley graph and denoted $Cay(G, S)$. If $G$ is infinite, then so is $Cay(G, S)$ and we can use it as a host graph. The result is the following:

**Theorem 5** (Perrin, 2020). Let $\Gamma = (V, E)$ be a Cayley graph with polynomial growth of order $d \geq 2$. There exists $\tilde{C}(\Gamma) > 0$ such that for any finite subgraph $(\Omega, B)$ of $\Gamma$, we have

$$\sigma_1(\Omega, B) \leq \frac{1}{|B|^\frac{1}{d-1}} \cdot \tilde{C}(\Gamma).$$

This theorem is way more general about the class of host graph $\Gamma$ but provides us control over the first non-trivial eigenvalue only, see [13] for details. An extension to this result is the following:

**Theorem 6.** Let $\Gamma = Cay(G, S)$ be a polynomial growth Cayley graph of order $d \geq 2$. Let $(\Omega, B)$ be a subgraph of $\Gamma$. Then there exists a constant $\tilde{C}(\Gamma) > 0$ such that for all $k < |B|$, we have

$$\sigma_k(\Omega, B) \leq \frac{1}{|B|^\frac{1}{d-1}} \cdot k^{\frac{d+2}{4}}.$$

As a corollary, we have:

**Corollary 7.** Let $\Gamma$ be a polynomial growth Cayley graph of order $d \geq 2$ and $(\Omega_i, B_i)_{i=1}^{\infty}$ be a sequence of subgraphs of $\Gamma$ such that $|\Omega_i| \to \infty$. Fix $k \in \mathbb{N}$. Then we have

$$\sigma_k(\Omega_i, B_i) \to 0 \quad \text{as} \quad i \to \infty.$$

The details of the proofs can be found in [14].

All these theorems follow from the investigation upon one class of host graphs $\Gamma$, which are Cayley graphs of polynomial growth groups. This consideration leads to a natural question: what can we say about the eigenvalues of subgraphs of a host graph $\Gamma$, whose growth rate is more than polynomial?

A first class of graphs we can think of is that of trees. In [6], the authors find upper bounds for the eigenvalues of a finite tree. Their investigations lead to the following result:

**Theorem 8** (He, Hua, 2020). Let $T$ be a finite tree with (uniformly) bounded degree $D$. Let $B$ be the boundary of the tree, i.e the set of vertices of degree one. Then we have

$$\sigma_1 \leq \frac{4D}{|B|}.$$

Higher Steklov eigenvalues are bounded as well: for all $k = 2, \ldots, |B| - 1$, we have

$$\sigma_k \leq \frac{8(D - 1)^2(k - 1)}{|B|}.$$

As stated by Remark 1.7 of [6], we can consider as host graph $\Gamma$ the Cayley graph of a free group and use this result to estimate the Steklov eigenvalues of a subgraph $(\Omega, B)$ of $\Gamma$. Since the growth rate of such a host graph is exponential, we now have a completely new class of host graphs for which we can estimate their subgraphs eigenvalues.

This paper’s objective is to study the subgraphs’s eigenvalues of a host graph $\Gamma$ which is roughly isometric to the hyperbolic plane $\mathbb{H}^2$ (see Definition 12). The hyperbolic plane is
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a Cartan-Hadamard manifold of constant sectional curvature $-1$. Then $\Gamma$ can be seen as a discrete equivalent of such a manifold. Because of its relation with $\mathbb{H}^2$, the growth rate of $\Gamma$ is exponential, and then $\Gamma$ does not enter the class of host graphs of Theorems 4, 5 and 6.

Despite a growth rate identical to that of the trees, the structure of $\Gamma$ is very different from the latter, because of its connection with $\mathbb{H}^2$. Therefore, the method we will use to obtain upper bounds has nothing to do with the one used in [6]. Indeed, He and Hua were able to work directly on the trees and use the great ease of disconnection of the trees as a tool to obtain the bounds of Theorem 8, while on our side we will use the proximity between $\Gamma$ and $\mathbb{H}^2$ to obtain upper bounds.

There are many graphs which are roughly isometric to the hyperbolic plane. This paper will focus on a particular class of such graphs, coming from a tiling of $\mathbb{H}^2$ associated with a triangle group. We shall refer to such a graph as triangle-tiling graph.

Triangle groups are part of the Coxeter groups, which can be seen as groups generated by reflections. These groups have been studied by many authors, see for instance [2, 7, 9]. Triangle groups are Coxeter groups with three generators, that can be regarded as reflections through the sides of a triangle. They lead to many beautiful geometric constructions and tiling, see [1, 11, 15, 16].

We will remind in Sect. 2 hereafter the notions that are required for the understanding of the paper.

Our main result is the following:

**Theorem 9.** Let $\Gamma$ be a triangle-tiling graph. Then there exists a constant $C = C(\Gamma) > 0$ such that for all subgraph $(\Omega, B)$ of $\Gamma$ and all $k < |B|$, we have

$$\sigma_k(\Omega, B) \leq C(\Gamma) \cdot \frac{1}{|B|} \cdot k^2.$$ 

As we will see in Sect. 2, the host graph $\Gamma$ is defined from the choice of three integers. As a consequence, we will see that there are infinitely many triangle-tiling graphs.

As a corollary, we obtain the interesting fact:

**Corollary 10.** Let $(\Omega_l, B_l)_{l \geq 1}$ be a family of subgraphs with boundary of $\Gamma$ such that $|\Omega_l| \to \infty$. Then for all $k \in \mathbb{N}$ fixed,

$$\sigma_k(\Omega_l, B_l) \to 0.$$

The number $\sigma_k(\Omega_l, B_l)$ is of course defined if and only if $|B_l| < k$. This condition is satisfied for $l$ big enough thanks to the assumption that $|\Omega_l| \to \infty$.

Our approach is sketched this way: we define a triangle-tiling graph $\Gamma$ that we use as a host graph and show that it is roughly isometric to $\mathbb{H}^2$ (see Definition 15). Thanks to the rough isometry, we can naturally associate to a subgraph $(\Omega, B)$ of $\Gamma$ a bounded domain $(N, \Sigma)$ of $\mathbb{H}^2$ ($N$ is the interior of the domain while $\Sigma$ is its boundary). We can then use results from [3] to give upper bounds for $\sigma_k(N, \Sigma)$.

Once this task is completed we use the work of Colbois et al. presented in [4] in order to discretize a Riemannian manifold with boundary $(N, \Sigma, g')$, obtained as a deformation of the domain $(N, \Sigma)$. This discretization will give us a path linking the eigenvalues of $(N, \Sigma)$ and the ones of $(\Omega, B)$, which will allow us to conclude.
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Notation. Throughout this paper, we shall work on graphs, on domains of $\mathbb{H}^2$ and on a manifold obtained from the domains. As stated before, the host graph will be denoted $\Gamma = (V, E)$. A subgraph of $\Gamma$ is denoted $(\Omega, B)$, while $(N, \Sigma)$ and $(\tilde{N}, \tilde{\Sigma})$ are used to speak about domains of $\mathbb{H}^2$. We use $g$ to denote the metric of $\mathbb{H}^2$ and $g'$ the one of the manifold; hence $(N, \Sigma, g')$ is the notation we will use to speak about the manifold. A discretization of the manifold will be called $(\tilde{V}, \tilde{E}, \tilde{V}_\Sigma)$. We shall use the variables $v, w$ to speak about vertices of graphs and $x, y, z$ for elements of the domains or manifold. Several constants will appear, we shall call them $C_1, C_2, \ldots$; each $C_l$ is used exactly once.

Plan of the paper. In Sect. 2, we define precisely what is a triangle-tiling graph. In Sect. 3, we make the constructions. The leading idea is actually simple: we want to associate a domain to a subgraph. However, we encounter some difficulties for different reasons. One of them is the question of the isolated boundary vertices, also called bad boundary vertices in \cite{5}, Def. 3.1. We solve this problem in Sect. 3.1. Another difficulty comes from the fact that we want the domain to have a smooth boundary. This is the object of Sect. 3.2. In Sect. 4 we prove Theorem 9. In order to do so, we want to use Theorem 3 of \cite{4}. Therefore we have to make sure that the hypotheses of the Theorem are verified, which is the object of Sect. 4.1. Once it is done, we apply the Theorem and conclude the proof.

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2 Triangle groups and associated triangle-tiling graphs

Let us begin by explaining what triangle groups are and what links they have with tessellations of the model spaces $S^2, \mathbb{E}^2$ and $\mathbb{H}^2$. When it is done, we can explain how to associate a triangle-tiling graph $\Gamma$ to a triangle group.

Definition 11. Let $p, q, r \geq 2$ be integers. The triangle group $T^*(p, q, r)$ associated is

$$T^*(p, q, r) = \langle P, Q, R : P^2 = Q^2 = R^2 = (PQ)^r = (QR)^p = (RP)^q = 1 \rangle.$$  

In order to see the links between such an abstract group and a group of reflection, one can think about $P, Q, R$ as reflections through the opposite sides of a triangle with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ respectively.

It is well known that a triangle with angles $\alpha, \beta, \gamma$ satisfies $\alpha + \beta + \gamma > \pi$ in the spherical case, while we have $\alpha + \beta + \gamma = \pi$ in the Euclidean case and that $\alpha + \beta + \gamma < \pi$ in the hyperbolic case. Hence we can regroup the unordered triplets $p, q, r$ according to the value of $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$. If the number obtained is greater than 1 we have to think about a spherical triangle, if it is equal to 1 we have to think about a Euclidean one and if it is less than 1 we have to think about a hyperbolic one.
As said before, we want to work on graphs that have exponential growth rates, therefore we will only consider the third case in this paper. Then, from now on, $p, q, r \geq 2$ will be integers satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$  

**Definition 12.** We denote by $\mathbb{H}^2$ the hyperbolic plane, represented here by Poincaré’s disk model, which is

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

directed with the Riemannian metric

$$g(x, y) = 4 \cdot \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$  

We denote by $d_g(\cdot, \cdot)$ the distance induced by the metric $g$.

**Remark 13.** It is a known fact [1] that for any triplet $0 \leq \alpha, \beta, \gamma < \pi$ such that $\alpha + \beta + \gamma < \pi$, there exists a hyperbolic triangle with angles $\alpha, \beta, \gamma$. Moreover, there is a unique one up to isometry.

Hence, given $p, q, r$ as before, there exists a unique triangle which has angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$.

We state now Theorem 2.8 of [11]:

**Theorem 14.** Let $P, Q, R$ be the reflections in the sides of a hyperbolic triangle $\Delta_0$ with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$. The images of $\Delta_0$ under the action of the distinct elements of the group $T^*(p, q, r)$ generated by $P, Q, R$ fill the hyperbolic plane without gaps and overlapping.

This means that the choice of the numbers $p, q, r$ gives rise to a tessellation of the hyperbolic plane. Moreover, we know [11] that reflections through geodesics are isometries of $\mathbb{H}^2$.

Hence, each tile of the tessellation is a triangle which is isometric to the initial one.

![Figure 1: Tiling of the hyperbolic plane with congruent triangles of angles $\frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{\pi}{6}$](image)

From such a tiling associated to a triangle group $T^*(p, q, r)$, one can naturally define an infinite simple connected undirected graph $\Gamma = \Gamma(p, q, r)$, called a triangle-tiling graph and that we will use as a host graph. We explain here how to define $\Gamma$.

Each triangle contains a point that is the center of its inscribed circle [1]. We consider these points. They form the set $V$ of vertices of $\Gamma$. The graph structure of $\Gamma$ is defined as follows: two vertices $v_1, v_2 \in V$ are joined by an edge $\{v_1, v_2\}$ if and only if they belong to two adjacent triangles.

It is then obvious that $\Gamma = (V, E)$ is an infinite, 3-regular graph.
We can see $\Gamma$ as a metric space when endowed with the path metric: each edge is of length 1, the distance between two vertices $v_1, v_2 \in V$ is the minimal number of edges we have to cross to go from $v_1$ to $v_2$.

Because of its bonds with $H^2$, it is clear that $\Gamma$ has an exponential growth rate. Hence, as said in Sect. 1, $\Gamma$ does not enter the class of graphs concerned by Theorem 4, 5 and 6. Moreover, $\Gamma$ has cycles, therefore it is not a tree. Hence, it does not enter the class of graphs of Theorem 8 either.

Two roughly isometric graphs have the same number of Ends [10]. It is obvious that $\Gamma$ has 1 End while a Cayley graph of a free group have infinitely many. Therefore, as said before, the structure of $\Gamma$ is completely different from the graphs concerned by Theorem 8 and this difference will be felt in the way we solve the problem.

Definition 15. A rough isometry between two metric spaces $(X, d_X)$ and $(Y, d_Y)$ is a map $\phi : X \to Y$ such that there exist constants $C_1 > 1, C_2, C_3 > 0$ satisfying

$$C_1^{-1} \cdot d_X(x_1, x_2) - C_2 \leq d_Y(\phi(x_1), \phi(x_2)) \leq C_1 \cdot d_X(x_1, x_2) + C_2$$

for all $x_1, x_2 \in X$ and satisfying

$$\bigcup_{x \in X} B(\phi(x), C_3) = Y.$$ 

If there is such a map, we say that $X$ is roughly isometric to $Y$.

Proposition 16. $\Gamma$ is roughly isometric to $(H^2, g)$.

Proof. Take $\phi : \Gamma \to \mathbb{H}^2$ as the canonical injection and take the constants as the triangle's diameter. \hfill $\square$

This canonical injection will be helpful in this paper: we will see vertices of $\Gamma$ as points in $\mathbb{H}^2$ when we have the use for it. We will use the notation $v$ instead of $\phi(v)$ when no confusion can be made.

3 Construction of the domain $(N, \Sigma)$

We consider a finite subset of vertices $\Omega \subset V$, connected for $\Gamma$, giving birth to a subgraph with boundary $(\Omega, B)$ as in Definition 3. We remind that each vertex can be seen as the center of a triangle of the tiling and that all triangles are isometric.

This section aims to detail a method allowing us to associate a smooth bounded domain $(N, \Sigma)$ to the subgraph $(\Omega, B)$.

The relevance of $(N, \Sigma)$ lies within its structural links with the subgraph $(\Omega, B)$: we will transcribe the structure of $(\Omega, B)$ onto $(N, \Sigma)$.

Before starting, we want to give an overview of the problems that could happen and that we will avoid.

The structural information of $(\Omega, B)$ is of two types: the neighborhood structure and the interior/boundary structure. Hence, we have to make sure that the domain of $\mathbb{H}^2$ we will associate to $(\Omega, B)$ is able to reflect these two pieces of information.

In other words, for two $v_1, v_2 \in \Omega$, we want $v_1$ to be near $v_2$ in $(\Omega, B)$ if and only if $v_1$ is near $v_2$ in the domain. Moreover, for $v \in B$, we want to guarantee the existence of a part of $\Sigma$ near $v$. Reciprocally, for each $x \in \Sigma$, we want to guarantee the existence of a vertex $v \in B$ near $x$. The sense of the word near is the following: the proximity between
$x$ and $v$ does not depend on the subgraph $(\Omega, B)$. This proximity shall be quantified by Proposition 34.

As already spotted by Han and Hua in [5], one of the difficulties comes from the isolated boundary vertices. If $v \in B$ is isolated, we have to be tricky to make sure there is $x \in \Sigma$ which is near $v$, see Example 20.

A second difficulty is the following: we want the domain $(N, \Sigma)$ to be smooth. This will give us the opportunity to easily make a change of metric on $(N, \Sigma)$.

Hence the process contains two steps: at first we find a domain $(\tilde{N}, \tilde{\Sigma})$ which is structurally related to $(\Omega, B)$ but whose boundary $\Sigma$ is not smooth, and secondly we change this domain slightly by smoothing the angles in order to get the wanted domain $(N, \Sigma)$.

### 3.1 Construction of $(\tilde{N}, \tilde{\Sigma})$

Let us begin by considering a vertex $v \in \Omega$ and the triangle $T_v$ associated. In this section, $v$ will always refer to this particular triangle. We call $A_p, A_q, A_r$ the vertices of $T_v$, respectively at angles $\frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}$. We define a map $H : \{A_p, A_q, A_r\} \rightarrow \mathbb{H}^2$, that we call a homothety with ratio $\frac{9}{10}$ of $T_v$, as follows: $H(A_p)$ is the unique point of the geodesic segment $[v, A_p]$ such that $d_g(v, H(A_p)) = \frac{9}{10} \cdot d_g(v, A_p)$. The points $H(A_q)$ and $H(A_r)$ are defined similarly.

We then connect $H(A_p), H(A_q)$ and $H(A_r)$ with geodesic segments. This gives birth to a new triangle, denoted $T'_v$ which can be seen as a kind of homothety of $T_v$. By convexity, $T'_v$ is strictly contained inside the initial triangle $T_v$. It is also easy to see that $v$ is contained inside $T'_v$.

If $w \in \Omega$ is another vertex of the subgraph, then by construction there is a triangle $T_w$ of the tiling associated to $w$ and there is an isometry $\psi_{v,w} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ such that $\psi_{v,w}(T_v) = T_w$.

We apply this process to each vertex of $(\Omega, B)$. Hence we have now at our disposal $|\Omega|$ new triangles, disjoint from each other and isometric to each other.

If $v_1, v_2 \in \Omega$ are such that $v_1 \sim v_2$ in $(\Omega, B)$, then by definition of $\Gamma$, $v_1$ and $v_2$ represent the centers of two triangles, let us say $T_1$ and $T_2$, having one side in common. Thus $T_1$ has two vertices $x, y$ which are also vertices of the triangle $T_2$. As we said before, there is an isometry $\psi_{v,v_1}$ of $\mathbb{H}^2$ such that $\psi_{v,v_1}(T_v) = T_1$. Without loss of generality, say that $\psi_{v,v_1}(A_p) = x$ and $\psi_{v,v_1}(A_q) = y$.

We denote $x_1 := \psi_{v,v_1}(H(A_p))$ and $y_1 := \psi_{v,v_1}(H(A_q))$, which are vertices of the triangle $T'_1 = \psi_{v,v_1}(T_v)$. Similarly, we denote $x_2 := \psi_{v,v_2}(H(A_p))$ and $y_2 := \psi_{v,v_2}(H(A_q))$ which are vertices of the triangle $T'_2 = \psi_{v,v_2}(T_v)$.

We then connect $x_1$ to $x_2$ by a geodesic segment, and we do the same with $y_1$ and $y_2$, see Fig. 2.

We write $T'_1 \sim T'_2$ in order to say that we have connected the triangles $T'_1$ and $T'_2$.

This process connecting the triangles according to the structure of $(\Omega, B)$ allows us to notice the following relation: for two vertices $v_1, v_2 \in \Omega$ which are the centers of two triangles $T'_1, T'_2$, we have

$$v_1 \sim v_2 \iff T'_1 \sim T'_2.$$
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Figure 2: The vertices $x_1, y_1$ of $T_1'$ are connected respectively to the vertices $x_2, y_2$ of $T_2'$ because of the assumption that $v_1 \sim v_2$ in $(\Omega, B)$.

Let us suppose that $z$ is the common vertex of $2p$ triangles such that their centers $v_1, \ldots, v_{2p}$ satisfy $v_1 \sim v_2 \sim v_3 \sim \ldots \sim v_{2p} \sim v_1$ in $(\Omega, B)$. Without loss of generality, let us say that $\psi_{v, v_1}(A_p) = z$. We denote $z_1 = \psi_{v, v_1}(H(A_p)), \ldots, z_{2p} = \psi_{v, v_{2p}}(H(A_p))$ as before. By applying the process described above, we connect $z_1$ to $z_2$, $z_2$ to $z_3$, $z_3$ to $z_4$, and $z_4$ to $z_1$ by geodesic segments, see Fig. 3.

Of course, there is nothing specific about $p$ and the same holds for $q$ and $r$.

Remark 17. The previous construction naturally generates different simple polygons contained inside the hyperbolic plane $\mathbb{H}^2$, of which the exhaustive list is the following:

- Each vertex $w \in \bar{\Omega}$ adds one triangle $T_w'$;
- Each couple of vertices $v_1, v_2 \in \bar{\Omega}$ such that $v_1 \sim v_2$ adds one quadrilateral;
- Each vertex $z$ which is the common vertex of $2p$ triangles such that their centers $v_1, \ldots, v_{2p}$ satisfy $v_1 \sim v_2 \sim v_3 \sim \ldots \sim v_{2p} \sim v_1$ in $(\Omega, B)$ adds one $2p$-gon;
- Each vertex $z$ which is the common vertex of $2q$ triangles such that their centers $v_1, \ldots, v_{2q}$ satisfy $v_1 \sim v_2 \sim v_3 \sim \ldots \sim v_{2q} \sim v_1$ in $(\Omega, B)$ adds one $2q$-gon;

Figure 3: We connected $z_1$ to $z_2$, $z_2$ to $z_3$, $z_3$ to $z_4$ and $z_4$ to $z_1$ because of the assumption that $v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_1$ in $(\Omega, B)$.
• Each vertex $z$ which is the common vertex of $2r$ triangles such that their centers $v_1, \ldots, v_{2r}$ satisfy $v_1 \sim v_2 \sim v_3 \sim \ldots \sim v_{2r} \sim v_1$ in $(\Omega, B)$ adds one $2r$-gon.

**Definition 18.** We call $K$ the compact subset of $\mathbb{H}^2$ obtained by considering the closure of the union of all the simple polygons generated by the previous construction. We also call $(\bar{N}, \bar{\Sigma})$ the bounded domain of $\mathbb{H}^2$ defined by $\bar{N} = \bar{K}$ and $\bar{\Sigma}$ the boundary of $\bar{N}$.

![Figure 4: The crosses represent the interior $\Omega$ of the subgraph, the big dots represent the boundary $B$ of the subgraph. The polynomial curve in bold represent the boundary $\bar{\Sigma}$ while the polygon (of which $\bar{\Sigma}$ is the boundary) is the interior $\bar{N}$.](image)

**Remark 19.** We remind that, by construction, the domain $(\bar{N}, \bar{\Sigma})$ has the same neighborhood structure as $(\Omega, B)$. Indeed, we already saw that for $v_1, v_2 \in \bar{\Omega}$,

$$v_1 \sim v_2 \iff T'_1 \sim T'_2.$$ 

However, the boundary structure of $(\bar{N}, \bar{\Sigma})$ is not equivalent to the one of $(\Omega, B)$. We already have one implication: for all $x \in \bar{\Sigma}$, there exists $w \in B$ such that $w$ is near $x$.

The reciprocal is not verified. If $w \in B$, there is no guarantee that there exists $x \in \bar{\Sigma}$ such that $x$ is near $w$. To see that, one can look at Example 20.

**Example 20.** Choose $v^* \in V$ and define $\Omega$ as the ball of radius $n$ deprived of $v^*$. This will give rise to a subgraph $(\Omega, B)$, for which $v^* \in B$. However, there is no $x \in \bar{\Sigma}$ near $v^*$. Indeed, the bigger $n$ is, the bigger the distance between of $\bar{\Sigma}$ and $v^*$ is also. Hence the proximity between $\bar{\Sigma}$ and $v^*$ depends on the subgraph, which we want to avoid.

To remedy this problem, we proceed to a surgery of this domain $(\bar{N}, \bar{\Sigma})$: for each $w \in B$, we remove the ball centered at $w$ of radius $\frac{\rho}{2}$, where $\rho$ denotes the radius of the circle inscribed in $T'_w$, see Fig. 5.

**Definition 21.** We call $(\tilde{N}, \tilde{\Sigma})$ the domain obtain after the removal of the balls.

**Remark 22.** This last surgery obviously gives us the reciprocal we had not until now: for each $w \in B$, there exists $x \in \bar{\Sigma}$ such that $x$ is near $w$. 

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Figure 5: The crosses represent the vertices of $\Omega$, the dots represent the boundary $B$. The balls surrounding the boundary vertices are removed from the domain and the structure of the subgraph is readable on the domain.

This bounded domain $(\tilde{N}, \tilde{\Sigma})$ is not our final domain because we want one with a smooth boundary.

3.2 Smoothing of the domain $(\tilde{N}, \tilde{\Sigma})$

As we said in the introduction, we want to discretize the domain in order to find an upper bound for the Steklov spectrum of $(\Omega, B)$. One way to do this consists in using Theorem 3 of [4]. Of course, we have to make sure the assumptions of this theorem are verified before using it. However, the domain $(\tilde{N}, \tilde{\Sigma})$ does not satisfy all these assumptions, see Remark 26. This section is devoted to modify the domain $(\tilde{N}, \tilde{\Sigma})$ and get a new domain $(N, \Sigma)$ which has the advantage to have a smooth boundary.

Note that, as always in this paper, we have to make sure that the operations we make do not depend on the subgraph $(\Omega, B)$, but only on the host graph $\Gamma$.

Each connected component of $\tilde{\Sigma}$ is a simple closed $C^\infty$ piecewise curve, composed with geodesic segments. Note that there exist at most $4 \times 3 - 3 = 9$ different types of segments. We shall designate by conic singularity the intersection of two geodesic segments forming $\tilde{\Sigma}$. A conic singularity is therefore a point of the curve whose neighborhoods are of class $C^0$, but not of class $C^1$. By construction, a conic singularity is always located on a vertex of a triangle $T'$.

The regularity of the domain $(\tilde{N}, \tilde{\Sigma})$ allows us to state that it has at most $\binom{4}{2} \times 3 = 18$ different internal angles (two congruent angles are identified).

The interest of these comments is to simplify considerably the smoothing of the domain $(\tilde{N}, \tilde{\Sigma})$. Indeed, there are only 18 kinds of angles to smooth out.

Let us call $\lambda_1, \ldots, \lambda_9$ the length of the geodesic segments and let us denote

$$\lambda := \min\{\lambda_1, \ldots, \lambda_9\}.$$
If \( \tilde{\Sigma} \) has \( n \) conic singularities, let us call them \( z_1, \ldots, z_n \). For each conic singularity \( z_i \), there exist exactly two points \( x_i, x'_i \in \tilde{\Sigma} \) such that

\[
d_g(x_i, z_i) = d_g(x'_i, z_i) = \frac{\lambda}{10}.
\]

Let us consider a conic singularity \( z_i \) as well as the two points \( x_i, x'_i \) associated.

We then create a smooth curve

\[
\alpha_1 : [0, 1] \to \mathbb{H}^2
\]

such that

- \( \alpha_1(0) = x_i, \alpha_1(1) = x'_i \);
- For all \( t \in (0, 1) \) we have \( \alpha_1(t) \in \tilde{N} \);
- For all \( t \in [0, 1] \) we have \( d_g(\alpha_1(t), z_i) \leq \frac{\lambda}{10} \);
- A curve whose image is

\[
[z_{i-1}, x_i] \cup \alpha_1([0, 1]) \cup [x'_i, z_{i+1}]
\]

is smooth, see Fig. 6.

![Figure 6: The curve \( \alpha_1 \) can be seen as a smoothing of the angle at the conic singularity \( z_i \).](image)

Then suppose that \( z_i \) is a conic singularity associated with another kind of angle. We then create a smooth curve

\[
\alpha_2 : [0, 1] \to \mathbb{H}^2
\]

with the same four properties as the previous curve, see Fig. 7.
We continue the process and create a smoothing curve for each type of angle, at most 18 times as said before.

**Remark 23.** If $z_j$ is another conic singularity of the same type as $z_i$, meaning that the angle at $z_j$ is congruent to the angle at $z_i$, there is then an isometry $\Psi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ which sends the angle at $z_i$ onto the angle at $z_j$. The smoothing curve at angle $z_j$ is then given by $\Psi \circ \alpha_\mu$, where $\mu \in \{1, \ldots, 18\}$ depends on the nature of the angle.

Thus, we smooth out the domain $(\tilde{N}, \tilde{\Sigma})$ with these 18 curves and obtain a new connected domain with smooth boundary.

We obtain the domain $(N, \Sigma)$ that we wanted. By construction, the domain $(N, \Sigma)$ has the following characteristics:

- $(N, \Sigma)$ is connected;
- The boundary $\Sigma$ is smooth;
- $\Sigma$ is composed of at most 28 types of curve:
  - The 9 geodesic segments (coming from triangles and quadrilaterals);
  - The 18 smoothing curves $\alpha_1, \ldots, \alpha_{18}$;
  - The circles resulting from the removal of the balls.

Moreover, the domain $(N, \Sigma)$ is constructed in a way that the structure of the subgraph $(\Omega, B)$ is readable in it. Indeed, if we call *smoothed triangle* a region of $(N, \Sigma)$ of the form $N \cap T'_w$, for $w \in \Omega$, then

- A smoothed triangle $N \cap T'_v$ is connected to a neighbor $N \cap T'_u$ if and only if $v \sim u$ in $(\Omega, B)$;

- A vertex $w$ is part of $B$ if and only if there exists $x \in \Sigma$ such that $x$ is near $w$. As said before, Proposition 34 will clarify the sense of the word *near*.
Remark 24. Since each $w \in B$ adds one connected component of $\Sigma$ as a circle, we have the inequality

$$|\Sigma| \geq C_4 \cdot |B|,$$

where $C_4$ corresponds to the perimeter of a circle of radius $\rho_2^2$.

4 Proof of the main theorem

Let us begin by recalling Theorem 1.2 of $[3]$.

Theorem 25. There exists a constant $C_5$ such that for all bounded domain $(N, \Sigma)$ of the hyperbolic space $\mathbb{H}^2$ and for all $k \geq 0$,

$$\sigma_k(N, g) \leq C_5 \cdot \frac{k}{|\Sigma|}.$$ 

Actually, the result of Colbois et al. is more general than that, but this statement is enough for our needs.

The domain $(N, \Sigma)$ being structurally similar to the subgraph $(\Omega, B)$, we will show that a bound of the same type exists for the subgraph’s spectrum. The goal of this section is to transfer this result to the subgraph.

To do this, we want to discretize the domain $(N, \Sigma)$. Let us recall the conditions that the domain must satisfy to be discretized:

We have to assume the existence of constants $\kappa > 0$ and $r_0 \in (0, 1)$ such that

- The boundary $\Sigma$ admits a neighborhood which is isometric to the cylinder $[0, 1] \times \Sigma$, whose boundary corresponds to $\{0\} \times \Sigma$;
- The Ricci curvature of $N$ is bounded below by $-\kappa$;
- The Ricci curvature of $\Sigma$ is bounded below by $0$;
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- For all $x \in N$ such that $d_g(x, \Sigma) > 1$, we have $\text{inj}_M(x) > r_0$;
- For all $x \in \Sigma$, we have $\text{inj}_\Sigma(x) > r_0$.

For further investigation on this topic and to understand why these assumptions are made, one can look at [4].

**Remark 26.** The last four conditions are trivially satisfied by $(N, \Sigma)$. Moreover, the constants $\kappa, r_0$ do not depend on the subgraph $(\Omega, B)$. Indeed, the regularity of the construction of the domain $(N, \Sigma)$ allows to give constants $\kappa, r_0$ valid for any domain $(N, \Sigma)$ obtained by the process described above.

In other words, if we call $\mathcal{M} = \mathcal{M}(\kappa, r_0)$ the class of 2-dimensional manifolds which satisfy the last four properties, then $N \in \mathcal{M}$ whatever the chosen subgraph $(\Omega, B)$.

On the other hand, the first assumption is not satisfied by the domain. Indeed, $\Sigma$ does not have a neighborhood isometric to a cylinder. To remedy this, we will proceed to a change of metric on $N$ in order to obtain a new Riemannian manifold which satisfies the five properties.

### 4.1 Changing the metric on the domain

The main difficulty of this subsection is proceeding to a change of metric which is uniform for all domains $(N, \Sigma)$ obtained by the procedure described in Sect. 3. Here, the word *uniform* reflects the existence of a constant $C_0$ as in Proposition 28 which is valid for all domains, as pointed out in Remark 29.

Let us denote

$$N(\delta) = \{ x \in N : d_g(x, \Sigma) \leq \delta \}$$

the $\delta$-neighborhood of the boundary.

**Proposition 27** (Lemma 34 of [4]). There exist on $(N, \Sigma)$ a $\delta > 0$ (depending only on the 28 types of curves) and a Riemannian metric $g'$ such that

- $(N(\delta), g')$ is isometric to $[0, 1] \times \Sigma$;
- The metrics $g$ and $g'$ are homothetic on $N \setminus N(3\delta)$.

**Proof.** We will use the Fermi parallel coordinates: we parametrize each connected component of $\Sigma$ by arc-length and call $s$ the parameter. We then use the distance $t$ to $\Sigma$ as a second parameter to describe the points of $N$ lying in a close neighborhood of $\Sigma$. In these coordinates, the hyperbolic metric is expressed by

$$g(s, t) = \varphi(s, t) \cdot ds^2 + dt^2,$$

where $\varphi$ is a smooth positive function satisfying $\varphi(s, 0) = 1$.

Let $\delta > 0$ be small enough to have $\frac{1}{2} \leq \varphi(s, t) \leq 2$ on $N(3\delta)$ (such a $\delta$ exists because $\varphi$ is smooth).

We call $g_0$ the product metric which, in the Fermi coordinates $(s, t)$, is expressed by

$$g_0(s, t) = ds^2 + dt^2.$$

We then take a smooth function

$$\chi : [0, 3\delta] \longrightarrow [0, 1]$$

such that $\chi \equiv 0$ on $[0, \delta]$, $\chi \equiv 1$ on $[2\delta, 3\delta]$ and such that $\chi$ is strictly increasing on $[\delta, 2\delta]$. 

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Then we define the metric
\[ g_\delta(s, t) = \chi(t)g(s, t) + (1 - \chi(t))g_0(s, t). \]
This metric coincides with the hyperbolic metric on \( N(3\delta) \setminus N(2\delta) \), then it can be extended all over the domain \( N \) into a metric that we continue to call \( g_\delta \).
Moreover, endowed with this metric, \( N(\delta) \) is isometric to \([0, \delta] \times \Sigma \). We then define the metric
\[ g' := \frac{1}{\delta^2} g_\delta, \]
for the cylindrical neighborhood to have length 1.

The value of \( \delta \) depends only on the 28 types of curves composing \( \Sigma \). That is the reason we built the domain \((N, \Sigma)\) with such regularity. Thanks to the process, we can choose \( \delta \) independently of the subgraph \((\Omega, B)\) chosen, see Remark 29.

**Proposition 28** (Lemma 34 of [1]). There exists a constant \( C_6 > 1 \) such that for all \( x \in N \) and all \( v \in T_x N \), \( v \neq 0 \), we have
\[ \frac{1}{C_6} \leq \frac{g'(x)(v, v)}{g(x)(v, v)} \leq C_6. \]

Proof. We distinguish three cases:
- \( x \in N \setminus N(2\delta) \);
- \( x \in N(\delta) \);
- \( x \in N(2\delta) \setminus N(\delta) \).
Let us start with the first one. Let \( x \in N \setminus N(2\delta) \) and \( 0 \neq v \in T_x N \). We have
\[ \frac{g'(x)(v, v)}{g(x)(v, v)} = \frac{\frac{1}{\delta^2} g_\delta(x)(v, v)}{g(x)(v, v)} = \frac{\frac{1}{\delta^2} g(x)(v, v)}{g(x)(v, v)} = \frac{1}{\delta^2} \]
because on \( N \setminus N(2\delta) \), the metric \( g_\delta \) coincides with the hyperbolic metric \( g \).
For the second case, let \( x \in N(\delta) \) and \( 0 \neq v \in T_x N \). we have
\[ \frac{g'(x)(v, v)}{g(x)(v, v)} = \frac{g'(x)(v, v)}{(\varphi(s, t)ds^2 + dt^2)(v, v)} \leq \frac{g'(x)(v, v)}{(\frac{1}{2}ds^2 + \frac{1}{2}dt^2)(v, v)} \]
\[ = \frac{1}{2}(ds^2 + dt^2)(v, v) = \frac{1}{2}g_0(x)(v, v) \]
\[ = \frac{2}{\delta^2} \]
because \( g_\delta \) coincides with the product metric \( g_0 \) on \( N(\delta) \).
In a similar way, we have
\[ \frac{g'(x)(v, v)}{g(x)(v, v)} = \frac{g'(x)(v, v)}{(\varphi(s, t)ds^2 + dt^2)(v, v)} \geq \frac{g'(x)(v, v)}{(2ds^2 + 2dt^2)(v, v)} \]
\[ = \frac{1}{2}(ds^2 + dt^2)(v, v) = \frac{1}{2}g_0(x)(v, v) \]
\[ = \frac{1}{2\delta^2}. \]
Let us now look at the third case. Let \( x \in N(2\delta) \setminus N(\delta) \) and \( 0 \neq v \in T_x N \).
We recall that on \( N(2\delta) \setminus N(\delta) \), the metric \( g_\delta \) interpolates the product metric \( g_0 \) and the hyperbolic metric \( g \) with the help of a smooth increasing function \( \chi \).
Then we have
\[
g'(x)(v,v) \over g(x)(v,v) = \frac{\frac{1}{2} g_\delta(x)(v,v)}{g(x)(v,v)} = \frac{\frac{1}{2} (\chi(t)g(s,t) + (1 - \chi(t))g_0(s,t))(v,v)}{g(x)(v,v)}
\]
\[
= \frac{1}{\delta^2} \left( \chi(t) + (1 - \chi(t)) \frac{g_0(s,t)(v,v)}{g(x)(v,v)} \right) \geq \frac{1}{\delta^2} \left( \chi(t) + (1 - \chi(t)) \frac{g_0(s,t)(v,v)}{2g_0(x)(v,v)} \right)
\]
\[
= \frac{\chi(t)}{\delta^2} + \frac{1 - \chi(t)}{2\delta^2} \geq \frac{1}{2\delta^2}.
\]

Similarly, we have
\[
g'(x)(v,v) = \frac{1}{2} g_\delta(x)(v,v) = \frac{1}{2} (\chi(t)g(s,t) + (1 - \chi(t))g_0(s,t))(v,v)
\]
\[
= \frac{1}{\delta^2} \left( \chi(t) + (1 - \chi(t)) \frac{g_0(s,t)(v,v)}{g(x)(v,v)} \right) \leq \frac{1}{\delta^2} \left( \chi(t) + (1 - \chi(t)) \frac{g_0(s,t)(v,v)}{\frac{1}{2}g_0(x)(v,v)} \right)
\]
\[
= \frac{\chi(t)}{\delta^2} + \frac{1 - \chi(t)}{\frac{1}{2}\delta^2} \leq \frac{1}{\frac{1}{2}\delta^2} = \frac{2}{\delta^2}.
\]

Then the ratio is bounded for all \( x \in N \) and for all \( v \in T_x N \), \( v \neq 0 \), and we can choose

\[
C_6 := \frac{2}{\delta^2}.
\]

**Remark 29.** This constant \( C_6 \) does not depend on the chosen subgraph \((\Omega, B)\). Indeed, the function \( \varphi \) depends only on the, at most, 28 types of curves forming \( \Sigma \) (which we have fixed once and for all), and \( \delta \) depends only on \( \varphi \). Thus, as said before, the constant \( \delta > 0 \) can be chosen independently of the subgraph, which allows us to fix a universal value of \( C_6 > 1 \) for all the domains \((N, \Sigma)\) obtained thanks to the procedure described in Sect. 3.

We now have at our disposal a new Riemannian manifold with boundary, denoted \((N, \Sigma, g')\), which is related to \((N, \Sigma, g)\) in the sense of Proposition 28. We recall now Proposition 32 of [4]:

**Proposition 30.** Let \( N \) be a Riemannian manifold of dimension \( m \), compact with smooth boundary and let \( g, g' \) be two Riemannian metrics on \( N \). Let us assume that there exists a constant \( C_6 > 1 \) such that for all \( x \in N \) and for all \( v \in T_x N \), \( v \neq 0 \), we have

\[
\frac{1}{C_6} \leq g'(x)(v,v) \over g(x)(v,v) \leq C_6.
\]

Then we have

\[
\frac{1}{C_6^{2m+1}} \leq \frac{\sigma_k(N, g')}{\sigma_k(N, g)} \leq C_6^{2m+1}.
\]

The assumption is exactly what we prove at Proposition 28. Hence we can apply this result to \((N, \Sigma, g)\) and \((N, \Sigma, g')\) in order to get:

\[
\sigma_k(N, g') \leq C_6^5 \cdot \sigma_k(N, g).
\]
4.2 Discretization of the manifold \((N, \Sigma, g')\)

Let us recall that we proceeded to a change of metric on \(N\) in order to give it the ability to be discretized, according to constants \(r_0\) and \(\kappa\), as said in Remark 26. There exist several ways to discretize a manifold. In this paper, we apply the process described in [4], for we want the discretization to have a spectral link with the manifold.

**Definition 31.** An \(\varepsilon\)-discretization of a compact manifold with boundary \((N, \Sigma, g')\) is a process allowing us to associate a graph with boundary \((\tilde{V}, \tilde{E}, \Sigma')\) to this manifold.

This process is the following:

We choose \(\varepsilon \in (0, r_0/4)\) and we choose \(V_{\Sigma}'\) a maximal \(\varepsilon\)-separated subset of \(\Sigma\). Then we call \(V'_{\Sigma}\) the copy of \(V_{\Sigma}\) lying \(4\varepsilon\) away from the boundary:

\[ V'_{\Sigma} = \{4\varepsilon\} \times V_{\Sigma}. \]

Then we choose \(V_I\) a maximal \(\varepsilon\)-separated subset of \(N\setminus[0,4\varepsilon] \times \Sigma\) such that \(V'_{\Sigma} \subset V_I\).

Then we consider the subset \(\tilde{V} = V_{\Sigma} \cup V_I\) and grant it the structure of a graph by decreeing

- Two vertices \(v, w \in \tilde{V}\) are adjacents as soon as \(d_{\tilde{g}}(v, w) \leq 3\varepsilon\);

- A vertex \(v \in V_{\Sigma}\) is adjacent to its counterpart \(v' \in V'_{\Sigma}\).

This process gives a graph with boundary \((\tilde{V}, \tilde{E}, \Sigma)\), simply denoted \((\tilde{V}, V_{\Sigma})\) hereafter, whose boundary is \(V_{\Sigma}\) and that we call \(\varepsilon\)-discretization of \(N\).

Theorem 3 point 4) of [4] allows us to state:

**Theorem 32.** There exists a constant \(C_7 > 0\) depending only on \(\kappa, r_0\) and \(\varepsilon\) such that for all \(k \leq |V_{\Sigma}|\), we have

\[ \sigma_k(\tilde{V}, V_{\Sigma}) \leq C_7 \cdot \sigma_k(N, g') \cdot k. \] (4)

4.3 Rough isometry between \((\tilde{V}, V_{\Sigma})\) and \((\Omega, B)\)

We now want to exploit the graph \((\tilde{V}, V_{\Sigma})\) for which we have an upper bound relative to its spectrum to control the spectrum of our initial subgraph \((\Omega, B)\). In order to do it, we will have to deal with the concept of rough isometry once again. This will allow us to use Proposition 16 of [4] to compare the Steklov spectra of the graphs. The main difficulty here is that we have to make sure the constants of the rough isometry are independant of the subgraph \((\Omega, B)\). Let us begin by defining what is a rough isometry in the context of graphs with boundary.

**Definition 33.** A rough isometry \(\phi\) between two graphs with boundary \((\tilde{\Omega}_1, E'_1, B_1)\) and \((\tilde{\Omega}_2, E'_2, B_2)\) is a rough isometry which sends \(B_1\) onto \(B_2\) and such that the restriction of \(\phi\) to \(B_1\) is a rough isometry \(B_1 \to B_2\) when considering extrinsic distances on \(B_1\) and \(B_2\).

**Proposition 34.** There exists a rough isometry \(\tilde{\phi} : (\tilde{V}, V_{\Sigma}) \to (\Omega, B)\) whose constants \(C_1, C_2, C_3\) are independant from \((\Omega, B)\).

**Proof.** We have to define a map \(\tilde{\phi} : (\tilde{V}, V_{\Sigma}) \to (\Omega, B)\) and show that it is a rough isometry. Remark that the vertices \(v\) of \(\tilde{V}\) can be of different types. There are boundary vertices coming from the 28 different kind of curves forming \(\Sigma\), and there are interior vertices coming from \(N\). As a consequence, the definition of \(\tilde{\phi}\) is a little bit heavy, but the idea to define the rough isometry is very natural: each vertex \(v \in \tilde{V}\) is sent onto the vertex \(w\) of \((\Omega, B)\) which is of same nature (interior or boundary) and which is the nearest from it.

Let us define

\[ \tilde{\phi} : (\tilde{V}, V_{\Sigma}) \to (\Omega, B). \]

For the vertices of the boundary:
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• For $v \in V_\Sigma$ such that $v$ is part of a side of a triangle $T'$, we choose $\bar{\phi}(v) \in B$ the vertex at the center of $T'$;

• For $v \in V_\Sigma$ such that $v$ is part of the boundary of a ball that had been removed, we choose $\bar{\phi}(v) \in B$ the vertex at the center of the removed ball;

• For $v \in V_\Sigma$ such that $v$ is part of a side of a quadrilateral, we find the side of a triangle closest to $v$ and we choose $\bar{\phi}(v) \in B$ as if $v$ were on this triangle’s side;

• For $v \in V_\Sigma$ such that $v$ is part of a smoothing curve, we find the side of a triangle closest to $v$ and choose $\bar{\phi}(v) \in B$ as if $v$ were on this triangle’s side.

Figure 9: The vertices of $V_\Sigma$ are represented by diamonds, the dot vertex belongs to $B$. All of the diamonds are sent to the dot by $\bar{\phi}$.

And for the interior vertices:

• For $v \in V_I$ such that $v$ is part of a triangle whose center is $w \in \Omega$, we choose $\bar{\phi}(v) = w$;

• For $v \in V_I$ such that $v$ is part of a triangle whose center is $w \in B$, then there exists at least one $w' \in \Omega$ such that $w \sim w'$. We then choose $\bar{\phi}(v) = w'$. If there are several possibilities, we choose one once and for all;

• For $v \in V_I$ such that $v$ is part of a quadrilateral, then two opposite sides of this quadrilateral are the sides of two triangles $T_1', T_2'$. At least one of them has a center $w \in \Omega$. We then choose $\bar{\phi}(v) = w$. If there are two possibilities, we choose one once and for all;

• For $v \in V_I$ such that $v$ is part of a 2p-gon (respectively 2q-gon, 2r-gon), then this 2p-gon (resp. 2q-gon, 2r-gon) is surrounded by 2p (resp. 2q, 2r) triangles $T_1', \ldots, T_{2p}'$ (resp. $T_{2q}', T_{2r}'$) of which at least $p$ (resp. $q, r$) have a center $w \in \Omega$. We then choose $\bar{\phi}(v) = w$ once and for all.
In order to show that $\bar{\phi}$ is a rough isometry, let us partition the domain $(N, \Sigma)$ into cobblestones: a cobblestone $C$ is defined as the intersection of a triangle $T$ of the initial tiling with $N$. If $w \in \Omega$ is the center of a triangle $T_w$, we denote by $C_w$ the associated cobblestone. We also write $C_w \sim C_{w'}$ to say that two cobblestones are adjacent.

Then we choose $C_1$ as the cardinality of the biggest possible $\varepsilon$-separated set contained inside a cobblestone multiplied by $\max\{p, q, r\}$. Then we choose $C_2 = C_1$. Thus, if two vertices $v_1, v_2 \in \tilde{V}$ belongs to the same cobblestone, we have $d_{\tilde{V}}(v_1, v_2) \leq C_1$.

We recall that by our construction of the domain $N$, for $w, w' \in \bar{\Omega}$ we have

$$w \sim w' \iff C_w \sim C_{w'},$$

i.e the neighborhood structure of the subgraph is readable onto the domain. Therefore, for $w_1, w_2 \in \bar{\Omega}$, $w_1 \neq w_2$, the distance $d_{\bar{\Omega}}(w_1, w_2)$ represents the number of cobblestones that separate $w_1$ from $w_2$ plus one. Thus, if $v_1, v_2 \in \tilde{V}$ are such that $\bar{\phi}(v_1) = w_1$ and $\bar{\phi}(v_2) = w_2$, then we have

$$C_1^{-1}d_{\bar{\phi}}(v_1, v_2) - C_2 \leq d_{\bar{\Omega}}(w_1, w_2) \leq C_1d_{\bar{\phi}}(v_1, v_2) + C_2.$$

Moreover, $\bar{\phi}$ is a surjective map so we can choose $C_3 = 1$ and we get

$$\bigcup_{v \in \tilde{V}} B(\bar{\phi}(v), C_3) = \bar{\Omega}.$$

We can now recall Proposition 16 of [4]:

**Proposition 35.** Given $C_1 \geq 1, C_2, C_3 \geq 0$, there exist some constants $C_8, C_9$ depending only on $C_1, C_2, C_3$ and of the maximal degree of the vertices such that for all graphs with boundary $(\Gamma_1, B_1), (\Gamma_2, B_2)$ roughly isometric with constants $C_1, C_2, C_3$, we have

$$C_8 \leq \frac{\sigma_k(\Gamma_1, B_1)}{\sigma_k(\Gamma_2, B_2)} \leq C_9.$$  

Applied to this situation, we obtain

$$\sigma_k(\Omega, B) \leq \frac{1}{C_8} \sigma_k(\tilde{V}, V_\Sigma).$$  

(5)
4.4 Conclusion

In this section, we prove Theorem 9 and Corollary 10 by assembling the different results we got before.

Let us prove Theorem 9.

Proof.

\[
\sigma_k(\Omega, B) \leq \frac{1}{C_8} \cdot \sigma_k(\tilde{V}, V_\Sigma) \leq \frac{1}{C_8} \cdot C_7 \cdot \sigma_k(N, g') \cdot k \leq \frac{1}{C_8} \cdot C_7 \cdot C_6^5 \cdot \sigma_k(N, g) \cdot k \leq \frac{1}{C_8} \cdot C_7 \cdot C_6^5 \cdot C_6 \cdot \frac{k}{|\Sigma|} \cdot k \leq \frac{1}{C_8} \cdot C_7 \cdot C_6^5 \cdot C_6 \cdot \frac{k}{C_4 \cdot |B|} \cdot k =: C \cdot \frac{1}{|B|} \cdot k^2.
\]

Throughout the paper, we took care of specifying on which parameters the constants depend. It happens that they do not depend on the subgraph \((\Omega, B)\) chosen. They only depend on the host graph \(\Gamma\) and on \(\varepsilon\). Therefore, if we set a value for \(\varepsilon\), we can take the same constant \(C\) for all subgraph \((\Omega, B)\) of \(\Gamma\); it is now fixed once and for all.

As a consequence, for a choice of three integers \(p, q, r \geq 2\) such that \(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1\), giving birth to a tessellation of the hyperbolic plane and to a host graph \(\Gamma\), there exists a constant \(C = C(\Gamma)\) such that for any subgraph \((\Omega, B)\) of \(\Gamma\), we have

\[
\sigma_k(\Omega, B) \leq C(\Gamma) \cdot \frac{1}{|B|} \cdot k^2.
\]

\(\square\)

From this statement, let us prove Corollary 10.

Proof. It is enough to notice the following fact: for \((\Omega_l, B_l)_{l \geq 1}\) a family of subgraphs of \(\Gamma\) such that \(|\Omega_l| \rightarrow \infty\), then we also have \(|B_l| \rightarrow \infty\).

Therefore, for all \(k \in \mathbb{N}\) fixed, we have

\[
\sigma_k(\Omega_l, B_l) \leq C(\Gamma) \cdot \frac{1}{|B_l|} \cdot k^2 \rightarrow 0 \quad \text{as} \quad l \rightarrow \infty.
\]

\(\square\)

5 Consideration and interrogation

All the constructions above were about a host graph \(\Gamma\), which was a triangle-tiling graph. However, the reader would maybe have observed that the triangles were not necessary in this paper. The information we used is the finite number of possible situations, like the 28 different kinds of curves composing \(\Sigma\) or the 18 types of angles to smooth out.

All these constructions could have emerged from any tiling of the hyperbolic plane, as long as the tiles are compact and the number of different tiles in the tessellation is finite.
If we did so, the number of different possible situations would have been larger, and the constants would have been different. Nevertheless, the result would have been the same.

This comment shows that the result we get in this paper is more general than it primarily seems. Unfortunately, it has its limits. If we get interested in a tiling of the hyperbolic plane which has infinitely many kinds of tiles, then our construction is not relevant anymore. In the same way, if a tile of the tessellation is not compact, we cannot use our method either.

This consideration leads to an open question:

*If* $\Gamma$ *is a graph roughly isometric to the hyperbolic plane, is there a constant* $C = C(\Gamma)$ *such that a bound as in Theorem 9 exists?*

This question naturally leads to a more general interrogation. In order to properly define the problem, let us give a definition.

**Definition 36.** We say that a host graph $\Gamma$ has the property (P) if for each $k \in \mathbb{N}$ and each family $(\Omega_l, B_l)_{l \geq 1}$ of subgraphs of $\Gamma$, we have

$$|\Omega_l| \quad \text{as} \quad l \to \infty \Rightarrow \sigma_k(\Omega_l, B_l) \quad \to \quad 0.$$

Now we can ask the following open question:

*Let* $\Gamma_1, \Gamma_2$ *be two roughly isometric graphs. Let us assume that* $\Gamma_1$ *has the property (P). Does* $\Gamma_2$ *also have the property (P)?*

Reformulated in a geometric group theory way, the question becomes

*Is the property (P) a large scale invariant?*

This question, apparently not so hard, appears to be more thorny than expected. If positively answered, it would automatically generalise our result to any graph roughly isometric to the hyperbolic plane, and it would certainly have many other applications.

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The global pictures of Poincaré’s Disk $\mathbb{H}^2$ comes from [15].

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