QUANTUM UNCERTAINTY AND THE SPECTRA OF SYMMETRIC OPERATORS

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ABSTRACT. In certain circumstances, the uncertainty, $\Delta S[\phi]$, of a quantum observable, $S$, can be bounded from below by a finite overall constant $\Delta S > 0$, i.e., $\Delta S[\phi] \geq \Delta S$, for all physical states $\phi$. For example, a finite lower bound to the resolution of distances has been used to model a natural ultraviolet cutoff at the Planck or string scale. In general, the minimum uncertainty of an observable can depend on the expectation value, $t = \langle \phi, S\phi \rangle$, through a function $\Delta S_t$ of $t$, i.e., $\Delta S[\phi] \geq \Delta S_t$, for all physical states $\phi$ with $\langle \phi, S\phi \rangle = t$. An observable whose uncertainty is finitely bounded from below is necessarily described by an operator that is merely symmetric rather than self-adjoint on the physical domain. Nevertheless, on larger domains, the operator possesses a family of self-adjoint extensions. Here, we prove results on the relationship between the spacing of the eigenvalues of these self-adjoint extensions and the function $\Delta S_t$. We also discuss potential applications in quantum and classical information theory.

Key words and phrases: self-adjoint extensions of symmetric operators, generalized observables, finite minimum uncertainty, spectra of symmetric operators

2000 Mathematics Subject Classification — 81Q10 (self-adjoint operator theory in quantum theory, including spectral analysis), 47A10 (general theory of linear operators; spectrum, resolvent)

1. INTRODUCTION

The uncertainty, $\Delta S[\phi]$, of a quantum observable, $S$, can possess a finite lower bound $\Delta S > 0$, i.e., $\Delta S[\phi] \geq \Delta S$, for all physical states $\phi$. A simple example is the momentum operator of a particle confined to a box with Dirichlet boundary conditions. Since the position uncertainty is bounded from above, the uncertainty relation implies that the momentum uncertainty is finitely bounded from below. Another example arises from general arguments of quantum gravity and string theory, which point towards corrections to the uncertainty relations which are of the type $\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + \ldots)$. For positive $\beta$, this type of uncertainty relation implies a finite lower bound to the position uncertainty. A Hilbert space representation and functional analytic investigation of the underlying type of generalized commutation relations first appeared in the context of quantum group symmetric quantum mechanics and quantum field theory, followed by representations in quantum mechanics and quantum field theory with undeformed symmetries. This made it possible to implement this type of ultraviolet cutoff in various quantum mechanical systems, see e.g., as well as in quantum field theory, with applications, e.g., in the study of black hole radiation and inflationary cosmology, see e.g.
Our aim in the present paper is to extend the basic functional analytic understanding of observables whose uncertainty is finitely bounded from below. We will consider the general case where $\Delta S$ can be a function $\Delta S_t$ of the expectation value, $t = \langle \phi, S\phi \rangle$, i.e., $\Delta S[\phi] \geq \Delta S_t$, for all physical states $\phi$ with $t = \langle \phi, S\phi \rangle$. As we will explain below, such observables are necessarily described by operators that are merely symmetric rather than self-adjoint on their domain in the Hilbert space. Each such symmetric operator possesses, nevertheless, a family of self-adjoint extensions to larger domains in the Hilbert space. The spectra of these self-adjoint extensions are discrete. Our aim here will be to prove results on the close relationship between the spacing of the eigenvalues of these self-adjoint extensions and the function $\Delta S_t$.

2. Symmetric operators

Let $S$ be a closed, symmetric operator defined on a dense domain, $\text{Dom}(S)$, in a separable Hilbert space $\mathcal{H}$. Recall that the deficiency indices $(n_+, n_-)$ of $S$ are defined as the dimensions of the subspaces $\text{Ran}(S + z)^\perp = \ker(S^* - z)$ and $\text{Ran}(S - z)^\perp = \ker(S^* + z)$ respectively where $z$ belongs to the open complex upper half plane (UHP). The dimensions of these two subspaces are constant for $z$ within the upper and lower half plane respectively (6, section 78). For $z = i$ we will call $\mathcal{D}_+ := \ker(S^* - i)$ and $\mathcal{D}_- := \ker(S^* + i)$ the deficiency subspaces of $S$.

We will let $\sigma(S)$, $\sigma_p(S)$, $\sigma_c(S)$, $\sigma_r(S)$, and $\sigma_e(S)$ denote the spectrum, and the point, continuous, residual and essential spectrum of $S$ respectively. Recall that $\sigma(S)$ is defined as the set of all $\lambda \in \mathbb{C}$ such that $(S - \lambda)$ does not have a bounded inverse defined on all of $\mathcal{H}$. The point spectrum $\sigma_p(S)$ is defined as the set of all eigenvalues, $\sigma_c(S)$ is here defined as the set of all $\lambda$ such that $\text{Ran}(S - \lambda)$ is not closed, $\sigma_r(S)$ is defined as the set of all $\lambda$ such that $\lambda \notin \sigma_p(S)$ and $\text{Ran}(S - \lambda)$ is not dense, and $\sigma_e(S)$ is the set of all $\lambda$ such that $S - \lambda$ is not Fredholm. Recall that a closed, densely defined operator $T$ is called Fredholm if $\text{Ran}(T)$ is closed and if the dimension of $\ker(T)$ and the co-dimension of $\text{Ran}(T)$ are both finite. If $T$ is unbounded, we include the point at infinity as part of the essential spectrum. Clearly all the above sets are subsets of $\sigma(S)$, and $\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S)$.

If $S$ is symmetric, and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then it is easy to see that $S - z$ is bounded below by $\frac{1}{\min|z|}$. This shows that any non-real $z \in \sigma(S)$ must belong to the residual spectrum $\sigma_r(S)$ of $S$. If $S$ has finite deficiency indices, then the orthogonal complement of $\text{Ran}(S - z)$ is finite dimensional for any $z \in \mathbb{C} \setminus \mathbb{R}$, which shows that $\sigma_e(S) \subset \mathbb{R}$.

The domain of the adjoint $S^*$ of $S$ can be decomposed as (6, pg. 98):

\begin{equation}
\text{Dom}(S^*) = \text{Dom}(S) + \mathcal{D}_+ + \mathcal{D}_-.
\end{equation}

Here the linear manifolds $\text{Dom}(S)$, $\mathcal{D}_+$ and $\mathcal{D}_-$ are non-orthogonal, linearly independent, non-closed subspaces of $\mathcal{H}$. The notation $+$ denotes the non-orthogonal direct sum of these linear subspaces. If $S$ has finite deficiency indices, and if the co-dimension of $\text{Ran}(S - \lambda)$ is infinite, then $\lambda \in \mathbb{R}$. Furthermore if $\lambda \in \text{Ran}(S - \lambda)^\perp$ then $\lambda$ is an eigenvalue to $S^*$. This and the fact that the dimension of $\text{Dom}(S^*)$ modulo $\text{Dom}(S)$ is finite (by the above equation (1)) allows one to conclude that $\lambda$ must be an eigenvalue of infinite multiplicity to $S$. Hence if $\lambda \in \sigma_e(S)$ then either it is an eigenvalue of infinite multiplicity or it belongs to the continuous spectrum of $S$. 

Claim 1. If $S$ is a symmetric operator with finite and equal deficiency indices then $\sigma_e(S) = \sigma_e(S')$, and $\sigma_c(S) = \sigma_c(S')$ for any self-adjoint extension $S'$ of $S$ within $\mathcal{H}$.

The domain of any self-adjoint extension $S'$ of $S$ can be written as ([6], pg. 100):
\[ \text{Dom}(S') = \text{Dom}(S) + (U - 1)D_+. \]

Here, $U$ is the isometry from $D_+$ onto $D_-$ that defines the self-adjoint extension $S'$. Since the domain of $S$ and $S'$ differ by a finite dimensional subspace, so do the range of $S'$ and $S$. Using these facts it is straightforward to establish the claim.

3. Minimum Uncertainty and Spectra of Self-Adjoint Extensions

As was first pointed out in [9], there exists a close relationship between the finite lower bound $\Delta S_\lambda$ on the uncertainty of a symmetric operator and the spectra of its self-adjoint extensions. Our aim now is to refine those results, and to include new results, in particular, concerning the dependence of the density of eigenvalues on the operator’s deficiency indices.

Definition 1. We denote the expectation value and the uncertainty of a symmetric operator $S$ with respect to a unit-length vector $\phi \in \text{Dom}(S)$ by $S_\phi := \langle \phi, S\phi \rangle$ and by $\Delta S[\phi] := \sqrt{\langle S\phi, S\phi \rangle - \langle (S\phi), \phi \rangle^2}$ respectively. For a fixed expectation value $t \in \mathbb{R}$, the quantity $\Delta S_t := \inf_{\phi \in \text{Dom}(S), \langle S\phi, \phi \rangle = t, \|\phi\| = 1} \Delta S[\phi]$ will be called the minimum uncertainty of $S$ at $t$. The overall lower bound on the uncertainty of $S$ will be denoted by $\Delta S := \inf_{t \in \mathbb{R}} \Delta S_t$.

Recall that a symmetric operator $S$ is said to be simple if there is no subspace $\mathcal{S} \subset \mathcal{H}$ such that $S|\mathcal{S}$ is self-adjoint. A point $z \in \mathbb{C}$ is said to be a point of regular type for $S$ if $(S - z)$ has a bounded inverse defined on $\text{Ran}(S - z)$. As discussed in the introduction, every point $z \in \mathbb{C} \setminus \mathbb{R}$ is a point of regular type for a symmetric operator $S$. $S$ is then said to be regular, if every $z \in \mathbb{C}$ is a point of regular type for $S$. It is clear that if $\phi$ is an eigenvector of $S$, then $\Delta S[\phi] = 0$. Furthermore, if $\lambda \in \mathbb{R}$ belongs to $\sigma_e(S)$ then $\Delta S_\lambda = 0$. Hence, if $\Delta S \geq \epsilon > 0$ this implies that $S$ has no continuous or point spectrum on the real line. This means, in particular, that such an $S$ is not self-adjoint, and is both simple and regular. In addition, the theorem below shows that if $\Delta S > 0$ then $S$ must have equal deficiency indices.

Theorem 1. If $S$ is a symmetric operator with unequal deficiency indices, then $\Delta S = 0$. Furthermore, $\Delta S_t = 0$ for all $t \in \mathbb{R}$.

In the proof of this theorem, it will be convenient to use the Cayley transform of the symmetric operator $S$. Given $\lambda$ in the upper half plane ($UHP$), consider $\kappa_\lambda(z) := \frac{z - \lambda}{\overline{z} - \lambda}$. If $S$ is self-adjoint, then $\kappa(S)$ is unitary by the functional calculus. More generally if $S$ is symmetric, $\kappa_\lambda(S)$ is a partially defined transformation which is an isometry from $\text{Ran}(S - \overline{\lambda})$ onto $\text{Ran}(S - \lambda)$ ([6], sections 67 and 79). For convenience we will take $\lambda = i$, and write $\kappa_i(z) =: \kappa(z)$. The linear map $V := \kappa(S) : D_+ \rightarrow D_+$ is called the Cayley transform of $S$. One can further show that if $\kappa^{-1}(z) := \frac{z + \lambda}{\overline{z} + \lambda}$ and $V = \kappa(S)$, then $\kappa^{-1}(\kappa(z)) = z$ and $S = \kappa^{-1}(V)$. Recall that all symmetric extensions of the symmetric operator $S$ can be constructed by taking the inverse Cayley transforms of partial isometric extensions of the Cayley transform $V = \kappa(S)$. For example, if $S$ has deficiency indices $(n, m)$, one can define an arbitrary partial isometry $W$ from $D_+$ into $D_-$, and the inverse Cayley
transform \( \kappa^{-1}(V') =: S' \) of \( V' := V \oplus W \) on \( H = \text{Ker}(S^* - i) \oplus D_+ \) will be a symmetric extension of \( S \).

The proof of the above theorem will also make use of the Wold decomposition for isometries. Recall that the Wold decomposition theorem states that any isometry on a Hilbert space \( H \) is isometrically isomorphic to an operator \( U \oplus (\oplus_{t \in \Lambda} R) \) on \( H_0 \oplus (\oplus_{t \in \Lambda} l^2(\mathbb{N})) \) where \( U \) is some unitary on \( H_0 \), \( \Lambda \) is some index set, and \( R \) is the right shift operator on the Hilbert space of square summable sequences, \( l^2(\mathbb{N}) \) (see e.g. [7], pg. 2).

**Proof.** Suppose \( S \) has deficiency indices \((n_+, n_-)\), \( n_+ \neq n_- \), \( n_+ = \text{dim}(D_+) \). Then \( S \) has a symmetric extension \( S' \) with deficiency indices either \((0, j)\) or \((j, 0)\), where \( j := |n_+ - n_-| \). Recall that such an extension is obtained as follows. Take the Cayley transform \( V \) of \( S \), and in the case where \( n_- > n_+ \), extend it by an arbitrary isometry from \( D_+ \) into \( D_- \) to obtain an isometry \( V' \) with \( \text{dim}(\text{Ran}(V')^{-1}) = j \). The inverse Cayley transform of \( V' \) yields the desired symmetric extension \( S' \) with deficiency indices \((0, j)\). In the case where \( n_+ > n_- \), extend \( V \) by an arbitrary isometry from an arbitrary \( n_- \)-dimensional subspace of \( D_+ \) onto \( D_- \) to obtain a partial isometry \( V' \) with \( \text{dim}(\text{Ker}(V')) = j \) and \( \text{dim}(\text{Ran}(V')^{-1}) = 0 \). Again, the inverse Cayley transform of \( V' \) yields the desired symmetric extension \( S' \) for this case with deficiency indices \((j, 0)\).

Accordingly, the Cayley transform \( V' \) of \( S' \) is either an isometry with \( \text{dim}(\text{Ran}(V')^{-1}) = j \) or the adjoint of an isometry, with \( \text{dim}(\text{Ker}(V')) = j \). By the Wold decomposition theorem, \( V' \) is isometrically isomorphic to the direct sum of a unitary operator \( U \) and either \( j \) copies of the right shift operator or \( j \) copies of the left shift operator on \( H_0 \oplus \bigoplus_{n=1}^{j} l^2(\mathbb{N}) \). It follows that \( \sigma(V') \supset \sigma(R) \) or \( \supset \sigma(L) \) respectively, where \( R \) and \( L \) are the right and left shift operators on \( l^2(\mathbb{N}) \). It is known that the unit circle lies in the continuous spectrum of both the right and left shift operators. It is not difficult to see that \( \lambda \in \sigma_c(V') \setminus \{1\} \) where \( V' \) is the Cayley transform of \( S' \) if and only if \( \kappa^{-1}(\lambda) \in \sigma_c(S') = \sigma_c(S) \). It follows that the continuous spectrum of \( S \) (which is a subset of \( \mathbb{R} \)) is non-empty and hence there exist \( \phi \in \text{Dom}(S) \) for which \( \Delta S[\phi] \) is arbitrarily small.

Furthermore, the above shows that the continuous spectrum of \( S \) is all of \( \mathbb{R} \). Using this fact, it is not difficult to show that \( \Delta S_t = 0 \) for all \( t \in \mathbb{R} \). First, given any fixed \( t \in \mathbb{R} \), since \( t \in \sigma_c(S) \), one can find a sequence \((\phi_n)_{n \in \mathbb{N}} \subset \text{Dom}(S) \) such that \( \| (S - t)\phi_n \| \to 0 \), \( \| \phi_n \| = 1 \), and \( t_n := \langle S\phi_n, \phi_n \rangle \geq t \). Again, since \( \sigma_c(S) = \mathbb{R} \), one can find a unit norm \( \psi \in \text{Dom}(S) \) such that \( \overline{S}\psi = t' < t \). Let \( P_t \) denote the projectors onto the orthogonal complements of the one dimensional subspaces spanned by the \( \phi_n \). Then each vector \( P_t\psi \in \text{Dom}(S) \), and it is easy to verify that if \( \psi_n := \frac{P_t\psi}{\|P_t\psi\|} \), then \( \overline{S}\psi_n =: t_n' \leq t \). For each \( n \in \mathbb{N} \), let \( \varphi_n \) be the linear combination of the \( \psi_n \) and \( \phi_n \) such that \( \| \varphi_n \| = 1 \) and \( \overline{S}\varphi_n = t \). It is straightforward to verify that \( (S - t)\varphi_n \to 0 \) so that \( \Delta S_t = 0 \).

Note that the above theorem implies that if \( S \) is any symmetric operator with unequal deficiency indices that represents a quantum mechanical observable, then even though \( S \) is not self-adjoint, it is possible to measure that observable as precisely as one likes in the sense that \( \Delta S_t = 0 \) for all \( t \in \mathbb{R} \). Nevertheless, despite the fact that \( \Delta S_t = 0 \) for all \( t \in \mathbb{R} \), the situation is physically different from the case of a self-adjoint observable. This is because, in this case, the formal, non-normalizable quasi-eigenstates of the symmetric operator \( S \) are non-orthogonal. For example,
consider the case of the symmetric derivative operator $D := i \frac{d}{dx}$ defined with domain $\text{Dom}(D)$ in $L^2[0, \infty)$, $\text{Dom}(D) := \{ \phi \in L^2[0, \infty) \mid \phi \in AC_{\text{loc}}[0, \infty); D\phi \in L^2[0, \infty); \phi(0) = 0 \}$. Here, $AC_{\text{loc}}[0, \infty)$ denotes the set of all functions which are absolutely continuous on any compact subinterval of $[0, \infty)$. It is straightforward to check that $D$ has deficiency indices $(0, 1)$. If $\phi_\lambda(x) := \frac{e^{-i\lambda x}}{\sqrt{2\pi}}$, for $x \in [0, \infty)$, then $\phi_\lambda$ can be thought of as a formal, non-normalizable quasi-eigenstate for $S$, since if $f \in L^2[0, \infty)$, the formal inner product of $f$ with the $\phi_\lambda$, 
\[(3) \quad \int_0^\infty f(x) \frac{1}{\sqrt{2\pi}} e^{i\lambda x} =: F(\lambda),\]
generates a unitary transformation, i.e. the Fourier transform, of $L^2[0, \infty)$ onto a subspace of $L^2(\mathbb{R})$, and under this transformation $S$ is transformed into multiplication by the independent variable. These quasi-eigenstates are non-orthogonal in the following sense. For $\varepsilon > 0$ and $\lambda \in \mathbb{R}$, let $\phi(\varepsilon, \lambda; x) := \frac{1}{\sqrt{2\pi}} \left(e^{-i\lambda x - \varepsilon x} - e^{-i\lambda x}\right)$. Then $\phi(\varepsilon, \lambda; x) \in \text{Dom}(D)$ for any $\varepsilon > 0$, and as $\varepsilon \to 0$, $\phi(\varepsilon, \lambda; x)$ converges to $\phi_\lambda$ in $L^2$ norm on any compact subinterval of $[0, \infty)$. Furthermore, it is straightforward to check that $\frac{\langle D\phi(\varepsilon, \lambda; \cdot), \phi(\varepsilon, \lambda; \cdot) \rangle}{\|\phi(\varepsilon, \lambda; \cdot)\|^2} \to \lambda^2$ as $\varepsilon \to 0$. However, if $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then the inner product $\langle \phi(\varepsilon, \lambda_1; \cdot), \phi(\varepsilon, \lambda_2; \cdot) \rangle$ converges to $\frac{1}{2\pi(\lambda_2^2 - \lambda_1^2)} \neq 0$ in the limit as $\varepsilon \to 0$. In this sense, the formal non-normalizable quasi-eigenstates $\phi_\lambda$ are not orthogonal. Compare this to the case of the self-adjoint derivative operator $D' := i \frac{d}{dx}$ in $L^2(\mathbb{R})$. In this case the non-normalizable eigenstates to eigenvalues $\lambda \in \mathbb{R}$ are again $\phi_\lambda(x) = \frac{1}{\sqrt{2\pi}} e^{-i\lambda x}, x \in \mathbb{R}$. If one chooses, for example, $\phi(\varepsilon, \lambda; x) := \frac{1}{\sqrt{2\pi}} e^{-i\lambda x - \varepsilon x} \in \text{Dom}(D')$, then it is straightforward to check that the inner product $\langle \phi(\varepsilon, \lambda_1; \cdot), \phi(\varepsilon, \lambda_2; \cdot) \rangle$ vanishes as $\varepsilon \to 0$ if $\lambda_1 \neq \lambda_2$, so that the non-normalizable eigenstates of this self-adjoint operator are indeed orthogonal.

For a concrete physical example, consider a telescope with some finite aperture. The accurate measurement of the arriving photons’ momentum orthogonal to the telescope is essential for the production of a sharp image. This amounts to the measurement of the momentum of a particle in a box. The momentum operator (which is just $i$ times the first derivative operator) acting on a particle in a box is a symmetric operator with deficiency indices $(1, 1)$. In this case the finite aperture of the telescope causes a minimum uncertainty in the angle measurements $\Theta$. The case of a telescope is the case of light being diffracted as it passes through a slit. The case of the symmetric derivative operator $D$ on the half line, $L^2[0, \infty)$, is that of light being diffracted at a single edge, i.e., passing a single wall. The fact that the quasi eigenstates are not orthogonal means, physically, that there is a diffraction pattern in this case as well.

If $S$ is a simple symmetric operator with deficiency indices $(j, 0)$, or $(0, j)$, then it is straightforward to verify that $S$ is isometrically isomorphic to $j$ copies of the differentiation operator $i \frac{d}{dx}$ on $L^2(0, \infty)$ or $L^2(-\infty, 0)$ respectively. This fact was first proven by von Neumann, see for example, (6). Section 82). Hence, if $S$ has deficiency indices $(j, 0)$, or $(0, j)$, it generates a semi-group of isometries or co-isometries which is isometrically isomorphic to $j$ copies of right translation on $L^2(0, \infty)$ or $L^2(-\infty, 0)$, respectively. It follows that if $S$ is a symmetric quantum mechanical Hamiltonian operator, which has deficiency indices $(m, n)$, and $j := |n - m|$, then any maximal symmetric extension of $S$ will generate either an isometric
or co-isometric time evolution of the quantum mechanical system. Furthermore, the Hilbert space can be decomposed into \( j + 1 \) subspaces such that the time-evolution on the first subspace is unitary, and such that the time evolution on each of the other subspaces is purely isometric or co-isometric. If the state of the system begins in one of these subspaces, its image at any later time will be confined to that subspace, so that there are, in general, subspaces of the Hilbert space which will be inaccessible to the time evolution of the system once the initial state is fixed.

**Theorem 2.** Let \( S \) be a densely defined, closed symmetric operator with finite and equal deficiency indices \((n,n)\). If \( \Delta S > 0 \), then any self-adjoint extension \( S' \) of \( S \) has a purely discrete spectrum, \( \sigma(S') = \sigma_e(S) \). In particular, if \( \Delta S_i > \epsilon > 0 \) for all \( t \in I \subset \mathbb{R} \), then \( S' \) can have no more than \( n \) eigenvalues (including multiplicities) in any interval \( J \subset I \) of length less than or equal to \( \epsilon \), and if \( n = 1 \), then \( S' \) can have no more than one eigenvalue in any interval \( J \subset I \) of length less than or equal to \( 2\epsilon \).

This theorem shows, in particular, that if \( \Delta S > \epsilon \), then any self-adjoint extension of \( S \) has no more than \( n \) eigenvalues in any interval of length \( \epsilon \). The authors are currently investigating whether the improved result that holds for the \( n = 1 \) case generalizes to the case of higher deficiency indices.

**Proof.** If \( \Delta S > 0 \), then as in the discussion preceding the proof of Theorem 1 we conclude that every \( z \in \mathbb{C} \) is a point of regular type for \( S \). Since \( S \) has finite and equal deficiency indices, if \( S' \) is any self-adjoint extension of \( S \), it follows that \( \sigma_e(S') = \sigma_e(S) \) consists only of the point at infinity. This implies that the spectrum of \( S' \) consists solely of eigenvalues of finite multiplicity with no finite accumulation point.

Suppose that there is a self-adjoint extension \( S' \) of \( S \) which has \( n+1 \) eigenvectors \( \phi_i \) to eigenvalues \( \lambda_i \) where \( \lambda_i \in J \subset I \), and the length of \( J \) is less than or equal to \( \epsilon \). Then since the dimension of \( \text{Dom}(S') \) modulo \( \text{Dom}(S) \) is \( n \), there is a non-zero linear combination of these orthogonal eigenvectors, \( \psi = \sum_{i=1}^{n+1} c_i \phi_i \) which has unit norm and which belongs to \( \text{Dom}(S) \). The expectation value of the symmetric operator \( S \) in the state \( \psi \) lies in \( J \), \( t := \overline{S}_\psi \in J \) since \( \psi \) is a linear combination of eigenvectors to \( S' \) whose eigenvalues all lie in \( J \). Now it is straightforward to verify that since \( |\lambda_i| < |t| + \epsilon \) for all \( 1 \leq i \leq n + 1 \), that

$$ (\Delta S(\psi))^2 = \sum_{i=1}^{n+1} \lambda_i^2 |c_i|^2 - t^2 \leq \sum_{i=1}^{n+1} (|t| + \epsilon)^2 |c_i|^2 - t^2 = 2 |t| \epsilon + \epsilon^2 $$

Now first suppose that \( 0 \in J \) and that \( t := \overline{S}_\psi = 0 \). Then in this case equation (4) contradicts the fact that \( \Delta S_\psi > \epsilon \), proving the claim for this case.

If \( t \neq 0 \), then consider the symmetric operator \( S(t) := S - t \) on \( \text{Dom}(S) \). Given any \( \phi \in \text{Dom}(S) \) which has unit norm and expectation value \( \overline{S}_\phi = \langle S\phi, \phi \rangle = t \), it is not hard to see that \( \overline{S}(t) \phi = (S(t)\phi, \phi) = 0 \) and that

$$ \Delta S(t)[\phi] = \langle S(t)\phi, S(t)\phi \rangle - 2t \langle S\phi, \phi \rangle + t^2 $$

$$ = \langle S\phi, S\phi \rangle - t^2 = \Delta S[\phi]. $$

This shows that \( \Delta S(t)_0 = \Delta S_t > \epsilon \). Now let \( S' \) be any self-adjoint extension of \( S \). Applying the above result for the expectation value 0 to the symmetric operator \( S(t) \), we conclude that the self-adjoint extension \( S'(t) := S' - t \) of \( S(t) \) can have
no more than \( n \) eigenvalues in the interval \( J - t \). This in turn implies that \( S' \) can have no more than \( n \) eigenvalues in the interval \( J \).

Now suppose that \( n = 1 \). As in the above, if \( S' \) is a self-adjoint extension of \( S \) that has two eigenvectors \( \phi_1, \phi_2 \) to eigenvalues \( \lambda_1, \lambda_2 \), in a subinterval \( J \subset I \) of length less than or equal to \( 2\epsilon \), then there is a unit norm vector \( \psi = c_1 \phi_1 + c_2 \phi_2 \), that belongs to \( \text{Dom}(S) \). The expectation value of \( \psi \), \( t := \langle S\psi, \psi \rangle \) will also belong to \( J \). The expectation value \( t \) and the fact that \( \psi \) has unit norm, uniquely determines the constants \( c_1 \) and \( c_2 \) up to complex numbers of modulus one:

\[
|c_1| = \sqrt{\frac{|\lambda_2 - t|}{|\lambda_1 - \lambda_2|}} \quad \text{and} \quad |c_2| = \sqrt{\frac{|\lambda_1 - t|}{|\lambda_1 - \lambda_2|}}.
\]

It is now straightforward to calculate that \( \Delta S[\psi] = \sqrt{\lambda_1 - t} ||\lambda_2 - t|| \). Assume, without loss of generality, that \( \lambda_1 < \lambda_2 \), so that \( \lambda_1 < t < \lambda_2 \). Since \( \lambda_1, \lambda_2 \) belong to the same interval \( J \) with length less than or equal to \( 2\epsilon \), it follows that \( |\lambda_2 - t| \leq 2\epsilon - |\lambda_1 - t| \), so that \( \Delta S[\psi] \leq \sqrt{(2\epsilon |\lambda_1 - t| - |\lambda_1 - t|^2)} \). It is simple to check that the function \( f(x) = 2\epsilon x - x^2 \) has a global maximum of \( \epsilon^2 \) when \( x = \epsilon \), so that \( \Delta S[\psi] \leq \epsilon \). Since \( t \in I \), this contradicts the assumption that \( \Delta S > \epsilon \).

\[ \square \]

**Corollary 1.** If \( S \) is a symmetric operator with finite deficiency indices such that \( \Delta S = \epsilon > 0 \), then \( S \) is simple, regular, the deficiency indices \( (n, n) \) of \( S \) are equal, and the spectrum of any self-adjoint extension of \( S \) is purely discrete and consists of eigenvalues of finite multiplicity at most \( n \) with no finite accumulation point.

It is known (\[9\], section 83) that if \( S \) is a closed, densely defined simple symmetric operator with equal and finite deficiency indices \( (n, n) \), then the multiplicity of any eigenvalue of any self-adjoint extension \( S' \) of \( S \) does not exceed \( n \). Corollary \[1\] is an immediate consequence of this fact and Theorems \[1\] and \[2\].

For example, consider the symmetric differential operator \( S' := \frac{d^2}{dx^2} + x \) defined on the dense domain \( C_0^\infty(0, \infty) \subset L^2[0, \infty) \) of infinitely differentiable functions with compact support in \( (0, \infty) \). Let \( S \) be the closure of \( S' \). Let \( D \) be the closed symmetric derivative operator on \( L^2[0, \infty) \) which is the closure of the symmetric derivative operator \( D' := i\frac{d}{dx} \) on the domain \( \mathcal{D} := C_0^\infty(0, \infty) \). It follows that \( \mathcal{D} \) is a core for both \( D \) and for \( S \). Recall that a dense set of vectors \( \mathcal{D} \) is called a core for a closable operator \( T \), if \( \overline{T} \mathcal{D} = \mathcal{D} \). For all \( \phi \in \mathcal{D} \), it is easy to verify that \( [D, S]|\phi := (DS - SD)\phi = i(D^2 + 1)\phi \). By the uncertainty principle, it follows that for any \( \phi \in \mathcal{D} \),

\[
\Delta S[\phi] \Delta D[\phi] \geq \frac{1}{2} ||[\phi, [D, S]|\phi]|| = \frac{1}{2} (\phi, (D^2 + 1)\phi)
\]

Using the fact that the function \( f(t) = \frac{t^2 + 1}{2} \) is concave up for all \( t \in (0, \infty) \) and has a global minimum \( f(1) = 1 \), we conclude that \( \Delta S[\phi] \geq 1 \) for any \( \phi \in \mathcal{D} \). Since \( \mathcal{D} \) is a core for \( S \), given any \( \psi \in \text{Dom}(S) \) we can find a sequence \( \psi_n \in \mathcal{D} \) such that \( \psi_n \to \psi \) and \( S\psi_n \to S\psi \). It follows that \( \Delta S[\psi] = \lim_{n \to \infty} \Delta S[\psi_n] \geq 1 \). This shows that \( \Delta S \geq 1 \). Now \( S \) is a second order symmetric differential operator. It is known that the deficiency indices of such an operator are equal and do not exceed \((2, 2)\) (\[8\], Section 17). Since \( \Delta S \geq 1 \), Corollary \[1\] also implies that the deficiency indices
of $S$ must be equal and non-zero. Hence $D$ has deficiency indices $(1, 1)$ or $(2, 2)$. Applying Theorem 2 one can now conclude that any self-adjoint extension of $S$ can have at most two eigenvalues in any interval of length 1.

Conversely, if $S$ has finite deficiency indices and is simple and regular, then $\Delta S > 0$.

**Theorem 3.** Suppose that $S$ is a regular, simple symmetric operator with finite and equal deficiency indices. Let $\mathcal{S}$ denote the set of all self-adjoint extensions of $S$ within $\mathcal{H}$. Then,

$$
\Delta S_t \geq \max_{S' \in \mathcal{S}} \Delta S'_t = \max_{S' \in \mathcal{S}} \left( \min_{\lambda, \mu \in \sigma(S')} \sqrt{|\lambda - t||\mu - t|} \right).
$$

**Proof.** Note that if $S$ is simple and regular with finite deficiency indices $(m, n)$, then these indices must be equal, otherwise $S$ would have continuous spectra and would not be regular.

Since $\text{Dom}(S) \subset \text{Dom}(S')$ and $S'|_{\text{Dom}(S)} = S$ for any $S' \in \mathcal{S}$, it is clear that $\Delta S_t \geq \max_{S' \in \mathcal{S}} \Delta S'_t$. It remains to prove that $\Delta S'_t = \min_{\lambda, \mu \in \sigma(S')} \sqrt{|\lambda - t||\mu - t|}$ for any $S' \in \mathcal{S}$. Since we assume $S$ is regular, simple, and has finite deficiency indices, the essential spectrum of $S$ is empty. Hence by Claim 1 $\sigma_e(S')$ is empty for any $S' \in \mathcal{S}$. This shows that the spectrum of any $S'$ consists solely of eigenvalues of finite multiplicity with no finite accumulation point. Order the eigenvalues as a non-decreasing sequence $(t_n)_{n \in \mathbb{N}}$ and let $\{b_n\}_{n \in \mathbb{N}}$ be the corresponding orthonormal eigenbasis such that $S'b_n = t_nb_n$. Here $\mathbb{M} = \pm \mathbb{N}$ or $= \mathbb{Z}$, depending on whether $S'$ is bounded above, below, or neither bounded above nor below. To calculate $\Delta S'_t$, we need to minimize the functional

$$
\Phi'[\phi] := \langle S'\phi, S'\phi \rangle - t^2
$$

over the set of all unit norm $\phi \in \text{Dom}(S')$ which satisfy $\langle S'\phi, \phi \rangle = t$. Let us assume that $t$ is not an eigenvalue of $S'$ as in this case $\Delta S'_t = 0$ and (3) holds trivially. Expanding $\phi$ in the basis $b_n$, $\phi = \sum_{n \in \mathbb{M}} \phi_n b_n$, we see that to find the extreme points of $\Phi'$ subject to these constraints we need to set the functional derivative of

$$
\Phi[\phi] := \sum_{n \in \mathbb{M}} \phi_n \overline{\phi}_n \left( \phi_n \overline{\phi}_n (t_n^2 - t^2) - \alpha_1 \phi_n \overline{\phi}_n t_n - \alpha_2 \right)
$$

(10)

to zero. Here $\alpha_1, \alpha_2$ are Lagrange multipliers. Setting the functional derivative of $\Phi$ with respect to $\phi$ to 0 yields:

$$
0 = \phi_n \left( (t_n^2 - t^2) - \alpha_1 t_n - \alpha_2 \right)
$$

(11)

Formula (11) leads to the conclusion that if $\phi$ is an extreme point, it must be a linear combination of two eigenvectors to $S'$ corresponding to two distinct eigenvalues.

To see this note that if the decomposition of $\phi$ in the eigenbasis $\{b_n\}$ had three non-zero coefficients, say $\phi_{j_i}$, $i = 1, 2, 3$, all of which correspond to eigenvectors $b_{j_i}$ with distinct eigenvalues, $t_i \neq t_j$, $1 \leq i, j \leq 3$, then Equation (11) leads to the conclusion that $\alpha_1 = t_{j_1} + t_{j_2} - t_{j_3}$ which would imply that $t_{j_1} = t_{j_3}$, a contradiction. Furthermore, $\phi$ cannot just be a linear combination of eigenvectors $b_j$ to one eigenvalue, as such a linear combination cannot satisfy the constraint $\langle S\phi, \phi \rangle = t$. So let $\lambda := t_i$ and $\mu := t_j$ for any $j, i \in \mathbb{Z}$ for which $t_i \neq t_j$. Choose $\varphi \in \text{Ker}(S^* - \lambda)$ and $\psi \in \text{Ker}(S^* - \mu)$. We have shown that $\phi$ has the form $\phi = c_1 \varphi + c_2 \psi$. As in the proof of Theorem 2 the constraints that $\langle \phi, \phi \rangle = 1$ and $\langle S\phi, \phi \rangle = t$ uniquely determine $c_1$ and $c_2$ up to complex numbers of modulus one:
by formula (8), \( \Delta S \) by Theorem 2, if \( \mu \) let \( S \)
the deficiency indices \( \Phi, \) then \( \Delta S \) is satisfied proves the first part of the claim.

It is known that if \( \lambda S > 0, \) then \( \max_{\lambda \in \sigma(S')} \Delta S' \geq \frac{\Delta S}{2}. \) If \( n = 1, \) then \( \max_{\lambda \in \sigma(S')} \Delta S' \geq \Delta S, \) so that

\[
\Delta S = \inf_{t \in \mathbb{R}} \max_{\lambda \in \sigma(S')} \min_{\lambda, \mu \in \sigma(S')} \sqrt{|\lambda - t||\mu - t|}.
\]

Proof. It is known that if \( \lambda \) is a regular point of a symmetric operator \( S \) with deficiency indices \( (n, n), \) then there exists a self-adjoint extension of \( S \) for which \( \lambda \) is an eigenvalue of multiplicity \( n \) \((6), \) pg. 109). Given any \( \lambda \) for which \( |\lambda - t| < \Delta S, \) let \( S' \) be the self-adjoint extension of \( S \) for which \( \lambda \) is an eigenvalue of multiplicity \( n. \) By Theorem 2, if \( \mu \neq \lambda \) belongs to \( \sigma(S'), \) it must be that \( |\mu - t| \geq \Delta S - |\lambda - t|. \) Again, by formula \( (6), \) \( \Delta S' \geq \sqrt{|\lambda - t||\mu - t|}. \) It is a simple calculus exercise to show that this is maximized when \( |\lambda - t| = \frac{\Delta S}{2}. \) Choosing \( \lambda \) so that this condition is satisfied proves the first part of the claim.

Using the result of Theorem 2 in the case where \( n = 1, \) and repeating the above arguments shows that, in the case where \( n = 1, \) \( \max_{\lambda \in \sigma(S')} \Delta S' \geq \Delta S. \) By Theorem 2, \( \Delta S \geq \max_{\lambda \in \sigma(S')} \Delta S', \) so that \( \Delta S \geq \max_{\lambda \in \sigma(S')} \Delta S' \geq \Delta S. \) Taking the infimum over \( t \in \mathbb{R} \) of both sides yields \( \Delta S = \inf_{t \in \mathbb{R}} \max_{\lambda \in \sigma(S')} \Delta S'. \) Combining this with formula \( (8) \) now yields the formula \( (12). \)

If the improved result of Theorem 2 that holds for the \( n = 1 \) case could be established for all values of \( n, \) then the stronger result of the above theorem for the \( n = 1 \) case would also hold for all \( n. \)

4. Outlook

Our new results on operators whose uncertainty is bounded from below are of potential interest in quantum gravity. This is because our results improve on the results of [9], where it was first pointed out that physical fields in theories with a finite lower bound on spatial resolution, \( \Delta x, \) possess the so-called sampling property: a field is fully determined by its amplitude samples taken at the eigenvalues of any one of the self-adjoint extensions of \( x. \) With this ultraviolet cutoff, physical theories can therefore be written, equivalently, as living on a continuous space, or as living on any one of a family of discrete lattices of points. This provides a new approach to reconciling general relativity’s requirement that spacetime be a continuous manifold with the fact that quantum field theories tend to be well-defined only on lattices. This approach has been extended to quantum field theory on flat and curved space see [10]. Representing Hamiltonians as symmetric operators with unequal deficiency indices may also be of physical significance, since the co-isometric time evolution of a quantum system generated by such a Hamiltonian could be
useful for describing information vanishing beyond horizons, e.g. the horizon of a black hole [11].

Finally, we remark that our results are also of potential interest in information theory. The theory of spaces of functions which are determined by their amplitudes on discrete points of sufficient density is a long-established field, called sampling theory. Sampling theory plays a central role in information theory, where it serves as the crucial link between discrete and continuous representations of information, see [12]. Our results on the relationship between the varying uncertainty bound $\Delta S_t$ and varying density of the eigenvalues of the self-adjoint extensions of $S$ therefore contribute new tools for handling the difficult case of sampling and reconstruction with a variable Nyquist rate.
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