Overdetermined Transforms in Integral Geometry

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Abstract. A simple example of an $n$-dimensional admissible complex of planes is given for the overdetermined $k$-plane transform in $\mathbb{R}^n$. For the corresponding restricted $k$-plane transform sharp existence conditions are obtained and explicit inversion formulas are discussed in the general context of $L^p$ functions. Similar questions are studied for overdetermined Radon type transforms on the sphere and the hyperbolic space. A theorem describing the range of the restricted $k$-plane transform on the space of rapidly decreasing smooth functions is proved.

1. Introduction

The $k$-plane Radon-John transform of a function $f$ on $\mathbb{R}^n$ is a mapping

$$R_k : f(x) \rightarrow \varphi(\tau) = \int_{\tau} f(x) d_\tau x,$$

where $\tau$ is a $k$-dimensional plane in $\mathbb{R}^n$, $1 \leq k \leq n - 1$, and $d_\tau x$ is the Euclidean volume element on $\tau$; see, e.g., [12, 11, 25, 26]. We denote by $\Pi_{n,k}$ the manifold of all non-oriented $k$-dimensional planes in $\mathbb{R}^n$. Since $\dim \Pi_{n,k} = (k + 1)(n - k)$ is greater than $n$ if $k < n - 1$, then the inversion problem for $R_k$ is overdetermined if we use information about $(R_k f)(\tau)$ for all $\tau \in \Pi_{n,k}$. The celebrated Gel’fand’s question is how to reduce this overdeterminedness or, more precisely, how to define an $n$-dimensional subset $\tilde{\Pi}_{n,k}$ of $\Pi_{n,k}$ so that $f(x)$ could be recovered.

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knowing $\varphi(\tau)$ only for $\tau \in \tilde{\Pi}_{n,k}$; see, e.g., [9]. We call the subsets $\tilde{\Pi}_{n,k}$ admissible complexes of $k$-planes.

The background of the theory related to this question was developed in Gel'fand's school; see, e.g., [10, 11, 13, 14, 15, 16, 17, 18, 20, 27, 29]. The construction of the admissible complexes in these works is usually given in topological terms, and the relevant inversion formulas rely on the fundamental concepts of the kappa-operator (for $k$ even) and the Crofton operator (for $k$ odd, when the inversion formulas are nonlocal). An alternate approach to nonlocal inversion formulas, which employs the Fourier integral operators, was suggested by Greenleaf and Uhlmann [23]. In all aforementioned works the Radon type transforms are studied mainly on smooth rapidly decreasing functions.

Our interest to the problem is motivated by the following.

1. Is it possible to construct an easily visualizable admissible complex $\tilde{\Pi}_{n,k}$ using relatively simple tools and derive the relevant explicit inversion formulas for $R_k f$? A progress in this direction might be helpful in convex geometry, where geometrically transparent analytic constructions are crucial; see, e.g., [8, 40].

2. It is known [42, 43, 39] that for $f \in L^p(\mathbb{R}^n)$, $(R_k f)(\tau)$ is finite for almost all $\tau \in \Pi_{n,k}$ provided that $1 \leq p < n/k$, and this condition is sharp. However, $\tilde{\Pi}_{n,k}$ has measure zero in $\Pi_{n,k}$. Thus, what can one say about the existence of the restricted transform

$$ (\tilde{R}_k f)(\tau) = (R_k f)(\tau)_{|\tau \in \tilde{\Pi}_{n,k}} $$

on functions $f \in L^p(\mathbb{R}^n)$?

3. How to describe the range $\tilde{R}_k(X)$ where $X = S(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing smooth functions or any other reasonable function space?

Similar questions can be posed in other contexts of integral geometry, for instance, in the elliptic or hyperbolic space.

All these questions are in the spirit of [9] and related publications, however, the settings are not identical, e.g., in the part related to the $L^p$ theory. Our tools are also different. In Section 2 we define the admissible complex $\tilde{\Pi}_{n,k}$ as a set of all $k$-dimensional planes in $\mathbb{R}^n$ which are parallel to a fixed $(k+1)$-dimensional subspace, for instance, a $(k+1)$-dimensional coordinate plane. We establish sharp conditions for the existence of the corresponding restricted $k$-plane transform $\tilde{R}_k$ on $L^p$ functions. The explicit inversion formulas for $\tilde{R}_k$ are simple consequences of the known inversion formulas for the case of hyperplanes of codimension 1. The case of smooth functions is also included. Similar
questions are studied in Sections 3 and 4 for geodesic Radon transforms on the sphere and the hyperbolic space. Here we use the same idea adapted for the corresponding group of motions.

Section 5 conceptually pertains to Section 2. We have put it at the end of the paper in order not to overload the reader with technicalities. Here the main result is Theorem 5.4 which describes the range of the restricted $k$-plane transform on the space $S(R^n)$ of rapidly decreasing smooth functions. The classical Helgason theorem for $k = n - 1$ [25, p. 5] stating that the Radon transform is a bijective map from $S(R^n)$ onto the corresponding space $S_H(S^{n-1} \times R)$, arises as a particular case of our result. The proof Theorem 5.4 follows the same scheme as in [25], but it is more detailed and, probably, simplified (here we employ some ideas from Carton-Lebrun [7]). Moreover, our proof gives not only the bijectivity, but also the continuity of $\tilde{R}_k$ and its inverse in the respective topologies.

It is worth mentioning that in the case of the overdetermined $k$-plane transform on $S(R^n)$, different range characterizations can be found in [21, 22, 24, 28, 31, 32, 33, 34].

It might be of interest to describe the ranges of the restricted transforms from Sections 3 and 4 by making use of the relevant tools of harmonic analysis. This topic can be addressed in future publications.

Some notation. The following notation will be used throughout the paper. We fix an orthonormal basis $e_1, \ldots, e_n$ in $R^{n+1}$; $S^n$ is the $n$-dimensional unit sphere in $R^{n+1}$. If $\theta \in S^n$ is the variable of integration, then $d\theta$ stands for the $O(n+1)$-invariant measure on $S^n$ and $\sigma_n = \int_{S^n} d\theta = 2\pi^{n+1}/\Gamma((n+1)/2)$ is the surface area of $S^n$. We write $d_\ast \theta = \sigma_n^{-1} d\theta$ for the corresponding normalized measure.

2. The $k$-plane transform on $R^n$

2.1. Definitions. In this section we denote

\begin{equation}
R^{k+1} = Re_1 \oplus \cdots \oplus Re_{k+1}, \quad R^{n-k-1} = Re_{k+2} \oplus \cdots \oplus Re_n.
\end{equation}

Let $\tilde{\Pi}_{n,k}$ be the manifold of all $k$-planes in $R^n$ which are parallel to $R^{k+1}$. We write $x \in R^n$ as $x = (x',x'')$ where $x' \in R^{k+1}$, $x'' \in R^{n-k-1}$. Every plane $\tau \in \tilde{\Pi}_{n,k}$ is parametrized by the triple $(\theta,s;x'') \in S^k \times R \times R^{n-k-1}$, where $S^k$ is the unit sphere in $R^{k+1}$. Specifically,

$\tau \equiv \tau(\theta,s;x'') = \tau_0 + x'', \quad \tau_0 = \{x' \in R^{k+1} : \theta \cdot x' = s\}.$

We denote $\tilde{Z}_{n,k} = S^k \times R \times R^{n-k-1}$ and equip this set with the measure $d\tau = d_\ast \theta ds dx''$. Clearly, dim $\tilde{\Pi}_{n,k} = n$ and

\begin{equation}
\tau(\theta,s;x'') = \tau(-\theta,-s;x'') \quad \forall (\theta,s;x'') \in \tilde{Z}_{n,k}.
\end{equation}
The $k$-plane transform (1.1) restricted to $\tilde{\Pi}_{n,k}$ has the form

$$\left(\tilde{R}_k f\right)(\theta, s; x''') = \int_{\theta \perp \cap \mathbb{R}^{k+1}} f(s\theta + u, x''') \, du,$$

where $du$ is the volume element of $\theta \perp \cap \mathbb{R}^{k+1}$. We shall also write

$$\left(\tilde{R}_k f\right)(\theta, s; x''') = (R f''')(\theta, s), \quad f'''(\cdot) = f(\cdot, x'''),$$

where $R$ is the usual hyperplane Radon transform in $\mathbb{R}^{k+1}$ of a function in the $x'$-variable, so that many properties of $R$ can be transferred to $\tilde{R}_k$. Below we review some of them.

2.2. Existence on $L^p$-functions.

**Theorem 2.1.** The integral $\tilde{R}_k f$ is finite a.e. on $\tilde{Z}_{n,k}$ if $f$ is locally integrable on $\mathbb{R}^n \setminus \{ x : x' = 0 \}$ and

$$\int_{|x'''| < a} \int_{|x'| > 1} \frac{|f(x', x''')|}{|x'|} \, dx' < \infty \quad \text{for any } a > 0.$$

This statement follows immediately from [41, Theorem 3.2].

**Corollary 2.2.** If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < (k+1)/k$, then $(\tilde{R}_k f)(\tau)$ is finite for almost all planes $\tau \in \tilde{\Pi}_{n,k}$. If $p \geq (k+1)/k$, then there is a function $f_0 \in L^p(\mathbb{R}^n)$ for which $(\tilde{R}_k f_0)(\tau) \equiv \infty$ on $\tilde{\Pi}_{n,k}$.

**Proof.** The first statement follows from (2.5) by Hölder’s inequality. For the second statement we can take, e.g.,

$$f_0(x) = \frac{(2 + |x'|)^{-(k+1)/p} e^{-|x'''|^2}}{\log^{1/p'}(2 + |x'|)}, \quad 0 < \delta < 1/p', \quad 1/p + 1/p' = 1.$$

**Open problem.** The condition $p < (k+1)/k$ differs from $p < n/k$ for the nonrestricted $k$-plane transform. It would be interesting to investigate restrictions on $p$ for other known admissible complexes; see references in Introduction.

2.3. Inversion formulas. Let $1 \leq p < (k+1)/k$. Suppose that

$$\int dx'' \int_{|x''| < a} |f(x', x'')|^p \, dx' < \infty \quad \text{for all } a > 0.$$
Then $\varphi = \tilde{R}_k f$ is well-defined by Theorem 2.1 and the function $f_{x''}(x') \equiv f(x', x'')$ belongs to $L^p(\mathbb{R}^{k+1})$ for almost all $x'' \in \mathbb{R}^{n-k-1}$. Furthermore, the function $\varphi_{x''}(\theta, s) \equiv \varphi(\theta, s; x'')$ has the form

\begin{equation}
\varphi_{x''}(\theta, s) = (R_{x''})_{x'}(\theta, s)
\end{equation}

where $R$ is the usual hyperplane Radon transform in $\mathbb{R}^{k+1}$. Hence, inverting $R$ by any known method (see, e.g., [39, 41]), we reconstruct $f_{x''}(x') \equiv f(x)$. For example, if $k = 1$ and $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then, by the formula (5.12) from [39] we have

\begin{equation}
f(x) = \frac{1}{\pi} \int_0^\infty \frac{1}{t^2} \left( R^*_{x'} \varphi_{x''}(x') - (R^*_{x'} \varphi_{x''})(x') \right) \, dt.
\end{equation}

Here $x' \in \mathbb{R}^2$, $x'' \in \mathbb{R}^{n-2}$.

The integral $\int_0^\infty (\cdot)$ is understood as $\lim_{\varepsilon \to 0} \int_0^{\varepsilon} (\cdot)$ for almost all $x = (x', x'')$ in $\mathbb{R}^n$ or in the $L^p(\mathbb{R}^2)$-norm for almost all $x'' \in \mathbb{R}^{n-2}$.

3. The Elliptic Case

3.1. Definitions. The $n$-dimensional elliptic space can be interpreted as the $n$-dimensional unit sphere $S^n$ with identified antipodal points. According to this interpretation, we assume all functions on $S^n$ to be even. The Funk transform $F$ integrates a function $f$ on $S^n$ over $(n-1)$-dimensional totally geodesic submanifolds of $S^n$ (great circles, if $n = 2$). For any $1 \leq k \leq n - 1$, the corresponding transform $F_k$ is similarly defined by integration over $k$-dimensional totally geodesic submanifolds of $S^n$. It can be realized as an integral

\begin{equation}
(F_k f)(\xi) = \int_{S^{n-2} \cap \xi} f(\theta) \, d\xi \theta, \quad \xi \in G_{n+1,k+1},
\end{equation}

where $G_{n+1,k+1}$ is the Grassmann manifold of $(k+1)$-dimensional linear subspaces of $\mathbb{R}^{n+1}$ and $d\xi \theta$ denotes the $O(n+1)$-invariant probability measure on $S^n \cap \xi$.

Suppose $k < n - 1$. Then $\dim G_{n+1,k+1} = (k+1)(n-k) > n$ and the inversion problem for $F_k f$ is overdetermined. Our aim is to define an $n$-dimensional admissible complex $\tilde{G}_n$ in $G_{n+1,k+1}$, so that $f$ could be explicitly reconstructed from $(F_k f)(\xi)$, $\xi \in \tilde{G}_n$. We will be using the
same idea as in the Euclidean case, but translations will be replaced by orthogonal transformations. Let

\[ \mathbb{R}^{n-k} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{n-k}, \quad \mathbb{R}^{k+1} = \mathbb{R}e_{n-k+1} \oplus \cdots \oplus \mathbb{R}e_{n+1}, \]

\[ S^{n-k-1} = S^n \cap \mathbb{R}^{n-k}. \]

Fix a point \( v \in S^{n-k-1} \) and denote

\[ \mathbb{R}^{k+2}_v = \mathbb{R}v \oplus \mathbb{R}^{k+1}_v, \quad S^{k+1}_v = S^n \cap \mathbb{R}^{k+2}_v. \]

Clearly, every point \( \theta = (\theta_1, \ldots, \theta_{n+1}) \in S^n \) belongs to some \( S^{k+1}_v \). Specifically, if \( \theta' = (\theta_1, \ldots, \theta_{n-k}) \neq 0 \), then \( v = \theta'/|\theta'| \). If \( \theta' = 0 \) then \( \theta \in S^{k+1}_v \) for all \( v \in S^{n-k-1} \). We define

\[ \tilde{G}_n = \{ \xi \in G_{n+1,k+1} : \xi \subset \mathbb{R}^{k+2}_v \text{ for some } v \in S^{n-k-1} \}, \]

which is a fiber bundle over \( S^{n-k-1} \) with fibers isomorphic to the Grassmannian \( G_{k+2,k+1} \). It will be shown that \( \tilde{G}_n \) is an admissible complex for \( \mathcal{F}_k \).

Clearly, \( \dim \tilde{G}_n = n \). Furthermore, every \( \xi \in \tilde{G}_n \) can be parametrized as \( \xi = \xi(v, w) \) where \( v \in S^{n-k-1} \), \( w \in S^{k+1}_v \), \( w \perp \xi \), so that \( \xi(v, w) = \xi(\pm v, \pm w) \) with any combination of pluses and minuses. We denote

\[ \tilde{S}_n = \{ (v, w) : v \in S^{n-k-1}, w \in S^{k+1}_v \}. \]

and equip \( \tilde{S}_n \) with the product measure \( d_v d_w \) where \( d_v \) and \( d_w \) stand for the corresponding probability measures on \( S^{n-k-1} \) and \( S^{k+1}_v \), respectively. The transformation (3.1) restricted to \( \tilde{G}_n \) can be realized as

\[ (\tilde{\mathcal{F}}_k f)(v, w) = \int_{\{ \theta \in S^{k+1}_v : \theta \cdot w = 0 \}} f(\theta) d_{v,w} \theta, \quad (v, w) \in \tilde{S}_n, \]

which is the usual Funk transform on \( S^{k+1}_v \) with the relevant normalized surface measure \( d_{v,w} \). Clearly, \( (\tilde{\mathcal{F}}_k f)(v, w) = (\tilde{\mathcal{F}}_k f)(\pm v, \pm w) \).

3.2. Existence on \( L^p \)-functions. Fix any \( v \in S^{n-k-1} \) and choose an orthogonal transformation \( \gamma_v \) in the coordinate plane \( \mathbb{R}^{n-k} \), so that \( \gamma_v e_{n-k} = v \). Let

\[ \tilde{\gamma}_v = \begin{bmatrix} \gamma_v & 0 \\ 0 & I_{k+1} \end{bmatrix} \in O(n+1). \]

Then

\[ (\tilde{\mathcal{F}}_k f)(v, \tilde{\gamma}_v \zeta) = (\mathcal{F} f_v)(\zeta), \quad f_v(\eta) = f(\tilde{\gamma}_v \eta), \]
where
\[(3.9)\quad (\mathcal{F} f_v)(\zeta) \equiv \int_{\eta \cdot \zeta = 0} f_v(\eta) \, d\zeta \eta\]
is the usual Funk transform on the sphere $S^{k+1}$ defined by
\[(3.10)\quad S^{k+1} = S^n \cap \mathbb{R}^{k+2}, \quad \mathbb{R}^{k+2} = \mathbb{R}_{n-k} \oplus \cdots \oplus \mathbb{R}_{n+1}.
Thus, the existence of $\tilde{F}_k f$ is equivalent to the existence of the Funk transform $(3.9)$. The latter is well-defined whenever $f_v \in L^1(S^{k+1})$ and may not exist otherwise (take, e.g., $f_v(\eta) = |\eta_{n+1}|^{-1} \notin L^1(S^{k+1})$, for which $(\mathcal{F} f_v)(\zeta) \equiv \infty$; cf. [37, formula (2.12)]). This observation yields the following.

**Theorem 3.1.** Let $1 \leq k \leq n-1$,
\[(3.11)\quad \int_{S^n} |f(\theta)| \frac{d_s \theta}{|\theta'|^{n-k-1}} < \infty,
where $\theta' = (\theta_1, \ldots, \theta_{n-k})$ is the orthogonal projection of $\theta$ onto the coordinate plane $\mathbb{R}^{n-k} = \mathbb{R}_1 \oplus \cdots \oplus \mathbb{R}_{n-k}$. Then
\[(3.12)\quad \int_{\tilde{S}_n} (\tilde{F}_k f)(v, w) \, d_s v d_s w = \frac{2 \sigma_n}{\sigma_{k+1} \sigma_{n-k-1}} \int_{S^n} f(\theta) \frac{d_s \theta}{|\theta'|^{n-k-1}}.
PROOF. If $k = n-1$, then (3.12) is a particular case of the duality for Radon-type transforms [25]. Denote the left-hand side of (3.12) by $I$. Since
\[\int_{S^{k+1}} (\mathcal{F} f_v)(\zeta) \, d_s \zeta = \int_{S^{k+1}} f_v(\eta) \, d_s \eta, \quad f_v(\eta) = f(\tilde{\gamma}_v \eta),\]
then, by (3.8),
\[(3.13)\quad I = \int_{S^{n-k-1}} \int_{S^{k+1}} (\tilde{F}_k f)(v, w) \, d_s v \, d_s w = \int_{S^{n-k-1}} \int_{S^{k+1}} (\mathcal{F} f_v)(\zeta) \, d_s \zeta
\quad = \int_{S^{n-k-1}} \int_{S^{k+1}} f_v(\eta) \, d_s \eta = \frac{1}{\sigma_{k+1}} \int_{S^{n-k-1}} \int_{S^{k+1}} f(\tilde{\gamma}_v \eta) \, d\eta
\quad = \frac{1}{\sigma_{k+1}} \int_{S^{n-k-1}} \int_{S^{k+1}} \sin^k \psi \, d\psi \int f(v \cos \psi + \omega \sin \psi) \, d\omega.
Using the bi-spherical coordinates [44, pp. 12, 22]
\[(3.14)\quad \theta = v \cos \psi + \omega \sin \psi, \quad d\theta = \sin^k \psi \, d\psi \, d\omega,\]
v \in S^{n-k-1} \subset \mathbb{R}^{n-k}, \quad \omega \in S^k \subset \mathbb{R}^{k+1}, \quad 0 \leq \psi \leq \pi/2,

and noting that \cos \psi = |\theta'|, we continue:

\[ I = \frac{2}{\sigma_{k+1}\sigma_{n-k-1}} \int_0^{\pi/2} \int_{S^{n-k-1}} \sin^k \psi d\psi \int_{S^k} dv \int_{S^{n-k-1}} f(v \cos \psi + \omega \sin \psi) d\omega \]

\[ = \frac{2}{\sigma_{k+1}\sigma_{n-k-1}} \int_{S^n} f(\theta) \frac{d\theta}{|\theta'|^{n-k-1}}, \]

as desired. \qed

**Corollary 3.2.** Let \( 1 \leq k < n - 1, f \in L^p(S^n), n - k < p \leq \infty. \) Then \((\tilde{\mathcal{F}}_k f)(v, w)\) is finite for almost all \((v, w) \in \tilde{S}_n\) and the operator \(\tilde{\mathcal{F}}_k\) is bounded from \(L^p(S^n)\) to \(L^1(\tilde{S}_n)\). If \( p \leq n - k \), then there is a function \(\tilde{f} \in L^p(S^n)\) for which \((\tilde{\mathcal{F}}_k \tilde{f})(v, w) = \infty\). Specifically,

\[ \tilde{f}(\theta) = |\theta'|^{-1}(1 - \log |\theta'|)^{-1}, \quad \theta' = (\theta_1, \ldots, \theta_{n-k}). \]

**Proof.** The first statement follows from (3.12) by Hölder’s inequality, because the integral

\[ \int_{S^n} \frac{d\theta}{|\theta'|^{(n-k-1)p'}} = \sigma_k \sigma_{n-k-1} \int_0^{\pi/2} \sin^k \psi \cos^{(n-k-1)(1-p')}(\cos \psi) \psi d\psi, \quad \frac{1}{p} + \frac{1}{p'} = 1, \]

is finite if \( n - k < p \). To prove the second statement, we have,

\[ \int_{S^n} |\tilde{f}(\theta)|^p d\theta = \sigma_k \sigma_{n-k-1} \int_0^{\pi/2} (1 - \log(\cos \psi))^{-p} \sin^k \psi \cos^{n-k-1-p}(\cos \psi) \psi d\psi \]

\[ = \sigma_k \sigma_{n-k-1} \int_0^1 (1 - \log s)^{-p}(1 - s^2)^{(k-1)/2} s^{n-k-1-p} ds. \]

This integral is finite if \( p \leq n - k, 1 \leq k < n - 1. \)

Let us show that \((\tilde{\mathcal{F}}_k \tilde{f})(v, w) = \infty\). By (3.8), it suffices to prove that \((\tilde{\mathcal{F}} \tilde{f}_v)(\zeta) = \infty\). For \( \eta = (\eta_{n-k}, \ldots, \eta_{n+1}) \in S^{k+1} \subset \mathbb{R}^{k+2} \) (see (3.10)), we have \( \tilde{f}_v(\eta) = |\eta_{n-k}|^{-1}(1 - \log |\eta_{n-k}|)^{-1}. \) Let \( r_\zeta \) be an orthogonal transformation in \( \mathbb{R}^{k+2} \) such that \( r_\zeta e_{n-k} = \zeta, \) and let \( S^k \) be the unit sphere in the plane \( \mathbb{R}^{k+1} = \mathbb{R}e_{n-k+1} \oplus \cdots \oplus \mathbb{R}e_{n+1}. \) Setting
\( \eta = r \zeta u \), we have
\[
(F \tilde{f}_v)(\zeta) = \int_{\eta \zeta = 0} |\eta_{n-k}|^{-1} (1 - \log |\eta_{n-k}|)^{-1} d\zeta \eta
\]
\[
= \int_{S^k} |r \zeta u \cdot e_{n-k}|^{-1} (1 - \log |r \zeta u \cdot e_{n-k}|)^{-1} du,
\]
where
\[
|r \zeta u \cdot e_{n-k}| = |u \cdot r^{-1} \zeta e_{n-k}| = |u \cdot \text{Pr}_{\mathbb{R}^{k+1}}(r^{-1} \zeta e_{n-k})| = h |u \cdot \sigma|,
\]
\[ h = |\text{Pr}_{\mathbb{R}^{k+1}}(r^{-1} \zeta e_{n-k})|, \quad \sigma = \text{Pr}_{\mathbb{R}^{k+1}}(r^{-1} \zeta e_{n-k})/h \in S^k. \]
Hence,
\[
(F \tilde{f}_v)(\zeta) = \int_{S^k} (h |u \cdot \sigma|)^{-1} (1 - \log(h |u \cdot \sigma|))^{-1} du
\]
\[
= \frac{2\sigma_{k-1}}{h} \int_0^1 \frac{(1 - t^2)^{k-1}}{t (1 - \log(ht))} dt \geq \frac{c_k}{h} \int_0^{1/2} \frac{dt}{t (1 - \log(ht))},
\]
where
\[ c_k = \text{const.} \] The last integral diverges. \( \square \)

3.3. Inversion formulas. To reconstruct \( f \) from \( \tilde{F}_k f \), it suffices to invert the usual Funk transform \( F \) in (3.8) by any known method; see, e.g., [12, 25, 30, 35, 36, 37, 41]. Let \( \varphi(v, w) = (\tilde{F}_k f)(v, w) \), \( \varphi_v(\zeta) = \varphi(v, \tilde{\gamma}_v \zeta) \), where \( \tilde{\gamma}_v \) has the form (3.7). Then
\[
(3.15) \quad f_v(\eta) \equiv f(\tilde{\gamma}_v \eta) = (F^{-1} \varphi_v)(\eta).
\]
If \( f \) is a continuous function, its value at a point \( \theta \in S^n \) can be found as follows. Interpret \( \theta \) as a column vector \( \theta = (\theta_1, \ldots, \theta_{n+1})^T \) and set
\[
(3.16) \quad \theta' = (\theta_1, \ldots, \theta_{n-k})^T \in \mathbb{R}^{n-k}, \quad \theta'' = (\theta_{n-k+1}, \ldots, \theta_{n+1})^T \in \mathbb{R}^{k+1},
\]
\[
(3.17) \quad v = \theta'/|\theta'| \in S^{n-k-1} \subset \mathbb{R}^{n-k},
\]
\[
(3.17) \quad \eta = (0, \ldots, 0, |\theta'|, \theta'')^T \in S^{k+1} \subset \mathbb{R}^{k+2}.
\]
If \( \theta' \neq 0 \), then \( \tilde{\gamma}_v \eta = \theta \) and we get
\[
(3.18) \quad f(\theta) = (F^{-1} \varphi_v)(\eta), \quad \varphi_v(\zeta) = \varphi(v, \tilde{\gamma}_v \zeta).
\]
If \( \theta' = 0 \), then \( f(\theta) \) can be reconstructed by continuity.
If $f$ is an arbitrary function satisfying (3.11), for instance, $f \in L^p(S^n)$, $n - k < p \leq \infty$, then the integral
\[
\int_{S^{n-k-1}} d_{\ast} v \int_{S^{k+1}} |f_{v}(\eta)| d_{\ast} \eta
\]
is finite; see calculations after (3.13). Hence, by (3.15), $f$ can be explicitly reconstructed at almost all points on almost all spheres $S^{k+1}_v$ by making use of known inversion formulas for the Funk transform on this class of functions; see, e.g., [36, 37, 41].

4. The Hyperbolic Case

The hyperbolic totally geodesic Radon transform assigns to each sufficiently good function $f$ on the real hyperbolic space $\mathbb{H}^n$ the collection of integrals of $f$ over $k$-dimensional totally geodesic submanifolds of $\mathbb{H}^n$. If $k < n - 1$, the inversion problem for this transform is overdetermined and we shall construct the corresponding admissible complex, using the same idea as in the spherical case. Realization of this idea relies on the geometry of $\mathbb{H}^n$.

4.1. Preliminaries. We recall basic facts. More details can be found in [4, 5, 38, 44]; see also [12, 19]. Let $\mathbb{E}^{n,1} \sim \mathbb{R}^{n+1}$, $n \geq 2$, be the real pseudo-Euclidean space of points $x = (x_1, \ldots, x_{n+1})$ with the inner product
\[
[x, y] = -x_1 y_1 - \cdots - x_n y_n + x_{n+1} y_{n+1}.
\]
The $n$-dimensional real hyperbolic space $\mathbb{H}^n$ will be realized as the “upper” sheet of the two-sheeted hyperboloid
\[
\mathbb{H}^n = \{ x \in \mathbb{E}^{n,1} : [x, x] = 1, x_{n+1} > 0 \};
\]
dist$(x, y) = \cosh^{-1}[x, y]$ is the geodesic distance between the points $x$ and $y$ in $\mathbb{H}^n$. The hyperbolic coordinates $\theta_1, \ldots, \theta_{n-1}, r$ of a point $x \in \mathbb{H}^n$ are defined by
\[
\begin{align*}
x_1 &= \sinh r \sin \theta_{n-1} \cdots \sin \theta_2 \sin \theta_1, \\
x_2 &= \sinh r \sin \theta_{n-1} \cdots \sin \theta_2 \cos \theta_1, \\
&\quad \vdots \\
x_n &= \sinh r \cos \theta_{n-1}, \\
x_{n+1} &= \cosh r,
\end{align*}
\]
where $0 \leq \theta_j < 2\pi; 0 \leq \theta_j < \pi, 1 < j \leq n - 1; 0 \leq r < \infty$. Every $x \in \mathbb{H}^n$ is representable as
\[
x = \theta \sinh r + e_{n+1} \cosh r
\]
where \( \theta \) is a point of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n \) with Euler’s angles \( \theta_1, \ldots, \theta_{n-1} \). We denote by \( SO_0(n,1) \) the connected group of pseudo-rotations of \( E^{n,1} \) which preserve the bilinear form (4.1); \( SO(n) \) is the rotation group in \( \mathbb{R}^n \) which is identified with the subgroup of all pseudo-rotations leaving \( e_{n+1} \) fixed. The \( SO_0(n,1) \)-invariant measure \( d\theta \) on \( S^{n-1} \) is defined by

\[
(4.5) \quad d\theta = J(\theta) \prod_{i=1}^{n-1} d\theta_i, \quad J(\theta) = \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \cdots \sin \theta_{n-1},
\]

so that

\[
\sigma_{n-1} = \int_{S^{n-1}} d\theta = \frac{2\pi^{n/2}}{\Gamma(n/2)}.
\]

The \( SO_0(n,1) \)-invariant measure \( dx \) on \( H^n \) can be defined by

\[
(4.6) \quad dx = \sinh^{n-1} r \, dr
\]

where \( d\theta \) has the form (4.5). Then

\[
(4.7) \quad \int_{H^n} f(x) \, dx = \int_0^\infty \sinh^{n-1} r \, dr \int_{S^{n-1}} f(\theta \sinh r + e_{n+1} \cosh r) \, d\theta.
\]

We will need a generalization of (4.4) of the form

\[
(4.8) \quad x = v \sinh r + u \cosh r,
\]

where \( v \in S^{n-k-1} \subset \mathbb{R}^{n-k} \), \( u \in \mathbb{H}^k \subset \mathbb{R}^{k+1} \), \( 0 \leq r < \infty \), \( 0 \leq k < n \),

where \( \mathbb{R}^{n-k} \) and \( \mathbb{R}^{k+1} \) have the same meaning as in (3.2). Then

\[
(4.9) \quad dx = dv \, du \, dv(r), \quad dv(r) = \sinh^{n-k-1} r \cosh^k r \, dr,
\]

d\( v \) and d\( u \) being the Riemannian measures on \( S^{n-k-1} \) and \( \mathbb{H}^k \), respectively; see [44, pp. 12, 23]. Owing to (4.8),

\[
(4.10) \quad \int_{H^n} f(x) \, dx = \int_0^\infty dv(r) \int_{S^{n-k-1}} dv \int_{\mathbb{H}^k} f(v \sinh r + u \cosh r) \, du.
\]

The case \( k = 0 \) agrees with (4.7). If \( k = n-1 \), then (4.10) yields

\[
(4.11) \quad \int_{H^n} f(x) \, dx = \int_{-\infty}^{\infty} \cosh^{n-1} r \, dr \int_{\mathbb{H}^{n-1}} f(e_1 \sinh r + u \cosh r) \, du.
\]

We will also need the one-sheeted hyperboloid

\[
(4.12) \quad \mathbb{H}^n = \{ y \in E^{n,1} : [y,y] = -1 \}.
\]

Every point \( y \in \mathbb{H}^n \) is representable as \( y = \sigma \cosh \rho + e_{n+1} \sinh \rho \) where \( -\infty < \rho < \infty \) and \( \sigma \in S^{n-1} \). In this notation the \( SO_0(n,1) \)-invariant measure \( dy \) on \( \mathbb{H}^n \) has the form

\[
(4.13) \quad dy = \cosh^{n-1} \rho \, d\sigma d\rho.
\]
The hyperbolic Radon transform of a sufficiently good function $f$ on $\mathbb{H}^n$ is defined as a function on $\mathbb{H}^n$ by the formula
\begin{equation}
(\mathcal{H}f)(y) = \int_{\xi_y} f(x) \, dy_x, \quad \xi_y = \{ x \in \mathbb{H}^n : [x, y] = 0 \}, \quad y \in \mathbb{H}^n,
\end{equation}
where the measure $dy_x$ is the image of the standard measure on the $(n - 1)$-dimensional hyperboloid $\xi_{e_n} = \{ x \in \mathbb{H}^n : x_n = 0 \}$ under the transformation $\omega_y \in SO_0(n, 1)$ satisfying $\omega_y \xi_{e_n} = \xi_y$.

Theorem 4.1. [4, Corollaries 3.7, 3.8] The integral (4.14) is finite for almost all $y \in \mathbb{H}^n$ whenever
\begin{equation}
\int_{\mathbb{H}^n} |f(x)| \, dx_{n+1} < \infty.
\end{equation}
In particular, (4.14) is finite a.e. if $f \in L^p(\mathbb{H}^n)$,
\begin{equation}
1 \leq p < (n - 1)/(n - 2).
\end{equation}
The condition (4.16) is sharp. Moreover, for every $\sigma \in S^{n-1}$,
\begin{equation}
\int_{-\infty}^{\infty} (\mathcal{H}f)(\sigma \cosh \rho + e_{n+1} \sinh \rho) \frac{d\rho}{\cosh \rho} = \int_{\mathbb{H}^n} f(x) \, dx_{n+1}
\end{equation}
and this expression does not exceed $c \|f\|_p$, $c = \text{const}$.

Explicit inversion formulas for $\mathcal{H}f$ and other properties of this transform can be found in [1, 2, 3, 4, 25, 36, 38, 43].

4.2. Radon transform over $k$-geodesics in $\mathbb{H}^n$. We denote by $\Xi_k$ the set of all $k$-dimensional totally geodesic submanifolds $\xi$ of $\mathbb{H}^n$, $1 \leq k \leq n - 1$. The corresponding Radon transform has the form
\begin{equation}
(\mathcal{H}_k f)(\xi) = \int_{\{ x \in \mathbb{H}^n : d(x, \xi) = 0 \}} f(x) \, d\xi_x, \quad \xi \in \Xi_k,
\end{equation}
where $d\xi_x$ is the volume element on $\xi$. To give this formula precise meaning, we set $\Xi^{n,1} = \mathbb{R}^{n-k} \times \mathbb{E}^{k,1}$ where
\begin{equation}
\mathbb{R}^{n-k} = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_{n-k}, \quad \mathbb{E}^{k,1} = \mathbb{R}e_{n-k+1} \oplus \ldots \oplus \mathbb{R}e_{n+1},
\end{equation}
and let $r_\xi \in SO_0(n, 1)$ be a pseudo-rotation which takes the $k$-dimensional hyperboloid $\mathbb{H}^k = \mathbb{H}^n \cap \mathbb{R}^{k+1}$ to $\xi$. Then
\begin{equation}
(\mathcal{H}_k f)(\xi) = \int_{\mathbb{H}^k} f(r_\xi x) \, d\xi_x
\end{equation}
where the measure $d\eta x$ on $\mathbb{H}^k$ is defined in a standard way. If $f \in L^p(\mathbb{H}^n)$, this integral is finite for almost all $\xi \in \Xi_k$ provided that
\begin{equation}
1 \leq p < (n-1)/(k-1),
\end{equation}
and this condition is sharp, [5, 43].

Suppose $k < n-1$ and define an $n$-dimensional admissible complex $\tilde{\Xi}_n$ in $\Xi_k$, so that $f$ can be explicitly reconstructed from $(H_k f)(\xi)$, $\xi \in \tilde{\Xi}_n$. We fix a point $v \in S^{n-k-1} \subset \mathbb{R}^{n-k}$ (see (4.19)) and denote
\begin{equation}
\mathbb{R}^k v + \mathbb{R}^{k+1} v = \mathbb{H}^n \cap \mathbb{R}^{k+2},
\end{equation}
cf. (3.3). Then we set
\begin{equation}
\tilde{\Xi}_n = \{ \xi \in \Xi_k : \xi \subset \mathbb{H}^{k+1} \text{ for some } v \in S^{n-k-1} \},
\end{equation}
cf. (3.4). Clearly, dim $\tilde{\Xi}_n = n$. Every $k$-geodesic $\xi \in \tilde{\Xi}_n$ can be indexed by the pair $(v, y)$ where $v \in S^{n-k-1}$ and $y$ is a point of the one-sheeted hyperboloid $\mathbb{H}^{k+1}_v = \mathbb{H}^n \cap \mathbb{R}^{k+2}$. We denote
\begin{equation}
(4.23) \quad \mathbb{H}_n = \{ (v, w) : v \in S^{n-k-1}, w \in \mathbb{H}^{k+1}_v \}
\end{equation}
and equip this set with the product measure $dvdw$, where $dv$ and $dw$ are canonical measures on $S^{n-k-1}$ and $\mathbb{H}^{k+1}_v$, respectively; cf. (4.5), (4.13). The Radon transform (4.20) restricted to $\tilde{\Xi}_n$ can be realized as
\begin{equation}
(4.24) \quad (\tilde{H}_k f)(v, w) = \int_{\{x \in \mathbb{H}_n^{k+1} : [x, w] = 0\}} f(x) d_v x, \quad (v, w) \in \mathbb{H}_n,
\end{equation}
where the measure $d_v x$ is defined as the image of the corresponding canonical measure on $\mathbb{H}^k$. Clearly, (4.24) is the usual hyperbolic Radon transform on the $(k+1)$-dimensional hyperboloid $\mathbb{H}^{k+1}_v$; cf. (4.14).

4.3 Existence on $L^p$-functions. Fix $v \in S^{n-k-1}$ and choose an arbitrary rotation $\gamma_v$ in $\mathbb{R}^{n-k}$, so that $\gamma_v e_{n-k} = v$. As in (3.7), let
\begin{equation}
(4.25) \quad \tilde{\gamma}_v = \begin{bmatrix} \gamma_v & 0 \\ 0 & I_{k+1} \end{bmatrix}
\end{equation}
and change variables in (4.24) by setting $x = \tilde{\gamma}_v \eta$, $w = \tilde{\gamma}_v \zeta$. Then $\eta \in \mathbb{H}_v^{k+1}$, $\zeta \in \mathbb{H}_{k+1}$.
\begin{equation}
\mathbb{H}^{k+1}_v = \mathbb{H}^n \cap \mathbb{R}^{k+2}, \quad \mathbb{R}^{k+2} = \mathbb{R} e_{n-k} \oplus \cdots \oplus \mathbb{R} e_{n+1}.
\end{equation}
We get
\begin{equation}
(4.27) \quad (\tilde{H}_k f)(v, \tilde{\gamma}_v \zeta) = (H f_v)(\zeta), \quad f_v(\eta) = f(\tilde{\gamma}_v \eta),
\end{equation}
where

\[(4.28) \quad (\mathcal{H}f_v)(\zeta) \equiv \int_{\{\eta \in \mathbb{H}^{k+1} : [\eta, \zeta] = 0\}} f_v(\eta) \, d\zeta \eta \]

is the usual hyperbolic Radon transform on \(\mathbb{H}^{k+1}\). Thus, the existence of \(\mathcal{H}_k f\) is equivalent to the existence of \((\mathcal{H}f_v)(\zeta)\). The latter is characterized by Theorem 4.1 which should be applied to \(f_v\). To reformulate the conditions of that theorem in terms of \(f\), we need the following

**Lemma 4.2.** The equality

\[(4.29) \quad \int_{S^{n-k-1}} dv \int_{\mathbb{H}^{k+1}} f_v(\eta) \, d\eta = 2 \int_{\mathbb{H}^n} \frac{f(x)}{|x'|^{n-k-1}} \, dx, \quad x' = (x_1, \ldots, x_{n-k}), \]

holds provided that either side of it is finite when \(f\) is replaced by \(|f|\).

**Proof.** Let \(\eta = \eta_{n-k} e_{n-k} + \tilde{\eta}, \tilde{\eta} = (\eta_{n-k+1}, \ldots, \eta_{n+1})\). Then \(\tilde{\gamma}_v \eta = v \eta_{n-k} + \tilde{\eta}\) and (4.11) yields

\[
\text{l.h.s} = \int_{S^{n-k-1}} dv \int_{\mathbb{H}^{k+1}} f(v \eta_{n-k} + \tilde{\eta}) \, d\eta \\
= \int_{S^{n-k-1}} dv \int_{\mathbb{H}^k} \cosh^k r \, dr \int_{\mathbb{H}^k} f(v \sinh r + u \cosh r) \, du \\
= 2 \int_0^\infty \frac{d\nu(r)}{\sinh^{n-k-1} r} \int_{S^{n-k-1}} dv \int_{\mathbb{H}^k} f(v \sinh r + u \cosh r) \, du, \]

\(d\nu(r) = \sinh^{n-k-1} r \cosh^k r \, dr\). By (4.10), the result follows. \(\square\)

**Theorem 4.3.** Let \(1 \leq k \leq n - 1\). The integral (4.24) is finite for almost all \((v, w) \in \mathbb{H}^n\) provided that

\[(4.30) \quad \int_{\mathbb{H}^n} |f(x)|^p \frac{dx}{|x'|^{n-k-1}} < \infty, \quad 1 \leq p < k/(k - 1). \]

**Proof.** If \(f\) satisfies (4.30), then, by (4.29), \(f_v \in L^p(\mathbb{H}^{k+1})\). Hence, by Theorem 4.1 and (4.27), \((\mathcal{H}_k f)(v, \tilde{\gamma}_v \zeta) = (\mathcal{H}f_v)(\zeta)\) is finite for almost all \(v \in S^{n-k-1}\) and \(\zeta \in \mathbb{H}^{k+1}\). It follows that \((\mathcal{H}_k f)(v, w)\) is finite for almost all \(v \in S^{n-k-1}\) and \(w \in \mathbb{H}_u^{k+1}\). \(\square\)

**Remark 4.4.** The restriction \(1 \leq p < k/(k - 1)\) is sharp, as in Theorem 4.1, and the bound \(k/(k - 1)\) is smaller than \((n - 1)/(k - 1)\) if \(k < n - 1\); cf. (4.21).
4.4. Inversion formulas. To reconstruct an arbitrary function $f$ satisfying (4.30) from $\varphi(v, w) = (\mathcal{H}_f)(v, w)$, it suffices to invert the usual hyperbolic Radon transform $(\mathcal{H}_f)(\zeta)$ from (4.28). Specifically, fix $v \in S^{n-k-1}$ and let $\varphi_v(\zeta) = \varphi(v, \tilde{\gamma}_v \zeta)$. Using any known inversion formula for $\mathcal{H}$ (see, e.g., [4, 25, 36, 38]), we get

$$f_v(\eta) \equiv f(\tilde{\gamma}_v \eta) = (\mathcal{H}^{-1}_v \varphi_v)(\eta).$$

Since $f_v \in L^p(\mathbb{H}^{k+1})$, $1 \leq p < k/(k - 1)$, then it can be evaluated at almost all points of almost all hyperboloids $\mathbb{H}^{k+1}$. If, in addition to (4.30), $f$ is continuous, then, to find the value of $f$ at a point $x \in \mathbb{R}^n$, we regard $x$ as a column vector $x = (x_1, \ldots, x_{n+1})^T$ and set

$$x' = (x_1, \ldots, x_{n-k})^T \in \mathbb{R}^{n-k}, \quad x'' = (x_{n-k+1}, \ldots, x_{n+1})^T \in \mathbb{R}^{k+1},$$

$$v = x'/|x'| \in S^{n-k-1} \subset \mathbb{R}^{n-k},$$

$$\eta = (0, \ldots, 0, |x'|, x'')^T \in \mathbb{H}^{k+1} \subset \mathbb{R}^{k+2}.$$

Then $x = \tilde{\gamma}_v \eta$ and we get $f(x) = (\mathcal{H}^{-1}_v \varphi_v)(\eta)$.

5. The Range of the Restricted k-plane Transform

5.1. Definitions and the main result. We will be using the same notation as in Section 2. In the following $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_+^n = \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+$ ($n$ times). The Schwartz space $S(\mathbb{R}^n)$ is defined in a standard way with the topology generated by the sequence of norms

$$||f||_m = \sup_{|\alpha| \leq m} (1 + |x|)^m |(\partial^\alpha f)(x)|, \quad m = 0, 1, 2, \ldots.$$

The Fourier transform of $f \in S(\mathbb{R}^n)$ has the form

$$\hat{Ff}(y) \equiv \hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{ix \cdot y} dx.$$

The corresponding inverse Fourier transform will be denoted by $\hat{f}$.

**Definition 5.1.** A function $g$ on the sphere $S^k \subset \mathbb{R}^{k+1}$ is called differentiable if the homogeneous function $\tilde{g}(x) = g(x/|x|)$ is differentiable in the usual sense on $\mathbb{R}^{k+1} \setminus \{0\}$. The derivatives of $g$ will be defined as restrictions to $S^k$ of the corresponding derivatives of $\tilde{g}(x)$:

$$\tilde{g}^\alpha(\theta) = (\partial^\alpha \tilde{g})(x)|_{x=\theta}, \quad \alpha \in \mathbb{Z}_+^{k+1}, \quad \theta \in S^k.$$
Definition 5.2. We denote by $S_c(\tilde{Z}_{n,k})$ the space of functions $\varphi(\theta, s; x'')$ on $\tilde{Z}_{n,k} = S^k \times \mathbb{R} \times \mathbb{R}^{n-k-1}$, which are infinitely differentiable in $\theta$, $s$ and $x''$, rapidly decreasing as $|s| + |x''| \to \infty$ together with all derivatives, and satisfy
\begin{equation}
\varphi(-\theta, -s; x'') = \varphi(\theta, s; x'') \quad \forall (\theta, s; x'') \in \tilde{Z}_{n,k}.
\end{equation}

The topology in $S_c(\tilde{Z}_{n,k})$ is defined by the sequence of norms
\begin{equation}
||\varphi||_m = \sup_{|\mu| + j + |\gamma| \leq m} \sup_{\theta, s, x''} (1 + |s| + |x''|)^m |(\partial^{\mu}_\theta \partial^j_s \partial^{\gamma}_{x''} \varphi)(\theta, s; x'')|.
\end{equation}

The space $S_c(Z_n)$ of rapidly decreasing even smooth functions $\tilde{\varphi}(\theta, s)$ on $Z_n = S^{n-1} \times \mathbb{R}$ is defined similarly.

Definition 5.3. Let $S_H(\tilde{Z}_{n,k})$ denote the subspace of all functions $\varphi \in S_c(\tilde{Z}_{n,k})$ satisfying the **moment condition**: For every $m \in \mathbb{Z}_+$ there exists a homogeneous polynomial
\begin{equation}
P_m(\theta, x'') = \sum_{|\alpha| = m} c_\alpha(x'') \theta^\alpha
\end{equation}
with coefficients $c_\alpha(x'')$ in $S(\mathbb{R}^{n-k-1})$ such that
\begin{equation}
\int_{\mathbb{R}} \varphi(\theta, s; x'') s^m ds = P_m(\theta, x'').
\end{equation}

We equip $S_H(\tilde{Z}_{n,k})$ with the induced topology of $S_c(\tilde{Z}_{n,k})$.

The main result of this section is the following

**Theorem 5.4.** The restricted $k$-plane transform $\tilde{R}_k$ acts as an isomorphism from $S(\mathbb{R}^n)$ onto $S_H(\tilde{Z}_{n,k})$.

### 5.2. Auxiliary statements.

**Lemma 5.5.**
(i) If $f \in C^k(\mathbb{R}^n)$, $t \in \mathbb{R}$, then for $|\alpha| \leq k$ and $j \leq k$,
\begin{equation}
\partial^\alpha_x [f(tx/|x|)] = |x|^{-|\alpha|} \sum_{|\gamma| = 1} |\gamma| t|\gamma| h_{\alpha,\gamma}(x/|x|) (\partial^\gamma f)(tx/|x|),
\end{equation}
\begin{equation}
\frac{\partial^j}{\partial t^j} [f(tx/|x|)] = \sum_{|\gamma| = j} h_\gamma(x/|x|) (\partial^\gamma f)(tx/|x|),
\end{equation}
where $h_{\alpha,\gamma}$ and $h_\gamma$ are homogeneous polynomials independent of $f$. 
(ii) If \( g \in C^k(\mathbb{R}_+) \), \( \mathbb{R}_+ = (0, \infty) \), then for \( 1 \leq |\beta| \leq k \) and \( x \neq 0 \),

\[
\partial_x^{|\beta|} g(|x|) = \sum_{k=1}^{[\beta]} |x|^{k-|\beta|} h_{\beta,k}(x/|x|) g^{(k)}(|x|),
\]

where \( h_{\beta,k} \) are homogeneous polynomials independent of \( g \).

**Proof.** We proceed by induction. Let \( |\alpha| = 1 \), that is, \( \partial_x^\alpha = \partial/\partial x_j \) for some \( j \in \{1, 2, \ldots, n\} \). Then

\[
\partial_{x_j} [f(tx/|x|)] = t \sum_{k=1}^n (\partial_k f)(tx/|x|) p_{j,k}(x),
\]

\[
p_{j,k}(x) = \partial_{x_j} \left[ \frac{x_k}{|x|} \right] = \frac{1}{|x|} \left\{ \begin{array}{ll}
-x_k x_j & \text{if } j \neq k, \\
\frac{x_k^2}{|x|^2} & \text{if } j = k.
\end{array} \right.
\]

This gives (5.6) for \( |\alpha| = 1 \). Now the routine calculation shows that if (5.6) holds for any \( |\alpha| = \ell \), then it is true for \( |\alpha| = \ell + 1 \).

The proof of (5.7) is easier. For \( j = 1 \),

\[
\partial_t [f(tx/|x|)] = \sum_{k=1}^n (\partial_k f)(tx/|x|) \frac{x_k}{|x|}.
\]

The general case follows by iteration. The proof of (5.8) is straightforward by induction.

**Corollary 5.6.** Let \( f \in S(\mathbb{R}^n) \), \( \tilde{f}(\theta, t) = f(t\theta) \), where \( t \in \mathbb{R} \), \( \theta \in S^{n-1} \). Then for any \( m \in \mathbb{Z}_+ \) there exist \( N \in \mathbb{Z}_+ \) and a constant \( c_{m,N} \) independent of \( f \) such that

\[
||\tilde{f}||_m \leq \sup_{|\alpha| + j \leq m} |(1 + |t|)^m (\partial_\theta^\alpha \partial_t^j \tilde{f})(\theta, t)| 
\]

\[
\leq c_{m,N} ||f||_N \sup_{|\gamma| \leq N} (1 + |y|)^N |(|\gamma| f)(y)|.
\]

In other words, \( f \to \tilde{f} \) is a continuous mapping from \( S(\mathbb{R}^n) \) to \( S_c(Z_n) \).

**Corollary 5.7.** The map \( F_1 \), which assigns to a function \( w(\theta, t) \in S_c(Z_n) \) its Fourier transform in the \( t \)-variable, is an automorphism of the space \( S_c(Z_n) \).

**5.3. Proof of Theorem 5.4.** We split the proof in several steps.

**Proposition 5.8.** If \( f \in S(\mathbb{R}^n) \), then \( \tilde{R}_k f \in S_H(\tilde{Z}_{n,k}) \) and the map \( f \to \tilde{R}_k f \) is continuous.
PROOF. By (2.3) and (2.8), the function
\[
\varphi(\theta, s; x'') = \int_{\theta \perp \mathbb{R}^{k+1}} f(s\theta + u, x'') \, du = (Rf_{x''})(\theta, s),
\]
is the usual hyperplane Radon transform in \(\mathbb{R}^{k+1}\) of \(f_{x''}(x') = f(x', x'')\). Hence, (5.5) follows from the equalities
\[
\int_{\mathbb{R}} \varphi(\theta, s; x'') \, s^m \, ds = \int(Rf_{x''})(\theta, s) \, s^m \, ds = \int f(x', x'')(x' \cdot \theta)^k \, dx'.
\]
The evenness property (5.3) is a consequence of (5.9). Furthermore, by the Projection-Slice Theorem,
\[
[(\varphi(\theta, \cdot; x''))^\wedge(\eta)] = [(Rf_{x''})(\theta, \cdot)]^\wedge(\eta) = [f(\cdot, x'')]^\wedge(\eta \theta).
\]
Hence, \(A : f \rightarrow \varphi = \tilde{R}_k f\) is a composition of three mappings, specifically, \(A = A_3 A_2 A_1\), where
\[
\begin{align*}
A_1 &: f(x) \rightarrow [f(\cdot, x'')]^\wedge(\xi') \equiv g(\xi', x''); \\
A_2 &: g(\xi', x'') \rightarrow g(\theta \eta, x'') \equiv w(\theta, \eta; x'''); \\
A_3 &: w(\theta, \eta; x'') \rightarrow [w(\theta, \cdot; x'')]^\vee(s) \equiv \varphi(\theta, s; x'').
\end{align*}
\]
The continuity of the operators
\[
A_1 : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1}), \quad A_3 : S_e(\tilde{Z}_{n,k}) \rightarrow S_e(\tilde{Z}_{n,k})
\]
is a consequence of the isomorphism property of the Fourier transform. The continuity of \(A_2\) from \(S(\mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1})\) to \(S_e(\tilde{Z}_{n,k})\) follows from Corollary 5.6 applied in the \(\xi'\)-variable. This gives the result. \(\square\)

The next proposition is the most technical.

PROPOSITION 5.9. If \(\varphi \in S_H(\tilde{Z}_{n,k})\), then the function
\[
\psi(x) \equiv \psi(x', x'') = \int_{\mathbb{R}} \varphi(x' / |x'|, s; x'') e^{is|x'|} \, ds
\]
belongs to \(S(\mathbb{R}^n)\) and the map \(\varphi \rightarrow \psi\) is continuous.

PROOF. We have to show that for every \(m \in \mathbb{Z}_+\) there exist \(M = M(m) \in \mathbb{Z}_+\) and a constant \(C_m > 0\) independent of \(\varphi\) such that \(||\psi||_m \leq C_m ||\varphi||_M\). To this end, it suffices to prove the following inequalities:
\[
\sup_{|p|+|\gamma| \leq m} \sup_{|x'| < 1} \sup_{x''} (1+|x''|)^m |\partial_x^p \partial_{x''}^\gamma \psi(x', x'')| \leq C_m ||\varphi||_M, \tag{5.11}
\]
\[
\sup_{|p|+|\gamma| \leq m} \sup_{|x'| > 1} \sup_{x''} (1+|x'|+|x''|)^m |\partial_x^p \partial_{x''}^\gamma \psi(x', x'')| \leq C_m ||\varphi||_M. \tag{5.12}
\]
In the following the letters \( c \) and \( C \) with subscripts stand for constants which are not necessarily the same in any two occurrences.

**STEP 1 (proof of (5.11)).** Fix any \( q \in \mathbb{N} \). By Taylor’s formula,

\[
(5.13) \quad e^z = \sum_{\nu=0}^{q-1} \frac{z^\nu}{\nu!} + e_q(z), \quad e_q(z) = \sum_{\nu=q}^{\infty} \frac{z^\nu}{\nu!}.
\]

Putting \( z = i\varepsilon |x'| \), we have

\[
\psi(x', x'') = \sum_{\nu=0}^{q-1} \frac{(i|x'|)^\nu}{\nu!} \int_R \varphi(x'/|x'|, s; x'') s^\nu \, ds + \int_R \varphi(x'/|x'|, s; x'') e_q(i\varepsilon |x'|) \, ds.
\]

By (5.5),

\[
(i|x'|)^\nu \int_R \varphi(x'/|x'|, s; x'') s^\nu \, ds = (i|x'|)^\nu P_\nu(x'/|x'|, x'')
\]

where \( P_\nu(\theta, x'') = \sum_{|\alpha|=\nu} c_\alpha(x'') \theta^\alpha \), \( c_\alpha(x'') \in S(\mathbb{R}^{n-k-1}) \), or, by the homogeneity,

\[
(5.14) \quad (i|x'|)^\nu \int_R \varphi(x'/|x'|, s; x'') s^\nu \, ds = P_\nu(ix', x'').
\]

Hence, for \( \gamma \in \mathbb{Z}_+^{n-k-1} \), we may write

\[
(5.15) \quad \partial_\nu^\gamma \psi(x', x'') = \sum_{\nu=0}^{q-1} \frac{P_{\nu,\gamma}(ix', x'')}{\nu!} + \Psi_{q,\gamma}(x', x''),
\]

\[
(5.16) \quad P_{\nu,\gamma}(ix', x'') = (i|x'|)^\nu \int_R (\partial_\nu^\gamma \varphi)(x'/|x'|, s; x'') s^\nu \, ds
\]

\[
= \sum_{|\alpha| = \nu} (\partial_\alpha^\gamma c_\alpha)(x'')(ix')^\alpha,
\]

\[
(5.17) \quad \Psi_{q,\gamma}(x', x'') = \int_R (\partial_\nu^\gamma \varphi)(x'/|x'|, s; x'') e_q(i\varepsilon |x'|) \, ds.
\]
Let us estimate the derivatives \((\partial_{x'}^\beta \Psi_{q,\gamma})(x', x'')\), assuming \(0 \leq |p| < q\). For \(|x'| > 0\) we have

\[
\partial_{x'}^\beta[(\partial_{x''}^\alpha \varphi)(x'/|x'|, s; x'') e_q(is|x'|)]
\]

(5.19) \\
\[
= \sum_{\alpha + \beta = p} c_{\alpha, \beta} \partial_{x'}^\alpha[(\partial_{x''}^\gamma \varphi)(x'/|x'|, s; x'')] \partial_{x'}^\beta[e_q(is|x'|)].
\]

By (5.6),

\[
\partial_{x'}^\alpha[(\partial_{x''}^\gamma \varphi)(x'/|x'|, s; x'')] = |x'|^{-|\alpha|} \sum_{|\mu| = 1} \partial_{x'}^\mu \partial_{x''}^\nu \varphi_0(x'/|x'|, s; x'')
\]

where \(\varphi_0(x', s; x'') = \varphi(x'/|x'|, s; x'')\) and \(h_{\alpha, \mu}\) are homogeneous polynomials independent of \(\varphi\). Thus, by (5.2),

\[
|\partial_{x'}^\alpha[(\partial_{x''}^\gamma \varphi)(x'/|x'|, s; x'')]| \leq c_p |x'|^{-|\alpha|} \sum_{|\mu| = 1} \sup_{\theta} |(\partial_{x'}^\mu \partial_{\theta}^\nu \varphi)(\theta, s; x'')|.
\]

To estimate \(\partial_{x'}^\beta[e_q(is|x'|)]\), we consider the cases \(\beta \neq 0\) and \(\beta = 0\) separately. If \(\beta \neq 0\), then, by (5.8),

\[
|\partial_{x'}^\beta[e_q(is|x'|)]| \leq \sum_{j=1}^{q} |x'|^{j-|\beta|} \sup_{\theta} |h_{\beta,j}(x'/|x'|)| |s|^j |e_q(j)(is|x'|)|
\]

(5.21)

where \(h_{\beta,j}\) are homogeneous polynomials. Since \(e_q(j)(z) = e_{q-j}(z)\) for any \(0 \leq j \leq q\), the function

\[
e_q(j)(z) = e_{q-j}(z) = \frac{1}{z^{q-j}} \left(e^z - \sum_{\nu=0}^{q-j-1} \frac{z^\nu}{\nu!}\right), \quad z \in \mathbb{C},
\]

is bounded (check the cases \(|z| \leq 1\) and \(|z| > 1\) separately). Hence, the expression \((is|x'|)^{j-q} e_q(j)(is|x'|)\) is bounded uniformly in \(s\) and \(x'\), and (5.21) yields

\[
|\partial_{x'}^\beta[e_q(is|x'|)]| \leq c_{\beta, q} \sum_{j=1}^{q} |x'|^{j-|\beta|} \sup_{\theta} |h_{\beta,j}(\theta, s; x'')| |s|^j |s x'|^{q-j}.
\]

This gives

\[
|\partial_{x'}^\beta[e_q(is|x'|)]| \leq c_q |x'|^{q-|\beta|}(1 + |s|)^q.
\]

(5.22)
The last estimate extends to $\beta = 0$, but the proof in this case is easier. Combining (5.20), (5.22) and (5.19), and keeping in mind that $|p| < q$, we obtain

$$|\partial_x^p [\partial_{x''} \varphi(x' / |x'|, s; x'')] e_q(i s |x'|)|$$

$$\leq c_q (1 + |s|)^q \left( \sum_{|\mu| = 1}^q \sup_{\theta} |(\partial_\theta^p \partial_{x''} \varphi)(\theta, s; x'')| \right) \sum_{|\alpha| + |\beta| = p} |x'|^{q - |\beta| - |\alpha|}$$

(5.23) $\leq \frac{\tilde{c}_q}{(1 + |s|)^2} |x'|^{q - |p|} \sup_{|\mu| \leq q} \sup_{\theta, s} (1 + |s|)^{q+2} |(\partial_\theta^p \partial_{x''} \varphi)(\theta, s; x'')|.$

Since $q, \gamma$ and $|p| < q$ are arbitrary, the latter means that we can differentiate under the sign of integration in (5.18) infinitely many times. Moreover, if we fix any $m \in \mathbb{Z}_+$ and any $q > m$, then, by (5.23), we obtain

$$\sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} \sup_{x''} (1 + |x''|)^m |\partial_x^p \Psi_{q, \gamma}(x', x'')|$$

$$\leq \sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} \sup_{x''} (1 + |x''|)^m$$

$$\times \sup_{|\mu| \leq q} \sup_{\theta, s} (1 + |s|)^{q+2} |(\partial_\theta^p \partial_{x''} \varphi)(\theta, s; x'')|$$

$$\leq C_q \sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} |x'|^{q - |p|}$$

$$\times \sup_{|\mu| \leq q} \sup_{\theta, s, x''} (1 + |s| + |x''|)^{m+q+2} |(\partial_\theta^p \partial_{x''} \varphi)(\theta, s; x'')|$$

$$\leq C_q \sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} |x'|^{q - |p|} |||\varphi|||_{m+q+2}.$$

Setting $q = m + 1$, we get

$$\text{(5.24) } \sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} (1 + |x''|)^m |\partial_x^p \Psi_{m+1, \gamma}(x', x'')| \leq C_m |||\varphi|||_{2m+3}.$$

Let us estimate the derivatives of the first term in (5.15). By (5.16) and (5.17),

$$(\partial_x^P \Psi_{\nu, \gamma})(ix', x'') = \partial_x^P \left[ (i|x'|)^\nu \int_{\mathbb{R}} (\partial_{x''} \varphi)(x' / |x'|, s; x'') s^\nu ds \right]$$

$$= \sum_{|\alpha| = \nu} (\partial^\alpha c_\alpha)(x'') \partial_x^P [((ix')^\alpha], \nu \leq q - 1.$$
Hence, if $|p| > \nu$, then $(\partial^p_{x'}, P_{\nu, \gamma})(ix', x'') = 0$. Suppose $|p| \leq \nu$. Then

\[
\sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} \sup_{x''} (1 + |x''|)^m |(\partial^p_{x'} P_{\nu, \gamma})(ix', x'')|
\leq c_p \sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} \sup_{x''} (1 + |x''|)^m 
\times \sum_{\alpha + \beta = p} |\partial^\beta_{x''} [(i|x'|)']^\nu| \int \partial^\alpha_{x'} [\partial^\gamma_{x''} \varphi](x'/|x'|, s, x'') s^\nu ds.
\]

By (5.8) and (5.20), the last expression does not exceed the following:

\[
c_{p, \nu} \sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} \sum_{\alpha + \beta = p} |x'|^{\nu - |\beta| - |\alpha|} \sup_{x''} (1 + |x''|)^m 
\times \int \left[ \sup_{|\theta| = 1} \left| (\partial^\gamma_{x''} \partial^\mu_{s} \varphi)(\theta, s; x'') \right| \right] s^\nu ds
\leq \tilde{c}_{p, \nu} \sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} \sup_{s, x''} |x'|^{\nu - |p|} \sup_{s, x''} (1 + |x''| + |s|)^{m + \nu + 2}
\times \sum_{|\alpha| = 1} \sup_{|\theta| = 1} \left| (\partial^\gamma_{x''} \partial^\mu_{s} \varphi)(\theta, s; x'') \right|
\leq C_{p, \nu} \sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} \sup_{|\mu| + |\gamma| + |\nu| + |p| + m + \nu}
\times \sup_{\theta, s, x''} (1 + |x''| + |s|)^{m + \nu + 2 + |p|} \left| (\partial^\gamma_{x''} \partial^\mu_{s} \partial^\nu_{\theta} \varphi)(\theta, s; x'') \right|
\leq C_{p, \nu} \sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} \sup_{|\mu| + |\gamma| + |\nu| + |p|}
\left| \varphi \right|_{2m + \nu + 2} \leq C_{m, q} \left| \varphi \right|_{2m + q + 1} = C_m \left| \varphi \right|_{3m + 2};
\]

here we choose $q = m + 1$, as in (5.24).

Combining the last estimate with (5.15) and (5.24), we obtain

\[
\sup_{|p| + |\gamma| \leq m} \sup_{|x'| < 1} \sup_{x''} (1 + |x''|)^m |\partial^p_{x'} \partial^\gamma_{x''} \psi(x', x'')| \leq C_m \left( \left| \varphi \right|_{2m + 3} + \left| \varphi \right|_{3m + 2} \right).
\]

This gives the first required inequality (5.11).

STEP 2. Let us prove (5.12). Fix any $m \in \mathbb{Z}_+$. Then integration by parts yields

\[
\varphi(\theta, \cdot; x'') \wedge (\eta) = (-i\eta)^{-m} \int_R (\partial^m_{s} \varphi)(\theta, s; x'') e^{i\eta s} ds.
\]
For arbitrary multi-indices $\gamma \in \mathbb{Z}^{n-k-1}_+$ and $p \in \mathbb{Z}^{k+1}_+$ satisfying $|p| + |\gamma| \leq m$, we have
\[
(\partial^p \partial^\gamma_{x'} \psi)(x', x'') = \partial^p_x [(-i|x'|)^{-m} \psi_{m,\gamma}(x', x'')],
\]
where
\[
\psi_{m,\gamma}(x', x'') = \int_{\mathbb{R}} (\partial^m_p \partial^\gamma_{x''} \varphi)(x'/|x'|, s; x'') e^{is|x'|} ds,
\]
and therefore,
\[
|(\partial^p \partial^\gamma_{x'} \psi)(x', x'')| \leq c_p \sum_{u+v=p} |\partial^u_x [|x'|^{-m}]||\psi_{m,\gamma}(x', x'')|.
\]
By (5.8),
\[
|(\partial^u_x [|x'|^{-m}]| \leq c_{m,u} |x'|^{-m-|u|}.
\]
To estimate $|(\partial^p \psi_{m,\gamma})(x', x'')|$, as in STEP 1, we have
\[
\partial^p_x [(\partial^m_p \partial^\gamma_{x''} \varphi)(x'/|x'|, s; x'') e^{is|x'|}]
\]
\[
= \sum_{\alpha+\beta=v} c_{\alpha,\beta} \partial^p_x [(\partial^m_p \partial^\gamma_{x''} \varphi)(x'/|x'|, s; x'')] \partial^\beta \partial^\gamma_{x'} e^{is|x'|},
\]
where
\[
|\partial^p_x [(\partial^m_p \partial^\gamma_{x''} \varphi)(x'/|x'|, s; x'')]| \leq c_v |x'|^{-|\alpha|} \sup_{|\mu|=1} |(\partial^\mu s \partial^\gamma_{x''} \tilde{\varphi})(\theta, s; x'')|;
\]
cf. (5.20). Furthermore, by (5.8), for $\beta \neq 0$ we have
\[
\partial^\beta_x e^{is|x'|} = \sum_{j=1}^{[\beta]} |x'|^{j-|\beta|} h_{\beta,j}(x'/|x'|) (is)^j e^{is|x'|},
\]
where $h_{\beta,j}$ are homogeneous polynomials. Hence, since $|x'| > 1$,
\[
|\partial^\beta_x e^{is|x'|}| \leq c_\beta \sum_{j=1}^{[\beta]} |x'|^{j-|\beta|} \sup_{|\mu|=1} |(\partial^\mu s \partial^\gamma_{x''} \tilde{\varphi})(\theta, s; x'')| \leq \tilde{c}_\beta (1 + |s|)^{|\beta|}.
\]
This estimate obviously holds if $\beta = 0$. Combining (5.27), (5.28), and (5.29), for $|v| \leq |p|$ we obtain
\[
|\partial^p_x [(\partial^m_p \partial^\gamma_{x''} \varphi)(x'/|x'|, s; x'') e^{is|x'|}]|
\]
\[
\leq c_v (1 + |s|)^{|v|} \sum_{|\mu|=1} \sup_{|\theta|} |(\partial^\mu s \partial^\gamma_{x''} \tilde{\varphi})(\theta, s; x'')| \leq \frac{c_p}{(1 + |s|)^2} \sup_{|\theta, s|} \sup_{|\mu|+|j+|\gamma|\leq 2m+2} (1 + |s| + |x''|)^{2m+2} |(\partial^\mu s \partial^\gamma_{x''} \tilde{\varphi})(\theta, s; x'')|.
\]
Hence, we can differentiate under the sign of integration in $\psi_{m,\gamma}(x', x'')$ and get

\[(5.30) \quad \left| (\partial_{x'}^\mu \psi_{m,\gamma})(x', x'') \right| \leq \tilde{c}_p \sup_{|p|+|\gamma| \leq m} \sup_{|x'| > 1} (1 + |s| + |x''|)^{2m+2} |(\partial_\theta^\mu \partial_s^\nu \partial_{x''}^\gamma \varphi)(\theta, s; x'')|, \]

Thus, (5.25), (5.26), and (5.30) yield

\[(5.31) \quad \sup_{|p|+|\gamma| \leq m} \sup_{|x'| > 1} (1 + |s| + |x''|)^m |\partial_{x'}^p \partial_{x''}^\gamma \psi(x', x'')| \leq c_m \sup_{|p|+|\gamma| \leq m} \sup_{|x'| > 1} (1 + |s| + |x''|)^{m} \sum_{u+v=p} |x'|^{-m-u} \times \sup_{|s| + |x''| \leq 2m+2} \sup_{\theta, s} (1 + |s| + |x''|)^{2m+2} |(\partial_\theta^\mu \partial_s^\nu \partial_{x''}^\gamma \varphi)(\theta, s; x'')|. \]

Since for $|x'| > 1$,

\[
\frac{1 + |x'| + |x''|}{|x'|} = 1 + \frac{1 + |x''|}{|x'|} < 2 + |x''|,
\]

then the expression in (5.31) does not exceed

\[
C_m \sup_{|p|+|\gamma| \leq 3m+2} \sup_{\theta, s, x''} (1 + |s| + |x''|)^{3m+2} |(\partial_\theta^\mu \partial_s^\nu \partial_{x''}^\gamma \varphi)(\theta, s; x'')|,
\]

which is $C_m ||\varphi||_{3m+2}$. This completes the proof of Proposition 5.9. □

5.3.1. The end of the proof of Theorem 5.4. In view of Proposition 5.8, it remains to prove that any $\varphi \in S_H(\tilde{Z}_{n,k})$ is uniquely represented as $\varphi = \tilde{R}_n f$ for some $f \in S(\mathbb{R}^n)$ and the map $\varphi \to f$ is continuous in the topology of the corresponding Schwartz spaces. In the following, dealing with a function of $x = (x', x'')$ on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k-1}$, we denote by $F_1[\cdot](y')$ and $F_2[\cdot](y'')$ the Fourier transform of this function in the first and the second variable, respectively. The corresponding $n$-dimensional Fourier transform will be denoted by $F_n[\cdot](y), y = (y', y'')$.

If $\varphi \in S_H(\tilde{Z}_{n,k})$, then the function $\psi(y', x'') = [\varphi(y'/|y'|, \cdot, x'')]^\wedge(|y'|)$ belongs to $S(\mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1}) = S(\mathbb{R}^n)$; see Proposition 5.9. We define a new function

\[
\psi_1(y) \equiv \psi_1(y', y'') = F_2[\psi(y', \cdot)](y'').
\]

Putting $y' = \eta \theta, \eta \in \mathbb{R}, \theta \in S^k$, according to (5.10) and (5.3), we have

\[(5.32) \quad F_2^{-1}[\psi_1(\eta \theta, \cdot)](x'') = \psi(\eta \theta, x'') = [\varphi(\theta, \cdot, x'')]^\wedge(\eta).\]
Now, let \( f \in S(\mathbb{R}^n) \) be the inverse \( n \)-dimensional Fourier transform of \( \psi_1 \), i.e., \( f = F_n^{-1} \psi_1 = F_1^{-1} F_2^{-1} \psi_1 \). Then by (2.4) and the Projection-Slice Theorem,

\[
[(\tilde{R}_k f)(\theta, \cdot; x'')]^\wedge(\eta) = [(Rf_{x''})(\theta, \cdot)]^\wedge(\eta) = [f(\cdot; x'')]^\wedge(\eta) \\
= [(F_1^{-1} F_2^{-1} \psi_1)(\cdot; x'')]^\wedge(\eta) \\
= [F_2^{-1} [\psi_1(\eta \theta, \cdot)](x'')] \quad \text{(use (5.32))} \\
= [\varphi(\theta, \cdot; x'')]^\wedge(\eta).
\]

It follows that \( \tilde{R}_k f = \varphi \). Moreover, the map \( \varphi \to f \) is continuous thanks to the continuity of the mappings

\[
\varphi \to \psi \to \psi_1 \to F_n^{-1} \psi_1 = f
\]

in the corresponding Schwartz spaces.

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