Convergence Rates of Variational Posterior Distributions

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Abstract

We study convergence rates of variational posterior distributions for nonparametric and high-dimensional inference. We formulate general conditions on prior, likelihood, and variational class that characterize the convergence rates. Under similar “prior mass and testing” conditions considered in the literature, the rate is found to be the sum of two terms. The first term stands for the convergence rate of the true posterior distribution, and the second term is contributed by the variational approximation error. For a class of priors that admit the structure of a mixture of product measures, we propose a novel prior mass condition, under which the variational approximation error of the generalized mean-field class is dominated by convergence rate of the true posterior. We demonstrate the applicability of our general results for various models, prior distributions and variational classes by deriving convergence rates of the corresponding variational posteriors.

Keywords. posterior contraction, mean-field variational inference, density estimation, Gaussian sequence model, piecewise constant model, empirical Bayes

1 Introduction

Variational Bayes inference is a popular technique to approximate difficult-to-compute probability posterior distributions. Given a posterior distribution $\Pi(\cdot | X^{(n)})$, and a variational family $S$, variational Bayes inference seeks a $\hat{Q} \in S$ that best approximates $\Pi(\cdot | X^{(n)})$ under the Kullback-Leibler divergence. Though it is not exact Bayes inference, the variational class $S$ often gives computational advantage and leads to algorithms such as coordinate ascent that can be efficiently implemented on large-scale data sets. Researchers in many fields have used variational Bayes inference to solve real problems. Successful examples include statistical genetics [7, 24], natural language processing [5, 18], computer vision [29], and network analysis [4, 35], to name a few. We refer the readers to an excellent recent review [6] on this topic.

The goal of this paper is to study the variational posterior distribution $\hat{Q}$ from a theoretic perspective. We propose general conditions on the prior, the likelihood and the variational class to characterize the convergence rate of the variational posterior to the true data generating process.
Before discussing our results, we give a brief review on the theory of convergence rates of the posterior distributions in the literature. In order that the posterior distribution concentrates around the true parameter with some rate, the “prior mass and testing” framework requires three conditions on the prior and the likelihood: a) The prior is required to put a minimal amount of mass in a neighborhood of the true parameter; b) Restricted to a subset of the parameter space, there exists a testing function that can distinguish the truth from the complement of its neighborhood; c) The prior is essentially supported on the subset described above. Rigorous statements of these three conditions can be found in seminal papers [14, 28, 13]. Earlier versions of these conditions go back to [27, 19, 3, 2]. We also mention another line of work [36, 32, 9, 15] that established posterior rates of convergence using other approaches.

In this paper, we show that under almost the same three conditions, the variational posterior $\hat{Q}$ also converges to the true parameter, and the rate of convergence is given by

$$\epsilon_n^2 + \frac{1}{n} \inf_{Q \in S} P_0^{(n)} D(Q \| \Pi(\cdot | X^{(n)})).$$  \hspace{1cm} (1)

The first term $\epsilon_n^2$ is the rate of convergence of the posterior distribution $\Pi(\cdot | X^{(n)})$. The second term is the variational approximation error with respect to the class $S$ under the data generating process $P_0^{(n)}$. It is interesting to note that when $\Pi(\cdot | X^{(n)}) \in S$, the second term of (1) is zero, and then our theory recovers the usual “prior mass and testing” framework in the literature. Moreover, since we are able to generalize the “prior mass and testing” theory with the same old conditions, many well-studied problems in the literature can now be revisited under our framework of variational Bayes inference with very similar proof techniques. This will be illustrated with several examples considered in the paper.

Remarkably, for a special class of prior distributions and a corresponding variational class, the second term of (1) will be automatically dominated by $\epsilon_n^2$ under a modified “prior mass” condition. We illustrate this result by a prior distribution of product measure

$$d\Pi(\theta) = \prod_j d\Pi_j(\theta_j),$$

and a mean-field variational class

$$S_{\text{MF}} = \left\{ Q : dQ(\theta) = \prod_j dQ_j(\theta_j) \right\}.$$

As long as there exists a subset $\otimes_j \tilde{\Theta}_j \subset \left\{ \theta : D_\theta \left( P_0^{(n)} \| P_\theta^{(n)} \right) \leq C_1 n \epsilon_n^2 \right\}$, such that the prior mass condition

$$\Pi \left( \otimes_j \tilde{\Theta}_j \right) \geq \exp \left( -C_2 n \epsilon_n^2 \right)$$  \hspace{1cm} (2)

holds together with the testing conditions, then the variational posterior distribution $\hat{Q}$ converges to the true parameter with the rate $\epsilon_n^2$. In other words, the variational approximation error term in (1) is dominated under this stronger prior mass condition (2). Here, $D_\rho(\cdot \| \cdot)$
stands for a Rényi divergence with some $\rho > 1$. The implication of the condition (2) is important. It says that as long as the prior satisfies a “prior mass” condition that is coherent with the structure of the variational class, the resulted variational approximation error will always be small compared with the statistical error from the true posterior. Therefore, the condition (2) offers a practical guidance on how to choose a good prior for variational Bayes inference. In addition, as a condition only on the prior mass, (2) is usually very easy to check. This mathematical simplicity is not just for independent priors and the mean-field class. In Section 4, a more general condition is proposed that includes the setting of (2) as a special case.

Besides the general formulation of conditions to ensure convergence of the variational posteriors, several interesting aspects of variational Bayes inference are also rigorously discussed in the paper. We show that for a Gaussian sequence model with a sieve prior, its mean-field variational approximation of the posterior distribution has an explicit formula that resembles that of an empirical Bayes procedure studied by [26]. Moreover, we show that the empirical Bayes procedure is exactly a variational Bayes approximation using a specially designed variational class. This connection between empirical Bayes and variational Bayes is interesting, and may suggest similar theoretical properties of the two.

Finally, we would like to remark that the general rate (1) for variational posteriors is only an upper bound. It is not always true that the variational posterior has a slower convergence rate than the true posterior. Even though the variational posterior may not be a good approximation to the true posterior, it can still be much closer to the true parameter if extra regularity is given by the variational class $S$. We construct an example in Section 5.2 and show that in the case where the prior and the posterior undersmooth the data, the variational class $S$ helps to improve estimation by reducing the extra variance from undersmoothing. The practical implication of this example can be profound. Given a certain budget of computational resources, we suggest to take advantage of both the prior II and the variational class $S$ and cleverly distribute the prior knowledge between the two to achieve both statistical accuracy and computational efficiency.

Related Work. Statistical properties of variational posterior distributions have also been studied in the literature. A recent work by [33] established Bernstein-von Mises type of results for parametric models. We refer the readers to [6, 33] for other related references on theories for parametric variational Bayes inference. For nonparametric and high-dimensional models, recent work by [1, 34] studied variational approximation to tempered posteriors, where the likelihood $dP^{(n)}_{\theta}/dP^{(n)}_{0}$ is replaced by $(dP^{(n)}_{\theta}/dP^{(n)}_{0})^\alpha$ for some $\alpha \in (0, 1)$. Though the papers are interesting, these results do not apply to the usual posterior distributions with $\alpha = 1$. The most relevant paper to ours is [36], where the results cover both posterior distributions and their variational approximations. However, the conditions in [36] are rather abstract and are not easy to check in applications.
The rest of the paper is organized as follows. In Section 2, we formulate the problem and introduce the general conditions that characterize convergence rates of variational posteriors. This section also includes results for the mean-field variational class, where the variational approximation error can be explicitly analyzed. In Section 3, we apply our general theory to three examples that use three different variational classes. Then, in Section 4, for a general class of prior distributions and variational sets, we propose corresponding prior mass conditions that lead to automatic control of the variational approximation error. In Section 5, we discuss the relation between variational Bayes and empirical Bayes through an example of Gaussian sequence model. We also discuss possible situations where variational posterior outperforms the true one in that section. Finally, in Section 6, we give all the proofs of the paper.

Notations. We close this section by introducing notations that will be used later. For \( a, b \in \mathbb{R} \), let \( a \vee b = \max(a, b) \) and \( a \wedge b = \min(a, b) \). For a positive real number \( x \), \( \lceil x \rceil \) is the smallest integer no smaller than \( x \) and \( \lfloor x \rfloor \) is the largest integer no larger than \( x \). For two positive sequences \( \{a_n\} \) and \( \{b_n\} \), we write \( a_n \lesssim b_n \) or \( a_n = O(b_n) \) if \( a_n \leq Cb_n \) for all \( n \) with some constant \( C > 0 \) that does not depend on \( n \). The relation \( a_n \approx b_n \) holds if both \( a_n \lesssim b_n \) and \( b_n \lesssim a_n \) hold. For an integer \( m \), \( \lfloor m \rfloor \) denotes the set \( \{1, 2, ..., m\} \). Given a set \( S \), \( |S| \) denotes its cardinality, and \( 1_S \) is the associated indicator function. The \( \ell^p \) norm of a vector \( v \in \mathbb{R}^m \) with \( 1 \leq m \leq \infty \) is defined as \( \|v\|_p = \left( \sum_{j=1}^{m} \left| v_j \right|^p \right)^{1/p} \) for \( 1 \leq p < \infty \) and \( \|v\|_\infty = \sup_{1 \leq k \leq m} |v_k| \). Moreover, we use \( \|v\| \) to denote the \( \ell_2 \) norm \( \|v\|_2 \) by convention. For any function \( f \), the \( \ell_p \) norm is defined in a similar way, i.e. \( \|f\|_p = \left( \int |f|^p dx \right)^{1/p} \). Specifically, \( \|f\|_\infty = \sup_x |f(x)| \). The notation \( N(\delta, S, d) \) is used to denote the \( \delta \)-covering number of a set \( S \) under a metric \( d \). We use \( \mathbb{P} \) and \( \mathbb{E} \) to denote generic probability and expectation whose distribution is determined from the context. The notation \( \mathbb{P} f \) also means expectation of \( f \) under \( \mathbb{P} \) so that \( \mathbb{P} f = \int f d\mathbb{P} \). Throughout the paper, \( C, c \) and their variants denote generic constants that do not depend on \( n \). Their values may change from line to line.

2 Main Results

2.1 Definitions and Settings

We start this section by introducing a class of divergence functions.

Definition 2.1 (Rényi divergence). Let \( \rho > 0 \) and \( \rho \neq 1 \). The \( \rho \)-Rényi divergence between two probability measures \( P_1 \) and \( P_2 \) is defined as

\[
D_\rho(P_1||P_2) = \begin{cases} 
\frac{1}{\rho-1} \log \int \left( \frac{dP_1}{dP_2} \right)^{\rho-1} dP_1, & \text{if } P_1 \ll P_2, \\
+\infty, & \text{otherwise.}
\end{cases}
\]

The relations between the Rényi divergence and other divergence functions are summarized below.

4
1. When $\rho \to 1$, the Rényi divergence converges to the Kullback-Leibler divergence, defined as

$$D_1(P_1 \parallel P_2) = \begin{cases} \int \log \left( \frac{dP_1}{dP_2} \right) dP_1, & \text{if } P_1 \ll P_2, \\ +\infty, & \text{otherwise.} \end{cases}$$

From now on, we use $D(P_1 \parallel P_2)$ without the subscript to denote $D_1(P_1 \parallel P_2)$.

2. When $\rho = 1/2$, the Rényi divergence is related to the Hellinger distance by

$$D_{1/2}(P_1 \parallel P_2) = -2 \log(1 - H(P_1, P_2)^2),$$

and the Hellinger distance is defined as

$$H(P_1, P_2) = \sqrt{\frac{1}{2} \int (\sqrt{dP_1} - \sqrt{dP_2})^2}.$$

3. When $\rho = 2$, the Rényi divergence is related to the $\chi^2$-divergence by

$$D_2(P_1 \parallel P_2) = \log(1 + \chi^2(P_1 \parallel P_2)),$$

and the $\chi^2$-divergence is defined as

$$\chi^2(P_1 \parallel P_2) = \int \frac{(dP_1)^2}{dP_2} - 1.$$

**Definition 2.2 (total variation).** The total variation distance between two probability measures $P_1$ and $P_2$ is defined as

$$\text{TV}(P_1, P_2) = \frac{1}{2} \int |dP_1 - dP_2|.$$

The relation among the divergence functions defined above is given by the following proposition (see [31]).

**Proposition 2.1.** With the above definitions, the following inequalities hold,

$$(P_1, P_2)^2 \leq 2H(P_1, P_2)^2 \leq D_{1/2}(P_1 \parallel P_2) \leq D(P_1 \parallel P_2) \leq D_2(P_1 \parallel P_2) \leq \chi^2(P_1 \parallel P_2).$$

Moreover, the Rényi divergence $D_\rho(P_1 \parallel P_2)$ is a non-decreasing function of $\rho$.

Now we are ready to introduce the variational posterior distribution. Given a statistical model $P_\theta^{(n)}$ parametrized by $\theta$, and a prior distribution $\theta \sim \Pi$, the posterior distribution is defined by

$$d\Pi(\theta|X^{(n)}) = \frac{dP_\theta^{(n)}(X^{(n)})d\Pi(\theta)}{\int dP_\theta^{(n)}(X^{(n)})d\Pi(\theta)}.$$ 

To address possible computational difficulty of the posterior distribution, variational approximation is a way to find the closest object in a class $S$ of probability measures to $\Pi(\cdot|X^{(n)})$. The class $S$ is usually required to be computationally or analytically tractable. The most popular mathematical definition of variational approximation is given through the KL-divergence.
**Definition 2.3** (variational posterior). Let $S$ be a family of distributions. The variational posterior is defined as

$$
\hat{Q} = \arg\min_{Q \in S} D(Q \| \Pi(\cdot | X^{(n)})).
$$

(3)

Just like the posterior distribution $\Pi(\cdot | X^{(n)})$, the variational posterior $\hat{Q}$ is a data-dependent measure that summarizes information from both the prior and the data. Moreover, $\hat{Q}$ is usually influenced by the choice of variational set $S$. For a variational set $S$, the corresponding variational posterior can be regarded as the projection of the true posterior onto $S$ under KL-divergence. When $S$ is the set of all distributions, $\hat{Q}$ turns out to be the true posterior $\Pi(\cdot | X^{(n)})$. The complexity of the class $S$ usually determines the complexity of the optimization (3). In this paper, our main goal is to study the statistical property of the data-dependent measure $\hat{Q}$ for a general $S$.

### 2.2 Results for General Variational Posteriors

Assume the observation $X^{(n)}$ is generated from a probability measure $P^{(n)}_0$, and $\hat{Q}$ is the variational posterior distribution driven by $X^{(n)}$. The goal of this paper is to analyze $\hat{Q}$ from a frequentist perspective. In other words, we study statistical properties of $\hat{Q}$ under $P^{(n)}_0$.

Theorem 2.1. Suppose $\epsilon_n$ is a sequence that satisfies $n \epsilon_n^2 \geq 1$. Consider a loss function $L(\cdot, \cdot)$, such that for any two probability measures $P_1$ and $P_2$, $L(P_1, P_2) \geq 0$. Let $C, C_1, C_2, C_3 > 0$ be constants such that $C > C_2 + C_3 + 2$. We assume

- For any $\epsilon > \epsilon_n$, there exists a set $\Theta_n(\epsilon)$ and a testing function $\phi_n$, such that
  $$
P^{(n)}_0 \phi_n + \sup_{\theta \in \Theta_n(\epsilon)} P^{(n)}_\theta (1 - \phi_n) \leq \exp(-Cn\epsilon^2).
  $$
  (A1)

- For any $\epsilon > \epsilon_n$, the set $\Theta_n(\epsilon)$ above satisfies
  $$
  \Pi(\Theta_n(\epsilon)^c) \leq \exp(-Cn\epsilon^2).
  $$
  (A2)

- For some constant $\rho > 1$,
  $$
P \left(D_{\rho}(P^{(n)} \| P^{(n)}_\theta) \leq C_3 n \epsilon_n^2\right) \geq \exp(-C_2 n \epsilon_n^2).
  $$
  (A3)

Then for the variational posterior $\hat{Q}$ defined in (3), we have

$$
P^{(n)}_0 \hat{Q} L(P^{(n)}_\theta, P^{(n)}_0) \leq Mn(\epsilon_n^2 + \gamma_n^2),
$$

(4)

for some constant $M$ only depending on $C_1, C$ and $\rho$, where the quantity $\gamma_n^2$ is defined as

$$
\gamma_n^2 = \frac{1}{n} \inf_{Q \in S} P^{(n)}_0 D(Q \| \Pi(\cdot | X^{(n)})).
$$
Conditions (A1)-(A3) resemble the three conditions of “prior mass and testing” in [14]. Interestingly, Theorem 2.1 shows that with a slight modification, these three conditions also lead to the convergence of the variational posterior. The testing conditions (A1) and (A2) are required to hold for all $\epsilon > \epsilon_n$. In the prior mass condition (A3), the neighborhood of $P_0^{(n)}$ is defined through a Rènyi divergence with a $\rho > 1$, compared with the KL-divergence used in [14]. According to Proposition 2.1, $D_\rho(P_1^n\|P_2^n) \geq D(P_1^n\|P_2^n)$ for $\rho > 1$, so the condition (A3) in our paper is slightly stronger than that in [14]. This stronger “prior mass” condition ensures that the loss $L(P_\theta^{(n)}, P_0^{(n)})$ is exponentially integrable under the true posterior $\Pi(\cdot|X^{(n)})$, which is a key step in the proof of Theorem 2.1. In all the examples considered in this paper, we will check (A3) with $D_2(P_0^{(n)}\|P_\theta^{(n)})$, which turns out to be a very convenient choice.

The convergence rate is the sum of two terms, $\epsilon_n^2$ and $\gamma_n^2$. The first term $\epsilon_n^2$ is the convergence rate of the true posterior $\Pi(\cdot|X^{(n)})$. The second term $\gamma_n^2$ characterizes the approximation error given by the variational set $\mathcal{S}$. A larger $\mathcal{S}$ means more expressive power given by the variational approximation, and thus the rate of $\gamma_n^2$ is smaller. When $\mathcal{S}$ is the set of all probability distributions, $\gamma_n^2$ is zero, and Theorem 2.1 recovers the classical posterior convergence result in [14].

It is worth mentioning that we characterize the convergence of the variational posterior $\hat{Q}$ through the expected loss $P_0^{(n)}\hat{Q}L(P_\theta^{(n)}, P_0^{(n)})$. This criterion is also used in previous work on theoretical analysis of posterior convergence in [9]. It is more convenient to work with this criterion using our proof technique. Moreover, convergence in $P_0^{(n)}\hat{Q}L(P_\theta^{(n)}, P_0^{(n)})$ also implies that the entire variational posterior distribution concentrates in a neighborhood of the true distribution $P_0^{(n)}$ with a radius of the same rate. When the loss function is convex, it also implies the existence of a point estimator that enjoys the same convergence rate. We summarize these results in the next corollary.

**Corollary 2.1.** Under the same setting of Theorem 2.1, for any diverging sequence $M_n \to \infty$, we have

$$P_0^{(n)}\hat{Q}\left(L(P_\theta^{(n)}, P_0^{(n)}) > M_n n(\epsilon_n^2 + \gamma_n^2)\right) \to 0.$$  

Furthermore, if the loss $L(P_\theta^{(n)}, P_0^{(n)})$ is convex respect to $\theta$, then the variational posterior mean $\tilde{\Theta} = \hat{Q}\theta$ satisfies

$$P_0^{(n)}L(P_\tilde{\Theta}^{(n)}, P_0^{(n)}) \leq M n(\epsilon_n^2 + \gamma_n^2),$$

where $M$ is the same constant in (4).

**Proof.** The first result is an application of Markov’s inequality

$$P_0^{(n)}\hat{Q}\left(L(P_\theta^{(n)}, P_0^{(n)}) > M_n n(\epsilon_n^2 + \gamma_n^2)\right) \leq \frac{P_0^{(n)}\hat{Q}L(P_\theta^{(n)}, P_0^{(n)})}{M_n n(\epsilon_n^2 + \gamma_n^2)} \leq \frac{M}{M_n} \to 0.$$  

The second result is directly implied by Jensen’s inequality that

$$P_0^{(n)}L(P_\tilde{\Theta}^{(n)}, P_0^{(n)}) \leq P_0^{(n)}\hat{Q}L(P_\theta^{(n)}, P_0^{(n)}) \leq M n(\epsilon_n^2 + \gamma_n^2).$$
To apply Theorem 2.1 to specific problems, we need to analyze the variational approximation error \( \gamma_n^2 = \frac{1}{n} \inf_{Q \in \mathcal{S}} P_0^{(n)} D(Q\|\Pi(L|X^{(n)})) \) in each individual setting. However, this task may not be trivial for many problems. Now we borrow a technique in [36] to get a useful upper bound for \( \gamma_n^2 \). For any \( Q \in \mathcal{S} \), we have

\[
\begin{align*}
n \gamma_n^2 & \leq P_0^{(n)} D(Q\|\Pi(L|X^{(n)})) = D(Q\|\Pi) + Q \left[ \int \log \left( \frac{dP_0^{(n)}}{dP_{\theta}^{(n)}} \right) dP_0^{(n)} \right] \\
& = D(Q\|\Pi) + Q \left[ D(P_0^{(n)}\|P_{\theta}^{(n)}) - D(P_0^{(n)}\|P_{H}^{(n)}) \right] \\
& \leq D(Q\|\Pi) + Q \left[ D(P_0^{(n)}\|P_{\theta}^{(n)}) \right],
\end{align*}
\]

where \( P_{H}^{(n)} = \int P_{\theta}^{(n)} d\Pi(\theta) \). Then, we obtain the upper bound

\[
\gamma_n^2 \leq \inf_{Q \in \mathcal{S}} R(Q),
\]

where

\[
R(Q) = \frac{1}{n} \left( D(Q\|\Pi) + Q \left[ D(P_0^{(n)}\|P_{\theta}^{(n)}) \right] \right).
\]

(5)

Now, it is easy to see that a sufficient condition for the variational posterior to converge at the same rate as the true posterior is

\[
\inf_{Q \in \mathcal{S}} R(Q) \lesssim \epsilon_n^2.
\]

(A4)

We incorporate this condition into the next theorem.

**Theorem 2.2.** Suppose \( \epsilon_n \) is a sequence that satisfies \( n\epsilon_n^2 \geq 1 \), for which the conditions (A1), (A2), (A3), (A4) hold. Then, for the variational posterior \( \hat{Q} \) that is defined in (3), we have

\[
P_0^{(n)} \hat{Q} L(P_0^{(n)}, P_0^{(n)}) \lesssim n\epsilon_n^2.
\]

(6)

We would like to remark that the quantity \( \inf_{Q \in \mathcal{S}} R(Q) \) is easier to analyze compared with the original definition of \( \gamma_n^2 \). According to its definition given by (5), it is sufficient to find a distribution \( Q \in \mathcal{S} \), such that

\[
D(Q\|\Pi) \lesssim n\epsilon_n^2 \quad \text{and} \quad Q \left[ D(P_0^{(n)}\|P_{\theta}^{(n)}) \right] \lesssim n\epsilon_n^2.
\]

One way to construct such a distribution \( Q \) that satisfies the above two inequalities is to focus on those whose supports are within the set \( C = \{ \theta : D(P_0^{(n)}\|P_{\theta}^{(n)}) \leq C n\epsilon_n^2 \} \) for some constant \( C > 0 \). We summarize this method into the following theorem.

**Theorem 2.3.** Suppose there exist constants \( C_1, C_2 > 0 \), such that

\[
\inf_{Q \in \mathcal{S} \cap \mathcal{E}} D(Q\|\Pi) \leq C_1 n\epsilon_n^2, \quad \text{(A4*)}
\]

where \( \mathcal{E} = \{ Q : \text{supp}(Q) \subseteq C \} \) with \( C = \{ \theta : D(P_0^{(n)}\|P_{\theta}^{(n)}) \leq C_2 n\epsilon_n^2 \} \). Then, we have

\[
\inf_{Q \in \mathcal{S}} R(Q) \leq (C_1 + C_2)\epsilon_n^2.
\]
2.3 Results for Mean-Field Variational Posteriors

A special choice of \( S \) is the mean-field class of distributions. Not only does this class lead to computationally efficient algorithms such as coordinate ascent, but in this section, we will also show that the structure of this class leads to a convenient convergence analysis. We begin with its definition.

**Definition 2.4** (mean-field class). For parameters in a product space that can be written as \( \theta = (\theta_1, \theta_2, ..., \theta_m) \) with some \( 1 \leq m \leq \infty \), the mean-field variational family is defined as

\[
S_{MF} = \left\{ Q : dQ(\theta) = \prod_{j=1}^{m} dQ_j(\theta_j) \right\}.
\]

The following theorem can be viewed as an application of Theorem 2.3 to the mean-field class.

**Theorem 2.4.** Suppose there exists a \( \tilde{Q} \in S_{MF} \) and a subset \( \otimes_{j=1}^{m} \tilde{\Theta}_j \), such that

\[
\otimes_{j=1}^{m} \tilde{\Theta}_j \subset \left\{ \theta : D(P_0^{(n)}||P_\theta^{(n)}) \leq C_1 n\epsilon_n^2, \quad \log \frac{d\tilde{Q}(\theta)}{d\Pi(\theta)} \leq C_2 n\epsilon_n^2 \right\}, \tag{7}
\]

and

\[
-\sum_{j=1}^{m} \log \tilde{Q}_j(\tilde{\Theta}_j) \leq C_3 n\epsilon_n^2, \tag{8}
\]

for some constants \( C_1, C_2, C_3 > 0 \). Then, we have

\[
\inf_{Q \in S_{MF}} R(Q) \leq (C_1 + C_2 + C_3)\epsilon_n^2.
\]

Note that the condition (8) can also be written as

\[
\tilde{Q} \left( \otimes_{j=1}^{m} \tilde{\Theta}_j \right) \geq \exp \left( -C_3 n\epsilon_n^2 \right).
\]

In other words, Theorem 2.4 gives an interesting “distribution mass” type of characterization for \( \inf_{Q \in S} R(Q) \). Checking (8) is very similar to checking the “prior mass” condition (A3), and is usually not hard in many examples. We only need to make sure that \( \tilde{Q} \) is not too far away from the prior \( \Pi \) in the sense of (7). In fact, if the prior \( \Pi \) belongs to the class \( S_{MF} \), then one can take \( \tilde{Q} = \Pi \), and the conditions of Theorem 2.4 are simply a “prior mass” condition \( \Pi \left( \otimes_{j=1}^{m} \tilde{\Theta}_j \right) \geq \exp \left( -C_3 n\epsilon_n^2 \right) \), with the choice of \( \otimes_{j=1}^{m} \tilde{\Theta}_j \) being a subset of the KL-neighborhood \( \left\{ \theta : D(P_0^{(n)}||P_\theta^{(n)}) \leq C_1 n\epsilon_n^2 \right\} \). A more general characterization of the variational approximation error through a prior mass condition will be studied Section 4.
3 Applications

In this section, we consider several examples to illustrate the theory developed in Section 2. We use a different variational set for every example. The results are summarized by the following table.

| models                                | variational sets | results                      |
|---------------------------------------|------------------|------------------------------|
| Gaussian sequence model               | mean-field $S_{\text{MF}}$ | Theorems 3.1 and 3.2         |
| infinite dimensional exponential families | Gaussian mean-field $S_{\text{G}}$ | Theorem 3.3                  |
| piecewise constant model              | Markov chain $S_{\text{MC}}$ | Theorem 3.5                  |

Table 1: Summary of results in Section 3.

3.1 Gaussian Sequence Model

In this section, we illustrate the application of our main results for the mean-field class $S_{\text{MF}}$. Consider observations generated by a Gaussian sequence model,

$$Y_j = \theta_j + \frac{1}{\sqrt{n}} Z_j, \quad Z_j \overset{i.i.d}{\sim} N(0, 1), \quad j \geq 1. \quad (9)$$

We use the notation $P_{\theta}^{(n)} = \otimes_j N(\theta_j, n^{-1})$ for the distribution above. Our goal is to use variational Bayes methods to estimate $\theta^*$ that belongs to the following Sobolev ball,

$$\Theta_\alpha(B) = \left\{ \theta = (\theta_j)_{j=1}^\infty : \sum_{j=1}^\infty j^{2\alpha} \theta_j^2 \leq B^2 \right\}. \quad (10)$$

Here, the smoothness $\alpha > 0$ and the radius $B > 0$ are considered as constants throughout the paper.

The prior distribution $\theta \sim \Pi$ is described through the following sampling process.

1. Sample $k \sim \pi$;
2. Conditioning on $k$, sample $\theta_j \sim f_j$ for all $j \in [k]$, and set $\theta_j = 0$ for all $j > k$.

In other words, the prior on $\theta$ is a mixture of product measures,

$$d\Pi(\theta) = \sum_{k=1}^\infty \pi(k) \prod_{j=1}^k f_j(\theta_j) \prod_{j>k} \delta_0(\theta_j) d\theta. \quad (11)$$

Priors of similar forms are also considered in [25, 10, 11, 26]. Direct calculation implies that the posterior is also in the form of a mixture of product measures.

The loss function for this problem is $L(P_{\theta}^{(n)}, P_{\theta^*}^{(n)}) = n ||\theta - \theta^*||^2$, which is a natural choice for the Gaussian sequence model.

Consider the variational posterior $\hat{Q}$ defined by (3) with $S = S_{\text{MF}}$. That is, we seek a data-dependent measure in a more tractable form of a product measure. We will show that
even though the posterior itself is not a product measure, using $\hat{Q}$ from the mean-field class still gives us a rate-optimal contraction result.

The conditions on the prior distributions are summarized below.

- There exist some constants $C_1, C_2 > 0$ such that
  \[
  \sum_{j=k}^{\infty} \pi(j) \leq C_1 \exp(-C_2 k), \text{ for all } k.
  \]  
  (12)

- There exist some constants $C_3, C_4 > 0$ such that for $k_0 = \left\lceil \left( \frac{n}{\log n} \right)^{\frac{1}{2\alpha + 1}} \right\rceil$,
  \[
  \pi(k_0) \geq C_3 \exp(-C_4 k_0 \log k_0).
  \]  
  (13)

- For the $k_0$ defined above, there exist some constants $c_0 \in \mathbb{R}$ and $c_1 > 0$ such that
  \[
  - \log f_j(x) \leq c_0 + c_1 j^{2\alpha + 1} x^2, \quad \text{for all } j \leq k_0 \text{ and } x \in \mathbb{R}.
  \]  
  (14)

These three conditions on $\Pi$ include a large class of prior distributions. We remark that even though (14) involves $\alpha$, it does not mean that one needs to know $\alpha$ when defining the prior $\Pi$. For example, the choice that $\pi(k) \propto e^{-\tau k}$ and $f_j$ being $N(0, \sigma^2)$ for some constants $\tau, \sigma^2 > 0$ easily satisfies all the three conditions (12)-(14).

Conditions (12)-(14) will be used to derive the four conditions in Theorem 2.2. To be specific, (A1) and (A2) are consequences of (12) (see Lemma 6.5), and (A3) and (A4) can be derived from (13) and (14) (see Lemma 6.6). Then, by Theorem 2.2, we obtain the following result.

**Theorem 3.1.** Consider the prior $\Pi$ that satisfies (12)-(14). Then, for any $\theta^* \in \Theta_\alpha(B)$, we have
\[
P_{\theta^*}^{(n)} \hat{Q} \|\theta - \theta^*\|^2 \lesssim n^{-\frac{2\alpha}{2\alpha + 1}} (\log n)^{\frac{2\alpha}{2\alpha + 1}},
\]
where $\hat{Q}$ is the variational posterior defined by (3) with $S = S_{\text{MF}}$.

It is well known that the minimax rate of estimating $\theta^*$ in $\Theta_\alpha(B)$ is $n^{-\frac{2\alpha}{2\alpha + 1}}$ [16]. Using a mean-field variational posterior, we achieve the minimax rate up to a logarithmic factor. In fact, the following proposition demonstrates that this rate cannot be improved without any modification of the conditions on the prior.

**Proposition 3.1.** Consider the prior $\Pi$ specified in (11). Assume that $\max_j \|f_j\|_\infty \leq a$ and $\pi(k)$ is nonincreasing over $k$. Then, we have
\[
\sup_{\theta^* \in \Theta_\alpha(B)} P_{\theta^*}^{(n)} \hat{Q} \|\theta - \theta^*\|^2 \gtrsim n^{-\frac{2\alpha}{2\alpha + 1}} (\log n)^{\frac{2\alpha}{2\alpha + 1}},
\]
where $\hat{Q}$ is the variational posterior defined by (3) with $S = S_{\text{MF}}$. 

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On the other hand, the extra logarithmic factor can actually be removed by a rescaling of the prior. Thus, the extra logarithmic factor is caused by the choice of the prior instead of the proof technique. Consider $\Pi$ that has the following sampling process.

1. Sample $k \sim \pi$;
2. Conditioning on $k$, sample $\sqrt{n}\theta_j \sim g_j$ for all $j \in [k]$, and set $\theta_j = 0$ for all $j > k$.

Obviously, this prior is the same as the previous one because $f_j(x) = \sqrt{n}g_j(\sqrt{n}x)$. However, the $\sqrt{n}$-scaling allows us to formulate conditions that help remove the logarithmic factor in Theorem 3.1. The same rescaling is also used in [10, 11] to achieve sharp minimax rates. The following two conditions will be used to replace (13) and (14).

- There exist some constants $C_3, C_4 > 0$ such that for $k_0 = \lceil n^{1/2\alpha + 1} \rceil$,
  $$\pi(k_0) \geq C_3 \exp(-C_4 k_0).$$  (15)

- For the $k_0$ defined above, there exist constants $c_0 \in \mathbb{R}, c_1 > 0$ and $0 < \beta < \frac{2}{2\alpha + 1}$, such that
  $$-\log g_j(x) \leq c_0 + c_1 |x|^\beta,$$
  for all $j \leq k_0$ and $x \in \mathbb{R}$.  (16)

The condition (15) is similar to (13), while (16) is stronger compared with (14). In general, one can choose $g_j$ to be a density with a heavy tail. As an example, one can easily check that $\pi(k) \propto e^{-\tau k}$ and $g_j(x) = \frac{1}{\pi\sigma(1+(x/\sigma)^2)}$ with constants $\tau, \sigma^2 > 0$ satisfy the two conditions. Conditions (A3) and (A4) can be derived from (15) and (16) (see Lemma 6.7). This leads to the following result.

**Theorem 3.2.** Consider the prior $\Pi$ that satisfies (12), (15) and (16). Then, for any $\theta^* \in \Theta_\alpha(B)$, we have

$$P^{(n)}_{\hat{Q}}\| \theta - \theta^* \|^2 \lesssim n^{-\frac{2\alpha}{2\alpha + 1}},$$

where $\hat{Q}$ is the variational posterior defined by (3) with $S = S_{\text{MF}}$.

Theorem 3.2 is interesting. It shows that the mean-field variational posterior $\hat{Q}$ is able to achieve the sharp minimax rate even though the true posterior distribution is not a product measure. In order to further understand the data-dependent measure $\hat{Q}$, we show in Section 5.1 that $\hat{Q}$ enjoys a very similar form to the empirical Bayes posterior that is studied by [26]. This surprising discovery explains the good property of $\hat{Q}$ from a different perspective. Details of the discussion are referred to Section 5.1.

### 3.2 Infinite Dimensional Exponential Families

In this section, we study another interesting variational family. The Gaussian mean-field family is defined as

$$S_G = \{ Q = \otimes_j N(\mu_j, \sigma_j^2) : \mu_j \in \mathbb{R}, \sigma_j^2 \geq 0 \}.  \quad (17)$$
This class offers better interpretability of the results because every distribution in $S_G$ is fully determined by a sequence of mean and variance parameters. Note that we allow $\sigma_j^2$ to be zero and $N(\mu_j, 0)$ is understood as the delta measure $\delta_{\mu_j}$ on $\mu_j$.

The application of $S_G$ is illustrated by an infinite dimensional exponential family model. We define the probability measure $P_\theta$ by

$$
\frac{dP_\theta}{d\ell} = \exp \left( \sum_{j=0}^{\infty} \theta_j \phi_j - c(\theta) \right),
$$

where $\ell$ denotes the Lebesgue measure on $[0, 1]$, $\phi_j$ is the $j$th Fourier basis function of $L^2[0, 1]$, and $c(\theta)$ is given by

$$
c(\theta) = \log \int_0^1 \exp \left( \sum_{j=0}^{\infty} \theta_j \phi_j(x) \right) dx.
$$

Since $\phi_0(x) = 1$ and $\theta_0$ can take arbitrary values without changing $P_\theta$, we simply set $\theta_0 = 0$. In other words, $P_\theta$ is fully parameterized by $\theta = (\theta_1, \theta_2, \ldots)$. Given i.i.d. observations from $P^*_n$, our goal is to estimate $P^{*}_\theta$, where $\theta^*$ is assumed to belong to the Sobolev ball $\Theta^{\alpha}(B)$ defined in (10). The loss function is chosen as the squared Hellinger distance $L(P^*_n, P^*_\theta) = nH^2(P_\theta, P^{*}_\theta)$.

We consider a prior distribution $\Pi$ that is similar to the one used in Section 3.1. Its sampling process is described as follows.

1. Sample $k \sim \pi$;

2. Conditioning on $k$, sample $\theta_j \sim f_j$ for all $j \in [k]$, and set $\theta_j = 0$ for all $j > k$.

We impose the following conditions on the prior $\Pi$.

- There exist some constants $C_1, C_2 > 0$ such that

$$
\sum_{j=k}^{\infty} \pi(j) \leq C_1 \exp(-C_2 k \log k), \text{ for all } k.
$$

- There exist some constants $C_3, C_4 > 0$ such that for $k_0 = \left\lceil \left( \frac{n}{\log n} \right)^{\frac{1}{2\alpha+1}} \right\rceil$

$$
\pi(k_0) \geq C_3 \exp(-C_4 k_0 \log k_0).
$$

- There exist some constants $c_0 \in \mathbb{R}$ and $c_1, \beta > 0$ such that

$$
-\log f_j(x) \geq c_0 + c_1 |x|^\beta,
$$

for all $x \in \mathbb{R}$ and $j \in [k_0]$ with $k_0$ defined above.

- For the $k_0$ defined above, there exist some constants $c'_0 \in \mathbb{R}$ and $c'_1 > 0$ such that

$$
-\log f_j(x) \leq c'_0 + c'_1 j^{2\alpha+1} x^2,
$$

for all $j \leq k_0$ and $x \in \mathbb{R}$.
The conditions (19)-(22) are satisfied by a large class of prior distributions. For example, one can choose \( k \sim \text{Poisson}(\tau) \) and \( f_j \) being the density of \( N(0, \sigma^2) \) for some constants \( \tau, \sigma^2 > 0 \), and then the four conditions are easily satisfied.

In order to apply Theorem 2.2, it is sufficient to check the conditions (A1)-(A4). We show that (A1) and (A2) are implied by (19) and (21) (see Lemma 6.8), and (A3) and (A4) can be derived from (20) and (22) (see Lemma 6.9). Thus, we have the following result.

**Theorem 3.3.** Consider the prior \( \Pi \) that satisfies (19)-(22). Then, for any \( \theta^* \in \Theta_{\alpha}(B) \) with some \( \alpha > 1/2 \), we have

\[
P^n_{\theta^*} \hat{Q} H^2(P_\theta, P_{\theta^*}) \lesssim n^{-\frac{2\alpha}{2\alpha+1}} (\log n)^{\frac{2\alpha}{2\alpha+1}},
\]

where \( \hat{Q} \) is the variational posterior defined by (3) with \( S = S_G \).

The theorem shows that the Gaussian mean-field variational posterior is able to achieve the minimax rate \( n^{-\frac{2\alpha}{2\alpha+1}} \) up to a logarithmic factor. We remark that the same result also holds for the mean-field variational posterior defined with \( S_{\text{MF}} \). This is because \( S_G \subset S_{\text{MF}} \), and thus \( \inf_{Q \in S_{\text{MF}}} R(Q) \leq \inf_{Q \in S_G} R(Q) \).

One may wonder if the extra logarithmic factor can be removed by using the same rescaling technique in Section 3.1. Unfortunately, this idea does not work for the model here. However, if we consider the mean-field variational class \( S_{\text{MF}} \), then it will be possible to achieve the sharp minimax rate by using an \( \ell_1 \)-modification technique in [10] without the extra logarithmic factor.

### 3.3 Piecewise Constant Model

The previous two sections consider examples of the mean-field variational set and its variant. In this section, we use another example to illustrate a situation where the mean-field variational set only gives a trivial rate. On the other hand, we show that an alternative variational class with an appropriate dependence structure is able to achieve the optimal rate.

We consider the following piecewise constant model,

\[
X_i = \theta_i + \sigma Z_i, \quad i \in [n],
\]

where \( Z_i \sim N(0, 1) \) independently for all \( i \in [n] \). The true parameter \( \theta^* \) is assumed to belong to the class \( \Theta_{\alpha}(B) = \{ \theta \in \Theta_{\alpha_*} : \|\theta\|_\infty \leq B \} \), where for a general \( k \in [n] \),

\[
\Theta_k = \left\{ \theta \in \mathbb{R}^n : \text{there exist } \{a_j\}^k_{j=0} \text{ and } \{\mu_j\}^k_{j=1} \text{ such that } 0 = a_0 \leq a_1 \leq \cdots \leq a_k = n, \text{ and } \theta_j = \mu_j \text{ for all } i \in (a_{j-1} : a_j) \right\}.
\]

Here for any two integers \( a < b \), we use \( (a : b) \) to denote all integers from \( a + 1 \) to \( b \). We assume both \( B > 0 \) and \( \sigma^2 > 0 \) are constants throughout this section. A vector \( \theta^* \in \Theta_{\alpha_*}(B) \)
is a piecewise constant signal with at most $k^*$ pieces. We use $P_{\theta}^{(n)}$ to denote the probability distribution of $N(\theta, \sigma^2 I_n)$ in this section.

The piecewise constant model is widely studied in the literature of change-point analysis. Recently, the minimax rate of the class $\Theta_{k^*}$ is derived by [12]. When $2 < k^* \leq n^{1-\delta}$ for some constant $\delta \in (0, 1)$, the minimax rate is $\inf_{\hat{\theta}} \sup_{\theta^*} \mathbb{E}_{P_{\theta}^{(n)}} \|\hat{\theta} - \theta^*\|^2 \approx k^* \log n$. With an extra constraint on the infinity norm, the minimax rate for $\Theta_{k^*}(B)$ is still $k^* \log n$, with a slight modification of the proof in [12]. Since $D_\rho(P_{\theta}^{(n)}, P_{\theta'}^{(n)}) = \frac{\rho^2}{2 \sigma^2} \|\theta - \theta'\|^2$ in this case, it is natural to choose the loss function as $L(P_{\theta}^{(n)}, P_{\theta'}^{(n)}) = \|\theta - \theta'\|^2$.

We put a prior distribution $\Pi$ on the parameter $\theta$. Consider $\Pi$ that has the following sampling process.

1. Sample $k \sim \pi$;
2. Conditioning on $k$, sample $k - 1$ change points uniformly from $\{2, 3, \ldots, n\}$. In other words, we uniformly sample a subset $A \subset \{2, 3, \ldots, n\}$ of size $k - 1$ with probability $(\frac{n - 1}{k - 1})^{-1}$;
3. Conditioning on $A$, sample $\theta_i$ according to $\theta_i \sim g_i$ for all $i \in A$, and $\theta_i = \theta_{i-1}$ for all $i \notin A$.

We first consider variational inference via the mean-field class. Define $\hat{Q}_{MF}$ by (3) with $S = S_{MF}$. Interestingly, for the piecewise constant model, $\hat{Q}_{MF}$ only gives a trivial rate.

**Theorem 3.4.** For the prior $\Pi$ specified above with any $g_i$’s absolutely continuous with respect to the Lebesgue measure, we have

$$\sup_{\theta^* \in \Theta_{k^*}(B)} P_{\theta^*}^{(n)} \hat{Q}_{MF} \|\theta - \theta^*\|^2 \gtrsim n,$$

for any $k^* \in [n]$, where $\hat{Q}_{MF}$ is the variational posterior defined by (3) with $S = S_{MF}$.

The result of Theorem 3.4 shows that the mean-field variational posterior $\hat{Q}_{MF}$ is unable to achieve a better rate than simply estimating $\theta^*$ by the naive estimator $\hat{\theta} = X$. The proof in Section 6.5 reveals the reason of this phenomenon. Since the independence structure of the class $S_{MF}$ fails to capture the underlying dependence structure of the parameter space $\Theta_{k^*}(B)$, the variational posterior distribution is equivalent to the posterior distribution induced by the prior $\Pi = \otimes_{i=1}^n g_i$.

In order to achieve the minimax rate of the space $\Theta_{k^*}(B)$, it is necessary to introduce some dependence structure in the variational class. One of the simplest classes of dependent distributions is the class of first-order Markov chains, defined by

$$S_{MC} = \left\{ Q : dQ(\theta) = dQ_1(\theta_1) \prod_{j=2}^n Q_j(\theta_j|\theta_{j-1}) \right\}.$$  

The class $S_{MC}$ introduces a natural dependence structure for the piecewise constant model, and it is compatible with the prior distribution $\Pi$, because conditioning on the change point

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pattern A, the prior distribution of \( \theta | A \) belongs to the class \( S_{MC} \). We define the variational posterior \( \hat{Q}_{MC} \) by (3) with \( S = S_{MC} \). In order to derive the rate for \( \hat{Q}_{MC} \), we impose the following conditions on the prior distribution \( \Pi \).

- There exist some constants \( C_1, C_2 > 0 \) such that
  \[
  \sum_{j=k}^{n} \pi(j) \leq C_1 n^{-C_2 k},
  \]
  for all \( k \geq 2 \).

- There exists some constants \( C_3, C_4 > 0 \) such that
  \[
  \pi(k^*) \geq C_3 n^{-C_4 k^*}.
  \]

- There exists a constant \( c > 0 \) such that
  \[
  g_i(x) \geq c, \text{ for all } i \in [n] \text{ and } |x| \leq B + 1.
  \]

A typical example of \( \pi \) that satisfies both (25) and (26) is \( \pi(k) \propto n^{-k} \). The condition (27) is very general, and one can basically choose any reasonable density \( g_j \) that has positive value on the interval \([-B-1, B+1]\). The condition (25) is used to derive (A1) and (A2) in Theorem 2.2 (see Lemma 6.12), while the conditions (26) and (27) are used to derive (A3) and (A4) (see Lemma 6.13). According to Theorem 2.2, we get the following result.

**Theorem 3.5.** Consider a prior distribution \( \Pi \) that satisfies (25)-(27). Then, for any \( \theta^* \in \Theta_k^*(B) \), we have

\[
P_{\theta^*} \hat{Q}_{MC} \| \theta - \theta^* \|^2 \lesssim k^* \log n,
\]

where \( \hat{Q}_{MC} \) is the variational posterior defined by (3) with \( S = S_{MC} \).

Theorem 3.5 shows that \( \hat{Q}_{MC} \) is able to achieve the minimax rate of the problem. This example illustrates the importance of the choice of the variational class. According to Theorem 2.1, the rate of a variational posterior is upper bounded by \( \epsilon_{n}^2 \), the rate of the true posterior, plus \( \gamma_{n}^2 \), the variational approximation error. The choice of \( S_{MF} \) for the piecewise constant model leads to a very large \( \gamma_{n}^2 \), and thus a trivial rate in Theorem 3.4. On the other hand, the variational approximation error given by the class \( S_{MC} \) is small, which is dominated by the minimax rate.

### 4 The Generalized Mean-Field Class

#### 4.1 General Settings

In this section, we consider a general form of probability models

\[
\mathcal{M} = \left\{ P_{k,\theta}^{(n)} : k \in K, \theta^{(k)} \in \Theta^{(k)} \right\}.
\]
Here, the probability $P_{k,\theta(k)}^{(n)}$ is determined by an index $k$ and a parameter $\theta^{(k)}$. We assume that the set $K$ is either countable or finite. For a given $k$, the probability $P_{k,\theta(k)}^{(n)}$ is parametrized by a $\theta^{(k)}$ in a parameter space $\Theta^{(k)}$ that is indexed by this $k$. Without loss of generality, we assume that the parameter $\theta^{(k)}$ can be written in a blockwise structure

$$\theta^{(k)} = (\theta_1^{(k)}, \ldots, \theta_{m_k}^{(k)}).$$

Note that the dimension of $\theta^{(k)}$ may vary with $k$.

The model $M$ is very natural for many applications. One can think of $k$ as a model dimension index, which determines the complexity of the parameter space $\Theta^{(k)}$. A leading example is the mixture density model, where $k$ stands for the number of components.

To model the hierarchical structure of $(k,\theta^{(k)})$, one naturally uses a hierarchical prior distribution, which is specified through the following sampling process:

1. Firstly, sample $k \sim \pi$ from $K$;
2. Conditioning on $k$, sample $\theta^{(k)}$ from the probability measure $\widetilde{\Pi}^{(k)}$, and $\widetilde{\Pi}^{(k)}$ has a product structure
   $$d\widetilde{\Pi}^{(k)}(\theta^{(k)}) = \prod_{j=1}^{m_k} d\widetilde{\Pi}_j^{(k)}(\theta_j^{(k)});$$
   (28)
3. With $k$ and $\theta^{(k)}$ sampled from previous steps, we obtain $P_{k,\theta^{(k)}}^{(n)}$, and we use $\Pi$ to denote the prior distribution that $P_{k,\theta^{(k)}}^{(n)} \sim \Pi$.

For variational inference, we consider a generalized mean-field class that naturally takes advantage of the structure of the prior distribution. For a given $k \in K$, the corresponding mean-field class is defined as

$$S_{\text{MF}}^{(k)} = \left\{ Q^{(k)} : Q^{(k)} \text{ is the pushforward of some } \widetilde{Q}^{(k)} \in S_{\text{MF}}^{(k)} \text{ induced by the map } \theta^{(k)} \rightarrow P_{k,\theta^{(k)}}^{(n)} \right\},$$

where

$$S_{\text{MF}}^{(k)} = \left\{ \widetilde{Q}^{(k)} : d\widetilde{Q}^{(k)}(\theta^{(k)}) = \prod_{j=1}^{m_k} d\widetilde{Q}_j^{(k)}(\theta_j^{(k)}) \right\}. \quad (29)$$

Then, the generalized mean-field class is defined as

$$S_{\text{GMF}} = \bigcup_{k \in K} S_{\text{MF}}^{(k)}. \quad (30)$$

Since the generalized mean-field class is a union of mean-field classes over $k \in K$, the variational posterior $\hat{Q}$ defined by (3) with $S = S_{\text{GMF}}$ is in the form of $\prod_{j=1}^{m_k} d\widetilde{Q}_j^{(k)}(\theta_j^{(k)})$ with a data-driven $\hat{k}$ determined by the optimization (3). A special case is when $K$ is a singleton set. Then, the prior distribution is a product measure and the generalized mean-field class is reduced to the mean-field class.
4.2 Results for the Generalized Mean-Field Class

Assume the observation \( X^{(n)} \) is generated from a probability measure \( P_0^{(n)} \), and \( \hat{Q} \) is the variational posterior driven by \( X^{(n)} \). For the general settings described above, we show that the variational approximation error can be automatically controlled by a prior mass condition.

**Theorem 4.1.** Suppose \( \epsilon_n \) is a sequence that satisfies \( n\epsilon_n^2 \geq 1 \). Let \( \rho > 1 \) be a constant and \( C_2, C_3 > 0 \) be constants. We assume that there exists a \( k_0 \in K \) and a subset \( \tilde{\Theta}^{(k_0)} = \bigotimes_j \tilde{\Theta}_j^{(k_0)} \subset \{ \theta^{(k_0)} : D_\rho(P_0^{(n)} \| P_{k_0, \theta^{(k_0)}}^{(n)}) \leq C_3 n\epsilon_n^2 \} \), such that

\[
- \log \pi(k_0) - \sum_{j=1}^{m_{k_0}} \log \tilde{\Pi}_j(\tilde{\Theta}_j^{(k_0)}) \leq C_2 n\epsilon_n^2, \tag{A3*}
\]

where \( \pi(k_0) \) and \( \tilde{\Pi}_j \) are defined in the prior sampling procedure. Moreover, assume that the conditions (A1) and (A2) hold for all \( \epsilon > \epsilon_n \) and some constant \( C > C_2 + C_3 + 2 \). Then for the variational posterior \( \hat{Q} \) defined in (3) with the variational set \( S_{\text{GMF}} \), we have

\[
P_0^{(n)} \hat{Q} L(P_0^{(n)}, P_{k_0, \theta^{(k_0)}}^{(n)}) \lesssim n\epsilon_n^2. \tag{32}
\]

Theorem 4.1 characterizes the convergence rate of the generalized mean-field variational posterior using the conditions (A1), (A2) and (A3*). It will be shown in the proof that both the conditions (A3) and (A4) are consequences of (A3*). Therefore, the variational approximation error \( \gamma_n^2 \) is dominated by the convergence rate \( \epsilon_n^2 \) of the true posterior distribution. Given the structure of the prior distribution, an equivalent way of writing (A3*) is

\[
\Pi \left( \{ P_{k, \theta^{(k)}} : k = k_0, \theta^{(k_0)} \in \tilde{\Theta}^{(k_0)} \} \right) \geq \exp(-C_2 n\epsilon_n^2).
\]

Therefore, our three conditions (A1), (A2) and (A3*) still fall into the “prior mass and testing” framework, and directly correspond to the three conditions in [14] for convergence rates of the true posterior. As a result, applications of Theorem 4.1 to various specific variational Bayes inference problems are as straightforward as the applications of the usual “prior mass and testing” framework in [14].

An interesting special case is when the set \( K \) is a singleton. Then, for a product prior measure and the mean-field variational class, the condition (A3*) is reduced to (2) discussed in Section 1.

4.3 Density Estimation via Location-Scale Mixtures

In this section, we consider the location-scale mixture model as an application of the theory. The location-scale mixture density is defined as

\[
f(x) = n(x; k, \mu, w, \sigma) = \sum_{j=1}^{k} w_j \psi_\sigma(x - \mu_j), \tag{33}
\]
where \( k \in \mathbb{N}_+ \), \( \sigma > 0 \), \( w_j \geq 0 \), \( \sum_{j=1}^{k} w_j = 1 \), \( \mu_j \in \mathbb{R} \) and

\[
\psi_\sigma(x) = \frac{1}{2\sigma \Gamma(1 + \frac{1}{p})} \exp(-(|x|/\sigma)^p),
\]

for some positive even integer \( p \). The kernel \( \psi_\sigma(\cdot) \) has a pre-specified form, for example, Gaussian density when \( p = 2 \), while the parameters \( k \in \mathbb{N}_+ \), \( \sigma > 0 \), \( w_j \geq 0 \), \( \sum_{j=1}^{k} w_j = 1 \), and \( \mu_j \in \mathbb{R} \) are to be learned from the data.

The location-scale mixture model (33) can be written as a special example of the general probability models introduced in Section 4.1. In this case, the countable set \( \mathcal{K} \) is the positive integer set \( \mathbb{N}_+ \). The parameter space indexed by \( k \) is defined as

\[
\Theta^{(k)} = \left\{ \theta^{(k)} = (\mu, w, \sigma) : \mu = (\mu_1, \cdots, \mu_k) \in \mathbb{R}^k, w = (w_1, \cdots, w_k) \in \Delta_k, \sigma \in \mathbb{R}^+ \right\}.
\]

where \( \Delta_k = \{ w \in \mathbb{R}^k : w_j \geq 0 \text{ for } j = 1, \cdots, k \text{ and } \sum_{j=1}^{k} w_j = 1 \} \).

Given i.i.d. observations \( X_1, \ldots, X_n \) sampled from some density function \( f_0 \), our goal is to estimate the density \( f_0 \) through the location-scale mixture model (33). For any density \( f \), we denote its probability measure as \( P_f \). In the paper [17], a Bayesian procedure is proposed and a nearly minimax optimal convergence rate is derived for the true posterior distribution. We will follow the same setting in [17], but analyze the generalized mean-field variational posterior.

We first specify the prior distribution \( \Pi \) through the following sampling process:

1. Sample the number of mixtures \( k \sim \pi \);

2. Conditioning on \( k \), sample the location parameters \( \mu_1, \cdots, \mu_k \) independently from \( p_{\mu} \), sample the weights \( w = (w_1, \cdots, w_k) \) from \( p_{(k)} \), and then sample the precision parameter \( \tau = \sigma^{-2} \) from \( p_{\tau} \);

3. Define the density \( f(\cdot) = m(\cdot; k, \mu, w, \sigma) \), which induces the prior \( f \sim \Pi \).

The generalized mean-field variational class is defined in the same way as (29)-(31), where the blockwise structure (30) in this case is specified as

\[
\tilde{S}^{(k)}_{\text{MF}} = \left\{ \tilde{Q}^{(k)} : d\tilde{Q}^{(k)}(\theta^{(k)}) = d\tilde{Q}_\sigma(\sigma) d\tilde{Q}_w(w) \prod_{j=1}^{k} d\tilde{Q}_{\mu_j}(\mu_j) \right\}.
\]

Note that we do not factorize \( d\tilde{Q}_w(w) \) because of the constraint \( \sum_{j=1}^{k} w_j = 1 \). The variational posterior distribution is defined as \( \tilde{Q} \) by (3) with \( \mathcal{S} = \mathcal{S}_{\text{GMF}} \). The loss function here is chosen as squared Hellinger distance, i.e., \( L(P_f^n, P_{f_0}^n) = nH^2(P_f, P_{f_0}) \).

In order that \( \tilde{Q} \) enjoys a good convergence rate, we need conditions on the prior distribution \( \Pi \) and the true density function \( f_0 \). We first list the conditions on the prior.
1. There exist constants $C_1, C_2 > 0$, such that
\[
\sum_{m=k}^{\infty} \pi(m) \leq C_1 \exp(-C_2 k \log k),
\]
for all $m > 0$. There exist constants $t, C_3, C_4 > 0$, such that
\[
\pi(k_0) \geq C_3 \exp(-C_4 k_0 \log k_0),
\]
for all $n^{\frac{1}{2^{\alpha + 1}}} \leq k_0 \leq n^{\frac{1}{2^{\alpha + 1}} + t}$.

2. There exist constants $c_1, c_2, c_3 > 0$, such that
\[
\int_{-\infty}^{-x_0} p_{\mu}(x) dx + \int_{x_0}^{\infty} p_{\mu}(x) dx \leq c_1 \exp(-c_2 x_0^{c_3}),
\]
for all $x_0 > 0$ and constants $c_4, c_5, c_6$, such that
\[
p_{\mu}(x) \geq c_4 \exp(-c_5 |x|^{c_6}),
\]
for all $x$.

3. There exist constants $t, d_1, d_2, d_3 > 0$, such that
\[
\int_{w \in \Delta_{k_0}(w_0, \epsilon)} p_{w}^{(k_0)}(x) dx \geq d_1 \exp\left(-d_2 k_0 (\log k_0)^{d_3} \log\left(\frac{1}{\epsilon}\right)\right),
\]
for all $w_0 \in \Delta_{k_0}$ and $n^{\frac{1}{2^{\alpha + 1}}} \leq k_0 \leq n^{\frac{1}{2^{\alpha + 1}} + t}$, where $\Delta_{k_0}(w_0, \epsilon) = \{w \in \Delta_{k_0} : \|w - w_0\|_1 \leq \epsilon\}$.

4. There exist constants $b_0, b_1, b_2, b_3 > 0$, such that
\[
\|p_\tau\|_\infty < b_0, \quad \int_{\tau_0}^{\infty} p_\tau(x) dx \leq b_1 \exp(-b_2 |\tau_0|^{b_3}),
\]
for all $\tau_0 > 0$. There exist constants $b_4, b_5 > 0$ and a constant $b_6 \in (0, 1]$ that satisfy
\[
p_\tau(x) \geq b_4 \exp(-b_5 |x|^{b_6}),
\]
for all $x > 0$.

The conditions on the prior distribution are quite general. For example, one can choose $k \sim \text{Poisson}(\xi_0)$, $\mu_j \sim N(0, \sigma_0^2)$, $w \sim \text{Dir}(\alpha_0, \alpha_0, \cdots, \alpha_0)$ and $\tau \sim \Gamma(a_0, b_0)$ for some positive constants $\xi_0, \sigma_0, \alpha_0, a_0, b_0$. Then, the conditions above are all satisfied.

Next, we list the conditions on the true density function $f_0$: }
B1 (Smoothness) The logarithmic density function $\log f_0$ is assumed to be locally $\alpha$-Hölder smooth. In other words, for the derivative $l_j(x) = \frac{d}{dx^j} \log f_0(x)$, there exists a polynomial $L(\cdot)$ and a constant $\gamma > 0$ such that,

$$|l_{\lfloor \alpha \rfloor}(x) - l_{\lfloor \alpha \rfloor}(y)| \leq L(x)|x - y|^{\alpha - \lfloor \alpha \rfloor},$$

(43)

for all $x, y$ that satisfies $|x - y| \leq \gamma$. Here, the degree and the coefficients of the polynomial $L(\cdot)$ are all assumed to be constants. Moreover, the derivative $l_j(x)$ satisfies the bound

$$\int |l_j|^2 \alpha + \epsilon_j f_0 < \text{max}$$

for all $j = 1, \ldots, \lfloor \alpha \rfloor$ with some constants $\epsilon, s_{\text{max}} > 0$.

B2 (Tail) There exist positive constants $T, \xi_1, \xi_2, \xi_3$ such that

$$f_0(x) \leq \xi_1 e^{-\xi_2 |x|^{\xi_3}},$$

(44)

for all $|x| \geq T$.

B3 (Monotonicity) There exist constants $x_m < x_M$ such that $f_0$ is nondecreasing on $(-\infty, x_m)$ and is nonincreasing on $(x_M, \infty)$. Without loss of generality, we assume $f_0(x_m) = f_0(x_M) = c$ and $f_0(x) \geq c$ for all $x_m < x < x_M$ with some constant $c > 0$.

These conditions are exactly the same as in [17] and similar conditions are also considered in [21]. The conditions allow a well-behaved approximation to the true density by a location-scale mixture. There are many density functions that satisfy the conditions (B1)-(B3), for which we refer to [17].

The convergence rate of the generalized mean-field variational posterior is given by the following theorem.

**Theorem 4.2.** Consider i.i.d. observations generated by $P^n_{f_0}$, and the density function $f_0$ satisfies conditions (B1)-(B3). For the prior $f(\cdot) = m(\cdot; k, \mu, w, \sigma) \sim \Pi$ that satisfies (36)-(42), we have

$$P^n_{f_0} \tilde{Q} H^2(P_f, P_{f_0}) \lesssim n^{- \frac{2\alpha}{2\alpha + r}} (\log n)^{\frac{2\alpha r}{2\alpha + r}},$$

where $\tilde{Q}$ is the variational posterior distribution defined by (3) with the set $S = S_{\text{GMF}}$, and $r = \frac{p}{\min(p, \xi_1)} + \max\{d_3 + 1, \frac{c_6}{\min(p, \xi_3)}\}$, with $p, \xi_3, c_6, d_3$ defined in (34), (44), (39) and (40), respectively.

The proof of Theorem 4.2 largely follows the arguments in [17] that are used to establish the corresponding result for the true posterior distribution, thanks to the fact that Theorem 4.1 requires three very similar “prior mass and testing” conditions to that of [14]. The only difference is that function approximations via location-scale mixtures need to be analyzed under a stronger divergence $D_\rho(\cdot\|\cdot)$ for some $\rho > 1$. For this reason, the proof of Theorem 4.2 relies on the construction of a surrogate density function $\tilde{f}_0$. We first apply Theorem 4.1 and establish a convergence rate under $\tilde{f}_0$. Then, the conclusion is transferred to $f_0$ with a change-of-measure argument. Details of the proof are given in Section 6.6.
5 Discussion

5.1 Variational Bayes and Empirical Bayes

In most cases, the variational posterior does not have a closed form and needs to be solved by coordinate ascent algorithms [6]. However, for the Gaussian sequence model (9) with the prior distribution (11) considered in Section 3, one can write down the exact form of the mean-field variational posterior distribution.

**Theorem 5.1.** Consider the variational posterior $\hat{Q}_{\text{VB}}$ induced by the likelihood (9), the prior (11) and the mean-field variational set $S_{\text{MF}}$. The distribution $\hat{Q}_{\text{VB}}$ is a product measure with the density of each coordinate specified by

$$q_j = \begin{cases} \tilde{f}_j, & j < \hat{k}, \\ \tilde{p}\delta_0 + (1 - \tilde{p})\tilde{f}_{\hat{k}}, & j = \hat{k}, \\ \delta_0, & j > \hat{k}. \end{cases}$$

(45)

where

$$\tilde{f}_j(\theta_j) \propto f_j(\theta_j) \exp\left(-\frac{n}{2}(\theta_j - Y_j)^2\right),$$

$$\tilde{p} = \frac{\pi(k - 1|Y)}{\pi(k - 1|Y) + \pi(k|Y)},$$

and

$$\hat{k} = \arg\max_k (\pi(k - 1|Y) + \pi(k|Y)).$$

(46)

The number $\pi(k|Y)$ is the posterior probability of the model dimension, and according to Bayes formula, it is

$$\pi(k|Y) \propto \pi(k) \prod_{j \leq \hat{k}} \int f_j(\theta_j) \exp\left(-\frac{n}{2}(\theta_j - Y_j)^2\right) d\theta_j \prod_{j > \hat{k}} \exp\left(-\frac{nY_j^2}{2}\right).$$

In other words, the mean-field variational posterior $\hat{Q}_{\text{VB}}$ is nearly equivalent to a thresholding rule. It estimates all $\theta^*_j$ by 0 after $\hat{k}$ and applies the usual posterior distribution for each coordinate before $\hat{k}$. A mixed strategy is applied to the $\hat{k}$th coordinate. The effective model dimension $\hat{k}$ is found in a data-driven way through (46).

Readers who are familiar with empirical Bayes procedures may recognize the similarity between the variational Bayes posterior $\hat{Q}_{\text{VB}}$ and the empirical Bayes posterior $\hat{Q}_{\text{EB}}$. In the framework of empirical Bayes, the hyper-parameter $k$ is found by maximizing the marginal likelihood. For the Gaussian sequence model (9), this corresponds to the distribution with each coordinate

$$q_j = \begin{cases} \tilde{f}_j, & j \leq \hat{k}, \\ \delta_0, & j > \hat{k}, \end{cases}$$

and the number $\hat{k}$ is determined by

$$\hat{k} = \arg\max_k \pi(k|Y).$$
In fact, the canonical form of empirical Bayes corresponds to the above definition of \( \hat{k} \) with a flat prior for \( \pi(k) \). Here, we present a form that involves a general prior weight.

The similarity between \( \hat{Q}_{VB} \) and \( \hat{Q}_{EB} \) is obvious. Both distributions are thresholding rules. The effective dimensions \( \tilde{k} \) and \( \hat{k} \) are also defined in similar ways. The main difference is that the variational posterior \( \hat{Q}_{VB} \) is uncertain whether to threshold the \( \tilde{k} \)th coordinate or not, and the outcome is determined by a coin flip with probability \( \tilde{p} \).

Given the similarity, interesting conclusions for \( \hat{Q}_{EB} \) are also expected to hold for \( \hat{Q}_{VB} \). In a recent interesting work by [26], it is shown that \[ P^{(n)}_{\theta^*} \tilde{k} \lesssim \left( \frac{n}{\log n} \right)^{\frac{1}{2\alpha+1}}. \] The same conclusion also holds for the model dimension \( \tilde{k} \) of the variational posterior.

**Theorem 5.2.** For the prior distribution \( \Pi \) defined in (11), we assume that \( \max_j \|f_j\|_\infty \leq a \) and \( \pi(k) \) is nonincreasing over \( k \). Then, we have

\[
P^{(n)}_{\theta^*} \hat{k} \lesssim \left( \frac{n}{\log n} \right)^{\frac{1}{2\alpha+1}},
\]

for any \( \theta^* \in \Theta_\alpha(B) \).

We remark that the ideal effective model dimension for a signal in the parameter space \( \Theta_\alpha(B) \) should be of order \( n^{\frac{1}{2\alpha+1}} \), which leads to optimal bias and variance trade-off [16]. The bounds for \( \hat{k} \) and \( \tilde{k} \) imply that both model dimensions are too small compared with the optimal order. This implies an extra bias and thus an extra logarithmic factor in the rate of convergence in Theorem 3.1. On the other hand, one can use a rescaling prior \( f_j(x) = \sqrt{n}g_j(\sqrt{n}x) \) as is used in Theorem 3.2 to avoid the extra logarithmic factor. For this prior, both \( \hat{k} \) and \( \tilde{k} \) will be of the optimal order of \( n^{\frac{1}{2\alpha+1}} \).

To close this section, we show that with a special variational class, the induced variational posterior is exactly the empirical Bayes posterior.

**Theorem 5.3.** Define the following set

\[
S_{EB} = \left\{ Q : Q = \prod_{j=1}^{k} dQ_j(\theta_j) \prod_{j > k} \delta_0(\theta_j)d\theta_j \text{ for some integer } k \text{ and } Q_j \ll \ell \text{ for all } j \leq k \right\},
\]

where \( \ell \) stands for the Lebesgue measure on the real line. Then, the empirical Bayes posterior \( \hat{Q}_{EB} \) is the variational posterior induced by the likelihood (9), the prior (11) and the variational class \( S_{EB} \).

We note that \( S_{EB} \) is equivalent to a generalized mean-field class considered in Section 4. The result of Theorem 5.3 is remarkable. Since empirical Bayes is a special form of variational Bayes, the general framework developed in this paper can potentially be applied to empirical Bayes procedures, and thus reproduce some of the results of [26].
5.2 Variational Posterior and True Posterior

According to Theorem 2.1, the convergence rate of the posterior is determined by the sum of \( \epsilon_n^2 \), the rate of the true posterior, and \( \gamma_n^2 \), the variational approximation error. Since \( \epsilon_n^2 + \gamma_n^2 \geq \epsilon_n^2 \), the variational posterior rate of convergence is always slower than that of the true posterior. However, Theorem 2.1 just gives an upper bound. In this section, we give an example, and we show that it is possible for a variational posterior to enjoy faster convergence rate than that of the true posterior.

We consider the setting of Gaussian sequence model (9). The true signal \( \theta^* \) that generates the data is assumed to belong to the Sobolev ball \( \Theta_\alpha(B) \). The prior distribution is specified as

\[
\theta \sim d\Pi = \prod_{j \leq n} dN(0, j^{-2\beta-1}) \prod_{j > n} \delta_0.
\]

Note that a similar Gaussian process prior is well studied in the literature [30, 8]. We force all the coordinates after \( n \) to be zero, so that the variational approximation through Kullback-Leibler divergence will not explode. For the specified prior, the posterior contraction rate is

\[
n^{-\frac{1}{2}(\alpha \wedge \beta)} \frac{2\alpha}{2\beta+1}, \quad k \leq n^{-\frac{1}{2}\beta + 1},
\]

\[
n^{-\frac{1}{2}} \epsilon_n^2, \quad k > n^{-\frac{1}{2}\beta + 1},
\]

where \( \hat{Q}_{[k]} \) is the variational posterior defined by (3) with \( S = S_{[k]} \).

For the variational Bayes procedure, we consider the following variational class

\[
S_{[k]} = \left\{ Q : dQ = \prod_{j \leq k} dQ_j \prod_{j=k+1}^{n} dN(0, e^{-jn}) \prod_{j>n} \delta_0 \right\},
\]

for a given integer \( k \). It is easy to see that the variational posterior \( \hat{Q}_{[k]} \) defined by (3) with \( S = S_{[k]} \) can be written as

\[
d\hat{Q}_{[k]} = \prod_{j \leq k} dN \left( \frac{n}{n + j^{2\beta + 1}} Y_j, \frac{1}{n + j^{2\beta + 1}} \right) \prod_{j=k+1}^{n} dN(0, e^{-jn}) \prod_{j>n} \delta_0.
\]

In other words, the class \( S_{[k]} \) does not put any constraint on the first \( k \) coordinates and shrink all the coordinates after \( k \) to zero. Ideally, one would like to use \( \delta_0 \) for the coordinates after \( k \). However, that would lead to \( D(Q\|\Pi(\cdot|Y)) = \infty \) for all \( Q \in S_{[k]} \) given that the support of \( \delta_0 \) is a singleton. That is why we use \( N(0, e^{-jn}) \) instead.

The rate of \( \hat{Q}_{[k]} \) for each \( k \) is given by the following theorem.

**Theorem 5.4.** For the variational posterior \( \hat{Q}_{[k]} \), we have

\[
\sup_{\theta^* \in \Theta_\alpha(B)} \mathbb{P}_{\theta^*}^{\hat{Q}_{[k]}(\cdot)} \| \theta - \theta^* \|^2 \asymp \begin{cases} \frac{k}{n} + k^{-2\alpha}, & k \leq n^{-\frac{1}{2}\beta + 1}, \\ n^{-\frac{2(\alpha \wedge \beta)}{2\beta + 1}}, & k > n^{-\frac{1}{2}\beta + 1}, \end{cases}
\]

where \( \hat{Q}_{[k]} \) is the variational posterior defined by (3) with \( S = S_{[k]} \).

Note that Theorem 5.4 gives both upper and lower bounds for \( \hat{Q}_{[k]} \). This makes the comparison between variational posterior and true posterior possible. Observe that when
If $k = \infty$, we have $\hat{Q}_{[\infty]} = \Pi(\cdot|Y)$, and the result is reduced to the posterior contraction rate $n^{-\frac{2(\alpha \wedge \beta)}{2\beta+1}}$ in [8].

Depending on the values of $\alpha, \beta$ and $k$, the rate for $\hat{Q}_{[k]}$ can be better than that of the true posterior. For example, when $\beta < \alpha$, the choice $k = \frac{1}{n^{1+\alpha}}$ leads to the minimax rate $n^{-\frac{2\alpha}{2\alpha+1}}$, which is always faster than $n^{-\frac{2(\alpha \wedge \beta)}{2\beta+1}}$. This is because for a $\beta < \alpha$, the true posterior distribution undersmooths the data, but the variational class $S_{[k]}$ with $k = \frac{1}{n^{1+\alpha}}$ helps to reduce the extra variance resulted from undersmoothing by thresholding all the coordinates after $k$. On the other hand, when $\beta \geq \alpha$, an improvement through the variational class $S_{[k]}$ is not possible. In this case, the true posterior has already overly smoothed the data, and the information loss cannot be recovered by the variational class. In general, we plot the exponent value of the rate of $\hat{Q}_{[k]}$ against the value of $\log n(k)$ in Figure 5.2.

![Diagram](image)

**Figure 1:** The exponent value of the rate of $\hat{Q}_{[k]}$ against the value of $\log n(k)$.

Though we only analyze the Gaussian sequence model here, the discovery in this section is potentially important. Bayesian statisticians tend to think of variational approximation only as a tool for computation, but as a “bad thing” to do for posterior inference. Our example shows that this is not true. In contrast, smartly designed variational class can serve as a regularizer in addition to its contribution to the computational issue. A good strategy is to distribute all the prior information between the prior $\Pi$ and the variational class $S$, while still keeping the computational efficiency.
5.3 Beyond the Kullback-Leibler Approximation

Modern variational approximation methods are not limited to the approximation by Kullback-Leibler divergence. For example, [20] proposed a generalized variational inference method using Rényi divergence and derived a corresponding evidence lower bound. Though alternative divergences may be hard to optimize, they may give better approximations [22, 23].

It is possible to generalize our results to variational approximation using other criterions. We first introduce a $D_*$-variational posterior.

**Definition 5.1** ($D_*$-variational posterior). Let $\mathcal{S}$ be a family of distributions. The $D_*$-variational posterior is defined as

$$
\hat{Q}_* = \arg\min_{Q \in \mathcal{S}} D_* (Q || \Pi(\cdot | X^{(n)})).
$$

(47)

Then we state a result that extends Theorem 2.1 to the $D_*$-variational posterior distribution.

**Theorem 5.5.** Suppose $D_*$ is a divergence such that $D_* (P_1 || P_2) \geq 0$ for all probability measures $P_1$ and $P_2$. Assume $D_* (P_1 || P_2) \geq D (P_1 || P_2)$ for any $P_1 \in \mathcal{S}$ and any $P_2$, and the conditions (A1)-(A3) in Theorem 2.1 hold. Then for the $D_*$-variational posterior $\hat{Q}_*$ defined in (47), we have

$$
P_0^{(n)} \hat{Q}_* L(P_0^{(n)}, P_0^{(n)}) \leq Mn^2 \epsilon^2 n + \gamma^2_n
$$

(48)

for some constant $M > 0$, where the quantity $\gamma^2_n$ is defined as

$$
\gamma^2_n = \frac{1}{n} \inf_{Q \in \mathcal{S}} P_0^{(n)} D_* (Q || \Pi(\cdot | X^{(n)})).
$$

Theorem 5.5 is a generalization of Theorem 2.1 for a divergence $D_*$ that is not smaller than the Kullback-Leibler divergence. Examples of applications include all Rényi divergence with $\rho \geq 1$ and the $\chi^2$-divergence. Divergence functions that are not necessarily larger than the Kullback-Leibler require new techniques to analyze, and will be considered as an interesting future project.

6 Proofs

6.1 Proof of Theorem 2.1

This section gives the proof of Theorem 2.1, which is divided into several lemmas. We first give an inequality that uses the basic property of the KL-divergence.

**Lemma 6.1.** For any function $f$ and two probability measure $P$ and $Q$, we have

$$
\int f(x) dQ(x) \leq D(Q || P) + \log \int \exp(f(x)) dP(x).
$$
Therefore, using Jensen’s inequality, we get

\[ Q_{\tilde{P}} \]

for all \( a > 0 \). By Lemma 6.1, we have

Proof.

The proof is complete by taking minimum over \( a > 0 \) and \( Q \in S \).

Lemma 6.2. For the \( \hat{Q} \) defined in (3), we have

\[ P_0^{(n)} \hat{Q} L(P_\theta^{(n)}, P_0^{(n)}) \leq \inf_{a > 0} \frac{1}{a} \left( \inf_{Q \in S} P_0^{(n)} D(Q \| \Pi(\cdot | X^{(n)})) + \log P_0^{(n)} \Pi(\exp(a L(P_\theta^{(n)}, P_0^{(n)})) | X^{(n)}) \right). \]

Proof. By Lemma 6.1, we have

\[ a \hat{Q} L(P_\theta^{(n)}, P_0^{(n)}) \leq D(\hat{Q} \| \Pi(\cdot | X^{(n)})) + \log \Pi(\exp(a L(P_\theta^{(n)}, P_0^{(n)})) | X^{(n)}), \]

for all \( a > 0 \). By the definition of \( \hat{Q} \), we have

\[ D(\hat{Q} \| \Pi(\cdot | X^{(n)})) \leq D(Q \| \Pi(\cdot | X^{(n)})), \]

for all \( Q \in S \). Taking expectation on both sides, we have

\[ a P_0^{(n)} \hat{Q} L(P_\theta^{(n)}, P_0^{(n)}) \leq P_0^{(n)} D(Q \| \Pi(\cdot | X^{(n)})) + P_0^{(n)} \log \Pi(\exp(a L(P_\theta^{(n)}, P_0^{(n)})) | X^{(n)}). \]

Using Jensen’s inequality, we get

\[ P_0^{(n)} \log \Pi(\exp(a L(P_\theta^{(n)}, P_0^{(n)})) | X^{(n)}) \leq \log P_0^{(n)} \Pi(\exp(a L(P_\theta^{(n)}, P_0^{(n)})) | X^{(n)}). \]

Therefore,

\[ P_0^{(n)} \hat{Q} L(P_\theta^{(n)}, P_0^{(n)}) \leq \frac{1}{a} \left( P_0^{(n)} D(Q \| \Pi(\cdot | X^{(n)})) + \log P_0^{(n)} \Pi(\exp(a L(P_\theta^{(n)}, P_0^{(n)})) | X^{(n)}) \right). \]

The proof is complete by taking minimum over \( a > 0 \) and \( Q \in S \).
In order to bound $P^{(n)}_0 \Pi(\exp(aL(P^{(n)}_\theta, P^{(n)}_0))|X^{(n)})$, we need the following lemma on the posterior tail probability. Its proof is similar to the one used in [14].

**Lemma 6.3.** Under the conditions of Theorem 2.1, we have

\[ P^{(n)}_0 \Pi \left( L(P^{(n)}_\theta, P^{(n)}_0) > C_1 n\epsilon^2 \big| X^{(n)} \right) \leq \exp(-Cn\epsilon^2) + \exp(-\lambda n\epsilon^2) + 2\exp(-n\epsilon^2), \]

for all $\epsilon \geq \epsilon_n$, where $\lambda = \rho - 1$ for $\rho$ in (A3).

**Proof.** We first define the sets

\[ U_n = \left\{ \theta : L(P^{(n)}_\theta, P^{(n)}_0) > C_1 n\epsilon^2 \right\}, \quad K_n = \left\{ \theta : D_{1+\lambda}(P^{(n)}_0 \| P^{(n)}_\theta) \leq C_3 n\epsilon^2 \right\}. \]

We also define the event

\[ A_n = \left\{ X^{(n)} : \int \frac{dP^{(n)}_\theta}{dP^{(n)}_0}(X^{(n)})d\tilde{\Pi}(\theta) \leq \exp(-(C_3 + 1)n\epsilon^2) \right\}, \]

where the probability measure $\tilde{\Pi}$ is defined as $\tilde{\Pi}(B) = \frac{\Pi(B \cap K_n)}{\Pi(K_n)}$. Let $\Theta_\epsilon(\epsilon)$ and $\phi_n$ be the set and the testing function in (A1). Then, we bound $P^{(n)}_0 \Pi(U_n|X^{(n)})$ by

\[ P^{(n)}_0 \Pi(U_n|X^{(n)}) \leq P^{(n)}_0 \phi_n + P^{(n)}_0(A_n) + P^{(n)}_0(1 - \phi_n)\Pi(U_n|X^{(n)})1_{A_n^c}. \]

We will give bounds for the three terms above respectively. By (A1),

\[ P^{(n)}_0 \phi_n \leq \exp(-Cn\epsilon^2). \]

Using the definitions of $A_n$, we have

\[ P^{(n)}_0(A_n) = P^{(n)}_0 \left( \left( \int \frac{dP^{(n)}_\theta}{dP^{(n)}_0}(X^{(n)})d\tilde{\Pi}(\theta) \right)^{-\lambda} > \exp(\lambda(C_3 + 1)n\epsilon^2) \right) \]

\[ \leq \exp(-\lambda(C_3 + 1)n\epsilon^2) P^{(n)}_0 \left( \int \frac{dP^{(n)}_\theta}{dP^{(n)}_0}(X^{(n)})d\tilde{\Pi}(\theta) \right)^{-\lambda} \]

\[ \leq \exp(-\lambda(C_3 + 1)n\epsilon^2) \int \left( \int \left( \frac{dP^{(n)}_0}{dP^{(n)}_\theta} \right)^{1+\lambda} \right) d\tilde{\Pi}(\theta) \quad \text{(Jensen’s Inequality)} \]

\[ = \exp(-\lambda(C_3 + 1)n\epsilon^2) \int \exp(\lambda D_{1+\lambda}(P^{(n)}_0 \| P^{(n)}_\theta)) d\tilde{\Pi}(\theta) \]

\[ \leq \exp(-\lambda(C_3 + 1)n\epsilon^2 + \lambda C_3 n\epsilon^2) \]

\[ \leq \exp(-\lambda n\epsilon^2). \]
Now we analyze the third term. On the event $A^c_n$, we have
\[
\int \frac{dP^{(n)}_\theta}{dP^{(n)}_0}(X^{(n)})d\Pi(\theta) \geq \Pi(K_n) \int \frac{dP^{(n)}_\theta}{dP^{(n)}_0}(X^{(n)})d\tilde{\Pi}(\theta) \geq \exp(-(C_2 + C_3 + 1)ne^2),
\]
where the last inequality is by (A3). Then, it follows that
\[
P_0^{(n)} \frac{\int_{U_n} \frac{dP^{(n)}_\theta}{dP^{(n)}_0}(X^{(n)})d\Pi(\theta)}{\int \frac{dP^{(n)}_\theta}{dP^{(n)}_0}(X^{(n)})d\Pi(\theta)} (1 - \phi_n) 1_{A^c_n} \leq \exp((C_3 + C_2 + 1)ne^2) P_0^{(n)} \int_{U_n} \frac{dP^{(n)}_\theta}{dP^{(n)}_0}(X^{(n)})(1 - \phi_n)d\Pi(\theta)
\]
\[
\leq \exp((C_3 + C_2 + 1)ne^2) \left[ \int_{U_n \cap \Theta_n(\epsilon)} P^{(n)}_\theta (1 - \phi_n)d\Pi(\theta) + \Pi(\Theta_n(\epsilon)^c) \right]
\]
\[
\leq \exp((C_3 + C_2 + 1)ne^2)(\exp(-Cne^2) + \exp(-Cne^2)),
\]
where the last inequality is by (A1) and (A2). Since $C > C_3 + C_2 + 2$, we obtain the bound
\[
P_0^{(n)} \Pi(U_n|X^{(n)}) \leq \exp(-Cne^2) + \exp(-\lambda ne^2) + 2\exp(-ne^2).
\]
Combining the bounds (49), (50) and (51), we have
\[
P_0^{(n)}\Pi(U_n|X^{(n)}) \leq \exp(-Cne^2) + \exp(-\lambda ne^2) + 2\exp(-ne^2).
\]

Next, we derive a moment generating function bound for a sub-exponential random variable.

**Lemma 6.4.** Suppose the random variable $X$ satisfies
\[
\mathbb{P}(X \geq t) \leq c_1 \exp(-c_2t),
\]
for all $t \geq t_0 > 0$. Then, for any $0 < a \leq \frac{1}{2}c_2$,
\[
\mathbb{E}\exp(aX) \leq \exp(at_0) + c_1.
\]

**Proof.** Set $Y = \exp(aX)$ for some $0 < a \leq \frac{1}{2}c_2$. Then,
\[
\mathbb{E}Y \leq M_0 + \int_{M_0}^{\infty} \mathbb{P}(Y \geq y)dy = M_0 + \int_{M_0}^{\infty} \mathbb{P}\left(X \geq \frac{1}{a} \log y\right) dy \leq M_0 + c_1 \int_{M_0}^{\infty} y^{-c_2/a}dy.
\]
Choose $M_0 = \exp(at_0)$, and then since $a \leq \frac{1}{2}c_2$, we have
\[
\mathbb{E}Y \leq \exp(at_0) + \frac{c_1a}{c_2-a} \exp((a - c_2)t_0) \leq \exp(at_0) + c_1 \exp(-at_0) \leq \exp(at_0) + c_1.
\]
Now we are ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** By Lemma 6.3, we have

\[ P_0^{(n)} \left( L(P_\theta^{(n)}, P_0^{(n)}) > t | X^{(n)} \right) \leq c_1 \exp(-c_2 t), \]

for all \( t \geq t_0 \). Here, \( c_1 = 4, c_2 = \min \{ C, \lambda, 1 \} / C_1 \) and \( t_0 = C_1 n \epsilon_n^2 \). Then, by Lemma 6.4, we have

\[ P_0^{(n)} \left( \exp \left( a L(P_\theta^{(n)}, P_0^{(n)}) \right) | X^{(n)} \right) \leq \exp \left( a C_1 n \epsilon_n^2 \right) + 4, \]

for all \( a \leq \min \{ C, \lambda, 1 \} / (2C_1) \). Taking \( a = \min \{ C, \lambda, 1 \} / (2C_1) \) and using Lemma 6.2, we get

\[ P_0^{(n)} \left( \hat{Q}_L(P_\theta^{(n)}, P_0^{(n)}) \right) \leq n \gamma_2 n + \log (4 + e^{a C_1 n \epsilon_n^2}), \]

with some \( M > 0 \) that only depends on \( C, C_1, \lambda \).

### 6.2 Proofs of Theorem 2.3, Theorem 2.4 and Theorem 4.1

**Proof of Theorem 2.3.** For any \( Q \in S \cap \mathcal{E} \), we have \( \text{supp}(Q) \subset C \), and thus \( QD(P_0^{(n)} || P_\theta^{(n)}) \leq C_2 n \epsilon_n^2 \). By \( \mathbf{(A4^*)} \), we have \( D(Q || \Pi) \leq C_1 n \epsilon_n^2 \). Therefore, \( R(Q) \leq (C_1 + C_2) n \epsilon_n^2 \), and the proof is complete.

**Proof of Theorem 2.4.** It is sufficient to find a \( Q \in S_{MF} \) and bound

\[ R(Q) = \frac{1}{n} \left( D(Q || \Pi) + QD(P_0^{(n)} || P_\theta^{(n)}) \right). \]

We choose \( Q \) to be the product measure \( dQ(\theta) = \prod_{j=1}^m dQ_j(\theta_j) \), with

\[ Q_j(B_j) = \frac{Q_j(B_j \cap \Theta_j)}{Q_j(\Theta_j)}. \]

Then, it is easy to see that \( Q \in S_{MC} \) and \( \text{supp}(Q) \subset \otimes_{j=1}^m \Theta_j \). By (7), we have

\[ QD(P_0^{(n)} || P_\theta^{(n)}) \leq C_1 n \epsilon_n^2. \]

Moreover, we can write \( D(Q || \Pi) \) as below

\[ D(Q || \Pi) = Q \log \frac{dQ}{d\Pi} + Q \log \frac{d\tilde{Q}}{d\Pi}, \]

where

\[ Q \log \frac{dQ}{d\Pi} = - \sum_{j=1}^m \log \tilde{Q}_j(\Theta_j) \leq C_3 n \epsilon_n^2, \]

by (8), and

\[ Q \log \frac{d\tilde{Q}}{d\Pi} \leq C_2 n \epsilon_n^2, \]

by (7). Hence, we obtain the desired bound.
Proof of Theorem 4.1. In view of Theorem 2.2, it is sufficient to show that (A3) and (A4) can be derived from (A3*). We first establish (A3). Note that
\[
\Pi \left( \left\{ P_{k,\theta(k)}^{(n)} : D_{\rho} \left( P_0^{(n)} \parallel P_{k,\theta(k)}^{(n)} \right) \leq C_3 n^{\epsilon_n^2} \right\} \right) \geq \Pi \left( \left\{ P_{k,\theta(k)}^{(n)} : k = k_0, \theta(k_0) \in \tilde{\Theta}^{(k_0)} \right\} \right).
\]
Then, (A3) follows (A3*). For (A4), we need to find a \( Q \in S_{\text{GMF}} \) and bound \( R(Q) \). Define the probability measure \( \bar{\Pi}^{(k_0)}(B) = \tilde{\bar{\Pi}}^{(k_0)}(B \cap \tilde{\Theta}^{(k_0)}) / \tilde{\bar{\Pi}}^{(k_0)}(\tilde{\Theta}^{(k_0)}) \).

By the structure of the prior distribution, we take the \( Q \in S_{\text{MF}}^{(k_0)} \) induced by \( \tilde{Q} = \bar{\Pi}^{(k_0)} \in \tilde{S}_{\text{MF}}^{(k_0)} \). Then,
\[
D(Q \parallel \Pi) \leq \log \frac{1}{\pi^{(k_0)}} + D \left( \bar{\Pi}^{(k_0)} \parallel \bar{\Pi}^{(k_0)} \right) \leq \log \frac{1}{\pi^{(k_0)}} + \log \frac{1}{\bar{\Pi}^{(k_0)}(\tilde{\Theta}^{(k_0)})} \lesssim n^{\epsilon_n^2},
\]
by (A3*). We also have
\[
QD \left( P_0^{(n)} \parallel P_{k,\theta(k)}^{(n)} \right) \leq QD_{\rho} \left( P_0^{(n)} \parallel P_{k,\theta(k)}^{(n)} \right) \lesssim n^{\epsilon_n^2},
\]
because \( Q \) is induced by \( \bar{\Pi}^{(k_0)} \), which is supported on
\[
\tilde{\Theta}^{(k_0)} \subseteq \left\{ \theta^{(k_0)} : D_{\rho} \left( P_0^{(n)} \parallel P_{k_0,\theta(k_0)}^{(n)} \right) \leq C_2 n^{\epsilon_n^2} \right\}.
\]
Therefore, \( R(Q) \lesssim n^{\epsilon_n^2} \), and the proof is complete.

6.3 Proofs of Theorem 3.1 and Theorem 3.2

The proofs of Theorem 3.1 and Theorem 3.2 will be split into the following three lemmas. Recall that we use the loss \( L(P_{\theta^*}^{(n)}, P_{\theta}^{(n)}) = n \| \theta - \theta^* \|^2 \) for this model.

Lemma 6.5. For the prior \( \Pi \) that satisfies (12), the conditions (A1) and (A2) hold for all \( \epsilon \geq n^{-1/2} \).

Proof. Given any \( \epsilon \geq n^{-1/2} \) and any \( C > 0 \), we define
\[
\Theta_n(\epsilon) = \left\{ \theta = (\theta_j) : \theta_j = 0, \text{ for all } j > Cn^{\epsilon^2}/C_2 \right\}.
\]
Then, by (12), we have
\[
\Pi(\Theta_n(\epsilon)^c) \leq \Pi(k > Cn^{\epsilon^2}/C_2) \lesssim \exp \left( -Cn^{\epsilon^2} \right).
\]
This proves (A2). To show (A1), we consider the following testing problem,
\[
H_0 : \theta = \theta^*, \quad H_1 : \theta \in \Theta_n(\epsilon) \text{ and } \| \theta - \theta^* \|^2 \geq \tilde{C} \epsilon^2.
\]
By Lemma 5 in [13] and Theorem 7.1 in [14], it is sufficient to establish the bound
\[ \log N(\epsilon/8, \{\theta \in \Theta_n(\epsilon) : \|\theta - \theta^*\| \leq \epsilon\}, \|\cdot\|) \lesssim n\epsilon^2. \]

This is obviously true given a standard volume ratio calculation in a Euclidean space of dimension \([Cn\epsilon^2/C2]\). Then, by Theorem 7.1 in [14], there exists a testing procedure \(\phi_n\) such that (A1) holds. Note that the testing error can be arbitrarily small given a sufficiently large \(\tilde{C} > 0\).

**Lemma 6.6.** Assume \(\theta^* \in \Theta_\alpha(B)\). For the prior \(\Pi\) that satisfies (13) and (14), the conditions (A3) and (A4) hold for \(\epsilon_n = n^{-\frac{\alpha}{2\alpha + 1}}(\log n)^{\frac{1}{2\alpha + 1}}\).

**Proof.** We first show (A4). We will apply Theorem 2.4 by constructing a \(\bar{Q} \in \mathcal{S}_{MF}\) and \(\otimes_j \tilde{\Theta}_j\) that satisfy the conditions (7) and (8). Define \(\tilde{\Theta}_j = [\theta_j^* - n^{-1/2}, \theta_j^* + n^{-1/2}]\) for all \(j \leq k_0\) and \(\tilde{\Theta}_j = \emptyset\) for all \(j > k_0\), where \(k_0 = \left\lceil \left(\frac{n}{\log n}\right)^{\frac{1}{2\alpha + 1}} \right\rceil\) is the same as defined in 13. We also define the measure \(\bar{Q}\) by
\[ d\bar{Q}(\theta) = \prod_{j=1}^{k_0} f_j(\theta_j) \prod_{j>k_0} \delta_0(\theta_j)d\theta. \]

It is easy to see that \(\bar{Q} \in \mathcal{S}_{MF}\). For any \(\theta \in \otimes_j \tilde{\Theta}_j\), we have
\[ D_2(P^{(n)}_{\theta^*}||P^{(n)}_{\theta}) = 2D(P^{(n)}_{\theta^*}||P^{(n)}_{\theta}) = n\|\theta - \theta^*\|^2 \leq k_0 \lesssim n\epsilon_n^2, \quad (52) \]
and
\[ \log \frac{d\bar{Q}(\theta)}{d\Pi(\theta)} \leq -\log \frac{\pi(k_0)}{n(\log n)^{\frac{1}{2\alpha + 1}}} - \log C_3 + C_4k_0 \log k_0 \lesssim n\epsilon_n^2. \quad (53) \]

Therefore, the condition (7) holds. To check the condition (8), we use the bound
\[ -\sum_{j=1}^{\infty} \log \bar{Q}_j(\tilde{\Theta}_j) = -\sum_{j=1}^{k_0} \log \bar{Q}_j(\tilde{\Theta}_j) = -\sum_{j=1}^{k_0} \log \int_{\theta_j^* - n^{-1/2}}^{\theta_j^* + n^{-1/2}} f_j(x)dx \leq -k_0 \log(2n^{-1/2}) - \frac{1}{2n^{-1/2}} \sum_{j=1}^{k_0} \int_{\theta_j^* - n^{-1/2}}^{\theta_j^* + n^{-1/2}} \log f_j(x)dx, \]
where we have used Jensen’s inequality above. We are going to bound each of the integral above using (14). For any \(j \leq k_0\), we have
\[ -\frac{1}{2n^{-1/2}} \int_{\theta_j^* - n^{-1/2}}^{\theta_j^* + n^{-1/2}} \log f_j(x)dx \leq c_0 + c_1 j^{2\alpha + 1}(3\theta_j^* + n^{-1}) \leq c_0 + 3c_kj^{2\alpha} + c_1 k_0^{2\alpha + 1} n^{-1} \leq c_0 + c_1 + 3c_k k_0^{2\alpha} \theta_j^2. \]

Hence, we get
\[ -\sum_{j=1}^{\infty} \log \bar{Q}_j(\tilde{\Theta}_j) \leq \frac{1}{2} k_0 \log n + (c_0 + c_1 - \log 2) k_0 + 3c_k \sum_j j^{2\alpha} \theta_j^2 \lesssim n\epsilon_n^2, \quad (54) \]
which implies that (8) holds. The condition (A4) is thus proved by applying Theorem 2.4.

Finally, we derive the condition (A3). In view of (52), there is a constant $C > 0$, such that

$$-\log \Pi \left( D_2(P_{\theta^*}^{(n)} \| P_{\theta}^{(n)}) \leq C n \epsilon_n^2 \right)$$

$$\leq -\log \pi(k_0) - \log \tilde{Q} \left( D_2(P_{\theta^*}^{(n)} \| P_{\theta}^{(n)}) \leq C n \epsilon_n^2 \right)$$

$$\leq -\log \pi(k_0) - \sum_{j=1}^{\infty} \log \tilde{Q}_j(\Theta_j) \lesssim n \epsilon_n^2.$$

The last inequality above is by (53) and (54). Hence, the proof is complete. 

**Lemma 6.7.** Assume $\theta^* \in \Theta_\alpha(B)$. For the prior $\Pi$ that satisfies (15) and (16), the conditions (A3) and (A4) hold for $\epsilon_n = n^{2/\alpha+1}$.

**Proof.** The proof is essentially the same as that of Lemma 6.6. We define $\tilde{Q} \in S_{MF}$ and $\otimes_j \tilde{\Theta}_j$ in the same way except that $k_0 = \lceil n^{2/\alpha+1} \rceil$. Then, by the same calculation, we have for any $\theta \in \otimes_j \tilde{\Theta}_j$,

$$D_2(P_{\theta^*}^{(n)} \| P_{\theta}^{(n)}) = 2D(P_{\theta^*}^{(n)} \| P_{\theta}^{(n)}) \lesssim n \epsilon_n^2, \text{ and } \log \frac{d\tilde{Q}(\theta)}{d\Pi(\theta)} \lesssim n \epsilon_n^2.$$

Therefore, the condition (7) holds. For any $j \leq k_0$,

$$-\log \tilde{Q}_j(\Theta_j) = -\log \int_{\sqrt{n\theta_j^*}^{(n)}}^{\sqrt{n\theta_j^*}+1} g_j(x) dx$$

$$\leq c_0 - \log 2 - \log \int_{\sqrt{n\theta_j^*}^{(n)}}^{\sqrt{n\theta_j^*}+1} \frac{1}{2} \exp(-c_1 |x|^\beta) dx$$

$$\leq c_0 - \log 2 + \frac{c_1}{2} \int_{\sqrt{n\theta_j^*}^{(n)}}^{\sqrt{n\theta_j^*}+1} |x|^\beta dx$$

By Hölder’s inequality,

$$\int_{\sqrt{n\theta_j^*}^{(n)}}^{\sqrt{n\theta_j^*}+1} |x|^\beta dx \leq \left( \int_{\sqrt{n\theta_j^*}^{(n)}}^{\sqrt{n\theta_j^*}+1} x^2 \right)^{\beta/2} \left( \int_{\sqrt{n\theta_j^*}^{(n)}}^{\sqrt{n\theta_j^*}+1} 1 \right)^{(2-\beta)/2}$$

$$= \left( 2(n\theta_j^*+1) \right)^{\beta/2} \cdot 2^{2-\beta} = 2(n\theta_j^*+1)^{\beta/2}$$

$$\leq 4(n\theta_j^*)^{\beta/2} + 4.$$
Using Hölder’s inequality again, we get

$$\sum_{j=1}^{k_0} |\theta_j^*|^\beta \leq \left( \sum_{j=1}^{k_0} j^{2\alpha} \theta_j^{*2} \right)^{\beta/2} \left( \sum_{j=1}^{k_0} j^{2\alpha - \frac{2\alpha \beta}{2-\beta}} \right)^{1-\beta/2}.$$  

Set $t = \frac{2\alpha \beta}{2-\beta}$. As $0 < \beta < \frac{2}{2\alpha + 1}$, we have $t \in (0, 1)$. Then

$$\sum_{j=1}^{k_0} j^{-t} \leq 1 + \int_1^{k_0} x^{-t} \, dx = \frac{k_0^{1-t} + t}{1-t} < c_2 k_0^{1-t}.$$  

Thus,

$$\sum_{j=1}^{k_0} |\theta_j^*|^\beta \leq B^\beta c_2 k_0^{(1-t)\frac{2-\beta}{2}} \lesssim n^{1/(2\alpha + 1) - \beta/2}.$$  

This leads to the desired bound $-\sum_{j=1}^{k_0} \log \tilde{Q}_j(\tilde{\Theta}_j) \lesssim n\epsilon_2^2$ in (8). The condition (A4) is thus proved by applying Theorem 2.4.

The condition (A3) can be derived in the same way as in the proof of Lemma 6.6. □

Proofs of Theorem 3.1 and Theorem 3.2. The results are directly implied by Lemma 6.5, Lemma 6.6 and Lemma 6.7. □

6.4 Proof of Theorem 3.3

For Theorem 3.3, the loss function is $L(P_\theta^n, P_{\theta^*}^n) = nH^2(P_\theta, P_{\theta^*})$. We split the proof of Theorem 3.3 into following two lemmas.

Lemma 6.8. Assume $\theta^* \in \Theta_\alpha(B)$ for $\alpha > 1/2$. For the prior $\Pi$ that satisfies (19) and (21), the conditions (A1) and (A2) hold for all $\epsilon \geq \left( \frac{\log n}{n} \right)^{2\alpha + 1}$.

Lemma 6.9. Assume $\theta^* \in \Theta_\alpha(B)$ for $\alpha > 1/2$. For the prior $\Pi$ that satisfies (20) and (22), the conditions (A3) and (A4) hold for $\epsilon_2^2 = \left( \frac{\log n}{n} \right)^{2\alpha + 1}$.

Before proving these two lemmas, we need the following two results that establish relations between different divergence functions for the exponential family model.

Lemma 6.10. If $||\theta - \theta'||_1 \leq \frac{1}{\sqrt{2}}$, then

$$H(P_\theta, P_{\theta'}) \leq 2\sqrt{2} ||\theta - \theta'||_1.$$  

Proof. We first give some uniform bounds that are well known for exponential family density functions (see [25]). For any $\theta, \theta'$, we have

$$\left\| \log \frac{dP_\theta}{dP_{\theta'}} \right\|_\infty \leq 2\sqrt{2} ||\theta - \theta'||_1. \quad (55)$$  

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We start from the left hand side of the inequality:

\[
H^2(P_\theta, P_{\theta'})^2 = \frac{1}{2} \int \left( \sqrt{\frac{dP_\theta}{dP_{\theta'}}} - 1 \right)^2 dP_{\theta'} \\
\leq \frac{1}{2} \int \left( \exp(\sqrt{2}\|\theta - \theta'\|_1) - 1 \right)^2 dP_{\theta'} + \frac{1}{2} \int \left( \exp(-\sqrt{2}\|\theta - \theta'\|_1) - 1 \right)^2 dP_{\theta'} \\
\leq \frac{1}{2} \int 8\|\theta - \theta'\|_1^2 dP_{\theta'} + \frac{1}{2} \int 8\|\theta - \theta'\|_1^2 dP_{\theta'} \\
= 8\|\theta - \theta'\|_1^2,
\]

where we have applied the property that \(e^{\frac{x^2}{2}}\) is monotonically increasing for all \(x\). Then it follows that \(H(P_\theta, P_{\theta'}) \leq 2\sqrt{2}\|\theta - \theta'\|_1\).

\[\text{Lemma 6.11.} \quad \text{For any } \theta \text{ and any } \theta^* \in \Theta_\alpha(B) \text{ with } \alpha > 1/2, \text{ we have}
\]

\[
C_0^{-1} \exp \left( -3\sqrt{2}\|\theta^* - \theta\|_1 \right) \|\theta^* - \theta\|^2 \leq 2H^2(P_{\theta^*}, P_\theta) \leq D(P_{\theta^*} \| P_\theta)
\]

\[
\leq D_2(P_{\theta^*} \| P_\theta) \leq C_0 \exp \left( 3\sqrt{2}\|\theta^* - \theta\|_1 \right) \|\theta^* - \theta\|^2,
\]

where the constant \(C_0 > 0\) only depends on \(\alpha\) and \(B\).

\[\text{Proof.} \quad \text{For any } \theta^* \in \Theta_\alpha(B), \text{ we have } \left\| \log \frac{dP_{\theta^*}}{d\ell} \right\|_\infty \leq 2\sqrt{2}\|\theta^*\|_1. \text{ Since}
\]

\[
\|\theta^*\|_1^2 \leq \left( \sum_{j=1}^{\infty} j^{-2\alpha} \right) \left( \sum_{j=1}^{\infty} j^{2\alpha} \theta_j^2 \right) \leq B^2 \gamma_\alpha,
\]

\[\text{where } \gamma_\alpha = \sum_{j=1}^{\infty} j^{-2\alpha} = O(1) \text{ for } \alpha > 1/2. \text{ This gives}
\]

\[
\left\| \log \frac{dP_{\theta^*}}{d\ell} \right\|_\infty \leq 2\sqrt{2}\gamma_{\alpha}^{1/2} B.
\]

Now we proceed to show Lemma 6.11. Given the result of Proposition 2.1, it is sufficient to prove the first and the last inequalities. Define

\[
V(P_{\theta^*}, P_\theta) = \int \left( \log \frac{dP_{\theta^*}}{dP_\theta} - D(P_{\theta^*} \| P_\theta) \right)^2 dP_{\theta^*}.
\]

Following the argument in the proof of Lemma 3.2 in [10], we have

\[
e^{-\left\| \log \frac{dP_{\theta^*}}{d\ell} \right\|_\infty} \|\theta^* - \theta\|^2 \leq V(P_{\theta^*}, P_\theta) \leq 4H^2(P_{\theta^*}, P_\theta)e^{3/2\left\| \log \frac{dP_{\theta^*}}{d\ell} \right\|_\infty}.
\]

By (55) and (57), we have

\[
C_0^{-1}\|\theta - \theta^*\|^2 \leq 2H^2(P_\theta, P_{\theta'}) \exp \left( 3\sqrt{2}\|\theta - \theta^*\|_1 \right),
\]

for \(C_0 = 2\exp(2\sqrt{2}\gamma_{\alpha}^{1/2} B)\), which implies the first inequality.
For the last inequality, we have
\[
D_2(P_{\theta^*} \| P_\theta) = \log \left( \int dP_{\theta^*} \exp \left( \log \frac{dP_{\theta^*}}{dP_\theta} \right) \right)
\]
\[
= \log \left( 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \int dP_{\theta^*} \left( \log \frac{dP_{\theta^*}}{dP_\theta} \right)^l \right)
\]
\[
\leq \log \left( 1 + D(P_{\theta^*} \| P_\theta) \sum_{l=1}^{\infty} \frac{1}{l!} \left\| \log \frac{dP_{\theta^*}}{dP_\theta} \right\|_\infty^{l-1} \right)
\]
\[
\leq D(P_{\theta^*} \| P_\theta) \exp \left( \left\| \log \frac{dP_{\theta^*}}{dP_\theta} \right\|_\infty \right)
\]
\[
\leq D(P_{\theta^*} \| P_\theta) e^{2\sqrt{2}\|\theta - \theta^*\|_1},
\]
where we have used the inequality that \( e^{x-1} \leq e^x \) for all \( x > 0 \) and the last inequality is by (55). By the same argument in the proof of Lemma 3.2 in [10], we have
\[
D(P_{\theta^*} \| P_\theta) \leq e^{3\sqrt{2}\|\theta - \theta^*\|_1 + 2\sqrt{2}\|\theta^*\|_1 \|\theta - \theta^*\|^2}.
\]
Therefore, we obtain the bound
\[
D_2(P_{\theta^*} \| P_\theta) \leq e^{3\sqrt{2}\|\theta - \theta^*\|_1 + 2\sqrt{2}\|\theta^*\|_1 \|\theta - \theta^*\|^2},
\]
which implies the desired result by (56).

Now we are ready to prove Lemma 6.8 and Lemma 6.9.

**Proof of Lemma 6.8.** Given any \( \epsilon \geq \left( \frac{\log n}{n} \right)^{\frac{2\alpha}{2\alpha+1}} \), we define the set
\[
\Theta_n(\epsilon) = \{ \theta = (\theta_j) : \theta_j \in [-w_n, w_n] \text{ for } 1 \leq j \leq k_n, \theta_j = 0 \text{ for } j > k_n \},
\]
where \( w_n = (\tilde{C}n\epsilon^2)^{1/\beta} \) and \( k_n = \left\lceil \frac{\tilde{C} n \epsilon^2}{\log(n \epsilon^2)} \right\rceil \). We bound \( \Pi(\Theta_n(\epsilon)^c) \) by
\[
\Pi(\Theta_n(\epsilon)^c) \leq \Pi(k > k_n) + \sum_{j=1}^{k_n} \Pi(k = j) \sum_{i=1}^{j} \Pi(|\theta_i| > w_n | k = j)
\]
\[
\leq \Pi(k > k_n) + \sum_{j=1}^{k_n} \Pi(k = j) \sum_{i=1}^{j} \int_{|x| > w_n} f_i(x) dx
\]
\[
\leq \Pi(k > k_n) + \sum_{j=1}^{k_n} \int_{|x| > w_n} f_j(x) dx
\]
\[
\leq \Pi(k > k_n) + \sum_{j=1}^{k_n} e^{-c_1 w_n^\beta/2} \int e^{c_1 |x|^{\beta/2}} f_j(x) dx
\]
\[
\leq \Pi(k > k_n) + \sum_{j=1}^{k_n} e^{-c_1 w_n^\beta/2} \int e^{c_1 |x|^{\beta/2} - c_0 - c_1 |x|^{\beta}} dx
\]
\[
\leq \exp(-C_2 k_n \log k_n) + k_n \exp \left( -c_1 \tilde{C} n \epsilon^2 / 2 \right),
\]

where we have used the conditions (19) and (21). Therefore, for any \( C > 0 \), we can choose a sufficiently large \( C \), such that \( \Pi(\Theta_n(\epsilon)) \lesssim \exp(-Cn\epsilon^2) \), which proves (A2).

To prove (A1), we consider the following testing problem,
\[
H_0 : \theta = \theta^*, \quad H_1 : \theta \in \Theta_n(\epsilon) \text{ and } H(P_0, P_{\theta^*}) \geq C'\epsilon.
\]
By Theorem 7.1 in [14], it is sufficient to establish the bound
\[
\log N(\epsilon, \{P_0 : \theta \in \Theta_n(\epsilon)\}, H) \lesssim n\epsilon^2.
\]
Note that for any \( \theta, \theta' \in \Theta_n(\epsilon) \), we have \( \|\theta - \theta'\|_1 \leq \sqrt{k_n}\|\theta - \theta'\| \). Therefore, by Lemma 6.10,
\[
H(P_\theta, P_{\theta'}) \lesssim \|\theta - \theta'\|_1 \leq \sqrt{k_n}\|\theta - \theta'\|,
\]
when \( \|\theta - \theta'\|_1 \leq \frac{1}{\sqrt{2}} \). This means as long as \( \|\theta - \theta'\| \leq k_n^{-1/2}(\epsilon \wedge 2^{-1/2}) \), we have \( H(P_\theta, P_{\theta'}) \lesssim \epsilon \).
Thus, there exists a constant \( c' \), such that
\[
\log N(\epsilon, \{P_0 : \theta \in \Theta_n(\epsilon)\}, H) \leq \log N\left(c'k_n^{-1/2}(\epsilon \wedge 2^{-1/2}), \left\{\theta \in \mathbb{R}^{k_n} : \|\theta\|_2 \leq k_n\epsilon^2\right\}, \|\cdot\|\right) \leq k_n \log\left(c'\frac{n\epsilon^2}{\|\cdot\|}\right) \lesssim k_n \log(n\epsilon^2) \asymp n\epsilon^2,
\]
where we have used the condition \( \epsilon \geq \left(\frac{\log n}{n}\right)^{\frac{\alpha}{2\alpha + 1}} \) in the last two steps above.

It implies the existence of a testing function that satisfies (A1). The testing error can be made arbitrarily small by choosing a sufficiently large \( C' \). Hence, the proof is complete. \( \square \)

**Proof of Lemma 6.9.** In the first part of the proof, we derive (A3). We take \( k_0 = \lceil (n/\log n)^{1/(2\alpha + 1)} \rceil \). Define \( \tilde{\Theta} = \bigotimes_j \tilde{\Theta}_j \), where \( \tilde{\Theta}_j = [\theta^*_j - n^{-1/2}, \theta^*_j + n^{-1/2}] \) for all \( j \leq k_0 \) and \( \tilde{\Theta}_j = \{0\} \) for all \( j > k_0 \). Then, by Lemma 6.10, for all \( \theta \in \tilde{\Theta} \),
\[
D_2(P_{\theta^*} \| P_\theta) \leq C_0 \exp(3\sqrt{2}\|\theta^* - \theta\|_1)\|\theta - \theta^*\|^2
= C_0 \exp\left(3\sqrt{2}\left(\frac{k_0}{\sqrt{n}} + \sum_{j > k_0} |\theta^*_j|\right)\right)\left(\frac{k_0}{n} + \sum_{j > k_0} \theta^2_j\right)
\leq C_0 \exp\left(3\sqrt{2}\left(n^{1 - 2\alpha}/n^{2\alpha} + B\gamma_n^{1/2}\right)\right)\left(\frac{k_0}{n} + k_0^{-2\alpha} B^2\right)
\lesssim n\epsilon_n^2.
\]
where we have use the condition \( \alpha > 1/2 \).

Therefore, it is sufficient to lower bound \( \Pi(\tilde{\Theta}) \), which has been done in the proof of Lemma 6.6.

Now we will derive (A4). Rather than using the results of Theorem 2.3 or Theorem 2.4, we will construct a \( Q \in \mathcal{S}_G \) and bound \( R(Q) \) directly. Note that in the current setting, we have
\[
R(Q) = \frac{1}{n} D(Q \| \Pi) + QD(P_{\theta^*} \| P_\theta).
\]
For $k_0 = [\frac{n}{\log n}]^{\frac{1}{\alpha+1}}$, define $Q = \otimes_j Q_j$, where $Q_j = N(\theta_j^*, n^{-1})$ for $j \leq k_0$ and $Q_j = N(0, 0)$ for $j > k_0$. Then, it is easy to see that $Q \in \mathcal{S}_G$.

We first give a bound for $D(Q||\Pi)$. Let $F_j$ denote the probability distribution with density function $f_j$. Then, we have

$$D(Q||\Pi) \leq \log \frac{1}{\pi(k_0)} + \sum_{j=1}^{k_0} D(N(\theta_j^*, n^{-1})||F_j),$$

where the first term on the right hand side above can be bounded as

$$\log \frac{1}{\pi(k_0)} \lesssim k_0 \log k_0 \lesssim n\epsilon_n^2,$$

according to the condition (20). For any $j \leq k_0$, we use $\psi_j$ to denote the density function of $N(\theta_j^*, n^{-1})$. Then, by (22), we have

$$D\left( N(\theta_j^*, n^{-1})||F_j \right) = \int \psi_j \log \psi_j - \int \psi_j \log f_j \leq \int \psi_j \log \psi_j + c_0' + c_1' j^{2\alpha+1} \int \phi_j(x)x^2 dx = \frac{1}{2} \log \left( \frac{n}{2\pi e} \right) + c_0' + c_1' j^{2\alpha+1} (n^{-1} + \theta_j^{*2}).$$

Since $\theta^* \in \Theta_\alpha(B)$, we have

$$\sum_{j=1}^{k_0} D\left( N(\theta_j^*, n^{-1})||F_j \right) \lesssim k_0 \log n \lesssim n\epsilon_n^2.$$

Therefore, we have obtained $D(Q||\Pi) \lesssim n\epsilon_n^2$.

We then derive a bound for $QD(P_{\theta^*}||P_\theta)$. For $j \leq k_0$, we write $\theta_j = \theta_j^* + \frac{1}{\sqrt{n}} Z_j$ where $Z_j \sim N(0, 1)$. Then according to Lemma 6.11, it follows that

$$QD(P_{\theta^*}||P_\theta) \lesssim Q \exp \left( 3\sqrt{2} ||\theta - \theta^*||_1 \right) ||\theta - \theta^*||^2$$

$$= Q \left[ e^{3\sqrt{2} \sum_{j=1}^{k_0} |\theta_j - \theta_j^*|} \left( \sum_{j=1}^{k_0} (\theta_j - \theta_j^*)^2 + \sum_{j > k_0} \theta_j^{*2} \right) \right]$$

$$= \mathbb{E} e^{3\sqrt{2} \sum_{j=1}^{k_0} |Z_j|/\sqrt{n}} \sum_{j=1}^{k_0} Z_j^2/n + \left( \sum_{j > k_0} \theta_j^{*2} \right) \mathbb{E} e^{3\sqrt{2} \sum_{j=1}^{k_0} |Z_j|/\sqrt{n}}, \quad (58)$$

where the last inequality is by (56). Suppose we can show

$$\mathbb{E} e^{3\sqrt{2} \sum_{j=1}^{k_0} |Z_j|/\sqrt{n}} = O(1), \quad (59)$$

and

$$\mathbb{E} Z_j^2 e^{3\sqrt{2} \sum_{j=1}^{k_0} |Z_j|/\sqrt{n}} = O(1). \quad (60)$$
Then, up to a constant, (58) can be bounded by
\[ \frac{k_0}{n} + \sum_{j > k_0} \theta_j^2 \lesssim \epsilon_n^2, \]
which further implies \( QD(P_{\theta^*} \| P_{\theta}) \lesssim \epsilon_n^2 \).

To complete the proof, we show (59). We have
\[
E e^{3\sqrt{2} \sum_{j=1}^{k_0} |Z_j|/\sqrt{n}} \leq E \exp \left( \frac{3\sqrt{2} k_0}{\sqrt{n}} \sum_{j=1}^{k_0} (1 + Z_j^2) \right) = \exp \left( \frac{3\sqrt{2} k_0}{\sqrt{n}} \right) \exp \left( \frac{3\sqrt{2} k_0}{\sqrt{n}} \lambda_{k_0}^2 \right) = \exp \left( \frac{3\sqrt{2} k_0}{\sqrt{n}} \right) \left( 1 - \frac{6\sqrt{2}}{\sqrt{n}} \right)^{-\frac{k_0}{2}}.
\]

Since \( \alpha > 1/2 \), we have \( k_0/\sqrt{n} = O(1) \), and thus (59) holds. For (60), we have
\[
E Z_1^2 e^{3\sqrt{2} \sum_{j=1}^{k_0} |Z_j|/\sqrt{n}} = \left( E Z_1^2 e^{3\sqrt{2} |Z_1|/\sqrt{n}} \right)^2 \left( E e^{3\sqrt{2} \sum_{j=2}^{k_0} |Z_j|/\sqrt{n}} \right).
\]
Note that \( E Z_1^2 e^{3\sqrt{2} |Z_1|/\sqrt{n}} = O(1) \), and \( E e^{3\sqrt{2} \sum_{j=2}^{k_0} |Z_j|/\sqrt{n}} \) shares the same bound for (59). This implies (60) also holds.

Proof of Theorem 3.3. The result is immediately implied by Lemma 6.8 and Lemma 6.9 in view of Theorem 2.2.

6.5 Proofs of Theorem 3.4 and Theorem 3.5

Proof of Theorem 3.4. Recall that \( \Theta_k \) is the space of piecewise constant vectors with at most \( k \) pieces. Then, we have the partition
\[ \mathbb{R}^n = \Theta_{n-1} \cup (\Theta_n \setminus \Theta_{n-1}). \]
Suppose the measure \( Q \in \mathcal{S}_{\text{MF}} \) and \( D(Q \| \Pi) < \infty \), then the support of \( Q \) is the subset of the support of \( \Pi \).

Note that the distributions \( g_i \)'s are all absolutely continuous. That is, for any singleton \( x \), \( \Pi(\theta_j = x) = 0 \), which indicates that \( Q(\theta_j = x) = 0 \) for any singleton \( x \). Thus, \( Q \) is continuous in each coordinate and for any \( j \in [n-1] \), \( Q(\theta_j = \theta_{j+1}) = \int Q(\theta_j = \theta_{j+1} = x) dx = \int Q(\theta_j = x) Q(\theta_{j+1} = x) dx = 0 \). Therefore,
\[ Q(\Theta_{n-1}) = Q \text{ (there exists a } j \in [n-1], \text{ such that } \theta_j = \theta_{j+1} = 0, \]
Therefore, the independent structure of $Q$ would imply a delta measure for some coordinate, which leads to $D(Q\|\Pi) = \infty$. This implies that $Q$ is supported on $\Theta_n \setminus \Theta_{n-1}$. Therefore,

$$D(Q\|\Pi) = \int \log \frac{dQ}{\pi(n) \prod_{i=1}^{n} g_i} dQ.$$ 

Then, by the definition of $S_{MF}$ and the independent structure of $P_{\theta}^{(n)}$, we have

$$\hat{Q}_{MF} = \arg\min_{Q \in S_{MF}} \left\{ D(Q\|\Pi) + QD(P_{\theta}^{(n)}\|P_{\theta}^{(n)}) \right\}$$
$$= \arg\min_{dQ=\prod_{i=1}^{n} q_i} \left\{ Q \sum_{i=1}^{n} \left( \log \frac{q_i}{g_i} + D\left( N(\theta_i^*, \sigma^2) \| N(\theta_i, \sigma^2) \right) \right) \right\}.$$ 

Hence,

$$d\hat{Q}_{MF} \propto \prod_{i=1}^{n} g_i(\theta_i) \exp \left( \frac{-(\theta_i - X_i)^2}{2\sigma^2} \right).$$ 

In other words, the mean-field variational posterior $\hat{Q}_{MF}$ is a product measure, and on each coordinate, it equals the posterior distribution induced by the prior $g_i$. Now we give a lower bound for $P_{\theta^*}^{(n)} \hat{Q}_{MF}\|\theta - \theta^*\|_2$. Since $\|\theta - \theta^*\|_2 = \sum_{i=1}^{n} (\theta_i - \theta_i^*)^2$, we have

$$P_{\theta^*}^{(n)} \hat{Q}_{MF}\|\theta - \theta^*\|_2 = \sum_{i=1}^{n} P_{\theta^*} \mathbb{E} \left( (\theta_i - \theta_i^*)^2 | X_i \right),$$

where we use $\mathbb{E}(\cdot|X_i)$ to stand for the posterior expectation of $\theta_i$ with the prior $\theta_i \sim g_i$. By Jensen’s inequality,

$$\mathbb{E} \left( (\theta_i - \theta_i^*)^2 | X_i \right) \geq (\mathbb{E}(\theta_i|X_i) - \theta^*)^2.$$ 

Therefore,

$$\sup_{\theta^* \in \Theta_{\mathbb{I}}(B)} P_{\theta^*}^{(n)} \hat{Q}_{MF}\|\theta - \theta^*\|_2 \geq \sup_{\theta^* \in \Theta_{\mathbb{I}}(B)} \sum_{i=1}^{n} P_{\theta^*} \mathbb{E}(\theta_i|X_i) - \theta_i^* \right)^2$$
$$\geq \frac{1}{2} \sum_{i=1}^{n} \left( P_{\theta^* = -B} \mathbb{E}(\theta_i|X_i) - \theta_i^* \right)^2 + \frac{1}{2} \sum_{i=1}^{n} P_{\theta^* = B} \mathbb{E}(\theta_i|X_i) - \theta_i^* \right)^2$$
$$= \frac{1}{2} \sum_{i=1}^{n} \left( P_{\theta^* = -B} \mathbb{E}(\theta_i|X_i) - \theta_i^* \right)^2 + P_{\theta^* = B} \mathbb{E}(\theta_i|X_i) - \theta_i^* \right)^2$$
$$\geq \sum_{i=1}^{n} B^2 \int \min \left( dN(B, \sigma^2), dN(-B, \sigma^2) \right)$$
$$\geq n.$$ 

The proof is complete. \qed

The proof of Theorem 3.5 is split into the following two lemmas.
Lemma 6.12. Assume \( \theta^* \in \Theta_{k^*}(B) \). For the prior \( \Pi \) that satisfies (25), the conditions (A1) and (A2) hold for all \( \epsilon > \sqrt{\frac{k^* \log n}{n}} \).

Lemma 6.13. Assume \( \theta^* \in \Theta_{k^*}(B) \). For the prior \( \Pi \) that satisfies (26) and (27), the conditions (A3) and (A4) hold for \( \epsilon = \sigma \sqrt{\frac{k^* \log n}{n}} \).

We need the following lemma to facilitate the proofs of Lemma 6.12 and Lemma 6.13.

Lemma 6.14. Suppose \( \theta^* \in \Theta_{k^*} \). For some integer \( m \geq k^* \), define
\[
\hat{\theta}_m = \arg\min_{\theta \in \Theta_m} \| \theta - X \|^2. \tag{61}
\]
Then for any \( t \geq 24\sigma^2 r \log \frac{en}{r} \) with \( r = \min\{n, m + k^*\} \), we have
\[
P_{\theta^*}^{(n)}(\| \hat{\theta}_m - \theta^* \|^2 > t) \leq \exp\left(-\frac{t}{16\sigma^2}\right).
\]

Proof. According to the definition,
\[
\| \hat{\theta}_m - X \|^2 \leq \| \theta^* - X \|^2.
\]
Using the identity \( \| \hat{\theta}_m - X \|^2 = \| \hat{\theta}_m - \theta^* \|^2 + \| \theta^* - X \|^2 + 2 \langle \hat{\theta}_m - \theta^*, \theta^* - X \rangle \), we get
\[
\| \hat{\theta}_m - \theta^* \|^2 \leq 2 \left\| \frac{\hat{\theta}_m - \theta^*}{\| \hat{\theta}_m - \theta^* \|} \right\| X - \theta^* \rangle.
\]
Since \( \frac{\hat{\theta}_m - \theta^*}{\| \hat{\theta}_m - \theta^* \|} \in \Theta_r \), we have
\[
\| \hat{\theta}_m - \theta^* \|^2 \leq 4\sigma^2 \sup_{\| u \| = 1: u \in \Theta_r} \sum_{i=1}^{n} u_i Z_i^2,
\]
where \( Z_i \sim N(0, 1) \). Then,
\[
P_{\theta^*}^{(n)}(\| \hat{\theta}_m - \theta^* \|^2 > t) \leq \mathbb{P}\left( \sup_{\| u \| = 1: u \in \Theta_r} \sum_{i=1}^{n} u_i Z_i^2 \geq \frac{t}{4\sigma^2} \right)
\]
\[
\leq \sum_{x_1 + x_2 + \cdots + x_r = n} \mathbb{P}\left( \sup_{\sum_{i=1}^{r} x_i u_i^2 = 1} \sum_{i=1}^{r} \sqrt{x_i} \bar{u}_i \bar{Z}_i \geq \frac{t}{4\sigma^2} \right)
\]
\[
= \sum_{x_1 + x_2 + \cdots + x_r = n} \mathbb{P}\left( \| \bar{Z} \|^2 \geq \frac{t}{4\sigma^2} \right),
\]
where \( r = \min\{m + k^*, n\} \), \( \bar{Z} = (\bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_r)^T \sim N(0, I_r) \). Then a standard chi-squared bound gives
\[
P_{\theta^*}^{(n)}(\| \hat{\theta}_m - \theta^* \|^2 > t) \leq \exp\left( r \log \frac{en}{r} + \frac{r}{2} \log 2 \right) \exp\left(-\frac{t}{8\sigma^2}\right)
\]
\[
\leq \exp\left( \frac{t}{16\sigma^2} \right) \exp\left(-\frac{t}{8\sigma^2}\right) = \exp\left(-\frac{t}{16\sigma^2}\right).
\]
The proof is complete. \( \square \)
Now we start to prove Lemma 6.12 and Lemma 6.13.

**Proof of Lemma 6.12.** For any \( \epsilon > \sqrt{\frac{k^* \log n}{n}} \), we set \( m = \lceil C_0 n \epsilon^2 \rceil \). Choose a sufficiently large \( C_0 \) so that \( m \geq 2k^* \geq 2 \). We consider \( \Theta_n(\epsilon) = \Theta_\epsilon \) with \( r = \min\{k^* + m, n\} \). Then by the condition (25), we have \( \Pi(\Theta_n(\epsilon)^c) \lesssim \exp(-Cn\epsilon^2) \) for a sufficiently large \( C_0 \), which implies (A2).

To show (A1), we consider the testing function \( \phi_n = \mathbb{I}\{ \| \hat{\theta}_m - \theta^* \| \geq 5\sigma \sqrt{(C^0 + 1)n} \epsilon \} \), where \( \hat{\theta}_m \) is defined in (61). Note that

\[
(5\sigma \sqrt{(C^0 + 1)n})^2 \geq 24(C^0 + 1)\sigma^2 n \epsilon^2 \geq 24(C^0 n \epsilon^2 + k^* \log n)\sigma^2 \geq 24(2m + k^*)\sigma^2 \log n,
\]

and we apply Lemma 6.14 to obtain

\[
P^{(n)}_{\hat{\theta}}(\phi_n) = P^{(n)}_{\hat{\theta}}(\| \hat{\theta}_m - \theta^* \|^2 \geq 25(C^0 + 1)\sigma^2 n \epsilon^2) \leq \exp\left( -\frac{25}{16} (C^0 + 1)n \epsilon^2 \right).
\]

Moreover, for any \( \theta \in \Theta_n(\epsilon) \) and \( \| \theta - \theta^* \|^2 \geq 10\sigma \epsilon \sqrt{(C^0 + 1)n} \), we have

\[
(5\sigma \sqrt{(C^0 + 1)n})^2 \geq 24(2m + k^*)\sigma^2 \log n \geq 24(2m + r)\sigma^2 \log n,
\]

and then,

\[
P^{(n)}_{\theta}(1 - \phi_n) = P^{(n)}_{\hat{\theta}}(\| \hat{\theta}_m - \theta^* \| \leq 5\sigma \epsilon \sqrt{(C^0 + 1)n})
\leq P^{(n)}_{\hat{\theta}}(\| \hat{\theta}_m - \theta \| \leq 5\sigma \epsilon \sqrt{(C^0 + 1)n})
\leq \exp\left( -\frac{25}{16} (C^0 + 1)n \epsilon^2 \right)
\]

Therefore, (A1) is satisfied with a sufficiently large \( C_0 \). \( \square \)

**Proof of Lemma 6.13.** We first use Theorem 2.3 to verify (A4). For \( \theta^* \in \Theta_{k^*}(B) \), there exists \( 0 = a_0 \leq a_1 \leq \cdots \leq a_{k^*} = n \), such that \( \theta^* \) is a piecewise constant vector with respect to the intervals \( (a_{j-1}, a_j] \). The set \( A(\theta^*) \) is defined by \( \{ a_0 + 1, a_1 + 1, ..., a_{k^* - 1} + 1 \} \). Define \( \tilde{\Theta}_{i} = \left( \theta^* + \sqrt{\frac{1}{n}} \right)^i \). Consider the distribution \( Q \) defined through the following sampling process: for each \( i \in [n] \), we sample \( \theta_i \sim \tilde{g}_i \) if \( i \in A(\theta^*) \) and set \( \theta_i = \theta_{i-1} \) if \( i \in A(\theta^*)^c \). The density \( \tilde{g}_i \) is defined by \( \tilde{g}_i(x) = \tilde{g}_i(x) \tilde{g}_i(x)dx \). According to the definition, we have \( Q \in S_{MC} \). Moreover, for any \( \theta \in \text{supp}(Q) \), we have

\[
D_2(P^{(n)}_{\hat{\theta}} || P^{(n)}_{\theta}) = \frac{\| \theta - \theta^* \|^2}{\sigma^2} \leq k^* \log n.
\]

This leads to \( QD(P^{(n)}_{\hat{\theta}} || P^{(n)}_{\theta}) \leq QD_2(P^{(n)}_{\hat{\theta}} || P^{(n)}_{\theta}) \leq k^* \log n \). By conditions (26) and (27), we have

\[
D(Q||\Pi) = -\log \pi(k^*) - \log \Pi(A = A(\theta^*)|k^*) - \sum_{j=0}^{k^*-1} \log \Pi(\theta_{a_j+1} \in \tilde{\Theta}_{a_j+1}|A = A(\theta^*)) \leq -\log C_3 + C_4 k^* \log n + 2k^* \log n - k^* \log \left( 2c\sigma \sqrt{\frac{1}{n}} \right) \lesssim k^* \log n,
\]

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which implies the condition (A4) by Theorem 2.3.

Finally, we check (A3). By (62), we have

$$\text{supp}(Q) \subset \left\{ D_2(P_{\theta}^{(n)} \| P_{\theta}^{(n)}) \leq k^* \log n \right\}.$$  

Thus,

$$- \log \Pi \left( D_2(P_{\theta}^{(n)} \| P_{\theta}^{(n)}) \leq k^* \log n \right) \leq - \log \Pi(\text{supp}(Q))$$

$$\leq - \log \pi(k^*) - \log \Pi(A = A(\theta^*) \| k^* - 1 \sum_{j=0}^{k^*-1} \log \Pi(\theta_{a_j} \in \tilde{\Theta}_{a_j+1} | A = A(\theta^*))$$

$$\lesssim k^* \log n.$$  

Thus, the condition (A3) holds.  

\[ \square \]

**Proof of Theorem 3.5.** The result is immediately implied by Lemma 6.12 and Lemma 6.13 in view of Theorem 2.2.  

\[ \square \]

### 6.6 Proof of Theorem 4.2

To prove Theorem 4.2, we first establish $P_{\tilde{f}_0}^{n} \hat{Q}H^2(f, \tilde{f}_0)$ by applying Theorem 4.1 for a $\tilde{f}_0$ that is constructed to be close to $f_0$. Then, with a change-of-measure argument, we derive a bound for $P_{\tilde{f}_0}^{n} \hat{Q}H^2(f, f_0)$. The construction of the surrogate density function $\tilde{f}_0$ is given by the following lemma.

**Lemma 6.15.** Suppose that the true density $f_0$ satisfies conditions (B1)-(B3). For a constant $H_1 > 2\alpha$, we define $\tilde{f}_0(x) = \frac{f_{\sigma}(x)}{\int_{E_\sigma f_{\sigma}(x)} dx}$ with $E_\sigma = \{ x : f_0(x) \geq \sigma H_1 \}$. For a constant $\xi_1 \leq \min\{\xi_3, p\}$ and a sufficiently small $\sigma > 0$, there exists a finite mixture $m = m(\cdot; k_\sigma, \mu_\sigma, w_\sigma, \sigma)$ with $k_\sigma = O(\sigma^{-1/2} | \log \sigma |^{p/\xi_1})$, such that

$$D_2(P_0 \| P_m) = O(\sigma^{2\alpha}).$$  

Moreover, (63) holds for all mixtures $m' = m(\cdot; k_\sigma, \mu, w, \sigma')$ such that $\sigma' \in [\sigma, \sigma + \sigma H_1 + 2\alpha + 2]$, $||\mu - \mu_\sigma||_1 \leq \sigma H_1 + 2\alpha + 2$ and $w \in \Delta_{K_\sigma}(w_\sigma, \sigma H_1 + 2\alpha + 1)$.

With the definition of $\tilde{f}_0$ and its property given by Lemma 6.15, we can bound $P_{\tilde{f}_0}^{n} \hat{Q}H^2(f, \tilde{f}_0)$ by checking the conditions (A1), (A2) and (A3*) in Theorem 4.1. This argument is split into the next two lemmas.

**Lemma 6.16.** For the prior $\Pi$ that satisfies conditions (36), (38) and (41), the conditions (A1) and (A2) hold for all $\epsilon > n^{\delta}$ with some constant $\delta > -1/2$ with respect to $P_0^{(n)} = P_{\tilde{f}_0}^{n}$.

**Lemma 6.17.** Suppose that the true density $f_0$ satisfies conditions (B1)-(B3), and the prior $\Pi$ satisfies conditions (37), (39), (40) and (42). Then the condition (A3*) holds with respect to $P_0^{(n)} = P_{\tilde{f}_0}^{n}$. Here, the density $\tilde{f}_0$ is defined in Lemma 6.15 and the rate is $\epsilon_n = n^{-\frac{\alpha}{2\alpha + 1}} (\log n)^{\frac{2\alpha}{2\alpha + 1}}$ with $r$ given in Theorem 4.2.
We first prove Lemma 6.15, and then prove Lemma 6.16 and Lemma 6.17. To facilitate the proof of Lemma 6.15, we introduce the following lemma, which is analogous to Theorem 1 in [17].

Lemma 6.18. Let \( f_0 \) be a density satisfying conditions (B1)-(B3), and let \( K_\sigma \) denote the convolution operator induced by the kernel \( \psi_\sigma \). Then there exists a density \( h_\alpha \) such that for a small enough \( \sigma > 0 \),

\[
\int \frac{f_0^2}{K_\sigma h_\alpha} = 1 + O(\sigma^{2\alpha}).
\]

Proof. We set \( G_\sigma = \{ x : f_0(x) \geq \sigma^{H_0} \} \) and

\[
A_\sigma = \{ x : |l_j(x)| \leq B\sigma^{-j}\log\sigma^{-j/p}, j = 1, \cdots, |\alpha|, |L(x)| \leq B\sigma^{-\alpha}\log\sigma^{-\alpha/p} \}.
\]

This is the same definition that appears in Lemma 1 of [17]. Note that \( \int \frac{f_0^2}{K_\sigma h_\alpha} dx - 1 \geq 0 \), and we only need to derive an upper bound for this integral. We first have the following decomposition

\[
\int \frac{f_0(x)^2}{K_\sigma h_\alpha(x)} dx = \int_{A_\sigma \cap G_\sigma} \frac{(f_0(x) - K_\sigma h_\alpha(x))^2}{K_\sigma h_\alpha(x)} dx + \int_{A_\sigma \cup G_\sigma} (K_\sigma h_\alpha(x) - f_0(x)) dx + \int_{A_\sigma \cap G_\sigma} f_0(x) dx.
\]

The first and third terms can be bounded by \( O(\sigma^{2\alpha}) \) according to the same argument in the proof of Theorem 1 in [17] when \( H_0 \) is chosen to be large enough. For the second term, according to Remark 1 in [17], we have \( \frac{f_0(x)}{K_\sigma h_\alpha(x)} \leq M_0 \) with some constant \( M_0 > 0 \) for all \( x \). Then Lemma 2 in [17] implies

\[
\int_{A_\sigma \cup G_\sigma} \frac{f_0(x)^2}{K_\sigma h_\alpha(x)} dx \leq M_0 \int_{A_\sigma \cup G_\sigma} f_0(x) dx = O(\sigma^{2\alpha}).
\]

The last term can be upper bounded by 1. Summing up all the terms, we obtain the desired conclusion.

Proof of Lemma 6.15. The proof uses a slightly modified argument in the proof of Lemma 4 in [17]. First of all, according to Lemma 6.18, there exists a density \( h_\alpha \) such that \( \int \frac{f_0^2}{K_\sigma h_\alpha} = 1 + O(\sigma^{2\alpha}) \). Define \( E'_\alpha = \{ x : f_0(x) \geq \sigma^{H_2} \} \), where \( H_2 > H_1 \) is chosen to be large enough. Set \( \tilde{h}_\alpha(x) = \frac{h_\alpha(x)}{\int_{E'_\alpha} h_\alpha(x) dx} \). Define the number \( a_\sigma = C_0 \log \sigma^{1/\xi_4} \), with \( \xi_4 \leq \min\{ \xi_3, p \} \) and some constant \( C_0 > 0 \). We choose \( m = m(\sigma; k_\sigma, \mu_\sigma, w_\sigma) \) to be the finite mixture given by Lemma 12 in [17] that satisfies

\[
\|K_\sigma \tilde{h}_\alpha - m\|_\infty \leq \sigma^{-1} \exp(-C_0|\log\sigma|^{p/\xi_4}),
\]

for \( x \in [-a_\sigma, a_\sigma] \). We will show that this mixture density satisfies (63). We write

\[
D_2(P_{f_0} \| P_m) = \int \frac{f_0^2}{m} = \int_{E_\sigma} \frac{f_0^2}{f_0^2 K_\sigma h_\alpha} K_\sigma \tilde{h}_\alpha.
\]

The four ratios will be bounded separately.
1. According to (B2), we know that \( \int_{E_\sigma} f_0(x) dx = O(1) \), for any constant \( b > 0 \). Since \( H_1 > 2\alpha \),

\[
\int_{E_\sigma} f_0(x) dx \leq (\sigma^{H_1})^{2\alpha} \int_{E_\sigma} f_0(x)^{1-2\alpha} dx = O(\sigma^{2\alpha}).
\]

This leads to

\[
\left| \frac{\tilde{f}_0^2(x)}{f_0^2(x)} - 1 \right| = \left| \frac{1}{(1 - \int_{E_\sigma} f_0(x) dx)^2} - 1 \right| \leq C_1\sigma^{2\alpha},
\]

for a constant \( C_1 > 0 \) and all \( x \in E_\sigma \).

2. For the second term, we have

\[
\int_{E_\sigma} \frac{f_0^2(x)}{K_\sigma h_\alpha} dx = \int \frac{f_0^2(x)}{K_\sigma h_\alpha} dx - \int_{E_\sigma} \frac{f_0^2(x)}{K_\sigma h_\alpha} dx.
\]

Since \( \frac{f_0(x)}{K_\sigma h_\alpha(x)} \leq M_0 \) for a constant \( M_0 \) uniformly over \( x \),

\[
\int_{E_\sigma} \frac{f_0^2(x)}{K_\sigma h_\alpha} dx \leq M_0 \int_{E_\sigma} f_0(x) dx = O(\sigma^{2\alpha}).
\]

Combining with Lemma 6.18, we conclude that

\[
\left| \int_{E_\sigma} \frac{f_0^2(x)}{K_\sigma h_\alpha} dx - 1 \right| \leq C_2\sigma^{2\alpha},
\]

for a constant \( C_2 > 0 \).

3. By the same argument in the proof of Lemma 4 in [17], we get

\[
\left| \frac{K_\sigma h_\alpha(x)}{K_\sigma h_\alpha(x)} - 1 \right| \leq C_3\sigma^{2\alpha},
\]

for a constant \( C_3 > 0 \) and all \( x \in E_\sigma \).

4. According to the proof of Lemma 4 in [17], we have \( E'_\sigma \subset \{ x : f_0(x) \geq c_0\sigma^{H_2} \} \) for some constant \( c_0 \). Because \( \xi_4 \leq \xi_3 \), \( E'_\sigma \subset [-a_\sigma, a_\sigma] \). This leads to the inequality \( \| K_\sigma h_\alpha - m \|_{\infty} \leq \sigma^{-1} \exp(-C_0|\log \sigma|^{p/\xi_4}) \). Note that for any \( x \in E_\sigma \), we have \( K_\sigma(x)\hat{h}_\alpha(x) \geq K_\sigma h_\alpha(x) \geq f_0(x) \geq \sigma^{H_1} \) uniformly over \( x \in E_\sigma \). Thus, for a sufficiently large \( C_0 \),

\[
\sigma^{-1} \exp(-C_0|\log \sigma|^{p/\xi_4}) = \sigma^{C_0|\log \sigma|^{p-\xi_4/\xi_4-1}} = O(\sigma^{H_1+2\alpha}),
\]

where we have used the condition \( \xi_4 \leq p \). Then we have

\[
\left| \frac{K_\sigma \hat{h}_\alpha(x)}{m(x)} - 1 \right| \leq \frac{\| K_\sigma \hat{h}_\alpha - m \|_{\infty}}{K_\sigma h_\alpha(x) - \| K_\sigma h_\alpha - m \|_{\infty}} \leq C_4\sigma^{2\alpha}.
\]

for all \( x \in E_\sigma \) with some constant \( C_4 > 0 \).
Combining the bounds of all terms above, we get
\[
\int \frac{f_0^2(x)}{m(x; k, \mu, w, \sigma)} \, dx = 1 + O(\sigma^{2\alpha}),
\]
which indicates that (63) holds. When \( \sigma' \in [\sigma, \sigma^{H_1+2\alpha+2}] \), \( \|\mu - \mu_{\sigma}\|_1 \leq \sigma^{H_1+2\alpha+2} \) and \( w \in \Delta_{k,\sigma}(w_{\sigma}, \sigma^{H_1+2\alpha+1}) \), according to Lemma 3 in [17], we have
\[
\|m(\cdot; k, \mu, w, \sigma') - m(\cdot; k, \mu_{\sigma}, w_{\sigma}, \sigma)\|_{\infty} = O(\sigma^{H_1+2\alpha}).
\]
Then the four points listed above also hold, which means that (63) is also satisfied for these \((k, \mu, w, \sigma)\). The proof is complete. \( \square \)

Now we prove Lemma 6.16 and Lemma 6.17.

**Proof of Lemma 6.16.** We consider the set
\[
\Theta_n(\epsilon) = \bigcup_{k=1}^{k_n} \Theta_k(\epsilon),
\]
where
\[
\Theta_k(\epsilon) = \left\{ m(\cdot; k, \mu, w, \sigma) : \mu \in \otimes_{j=1}^{k}\left[ -b_n, b_n \right], \sigma \in (m_\sigma, M_\sigma) \right\},
\]
with \( k_n = \left\lfloor \frac{n\epsilon^2}{\log(n\epsilon^2)} \right\rfloor \), \( b_n = (n\epsilon^2)^{\frac{1}{3}}, m_\sigma = (n\epsilon^2)^{-\frac{1}{3}} \) and \( M_\sigma = \exp\left(\frac{1}{2}n\epsilon^2\right) \). With slight abuse of notation, we use \( \Pi \) to denote the prior distribution on both the density and the parameters. We have
\[
\Pi(\Theta_n(\epsilon)) \leq \Pi(\sigma \notin (m_\sigma, M_\sigma)) + \Pi(k > k_n) + \Pi(\max_j |\mu_j| > b_n).
\]
Now we derive an upper bound for each term.

1. Set \( \tau = \sigma^{-2} \), and then
\[
\Pi(\sigma \notin (m_\sigma, M_\sigma)) \leq \Pi(\tau \leq \exp(-n\epsilon^2)) + \Pi(\tau > (n\epsilon^2)^{1/b_3}) \leq b_0 \exp(-n\epsilon^2) + b_1 \exp(-b_2n\epsilon^2),
\]
where we have used the condition (41).

2. By the condition (36), we have
\[
\Pi(k > k_n) \leq C_1 \exp(-C_2k_n \log(k_n)) \leq C_1 \exp(-\tilde{C}_2n\epsilon^2).
\]
3. According to the conditions (36) and (38),

$$\Pi(\max_j |\mu_j| > b_n) = \sum_{k=1}^{\infty} \pi(k) \Pi\left(\max_{1 \leq j \leq k} |\mu_j| > b_n|k\right)$$

$$\leq \sum_{k=1}^{\infty} \pi(k) k \left(\int_{-\infty}^{-b_n} p_\mu(x)dx + \int_{b_n}^{\infty} p_\mu(x)dx\right)$$

$$\leq c_1 \exp(-c_2 n\epsilon^2) \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} \pi(k)$$

$$\leq c_1 \exp(-c_2 n\epsilon^2) \sum_{m=1}^{\infty} C_1 \exp(-C_2 m \log m)$$

$$\leq \tilde{c}_1 \exp(-c_2 n\epsilon^2).$$

Summing up the three bounds above, we have \(\Pi(\Theta_n(\epsilon)^c) \lesssim \exp(-C_0 n\epsilon^2)\) for some constant \(C_0 > 0\). In order that the constant \(C_0\) can be arbitrarily large, one can replace \(\epsilon\) by \(\tilde{C}\epsilon\) for a sufficiently large \(\tilde{C}\) and use the same argument above. We therefore obtain (A2).

Now we start to show (A1). By Theorem 7.1 in [14], it is sufficient to bound the metric entropy

$$\log N(\epsilon, \Theta_n(\epsilon), H) \lesssim n\epsilon^2.$$

Since \(H^2(P_1, P_2) \leq TV(P_1, P_2)\), we have \(\log N(\epsilon, \Theta_n(\epsilon), H) \leq \log N(\epsilon^2, \Theta_n(\epsilon), TV)\). According to (64),

$$N(\epsilon^2, \Theta_n(\epsilon), TV) \leq \sum_{k=1}^{k_n} N(\epsilon^2, \Theta_k(\epsilon), TV),$$

and thus it is sufficient to bound \(N(\epsilon^2, \Theta_k(\epsilon), TV)\) for each \(k \in [k_n]\).

We use \(\psi\) to denote \(\psi_\sigma\) with \(\sigma = 1\) in short. According to Lemma 3 in [17], for any \(m = m(\cdot; k, \mu, w, \sigma)\) and \(\tilde{m} = m(\cdot; k, \tilde{\mu}, \tilde{w}, \tilde{\sigma})\) in \(\Theta_k(\epsilon)\), we have

$$TV(m, \tilde{m}) \leq \|w - \tilde{w}\|_1 + 2\|\psi\|_\infty \sum_{i=1}^{k} \frac{w_i \wedge \tilde{w}_i}{\sigma \wedge \tilde{\sigma}} |\mu_i - \tilde{\mu}_i| + \frac{|\sigma - \tilde{\sigma}|}{\sigma \wedge \tilde{\sigma}}.$$

Based on the fact that \(N(\epsilon, A \times B, d_1 + d_2) \leq N(t\epsilon, A, d_1) \times N((1-t)\epsilon, B, d_2)\), we have

$$N(\epsilon^2, \Theta_k(\epsilon), TV) \leq N\left(\frac{\epsilon^2}{3}, \Delta_k, \|\cdot\|_1\right) N\left(\frac{m_\sigma \epsilon^2}{6\|\psi\|_\infty}, [-b_n, b_n]^k, \|\cdot\|_1\right) N\left(\frac{m_\sigma \epsilon^2}{3}, (m_\sigma, M_\sigma), |\cdot|\right).$$

Then, we use Lemma 5 in [17], and obtain

$$N\left(\frac{\epsilon^2}{3}, \Delta_k, \|\cdot\|_1\right) \leq \exp\left((k-1) \log \frac{15}{\epsilon^2}\right) \leq \exp(C_1 k \log(n\epsilon^2)), $$

$$N\left(\frac{m_\sigma \epsilon^2}{6\|\psi\|_\infty}, [-b_n, b_n]^k, \|\cdot\|_1\right) \leq \frac{k!(b_n + \epsilon)^k}{\epsilon^k} \leq \exp(C_2 k (\log k + \log n\epsilon^2)).$$
where \( \epsilon = \frac{m_2\sigma^2}{6\|\psi\|_\infty} \), and

\[
N\left( \frac{m_2\epsilon^2}{3}, (m_\sigma, M_\sigma), |\cdot| \right) \leq \frac{M_\sigma}{m_\epsilon^2/3} \leq \exp(C_3n\epsilon^2),
\]

for some constants \( C_1, C_2, C_3 > 0 \). Note that we have used the condition \( \epsilon > n^\delta \) for some constant \( \delta > -1/2 \) to derive the above bounds. Finally, we have

\[
N(\epsilon^2, \Theta_n(\epsilon), TV) \leq k_n \exp \left( C_1k_n \log n\epsilon^2 + C_2k_n(\log k_n + \log n\epsilon^2) + C_3n\epsilon^2 \right),
\]

which leads to

\[
\log N(\epsilon^2, \Theta_n(\epsilon), TV) \lesssim k_n \log(n\epsilon^2) \lesssim n\epsilon^2.
\]

The proof is complete. \( \square \)

**Proof of Lemma 6.17.** According to Lemma 6.15, there exist \((k_\sigma, \mu_\sigma, w_\sigma, \sigma)\) such that (63) holds. Then we consider \( k_0 = k_\sigma \) and \( \Theta^{(k_\sigma)} = \Theta^{(k_\sigma)}(\mu_\sigma) \otimes \Theta^{(w_\sigma)}(w_\sigma) \otimes \Theta^{(k_\sigma)}(\sigma) \), where \( \Theta^{(k_\sigma)}(\mu_\sigma) = \otimes_{j=1}^{k_\sigma} \Theta_{\mu_j}^{(k_\sigma)} \).

To be specific, let \( H_1 \) be any fixed constant such that \( H_1 > 2\alpha \), and then we define

\[
\Theta^{(k_\sigma)}(\mu_j) = [\mu_{\sigma,j} - k_\sigma^{-1} \sigma H_1 + 2\alpha + 2, \mu_{\sigma,j} + k_\sigma^{-1} \sigma H_1 + 2\alpha + 2],
\]

and

\[
\Theta^{(k_\sigma)}(\omega) = [\sigma, \sigma + \sigma H_1 + 2\alpha + 2].
\]

The conclusion of Lemma 6.15 implies

\[
\Theta^{(k_\sigma)} \subset \left\{ (\mu, \omega, \sigma) : nD_2(P_{\mu_\sigma}\|P_{m_{\mu_\sigma, \sigma}}) \leq C_2n\sigma^{2\alpha} \right\},
\]

for a constant \( C_2 > 0 \). Choose \( \sigma = n^{-\alpha/2+1}(\log n)^{\frac{t}{2\alpha+1}} \), then \( n^{\frac{1}{2\alpha+1}} \leq k_\sigma \leq n^{\frac{1}{2\alpha+1}+t} \) for any \( t > 0 \) as \( n \to \infty \). Then the condition (37) implies

\[
- \log \pi(k_\sigma) \lesssim k_\sigma \log k_\sigma.
\]

We also have

\[
\Pi_{\mu_j}(\Theta^{(k_\sigma)}) \geq \int_{\mu_{\sigma,j} - k_\sigma^{-1} \sigma H_1 + 2\alpha + 2}^{\mu_{\sigma,j} + k_\sigma^{-1} \sigma H_1 + 2\alpha + 2} p_\mu(x) dx.
\]

According to the condition (39) and Lemma 6.15, we have \( |\mu_j| \lesssim |\log \sigma|^{1/\xi_4} \) with \( \xi_4 < \min\{\xi_3, p\} \) as in Lemma 6.15. Then,

\[
- \log \Pi_{\mu_j}(\Theta^{(k_\sigma)}) \lesssim |\log \sigma| + |\log \sigma|^{c_6/\xi_4} \lesssim |\log \sigma|^{\max\{1,c_6/\xi_4\}}.
\]

By (40), we have

\[
- \log \Pi_{\mu_j}(\Theta^{(k_\sigma)}) \lesssim k_\sigma (\log k_\sigma)^{d_3} |\log \sigma|.
\]
Finally, the condition (42) leads to

\[- \log \Pi_\sigma(\Theta^{(k_\sigma)}) \leq - \log \left( \int_{(\sigma + p_H^2 + 2\alpha + 1)^{-1}}^{\sigma^{-1}} p_\tau(x) dx \right) \lesssim | \log \sigma | + \sigma^{-b_0} \lesssim \sigma^{-1}.

With the choice \( \xi_4 = \min\{p, \xi_3\} \) and \( k_\sigma = O(\sigma^{-1} | \log \sigma |^{p/\xi_4}) \), we have

\[- \log \pi(k_\sigma) - \sum_{j=1}^{k_\sigma} \Pi_{\mu_j}(\Theta^{(k_\mu_j)}) - \log \Pi_{\mu}(\Theta^{(k_\mu)}) - \log \Pi_\sigma(\Theta^{(k_\sigma)}) \leq C_3 \sigma^{-1} (\log \sigma)^r.

where \( r = \frac{p}{\min\{p, \xi_3\}} + \max\{d_3 + 1, \frac{c_6}{\min\{p, \xi_3\}}\} \). Plug in \( \sigma = n^{-\frac{1}{2\alpha + 1}} (\log n)^{\frac{r}{2\alpha + 1}} \), we obtain (A3*) with respect to \( \hat{f}_0 \).

Finally we prove Theorem 4.2.

Proof of Theorem 4.2. We bound \( P^n_{f_0} \hat{Q} H^2(P_f, P_{f_0}) \) by

\[
P^n_{f_0} \hat{Q} H^2(P_f, P_{f_0}) \leq P^n_{f_0} \hat{Q} H^2(P_f, P_{f_0}) + \text{TV}(P^n_{f_0}, P^n_{f_0})
\leq 2P^n_{f_0} \hat{Q} H^2(P_f, P_{f_0}) + 2H^2(P_{f_0}, P_{f_0}) + \text{TV}(P^n_{f_0}, P^n_{f_0}).
\]

By Lemma 6.16, Lemma 6.17 and Theorem 4.1, we have

\[
P^n_{f_0} \hat{Q} H^2(P_f, P_{f_0}) \lesssim \sigma^{2\alpha},
\]

for \( \sigma \asymp n^{-\frac{1}{2\alpha + 1}} (\log n)^{\frac{r}{2\alpha + 1}} \). Note that \( \hat{f}_0(x) = \frac{f_0^1 E_\sigma(x)}{\int_{E_\sigma} f_0(x) dx} \) with \( E_\sigma = \{ x : f_0(x) \geq \sigma^{H_1} \} \), and

\[
R = \int_{E_\sigma} f_0(x) dx \leq \sigma^{H_1/2} \int_{E_\sigma} \sqrt{f_0(x)} dx = O(\sigma^{H_1/2}).
\]

Then,

\[
H^2(\hat{f}_0, f_0) = 1 - \int \sqrt{f_0(x)\hat{f}_0(x)} dx = 1 - \sqrt{1 - R} = O(\sigma^{H_1/2}).
\]

Moreover,

\[
\text{TV}(P^n_{f_0}, P^n_{f_0}) = 1 - (1 - R)^n = O(nR) = O(n^{\sigma^{H_1/2}}).
\]

With the choice \( H_1 = 8\alpha + 4 \), the proof is complete. \( \square \)

6.7 Proofs of Theorem 5.1, Theorem 5.2, Proposition 3.1 and Theorem 5.3

The proof of Theorem 5.1 requires the following lemma.

Lemma 6.19. The variational posterior \( \hat{Q} \) with respect to the set \( S_{\text{MF}} \) is a product measure, with each coordinate in the form of

\[
g_j = \begin{cases} 
g_j, & j < k, 
p\delta_0 + (1-p)g_k, & j = k, 
\delta_0, & j > k, \end{cases}
\]

where \( g_j \) is a density of a continuous distribution, \( k \) is some integer and \( p \in [0,1) \).
Proof. In order that $D(\hat{Q}\|\Pi(\cdot|Y)) < \infty$, the support of $\hat{Q}$ must be contained in that of $\Pi(\cdot|Y)$. Thus, for any $B$ that $\Pi(B|Y) = 1$, we must have $\hat{Q}(B) = 1$. Because for each coordinate, 0 is the only point that the prior may put positive mass on, we can assume that $q_j = p_j g_j + (1 - p_j) \delta_0$, where $g_j$ is a density of a continuous distribution.

We use the notation $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. For each $k$, define

$$B_k = \{\theta = (\theta_j)_{j=1}^\infty : \theta_j \in \mathbb{R}_0 \text{ for } j \in [k] \text{ and } \theta_j = 0 \text{ for } j > k\}.$$  

Observe that for $j \neq k$, $B_j \cap B_k = \emptyset$. Next, we define the set $B = \cup_{k=0}^\infty B_k$.

By Bayes formula, the posterior distribution can be written as

$$d\Pi(\theta|Y) = \sum_{k=0}^\infty \pi(k|Y) \prod_{j \leq k} \tilde{f}_j(\theta_j) \prod_{j > k} \delta_0(\theta_j) d\theta,$$

where the density $\tilde{f}_j$ is determined by

$$\tilde{f}_j(\theta_j) \propto f_j(\theta_j) \exp\left(-\frac{n(\theta_j - Y_j)^2}{2}\right).$$

Therefore, we can write $\Pi(\cdot|Y) = \sum_k \pi(k|Y) \Pi_k$, where the distribution $\Pi_k$ is determined by

$$d\Pi_k(\theta) = \prod_{j \leq k} \tilde{f}_j(\theta_j) \prod_{j > k} \delta_0(\theta_j) d\theta.$$  

By the definition of $B_k$, it is easy to see that $\Pi_k(B_j) = 1$ for $j = k$ and $\Pi_k(B_j) = 0$ for $j \neq k$. Hence,

$$\Pi(B|Y) = \sum_k \pi(k|Y) \Pi_k(B) = \sum_k \pi(k|Y) \Pi_k(B_k) = \sum_k \pi(k|Y) = 1,$$

which then implies that $\hat{Q}(B) = 1$.

Note that for each $k$,

$$\hat{Q}(B_k) = \prod_{j \leq k} p_j \prod_{j > k} (1 - p_j),$$

and then

$$1 = \hat{Q}(B) = \sum_{k=0}^\infty \prod_{j \leq k} p_j \prod_{j > k} (1 - p_j).$$

For any $0 < k < s$,

$$1 \geq (1 - p_k) p_s + \sum_{k=0}^\infty \prod_{j \leq k} p_j \prod_{j > k} (1 - p_j)$$

$$= (1 - p_k) p_s + 1.$$  

Therefore, $(1 - p_k)p_s = 0$ for all $0 < k < s$, and there are three possible cases:
• \( p_j = 0 \) for all \( j \).
• \( p_j = 1 \) for all \( j \).
• \( p_j = 0 \) for \( j < k \), \( p_j = 1 \) for \( j > k \), and \( p_k \in [0, 1) \) for some \( k \in \mathbb{N} \).

However, the first two cases do not satisfy the constraint (69). Thus, the variational posterior \( \tilde{Q} \) is limited to the form (67), which completes the proof.

**Proof of Theorem 5.1.** By Lemma 6.19, the variational posterior has the form

\[
\frac{p \prod_{j<k} g_j \prod_{j\geq k} \delta_0 + (1-p) \prod_{j\leq k} g_j \prod_{j> k} \delta_0}{\pi(k-1|Y) d\Pi_{k-1} + \pi(k|Y) d\Pi_k} dQ_k,
\]

We denote the above distribution by \( Q_k \). Then, it is easy to see that \( Q_k(B_{k-1} \cup B_k) = 1 \). This implies

\[
D(Q_k||\Pi(\cdot|Y)) = \int \log \frac{dQ_k}{\pi(k-1|Y) d\Pi_{k-1} + \pi(k|Y) d\Pi_k} dQ_k,
\]

by the form of the posterior distribution (68). Therefore,

\[
D(Q_k||\Pi(\cdot|Y)) = \int_{B_{k-1}} \log \frac{dQ_k}{\pi(k-1|Y) d\Pi_{k-1} + \pi(k|Y) d\Pi_k} dQ_k + \int_{B_k} \log \frac{dQ_k}{\pi(k-1|Y) d\Pi_{k-1} + \pi(k|Y) d\Pi_k} dQ_k
\]

\[
= p \log \frac{p}{\pi(k-1|Y)} + (1-p) \log \frac{1-p}{\pi(k|Y)}
\]

\[
+ D\left( \prod_{j<k} g_j \prod_{j\geq k} \delta_0 || \Pi_{k-1} \right) + (1-p) D\left( \prod_{j\leq k} g_j \prod_{j> k} \delta_0 || \Pi_k \right)
\]

\[
= p \log \frac{p}{\pi(k-1|Y)} + (1-p) \log \frac{1-p}{\pi(k|Y)}
\]

\[
+ (1-p) D(g_k || \hat{f}_k) + \sum_{j<k} D(g_j || \hat{f}_j).
\]

Minimize \( D(Q_k||\Pi(\cdot|Y)) \) over \( g_j \) and \( p \), and we get \( g_j = \hat{f}_j \) for all \( j \leq k \) and \( p = \frac{\pi(k-1|Y)}{\pi(k-1|Y) + \pi(k|Y)} \).

Finally, we minimize over \( k \), which leads to \( \hat{k} = \text{argmax}_k (\pi(k-1|Y) + \pi(k|Y)) \). Use the formula

\[
\pi(k|Y) \propto \pi(k) \prod_{j\leq k} f_j(\theta_j) \exp\left(-\frac{n(\theta_j - Y_j)^2}{2}\right) d\theta_j \prod_{j> k} \exp\left(-\frac{nY_j^2}{2}\right),
\]

and the proof is complete.

**Proof of Theorem 5.2.** We use the notation

\[
W_j = \int f_j(\theta_j) \exp\left(-\frac{n(\theta_j - Y_j)^2}{2}\right) d\theta_j.
\]
By the condition $\|f_j\|_\infty \leq a$, we have $W_j \leq a\sqrt{\frac{2\pi}{n}} \leq 1$. Define the objective function

$$L(k) = \sum_{j<k} \log \frac{1}{W_j} + \sum_{j>k} \frac{nY_j^2}{2} - \log \left( \pi(k-1) \exp \left( -\frac{nY_k^2}{2} \right) + \pi(k)Z_k \right).$$

It is easy to check that

$$\tilde{k} = \arg\max_k (\pi(k-1|Y) + \pi(k|Y)) = \arg\min_k L(k).$$

To give a bound for $\tilde{k}$, we first study the difference $L(k_1) - L(k_2)$ for any $k_1 < k_2$. We use the inequalities

$$\log \left( \frac{\pi(k-1) \exp(-\frac{nY_k^2}{2}) + \pi(k)W_k}{\pi(k-1) + \pi(k)} \right) \leq \max \left\{ -\frac{nY_k^2}{2}, \log W_k \right\} \leq 0,$$

and

$$\log \left( \frac{\pi(k-1) \exp(-\frac{nY_k^2}{2}) + \pi(k)W_k}{\pi(k-1) + \pi(k)} \right) \geq \min \left\{ -\frac{nY_k^2}{2}, \log W_k \right\} \geq -\frac{nY_k^2}{2} + \log W_k.$$

Then, we have

$$L(k_1) - L(k_2) \leq \sum_{j=k_1}^{k_2} \frac{nY_j^2}{2} + \sum_{j=k_1+1}^{k_2-1} \log W_j + \log \left( \frac{\pi(k_2 - 1) + \pi(k_2)}{\pi(k_1 - 1) + \pi(k_1)} \right)$$

$$\leq \sum_{j=k_1}^{k_2} \frac{nY_j^2}{2} - (k_2 - k_1 - 1) \left( \frac{1}{2} \log n - \log (a\sqrt{2\pi}) \right)$$

$$\leq n \sum_{j=k_1}^{k_2} \theta_j^2 + \sum_{j=k_1}^{k_2} Z_j^2 - (k_2 - k_1 - 1) \left( \frac{1}{2} \log n - \log (a\sqrt{2\pi}) \right)$$

$$\leq nB^2k_1^{-2\alpha} + \sum_{j=k_1}^{k_2} Z_j^2 - (k_2 - k_1 - 1) \left( \frac{1}{2} \log n - \log (a\sqrt{2\pi}) \right),$$

where $Z_j \sim N(0, 1)$. Now we bound $\mathbb{P}_{\theta^*}^{(n)}(\tilde{k})$ by

$$\mathbb{P}_{\theta^*}^{(n)}(\tilde{k}) \leq Ck_0 + \sum_{l>Ck_0} l\mathbb{P}_{\theta^*}^{(n)}(\tilde{k} = l), \quad (70)$$

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where \( k_0 = \lceil \left( \frac{n}{\log n} \right)^{\frac{1}{2+\alpha}} \rceil \), and \( C \) is some large constant. For each \( l > Ck_0 \),

\[
\mathbb{P}_{\theta^*}^{(n)} (\hat{k} = l) \leq \mathbb{P}_{\theta^*}^{(n)} (L(l) \leq L(k_0)) \\
\leq \mathbb{P} \left( nB^2 k_0^{-2\alpha} + \sum_{j=k_0}^l Z_j^2 - (l - k_0 - 1) \left( \frac{1}{2} \log n - \log (a\sqrt{2\pi}) \right) \geq 0 \right) \\
\leq \mathbb{P} \left( \sum_{j=k_0}^l Z_j^2 \geq (l - k_0 - 1) \left( \frac{1}{2} \log n - \log (a\sqrt{2\pi}) \right) - C_1 \left( \frac{n}{\log n} \right)^{\frac{2\alpha}{2\alpha+1}} \right) \\
\leq \mathbb{P} \left( \sum_{j=k_0}^l Z_j^2 \geq c(l - k_0 - 1) \log n \right),
\]

where the last inequality is by the fact that \( C_1 \left( \frac{n}{\log n} \right)^{\frac{2\alpha}{2\alpha+1}} \) is of a smaller order than \((l - k_0 - 1) \log n\). Finally, a standard chi-squared tail bound gives

\[
\mathbb{P}_{\theta^*}^{(n)} (\hat{k} = l) \lesssim \exp \left( -C'(l - k_0) \log n \right).
\]

Using (70) and summing over \( l \), we get \( \mathbb{P}_{\theta^*}^{(n)} \hat{k} \lesssim k_0 \), and the proof is complete. \( \square \)

Proof of Proposition 3.1. According to Theorem 5.1, the variational posterior \( \hat{Q} \) is a product measure, and for any coordinate after a \( \hat{k} \), the component is \( \delta_0 \). By Theorem 5.2, we know that \( \mathbb{P}_{\theta^*}^{(n)} \hat{k} \leq C \left( \frac{n}{\log n} \right)^{\frac{1}{2+\alpha}} \). Use the notation \( \hat{k} = C \left( \frac{n}{\log n} \right)^{\frac{1}{2+\alpha}} \). Then, we have \( \mathbb{P}_{\theta^*}^{(n)} \left( \hat{k} > 2\hat{k} \right) \leq 1/2 \) by Markov inequality. Consider a \( \theta^* \) with every entry zero except that \( \theta^*_{[2\hat{k}]} = B[2\hat{k}]^{-\alpha} \). It is easy to check that \( \theta^* \in \Theta_\alpha(B) \). For this \( \theta^* \), we have

\[
\mathbb{P}_{\theta^*}^{(n)} Q \| \theta - \theta^* \|^2 \geq \mathbb{P}_{\theta^*}^{(n)} \hat{Q}(\theta_{[2\hat{k}]}) - \theta^*_{[2\hat{k}]})^2 \mathbb{I}\{\hat{k} \leq 2\hat{k}\} \\
= \theta^*_{[2\hat{k}]^2} \mathbb{P}_{\theta^*}^{(n)} (\hat{k} \leq 2\hat{k}) \\
\geq \frac{1}{2} \theta^*_{[2\hat{k}]^2} \\
\asymp n^{-\frac{2\alpha}{2\alpha+1}} (\log n)^{\frac{2\alpha}{2\alpha+1}}.
\]

Thus, the proof is complete. \( \square \)

Proof of Theorem 5.3. By the definition of \( \mathcal{S}_{\text{EB}} \), we consider \( Q_k \) with

\[
g_j = \begin{cases} 
g_j, & j \leq k, \\
\delta_0, & j > k. \end{cases}
\]

Using the same argument in the proof of Theorem 5.1, we have

\[
D(Q_k \| \Pi(\cdot | Y)) = \sum_{j \leq k} D(g_j \| f_j) + \log \frac{1}{\pi(k | Y)}.
\]

Therefore, \( g_j = \bar{f}_j \) and the optimal \( k \) is \( \hat{k} = \arg\max_k \pi(k | Y) \). The proof is complete. \( \square \)
6.8 Proof of Theorem 5.4

Proof of Theorem 5.4. Recall that

\[ d \tilde{Q}_{[k]} = \prod_{j \leq k} dN \left( \frac{n}{n + j^{2\beta+1}}, \frac{1}{n + j^{2\beta+1}} \right) \prod_{j=k+1}^n dN(0, e^{-jn}) \prod_j \delta_0. \]

Then, we can decompose the risk into

\[ P^{(n)}_{\theta^{\star}} \tilde{Q}_{[k]} \| \theta - \theta^{\star} \|^2 \leq \sum_{j \leq k} \left( \frac{j^{2\beta+1}}{n + j^{2\beta+1}} \right)^2 \theta_j^2 + \sum_{j > k} \theta_j^2 + 2 \sum_{j \leq k} \frac{1}{n + j^{2\beta+1}} + 2e^{-kn}. \]

For the upper bound, we have

\[ P^{(n)}_{\theta^{\star}} \tilde{Q}_{[k]} \| \theta - \theta^{\star} \|^2 \leq \sum_{j \leq k} \left( \frac{j^{2\beta+1}}{n + j^{2\beta+1}} \right)^2 \theta_j^2 + \sum_{j > k} \theta_j^2 + 2 \sum_{j \leq k} \frac{1}{n + j^{2\beta+1}} + 2e^{-kn}. \]

Now we discuss in the two cases:

- When \( k \leq n^{\frac{1}{2\beta+1}} \), we have

  \[ \sum_{j > k} \theta_j^2 \leq k^{-2\alpha} B^2, \quad \sum_{j \leq k} \frac{1}{n + j^{2\beta+1}} \leq \frac{k}{n}, \]

  and

  \[ \sum_{j \leq k} \left( \frac{j^{2\beta+1}}{n + j^{2\beta+1}} \right)^2 \theta_j^2 \leq \sum_{j \leq k} \frac{j^{4\beta+2-2\alpha}}{n^2} j^{2\alpha} \theta_j^2 \leq \frac{1 + k^{4\beta+2-2\alpha}}{n^2} B^2. \]

  Therefore,

  \[ P^{(n)}_{\theta^{\star}} \tilde{Q}_{[k]} \| \theta - \theta^{\star} \|^2 \lesssim k^{-2\alpha} + \frac{k}{n}. \]

- When \( k > n^{\frac{1}{2\beta+1}} \), we have

  \[ \sum_{j \leq k} \frac{1}{n + j^{2\beta+1}} \leq \frac{n^{\frac{1}{2\beta+1}}}{n} + \sum_{j > n^{\frac{1}{2\beta+1}}} j^{-2\beta-1} \lesssim n^{-\frac{2\beta}{2\beta+1}}, \]
and
\[ \sum_{j \leq k} \left( \frac{j^{2\beta+1}}{n + j^{2\beta+1}} \right)^2 \theta_j^2 \leq \sum_{j \leq n^{2\beta+1}} \frac{j^{4\beta+2-2\alpha}}{n^2} j^{2\alpha} \theta_j^2 + \sum_{j > n^{2\beta+1}} \theta_j^2 \lesssim n^{-\frac{2\alpha}{2\beta+1}}. \]

Thus, we have
\[ P_{\theta^*}^{(n)} \hat{Q}_{[k]} \| \theta - \theta^* \|^2 \lesssim n^{-\frac{2(a \wedge \beta)}{2\beta+1}}. \]

Now we prove the lower bound. According to the risk decomposition, we have
\[ P_{\theta^*}^{(n)} \hat{Q}_{[k]} \| \theta - \theta^* \|^2 \geq \sum_{j \leq k} \left( \frac{j^{2\beta+1}}{n + j^{2\beta+1}} \right)^2 \theta_j^2 + \sum_{j > k} \theta_j^2 + \sum_{j \leq k} \frac{1}{n + j^{2\beta+1}}. \]

- When \( k \leq n^{\frac{1}{2\beta+1}} \), we consider a \( \theta^* \) with every coordinate 0 except that \( \theta_{k+1}^* = (k + 1)^{-\alpha} B \). It is easy to check that \( \theta^* \in \Theta_\alpha(B) \). Then, we have \( \sum_{j \leq k} \frac{1}{n + j^{2\beta+1}} \geq \frac{k}{2n} \) and \( \sum_{j > k} \theta_j^2 \geq B^2 (k + 1)^{-2\alpha} \). Therefore,
\[ \sup_{\theta^* \in \Theta_\alpha(B)} P_{\theta^*}^{(n)} \hat{Q}_{[k]} \| \theta - \theta^* \|^2 \gtrsim k^{-2\alpha} + \frac{k}{n}. \]

- When \( k > n^{\frac{1}{2\beta+1}} \), we consider a \( \theta^* \) with every coordinate 0 except that \( \theta_{\lfloor n^{\frac{1}{2\beta+1}} \rfloor}^* = (\lfloor n^{\frac{1}{2\beta+1}} \rfloor)^{-\alpha} B \), and it is easy to check that \( \theta^* \in \Theta_\alpha(B) \). Then, we have
\[ \sum_{j \leq k} \left( \frac{j^{2\beta+1}}{n + j^{2\beta+1}} \right)^2 \theta_j^2 \geq \frac{1}{4} \theta_{\lfloor n^{\frac{1}{2\beta+1}} \rfloor}^2 \gtrsim n^{-\frac{2\alpha}{2\beta+1}}, \]

and
\[ \sum_{j \leq k} \frac{1}{n + j^{2\beta+1}} \gtrsim \sum_{j \leq n^{\frac{1}{2\beta+1}}} \frac{1}{n + j^{2\beta+1}} \gtrsim n^{-\frac{2\beta}{2\beta+1}}. \]

This leads to the lower bound
\[ \sup_{\theta^* \in \Theta_\alpha(B)} P_{\theta^*}^{(n)} \hat{Q}_{[k]} \| \theta - \theta^* \|^2 \gtrsim n^{-\frac{2(a \wedge \beta)}{2\beta+1}}. \]

Now the proof is complete. \( \square \)

### 6.9 Proof of Theorem 5.5

**Proof of Theorem 5.5.** By Lemma 6.1, we have
\[ a\hat{Q}_* L(P_{\theta}^{(n)}, P_0^{(n)}) \leq D(\hat{Q}_* \| \Pi(\cdot | X^{(n)}) \) + log \Pi(exp(aL(P_{\theta}^{(n)}, P_0^{(n)}))) | X^{(n)}). \]

Then, under the conditions of Theorem 5.5, we have
\[ D(\hat{Q}_* \| \Pi(\cdot | X^{(n)})) \leq D_*(\hat{Q}_* \| \Pi(\cdot | X^{(n)}) \) \leq D_*(Q \| \Pi(\cdot | X^{(n)})), \]

for all \( Q \in \mathcal{S} \). Then, following the same argument in the proof of Theorem 2.1, we complete the proof. \( \square \)
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