Non-equilibrium, time-dependent effective theory of a weakly coupled superconductor at finite temperature.

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Abstract

We perform a well defined derivative expansion to obtain the time dependent effective theory for a BCS superconductor at finite temperature, using an arbitrary curve in the complex time plane. Our expansion is unique, being free of any problems related to the order in which analytical continuation and the derivative expansion are made, or equivalently to the order in which the zero momentum and zero energy limits are taken. In other words Landau damping terms are present but do not lead to the pathologies found in standard approaches. We discuss the reason for this, suggesting that these standard approaches may be internally inconsistent. The methods presented here are an attempt to overcome those inconsistencies, and finally, our results are compared to others in the literature.

1 Introduction

Effective actions have many applications in physics, since they simplify the description of highly complicated physical systems by integrating out all but the relevant degrees of freedom. As a result one is left with effective equations for the physically significant modes, with coefficients and couplings depending on the rest of the content of the theory. In the case of superconductors, static phenomena are well described by the effective Lagrangian of Ginzburg and Landau \cite{1}, and this can be derived from the microscopic BCS theory (see for example \cite{2, 3} and references within). In relativistic particle physics, effective actions play an important rôle in inflationary theories of cosmology \cite{4}, and can be used to describe low energy QCD and electro-weak phenomena \cite{5} to name but a few. In the simplest case of fields which are constant in space and time, the effective action reduces to the effective potential multiplied by a space-time volume factor.

However, to study the non-equilibrium dynamics say of a phase transition, be it in superconductors, inflation, or QCD, space and time dependent corrections to the effective potential are required. To obtain these, one often carries out a derivative expansion, the
lowest order of which gives the effective potential. Calculations of the derivative expansion have been successfully carried out at zero temperature $T = 0$ where there is no Landau-damping, e.g. see [3, 6, 7] in the case of BCS superconductors, and references [5, 8, 9] for relativistic contexts. But at finite temperature, $T > 0$, the problems arising from Landau-damping have led to a failure of the derivative expansion both for BCS theory [10, 11, 12, 13] and in relativistic models [14, 15, 16, 17, 18, 19, 20, 21, 22].

So far, suggested solutions to this problem have been limited. One common approximation which is often used to produce an effective theory is simply to replace the classical potential by the finite temperature effective potential, keeping the same classical kinetic terms. Indeed this approach is often used in simple models of inflation [4] and bubble nucleation in first order phase transitions (see for example [23]). It is very difficult to justify such methods, however, as the derivative expansions which should be used to verify them fail at $T > 0$ as commented above. In the case of BCS superconductors, Stoof [11] appreciated the problems but simply dropped the Landau damping terms. The same is true of [2]. Aitchison and Lee [13] try another approach. They include the effect of thermal damping, i.e. resum some effects from higher order terms into the propagators [13]. This solves the technical problem with the derivative expansion and is physically realistic, but experience suggests that it can be difficult to implement such resummations consistently [24, 25]. It is also technically very difficult to work with these methods. Many other authors have also tackled this question (see [3] and references within).

The purpose of this paper is to study once more the question of derivative expansions at $T > 0$, and here it is done in the context of BCS theory. As mentioned above, this question has been the subject of numerous papers, not only in the case of superconductors [3, 10, 11, 12, 13], but also in the application of derivative expansions to particle physics and cosmology [14, 15, 16, 17, 18, 19, 20, 21, 22]. Despite this, we feel that this paper is justified as it raises some questions and potential problems with those previous calculations. Indeed, we will argue that we believe those calculations to be internally inconsistent and therefore incorrect. It is then argued that this inconsistency is responsible for the badly defined and non-unique $T > 0$ derivative expansions which were previously obtained.

The paper is set up in the following way. In section 2 the effective action for the pair field is derived from the BCS Lagrangian. This effective action can only be evaluated within an approximation scheme, whose first component is a weak field expansion (section 3). The second component is a derivative expansion (sections 4-6). Section 4 discusses the reasons for which we believe the previous calculations of the derivative expansion to be internally inconsistent, whilst section 5 presents our own calculation scheme. Throughout we work at finite temperature and in arbitrary path ordered approaches to thermal field theory—that is, we do not work specifically in the imaginary- or real-time formalism, but with an arbitrary contour $C$ in the complex time plane. As usual, the contour starts at $t_i$ and ends at $t_i - i\beta$ (where $\beta = 1/T$), and our results are explicitly shown to be the same for any $C$. To carry out the calculation we work throughout in $(t,\vec{p})$ space, and furthermore on its implementation in section 5 we see that Landau-damping no longer leads to problems with our derivative expansion which turns out to be well defined. The explicit results of our calculation are given in section 6. There we work for simplicity in the degenerate gas approximation [2] and present results for the effective potential as well as higher order corrections in time and space gradients. From those results the effective equations of motion for the Cooper pair field are seen to break time translation invariance, having an explicit dependence on the initial time $t_i$ at which the system was set up. We believe this to be reasonable as by the very act of performing a derivative expansion one
is dealing with time-dependent non-equilibrium systems. Our effective equations are also seen to contain both oscillatory and dissipative terms for all times \( t > t_i \). Finally, though we believe that the method presented here is an improvement on previous ones, it raises some confusing issues and may still not be entirely satisfactory. Such issues are discussed with the conclusions in section 7.

## 2 The effective action

In this section we derive the effective action for the auxiliary (pair) field \( \Delta \) [2, 3]. We work in Minkowski space in (3+1) dimensions and use the notation \( x = (t, \vec{x}) \) where, as usual in finite temperature calculations, the times can be complex. The starting point is the microscopic BCS Lagrangian

\[
\mathcal{L} = \sum_{s=\uparrow, \downarrow} \psi_s^* \left( i\partial_t + \frac{\nabla^2}{2m} + \mu \right) \psi_s + g \psi^*_\uparrow \psi^*_\downarrow \psi_\downarrow \psi_\uparrow. \tag{2.1}
\]

Here \( \psi_s \) are anticommuting fermionic field operators, \( m \) is the electron mass and \( \mu \) is the chemical potential. The last term leads to the formation of bound states (Cooper pairs) for \( g > 0 \). As usual, it is useful to work with the two component fields

\[
\Psi = \begin{pmatrix} \psi_\uparrow \\ \psi^*_\downarrow \end{pmatrix}; \quad \Psi^\dagger = (\psi^*_\uparrow, \psi_\downarrow)
\]

so that in equilibrium at finite temperature \( T = 1/\beta \) the partition function is

\[
Z = \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger \exp \left[ i \int_C d^4x \mathcal{L} \right]. \tag{2.2}
\]

Here \( \int_C d^4x = \int_C dt \int_{-\infty}^{\infty} d^3x \) where \( C \) is any contour in the imaginary time plane starting at \( t_i \) and ending at \( t_i - i\beta \) [26, 27]: here we do not restrict the calculation to any particular thermal field theory formalism. In this path integral representation the \( \Psi \)'s are now fields (and no longer operators) which are anti-periodic;

\[
\Psi(t, \vec{x}) = -\Psi(t - i\beta, \vec{x}). \tag{2.3}
\]

In the usual way, one can rewrite the quartic term in (2.1) through the identity

\[
\exp \left[ ig \int_C d^4x \psi_\uparrow \psi^*_\downarrow \psi_\downarrow \psi_\uparrow \right] = \int \mathcal{D}\Delta^* \mathcal{D}\Delta \exp \left[ i \int_C d^4x \left( \Delta^*_\downarrow \psi_\uparrow + \Delta^*_\uparrow \psi^*_\downarrow - \frac{1}{g} |\Delta|^2 \right) \right]. \tag{2.4}
\]

which on substitution into (2.2) and performing the Gaussian integral over the Grassman fields gives

\[
Z = \int \mathcal{D}\Delta^* \mathcal{D}\Delta \exp \left[ i \tilde{S}_{\text{eff}}[\Delta^*, \Delta] \right]. \tag{2.5}
\]

Here

\[
\tilde{S}_{\text{eff}}[\Delta^*, \Delta] = S_{\text{eff}}[\Delta^*, \Delta] - \frac{1}{g} \int_C d^4x \Delta^* \Delta \tag{2.6}
\]

where the effective action \( S_{\text{eff}} \) is given by

\[
S_{\text{eff}}[\Delta^*, \Delta] = -i \text{Tr} \ln K^{-1} = -itr \int_C d^4x \ln K^{-1}(x, x) \tag{2.7}
\]
and the matrix $K^{-1}$ by

$$ K^{-1}(x,y) = \left( \begin{array}{cc} i \partial_t + \frac{\Sigma^2}{2m} + \mu & \Delta(t,\bar{x}) \\ \Delta^*(t,\bar{x}) & i \partial_t - \frac{\Sigma^2}{2m} - \mu \end{array} \right) \delta^4(x-y). \quad (2.8) $$

Finally, in equation (2.7), ‘tr’ means the trace over matrix components.

The purpose of the remainder of this paper is to calculate $S_{\text{eff}}$. Such a calculation must clearly be done within an approximation scheme, and this consists first of a weak field expansion (section 3) and then a derivative expansion (sections 4 and 5).

3 The Weak Field Expansion

The first component of the approximation scheme is a weak field expansion. We work in the symmetry broken phase, so that $0 \leq T < T_C$ where $T_C$ is the critical temperature, and then expand $\Delta(t,\bar{x})$ in the effective action $S_{\text{eff}}[\Delta^*,\Delta]$ by writing

$$ \Delta(t,\bar{x}) = (|\Delta_0| + \phi(t,\bar{x})) e^{i\theta_0}. \quad (3.1) $$

Here $\Delta_0 = |\Delta_0| e^{i\theta_0}$ is the $x$ independent solution of the gap equation

$$ \frac{\delta S_{\text{eff}}}{\delta \Delta^*} \bigg|_{\Delta=\Delta_0} = \frac{\delta S_{\text{eff}}}{\delta \Delta} \bigg|_{\Delta=\Delta_0} - \frac{1}{g} \Delta_0 = 0. \quad (3.2) $$

One can absorb the phase by making a symmetry transformation so without loss of generality we will take $\Delta_0$ real, $\theta_0 = 0$.

In order to expand about $\Delta_0$, first introduce the matrices $\Phi(x)$ and $G^{-1}(x,y)$ defined by

$$ \Phi(x) = \left( \begin{array}{cc} 0 & \phi(t,\bar{x}) \\ \phi^*(t,\bar{x}) & 0 \end{array} \right) $$

$$ G^{-1}(x,y) = \left( \begin{array}{cc} i \partial_t + \frac{\Sigma^2}{2m} + \mu & \Delta_0 \\ \Delta_0 & i \partial_t - \frac{\Sigma^2}{2m} - \mu \end{array} \right) \delta^4(x-y), \quad (3.3) $$

and also the inverse matrix $G(x,y)$ which satisfies

$$ \int d^4y \ G^{-1}(x,y) \ G(y,z) = \delta^4(x-z) \mathbb{I}. \quad (3.4) $$

Now, by definition of $K^{-1}$ in (2.8), it follows that

$$ K^{-1}(x,y) = G^{-1}(x,y) + \Phi(x) \delta^4(x-y) $$

$$ = \int d^4z \ G^{-1}(x,z) \left[ \delta^4(x-z) \mathbb{I} + G(z,y) \Phi(x) \right]. \quad (3.5) $$

Hence substitution of (3.5) into (2.7) followed by an expansion of the logarithm in powers of $\phi(x)$ gives

$$ S_{\text{eff}}[\Delta^*,\Delta] = S_{\text{eff}}^{(0)}[\phi^*,\phi] + S_{\text{eff}}^{(1)}[\phi^*,\phi] + S_{\text{eff}}^{(2)}[\phi^*,\phi] + \mathcal{O}(\phi^3) \quad (3.6) $$
where

\[
S_{\text{eff}}^{(0)}[\phi^*, \phi] = -i \text{tr} \int d^4x \ln G^{-1}(x, x)
\]

\[
S_{\text{eff}}^{(1)}[\phi^*, \phi] = -i \text{tr} \int d^4x \, G(x, x)\Phi(x)
\]

\[
S_{\text{eff}}^{(2)}[\phi^*, \phi] = \frac{i}{2} \text{tr} \int d^4x \int d^4y \, G(x, y)\Phi(y)G(y, x)\Phi(x).
\]  

(3.7)

(3.8)

Notice that whilst the first two terms in the expansion of \(S_{\text{eff}}\) are local, the third one \(S_{\text{eff}}^{(2)}\) is non-local, as indeed are the higher order terms in \(\phi\). The purpose of the derivative expansion, which is the second ingredient in the approximation scheme used to calculate \(S_{\text{eff}}\), is to remove this non-locality. Here we focus on \(S_{\text{eff}}^{(2)}\), and the derivative expansion is done by Taylor expanding \(\Phi(y)\) in (3.8) as

\[
\Phi(y) = \Phi(x) + (y - x)\frac{\partial \Phi(x)}{\partial x} + \ldots.
\]

It is this derivative expansion which brings us into contact with the technical problem of ill-defined zero energy and momentum limits of thermal Green functions, and these in turn cause the inconsistencies usually found in such expansions at finite temperature [10, 11, 12, 13].

4 The derivative expansion

Before carrying out the above mentioned derivative expansion on \(S_{\text{eff}}^{(2)}\), we try to summarise the different approaches to this expansion that have previously been used in the literature [2, 3, 10, 11, 13].

First consider the procedure carried out by authors who have worked in the imaginary time formalism of thermal field theory [2, 3, 10, 13]. Typically their calculations proceed through the following stages. Normally the first step is to Fourier transform equation (2.7) into momentum space to give

\[
S_{\text{eff}}^{(2)} = \frac{-i}{2} \text{tr} \sum_{k_0} \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_{p_0} \int \frac{d^3 \vec{p}}{(2\pi)^3} \Phi(p)G(k - p)\Phi(-p)G(k).
\]  

(4.1)

A simple but important point to note here is that the energies, \(k_0, p_0\), have been taken to be imaginary and discrete as is typical for systems in equilibrium. The second step is to expand expression (4.1) about \(p = 0\) in powers of \(p_0\) and \(\vec{p}\), and then evaluate the sums and integrals over \(\vec{k}\). Finally, the resulting expressions are converted back into real times (that is some analytic continuation is carried out) to give an expression for \(S_{\text{eff}}^{(2)}\) in terms of derivatives of \(\phi(x)\). Notice that in the first step of these calculations—that is, in writing down equation (4.1) with the discrete momenta—both space and time translation invariance have been assumed, and furthermore the fields and propagators involved in the rest of the calculation are those of equilibrium systems. It follows that in the second step, discrete momenta (or equivalently equilibrium fields) have been used within the derivative expansion itself—below we will comment further on this point.

The aim of this section is to discuss whether or not the operations outlined in the above paragraph are in fact consistent.
To that effect consider now expression (3.8) in \((t, \vec{x})\) space (rather than the Fourier transformed expression (4.1)). In particular, focus on the time component of (3.8) since it is this which changes between the \(T = 0\) and \(T > 0\) calculation, and also let \(x = (t, \vec{x})\) and \(y = (t', \vec{y})\). The derivative expansion of \(S_{\text{eff}}^{(2)}\) is obtained by writing

\[
\Phi(t') = \Phi(t) + (t' - t) \frac{\partial \Phi(t)}{\partial t} + \ldots
\]  

(4.2)

Substitute (4.2) into (3.8) and consider the contribution to \(S_{\text{eff}}^{(2)}\) which arises say from the term containing the time derivative \(\partial_t \Phi(t)\). Clearly this term is not periodic in \(t'\) in the imaginary time direction since \(t' - t - i\beta \neq t' - t\). Indeed this conclusion holds for every individual term in the derivative expansion of (2.7), apart from the \(t'\)-independent contribution.

Thus we arrive at the conclusion that each term in the derivative expansion of (3.8) is not periodic in imaginary time. (One should perhaps not be surprised at this—by the very essence of the derivative expansion one is dealing with slowly time varying and therefore non-equilibrium systems.) Hence it follows that at each order in the derivative expansion, taking a Fourier transform would not give rise to discrete energies, and hence it is not correct to use discrete energies (which correspond to periodic fields) if one were to work with Euclidean energies. However, we have stressed in the first paragraph of this section that authors who previously calculated derivative expansions in the ITF formalism did assume such discrete energies. We therefore believe that these previous calculations are inconsistent.

Before suggesting a method to overcome this inconsistency, note that the above discussion has emphasised the imaginary time approach to thermal field theory whilst some authors have used a real time approach [11]. Recall though that these alternative real-time methods always give the same answer as the imaginary time calculations for any physical quantity [28, 29, 30]. In particular, this is true when one compares results for thermal effective actions, provided note is taken of various technical issues [19, 31, 32]. We now rephrase the above arguments in a way suitable for real-time methods. This is done by discussing the periodicity of the fields—a boundary condition which is common to all path-integral approaches to thermal field theory. The generalisation of expression (4.2) to infinite order is

\[
\Phi(t') = e^{iE'(t'-t)}\Phi(t''\mid t''=t)
\]  

(4.3)

where

\[
E'' = -i \frac{\partial}{\partial t''}.
\]

If the left hand side, \(\Phi(t')\), is periodic in imaginary time, then for equality the right hand side must also be periodic. This is assumed in previous calculations [11] and is clearly only true if \(e^{\beta E''} = 1\), a condition which was imposed in those calculations. However, as was noted above, each individual term in the derivative expansion (in other words the expansion of the exponential in (4.3)), is not periodic. Furthermore, since one always truncates the series and works to a finite order in derivatives, it follows that the approximation to the field \(\Phi(t)\) used in the expansion cannot be periodic. Thus, we believe that setting \(e^{\beta E''} = 1\) is not consistent with the derivative expansion. Again, we stress that since this was done in previous calculations, we believe these to be inconsistent.

The approach to the derivative expansion presented in this paper therefore has one crucial difference with previous calculations. As shown above, each term in the derivative expansion is not periodic, and therefore our calculation does not assume periodicity. This
is equivalent to saying that we allow $e^{\beta E''} \neq 1$, or equivalently that we do not work with discrete momenta for the field $\Phi(t)$—as is typical of non-equilibrium systems. Given the lack of discrete momenta, it is difficult to work in $(p_0, \vec{p})$ space and instead we work in $(t, \vec{p})$ space. This is in fact a great simplification as it turns out that all the time integrations can be done exactly for any arbitrary thermal field theory formalism, and thus it avoids all the above mentioned issues. Furthermore, there are no problems with analytic continuation, which is another point we feel has been passed over rather hastily in the existing calculations. Finally, note that consistently with these discussions, we obtain the same results as in the existing literature [10, 13] if we set $e^{\beta E''} = 1$.

Our convention for Fourier transforms is

$$f(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} f(t, \vec{k}) e^{-i\vec{k}.\vec{x}}.$$  

5 The calculation

For the reasons discussed in the previous section, namely that each term in the derivative expansion leads to non-periodic functions, we work throughout in $(t, \vec{k})$ space. The first part of the calculation, section 5.1, consists of determining the propagator $G(t, t', \vec{k})$ which appears in $S_{\text{eff}}^{(2)}$. The gap equation is then calculated in section 5.2 whilst the derivative expansion of $S_{\text{eff}}^{(2)}$ is set up in section 5.3. There we will see that by working with the time variables, the integrations can indeed be done explicitly for an arbitrary thermal field theory formalism. At the same time this avoids the issues of analytic continuation which complicate some approaches [28, 30, 33].

5.1 The fermion propagator

As was discussed in section 4, the fermion propagator $G(t, t', \vec{k})$ is in fact an equilibrium propagator since the Fermi fields are in equilibrium by equation (2.3). Its calculation therefore follows standard methods which we outline here; the details are given in appendix A.

First use (3.3) to define the matrix $G^{-1}(x)$ through

$$G^{-1}(x)\delta^4(x - y) := G^{-1}(x, y)$$

so that in $(t, \vec{k})$ space

$$G^{-1}(t, \vec{k}) = \begin{pmatrix} i\partial_t - \frac{\vec{k}^2}{2m} + \mu & \Delta_0 \\ \Delta_0 & i\partial_t + \frac{\vec{k}^2}{2m} - \mu \end{pmatrix} = \begin{pmatrix} i\partial_t - \epsilon_{\vec{k}} & \Delta_0 \\ \Delta_0 & i\partial_t + \epsilon_{\vec{k}} \end{pmatrix}$$  \hspace{1cm} (5.1)

where $\epsilon_{\vec{k}}$ is the energy measured from the Fermi surface;

$$\epsilon_{\vec{k}} = \frac{\vec{k}^2}{2m} - \mu.$$  

1Regarding the discussions of (anti-)periodicity made above, it should be noted that $G(t, \vec{k})$ is the propagator for the Fermi fields, which by definition in equation (2.3) are anti-periodic in imaginary time. Hence $G(t, \vec{k})$ is anti-periodic; $G(t, \vec{k}) = -G(t - i\beta, \vec{k})$. Regarding $\Phi(t, \vec{k})$, or equivalently the field $\Delta(t, \vec{k})$, we have shown that for the derivative expansion to be consistent at any finite order, $\Delta(t, \vec{k})$ cannot be periodic. That this is the case could be achieved by adding a small time dependent perturbation to $\Delta$, and we are then watching the system respond to this. In addition, it should be noted that in equation (2.4) which introduces the $\Delta(t, \vec{k})$ field, there is nothing to constrain $\Delta$ to be periodic in the first place.
As usual in the BCS limit in which we are working here, the Fermi wavenumber is defined through the chemical potential \( \mu \) by \( k_F^2 = 2m\mu \).

Now, from equation (3.4), \( G(t,t',\vec{k}) \) satisfies

\[
G^{-1}(t,\vec{k})G(t,t',\vec{k}) = \delta(t - t')\mathbf{1}.
\]

In the usual way one can solve this equation for \( G(t,t',\vec{k}) \) by diagonalising the operator \( G^{-1}(t,\vec{k}) \) which is given in (5.1) (see appendix A). The result is

\[
G(t,t',\vec{k}) = \begin{pmatrix}
u_+^2 F_-(t,t',\vec{k}) + u_-^2 F_+(t,t',\vec{k}) & u_+ u_-(F_+(t,t',\vec{k}) - F_-(t,t',\vec{k})) \\ u_+ u_- (F_+(t,t',\vec{k}) - F_-(t,t',\vec{k})) & u_-^2 F_+(t,t',\vec{k}) + u_+^2 F_-(t,t',\vec{k}) \end{pmatrix}.
\]

Here \( u_\pm \) are functions of \( \vec{k} \) (this dependence has not been written in (5.2) for clarity), and are given by

\[
u_\pm(\vec{k}) = + \left( \frac{1}{2} \left[ 1 \pm \frac{\epsilon_\vec{k}}{E_\vec{k}} \right] \right)^{\frac{1}{2}}
\]

where \( E_\vec{k} \) is the dispersion relation

\[
E_\vec{k} = + \left[ |\Delta_0|^2 + \epsilon_\vec{k}^2 \right]^{\frac{1}{2}}.
\]

It is important to recognise that \( u_+(\vec{k}) \) and \( u_-(\vec{k}) \) are just the usual functions often called \( u(\vec{k}) \) and \( v(\vec{k}) \) respectively which, in the operator approach, are the coefficients of the Bogoliubov transformation which diagonalises the Lagrangian. We have chosen this different notation as later (in section 5) it will enable us to write our results in a significantly more compact form.

As a result of the diagonalisation procedure (appendix A), the functions \( F_\pm(t,t',\vec{k}) \) which appear in (5.2) are seen to satisfy the Feynman equation

\[
(i\partial_t \pm E_\vec{k})F_\pm(t,t',\vec{k}) = \delta(t - t')\mathbf{1}.
\]

Hence their solutions are known, and for our purposes they are best given in a Mills representation [34, 32];

\[
F_\pm(t,t',\vec{k}) = -i \int \frac{dk_0}{2\pi} e^{-ik_0(t-t')} (\theta_C(t,t') - N(k_0)) \rho_\pm(k_0,\vec{k}).
\]

Here \( N(k_0) \) is the Fermi-Dirac distribution:

\[
N(k_0) = \frac{1}{e^{\beta k_0} + 1},
\]

the \( \theta_C \) is the usual \( \theta \) function for the contour \( C \) [26, 27], and the spectral functions \( \rho \) are given by

\[
\rho_\pm(k_0,\vec{k}) = 2\pi \delta(k_0 \pm E_\vec{k}).
\]

Finally we note that \( F_+(t,t',\vec{k}) = F_-(t',t,\vec{k}) \).
5.2 The Linear terms and the Gap Equation

Before turning to the calculation of $S_{\text{eff}}^{(2)}$ in (3.8), we consider briefly the gap equation (3.2) or in other words the terms linear in $\phi$ in $\tilde{S}_{\text{eff}}$. As usual the gap equation is a necessary tool in the evaluation of $S_{\text{eff}}^{(2)}$, as it is used to remove UV divergences (see below).

Equation (3.7) gives $S_{\text{eff}}^{(1)}$ to be

$$S_{\text{eff}}^{(1)}[\Delta^*, \Delta] = -i \int_C d^4x (G_{12}(0)\phi^*(x) + G_{21}(0)\phi(x))$$

where from (5.2)

$$G_{12}(0) := G_{12}(x, x) = i \int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0}{2E_k} (F_+(t, t, k) - F_-(t, t, k))$$

$$= -i \int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0}{2E_k} (1 - 2N(E_k)).$$

Thus on substitution into (3.2) one obtains

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} (1 - 2N(E_k)) = -\frac{1}{g}$$

(5.5)

which agrees, for example, with Stoof [11]. Notice that the first term on the left hand side of (5.5) is linearly divergent since $E_k \propto k^2$ for large $k$.

5.3 Quadratic Terms

Now consider the important quadratic term $S_{\text{eff}}^{(2)}$ of the effective action (2.7). Substitute the propagator (5.2) into (3.8) Fourier transformed into $(t, \vec{k})$ space. On taking the trace in (3.8) one obtains four terms. Two of these are proportional to $\phi^*(t, \vec{p})\phi(t', -\vec{p})$ and $\phi(t, \vec{p})\phi^*(t', -\vec{p})$ and, as is common in the literature, we call these the diagonal terms. The non-diagonal terms are proportional to $\phi^*(t, \vec{p})\phi^*(t', -\vec{p})$ and $\phi(t, \vec{p})\phi(t', -\vec{p})$ and in fact the coefficients of these two non-diagonal terms will be seen to be identical. Thus we write

$$S_{\text{eff}}^{(2)} = S_{\text{eff}}^{(2)}[\text{Da}] + S_{\text{eff}}^{(2)}[\text{Db}] + S_{\text{eff}}^{(2)}[\text{ND}]$$

(5.6)

where the D and ND stand for diagonal and non-diagonal as explained above. The diagonal terms are given by

$$S_{\text{eff}}^{(2)}[\text{Da}] = \frac{i}{2} \int_C dt \int_C dt' \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{k}}{(2\pi)^3} \phi(t, \vec{p})\phi^*(t', -\vec{p}) \times$$

$$\times \left[u_+^2(\vec{k} - \vec{p}/2)F_+(t, t', \vec{k} - \vec{p}/2) + u_-^2(\vec{k} - \vec{p}/2)F_-(t, t', \vec{k} - \vec{p}/2)\right] \times$$

$$\times \left[u_+^2(\vec{k} + \vec{p}/2)F_-(t', t, \vec{k} + \vec{p}/2) + u_-^2(\vec{k} + \vec{p}/2)F_+(t', t, \vec{k} + \vec{p}/2)\right]$$

(5.7)

and

$$S_{\text{eff}}^{(2)}[\text{Db}] = \frac{i}{2} \int_C dt \int_C dt' \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{k}}{(2\pi)^3} \phi^*(t, \vec{p})\phi(t', -\vec{p}) \times$$

$$\times \left[u_+^2(\vec{k} + \vec{p}/2)F_+(t', t, \vec{k} + \vec{p}/2) + u_-^2(\vec{k} + \vec{p}/2)F_-(t', t, \vec{k} + \vec{p}/2)\right]$$

$$\times \left[u_+^2(\vec{k} - \vec{p}/2)F_-(t, t', \vec{k} - \vec{p}/2) + u_-^2(\vec{k} - \vec{p}/2)F_+(t, t', \vec{k} - \vec{p}/2)\right].$$

(5.8)
Observe that interchange of $t$ and $t'$ in (5.7) followed by the relabelling $\vec{p} \to -\vec{p}$ yields
$S_{\text{eff}}^{(2)}[\mathcal{B}] = S_{\text{eff}}^{(2)}[\mathcal{D}]$. However, consider now any given term in the derivative expansion of $S_{\text{eff}}^{(2)}[\mathcal{D}]$, which is obtained by writing $\phi^*(t', -\vec{p}) = \phi^*(t, -\vec{p}) + (t' - t)\partial_t \phi^*(t, -\vec{p}) + \ldots$ in (5.7). It is straightforward to see that the changes $t' \leftrightarrow t$ and $\vec{p} \to -\vec{p}$ for that term do not yield the corresponding term of $S_{\text{eff}}^{(2)}[\mathcal{B}]$ (the only exception is the $t'$ independent contribution). That is, whilst this change of variables will yield $S_{\text{eff}}^{(2)}[\mathcal{B}] = S_{\text{eff}}^{(2)}[\mathcal{D}]$ to infinite order in the derivative expansion, this is not true to any finite order. Since the derivative expansion is always truncated, it is therefore necessary to work with $S_{\text{eff}}^{(2)}[\mathcal{B}]$ and $S_{\text{eff}}^{(2)}[\mathcal{D}]$ separately. Below we will indeed verify that the results obtained in this way differ from those of a derivative expansion on $2S_{\text{eff}}^{(2)}[\mathcal{B}]$ (see equation (6.4)).

The non-diagonal term in (5.6) is given by

$$
S_{\text{eff}}^{(2)}[\mathcal{N}\mathcal{D}] = \frac{i}{2} \int_C \int_C dt \int_C \int_C dt' \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{k}}{(2\pi)^3} \left\{ \phi^*(t, \vec{p}) \phi^*(t', -\vec{p}) + \phi(t, \vec{p}) \phi(t', -\vec{p}) \right\} 
\times u_+(\vec{k} - \vec{p}/2)u_-(\vec{k} - \vec{p}/2)u_+(\vec{k} + \vec{p}/2)u_-(\vec{k} + \vec{p}/2) 
\times \left[ F_+(t', t, \vec{k} + \vec{p}/2) - F_-(t', t, \vec{k} + \vec{p}/2) \right] 
\times \left[ F_+(t, t', \vec{k} - \vec{p}/2) - F_-(t, t', \vec{k} - \vec{p}/2) \right].
$$

(5.9)

### 5.3.1 Derivative expansion of $S_{\text{eff}}[\mathcal{D}]$

As a specific example we now carry out a derivative expansion on $S_{\text{eff}}[\mathcal{D}]$ given in (5.7). The computation of $S_{\text{eff}}[\mathcal{B}]$ and $S_{\text{eff}}[\mathcal{N}\mathcal{D}]$ will follow very similar steps.

First Taylor expand $\phi(t', -\vec{p})$ about $t' = t$;

$$
\phi(t', -\vec{p}) = \phi(t, -\vec{p}) + (t' - t) \frac{\partial \phi(t, -\vec{p})}{\partial t} \bigg|_t + \frac{1}{2!}(t' - t)^2 \frac{\partial^2 \phi(t, -\vec{p})}{\partial t^2} \bigg|_t + \ldots
$$

(5.10)

$$
= e^{iE''(t' - t)} \phi(t'', -\vec{p}) \bigg|_{t'' = t}
$$

(5.11)

where $E'' = -i \partial_{t''}$. It is now possible to proceed in two different but equivalent ways. The first is to consider a given term in (5.10) (for example one proportional to $\partial^a \phi$), substitute it into (5.7) and then calculate its contribution to the effective action. Equivalently one can substitute (5.11) directly into (5.7), and then obtain the contribution from $\partial^a \phi$ by simply calculating the term proportional to $E^{a''}$ in the expansion of $S_{\text{eff}}^{(2)}[\mathcal{D}]$ in powers of $E^{a''}$. We follow this second more powerful method here.

Therefore, substitute (5.11) into (5.7) to obtain

$$
S_{\text{eff}}^{(2)}[\mathcal{D}] = \int_C \int_C dt \int \frac{d^3\vec{p}}{(2\pi)^3} \phi^*(t, \vec{p}) \mathcal{B}_{\mathcal{D}}(t, t_i, t''', \vec{p}) \phi(t'''', -\vec{p}) \bigg|_{t''' = t}
$$

where $\mathcal{B}_{\mathcal{D}}$ is given by

$$
\mathcal{B}_{\mathcal{D}}(t, t_i, t''', \vec{p}) := \frac{i}{2} \int \frac{d^3\vec{k}}{(2\pi)^3} \int_C dt' \ e^{iE''(t' - t)} \times
\times \left[ u_+^2(\vec{k} - \vec{p}/2)u_-^2(\vec{k} + \vec{p}/2)F_+(t, t', \vec{k} + \vec{p}/2)F_+(t', t, \vec{k} + \vec{p}/2)
+ u_+^2(\vec{k} - \vec{p}/2)u_+^2(\vec{k} + \vec{p}/2)F_-(t, t', \vec{k} - \vec{p}/2)F_-(t', t, \vec{k} + \vec{p}/2)
+ u_-^2(\vec{k} - \vec{p}/2)u_+^2(\vec{k} + \vec{p}/2)F_+(t, t', \vec{k} - \vec{p}/2)F_-(t', t, \vec{k} + \vec{p}/2)
\right].
$$
Observe that all the time dependence is in the function after similar manipulations for the other terms, we obtain letting \( \bar{\rho} \) may be evaluated for any \( J \) to obtain

\[
\frac{i}{2} \int \frac{d^3 \bar{k}}{(2\pi)^3} \sum_{s_0, s_1 = \pm} u^2_{-s_0}(\bar{k} - \bar{p}/2)u^2_{-s_1}(\bar{k} + \bar{p}/2) \bar{Z}_{s_0 s_1}(t, t_i, E''', \bar{k}, \bar{p}) \quad (5.12)
\]

and

\[
\bar{Z}_{s_0 s_1}(t, t_i, E'', \bar{k}, \bar{p}) := \int_C dt' F_{-s_0}(t, t', \bar{k} - \bar{p}/2) F_{s_1}(t', t, \bar{k} + \bar{p}/2) e^{iE''(t' - t)}. \quad (5.13)
\]

In traditional work [10, 13], one finds that the retarded thermal Green function [28, 29] appears instead of \( B_{Da} \). These are the same only in the limit \( e^{\beta E''} = 1 \) in accordance to the discussion in section 4. Below we will see that the so called Landau damping contributions come from the terms in (5.12) with \( s_0 = -s_1 \), and furthermore that they do not give rise to any problems with the derivative expansion.

The only dependence on the variable \( t' \) in (5.7) is now in \( \bar{Z} \), and we show that this may be evaluated for any contour \( C \). Thus our method includes both imaginary- and real-time approaches to thermal field theory. Substitute expressions (5.3) for \( F_\pm \) into (5.13) to obtain

\[
Z_{s_0 s_1} = \int_C dt' \int \frac{dk_0}{2\pi} \int \frac{dk_1}{2\pi} \rho_{-s_0}(k_0, \bar{k} - \bar{p}/2) \rho_{s_1}(k_1, \bar{k} + \bar{p}/2) \times \left[ \theta_C(t', t) - N(k_1) \right] \left[ \theta_C(t, t') - N(k_0) \right] e^{iE''(t' - t)} e^{ik_0(t' - t)} e^{-ik_1(t' - t)}
\]

\[
= \int_C dt' \int \frac{dk_0}{2\pi} \int \frac{dk_1}{2\pi} \rho_{-s_0}(k_0, \bar{k} - \bar{p}/2) \rho_{s_1}(k_1, \bar{k} + \bar{p}/2) J_C(k_0, k_1, t_i - t, E'')(5.14)
\]

Observe that all the time dependence is in the function \( J_C(k_0, k_1, t_i - t, E'') \) which, on letting \( \bar{A} = E'' + k_0 - k_1 \), is given by

\[
J_C(k_0, k_1, t_i - t, E'') = \int_C dt' \left[ \theta_C(t', t) - N(k_1) \right] \left[ \theta_C(t, t') - N(k_0) \right] e^{i\bar{A}(t' - t)}
\]

\[
= \int_C dt' N(k_0)N(k_1) e^{i\bar{A}(t' - t)} - N(k_1) \int_C dt' \theta_C(t, t') e^{i\bar{A}(t' - t)}
\]

\[
- N(k_0) \int_C dt' \theta_C(t', t) e^{i\bar{A}(t' - t)} + \int_C dt' \theta_C(t', t) \theta_C(t, t') e^{i\bar{A}(t' - t)}.
\]

Consider now an arbitrary contour \( C \) starting at \( t_i \) and ending at \( t_i - i\beta \). The integrals in (5.15) may be calculated explicitly for that contour \( C \): using the definition of \( \theta_C(t', t) \), the final term contributing to \( J_C(k_0, k_1, t_i - t, E'') \) vanishes; the only contribution to the second term is for \( t_i < t' < t \); and the contribution to the third term is for \( t < t' < t_i - i\beta \). Thus the second term, for example, is given by

\[
N(k_1) \int_C dt' \theta_C(t, t') e^{i\bar{A}(t' - t)} = N(k_1) e^{-i\bar{A}t} - i\bar{A} \int_{t_i}^t dt' e^{i\bar{A}t'}
\]

\[
= \frac{1}{i\bar{A}} \left[ N(k_1) e^{-i\bar{A}t} e^{i\bar{A}t_i} - e^{i\bar{A}t - i\bar{A}t_i} \right].
\]

After similar manipulations for the other terms, we obtain

\[
J_C(k_0, k_1, \delta, E'') = -\frac{N(k_0)N(k_1)}{i\bar{A}} \left[ e^{\beta \delta} \left( e^{i\bar{A}} - 1 \right) + e^{\beta k_0} \left( 1 - e^{i\bar{A}} \right) \left( 1 - e^{\beta E''} \right) \right] \quad (5.15)
\]
where $\delta = t_i - t$. This step is perhaps the most important one of our method—by working with the time variable $t$ itself rather than Fourier transforming, we have been able to do the integrals explicitly and without using the explicit form for the spectral functions $\rho$.

Substituting (5.15) back into (5.14) and using the definitions of $\rho$ in (5.4) gives

$$
\bar{Z}_{s_0s_1}(t, t_i, E'', \vec{k}, \vec{p}) = \frac{1}{(\exp\{\beta s_0 E_{\vec{k}-\vec{p}/2}\} + 1) (\exp\{\beta s_1 E_{\vec{k}+\vec{p}/2}\} + 1) iA} \left\{ (1 - e^{\beta A}) + \exp\{\beta (s_0 E_{\vec{k}-\vec{p}/2} + s_1 E_{\vec{k}+\vec{p}/2})\} (e^{i\delta A} - 1) \left( 1 - e^{\beta E''} \right) \right\} \tag{5.16}
$$

where we have used the shorthand notation

$$
A = E'' + s_0 E_{\vec{k}-\vec{p}/2} + s_1 E_{\vec{k}+\vec{p}/2}. \tag{5.17}
$$

Before going any further we make the following comments. First, observe that if we set $e^{\beta E''} = 1$ then our expression (5.16) reduces to the corresponding equation (number (10)) of [13]. This is consistent with the discussion of section 4.

Secondly, note that the time independent contributions to the effective action have $E'' = 0$ and hence $e^{\beta E''} = 1$. Thus all our time independent results will be identical to those of [11, 13].

Thirdly, equation (5.16) has an explicit dependence on the initial time $t_i$. Time translation invariance has therefore been lost as expected for such a time dependent non-equilibrium system. Furthermore, an important point to observe is that even at $t = t_i$ ($\delta = 0$), our expression (5.16) differs similar terms in [10, 13] by a factor of $e^{\beta E''}$. This point will be discussed further below.

Finally, consider the terms with $s_0 = -s_1$ in $B_{Da}$. From equations (5.12) and (5.13) and the expression for $\bar{Z}$ in (5.16), these terms correspond to Landau damping terms in the original formalism (when $e^{\beta E''} = 1$). The reason is that in this case, the denominators contain $E'' \pm (E_{\vec{k}-\vec{p}/2} - E_{\vec{k}+\vec{p}/2})$, which vanishes when $\vec{p} = 0$ and $E'' = 0$ thus leading to a potential divergence. It is this divergence in the bubble diagram which is sometimes ignored [11], and was the cause of the badly defined derivative expansion in [13]. However, observe that in our case we have $e^{\beta E''} \neq 1$, so that the $\vec{p} = 0$ and $E'' = 0$ limits are actually well defined: the zero in the denominator $B_{Da}$ when $s_0 = -s_1$ is exactly cancelled by a zero in the numerator of this expression. Thus here we are dealing with Landau damping terms that have a well defined derivative expansion. Similar comments will apply for the other diagonal term as well as the non-diagonal terms which we now calculate.\(^2\)

\(5.3.2\) Second diagonal term $S^{(2)}_{\text{eff}}[Db]$  

Calculation of $S^{(2)}_{\text{eff}}[Db]$ follows a very similar route to that above. Indeed, the function $\bar{Z}$ introduced above also appears in $S^{(2)}_{\text{eff}}[Db]$ as can be seen by comparing the temporal

\(^2\)Interestingly, the cuts at $E'' = \pm 2\Delta_0$ familiar from Green functions commonly encountered in field theory are also not present in the $B$'s when $s_0 = s_1$. However, one must be very precise when making the link between physics and the analytic properties of particular Green functions. For instance it is the theory of linear response which links cuts at $E'' = \pm 2\Delta_0$ in retarded Green functions to physical decay processes at those energy scales [27, 29]. Here we are not using linear response theory but rather looking at systems which are slowly varying in time. It is therefore not clear that $B$ should have the same analytic structure as the retarded Green functions of linear response [27, 29] as they found to be relevant to different physical problems.
dependence of $S^{(2)}_{\text{eff}}[\text{Da}]$ in (5.7) with that of $S^{(2)}_{\text{eff}}[\text{Db}]$ in (5.8). This leads to a switch from $u_-$ to $u_+$ functions and one obtains

$$S^{(2)}_{\text{eff}}[\text{Db}] = \int_C dt \int \frac{d^3 \vec{p}}{(2\pi)^3} \phi^*(t, \vec{p}) \mathcal{B}_{Db}(t, t, t'', \vec{p}) \phi(t'', -\vec{p}) \bigg|_{t''=t}$$

where the bubble diagram $\mathcal{B}_{Db}(t, t, t'', \vec{p})$ contains contributions from all the four terms in (5.8):

$$\mathcal{B}_{Db} = \frac{i}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_{s_0, s_1 = \pm} u_{s_0}^2 (\vec{k} - \vec{p}/2) u_{s_1}^2 (\vec{k} + \vec{p}/2) \bar{Z}_{s_0 s_1}.$$  

(5.18)

### 5.3.3 Non-diagonal term $S^{(2)}_{\text{eff}}[\text{ND}]$

The non-diagonal terms (5.9) again follow a similar pattern. Here, since

$$u_+ (\vec{k} - \vec{p}/2) u_+ (\vec{k} - \vec{p}/2) u_- (\vec{k} + \vec{p}/2) u_- (\vec{k} + \vec{p}/2) = \frac{|\Delta_0|^2}{4E_{\vec{k}+\vec{p}/2} E_{\vec{k}-\vec{p}/2}}$$

it follows that

$$S^{(2)}_{\text{eff}}[\text{ND}] = \int_C dt \int \frac{d^3 \vec{p}}{(2\pi)^3} \phi^*(t, \vec{p}) \mathcal{B}_{ND}(t, t, t'', \vec{p}) \phi(t'', -\vec{p}) \bigg|_{t''=t}$$

where the bubble diagram $\mathcal{B}_{ND}(t, t, t'', \vec{p})$ is

$$\mathcal{B}_{ND} = \frac{i}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{|\Delta_0|^2}{4E_{\vec{k}+\vec{p}/2} E_{\vec{k}-\vec{p}/2}} \sum_{s_0, s_1 = \pm} (-s_0 s_1) \bar{Z}_{s_0 s_1}.$$  

(5.19)

### 5.3.4 Total quadratic term and derivative expansion

To conclude, we have

$$S^{(2)}_{\text{eff}} = S^{(2)}_{\text{eff}}[\text{Da}] + S^{(2)}_{\text{eff}}[\text{Db}] + S^{(2)}_{\text{eff}}[\text{ND}]$$

where

$$S^{(2)}_{\text{eff}} = \int_C dt \int \frac{d^3 \vec{p}}{(2\pi)^3} \left( \phi^*(t, \vec{p}) \phi(t, \vec{p}) \right) \left( \begin{array}{cc} B_{Da} & B_{ND} \\ B_{ND} & B_{Db} \end{array} \right) \left( \begin{array}{c} \phi(t'', -\vec{p}) \\ \phi(t'', -\vec{p}) \end{array} \right) \bigg|_{t''=t}$$

with the $B$’s given in (5.12), (5.18) and (5.19).

It is now straightforward to expand this effective action in powers of $\partial^n \phi$ and $\nabla^n \phi$ where $n \geq 0$; it is done by expanding the $B$’s above about $E'' = 0$ and $\vec{p} = 0$. We now carry out this derivative expansion.

### 6 The derivative expansion in the degenerate gas approximation

Though it is not difficult to expand the expressions given above in $E''$ and $\vec{p}$ to obtain the derivative expansion and hence effective equations of motion, the resulting expressions are however very complicated. To simplify them we therefore work in the degenerate gas approximation.
approximation \([2]\) for which \(T < T_C\), and furthermore the gap \(\Delta_0\) is much bigger than the temperature.

Before clarifying further this approximation, note that our expansion also imposes \(\beta|\partial_t \ln(\phi)| \ll 1\) since the factors of \(\exp\{\beta E''\} \equiv \exp\{-i\beta \partial_t\phi\}\) must be expanded. It is fundamental to remember that by their very definition, derivative expansions are valid only for fields varying slowly in space and time, and we should not be surprised that our expressions differ from those used in linear response theory. The latter uses retarded Green functions which describe the response to a delta-function impulse, a very different physical problem. Also with \(|\partial_t \phi/\phi| \ll T\) for the expansion to be valid, clearly taking the zero temperature \(T \rightarrow 0\) limit makes no sense in our expressions.

To summarise, the results presented below are valid in the regime
\[ \mu \gg \Delta_0 \gg \beta^{-1} = T \gg |\partial_t \phi/\phi|, \]
and the gap \(\Delta_0\) will be used as the primary mass scale for the problem. Thus we introduce the following dimensionless parameters
\[ y = \frac{\epsilon}{\Delta_0}, \quad z^2 = (1 + y^2) = \frac{E^2}{\Delta_0^2}. \]

To exploit the degenerate gas approximation, observe that the integrals are sharply peaked around the Fermi surface \(k = k_f\). This is not true if there are UV divergent terms but we shall see that in the effective potential, these term cancel through the gap equation. The Fermi-Dirac function \(N_f\) then appears as
\[ N_f = [\exp\{\beta \Delta_0 z\} + 1]^{-1}, \]
and is even more sharply peaked than other parts of the expression. Thus we perform the following simplifications:-

- **Isolated factors of \(k\) not inside an \(\epsilon_{\vec{k}}\) or \(E_{\vec{k}}\) factor can be replaced by a constant \(k_f\).** Valid since \(k^2 = k_f^2(1 + y^2/Y^2)\) where \(Y := \mu/\Delta_0 \gg 1\).

- **In terms with at least one factor of \(N_f\), \(\epsilon_{\vec{k}} \approx 0, E_{\vec{k}} \approx \Delta_0\), i.e. \(y \approx 0, z \approx 1\).** Valid since \(\Delta_0 \gg T\).

The results then have a common factor of \(\rho\)
\[ \rho := \frac{mk_f}{2\pi^2}. \]

For example, the gap equation in this notation is
\[ -\frac{1}{g} = \rho \int \, dy \left( \frac{1}{2z} - N_f \right). \]

The expressions will also depend on the Minkowskii time \(t\) elapsed since the initial conditions were set at time \(t_i\).
6.1 Results

The notation in this section is as follows. From equations (2.6) and (3.6), the effective action is given by

\[ \tilde{S}_{\text{eff}} := -i \hbar \int d^4x \ln G^{-1}(x, x) - \frac{1}{g} \int d^4x \Delta_0^2 \]

\[ + \int d^4x \left( \phi^{*} \gamma^{D_0} \phi + \phi \gamma^{D_0} \phi^{*} + \phi^{*} \gamma^{ND} \phi^* + \phi \gamma^{ND} \phi \right) \]  

(6.1)

where \( \phi = \phi(t, \vec{x}) \), \( \phi^{*} = \phi(t, \vec{x}) \) and \( \gamma \)'s are of the form

\[ \gamma^X = \sum_{a,b} \gamma_{a,b}^X \left( -i \frac{\partial}{\partial t} \right)^a \left( i \nabla \right)^b . \]  

(6.2)

The zeroth order effective potential terms, \( \gamma_{0,0}^X \) are easily obtained by first setting \( E'' = 0 = \vec{p} \) in (5.13) and then using the gap equation to remove the UV divergences. In the degenerate gas approximation we have

\[ \gamma_{0,0}^{Da} = \gamma_{0,0}^{Db} = \gamma_{0,0}^{ND} = 0, \]

\[ = \frac{\rho}{8} \int dy \left[ \frac{1}{z^3} - 2N_f - 2(\Delta_0 / \beta)_N(I - N_f) \right] . \]  

(6.3)

The first order terms are slightly more complicated in part and are given by

\[ \gamma_{1,0}^{Da} = \gamma_{1,0}^{Db} = \gamma_{1,0}^{ND} = 0, \]

\[ = -\frac{\rho \beta}{16} \int dy \left[ \left( \frac{(z - y)^2}{z^3} - 2N_f \right) e^{-2i(t-t_f)\Delta_0} \right. \]

\[ - N_f^2 \left( e^{2i(t-t_f)\Delta_0} - e^{-2i(t-t_f)\Delta_0} \right) + 2(\Delta_0 / \beta)_N(I - N_f) \left( 1 - 2 \frac{i(t - t_f)}{\beta} \right) \]  

\[ \gamma_{1,0}^{Db} = -\frac{\rho \beta}{16} \int dy \left[ \left( \frac{(z + y)^2}{z^3} - 2N_f \right) e^{-2i(t-t_f)\Delta_0} \right. \]

\[ - N_f^2 \left( e^{2i(t-t_f)\Delta_0} - e^{-2i(t-t_f)\Delta_0} \right) + 2(\Delta_0 / \beta)_N(I - N_f) \left( 1 - 2 \frac{i(t - t_f)}{\beta} \right) \]  

\[ \gamma_{1,0}^{ND} = -\frac{\rho \beta}{16} \int dy \left[ \left( \frac{1}{z^3} + 2N_f \right) e^{-2i(t-t_f)\Delta_0} + N_f^2 \left( e^{2i(t-t_f)\Delta_0} - e^{-2i(t-t_f)\Delta_0} \right) \right. \]

\[ + 2(\Delta_0 / \beta)_N(I - N_f) \left( 1 - 2 \frac{i(t - t_f)}{\beta} \right) \]  

(6.4)

Notice that the only difference between these two diagonal contributions \( \gamma_{1,0}^{Da} \) and \( \gamma_{1,0}^{Db} \) are the terms odd in \( y \) (i.e. \( \xi_y \)). One should also note that for \( t > t_i \), the \( \gamma_{1,0} \) terms contain both real and imaginary components. In the equations of motion, which follow directly from (6.1), these correspond to oscillatory and dissipative terms respectively. The second order terms may be found in appendix B.

The above results may be rewritten in terms of the real Higgs \( \phi_h \) and Goldstone fields \( \phi_g \) where

\[ \phi = \frac{1}{\sqrt{2}}(\phi_h + i\phi_g). \]
One obtains
\[
S_{\text{eff}} := -i \text{tr} \int d^4x \ln G^{-1}(x,x) - \frac{1}{g} \int d^4x \Delta_0^2 + \int d^4x \left( \phi_h \gamma^H \phi_h + \phi_g \gamma^G \phi_g + i \phi_h \gamma^{HG} \phi_g - i \phi_g \gamma^{HG} \phi_h \right).
\]

The coefficients $\gamma^h$ and $\gamma^g$ are defined in the derivative expansion as in 6.2 and their explicit forms are
\[
\begin{align*}
\gamma^H_{0,0} & = 2 \gamma^D_{0,0} = \frac{\rho}{4} \int dy \left[ \frac{1}{z^3} - 2 \frac{N_f}{z^2} - 2(\Delta_0 \beta) N_f (1 - N_f) \right] \\
\gamma^G_{0,0} & = 0 \\
\gamma^H_{1,0} & = -\frac{\rho \beta}{8} \int dy \left[ \frac{y^2}{z} e^{-2i(t-t_i) \Delta_0 z} + 2(\Delta_0 \beta) N_f (1 - N_f) \cdot \frac{i(t-t_i)}{\beta} \right] \\
\gamma^H_{0,1} & = 0 \\
\gamma^G_{0,1} & = \gamma^G_{0,1} = 0 \\
\gamma^G_{1,0} & = -\frac{\rho \beta}{8} \int dy \left[ \left( \frac{1}{z^2} - 2 \frac{N_f}{z} \right) e^{-2i(t-t_i) \Delta_0 z} - \frac{N_f}{z^2} \left( e^{2i(t-t_i) \Delta_0 z} - e^{-2i(t-t_i) \Delta_0 z} \right) \right] \\
\gamma^{HG}_{0,0} & = \gamma^{HG}_{0,1} = 0 \\
\gamma^{HG}_{1,0} & = \frac{\rho \beta}{8} \int dy \frac{y}{z^2} e^{-2i(t-t_i) \Delta_0 z}. 
\end{align*}
\]

\[\text{(6.5)}\]

### 7 Discussion and Conclusions

Several comments can be made about these results. First, for any static field configurations $E'' = 0$, and hence $\exp\{\beta E''\} = 1$ as assumed in all the previous calculations (see the discussion in section 4). Thus for static configurations we recover the static limit of retarded equilibrium Green function results used by others, so that our effective action is identical to standard calculations in the static limit. In particular, our lowest order term in the derivative expansion, the effective potential, is identical to the one which would be calculated using pure equilibrium methods. The differences are in the non-static terms: we have a unique derivative expansion which does not depend on the order in which the spatial and temporal derivative expansions are made. Thus only by using our expansion can one justify the use of the traditional effective potential as the potential for some effective theory, since here we have a fully defined and controlled expansion. Following on from this, since the physical masses for the bosonic field $\Delta$ are encoded by the second derivative of the effective potential, we expect to reproduce Goldstone’s theorem. We indeed find that the mass term for the Goldstone mode, $\gamma^G_{0,0}$, is zero.

As discussed in section 4, our calculation takes into account the fact that time translation invariance is broken by the non-equilibrium derivative expansion. This is reflected in the final effective action obtained in (6.1) where the coefficients depend explicitly on the initial time $t_i$ at which the system was perturbed.

Though we believe that the methods presented here are more consistent than previous methods used to calculate the derivative expansion, there are still a number of confusing aspects in this work. One of these (which we believe also exists in the previous calculations, though it is not explicitly commented upon there) is the question of the difference between $S^{(2)}_{\text{eff}}[Da]$ and $S^{(2)}_{\text{eff}}[Db]$ of (5.7) and (5.8). We believe to have argued correctly in
section 5 that the change of variables $t \leftrightarrow t'$ and $\vec{p} \rightarrow -\vec{p}$ should not be carried out before the derivative expansion. However, this in turn is responsible for the coupling between the Higgs and Goldstone modes in (6.5). Another interesting aspect of our results is that apart from breaking time-translation invariance (which we do believe to be correct), observe that another symmetry has been broken: since $(\gamma_{1,0}^{Da})^* \neq \gamma_{1,0}^{Db}$ then to this order in the derivative expansion $S_{\text{eff}}$ is not invariant under $\phi \leftrightarrow \phi^*$. However, one fully expects this to be broken if there is a non-zero charge in the system. This corresponds to having an imaginary initial time derivative for the scalar field $\phi$. Finally, note that the first terms of $\gamma_{1,0}^{Da}$ and $\gamma_{1,0}^{Db}$ are actually divergent at $t = t_i$. This infinity is exactly the one which arises in (5.13) if one were to set $t = t_i$ and $\vec{p} = 0$. It should be noted that this divergence is not present in the usual calculations. As commented on just after equation (5.17) it stems from the fact that at $t = t_i$ our expressions differ from similar terms in [11, 13] by the factor of $e^{\beta E''} \neq 1$. In this non-relativistic case such infinities are not too worrying one always has a UV cutoff in mind, but the same is not true in relativistic cases where they would have to be renormalised in some way [35].

To conclude, we have argued that the traditional approach to producing derivative expansions of thermal effective actions are inconsistent since they try to use equilibrium fields in constructing the derivative expansion. We have argued that this is technically not correct, and that it is incompatible with the physical problem being described—that of slowly varying fields. We have shown that letting the fields be out of equilibrium leads to a well behaved derivative expansion. It means that the Green functions at the heart of our calculations, our $B$’s, are not the retarded thermal Green functions normally encountered in such work. Indeed, as linear response analysis shows [27], those are relevant to a very different type of physical problem where the system is hit with a sudden change rather than responding to a slow variation. Finally, we have used these ideas to calculate the effective action for a BCS superconductor.

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A Calculation of the fermionic propagator

Calculation of the fermionic propagator in $(t, \vec{k})$ variables proceeds as follows. Throughout it is vital to keep track of the delta functions and boundary conditions so that the solution is valid for any path-ordered formalism of thermal field theory.

As in section 5, define the matrix $G^{-1}(x)$ through

$$G^{-1}(x)\delta^4(x-y) := G^{-1}(x,y)$$

(A.1)

so that in $(t, \vec{k})$ space

$$G^{-1}(t, \vec{k}) = \begin{pmatrix} i\partial_t - \frac{\vec{k}^2}{2m} + \mu & \Delta_0 \\ \Delta_0 & i\partial_t + \frac{\vec{k}^2}{2m} - \mu \end{pmatrix} = \begin{pmatrix} i\partial_t - \epsilon_\vec{k} & \Delta_0 \\ \Delta_0 & i\partial_t + \epsilon_\vec{k} \end{pmatrix} = i\partial_t \mathbb{I} + A_\vec{k}.$$
Here \( \epsilon_{\vec{k}} = \frac{k^2}{2m} - \mu \),

\[
A_{\vec{k}} = \begin{pmatrix}
-\epsilon_{\vec{k}} & \Delta_0 \\
\Delta_0 & +\epsilon_{\vec{k}}
\end{pmatrix}
= E_{\vec{k}} \begin{pmatrix}
-\cos(2\theta_{\vec{k}}) & \sin(2\theta_{\vec{k}}) \\
\sin(2\theta_{\vec{k}}) & \cos(2\theta_{\vec{k}})
\end{pmatrix},
\tag{A.3}
\]

and \( E_{\vec{k}} \) is the dispersion relationship \( E_{\vec{k}}^2 = |\Delta_0|^2 + \epsilon_{\vec{k}}^2 \). The angle \( \theta_{\vec{k}} \) is defined through

\[
\cos(2\theta_{\vec{k}}) = \frac{\epsilon_{\vec{k}}}{E_{\vec{k}}}, \quad \sin(2\theta_{\vec{k}}) = \frac{\Delta_0}{E_{\vec{k}}} = -2|u_+(\vec{k})||u_-(\vec{k})|
\tag{A.4}
\]

where, as in section 5,

\[
u_{\pm}(\vec{k}) = \left(\frac{1}{2} \left[1 \pm \frac{\epsilon_{\vec{k}}}{E_{\vec{k}}}\right]\right)^{\frac{1}{2}} \tag{A.5}
\]

so that \( \cos(\theta_{\vec{k}}) = u_+(\vec{k}) \) and \( -\sin(\theta_{\vec{k}}) = u_-(\vec{k}) \).

Now, from equation (3.4),

\[
G^{-1}(x)G(x, z) = \delta^4(x - z) \mathbb{I}
\implies
(i \partial_t \mathbb{I} + A_{\vec{k}}) G(t, t', \vec{k}) = \delta(t - t') \mathbb{I}. \tag{A.6}
\]

The diagonalised form of \( A_{\vec{k}} \) is

\[
A_{\vec{k}} = B_{\vec{k}} \cdot \Lambda_{\vec{k}} \cdot B_{\vec{k}}^{-1}\tag{A.7}
\]

where \( \Lambda_{\vec{k}} \) contains the eigen-values of \( A_{\vec{k}} \);

\[
\Lambda_{\vec{k}} = \begin{pmatrix}
E_{\vec{k}} & 0 \\
0 & -E_{\vec{k}}
\end{pmatrix}, \tag{A.8}
\]

and \( B_{\vec{k}} \) its eigen-vectors;

\[
B_{\vec{k}} = \begin{pmatrix}
-\sin(\theta_{\vec{k}}) & \cos(\theta_{\vec{k}}) \\
\cos(\theta_{\vec{k}}) & -\sin(\theta_{\vec{k}})
\end{pmatrix} = \begin{pmatrix}
u_-(\vec{k}) & u_+(\vec{k}) \\
u_+(\vec{k}) & u_-(\vec{k})
\end{pmatrix}. \tag{A.9}
\]

Now equation (A.6) is equivalent to

\[
(i \partial_t B_{\vec{k}}^{-1} + B_{\vec{k}}^{-1} A_{\vec{k}}) G(t, t', \vec{k}) B_{\vec{k}} = \delta(t - t') \mathbb{I}\tag{A.10}
\]

so that on defining the matrix

\[
F(t, t', \vec{k}) = B_{\vec{k}}^{-1} G(t, t', \vec{k}) B_{\vec{k}} \tag{A.11}
\]

one obtains

\[
(i \partial_t \mathbb{I} + \Lambda_{\vec{k}}) F(t, t', \vec{k}) = \delta(t - t') \mathbb{I}. \tag{A.12}
\]

Thus \( F_{\vec{k}} \) satisfies the usual Feynman equation and is given by

\[
F(t, t', \vec{k}) = \begin{pmatrix}
F_+(t, t', \vec{k}) & 0 \\
0 & F_-(t, t', \vec{k})
\end{pmatrix}. \tag{A.13}
\]
where in the Mills representation [34]

\[ F_\pm(t, t', \vec{k}) = -i \int \frac{dk_0}{2\pi} e^{-ik_0(t-t')} (\theta_C(t, t') - N(k_0)) \rho_\pm(k_0, \vec{k}) \]

so that \( F_-(t, t', \vec{k}) = F_+(t', t, \vec{k}) \). Here \( N(k_0) \) is the Fermi-Dirac distribution:

\[ N(k_0) = \frac{1}{e^{\beta k_0} + 1}, \quad (A.14) \]

the \( \theta_C \) is the usual \( \theta \) function for the contour \( C \) [26, 27], and the spectral functions \( \rho_\pm \) are given by

\[ \rho_\pm(k_0, \vec{k}) = 2\pi \delta(k_0 \pm E_\vec{k}). \quad (A.15) \]

Finally, these results may be used to compute the explicit form of \( G(t, t', \vec{k}) \): inverting equation (A.11), using (A.9) and (A.13) gives, after some manipulation,

\[ G(t, t', \vec{k}) = \begin{pmatrix} u_+^2 F_-(t, t', \vec{k}) + u_+^2 F_+(t, t', \vec{k}) & u_+ u_- (F_+(t, t', \vec{k}) - F_-(t, t', \vec{k})) \\ u_+ u_- (F_+(t, t', \vec{k}) - F_-(t, t', \vec{k})) & u_+^2 F_+(t, t', \vec{k}) + u_-^2 F_-(t, t', \vec{k}) \end{pmatrix} \quad (A.16) \]

where the \( \vec{k} \) dependence of \( u_\pm(\vec{k}) \) has been omitted but is understood.

\section*{B Second Order Terms}

Here we write out the results for the second order terms.

\[ \gamma_{2,0}^{\text{Da}} = -\frac{\rho}{16\Delta_0^2} \int dy \left\{ \frac{\tilde{z}^2 + y^2}{z^5} + \frac{(z-y)^2}{z^4} (\beta \Delta_0) e^{-2i(t-t_i)} \Delta_0 z \right\} \]

\[ \gamma_{2,0}^{\text{Db}} = \gamma_{2,0}^{\text{Da}} \bigg|_{y \rightarrow -y} \]

\[ \gamma_{2,0}^{\text{ND}} = -\frac{\rho}{16\Delta_0^2} \int dy \left\{ \frac{1}{z^5} + \frac{1}{z^4} (\beta \Delta_0) e^{-2i(t-t_i)} \Delta_0 z \right\} \]

\[ \gamma_{0,2} = \gamma_{0,2}^{\text{Da}} = \frac{\rho \mu}{48m\Delta_0^2} \int dy \left\{ \frac{1}{z^7} \left[ (-6z^2 + 10) + \frac{\Delta_0}{\mu} y(-6z^4 - 23z^2 + 20) \right] \right\} \]

\[ \gamma_{0,2}^{\text{Db}} = \gamma_{0,2}^{\text{ND}} \]
\[
+ \left( \frac{\Delta_0}{\mu} \right)^2 (-6z^6 - z^4 + 17z^2 - 10) \\
+ 4N_f \left[ 3 + 3(\beta\Delta_0) - (\beta\Delta_0)^2 \right] + 12N_f^2(\beta\Delta_0)(-1 + (\beta\Delta_0)) \}
\]

\[
\gamma^\text{ND}_{0,2} = \frac{\rho\mu}{48m\Delta_0^2} \int dy \left\{ \frac{1}{z^7} \left[ (-4z^2 + 10) + \frac{\Delta_0}{\mu} y(z^2 + 20) + 5 \left( \frac{\Delta_0}{\mu} \right)^2 (z^4 + z^2 - 2) \right] \\
- 4N_f \left[ 3 + 3(\beta\Delta_0) + (\beta\Delta_0)^2 \right] + 12N_f^2(\beta\Delta_0)(1 + (\beta\Delta_0)) \}.
\]

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