ON MEAN CURVATURE INTEGRALS OF THE OUTER PARALLEL CONVEX BODY OF CONSTANT WIDTH

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Abstract. In this paper, we obtain some results about the mean curvature integrals of the outer parallel convex body of constant width.

1. Introduction

The mean curvature integrals is a basic concept in Integral Geometry. It connects many geometric invariants, such as area, the degree of the spherical Gauss map, the Euler-Poincaré characteristic, the Gauss-Kronecker curvature and so on. Also it is closely related to the Minkowski quermassintegral of convex body and plays an important role in the theory of convex body.

Under the assumptions that $\mathbb{R}^n$ is the $n$-dimensional Euclidean space and $L_{r|i|O}$ is an $r$-dimensional linear subspace through a fixed point $O$, Santaló \cite{1} studied the $l$th mean curvature integral $M_l^{(n)}(\Phi)$ of a flattened convex body $K$ in $\mathbb{R}^n$ and established the expression of $M_l^{(n)}(\Phi)$ in terms of $M_l^{(r)}(\Phi)$, where $M_l^{(r)}(\Phi)$ is the $l$th mean curvature integral of $K$ in $L_{r|i|O}$. Later, Zhou-Jiang \cite{2}, Jiang-Zeng \cite{3} and Zeng-Ma-Xia \cite{4} studied the mean curvature integrals $M_l^{(n)}(\Phi)$ of the outer parallel convex body $K_\rho$, which generalized the results of Santaló.

Let $\Phi$ be a convex body of constant width $h$ with $C^2$ boundary $\partial \Phi$ in $\mathbb{R}^n$ (any two parallel support hyperplanes of $\Phi$ are always separated by a constant $h$). Let $(\Phi'_\rho)^{(n)}(\Phi)$ be the outer parallel body of $\Phi'_\rho$ in the distance $\rho$ in $\mathbb{R}^n$, where $\Phi'_\rho$ is the orthogonal projection of $\Phi$ on the $r$-dimensional linear subspace $L_{r|i|O} \subseteq \mathbb{R}^n$. Denote by $M_l^{(n)}(\partial(\Phi'_\rho)^{(n)}(\Phi))$ $(l = 0, 1, \ldots, n - 1)$ the mean curvature integrals of $(\Phi'_\rho)^{(n)}(\Phi)$ and by $M_l^{(r)}(\partial(\Phi'_\rho))$ $(l = 0, 1, \ldots, r - 1)$ the mean curvature integrals of $\Phi'_\rho$ in $L_{r|i|O}$. In this paper, we shall prove the following results.

Theorem 1.1. (1) If $l \geq n - r$, then

$$ M_l^{(n)}(\partial(\Phi'_\rho)^{(n)}(\Phi)) = \sum_{j=0}^{n-l-1} \sum_{i=0}^{n-l-j-1} (-1)^i \frac{(n-l-1)!}{(n-1)!} \frac{(n-l-j-1)!}{(n-i-1)!} \frac{(r-1)!}{(r-1-p)!} \frac{O_{r-i-1}}{O_{r-i-1}} M_l^{(r)}(\partial(\Phi'_\rho)) \rho^p h^{n-l-j-i-1}. $$

(1.1)

(2) If $l = n - r - 1$, then

$$ M_l^{(n)}(\partial(\Phi'_\rho)^{(n)}(\Phi)) = (-1)^r \frac{(n-1)!}{(n-r-1)!} \frac{(n-1)!}{(n-r-1)!} \frac{(r-1)!}{(r-1-p)!} \frac{O_{r-i-1}}{O_{r-i-1}} M_l^{(r)}(\partial(\Phi'_\rho)) h^{r-i} $$

$$ + \sum_{j=1}^{r} \sum_{i=0}^{r-j} (-1)^i \frac{(r-j)!}{(r-j-i)!} \frac{(r-1)!}{(r-1-p)!} \frac{O_{r-i-1}}{O_{r-i-1}} M_l^{(r)}(\partial(\Phi'_\rho)) \rho^p h^{r-j-i}. $$

(1.2)

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(3) If \( l < n - r - 1 \), then

\[
M_l^{(n)} \left( \partial(\Phi'_r)^{(n)} \right) = \sum_{j=0}^{n-r-l-1} (-1)^j \frac{(n-l-j-1)!}{j!} O_{n-r-1} V_r(\Phi'_r) \rho^j h^{n-l-j-r-1}
\]
\[
+ \sum_{j=0}^{n-r-l-1} \sum_{i=0}^{n-j-1} (-1)^i \frac{(n-l-j-1)!}{i!} \frac{(r-1)!}{(r-i-1)!} O_{r-i-1} M_{r-i-1}^{(r)} \rho^i h^{n-l-j-i-1}, \tag{1.3}
\]

where \( V_r(\Phi'_r) \) denotes the \( r \)-dimensional volume of \( \Phi'_r \).

Based on Theorem 1.1, we continue to calculate the integral of \( M_l^{(n)} \left( \partial(\Phi'_r)^{(n)} \right) \) on Grassmann manifold \( G_{r,n-r} \) and get the following result.

**Theorem 1.2.**

(1) If \( l \geq n - r \), then

\[
\int_{G_{r,n-r}} M_l^{(n)} \left( \partial(\Phi'_r)^{(n)} \right) dL_r[0] =
\]
\[
\sum_{j=0}^{n-r-l-j-1} \sum_{i=0}^{n-i-1} (-1)^j \frac{(n-l-j-1)!}{i!} \frac{(r-1)!}{(r-i-1)!} O_{r-i-1} M_{r-i-1}^{(r)} \rho^i h^{n-l-j-i-1} M_{n-i-1}^{(n)} \left( \partial\Phi \right). \tag{1.4}
\]

(2) If \( l = n - r - 1 \), then

\[
\int_{G_{r,n-r}} M_l^{(n)} \left( \partial(\Phi'_r)^{(n)} \right) dL_r[0] =
\]
\[
(-1)^r \frac{(n-1)!}{(r-1)!} \frac{(r-1)!}{(r-i-1)!} O_{r-i-1} M_{r-i-1}^{(r)} \rho^i h^{n-l-j-i-1} M_{n-i-1}^{(n)} \left( \partial\Phi \right). \tag{1.4}
\]

(3) If \( l < n - r - 1 \), then

\[
\int_{G_{r,n-r}} M_l^{(n)} \left( \partial(\Phi'_r)^{(n)} \right) dL_r[0] =
\]
\[
\sum_{j=0}^{n-r-l-1} (-1)^j \frac{(n-l-j-1)!}{i!} \frac{(r-1)!}{(r-i-1)!} O_{r-i-1} M_{r-i-1}^{(r)} \rho^i h^{n-l-j-i-1} M_{n-i-1}^{(n)} \left( \partial\Phi \right).
\]
Note that the Minkowski quermassintegrals \( W_r(n)(K) \) of a convex body \( K \) are well defined for any convex figure, whereas \( W_r(n)(\partial K) \) makes sense only if \( \partial K \) is of class \( C^2 \).

Let \( K \) be a convex body in \( \mathbb{R}^n \), then \( \partial K \) is a \((n-1)\)-dimensional convex hypersurface. Assuming that \( \partial K \) is of class \( C^2 \) and \( P \) is a point of \( \partial K \), we choose \( e_1, \ldots, e_{n-1} \) to be the principal curvature directions at \( P \).
directions at the point $P$. Further, we suppose that $\kappa_1, \cdots, \kappa_{n-1}$ are the principal curvatures at
the point $P$, which correspond to the principal curvature directions. Consider the Gauss map $G : p \to N(p)$, whose differential
\[
d G_p : x'(t) \to N'(t) \quad (x(0) = p)
\]
satisfies Rodrigues’ equations,
\[
d G_p(e_i) = -\kappa_i e_i, \quad i = 1, \cdots, n-1.
\]
Then we have the mean curvature
\[
H = \frac{1}{n-1}(\kappa_1 + \cdots + \kappa_{n-1}) = -\frac{1}{n-1}\text{trace}(dG_p),
\]
and the Gauss-Kronecker curvature
\[
K = \kappa_1 \cdots \kappa_{n-1} = (-1)^{n-1}\text{det}(dG_p).
\]
The $i$th order mean curvature is the $i$th order elementary symmetric function of the principal curvatures. We denote by $H_i$ the $i$th order mean curvature normalized such that
\[
\prod_{i=1}^{n-1}(1 + tk_i) = \sum_{i=0}^{n-1} H_i t^i.
\]
Thus, $H_1 = H$ is the mean curvature and $H_{n-1}$ is the Gauss-Kronecker curvature.

The $i$th order mean curvature integral $M_i^{(n)}$ of $\partial K$ at $P$ is defined by
\[
M_i^{(n)}(\partial K) = \int_{\partial K} H_i d\sigma = \left(\begin{array}{c} n-i \\ j \end{array}\right)^{-1} \int_{\partial K} \kappa_{j_1, \cdots, j_i} d\sigma, \quad i = 1, \cdots, n-1,
\]
where $\kappa_{j_1, \cdots, j_i}$ denotes the $i$th elementary symmetric function of the principal curvatures and $d\sigma$ is the area element of $\partial K$. Let $M_0^{(n)}(\partial K) = F$, the area of $\partial K$, for completeness.

Let $K$ be a convex body in $\mathbb{R}^n$ and $L_{r[0]}(r < n)$ $r$-dimensional plane through fixed point $O$ in $\mathbb{R}^n$. Denote by $K'_r$ the orthogonal projection of $K$ onto $L_{r[0]}$. Now, let $M_q^{(r)}(\partial K'_r)(q = 0, 1, \cdots, r-1)$ be the mean curvature integrals of $\partial K'_r$ as a convex surface in $L_{r[0]}$ and $M_q^{(n)}(\partial K'_r)(q = 0, 1, \cdots, n-1)$ the mean curvature integrals of $K'_r$ as a flattened convex body in $\mathbb{R}^n$, then we have the relations between them obtained by Santaló (see [5], [6]).

**Lemma 2.1.** (1) If $q \geq n - r$, then
\[
M_q^{(n)}(\partial K'_r) = \left(\begin{array}{c} r-1 \\ q-n+r \end{array}\right) O_q O_{q-n+r} M_q^{(r)}(\partial K'_r).
\]

(2) If $q = n - r - 1$, then
\[
M_{n-r-1}^{(n)}(\partial K'_r) = \left(\begin{array}{c} n-1 \\ n-r-1 \end{array}\right)^{-1} O_{n-r-1} V_r(K'_r),
\]
where $V_r(K'_r)$ denotes the $r$-dimensional volume of $K'_r$.

(3) If $q < n - r - 1$, then
\[
M_q^{(n)}(\partial K'_r) = 0.
\]

We also need the following result (see [8]).
Lemma 2.2. Let $\Phi$ be an convex body of constant width $h$ in $\mathbb{R}^n$. Then
\[
W_s^{(n)}(\Phi) = \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} W_{n-i}^{(n)}(\Phi) h^{n-s-i}, \quad s = 0, 1, \ldots, n, \tag{2.11}
\]
where $W_s^{(n)}(\Phi)$ is the quermassintegral of $\Phi$.

3. The proof of main Theorems

The proof of Theorem 1.1

Proof. By (2.6) and (2.11) we have
\[
W_l^{(n)}(\Phi_p) = \sum_{j=0}^{n-l} \binom{n-l}{j} W_{l+j}^{(n)}(K) \rho^j = \sum_{j=0}^{n-l} \sum_{i=0}^{n-l-j} (-1)^i \binom{n-l}{j} \binom{n-l-j}{i} W_{n-i}^{(n)}(\Phi) \rho^i h^{n-l-j-i}, \quad l = 0, 1, \ldots, n. \tag{3.1}
\]
Applying this formula to the convex body $(\Phi_r)^{(n)}_\rho$, we have
\[
W_l^{(n)}((\Phi_r)^{(n)}_\rho) = \sum_{j=0}^{n-l} \sum_{i=0}^{n-l-j} (-1)^i \binom{n-l}{j} \binom{n-l-j}{i} W_{n-i}^{(n)}(\Phi) \rho^i h^{n-l-j-i}, \quad l = 0, 1, \ldots, n. \tag{3.2}
\]
By (2.7), we obtain
\[
M_l^{(n)}(\partial (\Phi_r)^{(n)}_\rho) = n W_{l+1}^{(n)}((\Phi_r)^{(n)}_\rho), \quad l = 0, 1, \ldots, n-1. \tag{3.3}
\]
We apply (3.2) and (3.3) to get for $l = 0, 1, \ldots, n$,
\[
M_l^{(n)}(\partial (\Phi_r)^{(n)}_\rho) = n \sum_{j=0}^{n-l-1} \sum_{i=0}^{n-l-j-1} (-1)^i \binom{n-l-1}{j} \binom{n-l-j-1}{i} W_{n-i}^{(n)}(\Phi) \rho^i h^{n-l-j-i-1}
= \sum_{j=0}^{n-l-1} \sum_{i=0}^{n-l-j-1} (-1)^i \binom{n-l-1}{j} \binom{n-l-j-1}{i} M_{n-i-1}^{(n)}(\partial (\Phi_r)^{(n)}_\rho) \rho^i h^{n-l-j-i-1}. \tag{3.4}
\]
Now, we are ready to compute the mean curvature integral of $\partial (\Phi_r)^{(n)}_\rho$ from the below three cases.

Case 1. If $l \geq n - r$, we have $n - i - 1 \geq n - (n - l - j - 1) - 1 \geq l \geq n - r$. Then by (3.4) and Santaló’s result (2.8), we can get
\[
M_l^{(n)}(\partial (\Phi_r)^{(n)}_\rho) = \sum_{j=0}^{n-l-1} \sum_{i=0}^{n-l-j-1} (-1)^i \binom{n-l-1}{j} \binom{n-l-j-1}{i} (r-j) M_{n-i-1}^{(n)}(\partial (\Phi_r)^{(n)}_\rho) \rho^i h^{n-l-j-i-1}. \tag{3.4}
\]
Case 2. If $l = n - r - 1$, then (3.4) can be expressed as
\[
M_l^{(n)}(\partial (\Phi_r)^{(n)}_\rho) = \sum_{j=0}^{r} \sum_{i=0}^{r-j} (-1)^i \binom{r-j}{i} M_{n-i-1}^{(n)}(\partial (\Phi_r)^{(n)}_\rho) \rho^i h^{r-j-i}. \tag{3.5}
\]
Note that \( n - i - 1 \geq n - r - 1 \), thus the right side of the above equation can be decomposed into three parts

\[
M_l^{(n)} \left( \partial (\Phi'_r)_{\rho}^{(n)} \right) = (-1)^r M_{n-r-1}^{(n)}(\partial \Phi'_r) + \sum_{i=0}^{r-1} (-1)^i {r \choose i} M_{n-i-1}^{(n)}(\partial \Phi'_r) h^{r-i}
\]

\[
+ \sum_{j=1}^{r} \sum_{i=0}^{r-j} (-1)^i {r \choose j} (-1)^j M_{n-i-1}^{(n)}(\partial \Phi'_r) \rho^j h^{r-j-i}
\]

\[
= (-1)^r (n-r-1) O_{n-r-1} V_1(\Phi'_r) + \sum_{i=0}^{r-1} (-1)^i {r \choose i} O_{n-i-1} (r-i-1) M_{n-i-1}^{(n)}(\partial \Phi'_r) h^{r-i}
\]

\[
+ \sum_{j=1}^{r} \sum_{i=0}^{r-j} (-1)^i {r \choose j} (-1)^j \frac{(r-i-1)}{(n-i-1)} O_{n-i-1} M_{n-i-1}^{(n)}(\partial \Phi'_r) \rho^j h^{r-j-i}.
\]

where the last equation follows from (2.8) and (2.9).

**Case 3.** If \( l < n - r - 1 \), we first analyze the lower index of mean curvature integral \( M_{n-1-i}^{(n)} \) in the equation (3.4) and get the following table without considering other coefficients for the time being.

| \( j \) | 0 | 1 | \ldots | \( n-r-2 \) | \( n-r-1 \) | \( n-r \) | \( n-r+1 \) | \ldots | \( n-l-2 \) | \( n-l-1 \) |
|------|---|---|------|--------|--------|--------|--------|------|--------|--------|
| 0    | \( M_{n-1}^{(n)} \) | \( M_{n-2}^{(n)} \) | \ldots | \( M_{n-r+1}^{(n)} \) | \( M_{n-r}^{(n)} \) | \( M_{n-r-1}^{(n)} \) | \( M_{n-r-2}^{(n)} \) | \ldots | \( M_{n-l-2}^{(n)} \) | \( M_{n-l-1}^{(n)} \) |
| 1    | \( M_{n-1}^{(n)} \) | \( M_{n-2}^{(n)} \) | \ldots | \( M_{n-r+1}^{(n)} \) | \( M_{n-r}^{(n)} \) | \( M_{n-r-1}^{(n)} \) | \( M_{n-r-2}^{(n)} \) | \ldots | \( M_{n-l-2}^{(n)} \) | \( M_{n-l-1}^{(n)} \) |
| \ldots | \( M_{n-1}^{(n)} \) | \( M_{n-2}^{(n)} \) | \ldots | \( M_{n-r+1}^{(n)} \) | \( M_{n-r}^{(n)} \) | \( M_{n-r-1}^{(n)} \) | \( M_{n-r-2}^{(n)} \) | \ldots | \( M_{n-l-2}^{(n)} \) | \( M_{n-l-1}^{(n)} \) |
| \( n-r-l-1 \) | \( M_{n-1}^{(n)} \) | \( M_{n-2}^{(n)} \) | \ldots | \( M_{n-r+1}^{(n)} \) | \( M_{n-r}^{(n)} \) | \( M_{n-r-1}^{(n)} \) | \( M_{n-r-2}^{(n)} \) | \ldots | \( M_{n-l-2}^{(n)} \) | \( M_{n-l-1}^{(n)} \) |
| \( n-r-l \) | \( M_{n-1}^{(n)} \) | \( M_{n-2}^{(n)} \) | \ldots | \( M_{n-r+1}^{(n)} \) | \( M_{n-r}^{(n)} \) | \( M_{n-r-1}^{(n)} \) | \( M_{n-r-2}^{(n)} \) | \ldots | \( M_{n-l-2}^{(n)} \) | \( M_{n-l-1}^{(n)} \) |
| \( n-r-l+1 \) | \( M_{n-1}^{(n)} \) | \( M_{n-2}^{(n)} \) | \ldots | \( M_{n-r+1}^{(n)} \) | \( M_{n-r}^{(n)} \) | \( M_{n-r-1}^{(n)} \) | \( M_{n-r-2}^{(n)} \) | \ldots | \( M_{n-l-2}^{(n)} \) | \( M_{n-l-1}^{(n)} \) |
| \ldots | \( M_{n-1}^{(n)} \) | \( M_{n-2}^{(n)} \) | \ldots | \( M_{n-r+1}^{(n)} \) | \( M_{n-r}^{(n)} \) | \( M_{n-r-1}^{(n)} \) | \( M_{n-r-2}^{(n)} \) | \ldots | \( M_{n-l-2}^{(n)} \) | \( M_{n-l-1}^{(n)} \) |
| \( n-l-2 \) | \( M_{n-1}^{(n)} \) | \( M_{n-2}^{(n)} \) | \ldots | \( M_{n-r+1}^{(n)} \) | \( M_{n-r}^{(n)} \) | \( M_{n-r-1}^{(n)} \) | \( M_{n-r-2}^{(n)} \) | \ldots | \( M_{n-l-2}^{(n)} \) | \( M_{n-l-1}^{(n)} \) |
| \( n-l-1 \) | \( M_{n-1}^{(n)} \) | \( M_{n-2}^{(n)} \) | \ldots | \( M_{n-r+1}^{(n)} \) | \( M_{n-r}^{(n)} \) | \( M_{n-r-1}^{(n)} \) | \( M_{n-r-2}^{(n)} \) | \ldots | \( M_{n-l-2}^{(n)} \) | \( M_{n-l-1}^{(n)} \) |

By (2.10), we know that when \( i > r \), \( M_{n-1-i}^{(n)}(\partial \Phi'_r) = 0 \). Therefore, combined with the above table, (3.4) can be rewritten as

\[
M_l^{(n)} \left( \partial (\Phi'_r)_{\rho}^{(n)} \right) = \begin{align*}
&= \sum_{j=0}^{n-r-1} \sum_{i=0}^{r-1} (-1)^i \binom{n-l-1}{j} \binom{n-l-j-1}{i} M_{n-i-1}^{(n)}(\partial \Phi'_r) \rho^j h^{n-l-j-i-1} \\
&+ \sum_{j=n-r-l+1}^{n-r-1} \sum_{i=0}^{r-1} (-1)^i \binom{n-l-1}{j} \binom{n-l-j-1}{i} M_{n-i-1}^{(n)}(\partial \Phi'_r) \rho^j h^{n-l-j-i-1} \\
&+ \sum_{j=0}^{n-r-1} (-1)^r \binom{n-l-1}{j} M_{n-r-1}^{(n)}(\partial \Phi'_r) \rho^j h^{n-l-j-r-1} 
\end{align*}
\]
\[
= \sum_{j=0}^{n-r-l} \sum_{i=0}^{r-1} (-1)^i \binom{n-l-1}{j} \binom{n-l-j-1}{i} \frac{O_{n-i-1}}{O_{n-i-1}} M^{(r)}_{r-i-1}(\partial \Phi'_r) \rho^i h^{n-l-j-i-1}
+ \sum_{j=n-r-l+1}^{n-r-1} \sum_{i=0}^{r-1} (-1)^i \binom{n-l-1}{j} \binom{n-l-j-1}{i} \frac{O_{n-i-1}}{O_{n-i-1}} M^{(r)}_{r-i-1}(\partial \Phi'_r) \rho^i h^{n-l-j-i-1}
+ \sum_{j=0}^{n-r-l} (-1)^r \binom{n-l-1}{j} \binom{n-l-j-1}{n-r-1} O_{n-r-1} V_r(\Phi'_r) \rho^i h^{n-l-j-r-1}
\]

where the last equation follows from (2.8) and (2.9). This completes the proof Theorem (1.1). \(\square\)

**The proof of Theorem 1.2**

*Proof. By (2.2), we have
\[
I_{n-r}(\Phi) = \int_{Gr,n-r} V(\Phi'_r) dL_r[\Phi] = \int_{G_{n-r,r}} V(\Phi'_r) dL_{n-r}[\Phi].
\]

Divided by \(m(G_{r,n-r})\), the volume of Grassmann manifold \(G_{r,n-r}\), we obtain the mean value of the projection volumes \(V(\Phi'_r)\)
\[
E(V(\Phi'_r)) = \frac{I_{n-r}(\Phi)}{m(G_{r,n-r})} = \frac{O_{r-1} \cdots O_1 O_0}{O_{n-1} \cdots O_{n-r}} I_{n-r}(\Phi).
\]

Recalling the definition of quermassintegral (2.1), we have
\[
W^{(n)}_{n-r}(\Phi) = \frac{r O_{n-1}}{n O_{r-1}} E(V(\Phi'_r)) = \frac{r O_{r-2} \cdots O_0}{n O_{n-2} \cdots O_{n-r}} I_{n-r}(\Phi), \ r = 1, \cdots, n-1.
\]

Apply the above equation to \(\Phi'_r\) and use the fact \((\Phi'_r)_r = (\Phi'_r)^{(r)}(\Phi)\) to get
\[
W^{(n)}_{n-r}(\Phi'_r) = \frac{r O_{r-2} \cdots O_0}{n O_{n-2} \cdots O_{n-r}} \int_{Gr,n-r} V((\Phi'_r)^{(r)}) dL_r[\Phi], \ r = 1, \cdots, n-1. \tag{3.6}
\]

On the other hand, by (3.1), we get
\[
W^{(n)}_{n-r}(\Phi'_r) = \sum_{j=0}^{r} \binom{r}{j} W^{(n)}_{n-r+j}(\Phi) \rho^j, \ r = 0, 1, \cdots, n-1. \tag{3.7}
\]

And by Steiner formula (2.5) we have
\[
V((\Phi'_r)^{(r)}) = \sum_{j=0}^{r} \binom{r}{j} W^{(r)}_{j}(\Phi'_r) \rho^j, \ r = 0, 1, \cdots, n-1. \tag{3.8}
\]

Combined with (3.6), (3.6) and (3.7), we get
\[
\int_{Gr,n-r} W^{(r)}_{j}(\Phi'_r) dL_r[\Phi] = \frac{n O_{n-2} \cdots O_{n-r}}{r O_{r-2} \cdots O_0} W^{(n)}_{n-r+j}(\Phi).
\]

Using (2.7) yields
\[
\int_{Gr,n-r} M^{(r)}_{t}(\partial \Phi'_r) dL_r[\Phi] = \frac{O_{n-2} \cdots O_{n-r}}{O_{r-2} \cdots O_0} M^{(n)}_{n-r+t}(\partial \Phi), \ t = 0, 1, \cdots, r-1. \tag{3.9}
\]

Now, we are ready to compute the integral of \(M^{(n)}_{t}(\partial (\Phi'_r)^{(n)}(\Phi))\) on Grassmann manifold \(G_{r,n-r} \).
Case 1. When \( q \geq n - r \), by (3.11) and (3.9)

\[
\int_{G_{r,n-r}} M_{l}^{(n)} (\partial (\Phi_{r})^{(n)} ) dL_{r}[O] = \\
\sum_{j=0}^{n-l-1} \sum_{i=0}^{n-l-j-1} (-1)^{j} \left( \begin{array}{c} n-l-1 \\ j \end{array} \right) \Gamma \left( \begin{array}{c} r-i-1 \\ i \end{array} \right) \frac{\Omega_{n-i-1} M_{r-i-1}}{O_{r-i-1}} (\partial \Phi_{r})^{(n)} \rho j \h^{n-l-j-i-1} dL_{r}[O] \\
= \sum_{j=0}^{n-l-1} \sum_{i=0}^{n-l-j-1} (-1)^{j} \left( \begin{array}{c} n-l-1 \\ j \end{array} \right) \Gamma \left( \begin{array}{c} r-i-1 \\ i \end{array} \right) \frac{\Omega_{n-i-1} \Omega_{n-2} \cdots \Omega_{n-r}}{O_{r-i-1} O_{r-2} \cdots O_{0}} \rho j \h^{n-l-j-i-1} M_{n-i-1}^{(n)} (\partial \Phi).
\]

Case 2. When \( q = n - r - 1 \), by (3.12) and (3.9)

\[
\int_{G_{r,n-r}} M_{l}^{(n)} (\partial (\Phi_{r})^{(n)} ) dL_{r}[O] = \\
(-1)^{r} \left( \begin{array}{c} n-1 \\ r-1 \end{array} \right) \Omega_{n-r-1} \int_{G_{r,n-r}} V_{r}(\Phi_{r}) dL_{r}[O] \\
+ \sum_{i=0}^{r-1} (-1)^{i} \left( \begin{array}{c} r-i-1 \\ n-i-1 \end{array} \right) \frac{\Omega_{n-i-1} h^{-i}}{O_{r-i-1}} \int_{G_{r,n-r}} M_{r-i-1}^{(r)} (\partial \Phi_{r})^{(n)} dL_{r}[O] \\
+ \sum_{j=0}^{r} \sum_{i=0}^{r-j} (-1)^{j} \left( \begin{array}{c} r-j \\ n-1 \\ i \end{array} \right) \frac{\Omega_{n-i-1} \rho j h^{-j-i}}{O_{r-i-1} O_{r-2} \cdots O_{0}} \int_{G_{r,n-r}} M_{r-i-1}^{(r)} (\partial \Phi_{r})^{(n)} dL_{r}[O].
\] (3.10)

Note that \( \int_{G_{r,n-r}} V_{r}(\Phi_{r}) dL_{r}[O] = I_{n-r}(\Phi) \) and \( W_{n-r}(\Phi) = \frac{\rho O_{n-2} \cdots O_{n-r}}{n O_{n-2} \cdots O_{n-r}} I_{n-r}(\Phi) \), then we have

\[
\int_{G_{r,n-r}} V_{r}(\Phi_{r}) dL_{r}[O] = \frac{n O_{n-2} \cdots O_{n-r}}{r O_{r-2} \cdots O_{0}} W_{n-r}(\Phi) = \frac{O_{n-2} \cdots O_{n-r}}{r O_{r-2} \cdots O_{0}} M_{n-r-1}^{(n)} (\partial \Phi). \] (3.11)

Inserting (3.11) to (3.10) and using (3.9), we get

\[
\int_{G_{r,n-r}} M_{l}^{(n)} (\partial (\Phi_{r})^{(n)} ) dL_{r}[O] = \\
(-1)^{r} \left( \begin{array}{c} n-1 \\ r-1 \end{array} \right) \Omega_{n-r-1} \frac{O_{n-2} \cdots O_{n-r}}{r O_{r-2} \cdots O_{0}} M_{n-r-1}^{(n)} (\partial \Phi) \\
+ \sum_{i=0}^{r-1} (-1)^{i} \left( \begin{array}{c} r-i-1 \\ n-i-1 \end{array} \right) \frac{O_{n-i-1} O_{n-2} \cdots O_{n-r}}{O_{r-i-1} O_{r-2} \cdots O_{0}} h^{-i} M_{n-i-1}^{(n)} (\partial \Phi) \\
+ \sum_{j=0}^{r} \sum_{i=0}^{r-j} (-1)^{j} \left( \begin{array}{c} r-j \\ n-1 \\ i \end{array} \right) \frac{O_{n-i-1} O_{n-2} \cdots O_{n-r}}{O_{r-i-1} O_{r-2} \cdots O_{0}} \rho j h^{-j-i} M_{n-i-1}^{(n)} (\partial \Phi). \] (3.12)

Case 3. When \( q < n - r - 1 \), by (3.13), (3.9) and (3.11), we have

\[
\int_{G_{r,n-r}} M_{l}^{(n)} (\partial (\Phi_{r})^{(n)} ) dL_{r}[O] = \\
\sum_{j=0}^{n-r-l-1} (-1)^{j} \left( \begin{array}{c} n-l-1 \\ r \end{array} \right) \frac{O_{n-r-1} \rho j h^{n-l-j-r-1}}{O_{r} (\partial \Phi_{r})^{(n)} } \int_{G_{r,n-r}} V_{r}(\Phi_{r}) dL_{r}[O]
\]
\[ + \sum_{j=0}^{n-r-l} \sum_{i=0}^{r-1} (-1)^i \frac{(n-l-1)(n-l-j-1)(r-1)}{(n-1)i} \frac{Q_{n-i-1}}{O_{r-i-1}} \rho^j h^{n-l-j-i-1} \int_{G_{r-n-r}} M_{r-i-1}^{(r)} (\partial \phi') dL_{r-i-1} \]

\[ + \sum_{j=n-r-l+1}^{n-r-l-1} \sum_{i=0}^{r-1} (-1)^i \frac{(n-l-1)(n-l-j-1)(r-1)}{(n-1)i} \frac{Q_{n-i-1}}{O_{r-i-1} O_{r-2} \cdots O_0} \rho^j h^{n-l-j-i-1} \int_{G_{r-n-r}} M_{r-i-1}^{(r)} (\partial \phi') dL_{r-i-1} \]

\[ = \sum_{j=0}^{n-r-l-1} (-1)^i \frac{(n-l-1)(n-l-j-1)(r-1)}{(n-r-1)i} \frac{Q_{n-i-1} O_{r-2} \cdots O_0}{O_{r-i-1} O_{r-2} \cdots O_0} \rho^j h^{n-l-j-i-1} M_{n-i-1}^{(n)} (\partial \phi) \]

\[ + \sum_{j=n-r-l+1}^{n-r-l-1} \sum_{i=0}^{n-l-j-1} (-1)^i \frac{(n-l-1)(n-l-j-1)(r-1)}{(n-1)i} \frac{Q_{n-i-1} O_{r-2} \cdots O_0}{O_{r-i-1} O_{r-2} \cdots O_0} \rho^j h^{n-l-j-i-1} M_{n-i-1}^{(n)} (\partial \phi) \]

This completes the proof of Theorem 1.2. \[\square\]

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