Asymptotic behavior of a series of Euler’s totient function $\varphi(k)$ times the index of $1/k$ in a Farey sequence

R. Tomás *
CERN, CH 1211 Geneva 23, Switzerland

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Abstract
Motivated by studies in accelerator physics this paper computes the asymptotic behavior of the series $\sum_{k=1}^{N} \varphi(k)I_N \left( \frac{1}{k} \right)$, where $\varphi(k)$ is Euler’s totient function and $I_N \left( \frac{1}{k} \right)$ is the position that $1/k$ occupies in the Farey sequence of order $N$. To this end an exact formula for $I_N \left( \frac{1}{k} \right)$ is derived when all integers in $\left[ 2, \left\lceil \frac{N}{k} \right\rceil \right]$ are divisors of $N$.

1 Results
Let $I_N \left( \frac{1}{k} \right)$, with $k \in \mathbb{N}$ and $k \leq N$, be the position that $1/k$ occupies in the Farey sequence of order $N$. Some useful facts follow:

\[
I_N \left( \frac{0}{1} \right) = 1, \quad (1)
\]
\[
I_N \left( \frac{1}{1} \right) = \left| F_N \right|, \quad (2)
\]
\[
I_N \left( \frac{1}{k} \right) = 1 + \sum_{j=k}^{N} \phi \left( j; \left\lceil \frac{j}{k} \right\rceil \right), \quad (3)
\]
\[
I_N \left( \frac{1}{k} \right) = 1 + \left| F_N \right| - I_N \left( \frac{k-1}{k} \right), \quad (4)
\]

*rogelio.tomas@cern.ch
where |\(F_N\)| stands for the cardinality of the Farey sequence of order \(N\) and Eq. 3 is found at Remark 7.10 (there fractions are indexed starting with 0); \(\phi(n;[\cdot,\cdot])\) is defined as the number of elements from an interval of integers that are relatively prime to \(n\). Consequently \(\phi(n;[1,n]) = \phi(n)\) and \(|F_N| = 1 + \sum_{j=1}^{N} \phi(j;[1,j])\). Efficient algorithms for the computation of \(I_N(x)\) were recently developed in [2].

**Theorem 1.1.** Given \(k \in \mathbb{N}\) such that \(k \leq N\), then

\[
I_N \left( \frac{1}{k} \right) \leq \frac{N^2 + N}{2k} .
\]

**Proof.** In Eq. (3) the set \([1, \frac{1}{k}]\) has a maximum of \(j/k\) elements with \(j\) running between \(k\) and \(N\). Therefore \(\phi\) tests a maximum of

\[
\sum_{j=k}^{N} \frac{j}{k} = \frac{(N - k + 1)(N + k)}{2k} = \frac{N^2 - k^2 + N + k}{2k}
\]

elements.

Let the subsequence of \(F_N\), \(F_{N}^{1/a,1/b}\) be defined as all the fractions of \(F_N\) in \([1/a, 1/b]\) with \(1 \leq b \leq a \leq N\).

**Theorem 1.2.** If \(N\) is a multiple of \(i\) and \(i + 1\) there is a bijective and order-preserving map between \(F_i\) and \(F_{N}^{1/q,1/(q-1)}\), with \(q\) being an integer fulfilling \(N/(i + 1) < q \leq N/i\), given by

\[
F_i \rightarrow F_{N}^{1/q,1/(q-1)} , \quad \frac{h}{k} \mapsto \frac{k}{kq - h} .
\]

**Proof.** The demonstration is given in two steps. The first step is shown in Fig. 1 where by construction it is clear that all Farey fractions in \([1/q, 1/(q - 1)]\) at any order are connected to the Farey fractions in \([0/1, 1/1]\) by the application in Eq. (7).

A second step is needed to show that the application is bijective between the sets \(F_i\) and \(F_{N}^{1/q,1/(q-1)}\). The fraction \(h/k\) belongs to \(F_i\) if \(k \leq i\), and similarly \(k/(kq - h)\) belongs to \(F_{N}^{1/q,1/(q-1)}\) if \((kq - h) \leq N\). Since \(N\) is a multiple of \(i\) and \(i + 1\) and that \(N/(i + 1) < q \leq N/i\), the largest value \((kq - h)\) takes is \((kN - h)\). Therefore \(k/(kq - h)\) belongs to \(F_{N}^{1/q,1/(q-1)}\) if \(k \leq i\), i.e., if \(h/k\) belongs to \(F_i\).

In the opposite direction, if \((kq - h) \leq N\) the largest possible \(k\) is obtained by inserting the smallest \(q\), which is \(N/(i + 1) + 1\), yielding

\[
k \left( \frac{N}{i + 1} + 1 \right) - h \leq N .
\]
Figure 1: Application between the Farey fractions in $[1/q, 1/(q - 1)]$ and $[0, 1]$ demonstrated by using that the next Farey fraction appearing between $h/k$ and $h'/k'$ is $(h+h')/(k+k')$. By applying this rule independently to $1/q$ and $1/(q-1)$ (top) and $0/1$ and $1/1$ (bottom) the map of Eq. (7) is apparent. The second and the last to the last terms of $F_i$ (bottom) and $F_{N+1}/q, 1/(q-1)$ (top) are also shown. They can be computed using Corollary 3.2 of [3].
If \( k = h \) this corresponds to the trivial case \( h/k = 1/1 \), which clearly satisfies Eq. \(1\). Else, \( k > h \) and Eq. \(8\) becomes
\[
\frac{k}{i+1} < N, \quad k \leq i, \quad (9) \]
concluding that if \( k/(kq - h) \) belongs to \( F_N^{1/q,1/(q-1)} \), \( h/k \) belongs to \( F_i \).

Note that in the case \( q = 2 \) the map in Eq. \(7\) can be viewed as a map from Lemma 1.1 that reflects a Farey sequence to the right half sequence of a Farey subsequence arising in the combinatorics of finite sets.

**Theorem 1.3.** Let \( N/(i+1) \leq k \leq N/i \) and \( N \) be a multiple of all integers in \([2, i]\), then
\[
I_N \left( \frac{1}{k} \right) = 2 + N \sum_{j=1}^{i} \frac{\varphi(j)}{j} - k \Phi(i), \quad (11)
\]
where \( \Phi(i) \) is the totient summatory function, \( \Phi(i) \equiv \sum_{j=1}^{i} \varphi(j) \equiv |F_i| - 1 \).

**Proof.** Express \( I_N(1/(N/i)) \) as the sum of the cardinalities of all subsequences of the form \( F_N^{1/q,1/(q-1)} \) such that \( q > N/i \),
\[
I_N \left( \frac{1}{N/i} \right) = 2 + \sum_{q=N/i+1}^{N} \left( \left| F_N^{1/q,1/(q-1)} \right| - 1 \right). \quad (12)
\]
By virtue of Theorem 1.2, \( \left| F_N^{1/q,1/(q-1)} \right| = |F_i| \) when \( N/(i+1) < q \leq N/i \) and Eq. \(12\) is directly re-written as
\[
I_N \left( \frac{1}{N/i} \right) = 2 + N \sum_{j=1}^{i-1} \frac{\Phi(j)}{j(j+1)}. \quad (13)
\]
\( I_N(1/k) \) is computed by adding to the expression above the cardinality of the remaining subsequences between \( N/i \) and \( k \), yielding
\[
I_N \left( \frac{1}{k} \right) = \Phi(i) \left( \frac{N}{i} - k \right) + I_N \left( \frac{1}{N/i} \right). \quad (14)
\]
After some algebra Eq. \(11\) is obtained.

**Theorem 1.4.**
\[
\sum_{k=1}^{N} \varphi(k) I_N(1/k) = \frac{N^3}{6\zeta(3)} + O \left( \frac{N^3}{\log N} \right) \quad (15)
\]
Proof. Let $N = \text{lcm}(2, 3, 4, \ldots, i_{\text{max}})$ be the least common multiple of the first $i_{\text{max}}$ numbers. The summation in the left hand side of Eq. (15) is split into two parts, the first part being for $k \leq N/i_{\text{max}}$. Equation (5) is used to give an upper bound to the summation corresponding to this first part. In the second part, $k > N/i_{\text{max}}$, Eq. (11) is used. The two contributions are given by

\[
\sum_{k=1}^{N/i_{\text{max}}} \varphi(k)I_N(1/k) \leq (N^2 + N) \sum_{k=1}^{N/i_{\text{max}}} \frac{\varphi(k)}{2k} ,
\]

\[
\sum_{k>N/i_{\text{max}}} \varphi(k)I_N(1/k) = \sum_{i=1}^{i_{\text{max}}-1} \sum_{k=N/i}^{N} \varphi(k) \left( 2 + N \sum_{j=1}^{i} \frac{\varphi(j)}{j} - k\Phi(i) \right)
\]

After some algebra using the following relations,

\[
\sum_{k=1}^{N} \varphi(k) = \frac{3}{\pi^2} N^2 + O(N \log N) ,
\]

\[
\sum_{k=1}^{N} \frac{\varphi(k)}{k} = \frac{6}{\pi^2} N + O((\log N)^{2/3}(\log \log N)^{4/3}) ,
\]

\[
\sum_{k=1}^{N} \varphi(k)k = \frac{2}{\pi^2} N^3 + O(N^2) ,
\]

\[
\sum_{k=1}^{\infty} \frac{\varphi(k)}{k^3} = \frac{\zeta(2)}{\zeta(3)} = \frac{\pi^2}{6\zeta(3)} ,
\]

one obtains

\[
\sum_{k=1}^{N/i_{\text{max}}} \varphi(k)I_N(1/k) \leq \frac{3}{\pi^2 i_{\text{max}}} (N^3 + N^2) ,
\]

\[
\sum_{k>N/i_{\text{max}}} \varphi(k)I_N(1/k) = \frac{N^3}{\pi^2} \sum_{i=1}^{i_{\text{max}}} \frac{\varphi(i)}{i^3} + O(N^2 \log N) .
\]

When taking the limit $i_{\text{max}} \to \infty$, $N$ tends to $e^{i_{\text{max}}}$ and, inversely, $i_{\text{max}}$ tends to $\ln(N)$. Equations (21) and (22) become

\[
\sum_{k=1}^{N/i_{\text{max}}} \varphi(k)I_N(1/k) \leq \frac{3}{\pi^2} \frac{N^3 + N^2}{\ln(N)} ,
\]

\[
\sum_{k>N/i_{\text{max}}} \varphi(k)I_N(1/k) = \frac{N^3}{6\zeta(3)} + O(N^2 \log N) .
\]

Combining these two last equations Eq. (15) is obtained. \qed
2 Discussion

Let $T^D_N$ be the number of linear, integral and irreducible polynomials of dimension $D$ of the form

$$
\sum_{i=1}^{D} a_i x_i - a_{D+1},
$$

such that $\sum_{i=1}^{D} |a_i| \leq N$ and having at least one root in the unitary $D$-cube. $D = 0$ corresponds to the null polynomial $0$. $D = 1$ corresponds to Farey sequences. The result of the previous section serves to compute $T^D_N$, see [4, 5]. The asymptotic behavior in $N$ of $T^D_N$, for $D = 0, 1, 2$ follow,

$$
T^0_N = 1,
$$

$$
T^1_N = \frac{3}{\pi^2} N^2 + O(N \log N),
$$

$$
T^2_N = \frac{2N^3}{3\zeta(3)} + O\left(\frac{N^3}{\log N}\right).
$$


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