PERFECTNESS OF KIRILLOV–RESHETIKHIN CRYSTALS FOR NONEXCEPTIONAL TYPES

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Abstract. For nonexceptional types, we prove a conjecture of Hatayama et al. about the prefectness of Kirillov–Reshetikhin crystals.

1. Introduction

Kirillov–Reshetikhin (KR) crystals $B^{r,s}$ are finite affine crystals corresponding to finite-dimensional $U'_q(g)$-modules [3, 4], where $g$ is an affine Kac–Moody algebra. Recently, a lot of progress has been made regarding these crystals which appear in mathematical physics and the path realization of affine highest weight crystals [12]. In [19, 20] it was shown that the KR crystals exist and in [5] combinatorial realizations for these crystals were provided for all nonexceptional types. In this paper, we prove a conjecture of Hatayama et al. [7, Conjecture 2.1] about the perfectness of KR crystals.

Conjecture 1.1. [7, Conjecture 2.1] The Kirillov-Reshetikhin crystal $B^{r,s}$ is perfect if and only if $\frac{s}{c_r}$ is an integer with $c_r$ as in Table 1. If $B^{r,s}$ is perfect, its level is $\frac{s}{c_r}$.

In [13], this conjecture was proven for type $A_n^{(1)}$, for $B_{1,s}$ for nonexceptional types (except for type $C_n^{(1)}$), for $B_{n-1,s}$, $B_n$, $s$ of type $D_n^{(1)}$, and $B_n$, $s$ for types $C_n^{(1)}$ and $D_{n+1}^{(2)}$. When the highest weight is given by the highest root, level-1 perfect crystals were constructed in [1]. For $1 \leq r \leq n-2$ for type $D_n^{(1)}$, $1 \leq r \leq n-1$ for type $B_n^{(1)}$, and $1 \leq r \leq n$ for type $A_{2n-1}^{(2)}$, the conjecture was proved in [21]. The case $G_2^{(1)}$ and $r = 1$ was treated in [23] and the case $D_4^{(3)}$ and $r = 1$ was treated in [15]. Naito and Sagaki [17] showed that the conjecture holds for twisted algebras, if it is true for the untwisted simply-laced cases.

In this paper we prove Conjecture 1.1 in general for nonexceptional types.

Theorem 1.2. If $g$ is of nonexceptional type, Conjecture 1.1 is true.

The paper is organized as follows. In Section 2 we give basic notation and the definition of perfectness in Definition 2.1. In Section 3 we review the realizations of the KR crystals of nonexceptional types as recently provided in [5]. Section 4 is reserved for the proof of Theorem 1.2 and an explicit description of the minimal elements $B_{min}^{r,c_r,s}$ of the perfect crystals. Several examples for KR crystals of type $C_3^{(1)}$ are given in Section 5.

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2. Definitions and perfectness

We follow the notation of [11, 5]. Let $\mathcal{B}$ be a $U_q'(\mathfrak{g})$-crystal [14]. Denote by $\alpha_i$ and $\Lambda_i$ for $i \in I$ the simple roots and fundamental weights and by $c$ the canonical central element associated to $\mathfrak{g}$, where $I$ is the index set of the Dynkin diagram of $\mathfrak{g}$ (see Table 2). Let $P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i$ be the weight lattice of $\mathfrak{g}$ and $P^+$ the set of dominant weights. For a positive integer $\ell$, the set of level-$\ell$ weights is

$$
P_\ell^+ = \{ \Lambda \in P^+ \mid \text{lev}(\Lambda) = \ell \},
$$

where $\text{lev}(\Lambda) := \Lambda(c)$. The set of level-0 weights is denoted by $P_0$.

We denote by $f_i, e_i : \mathcal{B} \rightarrow \mathcal{B} \cup \{ \emptyset \}$ for $i \in I$ the Kashiwara operators and by $\text{wt} : \mathcal{B} \rightarrow P$ the weight function on the crystal. For $b \in \mathcal{B}$ we define $\varepsilon_i(b) = \max \{ k \mid e_i^k(b) \neq \emptyset \}$, $\varphi_i(b) = \max \{ k \mid f_i^k(b) \neq \emptyset \}$, and

$$
\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i.
$$

Next we define perfect crystals, see for example [10].

**Definition 2.1.** For a positive integer $\ell > 0$, a crystal $\mathcal{B}$ is called perfect crystal of level $\ell$, if the following conditions are satisfied:

1. $\mathcal{B}$ is isomorphic to the crystal graph of a finite-dimensional $U_q'(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. There exists a $\lambda \in P_0$, such that $\text{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I \setminus \{0\}} \mathbb{Z}_{\leq 0} \alpha_i$ and there is a unique element in $\mathcal{B}$ of classical weight $\lambda$.
4. $\forall b \in \mathcal{B}, \ \text{lev}(\varepsilon(b)) \geq \ell$.

| $(c_1, \ldots, c_n)$ | $A_n^{(1)}$ | $(1, \ldots, 1)$ |
|----------------------|-------------|-----------------|
| $B_n^{(1)}$ | $(1, \ldots, 1, 2)$ |
| $C_n^{(1)}$ | $(2, \ldots, 2, 1)$ |
| $D_n^{(1)}$ | $(1, \ldots, 1)$ |
| $A_{2n-1}^{(2)}$ | $(1, \ldots, 1)$ |
| $A_{2n}^{(2)}$ | $(1, \ldots, 1)$ |
| $E_{n+1}$ | $(1, \ldots, 1)$ |

**Table 1. List of $c_r$**
Table 2. Dynkin diagrams

(5) \( \forall \Lambda \in P_\ell^+, \) there exist unique elements \( b_\Lambda, b^\Lambda \in B, \) such that
\[
\varepsilon(b_\Lambda) = \Lambda = \phi(b^\Lambda).
\]

We denote by \( B_{\text{min}} \) the set of minimal elements in \( B, \) namely
\[
B_{\text{min}} = \{ b \in B \mid \text{lev}(\varepsilon(b)) = \ell \}.
\]

Note that condition (5) of Definition 2.1 ensures that \( \varepsilon, \phi : B_{\text{min}} \to P_\ell^+ \) are bijections. They induce an automorphism \( \tau = \varepsilon \circ \phi^{-1} \) on \( P_\ell^+. \)

In [21, 5] \( \pm \)-diagrams were introduced, which describe the branching \( X_n \to X_{n-1} \)
where \( X_n = B_n, C_n, D_n. \) A \( \pm \)-diagram \( P \) of shape \( \Lambda/\lambda \) is a sequence of partitions \( \lambda \subseteq \mu \subseteq \Lambda \) such that \( \Lambda/\mu \) and \( \mu/\lambda \) are horizontal strips (i.e. every column contains at most one box). We depict this \( \pm \)-diagram by the skew tableau of shape \( \Lambda/\lambda \) in which the cells of \( \mu/\lambda \) are filled with the symbol \( + \) and those of \( \Lambda/\mu \) are filled with the symbol \( - \). There are further type specific rules which can be found in [5, Section 3.2]. There exists a bijection \( \Phi \) between \( \pm \)-diagrams and \( X_{n-1} \)-highest weight vectors.

3. Realization of KR-crystals

Throughout the paper we use the realization of \( B^{r,s} \) as given in [5, 20, 21]. In this section we briefly recall the main constructions.

3.1. KR crystals of type \( A_n^{(1)} \). Let \( \Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_n\Lambda_n \) be a dominant weight. Then the level is given by
\[
\text{lev}(\Lambda) = \ell_0 + \cdots + \ell_n.
\]
A combinatorial description of $B^{r,s}$ of type $A_{n}^{(1)}$ was provided by Shimozono [22]. As a $\{1,2,\ldots,n\}$-crystal

$$B^{r,s} \cong B(s\Lambda_{r}).$$

The Dynkin diagram of $A_{n}^{(1)}$ has a cyclic automorphism $\sigma(i) = i + 1 \pmod{n + 1}$ which extends to the crystal in form of the promotion operator. The action of the affine crystal operators $f_{0}$ and $e_{0}$ is given by

$$f_{0} = \sigma^{-1} \circ f_{1} \circ \sigma \quad \text{and} \quad e_{0} = \sigma^{-1} \circ e_{1} \circ \sigma.$$  

3.2. KR crystals of type $D_{n}^{(1)}$, $B_{n}^{(1)}$, $A_{2n-1}^{(2)}$. Let $\Lambda = \ell_{0}\Lambda_{0} + \ell_{1}\Lambda_{1} + \cdots + \ell_{n}\Lambda_{n}$ be a dominant weight. Then the level is given by

$$\text{lev}(\Lambda) = \ell_{0} + \ell_{1} + 2\ell_{2} + 2\ell_{3} + \cdots + 2\ell_{n-2} + \ell_{n-1} + \ell_{n} \quad \text{for type } D_{n}^{(1)}$$

$$\text{lev}(\Lambda) = \ell_{0} + \ell_{1} + 2\ell_{2} + 2\ell_{3} + \cdots + 2\ell_{n-2} + \ell_{n-1} + \ell_{n} \quad \text{for type } B_{n}^{(1)}$$

$$\text{lev}(\Lambda) = \ell_{0} + \ell_{1} + 2\ell_{2} + 2\ell_{3} + \cdots + 2\ell_{n-2} + \ell_{n-1} + 2\ell_{n} \quad \text{for type } A_{2n-1}^{(2)}.$$  

We have the following realization of $B^{r,s}$. Let $X_{n} = D_{n}, B_{n}, C_{n}$ be the classical subalgebra for $D_{n}^{(1)}$, $B_{n}^{(1)}, A_{2n-1}^{(2)}$, respectively.

**Definition 3.1.** Let $1 \leq r \leq n - 2$ for type $D_{n}^{(1)}$, $1 \leq r \leq n - 1$ for type $B_{n}^{(1)}$, and $1 \leq r \leq n$ for type $A_{2n-1}^{(2)}$. Then $B^{r,s}$ is defined as follows. As an $X_{n}$-crystal

(3.1)  

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda),$$

where the sum runs over all dominant weights $\Lambda$ that can be obtained from $s\Lambda_{r}$ by the removal of vertical dominoes. The affine crystal operators $e_{0}$ and $f_{0}$ are defined as

(3.2)  

$$f_{0} = \sigma \circ f_{1} \circ \sigma \quad \text{and} \quad e_{0} = \sigma \circ e_{1} \circ \sigma,$$

where $\sigma$ is the crystal automorphism defined in [21, Definition 4.2].

**Definition 3.2.** Let $B_{n}^{(s)}_{A_{2n-1}^{(2)}}$ be the $A_{2n-1}^{(2)}$-KR crystal. Then $B^{r,s}$ of type $B_{n}^{(1)}$ is defined through the unique injective map $S : B^{r,s} \to B_{n}^{(s)}_{A_{2n-1}^{(2)}}$ such that

$$S(e_{i}b) = e_{i}^{m_{i}}S(b), \quad S(f_{i}b) = f_{i}^{m_{i}}S(b) \quad \text{for } i \in I,$$

where $(m_{i})_{0 \leq i \leq n} = (2,2,\ldots,2,1)$.

In addition, the $\pm$-diagrams of $A_{2n-1}^{(2)}$ that occur in the image are precisely those which can be obtained by doubling a $\pm$-diagram of $B^{r,s}$ (see [5, Lemma 3.5]). $S$ induces an embedding of dominant weights of $B_{n}^{(1)}$ into dominant weights of $A_{2n-1}^{(2)}$, namely $S(\Lambda) = m_{i}\Lambda_{i}$. It is easy to see that for any $\Lambda \in P^{+}$ we have $\text{lev}(S(\Lambda)) = 2\text{lev}(\Lambda)$.

For the definition of $B^{r,s}$ and $B^{r-1,s}$ of type $D_{n}^{(1)}$, see for example [5, Section 6.2].
3.3. **KR crystal of type** $C_n^{(1)}$. The level of a dominant $C_n^{(1)}$ weight $\Lambda = \ell_0\Lambda_0 + \cdots + \ell_n\Lambda_n$ is given by

$$\text{lev}(\Lambda) = \ell_0 + \cdots + \ell_n.$$  

We use the realization of $B^{r,s}$ as the fixed point set of the automorphism $\sigma$ [21 Definition 4.2] (see Definition 3.1) inside $B_{A_{2n+1}^{(2)}}^{r,s}$ of [5] Theorem 5.7.

**Definition 3.3.** For $1 \leq r < n$, the KR crystal $B^{r,s}$ of type $C_n^{(1)}$ is defined to be the fixed point set under $\sigma$ inside $B_{A_{2n+1}^{(2)}}^{r,s}$ with the operators

$$e_i = \begin{cases} e_0e_1 & \text{for } i = 0, \\ e_{i+1} & \text{for } 1 \leq i \leq n, \end{cases}$$

where the Kashiwara operators on the right act in $B_{A_{2n+1}^{(2)}}^{r,s}$. Under the crystal embedding $S : B^{r,s} \rightarrow B_{A_{2n+1}^{(2)}}^{r,s}$ we have

$$\Lambda_i \mapsto \begin{cases} \Lambda_0 + \Lambda_1 & \text{for } i = 0, \\ \Lambda_{i+1} & \text{for } 1 \leq i \leq n. \end{cases}$$

Under the embedding $S$, the level of $\Lambda \in P^+$ doubles, that is $\text{lev}(S(\Lambda)) = 2\text{lev}(\Lambda)$. For $B^{r,s}$ of type $C_n^{(1)}$ we refer to [5] Section 6.1.

3.4. **KR crystals of type** $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$. Let $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_n\Lambda_n$ be a dominant weight. The level is given by

$$\text{lev}(\Lambda) = \ell_0 + 2\ell_1 + 2\ell_2 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + 2\ell_n$$

for type $A_{2n}^{(2)}$

$$\text{lev}(\Lambda) = \ell_0 + 2\ell_1 + 2\ell_2 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + \ell_n$$

for type $D_{n+1}^{(2)}$.

Define positive integers $m_i$ for $i \in I$ as follows:

$$m_0, m_1, \ldots, m_{n-1}, m_n = \begin{cases} (1, 2, \ldots, 2, 2) & \text{for } A_{2n}^{(2)}, \\ (1, 2, \ldots, 2, 1) & \text{for } D_{n+1}^{(2)}. \end{cases}$$

Then $B^{r,s}$ can be realized as follows.

**Definition 3.4.** For $1 \leq r \leq n$ for $g = A_{2n}^{(2)}$, $1 \leq r < n$ for $g = D_{n+1}^{(2)}$ and $s \geq 1$, there exists a unique injective map $S : B_{6}^{r,s} \rightarrow B_{C_{n+1}^{(2)}}^{r,2s}$ such that

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b)$$

for $i \in I$.

The $\pm$-diagrams of $C_n^{(1)}$ that occur in the image of $S$ are precisely those which can be obtained by doubling a $\pm$-diagram of $B^{r,s}$ (see [5] Lemma 3.5). $S$ induces an embedding of dominant weights for $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$ into dominant weights of type $C_n^{(1)}$, with $S(\Lambda_i) = m_i\Lambda_i$. This map preserves the level of a weight, that is $\text{lev}(S(\Lambda)) = \text{lev}(\Lambda)$.

For the case $r = n$ of type $D_{n+1}^{(2)}$ we refer to [5] Definition 6.2.
For type $A_n^{(1)}$, perfectness of $B^{r,s}$ was proven in \cite{13}. For all other types, in the case that $\frac{\alpha}{\tau}$ is an integer, we need to show that the 5 defining conditions in Definition 2.1 are satisfied:

1. This was recently shown in \cite{20}.
2. This follows from \cite{6} Corollary 6.1 under \cite{6} Assumption 1. Assumption 1 is satisfied except for type $A_n^{(2)}$. The regularity of $B^{r,s}$ is ensured by (1). The existence of an automorphism $\sigma$ was proven in \cite{6} Section 7, and the unique element $u \in B^{r,s}$ such that $\varepsilon(u) = s\Lambda_0$ and $\varphi(u) = s\Lambda_\nu$ (where $\nu = 1$ for $r$ odd for types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$; $\nu = r$ for $A_n^{(1)}$, and $\nu = 0$ otherwise) is given by the classically highest weight element in the component $B(0)$ for $\nu = 0$, $B(s\Lambda_1)$ for $\nu = 1$, and $B(s\Lambda_r)$ for $\nu = r$. Note that $\Lambda_0 = \tau(\Lambda_\nu)$, where $\tau = \varepsilon \circ \varphi^{-1}$. For type $A_n^{(2)}$, perfectness follows from \cite{17}.
3. The statement is true for $\lambda = s(\Lambda_r - \Lambda_s(c)\Lambda_0)$, which follows from the decomposition formulas \cite{2} \cite{8} \cite{9} \cite{18}.

Conditions (4) and (5) will be shown in the following subsections using case by case considerations: Section 4.1 for type $A_n^{(1)}$, Sections 4.2, 4.3, and 4.4 for types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, Sections 4.5 and 4.6 for type $C_n^{(1)}$, Section 4.7 for type $A_n^{(2)}$, and Sections 4.8 and 4.9 for type $D_n^{(2)}$.

When $\frac{\alpha}{\tau}$ is not an integer, we show in the subsequent sections that the minimum of the level of $\varepsilon(b)$ is the smallest integer exceeding $\frac{\alpha}{\tau}$, and provide examples that contradict condition (5) of Definition 2.1 for each crystal, thereby proving that $B^{r,s}$ is not perfect. In the case that $\frac{\alpha}{\tau}$ is an integer, we provide an explicit construction of the minimal elements of $B^{r,s}$.

4.1. Type $A_n^{(1)}$. It was already proven in \cite{13} that $B^{r,s}$ is perfect. We give below its associated automorphism $\tau$ and minimal elements. $\tau$ on $P$ is defined by

$$\tau\left(\sum_{i=0}^{n} k_i \Lambda_i\right) = \sum_{i=0}^{n} k_i \Lambda_{i-r \mod n+1}.$$  

Recall that $B^{r,s}$ is identified with the set of semistandard tableaux of $r \times s$ rectangular shape over the alphabet $\{1, 2, \ldots, n+1\}$. For $b \in B^{r,s}$ let $x_{ij} = x_{ij}(b)$ denote the number of letters $j$ in the $i$-th row of $b$ for $1 \leq i \leq r, 1 \leq j \leq n+1$. Set $r' = n + 1 - r$, then

$$x_{ij} = 0 \quad \text{unless} \quad i \leq j \leq i + r'.$$

Let $\Lambda = \sum_{i=0}^{n} \ell_i \Lambda_i$ be in $P^+_s$, that is, $\ell_0, \ell_1, \ldots, \ell_n \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^{n} \ell_i = s$. Then $x_{ij}(b)$ of the minimal element $b$ such that $\varepsilon(b) = \Lambda$ is given by

$$x_{ij} = \ell_0 + \sum_{\alpha = i}^{r-1} \ell_{\alpha + r'},$$  

$$x_{ij} = \ell_{j-i} \quad (i < j < i + r'),$$

$$x_{ij} = \sum_{\alpha = 0}^{i-1} \ell_{\alpha + r'}$$

for $1 \leq i \leq r$. 

4.2. Types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$. Conditions (4) and (5) of Definition 2.1 for $1 \leq r \leq n - 2$ for type $D_n^{(1)}$, $1 \leq r \leq n - 1$ for type $B_n^{(1)}$, and $1 \leq r \leq n$ for type $A_{2n-1}^{(2)}$ were shown in [21, Section 6]. We briefly review the construction of the minimal elements here since they are important in the construction of the minimal elements for type $C_n^{(1)}$.

To a given fundamental weight $\Lambda_k$ we may associate the following $\pm$-diagram

\[
\begin{array}{c}
\emptyset \quad \text{if } r \text{ is even and } k = 0 \\
\pm \quad \text{if } r \text{ is even and } k = 1 \\
\pm \quad \text{if } r \text{ is odd and } k = 0 \\
\pm \quad \text{if } r \text{ is odd and } k = 1 \\
k+1 \begin{cases}
\pm \quad \text{if } k \not\equiv r \mod 2 \text{ and } 2 \leq k \leq r
\end{cases}
\end{array}
\]

This map can be extended to any dominant weight $\Lambda = \ell_0 \Lambda_0 + \cdots + \ell_n \Lambda_n$ by concatenating the columns of the $\pm$-diagrams of each piece.
To every fundamental weight $\Lambda_k$ we also associate a string of operators $f_i$ with $i \in \{2, 3, \ldots, n\}$ as follows. Let $T(\Lambda_k)$ be the tableau assigned to $\Lambda_k$ as

$$
T(\Lambda_k) = \begin{cases}
    u & \text{if } r \text{ is even and } k = 0 \\
    1 & \text{if } r \text{ is even and } k = 1 \\
    1 & \text{if } r \text{ is odd and } k = 0 \\
    k & \text{if } r \text{ is odd and } k = 1 \\
    2 & \text{if } 2 \leq k \leq r \text{ and } k \not\equiv r \mod 2 \\
    1 & \text{if } 2 \leq k \leq r \text{ and } k \equiv r \mod 2 \\
    \vdots & \vdots \\
    1 & k \\
    n & \pi \\
    0 & \text{previous case with } n \leftrightarrow \pi \\
    r & \text{for } k = n - 1 \text{ for type } D_n^{(1)} \\
    r & \text{for } k = n \text{ for type } B_n^{(1)}
\end{cases}
$$

Then $f(\Lambda_k)$ for $0 \leq k \leq n$ is defined such that $T(\Lambda_k) = f(\Lambda_k)\Phi(\text{diagram}(\Lambda_k))$, where $\Phi$ is the bijection between $\pm$-diagrams and $X_{n-1}$-highest weight elements (see [21 5]). Note that in fact $f(\Lambda_0) = f(\Lambda_1) = 1$.

The minimal element $b$ in $B_{\sigma'}^{r,s}$ that satisfies $\varepsilon(b) = \Lambda$ can now be constructed as follows

$$
b = f(\Lambda_n)^{r_n} \cdots f(\Lambda_2)^{r_2}\Phi(\text{diagram}(\Lambda)).$$

From the condition that $\text{wt}(b) = \varphi(b) - \varepsilon(b)$ it is not hard to see that $\varphi(b) = \varepsilon(b)$ for $b \in B_{\sigma'}^{r,s}$ and $r$ even. For $r$ odd, we have $\varphi(b) = \sigma \circ \sigma' \circ \varepsilon(b)$ for $b \in B_{\min}^{r,s}$, where $\sigma$ is the Dynkin diagram automorphism interchanging nodes 0 and 1, $\sigma'$ is
the Dynkin diagram automorphism interchanging nodes 2 and 3 for type $D_3^{(1)}$, and $\sigma'$ is the identity for type $B_2^{(1)}$ and $A_2^{(2)}$. Hence, for $\Lambda = \sum_{i=0}^{n} \ell_i \Lambda_i \in P^+$, we have

$$\tau(\Lambda) = \begin{cases} 
\Lambda & \text{if } r \text{ is even}, \\
\ell_0 \Lambda_1 + \ell_1 \Lambda_0 + \sum_{i=2}^{n} \ell_i \Lambda_i & \text{if } r \text{ is odd}, \\
\ell_0 \Lambda_1 + \ell_1 \Lambda_0 + \sum_{i=2}^{n} \ell_i \Lambda_i + \ell_{n-1} \Lambda_n + \ell_n \Lambda_{n-1} & \text{if } r \text{ is odd, type } D_n^{(1)}.
\end{cases}$$

4.3. Type $D_n^{(1)}$ for $r = n - 1, n$. The cases when $r = n, n - 1$ for type $D_n^{(1)}$ were treated in [3]. We will give the minimal elements below. Since $B_n$ and $B_{n-1}$ are related via the Dynkin diagram automorphism interchanging $\Lambda_n$ and $\Lambda_{n-1}$, we only deal with $B_n$. As a $D_n$-crystal it is isomorphic to $B(s \Lambda_n)$. There is a description of an element in terms of semistandard tableau of $n \times s$ rectangular shape with letters from the alphabet $A = \{1, 2, \ldots, n, \pi, \ldots, \}$ with partial order

$$1 < 2 < \cdots < n - 1 < \frac{n}{n} < \frac{n-1}{n} < \cdots < \frac{1}{n}.$$ 

Moreover, each column does not contain both $k$ and $\overline{k}$. Let $c_i$ be the $i$th column. Then the number of barred letters in $c_i$ is even, and the action of $c_i f_i (i = 1, \ldots, n)$ is calculated through that of $c_i \circ \cdots \circ c_1$ of $B(\Lambda_n)$. With this realization the minimal element $b_\Lambda$ such that $v(b_\Lambda) = \Lambda = \sum_{i=0}^{n} \ell_i \Lambda_i$ ($\ell_i \in \mathbb{Z}_{\geq 0}, \text{lev}(\Lambda) = s$) is given as follows. Let $x_{ij} (1 \leq i \leq n, j \in A)$ be the number of $j$ in the $i$th row. $x_{ij} = 0$ unless $i \leq j \leq n - i + 1$. The other $x_{ij}$ of $b_\Lambda$ is given by

$$x_{11} = \ell_0 + \ell_2 + \ell_3 + \cdots + \ell_{n-2} + \begin{cases} 
\ell_{n-1} & \text{for } n \text{ even}, \\
\ell_n & \text{for } n \text{ odd},
\end{cases}$$

$$x_{ij} = \ell_{j-1} (2 \leq j \leq n - 1), \quad (x_{1n}, x_{n\overline{n}}) = \begin{cases} 
(0, \ell_n) & \text{for } n \text{ even}, \\
(\ell_{n-1}, 0) & \text{for } n \text{ odd},
\end{cases}$$

if $2 \leq i \leq n - 1$,

$$x_{ii} = \ell_0 + \ell_2 + \ell_3 + \cdots + \ell_{n-i}, \quad x_{ij} = \ell_{j-i} (i + 1 \leq j \leq n - 1),$$

$$(x_{in}, x_{\overline{n}i}) = \begin{cases} 
(\ell_{n-i} + \ell_{n-i+1}, 0) & n - i \text{ even}, \\
(0, \ell_{n-i} + \ell_{n-i+1}) & n - i \text{ odd},
\end{cases}$$

$$x_{i\overline{j}} = \ell_{2n+1-i-j} (n - i + 3 \leq j \leq n - 1), \quad x_{i\overline{n-i+2}} = \begin{cases} 
\ell_{n-1} & n \text{ even}, \\
\ell_n & n \text{ odd},
\end{cases}$$

$$x_{i\overline{n-i+1}} = \ell_{n-i+1} + \ell_{n-i+2} + \cdots + \ell_{n-2} + \begin{cases} 
\ell_n & n \text{ even}, \\
\ell_{n-1} & n \text{ odd},
\end{cases}$$

and

$$x_{nn} = \ell_0, \quad x_{n\overline{n}} = 0, \quad x_{n\overline{j}} = \ell_{n+1-j} (3 \leq j \leq n - 1),$$

$$x_{n\overline{1}} = \begin{cases} 
\ell_{n-1} & n \text{ even}, \\
\ell_n & n \text{ odd},
\end{cases} \quad x_{n\overline{1}} = \ell_1 + \ell_2 + \cdots + \ell_{n-2} + \begin{cases} 
\ell_n & n \text{ even}, \\
\ell_{n-1} & n \text{ odd}.\n\end{cases}$$
The automorphism \( \tau \) is given by
\[
\tau \left( \sum_{i=0}^{n} \ell_i \Lambda_i \right) = \ell_0 \Lambda_{n-1} + \ell_1 \Lambda_n + \sum_{i=2}^{n-2} \ell_i \Lambda_{n-i} + \begin{cases} 
\ell_{n-1} \Lambda_0 + \ell_n \Lambda_1 & n \text{ even}, \\
\ell_{n-1} \Lambda_1 + \ell_n \Lambda_0 & n \text{ odd}.
\end{cases}
\]

4.4. Type \( B_n^{(1)} \) for \( r = n \). In this section we consider the perfectness of \( B_n^{r,s} \) of type \( B_n^{(1)} \).

**Proposition 4.1.** We have
\[
\min \{ \text{lev}(\varepsilon(b)) \mid b \in B_n^{2s+1} \} \geq s + 1,
\]
\[
\min \{ \text{lev}(\varepsilon(b)) \mid b \in B_n^{2s} \} \geq s.
\]

**Proof.** Suppose, there exists an element \( b \in B_n^{2s+1} \) with \( \text{lev}(\varepsilon(b)) = p < s + 1 \). Since \( B_n^{2s+1} \) is embedded into \( B_n^{2s+1,1} \) by Definition 3.2, this would yield an element \( \tilde{b} \in B_n^{2s+1} \) with \( \text{lev}(\tilde{b}) < 2s + 1 \). But this is not possible, since \( B_n^{2s+1,1} \) is a perfect crystal of level \( 2s + 1 \).

Suppose there exists an element \( b \in B_n^{2s} \) with \( \text{lev}(\varepsilon(b)) = p < s \). By the same argument one obtains a contradiction to the level of \( B_n^{2s} \).

Hence to show that \( B_n^{2s+1} \) is not perfect, it is enough to provide two elements \( b_1, b_2 \in B_n^{2s+1,1} \) which are in the realization of \( B^{r,s} \) under \( S \) and satisfy \( \varepsilon(b_1) = \varepsilon(b_2) = \lambda \), where \( \text{lev}(\lambda) = 2s + 2 \).

**Proposition 4.2.** Define the following elements \( b_1, b_2 \in B_n^{2s+1,1} \): For \( n \) odd, let \( P_1 \) be the \( \pm \)-diagram corresponding to one column of height \( n \) with a \( + \), and \( 2s \) columns of height \( 1 \) with \( - \) signs, and \( P_2 \) the analogous \( \pm \)-diagram but with a \( - \) in the column of height \( n \). Set \( \tilde{a} = (n(n-1)^2n(n-2)^2(n-1)^2n \ldots 2^2 \ldots (n-1)^2n) \) and
\[
b_1 = f_{\tilde{a}}(\Phi(P_1)) \quad \text{and} \quad b_2 = f_{\tilde{a}}(\Phi(P_2)).
\]

For \( n \) even, replace the columns of height \( 1 \) with columns of height \( 2 \) and fill them with \( \pm \)-pairs. Then \( b_1, b_2 \in S(B_n^{2s+1}) \) and \( \varepsilon(b_1) = \varepsilon(b_2) = 2s \Lambda_1 + \Lambda_n \), which is of level \( 2s + 2 \).

**Proof.** It is clear from the construction that the \( \pm \)-diagrams corresponding to \( b_1 \) and \( b_2 \) can be obtained by doubling a \( B_n^{(1)} \) \( \pm \)-diagram (see [3] Lemma 3,5)). Hence \( \Phi(P_1), \Phi(P_2) \in S(B_n^{2s+1}) \). The sequence \( \tilde{a} \) can be obtained by doubling a type \( B_n^{(1)} \) sequence using \( (m_1, m_2, \ldots, m_n) = (2, \ldots, 2, 1) \), so by Definition 3.2 \( b_1 \) and \( b_2 \) are in the image of the embedding \( S \) that realizes \( B_n^{2s+1} \). The claim that \( \varepsilon(b_1) = \varepsilon(b_2) = 2s \Lambda_1 + \Lambda_n \) can be checked explicitly.

**Corollary 4.3.** The KR crystal \( B_n^{2s+1} \) of type \( B_n^{(1)} \) is not perfect.

**Proof.** This follows directly from Proposition 4.2 using the embedding \( S \) of Definition 3.2.

**Proposition 4.4.** There exists a bijection, induced by \( \varepsilon \), from \( B_n^{2s} \) to \( P_s^+ \). Hence \( B_n^{2s} \) is perfect of level \( s \).
Proof. Let $S$ be the embedding from Definition 3.2. Then we have an induced embedding of dominant weights $\Lambda$ of $B_n^{(1)}$ into dominant weights of $A_{2n-1}^{(2)}$ via the map $S$, that sends $\Lambda_i \mapsto m_i\Lambda_i$.

In [21, Section 6] (see Section 4.2) the minimal elements for $A_{2n-1}^{(2)}$ were constructed by giving a $\pm$-diagram and a sequence from the $\{2, \ldots, n\}$-highest weight to the minimal element. Since $(m_0, \ldots, m_{n-1}, m_n) = (2, \ldots, 2, 1)$ and columns of height $n$ in $B_n^{(1)}$ for type $A_{2n-1}^{(2)}$ are doubled, it is clear from the construction that the $\pm$-diagrams corresponding to weights $S(\Lambda)$ are in the image of $S$ of $\pm$-diagrams for $B_n^{(1)}$ (see [5, Lemma 3.5]). Also, since under $S$ all weights $\Lambda_i$ for $1 \leq i < n$ are doubled, it follows that the sequences are “doubled” using the $m_i$. Hence a minimal element of $B_r^{n,2s}$ of level $s$ is in one-to-one correspondence with those minimal elements in $B_r^{n,2s} A_{2n-1}^{(2)}$ that can be obtained from doubling a $\pm$-diagram of $B_r^{n,2s}$. This implies that $\varepsilon$ defines a bijection between $B_{\min}^{n,2s}$ and $P_s^+$. □

The automorphism $\tau$ of the perfect KR crystal $B^{n,2s}$ is given by

$$\tau(\sum_{i=0}^n \ell_i\Lambda_i) = \begin{cases} \sum_{i=0}^n \ell_i\Lambda_i & \text{if } n \text{ is even}, \\ \ell_0\Lambda_1 + \ell_1\Lambda_0 + \sum_{i=2}^n \ell_i\Lambda_i & \text{if } n \text{ is odd}. \end{cases}$$

4.5. Type $C_n^{(1)}$. In this section we consider $B^{r,s}$ of type $C_n^{(1)}$ for $r < n$.

Proposition 4.5. Let $r < n$. Then

$$\min\{\ell(\varepsilon(b)) \mid b \in B^{r,2s+1}\} \geq s + 1,$$

$$\min\{\ell(\varepsilon(b)) \mid b \in B^{r,2s}\} \geq s.$$

Proof. By Definition 3.3 the crystal $B^{r,s}$ is realized inside $B_r^{r,s} A_{2n+1}^{(2)}$. The proof is similar to the proof of Proposition 4.1 for type $B_n^{(1)}$. □

Hence to show that $B_r^{r,2s+1}$ is not perfect, it is suffices to give two elements $b_1, b_2 \in B_r^{r,2s+1} A_{2n+1}^{(2)}$ that are fixed points under $\sigma$ with $\varepsilon(b_1) = \varepsilon(b_2) = \Lambda$, where $\ell(\Lambda) = 2s + 2$.

Proposition 4.6. Let $b_1, b_2 \in B_r^{r,2s+1}$, where $b_1$ consists of $s$ columns of the form read from bottom to top $(1,2,\ldots,r)$, $s$ columns of the form $(r, r-1, \ldots, 1)$, and a column $(r+1, \ldots, 2)$. In $b_2$ the last column is replaced by $(r+2, 2r+2)$ if $2r + 2 \leq n$ and $(r+2, \ldots, n, \pi, \ldots, \bar{\pi})$ of height $n$ otherwise. Then

$$\varepsilon(b_1) = \varepsilon(b_2) = \begin{cases} s\Lambda_r + \Lambda_{r+1} & \text{if } r > 1, \\ s(\Lambda_0 + \Lambda_1) + \Lambda_2 & \text{if } r = 1, \end{cases}$$

which is of level $2s + 2$.

Proof. The claim is easy to check explicitly. □

Corollary 4.7. The KR crystal $B_r^{n,2s+1}$ of type $C_n^{(1)}$ is not perfect.

Proof. The $\{2, \ldots, n\}$-highest weight elements in the same component as $b_1$ and $b_2$ of Proposition 4.6 correspond to $\pm$-diagrams that are invariant under $\sigma$. Hence, by Definition 3.3 $b_1$ and $b_2$ are fixed points under $\sigma$. Combining this result with Proposition 4.5 proves that $B_r^{r,2s+1}$ is not perfect. □
Proposition 4.8. There exists a bijection, induced by $\varepsilon$, from $B^r_{\text{min}}$ to $P^+_s$. Hence $B^r_{\text{min}}$ is perfect of level $s$.

Proof. By Definition 3.3 $B^{r,s}$ of type $C_n^{(1)}$ is realized inside $B^{r,s}_{A_{2n+1}^{(2)}}$ as the fixed points under $\sigma$. Under the embedding $S$, it is clear that a dominant weight $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_{n+1}\Lambda_{n+1}$ of type $A_{2n+1}^{(2)}$ is in the image if and only if $\ell_0 = \ell_1$. Hence it is clear from the construction of the minimal elements for $A_{2n+1}^{(2)}$ as described in Section 4.2 that the minimal elements corresponding to $\Lambda$ with $\ell_0 = \ell_1$ are invariant under $\sigma$. By [21, Theorem 6.1] there is a bijection between all dominant weights $\Lambda$ of type $A_{2n+1}^{(2)}$ with $\ell_0 = \ell_1$ and $\text{lev}(\Lambda) = 2s$ and minimal elements in $B^{r,2s}_{A_{2n+1}^{(2)}}$ that are invariant under $\sigma$. Hence using $S$, there is a bijection between dominant weights in $P^+_s$ of type $C_n^{(1)}$ and $B^r_{\text{min}}$. \hfill \blacksquare

The automorphism $\tau$ of the perfect KR crystal $B^{r,2s}$ is given by the identity.

4.6. Type $C_n^{(1)}$ for $r = n$. This case is treated in [13]. For the minimal elements, we follow the construction in Section 4.2. To every fundamental weight $\Lambda_k$ we associate a column tableau $T(\Lambda_k)$ of height $n$ whose entries are $k+1, k+2, \ldots, n, \overline{n}, \ldots, \overline{n-k+1}$ (1, 2, \ldots, $n$ for $k = 0$) reading from bottom to top. Let $f(\Lambda_k)$ be defined such that $T(\Lambda_k) = f(\Lambda_k)b_1$, where $b_k$ is the highest weight tableau in $B(k\Lambda_n)$. Then the minimal element $b$ in $B^{n,s}$ such that $\varepsilon(b) = \Lambda = \sum_{i=0}^{n} \ell_i\Lambda_i \in P^+_s$ is constructed as

$$b = f(\Lambda_n)^{\ell_n} \cdots f(\Lambda_1)^{\ell_1} b_s.$$ 

The automorphism $\tau$ is given by

$$\tau \left( \sum_{i=0}^{n} \ell_i\Lambda_i \right) = \sum_{i=0}^{n} \ell_i\Lambda_{n-i}.$$ 

4.7. Type $A_{2n}^{(2)}$. For type $A_{2n}^{(2)}$ one may use the result of Naito and Sagaki [17, Theorem 2.4.1] which states that under their [17, Assumption 2.3.1] (which requires that $B^{r,s}$ for $A_{2n+1}^{(2)}$ is perfect) all $B^{r,s}$ for $A_{2n}^{(2)}$ are perfect. Here we provide a description of the minimal elements via the embedding $S$ into $B^{r,2s}_{C_n^{(1)}}$.

Proposition 4.9. The minimal elements of $B^{r,s}$ of level $s$ are precisely those that correspond to doubled $\pm$-diagrams in $B^{r,2s}_{C_n^{(1)}}$.

Proof. In Proposition 4.8 a description of the minimal elements of $B^{r,2s}_{C_n^{(1)}}$ is given. We have the realization of $B^{r,s}$ via the map $S$ from Definition 3.3. In the same way as in the proof of Proposition 4.8 one can show, that the minimal elements of $B^{r,2s}_{C_n^{(1)}}$ that correspond to doubled dominant weights are precisely those in the realization of $B^{r,s}$, hence $\varepsilon$ defines a bijection between $B^r_{\text{min}}$ and $P^+_s$. \hfill \blacksquare

The automorphism $\tau$ is given by the identity.

4.8. Type $D_{n+1}^{(2)}$ for $r < n$.

Proposition 4.10. Let $r < n$. There exists a bijection $B^r_{\text{min}}$ to $P^+_s$, defined by $\varepsilon$. Hence $B^r_{\text{min}}$ is perfect.

Proof. This proof is analogous to the proof of Proposition 4.9. \hfill \blacksquare
The automorphism \( \tau \) is given by the identity.

4.9. Type \( D_{n+1}^{(2)} \) for \( r = n \). This case is already treated in [13], which we summarize below. As a \( B_n \)-crystal it is isomorphic to \( B(s\Lambda_n) \). There is a description of its elements in terms of semistandard tableaux of \( n \times s \) rectangular shape with letters from the alphabet \( A = \{ 1 < 2 < \cdots < n < \pi < \cdots < \tau \} \). Moreover, each column does not contain both \( k \) and \( k \). Let \( c_i \) be the \( i \)th column, then the action of \( e_i, f_i \) \( (i = 1, \ldots, n) \) is calculated through that of \( c_s \otimes \cdots \otimes c_1 \) of \( B(\Lambda_n)^{\otimes s} \). With this realization the minimal element \( b_{\Lambda} \) such that \( \varepsilon(b_{\Lambda}) = \Lambda = \sum_{i=0}^{n} \ell_i \Lambda_i \in P_s^+ \) is given as follows. Let \( x_{ij} \) \( (1 \leq i \leq n, j \in A) \) be the number of \( j \) in the \( i \)th row. Note that \( x_{ij} = 0 \) unless \( i \leq j \leq n-i+1 \). The table \((x_{ij})\) of \( b_{\Lambda} \) is then given by \( x_{ii} = \ell_0 + \cdots + \ell_{n-i} \) \( (1 \leq i \leq n) \), \( x_{ij} = \ell_{i-j} \) \( (i+1 \leq j \leq n) \), \( x_{ij} = \ell_j + \cdots + \ell_n \) \( (n-i+1 \leq j \leq n) \). The automorphism \( \tau \) is given by

\[
\tau\left(\sum_{i=0}^{n} \ell_i \Lambda_i \right) = \sum_{i=0}^{n} \ell_i \Lambda_{n-i}.
\]

5. Examples for type \( C_3^{(1)} \)

In this section we present the affine crystal structure for \( B^{2,2} \) and \( B^{2,1} \) of type \( C_3^{(1)} \). We also list all minimal elements for \( B^{2,3} \) of type \( C_3^{(1)} \) to illustrate that \( \varepsilon \) is not a bijection and hence \( B^{2,3} \) is not perfect.

5.1. KR crystal \( B^{2,2} \). The KR crystal \( B^{2,2} \) has three classical components

\[
B^{2,2} \cong B(2\Lambda_2) \oplus B(2\Lambda_1) \oplus B(0).
\]

The unique element in \( B(0) \) is denoted by \( u \). Since \( f_0 \) commutes with \( f_2, f_3 \) and the classical \( C_3 \)-crystal structure is explicitly known by [16], it suffices to determine \( f_0 \) on each \( \{2,3\} \)-component. All \( \{2,3\} \)-highest weight crystal elements are given in Table 3 together with the action of \( f_0 \).

The bijection \( \varepsilon : B^{2,2}_{\min} \to P_1^+ \) is given by

| \( b \) | \( \varepsilon(b) \) |
|---|---|
| \( u \) | \( \Lambda_0 \) |
| \( 1 \ 1 \) | \( \Lambda_1 \) |
| \( 2 \ 1 \) | \( \Lambda_2 \) |
| \( 3 \ 2 \) | \( \Lambda_3 \) |
| \( 2 \ 3 \) |

5.2. KR crystal \( B^{2,1} \). The KR crystal graph for \( B^{2,1} \) of type \( C_3^{(1)} \) is given in Figure 4. It has only one classical component

\[
B^{2,1} \cong B(\Lambda_2).
\]
Figure 1. $B^{2,1}$ of type $C_3^{(1)}$

$B^{2,1}$ is not perfect, since $\varepsilon$ is not a bijection from minimal elements to level 1 dominant weights:

| $b$   | $\varepsilon(b)$ |
|-------|------------------|
| 2/1   | $\Lambda_0$      |
| 2/2   | $\Lambda_1$      |
| 2/3   | $\Lambda_2$      |
| 3/3   | $\Lambda_3$      |
Table 3. Action of $f_0$ on $\{2,3\}$-highest weight elements in $B^{2,2}$ of type $C_3^{(1)}$
5.3. **KR crystal** $B^{2,3}$. The KR crystal $B^{2,3}$ of type $C_4^{(1)}$ is also not perfect. The map $\varepsilon$ from the minimal elements to level 2 dominant weights is given below:

| $b$          | $\varepsilon(b)$ |
|--------------|------------------|
| $2$          | $2\Lambda_0$     |
| $1$          | $\Lambda_0 + \Lambda_1$ |
| $2$          | $\Lambda_0 + \Lambda_2$ |
| $1\ 1\ 1$   | $\Lambda_0 + \Lambda_3$ |
| $2\ 2\ 1$   | $2\Lambda_1$     |
| $2\ 1\ 1\ 1$| $\Lambda_1 + \Lambda_2$ |
| $3$          | $\Lambda_1 + \Lambda_3$ |
| $3\ 2\ 1$   | $2\Lambda_2$     |
| $2\ 3\ 1\ 1$| $\Lambda_2 + \Lambda_3$ |
| $2\ 1\ 2\ 1$| $2\Lambda_3$     |

Under the embedding $S : B^{2,3} \rightarrow B^{2,3}_{A_7^{(2)}}$ of Definition 3.3 we have

$$S \left( \begin{array}{c} 2 \\ 2 \ 2 \ 1 \end{array} \right) = \begin{array}{c} 2 \\ 2 \ 2 \ 1 \ 1 \ 3 \ 2 \ 1 \end{array} = b_1 \quad \text{and} \quad S \left( \begin{array}{c} 3 \\ 3 \ 1 \ 3 \ 1 \end{array} \right) = \begin{array}{c} 3 \\ 2 \ 4 \ 1 \ 1 \ 4 \ 2 \end{array} = b_2$$

which are precisely the two elements $b_1, b_2$ of Proposition 4.6 such that $\varepsilon(b_1) = \varepsilon(b_2) = \Lambda_2 + \Lambda_3$ in type $A_7^{(2)}$.

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