MODULI SPACES $M_{g,n}(W)$ FOR SURFACES

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0. Introduction

0.1. We recall the following coarse moduli spaces in the case of curves:

1. $M_g$, parameterizing nonsingular curves of genus $g \geq 2$ and its compactification $\overline{M}_g$, parameterizing Mumford-Deligne moduli-stable curves, see Mumford [21],
2. spaces $M_{g,n}$, $2g - 2 + n > 0$, for stable $n$-pointed curves, see Knudsen [10],
3. a moduli space $M_{g,n}(W)$ of stable maps from reduced curves to a variety $W$, see Kontsevich [16].

It is well known that $\overline{M}_g$ and $M_{g,n}$ are projective, $M_g$ is quasi-projective.

0.2. For surfaces, Gieseker [8] established the existence of a quasi-projective scheme parameterizing surfaces with at worst Du Val singularities, ample canonical class $K$ and fixed $K^2$, this is a straightforward analog of $M_g$ and we will denote it by $M_{K^2}$. A geometrically meaningful compactification of this space, $\overline{M}^{sm}_{K^2}$, was constructed by Kollár and Shepherd-Barron in [15] as a separated algebraic space. It is a moduli space of smoothable stable (not in the G.I.T. sense) surfaces of general type. In [12] Kollár has shown that if the class of smoothable stable surfaces with a fixed $K^2$ is bounded then $\overline{M}^{sm}_{K^2}$ is in fact a projective scheme (Corollary 5.6). Finally, the boundedness was proved by the author in [2].

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0.3. The main purpose of this paper is to construct analogs of $M_{g,n}$ and $M_{g,n}(W)$ in the case of surfaces, and to prove their projectiveness. After this is done, we touch on a connection between our moduli spaces and the standard moduli spaces of K3 and Abelian surfaces.

0.4. An idea of “$M_{g,n}$ for surfaces” occurred to me when I mentioned that my boundedness theorem 9.2 [2] is strictly stronger than what was used for $\overline{M}_{K3}$. Then, looking at the definition of $M_{g,n}(W)$ in [16] I realized that this is simply a relative version of the same scheme, and can be done for surfaces too.

0.5. The basic construction of a moduli space as an algebraic space used here is the same as in [22], [11], [26] and elsewhere. For the hardest question involved, proof of local closedness, we refer to a result of Kollár [14].

For proving that our moduli spaces are projective schemes, rather than mere algebraic spaces, we use Kollár’s Ampleness Lemma 3.9 [12], which can be applied in a straightforward way to a variety of complete moduli problems.

0.6. Kontsevich and Manin [17] use the moduli spaces $M_{g,n}(W)$ to define Gromov-Witten classes of varieties in the “quantum cohomology” theory. Hence one distant application of “$M_{g,n}(W)$ for surfaces” might be “higher” GW-classes of schemes.

Notation. All schemes are assumed to be at least Noetherian and defined over a fixed algebraically closed field $k$ of characteristic zero. Obstacles to extending the results to positive characteristic are discussed in the last section. In most cases, we prefer the additive notation to the multiplicative one, for divisors and line bundles. All moduli spaces in this paper are coarse.

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1. Overview

1.1. One possible approach to solving an algebro-geometric moduli problem goes through the following steps:

1. defining the objects that we are trying to parameterize,
2. giving the right definition for a moduli functor,
3. establishing properties of this functor,
4. constructing a moduli space in some fashion,
5. proving finer facts about this space.

In our treatment, we will follow two guiding principles well understood in the field:
Principle 1. Most moduli spaces exist in the category of algebraic spaces.

Principle 2. Most complete and separated moduli spaces are projective.

1.2. Moduli spaces of nonsingular curves $M_g$ and of Deligne-Mumford stable curves $\overline{M}_g$ of genus $g$ provide a textbook illustration of how this works in practice. Since nonsingular curves can degenerate into singular ones in an uncorrectable way, $M_g$ is not complete. There are many different ways to compactify it but the one we are interested in here is adding more curves and trying to solve an enlarged moduli problem. It turns out that the curves one has to add are Deligne-Mumford moduli-stable curves which are defined as connected and complete reduced curves with ordinary nodes only such that every smooth rational irreducible component intersects others in at least 3 points and every irreducible component of arithmetical genus one intersects the rest in at least 1 point. The latter condition has two equivalent meanings:

1. the automorphism group $\text{Aut}(X)$ is finite (and this property is a must for the Geometric Invariant Theory),
2. the dualizing sheaf $\omega_X$ is ample.

To arrive at this answer, one can look at the way the good limits are obtained. One considers a family $\mathcal{X}$ of curves over a marked curve, or the specter of a DVR ring, $(\mathcal{S},0)$ with a nonsingular general fibre and a degenerate special fibre. Then by the Semistable Reduction Theorem, after making a finite base change $\mathcal{S}' \to \mathcal{S}$ and resolving the singularities of $\mathcal{X}'$, the central fibre will be a reduced curve with simple nodes. Following (1) above one should contract all the rational curves $E$ in the central fiber that intersect the rest only at 1 or 2 points. These have self-intersection numbers $E^2 = -1$ and $E^2 = -2$ respectfully. Contracting $(-1)$-curves leaves the ambient space, which is a surface, nonsingular. Contracting $(-2)$-curves introduces very simple surface singularities, called Du Val or rational double. The central fiber has nodes only.

1.3. One can recognize that the above is a field of the Minimal Model Program. In fact, we have just constructed the canonical model, in dimension 2, of $\mathcal{X}'$ over $\mathcal{S}'$. So, to generalize $\overline{M}_g$ to the surfaces of general type we have to apply the Minimal Model Program in dimension 3. This was done by Kollár and Shepherd-Barron in [15]. By that time, the end of 1980-s, all the necessary for this construction tools from MMP in dimension 3 were already available. The new objects that one has to add are defined as connected reduced surfaces with semi-log canonical singularities and ample tensor power of the dualizing sheaf $(\omega_X^N)^{**}$, where $**$ means taking the self-dual. Using the additive notation, we say that a $\mathbb{Q}$-Cartier divisor $K_X$ is ample.

1.4. At the present time, the log Minimal Model Program in dimension 3 is in a pretty good shape. Let us see what kind of statements we can get using its principles. Keeping in line with what we did before, we now
consider pairs \((X, B)\) of surfaces \(X\) and divisors \(B = \sum_{j=1}^{n} B_j\) with ample \(K_X + B\). A construction very similar to the one above, with Semistable Reduction Theorem and, this time log canonical model, shows that we again get a complete moduli functor. What about singularities of the pair \((X, B)\)? Why, they ought to be semi-log canonical, of course!

What is the analog of this in dimension one? That is easy to answer and we get something very familiar. The divisor \(B = \sum_{j=1}^{n} B_j\) becomes a set of marked points. Semi-log canonical means that the curve has nodes only, and that marked points are distinct and lie in the nonsingular part. These are exactly the \(n\)-marked semistable curves of Knudsen [10].

1.5. Another possible generalization would be looking not at absolute curves (or surfaces) \(X\) (or pairs \((X, B)\)) but doing it in the relative setting. In other words, let us consider maps \(X \to W\) to a fixed projective scheme \(W\) with \(K_X\) (resp. \(K_X + B\)) relatively ample. The only modification in the above construction will be that we have to apply the relative version of the (log) Minimal Model Program over \(S' \times W\) instead of over \(S'\). What we get for curves is the moduli space \(M_{g,n}(W)\) introduced by Kontsevich in [16].

1.6. Now that we have outlined the objects we will be dealing with, let us return back to the basic example of \(\overline{M}_g\). We recall two different approaches to constructing it.

Approach 1 (G.I.T.). One first proves that moduli-stable curves are asymptotically Hilbert-stable, [21]. Then the standard G.I.T. machinery produces a quasi-projective moduli space. Since it is complete, it is actually projective.

Approach 2. Using a fairly general argument ([22], p.172) one proves the existence of a moduli space in the category of algebraic spaces. To a family of curves \(\pi : X \to S\) one can in a natural way associate line bundles on \(S\) which are defined as \(\text{det}(\pi_*\omega^k)\). They descend to \((\mathbb{Q}\text{-})\)line bundles \(\lambda_k\) on \(\overline{M}_g\) and one can further show that \(\lambda_k\) are ample for \(k \geq 1\).

1.7. As mentioned in [24] and [12], for surfaces of general type the first approach fails. By [21] 3.19 in order to be asymptotically Chow- or Hilbert-stable a surface has to have singularities of multiplicities at most 7. On the other hand, the semi-stable limits described above have semi-log canonical singularities and it looks like they must be included in any reasonable complete moduli problem. These semi-log canonical singularities include all quotient singularities, for example, and can have arbitrarily high multiplicities.

1.8. The second approach is what we will be using here. After establishing the existence as an algebraic space, we will use Kollár’s Ampleness Lemma
to prove that it is projective. The Ampleness Lemma is a general scheme that shows projectiveness once some good properties of the moduli functor are established: local closedness, completeness, separateness, semipositivity, finite reduced automorphism groups, and, crucially, boundedness. The projectiveness will be the only “finer” property of the obtained moduli spaces that we will consider.

2. The objects

2.1. The main objects into the consideration will be stable maps of pairs \( g : (X, B) \to W \), where

1. \( W \subset \mathbb{P} \) is a fixed projective scheme,
2. \( X \) is a connected projective surface,
3. \( B = \sum_{j=1}^{n} B_j \) is a divisor on \( X \), \( B_j \) are reduced but not necessarily irreducible,
4. the pair \( (X, B) \) has semi-log canonical singularities,
5. the divisor \( K_X + B \) is relatively \( g \)-ample.

The precise definitions follow.

2.2. For a normal variety \( X \), \( K_X \) or simply \( K \) will always denote the class of linear equivalence of the canonical Weil divisor. The corresponding reflexive sheaf \( \mathcal{O}_X(K_X) \) is defined as \( i_*\Omega_{\text{dim } X}^1 \), where \( i : U \to X \) is the embedding of the nonsingular part of \( X \).

Definition 2.3. An \( \mathbb{R} \)-divisor \( D = \sum d_j D_j \) is a linear combination of prime Weil divisors with real coefficients, i.e. an element of \( N^1 \otimes \mathbb{R} \). An \( \mathbb{R} \)-divisor is said to be \( \mathbb{R} \)-Cartier if it is a combination of Cartier divisors with real coefficients, i.e. if it belongs to the image of \( \text{Div}(X) \otimes \mathbb{R} \to N^0(X) \otimes \mathbb{R} \) (this map is of course injective for normal varieties). The \( \mathbb{Q} \)-divisors and \( \mathbb{Q} \)-Cartier divisors are defined in a similar fashion.

Definition 2.4. Consider an \( \mathbb{R} \)-divisor \( K + B = K_X + \sum b_j B_j \) and assume that

1. \( K + B \) is \( \mathbb{R} \)-Cartier,
2. \( 0 \leq b_j \leq 1 \).

For any resolution \( f : Y \to X \) look at the natural formula

\[
K_Y + B_Y = f^*(K_X + \sum b_j B_j) = K_Y + \sum b_j f^{-1}B_j + \sum b_i F_i \quad (1)
\]

or, equivalently,

\[
K_Y + \sum b_j f^{-1}B_j + \sum F_i = f^*(K_X + \sum b_j B_j) + \sum a_i F_i \quad (2)
\]

Here \( f^{-1}B_j \) are the proper preimages of \( B_j \) and \( F_i \) are the exceptional divisors of \( f : Y \to X \).

The coefficients \( b_i, b_j \) are called codiscrepancies, the coefficients \( a_i = 1 - b_i, a_j = 1 - b_j - \log \text{ discrepancies} \).
Remark 2.5. In fact, \(K + B\) is not a usual \(\mathbb{R}\)-divisor but rather a special gadget consisting of a linear class of a Weil divisor \(K\) (or a corresponding reflexive sheaf) and an honest \(\mathbb{R}\)-divisor \(B\). This, however, does not cause any confusion.

Definition 2.6. A pair \((X, B)\) (or a divisor \(K + B\)) is said to be
1. log canonical, if the log discrepancies \(f_k \geq 0\)
2. Kawamata log terminal, if \(f_k > 0\)
3. canonical, if \(f_k \geq 1\)
4. terminal, if \(f_k > 1\)
for every resolution \(f : Y \to X, \{k\} = \{i\} \cup \{j\}\).

2.7. The notion of semi-log canonical is a generalization of log canonical to the case of varieties that are singular in codimension 1. The basic observation here is that for a curve with a simple node the definition of the log discrepancies still makes sense and gives \(a_1 = a_2 = 0\), so it can also be considered to be (semi-)log canonical. No new Kawamata semi-log terminal singularities appear, however.

Recall that according to Serre’s criterion normal is equivalent to Serre’s condition \(S_2\) and regularity in codimension 1. So, if we do allow singularities in codimension 1, \(S_2\) will be exactly what we will need to keep.

Definition 2.8. Let \(X\) be a reduced (but not necessarily irreducible) equidimensional scheme which satisfies Serre’s condition \(S_2\) and is Gorenstein in codimension 1. Let \(B = \sum b_j B_j, 0 \leq b_j \leq 1\) be a linear combination with real coefficients of codimension 1 subvarieties none of irreducible components of which is contained in the singular locus of \(X\). Denote by \(\mathcal{O}(K_X)\) the reflexive sheaf \(i_* (\omega_U)\), where \(i : U \to X\) is the open subset of Gorenstein points of \(X\) and \(\omega_U\) is the dualizing sheaf of \(U\). We can again consider a formal combination of \(K_X\) and an \(\mathbb{R}\)-divisor \(B\), and there is a good definition for \(K_X + \sum b_j B_j\) to be \(\mathbb{R}\)-Cartier. It means that in a neighborhood of any point \(P \in X\) we can choose a section \(s\) of \(\mathcal{O}(K_X)\) such that the divisor \((s) + \sum b_j B_j\) is a formal combination with real coefficients of Cartier divisors with no components entirely in the singular set.

A pair \((X, B)\) (or a divisor \(K_X + B\)) is said to be semi-log canonical if, similar to the above,
1. \(K_X + B\) is \(\mathbb{R}\)-Cartier,
2. for any morphism \(f : Y \to X\) which is birational on every irreducible component, and with a nonsingular \(Y\), in the natural formula
\[f^*(K_X + B) = K_Y + f^{-1}B + \sum b_i F_i\]
with \(F_i\) being irreducible components of the exceptional set, all \(b_i \leq 1\) (resp. \(a_i = 1 - b_i \geq 0\)).

As before, the coefficients \(b_i, b_j\) are called codiscrepancies, the coefficients \(a_i, a_j\) – the log discrepancies.
Remarks 2.9. In the case when \((X, B)\) has a good semi-resolution (for example, for surfaces) this definition is equivalent to that of [13], [5] chapter 12. In our opinion, it is more natural to give a definition which is independent of the existence of a semi-resolution.

Remark 2.10. For surfaces the condition \(S_2\) is of course equivalent to Cohen-Macaulay.

Definition 2.11. By the Kleiman’s criterion, the ampleness for proper schemes is a numerical condition, hence it extends to \(\mathbb{R}\)-Cartier divisors. If coefficients of \(B\) are rational, \(K_X + B\) is \(g\)-ample iff for some positive integer \(n\) the divisor \(n(K_X + B)\) is Cartier and \(g\)-ample in the usual sense.

Remark 2.12. Below we will only consider the case when \(B\) is reduced, i.e. all the coefficients \(b_j = 1\). See the last section for the discussion on non-integral coefficients.

Example 2.13. If \(X\) is a curve then \((X, B)\) is semi-log canonical iff the only singularities of \(X\) are simple nodes and \(B\) consists of distinct points lying in the nonsingular part of \(X\). \(K_X + B\) is relatively ample iff every smooth rational component of \(X\) mapping to a point on \(W\) has at least 3 points of intersection with the rest of \(X\), or points in \(B_j\), and every component of arithmetical genus 1 has at least 1 such point. In the absolute case, i.e. when \(W\) is a point, this is the usual definition of a stable curve with marked points. Every \(B_j\) can also be considered as a group of unordered points.

Example 2.14. The only codimension 1 semi-log canonical singularities are normal crossings.

Example 2.15. If \(X\) is a nonsingular surface then \((X, B)\) is semi-log canonical iff \(B\) has only normal intersections.

Example 2.16. For the case when \(X\) is a surface and \(B\) is empty the semi-log canonical singularities over \(\mathbb{C}\) were classified in [13]. They are (modulo analytic isomorphism): nonsingular points, Du Val singularities, cones over nonsingular elliptic curves, cusps or degenerate cusps (which are similar to cones over singular curves of arithmetical genus 1), double normal crossing points \(xy = 0\), pinch points \(x^2 = y^2z\), and all cyclic quotients of the above. If \(B\) is nonempty then the singularities of \(X\) are from the same list and, in addition, there are different ways \(B\) can pass through them. For normal \(X\) the list could be found in [1] for example.

2.17. The following describes an easy reduction of semi-log canonical singularities to log canonical, cf. [5] 12.2.4.
Lemma 2.18. Let \((X, B)\) be as in the definition 2.8 and denote by \(\nu : X' \to X\) its normalization. Assume that \(K_X + B\) is \(\mathbb{R}\)-Cartier. Then \((X, B)\) is semi-log canonical iff \((X', \nu^{-1}B + \text{cond}(\nu))\) is log canonical, and they have the same log discrepancies.

Proof. Clear from the definition.

2.19. The next theorem explains how semi-log canonical surfaces appear in families (cf. [15] 5.1). But first we will need an auxiliary definition.

Definition 2.20. Let \(f : (X, B) \to S\) be a 3-dimensional one-parameter family. Let \(B = \sum b_j B_j\) with \(0 \leq b_j \leq 1\) be an \(\mathbb{R}\)-divisor and assume that \(X\) and all \(B_j\) are flat over \(S\) and that \(K_X + B\) is \(\mathbb{R}\)-Cartier. We will say that the pair \((X, B)\) (or the divisor \(K_X + B\)) is \(f\)-canonical if in the definition of log discrepancies for all exceptional divisors with \(f(F_i)\) a closed point on \(S\) one has for the corresponding log discrepancy \(a(F_i) \geq 1\) (resp. \(b(F_i) \leq 0\)). This condition does not say anything about log discrepancies of divisors that map surjectively onto \(S\).

Theorem 2.21. Let \(f : (X, B) \to S\) be a 3-dimensional one-parameter family over a pointed curve or a specter of a DVR (a discrete valuation ring). Let \(B = \sum b_j B_j\) with \(0 \leq b_j \leq 1\) be an \(\mathbb{R}\)-divisor and assume that \(X\) is irreducible, \(X\) and all \(B_j\) are flat over \(S\) and that the fibers satisfy Serre’s condition \(S_2\) and are Gorenstein in codimension 1 (note that this implies that \(X\) itself is Cohen-Macaulay and is Gorenstein in codimension 1). Further assume that \(K_X + B\) is \(\mathbb{R}\)-Cartier. Then the following is true:

1. If \(K_{X_0} + B_0\) is semi-log canonical then \(K_X + B\) is log canonical and \(f\)-canonical.
2. Under assumptions of (1), the general fiber is also semi-log canonical.
3. Suppose that there exists a birational morphism \(\mu : Y \to X\) with a non-singular \(Y\) such that all exceptional divisors of \(\mu\) and strict preimages of \(B_i\) have normal crossings and such that the central fiber is reduced. Then the opposite to (1) is true.

Proof. The proof is an application of the adjunction formula.

(1) The log adjunction theorem [15] 17.12 and 2.18 imply that \(K_{X_0} + B_0\) is semi-log canonical iff \(K_X + B + X_0\) is. Now, the connection between the log discrepancies of the divisors \(K_X + B\) and \(K_X + B + X_0\) is clear. For a divisor \(E\) mapping onto \(S\) the log discrepancies are the same. For \(E\) mapping to a central point of \(S\) the difference is the coefficient of \(E\) in the central fiber of \(Y \to S\), and so is at least 1.

(3) Here the differences between the log discrepancies over the central fiber are all equal to 1.

(2) is by adjunction.

2.22. Finally, we show how to pass from a relatively ample \(K + B\) to an ample divisor.
Lemma 2.23. Let $g : (X, B) \to W \subset \mathbb{P}$ be a map, where $X$ is a projective surface and $B = \sum b_j B_j$ is an $\mathbb{R}$-divisor on $X$. Assume that $K_X + B$ is semi-log canonical and is relatively g-ample. Then $K_X + B + 4H$ is ample, where $H = g^*\mathcal{O}(1)$.

Proof. It is enough to prove that the restriction on the normalization of $X$ is ample, therefore by Lemma 2.18 we can assume that $X$ is normal and that $(X, B)$ is log canonical.

We show that $K_X + B + 3H$ is nef (numerically effective) and this implies the statement. Indeed, $K_X + B + MH$ is ample for $M \gg 0$ and $K_X + B + 4H$ is a weighted average of the above two divisors.

Assume that $K_X + B + 3H$ is not nef. Then the Cone Theorem, which holds for arbitrary normal surfaces, tells us that there exists an irreducible curve $C$ generating an extremal ray and such that

$$(K_X + B)C < 0$$

This is possible only if $C$ does not map to a point. But then $C \cdot 3H \geq 3$ and $(K_X + B)C \geq -3$ by a theorem on the length of extremal curves, see [19].

In dimension 2 the latter statement is very elementary. Let $f : Y \to X$ be a minimal resolution of singularities of $X$. Then, if $X \neq \mathbb{P}^e$, one necessarily has $(f^{-1}C)^2 \leq 0$ and

$$(K_X + B)C \geq (K_Y + f^{-1}B)f^{-1}C \geq (K_Y + f^{-1}C)f^{-1}C = 2p_a(f^{-1}C) - 2 \geq -2.$$ 

And the case of $X = \mathbb{P}^e$ is clear. $\square$

3. Definition and properties of a moduli functor

3.1. Below we give a few general definitions for moduli functors. They are fairly standard (see e.g. [20], [12]) but we need to make slight modifications to adapt them to our situation.

3.2. The moduli functor for a moduli problem of polarized schemes is normally constructed in the following way. One fixes a class $\mathcal{C}$ of schemes $X/k$ with a polarization, i.e. an ample line bundle, $L$ and some extra structure and subject to some conditions. Then for an arbitrary scheme $S/k$ one defines $\mathcal{M}(\mathcal{C}, S)$ as the set of all (relatively) polarized flat families over $S$ with all geometric fibers from $\mathcal{C}$ and, possibly, subject to more conditions. The families are considered modulo an equivalence relation. Usually it is an isomorphism between $X_1/S$ and $X_2/S$ with whatever extra structure they have and a fiber-wise linear equivalence between $L_1$ and $L_2$. In other cases it is an algebraic or a numerical, or a numerical up to a scalar equivalence, instead of linear.

Sometimes, it is also useful considering a $\mathbb{Q}$-polarization $L$ on $X$, i.e. a reflexive sheaf such that $(L^{\otimes N})^\vee$ is an ample line bundle.
The above definition is intentionally vague since extra structures and conditions vary greatly from one moduli problem to another. Instead of trying to cover all future generalizations, we will formulate general principles and, when nontrivial, say exactly how they specialize to our situation.

Definition 3.4. The class $S$ is said to be bounded if there exists a scheme $(X, L)$ with an extra structure, and a morphism $F : X \to S$ to a scheme $S$ of finite type such that all elements of $C$ appear as geometric fibers of $F$, not necessarily in a one-to-one way.

There are two important variations of this definition. There is the polarized boundedness, when one requires $F$ to be projective and $L$ to restrict to the given polarization $L$ on a fiber, versus non-polarized. One can also consider boundedness in the narrow sense, requiring that all fibers of $F$ belong to $C$, or in the wide sense, asking only for some of the fibers to be from $C$.

Here we make the choice of the polarized boundedness in the wide sense.

Definition 3.5. A moduli functor $\mathcal{MC}$ is said to be separated if every one-parameter family in $\mathcal{MC}(S_{\text{gen}})$, where $S_{\text{gen}}$ is a generic point of a DVR, has at most one extension to $S$.

Definition 3.6. A moduli functor $\mathcal{MC}$ is said to be complete if every one-parameter family in $\mathcal{MC}(S_{\text{gen}})$, where $S_{\text{gen}}$ is a generic point of a DVR, has at least one extension after a finite cover $S' \to S$.

Definition 3.7. A class $C$ is said to be locally closed if for every flat family $F : (X, L) \to S$ with an extra structure there exist locally closed subschemes $S_l \subset S$ with the following universal property:

- A morphism of schemes $T \to S$ factors through $\bigsqcup S_l$ iff $(X, L) \times_S T \to T$ belongs to $\mathcal{MC}(T)$.

Definition 3.8. The class $C$ is said to have finite reduced automorphisms if every object in $C$ has a finite and reduced (the latter is automatic in characteristic 0) group of automorphisms.

Definition 3.9. A moduli functor $\mathcal{MC}$ is said to be functorially polarizable if for every family $(X, L)$ in $\mathcal{MC}(S)$ there exists an equivalent family $(X', L')$ such that

1. if $(X_1, L_1)$ and $(X_2, L_2)$ are equivalent, then $(X_1, L'_1)$ and $(X_2, L'_2)$ are isomorphic,
2. for any base change $h : S' \to S$, $(X', L')$ and $(X', h^*(L'))$ are isomorphic.

The main example of a functorial polarization is delivered by the polarization $\omega_{X/S}$ for canonically polarized manifolds.
Definition 3.10. A functorial polarization $\mathcal{L}^c$ is said to be semipositive if there exists a fixed $k_0$ such that whenever $\mathcal{S}$ is a complete smooth curve and $f : (X, \mathcal{L}) \to \mathcal{S}$ an element in $\mathcal{MC}(\mathcal{S})$, then for all $k \geq k_0$ the vector bundles $f_*(k\mathcal{L}^c)$ are semipositive, i.e. all their quotients have nonnegative degrees.

This definition will be slightly modified for our purposes, we will also require semipositiveness of restrictions of $\mathcal{L}^c$ to certain divisors $B_j$ on $X$.

3.11. The following is the class that we will be considering from now on.

Definition 3.12. The elements of the class $\mathcal{C}^N = \mathcal{C}^N_{(K+B)^2,(K+B)H,H^2}$ are stable maps of pairs $g : (X, B, L_N) \to W$, where

1. $W \subset \mathbb{P}$ is a fixed projective scheme,
2. $X$ is a connected projective surface,
3. $B = \sum_{j=1}^n B_j$ is a divisor on $X$, $B_j$ are reduced but not necessarily irreducible,
4. the pair $(X, B)$ has semi-log canonical singularities,
5. the divisor $K_X + B$ is relatively $g$-ample,
6. $(K_X + B)^2 = C_1, (K_X + B)H = C_2, H^2 = C_3$ are fixed,
7. $L_N = \mathcal{O}(N(K_X + B + 5H))$, where $H = g^*\mathcal{O}_W(1)$. Here $N$ is a positive integer such that for every map as above $L_N$ is a line bundle. For example, we can choose $N$ to be the minimal positive integer satisfying this condition. The existence of such an $N$ will be proved in 3.14, and it is ample by [2,23].

3.13. The classes $\mathcal{C}^N$ and $\mathcal{C}^M$ for different $N, M$ are in a one-to-one correspondence between each other, and the only difference is the polarizations.

As a consequence, the polarization in our functor plays a secondary role. We will switch from a polarization $L_N$ to its multiple $L_M$ when it will be convenient.

Theorem 3.14. For some $M > 0$ the class $\mathcal{C}^M$ is bounded.

Proof. We start with the boundedness theorem which gives what we want in the absolute case.

Theorem 3.15 ([2], 9.2). Fix a constant $C$ and a set $\mathcal{A}$ satisfying the descending chain condition. Consider all surfaces $X$ with an $\mathbb{R}$-divisor $B = \sum b_j B_j$ such that the pair $(X, B)$ is semi-log canonical, $K_X + B$ is ample, $b_j \in \mathcal{A}$ and $(K_X + B)^2 = C$. Then the class $\{(X, \sum b_j B_j)\}$ is bounded.

Apply this theorem with the set $\mathcal{A} = \{1\}$ to $K_X + B + D$, where $D$ is a general member of the linear system $|4H|$. Since this linear system is base point free, the pair $(X, B + D)$ also has semi-log canonical singularities. Therefore, all pairs $(X, B)$ satisfying the conditions of the theorem can be embedded by a linear system $|M(K_X + B + 4L)|$ for a fixed large divisible $M$ in a fixed projective space $\mathbb{P}^e$. Every map $g : X \to W$ is defined by its graph $\Gamma_g$. Consider a Veronese embedding of $W$ by $|\mathcal{O}_W(M)|$ in some $\mathbb{P}^e$.
and then look at the graphs \( \Gamma_g \) in a Segre embedding \( \mathbb{P}^r \times \mathbb{P}^r \subset \mathbb{P}^r \). Note that \( \mathcal{O}_{\mathbb{P}^r}(1) \) restricted on \( X \simeq \Gamma_g \) is \( L_M = M(K_X + B + 5H) \).

\( L^2 \) is fixed, hence by the boundedness theorem 3.14 above there are only finitely many possibilities for Hilbert polynomials \( \chi(\mathcal{O}_{\Gamma_g}(t)) \). By the same theorem, there are also only finitely many possibilities for Hilbert polynomials \( \chi(\mathcal{O}_{B_j}(t)) \). Therefore, all elements of our class \( g : (X, B) \to W \) are parameterized by finitely many products of Hilbert schemes. In each product, we have to extract a subscheme parameterizing subschemes of \( \mathbb{P}^r \times W \) and with fixed \( \mathcal{O}_{\mathbb{P}^r}(1) \), \( \mathcal{O}_{\mathbb{P}^r}(1) \cdot \mathcal{O}_{\mathbb{P}^r}(1) \) and \( \mathcal{O}_{\mathbb{P}^r}(1)^2 \), and these are obviously closed algebraic conditions. We also need to extract the graphs, i.e subschemes mapping isomorphically to \( \mathbb{P}^r \), and this is an open condition.

The resulting scheme will parameterize the maps, including all maps from the class \( \mathcal{C}^M \). This proves the theorem. \( \square \)

3.16. We won’t need the boundedness of the class \( \mathcal{C}^N \) itself, although it will follow from the proof of the local closedness 3.26.

**Definition 3.17.** There are several ways to define the moduli functor for our class. The one we use here is the most straightforward one (cf. [15], [20] in the absolute case with \( B = \emptyset \)). For any scheme \( S/k \), \( \mathcal{M}^N = \mathcal{M}^{(K+B)^2,(K+B)H} \) is given by

\[
\mathcal{M}^N(S) = \left\{ \begin{aligned}
\text{all families } f : (X, \mathcal{L}) \to S \\
\text{with a divisor } B = \sum_{j=1}^N B_j \text{ on } X, \\
\text{a map } g : X \to W \text{ and a line bundle } \mathcal{L} \text{ such that every } \\
\text{geometric fiber belongs to } \mathcal{C}, X \text{ and all } B_j \text{ are flat over } S
\end{aligned} \right\}
\]

Two families over \( S \) are equivalent if they are isomorphic fiber-wise.

In this functor we consider a sub-functor \( \mathcal{M}^{N,\text{irr}} \), requiring in addition that for each \( s \) there exists a 1-dimensional family from \( \mathcal{M}^N \) with the central fiber \( X_s \) and an irreducible general fiber \( X_g \) such that:

1. \( X_g \) is irreducible,
2. the pair \( (X_g, 0) \) is (Kawamata) log terminal.

This is similar to the smoothability condition for \( \overline{M}^{mn}_{K^2} \subset \overline{M}_{K^2} \) (see [12]) and is necessary due to the technical reasons. Consider a one parameter family of maps. Then we would like the ambient 3-fold to be irreducible since MMP is not developed for non-irreducible varieties yet. We would also want the 3-fold to have log terminal singularities because they are Cohen-Macaulay in characteristic 0.

3.18. A little disadvantage of the above definition is that even though \( \mathcal{M}^{N,\text{irr}} \) and, say, \( \mathcal{M}^{2N,\text{irr}} \) are the same on the closed points, the corresponding moduli spaces can potentially have different scheme structures, the second one could be strictly larger. So, in fact, we have not one but infinitely many moduli spaces. It would be better if we had a formula for
the minimal $N$ in terms of $(K + B)^2, (K + B)H, H^2$. We know, however, only that such an $N$ exists.

3.19. A different solution was suggested (again, in the absolute case with $B = \emptyset$) by Kollár in [12,14]. In a sense, it produces a moduli space with the “minimal” scheme structure.

We introduce some necessary notation first.

**Definition 3.20.** Let $F: \mathcal{X} \to S$ be a projective family of graphs of maps $(X,B) \to W$. Assume that every fiber is Gorenstein in codimension 1 and satisfies Serre’s condition $S_2$. Denote by $i: \mathcal{U} \hookrightarrow \mathcal{X}$ the open subset where $f$ is Gorenstein and the divisors $B_j$ are Cartier. Note that on every fiber one has $\text{codim}_{X_s}(X_s - \mathcal{U}_s) \geq 2$. Define the sheaves $L_{\mathcal{U},k}$ and $L_k$ by

$$L_{\mathcal{U},k} = \mathcal{O}_{\mathcal{U}}(k(K_{\mathcal{U}}/S + B + g^*\mathcal{O}_W(5)))$$

$$L_k = i_*L_{\mathcal{U},k}$$

It follows that the sheaves $L_k$ on $\mathcal{X}$ are coherent.

**Notation 3.21.** Let $f: \mathcal{X} \to S$ be a morphism of schemes, $i: \mathcal{U} \hookrightarrow \mathcal{X}$ be the immersion of an open set and $\mathcal{F}$ be a coherent sheaf on $\mathcal{U}$ which is flat over $S$. For a base change $h: \mathcal{S}' \to \mathcal{S}$ we obtain $\mathcal{X}' := \mathcal{X} \times_{\mathcal{S}} S'$, $h^h := h \times S'$ etc. Denote the induced morphism $\mathcal{U}' \to \mathcal{U}$ by $h_{\mathcal{U}}$ and set $\mathcal{F}' := h_*^h \mathcal{F}$. The induced morphism $\mathcal{X}' \to \mathcal{X}$ is denoted by $h_X$

One says that the push forward of $\mathcal{F}$ commutes with a base change $h: \mathcal{S}' \to \mathcal{S}$ if the natural map $h_X^*(i_*\mathcal{F}) \to i^!h_*^h \mathcal{F}$ is an isomorphism.

**Definition 3.22.** Define $\mathcal{MC}^{all} = \mathcal{MC}_{(K + B)^2,(K + B)H,H^2}$ by

$$\mathcal{MC}^{all}(\mathcal{S}) = \begin{cases} 
\text{all families } f: \mathcal{X} \to \mathcal{S} \text{ with a divisor } B = \sum_{j=1}^{N} B_j \text{ on } \mathcal{X} \text{ and} \\
\text{a map } g: \mathcal{X} \to W \text{ such that every geometric fiber belongs to } \mathcal{C}, \\
\mathcal{X} \text{ and all } B_j \text{ are flat over } \mathcal{S}, \text{ and for each } k \\
i_*L_{\mathcal{U},k} \text{ commutes with arbitrary base changes} 
\end{cases}$$

As above, one can consider a sub-functor $\mathcal{MC}^{all} \subset \mathcal{MC}^{all}$.

We will not go into detailed discussion of this functor.

3.23. One can see that if we require that $i_*L_{\mathcal{U},k}$ commutes with arbitrary base changes only for $k = N$ instead of all positive $k$, then we get the previous definition of the moduli functor. Indeed, if a line bundle $\mathcal{L}$ exists, then $L_N = \mathcal{L} + f^*E$ for some invertible sheaf $E$ on $S$. Then for every $h: \mathcal{S}' \to \mathcal{S}$ the two sheaves $i_*h_*^h L_{\mathcal{U},N}$ and $h_X^*(i_*L_{\mathcal{U},N}) = h_X^*(L_N)$ on $\mathcal{X}'$ are both reflexive and coincide on $h_X^{-1}(\mathcal{U})$, hence everywhere.
Vice versa, if \( i_*L_{U,N} \) commutes with base changes, then \( \mathcal{L}_N \) is flat and for every closed point \( s \in S \)
\[
\mathcal{L}_N|_{X_s} = \mathcal{O}_{X_s}(N(K + B + g^*\mathcal{O}_W(5)))
\]
Since the latter restriction is locally free for every \( s \) and the sheaves \( \mathcal{O}_{X}, \mathcal{L}_N \) are coherent and flat over \( S \), it follows by [18] 22.5, 22.3 that \( \mathcal{L}_N \) is locally free.

3.24. Now let us show that our moduli functor \( \mathcal{M}C^N \) has all the good properties listed above. We start with the local closedness. The main technical result we will be using is the following theorem.

**Theorem 3.25** (Kollár [14]). With the above notations, assume that \( f : X \to S \) is projective, \( i_*F \) is coherent and that for every point \( s \in S \) the sheaf \( F_s \) on the fiber \( X_s \) satisfies Serre’s condition \( S_2 \). Then there exist locally closed subschemes \( S_l \subset S \) such that for any morphism \( h : T \to S \) the following are equivalent:

1. \( h \) factors through \( T \to \bigsquare S_l \to S \),
2. \( i^h_*F^h \) commutes with all future base changes.

**Theorem 3.26.** The functors \( \mathcal{M}C^N \) and \( \mathcal{M}C'^N \) are locally closed.

**Proof.** Let \( F : \mathcal{X} \to S \) be an arbitrary projective family of graphs of maps \( (X, B) \to W \). First, after the flattening decomposition (see [20] lecture 8) of \( S \) into locally closed subschemes, we can assume that \( \mathcal{X}' \) and \( B_j \) are flat over \( S \) if they are not already.

Consider a one-parameter sub-family \( \mathcal{X}_R \to \mathcal{R} \) and a point \( P \) on the central fiber \( \mathcal{X}_0 \). Then \( \mathcal{X}_0 \) is Cohen-Macaulay at \( P \) iff the 3-fold \( \mathcal{X}_R \) is. The property of a local ring to be Cohen-Macaulay is open ([18] 24.5) and the morphism \( F \) is projective. Therefore, if \( \mathcal{X}_0 \) is Cohen-Macaulay then there exists an open neighborhood of \( \mathcal{R} \), and also of \( S \), that contains exactly the points over which the fibers are Cohen-Macaulay.

The property of a local ring to be Gorenstein is also open ([18] 24.6) and by the same argument there exists a closed subset \( Z \) of non-Gorenstein points in \( \mathcal{X} \). Give it the structure of a reduced scheme. Then we have to throw away all fibers on which the Hilbert polynomial of \( \mathcal{O}_Z \otimes \mathcal{O}_S(s) \) has degree \( \geq 1 \). There are only finitely many possible Hilbert polynomials and the condition on the degree is obviously closed.

At this point we use the previous theorem 3.25 to the sheaf \( L_{U,N} \) to conclude that there exist locally closed subschemes \( S_l \subset S \) such that every map \( h : T \to S \) with \( \mathcal{X} \times T \in \mathcal{M}C(T) \) factors through \( \bigsquare S_l \). \( S_l \) are disjoint, so we can concentrate on one of them. If \( P \) is a point of \( S \) and some \( h \) as in the definition does not factor through \( S - P \), then the fiber of \( F \) over \( P \) has to be a pair \( (X, B) \) from our class. The sheaf \( \mathcal{L}_N \) on \( X \times S_l \) is flat over \( S_l \) and its restriction to the fiber over \( P \) is locally free. Hence, it has to be locally
free in a neighborhood of the fiber. Therefore, for each $S_l$ if we denote by $U_l \subset S_l$ the open set over which $L_N$ is locally free, then $h : T \to S$ has to factor through $\bigsqcup U_l$. Now we can apply [2.21](2) to conclude that there exist open subsets $V_l \subset U_l$ containing all the points over which the fibers have semi-log canonical singularities. Also, $\mathcal{MC}'^N \subset \mathcal{MC}^N$ is evidently closed and we end up with a disjoint union of locally closed subschemes.

There is one more thing one has to take care of: the polarization $\mathcal{O}_{\mathcal{P}_x}(1)$ on the fibers has to coincide with $L_N$ or its fixed multiple $L_M$. Standard semi-continuity theorems for $h^0$ in flat families show that there exists a closed subset where the two sheaves are the same. One can also define the scheme structure on it, see lemma 1.26 [26].

**Lemma 3.27.** For the functors $\mathcal{MC}^N$ and $\mathcal{MC}'^N$ the polarization $L_N$ is functorial.

*Proof.* $K_{U/S}$ of a flat family commutes with base changes, and so do $\mathcal{O}(\mathcal{B}_j)$ and $g^*\mathcal{O}_W(1)$. Therefore, $L_{U,k}$ are functorial. By the definitions of the functor $\mathcal{MC}$ the same is true for $L_k$ (resp. $L_N$).

**Theorem 3.28.** $\mathcal{MC}'^N$ is

1. separated,
2. complete,
3. have finite and reduced automorphisms.

*Proof.* The first two properties have code names in the Minimal Model Program: “uniqueness and existence of the log canonical model”. It is enough to check them in the case when the general fiber is irreducible and has log terminal singularities.

1. Let $S$ be a specter of a DVR or a pointed curve. Two families in $\mathcal{MC}(S)$ that coincide outside of 0 are birationally isomorphic. [2.21](1) implies that they are both log canonical and both are relative log canonical models over $S \times W$ for the same divisor, hence isomorphic. If $Y \to S$ is a common resolution then the divisor is

$$K_Y + f^{-1}B + \sum \mathcal{E}_i$$

where $\mathcal{E}_i$ are exceptional divisors that do not map to a central point $0 \in S$.

2. If there is a family over $S - 0$, we can complete it over 0 somehow. Then by a variant of the Semistable Reduction Theorem, after a finite base change, there is a resolution $Y$ of singularities such that the central fiber is reduced and all exceptional divisors and $\mathcal{B}_j$ have normal crossings. Consider the log canonical model for the same divisor as above, relative over $S \times W$. It exists by [9] for example. This log canonical model has the same fibers as $(X, B)$ outside 0. It has log terminal singularities only, which are Cohen-Macaulay in dimension 3 and characteristic 0. Therefore, the central fiber is also Cohen-Macaulay and it is from our class $C$ by [2.21](3).
We also have to show that the sheaf $L_N$ for this family is locally free. It amounts to proving that the Hilbert polynomials $h_1(t)$ of the sheaf $L_{N,0}$ on the special fiber, and $h_2(t)$ of the sheaf $L_{N,g}$ of the general fiber coincide. Both sheaves are locally free. But the log canonical model is constructed by applying the Base Point Freeness theorem, and by the very construction we have that some $L_M$ for a large divisible $M$ is locally free on $X$. Therefore the polynomials $h_1(M/Nt)$ and $h_2(M/Nt)$ are the same, and that means that $h_1(t)$ and $h_2(t)$ are also the same.

(3) In the absolute case, the fact that $K+B$ is ample and log canonical implies that the automorphism group is finite by \cite{7}. In the relative case we apply the same theorem to $K_X+D$, $D \in |4H|$ general, which is ample by lemma \ref{lem:ample}. We are working in characteristic 0 and so the group scheme $\text{Aut} X$ is reduced. \hfill \Box

**Theorem 3.29.** The functors $\mathcal{M}C^N$ and $\mathcal{M}C'^N$ are semipositive.

**Proof.** One has the following

**Theorem 3.30** (Kollár \cite{12} 4.12). Let $Z$ be a complete variety over a field of characteristic zero. Assume that $Z$ satisfies Serre’s condition $S_2$ and that it is Gorenstein in codimension one. Let $Z \to C$ be a map onto a smooth curve. Assume that the general fiber of $f$ has only semi-log canonical singularities, and further that $K$ of the general fiber is ample. Then $f_*\mathcal{O}(kK_{Z/C})$ is semipositive for $k \geq 1$.

For the sheaves $L_N = \mathcal{O}_X(N(K_{X/S} + B))$ with empty $B$ in the absolute case this is exactly what we need. Analyzing the proof of \ref{thm:semipositive} shows that it works with very minor changes in the case of a non-empty reduced $B$. In the relative case instead of $K_{X/S}+B$ we consider $K_{X/S}+B+5H$, $H = g^*\mathcal{O}_W(1)$. We can think of $5H$ simply as of an additional component of the boundary $B$. If a member of the linear system $|5H|$ is chosen generically, on the general fiber of $f$ the pair $(X, B+5H)$ will still be semi-log canonical.

For the positiveness of the sheaves $L_N|_{B_j}$ we use the log adjunction formula, see \cite{25} or \cite{3} chapter 16. We get the following semi-log canonical divisors on $B_j$:

$$K_X+B|_{B_j} = K_{B_j} + \sum (1-1/m_k)\mathcal{M}_k$$

for some Weil divisors $\mathcal{M}_k$ on $B_j$ and $m_k \in \mathbb{N} \cup \{\infty\}$. So, here we need a more general semipositivity theorem, with nonempty $B$ that has fractional coefficients. The situation is saved by the fact that the relative dimension of $B_j$ over $S$ equals 1, and the semipositivity for this case is proved in \cite{12} 4.7. \hfill \Box

4. Existence and projectivity of a moduli space

**Theorem 4.1.** The functor $\mathcal{M}C = \mathcal{M}C^N$ is coarsely represented by a proper separated algebraic space of finite type $\mathcal{M}C = \mathcal{M}C'^N$. 

Proof. The proof is essentially the same as in [22], p.172. We remind that we are working in characteristic zero, and over \( \mathbb{C} \) the argument is easier.

The class \( C^M \) is bounded, and we can embed all graphs \( \Gamma_g \) of the maps \( g \) by a linear system \( |M(K_X + B + 5H)| \) in \( \mathbb{P}^e \times \mathbb{P}^e \subset \mathbb{P}^e \) as in 3.14 for a large divisible \( M \). By taking \( M \) even larger we can assume that all \( X = \Gamma_g \) and all \( B_j \subset X \) are projectively normal, \( h^0(M(K_X + B + 5H)) \) is locally constant and there are no higher cohomologies. \((\Gamma_g, B)\) are parameterized, not in a one-to-one way, by some scheme that we will denote by \( H \). For any family in \( MC_N(T) \), the embedding by a relatively very ample linear system \( |M(K_X + B + 4H)| \) defines a non-unique map \( T \rightarrow H \). By 3.26 there exists a disjoint union of locally closed subschemes \( S = \bigsqcup S_l \hookrightarrow H \) with a universal property, and \( T \rightarrow H \) factors through \( S \). We conclude that the coarse moduli space \( MC \) is a categorial quotient of \( S \) by an equivalence relation \( R \), described as follows.

\( R \) is a set of pairs \((h, G)\), where \( h \in S \) and \( G \) corresponds to a different embedding of \( X \) in \( \mathbb{P}^e \), i.e. \( G \) varies in a group \( PGL(d_1 + 1) \). There is a natural map \( F : R \rightarrow S \times S \). Every fiber of \( \pi_1 \circ F \) is isomorphic to \( PGL(d_1 + 1) \) and this map is obviously smooth. The map \( F \) is quasi-finite and unramified because its fibers are automorphism groups of objects in \( C \), and these are finite reduced. The fact that \( MC \) is also proper implies that \( F \) is finite.

The rest of the proof is the same as in [22], p.172 verbatim. By taking the transversal sections locally the question is reduced to the case of a finite equivalence relation dominated by a map \( F' : R \rightarrow H' \times H' \) with \( \pi_1 \circ F' \) étale, and then the quotient is easily constructed as an algebraic space.

Finally, since \( MC \) is proper, so is \( MC \).

\[ \square \]

**Theorem 4.2.** The moduli space \( MC = MC'N \) is projective.

Proof. The proof follows the general scheme of [12]. By the very construction of \( MC \), there exists a subscheme \( S \subset H \) of a product of Hilbert schemes, with the corresponding universal family \( V_S \rightarrow S \), that maps to \( MC \). One starts by constructing a finite morphism from a scheme \( Y \rightarrow MC \) with a universal family \( f : V_Y \rightarrow Y \). This is done locally by cutting \( S \rightarrow MC \) transversally, then adding more copies of these sections, so that the automorphisms do not obstruct gluing the local pieces together, see [12] 2.7.

The only properties of the class \( C \) used in this construction are boundedness and finiteness of automorphisms, which we have.

Next step is to consider the line bundles

\[ \lambda_M = \det(f_* \mathcal{L}_M \oplus f_* \mathcal{L}_M|_{B_j}) \]

on \( Y \) for \( M \) large divisible, where

\[ \mathcal{L}_M = \mathcal{O}_V(M(K_V + B + g^* \mathcal{O}_W(5))). \]

These line bundles do not descend to \( MC \) because of automorphisms, but since the objects of \( C \) have finite groups of automorphisms and \( C \) is bounded,
for every $M$ there is a finite power of $\lambda_M$ that does come from a line bundle on $\mathcal{MC}$. To prove that $\mathcal{MC}$ is projective it is enough to show that one of $\lambda_M$ is ample, which is achieved by the following theorem. For simplicity we formulate it only in characteristic 0.

**Notation 4.3.** Let $Y$ be a scheme and let $W$ be a vector bundle of rank $w$ with structure group $\rho : G \to GL_w$. Let $q : W \to Q$ be a quotient vector bundle of rank $k$. Let $Gr(w,k)/G$ denote the set of $G$-orbits on the $k$-dimensional quotients of a $w$-dimensional vector space. The natural map of sets

$$u_{Gr} : \{\text{closed points of } X\} \to Gr(w,k)/G$$

is called the **classifying map**.

One says that the classifying map is **finite** if

1. every fiber of $u_{Gr}$ is finite, and
2. for every $y \in Y$ only finitely many elements of $G$ leave $\ker q_y$ invariant.

**Theorem 4.4** (Kollár’s Ampleness Lemma, [12] 3.9). Let $Y$ be a proper algebraic space and let $W$ be a semipositive vector bundle with structure group $G$. Let $Q$ be a quotient vector bundle of $W$. Assume that

1. $G$ is reductive,
2. the classifying map is finite.

Then $\det Q$ is ample. In particular, $Y$ is projective.

This is what it translates to in our situation. The sheaves are

$$W = \text{Sym}^j(f_*\mathcal{L}_M) \oplus \text{Sym}^j(f_*\mathcal{L}_M|_{B_j})$$

and

$$Q = f_*L_jM \oplus f_*L_jM|_{B_j},$$

$q$ is the multiplication map.

By [3,29] we already know that $Q$ is semipositive, and so is $W$ since symmetric powers of a semipositive sheaf are semipositive.

Recall that the universal family $U_Y$ over $Y$ is embedded into a product of $Y$ and

$$\mathbb{P}^e \times W \subset \mathbb{P}^e \times \mathbb{P}^e \subset \mathbb{P}^e$$

and that the sheaf $L_M$ is the restriction of $\mathcal{O}_{\mathbb{P}^e}(1)$ in this embedding.

The group $G$ acting on $W$ is $GL_{d_1+1} \times GL_1$. If every fiber $X = \Gamma_g$ together with all $B_j$ can be uniquely reconstructed from the map $W_s \to Q_s$, then the fibers of $u_{Gr}$ will be exactly the same as fibers of $Y \to \mathcal{MC}$, hence finite.

For this to be true we need the following:

1. every fiber in $\mathbb{P}^e$ is set-theoretically defined by degree $\leq j$ equations,
2. the multiplication maps $\text{Sym}^j(f_*\mathcal{L}_M) \to f_*L_jM$ and $\text{Sym}^j(f_*\mathcal{L}_M|_{B_j}) \to f_*L_jM|_{B_j}$ are surjective.
(1) holds if \( j \) is large enough. (2) is satisfied because we have chosen \( M \) so large that all \( X \) and \( B_j \) are projectively normal in \( \mathbb{P}^r \).

Finally, the second condition in the definition of finiteness of the classifying map is satisfied because all graphs \((\Gamma_g, B) = (X, B)\) in \( \mathbb{P}^r \) have finite groups of automorphisms.

5. RELATED QUESTIONS

5.1. Let us see how our moduli spaces are related to some others. For example, consider the moduli space \( M_{L^2} \) of K3 surfaces \( X \) with a polarization \( L \) with a fixed square. Compare it with \( M_{(K+B)^2} \), where \( W = \text{pt}, B = B_1 \) is one reduced divisor and \((K + B)^2 = L^2\) is the same number. \( M_{(K+B)^2} \) contains an open subset \( U \) parameterizing K3 surfaces with reduced divisors having normal intersections only, and we have a map \( F: U \to M_{H^2} \). A well-known result (Saint-Donat [23]) says that every ample linear system \(|L|\) on a K3 surface contains at least one reduced divisor with normal intersections, therefore \( F \) is surjective. In fact, \( M_{H^2} \) is a quotient of \( U \) modulo an obvious equivalence relation \( R: (X_1, B_1) \sim_R (X_2, B_2) \) iff \( X_1, X_2 \) are isomorphic and \( B_1, B_2 \) are linearly equivalent.

There is a natural map \( G: R \to U \times U \). \( \pi_1 \circ G \) is smooth and its fibers are open subsets in \( \mathbb{P}^r(\mathbb{P}^1 - \mathbb{P}^r) \). The situation is very similar to what we had in theorem 4.1, except this time the quotient \( U/R \) is not proper. The obvious way to try to obtain a compactification of \( M_{H^2} \) is to consider the closure \( \overline{U} \) of \( U \) in \( M_{(K+B)^2} \), then somehow define the closure \( \overline{R} \) of \( R \), and ask if it has good enough properties enabling one to construct \( \overline{U}/\overline{R} \) and to prove that it is projective. Alternatively, one can ask if the closure of \( G(R) \) in \( \overline{U} \times \overline{U} \) has good properties.

The situation resembles what happens for elliptic curves. The natural compactification of the moduli space \( M_1 = \mathbb{A}^2_{\mathbb{C}} \) is \( \mathbb{P}^2_{\mathbb{C}} \), and the infinite point corresponds not to one but to many degenerations: wheels of rational curves of lengths 1...n if we consider \( M_1 \) as a factor of \( M_{1,n} \). Similarly, the boundary points of \( M_{H^2} \) should correspond to many different degenerations of smooth K3 surfaces with geometric divisors, properly identified.

The first thing to ask on this way is:

**Question 5.2.** Is it possible to define an equivalence relation \( \overline{G}: \overline{R} \to \overline{U} \times \overline{U} \), so that the morphism \( \pi_1 \circ \overline{G} \) is smooth or at least flat?

Even if this is done, there are problems with taking the quotient. There does not seem to exist in the literature a ready-to-use method that would cover our situation. There is, on one hand, a theorem of M.Artin (see [3] 7.1, [4] 6.3) that shows that if \( \overline{G}: \overline{R} \to \overline{U} \times \overline{U} \) were a monomorphism (which it is not) with flat projections, then the quotient would be defined as an algebraic space. In this case it would also easily follow that the quotient is actually projective.
On the other hand, there is the method of [22], p.172 that we used in the previous section, in which the equivalence relation is smooth, and the map $\mathcal{O}$ is finite. Natural degenerations of K3 surfaces can have infinite groups of automorphisms, however.

I think that the question deserves a more detailed consideration.

5.3. Similarly to K3 surfaces, for any principally polarized Abelian variety $A$ with a theta divisor $\Theta$ the pair $(A, \Theta)$ has log canonical singularities, see [13]. So, the previous discussion applies to principally polarized Abelian surfaces too. One can also ask what happens if the polarization is not principal.

5.4. It goes without saying that the projectivity theorem 4.2 applies in the case of curves, with significant simplifications. Therefore, the moduli spaces $M_{g,n}(W)$ of [16] are also projective.

5.5. Most $\mathcal{M}_C(K+B)^2,(K+B)H,H^2$ are definitely not irreducible and not even connected. They are subdivided according to various invariants, such as the numerical or homological type of $g(X)$ and $g(B_j)$, intersection numbers $(K + B)B_j$ etc.

One can also get by fixing only one number, $(K + B + 4H)^2$. Then there are only finitely many possibilities for other invariants.

5.6. The boundedness theorem 3.15 is in fact even stronger than what we used here: it applies to the case when the coefficients $b_j$ belong to an arbitrary set $A$ that satisfies the descending chain condition. One, perhaps, would want to define even more general moduli spaces. There are two obstacles, however. First, the semipositiveness theorem 3.30 for the case of fractional coefficients seems to be quite hard to prove, but probably still possible. The second obstacle is a fundamental one: for proving the semipositive theorems for $L_k|B_j$ we used the log adjunction formula. It basically just says $K + B|_B = K_B$, and here the coefficient 1 of $B$ is important.

5.7. The places where assumption about the characteristic 0 was used:

1. MMP in dimension 3. This is not serious since we worked in the situation of the relative dimension 2. For surfaces log MMP is characteristic free, and perhaps it is true for families of surfaces in generality needed. For the case $B = \emptyset$ see [5]

2. The semipositiveness theorem 3.30 requires characteristic 0. Since we are dealing with a case of relative dimension 2 only, this also probably can be dealt with.

3. A group scheme in characteristic 0 is reduced, hence smooth. This was used in the proof of 3.11. Perhaps, the argument could be strengthened.

4. The argument of [22], p.172 is a whole lot more complicated in characteristic $p > 0$. 
5.8. It should be possible to prove the semipositiveness theorems and the Ampleness Lemma, as well as the [22] p.172 argument, entirely in the relative situation $/W$, without appealing to absolutely ample divisors. The moduli spaces obtained should be then projective over $W$.

5.9. One can see that most of the theorems that we proved for the functor $\mathcal{M}N^t$ apply to the functor $\mathcal{M}^\text{full}$ as well.

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