Research Article

Barycentric Rational Collocation Method for the Incompressible Forchheimer Flow in Porous Media

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Barycentric rational collocation method is introduced to solve the Forchheimer law modeling incompressible fluids in porous media. The unknown velocity and pressure are approximated by the barycentric rational function. The main advantages of this method are high precision and efficiency. At the same time, the algorithm and program can be expanded to other problems. The numerical stability can be guaranteed. The matrix form of the collocation method is obtained from the discrete numerical schemes. Numerical analysis and error estimates for velocity and pressure are established. Numerical experiments are carried out to validate the convergence rates and show the efficiency.

1. Introduction

Darcy flow in porous media is of great interest in many science and engineering fields such as oil recovery and groundwater pollution contamination. Darcy’s law,

\[ \frac{\mu}{k} u(x) = -\frac{d\bar{p}(x)}{dx} + \rho g(x), \quad x \in (a, b), \tag{1} \]

mainly describes the linear relationship between Darcy velocity \( u \) and derivative of pressure \( \bar{p} \). Here, symbols \( \mu, k, \rho \), and \( g(x) \) represent the viscosity coefficient, permeability, the density of the fluids, and the gravitational term, respectively. This model is widely used and suitable for low velocity, small porosity, and permeability fluids [1–4].

If the porosity is nonuniform and velocity is higher, a second-order term is needed to be added, the non-Darcy relationship has been researched by Forchheimer [1]. For example, the high-speed Forchheimer flow of single-phase incompressible fluid in porous medium is presented as follows:

\[ \frac{\mu}{k} u(x) + \beta \rho |u(x)| u(x) = -\frac{d\bar{p}(x)}{dx} + \rho g(x), \quad x \in (a, b). \tag{2} \]

Note that when Forchheimer number \( \beta = 0 \), nonlinear model (2) degenerates to linear Darcy’s law (1).

Model (2) is also called Darcy–Forchheimer law [5–10]. In [6], a block-centered finite difference method has been introduced to solve the Darcy–Forchheimer law. Discrete numerical scheme and error estimates were given. Mixed finite element method (MFEM) for equation (2) was studied in [7, 8]. Using this method, velocity and pressure can be approximated simultaneously. Two-grid and multigrid block-centered finite difference method (FDM) for the Darcy–Forchheimer flow in porous media was researched in [10, 11], respectively. This method can improve the efficiency of dealing with nonlinear problems. The barycentric formula is obtained by the Lagrange interpolation formula [12–16] and has been used to solve Volterra equation and Volterra integro-differential equation [12, 17, 18]. Floater and Hormann [19] have proposed a rational interpolation scheme which has higher accuracy on equidistant and special distributed nodes. Wang et al. [20–22] successfully applied the barycentric rational collocation method (BRCM) to solve initial value problem, boundary value problem, plane elasticity problem, and some nonlinear problems. These research studies extended the application fields of...
barycentric rational collocation method. In recent papers, Li et al. [23–27] have used the barycentric rational collocation method to solve heat conduction equation, biharmonic problem, and second-order Volterra integro-differential equation.

In this paper, barycentric rational collocation method is introduced to solve the incompressible Forchheimer flow. We demonstrate that barycentric rational collocation method is highly accurate for both velocity and pressure. $O(h^d)$ error estimates for velocity and pressure are given. Numerical experiments [28–32] are carried out to show the convergence rates. The paper is organized as follows. In Section 2, notations and barycentric formula are given. In Section 3, convergence analysis of barycentric rational collocation method for Forchheimer law and error estimates of velocity and pressure are presented. In Section 4, numerical examples are carried out to verify the convergence rates and show the efficiency. Throughout this paper, $C$ denotes a positive constant independent of $h$.

### 2. Notations and Barycentric Rational Algorithm

The partition of interval $\Omega = [a,b]$ is as follows:

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b. \quad (3)$$

Define

$$\Omega_i = [x_{i-1}, x_i], \quad 1 \leq i \leq n,$$

$$h_i = x_i - x_{i-1}, \quad 1 \leq i \leq n,$$

$$h = \max_{1 \leq i \leq n} h_i. \quad (4)$$

For the function $u(x)$, the interpolation function $r(x)$ $(d = 0, 1, \ldots, n)$ is given as

$$r(x) = \sum_{i=0}^{n-d} \lambda_i(x) p_i(x) \over \sum_{i=0}^{n-d} \lambda_i(x), \quad (5)$$

Symbol $p_i(x)$ denotes the $d$-order interpolation polynomial such that $p_i(x_k) = u(x_k)$ for $k = i, i + 1, \ldots, i + d$,

$$p_i(x) = \sum_{k=i}^{i+d} \prod_{j=i, j \neq k}^{i+d} \frac{x - x_j}{x_k - x_j}. \quad (6)$$

where $u_k = u(x_k)$ and $\lambda_i(x)$ is a blending function

$$\lambda_i(x) = \frac{(-1)^i}{(x - x_i) \cdots (x - x_{i+d})} \quad (7)$$

For the numerator term in (5), we deduce that

$$\sum_{i=0}^{n-d} \lambda_i(x) p_i(x) = \sum_{i=0}^{n-d} \left( \sum_{k=i}^{i+d} \frac{1}{x_k - x_j} \right) u_k = \sum_{k=0}^{n} \omega_k u_k. \quad (8)$$

Here,

$$\omega_k = \sum_{i=0}^{n-d} \prod_{j=i, j \neq k}^{i+d} \frac{1}{x_k - x_j} \quad (9)$$

and $I_k = \{i \in I; k - d \leq i \leq k\}$, $I = \{0, 1, \ldots, n - d\}$. Note that

$$\sum_{k=i}^{i+d} \prod_{j=i, j \neq k}^{i+d} x - x_j = 1, \quad (10)$$

and for the denominator term in (5),

$$\sum_{i=0}^{n-d} \lambda_i(x) = \sum_{k=0}^{n} \omega_k \frac{1}{x - x_k}. \quad (11)$$

Through further deduction, we get

$$r(x) = \sum_{j=0}^{n} \left( \omega_j / x - x_j \right) u_j / \sum_{j=0}^{n} \omega_j / x - x_j = \sum_{j=0}^{n} r_j(x) u_j. \quad (12)$$

Here, $\omega_k$ is described as (9). The basis function $r_j(x)$ of barycentric rational interpolation is

$$r_j(x) = \omega_j / x - x_j / \sum_{j=0}^{n} \omega_j / x - x_j, \quad j = 0, 1, \ldots, n. \quad (13)$$

Then, we get the derivative formula at node $x_i$ as

$$u^{(m)}(x_i) = u^{(m)} = \frac{d^m u(x_i)}{dx^m} = \sum_{j=0}^{n} r_j^{(m)}(x_i) u_j = \sum_{j=0}^{n} D_{ij}^{(m)} u_j, \quad m = 0, 1, 2, \ldots. \quad (14)$$

$$u^{(m)} = [u_0^{(m)}, u_1^{(m)}, \ldots, u_n^{(m)}],$$

$$D_{ij}^{(m)} = r_j^{(m)}(x_i). \quad (16)$$

The derivative formulation of the basis function $r_j(x)$ at node $x_i$ is
\begin{equation}
    r'_j(x_i) = \frac{\omega_j \omega_i}{x_i - x_j}, \quad j \neq i, \quad (17)
\end{equation}

\begin{equation}
    r'_i(x_i) = - \sum_{j \neq i} r'_j(x_i). \quad (18)
\end{equation}

According to induction (14)–(18), we obtain the recurrence formula of $D_{ij}^{(m)}$ as

\begin{equation}
\begin{aligned}
    D_{ij}^{(m)} &= m \left( D_{ij}^{(m-1)} - \frac{D_{ij}^{(m-1)}}{x_i - x_j} \right), \quad i \neq j, \\
    D_{ii}^{(m)} &= \sum_{j \neq i} D_{ij}^{(m)}.
\end{aligned} \quad (19)
\end{equation}

3. Convergence Rates and Error Estimates

Define the error between $u(x)$ and barycentric rational interpolation function $r(x)$ as

\begin{equation}
    e(x) = u(x) - r(x). \quad (20)
\end{equation}

According to rational interpolation error theory, we know

\begin{equation}
    e(x) = (x - x_i) \ldots (x - x_{i+d})u[x_i, x_{i+1}, \ldots, x_{i+d}, x]. \quad (21)
\end{equation}

Combining (21) with (5), we see

\begin{equation}
    e(x) = \sum_{i=0}^{n-d} \lambda_i(x) (u(x) - p_i(x)) = \frac{A(x)}{B(x)}. \quad (22)
\end{equation}

\begin{equation}
\begin{aligned}
    A(x) &= \sum_{i=0}^{n-d} (-1)^i u[x_i, x_{i+1}, \ldots, x_{i+d}, x], \\
    B(x) &= \sum_{i=0}^{n-d} \lambda_i(x).
\end{aligned} \quad (23)
\end{equation}

Define the error norm of $e(x)$ as

\begin{equation}
    |e(x)| = \max_{a \leq x \leq b} |e(x)|. \quad (24)
\end{equation}

The following lemma has been proved in [12].

**Lemma 1** (see [12]). *For the error $e(x)$ defined as (20), we have*

\begin{equation}
\begin{aligned}
    |e(x)| &\leq Ch^{d+1}, \quad u \in C^{d+2}[a, b], \\
    |e_x(x)| &\leq Ch^d, \quad u \in C^{d+3}[a, b].
\end{aligned} \quad (25)
\end{equation}

Now, we deal with the barycentric rational collocation schemes for the following Forchheimer equations:

\begin{align}
    \dot{p} &= \frac{\partial p(x)}{\partial x} = \frac{\mu}{k} u(x) - \beta p[u(x)] u(x) + \rho g(x), \quad x \in (a, b), \\
    \dot{u} &= \frac{\partial u(x)}{\partial x} = f(x), \quad x \in (a, b), \\
    u(0) &= u_0, \\
    \hat{p}(0) &= \hat{p}_0.
\end{align} \quad (26)

For the second equation of (26), the approximate formula is

\begin{equation}
    \sum_{j=0}^{n} r'_j(x_i) u_j = f(x_i), \quad i = 0, 1, \ldots, n. \quad (27)
\end{equation}

Taking $x = x_i$ in (27), the numerical scheme is

\begin{equation}
    \sum_{j=0}^{n} r'_j(x_i) u_j = f(x_i), \quad i = 0, 1, \ldots, n. \quad (28)
\end{equation}

For the first equation of (26), the approximate formula is as follows:

\begin{equation}
\begin{aligned}
    \sum_{j=0}^{n} r'_j(x_i) \hat{p}_j &= \frac{\mu}{k} u(x) - \beta p[u(x)] u(x) + \rho g(x), \\
    \hat{p}(0) &= \hat{p}_0.
\end{aligned} \quad (29)
\end{equation}

Then, the calculation scheme is

\begin{equation}
\begin{aligned}
    \sum_{j=0}^{n} r'_j(x_i) \hat{p}_j &= \frac{\mu}{k} u_i - \beta p[u_i] u_i + \rho g(x_i), \quad i = 0, 1, \ldots, n.
\end{aligned} \quad (30)
\end{equation}

Note that, in practical calculation, first step, we approximate the second equation of (26) and then the first equation of (26).

Let $u(x_n)$ denote the numerical solution of $u(x)$, then we have

\begin{equation}
\begin{aligned}
    \dot{u}(x_n) &= f(x), \\
    \lim_{n \to \infty} \dot{u}(x_n) &= f(x).
\end{aligned} \quad (31)
\end{equation}

Based on the above states, the next theorem gives the error analysis of Darcy velocity.

**Theorem 1.** Let $u(x_n): \dot{u}(x_n) = f(x)$ and $f(x) \in C[a, b]$. If $u(x) \in C^{d+3}[a, b]$, then we have

\begin{equation}
    |u(x) - u(x_n)| \leq Ch^d. \quad (32)
\end{equation}

**Proof.** For the second equation of (26), using the notation of differential matrix, the discrete form of the collocation method is

\begin{equation}
    L_1 u = D^{(1)} u = f, \quad (33)
\end{equation}

where
\[
L_1 = \begin{bmatrix}
D_{00}^{(1)} & \ldots & D_{0n}^{(1)} \\
\vdots & \ddots & \vdots \\
D_{n0}^{(1)} & \ldots & D_{nn}^{(1)}
\end{bmatrix},
\]
\[
u = (u_0, u_1, \ldots, u_n)^T,
\]
\[
f = (f_0, f_1, \ldots, f_n)^T.
\]

Furthermore, we have
\[
\mathcal{D}u(x) - \mathcal{D}u(x_n) = u_x(x) - u_x(x_n) = \frac{\sum_{j=0}^{n-d} (-1)^j u_j [x_j, x_{j+1}, \ldots, x_{j+d}, x]}{\sum_{j=0}^{n-d} A_j(x)} = e_x(x) = O(h^d).
\]

The following theorem presents the error analysis of pressure \( \bar{p} \).

**Theorem 2.** Let \( \bar{p}(x_n) \): \( \mathcal{D}\bar{p}(x_n) = -(\mu/k)u(x) - \beta\rho u(x)u(x) + \rho g(x) \) and \( f(x) \in C[a, b] \). If \( \bar{p}(x) \in C^{d+3}[a, b] \) and \( u(x) \in C^{d+3}[a, b] \), then we have
\[
|\bar{p}(x) - \bar{p}(x_n)| \leq Ch^d.
\]

**Proof.** For the first equation of (26), the discrete numerical scheme is
\[
\mathcal{D}\bar{p}(x) - \mathcal{D}\bar{p}(x_n) = \frac{\mu}{k} (u(x) - u(x_n)) - \beta\rho (u(x)u(x) - u(x_n)u(x_n)) = E_1 + E_2.
\]

As \( E_1 \), note that \( \mu \) and \( k \) are positive constants, we have
\[
|E_1| = \left| \frac{\mu}{k} (u(x) - u(x_n)) \right| \leq C|u(x) - u(x_n)| = O(h^d)
\]

Similarly, for \( E_2 \), according to the monotonicity of the nonlinear term, we know
\[
|E_2| = \left| -\beta\rho (u(x)u(x) - u(x_n)u(x_n)) \right| \leq C|u(x) - u(x_n)| = O(h^d).
\]

Remark 1. In the above proof of Theorem 2, coefficients \( \mu, k, \beta, \) and \( \rho \) are supposed to be positive constants. If they are functions that depend on variable \( x \) and bounded, the proof is similar.

4. Numerical Experiments

In this section, we carry out some numerical experiments using barycentric rational collocation method to solve the Forchheimer equations.
Example 1. Consider the following incompressible Forchheimer model with $\Omega = [0, 1]$:

$$
\begin{align*}
  &u(x) + \frac{3}{10} u(x)u(x) = -\frac{d\bar{p}(x)}{dx} + g(x), \quad x \in \Omega, \\
  &\frac{d u(x)}{dx} = \pi \cos(\pi x), \quad x \in \Omega, \\
  &u(0) = 0, \\
  &\bar{p}(0) = 0.
\end{align*}
$$

The analysis solution is chosen to be

$$
\begin{align*}
  u(x) &= 2 \sin(\pi x), \\
  \bar{p}(x) &= x^2 - x^{13}.
\end{align*}
$$

Gravitational term $g(x)$ is determined according to the first equation of (43). Define absolute error and relative error as

$$
\begin{align*}
  &e_1(u) = |u(x) - u(x_0)|, \\
  &e_{r1}(u) = \frac{|u(x) - u(x_0)|}{|u(x)|}, \\
  &e_2(u) = \left\|u(x) - u(x_0)\right\|_2, \\
  &e_{r2}(u) = \frac{\left\|u(x) - u(x_0)\right\|_2}{\|u(x)\|_2}, \\
  &e_1(\bar{p}) = |\bar{p}(x) - \bar{p}(x_0)|, \\
  &e_{r1}(\bar{p}) = \frac{|\bar{p}(x) - \bar{p}(x_0)|}{|\bar{p}(x)|}, \\
  &e_2(\bar{p}) = \left\|\bar{p}(x) - \bar{p}(x_0)\right\|_2, \\
  &e_{r2}(\bar{p}) = \frac{\left\|\bar{p}(x) - \bar{p}(x_0)\right\|_2}{\|ar{p}(x)\|_2}.
\end{align*}
$$

| $d$ | $e_1(u)$ | $e_{r1}(u)$ | $e_2(u)$ | $e_{r2}(u)$ |
|-----|-----------|-------------|-----------|-------------|
| 1   | 2.1667e-03 | 1.0834e-03 | 6.0702e-03 | 3.0351e-03 |
| 2   | 9.6414e-05 | 4.8207e-05 | 2.7904e-04 | 1.3952e-04 |
| 3   | 9.8776e-06 | 4.8438e-06 | 3.1901e-05 | 1.5951e-05 |
| 4   | 6.4868e-07 | 3.2434e-07 | 2.2867e-06 | 1.1433e-06 |
| 5   | 1.0361e-07 | 5.1806e-08 | 3.7468e-07 | 1.8734e-07 |
| 6   | 8.9755e-09 | 4.4877e-09 | 3.3227e-08 | 1.6611e-08 |
| 7   | 1.3998e-09 | 6.9989e-10 | 5.2848e-09 | 2.6424e-09 |
| 8   | 1.5167e-10 | 7.5836e-11 | 5.8167e-10 | 2.9083e-10 |
| 9   | 2.0588e-11 | 1.0294e-11 | 7.9930e-11 | 3.9966e-11 |
| 10  | 4.6402e-12 | 2.3201e-12 | 1.8986e-11 | 9.4929e-12 |
| 11  | 1.6687e-12 | 8.3433e-13 | 6.4924e-12 | 3.2462e-12 |

Numerical results are listed in Tables 1–4 . The corresponding approximate figures between analysis solution and numerical solution can be seen in Figures 1 and 2. We test the barycentric rational with the uniform nodes for the direct methods. Tables 3 and 4 show that the convergence rates of velocity and pressure are $O(h^d)$ with $d = 1, 2, 3, 4$. The theoretical convergence rate $O(h^d)$ is reflected.

Table 2: Errors of pressure $\bar{p}$ with $n = 20$ for Example 1.

| $d$ | $e_1(\bar{p})$ | $e_{r1}(\bar{p})$ | $e_2(\bar{p})$ | $e_{r2}(\bar{p})$ |
|-----|----------------|--------------------|----------------|--------------------|
| 1   | 1.5603e-01     | 2.5935e-01         | 6.0581e-01     | 1.0070e+00         |
| 2   | 5.5500e-02     | 9.2256e-02         | 2.2664e-01     | 3.7673e-01         |
| 3   | 2.0024e-02     | 3.3284e-02         | 8.3781e-02     | 1.3926e-01         |
| 4   | 7.0633e-03     | 1.1741e-02         | 3.0018e-02     | 4.9897e-02         |
| 5   | 2.3541e-03     | 3.9130e-03         | 1.0104e-02     | 1.6796e-02         |
| 6   | 7.3023e-04     | 1.2138e-03         | 3.1559e-03     | 5.2458e-03         |
| 7   | 2.0459e-04     | 3.4008e-04         | 8.8865e-04     | 1.4772e-03         |
| 8   | 5.1300e-05     | 8.5273e-05         | 2.2367e-04     | 3.7180e-04         |
| 9   | 1.0396e-05     | 1.7281e-05         | 4.5468e-05     | 7.5580e-05         |
| 10  | 2.0073e-06     | 3.3367e-06         | 8.7996e-06     | 1.4627e-05         |
| 11  | 1.3584e-07     | 2.2580e-07         | 5.9683e-07     | 9.9209e-07         |
Table 3: Errors of velocity $u$ and convergence rates for Example 1.

| $h$  | $d = 1$ | Rate | $d = 2$ | Rate | $d = 3$ | Rate | $d = 4$ | Rate |
|------|---------|------|---------|------|---------|------|---------|------|
| 1/20 | 2.1667e-03 | —    | 9.6414e-05 | —    | 9.6877e-06 | —    | 6.4868e-07 | —    |
| 1/40 | 4.7993e-04 | 2.17 | 9.7770e-06 | 3.30 | 4.9514e-07 | 4.29 | 1.2656e-08 | 5.68 |
| 1/80 | 1.0884e-04 | 2.14 | 1.0429e-06 | 3.23 | 2.5616e-08 | 4.27 | 2.8257e-10 | 5.49 |
| 1/160| 2.5226e-05 | 2.11 | 1.1568e-07 | 3.17 | 1.3571e-09 | 4.24 | 6.7115e-12 | 5.40 |
| 1/320| 5.9548e-06 | 2.08 | 1.3223e-08 | 3.13 | 7.3897e-11 | 4.20 | 1.7519e-13 | 5.26 |

Table 4: Errors of pressure $\tilde{p}$ and convergence rates for Example 1.

| $h$  | $d = 1$ | Rate | $d = 2$ | Rate | $d = 3$ | Rate | $d = 4$ | Rate |
|------|---------|------|---------|------|---------|------|---------|------|
| 1/20 | 1.5603e-01 | —    | 5.5500e-02 | —    | 2.0024e-02 | —    | 7.0633e-03 | —    |
| 1/40 | 5.3988e-02 | 1.53 | 1.0784e-02 | 2.36 | 2.1501e-03 | 3.22 | 4.1075e-04 | 4.10 |
| 1/80 | 1.8789e-02 | 1.52 | 1.9925e-02 | 2.44 | 2.0891e-04 | 3.36 | 2.0755e-05 | 4.31 |
| 1/160| 6.5786e-03 | 1.51 | 3.5968e-04 | 2.47 | 1.9340e-05 | 3.43 | 9.7970e-07 | 4.40 |
| 1/320| 2.3128e-03 | 1.51 | 6.4217e-05 | 2.49 | 1.7487e-06 | 3.47 | 4.4727e-08 | 4.45 |

Figure 1: Solution of fluid velocity $u$ for Example 1 ($d = 1, h = 1/320$).

Figure 2: Solution of pressure $\tilde{p}$ for Example 1 ($d = 1, h = 1/320$).
Tables 5 and 6. The corresponding approximate results between analysis solution and numerical solution can be seen in Figures 3 and 4. We test the barycentric rational with the uniform nodes for the direct methods. Tables 5 and 6 show that the convergence rates are $O(h^d)$ with $d = 1, 2, 3, 4$.

Remark 2. Numerical experiments using the barycentric rational collocation method (BRCM) for the Forchheimer equations show the consistency of the convergence rates with the theoretical analysis. The main advantages of this BRCM are high precision and efficiency. The algorithm and program can be expanded to similar initial value problem and boundary value problem. It can effectively avoid the oscillation of other interpolation collocation methods. The numerical stability is guaranteed. We demonstrate that the proposed numerical scheme is $O(h^d)$ accurate for both Darcy velocity and pressure. In practical simulation, if the parameter $d$ is further increased, we can get more accurate results. In the future, we will research higher-dimensional Forchheimer law and compressible fluids Forchheimer problems.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.
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