Framework for a novel mixed analytical/numerical approach for the computation of two-loop $N$-point Feynman diagrams

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Abstract

A framework to represent and compute two-loop $N$-point Feynman diagrams as double-integrals is discussed. The integrands are “generalised one-loop type” multi-point functions multiplied by simple weighting factors. The final integrations over these two variables are to be performed numerically, whereas the ingredients involved in the integrands, in particular the “generalised one-loop type” functions, are computed analytically. The idea is illustrated on a few examples of scalar three- and four-point functions.

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1 Introduction

A key ingredient in an automated evaluation of two-loop multileg processes is a fast and numerically stable evaluation of scalar Feynman integrals. The derivation of a fully analytic result remains beyond reach so far in the general mass case. On the opposite side, in particular for the calculation of two-loop three- and four-point functions in the general complex mass case relying on multidimensional numerical integration by means of sector decomposition [1–5], a reliable result has a high computing cost. Approaches based on Mellin-Barnes techniques [6–10] allow to perform part of the integrals analytically, yet, as far as we know, the number of integrals left over for numerical quadratures depends on the topologies considered and can remain rather costly. It would therefore be useful to perform part of the Feynman parameter integrations analytically in a systematic way to reduce the number of numerical quadratures.

This article aims at initiating such a working program, advocating the implementation of two-loop N-point functions in n dimensions \((2)I_N^N\) as (weighted sums of) double integrals in the form:

\[
(2)I_N^N \sim \sum \int_0^1 d\rho \int_0^1 d\xi W(\rho, \xi) (1)\tilde{I}_{N'}^N(\rho, \xi)
\]

where \(W(\rho, \xi)\) are weighting functions given analytically. The factors \((1)\tilde{I}_{N'}^N(\rho, \xi)\) are \(N'\)-point functions of some “generalised one-loop type” in a sense explained below. Once the \((1)\tilde{I}_{N'}^N(\rho, \xi)\) computed analytically, the \((2)I_N^N\) are obtained by numerical quadrature over the sole two remaining variables \(\rho\) and \(\xi\), which represents a substantial gain w.r.t. a fully numerical integration over the many Feynman parameter of the primary two-loop integral.

In this article, we first provide a general argument in sec. 2. In sec. 3, we show that the polytopes spanned by the Feynman parameters related to the “generalised one-loop functions” can always be partitioned into simplices. We then illustrate the general argument given in sec. 2 considering an example of three-point scalar diagram \((2)I_4^3\) with a non planar topology in sec. 4. and an example of four-point scalar diagram \((2)I_4^4\) with a non planar topology in sec. 5. Sec. 6 concludes this article with an overview of extensions of the present program to be presented in subsequent publications. Finally, Appendix A provides a proof that, for any general \(N\)-point two-loop diagram in a scalar theory with three-leg vertices with different topologies, a shrewd choice of parametrisation can always be found which leads to simplified building blocks in the “generalised one-loop” amplitude.

2 General argument

Let us consider an arbitrary two-loop Feynman diagram with topology \(T\) involving \(N\) external legs with external momenta \(\{p_i, i = 1, \ldots, N\}\) and \(I\) internal lines with internal masses \(\{m_k^2, k = 1, \ldots, I\}\). To simplify we stick here to a scalar function i.e. we ignore sophistications that may arise from spin-carrying internal lines and/or derivative couplings. We need not specify the type

3Since the two examples studied do not have UV divergences nor IR/collinear ones, the space-time dimension \(n\) is taken to 4 for these two cases.
of scalar vertices considered either. The integral representation of the diagram is given by:

\[
(2) I^\mu_N \left( \{p_i\}; \{m_k^2\}, \mathcal{T} \right) = \int \left[ \prod_{l=1}^{2} \frac{d^m k_l}{(2\pi)^n} \right] \prod_{k=1}^{I} \frac{1}{q_k^2 - m_k^2 + i\lambda} 
\]

where the internal momenta \( \{q_k\} \) are graded sums of the two loop momenta \( k_l, l = 1, 2 \) and the external momenta \( \{p_i\} \). In order to introduce our notations we recast eq. (2) into the following mixed parametric representation of this diagram:

\[
(2) I^\mu_N \left( \{p_i\}; \{m_k^2\}, \mathcal{T} \right) = (-i)^I \int_{(\mathbb{R}^+)^I} \left[ \prod_{k=1}^{I} d\tau_k \right] \delta \left( 1 - \sum_{l=1}^{I} \tau_l \right) \int_{0}^{+\infty} d\alpha \alpha^{I-1} 
\]

\[
\times \int \left[ \prod_{l=1}^{2} \frac{d^m k_l}{(2\pi)^n} \right] \exp \left\{ i\alpha \sum_{k=1}^{I} \tau_k \left( q_k^2 - m_k^2 + i\lambda \right) \right\} 
\]

The integration over the two loop momenta \( k_i \) is made easier rewriting the denominator in the integrand as follows:

\[
\sum_{k=1}^{I} \tau_k (q_k^2 - m_k^2) = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \cdot A \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + 2 \begin{bmatrix} r_1 & r_2 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + C 
\]

In eq. (4) the elements of the \( 2 \times 2 \) symmetric matrix \( A \) are sums of Feynman parameters \( \tau_k \)'s, whereas the \( n \)-vectors \( r_l \) are linear combinations of external momenta \( \{p_i\} \) weighted by Feynman parameters \( \{\tau_k\} \). The compact notations mean:

\[
\begin{bmatrix} k_1 & k_2 \end{bmatrix} \cdot A \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \sum_{j,l=1}^{2} A_{jl} (k_j \cdot k_l) \\
\begin{bmatrix} r_1 & r_2 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \sum_{l=1}^{2} (r_l \cdot k_l) 
\]

The term \( C \) is of the form

\[
C = \sum_{i,j} Q_{ij} (p_i \cdot p_j) - \sum_{j=1}^{I} \tau_j m_j^2 
\]

where the matrix \( Q \) is linear in the Feynman parameters \( \{\tau_k\} \). The integrations over the two loop momenta \( k_l \), then over the parameter \( \alpha \) yield\(^2\)

\[
(2) I^\mu_N \left( \{p_j\}; \mathcal{T} \right) = \int_{(\mathbb{R}^+)^I} \left[ \prod_{k=1}^{I} d\tau_k \right] \delta \left( 1 - \sum_{l=1}^{I} \tau_l \right) \left[ \det(A) \right]^{-\frac{1}{2}} [\mathcal{D}(\{\tau_k\}) - i\lambda]^{n-l} 
\]

where the term \( \mathcal{D} \) is given by:

\[
\mathcal{D}(\{\tau_k\}) = \left\{ \sum_{i,j=1}^{2} [A^{-1}]_{ij} (r_i \cdot r_j) \right\} - C
\]

\(^2\)up to a constant factor \((-1)^{I+1} (4\pi)^{-n} \Gamma(I - n)\) irrelevant here, which will be dropped in the following.
The elements of the matrix $A$ first term of eq. (4) is equal to the combination of internal lines common to the two overlapping loops.

Let us partition the set of Feynman parameter labels $\{\tau_k\}$ into three subsets $S_j$ and define three auxiliary parameters $\rho_j$, $j = 1, 2, 3$ accordingly as follows: i) $S_1$ contains the labels of the internal lines involving only $k_1$ not $k_2$, to $S_1$ is associated $\rho_1 \equiv \sum_{i \in S_1} \tau_i$; ii) $S_2$ contains the labels of internal lines involving only $k_2$ not $k_1$, to $S_2$ is associated $\rho_2 \equiv \sum_{i \in S_2} \tau_i$; iii) $S_3$ contains the labels of internal lines common to the two overlapping loops. Each of these lines involves the same combination $k_1 + k_2$, so that the matrix element $A_{12}$ weighting the scalar product $(k_1 \cdot k_2)$ in the first term of eq. (4) is equal to the combination $\rho_3 \equiv \sum_{i \in S_3} \tau_i$. The $\rho_j$’s thus fulfill the constraint

$$\rho_1 + \rho_2 + \rho_3 = \sum_{j=1}^{I} \tau_j = 1$$

The elements of the matrix $A$ read:

$$A_{12} = \rho_3, \quad A_{11} = \rho_1 + \rho_3, \quad \text{and} \quad A_{22} = \rho_2 + \rho_3$$

---

3It could alternatively involve $k_1 - k_2$ in every internal line common to the two overlapping loops, depending on the convention adopted for the orientations of the loop momenta.
Hence:
\[
\det(A) = \rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1
\]  
\[\text{(12)}\]

The determinant \(\det(A)\) is clearly non-negative. Let \(|S_j|\) be the number of elements of \(S_j\), with \(|S_1| + |S_2| + |S_3| = I\). Let us introduce \(|S_j|\) parameters \(u_{k_j}\) with \(k_j \in S_j\) so as to reparametrise the \(\tau_{k_j}\) summing up into \(\rho_j\) as follows:
\[
\tau_{k_j} = \rho_j u_{k_j} \quad \text{with the constraint} \quad \sum_{k_j \in S_j} u_{k_j} = 1
\]  
\[\text{(13)}\]

Accordingly the reparametrised integration measure takes the following factorised form:
\[
\left[ \prod_{j=1}^{I} d\tau_j \right] \delta \left( 1 - \sum_{i=1}^{I} \tau_i \right) = \prod_{k=1}^{3} \left\{ \rho_k \prod_{j_k \in S_k}^{\vert S_k \vert - 1} \int du_{j_k} \delta \left( 1 - \sum_{l \in S_k} u_l \right) \right\} \delta \left( 1 - \sum_{i=1}^{3} \rho_i \right)
\]  
\[\text{(14)}\]

With this reparametrisation, the elements of the \(A\) matrix depend only on the parameters \(\rho_j\) and on none of the \(u_i\)'s, so do Cof[\(A\)] and \(\det(A)\). In \(\mathcal{F}\), the dependence in the \(u_i\)'s enters through the factors \((\tau_i, \tau_j)\), quadratically, and through the term \(\mathcal{C}\), linearly. The term \(\mathcal{F}\) may thus be seen as a polynomial of second degree in the \(u_i\)'s and can thus be interpreted as building up the integrand of a “generalised” one-loop function represented as a Feynman integral over the \(u_i\)'s. The integral representation of the two-loop diagram \(\{2\} I^p_N(\{p_j\}; \mathcal{T})\) can thus be recast in the following form:
\[
\{2\} I^p_N(\{p_j\}; \mathcal{T}) = \int_{(\mathbb{R}^+)^3} \left[ \prod_{k=1}^{3} \int d\rho_k \rho_k^{\vert S_k \vert - 1} \right] \delta \left( 1 - \sum_{i=1}^{3} \rho_i \right) \left[ \rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1 \right]^{I - \frac{3n}{2}} \{1\} I^{n'}_{N'}
\]  
\[\text{(15)}\]

where we have introduced
\[
\{1\} I^{n'}_{N'} = \int_{(\mathbb{R}^+)^I} \left[ \prod_{k=1}^{3} \int \prod_{j_k \in S_k} du_{j_k} \delta \left( 1 - \sum_{l \in S_k} u_l \right) \right] \left[ \mathcal{F}(\{u_k\}, \{\rho_l\}) - i \lambda \right]^{n-I}
\]  
\[\text{(16)}\]

with \(\mathcal{F}(\{u_k\}, \{\rho_l\}) = \mathcal{F}(\{\tau_i(\{u_k\}, \{\rho_l\})\})\) and we have set \(N' = I - 2\) and \(n' = 2(n - 2)\). The reparametrisation of \(\{2\} I^p_N(\{p_j\}; \mathcal{T})\) according to eqs. (15), (16) has already been used in the literature \[\text{(12)}\text{[15]}\] in order to perform the integration over all Feynman parameters fully numerically. We alternatively wish to advocate here the separate identification of \(\{1\} I^{n'}_{N'}\) in eq. (16) with \(n - I = -N' + n'/2\) as a \(N'\)-point function of “generalised one-loop type” in \(n'\) dimensions, and the possibility to compute \(\{1\} I^{n'}_{N'}, \text{analytically}\).

The above qualitative “generalised one-loop type” refers to two kinds of generalisations.
1. After integrating over three of the \(u_i\)’s in order to eliminate the \(\delta(1 - \sum_{l \in S_k} u_l)\)-constraints, the effective kinematics of the “generalised” one-loop \(N'\)-point function in \(n'\) dimensions is encoded in a \((I - 3) \times (I - 3)\) matrix \(G = G(\{p_j\}, \{\rho_l\})\), a column \((I - 3)\)-vector \(V = V(\{p_j\}, \{\rho_l\})\) and a scalar function \(C = C(\{p_j\}, \{\rho_l\})\), all of which functions of the external momenta \(\{p_j\}\) and of the
integration variables \( \{ \rho_k \} \) seen as external parameters. The matrix \( G \) somehow plays the role of an “effective Gram matrix”, with which it shares a few features, namely it is real and symmetric and it does not depend on the (possibly complex) internal masses. Although not made explicit, \( V \) and \( C \) depend on the internal masses. Let us note that this effective kinematics of the “generalised” one-loop function depends on the \( \rho_j \) seen as “external” parameters beside the external momenta \( p_k \)’s, and that it may span a larger parameter space than the one involved in standard one-loop \( N’ \)-point functions involved in collider processes at one loop.

2) Unlike for the standard one-loop function, the integration domain of the parameters \( u_k \)’s is not the usual \((I-3)\)-simplex defined by \( \Sigma_{(I-3)} = \{ u_k \geq 0, k = 1, \cdots, I-3 \mid \sum_{k=1}^{I-3} u_k = 1 \} \) but instead the polysimplicial set \( \# \Sigma_{(|S_1|−1)} \times \Sigma_{(|S_2|−1)} \times \Sigma_{(|S_3|−1)} \); The quantity \( \mathcal{F} \) formally reads:

\[
\mathcal{F} = U^T \cdot G \cdot U - 2 \cdot V^T \cdot U - C
\]

(17)

where \( U \) is the column \((I−3)\)-vector gathering the yet unintegrated \((I−3)\) variables \( u_k \) parametrising the polysimplicial integration domain \( \Sigma_{(|S_1|−1)} \times \Sigma_{(|S_2|−1)} \times \Sigma_{(|S_3|−1)} \). The two-dimensional integral representation and the corresponding weighting function \( W \) advocated in eq. (1) are readily obtained from eq. (15) using a reparametrisation of the form \( \rho_1 = \rho \xi, \rho_2 = \rho (1-\xi), \rho_3 = (1-\rho) \)

with \( 0 \leq \rho, \xi \leq 1 \). A few illustrative examples are provided in the sections 4 and 5. More details will be provided in subsequent papers.

3 Decomposition of polytopes into simplices

Since the integration domain spanned by the \( u_i \)’s is not the simplex anymore, one might fear that the above functions of “generalised one-loop type” be not computable analytically because the standard one-loop methods could not be used. This fear is groundless, as general theorems on triangulation [16] ensure that any \((I−3)\)-polytope can be partitioned into \((I−3)\)-simplices; furthermore the polytope \( V = \Sigma_{|S_1|−1} \times \Sigma_{|S_2|−1} \times \Sigma_{|S_3|−1} \) spanned by the \( u_i \)’s being convex, the imaginary part \( \text{Im}(\mathcal{F}−i\lambda) \) keeps a constant (minus) sign over \( V \) even with internal general complex masses, hence it does also over each of the simplices involved in such a partition.

3.1 Two-loop three-point functions with three-leg vertices

In this section, we explicitly build such partitions in the cases of two-loop three-point functions involving three-leg vertices. In these cases, \( V \) being three-dimensional, such a decomposition can thus be readily performed and visualised. Anticipating on the next section, the domains spanned by the \( u_i \)’s for, respectively, the planar and non planar two-loop three-point functions are the wedge \( \Pi_{(3)}^{(0)} = \{ 0 \leq u_1, u_2, u_3 \leq 1; u_1 + u_2 \leq 1 \} \) and the cube \( K_{(3)}^{(3)} = \{ 0 \leq u_1, u_2, u_3 \leq 1 \} \). The cube \( K_{(3)}^{(3)} \), depicted in fig. 1, can be partitioned into the two wedges: \( \Pi_{(3)}^{(1)} = \{ 0 \leq u_1, u_2, u_3 \leq 1; u_1 + u_2 \geq 1 \} \)

and \( \Pi_{(3)}^{(0)} \) by the plane \( \{ u_1 + u_2 = 1 \} \). The mapping \( (u_1, u_2, u_3) \rightarrow (1−u_2, 1−u_1, u_3) \) exchanges \( \Pi_{(3)}^{(1)} \) and \( \Pi_{(3)}^{(0)} \).

\[\text{The polysimplicial set depends on the topology } \mathcal{T} \text{ of the two-loop diagram considered. It is understood that, in case some of the } |S_j| \text{ equals 1, the corresponding trivial set factor } \Sigma_{(|S_j|−1)} \text{ shall be omitted.}\]
Let us focus on $\Pi^{(0)}_3$ and partition it into the simplex $\Sigma^{(3)} = \{0 \leq u_1, u_2, u_3 \leq 1; u_1 + u_2 + u_3 \leq 1\}$ and the leftover volume denoted by $\Xi^{(3)}$. The latter can be partitioned into the following two tetrahedra: $\Theta_D^{ABH} = \{0 \leq u_1, u_2, u_3 \leq 1; u_1 + u_2 \leq 1; u_2 + u_3 \geq 1\}$ and $\Theta_F^{ABH} = \{0 \leq u_1, u_2, u_3 \leq 1; u_1 + u_2 \leq 1; u_2 + u_3 \leq 1; u_1 + u_2 + u_3 \geq 1\}$, by the plane $\{u_2 + u_3 = 1\}$. These two tetrahedra are not isometric to $\Sigma^{(3)}$ because, for example:

$$
\begin{align*}
l[DA] &= l[DH] = l[AB] = 1 \\
l[AH] &= l[DB] = \frac{1}{\sqrt{3}}
\end{align*}
$$

(18)

where $l[AB]$ is the distance between the points $A$ and $B$. Nevertheless these two tetrahedra can be mapped onto $\Sigma^{(3)}$ by affine transformations. Indeed, for any point $M$ in the tetrahedron $\Theta_D^{ABH}$, the vector $\overrightarrow{OM}$ can be written as:

$$
\overrightarrow{OM} = \overrightarrow{OD} + \overrightarrow{DM}
$$

(19)

Let us define the following basis of unit vectors: $\vec{e}_1 = \overrightarrow{OF}, \vec{e}_2 = \overrightarrow{OH}$ and $\vec{e}_3 = \overrightarrow{OA}$. The vectors $\overrightarrow{OD}$ and $\overrightarrow{DM}$ read $\overrightarrow{OD} = \sum_{i=1}^{3} u_i \vec{e}_i$ and $\overrightarrow{DM} = \vec{e}_2 + \vec{e}_3 \equiv u_{D2}\vec{e}_2 + u_{D3}\vec{e}_3$. The vector $\overrightarrow{DM}$ can be written $\overrightarrow{DM} = \sum_{i=1}^{3} y_i \vec{a}_i$ with $\vec{a}_1 \equiv \overrightarrow{DA} = -\vec{e}_2$, $\vec{a}_2 \equiv \overrightarrow{DB} = \vec{e}_1 - \vec{e}_2$ and $\vec{a}_3 \equiv \overrightarrow{DH} = -\vec{e}_3$. Let us define the $3 \times 3$ matrix $N$ such that $\vec{a}_i = \sum_{j=1}^{3} N_{ij} \vec{e}_j$. In terms of the $u_i$’s eq. (19) translates into:

$$
U = U_D + N^T \cdot Y
$$

(20)

with:

$$
U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad U_D = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

(21)

Using eq. (20), the constraints on the $u_i$’s of a point spanning the tetrahedron $\Theta_D^{ABH}$, which read $\{0 \leq u_1, u_2, u_3 \leq 1; u_1 + u_2 \leq 1; u_2 + u_3 \geq 1\}$, translate into $\{0 \leq y_1, y_2, y_3 \leq 1; 0 \leq y_1 + y_2 + y_3 \leq 1\}$ in terms of the $y_i$’s i.e. $Y$ spans $\Sigma^{(3)}$. Likewise, for the other tetrahedron $\Theta_F^{ABH}$ the constraints read $\{0 \leq u_1, u_2, u_3 \leq 1; u_1 + u_2 \leq 1; u_2 + u_3 \leq 1; u_1 + u_2 + u_3 \geq 1\}$. Performing the reparametrisation

$$
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 - y_1' - y_2' \\ y_2' \\ 1 - y_2' - y_3' \end{pmatrix}
$$

(22)
the column vector $Y'$ is readily seen to span $\Sigma_{(3)}$ too.

Figure 2: Decomposition of the wedge $\Pi_{(3)}^{(0)}$ into the simplex $\Sigma_{(3)}$ and two tetrahedra $\Theta_{ABH}^D$ and $\Theta_{ABH}^F$.

Under an affine transformation of the type (20): $U = U_0 + N^T \cdot Y$, the second order polynomial $\mathcal{F}(U)$ can be rewritten as:

$$\mathcal{F}(U) = U^T \cdot G \cdot U + 2V^T \cdot U + C$$

$$= Y^T \cdot (N \cdot G \cdot N^T) \cdot Y + 2[(U_0^T \cdot G + V^T) \cdot N^T] \cdot Y + U_0^T \cdot G \cdot U_0 + 2V^T \cdot U_0 + C$$

$$\equiv Y^T \cdot G' \cdot Y + 2V'^T \cdot Y + C'$$

The new variables span $\Sigma_{(3)}$ in all contributions of the partition, and the second order polynomials take similar forms yet with Gram matrices $G^{(t)}$, vectors $V^{(t)}$ and constants $C^{(t)}$ depending on the parameters of the reparametrisations $\{U_0, N\}$ and proper to each contribution.

### 3.2 Other types of two-loop functions

For two-loop four point functions with three-leg vertices, and other cases for which $V$ is four-dimensional or more, things are harder to visualise. Nevertheless, as already mentioned general theorems [16] on triangulation ensure that $V$ can be partitioned into simplicies — this is the higher dimensional generalisation of the property whereby any polygon can be partitioned into triangles — and each of these $(I - 3)$-simplices can be further mapped onto the normalised $(I - 3)$-simplex $\Sigma_{(I-3)}$ by an affine transformation of the $u_i$ coordinates. This proves that standard methods used to analytically compute ordinary one-loop $N'$-point functions can also be used to analytically compute $N'$-point functions of “generalised one-loop type”\(^5\). So long for the principles.

From a practical point of view, several triangulation methods are available, and the number of simplices as well as their sizes and shapes depend on the method used. The integration over each simplex will generate dilogarithms, some of which will cancel against other dilogarithms coming from other simplicies. Accordingly a question arises and practically matters regarding computational efficiency: what is the method which optimises the number of simplices so as to minimise the number of dilogarithms?

\(^5\)Strictly speaking, for UV or IR/collinear divergent cases, the expansion around $n' - 4$ of the standard method for one-loop computations has to be pushed further in order to keep all the relevant terms.
We will not elaborate more on this point in this article because we tackled the analytic computation and the issue of taming the proliferation of dilogarithms in an alternative way. In ref. [17] we have developed an alternative method to integrate over the \( u_i \)'s which makes the integration over each Feynman parameter \( u_i \) trivial, by representing the integrand as a derivative with respect to this Feynman parameter. For this purpose we use a “Stokes-type” identity and an integral representation requiring the introduction of an extra variable ranging from 0 to \(+\infty\) for each \( u_i \). This handling is iterated until no more Feynman parameter remains to be integrated over. These extra integral representations are then easily undone later on — this trick plays a role similar to a “catalyst” in chemistry.

The domain \([0, +\infty[^{I-3}\) on which the auxiliary variables are integrated over is independent of the initial volume spanned by the \( u_i \)'s. For example, for a four-point “generalised one-loop function” it is the principal octant. Thus, for a given number of external legs, whatever the volume spanned by the \( u_i \)'s is, the domain of the leftover integrations over the new variables are always the same as in the case of the genuine one-loop, only the number of terms to integrate differs, and for all these terms the integration is of the same type. In other words, the method developed in ref. [17] and applied to the usual one-loop case where the volume of integration is a simplex merely for sake of illustration, can be used for more general integration domains, the only change will be the number of terms in the integrand. This alternative method thus provides another practical way to analytically compute the “generalised one-loop functions”. The explicit application of this method to the computation of the “generalised one-loop functions” and its optimisation to tame the number of dilogarithms generated will be further elaborated in subsequent publications.

4 An example of scalar two-loop three-point topology

For illustrative purpose let us consider the non-planar diagram drawn on fig. 3. With \( N = 3 \), \( I = 6 \) and \( n = 4 \) this diagram has the following parametric integral representation:

\[
(2) I_3^4 \{\{p_j\}; T\} = \int_{(\mathbb{R}^+)^{6}} \prod_{k=1}^{6} d\tau_k \left[ \delta \left( 1 - \sum_{l=1}^{6} \tau_l \right) \right] \left[ \mathcal{F}\{\{\tau_k\}\} - i \lambda \right]^{-2}
\]  

(24)

The factor \( \mathcal{F} \), whose cumbersome expression is not made explicit here, involves the matrix \( A \) given by

\[
A = \begin{bmatrix}
\tau_1 + \tau_2 + \tau_5 + \tau_6 & \tau_5 + \tau_6 \\
\tau_5 + \tau_6 & \tau_3 + \tau_4 + \tau_5 + \tau_6
\end{bmatrix}
\]

(25)

Let us reparametrise the Feynman parameters \( \tau_k \)'s as follows:

\[
\rho_1 = \tau_1 + \tau_2, \quad \rho_2 = \tau_3 + \tau_4 \quad \text{and} \quad \rho_3 = \tau_5 + \tau_6
\]

(26)

further with

\[
\begin{align*}
\tau_1 &= \rho_1 u_1, \\
\tau_2 &= \rho_1 (1 - u_1), \\
\tau_3 &= \rho_2 (1 - u_2), \\
\tau_4 &= \rho_2 u_2, \\
\tau_5 &= \rho_3 u_3, \\
\tau_6 &= \rho_3 (1 - u_3)
\end{align*}
\]

(27)

\[^{6}\text{In this case — and similarly for the planar topology — } N - \frac{3}{2} n = 0 \text{ thus the factor } \det(A) \text{ present in eq. (9) does not appear in eq. (24).}\]
Figure 3: The diagram picturing the two-loop three-point function with a non planar topology.

Let us note $k_1$ and $k_2$ the four-momenta running into the loops, the internal four-momenta of fig. 3 are given by:

$$
q_1 = k_1 - p_1, \quad q_2 = k_1, \quad q_3 = k_2,
q_4 = k_2 - p_2, \quad q_5 = k_1 + k_2 - p_1 - p_2, \quad q_6 = k_1 + k_2
$$

(28)

In addition, a mass $m_i$ is associated to each internal line with four-momentum $q_i$. Each variable $u_1, u_2$ and $u_3$ spans the interval $\Sigma_{(1)} = [0, 1]$.

The matrix $G$ reads:

$$
G_{11} = \rho_1^2 (\rho_2 + \rho_3) p_1^2 \quad G_{12} = \frac{1}{2} \rho_1 \rho_2 \rho_3 (p_1^2 + p_2^2 - p_3^2) \quad G_{13} = \frac{1}{2} \rho_1 \rho_2 \rho_3 (p_3^2 - p_2^2 + p_1^2)
$$

$$
G_{22} = \rho_2^2 (\rho_1 + \rho_3) p_2^2 \quad G_{23} = \frac{1}{2} \rho_1 \rho_2 \rho_3 (p_2^2 + p_3^2 - p_1^2) \quad G_{33} = \rho_3^2 (\rho_1 + \rho_2) p_3^2
$$

(29)

whereas the vector $V$ reads:

$$
V_1 = \frac{1}{2} \rho_1 (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1) \left[ p_1^2 + m_2^2 - m_1^2 \right]
$$

$$
V_2 = \frac{1}{2} \rho_2 (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1) \left[ p_2^2 + m_3^2 - m_1^2 \right]
$$

$$
V_3 = \frac{1}{2} \rho_3 (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1) \left[ p_3^2 + m_6^2 - m_1^2 \right]
$$

(30)

and $C$ is given by:

$$
C = -(\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1) (\rho_1 m_2^2 + \rho_2 m_3^2 + \rho_3 m_6^2)
$$

(31)

The explicit expression for $\mathcal{F}$ then follows from eq. (17). We thereby get the advocated integral representation

$$
\mathcal{F}^{(2)} \left( \{ p_j \}; \mathcal{T} \right) = \int_{(\mathbb{R}^{+})^3} dp_1 dp_2 dp_3 \rho_1 \rho_2 \rho_3 \delta \left( 1 - \sum_{l=1}^{3} \rho_l \right) \mathcal{F}^{(1)}_4
$$

(32)
involving the four-point function of “generalised one-loop type” given by:

\[ \bar{I}_4^4 = \int_0^1 du_1 \int_0^1 du_2 \int_0^1 du_3 \left[ F(u_k, \{\rho_l\}) - i \lambda \right]^{-2} \]  (33)

The change of variables \( \rho_1 = \rho \xi, \rho_2 = \rho (1 - \xi), \rho_3 = (1 - \rho) \) amounts in eq. \( \text{(32)} \) to the replacement

\[ \int_{(R^+)^3} d\rho_1 d\rho_2 d\rho_3 \rho_1 \rho_2 \rho_3 \delta \left( 1 - \sum_{l=1}^{3} \rho_l \right) \times \rightarrow \int_0^1 d\rho \int_0^1 d\xi W(\rho, \xi) \times \]

where the weighting function \( W(\rho, \xi) \) is given by:

\[ W(\rho, \xi) = \rho^3 (1 - \rho) \xi^3 (1 - \xi) \]  (34)

We chose to illustrate our purpose with the three-point non-planar topology whose corresponding integral is usually considered more touchy to compute than for the planar topology with the same three-leg type vertices. The latter can all be worked out in a very similar way, and the domain of integration over the parameters \( u_1, u_2, u_3 \) is found to be the cylinder with triangular cross section \( \Sigma_{(2)} \times [0, 1] \) where \( \Sigma_{(2)} = \{0 \leq u_1, u_2, u_1 + u_2 \leq 1\} \) instead of the unit cube.

5 An example of scalar two-loop four-point topology

Let us now consider the non-planar diagram drawn on fig. 4. With \( N = 4, I = 7 \) and \( n = 4 \) this diagram has the following parametric integral representation:

\[ I^4_4 \left( \{p_j\}; T \right) = \int_{(R^+)^7} \left[ \prod_{k=1}^{7} d\tau_k \right] \delta \left( 1 - \sum_{l=1}^{7} \tau_l \right) \left[ \det(A) \right] \left[ F(\{\tau_k\}) - i \lambda \right]^{-3} \]  (35)

where the matrix \( A \) is given by

\[ A = \begin{bmatrix} \tau_1 + \tau_2 + \tau_3 + \tau_6 + \tau_7 & \tau_6 + \tau_7 \\ \tau_6 + \tau_7 & \tau_4 + \tau_5 + \tau_6 + \tau_7 \end{bmatrix} \]  (36)

The cumbersome expression of \( F \) is not made explicit here. Let us reparametrise the Feynman parameters \( \tau_k \)’s as follows:

\[ \rho_1 = \tau_1 + \tau_2 + \tau_3, \quad \rho_2 = \tau_4 + \tau_5 \quad \text{and} \quad \rho_3 = \tau_6 + \tau_7 \]  (37)

further with

\[ \tau_1 = \rho_1 u_1, \quad \tau_2 = \rho_1 u_2, \quad \tau_3 = \rho_1 (1 - u_1 - u_2), \]
\[ \tau_4 = \rho_2 (1 - u_3), \quad \tau_5 = \rho_2 u_3, \]
\[ \tau_6 = \rho_3 u_4, \quad \tau_7 = \rho_3 (1 - u_4) \]  (38)

Variables \( u_1, u_2 \) span the two-simplex \( \Sigma_{(2)} = \{0 \leq u_2, u_1 + u_2 \leq 1\} \) whereas each variable \( u_3 \) and \( u_4 \) spans the interval \( \Sigma_{(1)} = [0, 1] \).
Figure 4: The box picturing the two-loop four-point function with a non planar topology.

Again, let us note by $k_1$ and $k_2$ the loop momenta, the $q_i$’s are given by:

\[
\begin{align*}
q_1 &= k_1 - p_1 - p_2, & q_2 &= k_1 - p_2, & q_3 &= k_1, \\
q_4 &= k_2, & q_5 &= k_2 - p_3, & q_6 &= k_1 + k_2 + p_4 \\
q_7 &= k_1 + k_2
\end{align*}
\]  (39)

To each internal line with four-momentum $q_i$ is associated a mass $m_i$. Defining $s = (p_1 + p_2)^2$, $t = (p_1 + p_4)^2$, $u = (p_1 + p_3)^2$, the matrix $G$ reads:

\[
\begin{align*}
G_{11} &= \rho_1^2 (\rho_2 + \rho_3) s \\
G_{12} &= \frac{1}{2} \rho_1^2 (\rho_2 + \rho_3) (s - p_1^2 + p_2^2) \\
G_{13} &= \frac{1}{2} \rho_1 \rho_2 \rho_3 (s + p_3^2 - p_4^2) \\
G_{14} &= \frac{1}{2} \rho_1 \rho_2 \rho_3 (s - p_3^2 + p_4^2) \\
G_{22} &= \rho_1^2 (\rho_2 + \rho_3) p_2^2 \\
G_{23} &= \frac{1}{2} \rho_1 \rho_2 \rho_3 (p_2^2 + p_3^2 - t) \\
G_{24} &= \frac{1}{2} \rho_1 \rho_2 \rho_3 (p_2^2 + p_4^2 - u) \\
G_{33} &= \rho_2^2 (\rho_1 + \rho_3) p_3^2 \\
G_{34} &= \frac{1}{2} \rho_1 \rho_2 \rho_3 (p_3^2 + p_4^2 - s) \\
G_{44} &= \rho_3^2 (\rho_1 + \rho_2) p_4^2
\end{align*}
\]  (40)

whereas the vector $V$ reads:

\[
\begin{align*}
V_1 &= \frac{1}{2} \rho_1 (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1) \left[ s + m_3^2 - m_1^2 \right] \\
V_2 &= \frac{1}{2} \rho_1 (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1) \left[ p_2^2 + m_3^2 - m_2^2 \right] \\
V_3 &= \frac{1}{2} \rho_2 (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1) \left[ p_3^2 + m_1^2 - m_3^2 \right] \\
V_4 &= \frac{1}{2} \rho_3 (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1) \left[ p_4^2 + m_7^2 - m_5^2 \right]
\end{align*}
\]  (41)

and $C$ is given by:

\[
C = -(\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1) (\rho_1 m_3^2 + \rho_2 m_4^2 + \rho_3 m_7^2)
\]  (42)
The explicit expression for $\mathcal{F}$ then follows from eq. [17]. We thereby get the advocated integral representation

$$(2) I_4^4 (\{p_j\}; \mathcal{T}) = \int_{\mathbb{R}^3} d\rho_1 d\rho_2 d\rho_3 \rho_1 \rho_2 \rho_3 \delta \left(1 - \sum_{l=1}^{3} \rho_l\right) \left[\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1\right] (1) I_5^4$$

involving the five-point function of “generalised one-loop type” given by:

$$(1) I_5^4 = \int_0^1 du_1 \int_0^1 du_2 \int_{\Sigma_2} du_3 du_4 \left[\mathcal{F}(\{u_k\}, \{\rho_l\}) - i \lambda\right]^{-3}$$

The change of variables $\rho_1 = \rho \xi$, $\rho_2 = \rho (1 - \xi)$, $\rho_3 = (1 - \rho)$ amounts in eq. (43) to the replacement

$$\int_{\mathbb{R}^3} d\rho_1 d\rho_2 d\rho_3 \rho_1 \rho_2 \rho_3 \delta \left(1 - \sum_{l=1}^{3} \rho_l\right) \left[\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1\right] \times \rightarrow \int_0^1 d\rho \int_0^1 d\xi W(\rho, \xi) \times$$

where the weighting function $W(\rho, \xi)$ is given by:

$$W(\rho, \xi) = \rho^4 (1 - \rho)^2 \xi (1 - \xi) \left[(1 - \rho) + \rho \xi (1 - \xi)\right]$$

Here again the four-point function with planar topology with the same three-leg type vertices can be worked out in a quite similar way.

In the two above examples, the term $\det(A)$ noticeably factorises in the expressions of the quantities $V$ and $C$. This turns out to be a general feature at least for the type of three-leg vertices considered. More precisely, it can be shown that a parametrisation can always be found for which this property holds. A general proof is given in Appendix A for any $N$-point two-loop non planar topology, and a very similar proof holds true also for any planar topology with this type of vertex. This feature makes the discussion of both kinematic and fake singularities simpler and more transparent. In particular the Landau conditions to be fulfilled to encounter kinematic singularities take a simple form, whereas this parametrisation gives handles to circumvent possible numerical instabilities which might be induced by fake singularities. These issues will be more thoroughly discussed in a future publication.

### 6 Outlook

We do acknowledge that a long way shall still be scouted out to extend it to full-fledged two-loop tensor integrals appearing in general gauge theories with fermions and/or derivative couplings etc. covering the bestiary of all topologies and cases which appear in the general two-loop class of interest for precision collider physics. This marathon will not be undertaken any further in this account which intends to be a first step toward the completion of such a programme. Taking for granted that the approach advocated in this article already applies to a collection of relevant cases, a further issue consists in the analytical calculation of the above-coined “$N'$-point functions of generalised one-loop type” in closed form. As seen in sec. [2] the latter are generalised in two respects with respect to “ordinary” one-loop functions involved in collider processes at one loop. Firstly, the kinematics involved in the computation depends on the extra integration variables $\rho_j$’s or equivalently
\(\rho\) and \(\xi\) seen here as external parameters and may thus span a wider kinematical phase space than ordinarily met in genuine one-loop cases. Secondly, the \(N'\)-point functions of “generalised one-loop-type” are provided by Feynman-type parametric integrals over domains differing from the ordinary \((N' - 1)\)-simplex. Although long-tested standard techniques developed for the genuine one-loop case might be customised to treat the new ones at hand as shown in sec. 3, the above two issues motivate the development of a novel approach which tackles both these issues in a systematic and straightforward way while computing the “generalised one-loop type functions”. The presentation of such an approach and the exploration of its features is the subject of the publications \[17\]–\[19\].

More precisely, the method tailored for the computation of the “generalised one-loop functions” is implemented for the usual one-loop case with arbitrary kinematics for the real mass case in ref. \[17\], for the case of general complex masses in ref. \[18\], and lastly, for the IR/collinear divergent cases in ref. \[19\].

The extension of the strategy advocated above to more general cases will be elaborated in subsequent articles but let us try to discuss it more precisely. The complexity comes from the growing number of external legs as in the one-loop case but is also related to the number of internal legs, indeed the number of effective external legs for the underlying “generalised one-loop function” is the number of internal legs of the two-loop diagram minus 2. So the strategy will be to apply firstly our method to simple topologies, working with fixed number of external legs (two or three) and considering topologies with an increasing number of internal lines. Then, we will move to more complicated cases such as two-loop four-point functions involving “generalised one-loop functions” with up to five external legs. All what have been discussed so far is for scalar theories, in real life, we have to face gauge theories which lead to a “dressing” of the numerator because of spin 1/2 particles and derivative vertices. The method applied to these cases will involve “generalised one-loop functions” with a non-trivial numerator. Although reduction technics in principle apply to “generalised one-loop functions” because they are related to the invariance of the one-loop integrals under a shift of the internal 4-momentum running into the loop, their implementations in our case will be postponed after the completion of the scalar cases. There are ambiguities in our algorithm related to the parametrisation of the internal 4-momenta and also to the choice of the \(u\) parameters. It has been shown in appendix \[A\] that for some topologies \(\det(A)\) factorises from \(V\) and \(C\) reducing the complexity of these terms. Although this property does not remove the ambiguities, this is clearly a good guideline to follow. Nevertheless, the choice of a fixed algorithm for all the topologies is premature at this point because the experiences gained with the computation of different topologies will mature it. The UV divergent case will not cause problems because our method works also if the space-time dimension is away from 4 as shown in ref. \[19\]. In addition, by power counting, it is easy to show, for scalar theories having three- and four-leg vertices, that the number of internal lines for the potential UV divergent diagrams is 4 thus the “generalised one-loop functions” to be considered are two-point functions.

**In memoriam**

This work was initiated by Prof. Shimizu after a visit to LAPTh. He explained to us his ideas about the numerical computation of scalar two-loop three- and four-point functions. He shared his notes partly in English, partly in Japanese with us and he encouraged us to push forward the project presented here combining an analytical approach for the “generalised one-loop functions”
with a numerical computation of the left-over double parametric integration. J.-Ph. G. would like to thank Shimizu-sensei for giving him a taste of Japanese culture and for his kindness.

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A Factorisation of $\det(A)$ in $V$ and $C$

Different topologies for a general $N$-point two-loop diagram are discussed in the frame of a scalar theory with three-leg vertices. The first topology studied will be treated in details, the main formulae will be set and the reason of this factorisation will be explained. Then, the other topologies will be discussed succinctly.

Let us consider first the following diagram with a non planar topology depicted in fig. 5. The $n$-momentum carried by each internal line $j = 1, \ldots, N+3$ is labelled by $q_j$, which can be parametrised in a general way as $q_j = \hat{k}_j + \hat{r}_j$ where $\hat{r}_j$ is some shift depending on the external momenta $p_l$’s whereas $\hat{k}_j$ is a linear combination of the loop momenta $k_1$ and $k_2$. Let us hereby specify these linear combinations as:

$$
\hat{k}_1 = \hat{k}_2 = \ldots = \hat{k}_{i+1} \equiv k_1 \\
\hat{k}_{i+2} = \hat{k}_{i+3} = \ldots = \hat{k}_{N+1} \equiv k_2 \\
\hat{k}_{N+2} = \hat{k}_{N+3} = k_1 + k_2 
$$

Figure 5: diagram picturing a two-loop $N$-point function with a non planar topology.

$^7$In what follows the external momenta $p_j$, $j = 1, \ldots, N$ are assumed to be only constrained by the overall energy-momentum conservation $\sum_{j=1}^{N} p_j = 0$, otherwise arbitrary.
The $\hat{r}_j$'s can be conveniently rewritten as:

\[
\begin{align*}
\hat{r}_1 &= \hat{r}_{i+1} - t_1 \\
\hat{r}_2 &= \hat{r}_{i+1} - t_2 \\
\hat{r}_3 &= \hat{r}_{i+1} - t_3 \\
& \quad \vdots \\
\hat{r}_i &= \hat{r}_{i+1} - t_i \\
\hat{r}_{N+2} &= \hat{r}_{i+1} + \hat{r}_{i+2} - (t_1 + t_{N-1})
\end{align*}
\]

with

\[
\begin{align*}
t_1 &= p_{[1..i]} & t_{i+1} &= p_{[i+1]} \\
t_2 &= p_{[2..i]} & t_{i+2} &= p_{[i+1..i+2]} \\
t_3 &= p_{[3..i]} & t_{i+3} &= p_{[i+1..i+3]} \\
& \quad \vdots \\
t_i &= p_i & t_{N-1} &= p_{[i+1..N-1]}
\end{align*}
\]

where we have introduced the shorthand:

\[p_{[i..j]} \equiv \sum_{k=i}^{j} p_k\]  \(49\)

In eq. (47) $p_N$ has been traded for $-(t_1 + t_{N-1})$ using the overall momentum conservation $p_{[1..N]} = 0$. Energy-momentum conservation at each vertex implies that all the $\hat{r}_j$'s but two can be expressed in terms of two unfixed ones. These two arbitrary $\hat{r}$'s, which we implicitly chose above to be $\hat{r}_{i+1}$ and $\hat{r}_{i+2}$, reflect nothing but the invariance of the Feynman diagram under two independent shifts of the loop momenta $k_1$ and $k_2$ by arbitrary constants. The latter may be parametrised in a general way as:

\[\hat{r}_{i+1} = \sum_{i=1}^{N-1} \alpha_i t_i, \quad \hat{r}_{i+2} = \sum_{i=1}^{N-1} \beta_i t_i\]  \(50\)

This notational preamble being set, let us consider the quantity:

\[
\sum_{j=1}^{N+3} \tau_j \left(q_j - m_j^2\right) = \sum_{j=1}^{N+3} \tau_j \left[\hat{k}_j^2 + 2 (\hat{k}_j \cdot \hat{r}_j) + \hat{r}_j^2 - m_j^2\right] \tag{51}
\]

Using specification (46) we write

\[
\sum_{j=1}^{N+3} \tau_j \hat{k}_j^2 \equiv [k_1 \quad k_2] \cdot A \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}
\]

---

8This choice is not unique: any other choice of arbitrary duo $\hat{r}_j, \hat{r}_k$ associated to distinct loops e.g. with $j \in \{1, \ldots, i+1\}$ and $k \in \{i+2, \ldots, N+1\}$ would also fit the purpose.
This defines the $2 \times 2$ matrix $A$ whose elements are

$$A_{11} = \rho_1 + \rho_3, \quad A_{22} = \rho_2 + \rho_3 \quad \text{and} \quad A_{12} = \rho_3$$

(52)

where we introduced the three parameters $\rho_k, k = 1, 2, 3$ defined by:

$$\rho_k = \sum_{j \in S_k} \tau_j, \quad k = 1, 2, 3$$

(53)

$closely following the discussion in sec. 2. The second term of the r.h.s. of eq. (51) may be recast:

$$\sum_{j=1}^{N+3} \tau_i (\hat{k}_j \cdot \hat{r}_j) \equiv [k_1 \ k_2] \cdot B \cdot \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{N-1} \end{bmatrix}$$

(55)

where $B$ is a $2 \times (N - 1)$ matrix whose Feynman parameter dependent elements are read using eqs. (47) and parametrisation (50):

$$B_{1k} = \begin{cases} - (\tau_1 + \tau_{N+2}) + \alpha_k (\rho_1 + \rho_3) + \beta_k \rho_3 & \text{if } k = 1 \\ - \tau_k + \alpha_k (\rho_1 + \rho_3) + \beta_k \rho_3 & \text{if } 1 < k \leq i \\ \alpha_k (\rho_1 + \rho_3) + \beta_k \rho_3 & \text{if } i < k < N - 1 \\ - \tau_{N+2} + \alpha_k (\rho_1 + \rho_3) + \beta_k \rho_3 & \text{if } k = N - 1 \end{cases}$$

$$B_{2k} = \begin{cases} - \tau_{N+2} + \alpha_k \rho_3 + \beta_k (\rho_2 + \rho_3) & \text{if } k = 1 \\ \alpha_k \rho_3 + \beta_k (\rho_2 + \rho_3) & \text{if } 1 < k \leq i \\ - \tau_{k+2} + \alpha_k \rho_3 + \beta_k (\rho_2 + \rho_3) & \text{if } i < k < N - 1 \\ - (\tau_{N+2} + \tau_{N+1}) + \alpha_k \rho_3 + \beta_k (\rho_2 + \rho_3) & \text{if } k = N - 1 \end{cases}$$

(56)

For a general parametrisation of the Feynman diagram, $\alpha_j$ and $\beta_j$ are not all vanishing, the $B$ matrix thus depends on all the Feynman parameters $\tau_j$. We can clarify this Feynman parameter dependence noting that the matrix $B$ can be recast as follows:

$$B = \overline{B} + A \cdot \Delta$$

(57)

where $\overline{B}$ and $\Delta$ are defined by

$$\Delta = \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{N-2} & \alpha_{N-1} \\ \beta_1 & \beta_2 & \ldots & \beta_{N-2} & \beta_{N-1} \end{bmatrix}$$

(58)

and

$$\overline{B}_{1k} = \begin{cases} - (\tau_1 + \tau_{N+2}) & \text{if } k = 1 \\ - \tau_k & \text{if } 1 < k \leq i \\ 0 & \text{if } i < k < N - 1 \\ - \tau_{N+2} & \text{if } k = N - 1 \end{cases}$$

$$\overline{B}_{2k} = \begin{cases} - \tau_{N+2} & \text{if } k = 1 \\ 0 & \text{if } 1 < k \leq i \\ - \tau_{k+2} & \text{if } i < k < N - 1 \\ - (\tau_{N+2} + \tau_{N+1}) & \text{if } k = N - 1 \end{cases}$$

(59)
The important property of $\mathcal{B}$ is that it depends neither on $\tau_{i+1}$ nor on $\tau_{i+2}$ nor on $\tau_{N+3}$. Let us note $T$ the $(N - 1)$ column-vector whose elements are $t_1, \cdots, t_{N-1}$, which entered eq. (55). The bracketed term in the r.h.s. of eq. (8) can be re-written as:

$$\begin{bmatrix} r_1 & r_2 \end{bmatrix} \cdot \text{Cof}[A] \cdot \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = (B \cdot T)^T \cdot \text{Cof}[A] \cdot (B \cdot T)$$

$$= (\mathcal{B} \cdot T)^T \cdot \text{Cof}[A] \cdot (\mathcal{B} \cdot T) + \det(A) T^T \cdot [2 \Delta^T \cdot \mathcal{B} + \Delta^T \cdot A \cdot \Delta] \cdot T$$

(60)

The term $C$ in eq. (8) can in its turn be written using eqs. (47) and (50) as

$$C = [\hat{r}_{i+1} \hat{r}_{i+2}] \cdot A \cdot \begin{bmatrix} \hat{r}_{i+1} \\ \hat{r}_{i+2} \end{bmatrix} + 2 [\hat{r}_{i+1} \hat{r}_{i+2}] \cdot (\mathcal{B} \cdot T) + T^T \cdot \Gamma \cdot T - \sum_{j=1}^{I} \tau_j m_j^2$$

$$= T^T \cdot [\Delta^T \cdot A \cdot \Delta + 2 \Delta^T \cdot \mathcal{B} + \Gamma] \cdot T - \sum_{j=1}^{I} \tau_j m_j^2$$

(61)

where $\Gamma$ is a $(N - 1) \times (N - 1)$ matrix read on eq. (47) and given by

$$\Gamma = \begin{bmatrix}
\tau_1 + \tau_{N+2} & 0 & \cdots & 0 & 0 & \cdots & 0 & \tau_{N+2} \\
0 & \tau_2 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \tau_i & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \tau_{i+3} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & \tau_N & 0 \\
\tau_{N+2} & 0 & \cdots & 0 & 0 & \cdots & 0 & \tau_{N+1} + \tau_{N+2}
\end{bmatrix}$$

(62)

As $\mathcal{B}$, the matrix $\Gamma$ depends neither on $\tau_{i+1}$ nor on $\tau_{i+2}$ nor $\tau_{N+3}$. The $\Delta$-dependent terms in eqs. (60) and (61) cancel each other in $\mathcal{F}$, reflecting the independence of the Feynman diagram considered under the arbitrary shifts on loop momenta parametrised by $\hat{r}_{i+1}, \hat{r}_{i+2}$. The quantity $\mathcal{F}$ simplifies into:

$$\mathcal{F} = (\mathcal{B} \cdot T)^T \cdot \text{Cof}[A] \cdot (\mathcal{B} \cdot T) - \det(A) \left[ T^T \cdot \Gamma \cdot T - \sum_{i=1}^{I} \tau_i m_i^2 \right]$$

(63)

Here comes the key point. Since none of the three parameters $\tau_{i+1}$, $\tau_{i+2}$ and $\tau_{N+3}$ enters into the matrices $\mathcal{B}$ and $\Gamma$, the $\tau_j$’s may be conveniently reparametrised in three subsets corresponding respectively to $j \in S_k, k = 1, 2, 3$ defined in eq. (54), introducing $u_j$’s parameters such that the $\mathcal{B}$ matrix be homogeneous of degree 1 in the $u_j$’s, namely

$$\begin{align*}
\tau_k &= \rho_1 u_k \quad \text{for } k \in \{1, \ldots, i\} \\
\tau_{i+1} &= \rho_1 (1 - \sum_{j=1}^{i} u_j) \\
0 &\leq u_k, \sum_{j=1}^{i} u_j \leq 1, \quad k \in \{1, \ldots, i\}
\end{align*}$$

(64)

Had we chosen $\hat{r}_m, \hat{r}_n$ with any $m \in S_1, n \in S_2$ to be arbitrary instead of $\hat{r}_{i+1}, \hat{r}_{i+2}$ cf. footnote s, a similar reparametrisation could be obtained leading to similar features, *mutatis mutandis.*
\[\begin{align*}
\tau_k &= \rho_2 u_{k-2} \quad \text{for } k \in \{i + 3, \ldots, N + 1\} \\
\tau_{i+2} &= \rho_2 (1 - \sum_{j=i+1}^{N-1} u_j) \\
&\quad 0 \leq u_k, \sum_{j=1}^{N-1} u_j \leq 1, \quad k \in \{i + 3, \ldots, N + 1\} \\
\end{align*}\] (65)

\[\begin{align*}
\tau_{N+2} &= \rho_3 u_N \\
\tau_{N+3} &= \rho_3 (1 - u_N) \\
&\quad 0 \leq u_N \leq 1 \\
\end{align*}\] (66)

With this choice, the term $\mathbf{B}^T \cdot \text{Cof}[A] \cdot \mathbf{B}$ is homogeneous of degree 2 in the $u_i$’s and the corresponding term $(\mathbf{B} \cdot T)^T \cdot \text{Cof}[A] \cdot (\mathbf{B} \cdot T)$ in eq. 63 thus contributes only to the $U^T \cdot G \cdot U$ of eq. 17, whereas the other term in eq. 63 contributes to the terms $V$ and $C$ in eq. 17, which both appear to be $\propto \det(A)$. Furthermore the term $T^T \cdot \Gamma \cdot T$ being homogeneous of degree 1 in the $u_i$’s contributes only to $V$ but not to $C$: the $C$ term is thus a mere linear combination of $m_{i+1}^2$, $m_{i+2}^2$ and $m_{N+3}^2$.

Using these results we can proceed further and compute the matrix $G$, the vector $V$ and the scalar $C$ defined in eq. 17. The latter are the algebraic ingredients in terms of which the novel approach advocated in the Outlook and presented in 17–19 naturally proceeds. This will be the purpose of future publications.

Let us move now to the other topologies. For those topologies, once the parametrisation of the internal lines has been chosen, the sequences for the proof of the factorisation are identical to the previous case, so they will not be reproduced here and only the matrices $\mathbf{B}$ will be given.

The other non planar topology\textsuperscript{10} corresponds to the diagram depicted in fig. 6. Using the same

![Diagram](image_url)

Figure 6: diagram picturing a two-loop $N$-point function with a non planar topology.

\textsuperscript{10}Strictly speaking, this kind of topology can be planar in specific cases: for example if $i$ is chosen to be $N - 1$ in fig. 6 or in other words if the internal leg labelled by $N + 1$ connects two adjacent external legs. Note that this is always the case if $N = 3$. 
parametrisation for the $q_i$'s as in the preceding topology studied, the $\hat{k}_i$'s are given by:

\[
\begin{align*}
\hat{k}_1 &= \hat{k}_2 = \ldots = \hat{k}_i \equiv k_1 \\
\hat{k}_{i+1} &= \hat{k}_{i+2} = \ldots = \hat{k}_N \equiv k_2 \\
\hat{k}_{N+1} &= \hat{k}_{N+2} = \hat{k}_{N+3} = k_1 + k_2
\end{align*}
\]  \hspace{1cm} (67)

and the $\hat{r}_i$'s by:

\[
\begin{align*}
\hat{r}_1 &= \hat{r}_i - t_1 \\
\hat{r}_2 &= \hat{r}_i - t_2 \\
\hat{r}_3 &= \hat{r}_i - t_3 \\
&\vdots \\
\hat{r}_{i-1} &= \hat{r}_i - t_{i-1} \\
\hat{r}_{i+1} &= \hat{r}_i + \hat{r}_{i+1} - t_{N-1} \\
\hat{r}_{N+3} &= \hat{r}_i + \hat{r}_{i+1}
\end{align*}
\]  \hspace{1cm} (68)

with

\[
\begin{align*}
t_1 &= p_{[1..i-1]} \\
t_2 &= p_{[2..i-1]} \\
t_3 &= p_{[3..i-1]} \\
&\vdots \\
t_{i-1} &= p_{i-1} \\
t_{N-1} &= p_i
\end{align*}
\]  \hspace{1cm} (69)

The three sets of internal line labels become in this case:

\[
S_1 = \{1, \ldots, i\}, \quad S_2 = \{i+1, \ldots, N\} \quad \text{and} \quad S_3 = \{N+1, N+2, N+3\}
\]  \hspace{1cm} (70)

The 4-momenta $\hat{r}_i$ and $\hat{r}_{i+1}$ can be, in turn, parametrised with the $t_i$, $i = 1 \ldots N - 1$ as in eq. 50.

It is already clear from this point that, since from energy-momentum conservation we achieved to have one internal line for each set $S_i$, $i = 1, 2, 3$ whose $\hat{r}$ does not depend on the $t_i$'s. Thus, because of its definition, the matrix $\overline{B}$ will not depend on the Feynman parameter associated to these three internal lines. Indeed, a simple computation along the same lines as the first example treated shows that:

\[
\overline{B}_{1k} = \begin{cases} 
-(\tau_1 + \tau_{N+2}) & \text{if } k = 1 \\
-\tau_k & \text{if } 1 < k \leq i - 1 \\
0 & \text{if } i < k < N - 2 \\
-\tau_{N+2} & \text{if } k = N - 2 \\
-\tau_{N+1} & \text{if } k = N - 1 \\
\end{cases}
\]  \hspace{1cm} (71)

\[
\overline{B}_{2k} = \begin{cases} 
-\tau_k & \text{if } k = 1 \\
0 & \text{if } 1 < k < i \\
-\tau_{k+2} & \text{if } i \leq k < N - 2 \\
-(\tau_{N+2} + \tau_N) & \text{if } k = N - 2 \\
-\tau_{N+1} & \text{if } k = N - 1 \\
\end{cases}
\]
which confirms that the three Feynman parameters $\tau_i$, $\tau_{i+1}$ and $\tau_{N+3}$ do not appear in the $\mathcal{B}$ matrix. Since these three Feynman parameters belong to the different sets $S_i$, $i = 1, 2, 3$, it is always possible to choose the parameters $u_i$'s in such a way that $\mathcal{B}^T \cdot \text{Cof}[A] \cdot \mathcal{B}$ is homogeneous of degree 2 in these variables. Note that, in this case too, the $\Gamma$ matrix does not depend on $\tau_i$, $\tau_{i+1}$ and $\tau_{N+3}$.

For the third topology depicted in fig. 7, taking the same parametrisation for the $q_i$'s as in the first case studied, these linear combinations are specified as:

$$
\begin{align*}
\hat{k}_1 &= \hat{k}_2 = \ldots = \hat{k}_{i+1} \equiv k_1 \\
\hat{k}_{i+2} &= \hat{k}_{i+3} = \ldots = \hat{k}_{N+2} \equiv k_2 \\
\hat{k}_{N+3} &= k_1 + k_2
\end{align*}
$$

The $\hat{r}_j$'s can be conveniently rewritten as:

$$
\begin{align*}
\hat{r}_1 &= \hat{r}_{i+1} - t_1 & \hat{r}_{i+3} &= \hat{r}_{i+2} - t_{i+1} \\
\hat{r}_2 &= \hat{r}_{i+1} - t_2 & \hat{r}_{i+4} &= \hat{r}_{i+2} - t_{i+2} \\
\hat{r}_3 &= \hat{r}_{i+1} - t_3 & \hat{r}_{i+5} &= \hat{r}_{i+2} - t_{i+3} \\
\vdots & & \vdots \\
\hat{r}_i &= \hat{r}_{i+1} - t_i & \hat{r}_{N+1} &= \hat{r}_{i+2} - t_{N-1} \\
\hat{r}_{N+2} &= \hat{r}_{i+2} + t_1 & \hat{r}_{N+3} &= \hat{r}_{i+1} + \hat{r}_{i+2}
\end{align*}
$$

Figure 7: diagram picturing a two-loop $N$-point function with a planar topology.
with
\[ t_1 = p_{[1..i]} \quad t_{i+1} = p_{i+1} \]
\[ t_2 = p_{[2..i]} \quad t_{i+2} = p_{[i+1..i+2]} \]
\[ t_3 = p_{[3..i]} \quad t_{i+3} = p_{[i+1..i+3]} \]
\[ \vdots \]
\[ t_i = p_i \quad t_{N-1} = p_{[i+1..N-1]} \quad (74) \]

The sets of the internal line labels are given in this case by:
\[ S_1 = \{1, \cdots, i + 1\}, \quad S_2 = \{i + 2, \cdots, N + 2\}, \quad \text{and} \quad S_3 = \{N + 3\} \quad (75) \]

Again, for each set \( S_i \), \( i = 1, 2, 3 \), one internal line 4-momentum is independent on the \( t_i \)'s and so the property of factorisation of \( \det(A) \) holds also in this case as can be proven by an explicit computation:

\[ B_{1k} = \begin{cases} -\tau_k & \text{if} \quad 1 \leq k \leq i \\ 0 & \text{if} \quad i < k \leq N - 1 \end{cases} \]
\[ B_{2k} = \begin{cases} -\tau_{N+2} & \text{if} \quad k = 1 \\ 0 & \text{if} \quad 1 < k \leq i \\ -\tau_{k+2} & \text{if} \quad i < k \leq N - 1 \end{cases} \quad (76) \]

showing explicitly that \( B \) (and \( \Gamma \)) do not depend neither on \( \tau_{i+1} \) nor on \( \tau_{i+2} \) nor on \( \tau_{N+3} \).

This property of factorisation of \( \det(A) \) in the quantities \( V \) and \( C \) holds for every topology in a three-leg vertex scalar theory. Indeed, it is always possible to choose the internal 4-momenta in such a way that the matrix \( B \) is homogeneous of degree 1 in the \( u_i \)'s after the reparametrisation of the Feynman parameters. Let us conclude this appendix by remarking that this property does not always hold for scalar theory having four-leg vertices. Indeed, there exists some topologies where an internal line connects two four-leg vertices as depicted in fig. 8. In this case, the \( \hat{r}_i \)'s are given by:

\[ \hat{r}_1 = \hat{r}_i - t_1 \quad \hat{r}_{i+2} = \hat{r}_{i+1} - t_i \]
\[ \hat{r}_2 = \hat{r}_i - t_2 \quad \hat{r}_{i+3} = \hat{r}_{i+1} - t_{i+1} \]
\[ \hat{r}_3 = \hat{r}_i - t_3 \quad \hat{r}_{i+4} = \hat{r}_{i+1} - t_{i+2} \]
\[ \vdots \]
\[ \hat{r}_{i-1} = \hat{r}_i - t_i \quad \hat{r}_N = \hat{r}_{i+1} - t_{N-2} \]
\[ \hat{r}_{N+1} = \hat{r}_i + \hat{r}_{i+1} - t_{N-1} \quad (77) \]
Figure 8: diagram picturing a two-loop $N$-point function with a planar topology with four-leg vertices.

with

\[
\begin{align*}
t_1 &= p_{[1..i-1]} & t_i &= p_{i+1} \\
t_2 &= p_{[2..i-1]} & t_{i+1} &= p_{[i+1..i+2]} \\
t_3 &= p_{[3..i-1]} & t_{i+2} &= p_{[i+1..i+3]} \\
&\vdots & & \vdots \\
t_{i-1} &= p_{i-1} & t_{N-2} &= p_{[i+1..N-1]} \\
t_{N-1} &= -p_i \end{align*}
\] (78)

The three sets of internal line labels become in this case:

\[
S_1 = \{1, \cdots, i\}, \quad S_2 = \{i + 1, \cdots, N\} \quad \text{and} \quad S_3 = \{N + 1\} \quad (79)
\]

It is easy to realise that the $\mathcal{B}$ matrix will not depend on $\tau_i$ and $\tau_{i+1}$ but will depend on $\tau_{N+1}$. For this reason, after the reparametrisation of the Feynman parameters, the $\mathcal{B}$ matrix will not be homogeneous of degree 1 in the $u_j$‘s and thus the factorisation of $\det(A)$ in the $V$ and $C$ terms does not hold.

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