Perturbed Bernstein-type operators

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Abstract
The present paper deals with modifications of Bernstein, Kantorovich, Durrmeyer and genuine Bernstein–Durrmeyer operators. Some previous results are improved in this study. Direct estimates for these operators by means of the first and second modulus of continuity are given. Also the asymptotic formulas for the new operators are proved.

Keywords Approximation by polynomials · Bernstein operators · Kantorovich operators · Durrmeyer operators · Voronovskaya type theorem · First and second order moduli

Mathematics Subject Classification 41A25 · 41A36

1 Introduction

In 2018 Khosravian-Arab, Dehghan and Eslahchi introduced three modifications of the classical Bernstein operator. In this note we follow their approach, explain it and discuss further relevant, but truly different Bernstein-type operators which have been attracting attention in the past. Thus we will discuss the modifications of the classical Bernstein operators (pointwise defined, preserve linear functions, but not commutative), classical Kantorovich operators (defined on \(L_1\), do not preserve linear functions), Durrmeyer operators (globally defined, commutative, do not preserve linear functions) and genuine Bernstein–Durrmeyer operators (globally defined, also commutative, preserve linear functions). Only in the Bernstein case we will go one step further and add remarks on a second perturbation created by modifying the classical recursion twice.
The organization of this note follows the lines given above. Before we will give estimates we add two short sections on the recursion for the fundamental functions of the Bernstein operator and on the use of $\omega_2$.

2 On the recursion for the fundamental functions of Bernstein operators

For $f \in C[0, 1]$ the Bernstein operator $B_n : C[0, 1] \to \prod_n$ is given by

$$B_n(f; x) = \sum_{k=0}^{n} p_{n,k}(x) f \left( \frac{k}{n} \right), \ x \in [0, 1],$$

where the fundamental functions are defined by

$$p_{n,k}(x) := \begin{cases} \binom{n}{k} x^k (1-x)^{n-k}, & 0 \leq k \leq n, \ x \in [0, 1], \\ 0, & k < 0 \text{ or } n < k. \end{cases}$$

It is well-known that these functions satisfy the recursion

$$p_{n,k}(x) = (1-x) p_{n-1,k}(x) + x p_{n-1,k-1}(x), \ 0 \leq k \leq n. \quad (2.1)$$

In particular,

$$p_{n,0}(x) = (1-x)^n,$$
$$p_{n,n}(x) = x^n.$$

This recursion is closely related to the so-called de Casteljau algorithm and other methods to compute a value $B_n(f; x), x$ fixed. See [12] for details.

In [19] the recursion form (2.1) is perturbed by replacing it in the first modification $B_n^{M,1}$ by

$$p_{n,k}^{M,1}(x) = a(x,n) p_{n-1,k}(x) + a(1-x,n) p_{n-1,k-1}(x), \ 1 \leq k \leq n-1,$$
$$p_{n,0}^{M,1}(x) = (1-x)^n, \ p_{n,n}^{M,1}(x) = a(1-x,n) x^{n-1}.$$

Here

$$a(x,n) = a_1(n)x + a_0(n), \ n = 0, 1, \ldots,$$

replaces $(1-x)$ in the original formula. In the papers dealing with this modification (see [1,2,4,6,10,21]) the superscript “$M, 1$” refers to this first disorder in the recursion.
If we carry out the original recursion once again, we obtain
\[
p_{n,k}(x) = (1 - x)p_{n-1,k}(x) + xp_{n-1,k-1}(x) \\
= (1 - x)^2 p_{n-2,k}(x) + 2x(1 - x)p_{n-2,k-1}(x) + x^2 p_{n-2,k-2}(x).
\]

So for the second modification $B_n^{M,2}$
\[
(1 - x)^2 \text{ is replaced by } b(x, n) = b_2(n)x^2 + b_1(n)x + b_0(n), \\
2x(1 - x) \text{ is replaced by } d_0(n)x(1 - x), \text{ and} \\
x^2 \text{ is replaced by } b(1 - x, n).
\]

Fortunately enough, this disorder is always introduced in the last step/two steps only. This means that the first $n - 1/n - 2$ fundamental functions remain intact, and a somewhat arbitrary perturbation is only introduced in the last/last two step(s).

The present note is mostly written with the ambition to show how resistant the fundamental functions $p_{n,k}$ are with respect to such unexpected intrusions.

### 3 On the use of $\omega_2$

For many years researchers in approximation theory have been striving to give inequalities with $\omega_2$ being the dominant expression. Many people simply still ignore this. In this section we will provide a very brief explanation why the use of $\omega_2$ (or related quantities such as the Ditzian–Totik modulus of second order) is indeed the better and more powerful tool from the quantitative point of view.

Exemplarily we will discuss the classical Bernstein operator $B_n$ and start with a very good results by Păltănea [22] who confirmed an earlier conjecture of the second author [11], namely that one has
\[
\|B_n f - f\|_{\infty} \leq 1 \cdot \omega_2 \left( f; \frac{1}{\sqrt{n}} \right), \quad f \in C[0, 1], \quad n \in \mathbb{N}.
\]

Here the constant 1 is best possible.

This implies that the approximation by $B_n$ is of order $O \left( \frac{1}{n} \right)$ for $f \in C^2[0, 1]$, and of order $O \left( \frac{1}{\sqrt{n}} \right)$ for $f \in C^1[0, 1]$, and even for $f \in Lip1 = \{ f \in C[0, 1] : \omega_1(f; t) = O(t) \}$. An estimate in terms of $c \cdot \omega_1 \left( f; \frac{1}{\sqrt{n}} \right)$ only reaches $Lip1$. If $\omega_1(f; t) = o(t)$, then $f$ is a constant.

However, the inequality in terms of $\omega_2$ also shows that one has $O \left( \frac{1}{\sqrt{n}} \right)$ for
\[
f \in Lip^*1 = \{ f \in C[0, 1] : \omega_2(f; t) = O(t) \}.
\]
Moreover, \( \text{Lip}^* 1 \not\supset \text{Lip} 1 \), so the same order is true for the bigger set \( \text{Lip}^* 1 \). An example of a function \( g \in \text{Lip}^* 1 \setminus \text{Lip} 1 \) is

\[
g(x) = \begin{cases} 
0, & x = 0, \\
x \log |x|, & 0 < x \leq 1.
\end{cases}
\]

The problem is at \( x = 0 \). Moreover,

\[
\text{Lip}^* 1 \subset \{ f \in C[0, 1] : \omega_1(f; \delta) = O(\delta \cdot |\log \delta|), \delta \to 0 \},
\]
represents the Dini-Lipschitz class. Hence it follows that

\[
\text{Lip}^* 1 \subset \text{Lip} \alpha, \ 0 < \alpha < 1.
\]

All this happens inside \( C[0, 1] \). For \( k \geq 1 \) this story repeats between \( C^k[0, 1] \) and \( C^{k+2} \subset C^{k+1} \), a fact being important when dealing with simultaneous approximation. Much more can be found in the seminal paper of Zygmund [23].

4 The modified Bernstein operator \( B_{n}^{M, 1} \)

Recently, Khosravian-Arab et al. [19] have introduced modified Bernstein operators as follows:

\[
B_{n}^{M, 1}(f, x) = \sum_{k=0}^{n} p_{n,k}^{M, 1}(x) f \left( \frac{k}{n} \right), \ x \in [0, 1]. \tag{4.1}
\]

Note that throughout the paper we will assume that \( B_{n}^{M, 1}(e_0, x) = 1 \), namely the sequences \( a_i(n), i = 0, 1 \), verify the condition

\[
2a_0(n) + a_1(n) = 1. \tag{4.2}
\]

**Theorem 4.1** For \( B_{n}^{M, 1} \) given above, \( f \in C[0, 1], x \in [0, 1], n \geq 1, \) we have

\[
|B_{n}^{M, 1}(f; x) - f(x)| \leq |B_n(f; x) - f(x)| + \left( 1 + a_1(n) \right) \left( \frac{1}{2} - x \right) \omega_1 \left( f; \frac{1}{n} \right). \tag{4.3}
\]

**Proof** We have

\[
|B_{n}^{M, 1}(f; x) - f(x)| \leq |B_n(f; x) - f(x)| + |B_{n}^{M, 1}(f; x) - B_n(f; x)|. \tag{4.3}
\]
In the following we will give an estimate of the quantity $|B_{n}^{M,1}(f; x) - B_{n}(f; x)|$.

So,

$$B_{n}^{M,1}(f; x) - B_{n}(f; x) = \sum_{k=0}^{n} \{a(x, n) p_{n-1,k}(x)$$

$$+ a(1 - x, n) p_{n-1,k-1}(x)\} f\left(\frac{k}{n}\right)$$

$$- \sum_{k=0}^{n-1} \{(-1 - a_{1}(n)) x + a_{1}(n) + a_{0}(n)\} p_{n-1,k-1}(x) f\left(\frac{k}{n}\right)$$

$$= \sum_{k=0}^{n-1} \{(a_{1}(n) + 1) x$$

$$+ a_{0}(n) - 1\} p_{n-1,k}(x) f\left(\frac{k}{n}\right)$$

$$+ \sum_{k=1}^{n} \{(-1 - a_{1}(n)) x + a_{1}(n) + a_{0}(n)\} p_{n-1,k-1}(x) f\left(\frac{k}{n}\right)$$

$$= \sum_{k=0}^{n-1} \{(a_{1}(n) + 1) x + a_{0}(n) - 1\} p_{n-1,k}(x) f\left(\frac{k}{n}\right)$$

$$+ \sum_{k=0}^{n-1} \{(-1 - a_{1}(n)) x + a_{1}(n) + a_{0}(n)\} p_{n-1,k}(x) f\left(\frac{k + 1}{n}\right)$$

$$= \sum_{k=0}^{n-1} \left[ f\left(\frac{k + 1}{n}\right)$$

$$- f\left(\frac{k}{n}\right)\right] \{- (1 + a_{1}(n)) x + a_{0}(n) + a_{1}(n)\} p_{n-1,k}(x).$$

Therefore,

$$|B_{n}^{M,1}(f; x) - B_{n}(f; x)| \leq \left| - (1 + a_{1}(n)) x + a_{0}(n) + a_{1}(n)\omega_{1}\left( f; \frac{1}{n}\right)$$

$$= \left| (1 + a_{1}(n)) \left(\frac{1}{2} - x\right)\right| \omega_{1}\left( f; \frac{1}{n}\right)$$

and replacing this estimate in (4.3) the proof is complete. \qed

**Remark 4.1**

(i) For $a_{1}(n) = -1$ all the estimates for Bernstein operator $B_{n}$ hold.

(ii) If $a_{1}(n)$ is bounded, say $|a_{1}(n)| \leq A_{1}$, then

$$|B_{n}^{M,1}(f; x) - f(x)| \leq |B_{n}(f; x) - f(x)| + \frac{1}{2} (1 + A_{1}) \omega_{1}\left( f; \frac{1}{n}\right).$$
(iii) If \( f \in C^2[0, 1] \), then for \( a_1(n) \) bounded \( \| B_n^{M,1}(f) - f \|_\infty = O\left(\frac{1}{n}\right) \). This result is an improvement of [19, Theorem 9].

In order to prove a quantitative Voronovskaja theorem for \( B_n^{M,1} \) we first identify the limit.

**Proposition 4.1** Suppose that \( B_n^{M,1} \) is given as above, \( f \in C^2[0, 1], x \in [0, 1] \) and \( L_1 := \lim_{n \to \infty} a_1(n) \) exists. Then

\[
\lim_{n \to \infty} n \left[ B_n^{M,1}(f; x) - f(x) \right] = \frac{x(1-x)}{2} f''(x) + \frac{1-2x}{2} (1 + L_1) f'(x).
\]

**Proof** As above write

\[
n \left[ B_n^{M,1}(f; x) - f(x) \right] = n \left[ B_n(f; x) - f(x) \right] + n \left[ B_n^{M,1}(f; x) - B_n(f; x) \right]
\]

\[
= T_1(x) + T_2(x).
\]

The limit of \( T_1(x) \) is known, i.e., \( \frac{x(1-x)}{2} f''(x) \). Moreover,

\[
T_2(x) = \left[ -(1 + a_1(n))x + a_0(n) + a_1(n) \right] \sum_{k=0}^{n-1} \left[ f \left( \frac{k+1}{n} \right) - f \left( \frac{k}{n} \right) \right] p_{n-1,k}(x)
\]

\[
= \left[ -(1 + a_1(n))x + a_0(n) + a_1(n) \right] (B_n f)'(x).
\]

Hence,

\[
n \left[ B_n^{M,1}(f; x) - f(x) \right] = n \left[ B_n(f; x) - f(x) \right] + (1 + a_1(n)) \left( -x + \frac{1}{2} \right) (B_n f)'(x).
\]

Since \( \lim_{n \to \infty} n \left[ B_n(f; x) - f(x) \right] = \frac{x(1-x)}{2} f''(x) \) and \( \lim_{n \to \infty} (B_n f)'(x) = f'(x) \), the proof is complete. \( \square \)

**Theorem 4.2** Suppose that \( B_n^{M,1} \) is given as above, \( f \in C^2[0, 1], L_1 = \lim_{n \to \infty} a_1(n) \) exists. Then for \( x \in [0, 1] \) there holds

\[
\Delta_n^B := \left| n \left[ B_n^{M,1}(f; x) - f(x) \right] - \frac{x(1-x)}{2} f''(x) - \frac{1-2x}{2} (1 + L_1) f'(x) \right|
\]

\[
\leq X \left\{ \frac{5}{6} \sqrt{\frac{|X|}{3(n-2)X + 1}} \omega_1 \left( f''; \sqrt{\frac{3(n-2)X + 1}{n^2}} \right) + \frac{13}{16} \omega_2 \left( f''; \sqrt{\frac{3(n-2)X + 1}{n^2}} \right) \right\}
\]

\[
+ \left\{ \frac{|L_1 - a_1(n)| \cdot \| f' \|_\infty + |1 + L_1| \cdot \left( \frac{13}{4} \omega_2 \left( f'; \frac{1}{\sqrt{n}} \right) + \frac{1}{\sqrt{n}} \omega_1 \left( f'; \frac{1}{\sqrt{n}} \right) \right) \right\}.
\]

Here \( X := x(1-x), i.e., X' = 1 - 2x. \)
**Proof** For $\Delta^B_n$ the following inequality holds

\[
\Delta^B_n \leq n \left[ B_n(f; x) - f(x) \right] - \frac{x(1-x)}{2} f''(x) \\
+ \left| n \left[ B^M_n(f; x) - B_n(f; x) \right] - \frac{1-2x}{2} (1 + L_1) f'(x) \right|.
\] (4.4)

Gonska and Raşa [15] obtained a Voronovskaya estimate with first and second modulus of smoothness for Bernstein operator as follows

\[
\left| n \left[ B_n(f; x) - f(x) \right] - \frac{x(1-x)}{2} f''(x) \right| \\
\leq X \left\{ \frac{5}{6} \frac{|X'|}{\sqrt{3(n-2)X+1}} \omega_1 \left( f''; \sqrt{\frac{3(n-2)X+1}{n^2}} \right) \\
+ \frac{13}{16} \omega_2 \left( f''; \sqrt{\frac{3(n-2)X+1}{n^2}} \right) \right\}.
\] (4.5)

We estimate the second difference of (4.4) as follows

\[
\left| n \left[ B^M_n(f; x) - B_n(f; x) \right] - \frac{1-2x}{2} (1 + L_1) f'(x) \right| \\
= \left| (1 + a_1(n)) \left( -x + \frac{1}{2} \right) (B_n f)'(x) - \frac{1-2x}{2} (1 + L_1) f'(x) \right| \\
= \left| \frac{1-2x}{2} (a_1(n) - L_1) (B_n f)'(x) - \frac{1-2x}{2} (1 + L_1) \left[ f'(x) - (B_n f)'(x) \right] \right| \\
\leq \left| \frac{1-2x}{2} \right| \left\{ |L_1 - a_1(n)| \cdot |(B_n f)'(x)| + |1 + L_1| \left| f'(x) - (B_n f)'(x) \right| \right\}. 
\] (4.6)

Moreover, we use (see [14, Theorem 4.1])

\[
|f'(x) - (B_n f)'(x)| \leq \frac{13}{4} \omega_2 \left( f''; \frac{1}{\sqrt{n}} \right) + \frac{1}{\sqrt{n}} \omega_1 \left( f'; \frac{1}{\sqrt{n}} \right).
\] (4.7)

Also, we have

\[
|(B_n f)'(x)| = \left| \sum_{k=0}^{n-1} \frac{f \left( \frac{k+1}{n} \right) - f \left( \frac{k}{n} \right)}{\frac{1}{n}} p_{n-1,k}(x) \right| \leq n \omega_1 \left( f; \frac{1}{n} \right) \leq \|f'\|_{\infty}.
\] (4.8)

Using the relations (4.4)–(4.8) the theorem is proved.
Corollary 4.1 We have

\[ \Delta B_n \leq \begin{cases} 
O \left( \frac{1}{\sqrt{n}} \right) + \frac{1}{2} |L_1 - a_1(n)| \cdot \| f' \|_{\infty}, & \text{for } f \in C^3[0, 1], \\
O \left( \frac{1}{n} \right) + \frac{1}{2} |L_1 - a_1(n)| \cdot \| f' \|_{\infty}, & \text{for } f \in C^4[0, 1]. 
\end{cases} \]

Theorem 9 in [19] should be reformulated in the following way.

Proposition 4.2 If \( B_{n,1}^{M,1} \) is given as above (positive or non-positive), then for \( f \in B[0, 1] \) (bounded functions) holds

\[ \| B_{n,1}^{M,1} (f) - f \|_{\infty} \leq 2 \left( 3 |a_1(n)| + 1 \right) \omega_1 \left( f; \frac{1}{\sqrt{n}} \right), \quad n \geq 3. \]

If \( (a_1(n)) \) is bounded, then

\[ \| B_{n,1}^{M,1} (f) - f \|_{\infty} = O(1) \omega_1 \left( f; \frac{1}{\sqrt{n}} \right), \]

and

\[ \| B_{n,1}^{M,1} (f) - f \|_{\infty} = o(1), \quad \text{for } f \in C[0, 1]. \]

5 The modification \( B_{n,2}^{M,2} \)

Khosravian-Arab et al. [19] also introduced a second modification of the Bernstein operator as follows:

\[ B_{n,2}^{M,2} (f; x) = \sum_{k=0}^{n} p_{n,k}^{M,2} (x) f \left( \frac{k}{n} \right), \quad (5.1) \]

where

\[ p_{n,k}^{M,2} (x) = \left[ \frac{n}{2} x^2 + \left( -1 - \frac{n}{2} \right) x + 1 \right] p_{n-2,k} (x) + nx (1 - x) p_{n-2,k-1} (x) + \left[ \frac{n}{2} x^2 + \left( -\frac{n}{2} + 1 \right) x \right] p_{n-2,k-2} (x). \]

Lemma 5.1 [19] The moments of the operators \( B_{n,2}^{M,2} \) are given by

(i) \( B_{n,2}^{M,2} (e_0; x) = 1; \)
(ii) \( B_{n,2}^{M,2} (e_1; x) = x; \)
(iii) \( B_{n,2}^{M,2} (e_2; x) = x^2 + \frac{2x(1 - x)}{n^2}. \)
Theorem 5.1 Let $B_{n}^{M,2}$ be the modified Bernstein operator defined in (5.1). Then for $f \in C[0, 1]$ there holds

$$\|B_{n}^{M,2} - f\|_{\infty} \leq \begin{cases} \frac{1}{8} \omega_{1} \left( f'; \frac{1}{n} \right) + \frac{1}{\sqrt{n-2}} \omega_{1} \left( f''; \frac{1}{\sqrt{n-2}} \right) + \frac{2}{n} \|f''\|_{\infty} = o(1), & f \in C^{1}[0, 1], \\ O\left( \frac{1}{n} \right), & f \in C^{2}[0, 1]. \end{cases}$$

Proof We have

$$\left| B_{n}^{M,2}(f; x) - f(x) \right| = \sum_{k=0}^{n-2} p_{n-2,k}(x) \left\{ \frac{nx(x-1)}{2} \left[ f\left( \frac{k}{n} \right) - 2f\left( \frac{k+1}{n} \right) + f\left( \frac{k+2}{n} \right) \right] \\
\quad + (1-x)f\left( \frac{k}{n} \right) + xf\left( \frac{k+2}{n} \right) - f(x) \right\}$$

$$= \sum_{k=0}^{n-2} p_{n-2,k}(x) \left\{ \frac{nx(x-1)}{2} \left[ f\left( \frac{k}{n} \right) - 2f\left( \frac{k+1}{n} \right) + f\left( \frac{k+2}{n} \right) \right] \right. \\
\quad \left. + (1-x)\left[ f\left( \frac{k}{n-2} \right) - f(x) \right] + (1-x)\left[ f\left( \frac{k}{n} \right) - f\left( \frac{k}{n-2} \right) \right] \right. \\
\quad \left. + x\left[ f\left( \frac{k}{n-2} \right) - f(x) \right] + x\left[ f\left( \frac{k+2}{n} \right) - f\left( \frac{k}{n-2} \right) \right] \right\}. $$

Using the relation (see [22])

$$\|B_{n}(f) - f\|_{\infty} \leq \omega_{2}\left( f; \frac{1}{\sqrt{n}} \right),$$

we obtain

$$\left| B_{n}^{M,2}(f; x) - f(x) \right| \leq \frac{nx(1-x)}{2} \omega_{2}\left( f; \frac{1}{n} \right)$$

$$\quad + (1-x)\omega_{2}\left( f; \frac{1}{\sqrt{n-2}} \right) + x\omega_{2}\left( f; \frac{1}{\sqrt{n-2}} \right)$$

$$\quad + (1-x)\omega_{1}\left( f; \frac{2}{n} \right) + x\omega_{1}\left( f; \frac{2}{n} \right)$$

$$= \frac{nx(1-x)}{2} \omega_{2}\left( f; \frac{1}{n} \right)$$

$$\quad + \omega_{2}\left( f; \frac{1}{\sqrt{n-2}} \right) + \omega_{1}\left( f; \frac{2}{n} \right)$$

$$\leq \frac{n}{8} \omega_{2}\left( f; \frac{1}{n} \right) + \omega_{2}\left( f; \frac{1}{\sqrt{n-2}} \right)$$

$$\quad + \omega_{1}\left( f; \frac{2}{n} \right), \text{ for } f \in C[0, 1].$$
and the theorem is proved.

**Remark 5.1** The above inequality is an improvement and a generalization of [19, Theorem 14]. There a non-quantitative statement is obtained for \( f \in C^2[0, 1] \) only.

### 6 The modified Kantorovich operators \( K_{n}^{M,1} \)

An integral modification of Bernstein operators was introduced by Kantorovich [18] as follows:

\[
K_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{k \over n+1}^{(k+1) \over n+1} f(t) dt.
\]  

(6.1)

In a recent article of the present authors [3] the results on these mapping were supplemented. Here we only mention one result from there. Applying Păltănea’s result [22, Corollary 2.2.1] the following estimate in terms of the first and second modulus of continuity for the classical Kantorovich operators is obtained.

**Theorem 6.1** For \( n \geq 1 \) and all \( f \in C[0, 1] \) there holds

\[
\|K_n f - f\|_{\infty} \leq \frac{1}{2\sqrt{n} + 1} \omega_1(f; 1/\sqrt{n + 1}) + \frac{9}{8} \omega_2(f; 1/\sqrt{n + 1}).
\]

Recently, a Kantorovich variant of the modified Bernstein operators (4.1) was investigated in [6]. These operators are given by

\[
K_n^{M,1}(f; x) := (n + 1) \sum_{k=0}^{n} p_{n,k}^{M,1}(x) \int_{k \over n+1}^{(k+1) \over n+1} f(t) dt.
\]

A certain Stancu modification was introduced by Opriş [10,21].

**Theorem 6.2** For \( K_n^{M,1} \) given above, \( f \in C[0, 1], x \in [0, 1], n \geq 1 \), we have

\[
|K_n^{M,1}(f; x) - f(x)| \leq |K_n(f; x) - f(x)| + \left| (1 + a_1(n)) \left( {1 \over 2} - x \right) \right| \omega_1(f; 1/n + 1).
\]

**Proof** Again we start with

\[
|K_n^{M,1}(f; x) - f(x)| \leq |K_n(f; x) - f(x)| + \left| K_n^{M,1}(f; x) - K_n(f; x) \right|.
\]

(6.2)

In the following we will estimate the quantity \( K_n^{M,1}(f; x) - K_n(f; x) \).

One has

\[
K_n^{M,1}(f; x) - K_n(f; x) = (n + 1) \sum_{k=0}^{n} \left( (a(x, n)p_{n-1,k}(x)ight).
\]
we have

\[(1 - x) p_{n-1,k}(x) + x p_{n-1,k-1}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \]

\[
\left\{ \begin{array}{l}
(1 - x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \\
-(1 - x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \\
- \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt
\end{array} \right. 
\]

\[
= (n + 1) \sum_{k=0}^{n-1} \left[(a_1(n) + 1)x - (a_0(n) + a_1(n))\right] p_{n-1,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt 
\]

\[
= (n + 1) \sum_{k=0}^{n-1} \left[-(a_1(n) + 1)x + (a_0(n) + a_1(n))\right] p_{n-1,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt 
\]

\[
= (n + 1) \sum_{k=0}^{n-1} \left[-(a_1(n) + 1)x + (a_0(n) + a_1(n))\right] p_{n-1,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left[f(t) - f\left(t - \frac{1}{n+1}\right)\right] dt.
\]

Therefore,

\[
\left| K_n^{M,1}(f; x) - K_n(f; x) \right| \leq \left| 1 + a_1(n) \right| \left(\frac{1}{2} - x\right) \omega_1\left(f; \frac{1}{n+1}\right). \quad (6.3)
\]

From (6.2) and (6.3) it follows that for all cases of $K_n^{M,1}$ (positive and non-positive) we have

\[
\left| K_n^{M,1}(f; x) - f(x) \right| \leq \left| K_n(f; x) - f(x) \right| + \left| 1 + a_1(n) \right| \left(\frac{1}{2} - x\right) \omega_1\left(f; \frac{1}{n+1}\right).
\]

\[
\square
\]

**Remark 6.1**

(i) For $a_1(n) = -1$ all the estimates for Kantorovich operator $K_n$ hold.

(ii) If $a_1(n)$ is bounded, say $|a_1(n)| \leq A_1$, then

\[
\left| K_n^{M,1}(f; x) - f(x) \right| \leq \left| K_n(f; x) - f(x) \right| + \frac{1}{2}(1 + A_1) \omega_1\left(f; \frac{1}{n+1}\right).
\]
(iii) If \( f \in C^2[0, 1] \), then for \( a_1(n) \) bounded \( \| K_{n}^{M,1}(f) - f \|_{\infty} = O \left( \frac{1}{n} \right) \). This result is an improvement of [6, Theorem 2.6].

We will give next a Voronovskaya-type result for the modifications \( K_{n}^{M,1} \).

**Theorem 6.3** Suppose that \( f \in C^2[0, 1] \) and \( L_1 = \lim_{n \to \infty} a_1(n) \) exists. Then for \( x \in [0, 1] \) there holds

\[
\Delta_n^K := \left| n \left[ K_{n}^{M,1}(f; x) - f(x) \right] - \frac{X}{2} f''(x) - \frac{X'}{2} (2 + L_1) f'(x) \right|
\]

\[
\leq \frac{2}{3(n + 1)} \left( \frac{3}{4} \| f' \|_{\infty} + \| f'' \|_{\infty} \right) + \frac{9}{32} \left\{ \frac{2}{\sqrt{n + 1}} \omega_1 \left( f''; \frac{1}{\sqrt{n + 1}} \right) + \omega_2 \left( f''; \frac{1}{\sqrt{n + 1}} \right) \right\} + \frac{1}{2} |L_1 - a_1(n)| \cdot \| f' \|_{\infty} + \frac{1}{2} \left| L_1 + 1 \right| \left\{ \frac{1}{n + 1} \| f' \|_{\infty} + \frac{1}{\sqrt{n + 1}} \omega_1 \left( f'; \frac{1}{\sqrt{n + 1}} \right) + \frac{9}{8} \omega_2 \left( f'; \frac{1}{\sqrt{n + 1}} \right) \right\},
\]

where \( X := x(1 - x), \) i.e., \( X' = 1 - 2x. \)

**Proof** For \( \Delta_n^K \) the following inequality holds

\[
\Delta_n^K \leq \left| n \left[ K_{n}(f; x) - f(x) \right] - \frac{X'}{2} f'(x) - \frac{X}{2} f''(x) \right|
\]

\[
+ \left| n \left[ K_{n}^{M,1}(f; x) - K_{n}(f; x) \right] - \frac{X'}{2} (1 + L_1) f'(x) \right|. \tag{6.4}
\]

If \( a_1(n) = -1, \) i.e., \( L_1 = -1, \) the second summand cancels. So we have the “old” Voronovskaya-Kantorovich theorem with second modulus (see [3]):

\[
\left| n \left[ K_{n}(f; x) - f(x) \right] - \frac{X'}{2} f'(x) - \frac{X}{2} f''(x) \right| \leq \frac{2}{3(n + 1)} \left( \frac{3}{4} \| f' \|_{\infty} + \| f'' \|_{\infty} \right) + \frac{9}{32} \left\{ \frac{2}{\sqrt{n + 1}} \omega_1 \left( f''; \frac{1}{\sqrt{n + 1}} \right) + \omega_2 \left( f''; \frac{1}{\sqrt{n + 1}} \right) \right\}. \tag{6.5}
\]

The second summand of (6.4) can be written as

\[
\left| n \left[ K_{n}^{M,1}(f; x) - K_{n}(f; x) \right] - \frac{X'}{2} (1 + L_1) f'(x) \right|
\]

\[
= n \left\{ (n + 1) \sum_{k=0}^{n} P_{n,k}^{M,1}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \right.
\]

\[
- (n + 1) \sum_{k=0}^{n} P_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt - \frac{X'}{2} (L_1 + 1) f'(x) \right\}.
\]
From the above relation we obtain

\[
= n(n + 1) \sum_{k=0}^{n-1} \left( a_1(n) + 1 \right) \left( x - \frac{1}{2} \right) p_{n-1,k}(x) \left[ \int_{\frac{k}{n+1}}^{\frac{k+\frac{1}{n+1}}{2}} f(t) dt \right] - \frac{X'}{2} (L_1 + 1) f'(x)
\]

\[
= -\frac{X'}{2} (a_1(n) + 1)(K_n f)'(x) - \frac{X'}{2} (L_1 + 1) f'(x).
\]

So,

\[
\left| n \left[ K_n^{M,1}(f; x) - K_n(f; x) \right] - \frac{X'}{2} (1 + L_1) f'(x) \right| \leq \frac{1}{2} X' \left| (K_n f)'(x)(a_1(n) + 1) - (L_1 + 1) f'(x) \right|
\]

\[
= \frac{1}{2} X' \left| (K_n f)'(x)(a_1(n) - L_1) + (L_1 + 1) \left[ (K_n f)'(x) - f'(x) \right] \right|
\]

\[
\leq \frac{1}{2} \left\{ |L_1 - a_1(n)||K_n f)'(x)| + |L_1 + 1||(K_n f)'(x) - f'(x)| \right\}.
\]

But,

\[
|K_n f)'(x)| = n(n + 1) \left| \sum_{k=0}^{n-1} p_{n-1,k}(x) \left[ \int_{\frac{k}{n+1}}^{\frac{k+\frac{1}{n+1}}{2}} f(t) dt - \int_{\frac{k}{n+1}}^{\frac{k+\frac{1}{n+1}}{2}} f(t) dt \right] \right|
\]

\[
= n(n + 1) \sum_{k=0}^{n-1} p_{n-1,k}(x) \left| f(t) - f \left( t - \frac{1}{n + 1} \right) \right| dt
\]

\[
\leq \frac{n}{n + 1} \sum_{k=0}^{n-1} \|f'\|_{\infty} p_{n-1,k}(x) \leq \|f'\|_{\infty}.
\]

Moreover, from [13, Theorem 7] it follows

\[
|K_n f)'(x) - f'(x)| \leq \frac{1}{n + 1} \|f'\|_{\infty} + \frac{1}{\sqrt{n + 1}} \omega_1 \left( f'; \frac{1}{\sqrt{n + 1}} \right) + \frac{9}{8} \omega_2 \left( f'; \frac{1}{\sqrt{n + 1}} \right).
\]

From the above relation we obtain

\[
\left| n \left[ K_n^{M,1}(f; x) - K_n(f; x) \right] - \frac{X'}{2} (1 + L_1) f'(x) \right| \leq \frac{1}{2} |L_1 - a_1(n)| \cdot \|f'\|_{\infty}
\]

\[
+ \frac{1}{2} |L_1 + 1| \left\{ \frac{1}{n + 1} \|f'\|_{\infty} + \frac{1}{\sqrt{n + 1}} \omega_1 \left( f'; \frac{1}{\sqrt{n + 1}} \right) + \frac{9}{8} \omega_2 \left( f'; \frac{1}{\sqrt{n + 1}} \right) \right\}.
\]  (6.6)

Using the relations (6.4)–(6.6) we get the claim. \qed
Corollary 6.1 We have

\[ \Delta^K_n \leq \begin{cases} 
\mathcal{O}(1) + \frac{1}{2} |L_1 - a_1(n)| \cdot \|f'\|_{\infty}, & \text{for } f \in C^2[0, 1], \\
\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) + \frac{1}{2} |L_1 - a_1(n)| \cdot \|f'\|_{\infty}, & \text{for } f \in C^3[0, 1], \\
\mathcal{O}\left(\frac{1}{n}\right) + \frac{1}{2} |L_1 - a_1(n)| \cdot \|f'\|_{\infty}, & \text{for } f \in C^4[0, 1].
\end{cases} \]

7 The modified Durrmeyer operators \(D^M_n\)

The classical Durrmeyer operators were introduced by Durrmeyer [9] and, independently, by Lupaș [20]. These operators are defined as

\[ D^M_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{0}^{1} p_{n,k}(t) f(t) \, dt, \quad x \in [0, 1]. \]

In this section we study a Durrmeyer variant of the modified Bernstein operators introduced in a recent note of Acu et al. [5]:

\[ D^{M,1}_n(f; x) = (n + 1) \sum_{k=0}^{n} p^{M,1}_{n,k}(x) \int_{0}^{1} p_{n,k}(t) f(t) \, dt, \quad x \in [0, 1]. \quad (7.1) \]

Theorem 7.1 For \( n \geq 1 \) and \( f \in C^2[0, 1] \), one has

\[ \left\| n(D_n f - f) - (Xf')' \right\|_{\infty} \leq \frac{1}{n+2} \left( 2\|f'\|_{\infty} + 3\|f''\|_{\infty} \right) \]
\[ + \frac{5}{\sqrt{n+4}} \omega_1\left( f''; \frac{1}{\sqrt{n+4}} \right) + \frac{9}{8} \omega_2\left( f''; \frac{1}{\sqrt{n+4}} \right), \quad (7.2) \]

where \( X = x(1-x) \) and \( X' = 1 - 2x, \, x \in [0, 1] \).

Proof From [15, Theorem 3] we get

\[ \left| D_n(f; x) - f(x) - D_n(t-x;x) f'(x) - \frac{1}{2} D_n \left( (e_1 - x)^2; x \right) f''(x) \right| \]
\[ \leq D_n((e_1 - x)^2; x) \left\{ \frac{|D_n((e_1 - x)^3; x)|}{6h} \omega_1(f''; h) \right. \]
\[ + \left. \left( \frac{3}{4} + \frac{D_n((e_1 - x)^4; x)}{D_n((e_1 - x)^2; x) \cdot 16h^2} \right) \omega_2(f''; h) \right\}. \]
Using the central moments up to order 4 for Durrmeyer operators, namely

\[
D_n(t - x; x) = \frac{1 - 2x}{n + 2},
\]

\[
D_n((t - x)^2; x) = \frac{2[x(1 - x)(n - 3) + 1]}{(n + 2)(n + 3)},
\]

\[
D_n((t - x)^3; x) = \frac{6(1 - 2x)}{(n + 2)(n + 3)(n + 4)} [2x(1 - x)n + 2x^2 - 2x + 1],
\]

\[
D_n((t - x)^4; x) = \frac{12 [x^2(1 - x)^2 n^2 + 3x(1 - x)(7x^2 - 7x + 2)n - 10x(1 - x)(x^2 - x + 1) + 2]}{(n + 2)(n + 3)(n + 4)(n + 5)},
\]

we obtain

\[
\frac{|D_n((t - x)^3; x)|}{D_n((t - x)^2; x)} \leq \frac{6}{n + 4}; \quad \frac{|D_n((t - x)^4; x)|}{D_n((t - x)^2; x)} \leq \frac{6}{n + 4}.
\]

Therefore, the following inequality holds

\[
\left| D_n(f; x) - f(x) - \frac{1 - 2x}{n + 2} f'(x) - \frac{x(1 - x)(n - 3) + 1}{(n + 2)(n + 3)} f''(x) \right|
\]

\[
\leq \frac{1}{n + 2} \left\{ \frac{5}{h(n + 4)} \omega_1(f''; h) + \left( \frac{3}{4} + \frac{3}{8h^2(n + 4)} \right) \omega_2(f''; h) \right\}
\]

and for \( h = \frac{1}{\sqrt{n + 4}} \) we obtain, after multiplying both sides by \( n \),

\[
\left| n [D_n(f; x) - f(x)] - \frac{n(1 - 2x)}{n + 2} f'(x) - \frac{n [x(1 - x)(n - 3) + 1]}{(n + 2)(n + 3)} f''(x) \right|
\]

\[
\leq \frac{5}{\sqrt{n + 4}} \omega_1 \left( f''; \frac{1}{\sqrt{n + 4}} \right)
\]

\[
+ \frac{9}{8} \omega_2 \left( f''; \frac{1}{\sqrt{n + 4}} \right).
\]

We can write

\[
|n [D_n(f; x) - f(x)] - X f'(x) - X f''(x)|
\]

\[
\leq |n [D_n(f; x) - f(x)] - \frac{n}{n + 2} X f'(x) - \frac{n(n - 3)X}{(n + 2)(n + 3)} f''(x) - \frac{n}{(n + 2)(n + 3)} f''(x)|
\]
Let \( f \in C[0, 1] \), \( x \in [0, 1] \), \( n \geq 1 \). Then

\[
\left| D_{n}^{M,1}(f; x) - f(x) \right| \leq |D_{n}(f; x) - f(x)| + \left| (1 + a_1(n)) \left( \frac{1}{2} - x \right) \right| \left( 3\omega_2 \left( f; \sqrt{\sigma_n(x)} \right) + \frac{5}{(n + 2)\sqrt{\sigma_n(x)}} \omega_1(f; \sqrt{\sigma_n(x)}) \right),
\]

where \( \sigma_n(x) = \frac{2x(1-x)(n-1)(n-2) + 3n + 1}{2(n+2)^{2}(n+3)} \).

\[ \square \]

**Theorem 7.2** Let \( f \in C[0, 1] \), \( x \in [0, 1] \), \( n \geq 1 \). Then

\[
\left| D_{n}^{M,1}(f; x) - f(x) \right| \leq |D_{n}(f; x) - f(x)| + \left| (1 + a_1(n)) \left( \frac{1}{2} - x \right) \right| \left( 3\omega_2 \left( f; \sqrt{\sigma_n(x)} \right) + \frac{5}{(n + 2)\sqrt{\sigma_n(x)}} \omega_1(f; \sqrt{\sigma_n(x)}) \right),
\]

Next, we will give an estimate of the quantity \( |D_{n}^{M,1}(f; x) - D_{n}(f; x)| \). We have

\[
D_{n}^{M,1}(f; x) - D_{n}(f; x) = (n + 1) \sum_{k=0}^{n} a(x, n) p_{n-1,k}(x)
\]

\[
+ a(1-x, n) p_{n-1,1}(x) \int_{0}^{1} p_{n,k}(t) f(t) dt
\]

\[
- (n + 1) \sum_{k=0}^{n} \left( (1-x) p_{n-1,k}(x) + xp_{n-1,k-1}(x) \right) \int_{0}^{1} p_{n,k}(t) f(t) dt
\]

\[
= (n + 1)(a_1(n) + 1) \left( x - \frac{1}{2} \right) \left\{ \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_{0}^{1} p_{n,k}(t) f(t) dt - \sum_{k=1}^{n} p_{n-1,k-1}(x) \int_{0}^{1} p_{n,k}(t) f(t) dt \right\}
\]

\[
= (a_1(n) + 1) \left( x - \frac{1}{2} \right) [A_n(f; x) - B_n(f; x)],
\]
where
\[ A_n(f; x) = \sum_{k=0}^{n-1} p_{n-1,k}(x) F_k(f), \quad B_n(f; x) = \sum_{k=0}^{n-1} p_{n-1,k}(x) G_k(f), \]
\[ F_k(f; x) = (n+1) \int_0^1 p_{n,k}(t) f(t) dt, \quad G_k(f; x) = (n+1) \int_0^1 p_{n,k+1}(t) f(t) dt, \quad k = 0, \ldots, n-1. \]

For a positive linear functional \( F \) denote
\[ b_F := F(e_1) \quad \text{and} \quad \mu_F^2 := \frac{1}{2} F \left( e_1 - b_F e_0 \right)^2. \]

Using [7, Theorem 5] for \( f \in C[0, 1] \) and \( 0 < h \leq \frac{1}{2} \), we get
\[ |A_n(f; x) - B_n(f; x)| \leq \frac{3}{2} \left( 1 + \frac{\sigma_n(x)}{h^2} \right) \omega_2(f, h) + \frac{5\delta}{h} \omega_1(f, h), \quad (7.4) \]
where
\[ \sigma_n(x) := \sum_{k=0}^{n-1} \left( \mu_F^2 + \mu_G^2 \right) p_{n-1,k}(x), \quad \delta := \sup_k |b_{Fk} - b_{Gk}|. \]

In the present case
\[ b_{Fk} = \frac{k+1}{n+2}, \quad b_{Gk} = \frac{k+2}{n+2}, \quad \mu_{Fk}^2 = \frac{(k+1)(n-k+1)}{2(n+2)^2(n+3)}, \quad \mu_{Gk}^2 = \frac{(k+2)(n-k)}{2(n+2)^2(n+3)}, \]
so we obtain \( \sigma_n(x) = \frac{2x(1-x)(n-1)(n-2) + 3n + 1}{2(n+2)^2(n+3)} \) and \( \delta = \frac{1}{n+2}. \)

Choosing \( h := \sqrt{\sigma_n(x)} \) we get
\[ |D_n^{M,1}(f; x) - D_n(f; x)| \leq \left| (1 + a_1(n)) \left( \frac{1}{2} - x \right) \right| \left\{ 3\omega_2(f; \sqrt{\sigma_n(x)}) + \frac{5}{(n+2)\sqrt{\sigma_n(x)}} \omega_1(f; \sqrt{\sigma_n(x)}) \right\}. \quad (7.5) \]
Using relations \((7.3)\) and \((7.5)\) the proof is complete. \(\Box\)

**Remark 7.1**

(i) For \(a_1(n) = -1\) all the estimates for the Durrmeyer operator \(D_n\) hold.

(ii) If \(f \in C^2[0, 1]\), then for \(a_1(n)\) bounded \(\|D_n^{M,1}(f) - f\|_\infty = O\left(\frac{1}{n}\right)\).

**Theorem 7.3**

Suppose that \(D_n^{M,1}\) is given as above, \(f \in C^2[0, 1]\), \(L_1 = \lim_{n \to \infty} a_1(n)\) exists. Then for \(x \in [0, 1]\) there holds

\[
\Delta_n^D := \left| n \left[ D_n^{M,1}(f; x) - f(x) \right] - x(1 - x)f''(x) - \frac{1 - 2x}{2} (L_1 + 3) f'(x) \right|
\]

\[
\leq \frac{1}{n + 2} \left( 2 \|f'\|_\infty + 3 \|f''\|_\infty \right) + \frac{5}{\sqrt{n + 4}} \omega_1 \left( f'' : \frac{1}{\sqrt{n + 4}} \right) + \frac{9}{8} \omega_2 \left( f'' : \frac{2}{n + 2} \right)
\]

\[
+ \frac{1}{2} \left\{ |L_1 - a_1(n)| \cdot \|f'\|_\infty + |1 + L_1| \left[ \frac{2}{n + 2} |f'(x)| + \sqrt{\frac{2}{n + 2}} \omega_1 \left( f' : \sqrt{\frac{2}{n + 2}} \right) \right] \right\}
\]

\[
+ \frac{9}{8} \omega_2 \left( f' : \sqrt{\frac{2}{n + 2}} \right) \}.
\]

**Proof**

For \(\Delta_n^D\) the following inequality holds

\[
\Delta_n^D \leq \left| n \left[ D_n(f; x) - f(x) \right] - x(1 - x)f''(x) - (1 - 2x)f'(x) \right|
\]

\[
+ \left| n \left[ D_n^{M,1}(f; x) - D_n(f; x) \right] - (1 - 2x) \frac{L_1 + 1}{2} f'(x) \right|.
\] \hspace{1cm} (7.6)

The second difference of (7.6) can be estimated as follows

\[
\left| n \left[ D_n^{M,1}(f; x) - D_n(f; x) \right] - \frac{1 - 2x}{2} (1 + L_1) f'(x) \right|
\]

\[
= \left| n(n + 1)(a_1(n) + 1) \left( x - \frac{1}{2} \right) \left\{ \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_0^1 p_{n,k}(t) f(t) dt \right.ight.
\]

\[
- \sum_{k=1}^{n-1} p_{n-1,k-1}(x) \int_0^1 p_{n,k}(t) f(t) dt \right\} - \frac{1 - 2x}{2} (1 + L_1) f'(x) \right|.
\]
Using the relations (7.7), (7.9) and (7.10), we get

\[ (D_n f)'(x) = n \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_0^1 p_{n+1,k+1}(t)f'(t)dt \]

\[ = n(n+1) \left\{ \sum_{k=1}^n p_{n-1,k-1}(x) \int_0^1 p_{n,k}(t)f(t)dt - \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_0^1 p_{n,k}(t)f(t)dt \right\}. \]

From the above relation, we get

\[ \left| n \left[ D_n^{M,1}(f; x) - D_n(f; x) \right] - \frac{1 - 2x}{2} (1 + L_1) f'(x) \right| \]
\[ \leq \left| (a_1(n) + 1) \left( \frac{1}{2} - x \right) (D_n f)'(x) - \frac{1 - 2x}{2} (1 + L_1) f'(x) \right| \]
\[ \leq \frac{1}{2} - x \left\{ |L_1 - a_1(n)||D_n f'(x)| + |1 + L_1||f'(x) - (D_n f)'(x)| \right\}. \quad (7.7) \]

From [17, Theorem 2.45] we have

\[ |(D_n f)'(x) - f'(x)| \leq |(D_n e_1)'(x) - 1| |f'(x)| + \frac{1}{h} \gamma(x) \omega_1 (f'; h) \]
\[ + \left[ (D_n e_1)'(x) + \frac{1}{2h^2} \beta(x) \right] \omega_2 (f'; h), \quad (7.8) \]

where

\[ \gamma(x) := \left| \left(D_n \left( \frac{1}{2} e_2 - xe_1 \right) \right)'(x) \right| = \frac{2n|1 - 2x|}{(n+2)(n+3)} \leq \frac{2}{n+2}; \]
\[ \beta(x) := \left| \left(D \left( \frac{1}{3} e_3 - xe_2 + x^2 e_1 \right) \right)'(x) \right| = \frac{2n [x(1-x)(n-11) + 3]}{(n+2)(n+3)(n+4)} \leq \frac{1}{2(n+2)}. \]

Choosing \( h := \sqrt{\frac{2}{n+2}} \), we get

\[ |(D_n f)'(x) - f'(x)| \leq \frac{2}{n+2} |f'(x)| + \sqrt{\frac{2}{n+2}} \omega_1 \left( f'; \sqrt{\frac{2}{n+2}} \right) + \frac{9}{8} \omega_2 \left( f'; \sqrt{\frac{2}{n+2}} \right). \quad (7.9) \]

Also, we have

\[ |(D_n f)'(x)| \leq n \|f'\|_{\infty} \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_0^1 p_{n+1,k+1}(t)dt = \frac{n}{n+2} \|f'\|_{\infty} \leq \|f'\|_{\infty}. \quad (7.10) \]

Using the relations (7.7), (7.9) and (7.10), we get

\[ \left| n \left[ D_n^{M,1}(f; x) - D_n(f; x) \right] - \frac{1 - 2x}{2} (1 + L_1) f'(x) \right| \leq \frac{1}{2} \left| L_1 - a_1(n) \right| \|f'\|_{\infty} \]
\[ + |1 + L_1| \left[ \frac{2}{n+2} |f'(x)| + \sqrt{\frac{2}{n+2}} \omega_1 \left( f'; \sqrt{\frac{2}{n+2}} \right) + \frac{9}{8} \omega_2 \left( f'; \sqrt{\frac{2}{n+2}} \right) \right]. \]

(7.11)

From the relations (7.6), (7.11) and Theorem 7.1 the proof is complete. \(\square\)

Corollary 7.1 We have

\[
\Delta^D_n \leq \begin{cases} 
\mathcal{O} \left( \frac{1}{\sqrt{n}} \right) + \frac{1}{2} |L_1 - a_1(n)| \cdot \|f'\|_{\infty}, & \text{for } f \in C^3[0, 1], \\
\mathcal{O} \left( \frac{1}{n} \right) + \frac{1}{2} |L_1 - a_1(n)| \cdot \|f'\|_{\infty}, & \text{for } f \in C^4[0, 1].
\end{cases}
\]

8 The modified genuine Bernstein–Durrmeyer operators \(U_{n}^{M,1}\)

The genuine Bernstein–Durrmeyer operators were introduced by Chen [8] and Goodman and Sharma [16] as follows:

\[
U_n(f; x) = (1 - x)^n f(0) + x^n f(1) + (n - 1) \sum_{k=1}^{n-1} \left( \int_0^1 f(t) p_{n-2,k-1}(t) dt \right) p_{n,k}(x), \quad f \in C[0, 1].
\]

Using the fundamental polynomials \(p_{n,k}^{M,1}\) modified genuine Bernstein–Durrmeyer operators can be introduced as follows:

\[
U_{n}^{M,1}(f; x) = a(x, n)(1 - x)^{n-1} f(0) + a(1 - x, n)x^{n-1} f(1) + (n - 1) \sum_{k=1}^{n-1} p_{n,k}^{M,1}(x) \int_0^1 p_{n-2,k-1}(t) f(t) dt.
\]

(8.1)

This modification was also investigated in a recent note of Acu and Agrawal [4]. All the results given there will be improved in this section. Throughout this section we assume \(U_{n}^{M,1}(e_0) = 1\), namely the sequences \(a_0(n)\) and \(a_1(n)\) verify the condition (4.2).
Theorem 8.1 Let $f \in C[0, 1]$, $x \in [0, 1]$, $n \geq 1$. Then

$$
\left| U_{n}^{M,1}(f; x) - f(x) \right| \leq \left| U_{n}(f; x) - f(x) \right| \\
+ \left| (1 + a_{1}(n)) \left( \frac{1}{2} - x \right) \right| \left\{ 3 \omega_{2}(f; \sqrt{\sigma_{n}(x)}) \\
+ \frac{5}{n \sqrt{\sigma_{n}(x)}} \omega_{1}(f; \sqrt{\sigma_{n}(x)}) \right\}
$$

where $\sigma_{n}(x) = \frac{2nx(1-x) + (1-2x)^{2}}{n^{2}(n+1)} \leq \frac{1}{4n}$.

Proof We have

$$
\left| U_{n}^{M,1}(f; x) - f(x) \right| \leq \left| U_{n}(f; x) - f(x) \right| + \left| U_{n}^{M,1}(f; x) - U_{n}(f; x) \right|. \quad (8.2)
$$

In the following we will give an estimate of the quantity $\left| U_{n}^{M,1}(f; x) - U_{n}(f; x) \right|$. So,

$$
U_{n}^{M,1}(f; x) - U_{n}(f; x) = (n-1) \sum_{k=1}^{n-1} \left\{ a(x, n) p_{n-1,k}(x) \\
+ a(1-x, n) p_{n-1,k-1}(x) \right\} \int_{0}^{1} p_{n-2,k-1}(t) f(t) dt \\
+ a(x, n)(1-x)^{n-1} f(0) + a(1-x, n)x^{n-1} f(1) - (1-x)^{n} f(0) - x^{n} f(1) \\
- (n-1) \sum_{k=1}^{n-1} \left\{ (1-x) p_{n-1,k}(x) \\
+ x p_{n-1,k-1}(x) \right\} \int_{0}^{1} p_{n-2,k-1}(t) f(t) dt
$$

$$
= [(a_{1}(n) + 1)x + a_{0}(n) - 1] \left\{ (n-1) \sum_{k=1}^{n-1} p_{n-1,k}(x) \int_{0}^{1} p_{n-2,k-1}(t) f(t) dt \\
- (n-1) \sum_{k=1}^{n-1} p_{n-1,k-1}(x) \int_{0}^{1} p_{n-2,k-1}(t) f(t) dt \right\} (1-x)^{n-1} f(0) - x^{n-1} f(1)
$$

where

$$
A_{n}(f; x) := (n-1) \sum_{k=1}^{n-1} p_{n-1,k}(x) \int_{0}^{1} p_{n-2,k-1}(t) f(t) dt + (1-x)^{n-1} f(0); \\
B_{n}(f; x) := (n-1) \sum_{k=0}^{n-2} p_{n-1,k}(x) \int_{0}^{1} p_{n-2,k}(t) f(t) dt + x^{n-1} f(1).
$$
Note that the operators $A_n$ and $B_n$ can be written as follows

$$A_n(f; x) = \sum_{k=0}^{n-1} F_k(f) p_{n-1,k}(x), \quad B_n(f; x) = \sum_{k=0}^{n-1} G_k(f) p_{n-1,k}(x),$$

where

$$F_0(f; x) = f(0), \quad F_k(f; x) = (n-1) \int_0^1 p_{n-2,k-1}(t) f(t) \, dt, \quad k = 1, \ldots, n-1,$$

$$G_k(f; x) = (n-1) \int_0^1 p_{n-2,k}(t) f(t) \, dt, \quad k = 0, \ldots, n-2, \quad G_{n-1}(f; x) = f(1).$$

Let $F$ be a positive linear functional and

$$b^F := F(e_1), \quad \mu_2^F := \frac{1}{2} F\left(e_1 - b^Fe_0\right)^2,$$

$$\sigma_n(x) := \sum_{k=0}^{n-1} \left( \mu_2^F + \mu_2^G \right) p_{n-1,k}(x), \quad \delta := \sup_k \left| b^{F_k} - b^{G_k} \right|.$$

Using [7, Theorem 5] for $f \in C[0, 1]$ and $0 < h \leq \frac{1}{2}$, we get

$$|A_n(f; x) - B_n(f; x)| \leq \frac{3}{2} \left( 1 + \frac{\sigma_n(x)}{h^2} \right) \omega_2(f, h) + \frac{5\delta}{h} \omega_1(f, h). \quad (8.3)$$

Since

$$b^{F_k} = \frac{k}{n}, \quad b^{G_k} = \frac{k+1}{n}, \quad \mu_2^F = \frac{1}{2} \frac{k(n-k)}{n^2(n+1)}, \quad \mu_2^G = \frac{1}{2} \frac{(k+1)(n-k-1)}{n^2(n+1)},$$

we get $\sigma_n(x) = \frac{2nx(1-x) + (1-2x)^2}{2n^2(n+1)} \leq \frac{1}{4n}$ and $\delta = \frac{1}{n}$.

Choosing $h := \sqrt{\sigma_n(x)}$ we obtain

$$\left| U_n^{M,1}(f; x) - U_n(f; x) \right| \leq \left| (1 + a_1(n)) \left( \frac{1}{2} - x \right) \right| \left\{ 3\omega_2\left(f; \sqrt{\sigma_n(x)} \right) \right.$$

$$+ \frac{5}{n\sqrt{\sigma_n(x)}} \omega_1(f; \sqrt{\sigma_n(x)}) \bigg\}.$$

Using relations (8.2) and (8.4) we obtain the inequality claimed. □

Remark 8.1 (i) For $a_1(n) = 1$ estimates for genuine Bernstein–Durrmeyer operator $U_n$ are obtained; details are omitted here.

(ii) If $f \in C^2[0, 1]$, then for $a_1(n)$ bounded $\|U_n^{M,1}(f) - f\|_\infty = O\left(\frac{1}{n}\right)$. 
Theorem 8.2 Suppose that $U_{n}^{M,1}$ is given as above, $f \in C^{2}[0,1]$, $L_1 = \lim_{n \to \infty} a_1(n)$ exists. Then for $x \in [0,1]$ there holds

$$\Delta_{n}^{U} := \left| n \left[ U_{n}^{M,1}(f; x) - f(x) \right] - x(1-x)f''(x) - \frac{1-2x}{2}(1+L_1)f'(x) \right|$$

$$\leq \frac{5\sqrt{6}}{12} \omega_1 \left( f''; \sqrt{\frac{3}{n+2}} \right) + \frac{13}{32} \omega_2 \left( f''; \sqrt{\frac{3}{n+2}} \right) + \frac{9}{8} \omega_2 \left( f; \sqrt{\frac{2}{n+1}} \right)$$

$$+ \frac{1}{2} \left\{ |L_1 - a_1(n)| \| f' \|_{\infty} + |1 + L_1| \left( \frac{1}{\sqrt{n+1}} \omega_1 \left( f'; \frac{1}{\sqrt{n+1}} \right) \right) \right\}.$$

\textbf{Proof} For $\Delta_{n}^{U}$ the following inequality holds:

$$\Delta_{n}^{U} \leq \left| n \left[ U_{n}(f; x) - f(x) \right] - x(1-x)f''(x) \right|$$

$$+ \left| n \left[ U_{n}^{M,1}(f; x) - U_{n}(f; x) \right] - \frac{1-2x}{2}(1+L_1)f'(x) \right|.$$  \hspace{1cm} (8.5)

From [15, Theorem 5] a quantitative Voronovskaya-type theorem for genuine Bernstein–Durrmeyer operators can be given as follows:

$$\left| (n+1) \left[ U_{n}(f; x) - f(x) \right] - x(1-x)f''(x) \right|$$

$$\leq \frac{5\sqrt{6}}{12} \omega_1 \left( f''; \sqrt{\frac{3}{n+2}} \right) + \frac{13}{32} \omega_2 \left( f''; \sqrt{\frac{3}{n+2}} \right), n \geq 1.$$

Using the pointwise estimate of genuine Bernstein–Durrmeyer operator (see [17, Corollary 3.25])

$$|U_{n}(f; x) - f(x)| \leq \frac{9}{8} \omega_2 \left( f; \sqrt{\frac{2}{n+1}} \right),$$

we get

$$\left| n \left[ U_{n}(f; x) - f(x) \right] - x(1-x)f''(x) \right| \leq \frac{5\sqrt{6}}{12} \omega_1 \left( f''; \sqrt{\frac{3}{n+2}} \right)$$

$$+ \frac{13}{32} \omega_2 \left( f''; \sqrt{\frac{3}{n+2}} \right) + \frac{9}{8} \omega_2 \left( f; \sqrt{\frac{2}{n+1}} \right).$$  \hspace{1cm} (8.6)
The second difference of (8.5) can be estimated as follows

\[ n \left[ U_n^M,1(f; x) - U_n(f; x) \right] - \left| \frac{1 - 2x}{2} (1 + L_1) f'(x) \right| \]

\[ = n(a_1(n) + 1) \left( x - \frac{1}{2} \right) \left\{ (1 - x)^{n-1} f(0) - x^{n-1} f(1) \right. \]

\[ + (n - 1) \sum_{k=1}^{n-1} p_{n-1,k}(x) \int_0^1 p_{n-2,k-1}(t) f(t) dt \]

\[ - (n - 1) \sum_{k=1}^{n-1} p_{n-1,k-1}(x) \int_0^1 p_{n-2,k-1}(t) f(t) dt \left\} - \frac{1 - 2x}{2} (1 + L_1) f'(x) \right| . \]

But,

\[ (U_n f)'(x) = n \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_0^1 p_{n-1,k}(t) f'(t) dt \]

\[ = n \left\{ -(1 - x)^{n-1} f(0) + x^{n-1} f(1) \right. \]

\[ - (n - 1) \sum_{k=1}^{n-1} p_{n-1,k}(x) \int_0^1 p_{n-2,k-1}(t) f(t) dt \]

\[ + (n - 1) \sum_{k=1}^{n-1} p_{n-1,k-1}(x) \int_0^1 p_{n-2,k-1}(t) f(t) dt \left\} . \]

From the above relation, we get

\[ n \left[ U_n^{M,1}(f; x) - U_n(f; x) \right] - \left| \frac{1 - 2x}{2} (1 + L_1) f'(x) \right| \]

\[ \leq \left| (a_1(n) + 1) \left( \frac{1}{2} - x \right) (U_n f)'(x) - \frac{1 - 2x}{2} (1 + L_1) f'(x) \right| \]

\[ \leq \frac{1}{2} - x \left| \left\{ |L_1 - a_1(n)| |(U_n f)'(x)| + |1 + L_1||f'(x) - (U_n f)'(x)| \right\} \right| . \] (8.7)

From [17, Theorem 3.14] we have

\[ |(U_n f)'(x) - f'(x)| \leq \frac{1}{\sqrt{n + 1}} \omega_1 \left( f'; \frac{1}{\sqrt{n + 1}} \right) + \frac{5}{4} \omega_2 \left( f'; \frac{1}{\sqrt{n + 1}} \right) . \] (8.8)
Also, there holds
\[
\left| (U_n f)'(x) \right| \leq n \|f'\|_\infty \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_0^1 p_{n-1,k}(t) dt = \|f'\|_\infty. \tag{8.9}
\]

From the relations (8.5)–(8.9) the proof is complete. \qed

**Corollary 8.1** We have
\[
\Delta_n^U \leq \begin{cases} 
O\left(\frac{1}{\sqrt{n}}\right) + \frac{1}{2} |L_1 - a_1(n)| \cdot \|f'\|_\infty, & \text{for } f \in C^3[0, 1], \\
O\left(\frac{1}{n}\right) + \frac{1}{2} |L_1 - a_1(n)| \cdot \|f'\|_\infty, & \text{for } f \in C^4[0, 1].
\end{cases}
\]

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**References**

1. Acu, A.M., Acar, T., Muraru, C.V., Radu, V.A.: Some approximation properties by a class of bivariate operators. Math. Methods Appl. Sci. 42, 5551–5565 (2019)
2. Acu, A.M., Bascanbaz-Tunca, G., Cetin, N.: Approximation by certain linking operators. Ann. Funct. Anal. 11, 1184–1202 (2020)
3. Acu, A.M., Gonska, H.: Classical Kantorovich operators revisited. Ukr. Math. J. 71, 843–852 (2019)
4. Acu, A.M., Agrawal, P.: Better approximation of functions by genuine Bernstein–Durrmeyer type operators. Carpathian J. Math. 35(2), 125–136 (2019)
5. Acu, A.M., Gupta, V., Tachev, G.: Better numerical approximation by Durrmeyer type operators. Results Math. 74, 90 (2019)
6. Acu, A.M., Gupta, V., Tachev, G.: Modified Kantorovich operators with better approximation properties. Numer. Algorithms 81, 125–149 (2019)
7. Acu, A.M., Rasa, I.: New estimates for the differences of positive linear operators. Numer. Algorithms 73(3), 775–789 (2016)
8. Chen, W.: On the modified Durrmeyer–Bernstein operator (in handwritten Chinese). Report of the fifth Chinese conference on approximation theory, Zhen Zhou, China (1987)
9. Durrmeyer, J.L.: Une formule d’inversion de la transformée de Laplace: Applications à la théorie des moments. Thèse de 3e cycle, Paris (1967)
10. Gavrea, I., Opris, A.A.: Modified Kantorovich–Stancu operators (II). Stud. Univ. Babes-Bolyai Math. 64(2), 197–205 (2019)
11. Gonska, H.: Two problems on best constants in direct estimates. In: Ditzian, Z et al. (eds.) Problem section of proc. Edmonton conf. approximation theory, vol. 194. American Mathematical Society, Providence, RI (1983)
12. Gonska, H., Lupaș, A.: On an algorithm for Bernstein polynomials. In: Lyche, T., Mazure, M.-L., Schumaker, L. (eds.) Curve and Surface Design: Saint Malo, pp. 197–203. Nashboro Press, Brentwood (2002)
13. Gonska, H., Heilmann, M., Raşa, I.: Kantorovich operators of order k. Numer. Funct. Anal. Optim. 32(7), 717–738 (2011)
14. Gonska, H.: Quantitative Korovkin-type theorems on simultaneous approximation. Math. Z. 186, 419–433 (1984)
15. Gonska, H., Raşa, I.: A Voronovskaja estimate with second order of smoothness. In: Dumitru, A. et al. (ed) Proceedings of the 5th International Symposium “Mathematical Inequalities”, Sibiu, Romania, September 25–27, 2008, Sibiu, “Lucian Blaga” University Press. ISBN 978-973-739-740-9, pp. 76–90 (2008)
16. Goodman, T.N.T., Sharma, A.: A modified Bernstein–Schoenberg operator. In: Sendov, B. et al. Proceedings of the Conference on Constructive Theory of Functions, Varna 1987 (pp. 166–173). Publ. House Bulg. Acad. of Sci., Sofia (1988)
17. Kacsó, D.: Certain Bernstein–Durrmeyer type operators preserving linear functions. Habilitation Thesis, Duisburg-Essen University (2007)
18. Kantorovich, L.V.: Sur certains developpements suivant les polynômes de la forme de S. Bernstein I, II, Dokl. Akad. Nauk. SSSR, pp. 563–568, 595–600 (1930)
19. Khosravian-Arab, H., Dehghan, M., Eslahchi, M.R.: A new approach to improve the order of approximation of the Bernstein operators: theory and applications. Numer. Algorithms 77(1), 111–150 (2018)
20. Lupaş, A.: Die Folge der Betaoperatoren. Dissertation, Universität Stuttgart (1972)
21. Opris, A.A.: Approximation by modified Kantorovich–Stancu operators. J. Inequalities Appl. 2018, 346 (2018)
22. Păltănea, R.: Approximation Theory Using Positive Linear Operators. Birkhäuser, Boston (2004)
23. Zygmund, A.: Smooth functions. Duke Math. J. 12, 47–76 (1945)

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