ON TWO BLOCH TYPE THEOREMS FOR QUATERNIONIC
SLICE REGULAR FUNCTIONS

ZHENGHUA XU AND XIEPING WANG

Abstract. In this paper we prove two Bloch type theorems for quaternionic
slice regular functions. We first discuss the injective and covering properties
of some classes of slice regular functions from slice regular Bloch spaces and
slice regular Bergman spaces, respectively. And then we show that there exits
a universal ball contained in the image of the open unit ball
in quaternions
through the slice regular rotation $\tilde{f}_u$ of each slice regular function $f : \mathbb{H} \to \mathbb{H}$
with $f'(0) = 1$ for some $u \in \partial \mathbb{B}$.

1. Introduction

Let $\mathbb{C}$ be the complex plane and $D(z_0, R)$ the open disc centred at $z_0 \in \mathbb{C}$ with
radius $R > 0$. For simplicity, we denote by $\mathbb{D}$ the open unit disc $D(0, 1)$. Let $H(\mathbb{D})$
denote the class of holomorphic functions on $\mathbb{D}$. Given a function $F \in H(\mathbb{D})$, we
define $B_F$ to be the least upper bound of all positive numbers $R > 0$
such that there exists a number $z_0 \in \mathbb{C}$ and a domain $\Omega \subset \mathbb{D}$ which is mapped conformally
by $F$ onto $D(z_0, r)$. The Bloch constant $B_h$ is defined to be
$$B_h := \inf \{ B_F : F \in H(\mathbb{D}), F'(0) = 1 \}.$$ The Bloch’s theorem asserts that $B_h > 0$ in [3]. In [24], Landau showed that
(1.1) $B_h = \inf \{ B_F : F \in B_h, F(0) = 0, F'(0) = 1 \}$,
where $B_h$ denotes the class of holomorphic functions $F \in H(\mathbb{D})$ with the Bloch
seminorm
$$\|F\|_{B_h} := \sup_{z \in \mathbb{D}} (1 - |z|^2)|F'(z)| \leq 1.$$ In [1], Ahlfors proved that $B_h \geq 3/4$ using his far-reaching generalization of
the classical Schwarz-Pick lemma. The strict inequality $B_h > 3/4$ was established
by Heins [21] and Pommerenke [26], respectively. In [5], Bonk gave a new and
remarkable proof of this result and improved slightly it using the following theorem,
known as Bonk’s distortion theorem.

Theorem 1.1. Let $F \in B_h$ be such that $F'(0) = 1$. Then the inequality
$$\Re F'(z) \geq \frac{1 - \sqrt{3}|z|}{(1 - |z|/\sqrt{3})^3}$$
holds for all $z \in D(0, 1/\sqrt{3})$. 

2010 Mathematics Subject Classification. 30G35.
Key words and phrases. Quaternion, Slice regular functions,Bloch constant.
This work was supported by the NNSF of China (11071230), RFDP (20123402110068).
The finding of the precise value of $B_h$ is known to be the number one open problem in the geometric function theory of one complex variable since the confirmation of the Bieberbach conjecture by de Branges in 1985 [10]. To the authors’ knowledge, the best lower estimate for $B_h$ is by now in [6]. The Bloch’s theorem does not hold for general holomorphic mappings of several complex variables. The counterexample can be found in [20]. Thus, one needs to restrict the class of mappings to a more specific subclass to obtain a Bloch’s theorem. One of the well-known subclasses is the class of $K$-quasiregular mappings [31]. Recently, the Bloch’s theorem in the bicomplex number setting has been investigated successfully in [27].

In this paper, we first establish the quaternionic analogues of Bonk’s distortion theorem and Bloch type theorem for slice regular functions. The theory of slice regular functions over quaternions was initiated recently by Gentili and Struppa [17, 18]. It is significantly different from the more classical theory of regular functions in the sense of Cauchy-Fueter and has elegant applications to the functional calculus for noncommutative operators [9], Schur analysis [2] and the construction and classification of orthogonal complex structures on dense open subsets of $\mathbb{R}^4$ [13]. For the detailed up-to-date theory, we refer the reader to the monographs [16, 9].

In order to formulate precisely our main results, we first introduce a few notations. Let $\mathbb{H}$ be the skew field of quaternions and $S$ the unit sphere of purely imaginary quaternions, i.e.

$$S = \{q \in \mathbb{H} : q^2 = -1\}.$$ 

For each $R > 0$ and each point $q_0 \in \mathbb{H}$, we set

$$B(q_0, R) := \{q \in \mathbb{H} : |q - q_0| < R\},$$

the Euclidean ball centred at $q_0$ of radius $R$, and for each $I \in S$, we denote by $B(0, R)_I$ the intersection $B(0, R) \cap \mathbb{C}_I$. For simplicity, we denote by $B$ the open unit disc $B(0, 1)$. Also, we denote by $B_r$ the slice regular Bloch space on $\mathbb{B}$, that is, the space of slice regular functions $f$ on $\mathbb{B}$ with the Bloch seminorm

$$\|f\|_{B_r} := \sup_{q \in \mathbb{B}} (1 - |q|) |f'(q)| < \infty,$$

where $f'$ is the slice derivative of $f$. For each slice regular function $f$ on $\mathbb{B}$, we define $B_f$ to be the least upper bound of all positive numbers $R > 0$ such that there exists a number $q_0 \in \mathbb{H}$ and a domain $\Omega \subset \mathbb{B}$ such that the restriction

$$f|_\Omega : \Omega \to B(q_0, R)$$

is a homeomorphism. Similar to [11], we define the Bloch constant $B_r$ for $B_r$ as

$$B_r = \inf \{B_f : f \in B_r, f(0) = 0, f'(0) = 1, \|f\|_{B_r} \leq 1\}.$$ 

Now we can show that $B_r > 0.23$ by proving the following result.

**Theorem 1.2.** Let $f \in B_r$ be such that $f(0) = 0, f'(0) = 1$ and $\|f\|_{B_r} \leq 1$. Then

(a) the inequality

$$\text{Re} f'(q) \geq \frac{1 - \sqrt{3} |q|}{(1 - |q|/\sqrt{3})^3}$$

holds for all $q \in B(0, 1/\sqrt{3})$. In particular, $f|_{B(0, 1/\sqrt{3})} : B(0, 1/\sqrt{3}) \to \mathbb{H}$ is injective for every $I \in S.$
ON TWO BLOCH TYPE THEOREMS FOR QUATERNIONIC SLICE REGULAR FUNCTIONS

(b) \( f \) is injective on \( B(0, r_\mathbb{B}) \) and \( B(0, R_\mathbb{B}) \subset f(B(0, r_\mathbb{B})) \), where
\[
r_\mathbb{B} = \sup_{0 < r < 1} \left\{ \frac{1}{2} \log \frac{1+r}{1-r} - \sqrt{\frac{1}{4} \left( \log \frac{1+r}{1-r} \right)^2 - r^2} \right\} \approx 0.3552,
\]
and
\[
R_\mathbb{B} = \sup_{0 < r < 1} \left\{ \frac{1}{2r^2} \log \frac{1+r}{1-r} \left( \frac{1}{2} \log \frac{1+r}{1-r} - \sqrt{\frac{1}{4} \left( \log \frac{1+r}{1-r} \right)^2 - r^2} \right) \right\} \approx 0.2308.
\]
Moreover,
\[
B(0, \sqrt{3}/4) \subset f(B(0, 1/\sqrt{3})).
\]

We can also prove a Bloch type theorem for functions from slice regular Bergman space \( A^p(\mathbb{B})(1 \leq p < +\infty) \) of the second kind introduced in \[22\], that is, the Hilbert space of slice regular functions \( f \) on \( \mathbb{B} \) with the norm
\[
\|f\|_{A^p} := \sup_{t \in \mathbb{B}} \left( \int_{\mathcal{B}_I} |f_t(z)|^p d\sigma_I(z) \right)^{\frac{1}{p}} < \infty,
\]
where \( d\sigma_I \) stands for the normalised Lebesgue measure on the plane \( \mathbb{C}_I \), i.e
\[
d\sigma_I(z) = \frac{1}{\pi} dxdy
\]
for all \( z = x + yI \in \mathbb{C}_I \).

**Theorem 1.3.** Let \( p \in [1, +\infty) \) and \( f \in A^p(\mathbb{B}) \) be such that \( f(0) = 0, f'(0) = 1 \) and \( \|f\|_{A^p} \leq 1 \). Then \( f \) is injective on \( B(0, r_p) \) and \( B(0, R_p) \subset f(B(0, r_p)) \), where
\[
r_p = \sup_{0 < r < 1} \left\{ \left( 1 - r \right)^{-\frac{1}{p}} - \sqrt{\left( 1 - r \right)^{-\frac{1}{p}} - r^2} \right\},
\]
and
\[
R_p = \sup_{0 < r < 1} \left\{ r^{-2} \left( 1 - r \right)^{-\frac{1}{p}} \left( \left( 1 - r \right)^{-\frac{1}{p}} - \sqrt{\left( 1 - r \right)^{-\frac{1}{p}} - r^2} \right)^2 \right\}.
\]

To prove Theorems 1.2 and 1.3 we need to establish a quaternionic analogue of a classical result due to Landau (see \[22\] or \[22\] pp. 36–39) for slice regular functions. Let \( \alpha \in (0, 1) \). Define \( \mathcal{F}_\alpha \) to be a family of slice regular functions given by
\[
\mathcal{F}_\alpha := \{ f : \mathbb{B} \to \mathbb{B} \mid f \text{ is slice regular such that } f(0) = 0, |f'(0)| = \alpha \},
\]
and for every \( f \in \mathcal{F}_\alpha \), denote by \( r_\alpha(f) \) the radius of the largest ball \( B(r) \) on which \( f \) is injective. Namely,
\[
r_\alpha(f) = \sup \{ r : f|_{B(r)} \text{ is injective} \}.
\]
Set
\[
r(\mathcal{F}_\alpha) = \inf \{ r_\alpha(f) : f \in \mathcal{F}_\alpha \}.
\]
From [16 Theorem 8.13] it follows that for each \( f \in \mathcal{F}_\alpha \), the real differential of \( f \) at the origin 0 is non-degenerate, and thus the inverse function theorem implies that \( r_\alpha(f) \) is positive. Indeed, our result shows that \( r(\mathcal{F}_\alpha) \) is a positive number as well.

**Theorem 1.4.** With notations as above, the following statements hold:

(a) for each \( \alpha \in (0, 1) \),
\[
r(\mathcal{F}_\alpha) = \frac{\alpha}{1 + \sqrt{1 - \alpha^2}};
\]
(b) for each $\alpha \in (0, 1)$ and each $f \in \mathcal{F}_\alpha$,
\[ r_\alpha(f) \geq r(\mathcal{F}_\alpha) \]
with equality if and only if
\begin{equation}
(1.3) \quad f(q) = q(1 - q\alpha - t\theta)^{-*} (q - \alpha e^{i\theta}) v
\end{equation}
for some $I \in \mathbb{S}$, $\theta \in \mathbb{R}$ and $v \in \partial B$;  
(c) for each $0 < r \leq r(\mathcal{F}_\alpha)$,
\[ \bigcap_{f \in \mathcal{F}_\alpha} f(B(0, r)) = B(0, R_\alpha(r)), \]
where
\[ R_\alpha(r) = r^{\alpha - 1} \frac{\alpha - r}{1 - \alpha r}; \]
(d) for each $\alpha \in (0, 1)$, $r \in (0, r(\mathcal{F}_\alpha))$ and $f \in \mathcal{F}_\alpha$,
\[ B(0, R_\alpha(r)) \subset f(B(0, r)). \]
Moreover, $B(0, R_\alpha(r))$ is the largest ball contained in the image set $f(B(0, r))$ if and only if $f$ is of the form $(1.3)$.

In [3], Bloch discovered a theorem stating that if $F$ is a holomorphic function on the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$ such that $|F'(0)| = 1$, then the image domain contains discs of radius $3/2 - \sqrt{6}$. More recently, the corresponding result has been established for square-integrable monogenic functions in the open ball of the paravector space $\mathbb{R}^3$ with values in $\mathbb{R}^3 \subset \mathbb{H}$ as well (see [25, Theorem 8]), and for a kind of regular translations of slice regular functions on $\mathbb{B}$ in the quaternionic setting (see [28, Theorem 6]), respectively. In this paper, we also present another version of Bloch type theorem for slice regular functions. To this purpose, we need to consider the slice regular rotation $\tilde{f}_u$, instead of the regular translation in [28], of a slice regular function $f(q) = \sum_{n=0}^{\infty} q^n a_n$ on $\mathbb{B}$ given by
\[ \tilde{f}_u(q) = \sum_{n=0}^{\infty} q^n u^n a_n, \]
for some constant $u \in \partial \mathbb{B}$.

**Theorem 1.5.** Let $f$ be slice regular on $\overline{\mathbb{B}}$ such that $f'(0) = 1$. Then there exists $u \in \partial \mathbb{B}$ such that the image of $\mathbb{B}$ under the slice regular rotation $\tilde{f}_u$ of $f$ contains an open ball of radius $5/2 - \sqrt{6}$.

The remaining part of this paper is organized as follows. In Sect. 2, we set up basic notations and give some preliminary results from the theory of slice regular functions over quaternions. In Sect. 3, we shall give some useful lemmas, among which the Landau’s lemma is established by means of one Lindelöf type inequality for slice regular functions. Sect. 4 is devoted to the proofs of Theorems 1.2, 1.3 and 1.4. Finally, Theorem 1.5 will be proved in Sect. 5.
2. Preliminaries

In this section we recall some necessary definitions and preliminary results on slice regular functions. To have a more complete insight on the theory, we refer the reader to the monograph [16].

Let $\mathbb{H}$ denote the non-commutative, associative, real algebra of quaternions with standard basis \{1, $i$, $j$, $k$\}, subject to the multiplication rules
\[i^2 = j^2 = k^2 = ijk = -1.\]
Every element $q = x_0 + x_1i + x_2j + x_3k$ in $\mathbb{H}$ is composed by the real part $\text{Re}(q) = x_0$ and the imaginary part $\text{Im}(q) = x_1i + x_2j + x_3k$. The conjugate of $q \in \mathbb{H}$ is then $\bar{q} = \text{Re}(q) - \text{Im}(q)$ and its \textit{modulus} is defined by $|q|^2 = q\bar{q} = |\text{Re}(q)|^2 + |\text{Im}(q)|^2$.

We can therefore calculate the multiplicative inverse of each $q \neq 0$ as $q^{-1} = |q|^{-2}\bar{q}$. Every $q \in \mathbb{H}$ can be expressed as $q = x + yI$, where $x, y \in \mathbb{R}$ and
\[I = \frac{\text{Im}(q)}{|\text{Im}(q)|}\]
if $\text{Im} q \neq 0$, otherwise we take $I$ arbitrarily such that $I^2 = -1$. Then $I$ is an element of the unit 2-sphere of purely imaginary quaternions
\[\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}.

For every $I \in \mathbb{S}$ we will denote by $\mathbb{C}_I$ the plane $\mathbb{R} \oplus I\mathbb{R}$, isomorphic to $\mathbb{C}$, and, if $\Omega \subset \mathbb{H}$, by $\Omega_I$ the intersection $\Omega \cap \mathbb{C}_I$. Also, for $R > 0$, we will denote the open ball centred at $q_0 \in \mathbb{H}$ with radius $R$ by
\[B(q_0, R) = \{q \in \mathbb{H} : |q - q_0| < R\}.

And let $\mathcal{B}$ denote the open unit ball $B(0, 1)$ for simplicity.

We can now recall the definition of slice regularity.

**Definition 2.1.** Let $\Omega$ be a domain in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is called slice regular if, for all $I \in \mathbb{S}$, its restriction $f_I$ to $\Omega_I$ is holomorphic, i.e., it has continuous partial derivatives and satisfies
\[\partial_x f_I(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I\frac{\partial}{\partial y} \right) f_I(x + yI) = 0\]
for all $x + yI \in \Omega_I$. We will denote by $\mathcal{SR}(\mathcal{B})$ the set of slice regular functions on $\mathcal{B}$.

A wide class of examples of regular functions is given by polynomials and power series. Indeed, a function $f$ is slice regular on an open ball $B(0, R)$ if and only if $f$ admits a power series expansion $f(q) = \sum_{n=0}^{\infty} q^n a_n$ converging absolutely and uniformly on every compact subset of $B(0, R)$. As shown in [18], the natural domains of definition of slice regular functions are the so-called symmetric slice domains.

**Definition 2.2.** Let $\Omega$ be a domain in $\mathbb{H}$.

1. $\Omega$ is called a slice domain if it intersects the real axis and if for any $I \in \mathbb{S}$, $\Omega_I$ is a domain in $\mathbb{C}_I$.
2. $\Omega$ is called an axially symmetric domain if for any point $x + yI \in \Omega$, with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, the entire two-dimensional sphere $x + y\mathbb{S}$ is contained in $\Omega$.

A domain in $\mathbb{H}$ is called a symmetric slice domain if it is not only a slice domain, but also an axially symmetric domain. From now on, we will omit the term ‘slice’ when referring to slice regular functions and will focus mainly on regular functions.
on an open ball $B(0, R)$ which is a typical axially symmetric slice domain. For regular functions the natural definition of derivative is given by the following (see [17], [18]).

**Definition 2.3.** Let $f : \mathbb{B} \rightarrow \mathbb{H}$ be a regular function. For each $I \in S$, the $I$-derivative of $f$ at $q = x + yI$ is defined by

$$\partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI)$$

on $\mathbb{B}_I$. The slice derivative of $f$ is the function $f'$ defined by $\partial_I f$ on $\mathbb{B}_I$ for all $I \in S$.

The definition is well-defined because, by direct calculation, $\partial_I f = \partial_J f$ in $\mathbb{B}_I \cap \mathbb{B}_J$ for any choice of $I, J \in S$. Furthermore, notice that the operators $\partial_I$ and $\partial_I$ commute, and $\partial_I f = \partial f/\partial x$ for regular functions. Therefore, the slice derivative of a regular function is still regular so that we can iterate the differentiation to obtain the $n$-th slice derivative

$$\partial^n_I f(x + yI) = \frac{\partial^n f}{\partial x^n}(x + yI), \quad \forall n \in \mathbb{N}.$$ 

In what follows, for the sake of simplicity, we will denote the $n$-th slice derivative by $f^{(n)}$ for every $n \in \mathbb{N}$.

In the theory of regular functions, the following splitting lemma (see [18]) relates closely slice regularity to classical holomorphy.

**Lemma 2.4 (Splitting Lemma).** Let $f$ be a regular function on $\mathbb{B}$. Then for any $I \in S$ and any $J \in S$ with $J \perp I$, there exist two holomorphic functions $F, G : \mathbb{B}_I \rightarrow \mathbb{C}_I$ such that

$$f_I(z) = F(z) + G(z)J, \quad \forall z = x + yI \in \mathbb{B}_I.$$ 

Since the regularity does not keep under point-wise product of two regular functions a new multiplication operation, called the regular product (or $*$-product), appears via a suitable modification of the usual one subject to noncommutative setting. The regular product plays a key role in the theory of slice regular functions. On open balls centred at the origin, the $*$-product of two regular functions is defined by means of their power series expansions (see, e.g., [15], [8]).

**Definition 2.5.** Let $f, g : \mathbb{B} \rightarrow \mathbb{H}$ be two regular functions and let

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad g(q) = \sum_{n=0}^{\infty} q^n b_n$$

be their series expansions. The regular product (or $*$-product) of $f$ and $g$ is the function defined by

$$f \ast g(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^{n} a_k b_{n-k} \right)$$

and it is regular on $\mathbb{B}$.

Notice that the $*$-product is associative and is not, in general, commutative. Its connection with the usual pointwise product is clarified by the following result [15], [8].
Proposition 2.6. Let \( f \) and \( g \) be regular on \( \mathbb{B} \). Then for all \( q \in \mathbb{B} \),

\[
f \ast g(q) = \begin{cases} 
  f(q)g(f(q)^{-1}qf(q)) & \text{if } f(q) \neq 0; \\
  0 & \text{if } f(q) = 0.
\end{cases}
\]

We remark that if \( q = x + yI \) and \( f(q) \neq 0 \), then \( f(q)^{-1}qf(q) \) has the same modulus and same real part as \( q \). Therefore \( f(q)^{-1}qf(q) \) lies in the same 2-sphere \( x + yS \) as \( q \). Notice that a zero \( x_0 + y_0I \) of the function \( g \) is not necessarily a zero of \( f \ast g \), but some element on the same sphere \( x_0 + y_0S \) does. In particular, a real zero of \( g \) is still a zero of \( f \ast g \).

Definition 2.7. Let \( f(q) = \sum_{n=0}^{\infty} q^n a_n \) be a regular function on \( \mathbb{B} \). We define the regular conjugate of \( f \) as

\[ f^c(q) = \sum_{n=0}^{\infty} q^n \bar{a}_n, \]

and the symmetrization of \( f \) as

\[ f^s(q) = f \ast f^c(q) = f^c \ast f(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^{n} a_k \bar{a}_{n-k} \right). \]

Both \( f^c \) and \( f^s \) are regular functions on \( \mathbb{B} \).

We are now able to define the inverse element of a regular function \( f \) with respect to the *-product. Let \( Z_{f^s} \) denote the zero set of the symmetrization \( f^s \) of \( f \).

Definition 2.8. Let \( f \) be a regular function on \( \mathbb{B} \). If \( f \) does not vanish identically, its regular reciprocal is the function defined by

\[ f^{-\ast}(q) := f^s(q)^{-1} f^c(q) \]

and it is regular on \( \mathbb{B} \setminus Z_{f^s} \).

The following result shows that the regular quotient is nicely related to the pointwise quotient (see \([29, 30]\)).

Proposition 2.9. Let \( f \) and \( g \) be regular on \( \mathbb{B} \). Then for all \( q \in \mathbb{B} \setminus Z_{f^s} \),

\[ f^{-\ast} \ast g(q) = f(T_f(q))^{-1} g(T_f(q)), \]

where \( T_f : \mathbb{B} \setminus Z_{f^s} \to \mathbb{B} \setminus Z_{f^s} \) is defined by \( T_f(q) = f^c(q)^{-1} qf^c(q) \). Furthermore, \( T_f \) and \( T_{f^s} \) are mutual inverses so that \( T_f \) is a diffeomorphism.

We now recall a useful result for regular functions \([12, 14]\).

Theorem 2.10. Let \( \Omega \) be a symmetric slice domain and let \( f : \Omega \to \mathbb{H} \) be a nonconstant regular function. If \( U \) is an axially symmetric open subset of \( \Omega \), then \( f(U) \) is open. In particular, the image \( f(\Omega) \) is open.

3. Some lemmas

In this section, we shall give some useful lemmas, which will be used in Sections 4 and 5. We begin with the following simple proposition.
**Proposition 3.1.** Let $\Omega \subset \mathbb{H}$ be a bounded domain and $f: \Omega \to \mathbb{H}$ a continuous function such that $f(\Omega)$ is open in $\mathbb{H}$. Let $a \in \Omega$ be a point such that

\[(3.1) \quad s := \liminf_{q \to \partial \Omega} |f(q) - f(a)| > 0.\]

Then $B(f(a), s) \subset f(\Omega)$.

**Proof.** For each point $w$ on the boundary $\partial f(\Omega)$ of $f(\Omega)$, there is a sequence $\{q_n\}_{n=1}^{\infty}$ in $\Omega$ such that $\lim_{n \to \infty} f(q_n) = w$. Since $\overline{\Omega}$ is compact, we may assume that $\{q_n\}_{n=1}^{\infty}$ converges to a point, say $q_\infty \in \overline{\Omega}$. If $q_\infty \in \Omega$, then, by the continuity of $f$, $w = f(q_\infty) \in f(\Omega)$, which contradicts with the openness of $f(\Omega)$. Therefore, $q_\infty \in \partial \Omega$. This together with (3.1) implies that

\[|w - f(a)| = \lim_{n \to \infty} |f(q_n) - f(a)| \geq \liminf_{q \to \partial \Omega} |f(q) - f(a)| = s > 0.\]

In other words, the boundary $\partial f(\Omega)$ of the open set $f(\Omega)$ lies outside of the ball $B(f(a), s)$. Consequently, $f(\Omega)$ must contain the ball $B(f(a), s)$. \qed

**Remark 3.2.** From the proof of Proposition 3.1 above, the result holds naturally under the conditions as described in Proposition 3.1 for general setting $\mathbb{R}^n$, instead of quaternions $\mathbb{H} \cong \mathbb{R}^4$.

The so-called *Apollonius circle* also can be generalized trivially to the quaternionic setting.

**Lemma 3.3 (Apollonius).** Let $a, b \in \mathbb{H}$, and $R \ni t \neq 1$. Then the set $\{q \in \mathbb{H} : |q - a| \leq t|q - b|\}$ is the closed ball $\overline{B(c, r)}$ with center and radius given by

\[c = \frac{a - t^2b}{1 - t^2}, \quad r = \frac{|a - b|}{1 - t^2}.\]

Let $\mathcal{SR}(\mathbb{B}, \mathbb{B})$ be the class of regular function $f \in \mathcal{SR}(\mathbb{B})$ with values in $\mathbb{B}$. A typical example of $\mathcal{SR}(\mathbb{B}, \mathbb{B})$ is a regular M"{o}bius transformation of $\mathbb{B}$ onto $\mathbb{B}$ given by

\[f(q) = (1 - qv)^{-1} * (q - u)v\]

with $u \in \mathbb{B}$ and $v \in \partial \mathbb{B}$ (cf. [16, Corollary 9.17]).

**Proposition 3.4 (Lindel"{o}f).** Let $f \in \mathcal{SR}(\mathbb{B}, \mathbb{B})$. Then for all $q \in \mathbb{B}$, the following inequalities hold:

\[(3.2) \quad \left| f(q) - \frac{1 - |q|^2}{1 - |q||f(0)|} f(0) \right| \leq \frac{|q|(1 - |f(0)|^2)}{1 - |q|^2|f(0)|^2};\]

\[(3.3) \quad \frac{|f(0)| - |q|}{1 - |q||f(0)|} \leq |f(q)| \leq \frac{|q| + |f(0)|}{1 + |q||f(0)|};\]

\[(3.4) \quad |f(q) - f(0)| \leq \frac{|q|(1 - |f(0)|^2)}{1 - |q|^2|f(0)|^2}.

Equality holds for one of inequalities in (3.2) - (3.3) at some point $q_0 \in \mathbb{B} \setminus \{0\}$ if and only if $f$ is a regular M"{o}bius transformation of $\mathbb{B}$ onto $\mathbb{B}$. 
Proof. The Schwarz-Pick theorem (see [4]) gives, for all \( q \in \mathbb{B} \),
\[
(3.5) \quad \left| (1 - f(q)f(0))^{-1} * (f(q) - f(0)) \right| \leq |q|,
\]
which together with Proposition 2 implies that
\[
\left| \frac{f \circ T_{1 - f(0)}(q) - f(0)}{1 - f \circ T_{1 - f(0)}(q)f(0)} \right| \leq |q|.
\]

According to Lemma 3.3, the preceding inequality is equivalent to
\[
\left| f \circ T_{1 - f(0)}(q) - \frac{1 - |q|^2}{1 - |q|^2|f(0)|^2}f(0) \right| \leq \frac{|q|(1 - |f(0)|^2)}{1 - |q|^2|f(0)|^2}.
\]

Since \( f(\mathbb{B}) \subset \mathbb{B} \), it follows from Proposition 2 that \( T_{1 - f(0)} \) is a homeomorphism with inverse \( T_{1 - f(0)*r'} \). Replacing \( q \) by \( T_{1 - f(0)*r'}(q) \) in the preceding inequality gives that
\[
\left| f(q) - \frac{1 - |q|^2}{1 - |q|^2|f(0)|^2}f(0) \right| \leq \frac{|q|(1 - |f(0)|^2)}{1 - |q|^2|f(0)|^2}, \quad \forall q \in \mathbb{B}.
\]

If equality achieves at some point \( 0 \neq q_0 \in \mathbb{B} \) in the last inequality, then it also achieves at point \( 0 \neq q'_0 = T_{1 - f(0)*r'}(q_0) \) in inequality (3.5). It thus follows from the Schwarz-Pick theorem that \( f \) is a regular Möbius transformations of \( \mathbb{B} \) onto \( \mathbb{B} \).

Inequalities in (3.3) and (3.4) follow from (3.2) as well as the triangle inequality. The proof is complete. \( \square \)

For each \( p = x + yI \in \mathbb{H} \setminus \mathbb{R} \) with \( x, y \in \mathbb{R} \) and \( I \in \mathbb{S} \), we denote by \( S_p \) the 2-dimensional sphere given by
\[
S_p := \{ x + yJ : J \in \mathbb{S} \}.
\]
The next result is a crucial step towards Theorem 1.4.

Lemma 3.5 (Landau). Let \( f \in \mathcal{SR}(\mathbb{B}, \mathbb{B}) \) with \( f(0) = 0 \) and \( f'(0) = \alpha > 0 \). Then
(a) \( f \) is injective on \( B(0, r_0) \), where
\[
|f(q)| \geq |q| \frac{\alpha - |q|}{1 - \alpha |q|}
\]
(b) for each positive number \( r \leq r_0 \), \( f(0, B(r)) \) contains the ball \( B(0, R(r)) \),
where
\[
R(r) = r \alpha - r \frac{\alpha - r}{1 - \alpha r} \geq rr_0.
\]

Proof. Clearly, \( \alpha \in (0, 1] \). If \( \alpha = 1 \), then \( f(q) = q \) for all \( q \in \mathbb{B} \) and the results hold trivially. Now we assume that \( \alpha \in (0, 1) \). To prove (a), it suffices to show that if there exist two distinct points mapped by \( f \) to one common point, then one of these two points must lie outside of the open ball \( B(r_0) \). We proceed as follows. Suppose that there exists one point \( q_0 \in \mathbb{B} \) such that \( r_0 := |q_0| > 0 \) and \( f(q_0) = 0 \). From the maximum principle (see cf. [10] Theorem 7.1) and \( \alpha \in (0, 1) \), it follows that the function \( g(q) := q^{-1}f(q) \) is a regular self-mapping of \( \mathbb{B} \). Applying the first inequality in (3.3) to \( g \) shows that the following inequality
\[
(3.6) \quad |f(q)| \geq |q| \frac{\alpha - |q|}{1 - \alpha |q|}
\]
holds for every \( q \in \mathbb{B} \). Now it follows from inequality (3.3) with \( q = q_0 \) that
\[
\rho_0 \geq \alpha > r_0. 
\]

Now suppose that there exist two distinct points \( q_1, q_2 \in \mathbb{B} \) such that \( 0 < |q_1| \leq |q_2| =: \rho < 1 \) and \( f(q_1) = f(q_2) =: w_0 \neq 0 \). We claim that
\[
|w_0| \leq \rho^2. 
\]
Set
\[
\psi(q) = (f(q) - w_0) \ast (1 - f(q) \overline{w_0})^{-1} 
\]
We must consider the following two cases. The first case is that \( q_1, q_2 \) lie in a same 2-dimensional sphere, i.e. \( S_{q_1} = S_{q_2} \). In this case, the sphere \( S_{q_1} = S_{q_2} \) is contained in the zero set of \( \psi \), and hence the function
\[
h(q) := \left( \left((1 - q \overline{q_1})^{-1} \ast (q - q_1)\right)^{\ast}\right)^{-1} \psi(q) 
\]
is regular on \( \mathbb{B} \) and is bounded in modulus by one, in virtue of the maximum principle (see cf. [10, Theorem 7.1]). In particular, \(|h(0)| = |w_0|/|q_1|^2 \leq 1\), and thus
\[
|w_0| \leq |q_1|^2 = \rho^2 
\]
as claimed.

The second case is that \( q_1, q_2 \) lie in two different 2-dimensional spheres, i.e. \( S_{q_1} \cap S_{q_2} = \emptyset \). Set
\[
\varphi(q) = (q - q_1) \ast (1 - q \overline{q_1})^{-1}, \quad \text{and} \quad \phi(q) = (q - \tilde{q}_2) \ast (1 - q \overline{\tilde{q}_2})^{-1}, 
\]
where
\[
\tilde{q}_2 = T_{\varphi}(q_2). 
\]
Then \( \varphi \ast \phi \) has precisely two zeros \( q_1 \) and \( q_2 \). Reasoning as before, \( (\varphi \ast \phi)^{-1} \ast \psi \) is also a regular function on \( \mathbb{B} \) with values in \( \mathbb{B} \). In particular,
\[
|w_0|/|q_1||\tilde{q}_2| = |(\varphi \ast \phi)^{-1} \ast \psi(0)| \leq 1, 
\]
implying that
\[
|w_0| \leq |q_1||\tilde{q}_2| = |q_1||q_2| \leq \rho^2 
\]
as desired.

Now applying inequality (3.5) with \( q = q_2 \) once more, together with inequality (3.8) implies that
\[
\frac{\alpha - \rho}{1 - \alpha \rho} \leq \rho, 
\]
which is equivalent to
\[
\rho \geq r_0 = \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}. 
\]
Now from inequalities (3.7) and (3.9) we conclude that \( f|_{B(0, r_0)} \) is injective. This completes the proof of (a).

From (a) and the open mapping theorem (see [10, Theorem 7.7]), it follows that \( f|_{B(0, r_0)} : B(0, r_0) \to f(B(0, r_0)) \) is a homeomorphism. Thus for an arbitrary fixed \( r \in (0, r_0] \), we easily deduce from inequality (3.6) that the boundary \( \partial f(B(0, r)) \) of the open set \( f(B(0, r)) \) (containing the origin 0) lies outside of the ball \( B(R(0, r)) \) and (b) immediately follows. \( \square \)
Now we use the lemmas established in the preceding section to prove Theorems 1.2, 1.3 and 1.4.

**Proof of Theorem 1.2.** Recalling the classical Wolff-Warschawski-Noshiro theorem for holomorphic functions on convex domains in \( \mathbb{C} \) (cf. [19, lemma 2.4.1]), the assertion (a) follows directly from the splitting lemma (Lemma 2.4) and Theorem 1.1.

Now we turn to the assertion (b). We first prove that
\[
B(0, \sqrt{3}/4) \subset f(B(0, 1/\sqrt{3})).
\]
From Theorem 2.10, we deduce that the image set \( f(B(0, 1/\sqrt{3})) \) is open in \( \mathbb{H} \). Since that \( f(0) = 0 \), according to Proposition 3.1, it suffices to show that
\[
\min_{|q|=\sqrt{3}/4} |f(q)| = \frac{\sqrt{3}}{4}.
\]
Indeed, for each point \( q = e^{i\theta}/\sqrt{3} \) with \( I \in \mathbb{S} \) and \( \theta \in \mathbb{R} \), from inequality (1.2) it follows that
\[
|f(q)| \geq \text{Re} f(q) = \text{Re} \int_0^{\sqrt{3}/4} f'(te^{i\theta}) dt = \int_0^{\sqrt{3}/4} \frac{1 - \sqrt{3}t}{(1 - \sqrt{1/3}t)^4} dt = \frac{\sqrt{3}}{4}
\]
as desired. Now it remains to prove the injectivity of \( f|_{B(0,r\rho)} \) and the relation \( B(0,R\rho) \subset f(B(0,r\rho)) \). To this end, applying the fundamental theorem of calculus to \( f \) yields that
\[
|f(q)| \leq \frac{1}{2} \log \frac{1+|q|}{1-|q|}
\]
for all \( q \in \mathbb{B} \). For each \( r \in (0,1) \), consider the regular function \( g \) given by \( g(q) = f(rq) \), which satisfies both \( g(0) = 0 \) and \( g'(0) = r \). Thus it follows from Lemma 3.5 that \( g \) is injective in \( B(0,\rho_0) \) and \( B(0,R) \subset g(\mathbb{B}) \), where
\[
\rho_0 = \frac{r}{\frac{1}{2} \log \frac{1+r}{1-r} + \sqrt{\frac{1}{2} \log \frac{1+r}{1-r} - r^2}}
\]
and
\[
R = \frac{1}{2} \log \frac{1+r}{1-r} \rho_0^2.
\]
This means that \( f \) is injective in \( B(0,r\rho_0) \) and \( B(0,R) \subset f(B(0,r\rho_0)) \) for all \( 0 < r < 1 \). Hence \( f \) is injective in \( B(0,r_{\mathbb{B}}) \) and \( B(0,R_{\mathbb{B}}) \subset f(B(0,r_{\mathbb{B}})) \), where
\[
r_{\mathbb{B}} = \sup_{0<r<1} \left\{ \frac{1}{2} \log \frac{1+r}{1-r} - \frac{1}{4} \left( \log \frac{1+r}{1-r} \right)^2 - r^2 \right\} \approx 0.3552,
\]
and
\[
R_{\mathbb{B}} = \sup_{0<r<1} \left\{ \frac{1}{2r^2} \log \frac{1+r}{1-r} \left( \frac{1}{2} \log \frac{1+r}{1-r} - \frac{1}{4} \left( \log \frac{1+r}{1-r} \right)^2 - r^2 \right) \right\} \approx 0.2308.
\]
\( \square \)
Proof of Theorem 1.3. The proof is similar to that of Theorem 1.2. The only difference is that, instead of (4.1), we use the following estimate:

\[
|f(q)| \leq \frac{1}{(1 - |q|)^{\frac{p}{2}}}
\]

for every \( f \in A^p(\mathbb{B}) \) and all \( q \in \mathbb{B} \). The proof of inequality (4.2) is standard, which goes as follows. For \( p \in [1, +\infty) \), by Lemma 2.4, \(|f_I|^p\) is subharmonic on \( \mathbb{B}_I \) for all \( I \in \mathcal{S} \). Then, for \( r \in (0, 1 - |z|) \), we have

\[
|f_I(z)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f_I(z + re^{i\theta})|^p d\theta.
\]

Integration gives that

\[
(1 - |z|)^2 |f_I(z)|^p \leq \frac{1}{\pi} \int_0^{1-|z|} \int_0^{2\pi} |f_I(z + re^{i\theta})|^p r d\theta dr,
\]

which together with the prescribed condition \( \|f\|_{A^p} \leq 1 \) implies that

\[
(1 - |z|)^2 |f_I(z)|^p \leq \int_{\mathbb{B}_I} |f_I|^p d\sigma_I \leq 1
\]

for all \( I \in \mathcal{S} \) and \( z \in \mathbb{B}_I \). This leads to inequality (4.2). □

Proof of Theorem 1.4. With notations introduced in the introduction in mind, we consider the regular function \( f_\alpha : \mathbb{B} \to \mathbb{B} \) given by

\[
f_\alpha(q) = q + \alpha (1 + qa),
\]

which belongs to \( \mathcal{F}_\alpha \). Therefore, by the preceding theorem the assertions (a) and (b) in Lemma 3.5 holds for this function \( f_\alpha \). Note also that the slice derivative function

\[
f'_\alpha(q) = \frac{q^2 + 2q + \alpha}{(1 + qa)^2}
\]

has a zero at \( q = -r(\mathcal{F}_\alpha) \) and

\[
f_\alpha(-r) = -r \frac{\alpha - r}{1 - \alpha r}
\]

for every \( r \in (0, r(\mathcal{F}_\alpha)) \). This fact together with Lemma 3.5 implies the statements of (a) and (c) in this theorem.

To complete the proof, it remains to consider the extremal cases in (b) and (d). If \( r_\alpha(f) = r(\mathcal{F}_\alpha) \), then from the proof of Lemma 3.5 it follows that equality holds for inequality in (3.9). Consequently, equality has to hold for inequality in (3.6) at the point \( q = q_2 \). By Proposition 3.4 this is possible only if \( f \) is of the form in (1.3). Conversely, according to Lemma 3.5 each regular function \( f \) given in (1.3) is indeed injective on the ball \( B(0, r(\mathcal{F}_\alpha)) \). (It seems not easy to verify this fact directly.) Therefore, \( r_\alpha(f) \geq r(\mathcal{F}_\alpha) \). Moreover, the slice derivative function \( f' \) has a zero at the point

\[
q = r(\mathcal{F}_\alpha)e^{i\theta} = \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}e^{i\theta},
\]

and hence the radius of the largest disc \( B(0, r)_I \) on which \( f_I \) is injective is precisely \( r(\mathcal{F}_\alpha) \). Thus we conclude that \( r_\alpha(f) = r(\mathcal{F}_\alpha) \) for each regular function \( f \) given in (1.3). This completes the proof of (b). (d) can be proved similarly and its proof is left to the interested reader. □
5. Proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5. Given a constant $u \in \partial B$ and a regular function $f$ on $\overline{B}$ with power series expansion
\[
f(q) = \sum_{n=0}^{\infty} q^n a_n.
\]
Then the slice regular rotation $\tilde{f}_u$ of $f$ is defined to be the regular function on $\overline{B}$ given by
\[
\tilde{f}_u(q) = \sum_{n=0}^{\infty} q^n u^n a_n.
\]

Following the idea of Estermann in [11], we can prove the following lemma.

**Lemma 5.1.** Let $f \in \mathcal{SR}(B)$ be nonconstant and satisfy
\[
|f'(q)| \leq 2|f'(a)|
\]
for all $q \in \overline{B(a,r)}$ with some $a \in B$ and $r \in (0, 1 - |a|)$. Then there exists $u \in \partial B$ such that
\[
B(f(a), R) \subset \tilde{f}_u(B),
\]
where $R = (5 - 2\sqrt{6})r|f'(a)|$.

**Proof.** Firstly, let us consider the case that $a \in (-1, 1)$. Set
\[
g(q) = f(q) - f(a) - (q - a)f'(a).
\]
Then $g$ is a regular function on $B$. Applying the fundamental theorem of calculus to $g$ yields that, for all $q \in B$,
\[
g(q) = \int_{q-a}^{1} \frac{d}{dt}g(tq + (1-t)a)dt
\]
\[
=(q-a)\int_{0}^{1} f'(tq + (1-t)a) - f'(a)dt.
\]
Fix an arbitrary point $q = x + yI \in \overline{B(a,r)} \subset B$ with some $I \in \mathbb{S}$. By Lemma 2.3 and Cauchy integral formula for holomorphic functions, the integrand in the second integral in (5.2) can be represented as
\[
f'(tq + (1-t)a) - f'(a) = \frac{1}{2\pi i} \int_{\partial B(a,r)} \left( \frac{1}{z - (tq + (1-t)a)} - \frac{1}{z - a} \right) dz f'(z),
\]
from which and (5.1) it follows that, for all $t \in [0, 1],
\[
|f'(tq + (1-t)a) - f'(a)| \leq \frac{t|q-a|}{r-t|q-a|} \max_{\overline{B(a,r)}} |f'| \leq \frac{2t|q-a|}{r-t|q-a|}|f'(a)|.
\]
Therefore,
\[
|g(q)| \leq 2|q - a|^2 |f'(a)| \int_{0}^{1} \frac{t}{r-t|q-a|} dt \leq \frac{|q - a|^2 |f'(a)|}{r - |q - a|},
\]
and hence
\[
|f(q) - f(a)| \geq |(q - a)f'(a)| - |g(q)| \geq |q - a| \left( 1 - \frac{|q - a|}{r - |q - a|} \right) |f'(a)|
\]
for all $q \in B(a, r)$. In particular, the inequality

$$|f(q) - f(a)| \geq (3 - 2\sqrt{2})r|f'(a)|$$

holds for all $q \in \partial B(a, (1 - \sqrt{2}/2)r)$. Note that $a \in (-1, 1)$, $B(a, r)$ is an axially symmetric open subset of $\mathbb{B}$. Hence, by Theorem 2.10, $f(B(a, r))$ is open. This together with Proposition 3.1 implies that

$$B(f(a), R_1) \subset f(B(a, r)) \subset f(\mathbb{B}),$$

(5.3)

where $R_1 = (3 - 2\sqrt{2})r|f'(a)|$.

Secondly, let us consider the case that $a \notin \mathbb{B} \setminus \mathbb{R}$. Let $u = \frac{a}{||a||} \in \partial \mathbb{B}_I$ with $I \in \mathbb{S}$ and consider the regular function $\tilde{f}_u(q)$ on $\mathbb{B}$. It is obvious that

$$\tilde{f}_u(|a|) = f(a) \text{ and } \tilde{f}_u(|a|) = uf'(a).$$

The representation formula for regular functions in [8, Theorem 3.1] gives that

$$\tilde{f}'_u(q) = \frac{1}{2}(1 - I_q)\tilde{f}'_u(z) + \frac{1}{2}(1 + I_q)\tilde{f}'_u(\overline{z}),$$

where $q = x + yI_q$ for some $I_q \in \mathbb{S}$ and $z = x + yI$.

Notice that $\tilde{f}'_u(\overline{z}) = uf'(\overline{zu})$ and $\tilde{f}_u'(z) = uf'(zu)$, then we obtain that

$$|\tilde{f}'_u(q)| \leq |f'(zu)| + |f'(\overline{zu})|,$$

and hence, for $|q - |a|| \leq r$, which implies that $|zu - a| \leq r$ and $|\overline{zu} - a| \leq r$,

$$\max_{q \in B(|a|, r)} |\tilde{f}'_u(q)| \leq \max_{zu \in B(a, r)} (|f'(zu)| + \max_{zu \in B(a, r)} |f'(\overline{zu})|) \leq 4|f'(a)| = 4|\tilde{f}'_u(|a|)|.$$ 

Here the second inequality follows from 6.1.

Now a similar argument as in the first case gives that

$$|\tilde{f}_u(q) - \tilde{f}_u(|a|)| \geq |q - |a||\left(1 - \frac{2|q - |a||}{r - |q - |a||}\right)|\tilde{f}'_u(|a|)|, \quad \forall q \in B(|a|, r).$$

In particular, for any $q \in \partial B(|a|, (1 - \sqrt{6}/3)r)$,

$$|\tilde{f}_u(q) - \tilde{f}_u(|a|)| \geq (5 - 2\sqrt{6})r|\tilde{f}'_u(|a|)| = (5 - 2\sqrt{6})r|f'(a)| =: R_2.$$

By Proposition 3.1 and Theorem 2.10 again, we have

$$B(\tilde{f}_u(|a|), R_2) \subset \tilde{f}_u(B(|a|, r)) \subset \tilde{f}_u(\mathbb{B}),$$

i.e.,

$$B(f(a), R_2) \subset \tilde{f}_u(\mathbb{B}),$$

(5.4)

Now the desired result follows immediately from 6.3 and 6.1. \(\square\)

Finally we have all the tools to prove the second Bloch type theorem for regular functions.
ON TWO BLOCH TYPE THEOREMS FOR QUATERNIONIC SLICE REGULAR FUNCTIONS

Proof of Theorem 1.5. Let $f$ be as described. We consider the continuous function $\psi$ on $\mathbb{B}$ given by
$$
\psi(q) = (1 - |q|)|f'(q)|,
$$
which vanishes on the boundary $\partial \mathbb{B}$. Then there exists some $a \in \mathbb{B}$ such that
$$
\psi(a) = \max_{q \in \mathbb{B}} \psi(q).
$$
Set $r := \frac{1}{2}(1 - |a|)$. Then it is evident that
$$
B(a, r) \subset \mathbb{B}, \quad r|f'(a)| \geq \frac{1}{2}\psi(0) = \frac{1}{2},
$$
and
$$
r \leq 1 - |q|, \quad \forall q \in \overline{B(a, r)}.
$$
Whence
$$
|f'(q)| \leq 2|f'(a)|, \quad \forall q \in \overline{B(a, r)} \subset \mathbb{B}.
$$
From Lemma 5.1 there exists $u \in \partial \mathbb{B}$ such that
$$
B(f(a), R) \subset \tilde{f}_u(\mathbb{B}),
$$
where $R = (5 - 2\sqrt{6})r|f'(a)| \geq \frac{5}{2} - \sqrt{6}$. The proof is complete. \qed

References
1. L. V. Ahlfors, An extension of Schwarz’s lemma. Trans. Amer. Math. Soc. 43 (1938), no. 3, 359-364.
2. D. Alpay, F. Colombo, I. Sabadini, Slice Hyperholomorphic Schar Analysis, preprint 2015, available at: https://www.mate.polimi.it/biblioteca/add/quaderni/qdd209.pdf
3. A. Bloch, Les théorèmes de M. Valiron sur les fonctions entières et la théorie de l’uniformisation. Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. 3 (1925), 1-22.
4. C. Bisi, C. Stoppato, The Schwarz-Pick lemma for slice regular functions. Indiana Univ. Math. J. 61 (2012), 297–317.
5. M. Bonk, On Bloch’s constant. Proc. Amer. Math. Soc. 110 (1990), no. 4, 889-894.
6. H. Chen, P. M. Gauthier, On Bloch’s constant. J. Anal. Math. 69 (1996), 275-291.
7. F. Colombo, J. O. González-Cervantes, M. E. Luna-Elizarrahas, I. Sabadini, M. Shapiro, On two approaches to the Bergman theory for slice regular functions, Advances in hypercomplex analysis, 39-54, Springer INdAM Ser., 1, Springer, Milan, 2013.
8. F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, Extension results for slice regular functions of a quaternionic variable. Adv. Math. 222 (2009), 1793–1808.
9. F. Colombo, I. Sabadini, D.C. Struppa, Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions. Progress in Mathematics, vol. 289. Birkhäuser/Springer, Basel, 2011.
10. L. de Branges, A proof of the Bieberbach conjecture. Acta Mathematica, 154 (1985), 137-152.
11. T. Estermann. Notes on Landau’s proof of Picard’s “Great” Theorem. Studies in Pure Mathematics presented to R. Rado, ed. L. Mirsky. Acad. Press London, New York (1971), 101-106.
12. G. Gentili, C. Stoppato, The zero sets of slice regular functions and the open mapping theorem. Hypercomplex analysis and applications, 95-107, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2011.
13. G. Gentili, S. Salamon, C. Stoppato, Twistor transforms of quaternionic functions and orthogonal complex structures. J. Eur. Math. Soc. 16 (2014), 2323–2353.
14. G. Gentili, C. Stoppato, The open mapping theorem for regular quaternionic functions. Ann. Sc. Norm. Super. Pisa Cl. Sci. 5 (8) (2009), no. 4, 805-815.
15. G. Gentili, C. Stoppato, Zeros of regular functions and polynomials of a quaternionic variable. Mich. Math. J. 56 (2008), 655–667.
16. G. Gentili, C. Stoppato, D.C. Struppa, Regular functions of a quaternionic variable. Springer Monographs in Mathematics, Springer, Berlin-Heidelberg, 2013.
17. G. Gentili, D.C. Struppa, A new approach to Cullen-regular functions of a quaternionic variable. C. R. Math. Acad. Sci. Paris, 342 (2006), 741–744.
18. G. Gentili, D.C. Struppa, *A new theory of regular functions of a quaternionic variable*. Adv. Math., **216** (2007), 279–301.
19. I. Graham, G. Kohr, *Geometric function theory in one and higher dimensions*. Monographs and Textbooks in Pure and Applied Mathematics, 255. Marcel Dekker, Inc., New York, 2003.
20. L. A. Harris, *On the size of balls covered by analytic transformations*. Monatsh. Math. **83** (1977), no. 1, 9-23.
21. M. Heins, *On a class of conformal metrics*. Nagoya Math. J. **21** (1962), 1-60.
22. M. Heins, *Selected Topics in the Classical Theory of Functions of a Complex Variable*. Holt, Rinehart and Winston, New York, 1962.
23. E. Landau, *Der Picard-Schottysche Satz und die Blochsche Konstanten*. Sitzungsber Berl Akad. Wiss. Berlin Phys.-Math. Kl., 1926, pp. 467-474.
24. E. Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*. Math. Z. **30** (1929), no. 1, 608-634.
25. J. Morais, K. Gürlebeck, *Bloch’s theorem in the context of quaternion analysis*. Comput. Methods Funct. Theory 12 (2012), no. 2, 541-558.
26. Ch. Pommerenke, *On Bloch functions*. J. London Math. Soc. (2) **2** (1970) 689-695.
27. D. Rochon, *A Bloch constant for hyperholomorphic functions*. Complex Variables Theory Appl. **44** (2001), no. 2, 85-101.
28. C. Della Rocchetta, G. Gentili, G. Sarfatti, *A Bloch-Landau theorem for slice regular functions*. Advances in hypercomplex analysis, 55-74, Springer INdAM Ser., 1, Springer, Milan, 2013.
29. C. Stoppato, *Poles of regular quaternionic functions*. Complex Var. Elliptic Equ. **54** (2009), 1001–1018.
30. C. Stoppato, *Singularities of slice regular functions*. Math. Nachr. **285** (2012), 1274–1293.
31. H. Wu, *Normal families of holomorphic mappings*. Acta Math. **119** (1967), 193-233.

Zhenghua Xu, Department of Mathematics, University of Science and Technology of China, Hefei 230026, China

E-mail address: xzhengh@mail.ustc.edu.cn

Xieping Wang, Department of Mathematics, University of Science and Technology of China, Hefei 230026, China

E-mail address: pwx@mail.ustc.edu.cn