On Hilbert’s Tenth Problem

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Abstract

Using an iterated Horner schema for evaluation of diophantine polynomials, we define a partial \( \mu \)-recursive “decision” algorithm \( \text{decis} \) as a “race” for a first nullstelle versus a first (internal) proof of non-nullity for such a polynomial – within a given theory \( T \) extending Peano Arithmétique \( \text{PA} \). If \( T \) is diophantine sound, i.e. if (internal) provability implies truth – for diophantine formulae –, then the \( T \)-map \( \text{decis} \) gives correct results when applied to the codes of polynomial inequalities \( D(x_1, \ldots, x_m) \neq 0 \). The additional hypothesis that \( T \) be diophantine complete (in the syntactical sense) would guarantee in addition termination of \( \text{decis} \) on these formula, i.e. \( \text{decis} \) would constitute a decision algorithm for diophantine formulae in the sense of Hilbert’s 10th problem. From Matiyasevich’s impossibility for such a decision it follows, that a consistent theory \( T \) extending \( \text{PA} \) cannot be both diophantine sound and diophantine complete. We infer from this the existence of a diophantine formulae which is undecidable by \( T \). Diophantine correctness is inherited by the diophantine completion \( \tilde{T} \) of \( T \), and within this extension \( \text{decis} \) terminates on all externally given diophantine polynomials, correctly. Matiyasevich’s theorem – for the strengthening \( \tilde{T} \) of \( T \) – then shows that \( \tilde{T} \), and hence \( T \), cannot be diophantine sound. But since the internal consistency formula \( \text{Con}_T \) for \( T \) implies – within \( \text{PA} \) – diophantine soundness of \( T \), we get \( \text{PA} \vdash \neg \text{Con}_T \), in particular \( \text{PA} \) must derive its own internal inconsistency formula.

Overview

(i) Consider a theory \( T \) with quantifiers and having terms for all primitive recursive maps (“p.r. maps”): so \( T \) is to be Peano Arithmétique \( \text{PA} \) or one of \( \text{PA} \)’s extensions, e.g. \( \text{ZF} \) or \( \text{NGB} \).

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(ii) Obtain the theory $\tilde{T}$ by adding to $T$ the axiom $\neg\text{Con}_T$ of internal inconsistency. By Gödel’s second incompleteness theorem, $\tilde{T}$ is consistent relative to $T$.

(iii) $T$ admits a $\mu$-recursive, partially defined “algorithm” $\text{decis}$ aimed at deciding $T$-internal (Gödel numbers of) p.r. predicates.

(iv) By internal semantical completeness of $\tilde{T}$ with respect to p.r. predicates – involving evaluation of (Gödel numbers of) internal p.r. predicates – it is shown that in $\tilde{T}$ the partial $\mu$-recursive $T$-map $\text{decis}$ is in fact total, and that it gives correct results – the latter for arguments $p$ of form $p = \langle \varphi \rangle$, $\varphi = \varphi(n)$ a p.r. predicate, $\langle \varphi \rangle \in \mathbb{N}$ its internal Gödel number.

(v) within $\tilde{T}$, $\text{decis}$ decides in particular (systems of) diophantine equations.

(vi) Matiyasevich’s negative result concerning this decision problem of Hilbert is a theorem of $T$, a fortiori of $\tilde{T}$.

(vii) This contradiction shows $\tilde{T}$, hence also $T$, to be inconsistent: “unbounded formal quantification is incompatible with infinity.”

1 Decision

Crucial for the present approach to Hilbert’s decision problem is availability – within $T$ – of a (µ-recursive) evaluation map $ev : \mathbb{N} \times \mathbb{N} \supset [\mathbb{N}, 2|_{\text{PR}}] \times \mathbb{N} \rightarrow 2$ on the $T$-internal (primitive recursively decidable) set $[\mathbb{N}, 2|_{\text{PR}}] \subset \mathbb{N}$ of Gödel numbers (“codes”) of p.r. predicates. (Primitive recursive predicates are viewed as p.r. map terms with codomain $2 \subset \mathbb{N}$). This evaluation map $ev$ is defined in $T$ by (nested) double recursion à la Ackermann, see Péter 1967, and satisfies the characteristic equation

$ev(\langle \varphi \rangle, n) = \varphi(n)$

for p.r. predicates $\varphi = \varphi(n)$ of $T$, cf. Appendix. Here $\langle \varphi \rangle = \varphi(n)$ is $\varphi$’s $T$-internal Gödel number.

Define now the partial $\mu$-recursive “decision” $\tilde{T}$-map

$\text{decis} = \text{decis}(p) : [\mathbb{N}, 2|_{\text{PR}}] \rightarrow 2$

hoped for deciding (internal) p.r. predicates $p$, i.e. $p \in [\mathbb{N}, 2|_{\text{PR}}] \subset \text{formulae}_T = \text{formulae}_T \subset \mathbb{N}$, via the two “antagonistic” termination indices

$\mu_{ex}(p), \mu_{thm}(p) : [\mathbb{N}, 2|_{\text{PR}}] \rightarrow \mathbb{N} \cup \{\infty\}$ as follows:
\[ \mu_{ex}(p) := \mu\{ n : ev(p, n) = 0 \} \quad \text{“minimal counterexample”} \]
\[
= \begin{cases} 
\min\{ n : ev(p, n) = 0 \} & \text{if } \exists n(ev(p, n) = 0) \\
\infty & \text{if } \forall n(ev(p, n) = 1); 
\end{cases}
\]

the *theorem index* \( \mu_{thm}(p) \in \mathbb{N} \cup \{ \infty \} \) of \( p \in |\mathbb{N}, 2|_{PR} \) is defined by
\[
\mu_{thm}(p) := \mu\{ k : \text{thm}_\overline{T}(k) = p \};
\]

here the p.r. enumeration \( \text{thm}_\overline{T} = \text{thm}_\overline{T}(k) : \mathbb{N} \to \text{formulae}_T \subset \mathbb{N} \) is the \( \overline{T} \)-internal version of the metamathematical enumeration of all (Gödel numbers of) \( \overline{T} \)-theorems; enumeration is lexicographic by “length of shortest proof”.

Finally, we define the – a priori partial – \( \mu \)-recursive \( \overline{T} \)-map
\[
decis = \text{decis}(p) : |\mathbb{N}, 2|_{PR} \to 2 \quad \text{by}
\]
\[
decis(p) = \begin{cases} 
0 & \text{if } \mu_{ex}(p) < \infty \quad \text{ (“counterexample”)} \\
1 & \text{if } \mu_{ex}(p) = \infty \text{ and } \mu_{thm}(p) < \infty \quad \text{ (“theorem”)} \\
\infty & \text{otherwise, i.e. if } \mu_{thm}(p) = \mu_{ex}(p) = \infty.
\end{cases}
\]

For proving \( \text{decis} \) to be totally defined within \( \overline{T} = T + \neg \text{Con}_T \) we rely on the following

**Lemma** (Internal Semantical Completeness):
\[
\overline{T} \vdash \forall n(ev(p, n) = 1) \implies \exists k(\text{thm}_\overline{T}(k) = p)
\]

with \( p \) free on \( |\mathbb{N}, 2|_{PR} \), in closed form:
\[
\overline{T} \vdash (\forall p \in |\mathbb{N}, 2|_{PR})(\forall n(ev(p, n) = 1) \implies \exists k(\text{thm}_\overline{T}(k) = p)].
\]

**Proof:** One of the equivalent \( T \)-formulae expressing internal inconsistency of \( T \) is
\[
\neg \text{Con}_T = (\forall f \in \text{formulae}_T)(\exists k)(\text{thm}_T(k) = f) :
\]

“every internal formula (its Gödel number in \( T \)) is provable” (emphasis from Gödel). This gives in particular
\[
\overline{T} \vdash \exists k(\text{thm}_\overline{T}(k) = p),
\]

\( p \) free on \( |\mathbb{N}, 2|_{PR} \subset \text{formulae}_T \subset \mathbb{N} \), and hence – trivially – the assertion of the Lemma.

**Decision Lemma:**
(i) within $\tilde{T} = T + \neg \text{Con}_T$, the (a priori partial) $\mu$-recursive decision-
"algorithm"

$$\text{decis}(p) : \mathbb{N}, 2 |_{\text{PR}} \rightarrow 2$$

is in fact totally defined, with other words it terminates on all internal Gödel numbers $p \in \mathbb{N}, 2 |_{\text{PR}}$.

(ii) For $\varphi = \varphi(n)$ a p. r. predicate, $\vdash \psi \in \mathbb{N}, 2 |_{\text{PR}} \subset \mathbb{N}$ its $T$-
internal Gödel number, $\text{decis}(\psi)$ gives – in $T$ – the correct result:

- $\tilde{T} \vdash \text{decis}(\psi) = 0 \iff \exists n(\neg \varphi(n))$,
- $\tilde{T} \vdash \text{decis}(\psi) = 1 \iff \forall n(\varphi(n))$.

**Proof** of (i):

$$\tilde{T} \vdash [ \mu_{\text{thm}}(p) = \infty$$

$$\iff \forall n(ev(p,n) = 1)$$

$$\Rightarrow \exists k(\text{thm}T(k) = p)$$

by internal semantical completeness of $\tilde{T}$ above

$$\iff \mu_{\text{thm}}(p) < \infty \] .$$

Hence not both of $\mu_{\text{ex}}(p), \mu_{\text{thm}}(p)$ can be undefined. This shows termination

$$\text{decis}(p) \in \{0, 1\}$$

of $\text{decis}$ within $\tilde{T}$ for all (internal) p. r. predicates $p$ (Gödel numbers thereof).

**Proof of (ii):**

$$\tilde{T} \vdash [ \text{decis}(\psi) = 0$$

$$\iff \mu_{\text{ex}}(\psi) < \infty$$

$$\iff \exists n(ev(\psi, n) = 0)$$

$$\iff \exists n(\varphi(n) = 0) \quad \text{by ev’s evaluation property}$$

$$\iff \exists n(\neg \varphi(n)) \] \text{ as well as}$$

$$\tilde{T} \vdash [ \text{decis}(\psi) = 1$$

$$\Rightarrow \mu_{\text{ex}}(\psi) = \infty$$

$$\iff \forall n(\varphi(n) = 1)$$

$$\iff \forall n(\neg \varphi(n)) \] \quad \text{q.e.d.}$$
2 Hilbert’s 10th Problem revisited

A system

\[ D^L_1(x_1, \ldots, x_m) = D^R_1(x_1, \ldots, x_m) \]
\[ \vdots \]
\[ D^L_k(x_1, \ldots, x_m) = D^R_k(x_1, \ldots, x_m) \]

of \( k \) \textit{diophantine equations} – see Matiyasevich 1993, 1.1, 1.2, and 1.3 – gives rise to a p. r. predicate

\[ \varphi_{\psi} = \varphi(x_1, \ldots, x_m) : \mathbb{N}^m \to 2 \]

having the property that \((x_1, \ldots, x_m) \in \mathbb{N}^m\) is a solution to system \((D)\) iff it is a \textit{counterexample} to \(\varphi\), and \((D)\) has no solution (in natural numbers) iff \(\varphi\) holds for \((x_1, \ldots, x_m)\) free in \(\mathbb{N}^m\).

CANTOR’S p. r. enumeration \(\text{cantor}_m : \mathbb{N} \to \mathbb{N}^m\) having a p. r. inverse \(\text{cantor}^{-1}_m : \mathbb{N}^m \to \mathbb{N}\),

\[ \psi = \psi(n) := \varphi(\text{cantor}_m(n)) : \mathbb{N} \to 2 \]

is a p. r. predicate of \(T\) such that \((x_1, \ldots, x_m) \in \mathbb{N}^m\) solves \((D)\) iff \(\text{cantor}^{-1}_m(x_1, \ldots, x_m) \in \mathbb{N}\) is a \textit{counterexample} to \(\psi\), and \((D)\) is \textit{unsolvable} iff \(\psi(n)\) holds for \(n\) free in \(\mathbb{N}\). So from the Decision Lemma (for p. r. predicates) above we obtain:

**Decision Theorem:**

(i) Within the – somewhat strange – theory \(\tilde{T} = T + \neg \text{Con}_T\), the (partial) \(\mu\)-recursive map (the “algorithm”) \(\text{decis} : [\mathbb{N}, 2]_{\text{PR}} \to 2\) decides all (internal) primitive recursive predicates, in particular all (internal, a fortiori external) GÖDEL numbers coding “diophantine” predicates as considered above, and hence decides internal, a fortiori external (systems of) \textit{Diophantine equations}.

(ii) Since \(\mu\)-recursion and TURING-machines have equal \textit{computation power} – by the verified part of CHURCH’s thesis – this means: Within \(\tilde{T}\), \(\text{decis} \) gives rise to a TURING machine \(TM\) deciding all internally given as well as all externally given Diophantine equations, i. e. \(\tilde{T}\) admits a \textit{positive} solution to HILBERT’s 10th problem.

(iii) On the other hand, MATIYASEVICH’s \textit{negative} solution to this problem needs as a formal framework \(T\) just Arithmétique +\(\exists\).
(iv) The latter two results – MATIYASEVICH’s negative $\mathbf{T}$-theorem and our positive $\mathbf{T}$-theorem contradict each other in the stronger theory $\mathbf{T}$. This shows $\mathbf{T}$ to be inconsistent.

(v) Gödel’s consistency of $\neg \text{Con}_\mathbf{T}$ relative to $\mathbf{T}$ then entails inconsistency of $\mathbf{T}$, whence in particular inconsistency of Peano Arithmétique $\mathbf{PA}$ and of the classical set theories.

**Corollary:** Since MATIYASEVICH 1993 makes essential use of formal (existential) quantification for “unsolving” HILBERT’s 10th problem, this only decision problem on HILBERT’s list is again open – for treatment within the framework of a suitable constructive foundation for Arithmetic.

3 Appendix: Evaluation

In section 2 we made appeal to availability in $\mathbf{T}$ of an evaluation $ev = ev(p,n)$ of (internal) p.r. predicate codes $p$ satisfying

$$ev(\, \, ^\sim \varphi \, , n) = \varphi(n)$$

for (“external”) p.r. predicates $\varphi : \mathbb{N} \to 2$ in $\mathbf{T}$. We identify a p.r. predicate $\varphi = \varphi(n)$ of $\mathbf{T}$ with its associated p.r. map term $\varphi = \varphi(n) : \mathbb{N} \to 2$, since we want to define the evaluation of (internal) p.r. predicates by restriction of an evaluation of all internal p.r. map terms out of the set $[\mathbb{N}, 2]_{\mathbb{PR}} \subset \mathbb{N}$ of (internal) p.r. map terms from $\mathbb{N}$ to 2.

For defining this map term evaluation $ev$ by (nested) double recursion à la Ackermann (cf. Péter 1967) we need a universal set (object)

$$U = \mathbb{N}^{(*)}$$

of all nested pairs of natural numbers, and hence containing all PR-objects $1, \mathbb{N}, \ldots, A, \ldots, B, A \times B, \ldots$ as disjoint (exception: $1 \subset \mathbb{N}$) p.r. decidable subsets.

This set $\mathbb{N}^{(*)}$ is directly available in set theory. Within Peano Arithmétique, it can be “constructed” via coding as a decidable subset of $\mathbb{N}$.

**Definition:** Evaluation

$$ev = ev(u,a) : \mathbb{N} \times \mathbb{N}^{(*)} \supset PR \times \mathbb{N}^{(*)} \to \mathbb{N}^{(*)}$$

of the internal (Gödel numbers of) p.r. maps $u,v,w \in PR \subset \mathbb{N}$, on binary nested tuples $a,b,c \in \mathbb{N}^{(*)}$ of natural numbers is now defined by (nested) double recursion with principal recursion parameter “operator-depth” $\text{depth}(u)$ of $u$ as follows:
- basic internal map terms \(\gamma 0\), \(\gamma s\), \(\gamma id\), \(\gamma !\), \(\gamma \Delta\), \(\gamma \Theta\), \(\gamma \ell\):
  - \(ev(\gamma 0, 0) = 0 = 0(0) \in \mathbb{N}\) “zero map”,
  - \(ev(\gamma s, n) = n + 1 = s(n) \in \mathbb{N}\) “successor map”,
  - \(ev(\gamma id, a) = a = id(a)\) “identity”,
  - \(ev(\gamma !, a) = 0 = !(a) \in 1 \subset \mathbb{N}\) “terminal map”,
  - \(ev(\gamma \Delta, a) = (a, a) = \Delta(a)\) “diagonal”,
  - \(ev(\gamma \Theta, (a, b)) = (b, a) = \Theta(a, b)\) “transposition”,
  - \(ev(\gamma \ell, (a, b)) = a = \ell(a, b)\) “left projection”.

This defines \(ev\) on \(PR\)’s (map-)constants, depth of these “basic” map terms is set to 1.

We now define \(ev\) on compound internal p. r. map terms:

- internally composed \(v \circ \gamma\)
  \(ev(v \circ \gamma u, a) = ev(v, ev(u, a)).\)

This definition is legitimate, since

\[
\text{depth}(u), \text{depth}(v) < \text{depth}(v \circ \gamma u)
=_{\text{def}} \max(\text{depth}(u), \text{depth}(v)) + 1 \in \mathbb{N};
\]

Example:

\[
ev(\gamma s \circ \gamma \gamma s \circ \gamma s \circ \gamma s, s(0))
= ev(\gamma s, ev(\gamma s, ev(\gamma s, s(0))))
= ((s(0) + 1) + 1) + 1 = 4.
\]

- cylindrified \(\gamma id \circ \gamma \times\)
  \(ev(\gamma id \circ \gamma \times v, (a, b)) = (a, ev(v, b)),\)

“evaluation in the second component”.

legitimacy of this definition:

\[
\text{depth}(v) < \text{depth}(\gamma id \circ \gamma \times v) =_{\text{def}} \text{depth}(v) + 1.
\]

- internally iterated \(u^\delta\):
  \[ev(u^\delta, (a, 0)) = a,\]
  \[ev(u^\delta, (a, n + 1)) = ev(u, ev(u^\delta, (a, n))).\]

This last case is in fact a (nested) double recursion à la ACKERMANN, since the internally iterated \(u^\delta\) of \(u\) is evaluated in a p. r. manner with respect to the second parameter \(n \in \mathbb{N}\) – which is to count
the iteration loops still to be performed. The principal recursion parameter is (internal) operator-depth \( \text{depth} = \text{depth}(u) : \mathbb{N} \supset \text{PR} \rightarrow \mathbb{N} \), in particular in this last case \( \text{depth}(u^b) = \text{def} \text{depth}(u) + 1 \).

Each primitive recursive map can be generated from the basic maps \( 0, s, \text{id}, !, \Delta, \Theta, \text{\&} \) by composition, cylindrification and iteration: \( \text{substitution} \) is realized via composition with the induced \( (f, g) = (f, g)(c) = (f(c), g(c)) \) which in turn is obtained via diagonal, cylindrification, transposition, and composition. Since iteration \( g^b \) then gives the (“full”) schema of primitive recursion (see Freyd 1972, Pfender et al. 1994), \( ev \) in fact evaluates all Gödel numbers of (internal) p.r. map terms, recursively given in the above way.

Let us call \( \text{PR} + ev \) the extension of \( \text{PR} \) by a (formal) map

\[
ev = ev(u, a) : \mathbb{N} \times \mathbb{N}^(*) \supset \text{PR} \times \mathbb{N}^(*) \rightarrow \mathbb{N}^(*),
\]

satisfying the above 2-recursive system for \( ev \).

For our “set” theory \( T \) we now prove the following

**Evaluation Lemma:** For primitive recursive \( f : \mathbb{N}^(*) \supset A \rightarrow B \subset \mathbb{N}^(*) \) in \( T \), \( T \) extending \( \text{PR} + ev \), we have

\[
ev(\varphi^\gamma, n) = \varphi(n) : \mathbb{N} \rightarrow 2,
\]

in particular for \( \varphi : \mathbb{N} \rightarrow 2 \) (the map term representing) a p.r. predicate of \( T \):

\[
ev(\varphi^\gamma, n) = \varphi(n) : \mathbb{N} \rightarrow 2, \quad n \text{ free variable on } \mathbb{N}.
\]

**Proof** by external (“metamathematical”) induction on the operator-depth \( \text{depth}(f) \in \mathbb{N} \) of \( f \) varying on \( \text{PR} \subset \mathbb{N} \), in case of an iterated \( f = g^b(a, n) : A \times \mathbb{N} \rightarrow A \) this external induction will be combined with an internal induction on the iteration parameter \( n \in \mathbb{N} \). \( \text{depth} : \text{PR} \rightarrow \mathbb{N} \) is the external primitive recursive “twin” of \( \text{depth} : \text{PR} \rightarrow \mathbb{N} \) above; it is characterised by \( \text{depth}(\varphi^\gamma) = \text{num}(\text{depth}(f)) \) for \( f : A \rightarrow B \) in \( \text{PR} \subset T \). Here \( \text{num} = \text{num}(n) : \mathbb{N} \rightarrow T(1, \mathbb{N}) \) maps each external natural number \( n \) into its corresponding \( T \)-numeral, as defined e.g. in set theory by associating von Neumann numerals.

- Anchoring: the assertion holds for the basic maps \( 0, \ldots, \ell \) (with \( \text{depth} \) set to \( 1 \in \mathbb{N} \)) just by definition of \( ev \).
- composition case $f = h \circ g : A \to B \to C$:

$$
ev(\gamma f^\gamma, a) = ev(\gamma h \circ g^\gamma, a)
= ev(\gamma h^\gamma \gamma g^\gamma, a) \text{ since } \gamma \circ \gamma \text{ internalizes } \gamma'
= ev(\gamma h^\gamma, ev(\gamma g^\gamma, a)) \text{ by definition of } ev
= ev(\gamma h^\gamma, g(a)) \text{ by recursion hypothesis on } g
$$
since $\text{depth}(g) < \text{depth}(f)$

$$
ev(\gamma h^\gamma, g(a)) \text{ by recursion hypothesis on } h
$$
since $\text{depth}(h) < \text{depth}(f)$

$$
= (h \circ g)(a) = f(a).
$$

- case $f = id \times g : A \times B \to A \times C$ a cylindrified map:

$$
ev(\gamma f^\gamma, (a, b)) = ev(\gamma id \times g^\gamma, (a, b))
= ev(\gamma id^\gamma \gamma \times^\gamma \gamma g^\gamma, (a, b))
\quad \text{since } \gamma \times^\gamma \text{ is to internalize } \times
= (a, ev(\gamma g^\gamma, b)) \text{ by definition of } ev
= (a, g(b)) \text{ by recursion hypothesis on } g
$$
since $\text{depth}(g) < \text{depth}(f)$

$$
= (id \times g)(a, b) = f(a, b).
$$

- The remaining case – not quite so simple – is that of an \textit{iterated} $f = g^\delta : A \times \mathbb{N} \to A$ of a (p.r.) endo map $g : A \to A$, $g^\delta$ characterized by

$$
g^\delta(a, 0) = a, \quad g^\delta(a, n + 1) = g(g^\delta(a, n)) :$

the assertion of the Lemma holds in this last case too, since – “anchoring” $n = 0$ for internal induction:

$$
ev(\gamma f^\gamma, (a, 0)) = ev(\gamma g^\delta^\gamma, (a, 0))
= ev(\gamma g^\delta^\gamma, (a, 0)) = a \text{ since } (\_)^\delta \text{ internalizes } (\_)^\gamma
= g^\delta(a, 0) = f(a, 0)
$$

– as well as (internal induction step, using the external recursion
hypothesis):

\[ ev(⌜f^\gamma, (a, n + 1)⌝) = ev(⌜g^\delta, (a, n + 1)⌝) \]
\[ = ev(⌜g^\delta, (a, n + 1)⌝) \text{ since } (\omega)^\delta \text{ internalizes } (\omega)^\gamma \]
\[ = ev(⌜g^\delta, ev(⌜g^\delta, (a, n)⌝)⌝) \text{ by (internal) inductive definition of } ev \]
\[ \text{in the present case } v = u^\delta = r^\gamma \delta \]
\[ = ev(⌜g^\gamma, ev(⌜g^\delta, (a, n)⌝)⌝) \text{ by } (\omega)^\delta \text{ internalizing } (\omega)^\gamma \]
\[ = ev(⌜g^\gamma, g^\delta(a, n)⌝) \text{ by (internal) induction hypothesis on } n \]
\[ = g(g^\delta(a, n)) \text{ by (external) recursion hypothesis on } g \]
\[ \text{since } \text{depth}(g) < \text{depth}(f) \]
\[ = g^\delta(a, n + 1) = f(a, n + 1) \text{ by definition of the iterated } g^\delta \text{ q.e.d.} \]

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