PISIER TYPE INEQUALITIES FOR $K$-CONVEX SPACES

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Abstract. We generalize several theorems of Hytönen-Naor [HN] using the approach from [IVHV]. In particular, we give yet another necessary and sufficient condition (see (3.2)) to be a $K$-convex space, where the sufficiency was proved by Naor–Schechtman [NS]. This condition is in terms of the boundedness of the second order Riesz transforms $\{\Delta^{-1}D_i\}_{i=1}^n$ in $L^p(\Omega_n, X)$.

1. A variant of Pisier’s inequality in spaces of finite co-type

In this note we consider functions on Hamming cube $\Omega_n := \{-1, 1\}^n$. It is provided with natural measure giving $2^{-n}$ weight to each point, the integration will be denoted by $E$. There are differentiations: $\partial_i$ denotes the usual partial derivative with respect to $i$-th variable $\varepsilon_i$, but $D_i$ is often more convenient:

$$D_i = \varepsilon_i \partial_i.$$  

Laplacian is $\Delta := D_1 + \cdots + D_n$. Notice that $D_i^2 = D_i$.

The next theorem is basically proved in [IVHV], but for the convenience of reading we prove it here as it plays an important part in what follows.

**Theorem 1.1.** For any functions $f_i : \{-1, 1\}^n \to X$, $i = 1, \ldots, n$, $p \in [1, \infty)$, we have

$$E\|\sum_{i=1}^n D_if_i\|^p \leq C(q,p) E\|\sum_{i=1}^n \delta_i \Delta f_i\|^p$$  \hspace{1cm} (1.1)

if and only if $(X, \| \cdot \|)$ be a Banach space of finite co-type $q$.

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Here $\delta_i$ are standard Rademacher random variables. If one chooses $f_i = \Delta^{-1}D_i(f - Ef)$, then one restores Proposition 4.2 of [IVHV]. In other words, this becomes Pisier inequality [P] with constant independent of $n$.

**Remark 1.2.** In [HN] it was proved for UMD Banach spaces (and a bit more general case).

**Proof.** The proof of (1.1) for all finite co-type Banach spaces $X$ follows almost immediately from the formula

$$e^{-t\Delta}D_jf(\varepsilon) = \frac{e^{-t}}{\sqrt{1 - e^{-2t}}}E\xi\left[\frac{\xi_j(t) - e^{-t}}{\sqrt{1 - e^{-2t}}}f(\varepsilon\xi(t))\right],$$

(1.2)

where $\xi_i(t)$ assume values $\pm 1$ with probability $\frac{1}{2}(1 \pm e^{-t})$, and are mutually independent and independent from $\varepsilon_j$, $j = 1, \ldots, n$. The formula can be checked by direct calculation, see also Lemma 2.1 of [IVHV].

Adding and applying $\Delta = \Delta\varepsilon$, we get (we denote $\delta_j(t) := \frac{\xi_j(t) - e^{-t}}{\sqrt{1 - e^{-2t}}}$)

$$\Delta e^{-t\Delta}\sum_{j=1}^n D_jf_j(\varepsilon) = \frac{e^{-t}}{\sqrt{1 - e^{-2t}}}E\xi\left[\sum_{j=1}^n \delta_j(t)[\Delta f_j](\varepsilon\xi(t))\right].$$

After integrating in $t$, we get

$$-\sum_{j=1}^n D_jf_j(\varepsilon) = \int_0^\infty E\xi\left[\sum_{j=1}^n \delta_j(t)[\Delta f_j](\varepsilon\xi(t))\right] \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} dt.$$

Hence, we write

$$(E\xi\|\sum_{j=1}^n D_jf_j(\varepsilon)\|_p)^{1/p} \leq \int_0^\infty \left(E\xi E_{\varepsilon}\|\sum_{j=1}^n \delta_j(t)[\Delta f_j](\varepsilon\xi(t))\|_p\right)^{1/p} \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} dt =$$

$$\int_0^\infty \left(E\xi E_{\varepsilon}\left\|\sum_{j=1}^n \delta_j(t)[\Delta f_j](\varepsilon)\right\|_p\right)^{1/p} \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} dt,$$

where we used that for every fixed $t$, $\xi(t)$ the distribution of $\varepsilon \rightarrow [\Delta f_j](\varepsilon\xi(t))$ is the same as that of $\varepsilon \rightarrow [\Delta f_j](\varepsilon)$. We continue by introducing symmetrization by means of $\{\xi_i(t)\}$ independent from all $\{\xi_j\}_{j=1}^n$ and having the same distribution as $\xi_i$, $i = 1, \ldots, n$,

$$\int_0^\infty \left(E\xi E_{\xi^i}\left\|\sum_{j=1}^n \frac{\xi_j(t) - \xi_i^j(t)}{\sqrt{1 - e^{-2t}}}[\Delta f_j](\varepsilon)\right\|_p\right)^{1/p} \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} dt \leq$$

$$C(q, p) \int_0^\infty \left(E\xi E_{\delta}\left\|\sum_{j=1}^n \delta_j[\Delta f_j](\varepsilon)\right\|_p\right)^{1/p} \frac{e^{-t}}{(1 - e^{-2t})^{1 - \max(q, p)}} dt,$$
where we used Theorem 4.1 from [IVHV] and the fact that the co-type of $X$ is $q < \infty$. The last expression is bounded by

$$C(q, p) \max(q, p) \left( \mathbb{E}_{\varepsilon} \mathbb{E}_\delta \left\| \sum_{j=1}^n \delta_j [\Delta f_j](\varepsilon) \right\|^p \right)^{1/p},$$

and we are done.

Notice that, as it follows from [IVHV], the condition of having finite co-type is not only sufficient for inequality (1.1) to hold, but it is also necessary.

□

2. Pisier inequality’s constant

For any Banach space $X$ with no restriction Pisier’s inequality claims

$$(\mathbb{E}\|f - \mathbb{E}f\|^p)^{1/p} \leq C(n)(\mathbb{E}\left\| \sum_{i=1}^n \delta_i D_i f \right\|^p)^{1/p},$$

(2.1)

where

$$C(n) \leq C \log n.$$  

In [HN] it was shown that

$$C(n) \leq \log n + C.$$  

(2.2)

Remark 2.1. Looking at Section 6 in Talagrand’s paper [T] one can notice, that one gets the estimate from below of Pisier’s constant if $X = L^\infty(\Omega_n)$:

$$C(n) \geq \left( \frac{1}{2} - \delta \right) \log n - C_\delta.$$  

3. Yet another generalization of Pisier’s inequality

Let $F$ be a function of $\{-1, 1\}^n \times \{-1, 1\}^n$ with values in the Banach space $X$, and $F_j(\varepsilon) = \mathbb{E}_\delta \delta_j F(\varepsilon, \delta)$. For the special case $F(\varepsilon, \delta) = \sum_{j=1}^n f_j(\varepsilon)\delta_j$, inequality

$$(\mathbb{E}_\varepsilon\left\| \sum_{j=1}^n \Delta^{-1} D_j F_j \right\|^p)^{1/p} \leq C(p, n)(\mathbb{E}_{\delta, \varepsilon}\|F\|^p)^{1/p}, 1 < p < \infty,$$

(3.1)

is exactly (1.1). For such very special $F = \sum_{j=1}^n f_j(\varepsilon)\delta_j$ we know three things:

1. for general Banach space $X$, $C(p, n) \lesssim \log n$,
2. this is sharp growth,
3. $C(p, n) \leq C(p, q) < \infty$ iff $X$ is of finite co-type, and if $X$ is not of finite co-type, constant can grow logarithmically in $n$, [T].
However, it is interesting to ask for general function $F(\varepsilon, \delta)$ not just for functions of the type $F = \sum_{j=1}^{n} f_j(\varepsilon)\delta_j$ what happens with (3.1), namely,
A) for what Banach spaces constant does not depend on $n$?
B) What is the worst growth of constant with $n$ for general Banach space $X$?
C) What is the worst growth of constant with $n$ for special classes of Banach spaces, e.g. for $X$ of finite co-type?

Remark 3.1. In [HN] the example of co-type 2 space is considered, namely, $X = L^1([-1,1]^n)$, for which constant grows at least as $\sqrt{n}$. Below we show that this is the worst behavior for an arbitrary Banach space of finite co-type. Thus, conceptually, (3.1) turns out to be very different from (1.1) or from the original Pisier inequality.

Before formulating theorem let us consider the dual inequality to (3.1):

$$ E_{\delta,\varepsilon} \| \sum_{j=1}^{n} \delta_j \Delta^{-1} D_j g(\varepsilon) \|^p \leq C(p, n) E_{\varepsilon} \| g \|^p, \quad 1 < p < \infty. \quad (3.2) $$

In cases when $C(p, n) < \infty$ independent of $n$ and for $1 < p < \infty$, this is one of the typical Riesz transforms inequalities.

Theorem 3.2. Let $X$ be of finite co-type $q$. Then inequality (3.1) (and thus (3.2) for the dual space) holds with constant $C(p, q)\sqrt{n}$. The growth of constant cannot be improved in this class of $X$.

Proof. We again use the same formula (1.2), now in the following form:

$$ \Delta e^{-t\Delta} \sum_{j=1}^{n} \Delta^{-1} D_j F_j(\varepsilon) = \frac{e^{-t}}{1 - e^{-2t}} E_{\varepsilon} \left[ \sum_{j=1}^{n} \delta_j(t) F_j(\varepsilon \xi(t)) \right], $$

Hence

$$ E_{\delta,\varepsilon} \| \sum_{j=1}^{n} \Delta^{-1} D_j F_j \|^p \lesssim \int_{0}^{\infty} \frac{e^{-t}}{1 - e^{-2t}} E_{\delta} E_{\xi} E_{\varepsilon} \| \sum_{j=1}^{n} \delta_j(t) F(\varepsilon \xi(t), \delta) \|^p dt = $$

$$ \int_{0}^{\infty} \frac{e^{-t}}{1 - e^{-2t}} E_{\delta} E_{\xi} E_{\varepsilon} \| \sum_{j=1}^{n} \delta_j(t) F(\varepsilon, \delta) \|^p dt, $$
where we used that for every fixed $\xi$ the distribution of $\varepsilon \to F(\varepsilon \xi, \delta)$ is the same as the distribution of $\varepsilon \to F(\varepsilon, \delta)$. Using now finite co-type as before, we continue to write (below $\{\delta'_j\}$ are independent and independent of $\{\delta_j\}$ Rademacher random variables):

$$\int_0^\infty \frac{e^{-t}}{(1 - e^{-2t})^{1 - \min(1/p, 1/q)}} \mathbb{E}_{\delta} \mathbb{E}_{\varepsilon} \| \sum_{j=1}^n \delta_j \delta'_j F(\varepsilon, \delta) \|^p dt \lesssim (3.3)$$

$$C(p, q) \mathbb{E}_{\delta} \left[ \mathbb{E}_{\varepsilon} \| F(\varepsilon, \delta) \|^p \mathbb{E}_{\delta'} \| \sum_{j=1}^n \delta_j \delta'_j \|^p \right] \leq C'(p, q) n^{p/2} \mathbb{E}_{\varepsilon} \mathbb{E}_{\delta} \| F(\varepsilon, \delta) \|^p .$$

Theorem is proved. \qed

3.1. **$K$-convex Banach spaces.** Naor and Schechtman [NS] proved that if $(3.2)$ holds with constant independent of $n$, then $X$ is $K$-convex. However, it seems that the converse statement was open till now.

Let us quote [EI1]: “Nevertheless, $(3.2)$ with $C(p, n) = C(\log n + 1)$ is the best known bound to date for general $K$-convex spaces. . . . Under additional assumptions (e.g. when $X$ is a $UMD^+$ space or when $X$ is a $K$-convex Banach lattice), inequality $(3.2)$ is known to hold true with a constant $C(p, X)$ independent of the dimension $n$ for functions of arbitrary degree $d$, see [HN].”

Proposition 34 of [EI1] has the bound for $K$-convex spaces, and this bound is logarithmic in $n$ (logarithmic in $d$ for functions $f$ such that $\deg f \leq d$). As [EI1] mentions, under extra assumption that $X$ is a $UMD^+$ space or $K$-convex Banach lattice inequality $(3.2)$ was proved with constant independent of $n$, see [HN].

We prove $(3.2)$ with constant independent of $n$ for all $K$-convex $X$, thus making $K$-convexity to be equivalent to $(3.2)$.

Recall that $K$-convexity is equivalent to $B$-convexity, which is equivalent to being of type $> 1$, see Theorem 2.1 and Remark 2.4 of [P1].

**Theorem 3.3.** Let $X$ be of non-trivial type (which is the same as $K$-convex). Let $1 < p < \infty$. Then inequality $(3.2)$ holds with constant $C(p) < \infty$ independently of $n$.

**Proof.** We will be proving the dual inequality $(3.1)$ for $X^*$. As $X$ is of non-trivial type, it is $K$-convex by Pisier’s theorem (see [P1] or Theorem 7.4.28 of [HVNVW2]) (and is of finite co-type by König–Tzafriri theorem 7.1.14 in [HVNVW2], this we will not use). Then $X^*$ is of finite co-type $q$ and it is also $K$-convex (see [G] for self-duality of the class of $B$-convex Banach spaces).
Choose $1 < s \leq p$. We will use that $K$-convexity means that the Rademacher projection is bounded on functions in $L^s(X^*)$. As $X^*$ is of finite co-type $q$, Theorem 1.1 implies the following:

$$
E_\varepsilon \left\| \sum_{i=1}^{n} \Delta^{-1} D_i F_i \right\|_{X^*}^p \leq C(q,p) E_\delta,\varepsilon \left\| \sum_{i=1}^{n} \delta_i F_i \right\|_{X^*}^p .
$$

Now we use Kahane–Khintchine inequality to write for each fixed $\varepsilon_0 \in \Omega_n$:

$$
E_\delta \left\| \sum_{i=1}^{n} \delta_i F_i(\varepsilon_0) \right\|_{X^*}^p \leq C(s,q,p) \left( E_\delta \left\| \sum_{i=1}^{n} \delta_i F_i(\varepsilon_0) \right\|_{X^*}^s \right)^{p/s} .
$$

But the expression $\sum_{i=1}^{n} \delta_i F_i(\varepsilon_0)$ is the Rademacher projection of function $\delta \rightarrow F(\varepsilon_0, \delta)$. So $K$-convexity of $X^*$ implies

$$
\left( E_\delta \left\| \sum_{i=1}^{n} \delta_i F_i(\varepsilon_0) \right\|_{X^*}^s \right)^{p/s} \leq C'(K) \left( E_\delta \left\| F(\varepsilon_0, \delta) \right\|_{X^*}^s \right)^{p/s} \leq
\text{C}(K) E_\delta \left\| F(\varepsilon_0, \delta) \right\|_{X^*}^p .
$$

Now we combine that inequality with (3.5) for a fixed $\varepsilon = \varepsilon_0$. We are left to integrate in $\varepsilon_0$ and to use (3.4).

\[\Box\]

**Remark 3.4.** In [HN] inequality (3.1) was proved for $X$ such that $X^* \in UMD$ (in fact a potentially bigger class $UMD^+$ was involved). As non-trivial-type class is self dual, and also strictly wider than $UMD$, so the latter theorem generalizes Theorem 1.4 of [HN].

**Remark 3.5.** It is interesting to notice that the proof in [HN] is based on a formula that means that operators $\Delta^{-1} D_j$ are “averages of martingale transforms”. As these operators are “the second order Riesz transforms” on Hamming cube (in fact, $\Delta^{-1} D_j = \Delta^{-1} D_j^2 = (\Delta^{-1/2} D_j)^2$), it is natural to compare this averaging of martingale transforms to Riesz transforms with the same idea recently widely used in harmonic analysis, see, e. g. [DV], [PTV], [NTV1], [NTV2]. Paper [DV] is devoted to representing second order Riesz transforms in euclidean space as averaging of martingale transforms, in [PTV] the similar result is proved for the first order Riesz transforms.

**Theorem 3.6.** Let $X$ be an arbitrary Banach space, and $1 \leq p < \infty$. Then inequality (3.1) holds with constant $C(p,n) \lesssim n \log n$. 
Proof. We write

\[
\left( E_{\delta, \epsilon} \left\| P_{\tau} \sum_{j=1}^{n} \Delta^{-1} D_j F_j \right\|_p \right)^{1/p} \lesssim \frac{\int_{\tau}^{\infty} \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} (E_{\delta, \epsilon} E_{\epsilon}) \left\| \sum_{j=1}^{n} \delta_j \delta_j(t) F(\varepsilon \xi, \delta) \right\|_p \right)^{1/p} dt = \frac{\int_{\tau}^{\infty} \frac{e^{-t}}{1 - e^{-2t}} (E_{\delta, \epsilon} E_{\epsilon}) \left\| \sum_{j=1}^{n} \delta_j (\xi_j(t) - \xi_j'(t)) F(\varepsilon, \delta) \right\|_p \right)^{1/p} dt},
\]

Now we use Kahane contraction principle:

\[
\left( E_{\delta, \epsilon} \left\| P_{\tau} \sum_{j=1}^{n} \Delta^{-1} D_j F_j \right\|_p \right)^{1/p} \lesssim \log \frac{1 + e^{-\tau}}{1 - e^{-\tau}} \left[ E_{\epsilon} E_{\epsilon} \left( \left\| F(\varepsilon, \delta) \right\|_p \left( \sum_{j=1}^{n} \delta_j \left\| p \right\| \right) \right)^{1/p} \right] \leq \frac{n \log 1 + e^{-\tau}}{1 - e^{-\tau}} \left( E_{\epsilon, \delta} \left\| F(\varepsilon, \delta) \right\|_p \right)^{1/p}.
\]

Now using that \( \left\| f \right\|_p \leq e^{\tau n} \left\| P_{\tau} f \right\|_p \), we get

\[
\left( E_{\delta, \epsilon} \left\| P_{\tau} \sum_{j=1}^{n} \Delta^{-1} D_j F_j \right\|_p \right)^{1/p} \leq n \left( \min_{0 < r < 1} r^{-m} \log \frac{1 + r}{1 - r} \right)^{1/p} \left( E_{\epsilon, \delta} \left\| F(\varepsilon, \delta) \right\|_p \right)^{1/p} \lesssim n \log n \left( E_{\epsilon, \delta} \left\| F(\varepsilon, \delta) \right\|_p \right)^{1/p}.
\]

\[ \square \]

Remark 3.7. This theorem sounds a bit silly. It should be \( \lesssim \sqrt{n} \) for all Banach spaces. It can be that we missed something simple. On the other hand, it may be a worthwhile exercise to “marry” the example giving \( \sqrt{n} \) in [HN] and Talagrand’s example from [T], to possibly have a Banach space with behavior of constant in (3.1), which is worse than \( \sqrt{n} \). I did not try so far.

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