VECTOR BUNDLES ON PROJECTIVE VARIETIES WHOSE RESTRICTION TO AN AMPLE SUBVARIETY IS SPLIT

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Abstract. The purpose of this paper is to systematically study the splitting of vector bundles on smooth, projective varieties, whose restriction to the zero locus of a regular section of an ample vector bundle splits. We find ampleness and genericity conditions which ensure that the splitting of the vector bundle along the subvariety implies its global splitting.

Introduction

We say that a vector bundle splits if it is isomorphic to a direct sum of line bundles. Horrocks proved in [9] his celebrated splitting criterion for vector bundles on projective spaces, and his ideas gave rise to two main methods for proving the splitting of a vector bundle: either by imposing cohomological conditions, or by restricting to hypersurfaces in the ambient space.

In this article we will follow the latter path. Although there are several splitting criteria obtained by restricting to divisors, there seem to be no similar results for restrictions to higher co-dimensional subvarieties. Horrocks' result implies that a vector bundle on the projective space $\mathbb{P}^d_k$, where $\mathbb{k}$ is an algebraically closed field and $d \geq 3$, splits if and only if its restriction to a plane $\mathbb{P}^2_k \subset \mathbb{P}^d_k$ does so. Clearly, any plane is an ample subvariety of $\mathbb{P}^d_k$. In this article we will generalize this observation. Given a vector bundle $\mathcal{V}$ on a smooth, projective variety $X$, we ask under which assumptions the splitting of $\mathcal{V}$ along the zero locus of a regular section $s$ of an ample vector bundle $\mathcal{N} \to X$ implies its global splitting. We investigate this issue from two points of view, and we obtain respectively sufficient conditions, each being interesting in its own right.

First we prove (see theorem 2.2) that, by imposing sufficient ampleness on $\mathcal{N}$, the splitting of $\mathcal{V}$ along the zero locus of an arbitrary regular section $s \in \Gamma(X, \mathcal{N})$ implies its global splitting. The proof requires the vanishing of various cohomology groups, and we carefully control the amount of ampleness of $\mathcal{N}$ necessary to achieve this.

Second, we avoid imposing ampleness on $\mathcal{N}$, but rather we focus on genericity conditions on the choice of the section $s \in \Gamma(X, \mathcal{N})$. Theorem 3.8 states that the splitting of $\mathcal{V}$ along the zero locus $Y \subset X$ of $s$ implies its global splitting under the following hypotheses:

(a) the ample vector bundle $\mathcal{N}$ is globally generated, its rank is not too high (see the upper bound (3.1)), and $s$ is sufficiently general (in a precise sense);

(b) either the first cohomology group of any line bundle on $X$ vanishes, or the (finite dimensional) algebra of endomorphisms $\mathcal{V} \otimes \mathcal{O}_Y$ is semi-simple.

Clearly, the first condition is necessary. If we drop the assumption (b) and keep only (a), then we can only prove that $\mathcal{V}$ is obtained as a successive extension of line bundles on $X$.

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Finally, in the last section we specialize to the case when \( \mathcal{N} \) is an ample line bundle, and we extend the previous splitting criteria to vector bundles on varieties defined over ground fields of sufficiently large characteristic. The essential technical ingredient used here is Kodaira’s vanishing theorem in positive characteristic proved by Deligne and Illusie [6].

Let us briefly elaborate on these results. Their common root is proposition 1.9, which states without restrictions that \( \mathcal{V} \rightarrow X \) splits if and only if it does so on the formal completion \( \hat{X}_{Y_s} \) of \( X \) along the ample subvariety \( Y_s \) (of dimension at least one). Special attention is given to make this statement effective: we determine the order of the thickening of \( Y_s \) in \( X \) for which the splitting of \( \mathcal{V} \) along the thickening implies its global splitting. The proofs rely on the Buchsbaum-Eisenbud generic free resolutions [4] combined with the vanishing theorems of Laytimi-Nahm [10] and Manivel [11]. The conditions enumerated above (ampleness and genericity) can be checked in practice, and respectively correspond to situations when the splitting of \( \mathcal{V} \) along \( Y_s \) extends to a splitting along \( \hat{X}_{Y_s} \). We should also mention that, despite sharing a common root, the proofs of the two results mentioned before are very different in nature. While the cohomological criterion is based on effective cohomology vanishing theorems, the genericity criterion is essentially a gluing argument.

Finally, the appendix contains a descent result (see theorem A.2) of (hopefully) independent interest, which can be interpreted as the invariance of the classical Krull-Schmidt decomposition of a vector bundle (see [1]) under the change of the ground field. The theorem is necessary for the base change arguments in section 3.

1. The setup

We start by introducing the notations.

**Notations 1.1.** Throughout the article \( X \) stands for an irreducible, smooth, projective variety, defined over an algebraically closed field \( k \) of characteristic zero. (The only exception is section \( \S \) where we will also consider ground fields of positive characteristic.) By a vector (resp. line) bundle we mean a locally free (resp. invertible) sheaf.

We consider two vector bundles \( \mathcal{V} \) and \( \mathcal{N} \) on \( X \), of rank \( r \) and \( \nu \) respectively, and assume that \( \mathcal{N} \) is ample. We let \( \mathcal{E} := \text{End}(\mathcal{V}) \) be the vector bundle of endomorphisms of \( \mathcal{V} \). For a closed subscheme \( S \subset X \), we denote \( \mathcal{F}_S := \mathcal{V} \otimes \mathcal{O}_S \) and similarly for \( \mathcal{E} \).

The zero locus of a section \( s \in \Gamma(X, \mathcal{N}) \) is the subscheme \( Y_s \) of \( X \) defined by the ideal sheaf \( \mathcal{I}_{Y_s} := \text{Image}(\mathcal{N}^r \xrightarrow{s} \mathcal{O}_X) \). We say that \( s \) is a regular section if its zero locus \( Y_s \) is a locally complete intersection in \( X \).

Let \( Y \subset X \) be the zero locus of a regular section \( s \in \Gamma(X, \mathcal{N}) \). Then its ideal sheaf \( \mathcal{J}_Y \subset \mathcal{O}_X \) admits the following well-known Koszul resolution:

\[
0 \rightarrow \bigwedge^\nu \mathcal{N}^r \xrightarrow{\mathfrak{d}} \bigwedge^{\nu-1} \mathcal{N}^r \rightarrow \ldots \rightarrow \mathcal{N}^r \xrightarrow{\mathfrak{d}} \mathcal{J}_Y \rightarrow 0. \tag{1.1}
\]

(Here \( \mathfrak{d} \) stands for the contraction operation.) More generally, locally free resolutions of the powers of \( \mathcal{J}_Y \) are constructed in [4, Theorem 3.1]. For any \( m \geq 1 \), we have the resolution

\[
0 \rightarrow L_m^\nu(\mathcal{N}^r) \rightarrow L_m^{\nu-1}(\mathcal{N}^r) \rightarrow \ldots \rightarrow L_m^1(\mathcal{N}^r) \rightarrow \ldots \rightarrow \text{Sym}^m(\mathcal{N}^r) \xrightarrow{s^m \mathfrak{d}} \mathcal{J}_Y^m \rightarrow 0, \tag{1.2}
\]
where the vector bundles $L^j_m(N^\nu)$, $1 \leq j \leq \nu$, are defined as follows:

$$
L^j_m(N^\nu) := \text{Ker} \left( \text{Sym}^m(N^\nu) \otimes \bigwedge^{j-1} N^\nu \rightarrow \text{Sym}^{m+1}(N^\nu) \otimes \bigwedge^j N^\nu \right)
= \text{Im} \left( \text{Sym}^{m-1}(N^\nu) \otimes \bigwedge^j N^\nu \rightarrow \text{Sym}^m(N^\nu) \otimes \bigwedge^{j-1} N^\nu \right).
$$

(1.3)

Actually $L^j_m(N^\nu)$ is a direct summand in both $\text{Sym}^m(N^\nu) \otimes \bigwedge^{j-1} N^\nu$ and $\text{Sym}^{m-1}(N^\nu) \otimes \bigwedge^j N^\nu$ because the homomorphisms used for defining $L^j_m(N^\nu)$ are $\text{Aut}(N^\nu)$-invariant and the general linear group is linearly reductive in characteristic zero. The long exact sequence (1.2) breaks up into $\nu - 1$ short exact sequences of the form

$$
0 \rightarrow S^{(m)}_{j+1} \rightarrow L^j_m(N^\nu) \rightarrow S^{(m)}_j \rightarrow 0, \quad j = 1, \ldots, \nu - 1,
$$

with $S^{(m)}_1 = \mathcal{F}_Y$ and $S^{(m)}_{\nu} = \text{Sym}^{m-1}(N^\nu) \otimes \text{det}(N^\nu)$.

**Lemma 1.2.** Let the notations be as in (1.1) and let $\mathcal{F} \rightarrow X$ be a vector bundle of rank $f$.

(i) (arbitrary $\nu$, lot of positivity for $N$)

Let $m \geq 0$ be such that $\text{Sym}^{1+f}(\mathcal{F}^\vee) \otimes \text{det}(\mathcal{F}) \otimes \text{Sym}^{m+\nu}(N) \otimes \text{det}(N)^{-1}$ is ample. Then holds:

$$
H^i(X, \mathcal{F} \otimes \text{Sym}^m(N^\nu) \otimes \bigwedge^j N^\nu) = 0, \forall t < \dim X - \nu + j.
$$

In particular, if $\nu \leq \dim X - 2$ and $\text{Sym}^{1+f}(\mathcal{F}^\vee) \otimes \text{det}(\mathcal{F}) \otimes \text{Sym}^{1+\nu}(N) \otimes \text{det}(N)^{-1}$ is ample, then $H^j(X, \mathcal{F} \otimes \bigwedge^j N^\nu) = 0, \forall j = 1, \ldots, \nu$.

(ii) (low $\nu$, little positivity for $N$)

Assume that $\mathcal{F}^\vee \otimes N$ is ample, and $(\nu + 1)^2 \leq \dim X - f$. Then holds $H^j(X, \mathcal{F} \otimes \bigwedge^j N^\nu) = 0$, for all $j = 1, \ldots, \nu$.

**Proof.** (i) We consider the diagram

$$
\begin{array}{cccccc}
\mathcal{O}_{p_F}(1) & \xrightarrow{p} & \mathbb{P}(\mathcal{F}) & \xrightarrow{p} & \mathbb{P}(N^\nu) & \xrightarrow{p} \mathcal{O}_{p_N}(1)
\end{array}
$$

where $Z := \mathbb{P}(\mathcal{F}) \times_X \mathbb{P}(N^\nu)$ and $\mathcal{O}_{p_F}(1)$ and $\mathcal{O}_{p_N}(1)$ stand for the corresponding relatively ample line bundles. Then the relative canonical bundle of $p_N$ satisfies

$$
\kappa_{p_N} = \text{det}(N) \otimes \mathcal{O}_{p_N}(-\nu) \Rightarrow \text{Sym}^m(N) = (p_N)_* (\kappa_{p_N} \otimes \mathcal{O}_{p_N}(m + \nu) \otimes \text{det}(N)^{-1}), \forall m \geq 0.
$$

Similar conclusion holds for the relative canonical bundle $\kappa_{p_F}$ of $p_F$. By hypothesis, the line bundle $\mathcal{L} := (\mathcal{O}_{p_F}(1 + f) \otimes \text{det}(\mathcal{F})) \otimes (\mathcal{O}_{p_N}(m + \nu) \otimes \text{det}(N)^{-1})$ on $Z$ is ample, and $\kappa_X \otimes \mathcal{F}^\vee \otimes \text{Sym}^m(N) = p_* (\kappa_Z \otimes \mathcal{L})$. Furthermore, the projection formula implies

$$
H^i(X, \kappa_X \otimes \mathcal{F}^\vee \otimes \text{Sym}^m(N) \otimes \bigwedge^j N) = H^i(Z, \kappa_Z \otimes \mathcal{L} \otimes \bigwedge^j p^* N).
$$

On the right-hand-side, $p^* N \rightarrow Y$ is nef and $\mathcal{L} \rightarrow Z$ is ample. The vanishing theorem [11] implies that the cohomology group above vanishes for $i > \nu - j$. We obtain the desired conclusion by applying the Serre duality on $X$.

For the second claim, observe that the $j^{th}$ cohomology group of $\mathcal{F}^\vee \otimes N^\nu \otimes \bigwedge^j N^\nu$ vanishes, as $j < \dim X - \nu + j - 1$, and use that $\bigwedge^j N$ is a direct summand in $N \otimes \bigwedge^j N$. 


(ii) If $\mathcal{F} \otimes \mathcal{N}$ is ample, then $\mathcal{F} \otimes \hat{\bigwedge}^j \mathcal{N}$ is ample too, being a direct summand of $\mathcal{F} \otimes \mathcal{N}^{\otimes j}$.

The vanishing theorem [10, Theorem 2.1] yields $H^{\dim X - j}(X, \kappa_X \otimes \mathcal{F} \otimes \hat{\bigwedge}^j \mathcal{N}) = 0$, for all $j = 1, \ldots, \nu$. □

**Proposition 1.3.** Let $\mathcal{N} \to X$ be an ample vector bundle of rank $\nu$, and let $s \in \Gamma(X, \mathcal{N})$ be a regular section. We denote by $Y$ the zero locus of $s$, and by $\mathcal{I}_Y$ its ideal sheaf. We consider an arbitrary, locally free sheaf $\mathcal{F} \to X$ of rank $f$. Then hold:

(i) There is an integer $m_\mathcal{F} \geq 1$ such that

$H^t(X, \mathcal{F} \otimes \mathcal{I}_Y^m) = 0, \quad \forall m \geq m_\mathcal{F}, \forall t \leq \dim Y = \dim X - \nu.$

The integer $m_\mathcal{F}$ can be taken as the smallest positive integer such that the vector bundle $\text{Sym}^{1+j}(\mathcal{F}) \otimes \det(\mathcal{F}) \otimes \text{Sym}^{m-1+\nu}(\mathcal{N}) \otimes \det(\mathcal{N})^{-1}$ is ample.

(ii) Assume that $f + \frac{(\nu+1)^2}{4} \leq \dim X$, and $\mathcal{F} \otimes \mathcal{N}$ is ample. Then $H^1(X, \mathcal{F} \otimes \mathcal{I}_Y) = 0$.

**Proof.** (i) We tensor the short exact sequences (1.3) by $\mathcal{F}$. As the middle term $\mathcal{F} \otimes L_m^j(\mathcal{N}^\nu)$ is a direct summand in $\mathcal{F} \otimes \text{Sym}^{m-1+\nu}(\mathcal{N}^\nu) \otimes \hat{\bigwedge}^j \mathcal{N}^\nu$ for all $j$, Lemma 1.2(i) implies that for all $t \leq \dim X - \nu$ holds $H^{t+j-1}(\mathcal{F} \otimes L_m^j(\mathcal{N}^\nu)) = 0$, hence

$H^t(\mathcal{F} \otimes \mathcal{I}_Y^m) \subset H^{t+1}(\mathcal{F} \otimes S_2^{(m)}) \subset \cdots \subset H^{t+\nu-2}(\mathcal{F} \otimes S_\nu^{(m)}) \subset H^{t+\nu-1}(\mathcal{F} \otimes L_m^\nu(\mathcal{N}^\nu)) = 0$.

(ii) The same argument as above, together with lemma 1.2(ii) yields:

$H^1(X, \mathcal{F} \otimes \mathcal{I}_Y) \subset H^2(X, \mathcal{F} \otimes S_2^{(1)}) \subset \cdots \subset H^\nu(X, \mathcal{F} \otimes \det(\mathcal{N}^\nu)) = 0$. □

Let $Y \subset X$ be the zero locus of a regular section of $\mathcal{N}$, whose ideal sheaf is $\mathcal{I}_Y$. For $m \geq 1$, the $m$-th order thickening $Y_m$ of $Y$ is the closed subscheme defined by the sheaf of ideals $\mathcal{I}_Y^{m+1}$. Notice that $Y_0 = Y$ with this convention. The completion of $X$ along $Y$ is defined as the direct limit $\lim_{\to} Y_m$, and is denoted by $\hat{X}_Y$. When no confusion is possible, we simply write $\hat{X}$. The structure sheaves of two consecutive thickenings of $Y$ fit into the exact sequence

$0 \to \text{Sym}^m(\mathcal{N}^\nu_Y) \cong \mathcal{I}_Y^{m+1}/\mathcal{I}_Y^m \to \mathcal{O}_{Y_m} \to \mathcal{O}_{Y_{m-1}} \to 0$. \hspace{1cm} (1.5)$

When the ground field $k = \mathbb{C}$, we may consider two kinds of thickenings and completions: one by using germs of regular functions and another one using germs of analytic functions. However, the exact sequence (1.5) is valid in both cases.

**Corollary 1.4.** Let the situation be as in proposition 1.3, and let $\hat{X}$ be the (algebraic or analytic) formal completion of $X$ along $Y$. Then $H^j(X, \mathcal{F}) \to H^j(\hat{X}, \mathcal{F})$ is an isomorphism, for all $j = 0, \ldots, \dim Y - 1$.

**Proof.** This is a direct consequence of proposition 1.3. Regardless whether the computations are done algebraically or analytically, the generators of $\mathcal{I}_Y^{m+1}$ are the same for all $m$, by Chow’s theorem. Hence the resolution (1.2) is valid in both settings, and this is what was necessary for deducing the vanishing of $H^j(\mathcal{F} \otimes \mathcal{I}_Y^{m+1})$, for $j \leq \dim Y$ and $m \geq 0$. □

**Corollary 1.5.** Let $k = \mathbb{C}$, and the situation be as in proposition 1.3, with $\dim Y \geq 2$. Consider a sufficiently small (analytic or Zariski) open neighbourhood $U$ of $Y$ in $X$, and let $\mathcal{A}, \mathcal{C} \to X$ be two vector bundles. Then the following statements hold:

(i) Any extension of vector bundles

$0 \to \mathcal{A} \otimes \mathcal{O}_U \to \mathcal{S}_U \to \mathcal{C} \otimes \mathcal{O}_U \to 0 \hspace{1cm} (G)$
on $\mathcal{U}$ can be extended to an extension $0 \to A \to B \to C \to 0$ of vector bundles on $X$, and $B$ is uniquely defined, up to isomorphism.

(ii) Assume that the restriction to $\mathcal{U}$ of three vector bundles $A, C, S$ on $X$ fit in the extension $(G)$. Then $S$ is an extension of $C$ by $A$ on $X$.

Proof. (i) The extension $(G)$ corresponds to an element $\eta_{\mathcal{U}} \in H^1(\mathcal{U}, C^\vee \otimes A)$, and its restriction to $\hat{X}$ corresponds to the image $\eta_{\hat{X}} \in H^1(\hat{X}, C^\vee \otimes A)$ of $\eta_{\mathcal{U}}$. Corollary 1.4 implies that $H^1(X, C^\vee \otimes A) \to H^1(\hat{X}, C^\vee \otimes A)$ is an isomorphism, so $\eta_{\hat{X}}$ can be uniquely lifted to $\eta \in H^1(X, C^\vee \otimes A)$, which in turn defines the extension of vector bundles $0 \to A \to B \to C \to 0$ on $X$.

Let us prove the uniqueness of $B$. Its restriction to $\hat{X}$ is isomorphic to $\mathcal{U}_\mathcal{U} \otimes \mathcal{O}_{\hat{X}}$, so for two extensions $B$ and $B'$ as above there are homomorphisms $\hat{h} : B'_X \to B_X$ and $\hat{h}' : B_X \to B'_X$, inverse to each other. By applying corollary 1.4 again, we deduce that

$$
\Gamma(X, \text{Hom}(B', B)) \to \Gamma(\hat{X}, \text{Hom}(B'_X, B_X))
$$

are isomorphisms, thus $\hat{h}$ and $\hat{h}'$ uniquely extend to $h : B' \to B$ and $h' : B \to B'$. The compositions $h \circ h'$ and $h' \circ h$ are the (unique) extensions of $1_B|_X$ and $1_{B'}|_X$, so $h \circ h' = 1_B$ and $h' \circ h = 1_{B'}$, that is $B$ and $B'$ are isomorphic.

(ii) It is just a reformulation of the uniqueness statement above. \qed

**Definition 1.6.** We say that the vector bundle $\mathcal{V} \to X$ of rank $r$ splits (or is split) if there are $r$ line sub-bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r \in \text{Pic}(X)$ of $\mathcal{V}$ such that $\mathcal{V} = \bigoplus_{i=1}^r \mathcal{L}_i$. Thus $\mathcal{V}$ splits if and only if there are pairwise non-isomorphic line sub-bundles $\mathcal{L}_j$, $j \in J$, of $\mathcal{V}$ such that

$$
\mathcal{V} = \bigoplus_{j \in J} \mathcal{L}_j \otimes \mathcal{k}^{m_j} \quad \text{with} \quad \sum_{j \in J} m_j = r. \quad (1.6)
$$

We call the vector sub-bundles $\mathcal{V}_j := \mathcal{L}_j \otimes \mathcal{k}^{m_j}$, $j \in J$, the isotypical components of $\mathcal{V}$ corresponding to the splitting above.

If $\bigoplus_{j \in J} \mathcal{L}_j \otimes \mathcal{k}^{m_j}$ and $\bigoplus_{j' \in J'} \mathcal{L}_{j'} \otimes \mathcal{k}^{m_{j'}}$ are two splittings of $\mathcal{V}$, then there is a bijective function $\sigma : J \to J'$ such that $\mathcal{L}_{\sigma(j)} \cong \mathcal{L}_j$ and $m_{\sigma(j)} = m_j$ for all $j \in J$. (See [1], Theorem 1 and 2.)

Unfortunately, the isotypical components are not uniquely defined, as they depend on the choice of the splitting. Indeed, the global automorphisms of $\mathcal{V}$ send a splitting into another one. We define the relation ‘$<$’ on the index set $J$ as follows:

$$
i < j \quad \text{if} \quad i \neq j \quad \text{and} \quad \Gamma(X, \mathcal{L}_i^{-1} \mathcal{L}_j) \neq 0. \quad (1.7)
$$

It is straightforward to check that ‘$<$’ is a partial order. The maximal elements with respect to $<$ have the property that the corresponding isotypical components are uniquely defined.

**Lemma 1.7.** Let $M \subset J$ be the subset of maximal elements with respect to $<$. Then there is a natural, injective homomorphism of vector bundles

$$
ev_M : \bigoplus_{j \in M} \mathcal{L}_j \otimes \Gamma(X, \mathcal{L}_j^{-1} \mathcal{V}) \to \mathcal{V}. \quad (1.8)
$$

Proof. Clear, by the very definition. \qed

The following elementary lemma is the key for testing the splitting of a vector bundle. It allows to lift the splitting along a sub-variety to a splitting on the ambient space.
Lemma 1.8. Let $\Phi \in \Gamma(X, \mathcal{E})$, with $\mathcal{E} := \mathcal{E}(\mathcal{V})$, be a global endomorphism of the vector bundle $\mathcal{V} \to X$. Then the following statements hold:

(i) The eigenvalues of $\Phi_x$ do not depend on $x \in X$.

(ii) Let $S \subset X$ be a closed subscheme such that $\operatorname{res}_S : \Gamma(X, \mathcal{E}) \to \Gamma(S, \mathcal{E}_S)$ is surjective, and the restriction $\mathcal{V}_S$ splits. Then $\mathcal{V}$ is itself split.

Proof. (i) Notice that $\det(\Phi) \in \Gamma(X, \det(\mathcal{V})) = \Gamma(X, \mathcal{O}_X)$. Since $X$ is projective, $\det(\Phi)$ is the multiplication by a scalar. Hence either $\Phi_x : \mathcal{V}_x \to \mathcal{V}_x$ is an isomorphism for all $x \in X$, or is not invertible for any $x \in X$. Now we apply this remark to the endomorphism $\varepsilon \mathbb{I} - \Phi$, where $\varepsilon \in \mathbb{k}$ and $\mathbb{I}$ stands for the identity.

(ii) The hypothesis says that $\mathcal{V}_S \cong \ell_1 \oplus \ldots \oplus \ell_r$, where $r := \operatorname{rk}(\mathcal{V})$ and $\ell_1, \ldots, \ell_r \in \operatorname{Pic}(S)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be pairwise distinct scalars, and consider the diagonal endomorphism $\phi \in \Gamma(S, \mathcal{E}_S)$ given by multiplication by $\varepsilon_p$ on $\ell_p$. Since $\operatorname{res}_S$ is surjective, $\phi$ extends to $\Phi \in \Gamma(X, \mathcal{E})$. But the eigenvalues of $\Phi$ are independent of $x \in X$, so they are precisely $\varepsilon_1, \ldots, \varepsilon_r$. Overall, we have an endomorphism $\Phi$ of $\mathcal{V}$ with $\operatorname{rk}(\mathcal{V})$ distinct eigenvalues. Hence $\Phi_x$ is diagonalizable for all $x \in X$, and $L_p := \ker(\varepsilon_p \mathbb{I} - \Phi)$ are line bundles on $X$ such that $\mathcal{V} = L_1 \oplus \ldots \oplus L_r$. $\square$

The following is a general splitting criterion for vector bundles on arbitrary projective varieties. It is the common root of all the results obtained in this article.

Proposition 1.9. (i) Let $\mathcal{N}$ be an ample vector bundle of rank $\nu \leq \dim X - 1$, and let $Y$ be the zero locus of a regular section of $\mathcal{N}$. Then a vector bundle $\mathcal{V} \to X$ splits if and only if its restriction $\mathcal{V}|_Y \to \hat{X}$ to the formal completion of $X$ along $Y$ splits.

(ii) Let $m_\mathcal{V}$ be the smallest positive integer such that $\operatorname{Sym}^{1+r^2} (\mathcal{E} \otimes \operatorname{Sym}^{m_\nu}(\mathcal{N}) \otimes \det(\mathcal{N})^{-1}$ is ample. Then $\mathcal{V}$ splits if and only if $\mathcal{V}|_{Y_m}$, with $m \geq m_\mathcal{V}$, splits.

(iii) In particular, assume that the ground field is the field of complex numbers. Then $\mathcal{V}$ splits if and only if there is an open analytic neighbourhood $U$ of $Y$ such that $\mathcal{V} \otimes \mathcal{O}_U$ splits.

Proof. Proposition [1.8] implies that $H^1(\mathcal{E} \otimes \mathcal{O}^m_{Y}) = 0$, for all $m \geq m_\mathcal{V}$, hence the homomorphism $\Gamma(X, \mathcal{E}) \to \Gamma(Y_m, \mathcal{E}|_{Y_m})$ is surjective. The conclusion follows from lemma [1.8]. $\square$

In the subsequent sections we will obtain sufficient conditions for this general splitting criterion. The result is false if we require only the splitting of $\mathcal{V}|_Y$, instead of $\mathcal{V}|_{Y_m}$. Simply take $X = \mathbb{P}^n$ and $Y \cong \mathbb{P}^1$ a straight line in it. Then $Y$ is the zero locus of a section in $\mathcal{O}_{\mathbb{P}^n}(1)^{\otimes (n-1)}$, and any vector bundle on $\mathbb{P}^n$ splits along $Y$. However, according to the proposition, a vector bundle on $\mathbb{P}^n$ splits if its restriction to a sufficiently high order thickening of a line does so.

We conclude by remarking that [1.9(i), (ii)] can be strengthened. In [12], a subvariety $Y \subset X$ of codimension $\nu$ is called ample if the exceptional divisor $E$ of the blow-up $X' := \text{Bl}_Y(X)$ of $X$ along $Y$ is $(\nu - 1)$-ample. (This is an intrinsic property of $Y$, independent of additional data. However, the zero loci of regular sections of ample vector bundles are the prototypes of ample subvarieties. See [12] Proposition 4.5.) The condition above amounts to requiring (see Lemma 2.1 and 2.2 in loc. cit.) that for all vector bundles $\mathcal{F} \to X'$ holds

$$H^t(X', \mathcal{F} \otimes \mathcal{O}_{X'}(-mE)) = 0, \forall t \leq \dim X - \nu, \forall m \gg 0.$$  

By applying this to the pull-back to $X'$ of $\mathcal{E} = \mathcal{E}(\mathcal{V})$, one obtains [12 Proposition 6.2] that $H^t(X, \mathcal{E} \otimes \mathcal{O}^m_X) = 0, \forall t \leq \dim X, \forall m \gg 0$, and consequently holds:

Proposition 1.10. For an ample subvariety $Y \subset X$ as in [12], the statements (i), (iii) of proposition [1.9] above are still valid.

Unfortunately, we will not be able to use this strengthening since the forthcoming sections will require effective cohomology vanishing properties.
Vector bundles whose restriction to ample subvarieties are split

2. The first criterion: ampleness conditions for $N$

In this section we fix an arbitrary regular section of $N$ such that $\mathcal{V}$ splits along its zero locus, and we impose sufficient ampleness on $N$ in order to deduce the global splitting of $\mathcal{V}$. The strategy is to successively lift the splitting of $\mathcal{V}_{Y_m-1}$ to a splitting of $\mathcal{V}_{Y_m}$, for all $m \geq 1$.

**Proposition 2.1.** Let $Y$ be the zero locus of a regular section $s$ of $N$ such that $\mathcal{V}_Y \to Y$ splits. Then $\mathcal{V} \to X$ is split as soon as either one of the following conditions is satisfied:

(i) $\nu \leq \dim X - 1$, and $H^j(X, \mathcal{E} \otimes \mathcal{N}^\vee) = 0$, for $j = 1, \ldots, \nu$.

or (ii) $\nu \leq \dim X - 2$, and $H^1(Y, \text{Sym}^m(\mathcal{N}_Y^\vee) \otimes \mathcal{E}_Y) = 0$, for all $m \geq 1$.

We observe that the twist of any vector bundle by a sufficiently ample line bundle on $\mathcal{N}$ satisfies the previous conditions. Also, Horrocks’ splitting criterion for $X = \mathbb{P}^d$ is a particular case: just take $N := \mathcal{O}_{\mathbb{P}^d}(1)^{\otimes \nu}$ and apply (ii). Furthermore, we remark that the condition (ii) above involves only the vanishing of certain cohomology groups of $Y$, where $\mathcal{V}$ splits. This allows us to obtain below some (we believe) interesting and new splitting criteria.

**Proof.** (i) We prove that $H^1(X, \mathcal{E} \otimes \mathcal{J}_Y) = 0$, and the conclusion will follow from lemma [1.8](#). By using the hypothesis in [1.4], we obtain $H^j(X, \mathcal{E} \otimes S^{(i)}_j) \subset H^{j+1}(X, \mathcal{E} \otimes S^{(i)}_{j+1})$, for $j = 1, \ldots, \nu - 1$, so $H^1(X, \mathcal{E} \otimes \mathcal{J}_Y) \subset H^\nu(X, \mathcal{E} \otimes \det \mathcal{N}_Y^\vee) = 0$.

(ii) The exact sequence [1.5] implies that $\text{res}_{Y_{m-1}}^{Y_m} : \Gamma(Y_m, \mathcal{E}) \to \Gamma(Y_{m-1}, \mathcal{E})$ is surjective, for all $m \geq 1$. As $\Gamma(X, \mathcal{E}) = \lim \Gamma(Y_m, \mathcal{E})$, we deduce that $\Gamma(X, \mathcal{E}) \to \Gamma(Y, \mathcal{E})$ is surjective, so $\Gamma(X, \mathcal{E}) \to \Gamma(Y, \mathcal{E})$ is surjective by corollary [1.4].

**Theorem 2.2.** Assume that the rank of $N$ is $\nu \leq \dim X - 2$, and let $s \in \Gamma(X, N)$ be a regular section of $N$ with zero locus $Y$ such that the restriction $\mathcal{V}_Y \to Y$ splits. Then the vector bundle $\mathcal{V} \to X$ splits as soon as either one of the following conditions is satisfied:

(i) $\text{Sym}^{1+\nu}(\mathcal{E}) \otimes \text{Sym}^{1+\nu}(N) \otimes \det(N)^{-1}$ is ample;

(ii) $\frac{(\nu+1)^2}{4} \leq \dim X - \nu^2$ and $\mathcal{E} \otimes N$ is ample;

(iii) $Y$ is smooth, and $\mathcal{E} \otimes \text{Sym}^{1+\nu}(N) \otimes (\det N)^{-1}$ is ample.

(iv) $Y$ is smooth, and moreover $\nu \leq \dim X - 1$, $N = \mathcal{O}(A) \otimes A$ a globally generated vector bundle of rank $\nu$, and $A$ an ample line bundle such that $\mathcal{E}_Y \otimes A_Y$ is ample.

**Proof.** (i, ii) Lemma [1.2](#) respectively proposition [1.3](#), together with the previous proposition, imply that in both cases $H^1(X, \mathcal{E} \otimes \mathcal{J}_Y) = 0$. It remains to apply lemma [1.8](#).

(iii) By hypothesis, the relative hyperplane bundle $\mathcal{O}_q(1) \to \mathbb{P}(\mathcal{N}^\vee)$ is ample, so its restriction to $\mathbb{P}(\mathcal{N}_Y^\vee)$ is still ample. The projection formula implies

$$H^1(Y, \text{Sym}^m(\mathcal{N}_Y^\vee) \otimes \mathcal{E}_Y) \cong H^\nu(\mathbb{P}(\mathcal{N}_Y^\vee), (\mathcal{O}_q(m+\nu) \otimes (q^*\det N_Y)^{-1} \otimes q^*\mathcal{E}_Y)^\vee).$$

(2.1)

For $\mathcal{V}_Y \cong \bigoplus_{j=1}^r \ell_j$, we have $\mathcal{E}_Y \cong \bigoplus_{i,j} \ell_j \ell_i^{-1}$, hence

$$H^\nu(\mathbb{P}(\mathcal{N}_Y^\vee), (\mathcal{O}_q(m+\nu) \otimes (q^*\det N_Y)^{-1} \otimes q^*\mathcal{E}_Y)^\vee)$$

$$= \bigoplus_{\ell} H^\nu(\mathbb{P}(\mathcal{N}_Y^\vee), (\mathcal{O}_q(m+\nu) \otimes (q^*\det N_Y)^{-1} \otimes q^*\ell)^{-1}).$$

where $\ell$ runs over the direct summands of $\mathcal{E}_Y$. But $\text{Sym}^{\nu+1}N \otimes (\det N)^{-1} \otimes \mathcal{E}$ is ample, so $\mathcal{O}_q(m+\nu) \otimes (q^*\det N_Y)^{-1} \otimes q^*\ell$ is ample for all $\ell$ and $m \geq 1$. As $Y$ is smooth,
the Kodaira vanishing implies that all terms vanish in the equation above. Consequently $H^1(Y, \text{Sym}^m(N_Y) \otimes \mathcal{E}_Y) = 0$, for all $m \geq 1$, and we conclude by 2.1(ii).
(iv) For all the direct summands $\ell$ of $\mathcal{E}_Y$, the line bundle $\ell \otimes \mathcal{A}_Y$ is ample. Then holds
\[ H^1(Y, \text{Sym}^m(N_Y) \otimes \mathcal{E}_Y)^\vee = \bigoplus_{\ell} H^1(Y, \kappa_Y \otimes \text{Sym}^m(\mathcal{O}_Y) \otimes (\ell \otimes \mathcal{A}_Y^m)), \]
and the latter terms vanish by [10, Theorem 2.4]. Proposition 2.1(ii) yields the conclusion. □

In some cases (see [3] and the references therein) one is interested in proving the triviality of certain vector bundles. In this direction we obtain the following:

**Corollary 2.3.** Assume that $\nu \leq \frac{\dim X - 1}{2}$ and $N = \mathfrak{g} \otimes A$, with $\mathfrak{g} \to X$ globally generated and $A \to X$ ample, and that this zero locus is smooth. Then $\mathcal{V}$ is trivializable on $X$.

Proof. Indeed, in this case $\mathcal{E}_Y \cong \mathcal{O}_{\mathfrak{g}}^{\oplus r^2}$, and we apply theorem 2.2(iv) above. □

**Example 2.4.** (i) Probably the most down-to-earth application occurs for $N$ a line bundle such that $\mathcal{E} \otimes N$ is ample. Then the splitting of $\mathcal{V}$ follows from its splitting along a smooth divisor in $\lfloor N \rfloor$. Actually, for $r^2 \leq \dim X - 1$, the smoothness assumption on $\mathfrak{g}$ can be dropped. (A similar statement holds in positive characteristic too, c.f. theorem 4.3).
(ii) Let $V$ be a $(u + \nu)$-dimensional vector space over $\mathbb{C}$ such that $u > \nu \geq 2$, and let $X := \text{Grass}(V, \nu)$ the Grassmann variety of $\nu$-dimensional quotients of $V$. We denote by $\mathcal{Q}$ the universal quotient bundle on $X$, and let $\mathcal{O}_X(1) := \det \mathcal{Q}$. Furthermore, we consider a vector bundle $\mathcal{V} \to X$ of rank $\nu$. Then the following statements hold:
(a) Assume that $\mathcal{E}(a)$ is ample for some $a \geq 1$, and that $\mathcal{V}$ splits along the (smooth) zero locus of a regular section in $\mathcal{Q}(a)$. Then $\mathcal{V} \to X$ splits too.
(b) Assume that the restriction of $\mathcal{V}$ to the (smooth) zero locus of a regular section in $\mathcal{Q}(1)$ is trivializable. Then we have $\mathcal{V} \cong \mathcal{O}_{\mathcal{V}}^{\oplus \nu}$.

3. **THE SECOND CRITERION:**

**SPLITTING ALONG ZERO LOCI OF GENERIC SECTIONS OF $N$**

Now we change our point of view. Instead of imposing ampleness on $N$, we will prove that the splitting of a vector bundle along the zero locus of a very general section of a globally generated ample vector bundle implies its global splitting.

Throughout this section we assume that $N$ is globally generated, and furthermore:
\[ \nu \leq \min\{\frac{\dim X - 3}{2}, \frac{\dim X - 1}{3}\} \quad \text{that is} \quad \begin{cases} \nu = 1 & \text{for } \dim X = 5, 6, \\ \nu \leq \frac{\dim X - 1}{3} & \text{for } \dim X \geq 7, \end{cases} \]

or $\nu = 1$, $\dim X = 4$, and $\kappa_X \otimes N^2$ is globally generated too, where $\kappa_X$ stands for the canonical bundle of $X$.

Our goal is to prove that the splitting of $\mathcal{V}$ along the geometric generic section of $N$ implies its global splitting over $X$. The proof uses base change arguments, so we start with general considerations. First we notice that $\mathcal{Q} \hookrightarrow k$, as char($k$) = 0. The variety $X$ and the vector bundles $N, \mathcal{V}$ are defined by equations involving finitely many coefficients in $k$. After adjoining them to $\bar{Q}$, we obtain a field extension of finite type $\bar{Q} \hookrightarrow k_0$. In particular, $k_0$ is countable, so we can realize it as a sub-field of $\mathbb{C}$.

\[
\begin{align*}
\kappa_0 & \hookrightarrow k \quad \text{alg. closed} \quad \bar{k}_0 \hookrightarrow k \\
\kappa_0 & \hookrightarrow \mathbb{C} \quad \text{alg. closed} \quad \bar{k}_0 \hookrightarrow \mathbb{C}.
\end{align*}
\]
After replacing $k_0$ by $\bar{k}_0$, we find a countable, algebraically closed field $k_0$, which is simultaneously a sub-field of $k$ and of $C$, such that $X, N, \mathcal{V}$ are defined over $k_0$. In this situation we have the Cartesian, base change diagram

$$
X = X_k \xrightarrow{b} X_0 := X_{k_0}
$$

and there are vector bundles $N_0, \mathcal{V}_0$ on $X_0$ such that $N = N_0 \times_{k_0} k$ and $\mathcal{V} = \mathcal{V}_0 \times_{k_0} k$. We let $\mathcal{E}_0 := \mathcal{E}_{nd}(\mathcal{V}_0)$, as usual. As $\Gamma(X, \mathcal{N}) = \Gamma(X, N_0) \otimes_{k_0} k$ and $N \to X$ is globally generated, it follows that $N_0 \to X_0$ is globally generated too. We denote $\mathbb{P}^N_k := \mathbb{P}(\Gamma(X, N)) = \text{Proj}(\text{Sym}^*_{N_0}(\Gamma(X, N)_{\mathcal{V}}))$, and similarly for $k_0$, and we consider the trace morphism

$$
\mathbb{P}^N_k \to \mathbb{P}^N_{k_0}, \quad \mathfrak{p} \mapsto \mathfrak{p} \cap \text{Sym}^*_{N_0}(\Gamma(X_0, N_0)_{\mathcal{V}}).
$$

The sheaf $\mathcal{K}$ defined by

$$
0 \to \mathcal{K} := \text{Ker}(\text{ev}) \to \Gamma(X, N) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} N \to 0,
$$

is locally free, and the incidence variety $\mathcal{Y} := \{(s, x) \mid s(x) = 0\} \subset \mathbb{P}^N_k \times X$ is naturally isomorphic to the projective bundle $\mathbb{P}(\mathcal{K})$ over $X$. We denote by $\pi$ and $\varphi$ respectively the projections of $\mathcal{Y}$ onto $\mathbb{P}^N_k$ and $X$. For any open subset $S$ of $\mathbb{P}^N_k$, we let $\mathcal{Y}_S := \pi^{-1}(S)$. If the ground field $k = C$, we will consider open subsets of $\mathbb{P}^N_C$ in the analytic topology.

Definition 3.1.

(i) We denote by $k$ the quotient field of $\mathbb{P}^N_k$, and by $\bar{k}$ its algebraic closure.

(ii) The geometric generic section $\mathcal{Y}$ of $N$ is defined by the Cartesian diagram:

$$
\begin{array}{ccc}
\mathcal{Y} := \mathcal{Y}_{\bar{k}} & \xrightarrow{\psi} & \mathcal{Y} \\
\downarrow \varphi & & \downarrow \pi \\
\text{Spec}(\bar{k}) & \xrightarrow{} & \mathbb{P}^N_{\bar{k}}
\end{array}
$$

Lemma 3.2. Let $N \to X$ be globally generated, and assume that the restriction of $\mathcal{V}$ to the geometric generic section of $N$ is split. Then there is a non-empty Zariski open subset $S$ of $\mathbb{P}^N_k$, and a Galois cover $S' \to S$ such that $q^*\mathcal{V} \times_S S' \to \mathcal{Y}'_S$ splits, and $Y_s$ is smooth for all $s \in S$. If $k = C$, there is an open analytic subset $B \subset \mathbb{P}^N_C$ with the previous two properties.

Proof. Let $(q^*\mathcal{V})_Y$ be the pull-back of $q^*\mathcal{V}$ to $Y$. By hypothesis, there are $\ell'_1, \ldots, \ell'_r \in \text{Pic}(Y)$ such that $(q^*\mathcal{V})_Y = \ell'_1 \oplus \ldots \oplus \ell'_r$. Actually $\ell'_1, \ldots, \ell'_r$ are defined over an intermediate field $k \to k' \to k$ finitely generated over $k$, that is $(q^*\mathcal{V})_{k'} = \ell'_1 \oplus \ldots \oplus \ell'_r$. After enlarging $k'$, we may assume that $k'$ is the splitting field of an irreducible polynomial $p \in k[\xi]$ whose leading coefficient equals one. Then $k \to k'$ is a Galois extension with Galois group $G := \text{Gal}(k'/k)$. After inverting the denominators of the coefficients of $p$ and of its discriminant, we find an open subset $S = \text{Spec}(k[S]) \subset \mathbb{P}(\Gamma(X, N))$ such that $p \in k[S]$. The group $G$ acts on $S' := \text{Spec}((k[S])[\xi]/(p))$, and $\sigma : S' \to S$ is $G$-equivariant and étale. Actually $S = S'/G$ is a geometric quotient, as $k[S'/G] = k[S]$. Overall, we obtain the diagram

$$
\begin{array}{ccc}
\mathcal{Y}_{S'} & \xrightarrow{\psi_{S'}} & \mathcal{Y}_S \\
\downarrow \pi_{S'} & & \downarrow \pi_S \\
S' & \xrightarrow{\sigma} & S \subset \mathbb{P}^N_k
\end{array}
$$

\text{End of proof.}
After shrinking $S$ further, $\ell'_1, \ldots, \ell'_n$ are defined over $k[S']$, so $(q^*\mathcal{Y})_{S'} \to \mathcal{Y}_{S'}$ splits, and also $Y_s$ is smooth for all $s \in S$, by Bertini’s theorem.

If $k = \mathbb{C}$, there are open balls $B' \subset S'$ and $B \subset S$ such that $\sigma : B' \to B$ is an analytic isomorphism. Then the splitting of $(q^*\mathcal{Y})_{B'} \to \mathcal{Y}_{B'}$ descends to $(q^*\mathcal{Y})_B \to \mathcal{Y}_B$. 

Henceforth we assume $k = \mathbb{C}$. Let $B \subset \mathbb{P}(\Gamma(X, N))$ be an open ball as above, and choose an isotypical decomposition $(q^*\mathcal{Y})_B = \bigoplus_{j \in J} \ell_j \otimes \mathbb{C}^{m_j}$, with $\ell_j \in \text{Pic}(\mathcal{Y}_B)$ pairwise non-isomorphic. The restriction $\text{res}^X_{Y_s} : \text{Pic}(X) \to \text{Pic}(Y_s)$ is an isomorphism, for all $s \in B$, by the Lefschetz-Sommese theorem (see [14]). Hence, for all $j \in J$, we have the morphism $\lambda_j : B \to \text{Pic}(\mathcal{Y}_B/B) \xrightarrow{\text{res}} \text{Pic}(X)$, $\lambda_j(s) := (\text{res}^X_{Y_s})^{-1}(\ell_j \otimes \mathcal{O}_{Y_s})$.

**Lemma 3.3.** The morphism $\lambda_j$ is constant, for all $j \in J$. Consequently, we have $(q^*\mathcal{Y})_B \cong q^* \left( \bigoplus_{j \in J} \ell_j \otimes \mathcal{O}_{\mathcal{Y}_B} \right)$, with $\ell_j := \text{Im}(\lambda_j) \in \text{Pic}(X)$.

**Proof.** Case $\dim X \geq 5$. Let $s, t \in B$ be two sections whose zero loci $Y_s$ and $Y_t$ meet transversally. Then their intersection $Y_{st}$ is the zero locus of $(s, t) \in \Gamma(X, N^{\otimes 2})$ and $\dim Y_{st} \geq 3$, by (3.1). The Lefschetz-Sommese theorem implies that we have the following commutative diagram, whose arrows are all isomorphisms:

\[
\begin{array}{ccc}
\text{Pic}(X) & \xrightarrow{\text{res}^X_{Y_s}} & \text{Pic}(Y_s) \\
\downarrow & & \downarrow \\
\text{Pic}(Y_t) & \xrightarrow{\text{res}^X_{Y_{st}}} & \text{Pic}(Y_{st})
\end{array}
\]

As $\left((q^*\mathcal{Y}) \otimes \mathcal{O}_{Y_s}\right) \otimes \mathcal{O}_{Y_{st}} = \left((q^*\mathcal{Y}) \otimes \mathcal{O}_{Y_t}\right) \otimes \mathcal{O}_{Y_{st}}$, we deduce that

\[
\bigoplus_{j \in J} \left[\text{res}^X_{Y_{st}}(\ell_j \otimes \mathcal{O}_{Y_s})\right]^{\otimes m_j} = \bigoplus_{j \in J} \left[\text{res}^X_{Y_{st}}(\ell_j \otimes \mathcal{O}_{Y_t})\right]^{\otimes m_j}.
\]

But $Y_{st}$ is connected, so $\text{res}^X_{Y_{st}}(\ell_j \otimes \mathcal{O}_{Y_s})$ and $\text{res}^X_{Y_{st}}(\ell_j \otimes \mathcal{O}_{Y_t})$ coincide on $Y_{st}$, up to a permutation. As $\text{res}^X_{Y_{st}}$ is an isomorphism, the line bundles $\lambda_j(s)$ and $\lambda_j(t)$ on $X$ coincide, up to a permutation of the index set $J$.

We fix a base point $o \in B$. Since $N$ is globally generated, Bertini’s theorem implies that $B^c := \{t \in B \mid Y_o$ and $Y_t$ meet transversally$\} \subset B$ is open, non-empty and dense. The previous assignment defines an analytic map from $B^c$ to the (finite) set of permutations of $J$. This map is constant on each connected component $B^c_{\text{conn}}$ of $B^c$, so there is a permutation $\tau$ of $J$ such that $\ell_j \otimes \mathcal{O}_{Y_{st}} \cong \ell_{\tau(j)} \otimes \mathcal{O}_{Y_{st}}$ for all $t \in B^c_{\text{conn}}$. Since $\text{res}^X_{Y_{st}}$ is an isomorphism, and the line bundles $\ell_j$, $j \in J$, are pairwise non-isomorphic, we deduce that $\ell_j \cong \ell_{\tau(j)}$, so $\tau = 1$.

**Case** $\dim X = 4$. In this case $Y_s$ is a smooth threefold, and $\text{res}^X_{Y_s}$ is an isomorphism, for all $s \in B$. Also, the twisted canonical bundle $\kappa_{Y_s} \otimes N = (\kappa_X \otimes N) \otimes \mathcal{O}_{Y_s}$ is globally generated. By the Noether-Lefschetz theorem [13], the restriction $\text{res}^X_{Y_s}$ is an isomorphism, for almost all $t$, so there is a dense subset of pairs $(s, t) \in B \times B$ such that all the arrows in (3.3) are isomorphisms, and this is needed for proving that $\lambda_j$, $j \in J$, are constant.

Let $B \subset \mathbb{P}_C^N$ be as above. For all $s \in B$, let $M_s \subset J$ be the subset of maximal elements with respect to (1.7), corresponding to the splitting of $\mathcal{Y} \otimes \mathcal{O}_{Y_s}$. By semi-continuity, for any
s ∈ B, there is a neighbourhood $B_s ⊂ B$ of s such that $M_s ⊂ M_{s'}$ for all $s' ∈ B_s$. Thus there is a largest subset $M ⊂ J$, and an open subset $B' ⊂ B$ such that $M = M_s$ for all $s ∈ B'$.

**Lemma 3.4.** Let $k = C$ and assume that (3.1) is satisfied. Furthermore, let $B ⊂ \mathbb{P}^N_C$ be a ball such that $Y_s$ is smooth for all $s ∈ B$, $(q^*\mathcal{Y})_B$ splits over $\mathcal{Y}_B$, and the set of maximal elements $M ⊂ J$ with respect to $<$ is the same for all $s ∈ B$.

We consider the (analytic) open subset $U := q(\mathcal{Y}_B) ⊂ X$. Then there is an injective homomorphism of vector bundles $(\bigoplus_{\mu \in M} \mathcal{L}_\mu \otimes \mathcal{O}_U) \otimes \mathcal{O}_U → \mathcal{Y} \otimes \mathcal{O}_U$ whose restriction to $Y_s$ is the natural evaluation (1.8), for all $s ∈ B$.

**Proof.** The restriction to $Y_s$ of $ev : \bigoplus_{\mu \in M} q^*\mathcal{L}_\mu \otimes \pi^*\pi_*q^*\mathcal{L}_{\mu}^{-1} \otimes \mathcal{Y} → (q^*\mathcal{Y})_B$ is the homomorphism (1.8), for all $s ∈ B$. The maximality of $\mu ∈ M$ implies $\pi_*q^*(\mathcal{L}_{\mu}^{-1} \otimes \mathcal{Y}) \cong \mathcal{O}_{B^\mathbb{C}}$, for all $\mu ∈ M$, and that $ev$ is injective. We prove that, after suitable choices of bases in $\pi_*q^*(\mathcal{L}_{\mu}^{-1} \otimes \mathcal{Y})$, $\mu ∈ M$, the homomorphism $ev$ descends to $U$. We will deal with each $\mu ∈ M$ separately, the overall basis being the direct sum of the individual ones.

Consider $\mu ∈ M$, and a base point $o ∈ B$. Then $\mathcal{Y} := \mathcal{L}_{\mu}^{-1} \otimes \mathcal{Y}$ has the following properties:

- $q^*\mathcal{Y} \cong \mathcal{O}_{\mathcal{Y}_B}^\otimes \bigoplus_{j \in J \setminus \{\mu\}} q^*(\mathcal{L}_j \otimes \mathcal{Y}_B)^\otimes \mathcal{O}_{\mathcal{Y}_B}$
- $\pi_*q^*(\mathcal{Y}) \cong \mathcal{O}_{\mathcal{Y}_B}^\otimes$.
- $\pi_*q^*(\mathcal{Y}) \cong \mathcal{O}_{\mathcal{Y}_B}^\otimes$ is pointwise injective. We let $T ⊂ q^*\mathcal{Y}$ be its image.

We choose a complement $W = \bigoplus_{j \in J \setminus \{\mu\}} q^*(\mathcal{L}_j \otimes \mathcal{Y}_B)^\otimes$ of $T$ in $q^*\mathcal{Y}$, that is

$$\left(q^*\mathcal{Y}\right)_B = T \oplus W.$$

The isomorphism $\alpha_B$ above determines the pointwise injective homomorphism $\alpha : \mathcal{O}_{\mathcal{Y}_B}^\otimes → (q^*\mathcal{Y})_B = T \oplus W$ whose second component vanishes, since $\Gamma(\mathcal{Y}_B, W) = 0$. We let $\beta : q^*\mathcal{Y} → \mathcal{O}_{\mathcal{Y}_B}^\otimes$ be the left inverse of $\alpha$ with respect to the splitting (3.1), and notice that $\alpha \circ \beta|_{\mathcal{Y}} = \mathbb{1}_\mathcal{Y}$.

**Claim** After possibly shrinking $B$ and suitably changing the coordinates in $\mathcal{O}_{\mathcal{Y}_B}^\otimes$, the homomorphisms $\alpha$ descends to $q(\mathcal{Y}_B) ⊂ X$. Indeed, for any $s ∈ B$, we consider the diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathcal{Y}_B}^\otimes & -\xrightarrow{\alpha} & \mathcal{Y}_B^\otimes \\
\alpha \downarrow & \cong & \alpha \downarrow \\
\mathcal{O}_{\mathcal{Y}_s}^\otimes & -\xrightarrow{\alpha_s} & \mathcal{Y}_s^\otimes
\end{array}$$

with $a_s := \beta_s \circ \alpha_s ∈ \text{End}(\mathbb{C}^m)$.

Similarly, we let $\alpha_s' := \beta_s \circ \alpha_s$. It holds $\alpha'_s \circ \alpha_s = \beta_s \circ \alpha_s \circ \alpha_s = \beta_s \circ \alpha_s = \mathbb{1}$ (the second equality holds because $\text{Im}(\alpha_s|_{\mathcal{Y}_s}) = \mathcal{Y}_s$, $\text{Im}(\alpha_s|_{\mathcal{Y}_B})$, and similarly $\alpha_s \circ \alpha_s = \mathbb{1}$). Thus $a_s ∈ \text{Gl}(m; \mathbb{C})$ for all $s ∈ B$, and the new trivialization $\tilde{\alpha} := \alpha \circ a$ of $\mathcal{Y}$ satisfies

$$\tilde{\alpha}_s = \tilde{\alpha}_o,$$

indeed, we have $\tilde{\alpha}_s|_{\mathcal{Y}_s} = (\alpha_s \circ a)\circ \alpha|_{\mathcal{Y}_s} = \alpha_o|_{\mathcal{Y}_s} = \tilde{\alpha}_o|_{\mathcal{Y}_s}$. Moreover, for all $s, t ∈ B$, the trivializations of $\mathcal{Y}_{st}$ induced by $\tilde{\alpha}$, coming from $Y_s$ and $Y_t$, coincide. Equivalently, the
following diagram commutes:

\[ \begin{array}{ccc}
\mathcal{O}_{Y_{st}}^\oplus & \xrightarrow{\tilde{\alpha}_s} & \mathcal{I}_{Y_{st}} \subset \mathcal{Y}_{Y_{st}}' \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{O}_{Y_{st}}^\oplus & \xrightarrow{\tilde{\alpha}_t} & \mathcal{I}_{Y_{st}} \subset \mathcal{Y}_{Y_{st}}' \\
\end{array} \quad \Leftrightarrow \tilde{\alpha}_t^{-1} \circ \tilde{\alpha}_s|_{Y_{st}} = 1 \in \text{Gl}(r; \mathbb{C}). \tag{3.6} \]

The triple intersection \( Y_{ost} \) (that is the zero locus of \((a, s, t) \in \Gamma(X, N^\otimes 3)\)) is a non-empty, connected subscheme of \( X \), as \( \dim X - 3\nu \geq 1 \). Hence is enough to prove that the restriction of \((3.6)\) to \( Y_{ost} \) is the identity. After restricting \((3.5)\) to \( Y_{ost} \), we deduce

\[ \tilde{\alpha}_s|_{Y_{ost}} = \tilde{\alpha}_o|_{Y_{ost}} = \tilde{\alpha}_t|_{Y_{ost}} \quad \Rightarrow \quad \tilde{\alpha}_t^{-1} \circ \tilde{\alpha}_s|_{Y_{ost}} = 1. \]

Now we can conclude that the trivialization \( \tilde{\alpha} \) of \( \pi_* q^*(L^{-1}_\mu \otimes \mathcal{V}) \) descends to \( \mathcal{U} := q(Y_B) \), as announced. Indeed, we define \( \tilde{\alpha} : \mathcal{O}_{\mathcal{U}}^{\oplus m} \to \mathcal{V} \otimes \mathcal{O}_{\mathcal{U}}, \tilde{\alpha}(x) := \tilde{\alpha}_s(x) \) for some \( s \in B \) such that \( x \in Y_s \). The diagram \((3.6)\) implies that \( \tilde{\alpha}(x) \) is independent of \( s \in B \) with \( s(x) = 0 \). \[ \square \]

**Lemma 3.5.** Let the situation be as in lemma 3.4. Then \( \mathcal{V} \) is obtained as a successive extension of line bundles on \( X \).

**Proof.** Let \((q^* \mathcal{V})_B = \bigoplus_{j \in J} q^* \mathcal{L}_j \otimes \mathbb{C}^{m_j}\) be an isotypical decomposition, with \( \mathcal{L}_j \in \text{Pic}(X) \). First we prove the lemma over \( \mathcal{U} \), by induction on the cardinality of \( J \). For \(|J| = 1\), we have \((q^*(L^{-1} \otimes \mathcal{V}))/B \cong \mathcal{O}_{\mathcal{Y}_{B}}^{\oplus m} \) for some \( \mathcal{L} \in \text{Pic}(X) \). Lemma 3.4 implies \( \mathcal{Y}_{\mathcal{U}} \cong \mathcal{L} \otimes \mathcal{O}_{\mathcal{U}}^{\oplus m} \).

Now suppose that the lemma holds for \(|J| \leq n\), and prove that is still valid for \(|J| = n + 1\). For the maximal elements \( M \subset J \), lemma 3.4 states that there is an injective homomorphism \( \bigoplus_{\mu \in M} \mathcal{L}_\mu \otimes \mathcal{O}_{\mathcal{U}}^{\oplus m_\mu} \to \mathcal{Y}_{\mathcal{U}} \). Its cokernel \( \mathcal{W}_{\mathcal{U}} \) is locally free over \( \mathcal{U} \), and \( q^* \mathcal{W}_{\mathcal{U}} \cong \bigoplus_{j \in J \setminus M} q^* \mathcal{L}_j^{\oplus m_j} \). By the induction hypothesis, \( \mathcal{W}_{\mathcal{U}} \) is obtained by successive extensions from \( \mathcal{L}_j \to X, j \in J \setminus M \), so the same holds for \( \mathcal{Y}_{\mathcal{U}} \).

It remains to prove that \( \mathcal{V} \) itself is a successive extension of line bundles on \( X \). This follows by repeatedly applying corollary 1.5. Indeed, each of the successive extensions involved in \( \mathcal{Y}_{\mathcal{U}} \) can be uniquely extended to the whole \( X \), because \( \mathcal{U} \) is an open neighbourhood of the ample subvarieties \( Y_s, s \in B \). As \( \mathcal{Y}_{\mathcal{U}} \) is the result of this process over \( \mathcal{U} \), and \( \mathcal{V} \) is already defined on the whole \( X \), the uniqueness part of corollary 1.5 yields the conclusion. \[ \square \]

**Theorem 3.6.** Let \( k \) be an algebraically closed field of characteristic zero, \( X \) an irreducible, smooth, projective \( k \)-variety, and \( N \to X \) a globally generated, ample vector bundle satisfying (3.1). We assume that the restriction of \( \mathcal{V} \to X \) to the geometric generic section \( \mathcal{Y} \) of \( N \) splits. The following statements hold:

(i) If \( k \) is uncountable, then \( \mathcal{V} \) is a successive extension of line bundles on \( X \).

(ii) Assume \( k \) is arbitrary, and either one of the following two conditions is fulfilled:

\[ H^1(X, \mathcal{L}) = 0 \quad \text{for all } \mathcal{L} \in \text{Pic}(X) \]

\[ \Gamma(\mathcal{Y}, \text{End}(\mathcal{Y}_{\mathcal{Y}})) \quad \text{is a semi-simple, finite dimensional algebra.} \]

Then \( \mathcal{V} \) is actually a split vector bundle on \( X \).

**Proof.** The proof is done in two steps.

**Case** \( k = \mathbb{C} \). Let \( B \subset \mathbb{P}(\Gamma(X, N)) \) be as in lemma 3.4 and choose an isotypical decomposition \((q^* \mathcal{V})_B = \bigoplus_{j \in J} q^* \mathcal{L}_j \otimes \mathbb{C}^{m_j}\). Then lemma 3.5 says that \( \mathcal{V} \) is a successive extension of \( \mathcal{L}_j, j \in J \). Now assume that either one of the two conditions (H1) or (SS) is satisfied. On one hand, if \( H^1(X, \mathcal{L}) = 0 \) for all \( \mathcal{L} \in \text{Pic}(X) \), then any extension of line bundles is trivial,
so \( \mathcal{V} \) is isomorphic to \( \bigoplus_{j \in J} \mathbb{L}_j^{\otimes m_j} \). On the other hand, \( \Gamma(\mathbb{Y}, \mathcal{O}_\mathbb{Y}) \) is semi-simple if and only if \( \Gamma(\mathbb{Y}, L_j^{-1}L_j) = 0 \) \( \forall j \in J \). In this case all the elements of \( J \) are maximal with respect to \( (\ref{eq:12}) \), and the conclusion follows from lemma \( \ref{lem:3.4} \).

Case \( k \) arbitrary. Let \( k_0 \subset k \cap \mathbb{C} \) be a countable, algebraically closed field, such that \( X, N, \mathcal{V} \) are defined over \( k_0 \), and let \( X_0, N_0, \mathcal{V}_0 \) be the corresponding objects. Then the geometric generic fibre fits into the Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{V}_{k_0} & \xrightarrow{\psi} & \mathcal{V}_{k_0} \\
\downarrow & & \downarrow \pi \\
\Spec(k) & \longrightarrow & \Spec(k_0),
\end{array}
\]

and \( (q^* \mathcal{V})_{k_0} \) splits by theorem \( \ref{thm:A.2} \). Thus \( (q^* \mathcal{V})_{k_0} \times_{k_0} \mathbb{C} \) splits too. But this latter is the restriction of \( \mathcal{V}_\mathbb{C} := \mathcal{V} \times_{k_0} \mathbb{C} \) to the geometric generic section of \( N_\mathbb{C} \), hence \( \mathcal{V}_\mathbb{C} \to X_\mathbb{C} \) is a successive extension of line bundles \( \mathcal{L}_j \to X_\mathbb{C}, j \in J \), by the previous step. There is an intermediate field \( k_0 \to k_1 \to \mathbb{C} \) of finite type over \( k_0 \), such that \( \mathcal{L}_j, j \in J \), are defined over \( k_1 \). It follows that \( \mathcal{V}_0 \times_{k_0} k_1 \) is a successive extension of line bundles on \( X_0 \times_{k_0} k_1 \).

On one hand, if \( k \) is uncountable, the transcendence degree of \( k \) over \( k_0 \) is infinite because \( k_0 \) is countable. Hence we can realize \( k_1 \) as a sub-field of \( k \), and the conclusion follows.

On the other hand, if either of the conditions (H1) or (SS) is fulfilled (over \( k \)), then (by base change) the same holds over \( k_0 \) and \( \mathbb{C} \). It follows that \( \mathcal{V}_\mathbb{C} \) splits. By applying theorem \( \ref{thm:A.2} \) once more, we deduce the splitting of \( \mathcal{V}_0 \to X_0 \) and of \( \mathcal{V} \to X \).

\( \Box \)

**Remark 3.7.** The previous theorem raises a couple of questions. First, if the algebra of global endomorphisms of \( \mathcal{V}_\mathbb{C} \) is not semi-simple, we proved only that \( \mathcal{V} \) is a successive extension of line bundles on \( X \), and we don’t know whether \( \mathcal{V} \) actually splits. The difficulty is that unipotent automorphisms of \( \mathcal{V}_\mathbb{C} \) act on the isotypical decompositions of \( \mathcal{V}_\mathbb{C} \), and are mixing them (except the maximal components).

Second, it is not clear whether the upper bound \( \ref{thm:3.1} \) for the rank of \( N \) is optimal or not. The factor of 1/3 in the inequality is somewhat unpleasant. However, the following example illustrates the importance of the fact, used in the proof of the lemma \( \ref{lem:3.4} \), that the triple intersections \( Y_{ost} \), with \( o, s, t \in \Gamma(X, N) \), are non-empty and connected.

Let \( \mathcal{V} = \mathcal{F}_\mathbb{P}_2 \) be the tangent bundle of \( X = \mathbb{P}_\mathbb{C}^2 \). It is a non-split, uniform vector bundle of rank two, and its restriction to any line \( Y \subset \mathbb{P}_\mathbb{C}^2 \) is isomorphic to \( \mathcal{O}_Y(2) \oplus \mathcal{O}_Y(1) \). The incidence variety \( \mathcal{V} \) is the variety of full flags in \( \mathbb{C}^3 \), and we have the diagram

\[
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{q} & \mathbb{P}_\mathbb{C}^2 \\
\pi | & | & \text{\( \mathcal{P}_\mathbb{C} \)-fibration} \\
|\mathcal{O}_{\mathbb{P}_\mathbb{C}^2}(1)| & \cong & \mathbb{P}_\mathbb{C}^2
\end{array}
\]

The geometric generic fibre \( \mathbb{Y} \) of \( \pi \) is isomorphic to the projective line defined over the algebraic closure of the quotient field of \( \mathbb{P}_\mathbb{C}^2 \), so \( q^* \mathcal{F}_\mathbb{P}_2 \to \mathbb{Y} \) splits and there is a ball \( B \subset |\mathcal{O}_{\mathbb{P}_\mathbb{C}^2}(1)| \) such that \( (q^* \mathcal{F}_\mathbb{P}_2)|_{\pi^{-1}(B)} \) splits. However, this splitting does not descend to \( q(\pi^{-1}(B)) \subset \mathbb{P}_\mathbb{C}^2 \), for no such \( B \). Otherwise, proposition \( \ref{prop:1.13} \) would imply that \( \mathcal{F}_\mathbb{P}_2 \) splits, a contradiction.

An interesting consequence is that, for an arbitrary variety defined over an uncountable ground field (e.g. \( k = \mathbb{C} \)), one can deduce the splitting of a vector bundle from the splitting of its restriction to a single sufficiently general ample subvariety.
Theorem 3.8. Assume that (3.1) is satisfied, and $k$ is uncountable. Let $k_0 \subset k$ be a countable, algebraically closed sub-field such that $X, N, \mathcal{V}$ are defined over $k_0$. Consider a regular section $s \in \Gamma(X, N)$ with the following properties:

- $\mathcal{V}_s$ is split;
- in some affine chart induced from $\mathbb{P}^N_{k_0}$, the coordinates of the point $[s] \in \mathbb{P}^N_k$ are algebraically independent over $k_0$.

Assume furthermore that either one of the following two conditions is satisfied:

(H1) $H^1(X, \mathcal{L}) = 0$ for all $\mathcal{L} \in \text{Pic}(X)$;

(SS) $\Gamma(Y_s, \text{End}(\mathcal{V}_s))$ is a semi-simple, finite dimensional algebra.

Then the vector bundle $\mathcal{V} \to X$ splits into a direct sum of line bundles on $X$.

As $k_0$ is countable, the points $[s] \in \mathbb{P}^N_k$ with the previous properties lie in the complement of a countable union of proper subvarieties of $\mathbb{P}^N_k$, so we can reformulate as follows:

If $k$ is uncountable, and the restriction to the zero locus of a very general section of $N$ splits, then the vector bundle $\mathcal{V} \to X$ does the same.

Proof. Let $(c_1, \ldots, c_N)$ be the coordinates of $[s]$ in the affine chart $c_0 \neq 0$ on $\mathbb{P}^N_k$. By assumption, $c_1, \ldots, c_N$ are algebraically independent over $k_0$, which implies

$$k_0 \subset \hat{k}_0 := k_0(\xi_1, \ldots, \xi_N) \cong k_0(c_1, \ldots, c_N) \subset k \quad \text{closed} \quad \hat{k}_0 \subset k,$$

and therefore the closed point $[s] \in \mathbb{P}^N_k$ maps to the generic point of $\mathbb{P}^N_{k_0}$. (Here $\xi_1, \ldots, \xi_N$ are indeterminates.) Moreover, as $k_0$ is countable, it can be realized as a sub-field of $C$.

We consider the following diagram with Cartesian front and rear faces

Now we focus on the diagonal base change rectangle with dotted sides. Our hypothesis is that the restriction of $\mathcal{V}$ to $Y_s$ splits. Since $\mathcal{V}_s = \mathcal{V}_{\hat{k}_0} \times_{\hat{k}_0} Y_s$, and both $\hat{k}_0$ and $k$ are algebraically closed, theorem A.2 implies that $(q_s^* \mathcal{V})_{\hat{k}_0} \to \mathcal{V}_{\hat{k}_0}$ splits, so the vector bundle $(q_s^* \mathcal{V})_{\hat{k}_0} \times_{k_0} C \to \mathcal{V}_{\hat{k}_0} \times_{k_0} C$ splits too. Theorems 3.6 and A.2 imply that both $\mathcal{V}_C \to X_C$ and $\mathcal{V} \to X_0$ split. We conclude that the initial $\mathcal{V} \to X$ is split.

We illustrate this theorem with the following application.

Example 3.9. Let $\mathcal{V}$ be a vector bundle on $X := \mathbb{P}^n_C \times \mathbb{P}^n_C$, with $n \geq 2m + 1$ and $m \geq 2$. We consider the ample vector bundle $N := \mathcal{F}_C \boxtimes 0_{\mathbb{P}^n_C}^{(1)}$, and a very general section $s \in \Gamma(X, N)$. Then $\mathcal{V} \to X$ splits if and only if its restriction to $Y_s$ does so. (Notice that $Y_s \subset X$ has codimension $m$, and $Y_s \to \mathbb{P}^n_C$ is a $(m + 1)$-sheeted ramified covering.)

4. Splitting along divisors

Here we discuss in detail the case when $Y$ is a divisor, that is $N$ is an ample line bundle.
Notations 4.1. Throughout this section $X$ stands for a smooth projective variety, defined over an algebraically closed field $k$, $\dim_k X \geq 3$, and $\mathcal{O}_X(1) \to X$ is an ample line bundle. If the characteristic of the ground field is positive, we assume moreover:

(i) $p = \text{char}(k) > \dim_k X$;
(ii) The pair $(X, \mathcal{O}_X(1))$ admits a $W_2(k)$-lifting (see [7] Section 8), so the Kodaira vanishing theorem in positive characteristics [6] holds for $X$.

This latter condition is satisfied (see [6] pp. 257 and [7] Lemma 8.14) if there is an affine scheme $T$, étale over an open subset $\mathcal{U} \subset \text{Spec}(\mathbb{Z})$, with $<p> \in \mathcal{U}$, such that:

(iia) the pair $(X, \mathcal{O}_X(1))$ is a geometric fibre of a smooth family $(\mathcal{Z}, \mathcal{O}_\mathcal{Z}(1)) \to T$;
(ii) $\pi_* \mathcal{O}_\mathcal{Z}(m) \to T$ is locally free.

Although the results in this section are particular cases of those obtained so far, we must reprove lemma [12] because of the issues related to the Kodaira vanishing.

Lemma 4.2. Let $D \in |\mathcal{O}_X(m)|$, with $m \geq 1$, be a divisor such that

$$H^1(D, \mathcal{E}_D(-a)) = 0, \quad \forall a \geq c. \quad \text{(Recall that $\mathcal{E}$ stands for $\mathcal{E}nd(\mathcal{V})$.)} \quad (4.1)$$

Then hold:

(i) The cohomology group $H^1(X, \mathcal{E}(-a))$ vanishes for all $a \geq c$.
(ii) Assume moreover that $m \geq c$ and $\mathcal{V}_D$ splits. Then $\mathcal{V} \to X$ splits too.

Proof. (i) Let $a_0 := \min\{a \geq c \mid H^1(X, \mathcal{E}(-a)) = 0\}$. The Serre vanishing theorem implies that $a_0 < \infty$. Let us assume that $a_0 \geq c + 1$. The short exact sequence $0 \to \mathcal{O}_X(-m) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ yields

$$\ldots \to H^1(\mathcal{E}(-m - a_0 + 1)) \to H^1(\mathcal{E}(-a_0 + 1)) \to H^1(\mathcal{E}_D(-a_0 + 1)) \to \ldots$$

Now we notice that $-m - a_0 + 1 \leq -a_0$ and $a_0 - 1 \geq c$. The definition of $a_0$ together with the hypothesis imply that the first and last terms vanish, hence the middle term vanishes too. This contradicts the minimality of $a_0$.

(ii) As $m \geq c$, the first step implies that $\text{res}_D : \Gamma(X, \mathcal{E}) \to \Gamma(D, \mathcal{E}_D)$ is surjective. Hence $\mathcal{V} \to X$ splits, by lemma [12(ii)].

Theorem 4.3. Let the situation be as in [4.1]. Assume that $D \in |\mathcal{O}_X(m)|$ is smooth, and $\mathcal{E}(m) \to X$ is ample. Then $\mathcal{V} \to X$ splits if and only if $\mathcal{V}_D$ splits.

Applying repeatedly this criterion, one deduces that $\mathcal{V}$ splits over $X$ if and only if its restriction to a complete intersection surface in $X$ of sufficiently high degree splits.

Proof. (i) By hypothesis $\mathcal{V}_D = \bigoplus_{j=1}^r \ell_j$ for some line bundles $\ell_j \to D$. Since $\mathcal{E}(m)$ is ample, its restriction to $D$ is ample too, thus the direct summands $\ell_i^{-1} \ell_j \otimes \mathcal{O}_D(m + a)$ of $\mathcal{E}_D(m + a)$ are ample, for all $i, j$ and $a \geq 0$. The Kodaira vanishing theorem applied to the smooth divisor $D$ yields $H^1(\mathcal{E}_D(-m - a)) = 0$ for all $a \geq 0$, which is the condition [4.1].

Varieties enjoying additional cohomological properties admit stronger splitting criteria.

Definition 4.4. Let $X$ be a scheme and $h \geq 1$ an integer. We say that $X$ is an $h$-splitting scheme if $H^1(X, \mathcal{L}) = \ldots = H^h(X, \mathcal{L}) = 0$ for all line bundles $\mathcal{L} \to X$.

The cases $h = 1, 2$ correspond respectively to the notions of splitting, and of Horrocks scheme introduced in [7].
**Example 4.5.** If \((X, \mathcal{O}_X(1))\) is a \(d\)-dimensional arithmetically Cohen-Macaulay (aCM for short) variety with \(\text{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)\), then \(X\) is \((d - 1)\)-splitting. Indeed, we have

\[
H^j(X, \mathcal{O}_X(k)) = 0, \quad \forall \ k \geq 0 \text{ and } j = 1, \ldots, d - 1 \text{ by the aCM property,}
\]

\[
H^j(X, \mathcal{O}_X(k)) = 0, \quad \forall \ k < 0 \text{ and } j = 1, \ldots, d - 1 \text{ by the Kodaira vanishing.}
\]

Examples of aCM varieties with cyclic Picard group include Fano varieties with cyclic Picard groups (e.g., homogeneous spaces \(G/P\), with \(P \subset G\) a maximal parabolic subgroup), and smooth (resp. very general) complete intersections of dimension \(d \geq 4\) (resp. \(d \geq 3\)) in them.

The result below generalizes [2, Corollary 4.14] in several directions. On one hand, in characteristic zero, we allow 1- rather than 2-splitting varieties. On the other hand, we extend loc. cit. in positive characteristics.

**Theorem 4.6.** Let the situation be as in 4.1.

(i) Let \(\text{char}(k) = 0\), with \(k\) uncountable, and assume that \(\mathcal{O}_X(m)\) is globally generated, and \(D \in |\mathcal{O}_X(m)|\) is very general. (So \(D\) is smooth). Moreover, suppose that either one of the following conditions is satisfied:

(a) \(X\) is a 2-splitting variety of dimension \(\dim_k(X) = 3\), and \(\kappa_X(m)\) is generated by global sections. (Here \(\kappa_X\) stands for the canonical line bundle.)

(b) \(X\) is a 1-splitting variety of dimension \(\dim_k(X) = 4\), and \(\kappa_X(2m)\) is generated by global sections.

(c) \(X\) is a 1-splitting variety of dimension \(\dim_k(X) \geq 5\).

Then \(\mathcal{V} \to X\) splits if and only if its restriction \(\mathcal{V}_D\) is split over \(D\).

(ii) Assume that \(p > \dim_k(X) \geq 4\), \(X\) is a 2-splitting variety, and \(D \in |\mathcal{O}_X(m)|\), \(m \geq 1\). Then \(\mathcal{V} \to X\) splits if and only if \(\mathcal{V}_D \to D\) is split.

**Proof.** (i) **Case (a).** The Noether-Lefschetz theorem [13] states that \(\text{Pic}(X) \to \text{Pic}(D)\) is an isomorphism. Thus for any \(\ell \in \text{Pic}(D)\) there is \(\mathcal{L} \in \text{Pic}(X)\) such that \(\mathcal{L} \otimes \mathcal{O}_D = \ell\). The long exact sequence in cohomology associated to \(0 \to \mathcal{O}_X(-m) \to \mathcal{O}_X \to \mathcal{O}_D \to 0\) twisted by \(\mathcal{L}\) yields \(H^1(D, \ell) = 0\) for all \(\ell \in \text{Pic}(D)\).

By hypothesis, \(\mathcal{V}_D = \bigoplus_{j=1}^r \ell_j\) for some line bundles \(\ell_j \to D\). Then \(\mathcal{E}_D\) is still a direct sum of line bundles, so \(H^1(D, \mathcal{E}_D(-a)) = 0, \forall a \geq 0\). The conclusion follows from lemma 4.2(ii).

**Cases (b), (c).** The statements are particular cases of theorem 3.8.

(ii) First, we claim that \(\text{Pic}(X) \to \text{Pic}(D)\) is an isomorphism. Indeed, [8, Theorem 3.1, pp. 178] holds in arbitrary characteristics. It states that \(\text{Pic}(X) \to \text{Pic}(D)\) is an isomorphism as soon as the following three conditions are fulfilled:

- the effective Lefschetz condition \(\text{Eff}(X, D)\) holds;
- \(D\) meets every effective divisor;
- \(H^1(D, \mathcal{O}_D(-am)) = H^2(D, \mathcal{O}_D(-am)) = 0, \forall a \geq 1\).

The first condition is satisfied by [8, IV, Theorem 2.1.5], while the second holds because \(D\) is ample. For the third property, we apply the Kodaira vanishing theorem [8] for \(X\) in \(0 \to \mathcal{O}_X(-(a+1)m) \to \mathcal{O}_X(-am) \to \mathcal{O}_D(-am) \to 0, a \geq 1\). We conclude by repeating word-for-word the proof of (i)(a) above: as \(X\) is 2-splitting, first we obtain \(H^1(D, \ell) = 0\) for all \(\ell \in \text{Pic}(D)\), and after that \(H^1(D, \mathcal{E}_D(-a)) = 0\) for all \(a \geq 0\). □

**Appendix A. Splitting and base change**

This section is independent of the rest of the article. We will prove that the property of a vector bundle to be split is unaffected by changing the (algebraically closed) ground field. It
can be interpreted as the invariance of the Krull-Schmidt decomposition (see [1]) under the change of the ground field.\footnote{The author could not find any reference for this apparently classical problem.} The main result of this section, Theorem A.2 below, is necessary in section \[3\].

Throughout this section \(X\) stands for an irreducible, projective scheme, and we let \( \mathcal{V} = \bigoplus_{j \in J} \mathcal{L}_j \otimes k^{m_j} \) be a split vector bundle on it. Then the bundle of endomorphisms of \( \mathcal{V} \) splits too, \( \mathcal{E} = \bigoplus_{i,j \in J} \mathcal{L}_i^{-1} \mathcal{L}_j \otimes \text{Hom}_k(k^{m_i}, k^{m_j}) \), and the global endomorphisms of \( \mathcal{V} \) are:

\[
\Gamma(X, \mathcal{E}) = \bigoplus_{i<j} \left[ \Gamma(X, \mathcal{L}_i^{-1} \mathcal{L}_j) \otimes_k \text{Hom}(k^{m_i}, k^{m_j}) \right] \oplus \bigoplus_j \text{End}_k(k^{m_j}). \tag{A.1}
\]

(See (1.7) for the definition of ‘\(\prec\)’.) The following is an essential remark.

**Lemma A.1.** Let \( \mathcal{V} \) be a split vector bundle as in (1.6). Then \( A := \Gamma(X, \mathcal{E}) \) is a finite dimensional algebra over the ground field \( k \), and admits the following Wedderburn-Malcev (WM for shorthand) decomposition (see [5], Theorems 25.15, 26.4, and 72.19):

\[
A = \text{Rad}(A) \oplus S \quad \text{with} \quad \begin{cases} 
\text{Rad}(A) = \bigoplus_{i<j} \Gamma(X, \mathcal{L}_i^{-1} \mathcal{L}_j) \otimes_k \text{Hom}(k^{m_i}, k^{m_j}), \\
S = \bigoplus_j \text{End}_k(k^{m_j}).
\end{cases}
\]

**Proof.** The algebra \( A \) is finite dimensional because \( X \) is projective, and admits the decomposition (A.1). The sub-algebra \( S = \bigoplus_j \text{End}_k(k^{m_j}) \) of \( A \) is a direct sum of matrix algebras, hence it is semi-simple. With respect to the partial order \( \prec \), the elements of \( A \) can be represented as upper triangular block matrices, and \( S \) consists of the block diagonal matrices. Furthermore, is easy to check that \( \bigoplus_{i<j} \Gamma(X, \mathcal{L}_i^{-1} \mathcal{L}_j) \otimes_k \text{Hom}(k^{m_i}, k^{m_j}) \) is an ideal which consists of strictly upper triangular matrices, so it is nilpotent. As \( S \) is semi-simple, this nilpotent ideal is maximal, thus it equals \( \text{Rad}(A) \). \( \square \)

The next result is a ‘going-down’ criterion for split vector bundles.

**Theorem A.2.** Let \( h, k \) be two algebraically closed fields of characteristic zero, such that \( h \subset k \). Furthermore, let \( X_h \) be an irreducible projective scheme over \( h \), and let \( \mathcal{V}_h \) be a vector bundle over it. We define \( X_k := X_h \times_h k \) and \( \mathcal{V}_k := \mathcal{V}_h \times_{X_h} X_k \). Then \( \mathcal{V}_h \to X_h \) splits if and only if \( \mathcal{V}_k \to X_k \) splits.

**Proof.** The necessity is obvious. We are going to prove the sufficiency: if \( \mathcal{V}_k \) splits then \( \mathcal{V}_h \) splits too. Let \( \mathcal{E}_h, \mathcal{E}_k \) be the endomorphism bundles of \( \mathcal{V}_h \) and \( \mathcal{V}_k \) respectively, and define \( A_h := \Gamma(X_h, \mathcal{E}_h) \) and \( A_k := \Gamma(X_k, \mathcal{E}_k) \). Both \( A_h \) and \( A_k \) are finite dimensional algebras over \( h \) and \( k \) respectively, and \( A_k = A_h \otimes_h k \). The strategy to prove the splitting of \( \mathcal{V}_h \), is to construct \( \Phi \in A_h \) with \( \text{rk}(\mathcal{V}_h) \) distinct eigenvalues in \( h \). This is achieved in several steps.

**Step 1** Let us write \( \mathcal{V}_k = \bigoplus_{j \in J} \mathcal{L}_j \otimes_k k^{m_j} \) for some line bundles \( \mathcal{L}_j \to X_k \), \( j \in J \), defined over \( k \). In this case, the algebra \( A_k \) has the WM-decomposition

\[
A_k = \Gamma(X_k, \bigoplus_{i \prec j} \text{Hom}(\mathcal{V}_i, \mathcal{V}_j)) \oplus \bigoplus_j \text{End}(\mathcal{V}_j). \tag{A.2}
\]
We choose a WM-decomposition $A_h = \text{Rad}(A_h) \oplus S_h$, with $S_h \cong A_h / \text{Rad}(A_h)$ a semi-simple algebra over $k$, so $A_k = (\text{Rad}(A_h) \otimes_k k) \oplus (S_h \otimes_k k)$. By comparing this with \( A.2 \), the uniqueness of the WM-decomposition implies:

$$\text{Rad}(A_h) \otimes_k k = \text{Rad}(A_k) = \Gamma(X_k, \bigoplus_{i<j} \text{Hom}(Y_i, Y_j)),$$

it is a sub-algebra of $A_k$

$$S_h \otimes_k k \cong \Gamma(X_k, \bigoplus_j \text{End}(Y_j)) \cong \bigoplus_j \text{End}_k(k^{m_j}).$$

As $h$ is algebraically closed, it follows that $S_h \cong \bigoplus_j \text{End}_h(h^{m_j})$ and that there is an element $n \in \text{Rad}(A_k) \subset \Gamma(X_k, \text{End}(Y_k))$ such that

$$S_h \otimes_k k = (1-n) \cdot \left[ \Gamma(X_k, \bigoplus_j \text{End}(Y_j)) \right] \cdot (1-n)^{-1}$$

$$= \bigoplus_j \left[ \Gamma(X_k, (1-n) \cdot \text{End}(Y_j) \cdot (1-n)^{-1}) \right] = \bigoplus_j \Gamma(X_k, \text{End}((1-n)(Y_j))).$$

The automorphism $1-n$ of $Y_k$ sends $Y_k = \bigoplus Y_j$ into the new splitting $Y_k = \bigoplus (1-n)(Y_j)$. Thus, after a global change of coordinates in $Y_k \rightarrow X_k$, we may assume that $Y_k = \bigoplus Y_j$ has the property that the corresponding WM-decomposition \( A_k = \Gamma(X_k, \delta_k) \) coincides with the WM-decomposition of $A_h = \Gamma(X_h, \delta_h)$ tensored by $k$.

**Step 2** We choose $\alpha : \bigoplus_j \text{End}_h(h^{m_j}) \cong S_h$, and obtain the commutative diagram

$$\bigoplus_j \text{End}_h(h^{m_j}) \xrightarrow{\alpha} S_h \cong S_h \otimes_k k \xrightarrow{\alpha \otimes_k k} \bigoplus_j \text{End}_k(k^{m_j}) \cong S_k.$$

But $S_k = \bigoplus_j \Gamma(X_k, \text{End}(Y_j))$, with $Y_j = L_j \otimes k^{m_j}$, is canonically isomorphic to $\bigoplus_j \text{End}_k(k^{m_j})$ up to multiplication by an element in $(k \setminus \{0\})^J$. Then $\alpha \otimes_k k \in \text{Aut} \left( \bigoplus_j \text{End}_k(k^{m_j}) \right)$ respects the direct sum decomposition, that is $\alpha \otimes_k k = (\alpha_j)_{j \in J}$ with $\alpha_j \in \text{Aut} \left( \text{End}_k(k^{m_j}) \right)$. By the Skolem-Noether theorem, any automorphism of $\text{End}_k(k^{m_j})$ is inner, so each $\alpha_j$ is the conjugation by some $\phi_j \in \text{Aut}(k^{m_j})$. We use $(\phi_j)_{j \in J}$ to change the coordinates in $Y_k$ once more. (Observe that the isotypical components are preserved.) In these new coordinates, $\alpha \otimes_k k$ above becomes the identity.

**Step 3** As $S_h \subset \Gamma(X_h, \delta_h)$, there is a natural evaluation $\text{ev}_h : S_h \otimes \mathcal{O}_{X_h} \rightarrow \delta_h = \text{End}(Y_h)$. After tensoring it by $k$ we obtain the homomorphism $\text{ev}_h : S_k \otimes \mathcal{O}_{X_k} \rightarrow \delta_k = \text{End}(Y_k)$. This latter is injective, that is the left- is a vector sub-bundle of the right-hand-side, so the same holds for $\text{ev}_h$. We obtain the following commutative diagram:

$$\bigoplus_j \text{End}_h(h^{m_j}) \otimes \mathcal{O}_{X_h} \xrightarrow{\alpha \otimes \mathcal{O}_{X_h}} S_h \otimes \mathcal{O}_{X_h} \xrightarrow{\text{ev}_h \text{ injective}} \text{End}(Y_h)$$

$$\bigoplus_j \text{End}_k(k^{m_j}) \otimes \mathcal{O}_{X_k} \xrightarrow{\text{Step 2}} S_k \otimes \mathcal{O}_{X_k} \xrightarrow{\text{ev}_h \text{ injective}} \text{End}(Y_k)$$
Finally we choose a diagonal endomorphism $\Phi = (\Phi_j)_{j \in J} \in \bigoplus_j \text{End}_h(h^{m_j})$ which respects the direct sum decomposition, and which has pairwise distinct diagonal entries $\varepsilon_1, \ldots, \varepsilon_r \in h$. It defines an endomorphism of $\mathcal{V}_h$ with pairwise distinct eigenvalues in the field $h$. The desired splitting of $\mathcal{V}_h$ into a direct sum of line bundles is $\bigoplus_{j=1}^r \text{Ker}(\varepsilon_j \mathbb{1} - \Phi)$.

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