AN ANALYTICAL DESCRIPTION OF SPIN EFFECTS IN HADRON-HADRON SCATTERING VIA PMD-SQS OPTIMUM PRINCIPLE

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Abstract

In this paper an analytical description of spin-effects in hadron-hadron scattering is presented by using PMD-SQS-optimum principle in which the differential cross sections in the forward and backward c.m. angles are considered fixed from the experimental data. The experimental tests of the optimal predictions, obtained by using the available phase shifts, are discussed.

1 Introduction

Recently, in Ref. [1], by using reproducing kernel Hilbert space (RKHS) methods [2-4], we described the quantum scattering of the spinless particles by a principle of minimum distance in the space of quantum states (PMD-SQS). Some preliminary experimental tests of the PMD-SQS, even
in the crude form [1] when the complications due to the particle spins are neglected, showed that the actual experimental data for the differential cross sections of all principal hadron-hadron [nucleon-nucleon, antiproton-proton, meson-nucleon] scatterings at all energies higher than 2 GeV, can be well systematized by PMD-SQS predictions (see the papers [1]). Moreover, connections between the PMD-SQS and the maximum entropy principle for the statistics of the scattering quantum channels was also recently established by introducing quantum scattering entropies: $S_\theta$ and $S_J$ [5]-[7]. Then, it was shown that the experimental pion-nucleon as well as pion-nucleus scattering entropies are well described by optimal entropies and that the experimental data are consistent with the principle of minimum distance in the space of quantum states (PMD-SQS) [1]. However, the PMD-SQS in the crude form [1] cannot describe the polarization J-spin effects.

In this paper an analytical description of spin-effects in hadron-hadron scattering is presented by using PMD-SQS-optimum principle in which the differential cross sections in the forward ($x=+1$) and backward ($x=-1$) directions are considered fixed from the experimental data. An experimental test of the optimal prediction on the logarithmic slope $b$ is performed for the pion-nucleon scattering at the forward c.m. angles.

2 Optimal state description of pion-nucleon scattering

First we present the basic definitions on the the pion-nucleon scattering:

$$\pi + N \rightarrow \pi + N$$

Therefore, let $f^{++}(x)$ and $f^{-+}(x)$ be the scattering helicity amplitudes of the meson-nucleon scattering process (see ref.[14]) written in terms of the partial helicities $f_{J-}$ and $f_{J+}$ as follows

$$f_{++}(x) = \sum_{J=\frac{1}{2}}^{J_{\text{max}}} \left( J + \frac{1}{2} \right) (f_{J-} + f_{J+}) \frac{1}{2} \frac{1}{2} d^J \frac{1}{2} \frac{1}{2} (x)$$

$$f_{+-}(x) = \sum_{J=\frac{1}{2}}^{J_{\text{max}}} \left( J + \frac{1}{2} \right) (f_{J-} - f_{J+}) \frac{1}{2} \frac{1}{2} d^J \frac{1}{2} \frac{1}{2} (x)$$

(2)

where the rotation functions are defined as
\begin{equation}
\begin{aligned}
    d_{\ell+\frac{1}{2}}^{J} (x) &= \frac{1}{\ell+1} \cdot \left[ \frac{1+\ell}{2} \right]^\frac{1}{2} \left[ \hat{P}_{\ell+1}^{\ell}(x) - \hat{P}_{\ell}^{\ell}(x) \right] \\
    d_{\ell-\frac{1}{2}}^{J} (x) &= \frac{1}{\ell+1} \cdot \left[ \frac{1-\ell}{2} \right]^\frac{1}{2} \left[ \hat{P}_{\ell+1}^{\ell}(x) + \hat{P}_{\ell}^{\ell}(x) \right]
\end{aligned}
\end{equation}

where $P_{\ell}(x)$ are Legendre polynomials, $\hat{P}_{\ell}(x) = \frac{d}{dx} P_{\ell}(x)$, $x$ being the c.m. scattering angle. The normalization of the helicity amplitudes $f^{++}(x)$ and $f^{+-}(x)$, is chosen such that the c.m. differential cross section is given by

\begin{equation}
\frac{d\sigma}{d\Omega} (x) = |f^{++}(x)|^2 + |f^{+-}(x)|^2
\end{equation}

Then, the elastic integrated cross section is given by

\begin{equation}
\frac{\sigma_{el}}{2\pi} = \sum (j + \frac{1}{2}) \left[ |f_{j+}|^2 + |f_{j-}|^2 \right]
\end{equation}

Now, let us consider the following optimization problem:

\begin{equation}
\min \left\{ \sum (j + \frac{1}{2}) \left[ |f_{j+}|^2 + |f_{j-}|^2 \right] \right\}
\end{equation}

when $\frac{d\sigma}{d\Omega} (+1)$ and $\frac{d\sigma}{d\Omega} (-1)$ are fixed.

We proved that the solution of this optimization problem is given by the following results:

\begin{equation}
f_{o}^{++}(x) = f^{++}(+1) \frac{K_{\frac{1}{2}+\frac{1}{2}}(x, y)}{K_{\frac{1}{2}+\frac{1}{2}}(+1, +1)}
\end{equation}

\begin{equation}
f_{o}^{+-}(x) = f^{+-}(-1) \frac{K_{\frac{1}{2}-\frac{1}{2}}(x, y)}{K_{\frac{1}{2}-\frac{1}{2}}(-1, -1)}
\end{equation}

where the functions $K(x, y)$ are the reproducing kernels [2] expressed in terms of the rotation function by

\begin{equation}
K_{\frac{1}{2}+\frac{1}{2}}(x, y) = \sum_{1/2}^{J_{o}} (j + \frac{1}{2}) d_{\frac{1}{2}+\frac{1}{2}}^{j} (x) d_{\frac{1}{2}+\frac{1}{2}}^{j} (y)
\end{equation}

\begin{equation}
K_{\frac{1}{2}-\frac{1}{2}}(x, y) = \sum_{1/2}^{J_{o}} (j + \frac{1}{2}) d_{\frac{1}{2}-\frac{1}{2}}^{j} (x) d_{\frac{1}{2}-\frac{1}{2}}^{j} (y)
\end{equation}
while the optimal angular momentum is given by

\[ J_0 = \sqrt{\frac{4\pi}{\sigma_{el}} \left[ \frac{d\sigma}{d\Omega}(+1) + \frac{d\sigma}{d\Omega}(-1) \right] + \frac{1}{4} - 1} \]  

(11)

Now, let us consider the logarithmic slope \( b \) of the forward diffraction peak defined by

\[ b = \frac{d}{dt} \left[ \ln \frac{d\sigma}{dt}(s, t) \right]_{t=0} \]  

(12)

Then, using the definition of the rotation functions, from (70-70) we obtain the following optimal slope \( b_o \):

\[ b_o = \frac{\lambda^2}{4} \left[ \frac{4\pi}{\sigma_{el}} \left( \frac{d\sigma}{d\Omega}(+1) + \frac{d\sigma}{d\Omega}(-1) \right) - 1 \right] \]  

(13)

Optimal predictions on the differential cross section \( \frac{d\sigma_o}{d\Omega}(x) \) and also for the spin-polarization parameters (\( P_o, R_o, A_o \)), are as follows.

\[ \frac{d\sigma_o}{d\Omega}(x) = \frac{d\sigma}{d\Omega}(+1) \left[ \frac{K_{\frac{1}{2}\frac{1}{2}}(x, +1)}{K_{\frac{1}{2}\frac{1}{2}}(+1, +1)} \right]^2 + \frac{d\sigma}{d\Omega}(-1) \left[ \frac{K_{\frac{1}{2}\frac{1}{2}}(x, -1)}{K_{\frac{1}{2}\frac{1}{2}}(-1, -1)} \right]^2 \]  

(14)

\[ P_o \frac{d\sigma_o}{d\Omega}(x) = 2 \sqrt{\frac{d\sigma}{d\Omega}(+1)} \sqrt{\frac{d\sigma}{d\Omega}(-1)} \left[ \frac{K_{\frac{1}{2}\frac{1}{2}}(x, +1)}{K_{\frac{1}{2}\frac{1}{2}}(+1, +1)} \right] \left[ \frac{K_{\frac{1}{2}\frac{1}{2}}(x, -1)}{K_{\frac{1}{2}\frac{1}{2}}(-1, -1)} \right] \sin \phi(x) \]  

(15)

\[ R_o \frac{d\sigma_o}{d\Omega}(x) = 2 \sqrt{\frac{d\sigma}{d\Omega}(+1)} \sqrt{\frac{d\sigma}{d\Omega}(-1)} \left[ \frac{K_{\frac{1}{2}\frac{1}{2}}(x, +1)}{K_{\frac{1}{2}\frac{1}{2}}(+1, +1)} \right] \left[ \frac{K_{\frac{1}{2}\frac{1}{2}}(x, -1)}{K_{\frac{1}{2}\frac{1}{2}}(-1, -1)} \right] \cos \phi(x) \]  

(16)

\[ A_o \frac{d\sigma_o}{d\Omega}(x) = \frac{d\sigma}{d\Omega}(+1) \left[ \frac{K_{\frac{1}{2}\frac{1}{2}}(x, +1)}{K_{\frac{1}{2}\frac{1}{2}}(+1, +1)} \right]^2 - \frac{d\sigma}{d\Omega}(-1) \left[ \frac{K_{\frac{1}{2}\frac{1}{2}}(x, -1)}{K_{\frac{1}{2}\frac{1}{2}}(-1, -1)} \right]^2 \]  

(17)

where

\[ \cos \phi(x) = \frac{\text{Re}\{f^{++}(+1) \ast f^{+-}(-1)\}}{|f^{++}(+1)||f^{+-}(-1)|}, \quad \sin \phi(x) = \frac{\text{Im}\{f^{++}(+1) \ast f^{+-}(-1)\}}{|f^{++}(+1)||f^{+-}(-1)|} \]  

(18)
Finally, we note that in ref. [15] we proved the following optimal inequality

\[ b_o = \frac{\chi^2}{4} \left[ \frac{4\pi}{\sigma_{el}} \left( \frac{d\sigma}{d\Omega}(+1) + \frac{d\sigma}{d\Omega}(-1) \right) - 1 \right] \leq b_{\text{exp}} \]  

which includes in a more general and exact form the unitarity bounds derived by Martin [8] and Martin-MacDowell [9] (see also ref.[10]) and Ion [1],[11].

3 Experimental test of the PMD-SQS-optimal predictions

For an experimental test of the optimal result (13) the numerical values of the slopes \( b_o \) and \( b_{\text{exp}} \) are calculated directly by reconstruction of the helicity amplitudes from the experimental phase shifts (EPS) solutions of Holer et al. [14]. These results are given in the Tables 1-2 and are displayed in Fig 1-2.

4 Conclusions

The main results and conclusions obtained in this paper can be summarized as follows:

(i) The optimal state dominance in hadron-hadron scattering at small transfer momenta for \( p_{LAB} \geq 2 \text{ GeV/c} \) is a fact well evidenced experimentally by the results presented in Figs. 1-2.

(ii) In the low energy region, the optimal slope (13) is in good agreement with the experimental data at discrete values of energy between the resonances positions or/and in the region corresponding to the diffractive resonances see Figs. 1-2.

Finally, we hope that our results as well as those from ref. [15] are new steps towards an analytic description of the quantum scattering in terms of an optimum principle, namely, the principle of minimum distance in space of quantum state (PMD-SQS) [1].
References

[1] D. B. Ion, Phys. Lett. B 376, 282 (1996), and quoted therein references.

[2] N. Aronsjain, Proc. Cambridge Philos. Soc. 39 (1943) 133, Trans. Amer. Math. Soc. 68 (1950) 337; S. Bergman, The Kernel Function and Conformal mapping, Math. Surveys No 5. AMS, Providence, Rhode Island, 1950; S. Bergman, and M. Schiffer Kernel Functions and Elliptic Differential Equations in Mathematical Physics, Academic Press, New York, 1953; A. Meschkowski, Hilbertische Raume mit Kernfunction, Springer Berlin, 1962; H.S. Shapiro, Topics in Approximation Theory, Lectures Notes in Mathematics, No 187, Ch. 6, Springer, Berlin, 1971.

[3] D.B.Ion and H.Scutaru, International J.Theor.Phys. 24, 355 (1985).

[4] D.B.Ion, International J.Theor.Phys. 24 1217 (1985) ; D.B.Ion, International J.Theor.Phys. 25, 1257 (1986).

[5] D. B. Ion and M. L. Ion, Phys. Lett. B 352, 155 (1995).

[6] D. B. Ion and M. L. D. Ion, Phys. Rev. Lett. 81, 5714 (1998).

[7] D. B. Ion and M. L. D. Ion, Phys. Rev. Lett. 83, 463 (1999); Phys. Rev. E 60, 5261 (1999); Phys. Lett. B 482, 57 (2000); Phys. Lett. B 503, 263 (2001); Phys. Lett. B 519, 63 (2001); Chaos Solitons and Fractals, 13, 547 (2002).

[8] A. Martin, Phys. Rev. 129, 1432 (1963).

[9] S. W. MacDowell and A. Martin, Phys. Rev. 135 B, 960 (1964).

[10] S. M. Roy, Phys. Rep. 5C, 125 (1972).

[11] D. B. Ion, St. Cerc. Fiz. 43, 5 (1991).

[12] See the books: N. R. Hestenes, Calculus of variations and optimal control theory, John Wiley&Sons, Inc., 1966, and also V. M. Alecseev, V. M. Tihomirov and S. V. Fomin, Optimaloe Upravlenie, Nauka, Moskow,1979.

[13] D. B. Ion and M. L. Ion, Phys. Lett. B 379, 225 (1996).
Table 1: PMD-SQS optimal predictions for $\pi^+ P \rightarrow \pi^+ P$, calculated using the experimental phase shifts solutions from ref. [14]

| $P_{LAB}$ (GeV/c) | $(1/p_{cm})$ (GeV$^{-2}$) | $\frac{d\sigma}{d\Omega}$ (+1) (mb/GeV$^{-2}$) | $\frac{d\sigma}{d\Omega}$ (-1) (mb/GeV$^{-2}$) | $\sigma_{el}$ (mb) | $J_0$ | $b_0$ (GeV$^{-2}$) | $b_{o+1}$ (GeV$^{-2}$) | $b_{exp}$ (GeV$^{-3}$) |
|-------------------|--------------------------|--------------------------|--------------------------|-------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 0.020             | 3305.47                  | 0.185                    | 0.185                    | 2.329             | 0.500                    | 826.369                  | 0.000                    | 0.000                    |
| 0.040             | 831.948                  | 0.117                    | 0.273                    | 2.426             | 0.507                    | 212.175                  | -81.659                  | -255.100                  |
| 0.060             | 373.735                  | 0.042                    | 0.473                    | 2.955             | 0.562                    | 111.242                  | -76.667                  | -655.969                  |
| 0.079             | 218.466                  | 0.003                    | 0.834                    | 4.255             | 0.650                    | 80.360                   | -54.150                  | -6650.37                  |
| 0.097             | 147.078                  | 0.027                    | 1.254                    | 6.057             | 0.705                    | 60.956                   | -34.742                  | -382.911                  |
| 0.112             | 111.845                  | 0.133                    | 1.831                    | 8.638             | 0.763                    | 51.949                   | -22.531                  | 14.998                    |
| 0.130             | 84.501                   | 0.425                    | 2.911                    | 13.489            | 0.832                    | 44.518                   | -12.769                  | 46.680                    |
| 0.153             | 62.493                   | 1.294                    | 4.971                    | 24.018            | 0.878                    | 35.500                   | -5.047                   | 45.665                    |
| 0.172             | 50.486                   | 2.621                    | 7.570                    | 37.559            | 0.913                    | 30.414                   | -1.554                   | 36.764                    |
| 0.185             | 44.276                   | 4.091                    | 9.466                    | 49.167            | 0.927                    | 27.285                   | 0.505                    | 31.826                    |
| 0.200             | 38.530                   | 6.621                    | 12.743                   | 68.315            | 0.952                    | 24.677                   | 2.099                    | 28.020                    |
| 0.218             | 33.100                   | 11.275                   | 18.196                   | 101.316           | 0.976                    | 21.973                   | 3.297                    | 24.066                    |
| 0.247             | 26.651                   | 22.300                   | 28.265                   | 169.036           | 1.002                    | 18.383                   | 4.383                    | 19.386                    |
| 0.267             | 23.333                   | 30.173                   | 32.196                   | 203.762           | 1.024                    | 16.604                   | 5.021                    | 16.972                    |
| 0.280             | 21.531                   | 32.650                   | 31.802                   | 207.611           | 1.037                    | 15.616                   | 5.255                    | 15.690                    |
| 0.290             | 20.299                   | 33.087                   | 30.510                   | 202.992           | 1.046                    | 14.905                   | 5.320                    | 14.776                    |
| 0.295             | 19.727                   | 33.442                   | 29.397                   | 199.639           | 1.051                    | 14.575                   | 5.450                    | 14.316                    |
| 0.301             | 19.076                   | 33.287                   | 27.945                   | 193.442           | 1.056                    | 14.201                   | 5.543                    | 13.785                    |
| 0.305             | 18.662                   | 32.962                   | 26.816                   | 188.203           | 1.059                    | 13.956                   | 5.603                    | 13.460                    |
| 0.310             | 18.166                   | 32.487                   | 25.413                   | 180.947           | 1.067                    | 13.720                   | 5.705                    | 13.111                    |
| 0.320             | 17.238                   | 31.198                   | 22.436                   | 165.767           | 1.077                    | 13.213                   | 5.883                    | 12.391                    |
| 0.331             | 16.308                   | 28.798                   | 19.733                   | 150.463           | 1.074                    | 12.448                   | 5.729                    | 11.601                    |
| 0.351             | 14.822                   | 25.715                   | 14.507                   | 122.777           | 1.090                    | 11.550                   | 6.047                    | 10.492                    |
| 0.378             | 13.156                   | 20.937                   | 10.134                   | 92.640            | 1.113                    | 10.573                   | 6.052                    | 9.381                     |
| 0.408             | 11.654                   | 16.983                   | 7.542                    | 71.111            | 1.141                    | 9.713                    | 5.830                    | 8.490                     |
| $P_{LAB}$ (GeV/c) | $(1/p_{cm})^2$ (GeV⁻²) | $\frac{d\sigma}{d\Omega}$ (+1) (mb/sr) | $\frac{d\sigma}{d\Omega}$ (−1) (mb/sr) | $\sigma_{el}$ (mb) | $J_0$ | $b_o$ (GeV⁻²) | $b_{o+1}$ (GeV⁻²) | $b_{exp}$ (GeV⁻³) |
|------------------|--------------------------|-------------------------------|-------------------------------|-----------------|-----|------------|------------|----------------|
| 0.427            | 10.850                   | 15.042                        | 5.519                         | 60.807          | 1.121 | 8.813      | 5.720      | 7.679          |
| 0.456            | 9.797                    | 12.345                        | 3.528                         | 47.050          | 1.119 | 7.934      | 5.626      | 6.381          |
| 0.490            | 8.774                    | 10.059                        | 2.446                         | 37.329          | 1.112 | 7.041      | 5.234      | 5.382          |
| 0.532            | 7.748                    | 8.407                         | 1.372                         | 28.181          | 1.147 | 6.510      | 5.325      | 4.729          |
| 0.573            | 6.936                    | 6.766                         | 0.700                         | 21.827          | 1.133 | 5.720      | 5.020      | 4.864          |
| 0.614            | 6.266                    | 5.409                         | 0.233                         | 17.710          | 1.062 | 4.706      | 4.447      | 4.805          |
| 0.658            | 5.668                    | 4.386                         | 0.089                         | 14.446          | 1.035 | 4.099      | 3.989      | 5.937          |
| 0.675            | 5.463                    | 3.896                         | 0.082                         | 13.190          | 1.010 | 3.810      | 3.704      | 5.662          |
| 0.705            | 5.134                    | 3.558                         | 0.057                         | 11.125          | 1.027 | 3.667      | 3.585      | 7.180          |
| 0.725            | 4.934                    | 2.959                         | 0.037                         | 10.166          | 0.988 | 3.335      | 3.279      | 7.121          |
| 0.750            | 4.704                    | 2.523                         | 0.021                         | 8.767           | 0.974 | 3.112      | 3.076      | 7.092          |
| 0.777            | 4.476                    | 2.728                         | 0.081                         | 11.235          | 0.842 | 2.397      | 2.295      | 4.894          |
| 0.800            | 4.298                    | 1.981                         | 0.003                         | 7.050           | 0.946 | 2.726      | 2.720      | 2.383          |
| 0.822            | 4.139                    | 1.851                         | 0.002                         | 6.467           | 0.962 | 2.691      | 2.687      | 5.419          |
| 0.851            | 3.946                    | 1.868                         | 0.016                         | 6.383           | 0.989 | 2.671      | 2.640      | 3.371          |
| 0.875            | 3.798                    | 2.035                         | 0.060                         | 6.818           | 1.028 | 2.719      | 2.613      | 4.051          |
| 0.895            | 3.683                    | 2.183                         | 0.159                         | 7.683           | 1.020 | 2.606      | 2.367      | 3.029          |
| 0.923            | 3.532                    | 2.458                         | 0.315                         | 8.999           | 1.030 | 2.536      | 2.147      | 2.115          |
| 0.954            | 3.378                    | 2.812                         | 0.505                         | 10.365          | 1.067 | 2.551      | 2.034      | 2.057          |
| 0.975            | 3.281                    | 3.012                         | 0.618                         | 11.061          | 1.091 | 2.562      | 1.986      | 2.634          |
| 1.000            | 3.171                    | 3.507                         | 0.754                         | 11.758          | 1.192 | 2.818      | 2.179      | 1.543          |
| 1.030            | 3.049                    | 3.579                         | 0.986                         | 11.731          | 1.267 | 2.965      | 2.160      | 2.684          |
| 1.055            | 2.954                    | 3.687                         | 1.034                         | 11.903          | 1.288 | 2.942      | 2.136      | 3.021          |
| 1.080            | 2.864                    | 3.816                         | 1.093                         | 12.011          | 1.321 | 2.961      | 2.142      | 2.310          |
| 1.113            | 2.753                    | 4.249                         | 1.326                         | 12.298          | 1.439 | 3.233      | 2.300      | 4.100          |
| 1.154            | 2.625                    | 4.902                         | 1.594                         | 12.951          | 1.560 | 3.481      | 2.466      | 5.165          |
| 1.174            | 2.567                    | 5.248                         | 1.714                         | 13.390          | 1.605 | 3.552      | 2.520      | 6.220          |
| 1.210            | 2.469                    | 6.112                         | 2.065                         | 14.198          | 1.736 | 3.850      | 2.722      | 6.513          |
| 1.235            | 2.405                    | 6.754                         | 2.239                         | 14.778          | 1.810 | 3.996      | 2.851      | 6.239          |
| 1.280            | 2.297                    | 8.371                         | 2.494                         | 15.826          | 1.979 | 4.379      | 3.242      | 6.929          |
| 1.324            | 2.200                    | 10.306                        | 2.774                         | 16.680          | 2.179 | 4.869      | 3.720      | 7.720          |
| 1.360            | 2.126                    | 11.929                        | 2.728                         | 17.133          | 2.317 | 5.182      | 4.119      | 8.469          |
| 1.400            | 2.049                    | 13.468                        | 2.608                         | 17.546          | 2.430 | 5.387      | 4.430      | 8.160          |
| $P_{LAB}$ (GeV/c) | $(1/p_{cm})$ | $\frac{d\sigma}{d\Omega}$ (+1) (mb/sr) | $\frac{d\sigma}{d\Omega}$ (-1) (mb/sr) | $\sigma_{el}$ (mb) | $J_0$ | $b_o$ (GeV$^{-2}$) | $b_{o+1}$ (GeV$^{-2}$) | $b_{exp}$ (GeV$^{-3}$) |
|-------------------|--------------|---------------------------------|---------------------------------|-----------------|------|----------------|----------------|----------------|----------------|
| 1.430             | 1.995        | 14.381                          | 2.487                           | 17.605          | 2.506| 5.508          | 4.622          | 8.869          |
| 1.473             | 1.923        | 15.282                          | 2.189                           | 17.401          | 2.587| 5.584          | 4.824          | 8.308          |
| 1.505             | 1.872        | 15.777                          | 1.912                           | 17.265          | 2.623| 5.557          | 4.906          | 7.235          |
| 1.550             | 1.804        | 15.023                          | 1.591                           | 16.169          | 2.628| 5.373          | 4.816          | 7.097          |
| 1.590             | 1.748        | 14.457                          | 1.236                           | 15.287          | 2.626| 5.201          | 4.757          | 6.626          |
| 1.640             | 1.683        | 13.875                          | 0.879                           | 14.218          | 2.646| 5.065          | 4.738          | 6.513          |
| 1.680             | 1.634        | 13.428                          | 0.602                           | 13.232          | 2.684| 5.033          | 4.799          | 6.425          |
| 1.720             | 1.587        | 13.034                          | 0.450                           | 12.729          | 2.683| 4.885          | 4.709          | 6.383          |
| 1.760             | 1.543        | 12.617                          | 0.324                           | 12.065          | 2.705| 4.814          | 4.684          | 6.210          |
| 1.800             | 1.502        | 12.279                          | 0.241                           | 11.602          | 2.716| 4.715          | 4.617          | 5.858          |
| 1.840             | 1.462        | 11.843                          | 0.211                           | 11.032          | 2.739| 4.653          | 4.566          | 5.896          |
| 1.880             | 1.425        | 11.883                          | 0.135                           | 10.757          | 2.780| 4.644          | 4.587          | 5.958          |
| 1.920             | 1.389        | 11.822                          | 0.098                           | 10.320          | 2.843| 4.693          | 4.651          | 5.883          |
| 1.980             | 1.338        | 11.836                          | 0.054                           | 9.958           | 2.906| 4.686          | 4.663          | 5.865          |
| 2.03              | 1.299        | 11.976                          | 0.043                           | 9.603           | 2.997| 4.783          | 4.765          | 6.044          |
| 2.07              | 1.269        | 12.061                          | 0.037                           | 9.465           | 3.039| 4.779          | 4.763          | 5.881          |
| 2.15              | 1.213        | 12.500                          | 0.054                           | 9.179           | 3.176| 4.910          | 4.887          | 5.919          |
| 2.20              | 1.181        | 12.973                          | 0.086                           | 9.075           | 3.282| 5.043          | 5.007          | 5.997          |
| 2.28              | 1.132        | 13.706                          | 0.154                           | 8.894           | 3.453| 5.258          | 5.197          | 6.228          |
| 2.34              | 1.098        | 14.547                          | 0.187                           | 8.829           | 3.607| 5.482          | 5.409          | 6.387          |
| 2.46              | 1.036        | 15.840                          | 0.273                           | 8.702           | 3.849| 5.765          | 5.663          | 6.485          |
| 2.56              | 0.989        | 16.725                          | 0.274                           | 8.585           | 4.013| 5.903          | 5.804          | 6.458          |
| 2.75              | 0.910        | 18.072                          | 0.239                           | 8.328           | 4.280| 6.059          | 5.977          | 6.479          |
| 3.00              | 0.824        | 19.115                          | 0.167                           | 7.803           | 4.595| 6.190          | 6.135          | 6.848          |
| 3.40              | 0.715        | 20.284                          | 0.093                           | 7.098           | 5.027| 6.269          | 6.240          | 6.942          |
| 3.65              | 0.660        | 21.180                          | 0.053                           | 6.886           | 5.245| 6.231          | 6.215          | 6.708          |
| 4.00              | 0.596        | 23.088                          | 0.062                           | 6.590           | 5.663| 6.432          | 6.415          | 7.018          |
| 5.00              | 0.467        | 26.653                          | 0.042                           | 5.751           | 6.654| 6.690          | 6.680          | 6.989          |
| 6.00              | 0.383        | 30.776                          | 0.031                           | 5.259           | 7.594| 6.957          | 6.950          | 7.302          |
| 10.00             | 0.223        | 44.885                          | 0.097                           | 4.414           | 10.328| 7.087          | 7.072          | 7.369          |
Table 2: PMD-SQS optimal predictions for $\pi^+P \rightarrow \pi^+P$, calculated using the experimental phase shifts solutions from ref. [14]

| $P_{LAB}$ (GeV/c) | $(1/P_{cm})$ (GeV$^{-2}$) | $\frac{d^2 \sigma}{d\Omega} (+1)$ (mb/sr) | $\frac{d^2 \sigma}{d\Omega} (-1)$ (mb/sr) | $\sigma_{el}$ (mb) | $J_0$ | $b_0$ (GeV$^{-2}$) | $b_{0+1}$ (GeV$^{-2}$) | $b_{exp}$ (GeV$^{-3}$) |
|-------------------|-----------------------------|------------------------------------------|------------------------------------------|-------------------|------|------------------|---------------------|-------------------|
| 0.020             | 3305.47                     | 0.146                                    | 0.146                                    | 1.840             | 0.500| 826.369          | 0.000               | 0.000             |
| 0.040             | 831.948                     | 0.135                                    | 0.144                                    | 1.755             | 0.498| 206.915          | -7.183              | -17.101           |
| 0.060             | 373.735                     | 0.147                                    | 0.124                                    | 1.739             | 0.486| 89.461           | 5.960               | 3.735             |
| 0.079             | 218.466                     | 0.214                                    | 0.099                                    | 2.024             | 0.481| 51.553           | 17.938              | 22.395            |
| 0.097             | 147.078                     | 0.240                                    | 0.072                                    | 2.076             | 0.463| 32.714           | 16.711              | 17.401            |
| 0.112             | 111.845                     | 0.277                                    | 0.038                                    | 2.095             | 0.461| 24.770           | 18.437              | 18.375            |
| 0.130             | 84.501                      | 0.360                                    | 0.007                                    | 2.247             | 0.519| 22.326           | 21.460              | 25.665            |
| 0.153             | 62.493                      | 0.510                                    | 0.011                                    | 2.853             | 0.596| 20.242           | 19.456              | 24.444            |
| 0.172             | 50.486                      | 0.766                                    | 0.070                                    | 3.982             | 0.699| 20.666           | 17.890              | 24.759            |
| 0.185             | 44.276                      | 0.984                                    | 0.168                                    | 5.009             | 0.772| 20.934           | 16.255              | 24.908            |
| 0.200             | 38.530                      | 1.371                                    | 0.390                                    | 6.860             | 0.864| 21.439           | 14.564              | 25.689            |
| 0.218             | 33.100                      | 1.985                                    | 0.867                                    | 10.142            | 0.945| 20.963           | 12.078              | 25.069            |
| 0.247             | 26.651                      | 3.357                                    | 1.867                                    | 17.184            | 1.017| 18.788           | 9.693               | 20.694            |
| 0.267             | 23.333                      | 4.112                                    | 2.667                                    | 21.502            | 1.052| 17.276           | 8.184               | 17.930            |
| 0.280             | 21.531                      | 4.144                                    | 2.977                                    | 22.281            | 1.066| 16.236           | 7.198               | 16.414            |
| 0.290             | 20.299                      | 4.054                                    | 3.079                                    | 22.208            | 1.070| 15.409           | 6.568               | 15.127            |
| 0.295             | 19.727                      | 3.956                                    | 3.128                                    | 21.966            | 1.074| 15.055           | 6.231               | 14.511            |
| 0.301             | 19.076                      | 3.842                                    | 3.140                                    | 21.569            | 1.078| 14.632           | 5.906               | 13.870            |
| 0.305             | 18.662                      | 3.716                                    | 3.100                                    | 21.084            | 1.077| 14.288           | 5.668               | 13.312            |
| 0.310             | 18.166                      | 3.526                                    | 3.029                                    | 20.248            | 1.078| 13.932           | 5.396               | 12.668            |
| 0.320             | 17.238                      | 3.251                                    | 2.900                                    | 19.093            | 1.073| 13.136           | 4.911               | 11.270            |
| 0.331             | 16.308                      | 2.900                                    | 2.696                                    | 17.726            | 1.054| 12.096           | 4.305               | 9.950             |
| 0.351             | 14.822                      | 2.314                                    | 2.301                                    | 15.157            | 1.019| 10.474           | 3.403               | 7.325             |
| 0.378             | 13.156                      | 1.651                                    | 1.806                                    | 12.043            | 0.964| 8.574            | 2.377               | 3.563             |
| 0.408             | 11.654                      | 1.210                                    | 1.483                                    | 10.323            | 0.879| 6.640            | 1.380               | 0.334             |
| 0.427             | 10.850                      | 1.079                                    | 1.050                                    | 10.050            | 0.706| 4.507            | 0.946               | -3.973            |
| 0.456             | 9.797                       | 1.185                                    | 1.067                                    | 9.564             | 0.791| 4.798            | 1.364               | -0.107            |
| 0.490             | 8.774                       | 1.637                                    | 1.071                                    | 10.688            | 0.853| 4.790            | 2.028               | 2.468             |
| 0.532             | 7.748                       | 1.866                                    | 1.246                                    | 10.636            | 0.982| 5.185            | 2.334               | 2.193             |
| 0.573             | 6.936                       | 2.495                                    | 1.379                                    | 11.918            | 1.082| 5.349            | 2.829               | 4.136             |
| 0.614             | 6.266                       | 3.341                                    | 1.320                                    | 13.260            | 1.160| 5.354            | 3.394               | 6.863             |
Table 2 – continued from previous page

| $P_{LAB}$ (GeV/c) | $(1/P_{cm})$ | $\frac{d\sigma}{d\Omega}$ (+1) (mb/sr) | $\frac{d\sigma}{d\Omega}$ (-1) (mb/sr) | $\sigma_{el}$ (mb) | $J_0$ | $b_o$ (GeV$^{-2}$) | $b_{o+1}$ (GeV$^{-2}$) | $b_{exp}$ (GeV$^{-3}$) |
|------------------|--------------|-----------------------------------|-----------------------------------|-------------------|------|------------------|------------------|------------------|
| 0.658            | 5.668        | 5.251                             | 1.516                             | 15.402            | 1.402| 6.406            | 4.654            | 11.622           |
| 0.675            | 5.463        | 5.869                             | 1.585                             | 16.407            | 1.441| 6.431            | 4.774            | 10.935           |
| 0.705            | 5.134        | 7.139                             | 1.566                             | 17.806            | 1.529| 6.602            | 5.183            | 10.529           |
| 0.725            | 4.934        | 7.238                             | 1.388                             | 17.900            | 1.519| 6.283            | 5.073            | 10.413           |
| 0.750            | 4.704        | 6.794                             | 1.041                             | 16.931            | 1.463| 5.663            | 4.754            | 8.870            |
| 0.777            | 4.476        | 5.950                             | 0.719                             | 14.958            | 1.419| 5.151            | 4.475            | 9.318            |
| 0.800            | 4.298        | 5.567                             | 0.424                             | 13.119            | 1.447| 5.092            | 4.655            | 11.247           |
| 0.822            | 4.139        | 5.332                             | 0.296                             | 12.151            | 1.464| 4.989            | 4.671            | 12.973           |
| 0.851            | 3.946        | 5.873                             | 0.189                             | 12.277            | 1.541| 5.135            | 4.944            | 14.188           |
| 0.875            | 3.798        | 7.117                             | 0.151                             | 13.584            | 1.641| 5.435            | 5.303            | 14.491           |
| 0.895            | 3.683        | 8.524                             | 0.131                             | 15.314            | 1.712| 5.619            | 5.520            | 13.478           |
| 0.923            | 3.532        | 11.592                            | 0.135                             | 18.379            | 1.875| 6.197            | 6.116            | 13.609           |
| 0.954            | 3.378        | 15.096                            | 0.189                             | 21.664            | 2.019| 6.643            | 6.550            | 13.139           |
| 0.975            | 3.281        | 17.059                            | 0.237                             | 23.422            | 2.087| 6.790            | 6.686            | 13.470           |
| 1.000            | 3.171        | 19.475                            | 0.253                             | 25.402            | 2.164| 6.945            | 6.846            | 11.733           |
| 1.030            | 3.049        | 18.243                            | 0.350                             | 24.370            | 2.137| 6.546            | 6.408            | 11.839           |
| 1.055            | 2.954        | 16.727                            | 0.203                             | 22.918            | 2.088| 6.116            | 6.034            | 10.378           |
| 1.080            | 2.864        | 14.490                            | 0.177                             | 20.993            | 2.005| 5.569            | 5.494            | 8.474            |
| 1.113            | 2.753        | 12.188                            | 0.105                             | 18.286            | 1.949| 5.126            | 5.076            | 7.380            |
| 1.154            | 2.625        | 10.354                            | 0.063                             | 15.651            | 1.935| 4.833            | 4.800            | 6.605            |
| 1.174            | 2.567        | 9.985                             | 0.085                             | 14.774            | 1.969| 4.856            | 4.810            | 7.245            |
| 1.210            | 2.469        | 8.142                             | 0.072                             | 11.935            | 1.983| 4.721            | 4.674            | 8.381            |
| 1.235            | 2.405        | 9.990                             | 0.131                             | 13.495            | 2.110| 5.064            | 4.991            | 7.676            |
| 1.280            | 2.297        | 10.063                            | 0.188                             | 12.973            | 2.190| 5.127            | 5.022            | 7.619            |
| 1.324            | 2.200        | 10.496                            | 0.258                             | 12.483            | 2.328| 5.403            | 5.261            | 7.343            |
| 1.360            | 2.126        | 10.588                            | 0.280                             | 11.947            | 2.418| 5.544            | 5.388            | 8.192            |
| 1.400            | 2.049        | 10.944                            | 0.304                             | 11.481            | 2.544| 5.795            | 5.625            | 8.576            |
| 1.430            | 1.995        | 11.214                            | 0.369                             | 11.298            | 2.624| 5.928            | 5.724            | 9.010            |
| 1.473            | 1.923        | 11.226                            | 0.309                             | 10.762            | 2.704| 5.993            | 5.820            | 8.693            |
| 1.505            | 1.872        | 11.443                            | 0.308                             | 10.893            | 2.716| 5.876            | 5.709            | 7.948            |
| 1.550            | 1.804        | 11.496                            | 0.298                             | 10.650            | 2.764| 5.826            | 5.667            | 7.940            |
| 1.590            | 1.748        | 10.504                            | 0.443                             | 10.029            | 2.737| 5.558            | 5.315            | 8.458            |
| 1.640            | 1.683        | 11.858                            | 0.206                             | 10.190            | 2.889| 5.838            | 5.731            | 8.038            |
Table 2 – continued from previous page

| $P_{LAB}$ (GeV/c) | $(1/P_{cm})^2$ (GeV$^{-2}$) | $\frac{d\sigma}{d\Omega}(+1)$ (mb/sr) | $\frac{d\sigma}{d\Omega}(-1)$ (mb/sr) | $\sigma_{el}$ (mb) | $J_0$ | $b_o$ (GeV$^{-2}$) | $b_{o+1}$ (GeV$^{-2}$) | $b_{exp}$ (GeV$^{-3}$) |
|------------------|-----------------|----------------|----------------|-------------|-----|-----------------|----------------|---------------|
| 1.680            | 1.634           | 12.158         | 0.177          | 9.899       | 2.988| 5.986           | 5.894          | 8.250         |
| 1.720            | 1.587           | 12.398         | 0.141          | 9.833       | 3.034| 5.962           | 5.890          | 8.025         |
| 1.760            | 1.543           | 12.827         | 0.127          | 9.688       | 3.129| 6.097           | 6.033          | 8.204         |
| 1.800            | 1.502           | 13.270         | 0.102          | 9.649       | 3.203| 6.163           | 6.113          | 8.103         |
| 1.840            | 1.462           | 13.716         | 0.020          | 9.393       | 3.316| 6.351           | 6.341          | 8.462         |
| 1.880            | 1.425           | 14.604         | 0.069          | 9.731       | 3.382| 6.393           | 6.361          | 8.368         |
| 1.920            | 1.389           | 15.142         | 0.049          | 9.738       | 3.456| 6.459           | 6.437          | 8.194         |
| 1.980            | 1.338           | 16.207         | 0.027          | 9.768       | 3.597| 6.654           | 6.642          | 8.299         |
| 2.030            | 1.299           | 16.747         | 0.014          | 9.641       | 3.701| 6.770           | 6.764          | 8.233         |
| 2.070            | 1.269           | 17.193         | 0.006          | 9.587       | 3.774| 6.836           | 6.834          | 8.181         |
| 2.150            | 1.213           | 17.646         | 0.000          | 9.328       | 3.901| 6.907           | 6.907          | 7.983         |
| 2.200            | 1.181           | 18.039         | 0.003          | 9.112       | 4.013| 7.050           | 7.048          | 8.226         |
| 2.280            | 1.132           | 18.320         | 0.005          | 8.908       | 4.109| 7.032           | 7.030          | 8.119         |
| 2.340            | 1.098           | 18.611         | 0.011          | 8.732       | 4.201| 7.081           | 7.077          | 8.103         |
| 2.460            | 1.036           | 18.712         | 0.021          | 8.401       | 4.317| 6.996           | 6.988          | 7.853         |
| 2.560            | 0.989           | 18.970         | 0.031          | 8.233       | 4.409| 6.921           | 6.910          | 7.782         |
| 2.750            | 0.910           | 19.759         | 0.033          | 7.824       | 4.660| 7.006           | 6.994          | 7.648         |
| 3.000            | 0.824           | 21.451         | 0.025          | 7.593       | 4.983| 7.115           | 7.106          | 7.919         |
| 3.400            | 0.715           | 23.638         | 0.014          | 7.095       | 5.492| 7.309           | 7.305          | 8.036         |
| 3.650            | 0.660           | 24.337         | 0.008          | 6.800       | 5.726| 7.262           | 7.259          | 8.097         |
| 4.000            | 0.596           | 25.774         | 0.014          | 6.544       | 6.055| 7.234           | 7.230          | 7.829         |
| 5.000            | 0.467           | 30.771         | 0.006          | 5.818       | 7.169| 7.641           | 7.639          | 8.281         |
| 6.000            | 0.383           | 34.735         | 0.028          | 5.329       | 8.068| 7.759           | 7.753          | 8.507         |
| 10.000           | 0.223           | 52.147         | 0.019          | 4.601       | 10.947| 7.891          | 7.888          | 8.418         |
Figure 1: The experimental logarithmic slopes ($b_{\text{exp}}$) of the diffraction peak, for the forward $\pi^+P \rightarrow \pi^+P$ scattering, are compared with the PMD-SQS-optimal predictions $b_\circ$ (13) (see the text and Table 1).
Figure 2: The experimental logarithmic slopes ($b_{exp}$) of the diffraction peak, for the forward $\pi^+P \rightarrow \pi^+P$ scattering, are compared with the PMD-SQS-optimal predictions $b_o$ (13) (see the text and Table 2).