Free vibration of non-prismatic beam on variable Winkler elastic foundations

Adel A Al-Azzawi¹ and Khalida A Daud²

¹Assist Prof, College of Engineering, Al-Nahrain University, Baghdad, Iraq
²Lecturer, College of Engineering, Al-Nahrain University, Baghdad, Iraq

E-Mail: dr_adel_azzawi@yahoo.com

Abstract. In this paper, the finite difference and the finite element methods are applied to evaluate natural frequencies of non-prismatic and non-homogeneous beams, with different boundary conditions and resting on variable Winkler foundation. The finite difference method is used for solving differential equation of motion, especially with variable coefficients. This technique requires a lesser computing effort and is used in situations where the exact solution is very difficult to obtain. The main idea of this method is replacing derivatives present in the free vibration equation and boundary condition equations with finite difference expressions. The natural frequencies are determined by solving the eigenvalue problem of the obtained algebraic system resulting from finite difference method. In order to illustrate the correctness and performance of the method, a comprehensive numerical example of non-prismatic beams is presented. The results are compared with the finite element results using ABAQUS software and other available numerical and analytical solutions.

1. Introduction

Special members resting on elastic sub-grade or foundations is considered as important issue facing the designer in civil engineering constructions from a general perspective. Closed form solutions, which are limited to very simplified cases are restricted. While numerical solutions has become the most preferred one for solving complicated soil-structure interaction problems. On the other hands, free vibration analysis, which is an eigenvalue analysis has become the major factor in the buildings structural. In the analysis and design, the building free vibration behavior will affect their response to dynamic loadings such as seismic and wind. Several researches and studies are made during the past and nowadays for tracing the free vibration response in different civil engineering constructions.

In 2007, Ece et al. [1] derived exact solution for free vibration of flexural members with constant height and variable width (exponential curve) with different end conditions such as free, hinged and fixed ends. In the same year, Firouz-Abadi et al. [2] used equation of motion of variable cross-section beam to obtain a particular or singular differential expression with the vibration natural frequency term and applied Wentzel, Kramers, Brillouin (WKB) method, which is based on series solution to find the analysis curve. In 2011, Nikkhah Bahrami et al. [3] developed another approach to estimate the system mode shapes and natural frequencies for the non-uniform flexural members. Two methods were selected to obtain the solution for the free vibration of non-uniform flexural member on elastic sub-grades. The first one is the Variational Iteration Method and and the second one is the Homotopy Perturbation Method [4-6].

In 2011, Motaghian et al. [7] performed free vibration response analysis for flexural member on discontinuous sub-grade. The differential equation for the problem was solved in closed form using...
Fourier series. The results revealed that the selected solution can be successfully used to obtain the solution of free vibration of flexural member on discontinuous sub-grade problem with different end conditions.

In this research paper, a simplified numerical solution for free vibration response of non-prismatic and non-homogeneous beams with different edge conditions resting on variable Winkler springs is selected and proposed. Governing differential equation of motion for beams placed on elastic springs were derived and solved using finite differences, while the finite elements are used to verify the model.

2. The Equation of motion and boundary conditions

In this study, a vibrating beam, which is represented by thin, non-prismatic and non-homogenous Euler-Bernoulli beam resting on a non-uniform Winkler sub-grade is investigated as shown in figure 1.

![Figure 1. Non-prismatic beam rest on variable Winkler foundation](image)

The governing (partial) differential expression for the free un-damped vibration of thin non-homogeneous and non-prismatic members resting on variable Winkler sub-grade is given by [8] is modified herein:

$$\frac{\partial^2}{\partial x^2} \left( E(x) I(x) \frac{\partial^2 w}{\partial x^2} \right) + K(x) \ddot{w}(x,t) = -\mu(x) \frac{\partial^2 \ddot{w}}{\partial t^2}$$

where,
- $E(x)$ is the Young modulus of the member material along its length
- $I(x)$ is second moment of area of the member section,
- $\ddot{w}(x,t)$ is the deflection of the member at any position and time,
- $K(x)$ is the modulus of subgrade reaction for the supporting soil
- $\mu(x) = \rho(x) A(x)$ is the member mass per unit length
- $\rho(x)$ is the mass density for the beam material
- $A(x)$ is the beam cross sectional area
- $x$ is the distance along the beam
- $t$ is the time

Writing the deflection ($\ddot{w}$) by using the method of separation of variables, then

$$\ddot{w}(x,t) = \omega(x) e^{i\omega t}$$

where $\omega$ is the natural frequency of the system
Substitute equation (2) into equation (1) and simplifying to obtain the governing equation for the system

\[
\frac{d^2}{dx^2} \left( E(x) I(x) \frac{d^2 w}{dx^2} \right) + K(x). w(x) - \mu(x) \omega^2 w(x) = 0
\]

(3)

Simplifying, then

\[
E(x) \frac{d^2 I(x)}{dx^2} \frac{d^w}{dx^4} + I(x) \frac{d^2 E(x)}{dx^2} \frac{d^w}{dx^4} + \frac{dE(x)}{dx} \frac{dI(x)}{dx} \frac{d^w}{dx^4} + (K(x) - \mu(x) \omega^2) w(x) = 0
\]

(4)

Let

\[
c_1 = E(x) \frac{d^2 I(x)}{dx^2} \\
c_2 = \frac{dI(x)}{dx} \frac{dE(x)}{dx} + I(x) \frac{d^2 E(x)}{dx^2} \\
c_3 = \frac{dE(x)}{dx} \frac{dI(x)}{dx} \\
c_4 = K(x) - \mu(x) \omega^2
\]

(5)

Therefore,

\[
c_1 \frac{d^4 w}{dx^4} + c_2 \frac{d^3 w}{dx^3} + c_3 \frac{d^2 w}{dx^2} + c_4 w(x) = 0
\]

(6)

The boundary conditions for solving the differential equation is as follows

Fixed support \( w = \frac{dw}{dx} = 0 \)

Simple support \( w = \frac{d^2 w}{dx^2} = 0 \)

Free support \( \frac{d^2 w}{dx^2} = \frac{d^3 w}{dx^3} = 0 \)

3. Finite Difference Method

To solve equation (6), the Finite Difference Method is selected. This method is a numerical one that helps in solving systems with complicated geometry and loading. The member is discretized into number of segments and a system of equations is obtained then put in a matrix form as shown in figure 2. The determinate of such matrix is made zero in order to solve the eigenvalue problem and to obtain the natural frequency and the mode shapes.

Rewriting equation (6) into finite differences, then

\[
\text{Figure 2. Finite difference mesh}
\]
The final simultaneous equations are made in matrix form

\[
[C]\{w\} = 0
\]  

(8)

Here \([C]\) is the equation coefficient matrix while \(\{w\}\) is the unknown vector. Make \(|C| = 0\) to obtain the natural frequency and mode shapes.

4. Finite element analysis

The brick and beam elements are used to simulate the non-uniform member using ABAQUS Software and springs to simulate the nonhomogeneous soil. ABAQUS is a general software, which could be used to model complicated models with different nonlinearity sources under static and dynamic loadings. The model is shown in figure 3.

Figure 3. One and three dimensional finite elements under vibration.
5. Verification and application for the numerical study

5.1 Free vibration of simply supported prismatic beam

The member following properties with simple support will be considered to obtain natural frequency and draw the mode shape of deflection: L=4m, E=20000 MPa, I=0.000675 m$^4$, $\mu$=225 kg/m, K=0.0N/m. The modes shapes and natural frequency obtained numerically are shown in figure 4. The results are very close to Timoshenko et. al (1974)(exact solution) [8].

5.2 Free vibration of prismatic beam on Winkler foundation

The following properties of a prismatic beam with simply supported or fixed edges will be considered to obtain natural frequency: assuming L=1m, E=1 MPa, I=1 m$^4$, $\mu$=1kg/m, K=1.0 N/m$^2$ for comparison purposes. The natural frequencies obtained numerically are shown in table 1. The results are identical to that obtained by Chen (2000) (exact solution) [9].

Table 1. Natural frequency for the simply supported system

| Mode | Exact [9] | Finite differences | Finite elements |
|------|-----------|--------------------|-----------------|
| 1    | 9.9201    | 9.821              | 9.872           |
| 2    | 39.4911   | 39.192             | 38.643          |
| 3    | 88.8321   | 86.234             | 85.543          |

The effect of support conditions on the natural frequency for the system is shown in table 2. It is found that the natural frequency is enlarged when beam restrained condition is increased and this increment is reduced for higher mode shape.

Table 2. Natural frequency for different end support (finite differences)

| Mode | Simply supported | Fixed    | Percentage difference |
|------|------------------|----------|-----------------------|
| 1    | 9.821            | 22.143   | 125%                  |
| 2    | 39.192           | 60.896   | 55%                   |
| 3    | 86.234           | 120.763  | 40%                   |

The effect of subgrade value, which ranges from 1 to 100 on natural frequency for the simply supported beam on Winkler foundation was investigated. Figure 5 shows that the effect reduces for higher mode shapes. It was found that the increases in natural frequency values were 44%, 3% and 0.6% for first, second and third mode shapes, respectively.
5.3 Free vibration of non-prismatic beam on Winkler foundation

The following properties of a varying width (non-prismatic) cantilever beam on Winkler foundation will be considered to obtain natural frequency: assuming $L=1$ m, $E=1$ MPa, $I_o = 1$ m$^4$, $\mu = (1 - \alpha x)$ kg/m, $K=1.0$ N/m$^2$ for comparison purposes. Here a rectangular cross section is assumed.

The beam width is given by

$$b(x) = b_o (1 - \alpha x)$$

(9)

Therefore the cross sectional area is given by

$$A(x) = h \times b_o (1 - \alpha x) = A_o (1 - \alpha x)$$

(10)

And the beam bending rigidity is given by

$$EI(x) = E \frac{h^3}{12} b_o (1 - \alpha x) = EI_o (1 - \alpha x)$$

(11)

Where $A_o$ and $I_o$ are the beam section area and second moment of area at the origin. And $\alpha$ is tapered ratio, which is the ratio of beam width at the end ($x=L$) to the beam width at origin ($x=0$).

The natural frequencies obtained numerically are shown in table 3. The results are identical to that obtained by Mutman and Coskun (2013) [10] (Homotopy perturbation method). The maximum difference between the Mutman and Coskun (2013) [10] and finite difference or finite elements solutions was 1% or 3% respectively.

Table 3. Natural frequency for the simply supported system ($\alpha = 0.5$)

| Mode | Homotopy perturbation method | Finite differences | Finite elements |
|------|------------------------------|--------------------|----------------|
| 1    | 4.50571                      | 4.475              | 4.364          |
| 2    | 23.5506                      | 22.943             | 21.764         |
| 3    | 63.2104                      | 64.112             | 62.327         |
The effect of tapered ratio $\alpha$ on the system natural frequency is investigated by changing their value from 0 to 0.5 for the same studied beam as shown in figure 6. As the tapered ratio increased from 0 to 0.5 the natural frequency increased for the first, second and third mode by 31%, 7% and 2%, respectively. Similar behaviour was obtained through the previous study of Mutman and Coskun (2013) [10], in which the effect of tapered ratio may be neglected for higher mode shapes.

![Figure 6. Effect of tapered ratio on natural frequency](image)

5.4 Free vibration of prismatic beam on discontinuous Winkler foundation

The following properties of a prismatic cantilever beam on discontinuous Winkler foundation will be considered to obtain natural frequency: assuming $L=4\text{m}$, $EI=408400 \text{N m}^2$, $\mu=30.394 \text{kg/m}$, $K=8000 \text{N/m}^2$ and $\beta = L_s/L$ is the supported length ratio (0, 0.5 and 1) as shown in figure 7.

![Figure 7. Cantilever beam partially supported on Winkler foundation](image)

The effect of the supported length ratio $\beta$ on the system natural frequency is investigated by changing their value from 0 to 1.0 for the same studied beam of Kukla (1991) [11] and Motaghian et al. (2011) [7] as shown in figure 8. As the supported length ratio increased from 0 to 1.0, the natural frequency increased for the first, second and third mode by 690%, 238% and 55%, respectively. Similar behaviour was obtained through the previous study of Kukla (1991) [11] and Motaghian et al. (2011) [7].
5.5 Free vibration of prismatic beam on variable Winkler foundation

The following properties of a prismatic cantilever beam on variable Winkler foundation will be considered to obtain natural frequency: assuming $L=3\text{m}$, $EI=150000\text{ N m}^2$, $\mu=1500\text{ kg/m}$, $K(x) = (4x - 3x^2 + x^3) \times 10^5\text{ N/m}$.

The obtained results are confirmed with exact solution developed by Eisenberger (1994) [12] with maximum difference of 2 % and 3 % for finite differences and finite elements respectively as shown in table 4.

Table 4. Natural frequency for beam foundation system

| Mode | Exact solution [12] | Finite differences | Finite elements |
|------|---------------------|--------------------|-----------------|
| 1    | 10.00795            | 9.754              | 9.713           |
| 2    | 27.63314            | 26.497             | 25.954          |
| 3    | 70.0664             | 68.653             | 69.619          |

6. Conclusions

The equation of motion for non-prismatic and non-homogeneous beams resting on variable Winkler foundation was derived. This equation was converted into finite differences and solved for the free vibration case to obtain system natural frequencies and mode shapes. Also, the one- and three-dimensional finite elements were used to analyze the problems, which were solved by finite differences for comparison purposes. Good correlation was obtained between the present study numerical solution and previous researches. The following conclusions can be drawn:

a) The effect of support conditions on the natural frequency for the beam foundation is investigated. It is found that the natural frequency enlarged when beam restrained condition was increased and this increment is reduced for higher mode shape.

b) The effect of subgrade value, which ranges from 1 to 100 on natural frequency for the simply supported beam on Winkler foundation is investigated. It is found that the effect reduces for higher mode shapes. It was found that the increases in natural frequency values were 44%, 3% and 0.6% for first, second and third mode shapes, respectively.
c) The effect of tapered ratio $\alpha$ on the system natural frequency is investigated by changing its value from 0 to 0.5. As the tapered ratio increases from 0 to 0.5, the system natural frequency gets enlarged for the first, second and third mode by 31%, 7% and 2%, respectively and the effect of tapered ratio may be neglected for higher mode shapes.

d) The effect of the supported length ratio $\beta$ on the system natural frequency is investigated by changing its value from 0 to 1.0. As the supported length ratio increased from 0 to 1.0, the system natural frequency gets enlarged for the first, second and third mode by 690%, 238% and 55%, respectively.

References

[1] Ece M C, Aydogdu M and Taskin V 2007 Vibration of a variable cross-section beam Mechanics Research Communications; 34(1): pp 78-84
[2] Firouz-Abadi R, Haddadpour H and Novinzadeh A 2007 An asymptotic solution to transverse free vibrations of variable section beams Journal of Sound and Vibration; 304(3) pp 530-540
[3] Nikkhah Bahrami M, Khoshbayani Arani M and Rasekh Saleh N 2011 Modified wave approach for calculation of natural frequencies and mode shapes in arbitrary non-uniform beams Journal of Scientia Iranica; 18(5) pp 1088-1094
[4] Ozturk B 2009 Free vibration analysis of beam on elastic foundation by the variational iteration method International Journal of Nonlinear Sciences and Numerical Simulation; 10(10) pp 1255–1262
[5] Ozturk B, Coskun S B, Koc M Z and Atay M T 2010 Homotopy Perturbation Method for Free Vibration Analysis of Beams on Elastic Foundations. IOP Conference Series; 10(1).
[6] Coskun S B, Atay M T and Ozturk B 2011 Transverse Vibration Analysis of Euler-beroulli Beams using Analytical Approximate Techniques Advances in Vibration Analysis Research, InTech, Vienna, Austria
[7] Motaghin S E, Mofid M and Alanjari 2011 Exact solution to free vibration of beam partially supported by an elastic foundation Journal of Scientia Iranica; 8(4): pp 861-866
[8] Timoshenko S, Young D H and Weaver W 1974 Vibration Problems in Engineering John Wiley and sons, Inc.
[9] Chen C N 2000 Vibration of prismatic beam on an elastic foundation by the differential quadrature element methods Journal of Computers and Structures; 77 pp 1-9
[10] Mutman U and Coskun S B 2013 Free vibration analysis of non-uniform Euler beams on elastic foundation via Homotopy perturbation method, International Journal of Mechanical and Mechatronic engineering; 7(7)
[11] Kukla S 1991 Free vibration of a beam supported on a stepped elastic foundation Journal of Sound and Vibration; 149(2) pp 259-265
[12] Eisenberger M 1994 Vibration frequencies for beams on variable one and two parameter elastic foundations Journal of Sound and Vibration 176(5): pp 577-584