Comparison of different Tate conjectures

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1 Introduction

1.1 Tate conjectures in different forms

In [Tat1], Tate formulated [Tat1, Conj. 1] and [Tat1, Conj. B+SD, Conj. 2, Conj. B+SD+2] in terms of $L$-functions. He asked about the relations between these conjectures and proved some equivalences between them for smooth projective varieties over finite fields (cf. [Tat2, Thm. 2.9]). In this paper, We will try to answer his questions on the relations between these conjectures for a smooth projective variety over a finitely generated field of positive characteristic.

Let $X$ be a smooth projective geometrically connected variety over a finitely generated field $K$ (i.e. finitely generated over a finite field or $\mathbb{Q}$). One can then construct a projective and smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ of schemes of finite type over $\mathbb{Z}$, with $\mathcal{Y}$ irreducible and regular, whose generic fiber is $X/K$. For each closed point $y \in \mathcal{Y}$, let $k(y)$ denote the residue field of $y$ and $N(y) = q_y$ denote the cardinality of $k(y)$.

Set

$$P_y,i(T) := \det(1 - \sigma_y^{-1}T|H^i(\mathcal{X}_y, \mathbb{Q}_\ell)),$$

where $\sigma_y \in \text{Gal}(k(y)/k(y))$ is the arithmetic Frobenius element and $\ell \neq \text{char}(k(y))$ be a prime. Set

$$\Phi_i(s) := \prod_{y \in \mathcal{Y}_o} \frac{1}{P_y,i(q_y^{-s})},$$

where $\mathcal{Y}_o$ denotes the set of closed points. One can show that the product (1) converges absolutely for $\text{Re}(s) > \dim \mathcal{Y} + i/2$. If we replace $\mathcal{Y}$ by a nonempty open subscheme in (1), we divide $\Phi_i(s)$ by a product which converges absolutely for $\text{Re}(s) > \dim \mathcal{Y} + i/2 - 1$, so the zeros and poles of $\Phi_i$ in the strip

$$\dim \mathcal{Y} + i/2 - 1 < \text{Re}(s) \leq \dim \mathcal{Y} + i/2$$

depend only on $X/K$ and not on our choice of $\mathcal{X}/\mathcal{Y}$ (cf. Proposition 2.1). The zeta function of $\mathcal{X}$ is defined as

$$\zeta(\mathcal{X}, s) := \prod_{x \in \mathcal{X}_o} \frac{1}{1 - N(x)^{-s}}.$$

It follows that

$$\zeta(\mathcal{X}, s) = \prod_{y \in \mathcal{Y}_o} \zeta(\mathcal{X}_y, s),$$

where $\mathcal{X}_y$ denotes the special fiber over $y$. By Grothendieck trace formula, we have

$$\zeta(\mathcal{X}, s) = \prod_{i=0}^{2\dim X} \Phi_i(s)^{(-1)^i}.$$
Let \( \ell \neq \text{char}(K) \) be a prime number. Let \( A^i(X) \) be the group of classes of algebraic cycles of codimension \( i \) on \( X \), with coefficients in \( \mathbb{Q} \), for \( \ell \)-adic homological equivalence. Let \( N^i(X) \subseteq A^i(X) \) denote the group of classes of cycles that are numerically equivalent to zero. Let \( d \) denote the dimension of \( X \).

**Conjecture 1.1.** (Conjecture \( T^i(X, \ell) \)) the cycle class map

\[
A^i(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \longrightarrow H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^G_K
\]

is surjective.

**Conjecture 1.2.** (Conjecture \( E^i(X, \ell) \)) \( N^i(X) = 0 \), i.e. numerical equivalence is equal to \( \ell \)-adic homological equivalence for algebraic cycles of codimension \( i \) on \( X \).

**Conjecture 1.3.** (BSD) The rank of Pic\(^0\) \( X/K \) is equal to the order of the zero of \( \Phi_1(s) \) at \( s = \dim(\mathcal{Y}) \) (and of \( \Phi_{2d-1}(s) \) at \( s = \dim \mathcal{X} - 1 \) by duality).

**Conjecture 1.4** (Conjecture 2). The dimension of \( A^i(X)/N^i(X) \) is equal to the order of the poles of \( \Phi_2(s) \) at \( s = \dim \mathcal{Y} + i \) (and of \( \Phi_{2d-2}(s) \) at \( s = \dim \mathcal{X} - i \) by duality).

Let \( \mathcal{X} \) be a regular scheme of finite type over \( \mathbb{Z} \) whose zeta function \( \zeta(\mathcal{X}, s) \) can be meromorphically continued to the point \( s = \dim \mathcal{X} - 1 \). Let \( e(\mathcal{X}) \) be the order of \( \zeta(\mathcal{X}, s) \) at that point, and put

\[
z(\mathcal{X}) = \text{rank } H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) - \text{rank } H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) - e(\mathcal{X}).
\]

One can show that \( z(\mathcal{X}) \) is a birational invariant (cf. Proposition 2.2). Let \( f : \mathcal{X} \longrightarrow \mathcal{Y} \) be a morphism as discussed at the beginning, by Proposition 2.2, any two of the following statements imply the third:

(i) BSD for \( X/K \).

(ii) The Conjecture 1.4, for \( i = 1 \), for \( X/K \).

(iii) The equality \( z(\mathcal{X}) = z(\mathcal{Y}) \).

One can show that \( z(\mathcal{Y}) = 0 \) if \( \mathcal{Y} \) is of dimension 1, this leads to the following conjecture:

**Conjecture 1.5** (BSD+2). If \( \mathcal{X} \) is a regular scheme of finite type over \( \mathbb{Z} \), then the order of \( \zeta(\mathcal{X}, s) \) at the point \( s = \dim \mathcal{X} - 1 \) is equal to \( \text{rank } H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) - \text{rank } H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \).

For a morphism \( f : \mathcal{X} \longrightarrow \mathcal{Y} \) as above with \( \dim \mathcal{Y} = 1 \), Tate conjectured that the above statement is equivalent to BSD for \( X/K \) and the Conjecture 1.4, for \( i = 1 \), for \( X/K \). By our result Corollary 1.13, this is true when \( \mathcal{Y} \) is a curve over a finite field.
1.2 Main Theorems

For any noetherian scheme \(X\), the cohomological Brauer group

\[ \text{Br}(X) := H^2(X, \mathbb{G}_m)_{\text{tor}} \]

is defined to be the torsion part of the etale cohomology group \(H^2(X, \mathbb{G}_m)\).

**Theorem 1.6.** Let \(X\) be a smooth projective variety over a finitely generated field \(K\) of characteristic \(p > 0\). Let \(m\) denote the transcendence degree of \(K\) over its prime field. Then

\[ \dim_q(A^i(X)/N^i(X)) \leq -\text{ord}_{s=m+i} \Phi_2(s) \]

with equality if and only if \(E^i(X, \ell)\) and \(T^i(X, \ell)\) hold.

This was proved by Tate for the case that \(K\) is a finite field (cf. [Tat2, Thm. 2.9]). Our proof is based on the idea of his proof for finite fields and Deligne’s theory of weights.

**Corollary 1.7.** The following statements are equivalent.

1. \(\text{Br}(X_{K^s})^{G_K}(\ell)\) is finite for some prime \(\ell \neq p\).
2. \(\text{Br}(X_{K^s})^{G_K}(\ell)\) is finite for all primes \(\ell \neq p\).
3. \(T^1(X, \ell)\) holds for all primes \(\ell \neq p\).
4. The rank of \(\text{NS}(X)\) is equal to \(-\text{ord}_{s=m+1} \Phi_2(s)\).

**Remark 1.8.** The equivalence between (1) and (3) is well-known (cf. Proposition 3.2) and the above result was proved by Tate in the case that \(K\) is a finite field.

**Theorem 1.9.** Let \(A\) be an abelian variety over a finitely generated field \(K\) of characteristic \(p > 0\). Let \(Y\) be a smooth and irreducible variety over a finite field with function field \(K\). Set \(m = \dim(Y)\). Define

\[ \Pi_Y(A) := \text{Ker}(H^1(K, A) \rightarrow \prod_{s \in Y^1} H^1(K_s^{sh}, A)), \]

where \(Y^1\) denotes the set of points of codimension 1 and \(K_s^{sh}\) denotes the quotient field of a strict local ring at \(s\), then we have

\[ \text{rank } A(K) \leq \text{ord}_{s=m} \Phi_1(s), \]

and the following statements are equivalent.

1. \(\Pi_Y(A)(\ell)\) is finite for some prime \(\ell \neq p\).
2. \(\Pi_Y(A)(\ell)\) is finite for all primes \(\ell \neq p\).
3. The BSD conjecture for \(A\) holds i.e. the rank of \(A(K)\) is equal to \(\text{ord}_{s=m} \Phi_1(s)\).
Remark 1.10. This kind of Tate-Shafarevich group was studied by Keller in [Kel1]. The above theorem was proved for the case \( m = 1 \) by Schneider [Sch] and for the case that \( A \) can extend to an abelian scheme over a smooth projective variety by Keller [Kel2]. Our proof is based on a combination of their ideas. We will show that \( V_1 \breve{\Pi}_Y(A) \) is independent of the choice of \( Y \) (cf. Lemma 5.2), so we will write \( V_1 \breve{\Pi}_Y(A) \) for \( V_1 \breve{\Pi}_Y(A) \).

**Theorem 1.11.** Let \( \pi : \mathcal{X} \rightarrow \mathcal{Y} \) be a dominant morphism between smooth geometrically connected varieties over a finitely generated field \( k \). Let \( K \) be the function field of \( \mathcal{Y} \). Assume that the generic fiber \( X \) of \( \pi \) is smooth projective geometrically connected over \( K \). Set \( K' := Kk^\times \). Let \( \ell \neq \text{char}(k) \) be a prime, and set
\[
K := \ker(V_1 \text{Br}(\mathcal{X}_{k^s})^{G_k} \rightarrow V_1 \text{Br}(X_{K^s})^{G_K}).
\]
Then we have canonical exact sequences
\[
0 \rightarrow V_1 \text{Br}(\mathcal{Y}_{k^s})^{G_k} \rightarrow K \rightarrow V_1 \breve{\Pi}_{K'}(\text{Pic}^0_{X/K})^{G_k} \rightarrow 0,
\]
\[
0 \rightarrow K \rightarrow V_1 \text{Br}(\mathcal{X}_{k^s})^{G_k} \rightarrow V_1 \text{Br}(X_{K^s})^{G_K} \rightarrow 0.
\]
Here we write \( V_1 \breve{\Pi}_{K'}(\text{Pic}^0_{X/K})^{G_k} \) for \( V_1 \breve{\Pi}_{Y_{k^s}}(\text{Pic}^0_{X/K})^{G_k} \) which only depends on \( \text{Pic}^0_{X/K} \) (cf. Lemma 5.2). Moreover, if \( \text{char}(k) > 0 \), and \( B \) denotes the \( K/k \)-trace of \( \text{Pic}^0_{X/K} \), then we have a canonical exact sequence
\[
0 \rightarrow V_1 \breve{\Pi}_K(B) \rightarrow V_1 \breve{\Pi}_K(\text{Pic}^0_{X/K}) \rightarrow V_1 \breve{\Pi}_{K'}(\text{Pic}^0_{X/K})^{G_k} \rightarrow 0.
\]
In particular, if \( k \) is finite, we have \( V_1 \breve{\Pi}_K(\text{Pic}^0_{X/K}) = V_1 \breve{\Pi}_{K'}(\text{Pic}^0_{X/K})^{G_k} \).

Remark 1.12. The relation between Tate-Shafarevich group and Brauer group was studied by Artin and Grothendieck (cf. [Gro3, §4] or [Ulm, Prop. 5.3]), Milne [Mil2], Gonzales-Aviles [Goa] and Geisser [Gei1] for fibrations of relative dimension 1 over curves, by Keller [Kel1, Thm. 4.27] for relative dimension 1 over high dimensional bases, by Geisser [Gei2] for high relative dimension over curves. For fibrations over arbitrary finitely generated fields, Ulmer [Ulm, §7.3] formulated some questions on relations between Tate conjecture and BSD conjecture in terms of \( L \)-functions. The key idea of our proof is using a Lefschetz hyperplane argument developed by Colliot-Thélène and Skorobogatov (cf. [CTS] or [Yua2]).

**Corollary 1.13.** Let \( \pi : \mathcal{X} \rightarrow \mathcal{Y} \) be as in the above theorem, and assume that \( k \) is a finite field. Then Conjecture 1.5 for \( \mathcal{X} \) is equivalent to
\[
\text{Conjecture 1.5 for } \mathcal{Y} + \text{BSD for } \text{Pic}^0_{X/K} + \text{Conjecture 1.4 for } \mathcal{X} \text{ when } i = 1.
\]

Remark 1.14. For the case that \( \mathcal{Y} \) is a smooth projective curve and \( \pi \) is proper, this result was firstly proved by Geisser (cf. [Gei2, Thm. 1.1]).

**Corollary 1.15.** Assuming that \( T^1(X, \ell) \) holds for all smooth projective varieties \( X \) over finite fields of characteristic \( p \), then Conjecture 1.1 for \( i = 1 \), Conjecture 1.3, Conjecture 1.4 for \( i = 1 \) and Conjecture 1.5 hold for the case in characteristic \( p \).

Remark 1.16. \( T^1(X, \ell) \) for smooth projective varieties over finite fields has been reduced to \( T^1(X, \ell) \) for smooth projective surfaces over finite fields with \( H^1(X, \mathcal{O}_X) = 0 \) by the work of de Jong [deJ], Morrow [Mor, Thm. 4.3] and Yuan [Yua1, Thm. 1.6].
1.3 Notation and Terminology

Fields

By a \textit{finitely generated field}, we mean a field which is finitely generated over a prime field. For any field \( k \), denote by \( k^s \) the separable closure. Denote by \( G_k = \text{Gal}(k^s/k) \) the absolute Galois group of \( k \).

Henselization

Let \( R \) be a noetherian local ring, denote by \( R^h \) (resp. \( R^{sh} \)) the henselization (resp. strict henselization) of \( R \) at the maximal ideal. If \( R \) is a domain, denote by \( K^h \) (resp. \( K^{sh} \)) the fraction field of \( R^h \) (resp. \( R^{sh} \)).

Schemes

All schemes are assumed to be separated over their bases. For a noetherian scheme \( S \), denote by \( S^0 \) the set of closed points and by \( S^1 \) the set of points of codimension 1.

By a \textit{variety} over a field \( k \), we mean a scheme which is separated and of finite type over \( k \).

For a smooth proper geometrically connected variety \( X \) over a field \( k \), we use \( \text{Pic}^0_{X/k} \) to denote the underlying reduced closed subscheme of the identity component of the Picard scheme \( \text{Pic}_{X/k} \).

Cohomology

The default sheaves and cohomology over schemes are with respect to the small étale site. So \( H^i \) is usually the abbreviation of \( H^i_{\text{ét}} \) unless otherwise instructed. We use \( H^i_{\text{fpf}} \) to denote flat cohomology for sheaves on fppf site.

Brauer groups

For any noetherian scheme \( X \), denote the \textit{cohomological Brauer group}

\[
\text{Br}(X) := H^2(X, \mathbb{G}_m)_{\text{tor}}.
\]

Abelian group

For any abelian group \( M \), integer \( m \) and prime \( \ell \), we set

\[
M[m] = \{ x \in M | mx = 0 \}, \quad M_{\text{tor}} = \bigcup_{m \geq 1} M[m], \quad M(\ell) = \bigcup_{n \geq 1} M[\ell^n],
\]

\[
T_\ell M = \text{Hom}_\mathbb{Z}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, M) = \lim_{\leftarrow n} M[\ell^n], \quad V_\ell M = T_\ell(M) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]
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\section{L-functions and convergences}

\textbf{Proposition 2.1.} Notations as in the introduction. Then \( \Phi_i(s) \) defined by (1) converges absolutely for \( \text{Re}(s) > \dim \mathcal{Y} + i/2 \) and the zeros and poles of \( \Phi_i \) in the strip

\[
\dim \mathcal{Y} + i/2 - 1 < \text{Re}(s) \leq \dim \mathcal{Y} + i/2
\]

does not depend on the choice of a smooth projective model \( \mathcal{X} \to \mathcal{Y} \).

\textit{Proof.} By Deligne’s theorem, \( P_{y,i}(T) \) are of fixed degree with reciprocal roots of absolute value \( q_i^{-y/2} \), therefore, by [Ser, Thm. 1] the product (1) converges absolutely for \( \text{Re}(s) > \dim \mathcal{Y} + i/2 \). If we replace \( \mathcal{Y} \) by a nonempty open subscheme in (1), and let \( Z \) denote the complement, we divide \( \Phi_i(s) \) by

\[
\prod_{z \in Z} P_{z,i}(q_z^{-s}),
\]

which converges absolutely for \( \text{Re}(s) > \dim \mathcal{Y} + i/2 - 1 \) since \( \dim Z \leq \dim \mathcal{Y} - 1 \). It follows that the zeros and poles of \( \Phi_i \) in the strip

\[
\dim \mathcal{Y} + i/2 - 1 < \text{Re}(s) \leq \dim \mathcal{Y} + i/2
\]

depend only on \( X/K \) and not on our choice of \( \mathcal{X}/\mathcal{Y} \).

\textbf{Proposition 2.2.} Let \( \mathcal{X} \) be a regular scheme of finite type over \( \mathbb{Z} \). Assume that its zeta function \( \zeta(\mathcal{X}, s) \) can be meromorphically continued to the point \( s = \dim \mathcal{X} - 1 \). Let \( e(\mathcal{X}) \) be the order of \( \zeta(\mathcal{X}, s) \) at that point, and put

\[
z(\mathcal{X}) = \text{rank } H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) - \text{rank } H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) - e(\mathcal{X}).
\]

Then \( z(\mathcal{X}) \) is a birational invariant. Let \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism as discussed at the beginning, then any two of the following statements imply the third:

(i) BSD for \( X/K \).

(ii) The Conjecture 1.4, for \( i = 1 \), for \( X/K \).

(iii) The equality \( z(\mathcal{X}) = z(\mathcal{Y}) \).

\textit{Proof.} If one removes from \( \mathcal{X} \) a closed irreducible subscheme \( \mathcal{Z} \), then we have

\[
\zeta(\mathcal{X} - \mathcal{Z}, s) = \zeta(\mathcal{X}, s)/\zeta(\mathcal{Z}, s).
\]

Since \( \zeta(\mathcal{Z}, s) \) converges absoultely for \( \text{Re}(s) > \dim \mathcal{Z} \) and has a simple pole at \( s = \dim \mathcal{Z} \) (cf. [Ser, Thm. 1, 2, 3]), the order of \( \zeta(\mathcal{Z}, s) \) at \( \dim \mathcal{X} - 1 \) is 0 if the codimension of \( \mathcal{Z} \) is at least 2 and is 1 if the codimension of \( \mathcal{Z} \) is 1. If \( \mathcal{Z} \) is a divisor, set \( U = \mathcal{X} - \mathcal{Z} \), there is an exact sequence

\[
0 \to H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \to H^0(U, \mathcal{O}_U^*) \to \mathcal{Z} \to \text{Pic}(\mathcal{X}) \to \text{Pic}(U) \to 0.
\]
It follows that \( e(\mathcal{X}) = e(U) - 1 \), so \( z(\mathcal{X}) = z(U) \).

For the second claim, since \( f \) is proper and the generic fiber is geometrically irreducible and \( \mathcal{Y} \) is normal, we have \( f_*\mathcal{O}_\mathcal{X} = \mathcal{O}_{\mathcal{Y}} \). Thus \( H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}) = H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \). By [Har, Chap. III, Cor. 11.3], all fibers of \( f \) are geometrically connected. Let \( D \) be a vertical prime divisor on \( \mathcal{X} \). Since \( f \) is smooth proper and all fibers are connected, one can show that \( D = f^{-1}(f(D)) \) as a Weil divisor. Thus, any vertical divisor on \( \mathcal{X} \) is the pullback of some divisor on \( \mathcal{Y} \). By [Har, Chap. III, Cor. 11.3], all fibers of \( f \) are geometrically connected. Let \( D \) be a vertical prime divisor on \( \mathcal{X} \). Since \( f \) is smooth proper and all fibers are connected, one can show that \( D = f^{-1}(f(D)) \) as a Weil divisor. Thus, any vertical divisor on \( \mathcal{X} \) is the pullback of some divisor on \( \mathcal{Y} \). By the Leray spectral sequence

\[
H^p(\mathcal{Y}, R^q f_* \mathbb{G}_m) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{G}_m),
\]

and \( f_* \mathbb{G}_m = \mathbb{G}_m \), we get an injection

\[
H^1(\mathcal{Y}, \mathbb{G}_m) \hookrightarrow H^1(\mathcal{X}, \mathbb{G}_m).
\]

Thus, we get an exact sequence

\[
0 \rightarrow \text{Pic}(\mathcal{Y}) \rightarrow \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(X) \rightarrow 0.
\]

So \( z(\mathcal{X}) = z(\mathcal{Y}) \iff e(\mathcal{X}) - e(\mathcal{Y}) = -\text{rank Pic}(X) \). Since \( \Phi_{2d}(s) = \zeta(\mathcal{Y}, s - d) \), comparing the order of both sides of the equation (2) at \( s = \dim \mathcal{X} - 1 \), it suffices to show that \( \Phi_{i}(s) \) has no pole or zero at \( s = \dim \mathcal{X} - 1 \) for \( i \neq 2d - 1, 2d - 2, 2d \). Since all reciprocal roots of \( P_{y,i}(T) \) have absolute values \( q_y^{\frac{i}{2}} \), the product (1) converges absolutely for \( \text{Re}(s) > \dim \mathcal{Y} + i/2 \), the claim follows.

### 3 Tate conjecture for divisors and Brauer groups

Recall Conjecture 1.1 for \( i = 1 \),

**Conjecture 3.1.** (Conjecture \( T^1(X, \ell) \)). Let \( X \) be a projective and smooth variety over a finitely generated field \( k \) of characteristic \( p \geq 0 \), and \( \ell \neq p \) be a prime number. Then the cycle class map

\[
\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \rightarrow H^2(X_{k^s}, \mathbb{Q}_\ell(1))^{G_k}
\]

is surjective.

This conjecture is called Tate conjecture for divisors on \( X \). By the proposition below, it is equivalent to the finiteness of \( \text{Br}(X_{k^s})^{G_k}(\ell) \).

**Proposition 3.2.** Let \( X \) be a smooth projective geometrically connected variety over a field \( k \). Let \( \ell \) be a prime different from \( \text{char}(k) \), then the exact sequence of \( G_k \)-module

\[
0 \rightarrow \text{NS}(X_{k^s}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \rightarrow H^2(X_{k^s}, \mathbb{Q}_\ell(1)) \rightarrow V_\ell \text{Br}(X_{k^s}) \rightarrow 0
\]

is split. Taking \( G_k \)-invariant, there is a canonical exact sequence

\[
0 \rightarrow \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \rightarrow H^2(X_{k^s}, \mathbb{Q}_\ell(1))^{G_k} \rightarrow V_\ell \text{Br}(X_{k^s})^{G_k} \rightarrow 0.
\]
Proof. See [Yua2, §2.2]. □

In the following, we will show that $V_\ell \text{Br}(X_{k^s})^{G_k}$ is a birational invariant for smooth varieties over finitely generated fields.

**Proposition 3.3.** ([Tat2, Thm. 5.2]) Let $X$ be a smooth geometrically connected variety over a finitely generated field $k$ of characteristic $p \geq 0$ and $U \subseteq X$ be an open dense subvariety. Let $\ell \neq p$ be prime. Then the natural map

$$V_\ell \text{Br}(X_{k^s})^{G_k} \rightarrow V_\ell \text{Br}(U_{k^s})^{G_k}$$

is an isomorphism.

Proof. Extending $k$ to a finite extension will not affect the results. By purity for Brauer groups [ˇCes, Thm. 1.1], removing a closed subscheme of codimension $\geq 2$ will not change the Brauer group. Therefore, we might assume that $X - U = \bigsqcup D_i$, where $D_i$ is a smooth geometrically connected subvariety over $k$ of codimension 1. There is an exact sequence

$$H^2(X_{k^s}, \mathbb{Q}_\ell(1)) \rightarrow H^2(U_{k^s}, \mathbb{Q}_\ell(1)) \rightarrow \bigoplus_i H^3_{D_i}(X_{k^s}, \mathbb{Q}_\ell(1)).$$

By the theorem of cohomological purity (cf. [Fuj]),

$$H^3_{D_i}(X_{k^s}, \mathbb{Q}_\ell(1)) \cong H^1((D_i)_{k^s}, \mathbb{Q}_\ell)$$

is of weight $\geq 1$. Consider the commutative diagram

$$
\begin{array}{ccc}
H^2(X_{k^s}, \mathbb{Q}_\ell(1)) & \longrightarrow & V_\ell \text{Br}(X_{k^s}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
H^2(U_{k^s}, \mathbb{Q}_\ell(1)) & \longrightarrow & V_\ell \text{Br}(U_{k^s}) & \longrightarrow & 0
\end{array}
$$

Since the cokernel of the first column is of weight $\geq 1$, the cokernel of the second column is also of weight $\geq 1$. Since the second column is injective, taking $G_k$ invariants, the claim follows. □

The following lemma generalizes Proposition 3.2 to noncompact varieties over finite fields. We will use this lemma to prove the equivalence between Conjecture 1.5 for smooth varieties over finite fields and the finiteness of $\text{Br}(X_{\overline{k}})^{G_k}(\ell)$.

**Lemma 3.4.** Let $X$ be a smooth geometrically connected variety over a finite field $k$ of characteristic $p > 0$. Let $\ell \neq p$ be a prime. Then, there is a split exact sequence of $G_k$-representations

$$0 \rightarrow \text{Pic}(X_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \rightarrow H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)) \rightarrow V_\ell \text{Br}(X_{\overline{k}}) \rightarrow 0.$$
Proof. Firstly, if the claim holds for $G_{k_1}$-actions where $k_1/k$ is a finite Galois extension, then it also holds for $G_k$-actions. Let $V$ be a sub-representation of some $G_k$-representation $W$ over $\mathbb{Q}_\ell$. Let $p : W \to V$ be a $G_{k_1}$-equivariant projection. Then

$$
\frac{1}{|G_k/G_{k_1}|} \sum_{\sigma \in G_k/G_{k_1}} p^\sigma
$$

is a $G_k$-equivariant projection.

Secondly, we will show that the claim holds for open subvarieties of a smooth projective geometrically connected variety over $k$. Let $U \subseteq X$ be an open subset and $X$ is projective. There is a commutative diagram

$$
\begin{array}{c}
0 \longrightarrow \text{Pic}(X_\bar{k}) \otimes \mathbb{Z} \mathbb{Q}_\ell \longrightarrow H^2(X_\bar{k}, \mathbb{Q}_\ell(1)) \\
\downarrow a \quad \quad \downarrow b \\
0 \longrightarrow \text{Pic}(U_\bar{k}) \otimes \mathbb{Z} \mathbb{Q}_\ell \longrightarrow H^2(U_\bar{k}, \mathbb{Q}_\ell(1))
\end{array}
$$

By the theorem of cohomological purity (cf. [Fuj]), $\text{Ker}(b) = \text{Ker}(a)$ is generated by cycle classes of divisors contained in $X - U$. Since the first row is split, it follows that

$$
0 \longrightarrow \text{Pic}(U_\bar{k}) \otimes \mathbb{Z} \mathbb{Q}_\ell \longrightarrow \text{Im}(b)
$$

is also split. By extending $k$, we might assume that the $G_k$-action on $\text{Pic}(U_\bar{k}) \otimes \mathbb{Z} \mathbb{Q}_\ell$ is trivial. Then $\text{Pic}(U_\bar{k}) \otimes \mathbb{Q}_\ell$ is contained in the generalized 1-eigenspace of the Frobenius action on $H^2(U_\bar{k}, \mathbb{Q}_\ell(1))$. This eigenspace is a direct summand and is also contained in $\text{Im}(b)$ by the proof of Proposition 3.3. It follows that $\text{Pic}(U_\bar{k}) \otimes \mathbb{Q}_\ell$ is a direct summand of the generalized 1-eigenspace, and therefore, it is also a direct summand of $H^2(U_\bar{k}, \mathbb{Q}_\ell(1))$.

Thirdly, by de Jong's theorem, there exists an alteration $f : U \to X$ such that $U$ admits a smooth projective compactification. Since $f$ is proper and generic finite, there is a pushforward map

$$
f_* : H^2(U_\bar{k}, \mathbb{Q}_\ell(1)) \longrightarrow H^2(X_\bar{k}, \mathbb{Q}_\ell(1))
$$

such that $f_* f^* = \text{deg}(f)$. It follows that $f^* : H^2(X_\bar{k}, \mathbb{Q}_\ell(1)) \longrightarrow H^2(U_\bar{k}, \mathbb{Q}_\ell(1))$ is injective. Through $f^*$, $\text{Pic}(X_\bar{k}) \otimes \mathbb{Q}_\ell$ is a direct summand of $\text{Pic}(U_\bar{k}) \otimes \mathbb{Q}_\ell$, and therefore, is also a direct summand of $H^2(U_\bar{k}, \mathbb{Q}_\ell(1))$. It follows that $\text{Pic}(X_\bar{k}) \otimes \mathbb{Q}_\ell$ is a direct summand of $H^2(X_\bar{k}, \mathbb{Q}_\ell(1))$. \hfill $\square$

**Theorem 3.5.** Let $\mathcal{X}$ be a smooth and irreducible variety over a finite field of characteristic $p$. Let $\ell \neq p$ be a prime. Then the Conjecture 1.5 for $\mathcal{X}$ is equivalent to the finiteness of $\text{Br}(X_{\bar{k}})^{G_\kappa}(\ell)$.

**Proof.** Assuming that $\dim \mathcal{X} = n$ and $k = \mathbb{F}_q$, let $F$ denote the geometric Frobenius element in $G_k$. By Grothendieck’s formula, we have

$$
\zeta(\mathcal{X}, s) = \prod_{i=0}^{2n} \det(1 - q^{-s} F| H^i_c(\mathcal{X}_\bar{k}, \mathbb{Q}_\ell))^{(-1)^{i+1}}.
$$
By Deligne’s theorem, $H_c^i(X_k, \mathbb{Q}_\ell)$ is mixed of weight $\leq i$ and $H_c^{2n}(X_k, \mathbb{Q}_\ell)$ is pure of weight $2n$. Therefore, only $\det(1 - q^{-s}F|H_c^{2n-2}(X_k, \mathbb{Q}_\ell))$ and $\det(1 - q^{-s}F|H_c^{2n-1}(X_k, \mathbb{Q}_\ell))$ have contributions to the order of the pole of $\zeta(\mathcal{X}, s)$ at $s = n - 1$. By Poincaré duality,

$$H_c^{2n-1}(X_k, \mathbb{Q}_\ell) \cong H^1(X_k, \mathbb{Q}_\ell)^\vee(-n)$$

and

$$H_c^{2n-2}(X_k, \mathbb{Q}_\ell) \cong H^2(X_k, \mathbb{Q}_\ell)^\vee(-n).$$

Next, we will show that the dimension of the generalized $q^{n-1}$-eigenspace of $H^1(X_k, \mathbb{Q}_\ell)^\vee(-n)$ is equal to rank $H^0(X, \mathcal{O}_X^*)$. By Kummer exact sequence, we get an exact sequence

$$0 \longrightarrow H^0(X_k, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^1(X_k, \mathbb{Q}_\ell(1)) \longrightarrow V_tH^1(X_k, \mathbb{G}_m).$$

There exists an alteration $f : \mathcal{X} \longrightarrow \mathcal{X}$ such that $\mathcal{X}$ admits a smooth projective compactification over $k$. Since $\mathcal{z}(\mathcal{X})$ is a birational invariant, we might shrink $\mathcal{X}$ such that $\mathcal{f}$ is finite flat. There is a canonical norm map $f_*\mathbb{G}_m \to \mathbb{G}_m$ whose composition with the canonical map $\mathbb{G}_m \to f_*\mathbb{G}_m$ is equal to the multiplication by $\deg(f)$. Since $H^p(\mathcal{X}', \mathbb{G}_m) = H^p(\mathcal{X}, f_*\mathbb{G}_m)$, it follows that the composition

$$H^p(\mathcal{X}, \mathbb{G}_m) \longrightarrow H^p(\mathcal{X}', \mathbb{G}_m) = H^p(\mathcal{X}, f_*\mathbb{G}_m) \longrightarrow H^p(\mathcal{X}, \mathbb{G}_m)$$

is equal to the multiplication by $\deg(f)$. It follows that the natural map

$$V_tH^1(X_k, \mathbb{G}_m) \longrightarrow V_tH^1(X_k', \mathbb{G}_m)$$

is injective. Assuming that $\mathcal{X}' \subset \mathcal{X}$ where $\mathcal{X}$ is irreducible and smooth projective over $k$. Since $\mathcal{X}$ is regular, the natural map

$$\text{Pic}(\mathcal{X}_k) \longrightarrow \text{Pic}(\mathcal{X}_k')$$

is surjective and has a finitely generated kernel. Since $\text{Pic}(\mathcal{X}_k)$ is an extension of a finitely generated abelian group by a $\ell$-divisible torsion group $\text{Pic}^0_{\mathcal{X}/k}(\mathcal{X}_k)$, it follows that the natural map

$$V_t\text{Pic}(\mathcal{X}_k) \longrightarrow V_t\text{Pic}(\mathcal{X}_k')$$

is an isomorphism. Since $V_t\text{Pic}(\mathcal{X}_k) \cong H^1(\mathcal{X}_k, \mathbb{Q}_\ell(1))$, so $V_tH^1(X_k, \mathbb{G}_m)$ is pure of weight $-1$. It follows that the generalized 1-eigenspace of $H^1(X_k, \mathbb{Q}_\ell(1))$ can be identified with

$$H^0(\mathcal{X}_k, \mathbb{G}_m)^G_k \otimes_{\mathbb{Z}} \mathbb{Q}_\ell = H^0(\mathcal{X}, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$ 

This implies that the dimension of the generalized $q^{n-1}$-eigenspace of $H^1(X_k, \mathbb{Q}_\ell)^\vee(-n)$ is equal to rank $H^0(\mathcal{X}, \mathbb{G}_m)$.

Thus, it suffices to show that the dimension of the generalized 1-eigenspace of $H^2(X_k, \mathbb{Q}_\ell(1))$ is equal to rank $\text{Pic}(\mathcal{X})$ if and only if $V_t\text{Br}(\mathcal{X}_k)^{G_k}$ vanishes. By Lemma 3.4, there is an exact sequence

$$0 \longrightarrow \text{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2(X_k, \mathbb{Q}_\ell(1))^{G_k} \longrightarrow V_t\text{Br}(\mathcal{X}_k)^{G_k} \longrightarrow 0.$$
Assuming that the dimension of the generalized 1-eigenspace of $H^2(\mathcal{X}_k, \mathbb{Q}_\ell(1))$ is equal to \( \text{rank \ Pic}(\mathcal{X}) \), then
\[
\text{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \rightarrow H^2(\mathcal{X}_k, \mathbb{Q}_\ell(1))^{G_k}
\]
is an isomorphism. It follows that $V_\ell \text{Br}(\mathcal{X}_k)^{G_k} = 0$. Assuming that $V_\ell \text{Br}(\mathcal{X}_k)^{G_k} = 0$, then
\[
\text{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \rightarrow H^2(\mathcal{X}_k, \mathbb{Q}_\ell(1))^{G_k}
\]
is an isomorphism. Since the $G_k$-action on $\text{Pic}(\mathcal{X}_k) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ is semisimple. By Lemma 3.4, $H^2(\mathcal{X}_k, \mathbb{Q}_\ell(1))^{G_k}$ is a direct summand of $H^2(\mathcal{X}_k, \mathbb{Q}_\ell(1))$ as a $G_k$-representation. It follows that the generalized 1-eigenspace of $H^2(\mathcal{X}_k, \mathbb{Q}_\ell(1))$ is same as the 1-eigenspace which is equal to $\text{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. This completes the proof.

Corollary 3.6. Assuming that $T^1(\mathcal{X}, \ell)$ holds for all smooth projective varieties over a finite field $k$, then Conjecture 1.5 holds for all smooth varieties over $k$.

Proof. Let $\mathcal{X}$ be a smooth variety over $k$. By shrinking $\mathcal{X}$, we might assume that there is a finite flat morphism $f : \mathcal{X}' \rightarrow \mathcal{X}$ such that $\mathcal{X}'$ admits a smooth projective compactification $\overline{\mathcal{X}}'$. Assuming $T^1(\overline{\mathcal{X}}', \ell)$, we have $V_\ell \text{Br}(\mathcal{X}_k')^{G_k} = 0$. By Proposition 3.3, $V_\ell \text{Br}(\mathcal{X}_k')^{G_k} = V_\ell \text{Br}(\mathcal{X}_k')^{G_k} = 0$. Since $V_\ell \text{Br}(\mathcal{X}_k')^{G_k} \rightarrow V_\ell \text{Br}(\mathcal{X}_k')^{G_k}$ is injective, so $V_\ell \text{Br}(\mathcal{X}_k')^{G_k} = 0$. By the above theorem, Conjecture 1.5 holds for $\mathcal{X}$.

4 Proof of Theorem 1.6

Lemma 4.1. Let $k$ be a field and $V$ be a finite-dimensional $\mathbb{Q}_\ell$-linear continuous representation of $G_k$. Let $W \subseteq V$ be a subrepresentation. Assuming that there exists a $G_k$-equivariant paring $V \times V' \rightarrow \mathbb{Q}_\ell$ where $V'$ is a $\mathbb{Q}_\ell$-linear representation of a finite quotient of $G_k$ such that the restriction of the paring to $W \times V'$ is left non-degenerate, then $W$ is a direct summand of $V$ as a $G_k$-representation.

Proof. Since the pairing $W \times V'$ is left non-degenerate, it induces a surjection
\[
V' \rightarrow W^*,
\]
where $W^*$ is the dual representation of $W$. Since the $G_k$-action on $V'$ factors through a finite quotient, so $V'$ is semisimple. Thus, there exists a subrepresentation $W'$ of $V'$ such that the paring
\[
W \times W'
\]
is perfect. Define
\[
(W')^\perp := \{v \in V | (v, W') = 0\}.
\]
It follows that $V = W \oplus (W')^\perp$. This proves the claim.
Theorem 4.2. Let $X$ be a smooth projective geometrically connected variety over a finitely generated field $K$ of characteristic $p > 0$. Let $\pi : X \to Y$ be a smooth projective model of $X/K$ as in section 1, where $Y$ is a smooth geometrically connected variety over a finite field $k$. Set $m = \dim Y$, $d = \dim X$ and let $\ell \neq p$ be a prime. Then we have

$$\dim_{\mathbb{Q}}(A^i(X)/N^i(X)) \leq \dim_{\mathbb{Q}_\ell} H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^G \leq -\text{ord}_{s=m+i} \Phi_2(s).$$

Moreover, $\dim_{\mathbb{Q}}(A^i(X)/N^i(X)) = -\text{ord}_{s=m+i} \Phi_2(s)$ if and only if $T^i(X, \ell)$ and $E^i(X, \ell)$ hold.

Proof. The cycle classes of $A^{d-i}(X)$ generate a $\mathbb{Q}_\ell$-subspace in $H^{2d-2i}(X_{K^s}, \mathbb{Q}_\ell(d-i))$. Let $b_1, ..., b_s \in A^{d-i}(X)$ such that their cycle classes form a basis of this subspace. Let $B \subseteq A^{d-i}(X)$ be the subspace spanned by $b_i$. Then the intersection pairing

$$A^i(X)/N^i(X) \times B \to \mathbb{Q}$$

is left non-degenerate. Thus, the induced map $A^i(X)/N^i(X) \to B^*$ is injective. It follows that $A^i(X)/N^i(X)$ has a finite dimension. Let $a_1, ..., a_t \in A^i(X)$ such that they form a basis in $A^i(X)/N^i(X)$. Then there exist $a_i^1, ..., a_i^s \in B$ such that $a_i^1 a_i^j = \delta_{ij}$. So $cl(a_i)^t cl(a_i^*) = \delta_{ij}$. Thus $cl(a_1), ..., cl(a_t)$ are $\mathbb{Q}_\ell$-linear independent in $H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^G$. This proves the first inequality.

Let $a \in N^i(X)$ with $cl(a) \neq 0$, then $cl(a_1), ..., cl(a_t), cl(a)$ are $\mathbb{Q}_\ell$-linear independent. Let $\lambda_i \in \mathbb{Q}_\ell$ such that

$$\lambda_1 cl(a_1) + ... + \lambda_t cl(a_t) + \lambda_{t+1} cl(a) = 0$$

Intersecting with $a_i^*$, we get $\lambda_j = 0$ for $1 \leq j \leq t$. Since $cl(a) \neq 0$, so $\lambda_{t+1} = 0$. This implies that

$$\dim_{\mathbb{Q}}(A^i(X)/N^i(X)) < \dim_{\mathbb{Q}_\ell} H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^G$$

if $N^i(X) \neq 0$. Assuming the second inequality, we proved the “$\Rightarrow$” direction of the second claim.

Next we will prove the second inequality. Let $q$ denote the cardinality of $k$ and $K'$ denote $Kk^s$. Let $\mathcal{F}$ denote $R^{2d-2i} \pi_* \mathbb{Q}_\ell$. $\mathcal{F}$ is lisse since $\pi$ is smooth and proper. By Deligne’s theorem, $\mathcal{F}$ is pure of weight $2d-2i$. By Grothendieck’s formula, we have

$$\Phi_{2d-2i}(s) = L(Y, \mathcal{F}, s) = \prod_{j=0}^{2m} \det (1 - q^{-s} F|H^j_c(Y_k, \mathcal{F}))^{(-1)^{i+1}},$$

where $F$ denotes the geometric Frobenius element in $G_k$. Since $H^j_c(Y_k, \mathcal{F})$ is mixed of weight $\leq j+2d-2i$, thus $-\text{ord}_{s=d+m-i} \Phi_{2d-2i}(s)$ is equal to the dimension of the generalized $q^{d+m-i}$-eigenspace of $F$ on $H^{2m}_c(Y_k, \mathcal{F})$. By Poincaré duality,

$$H^{2m}_c(Y_k, \mathcal{F}) \cong H^0(Y_k, \mathcal{F}^\vee)^*(-m).$$

Therefore $-\text{ord}_{s=d+m-i} \Phi_{2d-2i}(s)$ is equal to the dimension of the generalized 1-eigenspace of $H^0(Y_k, \mathcal{F}^\vee)(i-d)$. $\mathcal{F}^\vee$ corresponds to the $G_K$-representation on $H^{2d-2i}(X_{K^s}, \mathbb{Q}_\ell)^\vee$.
which is isomorphic to $H^{2i}(X_{K^s}, \mathbb{Q}_\ell)(d)$. It follows that $-\text{ord}_{s=d+m-i}\Phi_{2d-2i}(s)$ is equal to the dimension of the 1-eigenspace of $H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K'}}$. By Poincaré duality, we have

$$\Phi_{2d-2i}(s) = \Phi_{2i}(s - d + 2i).$$

Thus $-\text{ord}_{s=m+i}\Phi_{2i}(s)$ is equal to the dimension of the generalized 1-eigenspace of $H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K'}}$ which contains $H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K'}}$. This proves the second inequality.

For “$\Leftarrow$” direction, assuming $T^i(X, \ell)$ and $E^i(X, \ell)$, then we have

$$A^i(X)_{\mathbb{Q}_\ell} = H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K'}}.$$

Consider the $G_{K}$-equivariant pairing

$$(3) \quad H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K'}} \times A^{d-i}(X)_{\mathbb{Q}_\ell} \rightarrow \mathbb{Q}_\ell.$$

$E^i(X, \ell)$ implies that

$$A^i(X)_{\mathbb{Q}_\ell} \times A^{d-i}(X)_{\mathbb{Q}_\ell} \rightarrow \mathbb{Q}_\ell$$

is left nondegenerate. Thus, the restriction of $\text{(3)}$

$$H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K}} \times A^{d-i}(X)_{\mathbb{Q}_\ell} \rightarrow \mathbb{Q}_\ell$$

is also left non-degenerate. By Lemma 4.1, $H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K}}$ is a direct summand of $H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K'}}$ as a $G_{K}$-representation. This implies that the 1-eigenspace of $F$ on $H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K'}}$ is a direct summand, and therefore, is equal to the generalized 1-eigenspace of $F$. It follows that

$$\dim_{\mathbb{Q}_\ell} H^{2i}(X_{K^s}, \mathbb{Q}_\ell(i))^{G_{K}} = -\text{ord}_{s=m+i}\Phi_{2i}(s).$$

This completes the proof. \qed

**Corollary 4.3.** The following statements are equivalent.

1. $\text{Br}(X_{K^s})^{G_{K}}(\ell)$ is finite for some prime $\ell \neq p$.
2. $\text{Br}(X_{K^s})^{G_{K}}(\ell)$ is finite for all primes $\ell \neq p$.
3. $T^1(X, \ell)$ holds for all primes $\ell \neq p$.
4. The rank of $\text{NS}(X)$ is equal to $-\text{ord}_{s=m+1}\Phi_{2}(s)$.

**Proof.** By Proposition 3.2, (1) $\iff T^1(X, \ell)$. Since it is well-known that $E^1(X, \ell)$ is true (cf., e.g., SGA 6 XIII, Theorem 4.6), by the theorem above, we have $T^1(X, \ell) \iff (4)$. Since $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$, the claim follows. \qed
5 Tate-Shafarevich group for abelian varieties over finitely generated fields

In this section, we will define Tate-Shafarevich group for abelian varieties over finitely generated fields of positive characteristics and study a geometric version of Tate-Shafarevich group. The Tate-Shafarevich group for abelian schemes over high dimensional bases was studied by Keller in [Kel1].

**Definition 5.1.** Let $S$ be an integral regular noetherian scheme with function field $K$. Let $A$ be an abelian variety over $K$. Define

$$\Sha_S(A) := \ker(H^1(K, A) \to \prod_{s \in S^1} H^1(K_{s^h}, A)),$$

where $S^1$ denotes the set points of codimension 1.

**Lemma 5.2.** Let $S$ be a smooth geometrically connected variety over a finitely generated field $k$. Let $K$ denote the function field of $S$ and let $A$ be an abelian variety over $K$. Write $K'$ for $K^k$. Let $U \subseteq S$ be an open dense subscheme. Let $\ell \neq \text{char}(k)$ be a prime. Then the inclusion

$$\Sha_S(A)_{G_k}^{G_k}(\ell) \to \Sha_U(A)_{G_k}(\ell)$$

has a finite cokernel. Furthermore, if $A$ extends to an abelian scheme $A \to U$, then we have a canonical isomorphism

$$\Sha_U(A)_{G_k}(\ell) \cong H^1(U_{k^s}, A)(\ell).$$

And we have

$$V_{\ell} \Sha_S(A)^{G_k} = V_{\ell} H^1(K', A)^{G_k},$$

which only depends on $A/K$. We will denote it by $V_{\ell} \Sha(A)^{G_k}$. If $k$ is a finite field, we have

$$V_{\ell} \Sha(S) = V_{\ell} H^1(K, A) \cong V_{\ell} \Sha(A)^{G_k}.$$  

We will write $V_{\ell} \Sha(K)(A)$ for $V_{\ell} \Sha(S)(A)$.

**Proof.** It is easy to see that

$$\Sha_S(A)^{G_k} = \ker(H^1(K', A)^{G_k} \to \prod_{s \in S^1} H^1(K_{s^h}^{k_s}, A)^{G_k^h}).$$

By Lemma 5.4 below, $H^1(K_{s^h}^{k_s}, A)^{G_k^h}(\ell)$ is finite. It follows that the inclusion

$$\Sha_S(A)^{G_k}(\ell) \to \Sha_U(A)^{G_k}(\ell)$$

has a finite cokernel and $V_{\ell} \Sha_S(A)^{G_k} = V_{\ell} H^1(K', A)^{G_k}$. In the case that $k$ is a finite field, by the same argument, the inclusion

$$\Sha(S)(\ell) \to \Sha(U)(\ell)$$
has a finite cokernel and $V_i\mathcal{H}_S(A) = V_iH^1(K, A)$.

If $A/K$ extends to an abelian scheme $\mathcal{A} \to U$, we will show that

$$H^1(U_{k^s}, \mathcal{A}[\ell^n]) = \text{Ker}(H^1(K', A[\ell^n])) \to \prod_{s \in U_{k^s}^1} H^1(K_{s^h}^1, A[\ell^n])).$$

This will imply that the composition of

$$H^1(U_{k^s}, \mathcal{A}[\ell^n]) \to H^1(U_{k^s}, j_* j^* \mathcal{A}) \to \mathcal{H}_U(A)[\ell^n]$$

is surjective, where $j : \text{Spec} K' \to U_{k^s}$ is the generic point. Since $\mathcal{A}[\ell^n]$ is a locally constant sheaf, we have $\mathcal{A}[\ell^n] \cong j_* j^* \mathcal{A}[\ell^n]$.

Thus, we get an exact sequence

$$0 \to H^1(U_{k^s}, \mathcal{A}[\ell^n]) \to H^1(K', A[\ell^n]) \to \prod_{s \in U_{k^s}^1} H^1(K_{s^h}^1, A).$$

Thus

$$H^1(U_{k^s}, \mathcal{A}[\ell^n]) \subseteq \text{Ker}(H^1(K', A[\ell^n])) \to \prod_{s \in U_{k^s}^1} H^1(K_{s^h}^1, A[\ell^n])).$$

Next, we will show that the inclusion is actually an equality. Since

$$H^1(K', A[\ell^n]) = \lim_{V \subseteq U_{k^s}} H^1(V, \mathcal{A}[\ell^n]),$$

let $x \in H^1(K', A[\ell^n])$, then there exists an open subscheme $V \subseteq U_{k^s}$ such that $x \in H^1(V, \mathcal{A}[\ell^n])$. Let $Y$ denote the reduced closed scheme $U_{k^s} - V$. Since $\mathcal{A}[\ell^n]$ is a locally constant sheaf, by the semi-purity [Fuj, §8], removing a closed subscheme of codimension at least 2 from $U_{k^s}$ will not change $H^1(U_{k^s}, \mathcal{A}[\ell^n])$. So by shrinking $Y$, we might assume that $Y$ is regular and of codimension 1 in $U_{k^s}$, then $H^1_Y(U_{k^s}, \mathcal{A}[\ell^n]) \cong H^0(Y, \mathcal{A}[\ell^n](-1))$ by purity [Fuj, §0]. Let $s$ be a generic point of $Y$, by purity, we have $H^1(K_{s^h}^1, A[\ell^n]) \cong (\mathcal{A}[\ell^n](-1))_s$. Thus, we get an exact sequence

$$0 \to H^1(U_{k^s}, \mathcal{A}[\ell^n]) \to H^1(V, \mathcal{A}[\ell^n]) \to \prod_{s \in Y^0} H^1(K_{s^h}^1, A[\ell^n]).$$

Therefore, if $x$ lies in

$$\text{Ker}(H^1(K', A[\ell^n])) \to \prod_{s \in U_{k^s}^1} H^1(K_{s^h}^1, A[\ell^n]),$$

then it also lies in $H^1(U_{k^s}, \mathcal{A}[\ell^n])$. This proves (4). By [Kel1, Thm. 3.3], we have $\mathcal{A} \cong j_* j^* \mathcal{A}$. Thus

$$H^1(U_{k^s}, \mathcal{A}) = \text{Ker}(H^1(K', A) \to \prod_{s \in U_{k^s}^1} H^1(K_{s^h}^1, A)).$$
It follows that $H^1(U_{k^s}, \mathcal{A})(\ell) \subseteq \Sha_{U_{k^s}}(A)(\ell)$. Since the natural map $H^1(U_{k^s}, \mathcal{A})(\ell) \to \Sha_{U_{k^s}}(A)(\ell)$ factors through $H^1(U_{k^s}, \mathcal{A})(\ell)$ and $H^1(U_{k^s}, \mathcal{A})(\ell) \to \Sha_{U_{k^s}}(A)(\ell)$ is surjective, thus

$$H^1(U_{k^s}, \mathcal{A})(\ell) = \Sha_{U_{k^s}}(A)(\ell).$$

By the same argument, one can show that

$$H^1(U, \mathcal{A})(\ell) = \Sha_U(A)(\ell).$$

It remains to prove that $V_{i\Sha_K}(A) \cong V_{i\Sha_{K'}(A)}^{G_k}$ when $k$ is a finite field. Firstly, we will show that the natural map

$$V_{i\Sha_K}(A) \to V_{i\Sha_{K'}(A)}^{G_k}$$

is surjective. Since there exists an abelian variety $B/K$ such that $A \times B$ is isogenous to $\text{Pic}^0_{X/K}$ for some smooth projective geometrically connected curve $X/K$, we might assume that $A = \text{Pic}^0_{X/K}$. One can spread out $A/K$ to a smooth projective relative curve $\pi : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{Y}$ is smooth and geometrically connected over $k$. In the following, we will show that there is commutative diagram with exact second row

$$
\begin{array}{ccc}
\text{Br}(\mathcal{X}) & \longrightarrow & V_{i\Sha_K}(\text{Pic}^0_{X/K}) \\
\downarrow & & \downarrow \\
V_{i\Sha_K}(\text{Pic}^0_{X/K})^{G_k} & \longrightarrow & V_{i\Sha_{K'}(\text{Pic}^0_{X/K})}^{G_k} & \longrightarrow & 0
\end{array}
$$

Then the surjectivity of the second column will follow from the surjectivity of the first column. By the Leray spectral sequence

$$E_2^{p,q} = H^p(\mathcal{Y}, R^q\pi_*\mathbb{G}_m) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{G}_m)$$

we get an exact sequence

$$H^2(\mathcal{Y}, \mathbb{G}_m) \to \text{Ker}(H^2(\mathcal{X}, \mathbb{G}_m) \to H^0(\mathcal{Y}, R^2\pi_*\mathbb{G}_m)) \to H^1(\mathcal{Y}, R^1\pi_*\mathbb{G}_m).$$

By Lemma 6.1 below, $H^0(\mathcal{Y}, R^2\pi_*\mathbb{G}_m)(\ell) \cong \text{Br}(X_{K^s})^{G_K}(\ell)$. Since $X$ is a smooth projective curve over $K$, we have $\text{Br}(X_{K^s}) = 0$. Thus, we get a natural map

$$V_{i\text{Br}}(\mathcal{X}) \to V_{i}H^1(\mathcal{Y}, R^1\pi_*\mathbb{G}_m).$$

Let $j : \text{Spec} K \to \mathcal{Y}$ be the generic point. By Lemma 6.1, we have

$$R^1\pi_*\mathbb{G}_m \cong j_*j^*R^1\pi_*\mathbb{G}_m.$$ 

Thus, we have

$$H^1(\mathcal{Y}, R^1\pi_*\mathbb{G}_m)(\ell) \cong H^1(\mathcal{Y}, j_*\text{Pic}_{X/K}^0)(\ell) \cong H^1(\mathcal{Y}, j_*\text{Pic}_{X/K}^0)(\ell) \cong \Sha(\text{Pic}_{X/K}^0)(\ell).$$

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Therefore,
\[ V_\ell H^1(\mathcal{Y}, R^{1}\pi_* \mathbb{G}_m) \cong V_\ell \mathfrak{III}_y(\text{Pic}^0_{X/K}) \]

By compositions, we get a natural map
\[ V_\ell \text{Br}(\mathcal{X}) \rightarrow V_\ell \mathfrak{III}_K(\text{Pic}^0_{X/K}) \].

By Theorem 6.2, there is a surjective map compatible with the above map
\[ V_\ell \text{Br}(\mathcal{X}_{k^s})^{G_k} \rightarrow V_\ell \mathfrak{III}_{K'}(\text{Pic}^0_{X/K})^{G_k} \].

It suffices to show that \( V_\ell \text{Br}(\mathcal{X}) \rightarrow V_\ell \text{Br}(\mathcal{X}_{k^s})^{G_k} \) is surjective. By Lemma 3.4, the natural map
\[ H^2(\mathcal{X}_{k^s}, \mathbb{Q}_\ell(1))^{G_k} \rightarrow V_\ell \text{Br}(\mathcal{X}_{k^s})^{G_k} \]

is surjective. Since \( H^2(\mathcal{X}, \mathbb{Q}_\ell(1)) \rightarrow V_\ell \text{Br}(\mathcal{X}) \) is surjective by definition, it suffices to show that
\[ H^2(\mathcal{X}, \mathbb{Q}_\ell(1)) \rightarrow H^2(\mathcal{X}_{k^s}, \mathbb{Q}_\ell(1))^{G_k} \]

is surjective. Consider the spectral sequence
\[ H^p(G_k, H^q(\mathcal{X}_{k^s}, \mu_{\ell^n})) \Rightarrow H^{p+q}(\mathcal{X}, \mu_{\ell^n}) \]

Since \( H^q(G_k, -) \) vanishes when \( q \geq 2 \), we get a surjection
\[ H^2(\mathcal{X}, \mu_{\ell^n}) \rightarrow H^2(\mathcal{X}_{k^s}, \mu_{\ell^n})^{G_k} \].

Taking limit, we get
\[ H^2(\mathcal{X}, \mathbb{Q}_\ell(1)) \rightarrow H^2(\mathcal{X}_{k^s}, \mathbb{Q}_\ell(1))^{G_k} \rightarrow 0. \]

Next, we will show that the natural map
\[ V_\ell \mathfrak{III}_K(A) \rightarrow V_\ell \mathfrak{III}_{K'}(A)^{G_k} \]

is injective. Consider the exact sequence
\[ 0 \rightarrow H^1(G_k, A(K')) \rightarrow H^1(K, A) \rightarrow H^1(K', A). \]

Let \( \text{Tr}_{K/k}(A) \) be the \( K/k \) trace of \( A \). By the Lang-Néron theorem, \( A(K')/\text{Tr}_{K/k}(A)(k^s) \) is finitely generated. Since \( H^1(G_k, \text{Tr}_{K/k}(A)) = 0 \) by Lang’s theorem, we have that \( H^1(G_k, A(K')) \) is finite. It follows that the natural map
\[ V_\ell H^1(K, A) \rightarrow V_\ell H^1(K', A) \]

is injective. So \( V_\ell \mathfrak{III}_K(\text{Pic}^0_{X/K}) \rightarrow V_\ell \mathfrak{III}_{K'}(\text{Pic}^0_{X/K})^{G_k} \) is also injective. This completes the proof. \[ \square \]
Lemma 5.3. Let $G$ be a smooth connected algebraic group over a finitely generated field $K$. Let $\ell \neq \text{char}(K)$ be a prime. Then the size of

$$\text{Hom}(G[\ell^n], \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G$$

is bounded by a constant independent of $n$.

Proof. Without loss of generality, we can replace $K$ by its finite extension. By Chevally’s Theorem, $G_\bar{K}$ is an extension of an abelian variety by a linear algebraic group. By extending $K$, we can assume that this extension is defined over $K$. So we have an exact sequence

$$0 \to H \to G \to A \to 0.$$  

Since $H$ is commutative, by extending $K$, we can assume that $H \cong \mathbb{G}_m \times \mathbb{G}_a$ over $K$. Thus, we have

$$0 \to H[\ell^n] \to G[\ell^n] \to A[\ell^n] \to 0.$$  

Taking dual, we get

$$0 \to A[\ell^n]^\vee \to G[\ell^n]^\vee \to H[\ell^n]^\vee \to 0.$$  

Since $H[\ell^n] = \mathbb{G}_m^s[\ell^n]$, so it suffices to prove the claim for abelian varieties and $\mathbb{G}_m$. Let $S$ be an irreducible regular scheme of finite type over $\text{Spec} \mathbb{Z}[\ell^{-1}]$ with function field $K$. Let $G$ be an abelian variety (resp. $\mathbb{G}_m$) over $K$, by shrinking $S$, we can assume that $G$ extends to an abelian scheme $G_S$ (resp. $\mathbb{G}_m,S$). Let $s \in S$ be closed point with finite residue field $k$. Write $\mathcal{F}$ for the etale sheaf represented by $G_S[\ell^n]$ on $S$. Since $G_S[\ell^n]$ is finite etale over $S$, $\mathcal{F}$ is a locally constant sheaf of $\mathbb{Z}/\ell^n$-module on $S$. Let $\eta$ be the generic point of $S$. Since $S$ is connected, we have $\mathcal{F}_\eta \cong \mathcal{F}_s$. Since $\mathcal{F}_\eta = G[\ell^n]$, we have $G[\ell^n] \cong G_s[\ell^n]$ as $G_k$-modules. Through this isomorphism, there is an inclusion

$$(G[\ell^n]^\vee)^G \subseteq (G_s[\ell^n]^\vee)^G.$$  

Therefore, it suffices to prove the claim for the case that $K$ is a finite field. In this case,

$$|\langle G[\ell^n]^\vee \rangle^G| = |G[\ell^n]_G| = |G[\ell^n]^G|.$$  

Since $G[\ell^n]^G \subseteq G(K)$, so the size is bounded by $|G(K)|$. This completes the proof.

\[\square\]

Lemma 5.4. Let $R$ be a henselian DVR with residue field $k$ and quotient field $K$. Assume that $k$ is a finitely generated field. Let $R^{sh}$ denote a strict henselization of $R$ and $K^{sh}$ denote its quotient field. Let $A$ be an abelian variety over $K$ and $\ell$ be a prime different from $\text{char}(k)$. Set $G := \text{Gal}(k^s/k) = \text{Gal}(K^{sh}/K)$. Then the size of

$$H^1(K^{sh}, A[\ell^n])^G$$

is bounded by a constant independent of $n$. And $H^1(K^{sh}, A)^G(\ell)$ is a finite group.
Proof. Let $I$ denote $\text{Gal}(K^s/K^\text{sh})$ and $I_1$ denote the wild inertia subgroup. Since $H^1(I_1, A[\ell^n]) = 0$, by [Mil3, Chap I, Lem. 2.18], we have

$$H^1(K^\text{sh}, A[\ell^n]) = (A[\ell^n](-1))_I.$$ 

Its dual $\text{Hom}((A[\ell^n](-1))_I, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \text{Hom}(A[\ell^n](-1), \mathbb{Q}_\ell/\mathbb{Z}_\ell)^I$ is canonically isomorphic to $A^t[\ell^n]^I$ by Weil pairing, where $A^t$ denotes the dual abelian variety of $A$. Let $\mathcal{A}^t$ be a Néron model of $A^t/K$ and $s : \text{Spec } k \longrightarrow S = \text{Spec } R$ be the closed point. Then

$$A^t[\ell^n]^I = \mathcal{A}^t_s[\ell^n].$$ 

There is an exact sequence

$$0 \longrightarrow (\mathcal{A}^t_s)^0 \longrightarrow \mathcal{A}^t_s \longrightarrow \Phi(A^t) \longrightarrow 0,$$

where $\Phi(A^t)$ denotes the Néron component group of $A^t$ which is a finite etale scheme over $k$. Taking dual and then taking $G_k$-invariants, we get an exact sequence

$$0 \longrightarrow (\Phi(A^t) \vee)^G \longrightarrow ((\mathcal{A}^t_s[\ell^n]) \vee)^G \longrightarrow (((\mathcal{A}^t_s)^0[\ell^n]) \vee)^G$$

Note that the size of $(\Phi(A^t) \vee)^G$ is bounded by a constant independent of $n$. Since $H^1(K^\text{sh}, A[\ell^n])^G \cong (((\mathcal{A}^t_s)^0[\ell^n]) \vee)^G$, it suffices to show that the size of $(((\mathcal{A}^t_s)^0[\ell^n]) \vee)^G$ is bounded independent of $n$. It follows directly from Lemma 5.3. This proves the first claim.

By Kummer exact sequence, we get an exact sequence

$$0 \longrightarrow A(K^\text{sh}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow H^1(K^\text{sh}, A(\ell)) \longrightarrow H^1(K^\text{sh}, A)(\ell) \longrightarrow 0.$$ 

Let $\mathcal{A}$ be a Néron model of $A$ over $S$ and $\mathcal{A}^0$ be the identity component of $\mathcal{A}$. We have $A(K^\text{sh}) = \mathcal{A}(S)$ and an exact sequence

$$0 \longrightarrow \mathcal{A}^0(S) \longrightarrow \mathcal{A}(S) \longrightarrow \Phi(A) \longrightarrow 0.$$ 

Since $\mathcal{A}^0(S)$ is $\ell$-divisible and $\Phi(A)$ is finite, by tensoring $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, we have

$$A(K^\text{sh}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0.$$ 

It follows

$$H^1(K^\text{sh}, A(\ell)) \cong H^1(K^\text{sh}, A)(\ell).$$

Since

$$H^1(K^\text{sh}, A(\ell))^G = \lim_{\rightarrow n} H^1(K^\text{sh}, A[\ell^n])^G,$$

by the first claim, the right side is finite. It follows that $H^1(K^\text{sh}, A)^G(\ell)$ is also finite. □
6 Fibrations and Brauer groups

Lemma 6.1. Let \( U \) be an irreducible regular scheme with function field \( K \). Let \( \pi : \mathcal{X} \to U \) be a smooth proper morphism with a generic fiber geometrically connected over \( K \). Let \( j : \text{Spec} \, K \to U \) be the generic point of \( U \). Then we have

(a) the natural map
\[
R^1\pi_*\mathbb{G}_m \to j_*(j^*R^1\pi_*\mathbb{G}_m)
\]
is an isomorphism,

(b) the natural map
\[
R^2\pi_*\mathbb{G}_m(\ell) \to j_*(j^*R^2\pi_*\mathbb{G}_m(\ell))
\]
is an isomorphism for any prime \( \ell \) invertible on \( U \).

Proof. It suffices to show that the induced maps on stalks are isomorphism. Thus, we might assume that \( U = \text{Spec} \, R \) where \( R \) is a strictly henselian regular local ring.

Let \( X \) denote the generic fiber. Let \( s \in U \) be the closed point. Then we have
\[
(R^1\pi_*\mathbb{G}_m)_s = \text{Pic}(\mathcal{X}) \quad \text{and} \quad (j_*(j^*R^1\pi_*\mathbb{G}_m))_s = \text{Pic}_{X/K}(K).
\]
Since \( \mathcal{X}_s \) admits a section \( s \to \mathcal{X}_s \) and \( \pi \) is smooth, the section can be extended to a section \( U \to \mathcal{X} \). Thus \( X(K) \) is not empty. So
\[
\text{Pic}_{X/K}(K) = \text{Pic}(X).
\]
Since \( \mathcal{X} \) is regular, the natural map
\[
\text{Pic}(\mathcal{X}) \to \text{Pic}(X)
\]
is surjective and has a kernel generated by vertical divisors. Since \( \pi \) is smooth and proper with a geometrically connected generic fiber, by the same arguments as in the proof of Proposition 2.2, there is an exact sequence
\[
0 \to \text{Pic}(	ext{Spec} \, R) \to \text{Pic}(\mathcal{X}) \to \text{Pic}(X) \to 0.
\]
Since \( R \) is an UFD, we have \( \text{Pic}(	ext{Spec} \, R) = 0 \). It follows that
\[
\text{Pic}(\mathcal{X}) \cong \text{Pic}(X).
\]
This proves (a).

Let \( I \) denote \( G_K \). For (b), the induced map on the stalk at \( s \) is
\[
\text{Br}(\mathcal{X})(\ell) \to \text{Br}(X_{K^s})^I(\ell).
\]
Since \( \pi \) is smooth and proper, we have
\[
H^2(\mathcal{X}, \mu_{\ell^\infty}) \cong H^2(X_{K^s}, \mu_{\ell^\infty}) = H^2(X_{K^s}, \mu_{\ell^\infty})^I.
\]
Consider

\[
0 \longrightarrow \text{Pic}(\mathcal{X}) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \longrightarrow H^2(\mathcal{X}, \mu_{\ell^\infty}) \longrightarrow \text{Br}(\mathcal{X})(\ell) \longrightarrow 0
\]

\[
0 \longrightarrow \text{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \longrightarrow H^2(X_{K^s}, \mu_{\ell^\infty}) \longrightarrow \text{Br}(X_{K^s})(\ell) \longrightarrow 0
\]

Since \(\text{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell\) is \(I\)-invariant and \(\text{Pic}(\mathcal{X}) = \text{Pic}(X)\). It suffices to show that \(\text{Pic}(X) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \rightarrow (\text{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell)^I\) is surjective. Write \(\text{NS}(X_{K^s})_{\text{free}}\) for \(\text{NS}(X_{K^s}) / \text{NS}(X_{K^s})_{\text{tor}}\). The action of \(I\) on \(\text{NS}(X_{K^s})\) factors through a finite quotient \(I'\). Consider the exact sequence

\[
0 \longrightarrow (\text{NS}(X_{K^s})_{\text{free}})_{I'} \otimes \mathbb{Z}_\ell \longrightarrow \text{NS}(X_{K^s})_{I''} \otimes \mathbb{Q}_\ell \longrightarrow (\text{NS}(X_K) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell)^I
\]

\[
\longrightarrow H^1(I', \text{NS}(X_{K^s})_{\text{free}} \otimes \mathbb{Z}_\ell).
\]

\(H^1(I', \text{NS}(X_{K^s})_{\text{free}} \otimes \mathbb{Z}_\ell)\) is killed by the order of \(I'\). Since \((\text{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell)^I = \text{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell\) is divisible, so the image of the last map is zero. Since

\[
\text{Pic}(X) \otimes \mathbb{Q}_\ell \rightarrow \text{NS}(X_{K^s})^I \otimes \mathbb{Q}_\ell
\]

is surjective (cf. [Yua2, §2.2]), the claim follows. By Snake Lemma, the natural map

\[
\text{Br}(\mathcal{X})(\ell) \rightarrow \text{Br}(X_{K^s})^I(\ell)
\]

is an isomorphism.

\[\square\]

**Theorem 6.2.** Let \(\pi : \mathcal{X} \rightarrow \mathcal{Y}\) be a dominant morphism between smooth geometrically connected varieties over a finitely generated field \(k\) of characteristic \(p \geq 0\). Let \(K\) be the function field of \(\mathcal{Y}\). Assume that the generic fiber \(X\) of \(\pi\) is smooth projective geometrically connected over \(K\). Set \(K' := K_{k^s}\) and let \(\ell \neq p\) be a prime. Set

\[
\mathcal{K} := \text{Ker}(V_{l}(\text{Br}(\mathcal{X}_{k^s})^{G_k}) \rightarrow V_{l}(\text{Br}(X_{K^s})^{G_K}),
\]

then we have canonical exact sequences

\[
0 \rightarrow V_{l}(\text{Br}(\mathcal{Y}_{k^s})^{G_k}) \rightarrow \mathcal{K} \rightarrow V_{l}(\text{Br}(X_{K^s})^{G_K}) \rightarrow 0,
\]

\[
0 \rightarrow \mathcal{K} \rightarrow V_{l}(\text{Br}(\mathcal{X}_{k^s})^{G_k}) \rightarrow V_{l}(\text{Br}(X_{K^s})^{G_K}) \rightarrow 0.
\]

**Proof.** By shrinking \(\mathcal{Y}\), we might assume that \(\pi\) is projective and smooth. Firstly, we will show that the natural map

\[
V_{l}(\text{Br}(\mathcal{X}_{k^s}) \rightarrow V_{l}(\text{Br}(X_{K^s})^{G_{K'}})
\]

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is surjective. Consider the Leray spectral sequence
\[ E_2^{p,q} = H^p(Y_{k^s}, R^q\pi_*\mathbb{Q}_\ell(1)) \Rightarrow H^{p+q}(X_{k^s}, \mathbb{Q}_\ell(1)). \]

By Deligne’s Lefschetz criteria (cf. [Del1]), the above spectral sequence degenerates at \( E_2 \). Thus, we get a canonical surjective map
\[ H^2(X_{K^s}, \mathbb{Q}_\ell(1)) \rightarrow H^0(Y_{k^s}, R^2\pi_*\mathbb{Q}_\ell(1)). \]

Since \( R^2\pi_*\mathbb{Q}_\ell(1) \) is lisse, we have
\[ H^0(Y_{k^s}, R^2\pi_*\mathbb{Q}_\ell(1)) = H^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_{K'}}. \]

It follows that the canonical map
\[ H^2(X_{k^s}, \mathbb{Q}_\ell(1)) \rightarrow H^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_{K'}} \]
is surjective. Consider the following commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
H^2(X_{k^s}, \mathbb{Q}_\ell(1)) & \rightarrow & V_{\ell}\text{Br}(X_{k^s}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \\
H^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_{K'}} & \rightarrow & V_{\ell}\text{Br}(X_{K^s})^{G_{K'}} & \rightarrow & 0
\end{array}
\]

Since the first column is surjective, the map
\[ V_{\ell}\text{Br}(X_{k^s}) \rightarrow V_{\ell}\text{Br}(X_{K^s})^{G_{K'}} \]

is also surjective. Secondly, we will show the exactness of the first sequence in the statement of theorem. By the Leray spectral sequence
\[ H^p(Y_{k^s}, R^q\pi_*\mathbb{G}_m) \Rightarrow H^{p+q}(X_{k^s}, \mathbb{G}_m), \]

we get a long exact sequence
\[
\text{Br}(Y_{k^s}) \rightarrow \text{Ker}(\text{Br}(X_{k^s}) \rightarrow H^0(Y_{k^s}, R^2\pi_*\mathbb{G}_m)) \rightarrow H^1(Y_{k^s}, R^1\pi_*\mathbb{G}_m) \rightarrow H^3(Y_{k^s}, \mathbb{G}_m).
\]

Note that without loss of generality, we can always replace \( k \) by its finite Galois extension. There exists a finite Galois extension \( L/K \) such that \( X(L) \) is not empty. Choose a \( K \)-morphism \( \text{Spec} L \rightarrow X \) and we might assume that \( L \) is the function field of smooth variety \( Z \) over \( k \). By shrinking \( Y \) and \( Z \), we can assume that \( \text{Spec} L \rightarrow X \) extends to a morphism \( Z \rightarrow \mathcal{X} \) such that \( Z \rightarrow Y \) is finite flat. Let \( \pi' \) denote \( Z \rightarrow Y \). Then we have a commutative diagram induced by the Leray spectral sequences for \( \pi \) and \( \pi' \)
\[
\begin{array}{ccccccccc}
H^1(Y_{k^s}, R^1\pi_*\mathbb{G}_m) & \rightarrow & H^3(Y_{k^s}, \mathbb{G}_m) \\
\downarrow & & \downarrow & & \\
H^1(Y_{k^s}, R^1\pi'_*\mathbb{G}_m) & \rightarrow & H^3(Y_{k^s}, \pi'_*\mathbb{G}_m)
\end{array}
\]
Since $\pi'$ is finite, we have $R^i\pi'_*\mathbb{G}_m = 0$ for $i > 0$ and $H^3(Y_{k^*}, \pi'_*\mathbb{G}_m) = H^3(Z_{k^*}, \mathbb{G}_m)$.

There is a canonical norm map $\pi'_*\mathbb{G}_m \to \mathbb{G}_m$ which induces a norm map

$$N : H^i(Z_{k^*}, \mathbb{G}_m) \to H^i(Y_{k^*}, \mathbb{G}_m)$$

such that the composition of $N$ with the pull back map $H^i(Y_{k^*}, \mathbb{G}_m) \to H^i(Z_{k^*}, \mathbb{G}_m)$ is equal to the multiplication by $\deg(\pi')$. It follows that the second column in the above diagram has a kernel killed by $\deg(\pi')$. Therefore, the first row has an image killed by $\deg(\pi')$ and the sequence $0 \to V_\ell \text{Br}(Y_{k^*}) \to V_\ell \text{Br}(X_{k^*})$ is split as $G_{k^*}\text{-representations}$. Thus, we get a split exact sequence of $G_{k^*}\text{-representations}$

$$0 \to V_\ell \text{Br}(Y_{k^*}) \to \text{Ker}(V_\ell \text{Br}(X_{k^*}) \to V_\ell H^0(Y_{k^*}, R^2\pi'_*\mathbb{G}_m)) \to V_\ell H^1(Y_{k^*}, R^1\pi'_*\mathbb{G}_m) \to 0.$$

In fact, the norm map $N$ induces a projection

$$\text{Ker}(V_\ell \text{Br}(X_{k^*}) \to V_\ell H^0(Y_{k^*}, R^2\pi'_*\mathbb{G}_m)) \to V_\ell \text{Br}(Y_{k^*}).$$

So we get an isomorphism

$$\text{Ker}(V_\ell \text{Br}(X_{k^*}) \to V_\ell H^0(Y_{k^*}, R^2\pi'_*\mathbb{G}_m)) \cong V_\ell \text{Br}(Y_{k^*}) \oplus V_\ell H^1(Y_{k^*}, R^1\pi'_*\mathbb{G}_m),$$

where the direct sum decomposition depends on the choice of $\pi'$. Next, we will show that there is a canonical isomorphism

$$V_\ell H^1(Y_{k^*}, R^1\pi'_*\mathbb{G}_m) \cong V_\ell \text{III}_{Y_{k^*}}(\text{Pic}^0_{X/K}).$$

Let $j : \text{Spec} \, K \to Y$ be the generic point. By Lemma 6.1, the natural map

$$R^1\pi'_*\mathbb{G}_m \to j_*j^*R^1\pi'_*\mathbb{G}_m$$

is an isomorphism. It follows that

$$H^1(Y_{k^*}, R^1\pi'_*\mathbb{G}_m) \cong H^1(Y_{k^*}, j_*j^*R^1\pi'_*\mathbb{G}_m).$$

There is an exact sequence

$$0 \to H^1(Y_{k^*}, j_*j^*R^1\pi'_*\mathbb{G}_m) \to H^1(K', \text{Pic}_{X/K}) \to \prod_{s \in Y_{k^*}} H^1(K_{s}^{\text{sh}}, \text{Pic}_{X/K}).$$

Let $\text{III}_{Y_{k^*}}(\text{Pic}_{X/K})$ denote the kernel of the third arrow. It suffices to show that

$$V_\ell \text{III}_{Y_{k^*}}(\text{Pic}^0_{X/K}) \cong V_\ell \text{III}_{Y_{k^*}}(\text{Pic}_{X/K}).$$

By the canonical exact sequence

$$0 \to \text{Pic}^0_{X/K} \to \text{Pic}_{X/K} \to \text{NS}(X_{k^*}) \to 0,$$
we get a long exact sequence

\[ H^0(K', \text{NS}(X_{K^s})) \rightarrow H^1(K', \text{Pic}^0_{X/K}) \rightarrow H^1(K', \text{Pic}_{X/K}) \rightarrow H^1(K', \text{NS}(X_{K^s})). \]

Let \( L/K \) be a finite Galois extension such that \( \text{Pic}(X_L) \rightarrow \text{NS}(X_{K^s}) \) is surjective. One can show that \( H^1(K', \text{NS}(X_{K^s})) \) is killed by \([L : K] \# \text{NS}(X_{K^s})_\text{tor}\) and the cokernel of \( H^0(K', \text{Pic}_{X/K}) \rightarrow H^0(K', \text{NS}(X_{K^s})) \) is killed by \([L : K]\). Thus, the kernel and the cokernel of

\[ H^1(K', \text{Pic}^0_{X/K}) \rightarrow H^1(K', \text{Pic}_{X/K}) \]

are killed by \([L : K] \# \text{NS}(X_{K^s})_\text{tor}\). By the same argument, the kernel and cokernel of

\[ \prod_{s \in \mathcal{Y}_{k^s}} H^1(K'^{sh}, \text{Pic}^0_{X/K}) \rightarrow \prod_{s \in \mathcal{Y}_{k^s}} H^1(K'^{sh}, \text{Pic}_{X/K}) \]

are also killed by \([L : K] \# \text{NS}(X_{K^s})_\text{tor}\). Taking \( V_\ell \), they become isomorphisms. Therefore, we have a canonical isomorphism

\[ V_\ell \text{III}_{k^s}(\text{Pic}^0_{X/K}) \cong V_\ell \text{III}_{k^s}(\text{Pic}_{X/K}). \]

Thus, we get a canonical isomorphism

\[ V_\ell H^1(Y_{k^s}, R^1\pi_*\underline{\mathbb{G}}_m) \cong V_\ell \text{III}_{k^s}(\text{Pic}^0_{X/K}). \]

By Lemma 6.1, the natural map

\[ R^2\pi_*\underline{\mathbb{G}}_m(\ell) \rightarrow j_*j^* R^2\pi_*\underline{\mathbb{G}}_m(\ell) \]

is an isomorphism. It follows directly

\[ H^0(Y_{k^s}, R^2\pi_*\underline{\mathbb{G}}_m)(\ell) \cong \text{Br}(X_{K^s})^{G_{K'}}(\ell). \]

Thus, there is an exact sequence

\[ 0 \rightarrow V_\ell \text{Br}(Y_{k^s}) \oplus V_\ell \text{III}_{K'}(\text{Pic}^0_{X/K}) \rightarrow V_\ell \text{Br}(X_{k^s}) \rightarrow V_\ell \text{Br}(X_{K^s})^{G_{K'}} \rightarrow 0, \]

where the second arrow depends on the choice of \( \pi' \). Taking \( G_k \)-invariants, we proved the exactness of the first sequence in the theorem.

It remains to show that the natural map

\[ V_\ell \text{Br}(X_{k^s})^{G_k} \rightarrow V_\ell \text{Br}(X_{K^s})^{G_K} \]

is surjective. To prove this, we will use a pull-back trick (cf. [CTS] or [Yua2]). Let \( W \) be a smooth projective geometrically connected curve over \( K \) contained in \( X \) which is a complete intersection of hyperplane sections. By Shrinking \( \mathcal{Y} \), we might assume that \( W \) admits a smooth projective model \( \mathcal{W} \rightarrow \mathcal{Y} \) where \( \mathcal{W} \) is the Zariski closure of \( W \) in \( \mathcal{X} \).
By shrinking $\mathcal{Y}$, we might assume that the map $Z \to X$ chosen before factors through $Z \to W$. Thus, we get a commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & V_{\ell}\text{Br}(Y_{k^s}) \oplus V_{\ell}\text{Pic}^0_{X/K} & \longrightarrow & V_{\ell}\text{Br}(X_{k^s}) & \longrightarrow & V_{\ell}\text{Br}(X_{k^s})^{G_{K'}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_{\ell}\text{Br}(Y_{k^s}) \oplus V_{\ell}\text{Pic}^0_{W/K} & \longrightarrow & V_{\ell}\text{Br}(W_{k^s}) & \longrightarrow & V_{\ell}\text{Br}(W_{k^s})^{G_{K'}} & \longrightarrow & 0
\end{array}
$$

Taking $G_k$ invariants, we get

$$
\begin{array}{ccccccc}
V_{\ell}\text{Br}(X_{k^s})^{G_k} & \longrightarrow & V_{\ell}\text{Br}(X_{k^s})^{G_k} & \longrightarrow & H^1(G_k, V_{\ell}\text{Br}(Y_{k^s}) \oplus V_{\ell}\text{Pic}^0_{X/K}) & \longrightarrow & c \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
V_{\ell}\text{Br}(W_{k^s})^{G_k} & \longrightarrow & V_{\ell}\text{Br}(W_{k^s})^{G_K} & \longrightarrow & H^1(G_k, V_{\ell}\text{Br}(Y_{k^s}) \oplus V_{\ell}\text{Pic}^0_{W/K}) & \longrightarrow &
\end{array}
$$

Since $V_{\ell}\text{Br}(W_{k^s})^{G_K} = 0$, the second arrow in the bottom vanishes. Thus, to show that $a$ is surjective, it suffices to show that $c$ is injective. This actually follows from the fact

$$
V_{\ell}\text{Pic}^0_{X/K} \longrightarrow V_{\ell}\text{Pic}^0_{W/K}
$$

is split as $G_k$-representations. By our choice of $W$ and the Lefschetz hyperplane section theorem, the pullback map

$$
H^1(X_{k^s}, \mathbb{Q}_\ell) \longrightarrow H^1(W_{k^s}, \mathbb{Q}_\ell)
$$

is injective. This implies that the induced map $\text{Pic}^0_{X/K} \longrightarrow \text{Pic}^0_{W/K}$ has a finite kernel. Therefore, there exists an abelian variety $A/K$ and an isogeny $\text{Pic}^0_{X/K} \times A \to \text{Pic}^0_{W/K}$. It follows that

$$
V_{\ell}\text{Pic}^0_{W/K} \oplus V_{\ell}\text{Pic}^0_{X/K} \cong V_{\ell}\text{Pic}^0_{W/K}.
$$

This proves the splitness. Thus, the natural map

$$
V_{\ell}\text{Br}(X_{k^s})^{G_k} \longrightarrow V_{\ell}\text{Br}(X_{k^s})^{G_K}
$$

is surjective.

In the theorem above, we assume that the generic fiber is projective over $K$. In the following corollary, we will deduce the surjectivity of $V_{\ell}\text{Br}(X_{k^s})^{G_k} \longrightarrow V_{\ell}\text{Br}(X_{k^s})^{G_K}$ from the above theorem for any fibration with a smooth generic fiber.

**Corollary 6.3.** Let $k$ be a finite field or a number field. Let $\pi : X \to Y$ be a dominant morphism between smooth geometrically connected varieties over $k$. Let $K$ be the function field of $Y$ and $X$ be the generic fiber of $\pi$. Let $\ell \neq \text{char}(k)$ be a prime. Assuming that $X$ is smooth over $K$, then the natural map

$$
V_{\ell}\text{Br}(X_{k^s})^{G_k} \longrightarrow V_{\ell}\text{Br}(X_{k^s})^{G_K}
$$

is surjective.
Proof. Firstly, note that without loss of generality, we can always extend \( k \) to its finite extension.

Secondly, we might assume that \( X \) is geometrically irreducible over \( K \). Let \( L \) be the algebraic closure of \( K \) in \( K(X) \). Then \( L/K \) is a finite separable extension and \( X \) is smooth and geometrically irreducible over \( \text{Spec} \, L \). By spreading out \( X \to \text{Spec} \, L \), we get a morphism with a smooth geometrically connected generic fiber. Since \( \text{Br}(X \otimes_L L^s)^{G_L} = \text{Br}(X_{K^s})^{G_K} \), it suffices to prove the claim for this morphism.

Thirdly, if \( X/K \) is birational equivalent to a smooth projective variety \( X' \) over \( K \), then the claim will follow from Theorem 6.2 since \( V_l \text{Br}(X_{K^s})^{G_K} \) is a birational invariant.

Next, we will use de Jong’s alteration theorem to reduce the question to this case. Let \( X'_{\bar{K}} \to X_{\bar{K}} \) be an alteration such that \( X'_{\bar{K}} \) admits a smooth projective compactification. We might assume that \( X' \) and the alteration are defined over a finite normal extension \( L \) of \( K \). By shrinking \( X \), we can assume that the alteration \( X' \to X_L \) is finite flat. By spreading out \( X' \to \text{Spec} \, L \), we get a commutative diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow f & & \downarrow \\
X & \longrightarrow & Y 
\end{array}
\]

where \( Y' \) has function field \( L \). By shrinking \( Y \), we can assume that \( f \) is finite flat. Thinking \( f \) as a \( Y \)-morphism and base change to \( \text{Spec} \, K^s \), we get \( X'_{K^s} \to X_{K^s} \) which is also finite flat. It induces a norm map \( \text{Br}(X'_{K^s}) \to \text{Br}(X_{K^s}) \) which is compatible with the norm map \( \text{Br}(X'_{k^s}) \to \text{Br}(X_k) \). Therefore we get a commutative diagram

\[
\begin{array}{ccc}
V_l \text{Br}(X'_{k^s})^{G_k} & \longrightarrow & V_l \text{Br}(X'_{K^s})^{G_K} \\
\downarrow & & \downarrow \\
V_l \text{Br}(X'_{k^s})^{G_k} & \longrightarrow & V_l \text{Br}(X_{K^s})^{G_K}
\end{array}
\]

Since vertical maps are surjective, it suffices to show that the first row is surjective. Since the claim holds for \( X' \to Y' \), the natural map

\[
V_l \text{Br}(X'_{k^s})^{G_k} \to V_l \text{Br}(X'_{L^s})^{G_{L^s}}
\]

is surjective. Thus, it suffices to show that there is a natural isomorphism

\[
\text{Br}(X'_{K^s})^{G_K} \cong \text{Br}(X' \otimes_L L^s)^{G_{L^s}}.
\]

Fix an algebraic closure \( \bar{K} \) of \( K \) and assume that \( L \) and \( L^s \) are contained in \( \bar{K} \). Since \( X'_{K^s} = X' \otimes_L L \otimes_K K^s \), the natural map \( L \otimes_K K^s \to L^s \) induces a map

\[
\text{Br}(X' \otimes_L L \otimes_K K^s) \to \text{Br}(X' \otimes_L L^s).
\]

Set \( K_1 = L \cap K^s \), by our assumption that \( L/K \) is a normal extension, so \( K_1/K \) is finite Galois. Write \( G = G_K \) and \( H = G_{K_1} \). We have

\[
L \otimes_K K^s = L \otimes_{K_1} K_1 \otimes_K K^s = \prod_{\sigma:K_1 \to K^s} L \otimes_{K_1,\sigma} K^s
\]
One can show that $L \otimes_{K_1,\sigma} K^s$ is a separable closure of $L$ and admits a $H$-action (acting on $K^s$). We denote it by $L^s_\sigma$. Then

$$\text{Br}(X' \otimes_L L \otimes_K K^s) \cong \prod_{\sigma: K_1 \hookrightarrow K^s} \text{Br}(X'_{L^s_\sigma}).$$

Taking $H$-invariant, we get

$$\text{Br}(X' \otimes_L L \otimes_K K^s)^H \cong \prod_{\sigma: K_1 \hookrightarrow K^s} \text{Br}(X'_{L^s_\sigma})^H.$$

Now $G/H$ acts as permutations on the right side, so the $G/H$-invariant of right side can be identified with the factor with the index $\sigma$ equal to the inclusion map $K_1 \subset K^s$. The factor can be identified with $\text{Br}(X' \otimes_L L \otimes_K K^s)^{G_k}$ through the natural isomorphism $L \otimes_{K_1} K^s \cong L^s$.

It follows that there is a natural isomorphism

$$\text{Br}(X' \otimes_L L \otimes_K K^s)^{G_k} \cong \text{Br}(X' \otimes_L L \otimes_K K^s)^{G_L}.$$

This completes the proof.

**Corollary 6.4.** Let $\mathcal{Y}$ be a smooth geometrically connected variety over a finitely generated field $k$ with function field $K$ of characteristic $p > 0$. Let $A$ be an abelian variety over $K$. Let $B = \text{Tr}_{K/k}(A)$ denote the $K/k$ trace of $A$. Let $\ell \neq p$ be a prime. Then there is a canonical exact sequence

$$0 \rightarrow V_{\ell \Pi_k}(B) \rightarrow V_{\ell \Pi_K}(A) \rightarrow V_{\ell \Pi_{K'}}(A)^{G_k} \rightarrow 0.$$

**Proof.** Consider the exact sequence

$$0 \rightarrow H^1(G_k, A(K')) \rightarrow H^1(K, A) \rightarrow H^1(K', A),$$

we will show that $V_{\ell}H^1(G_k, A(K'))$ is naturally isomorphic to $V_{\ell}H^1(k, B)$. This will imply the left exactness of the sequence in the claim. By the Lang-Néron theorem, $A(K')/B(k^s)$ is finitely generated, so $H^1(G_k, A(K')/B(k^s))$ is of finite exponent. By the exact sequence

$$0 \rightarrow B(k^s) \rightarrow A(K') \rightarrow A(K')/B(k^s) \rightarrow 0,$$

we get

$$V_{\ell}H^1(G_k, A(K')) = V_{\ell}H^1(G_k, B(k^s)) = V_{\ell \Pi_k}(B),$$

where the last equality follows from Lemma 5.2. It follows that

$$0 \rightarrow V_{\ell \Pi_k}(B) \rightarrow V_{\ell \Pi_K}(A) \rightarrow V_{\ell \Pi_{K'}}(A)^{G_k}$$

is exact.

To show the surjectivity of the last arrow, we might assume $A = \text{Pic}^0_{X/K}$ where $X$ is a smooth projective geometrically connected curve over $K$. Spreading out $X/K$, we get
a smooth projective relative curve $\pi : \mathcal{X} \rightarrow \mathcal{Y}$. Since $\text{Br}(\mathcal{X}_{k^s}) = 0$, by Theorem 6.2, we get a surjection

$$\text{Br}(\mathcal{X}_{k^s})^{G_k} \rightarrow V_{l\text{III}'}(A)^{G_k}.$$ 

Let $k_0$ be the algebraic closure of the prime field of $k$ in $k$. Then there exists a smooth geometrically connected variety $\mathcal{Z}$ over $k_0$ with function field $k$. By shrinking $\mathcal{Z}$, we can spread out $\pi$ to a smooth projective morphism $\pi: \mathcal{X}_{\mathcal{Z}} \rightarrow \mathcal{Y}_{\mathcal{Z}}$. By shrinking $\mathcal{Z}$ further, we can assume that $\mathcal{Y}_{\mathcal{Z}}$ is smooth over $\mathcal{Z}$. Using Theorem 6.2 for $\pi$, we get a surjection

$$V_{l\text{Br}}(\mathcal{X}_{k_0})^{G_{k_0}} \rightarrow V_{l\text{III}}(\mathcal{X}_{k_0})(A)^{G_{k_0}}.$$ 

It remains to show that the natural map

$$V_{l\text{Br}}(\mathcal{X}_{k_0})^{G_{k_0}} \rightarrow \text{Br}(\mathcal{X}_{k^s})^{G_k}$$

is surjective. Since $\mathcal{X}$ can be identified with the generic fiber of $\mathcal{X}_{\mathcal{Z}} \rightarrow \mathcal{Z}$, then the claim follows from Corollary 6.3.

\[\square\]

### 6.1 Proof of Theorem 1.11

It follows directly from Theorem 6.2 and Corollary 6.

### 7 BSD conjecture and Tate-Shafarevich group

#### 7.1 Cohomological interpretation of the vanishing order of $L$ function for an abelian variety

**Lemma 7.1.** Let $A$ be an abelian variety defined over a finitely generated field $K$ of characteristic $p > 0$. Assume that $A$ extends to an abelian scheme $\pi : \mathcal{A} \rightarrow \mathcal{Y}$, where $\mathcal{Y}$ is a smooth geometrically connected variety over a finite field $k$ with function field $K$. Set $m = \dim \mathcal{Y}$ and let $\ell \neq p$ be a prime. Then

$$\text{rank } A(K) \leq \text{ord}_{s=m} \Phi_1(s),$$

and $\text{ord}_{s=m} \Phi_1(s)$ is equal to the dimension of the generalized 1-eigenspace of the Frobenius action on $H^1(\mathcal{Y}_k, V_{\ell}\mathcal{A})$.

**Proof.** Let $q$ denote the cardinality of $k$ and $F$ denote the geometric Frobenius element in $G_k$. Set $\mathcal{F} = R^1\pi_*\mathbb{Q}_\ell$, $\mathcal{F}$ is lisse since $\pi$ is smooth and proper. By definition,

$$\Phi_1(s) = \prod_{y \in \mathcal{Y}^0} \det(1 - q^{-s}\sigma_y^{-1}|\mathcal{F}_y)^{-1}.$$ 

By Grothendieck’s formula,

$$\Phi_1(s) = \prod_{i=0}^{2m} \det(1 - q^{-s}F|H^i_c(\mathcal{Y}_k, \mathcal{F}))^{(-1)^{i+1}}.$$
By Deligne’s theorem, \( H^i_c(Y_k, \mathcal{F}) \) is of mixed weight \( \leq i + 1 \) and \( H^{2m}_c(Y_k, \mathcal{F}) \) is of pure weight \( 2m+1 \) by Poincaré duality. Therefore only \( \det(1-q^{-s}F|H^{2m-1}_c(Y_k, \mathcal{F})) \) contributes to the order of zero at \( s = m \). By Poincaré duality,

\[
H^{2m-1}_c(Y_k, \mathcal{F}) \cong H^1(Y_k, \mathcal{F}^\vee)(-m).
\]

It follows that \( \text{ord}_{s=m} \Phi_1(s) \) is equal to the dimension of the generalized 1-eigenspace of \( F \) on \( H^1(Y_k, \mathcal{F}^\vee) \). Thus

\[
\dim H^1(Y_k, \mathcal{F}^\vee)^{G_k} \leq \text{ord}_{s=m} \Phi_1(s).
\]

The lisse sheaf \( \mathcal{F} \) corresponds to the representation of \( G_K \) on \( H^1(A_K, \mathbb{Q}_\ell) \), therefore \( \mathcal{F}^\vee \) corresponds to \( H^1(A_K, \mathbb{Q}_\ell)^\vee \) which can be identified with \( V_\ell A \). Since \( \pi \) is an abelian scheme, there is a Kummer exact sequence of etale sheaves on \( Y \)

\[
0 \longrightarrow \mathcal{A}[\ell^n] \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}_0 \longrightarrow 0.
\]

Since \( \mathcal{A}[\ell^n] \) is a finite etale group scheme over \( Y \), it represents a locally constant sheaf on \( Y \). Let \( V_\ell \mathcal{A} \) be the \( \mathbb{Q}_\ell \)-sheaf associated to the inverse system \( \mathcal{A}[\ell^n] \). It corresponds to the \( G_K \) representation on \( V_\ell A \). Therefore \( V_\ell \mathcal{A} \) can be identified with \( \mathcal{F}^\vee \). From the Kummer exact sequence, we get

\[
0 \longrightarrow H^0(Y_k, \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^1(Y_k, V_\ell \mathcal{A}) \longrightarrow V_\ell H^1(Y_k, \mathcal{A}) \longrightarrow 0.
\]

By [Kell, Thm. 3.3], \( \mathcal{A} \) satisfies the Neron property

\[
\mathcal{A} \cong j_\ast j^* \mathcal{A},
\]

where \( j = \text{Spec } K \longrightarrow Y \) is the generic point of \( Y \). Write \( K' \) for \( K \bar{k} \), it follows

\[
H^0(Y_k, \mathcal{A}) = A(K').
\]

Taking \( G_k \)-invariants for the above exact sequence, we get

\[
0 \longrightarrow H^0(Y_k, \mathcal{A})^{G_k} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^1(Y_k, V_\ell \mathcal{A})^{G_k} \longrightarrow V_\ell H^1(Y_k, \mathcal{A})^{G_k}.
\]

Since \( H^0(Y_k, \mathcal{A})^{G_k} = A(K) \), it follows

\[
\text{rank } A(K) \leq \dim H^1(Y_k, V_\ell \mathcal{A})^{G_k} \leq \text{ord}_{s=m} \Phi_1(s).
\]

\[\square\]

### 7.2 Proof of Theorem 1.9

**Theorem 7.2.** Let \( A \) be an abelian variety over a finitely generated field \( K \) of characteristic \( p > 0 \). Let \( \ell \neq p \) be a prime. Then the BSD conjecture (i.e. Conjecture 1.4) for \( A \) is equivalent to the finiteness of \( \Sha_K(A)(\ell) \).
Proof. Assuming the BSD conjecture for $A$, by Lemma 7.1 and 7.3 below, we have $V_ℓ\Pi_K′(A)^G_k = 0$. By Lemma 5.2, $V_ℓ\Pi_K(A) = V_ℓ\Pi_K'(A)^G_k$, so $V_ℓ\Pi_K(A) = 0$. Thus $\Pi_K(A)(ℓ)$ is finite.

Assuming that $\Pi_K(A)(ℓ)$ is finite, then $V_ℓ\Pi_K′(A)^G_k = 0$. By Lemma 7.3 below,

$$H^0(\mathcal{Y}, \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}_ℓ \cong H^1(\mathcal{Y}_k, V_ℓ\mathcal{A})^{G_k}$$

and $H^0(\mathcal{Y}, \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}_ℓ$ is a direct summand of $H^1(\mathcal{Y}_k, V_ℓ\mathcal{A})$ as $G_k$-representations. It follows that $H^1(\mathcal{Y}_k, V_ℓ\mathcal{A})^{G_k}$ is direct summand of $H^1(\mathcal{Y}_k, V_ℓ\mathcal{A})$ as $G_k$-representations. This implies that $H^1(\mathcal{Y}_k, V_ℓ\mathcal{A})^{G_k}$ is equal to the generalized 1-eigenspace of $H^1(\mathcal{Y}_k, V_ℓ\mathcal{A})$. Then the claim follows from Lemma 7.1.

Lemma 7.3. Let $A$ be an abelian variety defined over a finitely generated field $K$ of characteristic $p > 0$. Assume that $A$ extends to an abelian scheme $\pi : \mathcal{A} \to \mathcal{Y}$, where $\mathcal{Y}$ is a smooth geometrically connected variety over a finite field $k$ with function field $K$. Let $ℓ \neq p$ be a prime. Then there is a split exact sequence of $G_k$-representations

$$0 \to H^0(\mathcal{Y}_k, \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}_ℓ \to H^1(\mathcal{Y}_k, V_ℓ\mathcal{A}) \to V_ℓH^1(\mathcal{Y}_k, \mathcal{A}) \to 0.$$

Proof. Write $K'$ for $Kk$. It suffices to show that $H^0(\mathcal{Y}_k, \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}_ℓ = A(K') \otimes_{\mathbb{Z}} \mathbb{Q}_ℓ$ is a direct summand of $H^1(\mathcal{Y}_k, V_ℓ\mathcal{A})$ as $G_k$-representations. Note that without loss of generality, we might extend $k$ or shrink $\mathcal{Y}$. By shrinking $\mathcal{Y}$, we can assume that $\mathcal{Y}$ is an open subvariety of a projective normal variety $\bar{\mathcal{Y}}$ over $k$. Let $V$ denote the smooth locus of $\bar{\mathcal{Y}}/k$. Then $\bar{\mathcal{Y}} - V$ has codimension $\geq 2$ in $\bar{\mathcal{Y}}$. Let $j$ denote the open immersion $\mathcal{Y} \hookrightarrow V$ and $\mathcal{F}$ denote $V_ℓ\mathcal{A}$ on $\mathcal{Y}$. By [Fu, Prop. 10.1.18 (iii)], $Rq^*j_*\mathcal{F}$ are constructible $\mathbb{Q}_ℓ$-sheaves for all $q \geq 0$. Set $D = V - \mathcal{Y}$, by removing a closed subset $Z$ of codimension $\geq 2$ from $V$, we might assume that $D$ is a smooth divisor and $R^1j_*\mathcal{F}$ and $j_*R^1\pi_*\mathcal{Q}_ℓ$ are lisse on $D$. There is a canonical exact sequence

$$0 \to H^1(V_k, j_*\mathcal{F}) \to H^1(\mathcal{Y}_k, \mathcal{F}) \to H^0(D_k, R^1j_*\mathcal{F}).$$

We will show that $H^0(D_k, R^1j_*\mathcal{F})$ is of weight $\geq 1$. This will imply that $H^1(V_k, j_*\mathcal{F})$ and $H^1(\mathcal{Y}_k, \mathcal{F})$ have the same generalized 1-eigenspace of the Frobenius action. Then the question will be reduced to show the splitness of

$$0 \to H^0(\mathcal{Y}_k, \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}_ℓ \to H^1(V_k, j_*V_ℓ\mathcal{A}),$$

and this will follow from Lemma 7.5 and Corollary 7.10 below.

Let $\eta$ be a generic point of $D$ and $k(\eta)$ be the residue field. Then we have

$$(R^1j_*\mathcal{F})_\eta = H^1(K^{\text{sh}}_\eta, V_ℓA).$$

Set $I = \text{Gal}(K^*/K^{\text{sh}}_\eta)$, by [Mil3, Chap I, Lem. 2.18], there is an isomorphism of $G_{k(\eta)}$-representations

$$H^1(K^{\text{sh}}_\eta, V_ℓA) \cong (V_ℓA)_I(-1).$$
Since \((V_\ell A)^{\nu}_I = (H^{1}(A_{K^*}, \mathbb{Q}_\ell)^{\nu})\), it follows
\[ H^{1}(K_{\eta}^{sh}, V_\ell A) \cong (H^{1}(A_{K^*}, \mathbb{Q}_\ell)^{\nu})(-1) \]
as \(G_{k(\eta)}\)-representations. Let \(s \in \{\eta\}\) be a closed point. By assumption, \(j_*R^1\pi_*\mathbb{Q}_\ell\) is lisse on \(D\). So \(G_{k(s)}\) acts on \(\langle j_*R^1\pi_*\mathbb{Q}_\ell \rangle = H^{1}(A_{K^*}, \mathbb{Q}_\ell)^{\nu}\). By [Del2, Cor. 1.8.9], the \(G_{k(s)}\)-action on \(H^{1}(A_{K^*}, \mathbb{Q}_\ell)^{\nu}\) is of weight \(\geq 1\). Thus, the \(G_{k(s)}\)-action on \(H^{1}(A_{K^*}, \mathbb{Q}_\ell)^{\nu}(-1)\) is of weight \(\geq 1\). It follows that \((R^1 j_*\mathcal{F})_{\tilde{\eta}}\) is of weight \(\geq 1\). Since this holds for all closed points in \(D\), it follows that \(H^0(D_k, R^1 j_*\mathcal{F})\) is of weight \(\geq 1\). This completes the proof.

\[\square\]

### 7.3 The Néron-Tate height paring and Yoneda pairing

Let \(S\) be a projective normal geometrically connected variety over a finite field \(k\) with function field \(K\). Let \(V \subseteq S\) be a regular open subscheme with \(S - V\) of codimension \(\geq 2\). Let \(A\) be an abelian variety over \(K\). Let \(U \subseteq V\) be an open dense subscheme such that \(A\) extends to an abelian scheme \(\pi : \mathcal{A} \rightarrow U\). Let \(j : U \hookrightarrow V\) be the inclusion and \(K'\) denote \(k\). The aim of this section is to show the splitness of

\[ 0 \rightarrow A(K') \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \rightarrow H^{1}(V_{k^*}, j_*V_\ell\mathcal{A}). \]

In the case that \(\dim S = 1\), \(V\) is equal to \(S\). \(j_*\mathcal{A}\) is the etale sheaf represented by a Néron model of \(A\). The claim was proved by Schneider [Sch]. In the case that \(A\) extends to an abelian scheme over a smooth projective variety \(S\) over \(k\), we have \(j_*\mathcal{A} = \mathcal{A}\). This case was proved by Keller [Kel2]. The idea is to construct compatible \(G_k\)-equivariant pairings

\[
\begin{align*}
H^0(V, \mathcal{A}^0) \otimes_{\mathbb{Z}} \mathbb{Q} & \times \text{Ext}^1_{V_{ppf}}(\mathcal{A}^0, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q} \\
H^1(V_k, V_\ell\mathcal{A}^0) & \times \lim_{\rightarrow m} \text{Ext}^1_{V_{et}}(\mathcal{A}^0[\ell^m], \mu_{\ell^m}) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_\ell \rightarrow \mathbb{Q}_\ell
\end{align*}
\]

where \(\mathcal{A}^0\) is a smooth group scheme over \(V\) with connected fibers (\(\mathcal{A}^0\) is taken to be the identity component of a Néron model of \(A\) for two special cases above) and the top pairing is left non-degenerate. The the splitness of (5) will follow from Lemma 4.1. In the following, we will explain the construction of \(\mathcal{A}^0\) and the two pairings.

#### 7.3.1 Néron model

**Lemma 7.4.** Let \(A\) be an abelian variety over a finitely generated field \(K\) of characteristic \(p > 0\). Let \(S\) be a projective normal geometrically connected variety over a finite field \(k\) with function field \(K\). Let \(S^1\) denote the set of points of codimension 1. Then there exists a regular open subscheme \(V \subseteq S\) with \(\text{codim}(S - V) \geq 2\) and a smooth commutative \(V\)-group scheme \(\mathcal{A}\) of finite type satisfying the following assumptions:
(i) $A$ is the generic fiber of $\mathcal{A}$ and the restriction of $\mathcal{A}$ to $\text{Spec} \mathcal{O}_{S,s}$ is a Néron model of $A/K$ for all $s \in S^1$.

(ii) $\mathcal{A}$ admits an open subgroup scheme $\mathcal{A}^0$ which equals to the identity component of $\mathcal{A}$ when restricted to $\text{Spec} \mathcal{O}_{S,s}$ for all $s \in S^1$ and there exists a closed reduced subscheme $Y$ of $V$ such that $\mathcal{A}$ is abelian over $V - Y$ and the quotient of $\mathcal{A}_Y$ by $\mathcal{A}_Y^0$ is a finite étale group scheme over $Y$.

(iii) All fibers of $\mathcal{A}^0 \rightarrow V$ are geometrically connected.

Proof. For the case $\dim(S) = 1$, we can take $V = S$ and take $\mathcal{A}$ to be a Néron model of $A/K$ over $S$ and $\mathcal{A}^0$ to be its identity component. In general, if $\dim(S) > 1$, the Néron model of $A$ over $S$ might not exist. The idea is that extending $A$ to a Néron model over $\text{Spec} \mathcal{O}_{S,s}$ for all $s \in S^1$, then spreads out. Since a Néron model of an abelian variety over a DVR is of finite type, therefore a Néron model of $A$ over $\text{Spec} \mathcal{O}_{S,s}$ can spread out to a smooth group scheme of finite type over an open neighborhood of $s \in S^1$. There exists a regular open dense subscheme $U$ of $S$ such that $A$ extends to an abelian scheme $\mathcal{A}_U \rightarrow U$. By [Kel1, Thm. 3.3], $\mathcal{A}_U \rightarrow U$ is a Néron model of $A$ over $U$. Since $S^1 - U$ is finite, therefore we can glue Néron models of $A$ over $\text{Spec} \mathcal{O}_{S,s}$ for all $s \in S^1 - U$ with $\mathcal{A}_U \rightarrow U$ to get a smooth commutative group scheme $\mathcal{A}$ of finite type over some open subset $V$ that contains $S^1$ satisfying the condition (i). For each $s \in S^1 - U$, by removing the closure of the complement of the identity component of $\mathcal{A}_s$, we get an open subgroup scheme $\mathcal{A}_s^0$ of $\mathcal{A}$, then shrink $V$, this subscheme becomes an open subgroup scheme. Since $\mathcal{A}_s/\mathcal{A}_s^0$ is finite étale over $k(s)$. Taking $Y = V - U$, by shrinking $V$, we will show that the quotient $\mathcal{A}_Y/\mathcal{A}_Y^0$ exists and is finite étale over $Y$. Let $s \in Y \cap S^1$. There is an exact sequence

$$0 \rightarrow \mathcal{A}_s^0 \rightarrow \mathcal{A}_s \rightarrow \mathcal{F}_s \rightarrow 0,$$

where $\mathcal{F}_s$ is a finite étale group scheme over $k(s)$. By shrinking $Y$, $\mathcal{F}_s$ extends to a finite étale group scheme $\mathcal{F}$ over $Y$ and the map $\mathcal{A}_s \rightarrow \mathcal{F}_s$ extends to a faithful flat morphism of finite type $\mathcal{A}_Y \rightarrow \mathcal{F}$. By shrinking $Y$ further, the kernel of $\mathcal{A}_Y \rightarrow \mathcal{F}$ can be identified with $\mathcal{A}_Y^0$. We get an exact sequence of fppf sheaves on $Y$

$$0 \rightarrow \mathcal{A}_Y^0 \rightarrow \mathcal{A}_Y \rightarrow \mathcal{F} \rightarrow 0.$$

Therefore, $\mathcal{A}^0 \rightarrow V$ satisfies condition (ii). Since $\mathcal{A}_s^0$ is geometrically connected for all $s \in S^1$, by shrinking $Y$, we can assume that all fibers of $\mathcal{A}_Y^0 \rightarrow Y$ are geometrically connected.

In the lemma below, we will show that the $\mathcal{A}^0$ constructed as above satisfies

$$V_\ell \mathcal{A}^0 \cong j_* V_\ell (\mathcal{A}|_U).$$

As a result, there is a canonical isomorphism

$$H^1(V_\ell, j_* V_\ell (\mathcal{A}|_U)) \cong H^1(V_\ell, V_\ell \mathcal{A}^0).$$
Lemma 7.5. Notations as above, let $U \subseteq V$ be an open dense subset such that $\mathcal{A}$ is an abelian scheme over $U$. Let $j : U \to V$ be the inclusion. Denote by $T_\ell \mathcal{A}^0$ (resp. $T_\ell \mathcal{A}$) the inverse system of $\ell$-torsion sheaves $(\mathcal{A}^0[l^n])_{n \geq 0}$ (resp. $(\mathcal{A}[l^n])_{n \geq 0}$). By shrinking $V$, there is a canonical isomorphism between $A$-R $\ell$-adic sheaves (cf. [Fu, Chap. 10])

$$T_\ell \mathcal{A}^0 \cong j_*(T_\ell \mathcal{A}|_U).$$

Proof. By the lemma below, $T_\ell \mathcal{A}^0$ is an A-R $\ell$-adic sheaf. There is an exact sequence

$$0 \to \mathcal{A}^0[l^n] \to \mathcal{A}[l^n] \to \mathcal{F}[l^n] \to 0.$$ 

Since $\mathcal{F}$ is finite etale, $T_\ell \mathcal{F}$ is A-R zero. Therefore $T_\ell \mathcal{A}^0$ is A-R isomorphic to $T_\ell \mathcal{A}$. Since $T_\ell \mathcal{A}|_U$ is a $\ell$-adic sheaf, $j_*(T_\ell \mathcal{A}|_U)$ is also A-R $\ell$-adic. Since $\mathcal{A}$ is a Néron model of $A$ when restricted to Spec $\mathcal{O}_{S,s}$ for all $s \in S^1$, we have $\mathcal{A} \cong j_*(\mathcal{A}|_U)$ over Spec $\mathcal{O}_{S,s}$. It follows

$$\mathcal{A}[l^n],s \cong (j_*(\mathcal{A}[l^n]|_U),s).$$

It follows that the kernel and cokernel of

$$T_\ell \mathcal{A} \to j_*(T_\ell \mathcal{A}|_U)$$

are supported on a closed subset of codimension $\geq 2$. Removing this closed subset from $V$, we have

$$T_\ell \mathcal{A}^0 \cong j_*(T_\ell \mathcal{A}|_U).$$

\[\square\]

Lemma 7.6. Let $S$ be a noetherian scheme and $G$ be a smooth commutative group scheme over $S$ of finite type. Let $\ell$ be a prime invertible on $S$. Then the inverse system of $\ell$-torsion sheaves $T_\ell G = (G[l^n])_{n \geq 1}$ is A-R $\ell$-adic. If all fibers of $G/S$ are geometrically connected, then

$$G \to G$$

is surjective as a morphism between etale sheaves on $S$.

Proof. Firstly, note that $G[l^n]$ is etale and of finite type over $S$. Thus the etale sheaf represented by $G[l^n]$ is constructible.

Secondly, let $f : S_1 \to S$ be a surjective between noetherian schemes. If the statement holds for $G_{S_1}/S_1$, then it holds for $G/S$. Set $\mathcal{F}_n = G[l^n]$ and $\mathcal{F} = (\mathcal{F}_n)$. Since $f^*$ is an exact functor, one can show that if $f^* \mathcal{F}$ satisfies the criterion of [Fu, Prop. 10.1.1], so does $\mathcal{F}$.

Thirdly, we might assume that $S$ is integral. By noetherian induction, it suffices to prove the statement for an open sense subscheme of $S$. Let $K$ denote the function field of $S$. $G_K$ is smooth and of finite type over $K$. Let $G^0_K$ be the identity component of $G_K$, then there is an exact sequence

$$0 \to G^0_K \to G_K \to \pi_0(G) \to 0,$$
where \( \pi_0(G) \) is a finite etale group scheme. By Chevalley’s theorem, \( G^0_K \) is an extension of an abelian variety \( A \) by \( \mathbb{G}_m \times \mathbb{G}_a \) i.e.

\[
0 \rightarrow \mathbb{G}_m \times \mathbb{G}_a \rightarrow G^0_K \rightarrow A \rightarrow 0.
\]

The above exact sequences can descend to a finite extension \( L \) of \( K \). Choose a model \( S_1 \) for \( L \). By shrinking \( S_1 \) and \( S \), we can assume that there exists a flat surjective morphism \( S_1 \rightarrow S \) of finite type whose function field extension corresponds to \( L/K \) and exact sequences

\[
0 \rightarrow G^0_{S_1} \rightarrow G_{S_1} \rightarrow \pi_0(G) \rightarrow 0
\]

and

\[
0 \rightarrow \mathbb{G}_m^{r} \times \mathbb{G}_a^{s} \rightarrow G^0_{S_1} \rightarrow \mathcal{A}_{S_1} \rightarrow 0,
\]

where \( \pi_0(G) \) is finite etale over \( S_1 \) and \( \mathcal{A}_{S_1} \) is an abelian scheme. We get exact sequences of systems of \( \ell \)-torsion sheaves

\[
0 \rightarrow \mathcal{H}_S \rightarrow \mathcal{H}_{S_1} \rightarrow \pi_0(G) \rightarrow 0
\]

and

\[
0 \rightarrow \mathcal{H}_S \rightarrow \mathcal{H}_{S_1} \rightarrow \mathcal{A}_{S_1} \rightarrow 0.
\]

Since \( \mathcal{H}_S = \mathcal{H}_{S_1} \) are \( \ell \)-adic sheaves, by [Fu, Prop. 10.1.7(iii)], \( \mathcal{H}_{S_1} \) is A-R \( \ell \)-adic. It follows that \( \mathcal{H}_S \) is A-R \( \ell \)-adic on \( S \). This proves the first claim.

For the second claim, we might assume \( S = \text{Spec} \ R \) where \( R \) is a strict local ring with residue field \( k \). It suffices to show that \( G(S) \) is \( \ell \)-divisible. Let \( \tau : S \rightarrow G \) be a section. \( \ell - 1(\tau) \) is etale over \( S \). \( \ell - 1(\tau)(k) \) is not empty since \( G(k) \) is \( \ell \) divisible. By Hensel’s Lemma, it follows that \( \ell - 1(\tau)(S) \) is not empty. Thus, \( G(S) \) is \( \ell \) divisible.

7.3.2 Yoneda pairing

Following notations in [Sch, §2], let \( V_{\text{fppf}} \) (resp. \( (\ell^n) - \text{V_{et}} \)) denote the category of fppf-sheaves (resp. etale sheaves of \( \mathbb{Z}/\ell^n\mathbb{Z} \)-modules) on \( V \).

**Lemma 7.7.** Notations as before, assuming that \( V \) is regular, then there is a commutative diagram

\[
\begin{array}{ccc}
H^0(V, \mathcal{A}^0) & \times & \text{Ext}^1_{V_{\text{fppf}}} (\mathcal{A}^0, \mathbb{G}_m) \\
\downarrow \delta & & \downarrow r_n \\
H^1(V, \mathcal{A}^0[\ell^n]) & \times & \text{Ext}^1_{(\ell^n) - \text{V_{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \\
\end{array} \rightarrow H^2(V, \mu_{\ell^n})
\]

where \( \delta \) is induced by the exact sequence \( 0 \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^0 \rightarrow 0 \) and \( r_n \) is defined by

\[
(0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{X} \rightarrow \mathcal{A}^0 \rightarrow 0) \leftrightarrow (0 \rightarrow \mu_{\ell^n} \rightarrow \mathcal{X}^{[\ell^n]} \rightarrow \mathcal{A}^0[\ell^n] \rightarrow 0).
\]
The natural map

\[ \text{Ext}^1_{\mathcal{V}_{	ext{fppt}}} (\mathcal{A}^0[\ell^n], \mathbb{G}_m) \rightarrow \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \]

defined by

\[ (0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{X} \rightarrow \mathcal{A}^0[\ell^n] \rightarrow 0) \mapsto (0 \rightarrow \mu_{\ell^n} \rightarrow \mathcal{X}[\ell^n] \rightarrow \mathcal{A}^0[\ell^n] \rightarrow 0) \]

is an isomorphism. And it induces a natural map

\[ \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^{n+1}], \mu_{\ell^{n+1}}) \rightarrow \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \]

compatible with pairings in the above diagram. As a result, by taking limit, there is a commutative diagram

\[
\begin{array}{ccc}
H^0(V, \mathcal{A}^0) & \times & \text{Ext}^1_{\mathcal{V}_{	ext{fppt}}} (\mathcal{A}^0, \mathbb{G}_m) \\
\downarrow & & \downarrow \text{lim}_{\ell^n} \\
H^1(V, T_\ell(\mathcal{A}^0)) & \times & \text{lim}_{\ell^n} \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \\
& & \downarrow \\
& & H^2(V, \mathbb{Z}_\ell(1))
\end{array}
\]

Proof. The two pairings in the first diagram are defined as Yoneda product of extensions and the commutativity of the diagram can be checked explicitly. We will construct a natural map

\[ \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \rightarrow \text{Ext}^1_{\mathcal{V}_{	ext{fppt}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \]

which induces a map

\[ \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \rightarrow \text{Ext}^1_{\mathcal{V}_{	ext{fppt}}} (\mathcal{A}^0[\ell^n], \mathbb{G}_m). \]

And it is easy to check the induced map is the inverse of (6) by definition. Let \((0 \rightarrow \mu_{\ell^n} \rightarrow \mathcal{Y} \rightarrow \mathcal{A}^0[\ell^n] \rightarrow 0) \in \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n})\). \(\mathcal{Y}\) may be regarded as a \(\mu_{\ell^n}\)-torsor over \(\mathcal{A}^0[\ell^n]\) and so is representable by an etale scheme of finite type over \(V\) (cf. [Mil81, Chap. III, Thm. 4.3]). Therefore the exact sequence can be regarded as an exact sequence of fppt sheaves on \(V\) which gives an element in \(\text{Ext}^1_{\mathcal{V}_{\text{fppt}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n})\). This defines a map

\[ \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \rightarrow \text{Ext}^1_{\mathcal{V}_{\text{fppt}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}). \]

By composing with the natural map

\[ \text{Ext}^1_{\mathcal{V}_{\text{fppt}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \rightarrow \text{Ext}^1_{\mathcal{V}_{\text{fppt}}} (\mathcal{A}^0[\ell^n], \mathbb{G}_m), \]

we get the desired map. Define

\[ \text{Ext}^1_{\mathcal{V}_{\text{fppt}}} (\mathcal{A}^0[\ell^{n+1}], \mathbb{G}_m) \rightarrow \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \]

as

\[ (0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{X} \rightarrow \mathcal{A}^0[\ell^{n+1}] \rightarrow 0) \mapsto (0 \rightarrow \mu_{\ell^n} \rightarrow \mathcal{X}[\ell^n] \rightarrow \mathcal{A}^0[\ell^n] \rightarrow 0). \]

Then the composition

\[ \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^{n+1}], \mu_{\ell^{n+1}}) \rightarrow \text{Ext}^1_{\mathcal{V}_{\text{fppt}}} (\mathcal{A}^0[\ell^{n+1}], \mathbb{G}_m) \rightarrow \text{Ext}^1_{(\ell^n) - \mathcal{V}_{\text{et}}} (\mathcal{A}^0[\ell^n], \mu_{\ell^n}) \]

gives a map compatible with the first diagram.
7.3.3 Compatibility between the height pairing and Yoneda paring

**Theorem 7.8.** Notations as before. Let \( S \) be a projective normal geometrically connected variety over a finite field \( k \). Fix a closed \( k \)-immersion \( \iota : S \to \mathbb{P}^m_k \). Let \( V \) be a regular open subscheme of \( S \) with \( \operatorname{codim}(S - V) \geq 2 \) satisfying conditions in the lemma above. Define a degree map \( \operatorname{Pic}(S) \to \mathbb{Z} \) by sending a prime Weil divisor to its degree as a subvariety in \( \mathbb{P}^m_k \). Composing it with the Yoneda pairing in the lemma above, we get a pairing

\[ H^0(V, \mathcal{O}) \times \operatorname{Ext}^1_{\mathcal{V}_{\text{fppf}}}(\mathcal{O}, \mathbb{G}_m) \to \mathbb{Z}. \tag{7} \]

There is a natural inclusion \( H^0(V, \mathcal{O}) \to A(K) \) and a natural map \( \operatorname{Ext}^1_{\mathcal{V}_{\text{fppf}}}(\mathcal{O}, \mathbb{G}_m) \to \operatorname{Ext}^1_K(A, \mathbb{G}_m) = \tilde{A}(K) \), where \( \tilde{A} \) is the dual abelian variety of \( A \). Then, up to a normalizing constant, the above pairing is compatible with the Néron-Tate height pairing defined in [Con, Cor. 9.17]

\[ A(K) \times \tilde{A}(K) \to \mathbb{R}. \]

**Remark 7.9.** This is a generalization of results in [Kel2, §3] and [Sch, §3]. The proof is just an imitation of arguments in [Blo, §2], [Sch, §3] and [Kel2, §3]. As a consequence, the pairing (7) is left non-degenerate after tensoring \( \mathbb{Q} \).

**Proof.** Firstly, we define the adele ring associated to \( S \) (cf.[Kel2, §3.1]. Define for \( T \subset S^1 \) finite the \( T \)-adele ring of \( S \) as the restricted product

\[ \mathcal{A}_{K,T} = \prod_{s \in T} K_s \times \prod_{s \in S^1 \setminus T} \hat{\mathcal{O}}_{S,s}, \]

where \( \hat{\mathcal{O}}_{S,s} \) is the completion of the local ring \( \mathcal{O}_{S,s} \) and \( K_s \) is the quotient field of \( \hat{\mathcal{O}}_{S,s} \), and the adele ring of \( S \) as

\[ \mathcal{A}_K = \lim_{T \subset S^1} \mathcal{A}_{K,T}. \]

For each \( s \in S^1 \), we define \( \operatorname{deg}(s) \) as the degree of the close subvariety \( \{s\} \) in \( \mathbb{P}^m_k \). This gives an absolute value on \( K_s \) i.e. \( | \cdot |_s = q^{-\operatorname{deg}(s),v_s(\cdot)} \), where \( q \) denotes the cardinality of \( k \). Since a principle Weil divisor has degree zero, thus the absolute value \( | \cdot |_s \) satisfies product formula. We call the field \( K \) equipped with the set of absolute values \( | \cdot |_s \) a generalized global field(cf.[Con, Def. 8.1]).

Secondly, we describe the first pairing explicitly. Let \( \tilde{a} = (0 \to \mathbb{G}_m \to \mathcal{X} \to \mathcal{O} \to 0) \in \operatorname{Ext}^1(\mathcal{O}, \mathbb{G}_m) \). One can think \( \mathcal{X} \) as a \( \mathbb{G}_m \)-torsor on \( \mathcal{O} \) and so it is representable by a smooth commutative \( S \)-group scheme of finite type (cf. [Mil1, Chap III, Thm. 4.3]). By Hilbert’s theorem 90, there are exact sequences

\[ 0 \to \mathbb{G}_m(K) \to \mathcal{X}(K) \to A(K) \to 0 \]

and

\[ 0 \to \mathbb{G}_m(A_K) \to \mathcal{X}(A_K) \to \mathcal{O}(A_K) \to 0. \]

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There is a natural homomorphism, the logarithmic modulus map,

\[ l : \mathbb{G}_m(A_K) \longrightarrow \log q \cdot \mathbb{Z} \subseteq \mathbb{R}, (a_s) \mapsto \sum \log |a_s|_s = -\log q \cdot \sum \deg(s) \cdot v_s(a_s). \]

By the product formula, \( l(\mathbb{G}_m(K)) = 0 \). Define \( G^1_m \) as the kernel of \( l \), and \( \mathcal{X}^{-1} \) as

\[ \mathcal{X}^{-1} = \{ a \in \mathcal{X}(A_K) : \exists n \in \mathbb{Z}_{\geq 1}, na \in \mathcal{X}^{-1}(A_K) \}, \]

the rational saturation of \( \mathcal{X}^{-1} \) with

\[ \mathcal{X}^{-1} = G^1_m \cdot \prod_{s \in S_1} \mathcal{X}(\hat{\mathcal{O}}_{S,s}) \subseteq \mathcal{X}(A_K). \]

Then there exists a unique extension \( l_\hat{a} : \mathcal{X}(A_K) \rightarrow \mathbb{R} \) of \( l \) vanishing on \( \mathcal{X}^{-1} \) (cf. [Kel2, Lem. 3.1.4] or [Blo, Lem. 1.8]). It induces by restriction to \( \mathcal{X}(K) \) a homomorphism

\[ l_\hat{a} : A(K) \longrightarrow \mathbb{R}. \]

Next, we will show that for any \( a \in \mathcal{A}^0(S) \subseteq A(K) \), we have

\[ l_\hat{a}(a) = -\log q \cdot \deg(a \vee \tilde{a}), \]

where \( \vee \) denotes the Yoneda pairing \( H^0(V, \mathcal{A}^0) \times \text{Ext}_V^1(\mathcal{A}^0, \mathbb{G}_m) \rightarrow H^1(V, \mathbb{G}_m) \). By definition, \( a \vee \tilde{a} \) is defined by the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{G}_m & \rightarrow & \mathcal{Y} & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
& & \downarrow{id} & & \downarrow{a} & & \downarrow{a} & & \\
0 & \rightarrow & \mathbb{G}_m & \rightarrow & \mathcal{X} & \rightarrow & \mathcal{A}^0 & \rightarrow & 0 \\
\end{array}
\]

By composition, one gets an extension

\[ l_{a \vee \tilde{a}} : \mathcal{Y}(A_K) \longrightarrow \mathcal{X}(A_K) \xrightarrow{l_\hat{a}} \mathbb{R} \]

of \( l : \mathbb{G}_m(A_K) \rightarrow \mathbb{R} \) to \( \mathcal{Y}(A_K) \), which induces because of \( l(\mathbb{G}_m(K)) = 0 \) in the exact sequence \( a \vee \tilde{a} \) by restriction to \( \mathcal{Y}(K) \) a homomorphism

\[ l_{a \vee \tilde{a}} : \mathbb{Z} \xrightarrow{a} A(K) \xrightarrow{l_\hat{a}} \mathbb{R}, \]

so one obviously has \( l_\hat{a}(a) = l_{a \vee \tilde{a}}(1) \). Since

\[ l_\hat{a}(\prod_{s \in S_1} \mathcal{X}(\hat{\mathcal{O}}_{S,s})) = 0, \]

hence

\[ l_{a \vee \tilde{a}}(\prod_{s \in S_1} \mathcal{Y}(\hat{\mathcal{O}}_{S,s})) = 0. \]
Set $e = a \lor \tilde{a}$, $e$ is represented by an exact sequence of fpf sheaves on $V$

$$0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{Y} \rightarrow \mathbb{Z} \rightarrow 0,$$

and $l_\mathcal{Y}$ is an extension of $l$ and vanishes on $\prod_{s \in s^1} \mathcal{Y}(\hat{\mathcal{O}}_{S,s})$. By [Kel2, Lem 3.3.2] replacing $X$ by $V$ in his argument) or [Sch, Lem. 12],

$$l_\mathcal{Y}(a) = l_\mathcal{Y}(1) = -\log q \cdot \deg(e).$$

Thirdly, we will show that the pairing $h(a, \tilde{a}) := l_\mathcal{Y}(a)$ can be written as a sum of the local Néron pairings which therefore coincides with the canonical Néron-Tate height pairing by [BG, p.307, Cor. 9.5.14]. The proof for our case is exactly same as the argument in [Blo, §2] and [Kel2, §3.4].

Given an extension $(0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{X} \rightarrow \mathcal{A}^0 \rightarrow 0) \in \text{Ext}^1_{\text{fppf}}(\mathcal{A}^0, \mathbb{G}_m)$, its restriction to Spec $\mathcal{O}_{S,a}$ for $s \in S^1$ is still an extension. Since $\mathcal{O}_{S,a}$ is a Dedekind domain, by [MW, p.53, Lem. 5.1], the push-forward of the sheaf

$$\mathcal{E} = \mathcal{E}_1^{(\text{Spec } \mathcal{O}_{S,a})_{\text{fppf}}} \otimes (\mathcal{A}^0, \mathbb{G}_m)$$

to the smooth site of Spec $\mathcal{O}_{S,a}$ is represented by the Néron model of $\bar{A} := \text{Pic}^0_{A/K}$ over Spec $\mathcal{O}_{S,a}$. By [Sch, Lem. 9], $\mathscr{H}om_{(\text{Spec } \mathcal{O}_{S,a})_{\text{fppf}}} (\mathcal{A}^0, \mathbb{G}_m) = 0$, it follows that

$$\bar{A}(K) = H^0(\hat{\mathcal{O}}_{S,a}, \mathcal{E} = \mathcal{E}_1^{(\text{Spec } \mathcal{O}_{S,a})_{\text{fppf}}} \otimes (\mathcal{A}^0, \mathbb{G}_m)) = \text{Ext}^1_{(\text{Spec } \mathcal{O}_{S,a})_{\text{fppf}}} (\mathcal{A}^0, \mathbb{G}_m).$$

Therefore, the restriction of $(0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{X} \rightarrow \mathcal{A}^0 \rightarrow 0)$ to Spec $\hat{\mathcal{O}}_{S,a}$ gives an element in $\bar{A}(K)$. Then one can define the local Néron pairings as $[\bar{A}, (2.6)]$ for each $s \in S^1$. By [Blo, Thm. 2.7] or [Kel2, Thm. 3.4.2], $h(\cdot, \cdot)$ is equal to the sum of the local Néron pairings. This completes the proof of the theorem.

**Corollary 7.10.** Notations as before. Set $d = \text{dim}(V)$ and let $K'$ denote $K\bar{k}$. Then, one can shrink $V$ (with Codim $(S - V) \geq 2$ ), such that

$$H^0(V_k, \mathcal{A}^0) \otimes \mathbb{Q}_\ell = A(K') \otimes \mathbb{Q}_\ell$$

and the injective map of $G_k$-representations

$$H^0(V_k, \mathcal{A}^0) \otimes \mathbb{Q}_\ell \rightarrow H^1(V_k, \mathcal{A}^0)$$

is split.

**Proof.** By the Lang-Néron theorem [Con, Thm.7.1], the quotient group $A(K')/\text{Tr}_{K/k}(A)\bar{k}$ is finitely generated where $\text{Tr}_{K/k}(A)$ is a $K/k$-trace of $A/K$. Since $\text{Tr}_{K/k}(A)\bar{k}$ is a $\ell$-divisible torsion group, we have that

$$A(K') \otimes \mathbb{Q}_\ell = A(K') \otimes \mathbb{Q}_\ell$$

has a finite dimension over $\mathbb{Q}_\ell$. Thus, there exists a finite extension $l/k$ such that $A(Kl) \otimes \mathbb{Q}_\ell = A(K') \otimes \mathbb{Q}_\ell$. Replacing $k$ by $l$, we might assume $A(K) \otimes \mathbb{Q}_\ell = A(K') \otimes \mathbb{Q}_\ell$. Since
A(K) is finitely generated, by shrinking V, we can assume \( \mathcal{A}(V) = A(K) \). Shrinking V further, since \( \mathcal{A}(V)/\mathcal{A}(0)V \) is finite, hence \( \mathcal{A}(0)V \otimes \mathbb{Q}_\ell = A(K) \otimes \mathbb{Q}_\ell \). Assume that \( \tilde{A}(K) \) is generated by finitely many elements \( \tilde{a}_i \). Since \( \tilde{A}(K) = \text{Ext}^1_{(\text{Spec } S, K)}(\mathcal{A}, \mathbb{G}_m) \)

for any \( s \in S^1 \), so for a given \( \tilde{a} \in \tilde{A}(K) \), it corresponds to an extension \( \tilde{a}_s : (0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{X} \rightarrow \mathcal{A} \rightarrow 0) \) over Spec \( OS,s \) whose restriction to Spec \( K \) is independent of \( s \). One can think \( X \) as a \( \mathbb{G}_m \)-torsor on \( A \) and so it is representable by a smooth commutative group scheme of finite type over Spec \( OS,s \) (cf. [Mil1, Chap III, Thm. 4.3]). Therefore, by shrinking V, one can glue \( \tilde{a}_s \) to get an extension \( (0 \rightarrow \mathbb{G}_m \rightarrow X \rightarrow A \rightarrow 0) \) over \( V \). Thus, by shrinking V, we can assume that the natural map

\[
\text{Ext}^1_{V_{\text{fpf}}}(\mathcal{A}^0, \mathbb{G}_m) \rightarrow \text{Ext}^1_K(A, \mathbb{G}_m) = \tilde{A}(K)
\]

is surjective. There is a commutative diagram

\[
\begin{array}{ccc}
H^0(V, \mathcal{A}^0) & \times & \text{Ext}^1_{V_{\text{fpf}}}(\mathcal{A}^0, \mathbb{G}_m) \\
\downarrow \delta & & \downarrow \lim_{\text{et}} \\
H^1(V_k, \mathcal{A}^0) & \times & \lim_{\eta \rightarrow n} \text{Ext}^1_{(\mathcal{A}^0[\ell^n], \mu_{\ell^n})}(\mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell \\
\downarrow & & \downarrow \\
& & H^2(V_k, \mathbb{Q}_\ell(1)) \\
\end{array}
\]

where \( \eta \) denotes the cycle class of \( OS,1 \) and \( \text{tr} \) denotes the trace map \( H^{2d}(S_k, \mathbb{Q}_\ell(d)) \simeq \mathbb{Q}_\ell \). The commutativity of the second square follows from the definition of degree. By Lemma 7.7, the first square commutes. By Theorem 7.8, the top pairing is compatible with the Néron-Tate pairing which is non-degenerate after tensoring \( \mathbb{Q} \) (cf. [Con, Cor. 9.17]). Assuming the surjectivity of \( \text{Ext}^1_{V_{\text{fpf}}}(\mathcal{A}^0, \mathbb{G}_m) \rightarrow \text{Ext}^1_K(A, \mathbb{G}_m) = \tilde{A}(K) \), then the top pairing is left non-degenerate after tensoring \( \mathbb{Q}_\ell \). By Lemma 4.1, the injective map of \( G_k \)-representations

\[
H^0(V_k, \mathcal{A}^0) \otimes \mathbb{Q}_\ell \rightarrow H^1(V_k, \mathcal{A}^0)
\]

is split.

\[\square\]

8 Applications

8.1 Proof of Corollary 1.13

Let \( \pi : \mathcal{X} \rightarrow \mathcal{Y} \) be a dominant morphism between smooth geometrically connected varieties over a finite field \( k \). Let \( K \) be the function field of \( \mathcal{Y} \). Assuming that the generic fiber \( X \) of \( \pi \) is smooth projective geometrically irreducible over \( K \), then Conjecture 1.5 for \( \mathcal{X} \) is equivalent to

Conjecture 1.5 for \( \mathcal{Y} + \text{BSD for Pic}^0_{X/K} + \text{Conjecture 1.4 for } X \) for \( i = 1 \).
Proof. By Theorem 3.5, Conjecture 1.5 for $\mathcal{X}$ is equivalent to the vanishing of $V_{\ell}\text{Br}(\mathcal{X}_k)^{G_k}$ which is equivalent to the vanishing of $V_{\ell}\text{Br}(\mathcal{Y}_k)^{G_k}$, $V_{\ell}\text{III}_K(\text{Pic}^0_{\mathcal{X}/K})$ and $V_{\ell}\text{Br}(\mathcal{X}_K)^{G_K}$ by Theorem 6.2. By Corollary 1.7, Theorem 1.9 and Theorem 3.5, this is equivalent to

\begin{equation*}
\text{Conjecture 1.5 for } \mathcal{Y} + \text{BSD for } \text{Pic}^0_{\mathcal{X}/K} + \text{Conjecture 1.4 for } \mathcal{X} \text{ for } i = 1
\end{equation*}

\hfill \square

8.2 Proof of Corollary 1.15

Proof. Assume that $T^1(X, \ell)$ holds for all smooth projective varieties $X$ over finite fields of characteristic $p$. Let $X$ be a smooth projective variety over a finitely generated field $K$ of characteristic $p$. We can spread out $X/K$ to get a smooth projective morphism $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ as in Theorem 6.2. By Theorem 3.5, Conjecture 1.5 holds for $\mathcal{X}$. By Corollary 1.13, we have Conjecture 1.1 for $i = 1$ holds for $X$, Conjecture 1.3 holds for $\text{Pic}^0_{\mathcal{X}/K}$, Conjecture 1.4 for $i = 1$ holds for $X$ by Theorem 1.6.

\hfill \square

8.3 The Picard number of Abelian varieties and K3 surfaces

Proposition 8.1. Let $X$ be an abelian variety (resp. a K3 surface) over a finitely generated field $K$ of characteristic $p > 0$ (resp. $p > 2$). Let $m$ be the transcendence degree of $K$ over its prime field. Then

\begin{equation*}
\text{rank NS}(X) = -\text{ord}_{s=m+1} \Phi_2(s).
\end{equation*}

Proof. $T^1(X, \ell)$ holds for abelian varieties over finitely generated fields of positive characteristic (cf.[Zar]) and for K3 surfaces over finitely generated fields of characteristic $p > 2$ (cf. [Cha] or [MP]). So the claim follows from Corollary 1.7.

\hfill \square

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