Game-theoretic Investigation of Intensional Equalities

Norihiro Yamada
norihiro1988@gmail.com
University of Oxford
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Abstract

This article presents a new game semantics for Martin-Löf type theory (MLTT), in which each game is equipped with selected isomorphism strategies that represent (computational) proofs for (intensional) equality between strategies on the game. These isomorphism strategies interpret propositional equalities in MLTT. As a main result, we have obtained a first game semantics for MLTT that refutes the principle of uniqueness of identity proofs (UIP) and validates univalence axiom (UA) though it does not model non-trivial higher equalities. Categorically, our model forms a substructure of the classic groupoid model of MLTT by Hofmann and Streicher. Similarly to the path from the groupoid model to the \( \omega \)-groupoid model, we are planning to generalize the game semantics to give rise to an \( \omega \)-groupoid structure to interpret non-trivial higher equalities in homotopy type theory (HoTT).

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1 Introduction

2 Games with Equalities

- Convention. The words games and strategies refer to predicative games and strategies on them, respectively, defined in Yam16.
- Notation. Given a game $G$, we write $\text{st}(G)$ for the set of all strategies on $G$.
- Notation. We write $A \models B$ for the subgame of $A \models B$ whose strategies are all isomorphisms.

2.1 Games with Equalities

- Theorem 2.1.1 (Isomorphism theorem). If $\phi : A \sim B$ or $\phi : A \models B$, then $P_A \cong P_B$.

Proof. Assume that $\phi : A \sim B$; the case $\phi : A \models B$ is analogous, and so we omit it. It is straightforward to see that $\phi$ and the inverse $\phi^{-1} : B \to A$ both behave like copy-cat strategies in the sense that $sb \in P_\phi^{\text{odd}}$ (resp. $sb \in P_\phi^{\text{odd}}$) with $b \in M_B$ implies $sba \in P_\phi$ (resp. $sba \in P_\phi$) for some $a \in M_A$ and $ta' \in P_\phi^{\text{odd}}$ (resp. $ta' \in P_\phi^{\text{odd}}$) with $a' \in M_A$ implies $ta'b' \in P_\phi$ (resp. $ta'b' \in P_\phi$) for some $b' \in M_B$ (since otherwise either $\phi \circ \phi^{-1}$ or $\phi^{-1} \circ \phi$ would not be a copy-cat strategy, a contradiction). Hence, we may define the function $\mathcal{F}(\phi) : P_A \to P_B$ by:

$$
\begin{align*}
\mathcal{F}(\phi) & : P_A \to P_B \\
\epsilon & \mapsto \epsilon \\
a_1a_2 \ldots a_{2n+1} & \mapsto b_1b_2 \ldots b_{2n+1} \\
 & \quad \text{(if } a_1a_2 \ldots a_{2n} \mapsto b_1b_2 \ldots b_{2n} \land a_1b_1b_2a_2 \ldots a_{2n-1}b_{2n-1}b_{2n}a_{2n+1}b_{2n+1} \in P_\phi^{-1}) \\
a_1a_2 \ldots a_{2n+2} & \mapsto b_1b_2 \ldots b_{2n+2} \\
 & \quad \text{(if } a_1a_2 \ldots a_{2n+1} \mapsto b_1b_2 \ldots b_{2n+1} \land b_1a_1a_2b_2 \ldots b_{2n+1}a_{2n+1}a_{2n+2}b_{2n+2} \in P_\phi). 
\end{align*}
$$

Similarly, we may define another function $\mathcal{G}(\phi) : P_B \to P_A$ by:

$$
\begin{align*}
\mathcal{G}(\phi) & : P_B \to P_A \\
\epsilon & \mapsto \epsilon \\
b_1b_2 \ldots b_{2n+1} & \mapsto a_1a_2 \ldots a_{2n+1} \\
 & \quad \text{(if } b_1b_2 \ldots b_{2n} \mapsto a_1a_2 \ldots a_{2n} \land b_1a_1a_2b_2 \ldots b_{2n-1}a_{2n-1}b_{2n}b_{2n+1}a_{2n+1} \in P_\phi) \\
b_1b_2 \ldots b_{2n+2} & \mapsto a_1a_2 \ldots a_{2n+2} \\
 & \quad \text{(if } b_1b_2 \ldots b_{2n+1} \mapsto a_1a_2 \ldots a_{2n+1} \land a_1b_1b_2a_2 \ldots a_{2n+1}b_{2n+1}a_{2n+2}b_{2n+2} \in P_\phi^{-1}). 
\end{align*}
$$

By induction on the length of input, it is easy to see that $\mathcal{F}(\phi)$ and $\mathcal{G}(\phi)$ are mutually inverses (and so we write $\mathcal{F}(\phi)^{-1}$ for $\mathcal{G}(\phi)$), completing the proof. $\blacksquare$

It is also clear from the proof of Theorem 2.1.1 that $\mathcal{F}(\phi^{-1}) = \mathcal{G}(\phi)^{-1}$ and $|\mathcal{F}(\phi)(s)| = |s|$ for all $s \in P_A$. This result clarifies the fact that isomorphic games are “essentially the same up to concrete implementation of positions”. Therefore it makes sense to regard isomorphism strategies between games (resp. strategies) as equivalences between the games (resp. strategies). This is the main idea behind the following definition:

- Definition 2.1.2 (GwEs). A predicative game with equality (GwE) is a groupoid whose objects are strategies on a fixed game $G$ and morphisms are selected strategies on $G \models G$.

- Notation. We usually specify a GwE by a pair $G = (G, =_G)$ of an underlying game $G$ and an assignment $=_G$ of a subgame $\sigma_1 =_G \sigma_2 \models G \models G$ to each pair $\sigma_1, \sigma_2 : G$, intended to mean $\text{ob}(G) \models \text{st}(G), G(\sigma_1, \sigma_2) \models \text{st}(\sigma_1 =_G \sigma_2)$. We usually write $\rho : \sigma_1 =_G \sigma_2$ for $\rho \in G(\sigma_1, \sigma_2)$. 

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 Remark. One may wonder if the subgame relation \( \sigma_1 =_G \sigma_2 \leq \sigma_1 \Rightarrow \sigma_2 \) is rather correct. However, it would not be general enough to define equalities in \( \Sigma \), \( \prod \)-spaces as we shall see.

**Definition 2.1.3** (Equality-preservation). A strategy \( \phi : A \Rightarrow B \) between GwEs \( A \) and \( B \) is **equality-preserving** if it is equipped with another strategy \( \phi^n : (A \Rightarrow A) \Rightarrow (B \Rightarrow B) \), called its **equality-preservation**, such that the maps \( (\sigma : A) \mapsto \phi \bullet \sigma \), \( (\rho : \sigma =_A \sigma') \mapsto \phi^n \bullet \rho \) form a functor \( \text{fun}(\phi) : (A, =_A) \Rightarrow (B, =_B) \).

Explicitly, an equality-preserving strategy \( \phi : A \Rightarrow B \) is a pair \( \phi = (\phi, \phi^n) \) of strategies \( \phi : A \Rightarrow B, \phi^n : (A \Rightarrow A) \Rightarrow (B \Rightarrow B) \) that satisfies the following three conditions:

1. \( \phi^n \bullet \rho_1 : \phi \bullet \sigma_1 =_B \phi \bullet \sigma_2 \)
2. \( \phi^n \bullet (\rho_2 \bullet \rho_1) = (\phi^n \bullet \rho_2) \bullet (\phi^n \bullet \rho_1) \)
3. \( \phi^n \bullet \text{id}_{\sigma_1} = \text{id}_{\phi \bullet \sigma_1} \)

for all \( \sigma_1, \sigma_2, \sigma_3 : A, \rho_1 : \sigma_1 =_A \sigma_2, \rho_2 : \sigma_2 =_A \sigma_3 \).

**Definition 2.1.4** (The category \( \mathcal{PG} \)). The category \( \mathcal{PG} \) of (predicative) games with equality is defined as follows:

- Objects are GwEs
- Morphisms \( A \to B \) are equality-preserving strategies \( \phi : A \Rightarrow B \)
- The composition \( \psi \bullet \phi : A \to C \) of morphisms \( \phi : A \to B, \psi : B \to C \) is defined by \( \psi \bullet \phi \overset{\text{df}}{=} \psi \circ \phi^1 \) for the object-map and \( (\psi \bullet \phi)^n \overset{\text{df}}{=} \psi^n \circ \phi^n \) for the arrow-map
- The identity \( \text{id}_A \) on each object \( A \) is the dereliction \( \text{der}_A : A \Rightarrow A \) equipped with the equality-preservation \( \text{der}_A^n = \text{id} \). 

**Proposition 2.1.5** (Well-defined \( \mathcal{PG} \)). The structure \( \mathcal{PG} \) forms a well-defined category.

Proof. First, the identities clearly satisfy the unit law. For the associativity of the composition, let \( \phi : A \to B, \psi : B \to C, \varphi : C \to D \) be morphisms in \( \mathcal{PG} \). Note that \( \varphi \bullet (\psi \bullet \phi) = (\varphi \bullet \psi) \bullet \phi \) has been established in the literature; see [AM99, McC98, Yam16] for the proof. From this, the equality between the equality-preservation immediately follows: \( (\varphi \bullet (\psi \bullet \phi))^n = \varphi^n \bullet (\psi \bullet \phi)^n = \varphi^n \bullet (\psi^n \bullet \phi^n) = (\varphi^n \bullet \psi^n) \bullet \phi^n = (\varphi \bullet (\psi^n \bullet \phi^n))^n \), which completes the proof. 

Seeing morphisms in \( \mathcal{PG} \) extensionally as functors, the composition \( \bullet \) in \( \mathcal{PG} \) corresponds to the composition of functors. More precisely, if \( \phi : A \Rightarrow B, \psi : B \Rightarrow C \) in \( \mathcal{PG} \), then we have \( \text{fun}(\psi \bullet \phi) = \text{fun}(\psi) \circ \text{fun}(\phi) \).

**Notation.** Let us write \( \downarrow : \mathcal{PG} \to \mathcal{WP} \) for the obvious forgetful functor, i.e., \( [(A, =_A)] \overset{\text{df}}{=} A \), \( [\phi, \phi^n] \overset{\text{df}}{=} \phi \) for all GwEs \( (A, =_A) \) and equality-preserving strategies \( (\phi, \phi^n) \).

### 2.2 Dependent Games with Equalities

**Definition 2.2.1** (DGwEs). A **dependent game with equality** (DGwE) over a GwE \( A \) is any functor \( B : A \to \mathcal{PG} \) that is uniform:

\[
\text{sab} \in B(\rho) \iff \text{sab} \in B(\tilde{\rho}) \\
\text{tmm} \in B(\rho)^m \iff \text{tmn} \in B(\tilde{\rho})^m
\]

for all \( \rho : \gamma =_? \tilde{\rho}, \gamma =_? \tilde{\gamma}, \text{sab} \in B(\rho) \cap B(\tilde{\rho}), \text{sab} \in \cup B \Rightarrow \cup B, \text{tm} \in B(\rho)^m \cap B(\tilde{\rho})^m, \text{tmn} \in (\cup B \Rightarrow \cup B) \Rightarrow (\cup B \Rightarrow \cup B) \).
Notation. For any $A \in \mathcal{P}\mathcal{G}_E$, we write $\mathcal{D}(A)$ for the set of all DGwEs over $A$.

**Definition 2.2.2** (Dependent union). Given a DGwE $B : A \to \mathcal{P}\mathcal{G}_E$, its dependent union $\sqcup B \in \mathcal{P}\mathcal{G}_E$ is defined by:

- The underlying game $\sqcup B$ is the dependent union of the dependent game $|B| \overset{\text{df}}{=} \{ B(\sigma) | \sigma : A \}$ over the game $|A|$ [Yam10].

- For any $\tau, \tau' : \sqcup B$, where $\tau : B(\sigma), \tau' : B(\sigma')$, $\sigma, \sigma' : A$, the game $\tau =_{\sqcup B} \tau'$ is defined by:

$$\text{st}(\tau =_{\sqcup B} \tau') \overset{\text{df}}{=} \bigcup_{\rho : \sigma =_A \sigma'} \text{st}(B(\rho) \bullet \tau = B(\sigma') \tau')$$

- The composition $\rho_2 \circ \rho_1 : \tau_1 =_{\sqcup B} \tau_3$ of morphisms $\rho_1 : \tau_1 =_{\sqcup B} \tau_2, \rho_2 : \tau_2 =_{\sqcup B} \tau_3$, where $\tau_1 : B(\sigma_1), \sigma_1 : A$ for $i = 1, 2, 3$, $\rho_1 : B(\rho_1) \bullet \tau_1 =_{B(\sigma_2)} \tau_2, \rho_2 : B(\rho_2) \bullet \tau_2 =_{B(\sigma_3)} \tau_3$, $\rho_1 : \sigma_1 =_A \sigma_2, \rho_2 : \sigma_2 =_A \sigma_3$, is defined by:

$$\rho_2 \circ \rho_1 \overset{\text{df}}{=} \rho_2 \circ \rho_1 \circ \tau_1 =_{\sqcup B} \tau_3$$

- The identity $\text{id}_\tau$ on each $\tau : \sqcup B$, where $\tau : B(\sigma), \sigma : A$, is the identity on $\tau$ in $B(\sigma)$.

**Remark.** Note that morphisms $\rho : \tau =_{\sqcup B} \tau'$, where $\tau : B(\sigma), \tau' : B(\sigma'), \sigma, \sigma' : A$, are strategies not between $\tau$ and $\tau'$ but between $B(\rho) \bullet \tau$ and $\tau'$ for some $\rho : \sigma =_A \sigma'$.

**Lemma 2.2.3** (Well-defined dependent union). For any DGwE $B : A \to \mathcal{P}\mathcal{G}_E$, its dependent union $\sqcup B$ forms a well-defined GwE.

**Proof.** First, it is straightforward to see that the composition and the identities in $\sqcup B$ are well-defined, where note that $B(\rho_2) \circ \rho_1 : B(\rho_2) \bullet B(\rho_1) \bullet \tau_1 =_{B(\sigma_2)} \tau_2, B(\rho_2) \bullet B(\rho_1) = B(\rho_2 \bullet \rho_1), id_{\tau} : B(id_\sigma) \bullet \tau =_{B(\sigma)} \tau$ and $B(id_\sigma) = id_{B(\sigma)}$ hold for any $\tau : B(\sigma), \tau_1 : B(\sigma_1)$, $\tau_2 : B(\sigma_2), \sigma, \sigma_1, \sigma_2, \sigma_3 : A, \rho_1 : \sigma_1 =_A \sigma_2, \rho_2 : \sigma_2 =_A \sigma_3, \rho_1 : B(\rho_1) \bullet \tau_1 =_{B(\sigma_2)} \tau_2$.

For the associativity of the composition, let $\rho_2 : \tau_2 =_{\sqcup B} \tau_3, \rho_3 : \tau_3 =_{\sqcup B} \tau_4$, where $\tau_1 : B(\sigma_1), \tau_4 : B(\sigma_4), \tau_4 : A, \rho_2 : B(\rho_2) \bullet \tau_2 =_{B(\sigma_2)} \tau_3, \rho_3 : B(\rho_3) \bullet \tau_3 =_{B(\sigma_4)} \tau_4$. Then observe that:

$$\rho_3 \circ (\rho_2 \circ \rho_1) = \rho_3 \circ (\rho_2 \bullet B(\rho_2)^- \bullet \rho_1)$$

$$= \rho_3 \bullet B(\rho_3)^- \bullet (\rho_2 \bullet B(\rho_2)^- \bullet \rho_1)$$

$$= \rho_3 \bullet (B(\rho_3)^- \bullet \rho_2) \bullet (B(\rho_2)^- \bullet \rho_1) \text{ (by the functoriality of $B$)}$$

$$= (\rho_3 \bullet B(\rho_3)^- \bullet \rho_2) \circ (B(\rho_2)^- \bullet \rho_1) \text{ (by the functoriality of $B$)}$$

$$= (\rho_3 \circ \rho_2) \bullet B(\rho_3 \bullet \rho_2)^- \bullet \rho_1 \text{ (by the functoriality of $B$)}$$

$$= \rho_3 \circ \rho_2 \circ \rho_1.$$
where note that \( \varrho^{-1} \) is the inverse of \( \varrho \) in \( B(\sigma') \). In fact, \( \varrho^* \) is the inverse of \( \varrho \):

\[
\varrho^* \circ \varrho = B(\rho)^* \circ B(\rho)^{-1} = \varrho^* \circ \varrho
\]

\[
= B(\rho)^{-1} \circ (\varrho^{-1} \circ \varrho) \text{ (by the functoriality of } B(\rho)^{-1})
\]

\[
= B(\rho)^{-1} \circ (\text{id}_{B(\rho) \circ \tau})
\]

\[
= \text{id}_{B(\rho) \circ \tau} \text{ (by the functoriality of } B)\]

\[
= \text{id}_\tau.
\]

and

\[
\varrho \circ \varrho^* = \varrho \circ B(\rho)^{-1} \circ \varrho
\]

\[
= \varrho \circ B(\rho \circ \rho^{-1}) \circ \varrho \text{ (by the functoriality of } B)
\]

\[
= \varrho \circ B(\text{id}_{\sigma'}) \circ \varrho
\]

\[
= \varrho \circ \varrho^{-1} \text{ (by the functoriality of } B)
\]

\[
= \varrho \circ \varrho^{-1}
\]

\[
= \text{id}_\tau.
\]

which completes the proof.

\[\Box\]

### 2.3 Dependent Function Space

**Definition 2.3.1** (Dependent function space). Given a DGwE \( B : A \to \mathcal{P}G\mathcal{E} \), the **dependent function space** \( \prod(A, B) \in \mathcal{P}G\mathcal{E} \) from \( A \) to \( B \) is defined as follows:

- The underlying game \( \prod(A, B) \) is the subgame of the dependent function space \( \prod(|A|, |B|) \) from the dependent game \( |B| = \{ B(\sigma) | \sigma : A \} \) to the game \( |A| \) defined in [Yam16] whose strategies \( \phi : \prod(|A|, |B|) \) are equality-preserving that satisfy

\[
\phi = \rho \circ B(\rho) \circ \phi \circ \sigma = B(\sigma') \circ \phi \circ \sigma'
\]

for all \( \sigma, \sigma', \rho : \sigma = A \sigma' \).

- For any \( \phi_1, \phi_2 : \prod(A, B) \), the subgame \( \phi_1 = \prod(A, B) \phi_2 \leq \prod(A, B) \) consists of strategies \( \mu : \prod(A[1], B[1]) \Rightarrow \prod(A[2], B[2]) \) that satisfy the following two conditions:

1. \( \mu_\sigma \overset{\text{df}}{=} \{ s \in B[1] \mid \sigma \in A[2] \in \mu \} \) for all \( \sigma : A \)
2. \( \text{nat}(\mu) \overset{\text{df}}{=} \{ \mu_\sigma | \sigma : A \} \) forms a natural transformation from fun(\( \phi_1 \)) to fun(\( \phi_2 \)).

- The composition, identities, and inverses of morphisms are the ones for strategies.

**Proposition 2.3.2** (Well-defined \( \prod \)). For any DGwE \( B : A \to \mathcal{D}G\mathcal{E} \), the dependent function space \( \prod(A, B) \) forms a well-defined GwE.
Proof. For the composition $\bullet$, let $\mu : \phi_1 \hat{=} \prod_{A,B} \phi_2, \nu : \phi_2 \hat{=} \prod_{A,B} \phi_3$. By the definition, $\langle \nu \bullet \mu \rangle = \nu \circ \mu_{\sigma} : \phi_1 \sigma = \mu_{\phi_1} \bullet \phi_2 \sigma$ for all $\sigma : A$, and so $\text{nat}(\nu \bullet \mu) = \{\mu_{\sigma} \bullet \phi_2\sigma : \sigma : A\}$ satisfies the naturality condition as $\text{nat}(\nu \bullet \mu)$ is just the vertical composition $\text{nat}(\nu) \circ \text{nat}(\mu)$ of natural transformations. Thus, $\nu \bullet \mu : \phi_1 \hat{=} \prod_{A,B} \phi_3$, and so the composition $\bullet$ is well-defined. Also, the identity $id_\sigma$ on each $\phi : \prod_{A,B} \phi$ clearly satisfies the two conditions, and so $id_\sigma : \phi = \prod_{A,B} \phi$. Note that the associativity of the composition and the unit law of the identities are just the corresponding properties of strategies.

Next, in light of Theorem 2.1.1 it is clear that the inverse $\mu^{-1} : \prod_{A,B} \sigma_1 \Rightarrow \prod_{A,B} \sigma_2$ satisfies $\langle \mu^{-1} \rangle = \mu^{-1} : \phi_2 \sigma_2 = \phi_1 \sigma_1$ for all $\sigma : A$. And also, it satisfies the naturality: Given $\rho : \sigma = \sigma'$, we have $\phi_2 \rho_{\sigma} \bullet \mu_{\sigma} = \mu_{\phi_2} \phi_1 \rho_{\sigma}$, whence $\mu_{\phi_2} \phi_1 \rho_{\sigma} = \rho' \mu_{\phi_2} \phi_1 \rho_{\sigma} = \rho_{\sigma} \phi_2 \rho_{\sigma}$, i.e., $\mu_{\phi_2} \phi_1 = \rho \circ \rho' \circ \rho_{\sigma}$, which completes the proof. 

Proposition 2.3.3 (Dependent functionality). Let $B : A \rightarrow \mathcal{P} \mathcal{G} \mathcal{E}$, and $\phi_1, \phi_2 : \prod_{A,B} B$. If $\phi_1 \hat{=} \prod_{A,B} \phi_2$ is true, then so is $\phi_1 \bullet \sigma_1 = \phi_2 \bullet \sigma_2$ for any $\sigma_1, \sigma_2 : A$ such that $\sigma_1 = \sigma_2$ is true.

Proof. Assume that $\psi : \phi_1 \hat{=} \prod_{A,B} \phi_2, \rho : \sigma_1 = \sigma_2$. Then we have at least two morphisms $\phi_2 \circ \theta, \phi_2 \circ \phi_1 : B(\rho) \bullet \phi_1 \sigma = B(\sigma_2) \phi_2 \sigma_2$. 

2.4 Dependent Pair Space

Definition 2.4.1 (Dependent pair space). Given a DGwE $B : A \rightarrow \mathcal{P} \mathcal{G} \mathcal{E}$, the dependent pair space $\sum(A,B) \in \mathcal{P} \mathcal{G} \mathcal{E}$ of $A$ and $B$ is defined as follows:

- The underlying game $\sum(A,B)$ is the dependent pair space $\sum(|A|, |B|)$ defined in [Yam16].
- For each pair $\langle \sigma, \tau \rangle, \langle \sigma', \tau' \rangle : \sum(A,B)$, the subgame $\langle \sigma, \tau \rangle = \sum_{A,B} \langle \sigma', \tau' \rangle \leq \sum(A,B)$ is defined by:

$$\text{st}(\langle \sigma, \tau \rangle = \sum_{A,B} \langle \sigma', \tau' \rangle) \overset{df}{=} \{\rho \circ \psi | \rho : \sigma = \sigma', \psi : B(\rho) \bullet \tau = B(\sigma') \tau'\}$$

where $\rho \circ \psi \overset{df}{=} (\rho \bullet \tau). \psi \circ \pi_2 : \sum(A,B) \Rightarrow \sum(A,B)$.
- The composition $(\rho_2 \circ \phi_2) \bullet (\rho_1 \circ \phi_1) : \langle \sigma_1, \tau_1 \rangle = \sum_{A,B} \langle \sigma_3, \tau_3 \rangle$ of morphisms $\rho_1 \circ \phi_1 : \langle \sigma_1, \tau_1 \rangle = \sum_{A,B} \langle \sigma_2, \tau_2 \rangle, \rho_2 \circ \phi_2 : \langle \sigma_2, \tau_2 \rangle = \sum_{A,B} \langle \sigma_3, \tau_3 \rangle$ is defined by:

$$(\rho_2 \circ \phi_2) \bullet (\rho_1 \circ \phi_1) \overset{df}{=} (\rho_2 \bullet \rho_1) \circ (\phi_2 \circ \phi_1).$$
- The identity $id_{\sigma \bullet \tau}$ on each object $\sigma \bullet \tau : \sum(A,B)$ is $id_{\sigma} \bullet id_{\tau}$.

Proposition 2.4.2 (Well-defined $\sum$). For any DGwE $B : A \rightarrow \mathcal{P} \mathcal{G} \mathcal{E}$, the dependent pair space $\sum(A,B)$ forms a well-defined GwE.

Proof. It is straightforward to see that $\sum(A,B)$ is a well-defined category, so we omit the verification, and focus to show that each morphism is an isomorphism. Let $\rho \circ \phi : \langle \sigma, \tau \rangle = \sum_{A,B} \langle \sigma', \tau' \rangle$; we define $(\rho \circ \phi)^{-1} \overset{df}{=} \rho^{-1} \circ \phi^*$. Note that $\rho^{-1} \circ \phi^* : \langle \sigma', \tau' \rangle = \sum_{A,B} \langle \sigma, \tau \rangle$ as $\rho^{-1} : \sigma' = \rho \circ \phi^* : B(\rho^{-1}) \bullet \tau = \mu_{\sigma} \tau$. Then $(\rho \circ \phi) \circ (\rho^{-1} \circ \phi^*) = (\rho \bullet \rho^{-1}) \circ (\phi \circ \phi^*) = id_{\sigma} \bullet id_{\tau} = id_{\langle \sigma', \tau' \rangle}$ and $(\rho^{-1} \circ \phi^*) \circ (\rho \circ \phi) = (\rho^{-1} \bullet \rho) \circ (\phi \circ \phi^*) = id_{\sigma} \bullet id_{\tau} = id_{\langle \sigma, \tau \rangle}$, completing the proof.
2.5 Identity Space

Definition 2.5.1 (Identity space). Given \( G \in \mathcal{PGE} \), \( \sigma_1, \sigma_2 : G \), the identity space \( \text{Id}_G(\sigma_1, \sigma_2) \in \mathcal{PGE} \) between \( \sigma_1 \) and \( \sigma_2 \) is the discrete groupoid on the set \( \text{ob} \{ \text{Id}_G(\sigma_1, \sigma_2) \} \equiv \text{st}(\sigma_1 = G \sigma_2) \).

3 Game-theoretic Groupoid Interpretation of MLTT

3.1 Game-theoretic Category with Families

Definition 3.1.1 (The CwF \( \mathcal{PGE} \)). The category with families \( \mathcal{PGE} \) is the tuple \( \mathcal{PGE} = (\mathcal{PGE}, T_y, T_m, \cdot, I, p, v, \langle \cdot, \cdot \rangle) \), where:

- The underlying category \( \mathcal{PGE} \) has been defined in Definition 2.4.
- For each \( \Gamma \in \mathcal{PGE} \), we define \( T_y(\Gamma) \equiv \mathcal{P}(\Gamma) \).
- For each \( A \in \mathcal{P}(\Gamma) \), we define \( T_m(\Gamma, A) \equiv \text{st}(\prod(\Gamma, A)) \).
- For each \( \phi : \Delta \to \Gamma \in \mathcal{PGE} \), the function \( \mathcal{P}(\Gamma) \to \mathcal{P}(\Delta) \) is defined by \( A \mapsto A \circ \text{fun}(\phi) \) (i.e., the composition of functors) for all \( A \in \mathcal{P}(\Gamma) \), and the function \( \mathcal{P}(\Gamma, A) : \text{st}(\prod(\Gamma, A)) \to \text{st}(\prod(\Delta, A)) \) for each \( A \in \mathcal{P}(\Gamma) \) is defined by \( \phi \mapsto \phi \circ \delta \) (i.e., the composition in \( \mathcal{PGE} \) for all \( \phi \in \mathcal{P}(\Gamma, A) \)).
- \( I \) is the discrete CwE with the underlying game \( I = \{0, 0, \epsilon\} \).
- We define \( \Gamma, \Delta, A = \sum(\Gamma, A) \) with \( p(A) : \sum(\Gamma, A) \to \Gamma \), \( v_A : \prod(\sum(\Gamma, A), A) \) are the respective identities \( \text{id}_\Gamma, \text{id}_{\omega A} \) in the category \( \mathcal{PGE} \) up to tags for disjoint union.

- For each \( \phi : \prod(\Delta, A) \), we define \( \langle \phi, \psi \rangle_A : \Delta \to \sum(\Gamma, A) \) to be the pairing \( \langle \phi, \psi \rangle \) equipped with the equality-preservation

\[
\langle \phi, \psi \rangle = \langle (\Delta \Rightarrow \Delta \Rightarrow \Delta), (\Gamma \Rightarrow \Gamma \Rightarrow \Gamma) \rangle \end{equation}

where \( \sigma_{\Gamma, A} : (\Gamma_1 \Rightarrow \Gamma_2 \Rightarrow \Gamma_3) \Rightarrow \text{id}_{\Gamma} \Rightarrow \text{id}_{\omega A} \) is the dereliction up to tags for disjoint union.

Theorem 3.1.2 (Well-defined CwF \( \mathcal{PGE} \)). The structure \( \mathcal{PGE} \) forms a well-defined CwF.

Proof. It is immediate that each component of \( \mathcal{PGE} \) is well-defined except the substitution of terms and the extension. Let \( \Gamma, \Delta \in \mathcal{PGE}, A \in \mathcal{P}(\Gamma) \), \( \phi : \Delta \to \Gamma, \varphi : \prod(\Gamma, A) \). Note that it has been shown in [Yam16] that \( \varphi \mapsto \varphi \circ \phi \) forms a strategy on the game \( \prod(\Delta, A) \Rightarrow (\omega A \Rightarrow \omega A) \) satisfies:

\[
(\varphi \circ \phi) \circ p = \varphi \circ (\phi \circ p) \end{equation}

for all \( \delta, \delta' : \Delta, p : \delta = \Delta \delta' \). Therefore we may conclude that \( \varphi \mapsto (\varphi \circ \phi, \varphi \circ \phi) : \prod(\Delta, A) \), showing that the substitution of terms is well-defined.
Next, for the context extension, let \( \psi : \prod (\Delta, A \{\phi\}) \). Again, it has been shown in [Yam16] that the pairing \( \langle \phi, \psi \rangle \) forms a strategy on the game \( |\Delta| \to \sum (\Gamma, |A|) \); thus, it remains to show that it is equality-preserving. Then for any \( \delta, \delta' : \Delta, p : \delta = \Delta \delta' \), we have:

\[
\langle \phi, \psi \rangle \bullet p = \vartheta_{\Gamma,A} \circ (\phi^m, \psi^m) \bullet p = \vartheta_{\Gamma,A} \circ \langle \phi^m \bullet p, \psi^m \bullet p \rangle = (\phi^m \bullet p) \& (\psi^m \bullet p)
\]

where \( \phi^m \bullet p : \phi \bullet \delta = \Gamma \phi \bullet \delta', \psi^m \bullet p : A(\phi \bullet p) \bullet (\psi \bullet \delta) = A(\phi \bullet p) \bullet (\psi \bullet \delta') \), whence

\[
\langle \phi, \psi \rangle \bullet \rho : (\phi \bullet \delta, \psi \bullet \delta') = \frac{\vartheta_{\Gamma,A}}{\sum (\Gamma, A)} (\phi \bullet \delta', \psi \bullet \delta')
\]

which shows that the context extension is well-defined.

Finally, we verify the required equations. Let \( \Gamma, \Delta, \Theta \in \mathcal{PGF}, A \in \mathcal{D}(\Gamma), \phi : \Delta \to \Gamma, \zeta : \Gamma \to \Theta, \varphi : \prod (\Gamma, A), \psi : \prod (\Delta, A \{\phi\}) \). Then we have:

- ** Ty-Id. ** \( A\{id_{\Gamma}\} = A \circ \text{fun}(\text{der}_{\Gamma}) = A \)

- ** Ty-Comp. ** \( A\{\phi \bullet \zeta\} = A \circ \text{fun}(\phi \bullet \zeta) = A \circ (\text{fun}(\phi) \circ \text{fun}(\zeta)) = (A \circ \text{fun}(\phi)) \circ \text{fun}(\zeta) = A(\phi) \circ \text{fun}(\zeta) = A(\phi \bullet \zeta) \)

- ** Tm-Id. ** \( \varphi\{id_{\Gamma}\} = \varphi \circ \text{der}_{\Gamma} = \varphi \land \varphi\{id_{\Gamma}\} \circ \varphi = \varphi^m \circ \text{der}_{\Gamma} = \varphi^m \circ \text{der}_{\Gamma \Rightarrow \Gamma} = \varphi^m \)

- ** Tm-Comp. ** \( \varphi\{\phi \bullet \zeta\} = \varphi \bullet (\phi \bullet \zeta) = (\varphi \bullet \phi) \bullet \zeta = \varphi\{\phi\} \bullet \zeta \land \varphi\{\phi \bullet \zeta\} = \varphi^m \bullet (\phi \bullet \zeta) = \varphi^m \circ \varphi\{\phi\} \circ \zeta^m = \varphi^m \circ \varphi\{\phi\} \circ \zeta^m = \varphi^m \circ \varphi\{\phi\} \circ \zeta^m \)

- ** Cons-L. ** \( p(A) \bullet \langle \phi, \psi \rangle = \psi \land p(A) = p(A)^m \circ \vartheta_{\Gamma,A} \circ \langle \phi^m, \psi^m \rangle = \phi^m \)

- ** Cons-R. ** \( v_A \{\langle \phi, \psi \rangle\} = v_A \bullet \langle \phi, \psi \rangle = \psi \land v_A \{\langle \phi, \psi \rangle\} = v_A^m \bullet \langle \phi, \psi \rangle = v_A^m \circ \vartheta_{\Gamma,A} \circ \langle \phi^m, \psi^m \rangle = \psi^m \)

- ** Cons-Nat. ** \( \langle \phi, \psi \rangle \bullet \gamma = \langle \phi \bullet \gamma, \psi \bullet \gamma \rangle \land \langle \phi, \psi \rangle = \langle \phi \bullet \gamma, \psi \bullet \gamma \rangle \land \zeta^m = \vartheta_{\Gamma,A} \bullet \langle \phi^m, \psi^m \rangle \bullet \zeta^m = \vartheta_{\Gamma,A} \bullet \langle \phi^m, \psi^m \rangle \bullet \zeta^m = \vartheta_{\Gamma,A} \bullet \langle \phi \bullet \gamma, \psi \bullet \gamma \rangle = \langle \phi \bullet \gamma, \psi \bullet \gamma \rangle \)

- ** Cons-Id. ** \( (p(A), v_A) = \text{der}_{\sum (\Gamma, A)} \land (p(A), v_A) = \vartheta_{\Gamma,A} \circ \langle p^m, v_A^m \rangle = \vartheta_{\Gamma,A} \circ \langle p^m, v_A^m \rangle = \text{der}_{\sum (\Gamma, A)} \circ \sum (\Gamma, A) \)

which completes the proof.

### 3.2 Game-theoretic Type Formers

**Theorem 3.2.1 (\( \mathcal{PGF} \) supports \( \Pi \)-types).** The CwF \( \mathcal{PGF} \) supports \( \Pi \)-types in the strict sense.

**Proof.** Let \( \Gamma \in \mathcal{PGF}, A \in \mathcal{D}(\Gamma), B \in \mathcal{D}(\sum (\Gamma, A)) \).

- ** \( \Pi \)-Form. ** First, for each \( \gamma : \Gamma \), we define \( B_\gamma \in \mathcal{D}(A(\gamma)) \) by:
  
  \[
  B_\gamma (\sigma) \overset{\text{df}}{=} B(\gamma, \sigma) \\
  B_\gamma (\varrho) \overset{\text{df}}{=} B(id_{\gamma} \& \varrho) : B(\gamma, \sigma) \to B(\gamma, \sigma')
  \]

  for all \( \sigma, \sigma' : A(\gamma), \varrho : \sigma = A(\gamma) \sigma' \). We then define \((\Pi, B) \in \mathcal{D}(\Gamma) \) by:

  \[
  \Pi(A, B)(\gamma) \overset{\text{df}}{=} \prod (A(\gamma), B_\gamma) \\
  \Pi(A, B)(\varrho) \overset{\text{df}}{=} \rho_{\Pi(A, B)} : \Pi(A(\gamma), B_\gamma) \to \Pi(A(\gamma'), B_{\gamma'})
  \]
for all \( \gamma, \gamma' : \Gamma \), \( \rho : \gamma = \gamma' \), where the morphism \( \rho_{\Pi(A,B)} : \prod(A(\gamma), B_\gamma) \to \prod(A(\gamma'), B_{\gamma'}) \) in the category \( \mathcal{P}\mathcal{G}\mathcal{E} \) is the strategy

\[
\rho_{\Pi(A,B)} \overset{\text{df}}{=} \&_{\phi, \Pi(A(\gamma), B_\gamma)} \phi \overset{\text{df}}{=} A(p^{-1})^\dagger \circ \nu_1; B(\rho \& \text{id})
\]

for which we define

\[
\phi \overset{\text{df}}{=} A(p^{-1})^\dagger; \nu_1; B(\rho \& \text{id}) = \begin{cases} s \mapsto \phi^\dagger, A(\gamma'), B_{\gamma'} & | s \in \phi^\dagger \to A(p^{-1})^\dagger(\phi^\dagger)^\dagger \circ \nu_1; B(\rho \& \text{id}), \\
\forall m \in \mathcal{S} \exists s. & \text{even}(t) \land m \in \mathcal{M} \to \{ t \mapsto \phi^\dagger(\phi^\dagger)^\dagger \circ \nu_1; B(\rho \& \text{id}) \}, \\
\end{cases}
\]

B(\rho \& \text{id}) = \bigcup_{\sigma : A(\gamma') \to A(\gamma')} B(\rho \& \text{id}_{A(p^{-1})^\dagger \circ \nu_1; B_{\gamma'}}) : \forall B_{\gamma'} \implies \forall B_{\gamma'}

equipped with the equality-preservation

\[
\rho^\sim_{\Pi(A,B)} \overset{\text{df}}{=} \&_{\phi_1, \phi_2, \Pi(A(\gamma), B_\gamma), \nu : \prod(A(\gamma), B_\gamma)} \nu \overset{\text{df}}{=} \prod(A(p^{-1})^\dagger; \nu, B_{\gamma'})
\]

for which we define \( \prod(A(p^{-1})^\dagger; \nu, B_{\gamma'}) = \rho^\sim_{\Pi(A,B)} \bullet \phi_1 = \Pi(A(\gamma), B_\gamma) \circ \rho^\sim_{\Pi(A,B)} \bullet \phi_2 \). Here we have used the “from-left-to-right” composition “\( \circ \)” than the usual “from-right-to-left” one “\( \circ \)” for readability. Also, the superscripts \([1], [2]\) on \( \gamma, \phi \) and \( \nu \) are just to distinguish different copies of them.

Note that, for any \( \phi : \Pi(A(\gamma), B_\gamma), \sigma' : A(\gamma') \), we have:

\[
\rho_{\Pi(A,B)} \bullet \phi : A(\gamma') \Rightarrow \forall B_{\gamma'}
\]

\[
(\rho_{\Pi(A,B)} \bullet \phi) \circ \sigma' = B(\rho \& \text{id}_{\sigma'} \bullet \nu; A(p^{-1}) \bullet \sigma : B_{\gamma'}(\gamma'))
\]

showing that \( \rho_{\Pi(A,B)} \bullet \phi : \Pi(A(\gamma), B_\gamma) \Rightarrow \Pi(A(\gamma'), B_{\gamma')} \). From this, it is clear that \( \rho_{\Pi(A,B)} \) is a strategy on the game \( \Pi(A(\gamma), B_\gamma) \Rightarrow \Pi(A(\gamma'), B_{\gamma'}) \). By a symmetric argument, \( \rho^\sim_{\Pi(A,B)} \) is a strategy on \( \Pi(A(\gamma), B_\gamma) \Rightarrow \Pi(A(\gamma'), B_{\gamma'}) \). Hence, it is easy to see that \( \rho^\sim_{\Pi(A,B)} \) is a strategy on \( (\Pi(A(\gamma), B_\gamma) \Rightarrow \Pi(A(\gamma'), B_{\gamma'}) \Rightarrow \Pi(A(\gamma'), B_{\gamma'})) \).

Now, we need to establish the functoriality of \( \rho^\sim_{\Pi(A,B)} \). Fix arbitrary \( \phi_1, \phi_2 : \Pi(A(\gamma), B_\gamma), \nu : \phi_1 = \Pi(A(\gamma), B_\gamma) \circ \phi_2 \). Note that we clearly have:

\[
\rho^\sim_{\Pi(A,B)} \bullet \nu = (\rho^\sim_{\Pi(A,B)} \circ \nu_1; \nu_1; \rho_{\Pi(A,B)} \bullet \phi_1 = \rho_{\Pi(A,B)} \bullet \phi_2.
\]

Also, for any \( \sigma' : A(\gamma') \), we have:

\[
(\rho^\sim_{\Pi(A,B)} \bullet \nu)_{\sigma'} = ((\rho^\sim_{\Pi(A,B)} \circ \nu_1; \nu_1; \rho_{\Pi(A,B)} \bullet \phi_1 = \rho_{\Pi(A,B)} \bullet \phi_2). \sigma'
\]

\[
= B(\rho \& \text{id}_{\sigma'} \bullet \nu; A(p^{-1}) \bullet \sigma : B_{\gamma'}(\gamma'))
\]

\[
= B(\rho \& \text{id}_{\sigma'} \bullet \nu; A(p^{-1}) \bullet \sigma : B_{\gamma'}(\gamma'))
\]

by the definition of \( ((\rho^\sim_{\Pi(A,B)} \circ \nu_1; \nu_1; \rho_{\Pi(A,B)} \bullet \phi_1 = \rho_{\Pi(A,B)} \bullet \phi_2). \sigma' \).

To establish \( \rho^\sim_{\Pi(A,B)} \bullet \nu : \Pi(A(\gamma), B_\gamma) \bullet \phi_1 = \Pi(A(\gamma'), B_{\gamma'}) \bullet \phi_2 \), it remains to show the naturality of the family \( \text{nat}(\rho^\sim_{\Pi(A,B)} \bullet \nu) = \{ (\rho^\sim_{\Pi(A,B)} \bullet \nu)_{\sigma'} : A(\gamma') \} \). Note that for all \( \sigma' : A(\gamma') \) we have:
It then immediately follows that:

\[
(B(p \& q')) = (\phi_2 \bullet A(\rho^{-1}) = g') \bullet (B(p \& id_{\sigma'}) \bullet B(p \& id_{\sigma'})^{-1}
\]

\[
= (B(p \& id_{\sigma'}) = (\phi_2 \bullet A(\rho^{-1}) = g') \bullet \nu_{A(\rho^{-1}) \sigma'} \bullet B(p \& id_{\sigma'})^{-1}
\]

\[
= (B(p \& id_{\sigma'}) = ((\phi_2 \bullet A(\rho^{-1}) = g') \bullet \nu_{A(\rho^{-1}) \sigma'} \bullet B(p \& id_{\sigma'})^{-1}
\]

\[
= (B(p \& id_{\sigma'}) = (\nu_{A(\rho^{-1}) \sigma'} \bullet (\phi_1 \bullet A(\rho^{-1}) = g')) \bullet B(p \& id_{\sigma'})^{-1}
\]

\[
= (B(p \& id_{\sigma'}) = (\nu_{A(\rho^{-1}) \sigma'} \bullet (B(p \& id_{\sigma'})^{-1} \bullet (\phi_1 \bullet A(\rho^{-1}) = g'))
\]

\[
= (B(p \& id_{\sigma'}) = (\nu_{A(\rho^{-1}) \sigma'} \bullet (B(p \& id_{\sigma'})^{-1} \bullet (\phi_1 \bullet A(\rho^{-1}) = g'))
\]

for all \( \sigma', \sigma_2 : A(\gamma), g' : \sigma_1 = A(\gamma) \sigma_2', i.e., \text{nat}(\rho_{\Pi(A,B)} \bullet \nu) \) is natural.

showing that \( \rho_{\Pi(A,B)} \) respects domain and codomain. It is simpler to see that it also respects composition and identities:

\[
\begin{align*}
\triangleright & \quad \rho_{\Pi(A,B)} \bullet (\nu_2 \bullet \nu_1) = \rho_{\Pi(A,B)} \bullet \nu_2 \bullet \nu_1 \quad = \rho_{\Pi(A,B)} \bullet \nu_2 \bullet \nu_1 \quad = \rho_{\Pi(A,B)} \bullet \nu_2 \bullet \nu_1 \\
& \quad \nu_1 \bullet \rho_{\Pi(A,B)} = (\rho_{\Pi(A,B)} \bullet \nu_2) \bullet (\rho_{\Pi(A,B)} \bullet \nu_1)
\end{align*}
\]

\[
\begin{align*}
\triangleright & \quad \rho_{\Pi(A,B)} \bullet (id_{\phi}) = \rho_{\Pi(A,B)} \bullet id_{\phi} \bullet \rho_{\Pi(A,B)} = \rho_{\Pi(A,B)} \bullet \rho_{\Pi(A,B)} = id_{\Pi(A(\gamma),B_{\gamma})}
\end{align*}
\]

for all \( \phi, \phi_1, \phi_2, \phi_3, \nu_1 : \phi_1 = \Pi(A(\gamma),B_{\gamma}) \phi_2, \nu_2 : \phi_2 = \Pi(A(\gamma),B_{\gamma}) \phi_3. \) Now, fix \( \phi_1, \phi_2, \nu : \phi_1 = \Pi(A(\gamma),B_{\gamma}) \phi_2; \)

\[\blacksquare\]

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