An improved existence criterion and an optimal result

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Abstract

We are concerned with a semi-linear elliptic equation on a smooth bounded domain \( \Omega \) of \( \mathbb{R}^n, n \geq 5 \), which involves a critical nonlinearity and a linear term of the form \( K(x)u^{(n+2)/(n-2)} \) and \( \mu u \), respectively. By using a test function procedure, we give an existence criterion involving the parameter \( \mu \) and the function \( K(x) \). For a particular case of \( \Omega, K(x) \) and \( n \), we prove its optimality through a Pohozaev type identity.

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1 Introduction and main results

This paper is a second part devoted to the study of the following nonlinear elliptic partial differential equation with zero Dirichlet boundary condition

\[
-\Delta u = K(x)u^q + \mu u \quad \text{in } \Omega, \\
u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

where \( \Omega \subset \mathbb{R}^n, n \geq 5 \), is a bounded domain with a smooth boundary \( \partial \Omega \), \( K(x) \) is a \( C^2 \)-function in \( \Omega \), \( q + 1 = \frac{2n}{n-2} \) is the critical exponent for the embedding \( H_0^1(\Omega) \) into \( L^{q+1}(\Omega) \) and \( 0 < \mu < \mu_1(\Omega) \), where \( \mu_1(\Omega) \) denotes the first eigenvalue of \( (-\Delta) \) in \( H_0^1(\Omega) \).

In [4, Theorem 1.1], we were interested on the existence of at least one solution to (1.1). This result was centered on a Lions’ condition. Namely, by using, we have proved the following theorem. Denote \( \sup_{\Omega}(K(x)) := K_\infty \) and \( S := \inf\{\|u\|^2, \ u \in H_0^1(\Omega) \text{ and } \|u\|_{q+1} = 1 \} \) is the best Sobolev constant, where \( J(u) := \int_\Omega K(x)|u(x)|^{q+1} \, dx \) and \( \|u\|_{p}^p := \int_\Omega |u(x)|^p \, dx \) for any \( p > 1 \).

Theorem 1.1 ([4])

\[
(1.2)
\]
When \( K(x) \equiv 1 \), we recognize the Brezis–Nirenberg existence result [5] Lemma 1.2. In order to establish the condition (1.2), Brezis and Nirenberg [5] Lemma 1.1 follow an original idea due to Aubin [11]. By considering the following test function

\[
u = \frac{\lambda}{1 + \lambda^2 |x - y_0|^2} =: \varphi(x) \cdot \delta_{y_0, \lambda}(x), \tag{1.3}\]

where \( \mu = n^2 - 2n \), \( y_0 \in \Omega \), \( \lambda > 0 \), \( \delta_{y_0, \lambda} \) are the positive solutions in \( \mathbb{R}^n \), concentrated at \( y_0 \), of \(-\Delta u = u^{n-2} \) and \( \varphi \) is a cut-off function, they proved that the condition (1.2) is satisfied for any \( \mu > 0 \).

When \( K(x) \neq 1 \), the situation becomes extremely different: Indeed the behavior of \( K(x) \) plays a crucial role in establishing existence results; see, e.g., [3] for the case \( \mu = 0 \) and \( K(x) \) is positive everywhere. But for \( \mu = 0 \) and, of course, \( K_\infty > 0 \), the following Pohozaev identity [8]

\[
\frac{1}{2} \int_{\partial \Omega} |\frac{\partial u}{\partial \nu}(x)|^2 \langle x, \nu(x) \rangle \, dx = \frac{n - 2}{2n} \int_{\Omega} \langle x, \nabla K(x) \rangle u^{\frac{2n}{n-2}}(x) \, dx + \mu \int_{\Omega} u^2(x) \, dx \tag{1.4}
\]

asserts that the problem (1.1) has no solution provided that \( \Omega \) is star-shaped with respect to the origin \( o \) of \( \mathbb{R}^n \) and \( \langle x, \nabla K(x) \rangle \leq 0 \) in \( \Omega \). Here \( \nu(x) \) denotes the outward normal vector at \( x \) to \( \partial \Omega \) and \( u \) is supposed to be a solution of (1.1). (This identity (1.4) is obtained by multiplying the equation given in (1.1) on the one hand by \( u \) and on the other hand by \( \sum_{i=1}^n x_i (\partial u / \partial x_i) \), and using an integration by parts and the fact that on \( \partial \Omega \) we have \( \nabla u = (\partial u / \partial \nu) \nu \). In fact, in this case the condition (1.2) is not satisfied. In view of this nonexistence result, naturally one can ask: What is the concrete condition that can we impose on \( \mu \) and an absolute maximum \( y_0 \) of \( K(x) \) in \( \Omega \) so that (1.2) becomes satisfied? In the case \( \mu > 0 \), Lions [7] Remark 4.7] considered the test function (1.3) and he showed that the condition (1.2) is satisfied provided that

\[
K_\infty = K(y_0) > 0 \quad \text{with} \quad y_0 \in \Omega,
\]

\[
-\frac{(n - 2)^2 \bar{c}_2 \Delta K(y_0)}{2n K(y_0)} < \mu \bar{c}_3,
\]

where \( \bar{c}_2 \) and \( \bar{c}_3 \) are two positive constants depending only on \( n \); see Proposition 1.1 below.

Convinced to expand the validity of the condition (1.2) to more large class of functions \( K(x) \) when \( \mu \) is fixed, a choice of a test function taking care of the geometry of \( \Omega \) becomes useful. To this end, let \( P \) be the projection from \( H^1(\Omega) \) onto \( H^1_0(\Omega) \); that is, \( u = Pf \) is the unique solution of \( \Delta u = \Delta f \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \). Denote by \( H \) the regular part of the Green’s function of \( (\Delta) \) on \( \Omega \). By using the test function \( Pf \delta_{y_0, \lambda} \), we are able to prove the following proposition:

**Proposition 1.1** Let \( n \geq 5 \). Let \( K(x) \in C^2(\Omega) \) satisfying \( K_\infty = K(y_0) > 0 \) with \( y_0 \in \Omega \) and let \( \mu > 0 \). Then the condition (1.2) holds true provided that one of the following two conditions is satisfied:

1. \[-\frac{(n - 2)^2 \bar{c}_2 \Delta K(y_0)}{2n K(y_0)} < \mu \bar{c}_3,\]
\[ ii) \quad -\frac{(n-2)^2 \bar{c}_2 \Delta K(y_0)}{2nK(y_0)} = \mu \bar{c}_3 \text{ and} \]

\[
\lim_{\lambda \to +\infty} \lambda^{n-2} \left[ -\int_{B_0} \frac{K(x)}{K(y_0)} - 1 - \frac{\Delta K(y_0)}{2nK(y_0)} |x|^2 \delta_{y_0, \lambda}^2 \right] + S_n \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} a_{n,k} \frac{\mu^k}{\lambda^2} < \frac{n \bar{c}_4}{n-2},
\]

where \( d_0 := \text{dist}(y_0, \partial \Omega) \), \( B_0 \) is the ball of center \( y_0 \) and radius \( d_0 \), \( S_n := \int_{\mathbb{R}^n} (1 + |x|^2)^{-n} dx \), \( \bar{c}_2 = \int_{\mathbb{R}^n} |x|^2/(1 + |x|^2)^n dx \), \( \bar{c}_3 = \int_{\mathbb{R}^n} 1/(1 + |x|^2)^{n-2} dx \), \( a_{n,k} \)'s are the constants defined by the following Taylor expansion

\[
(1 - \frac{\bar{c}_3}{\bar{c}_n S_n} t)^{n-2} = 1 - \frac{n \bar{c}_3}{(n-2)c_n S_n} t + \sum_{k=2}^{\left\lfloor \frac{n-2}{2} \right\rfloor} a_{n,k} t^k + o(t^{n-2}) \quad \text{as} \quad t \to 0,
\]

\[
\bar{c}_4 := -H(y_0, y_0) \int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^2)^{n+2}} + \mu \bar{c}_3^{-1} \left[ 2 \int_{\Omega} H(y_0, x) \frac{dx}{|x-y_0|^{n-2}} \right. \\
- \int_{\Omega} H^2(y_0, x) dx + \int_{\mathbb{R}^n \setminus \Omega} \frac{dx}{|x-y_0|^{2n-4}} \right]
\]

Remark 1.1 If we use the test function \( u_{\lambda, y_0} \) instead of \( P\delta_{y_0, \lambda} \), then the corresponding constant \( \bar{c}_4 \) can not be specified. This is due to the fact that \( u_{\lambda, y_0} \) does not deal with the boundary of \( \Omega \).

Example 1.1 Let \( \mu > 0 \) be fixed. To verify if a function \( K(x) \) satisfies the condition (ii), we need to know its Taylor expansion, near \( y_0 \), of order greater than 2. For example, let us take the case \( \Omega = B \) is the unit ball of \( \mathbb{R}^5 \) and \( y_0 \) is the origin of \( \mathbb{R}^5 \). Assume that \( K(x) = f(|x|) \) is radial and radially non-increasing function with

\[ f(t) = f(0) + at^2 + bt^3 + o(t^3) \quad \text{as} \quad t \to 0. \]

Then the condition (ii) is satisfied provided that

\[ -9 \bar{c}_2 a = \mu \bar{c}_3 f(0) \quad \text{and} \quad -3b \int_{\mathbb{R}^5} |x|^3/(1 + |x|^2)^5 dx < 5 \bar{c}_4 f(0). \]

In a second part of this work, we will try to analyze the optimality of the condition (i) for some class of functions when \( \Omega \) is a ball, \( n \) is odd with \( n = 5 \) or \( n > 19 \) and \( K(x) \) is close to a constant, radial and radially non-increasing. To this end, let us state the following assumptions: Assume that \n
\[ (K_1) \quad \Omega = B(y_0, \gamma) \text{ is the ball of center } y_0 \text{ and radius } \gamma \text{ in } \mathbb{R}^n. \]
$(K_\eta) \ K(x) = K(y_0) + \eta f_1(|x - y_0|)$ is a non-negative $C^2$-function in $\bar{\Omega}$, where $\eta, K(y_0) > 0$ are fixed constants and $f_1$ is a non-increasing function on $[0, \gamma]$ independent of $\eta$.

In this case, we will refer to the problem (1.1) as $(BN)_{\eta}$.

$(K_3) \ \limsup_{t \to 0} \frac{f'_1(t) - f''(0)t}{t^{n-3}} < +\infty$.

Our optimal result is the following:

**Theorem 1.2** Let $n$ be an odd integer with $n = 5$ or $n > 19$ and let $0 < \mu < \mu_1(\Omega)$. Assume that $\Omega$ and $K(x)$ satisfy the assumptions $(K_1), (K_\eta)$ and $(K_3)$. Then there exists a constant $\bar{\eta}$ depending on $n, f_1(t)$ and $K(y_0)$ such that if $0 < \eta \leq \bar{\eta}$, then the problem $(BN)_{\eta}$ admits a solution if and only if

$$-(n-2)^2 \bar{c}_2 \Delta K(y_0) < 2n K(y_0) \mu \bar{c}_3.$$

(1.5)

The proof of the sufficiency is obtained by a combination of the results Theorem 1.1 and Proposition 1.1. For the necessity of the condition (1.5), we argue by contradiction: The key point is to establish an adequate Pohozaev type identity for the desired solution of (1.1); this identity is a natural extension to that given in the proof of [5, Lemma 1.4]. To conclude, we need to investigate a constant $\bar{\eta}$ depending only on $n, f_1(t)$ and $K(y_0)$ such that if $\eta \leq \bar{\eta}$ and $[-(n-2)^2 \bar{c}_2 \Delta K(y_0)]/2n K(y_0) \geq \mu \bar{c}_3$, then this identity becomes impossible.

**2 Proof of the results**

*Proof of Proposition 1.1.* Let $y_0 \in \Omega$ be such that $K_\infty = K(y_0) > 0$. Denoting, for $\lambda > 0$ a fixed constant large enough,

$$A_{y_0, \mu}(\lambda) := \frac{\int_{\Omega} |\nabla P\delta_{y_0, \lambda}|^2 - \mu \int_{\Omega} (P\delta_{y_0, \lambda})^2}{\left(\int_{\Omega} K(P\delta_{y_0, \lambda})^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}.$$

(2.1)

In order to get the claim of Proposition 1.1 it is sufficient to prove that, for $\lambda$ large enough,

$$[A_{y_0, \mu}(\lambda)]^{\frac{n}{n-2}} < \frac{1}{K(y_0)} S^{\frac{n}{n-2}}$$

(2.2)

provided that one of the conditions (i) and (ii) is satisfied. To this end, we need an estimation of the following three quantities:

$$\int_{\Omega} (P\delta_{y_0, \lambda})^2, \quad \int_{\Omega} K(P\delta_{y_0, \lambda})^{\frac{2n}{n-2}} \quad \text{and} \quad \int_{\Omega} |\nabla P\delta_{y_0, \lambda}|^2.$$
The last two quantities were estimated in [2] (2.67), (5.31) and Estimate F8, and we have

\[
\int_{\Omega} K(P\delta_{y_0,\lambda})^{2n} = K(y_0)c_n^2 \left[ S_n + \frac{c_2}{2n} \frac{\Delta K(a)}{K(y_0)\lambda^2} - \frac{2n\bar{c}_1}{n - 2} \frac{H(y_0, y_0)}{\lambda^{n-2}} \right] + \int_{B_0} \left( \frac{K(x)}{K(y_0)} - 1 - \frac{\Delta K(y_0)}{2nK(y_0)}|x|^2 \right) \delta_{y_0,\lambda}^{2n} dx \\
+ o(\frac{1}{\lambda^{n-2}}) + O\left(\frac{\log(\lambda d_0)}{(\lambda d_0)^n}\right),
\]

where \( d_0 := \text{dist}(y_0, \partial\Omega), \) \( B_0 \) is the ball of center \( y_0 \) and radius \( d_0, \) \( \bar{c}_1 = \int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^2)^\frac{n}{2}} \) and \( \bar{c}_2 = \int_{\mathbb{R}^n} \frac{|x|^2}{(1 + |x|^2)^\frac{n}{2}} dx. \) Then we are left with the first quantity:

\[
\int_{\Omega} (P\delta_{y_0,\lambda})^2(x) dx = \int_{\Omega} \delta_{y_0,\lambda}^2 dx + \int_{\Omega} \theta_{y_0,\lambda}(x) dx + 2 \int_{\Omega} \delta_{y_0,\lambda}\theta_{y_0,\lambda}(x) dx,
\]

where \( \theta_{y_0,\lambda} := \delta_{y_0,\lambda} - P\delta_{y_0,\lambda}. \) First we recall that from [2] (5.25) we have the following estimate

\[
\theta_{y_0,\lambda}(x) = \frac{c_n^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} H(y_0, x) + \frac{1}{\lambda^{\frac{n-2}{2}}} \frac{1}{d_0^{2n}} \cdot O(1), \quad \forall \: x \in \Omega,
\]

where \(|O(1)|\) is a quantity upper-bounded by a positive constant \( M \) independent of \( x \in \Omega. \) This, together with Lebesgue’s dominated convergence theorem, implies that

\[
\int_{\Omega} \delta_{y_0,\lambda}\theta_{y_0,\lambda}(x) dx = \frac{c_n^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \int_{\Omega} H(y_0, x) \frac{\lambda^{n-2}}{(1 + \lambda^2|x - y_0|^2)^\frac{n-2}{2}} dx + o\left(\frac{1}{\lambda^{n-2}}\right) \\
= \frac{c_n^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \int_{\Omega} H(y_0, x) \frac{1}{|x - y_0|^{n-2}} dx + o\left(\frac{1}{\lambda^{n-2}}\right),
\]

\[
\int_{\Omega} \theta_{y_0,\lambda}^2 dx = \frac{c_n^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \int_{\Omega} H^2(y_0, x) dx + o\left(\frac{1}{\lambda^{n-2}}\right),
\]

On the other hand, by using, again, Lebesgue’s dominated convergence theorem we obtain

\[
\int_{\Omega} \delta_{y_0,\lambda}^2 dx = c_n^{\frac{n-2}{2}} \left( \frac{c_3}{\lambda^2} - \int_{\mathbb{R}^n \setminus \Omega} \frac{\lambda^{n-2}}{(1 + \lambda^2|x - y_0|^2)^{n-2}} dx \right) \\
= c_n^{\frac{n-2}{2}} \left( \frac{c_3}{\lambda^2} - \frac{c_5}{\lambda^{n-2}} + o\left(\frac{1}{\lambda^{n-2}}\right) \right),
\]
where \( \bar{c}_3 = \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^n} dx \) and \( \bar{c}_5 = \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{|x-y_0|^n} dx \). Combining (2.5)–(2.8) we obtain

\[
\int_{\Omega} (P\delta_{y_0,\lambda})^2(x)dx = \frac{c_n}{\lambda^{n-2}} \left[ -2 \int_{\Omega} H(y_0, x) \frac{1}{|x-y_0|^{n-2}} dx + \int_{\Omega} H^2(y_0, x)dx \right] + o\left( \frac{1}{\lambda^{n-2}} \right)
\]

\[
= \frac{c_n}{\lambda^{n-2}} \left[ \bar{c}_3 - \bar{c}_5 + o\left( \frac{1}{\lambda^{n-2}} \right) \right], \tag{2.9}
\]

where \( \bar{c}_5 = \int_{\Omega} \int (n-2)H^2(y_0, x)dx + \bar{c}_5 \). Combining (2.9) and (2.4) we get, for \( \lambda \) large enough,

\[
\left[ \int_{\Omega} |\nabla P\delta_{y_0,\lambda}|^2 - \mu \int_{\Omega} (P\delta_{y_0,\lambda})^2(x)dx \right]^{\frac{n}{n-2}}
\]

\[
= (S_n c^\#_n)^{\frac{n}{n-2}} \left[ 1 - \frac{\mu \bar{c}_3}{c_n S_n \lambda^2} - \frac{n \bar{c}_0}{(n-2)c_n S_n \lambda^2} + \frac{1}{(n-2)\lambda^{n-2}} + \sum_{k=2}^{[\frac{n-2}{2}]} a_{n,k} \frac{\mu^k}{\lambda^{2k}} \right] + o\left( \frac{1}{\lambda^{n-2}} \right), \tag{2.10}
\]

where \( a_{n,k} \)'s are fixed constants defined by the following Taylor expansion

\[
(1 - \frac{\bar{c}_3}{c_n S_n t})^{\frac{n}{n-2}} = 1 - \frac{n \bar{c}_3}{(n-2)c_n S_n} t + \sum_{k=2}^{[\frac{n-2}{2}]} a_{n,k} t^k + o(t^{\frac{n-2}{2}}) \quad \text{as} \quad t \to 0
\]

\(([(n-2)/2])\) denotes the integer part of \((n-2)/2\) and the sum \( \sum_{k=2}^{[\frac{n-2}{2}]} \) is omitted when \( n = 5 \) and \( \bar{c}_0 := (\bar{c}_1 H(y_0, y_0) + \mu c_n^{-1} \bar{c}_6) / S_n \). (2.1), (2.3) and (2.10) imply that, for \( \lambda \) large enough,

\[
\left[ A_{y_0,\mu}(\lambda) \right]^{\frac{n}{n-2}}
\]

\[
= \frac{S_n^{n-2} c^\#_n}{K(y_0)} \left[ 1 - \frac{\bar{c}_2 \Delta K(a)}{2nK(y_0)} + \frac{\mu \bar{c}_3}{(n-2)\lambda^2} + \frac{1}{2} \frac{n \bar{c}_4}{(n-2)S_n \lambda^{n-2}} \right] + o\left( \frac{1}{\lambda^{n-2}} \right) \tag{2.11}
\]

\[
- \frac{1}{S_n} \int_{B_0} \left[ \frac{K(x)}{K(y_0)} - 1 - \Delta K(y_0) \frac{1}{2nK(y_0)} |x|^2 \right] \delta_{y_0,\lambda}^{\frac{2n}{2n-2}} dx + \sum_{k=2}^{[\frac{n-2}{2}]} a_{n,k} \frac{\mu^k}{\lambda^{2k}},
\]

where \( \bar{c}_4 := -\bar{c}_1 H(y_0, y_0) + \mu c_n^{-1} \bar{c}_6 \). On the other hand, observe that since \( K(x) \in C^2(\Omega) \), then

\[
\int_{B_0} \left( \frac{K(x)}{K(y_0)} - 1 - \Delta K(y_0) \frac{1}{2nK(y_0)} |x|^2 \right) \delta_{y_0,\lambda}^{\frac{2n}{2n-2}} dx = o\left( \frac{1}{\lambda^2} \right). \tag{2.12}
\]

Observe also that

\[
S = c_n S_n^2. \tag{2.13}
\]
Thus under the condition (i), the claim (2.2) follows by combining (2.11)–(2.13) and taking \( \lambda \) large enough. If the condition (ii) is satisfied instead of (i), then (2.2) follows by taking \( \lambda \) large enough in the right hand side of (2.11) and using (2.13). This finishes the proof of Proposition 1.1.

Proof of Theorem 1.2. Sufficiency of the condition (1.5) : From (K_1) and (K_η) we get \( K_\infty = K(y_0) > 0 \) with \( y_0 \in \Omega \). This, together with the condition (1.5) and the result of Proposition 1.1 implies that (1.2) is satisfied. Thus a solution to problem (1.1) is obtained by applying Theorem 1.1.

Necessity of the condition (1.5) : Arguing by contradiction, assuming that the problem (1.1) has a solution \( u \) under the condition

\[
- (n - 2) \bar{c} \Delta K(y_0) \geq \mu \bar{c}_3.
\]

(2.14)

In particular, we have \( \Delta K(y_0) \neq 0 \). Up to a translation and a dilatation in the space, we can suppose that

\[
\Omega = B^{n} \quad \text{is the unit ball of } \mathbb{R}^n.
\]

Now, by a result of Gidas–Ni–Nirenberg [6, Theorem 1'], (K_1) and (K_η) imply that \( u \) is necessarily spherically symmetric. We write \( u(x) =: u(t) \) and \( K(x) =: f(t) \), where \( t = |x| \in [0, 1] \). Thus \( u \) satisfies the following ordinary differential equation

\[
\begin{align*}
-u'' - \frac{n - 1}{t} u' &= f(t) u^{\frac{n + 2}{n - 2}} + \mu u \quad \text{on } (0, 1), \\
u'(0) &= u(1) = 0.
\end{align*}
\]

(2.15)

(Note that \( u \in C^2(\Omega) \)). Let \( \psi \) be a smooth function on \( [0, 1] \) such that \( \psi(0) = 0 \). Multiplying the equation (2.15) by \( t^{n-1} \psi u' \) and \( (t^{n-1} \psi'(t) - (n - 1) t^{n-2} \psi(t)) u \) and integrating by parts several times in order to obtain

\[
\begin{align*}
- \frac{1}{2} |u'(1)|^2 \psi(1) + \frac{1}{2} \int_0^1 |u'(t)|^2 \left( t^{n-1} \psi'(t) - (n - 1) t^{n-2} \psi(t) \right) dt \\
= - \bar{c}_n \int_0^1 u^{\frac{2n}{n-2}} \left[ f(t) \left( t \psi'(t) + (n - 1) \psi(t) \right) + f'(t) \psi(t) \right] t^{n-2} dt \\
- \frac{\mu}{2} \int_0^1 u^2 \left( t^{n-1} \psi'(t) + (n - 1) t^{n-2} \psi(t) \right) dt, \\
\int_0^1 \left[ f(t) u^{\frac{2n}{n-2}} \left( t \psi'(t) - (n - 1) \psi(t) \right) + \mu u^2 \left( t \psi'(t) - (n - 1) \psi(t) \right) \right] t^{n-2} dt \\
= - \frac{1}{2} \int_0^1 u^2 \left[ t^3 \psi(3)(t) + (n - 1)(n - 3) \left( \psi(t) - t \psi'(t) \right) \right] t^{n-4} dt \\
+ \int_0^1 |u'(t)|^2 \left( t \psi'(t) - (n - 1) \psi(t) \right) t^{n-2} dt,
\end{align*}
\]

(2.16)

(2.17)
respectively, where \( c_n := \frac{n-2}{2n} \). Combining (2.16) and (2.17) we get
\[
- \frac{1}{2} |u'(1)|^2 \psi(1) + \int_0^1 u^2 \left[ \mu \psi'(t) + \frac{1}{4} \psi'''(t) + \frac{1}{4} \frac{(n-1)(n-3)}{t^3} (\psi(t) - t \psi'(t)) \right] t^{-1} dt \\
= \int_0^1 u^2 \left[ -c_n tf'(t)\psi(t) + \frac{(n-1)}{n} f(t) \left( \psi(t) - r \psi'(t) \right) \right] t^{-2} dt.
\]
(2.18)
(Note that the Pohozaev identity [1.4] corresponds to the case where \( \psi(t) = t \). In order to get the desired contradiction, we need to choose a suitable function \( \psi \) as a solution of the following ordinary differential equation
\[
\mu \psi' + \frac{1}{4} \psi''' + \frac{1}{4} \frac{(n-1)(n-3)}{t^3} (\psi - t \psi') = 0, \quad \forall \, t \in (0, 1].
\]
(2.19)
A straightforward computation shows that the equation (2.19) has two solutions defined on \([0, 1]\) by a series \( \psi_1(t) = \sum_{p=0}^{+\infty} a_{2p+1} t^{2p+1} \) and \( \psi_2(t) = \sum_{p=0}^{+\infty} a_{2p} t^{2p} \), where
\[
a_{2p+1} = -\frac{2(2p-1)}{p[(2p+1)(2p-1) - (n-1)(n-3)]} a_{2p-1}, \quad \forall \, p \geq 1,
\]
(2.20)
\[
\left\{ \begin{array}{ll}
a_{2p} = 0, & \forall \, 0 \leq p < \frac{n-1}{2}; \\
a_{2p} = -\frac{8(p-1)}{(2p-1)[4(p-1) - (n-1)(n-3)]} a_{2p-2}, & \forall \, p \geq \frac{n+1}{2}.
\end{array} \right.
\]
(2.21)
Let \( a_1 > 0 \) and \( a_{n-1} < 0 \) be fixed. Note that \( \psi_1 \) and \( \psi_2 \) are smooth on \([0, 1]\). On the other hand, we claim that, for \( \mu \) small enough, we have
\[
\psi_1(t) > 0 \quad \text{and} \quad \psi_2(t) < 0, \quad \forall \, t \in (0, 1].
\]
(2.22)
Indeed, it is sufficient to remark that \( \psi_1 \) and \( \psi_2 \) satisfy the hypotheses of the alternating series theorem for \( \mu \) small enough. Thus there exists a constant \( \mu(n) > 0 \) depending only on \( n \), such that the claim (2.22) is valid for every \( \mu < \mu(n) \). Denoting \( \bar{\eta} \equiv -2nc_2K(y_0)\mu(n)/(n - 2)^2c_2k_2k(y_0) \), (2.22) enables us to choose \( a_1 \) and \( a_{n-1} \) such that
\[
\bar{\psi}(t) := \psi_1(t) + \psi_2(t) \geq 0, \quad \forall \, t \in [0, 1], \forall \, \mu \leq \mu(n).
\]
(2.23)
Regarding the identities (2.18) and (2.23) and in order to get the desired contradiction, it is sufficient to investigate a constant \( \bar{\eta} > 0 \) such that if \( \eta \leq \bar{\eta} \), then for any \( t \in (0, 1] \),
\[
- \frac{n-2}{2n} tf_1(t) \bar{\psi}(t) + \frac{n-1}{n} \frac{1}{f_1(t)} \left( \bar{\psi}(t) - t \bar{\psi}'(t) \right) > 0.
\]
(2.24)
Let \( 0 < \delta \leq 1 \) be a fixed constant and \( \delta \leq t \leq 1 \). Combining (2.13), (2.20) and (2.21) and using the fact that \( \mu \leq \mu(n) \) we obtain
\[
- \frac{n-2}{2n} tf_1(t) \bar{\psi}(t) + \frac{n-1}{n} f(t) \left( \bar{\psi}(t) - t \bar{\psi}'(t) \right) \\
= \frac{n-1}{n} a_1 f(t) \left( \frac{\eta}{f(0)} O_n(1) - \frac{(n-2)a_{n-1}}{a_1} t^{p-4} \left( 1 + \frac{\eta}{f(0)} O_n(1) \right) \right) t^3 \\
- \frac{n-2}{2n} t f_1(t) \bar{\psi}(t),
\]
(2.25)
where $|O_n(1)|$ is upper-bounded by a fixed constant $M$ depending only on $n$. Let $\eta_2 > 0$ be a constant such that, for any $0 < \eta \leq \eta_2$, (2.23) is satisfied and

$$- \frac{\eta}{f(0)} |O_n(1)| - \frac{(n-2)a_{n-1}}{a_1} \delta^n \left(1 - \frac{\eta}{f(0)} |O_n(1)| \right) > 0. \quad (2.26)$$

Combining (2.23), (2.25), (K_\eta), and (2.26) we obtain (2.24) for any $\delta \leq t \leq 1$ and any $0 < \eta \leq \min(\eta_2, \eta_3)$. Observe that if we let $\delta$ tend to 0, then to regain (2.26) for $\eta \leq \eta_2$, $\eta_2$ must go to 0; this fact leads to the loss of (2.24). Thus we have to fix the constant $\delta$ and we need another argument for the case $0 < t \leq \delta$. To this end, we will take care of the local information about the function $f_1(t)$ near its critical point 0. First, let us observe that the condition (K_3) implies the existence of two constants $\delta, M_0 > 0$ such that, for any $0 < t \leq \delta$,

$$0 \leq f_1''(t) - tf_1''(0) \leq M_0 t^{n-3} \quad \text{or} \quad f_1''(t) - tf_1''(0) \leq 0. \quad (2.27)$$

In particular, we deduce from (2.23) and (2.27) that, for any $0 < t \leq \delta$,

$$- (f_1''(t) - tf_1''(0)) \psi(t) \geq -M_0 t^{n-3} \psi(t) \geq -M_0 t^{n-2} (a_1 - a_{n-1}) |O_n(1)|, \quad (2.28)$$

where $|O_n(1)|$ is upper-bounded by a constant $M_n$ depending only on $n$. Now, by combining (2.14), (2.20), (2.21) and (2.23) and using the fact that $\mu \leq \mu(n)$ we obtain

$$- \frac{n-2}{2n} \eta \eta f_1''(0) \psi(t) + \frac{n-1}{n} (f(0) + \eta f_1(t))(\psi(t) - t \psi'(t))$$

$$= \left( - \frac{n-2}{2n} \eta f_1''(0) a_1 - 2 f(0) \frac{n-1}{n} a_3 \right) t^3 - \frac{n-2}{2n} \eta \eta f_1''(0) (f_1'(t) - tf_1''(0)) \psi(t)$$

$$+ \left( - \frac{n-2}{2n} + \frac{n-1}{n} \right) \eta f_1''(0) a_3 - 4 f(0) \frac{n-1}{n} a_5 + \frac{\eta^2 \mu}{f(0) a_1 O_{n,f_1}(1)} \right) t^5$$

$$+ f(0) \left( - \frac{n-2(n-1)}{2n} a_{n-1} + \frac{\eta}{f(0)} O_{n,f_1}(1)(a_{n-1} + a_1) \right) t^{n-1}, \quad (2.29)$$

where $|O_{n,f_1}(1)|$ is upper-bounded by a fixed constant $M_{n,f_1}$ depending only on $n$ and the function $f_1(x)$. Finally, by using (2.20), (2.14) and the fact that $n \neq 7-19$ and that $c_2/c_3 = n(n-4)/4(n-1)(n-2)$ we get

$$- \frac{n-2}{2n} \eta f_1''(0) a_1 - 2 f(0) \frac{n-1}{n} a_3 \geq 0, \quad (2.30)$$

$$\frac{1}{\eta a_1} \left( - \frac{n-2}{2n} + \frac{n-1}{n} \right) \eta f_1''(0) a_3 - 4 f(0) \frac{n-1}{n} a_5 \geq M > 0, \quad (2.31)$$

where $M$ is a constant depending only on $n$. Combining (2.28)–(2.31) and taking $\eta_1 > 0$ small enough such that, for any $0 < \eta \leq \eta_1$,

$$- \frac{(n-2)(n-1)}{a_{n-1}} - \frac{\eta}{f(0)} (-a_{n-1} + a_1) \left( |O_{n,f_1}(1)| + \frac{n-2}{2n} M_0 |O_n(1)| \right) > 0,$$

$\quad M - \frac{\eta}{f(0)} |O_{n,f_1}(1)| > 0,$
we get (2.24) for any $0 < t \leq \delta$ and any $0 < \eta \leq \min(\bar{\eta}_1, \bar{\eta}_3)$. The proof of Theorem 1.2 follows by choosing $\bar{\eta} = \min(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3)$.

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