Bottom crossing probability for symmetric jump processes

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Abstract

We determine the decay rate of the bottom crossing probability for symmetric jump processes under the condition on heat kernel estimates. Our results are applicable to symmetric stable-like processes and stable-subordinated diffusion processes on a class of (unbounded) fractals and fractal-like spaces.

1 Introduction

We are concerned with the quantitative characterizations of transience and (non-point) recurrence for symmetric jump processes generated by regular Dirichlet forms. Such characterizations are expressed in terms of lower rate functions. In this paper, we discuss the long time asymptotic estimates of the bottom crossing probability related to lower rate functions.

For $\alpha \in (0, 2]$, let $M = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d})$ be the symmetric $\alpha$-stable process on $\mathbb{R}^d$. If $\alpha = 2$, then $M$ is the Brownian motion on $\mathbb{R}^d$. As is well known, if $d > \alpha$, then $M$ is transient in the sense that the particle escapes to infinity eventually with probability one. On the other hand, if $d = \alpha (= 1$ or $2$), then $M$ is non-point recurrent in the sense that the particle comes arbitrarily close to the origin but never hits it with probability one. We can characterize these two properties quantitatively as follows: Assume that $d \geq \alpha$.

If $g(t)$ is a positive decreasing function on $(0, \infty)$ such that $g(t) \to 0$ as $t \to \infty$, then the function $r(t) = t^{1/\alpha} g(t)$ satisfies

$$P \left( \text{there exists } T > 0 \text{ such that } |X_t| \geq r(t) \text{ for all } t \geq T \right) = 1 \text{ or } 0 \quad (1.1)$$

according as

$$\int_0^\infty h_g(t) \frac{dt}{t} < \infty \text{ or } = \infty \quad (1.2)$$

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for $P = P_0$ and

$$h_g(t) = \begin{cases} 
  g(t)^{d-\alpha} & (d > \alpha), \\
  \frac{1}{|\log g(t)|} & (d = \alpha).
\end{cases}$$

When $\alpha = 2$, Dvoretzky and Erdös [14] and Spitzer [33] established this integral test for $d \geq 3$ and $d = 2$, respectively (see also [23, 4.12]). When $0 < \alpha < 2$, J. Takeuchi [35] and J. Takeuchi and S. Watanabe [36] obtained the test for $d > \alpha$ and $d = \alpha = 1$, respectively. If the probability in (1.1) is one, then the function $r(t)$ is called a lower rate function of $M$. This function expresses how fast the particle escapes to infinity for $d > \alpha$, and how arbitrarily close it comes to the origin for $d = \alpha$. We can regard this function as the bottom of $M$ for all sufficiently large time.

Wichura [38] (see also [37] in the Brownian case) further proved that if the integral in (1.2) is convergent, then there exists a positive constant $L_{d,\alpha}$ such that

$$P (|X_t| < r(t) \text{ for some } t > T) = (1 + o(1)) \int_T^\infty h_g(s) \frac{ds}{s} \quad (T \to \infty) \quad (1.3)$$

under some additional condition on the function $g(t)$. This equality gives the precise asymptotic behavior of the bottom crossing probability and related it to the integral in (1.2).

The integral tests on lower rate functions are extended to more general symmetric diffusion processes (see, e.g., [6, 22, 17]) and symmetric jump processes (see, e.g., [21, 25, 31, 32]). Among them, the full heat kernel estimates are utilized in [6, 22, 32] to establish zero-one law type results. Our purpose in this paper is to determine the decay rate of the bottom crossing probability for a class of symmetric jump processes with no scaling property (see Theorems 2.4 and 2.6 below).

Our approach here is based on that of Wichura [38]. However, the scaling property and the rotation invariance of symmetric stable processes on $\mathbb{R}^d$ played a crucial role in his proof. Instead of these properties, we make use of the full heat kernel estimates by following [26] and [32]. Our results are applicable to symmetric stable-like processes (see [9, 11, 12]) and a class of symmetric jump processes on (unbounded) fractals and fractal-like spaces (see Section 3 below for details).

The rest of this paper is organized as follows: In Section 2, we first make assumptions on heat kernels and introduce the notion of lower rate functions. We then state our main results in this paper. In Section 3, we first calculate the decay rate of the bottom crossing probability for some lower rate functions. We then give examples to which our main results are applicable. In Section 4, we give estimates on the hitting time distributions of a process to closed balls during finite time interval. Using these estimates, we prove our result for the transient case in Section 5. The proof for the non-point recurrent case is given in Appendix A because this proof is similar to that for the transient case. Appendix B is devoted to the calculation of Dirichlet forms and heat kernels for a class of subordinated diffusion processes, which will be mentioned in Section 3.

Throughout this paper, the letters $c$ and $C$ (with subscript) denote finite positive constants which may vary from place to place. For positive functions $f(t)$ and $g(t)$ on
(1, ∞), we write \( f(t) \asymp g(t) \ (t \to \infty) \) if there exist positive constants \( T, c_1 \) and \( c_2 \) such that \( c_1 g(t) \leq f(t) \leq c_2 g(t) \) for all \( t \geq T \). For nonnegative functions \( f(x) \) and \( g(x) \) on a space \( S \), we write \( f(x) \lesssim g(x) \) (or \( g(x) \gtrsim f(x) \)) if there exists a positive constant \( c \) such that \( f(x) \leq cg(x) \) for all \( x \in S \). We also write \( f(x) \asymp g(x) \) if \( f(x) \lesssim g(x) \) and \( g(x) \lesssim f(x) \).

2 Results

We first recall the notion of Dirichlet forms from \[10\] and \[16\]. Let \((M, d)\) be a locally compact separable metric space and \( m \) a positive Radon measure on \( M \) with full support. We write \( C(M) \) for the totality of continuous functions on \( M \) and \( C_0(M) \) for that of continuous functions on \( M \) with compact support. Let \((\mathcal{E}, \mathcal{F})\) be a Dirichlet form on \( L^2(M; m) \), that is, \((\mathcal{E}, \mathcal{F})\) is a closed Markovian symmetric form on \( L^2(M; m) \). In what follows, we suppose that \((\mathcal{E}, \mathcal{F})\) is regular: \( \mathcal{F} \cap C_0(M) \) is dense both in \( L^2(M; m) \) with respect to the norm \( \| u \|_{\mathcal{E}} = \sqrt{\mathcal{E}_1(u, u)} \), and in \( C_0(M) \) with respect to the uniform norm \( \| \cdot \|_{\infty} \). Here

\[ \mathcal{E}_1(u, u) = \mathcal{E}(u, u) + \| u \|_{L^2(M; m)}^2, \quad u \in \mathcal{F}. \]

Let \( \mathcal{O} \) be the family of all open subsets of \( M \). For \( A \in \mathcal{O} \), we let

\[ \mathcal{L}_A = \{ u \in \mathcal{F} \mid u \geq 1 \text{ m.a.e. on } A \} \]

and

\[ \text{Cap}(A) = \begin{cases} \inf_{u \in \mathcal{L}_A} \mathcal{E}_1(u, u), & \text{if } \mathcal{L}_A \neq \emptyset, \\ \infty, & \text{if } \mathcal{L}_A \neq \emptyset. \end{cases} \]

For any \( A \subset M \), we define the (1-)capacity of \( A \) by

\[ \text{Cap}(A) := \inf_{B \in \mathcal{O}, A \subset B} \text{Cap}(B). \]

For \( A \subset M \), a statement depending on \( x \in A \) is said to hold q.e. on \( A \) if there exists a set \( N \subset A \) of zero capacity such that the statement holds for every \( x \in A \setminus N \). Here q.e. is an abbreviation for quasi everywhere.

We suppose that the Beurling-Deny expression of \((\mathcal{E}, \mathcal{F})\) (see \[16\], Theorem 3.2.1, Lemma 4.5.4)) is given by

\[ \mathcal{E}(u, v) = \iint_{M \times M \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y)) J(dx dy) \quad \text{for } u, v \in \mathcal{F} \cap C_0(M), \quad (2.1) \]

where \( \text{diag} = \{(x, y) \in M \times M \mid x = y\} \) and \( J(dx dy) \) is a symmetric positive Radon measure on \( M \times M \setminus \text{diag} \). We call \( J \) the jumping measure associated with \((\mathcal{E}, \mathcal{F})\).

We write \( \mathcal{B}(M) \) for the family of all Borel measurable subsets of \( M \). Let \( M_\Delta = M \cup \{ \Delta \} \) be the one point compactification of \( M \) and \( \mathcal{B}(M_\Delta) = \mathcal{B}(M) \cup \{ B \cup \{ \Delta \} \mid B \in \mathcal{B}(M) \} \). Let \( M = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in M}) \) be an \( m \)-symmetric Hunt process on \( M \) generated by \((\mathcal{E}, \mathcal{F})\) (\[16\]...
Theorem 7.2.1]). A set $B \subset M_\Delta$ is called nearly Borel measurable if for any probability measure $\mu$ on $M_\Delta$, there exist $B_1, B_2 \in \mathcal{B}(M_\Delta)$ such that $B_1 \subset B \subset B_2$ and
\[ P_\mu(X_t \in B_2 \setminus B_1 \text{ for some } t \geq 0) = 0. \]

By [16, Theorem 4.2.1], a set $N \subset M$ is of zero capacity if and only if $N$ is exceptional, that is, there exists a nearly Borel measurable set $\tilde{N} \supset N$ such that $P_m(\sigma_{\tilde{N}} < \infty) = 0$. Here $\sigma_{\tilde{N}} = \inf\{t > 0 \mid X_t \in \tilde{N}\}$ is the hitting time of $M$ to $\tilde{N}$. We say that a set $N \subset M$ is properly exceptional if $N$ is nearly Borel measurable such that $m(N) = 0$ and $M \setminus N$ is $M$-invariant, that is,
\[ P_x(X_t \in (M \setminus N)_\Delta \text{ and } X_{t-} \in (M \setminus N)_\Delta \text{ for any } t > 0) = 1 \quad \text{for any } x \in M \setminus N. \]
Here $(M \setminus N)_\Delta = (M \setminus N) \cup \{\Delta\}$ and $X_{t-} = \lim_{s \uparrow t} X_s$. Note that any properly exceptional set $N$ is exceptional and thus $\text{Cap}(N) = 0$ by [16, Theorem 4.2.1].

For $x \in M$ and $r > 0$, let $B(x, r) = \{y \in M \mid d(y, x) < r\}$ be an open ball with radius $r$ centered at $x$ and let $V(x, r) = m(B(x, r))$. We make the following three assumptions:

**Assumption 2.1.** For any $x \in M$ and $r > 0$, $V(x, r) < \infty$. Moreover, there exist constants $c_1 \in (0, 1]$, $c_2 \in [1, \infty)$, $d_1 > 0$ and $d_2 \geq d_1$ such that
\[ c_1 \left( \frac{R}{r} \right)^{d_1} \leq \frac{V(x, R)}{V(x, r)} \leq c_2 \left( \frac{R}{r} \right)^{d_2} \quad \text{for all } x \in M \text{ and } 0 < r < R. \tag{2.2} \]

Under Assumption 2.1, the diameter of $M$ is infinity. We see from [19, Proposition 5.2] that if $M$ is non-compact and connected, and if $B(x, r)$ is relatively compact for any $x \in M$ and $r > 0$, then Assumption 2.1 is fulfilled under the condition that for some $c > 0$ and $d > 0$,
\[ \frac{V(x, R)}{V(x, r)} \leq c \left( \frac{R}{r} \right)^d \quad \text{for all } x \in M \text{ and } 0 < r < R. \]

**Assumption 2.2.** There exist a properly exceptional Borel set $N$ and a nonnegative symmetric kernel $p(t, x, y)$ on $(0, \infty) \times (M \setminus N) \times (M \setminus N)$ such that
\[ P_x(X_t \in A) = \int_A p(t, x, y) \, m(dy), \quad t \geq 0, \quad A \in \mathcal{B}(M) \]
for any $x \in M \setminus N$ and
\[ p(t + s, x, y) = \int_M p(t, x, z)p(s, z, y) \, m(dz) \]
for any $x, y \in M \setminus N$ and $t, s > 0$.

The nonnegative symmetric kernel $p(t, x, y)$ in Assumption 2.2 is called the heat kernel of $M$. 

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Assumption 2.3. There exist positive constants $L_1$ and $L_2$ such that for any $(t, x, y) \in (0, \infty) \times (M \setminus N) \times (M \setminus N)$,

$$L_1 \min \left\{ \frac{1}{V(x, \phi^{-1}(t))}, \frac{t}{V(x, \phi^{-1}(t))} \right\} \leq p(t, x, y) \leq L_2 \min \left\{ \frac{1}{V(x, \phi^{-1}(t))}, \frac{t}{V(x, \phi^{-1}(t))} \right\}.$$  \hspace{1cm} (2.3)

Here $\phi(r)$ is some positive increasing function on $[0, \infty)$ such that

$$\phi(0) = 0, \quad c_3 \left( \frac{R}{r} \right)^{d_3} \leq \frac{\phi(R)}{\phi(r)} \leq c_4 \left( \frac{R}{r} \right)^{d_4} \quad (0 < r < R),$$  \hspace{1cm} (2.4)

for some constants $c_3 \in (0, 1]$, $c_4 \in [1, \infty)$, $d_3 > 0$ and $d_4 \geq d_3$.

It follows by (2.4) that

$$\frac{1}{c_4^{1/d_4}} \left( \frac{R}{r} \right)^{1/d_4} \leq \frac{\phi^{-1}(R)}{\phi^{-1}(r)} \leq \frac{1}{c_3^{1/d_3}} \left( \frac{R}{r} \right)^{1/d_3} \quad (0 < r < R).$$  \hspace{1cm} (2.5)

A positive function $r(t)$ on $(0, \infty)$ is called a lower rate function of $M$ if

$$P_x \left( \text{there exists } T > 0 \text{ such that } d(X_t, x) > r(t) \text{ for all } t > T \right) = 1$$

for q.e. $x \in M$. For a lower rate function $r(t)$, the probability

$$q_r(t, x) := P_x \left( d(X_u, x) \leq r(u) \text{ for some } u > t \right) \quad (x \in M, \ t > 0),$$

tends to 0 as $t \to \infty$ for q.e. $x \in M$. In what follows, we find the decay rate of $q_r(t, x)$ as $t \to \infty$.

2.1 Transient case

We first assume that $d_1 > d_4$. Then $M$ is transient in the sense that

$$P_x \left( \lim_{t \to \infty} X_t = \Delta \right) = 1 \quad \text{for q.e. } x \in M.$$  

If $B(x, r)$ is relatively compact for any $x \in M$ and $r > 0$, then the transience is equivalent to

$$P_x \left( \lim_{t \to \infty} d(x, X_t) = \infty \right) = 1 \quad \text{for q.e. } x \in M.$$  

If we assume in addition that $M$ is connected and

$$V(x, r) \asymp V(r)$$

for some positive function $V(r)$ on $(0, \infty)$, then the following integral test on the lower rate functions is derived in [32]: let $g(t)$ be a positive decreasing function on $(0, \infty)$ such that $g(t) \to 0$ as $t \to \infty$ and $\phi(t) = \phi^{-1}(t)g(t)$. Then

$$P_x \left( \text{there exists } T > 0 \text{ such that } d(X_u, x) > \phi(u) \text{ for all } u \geq T \right) = 1 \text{ or } 0$$  \hspace{1cm} (2.6)
for q.e. \( x \in M \) according as
\[
\int_{1}^{\infty} \frac{V(\varphi(t))}{\phi(\varphi(t))} \frac{dt}{V(\phi^{-1}(t))} < \infty \text{ or } = \infty. \tag{2.7}
\]

Fix a positive decreasing function \( g(t) \) on \((0, \infty)\) such that \( g(t) \to 0 \) as \( t \to \infty \). Define for \( t > 0 \) and \( c > 1 \),
\[
R_{c,t} = \inf \left\{ \frac{g(u)}{g(v)} \mid 1 \leq \frac{u}{v} \leq c, \ v \geq t \right\}.
\]

We then have

**Theorem 2.4.** Suppose that Assumptions 2.1–2.3 are fulfilled and \( d_1 > d_4 \). Let \( g(t) \) be a positive decreasing function on \((0, \infty)\) such that \( g(t) \to 0 \) as \( t \to \infty \) and \( \varphi(t) = \phi^{-1}(t)g(t) \). For q.e. \( x \in M \), if
\[
\int_{1}^{\infty} \frac{V(x, \varphi(t))}{\phi(\varphi(t))} \frac{dt}{V(x, \phi^{-1}(t))} < \infty
\]
and \( \lim_{c \to 1+0} (\lim_{t \to \infty} R_{c,t}) = 1 \), then
\[
q \varphi(t, x) \asymp \int_{t}^{\infty} \frac{V(x, \varphi(s))}{\phi(\varphi(s))} \frac{ds}{V(x, \phi^{-1}(s))} \quad (t \to \infty).
\]

**Remark 2.5.** Under Assumptions 2.1–2.3, \((E, F)\) is conservative and the heat kernel \( p(t, x, y) \) satisfies (2.3) for all \((t, x, y) \in (0, \infty) \times M \times M\) by [13, Theorem 1.13 and Proposition 3.1]. We can thus take a version of the process \( M \) such that Theorem 2.4 is valid for any \( x \in M \). Moreover, there exists a nonnegative measurable function \( J(x, y) \) on \( M \times M \setminus \text{diag} \) such that \( J(dx dy) = J(x, y) m(dx)m(dy) \) and
\[
J(x, y) \asymp \frac{1}{V(x, d(x, y))\phi(d(x, y))}.
\]

### 2.2 Recurrent case

We next assume that for some positive constants \( c_{v,1} \) and \( c_{v,2} \),
\[
c_{v,1}\phi(r) \leq V(x, r) \leq c_{v,2}\phi(r) \quad \text{for all } x \in M \text{ and } r > 0. \tag{2.8}
\]

Then \( M \) is irreducible recurrent by [32, Remark 2.2]. Moreover, we can show that \( M \) can not hit any point by following the proof of [32, Proposition 4.24] and using Lemma 4.2 below. These two properties imply that
\[
P_x \left( \liminf_{t \to \infty} d(x, X_t) = 0 \text{ and } d(x, X_t) > 0 \text{ for all } t > 0 \right) = 1 \quad \text{for q.e. } x \in M.
\]

We proved in [32] that if \( M \) is, in addition, connected, then (2.6) is valid according as
\[
\int_{1}^{\infty} \frac{dt}{t|\log g(t)|} < \infty \text{ or } = \infty. \tag{2.9}
\]

For \( t > 0 \) and \( c > 1 \), we define
\[
R_{c,t} = \sup \left\{ \frac{|\log g(u)|}{|\log g(v)|} \mid 1 \leq \frac{u}{v} \leq c, \ v \geq t \right\}.
\]
Theorem 2.6. Suppose that Assumptions 2.1, 2.3 and (2.8) are fulfilled. Let \( g(t) \) be a positive decreasing function on \( (0, \infty) \) such that \( g(t) \to 0 \) as \( t \to \infty \) and \( \varphi(t) = \phi^{-1}(t)g(t) \). If the integral in (2.9) is convergent and \( \lim_{c \to 1+0} \left( \lim_{t \to \infty} R_{c,t} \right) = 1 \), then

\[
q_{\varphi}(t, x) \asymp \int_{t}^{\infty} \frac{ds}{s|\log g(s)|} \quad (t \to \infty)
\]

for q.e. \( x \in M \).

Remarks similar to those just after Theorem 2.4 are also valid in Theorem 2.6.

3 Examples

We first apply Theorems 2.4 and 2.6 to some lower rate functions.

Example 3.1. Suppose that for some positive constants \( \alpha_1 \) and \( \alpha_2 \),

\[
V(x, r) \asymp r^{\alpha_1}1_{\{r < 1\}} + r^{\alpha_2}1_{\{r \geq 1\}}. \tag{3.1}
\]

Then Assumption 2.1 is fulfilled by \( d_1 = \alpha_1 \wedge \alpha_2 \) and \( d_2 = \alpha_1 \vee \alpha_2 \). We also suppose that Assumptions 2.2 and 2.3 are satisfied by the functions

\[
\phi(r) = r^{\beta_1}1_{\{r < 1\}} + r^{\beta_2}1_{\{r \geq 1\}} \tag{3.2}
\]

for some positive constants \( \beta_1 \) and \( \beta_2 \). Then \( d_3 = \beta_1 \wedge \beta_2 \) and \( d_4 = \beta_1 \vee \beta_2 \). Let \( g(t) \) be a positive decreasing function on \( (0, \infty) \) such that \( g(t) \to 0 \) as \( t \to \infty \) and \( \varphi(t) = \phi^{-1}(t)g(t) \).

We first assume that \( \alpha_1 \wedge \alpha_2 > \beta_1 \vee \beta_2 \). If \( g(t) \) satisfies the full assumptions in Theorem 2.4, then as \( t \to \infty \),

\[
q_{\varphi}(t, x) \asymp \begin{cases}
\int_{t}^{\infty} \varphi(s)^{\alpha_2 - \beta_2} \frac{ds}{s^{\alpha_2}} & \text{if } \varphi(t) \to \infty \quad (t \to \infty), \\
\frac{1}{t^{\frac{\alpha_2 - \beta_2}{\alpha_2}}} & \text{if } \varphi(t) \asymp 1 \quad (t \to \infty), \\
\int_{t}^{\infty} \varphi(s)^{\alpha_1 - \beta_1} \frac{ds}{s^{\alpha_1}} & \text{if } \varphi(t) \to 0 \quad (t \to \infty).
\end{cases}
\]

This implies that if \( \varphi(t) = t^{1/\beta_2}/(\log t)^{\frac{1+\varepsilon}{\beta_2}} \) for some \( \varepsilon > 0 \), then

\[
q_{\varphi}(t, x) \asymp \frac{1}{\varepsilon(\log t)^{\varepsilon}} \quad (t \to \infty). \tag{3.3}
\]

On the other hand, if \( \varphi(t) = t^p \) for some \( p < 1/\beta_2 \), then

\[
q_{\varphi}(t, x) \asymp \begin{cases}
\frac{1}{t^{\left(\frac{1}{\beta_2} - p\right)(\alpha_2 - \beta_2)}} & \text{if } 0 \leq p < \frac{1}{\beta_2}, \\
\frac{1}{t^{\frac{1}{\beta_2}((\alpha_2 - \beta_2) - p(\alpha_1 - \beta_1))}} & \text{if } p < 0.
\end{cases}
\]
We next assume that \( \alpha_1 = \beta_1 \) and \( \alpha_2 = \beta_2 \). If \( g(t) \) satisfies the full assumptions in Theorem 2.6, then
\[
q_{\varphi}(t, x) \asymp \int_t^{\infty} \frac{ds}{s|\log g(s)|} \quad (t \to \infty).
\]
Hence if \( \varphi(t) = e^{-(\log t)^{1+\varepsilon}} \) for some \( \varepsilon > 0 \), then we get (3.6). On the other hand, if \( \varphi(t) = e^{-tp} \) for some \( p > 0 \), then
\[
q_{\varphi}(t, x) \asymp \frac{1}{t^p} \quad (t \to \infty).
\]

We now provide examples satisfying Assumptions 2.2 and 2.3. Suppose that

- \( B(x, r) \) is relatively compact for any \( x \in M \) and \( r > 0 \);
- the distance \( d \) on \( M \) is geodesic: for any \( x, y \in M \), there exists a continuous map \( \gamma : [0, 1] \to M \) such that \( \gamma(0) = x, \gamma(1) = y \) and
  \[
d(\gamma(t), \gamma(s)) = |t - s|d(x, y) \quad \text{for any } t, s \in [0, 1].
\]

Let \( V(r) \) and \( \phi(r) \) satisfy (3.1) and (3.2) with \( \beta_1, \beta_2 \in (0, 2) \), respectively. Let \( c(x, y) \) be a uniformly positive and bounded function on \( M \) and
\[
\mathcal{E}(u, v) = \int\int_{M \times M \setminus \text{diag}} \frac{(u(x) - u(y))(v(x) - v(y))}{V(d(x, y))\phi(d(x, y))} c(x, y) \, m(dx)m(dy)
\]
for \( u, v \in L^2(\mathcal{M}; m) \) if the right hand side above makes sense. Denote by \( \mathcal{F} \) the \( \mathcal{E}_1 \)-closure of the totality of Lipschitz continuous functions on \( M \) with compact support. Then \( (\mathcal{E}, \mathcal{F}) \) is a regular Dirichlet form on \( L^2(\mathcal{M}; m) \) so that there exists an associated \( m \)-symmetric Hunt process \( \mathcal{M} \) on \( M \) of pure jump type. Furthermore, Assumptions 2.2 and 2.3 are fulfilled according to [12] (see also [9, 11, 26]).

**Example 3.2.** Suppose that Assumption 2.1 is fulfilled. Let \( \mathcal{M} \) be a conservative \( m \)-symmetric diffusion process on \( M \) such that the associated Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) is regular on \( L^2(\mathcal{M}; m) \). Suppose further that \( \mathcal{M} \) admits the heat kernel \( p(t, x, y) \) such that for some positive constants \( c_i, C_i \) \( (1 \leq i \leq 4) \) and \( \beta_1, \beta_2 \in [2, \infty) \), the following hold:

- if \( 0 < t \leq 1 \lor d(x, y) \), then
  \[
  \frac{c_1}{V(x, t^{1/\beta_1})} \exp\left\{ -C_1 \left( \frac{d(x, y)^{\beta_1}}{t} \right)^{\frac{1}{\beta_1 - 1}} \right\} \leq p(t, x, y)
  \]
  \[
  \leq \frac{c_2}{V(x, t^{1/\beta_1})} \exp\left\{ -C_2 \left( \frac{d(x, y)^{\beta_1}}{t} \right)^{\frac{1}{\beta_1 - 1}} \right\};
  \]

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• if \( t \geq 1 \lor d(x, y) \), then

\[
\frac{c_3}{V(x, t^{1/\beta_2})} \exp \left\{ -C_3 \left( \frac{d(x, y)^{\beta_2}}{t} \right)^{\frac{1}{\beta_2 - 1}} \right\} \leq p(t, x, y) \\
\leq \frac{c_4}{V(x, t^{1/\beta_2})} \exp \left\{ -C_4 \left( \frac{d(x, y)^{\beta_2}}{t} \right)^{\frac{1}{\beta_2 - 1}} \right\}.
\]

If \( \beta_1 = \beta_2 = 2 \), then the heat kernel \( p(t, x, y) \) admits the so-called Gaussian estimates. Here we allow \( \beta_1 \) and \( \beta_2 \) to be different and greater than 2. Such situation occurs for a class of symmetric diffusion processes on (unbounded) fractals and fractal-like spaces such as Sierpiński carpets and pre-carpets (see, e.g., [1, 2, 3, 4, 29]. See also the summary just after [24, Proposition 5.5] on the history of the analysis on Sierpiński carpets).

For \( \gamma \in (0, 1) \), we let \( M^{(1)} \) be a \( \gamma \)-stable subordinated diffusion process of \( M \) (see Appendix B below for definition). According to [28, Theorem 2.1], the associated Dirichlet form \( (E^{(1)}, F^{(1)}) \) is regular on \( L^2(M; m) \) and

\[
E^{(1)}(u, u) \asymp \int_{M \times M \setminus \text{diag}} \frac{(u(x) - u(y))^2}{V(x, d(x, y))\phi(d(x, y))} m(dx)m(dy)
\]  

for any \( u \in F^{(1)} \cap C_0(M) \), where

\[
\phi(r) = r^{\gamma \beta_1}1_{\{r < 1\}} + r^{\gamma \beta_2}1_{\{r \geq 1\}}.
\]

Furthermore, \( M^{(1)} \) admits the heat kernel \( q(t, x, y) \) such that

\[
q(t, x, y) \asymp \min \left\{ \frac{1}{V(x, \phi^{-1}(t))}, \frac{t}{V(x, d(x, y))\phi(d(x, y))} \right\}.
\]  

Therefore, Assumptions 2.2 and 2.3 are valid for \( M^{(1)} \). We note that if \( \alpha_1 = \alpha_2 \) and \( \beta_1 = \beta_2 \), then \( q(t, x, y) \) is already computed in [8] and [27]. We show (3.4) and (3.5) in Appendix B below.

We finally apply Theorem 2.4 to subordinated diffusion processes under the non-uniform volume growth condition.

**Example 3.3.** Suppose that \( M = \mathbb{R}^d \) and \( m \) is the Lebesgue measure on \( \mathbb{R}^d \) (\( dx \) in notation). Let \( h(x) \) be a positive function on \( \mathbb{R}^d \) such that \( h(x) \asymp (1 + |x|^2)^{\alpha/2} \) for some \( \alpha > -d/2 \) and \( \mu(dx) = h(x)^2 dx \). We denote by \( C_0^\infty(\mathbb{R}^d) \) the totality of smooth functions on \( \mathbb{R}^d \) with compact support. Let \( \mathcal{E} \) be a bilinear form on \( C_0^\infty(\mathbb{R}^d) \times C_0^\infty(\mathbb{R}^d) \) defined by

\[
\mathcal{E}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x) \mu(dx), \quad u, v \in C_0^\infty(\mathbb{R}^d)
\]

and let \( \mathcal{F} \) be the closure of \( C_0^\infty(\mathbb{R}^d) \) with respect to the norm

\[
\|u\|_{\mathcal{E}^1} = \sqrt{\mathcal{E}(u, u) + \|u\|^2_{L^2(\mathbb{R}^d; \mu)}}.
\]
Then \((\mathcal{E}, \mathcal{F})\) is a regular Dirichlet form on \(L^2(\mathbb{R}^d; \mu)\) such that there exists a \(\mu\)-symmetric diffusion process \(M\) on \(\mathbb{R}^d\). According to [20 Subsection 4.3] and [34 Section 4] (see also [18 Corollary 6.11]), we have

\[
V(x, r) = \mu(B(x, r)) = r^d(1 + r + |x|)^{2\alpha}
\]

so that Assumption [2.1] is valid for \(d_1 = d\) and \(d_2 = d + 2\alpha\). Furthermore, the associated heat kernel \(p(t, x, y)\) satisfies for any \(x, y \in \mathbb{R}^d\) and \(t > 0\),

\[
\frac{c_1}{t^{d/2}(1 + \sqrt{t} + |x|)^{2\alpha}} \exp\left(-C_1 \frac{d(x, y)^2}{t}\right) \leq p(t, x, y)
\]

\[
\leq \frac{c_2}{t^{d/2}(1 + \sqrt{t} + |x|)^{2\alpha}} \exp\left(-C_2 \frac{d(x, y)^2}{t}\right).
\]

For \(\gamma \in (0, 1)\), we let \(M^{(1)}\) be a \(\gamma\)-stable subordinated diffusion process of \(M\). Then by the same argument as in Example [3.2], the associated Dirichlet form \((\mathcal{E}^{(1)}, \mathcal{F}^{(1)})\) is regular on \(L^2(\mathbb{R}^d; \mu)\) and

\[
\mathcal{E}^{(1)}(u, u) \asymp \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}} \frac{(u(x) - u(y))^2}{|x - y|^{d+2\gamma} (1 + |x - y| + |x|)^{2\alpha}} \mu(dx)\mu(dy)
\]

for any \(u \in C_0^\infty(\mathbb{R}^d)\). Furthermore, \(M^{(1)}\) admits the heat kernel \(q(t, x, y)\) such that

\[
q(t, x, y) \asymp \min\left\{\frac{1}{t^{d/(2\gamma)} (1 + t^{1/(2\gamma)} + |x|)^{2\alpha}}, \frac{t}{(1 + |x - y| + |x|)^{2\alpha}|x - y|^{d+2\gamma}}\right\}.
\]

Therefore, \(M^{(1)}\) is transient for \(d + 2\alpha > 2\gamma\).

Let \(q(t)\) be a positive decreasing function on \((0, \infty)\) such that \(q(t) \to 0\) as \(t \to \infty\) and \(\varphi(t) = t^{1/(2\gamma)} q(t)\). If \(d > 2\gamma\) and \(d + 2\alpha > 2\gamma\), then under the full conditions in Theorem [2.4]

\[
q_\varphi(t, x) \asymp \int_t^\infty \frac{\varphi(s)^{d-2\gamma}}{s^{d/(2\gamma)}} \left(\frac{1 + \varphi(s) + |x|}{1 + s^{1/(2\gamma)} + |x|}\right)^{2\alpha} ds \quad (t \to \infty).
\]

This implies that if \(\varphi(t) = t^{1/(2\gamma)} / (\log t)^{\frac{1 + \varepsilon}{d + 2\alpha - 2\gamma}}\) for some \(\varepsilon > 0\), then

\[
q_\varphi(t, x) \asymp \frac{1}{\varepsilon (\log t)^{\varepsilon}} \quad (t \to \infty).
\]

(3.6)

On the other hand, if \(\varphi(t) = t^p\) for some \(p < 1/(2\gamma)\), then

\[
q_\varphi(t, x) \asymp \begin{cases} \frac{1}{t^{\frac{d}{2\gamma} - p(d + 2\alpha - 2\gamma)}}, & \text{if } 0 \leq p < \frac{1}{2\gamma}, \\ \frac{1}{t^{\frac{1}{2\gamma}(d + 2\alpha - 2\gamma) - p(d - 2\gamma)}}, & \text{if } p < 0. \end{cases}
\]
4 Hitting time distributions

Throughout this section, we assume the full conditions in Theorem 2.4. For simplicity, we also assume that $N = \emptyset$ in Assumptions 2.2 and 2.3. For the proof of Theorem 2.4, we first give estimates of the hitting time distributions to closed balls during finite time interval. To do so, we use the next lemma which goes back to [25].

Lemma 4.1. ([32 Lemma 4.19]). Let $a$, $b$, $c$ and $r$ be positive constants. Then for any $x \in M$,

\[
\frac{\int_a^b P_x(d(X_u, x) \leq r) \, du}{2 \int_0^{b-a} \sup_{d(y,x) \leq r} P_y(d(X_u, y) \leq 2r) \, du} \leq P_x(d(X_s, x) \leq r \text{ for some } s \in (a,b]) \leq \frac{\int_a^{b+c} P_x(d(X_u, x) \leq 2r) \, du}{\int_0^{c} \inf_{d(y,x) \leq r} P_y(d(X_u, y) \leq r) \, du}.
\]

By the same way as in [32 Lemma 4.20], we have

\[
L_1 \min \left\{ 1, \frac{V(x, r)}{V(x, \phi^{-1}(t))} \right\} \leq P_x(d(X_t, x) \leq r) \leq L_2 \min \left\{ 1, \frac{V(x, r)}{V(x, \phi^{-1}(t))} \right\} \tag{4.1}
\]

for any $x \in M$, $r > 0$ and $t > 0$. Using this inequality, we have the following two lemmas.

Lemma 4.2. Let $a$, $b$, $c$ and $r$ be positive constants. If $\phi(r) \leq a \wedge c$, then for any $x \in M$,

\[
P_x(d(X_s, x) \leq r \text{ for some } s \in (a,b]) \leq K_1 \frac{V(x, r)}{\phi(r)} \int_a^{b+c} \frac{du}{V(x, \phi^{-1}(u))},
\]

where $K_1 = 2^{d_2} c_2 L_2 / L_1$.

Proof. Suppose that $\phi(r) \leq a \wedge c$. Then by (4.1) and (2.2), we have

\[
\int_a^{b+c} P_x(d(X_u, x) \leq 2r) \, du \leq L_2 \int_a^{b+c} \min \left\{ 1, \frac{V(x, 2r)}{V(x, \phi^{-1}(u))} \right\} \, du \leq 2^{d_2} c_2 L_2 \int_a^{b+c} \min \left\{ 1, \frac{V(x, r)}{V(x, \phi^{-1}(u))} \right\} \, du = 2^{d_2} c_2 L_2 V(x, r) \int_a^{b+c} \frac{du}{V(x, \phi^{-1}(u))}. \tag{4.2}
\]

We also see by (4.1) that if $0 < u \leq \phi(r)$, then for any $y \in M$,

\[
P_y(d(X_u, y) \leq r) \geq L_1 \min \left\{ 1, \frac{V(y, r)}{V(y, \phi^{-1}(u))} \right\} = L_1,
\]

which implies that

\[
\int_0^{c} \inf_{d(y,x) \leq r} P_y(d(X_u, y) \leq r) \, du \geq \int_0^{\phi(r)} \inf_{d(y,x) \leq r} P_y(d(X_u, y) \leq r) \, du \geq L_1 \phi(r).
\]

Therefore, the proof is completed by Lemma 4.1. \qed
Lemma 4.3. Let $a$, $b$ and $r$ be positive constants. If $\phi(r) \leq a$ and $\phi(2r) \leq b - a$, then for any $x \in M$,

$$P_x(d(X_s, x) \leq r \text{ for some } s \in (a, b]) \geq K_2 \frac{V(x, r)}{\phi(r)} \int_a^b \frac{du}{V(x, \phi^{-1}(u))}; \quad (4.3)$$

where

$$K_* = \frac{c_1 d_1/d_4}{c_1 d_1 - d_4}, \quad K_2 = \frac{L_1}{L_2 c_4 2^{d_4+1} (1 + (c_2)^2 3^d K_*)}.$$

Proof. Suppose that $\phi(r) \leq a$ and $\phi(2r) \leq b - a$. Then by (4.1),

$$\int_a^b P_x(d(X_u, x) \leq r) \, du \geq L_1 \int_a^b \min \left\{1, \frac{V(x, r)}{V(x, \phi^{-1}(u))}\right\} \, du$$

$$= L_1 V(x, r) \int_a^b \frac{du}{V(x, \phi^{-1}(u))}.$$

If $d(y, x) \leq r$, then we have $B(y, 2r) \subset B(x, 3r)$ by the triangle inequality so that $V(y, 2r) \leq V(x, 3r)$. If we assume in addition that $u \geq \phi(2r)$, then

$$\phi^{-1}(u) - d(y, x) \geq \phi^{-1}(u) - r \geq \frac{1}{2} \phi^{-1}(u)$$

so that

$$V(y, \phi^{-1}(u)) \geq V(x, \phi^{-1}(u) - d(y, x)) \geq V(x, \phi^{-1}(u)/2) \geq \frac{1}{c_2 2^d} V(x, \phi^{-1}(u))$$

by the triangle inequality and (2.2). Hence (4.1) implies that

$$P_y(d(X_u, y) \leq 2r) \leq L_2 \frac{V(y, 2r)}{V(y, \phi^{-1}(u))} \leq L_2 c_2 2^d \frac{V(x, 3r)}{V(x, \phi^{-1}(u))}.$$

Since this inequality yields that

$$\int_0^{b-a} \sup_{d(y, x) \leq r} P_y(d(X_u, y) \leq 2r) \, du \leq L_2 \left(\phi(2r) + c_2 2^d V(x, 3r) \int_{\phi(2r)}^{b-a} \frac{du}{V(x, \phi^{-1}(u))}\right),$$

we have by Lemma 4.1

$$P_x(d(X_s, x) \leq r \text{ for some } s \in (a, b])$$

$$\geq \frac{L_1}{2L_2} V(x, r) \int_a^b \frac{du}{V(x, \phi^{-1}(u))} \frac{1}{\phi(2r) + c_2 2^d V(x, 3r) \int_{\phi(2r)}^{b-a} \frac{du}{V(x, \phi^{-1}(u))}}. \quad (4.4)$$

Fix $\theta > 1$ and take $t_m = t \theta^m \ (m \geq 0)$. Then

$$\int_t^{\infty} \frac{du}{V(x, \phi^{-1}(u))} = \sum_{m=0}^{\infty} \int_{t_m}^{t_{m+1}} \frac{du}{V(x, \phi^{-1}(u))} \leq \sum_{m=0}^{\infty} \frac{t_{m+1} - t_m}{V(x, \phi^{-1}(t_m))}$$

$$= t \sum_{m=0}^{\infty} \frac{\theta^{m+1} - \theta^m}{V(x, \phi^{-1}(t_m))}.$$
Since \( d_1 > d_4 \) by assumption and

\[
\frac{V(x, \phi^{-1}(t_m))}{V(x, \phi^{-1}(t))} \geq \frac{c_1}{c_4^{d_1/d_4}} (\theta^m)^{d_1/d_4} 
\]

by (2.2) and (2.5), the last expression of (4.5) is less than

\[
\frac{c_4^{d_1/d_4}}{c_1} \frac{t}{V(x, \phi^{-1}(t))} (\theta - 1) \sum_{m=0}^{\infty} (\theta^m)^{1-d_1/d_4} = \frac{c_4^{d_1/d_4}}{c_1} \frac{\theta - 1}{1 - \theta^{1-d_1/d_4}} \frac{t}{V(x, \phi^{-1}(t))}. 
\]

Then by letting \( \theta \to 1 + 0 \), we get

\[
\int_{t}^{\infty} \frac{du}{V(x, \phi^{-1}(u))} \leq K_* \frac{t}{V(x, \phi^{-1}(t))} 
\]

so that

\[
\int_{\phi(2r)}^{b-a} \frac{du}{V(x, \phi^{-1}(u))} \leq \int_{\phi(2r)}^{\infty} \frac{du}{V(x, \phi^{-1}(u))} \leq K_* \frac{\phi(2r)}{V(x, 2r)}. 
\]

Since (2.2) and (2.4) imply that

\[
\frac{V(x, 3r)}{V(x, 2r)} \leq c_2 \left( \frac{3}{2} \right)^{d_2} 
\]

and \( \phi(2r) \leq c_4 2^{d_4} \phi(r) \), respectively, we obtain (4.3) by (4.4). \( \square \)

5 Proof of Theorem 2.4

In this section, we prove Theorem 2.4 by using the results in Section 4. More precisely, we show that

\[
\limsup_{t \to \infty} \frac{q_\varphi(t, x)}{\int_t^\infty \frac{V(x, \varphi(s))}{\phi(\varphi(s))} \frac{ds}{V(x, \phi^{-1}(s))}} \leq \frac{L_2}{L_1} \frac{2^{d_2}(c_2)^2}{c_3^{d_2/d_3}} \frac{d_1 - d_4}{(d_1 - d_4)c_1 + 3^{d_2}d_4(c_2)^2c_4^{d_1/d_4}}. 
\] (5.1)

and

\[
\liminf_{t \to \infty} \frac{q_\varphi(t, x)}{\int_t^\infty \frac{V(x, \varphi(s))}{\phi(\varphi(s))} \frac{ds}{V(x, \phi^{-1}(s))}} \geq \frac{L_1}{L_2} \frac{(c_1)^2(c_3)^{3d_2/d_3-1}}{2^{d_4+1}(c_2c_4)^2} \frac{d_1 - d_4}{(d_1 - d_4)c_1 + 3^{d_2}d_4(c_2)^2c_4^{d_1/d_4}}. 
\] (5.2)

Throughout this section, we keep the same setting as in Section 4.
5.1 Proof of (5.1)

For fixed constants \( t > 0 \) and \( c \in (1, 2) \), we define a sequence \( \{n_k\}_{k=0}^\infty \) by \( n_k = tc^k \) \((k \geq 0)\).

Then

\[
q_x(t, x) = P_x \left( d(X_u, x) \leq \varphi(u) \text{ for some } u > t \right)
\]

\[
= P_x \left( \bigcup_{k=0}^\infty \{d(X_u, x) \leq \varphi(u) \text{ for some } u \in (n_k, n_{k+1}]\} \right) \tag{5.3}
\]

\[
\leq \sum_{k=0}^\infty P_x \left( d(X_u, x) \leq \varphi(u) \text{ for some } u \in (n_k, n_{k+1}] \right).
\]

To obtain an upper bound of the last term of (5.3), we show

\textbf{Lemma 5.1.} For each \( c \in (1, 2) \), there exists \( T_c > 0 \) such that for all \( t \geq T_c \),

\[
P_x \left( d(X_u, x) \leq \varphi(u) \text{ for some } u \in (n_k, n_{k+1}] \right)
\]

\[
\leq K_1 c_2 \frac{c^{1+d_2/d_3}}{c_3^{d_2/d_3}} \frac{\int_{n_k}^{n_{k+1}} V(x, \varphi(u)) \, du}{\varphi(\varphi^{-1}(u)) V(x, \varphi^{-1}(u))} \tag{5.4}
\]

for any \( x \in M \) and \( k \geq 0 \).

\textit{Proof.} For any \( u \in (n_k, n_{k+1}] \), we have by (2.5),

\[
\varphi(u) = \phi^{-1}(u)g(u) \leq \left( \frac{c}{c_3} \right)^{1/d_3} \phi^{-1}(n_k)g(n_k) = \left( \frac{c}{c_3} \right)^{1/d_3} \phi(n_k),
\]

which implies that

\[
P_x \left( d(X_u, x) \leq \varphi(u) \text{ for some } u \in (n_k, n_{k+1}] \right)
\]

\[
\leq P_x \left( d(X_u, x) \leq \left( \frac{c}{c_3} \right)^{1/d_3} \phi(n_k) \text{ for some } u \in (n_k, n_{k+1}] \right). \tag{5.6}
\]

We first give an upper bound of the last expression above by using Lemma 4.2. By (2.5),

\[
\frac{\phi^{-1}((c-1)^2u)}{\phi^{-1}(u)} \geq c_3^{1/d_3} \left( \frac{(c-1)^2u}{u} \right)^{1/d_3} = c_3^{1/d_3} (c-1)^{2/d_3} \tag{5.7}
\]

for any \( u > 0 \). Since \( g(t) \to 0 \) as \( t \to \infty \), there exists \( T_c > 0 \) such that

\[
g(u) \leq \left\{ \frac{c_3^2(c-1)^2}{c} \right\}^{1/d_3} \text{ for all } u \geq T_c.
\]

By this inequality and (5.7), we obtain

\[
\left( \frac{c}{c_3} \right)^{1/d_3} \varphi(u) = \left( \frac{c}{c_3} \right)^{1/d_3} \phi^{-1}(u)g(u) \leq \phi^{-1}((c-1)^2u) \text{ for all } u \geq T_c. \tag{5.8}
\]
We now suppose that \( t \geq T_c \). Since \( n_k \geq t \geq T_c \) and

\[
(c - 1)^2 n_k = (c - 1)(n_{k+1} - n_k) \leq n_k,
\]

we see from (5.8) that

\[
\left( \frac{c}{c_3} \right)^{1/d_3} \varphi(n_k) \leq \phi^{-1}((c - 1)^2 n_k) = \phi^{-1}((c - 1)(n_{k+1} - n_k)) \leq \phi^{-1}(n_k).
\]

Hence by Lemma 4.2 we get

\[
P_x \left( d(X_u, x) \leq (c/c_3)^{1/d_3} \varphi(n_k) \right) \text{ for some } u \in (n_k, n_{k+1})
\]

\[
\leq K_1 \frac{V(x, (c/c_3)^{1/d_3} \varphi(n_k))}{\phi((c/c_3)^{1/d_3} \varphi(n_k))} \int_{n_k}^{n_{k+1} + (c-1)(n_{k+1} - n_k)} \frac{du}{V(x, \phi^{-1}(u))}.
\]

We next evaluate the last expression above. Since

\[
\int_{n_k}^{n_{k+1} + (c-1)(n_{k+1} - n_k)} \frac{du}{V(x, \phi^{-1}(u))} \leq \frac{(c - 1)(n_{k+1} - n_k)}{V(x, \phi^{-1}(n_{k+1}))} \leq (c - 1) \int_{n_k}^{n_{k+1}} \frac{du}{V(x, \phi^{-1}(u))},
\]

we obtain

\[
\int_{n_k}^{n_{k+1} + (c-1)(n_{k+1} - n_k)} \frac{du}{V(x, \phi^{-1}(u))} = \int_{n_k}^{n_{k+1}} \frac{du}{V(x, \phi^{-1}(u))} + \int_{n_{k+1}}^{n_{k+1} + (c-1)(n_{k+1} - n_k)} \frac{du}{V(x, \phi^{-1}(u))} \leq c \int_{n_k}^{n_{k+1}} \frac{du}{V(x, \phi^{-1}(u))}.
\]

If \( n_k < u \leq n_{k+1} \), then \( \phi^{-1}(n_k) \leq \phi^{-1}(u) \) and \( g(n_k) \leq g(u)/R_{c,t} \) so that

\[
\varphi(n_k) = \phi^{-1}(n_k)g(n_k) \leq \frac{1}{R_{c,t}} \phi^{-1}(u)g(u) = \frac{1}{R_{c,t}} \varphi(u).
\]

Therefore, we have

\[
V \left( x, \left( \frac{c}{c_3} \right)^{1/d_3} \varphi(n_k) \right) \leq V \left( x, \left( \frac{c}{c_3} \right)^{1/d_3} \frac{1}{R_{c,t}} \varphi(u) \right) \leq c_2 \left( \frac{c}{c_3} \right)^{d_2/d_3} \left( \frac{1}{R_{c,t}^{d_2}} \right) V \left( x, \varphi(u) \right)
\]

by (2.2). Since \( \phi((c/c_3)^{1/d_3} \varphi(n_k)) \geq \phi(\varphi(u)) \) by (5.3), we see from (5.10) that

\[
\frac{V(x, (c/c_3)^{1/d_3} \varphi(n_k))}{\phi((c/c_3)^{1/d_3} \varphi(n_k))} \int_{n_k}^{n_{k+1} + (c-1)(n_{k+1} - n_k)} \frac{du}{V(x, \phi^{-1}(u))} \leq c c_2 \left( \frac{c}{c_3} \right)^{d_2/d_3} \left( \frac{1}{R_{c,t}^{d_2}} \right) \frac{1}{V(x, \phi^{-1}(u))} \int_{n_k}^{n_{k+1}} \frac{V(x, \varphi(u))}{\phi(\varphi(u))} \frac{du}{V(x, \phi^{-1}(u))}.
\]
Combining this with (5.6) and (5.9), we arrive at the inequality (5.4).

We can finish the proof of (5.1) by Lemma 5.1. In fact, it follows by (5.3) and (5.4) that

\[ q_\varphi(t, x) \leq K \frac{c_2^{1+d/d_3}}{c_3^{d_2/d_3}} \sum_{k=0}^{n_k+1} \int_{n_k} V(x, \varphi(u)) \frac{\phi'(u)}{\phi(u)} V(x, \phi^{-1}(u)) \]

and thus

\[ \int_t^{\infty} V(x, \varphi(u)) \frac{\phi'(u)}{\phi(u)} V(x, \phi^{-1}(u)) \leq K \frac{c_2^{1+d/d_3}}{c_3^{d_2/d_3}} \frac{1}{(R_{c,t})^{d_2}}. \]

Since \( \lim_{c \to 1+0} (\lim_{t \to \infty} R_{c,t}) = 1 \) by assumption, we get (5.1) by letting \( t \to \infty \) and then \( c \to 1 + 0 \) in the inequality above.

### 5.2 Proof of (5.2)

Fix positive constants \( t, k, \) and \( l \) with \( 1 < l < k < 2 \) and define a sequence \( \{n_m\}_{m=0}^{\infty} \) by

\[ n_0 = t, \ n_{2m+1} = kn_{2m}, \ n_{2m+2} = ln_{2m+1} \quad (m \geq 0). \]

If \( n_{2m} \leq u \leq n_{2m+1} \), then a calculation similar to (5.5) shows that

\[ \varphi(u) \geq \left( \frac{c_3}{k} \right)^{1/d_3} \varphi(n_{2m+1}). \quad (5.11) \]

We now define the event \( A_{2m} \) \( (m = 0, 1, 2, \ldots) \) by

\[ A_{2m} = \left\{ d(X_u, x) \leq \left( \frac{c_3}{k} \right)^{1/d_3} \varphi(n_{2m+1}) \text{ for some } u \in [n_{2m}, n_{2m+1}] \right\}. \quad (5.12) \]

Then (5.11) yields that

\[ A_{2m} \subset \left\{ d(X_u, x) \leq \varphi(u) \text{ for some } u \in [n_{2m}, n_{2m+1}] \right\} \]

and hence

\[ q_\varphi(t, x) = P_x \left( \bigcup_{m=0}^{\infty} \left\{ d(X_u, x) \leq \varphi(u) \text{ for some } u \in (n_m, n_{m+1}] \right\} \right) \]

\[ \geq P_x \left( \bigcup_{m=0}^{\infty} A_{2m} \right) \geq \sum_{i=0}^{\infty} \left( P_x(A_{2i}) - \sum_{j=i+1}^{\infty} P_x(A_{2i} \cap A_{2j}) \right). \quad (5.13) \]
The last inequality above is the so-called Bonferroni inequality (see, e.g., [15, Exercise 1.6.10]).

Let $\kappa_t = \min \{1, g(t)^{d_3}/c_3\}$ for $t > 0$. To calculate the last expression of (5.13), we first show

**Proposition 5.2.** If $1 < k < 3/2$, $1 < l < 2 - 1/k$ and $\kappa_t < k(l - 1)/2$, then there exists a positive constant $A(k, l)$ such that for any $i \geq 0$ and $j \geq i + 1$,

$$P_x(A_{2i} \cap A_{2j}) \leq A(k, l) P_x(A_{2i}) \int_{n_{2j-1}}^{n_{2j+1}} \frac{V(x, \varphi(u))}{\phi(\varphi(u))} \frac{du}{V(x, \phi^{-1}(u))} \text{ for any } x \in M.$$  

The constant $A(k, l)$ will be given in (5.28) below. For the proof of Proposition 5.2, we calculate $P_x(A_{2i} \cap A_{2j})$ by using the strong Markov property. For $i \geq 0$, we define

$$\sigma_{2i} = \left\{ \begin{array}{ll} \inf \{ u \in (n_{2i}, n_{2i+1}] \mid d(X_u, X_0) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1}) \}, & \text{if } \{ \} \neq \emptyset, \\ \infty, & \text{if } \{ \} = \emptyset. \end{array} \right.$$  

If $j \geq i + 1$, then by the strong Markov property,

$$\begin{align*}
P_x(A_{2i} \cap A_{2j}) &= P_x\left( \sigma_{2i} \leq n_{2i+1}, \ d(X_u, x) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1}) \text{ for some } u \in (n_{2j}, n_{2j+1}] \right) \\
&= E_x \left[ F_j(X_{\sigma_{2i}}, n_{2j} - \sigma_{2i}, n_{2j+1} - \sigma_{2i}) \mid \sigma_{2i} \leq n_{2i+1} \right] \\
&\leq P_x(\sigma_{2i} \leq n_{2i+1}) \sup_{d(z,x) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1})} F_j(z, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) \\
&= P_x(A_{2i}) \sup_{d(z,x) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1})} F_j(z, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}),  \\
\end{align*}$$

where

$$F_j(y, s_1, s_2) = P_y\left( d(X_u, x) \leq (c_3/k)^{1/d_3} \varphi(n_{2j+1}) \text{ for some } u \in (s_1, s_2] \right).$$

To obtain an upper bound of (5.14), we use the comparison of heat kernels:

**Lemma 5.3.** ([32, Lemma 4.2]). For any $t > 0$ and $x, z \in M$ such that $d(x, z) \leq \varphi^{-1}(t)$,

$$p(t, z, y) \leq H_1 p(t, x, y)$$

for any $y \in M$, where $H_1 = c_2 c_4 2^{d_2 + d_4} L_2 / L_1$.

**Proof.** Suppose that $d(x, z) \leq \varphi^{-1}(t)$. Since the heat kernel is symmetric by assumption, (2.3) implies that

$$p(t, z, y) = p(t, y, z) \leq \frac{L_2}{V(y, \varphi^{-1}(t))}.$$
for any $t > 0$ and $y, z \in M$. If $d(z, y) \leq \phi^{-1}(t)$, then we have $d(y, x) \leq d(y, z) + d(z, x) \leq 2\phi^{-1}(t)$ so that by (2.2) and (2.4),

\[
\frac{L_2}{V(y, \phi^{-1}(t))} = \frac{L_2 t}{V(y, \phi^{-1}(t))\phi(\phi^{-1}(t))} \leq \frac{L_2 t}{V(y, d(y, x)/2)\phi(d(y, x)/2)} \leq c_2c_42^{d_2+d_4}L_2\frac{1}{V(y, d(y, x))\phi(d(y, x))}.
\]

We thus get

\[
p(t, z, y) \leq c_2c_42^{d_2+d_4}L_2\min\left\{\frac{1}{V(y, \phi^{-1}(t))}, \frac{t}{V(y, d(y, x))\phi(d(y, x))}\right\}
\]

\[
\leq c_2c_42^{d_2+d_4}\frac{L_2}{L_1}p(t, y, x) = H_1p(t, x, y).
\]

The last equality above follows from the symmetry of the heat kernel. On the other hand, if $d(z, y) > \phi^{-1}(t)$, then $d(y, x) \leq 2d(y, z)$ so that we obtain (5.16) by the same way as for the former case.

The following lemma gives an upper bound of (5.14).

**Lemma 5.4.** If $\kappa_t < k(l - 1)$, then for any $x, z \in M$ with $d(x, z) \leq (c_3/k)^{1/d_3}\varphi(n_{2i+1})$,

\[
F_j(z, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) \leq H_1F_j(x, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}).
\]

**Proof.** Suppose that $\kappa_t < k(l - 1)$. Then for any $j \geq i + 1$,

\[
\kappa_t n_{2i} \leq k(l - 1)n_{2i} = n_{2i+2} - n_{2i+1} \leq n_{2j} - n_{2i+1}.
\]

On the other hand, since $g(u) \leq (c_3\kappa_t)^{1/d_3}$ for all $u \geq t$ and

\[
\phi^{-1}(u) \leq \phi^{-1}(\kappa_t u) \left(\frac{1}{c_3\kappa_t}\right)^{1/d_3}
\]

by (2.5), we have

\[
\varphi(u) = \phi^{-1}(u)g(u) \leq \phi^{-1}(\kappa_t u)
\]

for any $u \in [n_{2i}, n_{2i+1}]$. Therefore, we obtain by (5.11) and (5.18),

\[
\left(\frac{c_3}{k}\right)^{1/d_3} \varphi(n_{2i+1}) \leq \varphi(n_{2i}) \leq \phi^{-1}(\kappa_t n_{2i}) \leq \phi^{-1}(n_{2j} - n_{2i+1}).
\]

This inequality and Lemma 5.3 show that if $d(x, z) \leq (c_3/k)^{1/d_3}\varphi(n_{2i+1})$, then

\[
p(n_{2j} - n_{2i+1}, z, w) \leq H_1p(n_{2j} - n_{2i+1}, x, w)
\]

for any $w \in M$. 

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By the Markov property,
\[ F_j(z, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) = E_z \left[ F_j(X_{n_{2j} - n_{2i+1}}, 0, n_{2j+1} - n_{2i} - (n_{2j} - n_{2i+1})) \right] \]
\[ = \int_M p(n_{2j} - n_{2i+1}, z, w) F_j(w, 0, n_{2j+1} - n_{2i} - (n_{2j} - n_{2i+1})) m(dw). \]

Then (5.21) implies that for any \( x, z \in M \) with \( d(x, z) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1}) \), the last expression above is less than
\[ H_1 \int_M p(n_{2j} - n_{2i+1}, x, w) F_j(w, 0, n_{2j+1} - n_{2i} - (n_{2j} - n_{2i+1})) m(dw) \]
\[ = H_1 F_j(x, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}), \]
which is our assertion. \( \square \)

We further derive an upper bound of (5.17) by using the next lemma.

**Lemma 5.5.** Suppose that \( 1 < k < 3/2 \) and \( 1 < l < 2 - 1/k \). If \( \kappa_t < k(l - 1)/2 \), then for any \( x \in M \), \( i \geq 0 \) and \( j \geq i + 1 \),
\[ F_j(x, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) \]
\[ \leq H_2 \left( \frac{kl}{l-1} \right)^{d_2/d_3} V(x, (c_3/k)^{1/d_3} \varphi(n_{2j+1})) \int_{n_{2j-1}}^{n_{2j+1}} \frac{du}{V(x, \phi^{-1}(u))} \] 
(5.22)

for \( H_2 = 3K_1c_2/c_3^{d_2/d_3} \).

**Proof.** We first check that Lemma (4.2) is applicable to the calculation of the left hand side of (5.22). Suppose that \( \kappa_t < k(l - 1)/2 \). Since
\[ \kappa_t n_{2j} \leq \frac{1}{2} k(l - 1)n_{2j} = \frac{1}{2} (n_{2j+2} - n_{2j}), \]
we get
\[ \left( \frac{c_3}{k} \right)^{1/d_3} \varphi(n_{2j+1}) \leq \phi^{-1}(\kappa_t n_{2j}) \leq \phi^{-1} \left( \frac{1}{2} (n_{2j+2} - n_{2j+1}) \right) \]
by the same way as in (5.20). We further suppose that \( 1 < k < 3/2 \) and \( 1 < l < 2 - 1/k \). Then
\[ n_{2j+2} - n_{2j+1} \leq (n_{2j+1} - n_{2i}) - (n_{2j} - n_{2i+1}) \]
by direct calculation. Since \( kl < 2 \) by assumption, we also have
\[ n_{2j+2} - n_{2j+1} = kl(n_{2j} - n_{2j-1}) \leq 2(n_{2j} - n_{2i+1}) \]
so that
\[ \left( \frac{c_3}{k} \right)^{1/d_3} \varphi(n_{2j+1}) \leq \phi^{-1} \left( \min \left\{ n_{2j} - n_{2i+1}, \frac{1}{2} \left( (n_{2j+1} - n_{2i}) - (n_{2j} - n_{2i+1}) \right) \right\} \right). \]
Accordingly, we can apply Lemma 4.2 to the left hand side of (5.22) to get

\[
F_j(x, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) \\
= P_x \left( d(X_u, x) \leq (c_3/k)^{1/d_3} \varphi(n_{2j+1}) \text{ for some } u \in (n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) \right) \\
\leq K_1 \frac{V(x, (c_3/k)^{1/d_3} \varphi(n_{2j+1}))}{\phi((c_3/k)^{1/d_3} \varphi(n_{2j+1}))} \int_{n_{2j} - n_{2i+1}}^{n_{2j+1} - n_{2i} + n_{2j+1} - n_{2i}/2} \frac{du}{V(x, \varphi^{-1}(u))}. \\
\tag{5.23}
\]

We next evaluate the integral in the last expression above. For \( j \geq i + 1 \), since

\[
n_{2j} - n_{2i+1} \geq n_{2j} - n_{2j-1} = (l - 1)n_{2j-1}
\]

and \( n_{2j+1} = kln_{2j-1} \), it follows from (2.2) and (2.4) that

\[
V(x, \varphi^{-1}(n_{2j} - n_{2i+1})) \geq V(x, \varphi^{-1}((l - 1)n_{2j-1})) \geq \frac{d_2/d_3}{c_3} \frac{V(x, \varphi^{-1}(n_{2j+1}))}{c_2} \left( \frac{l - 1}{kl} \right)^{d_2/d_3} \frac{V(x, \varphi^{-1}(n_{2j+1}))}{c_2}
\]

Noting that

\[
n_{2j+1} - n_{2i} - (n_{2j} - n_{2i+1}) \leq 2(n_{2j+1} - n_{2j-1}),
\]

we obtain

\[
\int_{n_{2j} - n_{2i+1}}^{n_{2j+1} - n_{2i} + n_{2j+1} - n_{2i}/2} \frac{du}{V(x, \varphi^{-1}(u))} \leq \frac{3n_{2j+1} - n_{2i} - (n_{2j} - n_{2i+1})}{2} \frac{V(x, \varphi^{-1}(n_{2j} - n_{2i+1}))}{c_3} \int_{n_{2j-1}}^{n_{2j+1} - n_{2i+1}} \frac{du}{V(x, \varphi^{-1}(u))}. \\
\]

By this inequality with (5.23), we have (5.22). \( \Box \)

We see from Lemmas 5.4 and 5.5 that if \( 1 < k < 3/2, \ 1 < l < 2 - 1/k \) and \( \kappa_t < k(l - 1)/2 \), then for any \( x, z \in M \) with \( d(x, z) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1}) \),

\[
F_j(z, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) \\
\leq H_1H_2 \left( \frac{kl}{l - 1} \right)^{d_2/d_3} \frac{V(x, (c_3/k)^{1/d_3} \varphi(n_{2j+1}))}{\phi((c_3/k)^{1/d_3} \varphi(n_{2j+1}))} \int_{n_{2j-1}}^{n_{2j+1} - n_{2i+1}} \frac{du}{V(x, \varphi^{-1}(u))}. \tag{5.24}
\]

Proof of Proposition 5.2  Since \( d_1 > d_4 \) by assumption, we have for any \( x \in M \) and \( 0 < s < r \),

\[
\frac{V(x, s)}{\phi(s)} = \frac{V(x, s) \phi(r) V(x, r)}{V(x, r) \phi(s) \phi(r)} \leq \frac{c_4}{c_1} \left( \frac{r}{s} \right)^{d_4 - d_1} \frac{V(x, r)}{\phi(r)} \leq \frac{c_4 V(x, r)}{\phi(r)}
\]

by (2.2) and (2.4). Hence if \( 0 < s < cr \) for some \( c > 1 \), then

\[
\frac{V(x, s)}{\phi(s)} \leq \frac{c_4 V(x, cr)}{c_1 \phi(cr)} \leq \frac{c_2 c_4}{c_1 c_3} \left( \frac{d_2 - d_3}{\phi(r)} \right) \frac{V(x, r)}{\phi(r)}. \tag{5.25}
\]
For any $u \in [n_{2j-1}, n_{2j+1}]$, since we obtain
\[
\left( \frac{c_4}{k} \right)^{1/d_3} \varphi(n_{2j+1}) \leq \frac{1}{l^{1/d_3}} \varphi(u)
\]
by the same way as in (5.11), it follows from (5.25) that
\[
\frac{V(x, (c_3/k)^{1/d_3} \varphi(n_{2j+1}))}{\phi((c_3/k)^{1/d_3} \varphi(n_{2j+1}))} \leq \frac{c_2 c_4}{c_1 c_3} \frac{l(d_2-d_3)/d_3}{V(x, \varphi(u))}.
\]
Therefore, for any $t > 0$ and $k, l > 0$ with $1 < l < k < 2$, and for any $j \geq 1$,
\[
\frac{V(x, (c_3/k)^{1/d_3} \varphi(n_{2j+1}))}{\phi((c_3/k)^{1/d_3} \varphi(n_{2j+1}))} \leq \frac{c_2 c_4}{c_1 c_3} \frac{l(d_2-d_3)/d_3}{V(x, \varphi(u))} \int_{n_{2j-1}}^{n_{2j+1}} \frac{du}{V(x, \varphi^{-1}(u))}.
\] (5.26)

We see by (5.24) and (5.26) that if $1 < k < 3/2$, $1 < l < 2 - 1/k$ and $\kappa_t < k(l-1)/2$, then for any $x, z \in M$ with $d(x, z) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1})$,
\[
F_j(z, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) \leq A(k, l) \int_{n_{2j-1}}^{n_{2j+1}} \frac{V(x, \varphi(u))}{\phi(\varphi(u)) V(x, \varphi^{-1}(u))} \frac{du}{V(x, \varphi^{-1}(u))}.
\] (5.27)

for
\[
A(k, l) = H_1 H_2 \frac{c_2 c_4}{c_1 c_3} \left( \frac{kl}{l-1} \right)^{d_2/d_3} l^{(d_2-d_3)/d_3}.
\] (5.28)

By (5.14) and (5.27), the proof is complete. □

Under the full conditions in Proposition 5.2 we have
\[
\sum_{j=i+1}^{\infty} P_x(A_{2i} \cap A_{2j}) \leq A(k, l) P_x(A_{2i}) \int_{n_{2i+1}}^{\infty} \frac{V(x, \varphi(u))}{\phi(\varphi(u)) V(x, \varphi^{-1}(u))} \frac{du}{V(x, \varphi^{-1}(u))}.
\]
\[
\leq A(k, l) P_x(A_{2i}) \int_{t}^{\infty} \frac{V(x, \varphi(u))}{\phi(\varphi(u))} V(x, \varphi^{-1}(u)) \frac{du}{V(x, \varphi^{-1}(u))}.
\]

Therefore, (5.13) shows that
\[
q_x(t, x) \geq \left( 1 - A(k, l) \int_{t}^{\infty} \frac{V(x, \varphi(u))}{\phi(\varphi(u))} V(x, \varphi^{-1}(u)) \frac{du}{V(x, \varphi^{-1}(u))} \right) \sum_{i=0}^{\infty} P_x(A_{2i}).
\] (5.29)

We next derive a lower bound of $P_x(A_{2i})$.

Proposition 5.6. If $g(t) < 1$ and $\kappa_t < (k-1)/(2d_4 c_4)$, then for any $x \in M$ and $i \geq 0$,
\[
P_x(A_{2i}) \geq B(k, l) \min \left\{ R_{k, t}, \left( \frac{c_3}{T} \right)^{1/d_3} \right\} \int_{n_{2i}}^{n_{2i+2}} \frac{V(x, \varphi(u))}{\phi(\varphi(u))} V(x, \varphi^{-1}(u)) \frac{du}{V(x, \varphi^{-1}(u))}.
\] (5.30)

for
\[
B(k, l) = K_2 \frac{c_1 c_3^{2d_2/d_3}}{(c_2)^2 c_4} \frac{1}{kl - 1} \frac{1}{k^{2d_2/d_3 - 1}}.
\]

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To show Proposition 5.6, we first evaluate $P_x(A_{2i})$ by using Lemma 4.3.

**Lemma 5.7.** If $g(t) < 1$ and $\kappa_i < (k - 1)/(2^d c_4)$, then for any $x \in M$ and $i \geq 0$,

$$P_x(A_{2i}) \geq K_2 \frac{V(x, (c_3/k)^{1/d_3} \varphi(n_{2i} + 1))}{\phi((c_3/k)^{1/d_3} \varphi(n_{2i} + 1))} \int_{n_{2i}}^{n_{2i+1}} \frac{du}{V(x, \phi^{-1}(u))}. \quad (5.31)$$

**Proof.** Let $A_{2i}$ be the event defined by (5.12). Then

$$P_x(A_{2i}) = P_x \left( d(X_u, x) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1}) \text{ for some } u \in (n_{2i}, n_{2i+1}) \right). \quad (5.32)$$

Suppose that $g(t) < 1$ and $\kappa_i < (k - 1)/(2^d c_4)$. Then (5.11) shows that

$$\phi((c_3/k)^{1/d_3} \varphi(n_{2i+1})) \leq \phi(\varphi(n_{2i})) \leq \phi(\phi^{-1}(n_{2i})) = n_{2i}.$$  

We also have by (5.11) and (5.19),

$$\phi(2(c_3/k)^{1/d_3} \varphi(n_{2i+1})) \leq \phi(2\varphi(n_{2i})) \leq c_4^2 d_4 \phi(\phi^{-1}(\kappa_i n_{2i})) = c_4^2 d_4 \kappa_i n_{2i} \leq n_{2i+1} - n_{2i}.$$  

Hence we get (5.31) by applying Lemma 4.3 to the right hand side of (5.32).  

In order to give a lower bound of the right hand side of (5.31), we next show

**Lemma 5.8.** For any $i \geq 0$,

$$\frac{V(x, (c_3/k)^{1/d_3} \varphi(n_{2i+1}))}{\phi((c_3/k)^{1/d_3} \varphi(n_{2i+1}))} \int_{n_{2i}}^{n_{2i+1}} \frac{du}{V(x, \phi^{-1}(u))} \geq c_1 (c_3/k)^{2d_2/d_3} \frac{1}{(c_2)^2 c_4} \frac{k - 1}{k l - 1} \frac{1}{k^2 d_2/d_3 - 1} \min \left\{ R_{k, t}, \left( \frac{c_3}{c_4} \right)^{1/d_3} \right\} \int_{n_{2i}}^{n_{2i+2}} \frac{V(x, \varphi(u))}{\phi(\varphi(u))} \frac{du}{V(x, \phi^{-1}(u))}. \quad (5.33)$$

**Proof.** We begin by evaluating the integral in the left hand side of (5.33). By the definition of the sequence $\{n_m\}_{m=0}^\infty$, we have

$$\int_{n_{2i}}^{n_{2i+1}} \frac{du}{V(x, \phi^{-1}(u))} \geq \frac{n_{2i+1} - n_{2i}}{V(x, \phi^{-1}(n_{2i+1}))} = \frac{k - 1}{kl - 1} \frac{n_{2i+2} - n_{2i}}{V(x, \phi^{-1}(n_{2i+1}))}.$$  

$$= \frac{k - 1}{kl - 1} \left( \frac{n_{2i+2} - n_{2i+1}}{V(x, \phi^{-1}(n_{2i+1}))} + \frac{n_{2i+1} - n_{2i}}{V(x, \phi^{-1}(n_{2i+1}))} \right). \quad (5.34)$$

Then

$$\frac{n_{2i+2} - n_{2i+1}}{V(x, \phi^{-1}(n_{2i+1}))} \geq \int_{n_{2i+1}}^{n_{2i+2}} \frac{du}{V(x, \phi^{-1}(u))}$$

and

$$\frac{n_{2i+1} - n_{2i}}{V(x, \phi^{-1}(n_{2i+1}))} \geq \frac{d_2/d_3}{c_3 k^d_2/d_3} \frac{n_{2i+1} - n_{2i}}{V(x, \phi^{-1}(n_{2i}))} \geq \frac{d_2/d_3}{c_3 k^d_2/d_3} \int_{n_{2i}}^{n_{2i+1}} \frac{du}{V(x, \phi^{-1}(u))}.$$
by \((2.2)\) and \((2.3)\). At the first inequality above, we used the fact that \(c_3 \leq 1\) and \(k > 1\). Since
\[
\frac{n_{2i+2} - n_{2i+1}}{V(x, \phi^{-1}(n_{2i+1}))} + \frac{n_{2i+1} - n_{2i}}{V(x, \phi^{-1}(n_{2i+1}))} \\
\geq \int_{n_{2i+1}}^{n_{2i+2}} \frac{du}{V(x, \phi^{-1}(u))} + \frac{c_3^{d_2/d_3}}{c_2 k^{d_2/d_3}} \int_{n_{2i}}^{n_{2i+1}} \frac{du}{V(x, \phi^{-1}(u))} \geq \frac{c_3^{d_2/d_3}}{c_2 k^{d_2/d_3}} \int_{n_{2i}}^{n_{2i+2}} \frac{du}{V(x, \phi^{-1}(u))},
\]
\((5.34)\) implies that
\[
\int_{n_{2i}}^{n_{2i+1}} \frac{du}{V(x, \phi^{-1}(u))} \geq \frac{c_3^{d_2/d_3}}{c_2} k - 1 \frac{1}{kl - 1 k^{d_2/d_3}} \int_{n_{2i}}^{n_{2i+2}} \frac{du}{V(x, \phi^{-1}(u))}. \tag{5.35}
\]
On account of \((5.35)\), the proof of \((5.38)\) is completed by showing that
\[
V(x, (c_3/k)^{1/d_3} \varphi(n_{2i+1})) \int_{n_{2i}}^{n_{2i+2}} \frac{du}{V(x, \phi^{-1}(u))} \geq \frac{c_3}{c_2} \frac{c_3^{(d_2-d_3)/d_3}}{c_2 k^{d_2/d_3}} \min \left\{ R_{k,t}, \left( \frac{c_3}{l} \right)^{1/d_3} \right\} \int_{n_{2i}}^{n_{2i+2}} \frac{V(x, \varphi(u))}{\phi(\varphi(u))} \frac{du}{V(x, \phi^{-1}(u))}. \tag{5.36}
\]
If \(n_{2i} \leq u \leq n_{2i+1}\), then
\[
\varphi(n_{2i+1}) = \phi^{-1}(n_{2i+1}) g(n_{2i+1}) \geq R_{k,t} \phi^{-1}(u) g(u) = R_{k,t} \varphi(u).
\]
On the other hand, if \(n_{2i+1} \leq u \leq n_{2i+2}\), then
\[
\varphi(n_{2i+1}) \geq \left( \frac{c_3}{l} \right)^{1/d_3} \varphi(u)
\]
by \((5.5)\). Hence for any \(u \in [n_{2i}, n_{2i+2}]\), we have
\[
\varphi(n_{2i+1}) \geq \min \left\{ R_{k,t}, \left( \frac{c_3}{l} \right)^{1/d_3} \right\} \varphi(u)
\]
so that by \((5.25)\),
\[
V(x, (c_3/k)^{1/d_3} \varphi(n_{2i+1})) \geq \frac{c_1 c_3}{c_2 c_4} \frac{c_3^{(d_2-d_3)/d_3}}{k^{d_2/d_3}} \min \left\{ R_{k,t}, \left( \frac{c_3}{l} \right)^{1/d_3} \right\} \int_{n_{2i}}^{n_{2i+2}} \frac{V(x, \varphi(u))}{\phi(\varphi(u))} \frac{du}{V(x, \phi^{-1}(u))}.
\]
This inequality yields \((5.36)\).

**Proof of Proposition 5.6** The assertion follows by Lemmas 5.7 and 5.8.

We are now in a position to finish the proof of \((5.2)\). Under the full conditions in both Propositions 5.2 and 5.6 since
\[
\sum_{i=0}^{\infty} P_{\epsilon}(A_{2i}) \geq B(k, l) \min \left\{ R_{k,t}, \left( \frac{c_4}{l} \right)^{1/d_3} \right\} \int_{t}^{\infty} \frac{V(x, \varphi(u))}{\phi(\varphi(u))} \frac{du}{V(x, \phi^{-1}(u))}
\]

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by (5.31), we see from (5.29) that
\[
q_{\varphi}(t, x) \geq B(k, l) \min \left\{ R_{k, t}, \left( \frac{c_3}{l} \right)^{1/d_3} \right\} ^{d_2 - d_3} \left( 1 - A(k, l) \int_t^\infty \frac{V(x, \varphi(u))}{\phi(u)} \frac{du}{V(x, \varphi^{-1}(u))} \right).
\]

Namely,
\[
\frac{q_{\varphi}(t, x)}{\int_t^\infty \frac{V(x, \varphi(u))}{\phi(u)} \frac{du}{V(x, \varphi^{-1}(u))}} \geq B(k, l) \min \left\{ R_{k, t}, \left( \frac{c_3}{l} \right)^{1/d_3} \right\} ^{d_2 - d_3} \left( 1 - A(k, l) \int_t^\infty \frac{V(x, \varphi(u))}{\phi(u)} \frac{du}{V(x, \varphi^{-1}(u))} \right).
\]

By letting first \( t \to \infty \) and then \( l \to 1 + 0 \) and \( k \to 1 + 0 \), we arrive at (5.2).

A Proof of Theorem 2.6

In this appendix, we prove Theorem 2.6. More precisely, we show that
\[
\limsup_{t \to \infty} \frac{q_{\varphi}(t, x)}{\int_t^\infty \frac{V(x, \varphi(u))}{\phi(u)} \frac{du}{V(x, \varphi^{-1}(u))}} \leq \frac{L_2}{L_1} \frac{2d_2 c_2 (c_{v, 2})^2}{d_3 (c_{v, 1})^2}, \tag{A.1}
\]
and
\[
\liminf_{t \to \infty} \frac{q_{\varphi}(t, x)}{\int_t^\infty \frac{V(x, \varphi(u))}{\phi(u)} \frac{du}{V(x, \varphi^{-1}(u))}} \geq \frac{L_1}{L_2} \frac{(c_{v, 1})^2}{2d_2 + 1d_4 c_4 (c_{v, 2})^2}. \tag{A.2}
\]

Our approach here is similar to that of Theorem 2.4. Throughout this appendix, we assume the full conditions in Theorem 2.6. For simplicity, we also assume \( N = \emptyset \) in Assumptions 2.2 and 2.3.

A.1 Hitting time distributions

We first note that Lemmas 4.1 and 4.2 are valid under the setting in this section. Using these lemmas, we have

Lemma A.1. (32, Lemma 4.22). Let \( a, b, c \) and \( r \) be positive constants.

(i) If \( \phi(r) \leq a \wedge c \), then for any \( x \in M \),
\[
P_x \left( d(X_s, x) \leq r \text{ for some } s \in (a, b] \right) \leq K_3 \frac{\log \left( \frac{b + c}{a} \right)}{\log \left( \frac{c}{\phi(r)} \right)} \leq K_3 \frac{\log \left( \frac{b + c}{a} \right)}{1 + \log \left( \frac{c}{\phi(r)} \right)},
\]
where \( K_3 = K_1 (c_{v, 2}/c_{v, 1})^2 \).
(ii) If $\phi (r) \leq a$ and $\phi (2r) \leq b - a$, then for any $x \in M$,

$$P_x (d(X_s, x) \leq r \text{ for some } s \in (a, b]) \geq K_4 \frac{\log \left( \frac{b}{a} \right)}{1 + \log \left( \frac{b - a}{\phi (2r)} \right)},$$

where

$$K_4 = \frac{L_1}{L_2 2^{d_4 + 1} c_4 (c_{v,2})^2}.$$

**Proof.** We first suppose that $\phi (r) \leq a \wedge c$. Then by assumption and (4.2),

$$\int_a^{b + c} P_x (d(X_u, x) \leq 2r) \, du \leq \frac{2^{d_2} c_2 c_{v,2} L_2}{c_{v,1}} \phi (r) \int_a^{b + c} \frac{du}{u} = \frac{2^{d_2} c_2 c_{v,2} L_2}{c_{v,1}} \phi (r) \log \left( \frac{b + c}{a} \right).$$

We also see by assumption and (4.1) that

$$P_y (d(X_u, y) \leq r) \geq L_1 \min \left\{ 1, \frac{V(y, r)}{V(y, \phi^{-1}(u))} \right\} \geq L_1 \min \left\{ 1, \frac{c_{v,1} \phi (r)}{c_{v,2} u} \right\} \geq \frac{c_{v,1} L_1}{c_{v,2}} \min \left\{ 1, \frac{\phi (r)}{u} \right\}$$

for any $y \in M$ and $u > 0$, and therefore

$$\int_0^c \inf_{d(y, x) \leq r} P_y (d(X_u, y) \leq r) \, du \geq \frac{c_{v,1} L_1}{c_{v,2}} \phi (r) \left( \int_0^{\phi (r)} \frac{du}{u} + \phi (r) \int_{\phi (r)}^c \frac{du}{u} \right) = \frac{c_{v,1} L_1}{c_{v,2}} \phi (r) \left( 1 + \log \left( \frac{c}{\phi (r)} \right) \right).$$

Hence the proof of (i) is completed by Lemma 4.1.

We next suppose that $\phi (r) \leq a$ and $\phi (2r) \leq b - a$. Then by (A.3),

$$\int_a^b P_x (d(X_u, x) \leq r) \, du \geq \frac{c_{v,1} L_1}{c_{v,2}} \phi (r) \int_a^b \frac{du}{u} = \frac{c_{v,1} L_1}{c_{v,2}} \phi (r) \log \left( \frac{b}{a} \right).$$

In a way similar to (A.3), we also have

$$P_y (d(X_u, y) \leq 2r) \leq \frac{c_{v,2} L_2}{c_{v,1}} \min \left\{ 1, \frac{\phi (2r)}{u} \right\}$$

for any $y \in M$ and $u > 0$. Since $\phi (2r) \leq c_4 2^{d_4} \phi (r)$ by (2.4), we get

$$\int_0^{b-a} \sup_{d(y, x) \leq r} P_y (d(X_u, y) \leq 2r) \, du \leq \frac{c_{v,2} c_4 2^{d_4} L_2}{c_{v,1}} \phi (r) \left( 1 + \log \left( \frac{b - a}{\phi (2r)} \right) \right)$$

by the same way as in (A.4). Hence the assertion (ii) follows by Lemma 4.1. □
A.2 Proof of (A.1)

Since \( g(t) \to 0 \) as \( t \to \infty \), we always take \( t > 0 \) such that \( g(t) < 1 \). For a fixed constant \( c \in (1, 2) \), we define a sequence \( \{n_k\}_{k=0}^{\infty} \) by \( n_k = tc^k \) (\( k \geq 0 \)). In order to give an upper bound of the last expression in (5.3), we show

**Lemma A.2.** For any \( \varepsilon \in (0, d_3) \) and \( c \in (1, 2) \), there exists \( T_{\varepsilon, c} > 0 \) such that for all \( t \geq T_{\varepsilon, c} \),

\[
P_x(d(X_u, x) \leq \varphi(u) \text{ for some } u \in (n_k, n_{k+1}]) \leq K_3 \frac{\log((c - 1)^2 + c)}{c - 1} \int_{n_k}^{n_{k+1}} \frac{du}{c + \log g(u)}
\]

for any \( x \in M \) and \( k \geq 0 \).

**Proof.** We first note that

\[
P_x(d(X_u, x) \leq \varphi(u) \text{ for some } u \in (n_k, n_{k+1}]) \leq K_3 \frac{\log(n_{k+1} + (c - 1)(n_{k+1} - n_k))}{n_k} \frac{n_k}{1 + \log((c - 1)(n_{k+1} - n_k)) \phi((c/c_3)^{1/d_3} \varphi(n_k))}.
\]

We can show this inequality by following the proof of Lemma 5.1 and using Lemma A.1 instead of Lemma 4.2.

We next evaluate the right hand side of (A.6). Since \( n_{k+1} = cn_k \), we have

\[
\log \left( \frac{n_{k+1} + (c - 1)(n_{k+1} - n_k)}{n_k} \right) = \log((c - 1)^2 + c).
\]

Moreover, since (2.4) implies that

\[
\phi \left( \left( \frac{c}{c_3} \right)^{1/d_3} \varphi(n_k) \right) \leq c_4 \left( \frac{c}{c_3} \right)^{d_4/d_3} \varphi(n_k) \leq c_4 \left( \frac{c}{c_3} \right)^{d_4/d_3} \frac{1}{c_3} g(n_k) \phi^{-1}(n_k)
\]

we get

\[
\log \left( \frac{(c - 1)(n_{k+1} - n_k)}{\phi((c/c_3)^{1/d_3} \varphi(n_k))} \right) \geq \log \left( \frac{\left( \frac{c}{c_3} \right)^{1+d_4/d_3} (c - 1)^2 n_k}{c_3 d_4/d_3 c_4 g(n_k)^{d_3} n_k} \right)
\]

\[
= \log \left( \frac{c_3^{1+d_4/d_3} (c - 1)^2}{c_4 d_4/d_3} \right) + \log \left( \frac{c_3^{1+d_4/d_3}}{c_4} \right) + d_3 |\log g(n_k)|
\]

\[
\geq \log \left( \frac{c_3^{1+d_4/d_3}}{c_4} \right) + \log \left( \frac{c_3^{1+d_4/d_3}}{c_4} \right) + d_3 |\log g(n_{k+1})|.
\]
In particular, for any $\varepsilon \in (0, d_3)$ and $c \in (1, 2)$, there exists $T_{\varepsilon, c} > 0$ such that
\[
1 + \log \left( \frac{(c - 1)^2}{e^{d_3/d_3}} + \log \frac{c_{1+d_3/d_3}}{c_4} \right) \geq -\frac{\varepsilon}{R_{c,t}} \left| \log g(n_{k+1}) \right| \quad \text{for all } t \geq T_{\varepsilon, c},
\]
and hence
\[
\log \left( \frac{n_{k+1} + (c - 1)(n_{k+1} - n_k)}{1 + \log \frac{(c - 1)(n_{k+1} - n_k)}{\phi((c/c_3)^{1/d_3} \varphi(n_k))}} \right) \leq \frac{R_{c,t}}{d_3 - \varepsilon} \log((c - 1)^2 + c) \quad \text{for all } t \geq T_{\varepsilon, c}.
\]

for all $t \geq T_{\varepsilon, c}$. The proof is completed by (A.6) and (A.7).

By (5.3) and Lemma A.2, we see that for any $\varepsilon \in (0, d_3)$ and $c \in (1, 2)$, there exists $T_{\varepsilon, c} > 0$ such that for all $t \geq T_{\varepsilon, c}$,
\[
q_\varepsilon(t, x) \leq \frac{K_3}{d_3 - \varepsilon} \frac{c \log((c - 1)^2 + c)}{c - 1} R_{c,t} \int_t^\infty \frac{du}{u \log g(u)},
\]
and thus
\[
\int_t^\infty \frac{du}{u \log g(u)} \leq \frac{K_3}{d_3 - \varepsilon} \frac{c \log((c - 1)^2 + c)}{c - 1} R_{c,t}.
\]

By letting $t \to \infty$ and then $c \to 1 + 0$ and $\varepsilon \to +0$, we arrive at (A.1).

### A.3 Proof of (A.2)

Fix positive constants $t$, $k$ and $l$ with $1 < l < k < 2$. We define a sequence $\{n_m\}_{m=0}^\infty$ by
\[
n_0 = t, \quad n_{2m+1} = kn_{2m}, \quad n_{2m+2} = ln_{2m+1} \quad (m \geq 0).
\]

Let $A_{2m}$ be the event defined by (5.12). By the same way as in the proof of (5.2) (see Subsection 5.2), we first give an upper bound of the probability $P_x(A_{2i} \cap A_{2j})$.

**Proposition A.3.** Suppose that $1 < k < 3/2$ and $1 < l < 2 - 1/k$. Then there exists $T_{k,l} > 0$ such that for all $t \geq T_{k,l}$,
\[
P_x(A_{2i} \cap A_{2j}) \leq A'(k,l) P_x(A_{2i}) \int_{n_{2j-1}}^{n_{2j+1}} \frac{du}{u \log g(u)}
\]
for any $x \in M$, $i \geq 0$ and $j \geq i + 1$, where
\[
A'(k,l) = \frac{2H_1 K_3}{d_3 k l} \left( \frac{k l}{kl - 1} \right). \]
Recall the notation $F_j(y, s_1, s_2)$ in (5.13). For the proof of Proposition [A.3] it is enough to show the next lemma.

**Lemma A.4.** Suppose that $1 < k < 3/2$ and $1 < l < 2 - 1/k$. Then there exists $T_{k,l} > 0$ such that if $d(x, z) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1})$, then for all $t \geq T_{k,l}$,

$$F_j(z, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) \leq A'(k, l) \int_{n_{2j-1}}^{n_{2j+1}} \frac{du}{u | \log g(u)|}$$

for any $i \geq 0$ with $j \geq i + 1$.

**Proof.** In a way similar to the proof of Lemma 5.5, we can apply Lemmas A.1 and B.4 to show that if $d(x, z) \leq (c_3/k)^{1/d_3} \varphi(n_{2i+1})$ and $\kappa_l < k(1-l)/2$, then for any $i \geq 0$ and $j \geq i + 1$,

$$F_j(z, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i}) \leq H_1 F_j(x, n_{2j} - n_{2i+1}, n_{2j+1} - n_{2i})$$

$$\log \left( \frac{n_{2j+1} - n_{2i} + (n_{2j+1} - n_{2j} + n_{2i+1} - n_{2i})/2}{n_{2j} - n_{2i+1}} \right)$$

$$\leq H_1 K_3 \left( 1 + \log \left( \frac{n_{2j+1} - n_{2j} + n_{2i+1} - n_{2i}}{2 \phi((c_3/k)^{1/d_3} \varphi(n_{2j+1}))} \right) \right).$$

(A.9)

Since

$$n_{2j+1} - n_{2i} \leq n_{2j+1} = kl n_{2j-1}$$

and

$$n_{2j} - n_{2i+1} \geq n_{2j} - n_{2j-1} = (l-1)n_{2j-1},$$

we have

$$\frac{n_{2j+1} - n_{2i}}{n_{2j} - n_{2i+1}} \leq \frac{kl}{l-1}$$

so that

$$\frac{n_{2j+1} - n_{2i} + (n_{2j+1} - n_{2j} + n_{2i+1} - n_{2i})/2}{n_{2j} - n_{2i+1}} = 3 \left( \frac{n_{2j+1} - n_{2i}}{n_{2j} - n_{2i+1}} \right) - \frac{1}{2} \leq \frac{3}{2} \left( \frac{kl}{l-1} \right).$$

On the other hand, we obtain

$$\phi \left( \left( \frac{c_3}{k} \right)^{1/d_3} \varphi(n_{2j+1}) \right) = \phi \left( \left( \frac{c_3}{k} \right)^{1/d_3} \varphi(n_{2j+1}) \right)$$

$$\leq \frac{1}{c_3} \left( \left( \frac{c_3}{k} \right)^{1/d_3} g(n_{2j+1}) \right)^{d_3} n_{2j+1} = g(n_{2j+1})^{d_3} n_{2j}$$

by (2.4). Noting that

$$n_{2j+1} - n_{2j} + n_{2i+1} - n_{2i} \geq n_{2j+1} - n_{2j} = (k-1)n_{2j},$$

we get

$$\frac{n_{2j+1} - n_{2j} + n_{2i+1} - n_{2i}}{2 \phi((c_3/k)^{1/d_3} \varphi(n_{2j+1}))} \geq \frac{k-1}{2 g(n_{2j+1})^{d_3}}.$$
Hence if \( g(t) < 1 \), then
\[
\log \left( \frac{n_{2j+1} - n_{2j} + n_{2l+1} - n_{2l}}{2\phi((c_3/k)^{1/d_3}\varphi(n_{2j+1}))} \right) \geq \log \left( \frac{k - 1}{2g(n_{2j+1})^{d_3}} \right) = \log \frac{k - 1}{2} + d_3|\log g(n_{2j+1})|.
\]

(A.10)

Here we note that there exists \( T_{k,l} > 0 \) such that for all \( t \geq T_{k,l} \), we have \( g(t) < 1 \), \( \kappa_t < k(l - 1)/2 \) and
\[
1 + \log \frac{k - 1}{2} \geq -\frac{d_3}{2}|\log g(n_{2j+1})|.
\]

We thus obtain for all \( t \geq T_{k,l} \),
\[
1 + \log \left( \frac{n_{2j+1} - n_{2j} + n_{2l+1} - n_{2l}}{2\phi((c_3/k)^{1/d_3}\varphi(n_{2j+1}))} \right) \geq \frac{d_3}{2}|\log g(n_{2j+1})|
\]
by (A.10), whence
\[
\frac{\log \left( \frac{n_{2j+1} - n_{2j} + (n_{2l+1} - n_{2j} + n_{2l+1} - n_{2l})/2}{n_{2j} - n_{2l+1}} \right)}{1 + \log \left( \frac{n_{2j+1} - n_{2j} + n_{2l+1} - n_{2l}}{2\phi((c_3/k)^{1/d_3}\varphi(n_{2j+1}))} \right)} \leq \frac{2}{d_3} \log \left( \frac{3}{2} \frac{k}{l - 1} \right)
\]
\[
= \frac{2}{d_3} \frac{k}{kl - 1} \log \left( \frac{3}{2} \frac{k}{l - 1} \right) \left| \log g(n_{2j+1}) \right| \frac{n_{2j+1} - n_{2j-1}}{n_{2j+1}}
\]
\[
\leq \frac{2}{d_3} \frac{k}{kl - 1} \log \left( \frac{3}{2} \frac{k}{l - 1} \right) \int_{n_{2j-1}}^{n_{2j+1}} \frac{du}{u|\log g(u)|}.
\]

By this inequality and (A.9), we complete the proof. \( \square \)

**Proof of Proposition A.3** The assertion follows from (5.14) and Lemma A.4 \( \square \)

Under the full conditions in Proposition A.3 we have for all \( t \geq T_{k,l} \),
\[
\sum_{j=t+1}^{\infty} P_x(A_{2i} \cap A_{2j}) \leq A'(k, l) P_x(A_{2i}) \int_{n_{2i+1}}^{\infty} \frac{du}{u|\log g(u)|} \leq A'(k, l) P_x(A_{2i}) \int_{t}^{\infty} \frac{du}{u|\log g(u)|}
\]
so that by (5.13),
\[
q_x(t, x) \geq \left( 1 - A'(k, l) \int_{t}^{\infty} \frac{du}{u|\log g(u)|} \right) \sum_{i=0}^{\infty} P_x(A_{2i}).
\]

(A.11)

We next give a lower bound of \( P_x(A_{2i}) \).

**Proposition A.5.** For any \( \varepsilon > 0 \), \( k \in (1, 2) \) and \( l \in (1, 2) \) with \( k > l \), there exists \( T_{\varepsilon, k,l} > 0 \) such that for all \( t \geq T_{\varepsilon, k,l} \),
\[
P_x(A_{2i}) \geq \frac{B_x(k, l)}{R_{k, t}} \int_{n_{2i}}^{n_{2i+2}} \frac{du}{u|\log g(u)|}
\]
for any \( x \in M \) and \( i \geq 0 \), where

\[
B'_\varepsilon(k, l) = \frac{K_4}{d_4 + \varepsilon kl} \log k.
\]

Proof. We assume that \( g(t) < 1 \) and \( \kappa_t < (k-1)/(2^{d_4}c_4) \). In a way similar to the proof of Lemma \( 5.7 \), we can apply Lemma \( A.1 \) to show that

\[
P_x(A_{2i}) \geq K_4 \frac{\log k}{1 + \log \left( \frac{(k-1)n_{2i}}{\phi(2(c_3/k)^{1/d_3} \varphi(n_{2i+1}))} \right)}.
\]

Since

\[
\phi \left( 2 \left( \frac{c_3}{k} \right)^{1/d_3} \varphi(n_{2i+1}) \right) \geq 2^{d_3}c_3 \phi \left( \left( \frac{c_3}{k} \right)^{1/d_3} \varphi^{-1}(n_{2i+1}) g(n_{2i+1}) \right) \\
\geq 2^{d_3} c_3^{1+4/d_3} \frac{n_{2i+1}}{d_3} g(n_{2i+1})^{d_4} n_{2i+1}
\]

by (2.4), we have

\[
\log \left( \frac{(k-1)n_{2i}}{\phi(2(c_3/k)^{1/d_3} \varphi(n_{2i+1}))} \right) \leq \log \left( \frac{c_4 k^{d_4/d_3}(k-1)n_{2i}}{2^{d_3} c_3^{1+4/d_3} g(n_{2i+1})^{d_4} n_{2i+1}} \right) \\
= \log \left( \frac{c_4}{2^{d_3} c_3^{1+4/d_3}} \right) + \log k^{d_4/d_3-1}(k-1) + d_4 \log g(n_{2i+1}).
\]

Note that for any \( \varepsilon > 0 \), \( k \in (1, 2) \) and \( l \in (1, 2) \) with \( k > l \), there exists \( T_{\varepsilon,k,l} > 0 \) such that for all \( t \geq T_{\varepsilon,k,l} \), we obtain \( g(t) < 1 \), \( \kappa_t < (k - 1)/(2^{d_4}c_4) \) and

\[
1 + \log \left( \frac{c_4}{2^{d_3} c_3^{1+4/d_3}} \right) + \log k^{d_4/d_3-1}(k-1) \leq \varepsilon \log g(n_{2i+1}).
\]

Hence for all \( t \geq T_{\varepsilon,k,l} \),

\[
1 + \log \left( \frac{c_4}{2^{d_3} c_3^{1+4/d_3}} \right) + \log k^{d_4/d_3-1}(k-1) + d_4 \log g(n_{2i+1}) \leq (d_4 + \varepsilon) \log g(n_{2i+1})
\]

so that

\[
P_x(A_{2i}) \geq K_4 \frac{\log k}{d_4 + \varepsilon |\log g(n_{2i+1})|} = B'_\varepsilon(k, l) \frac{1}{|\log g(n_{2i+1})|} \left( \frac{n_{2i+1} - n_{2i}}{n_{2i}} + \frac{n_{2i+2} - n_{2i+1}}{n_{2i}} \right).
\]

In what follows, we assume that \( t \geq T_{\varepsilon,k,l} \). Let us evaluate the last expression of (A.12). By the definition of \( R_{k,t} \),

\[
\frac{1}{|\log g(n_{2i+1})|} \frac{n_{2i+1} - n_{2i}}{n_{2i}} = \frac{|\log g(n_{2i+1})|}{|\log g(n_{2i+1})| \frac{n_{2i+1} - n_{2i}}{n_{2i}} \log g(n_{2i+1})} \\
\geq \frac{1}{R_{k,t}} \int_{n_{2i}}^{n_{2i+1}} \frac{du}{u|\log g(u)|}.
\]
Since
\[
\frac{1}{|\log g(n_{2i+1})|} \frac{n_{2i+2} - n_{2i+1}}{n_{2i}} \geq \int_{n_{2i+1}}^{n_{2i+2}} \frac{du}{u|\log g(u)|},
\]
we get
\[
\frac{1}{|\log g(n_{2i+1})|} \left( \frac{n_{2i+1} - n_{2i}}{n_{2i}} + \frac{n_{2i+2} - n_{2i+1}}{n_{2i}} \right) \geq \frac{1}{R_{k,t}} \int_{n_{2i}}^{n_{2i+2}} \frac{du}{u|\log g(u)|}.
\]
Combining this with (A.12), we complete the proof.

We are now in a position to finish the proof of (A.2). Under the full conditions in Propositions A.3 and A.5, we obtain for all \( t \geq \max\{T_{k,l}, T_{T_{k,l}}\} \),
\[
\sum_{i=0}^{\infty} P_x(A_{2i}) \geq \frac{B'(k, l)}{R_{k,t}} \int_t^{\infty} \frac{du}{u|\log g(u)|}.
\]
Therefore, it follows by (A.11) that
\[
\frac{q_{\varphi}(t, x)}{\int_t^{\infty} \frac{du}{u|\log g(u)|}} \geq \frac{B'(k, l)}{R_{k,t}} \left( 1 - A'(k, l) \int_t^{\infty} \frac{du}{u|\log g(u)|} \right).
\]
By letting first \( t \to \infty \) and then \( l \to 1 + 0, k \to 1 + 0, \) and \( \varepsilon \to +0 \), we get (A.2).

B Derivation of (3.4) and (3.5)

In this appendix, we show (3.4) and (3.5) in Example 3.2 above.

B.1 Subordination

We first recall the notion of subordinators according to [7] and [30]. An increasing Lévy process on \([0, \infty)\) is called a subordinator. By [30, Theorem 21.5], subordinators are characterized by the following Laplace transform: if we denote by \( \pi_t(ds) \) the transition function of a subordinator, then
\[
\int_0^{\infty} e^{-\lambda s} \pi_t(ds) = e^{-t\psi(\lambda)} \quad (\lambda > 0, \ t > 0)
\]
for
\[
\psi(\lambda) = b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda s})\nu(ds).
\]
Here \( b \) is a nonnegative constant and \( \nu \) is a positive Radon measure on \((0, \infty)\) such that
\[
\int_{(0, \infty)} (s \wedge 1)\nu(ds) < \infty.
\]
For $\gamma \in (0,1)$, we say that a subordinator $\{\tau_t\}_{t \geq 0}$ is $\gamma$-stable if $\psi(\lambda) = \lambda^\gamma$, that is,

$$b = 0, \quad \nu(ds) = \frac{\gamma}{\Gamma(1-\gamma)} \frac{ds}{s^{1+\gamma}}$$

(B.1)

(see, e.g., [30] Examples 21.7 and 24.12).

We next introduce the subordination of symmetric Markov processes. Let $(M,d)$ be a locally compact separable metric space and $m$ a positive Radon measure on $M$ with full support. Let $M = (\Omega, \mathcal{F}, \{X_t\}_{t \geq 0}, \{P_x\}_{x \in M})$ be an $m$-symmetric Hunt process on $M$ such that the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular on $L^2(M;m)$. Fix a subordinator $\{\tau_t\}_{t \geq 0}$ defined on $(\Omega, \mathcal{F})$ such that it is independent of $\{X_t\}_{t \geq 0}$ under $P_x$ for every $x \in M$. Let $M^{(1)} = \{\{Y_t\}_{t \geq 0}, \{P_x\}_{x \in M}\}$ be a subordinated process of $M$ defined by

$$Y_t = X_{\tau_t} \quad \text{for } t \geq 0.$$

If we denote by $p(t, x, dy)$ the transition function of $M$, then the transition function of $M^{(1)}$ is given by

$$q(t, x, dy) = \int_0^\infty p(s, x, dy) \pi_t(ds)$$

(B.2)

([30] Theorem 30.1). According to [28] Theorem 2.1, $M^{(1)}$ associates a regular Dirichlet form $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ on $L^2(M;m)$ such that if $b > 0$, then $\mathcal{F}^{(1)} = \mathcal{F}$ and

$$\mathcal{E}^{(1)}(u, v) = b\mathcal{E}(u, v) + \iint_{M \times M \backslash \text{diag}} (u(x) - u(y))(v(x) - v(y)) J(dx,dy) + \int_M u(x)v(x) \ k(dx)$$

for

$$J(dx,dy) = \frac{1}{2} 1_{\{x \neq y\}} m(dx) \int_{(0,\infty)} \nu(ds) p(s, x, dy) \quad \text{(B.3)}$$

and

$$k(dx) = m(dx) \int_{(0,\infty)} (1 - p(s, x, M)) \nu(ds).$$

On the other hand, if $b = 0$, then the form $\mathcal{E}^{(1)}$ is the same as before and

$$\mathcal{F}^{(1)} = \left\{ u \in L^2(M;m) \mid \int_{(0,\infty)} \left( \int_M (u(x) - T_s u(x)) u(x) m(dx) \right) \nu(ds) < \infty \right\}.$$

We note that $\mathcal{F} \subset \mathcal{F}^{(1)}$.

**B.2 Derivation of (3.4)**

Throughout this subsection, we assume the full conditions in Example 3.2. We take the measure $\nu$ as in (B.1) for some $\gamma \in (0,1)$. We recall that

$$\phi(r) = r^{\gamma_1} 1_{\{r < 1\}} + r^{\gamma_2} 1_{\{r \geq 1\}}.$$
On account of (B.3), it is enough for the proof of (3.4) to show that

\[ \int_0^\infty p(s, x, y) \nu(ds) \asymp \frac{1}{V(x, d(x, y)) \phi(d(x, y))}. \]  

(B.4)

We only show the upper bound of the left side in (B.4) because the lower bound follows by the same way. Define

\[ (A) = \int_0^{1 \vee d(x, y)} p(s, x, y) \nu(ds) \quad \text{and} \quad (B) = \int_0^\infty p(s, x, y) \nu(ds). \]

Then by the change of variables \( u = d(x, y)^{\beta_1}/s \),

\[ (A) \lesssim \int_0^{1 \vee d(x, y)} \frac{1}{V(x, s^{1/\beta_1})} \exp \left\{ -C_2 \left( \frac{d(x, y)^{\beta_1}}{s} \right)^{1/(\beta_1-1)} \right\} \frac{ds}{s^{1+\gamma}} \]

\[ \lesssim \frac{1}{d(x, y)^{\gamma_{\beta_1}}} \int_{d(x, y)^{\beta_1} \vee d(x, y)^{\beta_1-1}}^\infty \frac{1}{V(x, d(x, y)/u^{1/\beta_1})} e^{-C_2 u^{1/(\beta_1-1)}} u^{\gamma-1} du =: (A'). \]

Since

\[ \frac{V(x, d(x, y)/u^{1/\beta_1})}{V(x, d(x, y))} \geq \begin{cases} \frac{C_3}{u^{d_1/\beta_1}}, & (0 < u < 1), \\ \frac{1}{C_4 u^{d_2/\beta_1}}, & (u \geq 1) \end{cases} \]  

(B.5)

by (2.2), we have

\[ (A') \lesssim \frac{1}{V(x, d(x, y)) d(x, y)^{\gamma_{\beta_1}}} \]

\[ \times \int_{d(x, y)^{\beta_1} \vee d(x, y)^{\beta_1-1}}^\infty \left( u^{d_1/\beta_1} 1_{\{u < 1\}} + u^{d_2/\beta_1} 1_{\{u \geq 1\}} \right) \exp \left( -C_2 u^{1/(\beta_1-1)} \right) u^{\gamma-1} du. \]

By the same way, we obtain

\[ (B) \lesssim \frac{1}{V(x, d(x, y)) d(x, y)^{\gamma_{\beta_2}}} \]

\[ \times \int_0^{d(x, y)^{\beta_2} \vee d(x, y)^{\beta_2-1}} \left( u^{d_1/\beta_2} 1_{\{u < 1\}} + u^{d_2/\beta_2} 1_{\{u \geq 1\}} \right) \exp \left( -C_4 u^{1/(\beta_2-1)} \right) u^{\gamma-1} du \]

so that

\[ \int_0^\infty p(s, x, y) \nu(ds) = (A) + (B) \lesssim \frac{1}{V(x, d(x, y)) \phi(d(x, y))}. \]

**B.3 Derivation of (3.5)**

Throughout this subsection, we keep the same setting as in Subsection B.2. Let \( \pi_t(s) \) be the density of the transition function for the \( \gamma \)-stable subordinator. Then the following relations hold (see, e.g., [3, Theorem 3.1] and [30, Remark 14.18]):
For each $t > 0$, $\pi_t(s)$ is a bounded continuous function on $(0, \infty)$ such that

$$\pi_t(s) = \frac{1}{t^{1/\gamma}} \pi_1 \left( \frac{s}{t^{1/\gamma}} \right)$$

for any $s > 0$ and $t > 0$;

- There exists $c > 0$ such that
  $$\pi_t(s) \leq c \frac{t}{s^{1+\gamma}} \exp \left( -\frac{t}{s^\gamma} \right) \leq c \frac{t}{s^{1+\gamma}}$$
  for any $s > 0$ and $t > 0$;

- There exists $c > 0$ such that if $s \geq t^{1/\gamma}$, then
  $$\pi_t(s) \geq c \frac{t}{s^{1+\gamma}}.$$  
(B.8)

By (B.2), the $\gamma$-stable subordinated diffusion process admits the heat kernel $q(t, x, y)$ such that

$$q(t, x, y) = \int_0^\infty p(s, x, y) \pi_t(s) \, ds.$$  
(B.2)

To get the upper bound of (3.5), we first show that

$$q(t, x, y) \lesssim \frac{t}{V(x, d(x, y)) \phi(d(x, y))}.$$  
(B.9)

We divide $q(t, x, y)$ into

$$I = \int_0^{1 \wedge d(x, y)} p(s, x, y) \pi_t(s) \, ds \quad \text{and} \quad II = \int_{1 \wedge d(x, y)}^{\infty} p(s, x, y) \pi_t(s) \, ds.$$  
(B.7)

By (B.7) and the change of variables $s = d(x, y)^{\beta_1}/u$,

$$I \lesssim t \int_0^{1 \wedge d(x, y)} \frac{1}{V(x, s^{1/\beta_1})} \exp \left\{ -C_2 \left( \frac{d(x, y)^{\beta_1}}{s^{1/(\beta_1-1)}} \right)^{1/(\beta_1-1)} \right\} \frac{ds}{s^{1+\gamma/2}}$$

$$= \frac{t}{d(x, y)^{\gamma \beta_1/2}} \int_{d(x, y)^{\beta_1} \wedge d(x, y)^{\beta_1-1}}^{\infty} \frac{1}{V(x, d(x, y)/u^{1/\beta_1})} \exp \left( -C_2 u^{1/(\beta_1-1)} \right) u^{\gamma-1} \, du =: (I)'.$$  
(B.7)

Then by (B.5),

$$(I)' \lesssim \frac{t}{V(x, d(x, y))d(x, y)^{\gamma \beta_1}}$$

$$\times \int_{d(x, y)^{\beta_1} \wedge d(x, y)^{\beta_1-1}}^{\infty} (u^{d_1/\beta_1} \mathbf{1}_{u<1} + u^{d_2/\beta_1} \mathbf{1}_{u \geq 1}) \exp \left( -C_2 u^{1/(\beta_1-1)} \right) u^{\gamma-1} \, du.$$  
(B.5)
By the same way, we obtain

\[
(II) \lesssim \frac{t}{V(x, d(x, y))d(x, y)^{\gamma \beta_2}} \times \int_0^{d(x, y)^{\beta_2} \land d(x, y)^{\beta_2-1}} (u^{d_1/\beta_2}1_{\{u<1\}} + u^{d_2/\beta_2}1_{\{u \geq 1\}}) \exp (-C_4 u^{1/(\beta_2-1)}) u^{\gamma-1} du
\]

so that (B.9) follows.

We next show that

\[
q(t, x, y) = (I) + (II) \lesssim \frac{1}{V(x, \phi^{-1}(t))}.
\] (B.10)

Suppose first that \(d(x, y) < 1\). By (B.7) and the change of variables \(u = t/s^\gamma\),

\[
(I) \lesssim t \int_0^1 \frac{1}{V(x, s^{1/\beta_1})} \exp \left(-\frac{t}{s^\gamma}\right) \frac{ds}{s^{1+\gamma}} \lesssim \int_t^\infty \frac{e^{-u}}{V(x, (t/u)^{1/(\gamma \beta_1)})} du =: (I').
\]

Then by (B.5),

\[
(I') \lesssim \frac{1}{V(x, t^{1/(\gamma \beta_1)})} \int_t^\infty (u^{d_1/(\gamma \beta_1)}1_{\{u<1\}} + u^{d_2/(\gamma \beta_1)}1_{\{u \geq 1\}}) e^{-u} du.
\]

By the same way, we get

\[
(II) \lesssim t \int_1^\infty \frac{1}{V(x, s^{1/\beta_2})} \exp \left(-\frac{t}{s^\gamma}\right) \frac{ds}{s^{1+\gamma}} \lesssim \int_0^t \frac{e^{-u}}{V(x, (t/u)^{1/(\gamma \beta_2)})} du
\]

\[
\lesssim \frac{1}{V(x, t^{1/(\gamma \beta_2)})} \int_0^t (u^{d_1/(\gamma \beta_2)}1_{\{u<1\}} + u^{d_2/(\gamma \beta_2)}1_{\{u \geq 1\}}) e^{-u} du.
\]

Hence if \(0 < t < 1\), then

\[
(I) + (II) \lesssim t \int_0^1 \frac{1}{V(x, s^{1/\beta_1})} \exp \left(-\frac{t}{s^\gamma}\right) \frac{ds}{s^{1+\gamma}} \lesssim \frac{1}{V(x, t^{1/(\gamma \beta_1)})}.
\]

On the other hand, if \(t \geq 1\), then

\[
(I) + (II) \lesssim \frac{1}{V(x, t^{1/(\gamma \beta_2)})} \int_0^t (u^{d_1/(\gamma \beta_2)}1_{\{u<1\}} + u^{d_2/(\gamma \beta_2)}1_{\{u \geq 1\}}) e^{-u} du \lesssim \frac{1}{V(x, t^{1/(\gamma \beta_2)})}.
\]

Therefore, (B.10) holds.

Suppose next that \(d(x, y) \geq 1\). If \(t < \phi(d(x, y))(= d(x, y)^{\gamma \beta_2})\), then (B.10) follows by (B.9). In what follows, we assume that \(t \geq \phi(d(x, y))\). By (B.7) and the change of variables \(u = t/s^\gamma\),

\[
(I) \lesssim t \int_0^{d(x, y)} \frac{1}{V(x, s^{1/\beta_1})} \exp \left(-\frac{t}{s^\gamma}\right) \frac{ds}{s^{1+\gamma}} \lesssim \int_{t/d(x, y)^\gamma}^\infty \frac{e^{-u}}{V(x, (t/u)^{1/(\gamma \beta_1)})} du =: (I'').
\]
Then by (B.5),
\[
(\text{I})'' \lesssim \frac{1}{V(x, t^{1/(\gamma_1)})} \int_{t/d(x,y)\gamma}^{\infty} (u^{d_1/(\gamma_1)} 1_{\{u<1\}} + u^{d_2/(\gamma_1)} 1_{\{u\geq1\}}) e^{-u} \, du.
\]
By the same way, we get
\[
(\text{II}) \lesssim \frac{1}{V(x, t^{1/(\gamma_2)})} \int_{0}^{t/d(x,y)\gamma} (u^{d_1/(\gamma_2)} 1_{\{u<1\}} + u^{d_2/(\gamma_2)} 1_{\{u\geq1\}}) e^{-u} \, du.
\]
Since \( t > t/d(x,y)\gamma \geq t^{1-1/\beta_2} \) by assumption, we obtain (B.10). As a consequence of the argument above, we get the upper bound of (3.3).

We next discuss the lower bound of (3.3). Suppose first that \( 0 < t < 1 \) and \( d(x,y) \beta_1 \leq 1 \vee d(x,y) \), it follows by (3.6) and the change of variables \( s = Tu \) \((T = t^{1/\gamma} \vee d(x,y) \beta_1)\) that
\[
\int_{0}^{t^{1/\gamma} \vee d(x,y) \beta_1} p(s, x, y) \pi_\epsilon(s) \, ds \\
\geq \int_{0}^{t^{1/\gamma} \vee d(x,y) \beta_1} \frac{1}{V(x, s^{1/\beta_1})} \exp \left\{ - \left( C_1 \frac{d(x,y) \beta_1}{s} \right)^{1/(\beta_1-1)} \right\} \pi_\epsilon(s) \, ds \\
= \frac{1}{t^{1/\gamma}} \int_{0}^{t^{1/\gamma} \vee d(x,y) \beta_1} \frac{1}{V(x, s^{1/\beta_1})} \exp \left\{ - \left( C_1 \frac{d(x,y) \beta_1}{s} \right)^{1/(\beta_1-1)} \right\} \pi_1 \left( \frac{s}{t^{1/\gamma}} \right) \, ds \\
= \frac{t}{t^{1/\gamma}} \int_{0}^{\infty} \frac{1}{V(x, (Tu)^{1/\beta_1})} \exp \left\{ - \left( C_1 \frac{d(x,y) \beta_1}{Tu} \right)^{1/(\beta_1-1)} \right\} \pi_1 \left( \frac{Tu}{t^{1/\gamma}} \right) \, du.
\]
By (2.2), we have
\[
V(x, (Tu)^{1/\beta_1}) \leq u^{d_1/\beta_1} V(x, T^{1/\beta_1}) / c_1 \text{ for any } u \in (0,1) \text{ so that the last expression of (B.11) is greater than}
\]
\[
c_1 \frac{T}{t^{1/\gamma}} \frac{1}{V(x, T^{1/\beta_1})} \int_{0}^{1} \frac{1}{u^{d_1/\beta_1}} \exp \left\{ - \left( \frac{C_1}{u^{1/(\beta_1-1)}} \right) \right\} \pi_1(u) \, du \\
\geq \min \left\{ \frac{1}{V(x, t^{1/\beta_1})}, \frac{1}{V(x, d(x,y))} \right\} \geq \min \left\{ \frac{1}{V(x, t^{1/\beta_1})}, \frac{1}{V(x, d(x,y))} d(x,y)^{\gamma_1} \right\}.
\]
Suppose next that \( 0 < t < 1 \) and \( d(x,y) \geq 1 \), or \( t \geq 1 \). Since \( t^{1/\gamma} \vee d(x,y) \beta_2 \geq 1 \vee d(x,y) \), it follows by (B.8) that
\[
\int_{t^{1/\gamma} \vee d(x,y) \beta_2}^{\infty} p(s, x, y) \pi_\epsilon(s) \, ds \\
\geq \int_{t^{1/\gamma} \vee d(x,y) \beta_2}^{\infty} \frac{1}{V(x, s^{1/\beta_2})} \exp \left\{ - \left( C_3 \frac{d(x,y) \beta_2}{s} \right)^{1/(\beta_2-1)} \right\} \, ds \\
\geq \int_{t^{1/\gamma} \vee d(x,y) \beta_2}^{\infty} \frac{1}{V(x, s^{1/\beta_2})} \, ds \leq \min \left\{ \frac{1}{V(x, t^{1/\beta_2})}, \frac{1}{V(x, d(x,y))} d(x,y)^{\gamma_2} \right\}.
\]

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At the last relation above, we used the fact that for any $p > 0$,
\[
\int_t^\infty \frac{ds}{V(x, s^{1/\beta_2}) s^{1+p}} \approx \frac{1}{V(x, t^{1/\beta_2}) t^p} \quad (t > 0),
\]
which follows by the same way as in (4.5). We thus get the lower bound of (3.5).

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