Cannon–Thurston maps, subgroup distortion, and hyperbolic hydra

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May 2, 2014

Abstract

We prove that Cannon–Thurston maps are well–defined (that is, subgroup inclusion induces a map of the boundaries) for heavily distorted free subgroups inside the family of hyperbolic groups known as hyperbolic hydra. Whilst this indicates that distortion is not an obstacle to the map being well–defined, we show that heavy subgroup distortion always manifests in Cannon–Thurston maps (when they are well–defined) in that their continuity is quantifiably wild.

2010 Mathematics Subject Classification: 20F67

Key words and phrases: Cannon–Thurston map, hyperbolic group, subgroup distortion, hydra, Ackermann’s function

1 Introduction

A Cannon–Thurston map $\partial \Lambda \to \partial \Gamma$ is a map between Gromov–boundaries that is induced by the inclusion $\Lambda \to \Gamma$ of an infinite hyperbolic subgroup $\Lambda$ in a hyperbolic group $\Gamma$. (More details are in Section 2.)

For a finitely generated subgroup $\Lambda$ of a finitely generated group $\Gamma$, define the distortion function

$$\text{Dist}_\Lambda(n) := \max \{ d_\Lambda(e, h) \mid h \in \Gamma, d_\Gamma(e, h) \leq n \},$$

where $d_\Gamma$ and $d_\Lambda$ are word metrics with respect to some finite generating sets. We say that $f \preceq g$ for $f, g : \mathbb{N} \to \mathbb{N}$ when there exists $C > 0$ such that $f(n) \leq C g(C n + C) + C n + C$ for all $n \geq 0$. We say $f \simeq g$ when $f \preceq g$ and $g \preceq f$. Up to $\simeq$, $\text{Dist}_\Lambda(n)$ does not depend on the choices of finite generating sets.

If $\Lambda$ is an undistorted subgroup in a hyperbolic group $\Gamma$ (that is, $\text{Dist}_\Lambda(n) \preceq n$), then $\Lambda$ is also hyperbolic (e.g. [4, page 461]) and the induced map $\partial \Lambda \to \partial \Gamma$ is readily seen to be well–defined.
Generalizing an example of Cannon & Thurston [7], Mitra showed it also is well–defined when \( \Lambda \) is an infinite normal subgroup of \( \Gamma \) [16], and when \( \Gamma \) is a finite graph of groups where the edge inclusions are quasi–isometric embeddings and \( \Lambda \) is one of the (infinite) vertex– or edge–groups [17]. We recently gave the first example where the Cannon–Thurston map is known not to be well–defined [1], and Matsuda & Oguni showed it leads to examples where the subgroup in question can be any non–elementary hyperbolic group or, even, relatively hyperbolic group [15].

In this article we show that extreme distortion is no barrier to the Cannon–Thurston map being well–defined, but such distortion manifests in extreme wildness of the map.

The **hyperbolic hydra** \( \Gamma_k \) \((k = 1, 2, \ldots)\) of [3] are a family of hyperbolic groups with finite–rank free subgroups \( \Lambda_k \) exhibiting the fastest–growing distortion functions known for hyperbolic subgroups of hyperbolic groups: \( \text{Dist}_k^{\Lambda_k} \) grows at least like \( A_k \), the \( k \)–th of Ackermann’s family of increasingly fast–growing functions that begins with \( A_1(n) = 2^n, A_2(n) = 2^n \), and \( A_3(n) \) equalling the height–\( n \) tower of powers of 2. We prove:

**Theorem 1.1.** Hyperbolic hydra have Cannon–Thurston maps \( \partial \Lambda_k \rightarrow \partial \Gamma_k \) for all \( k \).

Wildness is apparent in Cannon & Thurston’s example [7], a closed hyperbolic 3–manifold \( M \) fibering over the circle with fiber a hyperbolic surface \( S \) and pseudo–Anosov monodromy. They show that the Cannon–Thurston map \( \partial(\pi_1 S) = S^1 \rightarrow S^2 = \partial(\pi_1 M) \) for \( \pi_1 S \hookrightarrow \pi_1 M \) is well–defined but is a group–equivariant space–filling curve.

We detect wildness in Cannon–Thurston maps in the relationship between “\( \varepsilon \)” and “\( \delta \)” in their continuity (having given the boundaries their natural “visual” metrics—see Section 2).

The quality of a function \( f : U \rightarrow V \) between metric spaces is measured by its **modulus of continuity** \( \varepsilon : [0, \infty) \rightarrow [0, \infty] \),

\[
\varepsilon(\delta) := \sup \{ d_V(f(a), f(b)) \mid a, b \in U \text{ with } d_U(a, b) \leq \delta \},
\]

a notion going back at least to Lebesgue in 1909 [14]. An upper bound on \( \varepsilon(\delta) \) expresses a degree of good behaviour: \( f \) is uniformly continuous if \( \varepsilon(\delta) \rightarrow 0 \) as \( \delta \rightarrow 0 \); is Lipschitz if \( \varepsilon(\delta) \leq C\delta \) for a constant \( C > 0 \); and is \( \alpha \)–Hölder if \( \varepsilon(\delta) \leq C\delta^\alpha \) for a constant \( C > 0 \). Wildness manifests in lower bounds on \( \varepsilon(\delta) \), expressing that \( \varepsilon(\delta) \) is extravagantly larger than \( \delta \) when \( \delta \) is close to 0. This happens for the Cannon–Thurston map (if it is well–defined) when the subgroup in question is heavily distorted:

**Theorem 1.2.** Suppose \( \Lambda \leq \Gamma \) are hyperbolic, \( \Lambda \) is non–elementary, the Cannon–Thurston map \( \partial \Lambda \rightarrow \partial \Gamma \) is well–defined, and \( r, s > 1 \) are any respective visual parameters for visual metrics on \( \partial \Lambda \) and \( \partial \Gamma \). Then there exist \( \alpha, \beta > 0 \) so that for all \( n \geq 0 \) the modulus of continuity satisfies

\[
\varepsilon\left(\frac{\beta}{r\text{Dist}_k^{\Lambda_k}(n)}\right) \geq \frac{\alpha}{s^n}.
\]

The reason we insist that \( \Lambda \) be non–elementary (i.e. contains an \( F_2 \) subgroup; see e.g. [11, Theorem 2.28]) in this theorem is that elementary \( \Lambda \) are not interesting in this context: if \( \Lambda \) is finite, then \( \partial \Lambda \) is empty; if \( \Lambda \) is virtually \( \mathbb{Z} \), then \( \Lambda \) is quasiconvex in \( G \), its boundary \( \partial \Lambda \) is two points, and the Cannon–Thurston map is well–defined and is an embedding.

The moduli of continuity of Cannon–Thurston maps have received attention before. Cannon–Thurston maps were originally defined from the Gromov–boundary of a Kleinian group to \( \partial \mathbb{H}^3 = S^2 \) — see [17]. (The Cannon–Thurston maps we study here for hyperbolic groups are a natural generalization of the cocompact case.) When the Kleinian group is a finitely generated Fuchsian group of the first type with bounded geometry and no parabolic elements, Miyachi [18] gives an upper bound on the modulus of continuity for the Cannon–Thurston map, and shows it is not Hölder continuous when the group is geometrically infinite.

Theorems 1.1 and 1.2 combine with [3, Theorem 1.1], which says that \( \text{Dist}_k^{\Lambda_k} \geq A_k \), to give the following corollary which we will make precise and prove in Section 5.

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Corollary 1.3. For all $k \geq 2$, the modulus of continuity $\epsilon(\delta)$ for the Cannon–Thurston map $\partial \Lambda_k \to \partial \Gamma_k$ for hyperbolic hydra grows at least like $1/n$ when $\delta$ grows like $1/A_k(n)$.

By Lemma 3.2, this result neither depends on choices of finite generating sets for $\Gamma_k$ and $\Lambda_k$, nor of visual metrics on $\partial \Gamma_k$ and $\partial \Lambda_k$ (but the constants involved may depend on such choices).

Remark 1.4. A detailed understanding of the Cannon–Thurston Map $\partial \Lambda_k \to \partial \Gamma_k$ appears hard to obtain, since, whilst $\partial \Lambda_k$ is a Cantor set (as $\Lambda_k$ is free), $\partial \Gamma_k$ is not so readily identified. (I. Kapovich & M. Lustig [12] recently made advances in the understanding of Cannon–Thurston maps for certain free–by–cyclic groups, but $\Lambda_k \leq \Gamma_k$ do not fall within the scope of their work.) Here is what we can say about $\partial \Gamma_k$.

Splittings of hyperbolic free–by–cyclic groups $F \rtimes_{\varphi} \mathbb{Z}$ are studied in [13] and [5], the former dealing with the case where $\varphi$ is an irreducible hyperbolic free group automorphism, and the latter with $\varphi$ a general hyperbolic free group automorphism. The argument preceding Corollary 15 in [13] shows that any hyperbolic free–by–cyclic group has one–dimensional boundary: the cohomological dimension of any (finitely generated free)–by–cyclic group is 2 (see e.g. [6, pp.185–7]), so [2, Corollary 1.4(d)] implies $\partial \Gamma_k$ has dimension $2 - 1 = 1$.

The argument of [13, Corollary 15] shows that any hyperbolic free–by–cyclic group has connected, locally connected boundary without global cut points. To see this, it suffices by [11, Theorems 7.1 and 7.2] to check that $F \rtimes_{\varphi} \mathbb{Z}$ is freely indecomposable, which is true for any free group automorphism $\varphi$. Indeed, the Bass–Serre tree $T$ for any graph of groups decomposition of $F \rtimes_{\varphi} \mathbb{Z}$ admits a minimal action by the normal subgroup $F$ with quotient a finite graph by Grushko’s Theorem. This shows the edge stabilizers for the action of $F \rtimes_{\varphi} \mathbb{Z}$ on $T$ are non–trivial, so the decomposition cannot be free.

On the other hand, $\Gamma_k$ splits as an HNN–extension over $\mathbb{Z}$ for every $k$, so [11, Theorem 7.2] implies $\partial \Gamma_k$ has local cut points. Indeed, $\Gamma_k$ splits over $\mathbb{Z}$ as $(B, a_1) \ast (a_0^{-1} t ^{-1} a_1 a_0 t, a_1^{-1} t ^{-1})$, where $B$ denotes the subgroup generated by all the defining generators other than $a_0$ and $a_1$ (then $a_0$ appears as $t^{-1}a_1 t$), and for $k \geq 2$, $\Gamma_k$ splits as an HNN–extension over $\mathbb{Z}$ with the stable letter $a_k$ conjugating $t$ to $t a_k^{-1}$.

The organization of the remainder of this article. In Section 2 we give background on hyperbolic groups and their boundaries. In Section 3 we define Cannon–Thurston maps and prove an embellished version of a lemma of Mitra giving necessary and sufficient conditions for their being well–defined. In Section 4 we review hyperbolic hydra groups and prove Theorem 1.1. In Section 5 we prove Theorem 1.2 and Corollary 1.3.

Acknowledgment. We thank Mahan Mj (formerly Mitra) for conversations which fueled an interest in the relationship between subgroup distortion and Cannon–Thurston maps.

2 Hyperbolic groups and their boundaries

This section contains a brief account of some pertinent background. More general treatments can be found in, for example, [4, 9, 11, 19] and Gromov’s foundational article [10].

For a metric space $X$, the Gromov product $(a \cdot b)_e$ (or $(a \cdot b)^X$ to avoid ambiguity) of $a, b \in X$ with respect to $e \in X$ is

$$(a \cdot b)_e = \frac{1}{2} (d(a, e) + d(b, e) - d(a, b)).$$

One says $X$ is $(\delta)$–hyperbolic when

$$(a \cdot b)_e \geq \min \{(a \cdot c)_e, (b \cdot c)_e\} - \delta$$

for all $e, a, b, c \in X$, and $X$ is hyperbolic when it is $(\delta)$–hyperbolic for some $\delta \geq 0$. When $X$ is a geodesic space this is equivalent to other standard definitions of hyperbolicity (such as $\delta$–thin or $\delta$–slim triangles), although the $\delta$ involved may not agree.
When $X$ is $(0)$–hyperbolic and geodesic—that is, an $\mathbb{R}$–tree—$(a \cdot b)_e$ is the distance from $e$ to the geodesic between $a$ and $b$. Correspondingly, in a $(\delta)$–hyperbolic geodesic space every pair of geodesics from $e$ to $a$ and to $b$, both parametrized by arc–length, $6\delta$–fellow–travel for approximately $(a \cdot b)_e$, and then diverge (by the insize characterization of hyperbolicity of [4, page 408]). Indeed:

**Lemma 2.1.** In a $(\delta)$–hyperbolic geodesic metric space, for every geodesic $[a, b]$ connecting $a$ and $b$
\[|d(e, [a, b]) − (a \cdot b)_e| ≤ 6\delta.\]

**Proof.** See [4]: the proof of Proposition 1.22 on page 411 shows that insizes of geodesic triangles are at most $6\delta$, and the proof of Proposition 1.17(3) $\implies$ (2) on page 409 shows that all geodesic triangles are $6\delta$–thin, and the claimed inequality follows.\[\square\]

The (Gromov–) boundary $\partial X$ of a hyperbolic metric space $X$ is defined with reference to, but is in fact independent of, a point $e \in X$. It is the set of equivalence classes of sequences $(a_n)$ in $X$ such that $(a_m \cdot a_n)_e \to \infty$ as $m, n \to \infty$, where two such sequences $(a_n)$ and $(b_n)$ are equivalent when $(a_m \cdot b_n)_e \to \infty$ as $m, n \to \infty$. Indeed, they are equivalent when $(a_n \cdot b_n)_e \to \infty$ as $n \to \infty$ since
\[(a_n \cdot b_m)_e \geq \min\{(a_n \cdot b_n)_e, (b_m \cdot b_n)_e\} - \delta\]
by $(\delta)$–hyperbolicity. Denote the equivalence class of $(a_n)$ by $\lim a_n$.

When $X$ is a geodesic hyperbolic metric space, there are equivalent definitions of $\partial X$, such as $\partial X$ is the set of equivalence classes of geodesic rays emanating from $x$, where two such rays are equivalent when they stay uniformly close. The condition $(a_m \cdot a_n)_e \to \infty$ is what makes a sequence $(a_n)$ ray–like, and the condition $(a_m \cdot b_n)_e \to \infty$ corresponds to uniform closeness.

Extend the Gromov product to $\overline{X} := X \cup \partial X$ by
\[(a \cdot b)_e = \sup_{m, n \to \infty} \liminf (a_m \cdot b_n)_e\]
where the sup is over all sequences $(a_m)$ and $(b_n)$ in $X$ representing (when in $\partial X$) or tending to (when in $X$) $a$ and $b$, respectively. (The “$\sup \liminf$” is necessary—see [4, page 432].)

We note, for (3) in the following lemma, that in a proper geodesic hyperbolic metric space $X$, each pair of distinct points $a, b \in \partial X$ is joined by a bi–infinite geodesic line $[a, b]$ (Lemma 3.2 on page 428 of [4]).

**Lemma 2.2.** Suppose $X$ is a proper geodesic $(\delta)$–hyperbolic metric space.

(0). If $x, y \in \overline{X}$ and $e \in X$, then there exist sequences $(x_n)$ and $(y_n)$ in $X$ with $x = \lim x_n, y = \lim y_n,$ and $(x \cdot y)_e = \lim (x_n \cdot y_n)_e$.

(1). If $a, b, c \in \overline{X}$ and $e \in X$, then $(a \cdot b)_e \geq \min\{(a \cdot c)_e, (c \cdot b)_e\} - 2\delta$.

(2). If $a, b \in X$ and $c \in \partial X$, then $|d(a, b) − (a \cdot c)_b − (b \cdot c)_a| \leq \delta$.

(3). If $e \in X$ and $[a, b]$ is any geodesic joining any $a, b \in \partial X$, then $|d(e, [a, b]) − (a \cdot b)_e| \leq 8\delta$.

**Proof.** (0) and (1) are parts 3 and 4 of [4, page 433, Remark 3.17]. (Alternately, see parts 3 and 5 of [19, Lemma 4.6] but note that there infrim is used in place of $\sup \liminf$ and so the constants differ.)

For (2), using (0) take sequences $c_n, c'_n$ both approaching $c$ such that $(a \cdot c)_b = \lim (a \cdot c_n)_b$ and $(b \cdot c)_a = \lim (b \cdot c'_n)_a$. Now, $(\delta)$–hyperbolicity yields $(a \cdot c_n)_b \geq \min\{(a \cdot c'_n)_b, (c_n \cdot c'_n)_b\} - \delta$. As $n \to \infty$, we have $(c_n \cdot c'_n)_b \to \infty$, but $(a \cdot c_n)_b$ is bounded above by $d(a, b)$. So $(a \cdot c'_n)_b \leq (a \cdot c_n)_b + \delta$ for all sufficiently large $n$. Interchanging the roles of $c_n$ and $c'_n$ we find $|(a \cdot c'_n)_b − (a \cdot c_n)_b| \leq \delta$ for all sufficiently large $n$. Hence:
\[|d(a, b) − (a \cdot c_n)_b − (b \cdot c'_n)_a| \leq \delta + |d(a, b) − (a \cdot c'_n)_b − (b \cdot c'_n)_a| = \delta + 0 = \delta\]
for all sufficiently large $n$. Taking the limit as $n \to \infty$ gives the result.

For (3) (cf. Exercise 3.18(3) [4, page 433]), choose sequences $a_n \to a$ and $b_n \to b$ along $[a, b]$. Also choose $a_n', b_n' \in X$ in (0) so that $(a \cdot b)_e = \lim (a_n' \cdot b_n')_e$. For large enough $n$, the closest point of $[a, b]$ to $e$ lies on $[a_n, b_n]$, so
\[
|d(e, [a, b]) - (a_n \cdot b_n)_e| = |d(e, [a_n, b_n]) - (a_n \cdot b_n)_e| \leq 6\delta,
\]
with the inequality being by Lemma 2.1. By the $(\delta)$-hyperbolicity condition,
\[
(a_n \cdot b_n)_e \geq \min\{(a_n \cdot a_n')_e, (a_n' \cdot b_n')_e, (b_n \cdot b_n')_e\} - 2\delta,
\]
and
\[
(a_n' \cdot b_n')_e \geq \min\{(a_n \cdot a_n')_e, (a_n \cdot b_n)_e, (b_n \cdot b_n')_e\} - 2\delta.
\]
As $n \to \infty$ both $(a_n \cdot a_n')_e \to \infty$ and $(b_n \cdot b_n')_e \to \infty$, but $\limsup (a_n \cdot b_n)_e$ and $\limsup (a_n' \cdot b_n')_e$ are bounded above by $(a \cdot b)_e + 1$ (else, passing to subsequences, we can assume $(a_n \cdot b_n)_e > (a \cdot b)_e + 1/2$ for all $n$, and so $\liminf (a_n \cdot b_n)_e \geq (a \cdot b)_e + 1/2$ contrary to the definition of $(a \cdot b)_e$). So these two inequalities together give $|(a_n \cdot b_n)_e - (a_n' \cdot b_n')_e| \leq 2\delta$ for all sufficiently large $n$. Combining this with (1) gives the result. \hfill \Box

Visual metrics are natural metrics on the boundary $\partial X$ of a $(\delta)$--hyperbolic space $X$. Their essence is that $a, b \in \partial X$ are close when geodesics from a basepoint $e \in X$ to $a$ and from $e$ to $b$ fellow travel for a long distance. One might hope that if $r > 1$, then $d(a, b) = r^{-(a \cdot b)}$ would define such a metric, but unfortunately, as such, transitivity can fail. Instead, say that a metric $d$ on $\partial X$ is a visual metric with visual parameter $r > 1$ when there exist $k_1, k_2 > 0$ such that for all $a, b \in \partial X$,
\[
k_1 r^{-(a \cdot b)} \leq d(a, b) \leq k_2 r^{-(a \cdot b)}.
\]
By [4, Proposition 3.21, page 435] (existence) and [11, Theorem 2.18] (Hölder--equivalence):

**Lemma 2.3.** Suppose $X$ is a $(\delta)$--hyperbolic space, $r > 1$, and $e \in X$ is the base point with respect to which $\partial X$ is defined. Then there is a visual metric on $\partial X$ with parameter $r$. Moreover, any two visual metrics $d$ and $d'$ on $\partial X$ (perhaps with different $r$ and $e$) are Hölder--equivalent in that there exists $\alpha > 0$ such that the identity map $(\partial X, d) \to (\partial X, d')$ is $\alpha$--Hölder and its inverse is $(1/\alpha)$--Hölder.

We will need that $\overline{X} := X \cup \partial X$ is a compactification of $X$:

**Lemma 2.4.** If $X$ is a proper $(\delta)$--hyperbolic geodesic metric space, then there is compact, sequentially compact metric on $\overline{X} := X \cup \partial X$ such that the inclusions of $X$ and of $\partial X$ are homeomorphic onto their images. Under this topology, $\partial X$ is closed, and a sequence $x_n$ in $X$ converges to $x \in \partial X$ if and only if $(x_n)$ is in the equivalence class of $x$.

**Proof:** Sequential compactness is proved in [4, page 430, III.H.3.7] and compactness in [4, page 433 III.H.3.18(4)]. The last sentence follows from [4, page 431, III.H.3.13]. \hfill \Box

### 3 Cannon–Thurston maps and Mitra’s Lemma

Consider hyperbolic metric spaces $(X, d_X)$ and $(Y, d_Y)$ and a map $f : Y \to X$. The induced map $\hat{f} : \partial Y \to \partial X$, if it is well--defined, sends the point in $\partial Y$ represented by the sequence $(a_n)$ to the point in $\partial X$ represented by $(f(a_n))$. In the case where $X$ is a hyperbolic group, $Y$ is a hyperbolic subgroup, both with word metrics associated to finite generating sets, and $f$ is the inclusion map, $\hat{f}$ is the Cannon–Thurston map.

The following lemma is an embellished version of Mitra’s criterion for the Cannon–Thurston map to be well--defined ([16, Lemma 2.1] and [17, Lemma 2.1]). It also addresses continuity: if the map is well--defined, then it is necessarily continuous.
Our notation is that \( B_X(e, R) = \{ x \in X \mid d_X(e, x) < r \} \) and \( \overline{B}_X(e, R) = \{ x \in X \mid d_X(e, x) \leq r \} \), and we write \( \gamma = [x, y]_X \) to mean \( \gamma \) is a geodesic in \( X \) from \( x \) to \( y \). The metrics on \( \partial X \) and \( \partial Y \) implicit in this lemma are any visual metrics \( d_{\partial X} \) and \( d_{\partial Y} \).

The additional hypothesis for (e) can be removed if \( \delta_Y = 0 \); for \( \delta_Y > 0 \), we do not know whether (e) is equivalent to or strictly weaker than (a)–(d) in its absence. An inclusion map of a subgroup into an ambient group is Lipschitz when both have word metrics coming from some finite generating sets, and so it is satisfied in that setting.

**Lemma 3.1.** Suppose \((X, d_X)\) and \((Y, d_Y)\) are infinite proper geodesic hyperbolic metric spaces, and \(f : Y \to X\) is a proper map. Fix a basepoint \(e \in Y\). Define \( M, M', M'' : [0, \infty) \to [0, \infty) \) by

\[
M(N) := \inf \left\{ \left( (f(x) \cdot f(y))^X \right)_{f(e)} \mid x, y \in Y \text{ and } (x \cdot y)^Y \geq N \right\},
\]

\[
M'(N) := \inf \left\{ d_X(f(e), \gamma) \mid \gamma = [f(x), f(y)]_X \text{ for some } [x, y]_Y \in Y \setminus B_Y(e, N) \right\},
\]

\[
M''(N) := \inf \left\{ d_X(f(e), \gamma) \mid \gamma = [f(z), f(y)]_X \text{ for some } z \text{ on some } [e, y]_Y \text{ with } d_Y(e, z) \geq N \right\}.
\]

The following are equivalent:

(a). The induced map \( \hat{f} : \partial Y \to \partial X \) is well-defined.

(b). The induced map \( \hat{f} : \partial Y \to \partial X \) is well-defined and is uniformly continuous.

(c). \( M(N) \to \infty \) as \( N \to \infty \).

(d). \( M'(N) \to \infty \) as \( N \to \infty \).

Moreover, if \( \sup \{ d_X(f(x), f(y)) \mid d_Y(x, y) \leq r \} < \infty \) for all \( r \geq 0 \), then these are also equivalent to

(e). \( M''(N) \to \infty \) as \( N \to \infty \).

**Proof.** Each of \( M(N) \), \( M'(N) \), and \( M''(N) \) is a non-decreasing function, so is either bounded of tends to \( \infty \) as \( N \to \infty \).

That (b) \( \implies \) (a) is immediate.

Let us prove (a) \( \implies \) (c). Suppose \( M(N) \leq C \) for all \( N \). So there are sequences \((p_n)\) and \((q_n)\) in \( Y \) with \((p_n \cdot q_n)^Y \to \infty \) but \( (f(p_n) \cdot f(q_n))_{f(e)} \leq C \) for all \( n \). As \( Y \cup \partial Y \) is sequentially compact by Lemma 2.4, both \((p_n)\) and \((q_n)\) have convergent subsequences. But the condition \((p_n \cdot q_n)^Y \to \infty \) precludes any such subsequence from converging in \( Y \), so those subsequences converge in \( \partial Y \), indeed to the same point.

Next we prove (c) \( \implies \) (b). Suppose sequences \((p_n)\) and \((q_n)\) in \( Y \) both represent the same point in \( \partial Y \). Then \((p_n \cdot q_n)^Y \to \infty \) as \( n \to \infty \), and so \((f(p_n) \cdot f(q_n))_{f(e)} \to \infty \), since \( M(N) \to \infty \) as \( N \to \infty \). Thus if \((f(p_n))\) and \((f(q_n))\) converge to points in \( \partial X \), then those points are the same.

So, to prove \( \hat{f} \) is well-defined, it suffices to show that if a sequence \((a_n)\) in \( Y \) represents a point in \( \partial Y \) (and so \( d_Y(e, a_n) \to \infty \) by properness of \( Y \)), then \((f(a_n))\) represents a point in \( \partial X \). Indeed, it suffices to show that a subsequence of \((f(a_n))\) represents a point in \( \partial X \). By sequential compactness of \( X \cup \partial X \) (Lemma 2.4) a subsequence of \((f(a_n))\) converges. If it converges to a point in \( X \), then a subsequence of \((f(a_n))\) is in some compact (by properness of \( X \)) ball \( \overline{B}_X(e, R) \). But then, by properness of \( f \), a subsequence of \((a_n)\) would be contained in some ball \( B_Y(e, R') \), which would contradict \( d_Y(e, a_n) \to \infty \). So some subsequence of \((f(a_n))\) converges to (that is, represents — see Lemma 2.4) a point in \( \partial X \).

To establish continuity, suppose \( p, q \in \partial X \). By definition of the visual metrics \( d_{\partial X} \) and \( d_{\partial Y} \), there exist constants \( r, s > 1 \) and \( k, l > 0 \) such that

\[
d_{\partial X}(\hat{f}(p), \hat{f}(q)) \leq k r^{-\frac{d_{\partial Y}(\hat{f}(p), \hat{f}(q))^X_{f(e)}}{r l}},
\]
and
\[ d_{\gamma Y}(p, q) \geq l s^{-\lambda(p,q)^Y}. \] (4)

Since \((p \cdot q)^Y = \sup \inf_{m,n \to \infty} (p_m \cdot q_n)^Y\), there exist sequences \((p_m),(q_m)\) in \(Y\) representing \(p,q\) respectively, with \(\lim \inf_{m,n \to \infty} (p_m \cdot q_m)^Y \geq (p \cdot q)^Y - 1\). So \((p_m \cdot q_m)^Y \geq (p \cdot q)^Y - 2\) for all sufficiently large \(m,n\). By definition of \(M\) we have \((f(p_m) \cdot f(q_m))_{f(e)}^X \geq M((p \cdot q)^Y - 2)\) for such \(m,n\), and hence
\[ (\hat{f}(p) \cdot \hat{f}(q))_{f(e)}^X \geq M((p \cdot q)^Y - 2). \] (5)

Combining (3) and (5), we have
\[ d_{\gamma Y}(\hat{f}(p), \hat{f}(q)) \leq k r^{-M(p \cdot q)^Y}. \] (6)

So, by (4), if we make \(d_{\gamma Y}(p, q)\) sufficiently small, we can make \(d_{\gamma Y}(p, q)^Y\) arbitrarily large, so by hypothesis make \(M((p \cdot q)^Y - 2)\) arbitrarily large, and so by (6) make \(d_{\gamma X}(\hat{f}(p), \hat{f}(q))\) arbitrarily small.

The equivalence of (c) and (d) comes from Lemma 2.1, which implies that there exists \(C > 0\) such that \(M'(N) \leq M(N + C) + C\) and \(M(N) \leq M'(N + C) + C\) for all \(N\).

That (d) \(\implies\) (e) is immediate as \([z, \gamma]_Y\) is a geodesic segment in \(Y\) lying outside \(B_Y(e, N)\).

Here is a proof that (e) \(\implies\) (d) under the assumption that \(\sup\{d_X(f(x), f(y)) \mid d_Y(x, y) \leq r\} < \infty\) for all \(r \geq 0\). Suppose \(\lambda = [h_1, h_2]^Y\). As \(\tau := (h_1 \cdot h_2)^Y\) approximates \(d_Y(\lambda, e)\) with error at most a constant (Lemma 2.1), it is enough to show \(d_X([f(h_1), f(h_2)]_{\tau}, e) \to \infty\) as \(\tau \to \infty\). Let \(a_i = [e, h_i]^Y\) for \(i = 1, 2\).

By the thin–triangles condition, \([f(h_1), f(h_2)]_{a_i}\) lies in a \(C\)-neighborhood of a piecewise–geodesic path \([f(h_1), f(a_i(t))]_{a_i} \cup [f(a_i(t)), f(a_i(t))]_{a_i} \cup [f(a_i(t)), f(h_2)]_{a_i}\) for some constant \(C\). So it is enough to show that the distance of each of these three segments from \(f(e)\) in \(X\) tends to \(\infty\) as \(\tau \to \infty\). This is so for \([f(h_1), f(a_i(t))]_{a_i}\) and \([f(a_i(t)), f(h_2)]_{a_i}\) by (e). By the thin–triangles condition (see e.g. [4, pages 408–409]), \(d_Y(a_i(t), a_i(l))\) is at most a constant for all \(0 \leq l \leq t\), and so in particular \(d_Y(a_i(t), a_i(t))\) is at most a constant. The assumption that \(\sup\{d_X(f(x), f(y)) \mid d_Y(x, y) \leq r\} < \infty\) for all \(r \geq 0\) gives an upper bound, independent of \(t\), on the length of \([f(a_i(t)), f(a_i(t))]_{a_i}\). Since the distances of the endpoints of this segment from \(f(e)\) in \(X\) tend to \(\infty\) as \(\tau \to \infty\), so does the distance of the whole segment. □

The first part of the following lemma shows that Theorem 1.1 is not a quirk of the choice of generating sets. The second establishes the sense in which the function \(\epsilon(\delta)\) of Section 1 is an invariant for Cannon–Thurston maps. The third will allow us to reinterpret Lemma 3.1 (as Corollary 3.3) in a manner well suited to analyzing hyperbolic hydra.

**Lemma 3.2.** Suppose \(\Lambda\) is a hyperbolic subgroup of a hyperbolic group \(\Gamma\).

(i). Whether the Cannon–Thurston map \(\partial \Lambda \to \partial \Gamma\) is well-defined does not depend on the choice of finite generating sets giving the word metrics.

(ii). If the Cannon–Thurston map \(\iota : \partial \Lambda \to \partial \Gamma\) is well–defined for a hyperbolic subgroup \(\Lambda\) of a hyperbolic group \(\Gamma\), the modulus of continuity for \(\iota\) does not depend on the finite generating sets and the choices of visual metrics up to the following Hölder–type equivalence. If \(\epsilon(\delta)\) and \(\epsilon'(\delta)\) are the moduli of continuity of \(\iota\) defined with respect to different such choices, then there are functions \(f_1, f_2 : (0, \infty) \to (0, \infty)\), each of the form \(x \mapsto C_i x^{a_i}\), for some \(C_i, a_i > 0\), such that \(\epsilon'(\delta) \leq (f_1 \circ \epsilon \circ f_2)(\delta)\) for all \(\delta > 0\).

(iii). Suppose \(A\) and \(B\) are finite generating sets for \(\Gamma\) and \(\Lambda\), respectively. Suppose \(f : C_B(\Lambda) \to C_A(\Gamma)\) is any map between the respective Cayley graphs which restricts to the inclusion \(\Lambda \hookrightarrow \Gamma\) on the vertices of \(C_B(\Lambda)\) and sends edges to geodesics in \(C_A(\Gamma)\). The Cannon–Thurston map \(\partial \Lambda \to \partial \Gamma\), defined in terms of finite generating sets \(A\) for \(\Gamma\) and \(B\) for \(\Lambda\), is well–defined if and only if the Cannon–Thurston map \(\partial C_B(\Lambda) \to \partial C_A(\Gamma)\) is.
Proof. Suppose \( \partial \Gamma \), \( \partial_2 \Gamma \), \( \partial \Lambda \), and \( \partial_2 \Lambda \) are boundaries of \( \Gamma \) and \( \Lambda \) defined with respect to different finite generating sets. The composition \( \partial_1 \Lambda \to \partial_2 \Lambda \to \partial_2 \Gamma \to \partial_1 \Gamma \) of Cannon–Thurston maps is \( \partial_1 \Lambda \to \partial_1 \Gamma \), so (i) follows since \( \partial_1 \Lambda \to \partial_2 \Lambda \) and \( \partial_2 \Gamma \to \partial_1 \Gamma \) are well–defined since they are induced by identity maps on \( \Lambda \) and \( \Gamma \) changing the word metrics, which are Lipschitz.

If \( X, Y, Z \) are metric spaces and \( g : Y \to Z \) and \( h : X \to Y \) are maps with moduli of continuity \( \varepsilon_g \) and \( \varepsilon_h \), respectively, then it follows from the definition that \( \varepsilon_{g \circ h}(\delta) \leq (\varepsilon_g \circ \varepsilon_h)(\delta) \) for all \( \delta > 0 \). Specializing to the case of \( \partial_1 \Lambda \to \partial_2 \Lambda \) and \( \partial_2 \Gamma \to \partial_1 \Gamma \) in the previous paragraph, (ii) follows from the fact (Lemma 2.3) that \( \partial_1 \Lambda \to \partial_2 \Lambda \) and \( \partial_2 \Gamma \to \partial_1 \Gamma \) are Hölder.

For (iii), note that the map identifying \( \Gamma \) (resp. \( \Lambda \)) with the vertex set of \( C_\partial(\Gamma) \) (resp. \( C_\partial(\Lambda) \)) induces an isometry on their boundaries. \( \square \)

**Corollary 3.3.** Suppose \( \Lambda \) is a finite–rank free subgroup of a hyperbolic group \( \Gamma \). Suppose \( A \) is a finite generating set for \( \Gamma \) and \( B \) is a free basis for \( \Lambda \). The Cannon–Thurston map \( \partial \Lambda \to \partial \Gamma \) is well–defined if and only if for all \( M'' > 0 \), there exists \( N \) such that whenever \( \alpha \beta \) is a reduced word on \( B \) with \( |\alpha| \geq N \), any geodesic in the Cayley graph \( C_\Lambda(\Gamma) \) joining \( \alpha \) to \( \alpha \beta \) lies outside the ball of radius \( M'' \) about \( e \).

**Proof.** By Lemma 3.2(iii), the Cannon–Thurston map \( \partial \Lambda \to \partial \Gamma \) is well–defined if and only if the Cannon–Thurston map \( \partial C_\partial(\Lambda) \to \partial C_\partial(\Gamma) \) for \( f \) (as defined in that lemma) is. Now applying condition (e) of Lemma 3.1 to \( f \) gives the result, since reduced words correspond to geodesics in \( C_\partial(\Lambda) \). \( \square \)

## 4 Cannon–Thurston maps for hyperbolic hydra groups

In this section we will show that Cannon–Thurston maps for hyperbolic hydra are well–defined. Throughout we fix an integer \( k \geq 1 \).

The **hyperbolic hydra** \( \Gamma_k \) of [3] is a hyperbolic group

\[
\Gamma_k = F \rtimes_{\theta} \mathbb{Z}
\]

where \( F := F(a_0, \ldots, a_k, b_1, \ldots, b_l) \), and \( l \geq 1 \) is a certain integer, and \( \theta \) is a certain automorphism of \( F \) whose restriction to \( F(b_1, \ldots, b_j) \) is an automorphism and

\[
\theta(a_i) = \begin{cases} 
\theta(a_i) & i = 0, \\
a_0 & i = 1, \\
a_i a_{i-1} & 1 < i \leq k,
\end{cases}
\]

for certain words \( u \) and \( v \) on \( b_1, \ldots, b_l \). (In fact \( u \) and \( v \) depend on \( k \), but \( l \) and \( \theta |_{F(b_1, \ldots, b_j)} \) do not.)

Let \( t \) denote a generator of the \( \mathbb{Z} \)-factor, so \( t^{-1} a_i t = \theta(a_i) \) and \( t^{-1} b_j t = \theta(b_j) \) for all \( i \) and \( j \). For \( 1 \leq r \leq k \), let \( \Lambda_r \) be the subgroup \( \langle a_0 t, \ldots, a_r t, b_1, \ldots, b_l \rangle \) of \( \Gamma_k \). It is proved in [3] that \( \Lambda_k \) is free of rank \( k + l + 1 \) and is distorted so that \( \text{Dist}_{\Lambda_k}^t = \Lambda_k \). Let \( \Lambda_0 = \langle b_1, \ldots, b_l \rangle \). (Caution: here we have that \( \Lambda_r \leq \Lambda_k \leq \Gamma_k \) in contrast to [3] where \( \Lambda_r \leq \Gamma_r \).)

The hyperbolic hydra \( \Gamma_k \) is an elaboration of the **hydra group**

\[
G_k = F(a_1, \ldots, a_k) \rtimes_{\varphi} \mathbb{Z}
\]

from [8], where \( t \) generates the \( \mathbb{Z} \)-factor and \( \varphi \) is the automorphism of \( F(a_1, \ldots, a_k) \) such that

\[
\varphi(a_i) = t^{-1} a_i t = \begin{cases} 
a_1 & i = 1, \\
a_i a_{i-1} & 1 < i \leq k.
\end{cases}
\]
In [8] it is proved that $G_k$ is CAT(0) and has a rank--k free subgroup $H_k = \langle a_1, \ldots, a_k \rangle$, distorted so that $\text{Dist}_{H_k}^\theta \approx A_k$. While $G_k$ is easier to work with than $\Gamma_k$, it fails to be hyperbolic. The groups are related by the map $\Phi: \Gamma_k \to G_k$ which sends $a_i \to a_{\text{max}[i,j]}$ and $t \to t$ for all $i, j$.

The normal form of $g$ in $G_k$ (resp. $\Gamma_k$) is the unique $\hat{w}t^m$ such that $\hat{w}$ is a reduced word on $a_1, \ldots, a_k$ (resp. $a_0, \ldots, a_k, b_1, \ldots, b_l$) and $g = \hat{w}t^m$ in $G_k$ (resp. $\Gamma_k$). For $g$ in $G_k$ (resp. $\Gamma_k$) we denote the length of a shortest word on $a_1, \ldots, a_k, t$ (resp. $a_0, \ldots, a_k, b_1, \ldots, b_l, t$) representing $g$ in $G_k$ (resp. $\Gamma_k$) by $|g|_{G_k}$ (resp. $|g|_{\Gamma_k}$). An $H_k$-word (resp. $\Lambda_k$-word) is a reduced word on $a_1, \ldots, a_r, t$ (resp. on $a_0, \ldots, a_r, t, b_1, \ldots, b_l$). For $g$ in $H_k$ (resp. $\Lambda_k$) we denote the length of a shortest $H_k$-word (resp. $\Lambda_k$-word) representing $g$ by $|g|_{H_k}$ (resp. $|g|_{\Lambda_k}$).

The following lemma is a consequence of Proposition 4.8 of [3] (resp. Lemma 6.1 of [8]), which says that $\Lambda_k \cap \{t\} = \{1\}$ (resp. $H_k \cap \{t\} = \{1\}$).

**Lemma 4.1.** Given $g \in \Gamma_k$ (resp. $G_k$), if there exists $j$ such that $gt^j \in \Lambda_k$ (resp. $H_k$), then that $j$ is unique.

**Corollary 4.2.** If $g_1, g_2 \in \Lambda_k$ (resp. $G_k$) have normal forms $\hat{w}t^{n_1}$ and $\hat{w}t^{n_2}$, respectively, then $g_1 = g_2$.

Next we give a technical lemma comparing the location of the final $a_r$ or $a_r^{-1}$ in a $\Lambda_k$-word $w$ to that in its normal form $\hat{w}t^{m}$, and moreover in $\theta^n(\hat{w})$. (When we refer to $\theta^n(\hat{w}) \in F$ as a word, we mean the reduced word on $a_0, \ldots, a_k, b_1, \ldots, b_l$ that represents it.)

**Lemma 4.3.** For all integers $A, B \geq 0$ and $r$ with $k \geq r > 1$, there exists $N$ such that if $|n| \leq B$ and $w = u(a_r t)^{\pm 1}$ is a reduced $\Lambda_r$-word with $\ell_{\Lambda_r}(u) \geq N$, then the final $a_r^{\pm 1}$ in $\theta^n(\hat{w})$ occurs at least $A$ symbols in from the beginning of $\theta^n(\hat{w})$.

Similarly, such an $N$ exists for the final $a_0^{\pm 1}$ or $a_1^{\pm 1}$ in $\theta^n(\hat{w})$ when $r = 1$ and $w$ is $u(a_0 t)^{\pm 1}$ or $u(a_1 t)^{\pm 1}$.

**Proof.** Assume $r > 1$.

First we explain why there is an $a_r^{\pm 1}$ in $\theta^n(\hat{w})$ and how it relates to the $a_r^{\pm 1}$ in $w$. This stems from properties of $H_k$ and $G_k$. If $\hat{x} t^m$ is the normal form of a reduced $H_r$-word $x$, then the relative locations of the $a_r^{\pm 1}$ in $\hat{x}$ agree with those in $x$. This is because when each $t^{\pm 1}$ is shuffled to the end of $x$, each $a_i$ it passes is changed to $a_i^{\mp 1}$, and this neither introduces nor destroys any $a_r^{\pm 1}$, and it causes no cancellations between any $a_r^{\pm 1}$.

(See Lemma 6.2 in [8] for more details.)

The same is true if $\hat{x} t^m$ is the normal form of a reduced $\Lambda_r$-word $x$ (when $r > 1$). After all, changing letters $a_i$ to $\theta^{\mp 1}(a_i)$ again neither introduces nor destroys any $a_r^{\pm 1}$, and there can be no cancellations between any $a_r^{\pm 1}$, because the $\Phi: \Gamma_k \to G_k$ converts the process to one in $G_k$ and there are no cancellations between any $a_r^{\pm 1}$ there.

Likewise, the locations of the $a_r^{\pm 1}$ in $\theta^n(\hat{w})$, for any $n \in \mathbb{Z}$, correspond to their locations in $w$.

Now we will prove the lemma in the case where $w = u(a_r t)^{-1}$. Since the final letter of $\theta^{-1}(a_r^{-1})$ is $a_r^{-1}$, the $a_r^{-1}$ in $\theta^n(\hat{w})$ that corresponds to the final $a_r^{-1}$ in $w$ is the final letter of $\theta^n(\hat{w})$. So we are seeking to prove that $|n(\hat{w})|_F \geq A$. Since $\theta$ is an automorphism, only finitely many $v \in F$ satisfy $|\theta^n(v)|_F < A$. By Corollary 4.2, each such $v$ equals $\hat{w}$ for at most one $A$-word $w$, so the lemma is proved by taking $N$ sufficiently large to avoid these finitely many $w$.

Next we address the case where $w = u(a_r t)$. The normal forms $\hat{w} t^{i+1}$ of $w$ and $\hat{u} t^j$ of $u$ are related in that

$$w = u(a_r t) = \hat{u} t^j a_r t = \hat{u} \theta^{-j}(a_r) t^{j+1} = \hat{w} t^{j+1}.$$ 

Since there is no cancellation between $u$ and $a_r t$ in $w$ and the first letter of $\theta^{-j}(a_r)$ is $a_r$, there is no cancellation between $\hat{u}$ and $\theta^{-j}(a_r)$. Similarly, there is no cancellation between $\theta^n(\hat{u})$ and $\theta^{-j}(a_r)$ in $\theta^n(\hat{w}) = \theta^n(\hat{u}) \theta^{-j}(a_r)$. So we are now seeking to prove that $|n(\hat{u})|_F + 1 \geq A$, and this can be done as before. This completes the proof when $r > 1$. 

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The same proof works when \( r = 1 \), taking into account that \( \theta \) interchanges \( a_0 \) and \( a_1 \) (introducing some \( b_1, \ldots, b_l \) in the process). This is only a superficial complication: the locations of the \( a_n^e \) and \( a_n^{1} \) in \( \theta^n(\tilde{w}) \), for any \( n \in \mathbb{Z} \), correspond to their locations in \( w \), but some \( a_n^e \) may correspond to \( a_n^{1} \) and vice versa.

Let \( C(F) \) and \( C(\Gamma_k) \) denote the Cayley graphs of \( F \) and \( \Gamma_k \) with respect to \( a_0, \ldots, a_k, b_1, \ldots, b_l \) and \( a_0, \ldots, a_k, b_1, \ldots, b_l, t \). Let \( B_F(e, R) \) denote the open ball of radius \( R \) about \( e \) in \( C(F) \). Write \([x, y]_F\) or \([x, y]_{\Gamma_k}\) for a geodesic between \( x \) and \( y \) in \( C(F) \) or \( C(\Gamma_k) \). In the case of \( C(F) \), which is a tree, geodesics between any given pair of points are unique. Let \( d_F \) and \( d_{\Gamma_k} \) be the associated metrics.

The **shadow** of the suffix \( \beta \) of a reduced \( \Lambda_k \)-word \( \alpha\beta \) is the set of all geodesic segments \([\alpha \beta_j, \alpha \beta_{j+1}]_F \) where \( \beta_j \) denotes the length-\( i \) prefix of \( \beta \) and \( 0 \leq i < |\beta|_{\Lambda_k} \).

**Lemma 4.4.** For all \( K > 0 \), there exist integers \( C, R > 0 \) such that if \( \alpha\beta \) is a reduced \( \Lambda_k \)-word and \( |\alpha|_{\Lambda_k} \geq C \) and the shadow of \( \beta \) is outside \( B_F(e, R) \), then every \([\alpha, \alpha\beta]_{\Gamma_k} \) satisfies \( d_{\Gamma_k}([\alpha, \alpha\beta]_{\Gamma_k}, e) \geq K \).

**Proof.** As \( \Gamma_k \) is hyperbolic, there is some \( \delta > 0 \) such that every geodesic triangle in \( C(\Gamma_k) \) is \( \delta \)-slim.

The geodesic segment \([\alpha, \alpha\beta]_{\Gamma_k} \) is in a \( 2\delta \)-neighborhood of the piecewise-geodesic path \([\alpha, \tilde{a}]_{\Gamma_k} \cup [\tilde{a}, \alpha\beta]_{\Gamma_k} \cup [\alpha\beta, \alpha\beta]_{\Gamma_k} \) in \( C(\Gamma_k) \). So it suffices to have \([\alpha, \tilde{a}]_{\Gamma_k} \cup [\tilde{a}, \alpha\beta]_{\Gamma_k} \cup [\alpha\beta, \alpha\beta]_{\Gamma_k} \) at least \( K + 2\delta \) from \( e \).

To ensure \([\alpha, \tilde{a}]_{\Gamma_k} \) is at least \( K + 2\delta \) from \( e \), we need it to avoid finitely many elements of \( \Gamma_k \), say \( g_1, \ldots, g_m \).

Since \( \alpha = a\hat{t}^n \) for some \( n \), there is a unique geodesic \([\alpha, \tilde{a}] \) joining \( \alpha \) to \( \tilde{a} \) in \( C(\Gamma_k) \) and it is a succession of edges all labelled \( t \). So if \( g_i \) is on \([\alpha, \tilde{a}] \) then \( g_i t^j = a \in \Lambda_k \) for some \( j \). But then by Lemma 4.1, it suffices for \( \alpha \) not to be one of at most \( m \) elements of \( \Lambda_k \). So it suffices to ensure \(|\alpha|_{\Lambda_k} \) is sufficiently long.

Since \(|\alpha\beta|_{\Lambda_k} \geq |\alpha|_{\Lambda_k} \), the geodesic \([\tilde{a}, \alpha\beta]_{\Gamma_k} \) is then also at least \( K + 2\delta \) from \( e \).

Finally we consider \([\tilde{a}, \alpha\beta]_{\Gamma_k} \). As the shadow of \( \beta \) stays outside \( B_F(e, R) \), do so \([\tilde{a}, \alpha\beta]_F \), which is a subset of the shadow since \( C(F) \) is a tree. Mita shows in his proof of Theorem 4.3 in [16] that if \( 1 \to H \to G \to G/H \to 1 \) is a short exact sequence of finitely generated groups and \( H \) and \( G \) are infinite and hyperbolic, then condition \( \text{(b)} \) of our Lemma 3.1 applies (with \( X \) and \( Y \) being Cayley graphs of \( G \) and \( H \)). Applying this to \( G = \Gamma_k \)

We are now ready to use Corollary 3.3 to show the Cannon–Thurston map \( \partial \Lambda_k \to \partial \Gamma_k \) is well-defined.

**Proof of Theorem 1.1.** Fix some \( \delta > 0 \) so that all geodesic triangles in \( C(\Gamma_k) \) are \( \delta \)-slim.

For integers \( A, B \geq 0 \) and \( r \) with \( k \geq r \geq 1 \), let \( N(r, A, B) \) be the least integer \( N \) as per Lemma 4.3. Given an integer \( R > 0 \), recursively define a sequence \( N_k(R), N_{k-1}(R), \ldots, N_1(R) \) of positive integers by:

\[
N_k(R) := N(k, R, 0),
\]

and for \( r = k - 1, k - 2, \ldots, 1 \)

\[
N_r(R) := N(r, R + \max \left\{ |\tilde{u}|_{\Gamma_k} \mid \Lambda_k\text{-words } \tilde{u} \text{ with } |\tilde{u}|_{\Lambda_k} = \sum_{j=r+1}^{k} N_j(R) \right\}, \sum_{j=r+1}^{k} N_j(R)).
\]

Suppose \( M'' > 0 \) is given. Let \( K = M'' + (2\delta + 1)k \). Let \( R, C > 0 \) be obtained from \( K \) as per Lemma 4.4. Recall the map \( \Phi \) from the hyperbolic hydra group \( \Gamma_k \) to the hydra group \( G_k \) defined by \( a_i \to 0 \) for all \( i \). Let \( L \) be the maximum of \(|\theta^n(s)|_F\) ranging over all \( s \in \{b_1, \ldots, b_l\} \) and all \( n \in \mathbb{Z} \) for which there exists \( \tilde{u} \in F \) such that \( \tilde{u} t^n \in \Lambda_k \) and the reduced word representing \( \Phi(\tilde{u}) \) in \( F(a_1, \ldots, a_k) \) has length less than \( R \). (There are only finitely many such \( n \) since \( \tilde{u} t^n \in \Lambda_k \) implies \( \Phi(\tilde{u}) t^n \in H_k \), and for any \( x \in F(a_1, \ldots, a_k) \), there is at most one \( m \in \mathbb{Z} \) such that \( x t^m \in H_k \) by Lemma 4.1.) Choose \( C' > C \) so that every \( \Lambda_k\text{-word } u \) of length \(|u|_{\Lambda_k} \geq C' \) has free–by–cyclic normal form \( \tilde{u} t^n \) with \( |\tilde{u}|_F \geq R + (L/2) \).
Suppose \( \alpha \beta \) is a reduced \( \Lambda_k \)-word such that \( |a|_{\Lambda_k} \geq N := C' + \sum_{i=1}^{k} N_i(R) \). Express \( a \) as \( w_k \cdots w_0 \) where \( |w_r|_{\Lambda_k} = N_i(R) \) for \( 1 \leq r \leq k \). In particular, \( |a|_{\Lambda_k} \geq |w_0|_{\Lambda_k} \geq C' \geq C \). Let \( \hat{\beta}_r \) denote the longest prefix of \( \beta \) in \( \Lambda_r \) and let \( \gamma_r \) denote (any) geodesic \([a, \alpha \beta]_{\Gamma^r}\). In particular, \( \gamma_k \) is (any) geodesic \([a, \alpha \beta]_{\Gamma_k}\). We will show that \( \gamma_k \) lies at least a distance \( M' \) from \( e \) in \( C(\Gamma_k) \). Corollary 3.3 will then complete the proof.

Suppose, for a contradiction, that \( d_{\Gamma_k}(\gamma_k, e) > M'' \). Suppose \( \hat{a} \hat{r} a \) is the free–by–cyclic normal form of \( a \) in \( \Gamma_k \).

We show in this paragraph that the shadow of the suffix \( \hat{\beta}_0 \) of \( \alpha \hat{\beta}\) is outside of \( B_F(e, R) \). The endpoints of the geodesic segments in \( F \) comprising the shadow of \( \hat{\beta}_0 \) are all of the form \( a^{\hat{\theta}^{-n}}(x) \) for various \( x \in F(b_1, \ldots, b_t) \). There are two cases to consider: the length of the reduced word in \( F(a_1, \ldots, a_k) \) representing \( \Phi(\hat{a}) \) is at least \( R \) and is less than \( R \). In the former case, because \( \hat{a} \) contains at least \( R \) letters \( a \), the closest approach of any such geodesic to \( e \) (the Gromov product of its endpoints) is at least \( R \). In the latter case, \( L \) is an upper bound for the length of the constituent geodesics in the shadow of \( \hat{\beta}_0 \) and so, by definition of \( C' \), the shadow of \( \hat{\beta}_0 \) stays outside of \( B_F(e, R) \). In either case, the shadow stays outside of \( B_F(e, R) \).

On the other hand, the following claim, in the case \( r = 0 \), shows that

\[
d_{\Gamma_k}([a, \alpha \beta]_{\Gamma_k}, e) < M'' + (2\delta + 1)k,
\]

which equals \( K \), so Lemma 4.4 implies the shadow of \( \hat{\beta}_0 \) dips within \( B_F(e, R) \). This contradiction will prove the theorem.

**Claim.** For \( r = k, k-1, \ldots, 1, 0 \),

(i.) \( d_{\Gamma_k}(\gamma_r, e) < M'' + (2\delta + 1)(k-r) \), and

(ii.) \( w_r w_{r-1} \cdots w_0 \in \Lambda_r \).

We prove this claim using simultaneous downward induction on \( r \).

The base case \( r = k \) is straightforward: \( \gamma_k = [a, \alpha \beta]_{\Gamma_k} \) and \( w_k \cdots w_0 = a \in \Lambda_k \) by definition, and \( d_{\Gamma_k}(\gamma_k, e) < M'' \) by hypothesis.

Now we prove that (i) and (ii) implies (i) and (ii) for \( r = k-1, \ldots, 0 \), except we will not give the case \( r = 0 \) explicitly; it is similar to the following \( r > 0 \) case but has the superficial complication that \( \theta \) interchanges \( a_0^{\pm} \) and \( a_1^{\pm} \) (as well as introducing some letters \( b_j^{\pm} \)), which makes the notation more awkward.

We will make repeated use of the following lemma.

**Lemma 4.5.** Suppose \( w_k, \ldots, w_{r+1} \) are as defined earlier, and \( w_k \cdots w_{r+1} x(a_{r+1} t)^{\pm} y z \) is a reduced \( \Lambda_k \)-word in which the subwords \( x, y \) and \( z \) are \( \Lambda_{r+1} \)-words. If \( \gamma \) is any geodesic in \( C(\Gamma_k) \) from \( w_k \cdots w_{r+1} x(a_{r+1} t)^{\pm} y \) to \( w_k \cdots w_{r+1} x(a_{r+1} t)^{\pm} y z \) then \( d_{\Gamma_k}(\gamma, e) \geq K \).

To prove this lemma we consider the free–by–cyclic normal form \( w_k \cdots w_{r+2} t^{n} \). for \( w_k \cdots w_{r+2} \in \Gamma_k \). Now, \( |w_k \cdots w_{r+2}|_{\Lambda_k} = \sum_{j=r+2}^{k} N_j(R) \) by construction, so \( |n_r| \leq \sum_{j=r+2}^{k} N_j(R) \). So, by definition,

\[
N_{r+1}(R) \geq N(r+1, R + |w_k \cdots w_{r+2}|_{F}, |n_r|).
\]

By hypothesis, \( w_{r+1} x(a_{r+1} t)^{\pm} \in \Lambda_{r+1} \), and \( |w_{r+1}| = N_{r+1}(R) \) by construction, so the final \( a_{r+1}^{\pm} \) in \( \theta^{-n}(x(a_{r+1} t)^{\pm}) \) occurs at least \( R + |w_k \cdots w_{r+2}|_{F} \) symbols in from the start of the word. Therefore, the final \( a_{r+1}^{\pm} \) in \( w_k \cdots w_{r+2} \theta^{-n}(x(a_{r+1} t)^{\pm}) \) occurs at least \( (R + |w_k \cdots w_{r+2}|_{F}) = |w_k \cdots w_{r+2}|_{F} = R \) symbols in. Since \( y \) and \( z \) are \( \Lambda_{r+1} \)-words and \( w_k \cdots w_{r+1} x(a_{r+1} t)^{\pm} y z \) is reduced, this implies the shadow of \( z \) cannot dip inside \( B_F(e, R) \). So, by Lemma 4.4, \( d_{\Gamma_k}(\gamma, e) \geq K \), completing the proof of the lemma.

Returning to the proof of the claim, we will consider two cases: \( \hat{\beta}_r = \beta_{r+1} \) and \( \beta_r \neq \beta_{r+1} \). In the former case, we may assume \( \gamma_r = \gamma_{r+1} \). So (i) follows immediately from (i) and (ii). Since (i) implies \( d_{\Gamma_k}(\gamma_r, e) < K \), (ii) and Lemma 4.5 with \( \gamma = \beta_r \) shows that \( w_r \cdots w_0 \) cannot be expressed as \( x(a_{r+1} t)^{\pm} y \). So \( w_r \cdots w_0 \in \Lambda_r \) and we have (ii).
Next, assume \( \beta_r \neq \beta_{r+1} \). The (reduced) word \( \beta_{r+1} \) can be expressed as \( \beta_r(a_{r+1}t)^{\pm 1} \beta'_r \) for some \( \beta'_r \in \Lambda_{r+1} \).

Let \( p_1 \) be the geodesic segment in \( C(\Gamma) \) labelled \((a_{r+1}t)^{\pm 1}\) connecting \( a\beta_r \) to \( a\beta'_r(a_{r+1}t)^{\pm 1} \). Let \( \gamma'_r = [a\beta'_r(a_{r+1}t)^{\pm 1}, a\beta_{r+1}] \). Then by Lemma 4.5 with \( x = w_r \cdots w_0 \beta_r \), \( y \) the empty word, and \( z = \beta_{r+1} \),

\[
d_{\Gamma}(\gamma'_r, e) \geq K. \tag{7}
\]

If \((a_{r+1}t)^{\pm 1}\) occurs in \( w_r \cdots w_0 \), then Lemma 4.5 with \( x(a_{r+1}t)^{\pm 1}y = w_r \cdots w_0 \) and \( z = \beta_r \) shows that

\[
d_{\Gamma}(\gamma_r, e) \geq K. \tag{8}
\]

By the slim–triangles condition for \( C(\Gamma) \), \( \gamma_{r+1} \) is contained in the \( 2\delta \)–neighborhood of \( \gamma_r \cup p_r \cup \gamma'_r \) and hence in the \((2\delta+1)\)–neighborhood of \( \gamma_r \cup \gamma'_r \). So

\[
\min\{d(e, \gamma_r), d(e, \gamma'_r)\} \leq d(e, \gamma_{r+1}) + (2\delta + 1)
\]

\[
< M'' + (2\delta + 1)(k - (r + 1)) + (2\delta + 1)
\]

\[
= M'' + (2\delta + 1)(k - r),
\]

the second inequality coming from \((i_{r+1})\).

But by (7), \( d(e, \gamma'_r) \geq K \geq M'' + (2\delta + 1)(k - r) \), so \( \min\{d(e, \gamma_r), d(e, \gamma'_r)\} = d(e, \gamma_r) \) and \((i_r)\) follows.

Moreover, (8) cannot be true since it contradicts \((i_r)\), so \((a_{r+1}t)^{\pm 1}\) does not occur in \( w_r \cdots w_0 \) and \((ii_r)\) follows. This completes the induction step of the claim, so the theorem is proved.

\section{Wildness of Cannon–Thurston maps}

\textbf{Proof of Theorem 1.2.} We have that \( \Gamma \) and \( \Lambda \) are \((\delta_\Gamma)\)– and \((\delta_\Lambda)\)–hyperbolic, respectively, for some \( \delta_\Gamma, \delta_\Lambda > 0 \).

Let \( i : \Lambda \to \Gamma \) denote the inclusion map and \( \partial \Lambda \to \partial \Gamma \) denote the Cannon–Thurston map.

Since \( \Lambda \) is non–elementary, \( |i(\partial \Lambda)| = \infty \) by [11, Thm. 12.2(1)]. We may thus choose \( p_1, p_2, p_3 \in \partial \Lambda \) with

\[
i p_1, i p_2, i p_3 \in \partial \Gamma \text{ distinct. Let } C = 2\delta_\Lambda + \max\{|p_i \cdot p_j| \delta_\Lambda |1 \leq i < j \leq 3\}.
\]

By definition of the distortion function (see Section 1), we can take a sequence \( h_n \in \Lambda \) with \( d_{\Gamma}(e, h_n) \leq n \) and \( d_{\Lambda}(e, h_n) = \text{Dist}_{\Lambda}(n) \). By Lemma 2.2(1)

\[
(p_i \cdot p_j)^{\delta_\Lambda}_e = (h_n p_i \cdot h_n p_j)^{\delta_\Lambda}_e \geq \min\{(e \cdot h_n p_i)^{\delta_\Lambda}_{h_n}, (e \cdot h_n p_j)^{\delta_\Lambda}_{h_n}\} - 2\delta_\Lambda,
\]

so we can choose \( i = i(n) \) and \( j = j(n) \in \{1, 2, 3\} \) with \( i \neq j \) such that

\[
(e \cdot h_n p_i)^{\delta_\Lambda}_{h_n}, (e \cdot h_n p_j)^{\delta_\Lambda}_{h_n} \leq C.
\]

Combined with Lemma 2.2(2) this gives that for \( k = i, j \),

\[
(h_n \cdot h_n p_k)^{\delta_\Lambda}_e \geq \text{Dist}_{\Lambda}(n) - (e \cdot h_n p_k)^{\delta_\Lambda}_{h_n} - \delta_\Lambda \geq \text{Dist}_{\Lambda}(n) - C - \delta_\Lambda.
\]

So, by Lemma 2.2(1),

\[
(h_n p_i \cdot h_n p_j)^{\delta_\Lambda}_e \geq \min\{(h_n \cdot h_n p_i)^{\delta_\Lambda}_{h_n}, (h_n \cdot h_n p_j)^{\delta_\Lambda}_{h_n}\} - 2\delta_\Lambda \geq \text{Dist}_{\Lambda}(n) - C - 3\delta_\Lambda.
\]

Writing \( \beta := k_2 r^{C+3\delta_\Lambda} \), where \( k_2 \) is as per (2) in Section 2 applied to \( \partial \Lambda \), we get

\[
d_{\partial \Lambda}(h_n p_i, h_n p_j) \leq k_2 r^{-\delta_\Lambda} \leq \frac{\beta}{r^{\text{Dist}_{\Lambda}(n)}}, \tag{9}
\]
On the other hand, using Lemma 2.2(3) for the first inequality,

\[(i(h_n p_1), i(h_n p_j)) = (h_n i p_1, h_n i p_j)^\Gamma\]

\[\leq d_{\Gamma}(e, [h_n i p_1, h_n i p_j]) + 8\delta_{\Gamma} \]

\[\leq d_{\Gamma}(e, h_n) + d_{\Gamma}(h_n, [h_n i p_1, h_n i p_j]) + 8\delta_{\Gamma} \]

\[\leq n + d_{\Gamma}(e, [i p_1, i p_j]) + 8\delta_{\Gamma}.\]

Writing \(\alpha := k_1/s^\delta + \max|d_{\Gamma}(e, [i p_1, i p_j])|\), where \(k_1\) is as per (2) in Section 2 applied to \(\partial\Gamma\), we then get

\[d_{\partial\Gamma}(i(h_n p_1), i(h_n p_j)) \geq k_1 s^{-((i(h_n p_1), i(h_n p_j))^\Gamma)} \geq \frac{\alpha}{s^n}.\] (10)

Combining (9) and (10) yields the inequality claimed in Theorem 1.2. \(\square\)

Our final proposition provides precise details of what we mean by Corollary 1.3 via \(\delta = 1/n\). The inequality we obtain is not as clean as we might like, but for \(k = 3\) (when it is a tower of powers of 2) and beyond, the Ackermann function \(A_k\) is by far the dominant factor on the lefthand side, and can be considered to absorb the log and the exponential.

**Proposition 5.1.** For all \(k \geq 2\), the modulus of continuity \(\epsilon(\delta)\) of the Cannon–Thurston map \(\partial\Lambda_1 \to \partial\Gamma_k\) for hyperbolic hydra has the property that there exist \(C_0, C_1 > 0\) and \(C_2 > 1\) such that for all \(\eta \in (0, C_0)\),

\[\epsilon\left(\frac{1}{C_2 A_k([C, \log(1/\eta)])}\right) \geq \eta.\]

**Proof.** For convenience, extend the domains of the functions \(A_k : \mathbb{N} \to \mathbb{N}\) and \(\text{Dist}^\Gamma_{\Lambda_k} : \mathbb{N} \to \mathbb{N}\) to \([1, \infty)\) by declaring the functions to be constant on the half–open intervals \([n, n + 1)\).

From Theorem 1.2 we have

\[\epsilon\left(\frac{\beta}{r \text{Dist}^\Gamma_{\Lambda_k}(n)}\right) \geq \frac{\alpha}{s^n}\]

for all real \(n \geq 0\) and some constants \(\alpha, \beta > 0\) and \(r, s > 1\). Thus for all \(\eta \leq \alpha,\)

\[\epsilon\left(\frac{1}{\exp(\log(r)\text{Dist}^\Gamma_{\Lambda_k}(\log((\alpha/\eta)) - \log(\beta)))}\right) \geq \eta.\] (11)

As \(\text{Dist}^\Gamma_{\Lambda_k} \geq A_k\) by [3], there exists \(C > 0\) such that \(A_k(n) \leq C\text{Dist}^\Gamma_{\Lambda_k}(Cn + C) + Cn + C\) for all real \(n\), and therefore

\[\text{Dist}^\Gamma_{\Lambda_k}(N) \geq \frac{1}{C} A_k\left(\frac{N - C}{C}\right) - \frac{N}{C}\]

for \(N \geq 2C\). So

\[\log(r)\text{Dist}^\Gamma_{\Lambda_k}(\log((\alpha/\eta)) - \log(\beta)) \geq K_1 A_k(K_2 \log(1/\eta) - K_3) - K_4 \log(1/\eta) - K_5\]

for all \(\eta \in (0, K_0)\), for suitable constants \(K_0, \ldots, K_5 > 0\). By shrinking \(K_0\) if necessary, we can make \(\log(1/\eta)\) arbitrarily large, so that we may absorb the constant \(K_3\) into \(K_2\). Moreover, \(A_k\) grows faster than a linear function as \(k \geq 2\), so (by further shrinking \(K_0\)) the constants \(K_4\) and \(K_5\) can be absorbed into \(K_1\). Thus

\[\log(r)\text{Dist}^\Gamma_{\Lambda_k}(\log((\alpha/\eta)) - \log(\beta)) \geq K_1 A_k(K_2 \log(1/\eta)).\] (12)

So combining (11) and (12) and setting \(C_0 = K_0\), \(C_1 = K_2\), and \(C_2 = e^{K_1}\), we have the result claimed in the proposition. \(\square\)
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