Fractional Quantum Hall Effect in a Curved Space: Gravitational Anomaly and Electromagnetic Response

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We develop a general method to compute correlation functions of fractional quantum Hall (FQH) states on a curved space. In a curved space, local transformation properties of FQH states are examined through local geometric variations, which are essentially governed by the gravitational anomaly. Furthermore, we show that the electromagnetic response of FQH states is related to the gravitational response (a response to curvature). Thus, the gravitational anomaly is also seen in the structure factor and the Hall conductance in flat space. The method is based on iteration of a Ward identity obtained for FQH states.

Introduction

Important universal properties of fractional quantum Hall (FQH) states are evident in the quantization of kinetic coefficients in terms of the filling fraction. The most well-known kinetic coefficient is the Hall conductance [1], a transversal response to the electromagnetic field. Beside this, FQH states possess a richer structure evident through their response to changes in spatial geometry and topology, both captured by the gravitational response.

A kinetic coefficient which reflects a transversal response to the gravitational field is the odd viscosity (also referred as anomalous viscosity, Hall viscosity or Lorentz shear modulus) [2]. This coefficient also exhibits a quantization and reveals universal features of FQH states as much as the Hall conductance. While the Hall conductance is seen in an adiabatic response to homogeneous flux deformation [1], the anomalous viscosity is seen as an adiabatic response to homogeneous metric deformations [2]. However, even more universal features become apparent when one considers the adiabatic response to inhomogeneous deformations of the flux and the metric. This is the subject of this paper.

In this paper, we compute the response of the FQH states to local curvature and show that this response reveals corrections to physical quantities in a flat space that remain hidden otherwise. We compute the particle density through a gradient expansion in local curvature, explain the relation of the leading terms to the gravitational anomaly, and show that they are geometrical in nature. For this reason, we expect these terms to be universal (i.e., insensitive to the details of the underlying electronic interaction as long as the interaction gives rise to the FQH state). We develop a general method to compute these terms. Additionally, we show that the dependence on curvature determines the long wavelength expansion of the static structure factor in a flat background, linking the electromagnetic response to the gravitational anomaly. Furthermore, correlation functions computed on arbitrary surfaces provide information about the properties of FQH states under general covariant and, in particular, conformal transformations.

We consider only Laughlin states for which the filling fraction $\nu$ is the inverse of an integer. We restrict our analysis to FQH states without boundaries. Though our analysis is limited to the Laughlin wave function, we believe that our results capture the geometric properties of FQH states. As such, they may serve as universal bounds for response functions in realistic materials exhibiting the FQH effect. Generalization of our results to other FQH is possible and will be reserved for a subsequent paper. We start by formulating the main results.

Main Results

We consider electrons placed on a closed oriented curved surface, such as a deformed sphere, and assume that the magnetic flux through a differential volume element of the surface $d\Phi = BdV$ is uniformly proportional to the volume, where $B > 0$ is a uniform magnetic field. The total number of flux quanta $N_\phi = V(2\pi l^2)^{-1}$ piercing the surface is an integer equal to the area $V$ of the surface in units of $2\pi l^2$, where $l = \sqrt{\hbar/eB}$ is the magnetic length.

In this setting, the lowest Landau level (LLL) remains degenerate on a curved surface, with a gap to excitations on the order of the cyclotron energy [6]. The degeneracy of the level is determined by the Riemann-Roch theorem. Assuming that the surface possesses no singularities so that the Euler characteristic $\chi$ is an even integer, the degeneracy is $N_1 = N_\phi + \chi/2$ [5, 6]. If the number of particles is exactly equal to $N_1$, the ground state will form a droplet without a boundary, completely covering the surface.

This result readily extends to Laughlin states (for a sphere and torus see [6], for a general Riemann surface see [6, 8, 9]): the droplet has no boundary if the number of particles $N$ is equal to

$$N_\nu = \nu N_\phi + \chi/2,$$

assuming that $N_\nu$ is integer. We consider this case.

We focus on the particle density $\rho$ defined such that $\rho dV$ is the number of particles in the volume element $dV$. A locally coordinate invariant quantity, the density must be expressed locally through the (scalar) curvature $R$. In this paper, we compute the leading terms in the gradient expansion of the density of the ground state.
\[ (\rho) = \rho_0 + \frac{1}{8\pi} R - \frac{b}{8\pi} (-l^2 \Delta g) R, \quad b = \frac{1}{3} + \nu - \frac{1}{4\nu}, \quad (2) \]

where \( \rho_0 = \nu (2\pi l^2)^{-1} \) and \( \Delta g \) is the Laplace-Beltrami operator. We omit higher order terms in \( l^2 \). These are controlled by the short distance physics, and are at present not accessible by our methods.

The first two terms are a local version of the global relation \( (1) \) between the maximum particle number and the number of flux quanta. Eq. \( (1) \) is obtained by integrating \( (2) \) over the surface with the help of the Gauss-Bonnet theorem \( \int R dV = 4\pi \chi \). Higher order terms do not contribute to this expression.

The second term indicates that particles accumulate in regions of positive curvature and repell from regions of negative curvature. For example, it shows the excess number of particles accumulating at the tip of a cone. If the conical singularity is of the order \( \alpha > -1 \) such that the metric is locally \( |z|^{2\alpha} dz d\bar{z} \), the excess number of particles at the tip is \( -\alpha/2 \). This term appears in equivalent form in \( [3, 4] \).

The last term encodes the gravitational anomaly, which we explain in the body of the paper. We discuss its implications below.

For the case of free electrons at integer filling \( \nu = 1 \), Eq. \( (2) \) was obtained in \( [13-15] \). In equivalent form, it is known in mathematical literature as an asymptotic expansion of the Bergman kernel \( [16] \). Defining the linear response to curvature as

\[ \eta = (\rho_0 l^2)^{-1} \left. \frac{\delta \rho}{\delta R} \right|_{R=0}, \quad (3) \]

and passing to Fourier modes, Eq. \( (2) \) implies

\[ \eta(q) = \frac{1}{4\nu} (1 - bq^2 + O(q^4)), \quad q = kl. \quad (4) \]

In \( [11] \), one of the authors argued that the kinetic coefficient defined by \( [3] \) enters the hydrodynamics of a FQH incompressible quantum liquid (see also \( [12] \)) as the anomalous term in the momentum flux tensor representing kinematic odd-viscosity. The homogeneous part of the odd-viscosity, computed through alternative methods in \( [2, 4, 10] \), corresponds to the first term in \( [3] \). The leading gradient correction to the odd-viscosity for the integer case \( \nu = 1 \) was recently computed in \( [13] \). It is related to the second term in Eq. \( (4) \), and as we show below, receives a contribution from the gravitational anomaly.

We show that the following general relation between \( \eta(k) \) and the static structure factor \( s(k) = \langle \rho_k \rho_{-k} \rangle_c / \rho_0 \) is valid for Laughlin states, and likely for more general FQH states as well

\[ \frac{q^4}{2} \eta(q) = \left( 1 + \frac{q^2}{2} \right) s(q) - \frac{q^2}{2}, \quad q = kl. \quad (5) \]

Using these relations we obtain

\[ s(q) = \frac{1}{2} q^2 + s_2 q^4 + s_3 q^6 + O(q^8) \quad (6) \]

where

\[ s_2 = (\nu^{-1} - 2)/8, \quad s_3 = (3\nu^{-1} - 4)(\nu^{-1} - 3)/96. \]

The term of order \( q^4 \) in the structure factor goes back to \( [17] \). We find that it is controlled by \( \eta(0) \) and \( \lim_{q \rightarrow 0} s(q)/q^2 \). The next correction \( s_3 \) was recently obtained in \( [19] \) by means of a Mayer expansion. We provide an alternative derivation which emphasizes its connection to the gravitational anomaly. Curiously, \( s_2 \) vanishes at \( \nu = 1/2 \), the bosonic Laughlin state, while \( s_3 \) vanishes for the Laughlin state at \( \nu = 1/3 \) filling.

We also mention another general relation between the structure factor and the Hall conductance valid for the Laughlin wave function

\[ \sigma_H(k) = \frac{\nu^2}{\hbar} \eta(k) \quad (7) \]

We clarify it in the body of the paper (see also \( [12] \)). This relation links the Hall conductance to the response to curvature through \( [5, 18] \). Furthermore, knowledge of \( s_3 \) determines the Hall conductance up to order \( k^4 \).

These results follow from iteration of a Ward identity obtained for the Laughlin wave function in \( [21] \), combined with the gravitational anomaly. An important ingredient of the Ward identity is the two point function of the “Bose” field \( \varphi \) at merged points. The Bose field is defined as a potential of charges created by particles through the Poisson equation

\[ -\Delta_g \varphi = 4\pi \nu^{-1} \rho. \quad (8) \]

We show that in the leading \( 1/N \) approximation, the Bose field has Gaussian correlations. This means that (i) the connected correlation function of \( \varphi \) at large distances between points is the Green function of the Laplace-Beltrami operator

\[ \Delta_g G(z, z') = -4\pi \left[ \frac{1}{\sqrt{g}} \delta^{(2)}(z - z') - \frac{1}{V} \right] \]

and that (ii) at small distances between points the correlation function is the regularized Green function,

\[ \langle \varphi(1) \varphi(2) \rangle_c = \frac{1}{\nu} \left( \begin{array}{ll} G(1, 2) & \text{at large separation} \\ G_R(1, 2) & \text{at short distances} \end{array} \right) \quad (9) \]

The regularized Green function is defined as

\[ G_R(1, 2) = G(1, 2) + 2 \log d(1, 2), \quad (10) \]

where \( d(1, 2) \) is the geodesic distance between the points.

The apparent metric dependence of the two point correlation function at short distances is referred to as the gravitational anomaly.
The Laughlin State on a Riemann Surface

It is convenient to work in holomorphic coordinates where the metric is conformal to the Euclidean metric $ds^2 = \sqrt{g} dz d\bar{z}$. In these coordinates, the scalar curvature reads $R = -\Delta_g \log \sqrt{g}$, where the Laplace-Beltrami operator takes the form $\Delta_g = (4/\sqrt{g}) \partial \bar{\partial}$. The Kähler potential $K$, defined through the equation $\partial \bar{\partial} K = \sqrt{g}$, also plays an important role.

We choose a gauge potential with the anti-holomorphic component $A = \frac{1}{2}(A_1 + iA_2) = iB \bar{\partial} K/4$, such that $\nabla \times A = B \sqrt{g}$, where $B$ is a uniform magnetic field. The states in the LLL are defined as zero modes of the antiholomorphic component of the covariant momentum

$$\Pi = -i\hbar \partial - eA.$$  \hspace{1cm} (11)

The solutions to $\Pi \psi_n = 0$ are the single particle eigenstates given by $\psi_n(z) = s_n(z)e^{-K(z, \bar{z})/4}^2$, where the functions $s_n$ are called holomorphic sections, defined as solutions to $\partial \bar{\partial} s_n = 0$ such that $\psi_n$ is normalizable.

The many-body ground state wave function for free fermions is the Slater determinant of the single particle eigenstates $\Psi_1(z_1, ..., z_N) \propto e^{-\sum_{i<j} K(z_i, \bar{z}_j)/4} \det[s_n(z_i)]$, as shown in [14].

For simplicity, we specialize to a surface of genus zero, with a marked point chosen to be at infinity where $K \sim (V/\pi) \log |z|^2 + o(1)$ and $\log \sqrt{g} \sim -2 \log |z|^2$. In this case, the holomorphic sections $s_n(z)$ are polynomials of degree $n = 0, 1, ..., N_c$, and the Vandermonde identity implies $\det[s_n(z_i)] \propto \prod_{i<j} (z_i - z_j)$.

We construct Laughlin states at the filling fraction $\nu$ by raising the determinant to the power equal to the inverse filling fraction $\beta = 1/\nu$. On a surface of genus zero, the $N$ electron Laughlin wave function is

$$\Psi_\beta = \frac{1}{\sqrt{Z[g]}} \prod_{i<j}^N (z_i - z_j)^\beta e^{-\frac{1}{4\hbar} \sum_{i<j} K(z_i, \bar{z}_j)}$$ \hspace{1cm} (12)

where $Z[g]$ is a normalization factor. This wave function is normalizable only for $N \leq N_c$ given by [15]. We consider states with $N = N_c$, the only case in which the wave-function is modular invariant, indicating the electron droplet completely covers the surface.

Though we work only in genus zero, our formulas are local and therefore apply to more general surfaces. For a comprehensive discussion of the LLL on a surface of arbitrary genus see [16].

In the case of a sphere of radius $r$, inserting the Kähler potential $K = 4r^2 \log(1 + |z|^2/4r^2)$ into (12) reproduces the well-known wave-function on a sphere in stereographic coordinates [2]. In the limit that $r \rightarrow \infty$, $K = |z|^2$ and the planar wave function is recovered.

With this setup, we wish to evaluate equal time correlation functions in the limit $l \rightarrow 0, N_c \rightarrow \infty$ such that the area $V = 2\pi l^2 N_c$ is fixed.

Generating functional

The normalization factor $Z[g]$ encodes the geometry of the surface through its dependence on the metric. It can be used to generate response functions to deformations of the metric. From (12) the generating functional is defined as

$$Z[g] = \int \prod_{i<j}^N |z_i - z_j|^\beta \prod_{i}^N e^{W(z_i, \bar{z}_i)} d^2 z_i,$$ \hspace{1cm} (13)

where $W = -K/2l^2 + \log \sqrt{g}$. Each variation of $\log Z$ over $W(z, \bar{z})$ inserts a factor of $\sqrt{g(z)\rho(z)}$ into the integral (13), where

$$\rho(z) = \frac{1}{\sqrt{g}} \sum_1^N \delta^{g}(z - z_i),$$

is the particle density. Higher order connected correlation functions of the density can be generated in this manner.

More generally, if $A(z_1, ..., z_N)$ is a (metric independent) symmetric function of the coordinates, then

$$\frac{\delta \langle A \rangle}{\delta \rho} = \sqrt{g} \langle A \rho \rangle_c.$$ \hspace{1cm} (14)

This method for computing correlation functions is detailed in [21].

Relations between linear responses on the lowest Landau level

Using the explicit dependence of $W$ on $\sqrt{g}$, we observe a general relation for a linear response to area preserving variations of the metric

$$\frac{1}{2} (-l^2 \Delta_g) \frac{\delta \langle A \rangle}{\delta \sqrt{g}(\zeta)} = \left( 1 + \frac{1}{2} (-l^2 \Delta_g) \right) \langle A \rho(\zeta) \rangle_c.$$ \hspace{1cm} (15)

This relation is valid for any $N$ and any $\beta$ (including the integer case). With the choice of $A = \sum_i \delta^{g}(z - z_i)$ and the identity $\delta(\rho)/\delta \sqrt{g}|_{R=0} = -\Delta |\delta(\rho)/\delta R|_{R=0}$, we obtain (14).

This relation reflects a symmetry between gravity and electromagnetism specific to the LLL. It can be traced back to properties of zero modes of the operator [14].

Similar arguments lead to the relation between the static structure factor and the Hall conductance expressed in [21]. The generating functional (13) can be seen as the normalization factor of the Laughlin wave function in a flat space, but in a weakly inhomogeneous magnetic field. A key assumption is that the form of the wave function is the same as in the case of a uniform magnetic field where $B = -(\hbar/2e) \Delta W$, as in [22]. The two-point density correlation function

$$\langle \rho(z)\rho(z') \rangle_c = \delta(\rho(z))/\delta W(z'),$$

can then be connected to a variation of the density over magnetic field at fixed filling fraction. In Fourier modes, this identity becomes

$$\rho_0 s(k) = (\hbar/2e) k^2 \delta \rho_k/\delta B_k.$$
The inhomogeneous version of the Streda formula
\( \epsilon \delta \rho_b / \delta B_k = \sigma_H(k) \), yields the relation
\( (7) \) \[21\]. Then, computing \( \eta(k) \) allows us to extract \( s(k) \), and thus \( \sigma_H(k) \), from \[3\]. Moreover, once we compute \( \langle \rho \rangle \), we can recover the generating functional which we present in the end of the paper.

To compute the response to curvature we employ the Ward identity explained in the next section.

**Ward identity** The generating functional \( Z[g] \) is invariant under any transformation of coordinates of the integrand \[13\]. In particular, a holomorphic infinitesimal diffeomorphism \( z_i \to z_i + \epsilon (z - z_i) \) where \( \epsilon \) is a parameter, invokes a change of the integrand \[13\] by the factor
\[
\sum_{i} \frac{\partial z_i W}{z - z_i} + \sum_{j \neq i} \frac{\beta}{(z - z_i)(z - z_j)} + \sum_{i} \frac{1}{(z - z_i)^2}.
\]

The Ward identity states that the expectation value of this factor vanishes. Expressing the sum as an integral over the density \( \sum_{i} \to \int d^2 \xi \sqrt{g(\xi)} \rho(\xi) \), yields the relation connecting one- and two-point functions
\[
-2\beta \int \frac{\partial W}{z - \xi} \rho(\xi) \sqrt{g} d^2 \xi = \langle(\partial \varphi)^2 \rangle + (2 - \beta) \langle \partial^2 \varphi \rangle,
\]
where the Bose field \( \varphi = -\beta \sum_i \log |z - z_i|^2 \). Eq. \[9\] was obtained in \[21\]. Furthermore, it is convenient to define the field
\[
\hat{\varphi} = \varphi + \frac{K}{2t^2} - \frac{\beta}{2} \log \sqrt{g},
\]
that vanishes at \( z \to \infty \). The anti-holomorphic derivative of Eq. \[10\] eliminates the integral, by virtue of the \( \partial \)-bar formula \( \partial \bar{z} = \pi \bar{g}^{(2)}(z) \), to give
\[
\langle \rho \rangle \partial \langle \hat{\varphi} \rangle + \left( 1 - \frac{\beta}{2} \right) \partial \langle \rho \rangle = \frac{1}{2 \pi \beta \sqrt{g}} \partial \langle(\partial \hat{\varphi})^2 \rangle.
\]

**Iterating the Ward identity: the leading order** The Ward identity consists of terms of different orders in \( N \), and can be solved iteratively order by order. The first term on the l.h.s. of \[17\] is of the order \( N^2 \), the other two are of the order \( N \). To leading order we thus have \( \langle \hat{\varphi} \rangle = 0 \), which yields
\[
\langle \varphi \rangle = -\frac{K}{2t^2} + \frac{\beta}{2} \log \sqrt{g} + O(t^2).
\]

From this, using \[8\] we recover the first two terms in \[2\].

To proceed with the next iteration, we need to know \( \langle(\partial \hat{\varphi})^2 \rangle \) or rather the short distance behavior of the connected two-point correlation function \( \langle \varphi(z) \varphi(z') \rangle \).

**The Gravitational Anomaly** We obtain the two-point function from \[14\] by varying the one-point function of \( \varphi \) with respect to \( W \), which to leading order implies
\[
\Delta_g \langle \varphi(z) \varphi(z') \rangle = -4\pi \beta \left[ \frac{1}{\sqrt{g}} \delta^{(2)}(z - z') - \frac{1}{V} \right],
\]
where we made use of \[8\] to rewrite \( \rho \) in terms of \( \varphi \). Thus, we see that the two-point function is the Green function of the Laplace-Beltrami operator as in \[9\]. At short distances, the two-point correlation function \( \langle \varphi(z) \varphi(z') \rangle \) is regular, and general covariance requires regularization to be as in Eq. \[9\]. We save further discussion of this subtle point for a subsequent paper.

We are now in a position to compute the missing ingredient of the Ward identity \[17\]. Taking derivatives and merging points, we obtain the known result for the Green function
\[
\langle(\partial \hat{\varphi}(z))^2 \rangle = \frac{\beta}{6} \left[ \partial^2 \log \sqrt{g} - \frac{1}{2} (\partial \log \sqrt{g})^2 \right].
\]

Applying \( \tilde{\partial} \) we obtain the anomalous part of the Ward identity \[17\],
\[
\frac{1}{\sqrt{g}} \tilde{\partial} \langle(\partial \hat{\varphi}(z))^2 \rangle = -\frac{\beta}{24} \partial R,
\]
This formula represents the gravitational or trace anomaly. It shows that the connected correlation function of the holomorphic derivative of the Gaussian field is no longer holomorphic in a curved space.

**Iterating the Ward identity: subsequent orders** The anomalous contribution \[15\] allows us to extract \( b \) by computing the next order in the Ward identity. Inserting \[16\] into \[17\], matching terms of the same order (by replacing the first \( \langle \rho \rangle \) in \[17\] with its leading order) reduces \[17\] to linear form, which readily integrates to
\[
\rho_0 \langle \hat{\varphi} \rangle + \left( 1 - \frac{\beta}{2} \right) \langle \rho \rangle = -\frac{1}{48\pi} R.
\]

In this equation, all of the terms are proportional to the curvature. The r.h.s. is proportional to the trace anomaly of the free Gaussian field. Matching the coefficients determines the coefficient \( b \) in \( \langle \rho \rangle \).

**Generating functional and Polyakov’s Liouville action** Once we know the density \( \rho \), the generating functional can be computed by integrating \[12\] with \( A = Z[g] \), in a similar manner to what was done in \[21\]
\[
\frac{1}{2} (-t^2 \Delta_g) \frac{\delta \log Z[g]}{\delta \sqrt{g}} = \left( 1 + \frac{1}{2} (-t^2 \Delta_g) \right) \langle \rho \rangle.
\]

The result for \( \beta = 1 \) was presented in the recent paper \[14\]. The generating functional for an arbitrary filling fraction, developed as an expansion in \( 1/N_\phi \), reads
\[
\log \frac{Z[g]}{Z[g_0]} = \frac{N_\phi}{2} (N_\nu + 1) + N_\phi A^{(2)}[g] + N_\phi A^{(1)}[g] + A^{(0)}[g],
\]
\[
A^{(2)} = -\frac{1}{2\beta} \frac{1}{V^2} \int KdV, \quad A^{(1)} = \frac{1}{2V} \int \log \sqrt{g} dV;
\]
\[
A^{(0)} = \frac{1}{16\pi} \left( \frac{1}{3} + \frac{\beta - 1}{2} \right) \left( \int \log \sqrt{g} R dV + 16\pi \right),
\]
where \( Z[g_0] \) is the generating functional on a sphere.
The functionals $A^{(2)}$ and $A^{(1)}$ are known objects in Kähler geometry [14, 24]. Unlike the higher order terms, the first three terms cannot be expressed locally through the scalar curvature $R$. For this reason, they obey non-trivial co-cycle properties explained in [14, 24]. The variations of the first two functionals over the Kähler potential are the volume form and the curvature.

The generating functional encodes the gravitational and electromagnetic response of the FQH states. It shows how various correlation functions transform under variations of the geometry such as conformal transformations. In particular, the order zero (dimensionless) term $A^{(0)}$ reflects the gravitational anomaly. The functional $A^{(0)}$ is Polyakov’s Liouville action. Recall that Polyakov’s action represents the partition function of a Gaussian free field [23]. It appears as a normalized spectral determinant of the Laplace-Beltrami operator

$$-\frac{1}{2} \log \frac{\det (-\Delta_g)}{\det (-\Delta_{g_0})} = \frac{1}{96\pi} \int \log \sqrt{g} R dV + \frac{1}{6}. \quad (22)$$

Polyakov’s action transforms covariantly under conformal transformations.

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