A CENTRAL LIMIT THEOREM FOR STAR-GENERATORS OF $S_\infty$,
WHICH RELATES TO THE LAW OF A GUE MATRIX

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Abstract. It is well-known that, on a purely algebraic level, a simplified version of the
Central Limit Theorem (CLT) can be proved in the framework of a non-commutative
probability space, under the hypotheses that the sequence of non-commutative random
variables we consider is exchangeable and obeys a certain vanishing condition of some of
its joint moments. In this approach (which covers versions for both the classical CLT
and the CLT of free probability), the determination of the resulting limit law has to be
addressed on a case-by-case basis.

In this paper we discuss an instance of the above theorem that takes place in the
framework of the group algebra $\mathbb{C}[S_\infty]$ of the infinite symmetric group: the exchangeable
sequence is provided by the star-generators of $S_\infty$, and the expectation functional used on
$\mathbb{C}[S_\infty]$ depends in a natural way on a parameter $d \in \mathbb{N}$. We identify precisely the limit
distribution $\mu_d$ for this special instance of CLT, via a connection that
$\mu_d$ turns out to have with the average empirical eigenvalue distribution of a random $d \times d$
GUE matrix. Moreover, we put into evidence a multi-variate version of this result which follows from
the observation that, on the level of calculations with pair-partitions, the (non-centered)
star-generators are related to a (centered) exchangeable sequence of GUE matrices with
independent entries.

1. INTRODUCTION

1.1. A central limit theorem for star-generators of $S_\infty$.

It is well-known that, on a purely algebraic level, a simplified version of the Central Limit
Theorem (CLT) can be proved via a direct calculation of moments: one expresses the relevant moments as sums with terms indexed by set-partitions, and one keeps a record of which terms in those sums can actually contribute in the limit. This proof works without changes when one moves to the framework of a non-commutative probability space. Moreover, the argument still works when the considered sequence of non-commutative random variables is not required to satisfy some form of independence, but is only required to have a weaker property of exchangeability, together with an additional vanishing condition on some of the sequence’s joint moments (those joint moments which, in a certain sense, contain a “singleton”); see [5], Theorem 0 on page 138. A price to pay for these weakened hypotheses is that the resulting limit law is no longer universal, and its determination has to be addressed on a case-by-case basis. In particular, this approach covers simplified “combinatorial” versions of both the classical CLT, where the limit is the normal law, and of the CLT of free probability, where the limit is the semicircle law of Wigner.

In the present paper we examine an interesting instance of this limit theorem for exchangeable sequences, taking place in the framework of the infinite symmetric group

\begin{equation}
S_\infty := \{ \tau : \mathbb{N} \to \mathbb{N} \mid \tau \text{ is bijective and } \exists k_0 \in \mathbb{N} \text{ such that } \tau(k) = k \text{ for } k > k_0 \}.
\end{equation}

On the group algebra $\mathbb{C}[S_\infty]$ we consider a natural expectation functional, as follows: we fix a $d \in \mathbb{N} = \{1, 2, \ldots\}$ and we let $\varphi_d : \mathbb{C}[S_\infty] \to \mathbb{C}$ be the linear functional determined by

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the requirement that
\[
\varphi_d(\tau) = (1/d)^{||\tau||}, \quad \tau \in S_\infty,
\]
where the length \(||\tau||\) of a permutation \(\tau \in S_\infty\) is defined as the minimal number \(m\) of transpositions \(\rho_1, \ldots, \rho_m\) needed in order to achieve a factorization \(\tau = \rho_1 \cdots \rho_m\). The restriction of \(\varphi_d\) to \(S_\infty\) is an example of “block character” in the sense of [7], and is at the same time an example of extremal character of \(S_\infty\), parametrized by a very simple double-sequence in the Thoma classification of such extremal characters (see e.g. Section 3 of the survey paper [13], or Section 4.1 of the monograph [3]). In particular, the linear functional \(\varphi_d\) is positive; that is, one has
\[
\varphi_d(a^* a) \geq 0 \text{ for all } a \in C[S_\infty],
\]
where the \(*\)-operation on \(C[S_\infty]\) is introduced via the requirement that \(\tau^* = \tau^{-1}\) for all \(\tau \in S_\infty\). Hence \((C[S_\infty], \varphi_d)\) is an example of \(*\)-probability space (by which we mean a couple \((A, \varphi)\) where \(A\) is a unital \(*\)-algebra over \(C\) and \(\varphi : A \to C\) is a positive linear functional such that \(\varphi(1) = 1\)).

A natural example of exchangeable sequence in \((C[S_\infty], \varphi_d)\) is provided by its so-called “star-generators”, i.e. by the sequence of transpositions
\[
\gamma_1 = (1, 2), \gamma_2 = (1, 3), \ldots, \gamma_n = (1, n+1), \ldots
\]
This sequence of generators of \(S_\infty\) has received some attention in the combinatorics literature, see e.g. the exposition and references included in the introduction of [6]. The CLT for exchangeable sequences applies to the centerings of the \(\gamma_n\)’s and produces a probability distribution \(\mu_d\) on \(\mathbb{R}\), which is the limit in moments, for \(n \to \infty\), of the centered and normalized sums
\[
s_n = \frac{1}{\sqrt{n}} \left( (\gamma_1 + \cdots + \gamma_n) - \frac{n}{d} \right) \in C[S_\infty].
\]
In this paper we prove that the probability distribution \(\mu_d\) is intimately related to the empirical eigenvalue distribution of a Gaussian Hermitian (usually referred to as “GUE”) random matrix of size \(d \times d\). More precisely, one has the following theorem.

**Theorem 1.1.** Let \(\mu_d\) be the limit in moments of the sequence \((1.4)\) in \((C[S_\infty], \varphi_d)\). Then
\[
\mu_d * N(0, 1/d^2) = \nu_d \quad \text{(convolution of probability measures)},
\]
where \(N(0, 1/d^2)\) is the centered normal distribution of variance \(1/d^2\) and \(\nu_d\) is the average empirical eigenvalue distribution of a \(d \times d\) GUE matrix of variance 1.

The convolution formula from Theorem 1.1 can be combined with known facts about the average empirical eigenvalue distribution of a GUE matrix in order to obtain further information about \(\mu_d\). One gets in particular a precise description of Laplace transform. Indeed, it is known (see e.g. Section 3.3 of the monograph [1], or Section 2 of the survey paper [10]) that the Laplace transform of \(\nu_d\) is defined for all \(z \in \mathbb{C}\), and has the explicit formula
\[
\int_{\mathbb{R}} e^{zt} \, d \nu_d(t) = Q_d(z) \cdot e^{z^2/(2d)}, \quad z \in \mathbb{C},
\]
where \(Q_d(z)\) is the polynomial defined by
\[
Q_d(z) = \sum_{j=0}^{d-1} \frac{1}{d^j (j+1)!} \cdot \binom{d-1}{j} \cdot z^{2j}.
\]
Corollary 1.2. The Laplace transform (or equivalently, the exponential moment-generating function) of \( \mu_d \) is defined for all \( z \in \mathbb{C} \), and has the explicit formula
\[
\int e^{zt} \, d\mu_d(t) = Q_d(z) \cdot e^{(d-1)z^2/(2d^2)}, \quad z \in \mathbb{C},
\]
where the polynomial \( Q_d \) is picked from (1.7).

Remark 1.3. (1) From Corollary 1.2 and general considerations on the Laplace transform it follows that, for \( d \geq 2 \), the probability measure \( \mu_d \) is absolutely continuous with respect to Lebesgue measure, with density of the form \( P_d(t) e^{-t^2d^2/(2d^2)} \), where \( P_d \) is an even polynomial of degree \( 2d - 2 \). Moreover, the coefficients of \( P_d \) can be obtained out of those of the polynomial \( Q_d \) from Equation (1.7) by solving a triangular system of linear equations. For instance, for \( d = 2 \) and \( d = 3 \) one gets
\[
d\mu_2(t) = \frac{1}{\sqrt{2\pi}} 8t^2 e^{-2t^2} \, dt, \quad d\mu_3(t) = \frac{3}{64\sqrt{\pi}} (20 - 108t^2 + 243t^4) e^{-9t^2/4} \, dt.
\]

(2) The sum \( \gamma_1 + \cdots + \gamma_n \) appearing in (1.4) is unitarily conjugated to the sum of transpositions \( (1, n+1) + (2, n+1) + \cdots + (n, n+1) \in \mathbb{C}[S_\infty] \), which goes under the name of “Jucys-Murphy element”. Due to this fact, Theorem 1.1 can be re-stated as a result about the limit distribution for the sequence of centered and renormalized Jucys-Murphy elements.

(3) It is natural to ask what happens when in Theorem 1.1 we replace the expectation functional \( \varphi_d \) by a functional \( \varphi : \mathbb{C}[S_\infty] \to \mathbb{C} \) coming from a more general extremal character of \( S_\infty \). The centered star-generator will continue to be exchangeable in the newly considered *-probability space \((\mathbb{C}[S_\infty], \varphi)\), but will no longer satisfy the vanishing-moment condition. In order to understand what are the correct setting and limit theorem for this more general situation, it is likely that one needs some considerations of non-commutative dynamical systems, along the lines started in [8].

(4) In the limit case “\( d = \infty \)”, i.e. when in the considerations leading to Theorem 1.1 we replace \( 1/d \) by 0, the limit law \( \mu_d \) becomes the semicircle law, and we retrieve a result of Biane [2].

The paper [2] also puts into evidence a phenomenon of asymptotic free independence, going in a multi-variate framework. In connection to that, we prove that Theorem 1.1 has a multi-variate version, which is discussed in the next subsection.

1.2. Multi-variate extension of Theorem 1.1.
In order to state (in Theorem 1.7 below) the multi-variate version, we first introduce some relevant notation referring to non-commutative distributions.

Definition and Remark 1.4. Let \( r \) be in \( \mathbb{N} \).

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1 For \( d = 1 \) it is immediately inferred, either from Equation (1.5) or from Equation (1.8), that \( \mu_1 \) is the Dirac mass at 0.
Let $\mathbb{C}(X_1, \ldots, X_r)$ be the algebra of polynomials in non-commuting indeterminates $X_1, \ldots, X_r$. We turn $\mathbb{C}(X_1, \ldots, X_r)$ into a $*$-algebra by considering on it the $*$-operation uniquely determined by the requirement that $X_p^* = X_p$ for $1 \leq p \leq r$. We denote

$$D_r = \{ \mu : \mathbb{C}(X_1, \ldots, X_r) \to \mathbb{C}, \text{ linear } | \mu \text{ is positive and has } \mu(1) = 1 \}$$

(1.9) where the fact that $\mu$ is positive means, by definition, that $\mu(P^* P) \geq 0$ for all $P \in \mathbb{C}(X_1, \ldots, X_r)$). For a functional $\mu \in D_r$, the quantities of the form

$$\mu(X_{i(1)} \cdots X_{i(k)}), \text{ with } k \in \mathbb{N} \text{ and } (i(1), \ldots, i(k)) \in \{1, \ldots, r\}^k,$$

are called moments of $\mu$. On $D_r$ we have a natural notion of convergence in moments, where the convergence of a sequence $(\mu_n)_{n=1}^{\infty}$ to a limit $\mu$ amounts to the fact that

$$\lim_{n \to \infty} \mu_n(P) = \mu(P), \forall P \in \mathbb{C}(X_1, \ldots, X_r).$$

(2) Let $\mu$ be a functional in the space $D_r$ considered above. The commuting exponential generating function (or “commuting e.g.f”, for short) for the moments of $\mu$ is the power series

$$f(z_1, \ldots, z_r) := 1 + \sum_{k=1}^{\infty} \sum_{i(1), \ldots, i(k)=1}^{r} \frac{\mu(X_{i(1)} \cdots X_{i(k)})}{k!} \cdot z_{i(1)} \cdots z_{i(k)},$$

where $z_1, \ldots, z_r$ are commuting indeterminates.

We note that in the case $r = 1$, (1.11) gives precisely the usual e.g.f for the moments of a distribution $\mu$. When $r \geq 2$, the series $f(z_1, \ldots, z_r)$ from (1.11) still says something about the moments of $\mu$, but one can only retrieve certain symmetric sums of such moments. (E.g. the coefficient of $z_1 z_2$ allows us to find the sum $\mu(X_1 X_2) + \mu(X_2 X_1)$, but not the individual values of $\mu(X_1 X_2)$ and $\mu(X_2 X_1)$.)

(3) Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $a_1, \ldots, a_r$ be an $r$-tuple of selfadjoint elements of $\mathcal{A}$. The joint distribution of $a_1, \ldots, a_r$ in $(\mathcal{A}, \varphi)$ is the functional $\mu \in D_r$ defined via the requirement that for every $k \in \mathbb{N}$ and every tuple $(i(1), \ldots, i(k)) \in \{1, \ldots, r\}^k$ one has

$$\mu(X_{i(1)} \cdots X_{i(k)}) = \varphi(a_{i(1)} \cdots a_{i(k)}).$$

**Remark 1.5.** Some of the notions from Definition (1.4) can still be considered in connection to any linear functional $\mu : \mathbb{C}(X_1, \ldots, X_r) \to \mathbb{C}$ which has $\mu(1) = 1$, but is not assumed to be positive, hence does not necessarily belong to the space $D_r$ from (1.9). In particular, for such $\mu$ we will still talk about its moments, defined as in (1.10), and about its commuting e.g.f $f(z_1, \ldots, z_r)$, defined as in (1.11).

An important example of functional in $D_r$, which we will use is the joint distribution of an $r$-tuple of $d \times d$ GUE random matrices with independent entries.

**Definition 1.6.** Let $M_1, \ldots, M_r$ be a tuple of $d \times d$ GUE random matrices with independent entries. We will denote by $\nu^{(r,d)}$ the functional in $D_r$ which is defined via the requirement that for every $k \in \mathbb{N}$ and $(i(1), \ldots, i(k)) \in \{1, \ldots, r\}^k$ one has

$$\nu^{(r,d)}(X_{i(1)} \cdots X_{i(k)}) = (E \circ \text{tr}_d)(M_{i(1)} \cdots M_{i(k)}),$$

where “$(E \circ \text{tr}_d)$” is the expected normalized trace of a $d \times d$ random matrix. The details of this definition are reviewed in Remark (5.2) below.

**Theorem 1.7.** Let $d \in \mathbb{N}$ and consider the $*$-probability space $(\mathbb{C}[S_{\infty}], \varphi_d)$, with $\varphi_d$ defined as in (1.2) above. Let us also consider an $r \in \mathbb{N}$ and an injective function

$$\gamma : \{1, \ldots, r\} \times \mathbb{N} \to \{\gamma_n | n \in \mathbb{N}\} \subseteq \mathbb{C}[S_{\infty}],$$

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where \( \{ \gamma_n \mid n \in \mathbb{N} \} \) are the star-generators of \( S_\infty \). For every \( p \in \{1, \ldots, r\} \) and \( n \in \mathbb{N} \) we consider the selfadjoint element

\[
sp_n := \frac{1}{\sqrt{n}} \left( \gamma(p, 1) + \cdots + \gamma(p, n) - \frac{n}{d} \right) \in \mathbb{C}[S_\infty],
\]

then for every \( n \in \mathbb{N} \) we consider the joint distribution \( \mu_n \in \mathcal{D}_r \) of the \( r \)-tuple \( s_{1,n}, \ldots, s_{r,n} \) (in the sense of Definition 1.4(3)). One has that:

1. The sequence \( (\mu_n)_{n=1}^\infty \) defined above converges in moments to a limit \( \mu^{(r,d)} \in \mathcal{D}_r \).
2. Let \( f \) be the commuting e.g.f for the moments of \( \mu^{(r,d)} \), and let \( g \) be the commuting e.g.f for the moments of the joint GUE distribution \( \nu^{(r,d)} \) from Definition 1.6. Then

\[
f(z_1, \ldots, z_r) \cdot e^{\left( z_1^2 + \cdots + z_r^2 \right)/2d^2} = g(z_1, \ldots, z_r).
\]

In the special case when \( r = 1 \) and \( \gamma \) is defined by putting \( \gamma(1, n) := \gamma_n \), Theorem 1.7 retrieves the statement of Theorem 1.1. Indeed, in this case Equation (1.13) takes the form

\[
f(z) \cdot e^{z^2/(2d^2)} = g(z),
\]

and we recognize the two sides of this equality to be the e.g.f’s for moments of \( \mu_d \ast N(0, 1/d^2) \) and respectively of \( \nu_d \), with \( \mu_d \) and \( \nu_d \) as in the statement of Theorem 1.1. It follows that \( \mu_d \ast N(0, 1/d^2) \) and \( \nu_d \) have the same sequence of moments. It is easy to check that this sequence of moments grows slowly enough to ensure the uniqueness of the underlying probability distribution, and the equality \( \mu_d \ast N(0, 1/d^2) = \nu_d \) hence follows.

### 1.3. In the background: functions on set-partitions.

We now move to explain how Theorem 1.7 comes about. In order to do so, it is relevant that we review some ideas concerning functions on set-partitions that were championed by Bożejko and Speicher in [5]. A quick review of notation: in Definition 1.8 below we consider partially ordered sets of the form \( (\mathcal{P}(k), \preceq) \) where \( k \in \mathbb{N} \) and \( \mathcal{P}(k) \) denotes the set of all partitions of \( \{1, \ldots, k\} \), partially ordered by reverse refinement (for \( \pi = \{V_1, \ldots, V_q\} \) and \( \rho = \{W_1, \ldots, W_q\} \) in \( \mathcal{P}(k) \) we write \( \pi \preceq \rho \) to mean that every block \( V_i \) of \( \pi \) is contained in some block \( W_j \) of \( \rho \). We let \( \mathcal{P}_2(k) \) denote the set of all pair-partitions of \( \{1, \ldots, k\} \):

\[
\mathcal{P}_2(k) := \{ \pi \in \mathcal{P}(k) \mid \text{every block } V \text{ of } \pi \text{ has } |V| = 2 \}, \quad k \in \mathbb{N},
\]

with the convention that \( \mathcal{P}_2(k) = \emptyset \) when \( k \) is odd. Moreover, we will use the standard notation that

\[
\text{for any } k \in \mathbb{N} \text{ and any tuple } \underline{i} = (i(1), \ldots, i(k)) \in \mathbb{N}^k,
\]

\[
\text{Ker}(\underline{i}) \in \mathcal{P}(k) \text{ denotes the partition of } \{1, \ldots, k\} \text{ into level sets of } \underline{i}.
\]

This notation is useful when one deals with exchangeable sequences – indeed, one of the possible descriptions (cf. Remark 2.1 below) for the exchangeability of a sequence \( (a_n)_{n=1}^\infty \) in a \( * \)-probability space \( (\mathcal{A}, \varphi) \) is that

\[
\varphi(a_{i(1)} \cdots a_{i(k)}) = \varphi(a_{\underline{i}(1)} \cdots a_{\underline{i}(k)})
\]

for every \( k \in \mathbb{N} \) and \( \underline{i}, \underline{j} \in \mathbb{N}^k \) such that \( \text{Ker}(\underline{i}) = \text{Ker}(\underline{j}) \).

**Definition 1.8.** Let \( (\mathcal{A}, \varphi) \) be a \( * \)-probability space and let \( (a_n)_{n=1}^\infty \) be an exchangeable sequence of selfadjoint elements of \( \mathcal{A} \).
(1) The function on partitions associated to \((a_n)_{n=1}^\infty\) is \( t : \sqcup_{k=1}^\infty P(k) \to \mathbb{C} \) defined as follows: for every \( k \in \mathbb{N} \) and \( \pi \in P(k) \) we put
\[
\begin{align*}
\{ t(\pi) := \varphi(a^{(1)}_{\bar{i}}, \ldots, a^{(k)}_{\bar{i}}), & \text{ where } i \in \mathbb{N}^k \text{ is } \\
\text{any } k\text{-tuple such that } \text{Ker} (\bar{i}) = \pi. \}
\end{align*}
\]
\[(1.16)\]
This formula is unambiguous due to (1.15) above.

(2) Consider the algebra \( \mathbb{C}\{X_p \mid p \in \mathbb{N}\} \) of polynomials in a countable collection of non-commuting indeterminates. Following \([5]\), we use the function \( \gamma \) associated to \((\mu)\) define a linear functional \( \mu \) via the prescription that \( \mu(1) = 1 \) and that for every \( k \in \mathbb{N} \) and every tuple \( i = (\bar{i}(1), \ldots, \bar{i}(k)) \in \mathbb{N}^k \) we put
\[
\mu(X^{(1)}_{\bar{i}} \cdots X^{(k)}_{\bar{i}}) = \sum_{\pi \in P_2(k), \pi \leq \text{Ker} (\bar{i})} t(\pi).
\]
\[(1.17)\]
In \[(1.17)\] we use the convention that \( \mu(X^{(1)}_{\bar{i}} \cdots X^{(k)}_{\bar{i}}) = 0 \) whenever \( \{ \pi \in P_2(k) \mid \pi \leq \text{Ker} (\bar{i}) \} = \emptyset \), i.e. whenever the partition \( \text{Ker}(\bar{i}) \) has at least one block of odd cardinality.

Remark 1.9. It was shown in \([5]\) (see review in Section 2 below) that, in the framework of the preceding definition, the only additional requirement needed to ensure that the CLT works on the exchangeable sequence \((a_n)_{n=1}^\infty\) is that \( t \) from \[(1.16)\] satisfies:
\[
\begin{align*}
\{ t(\pi) &= 0 \text{ whenever the partition } \pi \in \sqcup_{k=1}^\infty P(k) \\
\text{has at least one block } V \text{ with } |V| = 1. \}
\end{align*}
\]
\[(1.18)\]
If this additional requirement is fulfilled, then for every \( r \in \mathbb{N} \) one gets a suitable “exchangeable CLT for \( r \)-tuples”, where the limit law is the restriction of the linear functional \( \mu \) from \[(1.17)\] to the subalgebra \( \mathbb{C}\{X_1, \ldots, X_r\} \) of \( \mathbb{C}\{X_p \mid p \in \mathbb{N}\} \). The precise statement of this exchangeable CLT for \( r \)-tuples is reviewed in Theorem 2.5 below.

In order to connect the notions introduced in Definition 1.8 to the framework of the present paper, we make the following notation.

Notation 1.10. Let \( d \in \mathbb{N} \) and consider the \( * \)-probability space \((\mathbb{C}[S_\infty], \varphi_d)\), with \( \varphi_d \) defined as in \[(1.2)\] above.

(1) Let \((\gamma_n)_{n=1}^\infty\) be the exchangeable sequence of star-generators from \[(1.3)\]. We denote the function on partitions associated to this exchangeable sequence by \( \varphi_d \), and we denote the corresponding linear functional on \( \mathbb{C}\{X_p \mid p \in \mathbb{N}\} \) by \( \nu^{(\infty,d)} \).

(2) Consider on the other hand the exchangeable sequence \((\gamma_n - \frac{1}{d})_{n=1}^\infty\) obtained by centering the star-generators \( \gamma_n \) with respect to \( \varphi_d \). We denote the function on partitions associated to \((\gamma_n - \frac{1}{d})_{n=1}^\infty\) by \( t_d \), and we denote the corresponding linear functional on \( \mathbb{C}\{X_p \mid p \in \mathbb{N}\} \) by \( \mu^{(\infty,d)} \).

We can now put forward three separate propositions which, when combined together, give the statement of Theorem 1.7.

Proposition 1.11. Let \( d, r \) be in \( \mathbb{N} \).

(1) The linear functional \( \mu^{(\infty,d)} : \mathbb{C}\{X_p \mid p \in \mathbb{N}\} \to \mathbb{C} \) from Notation 1.10(2) is positive, that is, one has \( \mu(P^*P) \geq 0 \) for all \( P \in \mathbb{C}\{X_p \mid p \in \mathbb{N}\} \). Consequently, the restriction
\[
\mu^{(r,d)} := \mu^{(\infty,d)} \mid \mathbb{C}\{X_1, \ldots, X_r\}
\]
belongs to the space of distributions \( \mathcal{D}_r \) introduced in Definition 1.4(1).
Remark 1.13. Proposition 1.11 is a direct application of the exchangeable CLT theorem – the only verifications that need to be done are that the sequence of centered star-generators \( (\gamma_n - \frac{1}{d})_{n=1}^\infty \) is indeed exchangeable in \((\mathbb{C}[S_\infty], \varphi_d)\), and that the associated function on partitions \( u_d : \sqcup_{k=1}^\infty \mathcal{P}(k) \to \mathbb{C} \) satisfies the requirement (1.18) mentioned in Remark 1.9. We will make these verifications in Section 4 below.

Proposition 1.12 is of a different nature (has to be, since the function on partitions \( u_d : \sqcup_{k=1}^\infty \mathcal{P}(k) \to \mathbb{C} \) associated to \( (\gamma_n - \frac{1}{d})_{n=1}^\infty \) clearly does not satisfy (1.18)). The proof of this proposition is obtained by the direct examination of certain joint moments of the \( \gamma_n \)'s, which we then compare against the corresponding joint moments of a relevant exchangeable sequence of GUE matrices. The review of GUE matrices and the proof of Proposition 1.12 are made in Section 5 of the paper.

Upon invoking the interpretations of the distributions \( \mu^{(r,d)}, \nu^{(r,d)} \) which follow from Propositions 1.11 and 1.12, there is only one item remaining to be proved in Theorem 1.7, namely Equation (1.13) in part (2) of the theorem. This item is just a statement about what happens to commuting e.g.f.’s when we perform a translation of an exchangeable sequence, and is covered by the following general result.

Proposition 1.14. Let \((A, \varphi)\) be a \(*\)-probability space. Let \((a_n)_{n=1}^\infty\) be an exchangeable sequence of selfadjoint elements of \( A \), with associated function \( t : \sqcup_{k=1}^\infty \mathcal{P}(k) \to \mathbb{C} \), such that \( t \) fulfills the vanishing condition indicated in (1.18). Pick \( \lambda \in \mathbb{R} \) and put
\[
(1.20) \quad b_n := a_n + \lambda, \quad n \in \mathbb{N}.
\]
This creates a new exchangeable sequence \( (b_n)_{n=1}^\infty \), which also has an associated function on partitions, \( u : \sqcup_{k=1}^\infty \mathcal{P}(k) \to \mathbb{C} \). Let \( \mu, \nu : \mathbb{C}(\{X_p \mid p \in \mathbb{N}\}) \to \mathbb{C} \) be the linear functionals obtained by using \( t \) and \( u \), respectively, via the recipe described in Definition 1.8(2). Finally, pick an \( r \in \mathbb{N} \) and let \( f, g \) denote the commuting e.g.f.’s of the functionals \( \mu | \mathbb{C}(X_1, \ldots, X_r) \) and respectively \( \nu | \mathbb{C}(X_1, \ldots, X_r) \). Then one has:
\[
(1.21) \quad f(z_1, \ldots, z_r) \cdot e^{\lambda^2(z_1^2 + \cdots + z_r^2)/2} = g(z_1, \ldots, z_r).
\]

Proposition 1.14 can in particular be applied to the case when \( a_n = \gamma_n - \frac{1}{d} \) in the \(*\)-probability space \((\mathbb{C}[S_\infty], \varphi_d)\), with \( \lambda = 1/d \), and this leads precisely to the statement of Theorem 1.7(2).

1.4. Organization of the paper.

In view of the discussion in Section 1.3, what we are left to do is show the proofs of the Propositions 1.11, 1.12 and 1.14.

Besides the present Introduction, the paper has four sections. Section 2 is devoted to the review of the relevant facts related to exchangeable sequences, and of the exchangeable CLT theorem. This is followed by the proof of Proposition 1.12 which is done in Section 3. In Section 4 we return to the framework of the \(*\)-probability space \((\mathbb{C}[S_\infty], \varphi_d)\), and we verify that the centered star-generators \( (\gamma_n - \frac{1}{d})_{n=1}^\infty \) have indeed the required properties which
ensure the validity of Proposition 1.11. In the final Section 5 we establish the connection (formalized in Proposition 5.6) between star-generators and an exchangeable sequence of \( d \times d \) GUE matrices with independent entries, and we prove Proposition 1.12.

2. Review of CLT for exchangeable sequences

**Definition and Remark 2.1.** We collect here a few basic notions and facts, some of them already mentioned in the Introduction, related to exchangeable sequences.

Let \((\mathcal{A}, \varphi)\) be a \(*\)-probability space and let \((a_n)_{n=1}^\infty\) be a sequence of selfadjoint elements of \(\mathcal{A}\). Quantities of the form

\[
\varphi(a_{i(1)} \cdots a_{i(k)}), \quad \text{with } k \in \mathbb{N} \text{ and } \vec{i} : \{1, \ldots, k\} \to \mathbb{N}
\]

go under the name of joint moments of \((a_n)_{n=1}^\infty\). We say that the sequence \((a_n)_{n=1}^\infty\) is exchangeable to mean that its joint moments are invariant under the natural action of \(S_\infty\), that is:

\[
\left\{ \begin{array}{l}
\varphi(a_{i(1)} \cdots a_{i(k)}) = \varphi(a_{\vec{j}(1)} \cdots a_{\vec{j}(k)}) \\
\quad \text{for every } k \in \mathbb{N} \text{ and } \vec{j} : \{1, \ldots, k\} \to \mathbb{N} \\
\quad \text{for which } \exists \tau \in S_\infty \text{ such that } \vec{j} = \tau \circ \vec{i}
\end{array} \right.
\]

(2.1)

Recall from Equation (1.14) in the Introduction that, upon treating a tuple \(\vec{i} \in \mathbb{N}^k\) as a function \(\vec{i} : \{1, \ldots, k\} \to \mathbb{N}\), one defines the kernel \(\text{Ker}(\vec{i})\) as the partition of \(\{1, \ldots, k\}\) into level-sets of \(\vec{i}\): two numbers \(p, q \in \{1, \ldots, k\}\) belong to the same block of \(\text{Ker}(\vec{i})\) if and only if \(\vec{i}(p) = \vec{i}(q)\). It is easily seen that for two tuples \(\vec{i}, \vec{j} : \{1, \ldots, k\} \to \mathbb{N}\), the existence of a permutation \(\tau \in S_\infty\) such that \(\vec{j} = \tau \circ \vec{i}\) is equivalent to the fact that \(\text{Ker}(\vec{j}) = \text{Ker}(\vec{i})\).

Hence the definition of what it means for \((a_n)_{n=1}^\infty\) to be exchangeable can also be phrased in the way it was done in Equation (1.15) of the Introduction, which leads naturally to considering the function on partitions \(t : \sqcup_{k=1}^\infty \mathcal{P}(k) \to \mathbb{C}\) associated to \((a_n)_{n=1}^\infty\) in Definition 1.8(1).

Note that if \((a_n)_{n=1}^\infty\) is exchangeable, then the \(a_n\)'s are in particular identically distributed. That is, for every \(k \in \mathbb{N}\) one has a common “moment of order \(k\)” for all the \(a_n\)'s:

\[
\varphi(a_{i(1)}^k) = \varphi(a_{i(2)}^k) = \cdots = t(1_k),
\]

where, following standard combinatorics literature, we let \(1_k \in \mathcal{P}(k)\) denote the partition of \(\{1, \ldots, k\}\) into only one block.

We now turn to the important vanishing condition that was mentioned in Equation (1.18) of Remark 1.9. For our purposes, it is convenient to phrase things in a way that also allows for translations of a given sequence of selfadjoint elements, and we thus proceed as follows.

**Definition 2.2.** Let \((\mathcal{A}, \varphi)\) be a \(*\)-probability space and let \((a_n)_{n=1}^\infty\) be a sequence of selfadjoint elements of \(\mathcal{A}\). We say that \((a_n)_{n=1}^\infty\) has the singleton-factorization property to mean that the following implication holds:

\[
\left\{ \begin{array}{l}
\text{Let } k \in \mathbb{N}, \vec{i} : \{1, \ldots, k\} \to \mathbb{N} \text{ and } j_0 \in \{1, \ldots, k\} \text{ be} \\
\quad \text{such that } \vec{i}(j) \neq \vec{i}(j_0) \text{ for all } j \neq j_0 \text{ in } \{1, \ldots, k\}. \\
\quad \text{Then } \varphi(a_{\vec{i}(1)} \cdots a_{\vec{i}(k)}) = \varphi(a_{\vec{i}(j_0)}) \cdot \varphi(a_{\vec{i}(1)} \cdots a_{\vec{i}(j_0)} \vec{i}(j_0) + 1 \cdots a_{\vec{i}(k)}).
\end{array} \right.
\]

(2.2)

The next proposition records the useful fact that both exchangeability and singleton-factorization property are preserved when one does a translation of the sequence of \(a_n\)'s. The proof of the proposition is straightforward (both statements (1) and (2) are verified via

\footnote{The sequence \((a_n)_{n=1}^\infty\) considered in this definition is not assumed to be exchangeable.}
induction arguments on the length of the relevant joint moments), and is left as exercise to the reader.

**Proposition 2.3.** Let \((A, \varphi)\) be a \(*\)-probability space, let \((a_n)_{n=1}^\infty\) be a sequence of selfadjoint elements of \(A\), and let \(\lambda\) be a real number. We put \(b_n := a_n + \lambda\), \(n \in \mathbb{N}\).

1. Suppose that \((a_n)_{n=1}^\infty\) is exchangeable. Then \((b_n)_{n=1}^\infty\) is exchangeable as well.

2. Suppose that \((a_n)_{n=1}^\infty\) has the singleton-factorization property. Then \((b_n)_{n=1}^\infty\) has the singleton-factorization property as well.

**Remark 2.4.** Let \((A, \varphi)\) be a \(*\)-probability space and let \((a_n)_{n=1}^\infty\) be a sequence of selfadjoint elements of \(A\) such that:

1. \((a_n)_{n=1}^\infty\) is exchangeable, and
2. \((a_n)_{n=1}^\infty\) has the singleton-factorization property.

Let \(t : \mathbb{U}_{k=1}^\infty \rightarrow \mathbb{C}\) be the function on partitions associated to \((a_n)_{n=1}^\infty\), in the way indicated in Definition 1.8(1). The additional assumption (ii) gives a factorization formula satisfied by \(t\): if \(\pi \in \mathcal{P}(k)\) has a block \(V\) with \(|V| = 1\) and if \(\pi_o \in \mathcal{P}(k-1)\) is obtained from the restriction of \(\pi\) to \(\{1, \ldots, k\} \setminus V\) (by relabelling the elements of \(\{1, \ldots, k\} \setminus V\) to become \(1, \ldots, k-1\), in increasing order), then we have

\[
t(\pi) = t(1_1) \cdot t(\pi_o),
\]

where \(1_1\) is the unique partition in \(\mathcal{P}(1)\). Repeated use of the formula (2.3) shows that, in this case, the function \(t\) is completely determined if we know its values on partitions \(\pi\) which have no blocks of cardinality 1 and if, in addition to that, we know what is \(t(1_1)\).

[A concrete example: the common value of all the joint moments \(\varphi(a_i a_j a_k)\) with \(i \neq j \neq k \neq i\) in \(\mathbb{N}\) is recorded as \(t(\pi)\) for \(\pi = \{ \{1, 3\}, \{2\}, \{4\} \} \in \mathcal{P}(4)\). The singleton-factorization property ensures that this particular \(t(\pi)\) factors as \(t(1_1)^2 \cdot t(1_2)\), where \(1_1\) is the unique partition of the set \(\{1\}\) and \(1_2\) is the partition of \(\{1, 2\}\) into a single block.]

Now, besides the assumptions (i) + (ii) above, suppose we also know that

(iii) the \(a_n\)'s are centered (we have \(\varphi(a_n) = 0\) for all \(n \in \mathbb{N}\)).

Clearly, the assumption (iii) amounts to the fact that \(t(1_1) = 0\). It is obvious that, in this case, the factorization rule from Equation (2.3) becomes precisely the vanishing condition that had been indicated in Equation (1.18) of Remark 1.9.

We next review the limit theorem for exchangeable sequences that we want to use. We follow the version that is indicated in [5] – see Theorem 0 on page 138 and also Theorem 2 on page 142 of that paper.

**Theorem 2.5.** (CLT for an exchangeable sequence, following [5].)

Let \((A, \varphi)\) be a \(*\)-probability space and let \((a_n)_{n=1}^\infty\) be a sequence of selfadjoint elements of \(A\) which satisfies the assumptions (i), (ii) and (iii) listed in the preceding remark. Let \(t : \mathbb{U}_{k=1}^\infty \mathcal{P}(k) \rightarrow \mathbb{C}\) be the function on partitions associated to \((a_n)_{n=1}^\infty\) as in Definition 1.8(1), and let \(\mu : \mathbb{C}\langle \{X_p \mid p \in \mathbb{N}\}\rangle \rightarrow \mathbb{C}\) be the linear functional constructed from \(t\) in the way described in Definition 1.8(2). Then:

1. \(\mu\) is positive (that is, \(\mu(P^* P) \geq 0\) for all \(P \in \{X_p \mid p \in \mathbb{N}\}\)).

2. Consider an \(r \in \mathbb{N}\) and an injective function \(\lambda : \{1, \ldots, r\} \times \mathbb{N} \rightarrow \mathbb{N}\). We denote \(a(p, n) := a_{\lambda(p, n)}\), for all \(p \in \{1, \ldots, r\}\) and \(n \in \mathbb{N}\). Moreover, for every \(p \in \{1, \ldots, r\}\) and
\[ n \in \mathbb{N} \text{ we put} \quad s_{p,n} := \frac{1}{\sqrt{n}} \left( a(p, 1) + \cdots + a(p, n) \right) \in \mathcal{A}, \]
and for every \( n \in \mathbb{N} \) we consider the joint distribution \( \mu_n \in \mathcal{D}_r \) of the \( r \)-tuple \( s_{1,n}, \ldots, s_{r,n} \) (in the sense of Definition [1,4,3]). Then the sequence \( (\mu_n)_{n=1}^{\infty} \) converges in moments to the restriction of the linear functional \( \mu \) from part (1) to the subalgebra \( \mathbb{C} \langle X_1, \ldots, X_r \rangle \) of \( \mathbb{C} \langle \{ X_p \mid p \in \mathbb{N} \} \rangle \). \[ \square \]

**Remark 2.6.** Let \( (\mathcal{A}, \varphi) \) be a \(*\)-probability space and let \( (a_n)_{n=1}^{\infty} \) be a sequence of selfadjoint elements of \( \mathcal{A} \) which satisfies assumptions (i) and (ii) from Remark [2.4], but where we do not make the assumption (iii) that the \( a_n \)'s are centered. Denoting \( \alpha_1 := \varphi(a_1) = \varphi(a_2) = \cdots \), we can still apply Theorem [2.5] to the sequence \( (\hat{a}_n)_{n=1}^{\infty} \) where \( \hat{a}_n := a_n - \alpha_1, \ n \in \mathbb{N} \). Indeed, the sequence \( (\hat{a}_n)_{n=1}^{\infty} \) still satisfies (i) and (ii), by Proposition [2.3] and in addition to that the \( \hat{a}_n \)'s are centered as well. Note that in this case, if we want to write the elements \( s_{p,n} \) involved in Theorem [2.5] directly in terms of the \( a_n \)'s, then they are
\[ s_{p,n} = \frac{1}{\sqrt{n}} \left( a(p, 1) + \cdots + a(p, n) - n\alpha_1 \right) , \quad p \in \{1, \ldots, r\} \text{ and } n \in \mathbb{N}. \]

3. **Translation of an exchangeable sequence, and proof of Proposition [1.4]**

We begin with a lemma which is, essentially, the special case “\( r = 1 \)” of the proposition we are aiming to prove.

**Lemma 3.1.** Let \( (\mathcal{A}, \varphi) \) be a \(*\)-probability space. Let \( (a_n)_{n=1}^{\infty} \) be an exchangeable sequence of selfadjoint elements of \( \mathcal{A} \), with associated function \( t : \sqcup_{k=1}^{\infty} \mathcal{P}(k) \to \mathbb{C} \), and suppose that \( t \) fulfills the vanishing condition indicated in [1.13] of Remark [1.7]. Pick a \( \lambda \in \mathbb{R} \) and put
\[ b_n := a_n + \lambda, \quad n \in \mathbb{N}. \]
This creates a new exchangeable sequence \( (b_n)_{n=1}^{\infty} \), which also has an associated function on partitions, \( u : \sqcup_{k=1}^{\infty} \mathcal{P}(k) \to \mathbb{C} \).

Consider the power series \( \tilde{f}_1, \tilde{g}_1 \in \mathbb{C}[[z]] \) defined by
\[ (3.1) \quad \tilde{f}_1(z) = \sum_{m=0}^{\infty} \frac{\alpha_{2m}}{(2m)!} z^m \text{ and } \tilde{g}_1(z) = \sum_{m=0}^{\infty} \frac{\beta_{2m}}{(2m)!} z^m, \]
where for every \( m \in \mathbb{N} \) we put
\[ \alpha_{2m} := \sum_{\pi \in \mathcal{P}_2(2m)} t(\pi) \text{ and } \beta_{2m} := \sum_{\pi \in \mathcal{P}_2(2m)} u(\pi), \]
and we also make the convention to put \( \alpha_0 = \beta_0 = 1 \). Then one has\[ (3.2) \quad \tilde{f}_1(z) \cdot e^{\lambda^2 z/2} = \tilde{g}_1(z). \]

**Proof.** The series appearing on both sides of [3.2] have constant terms equal to 1, while for an \( m \in \mathbb{N} \) the equality of their coefficients of order \( m \) amounts to
\[ (3.3) \quad \frac{\beta_{2m}}{(2m)!} = \sum_{\ell=0}^{m} \frac{\alpha_{2\ell}}{(2\ell)!} \cdot \frac{(\lambda^2 / 2)^{m-\ell}}{(m-\ell)!} \]
(where on the right-hand side we considered the Cauchy product of \( \tilde{f}_1(z) \) with the series expansion for \( e^{\lambda^2 z/2} \)). We fix for the whole proof an \( m \in \mathbb{N} \) for which we will verify that [3.3] holds.
Let $\pi$ be in $\mathcal{P}_2(2m)$ and let us consider a tuple $\tilde{u} = (\tilde{u}(1), \ldots, \tilde{u}(2m)) \in \mathbb{N}^{2m}$ such that $\text{Ker}(\tilde{u}) = \pi$. Then
\[
\mathbf{u}(\pi) = \varphi(b_{\tilde{u}(1)} \cdots b_{\tilde{u}(2m)}) = \varphi\left((a_{\tilde{u}(1)} + \lambda) \cdots (a_{\tilde{u}(2m)} + \lambda)\right) = \sum_{A \subseteq \{1, \ldots, 2m\}} \lambda^{2m-|A|} \varphi\left(\prod_{p \in A} a_{\tilde{u}(p)}\right) = \sum_{\emptyset \neq A \subseteq \{1, \ldots, 2m\}} \lambda^{2m-|A|} \cdot t(\pi|A),
\]
where for a non-empty set $A \subseteq \{1, \ldots, 2m\}$ we have denoted as "$\pi|A$" the partition in $\mathcal{P}(|A|)$ which is obtained by restricting $\tilde{u}$ to $A$ and then by redenoting the elements of $A$ as $\{1, \ldots, |A|\}$, in increasing order. We note that for every $\emptyset \neq A \subseteq \{1, \ldots, 2m\}$, all the blocks of $\pi|A$ have cardinality 1 or 2; and moreover, if at least one of those blocks has cardinality 1 then the hypothesis (1.18) assumed on $t$ entails that $t(\pi|A) = 0$. Observe, moreover, that the requirement on $\pi|A$ to only have blocks of cardinality 2 is equivalent to the requirement that $A$ is a union of blocks of $\pi$; we will refer to this situation by saying that "$A$ is a $\pi$-saturated set".

The conclusion of the preceding paragraph is that for any $\pi \in \mathcal{P}_2(2m)$ we have
\[
(3.4) \quad \mathbf{u}(\pi) = \lambda^{2m} + \sum_{\emptyset \neq A \subseteq \{1, \ldots, 2m\}, \pi \text{ is } \pi\text{-saturated}} \lambda^{2m-|A|} \cdot t(\pi|A).
\]
In this formula we sum over $\pi \in \mathcal{P}_2(2m)$ and then we reverse the order of summation in the ensuing double sum, to obtain:
\[
(3.5) \quad \beta_{2m} = \lambda^{2m} \cdot |\mathcal{P}_2(2m)| + \sum_{\emptyset \neq A \subseteq \{1, \ldots, 2m\}, A \text{ is } \pi\text{-saturated}} \lambda^{2m-|A|} \cdot \left(\sum_{\pi \in \mathcal{P}_2(2m), \pi \text{ is } \pi\text{-saturated}} t(\pi|A)\right).
\]
Now let us fix for a moment a non-empty set $A \subseteq \{1, \ldots, 2m\}$. If $|A|$ is odd then, clearly, there are no partitions $\pi \in \mathcal{P}_2(2m)$ such that $A$ is $\pi$-saturated. If $|A| = 2\ell$ for some $\ell \in \{1, \ldots, m\}$, then we claim that:
\[
(3.6) \quad \sum_{\pi \in \mathcal{P}_2(2m), \pi \text{ is } \pi\text{-saturated}} t(\pi|A) = |\mathcal{P}_2(2m - 2\ell)| \cdot \sum_{\rho \in \mathcal{P}_2(2\ell)} t(\rho) = |\mathcal{P}_2(2m - 2\ell)| \cdot \alpha_{2\ell}.
\]
The formula (3.6) is easily verified upon noticing that the pair-partitions $\pi \in \mathcal{P}_2(2m)$ such that $A$ is $\pi$-saturated are parametrized by couples $(\rho, \rho') \in \mathcal{P}_2(2\ell) \times \mathcal{P}_2(2m - 2\ell)$, where $\rho$ is turned into a pair-partition of $A$ while $\rho'$ is turned into a pair-partition of $\{1, \ldots, 2m\} \setminus A$, in the natural way.

By plugging (3.6) into (3.5), we find that:
\[
(3.7) \quad \beta_{2m} = \lambda^{2m} \cdot |\mathcal{P}_2(2m)| + \sum_{\ell=1}^{m} \binom{2m}{2\ell} \cdot |\mathcal{P}_2(2m - 2\ell)| \cdot \alpha_{2\ell}.
\]
In the latter equation we substitute $|\mathcal{P}_2(2m)| = (2m)!/(2^m m!)$ and $|\mathcal{P}_2(2m - 2\ell)| = (2m - 2\ell)!/(2^{m-\ell} (m - \ell)!)$, and then some straightforward algebra leads to the required formula (3.3). \hfill \Box

The next lemma provides a reduction from multi-variate case to univariate case in the commuting e.g.f.’s that are of interest in Proposition 1.14.

\footnote{The restriction of $\pi$ to $A$ is the partition of $A$ into the blocks $\{A \cap V \mid V \in \pi \text{ and } A \cap V \neq \emptyset\}$.
Lemma 3.2. Suppose we are given a function $t : \mathbb{U} \to \mathbb{C}$, and we use it to construct a linear functional $\mu : \mathbb{C}\langle \{X_p | p \in \mathbb{N}\} \to \mathbb{C}$ via the recipe described in Definition 1.8(2). Consider moreover an $r \in \mathbb{N}$, and let $f_r(\z_1, \ldots, \z_r)$ be the commuting e.g.f (as in Definition 1.4(2)) for the moments of the restriction of $\mu$ to $\mathbb{C}\langle X_1, \ldots, X_r \rangle \subseteq \mathbb{C}\langle \{X_p | p \in \mathbb{N}\} \rangle$.

On the other hand, consider the series of one variable $f_1(z) := \sum_{m=0}^{\infty} \frac{\alpha_{2m}}{(2m)!} z^m$, where

$$\alpha_{2m} := \sum_{\pi \in \mathcal{P}(2m)} t(\pi) \text{ for } m \in \mathbb{N}, \text{ and we also put } \alpha_0 := 1.$$ Then the series $f_r$ and $f_1$ are related by the formula

$$f_r(z_1, \ldots, z_r) = f_1(z_1^2 + \cdots + z_r^2).$$

Proof. We consider some integers $\ell_1, \ldots, \ell_r \geq 0$ for which we will examine the coefficient of $z_1^{\ell_1} \cdots z_r^{\ell_r}$ in the series $f_r$. Without loss of generality, we assume that not all of $\ell_1, \ldots, \ell_r$ are equal to 0 (since the series on both sides of (3.8) have constant terms equal to 1). We denote $\ell_1 + \cdots + \ell_r =: k \in \mathbb{N}$.

By direct inspection of the formula which defines the moments a commuting e.g.f, and then by invoking the specific formula for the moments of $\mu$, we see that:

$$\left( \begin{array}{c} \text{coefficient of} \\ z_1^{\ell_1} \cdots z_r^{\ell_r} \text{ in } f_r \\ \end{array} \right) = \frac{1}{k!} \cdot \sum_{i=\mathbb{P}(2k)} \mu(X_{i(1)} \cdots X_{i(k)})$$

$$= \frac{1}{k!} \cdot \sum_{i=\mathbb{P}(2k)} \sum_{\pi \in \mathcal{P}(k), \pi \subseteq \text{Ker}(i)} t(\pi).$$

In the latter expression we reverse the order of the two sums, and continue with

$$= \frac{1}{k!} \sum_{\pi \in \mathcal{P}(k)} t(\pi) \left\{ i \in \{1, \ldots, r\}^k \left| i^{-1}(1) = \ell_1, \ldots, i^{-1}(r) = \ell_r, \right. \right. \left. \right. \left. \text{and Ker}(i) \geq \pi \right\}$$

If not true that all of $\ell_1, \ldots, \ell_r$ are even, then it is immediate that

$$\left\{ i \in \{1, \ldots, r\}^k \left| i^{-1}(1) = \ell_1, \ldots, i^{-1}(r) = \ell_r, \right. \right. \left. \right. \left. \text{and Ker}(i) \geq \pi \right\} = \emptyset, \text{ for all } \pi \in \mathcal{P}(k),$$

and Equation (3.9) entails that the coefficient of $z_1^{\ell_1} \cdots z_r^{\ell_r}$ in $f_r$ is equal to 0.

Let us now assume that $\ell_1, \ldots, \ell_r$ are even, and let us write $\ell_1 = 2j_1, \ldots, \ell_r = 2j_r$. We also denote $j_1 + \cdots + j_r =: m$ (hence $m = k/2 \in \mathbb{N}$). Observe that for every $\pi \in \mathcal{P}(2k)$, we have

$$\left\{ i \in \{1, \ldots, r\}^k \left| i^{-1}(1) = \ell_1, \ldots, i^{-1}(r) = \ell_r, \right. \right. \left. \right. \left. \text{and Ker}(i) \geq \pi \right\} = \frac{m!}{j_1! \cdots j_r!}.$$ Indeed, the tuples $i$ which are being counted on the left-hand side of (3.10) are constant along every block of $\pi$, and can be thought of as “colourings” of the blocks of $\pi$, where $j_1$ blocks have colour 1, $j_2$ blocks have colour 2, and so on. The number of such colourings is given by the multinomial indexed on the right-hand side of (3.10). Upon plugging (3.10) into (3.9), we conclude that the coefficient of $z_1^{2j_1} \cdots z_r^{2j_r}$ in $f_r$ is equal to

$$\frac{1}{(2m)!} \cdot \frac{m!}{j_1! j_2! \cdots j_r!} \cdot \sum_{\pi \in \mathcal{P}(2m)} t(\pi), \text{ hence to } \sum_{\pi \in \mathcal{P}(2m)} \frac{t(\pi)}{(2m)!} \cdot \frac{m!}{j_1! j_2! \cdots j_r!}.$$
It is immediately verified that the coefficients of the series
\[ \tilde{f}_1(z_1^2 + \cdots + z_r^2) := 1 + \sum_{m=1}^{\infty} \frac{\alpha_{2m}(2m)!}{(2m)!} (z_1^2 + \cdots + z_r^2)^m \]
have exactly the same values as those found above for the coefficients of \( f_r \) (namely the coefficients of monomials \( z_1^{2j_1} \cdots z_r^{2j_r} \) are as shown in (3.11), while all the other coefficients are equal to 0). This concludes the verification of the required equality (3.8). □

3.3. Proof of Proposition 1.14. The series \( f(z_1, \ldots, z_r) \) in Proposition 1.14 is the same as the \( f_r(z_1, \ldots, z_r) \) from the above Lemma 3.2, and can therefore be expressed as \( \tilde{f}_1(z_1^2 + \cdots + z_r^2) \), with \( \tilde{f}_1 \in \mathbb{C}[[z]] \) as described in Lemma 3.2. Likewise, the series \( g(z_1, \ldots, z_r) \) in Proposition 1.14 can be written as “\( g_r(z_1, \ldots, z_r) \)” in the framework of Lemma 3.2, where we now start from the function on partitions \( u \) (instead of \( t \)); consequently, we have the formula
\[ g(z_1, \ldots, z_r) = \tilde{g}_1(z_1^2 + \cdots + z_r^2), \]
for the corresponding series \( \tilde{g}_1 \in \mathbb{C}[[z]] \) defined in the way described in Lemma 3.2. We are only left to invoke Lemma 3.1 which connects the series \( \tilde{f}_1 \) and \( \tilde{g}_1 \) – the formula (3.2) from Lemma 3.1 converts precisely into the formula (1.21) from Proposition 1.14. □

4. The Exchangeable Sequence of Star-Generators of \( S_\infty \), and Proof of Proposition 1.11

Notation and Remark 4.1. (1) Let \( S_\infty \) be the infinite symmetric group (as in (1.1) of the Introduction). We will write the permutations in \( S_\infty \) by using cycle notation, where we only indicate the cycles of length \( \geq 2 \) of the permutation – it is implicitly assumed that all the numbers in \( \mathbb{N} \) that are not indicated in the cycle notation are fixed points of the permutation in question. This convention was in particular used in Equation (1.3) of the Introduction, where we considered the star-generators
\[ \gamma_n := (1, n+1), \quad n \in \mathbb{N}. \]

(2) As mentioned in the Introduction, we will use the notation \( ||\tau|| \) for the minimal number of factors required in a factorization of \( \tau \) into transpositions and where, by convention, we have \( ||\tau|| = 0 \) if and only if \( \tau \) is the identity permutation of \( \mathbb{N} \). Note that, as a consequence of the fact that the set of transpositions in \( S_\infty \) is invariant under conjugation, the map \( \tau \mapsto ||\tau|| \) is constant on conjugacy classes of \( S_\infty \).

(3) We will use the notation “\#” for the number of cycles (including fixed points) of a permutation \( \tau \in S_\infty \) on a given invariant finite set. More precisely: if \( \tau \in S_\infty \) and if \( A \subseteq \mathbb{N} \) is a finite set such that \( \tau(A) = A \), then we denote
\[ \#(\tau \mid A) := \text{ (number of orbits into which } A \text{ is partitioned by the action of } \tau). \]
This notation is useful for giving an alternative description of the number \( ||\tau|| \) reviewed in (2) above; indeed, it is easy to verify that one has the formula
\[ ||\tau|| = |A| - \#(\tau \mid A), \]
holding for \( \tau \in S_\infty \) and with \( A \) being any finite subset of \( \mathbb{N} \) such that \( \tau(b) = b \) for all \( b \in \mathbb{N} \setminus A \).
(4) On the group algebra \( \mathbb{C}[S_\infty] \) we consider the \( * \)-operation determined by the requirement

\[
\tau^* := \tau^{-1}, \quad \forall \tau \in S_\infty.
\]

That is, every permutation \( \tau \) in \( S_\infty \) becomes a unitary element of \( \mathbb{C}[S_\infty] \). Note that if \( \tau \) is a product of disjoint transpositions (hence \( \tau = \tau^{-1} \)), then \( \tau \) is at the same time a selfadjoint element of \( \mathbb{C}[S_\infty] \). In particular, the star-generators \((\gamma_n)_{n=1}^\infty\) form a sequence of selfadjoint elements of \( \mathbb{C}[S_\infty] \).

(5) Let \( d \) be in \( \mathbb{N} \) and let \( \varphi_d : \mathbb{C}[S_\infty] \to \mathbb{C} \) be the linear functional defined in the way indicated in Equation (4.4) of the Introduction. It is obvious that \( \varphi_d(1) = 1 \), and it turns out that, moreover, \( \varphi_d \) is positive; hence \((\mathbb{C}[S_\infty], \varphi_d)\) is a \( * \)-probability space. The positivity property of \( \varphi_d \) can be directly verified by examining the action of permutations on words of finite length over the alphabet \( \{1, \ldots, d\} \); for a detailed presentation of how this goes, we refer the reader to [7].

We also note that \( \varphi_d \) has the trace property:

\[
(4.2) \quad \varphi_d(ab) = \varphi_d(ba), \quad \forall a, b \in \mathbb{C}[S_\infty].
\]

Indeed, this boils down to checking that \( ||\sigma\tau|| = ||\tau\sigma|| \) for all \( \sigma, \tau \in S_\infty \), and the latter equality follows from the fact that \( \sigma\tau \) and \( \tau\sigma \) belong to the same conjugacy class of \( S_\infty \). In connection to the trace property, we mention that the restriction of \( \varphi_d \) to the group \( S_\infty \) is what is called an “extremal character” of this group. In the well-known parametrization of Thoma for such characters (see e.g. Section 3 of the survey paper [13], or Section 4.1 of the monograph [3]), \( \varphi_d \) is the character of \( S_\infty \) parametrized by the Thoma double sequence \( (\alpha_n; \beta_n)_{n=1}^\infty \) having

\[
\alpha_1 = \cdots = \alpha_d = 1/d, \quad \alpha_n = 0 \text{ for } n > d, \quad \beta_n = 0 \text{ for all } n \in \mathbb{N}.
\]

This identification as an extreme character can also be used as an argument for the fact that \( \varphi_d \) is a positive functional on \( \mathbb{C}[S_\infty] \).

Remark 4.2. In connection to the framework from Notation (4.5), one may wonder what is so special about using a base of the form \( 1/d \), with \( d \in \mathbb{N} \), in the formula (4.2) which defines the linear functional considered on \( \mathbb{C}[S_\infty] \). Why doesn’t one consider a linear functional \( \varphi : \mathbb{C}[S_\infty] \to \mathbb{C} \) defined by the requirement that

\[
(4.3) \quad \varphi(\tau) = q^{||\tau||}, \quad \forall \tau \in S_\infty,
\]

where \( q \) is some arbitrary (but fixed) real number? The reason is that the functional defined by Equation (4.3) is positive on \( \mathbb{C}[S_\infty] \) if and only if \( q \) belongs to the special subset

\[
(4.4) \quad \left\{ \frac{1}{d} \mid d \in \mathbb{N} \right\} \cup \{0\} \cup \left\{ -\frac{1}{d} \mid d \in \mathbb{N} \right\} \subseteq \mathbb{R};
\]

so the choice \( q = 1/d \) is, in fact, not too restrictive. Concerning the possible alternative of using \( q = -1/d \), let us observe that the functionals corresponding to \( q = 1/d \) and \( q = -1/d \) in (4.3) only differ by a multiplication with the character \( \tau \mapsto \text{sign}(\tau) \) on \( S_\infty \); as a consequence of this fact, the whole layout of the present paper wouldn’t change much if we would choose to work with \( q = -1/d \) instead of \( q = 1/d \). We also note that the choice \( q = 0 \) in the set of possible values of \( q \) indicated in (4.4) would make the functional \( \varphi \) defined by (4.3) become the so-called “canonical trace” associated to the regular representation of \( S_\infty \); in this case, as mentioned in the Introduction, the counterpart of our main Theorem (1.7) is a result obtained by Biane in [2].
In order to prove that the positivity requirement restricts \( q \) to the set of values indicated in (4.3), one can use a direct argument, described as follows. The set of values

\[
\{ q \in \mathbb{R} \mid \text{the functional } \varphi \text{ defined in (1.3) is positive on } \mathbb{C}[S_\infty]\}
\]

is symmetric, because the functionals “\( \varphi \)” corresponding to \( q \) and to \(-q\) are obtained from each other by multiplication with the sign character of \( S_\infty \). It thus suffices to look at a \( q \) from the set (4.5) such that \( q < 0 \). For every \( n \in \mathbb{N} \), let \( S_n := \{ \tau \in S_\infty \mid \tau(k) = k \text{ for all } k > n \} \), and consider the square matrix (of size \( n! \)) \( G_n := [\varphi(\sigma^{-1}\tau)]_{\sigma,\tau \in S_n} \). It is easily seen that, since \( \varphi \) is positive, the matrices \( G_n \) have to be non-negative definite for all \( n \in \mathbb{N} \). But it is also easily seen that, for every \( n \in \mathbb{N} \), \( G_n \) has the eigenvalue

\[
\lambda_n = \sum_{\tau \in S_n} q ||\tau||
\]

(corresponding to the eigenvector that has all the components equal to 1), and that this eigenvalue can be factored as \( \lambda_n = (1 + q)(1 + 2q) \cdots (1 + (n - 1)q) \). Finally, for our \( q < 0 \), the condition that \( \lambda_n \geq 0 \) for all \( n \in \mathbb{N} \) forces \( q \) to be of the form \( q = -1/d \) with \( d \in \mathbb{N} \).

In the framework from Notation 4.1, we now consider the sequence \( (\gamma_n)_{n=1}^{\infty} \) of star-generators of \( S_\infty \), and we prove that it satisfies the hypotheses discussed in Theorem 2.5 and Remark 2.6.

**Proposition 4.3.** Let \( d \) be in \( \mathbb{N} \). Consider the *-probability space \( (\mathbb{C}[S_\infty], \varphi_d) \) and the sequence of selfadjoint elements \( (\gamma_n)_{n=1}^{\infty} \) in \( \mathbb{C}[S_\infty] \). Then:

(1) \( (\gamma_n)_{n=1}^{\infty} \) is exchangeable.

(2) \( (\gamma_n)_{n=1}^{\infty} \) has the singleton-factorization property.

**Proof.** (1) Consider, same as in Definition 2.1, a \( k \in \mathbb{N} \) and two tuples \( i, j : \{1, \ldots, k\} \rightarrow \mathbb{N} \) for which there exists a permutation \( \tau \in S_\infty \) such that \( j = \tau \circ i \). We have to verify the equality

\[
\varphi_d(\gamma_i(1) \cdots \gamma_i(k)) = \varphi_d(\gamma_j(1) \cdots \gamma_j(k)).
\]

Let \( \theta : \mathbb{N} \rightarrow \mathbb{N} \) be defined by putting \( \theta(1) = 1 \) and \( \theta(n) = 1 + \tau(n - 1) \) for \( n \geq 2 \). It is immediate that \( \theta \in S_\infty \). For every \( 1 \leq h \leq k \) we have:

\[
\theta \gamma_i(h) \theta^{-1} = \theta (1, i(h) + 1) \theta^{-1} = (1, j(h) + 1) = \gamma_j(h),
\]

where at the second equality sign we used the fact that \( \theta(1) = 1 \) and \( \theta(i(h) + 1) = 1 + \tau(i(h)) = 1 + j(h) \). From (4.7) we infer that

\[
\theta(\gamma_i(1) \cdots \gamma_i(k)) \theta^{-1} = \gamma_j(1) \cdots \gamma_j(k) \in \mathbb{C}[S_\infty],
\]

and the required equality (4.6) follows from the fact that \( \varphi_d \) has the trace property.

(2) Consider, same as in Definition 2.2, a \( k \in \mathbb{N} \), a tuple \( i : \{1, \ldots, k\} \rightarrow \mathbb{N} \) and an index \( j \in \{1, \ldots, k\} \) such that \( i(j) \neq i(\ell) \) for all \( \ell \neq j \) in \( \{1, \ldots, k\} \). We have to verify the equality

\[
\varphi_d(\gamma_i(1) \cdots \gamma_i(k)) = \varphi_d(\gamma_j(j)) \cdot \varphi_d(\gamma_i(1) \cdots \gamma_i(j-1) \gamma_i(j+1) \cdots \gamma_i(k)).
\]

Consider the permutations

\[
\sigma_1 := \gamma_i(1) \cdots \gamma_i(j-1), \quad \text{and} \quad \sigma_2 := \gamma_i(j+1) \cdots \gamma_i(k),
\]

and

\[
\sigma_3 := \gamma_j(1) \cdots \gamma_j(k).\]
where in the case \( j = 1 \) (respectively \( j = k \)) we make the convention that \( \sigma_1 \) (respectively \( \sigma_2 \)) is the identity permutation. The required formula (4.8) then amounts to
\[
\varphi_d(\gamma_{\tilde{i}(j)} \sigma_2) = \varphi_d(\gamma_{\tilde{i}(j)}) \cdot \varphi_d(\sigma_1 \sigma_2);
\]
Remembering how \( \varphi_d \) is defined, we thus see that what we have to verify is a relation between two lengths:
\[
\|\gamma_{\tilde{i}(j)} \sigma_2\| = 1 + \|\sigma_1 \sigma_2\|.
\]
It is convenient to replace this verification with the equivalent one that
\[
\|\gamma_{\tilde{i}(j)} \cdot (\sigma_2 \sigma_1)\| = 1 + \|\sigma_2 \sigma_1\|,
\]
where the equalities \( \|\gamma_{\tilde{i}(j)} \sigma_2\| = \|\gamma_{\tilde{i}(j)} \sigma_2 \sigma_1\| \) and \( \|\sigma_1 \sigma_2\| = \|\sigma_2 \sigma_1\| \) follow from the fact that \( \| \cdot \| \) is constant on the conjugacy classes of \( S_\infty \).

In the case when \( \sigma_2 \sigma_1 \) is the identity permutation, the equality (4.10) holds trivially; so we will assume that \( \sigma_2 \sigma_1 \) is not the identity permutation, and we will consider the unique factorization
\[
\sigma_2 \sigma_1 = \theta_1 \cdots \theta_p
\]
where \( p \geq 1 \) and \( \theta_1, \ldots, \theta_p \) are disjoint cycles of lengths \( \ell_1, \ldots, \ell_p \geq 2 \). In particular, this gives us the explicit formula
\[
\|\sigma_2 \sigma_1\| = \sum_{r=1}^{p} (\ell_r - 1)
\]
(following for instance from Equation (4.11) in Notation 4.1(3)).

Our hypothesis \( \tilde{i}(j) \neq \tilde{i}(\ell) \) for all \( \ell \neq j \) implies that \( \tilde{i}(j) + 1 \) is a fixed point of \( \sigma_2 \sigma_1 \), since it is fixed by all the transpositions in the products defining \( \sigma_1 \) and \( \sigma_2 \). Hence \( \tilde{i}(j) + 1 \) is not included in any of the cycles \( \theta_1, \ldots, \theta_p \) from (4.11). For further discussion we consider two cases, according to whether the number 1 is or is not included in one of those cycles.

Case 1. 1 is not a fixed point of \( \sigma_2 \sigma_1 \), hence it is included in one of the cycles \( \theta_1, \ldots, \theta_p \).

Since the product of cycles \( \theta_1, \ldots, \theta_p \) is a commuting one, we may assume without loss of generality (by relabeling the cycles, if needed) that 1 appears in the cycle \( \theta_1 \). Then
\[
\gamma_{\tilde{i}(j)} \sigma_2 \sigma_1 = (\gamma_{\tilde{i}(j)} \theta_1) \theta_2 \cdots \theta_p,
\]
where \( \gamma_{\tilde{i}(j)} \theta_1 \) is a cycle of length \( 1 + \ell_1 \) (cycling the numbers that were in \( \theta_1 \) and the number \( \tilde{i}(j) + 1 \)). The right-hand side of (4.13) is a disjoint cycle decomposition, and the counterpart of Equation (4.12) is thus
\[
\|\gamma_{\tilde{i}(j)} \cdot (\sigma_2 \sigma_1)\| = ((1 + \ell_1) - 1) + \sum_{r=2}^{p} (\ell_r - 1).
\]

By comparing to the right-hand side of (4.12) we see that we got indeed \( 1 + \|\sigma_2 \sigma_1\| \), as required.

Case 2. 1 is a fixed point of \( \tau \), hence is not included in any of the cycles \( \theta_1, \ldots, \theta_p \).

In this case, \( \gamma_{\tilde{i}(j)} = (1, \tilde{i}(j) + 1) \) commutes with the cycles \( \theta_1, \ldots, \theta_p \), hence the factorization of \( \gamma_{\tilde{i}(j)} \cdot (\sigma_2 \sigma_1) \) into a product of disjoint cycles is just \( \theta_0 \theta_1 \cdots \theta_p \) with \( \theta_0 = \gamma_{\tilde{i}(j)} \). The counterpart of Equation (4.12) is thus
\[
\|\gamma_{\tilde{i}(j)} \cdot (\sigma_2 \sigma_1)\| = \sum_{r=0}^{p} (\ell_r - 1), \text{ with } \ell_0 = 2,
\]
and the required equality (4.10) follows in this case as well. \( \square \)
4.4. Proof of Proposition 1.11. The sequence of selfadjoint elements \((\gamma_n)_{n=1}^\infty\) in the *-probability space \((\mathbb{C}[S_\infty], \varphi_d)\) fits in the framework of Remark 2.6 and we can therefore apply Theorem 2.5 to the centered sequence \((\gamma_n - \frac{1}{d})_{n=1}^\infty\). Upon doing so, we find precisely the statement of Proposition 1.11 \(\square\)

Remark 4.5. A direct combinatorial approach to the moments of the sums \(\gamma_1 + \cdots + \gamma_n\) in the *-probability space \((\mathbb{C}[S_\infty], \varphi_d)\) would involve the counting of factorizations of a given permutation \(\tau \in S_\infty\) as product of a specified number of star-generators. More precisely, for every \(p, n \in \mathbb{N}\) we have (immediately from the definitions) that
\[
\varphi_d\left((\gamma_1 + \cdots + \gamma_n)^p\right) = \sum_{\tau \in S_{n+1}} c_{n,p}(\tau) (1/d)^{||\tau||},
\]
where \(S_{n+1} = \{\tau \in S_\infty \mid \tau(k) = k\text{ for all } k > n + 1\}\) and where for \(\tau \in S_{n+1}\) we put
\[
c_{n,p}(\tau) := \left| \{ \hat{\lambda} : \{1, \ldots, p\} \to \{1, \ldots, n\} \mid \gamma_{\hat{\lambda}(1)} \cdots \gamma_{\hat{\lambda}(p)} = \tau \} \right|.
\]
In connection to this, we record here the intriguing fact that one has precise enumerative formulas for the related cardinalities
\[
\varphi_{c_{n,p}}(\tau) := \left| \{ \hat{\lambda} : \{1, \ldots, p\} \to \{1, \ldots, n\}, \text{ surjective} \mid \gamma_{\hat{\lambda}(1)} \cdots \gamma_{\hat{\lambda}(p)} = \tau \} \right|,
\]
counting the so-called “transitive” factorizations of \(\tau \in S_{n+1}\) into a product of \(p\) star-generators. (See Theorem 1.1 of [9] and the subsequent discussion in Section 1.2.1 of that paper, relating to previous results from [11, 14].) We were not able, however, to use this circle of combinatorial ideas in order to provide an alternative proof for Theorem 1.1 of the present paper. Such an alternative proof may be possible, but let us in any case note that in Equation (1.14) we need to pursue the case when \(p\) is fixed and \(n \to \infty\), while the factorizations into star-generators counted by \(c_{n,p}(\tau)\) cannot be transitive for \(n > p\).

5. Connection to GUE matrices and proof of Proposition 1.12

Notation 5.1. Throughout this section we fix a \(d \in \mathbb{N}\) and we continue to use the framework considered in Section 4. So we consider, same as in Proposition 1.3, the exchangeable sequence \((\gamma_n)_{n=1}^\infty\) in the *-probability space \((\mathbb{C}[S_\infty], \varphi_d)\). Moreover, same as in Notation 1.10 of the Introduction, we let \(u_d : \sqcup_{k=1}^\infty P(k) \to \mathbb{C}\) be the function on partitions associated to \((\gamma_n)_{n=1}^\infty\), and we let \(\nu^{(\infty,d)} : \mathbb{C}\langle \{X_p \mid p \in \mathbb{N}\} \rangle \to \mathbb{C}\) be the linear functional constructed by using the function \(u_d\).

The goal of the section is to point out a connection between the exchangeable sequence of (non-centered!) star-generators \((\gamma_n)_{n=1}^\infty\) and a (centered) exchangeable sequence which arises naturally in connection to GUE matrices.

Remark and Notation 5.2. (Sequence of \(d \times d\) GUE matrices.)
Let \((\Omega, \mathcal{F}, P)\) be a probability space, let \(L^\infty(\Omega, \mathcal{F}, P)\) be the algebra of complex random variables with finite moments of all orders on \(\Omega\), and let \(E : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{C}\) be the expectation functional. Given a \(d \in \mathbb{N}\) we can then consider the *-probability space
\[
\left(M_d\left(L^\infty(\Omega, \mathcal{F}, P)\right), E \circ \text{tr}_d\right),
\]
where \( \mathcal{M}_d(L^\infty(\Omega, \mathcal{F}, P)) \) is the \(*\)-algebra of \( d \times d \) matrices with entries from \( L^\infty(\Omega, \mathcal{F}, P) \) and where \( \text{tr}_d : \mathcal{M}_d(L^\infty(\Omega, \mathcal{F}, P)) \to L^\infty(\Omega, \mathcal{F}, P) \) is the normalized trace.

Suppose now that in \( L^\infty(\Omega, \mathcal{F}, P) \) we have a countable family of independent Gaussian random variables, denoted as

\[
\{\xi_{i,j}^{(n)} \mid 1 \leq i \leq j \leq d, \ n \in \mathbb{N}\} \cup \{\eta_{i,j}^{(n)} \mid 1 \leq i < j \leq d, \ n \in \mathbb{N}\},
\]

where all the random variables in (5.2) are centered and have variances given by

\[
\text{Var}(\xi_{i,i}^{(n)}) = \frac{1}{d}, \quad \forall 1 \leq i \leq d, \ n \in \mathbb{N}, \quad \text{Var}(\xi_{i,j}^{(n)}) = \frac{1}{2d}, \quad \forall 1 \leq i < j \leq d, \ n \in \mathbb{N}.
\]

For every \( n \in \mathbb{N} \), consider the selfadjoint matrix \( M_n \in \mathcal{M}_d(\mathbb{L}_\infty(\Omega, \mathcal{F}, P)) \) with entries described as follows:

- for every \( 1 \leq i \leq d \), the \((i, i)\)-entry of \( M_n \) is \( \xi_{i,i}^{(n)} \);
- for every \( 1 \leq i < j \leq d \), the \((i, j)\)-entry of \( M_n \) is \( \xi_{i,j}^{(n)} + \sqrt{-1}\eta_{i,j}^{(n)} \),
- and the \((j, i)\)-entry of \( M_n \) is \( \xi_{i,j}^{(n)} - \sqrt{-1}\eta_{i,j}^{(n)} \).

These \( M_n \)'s form what is called a sequence of random \( d \times d \) GUE matrices of variance 1, with independent entries.

For the present paper it will be of relevance to note that \( (M_n)_{n=1}^\infty \) is an exchangeable sequence of selfadjoint elements in the \(*\)-probability space \((\mathbb{L}_\infty, \mathcal{F}, P)\). One has, moreover, a precise formula for computing joint moments of \( M_n \)'s, which is obtained by invoking the so-called “Wick formula” for moments of Gaussian random variables, and is reviewed in Remark 5.4 below. The formula presented in Remark 5.4 will refer to some specific permutations in \( S_\infty \), so we first take a moment to give names to these permutations.

**Notation 5.3.** (1) Let \( k = 2h \) be an even positive integer and let \( \pi = \{a_1, b_1, \ldots, a_h, b_h\} \) be a pair-partition in \( \mathcal{P}_2(k) \). We will denote

\[
p_{\pi} := (a_1, b_1) \cdots (a_h, b_h) \in S_\infty
\]

(commuting product of \( h \) disjoint transpositions, corresponding to the \( h \) pairs in \( \pi \)).

(2) For every \( k \in \mathbb{N} \) we will denote

\[
c_{1 \to k} := (1, 2, \ldots, k) \in S_\infty \quad \text{and} \quad c_{k \to 1} := c_{1 \to k}^{-1} = (k, \ldots, 2, 1) \in S_\infty.
\]

In particular, \( c_{1 \to 1} \) is the identity permutation of \( \mathbb{N} \), and \( c_{1 \to 2} \) is the first star-generator \( \gamma_1 \).

**Remark 5.4.** (GUE moments via the Wick formula.) We now consider again the GUE matrices \( (M_n)_{n=1}^\infty \) from Notation 5.2 which we view as a sequence of selfadjoint elements in the \(*\)-probability space from (5.1). The joint moments of the \( M_n \)'s can be described as follows: for every \( k \in \mathbb{N} \) and \( \tilde{\pi} = (\tilde{\pi}(1), \ldots, \tilde{\pi}(k)) \in \mathbb{N}^k \), one has

\[
(E \circ \text{tr}_d)(M_{\tilde{\pi}(1)} \cdots M_{\tilde{\pi}(k)}) = \frac{1}{d^{(k+2)/2}} \sum_{\pi \in \mathcal{P}_2(k), \, \pi \leq \text{Ker}(\tilde{\pi})} \text{d}^\#(c_{1 \to k} P_{\pi}|\{1, \ldots, k\}).
\]

On the right-hand side of Equation (5.3) we have used the convention introduced in Notation 4.1.3: \( "\#(c_{1 \to k} P_{\pi}|\{1, \ldots, k\})" \) stands for the number of orbits into which the permutation \( c_{1 \to k} P_{\pi} \in S_\infty \) breaks the invariant finite set \( \{1, \ldots, k\} \).

Some concrete examples: when we evaluate \( (E \circ \text{tr}_d)(M_1 M_2 M_1 M_2) \), the sum on the right-hand side of (5.3) has only one term, which is \( d^1 \), as one sees upon multiplying \( (1, 2, 3, 4) \cdot (1, 3)(2, 4) = (4, 3, 2, 1) \in S_\infty \). But when we evaluate \( (E \circ \text{tr}_d)(M_1^4) \), the sum on the right-hand side of (5.3) has 3 terms, corresponding to the 3 pairings in \( \mathcal{P}_2(4) \); upon performing
the suitable multiplications in $S_\infty$ we find that these 3 terms add up to $2d^3 + d$. Equation (5.3) thus tells us that:
\[(E \circ \text{tr}_d)(M_1 M_2 M_1 M_2) = \frac{1}{d^2}, \text{ while } (E \circ \text{tr}_d)(M_1^4) = 2 + \frac{1}{d^2}.\]

Note that Equation (5.3) gives in particular an explicit writing for the moments of the common distribution of the $M_n$'s in the $*$-probability space (5.1). This distribution is called the average empirical eigenvalue distribution of a GUE and was denoted by $\nu_d$ in the statement of Theorem 1.1 in the Introduction. More precisely, one finds that for every $k \in \mathbb{N}$ one has
\[\int_{\mathbb{R}} t^k d\nu_d(t) = (E \circ \text{tr}_d)(M_1^k) = \frac{1}{d(k+2)/2} \sum_{\pi \in \mathcal{P}_2(k)} q^{\#(\{1, \ldots, k\})}.\]

Indeed, if $\rho \in P(k)$, then the sum on the right-hand side of (5.6) has only one term, corresponding to $\rho$ itself.

For a presentation of how the Wick formula is used in order to derive Equation (5.3), the reader can consult for instance the survey paper [15]. (See Theorem 2.7 and the calculation preceding it in [15], where one must also insert the superscript indices “$(n)$” used in our Notation 5.2 in the description of the entries of $M_n$.)

**Remark and Notation 5.5.** Since the summation on the right-hand side of Equation (5.3) only depends on the partition $\text{Ker}(\pi)$ (rather than depending on $\pi$ itself), it is clear that $(M_n)_{n=1}^{\infty}$ is an exchangeable sequence in the $*$-probability space (5.1). We will use the notation
\[\nu_d : \bigcup_{k=1}^{\infty} \mathcal{P}(k) \rightarrow \mathbb{C}\]
for the function on partitions associated to this exchangeable sequence. Equation (5.3) provides us with an explicit formula for $\nu_d$, which reads as follows: for every $k \in \mathbb{N}$ and $\rho \in \mathcal{P}(k)$, one has
\[\nu_d(\rho) = \sum_{\pi \in \mathcal{P}_2(k), \pi \leq \rho} (1/d)^{(k+2)/2 - \#(\{1, \ldots, k\})}.\]

As a special case of Equation (5.6), let us observe that $\nu_d(\rho) = 0$ whenever it is not true that all blocks of $\rho$ have even cardinality (since in that case the sum on the right-hand side of (5.6) is empty). In particular it follows that $\nu_d(\rho) = 0$ whenever $\rho$ has at least one block of cardinality 1, and we thus see that $(M_n)_{n=1}^{\infty}$ fulfills all the conditions which are needed in the CLT for exchangeable sequences, Theorem 2.5 in the Introduction. More precisely, one finds that for every $k \in \mathbb{N}$ and $\rho \in \mathcal{P}(k)$, one has
\[\nu_d(\rho) = \sum_{\pi \in \mathcal{P}_2(k), \pi \leq \rho} (1/d)^{(k+2)/2 - \#(\{1, \ldots, k\})}.\]

It comes in handy to also record here that for a pair-partition $\pi \in \mathcal{P}_2(k)$ one has
\[\nu_d(\pi) = (1/d)^{(k+2)/2 - \#(\{1, \ldots, k\})}.\]

Indeed, if $\pi$ is a pair-partition, then the sum on the right-hand side of (5.6) has only one term, corresponding to $\pi$ itself.

The main point of the present section is to observe a connection between the functions on partitions $\nu_d$ (which was just defined) and $\mu_d$ (corresponding to the star-generators $(\gamma_n)_{n=1}^{\infty}$ of $S_\infty$, and coming from the Notation 1.10(2) in the Introduction). Of course, $\nu_d$ and $\mu_d$ cannot coincide, since evaluating them at the unique partition $\pi_1 \in \mathcal{P}(1)$ gives $\nu_d(\pi_1) = 1/d$ (the common value $\varphi_d(\gamma_n)$ for all $n \in \mathbb{N}$) and $\nu_d(\pi_1) = 0$ (the common value $(E \circ \text{tr}_d)(M_n)$ for all $n \in \mathbb{N}$). But nevertheless, it turns out that we have the following proposition.
Proposition 5.6. The functions on partitions $v_d$ (from Equation (5.6)) and $u_d$ (from Notation 1.10(2)) coincide on $\bigcup_{k=1}^{\infty} P_2(k)$.

If we assume Proposition 5.6, then it is easy to write down the proof (announced in the title of the section) for Proposition 1.12.

5.7. Proof of Proposition 1.12 by assuming Proposition 5.6.

The linear functional $\nu^{(\infty,d)} : \mathbb{C}(\{X_p \mid p \in \mathbb{N}\}) \to \mathbb{C}$ was introduced in Notation 1.10(1) in reference to the function on partitions $u_d$. But the definition of $\nu^{(\infty,d)}$ (following the recipe from Definition 1.8(2)) actually uses only the values of $u_d$ on pair-partitions. Due to Proposition 5.6, the linear functional $\nu^{(\infty,d)}$ can then be viewed as coming from the function on partitions $v_d$, hence from the exchangeable sequence $(M_n)_{n=1}^{\infty}$. It was noticed in Remark 5.5 that $(M_n)_{n=1}^{\infty}$ and $v_d$ satisfy the hypotheses of the exchangeable CLT, Theorem 2.5. This theorem implies that $\nu^{(\infty,d)}$ is a positive linear functional.

Moreover, let us fix an $r \in \mathbb{N}$. For every even $k \in \mathbb{N}$ and $\underline{i} = (\underline{i}(1), \ldots, \underline{i}(k)) \in \{1, \ldots, r\}^k$, the recipe used for writing $\nu^{(\infty,d)}(X_{\underline{i}(1)} \cdots X_{\underline{i}(k)})$ in terms of $v_d$ (cf. Definition 1.8(2)), followed by the explicit formula for $v_d(\pi)$ in Equation (5.7) give us that

$$\nu^{(\infty,d)}(X_{\underline{i}(1)} \cdots X_{\underline{i}(k)}) = \sum_{\pi \in P_2(k), \pi \leq Ker(\underline{i})} (1/d)^{(k+2)/2-\#(c_{1\to k}P_\pi\{1,\ldots,k\})}.$$  

But the right-hand side of the latter equation can be continued with

$$= (E \circ tr_d)(M_{\underline{i}(1)} \cdots M_{\underline{i}(k)}) \quad \text{(by Equation (5.3))}$$
$$= \nu^{(r,d)}(X_{\underline{i}(1)} \cdots X_{\underline{i}(k)}) \quad \text{(by the definition of $\nu^{(r,d)}$, Equation (1.12)).}$$

The equality $\nu^{(\infty,d)}(X_{\underline{i}(1)} \cdots X_{\underline{i}(k)}) = \nu^{(r,d)}(X_{\underline{i}(1)} \cdots X_{\underline{i}(k)})$ also holds for any odd $k$ and $\underline{i} = (\underline{i}(1), \ldots, \underline{i}(k)) \in \{1, \ldots, r\}^k$, when both sides of the equality are equal to 0. This shows that the restriction of $\nu^{(\infty,d)}$ to $\mathbb{C}(X_1, \ldots, X_r)$ is equal to $\nu^{(r,d)}$, as required.

We are left with the job of proving Proposition 5.6. We start on this job by observing an explicit formula for how $u_d$ acts on pair-partitions. This will be, in a certain sense, parallel to the formula recorded for $v_d$ in Equation (5.7), only that we need to use a different way of constructing a permutation out of a pair-partition.

Notation 5.8. Let $k = 2h$ be an even positive integer, and let $\pi$ be a pair-partition in $P_2(k)$. We consider the unique way of writing $\pi$ in the form

$$\pi = \{V_1, \ldots, V_h\}, \quad \text{with} \quad V_1 = \{a_1, b_1\}, \ldots, V_h = \{a_h, b_h\}$$
$$\text{where} \quad a_1 < b_1, \ldots, a_h < b_h \quad \text{and} \quad a_1 < a_2 < \cdots < a_h.$$

We will use the notation $q_\pi$ for the permutation

$$q_\pi := \gamma_\underline{i}(1)\gamma_\underline{i}(2) \cdots \gamma_\underline{i}(2h) \in S_{\infty},$$

where the tuple $\underline{i} : \{1, \ldots, 2h\} \to \{1, \ldots, h\}$ is defined by putting

$$\underline{i}(a_1) = \underline{i}(b_1) = 1, \ldots, \underline{i}(a_h) = \underline{i}(b_h) = h.$$  

It is useful to keep in mind that: since the product defining $q_\pi$ in Equation (5.9) only uses the star-generators $\gamma_1 = (1, 2), \ldots, \gamma_\hbar = (1, h+1)$, one has $q_\pi(m) = m$ for all $m > h + 1$. 

which appeared in the construction of $p_\pi$ associated to $\pi$ in Notation 5.3, which was simply $p_\pi = (1,5)(2,4)(3,7)(6,8) \in S_\infty$.

**Lemma 5.9.** For every even $k = 2h \in \mathbb{N}$ and $\pi \in \mathcal{P}_2(k)$ one has:

$$u_d(\pi) = (1/d)^{(h+1)-\#(q_\pi|\{1,...,h\})}. \tag{5.11}$$

*Proof.* The tuple $\vec{i}$ which appeared in the construction of $q_\pi$ (cf. Equation 5.10) has $\text{Ker}(\vec{i}) = \pi$; hence, directly from the definition of $u_d$, we get that

$$u_d(\pi) = \varphi_d(\gamma_1(4) \cdots \gamma_2(2h)).$$

This can be followed with

$$= \varphi_d(q_\pi) = (1/d)^{|q_\pi|} = (1/d)^{(h+1)-\#(q_\pi|\{1,...,h\})},$$

where the third equality invokes the description of $|q_\pi|$ provided by Remark 4.1(3). \qed

**Remark 5.10.** (1) For any even $k = 2h \in \mathbb{N}$ and $\pi \in \mathcal{P}_2(k)$, we now have both values $v_d(\pi)$ and $u_d(\pi)$ written as powers of $1/d$, with non-negative integer exponents (in Equations (5.7) and (5.11), respectively). The proof of Proposition 5.9 is thus reduced to checking the equality of the said exponents. That is, we are left to verify the following combinatorial formula relating the permutations $p_\pi$ and $q_\pi$:

$$h+1 - \#(c_1 \rightarrow 2h p_\pi | \{1, \ldots, 2h\}) = h+1 - \#(q_\pi|\{1, \ldots, h\}), \tag{5.12}$$

(2) It is instructive to have a look at the special case when the pair-partition $\pi \in \mathcal{P}_2(k)$ is non-crossing. In this case one gets $v_d(\pi) = u_d(\pi) = 1$, that is, both sides of Equation (5.12) are equal to 0. This is immediate on the right-hand side, where it is easily seen that if $\pi$ is non-crossing then $q_\pi$ must be the identity permutation in $S_\infty$. (Indeed, if $\pi$ is non-crossing then in the canonical writing (5.8) of $\pi$ there has to be a block $V_i = \{a_i, b_i\}$ with $b_i = 1 + a_i$; so the product (5.9) which defines $q_\pi$ has two adjacent occurrences of $\gamma_i$—remove them and continue by induction.) The fact that the left-hand side of (5.12) is also equal to 0 for a non-crossing $\pi$ is less immediate; it can for instance be explained by using the notion of Kreweras complement of $\pi$, and the known formula for the number of blocks of the Kreweras complement (see e.g. pages 147-148 in Lecture 9).

(3) We mention that yet another function on pair-partitions, related to the same $*$-probability space $(\mathbb{C}[S_\infty], \varphi_d)$ as considered in the present paper, was studied in [4] (see Theorem 3.4 of [4], where the sequence $\alpha_1, \alpha_2, \ldots$ of the theorem has to be specialized to $\alpha_1 = \cdots = \alpha_d = 1/d$ and $\alpha_i = 0$ for $i > d$). Denoting the function on pair-partitions studied in [4] by $\text{“}w_d\text{”}$, it is also the case that for every even $k \in \mathbb{N}$ and $\pi \in \mathcal{P}_2(k)$ one has $w_d(\pi) = (1/d)^m$ for some $m \in \mathbb{N} \cup \{0\}$, with $m = 0$ when $\pi$ is non-crossing. But experimenting with small values of $k$ doesn’t suggest a direct connection between $w_d$ and $v_d = u_d$ (for instance in the example of $\pi \in \mathcal{P}_2(8)$ used for illustration at the end of Notation 5.8 one gets $v_d(\pi) = u_d(\pi) = (1/d)^2$ and $w_d(\pi) = (1/d)^3$).

In order to complete the proof of Proposition 5.6, the ingredient still needed is a good understanding of how the permutation $q_\pi$ works, for a pairing $\pi \in \mathcal{P}_2(k)$. We look more carefully at this, in the next remark.
Remark 5.11. Let $\pi \in \mathcal{P}_2(k)$ for an even $k = 2h \in \mathbb{N}$, in reference to which we use the notation considered above. As noticed at the end of Notation 5.8, $q_\pi$ can only move the numbers from $\{1, \ldots, h+1\}$. It is instructive to examine in detail how $q_\pi$ acts on a specified $j \in \{1, \ldots, h+1\}$. We will look at a $j \neq 1$ (the case $j = 1$ is only slightly different from the others).

Among $\gamma_1, \ldots, \gamma_h$, the only transposition that actually moves $j$ is $\gamma_{j-1}$, which appears in the product \((5.9)\) on positions $a_{j-1}$ and $b_{j-1}$ (here the block $V_{j-1} = \{a_{j-1}, b_{j-1}\}$ is as in the explicit writing of $\pi$ from \((5.8)\)). When we successively apply the factors $\gamma_i(2h), \gamma_i(2h-1), \ldots$ from \((5.9)\) to $j$, the first time when $j$ is actually moved thus occurs when we do \[(5.13)\]

$$
\gamma_{\ell}(b_{j-1})(j) = \gamma_{j-1}(j) = 1.
$$

The value 1 is then immediately moved by the next factor (reading from right to left) in the product, $\gamma_{\ell}(b_{j-1}-1)$, and there are several possible cases for how this can go:

Case 1. $b_{j-1} - 1$ still belongs to $V_{j-1}$, that is, we have $b_{j-1} - 1 = a_{j-1}$.

In this case we get $\gamma_{\ell}(b_{j-1}-1)(1) = \gamma_{\ell}(a_{j-1})(1) = \gamma_{j-1}(1) = j$, and it follows that $j$ is a fixed point of $q_\pi$, since none of the factors to the left of $\gamma_{\ell}(a_{j-1})$ in the product \((5.9)\) can move $j$.

Case 2. $b_{j-1} - 1$ belongs to $V_{i-1}$ for an $i \neq j$ in $\{2, \ldots, h+1\}$, and $b_{j-1} - 1 = a_{i-1}$.

In this case we get $\gamma_{\ell}(b_{j-1}-1)(1) = \gamma_{\ell}(a_{i-1})(1) = \gamma_{i-1}(1) = i$, and also that $q_\pi(j) = i$, since none of the factors to the left of $\gamma_{\ell}(a_{i-1})$ in the product \((5.9)\) can move $i$.

Case 3. $b_{j-1} - 1$ belongs to $V_{i-1}$ for an $i \neq j$ in $\{2, \ldots, h+1\}$, and $b_{j-1} - 1 = b_{l-1}$.

In this case we still get (same as in Case 2) $\gamma_{\ell}(b_{j-1}-1)(1) = i$, but we cannot yet decide what is $q_\pi(j)$, since the number $i$ will be moved by the later factor $\gamma_{\ell}(a_{i-1})$ of the product \((5.9)\). More precisely, Case 3 can be divided into subcases according to the status of $a_{i-1}$, as follows.

Case 3-1. $a_{i-1} = 1$ (which, according to the convention from \((5.8)\), means that $i = 2$).

Then $\gamma_{\ell}(a_{i-1})(i) = \gamma_{i-1}(i) = 1$, and we conclude that $q_\pi(j) = 1$.

Case 3-2. $a_{i-1} \neq 1$, and $a_{i-1} - 1 = a_{\ell-1}$ for \footnote{The value of $\ell$ appearing here is sure to be such that $\ell \neq i$, but it is not ruled out that we have $\ell = j$. In the latter case, the outcome of Case 3-2 is that $j$ is a fixed point for $q_\pi$.} an $\ell \in \{2, \ldots, h+1\}$.

In this subcase we find that $\gamma_{\ell}(a_{i-1})(i) = \gamma_{i-1}(i) = 1$, followed by $\gamma_{\ell}(a_{i-1})(1) = \gamma_{\ell-1}(1) = \ell$. At this point we can conclude that $q_\pi(j) = \ell$, because $\ell$ is no longer moved by the remaining factors (to the left of $\gamma_{\ell}(a_{i-1})$) to be considered in the product \((5.10)\).

Case 3-3. $a_{i-1} \neq 1$, and $a_{i-1} - 1 = b_{k-1}$ for an $\ell \in \{2, \ldots, h+1\}$.

In this subcase we find, same as in Case 3-2, that $\gamma_{\ell}(a_{i-1})(i) = 1$, which is now followed by $\gamma_{\ell}(b_{k-1})(1) = \ell$. But unlike in Case 3-2, in order to continue the discussion towards the determination of $q_\pi(j)$, we need to make a further subdivision into subcases. Indeed, what we must do is look at the number $a_{\ell-1} < b_{\ell-1}$ and break again into three subcases (which could be numbered as Cases 3-3-1, 3-3-2 and 3-3-3) according to whether $a_{\ell-1} = 1$, or $a_{\ell-1} - 1 = a_{m-1}$ for some $m \in \{2, \ldots, h+1\}$, or $a_{\ell-1} - 1 = b_{m-1}$ for some $m \in \{2, \ldots, h+1\}$.

The conclusion of all this discussion will be recorded in Lemma 5.13 below. In order to state the lemma, we first clarify the convention for how to restrict permutations to subsets that aren’t necessarily invariant.

Notation 5.12. Let $\tau$ be a permutation in $S_{\infty}$, and let $A$ be a finite non-empty subset of $\mathbb{N}$. We do not assume that $A$ is invariant for $\tau$, but let us note that there still exists a
natural bijection \( \theta : A \to A \) which one could call “permutation of \( A \) induced by \( \tau \)”, and is described as follows.

Let \( a \) be a number in \( A \). We look at the sequence of values \( v_1, v_2, \ldots, v_k, \ldots \) in \( \mathbb{N} \) obtained by putting

\[
(5.14) \quad v_1 = \tau(a), v_2 = \tau(v_1), \ldots, v_k = \tau(v_{k-1}), \ldots
\]

and we define \( \theta(a) := v_{k_0} \in A \) where \( k_0 := \min\{k \in \mathbb{N} \mid v_k \in A\} \). In words: the \( v_k \)'s in (5.14) follow the orbit of \( \tau \) which contains \( a \), and \( \theta(a) \) is the first re-entry in \( A \) which is encountered along that orbit.

We leave to the reader the (fairly straightforward but nevertheless tedious) job to verify that, upon re-reading and suitably expanding the multi-case discussion about “how to find out what is \( q_\pi(j) \)” from Remark 5.11 one arrives to the following formal statement.

**Lemma 5.13.** Let \( k = 2h \) be an even positive integer and let \( \pi = \{V_1, \ldots, V_h\} \) be a pair-partition in \( \mathcal{P}_2(2h) \), where we write explicitly \( V_1 = \{a_1, b_1\}, \ldots, V_h = \{a_h, b_h\} \), with \( a_1 < b_1, \ldots, a_h < b_h \) and with \( 1 = a_1 < \cdots < a_h \). We also put \( b_0 := 2h + 1 \). We consider two permutations \( \theta_1, \theta_2 \) of sets of cardinality \( h + 1 \), as follows.

- **On the one hand,** let \( q_\pi \in S_\infty \) be defined as in Equation (5.9) of Notation 5.8 and let \( \theta_1 \) be the restriction of \( q_\pi \) to its invariant set \( \{1, \ldots, h + 1\} \).

- **On the other hand** let us consider the permutations \( p_\pi \) and \( c_{2h+1} \) defined in Notation 5.12 and let \( \theta_2 \) be the permutation induced (in the sense of Notation 5.8) by the product \( p_\pi \cdot c_{2h+1} \in S_\infty \) on the finite set \( \{b_0, b_1, \ldots, b_h\} \subseteq \mathbb{N} \).

Then \( \theta_1 \) and \( \theta_2 \) are conjugated by the bijection \( \{1, 2, \ldots, h + 1\} \to \{b_0, b_1, \ldots, b_h\} \) which maps \( i \mapsto b_{i-1} \) for every \( 1 \leq i \leq h + 1 \).

In the framework of the preceding lemma, it will be useful to also have on record the following observation.

**Lemma 5.14.** Consider the same notations as in Lemma 5.13. Suppose \( R \) is an orbit of the permutation \( p_\pi \cdot c_{2h+1} \), such that \( R \subseteq \{1, 2, \ldots, 2h + 1\} \). Then \( R \cap \{b_0, b_1, \ldots, b_h\} \neq \emptyset \).

**Proof.** Assume for contradiction that \( R \cap \{b_0, b_1, \ldots, b_h\} = \emptyset \). This implies that

\[
R \subseteq \{1, 2, \ldots, 2h + 1\} \setminus \{b_0, b_1, \ldots, b_h\} = \{a_1, \ldots, a_h\}.
\]

Let us observe that, due to the convention of how \( a_1, \ldots, a_h \) are chosen, we have the implication

\[
\left( \frac{2 \leq i \leq h}{c_{2h+1}}(a_i) = a_j \right) \Rightarrow a_j < a_i \Rightarrow j < i.
\]

By starting with an \( a_i \in R \) and by iterating the above observation, we must eventually get that \( a_1 \in R \). But \( a_1 = 1 \), and \( (p_\pi \cdot c_{2h+1})(1) = p_\pi(2h + 1) = 2h + 1 = b_0 \). It follows that \( b_0 \in R \), contradiction.

\[
\square
\]

5.15. **Proof of Proposition 5.6.** We fix an even positive integer \( k = 2h \) and a pair-partition \( \pi \in \mathcal{P}_2(2h) \). We consider the canonical writing \( \pi = \{\{a_1, b_1\}, \ldots, \{a_h, b_h\}\} \) described in Notation 5.8. As observed in Remark 5.10(1), we are left to verify a combinatorial identity, coming to:

\[
(5.15) \quad \#(q_\pi \mid \{1, \ldots, h + 1\}) = \#(p_\pi \cdot c_{1 \to 2h} \mid \{1, \ldots, 2h\}).
\]
We will do this verification by checking that both sides of Equation (5.15) are equal to (5.16)
\[
\# \left( p_\pi \cdot c_{2h+1 \rightarrow 1} \mid \{1, \ldots, 2h+1\} \right).
\]

**Verification that the left-hand side of (5.15) is equal to (5.16).** Same as in Lemma 5.13 besides the numbers \(b_1, \ldots, b_h\) considered in the canonical writing of \(\pi\) we also put \(b_0 := 2h + 1\). Lemma 5.13 says that the permutation induced by \(p_\pi \cdot c_{2h+1 \rightarrow 1}\) on \(\{b_0, b_1, \ldots, b_h\}\) is precisely what one obtains by starting from \(q_\pi \mid \{1, \ldots, h+1\}\) and by doing the identification \(i \leftrightarrow b_{i-1}, 1 \leq i \leq h + 1\). This implies that \(\#(q_\pi \mid \{1, \ldots, h+1\})\) is equal to the number of orbits of \(p_\pi \cdot c_{2h+1 \rightarrow 1}\) which intersect \(\{b_0, b_1, \ldots, b_h\}\). However, Lemma 5.14 assures us that these are all the orbits of \(p_\pi \cdot c_{2h+1 \rightarrow 1} \mid \{1, \ldots, 2h + 1\}\) (and this concludes the required verification).

**Verification that the right-hand side of (5.15) is equal to (5.16).** Here we first observe the equality
\[
\#(p_\pi \cdot c_{2h+1 \rightarrow 1} \mid \{1, \ldots, 2h+1\}) = \#(p_\pi c_1 \rightarrow 2h+1 \mid \{1, \ldots, 2h+1\}),
\]
which is an immediate consequence of the fact that \(p_\pi\) is its own inverse. So it suffices to verify the equality between the right-hand side of (5.15) and the right-hand side of (5.17). The latter equality comes out of the following observation: the action of the permutation \(p_\pi \cdot c_1 \rightarrow 2h+1\) is such that \(2h \mapsto 2h + 1 \mapsto b_1\);
this shows that \(p_\pi \cdot c_1 \rightarrow 2h+1 \mid \{1, \ldots, 2h + 1\}\) can be obtained by starting from \(p_\pi \cdot c_1 \rightarrow 2h \mid \{1, \ldots, 2h\}\) and by inserting the number \(2h + 1\) in the cycle which contains \(2h\) (from "(\ldots, 2h, b_1, \ldots)\", that cycle becomes "(\ldots, 2h, 2h + 1, b_1, \ldots)\”). It follows that \(p_\pi \cdot c_1 \rightarrow 2h+1 \mid \{1, \ldots, 2h + 1\}\) and \(p_\pi \cdot c_1 \rightarrow 2h \mid \{1, \ldots, 2h\}\) have the same number of orbits, as required. \(\Box\)

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