GLOBAL EXISTENCE RESULTS FOR THE NAVIER-STOKES EQUATIONS IN THE ROTATIONAL FRAMEWORK

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ABSTRACT. Consider the equations of Navier-Stokes in \( \mathbb{R}^3 \) in the rotational setting, i.e. with Coriolis force. It is shown that this set of equations admits a unique, global mild solution provided the initial data is small with respect to the norm the Fourier-Besov space \( \dot{FB}^{3-3/p}_{p,r}(\mathbb{R}^3) \), where \( p \in (1, \infty] \) and \( r \in [1, \infty] \).

In the two-dimensional setting, a unique, global mild solution to this set of equations exists for non-small initial data \( u_0 \in L^p_\sigma(\mathbb{R}^2) \) for \( p \in [2, \infty) \).

1. Introduction and Main Results

Consider the flow of an incompressible, viscous fluid in \( \mathbb{R}^3 \) in the rotational framework which is described by the following set of equations

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \Delta u + \Omega e_3 \times u + \nabla \pi &= 0, \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\
\text{div} u &= 0, \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\
u(0) &= u_0, \quad \text{in } \mathbb{R}^3.
\end{aligned}
\]

Here, \( u \) and \( \pi \) represent the velocity and pressure of the fluid, respectively, and \( \Omega \in \mathbb{R} \) denotes the speed of rotation around the unit vector \( e_3 = (0, 0, 1) \) in \( x_3 \)-direction. If \( \Omega = 0 \), the system reduces to the classical Navier-Stokes system.

This set of equations recently gained quite some attention due to its importance in applications to geophysical flows. In particular, large scale atmospheric and oceanic flows are dominated by rotational effects, see e.g. [17] or [6].

If \( \Omega = 0 \), the classical Navier-Stokes equations have been considered by many authors in various scaling invariant spaces, in particular in

\[
\dot{H}^\frac{3}{p}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow B^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3) \hookrightarrow BMO^{-1}(\mathbb{R}^3) \hookrightarrow B^{-1}_\infty(\mathbb{R}^3),
\]

where \( 3 < p < \infty \). The space \( BMO^{-1}(\mathbb{R}^3) \) is the largest scaling invariant space known for which equation (1.1) with \( \Omega = 0 \) is well-posed.

It is a very remarkable fact that the equation (1.1) allows a global, mild solution for arbitrary large data in the \( L^2 \)-setting provided the speed \( \Omega \) of rotation is fast enough, see [2], [3] and [6]. More precisely, it was proved by Chemin, Desjardins, Gallagher and Grenier in [6] that for initial data \( u_0 \in L^2(\mathbb{R}^2)^3 + H^{1/2}(\mathbb{R}^3)^3 \) satisfying \( \text{div} u_0 = 0 \), there exists a constant \( \Omega_0 > 0 \) such that for every \( \Omega \geq \Omega_0 \) the equation (1.1) admits a unique, global mild solution. The case of periodic initial data was considered before by Babin, Mahalov and Nicolaenko in the papers [2] and [3].

It is now a natural question to ask whether, for given and fixed \( \Omega > 0 \), there exists a unique, global mild solution to (1.1) provided the initial data is sufficiently small with respect to the above or related norms. In this context it is natural to extend the classical Fujita-Kato approach for the Navier-Stokes equations to the rotational setting. Hieber and Shibata considered in [14] the case of initial data belonging to \( \dot{H}^\frac{3}{p}(\mathbb{R}^3) \) and proved a global well-posedness result for (1.1) for initial data being small.
with respect to $H^2_\Omega(\mathbb{R}^3)$. Generalizations of this result to the case of Fourier-Besov spaces are due to Konieczny and Yoneda [16] and Iwabuchi and Takada [18].

More precisely, Konieczny and Yoneda proved the existence of a unique global mild solution to (1.1) for initial data $u_0$ being small with respect to the norm of $\dot{FB}^{2-\frac{2}{p}}_{p,\infty}(\mathbb{R}^3)$, where $1 < p \leq \infty$. For the case $p = 1$ considered in [18], the existence of a unique global mild solution was proved provided the initial data $u_0$ are small with respect to $\dot{FB}^{1}_{1,\Omega}(\mathbb{R}^3)$. Moreover, it was shown in [18] that the space $\dot{FB}^{1}_{1,\Omega}(\mathbb{R}^3)$ is critical for the well-posedness of system (1.1). In fact, it was shown in [18] that equation (1.1) is ill-posed in $\dot{FB}^{-1}_{1,\Omega}(\mathbb{R}^3)$ whenever $2 < q \leq \infty$ and $\Omega \in \mathbb{R}$.

Giga, Inui, Mahalov and Saal considered in [11] the problem of non-decaying initial data and obtained the uniform global solvability of (1.1) in the scaling invariant space $FM^{-1}_0(\mathbb{R}^3)$. For details, see [11] and [12]. Note that all of these results rely on good mapping properties of the Stokes-Coriolis semigroups on these function spaces.

It seems to be unknown, whether global existence results are also true for initial data $u_0$ being small with respect to $L^p(\mathbb{R}^3)$ for $p \geq 3$. The main difficulty here is that Mikhlin’s theorem applied to the vorticity equation allows then for a global estimate in two dimensions which can used to control the term $\nabla u$. Our argument is based on applying the curl operator to equation (1.1). The resulting vorticity equation allows then for a global estimate in two dimensions which can used to control the term $\nabla u$ in the $L^p$-norm.

In order to formulate our first result, let us recall the definition of Fourier-Besov spaces. To this end, let $\varphi$ be a $C^\infty$ function satisfying $\supp \varphi \subset \{3/4 \leq |\xi| \leq 8/3\}$ and

$$
\sum_{k \in \mathbb{Z}} \varphi(2^{-k} \xi) = 1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.
$$

For $k \in \mathbb{Z}$, set $\varphi_k(\xi) = \varphi(2^{-k} \xi)$ and $h_k = \mathcal{F}^{-1} \varphi_k$. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the space $\dot{FB}^s_{p,r}(\mathbb{R}^3)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\hat{f} \in L^1_{loc}(\mathbb{R}^3)$ and

$$
\|f\|_{\dot{FB}^s_{p,r}} := \left\| \left\{ 2^{js} \| \varphi_j \hat{f} \|_{L^p(\mathbb{R}^3)} \right\}_{j \in \mathbb{Z}} \right\|_r < \infty.
$$

Given $1 \leq q \leq \infty$ and $T \in (0, \infty]$, we also make use of Chemin-Lerner type spaces $\dot{L}^q([0, T]; \dot{FB}^s_{p,r}(\mathbb{R}^3))$, which are defined to be the completion of $C([0, T]; S(\mathbb{R}^3))$ with respect to the norm

$$
\|f\|_{\dot{L}^q([0, T]; \dot{FB}^s_{p,r}(\mathbb{R}^3))} := \left\| \left\{ 2^{js} \| \varphi_j \hat{f} \|_{L^q([0, T]; L^p(\mathbb{R}^3))} \right\}_{j \in \mathbb{Z}} \right\|_r.
$$

We are now in the position to state our first result.

**Theorem 1.1.** Let $\Omega \in \mathbb{R}$ and $1 < p \leq \infty$, $1 \leq r \leq \infty$. Then there exist constants $C > 0$ and $\varepsilon > 0$, independent of $\Omega$, such that for every $u_0 \in \dot{FB}^{2-\frac{2}{p}}_{p,\infty}(\mathbb{R}^3)$ satisfying $\text{div} \, u_0 = 0$ and $\|u_0\|_{\dot{FB}^{2-\frac{2}{p}}_{p,\infty}} \leq \varepsilon$, the
equation (1.1) admits a unique, global mild solution \( u \in X \), where \( X \) is given by
\[
X = \{ u \in C([0, \infty); F^2_{p,r} B_{p,r}^2(\mathbb{R}^3)) : \| u \|_X \leq C\varepsilon, \text{div}\ u = 0 \}
\]
with
\[
\| u \|_X = \| u \|_{L^\infty([0, \infty); F^2_{p,r} B_{p,r}^2(\mathbb{R}^3))} + \| u \|_{L^3([0, \infty); F^2_{p,r} B_{p,r}^2(\mathbb{R}^3))}.
\]

**Remarks 1.2.** a) Observe that due to the results in [18], the above system (1.1) is ill-posed provided \( p = 1 \) and \( r > 2 \).
b) Note that the case \( r = \infty \) coincides with the result of Konieczny and Yoneda in [16].
c) Iwabuchi and Takada [18] recently proved the existence of a unique, global mild solution to equation (1.1) for initial data small with respect to the norm of \( F^2_{1,2} B^{-1} \).
d) Note that neither \( F^2_{1,2} B^{-1} \mathbb{R}^3 \subset F^2_{p,r} B^{-3/p} \mathbb{R}^3 \) for \( r \in [1, \infty] \) nor \( F^2_{p,r} B^{-3/p} \mathbb{R}^3 \subset F^2_{1,2} B^{-1} \mathbb{R}^3 \) for \( r > 2 \).

Our second result concerning non-small data in the \( L^p(\mathbb{R}^2) \)-setting reads as follows. We denote by \( L^p_\#(\mathbb{R}^2) \) the solenoidal subspace of \( L^p(\mathbb{R}^2) \).

**Theorem 1.3.** Let \( 2 \leq p < \infty \) and \( u_0 \in L^p_\#(\mathbb{R}^2) \). Then equation (1.1) admits a unique, global mild solution \( u \in C([0, \infty), L^p_\#(\mathbb{R}^2)) \).

## 2. Linear and Bilinear Estimates

We start this section by considering the linear Stokes problem with Coriolis force
\[
\begin{cases}
\partial_t u - \Delta u + \Omega e_3 \times u + \nabla \pi = 0, & \text{in } \mathbb{R}^3 \times (0, \infty), \\
\text{div}\ u = 0, & \text{in } \mathbb{R}^3 \times (0, \infty), \\
u(0) = u_0, & \text{in } \mathbb{R}^3.
\end{cases}
\]
(2.1)

It was shown in [14] that the solution of (2.1) is given by the Stokes-Coriolis semigroup \( T \), which has the explicit representation
\[
T(t)f := F^{-1} \left[ \cos \left( \frac{\xi_3 t}{|\xi|} \right) e^{-|\xi|^2 t} \text{Id} f(\xi) + \sin \left( \frac{\xi_3 t}{|\xi|} \right) e^{-|\xi|^2 t} R(\xi) \hat{f}(\xi) \right], \quad t > 0,
\]
(2.2)

for divergence free vector fields \( f \in \mathcal{S}(\mathbb{R}^3) \). Here \( \text{Id} \) is the identity matrix in \( \mathbb{R}^3 \) and \( R(\xi) \) is the skew symmetric matrix defined by
\[
R(\xi) := \frac{1}{|\xi|} \begin{pmatrix}
0 & \xi_3 & -\xi_2 \\
-\xi_3 & 0 & \xi_1 \\
\xi_2 & -\xi_1 & 0
\end{pmatrix}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.
\]

In order to solve equation (1.1), consider the integral equation
\[
\Phi(u) := T(t)u_0 - \int_0^t T(t - \tau) \text{div}(u \otimes u)(\tau) d\tau,
\]
where \( \mathbb{P} := (\delta_{ij} + R_i R_j)_{1 \leq i,j \leq 3} \) denotes the Helmholtz projection from \( L^p(\mathbb{R}^3) \) onto its divergence free vector fields. Here \( R_i \) denotes the Riesz transforms for \( i = 1, 2, 3 \). Since the Riesz transforms \( R_i \) are bounded operators on \( F^2_{p,q} \) for all values of \( p, q \in [1, \infty] \) and \( s \in \mathbb{R} \), we see that \( \mathbb{P} \) defines a bounded operator also on these spaces.

Our first estimate concerns the above convolution integral.
Lemma 2.1. Let $1 \leq p, q, a, r \leq \infty$, $s \in \mathbb{R}$ and $f \in L^a([0, \infty); F^s_{p,r}(\mathbb{R}^3))$. Then there exists a constant $C > 0$ such that

$$
\left\| \int_0^t T(t - \tau) f(\tau) d\tau \right\|_{L^s([0, \infty); F^s_{p,r}(\mathbb{R}^3))} \leq C \| f \|_{L^a([0, \infty); F^{s-\frac{2}{q}+\frac{3}{4}}_{p,r}(\mathbb{R}^3))}.
$$

Proof. By the definition of the norm of $L^s([0, \infty); F^s_{p,r}(\mathbb{R}^3))$, and by Young’s inequality

$$
\left\| \int_0^t T(t - \tau) f(\tau) d\tau \right\|_{L^s([0, \infty); F^s_{p,r}(\mathbb{R}^3))} = \left( \sum_k 2^{ksr} \left( \int_0^\infty \left\| f(\tau) \|_{L^p} \right\|^{\frac{q}{s}} \right)^\frac{1}{q} \right)^{\frac{1}{r}}
$$

$$
\leq \left( \sum_k 2^{ksr} \left( \int_0^\infty \left( \int_0^\tau e^{-t-2^k \tau} \left\| f(\tau) \|_{L^p} \right\|^{\frac{q}{s}} \right)^\frac{1}{q} \right)^\frac{1}{r} \right)^{\frac{1}{s}}
$$

where $\tilde{q}$ satisfies $1 + \frac{1}{\tilde{q}} = \frac{1}{q} + \frac{1}{a}$. We hence obtain

$$
\left\| \int_0^t T(t - \tau) f(\tau) d\tau \right\|_{L^s([0, \infty); F^s_{p,r}(\mathbb{R}^3))} \leq C \left( \sum_k 2^{ksr} 2^{-2k(1+\frac{1}{\tilde{q}}-\frac{1}{q})r} \left\| f \|_{L^a([0, \infty); L^p)} \right\|_{L^s([0, \infty); F^{s-\frac{2}{q}+\frac{3}{4}}_{p,r}(\mathbb{R}^3))} \right)^{\frac{1}{r}}
$$

$$
\leq C \| f \|_{L^a([0, \infty); F^{s-\frac{2}{q}+\frac{3}{4}}_{p,r}(\mathbb{R}^3))}.
$$

Lemma 2.2. Let $1 < p \leq \infty$ and $1 \leq p, q, a, r \leq \infty$ and assume that $-1 < s < 3 - \frac{2}{r}$. Set

$$
Y := \tilde{L}^\infty([0, \infty); F^s_{p,r}(\mathbb{R}^3)) \cap \tilde{L}^1([0, \infty); F^{1-s-\frac{2}{r}}_{p,r}(\mathbb{R}^3)).
$$

Then there exists a constant $C > 0$ such that

$$
\|uv\|_{\tilde{L}^1([0, \infty); F^{s+1}_{p,r}(\mathbb{R}^3))} \leq C \|u\|_{Y} \|v\|_{Y}.
$$

Proof. Let $\varphi$ and $h_k$ be defined as in Section 1 for $k \in \mathbb{Z}$. Define the homogeneous dyadic blocks $\hat{\Delta}_k$ by

$$
\hat{\Delta}_k u := \varphi(2^{-k}D) u = \int_{\mathbb{R}^3} h_k(y) u(x-y) dy, \quad k \in \mathbb{Z},
$$

and for $j \in \mathbb{Z}$, set $\hat{S}_j u := \sum_{k=-\infty}^{j} \hat{\Delta}_k u$. We then obtain

$$
\|uv\|_{\tilde{L}^1([0, \infty); F^{s+1}_{p,r}(\mathbb{R}^3))} = \left( \sum_j 2^{j(s+1)r} \left( \int_0^\infty \left\| \hat{\Delta}_j(uv) \|_{L^p} dt \right\| \right)^{\frac{1}{r}}
$$

Using the Bony decomposition [3], [4], and [5], we rewrite $\hat{\Delta}_j(uv)$ as

$$
\hat{\Delta}_j(uv) = \sum_{|k-j| \leq 4} \hat{\Delta}_j(\hat{S}_{k+1} u \hat{\Delta}_k v) + \sum_{|k-j| \leq 4} \hat{\Delta}_j(\hat{S}_{k+1} v \hat{\Delta}_k u) + \sum_{k \geq j-2} \hat{\Delta}_j(\hat{\Delta}_k u \hat{\Delta}_k v) =: I + II + III.
$$
Then, by triangle inequalities in $L^p(\mathbb{R}^3)$ and $l^r(\mathbb{Z})$, we have

$$
\|uv\|_{L^1([0,\infty);FB^{s,r}_{p,r}(\mathbb{R}^3))} \leq \left( \sum_j 2^{j(s+1)r} \left( \int_0^\infty \| \hat{\Delta}_j I \|_{L^p} dt \right)^r \right)^{\frac{1}{r}} + \left( \sum_j 2^{j(s+1)r} \left( \int_0^\infty \| \hat{\Delta}_j II \|_{L^p} dt \right)^r \right)^{\frac{1}{r}} + \left( \sum_j 2^{j(s+1)r} \left( \int_0^\infty \| \hat{\Delta}_j III \|_{L^p} dt \right)^r \right)^{\frac{1}{r}} \\
= J_1 + J_2 + J_3.
$$

For the term $J_1$, we note

$$
J_1 = \left( \sum_j 2^{j(s+1)r} \left( \int_0^\infty \sum_{|k-j| \leq 4} \hat{\Delta}_j (\hat{S}_k u \Delta_k v) \|_{L^p} dt \right)^r \right)^{\frac{1}{r}}.
$$

For fixed $j$, Lemma 24.1 yields

$$
2^{j(s+1)} \int_0^\infty \sum_{|k-j| \leq 4} \hat{\Delta}_j (\hat{S}_k u \Delta_k v) \|_{L^p} dt \\
\leq 2^{j(s+1)} \int_0^\infty \sum_{|k-j| \leq 4} \| \varphi_j (\chi_k \hat{\mu} \ast \varphi_k \hat{\nu}) \|_{L^p} dt \\
\leq 2^{j(s+1)} \int_0^\infty \sum_{|k-j| \leq 4} \| \chi_k \hat{\mu} \|_{L^1} \| \varphi_k \hat{\nu} \|_{L^p} dt \\
\leq 2^{j(s+1)} \int_0^\infty \sum_{|k-j| \leq 4} \| \varphi_k \hat{\nu} \|_{L^p} \| \varphi_k \hat{\nu} \|_{L^p} \| \varphi_k \hat{\nu} \|_{L^p} dt \\
\leq 2^{j(s+1)} \int_0^\infty \sum_{|k-j| \leq 4} \left( \sum_{k' \leq k} \| \varphi_k \hat{\nu} \|_{L^p} \| \varphi_k \hat{\nu} \|_{L^p} \right) \left( \sum_{k' \leq k} 2^{k' r' (3-\frac{3}{r'})} \right)^{\frac{1}{r'}} \| \varphi_k \hat{\nu} \|_{L^p} dt \\
\leq C \|u\|_{\dot{L}^{\infty}([0,\infty);FB^s_{p,r})} 2^{j(s+1)} \int_0^\infty \sum_{|k-j| \leq 4} 2^{k(3-\frac{3}{r'})} \| \varphi_k \hat{\nu} \|_{L^p} dt.
$$

Hence, by Young’s inequality,

$$
J_1 \leq C \|u\|_{\dot{L}^{\infty}([0,\infty);FB^s_{p,r})} \left( \sum_j \left( \sum_{|k-j| \leq 4} 2^{k(3-\frac{3}{r'})} \| \varphi_k \hat{\nu} \|_{L^1([0,\infty);L^p)} \right)^r \right)^{\frac{1}{r}} \\
\leq C \|u\|_{\dot{L}^{\infty}([0,\infty);FB^s_{p,r})} \|v\|_{\dot{L}^1([0,\infty);FB^s_{p,r})}.
$$

The term $J_2$ is estimated in the same way as $J_1$. In fact,

$$
J_2 \leq C \|v\|_{\dot{L}^{\infty}([0,\infty);FB^s_{p,r})} \left( \sum_j \left( \sum_{|k-j| \leq 4} 2^{k(3-\frac{3}{r'})} \| \varphi_k \hat{\nu} \|_{L^1([0,\infty);L^p)} \right)^r \right)^{\frac{1}{r}} \\
\leq C \|v\|_{\dot{L}^{\infty}([0,\infty);FB^s_{p,r})} \|u\|_{\dot{L}^1([0,\infty);FB^{4-\frac{2}{3}}_{p,r})}.
$$
Finally, we focus on the third term $J_3$. As in the estimate to $J_1$, for fixed $j$, we obtain
\[
2^{j(s+1)} \int_0^\infty \left\| \sum_{k \geq j-2} \sum_{|k-k'| \leq 1} \varphi_j(\varphi_k \hat{u} * \varphi_k \hat{v}) \right\|_{L^p} dt
\]
\[
\leq 2^{j(s+1)} \int_0^\infty \left\| \sum_{k \geq j-2} \sum_{|k-k'| \leq 1} \varphi_k \hat{v} \right\|_{L^p} \left\| \varphi_k \hat{u} \right\|_{L^p} 2^{k(3-\frac{2}{p})} dt
\]
\[
\leq C 2^{j(s+1)} \int_0^\infty \left( \sum_{k \geq j-2} \left( \sum_{|k-k'| \leq 1} \left\| \varphi_k \hat{v} \right\|_{L^p} \right)^r \right)^{\frac{1}{r}} \left( \sum_{k \geq j-2} \left\| \varphi_k \hat{u} \right\|_{L^p} \right) dt
\]
\[
\leq C \|v\|_{\tilde{L}^{\infty}([0,\infty);FB_{p,r}^{2-\frac{2}{p}}(\mathbb{R}^3))} \sum_{k \geq j-2} 2^{(j-k)(s+1)} 2^{k(3-\frac{2}{p})} \int_0^\infty \left\| \varphi_k \hat{u} \right\|_{L^p} dt.
\]
Thus, by Young’s inequality
\[
J_3 \leq C \|v\|_{\tilde{L}^{\infty}([0,\infty);FB_{p,r}^{2-\frac{2}{p}}(\mathbb{R}^3))} \|u\|_{\tilde{L}^1([0,\infty);FB_{p,r}^{4-\frac{2}{p}}(\mathbb{R}^3))}
\]
since $s > -1$.

Summing up, we see that
\[
\|uv\|_{\tilde{L}^1([0,\infty);FB_{p,r}^{4-\frac{2}{p}}(\mathbb{R}^3))} \leq C \|u\|_{Y} \|v\|_{Y}.
\]
\]

We conclude this section with the following Lemma.

Lemma 2.3. [19] Let $1 \leq p, q \leq \infty$, $0 < r < R < \infty$, $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then, for any multiindex $\gamma \in \mathbb{N}^n$, the following estimates hold:

a) If $\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq R 2^j \}$, then $\| (i \cdot)^\gamma \hat{f} \|_{L^q(\mathbb{R}^n)} \leq 2^{|\gamma| + n j \left( \frac{1}{p} - \frac{1}{q} \right)} \| \hat{f} \|_{L^p(\mathbb{R}^n)}$.

b) If $\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^n : r 2^j \leq |\xi| \leq R 2^j \}$, then $\| \hat{f} \|_{L^q(\mathbb{R}^n)} \leq 2^{-j |\gamma|} \sup_{|\beta| = |\gamma|} \| (i \cdot)^\beta \hat{f} \|_{L^q(\mathbb{R}^n)}$.

3. Proof of Theorem [11]

For the proof of Theorem [11] we make use of the following standard fixed point result. For a proof, we refer e.g. to [7].

Proposition 3.1. Let $X$ be a Banach space and $B : X \times X \to X$ be a bounded bilinear form satisfying $\|B(x_1, x_2)\|_X \leq \eta \|x_1\|_X \|x_2\|_X$ for all $x_1, x_2 \in X$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < \frac{1}{4 \eta}$ and if $a \in X$ such that $\|a\|_X \leq \varepsilon$, the equation $x = a + B(x, x)$ has a solution in $X$ such that $\|x\|_X \leq 2 \varepsilon$. This solution is the only one in the ball $\mathcal{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on $a$ in the following sense: if $\|a\|_X \leq \varepsilon$, $\tilde{x} = a + B(\tilde{x}, \tilde{x})$, and $\|\tilde{x}\|_X \leq 2\varepsilon$, then
\[
\|x - \tilde{x}\|_X \leq \frac{1}{1 - 4\eta \varepsilon} \|a - \tilde{a}\|_X.
\]

In the following, we choose an underlying Banach space $X$ given by
\[
X := \tilde{L}^{\infty}([0, \infty); FB_{p,r}^{2-\frac{2}{p}}(\mathbb{R}^3)) \cap \tilde{L}^1([0, \infty); FB_{p,r}^{4-\frac{2}{p}}(\mathbb{R}^3)),
\]
and recall that $\Phi$ was defined by
\[
\Phi(u) = T(t)u_0 - \int_0^t T(t - \tau) \mathbb{P} \text{div}(u \otimes u)(\tau) d\tau.
\]
We estimate first the term $T(t)u_0$. 
Lemma 3.2. Let \( p, r \in [1, \infty], s = 2 - 3/p \) and \( u_0 \in F^2_{p, r} (\mathbb{R}^3) \). Then there exists a constant \( C > 0 \) such that

\[
\| T(t)u_0 \|_{L^\infty([0, \infty); F^2_{p, r})} \leq C \| u_0 \|_{F^2_{p, r}}, \quad t > 0,
\]

\[
\| T(t)u_0 \|_{L^1([0, \infty); F^2_{p, r})} \leq C \| u_0 \|_{F^2_{p, r}}, \quad t > 0.
\]

Proof. We prove first estimate (3.1). By the definition of the norm, we have

\[
\| T(t)u_0 \|_{L^\infty([0, \infty); F^2_{p, r})} \leq \left( \sum_k 2^{k(2 - \frac{3}{p})} \sup_{t \in [0, \infty)} \| \varphi_k \tilde{u}_0 \|_{L^p} \right)^{\frac{1}{p}} \leq C \| u_0 \|_{F^2_{p, r}}.
\]

In order to prove the second estimate (3.2), above, note that

\[
\| T(t)u_0 \|_{L^1([0, \infty); F^2_{p, r})} \leq \left( \sum_k 2^{k(4 - \frac{3}{p})} \left( \int_0^\infty e^{-t2^{2k}} \| \varphi_k \tilde{u}_0 \|_{L^p} dt \right)^{\frac{1}{p}} \right) \leq C \| u_0 \|_{F^2_{p, r}}.
\]

We next consider the bilinear operator \( B \) given by

\[
B(u, v) := \int_0^t T(t - \tau) \text{div}(u \otimes v) d\tau.
\]

By Lemma 2.4, Lemma 2.6 and Lemma 2.2 with \( s = 2 - \frac{3}{p} \), we obtain

\[
\| B(u, v) \|_{L^1([0, \infty); F^{4\frac{3}{4}}_{p, r} (\mathbb{R}^3))} = \left\| \int_0^t T(t - \tau) \text{div}(u \otimes v) d\tau \right\|_{L^1([0, \infty); F^{4\frac{3}{4}}_{p, r} (\mathbb{R}^3))}
\leq C \| \text{div}(u \otimes v) \|_{L^1([0, \infty); F^{2\frac{3}{4}}_{p, r})}
\leq C \| uv \|_{L^1([0, \infty); F^{3\frac{3}{4}}_{p, r})}
\leq C \| u \|_X \| v \|_X.
\]

Similarly,

\[
\| B(u, v) \|_{L^\infty([0, \infty); F^{2\frac{3}{4}}_{p, r} (\mathbb{R}^3))} = \left\| \int_0^t T(t - \tau) \text{div}(u \otimes v) d\tau \right\|_{L^\infty([0, \infty); F^{2\frac{3}{4}}_{p, r} (\mathbb{R}^3))}
\leq C \| \text{div}(u \otimes v) \|_{L^1([0, \infty); F^{2\frac{3}{4}}_{p, r})}
\leq C \| uv \|_{L^1([0, \infty); F^{3\frac{3}{4}}_{p, r})}
\leq C \| u \|_X \| v \|_X.
\]

Thus, combining these estimates with Lemma 3.2 yields

\[
\| \Phi(u) \|_X \leq C \| u_0 \|_{F^{2\frac{3}{4}}_{p, r}} + 4C \varepsilon^2,
\]

as well as

\[
\| \Phi(u) - \Phi(v) \|_X \leq C(\| u \|_X + \| v \|_X) \| u - v \|_X.
\]

Choosing now \( \varepsilon \leq \frac{1}{8C} \), for every \( u_0 \in F^2_{p, r} (\mathbb{R}^3) \) with \( \| u_0 \|_{F^{2\frac{3}{4}}_{p, r}} \leq \frac{1}{8} \), we finally obtain

\[
\| \Phi(u) \|_X \leq 2\varepsilon \quad \text{and} \quad \| \Phi(u) - \Phi(v) \|_X \leq \frac{1}{2} \| u - v \|_X.
\]

Applying Proposition 3.1 to the given situation completes the proof of Theorem 1.1.
4. Global existence for non-small data in $L^p_0(\mathbb{R}^2)$

In this section we consider equation (4.1) in the two-dimensional setting and in the case where the initial data $u_0$ belong to $L^p_0(\mathbb{R}^2)$ for $p > 2$. To this end, we note first that the equations of Navier-Stokes with Coriolis force are equivalent to the Navier-Stokes equations with linearly growing initial data. Indeed, we may rewrite equation (4.1) for a two-dimensional rotating fluid as

$$
\begin{align*}
\partial_t u - \Delta u + u \cdot \nabla u - 2M u + \nabla \pi &= 0, \quad y \in \mathbb{R}^2, \ t > 0, \\
\text{div } u &= 0, \quad y \in \mathbb{R}^2, \ t > 0, \\
{u(0)} &= {u_0}, \quad y \in \mathbb{R}^2,
\end{align*}
$$

where $M$ is given by

$$
M = \frac{\Omega}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

Then, by the change of variables $x = e^{-tM} y$ and by setting

$$
v(t, x) := e^{-tM} u(t, e^{tM} x), \quad q(t, x) := \pi(t, e^{tM} x),
$$

we obtain the following set of equations for $v$

$$
\begin{align*}
\partial_t v - \Delta v + v \cdot \nabla v - M x \cdot \nabla v - M v + \nabla q &= 0, \quad x \in \mathbb{R}^2, \ t > 0, \\
\text{div } v &= 0, \quad x \in \mathbb{R}^2, \ t > 0, \\
v(0) &= u_0, \quad x \in \mathbb{R}^2.
\end{align*}
$$

These are the usual equations of Navier-Stokes with linearly growing initial data. Indeed, setting $U = v - M x$, we have

$$
\begin{align*}
\partial_t U - \Delta U + U \cdot \nabla U + \nabla \tilde{\pi} &= 0, \quad x \in \mathbb{R}^2, \ t > 0, \\
\text{div } U &= 0, \quad x \in \mathbb{R}^2, \ t > 0, \\
U(0) &= u_0 - M x, \quad x \in \mathbb{R}^2,
\end{align*}
$$

with $\nabla \tilde{\pi} = \nabla q - M M x$. For initial data $u_0 \in L^p_0(\mathbb{R}^2)$, it was shown in [13] that there exists a unique, local mild solution $v$ to equation (4.2) in the space $C([0, T_0); L^p_0(\mathbb{R}^2))$, where $2 \leq p < \infty$. We note that if $u_0 \in L^p_0(\mathbb{R}^2)$, Theorem 2.1 in [13] implies that $\frac{t}{2} \partial_t \nabla v \in C([0, T_0); L^p(\mathbb{R}^2))$. Thus there exists $t_1 \in (0, T_0)$ such that $\nabla v(t_1) \in L^p(\mathbb{R}^2)$ which implies that $\text{rot } v(t_1) \in L^p(\mathbb{R}^2)$. Hence, in order to prove Theorem 1.3 it suffices to show an a priori estimate of the following form. In the sequel, we set $w := \text{rot } v$.

**Proposition 4.1.** Let $2 \leq p < \infty$ and $v(t_1) \in L^p_0(\mathbb{R}^2)$ such that $\text{rot } v(t_1) \in L^p(\mathbb{R}^2)$ for some $t_1 \in (0, T_0)$. Let $v$ be the mild solution of (4.2). Then there exists a constant $C > 0$ such that

$$
\|v(t)\|_{L^p} \leq C \|v(t_1)\|_{L^p} \exp \left(C t \|w(t_1)\|_{L^p} \right), \quad t > t_1,
$$

where $w(t_1) = \text{rot } v(t_1)$.

**Proof.** Consider the operator $A$ in $L^p_0(\mathbb{R}^2)$ given by

$$
A u := -\Delta u - < M \cdot \nabla u > + M u
$$

equipped with the domain $D(A) = \{ u \in W^{2,p}(\mathbb{R}^2) : < M \cdot \nabla u > \in L^p(\mathbb{R}^2) \}$. By the results in [13], the mild solution of (4.2) is represented by

$$
v(t) = e^{-tA} v(t_1) - \int_{t_1}^t e^{-(t-s)A} \mathbb{P}(v \cdot \nabla v)(s) ds + 2 \int_{t_1}^t e^{-(t-s)A} \mathbb{P}(M v)(s) ds,
$$

for $t > t_1$. Applying Proposition 3.4 in [13] yields

$$
(4.4) \|e^{-(t-s)A} \mathbb{P}(v \cdot \nabla v)(s)\|_{L^p} \leq \frac{C}{(t-s)^{\frac{\sigma}{p}}} \|v \cdot \nabla v(s)\|_{L^p} \leq \frac{C}{(t-s)^{\frac{\sigma}{p}}} \|v(s)\|_{L^p} \|\nabla v(s)\|_{L^p}, \quad t > s > t_1.
$$
Hence, we have

\[ \|v(t)\|_{L^p} \leq C \|v(t_1)\|_{L^p} + C \int_{t_1}^t \frac{C}{(t-s)^{\frac{2}{p}}} \|v(s)\|_{L^p} \cdot \|w(s)\|_{L^p} ds + C \int_{t_1}^t \|v(s)\|_{L^p} ds, \quad t > t_1. \]

Next, applying curl to equation (4.2), we verify that the vorticity \( w = \text{rot} v \) satisfies the equation

\[ \begin{cases} \partial_t w - \Delta w + v \cdot \nabla w - Mx \cdot \nabla w = 0, & x \in \mathbb{R}^2, \quad t > 0, \\ w(0) = \text{rot} u_0. & \end{cases} \]

A standard energy estimate allows us to show that

\[ \|w(t)\|_{L^p} \leq C \|w(t_1)\|_{L^p}, \quad t > t_1. \]

Hence, we have

\[ \|v(t)\|_{L^p} \leq C \|v(t_1)\|_{L^p} + C \|w(t_1)\|_{L^p} \int_{t_1}^t \left( \frac{1}{(t-s)^{\frac{2}{p}}} + 1 \right) \|v(s)\|_{L^p} ds, \quad t > t_1. \]

Finally, Gronwall’s inequality yields the desired estimate. This finishes the proof of Proposition 4.1. \( \square \)

By Proposition 4.1, we obtain a unique, global solution \( \tilde{v} \) of (4.2) on \([t_1, \infty)\) for the initial data \( v(t_1) \). A uniqueness argument ensures that \( v(t) = \tilde{v}(t) \) on \([t_1, T_0)\). Therefore, the local solution \( v \) on \([0, T_0)\) can be continued globally. This finishes the proof of Theorem 1.3.

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