A COMPACTNESS RESULT FOR A SYSTEM WITH WEIGHT AND BOUNDARY SINGULARITY.

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ABSTRACT. We give blow-up behavior for solutions to an elliptic system with Dirichlet condition, and, weight and boundary singularity. Also, we have a compactness result for this elliptic system with regular Hölderian weight and boundary singularity and Lipschitz condition.

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1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open analytic domain $\Omega$ of $\mathbb{R}^2$.

We consider the following equation:

\[
(P) \begin{cases}
-\Delta u = |x|^{2\beta} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\
-\Delta v = W e^u & \text{in } \Omega \subset \mathbb{R}^2, \\
u = 0 & \text{in } \partial \Omega, \\
v = 0 & \text{in } \partial \Omega.
\end{cases}
\]

Here, we assume that:

$0 \leq V \leq b_1 < +\infty$, $e^u \in L^1(\Omega)$ and $u \in W^{1,1}_0(\Omega)$,

$0 \leq W \leq b_2 < +\infty$, $|x|^{2\beta} e^v \in L^1(\Omega)$ and $v \in W^{1,1}_0(\Omega)$,

and,

$0 \in \partial \Omega$, $\beta \geq 0$. 

When \( u = v \) and \( \beta = 0 \), the above system is reduced to an equation which was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1-17], one can find some existence and compactness results, also for a system.

Among other results, we can see in [6] the following important Theorems (\( \beta = 0 \)):

**Theorem A.** (Brezis-Merle [6]). Consider the case of one equation: if \((u_i)_i = (v_i)_i \) and \((V_i)_i = (W_i)_i \) are two sequences of functions relatively to the problem \((P)\) with, \(0 < a \leq V_i \leq b < +\infty\), then, for all compact set \(K\) of \(\Omega\),

\[
\sup_K u_i \leq c = c(a, b, K, \Omega).
\]

**Theorem B** (Brezis-Merle [6]). Consider the case of one equation and assume that \((u_i)_i \) and \((V_i)_i \) are two sequences of functions relatively to the previous problem \((P)\) with, \(0 \leq V_i \leq b < +\infty\), and,

\[
\int_{\Omega} e^{u_i} \, dy \leq C,
\]

then, for all compact set \(K\) of \(\Omega\),

\[
\sup_K u_i \leq c = c(b, C, K, \Omega).
\]

Next, we call energy the following quantity:

\[
E = \int_{\Omega} e^{u_i} \, dy.
\]

The boundedness of the energy is a necessary condition to work on the problem \((P)\) as showed in [6], by the following counterexample (\( \beta = 0 \)):

**Theorem C** (Brezis-Merle [6]). Consider the case of one equation, then there are two sequences \((u_i)_i \) and \((V_i)_i \) of the problem \((P)\) with, \(0 \leq V_i \leq b < +\infty\), and,

\[
\int_{\Omega} e^{u_i} \, dy \leq C,
\]

and

\[
\sup_{\Omega} u_i \to +\infty.
\]
When $\beta = 0$, the above system have many properties in the constant and the Lipschitzian cases. Indeed we have (when $\beta = 0$):

In [12], Dupaigne-Farina-Sirakov proved (by an existence result of Montenegro, see [16]) that the solutions of the above system when $V$ and $W$ are constants can be extremal and this condition imply the boundedness of the energy and directly the compactness. Note that in [11], if we assume (in particular) that $\nabla\log V$ and $\nabla\log W$ and $V > a > 0$ or $W > a' > 0$ and $V, W$ are nonnegative and uniformly bounded then the energy is bounded and we have a compactness result.

Note that in the case of one equation (and $\beta = 0$), we can prove by using the Pohozaev identity that if $+\infty > b \geq V \geq a > 0$, $\nabla V$ is uniformly Lipschitzian that the energy is bounded when $\Omega$ is starshaped.

In [15] Ma-Wei, using the moving-plane method showed that this fact is true for all domain $\Omega$ with the same assumptions on $V$. In [11] De Figueiredo-do O-Ruf extend this fact to a system by using the moving-plane method for a system.

Theorem C, shows that we have not a global compactness to the previous problem with one equation, perhaps we need more information on $V$ to conclude to the boundedness of the solutions. When $\nabla\log V$ is Lipschitz function and $\beta = 0$, Chen-Li and Ma-Wei see [7] and [15], showed that we have a compactness on all the open set. The proof is via the moving plane-Method of Serrin and Gidas-Ni-Nirenberg. Note that in [11], we have the same result for this system when $\nabla\log V$ and $\nabla\log W$ are uniformly bounded. We will see below that for a system we also have a compactness result when $V$ and $W$ are Lipschitzian and $\beta \geq 0$.

Now consider the case of one equation. In this case our equation have nice properties.

If we assume $V$ with more regularity, we can have another type of estimates, a $\sup + \inf$ type inequalities. It was proved by Shafrir see [17], that, if $(u_i), (V_i)$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left( \frac{a}{b} \right) \sup_{\Omega} u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with $A$ the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see [5].

Here we are interested by the case of a system of this type of equation. First, we give the behavior of the blow-up points on the boundary, with weight and boundary singularity, and in the second time we have a proof of compactness of the solutions to Gelfand-Liouville type system with weight and boundary singularity and Lipschitz condition.

Here, we write an extention of Brezis-Merle Problem (see [6]) to a system:

**Problem.** Suppose that $V_i \to V$ and $W_i \to W$ in $C^0(\Omega)$, with, $0 \leq V_i$ and $0 \leq W_i$. Also, we consider two sequences of solutions $(u_i), (v_i)$ of $(P)$ relatively to $(V_i), (W_i)$ such that,

$$\int_{\Omega} e^{u_i} dx \leq C_1, \quad \int_{\Omega} |x|^{2\beta} e^{v_i} dx \leq C_2,$$
is it possible to have:

$$\|u_i\|_{L^\infty} \leq C_3 = C_3(\beta, C_1, C_2, \Omega)?$$

and,

$$\|v_i\|_{L^\infty} \leq C_4 = C_4(\beta, C_1, C_2, \Omega)?$$

In this paper we give a characterization of the behavior of the blow-up points on the boundary and also a proof of the compactness theorem when $V_i$ and $W_i$ are uniformly Lipschitzian and $\beta \geq 0$. For the behavior of the blow-up points on the boundary, the following conditions are enough,

$$0 \leq V_i \leq b_1, \ 0 \leq W_i \leq b_2,$$

The conditions $V_i \to V$ and $W_i \to W$ in $C^0(\overline{\Omega})$ are not necessary.

But for the proof of the compactness for the system, we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A_1, \ |\nabla W_i|_{L^\infty} \leq A_2, \ \beta \geq 0.$$ 

Our main result is:

**Theorem 1.1.** Assume that $\max_{\Omega} u_i \to +\infty$ and $\max_{\Omega} v_i \to +\infty$ where $(u_i)$ and $(v_i)$ are solutions of the problem $(P)$ with $\beta \geq 0$, and:

$$0 \leq V_i \leq b_1, \ \text{and} \ \int_{\Omega} e^{u_i} dx \leq C_1, \ \forall \ i,$$

and,

$$0 \leq W_i \leq b_2, \ \text{and} \ \int_{\Omega} |x|^{2\beta} e^{v_i} dx \leq C_2, \ \forall \ i,$$

then: after passing to a subsequence, there is a function $u$, there is a number $N \in \mathbb{N}$ and $N$ points $x_1, x_2, \ldots, x_N \in \partial\Omega$, such that,

$$\int_{\partial\Omega} \partial_{\nu} u_i \varphi \to \int_{\partial\Omega} \partial_{\nu} u \varphi + \sum_{j=1}^{N} \alpha_j \varphi(x_j), \ \alpha_j \geq 4\pi,$$

for any $\varphi \in C^0(\partial\Omega)$, and,

$$u_i \to u \ \text{in} \ C^1_{\text{loc}}(\overline{\Omega} - \{x_1, \ldots, x_N\}).$$
\[
\int_{\partial \Omega} \partial_{\nu} v_i \varphi \rightarrow \int_{\partial \Omega} \partial_{\nu} v \varphi + \sum_{j=1}^{N} \beta_j \varphi(x_j), \quad \beta_j \geq 4\pi,
\]
for any \( \varphi \in C^0(\partial \Omega) \), and,

\[v_i \rightarrow v \text{ in } C^{1}_{\text{loc}}(\bar{\Omega} - \{x_1, \ldots, x_N\}).\]

In the following theorem, we have a proof for the global a priori estimate which concern the problem \((P)\).

**Theorem 1.2.** Assume that \((u_i), (v_i)\) are solutions of \((P)\) relatively to \((V_i), (W_i)\) with the following conditions:

\[x_1 = 0 \in \partial \Omega, \beta \geq 0,\]

and,

\[0 \leq V_i \leq b_1, \|\nabla V_i\|_{L^\infty} \leq A_1, \text{ and } \int_{\Omega} e^{u_i} \leq C_1,\]

\[0 \leq W_i \leq b_2, \|\nabla W_i\|_{L^\infty} \leq A_2, \text{ and } \int_{\Omega} |x|^{2\beta} e^{v_i} \leq C_2,\]

We have,

\[\|u_i\|_{L^\infty} \leq C_3(b_1, b_2, \beta, A_1, A_2, C_1, C_2, \Omega),\]

and,

\[\|v_i\|_{L^\infty} \leq C_4(b_1, b_2, \beta, A_1, A_2, C_1, C_2, \Omega),\]

2. **Proof of the theorems**

**Proof of theorem 1.1:**

We have:

\[u_i, v_i \in W^{1,1}_0(\Omega).\]

Since \(e^{u_i} \in L^1(\Omega)\) by the corollary 1 of Brezis-Merle’s paper (see [6]) we have \(e^{v_i} \in L^k(\Omega)\) for all \(k > 2\) and the elliptic estimates of Agmon and the Sobolev embedding (see [11]) imply that:
\[ u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}). \]

And,

We have:

\[ v_i, u_i \in W^{1,1}_0(\Omega). \]

Since \(|x|^{2\beta} e^{u_i} \in L^1(\Omega)\) by the corollary 1 of Brezis-Merle’s paper (see [6]) we have \(e^{u_i} \in L^k(\Omega)\) for all \(k > 2\) and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

\[ v_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}). \]

Since \(|x|^{2\beta} V e^{v_i}\) and \(W e^{u_i}\) are bounded in \(L^1(\Omega)\), we can extract from those two sequences two subsequences which converge to two non-negative measures \(\mu_1\) and \(\mu_2\). (This procedure is similar to the procedure of Brezis-Merle, we apply corollary 4 of Brezis-Merle paper, see [6]).

If \(\mu_1(x_0) < 4\pi\), by a Brezis-Merle estimate for the first equation, we have \(e^{u_i} \in L^{1+\epsilon}\) around \(x_0\), by the elliptic estimates, for the second equation, we have \(v_i \in W^{2,1+\epsilon} \subset L^\infty\) around \(x_0\), and, returning to the first equation, we have \(u_i \in L^\infty\) around \(x_0\).

If \(\mu_2(x_0) < 4\pi\), then \(u_i\) and \(v_i\) are also locally bounded around \(x_0\).

Thus, we take a look to the case when, \(\mu_1(x_0) \geq 4\pi\) and \(\mu_2(x_0) \geq 4\pi\). By our hypothesis, those points \(x_0\) are finite.

We will see that inside \(\Omega\) no such points exist. By contradiction, assume that, we have \(\mu_1(x_0) \geq 4\pi\). Let us consider a ball \(B_R(x_0)\) which contain only \(x_0\) as nonregular point. Thus, on \(\partial B_R(x_0)\), the two sequence \(u_i\) and \(v_i\) are uniformly bounded. Let us consider:

\[ \begin{cases} -\Delta z_i = |x|^{2\beta} V e^{u_i} & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z_i = 0 & \text{in } \partial B_R(x_0). \end{cases} \]

By the maximum principle we have:

\[ z_i \leq u_i \]

and \(z_i \rightarrow z\) almost everywhere on this ball, and thus,

\[ \int e^{z_i} \leq \int e^{u_i} \leq C, \]

and,
\[ \int e^z \leq C. \]

but, \( z \) is a solution in \( W^{1,q}_0(B_R(x_0)) \), \( 1 \leq q < 2 \), of the following equation:

\[ \begin{cases} -\Delta z = \mu_1 & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z = 0 & \text{in } \partial B_R(x_0). \end{cases} \]

with, \( \mu_1 \geq 4\pi \) and thus, \( \mu_1 \geq 4\pi \delta_{x_0} \) and then, by the maximum principle in \( W^{1,q}_0(B_R(x_0)) \):

\[ z \geq -2 \log |x - x_0| + C \]

thus,

\[ \int e^z = +\infty, \]

which is a contradiction. Thus, there is no nonregular points inside \( \Omega \)

Thus, we consider the case where we have nonregular points on the boundary, we use two estimates:

\[ \int_{\partial \Omega} \partial_{\nu} u_i d\sigma \leq C_1, \quad \int_{\partial \Omega} \partial_{\nu} v_i d\sigma \leq C_2, \]

and,

\[ ||\nabla u_i||_{L^q} \leq C_q, \quad ||\nabla v_i||_{L^q} \leq C'_q, \quad \forall \quad i \text{ and } 1 < q < 2. \]

We have the same computations, as in the case of one equation.

We consider a points \( x_0 \in \partial \Omega \) such that:

\[ \mu_1(x_0) < 4\pi. \]

We consider a test function on the boundary \( \eta \) we extend \( \eta \) by a harmonic function on \( \Omega \), we write the equation:

\[ -\Delta ((u_i - u)\eta) = |x|^{2\beta} (V_i e^{u_i} - V e^v)\eta + \nabla (u_i - u) |\nabla \eta| = f_i \]

with,
\[ \int |f_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi, \]

\[ -\Delta ((v_i - v)\eta) = (W_i e^{u_i} - W e^u)\eta + \nabla (v_i - v) |\nabla \eta| = g_i, \]

with,

\[ \int |g_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi, \]

By the Brezis-Merle estimate, we have uniformly, \( e^{u_i} \in L^{1+\epsilon} \) around \( x_0 \), by the elliptic estimates, for the second equation, we have \( v_i \in W^{2,1+\epsilon} \subset L^\infty \) around \( x_0 \), and , returning to the first equation, we have \( u_i \in L^\infty \) around \( x_0 \).

We have the same thing if we assume:

\[ \mu_2(x_0) < 4\pi. \]

Thus, if \( \mu_1(x_0) < 4\pi \) or \( \mu_2(x_0) < 4\pi \), we have for \( R > 0 \) small enough:

\[ (u_i, v_i) \in L^\infty (B_R(x_0) \cap \Omega). \]

By our hypothesis the set of the points such that:

\[ \mu_1(x_0) \geq 4\pi, \quad \mu_2(x_0) \geq 4\pi, \]

is finite, and, outside this set \( u_i \) and \( v_i \) are locally uniformly bounded. By the elliptic estimates, we have the \( C^1 \) convergence to \( u \) and \( v \) on each compact set of \( \Omega - \{x_1, \ldots, x_N\} \).

Indeed,

By the Stokes formula we have,

\[ \int_{\partial \Omega} \partial_{\nu} u_i d\sigma \leq C, \]

We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure \( \mu_1 \) such that,

\[ \int_{\partial \Omega} \partial_{\nu} u_i \varphi d\sigma \to \mu_1(\varphi), \quad \forall \ \varphi \in C^0(\partial \Omega). \]

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We take an \( x_0 \in \partial \Omega \) such that, \( \mu_1(x_0) < 4\pi \). For \( \epsilon > 0 \) small enough set \( I_\epsilon = B(x_0, \epsilon) \cap \partial \Omega \) on the unit disk or one can assume it as an interval. We choose a function \( \eta_\epsilon \) such that,

\[
\begin{align*}
\eta_\epsilon &\equiv 1, \text{ on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\
\eta_\epsilon &\equiv 0, \text{ outside } I_{2\epsilon}, \\
0 &\leq \eta_\epsilon \leq 1, \\
||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} &\leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\end{align*}
\]

We take a \( \tilde{\eta}_\epsilon \) such that,

\[
\begin{align*}
-\Delta \tilde{\eta}_\epsilon &= 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \\
\tilde{\eta}_\epsilon &= \eta_\epsilon \quad \text{in } \partial \Omega.
\end{align*}
\]

**Remark:** We use the following steps in the construction of \( \tilde{\eta}_\epsilon \):

We take a cutoff function \( \eta_0 \) in \( B(0, 2) \) or \( B(x_0, 2) \):

1- We set \( \eta_\epsilon(x) = \eta_0(|x - x_0|/\epsilon) \) in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart \((f, \tilde{\Omega})\) with \( f(0) = x_0 \) and we take \( \mu_\epsilon(x) = \eta_0(f(|x|/\epsilon)) \) to have connected sets \( I_\epsilon \) and we take \( \eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y)) \). Because \( f, f^{-1} \) are Lipschitz, \( |f(x) - x_0| \leq k_2|x| \leq 1 \) for \( |x| \leq 1/k_2 \) and \( |f(x) - x_0| \geq k_1|x| \geq 2 \) for \( |x| \geq 2/k_1 > 1/k_2 \), the support of \( \eta \) is in \( I_{(2/k_1)\epsilon} \).

\[
\begin{align*}
\eta_\epsilon &\equiv 1, \text{ on } f(I_{(1/k_2)\epsilon}), \quad 0 < \epsilon < \delta/2, \\
\eta_\epsilon &\equiv 0, \text{ outside } f(I_{(2/k_1)\epsilon}), \\
0 &\leq \eta_\epsilon \leq 1, \\
||\nabla \eta_\epsilon||_{L^\infty(I_{(2/k_1)\epsilon})} &\leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\end{align*}
\]

3- Also, we can take: \( \mu_\epsilon(x) = \eta_0(|x|/\epsilon) \) and \( \eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y)) \), we extend it by 0 outside \( f(B_1(0)) \). We have \( f(B_1(0)) = D_1(x_0), f(B_\epsilon(0)) = D_\epsilon(x_0) \) and \( f(B^+_\epsilon) = D^+_\epsilon(x_0) \) with \( f \) and \( f^{-1} \) smooth diffeomorphism.

\[
\begin{align*}
\eta_\epsilon &\equiv 1, \text{ on a the connected set } J_\epsilon = f(I_\epsilon), \quad 0 < \epsilon < \delta/2, \\
\eta_\epsilon &\equiv 0, \text{ outside } J'_\epsilon = f(I_{2\epsilon}), \\
0 &\leq \eta_\epsilon \leq 1, \\
||\nabla \eta_\epsilon||_{L^\infty(J'_\epsilon)} &\leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\end{align*}
\]

And, \( H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon \), since \( f \) is Lipschitz. Here \( H_1 \) is the Hausdorff measure.

We solve the Dirichlet Problem:
\[ \begin{align*}
-\Delta \eta &= -\Delta \eta, & \text{in } \Omega \subset \mathbb{R}^2, \\
\eta &= 0 & \text{in } \partial \Omega.
\end{align*} \]

and finally we set \( \tilde{\eta} = -\bar{\eta} + \eta \). Also, by the maximum principle and the elliptic estimates we have:

\[ ||\nabla \eta||_{L^\infty} \leq C(||\eta||_{L^\infty} + ||\nabla \eta||_{L^\infty} + ||\Delta \eta||_{L^\infty}) \leq \frac{C_1}{\epsilon^2}, \]

with \( C_1 \) depends on \( \Omega \).

We use the following estimate, see [8],

\[ ||\nabla v_i||_{L^q} \leq C_q, \quad ||\nabla u_i||_q \leq C_q, \quad \forall \ i \text{ and } 1 < q < 2. \]

We deduce from the last estimate that, \((v_i)\) converge weakly in \( W^{1,q}_0(\Omega) \), almost everywhere to a function \( v \geq 0 \) and \( \int_{\Omega} |x|^{2\beta} e^v < +\infty \) (by Fatou lemma). Also, \( V_i \) weakly converge to a nonnegative function \( V \) in \( L^\infty \).

We deduce from the last estimate that, \((u_i)\) converge weakly in \( W^{1,q}_0(\Omega) \), almost everywhere to a function \( u \geq 0 \) and \( \int_{\Omega} e^u < +\infty \) (by Fatou lemma). Also, \( W_i \) weakly converge to a nonnegative function \( W \) in \( L^\infty \).

The function \( u, v \) are in \( W^{1,q}_0(\Omega) \) solutions of:

\[ \begin{align*}
-\Delta u &= |x|^{2\beta} V e^v \in L^1(\Omega), & \text{in } \Omega \subset \mathbb{R}^2, \\
\eta &= 0 & \text{in } \partial \Omega.
\end{align*} \]

And,

\[ \begin{align*}
-\Delta v &= W e^u \in L^1(\Omega), & \text{in } \Omega \subset \mathbb{R}^2, \\
v &= 0 & \text{in } \partial \Omega.
\end{align*} \]

According to the corollary 1 of Brezis-Merle’s result, see [6], we have \( e^{ku} \in L^1(\Omega), k > 1 \). By the elliptic estimates, we have \( v \in C^1(\Omega) \).

According to the corollary 1 of Brezis-Merle’s result, see [6], we have \( e^{kv} \in L^1(\Omega), k > 1 \). By the elliptic estimates, we have \( u \in C^1(\Omega) \).

For two vectors \( f \) and \( g \) we denote by \( f \cdot g \) the inner product of \( f \) and \( g \).

We can write:

\[ -\Delta ((u_i - u)\eta) = |x|^{2\beta} (V_i e^{v_i} - Ve^v)\tilde{\eta} - 2\nabla (u_i - u) \cdot \nabla \tilde{\eta}. \quad (1) \]
\[-\Delta((v_i - v)\tilde{\eta}) = (W_i e^{u_i} - W e^u)\tilde{\eta} - 2\nabla(v_i - v) \cdot \nabla\tilde{\eta}.\]

We use the interior estimate of Brezis-Merle, see [6],

**Step 1:** Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between \(\tilde{\eta}\) and \(u\), we obtain,

\[
\int_\Omega |x|^{2\beta} V e^{v} \tilde{\eta} \, dx = \int_{\partial \Omega} \partial_\nu u \eta \tilde{\eta} \, d\sigma \leq C'' \epsilon \eta L^\infty = C \epsilon.
\] (2)

We have,

\[
\begin{cases}
-\Delta u_i = |x|^{2\beta} V_i e^{v_i} & \text{in } \Omega \subset \mathbb{R}^2, \\
u_i = 0 & \text{in } \partial \Omega.
\end{cases}
\]

We use the Green formula between \(u_i\) and \(\tilde{\eta}\) to have:

\[
\int_\Omega |x|^{2\beta} V_i e^{v_i} \tilde{\eta} \, dx = \int_{\partial \Omega} \partial_\nu u_i \eta \tilde{\eta} d\sigma \rightarrow \mu_1(\eta) \leq \mu_1(J'') \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0
\] (3)

From (2) and (3) we have for all \(\epsilon > 0\) there is \(i_0(i_0(\epsilon))\) such that, for \(i \geq i_0\),

\[
\int_\Omega |x|^{2\beta} (V_i e^{v_i} - V e^v) \tilde{\eta} \, dx \leq 4\pi - \epsilon_0 + C \epsilon
\] (4)

**Step 2:** Estimate of integral of the second term of the right hand side of (1).

Let \(\Sigma_\epsilon = \{x \in \Omega, d(x, \partial \Omega) = \epsilon^3\}\) and \(\Omega_\epsilon = \{x \in \Omega, d(x, \partial \Omega) \geq \epsilon^3\}, \epsilon > 0\). Then, for \(\epsilon\) small enough, \(\Sigma_\epsilon\) is hypersurface.

The measure of \(\Omega - \Omega_\epsilon\) is \(k_2 \epsilon^3 \leq \text{meas}(\Omega - \Omega_\epsilon) = \mu_L(\Omega - \Omega_\epsilon) \leq k_1 \epsilon^3\).

**Remark:** for the unit ball \(\bar{B}(0, 1)\), our new manifold is \(\bar{B}(0, 1 - \epsilon^3)\).

(Proof of this fact; let’s consider \(d(x, \partial \Omega) = d(x, z_0), z_0 \in \partial \Omega\), this imply that \((d(x, z_0))^2 \leq (d(x, z))^2\) for all \(z \in \partial \Omega\) which it is equivalent to \((z - z_0) \cdot (2x - z - z_0) \leq 0\) for all \(z \in \partial \Omega\), let’s consider a chart around \(z_0\) and \(\gamma(t)\) a curve in \(\partial \Omega\), we have;

\[(\gamma(t) - \gamma(t_0) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0\] if we divide by \((t - t_0)\) (with the sign and tend \(t\) to \(t_0\)), we have \(\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0\), this imply that \(x = z_0 - s \nu_0\) where \(\nu_0\) is the outward normal of \(\partial \Omega\) at \(z_0\))
With this fact, we can say that $S = \{ x, d(x, \partial \Omega) \leq \epsilon \} = \{ x = z_0 - sv_{z_0}, z_0 \in \partial \Omega, -\epsilon \leq s \leq \epsilon \}$. It is sufficient to work on $\partial \Omega$. Let’s consider a charts $(z, D = B(z, 4\epsilon_z), \gamma_z)$ with $z \in \partial \Omega$ such that $\cup_z B(z, \epsilon_z)$ is cover of $\partial \Omega$. One can extract a finite cover $(B(z_k, \epsilon_k), k = 1, \ldots, m, \text{by the area formula the measure of } S \cap B(z_k, \epsilon_k) \text{ is less than a } k\epsilon \text{ (a } \epsilon \text{-rectangle)}. \text{ For the reverse inequality, it is sufficient to consider one chart around one point of the boundary.}

We write,

$$\int_{\Omega} \left| \nabla (u_i - u) \cdot \nabla \bar{\eta}_\epsilon \right| dx = \int_{\Omega_{z_3}} \left| \nabla (u_i - u) \cdot \nabla \bar{\eta}_\epsilon \right| dx + \int_{\Omega - \Omega_{z_3}} \left| \nabla (u_i - u) \cdot \nabla \bar{\eta}_\epsilon \right| dx. \quad (5)$$

**Step 2.1:** Estimate of $\int_{\Omega - \Omega_{z_3}} \left| \nabla (u_i - u) \cdot \nabla \bar{\eta}_\epsilon \right| dx$.

First, we know from the elliptic estimates that $\|\nabla \bar{\eta}_\epsilon\|_{L^\infty} \leq C_1/\epsilon^2$, $C_1$ depends on $\Omega$.

We know that $(|\nabla u_i|)_i$ is bounded in $L^q$, $1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle’s theorem), then, $h = |\nabla u|$ a.e. Let $q'$ be the conjugate of $q$.

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_{\Omega} |\nabla u_i| f dx \to \int_{\Omega} |\nabla u| f dx$$

If we take $f = 1_{\Omega - \Omega_{z_3}}$, we have:

$$\text{for } \epsilon > 0 \exists \ i_1 = i_1(\epsilon) \in \mathbb{N}, \ i \geq i_1, \ \int_{\Omega - \Omega_{z_3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{z_3}} |\nabla u| + \epsilon^3.$$ 

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega - \Omega_{z_3}} |\nabla u_i| \leq \text{meas}(\Omega - \Omega_{z_3})\|\nabla u\|_{L^\infty} + \epsilon^3 = \epsilon^3(k_1\|\nabla u\|_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega - \Omega_{z_3}} |\nabla (u_i - u) \cdot \nabla \bar{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1\|\nabla u\|_{L^\infty} + 1) \quad (6)$$

The constant $C_1$ does not depend on $\epsilon$ but on $\Omega$.

**Step 2.2:** Estimate of $\int_{\partial \Omega_{z_3}} |\nabla (u_i - u) \cdot \nabla \bar{\eta}_\epsilon| dx$. 

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We know that, $\Omega \subset \subset \Omega$, and (because of Brezis-Merle’s interior estimates) $u_i \rightarrow u$ in $C^1(\Omega_{\epsilon^3})$. We have,

$$||\nabla (u_i - u)||_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3,$$ for $i \geq i_3 = i_3(\epsilon)$.

We write,

$$\int_{\Omega_{\epsilon^3}} |\nabla (u_i - u) \cdot \nabla \hat{\eta}_\epsilon| dx \leq ||\nabla (u_i - u)||_{L^\infty(\Omega_{\epsilon^3})} ||\nabla \hat{\eta}_\epsilon||_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} |\nabla (u_i - u) \cdot \nabla \hat{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 ||\nabla u||_{L^\infty} + 2)$$

(7)

From (4) and (7), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} | - \Delta [(u_i - u)\hat{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 ||\nabla u||_{L^\infty} + 2 + C)$$

(8)

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\left\{ \begin{array}{l}
-\Delta [(u_i - u)\hat{\eta}_\epsilon] = g_{i, \epsilon} \text{ in } \Omega \subset \mathbb{R}^2, \\
(u_i - u)\hat{\eta}_\epsilon = 0 \text{ in } \partial \Omega.
\end{array} \right.$$  

with $||g_{i, \epsilon}||_{L^1(\Omega)} \leq 4\pi - \frac{\epsilon_0}{2}.$

We can use Theorem 1 of [6] to conclude that there are $q \geq \tilde{q} > 1$ such that:

$$\int_{V_{\epsilon}(x_0)} e^{\tilde{q}[u_i - u]} dx \leq \int_{\Omega} e^{q[u_i - u] \hat{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where, $V_{\epsilon}(x_0)$ is a neighborhood of $x_0$ in $\Omega$. Here we have used that in a neighborhood of $x_0$ by the elliptic estimates, $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$.

Thus, for each $x_0 \in \partial \Omega - \{\bar{x}_1, \ldots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0$, $q_{x_0} > 1$ such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0}[u_i - u]} dx \leq C, \forall i.$$  

(9)

Now, we consider a cutoff function $\eta \in C^\infty(\mathbb{R}^2)$ such that
\[ \eta \equiv 1 \text{ on } B(x_0, \epsilon x_0/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon x_0/3). \]

We write

\[ -\Delta(v_i \eta) = W_i e^{\alpha} \eta - 2\nabla v_i \cdot \nabla \eta - v_i \Delta \eta. \]

Because, by Poincaré and Gagliardo-Nirenberg-Sobolev inequalities:

\[ ||v_i||_{q^*} \leq c_q ||\nabla v_i||_q \leq C_q, \ 1 \leq q < 2, \]

with, \( q^* = \frac{2q}{2 - q} > 2 > 1. \)

By the elliptic estimates, \((v_i)_i\) is uniformly bounded in \(L^\infty(V_\epsilon(x_0))\). Finally, we have, for some \( \epsilon > 0 \) small enough,

\[ ||v_i||_{C^{0,\theta}[B(x_0, \epsilon)]} \leq c_3 \ \forall \ i. \]

Now, we consider a cutoff function \( \eta \in C^\infty(\mathbb{R}^2) \) such that

\[ \eta \equiv 1 \text{ on } B(x_0, \epsilon x_0/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon x_0/3). \]

We write

\[ -\Delta(u_i \eta) = |x|^{2\beta} V_i e^{\alpha} \eta - 2\nabla u_i \cdot \nabla \eta - u_i \Delta \eta. \]

By the elliptic estimates, \((u_i)_i\) is uniformly bounded in \(L^\infty(V_\epsilon(x_0))\) and also in \(C^{0,\theta}\) norm.

If we repeat this procedure another time, we have a boundedness of \((u_i)_i\) and \((v_i)_i\) in the \(C^{1,\theta}\) norm, because they are bounded in \(W^{2,q} \subset W^{1,q^*}\) norms with \(2q/(2 - q) = q^* > 2. \)

We have the same computations and conclusion if we consider a regular point \( x_0 \) for the measure \( \mu_2. \)

We have proved that, there is a finite number of points \( \bar{x}_1, \ldots, \bar{x}_m \) such that the sequence \((u_i)_i\) and \((v_i)_i\) are locally uniformly bounded (in \(C^{1,\theta}, \theta > 0\)) in \( \Omega - \{ \bar{x}_1, \ldots, \bar{x}_m \}. \)

**Proof of theorem 1.2:**

Without loss of generality, we can assume that \( 0 = x_1 \) is a blow-up point. Since the boundary is an analytic curve \( \gamma_1(t) \), there is a neighborhood of \( 0 \) such that the curve \( \gamma_1 \) can be extend to a holomorphic map such that \( \gamma_1'(0) \neq 0 \) (series) and by the inverse mapping one can assume that this map is univalent around \( 0. \) In the case when the boundary is a simple Jordan curve the domain is simply connected. In the case that the domains has a finite number of holes it is conformally equivalent to a disk with a finite number of disks removed. Here we consider a general domain. Without loss of generality one can assume that \( \gamma_1(B_{\epsilon_1}^+) \subset \Omega \)
and also $\gamma_1(B_1^-) \subset \bar{\Omega}$ and $\gamma_1(-1, 1) \subset \partial \Omega$ and $\gamma_1$ is univalent. This means that $(B_1, \gamma_1)$ is a local chart around 0 for $\Omega$ and $\gamma_1$ univalent. (This fact holds if we assume that we have an analytic domain, (below a graph of an analytic function), we have necessary the condition $\partial \bar{\Omega} = \partial \Omega$ and the graph is analytic, in this case $\gamma_1(t) = (t, \varphi(t))$ with $\varphi$ real analytic and an example of this fact is the unit disk around the point $(0, 1)$ for example).

By this conformal transformation, we can assume that $\Omega = B_1^+$, the half ball, and $\partial^+ B_1^+$ is the exterior part, a part which not contain 0 and on which $u_i$ converge in the $C^1$ norm to $u$. Let us consider $B_1^+$, the half ball with radius $\epsilon > 0$. Also, one can consider a $C^1$ domain (a rectangle between two half disks) and by charts its image is a $C^1$ domain) We know that:

$$u_i \in C^{2,\epsilon}(\bar{\Omega}).$$

Thus we can use integrations by parts (Stokes formula). The Pohozaev identity applied around the blow-up 0:

$$\int_{B_1^+} \Delta u_i < x|\nabla v_i > dx = -\int_{B_1^+} \Delta v_i < x|\nabla u_i > dx + \int_{\partial^+ B_1^+} g(\nabla u_i, \nabla v_i) d\sigma, \quad (10)$$

Thus,

$$\int_{B_1^+} |x|^{2\beta} V_i e^{v_i} < x|\nabla v_i > dx = -\int_{B_1^+} W_i e^{u_i} < x|\nabla u_i > dx - \int_{\partial^+ B_1^+} g(\nabla u_i, \nabla v_i) d\sigma, \quad (11)$$

After integration by parts, we obtain:

$$\int_{B_1^+} 2V_i(1 + \beta)|x|^{2\beta} e^{v_i} dx + \int_{B_1^+} < x|\nabla V_i > |x|^{2\beta} e^{v_i} dx + \int_{\partial B_1^+} < \nu |x > |x|^{2\beta} V_i e^{v_i} d\sigma +$$

$$+ \int_{B_1^+} W_i e^{u_i} dx + \int_{B_1^+} < x|\nabla W_i > e^{u_i} dx + \int_{\partial B_1^+} < \nu |x > W_i e^{u_i} d\sigma =$$

$$= -\int_{\partial^+ B_1^+} g(\nabla u_i, \nabla v_i) d\sigma,$$

Also, for $u$ and $v$, we have:

$$\int_{B_1^+} 2V(1 + \beta)|x|^{2\beta} e^v dx + \int_{B_1^+} < x|\nabla V > |x|^{2\beta} e^v dx + \int_{\partial B_1^+} < \nu |x > |x|^{2\beta} V e^v d\sigma +$$

$$+ \int_{B_1^+} W e^u dx + \int_{B_1^+} < x|\nabla W > e^u dx + \int_{\partial B_1^+} < \nu |x > W e^u d\sigma =$$
\[ = - \int_{\partial^{+} B^+_1} g(\nabla u, \nabla v) \, d\sigma, \]

If we take the difference, we obtain:

\[ \int_{\gamma_1(B^+_1)} |x|^{2\beta} V_i e^{u_i} \, dx + \int_{\gamma_1(B^+_1)} W_i e^{u_i} \, dx = o(\epsilon) + o(1) \]

But,

\[ \int_{\gamma_1(B^+_1)} |x|^{2\beta} V_i e^{u_i} \, dx + \int_{\gamma_1(B^+_1)} W_i e^{u_i} \, dx = \int_{\partial\gamma_1(B^+_1)} \partial_{x_i} u_i \, d\sigma + \int_{\partial\gamma_1(B^+_1)} \partial_{x_i} v_i \, d\sigma \]

and,

\[ \int_{\partial\gamma_1(B^+_1)} \partial_{x_i} u_i \, d\sigma + \int_{\partial\gamma_1(B^+_1)} \partial_{x_i} v_i \, d\sigma \to \alpha_1 + \beta_1 > 0 \quad (12) \]

a contradiction.

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