Almost Shortest Paths and PRAM Distance Oracles in Weighted Graphs

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Abstract

Let $G = (V, E)$ be a weighted undirected graph with $n$ vertices and $m$ edges, and fix a set of $s$ sources $S \subseteq V$. For any pair $u, v \in V$, let $W(u, v)$ denote the weight of the heaviest edge on the $u$ to $v$ shortest path. For any constant $0 < \epsilon < 1$, we compute $(1 + \epsilon, \beta(u, v))$-approximate shortest paths from all sources in $S$ in near linear (in $m + ns$) time, where $\beta(u, v) = O(W(u, v))$. That is, the multiplicative stretch is $1 + \epsilon$, and the additive stretch for any $u \in S$, $v \in V$ is $\beta(u, v)$. Previous results of this type [Coh00, Elk01] were only able to compute distance estimates and not paths, and had far inferior additive terms.

We also introduce distance oracles for parallel models of computation (PRAM). Specifically, for any parameter $\delta > 0$, we preprocess a given weighted graph in poly-logarithmic time and near linear work, and store a data structure of size $O(n^{1+\delta})$. Given any query $u \in V$, we return a $(1 + \epsilon, \beta(u, v))$-approximation to all distances $u \times V$ in $O_t(1)$ time, where $\beta(u, v) = O_t(W(u, v))$. Moreover, the dependence of both $\beta$ and the query time on $\delta$ can be made poly($1/\delta$), by increasing the multiplicative stretch (to some constant larger than $9$).

Our algorithms are based on new constructions of spanners, emulators and hopsets for weighted graphs. We devise a $(1 + \epsilon, \beta(u, v))$-spanner for weighted graphs of size $O(n^{1+1/t} + \log t \cdot n)$ and $\beta(u, v) = W(u, v) \cdot \left(\frac{\log t}{\epsilon}\right)^{O(\log t)}$. We can have an improved $\beta = W(u, v) \cdot t^{O(1)}$ at the cost of increasing the multiplicative stretch to a constant larger than $3$. In addition, we devise a $(c, t^{O(1)})$-hopset of size $O(n^{1+1/t} + \log t \cdot n)$ for any constant $c > 3$. 
1 Introduction

1.1 Shortest Paths from Multiple Sources

Computing efficiently approximate shortest paths in graphs is one of the most basic problems in algorithmic graph theory. In this paper we focus on the setting of weighted undirected graphs. Given a graph $G = (V,E)$ with nonnegative weights $w : E \to \mathbb{R}_+$, denote $n = |V|$ and $m = |E|$, and let $d_G$ be its shortest path metric. Let $S \subseteq V$ be a designated set of sources, and the goal is to quickly compute approximations to all distances from every vertex $u \in S$ to every vertex in $v \in V$. We say that the computed distance $\hat{d}(u,v)$ is an $(\alpha, \beta)$-approximation if it satisfies

$$d_G(u,v) \leq \hat{d}(u,v) \leq \alpha \cdot d_G(u,v) + \beta.$$  

(For instance, the case $S = \{u\}$ is the single-source shortest paths (SSSP); $S = V$ is the all-pairs shortest paths (APSP). We will refer to the general case as s-SSP.)

There are two main approaches to the s-SSP problem in the centralized setting. The first one is to solve (approximate) APSP. Cohen and Zwick [CZ01] devised a $(2,0)$-approximation algorithm for it in time $O(m^{1/2}n^{3/2})$, a $(7/3,0)$-approximation in time $O(n^{7/3})$, and a $(3,0)$-approximation in $O(n^2)$ time. Baswana and Kavitha [BK10] improved the latter result, and devised a $(2,W(u,v))$-approximation in $O(n^2)$ time, where $W(u,v)$ is the weight of the heaviest edge on the $u-v$ path (assuming all edge weights are at least 1). Zwick [Zwi02] came up with a $(1 + \epsilon, 0)$-approximation for APSP in time $O(n^{2八个} \cdot \log W_{\text{max}})$, where $W_{\text{max}}$ is the largest edge weight, and $\omega$ is the matrix multiplication exponent. Dor et al. [DHZ00] showed that whenever $\alpha < 2$, the $(\alpha,0)$-APSP problem is at least as hard as Boolean matrix multiplication. They have also shown that in unweighted graphs, for a parameter $t = 1, 2, \ldots$, one can compute $(1,2^t)$-approximate APSP in $O(n^{2八个} / t)$ time.

A general problem with this approach to s-SSP, is that when $s \ll n$, it is desirable to have running time of $\approx mn + s \cdot n$, rather than (typically much larger) $n^2$. The second approach to the s-SSP problem involves sparse spanners and emulators. A spanning subgraph $G' = (V,E')$ of $G$ is called an $(\alpha, \beta)$-spanner (also known as near-additive spanner), if for every pair $x, y \in V$,

$$d_G(x,y) \leq d_{G'}(x,y) \leq \alpha \cdot d_G(x,y) + \beta.$$  

The graph $G'$ is called an emulator rather than a spanner, if we drop the spanning condition (i.e., $E'$ may contain edges that are not in $E$). In the context of near-additive spanners and emulators, it is common to assume the graph is unweighted.

Cohen [Coh93], improving previous bounds of Awerbuch et al. [ABCP93], showed that, for any $t = 1, 2, \ldots$ and $\epsilon > 0$, $(2t + \epsilon)$-spanners with $O(n^{1+1/t})$ edges can be computed in $O(mn^{1/t} \log W_{\text{max}})$ time. As a result, she derived a $(2t + \epsilon)$-SSP algorithm with running time $O_s((m \log W_{\text{max}} + sn) \cdot n^{1/t})$. Soon afterwards, in [Coh00], she devised another algorithm which for any parameters $t \geq 1$, $\rho \geq 1/t, \epsilon > 0$, solves the s-SSP problem with approximation $(1 + \epsilon, (\log n / \epsilon)^{(\log t) / \rho}) \cdot W_{\text{max}}$ in time $O(m \cdot n^p + s \cdot n^{1+1/t})$. Her result is based on a certain “encoding of large distances”, that in hindsight can be viewed as a precursor of emulators. Explicit constructions of (near-additive) (i.e., with multiplicative stretch $1 + \epsilon$ and some additive stretch) spanners and emulators were given in [EP04, Elk01]. Specifically, [Elk01] showed that for any parameters $t, \rho, \epsilon$ as above, a $(1 + \epsilon, \beta \cdot W_{\text{max}})$-emulator (respectively, spanner) with $O(n^{1+1/t})$ (resp., $O(n^{1+1/t} \cdot W_{\text{max}})$) edges can be constructed in $O(mn^p)$ time, where

$$\beta_E = \beta(t, \epsilon, \rho) = (t / \epsilon)^{(\log t) / \rho}.$$  

Based on these emulators, [Elk01] derived an algorithm for s-SSP problem with running time $O(m \cdot n^p + sn^{1+1/t})$ and approximation $(1 + \epsilon, \beta \cdot W_{\text{max}})$. Using a spanner instead of emulator, this algorithm can also return paths (as opposed to just distances, as in Cohen’s algorithm [Coh00], and in the the algorithm of [Elk01] that employs emulator), albeit its running time becomes $O(m \cdot n^p + sn^{1+1/t} W_{\text{max}})$. The latter is prohibitively large when $W_{\text{max}}$ is large.
In this paper we devise an efficient construction of near-additive spanners that resolves this issue. Specifically, our algorithm constructs a \((1 + \epsilon, \beta \cdot W(u, v))-\)spanner with \(O(n^{1+1/t} + \log t \cdot n)\) edges, in time \(O(m \cdot n^\rho)\), where
\[
\beta = \beta_0 = \left(\frac{\log t}{\epsilon}\right)^{O(\log t + 1/\rho)}.
\]

Note that \(\beta_0\) is significantly smaller than \(\beta_E\), and we have no longer any dependence on \(W_{\text{max}}\) in the size of the spanner. Also, the additive term in our spanner is proportional to \(W(u, v)\) and not to \(W_{\text{max}}\) (as it is the case in [Elk01, Coh00]), and \(W(u, v)\) is typically much smaller than \(W_{\text{max}}\) for most pairs \(u, v\). Moreover, our new construction is universal in the sense that the same construction applies for all values of \(\epsilon > 0\) simultaneously, while it is not the case for the construction of [Elk01, Coh00]. Our resulting \(s\)-SSP algorithm computes a \((1 + \epsilon, \beta_0 W(u, v))-\)approximation in \(O(m \cdot n^\rho + sn^{1+1/t})\) time.

While our additive term \(\beta_0\) is much smaller than \(\beta_E\), its dependence on \(t\) is still superpolynomial (roughly \(t^{\log \log t}\)). To alleviate this issue, we devise an additional construction of spanners with stretch \((3 + \epsilon, \beta_1 W(u, v))\) for any constant \(\epsilon > 0\), where \(\beta_1 = t^{O(1)} \cdot 2^{O(1/\rho)}\), also with \(O(n^{1+1/t} + \log t \cdot n)\) edges. As a result, we also provide a \((3 + \epsilon, \beta_1 W(u, v))-\)approximate \(s\)-SSP algorithm within the same time \(O(m \cdot n^\rho + sn^{1+1/t})\). We remark that for unweighted graphs, a construction of \((3 + \epsilon, \beta)-\)spanners of size \(O(k^\varphi n^{1+1/F_k} \rho^{3\varphi})\) with \(\beta = 2^{O(k)}\), where \(\varphi \approx 1.61\) is the golden ratio and \(F_k\) is the \(k\)-th Fibonacci number, was given by Pettie in [Pet09, Pet10]. However, that construction does not apply to weighted graphs.

1.2 Distance Oracles

An (approximate) distance oracle for a graph \(G = (V, E)\), is a compact data structure that can (approximately) answer distance queries quickly. Thorup and Zwick [TZ01], for a parameter \(t = 1, 2, \ldots\), devised a distance oracle with stretch \((2t - 1, 0)\), size \(O(t \cdot n^{1+1/t})\), and query time \(O(t)\). The query time was subsequently improved to \(O(1)\) in [Wul13, Che14], and the size to \(O(n^{1+1/t})\) by [Che15].

It is natural to try to replace the multiplicative stretch of \(2t - 1\) of these oracles by a near-additive one. (A purely additive stretch for sparse spanners, emulators and distance oracles is impossible in view of a lower bound of Abboud and Bodwin [AB15]. So near-additive stretch is the best one can hope for in all these settings.) In fact, this question was explicitly raised by Thorup and Zwick [TZ06], Patrascu and Roditty [PR10] and Elkin and Pettie [EP15]. For unweighted graphs some results along these lines were shown by Patrascu and Roditty [PR10] and Abraham and Gavoille [AG11]. The latter authors showed, generalizing a result from [PR10], that for any \(t = 1, 2, \ldots\), one can have a distance oracle with stretch \((2t - 1, 1)\), query time \(O(t)\), and size \(O(n^{1+1/t} + \mu n)\). Elkin and Pettie [EP15] showed that using a much larger query time \(O(n^\rho)\), for a parameter \(\mu > 1\), one can also have a distance oracle with stretch \(((1/\mu) O(1), \beta(t, \epsilon) \cdot (1/\mu) O(1))\) and size \(O(n^{1+1/t} \log t)\), where \(\beta(t, \epsilon) = O(1/\epsilon^{\log t})\). In this result, the multiplicative stretch can be \(o(t)\), and thus the tradeoff between the multiplicative stretch and the size of the oracle is far below the lower bound given by Erdos-Simonits conjecture. (This happens, of course, at the expense of the additive term; see [EP15] for further details and discussion.)

For weighted graphs, however, no distance oracles with mixed stretch are currently known. We address this problem in the PRAM setting. In this model, multiple processors are connected to a single memory block, and the operations are performed in parallel by these processors in synchronous rounds. The running time is measured by the number of rounds, and the work by the number of processors multiplied by the number of rounds. We utilize the power of parallelization: our distance oracle, given a query \(u \in V\), reports in constant time all distances from \(u\) to every other point in \(V\). Specifically, we show how to preprocess a weighted graph \(G = (V, E)\) in polylogarithmic time, specifically, in \(\left(\frac{\log n}{\epsilon}\right)^{O(\log t + 1/\rho)}\) time and \(O(m \cdot n^\rho)\) work, and store a data structure of size \(O(n^{1+1/t} + \log t \cdot n) \cdot \log^* n\). Then, given any query \(u \in V\), our oracle reports \((1 + \epsilon, \beta \cdot W(u, \cdot))-\)approximate distances from \(u\) to all \(v \in V\) within time \(O\left(\frac{\log t + 1/\rho}{\epsilon}\right)^{\log t + 1/\rho}\) and work \(\tilde{O}(n^{1+1/t})\), where \(\beta = O\left(\frac{\log t + 1/\rho}{\epsilon}\right)^{\log t + 1/\rho}\). Observe that the query time is constant whenever \(t, \epsilon\) and \(\rho\) are. Hence one can have simultaneously a distance oracle of size arbitrarily close to linear (i.e., of size
We show there exists a $(1 + \epsilon, \beta)$-spanner of size $O(n^{1+1/t})$ of our query algorithm cannot be improved too much, as returning all $u \times V$ distances requires $O(n^t)$ work.

We remark that [EN19b] had a result on approximately answering single-source shortest path queries in PRAM. However, their space usage is at least as large as the size of the graph (only the preprocessing work is bounded).

From the technical viewpoint, this result is achieved via a combination of our new constructions of emulators and hopsets. For parameters $\alpha, \beta \geq 1$, we say that a set of edges $H$ is an $(\alpha, \beta)$-hopset for a (weighted) graph $G = (V,E)$, if by adding $H$ to the graph, every pair $x,y \in V$ has an $\alpha$-approximate shortest path consisting of at most $\beta$ hops; Formally,

$$d_G(x,y) \leq d_{G\cup H}^\beta(x,y) \leq \alpha \cdot d_G(x,y),$$

where $d_G^\beta(x,y)$ is the shortest path containing at most $\beta$ edges. The parameter $\alpha$ is called the stretch, and $\beta$ is the hopbound.

Hopsets and near-additive spanners are fundamental combinatorial constructs, and play a major role in efficient approximation of shortest paths in various computational model. These objects have been extensively studied in recent years [EP04, Elko04, TZ06, EZ06, Pet09, Pet10, BKMP10, EN19a, KS97, Coh00, Ber09, Nan14, HKN14, MPVX15, HKN16, FL16, EN16, ABP17, EN19b, HP19]. The main interest is to understand the triple tradeoff between the size of the hopset (respectively, spanner), to the stretch $\alpha$, and to the hopbound (resp., additive stretch) $\beta$. For algorithmic applications, it is also crucial to bound the construction time of the hopset/spanner.

We show a simple construction of hopsets, that given any integer parameter $t > 1$, provides a $(c, t^{O(1)})$-hopset of size $O(n^{1+1/t} + \log t \cdot n)$, for any constant $c > 3$. In addition, we can make $\beta$ as small as $O(t^2)$ at the cost of increasing the stretch to $c = 15$. This improves the previously known hopbound, which was $\Omega((\log t)^{\log t})$.

As mentioned above, we show near-additive spanners for weighted graphs, where the additive stretch for the pair $x, y$ may depend also on the largest edge weight on the corresponding path from $x$ to $y$, $W(x,y)$. We show there exists a $(1 + \epsilon, \beta \cdot W(\cdot, \cdot))$-spanner of size $O(n^{1+1/t} + \log t \cdot n)$ with $\beta \leq \left(\frac{\log t}{\epsilon}\right)^{O(\log t)}$. We also show how to analyze the construction so that it yields smaller additive stretch, while increasing the multiplicative one. Specifically, we get a $(c, t^{O(1)} \cdot W(\cdot, \cdot))$-spanner of size $O(n^{1+1/t} + \log t \cdot n)$, for every constant $c > 3$. Our emulators have a somewhat improved $\beta$.

Finally, our new algorithm for constructing weighted spanners immediately gives rise to an extension of the result of [EP15] to weighted graphs. Specifically, the distance oracle obtained by plugging our spanner into the distance oracle of [EP15] (in the centralized model) has stretch $((1/\mu)^{O(1)}, \beta(t, \epsilon)(1/\mu)^{O(1)} W(u,v))$, size $O(n^{1+1/t} + n \cdot \log t)$ and query time $O(n^\mu)$.

1.3 Technical Overview

Our constructions of the spanners and hopsets are a variation on the ones used by [TZ06, Pet09, EN19b, HP19], with certain subtle adaptations required to provide an improved hopbound (or additive stretch) $\beta$. The basic idea in all these constructions is to generate a random hierarchy of vertex sets $V = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_k = \emptyset$, where for each $0 \leq i < k-1$, each element in $A_i$ is sampled to $A_{i+1}$ with probability $\approx n^{-2^{i-k}}$ (one should think of $k = \log t$). For each $v \in V$, the pivot $p_i(v)$ is the closest vertex in $A_i$ to $v$. Then the set of edges $H$ is created by connecting, for every $0 \leq i \leq k-1$, every vertex $v \in A_i \setminus A_{i+1}$ to its bunch: all other vertices in $A_i$ that are closer to it than $p_{i+1}(v)$. One difference in our construction is that we also connect each vertex to all its (at most $k$) pivots. The main innovation in this work is the new analysis of this construction, yielding various hopsets, emulators and spanners with improved properties.
**Hopsets.** Our analysis of the stretch of some pair \( x, y \in V \) in \( G \cup H \) goes roughly as follows. We divide the \( x - y \) path to intervals, and try to connect these intervals using \( H \) (and some of the graph edges) with few hops. Each interval can either have a low hopbound and low stretch path; or fail, in which case that interval admits a nearby pivot (of some level \( i \)). Previous works considered two failed intervals (the leftmost and rightmost ones), and tried to find an \( x, y \) path via the pivots of these intervals. We deviate from this approach, and only use one such "failed" interval – we prove that \( x, y \) must have sufficiently nearby pivots, and find an appropriate path via the \( x, y \) pivots, rather than the intervals’ pivots.

Using two intervals would have given multiplicative stretch at least 5 (as indeed was the case in [Pet09]), while [Pet10] used a more involved construction and argument to lower the stretch to \( 3 + \epsilon \) (and only in the setting of spanners, for unweighted graphs). We show here that the simple construction suffices to that end.

**Emulators for weighted graphs.** Our new analysis of \( H \) as a near-additive emulator for weighted graphs extends the works of [TZ06, Pet09, Pet10, EN19b], who only handled unweighted graphs. Our main new idea, given a pair \( x, y \in V \), is to iteratively find a vertex \( z \) on the \( x - y \) shortest path (sufficiently far from \( x \)), that admits in \( H \) a path with low multiplicative stretch. When there is no such \( z \), we show that we can reach \( y \) via at most \( \beta(x, y) \) steps. This method replaces the partitioning of the \( x, y \) path to equal size intervals (used in previous works), which cannot work for weighted graphs. We also incorporate in this method the idea described above for the hopset analysis.

**Spanners for weighted graphs.** The main issue we need to overcome in designing spanners, rather than emulators, is that we must use the graph edges. So in order to connect vertices \( v \in A_i \setminus A_{i+1} \) to the vertices in their bunch \( B(v) \), we need to add paths of possibly many edges in the graph (rather than a single edge, as for emulators/hopsets). We thus change the construction, and present a simplified and improved version of the ideas of [Pet09], using the so called half-bunch rather than the standard one. We then argue that all the shortest paths added will have few pairwise intersections, and thus contain a small number of edges. To that end, we need a stricter bound on the size of bunches, hence we alter the sampling probabilities accordingly.

### 1.4 Organization

In Section 2 we describe the construction of our hopsets and emulators. In Section 3 we give the analysis for hopsets, and in Section 4 for emulators. In Section 5 we show the construction and analysis of our spanners. We conclude in Section 6 with the algorithmic applications to approximate shortest path and PRAM distance oracles.

**Bibliographical note.** Independently from us, related results on hopsets and near-additive spanners with improved hopbound and additive stretch are shown in [BLP19].

### 2 Construction

We use a similar construction to that of [TZ06, EN19b, HP19], the main difference is that every vertex connects to pivots in all levels. Let \( G = (V, E) \) be graph with \( n \) vertices, and fix an integer parameter \( k \geq 1 \). Let \( \nu = 1/(2^k - 1) \). Let \( A_0 \ldots A_k \) be sets of vertices such that \( A_0 = V \), \( A_k = \emptyset \), and for \( 0 \leq i \leq k - 2 \), \( A_{i+1} \) is created by sampling every element from \( A_i \) with probability \( q_i = n^{-2^i \nu} \cdot 2^{-2^i - 1} \). For every \( 0 \leq i \leq k - 1 \), the expected size of \( A_i \) is:

\[
N_i := E[|A_i|] = n \prod_{j=0}^{i-1} q_j = n^{1 - (2^i - 1) \nu} \cdot 2^{-2^i - i + 1}
\]

For every \( i \in [k-1] \), define the pivot \( p_i(v) \) to be the closest vertex in \( A_i \) to \( v \), breaking ties by lexicographic order. For every \( v \in A_i \setminus A_{i+1} \) define the bunch
\[ B(u) = \{ v \in A_i : d_G(u, v) < d_G(u, A_{i+1}) \} \cup \{ p_j(u) | i < j < k \}. \]

That is, the bunch \( B(u) \) contains all the vertices which are in \( A_i \) and closer to \( u \) than \( p_{i+1}(u) \), and at most \( k \) pivots. We then define \( H = \{ (u, v) : u \in V, v \in B(u) \} \), where the length of the edge \((u, v)\) is set as the length of the shortest path between \( u, v \) in \( G \).

**Size analysis.** The analysis of the size of \( H \) is very similar to previous works, we include it for completeness. If we order the vertices in \( A_i \) by their distance to \( u \), it is easy to see that the number of vertices which are in \( A_i \) and closer than \( p_{i+1}(u) \) is bounded by a random variable sampled from a geometric distribution with parameter \( q_i \). Hence \( E[|B(u)|] \leq k + 1/q_i = k + n^{2+e}2^{e+1} \). For \( u \in A_{k-1} \), since \( p_k(u) \) doesn’t exist, \( B(u) \) contains all the vertices in \( A_{k-1} \). The number of vertices in \( A_{k-1} \) is a random variable sampled from binomial Distribution with parameters \((n, \prod_{j=0}^{k-1} q_j) = (n, n^{-2^{k-1}-1} \cdot 2^{-2^{k-1}-k+2}) \). Hence, the expected number of edges added by bunches of vertices in \( A_{k-1} \) is

\[
E \left[ \left( \frac{|A_{k-1}|}{2} \right) \right] \leq E[|A_{k-1}|^2] = E[|A_{k-1}|^2] + Var(|A_{k-1}|) = n^2 \prod_{j=0}^{k-1} q_j^2 + n(1 - \prod_{j=0}^{k-1} q_j) \prod_{j=0}^{k-1} q_j
\]

\[
\leq n^2 - 2^{(2^{k-1}-1)v} \cdot 2^{2(2^{k-1}-k+2)} + n \cdot 2^{-2^{k-1}-k+2} \leq (n^{1+\nu} + n)2^{3-k}.
\]

Hence, the total expected number of edges in \( H \) is:

\[
\sum_{i=0}^{k-2} (N_i \cdot n^{2+e} \cdot 2^{e+1}) + E[|A_{k-1}|^2] + kn
\]

\[
= \sum_{i=0}^{k-2} (n^{1+\nu} \cdot 2^{-i+2}) + E[|A_{k-1}|^2] + kn
\]

\[
= O(kn + n^{1+\nu}).
\]

### 3 A \((3 + \epsilon, \beta)\)-Hopset

Here we show that the set \( H \) of Section 2 serves as a \((3 + \epsilon, \beta)\)-hopset, for all \( 0 < \epsilon < 12 \) simultaneously, with \( \beta = 2^{O(k \log(1/\epsilon))} \).

Denote by \( d^{(i)}_{G,H}(u, v) \) the length of the shortest path between \( u, v \) in \( G \) that contains at most \( t \) edges. The following lemma bounds the number of hops and the stretch of the constructed hopset:

**Lemma 3.1.** Fix any \( 0 < \delta \leq 1/4 \) and any \( x, y \in V \). Then for every \( 0 \leq i \leq k - 1 \), at least one of the following holds:

1. \( d^{(2(\frac{1}{\delta})^{i-1})}_{G,H}(x, y) \leq (3 + \frac{12\delta}{1-3\delta})d_G(x, y) \)
2. \( d^{(1)}_{G,H}(x, p_{i+1}(x)) \leq \frac{3}{1-3\delta}d_G(x, y) \)

**Proof.** The proof is by induction on \( i \). For the base case \( i = 0 \), if \( y \in B(x) \), then the edge \((x, y)\) was added to the hopset and the first item holds. If not, it means that \( d_G(x, y) \geq d_G(x, p_1(x)) \), so the second item holds (since the coefficient of the right hand size is between \( 3 \) and \( 12 \) for \( 0 < \delta \leq 1/4 \)).

Assume the claim holds for \( i \), and we will prove for \( i + 1 \). Partition the shortest path between \( x \) and \( y \) into \( J \leq 1/\delta \) segments \( \{L_j = [u_j, v_j]\}_{j \in [J]} \) each of length at most \( \delta \cdot d_G(x, y) \), and at most \((1/\delta - 1)\) edges \( \{(v_j, u_{j+1})\}_{j \in [J]} \) between consecutive segments. We can use the following: setting \( u_1 = x \), and for each \( j \in [J] \), set \( v_j \) as the vertex in the shortest path between \( u_j \) and \( y \) which is farthest from \( u_j \), but still \( d_G(u_j, v_j) \leq \delta \cdot d_G(x, y) \). If \( v_j \neq y \), set \( u_{j+1} \) as the vertex which follows \( v_j \) in the shortest path between \( x \) and
y. Otherwise set $u_{j+1} = y$. This partition satisfies our requirement $J \leq 1/\delta$ because for every $j \in [J - 1]$, $d_G(u_j, u_{j+1}) > \delta \cdot d_G(x, y)$ (otherwise, we could have chosen $v_j$ as $u_{j+1}$).

Next, apply the induction hypothesis for all the pairs $(u_j, v_j)$ with parameter $i$. If for all the pairs $(u_j, v_j)$ the first item holds, we can show that the first item holds for $(x, y)$ with parameter $i$. Consider the path from $x$ to $y$ which uses the guaranteed path in $G \cup H$ of the first item for all the pairs $(u_j, v_j)$, and the edges $(v_j, u_{j+1})$. The number of hops in this path is bounded by $(1/\delta) \cdot (2(1/\delta)^i - 1) + (1/\delta - 1) \leq 2 \cdot (1/\delta)^{i+1} - 1$. The length of the path is bounded by:

$$d_{G \cup H}^{(2(1/\delta)^{i+1} - 1)}(x, y) \leq \sum_{j \in [J]} (d_{G \cup H}^{(2(1/\delta)^{i} - 1)}(u_j, v_j) + d_G^{(1)}(v_j, u_{j+1})) \leq (3 + \frac{12\delta}{1 - 3\delta})d_G(x, y)$$

Otherwise, there exist at least one segment for which the first item doesn’t hold. Let $l \in [J]$ be the smallest index so that only the second item holds for the pair $(u_l, v_l)$. By the induction hypothesis

$$d_{G \cup H}^{(1)}(u_l, p_{l+1}(u_l)) \leq \frac{3}{1 - 3\delta}d_G(u_l, v_l) \leq \frac{3\delta}{1 - 3\delta}d_G(x, y).$$

Since we added the edges $(x, p_{l+1}(x)), (y, p_{l+1}(y))$ to the hopset $H$, by the triangle inequality,

$$d_{G \cup H}^{(1)}(x, p_{l+1}(x)) \leq d_G(x, u_l) + d_G(u_l, u_{l+1}(u_l)) \leq d_G(x, u_l) + \frac{3\delta}{1 - 3\delta}d_G(x, y), \tag{1}$$

$$d_{G \cup H}^{(1)}(y, p_{l+1}(y)) \leq d_G(y, u_l) + d_G(u_l, u_{l+1}(u_l)) \leq d_G(y, u_l) + \frac{3\delta}{1 - 3\delta}d_G(x, y). \tag{2}$$

If the edge $(p_{l+1}(x), p_{l+1}(y))$ exists in $H$, its length can be bounded by

$$d_{G \cup H}^{(1)}(p_{l+1}(x), p_{l+1}(y)) \leq d_{G \cup H}^{(1)}(p_{l+1}(x), x) + d_G(x, y) + d_{G \cup H}^{(1)}(y, p_{l+1}(y)). \tag{3}$$

Thus, the distance between $x$ and $y$ using 3 hops is

$$d_{G \cup H}^{(3)}(x, y) \leq d_{G \cup H}^{(1)}(x, p_{l+1}(x)) + d_{G \cup H}^{(1)}(p_{l+1}(x), p_{l+1}(y)) + d_{G \cup H}^{(1)}(p_{l+1}(y), y)$$

$$\leq 2d_{G \cup H}^{(1)}(x, p_{l+1}(x)) + d_G(x, y) + 2d_{G \cup H}^{(1)}(p_{l+1}(y), y)$$

$$\leq 2(d_G(x, u_l) + \frac{3\delta}{1 - 3\delta}d_G(x, y)) + d_G(x, y) + 2(d_G(y, u_l) + \frac{3\delta}{1 - 3\delta}d_G(x, y))$$

$$\leq (3 + \frac{12\delta}{1 - 3\delta})d_G(x, y),$$

therefore the first item holds.

If $(p_{l+1}(x), p_{l+1}(y)) \notin H$, then we know that $d_{G \cup H}^{(1)}(p_{l+1}(x), p_{l+2}(p_{l+1}(x))) \leq d_G(p_{l+1}(x), p_{l+1}(y))$. We can bound the distance $d_{G \cup H}^{(1)}(x, p_{l+2}(x))$ using the triangle inequality:

$$d_{G \cup H}^{(1)}(x, p_{l+2}(x)) \leq d_G(x, p_{l+1}(x)) + d_G(p_{l+1}(x), p_{l+2}(p_{l+1}(x)))$$

$$\leq 2d_G(p_{l+1}(x), x) + d_G(x, y) + d_G(y, p_{l+1}(y))$$

$$\leq 2(d_G(x, u_l) + \frac{3\delta}{1 - 3\delta}d_G(x, y)) + d_G(x, y) + d_G(y, u_l) + \frac{3\delta}{1 - 3\delta}d_G(x, y)$$

$$\leq 3d_G(x, y) + \frac{9\delta}{1 - 3\delta}d_G(x, y) = \frac{3}{1 - 3\delta}d_G(x, y).$$

Thus the second item holds.
We conclude by summarizing the main result of this section.

**Theorem 3.2.** For any weighted graph $G = (V, E)$ on $n$ vertices, and any $k \geq 1$, there exists $H$ of size at most $O(kn + n^{1+1/(2^{k-1})})$, which is a $(3 + \varepsilon, \beta)$-hopset for any $0 < \varepsilon \leq 12$, with $\beta = 2(3 + 12/\varepsilon)^k.$

**Proof.** Let $x, y \in V$. Apply lemma 3.1 for $x, y$ with $\delta = \frac{8}{12 + 3\varepsilon}$ and $i = k - 1$. Since $A_k = \emptyset$, the first item must hold:

$$d_{G,H}^{(2(3+12/\varepsilon)^k-1)}(x, y) \leq (3 + \varepsilon)d_G(x, y).$$

$\Box$

**Remark 3.3.** Note that at its lowest, the hopbound is $O(4^k)$, achieved with stretch 15.

## 4 Near-Additive Emulators for Weighted Graph

### 4.1 A $(3 + \varepsilon, \beta(\cdot, \cdot))$-Emulator

In this section we show that the same $H$ constructed in Section 2 can also serve as a $(3 + \varepsilon, \beta)$ emulator for weighted graphs.

Let $G = (V, E)$ be a weighted graph with non-negative weights $w : E \rightarrow \mathbb{R}_+$, and for $x, y \in V$ let $W(x, y) = \max\{w(e) : e \in P_{xy}\}$ (where $P_{xy}$ is a shortest path from $x$ to $y$ in $G$). Let $k \geq 1$ and $\Delta > 3$ be given parameters (think of $\Delta = 3 + O(1/\varepsilon)$). Fix a pair $x, y \in V$, define $D_{-1} = 0$ and for any integer $i \geq 0$, let $D_i = W(x, y) \cdot \sum_{j=0}^i \Delta^j$. We can easily verify that

$$D_{i+1} = \Delta \cdot D_i + W(x, y). \quad (4)$$

**Lemma 4.1.** Let $0 \leq i \leq k$ and let $x, y \in V$ such that $d_G(x, y) \leq D_i \text{ and } d_H(x, p_i(x)) \leq \frac{2\Delta}{\Delta - 3}D_{i-1}$. Define $m = \max\{\Delta D_{i-1}, d_G(x, y)\}$. Then at least one of the following holds:

1. $d_H(x, y) \leq (3 + \frac{8}{\Delta-3})m$.
2. $d_H(x, p_{i+1}(x)) \leq \frac{2\Delta}{\Delta - 3}D_i$.

**Proof.** The proof is by induction on $i$. For the base case $i = 0$, if $y \in B(x)$, the first item holds. Otherwise $d_H(x, p_1(x)) \leq d_G(x, y) \leq W(x, y) = D_0$, thus the second item holds.

Assume the claim holds for $i$ and prove for $i + 1$. By the triangle inequality:

$$d_H(y, p_{i+1}(y)) \leq d_G(y, x) + d_G(x, p_{i+1}(x)). \quad (5)$$

If $p_{i+1}(y) \in B(p_{i+1}(x))$, we have

$$d_H(p_{i+1}(x), p_{i+1}(y)) \leq d_G(p_{i+1}(x), x) + d_G(x, y) + d_G(y, p_{i+1}(y)). \quad (6)$$

Thus, the distance between $x$ and $y$ is

$$d_H(x, y) \leq d_H(x, p_{i+1}(x)) + d_H(p_{i+1}(x), p_{i+1}(y)) + d_H(p_{i+1}(y), y) \leq 2d_H(x, p_{i+1}(x)) + d_G(x, y) + 2d_H(p_{i+1}(y), y) \leq 2d_H(x, p_{i+1}(x)) + d_G(x, y) + 2(d_G(y, x) + d_G(x, p_{i+1}(x))) \leq 3d_G(x, y) + \frac{4 \cdot 2\Delta}{\Delta - 3}D_i \leq \left(3 + \frac{8}{\Delta-3}\right)m.$$

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therefore the first item holds.
If \( p_{i+1}(y) \notin B(p_{i+1}(x)) \), then we know that
\[
d_H(p_{i+1}(x), p_{i+1}(y)) \leq d_G(p_{i+1}(x), p_{i+1}(y)) .
\] (7)
We can bound the distance \( d_H(x, p_{i+2}(x)) \) as follows.
\[
d_H(x, p_{i+2}(x)) \leq d_G(x, p_{i+1}(x)) + d_G(p_{i+1}(x), p_{i+2}(x))
\leq 2d_G(p_{i+1}(x), x) + d_G(x, y) + d_G(y, p_{i+1}(y))
\leq 2d_G(x, y) + 3 \cdot 2\Delta_i D_i
\leq 2D_i + \frac{6}{\Delta} D_{i+1}
= \frac{2\Delta}{\Delta - 3} D_{i+1}.
\]
Hence the second item holds.

We are now ready to state the result of this section.

**Theorem 4.2.** For any weighted graph \( G = (V, E) \) on \( n \) vertices, and any \( k \geq 1 \), there exists \( H \) of size at most \( O(kn + n^{1+1/(2^k-1)}) \), which is a \((3 + \varepsilon, \beta(\cdot, \cdot))\)-emulator for any \( \varepsilon > 0 \) with \( \beta(x, y) = 2(3 + \varepsilon) \cdot (3 + 8/\varepsilon)^{k-1} \cdot W(x, y) \).

**Proof.** Let \( x, y \in V \). Recall that \( P_{xy} \) is the shortest path between \( x \) and \( y \) in \( G \), and fix \( \Delta = 3 + 8/\varepsilon \).
Initialize \( i = 0 \). Let \( z \) be the farthest vertex from \( x \) in \( P_{xy} \) satisfying \( d_G(x, z) \leq D_i \). Note the the requirement \( d_H(x, p_0(x)) \leq \frac{2\Delta}{\Delta - 3} D_{i-1} = 0 \) holds since \( p_0(x) = x \). Apply lemma 4.1 on \( x, z \) and \( i \). If the second item holds, we increase \( i \) by one, update \( z \) to be the last vertex in \( P(x, y) \) satisfying \( d_G(x, z) \leq D_i \), and apply the lemma again for \( x, z \) and \( i \) (since the second item held for \( i - 1 \), we have that \( d_H(x, p_i(x)) \leq \frac{2\Delta}{\Delta - 3} D_{i-1} \) indeed holds).

Consider now the index \( i \) such that the first item holds (we must find such an index, since at \( i = k - 1 \) there is no pivot in level \( k \)). If it is the case that \( d_G(x, y) \geq D_i \) then since \( D_i - \Delta D_{i-1} = W(x, y) \), it must be that \( d_G(x, z) \geq \Delta D_{i-1} \), as otherwise we could have taken a further away \( z \) (recall that every edge on this path has weight at most \( W(x, y) \)). Therefore \( m = d_G(x, z) \) and we found a path in \( H \) from \( x \) to \( z \) with stretch at most \( 3 + \frac{8}{\Delta - 3} \). Next we update \( x = z, i = 0 \) and repeat the same procedure all over again.

The last remaining case is that we found an index \( i \) such that the first item holds but \( d_G(x, y) < D_i \). Note that in such a case it must be that \( z = y \). The path in \( H \) we have from \( x \) to \( y \) is of length at most \( (3 + \frac{8}{\Delta - 3}) \cdot D_i = (3 + \varepsilon) \cdot D_i \).
As \( i \leq k - 1 \) and \( D_{k-1} \leq 2\Delta^{k-1} \cdot W(x, y) \), we have that
\[
d_H(x, y) \leq 2(3 + \varepsilon) \cdot (3 + 8/\varepsilon)^{k-1} \cdot W(x, y),
\]
which is our additive stretch \( \beta \).

**Remark 4.3.** We note that the analysis did not use the fact that the path from \( x \) to \( y \) is a shortest path. In particular, for every path \( P \) from \( x \) to \( y \) of length \( d \), we can obtain a path in \( H \) of length at most \( (3 + \varepsilon) \cdot d + \beta(P) \), where \( \beta(P) = 2(3 + \varepsilon) \cdot (3 + 8/\varepsilon)^{k-1} \cdot W(P) \), and \( W(P) \) is the largest edge weight in \( P \).
4.2 A \((1 + \varepsilon, \beta(\cdot, \cdot))\)-Emulator for Weighted Graphs

In this section we will show the construction of Section 2 is also a \(1 + \varepsilon\) emulator for weighted graphs. Recall that [TZ06, EN19b] showed it was an emulator for unweighted graphs. From a high level, the proof mimics the ideas presented in Section 4.1, i.e., for the pair \(x, y\), we try to find the farthest \(z\) that admits a low multiplicative stretch path from \(x\) in \(H\). The specifics are somewhat different, though, in the \(1 + \varepsilon\) multiplicative stretch regime, so we provide the full details.

The first lemma asserts that pairs which are sufficiently far apart, admit a low stretch path or a nearby pivot of a higher level.

**Lemma 4.4.** Fix \(\Delta > 3\). Let \(0 \leq i < k\) and let \(x, y \in V\) such that \(d_G(x, y) \geq (3\Delta)^i W(x, y)\). Then at least one of the following holds:

1. \(d_H(x, y) \leq (1 + \frac{4i}{\Delta - 3}) d_G(x, y)\)
2. \(d_H(x, p_{i+1}(x)) \leq \frac{\Delta}{\Delta - 3} d_G(x, y)\)

**Proof.** Denote \(W = W(x, y)\). The proof is by induction on \(i\). The base case \(i = 0\) can be easily verified as before. Assume the claim holds for \(i\) and prove for \(i + 1\).

Divide the shortest path between \(x\) and \(y\) into \(J\) segments \(\{L_j = [u_j, u_{j+1}]\}_{j \in [J]}\) of size at least \((3\Delta)^i W\) and at most \(d_G(x, y)/\Delta\). It can be done as follows: define \(u_1 = x, j = 2\) and walk on the shortest path from \(x\) to \(y\). Define \(u_j\) as the first vertex which \(d_G(u_{j-1}, u_j) \geq (3\Delta)^i W\) or define \(u_j = y\) if \(d_G(u_{j-1}, y) < (3\Delta)^i W\). Increase \(j\) by 1 and repeat. Finally, we join the last two segments. The length of the last segment is at most \((3\Delta)^i W + W + (3\Delta)^i W \leq 3^{i+1} \Delta^i W \leq d_G(x, y)/\Delta\). The length of any segments except the last is also at most \((3\Delta)^i W + W \leq d_G(x, y)/\Delta\).

Apply the induction hypothesis for every segment with parameter \(i\). If for all the segments the first item holds, then first item holds for \(x, y\) and \(i + 1\), since

\[
d_H(x, y) \leq \sum_{j \in J} d_H(u_j, u_{j+1}) \leq \sum_{j \in J} (1 + \frac{4i}{\Delta - 3}) d_G(u_j, u_{j+1}) \leq (1 + \frac{4i}{\Delta - 3}) d_G(x, y).
\]

Otherwise, for at least one segment the second item holds. Let \(l, r\) be the indices of the start of the first segment for which the second item holds, and the end of the last segment for which second item holds, respectively. By symmetry of the first item, the second also holds for the pair \(u_r, u_{r-1}\) with parameter \(i\). Hence

\[
d_H(u_r, p_{i+1}(u_r)) \leq \frac{\Delta}{\Delta - 3} d_G(u_{r-1}, u_r) \leq \frac{d_G(x, y)}{\Delta - 3},
\]

\[
d_H(u_l, p_{i+1}(u_l)) \leq \frac{\Delta}{\Delta - 3} d_G(u_l, u_{l+1}) \leq \frac{d_G(x, y)}{\Delta - 3}. \tag{8}
\]

If \(p_{i+1}(u_l) \in B(p_{i+1}(u_l))\), we have

\[
d_H(p_{i+1}(u_l), p_{i+1}(u_r)) \leq d_G(p_{i+1}(u_l), u_l) + d_G(u_l, u_r) + d_G(u_r, p_{i+1}(u_r)). \tag{9}
\]

By the triangle inequality,

\[
d_H(u_l, u_r) \leq d_H(u_l, p_{i+1}(u_l)) + d_H(p_{i+1}(u_l), p_{i+1}(u_r)) + d_H(p_{i+1}(u_r), u_r)
\]

\[
\leq 2d_H(u_l, p_{i+1}(u_l)) + d_G(u_l, u_r) + 2d_H(p_{i+1}(u_r), u_r)
\]

\[
\leq 4d_G(x, y) + d_G(u_l, u_r). \tag{10}
\]
Thus, the distance between $x$ and $y$ in $H$,

\[ d_H(x, y) \leq \sum_{j \in [l]} d_H(u_j, u_{j+1}) \]

\[ \leq \sum_{j=1}^{l-1} \left( 1 + \frac{4i}{3} \right) d_G(u_j, u_{j+1}) + d_H(u_l, u_r) + \sum_{j=r}^{l} \left( 1 + \frac{4i}{3} \right) d_G(u_j, u_{j+1}) \]

\[ \leq \left( 1 + \frac{4i}{3} \right) d_G(x, u_l) + d_H(u_l, u_r) + \left( 1 + \frac{4i}{3} \right) d_G(u_r, y) \]

\[ \leq \left( 1 + \frac{4(i + 1)}{3} \right) d_G(x, y), \]

therefore the first item holds.

If $p_{i+1}(u_r) \notin B(p_{i+1}(u_l))$ then

\[ d_H(p_{i+1}(u_l), p_{i+2}(p_{i+1}(u_l))) \leq d_G(p_{i+1}(u_l), p_{i+1}(u_r)) . \quad (11) \]

We can bound the distance $d_H(x, p_{i+2}(x))$ by

\[ d_H(x, p_{i+2}(x)) \leq d_G(x, u_l) + d_G(u_l, p_{i+1}(u_l)) + d_G(p_{i+1}(u_l), p_{i+2}(p_{i+1}(u_l))) \]

\[ \leq d_G(x, u_l) + d_G(u_l, p_{i+1}(u_l)) + d_G(p_{i+1}(u_l), p_{i+1}(u_r)) \]

\[ \leq d_G(x, u_l) + d_G(u_l, p_{i+1}(u_l)) + d_G(u_l, p_{i+1}(u_l)) + d_G(u_l, u_r) + d_G(p_{i+1}(u_r), u_r) \]

\[ \leq d_G(x, y) + 3d_G(x, y) = \frac{\Delta}{\Delta - 3} d_G(x, y). \]

Hence the second item holds. \hfill \Box

The previous lemma is useful for vertices which are very far from each other, since for $i = k - 1$ the first item must hold. For vertices which are close to each other, we will need the following lemma.

**Lemma 4.5.** Let $0 \leq i < k$ and fix $x, y \in V$. Let $m = \max \{d_G(x, p_i(x)), d_G(y, p_i(y)), d_G(x, y)\}$. Then at least one of the following holds:

1. $d_H(x, y) \leq 5m$
2. $d_H(x, p_{i+1}(x)) \leq 4m$

**Proof.** If $p_i(y) \in B(p_i(x))$ the first item holds, since

\[ d_H(x, y) \leq d_H(x, p_i(x)) + d_H(p_i(x), p_i(y)) + d_H(p_i(y), y) \]

\[ \leq d_G(x, p_i(x)) + d_G(p_i(x), x) + d_G(x, y) + d_G(y, p_i(y)) + d_G(p_i(y), y) \leq 5m . \]

If $p_i(y) \notin B(p_i(x))$, then $d_G(p_i(x), p_{i+1}(p_i(x))) \leq d_G(p_i(x), p_i(y))$, in this case the second item holds, as

\[ d_H(x, p_{i+1}(x)) \leq d_H(x, p_i(x)) + d_H(p_i(x), p_{i+1}(p_i(x))) \]

\[ \leq d_G(x, p_i(x)) + d_G(p_i(x), x) + d_G(x, y) + d_G(y, p_i(y)) \leq 4m . \] \hfill \Box

**Theorem 4.6.** For any weighted graph $G = (V, E)$ on $n$ vertices, and any integer $k > 1$, there exists $H$ of size at most $O(kn + n^{1+1/(2^k-1)})$, which is a $(1 + \varepsilon, \beta(\cdot, \cdot))$-emulator for any any $0 < \varepsilon < 1$, where $\beta(x, y) = O(9 + \frac{12(k-1)}{\varepsilon})^{k-1} \cdot W(x, y)$.  

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Proof. Fix $\Delta = 3 + \frac{4(k-1)}{\varepsilon}$. Let $x, y \in V$, and $W = W(x, y)$. If $d_G(x, y) \geq (3\Delta)^{k-1}W$, we can apply Lemma 4.4 for $x, y$ and $i = k - 1$. Since $p_k(x)$ doesn’t exists, the first item must hold. Thus,

$$d_H(x, y) \leq \left(1 + \frac{4(k-1)}{\Delta - 3}\right)d_G(x, y) = (1 + \varepsilon)d_G(x, y).$$

Otherwise, take the integer $0 \leq i < k - 1$ satisfying $(3\Delta)^iW \leq d_G(x, y) < (3\Delta)^{i+1}W$ (note that there must be such an $i$, since $d_G(x, y) \geq W$). Apply Lemma 4.4 for $x, y$ and $i$. If the first item holds, we will get a constant $1 + \varepsilon$ stretch as before.

Otherwise, the second item holds, and we know that $d_G(x, p_{i+1}(x)) \leq \frac{\Delta}{\Delta - 3}d_G(x, y) \leq 2d_G(x, y)$. By symmetry of $x, y$ in the first item of Lemma 4.4, we have $d_G(y, p_{i+1}(y)) \leq 2d_G(x, y)$ as well. Set $j = i + 1$ and apply Lemma 4.5 with $x, y, j$, noting that $m \leq 2d_G(x, y)$. If the first item holds, we found a path in $H$ from $x$ to $y$ of length at most $5m \leq 10d_G(x, y) \leq \beta(x, y)$.

If the second items holds, we increase $j$ by one and apply Lemma 4.5 again. We continue this procedure until the first item holds. The bound $m$ increases every iteration by a factor of at most 4. Since the first item must hold for $j = k - 1$, the path we found is of length at most $10 \cdot 4^{k-2}d_G(x, y) \leq 10 \cdot 4^{k-1} \cdot (3\Delta)^{i+1}$, which is maximized for $i = k - 2$. Hence the additive stretch is at most $O((3\Delta)^{k-1}) = O((9 + \frac{12(k-1)}{\varepsilon})^{k-1})$, as required.

\qed

5 Near-Additive Spanners for Weighted Graphs

In this section we devise our spanners for weighted graphs. We first described the new construction, which differ from that of Section 2 in several aspects, which are required in order to keep the size of the spanner under control (and independent of $W_{\max}$).

Construction. Let $G = (V, E)$ be a weighted graph with $n$ vertices, and fix an integer parameter $k \geq 1$. Let $\nu = \frac{1}{(4/3)^{k-1}}$. Let $A_0 \ldots A_k$ be sets of vertices such that $A_0 = V$, $A_k = \emptyset$, and for $0 \leq i \leq k - 2$, $A_{i+1}$ is created by sampling every element from $A_i$ with probability $q_i = n^{-4\nu / 3^{i+1}}$. For every $0 \leq i \leq k - 1$, the expected size of $A_i$ is:

$$N_i := E[|A_i|] = n \prod_{j=0}^{i-1} q_j = n^{1-\frac{i}{\nu} \sum_{j=0}^{i-1} (4/3)^j} = n^{1-((4/3)^i - 1)\nu}.$$

For every $i \in [k-1]$, define the pivot $p_i(v)$ to be the closest vertex in $A_i$ to $v$, breaking ties by lexicographic order. For every $v \in A_i \backslash A_{i+1}$ define the half bunch

$$B_{1/2}(u) = \{v \in A_i : d_G(u, v) < d_G(u, A_{i+1})/2\} \cup \{p_j(u) | i < j < k\}.$$

And also define the usual bunch $B(u)$ as before. Let $H = \{P_{uv} : u \in V, v \in B_{1/2}(u)\}$, where $P_{uv}$ is the shortest path between $u, v$ in $G$ (if there is more than one, break ties consistently, by vertex id, say).

Size analysis. The following lemma will be useful to bound the size of the spanner $H$.

Lemma 5.1. Fix $0 \leq i \leq k - 1$. Let $u, v, x, y \in A_i$ be such that $v \in B_{1/2}(u)$ and $y \in B_{1/2}(x)$, and $P_{uv} \cap P_{xy} \neq \emptyset$, then all four points are in $B(u)$, or all four are in $B(x)$.

Proof. Assume w.l.o.g. that $P_{uv}$ is not shorter than $P_{xy}$. Let $z \in V$ be a point in the intersection of the two shortest paths, then

$$d_G(u, x) \leq d_G(u, z) + d_G(z, x) \leq d_G(u, v) + d_G(y, x) \leq 2d_G(u, v) < d_G(u, A_{i+1}),$$

so $x \in B(u)$. The calculation showing $y \in B(u)$ is essentially the same. \qed
Fix $0 \leq i \leq k - 2$, and consider the graph $G_i$ containing all the shortest paths $P_{uv}$ with $u \in A_i$ and $v \in B_{1/2}(u)$. We claim that the number of edges in $G_i$ is at most $O(n + C_i)$, where $C_i$ is the number of pairwise intersections between these shortest paths. This is because vertices participating in at most 1 path have degree at most 2, and each intersection increases the degree of one vertex by at most 2 (recall that shortest paths can meet at most once).

**Claim 5.2.** $E[|C_i|] \leq O(n^{1+\nu})$.

**Proof.** By Lemma 5.1 each intersecting pair of paths $P_{uv}$ and $P_{xy}$ we have that all four points belong to the same bunch. Thus, each $u \in A_i$ can introduce at most $|B(u)|^2$ pairwise intersecting paths. Recall that $|B(u)|$ is a random variable distributed geometrically with parameter $q_i$, so

$$
E[|B(u)|^2] = \sum_{j=1}^{\infty} j^2 \cdot q_i \cdot (1 - q_i)^{j-1} \leq q_i \cdot \sum_{j=1}^{\infty} (1 - q_i)^{j-1} \cdot j(j+1)(j+2) \leq \frac{6}{q_i^2}.
$$

Thus the expected number of intersections at level $i$ is at most

$$
E\left[\sum_{u \in A_i} |B(u)|^2\right] \leq O(N_i/q_i^3) = O(n^{1-((4/3)^i-1)/((4/3)^i-1)/((4/3)^i-1)}) = O(n^{1+\nu}).
$$

It remains to bound path intersections in the last level $k - 1$. Recall that

$$
N_{k-1} = n^{1-((4/3)^{k-1-1}/((4/3)^{k-1-1}/((4/3)^{k-1-1})} = n^{1/4}.
$$

Since the random choices for each point are independent, we have by Chernoff bound that $Pr[N_{k-1} > 2n^{1/4}] \leq e^{-O(n^{1/4})}$, so with very high probability the last set $A_{k-1}$ contains $O(n^{1/4})$ points. It means that we have $O(\sqrt{n})$ paths connecting these points, and even if they all intersect, they can yield at most $O(n)$ intersections.

We conclude that the size of $H$ is at most $O(k \cdot n^{1+\nu})$ (we can slightly change the probabilities by introducing a factor of $2^{-4/3^{i+1}}$ to obtain size $O(kn + n^{1+\nu})$, as we did before.

**Stretch analysis.** The stretch analysis is very similar to that of Section 4.1, the main difference is in the use of half-bunches rather than the full ones, but this will increase the distance to pivots by a factor of 2, and affect the additive stretch only. Once again we distinguish between constant stretch and stretch close to 1.

### 5.1 A $(3 + \varepsilon, \beta(\cdot, \cdot))$-Spanner

We follow the analysis and notation presented in Section 4.1, but with $\Delta > 5$. We replace Lemma 4.1 with the following.

**Lemma 5.3.** Let $0 \leq i \leq k$ and let $x, y \in V$ such that $d_G(x, y) \leq D_i$ and $d_H(x, p_i(x)) \leq \frac{3\Delta}{\Delta - 2}D_{i-1}$. Define $m = \max\{\Delta D_{i-1}, d_G(x, y)\}$. Then at least one of the following holds:

1. $d_H(x, y) \leq (3 + \frac{16\Delta}{\Delta - 2})m$.
2. $d_H(x, p_{i+1}(x)) \leq \frac{4\Delta}{\Delta - 2}D_i$.

The main difference in the proof is in (7), which is replaced by

$$
d_H(p_{i+1}(x), p_{i+2}(p_{i+1}(x))) \leq 2d_G(p_{i+1}(x), p_{i+1}(y)) ,
$$

since we use half-bunches. One can then follow the calculations in the proof of Lemma 4.1, and check that the altered constants used in the 2 cases above suffice. We derive the following result.
Theorem 5.4. For any weighted graph $G = (V, E)$ on $n$ vertices, and any $k \geq 1$, there exists $H$ of size at most $O(kn + n^{1+1/(4/3)^k-1})$, which is a $(3 + \varepsilon, \beta(\cdot, \cdot))$-spanner for any $\varepsilon > 0$ with $\beta(x, y) = (3 + \varepsilon) \cdot (5 + 16/\varepsilon)^{k-1} \cdot W(x, y)$.

The proof is exactly the same as the proof of Theorem 4.2, the only differences are taking $\Delta = 5 + \frac{16}{\varepsilon}$ (which affects the value of $\beta$) and using the bounds of Lemma 5.3 rather than of Lemma 4.1.

5.2 A $(1 + \varepsilon, \beta(\cdot, \cdot))$-Spanner

Once again we use the corresponding analysis of the emulator, this time from Section 4.2. The use of half-bunches instead of bunches creates the following version of Lemma 4.4.

Lemma 5.5. Fix $\Delta > 5$. Let $0 \leq i < k$ and let $x, y \in V$ such that $d_G(x, y) \geq (3\Delta)^i W(x, y)$. Then at least one of the following holds:

1. $d_H(x, y) \leq (1 + \frac{8i}{\Delta-5}) d_G(x, y)$
2. $d_H(x, p_{i+1}(x)) \leq \frac{2\Delta}{\Delta} d_G(x, y)$

The main difference in the proof is in (11), which is replaced by

$$d_H(p_{i+1}(u), p_{i+2}(p_{i+1}(u))) \leq 2d_G(p_{i+1}(u), p_{i+1}(u_r)) .$$

The new bounds in the Lemma guarantee the calculations still go through. For Lemma 4.5 which takes care of small distances, we have the following change, with a very similar proof.

Lemma 5.6. Let $0 \leq i < k$ and fix $x, y \in V$. Let $m = \max\{d_G(x, p_i(x)), d_G(y, p_i(y)), d_G(x, y)\}$. Then at least one of the following holds:

1. $d_H(x, y) \leq 5m$
2. $d_H(x, p_{i+1}(x)) \leq 7m$

We conclude with the following theorem.

Theorem 5.7. For any weighted graph $G = (V, E)$ on $n$ vertices, and any integer $k > 1$, there exists $H$ of size at most $O(kn + n^{1+1/(4/3)^k-1})$, which is a $(1 + \varepsilon, \beta(\cdot, \cdot))$-spanner for any $0 < \varepsilon < 1$, where $\beta(x, y) = O(15 + \frac{24(k-1)}{k})^{k-1} \cdot W(x, y)$.

In the proof we set $\Delta = 5 + \frac{8(k-1)}{\varepsilon}$. The rest of the calculations follow analogously, one change is that when iteratively applying Lemma 5.6, the bound $m$ increases by a factor of 7 (rather than 4, as in Lemma 4.5), but as $7 \leq 3\Delta$ is still true, it does not change anything.

6 Efficient Implementation and Algorithmic Applications

Since we use very similar constructions to the ones in [EN19b], we can use their efficient implementations (connecting to all pivots, which is the difference between constructions, can be done efficiently in their framework as well). We consider here the standard model of computation, and the PRAM (CRCW) model. Given a parameter $1/k < \rho < 1/2$, we will want poly-logarithmic parallel time and $\tilde{O}(|E| \cdot n^\rho)$ work. This is achieved by adding additional $\lceil 1/\rho \rceil$ sets $A_i$, that are sampled with uniform probability $n^{-\rho}$, which in turn increases the exponent of $\beta$ by an additive $1/\rho + 1$. (In the case of multiplicative stretch $1 + \epsilon$, it also increases the base of the exponent in $\beta$.)

We summarize the efficient implementation results for hopsets and emulators in the following theorem.
Theorem 6.1. For any weighted graph $G = (V, E)$ on $n$ vertices, parameters $k \geq 1$ and $1/k < \rho < 1/2$, there is a randomized algorithm running in time $\tilde{O}(|E| \cdot n^\rho)$, that whp compute $H$ of size at most $O(kn + n^{1+1/(2^k-1)})$, such that for any $0 < \varepsilon < 1$ this $H$ is:

1. A $(3+\varepsilon, \beta)$-hopset with $\beta = O(1/\varepsilon)^{k+1/\rho}$.
2. A $(3+\varepsilon, \beta(\cdot, \cdot))$-emulator with $\beta(x, y) = O(1/\varepsilon)^{k+1/\rho} \cdot W(x, y)$.
3. A $(1+\varepsilon, \beta(\cdot, \cdot))$-emulator with $\beta(x, y) = O\left(\frac{k+1/\rho}{\varepsilon}\right)^{k+1/\rho} \cdot W(x, y)$.

Given $\varepsilon$ in advance, the algorithm can also be implemented in the PRAM (CRCW) model, in parallel time $\left(\frac{\log n}{\varepsilon}\right)^{O(k+1/\rho)}$ and work $\tilde{O}(|E| \cdot n^\rho)$, while increasing the size of $H$ by a factor of $O(\log^* n)$.

For spanners, recall that in Section 5 we have a somewhat different construction, and in the analysis we enforce a stricter requirement on the sampling probabilities $q_i$. To handle this, we start sampling with the uniform probability $n^{-\rho}$ only when $N_i \leq n^{1-3\rho}$ (and not when $N_i \leq n^{1-\rho}$ like before). Now the bound of Claim 5.2 still holds, as $N_i/q_i^3 \leq n$ even for these latter sets. The 'price' we pay for waiting until $N_i \leq n^{1-3\rho}$ is that the work will now be $|E| \cdot n^{3\rho}$. Rescaling $\rho$ by $3$, we get the following.

Theorem 6.2. For any weighted graph $G = (V, E)$ on $n$ vertices, parameters $k \geq 1$ and $1/k < \rho < 1/6$, there is a randomized algorithm running in time $\tilde{O}(|E| \cdot n^\rho)$, that whp compute $H$ of size at most $O(kn + n^{1+1/(2^k-1)})$, such that for any $0 < \varepsilon < 1$ this $H$ is:

1. A $(3+\varepsilon, \beta(\cdot, \cdot))$-spanner with $\beta(x, y) = O(1/\varepsilon)^{k+3/\rho} \cdot W(x, y)$.
2. A $(1+\varepsilon, \beta(\cdot, \cdot))$-spanner with $\beta(x, y) = O\left(\frac{k+1/\rho}{\varepsilon}\right)^{k+3/\rho} \cdot W(x, y)$.

Given $\varepsilon$ in advance, the algorithm can also be implemented in the PRAM (CRCW) model, in parallel time $\left(\frac{\log n}{\varepsilon}\right)^{O(k+1/\rho)}$ and work $\tilde{O}(|E| \cdot n^\rho)$, while increasing the size of $H$ by a factor of $O(\log^* n)$.

6.1 Shortest Path in Weighted Graphs

Given a weighted graph $G = (V, E)$ with $n$ vertices and a set $S \subseteq V$ of $s$ sources, fix parameters $k \geq 1$, $0 < \varepsilon < 1$ and $0 < \rho < 1/6$. Compute a $(1+\varepsilon, \beta(\cdot, \cdot))$-spanner $H$ of size $O(kn + n^{1+1/(2^k-1)})$ with $\beta(x, y) = O((k+3/\rho)/\varepsilon)^{k+3/\rho} \cdot W(x, y)$, in time $\tilde{O}(|E| \cdot n^\rho)$. Next, for every $u \in S$ run Dijkstra’s shortest path algorithm in $H$, which takes time $O(s \cdot (|E(H)| + n \log n)) = \tilde{O}(s \cdot n^{1+1/(2^k-1)})$.

The total running time for computing $(1+\varepsilon, \beta(\cdot, \cdot))$-approximate shortest path for all $S \times V$, is $\tilde{O}(|E| \cdot n^\rho + s \cdot n^{1+1/(2^k-1)})$.

If one desires improved additive stretch, simply use the $(3+\varepsilon, \beta(\cdot, \cdot))$-spanner with $\beta(x, y) = O(1/\varepsilon)^{k+3/\rho} \cdot W(x, y)$.

6.2 PRAM Distance Oracles

Given a weighted graph $G = (V, E)$ with $n$ vertices, fix parameters $k \geq 1$, $0 < \varepsilon < 1$ and $0 < \rho < 1/6$. The properties of the distance oracle we want to compute are:

- Has size $O(kn + n^{1+1/(2^k-1)}) \cdot \log^* n$.
- Given query $u \in V$, can report $\left(1+\varepsilon, O\left(\frac{k+1/\rho}{\varepsilon}\right)^{k+1/\rho} \cdot W(u, v)\right)$-approximation to all distances $u, v, v \in V$.
- Has query time $O\left(\frac{k+1/\rho}{\varepsilon}\right)^{k+1/\rho}$ and $\tilde{O}(n^{1+1/(2^k-1)})$ work.
The preprocessing time is $O\left(\log n\cdot \varepsilon\right)^{O(k+1/\rho)}$ and work $\tilde{O}(|E|\cdot n^\rho)$.

The first step is to compute a $(1+\varepsilon, \beta(\cdot, \cdot))$-emulator $G'$ of size $O\left(\frac{k+1/\rho}{\varepsilon}\cdot W(x, y)\right)$, in parallel time $\left(\log n\cdot \varepsilon\right)^{O(k+1/\rho)}$ and work $\tilde{O}(|E|\cdot n^\rho)$. Next, compute a $(1+\varepsilon, \beta)$-hopset $H$ size $O\left(\frac{1}{2^k-1}\cdot \log^* n\right)$ for $G'$ with $\beta = O\left(\frac{k+1/\rho}{\varepsilon}\cdot W(x, y)\right)$ within the same parallel time and work $\tilde{O}(|E|\cdot n^\rho)$.

Whenever a query $u \in V$ arrives, we run $\beta$ rounds of Bellman-Ford algorithm in the graph $G' \cup H$. Each round of Bellman-Ford can be implemented in PRAM (CRCW) in $O\left(\frac{k+1/\rho}{\varepsilon}\right)^{O(k+1/\rho)}$ time. Given that we need $\deg_G(v)$ processors at each vertex $v \in V$. So the total parallel time for the query is $O\left(\frac{k+1/\rho}{\varepsilon}\right)^{k+1/\rho}$. This computation gives for every $v \in V$ a value at most $(1+\varepsilon) \cdot [(1+\varepsilon) \cdot d_G(u, v) + \beta(u, v)]$. Rescaling $\varepsilon$ by 3, say, we get a $(3+\varepsilon, O\left(\frac{k+1/\rho}{\varepsilon}\right)^{k+1/\rho} \cdot W(u, v))$-approximation. If one desires an improved additive stretch $\beta(u, v)$, or improved query time $\beta$, simply use the $(3+\varepsilon, \beta(\cdot, \cdot))$-emulator, or the $(3+\varepsilon, \beta)$-hopset.

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