Classical Propagation in the Quantum Inverted Oscillator

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Abstract

We emphasize the fact the evolution of quantum states in the inverted oscillator (IO) is reduced to classical equations of motion, stressing that the corresponding tunnelling and reflexion coefficients addressed in the literature are calculated by considering only classically trajectories. The Wigner function formalism is employed to describe the IO classical dynamics, subsequently leading to the introduction of the Ambiguity function lying in the so-called Reciprocal phase space. Our findings, show that the Ambiguity function behavior, subjected to the IO, allude a classical propagation with an associated integral of motion, and complex conjugated doubly degenerate energy states.

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I. INTRODUCTION

The inverted oscillator (IO) is one of the few completely solvable physical systems in both quantum and classical mechanics. Its classical Newtonian solutions are expressed in terms of hyperbolic functions that diverge exponentially in time, while the quantum counterpart leads to continuous, doubly degenerate energy eigenstates with no ground state defined. Ever since Barton’s thesis [5], the IO has been studied with high interest motivated by several technological applications and theoretical developments such as fission dynamics [26], string theory [15] and universe models [24, 18].

The evolution of the IO is acknowledged in the literature as being classical; we revisit this subject under the light of a free-coordinate formulation of the time-evolution operator in the Hilbert phase space, introduced in the recent publications [9, 10, 12], that naturally allows to introduce the system’s features in the phase space and in the reciprocal phase space. The paper is structured as follows: In section 2, we review the problem from the point of view of Newtonian mechanics by describing the classical phase portrait. Quantum mechanics in the Hilbert phase space [9, 10, 12] is briefly outlined in section 3, with the aim to demonstrate the well-known equivalence between classical and quantum evolution under quadratic Hamiltonians. In addition, Wigner function is introduced due to its particularly hallmark of being helpful to gain insight in the role of both quantum and classical mechanics. In section 4, we bring up for discussion the controversy of the quantum tunneling coefficient associated to the IO as described in the literature [8, 25, 23]. In section 5, the IO classical and quantum reciprocal phase spaces are studied for the first time, revealing an additional integral of motion. Finally, in the last section we provide the conclusions.

II. NEWTONIAN PICTURE

The classical inverted harmonic oscillator is characterized by the classical Hamiltonian

\[ H(x, p) = \frac{p^2}{2m} - \frac{1}{2}m\omega^2x^2 = E, \]  

where \( x \) and \( p \) are the canonical position and momentum variables, \( m \) is the particle’s mass, \( \omega \) denotes the repulsion parameter, and \( E \) stands for the energy. This model is completely integrable and shows non periodic behavior

\[
\begin{align*}
  x(t) &= x_0 \cosh(\omega t) + p_0 \sinh(\omega t) / m\omega \\
  p(t) &= m\omega x_0 \sinh(\omega t) + p_0 \cosh(\omega t).
\end{align*}
\]  

Figure 1: (a) Inverted oscillator potential barrier. (b) Classical phase-space portrait: solid or dashed lines correspond to particles with positive or negative energies respectively. The direction of motion is represented by the arrows.

The classical phase space pictured by Fig. II(b) displays asymptotic lines emerging from the origin called...
separatrices $p = \pm m\omega x$, and sets of hyperbolas around the saddle point $(x = 0, p = 0)$. The separatrices portray phase space trajectories of zero energy $E = 0$, and divide the phase space into four quadrants: the upper & lower sets of hyperbolic lines that represent phase space trajectories of particles with positive energies $E > 0$, moving over the barrier; and the right & left sets correspond to phase space trajectories of particles of negative energies $E < 0$, reflected from the barrier.

The approach towards the saddle point demands infinite time. It can only be performed by particles settle down on the separatrix $p = -m\omega x$, often referred as the stable separatrix, according to

$$x(t) = x_0 e^{-\omega t},$$  \number{3}

and the unstable separatrix $p = m\omega x$, describes particles moving away the saddle point

$$x(t) = x_0 e^{\omega t}. \number{4}$$

### III. CLASSICAL PROPAGATION OF WAVE PACKETS UNDER QUADRATIC HAMILTONIANS

It is well-known that quantum dynamics of quadratic Hamiltonians can be exactly reduced to classical equations of motion. In this section, we arrive to the same conclusion by exact algebraic manipulation of the quantum equations of motion in the Hilbert phase space.

Consider the abstract form of the von Neumann equation in the Hilbert phase space \cite{9,10,12}

$$i\hbar \frac{d}{dt} |\rho(t)\rangle = [H(\hat{x}, \hat{p}) - H(\hat{x}', \hat{p}')] |\rho(t)\rangle.$$ \number{5}

where

$$\left[\hat{x}, \hat{p}\right] = i\hbar, \quad \left[\hat{x}', \hat{p}'\right] = -i\hbar,$$ \number{6}

while the commutator of the cross-terms vanish $\left[\hat{x}, \hat{x}'\right] = \left[\hat{x}', \hat{p}'\right] = \left[\hat{p}, \hat{x}\right] = \left[\hat{p}', \hat{x}'\right] = 0$. The set of position and momentum operators $(\hat{x}, \hat{p}, \hat{x}', \hat{p}')$ are rewritten in terms of a new set of operators $(\hat{x}, \hat{p}, \hat{\lambda}, \hat{\theta})$, called the extended four-operator algebra, through Bopp transformations \cite{11}

$$\hat{x} - \hat{x}' = \hat{x} + \frac{\hat{h}}{2\theta}, \quad \hat{p} + \frac{\hat{h}}{2\lambda} = \hat{p} + \frac{\hat{h}}{\theta}.$$ \number{7}

$$\hat{x} = \hat{x} - \frac{\hbar}{2\theta}, \quad \hat{x}' = \hat{x} + \frac{\hbar}{2\theta},$$ \number{8}

Commutators relations of the operators $(\hat{x}, \hat{p}, \hat{\lambda}, \hat{\theta})$ \cite{9} are constructed such that Eqs. \number{7} and \number{8} attain the standart commuting relations given in Eq. \number{9}

$$[\hat{x}, \hat{p}] = 0, \quad [\hat{x}, \hat{\lambda}] = i, \quad [\hat{p}, \hat{\theta}] = i, \quad [\hat{\lambda}, \hat{\theta}] = 0.$$ \number{9}

Then, the substitution of Eqs. \number{7} and \number{8} in Eq. \number{5}, enable to write von Neumann equation in the Hilbert
where \( a, b, \) and \( c \) are constant coefficients, and \( \hat{x}, \hat{p} \) are the standard position and momentum operators. Then, by means of Eq. (10) the related von Neumann equation in the Hilbert phase space is obtained

\[
i \frac{d}{dt} |\rho(t)\rangle = \left[ \frac{\hat{p}\lambda}{m} + (b + 2c\hat{x})\hat{\theta} \right] |\rho(t)\rangle,
\]

(17)

more precisely, for a certain quadratic Hamiltonian von Neumann equation is brought to the form [9]

\[
i \frac{d}{dt} |\rho(t)\rangle = \left[ \frac{\partial}{\partial \hat{p}} H(\hat{x}, \hat{p})\hat{\lambda} - \frac{\partial}{\partial \hat{x}} H(\hat{x}, \hat{p})\hat{\theta} \right] |\rho(t)\rangle.
\]

(18)

Thus equations (13), (17), and (18) are quantum compliant due to \( \hbar \) is inherently cancelled without taking the classical limit \( \hbar \to 0 \). So, quantum and classical evolution is identical as long as quadratic Hamiltonians are considered, however, be aware that the quantum state \( |\rho(t)\rangle \) among other quantum restrictions obeys the uncertainty principle while the classical Koopman-von Neumann wave function \( |\Psi(t)\rangle \) can be more arbitrary since classical states develop eventually higher and higher resolution without limit [40].

On the other hand, the phase space representation \([x,p]\) of the ket \( |xp\rangle |\rho(t)\rangle \) is proportional to the Wigner function [42] [10] [See Appendix B for more details.]

\[
W(x, p; t) = \frac{1}{\sqrt{2\pi\hbar}} (xp|\rho(t))
\]

(19)

The equation of motion for the Wigner function is known as Moyal’s equation [35, 45, 16, 12], and coincide with the classical Koopman-Von Neumann equation for quadratic Hamiltonians. Hereafter, within this environment the quantum inverted oscillator (IO) is treated

\[
H = \frac{\hat{p}^2}{2m} - \frac{1}{2} m\omega^2 x^2;
\]

(20)

where \( m \) accounts the particle’s mass, \( \omega \) is the repulsion parameter, and \( \hat{x}, \hat{p} \) are the position and momentum operators. Employing Eqs. (17) and (12), the IO Moyal’s equation is attained

\[
-\frac{\partial W(x, p; t)}{\partial t} = \left[ \frac{p}{m} \frac{\partial}{\partial x} + m\omega^2 x \frac{\partial}{\partial p} \right] W(x, p; t).
\]

(21)

Numerical propagation of this equation is carried out utilizing Pure Gaussian Wigner functions, as initial states

\[
W(x, p, t_0 = 0) = \frac{1}{\pi \hbar} e^{-(m\omega^2(x-x_0)^2 + (p-p_0)^2)/(\hbar m)}.
\]

(22)

Purity condition, Eq. (23), stipulates that \( W(x, p; t) \) might be faithfully represented in terms of a
Schrödinger’s wavefunction up to a global phase factor.

\[ 2\pi \hbar \int W^2(x,p) dx dp = 1. \quad (23) \]

For illustrative purposes, the numerical propagation of two Wigner functions of energies \( E_1 = -0.5 \) and \( E_2 = -8 \) were implemented using the spectral Split-operator method \[12\] (See Python code in \[1\]); screenshots of the evolution are displayed in Fig.\[2\]. The studied Wigner functions move along the classical phase space trajectories following the level sets of the classical Hamiltonian Eq.\[1\], rewarded by the fact that Moyal’s equation and the classical Koopman von Neumann equation are identical for quadratic Hamiltonians. Thus, proceed to the comparison with the classical phase-space portrait is completely natural, for example a state that approaches the barrier from the left or right side is located above or below the unstable separatrix, and according to the arriving direction the positive energy components of Wigner function might be located in the upper or lower quadrants (Positive energy trajectories of the classical phase portrait), the zero energy components are settled down over the stable separatrix, and the negative energy components might be placed on the right or left quadrants (Negative energy trajectories of the classical phase portrait).

More importantly, the positive, zero and negative energy components of the Wigner function weights the particle’s contribution: (i) to move over the barrier, (ii) to stop at the top, or (iii) to be reflected. Another theoretical argument that support the classically evolution description is that the positive-definite Gaussian Wigner functions, set as initial states, remain positive distributions throughout the evolution generated by Eq.\[21\]. This is ensured by the fact that for pure states, Gaussians are the only possible positive Wigner functions, according to Hudson’s theorem \[27\]. As a result, the evolution of the Wigner function under quadratic Hamiltonians might be well sketched out by Newtonian particles.

The previous arguments and simulations prove that the evolution of the Wigner function under the IO is effectively classical in the sense that the equation of motion is free from \( \hbar \). Nevertheless, Planck’s constant still enters as a parameter in the initial state ensuring that the state is consistent with the uncertainty principle among others quantum conditions \[19\] \[40\], which for quantum pure states remains valid all along the propagation. Complementarily, it is noteworthy mention that the Epistemically Restricted Liouville mechanics \[41\] \[28\] is able to reproduce many quantum phenomena of Gaussian Quantum mechanics \[39\] \[41\] by emulating the uncertainty principle on the canonical variables and setting up the maximum entropy principle. However, the scope of this classical treatment was recently investigated in Ref. \[2\] by coupling a classical oscillator with a gaussian quantum oscillator, both equivalent under this criteria; the evolution showed that the quantum sector of the former violates the uncertainty principle stating that the quantum features cannot be completely overshadowed.

IV. APPROACH TOWARDS THE IO BARRIER

Quantum tunneling is a fundamental quantum mechanical effect where a particle penetrates a potential barrier energetically higher than the particle’s total energy, entering in the classically forbidden region, thus, leading a measurable probability of crossing the other side of the barrier, otherwise prohibited by the classical mechanics.

For states approaching the IO barrier from the right (left) side, shown in Fig.\[1a\], the prohibited regions are displayed in the classical phase space portrait, Fig.\[1b\], laying within the lower-half portion of the left quadrant (the upper-half portion of the right quadrant). Notwithstanding, analyzing the IO energy eigenstates, the authors in Ref. \[3\] derived an analytic expression for the tunneling coefficient \( T \) considering only classically allowed phase space trajectories, corresponding to positive energy components of Wigner function, located above the top of the barrier. This result is in contradiction with the conventional WKB theory where this effect comes from the use of complex trajectories, nevertheless, it was justified by the presence of separatrices in the classical phase space. The unstable separatrix automatically prohibits the flow of Wigner functions across the forbidden regions while the stable separatrix spreads the Wigner functions into two separated branches of positive and negative energy components, whenever the states cross it, turning the Wigner function strongly non local, and since the interference forms the basis for the semiclassical evaluation, this leads to write the semiclassical approximation of the Wigner function as WKB waves undergoing interference. On this way, for states with total negative energy, Fig.\[2\] the positive energy components of the Wigner function were associated with the tunneling coefficient \( T \), and the negative energy components with the reflection coefficient \( R \). Nonetheless, further research in Refs. \[33\] \[32\] extended the study of the conventional semiclassical approach by using the path integral framework in both the time and energy domains. They found that for a complete and accurate semiclassical approximation of the quantum propagator applied to tunneling problems, the semi-
classical propagator must consider two contributions from: (i) above the barrier trajectories and (ii) below the barrier trajectories associated with the tunneling loops. Ultimately, demonstrating that the later contribution becomes dominant at long times and far endpoints in barriers that flatten out at large distances $\lim_{x \to \infty} V(x) = 0$, regarding the calculation of the tunneling coefficient. Moreover, as an example of barriers that do not flatten out at large distances, they studied the IO, for which the second contribution to the semiclassical propagator vanishes at all, implying that no tunneling trajectories will develop, since the semiclassical propagator is exact and only picks up classically allowed trajectories. In the light of these insights, they refute the findings of Ref. [8], however, more recently papers about the IO [25, 23] deal with tunneling and reflection coefficients $T$ and $R$ as a result of trajectories in the classical phase space that exactly draw the time-evolution of the Wigner function.

V. RECIPROCAL PHASE SPACE

Appealing to the Ambiguity function $A(\lambda, \theta)$ [14, 13, 12] the inverted oscillator dynamics is alternatively reformulated in the $\lambda - \theta$ representation, hereafter referred as the Reciprocal phase space, bearing that $A(\lambda, \theta)$ is obtained through a two dimensional Fourier transform on the Wigner function

$$A(\lambda, \theta) = \int W(x, p) e^{i(\lambda x + p \theta)} dx dp. \quad (24)$$

The motion equation for the Ambiguity function rewritten for the IO hamiltonian, is read as [See Appendix C for details.]

$$\frac{\partial A(\lambda, \theta; t)}{\partial t} = \left[ \frac{\lambda}{m} \frac{\partial}{\partial \theta} + m \omega^2 \theta \frac{\partial}{\partial \lambda} \right] A(\lambda, \theta; t), \quad (25)$$

where the characteristics of this partial differential equation provides the ensuing reciprocal classical phase-space trajectories

$$\begin{cases} \lambda(t) = \lambda_0 \cosh(\omega t) - m \omega \theta_0 \sinh(\omega t) \\ \theta(t) = -\lambda_0 \sinh(\omega t) + \theta_0 \cosh(\omega t). \end{cases} \quad (26)$$

It turns out that this system obeys the ordinary differential equations below

$$\frac{d\lambda}{dt} = \frac{\partial \mathbb{H}(\lambda, \theta)}{\partial \theta}, \quad \frac{d\theta}{dt} = -\frac{\partial \mathbb{H}(\lambda, \theta)}{\partial \lambda}. \quad (27)$$

where $\mathbb{H}(\lambda, \theta)$ is a function constructed in the reciprocal phase space, such that $\mathbb{H}(\lambda, \theta) = \mathcal{E}$

$$\mathbb{H}(\lambda, \theta) = -\frac{1}{2} m \omega^2 \theta^2 + \frac{\lambda^2}{2m} = \mathcal{T}(\theta) + \mathcal{V}(\lambda) = \mathcal{E}, \quad (28)$$

and $\mathcal{T}(\theta)$ is a scalar function related to the motion, whereas $\mathcal{V}(\lambda)$ plays the analogue role of the barrier.

In comparison to the classical phase space, the reciprocal phase space Fig.(3-b), also exhibits the stable and unstable separatrices for particles of $\mathcal{E} = 0$, described by the next asymptotes and displacement rules

$$\theta = \frac{\lambda}{m \omega}, \quad \lambda(t) = \lambda_0 e^{-\omega t}, \quad (30)$$

$$\theta = -\frac{\lambda}{m \omega}, \quad \lambda(t) = \lambda_0 e^{\omega t}. \quad (31)$$

It follows that particles arriving the $\mathcal{V}(\lambda)$ barrier from the left (right) side are portrayed down (up) the unstable separatrix $\theta = -\lambda/m \omega$, besides, the upper and lower sets of hyperbolas describe particles of $\mathcal{E} < 0$ that pass below the $\mathcal{V}(\lambda)$ while the right and left sets represent particles of $\mathcal{E} > 0$ reflected from $\mathcal{V}(\lambda)$.

The quantum scenery is developed in terms of the Ambiguity function, who transforms the real-valued Wigner functions of energies $E_1 = 0.5$ and $E_2 = -8$, into symmetric complex-valued functions, centered at the origin in the $\lambda - \theta$ plane, real part is shown in Fig. 4 [See Appendix C, Figs. 5 and 6 to observe the imaginary part and the absolute value squared.]. Those quantum states evolve along the level sets of the conservation law given by Eq. (28), since the generator of motion $\hat{G}$ besides to commute with $\mathcal{H}(\hat{x}, \hat{p})$, also commutes with $\mathbb{H}(\hat{\lambda}, \hat{\theta})$.

$$\hat{G} = \frac{1}{m} \hat{p} \hat{\lambda} + m \omega^2 \hat{x} \hat{\theta}, \quad (32)$$

$$[\hat{G}, \mathcal{H}(\hat{x}, \hat{p})] = [\hat{G}, \mathbb{H}(\hat{\lambda}, \hat{\theta})] = 0. \quad (33)$$

This proves that $\mathcal{H}(\hat{x}, \hat{p}) = E$ and $\mathbb{H}(\hat{\lambda}, \hat{\theta}) = \mathcal{E}$, are integrals of motion associated with the transformation $\hat{U}(t) = e^{-\frac{\hat{G}t}{\hbar}}$. Hence, the conservative dynamics established in the reciprocal phase space forbids tunneling across the $\mathcal{V}(\lambda)$ barrier, and leads us to understand the real and complex components of $A(\lambda, \theta)$ as probability amplitudes dragged along the well defined trajectories stated by the Eq. (26). Complementarily, notice that the only completely real and positive ambiguity function corresponds to a Gaussian Wigner state centered at the origin in the $x - p$ plane, depicting a state with the highest probability to be found at the top of the IO potential barrier.

Finally, exploiting the ambiguity function features, the doubly degenerate energy states that characterizes
VI. SUMMARY AND CONCLUSIONS

We have reviewed the well-known fact that the propagation of quantum states subjected to quadratic Hamiltonians is described by classical equations of motion. This applies to the IO, for which the tunneling and reflection coefficients in phase space are given by the classically allowed phase space trajectories of the Wigner function, corresponding to energy components above or below the stable separatrix. Furthermore, this result coincides with the path integral framework in which quadratic Hamiltonians generate exact semiclassical propagators, that prevents the flow of quantum states across the classically forbidden regions. Therefore, the propagation of quantum states under quadratic Hamiltonians are perfectly reproduced by the Liouville equation, as well as the Koopman-Von Neumann equation of motion that are ultimately equivalent to the propagation of Newtonian particles.

Moreover, the most relevant contribution of this paper is the treatment in the reciprocal phase space that leads us to elucidate the IO as a classical dynamical system with two conservation laws associated to the propagator \( \hat{U}(t) = e^{-i\hat{G}t} \), including the energy as one of them. Despite this characteristic, a natural question is raised: whether or not there are in general cases where both quantum and classical operators share exactly the same symmetry. However, this treatment goes beyond the scope of the topic and will be subjected to further research. Moreover, another insight on the Ambiguity function is that it relates the pairs of degenerate energy states of the IO, as complex conjugates.

In summary, for quadratic Hamiltonians, quantum dynamics can be described by the Koopman-Von Neumann equation of motion; determining that quantum states strictly evolve throughout classical trajectories.
A similar behavior is observed in the reciprocal phase space where the state represented by the ambiguity function evolves along well defined trajectories in the reciprocal classical phase space. Finally, we stress that even if a quantum propagation is equivalent to a classical evolution, quantum mechanics sets additional restrictions on the states in order to maintain consistency.

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APPENDIX

A. HILBERT PHASE-SPACE REPRESENTATIONS

Schrödinger equation is restricted to describe closed quantum systems, i.e., not-interacting with the environment, maintaining perfect coherence along the evolution, and entailing no-loss of information. In this formalism, the knowledge about the system is encoded on the density operator state, constructed from a ket $|\psi\rangle$, which might be rewritten as a linear combination of a given complete set of eigenstates of an Hermitian operator

$$|\psi(t)\rangle = \sum_n C_n(t)|\phi_n\rangle,$$

the modulus squared of the coefficient $|C_n(t)|^2$ define the probability of finding the system in the eigenstate $|\phi_n\rangle$, then it must be true that $\sum_n |C_n(t)|^2 = 1$. This means that coherent superpositions between states are permitted, in particular a coherent superposition of two states might be constructed as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle).$$

An alternative representation of quantum states recast on the density operator state, constructed from a ket
and its bra
\[ \rho^{\text{pure}} = |\psi\rangle \langle \psi| . \] (37)

Any density operator state represented on this form is called pure, because it basically contains the same information as the ket, up to a global phase. The advantage of the density operator state over the ket relies on it can describe statistical ensembles of pure states called mixed
\[ \hat{\rho} = \sum_i p_i \hat{\rho}_i = \sum_i p_i |\psi_i\rangle \langle \psi_i| , \] (38)

where \( p_i \) is the probability associated to the pure state \(|\psi_i\rangle\), on such a way that \( \sum_i p_i = 1 \). Thus, incoherent superpositions who characterize mixed states are interpreted as a collective description of an ensemble of pure quantum states, for example
\[ \hat{\rho} = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2) , \] (39)

Matrix elements of the density operator state \( \hat{\rho} \) in a certain basis set of kets is given by \(|x\rangle \langle y| = \rho_{xy}\), for example in the position representation the above equation reads
\[ i\hbar \frac{\partial}{\partial t} \langle x|\rho|x'\rangle = \left[ H \left( x, -i\hbar \frac{\partial}{\partial x} \right) ight. \]  
\[ \left. - H \left( x', i\hbar \frac{\partial}{\partial x'} \right) \right]\langle x'|\rho|x'\rangle . \] (47)

while in the momentum representation we have
\[ i\hbar \frac{\partial}{\partial t} \langle p|\rho|p'\rangle = \left[ H \left( i\hbar \frac{\partial}{\partial p}, p \right) \right. \]  
\[ \left. - H \left( -i\hbar \frac{\partial}{\partial p}, p' \right) \right]\langle p'|\rho|p'\rangle . \] (48)

However, the free-coordinate formulation of von Neumann equation in the Hilbert phase space \([9,10,12]\) is achieved by the mirror quantum operators \(x'\) and \(p'\)
\[ i\hbar \frac{d}{dt} |\rho\rangle = [H (x', \hat{p}), |\rho\rangle] = \left[ H (x', \hat{p}) - H (\hat{x}', \hat{p}) \right] |\rho\rangle . \] (49)

such that
\[ [\hat{x}, \hat{p}] = i\hbar \quad [\hat{x}', \hat{p}'] = -i\hbar , \] (50)
\[ [\hat{x}, \hat{x}'] = [\hat{p}', \hat{p}] = [\hat{p}', \hat{x}] = 0 . \] (51)

Then, the linear change of variables
\[ \hat{x} = \hat{x} - \frac{\hbar}{2} \hat{\theta} , \quad \hat{x}' = \hat{x} + \frac{\hbar}{2} \hat{\theta} , \] (52)
\[ \hat{p} = \hat{p} + \frac{\hbar}{2} \hat{\lambda} , \quad \hat{p}' = \hat{p} - \frac{\hbar}{2} \hat{\lambda} , \] (53)

leads to
\[ i\hbar \frac{d}{dt} |\rho\rangle = [H (\hat{x} - \frac{\hbar}{2} \hat{\theta}, \hat{p} + \frac{\hbar}{2} \hat{\lambda}), |\rho\rangle] \]  
\[ - H \left( \hat{x} + \frac{\hbar}{2} \hat{\theta}, \hat{p} - \frac{\hbar}{2} \hat{\lambda} \right) |\rho\rangle . \] (54)

This scheme also employs four operators \((\hat{x}, \hat{p}, \hat{\lambda}, \hat{\theta})\), with the following commuting relations \(9\)
\[ [\hat{x}, \hat{\lambda}] = i , \quad [\hat{p}, \hat{\theta}] = i , \] (55)
\[ [\hat{x}, \hat{p}] = 0 , \quad [\hat{x}, \hat{\theta}] = 0 , \quad [\hat{\lambda}, \hat{p}] = 0 , \quad [\hat{\lambda}, \hat{\theta}] = 0 . \] (56)

As a consequence, the Hilbert phase space is parametrized by the spectrums of two commuting operators, selected from Eq. \([50]\), due to each pair share a common basis set of orthogonal eigenvectors:
representation $B(x,\theta; t)$ introduced by Blokhintsev [31], the double-momentum-space representation $Z(\lambda, p; t)$ and the Ambiguity function $A(\lambda, \theta; t)$ [14, 13].

The “Double-configuration-space-function” or Blokhintsev function is defined as

$$B(x, \theta; t) = \langle x - \frac{\hbar}{2} \theta | \rho(t) | x + \frac{\hbar}{2} \theta \rangle,$$

(61)

and more precisely for pure states is reduced to

$$B(x, \theta; t) = \psi(x - \frac{\hbar}{2} \theta) \psi^*(x + \frac{\hbar}{2} \theta).$$

(62)

The motion equation for the Blokhintsev function, $B(x, \theta; t)$ is

$$i\hbar \frac{\partial}{\partial t} B(x, \theta; t) = \left[ H \left(x - \frac{\hbar}{2} \theta, i \frac{\partial}{\partial \theta} - \frac{\hbar}{2} \frac{\partial}{\partial x} \right) \right] B(x, \theta; t).$$

(63)

Then, Wigner function might be obtained through a inverse Fourier transform on $B(x, \theta; t)$

$$W(x, p; t) = \frac{1}{2\pi} \int B(x, \theta; t) e^{ip\theta} d\theta.$$

(64)

Wigner function’s motion equation is named Moyal’s equation, in honor to the physicist José Enrique Moyal (1910-1998)

$$i\hbar \frac{\partial}{\partial t} W(x, p; t) = \left[ H \left(x + i \frac{\hbar}{2} \theta, i \frac{\partial}{\partial \theta} + \frac{\hbar}{2} \frac{\partial}{\partial x} \right) \right] W(x, p; t).$$

(65)

Applying a Fourier transform on the Wigner function we get the Double-momentum-space representation $Z(\lambda, p; t)$, who name is owed to the fact that $\hbar \lambda$ has the dimension of momentum

$$Z(\lambda, p; t) = \int W(x, p; t) e^{-ix\lambda} dx,$$

(66)

obeying

$$i\hbar \frac{\partial}{\partial t} Z(\lambda, p; t) = \left[ H \left(i \frac{\partial}{\partial \lambda} + \frac{\hbar}{2} \frac{\partial}{\partial p}, p + \frac{\hbar}{2} \lambda \right) \right] Z(\lambda, p; t).$$

(67)

In brief, the connection among these functions are obtained through partial Fourier transforms, keeping in mind that $\lambda$ is the conjugate variable of $x$, and $\theta$ is the conjugate variable of $p$

$$\lambda \xrightarrow{\mathcal{F}_x} x, \quad \lambda \xrightarrow{\mathcal{F}_\lambda^{-1}} x,$$

$$\theta \xrightarrow{\mathcal{F}_p} p, \quad \theta \xrightarrow{\mathcal{F}_\theta^{-1}} p.$$
Table 1: This table shows all representations hold by the Hilbert phase space. The explicit form for the extended four operators is constructed such that the commutation relations given by the Eq. (9) are fulfilled.

| Space | Main commuting relation | Basis | Completeness identity | Extended four operator algebra |
|-------|-------------------------|-------|----------------------|-------------------------------|
| $xp$-representation | $[x, \hat{p}] = 0$ | $|xp\rangle$ | $1 = \int dxdp \, |xp\rangle \langle xp|$ | $\hat{x} = x, \quad \hat{p} = p, \quad \hat{\lambda} = -i \frac{\partial}{\partial x}, \quad \hat{\theta} = -i \frac{\partial}{\partial p}$ |
| $x\theta$-representation | $[\hat{x}, \hat{\theta}] = 0$ | $|x\theta\rangle$ | $1 = \int dx d\theta \, |x\theta\rangle \langle x\theta|$ | $\hat{x} = x, \quad \hat{p} = i \frac{\partial}{\partial \theta}, \quad \hat{\lambda} = -i \frac{\partial}{\partial x}, \quad \hat{\theta} = \theta.$ |
| $\lambda p$-representation | $[\hat{\lambda}, \hat{\rho}] = 0$ | $|\lambda p\rangle$ | $1 = \int d\lambda dp \, |\lambda p\rangle \langle \lambda p|$ | $\hat{x} = i \frac{\partial}{\partial \lambda}, \quad \hat{p} = p, \quad \hat{\lambda} = \lambda, \quad \hat{\theta} = -i \frac{\partial}{\partial \rho}$ |
| $\lambda \theta$-representation | $[\hat{\lambda}, \hat{\varphi}] = 0$ | $|\lambda \varphi\rangle$ | $1 = \int d\lambda d\theta \, |\lambda \varphi\rangle \langle \lambda \varphi|$ | $\hat{x} = i \frac{\partial}{\partial \lambda}, \quad \hat{p} = i \frac{\partial}{\partial \theta}, \quad \hat{\lambda} = \lambda, \quad \hat{\theta} = \theta.$ |

where $(\lambda p | x \theta) = \exp(i \theta \lambda - ix \lambda)/(2\pi)$.

For instance, the Ambiguity function $A(\lambda, \theta; t)$ given by Eq. (24) can be alternatively obtained by

$$A(\lambda, \theta; t) = \int B(x, \theta; t)e^{-i\lambda x} dx,$$  \hspace{1cm} (70)

or

$$A(\lambda, \theta; t) = \int Z(\lambda, p; t)e^{-ip\theta} dp.$$ \hspace{1cm} (71)

The ambiguity function obeys the following motion equation

$$i\hbar \frac{\partial}{\partial \theta} A(\lambda, \theta; t) = \left[ H \left( i \frac{\partial}{\partial \lambda} - \frac{\hbar}{2} \theta, i \frac{\partial}{\partial \theta} + \frac{\hbar}{2} \lambda \right) \right] A(\lambda, \theta; t).$$  \hspace{1cm} (72)

In addition, the four distributions functions presented are proportional to the different representations of the ket in the Hilbert phase space, through

$$B(x, \theta; t) = \frac{1}{\sqrt{\hbar}} (|x\theta|\rho(t)), \quad Z(\lambda, p; t) = \frac{1}{\sqrt{\hbar}} (|\lambda p|\rho(t)), \quad W(x, p; t) = \frac{1}{\sqrt{2\pi \hbar}} (|xp|\rho(t)), \quad A(\lambda, \theta; t) = \frac{1}{\sqrt{\hbar}} (|\lambda \theta|\rho(t)).$$ \hspace{1cm} (73)

The first equivalence is established by writing the completeness identity for the quantum observables $x$ and $x'$

$$1 = \int dx dx' |xx'\rangle \langle xx'|,$$ \hspace{1cm} (75)

then the spatial linear change given by Eq. (52) in the $x - \theta$ representation is applied

$$1 = \int |J(x, x')| dxd\theta \left| x - \frac{\hbar}{2} \theta, x + \frac{\hbar}{2} \theta \right\langle x - \frac{\hbar}{2} \theta, x + \frac{\hbar}{2} \theta \right|,$$ \hspace{1cm} (76)

with

$$|J(x, x')| = \left| \frac{\partial x}{\partial x} \frac{\partial x'}{\partial x} \frac{\partial x'}{\partial x} \right| = \left| 1 - \frac{\hbar}{2} \right| = \hbar,$$ \hspace{1cm} (77)

thus,

$$1 = \int h dxd\theta \left| x - \frac{\hbar}{2} \theta, x + \frac{\hbar}{2} \theta \right\langle x - \frac{\hbar}{2} \theta, x + \frac{\hbar}{2} \theta \right|.$$ \hspace{1cm} (78)

Moreover, from the completeness identity for the $x$ and $\theta$ quantum observables we have

$$1 = \int dxd\theta |x\theta\rangle \langle x\theta|.$$ \hspace{1cm} (79)

It follows that Eqs. (78) and (79) enable to deduce

$$|x\theta\rangle = \sqrt{\hbar} |x - \frac{\hbar}{2} \theta, x + \frac{\hbar}{2} \theta\rangle,$$ \hspace{1cm} (80)

hence, the ket $|\rho(t)\rangle$ projection on the above basis leads

$$|x\theta|\rho(t)\rangle = \sqrt{\hbar} (x - \frac{\hbar}{2} \theta, x + \frac{\hbar}{2} \theta)|\rho(t)\rangle =$$ \hspace{1cm} (81)

$$\sqrt{\hbar} (x - \frac{\hbar}{2} \theta)|\rho(t)x + \frac{\hbar}{2} \theta\rangle,$$
where \(B(x, \theta; t)\) is easily recognized
\[
\langle x \theta | \rho(t) \rangle = \sqrt{\hbar} \langle x - \frac{\hbar}{2} \theta | \rho(t) | x + \frac{\hbar}{2} \theta \rangle = \sqrt{\hbar} B(x, \theta; t), \tag{82}
\]

\[
B(x, \theta; t) = \frac{1}{\sqrt{\hbar}} \langle x \theta | \rho(t) \rangle. \tag{83}
\]

The second relation is deduced from
\[
1 = \int dp dp' \langle pp' \rangle | \langle pp' \rangle | , \tag{84}
\]
thereupon the linear change of variable given by Eq. (55) in the \(\lambda - p\) representation, brought the above relation to
\[
1 = \int d\lambda dp \langle p + \frac{\hbar}{2} \lambda, p - \frac{\hbar}{2} \lambda \rangle \langle p + \frac{\hbar}{2} \lambda, p - \frac{\hbar}{2} \lambda \rangle , \tag{85}
\]
\[
|J(p, p')| = \left| \frac{\partial p}{\partial \lambda} \frac{\partial p}{\partial p'} \right| = \left| -\frac{\hbar}{2} 1 \right| = \hbar, \tag{86}
\]
or
\[
1 = \int d\lambda dp \langle \lambda p | \lambda p \rangle , \tag{87}
\]
and from the completeness identity for \(x\) and \(\theta\)
\[
1 = \int d\lambda dp \langle \lambda p | \lambda p \rangle . \tag{88}
\]

Then, the \(|\lambda p\rangle\) basis is identified from a comparison between Eqs. (82) and (85)
\[
|\lambda p \rangle = \sqrt{\hbar} |p + \frac{\hbar}{2} \lambda, p - \frac{\hbar}{2} \lambda\rangle , \tag{89}
\]
therefore,
\[
\langle \lambda p | \rho(t) \rangle = \sqrt{\hbar} \langle p + \frac{\hbar}{2} \lambda, p - \frac{\hbar}{2} \lambda | \rho(t) \rangle
= \sqrt{\hbar} \langle p + \frac{\hbar}{2} \lambda | \rho(t) | p - \frac{\hbar}{2} \lambda \rangle = \sqrt{\hbar} Z(\lambda, p; t), \tag{90}
\]
\[
Z(\lambda, p; t) = \frac{1}{\sqrt{\hbar}} \langle \lambda p | \rho(t) \rangle . \tag{91}
\]

Finally, the remaining equivalences are easily proved by means of partial Fourier transforms.

**C. REPRESENTATIONS FOR THE IO**

Having deduced Moyal’s equation in Eq. (65) it is straightforward to obtain the particular form of the motion equation for the IO Hamiltonian Eq. (20), as follows
\[
i \hbar \frac{\partial}{\partial t} W(x, p; t) = \left[ \frac{1}{2m} \left( \frac{p - i \hbar \partial}{2 \partial x} \right)^2 - \frac{1}{2} m \omega^2 \left( x + i \frac{\hbar}{2} \frac{\partial}{\partial p} \right)^2 - \frac{1}{2m} \left( p + i \frac{\hbar}{2} \frac{\partial}{\partial x} \right)^2 + \frac{1}{2} m \omega^2 \left( x - i \frac{\hbar}{2} \frac{\partial}{\partial p} \right)^2 \right] W(x, p; t). \tag{92}
\]

Upon expanding and simplifying we arrive to Eq. (91). On the other hand, incorporating the IO Hamiltonian into Eq. (72), after simplification Eq. (55) is obtained
\[
i \hbar \frac{\partial}{\partial t} A(\lambda, \theta; t) = \left[ \frac{1}{2m} \left( \frac{\partial}{\partial \theta} + \frac{\hbar}{2} \lambda \right)^2 - \frac{1}{2} m \omega^2 \left( \frac{\partial}{\partial \lambda} - \frac{\hbar}{2} \frac{\partial}{\partial \theta} \right)^2 \right] A(\lambda, \theta; t). \tag{93}
\]

It is important to pointed out that the phase space and the reciprocal phase space are the only representations for which the IO is exactly solvable. In contrast, the complexity in the \(x - \theta\) and \(\lambda - p\) representations require the use of second order partial differential equations, as show below
\[
\frac{\partial B(x, \theta; t)}{\partial t} = \left[ \frac{1}{m} \frac{\partial^2}{\partial \theta \partial x} + m \omega^2 x \theta \right] B(x, \theta; t), \tag{94}
\]
\[
\frac{\partial}{\partial \theta} Z(\lambda, p; t) = \left[ \frac{p \lambda}{m} + m \omega^2 \frac{\partial^2}{\partial \lambda \partial p} \right] Z(\lambda, p; t). \tag{95}
\]
Figure 5: Ambiguity function’s imaginary part $\Im [A(\lambda, \theta)]$ for energies $E_1 = -0.5 \ [\text{(a)}]\&\ [\text{(b)}]$, and $E_2 = -8 \ [\text{(c)}]\&\ [\text{(d)}]$ given in Wigner phase space for states subjected to the IO at times $t_0 = 0 \ a.u.$ and $t = 1.5 \ a.u. \ (\hbar = \omega = 1 \ a.u.)$. Black dots depict particles moving along the reciprocal phase space trajectories Eq.(26), and solid lines display the level set of the new conservation law Eq.(28).

Figure 6: Ambiguity function’s absolute value square $|A(\lambda, \theta)|^2$ for energies $E_1 = -0.5 \ [\text{(a)}]\&\ [\text{(b)}]$, and $E_2 = -8 \ [\text{(c)}]\&\ [\text{(d)}]$ given in Wigner phase space for states subjected to the IO at times $t_0 = 0 \ a.u.$ and $t = 1.5 \ a.u. \ (\hbar = \omega = 1 \ a.u.)$. Black dots depict particles moving along the reciprocal phase space trajectories Eq.(26), and solid lines display the level set of the new conservation law Eq.(28).