Entropic c–functions in $\mathcal{T}\mathcal{T}, J\mathcal{T}, T\mathcal{J}$ deformations

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We study, under an assumption, the holographic entanglement entropy of an interval in a quantum field theory obtained by deforming a two–dimensional holographic conformal field theory via a general linear combination of irrelevant operators $\mathcal{T}\mathcal{T}$, $J\mathcal{T}$ and $T\mathcal{J}$, and compute the Casin–Huerta entropic c–function. We find that the entropic c–function is ultraviolet cutoff independent, and along the renormalization group flow towards the ultraviolet, it is non–decreasing.
1. Introduction

It has recently been shown in [1] that the deformation of the worldsheet string theory on the background $AdS_3 \times S^1 \times \mathcal{N}$ with Neveu–Schwartz two–form $B$ field via a general linear combination of truly marginal worldsheet current–current operators

$$\lambda J_{SL}^- \bar{J}_{SL}^- + \epsilon_+ K \bar{J}_{SL}^- + \epsilon_- \bar{K} J_{SL}^-,$$

is equivalent to the deformation of the conformal field theory $\mathcal{M}$, which constitutes the symmetric product boundary theory $\mathcal{M}^p/S_p$ that describes long string states, by a general linear combination of irrelevant double trace operators

$$-t \bar{T} - \mu_+ J T - \mu_- \bar{J} T,$$

where $J$ and $\bar{J}$ are the left and right moving $U(1)$ currents, respectively, and $T$ and $\bar{T}$ are the holomorphic and anti–holomorphic stress tensor components, respectively, of the conformal field theory $\mathcal{M}$.

The operators $J \bar{T}$ and $T \bar{J}$ have left and right scaling dimensions $(1, 2)$ and $(2, 1)$, respectively, and therefore, the deformation (1.2) results in a theory that breaks Lorentz invariance [3, 4, 5]. The spacetime couplings $t$, $\mu_+$ and $\mu_-$ are related to the worldsheet dimensionless couplings $\lambda$, $\epsilon_+$ and $\epsilon_-$ via the relations [1],

$$t = \pi \alpha' \lambda, \quad \mu_{\pm} = 2 \sqrt{2 \alpha' \epsilon_{\pm}}, \quad \alpha' = l_s^2,$$

where $\alpha'$ is the Regge slope, and $l_s$ is the intrinsic string length.

The deformation (1.2) is irrelevant and therefore the couplings grow as we ascend the renormalization group, however, it is shown that the theory is solvable at finite couplings, in the sense that the spectrum [1, 6, 7, 8] and the partition sum [9 , 10, 11] can be computed exactly, for instance, on a torus. It is shown in [1] that in the space of couplings in which the combination

$$A = \frac{t}{4 \pi \alpha'} - \frac{(\mu_+ + \mu_-)^2}{32 \alpha'},$$

is positive, that is, $A > 0$, the energies of states are real, and the density of states asymptotically exhibits Hagedorn growth. In the limit $A \rightarrow 0^+$ however the theory appears to be distinct. The density of states asymptotically exhibits an intermediate growth between Cardy and Hagedorn growths.

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1 $p$ is the number of fundamental long strings needed to construct the $AdS_3$ background [2].
In this paper we study the (Von Neumann) entanglement entropy for an interval of length $L$ in the spacetime deformed theory with $A \geq 0$ from its bulk string theory description and compute the Casin-Huerta entropic $c$–function. The rest of the paper is organized as follows. In section 2 we review the corresponding bulk string theory background obtained under the deformation (1.1). In section 3 we compute the entanglement entropy in the deformed dual spacetime theory using the holographic prescription. Following this, we discuss its large and small $L$ limits. In section 4 we compute the entropic $c$–function and consider its large and small $L$ limits. In section 5 we discuss the main results and future research directions. In the appendix we collect some of the intermediate results that are required in section 3.

2. The bulk deformed string background

We begin with string theory on

$$AdS_3 \times S^1 \times \mathcal{N},$$

with Neveu–Schwartz two–form $B$ field. Where the component $\mathcal{N}$ is an internal six–
dimensional compact manifold. Its presence is irrelevant in our discussion. The $S^1$ component gives rise in the boundary conformal field theory to a $U(1)$ current algebra generated by the spacetime currents $J$ and $\bar{J}$ [12, 13].

The bosonic part of the worldsheet theory on $AdS_3 \times S^1$ is described by the action

$$S = \frac{k}{2\pi} \int d^2 z (\partial \phi \overline{\partial} \phi + e^{2\phi} \partial \overline{\gamma} \overline{\partial} \gamma + \frac{1}{k} \partial \psi \overline{\partial} \psi),$$

where $(\phi, \gamma, \overline{\gamma})$ are the coordinates on $AdS_3$, and $\psi \sim \psi + 2\pi$ is the coordinate on $S^1$. The boundary of $AdS_3$ is located at $\phi = +\infty$. The coordinates $\gamma$ and $\overline{\gamma}$ are

$$l_s \gamma = t + x, \quad l_s \overline{\gamma} = -t + x.$$  

The level $k$ is given by

$$l^2 = l^2_s k,$$

where $l$ is the radius of curvature of $AdS_3$.

The action has an affine $SL(2,R)_L \times SL(2,R)_R \times U(1)_L \times U(1)_R$ symmetry with left mover worldsheet currents

$$J^+_{SL} = e^{2\phi} \partial \overline{\gamma}, \quad J^-_{SL} = -2\gamma \partial \phi - 2\overline{\gamma} + \gamma^2 e^{2\phi} \overline{\partial} \gamma, \quad J^3_{SL} = \gamma e^{2\phi} \partial \overline{\gamma} - \partial \phi, \quad K = \partial \psi.$$  

and similar expressions for the right movers \( \mathcal{J}^+_{SL}, \mathcal{J}^j_{SL}, \mathcal{K} \).

Consider deforming the worldsheet theory (2.2) by adding to its Lagrangian the deformation (1.1). The deformation (1.1) is truly marginal, and therefore, it preserves the conformal symmetry. It breaks the affine \( SL(2, R)_L \times SL(2, R)_R \times U(1)_L \times U(1)_R \) worldsheet symmetry down to \( U(1)_L \times U(1)_R \times U(1)_L \times U(1)_R \) affine symmetry.

The deformation corresponds to a deformation of the metric \( g \), dilaton \( \Phi \), and Neveu–Schwartz two–form \( B \) \([1, 4]\). We shall refer to the deformed background as \( M_4 \).

\[
\begin{align*}
ds^2 &= d\phi^2 + h d\gamma d\bar{\gamma} + \frac{2he_+}{\sqrt{k}} d\psi d\bar{\gamma} + \frac{2he_-}{\sqrt{k}} d\psi d\gamma + \frac{1}{h} h_f^{-1} d\psi^2, \\
e^{2\Phi} &= g_s^2 e^{-2\phi} h, \\
B_{\gamma\bar{\gamma}} &= g_{\gamma\bar{\gamma}}, \\
B_{\gamma\psi} &= g_{\gamma\psi}, \\
B_{\psi\bar{\gamma}} &= g_{\psi\bar{\gamma}},
\end{align*}
\]

where

\[
h^{-1} = e^{-2\phi} + \lambda - 4\epsilon_+ \epsilon_-, \quad f^{-1} = h^{-1} + 4\epsilon_+ \epsilon_-.
\]

For \( \lambda = 0, \epsilon_\pm = 0 \) \( M_4 \) reduces to our starting background \( AdS_3 \times S^1 \). For \( \lambda = 0 \) \( M_4 \) reduces to a warped \( AdS_3 \times S^1 \) background \([1, 5]\). For \( \epsilon_\pm = 0 \) \( M_4 \) reduces to a background that is asymptotically \( AdS_3 \times S^1 \) for large negative \( \phi \) and \( \mathbb{R}_\phi \times \mathbb{R}^{1,1} \times S^1 \) for large positive \( \phi \) \([15]\).

It is shown in \([1]\) that for a combination of the couplings

\[
\Psi = \lambda - (\epsilon_+ + \epsilon_-)^2,
\]

with \( \Psi \geq 0 \) the geometry is smooth and has no closed time like curves. This positivity condition on (2.10) is the dual analogue of the positivity condition on (1.4).

In this paper we consider the case in which

\[
\epsilon_+ = \frac{\epsilon}{2}, \quad \epsilon_- = \frac{\epsilon}{2}, \quad \Psi \geq 0.
\]

In this case the background \( M_4 \) is

\[
\begin{align*}
ds^2 &= \alpha' d\phi^2 - hdt^2 + h \left( dx + \epsilon \sqrt{\frac{\alpha'}{k}} d\psi \right)^2 + \frac{\alpha'}{k} d\psi^2, \\
e^{2\Phi} &= g_s^2 e^{-2\phi} h, \\
h^{-1} &= e^{-2\phi} + \Psi.
\end{align*}
\]

In what follows we work on this background to compute the entanglement entropy and the entropic \( c \)–function for an interval of length \( L \) in the deformed two–dimensional long string conformal field theory \( \mathcal{M} \) using the holographic prescription.

3
3. Holographic entanglement entropy

In this section we compute the entanglement entropy in the deformed boundary long string conformal field theory $\mathcal{M}$ for a spatial interval of length $L$ with endpoints at $x = -L/2$ and $x = +L/2$. In what follows we begin by stating the holographic entanglement prescription [16, 17].

Suppose we have a $d$–dimensional holographic quantum field theory $\mathcal{M}_d$. Suppose also the dual string theory is on a background $\mathcal{M}_{d+1}$. To compute the entanglement entropy of a given spatial region $\mathcal{R}$ in the boundary quantum field theory $\mathcal{M}_d$ one first needs to find a co–dimension two static surface $\mathcal{K}$ in the bulk geometry $\mathcal{M}_{d+1}$ ending on the boundary of $\mathcal{R}$ that minimizes the area functional. Then the entanglement entropy $S_\mathcal{R}$ in the $d$–dimensional boundary quantum field theory $\mathcal{M}_d$ is given by

$$S_\mathcal{R} = \frac{\text{Area}(\mathcal{K})}{4G_N^{(d+1)}}, \quad (3.1)$$

where $G_N^{(d+1)}$ is the $d + 1$–dimensional Newton’s constant of the $\mathcal{M}_{d+1}$ geometry.

Following the holographic prescription we now compute the entanglement entropy for the spatial interval of length $L$ in the deformation (1.2). The bulk string geometry (2.12) at a moment of time is

$$ds^2 = \alpha' d\phi^2 + hdy^2 + \frac{\alpha'}{k}d\psi^2; \quad h = e^{-2\phi} + \Psi, \quad (3.2)$$

where $y = x + \epsilon \sqrt{\alpha'/k} \psi$.

We now look for a two–dimensional surface $\phi(y, \psi)$ in the geometry (3.2) (and wrapping the internal $\mathcal{N}$ space) that minimizes globally (in the space of functions) the area functional (3.1) which taking into account the dilaton [17] is

$$S = \frac{\sqrt{k\alpha'}}{4G_N^{(4)}} \int_0^{2\pi} d\psi \int_{-\frac{L}{2} + k \sqrt{\alpha'/k} \psi}^{\frac{L}{2} + k \sqrt{\alpha'/k} \psi} dy \ e^{2\phi} \sqrt{\frac{1}{h} \left( 1 + \frac{\alpha'}{h} (\partial_y \phi)^2 + k (\partial_\psi \phi)^2 \right)}, \quad (3.3)$$

with the boundary conditions

$$\phi(\pm L/2 + \epsilon \sqrt{\alpha'/k} \psi, \psi) = \infty, \quad \phi(y, 0) = \phi(y + \epsilon \sqrt{\alpha'/k} \psi, 2\pi, 2\pi), \quad (3.4)$$

where $\psi$ is on $S^1$ that is $\psi \sim \psi + 2\pi$.

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2 Here we rescaled the metric (3.2) by the level $k$.  

We note that under the following continuous and discrete spacetime transformations

\[
\psi \rightarrow \psi + \delta, \quad y \rightarrow y + \epsilon \sqrt{\frac{\alpha'}{k}} \delta, \quad \text{and} \quad \psi \rightarrow -\psi + 2\pi, \quad y \rightarrow -y + \epsilon \sqrt{\frac{\alpha'}{k}} 2\pi, \quad (3.5)
\]

where \(\delta\) is an arbitrary constant, the bulk background \((3.2)\) and the boundary conditions \((3.4)\) are invariant. There is a surface

\[
\phi(y, \psi) = \phi(y - \epsilon \sqrt{\frac{\alpha'}{k}} \psi),
\]

that is invariant under the above symmetry transformations \((3.5)\) and that minimizes locally (in the space of functions) the area functional. We conjecture that the local minimum corresponding to \((3.6)\) is also the global minimum. We leave further details for future work. In this paper we assume that the conjecture is true.

The minimal surface \((3.6)\) is generated by translating the curve, for example at \(\psi = 0\), \(\phi(y)\), along the line \(y = \epsilon \sqrt{\frac{\alpha'}{k}} \psi\). This curve has the parity symmetry \(y \rightarrow -y\). The entanglement entropy is then obtained using \((3.3)\). We find

\[
S = \frac{\sqrt{k}}{4G_N^{(3)}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \sqrt{H(U) \sqrt{1 + \beta(U)(\partial_x U)^2}}, \quad (3.7)
\]

where \(2\pi l_s G_N^{(3)} = G_N^{(4)}\), and

\[
U = e^\phi, \quad U^2 h^{-1} = 1 + U^2 (\lambda - \epsilon^2), \quad U^{-2} H(U) = U^2 h^{-1}, \quad U^4 \beta(U) = (1 + U^2 \lambda) \alpha'.
\]

The boundary conditions \((3.4)\) now take the form

\[
U(\pm L/2) = U_\infty, \quad (3.9)
\]

where \(U_\infty\) is an ultraviolet cutoff.

We denote the value at which the curve \(U\) takes its minimum value by \(U_0\). This value is related to the length of the interval \(L\). This follows from the Euler’s variational equation of the action \((3.7)\) with the boundary conditions \((3.9)\). One finds

\[
L(U_0) = 2\sqrt{H(U_0)} \int_{U_0}^{U_\infty} dU \frac{\sqrt{\beta(U)}}{\sqrt{H(U) - H(U_0)}}, \quad (3.10)
\]

The entropy using the Euler’s equation of motion becomes

\[
S = \frac{\sqrt{k}}{2G_N^{(3)}} \int_{U_0}^{U_\infty} dU \sqrt{\frac{\beta(U)}{H(U) - H(U_0) H(U)}}, \quad (3.11)
\]
We rewrite the expression (3.10) of the interval length $L$ in terms of the minimum value $U_0$ as

$$L(U_0) = \frac{\sqrt{\alpha'}}{U_0} \int_1^{x_{\infty}} \frac{dx}{x} \sqrt{\frac{(1 + \alpha x)(1 + \alpha_-)}{x(x - 1)(\alpha_- x + \alpha_- + 1)}},$$

where

$$\alpha = \lambda U_0^2, \quad \alpha_- = \Psi U_0^2, \quad x_{\infty} = \frac{U_0^2}{U_{\infty}^2}. \quad (3.13)$$

The integral (3.12) is ultraviolet convergent and it solves to

$$\frac{L}{2\sqrt{\alpha'\lambda}} = \sqrt{\frac{1 + \alpha}{\alpha}} E \left( \arcsin \sqrt{\frac{1 + \alpha_-}{1 + 2\alpha_-}}, \sqrt{\frac{1 + 2\alpha_-}{(1 + \alpha)(1 + \alpha_-)}} \right), \quad (3.14)$$

where $E(\varphi, k)$ is the incomplete elliptic integral of the second kind.

In the limit in which $\Psi \to 0^+$ or, equivalently $\alpha_- \to 0^+$, we note that the interval $L$ (3.14) takes the following simpler form

$$\frac{L}{2\sqrt{\alpha'\lambda}} = \sqrt{\frac{1 + \alpha}{\alpha}} E \left( \arcsin \sqrt{\frac{1 + \alpha_-}{1 + 2\alpha_-}}, \sqrt{\frac{1 + 2\alpha_-}{(1 + \alpha)(1 + \alpha_-)}} \right), \quad (3.15)$$

where $E(k)$ is the complete elliptic integral of the second kind. Sending $\alpha \to 0^+$ in (3.13) yields

$$\frac{L}{\sqrt{\alpha'}} = \frac{2}{U_0}. \quad (3.16)$$

We rewrite the entanglement entropy (3.11) as

$$S = \frac{\sqrt{k\alpha'}}{4G_{N}^{(3)}} \int_1^{x_{\infty}} \frac{dx}{x} \sqrt{\frac{\alpha x + 1}{x(x - 1)(\alpha_- x + \alpha_- + 1)} \cdot (\alpha_- x + 1)}. \quad (3.17)$$

We note that for $\alpha_- \neq 0$ the entropy diverges as $S \sim x_{\infty}$, and in the limit in which we take $\alpha_- \to 0^+$ with $\alpha \neq 0$ it diverges as $S \sim \sqrt{x_{\infty}}$. We also note that in the case in which we take both $\alpha \to 0^+$, $\alpha_- \to 0^+$ it diverges as $S \sim \log(x_{\infty})$. The integral (3.17) solves with the ultraviolet cutoff $x_{\infty}$ to

$$S = \frac{\sqrt{k\alpha'}}{2G_{N}^{(3)}} \frac{1}{\sqrt{(\alpha + 1)(\alpha_- + 1)}} \left\{ \left( \alpha_- + \alpha - \alpha \frac{d}{d\xi} \right) \right. \right.$$

$$\left. \left[ \frac{1}{\xi + 1} \cdot \Pi \left( \arcsin \sqrt{\frac{\alpha_+ + 1}{2\alpha_+ + 1}} \cdot \frac{1 - \frac{1}{\alpha}}{\xi + 1} \right), \frac{2\alpha_- + 1}{(\alpha + 1)(\alpha_- + 1)}, \sqrt{\frac{2\alpha_- + 1}{(\alpha + 1)(\alpha_- + 1)}} \right]_{\xi = 0} \right.$$

$$+ F \left( \arcsin \sqrt{\frac{\alpha_+ + 1}{2\alpha_+ + 1}} \cdot \frac{1 - \frac{1}{\alpha}}{\xi_{\infty}}, \sqrt{\frac{2\alpha_- + 1}{(\alpha + 1)(\alpha_- + 1)}} \right) \right\},$$

$$\quad (3.18)$$
where \( \Pi(\varphi, n, k) \) is the incomplete elliptic integral of the third kind, and \( F(\varphi, k) \) is the incomplete elliptic integral of the first kind.

In the limit \( \Psi \to 0^+ \) or, equivalently \( \alpha_\perp \to 0^+ \), the entropy \( \text{(3.18)} \) gives

\[
S = \frac{\sqrt{k\alpha'}}{2G_N^{(3)}} \left[ F \left( \arcsin \sqrt{1 - \frac{1}{x_\infty}}, \sqrt{\frac{1}{1 + \alpha}} \right) - E \left( \arcsin \sqrt{1 - \frac{1}{x_\infty}}, \sqrt{\frac{1}{1 + \alpha}} \right) \right] \\
+ \frac{\sqrt{k\alpha'}}{2G_N^{(3)}} \sqrt{\left(\alpha x_\infty + 1\right) \left(1 - \frac{1}{x_\infty}\right)}. 
\]

(3.19)

Taking \( \alpha \to 0^+ \) in \( \text{(3.19)} \) gives

\[
S = \frac{\sqrt{k\alpha'}}{2G_N^{(3)}} \text{Log}(2\sqrt{x_\infty}). 
\]

(3.20)

In the rest of the current section we study the above results in turns.

3.1. Case \( \Psi = 0 : \lambda = 0, \ \epsilon = 0 \)

In this case we have for the interval length \( L \) and entanglement entropy \( S \) from \( \text{(3.16)} \) and \( \text{(3.20)} \)

\[
\frac{L}{\sqrt{\alpha'}} = \frac{2}{U_0}, \quad S = \frac{\sqrt{k\alpha'}}{2G_N^{(3)}} \text{Log} \left( \frac{2\ U_\infty}{U_0} \right). 
\]

(3.21)

We write the entropy as

\[
S = \frac{c}{3} \text{Log} \left( \frac{L}{L_\Lambda} \right), \quad \frac{L_\Lambda}{\sqrt{\alpha'}} := \frac{2}{U_\infty}, \quad c = \frac{3\sqrt{k\alpha'}}{2G_N^{(3)}}, 
\]

(3.22)

where \( L_\Lambda \) is an ultraviolet cutoff. This result is the well–known entanglement entropy for a two–dimensional holographic conformal field theory with (Brown–Henneaux) central charge \( c \) that is dual to pure \( AdS_3 \) \([18, 19, 20]\).

3.2. Case \( \Psi = 0 : \lambda = \epsilon^2 \neq 0 \)

In this case we have from \( \text{(3.15)} \) and \( \text{(3.19)} \) that the interval length \( L \) and entanglement entropy \( S \) are given by

\[
\frac{L}{2\sqrt{\alpha'\lambda}} = \sqrt{\frac{1 + \alpha}{\alpha}} E \left( \sqrt{\frac{1}{1 + \alpha}} \right), 
\]

(3.23)
\[ S = \frac{\sqrt{k\alpha'}}{2G_N^{(3)}} \left[ F \left( \arcsin \sqrt{1 - \frac{1}{x_\infty}}, \sqrt{\frac{1}{1 + \alpha}} \right) - E \left( \arcsin \sqrt{1 - \frac{1}{x_\infty}}, \sqrt{\frac{1}{1 + \alpha}} \right) \right] + \frac{\sqrt{k\alpha'}}{2G_N^{(3)}} \sqrt{\alpha x_\infty + 1} \cdot \left( 1 - \frac{1}{x_\infty} \right), \quad \alpha = \lambda U_0^2, \quad x_\infty = \frac{U_\infty^2}{U_0^2}. \] 

(3.24)

We find from (3.23) that in the large \( U_0 \) limit the interval length \( L \) asymptotes to a minimum value which we denote by \( L_0 \). It takes the value

\[ L_0 = \pi \sqrt{\alpha'}. \] 

(3.25)

We find using (3.23) the following large \( U_0 \) expansion of the interval length \( L \),

\[ \frac{L}{L_0} = 1 + \frac{1}{4} \frac{1}{\alpha} - \frac{3}{64} \frac{1}{\alpha^2} + O \left( \frac{1}{\alpha^3} \right), \quad \alpha = \lambda U_0^2. \] 

(3.26)

Inverting the above equation one finds

\[ \alpha = \frac{1}{4\xi} - \frac{3}{16} + O(\xi), \quad \xi = \frac{L}{L_0} - 1. \] 

(3.27)

The small \( U_0 \) expansion of the interval length \( L \) is

\[ \frac{L}{\sqrt{\alpha'}} = \frac{2}{U_0} \left( 1 - \frac{1}{4} \alpha \ln \alpha + \frac{1}{32} \alpha^2 \ln \alpha + O(\alpha^3) \right), \quad \alpha = \lambda U_0^2. \] 

(3.28)

The leading term corresponds to the deep AdS\(_3\) geometry (see (3.21)). Therefore, in the large \( L \) limit the surface is deep inside the bulk. We note also that the correction starts at order \( \lambda = \epsilon^2 \).

Inverting equation (3.28) we find

\[ \alpha = \left( \frac{2}{\pi} \cdot \frac{L_0}{L} \right)^2 \left[ 1 - \left( \frac{2}{\pi} \cdot \frac{L_0}{L} \right)^2 \text{Log} \left( \frac{2}{\pi} \cdot \frac{L_0}{L} \right) + O \left( \frac{2}{\pi} \cdot \frac{L_0}{L} \right)^4 \right]. \] 

(3.29)

We can similarly study the large and small \( U_0 \) or, equivalently, the large and small \( L \) limit of the entanglement entropy (3.24). One finds in the large interval length \( L \) limit

\[ S = \frac{c}{3} \left[ \sqrt{\alpha x_\infty} - \frac{1}{2} \text{Log}(\alpha) - \frac{\alpha}{8} \text{Log}(\alpha) + O(\alpha^2) \right], \quad \alpha = \lambda U_0^2, \quad x_\infty = \frac{U_\infty^2}{U_0^2}, \] 

(3.30)

which upon using (3.29) gives

\[ S = \frac{c}{3} \left[ \frac{L_0}{L_A} - \text{Log} \left( \frac{2}{\pi} \cdot \frac{L_0}{L} \right) + \frac{1}{4} \left( \frac{2}{\pi} \cdot \frac{L_0}{L} \right)^2 \text{Log} \left( \frac{2}{\pi} \cdot \frac{L_0}{L} \right) + O \left( \frac{2}{\pi} \cdot \frac{L_0}{L} \right)^4 \right], \] 

(3.31)
where
\[
L_\Lambda = \frac{\pi \sqrt{\alpha'}}{U_\infty},
\] (3.32)
and \(L_\Lambda\) is an ultraviolet cutoff. The leading logarithmic term is the contribution from the \(AdS_3\) region found deep inside the bulk. The coefficient of this term is \(-c/3\).

As we approach \(L_0\) we find
\[
S = \frac{c}{3} \left[ \sqrt{\alpha x_\infty} + \frac{\pi}{4} \alpha^{1/2} - \frac{\pi}{32} \alpha^{3/2} + O\left( \frac{1}{\alpha^{5/2}} \right) \right], \quad \alpha = \lambda U_0^2, \quad x_\infty = \frac{U_\infty^2}{U_0^2},
\] (3.33)
which simplifies using (3.27) to
\[
S = \frac{c}{3} \left[ \frac{L_0}{L_\Lambda} + \frac{\pi}{2} \left( \frac{L}{L_0} - 1 \right)^{3/2} - \frac{\pi}{16} \left( \frac{L}{L_0} - 1 \right)^{5/2} + O\left( \left( \frac{L}{L_0} - 1 \right)^{7/2} \right) \right].
\] (3.34)
We note that there is no logarithmically divergent term. In local and Lorentz–invariant even–dimensional quantum field theories, however, in general the presence of a logarithmically divergent term is generic, and its coefficient is expected to be universal [21].

3.3. Case \(\Psi > 0 : \epsilon = 0\)

This case is studied in [22]. Setting \(\alpha = \alpha_-\) in (3.14) and (3.18) we find for the length \(L\) and entanglement entropy \(S\)
\[
\frac{L}{2\sqrt{\alpha'} \lambda} = \sqrt{\frac{1 + \alpha}{\alpha}} E \left( \arcsin \sqrt{\frac{1 + \alpha}{1 + 2\alpha}} \sqrt{\frac{1 + 2\alpha}{1 + 2\alpha + \alpha^2}} \right),
\] (3.35)
\[
S = \frac{\sqrt{k\alpha'}}{2G_N^{(3)} \alpha + 1} \left\{ \left( 2\alpha - \alpha^2 \frac{d}{d\xi} \right) \left[ \frac{1}{\xi + 1} \cdot \Pi \left( \arcsin \sqrt{\frac{\alpha + 1}{2\alpha + 1}} \cdot \left( 1 - \frac{1}{x_\infty} \right), \frac{2\alpha + 1}{(\xi + 1)(\alpha + 1)}, \sqrt{\frac{2\alpha + 1}{\alpha^2 + 2\alpha + 1}} \right] \right|_{\xi=0} + F \left( \arcsin \sqrt{\frac{\alpha + 1}{2\alpha + 1}} \cdot \left( 1 - \frac{1}{x_\infty} \right), \sqrt{\frac{2\alpha + 1}{\alpha^2 + 2\alpha + 1}} \right) \right\}, \quad \alpha = \lambda U_0^2, \quad x_\infty = \frac{U_\infty^2}{U_0^2}.
\] (3.36)
We find that in the large \(U_0\) limit the interval length \(L\) asymptotes to a minimum value which we denote by \(L_0\) (this should cause no ambiguity)
\[
L_0 = \frac{\pi \sqrt{\alpha' \lambda}}{2}.
\] (3.37)
We note that there is a factor of 2 difference between (3.25) and (3.37). We find from (3.35) that the interval length $L$ has the following large $U_0$ expansion

$$\frac{L}{L_0} = 1 + \frac{2}{\pi \alpha} + \frac{3\pi - 16}{16\pi \alpha^2} + \mathcal{O}\left(\frac{1}{\alpha^3}\right), \quad \alpha = \lambda U_0^2. \quad (3.38)$$

Inverting the above equation one finds

$$\alpha = \frac{2}{\pi \xi} + \frac{3\pi - 16}{32} + \mathcal{O}(\xi), \quad \xi = \frac{L}{L_0} - 1. \quad (3.39)$$

The small $U_0$ expansion takes the form

$$L \sqrt{\alpha'} = 2 U_0 \left[ 1 - \frac{\alpha^2}{4} \log(\alpha) + \frac{3}{8} \alpha^3 \log(\alpha) + \mathcal{O}(\alpha^4) \right], \quad \alpha = \lambda U_0^2. \quad (3.40)$$

We note that for a long interval the surface is deep inside the bulk in the AdS$_3$ region. We also note that the term linear in $\alpha$ is zero. Therefore, the correction starts, in this case, at order $\lambda^2$.

Inverting the above equation we find

$$\alpha = \left(\frac{4 L_0}{\pi L}\right)^2 \left[ 1 - \left(\frac{4 L_0}{\pi L}\right)^4 \log\left(\frac{4 L_0}{\pi L}\right) + \mathcal{O}\left(\left(\frac{4 L_0}{\pi L}\right)^6\right) \right]. \quad (3.41)$$

In the large $L$ limit we find that the entanglement entropy $S$ has the following series expansion

$$S = c \left[ \frac{1}{2}\alpha x_{\infty} - \frac{1}{2} \log(\alpha) - \frac{1}{4} \alpha + \mathcal{O}(\alpha^2 \log(\alpha)) \right], \quad \alpha = \lambda U_0^2, \quad x_{\infty} = \frac{U_0^2}{U^2_{\infty}}, \quad (3.42)$$

which using (3.41) gives

$$S = c \left[ \frac{1}{2} \frac{L_0^2}{L^2} \frac{16}{\pi^2} - \log\left(\frac{L_0}{L} \cdot \frac{4}{\pi}\right) \frac{L_0^2}{L^2} \frac{16}{\pi^2} - \frac{1}{4} \left( \frac{L_0^2}{L^2} \frac{16}{\pi^2} \right)^2 + \mathcal{O}\left( \left( \frac{L_0^2}{L^2} \frac{16}{\pi^2} \right)^2 \right) \right], \quad U_{\infty} = \frac{2\sqrt{\alpha'}}{L_\Lambda}. \quad (3.43)$$

The (leading) logarithmic term is due to the deep AdS$_3$ region in the bulk. The coefficient of this term is $-c/3$.

As we approach $L_0$ the entropy $S$ takes the form

$$S = c \left[ \frac{1}{2}\alpha x_{\infty} - \frac{1}{2} \log(\alpha) + \mathcal{O}\left(\frac{1}{\alpha}\right) \right], \quad \alpha = \lambda U_0^2, \quad x_{\infty} = \frac{U_0^2}{U^2_{\infty}}. \quad (3.44)$$

Using (3.39) this gives

$$S = c \left[ \frac{1}{2} \frac{16}{\pi^2} \frac{L_0^2}{L^2} + \frac{1}{2} \log\left(\frac{\pi}{2} \left( \frac{L}{L_0} - 1 \right) \right) + \mathcal{O}\left( \frac{L}{L_0} - 1 \right) \right]. \quad (3.45)$$

In this limit the geometry is a linearly varying dilaton background. We note that in this case we have a logarithmically divergent term as opposed to the former $\Psi = 0$ case. The coefficient of this term is $c/6$. 

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3.4. Case $\Psi > 0 : \epsilon > 0$

In the large $U_0$ limit we find from (3.14) that the interval length approach a minimum value $L_0$

$$L_0 = \frac{\pi \sqrt{\alpha \lambda}}{2},$$

that is determined solely by $\lambda$.

The length $L$ has the following large $U_0$ expansion

$$\frac{L}{L_0} = 1 + \frac{2 - \delta^2}{\pi (1 - \delta^2) \alpha} + \frac{3\pi - 16 + 2(4 - \pi)\delta^2 - \pi \delta^4}{16 \pi (1 - \delta^2)^2 \alpha^2} + \mathcal{O}\left(\frac{1}{\alpha^3}\right), \quad \delta^2 = \frac{\epsilon^2}{\lambda}, \quad \alpha = \lambda U_0^2.$$ (3.47)

Inverting this we find

$$\alpha = \frac{2 - \delta^2}{\pi (1 - \delta^2) \xi} + \frac{3\pi - 16 + 2(4 - \pi)\delta^2 - \pi \delta^4}{32 - 48\delta^2 + 16\delta^4} + \mathcal{O}(\xi), \quad \xi = \frac{L}{L_0} - 1.$$ (3.48)

The small $U_0$ expansion takes the form

$$\frac{L}{\sqrt{\alpha^2}} = \frac{2}{U_0} \left[ 1 - \frac{\delta^2}{4} \alpha \log(\alpha) - \frac{\alpha^2}{4} (1 + \delta^2 p) \log(\alpha) + \mathcal{O}(\alpha^3) \right], \quad \delta^2 = \frac{\epsilon^2}{\lambda}, \quad \alpha = \lambda U_0^2,$$ (3.49)

here $p$ is a polynomial in $\delta^2$. The leading term corresponds to the deep $AdS_3$ region. We note that in this case the correction starts at order $\epsilon^2$.

Inverting the above equation one finds

$$\alpha = \left( \frac{4 L_0}{\pi L} \right)^2 \left[ 1 - \delta^2 \left( \frac{4 L_0}{\pi L} \right)^2 \log \left( \frac{4 L_0}{\pi L} \right) - \left( \frac{4 L_0}{\pi L} \right)^4 \left( 1 + \delta^2 p \right) \log \left( \frac{4 L_0}{\pi L} \right) + \mathcal{O} \left( \left( \frac{4 L_0}{\pi L} \right)^6 \right) \right].$$ (3.50)

In the large $L$ limit we find the following expansion for the entropy $S$

$$S = \frac{c}{3} \left[ \frac{1}{2} \alpha x_{\infty} \sqrt{1 - \delta^2} - \frac{1}{2} \log(\alpha) - \frac{1}{4} \alpha \left( \frac{1}{(1 - \delta^2)^2} + \frac{1}{2} \delta^2 \log(\alpha) \right) + \mathcal{O}(\alpha^2) \right],$$ (3.51)

which upon using (3.50) gives

$$S = \frac{c}{3} \left[ \left( \frac{4 L_0}{\pi L_\Lambda} \right)^2 \sqrt{\frac{1 - \delta^2}{4}} - \log \left( \frac{4 L_0}{\pi L} \right) - \frac{1}{4} \left( \frac{4 L_0}{\pi L} \right)^2 \left[ \frac{1}{(1 - \delta^2)^2} - \delta^2 \log \left( \frac{4 L_0}{\pi L} \right) \right] + \mathcal{O} \left( \frac{L_0}{L} \right)^4 \right],$$ (3.52)

where $L_\Lambda$ is defined in (3.43). The leading logarithmic term as in the previous cases corresponds to the $AdS_3$ region found deep inside the bulk. The coefficient of this term is $-c/3$. 

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As we approach $L_0$ we find

$$S = \frac{c}{3} \left[ \frac{1}{2} \alpha x_{\infty} - \frac{2 - \delta^2}{4\sqrt{1 - \delta^2}} \log(\alpha) + \mathcal{O}\left(\frac{1}{\alpha}\right) \right],$$

(3.53)

which using (3.48) gives

$$S = \frac{c}{3} \left[ \frac{1}{2} \cdot \frac{16}{\pi^2} \cdot \frac{L_0^2}{L_{\Lambda}^2} \cdot \sqrt{1 - \delta^2} + \frac{2 - \delta^2}{4\sqrt{1 - \delta^2}} \log \left( \left( \frac{L}{L_0} - 1 \right) \cdot \frac{\pi (1 - \delta^2)}{2 - \delta^2} \right) \right] + \mathcal{O}\left(\frac{L}{L_0} - 1\right).$$

(3.54)

We note that the leading logarithmic term has a coefficient that depends on $\delta^2$.

In the next section we study in the above cases the Casin–Huerta entropic $c$–function.

4. Casin–Huerta entropic $c$–function

In quantum field theory entanglement entropy $S$ is ultraviolet divergent. However, in two–dimensional local and Lorentz–invariant quantum field theories it is shown that the Casin–Huerta entropic $c$–function \[23\] which is defined as

$$C := L \frac{\partial S}{\partial L},$$

(4.1)

is finite, and at fixed points of renormalization group flow it is proportional to the corresponding central charges.

In this section we study the entropic $c$–function for the different cases we studied in the former section.

4.1. Case $\Psi = 0 : \lambda = 0, \epsilon = 0$

In this case we have

$$C = \frac{c}{3}.$$ 

(4.2)

This is the result for a two–dimensional holographic conformal field theory with central charge $c$ \[23\]. It is independent of the interval length $L$. Thus, the entropic $c$–function is non–negative and constant.
4.2. Case $\Psi = 0 : \lambda = \epsilon^2 \neq 0$

In this case we find that the entropic $c$–function $C(\alpha)$ is given by

$$C = \frac{c}{3} \sqrt{1 + \alpha E \left( \sqrt{\frac{1}{1 + \alpha}} \right)}, \quad \alpha = \lambda U_0^2. \quad (4.3)$$

We study the small and large $U_0$ limit, or equivalently the large and small $L$ limit of the entropic $c$–function (4.3).

In the large $L$ limit we find

$$C = \frac{c}{3} \left( 1 + \frac{2}{\pi^2 \xi^2} \log \left( \frac{2}{\pi} \cdot \xi \right) + \mathcal{O} \left( \frac{1}{\xi^4} \right) \right), \quad L := \xi L_0, \quad L_0 = \pi \sqrt{\alpha' \lambda}. \quad (4.4)$$

In the small $L$ limit we find

$$C = \frac{c}{3} \left( \frac{\pi}{4} \cdot \frac{1}{\xi^{7/2}} + \frac{5\pi}{32} \cdot \xi^{3/2} + \mathcal{O}(\xi^{3/2}) \right), \quad \xi = \frac{L}{L_0} - 1. \quad (4.5)$$

One can think of $L$ as a renormalization group scale. We note that the entropic $c$–function increases as we ascend the renormalization group, and it diverges in the ultraviolet at $L_0$. At short distances it diverges as

$$C \sim \xi^{-\frac{1}{2}}, \quad \xi = \frac{L}{L_0} - 1. \quad (4.6)$$

4.3. Case $\Psi > 0 : \epsilon = 0$

This case is studied in [22][3]. In this case we find that the entropic $c$–function $C(\alpha)$ in closed–form is given by

$$C = \frac{c}{3} (1 + \alpha) E \left( \arcsin \sqrt{\frac{1 + \alpha}{1 + 2\alpha}}, \sqrt{\frac{1 + 2\alpha}{(1 + \alpha)^2}} \right), \quad \alpha = \lambda U_0^2. \quad (4.7)$$

Using the results (3.39) and (3.41) for $U_0$ we find in the large $L$ limit

$$C = \frac{c}{3} \left( 1 + \frac{8}{\pi^2 \xi^2} + \mathcal{O} \left( \frac{1}{\pi \xi} \right)^4 \right), \quad \xi := \frac{L}{L_0}, \quad L_0 = \frac{\pi \sqrt{\alpha' \lambda}}{2}, \quad (4.8)$$

and in the small $L$ limit

$$C = \frac{c}{3} \left( \frac{1}{2} \left( 1 - \frac{1}{\xi} \right)^{-1} + \mathcal{O} \left( 1 - \frac{1}{\xi} \right) \right), \quad \xi := \frac{L}{L_0}. \quad (4.9)$$

We also note in this case that the entropic $c$–function is non–negative and increasing. At short distances it diverges as

$$C \sim \xi^{-1}, \quad \xi = \frac{L}{L_0} - 1. \quad (4.10)$$

Our results (4.8) and (4.9) are in agreement with the work [22].

3 It is also studied in a closely related work in [24].
4.4. Case $\Psi > 0 : \epsilon > 0$

In this case we find that the $c$–function $C(\alpha, \chi)$ is given by

$$C = \frac{c}{3} \sqrt{(1 + \alpha)(1 + \alpha\chi)} E \left( \arcsin \sqrt{\frac{1 + \alpha\chi}{1 + 2\alpha\chi}} \right), \quad (4.11)$$

where

$$\alpha = \lambda U_0^2, \quad \Psi = \lambda \chi. \quad (4.12)$$

In this case the large $L$ limit of the entropic $c$–function takes the form

$$C = \frac{c}{3} \left( 1 + \frac{8}{\pi^2 \xi^2} \left( \frac{1}{\chi^2} + (1 - \chi)\log \left( \frac{\pi \xi}{4} \right) \right) + \mathcal{O} \left( \frac{1}{\pi \xi} \right)^4 \right), \quad \xi := \frac{L}{L_0}, \quad L_0 = \frac{\pi \sqrt{\alpha'}}{2}. \quad (4.13)$$

In the small $L$ limit we find

$$C = \frac{c}{3} \left( \frac{1 + \chi}{4\sqrt{\chi}} \left( 1 - \frac{1}{\xi} \right)^{-1} + \mathcal{O} \left( 1 - \frac{1}{\xi} \right) \right), \quad \xi := \frac{L}{L_0}. \quad (4.14)$$

In this case also the entropic $c$–function is non–negative, ultraviolet cutoff independent and increasing. In the ultraviolet it diverges as

$$C \sim \chi^{-\frac{1}{2}} \cdot \xi^{-1}, \quad \xi = \frac{L}{L_0} - 1. \quad (4.15)$$

We note that setting $\chi = 1$ in (4.11) gives (4.7), and setting $\chi = 0$ gives (4.3). At $\alpha = 0$ it gives (4.2). The entropic $c$–function $C$ (4.11) and the interval length $L$ (3.14) satisfy the relation

$$\frac{C}{c_0} = \frac{L}{l_0} \cdot \sqrt{1 + \alpha\chi}, \quad l_0 = \frac{2\sqrt{\alpha'}}{U_0}, \quad c_0 = \frac{c}{3}, \quad \alpha = \lambda U_0^2, \quad (4.16)$$

where $l_0$ and $c_0$ are the interval length $L$ and the entropic $c$–function $C$ at $\lambda = 0$ or in the infrared, respectively.

5. Discussion

In this paper we conjectured that the minimal surface preserves the symmetry transformations (3.5), and we computed the entanglement entropy and the entropic $c$–function. We found that the entropic $c$–function is given by

$$C = \frac{c}{3} \sqrt{(1 + \alpha)(1 + \alpha\chi)} E \left( \arcsin \sqrt{\frac{1 + \alpha\chi}{1 + 2\alpha\chi}} \right), \quad (5.1)$$
where
\[ \alpha = \lambda U_0^2, \quad \chi = 1 - \delta^2, \quad \delta^2 = \frac{\epsilon^2}{\lambda}, \]  \hspace{1cm} (5.2)
and that it is non–negative and ultraviolet cutoff independent. The variable \( \alpha \) is related to the interval length \( L \) via
\[ \frac{L}{2\sqrt{\alpha'\lambda}} = \sqrt{\frac{1 + \alpha}{\alpha}} E \left( \arcsin \sqrt{\frac{1 + \alpha \chi}{1 + 2\alpha \chi}}, \sqrt{\frac{1 + 2\alpha \chi}{(1 + \alpha)(1 + \alpha \chi)}} \right). \]  \hspace{1cm} (5.3)

We also found that along the renormalization group flow towards the ultraviolet it is non–decreasing. At long distances it is proportional to the central charge of the original conformal field theory. At short distances it diverges.

In the case in which \( \Psi > 0 \), the entropic \( c \)–function diverges in the ultraviolet as
\[ C \sim \chi^{-\frac{1}{2}} \cdot \xi^{-1}, \quad \xi = \frac{L}{L_0} - 1, \quad L_0 = \frac{\pi \sqrt{\alpha'\lambda}}{2}, \quad \chi = 1 - \delta^2, \quad \delta^2 = \frac{\epsilon^2}{\lambda}. \]  \hspace{1cm} (5.4)
Note that \( L_0 \) is determined solely by \( \lambda \); it is independent of \( \epsilon \).

In the case in which \( \Psi = 0 \), the entropic \( c \)–function diverges at short distances as
\[ C \sim \xi^{-\frac{1}{2}}, \quad \xi = \frac{L}{L_0} - 1, \quad L_0 = \pi \sqrt{\alpha'\lambda}. \]  \hspace{1cm} (5.5)

We note that the exponent of \( \xi \) in the case in which \( \Psi > 0 \) is \(-1\), and in the case in which \( \Psi = 0 \) it is \(-1/2\). This is due to the fact that at short distances the entanglement entropy for the case in which \( \Psi > 0 \) \((3.54)\) contains a logarithmically divergent term (with a coefficient that depends on \( \chi \)) whereas in the case in which \( \Psi = 0 \) \((3.34)\) it does not contain a logarithmically divergent term.

It would be interesting to compute the entropic \( c \)–function directly in the boundary field theory, for example in perturbation theory, and compare it with the bulk calculation result \((5.1)\) (where \( \alpha \) is given by \((5.3)\)). It would be also nice to understand better the theory corresponding to the case in which \( \Psi = 0 \). We leave these for future work.

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Appendix A.

We collect the intermediate results required in section 3 to compute the interval length and the Von Neumann entanglement entropy, see [25].

We have (with $u > a > b > c > d$),

\[ \int_a^u dx \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} [(b-c)F(\varphi, k) + (a-b)\Pi(\varphi, n, k)], \quad (A.1) \]

\[ \int_a^u dx \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}} = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi(\varphi, n, k), \quad (A.2) \]

\[ \int_a^u dx \frac{x-c}{x-b} \sqrt{\frac{1}{(x-b)(x-a)(x-d)}} = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\varphi, k), \quad (A.3) \]

here

\[ \varphi = \arcsin \sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \quad n = \frac{a-d}{b-d}, \quad k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}. \quad (A.4) \]

We have (with $a > u \geq b > c$, $r \neq a$),

\[ \int_u^a dx \frac{1}{x-r} \sqrt{\frac{1}{(a-x)(x-b)(x-c)}} = \frac{2}{(a-r)\sqrt{a-c}} \Pi(\varphi, n, k), \quad (A.5) \]

here

\[ \varphi = \arcsin \sqrt{\frac{a-u}{a-b}}, \quad n = \frac{a-b}{a-r}, \quad k = \sqrt{\frac{a-b}{a-c}}. \quad (A.6) \]
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