Huygens’ construction in a dispersive medium moving at a constant velocity

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We extend the method of Huygens’ construction in a uniformly moving optical medium [Am. J. Phys. 72, 934 (2004)] to the case when the medium is dispersive in its rest frame of reference. The first-order Huygens’ construction analysis of the light drag in a transversely moving dispersive slab is in agreement with the results of the experiment by Jones (1975) and with the Player-Rogers formula for the downstream deflection of the beam. The derivation purports that the original Huygens’ principle remains valid in non-stationary situations if it is modified to include the relativistic effects on the secondary wavelets caused by the motion of the medium.

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I. INTRODUCTION

In recent paper [1], one of us (A.G.) presented a method for calculating the deflection of light refracted from a non-dispersive homogeneous optical material moving at a constant velocity parallel to the interface. The method was based on Huygens’ construction applied to the secondary wavelets, which were shown to be distorted, elliptic-shaped dragged ovals as a consequence of the motion of the medium. The obtained refraction law was used to explain the measurements for the transverse displacement of the light-beam probe in the experiment by Jones (1971-2), in which the beam was allowed to pass through a uniformly rotating disk made of a non-dispersive glass parallel to the axis of rotation [2, 3]. The agreement of the refraction formula with the results of the experiment and with the formula for the transverse Fresnel-Fizeau light drag confirmed the validity of Huygens’ construction as a ray-tracing tool in a dispersionless optical medium in uniform rectilinear motion.

In the same paper, it was noted that the rotating disk experiment was repeated by the same Jones a few years later (1975) using a highly dispersive glass material [4]. The deflection of the probe was significantly enhanced, and the value of the displacement was found to be different from the one predicted by the formula developed for the non-dispersive situation. It has been shown by Player [5] and Rogers [6] that the original formula for the transverse

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Fresnel-Fizeau deflection should be modified by the presence of an additional term due to dispersion, in agreement with the results of the repeated Jones experiment.

The purpose of the present paper is to extend the method of Huygens' construction in a uniformly moving optical medium to the case when the medium in question is dispersive in its rest frame of reference. In Sec. II we investigate the shape of a Huygens' wavelet in the presence of a dispersive medium moving at a constant velocity, and show that the shape of the wavelet is much more complicated than for a non-dispersive situation. We use these arguments in Sec. III to trace the advancement of a plane-polarized light beam normally incident upon a uniformly moving material slab, by applying the Huygens' construction to the deformed secondary wavelets. By limiting the analysis when the speed of the slab is much less than the speed of light in vacuum, we obtain the Player-Rogers formula for the transverse displacement of the beam.

II. THE SHAPE OF THE SECONDARY WAVELETS IN A UNIFORMLY MOVING DISPERSIVE MEDIUM

For the sake of simplicity, we will treat the problem two-dimensionally ($z = z' = 0$), although the same line of reasoning is valid in a more general three-dimensional case. Consider an observer in $S'$-frame, and with respect to him -- a homogeneous, isotropic, and transparent optical material at rest. The $S'$-observer measures the phase speed of light in the medium to be $c/n(\omega')$, which is a function of the locally measured frequency of the wave $\omega'$. Here, $n(\omega')$ is the phase refractive index, and $c$ is the speed of light in vacuum. With respect to $S'$, the space-time evolution of a secondary wavelet emanating from a given point in space is described by:

$$x'^2 + y'^2 = \left[\frac{c}{n(\omega')}\right]^2 t'^2,$$

where we choose the origin of the $x'y'$ coordinate system to overlap with the origin of the wavelet ($t' = 0$). By Lorentz-transforming Eq. (1), we find the shape of the elementary wavefront with respect to an observer in $S$-frame to whom the medium is moving at a constant speed $u$ in the positive direction of the $x$-axis:

$$(x - ut)^2 + y^2(1 - u^2/c^2) = n(\omega')^{-2}(ct - ux/c)^2.$$

But, the observer in $S$-frame uses $\omega$ instead of $\omega'$, and thus measures $n(\omega)$ instead of $n(\omega')$. To him, the frequency of the wave $\omega$ is not equal in every direction as for the $S'$-observer, but will vary with the angle $\varphi$ between the direction of the ray and the velocity of the medium. The transition from $n(\omega')$ to $n(\omega)$ can be accomplished by using the Doppler formula:

$$\omega' = \omega \frac{1 - [u n(\omega)/c] \cos \varphi}{\sqrt{1 - u^2/c^2}},$$

from which, to the first order in $u/c$, we have:

$$n(\omega') = n(\omega - \omega u n(\omega) \cos \varphi/c) \approx n(\omega) \left(1 - \frac{\omega u \partial n(\omega)}{c} \cos \varphi\right).$$

Expressing the angle $\varphi$ via the point of measurement $(x, y)$ of the wave characteristics, we have:

$$n(\omega') \approx n(\omega) \left(1 - \frac{\omega u \partial n(\omega)}{c} \frac{x}{\sqrt{x^2 + y^2}}\right).$$
FIG. 1: Huygens’ construction of the refracted wavefront $CD$ in the material slab that moves at a constant speed $u$ in the direction of the positive $x$-axis. The motion of the slab is causing the distortion of the elementary wavefronts originating along $AB$. The size of the wavelets are exaggerated for convenience.

We conclude that the shape of the elementary light pulse in $S$-frame described by Eq. (2) is generally a complicated curve, not necessarily an ellipse as in the case of a uniformly moving non-dispersive material.

**III. DERIVATION OF THE PLAYER-ROGERS FORMULA**

We will investigate the deflection of light from a dispersive material slab moving at a constant speed $u$ parallel to its surface (see Fig. 1). We consider the speed $u$ of the slab to be much less than $c$, and take the incident light to be normal to the velocity of the slab, therefore resembling the Jones’ setup. As a result of the motion of the slab, the incident beam will undergo a continual deflection inside the moving medium in the direction of its motion, and will eventually emerge from the slab displaced at a distance $q$ parallel to its original direction before the entrance. To find the displacement $q$ of the beam, we will use Huygens’ construction on the distorted secondary wavelets. At the instant when the incident wavefront $AB$ reaches the slab, the points along the interface will start radiating secondary wavelets in accord with Eq. (2). The envelope $CD$ of these wavelets forms the wavefront of the light beam at a later time $t$ counted from the beginning of the radiation of the elementary sources along the interface. The wavefront $CD$ is a tangent line of all the elementary wavefronts, and it touches the elementary wavefront emanating from the initially disturbed point $A$ at the point $C(x_0, y_0)$. The deflection angle $\theta$ of the beam is:

$$\tan \theta = \frac{x_0}{y_0},$$

(6)

Since the slope of the tangent line $CD$ is zero, we can find $x_0$ and $y_0$ by implicit differentiation of Eq. (2) with respect to $x$ and equating the terms $dy/dx$ to zero for the point $(x_0, y_0)$ in
question. Hence, we obtain:

\[ x_0 - ut = -\frac{u}{c} n(\omega) - 2 (ct - ux_0/c) - n(\omega) - 3 (ct - ux_0/c)^2 \frac{\partial n(\omega')}{\partial x} \bigg|_{x_0,y_0}, \]  

(7)

where we have taken into account that \( n(\omega') \) is a function on \( x \) and \( y \) according to Eq. (5). From Eq. (5), we obtain:

\[ n(\omega')^{-1} \approx n(\omega)^{-1} \left( 1 + \frac{\omega u}{c} \frac{\partial n(\omega)}{\partial \omega} \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \right), \]  

(8)

\[ \frac{\partial n(\omega')}{\partial x} \bigg|_{x_0,y_0} \approx -\frac{\omega u}{c} n(\omega) \frac{\partial n(\omega)}{\partial \omega} \frac{y_0^2}{(x_0^2 + y_0^2)^{3/2}}, \]  

(9)

By substitution of Eqs. (8) and (9) into Eq. (7), and neglecting the second and higher order terms in \( u/c \), we obtain:

\[ x_0 \approx ut \left( 1 - \frac{1}{n(\omega)^2} \right) + \frac{\omega cut}{n(\omega)^2} \frac{\partial n(\omega)}{\partial \omega} \frac{1}{y_0}, \]  

(10)

where we used the approximations \( y_0^2(x_0^2 + y_0^2)^{-3/2} \approx y_0^{-1} \) and \( x_0(x_0^2 + y_0^2)^{-1/2} \approx x_0/y_0 \rightarrow 0 \) from the fact that \( x_0 \ll y_0 \) when \( u \ll c \). Since the point of tangency \( C(x_0,y_0) \) is a solution of Eq. (2), we have:

\[ y_0 = \left( \frac{n(\omega')^{-2}(ct - ux_0/c)^2 - (x_0 - ut)^2}{(1 - u^2/c^2)} \right)^{1/2}, \]  

(11)

which in the limit \( x_0 \rightarrow 0 \) reduces to:

\[ y_0 \approx \frac{ct}{n(\omega)}. \]  

(12)

By putting Eqs. (10) and (12) for \( x_0 \) and \( y_0 \) into Eq. (6), we have:

\[ \tan \theta \approx \frac{u}{c} \left( n(\omega) - \frac{1}{n(\omega)} + \omega \frac{\partial n(\omega)}{\partial \omega} \right), \]  

(13)

which is an expression for the deflection angle \( \theta \) of the beam to the first order in \( u/c \). Taking into account that \( \tan \theta = q/D \), we obtain the Player-Rogers formula for the transverse displacement \( q \) of the beam:

\[ q \approx \frac{Du}{c} \left( n(\omega) - \frac{1}{n(\omega)} + \omega \frac{\partial n(\omega)}{\partial \omega} \right), \]  

(14)

where \( D \) is the thickness of the slab.
IV. CONCLUDING REMARKS

The theoretical and experimental investigations of light propagation in moving media are very topical, leading to a variety of novel and exotic effects in relativistic and quantum optics. In this and the preceding paper [1] we have limited our discussion to a uniformly moving media and show that Huygens’ construction can be used to trace the path of the beam inside the moving medium if we take into account that the secondary wavelets are distorted as a consequence of the motion of the medium. The obtained formulas for the deflection of the beam to the first order in \( u/c \) coincide with the Fresnel and the Player-Rogers formulas for the transverse drag in the non-dispersive and dispersive case, respectively, in agreement with the experiments by Jones (1971-75).

The described Huygens-construction analysis of light propagation in uniformly moving media can have certain implications into the standard textbook discussion of Huygens-Fresnel principle. It offers a simple geometrical method of approach in the framework of introductory physics courses that might bring the subject of optics of uniformly moving media into the typical undergraduate classroom.

[1] A. Gjurchinovski, Am. J. Phys. 72, 934 (2004).
[2] R. V. Jones, J. Phys. A 4, L1 (1971).
[3] R. V. Jones, Proc. R. Soc. Lond. A 328, 337 (1972).
[4] R. V. Jones, Proc. R. Soc. Lond. A 345, 351 (1975).
[5] M. A. Player, Proc. R. Soc. Lond. A 345, 343 (1975).
[6] G. L. Rogers, Proc. R. Soc. Lond. A 345, 345 (1975).
[7] M. Artoni, I. Carusotto, G. C. La Rocca and F. Bassani, Phys. Rev. Lett. 86, 2549 (2001).
[8] I. Carusotto, M. Artoni, G. C. La Rocca and F. Bassani, Phys. Rev. Lett. 87, 064801 (2001).
[9] M. Artoni, I. Carusotto, G. C. La Rocca and F. Bassani, J. Opt. B: Quantum Semiclass. Opt. 4, S345 (2002).
[10] M. Artoni, I. Carusotto, Phys. Rev. A 67, 011602(R) (2003).
[11] I. Carusotto, M. Artoni, G. C. La Rocca and F. Bassani, Phys. Rev. A 68, 063819 (2003).
[12] U. Leonhardt and P. Piwnicki, Phys. Rev. A 60, 4301 (1999).
[13] U. Leonhardt and P. Piwnicki, Contemp. Phys. 41, 301 (2000).
[14] U. Leonhardt and P. Piwnicki, J. Mod. Opt. 48, 977 (2001).
[15] U. Leonhardt, Nature (London) 415, 406 (2002).
[16] U. Leonhardt and T. G. Philbin, New J. Phys. 8, 247 (2006).