1. Introduction and preliminaries

In the whole paper, \( \mathbb{C} \) denotes the complex number field and the collection of all \( m \times n \) matrices over the complex number field \( \mathbb{C} \) is denoted by \( \mathbb{C}^{m \times n} \). The real number field is represented by \( \mathbb{R} \). The conjugate transpose matrix of \( A \) is written by \( A^* \) and a matrix is said to be anti-Hermitian if \( A^* = -A \). Both symbols \( * \) and \( rankA \) stand for the rank of \( A \). An identity matrix with feasible shape is denoted by \( I \). The Moore-Penrose inverse of \( A \) is represented by \( A^{\dagger} = U \) and is determined by the four Penrose’s equations:

\[
AUA = A, \quad UAU = U, \quad (AU)^* = AU, \quad (UA)^* = UA.
\]

Furthermore, \( L_A = I - A^*A \) and \( R_A = I - AA^* \) are projectors induced by \( A \), and

\[
L_A = (LA)^* = (LA)^2 = L_A^2, \quad R_A = (RA)^2 = (RA)^* = R_A^2, \quad L_A^* = R_A, \quad R_A^* = L_A.
\]

A lot of the problems in different areas from engineering like linear descriptor systems (Gavin & Bhattacharyya, 1983), system design (Syrmos & Lewis, 1994), singular system control (Darouach, 2006), perturbation theory (Li, 2000), feedback (Syrmos & Lewis, 1993), etc., to medical researches based on mathematics models with partial differential equations (Kumar, Kumar, Osman, & Samet, 2020; Osman & Machado, 2018) require solutions of Sylvester-type matrix equations. For instance, Bai carried out the iterative solution of \( A_1X + XA_2 = B \) in Bai (2011). The consistent condition of \( A_1X + YA_2 = B \) to have a solution was evaluated by Roth (1952) and its general solution was researched by Baksalary and Kala (1979).

Recently, the general solution of

\[
A_1X_1 + Z_1B_1 = C_1, \quad A_2X_2 + Z_1B_2 = C_2,
\]

was examined in Wang and He (2013). Some solvability conditions to (1.1) were also discussed in Lee and Vu (2012). The condition number of (1.1) was also evaluated in Lin and Wei (2007). The constraint solution of (1.1) was researched by Wang, Rehman, He, and Zhang (2016). Some necessary and sufficient conditions of (1.1) when \( X_2 = X_1 \) were given in Wimmer (1994). Wang and He obtained necessary and sufficient conditions for

\[
A_1X_1 + Z_1B_1 = C_1, \quad A_2Z_1 + X_2B_2 = C_2,
\]
to have a general solution in He and Wang (2014). Recently, some interesting researches related to finding the general solutions to Sylvester-type matrix equations were made. In particular, systems of coupled generalized Sylvester matrix equations were studied in complex matrices (Wang & He, 2014) and
in quaternion matrices (He, Wang, & Zhang, 2018), and the general ϕ-Hermitian solution to mixed pairs of the quaternion Sylvester matrix equations was explored in He, Liu, and Tam (2017). The study of the properties of the matrix rank is important in establishing the necessary and sufficient conditions for the existence of the solution of matrix equations. Recently, new results on matrix ranks have been derived in Ma (2021). Some iterative algorithms of solving coupled matrix equations can be found in Ding and Chen (2003, 2006). The numerical solution of bilinear systems (Li & Liu, 2018a, 2018b), gradient-based iterative algorithms (Ding, Liu, & Bao, 2012; Ding, Liu, & Chu, 2013; Ding & Zhang, 2014), the Newton iteration algorithm (Xu, 2015), its applications to the parameter estimation and identification of dynamical systems (Xu, 2016, 2017; Xu, Chen, & Xiong, 2015) and to communication architecture of power monitoring system (Che et al., 2022).

Recently, Hajarian (2015) has developed the algorithm to find out the solution of

\[ A_1 X A_1^* + B_1 Y B_1^* = C_1 \]  

have explored using direct methods by simultaneous decomposition of a matrix triple for of given complex matrices by Liu and Tian (2011), and by determinantal representations of given quaternion matrices by Kyrchei (2019). Some new investigations on (1.2) were obtained in Deng and Hu (2005).

The iterative methods to compute the solutions of some matrix equations, including nonlinear matrix equations, were found, among them, least squares algorithms (Ding, 2013a, 2013b) and their applications to bilinear systems (Li & Liu, 2018a, 2018b), gradient-based iterative algorithms (Ding, Liu, & Bao, 2012; Ding, Liu, & Chu, 2013; Ding & Zhang, 2014), the Newton iteration algorithm (Xu, 2015), its applications to the parameter estimation and identification of dynamical systems (Xu, 2016, 2017; Xu, Chen, & Xiong, 2015) and to communication architecture of power monitoring system (Che et al., 2022).

Recently, Hajarian (2015) has developed the algorithm to find out the solution of

\[ A_1 X B_1 + C_1 Y D_1 = E_1, \]
\[ A_2 Z B_2 + C_2 Y D_2 = E_2. \]  

Motivated by the above findings and the remarkable usage of generalized Sylvester matrix equations in various applied areas, in this paper we explore the anti-Hermitian coupled Sylvester matrix equations

\[ A_1 U A_1^* + B_1 V B_1^* = C_1, \quad C_1 = -C_1^*, \]
\[ A_2 W A_2^* + B_2 V B_2^* = C_2, \quad C_2 = -C_2^*. \]  

over the complex number field \( \mathbb{C} \). Solving (1.3) will definitely enrich the usage of anti-Hermitian Sylvester matrix equation into a large number of fields. We give the necessary and sufficient conditions for the existence of its solution by applying the properties of the matrix rank. The following lemma has crucial role in gaining these results.

**Lemma 1.1** (Marsaglia & Styan, 1974). Let \( K \in \mathbb{C}^{m \times n}, \ P \in \mathbb{C}^{m \times k}, \ Q \in \mathbb{C}^{k \times n} \). Then

\[
\begin{bmatrix}
K \\
Q
\end{bmatrix}
- \begin{bmatrix}
r(QL_K) = r(K), \\
r(PK) - r(R_PK) = r(P), \\
r(Q0) - r(P) - r(Q) = r(R_PKL_Q).
\end{bmatrix}
\]

The principal objective of this paper is to search out the general solution to (1.3) when this system is solvable. The general solution of

\[ A_4 X - (A_4 X)^* + B_4 Y B_4^* + C_4 Z C_4^* = D_4, \quad D_4 = -D_4^*, \]
\[ Y = -Y^*, \quad Z = -Z^*, \]

(1.4) has important function in achieving the core result of this paper having anti-Hermitian nature over \( \mathbb{C} \).

**Lemma 1.2** (Rehman, Kyrchei, Ali, Akram, & Shakoor, 2019). Let \( A_4, B_4, C_4, \) and \( D_4 = -D_4^* \) be known coefficient matrices in (1.4) over \( \mathbb{C} \) with agreeable sizes. Assume

\[ A = R_A B_A, \quad B = R_A C_A, \quad C = R_A D_A R_A^*, \quad M = R_A B, \quad S = B L_M. \]

Then the terms given below are alike:

(1) The system (1.4) has a solution \( (X, Y, Z) \), where \( Y \) and \( Z \) are anti-Hermitian matrices.

(2) The coefficient matrices in (1.4) satisfy:

\[ R_M R_A C = 0, \quad R_A C R_A^* = 0. \]

(3)

\[
\begin{bmatrix}
D_4 & A_4 & B_4 & C_4 \\
A_4^* & 0 & 0 & 0 \\
D_4 & A_4 & B_4 & 0 \\
C_4^* & 0 & 0 & 0
\end{bmatrix}
=r\begin{bmatrix}
A_4 & B_4 & C_4 \\
0 & 0 & 0 \\
A_4^* & B_4 & 0 \\
0 & C_4^* & 0
\end{bmatrix}+r\begin{bmatrix}
A_4 & B_4 & C_4 \\
0 & 0 & 0 \\
A_4^* & B_4 & 0 \\
0 & C_4^* & 0
\end{bmatrix}. \]

Under these conditions, \( X, Y^* = -Y, \) and \( Z^* = -Z \) are given below

\[ X = A_4^* \left[ D_4 - B_4 Y B_4^* - C_4 Z C_4^* - \frac{1}{2} A_4^* \left[ D_4 - B_4 Y B_4^* - C_4 Z C_4^* \right] A_4^* \right] - U_4^* \left[ A_4^* \right] A_4^* - A_4 \left[ U_4^* \right] A_4^* + Z_4^* U_4^* U_4, \]
\[ Y = -Y^* = A_4^* C_4^* - \frac{1}{2} A_4^* B_4^* Y^* \left[ B_4^* C_4^* \right] A_4^* - \frac{1}{2} A_4^* \left[ B_4^* S C_4^* \right] A_4^* - \frac{1}{2} A_4^* \left[ J + S B^* \right] \left[ C \left( M^* \right) B^* \left( A^* \right) - A^* \left[ S \right] U_4^* S^* \left( A^* \right) \right] + U_4^* U_4 - U_4^* \left[ U_4^* \right] \left[ U_4^* \right], \]
\[ Z = -Z^* = \frac{1}{2} \left[ B^* C^* \right] \left[ J + S \left( S^* S^* \right) \right] + \left[ J + S \left( S^* S^* \right) \right] \left[ B^* C^* \right] \left( M^* \right) \left( M^* \right)^* \]
\[ + U_4^* U_4 U_4^* - U_4^* U_4 U_4^* - U_4^* U_4 U_4^* - \left( U_4^* U_4 U_4^* \right)^*, \]

where \( U_1, ..., U_5, \) and \( U_6 = -U_6^* \) are free matrices with acceptable dimensions.

The anti-Hermitian solution to the system (1.3) will be expressed in terms of the Moore-Penrose (MP) inverse. The novelty of the given results is obtaining a formal representation of the solution in terms of generalized inverses and the construction of an algorithm to find its explicit expression as well. Due to the important role of generalized inverses in many application fields, considerable effort has been exerted toward the numerical algorithms for fast and
accurate calculation of matrix generalized inverse. In general, most existing methods for their obtaining are iterative algorithms for approximating generalized inverses of complex matrices (some recent papers, see, e.g. Artidiello, Cordero, Torregrosa, & Vassileva, 2019; Guo & Huang, 2010; Sayevand, Pourdarvish, Machado, & Erfanifar, 2021). There are only several direct methods finding MP-inverse for an arbitrary complex matrix $A \in \mathbb{C}^{m \times n}$. The most famous is method based on singular value decomposition (SVD), i.e. if $A = U \Sigma V^T$, then $A^+ = V \Sigma^+ U^T$. The computational cost of this method is dominated by the cost of computing the SVD, which is several times higher than matrix-matrix multiplication. Another approach is constructing determinantal representations of the MP-inverse $A^+$. A well-known determinantal representation of an ordinary inverse is the adjugate matrix with the cofactors in entries. It has an important theoretical significance and bring forth Cramer’s rule for systems of linear equations. The same is desirable to have for the generalized inverses. Due to looking for their more applicable explicit expressions, there are various determinantal representations of generalized inverses (for the MP-inverse, see, e.g. Bapat, Bhaskara, & Prasad, 1990; Stanimirovic, 1996). Because the complexity of the previously obtained expressions of determinantal representations of the MP-inverse, they have a little applicability.

In this paper, we will used the determinantal representations of the MP-inverse recently obtained in Kyrchei (2008).

**Lemma 1.3** (Kyrchei, 2008, Theorem 2.2). If $A \in \mathbb{C}^{m \times n}$ with rank $A = r$, then the Moore-Penrose inverse $A^+ = (a^+_{ij}) \in \mathbb{C}^{n \times m}$ possess the following determinantal representations

$$a^+_{ij} = \frac{\sum_{\beta \in L_{r,m} \setminus \{i\}} |(A^+ A)_i (A^+)_i|_{\beta}^{\beta}}{\sum_{\beta \in L_{r,n} \setminus \{i\}} |(A^+ A^+)_{ij}|_{\beta}^{\beta}},$$

(1.5)

Here $|A|_\beta^\alpha$ denote a principal minor of $A$ whose rows and columns are indexed by $\alpha := \{x_1, \ldots, x_k\} \subseteq \{1, \ldots, n\}$, $L_{r,m} := \{ \alpha : 1 \leq x_1 < \cdots < x_k \leq m \}$, and $L_{r,n} \setminus \{i\} := \{ \alpha : \alpha \in L_{r,n}, i \notin \alpha \}$.

Also, $a^+_{ij}$ and $a^+_{ij}$ denote the $j$th column and the $i$th row of $A^+$, and $A_i(b)$ and $A_j(c)$ stand for the matrices obtained from $A$ by replacing its $i$th row with the row vector $b \in \mathbb{C}^{1 \times n}$ and its $j$th column with the column vector $c \in \mathbb{C}^m$, respectively.

The formulas (1.5) give very simple and elegant determinantal representations of the MP-inverse. So, for any $A \in \mathbb{C}^{m \times n}$, we have sum of all principal minors of $r$ order of the matrices $(A^+ A)_i (A^+)_i$ in denominators and sum of principal minors of $r$ order of the matrices $(A^+ A)_i (A^+)_i$ that contain the $i$th column or the $j$th row, respectively, in numerators into (1.5).

Note that for an arbitrary full-rank matrix $A$, Lemma 1.3 gives a new way of finding an inverse matrix.

**Corollary 1.1.** If $A \in \mathbb{C}^{m \times n}$ with rank $A = \min\{m,n\}$, then the inverse $A^{-1} = (a^{-1}_{ij}) \in \mathbb{C}^{n \times m}$ possess the following determinantal representations:

$$a^{-1}_{ij} = \begin{cases} 
\frac{(A^+ A)_i (A^+)_i}{|A^+ A|} & \text{if } rank A = n, \\
\frac{(A^+ A)_i (A^+)_i}{|A^+ A^+|} & \text{if } rank A = m.
\end{cases}$$

Note that these new determinantal representations of the Moore-Penrose inverse have been obtained by the developed novel limit-rank method in the case of quaternion matrices (Kyrchei, 2011) as well. This method was successfully applied for constructing determinantal representations of other generalized inverses in both cases for complex and quaternion matrices (see e.g. Kyrchei, 2017a, 2017b). It also yields Cramer’s rules of various matrix equations (Kyrchei, 2012, 2018a, 2018b, 2019, 2021; Rehman, Kyrchei, Ali, Akram, & Shakoor, 2020, 2021).

Our paper is composed of four sections. The general solution to (1.3) is constituted in Section 2 with a special case. The algorithm and numerical example of finding the anti-Hermitian solution to (1.3) are presented in Section 3. A conclusion to this paper is given in Section 4.

### 2. Main result

**Theorem 2.1.** Let $A_1 \in \mathbb{C}^{m \times n}, A_2 \in \mathbb{C}^{m \times q}, B_i \in \mathbb{C}^{m \times k}$, and $C_i = -C_i^* \in \mathbb{C}^{m \times m}$ for $i = 1, 2$. Assign

$$M_i = R_i B_i, S_i = B_i L_i, M_2 = R_2 B_2, S_2 = B_2 L_2, A_1 = R_1 L_1, A_2 = R_2 L_2, A_3 = R_3 L_3, A_4 = R_4 L_4, A_5 = R_5 L_5, A_6 = R_6 L_6, A_7 = R_7 L_7, A_8 = R_8 L_8, A_9 = R_9 L_9, A_{10} = R_{10} L_{10}, A_{11} = R_{11} L_{11},$$

$$C_1 = V_02 - V_0, V_02 = \frac{1}{2} M_1^* C_1 (B_1) - (I + S_1^* S_2) + \frac{1}{2} (I + S_1^* S_2) B_1^* C_1 (M_1^*),$$

$$V_01 = \frac{1}{2} M_1^* C_1 (B_1) - (I + S_1^* S_2) + \frac{1}{2} (I + S_1^* S_2) B_1^* C_1 (M_1^*), C_i = R_i C_i R_{-i}.$$  

Then the following conditions are equivalent:

1. System (1.3) is consistent.
2. The following equalities hold:

$$R_{A_1} C_1 R_{B_1} = 0, R_{M_2} R_{A_2} C_1 = 0, \quad (2.1)$$

$$R_{A_3} C_2 R_{B_2} = 0, R_{M_3} R_{A_4} C_2 = 0. \quad (2.2)$$
\[ R_{A_4}C_4R_{A_6} = 0, \quad R_{M_5}R_{A_4}C_4 = 0. \]  
(2.3)

The following rank equalities hold:

\[ r \begin{bmatrix} C_1 & A_1 \\ B_1 & 0 \end{bmatrix} = r(A_1) + r(B_1), \quad r \begin{bmatrix} C_1 & B_1 & A_1 \\ B_1 & 0 & 0 \end{bmatrix} = r(A_1) + r(B_1). \]  
(2.4)

\[ r \begin{bmatrix} C_1 & A_2 \\ B_2 & 0 \end{bmatrix} = r(A_2) + r(B_2), \quad r \begin{bmatrix} C_1 & B_2 & A_2 \\ B_2 & 0 & 0 \end{bmatrix} = r(A_2) + r(B_2). \]  
(2.5)

\[ \begin{bmatrix} -B_1 & 0 & -B_1 & A_1 \\ B_2 & B_2 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 \\ -B_1 & -B_1 & 0 & 0 \\ B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} -B_1 & 0 & -B_1 & A_1 \\ B_2 & B_2 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 \\ -B_1 & -B_1 & 0 & 0 \\ B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + r \begin{bmatrix} 0 & 0 & 0 & A_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

\[ \begin{bmatrix} -B_1 & 0 & -B_1 & A_1 \\ B_2 & B_2 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 \\ -B_1 & -B_1 & 0 & 0 \\ B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + r \begin{bmatrix} 0 & 0 & 0 & A_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + r(B_1). \]

(2.7)

Under these conditions, the general solution to (1.3) is

\[ U = A_3^1C_3(A_3^1)^* - \frac{1}{2}A_3^1B_3M_3^3C_3[I + (B_3^1)^*S_3](A_3^1)^* \]
\[- \frac{1}{2}A_3^1[I + S_3B_3^3]C_3(M_3^3)^*B_3^1(A_3^1)^* - A_3^1S_3U_3S_3(A_3^1)^* \]
\[ + L_{A_3}V_3 - V_3L_{A_3}. \]

\[ W = A_3^2C_3(A_3^2)^* - \frac{1}{2}A_3^2B_3M_3^3C_3[I + (B_3^1)^*S_3](A_3^2)^* \]
\[- \frac{1}{2}A_3^2[I + S_3B_3^3]C_3(M_3^3)^*B_3^1(A_3^2)^* - A_3^2S_3U_3S_3(A_3^2)^* \]
\[ + L_{A_3}V_3 - V_3L_{A_3}. \]

\[ V = \frac{1}{2}M_3^3C_3B_3^1(I + S_3^1S_3) + \frac{1}{2}(I + S_3^1S_3)B_3^1C_3(M_3^3)^* \]
\[ + L_{U_3}U_3L_{U_3} + L_{M_3}L_{S_3}U_3 - U_3L_{S_3}L_{U_3} + U_3L_{B_3} - L_{B_3}U_3, \]

(2.8)

or

\[ V = \frac{1}{2}M_3^3C_3B_3^1(I + S_3^1S_3) + \frac{1}{2}(I + S_3^1S_3)B_3^1C_3(M_3^3)^* \]
\[ + L_{U_3}U_3L_{U_3} + L_{M_3}L_{S_3}U_3 - U_3L_{S_3}L_{U_3} + U_3L_{B_3} - L_{B_3}U_3, \]

(2.9)

where

\[ Z = A_3^1C_3 - L_{M_3}U_3L_{M_3} - L_{M_3}U_4L_{M_3}. \]

(2.10)

\[ U_1 = A_3^1C_3(A_3^1)^* - \frac{1}{2}A_3^1B_3M_3^3C_3[I + (B_3^1)^*S_3](A_3^1)^* \]
\[- \frac{1}{2}A_3^1[I + S_3B_3^3]C_3(M_3^3)^*B_3^1(A_3^1)^* - A_3^1S_3U_3S_3(A_3^1)^* \]
\[ + L_{A_3}U_1 - U_1L_{A_3}. \]

(2.11)

\[ U_4 = \frac{1}{2}M_3^3C_3B_3^1(I + S_3^1S_3) + \frac{1}{2}(I + S_3^1S_3)B_3^1C_3(M_3^3)^* \]
\[ + L_{M_3}U_1 - U_1L_{M_3} + L_{M_3}L_{S_3}U_1 - U_1L_{S_3}L_{M_3} + U_1L_{B_3} - L_{B_3}U_1, \]

(2.12)

where \( V_1, \ldots, U_{13}, U_9 = -U_{-2}, U_{-1} = -U_{-1} \) are any matrices of acceptable shapes over \( \mathbb{C} \).

Proof. We write the equations in (1.3) as follows:

\[ A_3U_3^1 + B_3V_3^1 = C_3, \quad C_1 = -C_1^*, \]
\[ A_3U_2^1 + B_3V_2^1 = C_2, \quad C_2 = -C_2^* \]

By Lemma 1.2, Equations (2.19) and (2.20) are consistent if and only if

\[ R_{A_3}C_3R_{B_3} = 0, \quad R_{M_3}R_{A_3}C_3 = 0, \]
\[ R_{A_3}C_3R_{B_3} = 0, \quad R_{M_3}R_{A_3}C_3 = 0, \]

In this case, the general solution to (2.19) and (2.20) can be written as

\[ U = A_3^1C_3(A_3^1)^* - \frac{1}{2}A_3^1B_3M_3^3C_3[I + (B_3^1)^*S_3](A_3^1)^* \]
\[- \frac{1}{2}A_3^1[I + S_3B_3^3]C_3(M_3^3)^*B_3^1(A_3^1)^* - A_3^1S_3U_3S_3(A_3^1)^* \]
\[ + L_{A_3}V_3 - V_3L_{A_3}. \]

(2.21)

\[ V = \frac{1}{2}M_3^3C_3B_3^1(I + S_3^1S_3) + \frac{1}{2}(I + S_3^1S_3)B_3^1C_3(M_3^3)^* \]
\[ + L_{U_3}U_3L_{U_3} + L_{M_3}L_{S_3}U_3 - U_3L_{S_3}L_{U_3} + U_3L_{B_3} - L_{B_3}U_3, \]

(2.22)
where \( V_1, U_1 = -U_1', U_2, U_3, U_4 = -U_4', V_2, V_3, U_5 = U_6 \) are free matrices of adequate shapes over \( \mathbb{C} \).

Denote \( X^* = \begin{bmatrix} U_6 & U_3 & U_2 & U_5 \end{bmatrix} \). Equating (2.21) and (2.22), we obtain
\[
A_3X - (A_3X)^* + L_{M_1}U_{1,1}L_{M_2}U_{4,1}L_{M_3} = C_3. \tag{2.23}
\]

By Lemma 1.2, Equation (2.23) has a solution if and only if the equalities in (2.3) are satisfied and in this case its general solution can be expressed by (2.12)–(2.18).

(2) \( \iff \) (3) : From Lemma 1.2, we have
\[
egin{align*}
& R_{A_3}C_1R_{B_1} = 0 \iff r \begin{bmatrix} C_1 & A_1 \ B_1^* & 0 \end{bmatrix} = r(A_1) + r(B_1), \\
& R_{M_1}R_{A_3}C_1 = 0 \iff r \begin{bmatrix} R_{A_3}C_1 & M_1 \ \end{bmatrix} = r(M_1) \iff r \begin{bmatrix} R_{A_3}C_1 & R_{A_3}B_1 \ \end{bmatrix} = r(R_{A_3}B_1) \\
& \iff r \begin{bmatrix} C_1 & B_1 & A_1 \ \end{bmatrix} = r(A_1) + r(B_1), \\
& R_{A_3}C_2R_{B_2} = 0 \iff r \begin{bmatrix} C_2 & A_2 \ B_2^* & 0 \end{bmatrix} = r(A_2) + r(B_2), \\
& R_{M_2}R_{A_3}C_2 = 0 \iff r \begin{bmatrix} R_{A_3}C_2 & M_2 \ \end{bmatrix} = r(M_2) \iff r \begin{bmatrix} R_{A_3}C_2 & R_{A_3}B_2 \ \end{bmatrix} = r(R_{A_3}B_2) \\
& \iff r \begin{bmatrix} C_2 & B_2 & A_2 \ \end{bmatrix} = r(A_2) + r(B_2), \\
& R_{A_3}C_4R_{B_4} = 0 \iff r(R_{A_3}C_4R_{B_4}) = 0 \\
& \iff r \begin{bmatrix} C_4 & A_4 \ B_4^* & 0 \end{bmatrix} = r(A_4) + r(B_4), \\
& R_{A_3}C_3R_{A_3}L_{M_2}R_{A_3} = 0 \iff r \begin{bmatrix} R_{A_3}C_3R_{A_3} & R_{A_3}L_{M_2} \ \end{bmatrix} = r(R_{A_3}L_{M_2}) + r(L_{M_2}R_{A_3}) \\
& \iff r \begin{bmatrix} C_3 & L_{M_2} & A_3 \ \end{bmatrix} = r(L_{M_2}A_3) + r(L_{M_2}A_3) \\
& \iff r \begin{bmatrix} C_3 & L_{M_2} & A_3 \ \end{bmatrix} = r \begin{bmatrix} C_3 & L_{M_2} & A_3 \ \end{bmatrix} + r \begin{bmatrix} C_3 & L_{M_2} & A_3 \ \end{bmatrix} \\
& \iff r \begin{bmatrix} V_{02} - V_{01} & L_{M_1} & L_{B_2} & -L_{B_1} & L_{M_2}L_{S_1} & L_{M_2}L_{S_2} \ \end{bmatrix} = r \begin{bmatrix} V_{02} - V_{01} & L_{M_1} & L_{B_2} & -L_{B_1} & L_{M_2}L_{S_1} & L_{M_2}L_{S_2} \ \end{bmatrix} \\
& \iff r \begin{bmatrix} V_{02} - V_{01} & L_{M_1} & L_{B_2} & -L_{B_1} & L_{M_2}L_{S_1} & L_{M_2}L_{S_2} \ \end{bmatrix} = r \begin{bmatrix} V_{02} - V_{01} & L_{M_1} & L_{B_2} & -L_{B_1} & L_{M_2}L_{S_1} & L_{M_2}L_{S_2} \ \end{bmatrix} \\
& \iff r \begin{bmatrix} V_{02} - V_{01} & L_{M_1} & L_{B_2} & -L_{B_1} & L_{M_2}L_{S_1} & L_{M_2}L_{S_2} \ \end{bmatrix} = r \begin{bmatrix} V_{02} - V_{01} & L_{M_1} & L_{B_2} & -L_{B_1} & L_{M_2}L_{S_1} & L_{M_2}L_{S_2} \ \end{bmatrix}.
On the same lines, $R^r_{M_i}R_{A_i}E_{22} = 0$ can be proved to be same as (2.7).

Now we discuss the particular case of our system.

If $A_2$, $B_2$ and $C_2$ are all equal to zero in Theorem 2.1, then we get the following outcome.
Corollary 2.1. Given that $A_i, B_i$, and $C_i$ are matrices of feasible shapes over $C$. Assign

$$M_1 = R_{A_i}B_1, \quad S_1 = B_1L_{M_1}.$$  

Then the following conditions are equivalent:

1. System (1.2) is consistent.
2. $R_{A_i}C_iR_{B_i} = 0, \quad R_{M_i}R_{A_i}C_i = 0$.
3. $r \begin{bmatrix} C_i & A_i \\ B_i & 0 \end{bmatrix} = r(A_1) + r(B_1), \quad r \begin{bmatrix} C_1 & B_1 & A_1 \end{bmatrix} = r \begin{bmatrix} A_1 & B_1 \end{bmatrix}$.

Under these conditions, the general solution to (1.2) can be represented as

$$X_1 = -X_1^* = A_i^tC_i(A_i^t)^* - \frac{1}{2}A_i^tB_iM_i^tC_1 \left[ I + (B_i^t)^*S_i^* \right] (A_i^t)^*$$

$$+ \frac{1}{2}A_i^t \left[ I + S_i^*B_i^t \right] C_i(M_i^t)^*B_i^t(A_i^t)^* - A_i^tS_iU_iS_i(A_i^t)^* + L_{A_i}V_i - V_i^*L_{A_i},$$

$$Y_1 = -Y^* = \frac{1}{2}M_i^tC_i(B_i^t)^* \left[ I + S_i^*S_i^* \right] + \frac{1}{2}(I + S_i^*S_i)B_i^tC_i(M_i^t)^*$$

$$+ L_{M_i}U_1L_{M_1} + U_{M_1}L_{M_i} + U_{M_1}L_{S_1} - U_{S_1}L_{M_1} + U_{S_1}L_{B_1} - B_1U_{S_1}^*,$$

where $U_1 = -U_1^*, \quad V_i, \quad U_2$, and $U_3$ are free matrices of feasible shapes over $C$.

3. Algorithm with example

In this section, we construct the algorithm for finding solutions to (1.3) that is induced by Theorem 2.1.

Algorithm 3.1.

1. By Lemma 1.3 find the matrices $A_i^t, B_i^t$, further by direct multiplication compute $R_{A_i} = I - A_i^t$, $A_i^t, R_{B_i} = I - B_iB_i^t$, $M_i = R_{A_i}B_i, \quad M_i^t = I - M_i^tM_i, \quad R_{M_i} = I - M_i^tM_i$.

2. Similarly, find the matrices $A_i, A_i^t, R_{A_i}$ for $i = 3, 4$ and $B_i, B_i^t, R_{B_i}$. After that, compute $M_{3_i}^t, M_{4_i}^t, L_{M_i}, \quad R_{M_i}, \quad S_{3_i}$ and $C_3$.

3. Verify the consistence equalities (2.1), (2.2), and (2.3). If these equalities are hold, then we find solutions by the next steps.

4. By (2.17) and (2.18), compute the matrices $U_1, \quad U_3$ respectively. After that find $U$ by (2.8), and $W$ by (2.9).

5. By (2.16), compute $Z$ and using it find $U_2$ by (2.14) and $U_3$ by (2.12), or $U_3$ by (2.15) and $U_3$ by (2.13).

6. Finally, find $V$ by (2.10) or (2.11) according to case selected in the previous point.

The following example will be considered by using Algorithm 3.1. Note that our goal is both to confirm correctness of main results from Theorem 2.1 and to demonstrate the technique of applying the determinant representations of the MP-inverse from Lemma 1.3 by using a not too complicated and understandable example.

Example 3.1. Given the matrices:

$$A_1 = \begin{bmatrix} i & 1 & 1 & -1 & 1 + i \\ 1 & 0 & i & -i & 1 \\ 0 & -i & 0 & -i & 1 \\ -i & 1 & 1 & 1 & -i \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & i & 2 \\ -i & 1 & 2 \\ i & -1 & 2 \\ 1 & i & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} i & -1 & 1 & i \\ 1 & i & -i & 1 \\ -1 & -i & i & -1 \\ i & -1 & 1 & i \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 + i & 2i & -1 + i \\ -1 + i & -2 & -1 - i \\ i & 0 & -i \\ 1 & 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 - i & i & -1 - 2i + 2 \\ 1 + i & i & -1 - 2i + 2 \\ i -1 & 1 + i & -2 - 2i \\ 1 - i & 1 - i & 2 + 2i \end{bmatrix}, \quad C_2 = \begin{bmatrix} i & -i & -1 & 1 \\ -i & i & -1 & -1 \\ 1 & -1 & i & -i \\ -1 & 1 & -i & i \end{bmatrix}.$$  

Let us find a solution to the system (1.3) with the given above matrices by Algorithm 3.1.
1. Thanks to Lemma 1.3, we calculate the Moore-Penrose inverses. So,

\[
A_1^* = \frac{1}{15} \begin{bmatrix}
-1 - 2i & 2 & -i & -1 + 2i \\
3 + i & -1 & 3i & 3 - i \\
-2 + i & -2i & -1 & 2 + i \\
2 - i & 1 & 2i & 2 + i
\end{bmatrix}, \quad B_1^* = \frac{1}{24} \begin{bmatrix}
1 & i & -i & 1 \\
-1 - 1 - i & 1 \\
1 & i & -i & 1
\end{bmatrix}, \\
A_2^* = \frac{1}{12} \begin{bmatrix}
-i & -1 & 1 - 3i & 3 + i \\
-2i & -2 & 2 & 2i \\
-i & -1 & 1 + 3i & -3 + i
\end{bmatrix}, \quad B_2^* = \frac{1}{48} \begin{bmatrix}
-1 + i & i & 1 - i & 1 + i \\
-1 - i & 1 & 1 - i & -1 + i \\
2 + 2i & -2 - 2i & -2 + 2i & 2 - 2i
\end{bmatrix}.
\]

Then,

\[
R_{A_1} = \frac{1}{3} \begin{bmatrix} 1 & -i & -i & 0 \\ i & 2 & 0 & -i \\ i & 0 & 2 & i \\ 0 & i & -i & 1 \end{bmatrix}, \quad R_{A_2} = \frac{1}{2} \begin{bmatrix} 1 & i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{bmatrix}, \quad L_{B_1} = \frac{1}{6} \begin{bmatrix} 5 & -i & 2 \\ -i & 5 & 2 \\ 2i & 2 & 2 \end{bmatrix}, \quad L_{B_2} = \frac{1}{4} \begin{bmatrix} 3 & -i & i & -1 \\ i & 3 & 1 & i \\ -i & 1 & 3 & -i \\ -i & 1 & i & 3 \end{bmatrix}, \quad R_{B_1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -i & i \\ 1 & 3 & i & -i \\ i & -i & 3 & 1 \\ -i & 1 & 1 & 3 \end{bmatrix},
\]

\[
M_1 = R_{A_1} B_1 = \frac{1}{3} \begin{bmatrix} 1 & 2i & -4i \\ 4i & -4 & 8i \\ 3 & 3i & 6 \end{bmatrix}, \quad M_1^* = \frac{1}{60} \begin{bmatrix} 1 & 2i & -4i & 3 \\ -i & 2 & -4 & -3i \\ 2 & 4i & -8i & 6 \end{bmatrix}, \quad M_2 = R_{A_2} B_2 = \begin{bmatrix} -1 & i & -2i \\ i & 1 & -i \\ -1 & i & -2i \\ 1 & -i & 2i \end{bmatrix}, \quad M_2^* = \frac{1}{24} \begin{bmatrix} -1 & -i & 1 & 1 \\ -i & 1 & 1 & 1 \\ i & 1 & -i & 1 \\ 2i & -2 & 2i & -2i \end{bmatrix}, \quad M_3^* = \frac{1}{6} \begin{bmatrix} 5 & i & -2i \\ -i & 5 & 2 \\ 2i & 2 & 2 \end{bmatrix}, \quad S_1 = 0, \ S_2 = 0.
\]

Further,

\[
Y_{01} = \frac{1}{36} \begin{bmatrix} i & -1 & 2i \\ 1 & i & 2 \\ 2i & -2 & 4i \end{bmatrix}, \quad Y_{02} = \frac{1}{72} \begin{bmatrix} i & 1 & -2 \\ -1 & i & -2i \\ 2 & -2i & 4i \end{bmatrix}, \quad C_3 = Y_{02} - Y_{01} = \frac{1}{72} \begin{bmatrix} -i & 3 & 2 - 4i \\ -3 & -i & 4 - 2i \\ 2 - 4i & 4 - 2i & -4i \end{bmatrix}.
\]

2. Since \( L_{S_1} = L_{S_2} = I \), then \( A_3 = \begin{bmatrix} L_{B_1} & -L_{B_1} & L_{M_1} & -L_{M_1} \end{bmatrix} \), \( A_1^* = A_3^{-1} \), \( A_3 A_1^* = I \), and \( R_{A_3} = 0 \). So, \( A_4 = B_4 = M_3 = S_3 = C_4 = 0 \), and \( R_{A_4} = L_{M_4} = R_{M_4} = L_{S_4} = R_{S_4} = I \).

3. Confirm that (2.1), (2.2), and (2.3) are true for given matrices.
4. To avoid a trivial singular case in (2.8), we put

\[ V_1 = \begin{bmatrix} 1 - i & 2i & 2 & 0 \\ 0 & -i & -i & -1 \\ 1 + i & 1 + i & 2 & 0 \\ -2 & -3 & i & 1 \end{bmatrix}, \]

and

\[ V_2 = \begin{bmatrix} 1 - i & 2i & i \\ -i & -1 + i & 1 + i \\ -i & -2i & i \end{bmatrix}. \]

then

\[
U = \frac{1}{5} \begin{bmatrix} -2i & 1 + i & 7 - 3i & 2 + 2i \\ -1 + i & 2i & 4 - 2i & 1 - i \\ -7 - 3i & -4 + 2i & 8i & 2 + 4i \\ -2 + 2i & -1 - i & -2 + 4i & 0 \end{bmatrix},
\]

\[
W = \frac{1}{3} \begin{bmatrix} -2i & 2 & -2 \\ -2 & 2i & -2i \\ 2 & -2i & 2i \end{bmatrix}.
\]

5. By putting \( U_i, i = 10, \ldots, 13, \) as zero matrices of acceptable shapes, we have \( U_i = U_i^* = 0. \) Similarly, by putting \( U_i, i = 7, 8, \) as zero matrices of acceptable shapes, we find \( Z \) that gives, respectively by (2.13) and (2.14)

\[
U_2 = -U_3^* = \frac{1}{1440} \begin{bmatrix} 6 + 17i & 1 + 18i & 6 + 4i \\ -1 + 18i & -6 + 17i & 4 + 6i \\ 6 - 8i & 8 - 6i & -4i \end{bmatrix}.
\]

6. Finally, by (2.10), we find

\[
V = \frac{1}{720} \begin{bmatrix} 50i & -14 + 30i & -6 + 34i \\ 14 + 30i & 50i & 34 - 6i \\ 6 + 34i & -34 - 6i & 80i \end{bmatrix}.
\]

Note that Maple 2021 was used to perform the numerical experiment.

4. Conclusions

A closed form formula for the anti-Hermitian solution of a classical system of matrix equations are constructed in this paper. Some viable necessary and sufficient conditions are also discussed when this system is consistent over a complex field \( \mathbb{C} \) by applying properties of matrix rank. Special case of the researched system is also discussed. To give an algorithm finding the explicit numerical expression of the solution, it is used the determinantal representations of the MP-inverse recently obtained by one of the authors. The novelty of the conducted research is obtaining necessary and sufficient conditions to exist a solution, its formal representation of by closed formula in terms of generalized inverses, and the construction of an algorithm to find its explicit expression. A numerical example is also given to interpret the results established in this paper.

It is hoped that the developed ways of obtaining necessary and sufficient conditions to existing of a solution, representation its by generalized inverses, and constructing of algorithms by using determinantal representations of generalized inverses have potential applications to solving of a wide class of matrix equations, which is an area deserving of further study.

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