INITIAL SUCCESSIVE COEFFICIENTS OF INVERSE FUNCTIONS
OF CERTAIN CLASSES OF UNIVALENT FUNCTIONS

VASUDEVARAO ALLU AND VIBHUTI ARORA

Abstract. We consider functions of the type $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ from a family of all analytic and univalent functions in the unit disk. Let $F$ be the inverse function of $f$, given by $F(w) = w + \sum_{n=2}^{\infty} A_n w^n$ defined on some $|w| \leq r_0(f)$. In this paper, we find the sharp bounds of $|A_{n+1}| - |A_n|$, for $n = 1, 2$, for some subclasses of univalent functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f$ analytic in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ with Taylor series

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots,$$

with $a_1 = 1$. Let $\mathcal{S}$ be the set all functions $f \in \mathcal{A}$ that are univalent in $D$. Denote by $\mathcal{S}^*$, the family of functions $f$ in $\mathcal{A}$ such that $f(D)$ is a starlike domain with respect to the origin. The family of all functions $f \in \mathcal{A}$ for which $f(D)$ is a convex domain is denoted by $\mathcal{C}$ (see [6, 26]). In 1985, de Branges [5] solved the popular Bieberbach conjecture, which was conjectured in 1916 by Bieberbach, which states that the Taylor coefficients $a_n$ of functions $f \in \mathcal{S}$ of the form (1.1) satisfy the inequality $|a_n| \leq n$ and furthermore, equality could occur if $f$ is some rotation of the Koebe function $k(z) := z/(1 - z)^2$. Similarly, the problem of estimating sharp bound for successive coefficients, namely, $|a_{n+1}| - |a_n|$, is also an interesting coefficient problem for a function to be in class $\mathcal{S}$. This problem was first studied by Goluzin [8] with an idea to solve the Bieberbach conjecture. Hayman [10] proved $|a_{n+1}| - |a_n| \leq A$ for $f \in \mathcal{S}$, where $A \geq 1$ is an absolute constant and the best known estimate as of now is 3.61 due to Grispan [9]. On the other hand, for the class $\mathcal{S}$ a sharp bound is known only for $n = 2$ (see [6, Theorem 3.11]), namely

$$-1 \leq |a_3| - |a_2| \leq 1.029\ldots.$$

For convex functions, Li and Sugawa [13] investigated the sharp upper bound of $|a_{n+1}| - |a_n|$ for $n \geq 2$, and sharp lower bounds for $n = 2, 3$. Several results are known in this direction [16]. These observations are also addressed in the recent papers (see [2, 4]).

For $f \in \mathcal{S}$ denote by $F$ the inverse of $f$ given by

$$F(w) = w + \sum_{n=2}^{\infty} A_n w^n,$$

2010 Mathematics Subject Classification. 30D30, 30C45, 30C50 30C55.

Key words and phrases. Inverse coefficients, Successive coefficients, Univalent functions.
valid on some disk $|w| \leq r_0(f)$. Since $f(f^{-1}(w)) = w$, we can easily obtain by equating the coefficients

\[(1.2) \quad A_2 = -a_2 \text{ and } A_3 = 2a_2^2 - a_3.\]

The inverse functions are studied by several authors in different perspective (see, for instance, [23, 26] and reference therein).

Although, the sharp bounds are known for $|A_n|$, for $n \geq 2$, when $f \in S$ (see [14]), but the successive coefficient problem for inverse functions, \textit{i.e.}, the bounds $|A_{n+1}| - |A_n|$, is still not known for several important class of functions including the whole class $S$. So, it seems reasonable and interesting to compute the bounds of $|A_{n+1}| - |A_n|$ for the class of univalent functions and its subclasses even for some particular values of $n$. In [23, 24] this problem was considered when $n = 2$ for various subclasses of $S$. In the present paper, we obtain the sharp bounds for $|A_2| - |A_1|$ and $|A_3| - |A_2|$ for functions belongs to some important subclasses of $S$.

In this sequence, we have some subclasses of $S$, which have been widely used by many authors in different prospective.

1.1. The class $G(\nu)$. In this paper, we also consider the class $G(\nu)$. A locally univalent function $f \in \mathcal{A}$ is said to belong to $G(\nu)$ for some $\nu > 0$, if it satisfies the condition

$$\Re \left( 1 + \frac{zf'''(z)}{f'(z)} \right) < 1 + \frac{\nu}{2}, \quad z \in \mathbb{D}.$$ 

Ozaki [15] introduced the class $G(1) =: G$ and proved that functions in $G$ are univalent in $\mathbb{D}$. Later Umezawa [27] studied the class $G$ and showed that this class contains the class of functions convex in one direction. Moreover, functions in $G$ are proved to be starlike in $\mathbb{D}$ (see [19], [21]). Thus, the class $G(\nu)$ is included in $S^*$ whenever $\nu \in (0, 1]$. It can be easily seen that functions in $G(\nu)$ are not necessarily univalent in $\mathbb{D}$ if $\nu > 1$. Recently, the radius of convexity for functions in the class $G(\nu)$, $\nu > 0$, is studied in [12].

1.2. The class $F(\lambda)$. For $-1/2 < \lambda \leq 1$, the class $F(\lambda)$ defined by

$$F(\lambda) = \left\{ f \in \mathcal{A} : \Re \left( 1 + \frac{zf'''(z)}{f'(z)} \right) > \frac{1}{2} - \lambda \quad \text{for } z \in \mathbb{D} \right\}.$$ 

We note that clearly $F(1/2) =: C$ is the usual class of convex functions. Moreover, for $\lambda = 1$, we obtain the class $F(1) =: C(-1/2)$ which considered by many researcher in the recent years. Also, functions in $C(-1/2)$ are not necessarily starlike but are convex in some direction. Other related results for $f \in C(-1/2)$ were also obtained in [3, 20]. Functions in $F(\lambda)$ are close-to-convex for $1/2 \leq \lambda \leq 1$ but $F(\lambda)$ contains non-starlike functions for all $1/2 < \lambda \leq 1$ (see [18]). The class $F(\lambda)$ was also considered for the restriction $1/2 \leq \lambda \leq 1$, denote by $F_0(\lambda)$, and further extensively studied in the literature (see for instance [19, 21]).

1.3. The class $C_\gamma(\alpha)$. The family $C_\gamma(\alpha)$ of $\gamma$-convex functions of order $\alpha$ is defined by

$$C_\gamma(\alpha) = \left\{ f \in \mathcal{A} : \Re \left( e^{-i\gamma} \left( 1 + \frac{zf'''(z)}{f'(z)} \right) \right) > \alpha \cos \gamma \right\}.$$
where $0 \leq \alpha < 1$ and $-\pi/2 < \gamma < \pi/2$. We may set $C_0(\alpha) = C(\alpha)$ which consists of the normalized convex functions of order $\alpha$. A function in $C_0(0)$ need not be univalent in $\mathbb{D}$ for general values of $\gamma$ ($|\gamma| < \pi/2$). For example, the function $f(z) = i(1 - z)^t - i$ is known to belong to $C_{\pi/4}\setminus S$. Robertson [22] showed that $f \in C_{\gamma}$ is univalent if $0 < \cos \gamma \leq 0.2315\cdots$. Finally, Pfaltzgraff [17] has shown that $f \in C_{\gamma}$ is univalent whenever $0 < \cos \gamma \leq 1/2$.

This settles the improvement of range of $\gamma$ for which $f \in C_{\gamma}$ is univalent. On the other hand, in [25] it was also shown that functions in $C_{\gamma}$ which satisfy $f''(0) = 0$ are univalent for all real values of $\gamma$ with $|\gamma| < \pi/2$.

Let $\mathcal{P}$ denote the class of all analytic functions $p$ having positive real part in $\mathbb{D}$, with the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

A member of $\mathcal{P}$ is called a Carathéodory function. It is known that $|c_n| \leq 2$ for a function $p \in \mathcal{P}$ and for all $n \geq 1$ (see [6]).

To prove our results, we need the following lemma.

**Lemma 1.1.** [23] Let $B_1$, $B_2$, and $B_3$ be numbers such that $B_1 > 0$, $B_2 \in \mathbb{C}$, and $B_3 \in \mathbb{R}$. Let $p \in \mathcal{P}$ be of the form (1.3). Define $\Psi_+(c_1, c_2)$ and $\Psi_-(c_1, c_2)$ by

$$\Psi_+(c_1, c_2) = |B_2 c_1^2 + B_3 c_2| - |B_1 c_1|,$$

and

$$\Psi_-(c_1, c_2) = -\Psi_+(c_1, c_2).$$

Then

$$\Psi_+(c_1, c_2) \leq \begin{cases} |4B_2 + 2B_3| - 2B_1, & \text{if } |2B_2 + B_3| \geq |B_3| + B_1, \\ 2|B_3|, & \text{otherwise}, \end{cases}$$

and

$$\Psi_-(c_1, c_2) \leq \begin{cases} 2B_1 - B_4, & \text{if } B_1 \geq B_4 + 2|B_3|, \\ 2B_1 \sqrt{\frac{2|B_3|}{B_4 + 2|B_3|}}, & \text{if } B_1^2 \leq 2|B_3|(B_4 + 2|B_3|), \\ 2|B_3| + \frac{B_3^2}{B_4 + 2|B_3|}, & \text{otherwise}, \end{cases}$$

where $B_4 = |4B_2 + 2B_3|$. All inequalities in (1.4) and (1.5) are sharp.

Our main aim of this paper is to estimate the sharp bounds of $|A_3| - |A_1|$ and $|A_3| - |A_2|$ for functions $f$ belong to $G(\nu)$, $F_0(\lambda)$, and $C_{\gamma}(\alpha)$. The organization of this paper is as follows: Section 2 is devoted to the statements of main results. The proof of main results are given in Section 3.
We now state our first main result which provides sharp bounds for $|A_2| - |A_1|$ when $f$ belongs the class $\mathcal{G}(\nu)$.

**Theorem 2.1.** Let $0 < \nu \leq 1$. For every $f \in \mathcal{G}(\nu)$ of the form (1.1), we have

$$-1 \leq |A_2| - |A_1| \leq \frac{2\lambda - 1}{2},$$

Both inequalities are sharp.

**Theorem 2.2.** Let $0 < \nu \leq 1$. For every $f \in \mathcal{G}(\nu)$ of the form (1.1), we have

$$|A_3| - |A_2| \leq \frac{\nu}{6}$$

and

$$|A_3| - |A_2| \geq \begin{cases} -\frac{\nu(8\nu + 17)}{48(\nu + 1)}, & \text{for } 0 < \nu \leq 1/8, \\
-\nu \frac{2\sqrt{2(\nu + 1)}}{2\sqrt{2(\nu + 1)}}, & \text{for } 1/8 \leq \nu \leq 1. \end{cases}$$

(2.1)

The inequalities are sharp.

Next, we obtain the sharp bounds for $|A_2| - |A_1|$ and $|A_3| - |A_2|$ when the functions $f$ are in $\mathcal{F}_0(\lambda)$.

**Theorem 2.3.** Let $1/2 \leq \lambda \leq 1$. For every $f \in \mathcal{F}_0(\lambda)$ be of the form (1.1), we have

$$-1 \leq |A_2| - |A_1| \leq \frac{2\lambda - 1}{2}.$$

The inequalities are sharp.

**Theorem 2.4.** Let $1/2 \leq \lambda \leq 1$. For every $f \in \mathcal{F}_0(\lambda)$ of the form (1.1), we have

$$-\frac{\sqrt{2\lambda + 1}}{2\sqrt{2}} \leq |A_3| - |A_2| \leq \begin{cases} \frac{2\lambda + 1}{6}, & \text{for } 1/2 \leq \lambda \leq 3/4, \\
(2\lambda + 1)(2\lambda - 1) \frac{3}{3}, & \text{for } 3/4 \leq \lambda \leq 1. \end{cases}$$

(2.2)

The inequalities are sharp.

In the next theorem, we will discuss about the sharp bounds for $|A_2| - |A_1|$ and $|A_3| - |A_2|$ when the functions $f$ are $\gamma$-convex of order $\alpha$.

**Theorem 2.5.** Let $-\pi/2 < \gamma < \pi/2$ and $0 \leq \alpha < 1$. For every $f \in C_\gamma(\alpha)$ be of the form (1.1), we have

$$-1 \leq |A_2| - |A_1| \leq (1 - \alpha) \cos \gamma - 1.$$

Both inequalities are sharp.
Theorem 2.6. Let $-\pi/2 < \gamma < \pi/2$ and $0 \leq \alpha < 1$. For every $f \in \mathcal{C}_\gamma(\alpha)$ of the form (1.1), we have

\begin{equation}
|A_3| - |A_2| \leq \frac{(1 - \alpha) \cos \gamma}{3}
\end{equation}

and

\begin{equation}
|A_3| - |A_2| \geq \begin{cases}
-\frac{(1 - \alpha) \cos \gamma}{\sqrt{|\tau|} + 1}, & \text{for } |\tau| \geq 5/4, \\
-\frac{(1 - \alpha) \cos \gamma}{12(|\tau| + 1)}, & \text{for } |\tau| \leq 5/4,
\end{cases}
\end{equation}

where $\tau := 4(1 - \alpha)\mu - 1$. Both inequalities are sharp.

If we put $\alpha = 0$ and $\gamma = 0$ in Theorem 2.6, then we obtain the following result for the class of convex functions:

Corollary 2.7. For every $f \in \mathcal{C}$ of the form (1.1), we have

\begin{equation}
\frac{1}{2} \leq |A_3| - |A_2| \leq \frac{1}{3}
\end{equation}

Both inequalities are sharp.

3. Proof of the main results

This section is devoted to the detailed discussion on our proof of the main results.

3.1. Proof of Theorem 2.1. Let $f \in \mathcal{G}(\nu)$. Then there exists a function $p(z) = 1 + c_1z + c_2z^2 + \cdots \in \mathcal{P}$ satisfying

\begin{equation}
p(z) = \frac{1}{\nu} \left( \nu - \frac{2z p''(z)}{p'(z)} \right).
\end{equation}

After writing $f$ and $p$ in the series form and by comparing the coefficients of $z$ and $z^2$ in the above equation, we obtain the relations

\begin{equation}
a_2 = -\frac{\nu c_1}{4} \quad \text{and} \quad a_3 = \frac{\nu^2 c_1^2 - 2\nu c_2}{24}.
\end{equation}

Thus from equation (3.2) and (1.2), we have

\begin{equation}
|A_2| - |A_1| = \frac{\nu |c_1|}{4} - 1 \leq \frac{\nu - 2}{2},
\end{equation}

where the last inequality is obtained by using $|c_n| \leq 2$ for $n \geq 1$. For the equality, let us consider the function $g_1 \in \mathcal{G}(\nu)$ satisfying (3.1) with $p_1(z) = (1 + z)/(1 - z)$. Then we have

\begin{equation}
g_1(z) = \frac{(1 + z)^{1+\nu} - 1}{\nu + 1}, \quad z \in \mathbb{D},
\end{equation}

for which $A_2 = -\nu/2$ and $A_1 = 1$. On the other hand,

\begin{equation}
|A_1| - |A_2| = 1 - \frac{\nu |c_1|}{4} \leq 1.
\end{equation}
It is easy to see that equality holds when \( g_2 \in \mathcal{G}(\nu) \) defined by (3.1) with \( p_2(z) = (1 + z^2)/(1 - z^2) \). In this case
\[
g_2(z) = \int_0^z (1 - t^2)^{\nu/2} dt = z + \frac{\nu}{6} z^3 + \cdots, \quad z \in \mathbb{D}
\]
and
\[
(3.3) \quad g_2^{-1}(w) = w - \frac{\nu}{6} w^3 + \cdots, \quad w \in \mathbb{D}_{r_0}.
\]
This completes the proof. □

3.2. Proof of Theorem 2.2. Let \( f \in \mathcal{G}(\nu) \). Then from equation (1.2) and (3.2) we obtain
\[
|A_3| - |A_2| = \frac{\nu}{12} \left( |c_2 + \nu c_1^2| - 3|c_1| \right)
\]
(3.4)
\[
= \frac{\nu}{12} \left( |B_3 c_2 + B_2 c_1^2| - |B_1 c_1| \right),
\]
where
\[
B_1 := 3, B_2 := \nu, \text{ and } B_3 := 1.
\]
As \( |A_3| - |A_2| \) is invariant under rotation, to simplify the calculation we assume that \( c_1 = c \in [0, 2] \). Therefore, we can apply Lemma 1.1. A simple calculation shows that, when \( 0 < \nu \leq 1 \), the first condition \( |2B_2 + B_3| \geq |B_3| + B_1 \) for \( \Psi_+(x, c) \) is not satisfied. Hence it follows from Lemma 1.1 and the equation (3.4) that
\[
|A_3| - |A_2| \leq \frac{\nu}{12} (2B_3) = \frac{\nu}{6}.
\]
Here equality holds for \( g_2^{-1} \) given by (3.3) in which the coefficient of \( w^2 \) is 0 and \( w^3 \) is \(-\nu/6\). Thus, the right-hand equality of the theorem has been proved.

We now proceed to prove the left-hand side inequality. By checking the conditions for \( \Psi_-(x, c) \) in Lemma 1.1, we conclude that \( B_1^2 \leq 2 |B_3| |(B_4 + 2B_3)| \) holds but \( B_1 \geq B_4 + 2 |B_3| \) does not hold for \( \nu \geq 1/8 \). Thus, Lemma 1.1 together with equation (3.4) leads to desired inequality (2.1).

We now show that the inequalities in (2.1) are sharp by constructing extreme functions for both cases. For the case \( 1/8 \leq \nu \leq 1 \), we consider a function \( g_3 \) satisfying (3.1) with \( p_3 \in \mathcal{P} \) defined by
\[
p_3(z) = \frac{1 - z^2}{1 - 2sz + z^2} = 1 + 2sz + (4s^2 - 2)z^2 + (8s^3 - 6s)z^3 + \cdots,
\]
where \( s = 1/\sqrt{2(\nu + 1)} \). Then it is easy to see that the coefficients of \( p_3 \) are given by \( c_1 = 2s \) and \( c_2 = -2\nu/\nu + 1 \). From (3.4), we obtain \( |A_3| - |A_2| = -\nu/(2\sqrt{2\nu + 2}) \) and so the inequality (2.1) in this case is sharp for \( g_3 \in \mathcal{G}(\nu) \).

In the similar way we can see that the inequality (2.1) in the case \( 0 < \nu \leq 1/8 \) is sharp for \( g_4 \in \mathcal{G}(\nu) \) defined by (3.1), where
\[
p_4(z) = \frac{1 - z^2}{1 - 2rz + z^2},
\]
where $r = 3/(4\nu + 4)$. Then it is easy to see that the coefficients of $p_3$ are given by $c_1 = 2r$ and $c_2 = (1-16\nu-8\nu^2)/(2\nu+2)^2$. From (3.4), we obtain $|A_3| - |A_2| = \nu(8\nu+17)/48(\nu+1)$, which completes the proof of Theorem 2.2. □

3.3. Proof of Theorem 2.3

Suppose $f \in F_0(\lambda)$. Then from definition we can write

$$1 + \frac{zf''(z)}{f'(z)} = \left(\frac{1}{2} + \lambda\right)p(z) + \frac{1}{2} - \lambda, \quad z \in \mathbb{D}.$$  

By using the Taylor series representations of the functions $f$ and $p$, and comparing the coefficients of $z^n$ ($n = 1, 2$) both the sides, we obtain

$$a_2 = \frac{(1+2\lambda)c_1}{4} \quad \text{and} \quad a_3 = \frac{(1+2\lambda)(2c_2 + (1+2\lambda)c_1^2)}{24}.$$  

By using (1.2) together with (3.6) we can write

$$|A_2| - |A_1| = \frac{(2\lambda+1)|c_1| - 2c_1}{2\lambda} - 1 \leq \frac{2\lambda-1}{2}.$$  

The last inequality holds since $|c_1| \leq 2$. In order to show that the inequality is sharp, first we consider the function $f_1 \in F_0(\lambda)$ defined by

$$f_1(z) = \frac{(1-z)^{-2\lambda} - 1}{2\lambda}, \quad z \in \mathbb{D}.$$  

Hence, $f_1^{-1}$ is given by

$$f_1^{-1}(w) = w - \frac{(1+2\lambda)}{2}w^2 + \frac{(2\lambda+1)(4\lambda+1)}{6}w^3 + \cdots$$  

for $w \in \mathbb{D}_{r_0}$. Thus, $|A_2| - |A_1| = (2\lambda - 1)/2$, which shows that right hand side inequality is sharp.

Secondly, we estimate the upper bound for $|A_1| - |A_2| = 1 - (2\lambda+1)|c_1|/2 \leq 1$. For the sharpness, let us consider the function $f_2$ given by

$$f_2(z) = z + \frac{(2\lambda+1)}{6}z^3 + \cdots$$  

and the corresponding inverse function is of the form

$$f_2^{-1}(w) = w - \frac{(2\lambda+1)}{6}w^3 + \cdots \quad w \in \mathbb{D}_{r_0}.$$  

Hence $|A_2| - |A_1| = -1$. This completes the proof. □

3.4. Proof of Theorem 2.4

Let $f \in F_0(\lambda)$. Then by means of equation (1.2) and (3.6), we see that

$$|A_3| - |A_2| = \frac{2\lambda+1}{24}(|(4\lambda+2)c_1^2 - 2c_2| - 6|c_1|)$$  

$$= \frac{2\lambda+1}{24}(|B_2c_1^2 + B_3c_2| - |B_1c_1|),$$  

where

$$B_1 := 6, \quad B_2 := 4\lambda + 2 \quad \text{and} \quad B_3 := -2.$$
We can see that the functional \(|A_3| - |A_2|\) is rotationally invariant, so we assume \(c_1 = c \in [0, 1]\). Thus, we can apply Lemma 1.1 and by checking the conditions for the bound \(\Psi_+(x, c)\) we obtain
\[
\Psi_+(c_1, c_2) \leq \begin{cases} 
|4B_2 + 2B_3| - 2B_1, & \text{for } 3/4 \leq \lambda \leq 1, \\
2|B_3|, & \text{for } 1/2 \leq \lambda \leq 3/4.
\end{cases}
\]
Therefore, Lemma (1.1) and equation (3.9) yields
\[
|A_3| - |A_2| \leq \begin{cases} 
\frac{2\lambda + 1}{6}, & \text{for } 1/2 \leq \lambda \leq 3/4, \\
\frac{(2\lambda + 1)(2\lambda - 1)}{3}, & \text{for } 3/4 \leq \lambda \leq 1.
\end{cases}
\]

We next find the lower bound of \(|A_3| - |A_2|\). We can apply Lemma 1.1 for \(\Psi(c_1, c_2)\) and we obtain that the condition \(B_1^2 \leq 2|B_3|(B_4 + 2|B_3|)\) satisfies for \(\lambda \in [1/2, 1]\). Using Lemma 1.1 and equation (3.9), we obtain
\[
|A_3| - |A_2| \geq -2B_1 \frac{2\lambda + 1}{24} \sqrt{\frac{2|B_3|}{B_4 + 2|B_3|}} = -\sqrt{2\lambda + 1} \frac{2\lambda + 1}{2\sqrt{2}}.
\]

We now show that inequalities are sharp. When \(1/2 \leq \lambda \leq 3/4\), equality holds in (2.2) for \(f_2\) given by (3.8) and for \(3/4 \leq \lambda \leq 1\) equality holds in (2.2) for \(f_1\) given by (3.7). For the left-hand side equality, let us consider the function \(f_3 \in \mathcal{F}_0(\lambda)\) satisfying (3.5) with
\[
p_3(z) = 1 + 2tz + z^2 - \frac{1}{1 - z^2},
\]
where \(t = 1/\sqrt{4\lambda + 2}\), for which
\[
f_3^{-1}(w) = w - \sqrt{\frac{1 + 2\lambda}{2\sqrt{2}}}w^2 + 0 \cdot w^3 + \ldots.
\]
This completes the proof. \(\square\)

3.5. **Proof of Theorem 2.5.** Suppose \(f \in C_\gamma(\alpha)\). Then there exists a function \(p \in \mathcal{P}\) such that
\[
p(z) = \frac{1}{1 - \alpha} \left\{ \frac{1}{\cos \gamma} \left( e^{-i\gamma} \left( 1 + \frac{z f''(z)}{f'(z)} \right) + i \sin \gamma \right) - \alpha \right\}.
\]
Equating the coefficients of \(z^n\) on both the sides of (3.11) for \(n = 1, 2\), we obtain
\[
a_1 = 1, \ 2a_2 = (1 - \alpha)\mu c_1 \ 	ext{and} \ 6a_3 = (1 - \alpha)^2 \mu^2 c_1^2 + (1 - \alpha)\mu c_2,
\]
where \(\mu = e^{i\gamma} \cos \gamma\). Thus from (1.2) and (3.12) we obtain
\[
|A_2| - |A_1| = \frac{(1 - \alpha)|\mu c_1|}{2} - 1 \leq (1 - \alpha) \cos \gamma - 1.
\]
The last inequality holds since \(|c_1| \leq 2\). It is easy to see that the equality holds for the function
\[
h_1(z) = \frac{1}{2(1 - \alpha)\mu - 1} \left( \frac{1}{1 - z} \right) \left( \frac{1}{2(1 - \alpha)\mu - 1} - 1 \right) = z + (1 - \alpha)\mu z^2 + \cdots, \quad z \in \mathbb{D}
\]
and corresponding inverse function is
\[
h_1^{-1}(w) = w - (1 - \alpha)\mu w^2 + \cdots, \quad z \in \mathbb{D}_{r_0}
\]
Next, we compute the upper bound for \(|A_1| - |A_2| = 1 - (1 - \alpha)|\mu c_1| / 2 \leq 1\). Now we shall easily see that the inequality is sharp for the function
\[
h_2(z) = z + \frac{(1 - \alpha)\mu}{3} z^3 + \frac{(1 - \alpha)\mu((1 - \alpha)\mu + 1)}{10} z^5 + \ldots
\]
and
\[
h_2^{-1}(w) = w - \frac{(1 - \alpha)\mu}{3} w^3 + \ldots.
\]
This completes the proof. \(\square\)

3.6. **Proof of Theorem 2.6** Let \(f \in C_\gamma(\alpha)\). Then from (3.12) and (1.2), we obtain
\[
|A_3| - |A_2| = \frac{(1 - \alpha)\cos \gamma}{6} \left( \frac{2(1 - \alpha)\mu c_1^2 - c_2 - 3|c_1|}{1} \right)
\]
(3.14)
where
\[
B_1 := 3, \quad B_2 := 2(1 - \alpha)\mu, \quad \text{and} \quad B_3 = -1.
\]
Since \(|A_3| - |A_2|\) is rotationally invariant, we may assume that \(c_1 = c \in [0, 2]\). Therefore, we can apply Lemma 1.1. A simple calculation shows that the first condition \(|2B_2 + B_3| \geq |B_3| + B_1\) for \(\Psi_+(x, c)\) is not satisfied since \(|4(1 - \alpha)\mu - 1| \leq 3\). By using Lemma 1.1 and the equation (3.14), we obtain that
\[
|A_3| - |A_2| \leq \frac{(1 - \alpha)\cos \gamma}{3}.
\]
This proves the inequality (2.3).

We next prove the lower bound in (2.4) by checking the condition of Lemma 1.1 for \(\Psi_-(c_1, c_2)\). Note that the inequality \(B_1 \geq B_4 + 2|B_3|\) does not hold as \(|4(1 - \alpha)\mu - 1| \leq 3\) and \(B_4^2 \leq 2|B_3|(B_4 + 2|B_3|)\) hold when \(|4(1 - \alpha)\mu - 1| \geq 5/4\). Thus, we can apply Lemma 1.1 and obtain
\[
\Psi_-(c_1, c_2) \leq \begin{cases} 2B_1 \frac{2|B_3|}{B_4 + 2|B_3|}, & \text{if } |4(1 - \alpha)\mu - 1| \geq 5/4, \\ 2|B_3| + \frac{B_1^2}{B_4 + 2|B_3|}, & \text{if } |4(1 - \alpha)\mu - 1| \leq 5/4. \end{cases}
\]
By substituting the above inequality in (3.14), we obtain the required inequality (2.4).

It is easy to see that equality holds in (2.3) when \(h_2^{-1}\) is defined by (3.13).
For $|\tau| \geq 5/4$, we construct a function $h_3$ defined by (3.11) with

\[ p(z) = \frac{1 + q_1(q_2 + 1)z + q_2z^2}{1 + q_1(q_2 - 1) - q_2z^2}, \]

where

\[ q_1 = \frac{1}{\sqrt{|\tau|} + 1} \quad \text{and} \quad q_2 = e^{i\arg \tau} \]

with $\tau = 4(1 - \alpha)\mu - 1$. Then $A_2 = -(1 - \alpha)\mu/\sqrt{|\tau|} + 1$ and $A_3 = 0$ which gives the equality in (2.4).

For $|\tau| \leq 5/4$, consider the function $h_4$ given by (3.15) where $p(z)$ is of the form (3.15) with

\[ q_1 = \frac{3}{2(|\tau| + 1)} \quad \text{and} \quad q_2 = e^{i\arg \tau}. \]

Then we have

\[ A_2 = -\frac{3(1 - \alpha)\mu}{2(|\tau| + 1)} \quad \text{and} \quad A_3 = \frac{(1 - \alpha)\mu\tau(-4|\tau| + 5)}{12|\tau|(|\tau| + 1)}. \]

This completes the proof of this theorem. □

Acknowledgement. The first author thank SERB-CRG and the second author thank IIT Bhubaneswar for providing Institute Post Doctoral fellowship.

References

[1] V. Allu, A. Lecko, and D. K. Thomas, Hankel, Toeplitz and Hermitian-Toeplitz Determinants for Ozaki Close-to-convex Functions, *Medi. J. Math.* (to appear).
[2] V. Arora, S. Ponnusamy, and S. K. Sahoo, Successive coefficients for spirallike and related functions, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* 113(4) (2019), 2969–2979.
[3] V. Arora and S. K. Sahoo, Meromorphic functions with small Schwarzian derivative, *Stud. Univ. Babeş-Bolyai Math.* 63(3) (2018), 355–370.
[4] V. Arora, Initial Successive coefficients for certain classes of univalent functions, arXiv:2107.13867.
[5] L. De Branges, A proof of the Bieberbach conjecture, *Acta Math.* 154 (1985), 137–152.
[6] P. L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.
[7] Md Firoz Ali and A. Vasudevarao, On Coefficient Estimates of Negative Powers and Inverse Coefficients for Certain Starlike Functions, *Proc. Indian Acad. Sci. (Math. Sci.)* 127(3) (2017), 449–462.
[8] G. M. Goluzin, On distortion theorems and coefficients of univalent functions, *Mat. Sb.* 19 (61) (1946), 183–202 (in Russian).
[9] A. Z. Grinspan, Improved bounds for the difference of adjacent coefficients of univalent functions (Russian), Questions in the modern theory of functions (Novosibirsk), *Sib. Inst. Mat.* 38 (1976), 41–45.
[10] W. K. Hayman, On successive coefficients of univalent functions, *J. London. Math. Soc.* 38 (1963), 228–243.
[11] S. Kumar and S. K. Sahoo, Preserving properties and pre-Schwarzian norms of nonlinear integral transforms, *Acta Math. Hungar.* 162 (2020), 84–97.
[12] S. Kumar and S. K. Sahoo, Radius of convexity for integral operators involving Hornich operations, *J. Math. Anal. Appl.* 502 (2) (2021), 125265.
[13] M. Li and T. Sugawa, A note on successive coefficients of convex functions, *Comput. Methods Funct. Theory* 17(2) (2017), 179–193.
[14] C. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I. Math. Ann. 89 (1923), 103–121.

[15] S. Ozaki, On the theory of multivalent functions. II, Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4 (1941) 45–87.

[16] Z. Peng and M. Obradović, The estimate of the difference of initial successive coefficients of univalent functions, J. Math. Inequal. 13 (2019), 301–314.

[17] J. A. Pfaltzgraff, Univalence of the integral of $f'(z)^3$, Bull. London Math. Soc. 7 (1975), 254–256.

[18] J. A. Pfaltzgraff, M. O. Reade, and T. Umezawa, Sufficient conditions for univalence, Ann. Fac. Sci. Univ. Nat. Zaïre (Kinshasa) Sect. Math.-Phys. 2(2) (1976), 211–218.

[19] S. Ponnusamy and S. Rajasekaran, New sufficient conditions for starlike and univalent functions, Soochow J. Math. 21(2) (1995), 193–201.

[20] S. Ponnusamy, S. K. Sahoo, and H. Yanagihara, Radius of convexity of partial sums of functions in the close-to-convex family, Nonlinear Anal. 95 (2014), 219–228.

[21] S. Ponnusamy and A. Vasudevarao, Region of variability of two subclasses of univalent functions, J. Math. Anal. Appl. 332(2) (2007), 1323–1334.

[22] M. S. Robertson, Univalent functions $f(z)$ for which $zf'(z)$ is spirallike, Michigan Math. J. 16 (1969), 97–101.

[23] Y. J. Sim and D. K. Thomas, On the difference of inverse coefficients of univalent functions, Symmetry 12(12) (2020).

[24] Y. J. Sim and D. K. Thomas, A note on spirallike functions, Bull. Aust. Math. Soc., https://doi.org/10.1017/S0004972721000198.

[25] V. Singh and P. N. Chichra, Univalent functions $f(z)$ for which $zf'(z)$ is $\alpha$-spirallike, Indian J. Pure Appl. Math. 8 (1977), 253–259.

[26] D. K. Thomas, Nikola Tuneski and Allu Vasudevarao Univalent Functions, De Gruyter, 2018.

[27] T. Umezawa, Analytic functions convex in one direction, J. Math. Soc. Japan 4 (1952), 195–202.

School of Basic Science, Indian Institute of Technology Bhubaneswar, India
Email address: avrao@iitbbs.ac.in

School of Basic Science, Indian Institute of Technology Bhubaneswar, India
Email address: vibhutiaraor1991@gmail.com