Pathologies of the KMR prescriptions for unintegrated PDFs
Which prescription should be preferred?

Benjamin Guiot∗
Departamento de Física, Universidad Técnica Federico Santa María; Casilla 110-V, Valparaíso, Chile

Abstract
We discuss the different Kimber-Martin-Ryskin (KMR) prescriptions for unintegrated parton distribution functions (uPDFs). We show that the strong-ordering (SO) and the angular-ordering (AO) cut-offs lead to strong discrepancies between the obtained cross sections. While the result obtained with the AO cut-off overestimates the heavy-flavor cross section by about a factor 3, the SO cut-off gives the correct answer. We also solve the issue of the KMR uPDFs definitions mentioned in [1], and show that, in the case of the AO cut-off, the KMR uPDFs are ill-defined.

∗benjamin.guiot@usm.cl
1 Introduction

Understanding transverse momentum dependent parton distribution functions has been a topic of increasing theoretical and experimental interest. Compared to the collinear PDFs, they provide an additional information on the transverse dynamics of a parton inside the hadron. Depending on the kinematical range, several formalisms exist. The TMD factorization \cite{2-5} is valid for $k_t/Q$ small, with $k_t$ the parton transverse momentum and $Q$ the hard scale of the process. The TMD PDFs, mainly studied in SIDIS and Drell-Yan experiments, provide a 3-dimensional information on the hadron structure and could help to solve the proton-spin crisis. The $k_t$-factorization, first developed in \cite{6-9}, is used at small-$x$. In this case, $k_t$ is not restricted to small values. It finds applications at the LHC, where the transverse momentum of incoming spacelike partons can indeed be large, due to partonic evolutions.

In the context of $k_t$-factorization, where the transverse momentum PDFs are generally refereed as unintegrated PDFs (uPDFs), a popular construction of these functions is given by the Kimber-Martin-Ryskin (KMR) and by the Watt-Martin-Ryskin (WMR) prescriptions \cite{10,11}. However, in \cite{12}, it has been shown that one of the KMR/WMR prescriptions gives an overestimation of the heavy-quark cross section by about a factor 3. The reason is that the KMR/WMR uPDFs, computed with the angular-ordering cut-off (see section 4), are too large for $k_t^2 > \mu^2$, where $\mu^2$ is the factorization scale\footnote{One should remember that in perturbative calculations, and in particular at leading order, the factorization scale should be chosen close to the hard scale of the process.}. It has been shown that cutting off the tail of these distributions at $k_t^2 \sim \mu^2$ and taking into account leading order contributions gives a good description of the heavy-quark cross section \cite{12}. The present work has been motivated...
by [1], in particular because the (strong ordering) KMR uPDFs shown in this paper are similar to the "cut" KMR uPDFs used in [12]. Consequently, they should lead to the same results and conclusions.

This work has two objectives. We will study in details the different KMR/WMR prescriptions, in order to point out which one should be preferred, and we will solve the issue of the KMR/WMR uPDFs definitions addressed in [1]. The overlay of the paper is as following. After a short review of the DGLAP equation in section 2, we will present separately the KMR and the WRM prescriptions in section 3. We will see that they are not equivalent, and that the former do not obey the correct DGLAP equation. In section 4, we discuss in details the issue of the KMR/WMR uPDFs definitions, related to the fact that apparently mathematically equivalent definitions give different numerical results. Finally, in section 5, we study further the differences between the KMR/WMR prescriptions, by discussing the angular-ordering (AO) and the strong-ordering (SO) cut-offs. Using different cut-offs leads to significant differences for the cross section, and we will see that the SO cut-off should be preferred. In particular, we will show by performing explicit calculations that the SO cut-off gives results compatible with those obtained in [12]. In the mentioned paper, the main conclusion is that, contrary to a common belief, the main contribution to heavy-quark production at leading order (in a variable-flavor-number scheme) is given by $Qg \rightarrow Qg$, not by $gg \rightarrow Q\bar{Q}$. Calculations in agreement with data for heavy-quark production, and taking into account only the $gg$ contribution are incorrect. It is generally the case for the KMR/WMR uPDFs when the AO cut-off is used, see section 5.

2 The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equation with unregularized splitting functions

In this section, following [13], we quickly remind a form of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation useful for numerical treatments. For small $\delta x$ and $\delta t$, centered on $x$ and $t$, the variation of a parton density with $t$ is given by:

$$
\delta f(x, t) = \delta f_{\text{in}}(x, t) - \delta f_{\text{out}}(x, t).
$$

The variable $t$ has the dimension of energy squared. Equation (1) simply expresses that the change of a quantity in a volume (here $\delta t \delta x$) is given by what is going in, minus what is going out. Working with one parton flavor,
\( \delta f_{\text{in}}(x, t) \) receives a contribution from the splitting of partons at \( x' > x \):

\[
\delta f_{\text{in}}(x, t) = \frac{\delta t}{t} \int_x^1 dx' \int_0^1 dz \frac{\alpha_s}{2\pi} \tilde{P}(z)f(x', t)\delta(x - zx')
\]

\[
= \frac{\delta t}{t} \int_0^1 dz \frac{\alpha_s}{z} \tilde{P}(z)f(x, t, t).
\]  

(2)

It is proportional to the parton density at \( x' \) multiplied by the probability for a splitting at \( t \), with the daughter parton having a fraction \( z \) (generally light-cone momentum fraction) of its mother. The delta function ensures that after the splitting, the parton arrives in the volume \( \delta t \delta x \). \( \tilde{P}(z) \) is the unregularized splitting function. Similarly, the outgoing part is given by

\[
\delta f_{\text{out}}(x, t) = \frac{\delta t}{t} f(x, t) \int_0^1 dz \frac{\alpha_s}{2\pi} \tilde{P}(z).
\]  

(3)

One of the differences with Eq. (2) is that the parton density is outside of the integral. Indeed, for partons inside the volume \( \delta t \delta x \), any splitting will bring them out. So the contribution is simply given by the parton density at \( x \) multiplied by the total splitting probability (\( t \) fixed).

We now consider the realistic case of QCD. The variation of the quark density at leading order reads:

\[
\delta q(x, t) = \frac{\delta t}{t} \int_0^1 dz \frac{\alpha_s}{z} \left\{ \tilde{P}_{qq}(z)q \left( \frac{x}{z}, t \right) + \tilde{P}_{qg}(z)g \left( \frac{x}{z}, t \right) \right\}
\]

\[- \frac{\delta t}{t} q(x, t) \int_0^1 dz \frac{\alpha_s}{2\pi} \tilde{P}_{qq}(z).
\]  

(4)

The case of the gluon density is more complicated. One can arrive in the volume from \( g \rightarrow gg \) or from \( q \rightarrow qg \), and one leaves the volume from \( g \rightarrow gg \) or from \( g \rightarrow q\bar{q} \). As explained in [13], one subtlety is that both gluons produced in the splitting \( g \rightarrow gg \) can participate, giving

\[
\delta g_{\text{in}}(x, t) = \frac{\delta t}{T} \int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \left\{ 2\tilde{P}_{gg}(z)g \left( \frac{x}{z}, t \right) + \tilde{P}_{qg}(z) \left[ q \left( \frac{x}{z}, t \right) + \bar{q} \left( \frac{x}{z}, t \right) \right] \right\}.
\]  

(5)

The unregularized splitting function are given in [13], equations (5.10) and (5.20):

\[
\hat{P}_{gg}(z) = C_A \left[ \frac{1 - z}{z} + \frac{z}{1 - z} + z(1 - z) \right]
\]  

(6)

\[
\hat{P}_{qg}(z) = \hat{P}_{qg}(1 - z) = C_F \frac{1 + (1 - z)^2}{z}
\]  

(7)

The outgoing part is given by:

\[
\delta g_{\text{out}}(x, t) = \frac{\delta t}{t} g(x, t) \int_0^1 dz \frac{\alpha_s}{2\pi} \left[ \hat{P}_{gg}(z) + n_f \hat{P}_{qg} \right].
\]  

(8)
Note the factors 2 and 1, in front of \( \hat{P}_{gg} \) in Eqs. \([5]\) and \([8]\). The regularized splitting functions are obtained after applying the plus-prescription \([13]\):

\[
P(z) = \hat{P}(z) + .
\]

In the case of the gluon-gluon splitting function the result is

\[
P_{gg}(z) = 2C_A \left[ \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \frac{1}{6} (11C_A - 4N_f TR) \delta(1-z).
\]

Note the factor 2 in front of \( C_A \), compared to the unregularized case. In the following, we will use the unregularized splitting function \( \hat{P}_{gg} \), Eq. \([6]\), with a factor \( 2C_A \), for reasons explained in the next section. In the rest of the paper, all the mentioned splitting functions are the unregularized one, and they will be written without the "hat", in order to fit with the literature on KMR unintegrated PDFs (uPDFs).

### 3 The KMR unintegrated PDFs

We first start by discussing some ambiguities, related to the fact that in the literature, "KMR formalism" can refer both to \([10]\) and \([11]\). However, the equations given in these papers are not equivalent and we will refer to the second one as the Watt-Martin-Ryskin (WMR) formalism. In \([10]\), the DGLAP equation is written

\[
\frac{\partial a(x, \mu^2)}{\partial \ln \mu^2} = \sum_{a'} \frac{\alpha_s}{2\pi} \left[ \int_x^{1-\Delta} P_{aa'}(z) \frac{a'(x, \mu^2) \, dz}{z} - a(x, \mu^2) \int_0^{1-\Delta} P_{a'a}(z) \, dz \right],
\]

with \( a(x, \mu^2) = xf_a(x, \mu^2) \) and \( f_a(x, \mu^2) \) the number density. The sum on \( a' \) runs on all possible parton flavours: quarks, anti-quarks and gluon. In the case of the gluon distribution function, Eq. \([11]\) is not correct since it misses the factor 2 in front of \( P_{gg} \), compared to Eq. \([5]\). Taking the unregularized splitting function, Eq. \([6]\), with a factor \( 2C_A \) will not help, the issue being that the coefficients in front of \( P_{gg} \) should not be the same. It is also incorrect for the quark distribution function since, in the last term of Eq. \([11]\), the sum on \( a' \) implies the contribution of both \( P_{qq}(z) \) and \( P_{gq}(z) \), in disagreement with Eq. \([4]\).

In the WMR case, the DGLAP equation is written with an additional \( z \) factor in the last term [see \([11]\), equation (17)]:

\[
\frac{\partial a(x, \mu^2)}{\partial \ln \mu^2} = \sum_{a'} \frac{\alpha_s}{2\pi} \left[ \int_x^{1-\Delta} P_{aa'}(z) \frac{a'(x, \mu^2) \, dz}{z} - a(x, \mu^2) \int_0^{1-\Delta} z P_{a'a}(z) \, dz \right].
\]

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Consequently, Eqs. (12) and (11) are not equivalent. This difference can be traced up to the definition of the Sudakov factor [equations (18) in [11] and (3) in [10]]. The extra \( z \) factor is justified saying that it "avoids double-counting the s- and t-channel partons". It is also mentioned that, after taking the integration over \( z \) and summing on \( a' \), it gives a factor \( 1/2 \). In that case, and using \( P_{gg} \) with a factor \( 2C_A \), Eq. (12) with \( a = g \) is equivalent to Eqs. (5) and (8). It gives also the correct DGLAP equation for the quark since:

\[
- \frac{1}{2}q(x, t) \int_0^1 dz \frac{\alpha_s}{2\pi} \left[ P_{qq}(z) + P_{gq}(z) \right] = -q(x, t) \int_0^1 dz \frac{\alpha_s}{2\pi} P_{qq}(z). \tag{13}
\]

Here, we used that fact that \( P_{qq} \) and \( P_{gq} \) are related by \( z \rightarrow 1 - z \).

An advantage of the WMR prescription is that the \( z \) factor regularizes the divergence of the splitting function \( P_{gg} \) when \( z \) goes to zero. In recent papers, the used KMR prescription is in fact the WMR one, and it is in particular the case in [1], which we discuss now.

As explained in the introduction, the present work has been motivated by [1]. One of our goals is to discuss of the analysis given in this paper. It is then useful to give a short and similar presentation of the WMR formalism, insisting on important details.

The goal is to built an unintegrated parton densities which obeys (at least approximately):

\[
f_a(x, Q^2) = \int_0^{Q^2} F_a(x, k_t^2; Q^2) dk_t^2. \tag{14}
\]

This equation is sometimes written with a factor \( x \) in the l.h.s. In this case the function \( F_a(x, k_t^2; Q^2) \) is the momentum density. However, the factor \( 1/z \) in [1], equation (2), indicates that the authors are working with the parton densities, so we use the relation (14).

The derivation starts with the DGLAP equation. The main trick in the WMR prescription is the observation that using the Sudakov factor

\[
T_a(Q, k_t) = \exp \left\{ -\int k_t^2 \frac{dP_t}{P_t^2} \sum_a' \int_0^{1-\Delta(p_t)} dz zP_{a'a}(z, p_t) \right\}, \tag{15}
\]

with \( P_{a'a}(z, \mu) \) defined by

\[
P_{a'a}(z, \mu) = \frac{\alpha_s(\mu^2)}{2\pi} P_{a'a}^{LO}(z), \tag{16}
\]
the DGLAP equation\footnote{Strictly speaking, this is not the DGLAP equation since there is an extra $z$ factor in the WMR prescription.} can be rewritten

\[
\frac{\partial}{\partial \ln k_t^2} [T_a(Q, k_t)f_a(x, k_t)] = T_a(Q, k_t) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z, k_t)f_{a'}\left(\frac{x}{z}, k_t\right).
\]

(17)

However, for this to be correct, one should be careful with the $k_t$ dependence of the Sudakov factor. In particular, as mentioned in \cite{1}, the cut-off $\Delta$ should not be a function of $k_t$ when used in the definition of $T_a$, Eq. (15). In this case, we have

\[
\frac{\partial T_a(Q, k_t)}{\partial \ln k_t^2} = T_a(Q, k_t) \sum_{a'} \int_0^{1-\Delta(k_t)} dz P_{aa'}(z, k_t),
\]

(18)

and after a straightforward calculation, Eq. (17) can be written

\[
T_a(Q, k_t) \frac{\partial f_a(x, k_t^2)}{\partial \ln k_t^2} =
T_a(Q, k_t) \sum_{a'} \left[ \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z, k_t)f_{a'}\left(\frac{x}{z}, k_t\right) - f_a(x, k_t^2) \int_0^{1-\Delta} z P_{a'a}(z, k_t) dz \right],
\]

(19)

which is the "DGLAP equation" multiplied by $T_a$. The WMR unintegrated PDFs are defined as

\[
F_a(x, k_t^2, Q^2) = \frac{1}{k_t^2} f_a(x, k_t^2, Q^2) = \frac{1}{k_t^2} \frac{\partial}{\partial \ln k_t^2} [T_a(Q, k_t)f_a(x, k_t)].
\]

(20)

Collinear and unintegrated PDFs can be distinguished by the number of their arguments. Integrating $F(x, k_t^2, Q^2)$ over $k_t^2$ gives:

\[
\int_{Q_0^2}^{Q^2} dk_t^2 F_a(x, k_t^2, Q^2) = f_a(x, Q^2) - T_a(Q^2, Q_0^2)f_a(x, Q_0^2),
\]

(21)

which, for $Q^2 \gg Q_0^2$, is numerically close to Eq. (14). Using Eq. (17), the WMR unintegrated PDFs can also be defined by

\[
f_a(x, k_t^2, Q) = T_a(Q, k_t) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z, k_t)f_{a'}\left(\frac{x}{z}, k_t\right).
\]

(22)

The main concern of \cite{1} is the fact that definitions (20) and (22) do not give the same numerical result.
4 Discussion of the KMR/WMR uPDFs definitions

As explained in [1], two cut-off are usually used. The strong ordering (SO) cut-off

\[ \Delta = \frac{kt}{Q}, \]  

(23)

and the angular ordering (AO) cut-off

\[ \Delta = \frac{k_t}{k_t + Q}. \]  

(24)

The authors have shown that, using a cut-off dependent parton density \([D_a(x, \mu^2, \Delta)]\) instead of the usual one, Eqs. (20) and (22) give the same numerical result. It implies that the unintegrated PDFs depend also on the cut-off, \([D_a(x, k_t^2, \mu^2, \Delta)]\). However, this is not really satisfactory since we started with Eq. (14). Moreover, it is not clear how this new object should be used in practice, in the phenomenology.

In fact, the reason why the two definitions give different results is because Eq. (17) is not always true. Let’s consider the case of the AO cut-off. In this case, \(k_t > Q\) is not forbidden and the Sudakov factor can be larger than 1. In order to avoid this situation, the authors defined:

\[ T_a(Q, k_t) = 1, \quad k_t > Q. \]  

(25)

This equation can be written

\[ \tilde{T}_a(Q, k_t) = \Theta(Q^2 - k_t^2)T_a(Q, k_t) + \Theta(k_t^2 - Q^2), \]  

(26)

with \(\Theta\) the Heaviside function. In the previous section, we mentioned that one has to be careful with the \(k_t\) dependence of the Sudakov factor. With the new Sudakov factor, the l.h.s of Eq. (17) gives:

\[
\frac{\partial}{\partial \ln k_t^2} \left[ \tilde{T}_a(Q, k_t) f_a(x, k_t) \right] = \left[ k_t^2 T_a(Q, k_t) \frac{\partial}{\partial k_t^2} \Theta(Q^2 - k_t^2) + \Theta(Q^2 - k_t^2) \right] f_a(x, k_t) + \tilde{T}_a(Q, k_t) \frac{\partial}{\partial \ln k_t^2} f_a(x, k_t).
\]  

(27)

Having in mind that \(\langle \frac{d}{dx} \Theta(x - y), \phi \rangle = -\langle \frac{d}{dx} \Theta(y - x), \phi \rangle = \langle \delta(x - y), \phi \rangle\) and that \(T_a(Q, Q) = 1\), we see that the first and third terms in the bracket will cancel. Taking the derivative of the second term in the bracket, and
rewriting it in terms of $\tilde{T}_a$ we have

$$\frac{\partial}{\partial \ln k_t^2} \left[ \tilde{T}_a(Q, k_t) f_a(x, k_t) \right] = \tilde{T}_a(Q, k_t) f_a(x, k_t) \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z, k_t)$$

$$- \Theta(k_t^2 - Q^2) f_a(x, k_t) \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z, k_t) + \tilde{T}_a(Q, k_t) \frac{\partial}{\partial \ln k_t^2} f_a(x, k_t).$$

(28)

Finally, using the DGLAP equation for the last term we get

$$\frac{\partial}{\partial \ln k_t^2} \left[ \tilde{T}_a(Q, k_t) f_a(x, k_t) \right] = \tilde{T}_a(Q, k_t) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{a'a}(z, k_t) f_{a'} \left( \frac{x}{z}, k_t \right)$$

$$- \Theta(k_t^2 - Q^2) f_a(x, k_t) \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z, k_t),$$

(29)

showing that definitions (20) and (22) (with $T_a$ replaced by $\tilde{T}_a$) are not equivalent. There is then no need for these definitions to give the same numerical result, and no need for the cut-off dependent distribution functions.

5 The true $k_t$ dependence of WMR uPDFs

In this section, we want to insist on the conclusion reached in [12]. That is, the main contribution to the $p_t$ distribution of one heavy flavor is given by $Qg \rightarrow Qg$, not $gg \rightarrow Q\bar{Q}$ (for variable-flavor-number schemes). Using the KMR/WMR parametrization and the AO cut-off, one gets a satisfying result with $gg \rightarrow Q\bar{Q}$ alone, because of the too large $k_t$-tail of the distribution. Of course, there is no reason for stopping the calculation at this point, and the $Qg \rightarrow Qg$ contribution should also be computed. Doing this, the cross section for heavy-quark production will completely overshoot the data (or FONLL calculations [14] for a bare heavy quark), as we will demonstrate below.

In the opposite, cutting artificially the WMR uPDFs at $k_t > Q$ and adding up the contributions $Qg$ and $gg$ gives an excellent result [12] (figure 11). The present work has been motivated by the fact that the $k_t$ distribution of the WMR uPDFs presented in [1] (for the SO cut-off; figure 1, left, red curve) is very similar to the cut-WMR uPDFs used in [12]. It implies that, using the SO cut-off, the $gg \rightarrow Q\bar{Q}$ contribution will not be sufficient, and taking into account $Qg \rightarrow Qg$ will be necessary to bring agreement with data, as it should be. Leaving this discussion for later, we continue with the

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9 This does not mean that this object is devoid of interest. In any case, a cut-off will appear in the numerical implementation of unintegrated PDFs based on Eq. (14).
analysis of Eq. (29) and of the AO cut-off.

We first note that \( \tilde{T}_a(Q, Q) = 1 \). Then, integrating the l.h.s of Eq. (29) gives a result numerically close to Eq. (14). Consequently, a possible correct definition of the WMR uPDFs is

\[
fa(x, k_t^2, Q) = \tilde{T}_a(Q, k_t) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z, k_t) f_{a'} \left( \frac{x}{z}, k_t \right)
- \Theta(k_t^2 - Q^2) f_a(x, k_t) \sum_{a'} \int_0^{1-\Delta} dz \, z P_{a'a}(z, k_t). \tag{30}
\]

This distribution is displayed in Fig. 1 for \( x = 10^{-3} \) and \( Q^2 = 10 \text{ GeV}^2 \). Compared to Eq. (22), it receives a negative contribution at \( k_t > Q \). Then,

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{WMR uPDFs, Eq. (30), as a function of \( k_t^2 \) showing a discontinuity at \( k_t^2 = Q^2 \). It is compared to the PB uPDFs [15], which give an accurate result for the heavy-quark \( p_t \) distribution [12].}
\end{figure}

it presents a discontinuity at \( k_t = Q \), identical to the result shown in [1] (figure 1, right, dashed blue line), obtained from the definition (20). It shows the equivalence of Eqs. (20) and (30). No need for a cut-off dependent parton density, the issue was that Eq. (17) is incorrect for the Sudakov factor defined in Eq. (26).

Using the AO cut-off, there is an infinite number of non-equivalent definitions of the KMR/WMR uPDFs which do agree with Eq. (14). Indeed, we can always add a \( \Theta(k_t^2 - Q^2) A(x, k_t^2, Q^2) \) to the definition (30), with
$A(x, k_t^2, Q^2)$ any function. In particular, another correct definition is

$$f_a(x, k_t^2, Q) = \tilde{T}_a(Q, k_t) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z, k_t) f_{a'} \left( \frac{x}{z}, k_t \right)$$

$$= \frac{\partial}{\partial \ln k_t^2} \left[ \tilde{T}_a(Q, k_t) f_a(x, k_t) \right] + \Theta(k_t^2 - Q^2) f_a(x, k_t) \times \sum_{a'} \int_0^{1-\Delta} dz P_{a'a}(z, k_t)$$

(31)

These definitions differ for $k_t > Q$. Finally, we can also choose the function $A(x, k_t^2, Q^2)$ such that $f_a(x, k_t^2, Q) = 0$ for $k_t > Q$. It is clear that Eq. (14) is not enough to fix the definition of the KMR/WMR uPDFs. An extra condition could be that we want the distribution and its first derivative to be continuous at large $k_t$. It corresponds to the definition given in Eq. (31). A better condition is that numerical calculations should be in agreement with data once all contributions have been taken into account at a given order.

However, these two conditions are not compatible. The distributions obtained from Eq. (31) or Eq. (30) are too large for $k_t > Q$. The contribution $gg \to Q\bar{Q} + Qg \to Qg$ overestimates the NLO calculations [14] for the heavy-quark $p_t$ distribution, as shown in Fig. 2. These results have been obtained with the KATIE event generator [16], with the set-up identical to the one described in [12]. The charm mass has been set to 0 in the process $cg \to cg$. "WMR old" and "WMR" refer to definitions (22) and (30), respectively. As expected, the latter gives a smaller $gg$ contribution, due to the smaller unintegrated gluon density at $k_t > Q$. However, we can see that the $gg + cg$ contribution still overestimates NLO calculations. In [12], it has been shown that the same calculations done with the PB uPDFs [15] do a good job.

We now discuss the KMR/WMR prescription with the SO cut-off, and we will see that it solves all these issues. In this case, the condition $x < 1 - \Delta$ implies that:

$$k_t \leq Q(1 - x) \leq Q,$$

(32)

giving a Sudakov factor smaller than 1. Note than in [1], two plots are presented for the SO cut-off (figure 1, left), corresponding to the use of definitions (20) and (22). The reason for the strong difference between these two plots is that, when using Eq. (20), the authors relaxed the condition 4

\[4\text{This is due to the fact that in Eq. (14), the uPDFs are integrated only up to } Q^2. \text{ In the parton model, as defined in [5], the relation is } f(\xi) = \int_0^{Q^2} d'k_t f(\xi, k_t^2).\]

5

\[5\text{In any case, the distribution has a discontinuity at small } k_t.\]

6

\[6\text{The event generator KATIE does not accept massive quark as initial state.}\]
However, the condition $x < 1 - \Delta$ is true whatever the uPDFs definition, and can be maintained explicitly by a factor $\Theta(Q^2 - k_t^2)$ in Eqs. (20) and (22). Then, both definitions give trivially the same result, namely, a distribution with a sharp cut-off for $k_t > Q$. Consequently, the SO cut-off eliminates the issue of the uPDFs multiple definitions. Moreover, using the SO cut-off, Eq. (25) is unnecessary and Eq. (17) is true.

The main goal of this paper is to discuss the fact that, with the SO cut-off, the $gg$ contribution alone will underestimate the NLO result for heavy-quark production by a factor $\sim 3$. In figure 3, we show the $k_t$ dependence of the WMR uPDFs computed with this cut-off. For $k_t > 1$ GeV, these distributions are quite similar to the PB uPDFs, and we can anticipate that they will give a similar result. It is indeed the case, as shown in Fig. 4. As expected, the $gg$ contribution undershoot the NLO calculations for the charm $p_t$ distribution. It is only after including the $cg$ contribution (the main one) that we get the agreement between both. Note that, we still have to include the $q\bar{q} \rightarrow Q\bar{Q}$ and $cq \rightarrow cq$ processes, which are respectively negligible and small [12] (at least in this kinematical range).

Note the small difference between the slope of the $gg + cg$ contribution (Fig. 4 green line) with the slope of the $gg$ contribution (Fig. 2 purple line), obtained with the AO cut-off. The former is harder and follows exactly NLO

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7Doing this, Eq. (25) is used again, and the relation (17) is incorrect.
Figure 3: Charm and gluon uPDFs obtained with the WMR prescription and the SO cut-off, compared to the PB uPDFs.

Figure 4: Charm distribution, obtained with KaTIE and the WMR uPDFs presented in Fig. 3.

calculations. However, this small difference should not be overinterpreted. As explained before, we have neglected small contributions, and the full calculation could present a slightly modified slope. Moreover, the slope also depends on the choice made for the factorization scale.
6 Conclusion

In this paper we discussed the KMR and WMR prescriptions for uPDFs, and we underlined the fact that several recent studies using the "KMR" prescription are in fact using the WMR one. We have seen that only the WMR prescription gives the correct DGLAP equation.

Then, we addressed the issue of the apparently mathematically equivalent uPDFs definitions, giving different numerical results, mentioned in [1]. We have demonstrated that, with the Sudakov factor used in [1], these definitions were in fact not equivalent, and we gave the correct relation, Eq. (29).

We have seen that the WMR prescription leads to significant differences for the cross sections , depending on the choice made for the cut-off. With the AO cut-off, the contribution $gg + cg$ completely overshoot the NLO calculations (Fig. 2) and the uPDFs are not uniquely defined by Eq. (14).

In the opposite, the SO cut-off avoid these issues. It gives satisfying numerical results (Fig. 4), in agreement with those obtained in [12]. In particular, using the (SO) WMR uPDFs, we confirmed that the main contribution to heavy-quark production is given by $Qg \rightarrow Qg$, the $gg$ contribution alone being a factor $\sim 3$ below NLO calculations. Compared to the AO cut-off, the obtained $k_t$ distributions are closer to other uPDFs sets, e.g. the PB uPDFs.

Unfortunately, the majority of phenomenological papers use the AO cut-off. Calculations are done including only the $gg$ contribution [with the gluon unintegrated density built from Eq. (31)], giving an (accidental) reasonable agreement with data. The fact that the other contributions are not including is even not mentioned. One of the unpleasant consequences is to convince the reader that the main contribution to heavy-quark production is the $gg$ contribution. Then, using another, correct, uPDFs set, e.g. the PB one [13], and including only the $gg$ contribution, it leads to the erroneous conclusion that this set is not working. This is for instance the case in [17], where the PB and KMR uPDFs are discussed. In this paper, we can read that "a new Parton-Branching (PB) uPDF strongly underestimates the same experimental data". However, it has been shown in [12] that, once all contributions have been added up, the PB uPDFs give in fact a good description of the heavy-quark $p_t$ distribution.

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