EQUIVARIANT LEFSCHETZ NUMBER OF DIFFERENTIAL OPERATORS

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Abstract. Let $G$ be a compact Lie group acting on a compact complex manifold $M$. We prove a trace density formula for the $G$-Lefschetz number of a differential operator on $M$. We generalize Engeli and Felder’s recent results to orbifolds.

1. Introduction

In this paper, we study a $G$-equivariant Lefschetz number formula for a compact Lie group action on a compact connected complex manifold $M$. We assume that the isotopy group at each point of $M$ is finite, which implies that the quotient space $M/G$ is an orbifold.

Let $E$ be a $G$-equivariant vector bundle on $M$. Both $G$ and a global holomorphic differential operator $D \in D_E(M)$ act on the sheaf cohomology group $H^i(M, E)$. Since $M$ is compact, the sheaf cohomology $H^i(M, E)$ is finite dimensional. We can consider the following $G$-Lefschetz number, for $\gamma \in G$

$$D \mapsto L(\gamma, D) = \sum_i (-1)^i \text{tr}(\gamma H^i(D)),$$

where $H^i(D)$ denotes the action of $D$ on $H^i(M, E)$. $L(\cdots, D)$ is a smooth function on $G$ which contains useful information of the $G$-action. When $D$ is the identity operator, the topological expression of this number is known as the $G$-equivariant Riemann-Roch-Hirzebruch formula [BeGeVe] for a $G$-equivariant bundle $E$.

In this paper, we express the number $L(\gamma, D)$ for any differential operator, by an integral over the $\gamma$ fixed point submanifold $M^\gamma$ of some differential form $\chi_{0, \gamma}$ determined by finitely many jets of $D$ on $M^g$ and a hermitian metric on $M$ and $E$,

$$L(\gamma, D) = \int_{M^\gamma} \frac{1}{(2\pi i)^{n-l(\gamma)/2}} \chi_{0, \gamma}(D),$$

where $l(\gamma)$ is the real codimension of $M^\gamma$ in $M$. The proof of this theorem is a generalization of Engeli-Felder’s theorem [EnFe] on a manifold, with the local results developed in the previous work [PPT]. The analysis of this proof is more involved than [EnFe] because we have to work with the $\gamma$ traces of a heat kernel. When $G$ is finite and $D$ is $G$ invariant, we are able to compute the $G$ average of $L(\gamma, D)$,

$$\frac{1}{|G|} \sum_{\gamma \in G} L(\gamma, D) = \sum_{(\gamma) \in C(\gamma)} \int_{M^\gamma/C(\gamma)} \frac{1}{m_\gamma} \frac{1}{(2\pi i)^{n-l(\gamma)/2}} \chi_{0, \gamma}(D),$$

(1.1)

where $(\gamma)$ stands for the conjugacy class in $G$ containing $\gamma$, $C(\gamma)$ is the centralizer group of $\gamma$ in $G$, and $m_\gamma$ is the number of isotopy of $C(\gamma)$ action on $M^\gamma$. The proof of this result is an application of the proof of the $G$-Lefschetz number formula.

A $G$ invariant differential operator $D$ on $M$ descends to a differential operator $\hat{D}$ on the orbifold $X = M/G$. Hence, the above equation (1.1) can be viewed as a Lefschetz number formula of $\hat{D}$. This inspires a question of computing the Lefschetz number of a general differential operator on an orbifold.

Date: February 1, 2008.
orbifold $X$. It turns out that there are at least two in-equivalent definitions of a differential operator on an orbifold. One definition of a differential operator on $X$ from a vector bundle $E$ to $F$ is a linear map of sections $\Gamma^\infty(E) \to \Gamma^\infty(F)$ expressed by a finite combination of bundle endomorphisms and their covariant derivatives; the other definition is that a linear map $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$ such that there is an integer $N$ and for any smooth functions $f_0, \cdots, f_N$ on $X$, (holomorphic functions if one considers a holomorphic differential operator), the commutator

$$[f_N, [f_{N-1}, \cdots, [f_0, P] \cdots]] = 0.$$  

In the case of a manifold, these two definitions are equivalent. However, in the case of an orbifold, it is quite easy to check that the second definition contains the first one, but not vice versa. We provide in Remark 4.3 an explicit example which is a differential operator in the sense of the second definition but not the first on the orbifold $\mathbb{C}/\mathbb{Z}_2$. In the following, we call operators in the first definition geometric differential operators, and those in the second definition algebraic ones.

We establish a Lefschetz number formula for a geometric differential operator on a general (maybe non-reduced) complex orbifold. Let $E$ be an vector bundle on a compact orbifold $X$. We consider a global geometric differential operator $D$ on $X$, which is a global section of the sheaf of geometric holomorphic differential operators acting on sections $E$. Since $X$ is compact, we have that $H^i(X, E)$ is finite dimensional, and the following orbifold Lefschetz number is well defined,

$$D \mapsto L(D) = \sum (-1)^i \text{tr}(H^i(D)).$$

We prove in this paper an integral formula to compute $L(D)$, i.e.

$$L(D) = \int_{\bar{X}} \frac{1}{m_O} \frac{1}{2\pi i n_0 + l(O) + 1/2} \chi_0,\sigma(D),$$

where $\bar{X}$ is the inertia orbifold associated to the orbifold $X$, and $l(O)$ is a local constant, the real codimension of $\bar{X}$ in $X$, and $m_O$ is also a local constant, the number of isotopy, and $\chi_0,\sigma(D)$ is a top differential form on $\bar{X}$. The proof of this result is an extension of the proof of the $G$-equivariant Lefschetz number formula. When $D$ is the identity operator, then the above theorem together with the local Riemann-Roch-Hirzebruch theorem in [PPT] computes the holomorphic Euler characteristic of $E$ on $X$, which was computed in [Dü].

We remark that the $G$ Lefschetz formula proved in this paper probably can be generalized and applied to study toric varieties. For a compact toric manifold, the canonical torus action does not always have finite isotopy groups but only has finitely many isolated fixed points and each fixed point comes with a nice coordinate system. Musson [MU] provided a description of the algebra of differential operators on toric varieties. We plan to study the Lefschetz number formula and its application on toric varieties in the future.

This paper is designed as follows. In Section 2, we briefly review the $\gamma$-twisted Hochschild (co)homology of differential operators and some formal differential geometry on an orbifold. In Section 3, we prove the Lefschetz number formula for $L(\gamma, D)$ for $G$ acting on a compact manifold $M$. In Section 4, we compute the $G$-average and orbifold Lefschetz number formula. And we end this paper with a remark on differential operators on orbifolds and an open question for future research.

**Acknowledgments:** We would like to thank the organizers of the trimester on “Groupoids and stacks in physics and geometry” to hosting their visits of Institut Henri Poincaré. This work has been partially supported by the MISGAM programme of the European Science Foundation. The first author was partially supported by the European Union through the FP6 Marie Curie RTN ENIGMA (Contract number MRTN-CT-2004-5652) and the Swiss National Science Foundation (grant 200020-105450). The second author was partially supported by the U.S. National Science Foundation (grant 0703775).
2. Hochschild cohomology and formal geometry

In this section, we review some known results about Hochschild (co)homology of the algebra of differential operators on an orbifold. In this paper, we assume that \( G \) is a compact group acting on a complex manifold \( M \) such that the isotopy group at each point of \( M \) is finite.

Let \( \gamma \) be an element of \( G \). We study the geometry near a fixed point \( x \) of \( \gamma \). By Bochner’s theorem [Bo], nearby \( x \), there is a \( \gamma \) invariant coordinate neighborhood \( U \) such that the \( \gamma \) action is linear. And because of the above assumption of finite isotopy, \( \gamma \) acts on \( U \) of finite order. This inspires the following consideration.

Let \( \gamma \) be an element of a compact group acting on an \( n \)-dim complex vector space. We consider \( \mathcal{O}_n = \mathbb{C}[[y_1, \ldots, y_n]] \) and \( \mathcal{D} = \mathcal{O}_n[\partial y_1, \ldots, \partial y_n] \) the algebra of formal power series in \( n \) variables and formal differential operators, and \( \mathcal{O}^{\text{pol}}_n = \mathbb{C}[y_1, \ldots, y_n] \) and \( \mathcal{D}^{\text{pol}} = \mathcal{O}^{\text{pol}}_n[\partial y_1, \ldots, \partial y_n] \) their subalgebras of polynomial functions and differential operators. The \( \gamma \)-twisted Hochschild homology \( \mathcal{H}_*(\mathcal{D}^{\text{pol}}, \mathcal{D}_\gamma^{\text{pol}}) \) of \( \mathcal{D}^{\text{pol}} \) is computed to be \( \big[\text{ALFALASQ}\big] \)

\[
\begin{cases}
C \cdot \gamma = 2n - l(\gamma), \\
0 \quad \text{else},
\end{cases}
\]

where \( l(\gamma) \) is the real codimension of the \( \gamma \)-fixed point subspace \( V^\gamma \). And by \( \mathcal{D}_\gamma^{\text{pol}} \), we mean the vector space \( \mathcal{D}^{\text{pol}} \) with the following bimodule structure

\[
f \cdot \xi = f \circ \xi, \quad \xi \cdot f = \xi \circ \gamma^{-1}(f).
\]

We decompose \( \mathbb{C}^n \) into \( V^\gamma \perp V^\perp \) where \( V^\perp \) is the \( \gamma \) invariant subspace of \( \mathbb{C}^n \) complement to \( V^\gamma \). Let \( y^\gamma_1, \ldots, y^\gamma_{l(\gamma)/2} \) be a basis of \( V^\gamma \). Then the following expression defines a generator for \( H_{2n-l(\gamma)}(D, D_\gamma) \),

\[
c_{2n-l(\gamma)}^\gamma = \sum_{\epsilon \in \mathcal{S}_{2n-l(\gamma)}} 1 \otimes u_{\epsilon(1)} \otimes \cdots \otimes u_{\epsilon(2n-l(\gamma))}, \quad u_{2i-1} = \partial y^\gamma_i, \quad u_{2i} = y^\gamma_i.
\]

An explicit Hochschild cocycle \( \tau_{2n-l(\gamma)}^\gamma \) on \( H^\bullet(D; D_\gamma^*) \) was constructed in \([PPT]\) as follows. Given any \( D \in \mathcal{D} \), we can decompose it in \( \sum_i D_i^\gamma \otimes D_i^\perp \) with \( D_i \) a differential operator on \( V^\gamma \) and \( D_i^\perp \) a differential operator on \( V^\perp \). Define

\[
\tau_{2n-l(\gamma)}^\gamma(D_0, \ldots, D_{2n-l(\gamma)}) = \sum_{i_0, \ldots, i_{2n-l(\gamma)}} \tau_{2n-l(\gamma)}(D_{0,i_0}^\gamma \otimes \cdots \otimes D_{2n-l(\gamma),i_{2n-l(\gamma)}}, D_{0,i_0}^\gamma \circ \cdots \circ D_{2n-l(\gamma),i_{2n-l(\gamma)}}),
\]

where we have written \( D_k = \sum_{i_k} D_{k,i_k}^\gamma \otimes D_{k,i_k}^\perp \), and \( \tau_{2n-l(\gamma)} \) is a cocycle in \( H^{2n-l(\gamma)}(D_{V^\gamma}, D_{V^\perp}) \) as is defined in \([PFESH]\), and \( \mathcal{H}_\tau \) is the \( \gamma \)-trace on \( D_{V^\perp} \) defined in \([FE00]\). We refer to \([PPT]\) [Section 3] for explicit formulas of the cocycles \( \tau_{2n-l(\gamma)} \) and \( \mathcal{H}_\tau \).

The above computation of Hochschild (co)homologies can be extended to matrix valued differential operators \( \mathcal{D}_E \) with \( E \) a finite dimensional representation of \( \gamma \). The cycle \( c_{E,2n-l(\gamma)}^\gamma \) and cocycle \( \tau_{E,2n-l(\gamma)}^\gamma \) can be defined to be cocycles on \( \mathcal{D}_E \).

We list some properties of \( \tau_{E,2n-l(\gamma)}^\gamma \), which were proved in \([PPT]\) [Section 3].

1. \( \tau_{E,2n-l(\gamma)}^\gamma \) is invariant under the action of \( (GL_n(\mathbb{C}) \times GL(E))^\gamma \), the subgroup of \( GL_n(\mathbb{C}) \times GL(E) \) consisting elements commuting with \( \gamma \);
2. \( \tau_{E,2n-l(\gamma)}^\gamma(c_{E,2n-l(\gamma)}^\gamma) = (\det(1 - \gamma^{-1}))^{-1} \text{tr}_E(\gamma) \), where we consider the determinant of \( 1 - \gamma^{-1} \) on \( V^\perp \).

Remark 2.1. From the above formula it is possible that if \( \text{tr}_E(\gamma) = 0 \), \( \tau_{E,2n-l(\gamma)}^\gamma(c_{E,2n-l(\gamma)}^\gamma) = 0 \). However, \( c_{E,2n-l(\gamma)}^\gamma \) of the same form as \( c_{2n-l(\gamma)}^\gamma \) is never 0, which makes the later arguments work.
The following theorem is a straight forward generalization of Brylinski-Getzler’s results to the \( \gamma \)-twisted case, whose proof is omitted.

**Theorem 2.2.** Let \( \gamma \) be an element of a compact group acting on \( M \) preserving its complex structure. At every \( \gamma \) fixed point \( x \), there is a \( \gamma \) invariant coordinate neighborhood \( U \) on which \( \gamma \) acts linearly. Let \( E \) be a \( \gamma \)-equivariant vector bundle on \( M \). On \( U \), \( H_\bullet(\mathcal{D}_E(U), \mathcal{D}_E(U)_\gamma) \) is spanned by \( c_{E,2n-\ell(\gamma)} \).

We consider the setting in Theorem 2.2 and a connected component of \( \gamma \) fixed point, which is denoted by \( M^\gamma \). Let \( E \) be a \( \gamma \)-equivariant complex vector bundle on \( M \), which induces a bundle \( E \rightarrow M^\gamma \). On \( M^\gamma \), there is a vector bundle \( N^\gamma \) which is the normal bundle associated to the embedding of \( M^\gamma \) in \( M \). We notice that \( \gamma \) acts on both \( N^\gamma \) and \( E \) fiberwisely. Let \( \mathcal{O}_{N,\gamma}(E) \) be a bundle over \( M^\gamma \) whose fiber at \( x \in M^\gamma \) is the algebra of \( \text{End}(E_x) \) valued holomorphic functions on \( N^\gamma_x \). We consider the sheaf of Lie algebra of \( \gamma \)-invariant vector fields, i.e.

\[
W_n^\gamma = \left( \oplus_i \mathcal{O}_n \partial_{\theta_i} \right)^\gamma, \quad \text{and} \quad W_n(E_x)^\gamma = \left( \left( \oplus_i \mathcal{O}_n \partial_{\theta_i} \right) \times \mathfrak{gl}_r(\mathcal{O}_n(E_x)) \right)^\gamma, \quad r = \text{rank}(E).
\]

The formal infinite jets \( J^\infty(\mathcal{O}_{N,\gamma}(E)) \) of \( \mathcal{O}_{N,\gamma}(E) \) is a principal \( W_n(E)^\gamma \)-space. Furthermore, there is a natural flat connection \( A_{E,\gamma} \) on \( J^\infty(\mathcal{O}_{N,\gamma}(E)) \) with

\[
dA_{E,\gamma} + \frac{1}{2} [A_{E,\gamma}, A_{E,\gamma}] = 0.
\]

The connection \( A_{E,\gamma} \) induces a flat connection on the associated bundle \( J_1(E) \times_G \mathcal{D}_{\mathcal{O}_{N,\gamma}(E)} \) over \( M^\gamma \), where \( G = (GL_{n-\ell(\gamma)/2} \times GL_r)^\gamma \), with \( r = \text{rank}(E) \). The flat sections of this bundle \( J_1(E) \times_G \mathcal{D}_{\mathcal{O}_{N,\gamma}(E)} \) are in one-to-one correspondence to the restriction to \( M^\gamma \) of global differential operators \( \mathcal{D}_E \) on \( M \). We denote this flat connection on \( J_1(E) \times_G \mathcal{D}_{\mathcal{O}_{N,\gamma}(E)} \) by \( \omega_{E,\gamma} \).

**Proposition 2.3.** Let \( \Omega^* \) be the \( l(\gamma) \)-shifted complex of sheaves of complex valued smooth differential forms on \( M^\gamma \) with the de Rham differential, and \( \mathcal{C}_{*,\gamma}(D) \) be the \( \gamma \)-twisted complex of Hochschild chains of \( D \). There is a homomorphism of complexes of sheaves on \( M^\gamma \)

\[
\chi_\gamma : \mathcal{C}_{*,\gamma}(D) \rightarrow \Omega^{2n-\bullet}(M^\gamma)[l(\gamma)].
\]

**Proof.** The proof of this proposition is a copy of the proof of Proposition 2.4.1 [ENFE]. We have the following sequence of maps

\[
\chi_{p,\gamma} : \mathcal{C}_{p,\gamma}(D) \rightarrow \Omega^{2n-\ell(\gamma)-p}(M^\gamma)
\]

by

\[
\chi_{p,\gamma}(D_0, \ldots, D_p) = (-1)^p \tau_{V,2n-\ell(\gamma)}(\mathfrak{sh}_{p,2n-\ell(\gamma)-p}(\hat{D}_0, \hat{D}_1, \ldots, \hat{D}_p, \omega_{E,\gamma}, \ldots, \omega_{E,\gamma})),
\]

where \( \mathfrak{sh}_{p,2n-\ell(\gamma)-p} \) is the sum over all \( (p, 2n-\ell(\gamma)-p) \) shuffles, and we have identified a differential operator \( D_i \) with a flat section \( \hat{D}_i \) of the previously considered bundle \( J_1(E) \times_G \mathcal{D}_{\mathcal{O}_{N,\gamma}(E)} \).

It is straightforward to check the following identities

\[
d \circ \chi_{p,\gamma} = \chi_{p-1,\gamma} \circ b^\gamma.
\]

And the proposition follows from this identity.

\[\square\]

We conclude this section by introducing two \( \gamma \)-traces on \( \mathcal{D}_E \). In the following section, we will prove that these two traces are actually equal.

1. The first trace \( \text{Tr}_{1,\gamma} : \mathcal{D}_E(M) \rightarrow \mathbb{C} \) is defined to be

\[
D \mapsto \sum_{j=0}^{n} (-1)^j \text{tr}_j(\gamma H^j(D)),
\]

where \( H^j(M, E) \) is the sheaf cohomology group with the induced \( \gamma \) action, and \( H^j(D) : \mathcal{D}_E(M) \rightarrow \text{End}(H^j(M, E)) \). We remark that as \( M \) is compact, \( H^j(M, E) \) is finite dimensional, and the number \( \text{Tr}_{1,\gamma} \) is a finite number.
(2) The second trace $\text{Tr}_{2, \gamma}$ is defined to be
\[
D \mapsto \int_{M^\gamma} \frac{1}{(2\pi i)^n} \chi_{0, \gamma}(D).
\]

We remark that in the above formula, if $M^\gamma$ has different components, we sum over all components. And because $M$ is compact, each component of $M^\gamma$ is compact and there are only finitely many components, and therefore the above integral is a finite number.

3. G-Lefschetz number

We prove in this section that $\text{Tr}_{1, \gamma} = \text{Tr}_{2, \gamma}$ by verifying that they are both equal to a third trace which will be introduced in this section. Many ideas of the proof in this section are originally from \textsc{Kasraei}. We adapt them to study the $\gamma$-twisted traces.

In this whole section, we assume that $\gamma$ is an element of a compact group $G$ acting on $M$ preserving the complex structure and the isotopy of $G$ action at each point of $M$ is finite.

3.1. The third trace. We introduce a third $\gamma$-twisted trace on $D_E(M)$ in this subsection, and will prove in the following subsections that it is equal to the first and second $\gamma$-twisted traces separately.

We choose $(U_i)$ a locally finite open cover of $M^\gamma$ and consider the sheaf $D_{O_{N, \gamma}(E)}$ of differential operators on the bundle $O_{N, \gamma}(E)$. We remark that flat global sections of this sheaf are one-to-one correspondent to restrictions of global differential operators $D_E$ to $M^\gamma$. We consider the $\check{C}$ech double complex $C^0(U, C_{-p}(D_E, D_E, \gamma))$, where $C^0(D_E, D_E, \gamma)$ is the complex of $\gamma$ twisted Hochschild chains on $D_{O_{N, \gamma}(E)}$.

Given any operator $D \in D_E(M)$, the restriction of $D$ to $M^\gamma$ defines a $(0, 0)$ cocycle in $C^0(U, C_0)$. By Theorem 2.2, $D|_{U_i}$ is a Hochschild boundary when restricted to a sufficiently small open set $U_i$. Therefore, there is a $(-1, 0)$-cochain $D^{(1)}$ in $C^{-1,0}$ such that $D|_{U_i} = b^\gamma D^{(1)}|_{U_i}$. Thereafter, we consider the $\check{C}$ech differential on $D^{(1)}$, i.e. $\delta(D^{(1)})|_{U_{ij}} = D^{(1)}|_{U_i} - D^{(1)}|_{U_j}$. As $\delta$ commutes with $b^\gamma$, $b^\gamma(\delta(D^{(1)})) = \delta(b^\gamma(D^{(1)})) = \delta(D) = 0$. This shows that $\delta(D^{(1)})$ is again a Hochschild cycle. And again by Theorem 2.2 we know that there is an element $D^{(2)}$ in $C^{-2,1}$ such that $b^\gamma D^{(2)} = \delta(D^{(1)})$. By repeating this induction step, we will have a sequence of cochains $D^{(j)} \in C^{-j,j-1}$, $j = 1, \ldots, 2n-l(\gamma)$ with

$$b^\gamma D^{(1)} = D, \quad \delta(D^{(j)}) = b^\gamma(D^{(j+1)}), \quad j = 1, \ldots, 2n-l(\gamma) - 1.$$ 

We need to keep in mind at the step $j = 2n-l(\gamma)$ the Hochschild homology is not trivial. Therefore, we have that

$$\delta(D^{(2n-l(\gamma))}) = s^\gamma + b^\gamma(D^{(2n+1-l(\gamma))}),$$

where $s^\gamma \in C^{2n-l(\gamma), -2n+l(\gamma)}$ is equal to

$$s^\gamma_{i_0, \ldots, i_{2n-l(\gamma)}}(D)c^\gamma_{E}(U_{i_0} \cap \cdots \cap U_{i_{2n-l(\gamma)})},$$

(3.1) for some $2n-l(\gamma)$ $\check{C}$ech cocycle $\lambda^\gamma(D)$ in $\check{C}^{2n-l(\gamma)}(U; \mathbb{C})$. Therefore, $\lambda^\gamma(D)$ is an element in $H^{2n-l(\gamma)}(M^\gamma)$. We define $\text{Tr}_{3, \gamma}(D)$ to be the Poincaré dual of $|\lambda^\gamma(D)|$, which is a number in $\mathbb{C}$.

3.2. Local expression of the trace density. We start by fixing a smooth and locally finite triangulation $|K^\gamma|$ to $M^\gamma$, with the underlying simplicial complex $K_0^\gamma$. We consider an open cover $(U_i)_{i \in K_0^\gamma}$ of $M^\gamma$, where $K_0^\gamma$ is the set of vertices of the triangulation, such that $U_i$ is the complement of the simplexes not containing the vertex $i$. Because of the construction, we see that the cover $(U_i)$ satisfies for any $i_1 < \cdots < i_p$

(1) $U_{i_0} \cap \cdots \cap U_{i_p}$ is contractible,

(2) If $p \gg 2n-l(\gamma)$, then $U_{i_0} \cap \cdots \cap U_{i_p}$ is empty.
We consider the cell decomposition $C_\bullet$ dual to the above triangulation $K^\gamma$. We denote $C_{i_0,\ldots,i_p}$ the $(2n-l(\gamma)-p)$-cells dual to the simplex $K_{i_0,\ldots,i_p}$. The orientation of $C_\bullet$ is set to require $C_{i_0,\ldots,i_p}, K_{i_0,\ldots,i_p}=1$.

We have the following analogous proposition as [ENFe Prop. 5.1].

**Proposition 3.1.** For any global differential operator $D$ on $M$. Let $s$ be the $(2n-l(\gamma),-2n+l(\gamma))$-Čech cocycle defined in Equation (3.4), and $\chi_{2n-l(\gamma)}$ be the map defined in Proposition 3.1. Then

$$\text{Tr}_{2,\gamma}(D) = (-1)^{n-l(\gamma)/2} \sum_{i_0<\cdots<i_{2n-l(\gamma)}} \int_{C_{i_0,\ldots,i_{2n-l(\gamma)}}} \chi_{2n-l(\gamma)}(s_{i_0,\ldots,i_{2n-l(\gamma)})}.$$  

**Proof.** The proof is same to the proof of Proposition 5.1, [ENFe]. □

With this proposition, we can compare $\text{Tr}_{2,\gamma}$ and $\text{Tr}_{3,\gamma}$ locally on each $C_{i_1,\ldots,i_{2n-l(\gamma)}}$ and verify directly that on each connected component

$$\text{Tr}_{2,\gamma} = (-1)^{n-l(\gamma)} \frac{\text{tr}_{E}(\gamma)}{\det(1-\gamma^{-1})} \sum_{i_0<\cdots<i_{2n-l(\gamma)}} \chi_{i_0,\ldots,i_{2n-l(\gamma)}}(D) = (-1)^{n-l(\gamma)} \frac{\text{tr}_{E}(\gamma)}{\det(1-\gamma^{-1})} \text{Tr}_{3,\gamma}(D).$$

**Remark 3.2.** Because $\gamma$ acts on the bundle $E|M$, and $N^\gamma|_M$ fiberwisely of finite order, the eigenvalues of $\gamma$ are discretely distributed. This implies that $\text{tr}_{E}(\gamma)$ and $\frac{\text{tr}_{E}(\gamma)}{\det(1-\gamma^{-1})}$ are both local constants on $M^\gamma$, and the above equation is well defined.

3.3. **Asymptotic pairing.** This subsection provides an important tool to prove in the next subsection that $\text{Tr}_{1,\gamma} = \text{Tr}_{3,\gamma}$.

We consider the Dolbeault complex $(\Omega^{0,\bullet}(M,E),\overline{\partial})$ with values in the holomorphic vector bundle $E$. Because of the properness assumption on $\gamma$ action, we can fix a $\gamma$-invariant hermitian metric on $T_M$ and $E$. Accordingly, we can consider the Hilbert space of $L^2$-integrable sections of $\Omega^{0,\bullet}(M,E)$. On this Hilbert space, there is a self-adjoint positive semi-definite operator $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, where $\overline{\partial}^*$ is the Hodge dual of $\overline{\partial}$.

Let $e^{-t\Delta_{\overline{\partial}}}$ be the heat operator with kernel $k_t$. According to [BEGEVE] Theorem 2.30, as $t \to 0$ the heat kernel has an asymptotic expansion,

$$k_t(z,z') \sim \frac{1}{(\pi t)^n} e^{-\frac{\text{dist}(z,z')^2}{t}} (\Phi_0(z,z') + t\Phi_1(z,z') + \cdots),$$  

where $\text{dist}(z,z')$ is the geodesic distance between $z$ and $z'$.

We have the following generalization of Proposition 6.1 in [ENFE].

**Proposition 3.3.** Let $U$ be a $\Gamma$-invariant open subset of $M$, $A = D_E(U)$, and $M_c = \Omega^{0,\bullet}_c(U) \otimes \mathcal{O}_M(U)$ $D_E(U)$. We consider the $k_t^N$ be the truncation at the $N$-th term of the asymptotic expansion (3.2) with support in a small neighborhood of the diagonal in $U \times U$. For $D_0 \in M^k_c = \Omega^{0,k}_c(U) \otimes \mathcal{O}_M(U)$ $D_E(U)$, $D_1, \ldots, D_k \in A$, the expression

$$\Psi^\gamma_k(D_0, \ldots, D_k) = (-1)^{\frac{k(k+1)}{2}} \int_{\Delta^k} \text{Str}(\gamma D_0 k^N_0 e^\overline{\partial}^* D_1 |k^N_1| \cdots |\overline{\partial}^* D_k |k^N_k| ds_1 \cdots ds_k)$$  

is independent of $N$ for $N \gg 1$ and defines a continuous cocycle

$$\Psi^\gamma = \sum_k \Psi^\gamma_k \in \bigoplus_{k=0}^n \text{Hom}(M^k_c \otimes A^k, \mathbb{C})[t^{-\frac{1}{2}}] = C^0(A, M^k_{c,\gamma})[t^{-\frac{1}{2}}].$$  

In the above formula for $\Psi^\gamma_k$, $\text{Str}$ is the super trace on $\wedge^* T^0,1 U \otimes E|_U$, and $[\cdots]_-$ takes the none positive $t$-power terms.

**Proof.** Once we notice that the appearance of $\gamma$ in $\Psi^\gamma$ leads to the twisted cocycle condition and the fact that $\overline{\partial}, \overline{\partial}^*, \Delta_{\overline{\partial}}$ are all $\gamma$-invariant, the proof of this proposition is a repeat of the proof of Proposition 6.1 in [ENFe]. □
In the following, we adapt Engel-Felder’s construction in [ENGFE] (Section 6) to the $\gamma$-twisted situation.

Given $\varphi_0, \cdots, \varphi_k \in C^\infty_c(U) \subset A$, we can view them as $0$-cochains in $C^*(A, M_c)$. Let $\delta$ be the differential on $C^*(A, M_c)$. We consider

$$Z^k = \varphi_0 \cup \delta \varphi_1 \cup \cdots \delta \varphi_k \in C^k(A, M_c),$$

where $\cup : C^p(A; M_c) \otimes C^q(A; M_c) \to C^{p+q}(A; M_c \otimes_A M_c) \equiv C^{p+q}(A; M_c)$ by $M_c \otimes_A M_c \to M_c$.

By $M_{c, \gamma}$, we mean the linear space $M_{c, \gamma}$, but the right action of $A$ is twisted by $\gamma$. We use the following cup product

$$\cup : C^*(A; M_{c, \gamma}) \otimes C^*(A; M_c) \to C^*(A; M_{c, \gamma} \otimes_A M_c) = C^*(A; A^*_c),$$

to construct an element $\sigma_k^\gamma(\varphi_0, \cdots, \varphi_k)$ in $C^*(A; A^*_c)$ by

$$\sigma_k^\gamma(\varphi_0, \cdots, \varphi_k) = \Psi^\gamma \cup Z^k(\varphi_0, \cdots, \varphi_k).$$

If $A = D_E(U)$, $B = C^\infty(U)$, the above $\sigma_k^\gamma$ defines a linear map

$$\sigma_k^\gamma : C_k(A; A_\gamma) \otimes C_k(B) \to C[t^{-\frac{1}{2}}].$$

We have the following properties of the map $\sigma_k^\gamma$.

**Proposition 3.4.**

1. $\sigma_k^\gamma$ vanishes on $(\varphi_0, \cdots, \varphi_k)$ with $\cap_{i=0}^k \text{supp}(\varphi_i) = \emptyset$;
2. If $\varphi = \varphi_0 \otimes \cdots \otimes \varphi_k$, $s(\varphi) = 1 \otimes \varphi_0 \otimes \cdots \otimes \varphi_k$, and $D \in C_{k+1}(A; A_\gamma)$, then $\sigma_k^\gamma(b^E D \otimes \varphi) = \sigma_{k+1}^\gamma(D \otimes s(\varphi))$ for $k \geq 0$;
3. $\sigma_k^\gamma(D, \varphi) = \left[ \sum_{j=0}^{2n-\ell(\gamma)} \text{tr}_{E_0} \left( \varphi \gamma D_{\ell(\gamma) - 1} \right) \right]_{\cdot \cdot \cdot}$;
4. If $\varphi_i, i = 1, \cdots, 2n - \ell(\gamma)$ are $\gamma$ invariant and constant along the normal directions of $U^\gamma$ within a tubular neighborhood of $U^\gamma$, then $\sigma_k^\gamma \left( \gamma^(-1)(c_E(U)); \varphi_0 \otimes \cdots \otimes \varphi_{2n-\ell(\gamma)} \right)$ is equal to

$$\frac{(-1)^{n-\ell(\gamma)/2} \text{tr}_{E}(\gamma)}{(2\pi i)^{n-\ell(\gamma)/2}} \int_{U^\gamma} \frac{1}{\det(1 - \gamma^{-1})} \varphi_0 d\varphi_1 \cdots d\varphi_{2n-\ell(\gamma)}.$$

**Proof.** We write the pairing $\sigma_k^\gamma$ in a more explicit way:

$$\sigma_k^\gamma(D_0, \cdots, D_k; \varphi_0, \cdots, \varphi_k) = \sum_{j=0}^{k} (-1)^{j(k-j)} \Psi_j^\gamma \left( \gamma^{-1}(Z_{k-j}(D_{j+1}, \cdots, D_k; \varphi_0, \cdots, \varphi_k)) D_0, \cdots, D_j \right),$$

where $Z_{k-j}(D_{j+1}, \cdots, D_k; \varphi_0, \cdots, \varphi_k)$ is equal to

$$\sum_{\pi \in S_{k-j}} \text{sign}(\pi) \varphi_0 B_{\pi(1)}(\varphi_1) \cdots B_{\pi(k)}(\varphi_k),$$

with $B_i(\varphi) = [D_{ji}, \varphi]$ for $i = 1, \cdots, k - j$, and $B_i(\varphi) = [\partial_i, \varphi]$, for $i = k - j + 1, \cdots, k$.

For (1), we see that if $\cap_{i=0}^k \text{supp}(\varphi_i) = \emptyset$, then $Z_j^\gamma(\varphi_0, \cdots, \varphi_k) = 0$ everywhere. This implies $\sigma_k^\gamma$ vanishes on $(\varphi_0, \cdots, \varphi_k)$.

For (2), we notice that $\delta Z^k(\varphi_0, \cdots, \varphi_k) = \delta \varphi_0 \cup \cdots \cup \delta \varphi_k = 1 \cup \delta \varphi_0 \cup \cdots \cup \delta \varphi_k = Z^{k+1}(1, \varphi_0, \cdots, \varphi_k)$. Hence, since $\Psi^\gamma$ is a $\gamma$-twisted cocycle, we have $b^E(\Psi^\gamma \cup Z(\varphi)) = \Psi^\gamma \cup \delta(Z(\varphi)) = \Psi^\gamma \cup Z(s(\varphi))$.

For (3), we can check it directly by definition using the above explicit formula [3.4] for $\sigma^\gamma$.

For (4), we recall that $c_{2n-\ell(\gamma)}^\gamma = 1 \otimes u$, where $u$ contains $(2n - \ell(\gamma))!$ terms with $\partial_{y'}$ or $y'$. We observe from Equation [3.4] and the definition of $Z_{j}^{2n-\ell(\gamma)}$ that if the multiplication operator by $y'$
appears in $Z_{n-j}^{2n-l(\gamma)}$, then $Z_{n-j}^{2n-l(\gamma)}$ vanishes because the commutator $[y^i, \phi] = 0$. This implies that the non-zero terms in $\sigma_{2n-l(\gamma)}$ contain only terms in the expression $\langle \mathcal{X} \rangle$ with $j \geq n - \frac{l(\gamma)}{2}$.

On the other hand, if $\alpha Z_U$ is compact, the function is supported in the following open set $U$. As $\dim C(U) = n - l(\gamma)/2$, inside this tubular neighborhood of $U$, $Z_{n-j}^{2n-l(\gamma)}$ cannot contain more than $n - l(\gamma)/2$ terms like $[\partial_i, \phi]$, because otherwise $Z_{n-j}^{2n-l(\gamma)}$ contains a product of more than $n - l(\gamma)/2$ terms of anti-holomorphic differential forms along $U$. This implies that when $j > n - l(\gamma)/2$, the term $Z_{n-j}^{2n-l(\gamma)}(...)$ is supported away from $U$. The support of $Z_{n-j}^{2n-l(\gamma)}(...)$ is compact, the function $\text{dist}(\gamma^{-1}(x), x)$ achieves its absolute minimum on $\text{supp}(Z_{n-j}^{2n-l(\gamma)}(...))$, which is strictly positive as $Z_{n-j}^{2n-l(\gamma)}(...)$ is supported away from $U$. We assume that this minimum to be $\alpha_0$. We prove that when $j > n - l(\gamma)/2$, the term $\Psi_j^\gamma(\gamma^{-1}Z_{n-j}^{2n-l(\gamma)}(...))$ is supported within the open set $dist(x, x_{i+1}) < \alpha_0/j + 1$. Therefore, the product

$$k_n^N(x_0, x_1) \prod_{i=1}^{j} [\partial_i^*, D_i] x_i k_n^N(x_i, x_{i+1})$$

is supported in the following open set

$$\text{dist}(\gamma^{-1}(x_0), x_1) < \alpha_0/j + 1, \text{dist}(x_i, x_{i+1}) < \alpha_0/j + 1, i = 1, \cdots, j - 1, \text{dist}(x_j, x_0) < \alpha_0/j + 1$$

which is a subset of the following open set

$$\text{dist}(\gamma^{-1}(x_0), x_1) < \text{dist}(\gamma^{-1}(x_0), x_1) + \text{dist}(x_1, x_2) + \cdots + \text{dist}(x_j, x_0) < \alpha_0.$$
are inside the closed set \( \text{dist}(\gamma^{-1}(x), x) \geq a_0 \). Therefore
\[
\sigma_{2n-l(\gamma)}^\gamma(c_{E}^Y(U); \varphi_0 \otimes \cdots \otimes \varphi_{2n-1}) = (-1)^{(n-l(\gamma))(n-l(\gamma)+1)/2} \sum_{\pi \in S_{2n-l(\gamma)/2}} \Psi_{n-l(\gamma)/2}^\gamma(\gamma B_{2n-l(\gamma)}) \gamma \pi(1), \cdots, \gamma \pi(n-l(\gamma)/2),
\]
where \( B_{2n-l(\gamma)} \) is a multiplication operator
\[
B_{2n-l(\gamma)} = \sum_{\pi \in S_{2n-l(\gamma)/2}} \text{sign}(\pi) \frac{\partial \varphi_0}{\partial y^1} \cdots \frac{\partial \varphi_{n-l(\gamma)/2}}{\partial y^{n-l(\gamma)/2}} \cdot \frac{\partial \varphi_{n-l(\gamma)/2+1}}{\partial y^{n-l(\gamma)/2+1}} \cdots \frac{\partial \varphi_{2n-l(\gamma)}}{\partial y^{2n-l(\gamma)}} dy^1 \wedge \cdots \wedge dy^{n-l(\gamma)/2}.
\]
We define \( B = \varphi_0 \frac{\partial \varphi_2}{\partial y^2} \cdots \frac{\partial \varphi_{n-l(\gamma)/2}}{\partial y^{n-l(\gamma)/2}} \frac{\partial \varphi_{n-l(\gamma)/2+1}}{\partial y^{n-l(\gamma)/2+1}} \cdots \frac{\partial \varphi_{2n-l(\gamma)}}{\partial y^{2n-l(\gamma)}} \). And as all \( \varphi_i \) and \( y^i \) are \( \gamma \) invariant, \( B \) is \( \gamma \) invariant. We look at the expression
\[
\Psi_{n-l(\gamma)/2}(B, y^1, \cdots, y^{n-l(\gamma)/2}) = \text{sign}(\gamma) \int_{t \Delta_{n-l(\gamma)/2}} \text{Str}(\gamma B k_i^N) \frac{\partial}{\partial y^1} \cdots \frac{\partial}{\partial y^{n-l(\gamma)/2}} ds_1 \cdots ds_{n-l(\gamma)/2},
\]
where \( \text{sign}(\gamma) \) is equal to \((-1)^{(n-l(\gamma)/2)(n-l(\gamma)/2+1)/2} \). As \( B, [\partial \varphi, y^i] \) are all differential operators of order 0, it is sufficient to compute the leading term as \( t \to 0 \).

When \( t \to 0 \), we are reduced to a neighborhood of the origin of \( \mathbb{C}^n \) with the standard metric. And we have the following formulas for the operators in coordinates
\[
\partial = \sum dy^i \frac{\partial}{\partial y^i}, \quad \partial^* = -\sum \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^i}, \quad \Delta_\partial = -\sum_{j=1}^n \frac{\partial^2}{\partial y^i \partial y^i},
\]
where \( i \) is the subtraction. And the heat kernel is
\[
k_t(y, y') = \frac{1}{(4\pi t)^n} e^{-|y-y'|^2}.
\]
Furthermore, we notice \([\partial^*, y^i] = t \partial / \partial y^i\), which commutes with the heat kernel. And the expression of \( \Psi_{n-l(\gamma)/2}(B, y^1, \cdots, y^{n-l(\gamma)/2}) \) is simplified to
\[
= \text{sign}(\gamma) \int_{t \Delta_{n-l(\gamma)/2}} \text{Str}(\gamma B k_i^N) ds_1 \cdots ds_{n-l(\gamma)/2}
= \text{sign}(\gamma) \text{Str}(\gamma B k_i^N) \int_{t \Delta_{n-l(\gamma)/2}} ds_1 \cdots ds_{n-l(\gamma)/2}
= \frac{(-1)^{n-l(\gamma)/2} t^{n-l(\gamma)/2}}{(n-l(\gamma)/2)! \pi^{-n-l(\gamma)/2}} \text{sign}(\gamma) \text{Str}(\gamma B k_i^N),
\]
where \( \overline{B} \) is defined by \( B = \overline{B} dy^1 \wedge \cdots \wedge dy^{n-l(\gamma)/2} \), and the first numerical factor in the last expression is the volume of \( t \Delta_{n-l(\gamma)/2} \).

By [BEFGIEVE][Theorem 6.11], if \( \text{supp}(\varphi_0) \cap \cdots \cap \text{supp}(\varphi_{2n-l(\gamma)}) \) does not contain any \( \gamma \) fixed point, then \( \text{Str}(\gamma B k_i^N) \) converges to 0 as \( t \to 0 \), and when \( \text{supp}(\varphi_0) \cap \cdots \cap \text{supp}(\varphi_{2n-l(\gamma)}) \) contains \( \gamma \) fixed points, then as \( t \to 0 \)
\[
\text{Str}(\gamma B k_i^N) \to \frac{\text{tr}_E(\gamma)}{t^{n-l(\gamma)/2}} \int_{M^\gamma} \frac{1}{\det(1 - \gamma^{-1})} \overline{B} dy^1 \cdots dy^{n-l(\gamma)/2}.
\]
Finally, as we notice that different order of $y^i$ does not change the limit, we have the conclusion
\[
\sigma^2_{2n-(l(\gamma))}(\mathcal{C}_k(U); \varphi_0 \otimes \cdots \otimes \varphi_{2n-(l(\gamma))}) = (-1)^{(n-l(\gamma)/2)(n-l(\gamma)/2+1)/2} \sum_{\pi \in S_n-l(\gamma)/2} \Psi^{l(\gamma)/2}(\gamma B_{2n-l(\gamma)}; y^{l(1)}, \cdots, y^{n-l(\gamma)})
\]
\[
= (n-l(\gamma)/2)! \Psi^{l(\gamma)/2}(B, y^1, \cdots, y^{n-l(\gamma)/2})
\]
\[
\frac{(-1)^{(n-l(\gamma)/2)2} \operatorname{tr}_E(\gamma)}{(2\pi)^{n-l(\gamma)/2}} \int_{M^\gamma} \frac{1}{\det(1-\gamma)} \varphi_0 d\varphi_1 \cdots d\varphi_{2n-l(\gamma)}.
\]

In the above Proposition 3.4 (4), we proved that $j$ can not be strictly greater than $n-l(\gamma)$. The same arguments also prove the following corollary.

**Corollary 3.5.** If $\text{supp}(\varphi_0) \cap \cdots \cap \text{supp}(\varphi_k) \cap U^\gamma = \emptyset$, then for any $D_0, \cdots, D_k \in A$

\[
\sigma^2_k(D_0, \cdots, D_k; \varphi_0 \otimes \cdots \otimes \varphi_k) = 0.
\]

3.4. **Local expression of $\gamma$-Lefschetz number.** In this subsection, we will use the results developed in the previous subsection to prove that the first $\gamma$-twisted trace is equal to the third one.

We first observe that as $\bar{\partial}$ is $\gamma$-invariant, the same argument as [LNEE][Proposition 4.1] proves that

\[
\sum_{i=0}^{n} (-1)^i \operatorname{tr}_{\Omega^{(0,i)}(M,E)}(\gamma D e^{-t \Delta_\delta})
\]

is independent of $t$ and is equal to $\operatorname{Tr}_{1,\gamma}$.

Let $(\hat{U}_i)$ be an open cover of $M$ chosen as follows. We start with the open cover $(U_i)$ of $M^\gamma$ as is chosen in Section 3.2. Fix $\epsilon_0 > 0$. Let $T^\gamma$ be a $2\epsilon_0$ tubular neighborhood of $M^\gamma$ in $M$ with the map $\pi : T^\gamma \rightarrow M^\gamma$. Define $\hat{U}_i = \pi^{-1}(U_i)$. $(\hat{U}_i)$ forms a cover of the tubular neighborhood $T^\gamma$. Then we extend $(\hat{U}_i)$ to a open cover of $M$ by requiring that the extra open sets will not intersect with the $\frac{\epsilon_0}{2}$ neighborhood of $M^\gamma$. We choose $(\varphi_i)$ to be a partition of unity subordinate to the cover $(\hat{U}_i)$ such that $(\varphi_i)$ restricts to become a partition of unity on $M^\gamma$ subordinate to the cover $(U_i)$. Furthermore, by choosing a proper cut-off function, we can require that $\varphi_i$ to be $\gamma$ invariant and constant along the direction orthogonal to $M^\gamma$ within the $\epsilon_0$ neighborhood of $M^\gamma$ if $M^\gamma \cap \text{supp}(\varphi_i) \neq \emptyset$. And we require $(\hat{U}_i)$ to have the following properties for $i_0 < \cdots < i_k$,

1. $U_{i_0} \cap \cdots \cap U_{i_k}$ is either 0 or contractible;
2. If $k > 0$, then $U_{i_0} \cap \cdots \cap U_{i_k}$ is empty.

We have the following localization property about $\operatorname{Tr}_{1,\gamma}(D)$ for a differential operator $D$.

**Proposition 3.6.** Let $D$ be a differential operator on $M$, which is not necessarily holomorphic. If the support of $D$ is away from $M^\gamma$, then

\[
\sum_{i} (-1)^i \left[\operatorname{tr}_{\Omega^{(0,i)}(M,E)}(\gamma D e^{-t \Delta_\delta})\right]_i = 0.
\]

**Proof.** According to [BEGEV][Proposition 2.46], the kernel $p_t(x, y)$ of $D e^{-t \Delta_\delta}$ has the following asymptotic expansion as $t \to 0$

\[
\|p_t(x, y) - h_t(x, y) \sum_{i=-m}^{N} t^i \Psi_i(x, y)\| = O(t^{N-n-m}),
\]

where $h_t(x, y) = (4\pi t)^{-n} \exp(-\text{dist}(x, y)^2/4t)\Psi(d(x, y)^2)$, and $\Psi$ is a cut-off function, and $m$ is the order of the operator $D$. 

We observe that as \(D\) is supported away from \(M^\gamma\), \(\Psi_i(x,y)\)'s support is away from \(M^\gamma \times M \subset M \times M\) for any \(i\).

Now the \(\gamma\) trace of \(De^{-t\Delta_\delta}\) is computed by

\[
\int_M \text{tr}(\gamma_x p_t(\gamma^{-1}(x),x))dx
= \int_M \text{tr}(\gamma_x)\psi_t(\gamma^{-1}(x),x) \sum_{i=-m}^\infty t^i \psi_i(\gamma^{-1}(x),x)dx. \tag{3.6}
\]

As \(\Psi_i(x,y)\)'s support is away from \(M^\gamma \times M\), \(\Psi_i(\gamma^{-1}(x),y)\)'s support is also away from \(M^\gamma \times M\), and therefore \(\psi_i(\gamma^{-1}(x),x)\)'s support is away from \(M^\gamma\).

As \(M\) is compact, we know that for each \(i\), the support of \(\psi_t(\gamma^{-1}(x),y)\psi_i(\gamma^{-1}(x),y)\) is a compact subset of \(M \times M\), and therefore the support of \(\psi_t(\gamma^{-1}(x),y)\psi_i(\gamma^{-1}(x),x)\) is a compact set of \(M\).

Accordingly on the support of \(\psi_t(\gamma^{-1}(x),y)\psi_i(\gamma^{-1}(x),x)\), the function \(\text{dist}(\gamma^{-1}(x), x)\) reaches its absolute minimum. As the support of \(\psi_i(\gamma^{-1}(x),x)\) is away from \(M^\gamma\), we know that there is a positive number \(\epsilon\) such that the support of \(\psi_t(\gamma^{-1}(x),y)\psi_i(\gamma^{-1}(x),x), \text{min}(\text{dist}(\gamma^{-1}(x),x)) = \epsilon > 0\).

On the other hand, we know that the support of the heat kernel \(\psi_t(x,y)\) can be chosen to be arbitrarily close to the diagonal in \(M \times M\). This forces the function \(\psi_t(\gamma^{-1}(x),x)\) to be supported in the neighborhood \(\text{dist}(\gamma^{-1}(x), x) < \epsilon\). Considering the above arguments which show that the support of \(\psi_t(\gamma^{-1}(x),y)\psi_i(\gamma^{-1}(x),x)\) is outside the open set \(\text{dist}(\gamma^{-1}(x), x) < \epsilon\) for any \(i\), we conclude that the function \(\psi_t(\gamma^{-1}(x),y)\psi_i(\gamma^{-1}(x),x)\) has to vanish for any \(i\). Therefore, we conclude that \(\text{Tr}_{1,\gamma}(D) = 0\). \qed

**Proposition 3.7.** The two \(\gamma\)-twisted traces are same, \(\text{Tr}_{1,\gamma} = \text{Tr}_{3,\gamma}\).

*Proof.* We compute \(\text{Tr}_{1,\gamma}\) using the following formula

\[
\sum_{j=1}^n (-1)^j [\text{tr}_{\Omega(i,j)}(M,E)(\gamma De^{-t\Delta_\delta})]_-. 
\]

By Proposition 3.6, we can use a cut-off function to localize \(D\) to be supported within the \(\epsilon_0\) neighborhood of \(M^\gamma\) without changing the value of the above \(\gamma\)-twisted trace.

By the partition of unity \((\varphi_i)\), we can express the above \(\gamma\)-twisted trace by

\[
\sum_i \sum_{j=1}^n (-1)^j [\text{tr}_{\Omega(i,j)}(M,E)(\gamma \varphi_i De^{-t\Delta_\delta})]_.
\]

By the assumption that \(D\) is supported with the \(\epsilon_0\) neighborhood of \(M^\gamma\), all \(\varphi_i\)'s having non-trivial contributions in the above sum are from pullbacks of a partition of unity of \(M^\gamma\).

Using the definition of the pairing \(\sigma_0^\gamma\), we have that

\[
\sum_{j=1}^n (-1)^j [\text{tr}_{\Omega(i,j)}(M,E)(\gamma \varphi_i De^{-t\Delta_\delta})]_-= \sigma_0^\gamma(D_i;\varphi_i), \quad D_i = D|_{\tilde{U}_i} \in \mathcal{D}_E(\tilde{U}_i).
\]
As the twisted Hochschild homology of $D_E$ has 0 cohomology in degree 0, we have that $D_i = b^i D_i^{(1)}$. Then by Proposition 3.3 we have that
\[
\text{Tr}_{1,\gamma} = \sum_i \sigma_i^\gamma (b^i D_i^{(1)}; \varphi_i)
\]
Use Proposition 3.4 (ii)
\[
= \sum_i \sigma_i^\gamma (D_i^{(1)}; 1, \varphi_i)
\]
Use the partition of unity
\[
= \sum_j \sum_i \sigma_i^\gamma (D_i^{(1)}; \varphi_j, \varphi_i)
\]
\[
= \sum_{i \neq j} \sigma_i^\gamma (D_i^{(1)}; \varphi_j, \varphi_i) + \sum_j \sigma_j^\gamma (D_j^{(1)}; \varphi_j, \varphi_j)
\]
Write $\varphi_j = 1 - \sum_{i \neq j} \varphi_i$
\[
= \sum_{i \neq j} \sigma_i^\gamma (D_i^{(1)} - D_j^{(1)}; \varphi_j, \varphi_i)
\]
\[
= \sum_{i \neq j} \sigma_i^\gamma (\delta D_{j,i}; \varphi_j, \varphi_i).
\]
According to Corollary 3.5, we know that if $\text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)$ is away from $M^\gamma$, then the pairing $\sigma_i^\gamma (\delta D_{j,i}; \varphi_j, \varphi_i) = 0$. Therefore, by the assumption on supports of the partition of unity on $M$, we conclude that in the above sum, all the nontrivial contributions are from $\varphi_i, \varphi_j$ which are pullbacks of a partition of unity on $M^\gamma$.

We can repeat this computation by induction and have the following equality,
\[
\text{Tr}_{1,\gamma}(D) = \sum_{i_0 < \cdots < i_{2n-1-\ell(\gamma)}} \sigma_i^\gamma_{2n-\ell(\gamma)} (s_{i_0, \ldots, i_{2n-1-\ell(\gamma)}}; \varphi_{i_0, \ldots, i_{2n-1-\ell(\gamma)}})
\]
\[
+ \sum_{i_0 < \cdots < i_{2n-1-\ell(\gamma)+1}} \sigma_i^\gamma_{2n+1-\ell(\gamma)} (\delta D_{i_0, \ldots, i_{2n+1-\ell(\gamma)}}; \varphi_{i_0, \ldots, i_{2n+1-\ell(\gamma)}}),
\]
where $\varphi_{i_0, \ldots, i_{2n-1-\ell(\gamma)}} = \sum_{\pi \in S_{2n-1-\ell(\gamma)+1}} \text{sign}(\pi) \varphi_{i,0} \otimes \cdots \otimes \varphi_{i,2n-1-\ell(\gamma)}$, and $\varphi_{i_0, \ldots, i_{2n-1-\ell(\gamma)}}$ are from pullbacks of a partition of unity on $M^\gamma$.

For the second term in the above equation since there is no further nontrivial Hochschild cycles, we can continue the induction step and have for $k \geq 2n + 1 - \ell(\gamma)$
\[
\sum_{i_0 < \cdots < i_{2n-1-\ell(\gamma)+1}} \sigma_i^\gamma_{2n+1-\ell(\gamma)} (\delta D_{i_0, \ldots, i_{2n+1-\ell(\gamma)}}; \varphi_{i_0, \ldots, i_{2n+1-\ell(\gamma)}}) = \sum_{i_0 < \cdots < i_k} \sigma_i^\gamma (\delta D_{i_0, \ldots, i_k}; \varphi_{i_0, \ldots, i_k}).
\]
When $k$ is large enough, we know that $\bigcap_{k=0}^k \text{supp}(\varphi_i) = \emptyset$ and by Proposition 3.3 (i), these terms vanish.

Hence, we have that
\[
\text{Tr}_{1,\gamma}(D) = \sum_{i_0 < \cdots < i_{2n-\ell(\gamma)}} \sigma_i^\gamma_{2n-\ell(\gamma)} (s_{i_0, \ldots, i_{2n-\ell(\gamma)}}; \varphi_{i_0, \ldots, i_{2n-\ell(\gamma)}}),
\]
As all $\varphi_{i_k}$ in the above sum are from pullbacks of a partition of unity on $M^\gamma$, according to our assumption at the beginning of this subsection, all these $\varphi_{i_k}$s are $\gamma$ invariant and constant along the normal direction to $M^\gamma$ within $\epsilon_0$ distance. We can apply Proposition 3.3 (iv) to evaluate
\[ \mathcal{O}_{2n-l(\gamma)}^\gamma(\cdot \cdot \cdot) \text{, and have} \]
\[ \text{Tr}_{1, \gamma}(D) = (2n+1-l(\gamma))(\pi E(\gamma)(-1)^{n-l(\gamma)}/2) \sum_{i_0 < \cdots < i_{2n-l(\gamma)}} \lambda_{i_0, \cdots, i_{2n-l(\gamma)}(D)} \int_{M^\gamma} \varphi_{i_0} d\varphi_{i_1} \cdots d\varphi_{i_{2n-l(\gamma)}}. \]

Since the restriction of \((\varphi_i)\) to \(M^\gamma\) forms a partition of unity subordinate to \((U_i)\), we can evaluate the integral
\[ \int_{\Delta_{2n-l(\gamma)}} \varphi_0 d\varphi_1 \cdots d\varphi_{2n-l(\gamma)} = \frac{1}{(2n-l(\gamma)+1)!}. \]

Hence we have that
\[ \text{Tr}_{1, \gamma} = \frac{\text{tr}_E(\gamma)(-1)^{n-l(\gamma)}/2}{(2\pi i)^{n-l(\gamma)/2}} \sum_{i_0 < \cdots < i_{2n-l(\gamma)}} \lambda_{i_0, \cdots, i_{2n-l(\gamma)}(D)} \text{sign}(i_0, \cdots, i_{2n-l(\gamma)}) \]
\[ = \frac{\text{tr}_E(\gamma)(-1)^{n-l(\gamma)}/2}{(2\pi i)^{n-l(\gamma)/2}} \text{Tr}_{3, \gamma}. \]

In summary, we have proved the following formula for the \(\gamma\)-twisted Lefschetz number.

**Theorem 3.8.** Let \(M\) be a compact complex manifold, and \(\gamma\) be an element of a compact group acting on \(M\) preserving the complex structure, and \(E\) be a \(\gamma\)-equivariant complex vector bundle on \(M\), and \(D\) be a differential operator acting on \(E\). Then
\[ \sum_i (-1)^i \text{tr}(\gamma H^i(D)) = \int_{M^\gamma} \frac{1}{(2\pi i)^{n-l(\gamma)/2}} \chi_{0, \gamma}(D), \]
where \(\chi_{0, \gamma}\) is as defined in Section 2.

4. Orbifold Lefschetz number

Let \(G\) be a compact group acting on a complex manifold \(M\) with finite isotopy subgroups. Assume that the quotient \(X = M/G\) to be compact. In this section, we want to provide a \(G\)-equivariant Lefschetz number formula for \(G\)-invariant differential operator \(D\) acting on a \(G\)-equivariant vector bundle \(E\). As \(G\) acts on the sheaf cohomology \(H^j(M, E)\), we denote \(H^j_G(M, E)\) to be the subspace of \(G\)-fixed points in \(H^j(M, E)\). Let \(p\) be the projection from \(M \to X\). We consider the pushforward vector bundle \(p_*(E)\) on \(X\). \(H^j_G(M, E)\) can be identified with \(H^j(X, p_*(E))\). Because \(D\) is \(G\)-invariant, \(D\) acts on \(H^j_G(M, E)\). Furthermore, as \(X\) is compact, we know that \(H^j_G(M, E)\) is finite dimensional. Therefore we can define the \(G\)-equivariant Lefschetz number of \(D\) to be
\[ \sum_i (-1)^i \text{tr}(H^i_G(D)). \]

We will use the geometric data on the orbifold to compute the Lefschetz number. Therefore, we recall some differential geometry on a complex orbifold \(X\). Given an orbifold \(X\), we have a naturally associated orbifold \(\hat{X}\), which is usually called the inertia orbifold for \(X\). Let us define this inertia orbifold locally. In a sufficiently small open subset \(U\) of \(X\), we can represent \(X\) by a global quotient \(V/\Gamma\), where \(V\) is an open subset of \(C^n\) and \(\Gamma\) is a finite group acting on \(V\) linearly. Accordingly, we introduce the following stratified space \(\hat{U}\)
\[ \prod_{(\gamma) \subset \Gamma} V^\gamma/C(\gamma), \]
where \((\gamma)\) stands for the conjugacy class of \(\gamma\), and \(C(\gamma)\) is the centralizer group of \(\gamma\) in \(\Gamma\), and \(V^\gamma\) is the \(\gamma\) fixed point subspace of \(V\). The stratified charts \(\hat{U}\) glue together to become a stratified complex orbifold, which is usually denoted by \(\hat{X}\). We look at the \(\chi_{0, \gamma}(D)\) as defined in Section 2. The pushforward of the collection \(\chi_{0, \gamma}\) for all \(\gamma\) from \(V^\gamma\) to \(V^\gamma/C(\gamma)\) forms a section of top forms.
on the inertia orbifold $\tilde{X}$. We remark that because the quotient map $V^\gamma \to V^\gamma/C(\gamma)$ is a proper locally embedding map, the pushforward map is well defined.

Let us first consider the situation that $G$ is a finite group.

**Theorem 4.1.** Let $G$ be a finite group, and $\chi_{0,\gamma}(D)$ be the form defined in Section 2. We have

$$\sum_i (-1)^i \text{tr}(H^i_G(D)) = \sum_{\gamma \in G} \frac{1}{|G|} \int_{M^\gamma/C(\gamma)} \frac{1}{m_\gamma} \frac{1}{(2\pi i)^{n-\ell(\gamma)/2}} \chi_{0,\gamma}(D),$$

where $m_\gamma$ is the order of $\gamma$, a local constant number on $M^\gamma$, and $\ell(\gamma)$ runs over all conjugacy classes of $G$.

**Proof.** We consider the $G$ averaging operator $P_G = \frac{1}{|G|} \sum_{\gamma \in G} \gamma$ acting on $H^i(X, E)$, where $|G|$ is the size of $G$. It is straightforward to check that $P_G$ is a projection from $H^i(X, E)$ to $H^i_G(X, E)$. Therefore, we have

$$\text{tr}(H^i_G(D)) = \text{tr}(P_G H^i(D) P_G) = \text{tr}(P_G H^i(D)) = \sum_{\gamma \in G} \frac{1}{|G|} \text{tr}(\gamma H^i(D)).$$

And applying Theorem 3.8 we obtain the following equality for the Lefschetz number

$$\sum_i (-1)^i \text{tr}(H^i_G(D)) = \sum_{\gamma \in G} \frac{1}{|G|} \sum_i (-1)^i \text{tr}(\gamma H^i(D)) = \sum_{\gamma \in G} \frac{1}{|G|} \int_{M^\gamma} \frac{1}{(2\pi i)^{n-\ell(\gamma)/2}} \chi_{0,\gamma}(D) = \sum_{\gamma \in G} \frac{1}{|G|} \sum_{\alpha \in [(\gamma)]} \int_{M^\alpha} \frac{1}{(2\pi i)^{n-\ell(\alpha)/2}} \chi_{0,\alpha}(D).$$

We notice that for different $\alpha, \alpha'$ in the same conjugacy class of $G$, $M^\alpha$ is diffeomorphic to $M^{\alpha'}$, $l(\alpha) = l(\alpha')$ and $\chi_{0,\alpha}(D) = \chi_{0,\alpha'}(D)$ as $D$ and $\chi_0$ are both $G$-invariant. Let $[(\gamma)]$ and $|G|$ be the sizes of $(\gamma)$ and $G$. We continue the above computation

$$\sum_i (-1)^i \text{tr}(H^i_G(D)) = \sum_{\gamma \in G} \int_{M^\gamma} \frac{|[(\gamma)]|}{|G|} \frac{1}{(2\pi i)^{n-\ell(\gamma)/2}} \chi_{0,\gamma}(D) = \sum_{\gamma \in G} \int_{M^\gamma} \frac{1}{C(\gamma)} \frac{1}{(2\pi i)^{n-\ell(\gamma)/2}} \chi_{0,\gamma}(D) = \sum_{\gamma \in G} \int_{M^\gamma/C(\gamma)} \frac{1}{m_\gamma} \frac{1}{(2\pi i)^{n-\ell(\gamma)/2}} \chi_{0,\gamma}(D),$$

where in the last line of the above computation, we have used the definition of integration over an orbifold.

In the last part of this section we consider a compact complex orbifold $X$. Let $O_X$ be the sheaf of holomorphic functions on $X$. By the sheaf of geometric holomorphic differential operator on $X$, we mean the module of $O_X$ generated by sections of the sheaf of holomorphic vector fields on $X$. A global geometric differential operator $D$ on $X$ is a section of the sheaf of geometric holomorphic differential operators on $X$. Let $E$ be an orbifold vector bundle on $X$. We can define geometric differential operators on $E$ in the same fashion. As $X$ is compact, the generalized Lefschetz formula is defined as same as on a manifold to be

$$\sum_i (-1)^i \text{tr}(H^i(D)).$$
On the other hand, locally an orbifold is like a quotient of \( \mathbb{C}^n \) by a finite linear group action. We can apply the construction of \( \chi_0 \) as in Theorem 3.1 to \( D \) and \( \chi_0(D) \) is a differential form on the inertia \( \tilde{X} \).

**Theorem 4.2.** Let \( D \) be a global geometric differential operator acting on a vector bundle \( E \) of a compact complex orbifold \( X \). Then

\[
\sum_i (-1)^i \text{tr}(H^i(D)) = \int_{\tilde{X}} \frac{1}{m_\mathcal{O}} \frac{1}{(2\pi i)^{n-\ell(\mathcal{O})/2}} \chi_{0,\mathcal{O}}(D),
\]

where \( m_\mathcal{O} \) is a local constant on \( \tilde{X} \) telling the size of isotopy, \( \ell(\mathcal{O}) \) is a local constant telling the codimension of \( \mathcal{O} \) inside \( X \), and \( \chi_{0,\mathcal{O}} \) is a top degree differential form on \( \tilde{X} \).

**Proof.** We start with the following local result from [ALFALASG]. For any \( x \in X \), there is a small enough neighborhood \( U \) which is holomorphic to the quotient of a complex open set \( V \) by a finite group \( \Gamma \) linear action. And the Hochschild (co)homology of \( \mathcal{D}_E(U) \) is spanned by the conjugacy classes of \( \Gamma \) with degree equal to the codimension of the fixed point subspace. A crucial observation to the proof of this result is that the \( \Gamma \)-invariant subalgebra of the Weyl algebra is Morita equivalent to the crossed product algebra of the Weyl algebra with \( \Gamma \). The generators of (co)homology are the sums of \( \gamma_{2n-\ell(\gamma)} \) and \( \tau_{2n-\ell(\gamma)} \) in the same conjugacy classes.

The proof of this theorem is essentially a copy of Section 3. We can choose a nice cover of \( X \) and the corresponding triangulation. The new ingredient is that we will have more than one contributions in the arguments of Proposition 3.1-3.7, and instead we will have one contribution for each conjugacy class. However, since we have 1-1 correspondence for each \( \gamma \) by Proposition 3.1-3.7, by taking the sum, we obtain the statement of this theorem.

\[\square\]

**Remark 4.3.** We remark that the class of geometric differential operators considered in Theorem 4.2 is quite restrictive. It excludes many interesting operators which should be viewed as differential operators in an algebraic way, where were called algebraic differential operators in Introduction.

For example, we consider the simplest complex orbifold \( \mathbb{C}/\mathbb{Z}_2 \), where \( \mathbb{Z}_2 \) acts on \( \mathbb{C} \) by \( z \mapsto -z \). The differential operators considered in Theorem 4.2 are all of the form \( \sum_i f_i \partial_z^i \), where \( f_i \) is an even polynomial when \( i \) is even, and an odd polynomial otherwise. These are \( \mathbb{Z}_2 \) invariant differential operators on \( \mathbb{C} \).

On the other hand, the algebra of polynomials on \( \mathbb{C} \) invariant under \( \mathbb{Z}_2 \) are even polynomials. The operator \( D = 4 \partial_z^2 \) acts on the space of even polynomials linearly satisfying the Leibniz rule

\[
\frac{1}{z} \partial_z (z^{2m}) = 2m z^{2m-2}, \quad \forall m \geq 0.
\]

But it is obvious that this operator cannot be written as \( \mathbb{Z}_2 \)-invariant differential operator on \( \mathbb{C} \). This operator is a \( \mathbb{Z}_2 \)-invariant meromorphic differential operator on \( \mathbb{C} \) which descends to a “differential operator” on \( \mathbb{C}/\mathbb{Z}_2 \). This is probably related to connections with logarithmic singularities [LA]. A Lefschetz formula for this type of operator is a future research project.

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