Fixed Points of $G$-CW-complex with Prescribed Homotopy Type

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Abstract

For a group $G$ of not prime power order, Oliver showed that the obstruction for a finite CW-complex $F$ to be the fixed point set of a contractible finite $G$-CW complex is the Euler characteristic $\chi(F)$. We show that the similar problem for $F$ to be fixed point set of a finite $G$-CW complex of some given homotopy type is still the Euler characteristic. We also give additional information on the obstruction in terms of the whole and the individual connected components of the fixed point set.

Let $G$ be a group. A $G$-map is a pseudo-equivalence if it is a homotopy equivalence without considering the group action. Given a $G$-map $f: F \to Y$, we ask whether it is possible to extend $F$ to a bigger $G$-space $X$, and extend

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f to a pseudo-equivalence $G$-map $g: X \to Y$. We call $g$ a pseudo-equivalence extension of $f$.

In this paper, we restrict our study to the following setting. The group $G$ is finite, and we assume the order of $G$ is not a prime power. All spaces are finite $G$-CW-complexes. Moreover, we only consider $F = X^G$ in the pseudo-equivalence extension. In other words, the extension from $F$ to $X$ is obtained by attaching non-fixed $G$-cells.

For the special case $Y$ is a point, the pseudo-equivalence extension becomes whether a given space $F$ can be the fixed point of a $G$-action on a contractible space $X$. The classical result of Oliver [6] says that, if the order of $G$ is not prime power, then the only obstruction is the Euler number.

**Theorem** (Oliver). Suppose $G$ is a group of not prime power order. Then there is an integer $n_G$, such that a finite CW-complex $F$ is the fixed set of a finite contractible $G$-CW-complex if and only if $\chi(F) = 1 \mod n_G$.

By [6, Theorem 5] and the subsequent corollary, we recall that $n_G = 1$ (i.e., there is no obstruction) if and only if $G$ is not of the form $P \triangleleft H \triangleleft G$, with $P$ and $G/H$ having prime power orders, and $H/P$ cyclic. We also recall that $n_G = 0$ (i.e., the obstruction is exactly the Euler number) if and only if $G$ has a normal subgroup $P$ of prime power order, such that $G/P$ is cyclic.

The assumption of not prime power order is due to the theory of Smith [9]. The theory says that, if $p$ is a prime and $\mathbb{Z}_p$ acts on contractible $X$, then the fixed set $X^\mathbb{Z}_p$ is $\mathbb{F}_p$-acyclic ($\mathbb{F}_p$ is the group $\mathbb{Z}_p$ regarded as a field). The theory can be applied to semi-free $G$-actions, for any prime factor $p$ of the order of $G$, and can also be applied to the case $G$ is a $p$-group. Lowell Jones [5] showed that the converse of Smith theory for semi-free actions by cyclic groups. In a companion paper [3], we study the pseudo-equivalence extension problem for the case that Smith theory plays a role, and a $K_0$-obstruction occurs.

Our main result extends Oliver’s theorem, where the Smith theory does not apply.

**Theorem.** Suppose $G$ is a group of not prime power order, $Y$ is a connected finite $G$-CW-complex, and the connected components of $Y^G$ are $Y^G_\alpha$, $\alpha \in A$. Then there is a subgroup $N_Y \subset \mathbb{Z}^A$, such that a map $f: F \to Y^G \subset Y$ from a finite CW-complex $F$ has a pseudo-equivalence extension $g: X \to Y$, with $X^G = F$, if and only if

$$(\chi(F_\alpha) - \chi(Y^G_\alpha))_{\alpha \in A} \in N_Y, \quad F_\alpha = f^{-1}(Y^G_\alpha).$$
Moreover, we have
\[ n_G Z^A \subset N_Y \subset \{(a_\alpha) \in Z^A : n_G \text{ divides } \sum a_\alpha\}. \]

The inclusion \( N_Y \subset \{(a_\alpha) \in Z^A : n_G \text{ divides } \sum a_\alpha\} \) means the global Euler number condition \( \chi(F) = \chi(Y^G) \mod n_G \). The condition is necessary for any pseudo-equivalence \( g : X \to Y \). Specifically, since \( g \) is a homotopy equivalence, the mapping cone \( cX \cup_g Y \) of \( g \) is a contractible \( G \)-CW-complex with fixed set \( (cX \cup_g Y)^G = cF \cup Y^G \). By Oliver’s theorem, we get \( \chi(cF \cup Y^G) = 1 + \chi(Y^G) - \chi(F) = 1 \mod n_G \).

The inclusion \( n_G Z^A \subset N_Y \) means that the local Euler number condition \( \chi(F_\alpha) = \chi(Y^G_\alpha) \mod n_G \) for all \( \alpha \) is sufficient for the existence of pseudo-equivalence extension. This is proved in Proposition \( \text{[3]} \). We note that condition does not involve the fundamental group. Moreover, if \( Y^G \) is connected, then this implies \( N_Y = n_G Z \), and the Euler number condition is necessary and sufficient.

The theorem allows \( F \) to be empty. For the case \( G \) acts trivially on \( Y \), this means that the group \( G \) acts on a CW-complex \( X \) without fixed points, together with a homotopy equivalence \( X \to Y \) (which factors through \( X/G \)), if and only if \( \chi(Y) = 0 \mod n_G \).

We point out a consequence of the main theorem.

**Corollary.** Suppose \( G \) is a group of not prime power order, and \( X \) is a finite \( G \)-CW complex with non-empty and connected fixed set \( X^G \). Then \( F \) is the fixed set of a \( G \)-action on a finite \( G \)-CW complex pseudo-equivalent to \( X \) if and only if \( \chi(F) = \chi(X^G) \mod n_G \).

The significance of the result is that no pseudo-equivalent \( G \)-maps between \( G \)-spaces are specifically mentioned. Note that a pseudo-equivalent \( G \)-map does not necessarily have an inverse pseudo-equivalent \( G \)-map. To make pseudo-equality into an equivalence relation, therefore, we need to allow two \( G \)-spaces \( X, X' \) to be pseudo-equivalent if they are related by a zig-zag sequence of pseudo-equivalent \( G \)-maps

\[ X \overset{f_1}{\leftarrow} Y_1 \overset{g_1}{\to} X_1 \overset{f_2}{\leftarrow} Y_2 \overset{g_2}{\to} X_2 \cdots X_{n-1} \overset{f_n}{\leftarrow} Y_n \overset{g_n}{\to} X'. \]

The pseudo-equivalence in the corollary is the equivalence in this sense. We know \( \chi(X^G) \mod n_G \) is a pseudo-equivalence invariant, by applying Oliver’s theorem to the mapping cone. Conversely, if \( \chi(F) = \chi(X^G) \mod n_G \), and
$X^G$ is connected, then by the main theorem, any map $F \to X^G$ can be extended to a pseudo-equivalence $X' \to X$.

We remark that, if $X^G = \emptyset$, then the Euler number condition is not sufficient for the corollary. For example, for any $p$-subgroup $P$ of $G$, the Smith theory can be applied to the fixed sets of $P$ to give $H_*(X^P; \mathbb{F}_p) \cong H_*(X'^P; \mathbb{F}_p)$. This implies that, if $G$ acts freely on $X$, then the $G$-action on $X'$ must also be free. This property cannot be the consequence of the Euler number condition. In general, some homotopy fixed point condition needs to be satisfied.

Oliver and Petrie [7] studied the extension problem for a more general setting, except they conclude quasi-equivalence instead of pseudo-equivalence. Here quasi-equivalence means isomorphism on the fundamental group and the integral homology groups. This is weaker than pseudo-equivalence, and is equivalent to pseudo-equivalence when $Y$ is simply connected. Proposition 3.8 of [7] is the quasi-equivalence version of our main theorem.

Finally, we also have a version of the main theorem for $G$-ANR, which is an important equivariant topological category. In [6], Oliver showed that there is a number $m_G$ (that divides $n_G$, see Lemma 4), such that a finite CW-complex $F$ has a $G$-CW-resolution if and only if $\chi(F) = 1 \mod m_G$.

**Theorem.** Suppose $G$ is a group of not prime power order, $Y$ is a connected $G$-ANR, and the connected components of $Y^G$ are $Y^G_\alpha$, $\alpha \in A$. Then there is a subgroup $M_Y \subset \mathbb{Z}^A$, such that a map $f: F \to Y^G \subset Y$ from an ANR $F$ has a pseudo-equivalence extension $g: X \to Y$, with $X^G = F$, if and only if

$$(\chi(F_\alpha) - \chi(Y^G_\alpha))_{\alpha \in A} \in M_Y, \quad F_\alpha = f^{-1}(Y^G_\alpha).$$

Moreover, we have

$$m_G \mathbb{Z}^A \subset M_Y \subset \{(a_\alpha) \in \mathbb{Z}^A : m_G \text{ divides } \sum a_\alpha\}.$$ 

## 1 Cell-wise Partition of Euler Number

The key idea of our proof is to inductively apply Oliver’s construction to cells of $Y^G$. Then we need to organise our proof into the obstruction group $N_Y$.

Recall that a CW-complex $Y$ is regular if every cell $\sigma$ is given by an embedding $D^k \to Y$. This implies that the boundary $\partial \sigma$ of the cell is a sphere $S^{k-1}$ embedded in $Y$. Then $\chi(\sigma) = \chi(D^{\dim \sigma}) = 1$, $\chi(\partial \sigma) = \chi(S^{\dim \sigma - 1}) = 1 - (-1)^{\dim \sigma}$. 
Lemma 1. Suppose $Y$ is a regular CW-complex, and $f: F \to Y$ is a map. If $\chi(f^{-1}(\sigma)) = 1 \mod n$ for every cell $\sigma$ of $Y$, then $\chi(F) = \chi(Y) \mod n$.

The lemma allows $n = 0$, which means dropping “mod $n$” in the statement. In the following proof, all numerical equalities are true mod $n$.

Proof. If $\dim Y = 0$, then $Y$ consists of finitely many points $y_1, y_2, \ldots, y_k$, and $F = \bigcup_{i=1}^k f^{-1}(y_i)$ is a disjoint union. By the assumption, we have $\chi(f^{-1}(y_i)) = 1$. Therefore $\chi(F) = \sum_{i=1}^k \chi(f^{-1}(y_i)) = k = \chi(Y)$.

Suppose $\dim Y = d$, and the lemma is proved for CW-complexes of dimension $< d$. We have $Y^{d-1} \subset Y_1 \subset \cdots \subset Y_{k-1} \subset Y_k = Y$, where $Y^{d-1}$ is the $(d-1)$-skeleton of $Y$, and $Y_i$ is obtained by attaching a $d$-cell to $Y_{i-1}$. By the inductive assumption, we have $\chi(f^{-1}(Y^{d-1})) = \chi(Y^{d-1})$. Suppose we already proved $\chi(f^{-1}(Y_{i-1})) = \chi(Y_{i-1})$. Let $Y_i = Y_{i-1} \cup \sigma$ for a $d$-cell $\sigma$. Then $Y_{i-1} \cap \sigma$ is a CW-complex of dimension $< d$. Therefore we have $\chi(f^{-1}(Y_{i-1} \cap \sigma)) = \chi(Y_{i-1} \cap \sigma)$ by the inductive assumption. Combined with $\chi(f^{-1}(\sigma)) = 1 = \chi(\sigma)$, we get

$$
\chi(f^{-1}(Y_i)) = \chi(f^{-1}(Y_{i-1})) + \chi(f^{-1}(\sigma)) - \chi(f^{-1}(Y_{i-1} \cap \sigma)) = \chi(Y_{i-1}) + \chi(\sigma) - \chi(Y_{i-1} \cap \sigma) = \chi(Y_i).
$$

Inductively, this proves $\chi(F) = \chi(f^{-1}(Y_k)) = \chi(Y_k) = \chi(Y)$. $\square$

The converse of Lemma 1 is true up to homotopy equivalence.

Lemma 2. Suppose $Y$ is a connected regular CW-complex. Suppose $F \neq \emptyset$ and $f: F \to Y$ is a map, such that $\chi(F) = \chi(Y) \mod n$. Then there is a homotopy equivalence $\phi: F \simeq \hat{F}$ and a map $\hat{f}: \hat{F} \to Y$, such that $\hat{f} \phi \simeq f$, and $\chi(f^{-1}(\sigma)) = 1 \mod n$ for every cell $\sigma$ of $Y$.

Proof. We call a cell top cell if it is not in the boundary of any other cell. Fix $c \in F$ and a top cell $\beta$ containing $f(c)$. We regard $\beta$ as the “base cell” of $Y$. For each top cell $\sigma$ different from $\beta$, there is a continuous path $\gamma: [-1, 1] \to Y$, such that $\gamma(-1) = f(c)$, $\gamma(-1, 0) \cap \sigma = \emptyset$, and $\gamma(0, 1] \subset \sigma$. This implies that $\gamma(0) \in \partial \sigma$, and $f(c)$ is the only other possible point on the path lying inside $\partial \sigma$.

For any two spaces $A$ and $B$, we glue cones $cA$ and $cB$ to $F$ by identifying the cone points with $c$. The new space $F' = F \cup_c (cA \cup cB)$ is homotopy equivalent to $F$. We further extend $f$ to $f': F' \to Y$ by mapping the cones
to the path $\gamma$ in $Y$. The map is “straightforward” on $cA$ and is “twisted” on $cB$, as illustrated by the picture. Let $\chi(A) = a$ and $\chi(B) = b$. Then we have

$$\chi(f'^{-1}(\sigma)) = \chi(f^{-1}(\sigma)) + a + 2b,$$

$$\chi(f'^{-1}(\partial\sigma)) = \chi(f^{-1}(\partial\sigma)) + a + 3b,$$

$$\chi(f'^{-1}(Y - \hat{\sigma})) = \chi(f^{-1}(Y - \hat{\sigma})) + b.$$ 

It is therefore possible to choose $a$ and $b$, such that $\chi(f'^{-1}(\sigma)) = \chi(\sigma) = 1$ and $\chi(f'^{-1}(\partial\sigma)) = \chi(\partial\sigma) = 1 - (-1)^{\dim \sigma}$. By $\chi(F) = \chi(Y)$ mod $n$, this implies that $\chi(f'^{-1}(Y - \hat{\sigma})) = \chi(Y - \hat{\sigma})$ mod $n$.

The basic construction above reduces the problem to the restriction map $f'|: f'^{-1}(Y - \hat{\sigma}) \to Y - \hat{\sigma}$, which still satisfies the Euler number condition in the lemma. This gives a potentially inductive argument. We make the stronger inductive assumption that $f'^{-1}(Y - \hat{\sigma})$ can be extended to $F''$ by gluing cones (identifying cone points with $c$), and $f'|: f'^{-1}(Y - \hat{\sigma}) \to Y - \hat{\sigma}$ can be extended to $f'': F'' \to Y - \hat{\sigma}$, such that $\chi(f''^{-1}(\tau)) = 1$ mod $n$ for each cell $\tau$ of $Y - \hat{\sigma}$. Then $\hat{F} = F'' \cup_c (cA \cup cB)$ is obtained by gluing cones to $F$ (identifying cone points with $c$). Therefore $\hat{F}$ is homotopy equivalent to $F$, and $\hat{f} = f' \cup f'': \hat{F} \to Y$ is homotopy equivalent to $f$. Moreover, we still have $\chi(\hat{f}^{-1}(\tau)) = \chi(f''^{-1}(\tau)) = 1$ for any cell $\tau$ of $Y - \hat{\sigma}$. By applying Lemma \[\text{II}\] to the restriction $\hat{f}|: \hat{f}^{-1}(\partial\sigma) \to \partial\sigma$, where cells of $\partial\sigma$ are cells of $Y - \hat{\sigma}$, we get

$$\chi(\hat{f}^{-1}(\partial\sigma)) = \chi(\partial\sigma) = 1 - (-1)^{\dim \sigma} = \chi(f'^{-1}(\partial\sigma)).$$

On the other hand, we have

$$\chi(\hat{f}^{-1}(\sigma)) - \chi(\hat{f}^{-1}(\partial\sigma)) = \chi(\hat{f}^{-1}(\hat{\sigma})) = \chi(f'^{-1}(\hat{\sigma})) = \chi(f'^{-1}(\sigma)) - \chi(f'^{-1}(\partial\sigma)).$$

Therefore $\chi(\hat{f}^{-1}(\sigma)) = \chi(f'^{-1}(\sigma)) = 1$. 

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There are two problems with the induction argument. The first is that $Y - \sigma$ may not be connected. The second is that there is only one top cell $\beta$.

The case $Y - \sigma$ not connected happens only when $\sigma$ is a 1-cell, with only one end $v_0$ attached to $Y - \sigma$, and other end $v_1$ being “free”. In addition to glueing $cA$ and $cB$, we may additionally glue a cone $cC$, with the map from $cC$ to $Y$ extending all the way to $v_1$. Then we have

$$\chi(f'\sigma) = \chi(f^{-1}(\sigma)) + a + 2b + c,$$
$$\chi(f'v_0) = \chi(f^{-1}(v_0)) + a + 3b + c,$$
$$\chi(f'v_1) = \chi(f^{-1}(v_1)) + c,$$
$$\chi(f'Y - \sigma) = \chi(f^{-1}(Y - \sigma)) + b.$$

It is then possible to choose $a, b, c$, such that $\chi(f'\sigma) = \chi(f'v_0) = \chi(f'v_1) = 1$. The rest of the inductive argument is the same.

Finally, we consider the case $Y$ has only one top cell $\beta$. In this case, the assumption already says $\chi(f^{-1}(\beta)) = 1$, and additional cone construction over $\beta$ does not change this fact. What we need to do is to improve $\chi(f^{-1}(\sigma))$ to 1 for cells $\sigma$ in $\partial \beta$. The problem is then reduced to the restriction map $f|: f^{-1}(\partial \beta) \to \partial \beta$. The induction may continue over $\partial \beta$.

\[ \square \]

2 Non-empty Fixed Point

We prove the following result, for the case $F_\alpha = f^{-1}(Y_\alpha)$ are not empty. The case some $F_\alpha$ is empty will be handled in the next section. The proposition is part of the main theorem.

**Proposition 3.** Suppose $G$ is a group of not prime power order, $Y$ is a connected finite $G$-CW complex, $F$ is a finite CW-complex, and $f: F \to Y$ is a $G$-map. If $\chi(F_\alpha) = \chi(Y_\alpha^G)$ mod $n_G$ for every connected component $Y_\alpha^G$ of $Y^G$, then $f$ has pseudo-equivalence extension.
We establish the homotopy invariance of the pseudo-equivalence extension problem. In other words, we assume that \( f: F \to Y^G \subset Y \) has pseudo-equivalence extension \( g: X \to Y \). Then we change \( f, F, Y \) by equivalent homotopy and argue that we still have pseudo-equivalence extension.

First, suppose \( f \) is homotopy equivalent to \( f': F \to Y^G \subset Y \). By the equivariant version of the homotopy extension property, the homotopy extends to a \( G \)-homotopy from \( g: X \to Y \) to another \( G \)-map \( g': X \to Y \). Then the \( G \)-map \( g' \) extends \( f' \), and \( g' \) is still a pseudo-equivalence.

Second, suppose \( \phi: F \to F' \) is a homotopy equivalence. Then there is a map \( f': F' \to Y^G \), such that \( f: F \to Y^G \) is homotopy equivalent to \( f' \circ \phi: F \to F' \to Y^G \). By the first argument above, \( f' \circ \phi \) has pseudo-equivalence extension \( g: X \to Y \). Then \( X' = X \cup_{\phi} F' \) (glueing \( F \subset X \) to \( F' \) by \( \phi \)) is a \( G \)-CW-complex with \( F' \) as the fixed set, and the \( G \)-map \( g \cup f': X' \to Y \) is a pseudo-equivalence extension of \( f' \).

Third, suppose \( \psi: Y \to Y' \) is a \( G \)-homotopy equivalence. Then \( \psi \circ f: F \to Y' \) is extended to a pseudo-equivalence \( \psi \circ g: X \to Y' \).

We conclude that our pseudo-equivalence extension problem is homotopy invariant. Therefore we may assume \( Y \) is a regular \( G \)-CW-complex. Moreover, in the setting of Proposition 3 and under the assumption that all \( F_\alpha \) are not empty, by Lemma 2 we may assume \( \chi(f^{-1}(\sigma)) = 1 \mod n_\alpha \) for every cell \( \sigma \) of \( Y^G \). Then we may apply the following technical result by Oliver to cells of \( Y^G \). The first statement is [6, Theorem 2], and he called the \( G \)-CW-complex \( X \) in the statement \( G \)-resolution. The second statement is essentially the corollary to [6, Theorem 3].

**Lemma 4.** For any group \( G \) of not prime power order, there are non-negative integers \( m_\alpha \) and \( n_\alpha \). Suppose \( K \) is a finite \( G \)-CW-complex with fixed set \( F \).

1. If \( \chi(F) = 1 \mod m_\alpha \), then \( K \) can be extended to a finite \( G \)-CW-complex \( X \), such that \( F = X^G \), \( X \) is \( (\dim X - 1) \)-connected, and \( H_{\dim X}(X; \mathbb{Z}) \) is a projective \( \mathbb{Z}G \)-module.

2. If \( \chi(F) = 1 \mod n_\alpha \), then \( K \) can be extended to a finite contractible \( G \)-CW-complex \( X \), such that \( F = X^G \).

**Proof of Proposition 3 in case \( F \neq \emptyset \).** We first extend \( f: F \to Z = Y^G \) to a pseudo-equivalence extension \( h: W \to Z \).

We assume \( F^{k-1} = f^{-1}(Z^{k-1}) \) is already extended to a \( G \)-CW-complex \( W^{k-1} \) with fixed set \( F^{k-1} \). Moreover, we further assume that \( f|_{F^{k-1}} \) is ex-
tended to a $G$-map $h_{k-1}: W^{k-1} \to Z^{k-1}$, such that $(W^{k-1})^G = F^{k-1}$, and $h_{k-1}^{-1}(\sigma)$ is contractible for each cell $\sigma$ of $Z^{k-1}$.

We have $Z^{-1} = F^{-1} = \emptyset$. The following argument for $k = 0$ can be regarded as the initial step of the induction.

Let $\sigma$ be a $k$-cell of $Z$. Then we have $\chi(f^{-1}(\sigma)) = 1 \mod n_G$. Taking $f^{-1}(\sigma)$ and $f^{-1}(\sigma) \cup h_{k-1}^{-1}(\partial \sigma)$ as $F$ and $K$ in the second part of Lemma 3, we may extend $f^{-1}(\sigma) \cup h_{k-1}^{-1}(\partial \sigma)$ to a finite contractible $G$-CW-complex $W_\sigma$ with $W_\sigma^G = f^{-1}(\sigma)$. We may further arrange (taking “average”) to extend $f|_\sigma \cup h_{k-1}|_{\partial \sigma}$ to a $G$-map $h_\sigma: W_\sigma \to \sigma$ ($G$ acts trivially on $\sigma$).

Let $W^k = W^{k-1} \cup (\cup_{\text{dim} \sigma = k} W_\sigma)$, where the union identifies $h_{k-1}^{-1}(\partial \sigma) \subset W_\sigma$ with the same subset in $W^{k-1}$. Then we have $G$-map $h_k = h_{k-1} \cup (\cup_{\text{dim} \sigma = k} h_\sigma): W^k \to Z^k$, such that $(W^k)^G = F^{k-1} \cup (\cup_{\text{dim} \sigma = k} f^{-1}(\sigma)) = F_k$, and $h_k^{-1}(\sigma)$ is contractible for every cell $\sigma$ of $Z^k$.

When $k = \dim Z$, we get $h = h_{\dim Z}: W = W^{\dim Z} \to Z$, such that $W^G = F$, and $h^{-1}(\sigma)$ is contractible for each cell $\sigma$ of $Z$. This implies that $h: W \to Z = Y^G$ is a homotopy equivalence.

Next, we further extend $h$ to a pseudo-equivalence $g: X \to Y$.

The equivariant neighborhood $\text{nd}(Z)$ of $Z$ in $Y$ is the mapping cylinder of a $G$-map $\lambda: E \to Z$. We try to factor $\lambda$ through a $G$-map $\bar{\lambda}: E \to W$. Then $\lambda = h \circ \bar{\lambda}$, and we have a $G$-map from the mapping cylinder of $\bar{\lambda}: E \to W$ to the mapping cylinder of $\lambda: E \to Z$. The $G$-map extends to a $G$-map $g = \text{id} \cup h: X = (Y - \text{nd}(Z)) \cup Z W \to Y = (Y - \text{nd}(Z)) \cup \lambda Z$. We have $X^G = W^G = F$, and $g$ extends $f$. Moreover, since $h$ is a pseudo-equivalence, $g$ is also a pseudo-equivalence.

It remains to construct the lifting $\bar{\lambda}$. Since $G$ has no fixed points on $E$, we can construct the lifting if $h: W \to Z$ is highly connected on the non-fixed part of $W$. Recall that we actually constructed $h: W \to Z$ in such a way that, for each cell $\sigma$ in the (regular) CW-complex $Z$, $h^{-1}(\sigma)$ is contractible. Let $S$ be a finite $G$-set without fixed points and sufficiently large, so that all the isotropies on $E$ appear in $S$. Then we may take the cell-wise join of $h: W \to Z$ with $S \times Z \to Z$ several times to get $h': W' \to Z$. This means $h'^{-1}(\sigma) = h^{-1}(\sigma) \ast S \ast \cdots \ast S$. Since $h^{-1}(\sigma)$ is contractible, $h'^{-1}(\sigma)$ is still contractible. Since $S$ has no fixed points, we get $W'^G = W^G = F$. Therefore $h'$ is still a pseudo-equivalence extension of $f$. On the other hand, the non-fixed part of $h'^{-1}(\sigma)$ becomes more and more highly connected as we repeat the join construction more and more times. Therefore we may construct the lifting $\bar{\lambda}$ by using $h'$ instead of $h$. $\square$
3 Empty Fixed Point

We still need to prove Proposition 3 for the case some $F_\alpha = \emptyset$. In this case, $\chi(F_\alpha) = \chi(Y_\alpha^G) \mod n_G$ means $\chi(Y_\alpha^G) = 0 \mod n_G$. A typical example is $Y_\alpha^G = S^1$, and our proof starts with this special case. In fact, we will concentrate on the case $Y = Y^G = S^1$.

Lemma 5. Suppose $G$ is a group of not prime power order. Then there is a pseudo-equivalence $X \to S^1$, with $X$ has no fixed point, and $G$ fixes $S^1$.

The idea is to find a simply connected $G$-space $Z$ without fixed points, and a $G$-map $h: Z \to Z$ inducing zero homomorphism on the reduced homology. Then the mapping torus $X$ of $h$ together with the natural map to $S^1$ gives what we want. We may take $Z = S(V)$ and take $h$ to be the self map of $S(V)$ in the following result.

Lemma 6. Suppose $G$ is a group of not prime power order, and $V$ is a linear $G$-representation. If $V^G = 0$ and all Sylow subgroups of $G$ are isotropy groups of $V$, then there is a degree 0 $G$-map from the unit sphere $S(V)$ to itself.

Proof. Let $P$ be a Sylow subgroup and let $N_P$ be its normalizer in $G$. Since $P$ is an isotropy subgroup, the fixed subspace $V^P$ is not a zero subspace. Let $D$ be a small equivariant disk neighborhood of a point $x \in S(V^P)$. Then $D$ is an $N_P$-representation. Moreover, since $V$ is a linear representation, the $N_P$-representation is independent of the size of $D$. This means that the radial extension gives an $N_P$-equivariant homeomorphism $D/\partial D \cong S(V)$ sending $\ast = \partial D/\partial D$ to $x$. Then we may construct an $N_P$-map

$$S(V) = (S(V) - D) \cup_{\partial D} D \to S(V) \cup_x D/\partial D \to S(V).$$

The first map collapses $\partial D$ to $x$, and the second map uses the $N_P$-equivariant homeomorphism $D/\partial D \cong S(V)$. The map can be extended to a $G$-map

$$h_x: S(V) \to S(V) \cup_{Gx} (G \times_{N_P} D/\partial D) \to S(V).$$

If we fix an orientation of $S(V)$, then $D$ inherits the orientation, and the homeomorphism $D/\partial D \cong S(V)$ has degree $\pm 1$. By composing with the $-1$ map along a 1-dimensional subspace of $V^P$, we may change the sign of the degree of the homeomorphism. Therefore we can arrange to have
the degree of $h_x$ to be $1 + |G/N_P|$ or to be $1 - |G/N_P|$. If we apply the construction at several points $x \in S(V^P)$ with disjoint orbits $Gx$, then we get a $G$-map $S(V) \to S(V)$ of degree $1 + a|G/N_P|$ for any integer $a$. If we apply the construction to the Sylow subgroups $P_1, P_2, \ldots, P_n$ for all the distinct prime factors of $|G|$, then we get a $G$-map $S(V) \to S(V)$ of degree $1 + \sum a_i|G/N_{P_i}|$, where $a_1, a_2, \ldots, a_n$ can be any prescribed integers. Since $|G/N_{P_1}|, |G/N_{P_2}|, \ldots, |G/N_{P_n}|$ are coprime, we get degree $0$ by suitable choice of the integers $a_i$.

The kernel of the augmentation $\epsilon(\sum_{g \in G} a_g g) = \sum a_g : \mathbb{R}G \to \mathbb{R}$ is a representation satisfying the condition of the lemma. The direct sum of several copies of this kernel also satisfies the condition of the lemma. We also note that, by taking the direct sum of sufficiently many copies, then $S(V)$ is highly connected.

The existence of degree 0 equivariant map from the unit sphere of a representation to itself also follows from Thomas Bartsch’s study [1] of the existence of Borsuk-Ulam theorems. The result of Bartsch implies that for a group of not prime power order, we often have equivariant map of a representation sphere without fixed points to a proper sub-representation sphere. The degree of such map is necessarily 0.

**Proof of Proposition 3 in case $F_\alpha = \emptyset$.** Assume $F_\alpha = \emptyset$ for some $\alpha$. Then the condition $\chi(F_\alpha) = \chi(Y^G_\alpha) \mod n_G$ means $\chi(Y^G_\alpha) = 0 = \chi(S^1) \mod n_G$. By replacing each $F_\alpha = \emptyset$ with $S^1$ and taking any map $S^1 \to Y^G_\alpha$, we update $f$ to $f' : F' \to Y^G$, which satisfies the Euler number condition in the proposition, and all $F'_\alpha$ are not empty. Since the proposition is already proved for the case all $F_\alpha \neq \emptyset$, $f'$ has a pseudo-equivalence extension $X' \to Y$, with $X^G = F'$.

It remains to homotopically replace the extra circles added to $F$ by something that have no fixed points. The equivariant neighborhood $nd(S^1)$ of one such circle in $X'$ is the mapping cylinder of a $G$-map $\lambda : E \to S^1$. By Lemma 5 and the remark after the proof of Lemma 6, there is a pseudo-equivalence $\mu : W \to S^1$, such that $W$ has no fixed point, and the fibre of $\mu$ is highly connected. Since the fibre of $\mu$ is highly connected, $\lambda$ can be lifted to a $G$-map $\bar{\lambda} : E \to W$ with respect to $\mu$. In other words, we have $\lambda = \mu \circ \bar{\lambda}$. Let $X = (X' - nd(S^1)) \cup \bar{\lambda} W$ be obtained by glueing the boundary $E$ of $nd(S^1)$ to $W$, and this is done for all the circles in $X'$ that were used to replace all $F_\alpha = \emptyset$. Then $\lambda = \mu \circ \bar{\lambda}$ induces a pseudo-equivalence $X \to X' = (X' - nd(S^1)) \cup \lambda S^1$, because $\mu$ is a pseudo-equivalence. The
composition \( X \to X' \to Y \) is then a pseudo-equivalence with \( X^G = F \) and extending \( f \).

4 Obstruction Group

We construct the obstruction group \( N_Y \) in the tautological way. For a fixed \( G \)-CW-complex \( Y \), we let \( N_Y \) be the collection \( \nu(g) = (\chi(F_\alpha) - \chi(Y^G_\alpha))_{\alpha \in A} \) for all pseudo-equivalences \( g: X \to Y \). Here \( F_\alpha = g^{-1}(Y^G_\alpha) \cap X^G \). To show \( N_Y \) is a subgroup of \( \mathbb{N}^A \), we need to show that it is closed under negative and addition operations.

For any pseudo-equivalence \( g: X \to Y \), we construct the negative \( \bar{g}: \bar{X} = Y \cup X \times [0, 1] \cup Y \to Y \) as the double mapping cylinder of \( g \). Then \( \bar{g} \) is still a pseudo-equivalence, with \( \bar{F}_\alpha = Y^G_\alpha \cup F_\alpha \times [0, 1] \cup Y^G_\alpha \), and \( \chi(F_\alpha) - \chi(Y^G_\alpha) = -(\chi(F_\alpha) - \chi(Y^G_\alpha)) \). This proves that \( N_Y \) is closed under negative operation.

We note the following properties of the negative construction:

1. \( \bar{X} \) contains a copy of \( Y \), and the non-equivalent homotopy equivalence can be a homotopy retraction of \( \bar{X} \) to \( Y \).

2. \( \bar{F}_\alpha \) is connected. Therefore the connected components of the fixed parts of \( Y \) and \( X \) are in one-to-one correspondence.

We call a pseudo-equivalence with the two properties retracting equivalence.

Since the double negative satisfies \( \nu(\bar{g}) = \nu(g) \), every element in \( N_Y \) is represented by a retracting equivalence.

For two retracting equivalences \( g_1: X_1 \to Y \) and \( g_2: X_2 \to Y \), the addition \( g_1 \cup g_2: X_1 \cup_Y X_2 \to Y \) is still a retracting equivalence. It is also easy to see that \( \nu(g_1 \cup g_2) = \nu(g_1) + \nu(g_2) \). Therefore \( N_Y \) is closed under addition operation.

This completes the proof that \( N_Y \) is an abelian subgroup. Next we prove that \( N_Y \) is indeed the obstruction to pseudo-equivalence extension.

The definition of \( N_Y \) says that \( (\chi(F_\alpha) - \chi(Y^G_\alpha))_{\alpha \in A} \in N_Y \) is necessary. Conversely, if \( f: F \to Y \) satisfies \( (\chi(F_\alpha) - \chi(Y^G_\alpha))_{\alpha \in A} \in N_Y \), then \( (\chi(F_\alpha) - \chi(Y^G_\alpha))_{\alpha \in A} = \nu(g') \) for some a pseudo-equivalence \( g': X' \to Y \). This means \( \chi(F_\alpha) = \chi(F'_\alpha) \), where \( F'_\alpha = g'^{-1}(Y^G_\alpha) \cap X'^G \).

As remarked earlier, we may further assume that \( g' \) is a retracting extension. Then we may regard \( f \) as mapped into \( Y \subset X' \). This means that \( f \) is
a composition \((i: Y \to X' \text{ is the inclusion})\)

\[ f = g' \circ (i \circ f): F \overset{iorf}{\to} X' \overset{g'}{\to} Y. \]

Since connected components of the fixed parts of \(Y\) and \(X'\) are in one-to-one correspondence, we have \(Y^G = Y \cap F_{a'}\), \(F_a = f^{-1}(Y \cap F_{a'}) = (i \circ f)^{-1}(F_{a'}),\) and

\[
\chi((i \circ f)^{-1}(F_{a'})) = \chi(F_a) = \chi(F_{a'}). 
\]

By Proposition 3, this implies that \(i \circ f\) has pseudo-equivalence extension \(h: X \to X'\). Then \(g' \circ h: X \to Y\) is a pseudo-equivalence extension of \(f\).

Next, we give more restrictions on the obstruction group.

Let \(p: \tilde{Y} \to Y\) be the universal cover, with free action by the fundamental group \(\pi = \pi_1 Y\). The \(G\)-actions on \(Y\) lift to self homeomorphisms of \(\tilde{Y}\). All the liftings form a group \(\Gamma\) fitting into an exact sequence

\[ 1 \to \pi \to \Gamma \to G \to 1, \]

Let \(\tilde{y} \in \tilde{Y}\), and \(y = p(\tilde{y})\). Then the induced homomorphism \(\Gamma_{\tilde{y}} \to G_y\) of isotropy groups is an isomorphism.

If \(y \in Y^G\), then the isomorphism gives a splitting \(G = G_y \cong \Gamma_{\tilde{y}} \subset \Gamma\) of \(\Gamma \to G\). The splitting depends only on the connected component \(C\) of \(p^{-1}(Y^G)\) that contains \(\tilde{y}\). The component \(C\) covers a connected component \(Y^G\) of \(Y^G\). Therefore we may denote \(\Gamma_C = \Gamma_{\tilde{y}}\). The other connected components of \(p^{-1}(Y^G)\) are \(aC\) for \(a \in \pi\), and the corresponding isotropy group \(\Gamma_{aC}\) is a conjugation of \(\Gamma_C\) by \(a \in \pi\). Therefore a connected component of \(Y^G\) gives a \(\pi\)-conjugation class \(\Gamma_a = \{ a \Gamma_{a^{-1}}: a \in \pi \}\) of splittings \(G \to \Gamma\).

If we fix a splitting, i.e., a semi-direct product \(\Gamma = \pi \rtimes G\), then we have a well defined conjugation action \(a^n = uau^{-1}\) of \(u \in G\) on \(a \in \pi\). The map \(u \mapsto ?^u\) is a homomorphism \(G \to \text{Aut} (\pi)\). Any other splitting of \(\Gamma \to G\) is then given by a map \(u \in G \mapsto (a(u), u) \in \Gamma\), with \(a: G \to \pi\) satisfying the cochain condition \(a(uv) = a(u)a(v)^u\). Two cochains \(a(u), b(u)\) are equivalent if their corresponding splittings are conjugate by some \(c \in \pi\), which means \(b(u)a^u = ac(u)\). The equivalence classes of cochains form the cohomology set \(H^1(\pi; G)\) (not necessarily a group because \(G\) may not be commutative). This is the set of \(\pi\)-conjugate classes of splittings.

A \(G\)-map \(g: X \to Y\) lifts to a \(\Gamma\)-map \(\tilde{g}: \tilde{X} \to \tilde{Y}\). This induces a map of \(\mathbb{Z} [\Gamma]\)-chain complexes \(\tilde{g}_*: C(\tilde{X}) \to C(\tilde{Y})\). If \(g\) is a pseudo-equivalence, then \(\tilde{g}_*\) has a \(\mathbb{Z} [\pi]\)-chain homotopy inverse \(\varphi\). If \(|G|\) is invertible in a ring.
\( R \), then we may use one splitting \( \Gamma = \pi \times G \) to get a \( R[\Gamma] \)-chain map 
\[
\frac{1}{|\Gamma|} \sum_{u \in G} u \varphi : C(\tilde{Y}) \to C(\tilde{X}).
\]
This is a \( R[\Gamma] \)-chain homotopy inverse of \( \tilde{g}_*: C(\tilde{X}) \otimes R \to C(\tilde{Y}) \otimes R \).

**Proposition 7.** Suppose \( G \) is a cyclic group acting on a finite \( G \)-CW complex \( Y \). If the connected components \( Y^G_\alpha \) of \( Y^G \) give non-conjugate isotropy subgroups in the group of lifted \( G \)-actions on the universal cover \( \tilde{Y} \), then a rational pseudo-equivalence \( g: X \to Y \) satisfies \( \chi(F_\alpha) = \chi(Y^G_\alpha) \) for each \( \alpha \).

The non-conjugation condition means distinct elements in \( H^1(\pi; G) \). Since all pseudo-equivalences are rational pseudo-equivalences, the proposition implies that the component-wise Euler characteristic condition in Proposition 3 is also necessary. In other words, the obstruction group \( N_Y \) equals the lower bound in the main theorem.

**Proof.** Since \( |G| \) is invertible in \( \mathbb{Q} \), the pseudo-equivalence \( g \) induces a homotopy equivalence of \( \mathbb{Q}[\Gamma] \)-chain complexes
\[
\tilde{g}_*: C(\tilde{X}) \otimes \mathbb{Q} \to C(\tilde{Y}) \otimes \mathbb{Q}.
\]
Moreover, since the isotropy groups of the \( \Gamma \)-action on \( \tilde{Y} \) are all finite, we know both sides consist of finitely generated projective \( \mathbb{Q}[\Gamma] \)-modules. Then the \( \mathbb{Q}[\Gamma] \)-chain homotopy equivalence implies both chain complexes have the same Euler characteristic in \( K_0(\mathbb{Q} \Gamma) \).

The \( G \)-cell \( G\sigma \) of \( \tilde{Y} \) are in one-to-one correspondence with \( \Gamma \)-cells \( \Gamma \tilde{\sigma} \) of \( \tilde{Y} \), where \( \tilde{\sigma} \) is any cell of \( \tilde{Y} \) over \( \sigma \). The Euler characteristic of \( C(\tilde{Y}) \otimes \mathbb{Q} \) is
\[
\chi_{\Gamma}(\tilde{Y}) = \sum_{G\text{-cells of } Y} (-1)^{\dim \sigma} [\mathbb{Q}[\Gamma \tilde{\sigma}]].
\]
Here \( \Gamma \tilde{\sigma} = \Gamma / \Gamma_{\tilde{\sigma}} \) is a \( \Gamma \)-orbit, and \( \mathbb{Q}[\Gamma \tilde{\sigma}] \) is a projective module. For finite subgroup \( H \), we know the rank of the projective \( \mathbb{Q}\Gamma \)-module \( \mathbb{Q}[\Gamma / H] \) ((\( \gamma \)) is the conjugation class of \( \gamma \) in \( \Gamma \))
\[
\text{rank}(\mathbb{Q}[\Gamma / H]) = \frac{1}{|H|} \sum_{h \in H} [h] \in \oplus_{(\gamma) \subset \Gamma} \mathbb{Q}(\gamma).
\]
We have \( \text{rank}(\chi_{\Gamma}(\tilde{X})) = \text{rank}(\chi_{\Gamma}(\tilde{Y})) \).

For a connected component \( C \) of \( \tilde{Y}^G_\alpha = p^{-1}(Y^G_\alpha) \), we let \( \gamma \in \Gamma_C \) correspond to the generator of the cyclic group \( G \cong \Gamma_C \). By the non-conjugate
assumption, $\Gamma_C$ is the only isotropy group of $\tilde{Y}$ where $\gamma$ appears. Therefore $C = \tilde{Y}^{\Gamma_C} = \tilde{Y}^\gamma$, and $(\gamma)$ appears only in the following part of $\text{rank}(\chi_{\Gamma}(\tilde{Y}))$

$$\sum_{G\text{-cell } G\sigma, \text{such that } \gamma\tilde{\sigma} = \tilde{\sigma}} (-1)^{\dim \sigma} \frac{1}{n} \sum_{i=0}^{n-1} (\gamma^i) = \frac{\chi(\tilde{Y}^G)}{n} \sum_{i=0}^{n-1} (\gamma^i), \quad n = |G|.$$ 

Since distinct elements of $\Gamma_C$ are not $\pi$-conjugate, this implies the coefficient of $(\gamma)$ in $\text{rank}(\chi_{\Gamma}(\tilde{Y}))$ is $\frac{\chi(\tilde{Y}^G)}{n}$.

We have the same calculation for the pullback $C' = \tilde{g}^{-1}(C) \cap \tilde{F}$ over $X$. In particular, the coefficient of $(\gamma)$ in $\text{rank}(\chi_{\Gamma}(\tilde{X}))$ is $\frac{\chi(\tilde{X}^G)}{n} = \frac{\chi(\tilde{F}^G)}{n}$. By $\text{rank}(\chi_{\Gamma}(\tilde{X})) = \text{rank}(\chi_{\Gamma}(\tilde{Y}))$, we have $\frac{\chi(F^G)}{n} = \frac{\chi(Y^G)}{n}$. Therefore we conclude $\chi(F^G) = \chi(Y^G)$. 

If we remove the non-conjugate assumption. Then the argument above gives the following equality

$$\sum_{\Gamma\alpha \text{ conjugate to } (\gamma)} \chi(F^\alpha) = \sum_{\Gamma\alpha \text{ conjugate to } (\gamma)} \chi(Y^G).$$

This can be regarded as additional “semi-local” constraint on the obstruction group $N_Y$.

Next we apply the idea to more general group actions. Suppose $n_G = 0$. This means there is a normal subgroup $P$, such that $|P| = p^l$ for a prime $p$, and $G/P = \mathbb{Z}_n$ is cyclic. We may further assume that $p$ and $n$ are coprime.

Suppose $Y^P_{\beta}$ is a connected component of $Y^P$, such that $Y^P_{\beta}$ contains fixed point of $G$. Then $Y^P_{\beta}$ is $G$-invariant, and $\Gamma$ acts on $\tilde{Y}^P_{\beta} = p^{-1}(Y^P_{\beta})$. Let $D$ be a connected component of $Y^P_{\beta}$. Then we have $P \cong \Gamma_D \subset \Gamma$.

By $Y^G = (Y^P)^H$, we know $(Y^P)^H$ is the disjoint union of several connected components $Y^G_{\alpha_1}, \ldots, Y^G_{\alpha_s}$ of $Y^G$. Then we may pick connected components $C_1, \ldots, C_s$ of $Y^G_{\alpha_1}, \ldots, Y^G_{\alpha_s}$, such that $C_i \subset D$. Then we have $G \cong \Gamma_{C_i} \supset \Gamma_D$. We may also assume that, if $\Gamma_{C_i}$ and $\Gamma_{C_j}$ are $\pi$-conjugate, then $\Gamma_{C_i} = \Gamma_{C_j}$. Let $\gamma_i \in \Gamma_{C_i}/\Gamma_D$ correspond to the generator of the cyclic group $G/P$. Then

$$\gamma_i, \gamma_j \text{ are } \pi\text{-conjugate } \iff \Gamma_{C_i}, \Gamma_{C_j} \text{ are } \pi\text{-conjugate } \iff \Gamma_{C_i} = \Gamma_{C_j}.$$ 

Therefore

$$D^\gamma_i = D^{\Gamma_{C_i}/\Gamma_D} = \cup_{\Gamma_{C_i} = \Gamma_{C_j}} C_j.$$ 

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Suppose a $G$-map $g: X \to Y$ is a pseudo-equivalence. We denote $X^P_\beta = g^{-1}(Y^P_\beta) \cap X^P$, and denote by $D', C'_1, \ldots, C'_s$ the pullbacks of $D, C_1, \ldots, C_s$. The restriction $h: D' \to D$ of $\tilde{g}$ is a $\Gamma_{C_i}/\Gamma_D$-map, with the fixed part $h^\gamma_n: D'^\gamma_n \to D^\gamma_n$ given by $\cup_{\gamma} = \cup_{\gamma} C'_j \to \cup_{\gamma} C_j$.

By Smith theory, we know $g^P: X^P \to Y^P$ is $\mathbb{F}_p[\Gamma]$-homology equivalence. This implies $\mathbb{Z}_{p^k}[\Gamma]$-homology equivalence for all $k$. The homology equivalence is a connected sum of homology equivalences on connected components. Therefore the map of $\mathbb{Z}_{p^k}[\Gamma_{C_i}/\Gamma_D]$-chain complexes induced by $h$

$$h_*: C(D') \otimes \mathbb{Z}_{p^k} \to C(D) \otimes \mathbb{Z}_{p^k}$$

is isomorphic on the homology. Since $p$ and $n$ are coprime, both chain complexes consist of projective $\mathbb{Z}_{p^k}[\Gamma_{C_i}/\Gamma_D]$-modules. Therefore the homology equivalence is a $\mathbb{Z}_{p^k}[\Gamma_{C_i}/\Gamma_D]$-chain homotopy equivalence. Then we may apply the same idea in the proof of Proposition 7, with $\mathbb{Q}$ replaced by $\mathbb{Z}_{p^k}$, and without the non-conjugate assumption. The result is

$$\sum_{\gamma} \chi(F_{\alpha_j}) = \sum_{\gamma} \chi(Y^G_{\alpha_j}) \mod p^k.$$ 

Since this holds for all $k$, we conclude the following.

**Proposition 8.** Suppose $P$ is a normal $p$-subgroup of $G$, and $G/P$ is a cyclic group of order prime to $p$. Suppose $Y$ is a connected finite $G$-CW complex, and a $G$-map $g: X \to Y$ is a pseudo-equivalence. For any connected component $Y^P_\beta$, let $Y^G_{\alpha_1}, \ldots, Y^G_{\alpha_t}$ be all the connected components of $Y^G \cap Y^P_\beta$ with the same $\pi$-conjugate classes of liftings. Then

$$\sum \chi(F_{\alpha_j}) = \sum \chi(Y^G_{\alpha_j}).$$

If $Y$ is simply connected, then the relative simply connected condition is satisfied. The “semi-local” Euler characteristic condition should already appeared in Oliver-Petrie.

**References**

[1] T. Bartsch. On the existence of Borsuk-Ulam theorems. *Topology*, 31(3):533-543, 1992.
[2] H. Bass. Euler characteristics and characters of discrete groups. *Invent. Math.*, 35:155-196, 1976.

[3] S. Cappell, S. Weinberger, M. Yan. Fixed points of semi-free $G$-CW-complex with prescribed homotopy type. *preprint*, 2020.

[4] A. Hattori. Rank element of a projective module. *Nagoya J. Math.*, 25:113-120, 1965.

[5] L. Jones. The converse to the fixed point theorem of P.A. Smith: I. *Ann. of Math.*, 94(1):52-68, 1971.

[6] R. Oliver. Fixed-point sets of group actions on finite cyclic complexes. *Comment. Math. Helvetici*, 50:155-177, 1975.

[7] R. Oliver, T. Petrie. $G$-CW-surgery and $K_0(ZG)$. *Math. Z.*, 179:11-42, 1982.

[8] P. A. Smith. Fixed-Point theorems for periodic transformations. *Amer. J. Math.*, 63(1):1-8, 1941.

[9] J. Stallings. Centerless groups - an algebraic formulation of Gottlieb’s theorem. *Topology*, 4:129-134, 1965.