Random periodic solutions and ergodicity for stochastic differential equations

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Abstract

In this paper, we establish some sufficient conditions for the existence of stable random periodic solutions of stochastic differential equations on $\mathbb{R}^d$ and ergodicity in the random periodic regime. The techniques involve the existence of Lyapunov type function, using two-point generator of the stochastic flow map, strong Feller argument and weak convergence.

Keywords: Strong Feller property; periodic measures; PS-ergodicity; random periodic solutions, two-point generator.

1 Introduction

Random dynamical system (RDS) is a dynamical system with some randomness, its idea was discussed in 1945 by Ulam and von Nuemann [34] and few years later by Kakutani [22] and continued in the 1970s in the framework of ergodic theory. Series of works by Elworthy, Meyer, Baxendale, Bismut, Ikeda, Kunita, Watanabe and others [7, 8, 10, 14, 21, 23, 24] showing that stochastic differential equations (SDEs) induce stochastic flows, gave a substantial push to the subject. Towards late 1980s, it became clear that the techniques from dynamical systems and probability theory could produce a theory of RDS. It was extensively developed by Arnold [3] and his ”Bremen group” based on two-parameter stochastic flows generated by stochastic differential equations due to Kunita [23, 24] and others.

To investigate the long time behaviour of RDS is of great interests both in applications and theory. There are two main issues that motivate the behaviour of a mathematical model in the long run with theoretical and practical consequences. One is to understand the random equilibrium and their distributions, to describe invariant property under the transformation of RDS and where the orbits (ensemble of trajectories) converge to, in the long run. Another one is to ascertain whether the limiting behaviour is still essentially the same after small changes to the evolution rule.

Intuitively, the limiting behaviour of a dynamical system is captured by the concept of stationary, periodic solutions or more general, invariant or quasi-invariant manifolds of stationary or periodic solutions. To understand and give the existence of such solutions attracted vast interest in theory and applications. Periodic solutions have been crucial in the qualitative theory
of dynamical systems and its systematic consideration was initiated by Poincaré in his work [31].
Periodic solutions have been studied for many fascinating physical problems, for example, van
der Pol equations [35], Liénard equations [26], etc. However, once the influence of noise on the
system is considered, which is evidently inevitable in many situations; the dynamics start to
depend on both time and the noise path, so stationary and periodic solutions in the usual sense
may not exist for randomly perturbed systems.

As in the deterministic systems setting, random stationary solutions are central in the long
time behaviour of RDS. For an RDS \( \Phi : \mathbb{T}^+ \times \Omega \times \mathbb{M} \to \mathbb{M} \), over a metric dynamical
system \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})\), a random stationary solution is an \( \mathcal{F} \)-measurable random variable \( Y : \Omega \to \mathbb{M} \)
such that
\[
\Phi(t, \omega, Y(\omega)) = Y(\theta_t \omega), \quad \text{for all } t \in \mathbb{T}^+, \quad \mathbb{P} - \text{a.s.}
\] (1.1)
Here \( \mathbb{T} \) is a two-sided time domain (discrete or continuous) and \( \mathbb{T}^+ = \mathbb{T} \cap [0, +\infty) \).
The notion of random stationary solutions of RDS is a natural extension of fixed point solution of deterministic
systems. It is a one-force, one-solution setting that describes the pathwise invariance of the
system over time along the base dynamical system \( \theta \) on noise space and the pathwise limits of
random dynamical systems (e.g., [9, 32, 39]).

Analogous to dynamical systems, the notion of random periodic solutions plays a similar role
to RDS. In the physical world around us (e.g., biology, chemical reactions, climatic dynamics,
finance, etc.), we encounter many phenomena which repeat after certain interval of time. Due to
the unavoidable random influences, many of these phenomena may be best described by random
periodic paths rather than periodic solutions. For example, the maximum daily temperature
in any particular region is a random process, however, it certainly has periodic nature driven
by divine clock due to the rotation of the earth around the sun. There had been few attempts
in physics to study random perturbation of limit cycle for some time (e.g., [25, 38]). One of
the challenges that hindered real progress was the lack of a rigorous mathematical definition of
random periodic solution and appropriate mathematical tools. For a random path with some
periodic property, it is not obvious what a reasonable mathematical relation between the random
positions \( S(s, \omega) \) at time \( s \) and \( S(s + \tau, \omega) \) at time \( s + \tau \) after a period \( \tau \) should be. However,
as \( S(t, \omega) \) is a true path, so it is not necessarily true that \( S(s, \omega) = S(s + \tau, \omega) \). To require
that \( S(s + \tau, \omega) \) is in the neighbourhood of \( S(s, \omega) \) by considering a small noise perturbation
was worthwhile attempt. However, this approach does not apply to many stochastic differential
equations and lack rigour. It is not even true for small noise and the scope of applications is
limited. Recently, in Zhao and Zheng [40], Feng, Zhao and Zhou [10], Feng and Zhao [17], it
has been observed that for fixed \( s \), \( (S(s + k\tau, \omega))_{k \in \mathbb{Z}} \) should be a random stationary solution
of the discrete RDS \( \Phi(k\tau, \omega) \). This then led to the rigorous definition of random periodicity
\( S(s + \tau, \omega) = S(s, \theta_{\tau} \omega) \). For an RDS \( \Phi \) over a metric dynamical systems \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_s)_{s \in \mathbb{T}})\), a
random periodic solution is an \( \mathcal{F} \)-measurable function \( S : \mathbb{T} \times \Omega \to \mathbb{M} \), of period \( \tau \) such that
\[
S(s + \tau, \omega) = S(s, \theta_{\tau} \omega) \quad \text{and} \quad \Phi(t, \theta_s \omega, S(s, \omega)) = S(t + s, \omega), \quad \text{for all } s \in \mathbb{T}, \ t \in \mathbb{T}^+. \quad (1.2)
\]
The study of random periodic solutions is more fascinating and difficult than deterministic
periodic solutions. An extra essential difficulty is from the fact that trajectory (solution path) of
the random dynamical systems does not follow a periodic path, but the pullback path \( \{ \eta(s, \omega) :=
S(s, \theta_{-s} \omega) : 0 \leq s \leq \tau \} \) is a periodic curve and random periodic path moves from one periodic
curve to another one corresponding to different \( \omega \). If one considers a family of tajectories starting
from different points on the closed curve $\eta(., \omega)$, then the whole family of trajectories at time $t \in \mathbb{T}$ will lie on a closed curve $\eta(., \theta_t \omega)$.

Existence of random periodic solutions of stochastic (partial) differential equations were investigated by Feng, Zhou and Zhao [16] and Feng and Zhao [17]. Their results are based on infinite horizon stochastic integral equations and an Wiener-Sobolev compact embedding argument. In fact, one of the technical assumptions in their works was some boundedness conditions on the vector fields associated with the stochastic (partial) differential equations, though these conditions can be removed given more conditions such as weak dissipativity [18]. We employ the Lyapunov function technique to characterize the boundedness conditions (dissipativity of the stochastic flow) to prove the existence of unique stable random periodic solution. The conditions of our results are quite natural to some applicable SDEs and verifiable in terms of their coefficients. It is proved in a number of works (e.g., [27] and references therein) that the technique of Lyapunov function could be used to provide a bound to the top Lyapunov exponent of stochastic flows. We previously employed semiuniform ergodic theory approach to prove the existence of random periodic solutions in [33] where the conditions on the bound of top Lyapunov exponent was central. The Lyapunov function technique discussed here makes it easier to establish ergodicity of periodic measures induced by the random periodic solutions.

The purpose of ergodicity is to study invariant measures and related problems. It is one of the well studied problems in dynamical systems, stochastic analysis, statistical physics and related areas. Roughly speaking, an ergodic dynamical system is one that its behaviour averaged over long time is the same as its behaviour over phase space. In the framework of RDS, ergodicity is the cornerstone in the investigation of long time behaviour, various techniques and their variants have been developed by many researchers (e.g., [2, 3, 5, 6, 11, 20, 28, 29, 30] and references therein). Many important results were established in the regime of (random) stationary measures and (random) stationary process. However, various assumptions involved, automatically exclude random periodic regime. Feng and Zhao [18] defined periodic measures and proved the Krylov-Bogolyubov procedure as a variant of Poincaré Bendixson theorem in the RDS framework. The Krylov-Bogolyubov procedure for periodic measures was also investigated by Hasminksii [20], in terms of transition probability function. We shall recover the ergodicity of periodic measures in [18] for some SDEs using the Lyapunov function technique.

The outline of the paper is as follows. In Section 2 we present some standard notation and definitions that will be employed in our proofs. In Section 3 we prove the existence of random periodic solutions by employing the two-point generator and Lyapunov function techniques. Section 4 is about periodic measures induced by the random periodic solutions and their ergodicity in a certain Poincaré section.

2 Preliminaries

In this section, we fix notation that will be frequently used throughout this paper. We also, introduce random periodic solutions for stochastic flows generated by time dependent SDEs with a simple example to fix the idea (for more examples, see [16, 17, 18, 33]).

On a complete separable metric space $(\mathbb{M}, d)$, we denote the set of bounded measurable real-valued functions by $\mathcal{B}_b(\mathbb{M})$ with the norm $\|f\|_\infty := \sup_{x \in \mathbb{M}} |f(x)|$; and the set of bounded
continuous real-valued functions by $C_b(M)$: Let $C^l_b(\mathbb{R}^d; \mathbb{R}^d)$ be the Banach space of the functions $f : \mathbb{R}^d \to \mathbb{R}^d$ which has $l$-th derivative being continuous with the norm
\[ \|f\|_l := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{1 + |x|} + \sum_{1 \leq \alpha \leq l} \sup_{x \in \mathbb{R}^d} |D^\alpha f(x)|. \]

Let $(M, \mathcal{B}(M), \mu)$ be a Borel measure space, we denote $L^p(M), 1 \leq p < \infty$ as the set of real-valued Lebesgue integrable functions with the norm $\|f\|_p = (\int_M |f|^p d\mu)^{1/p}$.

In what follows, we will consider the case $M = \mathbb{R}^d$ from time to time without causing confusions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathcal{G} \subseteq \mathcal{F}$, we denote $L^p(\Omega, \mathcal{G}, \mathbb{P}), p \geq 1$ as the space of $\mathcal{G}$-measurable random variables $X : \Omega \to \mathbb{R}^d, d \in \mathbb{N}$, such that $\mathbb{E}|X|^p < \infty$, equipped with the $L^p$ norm $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$.

We shall fix the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as the classical Wiener space, i.e., $\Omega = C_0(\mathbb{R}; \mathbb{R}^m), m \in \mathbb{N}$, is a linear subspace of continuous functions that take zero at $t = 0$, endowed with compact open topology defined via
\[ d(\omega, \hat{\omega}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|\omega - \hat{\omega}\|_n}{1 + \|\omega - \hat{\omega}\|_n}, \quad \|\omega - \hat{\omega}\|_n = \sup_{t \in [-n,n]} |\omega(t) - \hat{\omega}(t)|. \]

The sigma algebra $\mathcal{F}$ is the Borel sigma algebra generated by open subsets of $\Omega$ and $\mathbb{P}$ is the Wiener measure, i.e., that the law of the process $\omega \in \Omega$ with $\omega(0) = 0$. Let the $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable flow $\theta : \mathbb{R} \times \Omega \to \Omega$, be defined by
\[ \theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t). \tag{2.1} \]

It is well known that the measurable flow $(\theta_t)_{t \in \mathbb{R}}$ is ergodic (e.g., [3, 4]). Let $\mathcal{F}_s^t := \sigma(\omega(u) - \omega(v) : s \leq u, v \leq t) \cup \mathcal{N}$, where $\mathcal{N}$ is a collection of $\mathbb{P}$-null sets of $\mathcal{F}$, then $\mathcal{F}_s^t$ is a two parameter filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ and the metric dynamical system $\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_s^t)_{t \geq s}, (\theta_t)_{t \in \mathbb{R}})$ is a filtered dynamical system. Note $\theta_{-r}^{-1}(\mathcal{F}_s^t) = \mathcal{F}_{s+tr}^t, r \in \mathbb{R}, -\infty < s \leq t < \infty$.

We consider time dependent SDE on $\mathbb{R}^d$ of the form
\[
    dX = f_0(t, X)dt + \sum_{k=1}^{m} f_k(t, X)dW^k_t, \quad X(t_0) = x \in \mathbb{R}^d, \quad t \geq t_0. \tag{2.2}
\]

**Proposition 2.1 (Stochastic flows [23])** Suppose that the coefficients $f_k(t, x), 0 \leq k \leq m,$ of SDE (2.2) are continuous in $t$ and uniformly Lipschitz continuous with respect to $x \in \mathbb{R}^d$. Then there exists a modification of the solution of SDE (2.2), denoted by $X(t, t_0, \omega, x)$ which satisfies the following properties:

1. For each $t, t_0 \in \mathbb{T} \subseteq \mathbb{R}, t \geq t_0$ and $x, X(t, t_0, \cdot, x)$ is $\mathcal{F}_{t_0}^t$-measurable,

2. For almost all $\omega, X(t, t_0, \omega, x)$ is continuous in $(t, t_0, x)$ and satisfies
\[ X(t_0, t_0, \omega, x) = x, \]
for almost all $\omega$,
\[ X(t + u, t_0, \omega, x) = X(t + u, t, \omega, X(t, t_0, \omega, x)) \]
(2.3)
is satisfied for all $t, t_0 \in T \subseteq \mathbb{R}$, $t \geq t_0$ and $u > 0$,
(4) for almost all $\omega$, $X(t, t_0, \omega, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a homeomorphism for all $t \geq t_0$,
(5) if in addition, the coefficients $f_0, f_1, \ldots, f_m$ are differentiable in $x$ and their first derivatives are continuous and bounded with respect to $(t, x)$, then, for almost all $\omega$, $X(t, t_0, \omega, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism for all $t \geq t_0$.

**Definition 2.2 (Random periodic solution for stochastic flows [16, 17, 40])** A random periodic solution of period $\tau$ of a stochastic flow $X : \Delta \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is an $\mathcal{F}$-measurable function $S : T \times \Omega \to \mathbb{R}^d$ such that
\[ S(t + \tau, \omega) = S(t, \theta_{\tau} \omega) \quad \text{and} \quad X(t + s, s, \omega, S(t, \omega)) = S(t + s, \omega), \quad \text{a.s.,} \]
for any $t, s \in T; t \geq s$ and $\omega \in \Omega$, where $\Delta := \{(t, s) \in T^2; t \geq s\}$.

**Remark 2.3** Suppose that $\{X(t, s, \omega, \cdot); t \geq s\}$ is a time homogeneous flow, for example, a solution of an autonomous SDEs driven by independent Brownian motions. In this case, we can write
\[ X(t, s, \omega, x) = X(t - s, 0, \theta_{s \omega}, x) \quad \text{and} \quad \Phi(t, \omega, x) := X(t, 0, \omega, x). \]

(i) Let $S(s, \omega)$ be a random periodic solution with period $\tau$ in the sense of definition 2.2 we have
\[ \Phi(t, \omega, S(s, \omega)) = X(t, 0, \theta_{s \omega}, S(s, \omega)) = X(t + s - s, 0, \theta_{s \omega}, S(s, \omega)) = X(t + s, s, \omega, S(s, \omega)) = S(t + s, \omega), \]
(2.4)
corresponding to the definition of random periodic solution we have in the introduction (equation (1.2)) for a cocycle.

(ii) On the other hand, if we set $\eta(s, \omega) := S(s, \theta_{-s} \omega)$, so that,
\[ \eta(s + \tau, \omega) = S(s + \tau, \theta_{-\tau} \omega) = S(t, \theta_{\tau} \circ \theta_{-\tau} \omega) = \eta(t, \omega), \]
and using the fact that $S(s, \omega)$ is a random periodic solution of a homogeneous flow $X(t, s, \omega, \cdot)$, we have that
\[ \eta(t + s, \theta_{s \omega}) = S(t + s, \theta_{s \omega}) = X(t + s, s, \theta_{s \omega}, S(s, \theta_{s \omega})) = X(t, 0, \omega, \eta(s, \omega)) = \Phi(t, \omega, \eta(s, \omega)). \]

Thus,
\[ \eta(s + \tau, \omega) = \eta(s, \omega) \quad \text{and} \quad \Phi(t, \omega, \eta(s, \omega)) = \eta(t + s, \theta_{s \omega}), \]
(2.5)
corresponding to random periodicity in the pullback sense considered in [40] and in the more recent works [33, 36, 37].
Example 2.4 Consider the following SDE
\[ dX = -\alpha(t)X dt + dW_t, \quad X(t_0) = x_0 \in \mathbb{R}^d, \quad t \geq t_0, \quad (2.6) \]
where \( \alpha : \mathbb{R} \to \mathbb{R} \) is a continuous function and there exists \( \tau > 0 \) such that \( \alpha(t + \tau) = \alpha(t) \) with
\[ \int_{-\infty}^{t} e^{-2 \int_{\tau}^{t} \alpha(u) du} ds < \infty, \quad \text{for} \quad 0 \leq t \leq \tau. \]
The random variable \( S(s, \omega) \) defined by
\[ S(s, \omega) = \int_{-\infty}^{s} e^{-\int_{\tau}^{t} \alpha(u) du} dW_r(\omega) \]
is a random periodic solution of the stochastic flow \( X(t, t_0, \omega, x_0) \) generated by SDE (2.6) defined by
\[ X(t, t_0, \omega, x_0) = x_0 e^{-\int_{t_0}^{t} \alpha(u) du} + \int_{t_0}^{t} e^{-\int_{s}^{t} \alpha(u) du} dW_s(\omega). \]
Indeed, by suitable change of variable and the periodicity of \( \alpha \), we have that
\[ S(s, \theta_{\tau} \omega) = \int_{-\infty}^{s} e^{-\int_{\tau}^{s} \alpha(u) du} dW_r(\omega) = S(s + \tau, \omega), \]
and
\[ X(t + s, s, \omega, S(s, \omega)) = e^{-\int_{t}^{t+s} \alpha(u) du} \int_{-\infty}^{s} e^{-\int_{\tau}^{s} \alpha(u) du} dW_r(\omega) + \int_{s}^{t+s} e^{-\int_{s}^{t} \alpha(u) du} dW_r(\omega) \]
\[ = \int_{-\infty}^{t+s} e^{-\int_{\tau}^{t} \alpha(u) du} dW_r(\omega) = S(t + s, \omega). \quad \square \]

3 Existence of random periodic solutions

We adopt the approach of studying the infinitesimal separation of trajectories of SDEs via their Markov evolution to prove the existence of stable random periodic solutions. For this, we recall a standard notion of the transition probability function \( P(t_0, x; t, A) \) induced by solutions of SDE (2.6),
\[ P(t_0, x; t, A) = \mathbb{P}\{ \omega \in \Omega : X(t, t_0, \omega, x) \in A \}, \quad t \geq t_0, \quad A \in \mathcal{B}(\mathbb{R}^d). \quad (3.1) \]
The Markov evolution \( T_{t,t_0} : \mathcal{C}_b(\mathbb{R}^d) \to \mathcal{C}_b(\mathbb{R}^d) \) is given by
\[ T_{t,t_0} h(x) = \int_{\mathbb{R}^d} h(y) P(t_0, x; t, dy) = \mathbb{E}[h(X(t, t_0, \omega, x))] \]
and for any probability measure \( \mu \) on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \),
\[ (T_{t,t_0}^* \mu)(A) = \int_{\mathbb{R}^d} P(t_0, x; t, A) \mu(dx), \quad \text{for any} \quad t \geq t_0 \quad \text{and} \quad B \in \mathcal{B}(\mathbb{R}^d). \quad (3.2) \]
The pioneeering work of Has'minskii [20] championed the stability theory of SDEs, by systematically adopting the concept Lyapunov function $V$ for the SDEs. The flavour in this concept is the fact that the average growth of a function $V$ along the trajectory $X(t, t_0, \omega, x)$ is expressed by
\[
\mathcal{L}V(t_0, x) = \lim_{t \downarrow t_0} \frac{E[V(t, X(t, t_0, \omega, x))] - V(t_0, x)}{|t - t_0|}. \tag{3.3}
\]
For $V \in C^{1,2}_b(\mathbb{R} \times \mathbb{R}^d)$, we can use Itô’s formula to write $\mathcal{L}$ as
\[
\mathcal{L}V(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^d f_i^0(t, x)\frac{\partial V(t, x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \sum_{k=1}^m f_k^i(t, x)f_k^j(t, x)\frac{\partial^2 V(t, x)}{\partial x_i \partial x_j}.
\]
The generator $\mathcal{L}$ determines the law of the one-point motions $\{X(t, t_0, \omega, x) : x \in \mathbb{R}^d, t \geq t_0\}$. Has'minskii [20] established the stability of solution of SDEs using the differential operator $\mathcal{L}$, more work in this direction can be found in (e.g., [27]). The idea in Has’minskii’s technique is to investigate the difference between two trajectories of a stochastic differential equation, where one of the trajectories is a deterministic (zero) solution. However, due to random fluctuations, zero solution of an SDE may not exist in general, for instance, the simple SDE in example [1, 4]. In order to investigate the infinitesimal separation of two nontrivial trajectories, one requires a differential operator that gives information about the joint distribution between these trajectories. From the definition of $\mathcal{L}$, the term $f_k^i(t, x)f_k^j(t, x)$ in the sum only contains the diagonal entries of the infinitesimal covariance of the stochastic flow $\{X(t, t_0, \omega, .) : t \geq t_0\}$, so $\mathcal{L}$ does not determine the law of stochastic flows (cf. [3, 5, 24]).

It is known (e.g., [3, 5, 24]) that the law of stochastic flows driven by Brownian motion is determined by the generator $\mathcal{L}^{(2)}$ of the two-point motions $\{(X(t, t_0, \omega, x), X(t, t_0, \omega, y)) : (x, y) \in \mathbb{R}^{2d}, t \geq t_0\}$. We denote the transition probability function and the Markov evolution corresponding to the two point motions $\{(X(t, t_0, \omega, x), X(t, t_0, \omega, y)) : (x, y) \in \mathbb{R}^{2d}, t \geq t_0\}$ by $P^{(2)}(t_0, (x, y); t, .)$ and $T^{(2)}_{t,t_0}$ respectively, and are defined by
\[
P^{(2)}(t_0, (x, y); t, E) := \mathbb{P}\{\omega : (X(t, t_0, \omega, x), X(t, t_0, \omega, y)) \in E\}, \quad t \geq t_0, \quad E \in B(\mathbb{R}^{2d})
\]
and
\[
T^{(2)}_{t,t_0}h(x, y) := \int_{\mathbb{R}^{2d}} h(u, z)P^{(2)}(t_0, (x, y); t, dz \otimes du)
\]
\[
= E[h(X(t, t_0, \omega, x), X(t, t_0, \omega, y))), \quad h \in C_b(\mathbb{R}^{2d}).
\]
For $V \in C^{1,2}_b(\mathbb{R} \times \mathbb{R}^{2d})$, the generator $\mathcal{L}^{(2)}$ is defined by
\[
\mathcal{L}^{(2)}V(t_0, x, y) = \lim_{t \downarrow t_0} \frac{E[V(t, X(t, t_0, \omega, \omega, x), X(t, t_0, \omega, \omega, y))] - V(t_0, x, y)}{|t - t_0|}
\]
and, by Itô’s formula, we have $\mathcal{L}^{(2)}$ in the form
\[
\mathcal{L}^{(2)} := \frac{\partial}{\partial t} + \sum_{i=1}^d f_i^0(t, x)\frac{\partial}{\partial x_i} + \sum_{i=1}^d f_i^1(t, y)\frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^d \sum_{k=1}^m f_k^i(t, x)f_k^j(t, y)\frac{\partial^2}{\partial x_i \partial y_j},
\]
where \( f^{i}_{k}(t, x, y) := f^{i}_{k}(t, x) \) and \( f^{i+d}_{k}(t, x, y) := f^{i}_{k}(t, y) \), \( i = 1, 2, \ldots, d \).

In particular, considering the difference between two solutions starting from two different initial values \( x, y \in \mathbb{R}^d \) i.e., the process \( \{X(t, t_0, \omega, x) - X(t, t_0, \omega, y) : x \neq y, t \geq t_0\} \), the two-point generator \( \mathcal{L}^{(2)} \) simplifies to

\[
\mathcal{L}^{(2)} V(t, x - y) = V_{t}(t, x - y) + \sum_{i=1}^{d} V_{x_{i}}(t, x - y) \left( f_{0}(t, x) - f_{0}(t, y) \right)
+ \frac{1}{2} \text{tr} \left( \left[ \sigma(t, x) - \sigma(t, y) \right]^{T} \mathcal{H} \left( t, x - y \right) \left[ \sigma(t, x) - \sigma(t, y) \right] \right),
\]

(3.4)

where \( \sigma(t, x) = (f_{1}(t, x), f_{2}(t, x), \cdots, f_{m}(t, x)) \), \( V_{x} = \left( \frac{\partial V}{\partial x_{i}} \right)_{1 \leq i \leq d} \) and \( \mathcal{H} = \left( \frac{\partial^{2} V}{\partial x_{i} \partial x_{j}} \right)_{1 \leq i, j \leq d} \).

Some stability and convergence results for stochastic flows from nontrivial reference solutions on a smooth manifold are normally via two-point generator (e.g., [5, 6, 11]). Evidently, the two-point generator \( \mathcal{L}^{(2)} \) is related to the one-point generator \( L \) of the derivative flow \( \{DX(t, t_0, \omega, x) : t \geq t_0\} \) (e.g., [2, 3, 5, 11]). The consideration of two-point generator \( \mathcal{L}^{(2)} \) is also one of the ways of making SDEs amenable to smooth ergodic theory. Schmalfuss in [32] studied the existence of random stationary solutions and random attractors by Lyapunov function defined via two-point generator \( \mathcal{L}^{(2)} \), our argument in this section, follows similar ideas in [32].

Before presenting our results, we recall a variant of Doob’s local martingale inequality that will be useful in the proofs.

**Proposition 3.1 (Exponential local martingale inequality (cf. [27]))** Let \( M = (M_t)_{t \geq 0} \) be a continuous local martingale. Then for any positive constants \( \tau, \gamma, \delta \), we have

\[
P \left\{ \omega : \sup_{t \leq \tau} \left( M_{t} - \frac{\gamma}{2} \langle M \rangle_{t} \right) > \delta \right\} \leq \exp(-\gamma \delta).
\]

In particular, let \( (M_t)_{t \geq 0} \) be a continuous real-valued local martingale vanishing at \( t = 0, (\tau_{k})_{k \geq 1} \) and \( (\gamma_{k})_{k \geq 0} \) be two sequences of positive numbers with \( \tau_{k} \to \infty, g(t) \) be a positive increasing function on \( \mathbb{R}^{+} \) such that

\[
\sum_{k=1}^{\infty} g(k)^{-\theta} < \infty, \quad \text{for some } \theta > 1.
\]

Then, for almost all \( \omega \in \Omega \) there is a random integer \( k_{0}(\omega) \) such that for all \( k \geq k_{0}(\omega) \)

\[
M_{t} \leq \frac{\gamma_{k}}{2} \langle M \rangle_{t} + \frac{\theta}{\gamma_{k}} \log(g(k)) \quad \text{on } 0 \leq t \leq \tau_{k}.
\]

**Theorem 3.2 (Existence of random periodic solution)** Let \( X(t, t_0, \omega, \cdot) \) be a stochastic flow of diffeomorphisms induced by the SDE \( \{\text{SDE} \} \) and suppose that \( f_{0}, f_{1}, f_{2}, \ldots, f_{m} \) are periodic in \( t \) with period \( \tau > 0 \). Let \( V \in C^{1,2}(\mathbb{R}^{+} \times \mathbb{R}^d, \mathbb{R}^{+}) \) with \( V(t, 0) = 0 \). Suppose there exist a function \( \lambda : \mathbb{R} \to \mathbb{R} \) such that

\[
\lim_{t \to \infty} \sup_{\frac{1}{2t}} \int_{t_0}^{t} \lambda(s)ds < \alpha < 0,
\]

(3.5)
\[ \mathcal{L}^{(2)}V(t, x - y) \leq \lambda(t)V(t, x - y), \quad \text{and} \quad |x|^p \leq V(t, x), \quad (3.6) \]

for all \( t \in \mathbb{R}, x, y \in \mathbb{R}^d, p \geq 1 \). Suppose further that

\[ \mathbb{E}\sup_{t > t_0} \ln V(t_0, X(t_0, \omega, x) - x) < \infty. \quad (3.7) \]

Then, there exists an \( \mathcal{F}_{-\infty}^t \)-measurable random variable \( S(t, \omega) \) which is the random periodic solution in the sense of Definition 2.2.

**Proof.** The idea is to show that \( \{X(t, t - n\tau, \omega, x)\}_{n \in \mathbb{N}} \) is a Cauchy sequence in a space of continuous function \( C([t_0, \infty); \mathbb{R}^d) \). For this, we shall first modify the initial value by a random variable in such a way that \( X(t, t_0, \omega, x) \neq X(t, t_0, \omega, y) \) for some \( y \in \mathbb{R}^d \) and all \( t > t_0 \). This modification is necessary to avoid difficulties in the definiteness of some integrals and logarithms. We can disregard this modification at the end, using the fact \( X(t, t_0, \omega, x) \) is a homeomorphism (e.g., [23, 24]) so that \( X(t, t_0, \omega, x) = X(t, t_0, \omega, y) \) if and only if \( x = y \), (see [32] for the case of time homogeneous flows, where such modification was made explicitly).

Denote \( X^x(t) := X(t, t_0, \omega, x) \), \( X^y(t) := X(t, t_0, \omega, y) \), we apply Itô’s formula on \( \ln(V(t, X^x(t) - X^y(t))) \) to get

\[
\ln V(t, X^x(t) - X^y(t)) = \ln V(t_0, x - y) + \int_{t_0}^{t} \frac{\mathcal{L}^{(2)}V(s, X^x(s) - X^y(s))}{V(s, X^x(s) - X^y(s))} ds + N(t, \omega) - \frac{1}{2}q(t, \omega),
\]

where

\[
N(t, \omega) = \int_{t_0}^{t} (\mathcal{G}V(s, X^x(s), X^y(s))) dW_s, \quad q(t, \omega) = \int_{t_0}^{t} (\mathcal{G}V(s, X^x(s), X^y(s)))^2 ds,
\]

\[
\mathcal{G}V(t, x, y) = \frac{V_x(t, x - y)(g(t, x) - g(t, y))}{V(t, x - y)}, \quad W_t = (W^1_t, \ldots, W^m_t).
\]

We observe that \( q(t, \omega) \) is a quadratic variation of \( N(t, \omega) \) and as a consequence of Proposition 3.1 we have that

\[
\mathbb{P}\left\{ \omega \in \Omega : \sup_{t \in [t_0, t_0 + k]} \left( N(t, \omega) - \frac{1}{2}q(t, \omega) \right) > 2 \ln k \right\} \leq \frac{1}{k^2}.
\]

An application of Borel-Cantelli lemma yields that for almost all \( \omega \in \Omega \), there exists a random integer \( n_0 = n_0(\omega) \) such that for any \( n \geq n_0 \)

\[
\sup_{t_0 \leq t \leq t_0 + n} \left( N(t, \omega) - \frac{1}{2}q(t, \omega) \right) \leq 2 \ln n.
\]

In particular,

\[
\frac{1}{t} \left( N(t, \omega) - \frac{1}{2}q(t, \omega) \right) \leq \frac{2 \ln n}{t_0 + n - 1}, \quad \text{for} \quad t_0 + n - 1 \leq t \leq t_0 + n. \quad (3.8)
\]
Next, applying the assumptions on \( V \) and on the two point generator (3.6), then the integrability condition (3.5), we have that

\[
\frac{1}{k} \sup_{n-1 \leq t \leq n} \ln |X^x(t) - X^y(t)| \\
\leq \frac{1}{p(n-1)} \sup_{n-1 \leq t \leq n} \ln V(t, X^x(t) - X^y(t)) \\
\leq \frac{1}{p(n-1)} \ln V(t_0, x - y) + \frac{1}{p(n-1)} \sup_{n-1 \leq t \leq n} \int_{t_0}^{t} \frac{\mathcal{L}^2 V(s, X^x(s) - X^y(s))}{V(s, X^x(s) - X^y(s))} ds \\
+ \frac{1}{p(n-1)} \sup_{n-1 \leq t \leq n} \left( N(t, \omega) - \frac{1}{2} q(t, \omega) \right) \\
\leq \frac{1}{p(n-1)} \ln V(t_0, x - y) + \sup_{t \in [n-1, n]} \frac{1}{p(n-1)} \int_{t_0}^{t} \lambda(s) ds + 2 \frac{\ln n}{p(n-1)}.
\]

So, for \( k \) large enough and for \( \varepsilon > 0 \), we have that

\[
|X(t, t_0, \omega, x) - X(t, t_0, \omega, y)| \leq (V(t_0, x - y)) \exp((\frac{\alpha}{p} + \varepsilon)(t - t_0)), \quad \text{for all } t \geq t_0. \tag{3.9}
\]

Let \( n \geq N \), for \( 0 < \varepsilon < -\frac{\alpha}{2p} \) and by flow property we have that

\[
\frac{1}{n\tau} \ln |X(t, t - n\tau, \omega, x) - X(t, t - N\tau, \omega, x)| \\
= \frac{1}{n\tau} \ln |X(t, t - n\tau, \omega, x) - X(t, t - n\tau, \omega, x, X(t - n\tau, t - N\tau, \omega, x))| \\
\leq \frac{1}{p(n\tau)} \ln V(t - n\tau, x - X(t - n\tau, t - N\tau, \omega, x)) + \frac{\alpha}{p} + \varepsilon.
\]

From the assumption that \( \mathbb{E}\sup_{t \geq t_0} \ln V(t_0, X(t, t_0, \omega, x) - x) < \infty \), we have that for almost all \( \omega \in \Omega \), there exist a random variable \( \beta(\omega) \) with \( \mathbb{E}[|\beta(\omega)|] < \infty \) such that

\[
\sup_{t \geq t_0} |X(t, t - n\tau, \omega, x) - X(t, t - N\tau, \omega, x)| \leq \beta \exp(\frac{\alpha n\tau}{2p}), \quad \text{a.s.} \tag{3.10}
\]

So, the sequence \( \{X(t, t - n\tau, \omega, x)\}_{n \in \mathbb{N}} \) is Cauchy in the space \( C([t_0, \infty); \mathbb{R}^d) \). Let \( S(t, \omega) \) be the limit of the sequence \( \{X(t, t - n\tau, \omega, x)\}_{n \in \mathbb{N}} \), then by the continuity of the stochastic flow map \((t, t_0, x) \mapsto X(t, t_0, \omega, x)\), we deduce for \( \tau > 0 \) that

\[
X(t + \tau, t, \omega, S(t, \omega)) = X(t + \tau, t, \omega, \lim_{n \to \infty} X(t, t - n\tau, \omega, x)) \\
= \lim_{n \to \infty} X(t + \tau, t + \tau - (n - 1)\tau, \omega, x) = S(t + \tau, \omega). \tag{3.11}
\]

Now, since the coefficients \( f_k : 0 \leq k \leq m, \) of our SDE are time periodic with period \( \tau \), we have
that
\[ X(t, t - n\tau, \theta_t \omega, x) = x + \int_{t-n\tau}^{t} f_0(r, X(r, r - n\tau, \theta_t \omega, x))dr \]
\[ + \int_{t-n\tau}^{t} \sum_{k=1}^{m} f_k(r, X(r, r - n\tau, \theta_t \omega, x))dW^k_r(\omega) \]
\[ = x + \int_{t-n\tau}^{t+\tau} f_0(r, X(r - \tau, r - \tau - n\tau, \theta_t \omega, x))dr \]
\[ + \int_{t-n\tau}^{t+\tau} \sum_{k=1}^{m} f_k(r, X(r - \tau, r - \tau - n\tau, \theta_t \omega, x))dW^k_r(\omega), \]
and
\[ X(t + \tau, t + \tau - n\tau, \omega, x) = x + \int_{t+\tau-n\tau}^{t+\tau} f_0(r, X(r, r - n\tau, \omega, x))dr \]
\[ + \int_{t+\tau-n\tau}^{t+\tau} \sum_{k=1}^{m} f_k(r, X(r, r - n\tau, \omega, x))dW^k_r(\omega). \]

By uniqueness of solution of SDE and the invariance of \( \theta \) under \( P \), we have that
\[ X(t, t - n\tau, \theta_t \omega, x) = X(t + \tau, t + \tau - n\tau, \omega, x), \quad P-a.s. \]
and then taking limit of both sides, we have from equation (3.11) that
\[ S(t, \theta_t \omega) = S(t + \tau, \omega), \quad P-a.s. \]

Moreover, from the inequality (3.9), we see that for any \( \mathcal{F}_{-\infty}^\tau \)-measurable random variable \( \xi(r, \omega) \)
\[ \lim_{n \to \infty} \sup_{t \in [n-1, n]} |S(t, \omega) - X(t, r, \omega, \xi(r, \omega))| = 0, \quad P-a.s. \quad (3.12) \]
 exponentially fast. \( \square \)

We have the following corollary that captures cases of nonautonomous SDEs in many applications. They also arise in many theoretical study in stochastic analysis and nonlinear dynamical systems.

**Corollary 3.3** Let \( \{X(t, t_0, \omega, .) : t \geq t_0\} \) be a stochastic flow of diffeomorphisms induced by the SDE \( (2.2) \) with \( \tau \)-periodic coefficients in time such that
\[ \begin{cases} \langle f_0(t, x) - f_0(t, y), x - y \rangle \leq \beta(t)|x - y|^2, \\ |f_k(t, x) - f_k(t, y)| \leq L(t)|x - y|, \quad 1 \leq k \leq m, \end{cases} \quad (3.13) \]
for all \( x, y \in \mathbb{R}^d, \ t \in \mathbb{R} \) and for some integrable functions \( \beta : \mathbb{R} \to \mathbb{R} \) and \( L : \mathbb{R} \to \mathbb{R}^+ \), such that
\[ \lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \beta(s)ds < 0, \quad \lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} L(s)ds < \infty. \quad (3.14) \]
Then the stochastic flow \( \{X(t, t_0, \omega, .) : t \geq t_0\} \) generated by \( (2.2) \) has a random periodic solution.
Proof. From (3.14), it is easy to know that there exists a real number \( p > 1 \) such that function 
\[
\lambda(s) := \beta(s) + \frac{(p-1)}{2}mL^2(s)
\] satisfies 
\[
\limsup_{t \to \infty} \frac{1}{2t} \int_{t_0}^{t} \lambda(s)ds < 0. \tag{3.15}
\]
Take \( V(t, x) = |x|^p \) for some \( p > 1 \) and compute \( \mathcal{L}^2 V(t, x - y) \),
\[
\frac{\partial V(t, x)}{\partial x^i} = px^i\left( \sum_{n=1}^{d}(x^n)^2 \right)^{\frac{p}{2}-1} = px^i|x|^{p-2}, \quad \frac{\partial^2 V(t, x)}{\partial x^i\partial x^j} = 2\left( \frac{p}{2} - 1 \right)x^ix^j|x|^{p-4} + \delta_{i,j}p|x|^{p-2},
\]
where \( \delta_{i,j} \) is the Kronecker symbol. We now have for \( 1 \leq k \leq m, \)
\[
\sum_{k=1}^{m} \sum_{i,j=1}^{d} (x^i - y^i)(f^i_k(t, x) - f^i_k(t, y))(f^j_k(t, x) - f^j_k(t, y))(x^j - y^j) \leq mL^2(t)|x - y|^4,
\]
\[
\sum_{k=1}^{m} \sum_{i,j=1}^{d} (f^i_k(t, x) - f^i_k(t, y))(f^j_k(t, x) - f^j_k(t, y)) \leq mL^2(t)|x - y|^2,
\]
and
\[
\mathcal{L}^{(2)} V(t, x - y) = \sum_{i=1}^{d} \left( f^i_0(t, x) - f^i_0(t, y) \right)p(x^i - y^i)|x - y|^{p-2}
\]
\[
+ \frac{1}{2} \sum_{i,j=1}^{d} \left( \sum_{k=1}^{m} (x^i - y^i)(f^i_k(t, x) - f^i_k(t, y))(f^j_k(t, x) - f^j_k(t, y))(x^j - y^j) \right) \times
\]
\[
\left\{ 2\left( \frac{p}{2} - 1 \right)p|x - y|^{p-4} \right\}
\]
\[
+ \left( \sum_{k=1}^{m} (f^i_k(t, x) - f^i_k(t, y))(f^j_k(t, x) - f^j_k(t, y))\delta_{i,j}p|x - y|^{p-2} \right)
\]
\[
\leq p\beta(t)|x - y|^p + \frac{pmL^2(t)}{2}(p-1)|x - y|^p = p\lambda(t)|x - y|^p.
\]
Take \( \hat{\lambda}(t) = p\lambda(t) \), it then follows from condition (3.14) that
\[
\limsup_{t \to \infty} \frac{1}{2t} \int_{t_0}^{t} \hat{\lambda}(s)ds < 0.
\]
As \( V(t, x) = |x|^p \), a one-point motion argument leads to if \( X_0 \in L^p(\Omega) \) then \( X(t, t_0, ., X_0) \in L^p(\Omega) \), it the follows that \( \mathbb{E}[V(t_0, X(t, t_0, \omega, x))] < \infty \), which gives us the temperedness assumption (3.17) on the random variable \( V(X(t, t_0, \omega, x)) \) (see Lemma 1 in [32]).

Remark 3.4 (i) If the functions \( \beta \) and \( L \) are continuous and periodic with period \( \tau \), then (3.14) can be replaced by a simple condition
\[
\int_{0}^{\tau} \beta(s)ds < 0. \tag{3.16}
\]
(ii) Random periodic processes arise naturally in stochastic dynamical models in climatology, neuroscience, economics, molecular dynamics, etc. This is due to the nonlinearity of the underlying vector fields and the onset of time-dependent random invariant sets, even in the case of temporal homogeneous vector fields. We proved the existence of stable random periodic solutions and ergodicity of periodic measures for dissipative stochastic stochastic differential equations. The assumption (3.16) we imposed is given in the sense of average, which is weaker than pointwise dissipativity. This is natural in some physical models and can be verified in terms of the coefficients. For example the following stochastic differential equations

\[ dx = [(-1 + \gamma \sin t)x - \delta x^3]dt + dW(t), \quad (3.17) \]

where \( W(t) \) is a one-dimensional Brownian motion and \( \delta \geq 0 \) and \( \gamma \in \mathbb{R} \) are real numbers, has a random periodic solution of period \( 2\pi \) according to Corollary 3.3. This natural result has not been discovered before.

4 Ergodicity in the random periodic regime

In this section, we discuss ergodicity in the random periodic regime by considering probability measures induced by random periodic solutions. As we noted in the introduction, RDS theory is a systematic mix of stochastic analysis and dynamical systems theory. From stochastic analysis perspective, invariant probability measures are investigated via Markov transition probability function. In this sense, ergodic theory is based on the dynamics of Markov evolution. From dynamical systems point of view, one studies random invariant probability measures whose conditional expectation with respect to a subalgebra of \( \mathcal{F} \) has one to one correspondence with the invariant measure of Markov evolution.

Here, we are interested in capturing ergodicity of the transition probability function in the random periodic regime (ergodicity of the law of random periodic solutions). PS-ergodicity is a new form of ergodicity for stochastic dynamical systems in the random periodic setting, recently developed by Feng and Zhao in [18]. It gives a new perspective and generalised form of Poincaré-Bendixson theorem for stochastic dynamical systems. We would like to argue using the information provided to us by the two-point generator, that the law of random periodic solutions form a family of PS-ergodic periodic measures.

**Definition 4.1 (Periodic measure [18])** Let \( \mathcal{M} \) be a Polish space, a measure \( \mu : \mathbb{R} \rightarrow \mathcal{P}(\mathcal{M}) \) is called a periodic measure of period \( \tau \) on the phase space \( (\mathcal{M}, \mathcal{B}(\mathcal{M})) \) for the Markovian stochastic flow \( \{X(t, s, \omega, \cdot) : t \geq s\} \) if for \( B \in \mathcal{B}(\mathcal{M}) \) we have that

\[ \mu_{s+t} = \mu_s \quad \text{and} \quad \mu_{t+s}(B) = \int_\mathcal{M} P(s, x; t+s, B)\mu_s(dx) = (T_{t+s}^s\mu_s)(B), \quad s \in \mathbb{R}, \quad t \in \mathbb{R}^+. \quad (4.1) \]

It is called a periodic measure with minimal period \( \tau \), if \( \tau > 0 \) is the smallest number such that (4.1) holds.

\(^{1}\)Ergodicity on Poincaré sections
Let $S : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ be the random periodic solutions of the stochastic flow \( \{X(t, s, \omega, \cdot) : t \geq s\} \). We consider the probability measure
\[
\mu_s(A) := \left( \mathbb{P} \circ S^{-1}(s, \cdot) \right)(A) = \mathbb{P}\{\omega : S(s, \omega) \in A\}, \quad A \in \mathcal{B}(\mathbb{R}^d).
\]
(4.2)

Then the measure \( \mu_s \) is \( \tau \)-periodic as
\[
\mu_{s+\tau}(A) = \mathbb{P}\{\omega : S(s + \tau, \omega) \in A\} = \mathbb{P}\{\omega : S(s, \theta_{\tau}\omega) \in A\} = \mathbb{P}\{\omega : S(s, \omega) \in A\} = \mu_s(A),
\]
(4.3)

Moreover, as it was shown in [18], \( \mu_s \) satisfies (4.1). Thus, the law of random periodic solution satisfies Definition 4.1.

**Definition 4.2 (Poincaré section for transition probability [18])** The collection of subsets \( \{L_s : s \geq 0\} \subset \mathcal{B}(\mathbb{R}^d) \), are called the Poincaré sections of the transition probability function \( P(s, x; t, \cdot) \) if
\[
L_{s+\tau} = L_s,
\]
and for any \( s \in [0, \tau), \ t \geq 0, \)
\[
P(s, x; s + \tau, L_{s+\tau}) = 1, \quad x \in L_s.
\]

**Remark 4.3** The choice of Poincaré section is not unique, example \( L_s = \mathbb{R}^d \) and \( L_s = \text{supp}(\mu_s) \) satisfy the definition of Poincaré section. However, the family \( L_s = \text{supp}(\mu_s) : s \in \mathbb{R} \) is a minimal Poincaré section [18] or \( n_\tau \)-irreducible Poincaré section. To see this, fix \( s \in [0, \tau) \) and any open set \( A_s \subset \text{supp}(\mu_s) \) with \( \mu_s(L_s \setminus A_s) > 0 \), we have for all \( x \in L_s, \)
\[
P(s, x; s + n_\tau, A_s) < 1, \quad n \in \mathbb{N}.
\]

This implies that \( A_s \) is not a Poincaré section for the transition probability \( P(s, x; s + n_\tau, \cdot) \), \( n \in \mathbb{N} \).

**Definition 4.4 (PS-ergodicity [18])** The family of \( \tau \)-periodic measures \( (\mu_s)_{s \in \mathbb{R}} \subset \mathcal{P}(\mathbb{R}^d) \) is PS-ergodic, if for each \( s \in [0, \tau) \), \( \mu_s \) as an invariant measure of the Markov evolution \( (T_{s+k\tau})_{k \in \mathbb{N}} \) at the integral multiples of the period on the Poincaré section \( L_s \) is ergodic.

In fact, a periodic measure \( \mu : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}^d) \) is PS-ergodic, if for any \( A \in \mathcal{B}(\mathbb{R}^d) \) with \( A \subset L_s \) and \( \mathcal{T} \subset [0, \tau) \), we have
\[
\lim_{N \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \int_{\mathbb{R}^d} \int_{\mathcal{T}} \left( \frac{1}{N} \sum_{n=0}^{N-1} P(s, x; t + n\tau, A) - \mu_s(A) \right) dt \mu_s(dy) ds = 0. \tag{4.4}
\]
The equation (4.4) is the Krylov–Bogolyubov scheme for periodic measure [20] [18].

Given the above preparation, we are now ready to prove the PS-ergodicity of periodic measures generated by a class of SDEs with time periodic coefficients satisfying conditions of Theorem 3.2. Precisely, we want to prove the convergence of Krylov–Bogolyubov scheme for periodic measures. However, strong Feller property of the Markov evolution \( (T_{s+n\tau})_{n \in \mathbb{N}} \) is crucial in the proof of the convergence of Krylov–Bogolyubov scheme. Recall that a Markov evolution \( (T_{s+n\tau})_{n \in \mathbb{N}} \) has strong Feller property if \( \mathcal{B}_c(\mathbb{R}^d) \ni \varphi \rightarrow T_{s+n\tau}\varphi \in \mathcal{C}_0(\mathbb{R}^d) \). Equivalently, the Markov evolution \( (T_{s+n\tau})_{n \in \mathbb{N}} \) has strong Feller property if and only if
This implies that for $t E$ have

where $\sigma$ that Theorem 3.2, there exists $f$ SDE. To this end, we start with following lemma.

First, we show for any $\alpha$ formula (Theorem 8.1 in [23]) we have

$$\sup_{t \in [s, s+n\tau]} ||\sigma^{-1}(t, x)||_{HS} \leq K, \quad \forall x \in \mathbb{R}^d, \quad n \geq 1, \quad (4.5)$$

where $\sigma^{-1}(t, x)$ is the right inverse of $\sigma(t, x) := (f_1(t, x), \ldots, f_m(t, x))$.

Then, there exists $0 < K_C < \infty$ such that for $x, y \in L_s$ with $s \in [0, \tau)$, and $h \in C_b(\mathbb{R}^d)$, we have

$$|T_{s+n\tau, s} h(x) - T_{s+n\tau, s} h(y)| \leq \frac{K_C}{\sqrt{n\tau}} \exp \left( \frac{1}{2} \int_s^{s+n\tau} \lambda(r)dr \right) ||h||_{\infty} |x - y|. \quad (4.6)$$

Proof. First, we show for any $Y, Z \in L_p(\Omega, \mathcal{F}_{-\infty}, \mathbb{P})$ for $p \geq 1$ and $t \geq s$,

$$\mathbb{E}|X(t, s, \omega, Y(s, \omega)) - X(t, s, \omega, Z(s, \omega))|^p \leq C \exp \left( \int_s^t \lambda(r)dr \right) \mathbb{E}|Y(s, \omega) - Z(s, \omega)|^p. \quad (4.7)$$

For this, set $\alpha(t) = \exp \left( -\int_s^t \lambda(r)dr \right)$ and $M(t, s, \omega, x) = \sum_{k=1}^m \int_s^t f_k(r, x)dW_r^k$, then by Itô’s formula (Theorem 8.1 in [23]) we have

$$d\left( \alpha(t)V(t, X(t, s, \omega, Y) - X(t, s, \omega, Z)) \right)$$

$$= -\lambda(t)\alpha(t)V(t, X(t, s, \omega, Y) - X(t, s, \omega, Z))dt$$

$$+ \alpha(t)\mathcal{L}^{(2)}V(t, X(t, \omega, Y) - X(t, \omega, Z))dt$$

$$+ \alpha(t)V(t, X(t, s, \omega, Y) - X(t, s, \omega, Z))d\left( M(t, s, \omega, Y) - M(t, s, \omega, Z) \right)$$

$$\leq \alpha(t)V(t, X(t, s, \omega, Y) - X(t, s, \omega, Z))d\left( M(t, s, \omega, Y) - M(t, s, \omega, Z) \right).$$

This implies that for $t \geq s$,

$$\mathbb{E}\left( |X(t, s, \omega, Y(s, \omega)) - X(t, s, \omega, Z(s, \omega))|^p \right) \leq C \exp \left( \int_s^t \lambda(r)dr \right) \mathbb{E}|Y(s, \omega) - Z(s, \omega)|^p.$$
In particular, for $x \neq y \in L_s$, $s \in [0, \tau)$, we obtain

$$\mathbb{E}|X(s + n\tau, s, \omega, x) - X(s + n\tau, s, \omega, y)|^p \leq C \exp \left( \int_s^{s+n\tau} \lambda(r)dr \right) \mathbb{E}|x - y|^p. \quad (4.8)$$

Since the coefficients $(f_k)_k^m$ are such that the derivative flow $v_{s+n\tau, s} := D_x X(s + n\tau, s, \omega, x)v$ at $x \in L_s$ in the direction $v \in \mathbb{R}^d$ exists for almost all $\omega \in \Omega$ and $s \in [0, \tau)$, $n \in \mathbb{N}$, then as $y \to x$, we have

$$\mathbb{E}|v_{s+n\tau, s}| = \mathbb{E}|D_x X(s + n\tau, s, \omega, x)v| \leq C^{1/p} \exp \left( \int_s^{s+n\tau} \lambda(r)dr \right) |v|. \quad (4.9)$$

Next, by Itô’s formula, we have for $h \in C^2_b(\mathbb{R}^d)$,

$$h(X(s + n\tau, s, \omega, x)) = T_{s+n\tau, s}h(x) + \int_s^{s+n\tau} D_x (T_{s+n\tau, r}h)(X(r, s, \omega, x)) \sigma(r, X(r, s, \omega, x))dW_r.$$  

Multiplying both sides by $\int_s^{s+n\tau} \sigma^{-1}(r, X(r, s, \omega, x))D\!X(r, s, \omega, x)v dW_r$, taking expectation and applying Itô isometry and Fubini’s theorem, we have

$$\mathbb{E} \left( h(X(s + n\tau, s, \omega, x)) \int_s^{s+n\tau} \sigma^{-1}(r, X(r, s, \omega, x))D\!X(r, s, \omega, x)v dW_r \right)$$

$$= \mathbb{E} \left( \int_s^{s+n\tau} D_x (T_{s+n\tau, r}h)(X(r, s, \omega, x)) \cdot D_x X(r, s, \omega, x)v dW_r \right)$$

$$= \int_s^{s+n\tau} D_x \mathbb{E}[T_{s+n\tau, r}h(X(r, s, \omega, x))] v dW_r.$$  

Using Markov property of the stochastic flow $\{X(t, s, \omega, \cdot) : t \geq s\}$, we have that

$$\mathbb{E}[T_{s+n\tau, r}h(X(r, s, \omega, x))] = T_{r,s} \circ T_{s+n\tau, r}h(x) = T_{s+n\tau, r}h(x),$$

so that, for any $h \in C^2_b(\mathbb{R}^d)$, we arrive at

$$D_x T_{s+n\tau, s}h(x) = \frac{1}{n\tau} \mathbb{E} \left[ h(X(s + n\tau, s, \omega, x)) \int_s^{s+n\tau} \sigma^{-1}(r, X(r, s, \omega, x))v_{r,s} dW_r \right]. \quad (4.10)$$

Next, since $C^2_b(\mathbb{R}^d)$ is dense in $C^1_b(\mathbb{R}^d)$, we obtain a version of Bismut-Elworthy-Li formula (e.g., [12, 13]) (4.10) for all $h \in C^1_b(\mathbb{R}^d)$.

Since $C^1_b(\mathbb{R}^d)$ is dense in $C_b(\mathbb{R}^d)$, we have $(h_m)_{m \in \mathbb{N}} \subset C^1_b(\mathbb{R}^d)$ such that $h_m \to h \in C_b(\mathbb{R}^d)$ and

$$\lim_{m \to \infty} T_{s+n\tau, s}h_m(x) = T_{s+n\tau, s}h(x),$$

$$\lim_{m \to \infty} D_x T_{s+n\tau, s}h_m(x) \cdot v = \frac{1}{n\tau} \mathbb{E} \left[ h(X(s + n\tau, s, \omega, x)) \int_s^{s+n\tau} \sigma^{-1}(r, X(r, s, \omega, x)) \cdot v_{r,s} dW_r \right].$$
convergence being uniform (cf. [12]). On the other hand, since $f_0 \in C^1_b(\mathbb{R} \times \mathbb{R}_d, \mathbb{R})$, $f_k \in C^2_b(\mathbb{R} \times \mathbb{R}_d, \mathbb{R})$, $1 \leq k \leq m$ with the non-degeneracy condition (4.5), there exists a function $0 < \rho_{s+n\tau, s} \in C^1_b(\mathbb{R}^d) \times C^1_b(\mathbb{R}^d)$ such that $P(s, x; s + n\tau, dy) = \rho_{s+n\tau, s}(x, y)dy$ (e.g., [12, 20, 24]). This implies that

$$
\lim_{m \to \infty} D_x T_{s+n\tau, s} h_m(x) \cdot v = \lim_{m \to \infty} \int_{\mathbb{R}^d} h_m(y) D_x (\rho_{s+n\tau, s})(x, y) \cdot v dy
$$

$$
= \int_{\mathbb{R}^d} h(y) D_x \rho_{s+n\tau, s}(x, y) \cdot v dy = D_x (T_{s+n\tau, s} h)(x),
$$

so (4.10) holds for all $h \in C_b(\mathbb{R}^d)$.

Next, by the equality (4.10), Cauchy–Schwartz inequality, Itô isometry and the condition (4.5), we have

$$
|D_x T_{s+n\tau, s} h(x)v|^2 \leq T_{s+n\tau, s} h^2(x) \frac{1}{(n\tau)^2} \mathbb{E} \left( \int_s^{s+n\tau} |\sigma^{-1}(r, X(r, s, \omega, x)) \cdot \nu_{r,\ell}|^2 dr \right)
$$

$$
\leq \|h\|_\infty^2 K^2 \frac{1}{n\tau} \mathbb{E} \|D_x X(s + n\tau, s, \omega, x)\|^2 |v|^2
$$

Comparing (4.11) with (4.9), we have

$$
|D_x T_{s+n\tau, s} h(x)v| \leq K_C \sqrt{\frac{1}{n\tau}} \mathbb{E} \left( \int_s^{s+n\tau} \lambda(r) dr \right) \|h\|_\infty |v|, \quad x \in L_s, \; v \in \mathbb{R}^d,
$$

where $K_C = K \sqrt{C}$. Finally, let $\eta^\ell(x, y) = \ell x + (1 - \ell)y$, $x, y \in L_s$, $\ell \in [0, 1]$, for $h \in C_b(\mathbb{R}^d)$, the mean value theorem, leads to

$$
|T_{s+n\tau, s} h(x) - T_{s+n\tau, s} h(y)| = \left| \int_0^1 (T_{s+n\tau, s} h)(\eta^\ell(x, y)) \cdot (x - y) d\ell \right|
$$

$$
\leq K_C \sqrt{\frac{1}{n\tau}} \mathbb{E} \left( \int_s^{s+n\tau} \lambda(r) dr \right) \|h\|_\infty |x - y|.
$$

\[ \square \]

**Theorem 4.6** Suppose the conditions of Lemma 4.5 are satisfied. Moreover, assume there exist $\tau$-periodic function $V \in C^{1,2}(\mathbb{R} \times \mathbb{R}_d; \mathbb{R}^+)$ satisfying the conditions of Theorem 3.2 and $C \geq 1$ such that $V(t, x) \leq C|x|^p$ and

$$
\mathbb{E} \left[ \sup_{t \geq t_0} V(t_0, X(t, t_0, \omega, x) - x) \right] < \infty
$$

Then, the family of periodic measures $(\mu_s)_{s \in [0, \tau)}$ induced by the random periodic path $S(s, \omega)$ is PS-ergodic.

**Proof.** **Step 1:** First, we show that there exists $0 < \tilde{K} < \infty$ such that for any initial value $x \in L_p(\Omega, \mathcal{F}_s, \mathbb{P})$, $p \geq 1$, we have

$$
\mathbb{E} (|X(s + n\tau, s, \omega, x) - S(s + n\tau, \omega)|^p) \leq \tilde{K} \exp \left( \int_s^{s+n\tau} \lambda(u) du \right), \quad \forall n \in \mathbb{N}.
$$

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For this, we recall from the definition of random periodic solution that $S(s + n\tau, \omega) = X(s + n\tau, s, S(s, \omega))$, $\mathbb{P}$ - a.s., then the estimate (4.7) yields,

$$
\mathbb{E}\left( |X(s + n\tau, s, \omega, x) - S(s + n\tau, \omega)|^p \right) = \mathbb{E}\left( |X(s + n\tau, s, \omega, x) - X(s + n\tau, s, \omega, S(s, \omega))|^p \right) \\
\leq C \exp\left( \int_s^{s+n\tau} \lambda(u)du \right) \mathbb{E}|x - S(s, \omega)|^p.
$$

Next, recall from the construction in Theorem 3.2 that $S(t, \omega)$ is the limit of the Cauchy sequence \{X(s, s - n\tau, \omega, x) : n \in \mathbb{N}\} in $C([s, \infty); \mathbb{R}^d)$ and $\mathbb{E}[S(s, \omega) - X(s, s - n\tau, \omega, x)]$ converges to zero exponentially according to equation (3.12) or (3.10). In view of these and the condition $|x|^p \leq V(t, x)$ with time periodicity of $V(t, x)$, together with triangle inequality, we have

$$
\mathbb{E}\left( |X(s + n\tau, s, \omega, x) - S(s + n\tau, \omega)|^p \right) \\
\leq C \exp\left( \int_s^{s+n\tau} \lambda(u)du \right) \{ \mathbb{E}|x - X(s, s - n\tau, \omega, x)|^p + \beta \exp\left( \frac{\alpha n\tau}{2} \right) \} \\
\leq C \exp\left( \int_s^{s+n\tau} \lambda(u)du \right) \{ \mathbb{E}[V(s, X(s, s - n\tau, \omega, x) - x)] + \beta \exp\left( \frac{\alpha n\tau}{2} \right) \} \\
= C \exp\left( \int_s^{s+n\tau} \lambda(u)du \right) \{ \mathbb{E}[V(s - n\tau, X(s, s - n\tau, \omega, x) - x)] + \beta \exp\left( \frac{\alpha n\tau}{2} \right) \},
$$

where $\beta$ and $\alpha$ are constants from the inequality (3.10).

Next, by the condition (4.13) we have a random variable $\gamma$ with $\mathbb{E}[\gamma(\omega)] < \infty$ such that

$$
\mathbb{E}\left( |X(s + n\tau, s, \omega, x) - S(s + n\tau, \omega)|^p \right) \leq \tilde{C}\mathbb{E}[\gamma(\omega)] \exp\left( \int_s^{s+n\tau} \lambda(u)du \right) \\
= \tilde{K} \exp\left( \int_s^{s+n\tau} \lambda(u)du \right).
$$

**Step II:** In this step, we show that there exists $K_1 > 0$ such that

$$
|T_{s+n\tau,s}\varphi(x) - \int_{L_s} \varphi d\mu_s| \leq K_1 ||\varphi||_{\infty} \exp\left( p^{-1} \int_s^{s+n\tau} \lambda(r)dr \right), \quad \forall \varphi \in C_b(L_s).
$$

To this end, we start by recalling that the periodic measure $\mu_s$ is invariant under the Markov evolution $(T_{s+n\tau,s})_{s \in \mathbb{N}}$, so that for any $h \in \text{Lip}_b(L_s)$

$$
\left| T_{s+n\tau,s}h(x) - \int_{L_s} h d\mu_s \right| = \left| \int_{L_s} (T_{s+n\tau,s}h(x) - T_{s+n\tau,s}h(y))d\mu_s(dy) \right| \\
\leq ||h||_{BL} \int_{L_s} \mathbb{E}|X(s + n\tau, s, \omega, x) - X(s + n\tau, s, \omega, y)| \mu_s(dy) \\
\leq \tilde{K} ||h||_{BL} \exp\left( p^{-1} \int_s^{s+n\tau} \lambda(u)du \right),
$$

where we have applied Hölder’s inequality and (4.14) in the last line. Let $\varphi \in C_b(L_s)$ be given. Setting $h = T_{s+n\tau,s}\varphi$ in (4.16), then, by Lemma 4.5 and by the invariance of $\mu_s$ under the Markov
Step III: To complete the proof, we employ density argument and use (4.15) to obtain the evolution \((T_{s+k\tau,s})_{k \in \mathbb{N}}\), we have
\[
|T_{s+s+n\tau,s}\varphi(x) - \int_{L_s} T_{s+s,n\tau,s}\varphi d\mu_s| = \left| \int_{L_s} (T_{s+s+n\tau,s}\varphi(x) - T_{s+n\tau,s}\varphi(y)) \mu_s(dy) \right|
\leq \tilde{K} \|T_{s+s,n\tau,s}\varphi\|_{BL} \exp \left( p^{-1} \int_s^{s+n\tau} \lambda(r) dr \right)
\leq K\tau \|\varphi\|_{\infty} \exp \left( p^{-1} \int_s^{s+n\tau} \lambda(r) dr \right),
\]
where \(K\tau = \frac{\tilde{K}K_m}{\sqrt{\tau}} \exp \left( \frac{1}{2} \int_s^{s+n\tau} \lambda(r) dr \right)\). It then follows that for any \(\varphi \in C_0(L_s)\) and \(n > 1\),
\[
|T_{s+n\tau,s}\varphi(x) - \int_{L_s} \varphi d\mu_s| \leq K\tau \|\varphi\|_{\infty} \exp \left( p^{-1} \int_s^{s+n\tau} \lambda(r) dr \right),
\]
this implies that there exists \(K_1 > 0\) such that (4.15) holds.

Step III: To complete the proof, we employ density argument and use (4.15) to obtain the convergence of Krylov–Bogolyubov scheme for periodic measure. Now, let \(A_s \subset L_s\) be a closed set, take \(\varphi = I_{A_s}\) and consider the sequence of functions \((\varphi_m)_{m \in \mathbb{N}}\) defined by
\[
\varphi_m(x) = \begin{cases} 1, & \text{if } x \in A_s, \\ 1 - 2^m d(x, A_s), & \text{if } d(x, A_s) \leq 2^{-m}, \\ 0, & \text{if } d(x, A_s) \geq 2^{-m}, \end{cases}
\]
where \(d(x, A_s) = \inf\{|x - y| : y \in A_s\}, x \in L_s\). Then,
\[
\varphi_m(x) \to \varphi(x), \quad \text{as } m \to \infty, \quad x \in L_s.
\]
Now, for all \(s \in [0, \tau)\) we have
\[
T_{s+n\tau,s}\varphi_m(x) \to T_{s+n\tau,s}\varphi(x) = T_{s+n\tau,s}I_{A_s}(x),
\]
this implies that \(T_{s+n\tau,s}\varphi_m \in C_0(L_s)\) and as \(\mu_s\) is invariant under \(T_{s+n\tau,s}\), then (4.15) leads to
\[
|P(s, x; s + n\tau, A_s) - \mu_s(A_s)| = |T_{s+n\tau,s}I_{A_s}(x) - \mu_s(A_s)| \leq K_1 \exp \left( p^{-1} \int_s^{s+n\tau} \lambda(u) du \right). \tag{4.17}
\]
By covering lemma (e.g., [11]), the inequality (4.17) holds for any \(A \in \mathcal{B}(\mathbb{R}^d)\) with \(A \subset L_s\), thus, for \(T \subset [0, \tau)\), we have
\[
\int_T |P(s, x; s + n\tau, A) - \mu_s(A)| ds \leq \int_0^\tau |P(s, x; s + n\tau, A) - \mu_s(A)| ds
\leq K_1 \int_0^\tau \exp \left( p^{-1} \int_s^{s+n\tau} \lambda(u) du \right) ds
= K_1 \int_0^\tau \exp \left( \frac{1}{2pm\tau} \int_s^{s+n\tau} \lambda(u) du \right)^{2n\tau} ds.
\]
Next, we use the Chapman–Kolmogorov equation for transition probability to obtain
\[ \left| \int_T [P(s, x; t + n\tau, A) - \mu_t(A)] dt \right| = \int_T \int_{\mathbb{R}^d} \left| P(t, y; t + n\tau, A) - \mu_t(A) \right| P(s, x; dy) dt \]
\[ \leq \int_0^T \int_{\mathbb{R}^d} K_1 \exp \left( \frac{1}{2p\mu} \int_t^{t+n\tau} \lambda(u) du \right)^{2n\tau} \left| P(s, x; t + n\tau, A) - \mu_t(A) \right| P(s, x; dy) dt \]
\[ = \int_0^T K_1 \exp \left( \frac{1}{2p\mu} \int_t^{t+n\tau} \lambda(u) du \right)^{2n\tau} \int_{\mathbb{R}^d} P(s, x; t + n\tau, A) dt \]
\[ = K_1 \int_0^T \exp \left( \frac{1}{2p\mu} \int_t^{t+n\tau} \lambda(u) du \right)^{2n\tau} dt. \]

By the condition \((3.5)\) of Theorem \(3.2\), we have there exist \(0 < \beta < 1\), \(0 < K_2 < \infty\), such that
\[ \left| \int_T \left( P(s, x; t + n\tau, A) - \mu_t(A) \right) dt \right| \leq \int_T \left| P(s, x; t + n\tau, A) - \mu_t(A) \right| dt \leq K_2 \beta^{n\tau}. \]

It then follows that
\[ \frac{1}{\tau} \int_0^\tau \int_{\mathbb{R}^d} \int_T \left\{ \frac{1}{N} \sum_{n=0}^{N-1} P(s, x; t + n\tau, A) - \mu_t(A) \right\} dt \left| \mu_s(dx) \right| ds \leq K_2 \sum_{k=0}^{N-1} \beta^{n\tau}. \]

This implies the convergence of the Krylov-Bogolyubov scheme for periodic measures. \(\square\)

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