ASYMPTOTICS AND ESTIMATES OF DEGREES OF CONVERGENCE IN THREE-DIMENSIONAL BOUNDARY VALUE PROBLEM WITH FREQUENT INTERCHANGE OF BOUNDARY CONDITIONS

Denis I. Borisov

Bashkir State Pedagogical University, October Revolution St., 3a, 450000, Ufa, Russia. E-mail: BorisovDI@ic.bashedu.ru, BorisovDI@bspu.ru

Abstract

We consider a singular perturbed eigenvalue problem for Laplace operator in a cylinder with frequent interchange of type of boundary condition on a lateral surface. These boundary conditions are prescribed by partition of lateral surface in a great number of narrow strips on those the Dirichlet and Neumann conditions are imposed by turns. We study the case of the homogenized problem containing Dirichlet condition on the lateral surface. When the width of strips varies slowly, we construct the leading terms of eigenvalues’ asymptotics expansions. We also estimate the degree of convergence for eigenvalues if the strips’ width varies rapidly.

Introduction

The present paper is devoted to the studying of a three-dimensional boundary value problem with frequent interchange of boundary condition. The main feature of formulation of such problems is partition of domain’s boundary in two parts, on the first the boundary condition of one type is imposed (ex. Dirichlet condition) while on the second the boundary condition of another type is prescribed (ex. Neumann condition). One of this parts is assumed to depend on a small parameter and consist of disjoint components; moreover, the small parameter going to zero, the number of components increases unboundedly while the measure of each component tends to zero. The question of homogenization for the problems of such kind are investigated well enough (see, for instance, [1]-[8]). The main homogenization result established in the papers cited can be formulated as follows. The solutions to the boundary value problem with frequent interchange of boundary conditions converges to ones of the problems with classic boundary conditions whose type is determined by a relationship between measured of parts of boundary with different boundary condition in the origin problem. The authors of [3], [4], [5], [9] considered the interchange between Dirichlet and Neumann or Robin

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condition and obtained the estimates of degrees of convergence provided each connected component with boundary condition of one of the types shrinks to a point. The asymptotics for the solutions of the problems with frequent interchange were constructed in [11]-[18]. Two-dimensional case was studied in [11]-[16]. In papers [17], [18] they constructed complete asymptotics expansions of Laplace operator’s eigenelements in a circular cylinder with frequent interchange between Dirichlet and Neumann condition imposed on narrow strips in a lateral surface; their width was constant. In [17] the author considered the case of the homogenized problem with Dirichlet condition on a lateral surface under additional assumption that the width of strips with Dirichlet and Neumann condition are of same order of smallness. In [18] they studied the case corresponding to the homogenized problem with Neumann or Robin condition on a lateral surface. In both cases it was shown that original perturbed problem has simple and double eigenvalues only. In addition, in [18] for cylinder of arbitrary cross-section and the width of strips varying slowly in the case of homogenized problem with Neumann or Robin condition on a lateral surface the author constructed the leading terms of asymptotics expansions for eigenelements, where eigenvalues were supposed to converge to simple limiting eigenvalues.

In the present paper we consider a singular perturbed eigenvalue problem for Laplacian in a cylinder of arbitrary cross-section. On the upper basis we impose Dirichlet condition while on the lower one we prescribe Neumann condition. The lateral surface is partitioned in a great number of narrow strips with varying width governed by two character parameters. On these strips the Dirichlet and Neumann conditions are imposed by turns. We study the case of homogenized problem with Dirichlet condition on the lateral surface. Provided the strips’ width varies slowly we construct the leading terms of the two-parametrical asymptotics expansions for the eigenelements. The form of these expansions allows us to maintain that in a general case the complete splitting of limiting multiply eigenvalues takes place and the perturbed problem has simple eigenvalues only. We also study the particular case of circular cylinder and show that depending on the strips’ width both the previous situation of the complete splitting of multiply eigenvalues and the situation of non-splitting may arise. We adduce the sufficient condition guaranteeing that the perturbed problem has at least one double eigenvalue. For the case of the strips’ width varying rapidly we estimate the degree of convergence for perturbed eigenvalues.

The result of this work were announced in [19].

1. Description of the problem and formulation of the results

Let \( x' = (x_1, x_2) \), \( x = (x', x_3) \) be Cartesian coordinates in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), \( \omega \subset \mathbb{R}^2 \) be a bounded simply connected domain whose boundary is infinitely differentiable, \( \Omega = \omega \times [0, H] \), \( H > 0 \), \( \omega_1 \), \( \omega_2 \) be upper and lower basis of the cylinder \( \Omega \), \( \omega_1 = \{ x : x' \in \omega, x_3 = H \} \), \( \omega_2 = \{ x : x' \in \omega, x_3 = 0 \} \). By \( s \) we denote the natural
parameter of the curve $\partial \omega$. We suppose that $N$ is a natural number, tending to infinity; $\varepsilon = H/(\pi N)$ is a small parameter. We define a set $\gamma_\varepsilon$ located in a lateral surface $\Sigma$ of the cylinder $\Omega$ and consisting of $N$ narrow strips:

$$\gamma_\varepsilon = \{ x : x' \in \partial \omega, |x_3 - \varepsilon \pi (j + 1/2)| < \varepsilon \eta_\varepsilon(s), j = 0, \ldots, N - 1 \},$$

where $\eta = \eta(\varepsilon), 0 < \eta(\varepsilon) < \pi/2, g_\varepsilon \in C^\infty(\partial \omega)$ is an arbitrary function obeying an estimate $0 < c \leq g_\varepsilon(s) \leq 1$ with constant $c$ independent on $\varepsilon$ and $s$ (cf. fig.)

In the paper we consider a singular perturbed eigenvalue problem:

$$-\Delta \psi_\varepsilon = \lambda_\varepsilon \psi_\varepsilon, \quad x \in \Omega, \quad (1.1)$$

$$\psi_\varepsilon = 0, \quad x \in \omega_1 \cup \gamma_\varepsilon, \quad (1.2)$$

$$\frac{\partial \psi_\varepsilon}{\partial \nu} = 0, \quad x \in \omega_2 \cup \Gamma_\varepsilon.$$

Here $\nu$ is the outward normal for the boundary $\partial \Omega$, and the set $\Gamma_\varepsilon$ is defined as a complement of $\Sigma_\varepsilon$ with respect to the lateral surface $\Sigma$.

Lobo and Pérez [4] studied the homogenization of the Poisson equation with the boundary condition (1.2) for the case when $\omega$ is a unit circle, $g_\varepsilon \equiv 1$. They established that under the equality

$$\lim_{\varepsilon \to 0} \varepsilon \ln \eta(\varepsilon) = 0 \quad (1.3)$$

the solution of such problem converges in $H^1(\Omega)$ norm to a solution of the same Poisson equation with the same boundary condition on the basis and with the
Theorem 1.1. Suppose the equality (1.3) holds. Then eigenvalues of the perturbed problem converge to eigenvalues of limiting one:

\[-\Delta \psi_0 = \lambda_0 \psi_0, \quad x \in \Omega, \quad \psi_0 = 0, \quad x \in \omega_1 \cup \Sigma, \quad \frac{\partial \psi_0}{\partial \nu} = 0, \quad x \in \omega_2, \quad (1.4)\]
as \(\varepsilon \to 0\). For each eigenfunction \(\psi_0\) associated with eigenvalue \(\lambda_0\) there exists converging to \(\psi_0\) in \(H^1(\Omega)\) linear combination of the perturbed eigenfunctions associated with eigenvalues converging to \(\lambda_0\). Total multiplicity of the perturbed eigenvalues converging to a same limiting eigenvalue coincides with the multiplicity of this limiting eigenvalue.

The problem (1.4) is easily solved by separation of variables: \(\lambda_0 = M^2 + \kappa\), \(\psi_0(x) = \phi_0(x') \cos Mx_3\), where \(M = \pi(m + 1/2)H^{-1}\), \(m \geq 0\) is an integer, \(\kappa\) and \(\phi_0\) are eigenvalues of two-dimensional problem

\[-\Delta x' \phi_0 = \kappa \phi_0, \quad x' \in \omega, \quad \phi_0 = 0, \quad x' \in \partial \omega. \quad (1.5)\]

We arrange the eigenvalues of both perturbed and limiting problem in an ascending order counting multiplicity:

\[\lambda_0^1 \leq \lambda_0^2 \leq \ldots \leq \lambda_0^k, \ldots, \quad \lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \ldots \leq \lambda_\varepsilon^k \ldots \quad (1.6)\]

Associated eigenfunctions \(\psi_\varepsilon^k\) are postulated to be orthonormalized in \(L_2(\Omega)\). We denote by \(M^k\), \(\kappa^k\) and \(\phi_0^k\) numbers \(M\), \(\kappa\) and functions \(\phi_0\) associated with \(\lambda_0^k\). Eigenfunctions of the problem (1.5) are supposed to be orthonormalized in \(L_2(\omega)\), moreover, the eigenfunctions associated with multiply eigenvalue are chosen in such a way their normal derivatives are to be orthogonal in \(L_2(\partial \omega)\) weighted by \((-\ln \sin \eta g_\varepsilon)\). The possibility of such orthogonalization follows from well-known theorem on diagonalization of two quadratic forms in a finite-dimensional space.

Observe, the problem (1.4) can have multiply eigenvalues. This situation takes place if the problem (1.5) has multiply eigenvalues or for some \(i\) and \(j\) the equality \(\lambda_0^i = M_i^2 + \kappa_i^2 = M_j^2 + \kappa_j^2 = \lambda_0^j\) holds. Clear, for each \(\kappa_i\) and \(\kappa_j\) we can always chose the height \(H\) in such way to achieve the equality \(\lambda_0^i = \lambda_0^j\).

Let us formulate the main results of the work.

Theorem 1.2. Suppose the equality (1.3) holds and there exists \(d > 0\) such that a Hölder norm \(\|g_\varepsilon\|_{C^{2+d}(\partial \omega)}\) is bounded with respect to \(\varepsilon\). Then the asymptotics for the eigenvalues of the perturbed problem have the form:

\[\lambda_\varepsilon^k = \lambda_0^k + \varepsilon \lambda_1^k(\eta(\varepsilon), \varepsilon) + O(\varepsilon^{3/2}(|\ln \eta|^{3/2} + 1)), \quad (1.7)\]

\[\lambda_1^k(\eta, \varepsilon) = \int_{\partial \omega} \left(\frac{\partial \phi_0^k}{\partial \nu}\right)^2 \ln \sin \eta g_\varepsilon \, ds, \quad (1.8)\]

where \(\nu\) is outward unit normal for \(\partial \omega\).
The statement about the asymptotics of the associated eigenfunctions under hypothesis of Theorem 1.2 will be formulated in the third section (see Theorem 3.1).

If for some $i \neq j$ the eigenvalues $\lambda^0_i$ and $\lambda^0_j$ does not coincide, then, as it follows from Theorem 1.2, the eigenvalues $\lambda^\varepsilon_i$ and $\lambda^\varepsilon_j$ does not coincide, too. If $\lambda^0_i = \lambda^0_j$, then for arbitrary domain $\omega$ and function $g_\varepsilon$ the quantities $\lambda^1_i$ and $\lambda^1_j$, generally speaking, are not equal. Thus, in general case the spectrum of the problem (1.1), (1.2) consists of simple eigenvalues only. At the same time, as it was shown in [17], for a circular cylinder with $g_\varepsilon \equiv 1$ the perturbed problem has also double eigenvalues. It is clear that even for a circular cylinder with an arbitrary function $g_\varepsilon$ the perturbed problem, generally speaking, does not have multiply eigenvalues. In the present paper for the case of circular cylinder we adduce sufficient conditions for the function $g_\varepsilon$ under those the perturbed problem has also multiply eigenvalues; in order to formulate them we introduce additional notations.

Let $\omega$ be a unit circle with center at the origin. Then the problem (1.3) admits the separation of the variables, its eigenvalues are roots of equations $J_n(\sqrt{\kappa}) = 0$, where $J_n$ are Bessel functions of integer order $n \geq 0$, associated eigenfunctions (not normalized in $L^2(\Omega)$) have the form $J_0(\sqrt{\kappa} r) \cos(n\theta)$, $J_n(\sqrt{\kappa} r) \sin(n\theta)$ ($n > 0$), where $(r, \theta)$ are polar coordinates, associated with the variables $x'$. All the roots of the equations $J_n(\sqrt{\kappa}) = 0$ being distinct [23], the problem (1.3) has simple ($n = 0$) and double ($n > 0$) eigenvalues only. We continue the function $g_\varepsilon(\theta)$ periodically to all values of $\theta$ by a period $2\pi$.

**Theorem 1.3.** Suppose the hypothesis of Theorem 1.2 holds, $\omega$ is a unit circle with center at the origin, the function $g_\varepsilon(\theta)$ is periodic on $\theta$ over the period $\pi/(2n)$, $n > 0$, $\lambda^k_0 = \kappa^2_k + M^2_k$ is a double eigenvalue of the problem (1.4), $\kappa_k$ is a root the equation $J_n(\sqrt{\kappa}) = 0$. Then the eigenvalue $\lambda^\varepsilon_k$ converging to $\lambda^k_0$ is double and has the asymptotics (1.7),

$$
\lambda^\varepsilon_k(\eta, \varepsilon) = \frac{2\kappa_k}{\pi} \int_0^{2\pi} \sin^2(n\theta + \alpha_\varepsilon) \ln \sin \eta g_\varepsilon(\theta) \, d\theta =
$$

$$
= \frac{2\kappa_k}{\pi} \int_0^{2\pi} \cos^2(n\theta + \alpha_\varepsilon) \ln \sin \eta g_\varepsilon(\theta) \, d\theta,
$$

where $\alpha_\varepsilon$ is chosen by the constraint

$$
\int_0^{2\pi} \sin(2n\theta + 2\alpha_\varepsilon) \ln \sin \eta g_\varepsilon(\theta) \, d\theta = 0. \tag{1.10}
$$

The asymptotics of the associated eigenfunctions are of the form (4.3).

The condition imposed in Theorem 1.2 to the function $g_\varepsilon$, are called to exclude bounded functions $g_\varepsilon$ having derivatives unbounded on $\varepsilon$. By this we don’t deal
with rapidly oscillating functions $g_\varepsilon$, those geometrically corresponds to the strips on the lateral surface of rapidly varying width. For these cases on the basis of Theorem 1.3 in the paper the degree of convergence for perturbed eigenvalues are estimated, the result is formulated in the following theorem.

**Theorem 1.4.** Suppose the equality (1.3) holds. Then the estimates

$$-C_k \varepsilon (|\ln \eta| + 1) \leq \lambda_\varepsilon^k - \lambda_0^k \leq 0,$$

are valid with positive constants $C_k$ independent on $\varepsilon$ and $\eta$.

2. Convergence of the perturbed eigenvalues

In this section we will prove Theorem 1.1 and auxiliary lemma which will be employed in the proof of Theorem 1.2.

Throughout this section the eigenvalues of perturbed and limiting problems are assumed to be arranged in accordance with (1.4), and associated eigenfunctions are supposed to be orthonormalized in $L^2(\Omega)$. The additional orthogonalization in $L^2(\partial \Omega)$ for limiting eigenfunctions is not assumed to take place.

To prove Theorem 1.1 we will use

**Lemma 2.1.** Let $Q$ be an arbitrary compact set in a complex plane containing no limiting eigenvalues. Then for $\lambda \in Q$ and $\varepsilon$ sufficiently small the boundary value problem

$$-\Delta u_\varepsilon = \lambda u_\varepsilon + f, \quad x \in \Omega, \quad u_\varepsilon = 0, \quad x \in \omega_1 \cup \gamma_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0, \quad x \in \omega_2 \cup \Gamma_\varepsilon, \quad (2.1)$$

is uniquely solvable for each function $f \in L^2(\Omega)$ and an uninform on $\varepsilon$, $\eta$, $\lambda$ and $f$ estimate

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \quad (2.2)$$

holds true. The function $u_\varepsilon$ converges in $H^1(\Omega)$ to the solution of the problem

$$-\Delta u_0 = \lambda u_0 + f, \quad x \in \Omega, \quad u_0 = 0, \quad x \in \partial \Omega \setminus \omega_2, \quad \frac{\partial u_0}{\partial \nu} = 0, \quad x \in \omega_2. \quad (2.3)$$

uniformly on $\lambda \in Q$ as $\varepsilon \to 0$.

**Proof.** Clear, the unique solvability of the problem (2.1) is an implication of the estimate (2.2). We will prove the latter by reductio ad absurdum. Suppose this estimate is wrong, then there exist sequences $\varepsilon_k \to 0$, $\lambda_k \in Q$, $f_k \in L^2(\Omega)$, such that for $\varepsilon = \varepsilon_k$, $\lambda = \lambda_k$, $f = f_k$ the solution of the problem (2.1) meets an inequality

$$\|u_{\varepsilon_k}\|_{H^1(\Omega)} \geq k \|f_k\|_{L^2(\Omega)}. \quad (2.4)$$
Without loss of generality we suppose the function \( u_{\varepsilon_k} \) is normalized in \( L^2(\Omega) \). Then, multiplying the equation in (2.1) by \( u_{\varepsilon_k} \) and integrating by parts once we get that

\[
\| u_{\varepsilon_k} \|_{H^1(\Omega)} \leq C \left( \| u_{\varepsilon_k} \|_{L^2(\Omega)} + \| f_k \|_{L^2(\Omega)} \right) = C \left( \| f_k \|_{L^2(\Omega)} + 1 \right),
\]

(2.5)

where constant \( C \) is independent on \( k \). From (2.4), (2.5) it follows the boundedness of \( u_{\varepsilon_k} \) in \( H^1(\Omega) \) norm:

\[
\| u_{\varepsilon_k} \|_{H^1(\Omega)} \leq C,
\]

(2.6)

where constant \( C \) is independent on \( k \). The inequalities (2.6) and (2.4) in an obvious way yield the convergence in \( L^2(\Omega) \):

\[
f_k \to 0 \quad \text{as} \quad k \to \infty.
\]

Next, treating (2.6) once again, bearing in mind the compactness of \( Q \) and extracting a subsequence from the sequence of indexes \( k \) if needed, we conclude that \( \lambda_k \) converges to \( \lambda_* \in Q \), and \( u_{\varepsilon_k} \) converges to \( u_* \) weakly in \( H^1(\Omega) \) and strongly in \( L^2(\Omega) \), moreover, the function \( u_* \) is nonzero due to normalization of \( u_{\varepsilon_k} \). Clearly, the function \( u_{\varepsilon_k} \) vanishes on a set \( \{ x : x' \in \partial \omega, |x_3 - \varepsilon \pi(j + 1/2)| < \varepsilon \eta, j = 0, N - 1 \} \cup \omega_1 \). Relying on this fact, by analogy with the proof of Theorem II.4 in [4] one can easily show that \( u_* \) vanishes on the lateral surface and the upper basis of the cylinder \( \Omega \). On the other hand, for each function \( v \in H^1(\Omega) \), vanishing on the lateral surface and on the upper basis of the cylinder \( \Omega \), the obvious integral equality

\[
\int_{\Omega} (\nabla u_{\varepsilon_k}, \nabla v) \, dx = \int_{\Omega} (\lambda_k u_{\varepsilon_k} + f_k) v \, dx,
\]

takes place, passing in which to a limit as \( k \to \infty \), we see that the function \( u_* \) is a nontrivial solution to the problem

\[
-\Delta u_* = \lambda_* u_*, \quad x \in \Omega, \quad u_* = 0, \quad x \in \partial \Omega \setminus \omega_2, \quad \frac{\partial u_*}{\partial \nu} = 0, \quad x \in \omega_2.
\]

Thus, \( \lambda_* \in Q \) is an eigenvalue of the limiting problem, what contradicts to lemma’s hypothesis. The proof of the estimate (2.2) is complete.

Employing now the estimate (2.2) instead of (2.6), by similar arguments it is easy to prove a strong in \( L^2(\Omega) \) and weak in \( H^1(\Omega) \) convergence of the solution of the problem (2.1) to the solution of the problem (2.3) for arbitrary converging sequences: \( \varepsilon_k \to 0, \lambda_k \to \lambda_* \) as \( k \to \infty \). By this convergence and continuity of \( u_0 \) on \( \lambda \in Q \) we deduce an uniform on \( \lambda \) convergence of \( u_\varepsilon \) to \( u_0 \) (strong in \( L^2(\Omega) \) and weak in \( H^1(\Omega) \)). Let us establish the strong convergence in \( H^1(\Omega) \). Clear, it is sufficient to prove the convergence of a norm \( \| u_\varepsilon \|_{H^1(\Omega)} \) to \( \| u_0 \|_{H^1(\Omega)} \). This fact follows from obvious assertions

\[
\| u_\varepsilon \|_{H^1(\Omega)}^2 = \lambda \| u_\varepsilon \|_{L^2(\Omega)}^2 + (u_\varepsilon, f)_{L^2(\Omega)} \xrightarrow{\varepsilon \to 0} \lambda \| u_0 \|_{L^2(\Omega)}^2 + (u_0, f)_{L^2(\Omega)} = \| u_0 \|_{H^1(\Omega)}^2.
\]

The proof is complete.
Proof of Theorem 1.1. It is known that the solutions of the problems (2.1) and (2.3) are meromorphic on $\lambda$ in the sense of $H^1(\Omega)$ norm, their singularities are simple poles coinciding with eigenvalues of perturbed and limiting problems respectively, residua at these poles are corresponding eigenfunctions.

Let $\lambda_0 = \lambda_0^q = \ldots = \lambda_0^{q+p-1}$ be a $p$-multiply eigenvalue of the limiting problem, $p \geq 1$, and $B_\delta(\lambda_0)$ be a closed circle of radius $\delta$ with center at a point $\lambda_0$ in a complex plane. We take $\delta$ sufficiently small such that the circle $B_\delta(\lambda_0)$ contains no limiting eigenvalues except $\lambda_0$. Then by analyticity of the solutions to the problems (2.1), (2.3) on the parameter $\lambda$ and Lemma 2.1 we derive the convergence in $H^1(\Omega)$

$$\frac{1}{2\pi i} \int_{\partial B_\delta} u_\varepsilon d\lambda \xrightarrow{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial B_\delta} u_0 d\lambda.$$  \hspace{1cm} (2.7)

Since a circle $B_\delta(\lambda_0)$ contains (simple) pole of the function $u_0$, it follows that the right side of (2.7) is nonzero. Therefore, the left side of (2.7) is nonzero, too, i.e., the circle $B_\delta(\lambda_0)$ contains (simple) pole of the function $u_\varepsilon$. This fact and an arbitrary choice of $\delta$ immediately imply that the eigenvalues of the perturbed problem converge to the eigenvalues of the limiting problem.

Let us establish the convergence of the eigenfunctions. By direct calculations we check that for $\lambda \in B_\delta(\lambda_0)$, $\lambda \neq \lambda_0$, $f = \psi_0^k$, $k = q, \ldots, q+p-1$ the solution to the problem (2.3) is a function

$$u_0 = \frac{\psi_0^k}{\lambda_0 - \lambda}.$$

Substituting this equality into (2.7) and calculating right side, we obtain that left side of (2.7) where $u_\varepsilon$ is a solution to the problem (2.1) with $f = \psi_0^k$, is the needed linear combination converging to $\psi_0^k$ in $H^1(\Omega)$.

Let us prove that perturbed eigenvalues $\lambda_\varepsilon^k$, $k = q, \ldots, q+p-1$, converge to $\lambda_0$. Suppose that eigenvalues $\lambda_j^k$, $j \in I_0$, converge to $\lambda_0$. We denote by $l$ the total multiplicity of all perturbed eigenvalues converging to $\lambda_0$: $l = |I_0|$. Showing, that $l = p$, we, clear, will prove the needed convergence. Since the eigenfunctions $\psi_0^k$, $k = q, \ldots, q+p-1$ are linear independent, the corresponding linear combinations of the functions $\psi_j^k$, $j \in I_0$, converging to $\psi_0^k$, are linear independent, too. The functions $\psi_\varepsilon^k$ are linear independent, therefore, by Steinitz theorem, the number $l$ can not be less than $p$. On the other hand, assuming, that $l > p$, by analogy with the proof of Lemma 2.1 one can show the existence of a sequence $\varepsilon_k \to 0$, on which each of (linear independent) functions $\psi_j^k$, $j \in I_0$, converges to a linear combinations of the functions $\psi_0^k$, and also, these combinations are linear independent. Therefore, the number $p$ does not exceed $l$, i.e., $l = p$. The proof of Theorem 1.1 is complete.

In proving Theorem 1.2 we will employ the following auxiliary statement.
Lemma 2.2. For \( \lambda \) close to \( p \)-multiply eigenvalue \( \lambda_0 = \lambda_0^q = \ldots = \lambda_0^{q+p-1} \) the solution of the boundary value problem (2.7) satisfies a representation

\[
   u_\varepsilon = \sum_{k=q}^{q+p-1} \frac{\psi^k_\varepsilon}{\lambda^k_\varepsilon - \lambda} \int_\Omega \psi^k_\varepsilon f \, dx + \tilde{u}_\varepsilon, \tag{2.8}
\]

where \( \tilde{u}_\varepsilon \) is a holomorphic on \( \lambda \) function orthogonal to all \( \psi^k_\varepsilon \) in \( L^2(\Omega) \), \( k = q, \ldots, q + p - 1 \). For the functions \( \tilde{u}_\varepsilon \) an uniform on \( \varepsilon, \eta, \lambda \) and \( f \) estimate

\[
   \|\tilde{u}_\varepsilon\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)} \tag{2.9}
\]

holds true.

Proof. As it was said in the proof of Theorem 1.1, \( u_\varepsilon \) is a meromorphic on \( \lambda \) function, having simple poles at the points \( \lambda^k_\varepsilon \), residua at these poles are corresponding eigenfunctions. Therefore, the equality

\[
   u_\varepsilon = \sum_{k=q}^{q+p-1} b_k \frac{\psi^k_\varepsilon}{\lambda^k_\varepsilon - \lambda} + \tilde{u}_\varepsilon, \tag{2.10}
\]

is correct, where \( \tilde{u}_\varepsilon \) is holomorphic on \( \lambda \in B_\delta(\lambda_0) \). We multiply the equation in the problem (2.1) by \( \psi^k_\varepsilon \) and integrate by parts. As a result we have

\[
   (\lambda^k_\varepsilon - \lambda)(\psi^k_\varepsilon, u_\varepsilon) = (f, \psi^k_\varepsilon).
\]

Substituting the representation (2.10) into the equalities obtained we deduce:

\[
   b_k = \int_\Omega \psi^k_\varepsilon f \, dx, \quad (\tilde{u}_\varepsilon, \psi^k_\varepsilon) = 0,
\]

what proves (2.8). It remains to establish the validity of the inequality (2.9). It is easy to see that \( \tilde{u}_\varepsilon \) is a solution of the problem (2.1) with right side

\[
   f - \sum_{k=q}^{q+p-1} \psi^k_\varepsilon(f, \psi^k_\varepsilon),
\]

and it is holomorphic on \( \lambda \in \partial B_\delta(\lambda_0) \). That’s why for \( \lambda \in \partial B_\delta(\lambda_0) \) a uniform on \( \varepsilon, \eta, \lambda \) and \( f \) estimate

\[
   \|\tilde{u}_\varepsilon\|_{H^1(\Omega)} \leq C\|f - \sum_{k=q}^{q+p-1} \psi^k_\varepsilon(f, \psi^k_\varepsilon)\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)},
\]

is valid, which by maximum principle for holomorphic functions takes place for \( \lambda \in B_\delta(\lambda_0) \), too. The proof is complete.
3. Asymptotics of the perturbed eigenelements

In this section we will prove Theorem 1.2 about asymptotics of the perturbed eigenvalues, and, under its hypothesis, Theorem 3.1 about asymptotics of associated eigenfunctions.

Proof of Theorem 1.2. We will construct the asymptotics relying on the method of composite expansions [20] and the multiscaled method [21]. Our strategy is, first, to construct these asymptotics expansions formally, and, second, to prove rigorously that these expansions formally constructed do provide asymptotics of the perturbed eigenelements. It is convenient to distinguish two cases in formal constructing, depending on whether the limiting or multiply eigenvalue of the problem \(1.5\) is associated with the limiting eigenvalue. In formal constructing we will dwell on the case of simple eigenvalue of the problem \(1.5\); the case of multiply eigenvalue has just small differences those will be clarified separately.

We start formal constructing. Let \(\lambda_0 = M^2 + \kappa^2\), where \(\kappa\) is a simple eigenvalue of the problem \(1.5\), \(\psi_0(x) = \phi_0(x') \cos Mx_3\) is the associated eigenfunction, \(\|\phi_0\|_{L^2(\omega)} = 1\), \(\lambda_\varepsilon\) is the perturbed eigenvalue converging to \(\lambda_0\).

We construct the asymptotics for \(\lambda_\varepsilon\) as follows:

\[
\lambda_\varepsilon = \lambda_0 + \varepsilon \lambda_1(\eta, \varepsilon). \tag{3.1}
\]

The asymptotics for associated eigenfunction is constructed as a sum of two expansions, outer expansion and boundary layer. Outer expansion looks as follows:

\[
\psi_\varepsilon^{ex}(x, \eta) = (\phi_0(x') + \varepsilon \phi_1(x', \eta, \varepsilon)) \cos Mx_3, \tag{3.2}
\]

and boundary layer is of the form

\[
\psi_\varepsilon^{bl}(\xi, s, x_3, \eta) = \varepsilon v_1^+(\xi, s, \eta, \varepsilon) \cos Mx_3, \tag{3.3}
\]

where \(\xi = (\xi_1, \xi_2) = (\tau \varepsilon^{-1}, x_3 \varepsilon^{-1} - \pi/2), \tau\) is a distance from a point to \(\partial \omega\) measured in the direction of inward normal. We introduce the boundary layer to satisfy boundary conditions on \(\gamma_\varepsilon\) and \(\Gamma_\varepsilon\). Moreover, in constructing of boundary layer we also employ the multiscaled method, the variable \(x_3\) plays ”slow time” role.

Let us proceed to the constructing of the asymptotics, i.e., to a determining of the functions \(\lambda_1, \phi_1, v_1^+\). First we substitute (3.1) and (3.2) into the equation (1.1) and gather the coefficients of the first power of \(\varepsilon\). This standard procedure implies the equation for the function \(\phi_1\):

\[
(\Delta_{x'} + \kappa)\phi_1 = -\lambda_1 \phi_0, \quad x' \in \omega. \tag{3.4}
\]

The boundary condition for the function \(\phi_1\) will be determined in constructing of the boundary layer. Let us derive the boundary condition for the function \(v_1^+\). In accordance with the method of composite expansions we require the sum of
the functions $\psi_{\varepsilon}^{ex}$ and $\psi_{\varepsilon}^{bl}$ to satisfy the boundary conditions (1.2) on $\gamma_\varepsilon$ and $\Gamma_\varepsilon$ asymptotically on $\varepsilon$. This constraint yields the boundary conditions for $v_1^+$:

$$v_1^+ = -\phi_1^D, \quad \xi \in \gamma(\eta_\varepsilon), \quad \frac{\partial v_1^+}{\partial \xi_2} = \phi_0^\nu, \quad \xi \in \Gamma(\eta_\varepsilon),$$

(3.5)

where $\gamma(a) = \{\xi : \xi_2 = 0, |\xi_1 - \pi j| < a, j \in \mathbb{Z}\}$, $\Gamma(a) = O_{\varepsilon}(\gamma(a))$,

$$\phi_1^D = \phi_1^D(s, \eta, \varepsilon) = \phi_1(x', \eta, \varepsilon), \quad \phi_0^\nu = \phi_0^\nu(s) = \frac{\partial}{\partial \nu} \phi_0(x'), \quad x' \in \partial \omega.$$

In order to deduce the equation for the function $v_1^+$, we first rewrite Laplace operator in the variables $(s, \tau, x_3)$:

$$\Delta_x = \frac{1}{H} \left( \frac{\partial}{\partial \tau} \left( H \frac{\partial}{\partial \tau} \right) + \frac{\partial}{\partial s} \left( \frac{1}{H} \frac{\partial}{\partial s} \right) \right) + \frac{\partial^2}{\partial x_3^2}, \quad H = 1 + \tau k,$$

(3.6)

$k = k(s) = (r''(s), v(s))_{\mathbb{R}^2}, \quad v = v(s), \quad r(s)$ is a two-dimensional vector-function prescribing the curve $\partial \omega, \quad k \in C^\infty(\partial \omega)$. Now we substitute (3.1), (3.3), (3.6) into (1.1), go over to the variables $\xi$ and write out the coefficient of smallest power of $\varepsilon$. As a result we have the equation for the function $v_1^+$:

$$\Delta_\xi v_1^+ = 0, \quad \xi_2 > 0.$$  

(3.7)

In accordance with the method of composite expansions, we should construct the solution to the problem (3.5), (3.7), decaying exponentially as $\xi_2 \to +\infty$.

We will employ the symbol $V(a)$ for the space of $\pi$-periodic on $\xi_1$ functions belonging to $C^\infty(\{\xi : \xi_2 > 0\} \setminus \{\xi : \xi_1 \neq (\pm a + \pi j), j \in \mathbb{Z}\})$ and decaying exponentially as $\xi_2 \to +\infty$ uniformly on $\xi_1$ together with all their derivatives. Denote $\Pi = \{\xi : \xi_2 > 0, |\xi_1| < \pi/2\}$.

We introduce the function

$$X(\xi, a) = \text{Re } \ln \left( \sin z + \sqrt{\sin^2 z - \sin^2 a} \right) - \xi_2,$$

$z = \xi_1 + i\xi_2$ is a complex variable. By direct calculations we check that $X(\xi, a) \in V(a) \cap H^1(\Pi)$ is a harmonic in a half-plane $\xi_2 > 0$ function being even on $\xi_1$ and satisfying boundary conditions

$$X = \ln \sin a, \quad x \in \gamma(a), \quad \frac{\partial X}{\partial \xi_2} = -1, \quad x \in \Gamma(a).$$

(3.8)

Thus, the solution of the problem (3.5), (3.7) is given by the formula:

$$v_1^+ \left( \xi, s, \eta, \varepsilon \right) = -\phi_0^\nu(s)X(\xi, \eta \varepsilon(s)).$$

(3.9)

Then, by virtue of the boundary condition (3.8),

$$v_1^+ \left( \xi, s, \eta, \varepsilon \right) = -\phi_0^\nu(s) \ln \sin \eta \varepsilon(s) \quad \text{on} \quad \gamma(\eta \varepsilon(s)).$$
In view of (3.5), last equality allows to obtain the boundary condition for \( \phi_1 \):
\[
\phi_1 = \phi_0^\nu \ln \sin \eta g_x, \quad x \in \partial \omega. \tag{3.10}
\]
The solvability condition of the boundary value problem (3.4), (3.10) is obtained in a standard way: we multiply both sides of the equation (3.4) by \( \phi_0 \) and integrate by parts. The equality obtained in this way and normalization condition for \( \phi_0 \) lead us to the formula (1.8).

In order to justify the leading terms of the asymptotics formally constructed we have to construct additional terms in the asymptotics for \( \psi_\varepsilon \). To the boundary layer one should add two terms; as a result the boundary layer becomes:
\[
\psi_\varepsilon^{bl}(\xi, s, x_3, \eta) = (\varepsilon v_+^1(\xi, s, \eta, \varepsilon) + \varepsilon^2 v_+^2(\xi, s, \eta, \varepsilon)) \cos M x_3 + \\
+ \varepsilon^2 v_-^0(\xi, s, \eta, \varepsilon) \sin(M x_3). \tag{3.11}
\]
The equations for the functions \( v_\pm^1 \) are got by substituting of (3.1), (3.6) and (3.11) into (1.1) and writing out the coefficients of the same powers of \( \varepsilon \) separately for \( \cos(M x_3) \) and \( \sin(M x_3) \):
\[
\Delta_\xi v_+^1 = -k \frac{\partial v_+^1}{\partial \xi_2}, \quad \Delta_\xi v_-^1 = 2M \frac{\partial v_+^1}{\partial \xi_1}, \quad \xi_2 > 0. \tag{3.12}
\]
We derive the boundary conditions for \( v_\pm^1 \) as well as (3.3):
\[
\frac{\partial v_+^1}{\partial \xi_2} = \phi_1^\nu, \quad \frac{\partial v_-^1}{\partial \xi_2} = 0, \quad \xi \in \Gamma(\eta g_x), \tag{3.13}
\]
where \( \phi_1^\nu \) is a value of normal derivative of the function \( \phi_1 \) on \( \partial \omega, \phi_1^\nu = \phi_1^\nu(s, \eta, \varepsilon) \).

We denote:
\[
Y(\xi, a) = \text{Im} \ln \left( \sin z + \sqrt{\sin^2 z - \sin^2 a} \right) - \frac{\pi}{2} + \xi_1.
\]
One can check that \( Y \in \mathcal{V}(a) \cap H^1(\Pi) \) is odd on \( \xi_1 \), harmonic function together with \( X \) satisfying Cauchy-Riemann conditions:
\[
\frac{\partial X}{\partial \xi_1} = \frac{\partial Y}{\partial \xi_2}, \quad \frac{\partial X}{\partial \xi_2} = -\frac{\partial Y}{\partial \xi_1}. \tag{3.14}
\]
The solutions of the problem (3.12), (3.13) can be obtained explicitly:
\[
v_+^1(\xi, s, \eta, \varepsilon) = \frac{1}{2} k(s) \phi_0^\nu(s) \left( \xi_2 X(\xi, \eta g_x(s)) + \int_{\xi_2}^{+\infty} X(\xi_1, t, \eta g_x(s)) \, dt \right) - \\
- \phi_1^\nu(s, \eta, \varepsilon) X((\xi, \eta g_x(s)), \tag{3.15}
\]
\[
v_-^1(\xi, s, \eta, \varepsilon) = -k(s) M \phi_0^\nu(s) \left( \xi_2 Y(\xi, \eta g_x(s)) + \int_{\xi_2}^{+\infty} Y(\xi_1, t, \eta g_x(s)) \, dt \right).\]
Clear, \( v_2^+ \in H^1(\Pi) \cap V(\eta g_\varepsilon) \). Below we will use following auxiliary lemmas.

It arises from the definition of the set \( V(a) \), belongings \( X, Y \in V(a) \), evenness \( X \) and oddness \( Y \) on \( \xi_1 \)

**Lemma 3.1.** The equalities

\[
\frac{\partial X}{\partial \xi_1} = 0, \quad Y = 0, \quad \xi_1 = \frac{\pi k}{2}, \quad k \in \mathbb{Z}
\]

are true.

**Lemma 3.2.** Suppose function \( v \in V(a) \cap L_2(\Pi) \) satisfies an equality

\[
\int_{-\pi/2}^{\pi/2} v(\xi) \, d\xi_1 = 0
\]

for each \( \xi_2 > 0 \) and \( \frac{\partial v}{\partial \xi_1} \in L_2(\Pi) \). Then an estimate

\[
\|v\|_{L_2(\Pi)} \leq \pi \left\| \frac{\partial v}{\partial \xi_1} \right\|_{L_2(\Pi)}
\]

is valid.

**Proof.** For \( \xi_2 > 0 \) by Poincaré inequality we have:

\[
\int_{-\pi/2}^{\pi/2} v^2 \, d\xi_1 \leq \frac{\pi^2}{2} \int_{-\pi/2}^{\pi/2} \left( \frac{\partial v}{\partial \xi_1} \right)^2 \, d\xi_1 \leq \frac{\pi^2}{2} \int_{-\pi/2}^{\pi/2} \left( \frac{\partial v}{\partial \xi_1} \right)^2 \, d\xi_1.
\]

Integrating now the inequality obtained over \( \xi_2 \in (0, +\infty) \), we arrive at the statement of the lemma. The proof is complete.

Throughout next lemma we denote by \( C \) various nonspecific constants independent on \( a \).

**Lemma 3.3.** As \( a \in (0, \pi/2) \) the functions \( X \) and \( Y \) posses following properties:

1. For each \( \xi_2 > 0 \) the equality

\[
\int_{-\pi/2}^{\pi/2} X(\xi, a) \, d\xi_1 = 0
\]

holds.

2. The assertions

\[
\|X\|_{L_2(\Pi)} = \|Y\|_{L_2(\Pi)} \leq C, \quad \|\xi_2 X\|_{L_2(\Pi)} = \|\xi_2 Y\|_{L_2(\Pi)} \leq \pi \|X\|_{L_2(\Pi)},
\]

\[
\|\nabla_\xi X\|_{L_2(\Pi)} = \sqrt{\pi} \ln \sin a^{1/2}, \quad \|\xi_2 \nabla_\xi X\|_{L_2(\Pi)} = \|\xi_2 \nabla_\xi Y\|_{L_2(\Pi)} = \|X\|_{L_2(\Pi)},
\]

\[
\left\| \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt \right\|_{L_2(\Pi)} = \left\| \int_{\xi_2}^{+\infty} Y(\xi_1, t, a) \, dt \right\|_{L_2(\Pi)} \leq \pi \|X\|_{L_2(\Pi)}
\]

are true.
(3). For functions \( \frac{\partial X}{\partial a}, \frac{\partial Y}{\partial a} \in \mathcal{V}(a) \cap L^2(\Pi) \),

\[
\frac{\partial}{\partial a} \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt, \quad \frac{\partial}{\partial a} \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt \in \mathcal{V}(a) \cap H^1(\Pi)
\]

the assertions

\[
\left\| \frac{\partial X}{\partial a} \right\|_{L^2(\Pi)} = \left\| \frac{\partial Y}{\partial a} \right\|_{L^2(\Pi)} = \frac{\sqrt{\pi} \cot a \ln \cos a}{\sqrt{2}},
\]

\[
\left\| \xi_2 \frac{\partial X}{\partial a} \right\|_{L^2(\Pi)} = \left\| \xi_2 \frac{\partial Y}{\partial a} \right\|_{L^2(\Pi)} \leq \pi \left\| \frac{\partial X}{\partial a} \right\|_{L^2(\Pi)},
\]

\[
\left\| \frac{\partial}{\partial a} \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt \right\|_{L^2(\Pi)} = \pi \left\| \frac{\partial X}{\partial a} \right\|_{L^2(\Pi)}
\]

hold.

**Proof.** Throughout the proof, not saying it specially, in various integrating by parts we will employ the boundary conditions for \( X \) and \( Y \) from Lemma 3.1.

The statement of the item (1) can be easily obtained by integrating by parts in equalities \((t > 0)\)

\[
\int_{\mathbb{P} \cap \{\xi_2 > t\}} \Delta_\xi X \, d\xi = 0, \quad \int_{\mathbb{P} \cap \{\xi_2 > t\}} \xi_2 \Delta_\xi X \, d\xi = 0.
\]

We proceed to the proof the items (2), (3). The belongings

\[
\frac{\partial X}{\partial a}, \frac{\partial Y}{\partial a}, \frac{\partial}{\partial a} \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt, \quad \frac{\partial}{\partial a} \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt \in \mathcal{V}(a) \cap L^2(\Pi)
\]

are established relying on the explicit form of \( X \) and \( Y \). The derivatives of these functions on \( \xi_1, \xi_2 \) equal to the functions \( \frac{\partial X}{\partial a}, \frac{\partial Y}{\partial a} \) due to (3.14), what proves the belongings to a space \( \mathcal{V}(a) \cap H^1(\Pi) \) for the functions \( \frac{\partial}{\partial a} \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt, \frac{\partial}{\partial a} \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt \in \mathcal{V}(a) \cap L^2(\Pi) \)

\[
\frac{\partial}{\partial a} \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt. \quad \text{The existence of other norms from the items (2), (3) follows from the explicit forms of the functions } X \text{ and } Y. \quad \text{Let us prove the coincidence of the corresponding norms of } X \text{ and } Y \text{ from (2), (3). Using Cauchy-Riemann conditions (3.14) and integrating by parts, for } \xi_2 > 0 \text{ we get:}
\]

\[
\frac{\partial}{\partial \xi_2} \int_{\pi/2}^{\pi/2} Y^2 \, d\xi_1 = 2 \int_{\pi/2}^{\pi/2} Y \frac{\partial Y}{\partial \xi_2} \, d\xi_1 = 2 \int_{\pi/2}^{\pi/2} Y \frac{\partial X}{\partial \xi_1} \, d\xi_1 = \n
\]

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\[
-2 \int_{\pi/2}^{\pi/2} \frac{\partial Y}{\partial \xi_1} X \, d\xi_1 = 2 \int_{\pi/2}^{\pi/2} \frac{\partial X}{\partial \xi_2} X \, d\xi_1 = \frac{\partial}{\partial \xi_2} \int_{\pi/2}^{\pi/2} X^2 \, d\xi_1,
\]

from what it follows that

\[
\int_{\pi/2}^{\pi/2} Y^2 \, d\xi_1 = \int_{\pi/2}^{\pi/2} X^2 \, d\xi_1,
\]

\(\xi_2 > 0\).

The equality obtained proves that \(\|X\|_{L^2(\Pi)} = \|Y\|_{L^2(\Pi)}\), \(\|\xi_2 X\|_{L^2(\Pi)} = \|\xi_2 Y\|_{L^2(\Pi)}\).

The coincidence of other norms for \(X\) and \(Y\) is established by analogy.

We proceed to the proof of the estimates and other equalities from the items \([2], [3]\). In \([13, \S 3]\) it was shown that \(\|X\|_{L^2(\Pi)}\) is continuous on \(a \in [0, \pi/2]\) function, from what it follows the needed estimate for this function. In \([11]\) it was proved that

\[
\int_{\gamma(a) \cap \Pi} \frac{\partial X}{\partial \xi_2} \, d\xi_1 = \pi - 2a, \quad \int_{\Gamma(a) \cap \Pi} X \, d\xi_1 = -2a \ln \sin a. \tag{3.16}
\]

Integrating by parts in an equality \(\int_{\Pi} X \Delta_\xi X \, d\xi = 0\), we obtain:

\[
\int_{\Pi} |\nabla_\xi X|^2 \, d\xi = -\ln \sin a \int_{\gamma(a) \cap \Pi} \frac{\partial X}{\partial \xi_2} \, d\xi_1 + \int_{\Gamma(a) \cap \Pi} X \, d\xi_1,
\]

from what and \((3.16)\) it follows the maintained formula for \(\|\nabla_\xi X\|_{L^2(\Pi)}\). Equalities

\[
0 = \int_{\Pi} \xi_2^2 X \Delta_\xi X \, d\xi = -\int_{\Pi} \xi_2^2 |\nabla_\xi X|^2 \, d\xi - 2 \int_{\Pi} \xi_2 X \frac{\partial X}{\partial \xi_2} \, d\xi = -\|\xi_2 \nabla_\xi X\|_{L^2(\Pi)}^2 + \|X\|_{L^2(\Pi)}^2 \tag{3.17}
\]

imply needed expression for \(\|\xi_2 \nabla_\xi X\|_{L^2(\Pi)}\). By Lemma \(\ref{lemma3}\) and the item \((1)\) we deduce:

\[
\|\xi_2 X\|_{L^2(\Pi)} \leq \pi \left\| \xi_2 \frac{\partial X}{\partial \xi_1} \right\|_{L^2(\Pi)} \leq \pi \|\xi_2 \nabla_\xi X\|_{L^2(\Pi)} = \pi \|X\|_{L^2(\Pi)}.
\]

Basing on the item \((1)\), Lemma \ref{lemma3}, \((3.14)\) and the proven equalities and estimates from the item \((2)\), we establish that

\[
\left\| \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt \right\|_{L^2(\Pi)} \leq \pi \left\| \int_{\xi_2}^{+\infty} \frac{\partial X}{\partial \xi_1}(\xi_1, t, a) \, dt \right\|_{L^2(\Pi)} = \pi \|Y\|_{L^2(\Pi)} = \pi \|X\|_{L^2(\Pi)}.
\]

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The proof of the item (2) is complete. By direct calculations one can easily check that
\[ X_1(\xi, a) = -\frac{1}{2} \xi_2 \int_{\xi_2}^{+\infty} \frac{\partial X}{\partial a}(\xi_1, t, a) \, dt \in H^1(\Pi) \cap V(a) \]
is an even on \( \xi_1 \) solution to a problem
\[ \Delta_\xi X_1(\xi) = \frac{\partial X}{\partial a}, \quad \xi_2 > 0, \]
\[ X_1 = 0, \quad \frac{\partial X_1}{\partial \xi_2} = -\frac{1}{2} \int_0^{+\infty} \frac{\partial X}{\partial a}(\xi_1, t, a) \, dt, \quad \xi_2 = 0. \tag{3.18} \]

Since for \( \xi_1 \in (a, \pi/2] \)
\[ \frac{\partial^2}{\partial \xi_1^2} \int_0^{+\infty} \frac{\partial X}{\partial a}(\xi_1, t, a) \, dt = -\int_0^{+\infty} \frac{\partial^2}{\partial t^2 \partial a} X(\xi_1, t, a) \, dt = 0, \]
in view of evenness and \( \pi \)-periodicity on \( \xi_1 \) of the function \( X \) we derive (\( \xi_1 \in (a, \pi/2] \)):
\[ \frac{\partial}{\partial a} \int_0^{+\infty} X(\xi_1, t) \, dt = \int_0^{+\infty} \frac{\partial X}{\partial a}(\pi/2, t, a) \, dt = \cot a \ln \cos a. \]

Relying on the statement of the item (1), integrating by parts and bearing in mind last equality and (3.18) we get:
\[
\begin{align*}
\int_{\Pi} \left( \frac{\partial X}{\partial a} \right)^2 \, d\xi &= \int_{\Pi} \left( \frac{\partial X}{\partial a} - \cot a \right) \frac{\partial X}{\partial a} \, d\xi = \int_{\Pi} \left( \frac{\partial X}{\partial a} - \cot a \right) \Delta_\xi \frac{\partial X_1}{\partial a} \, d\xi = \\
&= \int_a^{\pi/2} \left( \frac{\partial}{\partial a} X(\xi_1, 0, a) - \cot a \right) \frac{\partial}{\partial a} \int_0^{+\infty} X(\xi_1, \xi_2, a) \, d\xi_2 \, d\xi_1 = \\
&= \cot a \ln \cos a \int_a^{\pi/2} \left( \frac{\partial}{\partial a} X(\xi_1, 0, a) - \cot a \right) \, d\xi_1 = -\frac{\pi}{2} \cot^2 a \ln \cos a.
\end{align*}
\]

By analogy with (3.17) we prove the equality
\[ \left\| \xi_2 \nabla_\xi \frac{\partial X}{\partial a} \right\|_{L_2(\Pi)} = \left\| \frac{\partial X}{\partial a} \right\|_{L_2(\Pi)}, \]
what together with the estimate
\[ \left\| \frac{\partial X}{\partial a} \right\|_{L_2(\Pi)} \leq \pi \left\| \frac{\partial^2 X}{\partial \xi_1 \partial a} \right\|_{L_2(\Pi)} \leq \pi \left\| \xi_2 \nabla_\xi \frac{\partial X}{\partial a} \right\|_{L_2(\Pi)} \]
\[ \leq \pi \left\| \xi_2 \nabla_\xi \frac{\partial X}{\partial a} \right\|_{L_2(\Pi)} \]
implied by (1) and Lemma 3.2 lead to the second estimate of the item (3). Third estimate is established on the base of (1), Lemma 3.2 and Cauchy-Riemann conditions \((3.14)\):

\[
\left\| \frac{\partial}{\partial a} \int_{\xi_2}^{+\infty} X(\xi_1, t, a) \, dt \right\|_{L_2(\Pi)} \leq \pi \left\| \frac{\partial Y}{\partial a} \right\|_{L_2(\Pi)} = \pi \left\| \frac{\partial X}{\partial a} \right\|_{L_2(\Pi)}.
\]

The proof is complete.

**Lemma 3.4.** Suppose that there exists \(d > 0\), such that a Hölder norm \(\|g_\varepsilon\|_{C^{2+d}(\partial\omega)}\) is bounded on \(\varepsilon\). Then an uniform on \(\varepsilon\) and \(\eta\) estimate

\[
\|\phi_1\|_{C^2(\overline{\omega})} \leq C(|\ln \eta| + 1)
\]

is correct.

**Proof.** From \((3.8)\) and normalization condition for \(\phi_0\) it obviously arises an uniform on \(\varepsilon\) and \(\eta\) estimate:

\[
|\lambda_1| \leq C(|\ln \eta| + 1).
\]

(3.19)

Therefore, in accordance with general theory of elliptic boundary value problems and theorem on embedding \(H^2(\omega) \subset C(\overline{\omega})\) and due to orthogonality \(\phi_1\) and \(\phi_0\) an inequality

\[
\|\phi_1\|_{C^2(\overline{\omega})} \leq C \|\phi_0\|_{H^2(\omega)} \leq C (|\lambda_1| \|\phi_1\|_{L_2(\omega)} + \|\phi_0\| \ln \sin \eta \|g_\varepsilon\|_{C^2(\partial\omega)}) \leq C(|\ln \eta| + 1)
\]

takes place. Employing this estimate for \(C(\overline{\omega})\)-norm of \(\phi_1\) and Schauder inequalities \([22, \text{Chapter III, \S 1, formula (1.11)}]\), we deduce that

\[
\|\phi_1\|_{C^{2+d}(\overline{\omega})} \leq C (|\lambda_1| \|\phi_0\|_{C^d(\overline{\omega})} + \|\phi_1\|_{C(\overline{\omega})} + \|\phi_0\| \ln \sin \eta \|g_\varepsilon\|_{C^{2+d}(\overline{\omega})}),
\]

from what and \((3.19)\) the statement of the lemma follows. The proof is complete.

Let \(\chi(t)\) be an infinitely differentiable cut-off function, equalling to one as \(t < 1/4\) and vanishing as \(t > 3/4\), \(c_0\) is a sufficiently small fixed positive number such that in a domain \(\{x': |\tau| < c_0\}\) the variables \((s, \tau)\) are defined correctly. We denote:

\[
\tilde{\psi}_\varepsilon^{bl}(x, \eta) = \varepsilon^2 (v_2^+(\xi, s, \eta, \varepsilon) \cos Mx_3 + v_2^-(\xi, s, \eta, \varepsilon) \sin (Mx_3)) \chi(\tau/c_0).
\]

From the definition of the functions \(v_2^\pm\) and Lemmas \([3.1, 3.3, 3.4]\) it follows

**Lemma 3.5.** The function \(\tilde{\psi}_\varepsilon^{bl} \in H^1(\Omega) \cap C^\infty(\overline{\Omega \setminus (\gamma_\varepsilon \cap \Gamma_\varepsilon)})\) satisfies boundary conditions

\[
\tilde{\psi}_\varepsilon^{bl} = 0, \quad x \in \omega_1, \quad \frac{\partial}{\partial \nu} \tilde{\psi}_\varepsilon^{bl} = 0, \quad x \in \omega_2, \quad \frac{\partial}{\partial \nu} \tilde{\psi}_\varepsilon^{bl} = 0, \quad x \in \gamma_\varepsilon.
\]
Under hypothesis of Lemma 3.4 uniform on $\varepsilon$ and $\eta$ estimates
\[
\|\tilde{\psi}_e^l\|_{H^1(\Omega)} \leq C\varepsilon^{3/2} \left(|\ln \eta|^{1/2} + 1\right), \\
\|\psi_e^l\|_{H^1(\Omega)} \leq C\varepsilon^{1/2} \left(|\ln \eta|^{1/2} + 1\right), \\
\left\|\chi(\tau/c_0) \frac{\partial \psi_e^l}{\partial s}\right\|_{L^2(\Omega)} \leq C\varepsilon^{3/2}
\]
holds true.

**Lemma 3.6.** Suppose the hypothesis of Lemma 3.4 holds. Then there exists a solution $\psi_2 \in H^1(\Omega) \cap C_0^\infty(\Omega \setminus (\gamma_\varepsilon \cup \Gamma_\varepsilon))$ to the boundary value problem
\[(\Delta - 1)\psi_2 = -\varepsilon^2 \lambda_1 \psi_1 - \frac{\chi(\tau/c_0)}{H} \frac{\partial}{\partial s} \left(\frac{1}{H} \frac{\partial \psi_e^l}{\partial s}\right), \quad x \in \Omega,\]
\[
\psi_2 = 0, \quad x \in \omega_1, \quad \psi_2 = -\tilde{\psi}_e^l, \quad x \in \gamma_\varepsilon, \quad \frac{\partial \psi_2}{\partial \nu} = 0, \quad x \in \omega_2 \cup \Gamma_\varepsilon, \quad (3.20)
\]
This solution meets an uniform on $\varepsilon$ and $\eta$ estimate
\[
\|\psi_2\|_{H^1(\Omega)} \leq C\varepsilon^{3/2} \left(\varepsilon^{1/2}|\ln \eta|^2 + |\ln \eta|^{1/2} + 1\right).
\]
**Proof.** Following [22], by a solution of the problem (3.20) we mean a solution of an integral equation
\[-(\psi_2, v)_{H^1(\Omega)} = -\varepsilon^2 \lambda_1(\psi_1, v)_{L^2(\Omega)} + \left(\frac{\chi(\tau/c_0)}{H} \frac{\partial \psi_e^l}{\partial s} \frac{1}{H} \frac{\partial v}{\partial s}\right)_{L^2(\Omega)},\]
whose trace on $\omega_1$ is zero and trace on $\gamma_\varepsilon$ equals to $\tilde{\psi}_e^l$, where $v \in H^1(\Omega; \gamma_\varepsilon \cup \omega_1) \equiv \{v : v \in H^1(\Omega), v = 0 \text{ on } \gamma_\varepsilon \cup \omega_1\}$. The right side of this integral equality is estimated above by a quantity
\[
C \left(\varepsilon^2|\lambda_1||\phi_1||_{L^2(\omega)} + \left\|\chi(\tau/c_0) \frac{\partial \psi_e^l}{\partial s}\right\|_{L^2(\Omega)}\right) \|v\|_{H^1(\Omega)},
\]
where $C$ is independent on $\varepsilon$, $\eta$, $\lambda_1$, $\phi_1$, and $v$. By virtue of this estimate, following the ideas of [22], one can easily prove the existence of the solution to (3.20) in $H^1(\Omega)$ and an inequality
\[
\|\psi_2\|_{H^1(\Omega)} \leq C \left(\varepsilon^2|\lambda_1||\phi_1||_{L^2(\omega)} + \left\|\chi(\tau/c_0) \frac{\partial \psi_e^l}{\partial s}\right\|_{L^2(\Omega)}\right).
\]
The inequality obtained due to (3.19) and Lemmas 3.4, 3.5 yields the maintained estimate for $\|\psi_2\|_{H^1(\Omega)}$. The belonging $\|\psi_2\|_{H^1(\Omega)} \in C_0^\infty(\Omega \setminus (\gamma_\varepsilon \cap \Gamma_\varepsilon))$ is established by the theorems on the smoothness of solutions to elliptic boundary value problems. The proof is complete.

We set:
\[
\hat{\lambda}_\varepsilon = \lambda_0 + \varepsilon \lambda_1(\varepsilon, \eta), \quad \hat{\psi}_e(x) = \psi_e^l(x, \eta) + \chi(\tau/c_0)\psi_e^l(\xi, s, x_3, \eta) + \psi_2(x, \eta, \varepsilon).
\]
Next lemma maintains that formally constructed asymptotics for the eigenelements are formal asymptotics solution of the perturbed problem.
Lemma 3.7. Suppose the hypothesis of Theorem 1.2 holds. Then functions \( \hat{\lambda}_\varepsilon \) and \( \hat{\psi}_\varepsilon \) \( \in \mathcal{H}^1(\Omega) \cap C^\infty(\Omega \setminus (\Gamma_\varepsilon \cap \Gamma)) \) satisfy the boundary value problem (2.1) with \( u_\varepsilon = \psi_\varepsilon, \lambda = \lambda_\varepsilon, f = f_\varepsilon \), where for \( f_\varepsilon \) an uniform on \( \varepsilon \) and \( \eta \) estimate

\[
\| f_\varepsilon \|_{L^2(\Omega)} \leq C\varepsilon^{3/2}(|\ln \eta|^{3/2} + 1).
\]

holds. The equalities \( \hat{\lambda}_\varepsilon = \lambda_0 + o(1) \), \( \| \hat{\psi}_\varepsilon - \psi_0 \|_{H^1(\Omega)} = o(1) \) are correct as \( \varepsilon \to 0 \).

Proof. The convergence of \( \hat{\lambda}_\varepsilon \) to \( \lambda_0 \) follows from the estimate (3.19), and the equality \( \| \hat{\psi}_\varepsilon - \psi_0 \|_{H^1(\Omega)} = o(1) \) does from Lemmas 3.4-3.6. Boundary conditions for \( \hat{\psi}_\varepsilon \) follows from (3.5), (3.8), (3.10), (3.11), (3.13) and Lemmas 3.1, 3.6. Due to (1.5), (3.4), (3.6) and (3.20) the function \( f_\varepsilon = - (\Delta + \hat{\lambda}_\varepsilon) \hat{\psi}_\varepsilon \) meets a representation:

\[
f_\varepsilon = - \sum_{i=1}^3 f^{(i)}_\varepsilon, \quad f^{(1)}_\varepsilon = (\hat{\lambda}_\varepsilon + 1) \hat{\psi}_\varepsilon, \quad f^{(2)}_\varepsilon = \chi(\tau/c_0) \left( \frac{1}{H} \frac{\partial}{\partial \tau} \left( \frac{H}{\partial \tau} \right) + \frac{\partial^2}{\partial x_3^2} + \hat{\lambda}_\varepsilon \right) \hat{\psi}_\varepsilon^{bl}, \quad f^{(3)}_\varepsilon = 2(\nabla \chi(\tau/c_0), \nabla \hat{\psi}_\varepsilon^{bl})_{\mathbb{R}^3} + \hat{\psi}_\varepsilon^{bl} \Delta \chi(\tau/c_0).
\]

We obtain from Lemma 3.6 and (1.3), (3.19) that

\[
\| f^{(1)}_\varepsilon \|_{L^2(\Omega)} \leq C\varepsilon^{3/2}(|\ln \eta|^{3/2} + 1),
\]

where \( C \) is independent on \( \varepsilon \) and \( \eta \). Employing Lemmas 3.3, 3.4, the equality (1.3), equations (3.7), (3.12), formulae (3.9), (3.15), Cauchy-Riemann conditions (3.14) and an estimate \( \varepsilon \xi_2 < c_0 \) that is valid in a domain \( \{x' : \tau < c_0\} \), we get

\[
\| f^{(2)}_\varepsilon \|_{L^2(\Omega)} \leq C\varepsilon^{3/2}(|\ln \eta|^{1/2} + 1),
\]

where \( C \) is independent on \( \varepsilon \) and \( \eta \). Belongings \( X,Y \in \mathcal{V}(a) \), the definitions of \( \chi \) and explicit definition of \( X \) and \( Y \) imply

\[
\| f^{(3)}_\varepsilon \|_{L^2(\Omega)} \leq Ce^{-1/\varepsilon b},
\]

where \( b > 0 \) is a some fixed number, \( C \) is independent on \( \varepsilon \) and \( \eta \). Gathering together the inequalities for \( f^{(i)}_\varepsilon \) obtained, we arrive at the maintained estimate for \( f_\varepsilon \). The proof is complete.

Formal constructing in the case of multiply eigenvalue \( \xi \) actually does not differ from one given above almost in all details. Here we simultaneously asymptotics of several eigenvalues. The condition of additional orthogonalization in \( L^2(\partial \omega) \) for the eigenfunctions of the problem (1.3) associated with multiply eigenvalue described in the first section is a solvability condition of the problem (3.14), (3.16). All other arguments are not needed to be changed and are independent on the multiplicity of \( \xi \). Thus, in the case of \( p \)-multiply eigenvalue \( \xi = \xi_q = \ldots = \xi_{q+p-1} \)
as a result of constructing we have \(2p\) functions \(\lambda^k_\varepsilon\) and \(\psi^k_\varepsilon\), corresponding to \(\lambda_k\), \(\phi^k_0\) and defined as well as \(\lambda_\varepsilon\) and \(\psi_\varepsilon\). The functions \(\lambda^k_\varepsilon\) and \(\psi^k_\varepsilon\) obey Lemma 3.7, we indicate by \(f^k_\varepsilon\) the function \(f_\varepsilon\) from this lemma associated with \(\lambda^k_\varepsilon\) and \(\psi^k_\varepsilon\).

We start the justification of the asymptotics. Suppose \(\lambda_0 = \lambda^0_\varepsilon = \ldots = \lambda^{q+p-1}_\varepsilon\) is \(p\)-multiply limiting eigenvalue, \(p \geq 1\). Due to lemmas 2.3 and 3.7 the functions \(\psi^k_\varepsilon\) meet the representations \((k = q, \ldots, q + p - 1)\):

\[
\begin{align*}
\psi^k_\varepsilon &= \sum_{i=q}^{q+p-1} b^k_{ki} \psi^i_\varepsilon + \tilde{u}^k_\varepsilon, \\
b^k_{ki} &= \frac{1}{\lambda^k_\varepsilon - \psi^k_\varepsilon} \int_\Omega \psi^i_\varepsilon f^k_\varepsilon \, dx, 
\end{align*}
\]

where a function \(\tilde{u}^k_\varepsilon\) is orthogonal to the eigenfunctions \(\psi^i_\varepsilon\), \(i = q, \ldots, q + p - 1\), in \(L_2(\Omega)\) and satisfies an uniform on \(\varepsilon\) and \(\eta\) estimate:

\[
\|\tilde{u}^k_\varepsilon\|_{H^1(\Omega)} \leq C_\varepsilon^{3/2}(|\ln \eta|^{3/2} + 1). \tag{3.23}
\]

Now we multiply the representation \((3.21)\) by \(\psi^i_\varepsilon\) in \(L_2(\Omega)\) and bear in mind the orthonormalization condition for \(\psi^k_\varepsilon\) and orthogonality of \(\psi^i_\varepsilon\) and \(\tilde{u}^k_\varepsilon\), then we have:

\[
b^k_{ki} = (\psi^k_\varepsilon, \psi^i_\varepsilon)_{L_2(\Omega)}. \tag{3.24}
\]

Let us prove the correctness of the asymptotics \((1.7)\) by reductio ad absurdum. Suppose that at some sequence \(\varepsilon_j \to 0\) some of eigenvalues \(\lambda^k_\varepsilon, k = q, \ldots, q + p - 1\), does not satisfy the asymptotics \((1.7)\), i.e., the inequalities

\[
|\lambda^i_\varepsilon - \lambda^k_\varepsilon| \geq j(\varepsilon_j^{3/2}(\ln \varepsilon_j)^{3/2} + 1)), \quad i \in I_0, \quad k = q, \ldots, q + p - 1,
\]

hold, where \(I_0 \subseteq \{q, \ldots, q+p-1\}\) is a subset of indexes of the perturbed eigenvalues not satisfying the maintained asymptotics. From these estimates, \((3.22)\) and the estimates for \(f_\varepsilon^k\) from Lemma 3.7 we get:

\[
|b^k_{ki}| \leq C/j \to 0, \quad i \in I_0, \quad k = q, \ldots, q + p - 1.
\]

From \((3.24)\) and Lemma 3.7 the boundedness of \(b^k_{ki}\) follows, that’s why, extracting a subsequence from \(\varepsilon_j\) if needed, we assume that \(b^j_{ki} \to b^k_{ki}\) as \(j \to \infty\), \(k, i = q, \ldots, q + p - 1\), moreover, as it has been established, \(b^0_{k,i} = 0\), \(k = q, \ldots, q + p - 1\), \(i \in I_0\). By the numbers \(b^k_{ki}\) we compose \(p\) vectors \(b^k_\varepsilon\) following a rule: a vector \(b^k_\varepsilon\) consists of numbers \(b^k_{ki}\), we the index \(i\) consequently takes the values from the set \(\{q, \ldots, q + p - 1\} \setminus I_0\). By analogy we compose vectors \(b^0_\varepsilon\). Clear, the dimensions of these vectors are \((p - |I_0|) < p\) and the convergences \(b^j_{ki} \to b^0_{ki}\) hold. Taking into account this convergence, we multiply the representations \((3.21)\) each by other in \(L_2(\Omega)\) and employ the orthonormalization condition for \(\psi^k_\varepsilon\) and \(\tilde{u}^k_\varepsilon\), Lemma 3.7, and the estimate \((3.23)\); then we have:

\[
(b^0_\varepsilon, b^0_\varepsilon)_{\mathbb{R}^{p-|I_0|}} = \lim_{j \to \infty} (b^j_{ki}, b^j_{ki})_{\mathbb{R}^{p-|I_0|}} = \delta_{ki},
\]
where $\delta_{ki}$ is the Kronecker delta. Hence, $p$ vectors $b^0_k$ of dimensions $(p-|I_0|) < p$ is an orthonormalized system, a contradiction. The proof of Theorem 1.2 is complete.

Let us clarify the asymptotics behaviour of the perturbed eigenfunctions under hypothesis of Theorem 1.2. Let $\lambda_0 = \lambda_0^k$ be a simple limiting eigenvalue. It follows from (3.24) and Lemmas 3.5, 3.6 that

$$b^0_k = (\psi^k_0 + \varepsilon \phi^k \cos Mx_3, \psi^k_0)_{L^2(\Omega)} + O(\varepsilon^{3/2}(\ln \eta)^{3/2} + 1)).$$

Therefore, due to (3.21), (3.23) and Lemmas 3.5, 3.6 the eigenfunction of the perturbed problem

$$\tilde{\psi}^k_\varepsilon = (\psi^k_0 + \varepsilon \phi^k_1 \cos Mx_3, \psi^k_0)_{L^2(\Omega)} \psi^k_\varepsilon,$$  \hspace{1cm} (3.25)

associated with $\lambda^k_\varepsilon$, has the asymptotics

$$\tilde{\psi}^k_\varepsilon(x) = \psi^k_0(x) + \varepsilon \phi^k_1(x', \eta, \varepsilon) \cos Mx_3 +$$

$$+ \varepsilon \chi(\tau/c_0) \psi^k_0(s) X(\xi, \eta g_\varepsilon(s)) + O(\varepsilon^{3/2}(\ln \eta)^{3/2} + 1)),$$  \hspace{1cm} (3.26)

in $H^1(\Omega)$, where $\phi^k_0$ is a value of normal derivative of the function $\phi^k_0$ on the boundary $\partial \Omega$. The asymptotics obtained and Lemmas 3.4, 3.6 yield that $\|\tilde{\psi}^k_\varepsilon - \psi^k_0\|_{H^1(\Omega)} = o(1)$.

Let $\lambda_0 = \lambda^q_0 = \ldots = \lambda^{q+p-1}_0$ be a $p$-multiply limiting eigenvalue. Like before, by (3.24) and Lemmas 3.3, 3.6 we deduce:

$$b^k_{ki} = (\psi^k_0 + \varepsilon \phi^k \cos Mx_3, \psi^k_0)_{L^2(\Omega)} + O(\varepsilon^{3/2}(\ln \eta)^{3/2} + 1)).$$

From this fact, (3.21), (3.23) and Lemmas 3.3, 3.6 it follows that a linear combination of the perturbed eigenfunctions

$$\tilde{\psi}^k_\varepsilon = \sum_{i=q}^{q+p-1} (\psi^k_0 + \varepsilon \phi^k \cos Mx_3, \psi^k_0)_{L^2(\Omega)} \psi^k_\varepsilon,$$  \hspace{1cm} (3.27)

obeys asymptotics (3.26) in a sense of $H^1(\Omega)$ norm, where by $\tilde{\psi}^k_\varepsilon$ we mean the function (3.27). In particular, this fact implies that the functions $\tilde{\psi}^k_\varepsilon$ from (3.27) satisfies an equality $\|\tilde{\psi}^k_\varepsilon - \psi^k_0\|_{H^1(\Omega)} = o(1)$. Thus, we have proved

**Theorem 3.1.** Suppose the hypothesis of Theorem 1.2 holds. Then for each eigenfunction $\psi^k_0$ of the limiting problem there exists a perturbed eigenfunction $\tilde{\psi}^k_\varepsilon$ from (3.27) if limiting eigenvalue $\lambda^k_0$ is a simple and a linear combination $\tilde{\psi}^k_\varepsilon$ from (3.27) composed by eigenfunctions $\psi^k_i$, $i=q, \ldots, q+p-1$ if limiting eigenvalue $\lambda_0 = \lambda^q_0 = \ldots = \lambda^{q+p-1}_0$ is $p$-multiply, and this function or combination satisfies the equality $\|\tilde{\psi}^k_\varepsilon - \psi^k_0\|_{H^1(\Omega)} = o(1)$ and has the asymptotics (3.27) in $H^1(\Omega)$ norm.

4. **Proof of Theorems 1.3, 1.4**
Proof of Theorem 1.3. Throughout the proof, if it is not said specially, we keep the notations from the previous section. Since \( \lambda_0^k \) is a double eigenvalue, after the arranging (1.6) it will appear twice in the sequence \( \{ \lambda_j^0 \}_{j=1}^{\infty} \); assume that \( \lambda_0^k = \lambda_0^{k+1} \). Then \( \kappa \equiv \kappa_k = \kappa_{k+1} \), \( M_k = M_{k+1} \). Associated eigenfunctions counting all normalization and orthogonalization prescribed in the first section read as follows:

\[
\phi_0^k(x') = \frac{2}{\pi (\mathcal{J}_n(\sqrt{\mathfrak{g}}))^2} \mathcal{J}_n(\sqrt{\mathfrak{g}}) \cos(n\theta + \alpha_\varepsilon),
\]

\[
\phi_0^{k+1}(x') = \frac{2}{\pi (\mathcal{J}_n(\sqrt{\mathfrak{g}}))^2} \mathcal{J}_n(\sqrt{\mathfrak{g}}) \sin(n\theta + \alpha_\varepsilon),
\]

\[
\psi_0^k(x) = \phi_0^k(x') \cos M x_3, \quad \psi_0^{k+1}(x) = \phi_0^{k+1}(x') \cos M x_3.
\]

The equation (1.10) for \( \alpha_\varepsilon \), as one can easily check, is solvable and it is exactly the condition of orthogonality for normal derivatives of the functions \( \phi_0^k \) and \( \phi_0^{k+1} \) in \( L_2(\partial \omega) \) weighted by \( (-\ln \sin \mathfrak{g}_\varepsilon) \). Leading terms of the asymptotics for the eigenvalues \( \lambda_\varepsilon^k \) and \( \lambda_\varepsilon^{k+1} \), in accordance with Theorem 1.2 are

\[
\lambda_1^k(\eta, \varepsilon) = \frac{2\kappa}{\pi} \int_0^{2\pi} \sin^2(n\theta + \alpha_\varepsilon) \ln \sin \mathfrak{g}_\varepsilon(\theta) \, d\theta,
\]

\[
\lambda_1^{k+1}(\eta, \varepsilon) = \frac{2\kappa}{\pi} \int_0^{2\pi} \cos^2(n\theta + \alpha_\varepsilon) \ln \sin \mathfrak{g}_\varepsilon(\theta) \, d\theta.
\]

Let us prove that these terms are same:

\[
\lambda_1^k - \lambda_1^{k+1} = \frac{2\kappa}{\pi} \int_{\partial \omega} \cos(2n\theta + 2\alpha_\varepsilon) \ln \sin \mathfrak{g}_\varepsilon(\theta) \, d\theta =
\]

\[
= \frac{2\kappa}{\pi} \int_0^{2\pi} \sin(2nt + \alpha_\varepsilon) \ln \sin \mathfrak{g}_\varepsilon(t) \, dt = 0.
\]

In calculations we had made a change \( t = \theta - \pi/(2n) \), after that we used \( \pi/(2n) \)-periodicity of \( \mathfrak{g}_\varepsilon \) and the equality (1.10). Now we are going to prove that \( \lambda_\varepsilon^k = \lambda_\varepsilon^{k+1} \). Suppose it is wrong, then \( \lambda_\varepsilon^k, \lambda_\varepsilon^{k+1} \) are simple eigenvalues. According with Theorem 3.1, for the functions \( \psi_0^k, \psi_0^{k+1} \) there exist linear combination of the eigenfunctions \( \psi_\varepsilon^k, \psi_\varepsilon^{k+1} \), converging to \( \psi_0^k, \psi_0^{k+1} \) in \( H^1(\Omega) \):

\[
c_1 \psi_\varepsilon^k + c_2 \psi_\varepsilon^{k+1} \rightarrow \psi_0^k, \quad c_3 \psi_\varepsilon^k + c_4 \psi_\varepsilon^{k+1} \rightarrow \psi_0^{k+1}. \tag{4.1}
\]

From the hypothesis of the theorem it follows that \( \psi_\varepsilon^k(r, \theta + \pi/(2n), x_3) \) and \( \psi_\varepsilon^{k+1}(r, \theta + \pi/(2n), x_3) \) are perturbed eigenfunctions associated with \( \lambda_\varepsilon^k \) and \( \lambda_\varepsilon^{k+1} \), therefore

\[
\psi_\varepsilon^k(r, \theta + \pi/(2n), x_3) = c_5 \psi_\varepsilon^k(r, \theta, x_3), \quad \psi_\varepsilon^{k+1}(r, \theta + \pi/(2n), x_3) = c_6 \psi_\varepsilon^{k+1}(r, \theta, x_3).
\]
The equalities obtained and (4.1) yield
\[ c_1^k c_0^k \psi_\varepsilon^k + c_2^k c_0^k \psi_\varepsilon^{k+1} \to -\psi_0^{k+1}, \quad c_3^k c_1^k \psi_\varepsilon^k + c_4^k c_0^k \psi_\varepsilon^{k+1} \to \psi_0^k. \] (4.2)

Now we multiply first convergence from (4.1) by the second from (4.2) in \( L_2(\Omega) \) and we do the same with second convergence from (4.1) and the first from (4.2). The result reads as follows:
\[ c_1^k c_0^k + c_2^k c_0^k \psi_\varepsilon^{k+1} \to H/2, \quad c_3^k c_1^k + c_4^k c_0^k \psi_\varepsilon^{k+1} \to -H/2, \]
a contradiction, i.e., \( \lambda_\varepsilon = \lambda_\varepsilon^k = \lambda_\varepsilon^{k+1} \) is a double eigenvalue. The asymptotics of associated eigenfunctions can be easily obtained from Theorem 3.1: linear combinations converging to \( \psi_0^k \) and \( \psi_0^{k+1} \), owing to \( \lambda_\varepsilon \) being double are associated eigenfunctions. The main terms of the asymptotics from Theorem 3.1 depends on \( \varepsilon \), what happens because of additional orthogonalization in \( L_2(\partial \omega) \). At the same time, it is easy to eliminate this dependence: we just should consider suitable linear combinations of the functions \( \tilde{\psi}_\varepsilon^k \) and \( \tilde{\psi}_\varepsilon^{k+1} \) from Theorem 3.1, their main terms should be \( J_n(\sqrt{\varepsilon r}) \cos n\theta \) and \( J_n(\sqrt{\varepsilon r}) \sin n\theta \). As a result of these simple calculations we conclude that eigenfunctions associated with \( \lambda_\varepsilon \) can be chosen such that they converge to \( \tilde{\psi}_\varepsilon^k \) \( \cos n\theta \) \( \cos Mx_3 \) and \( \tilde{\psi}_\varepsilon^{k+1} \) \( \sin n\theta \) \( \cos Mx_3 \) in \( H^1(\Omega) \) and have in \( H^1(\Omega) \)-norm asymptotics
\[ \tilde{\psi}_\varepsilon^k(x) = \cos Mx_3 \left( J_n(\sqrt{\varepsilon r}) \cos n\theta + \varepsilon \tilde{\psi}_1^k(x', \eta, \varepsilon) + \varepsilon^3/2(|\ln \eta|^{3/2} + 1) \right), \] (4.3)
\[ \tilde{\psi}_\varepsilon^{k+1}(x) = \cos Mx_3 \left( J_n(\sqrt{\varepsilon r}) \sin n\theta + \varepsilon \tilde{\psi}_1^{k+1}(x', \eta, \varepsilon) + \varepsilon^3/2(|\ln \eta|^{3/2} + 1) \right), \]
where \( \tilde{\psi}_1^k \) and \( \tilde{\psi}_1^{k+1} \) are solutions to the problem (3.4), (3.10) with \( \lambda_1 = \lambda_\varepsilon^k = \lambda_\varepsilon^{k+1} \), \( \phi_0(x') = J_n(\sqrt{\varepsilon r}) \cos n\theta \) and \( \phi_0(x') = J_n(\sqrt{\varepsilon r}) \sin n\theta \), respectively. The proof is complete.

**Proof of Theorem 1.4.** Let
\[ \gamma_{\varepsilon, \ast} = \{ x : x' \in \partial \omega, |x_3 - \varepsilon \pi (j + 1/2)| < \varepsilon c\eta, j = 0, \ldots, N - 1 \}, \]
\( \lambda^k_{\varepsilon, \ast} \) indicate eigenvalues of the problem (1.4), (1.2) with \( \gamma_\varepsilon \) and \( \Gamma_\varepsilon \) replaced by \( \gamma_{\varepsilon, \ast} \) and \( \Sigma(\gamma_{\varepsilon, \ast}) \), respectively. The set \( \gamma_{\varepsilon, \ast} \) satisfies the hypothesis of Theorem 1.2 with the functions \( g_\varepsilon \equiv 1 \), hence, \( \lambda^k_{\varepsilon, \ast} \) meet asymptotics (1.7). The definition of \( \gamma_\varepsilon \) implies that \( \gamma_{\varepsilon, \ast} \subseteq \gamma_\varepsilon \); it is also clear that \( \gamma_\varepsilon \subseteq \Sigma \). Using these inclusions and minimax properties of eigenvalues for elliptic problem it is easy to show that
\[ \lambda^k_{\varepsilon, \ast} \leq \lambda^k_{\varepsilon, \ast} \leq \lambda^k_0, \]
from what, the asymptotics (1.7) for \( \lambda^k_{\varepsilon, \ast} \) and the inequalities (3.19) we get the maintained estimates for degree of convergence. The proof is complete.
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