Determination of Stresses in a Cylindrical Shell Taking Into Account Holes

Summary. Perforated elements are widely used in collectors of nuclear power plants, chemical devices, construction equipment. Perforation allows to reduce the material consumption of an expensive product, to facilitate the design and is performed for design reasons. At the same time, there is a concentration of stresses near the holes, which significantly reduces the durability of the perforated parts of the products, as well as determines the relevance of the study of the stress-strain state in the perforated parts. The paper presents an analytical description of the problem of the theory of elasticity for a perforated thick-walled cylinder in thermomechanical formulation. The problem of cylindrical bodies of rotation was considered in the axisymmetric three-dimensional formulation in displacements with the definition of stress and strain tensors. It is shown that numerical methods, in particular the finite element method, must be used to solve problems on the stress-strain state of perforated cylinders. For thick-walled perforated cylinders, it is advisable to use finite three-dimensional elements. Problems for homogeneous thick-walled and perforated cylinders were solved using three-dimensional and shell finite elements. Both four-node shell and three-dimensional eight-node prismatic finite elements were used in the numerical solution of this problem. The loading of the cylinder with holes was internal pressure. From the comparison of the results it follows that in the absence of holes it is enough to use two-dimensional shell finite elements, but in the presence of holes in perforated thick-walled shells there is a concentration of stress, so in this case it is advisable to consider the problem in three-dimensional form. It is shown in the paper that the solution of the problem taking into account the consolidated stiffness for the shell model gives an underestimated value of stresses, because it does not take into account the stress concentration on the inner surface of the shell. The coefficient of stress concentration obtained in the work due to the presence of holes was equal to two.

Keywords: shell, finite elements, thermoelastic problem, stress state, stress concentration factor.

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Визначення напружень в циліндричній оболонці з урахуванням отворів

Анотація. Перфоровані елементи широко застосовують у колекторах атомних енергетичних установок, хімічних апаратах, будівельній техніці. Перфорація дозволяє зменшити матеріаломісткість дорогого виробу, полегшити конструкцію і виконується з конструктивних міркувань. При цьому поблизу отворів має місце концентрація напружень, що істотно знизжує довговічність перфорованих частин виробів, а також обумовлює актуальність досягнення напружено-деформованого стану в перфорованих деталях. У роботі наведено аналітичний опис задачі теорії пружності для перфорованого товстостінного циліндра в термомеханічній постановці. Задача про циліндричні тіла обертання розглядалась у вісесиметричній тривимірній постановці у переміщеннях з визначеним тензором напружень та деформацій. Показано, що для розв’язання задачі про напружено-деформований стан перфорованих циліндрів необхідно використовувати чисельні методи, зокрема метод східнених елементів. Для товстостінних перфорованих циліндрів доцільно використовувати східнені тривимірні елементи. Були розв’язані задачі для однорідного товстостінного та перфорованого циліндра за допомогою тривимірних та оболонкових східнених елементів. При чисельному розв’язанні поставленої задачі використовувалися як чотирикутні оболонки, так і тривимірні восьминавові приладмірські східнені елементи. Навантаження циліндра з отворами відбувається внутрішнім тиском. Із порівняння результатів виявляється, що при відсутності отворів достатньо використовувати двовимірні оболонкові східнені елементи, проте при наявності отворів в перфорованих товстостінних оболонках виникає концентрація напружень, тому в цьому випадку доцільно розглядати задачу в тривимірній постановці і використовувати східнені тривимірні елементи. У роботі показано, що розв’язання задачі з урахуванням зведеного жорсткості для оболонкової моделі дає замінене значення напружень, оскільки не враховує концентрації напружень на внутрішній поверхні оболонки. Отриманий в роботі коефіцієнт концентрації напружень за рахунок наявності отворів дорівнює двом.

Ключові слова: оболонка, східнені елементи, термопружна задача, напружений стан, коефіцієнт концентрації напружень.

Формулировка задачи. Перфорированные элементы широко используются в сборках атомных энергетических установок, химических аппаратах, строительной технике. Перфорация позволяет сократить материалоемкость дорогого изделия, сделать конструкцию легче и выполняется из конструктивных соображений. При этом вблизи отверстий возникает концентрация напряжений, что существенно снижает долговечность перфорированных частей изделий, а также обусловливает актуальность достижения напряженно-деформированного состояния в перфорированных деталях. В работе приведено аналитическое представление задачи теории упругости для перфорированного толстостенного цилиндра в термомеханической постановке. Задача про цилиндрические тела вращения рассмотрена в восьмивузеловой трёхмерной постановке в перемещениях с заданным тензором напряжений и деформаций. Показано, что для решения задачи про напряженно-деформированное состояние перфорированных цилиндров необходимо использовать численные методы, в частности метод конечных элементов. Для толстостенных перфорированных цилиндров целесообразно использовать конечные трехмерные элементы. Были решены задачи для однородного толстостенного и перфорированного цилиндра с помощью трёхмерных и оболочечных конечных элементов. При численном решении поставленной задачи использовались как четырехугольные оболочки, так и трёхмерные восьминодные прикладные конечные элементы. Нагружение цилиндра с отверстиями произошло внутренним давлением. Из сравнения результатов выясняется, что при отсутствии отверстий достаточно использовать двухмерные оболочечные конечные элементы, однако при наличии отверстий в перфорированных толстостенных оболочках появляется концентрация напряжений, поэтому в этом случае целесообразно рассматривать задачу в трехмерной постановке и использовать конечные трехмерные элементы. В работе показано, что решение задачи с учитом уменьшенной жесткости для оболочечной модели даёт замененное значение напряжений, поскольку не учитывается концентрация напряжений на внутренней поверхности оболочки. Определенный в работе коэффициент концентрации напряжений, за счет наличия отверстий, равен двум.

Ключевые слова: оболочка, конечные элементы, термопружная задача, напряженное состояние, коэффициент концентрации напряжений.
rated parts of the products, which determines the relevance of the study of the stress-strain state in the perforated parts.

**Analysis of recent research and publications.** When using analytical approaches to solve the problem, we have to face the problem of summing infinite series and calculating singular integrals [2]. Most numerical procedures are based on the finite-element method of solving [3–5], which is more universal, but requires reforming the finite-element model in relation to the problem to be solved, so it is important in terms of labor costs which finite-element model del is used in calculations. This work is a development of previously performed calculations of stresses and strains in perforated pipes [6; 7].

**Selection of previously unsolved parts of the overall problem.** There are not enough studies comparing the numerical models used in the calculations of perforated elements, which are programmatically implemented, and tested on practical problems.

**The purpose of the article.** Determination of stresses in a perforated cylinder, study of stress concentration on the basis of comparison with the results for a smooth cylinder, as well as the choice to calculate a more rational finite element model.

**Presenting main material.** An important class of problems of thermoelasticity are the problems in which the bodies of rotation under conditions of axisymmetric thermomechanical loading are considered. It is convenient to consider these problems in a cylindrical coordinate system. Since the study area of the stress-strain state and the load conditions are axisymmetric, the components \( U_r \) and \( U_z \) velocity vectors, strain and stress tensors depend not only on the axial \( z = x_1 \) but also on the radial coordinate. There is no dependence on the angular coordinate \( \varphi \), the angular component \( U_\varphi \) of the displacement vector \( \vec{U} \) is zero. In this case, the Cauchy relations that connect the components of the strain tensor and the displacement vector take the form [8; 9]:

\[
\begin{align*}
\varepsilon_r &= \frac{\partial U_r}{\partial r}, \\
\varepsilon_z &= \frac{\partial U_z}{\partial r}, \\
\gamma_r &= \gamma_z = \left( \frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r} \right), \\
\gamma_{rz} &= \gamma_{zr} = \gamma_{rr} = \gamma_{zz} = 0.
\end{align*}
\]

(1)

For a linear-elastic isotropic material, we can obtain expressions for the components of the stress tensor:

\[
\begin{align*}
\sigma_{rr} &= \frac{2G}{1-\mu}\left( (1-\mu)\varepsilon_r + \mu(\varepsilon_z + \varepsilon_\varphi) \right), \\
\sigma_{zz} &= \frac{2G}{1-\mu}\left( (1-\mu)\varepsilon_z + \mu(\varepsilon_r + \varepsilon_\varphi) \right), \\
\sigma_{\varphi\varphi} &= \frac{2G}{1-\mu}\left( (1-\mu)\varepsilon_\varphi + \mu(\varepsilon_r + \varepsilon_z) \right), \\
\tau_{\varphi r} = \tau_{\varphi z} &= G\gamma_{\varphi r} = G\gamma_{\varphi z},
\end{align*}
\]

(2)

where \( G \) and \( \mu \) – shear modulus and Poisson’s ratio, \( \varepsilon' \) – temperature deformation. In this case, \( \gamma_{\varphi r} = \gamma_{\varphi z} = 0 \).

The three-dimensional case, only two equations remain when considering the axisymmetric problem:

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} \left( r \sigma_{rr} \right) + \frac{\partial \sigma_{\varphi r}}{\partial z} + f_0^r = 0, \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r \sigma_{\varphi r} \right) + \frac{\partial \sigma_{rr}}{\partial r} + f_0^\varphi = 0,
\end{align*}
\]

(3)

where \( f_0^r \) and \( f_0^\varphi \) are the radial and axial components of a given vector of distributed volumetric force. If the parts of the contour \( \Gamma' \) of the axial section of the body of rotation are given distributed surface forces with components \( p_r^\varphi \) and \( p_\varphi^r \), then the force limit conditions must be given in the form

\[
\begin{align*}
\sigma_{rr} n_r + \tau_{\varphi r} n_\varphi &= p_0^r, \\
\tau_{\varphi r} n_r + \sigma_{\varphi r} n_\varphi &= p_0^\varphi,
\end{align*}
\]

(4)

where \( n_r \), \( n_\varphi \) are the components of the unit vector of the external normal to the part of the circuit \( \Gamma' \). Kinematic boundary conditions on the part of the contour \( \Gamma'' \) of the axial section take the form

\[
U_r(p) = U_0^r(p), \quad U_\varphi(p) = U_0^\varphi(p), \quad p \in \Gamma'',
\]

(5)

where \( U_0^r(p) \) and \( U_0^\varphi(p) \) – components of a given vector of displacements \( U_0(p) \) points \( p \in \Gamma'' \).

Thus, to solve the axisymmetric problem of thermoelasticity, it is necessary to find ten functions: two components of the displacement vector and four components of strain and stress tensors, using two equilibrium equations (3), four Cauchy relations (1) and Hooke’s law (2), satisfying the boundary conditions (4) and (5).

Consider the solution of the axisymmetric problem in displacements. Volume deformation is expressed as in [3]:

\[
\theta = \frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r} = \frac{\partial U_\varphi}{\partial z} + \frac{\partial U_z}{\partial r} + \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) U_\varphi = \frac{\partial U_z}{\partial z} + \frac{DU_\varphi}{\partial z} + DU_r, \quad (6)
\]

where \( D = \frac{\partial}{\partial r} + \frac{1}{r} \) – the corresponding operator.

If we substitute (6) into physical equations (2), and then substitute the expressions obtained for stresses into conditions (3) with constant values of physical and mechanical properties of the structural material and the absence of bulk forces for the isothermal problem of elasticity, we obtain two differentials for equations for functions \( U_r(z,r) \) and \( U_\varphi(r,z) : 

\[
\begin{align*}
\frac{\partial^2 U_r}{\partial z^2} - 2\frac{\partial U_r}{\partial r} + \frac{\partial}{\partial r} \left( \frac{\partial U_r}{\partial r} \right) + \frac{1}{2(1-\mu)} \frac{\partial}{\partial r} \left[ (2 \mu + 1) \frac{\partial U_r}{\partial z} \right] &= 0, \\
\frac{\partial}{\partial r} \left( \frac{\partial U_r}{\partial r} \right) + \frac{1-2\mu}{2(1-\mu)} D \frac{\partial U_r}{\partial r} + \frac{1}{2(1-\mu)} \frac{\partial}{\partial r} \left( DU_r \right) &= 0.
\end{align*}
\]

(7)

Integrating equation (7), we find the functions \( U_r(z,r) \) and \( U_\varphi(r,z) \), which must satisfy equation (7) and the boundary conditions on the surface, which must be written in displacements. We introduce the operator \( D^1 \) so that

\[
D^1 \frac{\partial}{\partial r} = \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right).
\]

(8)

If we exclude the value \( U_r \) from conditions (7), we will have

\[
\frac{\partial^2 U_r}{\partial z^2} + 2D^1 \frac{\partial U_r}{\partial z} + D^1 D^1 U_r = 0.
\]

(9)

Similarly, we can write the differential equation of the fourth line, which must satisfy the displacement \( U_\varphi \).
\[
\frac{\partial^2 U}{\partial x^2} + 2 \frac{\partial^2 U}{\partial x \partial z} D \frac{\partial U}{\partial z} + \left( D \frac{\partial}{\partial r} \right)^2 U_z = 0 .
\]  
(10)

Thus, the solution of the axisymmetric isothermal problem of the theory of elasticity in displacements was reduced to the solution of two fourth-order differential equations (9) and (10).

In the case, when rotating bodies have a complex geometric shape or inhomogeneous physical and mechanical characteristics along the axis of rotation and radius, the use of analytical methods to solve the problem of determining the stress-strain state is inefficient, so it is necessary to focus on numerical methods, for example, finite element method (FEM) [3–5; 8].

In most cases, when solving an axisymmetric problem with the help of FEM, the solution is to find the nodal values of the two components of the displacement vector \( U = (U_x, U_z) \) on a two-dimensional grid of finite elements. This allows two-dimensional finite elements to be used, when solving an axisymmetric problem, as in the case of a plane problem. But in the case, when the axisymmetric structure has a complex shape, for example, perforated thick-walled cylindrical shells, which are widely used in energy, the question arises of setting the problem and choosing the type of finite elements. Given the fact that in the holes of perforated thick-walled shells there is a concentration of stresses, it is advisable to use finite three-dimensional elements and consider the problem in three-dimensional formulation.

In this work, both four-node shell and three-dimensional eight-node prismatic finite elements were used to calculate perforated shells. The choice of such an approach allows to determine with high accuracy the stress-strain state of structural elements taking into account the concentration of stresses near the holes.

Consider a thick-walled perforated cylinder with an outer diameter of 1072 mm, a wall thickness of 136 mm and a diameter of 13.2 mm perforating holes. The cylinder is loaded with an internal pressure of 10 MPa. Stainless steel 08X18H10T with the following mechanical properties was used as the material of the cylinder: Young’s modulus \( E = 2 \times 10^5 \) MPa, Poisson’s ratio \( \mu = 0.24 \). The cylinder can expand freely. One end of the cylinder is fixed in the axial direction.

When solving the problem with the help of shell finite elements, it is advisable to consider a cylindrical ring, the form, of which is shown in Figure 1, a–b. First, a perforated ring (Figure 1, b), loaded with internal pressure was considered. The displacement of the ring in the radial direction was 0.132 mm. This displacement corresponds to the thickness of the solid cylindrical ring 65 mm (Figure 1, a). The maximum stress in the solid ring was 59 MPa, and in the perforated ring – 101.5 MPa. In Figure 2 shows the stress concentration caused by perforation of the shell with round holes. The value of the stress concentration coefficient reaches 2.

**Conclusions and suggestions.** The paper presents an analytical description of the problem of the theory of elasticity for a perforated thick-walled cylinder in thermomechanical delivery. It is show, that numerical methods, in particular the finite element method, must be used to solve the problem of the stress-strain state of perforated cylinders. Moreover, for thick-walled perforated cylinders it is advisable to use three-dimensional finite elements. The paper shows that the solution of the problem taking into account the consolidated stiffness for the shell model gives an underestimated value of stresses, which does not take into account the concentration of stresses on the inner surface of the shell.

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