Stability of superflow in supersolid phases of lattice bosons with dipole-dipole interaction

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Abstract. We investigate the stability of superflow of bosons with isotropic dipole-dipole interactions in a two-dimensional optical lattice. We perform linear stability analyses for the dipolar Bose-Hubbard model in the hardcore boson limit, and show that the superflow can exist in a supersolid phase unless the velocity exceeds a certain critical value and that the critical value is remarkably smaller than that in the standard superfluid phase. Additionally, it is found that there exists a parameter range in which the SS phases are stabilized by a finite superflow. We also discuss the influence of quantum fluctuations on these results within the cluster mean-field approximation.

Recently, the physics of systems with strong dipole-dipole interactions has received considerable interest because of the successful experimental preparation of ultracold dipolar atoms [1] and polar molecules [2]. The long-range nature and anisotropy of the dipole-dipole interaction greatly enrich the variety of phenomena which can be observed in experiments with cold atoms and molecules. Recent theoretical studies predicted that various quantum phases, such as fermionic $p$-wave superfluids (SF) [3] and Haldane-Bose insulators [4], can be realized by controlling appropriately the strength of the dipole-dipole interaction and the orientation of the dipole moments. Furthermore, the systems of ultracold dipolar gases have the potential to answer the long-standing question of whether a solid can exhibit superfluidity [5–8]. The coexisting phase of solid order and superfluidity is usually called the “supersolid (SS)” phase. Recent quantum Monte Carlo (QMC) simulations for a two-dimensional Bose-Hubbard system with the isotropic long-range interactions have shown that an off-diagonal long-range order with a finite SF stiffness coexists with a crystalline long-range order in a wide range of parameters [7]. Hence, it is highly likely that the SS phases will be experimentally observed in the context of dipolar Bose gases loaded into optical lattices in the near future.

In order to prove the existence of SS phases in experiments, one has to verify that both the crystalline order and superfluidity are simultaneously present. While the existence of the crystalline order can be identified by using the Bragg scattering techniques in experiments of ultracold gases, it is difficult to measure directly the SF fraction. Instead, the superfluidity of the gases can be revealed by measuring the critical velocity above which superflow breaks down. In the experimental demonstrations of superfluidity of weakly- and strongly-interacting Bose gases [9, 10], and fermionic SF across the BEC-BCS crossover [11], the critical velocity of the superflow has been measured by using a moving optical lattice. In this work, we investigate the stability of superflow of dipolar Bose gases loaded into a two-dimensional moving optical
lattice, and propose that the superfluidity of the SS phases can be also identified by measuring the critical velocity of superflow.

Assuming that the dipole moments are polarized to the direction perpendicular to the lattice plane, we consider here the Bose Hubbard model with isotropic dipole-dipole interactions [5],

$$\hat{H} = -J \sum_{\langle j,l \rangle} (\hat{a}^\dagger_j \hat{a}_l + \text{h.c.}) + \frac{U}{2} \sum_j \hat{n}_j (\hat{n}_j - 1) + \sum_{j<l} V_{jl} \hat{n}_j \hat{n}_l - \mu \sum_j \hat{n}_j,$$  \hspace{1cm} (1)

where $\hat{a}^\dagger_j$ is the boson creation operator at site $j$ of a square lattice, $\hat{n}_j = \hat{a}^\dagger_j \hat{a}_j$, $J$ is the hopping between nearest-neighbor sites, and $U$ is the onsite interaction. The long-range part of the dipole-dipole interaction can be expressed by $V_{jl} = V d^3/|r_j - r_l|^3$. Here, $d$ is the lattice spacing and $r_j = (j_x d, j_y d)$ is a lattice vector, where $j_x$ and $j_y$ are integers. The average density of bosons, $n \equiv \sum_j \langle \hat{n}_j \rangle/M$, where $M$ is the total number of lattice sites, is controlled by the chemical potential $\mu$. In the hard-core boson limit, $U \to \infty$, only the occupation numbers $\hat{n}_j = 0$ or 1 are allowed, and thereby Eq. (1) can be mapped onto the following quantum spin-$1/2$ Hamiltonian:

$$\hat{H}_s = -J \sum_{\langle j,l \rangle} (\hat{S}^+_j \hat{S}^-_l + \text{h.c.}) + \sum_{j<l} V_{jl} \hat{S}^z_j \hat{S}^z_l - h \sum_j \hat{S}^z_j \hspace{1cm} (2)$$

with $\hat{S}^-_j = \hat{a}_j$ and $\hat{S}^+_j = \hat{n}_j - 1/2$. The shifted chemical potential $h = \mu - 1/2 \sum_{l \neq j} V_{jl}$ plays the role of the magnetic field acting on the spins. In the spin language, the density ordering and Bose-Einstein condensation are expressed by the spatial modulation of the expectation values of the longitudinal spin components and existence of non-zero transverse spin components, respectively [12].

The Weiss molecular-field approximation, or simply the mean-field (MF) approximation, is widely used to understand qualitatively the properties and behaviors of the spin systems. At zero temperature, the MF energy is obtained by replacing the spin operators at each lattice site by the classical vectors of length $S = 1/2$: $\mathbf{S}_j \to \mathbf{S}_j^0 = S(\cos \varphi_j \sin \theta_j, \sin \varphi_j \sin \theta_j, \cos \theta_j)$. For the case that the system does not have superflow, we can take $\varphi_j = 0$ without loss of generality. Minimizing the MF energy with respect to $\{\theta_j\}$ within an ansatz for some possible sublattice structures, we obtain the ground-state phase diagram in the $(V/J, h/J)$-plane as shown in Fig. 1.

One can see that in addition to the usual SF and Mott insulator (MI) phases, there exist the checkerboard solid (CS) and checkerboard supersolid (CSS) phases for $V/J > 3.024$. These features of the phase diagram, including the presence of stable CSS phases, are in qualitative agreement with the QMC results [7], which is surprising because the mean-field prediction of the existence of SS phases is not correct in the hard-core bosonic Hubbard model with nearest- and next-nearest-neighbor interactions [13]. This is attributed to the fact that the long-range nature of the dipole-dipole interactions sufficiently suppresses the quantum fluctuations. Other solid and SS phases with a more complicated structure appear in the region of large $V/J$. Since our main purpose here is to investigate superfluidity of the SF and CSS phases, in the figure we only show the region $V/J < 5.2$, where an additional SS phase, named SS2, is found.

**Figure 1.** The ground-state MF phase diagram of the dipolar Bose-Hubbard model in the hard-core boson limit. Second and first-order phase transitions are indicated by thin and thick solid lines, respectively. In the CS and CSS phases, the system has a checkerboard-type two-sublattice structure sketched in (I). The symmetry of SS2 state is sketched in (II). The dashed line represents the contour of $n = 0.45$. 


Now let us investigate the stability of superflow in the SF and CSS phases (see also Ref. 8). We consider here that the optical lattice confining hardcore dipolar bosons is moving at a constant velocity \(v\), and thereby in the frame where the lattice is at rest, the SF component of the hardcore bosons is flowing with quasi-momentum \(K = -mv\) [9,10], where \(m\) is the particle mass. Our stability analysis of superflow is based on the linear spin-wave (LSW) theory. First, minimizing the MF energy with \(\varphi_j = -K \cdot r_j\), we derive the current-carrying MF solutions of \(\{\theta_j\}\) for the SF and CSS phases. Next, we perform local rotations of the spin reference frame, so that at each site the new spin quantization axis is oriented along the direction of the classical spin vector. Introducing bosons via the standard Holstein-Primakoff transformation, we diagonalize the quadratic part of the resulting boson Hamiltonian by a Bogoliubov transformation. From the behavior of the excitation spectra obtained in this way, we examine the stability of superflow. Hereafter we assume \(K = (K,0)\), namely that the supercurrent flows along the \(x\) direction in the two-dimensional lattice. In Fig. 2, we show the excitation spectra \(\omega(q)\), where \(q\) is the quasi-momentum of the excitation, for \(V = 3.5J, n = 0.45\), and the different values of \(K\). When \(K = 0\), the system is in the stable CSS phase, and its excitation has two branches, reflecting the two-sublattice structure. The dispersion curves gradually deform (or tilt) with increasing \(K\). When \(K\) reaches a certain critical value, \(K_{CSS}^L\), the lower branch touches zero as shown in Figs. 2(b) and (d). When \(K > K_{CSS}^L\), a part of the excitation spectra has negative excitation energy, which signals the Landau instability (LI). Yet, since the LI is effective only in the presence of the dissipation, it cannot destabilize the system at (or near) zero temperature [14]. The existence of excitation modes with a complex frequency causes another type of instability, the dynamical instability (DI), which means exponential growth of the fluctuations in time and is relevant even at zero temperature. When \(K\) increases further and exceeds a critical value \(K_{CSS}^D\), long-wavelength phonons cause the DI, as shown in Figs. 2(c), (e), and (f), which reflects the fact that the effective mass in the \(x\)-direction is negative, resulting in the imaginary sound speed in the \(x\) direction.

The stability phase diagrams in the \((n, Kd)\)- and \((V/J, Kd)\)-planes are shown in Figs. 3(a) and (b). One can see that in Fig. 3(a), the critical quasi-momentum for DI in the CSS phase is remarkably smaller than that in the SF phase and independent on the density \(n\) within the MF and LSW calculations. The \(V/J\)-dependence of the critical quasi-momentum for DI exhibits a characteristic minimum structure near the phase boundary between SF and CSS as shown in Fig. 3(b). Additionally, it is found that there exists a parameter range in which the CSS phase is stabilized by a finite superflow.

Finally, let us discuss the influence of quantum fluctuations. To this end, we extend the cluster mean-field (CMF) theory [15] to the case where the system has finite superflow. Our calculations are based on a cluster of \(2 \times 2\) sites. The interactions between spins (hardcore bosons) in a cluster are treated exactly, while the influence of outside spins is replaced by effective fields whose values are determined self-consistently from the expectation values of inside spins in the

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**Figure 2.** Excitation spectra \(\omega(q)\) in the CSS phase for \(V = 3.5J, n = 0.45\), and different values of \(K\), where \(K = 0\) (a), \(K = K_{CSS}^L\) (b), and \(K = 0.83/d > K_{CSS}^D\) (c). (d) and (e) are magnification of (b) and (e) focused on the region of \(|q_x| \ll 1/d\) and \(q_y = 0\). In (f), the imaginary part of \(\omega(q)\) corresponding to (e) is shown.
usual manner. We determine the critical momenta for DI by the condition that the inverse effective mass becomes zero. As shown in Figs. 3(c) and (d), the overall values of the critical momenta for DI are smaller than those of the MF calculations, as a result of including the effects of the quantum fluctuations. However, the main qualitative features, e.g., the minimum structure in the $(V/J, Kd)$-plane and the existence of the flow-induced CSS phase, are not changed.

In conclusion, we have investigated the stability of superflow of dipolar Bose gases in a two-dimensional moving optical lattice. We have derived the mean-field phase diagram of the dipolar Bose-Hubbard model in the hardcore boson limit and have calculated the critical quasi-momenta for Landau and dynamical instabilities in the superfluid and checkerboard supersolid phases. It is expected that the superfluidity of the supersolid phase can be identified by measuring the critical quasi-momenta above which superflow breaks down.

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