KALEDIN'S DEGENERATION THEOREM AND TOPOLOGICAL
HOCHSCHILD HOMOLOGY

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ABSTRACT. We give a short proof of Kaledin’s theorem on the degeneration of the noncommutative Hodge-to-de Rham spectral sequence. Our approach is based on topological Hochschild homology and the theory of cyclotomic spectra. As a consequence, we also obtain relative versions of the degeneration theorem, both in characteristic zero and for regular bases in characteristic $p$.

1. INTRODUCTION

Let $X$ be a smooth and proper variety over a field $k$. A basic invariant of $X$ arises from the algebraic de Rham cohomology, $H^*_\text{DR}(X)$, given as the hypercohomology of the complex $\Omega^*_X$ of sheaves of algebraic differential forms on $X$ with the de Rham differential. Then $H^*_\text{DR}(X)$ is a finite-dimensional graded $k$-vector space, and is the abutment of the classical Hodge-to-de Rham spectral sequence $H^i(X, \Omega^j_X) \Rightarrow H^{i+j}_\text{DR}(X)$ arising from the naive filtration of the complex of sheaves $\Omega^*_X$. It is a fundamental fact in algebraic geometry that this spectral sequence degenerates when $k$ has characteristic zero. When $k = \mathbb{C}$ and $X$ is Kähler, the degeneration arises from Hodge theory.

After 2-periodization and in characteristic zero, the above invariants and questions have noncommutative analogs, i.e., they are defined for more generally for differential graded (dg) categories rather than only for varieties. Let $\mathcal{C}$ be a smooth and proper dg category over a field $k$ (e.g., $\mathcal{C}$ could be the derived category $D^b\text{Coh}(X)$ of a smooth and proper variety $X/k$). In this case, a basic invariant of $\mathcal{C}$ is given by the Hochschild homology $\text{HH}(\mathcal{C}/k)$, a noncommutative version of differential forms for $\mathcal{C}$. Hochschild homology takes values in the derived category $D(k)$ of $k$-vector spaces; it produces a perfect complex equipped with an action of the circle $S^1$, the noncommutative version of the de Rham differential. As a result, one can take the $S^1$-Tate construction to form $\text{HP}(\mathcal{C}/k) \overset{\text{def}}{=} \text{HH}(\mathcal{C}/k)^{S^1}$, called the periodic cyclic homology of $\mathcal{C}$ and often regarded as a noncommutative version of de Rham cohomology. One has a general spectral sequence, arising from the Postnikov filtration of $\text{HH}(\mathcal{C}/k)$, $\text{HH}_*(\mathcal{C}/k)[u^{\pm 1}] \Rightarrow \text{HP}_*(\mathcal{C}/k)$, called the (noncommutative) Hodge-to-de Rham spectral sequence. When $\mathcal{C} = D^b\text{Coh}(X)$ for $X$ in characteristic zero, this reproduces a 2-periodic version of the Hodge-to-de Rham spectral sequence.

The papers $\text{[Kal08, Kal16]}$ of Kaledin describe a proof of the following result, conjectured by Kontsevich and Soibelman $\text{[KS09, Conjecture 9.1.2]}$.

**Theorem 1.1** (Kaledin). Let $\mathcal{C}$ be a smooth and proper dg category over a field $k$ of characteristic zero. Then the Hodge-to-de Rham spectral sequence $\text{HH}_*(\mathcal{C}/k)[u^{\pm 1}] \Rightarrow \text{HP}_*(\mathcal{C}/k)$ degenerates at $E_2$. 

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An equivalent statement is that the $S^1$-action on $\text{HH}(\mathcal{C}/k)$, considered as an object of the derived category $D(k)$, is trivial; thus we may regard the result as a type of formality statement. Using the comparison between 2-periodic de Rham cohomology and periodic cyclic homology in characteristic zero, one recovers the classical result that the (commutative) Hodge-to-de Rham spectral sequence $H^i(X, \Omega^j_X) \implies H^{i+j}_{\text{dR}}(X)$ from Hodge cohomology to de Rham cohomology degenerates for a smooth proper variety $X$ in characteristic zero.

Kaledin’s proof of Theorem 1.1 is based on reduction mod $p$. Motivated by the approach of Deligne-Illusie [DI87] in the commutative case, Kaledin proves a formality statement for Hochschild homology in characteristic $p$ of smooth and proper dg categories which satisfy an amplitude bound on Hochschild cohomology and which admit a lifting mod $p^2$. Compare [Kal16, Th. 5.1] and [Kal16, Th. 5.5].

In this paper, we will give a short proof of the following slight variant of Kaledin’s characteristic $p$ degeneration results. Analogous arguments as in [Kal08, Kal16] show that this variant also implies Theorem 1.1.

**Theorem 1.2.** Let $\mathcal{C}$ be a smooth and proper dg category over a perfect field $k$ of characteristic $p > 0$. Suppose that:

1. $\mathcal{C}$ has a lift to a smooth proper dg category over $W_2(k)$.
2. $\text{HH}_i(\mathcal{C}/k)$ vanishes for $i \notin [-p, p]$.

Then the Hodge-to-de Rham spectral sequence $\text{HH}_*(\mathcal{C}/k)[u^{\pm 1}] \implies \text{HP}_*(\mathcal{C}/k)$ degenerates at $E_2$.

We will deduce Theorem 1.2 from the framework of topological Hochschild homology and in particular the theory of cyclotomic spectra as recently reformulated by Nikolaus-Scholze [NS17]. We give an overview of this apparatus in Section 2. The idea of using cyclotomic spectra here is, of course, far from new, and is already indicated in the papers of Kaledin.

Given $\mathcal{C}$, one considers the topological Hochschild homology $\text{THH}(\mathcal{C})$ as a module over the $E_\infty$-ring $\text{THH}(k)$, whose homotopy groups are given by $k[\sigma]$ for $|\sigma| = 2$. One has equivalences of spectra:

1. $\text{THH}(\mathcal{C})/\sigma \simeq \text{HH}(\mathcal{C}/k)$.
2. $\text{THH}(\mathcal{C})[1/\sigma]^{(1)} \simeq \text{HP}(\mathcal{C}/k)$ for smooth and proper $\mathcal{C}/k$. Here the superscript $^{(1)}$ denotes the Frobenius twist.

The first equivalence is elementary, while the second arises from the cyclotomic Frobenius and should compare to the “noncommutative Cartier isomorphisms” studied by Kaledin. These observations imply that the difference between 2-periodic Hochschild homology and periodic cyclic homology (i.e., differentials in the spectral sequence) is controlled precisely by the presence of $\sigma$-torsion in $\text{THH}_*(\mathcal{C})$. Under the above assumptions of liftability and amplitude bounds, the degeneration statement then follows from an elementary argument directly on the level of $\text{THH}$. We formulate this as a general formality statement in Proposition 3.9 below.

We also apply our methods to prove freeness and degeneration assertions in Hochschild homology for families of smooth and proper dg categories. We first review the commutative version. If $S$ is a scheme of finite type over a field of characteristic zero and $f : X \to S$ a proper smooth map, then one knows by a classical theorem of Deligne [Del68] that the relative Hodge cohomology sheaves $R^if_*\Omega_X^{\leq 0}/S$ form vector bundles on $S$, and that the relative Hodge-to-de Rham spectral sequence degenerates when $S$ is affine. When $S$ is smooth, this can be deduced by reduction mod $p$ and a relative version of the Deligne-Illusie constructions as in [Ill90].
There are noncommutative versions of these relative results, too. For example, in characteristic zero, one has the following result.

**Theorem 1.3.** Let $A$ be a commutative $\mathbb{Q}$-algebra and let $C$ be a smooth proper dg category over $A$. Then:

1. The Hochschild homology groups $\text{HH}_i(C/A)$ are finitely generated projective $A$-modules.
2. The relative Hodge-to-de Rham spectral sequence degenerates.

This result can be deduced from Kaledin’s theorem. When $A$ is smooth at least, the freeness of $\text{HH}_i(C/A)$ follows from the existence of a flat connection on periodic cyclic homology, due to Getzler [Get93], together with Theorem 1.1. Compare also [KS09, Remark 9.1.4] for a statement.

In fact, we will formulate the argument as a general formality (and local freeness) criterion for $S^1$-actions via cyclotomic spectra. This includes the argument for Kaledin’s theorem as well as additional input for the relative case.

**Formality criterion.** Let $A$ be a commutative $\mathbb{Q}$-algebra and let $M \in \text{Perf}(A)^{BS^1}$ be a perfect complex of $A$-modules equipped with an $S^1$-action. Suppose that there exists a finitely generated $\mathbb{Z}$-algebra $R \subset A$, a dualizable object $M'$ in the $\infty$-category $\text{Mod}_{\text{THH}(R)}^\otimes(\text{CycSp})$ of $\text{THH}(R)$-modules in cyclotomic spectra, and an equivalence $M \simeq M' \otimes_{\text{THH}(R)} A$ in $\text{Perf}(A)^{BS^1}$. Then the homology groups of $M$ are finitely generated projective $A$-modules and the $S^1$-action on $M$ is trivial.

In characteristic $p$, we can approach relative questions as well using the cyclotomic Frobenius, although our methods only apply when the base is smooth. Recent work of Petrov-Vaintrob-Vologodsky [PVV17] has obtained related statements using the methods of Kaledin and the Gauss-Manin connection in periodic cyclic homology. In particular, within the range $[-(p-3), (p-3)]$, they obtain a Fontaine-Laffaille structure on periodic cyclic homology, which they observe implies projectivity and degeneration.

**Theorem 1.4** (Cf. also [PVV17 Theorem 1]). Let $A$ be a regular noetherian $\mathbb{F}_p$-algebra such that the Frobenius map $A \rightarrow A$ is finite. Let $\tilde{A}$ be a flat lift of $A$ to $\mathbb{Z}/p^2$. Let $C$ be a smooth and proper dg category over $\tilde{A}$. Suppose that:

1. $C$ lifts to a smooth and proper dg category over $\tilde{A}$.
2. $\text{HH}_i(C/A) = 0$ for $i \notin [-1, (p-1)].$

Then the Hochschild homology groups $\text{HH}_i(C/A)$ are finitely generated projective $A$-modules and the relative Hodge-to-de Rham spectral sequence $\text{HH}_i(C/A)[u^{\pm 1}] \implies \text{HP}_*(C/A)$ degenerates at $E_2$.

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2. Cyclotomic spectra

Let $C$ be a $k$-linear stable $\infty$-category over a perfect field $k$ of characteristic $p > 0$. A basic invariant of $C$ which we will use essentially in this paper is the topological Hochschild homology $\text{THH}(C)$. The construction $\text{THH}(C)$ is one of a general class of additive invariants of stable $\infty$-categories, including algebraic $K$-theory, and about which there is a significant literature; compare for example [BGT13].

The construction $C \mapsto \text{THH}(C)$ is naturally a functor to the homotopy theory of spectra. By definition, $\text{THH}(C)$ is the Hochschild homology of $C$ relative to the sphere spectrum rather than to an ordinary ring. As we show below, $\text{THH}(C)$ contains significant information about the Hochschild homology $\text{HH}(C/k)$ and the spectral sequence for $\text{HP}(C/k)$. We begin by giving a brief overview of the relevant structure in this case. Compare also the discussion in numerous other sources, e.g., [Hes16, BM17, AMN17].

A basic input here is the calculation in the case when $C = \text{Perf}(k)$, recalled below (cf. [HM97, Sec. 5]).

**Theorem 2.1** (Bökstedt). $\text{THH}_*(k) \simeq k[\sigma]$, $|\sigma| = 2$.

Theorem 2.1 shows that $\text{THH}$ can be controlled in a convenient manner. A more naive variant of the construction $C \mapsto \text{THH}(C)$ is to consider the Hochschild homology $\text{HH}(C/k)$ over the integers. Since (by a straightforward calculation) $\text{HH}_*(\mathbb{F}_p/\mathbb{Z}) \simeq \Gamma(\sigma)$ is a divided power algebra on a degree two class, the construction of $\text{THH}$ should be regarded as an “improved” version of Hochschild homology over $\mathbb{Z}$.

We now describe more features of topological Hochschild homology. If $C$ is a $k$-linear stable $\infty$-category, then $\text{THH}(C)$ naturally acquires the structure of a module spectrum over the $E_\infty$-ring $\text{THH}(k)$. The construction $C \mapsto \text{THH}(C)$ yields a symmetric monoidal functor from $k$-linear stable $\infty$-categories to $\text{THH}(k)$-module spectra. If $C$ is smooth and proper over $k$, then $\text{THH}(C)$ is a perfect module over $\text{THH}(k)$. Furthermore, one has the relation

\[(1) \quad \text{THH}(C) \otimes_{\text{THH}(k)} k \simeq \text{HH}(C/k).\]

As a result of (1), $\text{THH}(C)$ can be thought of as a one-parameter deformation of $\text{HH}(C/k)$ over the element $\sigma$.

In addition, $\text{THH}(C)$ inherits an action of the circle $S^1$. The circle also acts on $\text{THH}(k)$ (considered as an $E_\infty$-ring spectrum), and $\text{THH}$ provides a symmetric monoidal functor

\[\{k\text{-linear stable } \infty\text{-categories}\} \rightarrow \text{Mod}_{\text{THH}(k)}(\text{Sp}^{B_{S^1}}),\]

i.e., into the $\infty$-category of spectra with $S^1$-action equipped with a compatible $\text{THH}(k)$-action. Using this, one can define the following (which can be thought of as a noncommutative version of crystalline cohomology).

**Definition 2.2** (Hesselholt [Hes16]). The periodic topological cyclic homology of $C$ is given by $\text{TP}(C) = \text{THH}(C)^{tS^1}$.

A result of [BMS] (see also [AMN17, Sec. 3]) shows that $\text{TP}$ provides a lift to characteristic zero of the periodic cyclic homology $\text{HP}(C/k)$. For example, $\text{TP}_*(k) \simeq W(k)[x^\pm 1]$ for $|x| = -2$, and in general one has a natural equivalence of $\text{TP}(k)$-modules

\[(2) \quad \text{TP}(C) \otimes_{\text{TP}(k)} \text{HP}(k) \simeq \text{HP}(C/k) \simeq \text{TP}(C)/p.\]

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1In the rest of this paper, we will generally use the language of stable $\infty$-categories [Lur14], and in particular work with $k$-linear stable $\infty$-categories rather than dg categories. We refer to [Coh16] for a comparison.
The construction $C \mapsto \text{TP}(C)$ is another extremely useful invariant one can extract from this machinery. It naturally provides a lax symmetric monoidal functor

$$\{\text{k-linear stable } \infty\text{-categories}\} \to \text{Mod}_{\text{TP}(k)}.$$ 

At least for smooth and proper $k$-linear $\infty$-categories, the construction $\text{TP}$ is actually symmetric monoidal, i.e., satisfies a Künneth theorem, by a result of Blumberg-Mandell [BM17] (see also [AMN17]).

In (2), we saw that periodic cyclic homology can be recovered from $\text{TP}$ by reducing mod $p$. Next, we show that we can reconstruct $H^p$ from $\text{THH}$ in another way. Note first that there is a natural map of $\text{E}_\infty$-rings $\text{TP}(k) \cong \text{THH}(k)^{tS^1} \to \text{THH}(k)^{tC_p}$.

**Proposition 2.3.** For $C$ a $k$-linear stable $\infty$-category, one has an equivalence of $\text{TP}(k)$-module spectra $\text{THH}(C)^{tC_p} \cong \text{TP}(C) \otimes_{\text{TP}(k)} \text{THH}(k)^{tC_p} \simeq H^p(C/k)$.

For future reference, we actually prove a more general statement.

**Proposition 2.4.** Let $X$ be an arbitrary object of the $\infty$-category $\text{Mod}_{\text{THH}(k)}(\text{Sp}^{BS_1})$ of modules over $\text{THH}(k)$ in the symmetric monoidal $\infty$-category of spectra equipped with an $S^1$-action. Then the natural map of $\text{TP}(k)$-modules

$$X^{tS^1} \otimes_{\text{TP}(k)} \text{THH}(k)^{tC_p} \to X^{tC_p}.$$ 

is an equivalence, and one has a natural equivalence of $\text{TP}(k)$-modules

$$X^{tS^1} \otimes_{\text{TP}(k)} \text{THH}(k)^{tC_p} \simeq (X \otimes_{\text{THH}(k)} k)^{tS^1}.$$

**Proof.** To see this, we note that there is an $S^1$-equivariant map of $\text{E}_\infty$-rings $\mathbb{Z} \to \text{THH}(\mathbb{F}_p)$, e.g., via the cyclotomic trace (cf. [NS17, IV.4]). One obtains a square of $\text{E}_\infty$-rings

$$\begin{array}{ccc}
\mathbb{Z}^{tS^1} & \mathbb{Z}^{tC_p} \\
\downarrow & \downarrow \\
\text{TP}(k) & \text{THH}(k)^{tC_p}
\end{array}$$

which one easily checks to be a pushout square. Now the equivalence (3) follows from [NS17, Lemma IV.4.12]. To see (4), we use the fact that $\text{THH}(k)^{tC_p} \simeq \text{TP}(k)/p$ as $\text{TP}(k)$-modules. This implies the result via the formula $(X \otimes_{\text{THH}(k)} k)^{tS^1} \simeq X^{tS^1} \otimes_{\text{TP}(k)} k^{tS^1} \simeq X^{tS^1}/p$, which holds because $k = \text{THH}(k)/\sigma$ belongs to the thick subcategory generated by the unit in $\text{Mod}_{\text{THH}(k)}(\text{Sp}^{BS_1})$ (and which is a generalization of (2)).

In addition to the parameter $\sigma$ arising from $\text{THH}(k)$, $\text{THH}$ comes with another crucial feature: namely, it has the structure of a cyclotomic spectrum. The first feature of the cyclotomic structure is the $S^1$-action on $\text{THH}(C)$. As explained in [NS17], the remaining datum of the cyclotomic structure can be encoded in a “Frobenius map” (which does not exist for $\text{HH}(C/k)$)

$$\varphi: \text{THH}(C) \to \text{THH}(C)^{tC_p}.$$ 

The map $\varphi$ has the structure of an $S^1$-equivariant map: $S^1$ acts on the source, $S^1/C_p$ acts on the target, and $S^1 \simeq S^1/C_p$ via the $p$th root. In [NS17], it is shown that in the bounded below (and $p$-local) setting, the entire datum of a cyclotomic spectrum (studied more classically using

\footnote{Compare the discussion in [AMN17] for a treatment.}
techniques of equivariant stable homotopy theory ([BHM93] [BML]).) can be constructed from the circle action and $\varphi$. Namely, they define:

**Definition 2.5** (Nikolaus-Scholze [NS17, Def. II.1.6]). The presentably symmetric monoidal stable $\infty$-category $\text{CycSp}$ of *cyclotomic spectra* is the $\infty$-category of tuples

$$\{ X \in \text{Fun}(BS^1, \text{Sp}), \varphi_p : X \to X^{tC_p} \}_{p=2,3,5,...}.$$ 

That is, to specify an object of $\text{CycSp}$ amounts to specifying a spectrum $X$ with an action of $S^1$, and $S^1 \simeq S^1/C_p$-equivariant Frobenius maps $\varphi_p : X \to X^{tC_p}$ for each prime number $p$. (When $X$ is $p$-local, we write $\varphi = \varphi_p$ and $\varphi_q = 0$ for $q \neq p$.)

The $\infty$-category $\text{CycSp}$ agrees with more classical approaches to cyclotomic spectra in the bounded-below case; see also [AMGR17].

**Example 2.6** (Cf. [NS17, IV.4] and [HM97]). Suppose $\mathcal{C} = \text{Perf}(k)$. In this case, the map $\varphi : \text{THH}(k) \to \text{THH}(k)^{tC_p}$ identifies the former with the connective cover of the latter, and $\pi_* (\text{THH}(k)^{tC_p}) \simeq k[t^{\pm 1}]$ is a Laurent polynomial ring with $|t| = 2$. The map $\varphi$ is given by the Frobenius on $\pi_0$ and sends $\sigma \mapsto t$. In particular, $\varphi$ induces an equivalence

$$\text{THH}(k)[1/\sigma] \simeq \text{THH}(k)^{tC_p}.$$ 

This computation was originally done by Hesselholt-Madsen [HM97], and we refer to [NS17, IV.4] for a complete description of $\text{THH}(k)$ as a cyclotomic spectrum.

Here $\text{THH}(k) \in \text{CA}l_*(\text{CycSp})$ is a commutative algebra object, and for $\mathcal{C}$ a $k$-linear stable $\infty$-category, then $\text{THH}(\mathcal{C})$ is a $\text{THH}(k)$-module. The functor $\text{THH}$ yields a symmetric monoidal functor

$$\{ k\text{-linear stable } \infty\text{-categories} \} \to \text{Mod}_{\text{THH}(k)}(\text{CycSp}).$$ 

Note in particular that for a smooth and proper $k$-linear stable $\infty$-category, $\text{THH}$ is therefore a dualizable object of $\text{Mod}_{\text{THH}(k)}(\text{CycSp})$. In this paper, all our degeneration arguments will take place in the latter $\infty$-category, and we will often state them in that manner.

We saw above that the cyclotomic Frobenius becomes an equivalence on connective covers for $\text{THH}(k)$. More generally, one can show (cf. [Hes96]) that for a smooth $k$-algebra, the cyclotomic Frobenius is an equivalence in high enough degrees. For our purposes, we need a basic observation that in the smooth and proper case, the cyclotomic Frobenius becomes an equivalence after inverting $\sigma$. This is a formal dualizability argument once one knows both sides satisfy a Künneth formula.

**Proposition 2.7.** Let $\mathcal{C}/k$ be a smooth and proper $k$-linear stable $\infty$-category. In this case, the cyclotomic Frobenius implements an equivalence

$$\text{THH}(\mathcal{C})[1/\sigma] \xrightarrow{\approx} \text{THH}(\mathcal{C})^{tC_p} \simeq H^p(\mathcal{C}/k).$$ 

More generally, if $X \in \text{Mod}_{\text{THH}(k)}(\text{CycSp})$ is a dualizable object, then the cyclotomic Frobenius implements an equivalence

$$X[1/\sigma] \xrightarrow{\approx} X^{tC_p} \simeq (X \otimes_{\text{THH}(k)} k)^{tS^1}.$$ 

The first equivalence is a $\varphi$-semilinear for the equivalence $\varphi : \text{THH}(k)[1/\sigma] \simeq \text{THH}(k)^{tC_p}$, while the second equivalence is $\text{TP}(k)$-linear.
Remark 2.8. Suppose differential forms on \( C \) are equivalent: observation and definition. crystalline (and de Rham) cohomology from THH. Morrow-Scholze \([\text{BMS}]\), where the Cartier isomorphism is an essential feature of their recovery of \([\text{Kal08}, \text{Kal16}]\). This relationship in the commutative case is made precise in the work of Bhatt-
tion 2.7 should be compared with the “noncommutative Cartier isomorphism” studied by Kaledin Cartier isomorphism. homology arising here is essentially the classical isomorphism

\[
\phi \colon k \to k
\]

is semilinear for the Frobenius. In particular, at the level of \( k \)-vector spaces, one has a natural isomorphism

\[
(\pi_i \text{THH}(C)[1/\sigma])^{(1)} \simeq \text{HP}_i(C/k).
\]

Remark 2.8. Suppose \( C = \text{Perf}(A) \) for \( A \) a smooth commutative \( k \)-algebra. In this case, \( \text{HP}(C/k) \) is related to 2-periodic de Rham cohomology of \( A \) while \( \text{THH}(C)[1/\sigma] \) is closely related to 2-periodic differential forms on \( C \) by \([\text{Hes96}]\). The relationship between differential forms and de Rham cohomology arising here is essentially the classical Cartier isomorphism. In particular, Proposition 2.7 should be compared with the “noncommutative Cartier isomorphism” studied by Kaledin \([\text{Kal08}, \text{Kal16}]\). This relationship in the commutative case is made precise in the work of Bhatt-Morrow-Scholze \([\text{BMS}]\), where the Cartier isomorphism is an essential feature of their recovery of crystalline (and de Rham) cohomology from THH.

3. The degeneration argument

In this section, we give the main degeneration argument. We begin with the following basic observation and definition.

Let \( R \) be an \( E_{\infty} \)-ring spectrum (in this section, \( R \) will be a field), and let \( M \) be an \( R \)-module spectrum equipped with an \( S^1 \)-action. Suppose the \( R \)-module \( M \) is projective. Then the following are equivalent:

1. The \( S^1 \)-Tate spectral sequence for \( \pi_*(M^{tS^1}) \) degenerates.
2. The \( S^1 \)-action on \( M \) (as an \( R \)-module) is trivial.

Clearly the second assertion implies the first. To see the converse, we observe that if the Tate spectral sequence degenerates, then by naturality, the homotopy fixed point spectral sequence for \( \pi_*(M) \) must degenerate too, so that the map \( \pi_*(M^{hS^1}) \to \pi_*(M) \) is surjective. Suppose \( M \), as an underlying \( R \)-module, is obtained as the summand \( Fe \) associated to an idempotent endomorphism \( e \) of a free \( R \)-module \( F \). If we give \( F \) the trivial \( S^1 \)-action, the degeneration of the homotopy fixed point spectral sequence shows that we can realize the map \( F \to M \) as an \( S^1 \)-equivariant map. Restricting now to the summand \( Fe \) of \( F \), we conclude that \( M \) is equivalent to \( Fe \) (with trivial action). This is the way in which we regard the degeneration of the \( S^1 \)-Tate spectral sequence as a formality statement.

Definition 3.1. Let \( k \) be a field. Let \( M \in \text{Perf}(k)^{B_S^1} \). We say that \( M \) is formal if the \( S^1 \)-Tate spectral sequence for \( M^{tS^1} \) (or equivalently the homotopy fixed point spectral sequence for \( M^{hS^1} \)) degenerates at \( E_2 \). This holds if and only if

\[
\dim_k \pi_{\text{even}}(M) = \dim_k \pi_0 M^{tS^1}, \quad \dim_k \pi_{\text{odd}}(M) = \dim_k \pi_1 M^{tS^1}
\]

For the rest of this section, \( k \) is a perfect field of characteristic \( p > 0 \). We will prove a formality criterion for objects of \( \text{Perf}(k)^{B_S^1} \). Our main interest, of course, is in the following example: in this section, we will state our arguments in the more general case of objects in \( \text{Perf}(k) \) with \( S^1 \)-action,
Proposition 3.4. Consider a smooth and proper $k$-linear stable ∞-category $C/k$ and its Hochschild homology $HH(C/k)$. One has that $\dim_k HH_*(C/k) < \infty$ and that $HH(C/k)$ inherits a circle action.

Definition 3.2. We say that the Hodge-to-de Rham spectral sequence degenerates for $C/k$ if $HH(C/k) \in \text{Perf}(k)^{BS_1}$ is formal. Equivalently, degeneration holds if and only if one has the numerical equalities $HH_{\text{even}}(C/k) = HP_0(C/k)$, $\dim_k HH_{\text{odd}}(C/k) = HP_1(C/k)$.

One source of objects of $\text{Perf}(k)^{BS_1}$ is the ∞-category of dualizable objects of $\text{Mod}_{THH(k)}(\text{CycSp})$. Given $X \in \text{Mod}_{THH(k)}(\text{CycSp})$, we have $X \otimes_{THH(k)} k \in \text{Mod}_{C}^{BS_1}$ and if $X$ is dualizable, then $X \otimes_{THH(k)} k$ is perfect as a $k$-module. For such objects, we will translate formality to a statement about $THH(k)$-modules. Note that $HH(C/k) \in \text{Perf}(k)^{BS_1}$ arises in this way, via $X = THH(C)$.

First, we need the following observation about module spectra over $THH(k)$, which follows from the classification of finitely generated modules over a principal ideal domain.

Proposition 3.3. Any perfect $THH(k)$-module spectrum is equivalent to a direct sum of copies of suspensions of $THH(k)$ and $THH(k)/\sigma^n$ for various $n$.

The following result now shows that degeneration is equivalent to a condition of torsion-freeness on $THH$.

Proposition 3.4. (1) Let $X \in \text{Mod}_{THH(k)}(\text{CycSp})$ be dualizable. Then $X \otimes_{THH(k)} k \in \text{Perf}(k)^{BS_1}$ is formal if and only if $X$ is free (equivalently, $\sigma$-torsion-free) as a $THH(k)$-module.

(2) If $C$ is smooth and proper over $k$, the Hodge-to-de Rham spectral sequence for $C$ degenerates if and only if $THH(C)$ is free (equivalently, $\sigma$-torsion-free) as a $THH(k)$-module.

Proof. Clearly, the second assertion is a special case of the first. It suffices to compare with [6]. In fact, by the equivalence given by Proposition [2,7], one sees that $\pi_* (X)[1/\sigma]$ is a finitely generated graded free $THH(k)_* [1/\sigma]$-module. Moreover, one has

$$\dim_k \pi_0 \left( (X \otimes_{THH(k)} k)^{S^1} \right) = \dim_k \pi_0 (X[1/\sigma]) = \text{rank}_{k[\sigma \pm 1]} \pi_{\text{even}}(X)[1/\sigma],$$

and similarly for the odd terms. Thus, formality holds if and only if the ranks agree, i.e.,

$$\text{rank}_{k[\sigma \pm 1]} \pi_{\text{even}}(X)[1/\sigma] = \dim_k \pi_{\text{even}}(X \otimes_{THH(k)} k), \quad \text{rank}_{k[\sigma \pm 1]} \pi_{\text{odd}}(X)[1/\sigma] = \dim_k \pi_{\text{odd}}(X \otimes_{THH(k)} k).$$

Note that $X \otimes_{THH(k)} k \simeq X/\sigma$. It follows (e.g., using Proposition [2,3]) that the ranks (over $\sigma = 0$ and $\sigma$ invertible, respectively) agree if and only if $X$ is (graded) free as a $THH(k)$-module spectrum.

It thus follows that, in order to verify degeneration of the Hodge-to-de Rham spectral sequence, one needs criteria for testing $\sigma$-torsion-freeness in $THH_* (C)$. We begin by observing that liftability to the sphere allows for a direct argument here. The general idea that liftability to the sphere should simplify the argument was well-known, and we are grateful to N. Rozenblyum for indicating it to us.

Example 3.5. Suppose $k = \mathbb{F}_p$ and suppose $C$ lifts to a stable ∞-category $\tilde{C}$ over the sphere $S^0$ (implicitly $p$-completed). Note that the map $S^0 \rightarrow THH(\mathbb{F}_p)$ factors through the natural map $\mathbb{F}_p \rightarrow THH(\mathbb{F}_p)$ given by choosing a basepoint in the circle $S^1$ via the equivalence $THH(\mathbb{F}_p) \simeq S^1 \otimes \mathbb{F}_p$ in $E_\infty$-rings [MSV97]. Then, as $THH(\mathbb{F}_p)$-module spectra, one has an equivalence

$$THH(C) \simeq THH(\tilde{C}) \otimes_{S^0} THH(\mathbb{F}_p) \simeq (THH(\tilde{C}) \otimes_{S^0} \mathbb{F}_p) \otimes_{\mathbb{F}_p} THH(\mathbb{F}_p).$$

3 Using the spectral version of the Witt vectors construction, this can be removed.
Since every $F_p$-module spectrum is (graded) free, this equivalence proves that $\text{THH}(C)$ is free as an $\text{THH}(F_p)$-module. Thus, degeneration holds for $C$.

We will now give the argument for a lifting to $W_2(k)$. If a $k$-linear stable $\infty$-category $C$ lifts to $W_2(k)$, then the $\text{THH}(k)$-module spectrum $\text{THH}(C)$ lifts to $\text{THH}(W_2(k))$. By considering the map $\text{THH}(W_2(k)) \to \text{THH}(k)$, we will be able to deduce $\sigma$-torsion-freeness (and thus degeneration) in many cases. The argument will require a small amount of additional bookkeeping and rely on an amplitude assumption. The basic input is the following fact about the homotopy ring of $\text{THH}(W_2(k))$. The entire computation is carried out in [Bru00], at least additively, but we will only need it in low degrees. For the reader’s convenience, we include a proof.

**Proposition 3.6** (Compare [Bru00]). Let $k$ be a perfect field.

1. We have
   \[ \pi_* \tau_{\leq 2p-2} \text{THH}(W_2(k)) \simeq W_2(k)[u]/u^p, \quad |u| = 2. \]
2. The map $\text{THH}_i(W_2(k)) \to \text{THH}_i(k)$ is zero for $0 < i \leq 2p - 2$. Furthermore, the map of $E_\infty$-rings $\text{THH}(W_2(k)) \to \text{THH}(k) \to \tau_{\leq 2p-2} \text{THH}(k)$ factors through the map $k \to \text{THH}(k) \to \tau_{\leq 2p-2} \text{THH}(k)$.

**Proof.** We compare with Hochschild homology over the integers. The map $S^0_{(p)} \to \mathbb{Z}_{(p)}$ induces an equivalence on degrees $< 2p - 3$. Thus, in the range stated in the theorem, we can compare $\text{THH}$ with Hochschild homology over $\mathbb{Z}_{(p)}$ or over $W(k)$. We have
   \[ \text{HH}_i(W_2(k)/\mathbb{Z}_{(p)}) \simeq \Gamma^*_W [u], \quad |u| = 2, \]
i.e., the divided power algebra on a class in degree two. Indeed, the Hochschild homology is the free simplicial commutative ring over $W_2(k)$ on a class in degree two.

It remains to check that the map $\text{THH}(W_2(k)) \to \text{THH}(k)$ vanishes on $\pi_2$. This, too, follows from the comparison with Hochschild homology over $\mathbb{Z}$. For a map of commutative rings $A \to B$, let $L_{B/A}$ denote the cotangent complex of $B$ over $A$. Using the classical Quillen spectral sequence from the cotangent complex to Hochschild homology (cf., e.g., [NS17, Prop. IV.4.1]), one has to show that the following map vanishes:
   \[ \pi_1 L_{W_2(k)/\mathbb{Z}_{(p)}} \to \pi_1 L_{k/\mathbb{Z}_{(p)}}. \]
Here one can replace the source $\mathbb{Z}_{(p)}$ with $W(k)$ since $k$ is perfect. Recall also that if $A$ is a ring and $a \in A$ a regular element, then one has a natural equivalence $L_{(A/a)/A} \simeq (a)/(a^2)[1]$. In our setting, one obtains for (6) the map of $W(k)$-modules
   \[ (p^2)/(p^4) \to (p)/(p^2), \]
which is zero. Finally, the factorization of the map of $E_\infty$-rings follows because $\tau_{\leq 2p-2} \text{THH}(W_2(k))$ is the truncation of the free $E_\infty$-ring over $W_2(k)$ on a class in degree two. \qed

We now give an argument that liftability together with a Tor-amplitude condition implies freeness. The observation is that if the Tor-amplitude is small, then any torsion has to occur in low homotopical degree.

**Proposition 3.7.** Let $M$ be a perfect $\text{THH}(k)$-module with Tor-amplitude contained in an interval $[a, b]$. Suppose that $M$ lifts to a perfect module over $\text{THH}(W_2(k))$. Then multiplication by $\sigma: \pi_{i-2}(M) \to \pi_i(M)$ is injective for $i \leq a + 2p - 2$. 

Proof. Without loss of generality, $a = 0$. By assumption, here we have $M \simeq \tilde{M} \otimes_{\mathrm{THH}(W_2(k))} \mathrm{THH}(k)$ for some connective and perfect $\mathrm{THH}(W_2(k))$-module $\tilde{M}$. Truncating, we find that there is a map of $\mathrm{THH}(k)$-modules
\begin{equation}
M \to \tau_{\leq 2p-2} \tilde{M} \otimes_{\tau_{\leq 2p-2} \mathrm{THH}(W_2(k))} \tau_{\leq 2p-2} \mathrm{THH}(k),
\end{equation}
which induces an isomorphism on degrees $\leq 2p - 2$. However, by Proposition 3.6 and the fact that any $k$-module spectrum is free, it follows that the right-hand-side is a free module over $\tau_{\leq 2p-2} \mathrm{THH}(k)$ on generators in nonnegative degrees. This shows that multiplication by $\sigma$ is an injection in this range of degrees.

Proposition 3.8. Let $M$ be a perfect $\mathrm{THH}(k)$-module with Tor-amplitude concentrated in $[-p, p]$. Suppose that $M$ lifts to a perfect module over $\mathrm{THH}(W_2(k))$. Then $M$ is free.

Proof. $M$ is a direct sum of $\mathrm{THH}(k)$-modules each of which is either free or equivalent to $M_{i,j} = \Sigma^i \mathrm{THH}(k)/\sigma^j$ for $-p \leq i \leq i + 2j + 1 \leq p$ as $M_{i,j}$ has Tor-amplitude $[i, i + 2j + 1]$. Note that $M_{i,j}$ has an element in $\pi_{i+2j-2}$ annihilated by $\sigma$, so we find $i + 2j - 2 \geq p - 3$ and therefore $i + 2j + 1 \geq p$ by Proposition 3.7. Therefore, $i + 2j + 1 = p$.

In particular, we find that if $M_{i,j}$ occurs as a summand, then $i + 2j + 1 = p$. We observe now that if the hypotheses of the lemma apply to $M$, then they apply to the $\mathrm{THH}(k)$-linear Spanier-Whitehead dual $\mathbb{D}M$: that is, $\mathbb{D}M$ is a perfect $\mathrm{THH}(k)$-module with Tor-amplitude concentrated in $[-p, p]$, and such that $\mathbb{D}M$ lifts to a perfect module over $\mathrm{THH}(W_2(k))$. If $M_{i,j}$ occurs as a summand of $M$, then its dual, which is given by $\Sigma^{-i-2j-1} \mathrm{THH}(k)/\sigma^j$, occurs as a summand of $\mathbb{D}M$. Applying the previous paragraph to $\mathbb{D}M$, it follows also that $-i = p$. Adding the two equalities, we find that $2j + 1 = 2p$, which is an evident contradiction.

Finally, we can state our general degeneration criterion in characteristic $p$, which will easily imply Theorem 1.2.

Proposition 3.9 (General formality criterion, characteristic $p$). Let $k$ be a perfect field of characteristic $p > 0$. Let $M \in \mathrm{Perf}(k)^{BS^1}$ be a perfect $k$-module with $S^1$-action whose amplitude is contained in $[-p, p]$. Suppose that there exists a dualizable object $M' \in \mathrm{Mod}_{\mathrm{THH}(k)}^{\mathrm{CycSp}}$ such that, as objects in $\mathrm{Perf}(k)^{BS^1}$, we have an equivalence $M \simeq M' \otimes_{\mathrm{THH}(k)} k$. Suppose furthermore that the underlying $\mathrm{THH}(k)$-module of $M'$ lifts to a perfect module over $\mathrm{THH}(W_2(k))$. Then $M$ is formal.

Proof. Combine Propositions 3.7 and 3.8.

Proof of Theorem 1.2. Let $\mathcal{C}$ be a smooth and proper stable $\infty$-category over $k$ satisfying the assumptions of the theorem. By assumption, there exists a smooth and proper lift $\tilde{\mathcal{C}}$ over $W_2(k)$ such that $\mathcal{C} \simeq \tilde{\mathcal{C}} \otimes_{W_2(k)} k$. Therefore, one has an equivalence of $\mathrm{THH}(k)$-modules
$$\mathrm{THH}(\mathcal{C}) \simeq \mathrm{THH}(\tilde{\mathcal{C}}) \otimes_{\mathrm{THH}(W_2(k))} \mathrm{THH}(k).$$

Furthermore, $\mathrm{THH}(\tilde{\mathcal{C}})$ is a perfect $\mathrm{THH}(W_2(k))$-module. Now, one can apply Proposition 3.9 with $M' = \mathrm{THH}(\tilde{\mathcal{C}})$.

Remark 3.10. The slight extension of the dimension range via duality goes back to the work of Deligne-Illusie [DI87] and appears in the recent work of Antieau-Vezzosi [AV17] on HKR isomorphisms in characteristic $p$. Note also that for a smooth and proper $k$-linear $\infty$-category $\mathcal{C}$, the
Hochschild homology $\text{HH}(\mathcal{C}/k)$ is always self-dual, cf. [Shk07]. Hence, it is no loss of generality to assume that the interval in which the amplitude of Hochschild homology is concentrated is symmetric about the origin.

We now describe the deduction of Theorem 1.1 from Theorem 1.2 as in [Kai08, Kal16]. We note that this is a standard argument and is also used in the commutative case [DI87]. We formulate the approach in the following formality criterion.

**Theorem 3.11** (General formality criterion, field case). Let $K$ be a field of characteristic zero. Let $M \in \text{Perf}(K)^{BS^1}$ be a perfect module equipped with an $S^1$-action. Suppose that there exists a finitely generated subring $R \subset K$ and a dualizable object $M' \in \text{Mod}_{\text{THH}(\mathcal{R})}(\text{CycSp})$ such that we have an equivalence in $\text{Perf}(K)^{BS^1}$, $M' \otimes_{\text{THH}(\mathcal{R})} K \simeq M$. Then $M$ is formal.

**Proof.** Any finitely generated field extension of $\mathbb{Q}$ is a filtered colimit of smooth $\mathbb{Z}$-algebras. Therefore, $K$ is a filtered colimit of its finitely generated subalgebras which are smooth over $\mathbb{Z}$. Enlarging $R$, we can assume that $R$ is smooth over $\mathbb{Z}$. Enlarging $R$ further, we can assume that the homology groups of $M' \otimes_{\text{THH}(\mathcal{R})} R$ (which is a perfect $R$-module spectrum) are finitely generated free $R$-modules and vanish for $i \notin [-p,p]$, for every prime $p$ which is noninvertible in $R$.

Suppose that the $S^1$-action on $M$ is nontrivial. Therefore, the $S^1$-action on $M' \otimes_{\text{THH}(\mathcal{R})} R$ is nontrivial too, and there exists a nontrivial differential in the Tate spectral sequence for $(M' \otimes_{\text{THH}(\mathcal{R})} R)^{tS^1}$. Then we can find a maximal ideal $m \subset R$ such that the first differential (which is a map of finitely generated free $R$-modules) remains nontrivial after base-change along $R \to R/m$ and thus after base-change along $R \to k \overset{\text{def}}{=} (R/m)_{\text{perf}}$.

Let $M'_k = M' \otimes_{\text{THH}(\mathcal{R})} \text{THH}(k) \in \text{Mod}_{\text{THH}(k)}(\text{CycSp})$, which is a dualizable object. Note that $k$ is a perfect field of characteristic $p > 0$. Moreover, the map $R \to k$ lifts to the length two Witt vectors because $R$ is smooth over $\mathbb{Z}$. It follows that the underlying $\text{THH}(k)$-module of $M'_k$ lifts to a perfect $\text{THH}((W_2(k))$-module. It follows that $M' \otimes_{\text{THH}(\mathcal{R})} k \in \text{Perf}(k)^{BS^1}$ is formal by Proposition 3.14. This contradicts the statement that there is a nontrivial differential in the Tate spectral sequence for $(M' \otimes_{\text{THH}(\mathcal{R})} k)^{tS^1}$ and proves the theorem. □

**Proof of Theorem 1.2.** Let $\mathcal{C}$ be a smooth and proper stable $\infty$-category over a field $K$ of characteristic zero. By the results of [Toe08], there exists a smooth and proper stable $\infty$-category $\tilde{\mathcal{C}}$ over a finitely generated subalgebra $\mathcal{R} \subset K$ such that $\mathcal{C} \simeq \tilde{\mathcal{C}} \otimes_{\mathcal{R}} K$. Then, one has the dualizable object $\text{THH}(\tilde{\mathcal{C}}) \in \text{Mod}_{\text{THH}(\mathcal{R})}(\text{CycSp})$ and by base-change, one has an equivalence in $\text{Perf}(K)^{BS^1}$, $\text{THH}(\tilde{\mathcal{C}}) \otimes_{\text{THH}(\mathcal{R})} K \simeq \text{HH}(\mathcal{C}/K)$. Now apply Theorem 3.11. □

We note that the above arguments actually enable a slight strengthening of Theorem 1.2 For example, Theorem 3.11 easily implies that if $F: \mathcal{C} \to \mathcal{D}$ is a functor of smooth and proper stable $\infty$-categories over $K$, then the $S^1$-action on the relative Hochschild homology $\text{fib}(\text{HH}(\mathcal{C}/K) \to \text{HH}(\mathcal{D}/K))$ is also trivial. More generally, this would work for any appropriately finite diagram. We formulate this as follows.

Let $K$ be a field of characteristic zero and let $N\text{Mot}_K$ denote the presentably symmetric monoidal $\infty$-category of noncommutative motives of $K$-linear stable $\infty$-categories introduced by Tabuada [Tab15] (see also [BGT13, HSS17]). Since Hochschild homology is an additive invariant, one has a symmetric monoidal, cocontinuous functor

\[ \text{HH}(\cdot/K): N\text{Mot}_K \to \text{Mod}_K^{BS^1}, \]
from \(N\text{Mot}_K\) into the \(\infty\)-category \(\text{Mod}^{BS^1}_{K}\) of \(K\)-module spectra equipped with an \(S^1\)-action. Let \(\text{Mot}^\omega_K \subset \text{Mot}_K\) denote the thick subcategory generated by the smooth and proper stable \(\infty\)-categories. Recall that if \(\mathcal{C}, \mathcal{D}\) are smooth and proper \(K\)-linear stable \(\infty\)-categories, then we have associated objects \([\mathcal{C}], [\mathcal{D}] \in N\text{Mot}^\omega_K\), and the mapping spectrum is given as

\[
\text{Hom}_{N\text{Mot}_K}([\mathcal{C}], [\mathcal{D}]) \simeq K(\text{Fun}(\mathcal{C}, \mathcal{D})),
\]

i.e., it is the connective algebraic K-theory spectrum of the \(\infty\)-category of \((K\)-linear) functors \(\mathcal{C} \to \mathcal{D}\). Note for instance that given a functor \(F: \mathcal{C} \to \mathcal{D}\), one can form the fiber of the associated map \([\mathcal{C}] \to [\mathcal{D}]\) of noncommutative motives, so that relative Hochschild homology is given by Hochschild homology of an object of \(N\text{Mot}^\omega_K\).

**Corollary 3.12.** For any \(X \in N\text{Mot}^\omega_K\), \(\text{HH}(X/K) \in \text{Perf}(K)^{BS^1}\) is formal.

**Proof.** By the results of [Toe08], and the fact that K-theory commutes with filtered colimits, it follows that \(N\text{Mot}^\omega_K\) is the filtered colimit of the stable \(\infty\)-categories \(N\text{Mot}^\omega_R\) of dualizable noncommutative motives of smooth and proper \(R\)-linear \(\infty\)-categories, as \(R\) ranges over the finitely generated subrings of \(K\). Thus, there exists \(R\) such that \(X\) arises via base-change from a dualizable object \(\tilde{X}\) in the \(\infty\)-category \(N\text{Mot}_R\). In this case, since \(\text{THH}\) is an additive invariant of \(R\)-linear stable \(\infty\)-categories into cyclotomic spectra (compare [BM12, BGT13, AMGR 17] for treatments), we can similarly form the dualizable object \(\text{THH}(\tilde{X}) \in \text{Mod}_{\text{THH}(R)}(\text{CycSp})\), which provides a lifting of \(\text{HH}(X/K)\). Now we can apply Theorem 3.11 as before. \(\square\)

### 4. Freeness results and degeneration in families

In this section, we will analyze Hodge-to-de Rham degeneration in families. In particular, we will give proofs of Theorems 1.3 and 1.4 showing that (under appropriate hypothesis) the relative Hodge-to-de Rham spectral sequence degenerates and that Hochschild homology is locally free. In characteristic zero, at least over a smooth base, this result follows from the existence of a connection [Get93] on periodic cyclic homology together with Theorem 1.1.

Throughout this section, we will need Künneth formulas, as in the form expressed in [AMN17]. If \((\mathcal{C}, \otimes, 1)\) is a symmetric monoidal stable \(\infty\)-category with biexact tensor product, then an object \(X \in \mathcal{C}\) is called perfect if it belongs to the thick subcategory generated by the unit. Perfectness is extremely useful to control objects in \(\mathcal{C}\) and their behavior. However, it can be tricky to check directly.

In [AMN17], the main result is that if \(k\) is a perfect field of characteristic \(p > 0\), in the \(\infty\)-category \(\text{Mod}_{\text{THH}(k)}(\text{Sp}^{BS^1})\) of modules over \(\text{THH}(k)\) in the \(\infty\)-category of spectra with an \(S^1\)-action, every dualizable object is perfect. This in particular implies the Künneth theorem for periodic topological cyclic homology proved by Blumberg-Mandell [BM17]. In this section, we will need variants of this result for non-regular rings in characteristic zero (Proposition 4.2) and in the perfect (but not necessarily field) case in characteristic \(p\) (Proposition 4.15). This will enable us to control Hochschild homology of stable \(\infty\)-categories over, respectively, local Artin rings in characteristic zero and large perfect rings in characteristic \(p\).

#### 4.1. Characteristic zero

In this subsection, we explain the deduction of Theorem 1.3 that the relative Hodge-to-de Rham spectral sequence degenerates for families of smooth and proper dg categories in characteristic zero, and that the relative Hochschild homology is locally free. We actually prove a result over connective \(E_\infty\)-rings and give a strengthening of the general formality criterion, Theorem 3.11.
The strategy will be to reduce to the local Artinian case, as is standard. We use the following definition.

**Definition 4.1.** A connective $E_{\infty}$-ring $A$ is *local Artinian* if $\pi_0(A)$ is a local Artinian ring, each homotopy group $\pi_i(A)$ is a finitely generated $\pi_0(A)$-module, and that $\pi_i(A) = 0$ for $i \gg 0$.

Fix a field $k$ of characteristic zero. Let $A$ be a local Artin $E_{\infty}$-ring with residue field $k$. Note that $A \to k$ admits a section unique up to homotopy by formal smoothness, compare, e.g., [Mat17 Prop. 2.14], and so we will consider $A$ as an $E_{\infty}$-algebra over $k$. Our first goal is to prove Künneth formulas for negative and periodic cyclic homology for smooth and proper stable $\infty$-categories over $A$.

Following [AMN17], we translate this into the following statement. As in section 2, $HH(A/k)$ defines a commutative algebra object in the $\infty$-category $\text{Sp}^{BS^1}$ of spectra with an $S^1$-action and we can consider the symmetric monoidal $\infty$-category of modules $\text{Mod}_{HH(A/k)}(\text{Sp}^{BS^1})$. Given an $A$-linear stable $\infty$-category $C$, the Hochschild homology $HH(C/k)$ defines an object in $\text{Mod}_{HH(A/k)}(\text{Sp}^{BS^1})$. The homotopy fixed points $HH(C/k)^{hS^1}$ are written $HC^{-}(C/k)$ and called the *negative cyclic homology* of $C$ (over $k$). See also [Hoy15] for comparisons with more classical definitions.

**Proposition 4.2.** Any dualizable object in the symmetric monoidal $\infty$-category $\text{Mod}_{HH(A/k)}(\text{Sp}^{BS^1})$ is perfect.

**Proof.** Let $M \in \text{Mod}_{HH(A/k)}(\text{Sp}^{BS^1})$ be a dualizable object. We have a lax symmetric monoidal functor

$$F : \text{Mod}_{HH(A/k)}(\text{Sp}^{BS^1}) \to \text{Mod}_{HC^{-}(A/k)}, \quad N \mapsto N^{hS^1}.$$  

By general results, this functor is fully faithful. Equivalently, the left adjoint functor

$$\text{Mod}_{HC^{-}(A/k)} \to \text{Mod}_{HH(A/k)}(\text{Sp}^{BS^1})$$

is a symmetric monoidal localization. Compare [AMN17 Sec. 7], which implies that $\text{Mod}_{k}(\text{Sp}^{BS^1})$ is identified with the $\infty$-category of $C^*(BS^1; k)$-modules complete with respect to the augmentation $C^*(BS^1; k) \to k$.

To check the equivalence, it suffices to prove that the functor is strictly symmetric monoidal on dualizable objects by [AMN17 Lemma 7.18]. That is, for dualizable objects $M, N \in \text{Mod}_{HH(A/k)}(\text{Sp}^{BS^1})$, one needs the map

$$(8) \quad F(M) \otimes_{HC^{-}(A/k)} F(N) \to F(M \otimes N)$$

to be an equivalence of $HC^{-}(A/k)$-module spectra. Note that we have an element $x \in \pi_{-2}HC^{-}(A/k)$ (i.e., a generator of $\pi_{-2}HC^{-}(k/k) \simeq \pi_{-2}C^*(BS^1; k)$) such that $HC^{-}(A/k)/x \simeq HH(A/k)$ and one has an equivalence of $HH(A/k)$-module spectra $F(M)/x \simeq M$ for any $M \in \text{Mod}_{HH(A/k)}(\text{Sp}^{BS^1})$ (cf. [AMN17 Sec. 7]). It thus follows that (8) becomes an equivalence after base-change $HC^{-}(A/k) \to HH(A/k)$.

It thus suffices to show that (8) becomes an equivalence after inverting $x$. Now we have

$$(F(M) \otimes_{HC^{-}(A/k)} F(N))[1/x] \simeq M^{tS^1} \otimes_{HP(A/k)} N^{tS^1}, \quad F(M \otimes N)[1/x] \simeq (M \otimes_{HH(A/k)} N)^{tS^1}.$$  

In other words, it suffices to show that the functor

$$F' : \text{Mod}_{HH(A/k)}(\text{Sp}^{BS^1}) \to \text{Mod}_{HP(A/k)}, \quad N \mapsto N^{tS^1}.$$  

---

*One could work in the derived $\infty$-category $D(k)$ in this subsection.*
is strictly symmetric monoidal on dualizable objects.

However, by Lemma 4.3 below, it follows that $F'$ can be identified with the functor $M \mapsto (M \otimes_{\text{HH}(A/k)} k)_{tS^1}$, i.e., $F'$ factors through the symmetric monoidal functor $\text{Mod}_{\text{HH}(A/k)}(\text{Sp}^{BS^1}) \to \text{Mod}_k(\text{Sp}^{BS^1})$ given by base-change $\text{HH}(A/k) \to k$. Furthermore, $\text{HP}(A/k) \simeq k^{tS^1}$. Since dualizable objects in $\text{Mod}_k(\text{Sp}^{BS^1})$ are perfect, it follows that $F'$ satisfies a Künneth formula. This implies the result.

**Lemma 4.3.** If $M$ is an object of $\text{Mod}_{\text{HH}(A/k)}(\text{Sp}^{BS^1})$ such that $M$ is bounded below, then the natural map $M \to M \otimes_{\text{HH}(A/k)} k$ induces an equivalence on $S^1$-Tate constructions.

**Proof.** Now $M \simeq \lim_{\tau \leq n} M$ and $M \otimes_{\text{HH}(A/k)} k \simeq \lim_{\tau \leq n} (M \otimes_{\text{HH}(A/k)} k)$. Both of these inverse limits become constant in any given range of dimensions. Therefore, they commute with $S^1$-Tate constructions. Therefore, it suffices to assume that $M$ is $n$-truncated, and by a filtration argument, discrete. By a further dévissage, we can assume that $M$ is actually a discrete $k$-module, considered as a $\text{HH}(A/k)$-module via the augmentation. We are thus reduced to showing that if $N$ is a discrete $k$-module, then the map

$$N \to N \otimes_{\text{HH}(A/k)} k \simeq N \otimes_k (k \otimes_{\text{HH}(A/k)} k) \simeq N \otimes_k \text{HH}(k \otimes_A k/k)$$

induces an equivalence on $S^1$-Tate constructions.

However, since the homology of $k \otimes_A k$ forms a connected graded, commutative Hopf algebra, it follows that $\pi_i(k \otimes_A k)$ is the tensor product of polynomial algebras on even-dimensional classes and exterior algebras on odd-dimensional classes. Therefore, $k \otimes_A k$ is a free $E_{\infty}$-$k$-algebra $\text{Sym}^*V$ for some $k$-module spectrum $V$ with $\pi_i(V) = 0$ for $i \leq 0$. Furthermore, $\text{HH}(k \otimes_A k/k) \simeq \text{Sym}^*(S^1 \otimes V)$. The desired equivalence now follows because for $i > 0$, $\text{Sym}^i(S^1 \otimes V)$ is a free module over the group ring $k[S^1]$, and so the terms for $i > 0$ do not contribute to the Tate construction.

**Corollary 4.4.** Let $A$ be a local Artin $E_{\infty}$-ring and let $C$ be a smooth and proper stable $\infty$-category over $A$. Then the map $\text{HP}(C/k) \to \text{HP}(C \otimes_A k/k)$ is an isomorphism.

Note that when $A = k$ itself, this recovers certain cases of the classical theorem of Goodwillie (Good85 Theorem II.5.1, Good86 Lemma I.3.3) about the nilinvariance of periodic cyclic homology. The corollary follows from Lemma 4.3 because one has an equivalence

$$\text{HH}(C \otimes_A k/k) \simeq \text{HH}(C/k) \otimes_{\text{HH}(A/k)} k.$$ 

**Corollary 4.5.** Let $A$ be a local Artin $E_{\infty}$-ring. Let $M \in \text{Mod}_{\text{HH}(A/k)}(\text{Sp}^{BS^1})$ be dualizable, and let $M_A = M \otimes_{\text{HH}(A/k)} A \in \text{Mod}_A(\text{Sp}^{BS^1})$ and $M_k = M \otimes_{\text{HH}(A/k)} k \in \text{Perf}(k^{BS^1})$. Then:

1. $M_A \in \text{Mod}_A(\text{Sp}^{BS^1})$ belongs to the thick subcategory generated by the unit.
2. $M^A_{tS^1} \otimes_A k \simeq M^A_{tS^1}$.
3. $M^A_{tS^1}$ is a graded free $A^{tS^1}$-module.

**Proof.** By Proposition 4.2 $M$ belongs to the thick subcategory generated by the unit in $\text{Mod}_{\text{HH}(A/k)}(\text{Sp}^{BS^1})$. It follows that $M_A \in \text{Mod}_A(\text{Sp}^{BS^1})$ belongs to the thick subcategory generated by the unit. Thus, we obtain the first claim. The second claim is implied by the first, as for any perfect object $X \in \text{Mod}_A(\text{Sp}^{BS^1})$, one has $(X \otimes A)^{tS^1} \simeq X^{tS^1} \otimes_A k$ by a thick subcategory argument. Finally, one has natural maps

$$M^{tS^1} \to M^A_{tS^1} \to M^A_{tS^1} \otimes_A k \simeq M^A_{k^{tS^1}},$$

where
such that the composite is an equivalence by Lemma 4.3. Thus, the map \( M'^{S^1}_A \rightarrow (M'^{S^1}_A) \otimes_A k \) has a section of \( k \)-module spectra. Lifting a basis, this implies that \( M'^{S^1}_A \) is free as an \( A^{S^1} \)-module. \( \square \)

**Lemma 4.6.** Let \( A \) be an augmented local Artin \( E_\infty \)-ring with residue field \( k \). Let \( M \) be a perfect \( A \)-module. Then

\[
\dim_k(\pi_*(M)) \leq (\dim_k \pi_*(A))(\dim_k \pi_*(M \otimes_A k)) ,
\]

and if equality holds \( M \) is free.

**Proof.** Since \( A \) has a filtration (in \( A \)-modules) by copies of \( k \), the inequality is evident. If equality holds, suppose that \( i \in \mathbb{Z} \) is minimal such that \( \pi_i(M) \neq 0 \). Choose \( x \in \pi_i(M) \) whose image in \( \pi_i(M \otimes_A k) \simeq \pi_i(M \otimes_{\pi_0(A)} k) \) is nonzero. Form a cofiber sequence \( \Sigma^i A \xrightarrow{\alpha} M \rightarrow N \) of \( A \)-modules. It follows that

\[
\dim_k(\pi_*(N \otimes_A k)) = \dim_k(\pi_*(M \otimes_A k)) - 1, \quad \dim_k(\pi_*(M)) \leq \dim_k(\pi_*(N)) + \dim_k(\pi_*(A)).
\]

Combining this with (9), we find that \( \dim_k \pi_*(N) = (\dim_k \pi_*(A))(\dim_k \pi_*(N \otimes_A k)) \). By an evident induction, \( N \) is free as an \( A \)-module. The long exact sequence in homotopy, which must reduce to a short exact sequence now shows that \( M \) is also free as an \( A \)-module. \( \square \)

We can now prove the main freeness and degeneration theorems of this section, which provides a substantial strengthening of Theorem 3.11. Let \( \text{CAlg}(\text{Sp}_{>0}) \) denote the \( \infty \)-category of connective \( E_\infty \)-rings. In the following argument, one could also work with simplicial commutative rings.

**Theorem 4.7** (General freeness criterion, relative case). Let \( A \) be a connective \( E_\infty \)-algebra over \( \mathbb{Q} \). Let \( M \in \text{Perf}(A)^{BS^1} \). Suppose that there exists a compact object \( R \in \text{CAlg}(\text{Sp}_{>0}) \) with a map \( R \rightarrow A \), a dualizable object \( M'_R \in \text{Mod}_{\text{THH}(R)}(\text{CycSp}) \), and an equivalence \( M'_R \otimes_{\text{THH}(R)} A \simeq M \in \text{Perf}(A)^{BS^1} \). Then \( M \) is a finitely generated projective \( A \)-module and the \( S^1 \)-action on \( M \) is formal.

**Proof.** We first treat the case where \( A \) is a local Artin \( E_\infty \)-ring with residue field \( k \). To see that \( M \) is free, it suffices to show that equality holds in (9). Our assumptions show that \( M \) lifts to a dualizable object of \( \text{Mod}_{\text{HH}(A/k)}(\text{Sp}^{BS^1}) \). Using the Tate spectral sequence, one obtains

\[
\dim_k(\pi_0(M^{S^1})) + \dim_k(\pi_1(M^{S^1})) \leq \dim_k(\pi_*(M)).
\]

Moreover, by Corollary 4.3, we know that \( M^{S^1} \) is a free \( A^{S^1} \)-module and that \( M^{S^1} \otimes_A k \simeq (M \otimes_A k)^{S^1} \). Note that \( \pi_0 R \) is a finitely generated \( \mathbb{Z} \)-algebra. Thus we can apply Theorem 3.11 and we find that \( M \otimes_A k \) is formal in \( \text{Perf}(k)^{BS^1} \). We obtain:

\[
\dim_k(\pi_0(M^{S^1})) + \dim_k(\pi_1(M^{S^1})) = \left( \dim_k(\pi_0 \left( (M \otimes_A k)^{S^1} \right)) + \dim_k(\pi_1 \left( (M \otimes_A k)^{S^1} \right)) \right) \dim_k(\pi_*(A))
\]

\[
= \dim_k(\pi_*(M \otimes_A k)) \dim_k(\pi_*(A)).
\]

Combining the above two inequalities, we obtain \( \dim_k \pi_*(M \otimes_A k) \dim_k(\pi_*(A)) \leq \dim_k(\pi_*(M)) \), which shows that the converse of (9) holds and \( M \) is free. Moreover, equality holds in (10), so that the \( S^1 \)-Tate spectral sequence for \( M \) degenerates and \( M \) is formal.

We now treat the general case. Clearly it suffices to treat the case where \( A \) is a compact object of the \( \infty \)-category of connective \( E_\infty \)-algebras over \( \mathbb{Q} \). In this case, \( \pi_0(A) \) is noetherian and the homotopy groups \( \pi_i(A) \) are finitely generated \( \pi_0(A) \)-modules. We thus suppose \( A \) is of this form.

To check the above statements, it suffices to replace \( A \) by its localization at any prime ideal of \( \pi_0(A) \). Thus, we may assume that \( \pi_0(A) \) is local. Let \( x_1, \ldots, x_n \in \pi_0(A) \) be a system of generators of the maximal ideal. For each \( r > 0 \), we let \( A'_r = A/(x_1^r, \ldots, x_n^r) \). Note moreover that
$A'_r \simeq \lim_{\tau \leq m} A'_r$ and that $\lim_{m} A'_r$ is the completion of $A$, which is in particular faithfully flat over $A$. By the above analysis, $\hat{M} \otimes_A \tau_{\leq m} A'_r$ is a free $\tau_{\leq m} A'_r$-module for each $m, r$ and the Tate spectral sequence degenerates. Now we can let $m, r \to \infty$. Since $M$ is perfect as an $A$-module, it follows that $M$ is free, as desired, and the $S^1$-action is formal. 

Let $A$ be a connective $E_{\infty}$-algebra over $\mathbb{Q}$. Similarly, one can construct $\mathcal{N}Mot_{A}$ the $\infty$-category of noncommutative motives of $A$-linear $\infty$-categories. We let $\mathcal{N}Mot_{A}$ denote the thick subcategory generated by the motives of smooth and proper $A$-linear $\infty$-categories. We have, again, a Hochschild homology functor $HH(\cdot/A) \to \text{Mod}^{BS^1}_A$. The next result gives a basic formality property of this functor; for smooth and proper $A$-linear $\infty$-categories, it includes the degeneration of the relative Hodge-to-de Rham spectral sequence.

**Corollary 4.8.** Let $X \in \mathcal{N}Mot^w_{A}$. Then $HH(X/A) \in \text{Mod}^{BS^1}_A$ is a finitely generated projective $A$-module and the $S^1$-action is formal.

**Proof.** Here we use a refinement of the results of [Toen18] for $E_{\infty}$-algebras. Namely, we claim that the functor which assigns to an $E_{\infty}$-ring spectrum $R$ the $\infty$-category of smooth and proper $R$-linear $\infty$-categories commutes with filtered colimits in $R$. Now, smooth and proper $R$-linear $\infty$-categories are compact; in fact, combine [AG14] Props. 3.5, 3.11. Therefore, it suffices to see that if $R$ is a filtered colimit of $E_{\infty}$-algebras $R_i$, then any smooth and proper $R$-linear $\infty$-category $\mathcal{C}$ descends to some $R_i$. To see this, we observe that $\mathcal{C}$ is equivalent to $\text{Perf}(B)$ for an associative $A$-algebra $B$ which is compact [AG14] Prop. 3.11 and we can descend the algebra to a compact algebra over some $R_i$ thanks to [Lur17] Lemma 11.5.7.17. Moreover, by compactness we can also descend the duality datum to some finite stage.

In view of this, we conclude that given $X \in \mathcal{N}Mot^w_{A}$, there exists a compact object $R \in \text{CAlg}(Sp_{>0})$ mapping to $A$ and a smooth and proper $R$-linear $\infty$-category $\mathcal{C}$ such that $\mathcal{C} \simeq \mathcal{C} \otimes_R A$. Using Theorem [Toen18] we can now conclude the proof as before. \hfill $\square$

### 4.2. Characteristic $p$.

The characteristic zero assertion essentially amounts to the idea that periodic cyclic homology should form a crystal over the base which is also coherent, and any such is necessarily well-known to be locally free. In characteristic $p$, one can appeal to an analogous argument: given a smooth algebra $R$ in characteristic $p$, any finitely generated $R$-module $M$ isomorphic to its own Frobenius twist is necessarily locally free [EK04] Prop. 1.2.3. In this subsection, we prove Theorem [3.4] from the introduction. In doing so, we essentially use the Frobenius-semilinearity of the cyclotomic Frobenius.

We first discuss what we mean by liftability. Let $A$ be a regular (noetherian) $F_p$-algebra. Recall that $A$ is $F$-finite if the Frobenius map $\varphi : A \to A$ is a finite morphism. We refer to [DM17] Sec. 2.2 for a general discussion of $F$-finite rings.

**Definition 4.9.** Given an $F$-finite regular noetherian ring $A$, a lift of $A$ to $\mathbb{Z}/p^2$ will mean a flat $\mathbb{Z}/p^2$-algebra $\tilde{A}$ with an isomorphism $\tilde{A} \otimes_{\mathbb{Z}/p^2} F_p \simeq A$.

Let $A$ be a regular noetherian $F_p$-algebra. By Popescu’s smoothing theorem (see [Sta17] Tag 07GC] for a general reference), $A$ is a filtered colimit of smooth $F_p$-algebras. It follows that the cotangent complex $L_{A/F_p}$ is concentrated in degree zero and identified with the Kähler differentials; in addition, they form a flat $A$-module. If $A$ is in addition $F$-finite, then the Kähler differentials are finitely generated and therefore projective as an $A$-module. Recall that the cotangent complex controls the infinitesimal deformation theory of $A$ [H17] Ch. III, Sec. 2]. Therefore, $A$ is formally smooth as an $F_p$-algebra, and a lift to $\mathbb{Z}/p^2$ exists. Given a lift $\tilde{A}$ to $\mathbb{Z}/p^2$, it follows that $\tilde{A}$ is
formally smooth over \( \mathbb{Z}/p^2 \). In particular, it follows that any two lifts to \( \mathbb{Z}/p^2 \) are (noncanonically) isomorphic. Moreover, if \( A \to B \) is a map of \( F \)-finite regular noetherian \( \mathbb{F}_p \)-algebras and \( \tilde{A}, \tilde{B} \) are respective lifts to \( \mathbb{Z}/p^2 \), then the map lifts to a map \( \tilde{A} \to \tilde{B} \).

Let \( A \) be a regular \( F \)-finite \( \mathbb{F}_p \)-algebra. Then the Frobenius \( \varphi: A \to A \) is a finite, flat morphism. We let \( A_{\text{perf}} \) denote the perfection of \( A \), i.e., the colimit of copies of \( A \) along the Frobenius map. Then we have inclusions

\[
A \subset A^{1/p} \subset A^{1/p^2} \subset \ldots A_{\text{perf}},
\]

such that all maps are faithfully flat and the colimit is \( A_{\text{perf}} \). Our strategy will essentially be descent to \( A_{\text{perf}} \). Unfortunately, \( A_{\text{perf}} \) is not noetherian. Thus, we will need the following result.

**Proposition 4.10.** Let \( A \) be a regular \( F \)-finite \( \mathbb{F}_p \)-algebra.

1. Then the ring \( A_{\text{perf}} \) is coherent, i.e., the finitely presented modules form an abelian category.
2. Let \( I \subset A \) be an ideal. Given a finitely presented \( A_{\text{perf}} \)-module \( M \), the submodule \( M' \subset M \) consisting of those elements annihilated by a power of \( I \) is also coherent and its annihilator in \( A_{\text{perf}} \) is finitely generated.

**Proof.** The first assertion follows because \( A_{\text{perf}} \) is the filtered colimit of copies of the noetherian ring \( A \) along the Frobenius map, which is flat by regularity. If \( M \) is a coherent \( A_{\text{perf}} \)-module, then \( M \) descends to \( A^{1/p^n} \) for some \( n \), i.e., there exists a finitely generated module \( M_n \) over \( A^{1/p^n} \) such that \( M \simeq A_{\text{perf}} \otimes_{A^{1/p^n}} A_{\text{perf}} \). Then \( M_n \) has an \( A^{1/p^n} \) submodule \( M'_n \) consisting of the \( I \)-power torsion, which is also finitely generated (and hence finitely presented), and such that the quotient has no \( I \)-power torsion. It follows from flatness that \( M'_n \otimes_{A^{1/p^n}} A_{\text{perf}} = M' \), which is thus coherent. Since \( M' \) is coherent, its annihilator ideal is also coherent. \( \square \)

We will also need to observe that analogs of Bökstedt’s calculation of \( \text{THH}(k) \) hold when \( k \) is any perfect \( \mathbb{F}_p \)-algebra, not only a field. Similarly, analogs of Propositions 3.6 and Proposition 3.7 hold with analogous arguments.

**Proposition 4.11.** Let \( k \) be a perfect \( \mathbb{F}_p \)-algebra. Suppose \( M \) is a perfect \( \text{THH}(k) \)-module and \( \pi_i(M/\sigma) = 0 \) for \( i \not\in [a, b] \) for \( b - a \leq 2p - 2 \). Suppose \( M \) lifts to a perfect \( \text{THH}(W_2(k)) \)-module. Then, as \( \pi_i \text{THH}(k) \simeq k[\sigma] \)-modules, one has \( \pi_i(M) \simeq \pi_i(M/\sigma) \otimes_k k[\sigma] \).

**Proof.** Without loss of generality, \( a = 0 \). For each \( j \), we need to argue that multiplication by \( \sigma \) is a split injection of \( k \)-modules \( \pi_{j-2}(M) \to \pi_j(M) \). Using the long exact sequence and the assumption on \( M/\sigma \), we find that multiplication by \( \sigma \) is an isomorphism for \( j \geq 2p - 1 \). For \( j \leq 2p - 2 \), the equivalence in the range \([0, 2p - 2]\) as in \([7]\) implies the result. \( \square \)

We can now state and prove the main formality statement in characteristic \( p \) over a regular \( F \)-finite base.

**Theorem 4.12** (Formality criterion, relative characteristic \( p \) case). Let \( A \) be a regular \( F \)-finite \( \mathbb{F}_p \)-algebra. Let \( \tilde{A} \) be a flat lift to \( \mathbb{Z}/p^2 \). Let \( M \in \text{Perf}(A)^{BS_1} \). Suppose that:

1. There exists a dualizable object \( M' \in \text{Mod}_{\text{THH}(A)\text{(CycSp)}} \) such that \( M' \otimes_{\text{THH}(A)} A \in \text{Perf}(A)^{BS_1} \).
2. \( \pi_i(M) \) vanishes for \( i \not\in [a, b] \) for some \( a, b \) with \( b - a \leq 2p - 2 \).
3. The underlying \( \text{THH}(A) \)-module of \( M' \) lifts to a perfect \( \text{THH}(\tilde{A}) \)-module.

Then \( M \) is a finitely generated (graded) projective \( A \)-module, and the \( S^1 \)-action on \( M \) is formal.
Proof. First, we can reduce to the case where \( A \) is an \( F \)-finite regular local ring with maximal ideal \( \mathfrak{m} \). In this case, we can induct on the Krull dimension \( d \) of \( A \). We can assume that the result holds for all \( F \)-finite regular local rings of Krull dimension less than \( d \). When \( d = 0 \), the claim is of course Theorem 1.2.

To verify the claims for \( A \), we can now replace \( A \) by its \( \mathfrak{m} \)-adic completion \( \widehat{A} \), which is faithfully flat over \( A \). Note that \( \widehat{A} \) is also an \( F \)-finite regular local ring of Krull dimension \( d \). Since \( \widehat{A} \) is complete, it contains a copy of its residue field \( k \) and is identified with \( \widehat{A} \simeq k[[x_1, \ldots, x_n]] \). We can consider the faithfully flat map \( \widehat{A} \to k_{\text{perf}}[[x_1, \ldots, x_d]] \). Replacing \( A \) with \( k_{\text{perf}}[[x_1, \ldots, x_d]] \), we will now simply assume that \( A \) is in addition complete and has perfect residue field. By the inductive hypothesis, all the differentials in the Hodge-to-de Rham spectral sequence are \( m \)-power torsion and it follows that the \( m \)-adic completion \( \widehat{A} \) is locally free away from \( \mathfrak{m} \).

Let \( A_{\text{perf}} \) denote the (colimit) perfection of \( A \), so one has a faithfully flat map \( A \to A_{\text{perf}} \). We form the base-changes \( M'_{\text{perf}} \overset{\text{def}}{=} M' \otimes_{\text{THH}(A)} \text{THH}(A_{\text{perf}}) \in \text{Mod}_{\text{THH}(A_{\text{perf}})}(\text{CycSp}) \) (which is a dualizable object) and \( M_{\text{perf}} = M \otimes_A A_{\text{perf}} \in \text{Perf}(A_{\text{perf}})^{BS^1} \). We claim that the cyclotomic Frobenius

\[
\varphi: M'_{\text{perf}}[1/\sigma] \to (M'_{\text{perf}})^{C_p} \simeq (M_{\text{perf}})^{tS^1}
\]

is an equivalence. This follows using the same arguments as in [AMN17] Sec. 4; again, one needs to know that both sides are symmetric monoidal functors in \( M'_{\text{perf}} \). For this, it suffices to show that \( M'_{\text{perf}} \) belongs to the thick subcategory generated by the unit in \( \text{Mod}_{\text{THH}(A_{\text{perf}})}(\text{Sp}^{BS^1}) \). We will check this in Proposition 5.15 below.

Note that \( M'_{\text{perf}} \) is an \( \text{THH}(A_{\text{perf}}) \)-module, and \( M'_{\text{perf}}/\sigma \simeq M_{\text{perf}} \). Under the liftability hypotheses, we conclude using Proposition 4.11 that there is an isomorphism of \( A_{\text{perf}}[\sigma] \)-modules

\[
\pi_* M'_{\text{perf}} \simeq \pi_* M_{\text{perf}}[\sigma].
\]

Combining, we find an isomorphism of \( A_{\text{perf}} \)-modules

\[
\pi_*(M_{\text{perf}})[\sigma^\pm 1]^{(1)} \simeq \pi_* \left( M_{\text{perf}}^{tS^1} \right).
\]

In addition, we have the Hodge-to-de Rham spectral sequence, which shows that \( \pi_0(M'_{\text{perf}}^{tS^1}) \) is a subquotient of \( \pi_{\text{even}}(M_{\text{perf}}) \) and is a coherent \( A_{\text{perf}} \)-module. Since the differentials are \( m \)-power torsion, it follows that the \( m \)-power torsion in \( \pi_0(M'_{\text{perf}}^{tS^1}) \) is a subquotient of the \( m \)-power torsion in \( \pi_{\text{even}}(M_{\text{perf}}) \).

Let \( I \) be the annihilator of the \( m \)-power torsion in \( \pi_{\text{even}}(M_{\text{perf}}) \), which by Proposition 4.10 is a finitely generated ideal. Then combining the above observations and [11], we find that \( I^{[p]} \) (i.e., the ideal generated by \( p \)th powers of elements in \( I \)) is the annihilator of the \( m \)-power torsion in \( \pi_0(M'_{\text{perf}}^{tS^1}) \). Since this is a subquotient of \( \pi_{\text{even}}(M_{\text{perf}}) \), it follows that \( I \subset I^{[p]} \), which is only possible for a finitely generated proper ideal if \( I = (0) \). Therefore, \( \pi_{\text{even}}(M_{\text{perf}}) \) (and similarly for the odd-dimensional Hochschild homology) is torsion-free.

Finally, it suffices to prove freeness. We have proved that \( \pi_*(M) \) consists of finitely generated, torsion-free \( A \)-modules. Let \( x \in \mathfrak{m} \setminus \mathfrak{m}^2 \), so that \( A/x \) is a regular local ring too. It follows that \( \pi_*(M/x) \) is \( x \)-torsion-free and that, by induction on the Krull dimension, \( \pi_*(M)/(x) \) is a free \( A/(x) \)-module. This easily implies that \( \pi_*(M) \) is free as an \( A \)-module. By comparing with the base-change from \( A \) to the perfection of its fraction field, it also follows that \( M \) is formal. \( \square \)
In the course of the above argument, we had to check a perfectness statement. In [AMN17], such results are proved when $A_{\text{perf}}$ is a field, but they depend on noetherianness hypotheses. One can carefully remove the noetherianness hypotheses in this case, but for simplicity, we verify this by using the technique of relative THH (also discussed in [AMN17 Sec. 3]). The starting point is a relative version of Bökstedt’s calculation. We denote by $S$ using the technique of relative THH (also discussed in [AMN17, Sec. 3]). The starting point is a $\Sigma^\infty_+ S^0_{\geq 0}$. The idea of considering THH relative to such $E_\infty$-rings is known to experts, and will play an important role in the forthcoming work [BMS].

**Proposition 4.13.** Let $A$ be an $F$-finite regular local ring with system of parameters $t_1, \ldots, t_n$ and perfect residue field $k$. Consider the map of $E_\infty$-rings $S^0[q_1, \ldots, q_n] \to A$, $q_i \mapsto t_i$. Then

$$\text{THH}(A/S^0[q_1, \ldots, q_n])_* \cong A[\sigma], \quad |\sigma| = 2.$$ 

**Proof.** Compare also the treatment in [AMN17 Sec. 3]. Since $A$ is $F$-finite and regular, the cotangent complex $L_{A/F_p}$ is a finitely generated free module in degree zero. By the transitivity sequence, $L_{A/F_p}$ is a perfect $A$-module. Thus, by the Quillen spectral sequence, the homotopy groups of $\text{HH}(A/Z[q_1, \ldots, q_n])$ and thus $\text{THH}(A/Z[q_1, \ldots, q_n])$ are finitely generated $A$-modules. Compare also [DM17] for general finite generation results.

Moreover, after base-change $S^0[q_1, \ldots, q_n] \to S^0$ sending $q_i \mapsto 0$, one obtains Bökstedt’s calculation $\text{THH}(k)_* \cong k[\sigma]$. Since the homotopy groups of $\text{THH}(A/S^0[q_1, \ldots, q_n])$ are finitely generated $A$-modules, and $A$ is local, the result follows. \qed

Let $A$ be as above. Given a smooth and proper $A$-linear stable $\infty$-category $C$, one can consider the invariant $\text{THH}(C/S^0[q_1, \ldots, q_n])$, which naturally takes values in the symmetric monoidal $\infty$-category $\text{Mod}_{\text{THH}(A/S^0[q_1, \ldots, q_n])}(\text{Sp}^{B_1 S})$. This produces a one-parameter deformation of Hochschild homology over $A$, and it is particularly well-behaved (at least for smooth and proper $A$-linear stable $\infty$-categories) by the following result.

**Proposition 4.14.** Let $A$ be an $F$-finite regular local ring with system of parameters $t_1, \ldots, t_n$ and perfect residue field $k$. Any dualizable object in $\text{Mod}_{\text{THH}(A/S^0[q_1, \ldots, q_n])}(\text{Sp}^{B_1 S})$ is perfect.

**Proof.** This follows by regularity from [AMN17 Theorem 2.15]. \qed

**Proposition 4.15.** Let $A$ be an $F$-finite regular local ring. Let $N \in \text{Mod}_{\text{THH}(A)}(\text{Sp}^{B_1 S})$ be dualizable. Then $N \otimes_{\text{THH}(A)} \text{THH}(A_{\text{perf}})$ is perfect.

**Proof.** In fact, we have a factorization of $E_\infty$-rings with $S^1$-action

$$\text{THH}(A) \to \text{THH}(A/S^0[q_1, \ldots, q_n]) \to \text{THH}(A_{\text{perf}}/S^0[q_1^{1/p^\infty}, \ldots, q_n^{1/p^\infty}] \cong \text{THH}(A_{\text{perf}}).$$

We have just seen that $N \otimes_{\text{THH}(A)} \text{THH}(A/S^0[q_1, \ldots, q_n])$ is perfect in $\text{Mod}_{\text{THH}(A/S^0[q_1, \ldots, q_n])}(\text{Sp}^{B_1 S})$;

base-changing up to $\text{THH}(A_{\text{perf}})$, the result follows. \qed

Once more, we make the statement for Hochschild homology of categories, or more generally for noncommutative motives. Let $A$ be an $F$-finite regular noetherian ring with lift $\tilde{A}$ to $\mathbb{Z}/p^2$. We use, again, the $\infty$-category $\mathcal{N}\text{Mot}_A$, its subcategory $\mathcal{N}\text{Mot}_A^+$ generated by the motives of smooth and proper $A$-linear $\infty$-categories, and the Hochschild homology functor $\text{HH}(\cdot/A) : \mathcal{N}\text{Mot}_A \to \text{Mod}_{A^{B_1 S}}$

**Corollary 4.16.** Let $X \in \mathcal{N}\text{Mot}_A^+$. Suppose that $X$ lifts to an object of $\mathcal{N}\text{Mot}_A^{\infty}$ and that $\text{HH}_i(X/A)$ vanishes for $i \not\in [a, b]$ for $b - a \leq 2p - 2$. Then the Hochschild homology groups $\text{HH}_i(X/A)$ are finitely generated projective $A$-modules and the $S^1$-action is formal.
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