COMBINATORICS AND TOPOLOGY OF STRAIGHTENING MAPS I: COMPACTNESS AND BIJECTIVITY

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ABSTRACT. We study the parameter space structure of degree \( d \geq 3 \) one complex variable polynomials as dynamical systems acting on \( \mathbb{C} \). We introduce and study straightening maps. These maps are a natural higher degree generalization of the ones introduced by Douady and Hubbard to prove the existence of small copies of the Mandelbrot set inside itself. We establish that straightening maps are always injective and that their image contains all the corresponding hyperbolic systems. Also, we characterize straightening maps with compact domain. Moreover, we give two classes of bijective straightening maps. The first produces an infinite collection of embedded copies of the \((d - 1)\)-fold product of the Mandelbrot set in the connectedness locus of degree \( d \geq 3 \). The second produces an infinite collection of full families of quadratic connected filled Julia sets in the cubic connectedness locus, such that each filled Julia set is quasiconformally embedded.

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Introduction

The Mandelbrot set consists of all parameters $c \in \mathbb{C}$ for which the filled Julia set $K(z^2 + c)$ of the quadratic polynomial $z^2 + c$ is connected. In any reasonable picture of the Mandelbrot set $\mathcal{M}$ one immediately observes the presence of small copies of it contained in itself. Douady and Hubbard [DH2] gave a mathematical proof of this fact. They established the existence of homeomorphisms $\chi$ from carefully chosen subsets $\mathcal{M}'$ of the Mandelbrot set onto the whole Mandelbrot set $\mathcal{M}$. These homeomorphisms $\chi : \mathcal{M}' \to \mathcal{M}$ are fundamental examples of what are usually called “straightening maps”. The degree $d \geq 3$ analogue of the Mandelbrot set is called the connectedness locus $C(d)$ of degree $d$. In this paper we propose a definition of straightening maps in the context of higher degree polynomial dynamics. This involves introducing appropriate domains as well as the corresponding target sets and the maps themselves. In contrast with quadratic polynomials, one encounters a large diversity of target sets. Therefore, a greater variety of structures replicate all over the corresponding connectedness loci. In particular, for cubic polynomials, the target sets include the cubic connectedness locus $C(3)$, the connectedness locus of biquadratic polynomials $((a, b) \in \mathbb{C}^2; K((z^2 + a)^2 + b))$ is connected), the full family of quadratic connected filled Julia sets $\mathcal{M}K = \{(c, z) \in \mathbb{C}^2; c \in \mathcal{M}, z \in K(z^2 + c)\}$, and the cross product of the Mandelbrot set with itself $\mathcal{M} \times \mathcal{M}$. In this paper we also establish the existence of a large collection of embeddings of $\mathcal{M} \times \mathcal{M}$ and inclusions of $\mathcal{M}K$ in the cubic connectedness locus. The latter restrict to quasiconformal embeddings of every quadratic filled Julia set.

We proceed with a brief overview of our main results. The next section (Section 1) contains a more detailed account of them, including the relevant definitions and statements.
The first objective of this paper is to propose a definition of straightening maps (i.e., to introduce the domains, the target sets, and the maps). We will work in the parameter space \( \text{Poly}(d) \) of monic centered polynomials of degree \( d \geq 2 \) with complex coefficients (i.e., polynomials of the form \( z^d + a_4z^{d-2} + \cdots + a_0 \)). The connectedness locus \( C(d) \) of degree \( d \) is the set formed by all the polynomials \( f \in \text{Poly}(d) \) which have connected filled Julia set \( K(f) \). We consider a post-critically finite polynomial \( f_0 \in C(d) \), with at least one Fatou critical point, and define a straightening map whose domain is, in a certain sense, centered at \( f_0 \) and whose target space is introduced via mapping schemata. Following Milnor [Mi4], one may associate to \( f_0 \) a combinatorial object, called the reduced mapping schema \( T \) of \( f_0 \), that encodes the dynamics along the Fatou critical orbits of \( f_0 \). Polynomial maps over \( T \) are certain polynomial dynamical systems which act on a disjoint union of finitely many copies of \( \mathbb{C} \). The parameter space formed by all monic centered polynomial maps over \( T \) is called the universal polynomial model space \( \text{Poly}(T) \) of \( T \). The notions of Fatou, Julia and filled Julia sets extend to maps in \( \text{Poly}(T) \). In \( \text{Poly}(T) \), the analogue of the connectedness loci is the fiberwise connectedness locus \( C(T) \). The fiberwise connectedness locus \( C(T) \) will be the target space for a straightening map defined around \( f_0 \). The domain \( \mathcal{R}(\lambda_0) \subset C(d) \) will be prescribed with the aid of the rational lamination \( \lambda_0 \) of \( f_0 \). The definition of the map involves extracting a polynomial-like map \( g \) over \( T \) from each polynomial \( f \in \mathcal{R}(\lambda_0) \) (i.e., a renormalization procedure). Polynomial-like maps over mapping schemata are a (straightforward) generalization of the classical ones introduced by Douady and Hubbard. The classical notion of hybrid conjugacy between polynomial-like maps also extends easily to our context. Straightening will assign to each polynomial \( f \in \mathcal{R}(\lambda_0) \) a map in the fiberwise connectedness locus \( C(T) \) which is hybrid equivalent to the polynomial-like map extracted from \( f \). However, in order to avoid ambiguities, we have to introduce “external markings” for polynomial-like maps over \( T \). After carefully introducing markings, we obtain the desired definition of the corresponding straightening map \( \chi_{\lambda_0} : \mathcal{R}(\lambda_0) \to C(T) \).

Our second objective is to study basic properties of straightening maps. In contrast with the quadratic case, higher degree straightening maps are often discontinuous (see [In4]). Nevertheless, straightening maps are still a natural and useful tool in higher degree polynomial dynamics. Basic questions that one may ask are the following. Are straightening maps injective? What is the image of a given straightening map? What is the domain compact? When is the domain connected? We will show the straightening map \( \chi_{\lambda_0} : \mathcal{R}(\lambda_0) \to C(T) \), briefly described above, is injective and its image contains all hyperbolic dynamical systems in the fiberwise connectedness locus \( C(T) \). Moreover, we give a characterization of straightenings with compact domains. These straightenings include those that arise from “primitive” hyperbolic post-critically finite polynomials.

Straightening maps that arise from “primitive” hyperbolic post-critically finite polynomials are natural candidates to be \emph{onto} the connectedness locus of the corresponding model space. We say that a polynomial \( f_0 \in C(d) \) is \emph{primitive} if, for all distinct and bounded Fatou components \( U, V \) of \( f_0 \), we have that \( U \cap \overline{V} = \emptyset \). Our third objective is to provide supporting evidence for the following conjecture:

**Conjecture.** If \( f_0 \) is a primitive hyperbolic post-critically finite polynomial with rational lamination \( \lambda_0 \) and reduced mapping schema \( T \), then \( \chi_{\lambda_0} : \mathcal{R}(\lambda_0) \to C(T) \) is a bijection with compact and connected domain.

We establish the above conjecture for two large classes of cubic straightening maps: “primitive disjoint type” and “primitive capture type”. Moreover, we show that the primitive disjoint type gives rise, via the inverse of straightening, to homeomorphically embedded copies of \( M \times M \) in the cubic connected locus. Similarly, each primitive capture type,
leads to an injective map from the full family of quadratic filled Julia sets $MK$ into the cubic connectedness locus. The inclusions of $MK$ in $C(3)$ have the nice extra property that restrict to a quasiconformal embeddings of every fiber $\{c\} \times K(Q_c)$.

Our results also show that primitive disjoint type straightenings are bijections in any degree (which establishes the existence of embedded copies of $M^{d-1}$ in $C(d)$) and, our techniques might generalize to degree $d \geq 4$ versions of primitive capture type straightenings. However, already for cubic polynomials, the above conjecture is open since our techniques do not fully apply for the two remaining types of cubic straightening maps (adjacent and bitransitive).

In [BHe], Buff and Henriksen showed that $K(z^2 + c)$ is quasiconformally embedded in the cubic connectedness locus provided that $z^2 + c$ has a non-repelling fixed point. The Buff and Henriksen embeddings are a particular case of the ones obtained via a straightening map of primitive capture type. In [In2], the first author showed that given a degree $d \geq 2$ polynomial $f \in C(d)$ with filled Julia $K(f)$, then there is a natural embedding of $K(f)$ into the connectedness locus of degree $d'$, for all $d' > d$. Epstein and Yampolsky [EY] obtained embedded copies of $M \times M$ by using the intertwining surgery technique.

As mentioned above, one of the main difficulties in the study of higher degree straightening maps stems from the lack of continuity (see [In4]). The other main difficulty stems from the insufficient understanding of the global structure of $C(d)$, for $d \geq 3$. In fact, the Douady and Hubbard techniques used in the study of the image of quadratic straightening maps break down since they are based on the “topological holomorphy” property of straightening (a very strong form of continuity) and on the knowledge of the global structure of the Mandelbrot set. As suggested by the previous paragraph, the problem of describing the image of higher degree straightening maps has been already addressed in the literature, sometimes in another equivalent language. One approach has been to construct the inverse of some straightening maps via the intertwining construction [EY]. The intertwining technique, which is certainly of intrinsic interest, may only solve the problem for a limited collection of straightening maps. Buff and Henriksen applied holomorphic motions techniques in appropriately chosen one dimensional slices of parameter space. Our approach involves a “combinatorial tuning” technique and obtaining suitable one dimensional restrictions of a given straightening map which behave as in the quadratic case. According to Lyubich [L], quadratic straightening may be also regarded as a holomorphic motion. Nevertheless, we only use the local properties of quadratic straightening to obtain surjectivity rather than the global properties of the holomorphic motions used by Buff and Henriksen.

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1. Outline of results

The aim of this outline is to provide the reader with a self-contained exposition of the statements of our results. These statements involve several definitions which are introduced in paragraphs 1.1 through 1.9. Paragraphs 1.10 through 1.14 contain the statements of theorems B, C, D, E and F, which are the main results of this paper.

Recall that $\text{Poly}(d)$ denotes the space of monic centered polynomials of degree $d \geq 2$. That is, polynomials of the form $z^d + a_d z^{d-1} + \cdots + a_0$. Therefore, $\text{Poly}(d)$ is naturally identified with $C^{d-1}$. Also, recall that we denote the connectedness locus by $C(d)$. According to Branner and Hubbard [BrH], $C(d)$ is a compact subset of $\text{Poly}(d)$. Although the
1.1. Combinatorially renormalizable polynomials. Given a polynomial \( f \in \mathbb{C}(d) \), the rational lamination \( \lambda_f \) of \( f \) is the equivalence relation in \( \mathbb{Q}/\mathbb{Z} \) that identifies two arguments \( \theta, \theta' \) if and only if the external rays of \( f \) with arguments \( \theta \) and \( \theta' \) land at a common point.

Post-critically finite polynomials in \( \mathbb{C}(d) \) are completely determined by their rational laminations \([\lambda_f]_0\). All the constructions in this paper only depend on a reference rational lamination \( \lambda_0 \). For this outline of results:

We let \( \lambda_0 \) be the rational lamination \( f_0 \).

We say that \( f \in \mathbb{C}(d) \) is combinatorially \( \lambda_0 \)-renormalizable if \( \lambda_f \supset \lambda_0 \). The set formed by all combinatorially \( \lambda_0 \)-renormalizable polynomials is denoted by \( \mathbb{C}(\lambda_0) \).

1.2. Mapping schemata. In [Mi4] a “model” for \( \mathbb{C}(\lambda_0) \) is suggested in terms of the reduced mapping schema \( \mathbb{T}(\lambda_0) \) for \( f_0 \).

In general, a (resp. reduced) mapping schema \( T \) is a triple \((|T|, \sigma, \delta)\) where \(|T|\) is a finite set, \( \sigma \) is a map from \(|T|\) into itself, and the degree function \( \delta \) is a map \(|T|\) into the integers with values (resp. \( \delta(v) \geq 2 \)) \( \delta(v) \geq 1 \) for all \( v \in |T| \).

Most of the mapping schemata considered in this paper are reduced.

The reduced mapping schema \( \mathbb{T}(\lambda_0) \) for \( \lambda_0 \) (or \( f_0 \)) is \((|T(\lambda_0)|, \sigma_{\lambda_0}, \delta_{\lambda_0})\) where:

(i) \(|T(\lambda_0)| = \text{Crit}(f_0) \cap F(f_0)\) is the set of Fatou critical points of \( f_0 \).

(ii) \( \sigma_{\lambda_0} : |T(\lambda_0)| \rightarrow |T(\lambda_0)| \) where \( \sigma_{\lambda_0}(v) = f_0^{\ell_v}(v) \) if \( \ell_v = \min\{k \in \mathbb{N} \mid f^k(v) \in |T(\lambda_0)|\} \).

(iii) \( \delta_{\lambda_0} : |T(\lambda_0)| \rightarrow \mathbb{N} \) where \( \delta_{\lambda_0}(v) \) is the local degree of \( f_0 \) at \( v \).

The number \( \ell_v \), as above, will be called the return time of \( v \) to \(|T(\lambda_0)|\) and will be consistently denoted by \( \ell_v \).

Note that apparently \( \mathbb{T}(\lambda_0) \) not only depends on \( \lambda_0 \) but also in \( f_0 \). However, we will define \( \mathbb{T}(\lambda_0) \) purely in terms of \( \lambda_0 \) in Section 3.3.

1.3. Universal polynomial model space. Each reduced mapping schema \( T = (|T|, \sigma, \delta) \) determines a complex affine space \( \text{Poly}(T) \) called the universal polynomial model space for \( T \). More precisely, \( \text{Poly}(T) \) consists of all maps \( f \) from \(|T| \times \mathbb{C}\) to itself such that the restriction of \( f \) to each component \(|v| \times \mathbb{C}\) is a monic centered polynomial map \( f_v \) of degree \( \delta(v) \), taking values in \(|\sigma(v)| \times \mathbb{C}\). For short we say that \( f \) is a polynomial map over \( T \). Given \( f \in \text{Poly}(T) \), the filled Julia set \( K(f) \) of \( f \) is the set of points in \(|T| \times \mathbb{C}\) with precompact forward orbit. The boundary \( \partial K(f) \) is called the Julia set \( J(f) \) of \( f \). We say that \( K(f) \) is fiberwise connected if the intersection of \( K(f) \) with every fiber \(|v| \times \mathbb{C}\) is connected. The subset of \( \text{Poly}(T) \) formed by the maps \( f \) with fiberwise connected filled Julia set \( K(f) \) is called the connectedness locus \( C(T) \) for \( T \). It follows that \( C(T) \) is a compact subset of \( \text{Poly}(T) \). We will denote the \( v \)-fiber of the filled Julia set \( K(f) \) by \( K(f, v) \). Also, we will abuse of notation and sometimes regard \( f \in \text{Poly}(T) \) as a collection \( f = (f_v : \mathbb{C} \rightarrow \mathbb{C})_{v \in |T|} \) of polynomials.
1.4. Polynomial-like maps over schemata. Now we generalize the notion of polynomial-like maps introduced by Douady and Hubbard [DH2].

Consider a mapping schema $T = ([T], \sigma, \delta)$. Let $U' \Subset U$ be open subsets contained in $[T] \times \mathbb{C}$ which are fiberwise topological disks. That is, for all $v \in [T]$ we have that $U' \cap (\{v\} \times \mathbb{C}) = \{v\} \times U'_v$ and $U \cap (\{v\} \times \mathbb{C}) = \{v\} \times U_v$ for some topological disks $U'_v, U_v \subset \mathbb{C}$. A proper and holomorphic skew product over $\sigma$,

$$g : U' \to U,$$

$$(v, z) \mapsto (\sigma(v), g_v(z))$$

is called a polynomial-like map over $T$ if the degree of $g_v : U'_v \to U_{\sigma(v)}$ is $\delta(v)$, for every $v \in [T]$. The filled Julia set $K(g)$ is the set of all $(v, z) \in U'$ such that $g^n((v, z))$ is well defined, for all $n \in \mathbb{N}$. Again we will abuse of notation and sometimes simply identify $g$ with the collection $(g_v : U'_v \to U_{\sigma(v)})_{v \in [T]}$. The $v$-fiber of $K(g)$ will be denoted by $K(g, v)$.

Simple examples of polynomial-like maps over a reduced mapping schema $T$ are obtained from polynomial maps in $C(T)$ after restriction to an appropriate neighborhood of their filled Julia sets.

Two polynomial-like maps $g_0$ and $g_1$ over a mapping schema $T$ with are said to be hybrid equivalent if there exists a fiberwise quasiconformal map $\psi$ defined on a neighborhood of $K(g_0)$ such that $\psi \circ g_0 = g_1 \circ \psi$ and

$$\frac{\partial \psi}{\partial z}(v, z) \equiv 0 \text{ a.e. on } K(g_0).$$

1.5. External markings. We will show that every polynomial-like map $g$ over a reduced mapping schema $T$ with fiberwise connected Julia set is hybrid equivalent to a polynomial map over $T$. However, in order to avoid ambiguities we are forced either to work with the moduli space of polynomial maps over $T$ (affine conjugacy classes of such maps) or to introduce external markings. For our purpose, the latter will be more convenient. Roughly speaking, an external marking of such a polynomial-like map $g$ over $T$ is a collection of accesses, one per fiber, to an appropriate periodic or preperiodic point in $J(g)$.

**Definition 1.1.** Let $g : U' \to U$ be a polynomial-like map over a reduced mapping schema $T = ([T], \sigma, \delta)$ with connected filled Julia set. A path to $K(g)$ is a path $\gamma : [0, 1] \to U'$ such that $\gamma(0, 1) \subset U' \setminus K(g)$ and $\gamma(0) \in J(g)$. We say two paths $\gamma_0$ and $\gamma_1$ to $K(f)$ are homotopic if there exists a continuous map $\tilde{\gamma} : [0, 1] \times [0, 1] \to U'$ such that

- $t \mapsto \tilde{\gamma}(s, t)$ is a path to $K(g)$ for any $s \in [0, 1]$;
- $\tilde{\gamma}(0, t) = \gamma_0(t)$ and $\tilde{\gamma}(1, t) = \gamma_1(t)$;
- $\tilde{\gamma}(s, 0) = \gamma_0(0)$ for any $s \in [0, 1]$.

An access to $K(g)$ is a homotopy class of paths to $K(g)$.

**Definition 1.2.** Let $g : U' \to U$ be a polynomial-like map over a reduced mapping schema $T = ([T], \sigma, \delta)$ with $K(g)$ connected. An external marking of $g$ is a collection $\Gamma = (\Gamma_v)_{v \in [T]}$ where each $\Gamma_v$ is an access in $\{v\} \times \mathbb{C}$ and such that $\Gamma$ is forward invariant in the following sense. For every $v \in [T]$ and every representative $\gamma_v \subset (\{v\} \times \mathbb{C}) \cap U'$ of $\Gamma_v$, the connected component of $g(\gamma_v) \subset U'$ that contains a point of $J(g)$ is a representative of $\Gamma_{\sigma(v)}$.

An externally marked polynomial-like map over $T$ is a pair $(g, \Gamma)$ of a polynomial-like map over $T$ and an external marking of it.

1.6. Standard external marking of polynomial maps in Poly($T$). Polynomial maps over a reduced mapping schema $T$ are naturally endowed with a standard marking which we introduce with the aid of appropriately defined Böttcher coordinates. More precisely, given
\( f \in \text{Poly}(T) \), the standard arguments for polynomials (e.g., see [Mi3]) easily generalize to prove that there exists a Böttcher coordinate \( \varphi \) (at infinity) for \( f \). That is, there exists a neighborhood \( U \) of \( |T| \times \{\infty\} \) in \( |T| \times \hat{\mathbb{C}} \) and a conformal map \( \varphi : U \to |T| \times \hat{\mathbb{C}} \) of the form \( \varphi(v, z) = (v, \varphi_v(z)) \) such that \( \varphi \circ f(v, z) = \sigma(v, (\varphi_v(z))^\delta(v)) \) and \( \varphi \) is asymptotic to the identity as \( z \to \infty \) on each fiber.

Let \( \Delta = \{z \in \mathbb{C}; |z| < 1\} \). If \( f \in \text{C}(T) \), then a Böttcher coordinate can be extended uniquely to a conformal isomorphism

\[
\varphi : (|T| \times \mathbb{C}) \setminus K(f) \to |T| \times (\mathbb{C} \setminus \Delta),
\]

which we also denote by \( \varphi \). In this case, \( \varphi \) is uniquely determined by \( f \in \text{C}(T) \).

For \( (v, \theta) \in |T| \times \mathbb{R}/\mathbb{Z} \), define the external ray for \( f \) by

\[
R_f(v, \theta) = \{\varphi^{-1}(v, r \exp(2\pi i \theta)); 1 < r < \infty\} \quad (v \in |T|, \theta \in \mathbb{R}/\mathbb{Z}).
\]

We say that \( R_f(v, \theta) \) lands at \((v, z) \in J(f)\) if

\[
\lim_{r \to 1} \varphi^{-1}(v, r \exp(2\pi i \theta)) = (v, z).
\]

The landing theorems easily generalize to this context (e.g., see [Mi3]). In particular, external rays with arguments in \( \mathbb{Q}/\mathbb{Z} \) always land (at eventually periodic points).

For \( f \in \text{C}(T) \), the collection

\[
\left( R_f(v, 0) \right)_{v \in |T|}
\]

of the external rays of angle 0 naturally induces an external marking of any polynomial-like restriction of \( f \). We call it the standard external marking of \( f \). Figure 1 is an example of external markings of polynomial of degree 4. It has three external markings, each corresponds to some invariant external ray. The standard marking corresponds to the 0-ray (i.e., the horizontal one).

1.7. **Straightening of polynomial-like maps over reduced schemata.** Consider two externally marked polynomial-like maps \( (g_0 : U'_0 \to U_0, \Gamma_0), (g_1 : U'_1 \to U_1, \Gamma_1) \) over \( T \) with connected filled Julia sets and assume there exists a topological conjugacy \( \psi : U_0 \to U_1 \) between \( g_0 \) and \( g_1 \). We say \( \psi \) respects external markings if \( \Gamma_1 = (\psi \circ \gamma_1)_{v \in |T|} \) where \( (\gamma_1)_{v \in |T|} \) is a representative of \( \Gamma_0 \). Note that this definition is independent of the choice of representative.

Section 2 contains the following generalization of Douady and Hubbard straightening theorem to our context.

**Theorem A** (Straightening). Let \( T \) be a reduced mapping schema and consider a polynomial-like map \( g \) over \( T \). Then, there exists a polynomial map \( f \in \text{Poly}(T) \) hybrid conjugate to \( g \). Moreover,

(i) If \( K(g) \) is fiberwise connected, then \( f \in \text{C}(T) \) and \( f \) is unique up to affine conjugacy.

(ii) If \( K(g) \) is fiberwise connected and \( g \) is externally marked by \( \Gamma \), then for exactly one \( f \in \text{C}(T) \) there exists a hybrid conjugacy that respects the external markings \( \Gamma \) and \( \Gamma' \), where \( \Gamma' \) is the standard external marking.

Based on the previous result which “straightens” a single polynomial-like map, we will define “straightening maps” from an appropriately defined domains contained in the connectedness locus.
1.8. Renormalizable polynomials. Recall from Section 1.1 that \( C(\lambda_0) \) denotes the \( \lambda_0 \)-combinatorially renormalizable locus. Under certain conditions it will be possible to extract a polynomial-like map over \( T(\lambda_0) \) from a given \( f \in C(\lambda_0) \). In order to be more precise, below, for every \( v \in |T(\lambda_0)| \), we identify a subset \( K_v(f) \) of \( K(f) \) which we call the \( v \)-small filled Julia set.

Given \( \theta, \theta' \in \mathbb{Q}/\mathbb{Z} \) so that the corresponding rays \( R_{\theta_0}(\theta), R_{\theta_0}(\theta') \) land at a common point (i.e., \( \theta, \theta' \) are \( \lambda_0 \)-equivalent) let \( \text{Sector}_{\theta_0}(\theta, \theta'; v) \) be the connected component of \( \mathbb{C} \setminus (R_{\theta_0}(\theta) \cup R_{\theta_0}(\theta')) \) that contains \( v \). If \( f \in C(\lambda_0) \), then we denote by \( \text{Sector}_f(\theta, \theta'; v) \) the connected component of \( \mathbb{C} \setminus (R_f(\theta) \cup R_f(\theta')) \) such that, for all \( t \in \mathbb{R}/\mathbb{Z} \),

\[
\text{Sector}_f(t) \subset \text{Sector}_f(\theta, \theta'; v) \quad \text{if and only if} \quad R_f(t) \subset \text{Sector}_f(\theta, \theta'; v).
\]

Now we define,

\[
K_v(f) = K(f) \cap \bigcap_{\theta - \theta' \neq 0} \text{Sector}_f(\theta, \theta'; v).
\]

It follows that \( K_v(f) \) is connected and \( f^{\ell_v}(K_v(f)) = K_{\sigma_\lambda(\lambda_0)}(f) \) (see Proposition 3.7).

We say that \( f \in C(\lambda_0) \) is \( \lambda_0 \)-renormalizable if for all \( v \in \text{Crit}(f_0) \cap F(f_0) \) there exist topological disks \( U'_v \Subset U_v \) such that \( K_v(f) \subset U'_v \) and \( f^{\ell_v} : U'_v \rightarrow U_{\sigma_{\lambda_0}(\lambda_0)}(v) \) is a proper map of degree \( \delta_{\ell_0}(v) \). We denote the set of \( \lambda_0 \)-renormalizable polynomials by \( \mathcal{R}(\lambda_0) \).
Corollary. Consider \( f \in \mathcal{R}(\lambda_0) \) and, using the notation above, we let
\[
U' = \{(v, z); z \in U_v'\}
\]
\[
U = \{(v, z); z \in U_v\}
\]
and \( g_v = f^\ell_v \). It follows that

\[
g : U' \to U
\]
\[
(v, z) \mapsto (\sigma_{\lambda_0}(v), g_v(z))
\]
is a polynomial-like map over \( T(\lambda_0) \) with (fiberwise connected) filled Julia set
\[
K(g) = \{(v, z); z \in K_v(f)\}.
\]
We say that \( g \) is a \( \lambda_0 \)-renormalization of \( f \). Note that \( g \) is uniquely defined over \( K(g) \), however there is a choice involved for the domain \( U' \).

1.9. **Internal angle systems and induced external markings.** We consistently mark the polynomial-like maps over \( T(\lambda_0) \) extracted from maps in \( \mathcal{R}(\lambda_0) \). More precisely, we introduce internal angle systems for \( f_0 \) and describe how an internal angle system determines an external marking for a \( \lambda_0 \)-renormalization of every \( f \in \mathcal{R}(\lambda_0) \).

For every \( v \in |T(\lambda_0)| \) denote by \( \gamma_v \) the boundary of the Fatou component of \( f_0 \) that contains \( v \). Since \( f_0 \) is post-critically finite, \( \gamma_v \) is a Jordan curve (e.g., see [Mi3]). Moreover, there exists a collection \( \alpha = (\alpha_v : \gamma_v \to \mathbb{R}/\mathbb{Z})_{v \in |T(\lambda_0)|} \) of homeomorphisms such that:
\[
\alpha_{\sigma_{\lambda_0}(v)}(f_0^\ell_v(z)) = \delta_{\lambda_0}(v)\alpha_v(z)
\]
for all \( z \in \gamma_v \).

We call \( \alpha = (\alpha_v)_{v \in |T(\lambda_0)|} \) an internal angle system for \( f_0 \).

An internal angle system determines an external marking of any \( \lambda_0 \)-renormalization of every \( f \in \mathcal{R}(\lambda_0) \). For each \( v \in |T(\lambda_0)| \) choose an argument \( \theta_v \) so that the external ray of \( f_0 \) with argument \( \theta_v \) lands at \( \alpha_v^{-1}(0) \). Given a \( \lambda_0 \)-renormalization \( g \) of a polynomial \( f \in \mathcal{R}(\lambda_0) \), let \( \Gamma_v \) be the access with representative the connected component of \( R_f(\theta_v) \cap U_v^\prime \) that contains a point of \( K_v(f) \). We say that \( \Gamma = (\Gamma_v)_{v \in |T(\lambda_0)|} \) is the external marking of \( g \) determined by the internal angle system \( \alpha : \gamma_v \to \mathbb{R}/\mathbb{Z} \). This external marking is independent of the choices involved (see Section 3.3).

1.10. **Straightening map and injectivity.** Finally we are ready to define the straightening maps under consideration. More precisely, as a consequence of Theorem A, we obtain the following result.

**Corollary.** Given an internal angle system \( \alpha \) for \( f_0 \) and \( f \in \mathcal{R}(\lambda_0) \), there exists a unique map \( P \in \mathcal{C}(T(\lambda_0)) \) such that there exists a hybrid equivalence between \( P \) and a \( \lambda_0 \)-renormalization \( g \) of \( f \), respecting external markings, where the external marking of \( g \) is the one determined by \( \alpha \) and the external marking of \( P \) is the standard external marking.

With the notation of the above corollary, given an internal angle system \( \alpha \) for \( f_0 \), we say that the associated straightening map \( \chi_{\lambda_0} : \mathcal{C}(\lambda_0) \to \mathcal{C}(T(\lambda_0)) \) is the function defined by \( \chi_{\lambda_0}(f) = P \). For example, consider \( f_0(z) = z^2 - 1 \) as in Figure 2. There are three polynomials \( f_1, f_2, \) and \( f_3 \in \mathcal{R}(\lambda_0) \) such that \( \chi_{\lambda_0}(f_i) \) is affinely conjugate to \( f_0 \) depending on the choice of the external marking of \( f_0 \). Figure 3 shows two of \( f_i \)’s and the other one is just the complex conjugate of the second one.

In Section 5 we prove the following result.
Figure 2. The Julia set and Yoccoz puzzles, and the rational lamination for $z^4 - 1$.

**Theorem B** (Injectivity of Straightening). Consider an internally angled post-critically finite polynomial $f_0$. Let $\chi_{\lambda_0} : R(\lambda_0) \to C(T(\lambda_0))$ be the associated straightening map. Then, $\chi_{\lambda_0}$ is injective.

1.11. **Onto hyperbolic maps.** To study the basic properties of the image of general straightening maps, we employ a “combinatorial tuning procedure” which is discussed in Section 6.

**Theorem C.** Let $f_0$ be an internally angled post-critically finite polynomial such that $R(\lambda_0) \neq \emptyset$ and $\chi_{\lambda_0} : R(\lambda_0) \to C(T(\lambda_0))$ be the associated straightening map. Denote by $\text{Hyp}(C(T(\lambda_0)))$ the set of hyperbolic maps contained in $C(T(\lambda_0))$. Then $\chi_{\lambda_0}(R(\lambda_0)) \supset \text{Hyp}(C(T(\lambda_0)))$ and

$$\chi_{\lambda_0}^{-1}(\text{Hyp}(C(T(\lambda_0)))) \to \text{Hyp}(C(T(\lambda_0)))$$

is biholomorphic.

An equivalent condition for $R(\lambda_0) \neq \emptyset$ is given in Proposition 6.7 (see also Theorem D).

1.12. **Compactness.** Compactness properties of the domain of a straightening map are useful to further study its image. Recall that we say that a post-critically finite polynomial $f \in C(d)$ is *primitive* if $f$ has at least one periodic critical point and, for every pair of distinct bounded Fatou components $V_1, V_2$ of $f_0$, we have that $\partial V_1 \cap \partial V_2 = \emptyset$.

**Theorem D.** Let $f_0$ be a post-critically finite polynomial. Then the following statements are equivalent:

1. $f_0$ is primitive.
2. $C(\lambda_0) = R(\lambda_0) \neq \emptyset$.
3. $R(\lambda_0)$ is compact and non-empty.
Figure 3. Tunings of $z^4 - 1$ with itself with different external markings, given by the external rays of angle $0$ and $1/3$.

1.13. **Cubic reduced mapping schemata.** There are exactly four types of cubic reduced mapping schemata.

- $T_{\text{adj}} = ((\{0\}, \sigma, \delta)$, where $\sigma(0) = 0$ and $\delta(0) = 3$, is called a reduced mapping schema of adjacent type. For example, if $f_0$ is polynomial of the form $z^3 + c$ for which the critical point $z = 0$ is periodic, then the reduced mapping schema $T(\lambda_0)$ is $T_{\text{adj}}$. Observe that $\text{Poly}(T_{\text{adj}}) = \text{Poly}(3)$ and $C(T_{\text{adj}}) = C(3)$.

- $T_{\text{bit}} = ((v_0, v_1), \sigma, \delta)$, where $\sigma(v_j) = v_{1-j}$ for $j = 0, 1$, and $\delta$ is constant ($= 2$), is called a reduced mapping schema of bitransitive type. If $f_0$ is a cubic polynomial with distinct critical points $v_0, v_1$ lying in the same periodic orbit, then $T(\lambda_0) = T_{\text{bit}}$. Note that $\text{Poly}(T_{\text{bit}})$ can be identified with the family of biquadratic polynomials $\text{Poly}(2 \times 2) = \{(z^2 + c_0)^2 + c_1; (c_0, c_1) \in \mathbb{C}^2\} \subset \text{Poly}(4)$. Thus the connectedness locus of $\text{Poly}(T_{\text{bit}})$ is identified with $\{(c_0, c_1) \in \mathbb{C}^2; (z^2 + c_0)^2 + c_1 \in C(4)\}$.

- $T_{\text{cap}} = ((v_0, v_1), \sigma, \delta)$, where $\sigma(v_j) = v_0$ for $j = 0, 1$, and $\delta$ is constant ($= 2$), is called a reduced mapping schema of capture type. If $f_0$ is a cubic polynomial with distinct
critical points $v_0, v_1$ such that $v_0$ is periodic and $v_1$ is not periodic but eventually lands in the orbit of $v_0$, then $T(d_0) = T_{\text{cap}}$. In this case, $\text{Poly}(T)$ is naturally identified with $\mathbb{C}^2$. In fact, given a map $f : \{v_0, v_1\} \times \mathbb{C} \to \{v_0, v_1\} \times \mathbb{C}$ in $\text{Poly}(T)$ there exists $(c_0, c_1)$ such that $f(v_j, z) = (v_j, z^2 + c_j)$. Note that $K(f)$ is fiberwise connected if and only if $c_0 \in \mathcal{M}$ and $c_1 \in K(z^2 + c_0)$. Therefore,

$$C(T_{\text{cap}}) = M \cup K = \{(c, z) \in \mathbb{C}^2; c \in \mathcal{M}, z \in K(z^2 + c)\}.$$ 

- $T_{\text{dis}} = \{(v_0, v_1), \sigma, \delta\}$ where $\sigma(v_j) = v_j$ for $j = 0, 1$, and $\delta$ is constant ($= 2$), is called a reduced mapping schema of disjoint type. If $f_0$ is a cubic polynomial with distinct critical points $v_0, v_1$ such that both critical points are periodic but belong to different orbits, then $T(d_0) = T_{\text{dis}}$. It follows that $\text{Poly}(T_{\text{dis}})$ is identified with $\text{Poly}(2) \times \text{Poly}(2)$ and $C(T_{\text{dis}}) = \mathcal{M} \times \mathcal{M}$.

1.14. **Capture and Disjoint type straightenings.** Using the compactness given by Theorem $D$ we are able to extend Theorem $C$ and describe the image of two classes straightening maps centered at a primitive hyperbolic post-critically finite polynomial $f_0$. The first class is the generalization of the notion of disjoint type; described above in the context of cubic polynomials. Namely, the reduced mapping schema $T$ consists of $d - 1$ elements which are pointwise fixed by the schema map. Equivalently, $f_0$ has exactly $d - 1$ superattracting periodic orbits. Note that the universal polynomial model space for $T$, consists of ordered $(d - 1)$-tuples of monic centered quadratic polynomials which act on $d - 1$ copies of $\mathbb{C}$. Thus, the corresponding connectedness locus is $\mathcal{M}^{d-1}$.

**Theorem E.** Let $f_0 \in C(d)$ be a polynomial with exactly $d - 1$ superattracting periodic orbits which is primitive. Then, given any internal angle system, the associated straightening map

$$\chi : \mathcal{R}(\lambda_0) \to \mathcal{M}^{d-1},$$

$$f \mapsto (\chi_1(f), \ldots, \chi_{d-1}(f)),$$

is a homeomorphism. Moreover, for all $1 \leq i_1 < \cdots < i_k \leq d - 1$ and all $c_1, \ldots, c_k \in \mathcal{M}$, the set

$$\{f \in \mathcal{R}(\lambda_0); \chi_{i_j}(f) = c_j \text{ for all } j = 1, \ldots, k\}$$

is contained in a codimension $k$ complex submanifold $S$ of $\mathcal{R}(d)$. Furthermore, if $k = d - 2$, then $\chi_j : S \cap \mathcal{R}(\lambda_0) \to \mathcal{M}$ extends to a quasiconformal map in a neighborhood of $S \cap \mathcal{R}(\lambda_0)$.

The second class is that of cubic straightening of capture type.

**Theorem F.** Let $f_0 \in C(3)$ be an internally angled primitive hyperbolic post-critically finite polynomial with rational lamination $\lambda_0$ and reduced mapping schema of capture type. Then $\mathcal{R}(\lambda_0)$ is connected, and the associated straightening map

$$\chi : \mathcal{R}(\lambda_0) \to \mathcal{M} \cup K$$

$$f \mapsto (\chi_1(f), \chi_2(f)),$$
is a bijection. Moreover, given \( c \in M \), the set

\[
\{ f \in R(\lambda_0) \colon \chi_1(f) = c \}
\]

is contained in a one dimensional complex analytic space \( S_c \) which is locally irreducible. Furthermore,

\[
\chi_2 : R(\lambda_0) \cap S_c \to K(z^2 + c)
\]

extends to a quasiconformal map from a neighborhood of \( R(\lambda_0) \cap S_c \) onto a neighborhood of \( K(z^2 + c) \).

Note that, in the previous theorem, \( S_c \) might be a singular analytic space. In order have well defined quasiconformal maps with domain in \( S_c \) we need to specify a conformal structure. In fact, there is only one such structure compatible with the embedding, since around each singular point of \( S_c \) there exists a punctured neighborhood \( U \) (in \( S_c \)) and a conformal isomorphism from the punctured disk onto \( U \) which has a unique continuous extension.

2. Polynomial-like maps

The aim of this section is to prove Theorem [A] (see [DH1]). That is, to generalize Douady and Hubbard straightening Theorem to the context of polynomial-like maps over reduced mapping schema.

2.1. Proof of the straightening theorem. Let \( g \) be a polynomial-like map over a reduced mapping schema \( T \). Following the ideas of Douady and Hubbard, we “paste” \( g \) with the dynamics of the polynomial map over \( T \) given by \( f_0(v, z) = (\sigma(v), z^\delta(v)) \).

Let \( \Delta(r) = \{|z| \leq r\} \) denote the closed disk of radius \( r > 0 \). Choose \( R' > R > 1 \). Let \( V = |T| \times (\hat{C} \setminus \Delta(R)) \) and

\[
V' = f_0(V) \cup \left( (|T| \setminus \sigma(|T|)) \times (\hat{C} \setminus \Delta(R')) \right).
\]

Restricting \( g \) to a smaller domain, if necessary, we may assume the domain \( U' \) and the range \( U \) have \( C^1 \) boundaries. Take a \( C^1 \) diffeomorphism \( h \) between \( A = U \setminus U' \) and \( B = V \setminus V' \) such that \( h(v, z) = (v, h_z(z)) \) for some \( h_z \) and \( h(g(v, z)) = f_0(h(v, z)) \) for all \( z \in \partial U' \). Then consider the \( C^1 \) manifold \( S = (U \cup V)/h \) and the map \( \hat{f} : S \to S \) defined by

\[
\hat{f}(v, z) = \begin{cases} 
g(v, z) & \text{if } (v, z) \in U', \\
f_0(v, z) & \text{if } (v, z) \in V. \end{cases}
\]

Applying the standard pull-back argument, we obtain a \( \hat{f} \)-invariant almost complex structure \( \mu_0 \) on \( S \). From the measurable Riemann mapping theorem, we obtain a biholomorphic map from \( (S, \mu_0) \) onto \( |T| \times \hat{C} \) which conjugates \( \hat{f} \) with a polynomial map \( f \) over \( |T| \). Moreover, since \( \mu_0 \) was obtained via the pull-back argument, it follows that \( g \) and \( f \) are hybrid conjugate.

The uniqueness part of Theorem [A] follows from the proposition below.

**Proposition 2.1.** If two polynomial maps \( f_0, f_1 \in \text{Poly}(T) \) over a reduced mapping schema \( T \) are hybrid equivalent, then they are affine conjugate.

Moreover, if \( f_0, f_1 \in \text{C}(T) \) and a hybrid conjugacy respects the standard external markings, then \( f_0 = f_1 \) and the affine conjugacy is the identity.

Before proving the proposition let us give a short discussion about external markings and affine conjugacies. Below, we show that, modulo affine conjugacy, there is only one external marking.
Given a polynomial map \( f \in \mathbb{C}(T) \) consider an external marking \( \Gamma \) for (a polynomial-like restriction of) \( f \). Since every access contains a unique external ray, by Lindel"of’s theorem, we can choose a collection of external rays \( (R_f(v, \theta_v))_{v \in |T|} \) as a representative of \( \Gamma \). It follows that,

\[
\delta(v) \theta_v = \theta_{e(v)}.
\]

Observe that there are only finitely many collections of angles \( (\theta_v) \) satisfying (1).

Now let 
\[
A(v, z) = (v, e^{2\pi i \theta_v} z),
\]
and \( \hat{f} = A^{-1} \circ f \circ A \). It is easy to check that \( \hat{f} \in \mathbb{C}(T) \) and \( A(R_f(v, \theta)) = R_f(v, \theta + \theta_v) \). In particular, \( A \) maps the standard external marking for \( f \) to the external marking \( (R_f(v, \theta_v)) \).

On the other hand, it is easy to check that if a collection \( (\theta_v) \) of angles satisfies (1), then \( (R_f(v, \theta_v)) \) defines an external marking \( \Gamma \) for \( f \).

Now we prove the proposition and, therefore, we finish the proof of Theorem A.

**Proof.** As in the statement of the proposition we consider two polynomial maps \( f_0, f_1 \) over \( T \) with hybrid conjugate polynomial-like restrictions \( f_j : U'_j \to U_j \) where \( j = 0, 1 \). Let us denote the hybrid conjugacy \( \psi : U_0 \to U_1 \). Replacing \( f_1 \) by an affinely conjugate polynomial map we may assume that \( \psi \) respects the standard external markings.

Define a homeomorphism \( \Phi \) by
\[
\Phi(v, z) = \begin{cases} 
\psi(v, z) & \text{if } (v, z) \in K(f_0), \\
\varphi_{f_1}^{-1} \circ \varphi_{f_0}(v, z) & \text{otherwise}.
\end{cases}
\]

Clearly, \( \Phi \circ f_0 = f_1 \circ \Phi \).

As in the proof of [DH2, Proposition 6], since \( \psi \) respects the standard external markings, it follows that \( \Phi \) is biholomorphic, hence affine. The affine map \( \Phi \) conjugates two monic centered polynomial maps over \( T \) and it is asymptotic to the identity at infinity in every fiber. Therefore, \( \Phi \) must be the identity. \( \square \)

3. **Laminations**

3.1. **Laminations.** Laminations were introduced to the context of polynomial dynamics in the early 1980’s by Thurston [Th]. Since then they have been proven to be an useful object to encode the landing pattern of external rays. In this section, we briefly review the basic notions and results about laminations. For our purposes we will need to consider the well known \( d \)-invariant laminations as well as forward invariant laminations with finite support. The latter type of laminations are used to define “combinatorial Yoccoz puzzles” (compare with [In3]).

For an integer \( d > 0 \), we let
\[
m_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \quad \theta \mapsto d\theta.
\]

Two subsets \( A, B \subset E \) are said to be **unlinked** if \( B \) is contained in a component of \( \mathbb{R}/\mathbb{Z} \setminus A \) (it is equivalent to \( A \) being contained in a component of \( \mathbb{R}/\mathbb{Z} \setminus B \)). Note that \( A, B \) are unlinked if and only if the Euclidean (or hyperbolic) convex hulls in \( \Delta = \{ |z| \leq 1 \} \) of \( \exp(2\pi i A) \) and \( \exp(2\pi i B) \) are disjoint.

**Definition 3.1** (Laminations without dynamics). Let \( E \subset \mathbb{R}/\mathbb{Z} \). An equivalence relation \( \lambda \) on \( E \) is called a lamination on \( E \) if the following three conditions hold.
(i) \( \lambda \) is closed in \( E \times E \).
(ii) Every equivalence class is finite.
(iii) Equivalence classes are pairwise unlinked.

The support \( \text{supp}(\lambda) \) of \( \lambda \) is the union of all non-trivial \( \lambda \)-equivalence classes.

A lamination on \( \mathbb{Q}/\mathbb{Z} \) (resp. \( \mathbb{R}/\mathbb{Z} \)) is called a rational (resp. real) lamination.

We will be mainly concerned with laminations that are compatible with the action of \( m_d \).

**Definition 3.2** (Forward invariant and \( d \)-invariant laminations). Let \( d > 1 \) be an integer and \( E \subset \mathbb{R}/\mathbb{Z} \). Let \( \lambda \) and \( \lambda' \) be laminations on \( E \) and \( E' = m_d^{-1}(E) \), respectively.

We say that \((m_d)_\ast \lambda = \lambda'\) if for any \( \lambda \)-equivalence class \( A \),

(i) \( m_d(A) \) is a \( \lambda' \)-equivalence class.
(ii) \( m_d : A \rightarrow m_d(A) \) is consecutive preserving, i.e., for any component \( (\theta, \theta') \) of \( \mathbb{R}/\mathbb{Z} \setminus A \), \((d\theta, d\theta')\) is a component of \( \mathbb{R}/\mathbb{Z} \setminus m_d(A) \).

A lamination \( \lambda \) is said to be forward invariant if \((m_d)_\ast \lambda = \lambda \). A forward invariant lamination \( \lambda \) on a \( m_d \)-completely invariant set \( E \subset \mathbb{R}/\mathbb{Z} \) is called \( d \)-invariant or simply invariant.

### 3.2. Invariant rational laminations: unlinked classes, fibers and critical elements.

Invariant rational laminations will be particularly useful in this paper. Recall that \( C(d) \) denotes connectedness locus in the space of monic centered degree \( d \geq 2 \) polynomials \( \text{Poly}(d) \). Given a monic centered polynomial \( f \in C(d) \) the rational lamination \( \lambda_f \) of \( f \) is the equivalence relation on \( \mathbb{Q}/\mathbb{Z} \) which identifies two rational arguments if and only if the corresponding rays land at a common point. According to [Ki], an equivalence relation \( \lambda \) on \( \mathbb{Q}/\mathbb{Z} \) is the rational lamination of some \( f \in C(d) \) if and only if \( \lambda \) is a \( d \)-invariant rational lamination.

Rational laminations allow us to establish when distinct polynomial share certain dynamical features. Therefore, it is natural to define some subsets of dynamical and parameter space in terms of rational laminations. Given a \( d \)-invariant rational lamination \( \lambda \), let

\[
C(\lambda) = \{ f \in C(d) ; \lambda_f \supset \lambda \}.
\]

Note that this set is always non-empty [Ki].

The sets \( C(\lambda) \) have already deserved a lot of attention. When \( \lambda \) is the rational lamination of a polynomial with all cycles repelling (equivalently, \( \lambda \) is maximal with respect to the partial order on \( d \)-invariant rational laminations), \( C(\lambda) \) is a “combinatorial class”. In degree 2, the MLC conjecture asserts that combinatorial classes are singletons. In degree 3, there are some combinatorial classes which are non-trivial continua [He]. However, it is natural to conjecture that in any degree, each combinatorial classes is contained in an analytic subspace of \( \text{Poly}(d) \) of codimension (at least) 1 (see [Le]).

This paper is mainly concerned with \( C(\lambda) \) for invariant rational laminations \( \lambda \) which are not maximal. In degree 2, following Douady and Hubbard, these sets coincide with homeomorphic copies of the Mandelbrot contained contained in the Mandelbrot set \( \mathcal{M} \). In order to compare \( C(\lambda) \) with its model we employ a renormalization procedure. A first step towards renormalization is to cut dynamical space into sectors and the filled Julia set into pieces.

**Definition 3.3** (Admissible). Consider a polynomial \( f \in C(d) \). We say that a lamination \( \lambda \) on \( E \subset \mathbb{Q}/\mathbb{Z} \) is admissible for \( f \) if for every pair of distinct arguments \( \theta \) and \( \theta' \) which are \( \lambda \)-related, we have that they are also \( \lambda_f \)-related.
Note that $C(\lambda)$ exactly consists of all the polynomials for which the invariant rational lamination $\lambda$ is admissible. Admissible laminations will allow us to cut the dynamical space into sectors and the filled Julia set into fibers (compare with [Sch]).

**Definition 3.4** (Sectors and fibers). Let $f \in C(d)$ and $\lambda$ be a lamination on $E \subset \mathbb{Q}/\mathbb{Z}$ that is admissible for $f$. Consider a set $L \subset \mathbb{R}/\mathbb{Z}$ which is unlinked with every non-trivial $\lambda$-class. If $\theta$ and $\theta'$ are distinct $\lambda$-equivalent arguments, we denote by

$$\text{Sector}(\theta, \theta'; L)$$

the connected component of $\mathbb{C} \setminus (\overline{R_f(\theta)} \cup \overline{R_f(\theta')})$ which contains every external ray $R_f(t)$ with $t \in L$. Moreover, we define the $\lambda$-fiber of $L$ by:

$$K_f(L) = K(f) \cap \bigcap_{\theta', \theta' \notin \theta} \text{Sector}(\theta, \theta'; L).$$

Our main interest now is when $\lambda$ is an invariant rational lamination and the sets $L$ above are, in a certain sense, as large as possible. That is, when $L$ are “unlinked classes”.

**Definition 3.5** (Unlinked classes). Let $\lambda$ be a $d$-invariant rational lamination. We say $\theta, \theta' \in \mathbb{R}/\mathbb{Z} \setminus \mathbb{Q}/\mathbb{Z}$ are $\lambda$-unlinked if $\theta = \theta'$ or for any $\lambda$-equivalence class $A$, $\theta$ and $\theta'$ lie in the same component of $\mathbb{R}/\mathbb{Z} \setminus A$.

Observe that, by definition, unlinked classes are contained in $\mathbb{R}/\mathbb{Z} \setminus \mathbb{Q}/\mathbb{Z}$. Moreover, the $\lambda$-unlinked relation is in fact an equivalence relation. Infinite $\lambda$-unlinked classes are closely related to the concept of gaps introduced by Thurston.

In order to visualize the unlinked classes of an invariant rational lamination it is convenient to consider the real extension of a rational lamination.

**Definition 3.6** (Real extension). Given a lamination $\lambda$ on $E \subset \mathbb{R}/\mathbb{Z}$ the real extension $\hat{\lambda}$ of $\lambda$ is the smallest equivalence relation that contains the closure $\overline{\lambda}$ of $\lambda$ in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.

According to [Ki], the real extension of an invariant rational lamination is an invariant real lamination.

For example, consider a polynomial $f$ with locally connected Julia set and such that the following technical condition is satisfied. No critical point with infinite forward orbit lies in the boundary of a bounded Fatou component (e.g., $f$ is post-critically finite). Then, the real extension of the rational lamination $\lambda_f$ encodes the landing pattern of all the external rays of $f$. That is $\hat{\lambda_f}$ identifies two arguments $\theta$ and $\theta'$ if and only if the external rays $R_f(\theta)$ and $R_f(\theta')$ land at a common point. In this case, $L$ is an infinite $\lambda$-unlinked if and only if there exists a bounded Fatou component $V$ such that $L$ is the set formed by the arguments of all the irrational rays landing at $\partial V$. Finite unlinked classes are irrational equivalence classes of $\hat{\lambda_f}$ and correspond to external rays landing at a common point, with infinite forward orbit.

The main properties of $\lambda$-unlinked classes and their fiber is summarized in the proposition below. Before we state the proposition let us agree that if $X, Y \subset \mathbb{C}$ and $f : X \to Y$ is a surjective map defined and holomorphic on a neighborhood of $X$, then we say that the degree of $f : X \to Y$ is $\delta \geq 1$ if every point in $Y$ has $\delta$ preimages on $X$, counting multiplicities.

**Proposition 3.7.** Consider a $d$-invariant rational lamination $\lambda$ and $f \in C(\lambda)$. Let $L$ be a $\lambda$-unlinked class. Then

1. $m_\lambda(L)$ is a $\lambda$-unlinked class.
2. $f(K_f(L)) = K_f(m_\lambda(L))$. 
(iii) If \( L \) is finite, then

(a) \( L \) is a \( \lambda \)-class. In particular, \( L \) is wandering (\( m^0(L) \neq m^n(L) \) for any \( n \neq m \)).
(b) \( m_\delta : L \rightarrow m_\delta(L) \) is \( \delta \)-to-one for some \( \delta \geq 1 \).
(c) The degree of \( f : K_f(L) \rightarrow K_f(m_\delta(L)) \) is well defined and exactly \( \delta(L) \).

(iv) If \( L \) is infinite, then

(a) \( L \) is eventually periodic (i.e., there exists \( n > 0 \) and \( \ell \geq 0 \) such that \( m_\ell^{n+\ell}(L) = m_\ell^{n}(L) \)).
(b) If \( m_\ell^{n}(L) = L \), then \( \delta(L) > 1 \), where \( \delta(L) = \prod_{k=0}^{n-1} \delta(m_\ell^k(L)) \).
(c) \( \overline{\mathcal{T}}/\lambda \) is homeomorphic to \( \mathbb{R}/\mathbb{Z} \). Moreover, if \( (\theta, \theta') \) is a connected component of \( \mathbb{R}/\mathbb{Z} \setminus \overline{\mathcal{T}} \), then \( \theta \) and \( \theta' \) are in \( \mathbb{Q}/\mathbb{Z} \) and \( \lambda \)-equivalent. Furthermore, \( (m_\ell(\theta), m_\ell(\theta')) \) is a connected component of \( m_\delta(L) \) or \( m_\delta(\theta) = m_\delta(\theta') \).
(d) \( m_\delta : \overline{\mathcal{T}} \rightarrow m_\delta(L) \) induces a map \( \hat{m}_\delta : \overline{\mathcal{T}}/\lambda \rightarrow m_\delta(L)/\lambda \), which is conjugate to the \( \delta \)-fold covering map \( m_\delta : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} \) for some \( \delta = \delta(L) \geq 1 \).
(e) Assume further that for any \( \lambda \)-class \( A \) and any \( \lambda \)-unlinked class \( L' \) such that \( A \cap \overline{L} = \emptyset \) and \( m_\delta(L') = m_\delta(L) \) we have \( A \cap \overline{L} = \emptyset \). Then the degree of \( f : K_f(L) \rightarrow K_f(m_\delta(L)) \) is exactly \( \delta(L) \).

The reader may find the proof of the above proposition at the end of this section.

The following examples show that the further assumptions in property (iv)(e) are necessary.

**Example 3.8.** 1. Consider a family \( P_a(z) = az^5 - (a + 1)z^3 + 1 \) of cubic polynomials having a period two superattractive cycle \([0,1]\). Let \( f_0 = P_{a_0} \) and \( f_1 = P_{a_1} \), where \( a_0 = -1/4 \) and \( a_1 = \frac{11 - \sqrt{77}}{20} \) (see Figure 1). The critical point \( c_0 = \frac{2(a + 1)}{3a} \) is mapped to 0 for \( f_0 \) and for \( f_1 \), it is mapped to the fixed point which is the intersection of the closures of the immediate basins of 0 and 1.

Although \( \lambda_0 = \lambda_{a_1} \) is admissible for \( f_1 \), the polynomial \( f_1 \) is not \( \lambda_0 \)-renormalizable. In fact, let \( L \) (resp. \( L' \)) be the \( \lambda_0 \)-unlinked class whose \( \lambda_0 \)-fiber for \( f_0 \) contains \( 0 \) (resp. \( c_{a_0} \)). Then, we have \( K_{f_0}(L) \cap K_{f_0}(L') = \{c_{a_0}\} \). Therefore, \( K_{f_0}(L) \) contains both of the critical points and any proper extension of \( f_1 : K_{f_0}(L) \rightarrow K_{f_0}(L') \) to a disk would have degree 3.

The assumptions of (iv)(e) do not hold. In fact, let \( L' \) be the \( \lambda \)-unlinked class distinct from \( L' \) such that \( m_\delta(L) = m_\delta(L') \). Note that \( \overline{L'} \cap \overline{L} \neq \emptyset \). Also, observe that \( A_0 = \overline{L} \cap \overline{L'} \) is a \( \lambda_0 \)-class. The \( \lambda_{a_1} \)-class

\[
A = A_0 \cup (\overline{L'} \cap \overline{L}),
\]

formed by the the angles landing at \( c_{a_1} \), intersects both \( \overline{L} \) and \( \overline{L'} \).

2. Let us now illustrate a more complicated situation in which the assumptions of (iv)(e) fail to hold. Namely, the assumption fails for a \( \lambda_0 \)-class \( A \) intersecting the closure of a \( \lambda_0 \)-unlinked class \( L \). However, \( L \) is the unique critical \( \lambda_0 \)-unlinked class whose closure intersects \( A \).

Let \( g_0(z) = 2.959...z^3 + 2.418...z^5 + 0.4590...z^7 \) and \( g_1(z) = -1.258...z + 0.06411...z^3 + 0.4047...z^5 + 0.08182...z^7 \) (see Figure 2). They are odd polynomials of degree 7. The critical points of \( g_0 \) are 0 (double critical point), \( \pm c_1 = \pm i \) and \( \pm c_2 = \pm 1.662...i \), which satisfy \( g_0(0) = 0, g_0(\pm c_1) = g_0(\mp c_2) = \mp c_1 \). In particular, \( g_0 \) is post-critically finite and hyperbolic. For \( g_1 \), the critical points are \( \pm c'_1 = \pm i, \pm c'_2 = \pm 1.793...i \) and \( \pm c_3 = \pm 0.8265...i \) and we have \( g_1(c_1) = \mp c_1, g_1(c_2) = 0 \) and \( g_2(c_3) = \mp c_3 \). Furthermore, we have \( \lambda_{c_1} \supset \lambda_{g_0} \). Let \( L \) be the \( \lambda_{c_i} \)-unlinked classes such that \( c_1 \in K_{g_0}(L) \) for \( i = 1,2 \). Then \( K_{g_0}(L) \) contains both \( c_1 \) and \( c_2 \), so similar to the previous example, this implies the conclusion of (iv)(e) does not hold.
Figure 4. Yoccoz puzzles and the rational laminations for the monic centered polynomials affinely conjugate to $f_a(z) = az^3 - (a + 1)z^2 + 1$ for $a = -1/4$ and $a = \frac{11 - 3\sqrt{17}}{4} = -0.3423...$ Dots indicate critical points.

Observe that there are two bifurcations from $g_0$ to $g_1$: The first one is that period two points collapse at the origin and new landing relations indicated by the light gray regions in the picture of the rational lamination for $g_1$ appear. The second is that the critical points $\pm c'_2$ hit the origin, the corresponding landing relations are indicated by the dark gray regions.

Below we combinatorially and abstractly describe the “location” and “orbit” of the critical points.

**Definition 3.9 (Critical elements and orbits).** Let $\lambda$ be a $d$-invariant rational lamination and $\hat{\lambda}$ its real extension (Definition 3.6). If $A$ is a $\lambda$-class, then $\delta(A)$ denotes the degree of $m_d : A \to m_d(A)$. If $A$ is a $\lambda$-class with $\delta(A) > 1$, then we say that $A$ is a *Julia critical element* of $\lambda$. 
If $L$ is an infinite unlinked $\lambda$-class, we say that $\delta(L)$ is the degree of $L$ where $\delta(L)$ is the number obtained in Proposition 3.7 (iv)(d). If $\delta(L) > 1$, then we say that $L$ is a Fatou critical element of $\lambda$.

We denote by $\text{Crit}(\lambda)$ the collection formed by all the critical elements for $\lambda$. The post-critical set and critical orbits of $\lambda$ are:

$$PC(\lambda) = \{m^n(C); \ C \in \text{Crit}(\lambda), n > 0\}$$

$$CO(\lambda) = \text{Crit}(\lambda) \cup PC(\lambda)$$

It follows that

$$d = 1 + \sum_{C \in \text{Crit}(\lambda)} (\delta(C) - 1)$$

[Ki] Lemma 4.10].
Definition 3.10 (Hyperbolic and post-critically finite laminations). An invariant rational lamination $\lambda$ is hyperbolic if it has no Julia critical element. An invariant rational lamination $\lambda$ is post-critically finite if every Julia critical element is contained in $\mathbb{Q}/\mathbb{Z}$.

It follows that a hyperbolic invariant rational lamination is the rational lamination of post-critically finite hyperbolic polynomial. Also, a post-critically finite invariant rational lamination is the rational lamination of a post-critically finite polynomial [Ki].

3.3. Renormalizations.

Definition 3.11 (Mapping schema associated to $\lambda_0$). Consider an invariant rational lamination $\lambda_0$.

- Let $|T(\lambda_0)|$ be the collection formed by the Fatou critical elements $v$ of $\lambda_0$.
- Let $\sigma_{\lambda_0} : |T(\lambda_0)| \to |T(\lambda_0)|$ be the map such that $\sigma_{\lambda_0}(v) = v'$ if $m^j_{\sigma}(v) = \lambda v'$ for some $\ell_v \geq 0$, and $m^j_{\sigma}(v) \notin |T(\lambda_0)|$, for all $0 < \ell_v < \ell_v$. We say that $\ell_v$ is the return time of $v$.
- Finally, let $\delta_{\lambda_0} : |T(\lambda_0)| \to \mathbb{N}$ be the corresponding degree map. That is, $\delta_{\lambda_0}(v) = \delta(v)$.

We call $T(\lambda_0) = (|T(\lambda_0)|, \sigma_{\lambda_0}, \delta_{\lambda_0})$ the reduced mapping schema of $\lambda_0$.

Definition 3.12 (Renormalizable and renormalization). We say $f \in C(\lambda_0)$ is $\lambda_0$-renormalizable if, for every $v \in |T(\lambda_0)|$ there exist topological disks $U'_v$ and $U_v$ such that

$$g = (f^{\ell_v} : U'_v \to U_{\sigma_{\lambda_0}(v)}; v \in |T(\lambda_0)|)$$

is a polynomial-like map over $T(\lambda_0)$ with fiberwise connected filled Julia set

$$K(g) = \bigcup_{v \in |T(\lambda_0)|} \{v\} \times K_f(v).$$

We call $g$ a $\lambda_0$-renormalization of $f$. We denote by $\mathcal{R}(\lambda_0)$ the subset of $C(\lambda_0)$ formed by all the $\lambda_0$-renormalizable polynomials.

The above definition is stronger than that in the outline (Section 1), but they are in fact equivalent.

Proposition 3.13. Let $f \in C(\lambda_0)$. If there exist topological disks $U'_v$ and $U_v$ for each $v \in |T(\lambda_0)|$ such that $K_v(f) \subset U'_v$ and $g = (f^{\ell_v} : U'_v \to U_{\sigma_{\lambda_0}(v)}; v \in |T(\lambda_0)|)$ is a polynomial-like map over $T(\lambda_0)$, then $g$ is a $\lambda_0$-renormalization of $f$.

Proof. It suffices to show $K_f(v) = \overline{K(g,v)}$. By construction, we have $K_f(v) \subset K(g,v)$. Since $v$ is an infinite $\lambda_0$-unlinked class, $K_f(v)$ is a full continuum. For $v$ periodic of period $n$ by $m_{\sigma}$, let $U'' = f^{-n}(U_v) \cap U'_v$. Then $f^n : U'' \to U_v$ is a polynomial-like map and its filled Julia set is $K(g,v)$. Hence $K(g,v)$ is the smallest full continuum in $U''$ completely invariant by $f^n : U'' \to U_v$. Therefore, $K_f(v) \subset K(g,v)$. For preperiodic $v \in |T(\lambda_0)|$, we need only take a backward image of $K(g,v')$ where $v' = \sigma^k(v)$ is periodic. □

In order to extract polynomial-like maps with an external marking it is convenient to “internally mark” the invariant rational lamination $\lambda_0$.

Definition 3.14 (Internally angled). An internally angled invariant rational lamination $\lambda_0$ is a pair $(\lambda_0, (\alpha_v : \overline{v} \to \mathbb{R}/\mathbb{Z})_{v \in |T(\lambda_0)|})$ such that

1) $\alpha_v$ induces a homeomorphism from $\overline{v}/\lambda_0$ to $\mathbb{R}/\mathbb{Z}$.
(ii) \( \alpha_{\sigma_v}(d\theta) = \delta_{\lambda_v}(\nu) \alpha_v(\theta) \) for \( \theta \in \mathbb{R} \).

We call \( (\alpha_v)_{v \in |\mathcal{T}(\lambda_0)|} \) an internal angle system.

The existence of internal angle system is guaranteed by Proposition 3.7. An internally angled invariant rational lamination determines an external marking the \( \lambda_0 \)-renormalization of every \( f \in \mathcal{R}(\lambda_0) \).

**Definition 3.15** (Induced external marking). Let \( (\lambda_0, (\alpha_v : \mathbb{R} \to \mathbb{R}/\mathbb{Z})_{v \in |\mathcal{T}(\lambda_0)|}) \) be an internally angled invariant rational lamination. For each \( v \in |\mathcal{T}(\lambda_0)| \) choose an argument \( \theta_v \) in \( \alpha_v^{-1}(0) \). Given a \( \lambda_0 \)-renormalization \( g \) of a polynomial \( f \in \mathcal{R}(\lambda_0) \), let \( \Gamma_v \) be the access with representative \( R_f(\theta_v) \cap U'_v \). We say that \( \Gamma = (\Gamma_v)_{v \in |\mathcal{T}(\lambda_0)|} \) is the external marking of \( g \) determined by the internal angle system \( (\alpha_v : \mathbb{R} \to \mathbb{R}/\mathbb{Z})_{v \in |\mathcal{T}(\lambda_0)|} \).

By Lindelöf Theorem, the external marking of \( g \) determined by an internal angle system is independent of the choices involved in the definition above.

**Lemma 3.16.** Let \( \lambda_0 \) be an invariant rational lamination. Let \( g \) be a \( \lambda_0 \)-renormalization of \( f \in \mathcal{R}(\lambda_0) \) and \( \Gamma \) an external marking of \( g \). Then there exists an internal angle system \( \alpha \) such that the external marking determined by \( \alpha \) is exactly \( \Gamma \).

**Proof.** Let \( \beta = (\beta_v) \) be an internal angle system for \( \lambda_0 \). Consider the straightening of \( g \) that maps the external marking induced by \( \beta \) onto the standard marking of \( P = \chi_{\lambda_0}(g) \). It follows that \( \Gamma \) is mapped onto an external marking of \( P \) determined by a collection of external rays \( (R_f(\theta_v, v)) \). Now the internal angle system \( \alpha = (\alpha_v = \beta_v + \theta_v) \) is such that the induced external marking \( \Gamma_\alpha \) of \( g \) maps onto the external marking of \( P \) determined by \( (R_f(\theta_v, v)) \). Therefore, \( \Gamma = \Gamma_\alpha \).

**3.4. Proof of Proposition 3.7** Most of this proposition is already proved in [Ki, Section 4] and [In3, Section 3]. In fact, it only remains to establish properties (ii), (iii)(c) and (iv)(e).

It is convenient to consider a (countable) set \( M \) contained in \( \lambda_0 \subset \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \) formed by the “image” of non-trivial relations. More precisely,

\[ M = \{(d\theta, d\theta') \in \lambda_0; \theta \neq \theta', (\theta, \theta') \in \lambda_0\} \]

Given a \( \lambda_0 \)-unlinked class \( L \), the invariance of \( \lambda_0 \) implies that:

\[ K_f(m_d(L)) = K(f) \cap \bigcup_{(\theta, \theta') \in M} \text{Sector}(\theta, \theta'; m_d(L)), \]

\[ K_f(L) = K(f) \cap \bigcup_{(\theta, \theta') \in M} S'(\theta, \theta'), \]

where

\[ S'(\theta, \theta') = \bigcup_{(\theta_1, \theta_2) \in \lambda_0} \text{Sector}(\theta_1, \theta_2; L) \]

is the intersection of all the components of \( f^{-1}(\text{Sector}(\theta, \theta'; m_d(L))) \) containing \( R_f(\theta) \) for some \( \theta \in L \).

Now (ii) follows, after countable intersection of the equation,

\[ f\left(S'(\theta, \theta')\right) = \text{Sector}(\theta, \theta'; m_d(L)), \]

which holds for all \( (\theta, \theta') \in M \).

To prove (iii)(c), we assume that \( L \) is finite and recall that, by (iii)(a), \( L \subset (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z} \). Hence \( K_f(L) \subset S'(\theta, \theta') \) for all \( (\theta, \theta') \in M \). Since, \( f : S'(\theta, \theta') \to \text{Sector}(\theta, \theta'; m_d(L)) \) is a proper map for all \( (\theta, \theta') \in M \), property (iii)(c) follows.
Now assume \( L \) is infinite and satisfies the assumption of \((\text{iv})(\text{e})\). Let \( M_L = \{ (\theta, \theta') \in M; \theta, \theta' \in m_d(L) \} \) and for each \((\theta, \theta') \in M_L\), let
\[
M'_{\lambda}(\theta, \theta') = \{ (\theta_1, \theta_2) \in M; \, d\theta_1 = \theta, \, d\theta_2 = \theta', \, \theta_j \in L, \, (j = 1, 2) \}.
\]
Then we have
\[
K_f(m_d(L)) = K(f) \cap \bigcap_{(\theta, \theta') \in M_L} \text{Sector}(\theta, \theta'; m_d(L)),
\]
\[
K_f(L) = K(f) \cap \bigcap_{(\theta, \theta') \in M_L} S''(\theta, \theta'),
\]
where
\[
S''(\theta, \theta') = \bigcup_{(\theta', \theta) \in M'_{\lambda}(\theta, \theta')} \text{Sector}(\theta, \theta'; L).
\]
The assumption guarantees that, for all \((\theta, \theta') \in M_L\), each component \( K \) of \( f^{-1}(K_f(m_d(L))) \) is either contained in or disjoint from \( S''(\theta, \theta') \). Hence, for all \((\theta, \theta') \in M_L\),
\[
f : f^{-1}(K_f(m_d(L))) \cap S''(\theta, \theta') \to K_f(m_d(L))
\]
has degree at least \( \delta(L) \). After taking a countable intersection, property \((\text{iv})(\text{e})\) follows. \(\square\)

### 4. Yoccoz puzzles

First, we introduce combinatorial Yoccoz puzzles in terms of laminations, then we define Yoccoz puzzles corresponding to a combinatorial Yoccoz puzzle. See [In3] for more details.

Throughout this section we fix an integer \( d \geq 2 \).

#### 4.1. Combinatorial Yoccoz Puzzle.

**Definition 4.1.** A **combinatorial Yoccoz puzzle** \( \Lambda = (\Lambda_k)_{k \geq 0} \) is a sequence of rational laminations with finite support such that:

(i) If \( \theta \in \text{supp}(\Lambda_0) \), then \( \theta \) is periodic under \( m_d \).

(ii) If \( \Lambda \) is a \( \Lambda_0 \)-class, then \( m_d(\Lambda) \) is a \( \Lambda_0 \)-class and \( m_d : \Lambda \to m_d(\Lambda) \) is continuous preserving.

(iii) \((m_d)_*\Lambda_{k+1} = \Lambda_k \) for any \( k \geq 0 \).

(iv) If \( \Lambda \) is a non-trivial \( \Lambda_k \)-class, then \( m_d(\Lambda) \) is also a non-trivial \( \Lambda_k \)-class.

We say that \( \text{supp}(\Lambda) = \bigcup_k \text{supp}(\Lambda_k) \) is the **support** of \( \Lambda \).

A \( \Lambda_k \)-unlinked class is a **combinatorial puzzle piece of depth \( k \)**. The collection of all depth \( k \) combinatorial puzzle pieces is denoted by \( \mathcal{L}_k(\Lambda) \).

We say that a combinatorial Yoccoz puzzle \( \Lambda \) is **admissible for a lamination \( \lambda \)** if \( \Lambda_k \subset \Lambda \) for all \( k \geq 0 \). We say \( \Lambda \) is **admissible for a polynomial** \( f \in \mathcal{C}(d) \) if \( \Lambda \) is admissible for \( \Lambda_f \).

We state, without proof, the following rather straightforward properties of combinatorial puzzles.

**Proposition 4.2.** Let \( \Lambda = (\Lambda_k)_{k \geq 0} \) be a combinatorial Yoccoz puzzle. Then

(i) \( \{ \text{supp}(\Lambda_k) \}_k \) forms an increasing sequence in \( \mathbb{Q}/\mathbb{Z} \) and \( \text{supp}(\Lambda) \) is completely invariant under \( m_d \).

(ii) For all \( k \geq 0 \), if \( L \) is combinatorial piece of a depth \( k + 1 \), then there exists a unique combinatorial piece \( L' \) of depth \( k \) such that \( L \subset L' \).
(iii) For all \( k \geq 0 \), if \( L' \) is a depth \( k + 1 \) combinatorial piece of a depth \( k + 1 \), then \( m_d(L') \) is a combinatorial piece of depth \( k \) and \( m_d : L' \to m_d(L) \) is \( \delta \)-to-one for some \( \delta \geq 1 \).

(iv) Let \( L \) be a combinatorial piece of depth \( k \), for some \( k \geq 0 \). If \( \theta \in \partial L \), then for all \( k' > k \), there exists a unique combinatorial piece \( L' \) of depth \( k' \) such that \( L' \subset L \) and \( \theta \in \partial L' \).

(v) Let \( L_1, \ldots, L_n \) be a complete list (without repetitions) of the combinatorial puzzle pieces of depth \( k \). Let \( C_1, \ldots, C_m \) be a complete list (without repetitions) of the \( \Lambda_k \)-critical classes. Then,

\[
\sum_{i=1}^{n}(\delta(L_i) - 1) + \sum_{j=1}^{m}(\delta(C_j) - 1) = d - 1,
\]

where, for \( A = L_\alpha \) or \( C_j \), the number \( \delta(A) \) denotes the degree of \( m_d : A \to m_d(A) \).

For a combinatorial puzzle piece \( L \), we denote by \( \delta(L) \) the degree of \( m_d : L \to m_d(L) \).

4.2. \textbf{Yoccoz Puzzle.} Let \( f \) be a monic centered polynomial with connected Julia set and associated Böttcher map \( \varphi : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus \Lambda \) where \( \Lambda = \{|z| \leq 1 \} \). For \( r > 0 \), the Jordan curve

\[
E_f(r) = \varphi^{-1}(\{z \in \mathbb{C} \setminus \overline{\Lambda}; |z| = \exp(r)\})
\]

is called the \textit{equipotential of level} \( r \). The open topological disk bounded by \( E_f(r) \) will be denoted by \( D_f(r) \).

Fix \( r > 0 \). Let \( \Lambda = (\Lambda_k) \) be an admissible combinatorial Yoccoz puzzle for \( f \in \mathcal{C}(d) \). We define the \textit{depth} \( k \) \textit{Yoccoz puzzle for} \( (f, \Lambda) \) as follows. Given a combinatorial puzzle piece \( L \) of depth \( k \), let

\[
P_k(f, L) = D_f(r/d^k) \cap \bigcap_{0 \neq \theta' \neq \theta \neq 0} \text{Sector}(\theta, \theta'; L)
\]

be the \textit{puzzle piece of depth} \( k \) \textit{for} \( (f, \Lambda) \) \textit{associated to} \( L \). We omit \( k \) and/or \( f \) when clear by context. Recall that we denote the collection of combinatorial puzzle pieces of depth \( k \geq 0 \) by \( \mathcal{L}_k(\Lambda) \). Similarly we let

\[
\mathcal{P}_k(f, \Lambda) = \{P(L); L \in \mathcal{L}_k(\Lambda)\}
\]

be the collection of all puzzle pieces of depth \( k \). That is, the \textit{Yoccoz puzzle of depth} \( k \) \textit{for} \( (f, \Lambda) \).

\textbf{Remark 4.3.} The interior of a puzzle piece is not connected in general. A “degenerate” puzzle appears when two distinct \( \Lambda_k \)-classes \( A, A' \) are contained in the same \( \lambda \)-class. Therefore, our definition of puzzle piece does not coincide with the usual one (i.e., the closure of a bounded component of the complement of a suitable graph constructed with equipotentials and external rays).

\textbf{Example 4.4.} 1. Let \( c = -0.1010... + 0.9562...i \) and \( f(z) = z^2 + c \), so that \( \alpha = f^4(c) \) is the alpha fixed point (i.e., the fixed point which is not a landing point of \( R_f(0) \)). Figure 5 shows its Julia set with its Yoccoz puzzles and (a part of) its lamination. The rational lamination \( \lambda_f \) of \( f \) properly contains the rational lamination of Douady’s rabbit (say \( \lambda_0 \)). In fact, the critical value \( c \) has the same landing angles of the alpha fixed point, i.e., \( 1/7, 2/7, 4/7 \). Thus the critical point \( 0 \) has six landing angles, consisting of two \( \lambda_0 \)-equivalence classes.

The lightgray region in the right picture indicates the landing relation for the critical point \( 0 \), which is not contained in the lamination of Douady’s rabbit. The Yoccoz puzzles
Figure 6. The Julia set and a “degenerate” puzzle for $z^2 + c$ with $c = -0.1010... + 0.9562...i$.

for $f$, determined by the lamination of Douady’s rabbit (namely, let $E_k$ be the set of landing angles for $f^{-k}(a)$ and $A_k = A_{E_k}$), is such that the puzzle piece of depth 4 containing 0 has two interior components.

2. Let $f_0(z) = z^3 + az^2$ where $a = 1.502... - 0.7790...i$ and let $f_1(z) = z^3 - 3/4z - \sqrt{7}/4i$ (Figure 7). There are two superattractive cycles of period one and two for $f_0$, and $f_1$ has two superattractive cycles of period two. Observe that $\lambda_{f_0} \subset \lambda_{f_1}$. Every non-trivial class of $\lambda_{f_0}$ eventually maps onto the fixed class formed by the angles $1/4, 5/8$. As above, we consider the Yoccoz puzzles $(\Lambda_k)$ determined by $\lambda_{f_0}$. The puzzle piece for $f_1$ of any depth corresponding to the fixed superattractive basin for $f_0$ has disconnected interior (and the number of components increases as depth increases).

3. Similarly, the polynomials $f_1$ and $g_1$ in Example 3.8 are also examples having degenerate puzzle pieces.

**Proposition 4.5.** Let $\Lambda = (\Lambda_k)$ be an admissible combinatorial Yoccoz puzzle for $f \in C(d)$. For all $k \geq 0$, let

$$\Gamma_k = E_f(r/d^k) \cup \bigcup_{\theta \in \text{supp}(\Lambda_k)} R_f(\theta).$$

Then the following statements hold:

(i) If $L$ is a combinatorial puzzle piece, then $P(L)$ is compact, connected and full. Moreover, the interior of $P(L)$ is the union of all the bounded components of $\mathbb{C} \setminus \Gamma_k$ which intersect $R_f(\theta)$ for some $\theta \in L$.

(ii) $P_k(f, \Lambda)$ is a partition of $D_f(r/d^k)$, i.e., depth $k$ puzzle pieces have mutually disjoint interior and

$$D_f(r/d^k) = \bigcup_{P \in P_k(f, \Lambda)} P.$$

(iii) If combinatorial puzzle pieces $L$ and $L'$ satisfy $L \subset L'$, then $P(L) \subset P(L')$.

(iv) If $L$ is a combinatorial puzzle piece of depth $k \geq 1$, then $f(P(L)) = P(m_\delta(L))$ and $f : \text{int } P(L) \to \text{int } P(m_\delta(L))$ is a proper map of degree $\delta(L)$.
Figure 7. Yoccoz puzzles and the rational laminations for $f_0(z) = z^3 + az^2$ ($a = 1.502... - 0.7790...i$) and $f_1(z) = z^3 - 3/4z - \sqrt{7}/4$.

For the proof, see [In3, Proposition 4.1]. This proposition implies that our puzzles have similar properties as the usual Yoccoz puzzles (defined by the closures of bounded components of $\mathbb{C} \setminus \Gamma_k$) and moreover, their properties can be described only in terms of $\Lambda$.

4.3. From combinatorial puzzles to laminations. Combinatorial Yoccoz puzzles are particularly suited to study the smallest invariant rational lamination for which they are admissible.

Definition 4.6. Let $\Lambda = (\Lambda_k)$ be a combinatorial Yoccoz puzzle. The rational lamination $\lambda(\Lambda)$ generated by $\Lambda$ is the smallest equivalence relation in $\mathbb{Q}/\mathbb{Z}$ containing the closure of $\bigcup_k \Lambda_k$ in $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$.

The previous definition is justified with the following result.

Proposition 4.7. For a combinatorial Yoccoz puzzle $\Lambda = (\Lambda_k)_{k \geq 0}$, the rational lamination $\lambda(\Lambda)$ generated by $\Lambda$ is an invariant rational lamination.
Proof. First, it is clear that $\lambda(A)$-classes are pairwise unlinked. Applying the argument for the claim in the proof of [Ki, Lemma 4.7] to our case, we have the following:

Claim. If $\theta, \theta' \in \mathbb{Q}/\mathbb{Z}$ are $\lambda$-related but not $\lambda$-equivalent, and if $\theta$ is periodic by $d$, then $\theta'$ is periodic of the same period.

If $A$ is a $\lambda(A)$-class, then $m_d^\varphi(A)$ contains a periodic angle under iterations of $m_d$. Therefore, $m_d^\varphi(A)$ (hence $A$) is a finite set since, by the claim above, $m_d^\varphi(A)$ consists of periodic angles of the same period.

Now let us prove that $\lambda(A)$ is closed. Assume that, for all $n \geq 0$, $\theta_n, \theta'_n$ are $\lambda(A)$-equivalent arguments such that $\theta_n \to \theta$ and $\theta'_n \to \theta'$, as $n \to \infty$. Then, since $\lambda(A)$-classes are pairwise unlinked, we may assume that these sequences are monotone. That is, $\theta_n \neq \theta$ and $\theta'_n \neq \theta'$. Moreover, we may also assume that, for all $n$, the $\lambda(A)$-class containing $\theta_n$ is contained in $[\theta_n, \theta_{n-1}) \cup (\theta'_{n-1}, \theta'_n)$.

It remains to prove that $(m_d)_\varphi(A) = \lambda(A)$. It is clear that for a $\lambda(A)$-class $A$, $m_d(A)$ is contained in a $\lambda(A)$-class $B$. Assume there exists $\theta'_0 \in B \setminus m_d(A)$. Take $\theta_1, \theta_2 \in A$ such that $(d\theta_1, d\theta_2)$ is the component of $\mathbb{R}/\mathbb{Z} \setminus m_d(A)$ containing $\theta'_0$. We may assume $\theta'_0$ and $d\theta_0$ are $\lambda$-related (because either $\theta_1 \sim_{\lambda(A)} \min([d\theta_0, d\theta_2) \cap B]$ or $\theta_2 \sim_{\lambda(A)} \max([d\theta_1, d\theta_2) \cap B]$ holds). Then there exists $\theta'_{0,n} \neq \theta'_n$ and $\theta'_{1,n} \neq d\theta_1$ such that $\theta'_{0,n}$ and $\theta'_{1,n}$ are $\lambda$-equivalent. Let $\theta_{0,n} \neq \theta_1$ be such that $d\theta_{0,n} = d\theta_{1,n}$. Then there exists $\theta_{0,n}$ such that $d\theta_{0,n} = \theta'_{0,n}$ and $\theta_{0,n}, \theta_{1,n}$ are $\lambda$-equivalent. Since $\lambda(A)$-classes are pairwise unlinked, $\theta_{0,n}, \theta_{1,n} \subseteq (\theta_1, \theta_2)$. Passing to a subsequence, we may assume $\theta_{0,n} \to \theta_0$ and $d\theta_0 = \theta'_0$. Then since $\theta_0$ and $\theta_1$ are $\lambda$-related and $\theta_0 \in [\theta_1, \theta_2]$, either $\theta_0 = \theta_1$ or $\theta_0 = \theta_2$, it is a contradiction. Hence $B = m_d(A)$ is a $\lambda(A)$-class. This argument also shows that $m_d : A \to m_d(A)$ is consecutive preserving. 

Definition 4.8. Let $\Lambda = (\Lambda_k)$ be an admissible combinatorial Yoccoz puzzle for a $d$-invariant rational lamination $\lambda$. We say $\Lambda$ is a generator of $\lambda$ if there exists $k \geq 0$ such that, for any combinatorial piece $L$ of depth $k$ which contains a $\lambda$-unlinked class in the forward orbit of a Fatou critical element of $\lambda$, we have that $\delta(L) = \delta(v)$. We call such $k$ a separation depth for $\lambda$.

The next proposition shows that a generator actually “generates” a substantial part of the corresponding rational lamination.

Proposition 4.9. Let $\Lambda = (\Lambda_k)_{k \geq 0}$ be a generator of an invariant rational lamination $\lambda$. If $L$ is a Fatou critical element of $\lambda$, then $L$ is a Fatou critical element of $\lambda(A)$.

Proof. Let $\lambda' = \lambda(A)$ and observe that $\lambda' \subset \lambda$. Therefore, given any infinite $\lambda$-unlinked class $L$, there exists a unique $\lambda'$-unlinked class $L'$ such that $L' \supset L$.

Let $L$ be a Fatou critical element of $\lambda$ and $L'$ the $\lambda$-unlinked class containing $L$. We claim that $L' = L$. For this purpose, let $k$ be a separation depth for $\lambda$. Denote by $L_k$ the combinatorial puzzle piece of depth $k$ containing $L'$. Thus, $\delta(L_k) = \delta(L') = \delta(L)$. Now assume that $L$ is periodic under $m_d$, say of period $p$. It follows that $m_d^p : L' \to L'$ and $m_d^p : L \to L$ have the same degree. Moreover, under both of these maps, the grand orbits of a given $\theta \in L$ coincide. Since such a grand orbit is dense in $L$ and in $L'$, we conclude that $L = L'$. In the case that $L$ is preperiodic, then there exists $\ell \geq 1$ such that $m_d^\ell(L) = m_d^\ell(L')$ is periodic and the degree of $m_d^\ell : L \to m_d^\ell(L)$ also coincides with the degree of $m_d^\ell : L' \to m_d^\ell(L')$. Hence, $L = L'$. 

\qed
Lemma 4.10. If \( \lambda \) be an invariant rational lamination, then there exists a generator of \( \lambda \).

Proof. Given \( v \) a \( \lambda \)-unlinked class in the forward orbit of a Fatou critical element of \( \lambda \), according to Proposition 3.7 (iv), for every critical element \( C \neq v \) of \( \lambda \) there exists a \( \lambda \)-class \( A_C \) such that the connected component of \( \mathbb{R}/\mathbb{Z} \setminus A_C \) that contains \( v \) is disjoint from \( C \). Hence, there exists a collection \( A_1, \ldots, A_n \) of \( \lambda \)-classes such that the following holds. If \( v \) is a \( \lambda \)-unlinked class in the forward orbit of a Fatou critical element of \( \lambda \) and \( C \) is a critical element of \( \lambda \), then the connected component of \( \mathbb{R}/\mathbb{Z} \setminus A_j \) that contains \( v \) is disjoint from \( C \), for some \( j = 1, \ldots, n \). Let \( E_0 \) be the periodic arguments in the \( m_d \)-forward orbit of \( A_1 \cup \cdots \cup A_n \). For \( k \geq 0 \), let \( E_k = m^{-k}(E_0) \) and \( A_k \) be the restriction of \( \lambda \) to \( E_k \times E_k \). It follows that the puzzle \( \Lambda = (A_k)_{k \geq 0} \) is a generator for \( \lambda \). \( \square \)

5. Injectivity of straightening maps

This section is devoted to establish that straightening maps are injective. That is, we prove Theorem B. The proof relies on the following main result.

Key Lemma 5.1. Let \( \lambda_0 \) be a post-critically finite \( d \)-invariant rational lamination with reduced schema \( T(\lambda_0) \). If \( f \in C(\lambda_0) \), then the set

\[
F = K(f) \setminus \bigcup_{n \geq 0} \bigcup_{v \in T(\lambda_0)} f^{-n}(K_f(v))
\]

has zero area.

Section 5.2 contains the proof of this lemma. In Section 5.1 we prove Theorem B assuming the Key Lemma.

Remark 5.2. Epstein and Yampolsky [EY] first proved this key lemma for cubic polynomials constructed by intertwining surgery of two quadratic polynomials. Haïssinsky [Ha3] also claimed the key lemma for a general intertwining surgery, but his proof contains a gap. (He claimed that the set denoted by \( F_1 \), in the proof below, is empty. However, \( F_1 \) is always nonempty in the case of intertwining surgery)

Remark 5.3. Theorem B often implies that the polynomial obtained through a quasiconformal construction from a collection of polynomials is uniquely determined by the original polynomials, up to affine conjugacy. For example, consider the case of intertwining surgery by Epstein and Yampolsky [EY]. Starting with two quadratic polynomials with connected Julia sets they construct a cubic polynomial (called an intertwining) having two quadratic-like restrictions whose Julia sets intersect at a repelling fixed point. The resulting cubic polynomial only depends on the choice of initial quadratic polynomials and not on the choices made throughout the surgery.

The first author [In2] used the intertwining construction to obtain a polynomial having a capture type renormalization. For such a construction, Theorem B can be applied to show that the resulting polynomial is uniquely determined by the initial data, up to affine conjugacy. Haïssinsky [Ha3] also constructed intertwinings at parabolic fixed points which produce a non-renormalizable polynomial. However, the Key Lemma 5.1 still holds and, therefore, the resulting polynomial of the construction is uniquely determined by the corresponding initial data, up to affine conjugacy.

5.1. Proof of Theorem B

First, we assume the key lemma and prove Theorem B.

Let \( \lambda_0 \) be a post-critically finite \( d \)-invariant rational lamination and let \( \alpha \) be an internal angle system for \( \lambda_0 \). Consider two renormalizable polynomials \( f_1, f_2 \in \mathcal{R}(\lambda_0) \) such that
\( \chi_{\lambda_0}(f_1) = \chi_{\lambda_0}(f_2) \), where \( \chi_{\lambda_0} \) is the corresponding straightening map. We must show that \( f_1 = f_2 \).

For the purpose of this proof it is better to extract “polynomial-like maps” \( g_1, g_2 \) over the \textit{unreduced mapping schema} for \( \lambda_0 \). More precisely, let \( |T^U(\lambda_0)| \) be the (forward) orbit of the Fatou critical elements of \( \lambda_0 \) and, for all \( v \in |T^U(\lambda_0)| \), let \( \delta(v) \) be the degree of \( m_d : v \to m_d(v) \). It follows that \( T^U(\lambda_0) = \{|T^U(\lambda_0)|, m_d, \delta\} \) is a mapping schema. If \( f \in \mathcal{R}(\lambda_0) \), then there exists domains \( U'_v \subseteq U_v \) such that \( g_v = f : U'_v \to U_{m_d(v)} \) is a proper map of degree \( \delta(v) \), for all \( v \in |T^U(\lambda_0)| \). Moreover, these domains may be chosen so that the forward orbit of \( (v, z) \in |T^U(\lambda_0)| \times \mathbb{C} \) under \( (v, z) \mapsto (m_d(v), f(z)) \) is well defined if and only if \( z \in K_\varepsilon(f) \). We say that \( g = (f : U'_v \to U_{m_d(v)})_{v \in |T^U(\lambda_0)|} \) is an \textit{unreduced} \( \lambda_0 \)-renormalization of \( f \).

Now let \( g_j = (f_j : U'_{j,v} \to U_{j,m_d(v)}, v \in |T^U(\lambda_0)|) \) be an unreduced \( \lambda_0 \)-renormalization of \( f_j \), for \( j = 1, 2 \). Given a hybrid conjugacy between the \( \lambda_0 \)-renormalizations of \( f_1 \) and \( f_2 \), that respects the external markings, determined by the internal angle system \( \alpha \), it is not difficult to extend \( \psi \) to a hybrid conjugacy between \( g_1 \) and \( g_2 \). That is, \( \psi = (\psi_v : U_{1,v} \to U_{2,v})_{v \in |T^U(\lambda_0)|} \) is such that, for all \( v \in |T^U(\lambda_0)| \):

\[
\frac{\partial \psi_v}{\partial z}(z) \equiv 0,
\]

for almost every \( z \in K_\varepsilon(f_1) \) and,

\[
f_2 \circ \psi_v(z) = \psi_{m_d(v)} \circ f_1(z),
\]

for all \( z \in U'_{1,v} \).

Let \( \theta_j \in \mathcal{T}(\lambda_0) \) be such that, for \( j = 1, 2 \), the external marking \( \Gamma_j \) of the \( \lambda_0 \)-renormalization of \( f_j \) is represented by \( (\gamma_{j,v}, v \in |T^U(\lambda_0)| \) where \( \gamma_{j,v} \subseteq R_{j,v}(\theta_j) \), for all \( v \in |T^U(\lambda_0)| \). Then, we may adjust the collection \( \{\theta_j, v \in |T^U(\lambda_0)| \) so that it extends to a larger one \( \{\theta_j, v \in |T^U(\lambda_0)| \) such that \( \theta_{m_d(v)} = \theta_{m_d(v)} \) for all \( v \in |T^U(\lambda_0)| \). It follows that \( \psi_v \) maps the access to \( K_{\varepsilon}(f_j) \) determined by \( R_j(\theta_j) \) onto the access to \( K_{\varepsilon}(f_2) \) determined by \( R_{j,v}(\theta_j) \), for all \( v \in |T^U(\lambda_0)| \). Hence, we can adjust the hybrid conjugacy \( \psi \) so that \( \psi_v \) maps \( R_{j,v}(\theta_j) \cap U_{1,v} \) onto \( R_{j,v}(\theta_j) \cap U_{2,v} \), for all \( v \).

Let \( \Lambda = (\Lambda_k) \) be a generator of \( \lambda_0 \) and \( k_0 \geq 0 \) be a separation depth. We may assume that every Julia critical element of \( \lambda_0 \) is a \( \Lambda_k \)-class for all \( k \geq k_0 \). As in the previous section, for each \( v \in |T^U(\lambda_0)| \), let \( \Lambda_k(v) \) be the combinatorial puzzle piece of depth \( k \geq 0 \) for \( \Lambda \) containing \( v \). Denote by \( P_k(f, v) \) the corresponding puzzle piece for \( (f, v) \). For each \( v \in |T^U(\lambda_0)| \), take a neighborhood \( V_{1,v} \) of \( K_j(v) \cap \text{int} P_{k_0}(f_1, v) \) and let \( V_{2,v} = \psi_v(V_{1,v}) \). We may choose \( V_{1,v} \) sufficiently small so that

\[
V_{j,v} \subseteq U_{j,v} \cap \text{int} P_{k_0}(f_j, v).
\]

Denote by \( \text{supp}(f_j, \Lambda_{k_0}) \) the set formed by the landing points of the external rays of \( f_j \) with arguments in \( \text{supp}(\Lambda_{k_0}) \). Shrinking \( V_{j,v} \), if necessary, we may also assume that

\[
\overline{V_{j,v}} \cap \partial P_{k_0}(f_j, v) \subseteq \text{supp}(f_j, \Lambda_{k_0}).
\]

In particular, for all \( v \neq v' \in |T^U(\lambda_0)| \), the neighborhoods \( V_{1,v} \) and \( V_{j,v'} \) are disjoint and

\[
\overline{V_{j,v}} \cap \overline{V_{j,v'}} \subseteq K_{f_j}(v) \cap K_{f_j}(v').
\]

For \( j = 1, 2 \) let \( V_j = \bigcup_{v \in |T^U(\lambda_0)|} V_{j,v} \). Define \( \Phi_0 : V_1 \to V_2 \) by

\[
\Phi_0 = \psi_v \circ \Phi_0 \text{ on } V_{1,v}.
\]

In view of the above, we may extend \( \Phi_0 \) quasiconformally to \( \mathbb{C} \) such that:

- \( \Phi_0(P_{k_0}(f_1, L)) = P_{k_0}(f_2, L) \) for all combinatorial pieces \( L \) of depth \( k_0 \).
- \( \Phi_0(z) = \varphi_j^{-1} \circ \varphi_j(z) \) for all \( z \in \mathbb{C} \setminus D_{f_j}(r/d_{k_0}) \) where \( \varphi_j \) denotes the Böttcher map of \( f_j \).
Moreover, constant of (quasiconformal) homeomorphism such that fact, the unreduced mapping schema of the arcwise connected set \( \Lambda = \lambda \), for \( \lambda > 0 \) asymptotic to the identity at infinity.

**Lemma 5.4.** The following hold for almost every \( z \in \Lambda \):

\[
\begin{align*}
\text{(i)} \quad & \lim_{n \to \infty} \| (f^n)'(z) \| = \infty \quad \text{with respect to the hyperbolic metric on } \mathbb{C} \setminus \text{PC}(f); \\
\text{(ii)} \quad & \lim_{n \to \infty} d(f^n(z), \mathcal{K}_f(\text{CO}^{\text{per}})) = 0; \\
\text{(iii)} \quad & \text{there exist } N = N(z) > 0 \text{ and } \mathcal{K} \in \text{Comp}(\mathcal{K}_f(\text{CO}^{\text{per}})) \text{ such that } f^{N+n}(z) \in \mathcal{N}_e(f^n(\mathcal{K})). \text{ Moreover, if } f^{N+n}(z) \in \mathcal{N}_e(f^n(\mathcal{K})) \text{ for } \nu \in \text{CO}^{\text{per}}, \text{ then } f^{N+n+1}(z) \in \mathcal{N}_e(f^n(\mathcal{K})(d(f^n(\mathcal{K})))).
\end{align*}
\]

**Proof.** The assertion (ii) is a stronger version of Mañé’s Lemma by McMullen (see [Mc1, Theorem 3.6]).
Figure 8. Yoccoz puzzles of a separation depth. There are two periodic cycle of period two and three of Fatou components, which are colored black. The period two cycles corresponds to $CO^0$, and the period three cycles corresponds to $CO^1$.

In view of [Mc1, Theorem 3.9]), for almost every point $z \in J(f)$ we have that

$$\lim_{n \to \infty} d(f^n(z), PC(f)) = 0.$$  

For all such $z \in F$,

$$\lim_{n \to \infty} d(f^n(z), K_f(CO^{i,\text{per}})) = 0$$

because $PC(f) \setminus K_f([T^U(\lambda_0)])$ is a finite set consisting only of repelling periodic points and their backward images, and $f^n(K_f([T^U(\lambda_0)]) = K_f(CO^{i,\text{per}})$ for some $n \geq 0$. Since $K_f(CO^{0,\text{per}})$ and $K_f(CO^{1,\text{per}})$ are disjoint forward invariant compact sets, we have

$$\lim_{n \to \infty} d(f^n(z), K_f(CO^{i,\text{per}})) = 0$$

for some $i \in \{0, 1\}$.

Now assume

(2)  

$$\lim_{n \to \infty} d(f^n(z), K_f(CO^{0,\text{per}})) = 0.$$  

For each $v \in CO^0$, $K_f(v)$ is contained in the interior of the puzzle piece $P(L_{k_0+1}(v))$. Therefore we have $K_f(v) \in \text{Comp}(K_f(CO^{0,\text{per}}))$ (i.e., $K_f(v)$ is a component of $K_f(CO^{0,\text{per}})$). By (2) and the continuity of $f$, it follows that there exist some $N \geq 0$ and $v \in CO^0$ such that
for any \( n \geq 0 \), \( f^{N(n)}(z) \in P(L_{k_0 + 1}(m_d^p(v))) \), which is a neighborhood of \( K_f(m_d^p(v)) \). Let \( p \) be a period of \( v \) under iterations of \( m_d \). Then \( f^{p(n)}(f^{N(n)}(z)) \) belongs to a neighborhood of \( K_f(v) \) for all \( n \geq 0 \). Since \( K_f(v) \) is the filled Julia set of a polynomial-like map, it follows that \( f^{N(n)}(z) \in K_f(v) \), which contradicts \( z \in F \). Therefore, we have proved (iii).

The assertion [iii] easily follows from [ii] and the continuity of \( f \). \( \square \)

Now we define, for each \( v \in CO^{1\text{-per}} \), an open set \( N'(v) \) (which is a slightly smaller set than \( N_{n_0}(K_f(v)) \)). Each \( K \in \text{Comp}(K_f(CO^{1\text{-per}})) \) can be written as

\[
K = \bigcup_{m=0}^{M} K_f(v_m), \quad v_m \in CO^{1\text{-per}}.
\]

Then the set

\[
I = \bigcup_{m \neq n} (K_f(v_m) \cap K_f(v_n)) \subset \text{supp}(f, \Lambda_{k_0 + 1})
\]

consists of repelling (pre)periodic points (see [Mc1, Theorem 7.3] and [In1, Proposition 3.4]). For each \( x \in I \), take \( \Theta(x) \subset \text{Angle}(x) \cap \text{supp}(\Lambda_{k_0}) \) such that \( d(\Theta(x)) = \Theta(f(x)) \) and each component of the complement of

\[
\Gamma(x) = \bigcup_{\theta \in \Theta(x)} R_f(\theta) \cup \{x\}
\]

intersects exactly one of the \( K_f(v_m) \) that contain \( x \). Let \( \Gamma = \bigcup_{x \in I} \Gamma(x) \) and let \( \Gamma_0 \) denote the union of the components of \( \Gamma \cap N_{n_0}(K) \) intersecting \( I \). Then each component of \( N_{n_0}(K) \setminus \Gamma_0 \) intersects exactly one of \( K_f(v_1), \ldots, K_f(v_M) \). Let us denote by \( N'(v_m) \) the component of \( N_{n_0}(K) \setminus \Gamma_0 \) containing \( K_f(v_m) \setminus I \).

By construction,

\[
\bigcup_{m=1}^{M} N'(v_m) = N_{n_0}(K)
\]

and \( N'(v) \ (v \in CO^{1\text{-per}}) \) are mutually disjoint.

**Lemma 5.5.** Let \( F_0 = \{z \in F; \text{Lemma 5.4 holds}\} \). For \( z \in F_0 \), we have

(i) For all \( n \geq N(z) \), there exists \( v(z, n) \in CO^{1\text{-per}} \) such that \( f^n(z) \in N'(v(z, n)) \);
(ii) there exists arbitrarily large \( n \) such that \( m_\Lambda(v(z, n)) \neq v(z, n + 1) \);
(iii) there exists \( \epsilon_1 > 0 \) independent of \( z \) such that if \( m_\Lambda(v(z, n)) \neq v(z, n + 1) \), then \( f^{p(n)}(z) \in N_{n_0}(I) \) and \( f^n(z) \in N_{n_0}(f^{-1}(I) \cap K_f(CO^{1\text{-per}} \setminus I)) \).

Furthermore, \( \epsilon_1 \) tends to zero as \( \epsilon_0 \) tends to zero.

**Proof.** The assertion [i] follows from the equation (3) above. If [ii] does not hold, then we would have that \( f^n(z) \in K_f(v(z, n)) \), for sufficiently large \( n \) (since \( \Lambda \) is a generator for \( \Lambda_0 \)). This is a contradiction with \( z \in F_0 \). Finally, [iii] follows from the fact that \( N_{n_0}(K_f(v)) \setminus N'(v) \) is contained in a small neighborhood of \( I \cap K_f(v) \). \( \square \)

Take a small neighborhood \( O \) of \( I \) such that \( f(O) \supset \overline{O} \) and \( f|_O \) is injective. Since \( CO(f) \setminus K_f(CO^{1\text{-per}}) \) is finite, we may also assume the following:

(i) No critical value lies in \( O^* = O \setminus I \).
(ii) If a component \( O_1 \) of \( f^{-1}(O) \) intersects \( K_f(CO^{1\text{-per}}) \), then \( O_1 \cap f^{-1}(I) \) and \( O_1 \cap CO(f) \) are contained in \( K_f(CO^{1\text{-per}}) \).
(iii) \( \hat{O} = O^*/f \) is a union of tori.

Figure 9 illustrates the statement of the following lemma.
Lemma 5.6. There exist open subsets $U_1$ and $U_2$ of $O$ such that the following statements hold:

(i) $U_2 \subsetneq U_1 \subsetneq O$ and $U_1$ does not intersect $\Gamma$.
(ii) Each one of the open sets $U_1$ and $U_2$ is a disjoint union of finitely many topological disks.
(iii) Each component $A$ of $U_1$ contains exactly one component $B$ of $U_2$ and $A \setminus \overline{B}$ is an annulus.
(iv) $U_2 / f \supset (O^* \cap K(f)) / f$. In other words, there exists $\varepsilon_2 > 0$ such that for any $z \in K(f) \cap N_\varepsilon(l)$, there exists some $N > 0$ such that $f^n(z) \in O$ for any $n$ with $0 \leq n < N$, $f^n(z) \in U_2$, and the branch of $f^{-N}$ sending $f^N(z)$ to $z$ can be univalently defined on the component of $U_1$ containing $f^N(z)$.

Proof. (See Figure 9) Let $K = (K(f) \cap O^*) / f$ and $\hat{\Gamma} = (\Gamma \cap O^*) / f$. Note that $\hat{\Gamma}$ and $\hat{K}$ are disjoint compact sets in $\hat{O}$. Take a fundamental domain $U$ for the covering projection $O \setminus \Gamma \rightarrow \hat{O} \setminus \hat{\Gamma}$. Let $U_2 \subsetneq U_1$ be neighborhoods of $U \cap K(f)$ in $O \setminus \Gamma$. It is easy to see that we can take $U_1$ and $U_2$ so that they satisfy the assertion. \hfill $\Box$

Proof of Key Lemma 5.7. It is enough to show that any $z \in F_0$ is not a Lebesgue density point of $K(f) \supset F_0$.

Let $\varepsilon_1$ and $\varepsilon_2$ be as in Lemma 5.5 and Lemma 5.6 respectively. We may assume $\varepsilon_1 \leq \varepsilon_2$. For $z \in F_0$, let

$$t(z) = \{ n \geq N(z); \ m_d(v(z,n)) = v(z,n+1) \}.$$ 

By Lemma 5.5, we have $t(z)$ contains infinitely many elements. For $n \in t(z)$, let $O_1(z,n)$ be the component of $f^{-1}(O \setminus \Gamma)$ containing $f^n(z)$. By the condition (1) on $O$ above, $f : O_1(z,n) \rightarrow O \setminus \Gamma$ is injective. (See Figure 9)

For $z \in F_0$ and $n \in t(z)$, $f^{n+1}(z)$ lies in a $\varepsilon_1$-neighborhood of $z$ by Lemma 5.5. Hence by Lemma 5.6, there exists some $n' > n$ such that $f^{n'}(z) \in U_2$ and the branch of $f^{-n'}(z)$ sending $f^n(z)$ to $f^{n+1}(z)$ is univalently defined on $U_1'(z,n)$, where $U_1'(z,n)$ is the component of $U_1$ containing $f^{n'}(z)$. Let $U_1(z,n)$ be the component of $f^{-n'}(U_1(z,n))$ containing $f^n(z)$. Then $U_1(z,n) \subset O_1(z,n)$ and $f^{n'} : U_1(z,n) \rightarrow U_1'(z,n)$ is a conformal isomorphism. Since $O_1(z,n) \cap PC(f) \subset O_1(z,n) \cap CO(f) = \emptyset$, there exists an univalent inverse branch $f^{-n'} : U_1'(z,n) \rightarrow U_1(z,n)$ sending $f^n(z)$ to $z$. Therefore, $f^n : U_1(z,n) \rightarrow U_1'(z,n)$ and $f^{n'} : U_1'(z,n) \rightarrow U_1(z,n)$ are conformal isomorphisms (condition (1)).

Denote by $i_{z,n} : U_1'(z,n) \hookrightarrow \mathbb{C} \setminus PC(f)$ the inclusion and consider the following diagram:

$$\xymatrix{ \mathbb{C} \setminus PC(f) \ar@{^{(}->}[rr]^{f^n} & & \mathbb{C} \setminus PC(f) \ar@{^{(}->}[rr]^{f^{n'}} & & U_1'(z,n) \ar@{_{(}->}[u]_{i_{z,n}} \ar@{_{(}->}[u]_{i_{z,n}} \ar@{=}[u]_e \ar@{=}[u]_e. }$$

Since $\lim_{n \rightarrow \infty} \|(f^n)'(z)\| = \infty$ with respect to the hyperbolic metric on $\mathbb{C} \setminus PC(f)$ and inclusion does not expand hyperbolic metric, we have $\lim_{n \rightarrow \infty} \|i_{z,n}'(z)\| = 0$ with respect to the corresponding hyperbolic metric. By Koebe distortion theorem, this implies that the diameter $U_2(z,n)$ shrinks to zero with bounded distortion. Furthermore, since $f^n : U_1(z,n) \rightarrow U_1'(z,n)$ is a conformal isomorphism sending $U_2(z,n)$ to $U_2'(z,n)$, by the Koebe distortion theorem applied to the inverse of the conformal isomorphism $f^{n'}|_{U_1'(z,n)}$, there exists some
Figure 9. Near $I$ and its inverse image. If $m_d(v(z, n)) \neq v(z, n + 1)$, $f^n(z)$ must be close to $f^{-1}(I) \setminus I$. 

\[
\begin{align*}
&f^{n+1}(z) \\
&\mathcal{N}'(v(z, n + 1)) \\
&\mathcal{N}'(v(z, n')) \\
&\mathcal{N}'(v(z, n)) \\
&\mathcal{N}'(v(z, n+1)) \\
&U_1 \\
&U_2 \\
&f(z) \\
&\{v(z, n), v(z, n + 1)\} \\
&\mathcal{N}'(v(z, n)) \\
&f^n(z) \\
&\mathcal{U}_1 \\
&\mathcal{U}_2 \\
&f_{n+1}(z) \\
\end{align*}
\]
\[ C, C' > 0 \text{ such that} \]
\[
\frac{\text{Area}(U_2(z,n) \setminus K(f))}{\text{Area}(U_2(z,n))} \geq C \frac{\text{Area}(U_2(z,n) \setminus K(f))}{\text{Area}(U_2(z,n))} \geq C'.
\]

Here, we have only finitely many choices for \( U_i(z,n) \) since each choice is a component of \( U_i \). This implies that \( C \) and \( C' \) are constants independent of \( z \in F_0 \) and \( n \in \Gamma(z) \).

Therefore,
\[
\lim_{n \to \infty} \frac{\text{Area}(U_2(z,n) \cap K(f))}{\text{Area}(U_2(z,n))} \leq 1 - C' < 1
\]
and \( z \) is not a Lebesgue density point of \( K(f) \).

6. Combinatorial Tuning

In this section we study the image of straightening and establish that, in a certain sense precised below, the straightening map is combinatorially onto.

**Definition 6.1.** A rational lamination over a mapping schema \( T = (|T|, \sigma, \delta) \) is a collection of rational laminations \( \lambda = (\lambda_v)_{v \in |T|} \) with \( \delta(v), \lambda_v = \lambda_{\sigma(v)} \).

For \( f \in C(T) \), the rational lamination of \( f \) is the collection \( \lambda_f = (\lambda_{f,v})_{v \in |T|} \) of equivalence relations in \( \mathbb{Q}/\mathbb{Z} \) such that: \( \theta \) and \( \theta' \) are \( \lambda_{f,v} \)-equivalent if \( R_f(v, \theta) \) and \( R_f(v, \theta') \) land at the same point.

We can similarly define unlinked classes and the reduced mapping schema of a rational lamination over \( T \). Hence we also define critical elements, post-critically finiteness and hyperbolicity of a rational lamination over \( T \).

**Definition 6.2.** Let \( \lambda_0 \) and \( \lambda \) be invariant rational laminations. Assume \( \lambda_0 \) has non-empty reduced schema \( T(\lambda_0) \) and \( \lambda_0 \subset \lambda \). Let \( \sigma = (\sigma_v : \mathbb{V} \to \mathbb{R}/\mathbb{Z}) \) be an internal angle system for \( \lambda_0 \).

For each \( v \in |T(\lambda_0)| \), let \( \lambda_v \) be the equivalence relation that identifies \( \theta \) and \( \theta' \) if and only if there exist \( t \in \alpha_v^{-1}(\theta) \) and \( s \in \alpha_v^{-1}(\theta') \) such that \( s \) and \( t \) are \( \lambda \)-equivalent. Then \( \lambda' = (\lambda_{f,v})_{v \in |T(\lambda_0)|} \) is a rational lamination over \( T(\lambda_0) \). We say \( \lambda \) is the combinatorial straightening of \( \lambda_f \) with respect to \( (\lambda_0, \alpha) \).

**Theorem 6.3.** Consider an internally angled invariant rational lamination \( \lambda_0 \) with non-empty reduced schema and such that \( \mathcal{R}(\lambda_0) \neq \emptyset \). Let \( \chi : \mathcal{R}(\lambda_0) \to C(T(\lambda_0)) \) be the corresponding straightening map.

If \( (\lambda_v) \) is a rational lamination over \( T(\lambda_0) \), then there exists \( f \in C(\lambda_0) \) such that the combinatorial straightening of \( \lambda_f \) is equal to \( \lambda_v \).

Moreover, we have the following:

- If \( f \) above is \( \lambda_0 \)-renormalizable, then the rational lamination of \( \chi(f) \) is \( (\lambda_v) \).
- There exists an algebraic set \( X = X(\lambda_0) \subset \text{Poly}(d) \) of pure codimension one such that \( \mathcal{R}(\lambda_0) \subset C(\lambda_0) \setminus X \).
- There exists an algebraic set \( Y = Y(\lambda_0) \subset \text{Poly}(T(\lambda_0)) \) of pure codimension one such that if \( P \in C(T(\lambda_0)) \) is post-critically finite, then there exists \( f \in \mathcal{R}(\lambda_0) \) such that \( P = \chi(f) \) or \( P \) is a non-hyperbolic map contained in \( Y \).

Note that \( C(\lambda_0) \) might be contained in a proper algebraic set, so it can happen that \( C(\lambda_0) \) is contained in \( X \) and \( \mathcal{R}(\lambda_0) = \emptyset \). Each irreducible component of the algebraic sets \( X \) and \( Y \) is defined by the existence of a parabolic periodic point of a given period, or a preperiodic critical point of given preperiod and eventual period.
The proof of the above theorem is at the end of this section, as well as that of Theorem 6.6. As preliminary work for the proof we start by studying the action of straightening on rational laminations (see Lemma 6.4). Then we find an inverse for this action in Theorem 6.6. The last of the necessary ingredients is the characterization of laminations such that \( \text{rel}(\lambda_0) \neq \emptyset \), contained in Proposition 6.7, which is based on the sufficient condition for a polynomial to belong to \( \text{rel}(\lambda_0) \) given in Lemma 6.10.

6.1. Combinatorial straightening and renormalized lamination.

**Lemma 6.4.** Assume that \( \lambda_0 \) is an invariant rational lamination with non-empty reduced schema. Let \( \alpha = (\alpha_v : \mathcal{V} \to \mathbb{R}/\mathbb{Z}) \) be an internal angle system for \( \lambda_0 \). Consider the corresponding straightening map \( \chi : \text{rel}(\lambda_0) \to C(T(\lambda_0)) \). Given \( f \in \text{rel}(\lambda_0) \) and let \( \lambda' = (\lambda_v)_{v \in T(\lambda_0)} \) be the combinatorial straightening of \( \lambda_f \).

Then, for all \( v \in [T(\lambda_0)] \),

\[
A_{\phi, f, v} = \lambda_v.
\]

**Proof.** Let \( g = (f^c : U \to U_{\sigma(v)}) \) be a \( \lambda_0 \)-renormalization. Let \( P = \chi(g) : [T(\lambda_0)] \times \mathbb{C} \to [T(\lambda_0)] \times \mathbb{C} \) be the straightening of \( g \) via the fiberwise quasiconformal conjugacy \( \psi_v : U_v \to \psi_v(U) \subset \{v\} \times \mathbb{C} \). Let \( \psi_v : \mathbb{C} \setminus K(P, v) \to \mathbb{C} \setminus \Delta \) be the Böttcher map. Given \( v \in [T(\lambda_0)] \) and \( \theta \in \mathbb{T} \cap \mathbb{Q}/\mathbb{Z} \), denote by \( \beta_v(\theta) \in \mathbb{R}/\mathbb{Z} \cong \partial \Delta \) the “landing point” of the arc \( \psi_v(\theta) \cap U_v \). That is, the closure of this arc intersects \( \partial \Delta \) at a point with argument \( \beta_v(\theta) \).

We claim that \( \beta_v \) and \( \alpha_v \) coincide in \( \mathbb{T} \cap \mathbb{Q}/\mathbb{Z} \). In fact, these maps at least coincide at one point, since \( \psi_v \) respects external markings. That is, if \( \alpha_v(\theta_0) = 0 \), then \( \beta_v(\theta_0) = 0 \). Now we observe that the set of arguments where \( \alpha_0 \) and \( \beta_0 \) coincide is forward invariant. More precisely, let \( A \subset \mathbb{T} \) be the largest set such that \( m^c_d(A) = \{0\} \) where \( \alpha_v(\theta_v) = \beta_v(\theta_v) = 0 \). Notice that \( \beta_{\psi_v^0} \circ m_d = m_{\psi_v^0} \circ \beta_v \) and \( \alpha_{\psi_v^0} \circ m_d = m_{\psi_v^0} \circ \alpha_v \). Therefore, \( \alpha_v(A) = \beta_v(A) \). Also, the cyclic order of the arguments in \( A \) is preserved by each of the maps \( \alpha_v \) and \( \beta_v \). Therefore, \( \alpha_v(\theta) = \beta_v(\theta) \), for all \( \theta \in A \). We may recursively repeat this argument in order to conclude that \( \alpha_v \) and \( \beta_v \) coincide at a set \( C \) such that \( \alpha_v(C) \) is dense. It follows that \( \alpha_v \) and \( \beta_v \) agree on \( \mathbb{T} \cap \mathbb{Q}/\mathbb{Z} \).

We conclude that \( \lambda_v(P) \supseteq \lambda_v \), by Lindelöf Theorem.

Now consider two distinct \( \lambda_v \)-classes \( A_1 \) and \( A_2 \) and denote by \( z_1 \) and \( z_2 \) the landing points, in \( K_f(v) \), of the rays with arguments in \( A_1 \) and \( A_2 \), respectively. Since \( \psi_v \) is injective, the landing points \( \psi_v(z_1) \) and \( \psi_v(z_2) \) of the external rays of \( P \) in the \( v \)-fiber, with arguments in \( \alpha_v(A_1) \) and \( \alpha_v(A_2) \) are distinct. Therefore, \( \lambda_v(P) = \lambda_v \). \( \square \)

6.2. Combinatorial Tuning. In the classical context of quadratic polynomials, “tuning” is the inverse of straightening. Below we will show that, in a combinatorial sense, it is possible to find such an inverse.

Let us first introduce the pull-back of a relation. For a map \( \varphi : A \to B \) and a relation \( \Lambda \) on a set \( B \), we say that \( a_1 \in A \) is \( \varphi^* \)-\( \Lambda \)-related to \( a_2 \in A \) if and only if \( \varphi(a_1) \) is \( \Lambda \)-related to \( \varphi(a_2) \).

**Definition 6.5.** Let \( \lambda_0 \) be a \( d \)-invariant lamination with internal angle system \( \alpha = (\alpha_v : \mathcal{V} \to \mathbb{R}/\mathbb{Z}) \). Also, let \( (\lambda_v)_{v \in T(\lambda_0)} \) be an invariant rational lamination over \( T(\lambda_0) \).

Given an infinite \( \lambda_0 \)-unlinked class \( \nu \), denote by \( n_\nu \geq 0 \) the smallest integer such that \( \nu^0 = m^\nu_d(\nu) \in [T(\lambda_0)] \). Let \( \pi_\nu = \alpha_\nu \circ m^\nu_d : \mathcal{V} \to \mathbb{R}/\mathbb{Z} \) and \( \lambda^\nu_\nu = \pi_\nu(\lambda_\nu) \).

We define the **combinatorial tuning** \( \mathcal{T}_\alpha(\lambda_0, (\lambda_v)) \) as the smallest equivalence relation in \( \mathbb{Q}/\mathbb{Z} \) containing:
(i) $\lambda_0$ and,
(ii) $\lambda'_v$, for all infinite $\lambda_0$-unlinked classes $v$.

**Theorem 6.6.** Let $\lambda_0$ be a $d$-invariant rational lamination, $\alpha$ be an internal angle system of $\lambda_0$ and $(\lambda_0)_{\in \mathcal{T}(\lambda_0)}$ be an invariant rational lamination over $T(\lambda_0)$. Then their combinatorial tuning $\lambda' = \mathcal{T}_\alpha(\lambda_0, (\lambda_v))$ is a $d$-invariant rational lamination containing $\lambda_0$. Moreover, the following statements hold:

(i) If $\lambda_0$ and $(\lambda_v)$ are hyperbolic, then $\lambda'$ is hyperbolic.
(ii) If $\lambda_0$ and $(\lambda_v)$ are post-critically finite, then $\lambda'$ is post-critically finite.
(iii) If $\lambda \in \mathcal{R}(\lambda_0)$ has rational lamination $\lambda'$, then the rational lamination of $\chi_{\lambda_0}(\lambda)$ is $(\lambda_v)$, where $\chi_{\lambda_0} : \mathcal{R}(\lambda_0) \to \mathcal{C}(T(\lambda_0))$ is the straightening map associated to the internal angle system $\alpha$.

**Proof.** Let $\mathcal{A}$ be the set of non-trivial $\lambda_0$-equivalence classes and let $\mathcal{B}$ be the union of the sets of non-trivial $\lambda'_v$-equivalence classes for all infinite $\lambda_0$-unlinked classes $v$. A non-trivial $\lambda'$-equivalence class $E$ is a maximal set of the form

$$E = \bigcup_{n=1}^N A_n$$

for $A_n \in \mathcal{A} \cup \mathcal{B}$ such that for any $n, m$, there exists a sequence $n_0 = n, n_1, \ldots, n_k = m$ such that $A_{n_j}$ and $A_{n_{j+1}}$ intersect for all $j = 1, \ldots, k - 1$. Finiteness of $E$ follows from the fact there exists $\ell$ and $p$ such that $m^{d}_{\ell}(\theta) = m^{d}_{\ell+p}(\theta)$ for all $\theta \in E$, since the same holds for $E \in \mathcal{A} \cup \mathcal{B}$. Thus it is clear that every $\lambda'$-equivalence class is finite.

Observe that if $A, B \in \mathcal{A} \cup \mathcal{B}$ are disjoint, then $A$ and $B$ are unlinked. Consider $A_1, A_2, B_1, B_2 \in \mathcal{A} \cup \mathcal{B}$ such that $A_1$ intersects $A_2$ and $B_1$ intersects $B_2$, and $A_1 \cup A_2$ and $B_1 \cup B_2$ are disjoint. Then since $A_j$ is unlinked with both $B_1$ and $B_2$, $B_1$ and $B_2$ are contained in the same component of $\mathbb{R} / \mathbb{Z} \setminus A_j$. Therefore, $A_1 \cup A_2$ and $B_1 \cup B_2$ are unlinked. We repeat this argument and conclude that $\lambda'$-equivalence classes are pairwise unlinked.

To see that $\lambda'$ is closed, consider a sequence of pairs $(\theta^*_n, \theta^*_n')$ such that $\theta^*_n \sim_{\lambda_0} \theta^*_n'$ and $\theta^*_n \to \theta^*_n, \theta^*_n \to \theta^*$ as $n \to \infty$. If $\theta^*_n \sim_{\lambda_0} \theta^*_n'$ for infinitely many $n$, then $\theta \sim_{\lambda_0} \theta^*$ because $\lambda_0$ is closed, so it follows that $\theta \sim_{\lambda_0} \theta^*$. If there exists an infinite $\lambda_0$-unlinked class $v$ such that $\theta^*_n, \theta^*_n' \in v$ for infinitely many $n$, then similarly we have $\theta \sim_{\lambda_0} \theta^*$ because $\lambda'_v$ is closed. Otherwise, by passing to a subsequence, we may assume that the convergences $\theta^*_n \to \theta$ and $\theta^*_n \to \theta^*$ are monotone, that $\theta^*_n$ and $\theta^*_n'$ lie in an infinite $\lambda_0$-unlinked class $v_n$, and that $v_n (n = 1, 2, \ldots)$ are pairwise disjoint. Then for each $n$, there exist $\ell_n \in (\theta^*_n, \theta^*_n')$, $\ell'_n \in (\theta^*_n, \theta^*_n')$ such that $\ell_n \sim_{\lambda_0} \ell'_n$ and that $v_n$ and $v_{n+1}$ lie in different components of $\mathbb{R} / \mathbb{Z} \setminus \{\ell_n, \ell'_n\}$. Therefore $\ell_n \sim_{\lambda_0} \ell'_n$ and $\ell'_n \setminus \theta^*$, thus $\theta \sim_{\lambda_0} \theta^*$.

To prove the $d$-invariance of $\lambda'$, we start with the following observation: Let $v$ be an infinite $\lambda_0$-unlinked class. Then $\pi_v : \overline{\mathcal{T}} \rightarrow \mathbb{R} / \mathbb{Z}$ induces a homeomorphism $\overline{\mathcal{T}} / \lambda_0 \rightarrow \mathbb{R} / \mathbb{Z}$. Furthermore, $\pi_{\mathcal{M}(v)} \circ m_d \circ \pi^{-1}_v : \mathbb{R} / \mathbb{Z} \to \mathbb{R} / \mathbb{Z}$ is well-defined equal to $m_{\mathcal{M}(v)}$, and $\delta(v), \lambda'_v = \lambda_{\alpha'}$, where $v' = m^{d^*}_{\ell}(v), w = m_d(v)$ and $w' = m^d_{\ell'}(w)$. Therefore, a $\lambda'_v$-class (resp. $\lambda'_v$-unlinked class) $A \subset \overline{\mathcal{T}}$ is mapped by $m_d$ to a $\lambda'_v$-class (resp. $\lambda'_v$-unlinked class) and it is one-to-one if $v$ is not critical, and $\delta(\alpha(v))-to-one if $v$ is critical.

Let $E$ be a $\lambda'$-class of the form $A$. Then its image

$$m_d(E) = \bigcup_{n=1}^N m_d(A_n)$$

is contained in some $\lambda'$-class $E'$. If $E' \setminus m_d(E)$ is nonempty, then there exist some $A' \in \mathcal{A} \cup \mathcal{B}$ intersecting both $E' \setminus m_d(E)$ and $m_d(E)$. If $A' \in \mathcal{A}$ then there exists a $\lambda_0$-class
A such that \( m_d(A) = A' \) and \( A \) intersects \( E \). Moreover, \( A \) is not contained in \( E \) because \( m_d(A) \neq m_d(E) \). This contradicts that \( E \) is maximal. Similarly, if \( A' \) is a \( \lambda' \)-class for an infinite \( \lambda_0 \)-unlinked class \( \nu' \), then there exist an infinite \( \lambda_0 \)-unlinked class \( \nu \) and a \( \lambda_0 \)-class \( A \) such that \( m_d(A) = A' \) and \( A \) intersects \( E \) but not contained in \( E \), that is a contradiction. Therefore, \( m_d(E) = E' \) is a \( \lambda' \)-equivalence class.

Now we show that \( m_d : E \to m_d(E) \) is consecutive preserving. First, consider a \( \lambda' \)-unlinked class \( L \). Then either \( L \) is a finite \( \lambda_0 \)-unlinked class, or \( L \) is contained in an infinite \( \lambda_0 \)-unlinked class. If \( L \) is a finite \( \lambda_0 \)-unlinked class, then \( m_d(L) \) is also a finite \( \lambda_0 \)-unlinked class, which is again a \( \lambda' \)-unlinked class. On the contrary, if \( L \) is contained in an infinite \( \lambda_0 \)-unlinked class \( v \), then \( L \) is a \( \lambda' \)-unlinked class. By construction, \( m_d(L) \) is a \( \lambda' \)-unlinked class, which is also a \( \lambda' \)-unlinked class.

Let \( (\theta_1, \theta_2) \) be a component of \( \mathbb{R}/\mathbb{Z} \setminus E \). Then there exist \( \theta_1' \in (\theta_1, \theta_2) \) arbitrarily close to \( \theta_1 \) such that \( \theta_1' \) and \( \theta_2' \) are \( \lambda' \)-unlinked. In fact, if otherwise, there exists some \( e > 0 \) such that any \( \theta_1' \in (\theta_1, \theta_1 + e) \) and \( \theta_2' \in (\theta_2 - e, \theta_2) \) are not \( \lambda' \)-unlinked. Hence there exist sequence \( t_n \in (\theta_1, \theta_1 + 1/n) \) and \( s_n \in (\theta_1 + e, \theta_2 - e) \) such that \( t_n \) and \( s_n \) are \( \lambda' \)-equivalent. By passing to a subsequence, we may assume \( s_n \) converges to some \( s \in (\theta_1, \theta_2) \). Since we already proved that \( \lambda' \) is closed, it follows that \( \theta_1 \) and \( s \) are \( \lambda' \)-equivalent and \( s \in E \), that is a contradiction.

Thus for any \( \theta \in (d\theta_1, d\theta_2) \), there exists some \( \theta_1', \theta_2' \in (\theta_1, \theta_2) \) such that \( \theta_1' \) and \( \theta_2' \) are \( \lambda' \)-unlinked and \( (\theta_1', \theta_2') \) and \( (\theta_1, \theta_2) \) are connected (not unlinked) for any \( \theta \in m_d^{-1}(\delta(\theta)) \cap (\theta_1, \theta_2) \).

Since \( d\theta_1' \) and \( d\theta_2' \) are also \( \lambda' \)-unlinked, \( \theta \) is not \( \lambda' \)-equivalent to \( d\theta_1 \). Therefore \( (d\theta_1, d\theta_2) \) is a component of \( \mathbb{R}/\mathbb{Z} \setminus m_d(E) \) and we have proved \( m_d : E \to m_d(E) \) is consecutive preserving.

It follows that, \( \lambda' \) is \( d \)-invariant.

For (i), let us further assume that \( \lambda_0 \) and \( (\lambda_\nu) \) are post-critically finite. Let \( \nu' \) be a finite \( \lambda' \)-unlinked class. Then \( \nu' \) is either a finite \( \lambda_0 \)-unlinked class, or a finite \( \lambda' \)-unlinked class for some infinite \( \lambda_0 \)-unlinked class \( v \). Hence \( \nu' \) is not critical by assumption. Therefore, \( \lambda' \) is post-critically finite.

For (ii), if \( \lambda_0 \) and \( (\lambda_\nu) \) are hyperbolic, then for each Fatou critical element \( \nu' \) of \( \lambda_\nu \), \( \nu' = \alpha_{\nu'}^{-1}(\nu') \) is a critical element of \( \lambda \) with \( \delta(\nu') = \delta(\nu') \). Also, since

\[
\sum_{\nu \in \text{Crit}(\lambda_\nu)} (\delta(\nu') - 1) = d - 1,
\]

all the critical elements of \( \lambda' \) are infinite \( \lambda' \)-unlinked classes. Therefore, \( \lambda' \) is also hyperbolic.

For (iii) assume that \( f \in \mathcal{R}(\lambda_0) \) is such that \( \lambda_f = \lambda' \). Then \( \alpha_{\nu'}^{-1}(\lambda_f) \nu = \lambda' \nu \) for \( \nu \in \mathcal{T}(\lambda_0) \) by Lemma 6.4. hence it suffices to show that \( \lambda'_\nu = \lambda'_\nu \). Since we have \( \lambda'_\nu \subset \lambda' \nu \) by definition, we need only show the following: Let \( \mathcal{E} \) be a \( \lambda' \)-class of the form \( (\mathcal{E}) \) and if \( A_1 \) intersects \( \nu \), then \( A_2, \ldots, A_N \) do not intersect \( \nu \).

For \( n \geq 2 \), take a sequence \( n_0 = 1, n_1, \ldots, n_K = n \) such that \( A_{n_k} \) intersects \( A_{n_{k+1}} \) for \( k = 0, \ldots, K - 1 \). We may assume \( A_{n_k} \neq A_{n_{k'}} \) if \( k \neq k' \). Furthermore, we may assume

\[
(5) \quad \#(A_{n_k} \cap A_{n_{k+1}}) = 2 \text{ for any } k.
\]

In fact, at least one of \( A_{n_k} \) and \( A_{n_{k+1}} \), say \( A_{n_k} \), is a \( \lambda' \nu \)-class for some \( \nu' \). Then \( A_{n_k} \) is not a \( \lambda' \nu \)-class, so it is a \( \lambda \)-class or a \( \lambda' \nu \)-class for some \( \nu'' \) whose boundary \( \partial \nu'' \) intersects \( \partial \nu' \). If \( A_{n_k} \) is a \( \lambda \)-class, then \( A_{n_k} \cap \partial \nu \) contains exactly two points and \( A_{n_{k+1}} \) is contained in \( A_{n_k} \). If \( A_{n_k} \) is a \( \lambda' \nu \)-class, then \( \partial \nu' \cap \partial \nu'' \) is contained in a \( \lambda \)-class \( A \), so we may add \( A \) in the sequence between \( A_{n_k} \) and \( A_{n_{k+1}} \) to satisfy the condition (5).
Since $A_{n_k}$ is a $\lambda_r$-class, $A_{n_k}$ is not. In particular, $A_{n_1}$ is contained in a component of $\mathbb{R}/\mathbb{Z} \setminus v$.

For $k \geq 1$, assume $A_{n_k}$ is contained in a component of $\mathbb{R}/\mathbb{Z} \setminus v$ and $A_{n_{k-1}}$ is contained in a component of $\mathbb{R}/\mathbb{Z} \setminus A_{n_k}$ containing $v$. Then $A_{n_{k-1}}$ lies in a component of $\mathbb{R}/\mathbb{Z} \setminus A_{n_k}$ different from the component containing $A_{n_{k-1}}$. Since otherwise we have $#(A_{n_{k-1}} \cap A_{n_k} \cap A_{n_{k-1}}) \geq 2$, which implies that two of them are equal. In particular, $A_{n_k}$ is contained in a component of $\mathbb{R}/\mathbb{Z} \setminus A_{n_{k-1}}$ containing $A_{n_{k-1}}$ and $v$. Therefore, by induction, $A_a = A_{n_k}$ is contained in a component of $\mathbb{R}/\mathbb{Z} \setminus v$ which contains $A_{n_1}$, that is, $A_a \cap v = \emptyset$. Therefore, the rational lamination of $\chi_{A_0}(f)$ is $(\lambda_v)$.

□

6.3. Renormalizable set. Our aim now is to characterize laminations with non-empty renormalizable set.

**Proposition 6.7.** Consider a $d$-invariant rational lamination such that $T(\lambda_0) \neq \emptyset$. Then the following statements are equivalent:

(i) $\mathcal{R}(\lambda_0) \neq \emptyset$.

(ii) If $A$ is a critical $\lambda_0$-class and $L$ is an infinite unlinked class in the forward orbit of a critical element, then $A \cap \overline{L} = \emptyset$.

Before proving the above result let us introduce an important class of rational laminations.

**Definition 6.8** (Primitive invariant laminations). We say a $d$-invariant rational lamination is **primitive** if there do not exist infinite $\lambda$-unlinked classes $L$ and $L'$ and $\lambda$-class $A$ such that $L \neq L'$ and both $\overline{L} \cap A$ and $\overline{L'} \cap A$ are nonempty.

As a direct consequence of the proposition we obtain the following result.

**Corollary 6.9.** If $\lambda_0$ is a primitive rational lamination and $|T(\lambda_0)|$, then $\mathcal{R}(\lambda_0) \neq \emptyset$.

To prove the proposition, we start with the lemma below, which gives sufficient conditions for a polynomial $f \in \mathcal{C}(\lambda_0)$ to be in $\mathcal{R}(\lambda_0)$. Then the reader may find the proof of the proposition after the proof of the lemma.

**Lemma 6.10.** Consider an invariant rational lamination $\lambda_0$ such that the following holds: if $A$ is a critical $\lambda_0$-class and $L$ is an infinite unlinked class in the forward orbit of a critical element, then $A \cap \overline{L} = \emptyset$. Let $\Lambda = (\Lambda_k)$ be a combinatorial Yoccoz puzzle which is a generator of $\lambda_0$.

There exists $K = K(\lambda_0) > 0$ such that if $f \in \mathcal{C}(\lambda_0)$ is such that,

- for all $\theta \in \supp \Lambda_0$, the external ray $R_f(\theta)$ lands at a repelling periodic point, and
- for all $\theta \in \supp \Lambda_K$, the landing point of the external ray $R_f(\theta)$ is not a critical point,

then $f \in \mathcal{R}(\lambda_0)$.

By this lemma, we conclude the existence of $X$ in Theorem 6.5.

**Corollary 6.11.** Let $\lambda_0$ be an invariant rational lamination. Then there exists an algebraic set $X \subset \text{Poly}(d)$ of pure codimension one such that $\mathcal{R}(\lambda_0) \supset \mathcal{C}(\lambda_0) \setminus X$.

**Proof.** This is a direct consequence of Lemma 6.10 if $\lambda_0$ satisfies the assumption.

Otherwise $\mathcal{C}(\lambda_0)$ is contained in a proper algebraic subset $X$ of $\text{Poly}(d)$ because every $f \in \mathcal{C}(\lambda_0)$ has a preperiodic critical point of preperiod and eventual period depending only on $\lambda_0$. (Indeed we have $\mathcal{R}(\lambda_0) = \mathcal{C}(\lambda_0) \setminus X = \emptyset$ by Proposition 6.7). □
Proof of Lemma 6.10. We claim that under the hypothesis of the lemma \( K_f(v) = \bigcap P_k(f,v) \). Directly from the definitions it follows that \( K_f(v) \subset \bigcap P_k(f,v) \). Now if \( z \notin K_f(v) \), then there exists a sector \( S \) bounded by \( \lambda_0 \)-equivalent rays with arguments in \( \mathcal{T} \) that is disjoint from \( K_f(v) \) and such that \( z \in S \). Since \( A \) is a generator, there exists \( k \) and a sector \( S' \) bounded by \( \lambda_k \)-equivalent arguments such that \( z \in S' \subset S \) by Proposition 4.9. Thus, \( z \notin P_k(f,v) \).

Now in order to show that \( f \in \mathcal{R}(\lambda_0) \) we apply the thickening procedure (cf. [M2]).

More precisely, let \( g \in \mathcal{R}(\lambda_0) \) in \([8]\), there exists a polynomial-like map \( \tilde{g} \) over \( \Lambda_0 \) which intersects \( \mathcal{L}_1(v) \), where \( L_k(v) \in \mathcal{L}_1(\lambda) \) is the combinatorial puzzle piece of depth \( k \) containing \( v \). Then for all \( v \) in the forward element of a Fatou critical element of \( \lambda_0 \), and all \( k > k_0 \), if the landing point of \( R_f(\theta) \) for every \( \theta \in \partial \mathcal{L}_1(v) \) is neither parabolic nor critical, then the thickening procedure gives Jordan domains \( \tilde{P}_k(f,v) \) such that:

\[
\begin{align*}
(i) & \quad P_k(f,v) \subset \tilde{P}_k(f,v) \subset \tilde{P}_{k-l}(f,v), \\
(ii) & \quad f : \tilde{P}_k(f,v) \to \tilde{P}_{k-l}(f,v) \text{ is a proper map of degree } \delta_k(v) \text{ (compare Proposition 3.7(iv)(e))}, \\
(iii) & \quad \bigcap P_k(f,v) = \bigcap \tilde{P}_k(f,v).
\end{align*}
\]

It follows that if we take \( U'_v = \tilde{P}_{k_\ell}'(f,v) \) and \( U_v = \tilde{P}_{k_\ell}(f,v) \) for all \( v \in |T(\lambda_0)| \), then \( g = (f^{\ell_\ell} : U'_v \to U_{\sigma_\ell}(v)) \) is the desired polynomial-like map over \( T(\lambda_0) \) with the appropriate fiberwise connected Julia set. Therefore if \( f \in \mathcal{C}(\lambda_0) \) satisfies the assumption for \( \lambda_0 \),

\[ K = k_0 + \max_{v \in |T(\lambda_0)|} \ell_v, \]

then \( f \in \mathcal{R}(\lambda_0) \). \( \square \)

Proof of Proposition 6.7. (i) \( \implies \) (ii) Assume that \( f \in \mathcal{R}(\lambda_0) \). Then there exists a polynomial-like map \( g \) over \( T(\lambda_0) \) such that \( K(g,v) = K_f(v) \) for all \( v \in |T(\lambda_0)| \). For each \( v \in |T(\lambda_0)| \), there exist a neighborhoods \( U'_v \subset U_v \) such that \( f^{\ell_\ell} : U'_v \to U_{\sigma_\ell}(v) \) is a proper map of degree \( \delta_k(v) \). In order to conclude that (ii) holds, we proceed by contradiction and suppose that there exists a critical \( \lambda_0 \)-class \( A \) such that \( A \cap m^\ell_\ell(\mathcal{T}) \neq \emptyset \) for some \( 0 \leq \ell < \ell_\ell \). Note that there are exactly \( \delta_k(v) \) \( \lambda_0 \)-classes \( B_1, \ldots, B_{\delta_k(v)} \) intersecting \( \mathcal{T} \) that map into the \( \lambda_0 \)-class \( m^\ell_\ell(\mathcal{T}) \). Let \( A_1, \ldots, A_{\delta_k(v)} \) be the corresponding \( \lambda_f \)-classes and \( t \in m^\ell_\ell(\mathcal{T}) \). It follows that the set

\[ \{ t' \in A_1 \cup \cdots \cup A_{\delta_k(v)} : m^\ell_\ell(t') = t \} \]

has cardinality strictly greater than \( \delta_k(v) \), since \( m^\ell_\ell(\mathcal{T}) = A \) for at least one \( j \). Therefore, the neighborhood \( U'_v \) of \( K_f(v) \) contains the portion inside \( D_f(r) \), for \( r \) sufficiently small, of at least \( \delta_k(v) + 1 \) rays which map onto the same ray under \( f^{\ell_\ell} \). Hence \( f^{\ell_\ell} : U'_v \to U_{\sigma_\ell}(v) \) is not a degree \( \delta_k(v) \) map.

(ii) \( \implies \) (i) Assume that (ii) holds, according to Lemma 7.2, Corollary 7.3 and the proof of Theorem 1.1 in [8], there exists a polynomial \( f \in \mathcal{C}(\lambda) \) with rational lamination \( \lambda_f = \lambda_0 \) and no neutral cycles. Hence, we may apply the previous lemma to a generator of \( \lambda_0 \) given by Lemma 4.10. \( \square \)

6.4. Proof of Theorem 6.3. Denote by \( \alpha \) the internal angle system. By Theorem 6.6 and Lemma 7.2, Corollary 7.3 and the proof of Theorem 1.1 in [8], there exists \( f \in \mathcal{C}(\lambda_0) \) with no neutral cycles and rational lamination \( \lambda_f = \lambda_0 \) such that every Fatou critical point has finite forward orbit. By Proposition 6.7 and Lemma 6.10 it follows that there
exists an algebraic set $Y \subset \text{Poly}(T(\lambda_0))$ of pure codimension one such that $f \in \mathcal{R}(\lambda_0)$, except when $P \in Y$. More precisely, the assumption of Lemma 6.10 does not hold if and only if the landing point of $R_P(v, \alpha_\theta(\theta))$ is a critical point for some $v \in \text{CO}(\lambda_0)$ and $\theta \in \text{supp} \Lambda_K \cap \partial v$, where $K$ is in Lemma 6.10. In particular, $P$ has a critical point in the Julia set, i.e., $P$ is not hyperbolic. Except for that case, from Theorem 6.6 we conclude that $\lambda_{\chi(f)} = \lambda_\theta$ for all $\chi \in [T(\lambda_0)]$.

Now if $P \in \mathcal{C}(T(\lambda_0))$ is post-critically finite, then the standard pull-back argument shows that $P$ is uniquely determined by its rational lamination. Taking $f$ as in the previous paragraph, such that $\lambda_{\chi(f)} = \lambda_\theta$ for all $\chi \in [T(\lambda_0)]$.

\section{Proof of Theorem C}

Let $\lambda_0$ be a post-critically finite rational lamination. Consider a component $\mathcal{H}$ of $\text{Hyp} \mathcal{C}(T(\lambda_0))$. Our aim now is to show that the image of $\chi_{\lambda_0}$ contains $\mathcal{H}$.

Consider the unique post-critically finite polynomial map $P_0 \in \mathcal{H}$ [Mi4 Corollary 4.2]. Since $P_0$ is hyperbolic, by Theorem 6.3 there exists $f_0 \in \mathcal{R}(\lambda_0)$ such that $\chi_{\lambda_0}(f_0) = P_0$. Note that $\mathcal{R}(\lambda_0) \subset \mathcal{C}(\lambda_0)$ is contained in an algebraic set $Y$ of $\text{Poly}(d)$. Moreover, the Julia set of $f_0$ over its Julia set is structurally stable in $Y$. Call $\mathcal{H}$' the component of structurally stable maps in $Y$ containing $f_0$. We claim that $\chi_{\lambda_0}(\mathcal{H}') = \mathcal{H}$.

In order to prove that $\chi_{\lambda_0}(\mathcal{H}') = \mathcal{H}$ we will employ Milnor's parametrization of $\mathcal{H}$. Let $T_0 = (|T_0|, \sigma_0, \delta_0)$ be the reduced mapping schema of $P_0$. Namely, $|T_0|$ is the set formed by the critical points of $P_0$, $\sigma_0 : |T_0| \to |T_0|$ is the first return map under iterations of $P_0$, and $\delta_0 : |T_0| \to \mathbb{N}$ is the local degree of $P_0$ at its critical points. Consider the space $B(T_0)$ of all proper holomorphic maps

$$\beta : |T_0| \times \Delta \to |T_0| \times \Delta$$

of the form $\beta(v, z) = (\sigma_0(v), \beta_v(z))$ where $\beta_v : \Delta \to \Delta$ is of degree $\delta_0(v)$ and the following hold:

- $\beta_v$ is boundary-rooted, i.e., the extension of $\beta_v$ to $S^1 = \partial \Delta$ satisfies $\beta_v(1) = 1$;
- if $v$ is periodic under $\sigma_0$, then $\beta_v$ is fixed point centered, i.e., $\beta_v(0) = 0$;
- if $v$ is not periodic, then $\beta_v$ is critically centered, i.e., the sum of its $\delta(v) - 1$ critical points (counted with multiplicity) is equal to zero.

According to Milnor [Mi4 Lemma 3.5] $B(T_0)$ is a topological cell. Now a map $\Phi : \mathcal{H} \to B(T_0)$ is introduced as follows. Given $P \in \mathcal{H}$, appropriate choices for the Riemann mappings, of the critical bounded Fatou components of $P$, “conjugate” the action of the first return map to that of an element $\Phi(P)$ of $B(T_0)$. Milnor showed that $\Phi : \mathcal{H} \to B(T_0)$ is a diffeomorphism [Mi4 Theorem 4.1’]. Using the fact that hybrid conjugacies between the $\lambda_0$-renormalization of $f \in \mathcal{H}'$ and the polynomial maps $\chi(f) \in \mathcal{H}$ restrict to conformal isomorphisms of bounded Fatou components, the standard quasiconformal surgery argument shows that $\Phi \circ \chi : \mathcal{H}' \to B(T_0)$ is onto. Thus, $\chi : \mathcal{H}' \to \mathcal{H}$ is also surjective and, therefore, a bijection.

Since we have already shown that $\chi_{\lambda_0} : \mathcal{H}' \to \mathcal{H}$ is a continuous bijection, along the same lines of the proof of [Mi4 Theorem 5.1], it follows that $\chi_{\lambda_0} : \mathcal{H}' \to \mathcal{H}$ is in fact biholomorphic.

\section{Compactness}

The main result of this section is to establish the following compactness result which generalizes [In3 Theorem 1.3].

\begin{theorem}
If $\lambda_0$ is a primitive invariant rational lamination, then $\mathcal{C}(\lambda_0) = \mathcal{R}(\lambda_0)$ and this set is compact.
\end{theorem}
The reader may find the proofs of Theorem 7.1 and Theorem 7.2 after the statements and proofs of the following two lemmas.

**Lemma 7.2.** Let \( \lambda_0 \) be a non-trivial invariant rational lamination such that the support of \( \lambda_0 \) is contained in the \( m_d \)-grand orbit of a finite set. Then \( \lambda_0 \) is not primitive.

**Proof.** By passing to an iterate, we may assume that the support of \( \lambda_0 \) is contained in the grand orbit of the finite set \( F = \{ j/(d - 1); j = 0, \ldots, d - 2 \} \). That is, \( t \in F \) if and only if \( m_d(t) = t \). We proceed by contradiction and assume that \( \lambda_0 \) is primitive.

Let \( A_1, \ldots, A_n \) be a complete list (without repetitions) of the \( \lambda_0 \)-classes contained in \( F \).

Let \( f = \) be a polynomial without neutral fixed points such that \( \lambda_f = \lambda_0 \). Let \( m \) be the number of non-trivial classes in \( F \) and observe that \( n + m \leq d - 1 \).

The union of the closure of the external rays with arguments in \( F \) cuts the complex plane into \( d - n \) regions \( U_1, \ldots, U_{d-n} \), since

\[
    d - n = 1 + \sum_{j=1}^{n} (|A_j| - 1).
\]

According to Goldberg and Milnor [GM], for all \( j \), there exists exactly one fixed point \( z_j \) in the region \( U_j \). These fixed points are not the landing point of rational rays since the lamination is supported in the grand orbit of \( F \). Therefore, \( z_j \) is an attracting fixed point, for all \( j \). For each \( j = 1, \ldots, d - n \), call \( L_j \) the fixed infinite \( \lambda_0 \)-unlinked class such that \( z_j \in K_f(L_j) \). By Proposition 3.1 (iv)(c) it follows that the boundary of \( L_j \) intersects a non-trivial class contained in \( F \), say \( A(L_j) \). Hence, for some \( j \neq k \), we have that \( A(L_j) = A(L_k) \). For otherwise, \( d - n \leq m \), but \( n + m \leq d - 1 \). Therefore, \( \lambda_0 \) is not primitive which gives us the desired contradiction.

**Lemma 7.3.** Consider a non-trivial primitive \( d \)-invariant rational lamination \( \lambda_0 \). Let \( E \) be the \( m_d \)-grand orbit of a finite set and \( \lambda = \lambda_0 \cap ((\mathbb{Q}/\mathbb{Z} \setminus E) \times (\mathbb{Q}/\mathbb{Z} \setminus E)) \). Then \( \lambda_0 \) is the smallest closed equivalence relation in \( \mathbb{Q}/\mathbb{Z} \) containing \( \lambda \).

**Proof.** Let \( \lambda' \) be the smallest closed equivalence relation in \( \mathbb{Q}/\mathbb{Z} \) containing \( \lambda \). It follows that \( \lambda' \) is a \( d \)-invariant rational lamination and \( \lambda' \subset \lambda_0 \).

Let \( L_0 \) be an infinite periodic \( \lambda' \)-unlinked class, say of period \( p \). We claim that \( L_0 \) is a \( \lambda_0 \)-unlinked class. For this, consider \( \pi : \mathbb{T}_0 \to \mathbb{R}/\mathbb{Z} \) such that \( \pi \circ m_d^p = m_d \circ \pi \) for some \( d' \geq 2 \).

Let \( \pi, \lambda_0 \) be the equivalence relation in \( \mathbb{Q}/\mathbb{Z} \) that identifies \( \theta \) and \( \theta' \) if and only if there exists \( t \in \pi^{-1}(\theta) \) and \( s \in \pi^{-1}(\theta') \) such that \( s \) and \( t \) are \( \lambda_0 \)-equivalent arguments. Observe that \( \pi, \lambda_0 \) is a \( d' \)-invariant rational lamination with support contained in \( \pi(E \cap \mathbb{T}_0) \). By the previous lemma, \( \pi, \lambda_0 \) is trivial or it is not primitive. The latter alternative is impossible, since taking the preimage of appropriate \( \pi, \lambda_0 \)-unlinked classes we would have that \( \lambda_0 \) is not primitive. Therefore, \( \pi, \lambda_0 \) is trivial. Hence, \( L_0 \) is a \( \lambda_0 \)-unlinked class.

Now let \( L_0 \) be a strictly preperiodic infinite \( \lambda' \)-unlinked class. We also claim that \( L_0 \) is a \( \lambda_0 \)-class. In fact, let \( \ell \geq 1 \) be such that \( m_d^\ell(L_0) \) is a periodic \( \lambda_0 \)-unlinked class. If \( \pi : \mathbb{T}_0 \to \mathbb{R}/\mathbb{Z} \) is such that \( \pi \circ m_d^\ell = m_d \circ \pi \) for some \( d' \geq 1 \), then \( m_d(\pi, \lambda_0) \) is the trivial rational lamination. In particular, every non-trivial \( \pi, \lambda_0 \)-class is mapped, under \( m_d \) onto a singleton. Since these classes are unlinked, there are at most \( d' - 1 \) non-trivial \( \pi, \lambda_0 \)-classes. Therefore, either \( \pi, \lambda_0 \) is trivial and \( L_0 \) is a \( \lambda_0 \)-unlinked class, or \( \lambda_0 \) is not primitive. Thus, \( L_0 \) is a \( \lambda_0 \)-unlinked class.

Now to prove that \( \lambda' = \lambda_0 \) just observe that otherwise we would have an infinite \( \lambda' \)-unlinked class that is not a \( \lambda_0 \)-unlinked class. 
\[\square\]
With the previous lemma we may now establish our compactness result.

**Proof of Theorem 7.1.** Assume that \( f \in C(\lambda_0) \). In order to show that \( C(\lambda_0) = \mathcal{R}(\lambda_0) \) we must prove that \( f \) is \( \lambda_0 \)-renormalizable (i.e., \( f \in \mathcal{R}(\lambda_0) \)). More precisely, we must extract a polynomial-like map \( g \) over \( T(\lambda_0) \) (see Definition 3.12). To extract a polynomial-like map from \( f \) we will apply the thickening procedure (Lemma 6.10) to an appropriate Yoccoz puzzle.

We start by finding an appropriate combinatorial Yoccoz puzzle. Let \( F \subset \mathbb{Q}/\mathbb{Z} \) be the set formed by the arguments of external rays of \( f \) landing at parabolic or critical points. Let \( m \) be the minimum common multiple of the periods, under \( m_d \), of the arguments in the forward orbit of \( F \). Denote by \( F' \subset \mathbb{Q}/\mathbb{Z} \) the set of all \( m_d \)-periodic arguments of period dividing \( m \) and, by \( E' \) the grand orbit of \( F' \). From the previous lemma, there exist \( \lambda_0 \)-classes \( A_1, \ldots, A_j \) contained in \( \mathbb{Q}/\mathbb{Z} \setminus E' \) that separate the critical elements of \( \lambda_0 \). The support of the depth 0 puzzle will be \( E_0 \subset \mathbb{Q}/\mathbb{Z} \), the set formed by all periodic arguments in the forward orbit of \( A_1 \cup \cdots \cup A_j \). More precisely, let \( \lambda \) be the restriction of \( \lambda_0 \) to \( E_0 \times E_0 \). For all \( \ell \geq 1 \), let \( E_\ell = m^{-\ell}_d(E_0) \) and define \( \Lambda_\ell \) as the restriction of \( \lambda_0 \) to \( E_\ell \times E_\ell \). It follows that \( \Lambda = (\Lambda_\ell) \) is a combinatorial Yoccoz puzzle admissible for \( \lambda_0 \). Note that, for some \( k \geq 0 \), we have that \( E_k \supset A_1 \cup \cdots \cup A_j \). In particular, \( \Lambda \) is a generator and such a number \( k \) is a separation depth for \( \lambda_0 \).

Since \( E_\ell \cap E' = \emptyset \), for all \( \theta \in E_\ell \), the landing points of the external ray of angle \( \theta \) is a non-critical eventually repelling periodic point. By Lemma 6.10 we have that \( f \in \mathcal{R}(\lambda_0) \).

Now we show that \( C(\lambda_0) \) is compact. Due to the compactness of \( C(d) \), we only have to show that \( C(\lambda_0) \) is closed in \( C(d) \). For this consider a sequence \( (f_n) \) of polynomials in \( C(\lambda_0) \) which converges to \( f \in C(d) \). Let \( F \subset \mathbb{Q}/\mathbb{Z} \) be the arguments of the rays landing at parabolic periodic points of \( f \) or at critical points of \( f \). Denote by \( E \) the \( m_d \)-grand orbit of \( F \).

We claim that

\[
\lambda_0 \cap ((\mathbb{Q}/\mathbb{Z} \setminus E) \times (\mathbb{Q}/\mathbb{Z} \setminus E)) \subset \lambda_f \cap ((\mathbb{Q}/\mathbb{Z} \setminus E) \times (\mathbb{Q}/\mathbb{Z} \setminus E)).
\]

Assume that two arguments \( \theta \) and \( \theta' \) in \( \mathbb{Q}/\mathbb{Z} \setminus E \) are \( \lambda_0 \)-equivalent. Since the landing points of the external rays with arguments \( \theta \) and \( \theta' \) are repelling for \( f \), they depend continuously on the polynomial in a neighborhood of \( f \). Hence for \( n \) sufficiently large, \( \theta \) and \( \theta' \) are \( \lambda_{f_n} \)-equivalent if and only if they are \( \lambda_f \)-equivalent. Since \( \lambda_{f_n} \supset \lambda_0 \), we have proved that \( \theta \) and \( \theta' \) are \( \lambda_f \)-equivalent.

From the previous lemma and the fact that \( \lambda_f \) is closed in \( \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \) we conclude that \( \lambda_f \supset \lambda_0 \). Hence, \( f \in C(\lambda_0) \).

**Proof of Theorem 7** Assume that \( \lambda_0 \) is post-critically finite but not primitive. From the previously proven theorem, it is sufficient to show that: \( \mathcal{R}(\lambda_0) \) is not compact and \( C(\lambda_0) \neq \mathcal{R}(\lambda_0) \).

Let \( f_0 \in \mathcal{R}(\lambda_0) \) be hyperbolic with \( \lambda_{f_0} = \lambda_0 \). Since \( \lambda_0 \) is not primitive, there exist two periodic Fatou components \( \Omega_1, \Omega_2 \) such that \( \partial \Omega_1 \cap \partial \Omega_2 = \{x_0\} \) where \( x_0 \) is a repelling periodic point of period dividing one of those of these two periodic Fatou components, say \( \Omega_1 \). We may assume that \( f_0 \) has an attracting (not superattracting) periodic point \( y_0 \) in \( \Omega_1 \). By Haïssinsky’s pinching theorem [Ha1] [Ha4], there exist a continuous path of quasiconformal deformations of polynomials \( (f_t)_{t \in [0,1]} \) such that

(i) \( f_t \) converges uniformly to a polynomial \( f_1 \), shrinking progressively some path \( \gamma \) connecting \( x_0 \) and \( y_0 \). There exists a semiconjugacy \( \phi : \mathbb{C} \to \mathbb{C} \) from \( f_0 \) to \( f_1 \) such
that \( \varphi \) is a homeomorphism outside \( \gamma \) and its preimages, and \( \varphi \) sends \( \gamma \) to a point, as well as each component of the preimages of \( \gamma \).

(ii) The point \( x_1 = \varphi(\gamma) \) is a parabolic periodic point \( f_1 \) and its immediate basin is equal to \( \varphi(\Omega_1 \setminus \Gamma) \) where \( \Gamma \) is the union of \( \gamma \) and all the preimages.

In particular, \( f_0 \) and \( f_1 \) are topologically conjugate on their Julia sets, by \( \varphi \). Therefore, \( f_{t} \in \mathcal{R}(\lambda_0) \) for \( t \in [0,1) \) but \( f_1 \in \mathcal{C}(\lambda_0) \) and \( f_1 \notin \mathcal{R}(\lambda_0) \). Otherwise, a \( \lambda_0 \)-renormalization \( g \) of \( f_1 \) would satisfy \( \{x_1\} = K(g, v_1) \cap K(g, v_2) \) where \( v_i \) is the infinite \( \lambda_0 \)-unlinked class such that \( K(f_0, v_i) = \Omega_i \). Therefore, \( x_1 \) would be repelling ([Mc1, Theorem 7.3] and [In1, Proposition 3.4]) which is a contradiction with (ii).

\[ \square \]

Remark 7.4. For postcritically finite invariant rational laminations \( \lambda_0 \), the remaining problem is to characterize when \( \mathcal{C}(\lambda_0) \) is compact. Note that \( \mathcal{C}(\lambda_0) \) is known to be compact in some cases and non-compact in some other cases. For example, if \( \lambda_0 \) is of degree two, then \( \mathcal{C}(\lambda_0) \) is always compact (it is a baby Mandelbrot set).

On the other hand, the following example shows that \( \mathcal{C}(\lambda_0) \) is not compact in general.

Example 7.5. Let \( \lambda_0 \) be a rational lamination of \( f_0(z) = z^3 + \frac{3}{2}z \). The critical points \( \pm i \sqrt{2} \) of \( f_0 \) are fixed and the origin lies in the boundaries of their immediate basin of attraction. Since \( f_0 \) is real and \( K(f_0) \cap \mathbb{R} = \{0\} \), the external rays of angles 0 and \( 1/2 \) are the positive and negative real line respectively, so 0 and \( \frac{1}{2} \) are \( \lambda_0 \)-equivalent.

For \( \mu = \zeta + i\xi \neq 1 \), let

\[
    f_\mu(z) = z^3 - \frac{2\xi}{\sqrt{2(1-\zeta)}}z^2 - \frac{1}{4}\left(2\zeta - 6 + \frac{2\xi^2}{\zeta - 1}\right)z.
\]

(Note that for \( \mu = 0 \), \( f_\mu \) is equal to the previously defined \( f_0 \).) Then the fixed points of \( f_\mu \)

\[ \text{Figure 10. The Julia sets of } f_\mu, \text{ and its limit.} \]

are 0, \( \alpha = \frac{\zeta}{\sqrt{2(1-\zeta)}} + \frac{1}{\sqrt{1-\zeta^2}}i \) and \( \tilde{\alpha} \) and the multipliers of \( \alpha \) and \( \tilde{\alpha} \) are \( \mu \) and \( \tilde{\mu} \) respectively. Hence if \( |\mu| < 1 \), then \( f_\mu \in \mathcal{C}(\lambda_0) \). Fix \( k > 1/2 \) and let \( \mu_n = 1 - k/n^2 + i/n \). Then \( |\mu_n| < 1 \) for sufficiently large \( n \), \( \mu_n \to 1 \), and

\[
    f_\mu(z) \to g_k(z) = z^3 - \sqrt{\frac{2}{k}}z^2 + \left(1 + \frac{1}{2k}\right)z.
\]
Clearly, $g_k$ is real and has a repelling fixed point at 0 and a parabolic fixed point at $1/\sqrt{2k} > 0$, hence $K(g_k) \cap \mathbb{R}$ is a closed interval and the external rays of angles 0 and $1/2$ do not land at the same point, hence $g_k \notin C(\lambda_0)$ (see Figure 10). In particular, $C(\lambda_0)$ is not compact.

Another example is given in Figure 11 which was proposed by Goldberg-Milnor [GM].

![Figure 11](image_url)  
Figure 11. Butterfly collapses to a period two parabolic cycle. See Goldberg-Milnor [GM].

8. Surjectivity of straightening maps

8.1. Preliminaries. To prove our surjectivity results we will employ the compactness of primitive renormalizable polynomials and the injectivity of straightening proven in previous sections together with the nice properties of straightening of quadratic-like families, established by Douady and Hubbard in [DH2]. We summarize some of these properties as follows (compare with [DH2, Proposition 13 and Chapter IV]).

**Theorem 8.1.** Let $X$ be a complex manifold and $\{f_\mu\}_{\mu \in X}$ be a holomorphic family of quadratic-like maps. Let

$$ M_X = \{ \mu \in X; K(f_\mu) \text{ is connected} \}, $$

and denote by

$$ \chi : M_X \to M $$

the corresponding straightening map. That is, $\chi(\mu) = c$ where $c \in M$ is such that $z \mapsto z^2 + c$ is hybrid equivalent to $f_\mu$. Then the following statements hold:

- $\chi$ is continuous.
- For all $c \in M$, we have that $\chi^{-1}(c)$ is a complex analytic subspace of $X$.
- If $X$ has complex dimension one and $\mu_0 \in M_X$, then there exists a neighborhood $V$ of $\mu_0$ in $M_X$ such that $\chi$ is either constant on $V$ or $\chi(V)$ contains a neighborhood of $\mu_0$ in $M$.

We will also need the following lemma on quasiconformal deformations:
Lemma 8.2. Let $\lambda_0$ be a d-invariant rational lamination. Let $f \in \mathcal{R}(\lambda_0)$ and $\chi_{\lambda_0}(f) = P \in \mathcal{C}(T(\lambda_0))$. Consider a quasiconformal deformation of $P$ on $K(P)$, i.e., there exists a quasiconformal map $\phi : C \to C$ whose complex dilatation is supported on $K(P)$ such that $\phi \circ P = P_1 \circ \phi$ for some $P_1 \in \mathcal{C}(T(\lambda_0))$.

Then there exists a quasiconformal deformation $f_1 \in \mathcal{R}(\lambda_0)$ of $f$ on $K(f)$ such that $\chi_{\lambda_0}(f_1) = P_1$.

Proof. Let us denote by $\sigma_0$ the standard complex structure on $C$ and let $\sigma = \phi^* \sigma_0$. Define an almost complex structure $\sigma_1$ on $C$ as follows:

$$\sigma_1 = \begin{cases} (\psi_v \circ f^n)^* \sigma & \text{on } f^{-n}(K_1(v)) \text{ for some } v \in |T(\lambda_0)| \text{ and } n \geq 0, \\ \sigma_0 & \text{otherwise}, \end{cases}$$

where $\psi = (\psi_v)$ is a hybrid conjugacy from a $\lambda$-renormalization of $f$ to $P$. Then $\sigma$ is well-defined because $\sigma$ is $g$-invariant. Its dilatation is uniformly bounded because $\sigma$ is defined by pullbacks of $\psi_v^*, \sigma = (\phi \circ \phi_0)^* \sigma_0$ by holomorphic maps and pullbacking by holomorphic map does not change the maximal dilatation. Furthermore, $\sigma$ is $f$-invariant by construction. Therefore, by the measurable Riemann mapping theorem, $f : (C, \sigma_1) \to (C, \sigma_1)$ is conformally conjugate to a polynomial $f_1 : C \to C$. Since this deformation preserves external rays and their landing points, we have $f_1 \in \mathcal{C}(\lambda_0)$ and $\lambda_0$ renormalization of $f$ gives the $\lambda_0$-renormalization of $f_1$. More precisely, let $\phi_1 : C \to C$ be the quasiconformal homeomorphism such that $\phi_1 \sigma_0 = \sigma_1$ and $f_1 = \phi_1 \circ f \circ \phi_1^{-1}$. Take a $\lambda_0$-renormalization $g = (f : U'_v \to U_{\sigma_0(v)}(v))$. Then $g_1 = (f_1 : \phi_1(U'_v) \to \phi_1(U_{\sigma_0(v)}))$ is a $\lambda_0$ renormalization of $f_1$ and it is hybrid equivalent to $P_1$ by $\phi \circ \psi \circ \phi_1^{-1}$. □

8.2. Surjectivity of straightening: primitive disjoint type.

Theorem 8.3. Let $f_0 \in \mathcal{C}(d)$ be an internally angled primitive hyperbolic post-critically finite polynomial with rational lamination $\lambda_0$. If $f_0$ has exactly $d - 1$ superattracting periodic orbits, then the corresponding straightening map $\chi_{\lambda_0} : \mathcal{R}(\lambda_0) \to \mathcal{M}^{d-1}$ is a homeomorphism.

Proof. Recall that the interior of the Mandelbrot set $M$ is dense in $M$. Therefore, the interior of $\mathcal{M}^{d-1}$ is also dense in $\mathcal{M}^{d-1}$. The components of the interior of $M$ are either hyperbolic (maps with an attracting cycle) or queer. The closure of the union of the hyperbolic components is the complement of the queer components. That is, $\partial M$ is contained in the closure of the union of the hyperbolic components.

Let $\lambda_0$ be an internally angled hyperbolic primitive rational lamination of degree $d$ with reduced mapping schema of disjoint type. Given $d - 1$ components $W_1, \ldots, W_{d-1}$ of the interior of $M$ we will show that $S_0 = W_1 \times \cdots \times W_{d-1}$ is in the image of the associated straightening map $\chi_{\lambda_0}$. Since $\mathcal{R}(\lambda_0)$ is compact and $\chi_{\lambda_0}$ is continuous, this implies that $\chi_{\lambda_0}$ is surjective.

Since hyperbolic maps are in the image of $\chi_{\lambda_0}$, we have that $(\partial M)^{d-1}$ is also contained in the image. In particular, $\partial W_1 \times \cdots \times \partial W_{d-1} \subset \chi_{\lambda_0}(\mathcal{R}(\lambda_0))$. Now, for $n \in \{1, \ldots, d - 1\}$, let

$$S_n = \partial W_1 \times \cdots \times \partial W_n \times W_{n+1} \times \cdots \times W_{d-1}.$$ 

Observe that $S_{d-1} \subset \chi_{\lambda_0}(\mathcal{R}(\lambda_0))$. To prove that $S_0$ is contained in $\chi_{\lambda_0}(\mathcal{R}(\lambda_0))$ it is enough to establish the following.

Claim. For all $1 \leq n \leq d - 1$, if $S_n \subset \chi_{\lambda_0}(\mathcal{R}(\lambda_0))$, then $S_{n-1} \subset \chi_{\lambda_0}(\mathcal{R}(\lambda_0))$.

Proof of the claim. Let $c = (c_1, \ldots, c_d) \in S_n$. Denote by $\pi : \mathbb{C}^{d-1} \to \mathbb{C}$ the projection which forgets the $n$th coordinate and by $\pi_n : \mathbb{C}^{d-1} \to \mathbb{C}$ the projection onto the $n$th coordinate. By Theorem 8.1, $(\pi \circ \chi_{\lambda_0})^{-1}(c)$ is a complex analytic space $Y$, since post-composition
of $\chi_{\lambda_0}$ by projection onto one coordinate is the straightening map of a quadratic-like family. After resolution of singularities, we may assume that $Y$ is a complex manifold. Moreover, $\pi_n \circ \chi_{\lambda_0} : R(\lambda_0) \cap Y \to M$ is injective. Since $R(\lambda_0)$ is compact (Theorem 8.1) and the uncountable set $\partial W_n$ is contained in $\pi_n \circ \chi_{\lambda_0}(R(\lambda_0) \cap Y)$, there exists $f \in R(\lambda_0)$ contained in a component of dimension at least one of $Y$ such that $\pi_n \circ \chi_{\lambda_0}(f) \in \partial W_n$. Theorem 8.1 implies that $\pi_n \circ \chi_{\lambda_0}$ is not constant along any one dimensional submanifold of $Y$ containing $f$, therefore $\pi_n \circ \chi_{\lambda_0}(R(\lambda_0) \cap Y)$ contains a neighborhood of $c_n$ in $M$, by Theorem 8.1. From Lemma 8.2, we conclude that $\pi_n \circ \chi_{\lambda_0}(R(\lambda_0) \cap Y)$ contains $W_n$. Hence, for all $c = (c_1, \ldots, c_d) \in S_n$ we have that $\{c_1 \times \cdots \times c_{n-1} \times W_n \times \{c_{n+1} \times \cdots \times \{c_{d-1} \times c_d \} \subset \chi_{\lambda_0}(R(\lambda_0))$ and the claim follows. 

\section{Surjectivity of straightening: cubic primitive capture type.}

\textbf{Theorem 8.4.} Let $f_0 \in C(3)$ be an internally angled primitive hyperbolic post-critically finite polynomial with rational lamination $\lambda_0$ and reduced mapping schema of capture type. Then the associated straightening map $\chi_{\lambda_0} : R(\lambda_0) \to M$ is a bijection.

The proof of Theorem 8.4 is similar to the disjoint case. However, we have to be careful since the straightening maps involved are discontinuous [In4]. Nevertheless, we use the fact that these maps are continuous along carefully chosen sequences.

\textbf{Lemma 8.5.} Let $f_n : U_n' \to U_n$ be a sequence of quadratic-like maps with connected Julia sets. Assume that the following hold:

(i) $f_n$ converges to a quadratic-like map $f : U' \to U$. More precisely, $f_n \to f$ uniformly on some neighborhood of $K(f; U', U)$ as $n \to \infty$;

(ii) $f_n$ is hybrid equivalent to $g_n \in M$.

(iii) $g_n \to g \in M$;

(iv) For any $z \in \text{int} \ K(g), z \in \text{int} \ K(g_n)$ for all sufficiently large $n$.

Then we can choose hybrid conjugacy $\psi_n$ between $f_n$ and $g_n$ such that $\psi_n$ converges to a hybrid conjugacy $\psi$ between $f$ and $g$.

In particular, let $\lambda_0$ be a hyperbolic 3-invariant rational lamination of $\mathcal{T}$. If $f_n \to f$ be a convergent sequence in $R(\lambda_0)$. If the quadratic renormalization $f_n^\prime : U_n' \to U_n$ of $f_n$ satisfies the above assumption, then $\chi_{\lambda_0}(f_n) \to \chi_{\lambda_0}(f)$.

\textbf{Remark 8.6.} The assumption (iv) is equivalent to $J(g_n) \to J(g)$ in the Hausdorff topology. Furthermore, (iv) holds if one of the following hold:

(i) $J(g) = K(g)$, i.e., $\text{int} \ K(g)$ is empty;

(ii) $g$ lies in the boundary of a hyperbolic component $W$ of $M$ and $g_n \in W \to g$ non-tangentially (see [Mc2] for the case $g$ parabolic and see [Yo] for the case $g$ has a Siegel disk).

\textbf{Proof.} Using the tubing construction of hybrid conjugacies [DH2], we can always choose hybrid conjugacies $\psi_n$ so that there exists some $K \geq 1$ such that $\psi_n$ is $K$-quasiconformal for any $n$. Therefore, by passing to a subsequence, $\psi_n$ converges to some quasiconformal map $\psi$. Furthermore, they also proved that hybrid conjugacy constructed by the tubing construction depends continuously outside the boundary of $M$.

Hence we may assume $g \in \partial M$. By Theorem 8.1, $f$ is hybrid equivalent to $g$ and $\psi$ is a quasiconformal conjugacy between $f$ and $g$. Let $\mu = \frac{\psi}{\psi}$ be the complex dilatation of $\psi$. Then, by the equation $\psi \circ f = g \circ \psi$, we have $f^* \mu = \mu$. Since $g$ does not carry an invariant line field on its Julia set (otherwise, $g$ must lie in a queer component), we may assume that
\( \mu \) vanishes on \( J(g) \). Furthermore, since \( \psi_n \) is holomorphic in the interior of \( K(f_n) \), \( \mu \) also vanishes on \( \text{int} K(g) \) by (iv). Therefore, \( \psi \) is a hybrid conjugacy between \( f \) and \( g \).

For \( f_n \to f \in \mathcal{R}(\lambda_0) \), let \( \omega_n \) be the captured critical point for \( f_n \) and \( k \) be the capture time. Then we have \( \chi_{\lambda_0}(f_n) = (\mathcal{G}_n, \mathcal{G}_n) \) where \( x_n = \psi_n(f_n^k(\omega_n)) \). If \( f_n : \mathcal{G}_n \to \mathcal{G}_n \) satisfy the assumption, then, after passing to the limit, we have \( x = \psi(f^k(\omega)) \), where \( \omega \) is the captured critical point for \( f \). Therefore, \( \chi_{\lambda_0}(f) = (g, x) = \lim \chi_{\lambda_0}(f_n) \).

**Proof of Theorem 8.4.** To simplify notation, here we identify the map \( h(z) = z^2 + c \) with \( c \in \mathbb{C} \). That is, straightening takes values in the set:

\[ \mathcal{MK} = \{(h, x); h(z) = z^2 + c \in \mathcal{M}, x \in K(h)\} \]

Observe that if \( h \) is hyperbolic and \( x \in \text{int} K(h_0) \), then the pair \((h, x)\) represents a hyperbolic dynamical system over the reduced schema of \( \lambda_0 \). By Theorem \( \mathcal{C} \), \((h, x)\) is in the image of \( \chi_{\lambda_0} \).

Now we consider \((h_0, x_0) \in \mathcal{MK} \). To prove that that \((h_0, x_0) \in \mathcal{R}(\lambda_0) \) we consider two cases according to whether \( h_0 \) is in a queer component or not.

**Case I:** \( h_0 \) does not lie in any queer component.

Take sequences \( h_n \to h_0 \) and \( x_n \to x_0 \) satisfying (iv) in Lemma 8.5 such that \( h_n \) is hyperbolic and \( x_n \in \text{int} K(h_n) \). Let \( f_n \in \mathcal{R}(\lambda_0) \) satisfy \( \chi_{\lambda_0}(f_n) = (h_n, x_n) \). Then, by Lemma 8.5, we have \( \chi_{\lambda_0}(f) = (h_0, x_0) \).

When \( h_0 \) is in a queer component we further subdivide into two cases according to whether the captured other critical forward orbit is finite or not.

**Case IIa:** Assume that \( h_0 \in \mathcal{W} \) for some queer component \( \mathcal{W} \) and that \( x_0 \) is eventually periodic for \( h_0 \).

In this case, we have \( h_j^j(x_0) = h_k^k(x_0) \) for some \( j \neq k \). Take \( h_1 \in \partial \mathcal{W} \subset \mathcal{M} \) and an eventually periodic point \( x_1 \) for \( h_1 \), which is the continuation of \( x_0 \) along a path in \( \mathcal{W} \). Take \( f_1 \in \mathcal{R}(\lambda_0) \) with \( \chi_{\lambda_0}(f_1) = (h_1, x_1) \). Let \( \omega_1 \) be the captured critical point for \( f_1 \) and let \( l > 0 \) be the capture time. Then we have \( f_1^{j+l}(\omega_1) = f_1^{j+l}(\omega_1) \).

Now consider the one-dimensional complex analytic sets

\[ V = \{(f, \omega); f \in \text{Poly}(3), \omega: \text{critical point for } f \text{ and } f^{j+l}(\omega) = f^{k+l}(\omega)\}, \]

\[ \bar{V} = \{(h, x); h \in \text{Poly}(2), h^{l}(x) = h^{l}(x)\} \]

Then \((f_1, \omega_1) \in V \) and \( \chi_{\lambda_0}(f_1) = (h_1, x_1) \) lies in \( \bar{V} \). Let \( \tilde{x}(f, \omega) = \pi_0 \circ \chi_{\lambda_0}(f) \). By Theorem 8.1, we can extend \( \tilde{x} \) continuously in a small neighborhood \( U \) of \( f_1 \) in \( V \). If \( \tilde{x} \) is constant on \( U \), then \( \chi_{\lambda_0} \) is also constant and it contradicts the injectivity of \( \chi_{\lambda_0} \). Hence \( \tilde{x} \) is non-constant, and we can find \((f_2, \omega_2) \in U \) with \( \tilde{x}(f_2, \omega_2) \in \mathcal{W} \). Furthermore, if we move \( f \) continuously from \( f_1 \) to \( f_2 \) in \( \tilde{x}^{-1}(\mathcal{W}) \cup \{f_1\} \), then every periodic point does not bifurcate, since they are all repelling for parameters in the closure of \( \tilde{x}^{-1}(\mathcal{W}) \). This implies that \( \omega_2 \) is still a captured critical point. Therefore, \( f_2 \in \mathcal{R}(\lambda_0) \) and \( \chi_{\lambda_0}(f_2) \in \bar{V} \). As before, deforming \( f_2 \) quasiconformally, we obtain a polynomial \( f \in \mathcal{R}(\lambda_0) \) with \( \chi_{\lambda_0}(f) = (h_0, x_0) \).

**Case IIb:** Assume that \( h_0 \in \mathcal{W} \) for some queer component \( \mathcal{W} \) and \( x_0 \in J(h_0)(= K(h_0)) \).

Since periodic points are dense in \( J(h_0) \), we can take a sequence \( x_n \to x_0 \) with \( x_n \) periodic. Take \( f_n \in \mathcal{R}(\lambda_0) \) which satisfy \( \chi_{\lambda_0}(f_n) = (h_0, x_n) \). Since \( \mathcal{R}(\lambda_0) \) is compact, we may assume \( f_n \) converges to some \( f \in \mathcal{R}(\lambda_0) \). By Lemma 8.5, we have \( \chi_{\lambda_0}(f) = (h_0, x_0) \).

8.4. **Complex submanifolds and quasiconformality of straightening.** Here we prove some regularity properties of straightening maps of primitive disjoint type and primitive cubic capture type. That is, we finish the proof of theorems \( \mathcal{C} \) and \( \mathcal{F} \).
8.4.1. Proof of Theorem In view of Theorem the straightening map \( \chi_{k_0} \) is surjective. We must show that for any \( 0 \leq k \leq d-1 \), given \( 1 \leq i_1 < \cdots < i_k \leq d-1 \) and \( c_1, \ldots, c_k \in M \), the set 
\[
\{ f \in \mathcal{R}(\lambda_0) ; \chi_{i_j}(f) = c_j \text{ for all } j = 1, \ldots, k \}
\]
is contained in a codimension \( k \) complex submanifold \( \mathcal{S} \) of \( \text{Poly}(d) \). For \( k = 0 \), then the statement above is clearly true. We proceed recursively and assume that we have proven 8.4.1. 

Let \( \mathcal{S} \) be the quadratic-like map obtained from \( \mathcal{Q} \mathcal{L} \) such that \( \mathcal{S} \) is contained in a codimension \( k \) complex submanifold \( \mathcal{S} \) of \( \text{Poly}(d) \). For all \( f \in \mathcal{R}(\lambda_0) \cap \mathcal{S} \) and for all \( j = 1, \ldots, k \), we have that \( \chi_{i_j}(f) = c_j \). We work in the space \( QL \) of quadratic-like germs introduced by [L]. Let
\[
\iota(f) = [f^\ell : U' \to U] \in QL
\]
be the quadratic-like map obtained from \( f \in \mathcal{U} \) which is hybrid conjugate to \( z^2 + \chi_{i_k}(f) \). Since \( \iota \) is an analytic map and by [L, Lemma 4.25], \( \iota(\mathcal{S}) \) is transverse to the foliation \( \mathcal{F} \) of hybrid classes at all \( f \in \mathcal{S} \cap \mathcal{R}(\lambda_0) \). Therefore, the subset of formed by polynomial \( f \in \mathcal{S} \) such that \( \iota(f) \) is hybrid conjugate to \( z \mapsto z^2 + c_{i_k} \) is a codimension one submanifold of \( \mathcal{S} \).

The quasiconformality of \( \chi \) along one dimensional slices as in the statement of Theorem follows directly from [L, Theorem 4.26].

8.4.2. Capture case. In order to complete the proof of Theorem we must establish that the straightening map involved is quasiconformal along appropriate slices. The proof is similar than that of Buff and Henriksen in [BHe]. We construct a holomorphic motion on the filled Julia set and use its quasiconformal extension, obtained via the \( \lambda \)-lemma, to show that the straightening map has the desired quasiconformal regularity.

Proof of Theorem Let \( c \in M \) and \( \mathcal{S} \) be as in the statement of the theorem. From Douady-Hubbard’s theorem it follows that \( \mathcal{S} \) is a complex analytic subspace of \( \text{Poly}(3) \). We take a point \( f_0 \in \mathcal{R}(\lambda_0) \cap \mathcal{S} \) and denote by \( \mathcal{S} \) a branch of \( \mathcal{S} \) at \( f_0 \). We will show that \( \chi \) is quasiconformal on \( \mathcal{S} \). Then, by injectivity, we will conclude that there is at most one branch of \( \mathcal{S} \) at \( f_0 \). Hence, it will immediately follow that \( \mathcal{S} \) is locally irreducible.

Consider a family \( (f^\omega : U'_{\omega} \to U_{\omega})_{\omega \in \mathcal{S}} \) of quadratic-like mappings. Since the hybrid class of the elements of this family is constant, it is stable on the whole parameter space \( \mathcal{S} \), in the sense of Mañé-Sad-Sullivan. It follows that the Julia set \( J_f(v_0) = \partial K_f(v_0) \) admits a holomorphic motion on \( \mathcal{S} \) [MC1]. Furthermore, we have a natural conformal conjugacy \( \psi_f : K_f(v_0) \to K_f(v_0) \) between \( f_0 \) and \( f \), which is also a holomorphic motion on \( \mathcal{S} \), where \( \psi_f \) is a hybrid conjugacy between \( f^\omega : U'_{\omega} \to U_{\omega} \) and \( Q(z) = z^2 + c \). Gluing both holomorphic motions together, we obtain a holomorphic motion \( h_f : K_f(v_0) \to K_f(v_0) \). For \( f \in \mathcal{S} \cap \mathcal{R}(\lambda_0) \), we have
\[
\chi(f) = (c, \psi_f(f^\omega(\omega_f))) = (c, \psi_{f_0} \circ h_f(f^\omega(\omega_f))),
\]
where \( \omega_f \) is the captured critical point.

By applying the \( \lambda \)-lemma [MSS] [SL], \( h_f \) extends to a holomorphic motion
\[
h : \mathcal{S} \times \mathbb{C} \to \mathbb{C},
\]
such that \( h_f : \mathbb{C} \to \mathbb{C} \) is quasiconformal. By the same argument as [BHe, Lemma 13], the map
\[
f \mapsto h_f(f^\omega(\omega_f))
\]
is locally quasiconformal on \( \mathcal{S} \). Taking a finite covering of \( \mathcal{R}(\lambda_0) \cap \mathcal{S} \), we conclude that \( \chi : \mathcal{R}(\lambda_0) \cap \mathcal{S} \to K(Q) \) extends to a quasiconformal homeomorphism.

To finish the proof we have to show that \( \mathcal{R}(\lambda_0) \) is connected. In fact, observe that if \( C \) is both open and closed in \( \mathcal{R}(\lambda_0) \) and \( \mathcal{S} \) is as above, then \( \mathcal{S} \) is either contained in \( C \) or
disjoint from $C$. Now $\chi_1 : \mathcal{R}(\lambda_0) \to M$ is continuous and $C$ is compact. Therefore, $\chi_1(C)$ and $\chi_1(\mathcal{R}(\lambda_0) \setminus C)$ are closed and disjoint subsets of $M$. Then one of these sets must be empty, since $M$ is connected. Hence, either $C$ is empty or $C = \mathcal{R}(\lambda_0)$.

\[ \Box \]

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