Light propagation in a plasma on Kerr spacetime: Separation of the Hamilton-Jacobi equation and calculation of the shadow

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We consider light propagation in a non-magnetized pressureless plasma around a Kerr black hole. We find the necessary and sufficient condition the plasma electron density has to satisfy to guarantee that the Hamilton-Jacobi equation for the light rays is separable, i.e., that a generalized Carter constant exists. For all cases where this condition is satisfied we determine the photon region, i.e., the region in the spacetime where spherical light rays exist. A spherical light ray is a light ray that stays on a sphere $r = \text{constant}$ (in Boyer-Lindquist coordinates). Based on these results, we calculate the shadow of a Kerr black hole under the influence of a plasma that satisfies the separability condition. More precisely, we derive an analytical formula for the boundary curve of the shadow on the sky of an observer that is located anywhere in the domain of outer communication. Several examples are worked out.

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I. INTRODUCTION

For most applications of general relativity light rays can be considered as lightlike geodesics of the spacetime metric, i.e., the influence of a medium on the light rays can be neglected. However, in the radio frequency range this is not always true. A well-known example is the influence of the Solar corona on the travel time and on the deflection angle of radio signals that come close to the Sun. This influence is routinely observed since the 1960s. In this case one may assume that the medium is a non-magnetized pressureless plasma and for the gravitational field the linearized theory is sufficient. The relevant equations have been determined by Muhlenman et al.\textsuperscript{1,2}. Gravitational lensing in this approximation was firstly discussed by Bliokh and Minakov\textsuperscript{3}. In these works both the gravitational and the plasma deflection are assumed to be small and they are calculated separately from each other.

There is good reason to assume that also black holes and other compact objects are surrounded by a plasma and it is an interesting question to investigate if the plasma could have an observable effect on radio signals that come close to such a compact object. In such cases the linearized theory is not sufficient. For a spherically symmetric uncharged black hole, one has to consider the Schwarzschild metric and for a rotating uncharged black hole one has to consider the Kerr metric. Some results on light propagation in a non-magnetized pressureless plasma on the Schwarzschild or Kerr spacetime are known. The influence of a spherically symmetric and time-independent plasma density on the light deflection in the Schwarzschild spacetime was calculated and discussed by Perlick\textsuperscript{4}. In this study neither the gravitational nor the plasma deflection is assumed to be small. Later, the influence of plasma effects on gravitational lensing was investigated, with different methods, by Bisnovatyi-Kogan and Tsupko\textsuperscript{5,7}. In the Kerr spacetime, light deflection in the equatorial plane was calculated by Perlick\textsuperscript{4} for the case of a rotationally symmetric and time-independent plasma density. Within the approximation of small deflection, lensing of light rays off the equatorial plane was considered by Morozova et al.\textsuperscript{8} who assumed a slowly rotating Kerr black hole. The influence of a plasma on the multiple imaging properties in the strong-bending regime was investigated by Bisnovatyi-Kogan and Tsupko\textsuperscript{7} for the Schwarzschild spacetime and recently extended to Kerr spacetime by Liu, Ding and Jing\textsuperscript{9}. For some possible astrophysical observations of plasma effects near compact objects we refer to Rogers\textsuperscript{10,12} and for plasma effects in strong lens systems to Er and Mao\textsuperscript{13}. For a review see\textsuperscript{14}.

Here we want to concentrate on the influence of a plasma on the shadow of a black hole. Roughly speaking, the shadow is the black disk an observer sees in the sky if a black hole is viewed against a backdrop of light sources that are distributed around the black hole but not between the observer and the black hole. For constructing the shadow we have to consider all past-oriented light rays that issue from a chosen observer position. Each of these light rays corresponds to a point on the observer’s sky. We assign darkness to a point if the corresponding light ray goes to the horizon and brightness otherwise. The idea is that there are light sources distributed in the spacetime, but not between the observer and the black hole and not inside a possible white-hole extension of the spacetime from where future-oriented light rays
could be sent across the horizon towards the observer. Without a plasma, the shadow of a Schwarzschild black hole was calculated by Synge [15] and the shape of the shadow of a Kerr black hole, for an observer at infin-
ity, was calculated by Bardeen [16]. In an earlier paper [17] we have generalized Synge’s formula to the case of a spherically symmetric and static plasma distribution on a spherically symmetric and static spacetime. As particular examples, we have worked out the results for the case that the underlying spacetime geometry is (a) that of a Schwarzschild black hole and (b) that of an Ellis wormhole. In the present paper we want to derive a similar result for a plasma around a Kerr black hole. More precisely, it is our goal to derive an analytical formula for the boundary curve of the shadow on the sky of an observer at an arbitrary position outside of the horizon of the black hole (i.e., in contrast to Bardeen we do not consider an observer at infinity). Without the influence of a plasma, such a formula has been derived by Grenzebach et al. [18, 19] for the case of a black hole of the Plebański-Demiański class (which includes the Kerr black hole as a special case). Their method made use of the fact that the equation of motion for vacuum light rays admits a fourth constant of motion, the Carter constant, in addition to the ones that follow from the symmetry. Therefore, an important first step in our analysis will be to find out under what conditions a Carter constant exists for light propagation in a plasma on a Kerr spacetime. Of course, this result is of interest not only for the calculation of the shadow but also for other problems.

The perspectives of actually observing the influence of a plasma on the shadow, e.g. for Sgr A* or for M87, have been discussed in our earlier paper [17].

The paper is organized as follows. In Sec. II we review the Hamilton formalism for light rays in a non-magnetized, pressureless plasma on a general-relativistic spacetime and we specialize the relevant equations to the Kerr metric. In Sec. III we derive the necessary and sufficient condition on the plasma electron density that guarantees separability of the Hamilton-Jacobi equation for light rays, i.e., that guarantees the existence of a Carter constant. In Sec. IV we determine the photon region for a plasma density around a Kerr black hole that satisfies the separability condition. The photon region is the spacetime region filled with spherical light rays, i.e., with light rays that stay on a sphere \( r = \text{constant} \) in Boyer-Lindquist coordinates. Knowledge on the location of the spherical geodesics and on the constants of motion associated with them is of crucial relevance for deriving the boundary curve of the shadow on the sky of an observer anywhere in the domain of outer communication of the black hole. This will be done, in close analogy to the procedure in Grenzebach et al. [18, 19], in Sec. V. In the rest of the paper we determine the photon regions and the shadow for some special cases where the separability condition is satisfied.

Our conventions are as follows. We use the summation convention for greek indices that take the values 0,1,2,3. Our choice of signature is \((-++++)\). We raise and lower greek indices with the spacetime metric. \( G \) is the gravitational constant and \( c \) is the vacuum speed of light. We use units such that \( h = 1 \), i.e., energies have the same unit as frequencies and momentum vectors have the same unit as wave vectors.

## II. HAMILTON FORMALISM FOR LIGHT RAYS IN A PLASMA ON KERR SPACETIME

Light propagation in a non-magnetized pressureless plasma can be characterized by the Hamiltonian

\[
H(x, p) = \frac{1}{2} (g^{\mu\nu}(x) p_\mu p_\nu + \omega_p(x)^2) .
\]  

(1)

Here \( g^{\mu\nu} \) are the contravariant components of the space-time metric tensor and \( \omega_p \) is the plasma electron frequency which equals, up to a scalar factor, the electron density,

\[
\omega_p(x) = \frac{4\pi e^2}{m_e} N_e(x) \tag{2}
\]

where \( e \) and \( m_e \) are the electron charge and mass, respectively, and \( N_e \) is the electron number density. \( x = (x^0,x^1,x^2,x^3) \) are the spacetime coordinates and \( p = (p_0,p_1,p_2,p_3) \) are the canonical momentum coordinates. The light rays are the solutions to Hamilton’s equations

\[
\frac{dx^\mu}{ds} = \frac{\partial H}{\partial p_\mu} , \quad \frac{dp_\mu}{ds} = - \frac{\partial H}{\partial x^\mu} , \quad H(x, p) = 0 , \tag{3}
\]

where \( s \) is a curve parameter which has no direct physical or geometrical meaning. A rigorous derivation from Maxwell’s equations of this Hamiltonian approach was given, even for the more general case of a magnetized plasma, by Breuer and Ehlers [22, 23]. For the much simpler case of a non-magnetized plasma, a similar derivation can be found in [11].

In regions where \( \omega_p \neq 0 \) we may use the Hamiltonian

\[
\tilde{H}(x, p) = \frac{1}{2} \left( \frac{c^2}{\omega_p(x)} g^{\mu\nu}(x) p_\mu p_\nu + c^2 \right) = \frac{c^2}{\omega_p(x)^2} H(x, p) \tag{4}
\]
because multiplying a Hamiltonian with a nowhere vanishing function does not affect the solution curves to \( H \), except for a reparametrization \( s \mapsto \tilde{s} \). Using the Hamiltonian \( H \) demonstrates that the rays are *timelike* geodesics of the conformally rescaled metric \( \tilde{g}_{\mu\nu}dx^\mu dx^\nu = c^{-2}\omega_p^2 g_{\mu\nu}dx^\mu dx^\nu \). The new curve parameter \( \tilde{s} \) is proper time with respect to this metric. Although such a conformal rescaling is often convenient, in the following we prefer to work with the Hamiltonian \( H \), rather than with the Hamiltonian \( \tilde{H} \), because then the equations can be immediately applied to all regions, independently of whether \( \omega_p \neq 0 \) or \( \omega_p = 0 \).

A plasma is a dispersive medium, i.e., light propagation depends on the frequency. To assign a frequency to a light ray, we have to choose an observer with four-velocity \( U^\mu(x) \), normalized to \( g_{\mu\nu}(x)U^\mu(x)U^\nu(x) = -c^2 \). Decomposing the momentum into a component parallel to \( U^\mu(x) \) and a component orthogonal to \( U^\mu(x) \),

\[
p^\mu = -\frac{1}{c} \omega(x) U^\mu(x) + k^\mu(x),
\]

defines the frequency

\[
\omega(x) = \frac{1}{c} p_\mu U^\mu(x)
\]

and the spatial wave vector

\[
k^\mu(x) = p^\mu + \frac{1}{c^2} p_\mu U^\mu(x) U^\mu(x).
\]

Note that \( \omega(x) \) is positive if \( U^\mu(x) \) is future-pointing and \( p^\mu \) is past-pointing. We have chosen this somewhat unusual convention because later, when we calculate the shadow, we will consider light rays issuing from the observer position into the past and we want to assign a positive frequency to those rays.

With the decomposition \( (5) \), the condition \( H = 0 \) may be solved for the frequency which gives the dispersion relation in the familiar form,

\[
\omega(x)^2 = k_\mu(x)k^\mu(x) + \omega_p(x)^2.
\]

As \( k^\mu(x) \) is spacelike, this equation implies

\[
\omega(x)^2 \geq \omega_p(x)^2,
\]

i.e., light propagation is possible only with a frequency that is bigger than the plasma frequency. If we introduce in the usual way the phase velocity

\[
v_p(x, \omega(x)) = \sqrt{\frac{\omega(x)^2}{k_\mu(x)k^\mu(x)}}
\]

and the index of refraction

\[
n(x, \omega(x)) = \frac{c}{v_p(x, \omega(x))},
\]

we see that the index of refraction, as a function of \( x \) and \( \omega(x) \), is of the same form for all \( U^\mu(x) \),

\[
n(x, \omega(x))^2 = 1 - \frac{\omega_p(x)^2}{\omega(x)^2}.
\]

The remarkable fact is that the index of refraction depends only on the function \( \omega_p \), i.e., on the plasma electron density, but not on the four-velocity field of the plasma. This demonstrates that a non-magnetized pressureless plasma is an example for the type of dispersive medium that was discussed in detail in the text-book by Synge [21], Chapter XI. Condition \( (9) \) guarantees that the index of refraction is real (and non-negative) for all allowed frequencies.

We will now specialize to the Kerr metric which is given, in Boyer-Lindquist coordinates \( x = (t, r, \vartheta, \varphi) \), by

\[
g_{\mu\nu}dx^\mu dx^\nu = -c^2 \left( 1 - \frac{2mr}{\rho^2} \right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + r^2 d\vartheta^2 + \frac{4mr \sin^2 \vartheta}{\rho^2} c dt d\varphi
\]

where

\[
\Delta = r^2 + a^2 - 2mr, \quad \rho^2 = r^2 + a^2 \cos^2 \vartheta.
\]

Here \( m \) is the mass parameter and \( a \) is the spin parameter,

\[
m = \frac{GM}{c^2}, \quad a = \frac{J}{Mc},
\]

where \( M \) is the mass and \( J \) is the spin. Throughout this paper, we assume that \( a^2 \leq m^2 \), i.e., we consider a black hole rather than a naked singularity. We restrict our consideration to the domain of outer communication, i.e., to the domain outside of the outer horizon, \( r > m + \sqrt{m^2 - a^2} \).

With the metric coefficients specified to the Kerr metric, the Hamiltonian \( H \) reads

\[
H = \frac{1}{2\rho^2} \left( -\frac{1}{\Delta} \left( a p_\varphi + (r^2 + a^2) p_t/c \right)^2 \right)
\]

\[
+ \left( \frac{p_\varphi}{\sin \vartheta} + a \sin \vartheta \frac{p_t}{c} \right)^2 + p_\varphi^2 + \Delta p_t^2 + \rho^2 p_\varphi^2.
\]

We assume that \( \omega_p \) is a function only of \( r \) and \( \vartheta \). Then \( \partial H/\partial t = 0 \) and \( \partial H/\partial \varphi = 0 \), i.e., \( p_t \) and \( p_\varphi \) are constants of motion. \( p_\varphi \) is the \( z \) component of the angular momentum. If we write

\[
p_t = c \omega_0,
\]

the physical meaning of the constant of motion \( \omega_0 \) becomes clear if we specify the frequency \( \omega \) to the case of an observer on a line, \( U^\mu(x) = c \delta^\mu_t (-g_{tt}(x))^{-1/2} \),

\[
\omega(x) = \frac{p_t}{\sqrt{-g_{tt}(x)}} = \frac{\omega_0}{\sqrt{1 - \frac{2mr}{\rho^2}}},
\]

where
which is possible everywhere outside the ergoregion where $2 m r < \rho^2$. We see that for a light ray that reaches infinity $\omega_0$ is the frequency measured by an observer at a line at infinity. In vacuum (i.e., if $\omega_p = 0$), the path of a light ray is independent of the frequency; correspondingly, there is no restriction on $\omega_0$. In a plasma, however, light propagation does depend on the frequency. If $\omega_0$ becomes too small in comparison to the plasma frequency, it is even impossible for a light ray with frequency $\omega_0$ to propagate in the plasma.

To derive the precise form of the condition on $\omega_0$, we read from (16) that the equation $H = 0$ can hold at a point with coordinates $(r, \vartheta)$ on the domain of outer communication only if

$$F(p_\varphi) := \left( a p_\varphi + (r^2 + a^2) \omega_0 \right)^2$$

$$- \Delta \left( \frac{p_\varphi}{\sin \vartheta} + a \sin \vartheta \omega_0 \right)^2 - \rho^2 \Delta \omega_0^2 \geq 0.$$ 

Here we have used that $\Delta > 0$ on the domain of outer communication. The function $F(p_\varphi)$ has an extremum at

$$p_{\varphi,e} = -\frac{2 m r a \sin^2 \vartheta \omega_0}{2 m r - \rho^2}.$$ 

Inside the ergoregion, where $2 m r > \rho^2$, this extremum is a minimum and $F(p_\varphi)$ takes arbitrarily large positive values, so the inequality (19) can be satisfied for any $\omega_0$ by choosing $p_\varphi$ appropriately. Outside the ergoregion, where $2 m r < \rho^2$, the extremum is a maximum. The inequality (19) can be satisfied only if $F(p_{\varphi,e}) \geq 0$ which is equivalent to

$$\omega_0^2 \geq \left( 1 - \frac{2 m r}{\rho^2} \right) \omega_p(r, \vartheta)^2.$$ 

As inside the ergoregion the inequality (21) is true for any real $\omega_p$, this inequality is the necessary and sufficient condition for the existence of a light ray with constant of motion $\omega_0$ anywhere on the domain of outer communication. If the plasma frequency is bounded on the domain outside of the ergoregion, $\omega_p(r, \vartheta) \leq \omega_e = \text{constant}$, light rays with $\omega_0^2 \geq \omega_e^2$ can travel through any point of the domain of outer communication.

If we have chosen a frequency $\omega_0$ that is allowed at a point with coordinates $r$ and $\vartheta$, the allowed values of $p_\varphi$, $p_\vartheta$ and $p_r$ are determined by the condition $H = 0$.

### III. SEPARABILITY OF THE HAMILTON-JACOBI EQUATION FOR LIGHT RAYS IN A PLASMA ON KERR SPACETIME

If $\omega_p$ depends only on $r$ and $\vartheta$, we have three constants of motion for light rays, $H = 0$, $p_t = c \omega_0$ and $p_\varphi$. We will now investigate for which special form of the function $\omega_p(r, \vartheta)$ the Hamilton-Jacobi equation can be separated; in this case the separation constant will give us a fourth constant of motion such that the equation for light rays becomes completely integrable.

By (16), the Hamilton-Jacobi equation

$$0 = H \left( x, \frac{\partial S}{\partial x} \right)$$

reads

$$0 = -\frac{1}{\Delta} \left( a \frac{\partial S}{\partial \varphi} + (r^2 + a^2) \frac{\partial S}{\partial t} \right)^2$$

$$+ \left( \frac{1}{\sin \vartheta} + a \sin \vartheta \frac{\partial S}{\partial \vartheta} \right)^2$$

$$+ \left( \frac{\partial S}{\partial \vartheta} \right)^2 + \Delta \left( \frac{\partial S}{\partial r} \right)^2 + \rho^2 \omega_p^2.$$ 

With the separation ansatz

$$S(t, \varphi, r, \vartheta) = c \omega_0 t + p_\varphi \varphi + S_r(r) + S_\vartheta(\vartheta)$$

the Hamilton-Jacobi equation takes the following form:

$$0 = -\frac{1}{\Delta} \left( a p_\varphi + (r^2 + a^2) \omega_0 \right)^2$$

$$+ \left( \frac{p_\varphi}{\sin \vartheta} + a \sin \vartheta \omega_0 \right)^2$$

$$+ S_\vartheta(\vartheta)^2 + \Delta S_r'(r)^2 + \omega_p^2 \left( r^2 + a^2 \cos^2 \vartheta \right).$$

Separability of the Hamilton-Jacobi equation requires that (25) can be rearranged in a way that the left-hand side is independent of $r$ and the right-hand side is independent of $\vartheta$. We read from (25) that this is possible if and only if the plasma frequency is of the form

$$\omega_p(r, \vartheta)^2 = \frac{f_\varphi(r) + f_\vartheta(\vartheta)}{r^2 + a^2 \cos^2 \vartheta}$$

with some functions $f_\varphi(r)$ and $f_\vartheta(\vartheta)$. Then the Hamilton-Jacobi equation reads

$$S_\vartheta'(\vartheta)^2 + \left( \frac{p_\varphi}{\sin \vartheta} + a \sin \vartheta \omega_0 \right)^2 + f_\vartheta(\vartheta)$$

$$= -\Delta S_r'(r)^2 + \frac{1}{\Delta} \left( a p_\varphi + (r^2 + a^2) \omega_0 \right)^2 - f_\varphi(r) =: K.$$ 

As the first expression is independent of $r$ whereas the second expression is independent of $\vartheta$, the quantity $K$ depends neither on $r$ nor on $\vartheta$, so it is a constant. $K$ is the generalized Carter constant.

With $S_\vartheta'(\vartheta) = p_\vartheta$ and $S_r'(r) = p_r$ we have found that

$$p_\vartheta^2 = K - \left( \frac{p_\varphi}{\sin \vartheta} + a \sin \vartheta \omega_0 \right)^2 - f_\vartheta(\vartheta),$$

$$\Delta p_r^2 = -K + \frac{1}{\Delta} \left( (r^2 + a^2) \omega_0 + a p_\varphi \right)^2 - f_\varphi(r).$$

With these expressions for $p_\vartheta$ and $p_r$ inserted into Hamilton’s equations

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu},$$

we see that...
for \( x^\mu = \vartheta \) and \( x^\mu = r \), respectively, we find

\[
\rho^4 \dot{\vartheta}^2 = K - \left( \frac{p_\varphi}{\sin \vartheta} + \alpha \sin \vartheta \omega_0 \right)^2 - f_\vartheta(\vartheta), \tag{31}
\]

\[
\rho^4 r^2 = -K \Delta + \left( (r^2 + \alpha^2) \omega_0 + a p_\varphi \right)^2 - f_r(\varrho) =: R(\varrho). \tag{32}
\]

The other two components, \( x^\mu = \varphi \) and \( x^\mu = t \), of \( (30) \) yield

\[
\rho^2 \dot{\varphi} = \frac{-2mr \sin^2 \vartheta \omega_0 + \left( r^2 - 2mr \right) p_\varphi}{\Delta \sin^2 \vartheta}, \tag{33}
\]

\[
\rho^2 \dot{t} = \frac{-\left( (r^2 + \alpha^2) \rho^2 + 2m \rho^2 \sin^2 \vartheta \right) \omega_0 - 2m \alpha p_\varphi}{c \Delta}. \tag{34}
\]

Formulas \( (31), (32), (33) \) and \( (34) \) give us the equations of motion for the rays in first-order form. To sum up, we have demonstrated that the condition \( (26) \) is necessary and sufficient for the existence of a generalized Carter constant, and that then complete integrability of the equations of motion is guaranteed. In the next sections we will calculate the photon regions and the shadow. As these calculations will be based on the existence of the Carter constant, we will have to restrict to the case that condition \( (26) \) is satisfied. For other plasma densities our mathematical methods will not work.

In terms of the index of refraction \( (12) \), the separability condition \( (26) \) reads

\[
n(x, \omega(x))^2 = 1 - \frac{f_r(r) + f_\vartheta(\vartheta)}{\omega(x)^2 \rho^2}. \tag{35}
\]

In regions where \( \omega_p \neq 0 \) the rays are timelike curves. The curve parameter is proper time with respect to the conformally rescaled metric \( \omega_p^{-2} g_{\mu \nu} dx^\mu dx^\nu \). Note that a homogeneous plasma, \( \omega_p = \omega_c = \) constant, satisfies the separability condition \( (26) \) with

\[
f_r(r) = \omega_c^2 r^2, \quad f_\vartheta(\vartheta) = \omega_c^2 \vartheta^2 \sin^2 \vartheta. \tag{36}
\]

In this case, the rays are timelike geodesics of the Kerr metric and the curve parameter is an affine parameter, i.e., it is affinely related to proper time of the Kerr metric.

\[\textbf{IV. PHOTON REGIONS IN A PLASMA ON THE KERR SPACETIME}\]

We consider a plasma distribution of the form of eq. \( (29) \) so that the \( \vartheta \) and the \( r \) components of the ray equation can be written in separated form, see \( (31) \) and \( (32) \).

The photon region is the region in spacetime filled with spherical light rays, i.e., with solutions to the ray equation that stay on a sphere \( r = \) constant. Unstable spherical light rays can serve as limit curves for light rays that approach them in a spiral motion. In the next section we will see that, for this reason, they are of crucial importance for the construction of the shadow.

Spherical light rays satisfy \( \dot{r} = 0 \) and \( \dot{\vartheta} = 0 \), i.e., in the notation of \( (32) \)

\[
0 = R(r) = -\left( K + f_r(r) \right) \Delta + \left( r^2 + \alpha^2 \right) \omega_0 + a p_\varphi, \tag{37}
\]

\[
0 = R'(r) = -\left( K + f_r(r) \right) 2(r - m) + \frac{4 \rho \rho_c}{c} \left( \omega_0^2 + a p_\varphi \right) - f'_r(r) \Delta. \tag{38}
\]

These two equations can be solved for the constants of motion \( a p_\varphi \) and \( K \),

\[
ap_\varphi = \frac{\omega_0}{r - m} \left( m^2 (a^2 - r^2) \pm r \Delta \frac{1}{2} f'_r(r) (r - m)^2 \right), \tag{39}
\]

\[
K = \frac{r^2 \Delta \omega_0^2}{(r - m)^2} \left( 1 \pm \sqrt{1 - f'_r(r) (r - m)^2 \frac{2r^2 \omega_0^2}{4r^2 \omega_0^2}} \right) - f_r(r). \tag{40}
\]

As the left-hand side of \( (31) \) is the square of a real quantity, the right-hand side must be non-negative,

\[
0 \leq K - \left( \frac{p_\varphi}{\sin \vartheta} + \alpha \sin \vartheta \omega_0 \right)^2 - f_\vartheta(\vartheta), \tag{41}
\]

hence

\[
\left( K - f_\vartheta(\vartheta) \right) a^2 \sin^2 \vartheta \geq \left( a p_\varphi + a^2 \sin^2 \vartheta \omega_0 \right)^2. \tag{42}
\]

Inserting \( (39) \) and \( (40) \) into \( (42) \) gives the photon region,

\[
\left( \frac{r^2 \Delta}{(r - m)^2} \left( 1 \pm \sqrt{1 - f'_r(r) (r - m)^2 \frac{2r^2 \omega_0^2}{4r^2 \omega_0^2}} \right) - f_r(r) + f_\vartheta(\vartheta) \right) a^2 \sin^2 \vartheta \geq \left( \frac{1}{(r - m)^2} \left( m^2 (a^2 - r^2) \pm r \Delta \frac{1}{2} f'_r(r) (r - m)^2 \right) + a^2 \sin^2 \vartheta \right)^2. \tag{43}
\]

Through each point with coordinates \( (r, \vartheta) \) where this inequality is satisfied, either with the plus or with the
minus sign, there is a spherical light ray.

In general, the photon region consists of stable and of unstable spherical light rays. For an unstable spherical light ray one must have

\[ 0 < R'(r) = -2K - 2f_r(r) + 4\omega_0 \left( r^2 + a^2 \right) \omega_0 = a p_\phi \]
\[ + 8r^2\omega_0^2 - f'_r(r) 4(r - m) - f''_r(r) \Delta \]  

(44)

where \( a, p_\phi \) and \( K \) have to be expressed in terms of the radius \( r \) and the frequency \( \omega_0 \) with the help of (39) and (40).

V. SHADOW OF A KERR BLACK HOLE IN A PLASMA

For constructing the shadow, we fix an observer at Boyer-Lindquist coordinates \((r_0, \vartheta_0)\) with \( r_0 > m + \sqrt{m^2 - a^2} \) and a tetrad

\[
e_0 = \frac{(r^2 + a^2) \partial_t + a c \partial_\vartheta}{c \rho \sqrt{\Delta}} \bigg|_{(r_0, \vartheta_0)},
\]

\[
e_1 = \frac{\partial_\varphi}{\rho \sin \vartheta} \bigg|_{(r_0, \vartheta_0)},
\]

\[
e_2 = -\frac{\partial_\varphi - a \sin^2 \vartheta \partial_t}{\rho \sin \vartheta} \bigg|_{(r_0, \vartheta_0)},
\]

\[
e_3 = -\frac{\sqrt{\Delta}}{\rho} \partial_r \bigg|_{(r_0, \vartheta_0)}.
\]

(45)

This tetrad is well-defined and orthonormal for any observer position in the domain of outer communication. It is chosen such that the ingoing and outgoing principal null directions of the Kerr spacetime are in the plane spanned by \( e_0 \) and \( e_3 \). We construct the shadow for the case that \( c e_0 \) is the 4-velocity of the observer, following closely the procedure of Grenzebach et al. [18].

We consider light rays issuing from the observer position into the past. For each light ray \( \lambda(s) \) with coordinate representation \((r(s), \vartheta(s), \varphi(s), t(s))\), we write the tangent vector as

\[
\hat{\lambda} = \dot{r} \partial_r + \dot{\vartheta} \partial_\vartheta + \dot{\varphi} \partial_\varphi + \dot{t} \partial_t
\]

(46)

where an overdot means derivative with respect to the curve parameter \( s \). On the other hand, the tangent vector at the observation event can be written as

\[
\hat{\lambda} = -\alpha e_0 + \beta \left( \sin \theta \cos \psi e_1 + \sin \theta \sin \psi e_2 + \cos \theta e_3 \right)
\]

(47)

where \( \alpha \) and \( \beta \) are positive factors. Recall that, by (1), we have parametrized the light rays such that \( g(\hat{\lambda}, \hat{\lambda}) = -\omega_p^2 \). Therefore, \( \alpha \) and \( \beta \) must be related by

\[
\alpha^2 - \beta^2 = \omega_p^2 \bigg|_{(r_0, \vartheta_0)}.
\]

(48)

Eq. (47) defines the celestial coordinates \( \theta \) and \( \psi \) for our observer. \( \theta \) is the colatitude and \( \psi \) is the azimuthal angle. The poles \( \theta = 0 \) and \( \theta = \pi \) correspond, respectively, to ingoing and outgoing past-oriented principal null rays. In this sense, we may say that \( \theta = 0 \) is “the direction towards the black hole” and \( \theta = \pi \) is “the direction away from the black hole” on the observer’s sky.

For each light ray, \( \alpha \) is determined by

\[
\alpha = g(\hat{\lambda}, e_0) = \frac{1}{\rho \sqrt{\Delta}} g(\hat{\lambda}, \left( r^2 + a^2 \right) \partial_t + a \partial_\varphi)
\]

(49)

\[
= \frac{(r^2 + a^2)}{\rho \sqrt{\Delta}} \left( t g_{tt} + \varphi g_{\varphi t} + \frac{a}{\rho \sqrt{\Delta}} (t g_{\varphi t} + \varphi g_{\varphi \varphi}) \right)
\]

\[
= \frac{(r^2 + a^2)}{\rho \sqrt{\Delta}} \omega_0 + \frac{a}{\rho \sqrt{\Delta}} p_\varphi
\]

hence

\[
\beta = \sqrt{\frac{1}{\rho^2 \Delta} \left( (r^2 + a^2) \omega_0 + a p_\varphi \right)^2 - \omega_p^2}
\]

(50)

where all expressions are to be evaluated at \((r_0, \vartheta_0)\). We will now determine how the constants of motion \( p_\varphi \) and \( K \) of a light ray are related to the celestial coordinates \( \theta \) and \( \psi \). To that end we compare coefficients of \( \partial_r \) in (46) and (47) which yields

\[
-\beta \cos \theta \frac{\sqrt{\Delta}}{\rho} = \dot{r}.
\]

(51)

Upon squaring both sides we find, with the help of (32) and (50),

\[
\left( \left( (r^2 + a^2) \omega_0 + a p_\varphi \right)^2 - \Delta \rho^2 \omega_p^2 \right) \left( 1 - \sin^2 \theta \right)
\]

\[
= -\left( K + f_r(r) \right) \Delta + \left( r^2 + a^2 \right) \omega_0^2 + a p_\varphi^2.
\]

(52)

Solving for \( \sin^2 \theta \) and taking the square root (using \( \sin \theta \geq 0 \) as \( 0 \leq \theta \leq \pi \)) results in

\[
\sin \theta = \left( \frac{\left( K - f_\theta(\vartheta) \right) \Delta}{\left( (r^2 + a^2) \omega_0 + a p_\varphi \right)^2 - \left( f_r(r) + f_\theta(\vartheta) \right) \Delta} \right)^{1/2}
\]

(53)

\[
\frac{\frac{\left( K - f_\theta(\vartheta) \right) \Delta}{\left( (r^2 + a^2) \omega_0 + a p_\varphi \right)^2 - \left( f_r(r) + f_\theta(\vartheta) \right) \Delta}}{\left( r_0, \vartheta_0 \right)}.
\]

Similarly, comparing coefficients of \( \partial_\varphi \) in (46) and (47) yields

\[
-\frac{\alpha a}{\rho \sqrt{\Delta}} - \frac{\beta \sin \theta \sin \psi}{\rho \sin \vartheta} = \dot{\varphi}.
\]

(54)

Upon inserting (33), (49), (50) and (53) into (54) we find

\[
\sin \psi = \frac{-p_\varphi - a \sin^2 \theta \omega_0}{\sin \theta \sqrt{K - f_\theta(\vartheta)}} \bigg|_{\vartheta_0}
\]

(55)
The shadow is the set of all points on the observer’s sky, i.e., of tangent vectors to light rays, such that past-oriented light rays with such a tangent vector go to the horizon. Light rays that correspond to boundary points of the shadow spiral asymptotically towards spherical light rays, so they must have the same values for $K$ and $p_x$ as the limiting spherical light rays. If $a \neq 0$, we can insert these values for $p_x$ and $K$ from (33) and (10), respectively, where $r = r_p$ runs over the radius values of the limiting spherical light rays. With these expressions for $K(r_p)$ and $p_x(r_p)$ inserted into (53) and (55) we get the boundary of the shadow as a curve on the observer’s sky parametrized by $r_p$. The boundary curve consists of a lower part where $\psi$ runs from $-\pi/2$ to $\pi/2$, and an upper part where $\psi$ runs from $\pi/2$ to $3\pi/2$. The parameter $r_p$ runs from a minimum value $r_p,\min$ to a maximum value $r_p,\max$ and then back to $r_p,\min$. The values $r_p,\min$ and $r_p,\max$ are determined by the property that then $\sin^2 \psi$ must be equal to 1, i.e., by (55), $r_p = r_{p,\min}/\max$ if

$$-p_x(r_p) - a \sin^2 \varnothing \omega_0 = \pm \sin \varnothing \sqrt{K(r_p) - f_\varnothing(\varnothing_O)}.$$  

Comparison with (43) shows that $r_{p,\min}/\max$ are the radius values of spherical light rays that have turning points at $\varnothing = \varnothing_O$. In other words, we get the interval of allowed $r_p$ values by intersecting the photon region with the cone $\varnothing = \varnothing_O$. Each value of $r_p$ in the interval $[r_{p,\min}, r_{p,\max}]$ corresponds to two points on the boundary curve of the shadow whose $\psi$ coordinates $\psi_1 \in [ -\pi/2, \pi/2]$ and $\psi_2 \in [\pi/2, 3\pi/2]$ are related by $\sin \psi_1 = \sin \psi_2$. The corresponding $\theta$ coordinates are the same, $\theta_1 = \theta_2$, because according to (53) they are uniquely determined by $K(r_p)$ and $p_x(r_p)$. This demonstrates the remarkable fact that the shadow is always symmetric with respect to a horizontal axis. This symmetry was not to be expected for an observer off the equatorial plane and a plasma density depending on $\varnothing$. 

Our equations (53) and (55) give us the shape and the size of the shadow for an observer anywhere in the domain of outer communication, providing that the observer’s four-velocity is proportional to our basis vector $e_0$. If the observer is moving with a different four-velocity, the shadow is distorted by aberration, see Grenzebach [21]. When plotting the shadow, below in Figs. 2, 3, and 4 we use stereographic projection onto a plane that is tangent to the celestial sphere at the pole $\theta = 0$, and in this plane we use (dimensionless) Cartesian coordinates,

$$X(r_p) = -2 \tan \left( \frac{\theta(r_p)}{2} \right) \sin \left( \psi(r_p) \right),$$

$$Y(r_p) = -2 \tan \left( \frac{\theta(r_p)}{2} \right) \cos \left( \psi(r_p) \right).$$

We indicate these Cartesian coordinate axes by cross-hairs in our plots. Recall that the pole $\theta = 0$, i.e., the origin of our Cartesian coordinate system, corresponds to a past-oriented ingoing principal null ray through the observer position. This method of plotting the shadow, which was also used in [18], is to be distinguished from the one introduced by Bardeen [16]. Firstly, Bardeen considers an observer at infinity while we allow any observer position in the domain of outer communication. Secondly, Bardeen plots the shadow on a plane where (dimensionful) impact parameters are used as the coordinates on the axes, while our (dimensionless) coordinates are directly related to angular measures on the observer’s sky. Thirdly, the origin of Bardeen’s coordinates corresponds to a null ray with $p_x = 0$ while our origin corresponds to a principal null ray which has $p_x = -a \sin \theta_O$; of course, the choice of the origin is a matter of convention. One just has to keep in mind that our way of plotting may be directly compared with Bardeen’s only if the observer is at a big radius coordinate and that the origin is horizontally shifted.

For the construction of the shadow we prescribe the constant of motion $\omega_0$. There are two different situations that may be considered. (a) Firstly, we may think of static light sources distributed at big radius coordinates that emit light rays monochromatically with this frequency $\omega_0$. Note that these light rays arrive at the observer, whose four-velocity $e_0c$ is given by (45), with different frequencies, i.e., in general this observer will see the shadow against a backdrop that is not monochromatic. This is true also for the shadow without a plasma. Only if the observer is static and far away from the black hole will the backdrop be monochromatic but, for any position of the observer, we may directly use our formula for the boundary curve of the shadow with the prescribed $\omega_0$. (b) Secondly, we may think of light sources that emit a wide range of frequencies, and of an observer that filters out a particular frequency $\omega_{\text{obs}}$. Then we have to express $\omega_0$ in our equations for the shadow in terms of $\omega_{\text{obs}}$. The desired relation between the two frequencies follows from (47) with $U^\mu = e_0^\mu c$ if $e_0^\mu$ is inserted from (45). We find that

$$\omega_{\text{obs}} = \frac{(r^2 + a^2)\omega_0 + a p_x}{\rho \sqrt{\Delta}} \bigg|_{(r_0, \varnothing_O)}.$$  

After solving this equation for $\omega_0$, we are able to replace $\omega_0$ in (53) and (55) by $\omega_{\text{obs}}$. The same substitution has to be made in (39) and (40) with $r = r_p$; inserting the latter expressions into (53) and (55) gives the boundary curve of the shadow parametrized by $r_p$, now for prescribed $\omega_{\text{obs}}$. The range of the curve parameter $r_p$ has to be determined for each value of $\omega_{\text{obs}}$ individually. Note that, by (58), $\omega_{\text{obs}} \to \omega_0$ for $r_0 \to \infty$, i.e., that there is no difference between case (a) and case (b) if the observer is at a big radius coordinate. 

For the reader’s convenience, we end this section by summarizing the construction method of the shadow in a step-by-step procedure.

1. Choose the mass parameter $m$ and the spin parameter $a$ with $a^2 \leq m^2$. The mass parameter $m$ gives a natural length unit, i.e., all other lengths may be given
in units of \( m \).

2. Choose a plasma frequency \( \omega_p(r, \vartheta) \) around the black hole which is related to the electron number density by the formula \( \omega_0 \). The plasma frequency has to satisfy the separability condition \( \omega_0 \), so the plasma distribution is characterized by two functions \( f_r(r) \) and \( f_\varphi(\vartheta) \). The refractive index of the plasma is then given by the expression \( \omega_0 \). Only for such plasma frequencies are the ray equations completely integrable and the shadow can be calculated analytically.

3. Choose a position of an observer anywhere in the domain of outer communication by prescribing its radial and angular coordinate \( r_O \) and \( \vartheta_O \). For an illustration see Fig. 7 in \[18\].

4. Choose the constant of motion \( \omega_0 \) for the rays that are to be considered. In the formulas for the shadow, \( \omega_0 \) will enter only in terms of the quotient \( \omega_p(r, \vartheta)^2/\omega_0 \). Therefore, it is convenient to give \( \omega_p(r, \vartheta) \) and \( \omega_0 \) as multiples of the same frequency unit \( \omega_c \) which will then drop out from all relevant formulas.

5. Write the celestial coordinates \( \sin \theta \) and \( \sin \psi \) in terms of the constants of motion of the corresponding ray by \( \left( \frac{33}{19} \right) \) and \( \left( \frac{35}{19} \right) \) with \( r = r_0 \) and \( \vartheta = \vartheta_0 \) substituted. For an illustration of the angles \( \theta \) and \( \psi \) see Fig. 8 in \[18\].

6. Substitute into these expressions for \( \sin \theta \) and \( \sin \psi \) the expressions \( K(r_p) \) and \( p_\varphi(r_p) \) according to formulas \( \left( \frac{39}{19} \right) \) and \( \left( \frac{40}{19} \right) \) with \( r = r_p \). Here \( r_p \) runs over an interval of radius coordinates for which unstable spherical light rays exist. This gives us \( \sin \theta \) and \( \sin \psi \) as functions of \( r_p \), i.e., it gives us a curve on the observer’s sky. This is the boundary curve of the shadow. In the next two steps we determine the range of the curve parameter \( r_p \).

(Here we assume \( a \neq 0 \). In the non-spinning case \( a = 0 \), the shadow is circular and the equation for \( \sin \theta \) gives us directly the angular radius of the shadow.) Note that, by \( \left( \frac{39}{19} \right) \) and \( \left( \frac{40}{19} \right) \), there is a sign ambiguity in the expressions for \( K(r_p) \) and \( p_\varphi(r_p) \). We will show below that for a wide range of plasma distributions, including all the examples considered in this paper, the equations can hold only with the plus sign. However, it is easy to construct mathematical examples where solutions with the minus sign occur.

7. Solve the equation \( \sin \psi(r_p) = 1 \) for \( r_p \). This gives us the minimal value \( r_{p,\text{min}} \). Note that there could be several solutions of this equation which are real and \( > m \). Then one has to determine, depending on the position of the light sources and of the observer, which one is relevant for the formation of the shadow. E.g., if there are two photon regions with unstable spherical light rays and if the observer is outside the outer one, there are two possible values for \( r_{p,\text{min}} \). If we stick with our general rule that there are light sources everywhere but not between the observer and the black hole (i.e., not in the region crossed by past-oriented light rays that approach the horizon), then we have to choose for \( r_{p,\text{min}} \) a value on the boundary of the inner photon region. However, if there are light sources only at big radius values, then we have to choose a value on the inner boundary of the outer photon region. If the plasma density is small and if the observer is in the equatorial plane, for small \( a \) it will be \( r_{p,\text{min}} \leq 3m \). If the plasma density is small, for small \( a \) it will be \( r_{p,\text{max}} \gtrsim 3m \). If a nearly extreme Kerr black hole \( (a \lesssim m) \) it will be \( r_{p,\text{max}} \gtrsim m \).

8. Solve the equation \( \sin \psi(r_p) = -1 \) for \( r_p \). This gives us the maximal value \( r_{p,\text{max}} \). As in the case of \( r_{p,\text{min}} \), there may be several solutions. If the plasma density is small, for small \( a \) it will be \( r_{p,\text{max}} \gtrsim 3m \). If a nearly extreme Kerr black hole \( (a \lesssim m) \) it will be \( r_{p,\text{max}} \gtrsim 4m \).

9. Calculate \( \sin \theta(r_p) \) and \( \sin \psi(r_p) \) where \( r_p \) ranges over the interval \( [r_{p,\text{min}}, r_{p,\text{max}}] \). Note that the \( \theta(r_{p,\text{min}}) + \theta(r_{p,\text{max}}) \) gives the horizontal diameter of the shadow.

10. Calculate the dimensionless Cartesian coordinates \( X \) and \( Y \) of the boundary curve of the shadow by formulas \( \left( \frac{57}{19} \right) \). Choosing \( -\pi/2 \leq \psi(r_p) \leq \pi/2 \) and letting \( r_p \) run from \( r_{p,\text{min}} \) to \( r_{p,\text{max}} \) gives the lower half of the boundary curve of the shadow. The upper half of the curve is the mirror image of the lower half with respect to a horizontal axis.

11. The preceding instruction applies to the case that \( \omega_0 \) is prescribed and that the observer has four-velocity \( e_0 \) as given in \( \left( \frac{45}{19} \right) \). If the shadow is to be determined for prescribed \( \omega_{\text{obs}} \), one has to express \( \omega_0 \) in terms of \( \omega_{\text{obs}} \) by \( \left( \frac{55}{19} \right) \), as outlined above. If the shadow is to be determined for an observer whose four-velocity is not equal to \( e_0 \), one has to apply the special-relativistic aberration formula, cf. \( \left( \frac{24}{19} \right) \) (and, in the case that \( \omega_{\text{obs}} \) is prescribed, the special-relativistic Doppler formula for \( \omega_{\text{obs}} \)).

VI. THE CASE OF A LOW DENSITY PLASMA

In this section we consider the case that the plasma frequency is small in comparison to the photon frequency. More precisely, we assume that the separability condition \( \omega_0 \) is satisfied and that the functions \( f_r(r), f_\varphi(\vartheta), f'_r(r), f'_\varphi(\vartheta) \) are so small that all expressions can be linearized with respect to these functions. (Actually, the function \( f'_\varphi(\vartheta) \) will not occur in the following calculations.)

We will first show that in this case \( \left( \frac{40}{19} \right) \) and, thus, \( \left( \frac{39}{19} \right) \) can hold only with the plus sign. By contradiction, let us assume that \( \left( \frac{40}{19} \right) \) holds with the minus sign. Then linearization of the square-root in \( \left( \frac{40}{19} \right) \) yields

\[
K = -f_r(r) + \ldots, \tag{59}
\]

hence

\[
K - f_\varphi(\vartheta) = -\omega_p(r, \vartheta) \frac{r^2}{c^2} + \ldots, \tag{60}
\]

This is a contradiction because, by \( \left( \frac{28}{19} \right) \), the left-hand side of \( \left( \frac{60}{19} \right) \) cannot be negative.

We are now ready to calculate a linear correction to the shadow due to the presence of a low density plasma. The boundary curve of the shadow for the observer at \( (r_0, \vartheta_0) \) is determined by equations \( \left( \frac{53}{19} \right) \) and \( \left( \frac{55}{19} \right) \) with \( K(r_p) \) and \( p_\varphi(r_p) \) given by \( \left( \frac{40}{19} \right) \) and \( \left( \frac{39}{19} \right) \) with the plus sign.
Here $r_p$ runs over the radius values of spherical light rays which satisfy \((43)\) with the plus sign. This gives us the angles $\theta$ and $\psi$ as functions of $r_p$, for a given observer position $(r_0, \vartheta_0)$.

Expanding (63) and (65), we obtain:

$$\sin \theta(r_p) = \frac{2 r_p \sqrt{\Delta_p \Delta_O}}{|Z|} + \frac{(r_p - m) \sqrt{\Delta_p \Delta_O} \left[ \Delta_p f'_p(r_p) Z + 2 \Delta_O f'_r(r_p) r_p (r_p - m) \right] - (r_p - m) \sqrt{\Delta_p \Delta_O} \left[ \Delta_p f'_p(r_p) + f_r(r_p) (r_p - m) + f_\vartheta(\vartheta_O) (r_p - m) \right]}{4 r_p \omega^2 \Delta_p |Z|},$$

$$\sin \psi(r_p) = -\frac{C_1}{a \sqrt{\Delta_p \sin \vartheta_O r_p}} - \frac{C_1 (r_p - m) \left[ \Delta_p f'_p(r_p) + f_r(r_p) (r_p - m) + f_\vartheta(\vartheta_O) (r_p - m) \right]}{16 \omega^2 a \Delta_p^{3/2} \sin \vartheta_O r_p^2} + \frac{\Delta_p f'_p(r_p) (r_p - m)}{8 \omega^2 a r_p^2 \sin \vartheta_O},$$

where

$$Z = r^2_O (r_p - m) + a^2 r_p - m r^2 + r_p \Delta, \quad \Delta = r^2_p + a^2 - 2 m r_p, \quad \Delta_O = r^2_O + a^2 - 2 m r_O. \quad (63) \quad (64)$$

Here $r_p$ is varying between a minimum value $r_{p, \text{min}}$ and a maximum value $r_{p, \text{max}}$ which are found from the equations $\sin \psi(r_p) = 1$ and $\sin \psi(r_p) = -1$, respectively.

In formulas (61) and (62) the first terms are vacuum terms, while the other terms are proportional to $1/\omega^2$. So in the case of high frequencies the plasma terms tend to zero.

### VII. THE SCHWARZSCHILD CASE

In the Schwarzschild case, $a = 0$, the separability condition \((26)\) requires the plasma frequency to be of the form

$$\omega_p(r, \vartheta)^2 = \frac{f_r(r) + f_\vartheta(\vartheta)}{r^2}. \quad (66)$$

This condition is, of course, in particular satisfied if $\omega_p$ depends only on $r$; this case was treated in our earlier paper \((17)\). However, the separability condition is also satisfied for some $\vartheta$ dependent distributions that may be considered, e.g., as reasonable models for a plasma that rotates about a Schwarzschild black hole. In the following we discuss this more general case.

For $a = 0$, the inequality \((13)\) reduces to an equality, as the right-hand side cannot be negative, so we get

$$0 = \frac{\omega^2_0}{(r - m)^2} \left[ -m r^2 \pm r^2 (r - 2 m) \sqrt{1 - f'_r(r) \frac{r - m}{2 r^2 \omega^2_0}} \right]^2 \Rightarrow \sin \theta = \frac{\sqrt{\frac{K - f_\vartheta(\vartheta)}{r^3 \omega^2_0} - f_r(r) - f_\vartheta(\vartheta)}}{(r_0, \vartheta_0)}. \quad (67)$$

Hence, the photon region degenerates to a photon sphere (or to several photon spheres) in this case. As we restrict to the domain of outer communication where $r > 2 m$, only the upper sign is possible and \((67)\) can be rewritten as

$$m = (r - 2 m) \sqrt{1 - f'_r(r) \frac{(r - m)}{2 r^2 \omega^2_0}}. \quad (68)$$

The same result follows from \((39)\). Upon squaring, the condition for a photon sphere can be rewritten as

$$(r - 3 m) 2 \omega^2_0 = \left(1 - \frac{2 m}{r} \right)^2 f'_r(r). \quad (69)$$

By \((40)\) and \((68)\), for a light ray on a photon sphere at $r = r_p$ the Carter constant takes the value

$$K = \frac{r^3_p \omega^2_0}{(r_p - 2 m)} - f_r(r_p). \quad (70)$$

The radius of the photon sphere and the corresponding Carter constant depend on $f_r(r)$ and on the frequency $\omega_0$ but not, remarkably, on $f_\vartheta(\vartheta)$. If $f'_r(r) = 0$, we can incorporate the constant $f_r(r)$ into the function $f_\vartheta(\vartheta)$, i.e., $\omega_p(r, \vartheta)^2 = f_\vartheta(\vartheta)/r^2$. Then the photon sphere is at $r = 3 m$, for any choice of $f_\vartheta(\vartheta)$ and all frequencies $\omega_0$, as can be read from \((69)\).

For the formation of a shadow we have to choose $f_r(r)$ and $f_\vartheta(\vartheta)$ such that there is an unstable photon sphere at a radius $r = r_p$. (If there are several unstable photon spheres, one has to determine which one is relevant for the shadow, depending on the position of the observer.) For $a = 0$ \((53)\) reduces to
Inserting into (71) the fixed value of $K$ from (70) gives the angular radius of the shadow which is circular,

$$\sin \theta = \sqrt{\frac{r_p^3 (r_O - 2m)}{r_O^3 (r_p - 2m)}} \times \left(1 - \frac{(f_r(r_p) + f_\vartheta(\vartheta_O)) \left(r_p - 2m \right)}{r_p^3 \omega_0^2} \right).$$

In the Schwarzschild case Eq. (72) gives the relation between the azimuthal angle $\psi$ and the constant of motion $p_\varphi$ but no further information on the shadow. From (72) we read that the plasma has an increasing effect of the shadow if and only if

$$\omega_p(r_O, \vartheta_O)^2 \left(1 - \frac{2m}{r_O} \right) > \omega_p(r_p, \vartheta_O)^2 \left(1 - \frac{2m}{r_p} \right).$$

This inequality is necessarily violated if the observer is in a vacuum region, i.e., if $\omega_p(r_O, \vartheta_O) = 0$. Moreover, if $\omega_p(r_O, \vartheta_O)$ goes to zero for $r_O \to \infty$, the inequality (73) is necessarily violated for observers who are sufficiently far away from the black hole. So in both cases the plasma either decreases the angular radius of the shadow or leaves it unchanged.

We summarize the results for a plasma with density $\omega_p(r, \vartheta)^2 = (f_r(r) + f_\vartheta(\vartheta))/r^2$ on the Schwarzschild spacetime: There is no photon region but a photon sphere (or several photon spheres), and the shadow is always circular, although the plasma distribution is not necessarily spherically symmetric. The violation of spherical symmetry is reflected by the fact that, by (72), the angular radius of the shadow depends on the $\vartheta$ coordinate of the observer.

We want to further specify the equations for the shadow in a plasma on the Schwarzschild spacetime for the case of a low density distribution. As before, by that we mean that all expressions may be linearized with respect to $f_r(r)$, $f_\vartheta(\vartheta)$ and $f'_r(r)$.

Linearization of (69) yields

$$r_p = 3m + \frac{f'_r(3m)}{18 \omega_0^2} \ldots$$

As $f_\vartheta(\vartheta)$ has no influence on the position of the photon sphere, this result is in agreement with Eq.(33) of our earlier paper [17] where we considered the case $f_\vartheta(\vartheta) = 0$. Using (74), linearization of (72) yields

$$\sin \theta = \frac{3 \sqrt{3} m \sqrt{r_O - 2m}}{r_O^{3/2}} \left(1 + \frac{\omega_p(r_O, \vartheta_O)}{2 \omega_0^2} \right) (1 - \frac{2m}{r_O}) - \frac{\omega_p(3m, \vartheta_O)^2}{6 \omega_0^2} + \ldots.$$
VIII. THE KERR SHADOW FOR SOME SPECIFIC PLASMA DISTRIBUTIONS

In this section we discuss a few examples of plasma distributions on the Kerr spacetime that satisfy the separability condition \((26)\). An interesting example would be the case that the plasma frequency depends on \(r\) and \(\vartheta\) as the mass density of a dust that is at rest at infinity. The analytical form of this matter distribution was found by Shapiro \([25]\) whose calculation even included a non-zero pressure. If one specializes his result to the pressureless case, one finds that the mass density is independent of \(\vartheta\) and proportional to \(r^{-3/2}\). In this case the separability condition \((26)\) is not satisfied on a Kerr spacetime with \(a \neq 0\), which means that our way of calculation can not be directly applied. However, our analytical formulas can be applied if we assume an additional \(\vartheta\) dependence of the plasma frequency, see Example 3 below. For the Schwarzschild case, \(a = 0\), the shadow in a plasma with plasma frequency proportional to \(r^{-3/2}\) was discussed in our earlier paper \([17]\).

In all the examples we are going to treat in this section we consider a plasma frequency that satisfies the separability condition \((26)\) with

\[
f_r(r) = C r^k, \quad f_\vartheta(\vartheta) \geq 0
\]  

(79)

where \(C \geq 0\) and \(0 \leq k \leq 2\). We will now show that in all these cases \((40)\) can hold only with the plus sign. We first treat the case \(C = 0\). If we assume that in this case \((40)\) holds with the minus sign we find that \(K = 0\). By \((28)\) this requires \(f_\vartheta(\vartheta) = 0\), i.e., we are in the vacuum case for which the solution set of \((43)\) with the lower sign is empty. We now treat the case \(C > 0\). We know from \((28)\) that \(K - f_\vartheta(\vartheta) \geq 0\), therefore our assumption \(f_\vartheta(\vartheta) \geq 0\) implies that \(K\) is non-negative. Let us assume that \((40)\) holds with the minus sign and write \(K\) as

\[
K = \frac{r^2 \Delta \omega_0^2}{(r-m)^2} (1 - \sqrt{1-x})^2 - f_r(r)
\]

(80)

with

\[
x = f'_r(r) \frac{r - m}{2r^2 \omega_0^2}, \quad 0 \leq x \leq 1.
\]

(81)

It can be easily shown that

\[
(1 - \sqrt{1-x})^2 \leq x^2 \leq x \quad \text{for} \quad 0 \leq x \leq 1.
\]

(82)

Hence

\[
K = \frac{r^2 \Delta \omega_0^2}{(r-m)^2} (1 - \sqrt{1-x})^2 - f_r(r)
\]

(83)

\[
\leq \frac{r^2 \Delta \omega_0^2}{(r-m)^2} x - f_r(r) = \frac{1}{2} \frac{\Delta f'_r(r)}{r - m} - f_r(r).
\]

We see that \(\omega_0^2\) has dropped out. As, by \((79)\),

\[
f'_r(r) = k \frac{f_r(r)}{r},
\]

(84)

this implies

\[
K \leq \frac{k}{2} \frac{\Delta f_r(r)}{r(r-m)} - f_r(r).
\]

(85)

As we assume \(0 \leq k \leq 2\), this implies

\[
K \leq \frac{\Delta f_r(r)}{r(r-m)} - f_r(r)
\]

(86)

\[
(\frac{r^2 - 2mr + a^2}{r(r-m)} f_r(r) - \frac{f_r(r)}{r})
\]

\[
\leq \frac{(r^2 - 2mr + m^2 f_r(r)}{r(r-m)} - f_r(r)
\]

\[
= \frac{r - m}{r} f_r(r) - \frac{m}{r} f_r(r) = - \frac{m}{r} f_r(r) < 0.
\]

We have thus obtained that \(K < 0\), which is the desired contradiction.

Therefore we have to consider \((40)\) only with the plus sign in the following examples.

As explained at the end of Sec. \([10]\) we use in all examples the same method of plotting the shadow as in \([15]\). When we are interested in the extreme case, \(a = m\), we actually choose \(a = 0.999m\) for our plots because our analytical formulas for the boundary curve of the shadow involve undetermined expressions when \(a\) is exactly equal to \(m\). Of course, the pictures of the shadow for \(a = 0.999m\) are practically indistinguishable from those for \(a = m\).

Example 1: The vacuum shadow on Kerr spacetime

For the sake of comparison, we briefly revisit the known case of the Kerr shadow in vacuum, \(\omega_p = 0\), which is equivalent to the limiting case of infinite photon frequency, \(\omega_0 \rightarrow \infty\), in a plasma of arbitrary density. The vacuum case is described by the first terms in formulas \((61)\) and \((62)\). These analytical formulas for the boundary curve of the shadow of a Kerr black hole in vacuum can be found as special cases in \([13]\) and \([19]\). In Fig. 2 we plot the photon region of an almost extreme Kerr black hole in vacuum and the shadow for different parameters \(a\).

Example 2: A plasma density with \(f_r(r) = 0\) on Kerr spacetime

As an example, we consider a plasma frequency which satisfies the separability condition with \(f'_r(r) = 0\). As a constant \(f_r(r)\) may be incorporated into \(f_\vartheta(\vartheta)\), we may choose \(f_r(r) = 0\), i.e.

\[
\omega_p(r, \vartheta)^2 = \frac{f_\vartheta(\vartheta)}{r^2 + a^2 \sin^2 \vartheta}.
\]

(87)

Then we must have \(f_\vartheta(\vartheta) \geq 0\) and, as shown in the introductory part of this section, the inequality \((43)\) can hold only with the upper sign, i.e., the photon region is given by

\[
\left( \frac{4r^2 \Delta}{(r-m)^2} - \frac{f_\vartheta(\vartheta)}{\omega_0^2} \right) a^2 \sin^2 \vartheta \geq 0.
\]

(88)
of a spherical light ray at \( r = r_p \) are given by

\[ a p_\varphi = -\frac{\omega_0}{(r - m)} \left( m(a^2 - r^2) + r \Delta \right), \quad (89) \]

By (39) and (40), respectively, the constants of motion with a constant \( \omega \), where \( \omega \) is a constant with the dimension of a frequency.

For the pictures in Fig. 3 we have chosen \( f_\vartheta(\vartheta) = \omega_c^2 m^2 (1 + 2 \sin^2 \vartheta) \), i.e.

\[ \omega_p(r, \vartheta)^2 = \frac{\omega_c^2 m^2 (1 + 2 \sin^2 \vartheta)}{r^2 + \omega_c^2 \cos^2 \vartheta} \quad (91) \]

Example 3: A plasma density with \( f_r(r) \propto \sqrt{r} \) and \( f_\vartheta(\vartheta) = 0 \) on Kerr spacetime

As our next example, we consider an inhomogeneous plasma where the plasma density is proportional to \( r^{-3/2} \), as for a dust; however, in order to satisfy the separability condition we have to assume a \( \vartheta \) dependence in addition,

\[ \omega_p(r, \vartheta)^2 = \frac{\omega_c^2 \sqrt{m^3 r}}{r^2 + a^2 \cos^2 \vartheta} \quad (92) \]

Example 4: A homogeneous plasma on Kerr spacetime

Finally let us consider the case of a homogeneous plasma, \( \omega_p(r, \vartheta) = \omega_c = \text{constant} \), for which the functions \( f_r(r) \) and \( f_\vartheta(\vartheta) \) are given by (39). The new feature, in comparison to the previous examples, is the existence of stable spherical light rays. This implies that from some observer positions there are light rays issuing into the past that go neither to infinity nor to the horizon; they rather stay inside a spatially compact region. If we stick with the rule that we assign darkness only to those past-oriented light rays that go to the horizon, we have to assign brightness to these light rays. The existence of bound photon orbits in a homogeneous plasma was also discussed by Kulsrud and Loeb [27] and by Bisnovatyi-Kogan and Tsupko [8]. Note that stable circular light rays exist also in some non-homogeneous plasma distributions, see Rogers [12].

Figure 5 shows the development of the stable and unstable photon regions and of the forbidden region. In this case the presence of the plasma always makes the shadow larger, in comparison to an observer at the same coordinate position in vacuum, see Fig. 6. This is similar to the Schwarzschild case, see Sec. VII. If \( \omega_c^2 / \omega_0^2 \) reaches a certain limit value, the shadow covers the entire sky.

**IX. CONCLUSIONS**

In this work we have investigated the propagation of light rays in a non-magnetized pressureless plasma on Kerr spacetime. We have worked in the framework of geometrical optics, considering the plasma as a medium with dispersive properties given by a frequency-dependent index of refraction. The gravitational field is determined by the mass and the spin of the Kerr black hole, i.e., the gravitational field of the plasma particles is not taken into account. In our model, the presence of the plasma manifests itself only in a change of the trajectories of light rays. Our investigation is valid for any value of the spin parameter \( a \) and for any light rays in the domain of outer communication. As the plasma is a dispersive medium, photons with different frequencies move along different trajectories. Therefore, all phenomena described here are chromatic.

We have utilized the Hamiltonian formalism for light rays in a plasma on Kerr spacetime and we have, in an important first step of our analysis, determined the necessary and sufficient condition for separability of the Hamilton-Jacobi equation, see Secs. II and III. We have demonstrated that the Hamilton-Jacobi equation is separable, i.e., that a generalized Carter constant exists, only for special distributions of the plasma electron density. The necessary and sufficient condition for separability is given in eq. (26).

For all cases where the condition of separability is satisfied we have determined the photon region, i.e., the region in spacetime where spherical light rays exist, see Sec. IV. Examples of photon regions are given in...
We have derived analytical formulas for the boundary curve of the shadow on the observers sky in terms of two angular celestial coordinates, see formulas (55) and (56) with (39) and (40). Our formulas are valid for any photon frequency at infinity $\omega_0$, for any value of the spin parameter $a$, for any position of the observer outside of the horizon of the black hole (i.e., with arbitrary distance from the black hole and arbitrary inclination) and for any plasma distribution which satisfies the separability condition. For the reader’s convenience, we have written a step-by-step procedure for the construction of the shadow, see the end of Sec. VII. We have worked out several examples, considering a low-density plasma in Sec. VI, a non-rotating black hole in Sec. VII and several specific plasma distributions on the Kerr spacetime in Sec. VIII.

In our approach, the plasma influences only the trajectories of light rays which results in a change of the geometrical size and shape of the shadow. We have not taken into account processes of absorption or scattering of photons; there are several ray tracing codes for light propagation in matter on the Kerr spacetime that take such effects into account, see e.g. James et al. [26]. In this paper, however, it was our goal to derive an analytical formula for the boundary curve of the shadow under idealized conditions. Our equations may be used as the zeroth order approximation for numerically studying more realistic situations.

In the presence of a plasma around a black hole, the size and the shape of the shadow differs from the vacuum case in a way that depends on the photon frequency. The relevant quantity is the ratio of the plasma frequency to the photon frequency. From our examples we conclude that if the plasma frequency is small in comparison with the photon frequency, the shadow is not very much different from the vacuum case. However, if the plasma frequency is close to the photon frequency the properties of the shadow are changed drastically because of significant changes in the photon regions.

In all applications to astrophysics, the difference of the plasma case to the vacuum case is significant only for radio frequencies. For an estimate of the effects in the case of Sgr $A^*$ and M87 we refer to our previous paper [17]. At low frequencies, i.e. at large wavelengths, where the influence of the plasma is most significant, scattering is expected to be non-negligible. As scattering will partly wash out the shadow, it is expected that the shadow can be observed only at wavelengths of approximately one millimeter or below where the plasma effects are very small.

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FIG. 2. Example 1, $\omega_p = 0$ (vacuum). Top: The photon region is shown for a black hole with $a = 0.999 \, m$. Here and in all the following pictures, the black disk is the region inside the horizon. For the chosen value of $a$ it is almost identical with the sphere of radius $m$. We use coordinates $x$, $y$ (not shown) and $z$ related to the Boyer-Lindquist coordinates by the usual transformation formulas from spherical polars to Cartesian coordinates. The dotted circles (here and in all the following pictures) intersect the boundary of the photon region where circular light rays in the equatorial plane exist; the one on the inner boundary is co-rotating while the one on the outer boundary is counter-rotating. For the chosen value of $a$ in vacuum the inner one is at a radius only slightly bigger than the radius of the horizon. Bottom: The picture shows the shadow for an observer at $r_O = 5 \, m$ and $\theta_O = \pi/2$, with the spin parameter $a$ equal to $0.02 \, m$ (leftmost, black), $0.4 \, m$ (green), $0.75 \, m$ (blue) and $0.999 \, m$ (red). The axes label $X$ and $Y$ refer to the Cartesian coordinates. The dashed circle is the celestial equator.
FIG. 3. Example 2, $\omega_p^2 = \omega_c^2 m^2 (1 + 2 \sin^2 \vartheta)/(r^2 + a^2 \cos^2 \vartheta)$. Top left: The surfaces $\omega_p = \text{constant}$. The next three pictures show the photon region for $a = 0.999 \text{ m}$. Top right: $\omega_c^2 / \omega_0^2 = 1$. The situation is qualitatively similar to the vacuum case, cf. Fig. 2 Middle left: $\omega_c^2 / \omega_0^2 = 7$. In addition to the two circular light rays in the equatorial plane (located where the dotted circles meet the photon region) there is a pair of circular light rays off the equatorial plane (located where the dashed circle meets the photon region). Middle right: $\omega_c^2 / \omega_0^2 = 10$. At $\omega_c^2 / \omega_0^2 \approx 9.000$ the photon region has become detached from the equatorial plane. A region (cross-hatched) has formed which is forbidden for light rays with the chosen $\omega_0$. Observers close to the equatorial plane do no longer see a shadow. At $\omega_c^2 / \omega_0^2 \approx 23.314$ the photon region vanishes, the forbidden region encloses the black hole completely. Bottom left: Boundary curve of the shadow for an observer at $r_0 = 5 \text{ m}$ and $\vartheta_0 = \pi/2$ with $a = 0.999 \text{ m}$ and $\omega_c^2 / \omega_0^2$ equal to 0 (outermost, red), 0.4 (blue), 1 (green), 1.5 (orange), 7 (magenta) and 8.9 (black). The shadow has shrunk to a point when the forbidden region reaches the observer position. Bottom right: Enlarged view of the first four boundary curves, stretched in the horizontal direction by a factor of 40.
FIG. 4. Example 3, $\omega_p(r, \theta)^2 = \omega_0^2 \sqrt{m \gamma (r^2 + a^2 \cos^2 \theta)}$. Top left: The surfaces $\omega_p = \text{constant}$. The next three pictures show the photon region for $a = 0.999 \text{ m}$. Top right: $\omega_p^2/\omega_0^2 = 7$. The situation is qualitatively similar to the vacuum case, cf. Fig. 2. Middle left: $\omega_p^2/\omega_0^2 = 14.5$. At $\omega_p^2/\omega_0^2 \approx 14.402$ the photon region has become detached from the axis. A region (cross-hatched) has formed which is forbidden for light rays with the chosen $\omega_0$. Observers close to the axis do no longer see a shadow. Middle right: $\omega_p^2/\omega_0^2 = 15.1$. The photon region has further shrunk. At $\omega_p^2/\omega_0^2 \approx 15.215$ it has vanished and the forbidden region encloses the black hole completely. Bottom left: Boundary curve of the shadow for an observer at $r_0 = 5 \text{ m}$ and $\theta_0 = \pi/2$ with $a = 0.999 \text{ m}$ and $\omega_p^2/\omega_0^2$ equal to 0 (outermost, red), 1 (blue), 3 (green), 6 (orange), 13 (magenta) and 15 (black). The shadow has shrunk to a point when the forbidden region reaches the equatorial plane. Bottom right: Enlarged view of the first four boundary curves, stretched in the horizontal direction by a factor of 40.
FIG. 5. Example 4, $\omega_p(r, \theta)^2 = \omega_0^2$. The pictures show the photon region for $a = 0.999\, m$ and $\omega_2^2/\omega_0^2$ equal to 1.079 (top), 1.085 (middle) and 1.16 (bottom). For $\omega_2^2/\omega_0^2 < 1$ the photon region is qualitatively similar to the vacuum case. At $\omega_2^2/\omega_0^2 = 1$ a forbidden region (cross-hatched) and a region with stable spherical light rays (hatched, green) come into existence. At $\omega_2^2/\omega_0^2 \approx 1.080$ the stable and the unstable photon regions come together. At $\omega_2^2/\omega_0^2 \approx 1.14$ the photon region becomes detached from the axis. At $\omega_2^2/\omega_0^2 \approx 2.29$ the photon region vanishes.
FIG. 6. Example 4, \( \omega_p(r, \vartheta)^2 = \omega_2^2 \). The picture shows the boundary curve of the shadow for an observer at \( r_O = 5 \text{ m} \) and \( \vartheta_O = \pi/2 \) with \( a = 0.999 \text{ m} \) and \( \omega_2^2/\omega_0^2 \) equal to 0 (innermost, red), 0.8 (green), 1.085 (blue), 1.2 (magenta) and 1.345 (black). When the forbidden region is formed at \( \omega_2^2/\omega_0^2 = 1 \), the shadow includes the point \( \theta = \pi \) (i.e., the direction pointing away from the black hole that corresponds to the point at infinity in the stereographic projection). Then, for a certain range of values of \( \omega_2^2/\omega_0^2 \), the shadow displays two structures similar to fish-tails, bright inside and dark outside. If \( \omega_2^2/\omega_0^2 \) is further increased, the shadow grows until it covers the entire sky.