Topological Aspects of Spin and Statistics in Nonlinear Sigma Models

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Abstract

We study the purely topological restrictions on allowed spin and statistics of topological solitons in nonlinear sigma models. Taking as space the connected d-manifold X, and considering nonlinear sigma models with the connected manifold M as target space, topological solitons are given by elements of $\pi_d(M)$. Any topological soliton $\alpha \in \pi_d(M)$ determines a quotient \( \text{Stat}_n(X, \alpha) \) of the group of framed braids on X, such that choices of allowed statistics for solitons of type $\alpha$ are given by unitary representations of $\text{Stat}_n(X, \alpha)$ when $n$ solitons are present. In particular, when $M = S^2$, as in the $O(3)$ nonlinear sigma model with Hopf term, and $\alpha \in \pi_2(S^2)$ is a generator, we compute that $\text{Stat}_n(R^2, \alpha) = \mathbb{Z}$, while $\text{Stat}_n(S^2, \alpha) = \mathbb{Z}_{2n}$. It follows that phase $\exp(i\theta)$ for interchanging two solitons of type $\alpha$ on $S^2$ must satisfy the constraint $\theta = k\pi/n$, $k \in \mathbb{Z}$, when $n$ such solitons are present.

1 Introduction

The behavior of quantum systems under the interchange of identical particles goes under the name of “statistics.” In elementary quantum mechanics, the interchange
of identical particles is assumed to affect the wavefunction of a many-particle system only by the introduction of a phase, and interchange is modelled in terms of the symmetric group, $S_n$. Thus for a system of $n$ identical particles one assumes that there is a character $\chi: S_n \to U(1)$ describing the statistics. For $n \geq 2$, $S_n$ has two characters: either $\chi(\sigma) = 1$ for all $\sigma \in S_n$, or $\chi(\sigma) = \text{sgn}(\sigma)$. Thus interchanging two particles must give rise to a phase of 1 or $-1$, corresponding to bosonic and fermionic statistics, respectively. Statistics in which particle interchange gives rise to higher-dimensional unitary representations of $S_n$ have also been studied; these go under the name of “parastatistics” or “nonabelian statistics” [1, 2].

More recently, in the physics of two-dimensional systems, it has been noted that a more precise description of the interchange of identical particles should keep track of the braid traced out by particles as they are interchanged [3]. For structureless point particles on the plane this amounts to replacing the symmetric group by the Artin braid group, $B_n$. The group $B_n$ has a circle’s worth of characters; that is, interchanging two particles gives rise to an arbitrary phase $\exp(i\theta) \in U(1)$. These statistics are known as “anyonic” or “fractional.” Anyonic statistics, as well as nonabelian anyonic statistics, appear to play a crucial role in the fractional quantum Hall effect [4, 5]. Anyons have also been proposed as a mechanism for superconductivity [6].

There are further variations on the theme of statistics which we would like to address here. If one considers structureless point particles on the space $X$, the proper analog of the Artin braid group is the group $B_n(X)$ of the braids on $X$ (defined below). The implications for statistics have been studied already by a number of authors [4, 5]. For particles with internal structure, however, one must also keep track of the rotation of the particles as one interchanges them. Thus it is natural to work with the group $FB_n(X)$ of framed braids on $X$. This has the braid group of $X$ as a quotient group, that is, there is a surjective homomorphism

$$\pi: FB_n(X) \to B_n(X).$$

A unitary representation of $FB_n(X)$ corresponds to a choice not only of statistics, but also certain a certain aspect of spin, namely, the phase induced by rotating a particle by $2\pi$. In the case $X = \mathbb{R}^2$, spin and statistics may be chosen independently, at least at the level of many-particle quantum mechanics. The mathematical reason is that in this case there is a surjective homomorphism

$$\iota: B_n(X) \to FB_n(X)$$

such that $\pi(\iota(g)) = g$ for all $g \in B_n(X)$. In general, however—for example, when $X = S^2$—this is not true, so certain relations between spin and statistics arise. These are not to be confused with the spin-statistics relations arising in quantum field theory.

For nonlinear sigma models, further constraints exist on the spin and statistics of topological solitons. This has been noted for the 2+1-dimensional $O(3)$ nonlinear sigma model with Hopf term by Wilczek and Zee [4], and subsequently elaborated on
by Wu and Zee [10], Wen [11], and others. Here we describe a general framework to
study fields from spacetime, \( \mathbb{R} \times X \), to a target manifold \( M \). (In the \( O(3) \) nonlinear
sigma model, \( M = S^2 \).) Topological solitons correspond to elements of \( \pi_d(M) \), where
\( d \) is the dimension of \( X \). Any topological soliton \( \alpha \) defines a certain quotient group
\( \text{Stat}_n(X, \alpha) \) of \( FB_n(X) \), and a representation \( \rho \) of \( FB_n(X) \) corresponds to an allowed
choice of spin and statistics for solitons of type \( \alpha \) if and only if \( \rho \) factors through
\( \text{Stat}_n(X, \alpha) \), that is, if there exists a representation \( \tilde{\rho} \) of \( \text{Stat}_n(X, \alpha) \) such that
\[
\rho = \tilde{\rho} j
\]
where \( j: FB_n(X) \to \text{Stat}_n(X, \alpha) \) is the quotient map.

An analysis along these lines leads to some interesting results. For example, using
a braid group analysis Thouless and Wu [7] concluded that the phase \( \exp(i\theta) \) from
interchanging two particles on \( S^2 \) must satisfy
\[
\theta = \frac{k\pi}{n-1}, \quad k \in \mathbb{Z}
\]
when \( n \) particles are present. This is correct for structureless point particles. However,
a study of the \( O(3) \) nonlinear sigma model gives a different result for solitons with
unit topological charge on \( S^2 \), namely:
\[
\theta = \frac{k\pi}{n}, \quad k \in \mathbb{Z}.
\]

As alluded to above, in quantum field theory the axioms of locality, Poincaré-
invariance, energy positivity, and so on give rise to additional relationships between
spin and statistics. These are of a different character than our results, which are
derived in the context of many-particle quantum mechanics by purely topological
methods. For spin-statistics theorems in quantum field theory, we refer the reader to
the original arguments of Fierz and Pauli [12], the theorems proved using the Garding-
Wightman [13] and C*-algebraic axioms [2, 14], and the more recent extensions to
field theory in 2 or 3 dimensions, in which anyonic statistics arise [15].

2 Spin, Statistics, and Framed Braids

The relations between spin, statistics, and braid groups treated here arise naturally
from considering quantization of systems for which the classical configuration space
\( C \) is not simply connected [16]. While the most obvious choice of the quantum
Hilbert space is simply the space \( L^2(C) \) of square-integrable functions on \( C \), one may
equally well use \( L^2(C, E) \), the space of \( L^2 \) sections of a flat line bundle \( E \) over \( C \).
Isomorphism classes of flat line bundles over \( C \) are in one-to-one correspondence with
characters of the fundamental group \( \pi_1(C) \). For any group \( G \), let \( G^* \) denote the group
of characters, or one-dimensional unitary representations, of \( G \). Then each element
of $\pi_1(C)^*$ gives a different quantization. (Of course, if $C$ is infinite-dimensional there are severe analytical difficulties in defining the appropriate $L^2$ spaces, which we do not address here.)

For a system of $n$ indistinguishable structureless point particles on a connected manifold $X$, where we assume that no two particles can be at the same place at the same time, the configuration space is $(X^n - \Delta)/S_n$, where $\Delta \subseteq X$ is given by

$$\Delta = \{(x_1, \ldots, x_n) : \exists j \neq k \ x_j = x_k\},$$

and the symmetric group $S_n$ acts on $X^n - \Delta$ by permutation of the points $(x_1, \ldots, x_n)$. Thus quantizing this system involves choosing a character of the braid group of $X$,

$$B_n(X) = \pi_1((X^n - \Delta)/S_n).$$

In other words elements of $B_n(X)^*$ correspond to choices of (abelian) statistics. Similarly, $k$-dimensional unitary representations of $B_n(X)$ correspond to nonabelian statistics, as they define flat $U(k)$-bundles over the configuration space $(X^n - \Delta)/S_n$.

If $\dim X > 2$, the braid group $B_n(X)$ equals $S_n$ when $X$ is simply connected [18], and in general is the wreath product of $S_n$ and $\pi_1(X)$ [8]. The two-dimensional case is more interesting; for example, $B_n(R^2) = B_n$ is the original braid group due to Artin [17], with generators $s_j$, $1 \leq j < n$, and relations

$$s_js_k = s_ks_j \quad |j - k| \geq 2,$$

$$s_js_{j+1}s_j = s_{j+1}s_j s_{j+1}.$$ 

Here $s_j$ corresponds to the interchange of the $j$th and $(j+1)$st particle in a counterclockwise manner. Later Fadell and Van Buskirk computed $B_n(S^2)$ [19], Van Buskirk computed $B_n(RP^2)$ [20], Birman computed $B_n(T^2)$ [18], and finally Scott computed $B_n(X)$ for $X$ any compact 2-manifold [21]. For example, $B_n(S^2)$ is the quotient of $B_n$ by the additional relation

$$s_1s_2 \ldots s_{n-1}s_{n-1}s_{n-2} \ldots s_1 = 1.$$  (1)

This is not surprising, since any element of $B_n(S^2)$ arises from a braid in $R^2$, which has $S^2$ as its one-point compactification, while the braid on the left side of equation (1) corresponds to moving the first particle in a counterclockwise fashion around the rest, and one may contract the loop traced out by the first particle around the south pole of $S^2$ (the point at infinity).

It is easy to see that $B_n^* = U(1)$, since all characters of $B_n$ are of the form

$$\chi(s_j) = e^{i\theta}.$$ 

Equation (1) implies that $e^{2i(n-1)\theta} = 1$ for any character of $B_n$ that factors through $B_n(S^2)$. Thus $B_n(S^2)^* = Z_{2(n-1)}$ for $n \geq 2$. This was noted by Thouless and Wu [7].
The situation changes when we consider particles with spin, for example, topological solitons in a nonlinear sigma model. As we shall see, in this context one must treat spin and statistics together using framed braids. A framed braid may be thought of as a “ribbon” [22], but here we prefer to think of it as a “thickened” braid. Physically, a thickened braid represents the world-tubes of a number of solitons. Let $X$ be an oriented connected manifold of dimension $d$, and let $D^d$ denote the closed unit ball in $\mathbb{R}^d$. Let $e_i: D^d \to X$, $1 \leq i \leq n$, be disjoint oriented balls embedded in $X$. Let a **framed braid** on $X$ be an oriented embedding $F$ of the disjoint union of $n$ solid cylinders $[0, 1] \times D^d$ in $[0, 1] \times X$, such that $F_i(0, \cdot) = F_i(1, \cdot) = e_i$, where $F_i$ denotes the embedding of the $i$th cylinder, and such that

$$F_i(t, x) = (t, F_{i,t}(x))$$

for some function $F_{i,t}: D^d \to X$. Let $FB_n(X)$ denote the set of homotopy classes of framed braids on $X$, where the homotopy is required to preserve the above conditions on $F$. One can check that $FB_n(X)$ is independent of the embeddings $e_i$, and there is a canonical quotient map

$$\pi: FB_n(X) \to B_n(X).$$

The framed braid group keeps track not only of the interchange of particles, but also their rotation in the process. For example, $FB_n = FB_n(\mathbb{R}^2)$ has generators $s_j$, $1 \leq j < n$, and $t_j$, $1 \leq j \leq n$, and relations

$$s_j s_k = s_k s_j \quad |j - k| \geq 2,$$

$$s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1},$$

$$s_j t_k = t_k s_j \quad k \neq j, j + 1,$$

$$t_{j+1} s_j = s_j t_j,$$

$$t_j s_j = s_j t_{j+1}.$$

The element $s_j$ corresponds to the interchange of the $j$th and $(j + 1)$st particles as before, while $t_j$ corresponds to a $2\pi$ rotation of the $j$th particle. In this case there is a natural inclusion $\iota: B_n \to FB_n$ such that $\pi \circ \iota$ is the identity on $B_n$. In other words, the exact sequence

$$FB_n(X) \to B_n(X) \to 1$$

splits in this case; this occurs whenever the tangent bundle of $X$ is trivializable. Thus every character of $FB_n$ restricts to a character of $B_n$, while conversely every character of $B_n$ extends to a character of $FB_n$. This allows us to describe characters $\chi$ of $FB_n$ in terms of two independent angles $\phi$ and $\theta$, related to spin and statistics, respectively:

$$\chi(t_j) = e^{i\phi}, \quad \chi(s_j) = e^{i\theta}.$$

Thus $FB_n^* = U(1) \times U(1)$ for $n \geq 2$. (Note that the character of $FB_n$ only depends on $\phi$ and $\theta$ modulo $2\pi$, so it detects only the spin mod $\mathbb{Z}$ of the particle in question.)
On $S^2$, however, spin and statistics are inextricably entangled, because $FB_n(S^2)$ is the quotient of $FB_n$ by the relation

$$s_1s_2 \ldots s_{n-1}s_{n-1}s_{n-2} \ldots s_1 = t_1^2.$$ (2)

In other words, the framed braid on $S^2$ in which the first particle moves around the rest but does not rotate about its own axis in the process is homotopic to one in which the first particle experiences a rotation by $4\pi$. This is easily visualized using a slight variant of the “belt trick” proof that $\pi_1(SO(3)) = \mathbb{Z}_2$, as in [23]. Thus there is no inclusion $\iota: B_n(S^2) \to FB_n(S^2)$ with $\pi \circ \iota$ equal to the identity on $B_n(S^2)$, for $n \geq 2$. Moreover, the character $\chi$ of $FB_n$ described above factors through a character of $FB_n(S^2)$ if and only if $e^{2i(n-1)\theta} = e^{2i\phi}$. It follows that $FB_n(S^2)^* \cong U(1) \times \mathbb{Z}_{2(n-1)}$.

3 Nonlinear Sigma Models

In a nonlinear sigma model, fields are described as maps from spacetime, $\mathbb{R} \times X$, to a target space $M$. The classical configuration space is thus the space of maps from $X$ to $M$, denoted Maps($X, M$). The configuration space is a disjoint union of connected components, one for each homotopy class in $[X, M]$. One may construct a flat line bundle on Maps($X, M$) from a flat line bundle on each component. A flat line bundle on the component of Maps($X, M$) containing a given map $f_0: X \to M$ is uniquely determined by a character on $\pi_1$ (Maps($X, M$), $f_0$). Thus, as described in the previous section, quantizing a given component of Maps($X, M$) depends on a choice of such a character.

We will study components of $[X, M]$ corresponding to collections of topological solitons. By a topological soliton, we mean a map $g: X \to M$ that is constant outside a small ball in $X$. With an appropriate Hamiltonian, solitons behave roughly like “point particles” with internal degrees of freedom. To work with topological solitons mathematically the Thom-Pontryagin construction [24] turns out to be quite important. Let $X$ be a compact oriented manifold of dimension $d$, and let $M$ be a connected manifold with a chosen basepoint $\ast$. (We will discuss the case where $X$ is noncompact below.) Each element of $\pi_d(M)$ defines an element $T(\alpha) \in [X, M]$ as follows. Let $e: D^d \to X$ be any oriented embedded ball in $X$. Representing $\alpha$ by a map $g: D^d \to M$ with $g|_{\partial D^d} = \ast$, we define $f_0: X \to M$ by setting $f_0 = ge^{-1}$ in the ball $e(D^d)$ and $f_0 = \ast$ outside the ball. The homotopy class of $f_0$ is obviously independent of the choice of the representative $g \in \alpha$ and the choice of $e$, so we may define the map

$$T: \pi_d(M) \to [X, M]$$

by letting $T(\alpha) = [f_0]$. Note that for any $n \geq 0$, $T(n\alpha)$ can be represented by a map from $X$ to $M$ that is constant outside of $n$ small balls, and looks like the map $f$ in each ball. Thus the path component of Maps($X, M$) corresponding to the element $TP(n\alpha)$ is the configuration space for the system consisting of $n$ solitons.
of type $\alpha$. Consequently, quantizing this system requires a choice of a character of $\pi_1(\text{Maps}(X, M), f_0)$ where $f_0 \in \text{Maps}(X, M)$ is any map with homotopy class $T(n\alpha)$.

We now show how any choice of topological soliton $\alpha \in \pi_d(M)$ determines a quotient group of the framed braid group $FB_n(X)$. Representations of this quotient group will correspond to allowed spin and statistics for solitons of type $\alpha$. First, choose a map $g: D^d \rightarrow M$ representing $\alpha$. Then, given any element $[F] \in FB_n(X)$, recall that for $1 \leq i \leq n$, $F_i: [0, 1] \times D^d \rightarrow [0, 1] \times X$ is an embedding; physically the image of $F_i$ will represent the world-tube of the $i$th soliton. We write

$$F_i(t, x) = (t, F_{i,t}(x))$$

where $(t, x) \in [0, 1] \times D^d$. We define a map $f: [0, 1] \times X \rightarrow M$ by letting $f(t, x) = g(F_{i,t}^{-1}(x))$ if $x \in X$ is in the image of $F_{i,t}$, and $f(t, x) = \ast$ otherwise. Note that $f(t, \cdot) = f_t$ is a loop of maps from $X$ to $M$. Thus $f$ defines an element $[f] \in \pi_1(\text{Maps}(X, M), f_0)$. Note also that $f_0 \in \text{Maps}(X, M)$ has homotopy class $[f_0] = T(n\alpha)$. Thus the group $\pi_1(\text{Maps}(X, M), f_0)$ only depends, up to isomorphism, on $n$ and $\alpha$. Moreover, it is easy to check that $[f]$ is independent of the choice of $g$ with $[g] = \alpha$ and the choice of framed braid $F \in [F]$. It follows that there is a map

$$\psi: FB_n(X) \rightarrow \pi_1(\text{Maps}(X, M), f_0).$$

We call the image of $\psi$ the \textit{spin-statistics group} of $n$ topological solitons of type $\alpha$, and write this group as $\text{Stat}_n(X, \alpha)$. It follows that the only allowed choices of (abelian) statistics $\chi \in FB_n(X)^*$ for solitons of type $\alpha$ are those which factor through $\text{Stat}_n(X, \alpha)$:

$$\chi = \tilde{\chi}\psi.$$

Since $\psi$ is onto, abelian statistics are in fact in one-to-one correspondence with elements of $\text{Stat}_n(X, \alpha)^*$. A similar remark holds for nonabelian statistics, as mentioned in the Introduction.

In the above we have assumed that space, $X$, is compact. In the most important noncompact case, $X = \mathbb{R}^n$, there would be no nontrivial topological solitons according to the definition above, as all maps $f_0: \mathbb{R}^n \rightarrow M$ are homotopic. In discussing solitons on $\mathbb{R}^n$ it is typical to work instead with its compactification, $S^n$. The reason usually given (which is admittedly somewhat heuristic), is that for non-topological terms in the action for the field $f: \mathbb{R} \times \mathbb{R}^n \rightarrow M$ to be finite, $f$ must be static at spatial infinity; that is, for some point $\ast \in M$,

$$\lim_{|\vec{x}| \rightarrow \infty} f(t, \vec{x}) = \ast \in M \quad (3)$$

for all times $t$. Thus for each $t$, the map $f_t: \mathbb{R}^n \rightarrow M$ extends uniquely to a continuous map from $S^n$ to $M$ sending the point at infinity to $\ast$. In other words, the physically relevant configuration space is the space $\text{Maps}_*(S^n, M)$ of \textit{basepoint preserving} maps from $S^n$ to $M$, where we take the point at infinity as the basepoint for $S^n$. The rest
of the analysis of spin and statistics of solitons need be only slightly changed in order to take this into account.

In general, when $X$ is noncompact, let $\overline{X}$ be the one-point compactification of $X$, with the point at infinity as basepoint. Then there is a Thom-Pontryagin map

$$T: \pi_d(M) \to [\overline{X}, M],$$

where $[\overline{X}, M]_*$ denotes the basepoint preserving homotopy classes of basepoint preserving maps from $\overline{X}$ to $M$. Let $\alpha \in \pi_d(M)$. Then for any $f_0: \overline{X} \to M$ with $[f_0] = T(n\alpha)$, there is a map

$$\psi: FB_n(X) \to \pi_1(\text{Maps}_*(X, M), f_0),$$

defined just as in the compact case, and we define the image of $\psi$ to be $\text{Stat}_n(X, \alpha)$. (In certain cases compactifications other than the one-point compactification would be more appropriate, but the necessary adjustments are easily made.)

We should note the mathematical resemblance between our techniques and those arising in the use of “configuration space models” in homotopy theory for computing the homology of iterated loop spaces $\Omega^n\Sigma^n M$ [23]. In particular it is interesting to note the use of creation and annihilation operators for topological solitons [26] in a purely mathematical context.

4 Examples

One nonlinear sigma model for which spin and statistics has been deeply studied is the $O(3)$ nonlinear sigma model with Hopf term [3]. Here fields are given by maps from $\mathbb{R} \times \mathbb{R}^2$ to $S^2$. In the study of this model it has been common to compactify space-time to $S^3$. This technique eliminates from consideration all fields with nonzero soliton number, a deficiency which seems so far only to have been addressed by J. Wen [11], although certain aspects of the mathematics seem to be foreshadowed by the work of Ringwood and Woodward [27]. The framework developed above, of course, treats spin and statistics for arbitrary soliton number. In this section we calculate $\text{Stat}_n(\mathbb{R}^2, \alpha)$ and $\text{Stat}_n(S^2, \alpha)$ for the soliton $\alpha$ corresponding to the map of degree one from $S^2$ to itself. In doing so we develop techniques applicable to general simply-connected target manifolds $M$. In particular, we compute the groups

$$\pi_1(\text{Maps}(S^2, M), f_0)$$

and

$$\pi_1(\text{Maps}_*(S^2, M), f_0)$$

for any $f_0: S^2 \to M$.

In what follows, we work in the category of spaces with basepoint, so “map” will mean “basepoint preserving map.” Let $M$ be a connected and simply connected
space with basepoint. First, note that a loop in Maps$_* (S^2, M)$ is the same as a map $f: S^1 \times S^2 / S^1 \times * \to M$. Let 

$$\iota: S^2 \to S^1 \times S^2 / S^1 \times *$$

denote the natural inclusion. This map induces a map 

$$\iota^*: [S^1 \times S^2 / S^1 \times *, M] \to \pi_2 (M).$$

In physical terms, $\iota^*[f] \in \pi_2 (M)$ represents the topological charge or “soliton number” of the field $f$. In fact the homotopy class of $f$ is completely determined by its soliton number together with its “instanton number,” an element of $\pi_3 (M)$.

**Theorem 1** For any simply connected space $M$, $[S^1 \times S^2 / S^1 \times *, M]$ is isomorphic to $\pi_3 (M) \oplus \pi_2 (M)$ in such a manner that the map

$$\iota^*: [S^1 \times S^2 / S^1 \times *, M] \to \pi_2 (M)$$

corresponds to the projection $p_2: \pi_3 (M) \oplus \pi_2 (M) \to \pi_2 (M)$.

**Proof** - Identify $S^3$ with the union of two solid tori, $S^3 = D^2 \times S^1 \cup_{S^1 \times S^1} S^1 \times D^2$. Let us write a point in $D^2 \times S^1$ as $(t\vec{x}, \vec{y})$ where $t \in [0, 1]$ and $\vec{x}, \vec{y}$ are unit vectors in $\mathbb{R}^2$, and similarly for $S^1 \times D^2$. Define the map $H: S^3 \to S^1 \times S^2 / S^1 \times *$ by

$$H(t\vec{x}, \vec{y}) = *$$
$$H(\vec{x}, t\vec{y}) = (\vec{x}, \rho(t\vec{y})).$$

Let $\vee$ denote the wedge of spaces with basepoint. The theorem is a consequence of the following lemma:

**Lemma 1** The map $\iota \vee H: S^2 \vee S^3 \to S^1 \times S^2 / S^1 \times *$ is a homotopy equivalence. Furthermore, $\pi_2 (\iota \vee H)$ and $\pi_1$ are homotopic as maps from $S^2 \vee S^3$ to $S^2$, and $\text{pinch}(\iota \vee H)$ and $\pi_2$ are homotopic as maps from $S^2 \vee S^3$ to $S^3$, where $\pi_1$ and $\pi_2$ are the natural quotient maps, and $\text{pinch}: S^1 \times S^2 / S^1 \times * \to S^3$ is the pinch map.

**Proof** - This follows easily from homology considerations. \qed

**Corollary 1** For any $f_0 \in \text{Maps}_*(S^2, M)$,

$$\pi_1 (\text{Maps}_*(S^2, M), f_0) \simeq \pi_3 (M).$$

**Proof** - This follows directly from the above theorem and the fact that elements of $\pi_1 (\text{Maps}_*(S^2, M), f_0)$ are the same as elements of $[S^1 \times S^2 / S^1 \times *, M]$ whose image under $\iota^*$ equals $[f_0] \in \pi_2 (M)$. \qed
In particular, to quantize the theory of maps \( f: \mathbb{R} \times \mathbb{R}^2 \to S^2 \) satisfying
\[
\lim_{|\vec{x}| \to \infty} f(t, \vec{x}) = *
\]
and having given soliton number requires a choice of a representation of \( \pi_3(S^2) = \mathbb{Z} \). This extends previous work that only treated the case of vanishing soliton number [3].

The group \( \pi_1(\text{Maps}(S^2, M), f_0) \) is a quotient of \( \pi_1(\text{Maps}_*(S^2, M), f_0) \), and computing it requires an analysis of soliton-instanton interactions. The key topological aspects of these interactions are encoded in the Whitehead product
\[
[\cdot, \cdot]: \pi_2(M) \times \pi_2(M) \to \pi_3(M).
\]

Let us briefly recall the definition of this product, referring the reader to standard texts on algebraic topology for more details [28, 29].

The universal Whitehead product is the map
\[
W: S^3 \to S^2 \vee S^2
\]
given by
\[
W(t\vec{x}, \vec{y}) = \rho(t\vec{x}) \vee *
\]
\[
W(\vec{x}, t\vec{y}) = * \vee \rho(t\vec{y}).
\]

Given \( \alpha = [f] \) and \( \beta = [g] \) in \( \pi_2(M) \), the Whitehead product \( [\alpha, \beta] \in \pi_3(M) \) is the class of the map \((f \vee g) \circ W\). In particular, for any \( \alpha \in \pi_2(M) \) the Whitehead product defines a homomorphism
\[
[\alpha, \cdot]: \pi_2(M) \to \pi_3(M).
\]

In our application, \( \alpha \) will be the class of \( f_0: S^2 \to M \), so we write this homomorphism as \([f_0, \cdot]\).

The natural inclusion \( \iota: S^2 \to S^1 \times S^2 \) induces a map \( \iota^*: [S^1 \times S^2, M] \to \pi_2(M) \). We have:

**Theorem 2** For any simply connected space \( M \) and any map \( f_0: S^2 \to M \), there is a bijection from the group \( \pi_3(M)/\text{Im}[f_0, \cdot] \) to the inverse image \( \iota^*[f_0] \subseteq [S^2 \times S^1, M] \) of the homotopy class \([f_0]\).

Proof - Let \( \Sigma A \) denote the suspension of a space \( A \) and \( CA \) the cone over \( A \). In general for a cofibration sequence \( A \hookrightarrow X \to X/A \) there is a left action of the group \([\Sigma A, M]\) on the set \([X/A, M]\), which has the property that there is a short exact sequence
\[
[X/A, M]/[\Sigma A, M] \to [X, M] \to [A, M]. \tag{4}
\]
That is, there is an injective map from the orbit space \([X/A, M]/[\Sigma A, M]\) to the set \([X, M]\), whose image is exactly the elements which are mapped to the trivial element.
in \([A, M]\) under \(\iota^*\). The action is defined by the coaction map \(\theta: X \cup_i CA \to X/A \vee \Sigma A\) together with the homotopy equivalence \(\gamma: X \cup_i CA \to X/A\). That is, given maps \(f: X/A \to M\) and \(\alpha: \Sigma A \to M\) there is a unique homotopy class \(\alpha \ast f: X/A \to M\) such that the following two maps are homotopic:

\[(\alpha \ast f) \circ \gamma \sim (f \vee \alpha) \circ \theta,\]  

see [28].

Now consider the cofibration sequence \(S^1 \times * \subset S^1 \times S^2 \to S^1 \times S^2/S^1 \times *\). Since \(\pi_1(M) = 0\), diagram (4) implies that there is an isomorphism

\[\left[ S^1 \times S^2/S^1 \times *, M \right] / \pi_2(M) \cong \left[ S^1 \times S^2, M \right].\]  

(6)

To compute the coaction map, let \(\tilde{H}: S^3 \to S^1 \times S^2 \cup C(S^1 \times *)\) be the map

\[
\begin{align*}
\tilde{H}(t\bar{x}, \bar{y}) &= t\bar{x} \in D^2 = C(S^1 \times *) \\
\tilde{H}(\bar{x}, t\bar{y}) &= (\bar{x}, \rho(t\bar{y}))
\end{align*}
\]

Then

\[i \vee H = (\iota \vee \tilde{H}) \circ j\]  

(7)

where

\[j: S^1 \times S^2 \cup C(S^1 \times *) \to S^1 \times S^2/S^1 \times *\]

is the natural homeomorphism.

The composite

\[(\pi_2 \vee \text{id}) \circ \theta \circ \tilde{H}: S^3 \to S^2 \vee S^2\]  

(8)

is clearly the universal Whitehead product \(W: S^3 \to S^2 \vee S^2\).

On the other hand the composite

\[(\text{pinch} \vee *) \circ \theta \circ (\iota \vee \tilde{H}): S^2 \vee S^3 \to S^3\]  

(9)

is homotopic to the projection \(\pi_2\) on the second factor. Therefore by Lemma 4, composites (8) and (9) and the Hilton-Milnor theorem [27],

\[(\iota \vee H \vee \iota) \circ (\iota_1 \vee ([\iota_1, \iota_3] + \iota_2)) \sim k \circ \iota \vee \tilde{H}\]  

(10)

as maps from \(S^2 \vee S^3\) to \(S^1 \times S^2/S^1 \times *\), where

\[k: S^1 \times S^2 \cup C(S^1 \times *) \to S^1 \times S^2/S^1 \times *\]

is the natural homotopy equivalence. Lemma 4 and diagram (10) computes the action of \(\pi_2(M)\) on \([S^1 \times S^2/S^1 \times *, M] \cong \pi_2(M)\pi_3(M)\) to be

\[\alpha \ast (f, \beta) = (f, \beta + [f, \alpha]).\]

By the exact sequence (3) we have

\[\pi_2(M)\pi_3(M) / \{(f, \beta) = (f, \beta + [f, \alpha])\} \cong \left[ S^1 \times S^2, M \right],\]

which implies Theorem 2. \(\square\)
Corollary 2  For any \( f_0 \in \text{Maps}(S^2, M) \),
\[
\pi_1(\text{Maps}(S^2, M), f_0) \cong \pi_3(M)/\text{Im}[f_0, \cdot].
\]

Proof - This follows from the theorem above and the fact that \( \pi_1(\text{Maps}(S^2, M), f_0) \cong \iota^*[f_0] \subseteq [S^1 \times S^2, M]. \) \( \Box \)

We now compute the spin-statistics groups of the \( O(3) \) nonlinear sigma model:

Corollary 3  Let \( \alpha \in \pi_2(S^2) \) be a generator and let \( f_0: S^2 \to S^2 \) have \( [f_0] = n\alpha \). Then
\[
\pi_1(\text{Maps}_*(S^2, S^2), f_0) = \mathbb{Z}, \quad \pi_1(\text{Maps}(S^2, S^2), f_0) = \mathbb{Z}_{2n},
\]
and for all \( n \geq 1 \),
\[
\text{Stat}_n(R^2, \alpha) = \mathbb{Z}, \quad \text{Stat}_n(S^2, \alpha) = \mathbb{Z}_{2n}.
\]

Proof - Note that \( \pi_1(\text{Maps}_*(S^2, S^2), f_0) \) is independent of \( f_0 \) and equals \( \pi_3(S^2) = \mathbb{Z} \) by Corollary 4. Thus our choice of \( \alpha \) determines a homomorphism \( \psi: FB_n(R^2) \to \mathbb{Z} \) as described in Section 3. Recall that \( t_1 \in FB(R^2) \) corresponds to the rotation of the first strand by \( 2\pi \) about its axis. We claim that \( \psi(t_1) = 1 \). It will follow that \( \text{Stat}_n(R^2, \alpha) \), the image of the map \( \psi \), is \( \mathbb{Z} \).

Associate to the framed braid \( t_1 \) a loop of basepoint preserving maps from \( S^2 \) to \( S^2 \), i.e. a map \( f: S^1 \times S^2 / S^1 \times * \to S^2 \). By Lemma 4, \( \psi(t_1) \in \pi_3(S^2) \) is represented by the map \( f \circ H: S^3 \to S^2 \). We may calculate the Hopf invariant of this map by taking the inverse images of two regular values in \( S^2 \) and computing their linking number in \( S^3 \). This is easily seen to be 1, so \( \psi(t_1) = 1 \).

Next note that by Corollary 5, \( \pi_1(\text{Maps}(S^2, S^2), f_0) \) is the quotient of \( \pi_3(S^2) \) by the subgroup generated by \( [\alpha, f_0] \). Moreover \( [\alpha, f_0] = n[\alpha, \alpha] = 2n \) by the bilinearity of the Whitehead product together with the fact that \( [\alpha, \alpha] = 2 \). Thus \( \pi(\text{Maps}(S^2, S^2), f_0) = \mathbb{Z}_{2n} \). Recall that \( FB_n(S^2) \) is a quotient of \( FB_n(R^2) \). Let \( t \in FB_n(S^2) \) be the image of \( t_1 \in FB(R^2) \). Then by the same argument as for \( t_1 \), \( \psi(t) = 1 \in \mathbb{Z}_{2n} \), so \( \text{Stat}_n(S^2, \alpha) = \mathbb{Z}_{2n} \). \( \Box \)

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