PERMANENCE AND EXTINCTION OF A STOCHASTIC SIS EPIDEMIC MODEL WITH THREE INDEPENDENT BROWNIAN MOTIONS

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Abstract. This paper is devoted to investigate the dynamics of a stochastic susceptible-infected-susceptible epidemic model with nonlinear incidence rate and three independent Brownian motions. By defining a threshold \( \lambda \), it is proved that if \( \lambda > 0 \), the disease is permanent and there is a stationary distribution. And when \( \lambda < 0 \), we show that the disease goes to extinction and the susceptible population weakly converges to a boundary distribution. Moreover, the existence of the stationary distribution is obtained and some numerical simulations are performed to illustrate our results. As a result, appropriate intensities of white noises make the susceptible and infected individuals fluctuate around their deterministic steady-state values; the larger the intensities of the white noises are, the larger amplitude of their fluctuations; but too large intensities of white noises may make both of the susceptible and infected individuals go to extinction.

1. Introduction. Medical research has shown that some diseases, in particular some sexually transmitted and bacterial diseases, do not have permanent immunity and individuals become susceptible again after infection. This type of diseases can be modelled appropriately by susceptible-infected-susceptible (SIS) epidemic models. Because the environmental randomness makes a big influence on the dynamics of epidemic models, many researchers suppose that some stochastic environmental factors act simultaneously on each individual in the population [2, 7, 8, 27, 29, 34]. In this paper, we shall investigate the following stochastic SIS epidemic model with nonlinear incidence rate

\[
\begin{align*}
\frac{dS(t)}{dt} &= [\Lambda - \beta g(S(t), I(t)) + \gamma I(t) - \mu S(t)]dt + aS(t)dB_1(t) - cg(S(t), I(t))dB_3(t), \\
\frac{dI(t)}{dt} &= [\beta g(S(t), I(t)) - (\mu + \gamma + \alpha)I(t)]dt + bI(t)dB_2(t) + cg(S(t), I(t))dB_3(t). \\
\end{align*}
\]

(1)

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where \( S(t) \) and \( I(t) \) denote the numbers of susceptible and infected individuals at \( t \), respectively. \( \Lambda > 0 \) is the recruitment rate of the population, \( \beta \geq 0 \) represents the disease transmission coefficient, \( \mu > 0 \) is the natural death rate of \( S \) and \( I \), \( \gamma \geq 0 \) is the recovery rate and \( \alpha \geq 0 \) is the disease-related death rate of \( I \). The transmission of the infection is governed by a nonlinear incidence rate \( \beta g(S(t), I(t)) \), where \( g(\cdot) \) is a nonnegative continuous function defined on \([0, \infty) \times [0, \infty)\) and satisfies that \( \lim_{S \to 0} g(S, I)/S \) exists for all \( I \geq 0 \), \( g(S, 0) = 0 \), \( b(S) = \lim_{I \to 0} g(S, I)/I \) is positive and finite for all \( S > 0 \), and \( g(S, I)/I \) is monotone nonincreasing with respect to \( I \in (0, \infty) \) for \( S \in (0, \infty) \). Obviously, the function \( g \) includes some special incidence rates \([6, 23, 35]\), such as bilinear incidence rate of the form \( g(S, I) = SI/(1 + mI) \) with a positive constant \( m \) denoting the half-saturation constant, Holling type II of the form \( g(S, I) = SI/(aS + bI + c) \) with positive constants \( a, b, \) and \( c \), and some other kinds of incidence rates like \( g(S, I) = e^{-mI}SI \) and \( g(S, I) = SI/(1 + mI^\theta) \) with positive constants \( m \) and \( \theta \). In system (1), \( B_1(t), B_2(t), \) and \( B_3(t) \) are three mutually independent Brownian motions defined on the complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) and nonnegative constants \( a, b, \) and \( c \) denote the intensities of the white noises. Throughout this paper, we assume that \( \mu > \frac{1}{2}a^2 \) (the reason shall be given in Section 2).

The objective of this paper is to study the extinction and permanence of the disease in (1). One of popular approaches to the investigation of SIS models is to introduce a reproduction value \([32, 34]\). For example, by defining a threshold \( R_0 = \Lambda \beta/[\mu(\mu + \gamma + \alpha)] \) for system (1) with \( g(S, I) = SI \), Zhou et al. \([32]\) obtained the existence of its stationary distribution when \( R_0 > 1 \) and found that the disease will go to extinction when either \( R_0 \leq 1 \) or some noise is sufficiently large, i.e., \( \beta^2 \leq 2c^2(\gamma + \alpha + \mu) + b^2c^2 \). In general, more researches focus on the models perturbed by only one noise. For example, Liu et al. \([19]\), Wen et al. \([28]\) investigated the existence, uniqueness, and stability of stationary distribution for system (1) with \( a = b = 0 \). Lan et al. \([15]\) obtained some analytic results and their related biological implications of dynamics of model (1) with \( a = b = 0 \) and \( g(S, I) = SI/[1 + b(S + I) + \sqrt{1 + 2m(S + I)}] \) and \( m \) is a positive constant. Du and Nhu \([5]\) obtained the sufficient and almost necessary conditions for permanence and extinction of the disease for model (1) with \( \gamma = \alpha = a = b = 0 \) and Beddington-DeAngelis incidence rate \( g(S, I) = SI/(1 + m_1S + m_2I) \). By introducing the following threshold value

\[
R_0(c) = \frac{2\beta f'(0)\Lambda \mu - c^2f''(0)\Lambda^2}{2\mu^2(\mu + \gamma + \alpha)},
\]

Teng and Wang \([27]\) investigated the following stochastic SIS epidemic model with nonlinear incidence rate

\[
\begin{aligned}
dS(t) &= [\Lambda - \beta S(t) f(I(t)) + \gamma I(t) - \mu S(t)]dt - cS(t) f(I(t)) dB(t), \\
dI(t) &= [\beta S(t) f(I(t)) - (\mu + \gamma + \alpha) I(t)]dt + cS(t) f(I(t)) dB(t),
\end{aligned}
\]

where \( f(I) \) is a continuously differentiable and nonnegative function of \( I \) and satisfies that \( f(0) = 0, f'(0) > 0, f'(I)/I \) is monotone decreasing on \( \mathbb{R}_+ \), \( B(t) \) is a one-dimensional standard Brownian motion defined on a probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), and \( c \) is the intensity. Obviously, system (1) includes (3) as a special case. Teng and Wang \([27]\) showed that the disease is extinct with probability one when \( R_0(c) < 1 \) and \( \beta \mu \geq c^2 f'(0)\Lambda \), but is weakly permanent with probability one when \( R_0(c) > 1 \). This result coincides with the deterministic case of system
(3), because Teng and Wang [27] proved that model (3) with $c = 0$ has a unique globally asymptotically stable equilibrium $(\frac{\Lambda}{\mu}, 0)$ when $R_0(0) \leq 1$, and has a unique globally asymptotically stable epidemic equilibrium $(S^*, I^*)$ when $R_0(0) > 1$. Teng and Wang [27] raised a question as to whether all results obtained in [27] still hold if the incidence rate function $\beta S f(I)$ given in model (3) is changed into more general form $\beta g(S, I)$ in system (1). Moreover, there is a gap in [27], from which we do not know what happens to system (3) under the following condition (see [27, Remark 1]):

$$R_0(c) < 1 \quad \text{and} \quad \frac{\beta \mu}{f'(0) \Lambda} < c^2 < \frac{\beta^2}{2(\mu + \gamma + \alpha)}. \quad (4)$$

Teng and Wang [27] thought under the assumption (4) the disease still dies out with probability one, but this conjecture is not proved and an affirmative answer is given by using the numerical simulations. In addition, whether the disease dies out when $R_0(c) = 1$ is left as an open problem. If the disease-related death rate $\alpha$ of $I$ is ignored, then system (3) is changed into the following stochastic differential equations (SDEs):

$$\begin{cases}
    dS(t) = [\Lambda - \beta S(t) f(I(t)) + \gamma I(t) - \mu S(t)] dt - cS(t) f(I(t)) dB(t), \\
    dI(t) = [\beta S(t) f(I(t)) - (\mu + \gamma) I(t)] dt + cS(t) f(I(t)) dB(t).
\end{cases} \quad (5)$$

It is obvious that the following set is a positive invariant set of stochastic model (5):

$$\left\{ (S(t), I(t)) \in \mathbb{R}_+^2 : S(t) + I(t) = N_0 \triangleq \frac{\Lambda}{\mu} \right\},$$

where $\mathbb{R}_+^2 \triangleq \{(x, y) : x > 0, y > 0\}$. The dynamics of stochastic model (5) is equivalent to that of the following one-dimensional SDE:

$$dI(t) = [\beta (N_0 - I(t)) f(I(t)) - (\mu + \gamma) I(t)] dt + c (N_0 - I(t)) f(I(t)) dB(t), \quad (6)$$

whose special forms have been investigated by Gray et al. [7] and Zhu [34]. More precisely, for system (5) with $f(I) = I$, Gray et al. [7] established a series of criteria on the existence of the unique global positive solution with any positive initial value, the threshold conditions determining whether the disease dies out or persists with probability one, the existence of unique stationary distribution, and the mean value and variance of the stationary distribution. These results are also extended by Zhu [34] to system (5) with $f(I) = I/(1 + mI)$ and $m > 0$. Xu [29] investigated the existence, uniqueness, global asymptotic stability and an explicit formula of the invariant density of the Fokker-Planck equation associated with SDE (6) with $f(I) = I$. If the disease-related death rate of $I$ cannot be ignored, that is, $\alpha \neq 0$, then even in the case where $a = b = 0$, we cannot turn the two-dimensional stochastic model (1) into a one-dimensional stochastic model. Therefore, we have to resort to a different approach from [7, 34].

What we are interested in is the existence of stationary distribution and convergence of the population in system (1) in terms of a new threshold, which is different from the basic reproduction number established in [19, 27, 28, 32, 34]. For this purpose, we first obtain some estimates of solutions in probability by using some standard inequalities. To obtain the bound of the total population, we have to estimate the stochastic terms in system (1). However, the presence of three different noises makes it difficult to estimate the stochastic terms, since sometimes they can not be neglected by the large number theorem for martingales [16]. We shall employ Chebyshev’s inequality and a method from Mao [20] to overcome this difficulty.
Motivated by [22], in order to determine whether the disease goes to extinction or not, we first pay attention to the equation on the boundary (that is, $I(t) = 0$) and then define a threshold $\lambda$ in terms of the boundary function. In fact, $\lambda$ is the Lyapunov exponent of $I(t)$ when $I(t)$ is small for a sufficiently long time. On the one hand, we define a stopping time to help us to show that the condition $\lambda > 0$ implies that the disease is permanent and system (1) has a stationary distribution. On the other hand, we shall see that the infected population converges to 0 and the susceptible population converges to the boundary distribution when $\lambda < 0$. We shall see that the extinction condition of disease (i.e., $\lambda < 0$) weakens the relevant conditions in [27]. In particular, the open problem given in Remark 1 of [27] is solved. Furthermore, we shall see the permanence condition of the disease (i.e., $\lambda > 0$) is much better than all the relevant results obtained in [19, 27, 28, 32, 34], for example, Teng and Wang [27] only proved weak permanence of the disease with probability one and permanence in the mean of the disease. Nevertheless, a main barrier in our analysis is that the monotonicity doesn’t work here, which makes us have to resort to Heine-Borel theorem and the ergodicity.

Apparently, our model (1) is more accurate since the speed of death might be affected by environmental factors, for example, potentially infectious contact. The previous threshold, for example, introduced in [7, 27, 34], depends on the upper bound of the total population and parameters of the system under investigation. By comparison, our threshold $\lambda$ can finely determine the extinction and permanence of the disease more because it relies on the parameters $(a, b, c, \mu, \gamma, \alpha)$ and the boundary distribution of the susceptible population and the susceptible population is permanent and plays an important part in system (1). Furthermore, we not only prove the permanence and extinction of the disease, but also give the existence of the stationary distribution and the weak convergence of the susceptible population. Most importantly, the stochastic SIS epidemic model (1) is more general than models in literature, our approach is applicable for a variety of other stochastic differential equations, and our main results improve the relevant results in [7, 27, 32, 34].

The paper is organized as follows. In Section 2, we review some basic concepts and results and give some estimates of the solution of (1). Section 3 is devoted to the permanence of the disease and the existence of stationary distribution of system (1). Section 4 is devoted to the extinction of the disease and the weak convergence of susceptible population. In Section 5, we perform some numerical simulations to illustrate our results. Finally, in Section 6 we summarize our contribution and also present some perspectives of this paper.

2. Preliminaries. In this section, we first recall some basic results on the existence of the solutions of (1). We denote that $z = (x, y) \triangleq (S(0), I(0))$ is the initial value in $\mathbb{R}_+^2$ and $(S_z(t), I_z(t))$ is the solution to (1) with initial value $z$.

We can follow similar arguments as [13] to obtain the following result, which shows that the solution process of system (1) is global and positive.

**Lemma 2.1.** For any initial value $(S(0), I(0))$ in $\mathbb{R}_+^2$, there exists a unique solution $(S(t), I(t))$ of system (1) for all $t \geq 0$, and the solution remains in $\mathbb{R}_+^2$ with probability one.

**Definition 2.2** (See, for example [17]). The infected population of model (1) is permanent if for any $\varepsilon \in (0, 1)$, there exist positive constants $N_2(\varepsilon)$ and $N_2^{-1}(\varepsilon)$
such that
\[
\liminf_{t \to \infty} \mathbb{P}\{I(u) \leq N_2(\varepsilon)\} \geq 1 - \varepsilon, \quad \liminf_{t \to \infty} \mathbb{P}\{I(u) \geq N_2^{-1}(\varepsilon)\} \geq 1 - \varepsilon.
\]

By applying Chebyshev’s inequality and comparison principle, we can obtain the upper and lower bound of the solution of (1) in probability.

**Lemma 2.3.** The solution \((S_z(t), I_z(t))\) of system (1) is ultimately bounded. More precisely, there exists a constant \(N_1 > 0\) such that
\[
N^0 \leq \liminf_{t \to \infty} (S_z(t) + I_z(t)) \leq \limsup_{t \to \infty} (S_z(t) + I_z(t)) \leq N_1 \quad \text{a.s.}
\]
(7)

and
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t (S_z(u) + I_z(u))du \leq N_0 \quad \text{a.s.}
\]
for all \(z \in \mathbb{R}_+^2\), where \(N_0 = \frac{\Lambda}{\mu}\) and \(N^0 = \frac{\Lambda}{\mu + \alpha}\). Furthermore, for any \(z \in \mathbb{R}_+^2\) and \(\varepsilon > 0\), there exist \(\bar{H} > 0\), \(\bar{H} > 0\) and \(T > 0\) such that
\[
\mathbb{P}\{S_z(t) \geq \bar{H} \text{ for all } t \geq 0\} \geq 1 - \varepsilon
\]
and
\[
\mathbb{P}\{I_z(t) \geq \bar{H} \text{ for all } t \in [0, T]\} \geq 1 - \varepsilon.
\]

**Proof.** Using a similar method as in [3], we have
\[
d(S_z(t) + I_z(t)) = [\Lambda - \mu S_z(t) - \mu I_z(t) - \alpha I_z(t)]dt + aS_z(t)dB_1(t) + bI_z(t)dB_2(t),
\]
and hence
\[
S_z(t) + I_z(t) = \frac{\Lambda}{\mu} + \left(z - \frac{\Lambda}{\mu}\right)e^{-\mu t} - \alpha \int_0^t e^{-\mu(t-s)}I_z(s)ds
\]
\[
+ a \int_0^t e^{-\mu(t-s)}S_z(s)dB_1(s) + b \int_0^t e^{-\mu(t-s)}I_z(s)dB_2(s)
\]
\[
\leq X_z(t),
\]
where \(X_z(t) = X(0) + A_z(t) - Q_z(t) + Y_z(t)\) and
\[
X(0) \triangleq z, \quad A(t) \triangleq \frac{\Lambda}{\mu} (1 - e^{-\mu t}), \quad Q_z(t) \triangleq z(1 - e^{-\mu t}),
\]
\[
Y_z(t) \triangleq a \int_0^t e^{-\mu(t-s)}S_z(s)dB_1(s) + b \int_0^t e^{-\mu(t-s)}I_z(s)dB_2(s).
\]

Note that \(A_z(t)\) and \(Q_z(t)\) are continuous adapted increasing process on \(t \geq 0\) with \(A_z(0) = Q_z(0) = 0\). By Theorem 1.3.9 in [20], we have
\[
\lim_{t \to \infty} X_z(t) < \infty \quad \text{a.s.,}
\]
which means that there exists a positive constant \(N_1\) such that
\[
S_z(t) + I_z(t) \leq N_1 \quad \text{a.s. for all } t > 0.
\]

On the other hand,
\[
S_z(t) + I_z(t) \geq z + \int_0^t [\Lambda - (\mu + \alpha)(S_z(s) + I_z(s))]ds
\]
\[
+ a \int_0^t S_z(s)dB_1(s) + b \int_0^t I_z(s)dB_2(s).
\]
From martingale property (see, for example, [20, Theorem 1.5.8]), we have
\[
E \left| \int_0^t aS_z(s)dB_1(s) + bI_z(s)dB_2(s) \right|^2 = E \int_0^t a^2S_z^2(s) + b^2I_z^2(s)ds \\
\leq \int_0^t (a^2 + b^2)N^2_1 ds
\]
for all \( t > 0 \). Then by Chebyshev’s inequality, for any \( \varepsilon > 0 \), we have \( P \{ \Omega_1 \} \geq 1 - \varepsilon \), where
\[
\Omega_1 = \left\{ \left| \int_0^t aS_z(s)dB_1(s) + bI_z(s)dB_2(s) \right| \leq \frac{\hat{M}}{\sqrt{\varepsilon}} \text{ for all } t > 0 \right\},
\]
where \( \hat{M} > 0 \) is a constant. Thus for \( \omega \in \Omega_1 \), we have
\[
S_z(t) + I_z(t) \geq x + \int_0^t [\Lambda - (\mu + \alpha)(S_z(s) + I_z(s))]ds - \frac{\hat{M}}{\sqrt{\varepsilon}} t.
\]
Using the comparison principle [10], we have
\[
S_z(t) + I_z(t) \geq \frac{\Lambda}{\mu + \alpha} + \left( \frac{\Lambda}{\mu + \alpha} - \frac{\Lambda}{\mu} \right) e^{-(\mu + \alpha)t} - \frac{\hat{M}}{\sqrt{\varepsilon}} \int_0^t s^{-\frac{1}{2}} e^{(\mu + \alpha)(s - t)}ds, \tag{9}
\]
which yields the result when \( t \) tends to infinity. It follows from (8) that
\[
\frac{1}{t} \int_0^t (S_z(u) + I_z(u))du \leq \frac{\Lambda}{\mu} + \frac{1}{t} \int_0^t \left( \frac{\Lambda}{\mu} - \frac{\Lambda}{\mu + \alpha} \right) e^{-\mu u}du \\
+ \frac{a}{t} \int_0^t \int_0^u e^{-\mu(u-s)}S_z(u)dB_1(u)du \\
+ \frac{b}{t} \int_0^t \int_0^u e^{-\mu(u-s)}I_z(u)dB_2(u)du,
\]
which together with the large number theorem for martingales yields the boundedness in mean when \( t \) tends to infinity.

Similarly, for any \( \varepsilon > 0 \), we have \( P(\Omega'_1) \geq 1 - \varepsilon \), where
\[
\Omega'_1 = \left\{ \left| \int_0^t aS_z(s)dB_1(s) - c\varphi(S_z(s), I_z(s))dB_2(s) \right| \leq \frac{\hat{M}}{\sqrt{\varepsilon}} \text{ for all } t > 0 \right\},
\]
and \( \hat{M} \) is a constant. By virtue of the continuity of \( g(S, I) \) with respect to \( (S, I) \), the existence of \( \lim_{S \to 0} g(S, I)/S \), and the upper boundedness of \( (S_z(t), I_z(t)) \), there exists \( M_1 > 0 \) such that \( g(S_z, I_z) \leq M_1S_z \) for all \( z \in \mathbb{R} \). Thus, for \( \omega \in \Omega'_1 \) and \( z \in \mathbb{R}^+ \), we have
\[
S_z(t) \geq x + \int_0^t [\Lambda - (\beta M_1 + \mu)S_z(s)]ds - \frac{\hat{M}}{2\sqrt{\varepsilon}} \int_0^t s^{-\frac{1}{2}}ds
\]
for all \( t \geq 0 \). Applying the comparison principle [10] yields that for all \( t \geq 0 \),
\[
S_z(t) \geq \frac{\Lambda}{\beta M_1 + \mu} + \left( x - \frac{\Lambda}{\beta M_1 + \mu} \right) e^{-(\beta M_1 + \mu)t} - \frac{\hat{M}}{2\sqrt{\varepsilon}} \int_0^t s^{-\frac{1}{2}} e^{(\beta M_1 + \mu)(s - t)}ds,
\]
from which there exists \( \overline{H} > 0 \) such that \( P \{ S_z(t) \geq \overline{H} \text{ for all } t \geq 0 \} \geq 1 - \varepsilon \). As for \( I_z(t) \), for any \( k > 0 \) and \( t \geq 0 \), by Itô’s formula (see [20, Theorem 6.2] for more
where \( \Phi \equiv \{ \text{there exist constants } r \} \), we have
\[
dI^{-k}_z(t) = - k \beta I^{-k}_z(t) \left( \frac{g(S_z(t), I_z(t))}{I_z(t)} \right) dt - k I^{-k}_z(t) \left( b dB_2 + \frac{cg(S_z(t), I_z(t))}{I_z(t)} dB_3 \right) \\
+ k I^{-k}_z(t) \left[ \mu + \gamma + \alpha + \frac{1}{2} (k + 1) \left( b^2 + \frac{c^2 g^2(S_z(t), I_z(t))}{I_z^2(t)} \right) \right] dt \\
\leq - \Phi I^{-k}_z(t) dt + \Psi I^{-k}_z(t) dt \\
- k I^{-k}_z(t) \left( b dB_2(t) + \frac{cg(S_z(t), I_z(t))}{I_z(t)} dB_3(t) \right),
\]
where \( \Phi \equiv k \beta m, \Psi \equiv k \left[ \mu + \gamma + \alpha + \frac{1}{2} (k + 1) \left( b^2 + c^2 M_2^2 \right) \right], \) and
\[
m \equiv \min \left\{ \frac{g(x, N_1)}{N_1}, \frac{H}{x} \leq N_1 \right\},
\]
\[M_2 \equiv \max \{ h(x) : 0 \leq x \leq N_1 \}, \]
and hence that
\[d\mathbb{E}[I^{-k}_z(t)] \leq \Psi I^{-k}_z(t) dt.\]
Then for \( t \in [0, T] \), there exists a constant \( M_3 > 0 \) such that
\[\mathbb{E}[I^{-k}_z(t)] \leq y^{-k} e^{\Psi t} \leq y^{-k} M_3.\]
Using Chebyshev’s inequality, for all \( t \in [0, T] \), we have
\[\mathbb{P} \{ I_z(t) \geq H \} = 1 - \mathbb{P} \{ H^{-k} < I^{-k}_z(t) \} \geq 1 - H^{-k} \mathbb{E}[I^{-k}_z(t)] \geq 1 - H^{-k} y^{-k} M_3,\]
then choosing \( H \) such that \( H^{-k} y^{-k} M_3 < \varepsilon \), we obtain that
\[\mathbb{P} \{ I_z(t) \geq H \text{ for all } t \in [0, T] \} \geq 1 - \varepsilon.\]
This completes the proof. \( \square \)

In model (1), all newborns are born as susceptible population and after a period the infectious individual becomes susceptible again. In order to take an in-depth study of susceptible population, we consider the equation on the boundary. When \( I(t) = 0 \), it follows from (1) and \( g(S, 0) = 0 \) that we have
\[d\varphi(t) = (\Lambda - \mu \varphi(t)) dt + a \varphi(t) dB_1(t) \quad \text{(10)}
\]
and define the generator
\[L \equiv (\Lambda - \mu x) \frac{d}{dx} + \frac{1}{2} \sigma^2 x^2 \frac{d^2}{dx^2}.\]
Take \( h(x) = x^2 \), it is obvious that \( \lim_{|x| \to \infty} h(x) = \infty \) and
\[L h(x) = (a^2 - 2 \mu) x^2 + 2 \Lambda x.\]
It is easy to see that if \( \mu < \frac{1}{2} a^2 \) then the diffusion of (10) is transient. If \( \mu > \frac{1}{2} a^2 \), there exist constants \( r_0, r_1 > 0 \) such that
\[L h(x) < -r_1 \text{ for all } |x| > r_0,\]
which implies that the diffusion of (10) is positive recurrent with stationary distribution \( \pi^* \) on \( \mathbb{R}^+ \). This is why we always assume that \( \mu > \frac{1}{2} a^2 \) throughout this
By Hölder’s inequality, it follows from (1) and Lemma 2.3 that

\[ f^*(x) = \left( \frac{2\Lambda}{a^2} \right)^{\frac{2\mu}{\alpha^2} + 1} \Gamma^{-1} \left( \frac{2\mu}{\alpha^2} + 1 \right) x^{-\left(\frac{2\alpha}{\alpha^2} + 2\right)} e^{-\frac{2\mu}{a^2} x^2}, \]

where \( \Gamma(\cdot) \) is the Gamma function (for more details, see [5, 21]).

By the ergodicity see [25, Theorem 3.16] for more details), for any \( p \in (-\infty, 3) \) such that \( \int_0^\infty \phi^p f^*(\phi) d\phi < \infty \),

\[ P \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi_x^p(t) dt = \int_0^\infty \phi^p f^*(\phi) d\phi \right\} = 1 \quad \text{for all } x > 0, \tag{11} \]

where \( \varphi_x \) is the solution to (10) starting at \( x \). The following result is necessary in the construction of sets to prove the permanence and extinction of the disease. Furthermore, it follows from (10) that

\[ \frac{\varphi_x(t) - x}{t} = \Lambda - \frac{1}{t} \int_0^t \mu \varphi_x(u) du + \frac{1}{t} \int_0^t a \varphi_x(u) dB_1(u), \]

which together with \( \liminf_{t \to \infty} \frac{\varphi_x(t) - x}{t} = 0 \) and the large number theorem for martingales yields that

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi_x(u) du = N_0 \quad \text{a.s.} \tag{12} \]

Remark 1. Indeed, (12) is a direct result of (11) by letting \( p = 1 < 1 + \frac{2\mu}{a^2} \), and the integral gives the expectation of the law of \( f^* \), which is \( N_0 \).

Lemma 2.4. For any \( T > 1 \), \( \epsilon > 0 \), \( A \in \mathcal{F} \), \( p^* \in (1, \frac{3}{2}) \) and \( \frac{1}{p^*} + \frac{1}{q^*} = 1 \), we have

\[ P \left\{ \left| \frac{S_z(T) - x}{T} \right| \geq \frac{\theta_1 (1 + M_1 N_1)^{1/p^*}}{\epsilon} \right\} \leq \epsilon \tag{13} \]

and

\[ P \left\{ \left| \frac{\ln I_z(T) - \ln y}{T} \right| \geq \frac{\theta_2 (1 + M_1 N_1)^{1/q^*}}{\epsilon} \right\} \leq \epsilon, \tag{14} \]

where constants \( \theta_1 \) and \( \theta_2 \) are independent of \( T \) and \( A \).

Proof. By Hölder’s inequality, it follows from (1) and Lemma 2.3 that

\[ \mathbb{E}1_A|S_z(T) - x| \leq \mathbb{E} \int_0^T 1_A |A + \beta g(S_z(t), I_z(t)) + \gamma I_z(t) + \mu S_z(t)| dt \]

\[ + \mathbb{E}1_A \left| \int_0^T aS_z(t) dB_1(t) - cg(S_z(t), I(t)) dB_3(t) \right| \]

\[ \leq (\mathbb{P}(A))^{\frac{1}{p^*}} \left( \mathbb{E} \int_0^T (A + \beta M_1 N_1 + \gamma I_z(t) + \mu S_z(t))^{p^*} dt \right)^{\frac{1}{p^*}} \]

\[ + \sqrt{\mathbb{P}(A)} \left( \mathbb{E} \int_0^T (a^2 N_1^2 + c^2 M_1^2 N_1^2) dt \right)^{\frac{1}{2}} \]

\[ \leq \theta_1 (\mathbb{P}(A))^{\frac{1}{p^*}} (1 + M_1 N_1)^{\frac{1}{p^*} T}, \]
where $\theta_1$ is a constant independent of $T$ and $A$. If $A = \Omega$, we have
\[
\mathbb{E} \left| \frac{S_z(T) - x}{T} \right| \leq \theta_1(1 + M_1N_1)^{1/\rho^*}.
\]
Hence by Chebyshev’s inequality, we obtain (13) immediately. Similarly, (14) can be obtained. □

**Theorem 2.5.** The susceptible population in system (1) is permanent in mean with probability one.

**Proof.** From Lemma 2.3, we have that for any $z \in \mathbb{R}_+^2$,
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t S_z(u) du \leq N_0 \quad \text{a.s.}
\]
Integrating the first equation of system (1), then for any $t \geq 0$
\[
\frac{S_z(t) - x}{t} = \Lambda - \frac{1}{t} \int_0^t \left[ \beta g(S_z(u), I_z(u)) - \gamma I_z(u) + \mu S_z(u) \right] du
\]
\[
+ \frac{1}{t} \int_0^t \left[ aS_z(u)dB_1(u) - c g(S_z(u), I_z(u))dB_3(u) \right]
\]
\[
\geq \Lambda - \frac{1}{t} \int_0^t \left[ (\beta M_1 + \mu)S_z(u) \right] du
\]
\[
+ \frac{1}{t} \int_0^t \left[ aS_z(u)dB_1(u) - c g(S_z(u), I_z(u))dB_3(u) \right].
\]
Note that $\lim_{t \to \infty} \frac{S_z(t) - x}{t} = 0$. Then by the large number theorem for martingales, we have
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t S_z(u) du \geq \frac{\Lambda}{\beta M_1 + \mu} \quad \text{a.s.}
\]
This completes the proof. □

3. **Permanence of disease.** In this section, we shall investigate when the disease is permanent and whether there is a stationary distribution of system (1). To this end, we shall investigate the Lyapunov exponent $\lambda$ given by
\[
\lambda \triangleq \int_0^\infty \left[ \beta h(\phi) - \frac{\gamma^2 h^2(\phi)}{2} - \left( \mu + \gamma + \alpha + \frac{b^2}{2} \right) \right] f^*(\phi) d\phi.
\]
By ergodicity, for any $\phi > 0$, we have
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t h(\varphi_x(s)) ds = \int_0^\infty h(\phi) f^*(\phi) d\phi
\]
and
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t h^2(\varphi_x(s)) ds = \int_0^\infty h^2(\phi) f^*(\phi) d\phi.
\]

**Lemma 3.1.** For any $T, D > 1, \varepsilon > 0$ and $\sigma > 0$, there is a $\delta = \delta(T, \varepsilon, \sigma) > 0$ such that
\[
\mathbb{P}\{\tau^*_\sigma \geq T\} \geq 1 - \varepsilon \quad \text{for all } z \in [D^{-1}, D] \times (0, \delta),
\]
where the stopping time is defined as $\tau^*_\sigma = \inf\{t \geq 0 : I_z(t) \geq \sigma\}$. 
Proof. Note that the exponential martingale inequality (see, for example, [20, Theorem 1.7.4])
\[
\mathbb{P} \left\{ \int_0^t G(s) dW(s) - \frac{p}{2} \int_0^t G^2(s) ds > q \text{ for all } t \geq 0 \right\} \leq e^{-pq}
\]
for any \( p, q > 0 \), if \( W(t) \) is a \( \mathcal{F}_t \)-adapted Brownian motion while \( G(t) \) is a real-valued \( \mathcal{F}_t \)-adapted process satisfying \( \int_0^t G^2(s) ds < \infty \) for all \( t \geq 0 \) almost surely. It follows that \( \mathbb{P}(\Omega_2) \geq 1 - \varepsilon \), where
\[
\Omega_2 = \left\{ \int_0^t b dB_2(u) + c \frac{g(S_z(u), I_z(u))}{I_z(u)} d B_3(u) \leq \frac{1}{2} \int_0^t (b^2 + c^2 g^2(S_z(u), I_z(u))) du + \ln \frac{1}{\varepsilon} \text{ for all } u \geq 0 \right\}.
\]
It follows from Itô’s formula that
\[
\ln I_z(t) = \ln y + \int_0^t \left( bdB_2(s) + \frac{cg(S_z(s), I_z(s))}{I_z(s)} dB_3(s) \right) ds + \int_0^t \left[ \beta g(S_z(s), I_z(s)) - \frac{c^2 g^2(S_z(s), I_z(s))}{2} - (\mu + \gamma + \alpha + \frac{b^2}{2}) \right] ds. \tag{15}
\]
When \( \omega \in \Omega_2 \) we have
\[
\ln I_z(t) \leq \ln y + \ln \frac{1}{\varepsilon} + \int_0^t \left[ \beta g(S_z(u), I_z(u)) I_z(t) \right] - (\mu + \gamma + \alpha) \right] du
\leq \ln y + \ln \frac{1}{\varepsilon} + [\beta M_2 - (\mu + \gamma + \alpha)]t.
\]
Let \( \delta = \sigma \varepsilon e^{(\mu + \gamma + \alpha - \beta M_2)}t \). Then \( I_z(t) < \sigma \) for all \( t \geq T, \omega \in \Omega_2 \) if \( y \leq \delta \). \( \Box \)

Lemma 3.2. For any \( D, T > 1, \varepsilon, \nu > 0 \), there is \( \sigma > 0 \) such that for all \( z \in [D^{-1}, D] \times (0, \sigma] \),
\[
\mathbb{P} \left\{ |\varphi_x(t) - S_z(t)| < \nu \text{ for all } t \in [0, T \wedge \tau_z^\sigma] \right\} \geq 1 - \varepsilon,
\]
where \( \wedge \) stands for minimum.

Proof. In view of Lemma 2.3, we have
\[
\mathbb{P} \left\{ \{ \varphi_x(t) \wedge S_z(t) \} \leq N_1 \text{ for all } t \leq T \right\} \geq 1 - \frac{\varepsilon}{2} \text{ for all } z \in [D^{-1}, D] \times (0, \delta],
\]
where \( \wedge \) stands for maximum. Applying Itô’s formula, we have
\[
|\varphi_x(s) - S_z(s)| \leq \int_0^s \mu |\varphi_x(u) - S_z(u)| du + \int_0^s \beta g(S_z(u), I_z(u)) du + \int_0^s \gamma I_z(u) du
+ a \left| \int_0^s (\varphi_x(u) - S_z(u)) dB_1(u) \right| + c \left| \int_0^s g(S_z(u), I_z(u)) dB_3(u) \right|,
\]
and hence
\[ \mathbb{E} \sup_{s \leq t} [\varphi_x(s \land \xi_z) - S_z(s \land \xi_z)]^2 \]
\[ \leq 4\mathbb{E} \left( \int_0^{t \land \xi_z} \mu |\varphi_x(u) - S_z(u)| du \right)^2 \]
\[ + 4\mathbb{E} \left( \int_0^{t \land \xi_z} (\beta g(S_z(u), I_z(u)) + \gamma I_z(u)) du \right)^2 \]
\[ + 4a^2 \mathbb{E} \sup_{s \leq t} \left| \int_0^{s \land \xi_z} (\varphi_x(u) - S_z(u)) dB_1(u) \right|^2 \]
\[ + 4c^2 \mathbb{E} \sup_{s \leq t} \left| \int_0^{s \land \xi_z} g(S_z(u), I_z(u)) dB_3(u) \right|^2, \]  

(16)

where \( \xi_z \triangleq \tau_z^\sigma \land \inf\{u \geq 0 : (\varphi_x(u) \lor S_z(u)) \geq N_1 \} \). By using Burkholer-Davis-Gundy inequality and Hölder’s inequality, we have that for all \( t \leq T \)
\[ \mathbb{E} \sup_{s \leq t} [\varphi_x(s \land \xi_z) - S_z(s \land \xi_z)]^2 \]
\[ \leq 4m_1 \mu^2 \mathbb{E} \int_0^{t \land \xi_z} (\varphi_x(u) - S_z(u))^2 du + 4m_2 (\beta M_2 + \gamma)^2 \sigma^2 T^2 \]
\[ + 4m_3 a^2 \mathbb{E} \int_0^{t \land \xi_z} (\varphi_x(u) - S_z(u))^2 du + 4m_4 c^2 M_3^2 \sigma^2 T \]
\[ \leq \overline{m} \left( \sigma^2 + \mathbb{E} \int_0^{t \land \xi_z} (\varphi_x(u) - S_z(u))^2 du \right) \]
\[ \leq \overline{m} \left( \sigma^2 + \int_0^t \mathbb{E} \sup_{s \leq u} (\varphi_x(s \land \xi_z) - S_z(s \land \xi_z))^2 du \right) \]

for some \( \overline{m} = \overline{m}(M_2, T) > 0 \), where \( m_i, i = 1, 2, 3, 4 \) are positive constants. Applying Grönwall’s inequality, we have
\[ \mathbb{E} \sup_{s \leq T} [\varphi_x(s \land \xi_z) - S_z(s \land \xi_z)]^2 \leq \overline{m} \sigma^2 e^{\overline{m}T}. \]

From Chebyshev’s inequality, it follows that when \( \sigma \) is sufficiently small,
\[ \mathbb{P} \left\{ \sup_{s \leq T} [\varphi_x(s \land \xi_z) - S_z(s \land \xi_z)]^2 \geq \nu^2 \right\} \leq \frac{\overline{m} \sigma^2 e^{\overline{m}T}}{\nu^2} < \frac{\varepsilon}{2}, \]

which together with
\[ \mathbb{P}\{s \land \xi_z = s \land \tau_z^\sigma \text{ for all } s \in [0, T] \} \geq \mathbb{P} \left\{ \sup_{s \leq T} \{\varphi_x(s) \lor S_z(s)\} \leq N_1 \right\} \geq 1 - \frac{\varepsilon}{2} \]
yields the result. This completes the proof. \( \square \)

**Lemma 3.3.** Assume that \( \lambda > 0 \), then for any \( \varepsilon > 0 \), \( D > 1 \), there are \( T = T(\varepsilon, D) > 0 \) and \( \delta_0 = \delta_0(\varepsilon, D) \) such that \( \mathbb{P}(\Omega_z) > 1 - 3\varepsilon \) for all \( z \in [D^{-1}, D] \times (0, \delta_0], \) where \( \Omega_z = \{\ln I_z(T) - \ln y \geq \frac{1}{2} |T\} \).
Proof. From the definition of \( \lambda \), we have
\[
\int_0^\infty \left[ \beta(h(\phi) - \eta_1) - \frac{\epsilon^2(h(\phi) + \eta_1)^2}{2} - \eta_0 \right] f^*(\phi) d\phi \geq \frac{3}{4} \lambda
\]
for sufficiently small \( \eta_1 \), where \( \eta_0 = \mu + \gamma + \alpha + \frac{\lambda^2}{2} \). Thus, for every \( \varepsilon > 0 \), using the ergodic theorem, there exists \( T(\varepsilon, x) \) such that \( \mathbb{P}(\Omega_2') \geq 1 - \frac{\varepsilon}{2} \) with
\[
\Omega_2' = \left\{ \frac{1}{T(\varepsilon, x)} \int_0^{T(\varepsilon, x)} \left[ \beta(h(\varphi_x(t)) - \eta_1) - \frac{\epsilon^2(h(\varphi_x(t)) + \eta_1)^2}{2} - \eta_0 \right] dt \geq \frac{5}{8} \lambda \right\}.
\]
Using the formula for the solution of the SDE (10) (see reference [20]), we have
\[
|\varphi_x(t) - \varphi_w(t)| = |w - x| \sup_{0 \leq t \leq T(\varepsilon, x)} \exp \left\{ - \left( \mu + \frac{\alpha^2}{2} \right) t + aB_1(t) \right\}.
\]
Choose \( \bar{\varepsilon} = \frac{\varepsilon}{T(\varepsilon, x)} \), then applying Doob’s martingale inequalities (Theorem 1.3.8 in [20]) yields
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T(\varepsilon, x)} |B_1(t)| \geq \frac{1}{\sqrt{\bar{\varepsilon}}} \right) \leq \bar{\varepsilon} \mathbb{E}(B_1(T(\varepsilon, x))^2) = \bar{\varepsilon} T(\varepsilon, x),
\]
and hence \( \mathbb{P}(\bar{\Omega}_2) \geq 1 - \frac{\varepsilon}{2} \), where
\[
\bar{\Omega}_2 = \left\{ \sup_{0 \leq t \leq T(\varepsilon, x)} |B_1(t)| < \frac{1}{\sqrt{\bar{\varepsilon}}} \right\}.
\]
Then \( |\varphi_w(t) - \varphi_x(t)| \leq \sigma \) for all \( \omega \in \bar{\Omega}_2 \) and \( w \in O(x, \delta_x) \), where \( \sigma = \delta_x \exp(a/\bar{\varepsilon}) \), and \( \delta_x \) to be chosen small enough so that
\[
\left| \beta(h(\varphi_x(t)) - \eta_1) - \frac{\epsilon^2(h(\varphi_x(t)) + \eta_1)^2}{2} - \left( \beta(h(\varphi_w(t)) - \eta_1) - \frac{\epsilon^2(h(\varphi_w(t)) + \eta_1)^2}{2} \right) \right| < \frac{1}{8} \lambda,
\]
whenever \( |\varphi_w(t) - \varphi_x(t)| \leq \sigma \). Then \( \Omega_2'' \cap \bar{\Omega}_2 \subseteq \Omega_2' \), where
\[
\Omega_2' = \left\{ \frac{1}{T(\varepsilon, x)} \int_0^{T(\varepsilon, x)} \left[ \beta(h(\varphi_x(t)) - \eta_1) - \frac{\epsilon^2(h(\varphi_x(t)) + \eta_1)^2}{2} - \eta_0 \right] dt \geq \frac{1}{2} \lambda \right\}
\]
for \( w \in O(x, \sigma_x) \). Hence, \( \mathbb{P}(\Omega_2') \geq \mathbb{P}(\Omega_2'') - (1 - \mathbb{P}(\bar{\Omega}_2)) \geq 1 - \varepsilon \).

By Heine-Borel theorem, the subset \( [D^{-1}, D] \) of \( \mathbb{R} \) can be covered by a set of neighborhoods of finite \( x_i \in [D^{-1}, D] \), i.e., \( [D^{-1}, D] \subset \bigcup_{i=1}^n O(x_i, \delta_{x_i}) \). Let \( T_1 \triangleq \max\{T(\varepsilon, x_i) : 1 \leq i \leq n\} \) and \( \delta_0 \triangleq \min_{1 \leq i \leq n} \delta_{x_i} \), then we have \( \mathbb{P}(\Omega_3) \geq 1 - \varepsilon \), where
\[
\Omega_3 = \left\{ \int_0^{T_1} \left[ \beta(h(\varphi_x(t)) - \eta_1) - \frac{\epsilon^2(h(\varphi_x(t)) + \eta_1)^2}{2} - \eta_0 \right] dt \geq \frac{1}{2} \lambda T_1 \text{ for all } x \in [D^{-1}, D] \right\}.
\]
It follows from Lemma 3.2 that we can choose $\sigma, \nu > 0$ such that $\mathbb{P}(\Omega_4) \geq 1 - \epsilon$ and that for all $\omega \in \Omega_4$, $t < T \wedge \tau_z^\omega$,

$$\frac{g(S_z(t), I_z(t))}{I_z(t)} - h(\varphi_z(t)) \leq \frac{g(S_z(t), I_z(t))}{I_z(t)} - h(S_z(t)) + |h(S_z(t)) - h(\varphi_z(t))| < \eta_1,$$

where $\Omega_4 = \{|\varphi_z(t) - S_z(t)| < \nu \text{ for all } t \in [0, T \wedge \tau_z^\omega]|\}$.

Since

$$\mathbb{E} \left[ \int_0^T bdB_2(t) + c \frac{g(S_z(t), I_z(t))}{I_z(t)} dB_3(t) \right] = 0,$$

then applying Chebyshev’s inequality, we have

$$\mathbb{P} \left\{ \left| \int_0^T bdB_2(t) + c \frac{g(S_z(t), I_z(t))}{I_z(t)} dB_3(t) \right| \leq \frac{M}{\sqrt{\epsilon}} \sqrt{T} \right\} \geq 1 - \epsilon,$$

where $M = M(b, c, M_2) > 0$ is a constant. Taking $T_2 > \frac{16MT^2}{\lambda^2 \epsilon^2}$, we have $\mathbb{P}(\Omega_5) \geq 1 - \epsilon$, where

$$\Omega_5 = \left\{ \left| \int_0^T bdB_2(t) + c \frac{g(S_z(t), I_z(t))}{I_z(t)} dB_3(t) \right| \leq \frac{\lambda T_2}{4} \right\}.$$

Recall (15), taking $T = T_1 \wedge T_2$ for $z \in [D^{-1}, D] \times (0, \delta_0]$ and $w \in \hat{\Omega}_z = \bigcap_{i=3}^5 \Omega_i$ yields

$$\ln I_z(T) - \ln y \geq \int_0^T \left[ \beta \frac{g(S_z(t), I_z(t))}{I_z(t)} - \frac{c^2 g(S_z(t), I_z(t))}{2} I_z^2(t) - \eta_0 \right] dt$$

$$- \left| \int_0^T (bdB_2(t) + c \frac{g(S_z(t), I_z(t))}{I_z(t)} dB_3(t)) \right|$$

$$\geq \int_0^T \left[ \beta (h(\varphi_z(t)) - \eta_1) - \frac{c^2(h(\varphi_z(t)) + \eta_1)^2}{2} - \eta_0 \right] dt - \frac{\lambda T_2}{4}$$

$$\geq \frac{\lambda T}{4}.$$

Consequently, we obtain $\mathbb{P}(\hat{\Omega}_z) = \mathbb{P}(\bigcap_{i=3}^5 \Omega_i) \geq 1 - 3\epsilon$, which completes the proof. \hfill \Box

**Theorem 3.4.** Assume that $\lambda > 0$, then the disease in system (1) is permanent.

**Proof.** Following similar argument as in Proposition 3.2 in [22] and noting Lemmas 2.3 and 3.3, we see that for any $\epsilon > 0$, there exist $T = T(\epsilon)$ and $\delta_1(\epsilon) > 0$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}\{I_z(kT) \leq \delta_1\} \leq \epsilon \quad \text{for all } z \in \mathbb{R}^2_+,$$

which together with Lemma 2.3 and taking $G = [\delta_1, N_1]$ yields that

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}\{I_z(kT) \in G\} \geq 1 - 2\epsilon \quad \text{for all } z \in \mathbb{R}^2_+.$$


By virtue of Lemma 2.3, there exists \( N_2 > 1 \) such that
\[
P\{N_2^{-1} \leq I_z(t) \leq N_2\} \geq 1 - \epsilon
\]
for all \( y \in G, t \leq T \). By the Markov property, for all \( t \leq T \)
\[
P\{N_2^{-1} \leq I_z(kT + t) \leq N_2\} \geq (1 - \epsilon)P\{I_z(kT) \in G\}.
\]
As a result, for any \( z \in \mathbb{R}^2_+ \),
\[
\liminf_{n \to \infty} \frac{1}{nT} \int_0^{nT} P\{N_2^{-1} \leq I_z(t) \leq N_2\} dt \geq (1 - 2\epsilon)(1 - \epsilon) \geq (1 - 3\epsilon),
\]
which implies that
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t P\{N_2^{-1} \leq I_z(u) \leq N_2\} du \geq 1 - 3\epsilon.
\]
This completes the proof. \( \square \)

In order to prove the existence of the stationary distribution of the system (1), we shall first cite a result in Khasminskii [12].

**Lemma 3.5.** The stochastic model (1) has a unique stationary distribution if there exists a strictly proper subinterval \((e, f)\) of \((0, \infty)\) such that
\[
E(\tau_z) < \infty \quad \text{for all } y \in (0, e] \cup [f, \infty),
\]
where \( \tau_z = \inf\{t \geq 0 : I_z(t) \in (e, f)\} \), and moreover,
\[
\sup_{y \in [e, f]} E(\tau_z) < \infty \quad \text{for every interval } [\tau, \overline{\tau}] \subset (0, \infty).
\]

**Theorem 3.6.** Assume that \( \lambda > 0 \), then system (1) has a unique stationary distribution.

**Proof.** In view of (7), it only suffices to show the conditions in Lemma 3.5 hold when \( \lambda > 0 \). First we take \( e = \delta_0 < N_1 \), where \( \delta_0 \) is defined in Lemma 3.3. For any \( y \in (0, e) \), it follows from the proof of Lemma 3.3 that
\[
E \ln I_z(t \wedge \tau_z) \geq \ln y + \frac{\lambda}{4} E(t \wedge \tau_z) \quad \text{for all } t \geq 0.
\]
By the definition of \( \tau_z \), we have
\[
\ln e \geq E \ln I_z(t \wedge \tau_z) \quad \text{for all } y \in (0, e),
\]
and hence
\[
E(t \wedge \tau_z) \leq \frac{4}{\lambda} \ln \frac{e}{y} < \infty \quad \text{for all } y \in (0, e).
\]
Letting \( t \to \infty \) yields that
\[
E(\tau_z) < \infty \quad \text{for all } y \in (0, e).
\]
Then take \( f = N_1 \). In virtue of Lemma 2.3, it is obvious that \( E(\tau_z) < \infty \). Thus, it follows from Lemma 3.5 and the continuity of \( \tau_z \) with respect to \( z \) that system (1) has a stationary distribution. This completes the proof. \( \square \)

**Remark 2.** In Theorem 3.6, we only obtain the existence of unique stationary distribution. In fact, we may employ the Markov semigroups theory and asymptotic properties (see, for example, Liu et al. [19], and Wen et al. [28]) to further show that the solutions can converge in \( L^1 \) to an invariant density.
Now, we apply the above theoretical results to system (3). Note that equation (10) with \( a = 0 \) (this is obviously an ordinary differential equation) has a unique stationary distribution with density given by \( f^*(\phi) = \delta(N_0) \), where \( \delta(\cdot) \) denotes the Dirac delta function. Thus, the threshold \( \lambda \) associated with (3) is given as follows:

\[
\lambda \triangleq \int_0^\infty \left[ \beta f'(0)\phi - \frac{c^2 f^2(0)\phi^2}{2} - (\mu + \gamma + \alpha) \right] f^*(\phi)d\phi
\]

\[
= \beta f'(0)N_0 - \frac{c^2 f^2(0)N_0^2}{2} - (\mu + \gamma + \alpha)
\]

\[
= (\mu + \gamma + \alpha) [R_0(c) - 1],
\]

where \( R_0(c) \) is given in (2). Thus, we immediately obtain the following result.

**Corollary 1.** Assume that \( R_0(c) > 1 \) then the disease in (3) is permanent and system (3) has a stationary distribution.

**Remark 3.** Corollary 1 obviously improves the relevant work by Teng and Wang [27] who only obtained the weak permanence and permanence in mean of the disease of (3) when \( R_0(c) > 1 \).

4. Extinction of disease. In this section, we discuss the extinction of disease and the convergence of the susceptible population.

**Lemma 4.1.** Assume that \( \lambda < 0 \). For any \( D > 1, \epsilon, \nu > 0 \), there exists \( \delta > 0 \) such that

\[
P \left( \limsup_{t \to \infty} \frac{\ln I_z(t)}{t} = \lambda \right) \cap \{ |\varphi_x(t) - S_z(t)| \leq \nu \text{ for all } t \geq 0 \} \geq 1 - 3\epsilon
\]

for all \( z \in [D^{-1}, D] \times [0, \delta] \).

**Proof.** For convenience, let \( \eta_0 = \mu + \gamma + \alpha + \frac{h^2}{T} \). Since

\[
\int_0^\infty \left[ -\beta h(\phi) + \frac{c^2 h^2(\phi)}{2} \right] f^*(\phi)d\phi = -\lambda - \eta_0 < \infty,
\]

then we can find \( \eta_2 > 0 \) such that

\[
\int_0^\infty \left[ -\beta (h(\phi) + \eta_2) + \frac{c^2 (h(\phi) - \eta_2)^2}{2} \right] f^*(\phi)d\phi \geq \frac{3}{4}\lambda - \eta_0.
\]

Similarly to the method in the proof of Lemma 3.3, using the ergodicity and Heine-Borel theorem, we obtain that there exists \( T_4 = T_4(\epsilon, D) \) such that for all \( t \geq T_4 \), \( P(\Omega_6) \geq 1 - \epsilon \), where

\[
\Omega_6 = \left\{ \frac{1}{T} \int_0^T \left[ \beta h(\varphi_x(s)) + \eta_2 \right] ds - \nu_0 \leq \frac{3}{4}\lambda \right\}.
\]

By the continuity of \( g(\cdot, \cdot) \), there exists positive constant \( \nu \) and \( \sigma \) such that

\[
\left| h(r) - \frac{g(p, q)}{q} \right| < \eta_2 \quad \text{whenever} \quad |p - r| < \nu \quad \text{and} \quad 0 < |q| < \sigma.
\]

Let \( \vartheta = \inf \{ t > 0 : |\varphi_x(t) - S_z(t)| \leq \nu \} \), then there exists \( \sigma \in (0, \sigma) \) such that

\[
\left| h(\varphi_x(t)) - \frac{g(S_z(t), I_z(t))}{I_z(t)} \right| < \eta_2
\]
for all $t \in [0, \vartheta_z \wedge \tau_z^\vartheta]$. Hence when $\omega \in \Omega_6 \cap \{\vartheta_z \geq T_4\}$,
$$
\frac{1}{\tau_i} \int_0^{\tau_i} \left[ \beta h(S_z(s)) - \frac{c^2 h^2(S_z(s))}{2} \right] ds - \eta_0 \leq \frac{3}{4} \lambda
$$
(18)
for all $t \in [T_4, \vartheta_z \wedge \tau_z^\vartheta]$. By strong law of large numbers for martingales, $\mathbb{P}(\Omega_7) \geq 1 - \varepsilon$, where
$$
\Omega_7 = \left\{ \int_0^{\tau_i} \left[ bdB_2(s) + c \frac{g(S_z(s), I_z(s))}{I_s(s)} dB_3(s) \right] \leq \frac{|\lambda|}{4} t \quad \text{for all} \; t \geq 0 \right\}.
$$
(19)
Applying (18) and (19) into (15), then for $\omega \in \Omega_6 \cap \Omega_7 \cap \{\vartheta_z \wedge \tau_z^\vartheta \geq T_4\}$, we have
$$
\ln I_z(t) \leq \ln y - \frac{|\lambda|}{2} t
$$
(20)
for all $t \in [T_4, \vartheta_z \wedge \tau_z^\vartheta]$. By virtue of Lemmas 3.1 and 3.2, we can find sufficiently small $\delta = \delta(\varepsilon, D)$ such that
$$
\ln \delta - \frac{|\lambda|}{2} T_4 < \ln \bar{\sigma}
$$
(21)
and $\mathbb{P}(\Omega_8) \geq 1 - \varepsilon$ for all $z \in [D^{-1}, D] \times (0, \delta]$, where
$$
\Omega_8 = \{ \zeta_z \triangleq \vartheta_z \wedge \tau_z^\vartheta \geq T_4 \}.
$$
Consequently, we obtain $\mathbb{P}(\bar{\Omega}_z) \geq 1 - 3\varepsilon$, where $\bar{\Omega}_z = \cap_{i=0}^{8} \Omega_i$. It follows from the definition of $\zeta_z$ that for $\omega \in \Omega_z$,
$$
|\varphi_x(t \wedge \zeta_z) - S_z(t \wedge \zeta_z)| \leq \nu.
$$
Then in $\bar{\Omega}_z$, $t \wedge \zeta_z < \vartheta_z$ for all $t \geq T_4$, which implies that $\bar{\Omega}_z \subset \{\zeta_z \leq \vartheta_z\}$. Since $\zeta_z = \vartheta_z \wedge \tau_z^\vartheta$, we have $\bar{\Omega}_z \subset \{\tau_z^\vartheta \leq \vartheta_z\}$. As a result, when $z \in [D^{-1}, D] \times (0, \delta]$ and $\omega \in \Omega_z$, it follows from (20) and (21) that
$$
\ln I_z(t \wedge \tau_z^\vartheta) \leq \ln y - \frac{|\lambda|}{2} (t \wedge \tau_z^\vartheta) < \ln \bar{\sigma} \quad \text{for all} \; t \geq T_4.
$$
This implies that $t \wedge \tau_z^\vartheta < \tau_z^\vartheta$ for all $t \geq T_4$, $z \in [D^{-1}, D] \times (0, \delta]$ and $\omega \in \bar{\Omega}_z$, and hence that $\tau_z^\vartheta = \vartheta_z = \infty$ for any $z \in [D^{-1}, D] \times (0, \delta]$ and $\omega \in \Omega$. If $\omega \in \Omega_z$, by ergodicity and strong law of large numbers for martingales, we have
$$
\limsup_{t \to \infty} \frac{\ln I_z(t)}{t} - \lambda \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left[ \beta \left| \frac{g(S_z(s), I_z(s))}{I_z(t)} - h(\varphi_x(s)) \right| + \frac{c^2}{2} \left| \frac{g^2(S_z(s), I_z(s))}{I_z(t)} - h^2(\varphi_x(s)) \right| \right] ds \leq c_0 \varepsilon,
$$
where $c_0$ is a positive constant. Thus, for any $\nu > 0$ and $z \in [D^{-1}, D] \times (0, \delta]$, we obtain
$$
\mathbb{P} \left\{ \limsup_{t \to \infty} \frac{\ln I_z(t)}{t} = \lambda \; \text{and} \; |\varphi_x(t) - S_z(t)| \leq \nu \; \text{for all} \; t \geq 0 \right\} \geq \mathbb{P}(\bar{\Omega}_z) \geq 1 - 3\varepsilon.
$$
The proof is completed.

**Theorem 4.2.** Assume that $\lambda < 0$, then the disease in system (1) is extinct and the susceptible population weakly converges to $\pi^\ast$.
Proof. By virtue of Lemma 4.1, we know that \( I(t) \) is not recurrent in \( \mathbb{R}_+ \). Note that the diffusion in (1) is nondegenerate, then \( I(t) \) is transient, and hence for any \( t \geq 0 \), there exists \( \delta > 0 \) such that

\[
\mathbb{E}(\tau'_2) < \infty \quad \text{for all } y \in (\delta, \infty),
\]

where \( \tau'_2 \) is defined as \( \inf \{ t \geq 0 : I_z(t) \in (0, \delta) \} \). Then it follows from Lemma 4.1 that

\[
\mathbb{P} \left\{ \limsup_{t \to \infty} \frac{\ln I_z(t)}{t} = \lambda \right\} = 1
\]

for all \( y > 0 \), which implies that as \( t \to \infty \), \( I_z(t) \to 0 \) a.s. that is, the disease in system (1) is extinct. In what follows, we prove that the susceptible population weakly converges to \( \pi^* \).

Let \( h(\cdot) \) be a Lipschitz function in \( \mathbb{R}_+ \) with constant \( K_h \) such that

\[
|h(x_1)| \leq K_h \text{ and } |h(x_1) - h(x_2)| \leq K_h|x_1 - x_2|
\]

for all \( x_1, x_2 \in \mathbb{R}_+ \). By the ergodicity (see [25], Theorem 3.16), for every function \( h(\cdot) : \mathbb{R} \to \mathbb{R} \) satisfying that \( \int_0^\infty |h(\phi)|f^*(\phi)d\phi < \infty \), we have

\[
\mathbb{P} \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T h(\varphi_z(t))dt = \int_0^\infty h(\phi)f^*(\phi)d\phi \right\} = 1 \quad \text{for all } x > 0.
\]

Then in order to prove the weak convergence of the distribution of \( S_z(t) \) to the measure \( \pi^* \), we only need to prove that

\[
\lim_{t \to \infty} \mathbb{E}_h(S_z(t)) = \pi^* \triangleq \int_0^\infty h(\phi)f^*(\phi)d\phi \quad \text{for all } z \in \mathbb{R}_+^2.
\]

Since \( |\mathbb{E}_h(S_z(t)) - h^*| \leq |\mathbb{E}_h(\varphi_z(t)) - h^*| + |\mathbb{E}_h(S_z(t)) - \mathbb{E}_h(\varphi_z(t))| \) and \( \varphi_z(t) \) converges weakly to \( \pi^* \), we have

\[
\limsup_{t \to \infty} \left| \mathbb{E}_h(S_z(t)) - h^* \right| \leq K_h\nu \limsup_{t \to \infty} \mathbb{P}\{|\varphi_z(t) - S_z(t)| \leq \nu\} \leq K_h\nu \limsup_{t \to \infty} \mathbb{P}\{|\varphi_z(t) - S_z(t)| \leq \nu\}.
\]

By the Markov property,

\[
\left| \mathbb{E}_h(S_z(t + i_0T_5)) - h^* \right| \leq \int_{\mathbb{R}_+^2} \left| \mathbb{E}_h(S_z(t)) - h^* \right| \mathbb{P}\{S_z(i_0T_5) \in dz\} \leq \int_{U_1} \left| \mathbb{E}_h(S_z(t)) - h^* \right| \mathbb{P}\{S_z(i_0T_5) \in dz\} + 2K_h\nu \mathbb{P}\{S(i_0T_5) \notin U_1\}.
\]

Applying (22), (23) and Fatou’s lemma into (24), we have

\[
\limsup_{t \to \infty} \left| \mathbb{E}_h(S_z(t + i_0T_5)) - h^* \right| \leq K_h\nu \leq K_h\nu + K_h^2\varepsilon + 6K_h\varepsilon.
\]

The proof is completed due to the arbitrariness of \( \varepsilon \) and \( \nu \). \( \square \)

The next result follows from Lemma 2.3 and Theorems 2.5, 4.2 immediately.

**Corollary 2.** If \( \lim_{t \to \infty} I_z(t) = 0 \), then \( \lim_{t \to \infty} \frac{1}{t} \int_0^t S_z(u)du = N_0 \).

**Proof.** From Lemma 2.3, we see that for any \( z \in \mathbb{R}_+^2 \),

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t S_z(u)du \leq N_0 \quad \text{a.s.}
\]
It is easy to see that \( g(S, I) \) is Lipschitz continuous with respect to \( I \) and \( 0 < g(S, I) \leq h(S)I \) for all \( I > 0 \). Thus, for any \( \varepsilon > 0 \), there exists \( T > 0 \) such that
\[
g(S_z(t), I_z(t)) \leq M_2 \varepsilon \quad \text{for all } t \geq T.
\]

Integrating the first equation of system (1) yields that for any \( t \geq 0 \)
\[
\frac{S_z(t) - x}{t} = \Lambda - \frac{1}{t} \int_0^t [\beta g(S_z(u), I_z(u)) - \gamma I_z(u) + \mu S_z(u)] du
\]
\[
+ \frac{1}{t} \int_0^t [aS_z(u)dB_1(u) - cg(S_z(u), I_z(u))dB_3(u)]
\]
\[
\geq \Lambda - \frac{1}{t} \int_0^t [\beta g(S_z(u), I_z(u)) + \mu S_z(u)] du - \frac{1}{t} \int_0^t [\beta M_2 \varepsilon + \mu S_z(u)] du
\]
\[
+ \frac{1}{t} \int_0^t [aS_z(u)dB_1(u) - cg(S_z(u), I_z(u))dB_3(u)].
\]

Note that \( \lim_{t \to \infty} \frac{S_z(t) - x}{t} = 0 \) and \( \lim_{t \to \infty} \frac{1}{t} \int_0^T [\beta g(S_z(u), I_z(u)) + \mu S_z(u)] du = 0 \).

Then by the large number theorem for martingales and the arbitrariness of \( \varepsilon \), we have
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t S_z(u) du \geq N_0 \quad \text{a.s.}
\]

This completes the proof.

Now, we apply Theorem 4.2 to system (3). Note that the threshold \( \lambda \) associated with (3) satisfies that \( \lambda = (\mu + \gamma + \alpha)[R_0(c) - 1] \), where \( R_0(c) \) is given in (2), then we can conclude that the disease in system (3) is extinct when \( R_0(c) < 1 \). In addition, it follows from (17) that
\[
\lambda = - \frac{c^2}{2} \left[ f'(0)N_0 - \frac{\beta}{c^2} \right]^2 + \frac{\beta^2}{2c^2} - (\mu + \gamma + \alpha),
\]
which is negative if
\[
c^2 > \frac{\beta^2}{2(\mu + \gamma + \alpha)}.
\]

Thus, we immediately obtain the following result.

**Corollary 3.** The disease in system (3) is extinct when either \( R_0(c) < 1 \) or (25) holds, where \( R_0(c) \) is given in (2).

**Remark 4.** Obviously, Corollary 3 improves the relevant work by Teng and Wang [27], Gray et al. [7] who showed that the disease is extinct with probability one when \( R_0(c) < 1 \) and \( \beta \mu \geq c^2 f'(0) \Lambda \). However, the restriction \( \beta \mu \geq c^2 f'(0) \Lambda \) is proved to be redundant in our paper.

**Remark 5.** The approach adopted by Teng and Wang [27] and their theoretical results are inapplicable to system (3) under the condition (4). As we stated before, condition (4) is actually a special case of our assumption \( \lambda < 0 \), this is to say, the disease still dies out with probability one under the condition (4).

5. **Numerical simulations.** In this section, we shall give an example and perform some numerical simulations to illustrate our results in Sections 3 and 4. Through this section, we take
\[
g(S, I) = \frac{SI}{1 + m_1S + m_2I}.
\]
and $h(S) = \frac{S}{1+m_1S}$, where $m_1, m_2 > 0$. By applying Milstein scheme in [9], we have the following discretization system of model (1):

$$S_{k+1} = S_k + \left( \Lambda - \frac{\beta S_k I_k}{1 + m_1 S_k + m_2 I_k} - \mu S_k + \gamma I_k \right) \Delta t + S_k \left[ a_1 \xi_1, k \sqrt{\Delta t} + \frac{1}{2} a_1^2 (\xi_1, k - 1) \Delta t \right]$$

$$+ \frac{S_k I_k}{1 + m_1 S_k + m_2 I_k} \left[ c_1, k \xi_2, k \sqrt{\Delta t} + \frac{1}{2} c_1^2 (\xi_2, k - 1) \Delta t \right]$$

$$I_{k+1} = I_k + \left( \frac{\beta S_k I_k}{1 + m_1 S_k + m_2 I_k} - (\mu + \gamma + \alpha) I_k \right) \Delta t + I_k \left[ b_1 \xi_2, k \sqrt{\Delta t} + \frac{1}{2} b_1^2 (\xi_2, k - 1) \Delta t \right]$$

$$+ \frac{S_k I_k}{1 + m_1 S_k + m_2 I_k} \left[ c_2, k \xi_3, k \sqrt{\Delta t} + \frac{1}{2} c_2^2 (\xi_3, k - 1) \Delta t \right],$$

where $\Delta t$ is the time increment and $\xi_1, k$, $\xi_2, k$ and $\xi_3, k$ ($k = 1, 2, 3, ...$) are independent Gaussian random variables which follow the standard Normal distribution $N(0, 1)$.

We first consider the following system

$$\begin{cases}
\quad dS(t) = \left[ 2 - \frac{0.4SI}{1 + 0.1S + I} + 0.2I(t) - 0.4S(t) \right] dt + aS(t)dB_1(t) - \frac{cSI}{1 + 0.1S + I} dB_3(t), \\
\quad dI(t) = \left[ \frac{0.4SI}{1 + 0.1S + I} - 0.8I(t) \right] dt + bI(t)dB_2(t) + \frac{cSI}{1 + 0.1S + I} dB_3(t),
\end{cases} \tag{26}$$

whose corresponding deterministic system of (26) has a unique globally asymptotically stable endemic equilibrium $(S^*, I^*) = \left( \frac{5}{8}, \frac{65}{16} \right)$ (for more details, see for example [27]).

First, we choose the following parameters $a = b = c = 0.1$. An easy calculation yields $\lambda = 0.4691 > 0$. This, together with Theorems 3.4 and 3.6, implies that the disease $I(t)$ is permanent and there exists a stationary distribution of system (1). This theoretical result is illustrated by Figures 1 and 2, from which we can see that the solution of model (26) oscillates around the deterministic endemic equilibrium $(S^*, I^*) = \left( \frac{5}{8}, \frac{65}{16} \right)$ after some initial transients. In fact, we can see that the larger the intensities of the white noises are, the larger the fluctuations of the solutions will be. From Figure 2 we can see that the values of $S(t)$ and $I(t)$ are distributed normally around the mean values $\frac{5}{8}$ and $\frac{65}{16}$, respectively.

In order to find out the effect of intensity of noises on the dynamical behaviours of $S$ and $I$, we shall increase the intensity $a, b$ and $c$ of environmental forcing, respectively. We first fix the two intensities $b = c = 0.1$ and increase intensity $a$ from 0.1 to 1. In this case, $\mu < \frac{1}{7} \alpha^2$, and hence the diffusion of (10) is transient as depicted in Figure 3. As we increase intensity $a$ further, we see from Figure 4 that $S$ is transient and $I$ is eventually extinct. This implies that too large intensities of white noises may make both of the susceptible and infected individuals go to
extinction while an appropriate intensities of white noises make them fluctuate around the deterministic steady-state values.

We next fix the two intensities \( a = c = 0.1 \) and increase intensity \( b \) from 0.1 to 1, or fix the two intensities \( a = b = 0.1 \) and increase intensity \( c \) from 0.1 to 0.31. By calculation, we obtain \( \lambda = -0.0259 < 0 \) and \( \lambda = -0.0092 < 0 \), respectively, which both imply that the disease \( I(t) \) is extinct and \( S(t) \) converges to a boundary distribution (i.e., the unique stationary solution of equation (10)) according to Theorem 4.2 (see Figures 5, 6 and 7). That is to say, too large intensities of white noises can lead the disease to extinction, which is a phenomenon different from its corresponding deterministic model of system (26).

In what follows, we consider the following stochastic system

\[
\begin{align*}
\frac{dS(t)}{dt} &= \left[ 1 - \frac{0.4SI}{1 + 0.1S + I} + 0.4I(t) - 0.4S(t) \right] dt \\
&\quad + aS(t)dB_1(t) - \frac{cSI}{1 + 0.1S + I}dB_3(t), \\
\frac{dI(t)}{dt} &= \left[ \frac{0.4SI}{1 + 0.1S + I} - 1.2I(t) \right] dt \\
&\quad + bI(t)dB_2(t) + \frac{cSI}{1 + 0.1S + I}dB_3(t),
\end{align*}
\]

(27)

whose corresponding deterministic system of (27) has a globally asymptotically stable disease-free equilibrium \((S^*, I^*) = (2.5, 0)\) (for more details, see for example [27]).

We first consider system (27) with parameters \( a = b = c = 0.1 \). Direct calculation yields that \( \lambda = -0.4267 < 0 \). Figures 8 shows that the solution of model (27) oscillates around the deterministic disease-free equilibrium \((S^*, I^*) = (2.5, 0)\) after some initial transients, that is, the disease \( I(t) \) is extinct and \( S(t) \) converges to a boundary distribution, which further illustrate the conclusion of Theorem 4.2.

If we increase the intensity of \( a, b \) and \( c \) of environmental forcing, respectively, we shall find out that \( I \) converges to 0 faster. For example, fix the two intensities \( b = c = 0.1 \) and increase intensity \( a \) from 0.1 to 1, the diffusion of (10) is transient and \( I \) is eventually extinct as depicted in Figure 9. Now, fix the two intensities \( a = c = 0.1 \) and increase intensity \( b \) from 0.1 to 1, or fix the two intensities \( a = b = 0.1 \) and increase intensity \( c \) from 0.1 to 0.31, then we can have \( \lambda = -0.9217 < 0 \) and \( \lambda = -2.4145 < 0 \), respectively. This, together with Theorems 4.2, implies that the disease \( I(t) \) is extinct and the susceptible population \( S(t) \) weakly converges to a boundary distribution (see Figures 10 and 11).

6. Conclusions and discussions. In this paper, we introduce a more accurate threshold which is sufficient and almost necessary condition of permanence and extinction of the disease in a class of stochastic SIS epidemic models with nonlinear incidence rate, which include the standard incidence, Beddington-DeAngelis incidence, and nonlinear incidence of the form \( f(S)h(I) \). Indeed, the case \( \lambda = 0 \) is not investigated and the relationship between \( \lambda = 0 \) and \( R_0 = 1 \) is not analyzed either in this paper. In this paper, we take several different noises into consideration and our model (1) is more general because many stochastic SIS models can be regarded as special cases of our model (for example, [5, 7, 27, 32, 34]). It is very interesting to notice that an appropriate intensities of white noises make the susceptible and infected individuals fluctuate around their deterministic steady-state values, that
Figure 1. Trajectories of solutions of model (26) with initial value $(S(0), I(0)) = (1.9, 1.9)$ and parameters $a = b = c = 0.1$.

Figure 2. Stationary distribution of model (26) with initial value $(S(0), I(0)) = (1.9, 1.9)$ and parameters $a = b = c = 0.1$: (a) the graph of the relative frequency densities of $S$ and $I$; (b) the joint density distribution of solution $(S, I)$.

the larger the intensities of the white noises are, the larger their fluctuations, and that too large intensities of white noises may make both of the susceptible and infected individuals go to extinction.

We should point out that our theoretical results can also be applied to the stochastic susceptible-infected-recovered (SIR) epidemic model

\[
\begin{align*}
\frac{dS(t)}{dt} &= [\Lambda - \beta g(S(t), I(t)) - \mu S(t)]dt \\
&\quad + aS(t)dB_1(t) - cg(S(t), I(t))dB_3(t), \\
\frac{dI(t)}{dt} &= [\beta g(S(t), I(t)) - (\mu + \gamma + \alpha)I(t)]dt \\
&\quad + bI(t)dB_2(t) + cg(S(t), I(t))dB_3(t), \\
\frac{dR(t)}{dt} &= [-\mu R(t) + \gamma I(t)]dt + \sigma R(t)dB_4(t),
\end{align*}
\]

(28)
Figure 3. Trajectories of solutions of model (26) with initial value $(S(0), I(0)) = (1.9, 1.9)$ and parameters $a = 1, b = 0.1, c = 0.1$.

Figure 4. Trajectories of solutions of model (26) with initial value $(S(0), I(0)) = (1.9, 1.9)$ and parameters $a = 2, b = 0.1, c = 0.1$.

Figure 5. Trajectories of solutions of model (26) with initial value $(S(0), I(0)) = (1.9, 1.9)$ and parameters $a = 0.1, b = 1, c = 0.1$. 
because the dynamics of recovered individuals have no influence on the disease transmission dynamics and we can omit to discuss the third equation of model (28), where $R(t)$ represents the number of recovered individuals, $\sigma \geq 0$, $B_i(t), i = 1, 2, 3, 4$ are mutually independent Brownian motions, and all the other parameters have the same meaning as them in (1). There are a number of results about permanence and extinction of the disease in the SIR model (28). For example, Du et al. [4] obtained sufficient conditions for the permanence and ergodicity in both of nondegenerate and degenerate cases for the SIR model (28) with $g(S, I) = SI/(1 + m_1 S + m_2 I)$ and $a = b = \sigma = 0$. Ji and Jiang [11] obtained a basic reproduction number to determine the extinction and permanence of the disease in system (28) with $g(S, I) = SI$ and $a = b = 0$. For the SIR model (28) with $g(S, I) = SI/(1 + mI)$ and $a = b = \sigma = 0$ and $m > 0$, Liu and Chen [18], Zhang et al. [30] also obtained a threshold

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6.png}
\caption{Trajectories of solutions of model (26) with initial value $(S(0), I(0)) = (1.9, 1.9)$ and parameters $a = 0.1, b = 0.1, c = 0.31$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.png}
\caption{Numerical simulations of model (26) with initial value $(S(0), I(0)) = (1.9, 1.9)$ and parameters $a = 0.1, b = 0.1, c = 0.31$: (a) Trajectories of the solution $S(t)$ of (26) and the solution $\varphi$ of (10) shows the convergence of $S$ to a boundary distribution; (b) The dynamics of $S(t)$ and $I(t)$ in time average.}
\end{figure}
determining whether the disease tends to zero exponentially with probability one or the disease is permanent in mean. By choosing appropriate Lyapunov functions for system (28) with $g(S, I) = SI$, Zhou et al. [33] investigated the existence of a stationary distribution, derived some sufficient conditions for the extinction of disease, and established how the solution spirals around the disease-free equilibrium of its associated deterministic system. It is easy to see that the relevant results obtained in [4, 11, 18, 30, 33] are completely covered by Corollaries 1 and 3.

The SIR epidemic model focuses on recovered individuals to be permanently immune, while the SIS epidemic model pays attention to recovered individuals to be immediately re-susceptible. However, some infections may not fall into either of these extreme categories. This is to say, it is natural to include the effects of immunity into the mathematical models in order to describe the actual dynamics of epidemic spread and to predict future outbreaks. We may propose the following continuous SIRS epidemic model with non-linear incidence rate and vaccination (see

Figure 8. Trajectories of solution $(S(t), I(t))$ of model (27) with initial value $(S(0), I(0)) = (1, 1)$ and parameters $a = 0.1$, $b = 0.1$, $c = 0.1$.

Figure 9. Trajectories of solution $(S(t), I(t))$ of model (27) with initial value $(S(0), I(0)) = (1, 1)$ and parameters $a = 1$, $b = 0.1$, $c = 0.1$. 
Figure 10. Trajectories of solution \((S(t), I(t))\) of model (27) with initial value \((S(0), I(0)) = (1, 1)\) and parameters \(a = 0.1, b = 1, c = 0.1\).

Figure 11. Trajectories of solution \((S(t), I(t))\) of model (27) with initial value \((S(0), I(0)) = (1, 1)\) and parameters \(a = 0.1, b = 0.1, c = 1\).

\[\begin{align*}
\text{d}S(t) &= [(1 - p)\Lambda - \beta g(S(t), I(t)) - \mu S(t) + \nu R]dt \\
&\quad + aS(t)dB_1(t) - cg(S(t), I(t))dB_3(t), \\
\text{d}I(t) &= [\beta g(S(t), I(t)) - (\mu + \gamma + \alpha)I(t)]dt \\
&\quad + bI(t)dB_2(t) + cg(S(t), I(t))dB_3(t), \\
\text{d}R(t) &= [p\Lambda - (\mu + \nu)R(t) + \gamma I(t)]dt + \sigma R(t)dB_4(t),
\end{align*}\]

(29)

where \(S(t)\) represents the number of susceptibles at time \(t\); \(I(t)\) represents the number of infective individuals and \(R(t)\) represents the number of recovered individuals with temporary immunity acquired from a disease. That is, after recovery, an individual loses immunity and therefore moves into the susceptible class. In model (29), \(\Lambda\) is the recruitment rate of the population of which a fraction \(p\) is vaccinated, so the fraction \((1 - p)\) is susceptible with \(0 \leq p \leq 1\), positive constant \(\nu\) is the
rate at which recovered individuals lose immunity and return to the susceptible class, \( B_i(t), i = 1, 2, 3, 4 \) are mutually independent Brownian motions and all the parameters have the same meaning as them in (1). It would be very interesting to see whether our analysis and methods in this paper are applicable to such a SIRS epidemic model as (29) because system (29) cannot be de-coupled any more. We look forward to investigating them in the future.

Finally, this paper is only a first step in introducing stochasticity into a SIS epidemic model. One may study a more realistic stochastic version of a SIS epidemic model by introducing a regime-switching diffusion process (see, e.g., [1]). This is interesting because the population may suffer sudden environmental changes such as climate changes, which may cause the disease to spread faster or slower. We also leave this topic for further investigation.

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