Non-Cohen-Macaulay canonical singularities

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Dedicated to Lawrence Ein on the occasion of his 60th birthday

1. INTRODUCTION

Since the first counter-example to Kodaira vanishing in positive characteristic was constructed by Raynaud [Ray78] many other counter-examples have been found satisfying various prescribed properties [DI87, Eke88, SB91, Kol96, Lau96, Muk13, DCF15, CT16a, CT16b]. An elementary counter-example for which the line bundle violating Kodaira vanishing is very ample was constructed by Lauritzen and Rao in [LR97]. Let us denote it by $X$. It is straightforward from the construction that $X$ is a rational variety and for $p = 2$ and $\dim X = 6$ it is Fano. Let $Z$ denote the cone over $X$ using the embedding given by the global sections of the very ample line bundle violating Kodaira vanishing. It is well-known that a cone over a Fano variety has klt singularities if $K_Z$ is $\mathbb{Q}$-Cartier. (See Definition 2.1.) The failure of Kodaira vanishing on $X$ implies that $Z$ will not have Cohen-Macaulay singularities, in particular it does not have rational singularities. As pointed out by Esnault and Kollár, although in this example $K_Z$ is not $\mathbb{Q}$-Cartier, one can easily find a boundary $\Delta$ on $Z$ that makes $K_Z + \Delta$ $\mathbb{Q}$-Cartier, and hence the pair $(Z, \Delta)$ klt. In other words Lauritzen and Rao’s counter-example to Kodaira vanishing produces a klt pair $(Z, \Delta)$ such that $Z$ is not Cohen-Macaulay. This provides a counter-example to the positive characteristic analogue of Elkik’s theorem [Elk81], [KM98, 5.22]. Examples of non-Cohen-Macaulay klt singularities were also given by Yasuda in [Yas14] and Cascini and Tanaka in [CT16a].

We will show that one can use the above $X$ to produce even more interesting singularities. I will demonstrate below that in fact the very ample line bundle $\omega_X^{-2}$ also violates Kodaira vanishing and hence leads to a cone, using the polarization given by $\omega_X^{-1}$, whose canonical sheaf is a line bundle, has canonical singularities, and is not Cohen-Macaulay. Of course, then it also does not have rational singularities. In other words, the purpose of this note is to prove the following.

**Theorem 1.1.** Let $k$ be a field of characteristic 2. Then there exists a Fano variety $X$ over $k$ such that

(i) $\dim X = 6$,

(ii) $\omega_X^{-1}$ is very ample, and

(iii) $\omega_X^{-2}$ violates Kodaira vanishing.

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Supported in part by NSF Grant DMS-1565352 and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics at the University of Washington.
One might ask if there is a similar example in smaller dimensions. It follows from [CT16b, A.1] that there are no Del Pezzo surfaces with this property. Hence a similar example in smaller dimension would be at least 3-dimensional. One might also ask if there is a similar example where $\omega_X^{-1}$ violates Kodaira vanishing. The example here is certainly not such and it is well-known that no such example exists for $\dim X = 2, 3$ [SB97, Sch07, Mad16]. While this is an interesting question, it is irrelevant for the purposes of the present article. The more interesting question is whether there are similar examples in all positive characteristics.

My main interest in the above result lies in the following application. By taking the cone over $X$ given by the embedding induced by the global sections of $\omega_X^{-1}$ we obtain the following.

**Theorem 1.2.** Let $k$ be a field of characteristic 2. Then there exists a variety $Z$ over $k$ with the following properties:

(a) $Z$ is of dimension 7, has a single isolated canonical singularity, and admits a resolution of singularities by a smooth variety over $k$,

(b) $\omega_Z$ is a line bundle,

(c) $Z$ is not Cohen-Macaulay, in particular, $Z$ is not Gorenstein and does not have rational singularities.

Again, one might ask if there are such singularities in smaller dimensions. Of course, if one finds examples such as in **Theorem 1.1** in smaller dimensions, that would provide smaller dimensional examples for **Theorem 1.2** as well. However, as mentioned above there are no examples similar to **Theorem 1.1** in dimension 2 which makes it an interesting question whether there exist 3-dimensional canonical singularities, perhaps even of index 1, that are not Cohen-Macaulay. And again, the possibly more interesting question is whether there are such singularities in all positive characteristics.

**Note added in proof.** While this paper was under review the above question has been answered. Bernasconi gave examples of non-Cohen-Macaulay klt singularities in characteristic 3 in [Ber17] and Totaro and Yasuda gave examples of non-Cohen-Macaulay terminal singularities in all positive characteristics in [Tot17] and [Yas17].

In the opposite direction Hacon and Witaszek [HW17] recently proved that in dimension 3 klt singularities are rational if the characteristic of the base field is sufficiently large.

**Acknowledgment.** I am grateful to János Kollár, Hiromu Tanaka, Burt Totaro, Takehiko Yasuda, and to the referee for useful comments.

2. **Non-Cohen-Macaulay singularities via failure of Kodaira vanishing**

**Definition 2.1.** Let $X$ be a smooth projective variety over $k$ and $L$ an ample line bundle on $X$. Then we will say that $L$ violates Kodaira vanishing if there exists an $i < \dim X$ such that $H^i(X, L^{-1}) \neq 0$. By Serre duality this is equivalent to that $H^{\dim X - 1}(X, L \otimes \omega_X) \neq 0$.

The canonical divisor of a normal variety $Z$ is denoted, as usual, by $K_Z$ and the associated reflexive sheaf of rank 1, the canonical sheaf, is denoted by $\omega_Z$. I.e., $\omega_Z \cong \mathcal{O}_Z(K_Z)$. A Weil divisor $D$ on $Z$ is $\mathbb{Q}$-Cartier if there exists a non-zero $m \in \mathbb{N}$ such that $mD$ is Cartier. A normal variety $Z$ is said to have rational singularities if for a resolution of singularities $\phi : \tilde{Z} \rightarrow Z$ the following conditions hold:
(i) $\mathcal{R}^i\phi_*\mathcal{O}_{\overline{Z}} = 0$ for $i > 0$, and
(ii) $\mathcal{R}^i f_*\omega_{\overline{Z}} = 0$ for $i > 0$.

In characteristic 0 (ii) is automatic by the Grauert-Riemenschneider vanishing theorem \cite{GR70, KM98, 2.68}. For the definition of klt and canonical singularities the reader is referred to \cite{Kol13, 2.8}.

Rational singularities are Cohen-Macaulay by the following well-known lemma. A very short proof is included for the convenience of the reader.

**Lemma 2.2.** Let $Z$ be a scheme with rational singularities. Then $Z$ is Cohen-Macaulay.

**Proof.** Let $d = \dim Z$ and let $\phi : \overline{Z} \to Z$ be a resolution of singularities of $Z$. This implies that $\mathcal{O}_Z \simeq \mathcal{R}\phi_*\mathcal{O}_{\overline{Z}}$ and $\omega_Z \simeq \mathcal{R}\phi_*\omega_{\overline{Z}}$. Then by Grothendieck duality

$$\omega_Z [d] \simeq \mathcal{R}\phi_*\omega_{\overline{Z}}[d] \simeq \mathcal{R}\phi_*\mathcal{R}\hom_{\overline{Z}}(\mathcal{O}_{\overline{Z}}, \omega_{\overline{Z}}) \simeq \mathcal{R}\hom_Z(\mathcal{R}\phi_*\mathcal{O}_{\overline{Z}}, \omega_Z) \simeq \omega_Z,$$

and hence $Z$ is Cohen-Macaulay. □

**Remark 2.3.** It follows easily that if $Z$ is not Cohen-Macaulay, then for any resolution of singularities $\phi : \overline{Z} \to Z$, there exists an $i > 0$ such that either $\mathcal{R}^i\phi_*\mathcal{O}_{\overline{Z}} \neq 0$ or $\mathcal{R}^i\phi_*\omega_{\overline{Z}} \neq 0$.

Next I will review the more-or-less well-known idea of constructing non-Cohen-Macaulay singularities as cones over varieties violating Kodaira vanishing.

(2.4) Let $X$ be a normal projective variety over a field $k$ of characteristic $p > 0$, $\mathcal{L}$ an ample line bundle on $X$, and $Z = C_a(X, \mathcal{L}) = \text{Spec}(\oplus_{m \geq 0} H^0(X, \mathcal{L}^m))$ the affine cone over $X$ with conormal bundle $\mathcal{L}$. (Here we follow the convention of \cite{Kol13, 3.8} on cones.)

Then we have the following well-known criterion cf. \cite{Kol13, 3.11}:

(2.5) $Z$ is Cohen-Macaulay if and only if $H^i(X, \mathcal{L}^q) = 0$ for all $0 < i < \dim X$ and $q \in \mathbb{Z}$.

This implies for example that cones over varieties whose structure sheaves have non-trivial middle cohomology, for instance abelian varieties of dimension at least 2, are not Cohen-Macaulay. It also implies that

(2.6) if some power of $\mathcal{L}$ violates Kodaira vanishing, then $Z$ is not Cohen-Macaulay.

Next recall that the canonical divisor of a canonical singularity is $\mathbb{Q}$-Cartier and observe that in the above construction

(2.7) if $\omega_X^r \simeq \mathcal{L}^q$ for some $r, q \in \mathbb{Z}$, $r \neq 0$, then $K_Z$ is $\mathbb{Q}$-Cartier of index at most $r$.

However, even if (2.7) fails, $Z$ may still provide an example of a klt singularity with an appropriate boundary as we will see in the next statement, which summarizes what we found in this section. Note that this statement is a simple consequence of the combination of \cite{Kol13, 3.1, 3.11}.

**Proposition 2.8.** In addition to the definitions in (2.4) assume that $X$ is a smooth Fano variety and that some power of $\mathcal{L}$ violates Kodaira vanishing. Then there exists a $\mathbb{Q}$-divisor $\Delta$ on $Z$ such that

(i) $(Z, \Delta)$ has klt singularities,
(ii) $Z$ is not Cohen-Macaulay, and hence in particular has non-rational singularities, and
(iii) if \( \omega_X \simeq L^q \) for some \( q \in \mathbb{Z} \), then \( Z \) has canonical singularities.

**Proof.** Since \( \omega_X^{-1} \) is ample, there is an \( r \in \mathbb{N}, r > 0 \), such that \( \mathcal{N} = L^{-1} \otimes \omega_X^{-r} \) is also ample. Let \( \mathcal{N} \) be a general member of the complete linear system corresponding to \( \mathcal{N}^m \) for some \( m \gg 0 \), \( N \subseteq Z \) the cone over \( N \), and \( \Delta := \frac{1}{rm} \tilde{N} \). Then \( \mathcal{O}_X(rm(K_X + \frac{1}{rm} N)) \simeq L^{-m} \), so \( K_Z + \Delta \) is a Cartier divisor on \( Z \) (cf. [Kol13, 3.14(4)]), and hence \( K_Z + \Delta \) is \( \mathbb{Q} \)-Cartier. Furthermore, \( N \) is smooth and hence \( (X, \frac{1}{rm} N) \) is klt, so (i) follows from [Kol13, 3.1(3)]. Now, if \( \omega_X \simeq L^q \) for some \( q \in \mathbb{Z} \), then \( \omega_Z \) is a line bundle and hence (iii) follows from (i). Finally, (ii) is simply a restatement of (2.6). \( \square \)

### 3. The Construction of Lauritzen and Rao

Next, I will recall the construction of Lauritzen and Rao from [LR97].

Let \( V \) be a vector space of dimension \( n + 1 \) over a field \( k \) of characteristic \( p \)
where \( p \geq n - 1 \geq 2 \), and let \( \mathbb{P}(V) \simeq \mathbb{P}^n \) be the associated projective space of
dimension \( n \). Let \( W := \mathbb{P}(V) \times \mathbb{P}(V^\vee) \) and for \( a, b \in \mathbb{Z} \) let \( \mathcal{O}_W(a, b) \) denote the line
bundle \( \mathcal{O}_{\mathbb{P}(V)}(a) \boxtimes \mathcal{O}_{\mathbb{P}(V^\vee)}(b) \) on \( W \). Next let \( \mathcal{A} \) be the locally free sheaf defined by
the short exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{A} & \longrightarrow & V \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & \mathcal{O}_{\mathbb{P}(V)}(1) & \longrightarrow & 0,
\end{array}
\]

and let \( \alpha : Y := \mathbb{P}(\mathcal{A}^\vee) \rightarrow \mathbb{P}(V) \) be the projective space bundle over \( \mathbb{P}(V) \) associated to \( \mathcal{A}^\vee \). Let \( \mathcal{O}_Y(1) \) denote the corresponding tautological line bundle on \( Y \). Then there exists another associated short exact sequence on \( Y \):

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \alpha^* \mathcal{A}^\vee & \longrightarrow & \mathcal{O}_Y(1) & \longrightarrow & 0,
\end{array}
\]

which defines the locally free sheaf \( \mathcal{G} \) on \( Y \). It is shown in [LR97, p.23] that \( Y \)
admits a closed embedding into \( W \simeq \mathbb{P}^n \times \mathbb{P}^n \) with bihomogenous coordinate ring

\[
k[x_0, \ldots, x_n; y_0, \ldots, y_n]/(\sum x_i y_i).
\]

In particular, the ideal sheaf of \( Y \) in \( W \) is \( \mathcal{O}_W(-1, -1) \). Let \( \mathcal{O}_Y(a, b) := \mathcal{O}_W(a, b)|_Y \). Then it follows easily that

\[
\omega_Y \simeq \mathcal{O}_Y(-n, -n), \quad \alpha^* \mathcal{O}_{\mathbb{P}(V)}(a) \simeq \mathcal{O}_Y(a, 0), \quad \text{and} \quad \mathcal{O}_Y(0, b) \simeq \mathcal{O}_Y(0, b).
\]

Let \( \eta \) be defined as the composition of the natural morphisms induced by the
morphisms in (3.1) and (3.2) using the isomorphisms in (3.4):

\[
\begin{array}{cccc}
V^\vee \otimes \mathcal{O}_Y & \xrightarrow{\alpha^* \mathcal{A}^\vee} & \mathcal{O}_Y(0, 1)
\end{array}
\]

\[
\begin{array}{cccc}
V^\vee \otimes \mathcal{O}_Y & \xrightarrow{\eta} & \mathcal{O}_Y(0, 1)
\end{array}
\]
Then we have the following commutative diagram, where $\mathcal{B} = \ker \eta$:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}_Y(-1,0) & \rightarrow & \mathcal{O}_Y(-1,0) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{B} \\
\downarrow & & \downarrow \\
V^\vee \otimes \mathcal{O}_Y & \rightarrow & \mathcal{O}_Y(0,1) & \rightarrow & 0 \\
\downarrow \eta & & \downarrow = \\
\alpha^*V^\vee & \rightarrow & \mathcal{O}_Y(0,1) & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

(3.6)

Finally, let $\mathcal{G}' = \mathcal{G} \otimes \mathcal{O}_\alpha(1)$ and $\pi : X := \mathbb{P}(F^*\mathcal{G}') \rightarrow Y$, where $F : Y \rightarrow Y$ is the absolute Frobenius morphism of $Y$. Note that by construction $\dim X = 3n - 3$ and $\dim Y = 2n - 1$.

(3.7) Again, it is shown in [LR97, p.23] that the tautological line bundle of $\pi$, denoted by $\mathcal{O}_\pi(1)$, is globally generated and the line bundle $\pi^*\mathcal{O}_Y(1,1) \otimes \mathcal{O}_\pi(1)$ is very ample. It follows that $\pi^*\mathcal{O}_Y(1,1) \otimes \mathcal{O}_\pi(q)$ is also very ample for any $q > 0$.

Using the formula for the canonical bundle of a projective space bundle, one obtains that

\[
\omega_X \simeq \pi^*\mathcal{O}_Y(p - n, p(n - 2) - n) \otimes \mathcal{O}_\pi(-n + 1)
\]

(3.8)

As it was pointed out by Hélène Esnault if one chooses the values $p = 2$ and $n = 3$, then $X$ is a Fano variety and hence there exists a klt pair $(Z, \Delta)$ where $Z$ is not Cohen-Macaulay, in particular, it does not have rational singularities cf. Proposition 2.8.

### 4. A Fano Variety Violating Kodaira Vanishing

We will use the above construction and prove that if $p = 2$ and $n = 3$, then the very ample line bundle $\omega_X^{-2}$ violates Kodaira vanishing. To do this, first we need to compute a few auxiliary cohomology groups. We will keep using the notation introduced in Section 3.

**Lemma 4.1.** Let $a, b \in \mathbb{Z}$. Then

\[
H^i(Y, \mathcal{O}_Y(a, b)) = 0
\]

if either

(i) $a$ and $b$ are arbitrary and $0 < i < n - 1$, or
(ii) $a, b > -n$ and $i > 0$, or
(iii) at least one of $a$ and $b$ is negative and $i = 0$.

**Proof.** By (3.3) we have the following short exact sequence:

\[
0 \rightarrow \mathcal{O}_W(a - 1, b - 1) \rightarrow \mathcal{O}_W(a, b) \rightarrow \mathcal{O}_Y(a, b) \rightarrow 0
\]
Since $W \simeq \mathbb{P}^n \times \mathbb{P}^n$, using the Künneth formula, the first two (non-zero) sheaves above have no cohomology in the following cases:

(a) for $0 < i < n$ and arbitrary $a$ and $b$,
(b) for $a, b > -n$ and $i > 0$, and
(c) at least one of $a$ and $b$ is negative and $i = 0$.

Then (a) implies (i), (b) implies (ii), and (a) and (c) together imply (iii).

**Corollary 4.2.** Under the same conditions as in Lemma 4.1,

$$H^i(Y, \mathcal{O}_Y(a, b) \otimes F^*(V^\vee \otimes \mathcal{O}_Y)) = 0.$$  

**Proof.** $F^*(V^\vee \otimes \mathcal{O}_Y)$ is a free $\mathcal{O}_Y$ sheaf, so this is straightforward from Lemma 4.1.

**Lemma 4.3.** Let $a, b \in \mathbb{Z}$. Then

$$H^1(Y, \mathcal{O}_Y(a, b) \otimes F^*\mathcal{B}) \simeq \text{coker } \left[ H^0(Y, \mathcal{O}_Y(a, b) \otimes F^*(V^\vee \otimes \mathcal{O}_Y)) \twoheadrightarrow H^0(Y, \mathcal{O}_Y(a, b + p)) \right]$$

where $\eta_1 = F^*\eta$ is induced by the morphism $\eta$ defined in (3.5). In particular, if either $a < 0$ or $b < -p$, then

$$H^1(Y, \mathcal{O}_Y(a, b) \otimes F^*\mathcal{B}) = 0.$$  

**Proof.** Consider the Frobenius pull-back of the middle row of the diagram in (3.6) twisted with $\mathcal{O}_Y(a, b)$:

$$0 \longrightarrow \mathcal{O}_Y(a, b) \otimes F^*\mathcal{B} \longrightarrow \mathcal{O}_Y(a, b) \otimes F^*(V^\vee \otimes \mathcal{O}_Y) \longrightarrow \mathcal{O}_Y(a, b + p) \longrightarrow 0.$$  

Then, since $n > 2$, both statements follow from Corollary 4.2.

**Lemma 4.4.** Let $a, b \in \mathbb{Z}$. Then the morphism induced by $\tau$ in (3.6) is an isomorphism:

$$H^1(\mathcal{O}_Y(a, b) \otimes F^*\mathcal{B}) \xrightarrow{\cong} H^1(\mathcal{O}_Y(a, b) \otimes F^*\mathcal{G}).$$

Furthermore, if $a < p$ or $b < -p$, then the natural morphism induced by the same morphism as above is an injection:

$$H^1(\mathcal{O}_Y(a, b) \otimes \text{Sym}^2 F^*\mathcal{B}) \hookrightarrow H^1(\mathcal{O}_Y(a, b) \otimes \text{Sym}^2 F^*\mathcal{G}).$$

**Proof.** Consider the Frobenius pull-back of the first vertical short exact sequence from (3.6):

$$0 \longrightarrow \mathcal{O}_Y(-p, 0) \longrightarrow F^*\mathcal{B} \longrightarrow F^*\mathcal{G} \longrightarrow 0.$$  

Since $n > 2$, this, combined with Lemma 4.1, implies the first statement.

Next, observe that this short exact sequence also implies that there exists a filtration

$$\text{Sym}^2 F^*\mathcal{B} \supseteq \mathcal{E} \supseteq \mathcal{O}_Y(-2p, 0)$$

such that (after twisting by $\mathcal{O}_Y(a, b)$) we have the short exact sequences

$$0 \longrightarrow \mathcal{O}_Y(a - 2p, b) \longrightarrow \mathcal{O}_Y(a, b) \otimes \mathcal{E} \longrightarrow \mathcal{O}_Y(a - p, b) \otimes F^*\mathcal{G} \longrightarrow 0.$$  

(4.4.1)
and
(4.4.2)  
\[ 0 \longrightarrow \mathcal{O}_Y(b,a) \otimes \mathcal{E} \longrightarrow \mathcal{O}_Y(b,a) \otimes \text{Sym}^2 F^* \mathcal{B} \longrightarrow \mathcal{O}_Y(b,a) \otimes \text{Sym}^2 F^* \mathcal{F} \longrightarrow 0 \]

Then by the first statement, Lemma 4.3, Lemma 4.1, and (4.4.1) it follows that if \( a < p \) or \( b < -p \), then
\[ H^1(Y, \mathcal{O}_Y(b,a) \otimes \mathcal{E}) = 0. \]

By (4.4.2) this implies the second statement.

Lemma 4.5. Let \( a, b \in \mathbb{Z} \) such that \( a \geq 0 \) and \( b > -n \). Then
\[ H^1(Y, \mathcal{O}_Y(a,b) \otimes \text{Sym}^2 F^* \mathcal{B}) \neq 0 \]

Proof. Observe that the middle horizontal short exact sequence in the diagram (3.6) implies that there exists a filtration
\[ \text{Sym}^2 (F^*(V^\vee \otimes \mathcal{O}_Y)) \supseteq \mathcal{F} \supseteq \text{Sym}^2 F^* \mathcal{B} \]

such that (after twisting by \( \mathcal{O}_Y(b,a) \)) we have the short exact sequences
(4.5.1)
\[ 0 \longrightarrow \mathcal{O}_Y(b,a) \otimes \text{Sym}^2 F^* \mathcal{B} \longrightarrow \mathcal{O}_Y(b,a) \otimes \mathcal{F} \longrightarrow \mathcal{O}_Y(b,a+p) \otimes F^* \mathcal{B} \longrightarrow 0 \]

and
(4.5.2)
\[ 0 \longrightarrow \mathcal{O}_Y(b,a) \otimes \mathcal{F} \longrightarrow \mathcal{O}_Y(b,a) \otimes \text{Sym}^2 (F^*(V^\vee \otimes \mathcal{O}_Y)) \longrightarrow \mathcal{O}_Y(b,a+2p) \longrightarrow 0. \]

Since \( n > 2 \), it follows from (4.5.2) and Lemma 4.1 that
\[ H^2(Y, \mathcal{O}_Y(b,a) \otimes \mathcal{F}) = 0 \]

and that
\[ H^1(Y, \mathcal{O}_Y(b,a) \otimes \mathcal{F}) \simeq \coker [H^0(Y, \mathcal{O}_Y(b,a) \otimes \text{Sym}^2 (F^*(V^\vee \otimes \mathcal{O}_Y))) \xrightarrow{\eta_2} H^0(Y, \mathcal{O}_Y(b,a+2p))]. \]

The morphism \( \eta_2 \) here is given by the matrix \([y_i^p y_j^p] \mid i, j = 0, \ldots, n\]. Furthermore, it follows from Lemma 4.3 that
\[ H^1(Y, \mathcal{O}_Y(b,a+p) \otimes F^* \mathcal{B}) \simeq \coker [H^0(Y, \mathcal{O}_Y(b,a+p) \otimes F^*(V^\vee \otimes \mathcal{O}_Y)) \xrightarrow{\eta_1} H^0(Y, \mathcal{O}_Y(b,a+2p))]. \]

The morphism \( \eta_1 \) here is given by the matrix \([y_i^p] \mid i = 0, \ldots, n\]. Note, that by assumption \( a \geq 0 \) and \( b \geq -n+1 \geq -p \), so \( H^0(Y, \mathcal{O}_Y(b,a+p) \otimes F^*(V^\vee \otimes \mathcal{O}_Y)) \neq 0 \). Then it is easy to see, for example from the description of \( \eta_1 \) and \( \eta_2 \) above, that
\[ \text{im} \eta_2 \subset \text{im} \eta_1, \]
and hence combined with (4.5.1) the above imply that
(4.5.3) \[ H^1(Y, \mathcal{O}_Y(b,a) \otimes \text{Sym}^2 F^* \mathcal{B}) \simeq \text{im} \eta_1 / \text{im} \eta_2 \neq 0. \]
Remark 4.6. Observe that the previous argument was the place where working in positive characteristic was crucial. The morphisms $\eta_1$ and $\eta_2$ are given by the $p^{th}$ powers of the global sections of $\mathcal{O}_Y(0, 1)$. We obtain the non-trivial cokernels and the “gap” between them from the fact that the global sections of $\mathcal{O}_Y(0, p)$ are not generated by these $p^{th}$ powers.

This argument fails for several reasons in characteristic 0. First of all, $p^{th}$ powers do not define an $\mathcal{O}_Y$-module homomorphism. Of course, they do not define one in any characteristic, which is the reason that we first have to pull-back everything by the Frobenius. However, the $p^{th}$ powers do give an $F^*\mathcal{O}_Y$-module homomorphism. There is of course no Frobenius in characteristic 0, but one might think that then one could use another finite morphism to pull-back these sections and thereby replacing the global sections by an appropriate power. However, in characteristic 0 this would mean switching to an actual cover many of whose properties would change. For instance, very likely that cover would no longer be Fano or even have negative Kodaira dimension and other parts of the proof would break down.

To summarize, the reason this argument works in positive characteristic is that there is a high degree endomorphism which is one-to-one on points. Then again, this is not surprising at all as this is usually the reason when a statement holds in positive characteristic but not in characteristic 0.

Theorem 4.7. If $p \leq n = 3$, then $\dim X = 6$, and $H^5(X, \omega_X^2) \neq 0$.

Proof. By Serre duality and (3.8) we have that

$$H^i(X, \omega_X^2)^* \simeq H^{6-i}(X, \omega_X^{-1}) \simeq H^{6-i}(X, \pi^*\mathcal{O}_Y(3-p, 3-p) \otimes \mathcal{O}_X(2))$$
$$\simeq H^{6-i}(Y, \mathcal{O}_Y(3-p, 3-p) \otimes \pi_*\mathcal{O}_X(2))$$
$$\simeq H^{6-i}(Y, \mathcal{O}_Y(3-p, 3-p) \otimes \text{Sym}^2 F^*G')$$
$$\simeq H^{6-i}(Y, \mathcal{O}_Y(3-p, 3+p) \otimes \text{Sym}^2 F^*G')$$

Since $p > a = 3-p \geq$ and $b = 3+p > -n = -3$, the statement follows from Lemma 4.4 and Lemma 4.5. \hfill \Box

This might seem to give a desired example in $p = 3$ as well, but this non-vanishing is only interesting when $X$ is Fano, i.e., when $\omega_X^{-1}$ is ample and that only holds when $p = 2$.

Corollary 4.8. If $n = 3$ and $p = 2$, then $X$ is a Fano variety on which $\omega_X^{-1}$ is very ample and $\omega_X^{-2}$ violates Kodaira vanishing. In particular, Theorem 1.1 follows.

Proof. If $n = 3$ and $p = 2$, then by (3.8) $\omega_X \simeq \pi^*\mathcal{O}_Y(-1, -1) \otimes \mathcal{O}_X(-2)$ and hence $\omega_X^{-1}$ is very ample by (3.7). By Theorem 4.7, $\omega_X^{-2}$ violates Kodaira vanishing. \hfill \Box

Corollary 4.9. Theorem 1.2 holds.

Proof. Let $Z = C_0(X, \omega_X^{-1})$. Then the statement follows from Corollary 4.8 and Proposition 2.8. \hfill \Box

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