ON THE MATRICES OF CENTRAL LINEAR MAPPINGS

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Summary. We show that a central linear mapping of a projectively embedded Euclidean n-space onto a projectively embedded Euclidean m-space is decomposable into a central projection followed by a similarity if, and only if, the least singular value of a certain matrix has multiplicity \( \geq 2m - n + 1 \). This matrix is arising, by a simple manipulation, from a matrix describing the given mapping in terms of homogeneous Cartesian coordinates.

Keywords: linear mapping, axonometry, singular values.

AMS classification: 51N15, 51N05, 15A18, 68U05.

1 Introduction

A linear mapping between projectively embedded Euclidean spaces is called central, if its exceptional subspace is not at infinity. Such a linear mapping is in general not decomposable into a central projection followed by a similarity. Necessary and sufficient conditions for the existence of such a decomposition have been given in [1] for arbitrary finite dimensions; cf. also [2], [3]. However, those results do not seem to be immediately applicable on a central axonometry, i.e., a central linear mapping given via an axonometric figure. On the other hand, in a series of recent papers [5], [6], [7] this problem of decomposition has been discussed for central axonometries of the Euclidean 3-space onto the Euclidean plane from an elementary point of view.

Loosely speaking, the concept of central axonometry is a geometric equivalent to the algebraic concept of a coordinate matrix for a linear mapping of the underlying vector spaces. However, from the results in [2] and [4] it is also not immediate whether or not a given matrix describes (in terms of homogeneous Cartesian coordinates) a mapping that permits the above-mentioned factorization. The aim of this communication is to give a criterion for this.

Let \( \mathbf{I}, \mathbf{J} \) be finite-dimensional Euclidean vector spaces. Given a linear mapping \( f : \mathbf{I} \rightarrow \mathbf{J} \) denote by \( f^{\text{ad}} : \mathbf{J} \rightarrow \mathbf{I} \) its adjoint mapping. Then \( f^{\text{ad}} \circ f \) is self-adjoint with eigenvalues

\[ v_1 \geq \cdots \geq v_r > v_{r+1} = \cdots = v_n = 0. \]

Here \( r \) equals the rank of \( f \) and \( n = \dim \mathbf{I} \). Moreover, each eigenvalue is written down repeatedly according to its multiplicity. The positive real numbers \( \sqrt{v_1}, \ldots, \sqrt{v_r} \) are frequently called the singular values of \( f \). The multiplicity of a singular value of \( f \) is defined via the multiplicity of the corresponding eigenvalue of \( f^{\text{ad}} \circ f \). It is immediate from the singular value decomposition that \( f \)

\footnote{A lot of further references can be found in the quoted papers.}

\footnote{For a self-adjoint mapping the algebraic and geometric multiplicities of an eigenvalue are identical. Hence we may unambiguously use the term ‘multiplicity’.}
and $f^{\text{ad}}$ share the same singular values (counted with their multiplicities). See, e.g., [3].

These results hold true, mutatis mutandis, when replacing $f$ by any real matrix, say $A$, and $f^{\text{ad}}$ by the transpose matrix $A^T$.

## 2 Decompositions

When discussing central linear mappings it will be convenient to consider Euclidean spaces embedded in projective spaces. Thus let $V$ be an $(n + 1)$-dimensional real vector space ($3 \leq n < \infty$) and $I$ one of its hyperplanes. Assume, furthermore, that $I$ is equipped with a positive definite inner product $\langle \cdot, \cdot \rangle$ so that $I$ is a Euclidean vector space. In the projective space on $V$, denoted by $\mathcal{P}(V)$, we consider the projective hyperplane $\mathcal{P}(I)$ as the hyperplane at infinity. The absolute polarity in $\mathcal{P}(I)$ is determined by the inner product on $I$. Hence $\mathcal{P}(V) \setminus \mathcal{P}(I)$ is a projectively embedded Euclidean space. Similarly, let $\mathcal{P}(W) \setminus \mathcal{P}(J)$ be an $m$-dimensional projectively embedded Euclidean space ($2 \leq m < n < \infty$). Given a linear mapping

$$f : V \to W$$  \hspace{1cm} (1)

of vector spaces then the associate (projective) linear mapping

$$\phi : \mathcal{P}(V) \setminus \mathcal{P}(\ker f) \to \mathcal{P}(W), \mathbb{R}x \mapsto \mathbb{R}(f(x))$$  \hspace{1cm} (2)

has $\mathcal{P}(\ker f)$ as its exceptional subspace. In the sequel we shall assume that

$$\ker f \not\subset I \quad \text{and} \quad f(V) = W,$$  \hspace{1cm} (3)

or, in other words, that $\phi$ is central and surjective.\(^4\) Obviously, (3) is equivalent to

$$f(I) = W.$$  \hspace{1cm} (4)

We recall some results [2], [4]: If $T$ is any complementary subspace of $\ker f$ in $V$, then denote by

$$\psi_T : \mathcal{P}(V) \setminus \mathcal{P}(\ker f) \to \mathcal{P}(T)$$  \hspace{1cm} (5)

the projection with the exceptional subspace $\mathcal{P}(\ker f)$ onto $\mathcal{P}(T)$. The restricted mapping

$$\phi_T := \phi \mid \mathcal{P}(T) : \mathcal{P}(T) \to \mathcal{P}(W)$$  \hspace{1cm} (6)

is a collineation and

$$\phi = \phi_T \circ \psi_T;$$  \hspace{1cm} (7)

every decomposition of $\phi$ into a projection and a collineation is of this form. In the Euclidean vector space $I$ we have the distinguished subspace

$$E := f^{-1}(J) \cap I.$$  \hspace{1cm} (8)

Write

$$f_E : E \to J, \; x \mapsto f(x);$$  \hspace{1cm} (9)

\(^3\)We do not endow this space with a unit segment.

\(^4\)This assumption of surjectivity is made ‘without loss of generality’ in most papers on this subject. It will, however, be essential several times in this paper.
this \( f_E \) is well-defined and surjective, since \( E \subset f^{-1}(J) \) and \( \ker f \not\subset E \). The subspace \( T \) can be chosen with \( \phi_T \) being a similarity if, and only if, the least singular value of \( f_E \) has multiplicity \( \geq 2m - n + 1 \).

Next, we assume that \( \mathcal{P}(T) \not\subset \mathcal{P}(I) \) is orthogonal to \( \mathcal{P} (\ker f) \). This means that \( (T \cap I) \perp \subset \ker f \cap I \) or \( (T \cap I) \not\subset \ker f \cap I \). Hence \( \psi_T \) is an orthogonal central projection\(^6\). It is easily seen from [2] that \( \phi \) permits a decomposition into an orthogonal central projection followed by a similarity if, and only if, all singular values of \( f_E \) are equal.

Finally, we are going to show that the crucial properties of \( f_E \) can be read off from another mapping: Denote by

\[
p : I \to E
\]

the orthogonal projection with the kernel \( E^\perp \subset I \). Then

\[
(f_E \circ p) \circ (f_E \circ p)^{ad} = f_E \circ p \circ p^{ad} \circ (f_E)^{ad} = f_E \circ (f_E)^{ad},
\]

since \( p^{ad} \) is the natural embedding \( E \to I \). Thus, by (11) and the results stated in Section \( \mathbb{1} \), \( f_E \) and \( (f_E \circ p)^{ad} \) have the same singular values (counted with their multiplicities). Hence, by the surjectivity of \( f_E \) and (11), all singular values of \( f_E \) are equal if, and only if, there exists a real number \( v > 0 \) such that

\[
(f_E \circ p) \circ (f_E \circ p)^{ad} = v \id_J.
\]

(12)

We shall use this in the next section.

### 3 A matrix characterization

Introducing homogeneous Cartesian coordinates in \( \mathcal{P}(V) \) is equivalent to choosing a basis \( \{b_0, \ldots, b_n\} \) of \( V \) such that \( \{b_1, \ldots, b_n\} \subset I \) is an orthonormal system. The origin is given by \( \mathbb{R}b_0 \) and the unit points are \( \mathbb{R}(b_0 + b_1), \ldots, \mathbb{R}(b_0 + b_n) \). In the same manner we are introducing homogeneous Cartesian coordinates in \( \mathcal{P}(W) \) via a basis \( \{c_0, \ldots, c_m\} \).

**Theorem 1** Suppose that \( f : V \to W \) is inducing a surjective central linear mapping \( \phi \) according to formula (2). Let

\[
A = \begin{pmatrix}
a_{00} & \cdots & a_{0n} \\
\vdots & \ddots & \vdots \\
a_{m0} & \cdots & a_{mn}
\end{pmatrix}
\]

be the coordinate matrix of \( f \) with respect to bases of \( V \) and \( W \) that are yielding homogeneous Cartesian coordinates. Write

\[
a_i := (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^n \text{ for all } i = 0, \ldots, m
\]

and

\[
\tilde{A} := \begin{pmatrix}
a_{1} - \frac{a_{0}}{a_{0} - a_{0}} a_{0} \\
\vdots \\
a_{m} - \frac{a_{0}}{a_{0} - a_{0}} a_{0}
\end{pmatrix}
\]

Then the following assertions hold true:

\(^5\)In [2, Satz 10] this multiplicity is printed incorrectly as \( 2m - n - 1 \).

\(^6\)The central projections used in elementary descriptive geometry are trivial examples of orthogonal central projections.
1. $\phi$ is decomposable into a central projection followed by a similarity if, and only if, the least singular value of the matrix $\tilde{A}$ has multiplicity $\geq 2m - n + 1$.

2. $\phi$ is decomposable into an orthogonal central projection followed by a similarity if, and only if, there exists a real number $v > 0$ such that

$$\tilde{A}\tilde{A}^T = \text{diag}(v, \ldots, v).$$

(16)

Proof. We read off from the top row of $A$ that

$$a_0x_0 + \cdots + a_nx_n = 0$$

is an equation of $f^{-1}(J) \neq I$ so that $a_0 \cdot a_0 \neq 0$. Write $\tilde{f} : I \to J$ for the linear mapping whose coordinate matrix with respect to $\{b_1, \ldots, b_n\}$ and $\{c_1, \ldots, c_m\}$ equals $\tilde{A}$. A straightforward calculation shows that

$$\tilde{f}(x) = f(x) \text{ for all } x \in E$$

and

$$\tilde{f}(a_0b_1 + \cdots + a_nb_n) = 0,$$

i.e., $E^\perp \subset \ker \tilde{f}$. Thus $\tilde{f}$ equals the mapping $f_E \circ p$ discussed above. Now the proof is completed by translating formulae (11) and (12) into the language of matrices.

We remark that (3) and the linear independence of $a_1, \ldots, a_m$ are equivalent conditions.

In contrast to the results in [5], [6], [7], the $\phi$-image of the origin $\mathbb{R}b_0$ does not appear in our characterization. On the other hand, we have

$$f(E^\perp) = \mathbb{R}(a_0 \cdot a_0)c_0 + \cdots + (a_0 \cdot a_m)c_m).$$

In projective terms this 1-dimensional subspace of $W$ gives the principal point of the mapping $\phi$. Exactly if the principal point of $\phi$ equals the origin $\mathbb{R}c_0$, then $\tilde{A}$ arises from $A$ merely by deleting the top row and the leading column.

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