Abstract. We prove affirmatively the conjecture raised by J. Mostovoy [3]: the space of short ropes is weakly homotopy equivalent to the classifying space of the topological monoid (or category) of long knots in $\mathbb{R}^3$. We make use of techniques developed by S. Galatius and O. Randal-Williams [2] to construct a manifold space model of the classifying space of the space of long knots, and we give an explicit map from the space of short ropes to the model in a geometric way.

1. Introduction

A long $j$--embedding in $\mathbb{R}^n$ is a smooth embedding $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$ that coincides with the standard inclusion outside a compact set. The space $\text{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ of all long $j$--embeddings in $\mathbb{R}^n$ equipped with the $C^\infty$--topology is widely studied in recent years, in particular in the metastable range of dimensions. Perhaps the space of long knots, long 1-embeddings in $\mathbb{R}^3$, is one of the most fascinating cases, but the dimension $(n, j) = (3, 1)$ is not in the stable range and some methods for studying $\text{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ in high (co)dimensional cases yield only information on $K := \pi_0(\text{Emb}(\mathbb{R}^1, \mathbb{R}^3))$ when applied to $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$. $K$ is just a free commutative monoid (and not a group) with respect to the connected-sum, and the group completion $\Omega \text{BEmb}(\mathbb{R}^1, \mathbb{R}^3)$ would be strictly better from homotopy-theoretic view than $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ itself. In fact the group completion is a 2-fold loop space, since the little 2--disks operad acts on $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ (Budney [1]). Moreover the group completion would be useful for study of (isotopy classes of) long knots since the natural map $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3) \to \Omega \text{BEmb}(\mathbb{R}^1, \mathbb{R}^3)$ induces a monomorphism on $\pi_0$.

From this viewpoint the result of Mostovoy [3] is very curious though it is also concerned with $K$. A parametrized short rope is a smooth embedding $\rho: [0, 1] \hookrightarrow \mathbb{R}^3$ of length $< 3$ such that $\rho(i) = (i, 0, 0)$ for $i = 0, 1$. Mostovoy has proved that the fundamental group of the space $B_2$ of parametrized short ropes is isomorphic to $\pi_1BK$, the group completion of $K$. This leads us to the question [3, Conjecture 1]: is the space $B_2$ the classifying space $B\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ of $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$? Our main result asserts that this is the case.

Theorem 1.1 (Corollary 3.7, Theorem 3.8). Mostovoy’s space of parametrized short ropes is weakly homotopy equivalent to the classifying space of the space of long knots.

One of the main ingredients in the proof of Theorem 1.1 is the technique of Galatius and Randal-Williams [2]. It enables us to construct a model of $B\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ The model is a space of certain 1--dimensional submanifolds in $\mathbb{R}^3$ whose connected components are non-compact closed subspaces of $\mathbb{R}^3$ (see Definition 3.1). We prove Theorem 1.1 by introducing the notion of reducible ropes (see Definition 3.1) and by comparing the manifold space model with the space of short ropes through reducible ropes:

Theorem 1.2 (Corollary 3.7, Theorem 3.8). The manifold space model and Mostovoy’s space of parametrized short ropes are both weakly homotopy equivalent to the space of reducible ropes.

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Notations.

2.1. As tools to study BEmb(\(\mathbb{R}^1, \mathbb{R}^3\)) we can realize the weak equivalence from the manifold space model to the space of reducible ropes as a "cut-off map" which is explicit and geometric. Therefore Mostovoy’s space of short ropes and the space of reducible ropes would serve as tools to study BEmb(\(\mathbb{R}^1, \mathbb{R}^3\)) in a geometric way.

2. Manifold space model of the classifying space of the space of long knots

2.1. Notations. Throughout this paper \(D^m\) and \(\overline{D}^m\) stand respectively for the open and closed unit \(m\)-disks;

\[
D^m := \{ p \in \mathbb{R}^m \mid |p| < 1 \}, \quad \overline{D}^m := \{ p \in \mathbb{R}^m \mid |p| \leq 1 \}.
\]

For a 1–dimensional manifold \(M \subset \mathbb{R}^1 \times D^2\) and a subset \(A \subset \mathbb{R}^1\), let

\[
M|_A := M \cap (A \times D^2).
\]

For a one point set \(A = \{ T \}\), we simply write \(M|_T\) for \(M|_{\{ T \}}\).

Definition 2.1. A 1–dimensional manifold \(M \subset \mathbb{R}^1 \times D^2\) is said to be

- reducible at \(T \in \mathbb{R}^1\) if \(M\) intersects \([T] \times D^2\) transversely in one point set.
- strongly reducible at \(T \in \mathbb{R}^1\) if \(M|_T\) is one point set and there exists an \(\epsilon > 0\) satisfying

\[
M|_{(T-\epsilon,T+\epsilon)} = (T-\epsilon, T+\epsilon) \times \{ p_{23}(M|_T) \},
\]

where \(p_{23} : \mathbb{R}^1 \times D^2 \to D^2\) is the projection.

Remark 2.2. The word “reducible” indicates that the manifold looks like a “connected sum” of two 1–manifolds. But the meaning is different from that in knot theory, in that a reducible manifold does not need to split into a connected sum of nontrivial knots.

2.2. The category \(\mathcal{K}\) of long knots. First we define the space \(\psi\) that we have referred to in Section 1 as the manifold space model.

Definition 2.3. Let \(\psi\) be the set of 1–dimensional submanifolds \(M \subset \mathbb{R}^1 \times D^2\) such that

- \(\partial M = \emptyset\),
- each connected component of \(M\) is a closed, non-compact subspace in \(\mathbb{R}^3\), and
- there exists at least one \(T \in \mathbb{R}\) such that \(M\) is reducible at \(T\) (see Figure 2.1). The above conditions imply that \(M \in \psi\) contains exactly one connected component \(M_0\) satisfying \(M_0|_t \neq \emptyset\) for any \(t \in \mathbb{R}^1\). Such a component is said to be long. It can also be seen that the other connected components (if they exist) are long on exactly one side; we say a component \(M_1\) is long on the left (resp. right) if there exists \(T \in \mathbb{R}^1\) such that \(M_1|_{s \leq T}\) (resp. \(s \geq T\)) but \(M_1|_{[T,\infty)} = \emptyset\) (resp. \(M_1|_{(-\infty,T]} = \emptyset\)). The set \(\psi\) is topologized as a subspace of \(\psi(3, 1)\) from Galatius and Randal-Williams \([2\text{ Section 3.1]}\) (without any “tangential data”).

Remark 2.4. Roughly speaking, two manifolds \(M, N \in \psi\) are “close to each other if they are close in a compact subspace of \(\mathbb{R}^3\)”. A bit more precisely, for \(M \in \psi\), the set of manifolds whose intersections with some compact subspace of \(\mathbb{R}^3\) are obtained by shifting \(M\) along small normal sections to \(M\) is a basic open neighborhood of \(M\) in \(\psi\). It is worth mentioning the following example: Let \(a : [0, 1) \to \mathbb{R}_{\geq 0}\) be a monotonically increasing

![Figure 2.1. An element of \(\psi\); the long component is drawn with a thick curve](image-url)
function with \( \alpha(0) = 0 \) and \( \lim_{t \to 1^-} \alpha(t) = \infty \), and \( M(t) \in \psi \) \((0 < t < 1)\) a continuous family satisfying \( M(t)|_{[\alpha(t), \alpha(t)]} = [-\alpha(t), \alpha(t)] \times (0, 0) \). Then \( M(t) \) converges to the trivial long knot \( \mathbb{R}^1 \times \{0, 0\} \) in this topology as \( t \) tends to 1 (see also [2 Example 2.2]).

**Remark 2.5.** For any \( M \in \psi \) there exists \( T \in \mathbb{R}^1 \) such that all the components of \( M \) that are long on the left (resp. right) are contained in \( (-\infty, T) \times D^2 \) (resp. \( (T, \infty) \times D^2 \)).

**Definition 2.6.** We define the category \( \mathcal{K} \) of long knots as follows. The space of objects of \( \mathcal{K} \) is \( D^2 \) with the usual topology. A non-identity morphism from \( p \) to \( q \) is a pair \((T, M)\), where \( T > 0 \) and \( M \in \psi \) is a long knot from \( p \) to \( q \), namely a connected 1-manifold (and hence long) that is strongly reducible at any \( t \in (-\infty, 0) \cup (T - 1, \infty) \) for some \( \epsilon > 0 \);

\[
M_{(-\infty,0)} = (-\infty, 0) \times \{p\}, \quad M_{(T-1,\infty)} = (T - 1, \infty) \times \{q\}.
\]

The identity morphism \( id: p \to p \) is given by \((0, \mathbb{R}^1 \times \{p\})\). The total space \( \bigcup_{p, q} \text{Map}_{\mathcal{K}}(p, q) \) of all morphisms is topologized as a subspace of \((\{0\} \sqcup \mathbb{R}^1_{\geq 0}) \times \psi\), where \( \{0\} \sqcup \mathbb{R}^1_{\geq 0} \) is a disjoint union. The composition \( \circ: \text{Map}_{\mathcal{K}}(q, r) \times \text{Map}_{\mathcal{K}}(p, q) \to \text{Map}_{\mathcal{K}}(p, r) \) is defined by

\[
(T_1, M_1) \circ (T_0, M_0) := (T_0 + T_1, M_0|_{(-\infty, T_0]} \cup (M_1|_{[0,0]) + T_0 e_1})),
\]

where \( e_1 = (1, 0, 0) \in \mathbb{R}^3 \) and \( +T e_1 \) stands for the translation by \( T \) in the direction of \( \mathbb{R}^1 \).

In this section we show that \( BK \) (see Section 2.3 for the definition) is weakly equivalent to \( \psi \). The following posets play roles as interfaces between them.

**Definition 2.7.** Define a poset \( \mathcal{D} \) by

\[
\mathcal{D} := \{(T, M) \in \mathbb{R}^1 \times \psi \mid M \text{ is reducible at } T\}
\]

and topologize \( \mathcal{D} \) as a subspace of \( \mathbb{R}^1 \times \psi \). Define the partial order \( \leq \) on \( \mathcal{D} \) so that \((T, M) < (T', M')\) if and only if \( M = M'\) and \( T < T'\). We regard \( \mathcal{D} \) as a small category in the usual way, namely \( \text{Map}_\mathcal{D}(x, y) \) is a one point set \( \{(x, y)\} \) if \( x \leq y \), and \( \emptyset \) otherwise. The total space of all morphisms is topologized as a subspace of \((\Delta \sqcup (\mathbb{R}^1 \times \mathbb{R}^1 \setminus \Delta)) \times \psi\), where \( \Delta := \{(x, x) \in \mathbb{R}^1 \times \mathbb{R}^1\} \) is the diagonal set.

Define \( \mathcal{D}^+ \) as a subposet of \( \mathcal{D} \) consisting of \((T, M)\) with \( M \) strongly reducible at \( T \).

2.3. **Classifying spaces of categories.** Here we recall the general definition of classifying spaces of topological categories.

For a topological category \( C \), its nerve is the simplicial space whose level \( l \) space \( N_lC \) consists of sequences of components of \( l \) morphisms \((x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_l} x_l) \) in \( C \) and is topologized as a subspace of the \( l \)-th power of the total space of all morphisms in \( C \). By definition \( N_0C \) is the space of objects in \( C \). The face maps are given by the compositions, and the degeneracy maps are given by inserting the identity morphisms. The classifying space \( BC \) of \( C \) is defined as the geometric realization of \( N_lC \):

\[
BC := |N_lC| := \left( \bigcup_{l \geq 0} \left( N_lC \times \Delta^l \right) \right)/\sim,
\]

where \( \Delta^l := \{(\lambda_0, \ldots, \lambda_l) \in [0, 1]^{l+1} \mid \sum \lambda_i = 1\} \) is the standard \( l \)-simplex. The relation \( \sim \) is defined so that, for any order preserving map \( \sigma: \{0, \ldots, l \pm 1\} \to \{0, \ldots, l\} \),

\[
N_{l+1}C \times \Delta^{l+1} \ni (\sigma^* f, \lambda) \sim (f, \sigma_* \lambda) \in N_lC \times \Delta^l
\]

where \( \sigma_* \) and \( \sigma^* \) are the induced maps on (co)simplicial spaces.

Recall from Segal [4] a sufficient condition for a simplicial map to induce a homotopy equivalence on geometric realizations.

**Definition 2.8 (4 Definition A.4).** We say a simplicial space \( A_* \) is good if \( s_i A_* \to A_{i+1} \)

is a closed cofibration for each \( l \) and \( 0 \leq i \leq l \), where \( s_i \) stands for the \( i \)-th degeneracy map.
Lemma 2.9 ([4] Proposition A.1). Let $A$, and $B$, be good simplicial spaces. Suppose there exists a simplicial map $f_i : A_i \to B_i$, which is a levelwise homotopy equivalence, that is $f_i : A_i \to B_i$ is a homotopy equivalence for each $i$. Then $f$ induces a homotopy equivalence $[f_*] : |A_*| \xrightarrow{\sim} |B_*|$ on the geometric realizations.

2.4. The classifying space of $\mathcal{K}$. Notice that any element of $N_1 \mathcal{D}$ (resp. $N_1 \mathcal{D}^+$), $l \geq 0$, can be expressed as a pair $(T_0 \leq \cdots \leq T_l ; M)$, where $M \in \psi$ is reducible (resp. strongly reducible) at each $T_i$. Similarly any element of $N_1 / \mathcal{K}$ ($l \geq 1$) is of the form $(0 \leq T_1 \leq \cdots \leq T_l ; M)$, where $M$ is a long knot that is strongly reducible at each $T_i$.

Lemma 2.10. The simplicial spaces $N_1 \mathcal{K}$, $N_1 \mathcal{D}$ and $N_1 \mathcal{D}^+$ are good.

Proof. For $0 \leq i \leq l$, $s_i N_1 \mathcal{K} = \{(0 \leq T_1 \leq \cdots \leq T_{l+1} ; M) \mid T_i = T_{i+1}\} \subset N_1 \mathcal{K}$ (here $T_0 := 0$) is a union of connected components of sequences involving identity morphisms, and hence the pair $(N_1 \mathcal{K}, s_i N_1 \mathcal{K})$ has the homotopy extension property. The proofs for $N_1 \mathcal{D}$ and $N_1 \mathcal{D}^+$ are the same. □

Proposition 2.11. There exists a zig-zag of levelwise homotopy equivalences $N_1 \mathcal{K} \rightarrow N_1 \mathcal{D} \rightarrow N_1 \mathcal{D}^+$, and $N_1 \mathcal{D}^+ \rightarrow \mathcal{B} \mathcal{D} \rightarrow \mathcal{B} \mathcal{D}^+$ are all homotopy equivalences.

Proof. The proof is the same as in Galatius and Randal-Williams [2] Theorem 3.9. That $\mathcal{B} \mathcal{D}^+ \rightarrow \mathcal{B} \mathcal{D}$ induced by the inclusion is a homotopy equivalence follows from [2] Lemma 3.4, which states that, for any $(T_0 \leq \cdots \leq T_l ; M) \in N_1 \mathcal{D}$, $M$ can be modified to be strongly reducible at $T_l$ in a canonical way.

Define the functor $F : \mathcal{D}^+ \rightarrow \mathcal{K}$ on objects by $(T, M) \mapsto M|_{[T]}$, and on morphisms by

$$F(T_0 \leq \cdots \leq T_l ; M) := (0 \leq T_1 - T_0 \leq \cdots \leq T_l - T_0; M|_{[T_1, T_l]} - T_0 e_1),$$

where $M|_{[T_1, T_l]}$ is the long-extension of $M|_{[T_0, T_l]}$ (see Figure 2.2), namely

$$(2.2) M|_{[T_0, T_l]} := \left(\{\infty_0, T_0\} \times \{p_{23}(M|_{T_0})\}\right) \cup M|_{[T_0, T_l]} \cup \left(\{T_l, \infty\} \times \{p_{23}(M|_{T_l})\}\right),$$

where $p_{23} : \mathbb{E} \times \mathcal{D}^2 \rightarrow \mathcal{D}^2$ is the second projection (2.2) is the same as $(\varphi_{\infty}(T_0, T_l) \times \text{id})^{-1}(M)$ in [2] Section 3.2]. Notice that $M|_{[T_0, T_l]}$ is a connected subspace of the long component of $M$ (see Remark 2.5), and its long extension is also connected. This induces a map $F : N_1 \mathcal{D}^+ \rightarrow N_1 \mathcal{K}$ of simplicial spaces.

We have a map $G : N_1 \mathcal{K} \rightarrow N_1 \mathcal{D}^+$, defined in level 0 by $G(p) := (0, \mathbb{R}, \times \{p\})$, and by the natural inclusion in positive levels (letting $T_0 := 0$). This is just a simplicial map up to homotopy (in levels 0 and 1), but is a levelwise homotopy inverse to $F$; the composite $F \circ G$ is the identity, and the other composite $G \circ F$ is given by

$$G \circ F(T_0 \leq \cdots \leq T_l ; M) = (0 \leq T_1 - T_0 \leq \cdots \leq T_l - T_0; M|_{[T_0, T_l]}),$$

which is isotopic to the identity via the same homotopy as the one exhibited in the last line in the proof of [2] Theorem 3.9 (as Figure 2.3). This homotopy firstly extends $M|_{[T_0, T_0]}$ and $M|_{[T_0, T_0]}$ respectively to left and right so that $M|_{[\infty_0, T_0]}$ and $M|_{[T_0, \infty]}$ (in which all the one-side long components are contained) escape respectively to $[\infty_0] \times \mathcal{D}^2$. Then they “vanish” at $s = 1$ by definition of the topology of $\psi$, see Remark 2.4. Simultaneously
this homotopy translates the manifold by $-T_0$ in the direction of $\mathbb{R}^1$. This homotopy keeps manifolds strongly reducible at each $T_t$.

Therefore $F : N_1 \mathcal{D}^+ \to N_1 \mathcal{K}$ is a levelwise homotopy equivalence of good simplicial spaces (Lemma 2.10), and $B\mathcal{D}^+ \to B\mathcal{K}$ is a homotopy equivalence by Lemma 2.9. \qed

Following Galatius and Randal-Williams [3], we denote the element of $B\mathcal{D}$ represented by $((T_0 \leq \cdots \leq T_l; M), (\lambda_0, \ldots, \lambda_l)) \in N_1 \mathcal{D} \times \Delta^l$ as a formal sum $\sum_{0 \leq i \leq l} \lambda_i T_i$ (this notation is compatible with the relation (2.11)).

**Theorem 2.12.** The forgetful map $u : B\mathcal{D} \to \psi$ given by $\sum_{i} \lambda_i T_i \mapsto M$ is a weak homotopy equivalence. Thus $B\mathcal{K}$ is weakly equivalent to $\psi$.

**Proof.** The proof is the same as that of [3, Theorem 3.10]: Given the following commutative diagram of strict arrows,

\[
\begin{array}{ccc}
\partial D^m & \xrightarrow{g} & B\mathcal{D} \\
\downarrow f & & \downarrow u \\
D^m & \xrightarrow{g} & \psi
\end{array}
\]

we find a dotted $g' : \overline{D}^m \to B\mathcal{D}$ that makes the diagram commutative. This means that the relative homotopy group $\pi_0(\psi', B\mathcal{D})$ ($\psi'$ is the mapping cylinder of $u$) vanishes for all $m$, and $u$ induces an isomorphism of homotopy groups in any dimension.

For $a \in \mathbb{R}$ let $U_a := \{ x \in \overline{D}^m \mid f(x) \in \psi \text{ is reducible at } a \}$. This is an open subspace of $\overline{D}^m$ and $\{U_{a\in \mathbb{R}}\}$ is an open covering of $\overline{D}^m$ because, by definition, such an $a$ exists for any $M \in \psi$. So by compactness we can pick finitely many $a_0 < \cdots < a_k$ such that $\{U_{a\in \mathbb{R}}\}_{0 \leq i \leq k}$ covers $\overline{D}^m$. Pick a partition of unity $\{\lambda_i : \overline{D}^m \to [0, 1]\}_{0 \leq i \leq k}$ subordinate to the cover. Using $\lambda_i$ as a formal coefficient of $a_i$ gives a map

\[\hat{g} : \overline{D}^m \to B\mathcal{D}, \quad \hat{g}(x) := \sum_{0 \leq i \leq k} \lambda_i(x) a_i\]

(illustrated by elements in $N_1 \mathcal{D} \times \Delta^l$) which lifts $f$, namely $u \circ \hat{g} = f$. Now we produce a homotopy $h : [0, 1] \times \partial D^m \to B\mathcal{D}$ such that $h(0, -) = \hat{g}|_{\partial D^m}(-)$, $h(1, -) = \hat{f}(-)$ and $h(s, -)$ lifts $f|_{\partial D^m}$ for all $s$; if such an $h$ exists, then we can define the desired map $g$ by

\[g(x) := \begin{cases} \hat{g}(2x) & |x| \leq 1/2, \\ h(2|x| - 1, x/|x|) & |x| \geq 1/2. \end{cases}\]

Since $\hat{f}$ is also a lift of $f|_{\partial D^m}$, we may suppose that $\hat{f}$ is of the form

\[\hat{f}(x) = \sum_{0 \leq i \leq l} \mu_i(x) b_i\]
for some $\mu_0, \ldots, \mu_l \geq 0$, $\sum_i \mu_i(x) = 1$ and $b_0 < \cdots < b_l$ (underlying manifolds $f(x)$ and $u(\hat{f}(x))$ are the same). Let $c_0 < \cdots < c_n$ be the re-ordering of the set $\{a_i\} \cup \{b_j\}$, in ascending order. Using the relation (2.1) we can write $\hat{g}_{\mid \mathcal{M}}$ and $\hat{f}$ as

$$\hat{g}_{\mid \mathcal{M}}(x) = \sum_{\alpha < \beta} \alpha(x)c_{\alpha} \quad \text{for some } \alpha_0, \ldots, \alpha_n \geq 0, \quad \sum_i \alpha_i = 1,$$

$$\hat{f}(x) = \sum_{\beta < \alpha} \beta(x)c_{\beta} \quad \text{for some } \beta_0, \ldots, \beta_n \geq 0, \quad \sum_i \beta_i = 1$$

(represented by elements in $N_n \mathcal{D} \times \Delta^0$). We define $h$ using the affine structure on the fibers of $u$:

$$h(s, x) := s \hat{g}_{\mid \mathcal{M}}(x) + (1 - s)\hat{f}(x) := \sum_{\alpha < \beta} (\alpha(x) + (1 - s)\beta(x))c_{\beta}.$$  \hfill \Box

Remark 2.13. We have topologized the spaces of morphisms of various categories so that the identity morphisms form disjoint components, as was also done in [3]. We may instead topologize the total space of morphisms in $\mathcal{K}$ (resp. $\mathcal{D}$) as a subspace of $[0, \infty) \times \psi$ (resp. $\mathbb{R} \times \mathbb{R} 	imes \psi$) and with the latter topology we can prove the similar results to the above. An advantage of the former topology is that it makes the proof of goodness of the nerves easier.

3. The space of reduced ropes

In this section we show that the conjecture of Mostovoy is true. We first characterize the weak homotopy type of $\psi$ as that of the space of reducible ropes, and then prove that the space of reducible ropes is weakly equivalent to the space of Mostovoy’s short ropes.

3.1. $B\mathcal{K}$ and the space of reducible ropes.

Definition 3.1 (Mostovoy [3]). A rope is a compact, connected 1–dimensional submanifold $r \subset \mathbb{R}^1 \times D^2$ with non-empty boundary $\partial r = [\partial_0 r, \partial_1 r], \partial r \in \partial \mathbb{R} \times D^2$. Let $R$ be the set of all ropes that are reducible at some $t \in (0, 1)$ (see Figure 3.1), topologized as a subspace of $\text{Emb}([0, 1], \mathbb{R} \times D^2) / \text{Diff}^+(\mathbb{R} \times [0, 1])$.

The function $f(t) := \tan \pi(t - (1/2))$ gives an orientation preserving diffeomorphism $f: (0, 1) \to \mathbb{R}$. Define the “cut-off” map $c: R \to \psi$ by

$$c(r) := (f \times \text{id}_{D^2})(r|_{(0, 1)}).$$

This map is defined since, for any reducible rope $r$, $c(r)$ has exactly one long component.

Our aim is to show that $c$ is a weak equivalence, and for this we introduce the following posets as interfaces between $R$ and $\psi$.

Definition 3.2. Define a poset $\mathcal{E}$ by

$$\mathcal{E} := \{(t, r) \in (0, 1) \times R | r \text{ is reducible at } t\}.$$  

Define the partial order $\preceq$ on $\mathcal{E}$ so that $(t, r) < (t', r')$ if and only if $r \preceq r'$ and $t < t'$. We regard $\mathcal{E}$ as a small category in the same way as $\mathcal{D}$. The total space of all morphisms is topologized as a subspace of $(\Delta \cup ((0, 1) \times (0, 1) \setminus \Delta)) \times R$, where $\Delta$ is the diagonal set.

Define $\mathcal{E}^+$ as a subposet of $\mathcal{E}$ consisting of $(t, r)$ with $r$ strongly reducible at $t$. 

![Figure 3.1. Reducible and non-reducible ropes](image-url)
Lemma 3.3. The simplicial spaces $N,E$ and $N,E^\perp$ are good.

Proof. The same as the proof of Lemma 2.10.

Any element in $N,E$ can be expressed as a pair $(t_0 \leq \cdots \leq t_i ; r)$ where $0 < t_i < 1$ and $r \in R$ is reducible at each $t_i$.

Proposition 3.4. There exists a zig-zag of levelwise homotopy equivalences $N,E \rightarrow N,E^\perp \rightarrow N,D^\perp$. Consequently $BE$ is weakly homotopy equivalent to $BD$.

Proof. That the inclusion $E^\perp \rightarrow E$ induces a homotopy equivalence $BE^\perp \simeq BE$ follows in the same way as [2, Theorem 3.9], using [2, Lemma 3.4].

Define a functor $\Phi: E^\perp \rightarrow D^\perp$ that induces a simplicial map $\Phi: N,E^\perp \rightarrow N,D^\perp$ by

$$\Phi(t;r) := (f(t),c(r))$$

(see Figure 3.2). Define the map in the reverse direction $\Gamma: N,D^\perp \rightarrow N,E^\perp$ by

$$\Gamma(T_0 \leq \cdots \leq T_i; M) := (t_0 \leq \cdots \leq t_i; (f^{-1} \times \text{id}_R)(\overline{M}|_{[T_0,T_i]})).$$

where $\overline{M}|_{[T_0,T_i]}$ is the long-extension of $M|_{[T_0,T_i]}$ (see (2.2)), and $t_i := f^{-1}(T_i) \in (0,1)$ (see Figure 3.2). Notice that $(f^{-1} \times \text{id}_R)(\overline{M})$ is not necessarily a tame (or regular) submanifold of $(0,1) \times D^2$ for some $M \in \psi$ (for example, a manifold $M$ that is “knotted” outside arbitrary compact set of $R^3$), but $(f^{-1} \times \text{id}_R)(\overline{M}|_{[T_0,T_i]})$ is indeed a tame submanifold in $(0,1) \times D^2$ since $\overline{M}|_{[T_0,T_i]}$ is a union of two straight half-lines outside $[T_0,T_i] \times D^2$.

We show that $\Phi$ is a levelwise homotopy equivalence, with a homotopy inverse $\Gamma$. The composite $\Phi \circ \Gamma$ is given by

$$\Phi \circ \Gamma(T_0 \leq \cdots \leq T_i; M) = (T_0 \leq \cdots \leq T_i; \overline{M}|_{[T_0,T_i]})$$

and a similar isomorphy from the proof of Proposition 2.11 proves that $\Phi \circ \Gamma \simeq \text{id}$. The other composite $\Gamma \circ \Phi$ is given by

$$\Gamma \circ \Phi(t_0 \leq \cdots \leq t_i; r) := (t_0 \leq \cdots \leq t_i; r|_{[t_0,t_i]}),$$

where

$$r|_{[t_0,t_i]} := ([0,t_0] \times \{p_{23}(r_{t_0})\}) \cup r|_{[t_0,t_i]} \cup ([t_i,1] \times \{p_{23}(r_{t_i})\}) \in R$$

is the “long-extension” of $r|_{[t_0,t_i]}$. The rope $r|_{[t_0,t_i]}$ can be obtained from $r$ by “unknotted” the edge parts $r|_{[t_0,t_i]} \cup r|_{[t_i,\infty)}$ (see Figure 3.2). This unknotted can be realized by applying Lemma 3.5 and its analogue respectively to $r|_{[t_i,\infty)}$ and $r|_{(-\infty,t_0]}$, keeping $r|_{[t_0,t_i]}$ unchanged (and hence keeping $r$ to be strongly reducible at each $t_i$). Thus $\Gamma \circ \Phi \simeq \text{id}$. □
Lemma 3.5 (Mostovoy [3] Lemma 10). Let $W$ be the subspace of $R$ consisting of $r$ that is “strongly reducible” at 0, that means $r|_{[-\varepsilon,\varepsilon]} = r|_{[0,\varepsilon]} = [0,\varepsilon) \times \{p_{23}(\partial_1 r)\}$ for some $\varepsilon > 0$. Then $W$ is contractible. In other words, there exists a canonical homotopy for any $r \in W$ that transforms $r$ to the trivial rope $[0,1] \times \{(0,0)\}$ keeping $r$ to be strongly reducible at 0.

Proof. Let $W' \subset W$ be the subspace consisting of $r \in W$ with $\partial_1 r = (i,0,0)$ for $i = 0, 1$. We show that the inclusion $W' \hookrightarrow W$ is a homotopy equivalence. This completes the proof since $W'$ is homeomorphic to the space $W'_L$ from [3] Lemma 10 via the diffeomorphism $\mathbb{R}^1 \times D^2 \rightarrow \mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$ defined by $(x,u) \mapsto (x,\tan(\pi |u|/2) \cdot u)$, and $W'_L$ has been shown to be contractible. In the proof of [3] Lemma 10 the contracting homotopy (denoted by $D''_r$) keeps ropes to be strongly reducible at 0.

A homotopy inverse $W \rightarrow W'$ can be realized as follows. For $p \in \mathbb{R}^2$ let $\xi_p : \mathbb{R} \rightarrow \mathbb{R}^2$ be the scaling by $1/2$ centered at $p$, namely $\xi_p(x) := (x + p)/2$. Notice that if $p \in D^2$ then $\xi_p(D^2) \subset D^2$. Let $b : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ be a monotonically increasing $C^\infty$-function satisfying $b(x) = 0$ for $x < 1/3$ and $b(x) = 1$ for $x > 2/3$. For $r \in W$, define $\Xi_r : \mathbb{R}^1 \times D^2 \rightarrow \mathbb{R}^1 \times D^2$ by

$$\Xi_r(x,(y,z)) := (x,\xi_{b(x)}(p_{23}(\partial_1 r) \cdot b(x)p_{23}(\partial_1 r)))(y,z).$$

Then $\Xi_r(r) \subset \mathbb{R}^1 \times D^2$ and $\Xi_r(\partial_1 r) = (i,\xi_{b(\partial_1 r)}(p_{23}(\partial_1 r))) = (i,0,0)$. Moreover $\Xi_r(r)$ is strongly reducible at 0 because for a small $0 < \varepsilon < 1/3$ such that $r|_{[-\varepsilon,\varepsilon]} = [0,\varepsilon) \times \{p_{23}(\partial_1 r)\}$ we have

$$\Xi_r(r)|_{[-\varepsilon,\varepsilon]} = \Xi_r([0,\varepsilon) \times \{p_{23}(\partial_1 r)\}] = [0,\varepsilon) \times \{\xi_{-\varepsilon}(\partial_1 r)\} \times \{p_{23}(\partial_1 r)\}] = [0,\varepsilon) \times \{(0,0)\}.$$ 

Thus we have a continuous map $\Xi_r : W \rightarrow W'$. The composite $W' \hookrightarrow W \xrightarrow{\Xi_r} W'$ is the scaling by $1/2$ in the $(y,z)$-direction and is homotopic to $i_{dyl}'$. The other composite $W \xrightarrow{\Xi_r} W' \hookrightarrow W$ is also homotopic to $i_{dyl}'$ because $\xi_p$ is homotopic to $i_{dyl}'$ for any $p \in D^2$. □

Theorem 3.6. The forgetful map induces a weak equivalence $v : BE \rightarrow R$. □

Proof. Replace $D$ with $E$ and take $a$ from (0, 1) in the proof of Theorem 2.12. □

Corollary 3.7. There exists a commutative diagram consisting of (weak) equivalences

$$\begin{array}{ccc}
R & \xrightarrow{i} & \psi \\
\downarrow u' \sim & \downarrow u' & \downarrow \\
BE & \xrightarrow{\Phi} & BD & \xrightarrow{f} & BK
\end{array}$$

where $u', v'$ are the composites of $u, v$ with the inclusions.

3.2. Reducible ropes and Mostovoy’s parametrized short ropes. In Corollary 3.7 we have seen that $BK$ is weakly equivalent to $R$. The following Theorem solves affirmatively the conjecture of Mostovoy. For a rope $r$ let $l(r)$ denote the length of $r$.

Theorem 3.8. Let $B_2$ be the space of embeddings $\rho : [0,1] \hookrightarrow \mathbb{R}^3$ satisfying $\rho(i) = (i,0,0)$ for $i = 0, 1$ and $l(\rho([0,1])) < 3$ (Mostovoy’s (parametrized) short ropes [3]). Then $B_2$ is weakly equivalent to $R$.

The rest of this paper is devoted to the proof of Theorem 3.8.

It is not difficult to see that the image of any $\rho \in B_2$ is in $\mathbb{R}^1 \times D^2(2\sqrt{2})$, where $D^2(\tau)$ is the open 2-disk centered at the origin and of radius $\tau$. Thus we may write $B_2$ as

$$B_2 = \{\rho : [0,1] \hookrightarrow \mathbb{R}^1 \times D^2(2\sqrt{2}) | \rho(i) = (i,0,0) \text{ for } i = 0, 1 \text{ and } l(\rho([0,1])) < 3\}.$$ 

Let $B^u_2 := B_2/\text{Diff}^u([0,1])$ (“$u$” indicates “unparametrized”), namely $B^u_2$ is the space of ropes in $\mathbb{R}^1 \times D^2(2\sqrt{2})$ with $\partial r = (\partial_1 r, \partial_2 r)$, $\partial_1 r = (i,0,0)$ and $l(r) < 3$. The following holds since $\text{Diff}^u([0,1])$ is contractible.
Lemma 3.9. $B_2 \to B_2^e$ is a homotopy equivalence.

We notice that $l(r) < 3$ implies that $r$ is a reducible rope, and hence we may regard $B_2^e$ as a subspace of $R(2\sqrt{2})$, where $R(r)$ is the space of reducible ropes in $\mathbb{R}^3 \times D^2(r)$.

Let $R'(r) \subset R(r)$ be the subspace consisting of $r \in R(r)$ with $l(r) < 3$ ("s" indicates "short"). By definition $B_2^s \subset R'(2\sqrt{2})$.

Lemma 3.10. The inclusion $B_2^s \hookrightarrow R'(2\sqrt{2})$ is a homotopy equivalence.

Proof. For $r \in R'(2\sqrt{2})$, let $\Xi_r : \mathbb{R}^1 \times D^2(2\sqrt{2}) \to \mathbb{R}^1 \times D^2(2\sqrt{2})$ be the map defined in (3.1) (notice that if $p \in D^2(r)$ then $\xi_p(D^2(r)) \subset D^2(r)$). Then $l(\Xi_r(r)) < 3$ because $\Xi_r$ is a shrinking map in the $(y,z)$-direction and hence does not increase the length, and $\Xi_r(\partial \tau r) = (i, \xi_{\partial \tau}(\partial \tau r)) = (i, 0, 0)$. Thus we have a continuous map $\Xi_r : R'(2\sqrt{2}) \to B_2^s$.

The composite $B_2^s \hookrightarrow R'(2\sqrt{2}) \xrightarrow{\Xi} B_2^e$ is the scaling by $1/2$ in the $(y,z)$-direction and is homotopic to $\text{id}_{B_2^e}$. The other composite $R'(2\sqrt{2}) \xrightarrow{\Xi} B_2^s \hookrightarrow R'(2\sqrt{2})$ is also homotopic to $\text{id}_{R'(2\sqrt{2})}$ because $\xi_p$ is homotopic to $\text{id}_{D^2(r)}$ for any $p \in D^2(r)$.

Next let $\mathcal{E}(r)$ be the poset consisting of $(t, r)$, where $t \in (0, 1)$ and $r \in R(r)$ such that $r$ is reducible at $t$. The partial order is defined in the same way as in Definition 3.2. Define $\mathcal{E}^e(r)$ to be a subposet of $\mathcal{E}(r)$ consisting of $(t, r)$ with $l(r) < 3$. Then we have a commutative diagram

$$
\begin{array}{ccc}
B\mathcal{E}^e(2\sqrt{2}) & \xrightarrow{\sim} & R'(2\sqrt{2}) \\
\downarrow & & \downarrow \text{Lemma 3.9} \downarrow \\
B\mathcal{E}(2\sqrt{2}) & \xrightarrow{\sim} & R(2\sqrt{2}) \\
\downarrow & & \downarrow \text{Theorem 3.12} \\
& & R
\end{array}
$$

where $B\mathcal{E}^e(2\sqrt{2}) \to B\mathcal{E}(2\sqrt{2})$ and $\sim$ are induced respectively by the inclusion and the forgetful map (see Theorem 3.12). That $\sim$ is a weak equivalence follows from the same argument as in the proof of Theorem 3.6. The homoeomorphism $R = R(1) \xrightarrow{\sim} R(r)$ is given by $r \mapsto (\text{id}_{\mathbb{R}^2} \times \tau)(r)$, where $\tau : D^2 \xrightarrow{\sim} D^2(r)$ is the scalar multiplication by $\tau$. The diagram together with the following Lemma completes the proof of Theorem 3.12.

Lemma 3.11. $B\mathcal{E}^e(\tau) \to B\mathcal{E}(\tau)$ is a homotopy equivalence.

Proof. Let $\mathcal{E}^e(\tau)$ be the subposet of $\mathcal{E}(\tau)$ consisting of $(t, r)$ with $r$ strongly reducible at $t$, and $\mathcal{E}^e(\tau) := \mathcal{E}^e(\tau) \cap \mathcal{E}(\tau)$. Then the inclusion $\mathcal{E}^e(\tau) \hookrightarrow \mathcal{E}(\tau)$ induces a homotopy equivalence $B\mathcal{E}^e(\tau) \to B\mathcal{E}(\tau)$. This follows in the same way as Galatius and Randall-Williams [2] Theorem 3.9, using [2] Lemma 3.4; modifying $r$ to be strongly reducible at each $t$ can be done keeping the length less than 3.

We show that $\mathcal{E}^e(\tau) \hookrightarrow \mathcal{E}(\tau)$ induces a levelwise homotopy equivalence $N_r\mathcal{E}^e(\tau) \to N_r\mathcal{E}(\tau)$. A homotopy inverse $N_r\mathcal{E}^e(\tau) \to N_r\mathcal{E}(\tau)$ is given as follows; firstly unknott $r|_{[0,\alpha \tau]} \cup r|_{[0, \infty)}$ similarly to the proof of Lemma 3.5 to obtain $r|_{[0, \infty)}$, then shrink $r|_{(0,0)}$ to

$$
\Theta(t, r) := \theta_{r,t} (r|_{[0,\alpha \tau]}) \cup ([l(r|_{[0,\alpha \tau]})^{-1}, 1] \times \{p_{23}(r|_{[0,\alpha \tau]})/l(r|_{[0,\alpha \tau]})\}),
$$

where $\theta_{r,t} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\theta_{r,t}(x) := x/l(r|_{[0,\alpha \tau]})$. It can be seen that $l(\Theta(t, r)) < 3$ since $l(\theta_{r,t}(r|_{[0,\alpha \tau]})) = 1$. The map $N_r\mathcal{E}^e(\tau) \to N_r\mathcal{E}(\tau)$

$$
(t_0 \leq \cdots \leq t_\tau ; r) \mapsto (t_0/l(r|_{[0,\alpha \tau]})), \ldots , t_\tau/l(r|_{[0,\alpha \tau]})) ; \Theta(t, r))
$$

gives a levelwise homotopy inverse.
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