A new properties of varieties of Leibnitz algebras

A. V. Shvetsova, T. V. Skoraya

The paper is devoted to the study of the new properties of varieties of Leibnitz algebras. The characteristic of base field $\Phi$ assumed to be zero. All undefined concepts can be found in [3]. The article presented two new results. The first result belongs to the second author and contains a proof of the sufficient conditions for the finiteness of the colength varieties of Leibnitz algebras. The second belongs to the first author. In it are found a basis of identities and a basis of the space of multilinear elements of variety $\widetilde{V}_3$ of Leibnitz algebras.

1. Introduction

A linear algebra with bilinear multiplication, which is satisfies to the Leibnitz identity $(xy)z \equiv (xz)y + x(yz)$, is called a Leibnitz algebra. Perhaps for the first time this concept was discussed in the article [2] as a generalization of Lie algebras. The Leibnitz identity allows any element expressed as a linear combination of elements in which the brackets are arranged from left to right. Therefore further agree omit brackets in left-normed products, i.e. $(((ab)c)\ldots d) = abc\ldots d$. A variety $\mathbf{V}$ of linear algebras over a field $\Phi$ is a set of algebras over this field that satisfy a fixed set of identities. Note, that the system of identities can be given implicitly. In this case the variety $\mathbf{V}$ is usually defined generating algebra given constructively.

Let $F(X, \mathbf{V})$ be a relatively free algebra of variety $\mathbf{V}$ with countable set of free generators $X = \{x_1, x_2, \ldots\}$. Consider the space of the multilinear elements of algebra $F(X, \mathbf{V})$. This space we will denote $P_n(\mathbf{V})$ and call multilinear part of variety $\mathbf{V}$. On this space naturally introduce the action of permutations that we can consider it as a $\Phi S_n$-module, where $S_n$ is a symmetric group. Since the field $\Phi$ has zero characteristic, then the space $P_n(\mathbf{V})$ is the direct sum of irreducible submodules. Denote by $\chi_\lambda$ the character of the irreducible representations of the symmetric group, which corresponds to the partition $\lambda$ of the number $n$. Then the character of module $P_n(\mathbf{V})$ is expressed by the formula

\begin{equation}
\chi(P_n(\mathbf{V})) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,
\end{equation}

where $m_\lambda$ are the multiplicities of irreducible submodules in this sum.

An important numerical characteristic of variety $\mathbf{V}$ of linear algebras is the colength $l_n(\mathbf{V})$, which is defined as the number of terms in the decomposition
of character in the sum of irreducible characters:

\[ l_n(V) = \sum_{\lambda \vdash n} m_\lambda. \]

We say that the colength of a variety is finite if there exists a constant \( C \) independent of \( n \) such that for any \( n \) the inequality \( l_n(V) \leq C \).

The right multiplication operator, for example, on an element \( z \), we denote by \( Z \), assuming that \( xz = xZ \). This designation allows the element \( x_1 y_2 \ldots y_n \) to write in the form \( xY^n \). Recall that the standard polynomial of degree \( n \) has the form:

\[ St_n(x_1, x_2, \ldots, x_n) = \sum_{q \in S_n} (-1)^q x_{q(1)} x_{q(2)} \ldots x_{q(n)}, \]

where the summation is carried out by elements of the symmetric group, and \((-1)^q\) is equal to +1 or -1 depending on the parity of permutation \( q \). Agree variables in standard polynomial denote with special characters above (below, wave and so on). For example, the standard polynomial of degree \( n \) in the variables \( x_1, x_2, \ldots, x_n \) we will write as follows: \( St_n = \mathfrak{f}_1 \mathfrak{f}_2 \ldots \mathfrak{f}_n \). It is clear that the standard polynomial is skew symmetric. Variables in different skew symmetric sets will be denoted by different symbols, for example:

\[ \sum_{q \in S_n, p \in S_m} (-1)^q (-1)^p x_{q(1)} x_{q(2)} \ldots x_{q(n)} y_{p(1)} y_{p(2)} \ldots y_{p(m)} = \]

\[ = \mathfrak{f}_1 \mathfrak{f}_2 \ldots \mathfrak{f}_n \mathfrak{y}_1 \mathfrak{y}_2 \ldots \mathfrak{y}_m. \]

## 2. Sufficient condition for the finiteness colength of varieties of Leibnitz algebras

Previously, in the article [7] are identified the necessary conditions for the finiteness colength of varieties of Leibnitz algebras. Further, we consider it sufficient conditions.

Following article [6] denote the variety of all Leibnitz algebras (Lie algebras), that satisfy the identity \((x_1 x_2) (x_3 x_4) \ldots (x_{2s+1} x_{2s+2}) \equiv 0\), by \( \widetilde{N}_s \mathcal{A} \) (relatively \( N_s \mathcal{A} \)). Let in addition \( V_1 = N_2 \mathcal{A} \) is a variety of all Lie algebras, commutator of which is nilpotent of class not more than two, and \( \tilde{V}_1 \) is a variety of Leibnitz algebras defined by the identity \( x_1 (x_2 x_3) (x_4 x_5) \equiv 0 \).

**Theorem 1.** Let \( V \) be a subvariety of variety \( \widetilde{N}_s \mathcal{A} \), which for any natural numbers \( k, m, \) and \( \alpha_1, \ldots, \alpha_k \in K \) satisfies the identity

\[ xY^k z Y^{m-k} \equiv \sum_{i=1}^k \alpha_i xY^{k-i} z Y^{m-k+i}. \]

Then the variety \( V \) has the final colength.

**Proof.** Because identity \((3)\) is not satisfied in the varieties \( V_1 \) and \( \tilde{V}_1 \), from conditions of the theorem follows that \( V_1, \tilde{V}_1 \not\subset V \subset \widetilde{N}_s \mathcal{A} \). Then by theorem 1 of article [4], there exists a constant \( C \) independent of \( n \) for such in the sum
is true the condition $(n - \lambda_1) < C$. In this case in the sum (2) the number of the non-zero terms is bounded by a constant independent of $n$. Thus, to prove the result, it suffices to establish that all multiplicities $m_\lambda$ are bounded by a constant, which also is independent of $n$.

The article [5] is proved that the multiplicity $m_\lambda(V)$ is equal to the number of linearly independent polyhomogeneous elements of special form. We will show that the dimension of the space of polyhomogeneous elements is bounded by a constant independent of $n$, which will complete the proof of the theorem.

Consider $\lambda$, for which $m_\lambda \neq 0$. For such partition true the condition $(n - \lambda_1) < C$ and to it correspond monomials of the form

$$g_s = Y^{\alpha_1}x_1 Y^{\alpha_2}x_2 Y^{\alpha_3} \ldots Y^{\alpha_s}x_s Y^{\alpha_{s+1}},$$

where $s < C$. Denote by $Q_{\lambda_1}$ the space generated by elements $g_s$. We prove that the number of linearly independent monomials $g_s$ bounded by a constant.

The proof is by induction on the number $s$ of generators $x_i$ and lexicographic order on lines of the form $(\alpha_1, \alpha_2, \ldots, \alpha_{s+1})$.

Consider the case $s = 1$. Then generating monomials of the space $Q_{\lambda_1}$ have the form: $Y^{\alpha_1}x_1 Y^{\alpha_2}$. If for these elements are true the conditions $\alpha_1 \geq m$ and $\alpha_2 \geq m$, then by the identity (3) they can be represented as a linear combination of the elements, in which $\alpha_1 < m$. Thus any monomial will be expressed through such monomials in which either only $\alpha_1 \geq m$ or $\alpha_2 \geq m$. The number of such monomials is bounded by the constant $2m$ independent of $n$.

In the general case the space $Q_{\lambda_1}$ will generate by elements, in which only one $\alpha_i$ is not less then $tm$. Note, that the general number of such elements is bounded by a constant independent of $n$.

Let $i$ will be a smallest index for which $\alpha_i \geq tm$. consider the corresponding element:

$$g_s = Y^{\alpha_1}x_1 Y^{\alpha_2} \ldots Y^{\alpha_i}x_i Y^{\alpha_{i+1}} \ldots Y^{\alpha_s}x_s Y^{\alpha_{s+1}}.$$

If $\alpha_{i+1} \geq tm$, then the identity (3) allows to bring the element $g_s$ to a linear combination of words, that are lexicographically smaller. If $\alpha_{i+1} < tm$, then modulo words, lexicographically smaller, the element $g_s$ can be written as

$$Y^{\alpha_1}x_1 \ldots Y^{\alpha_i}x_i Y^{\alpha_{i+1} - \alpha_i + 1} \left( y(y \ldots (yx_{i+1}) \ldots ) \right)^{\alpha_{i+1} + 1}_{\alpha_{i+1}}.$$

The Leibnitz identity allows to bring the last element to the sum of terms, that are lexicographically smaller, and term

$$Y^{\alpha_1}x_1 \ldots Y^{\alpha_i}x_i X' Y^{\alpha_{i+2}} \ldots Y^{\alpha_s}x_s Y^{\alpha_{s+1}},$$
where

\[ x' = (y(y \ldots (yx_{i+1}) \ldots ))(y(y \ldots (yx_{i+1}) \ldots )) \]  

We will contain element with fewer generators \( x_{i_r} \) covered by the induction assumption. The theorem is proved.

3. The basis of multilinear part of variety \( \widetilde{V}_3 \)

of Leibnitz algebras

The variety \( \widetilde{V}_3 \) of Leibnitz algebras is an equivalent to the well-known variety \( V_3 \) of Lie algebras. Previously, in the article [1] the growth of this variety was designated, and in the article [8] — its multiplicities and colength.

Let \( T = \Phi[t] \) be a ring of polynomial in the variable \( t \). Consider three-dimensional Heisenberg algebra \( H \) with the basis \( \{a, b, c\} \) and multiplication \( ba = -ab = c \), the product of the remaining basis elements is zero. Well known and easy to verify that the algebra \( H \) is nilpotent of the class two Lie algebra. Transform the polynomial ring \( T \) in the right module of algebra \( H \), in which the basis elements of algebra \( H \) act on the right on the polynomial \( f \) from \( T \) follows:

\[ fa = f', fb = tf, fc = f, \]

where \( f' \) is a partial derivative of a polynomial \( f \) in the variable \( t \). Consider the direct sum of vector spaces \( H \) and \( T \) with multiplication by the rule:

\[ (x + f)(y + g) = xy + fy, \]

where \( x, y \) are from \( H \); \( f, g \) are from \( T \). Denote it by the symbol \( \widetilde{H} \). Direct verification shows that \( \widetilde{H} \) is an algebra of Leibnitz. The algebra \( \widetilde{H} \) is the Leibnitz algebra, satisfies to the identity \( x(y(zt)) \equiv 0 \) and generates the variety \( \widetilde{V}_3 \) of Leibnitz algebras.

**Lemma.** The variety \( \widetilde{V}_3 \) satisfies to the identities:

\[
(4) \quad x(y(zt)) \equiv 0, \\
(5) \quad x_0A\overline{x_1}B\overline{x_2}C\overline{x_3}D\overline{x_4} \equiv 0, \\
(6) \quad x_0(x_1x_4)(x_2x_3) \equiv x_0(x_1x_2)(x_3x_4) + x_0(x_1x_3)(x_2x_4),
\]

where \( A, B, C, D \) are some words from generators.

**Proof.** The truth of identities (4) and (5) verified by arbitrary replacement generators by elements of algebra \( \widetilde{H} \) and was showed in the paper [1]. Consider the following special form of the second identity: \( x_0\overline{x_1}\overline{x_2}\overline{x_3}\overline{x_4} \equiv 0 \). Presenting it as a sum and using the identity \( xyz - xzy \equiv x(yz) \), we obtain:

\[ 2x_0(x_1x_2)(x_3x_4) - 2x_0(x_1x_3)(x_2x_4) + 2x_0(x_1x_4)(x_2x_3) \equiv 0. \]

Dividing this identity by 2 and moving the second term to the right, we obtain the identity (6). The lemma is proved.
Theorem 2. The set elements of form

\[ \theta(i, i_1, \ldots, i_m, j_1, \ldots, j_m) = x_i(x_{i_1}x_{j_1})(x_{i_2}x_{j_2})\cdots(x_{i_m}x_{j_m})x_{k_1}x_{k_2}\cdots x_{k_{n-2m-1}}, \]

where \( i_s < j_s, s = 1, 2, \ldots, m, i_1 < i_2 < \ldots < i_m, j_1 < j_2 < \ldots < j_m, k_1 < k_2 < \ldots < k - 2m - 1, \) is a basis of the space \( P_n(\tilde{V}_3). \)

Proof. Consider an arbitrary element of the space \( P_n(\tilde{V}_3). \) Using corollary \( x(yz) \equiv x(yz) \) from the Leibnitz identity and identity (1), move the all pairs as far right as possible.

We order the elements obtained using the lexicographic ordering of lines \((k_1, k_2, \ldots, k_{n-2m-1}). \) Let the considering element has a form:

\[ x_i(x_{i_1}x_{j_1})(x_{i_2}x_{j_2})\cdots(x_{i_m}x_{j_m})x_{k_1}x_{k_2}\cdots x_{k_{n-2m-1}} \]

and \( k_s > k_{s+1}. \) Using the identity \( x(yz) \equiv x(yz) \) we can write this element as a sum

\[ x_i(x_{i_1}x_{j_1})(x_{i_2}x_{j_2})\cdots(x_{i_m}x_{j_m})x_{k_1}x_{k_2}\cdots x_{k_{n-2m-1}} + x_i(x_{i_1}x_{j_1})(x_{i_2}x_{j_2})\cdots(x_{i_m}x_{j_m})x_{k_1}x_{k_2}\cdots x_{k_{n-2m-1}} \]

where the first term is lexicographically less, than parent element, and the second term has fewer number of single elements. Applying the same method to the resulting term, we eventually present our original element as a sum of terms, in that \( k_1 < k_2 < \ldots < k_{n-2m-1}. \)

Consider an arbitrary element, in which indexes of single elements are ordered. We choose the two lowest index in the considered element and redenote them through \( \alpha' \) and \( 2' \) relatively. We introduce the lexicographic order on lines \((j_1, j_2, \ldots, j_m). \) Using also the induction on the number of brackets, we will prove, that all received elements can be represented as a linear combination of elements \( \theta(i, i_1, \ldots, i_m, j_1, \ldots, j_m). \) The corollary \( x(yz) \equiv x(yz) \) of Leibnitz identity allows to order the indexes of elements in couples, and the identity \( x(yz) \equiv x(yz) \) allows to order the brackets by the indexes of first elements. According to these identities, the element can be written either in the form \( x_i(x_{i_1}x_{j_1})(x_{i_2}x_{j_2})\cdots(x_{i_m}x_{j_m})x_{k_1}x_{k_2}\cdots x_{k_{n-2m-1}} \), either in the form \( x_i(x_{i_1}x_{j_1})(x_{i_2}x_{j_2})\cdots(x_{i_m}x_{j_m})x_{k_1}x_{k_2}\cdots x_{k_{n-2m-1}} \). In the first case we can consider the ordering on \( m-1 \) brackets that runs by induction. In the second case we apply the identity (1) and obtain:

\[ x_i(x_{i_1}x_{j_1})(x_{i_2}x_{j_2})\cdots(x_{i_m}x_{j_m})x_{k_1}x_{k_2}\cdots x_{k_{n-2m-1}} + x_i(x_{i_1}x_{j_1})(x_{i_2}x_{j_2})\cdots(x_{i_m}x_{j_m})x_{k_1}x_{k_2}\cdots x_{k_{n-2m-1}} \]

where for the first term can be again apply the induction hypothesis, and the second term is lexicographically less. Therefore, any element of the space \( P_n(\tilde{V}_3) \) can be written as a linear combination of elements \( \theta(i, i_1, \ldots, i_m, j_1, \ldots, j_m) \) modulo \( \text{Id}(\tilde{V}_3). \)

We now prove, that the elements \( \theta(i, i_1, \ldots, i_m, j_1, \ldots, j_m) \) are linearly independent modulo \( \text{Id}(\tilde{V}_3). \) Consider the linear combination of these elements:

\[ \sum_{(i,i_1,\ldots,i_m,j_1,\ldots,j_m)} \alpha(i, i_1, \ldots, i_m, j_1, \ldots, j_m)\theta(i, i_1, \ldots, i_m, j_1, \ldots, j_m) = 0 \]
and show that all coefficient \( \alpha(i, i_1, ..., i_m, j_1, ..., j_m) \) are zero. Assume the contrary.

Choose an element \( \theta(i^*, i_{1*}, ..., i_{m*}, j_{1*}, ..., j_{m*}) \) with non-zero coefficient \( \alpha(i^*, i_{1*}, ..., i_{m*}, j_{1*}, ..., j_{m*}) \) such that the number of commutators \( m \) in it is the least and the index \( j_{1*}^* \) of element in the second position in the first bracket is the largest. Since each element is uniquely determined by the number \( m \), by the element \( x_i \) and by the sample \((i, i_1, ..., i_m, j_1, ..., j_m)\), then in the selected element \( \theta(i^*, i_{1*}, ..., i_{m*}, j_{1*}, ..., j_{m*}) \) these rates are fixed. We replace its generators on the basis elements of algebra \( \tilde{H} \) as follows: \( x_{i*} = f, x_{i_{s*}} = a, x_{j_{s*}} = b, s = 1, ..., m \), the rest generators we replace on element \( c \). After this substitution all elements, which are different from the chosen, will be equal to zero: if the element \( x_{i*} \) will be replace on the basis element of Heisenberg algebra, than this element will be zero (because \( x f \equiv 0 \) for any \( x \) from \( \tilde{H} \)); if the element will have more then \( m \) commutators, then it will be also zero (since the element \( c \) from the center of algebra fall into the commutator); a similar situation arises, if the element will have \( m \) commutators but the sample will be different from the fixed. Indeed, there are two kinds of elements containing \( m \) commutators at a fixed sample \((i^*, i_{1*}, ..., i_{m*}, j_{1*}, ..., j_{m*})\): these are the elements that contain in the second position and the first bracket \( x_{j_{1*}}^* \), and elements that contain in the second position and the first bracket \( x_{i_{s*}}^* \), where \( i_{s*}^* < j_{1*}^* (s = 2, ..., m) \).

All elements of the second kind are zero, as by described substitution the first bracket will be equal to \((aa)\). In one of brackets of the elements of the first type fall generators \( x_{i_{s*}} \) and \( x_{i_{t*}} \). As a result of described substitution this bracket also resets the element. Thus we obtained, that if \( f \neq 0 \), then \( \sum_{i, i_1, ..., i_m, j_1, ..., j_m} \alpha(i, i_1, ..., i_m, j_1, ..., j_m) \theta(i, i_1, ..., i_m, j_1, ..., j_m) = 0 \). Consequently, contrary to the assumption \( \alpha(i^*, i_{1*}, ..., i_{m*}, j_{1*}, ..., j_{m*}) \) is zero. The theorem is proved.

Note that in the proof of the theorem we used only the Leibnitz identity and corollaries from it, the identity \( x(y(zt)) \equiv 0 \), and lastly the identity (3). Consequently, any identity that runs in the variety \( \tilde{V}_3 \), is a corollary from these identities. Hence we obtain the following assertion.

**Corollary.** The identities (4) and (6) form a basis of identities of variety \( \tilde{V}_3 \).

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