ON THE STRUCTURE AND APPLICATION OF
POWER SET RING

ABOLFAZL TARIZADEH AND ZAHRA TAHERI AMIN

Abstract. In this paper, new progresses in the understanding
the structure of power set ring have been made and some new and
interesting results are obtained. Specially, it is shown that the
Booleanization of a commutative ring is isomorphic to the clopen
ring of its prime spectrum. The topological version of this result is
also proven which states that the space of connected components
of a compact space is homeomorphic to the prime spectrum of its
clopes. In particular, Stone’s Representation Theorem is gener-
alized from Boolean rings to arbitrary commutative rings. Finally,
as another application of power set ring, some important contribu-
tions to the field of fixed-point theorems have been made.

1. Introduction

In 1936, Marshall Stone published a long paper [7] that whose main
result was Stone’s Representation Theorem which states that every
Boolean ring is isomorphic to a certain subring of some power set ring,
an important result in mathematical logic, topology, universal alge-
bra and category theory. The theorem has been the starting point for
a whole new field of study, nowadays called Stone duality (or, Stone
spaces). It is worth to mention that “Boolean rings” and “Boolean
algebras” are the equivalent notions (that is, every Boolean ring can
be made into a Boolean algebra, and vice versa); though it is hard to
understand how this fact remained undiscovered for so long, it was not
until 1935 that Stone realized that the connection could be made for-
mal. For further studies on these topics we refer the interested reader
to [1]-[6].

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omeorem; Clopen.
In Theorem 3.1, we prove a general result which states that if \( R \) is a commutative ring then we have the canonical isomorphism of rings:
\[
\mathcal{B}(R) \simeq \text{Clop} \left( \text{Spec}(R) \right)
\]
where \( \mathcal{B}(R) \) denotes the ring of idempotents of \( R \) (we call \( \mathcal{B}(R) \) the Booleanization of \( R \), see §3). Stone’s Representation Theorem is a special case of the above isomorphism, see Corollary 3.2. Indeed, considering Clop(\( X \)) as a ring where \( X \) is a topological space, this simple idea was the beginning of the many novel results of this paper. In Theorem 5.2, we prove the topological version of Theorem 3.1 which states that if \( X \) is a compact space then we have the canonical isomorphism of topological spaces:
\[
\pi_0(X) \simeq \text{Spec} \left( \text{Clop}(X) \right)
\]
here \( \pi_0(X) \) denotes the space of connected components of \( X \). Moreover \( X \) is totally disconnected then we get the following isomorphism which is the topological version of the Stone’s Representation Theorem:
\[
X \simeq \text{Spec} \left( \text{Clop}(X) \right).
\]

We have also made further progresses in the understanding the structure of power set ring so that Theorems 4.1, 4.3, 5.6, 5.11, 5.13 and 5.15 are amongst the most important ones. Also, Theorems 6.1 and 6.2 provide important and non-trivial contributions to the field of fixed-point theorems.

2. Preliminaries

In this paper, all rings are commutative. If \( f \) is an element of a ring \( R \) then \( D(f) = \{ p \in \text{Spec}(R) : f \notin p \} \).

If \( X \) is a set then its power set \( \mathcal{P}(X) \) together with the symmetric difference \( A + B = (A \cup B) \setminus (A \cap B) \) as addition and the intersection \( A.B = A \cap B \) as multiplication forms a ring whose zero and unit are respectively the empty set and the whole set \( X \). The ring \( \mathcal{P}(X) \) is called the power set ring of \( X \). If \( f : X \to Y \) is a function then the map \( \mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X) \) defined by \( A \mapsto f^{-1}(A) \) is a morphism of rings. In fact, the assignments \( X \mapsto \mathcal{P}(X) \) and \( f \mapsto \mathcal{P}(f) \) form a faithful contravariant functor from the category of sets to the category of Boolean rings. We call it the power set ring functor.
A ring is called a Boolean ring if each element is an idempotent. Power set rings are typical examples of Boolean rings. It is easy to see that every Boolean ring is a commutative ring, and in a Boolean ring every prime ideal is a maximal ideal.

By a compact space we mean a quasi-compact and Hausdorff topological space.

If \( \varphi : A \to B \) is a morphism of rings then the induced map \( \text{Spec}(B) \to \text{Spec}(A) \) is denoted by \( \text{Spec}(\varphi) \) or by \( \varphi^* \).

**Remark 2.1.** If an ideal of a ring \( R \) is generated by a set of idempotents of \( R \) then it is called a regular ideal of \( R \). Every maximal element of the set of proper and regular ideals of \( R \) is called a max-regular ideal of \( R \). The set of max-regular ideals of \( R \) is called the Pierce spectrum of \( R \) and denoted by \( \text{Sp}(R) \). It is a compact totally disconnected space whose basis opens are of the form \( U_f = \{ M \in \text{Sp}(R) : f \notin M \} \) where \( f \in R \) is an idempotent. If \( p \in X = \text{Spec}(R) \) then the connected component \( [p] \) is equal to \( V(M) \) where \( M = (f \in p : f = f^2) \) is a max-regular ideal of \( R \). In fact, the map \( \pi_0(X) \to \text{Sp}(R) \) given by \( [p] \mapsto M \) is a homeomorphism.

### 3. Generalization of Stone Representation Theorem

In this section, Stone’s Representation Theorem is generalized from Boolean rings to arbitrary commutative rings.

If \( X \) is a topological space then by \( \text{Clop}(X) \) we mean the set of clopen (both open and closed) subsets of \( X \). Then \( \text{Clop}(X) \) is a subring of \( \mathcal{P}(X) \). If \( f : X \to Y \) is a continuous map of topological spaces then the map \( \text{Clop}(f) : \text{Clop}(Y) \to \text{Clop}(X) \) given by \( A \mapsto f^{-1}(A) \) is a morphism of rings. In fact, the assignments \( X \mapsto \text{Clop}(X) \) and \( f \mapsto \text{Clop}(f) \) form a contravariant functor from the category of topological spaces to the category of Boolean rings. We call it the Clopen functor.

Let \( \mathcal{B}(R) \) be the set of idempotents of a ring \( R \). Then it is easy to see that the set \( \mathcal{B}(R) \) by a new operation \( \oplus \) defined by \( f \oplus g := f + g - 2fg \) as the addition admits a commutative ring structure (whose multiplication is the restricted multiplication of \( R \)). Note that \( \mathcal{B}(R) \) is not necessarily a subring of \( R \). In fact, \( \mathcal{B}(R) \) is a subring of a non-zero ring
$R$ if and only if $\text{Char}(R) = 2$. We call $B(R)$ the Boolean ring (or, the Booleanization) of $R$. We have then:

**Theorem 3.1.** *(Generalized Stone Representation Theorem)* If $R$ is a ring then the map $B(R) \to \text{Clop} \left( \text{Spec}(R) \right)$ given by $f \sim D(f)$ is an isomorphism of rings.

**Proof.** It is well known that the above map is bijective, see e.g. [8, Proposition 3.1]. Clearly $D(fg) = D(f)D(g)$ and $D(1) = \text{Spec}(R)$. It remains to show that $D(f \oplus g) = D(f) + D(g)$. If $p \in D(f \oplus g)$ then clearly $p \in D(f) \cup D(g)$. But $1 - (f \oplus g) \in p$ because $f \oplus g$ is an idempotent. So $fg = fg(1 - (f \oplus g)) \in p$. This yields that $p \in D(f) + D(g)$. Conversely, assume that $p \in D(f) \cup D(g) \setminus D(fg)$. If $f \oplus g \in p$ then $f + g - fg \in p$. It follows that $f = f(f + g - fg) \in p$ and $g = g(f + g - fg) \in p$. But this is a contradiction and we win. □

**Corollary 3.2.** *(Stone Representation Theorem)* If $R$ is a Boolean ring then the map $R \to \text{Clop} \left( \text{Spec}(R) \right)$ given by $f \sim D(f)$ is an isomorphism of rings.

**Proof.** We have $f \oplus g = f + g$ for all $f, g \in R$ since $\text{Char}(R) = 2$. Thus $B(R) = R$ as rings and so by Theorem 3.1, $R \simeq \text{Clop} \left( \text{Spec}(R) \right)$. □

If $\varphi : R \to S$ is a morphism of rings then the map $B(\varphi) : B(R) \to B(S)$ given by $f \sim \varphi(f)$ is also a morphism of rings. In fact, the assignments $R \sim B(R)$ and $\varphi \sim B(\varphi)$ define a covariant functor from the category of commutative rings to the category of Boolean rings. We call it the Booleanization functor. In fact, the isomorphism of Theorem 3.1 is functorial that is, the Booleanization functor is isomorphic to $\text{Clop} \circ \text{Spec}$ where the contravariant functor $\text{Spec} : \text{C-Ring} \to \text{Top}$ is the usual spectrum functor (C-Ring denotes the category of commutative rings).

### 4. Booleanization versus Pierce spectrum

As stated before, $B(R)$ is not necessarily a subring of $R$. In spite of this we have the following topological identification.
Theorem 4.1. If $R$ is a ring then $\text{Spec}(\mathcal{B}(R))$ is canonically homeomorphic to the Pierce spectrum $\text{Sp}(R)$.

Proof. We show that the map $\mu : \text{Spec}(\mathcal{B}(R)) \to \text{Sp}(R)$ given by $P \rightsquigarrow (P)$ is a homeomorphism where $(P)$ is an ideal of $R$ generated by the subset $P$. If $P$ is a prime ideal of $\mathcal{B}(R)$ then $(P) \neq R$. Because if $1 \in (P)$ then we may write $1 = \sum_{i=1}^{m} r_i e_i$ where $r_i \in R$ and $e_i \in P$ for all $i$. It follows that $\prod_{i=1}^{m} (1 - e_i) = 0$. Thus $1 - e_i \in P$ for some $i$. But this is a contradiction. In fact, it is easy to see that $(P)$ is a max-regular ideal of $R$. The map $\mu$ is continuous since $\mu^{-1}(U_f) = D(f)$ for all $f \in \mathcal{B}(R)$. This map is also injective and closed map, see Remark [2.1]. Finally, if $M$ is a max-regular ideal of $R$ then the ideal $P = (f \in M : f = f^2)$ is a maximal ideal of $\mathcal{B}(R)$ generated by the subset $M \cap \mathcal{B}(R)$ is a proper ideal. If $P = \mathcal{B}(R)$ then we may write $1 = g_1 f_1 + \ldots + g_n f_n$ where $g_k \in \mathcal{B}(R)$ and $f_k \in M \cap \mathcal{B}(R)$ for all $k$. Then by induction on $n$ we get that $1 \in M$, which is a contradiction. Suppose there exist $f, g \in \mathcal{B}(R)$ such that $fg \in P$. Then we may write $fg = x_1 y_1 + \ldots + x_d y_d$ where $x_\ell \in \mathcal{B}(R)$ and $y_\ell \in \mathcal{B}(R) \cap M$. It follows that $fg \in M$. Suppose $f \notin M$ and $g \notin M$. This yields that $M + Rf = R$ and $M + Rg = R$ since $M + Rf$ and $M + Rg$ are regular ideals. Thus $1 = m + rf$ and $1 = m' + r'g$ where $m, m' \in M$. This implies that $1 = mm' + mrf + m' + rfg \in M$, a contradiction. Thus we may assume that, say, $f \in M$. So $f \in P$. Therefore $P$ is a prime ideal of $\mathcal{B}(R)$. It is easy to see that $\mu(P) = M$. 

Let $I$ be an ideal of a ring $R$ then $I^b = (f \in I : f = f^2)$ is the largest regular ideal of $R$ which is contained in $I$. We call $I^b$ the regular part of $I$. If $\varphi : R \to R'$ is a morphism of rings and $M$ a max-regular ideal of $R'$ then $(\varphi^{-1}(M))^b$ is a max-regular ideal of $R$. The map $\text{Sp}(\varphi) : \text{Sp}(R') \to \text{Sp}(R)$ given by $M \rightsquigarrow (\varphi^{-1}(M))^b$ is continuous because $(\text{Sp}(\varphi))^{-1}(U_f) = U_{\varphi(f)}$ for all $f \in \mathcal{B}(R)$. If $\psi : R' \to R''$ is a second morphism of rings then it is not hard to see that $\text{Sp}(\psi \circ \varphi) = \text{Sp}(\varphi) \circ \text{Sp}(\psi)$. In fact, the assignments $R \rightsquigarrow \text{Sp}(R)$ and $\varphi \rightsquigarrow \text{Sp}(\varphi)$ is a contravariant functor from the category of commutative rings to the category of compact totally disconnected spaces. We call it the Pierce functor. Then we have the following result.
Corollary 4.2. The Pierce functor is isomorphic to the contravariant functor $\text{Spec} \circ \mathcal{B}$.

Proof. It is easy to see that if $\varphi : R \to R'$ is a morphism of rings then the following diagram is commutative:

$$
\begin{array}{ccc}
\text{Spec}(\mathcal{B}(R')) & \xrightarrow{\mu_{R'}} & \text{Sp}(R') \\
\downarrow^{F(\varphi)} & & \downarrow^{\text{Sp}(\varphi)} \\
\text{Spec}(\mathcal{B}(R)) & \xrightarrow{\mu_R} & \text{Sp}(R)
\end{array}
$$

where $\mu_R$ and $\mu_{R'}$ are the canonical isomorphisms (see Theorem 4.1) and $F := \text{Spec} \circ \mathcal{B}$. □

The following result is inspired from Theorem 3.1.

Theorem 4.3. If $R$ is a ring then the map $\mathcal{B}(R) \to \text{Clop}(\text{Sp}(R))$ given by $f \mapsto U_f$ is an isomorphism of rings.

Proof. The above map clearly preserves units. Let $f, g \in \mathcal{B}(R)$. We have to show that $U_{fg} = U_f \cap U_g$ and $U_{f\oplus g} = U_f + U_g$. If $M \in U_f \cap U_g$ then $M + Rf = R$ since it is a regular ideal, and so $1 - f \in M$. It follows that $g(1 - f) = g - fg \in M$. This shows that $M \in U_{fg}$. Thus $U_{fg} = U_f \cap U_g$. If $M \in U_{f\oplus g}$ then clearly $M \in U_f \cup U_g$. Thus we may assume that, say, $M \in U_f$. It follows that $1 - f \in M$. If $M \in U_g$ then $1 - g \in M$. Thus $f + g = g(1 - f) = f(1 - g) \in M$, a contradiction. Hence $M \in U_f + U_g$. It is easy to see that $U_f + U_g \subseteq U_{f\oplus g}$. Therefore $U_{f\oplus g} = U_f + U_g$. Hence the above map is a morphism of rings. If $U_f = U_g$ then $\sqrt{Rf} = \sqrt{Rg}$. Because if $f \notin \sqrt{Rg}$ then there exists a prime ideal $p$ of $R$ such that $g \in p$ but $f \notin p$. It is easy to see that $M = (h \in p : h = h^2)$ is a max-regular ideal of $R$ and so $M \in U_f$. But this is a contradiction since $g \in M$. Hence $\sqrt{Rf} = \sqrt{Rg}$. Thus we may write $f = rg$ for some $r \in R$. It follows that $f = fg$. Similarly we get that $g = fg$. Therefore $f = g$. So the above map is injective. Finally, if $W$ is a clopen of $\text{Sp}(R)$ then it is quasi-compact, because in a quasi-compact space every closed is quasi-compact. Thus we may write $W = \bigcup_{k=1}^n U_{f_k}$ where $f_k \in \mathcal{B}(R)$ for all $k$. But it is easy to see that $f + g - fg$ is an idempotent and $U_f \cup U_g = U_{f + g - fg}$. Thus there exists an idempotent $e \in R$ such that $W = U_e$. Hence the above map
is surjective. □

5. Structure of power set ring with applications

If $A \in R = \mathcal{P}(X)$ then we may consider $\mathcal{P}(A)$ as an $R$–algebra whose structure morphism $\mathcal{P}(i) : R \to \mathcal{P}(A)$ given by $B \mapsto B \cap A$ is induced by the inclusion $i : A \to X$. If $A$ is a proper subset of $X$ then the ring $\mathcal{P}(A)$ is not a subring of $R$ since the units are not the same. But $\mathcal{P}(A)$ is an ideal of $R$ and the quotient ring $R/\mathcal{P}(A)$ is canonically isomorphic to the ring $\mathcal{P}(A^c)$. Specially, if $x \in X$ then $m_x := \mathcal{P}(X \setminus \{x\})$ is a maximal ideal of $R$.

It is easy to see that in a ring $R$ if an ideal is generated by a finite set of idempotents then it is generated by one idempotent. In fact, if $e, e' \in R$ are idempotents then the ideal $(e, e')$ is generated by the idempotent $e + e' - ee'$. In the power set ring the situation is a little more interesting:

Lemma 5.1. If $A_1, ..., A_n \in \mathcal{P}(X)$ then $(A_1, ..., A_n) = \mathcal{P}(\bigcup_{k=1}^{n} A_k)$.

Proof. First note that for each $A \in \mathcal{P}(X)$ then $(A) = \mathcal{P}(A)$. Because clearly $(A) \subseteq \mathcal{P}(A)$. Conversely, if $C \in \mathcal{P}(A)$ then $C = C \cap A = CA \in (A)$. Now if $B \in \mathcal{P}(X)$ then the ideal $(A, B)$ is generated by the element $A + B - AB = A \cup B$. Thus $(A, B) = \mathcal{P}(A \cup B)$. □

Theorem 5.2. Let $X$ be a compact space and $R = \text{Clop}(X)$. Then the map $\pi_0(X) \to \text{Spec}(R)$ given by $[x] \mapsto m_x \cap R$ is a homeomorphism.

Proof. We have $[x] = [y]$ if and only if $m_x \cap R = m_y \cap R$, because it is well known that if $X$ is a compact space then for each $x \in X$, the connected component $[x]$ is the intersection of all clopens of $X$ which contain $x$. Thus $\varphi$ is well-defined and injective. If $A \in R$ then $\varphi^{-1}(D(A))$ is an open of $\pi_0(X)$ since $\pi^{-1}(\varphi^{-1}(D(A))) = A$ where $\pi : X \to \pi_0(X)$ is the canonical projection. Hence $\varphi$ is continuous. It is also a closed map because $\pi_0(X)$ is quasi-compact and Spec$(R)$ is Hausdorff. It remains to show that it is surjective. Let $M$ be a maximal ideal of $R$. Suppose $M \neq m_x \cap R$ for all $x \in X$. Thus for each $x \in X$ there exists some $A_x \in R$ such that $A_x \in M$ and $A_x^c \in m_x \cap R$. It follows that $x \in A_x$ for all $x \in X$. But $X$ is quasi-compact, hence we
may write $X = \bigcup_{i=1}^{n} A_{x_i}$. There exists a maximal ideal $m$ of $\mathcal{P}(X)$ such that $M = m \cap R$, because it is easy to see that in an extension of rings, for every minimal prime of the subring then there exists a (minimal) prime of the extended ring which lies over it. Therefore $1 \in m$, see Lemma 5.1 (Even we may write $1 = \sum_{k=1}^{m} B_k$ where $B_k \in M$ for all $k$). But this is a contradiction and we win. □

Note that in the proof of Theorem 5.2, we have only used the quasi-compactness of $\pi_0(X)$. In fact we have:

**Corollary 5.3.** If $X$ is a compact space then $\pi_0(X)$ is a compact totally disconnected space.

**Proof.** It is an immediate consequence of Theorem 5.2 □

**Corollary 5.4.** Every compact totally disconnected topological space $X$ is homeomorphic to the prime spectrum of the Boolean ring $\text{Clop}(X)$.

**Proof.** It implies from Theorem 5.2 □

**Corollary 5.5.** If $X$ is a compact totally disconnected space then $\text{Clop}(X)$ is a basis for the opens of $X$.

*First Proof.* It implies from Corollary 5.4. *Second Proof.* Let $U$ be an open of $X$ and $x \in U$. Then $\{x\}$ is the intersection of all clopens of $X$ containing $x$. Thus there exists a finite number $A_1, \ldots, A_n \in \text{Clop}(X)$ such that $x \in A := \bigcap_{k=1}^{n} A_k \subseteq U$, because $U^c$ is quasi-compact. Thus $A$ is a clopen of $X$. □

**Theorem 5.6.** If $A$ and $B$ are Boolean rings then the map:

$$\text{Mor}_{\text{Ring}}(A, B) \rightarrow \text{Mor}_{\text{Top}}(\text{Spec}(B), \text{Spec}(A))$$

$$\varphi \mapsto \text{Spec}(\varphi)$$

is a one to one correspondence.
Proof. Let $\varphi, \psi : A \to B$ be two ring maps such that $\text{Spec}(\varphi) = \text{Spec}(\psi)$. To prove $\varphi = \psi$ it suffices to show that for each $f \in A$, $\varphi(f) - \psi(f)$ is a member of the radical Jacobson of $A$, because it is the zero ideal. Suppose there exists a maximal ideal $m$ of $A$ such that $\varphi(f) - \psi(f) \notin m$. It follows that $1 - (\varphi(f) - \psi(f)) \in m$. This yields that $\varphi(f)\psi(f) \in m$. We may assume that $\varphi(f) \in m$. This implies that $\psi(f) \notin m$. But we have $f \in \varphi^{-1}(m) = \psi^{-1}(m)$. This is a contradiction and we win. Conversely, let $h : \text{Spec}(B) \to \text{Spec}(A)$ be a continuous function. If $f \in A$ then $D(f)$ is a clopen of $\text{Spec}(A)$ and so $h^{-1}(D(f))$ is a clopen of $\text{Spec}(B)$. Hence there exists a unique (idempotent) $f' \in B$ such that $h^{-1}(D(f)) = D(f')$. Then consider the function $\varphi : A \to B$ defined by $f \mapsto f'$. This map is clearly multiplicative and preserves units. To prove its additivity it suffices to show that $h^{-1}(D(f+g)) = D(f'+g')$ where $g \in A$ and $\varphi(g) = g'$. If $p \in h^{-1}(D(f+g))$ then $1 - f - g \in h(p)$. Suppose $f' + g' \in p$. We have $fg \in h(p)$. Thus we may assume that, say, $f \in h(p)$. It follows that $g \notin h(p)$. Thus $p \in h^{-1}(D(g)) = D(g')$. But $(1 - f')g' \in p$. Therefore $1 - f' \in p$. This yields that $p \in D(f')$. But this is a contradiction. Hence, $h^{-1}(D(f+g)) \subseteq D(f'+g')$. The reverse inclusion is also proved by a similar argument. Thus $\varphi$ is a morphism of rings. It remains to show that $\text{Spec}(\varphi) = h$. If $p$ is a prime ideal of $B$ then $\varphi^{-1}(p) \subseteq h(p)$ and so $\varphi^{-1}(p) = h(p)$ since $\varphi^{-1}(p)$ is a maximal ideal of $A$. □

Note that the above theorem does not hold in general. For example, if $K$ is a field then every function $\text{Spec}(K) \to \text{Spec}(\mathbb{Z})$ is continuous, but there exists only one morphism of rings $\mathbb{Z} \to K$.

**Corollary 5.7.** The category of Boolean rings by the assignments $A \rightsquigarrow \text{Spec}(A)$ and $\varphi \rightsquigarrow \text{Spec}(\varphi)$ is antiequivalent to the category of compact totally disconnected spaces.

Proof. This functor by Theorem 5.6, is fully-faithful, and by Corollary 5.4 is essentially surjective. □

**Remark 5.8.** Here a second proof is given for Corollary 5.7 without using Theorem 5.6. The isomorphism of Corollary 5.2 is functorial. Hence, the identity functor of the category of Boolean rings is isomorphic to $\text{Clop} \circ \text{Spec}$. The isomorphism of Corollary 5.4 is also functorial. Therefore the identity functor of the category of compact totally
disconnected spaces is isomorphic to $\text{Spec} \circ \text{Clop}$.

**Proposition 5.9.** Let $A$ and $B$ be Boolean rings and $M$ a maximal ideal of $A$. Then the map $\varphi : A \to B$ which sends each $f \in A$ into 0 or 1, according as $f \in M$ or $f \notin M$, is a morphism of rings.

*First proof.* By Theorem 5.6, this map corresponds to the constant function $\text{Spec}(B) \to \text{Spec}(A)$ which sends each point of $\text{Spec}(B)$ into $M$. *Second proof.* To see the additivity of $\varphi$ it will be enough to show that if $f, g \notin M$ then $f + g \in M$. If $f + g \notin M$ then $f + fg = f(f + g) \notin M$, similarly we get that $g + fg = g(f + g) \notin M$. It follows that $0 = (f + fg)(g + fg) \notin M$, which is impossible. Hence $\varphi$ is a morphism of rings. □

Let $X$ be an infinite set and let $\text{Fin}(X)$ be the set of all finite subsets of $X$. Then $\text{Fin}(X)$ is an ideal of $\mathcal{P}(X)$ which is not a finitely generated ideal. Because if it is a finitely generated ideal then it is a principal ideal, say $\text{Fin}(X) = (A)$. Clearly $X \setminus A$ is non-empty, choose some $x$ in it, then we have $\{x\} \in \text{Fin}(X)$ and so $\{x\} = BA = B \cap A$ for some $B \in \mathcal{P}(X)$, but this is a contradiction. Hence, $\text{Fin}(X)$ is not a finitely generated ideal. Indeed, $\text{Fin}(X)$ is generated by the single-point subsets of $X$, since if $A \in \text{Fin}(X)$ then we may write $A = \sum_{x \in A} \{x\}$.

**Remark 5.10.** The power set ring functor is not full, see Theorem 5.11 or Proposition 5.14. At first, the following observation was intended to confirm it very quickly (though it was not successful but it deserves to mention at here). Let $X$ be a set and $S := \mathcal{P}(X)$. Then the map $\mathcal{P}(X) \to \mathcal{P}(S)$ given by $A \mapsto \mathcal{P}(A)$ is multiplicative and preserves units, but it is not additive. In fact, $\mathcal{P}(A) + \mathcal{P}(B) \subseteq \mathcal{P}(A + B)$. The question on the fullness of the above functor can be reformulated in the following form. Let $\varphi : \mathcal{P}(Y) \to \mathcal{P}(X)$ be a morphism of rings. Then it is natural to ask, does there exist a function $f : Y \to X$ such that $\varphi = \mathcal{P}(f)$? The answer is affirmative precisely when $Y$ is finite, see Theorem 5.11.

**Theorem 5.11.** Let $X$ and $Y$ be two sets with $Y$ non-empty. Then $X$ is a finite set if and only if every morphism of rings $\mathcal{P}(X) \to \mathcal{P}(Y)$ is of the form $\mathcal{P}(f)$ for some function $f : Y \to X$. 
**Proof.** Let \( X \) be a finite set. If \( y \in Y \) there exists a unique point \( x \in X \) such that \( \varphi^{-1}(m_y) = m_x \) because \( \bigcap_{x \in X} m_x = 0 \). Then the map \( f : Y \to X \) defined by \( y \mapsto x \) is the desired function. Conversely, if \( X \) is an infinite set then \( \text{Fin}(X) \) is a proper ideal of \( \mathcal{P}(X) \). Thus there exists a maximal ideal \( M \) of \( \mathcal{P}(X) \) such that \( \text{Fin}(X) \subseteq M \). Then the map \( \varphi : \mathcal{P}(X) \to \mathcal{P}(Y) \) which sends each \( A \in \mathcal{P}(X) \) into 0 or 1, according as \( A \in M \) or \( A \notin M \), is a morphism of rings, see Proposition 5.9. Thus by the hypothesis, there exists a function \( f : Y \to X \) such that \( \varphi = \mathcal{P}(f) \). But the image of \( f \) is nonempty, hence we may choose some point \( x \) in it. We have \( \{x\} \in M \) and so \( \varphi(\{x\}) = 0 \). But \( \varphi(\{x\}) = f^{-1}(\{x\}) \neq \emptyset \). This is a contradiction and we win. \( \square \)

Theorem 5.11 also leads us to the following general conclusion.

**Corollary 5.12.** Let \( X \) and \( Y \) be two sets. Then the image of the induced map:

\[
\text{Mor}_{\text{Set}}(X, Y) \to \text{Mor}_{\text{Ring}}(\mathcal{P}(Y), \mathcal{P}(X))
\]

\[
f \mapsto \mathcal{P}(f)
\]

is consisting of all morphisms of rings \( \varphi : \mathcal{P}(Y) \to \mathcal{P}(X) \) such that \( \text{Fin}(Y) + \varphi^{-1}(m_x) = \mathcal{P}(Y) \) for all \( x \in X \).

**Proof.** First we need to show that if \( f : X \to Y \) is a function then the induced ring map \( \varphi := \mathcal{P}(f) \) actually satisfies in the above condition. Suppose there is some \( x \in X \) such that \( \text{Fin}(Y) \subseteq \varphi^{-1}(m_x) \). This in particular yields that \( f^{-1}(\{y\}) \subseteq X \setminus \{x\} \) which is impossible where \( y := f(x) \). Conversely, let \( \varphi : \mathcal{P}(Y) \to \mathcal{P}(X) \) be any ring map which satisfies in the above condition. For each \( x \in X \) then by Theorem 5.13 there exists a unique point \( y \in Y \) such that \( \varphi^{-1}(m_x) = m_y \). Now it is easy to see that \( \varphi \) is induced by the function \( f : X \to Y \) which is defined as \( x \sim y \). \( \square \)

**Theorem 5.13.** Let \( X \) be an infinite set and let \( M \) be a maximal ideal of \( \mathcal{P}(X) \). Then the following are equivalent.

(i) \( \text{Fin}(X) \subseteq M \).

(ii) \( M \neq m_x \) for all \( x \in X \).

(iii) \( M \) is not a finitely generated ideal.

**Proof.** (i) \( \Rightarrow \) (ii): If \( M = m_x = \mathcal{P}(X \setminus \{x\}) \) for some \( x \in X \) then \( \{x\} \notin M \), which is a contradiction.
(ii) ⇒ (iii) : If $M$ is a finitely generated then it is a principal ideal, say $M = (A) = \mathcal{P}(A)$. But $A$ is a proper subset of $X$, hence we may choose some $x \in X \setminus A$. Then clearly $M \subseteq m_x$ and so $M = m_x$. This is a contradiction.

(iii) ⇒ (i) : It will be enough to show that $\{x\} \in M$ for all $x \in X$. Suppose there is some $x \in X$ such that $\{x\} \notin M$. If $A \subseteq X \setminus \{x\}$ then $A\{x\} = 0$ and so $A \in M$. This yields that $M = m_x$ is a principal ideal. But this is a contradiction and we win. \( \square \)

Theorem 5.13 tells us that the maximal ideals of $\mathcal{P}(X)$ are either infinitely generated or the principal ideals of the form $m_x$.

Let $X$ be a set and let $M$ be a maximal ideal of $\mathcal{P}(X)$. Then by Theorem 5.13 $M + \text{Fin}(X) = \mathcal{P}(X)$ if and only if $M = m_x$ for some $x \in X$.

**Proposition 5.14.** If $X$ is a set then the map $\text{Mor}_{\text{Ring}}(R, \mathbb{Z}_2) \to \text{Spec}(R)$ given by $\varphi \mapsto \varphi^{-1}(0)$ is a bijection where $R = \mathcal{P}(X)$. In particular, if $X$ is finite then $\text{Mor}_{\text{Ring}}(\mathcal{P}(X), \mathbb{Z}_2) \simeq X$.

**Proof.** It is clearly injective. If $M$ is a maximal ideal of $\mathcal{P}(X)$ then the ring $R = \mathcal{P}(X)/M$ is isomorphic to $\mathbb{Z}_2$ because it is easy to see that every Boolean local ring is isomorphic to $\mathbb{Z}_2$. Therefore $M = \pi^{-1}(0)$ where $\pi : \mathcal{P}(X) \to R$ is the canonical ring map. \( \square \)

**Theorem 5.15.** If $X$ is an infinite set then every maximal ideal of the ring $\mathcal{P}(X)$ modulo $\text{Fin}(X)$ is infinitely generated.

**Proof.** Let $M$ be a maximal ideal of $\mathcal{P}(X)$ such that $\text{Fin}(X) \subseteq M$. If $M/\text{Fin}(X)$ is finitely generated then it is a principal ideal. Thus there exists some $A \in \mathcal{P}(X)$ such that $M = \text{Fin}(X) + \mathcal{P}(A)$. But $A^c = X \setminus A$ is an infinite set because if it is finite then $1 = A^c + A \in M$, which is impossible. Thus we may choose two infinite and disjoint subsets $B$ and $C$ from the infinite set $A^c$. Therefore $BC = 0 \in M$ but non of them are in $M$, since if for example $B \in M$ then there exist a finite subset $F \subseteq X$ and some $A' \in \mathcal{P}(A)$ such that $B = F + A'$, but $B \cap A' = \emptyset$ and so $B \subseteq F$ which is impossible because $B$ is infinite. This is a contradiction and we win. \( \square \)

If $X$ is an infinite set then by Theorem 5.13 the ring $\mathcal{P}(X)$ modulo $\text{Fin}(X)$ is not a noetherian ring. In particular, $\text{Fin}(X)$ is not a maximal...
ideal of $\mathcal{P}(X)$.

6. Contribution to fixed-point theorems

The following result provides a general criterion guaranteeing that, if it is satisfied, then a function yields a fixed-point.

**Theorem 6.1.** A function $f : X \to X$ has a fixed-point if and only if the induced map $\mathcal{P}(f)^* : \text{Spec } \mathcal{P}(X) \to \text{Spec } \mathcal{P}(X)$ has a fixed-point.

**Proof.** If $f(x) = x$ for some $x \in X$ then $\varphi(m_x) = m_x$ where $\varphi = \mathcal{P}(f)^*$. Conversely, assume that $\varphi(M) = M$ for some $M \in \text{Spec } \mathcal{P}(X)$. If a function $f : X \to X$ does not have any fixed-point, then it is well known that there exists a function $g : X \to \{1, 2, 3\}$ such that $g(x) \neq g(f(x))$ for all $x \in X$. (The existence of such function is the key point of our proof, so we sketch its proof. First, Zorn’s lemma gives us a maximal function $g : A \to \{1, 2, 3\}$ satisfying in the above condition where $A \in \mathcal{P}(X)$ with $f(A) \subseteq A$. Then we show that $A = X$. If $x \in X \setminus A$, then by the induction we may define a function $h : A \cup \{x, f(x), f^2(x), \ldots\} \to \{1, 2, 3\}$ extending $g$ and still satisfying in the above condition as follows, for each natural number $n \geq 0$ we choose $h(f^n(x))$ to be a value in $\{1, 2, 3\}$ in a way that $h(f^n(x)) \neq h(f^{n-1}(x))$ and $h(f^n(x)) \neq h(f^{n+1}(x))$ if these values have already been defined. Finally, we reach to a contradiction with the maximality of $g$, hence $A = X$). If we setting $C_k := g^{-1}(\{k\})$ then it is obvious that $C_k \cap f(C_k) = \emptyset$ for $k = 1, 2, 3$. But $1 \notin M$ and so $C_k \notin M$ for some $k$. It follows that $f(C_k) \in M$. We have then $C_k \subseteq f^{-1}(f(C_k)) \in M$. Thus $C_k \in M$. But this is a contradiction and we win. □

As an application of Theorem 6.1, we obtain the following technical result.

**Theorem 6.2.** Let $f : X \to X$ be a function, $\varphi = \mathcal{P}(f)^*$ and $M \in \text{Spec } \mathcal{P}(X)$. Then $\varphi(M) = M$ if and only if $\{x \in X : f(x) \neq x\} \in M$.

**Proof.** Let $A = \{x \in X : f(x) \neq x\}$. If $A \in M$ then to prove the assertion it suffices to show that $M \subseteq \varphi(M)$, because $M$ is a maximal ideal. Suppose there exists some $B \in M$ such that $B \notin \varphi(M)$.
It follows that \( f^{-1}(B) \notin M \). Clearly \( f^{-1}(B) \cap A^c \subseteq B \). Therefore \( f^{-1}(B).A^c \in M \). But this is a contradiction. Conversely, assume that \( \varphi(M) = M \). If \( A \notin M \) then \( A \) is non-empty, hence we may choose a point \( p \) in it. Then consider the function \( g : X \to X \) which sends each \( x \in X \) into \( f(x) \) or \( p \), according as \( x \in A \) or \( x \in A^c \). Therefore the induced map \( \psi = \mathcal{P}(g)^* \) and \( \varphi \) agree on \( \eta(A) \) where \( \eta : X \to \text{Spec } \mathcal{P}(X) \) is the canonical map. But \( M \in \eta(A) \), because if \( M \in D(B) \) for some \( B \in \mathcal{P}(X) \) then \( A \cap B \notin M \) and hence \( A \cap B \) is nonempty, thus we may choose some \( x \in A \cap B \), this yields that \( m_x \in D(B) \cap \eta(A) \). So \( \psi(M) = M \). Thus by Theorem 6.1 \( g \) has a fixed-point. But this is a contradiction since \( g \) has no fixed-point. \( \square \)

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