1. Summary

Let \( \mathcal{P}(\{\}) \) be the linear system of degree \( d \) curves in \( \mathbb{P}^2 \). Then \( \mathcal{P}(\{\}) \) is a projective space of dimension \( \binom{d+2}{2} - 1 \). The Severi variety \( \mathcal{V}(\delta, \{\}) \subset \mathcal{P}(\{\}) \) is the subset corresponding to reduced, nodal curves with \( \delta \) nodes. The “well-known” problem, whether \( \mathcal{V}(\delta, \{\}) \) is irreducible or not, is solved affirmatively by Harris [H], and Ran [R1]. The next question about the Severi variety is to find its degree, \( N(\delta, d) \). In the paper [R2], Ran gave the recursive formulae. In this paper, we will give closed-form formulae for cogenus 3 and 4 cases using his method. These formulae coincide with those of I. Vainsencher [V], and for cogenus 3 case, that of J. Harris and R. Pandharipande [H-P]. Ran gave the formula of cogenus 2 case using his method in the paper [R3], namely,

\[
N(2, d) = \frac{3}{2}(d-1)(d-2)(3d^2 - 3d - 11)
\]

Therefore, this paper is an extension of that paper. One small benefit of our approach is that we can calculate, as a special case, the formulae of degree of the locus of all curves having less than three nodes and some tangency conditions to fixed line.

Another result of this paper is that we calculate the degree of the polynomial \( N(\pi, \delta, d) \) in \( d \), which is the degree of the locus of curves \( C \) in \( \mathbb{P}^2 \) with degree \( d \) and having \( \delta \) nodes and \( C \cap L \) is of type \( \pi \) to fixed line \( L \). Using this result, we also calculate the coefficients of two leading terms of Severi polynomial, \( N(\delta, d) \). The results are: for any \( \delta \) and any partition \( \pi = [\ell_1, \ldots, \ell_n] \),

\[
\deg N(\pi, \delta, d) = 2\delta + \sum_{i=2}^{n} \ell_i \quad \text{(Prop. 4.2)}
\]

\[
a_{2\delta}^{\delta} = \frac{3^\delta}{\delta!} \quad \text{(Cor. 4.3)}
\]

\[
a_{2\delta-1}^{\delta} = -2\delta a_{2\delta}^{\delta} = \frac{-2 \times 3^\delta}{(\delta-1)!} \quad \text{(Cor. 4.5)}
\]
where \( a_\beta^\delta \) is the coefficient of the degree \( \beta \) term of the polynomial \( N(\delta, d) \). So,

\[
N(\delta, d) = \frac{3^\delta}{\delta!}d^{2\delta} - \frac{2 \times 3^\delta}{(\delta - 1)!}d^{2\delta - 1} + \text{lower degree terms.}
\]

For \( \delta \leq 6 \), these two coefficients coincide with those of the polynomials of I. Vainsencher \([V]\).

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2. Set of divisors of \( P^1 \)

In this section, we calculate the degree of set of divisors of \( P^1 \) because we need the degree of the locus of smooth curves with tangency conditions to fixed line.

Definition 1. A partition \( \pi = [\ell_1, ..., \ell_n] \) is a sequence of nonnegative integers.

Definition 2. A divisor \( D \) of \( P^1 \) is of type \( \pi \) if it has the form \( D = \sum_{i=1}^n \sum_{j=1}^{\ell_i} iP_{ij} \) for some distinct points \( P_{ij} \).

Remark 2.1. 1. Degree of a divisor \( D \) of type \( \pi \) is \( \sum_{i=1}^n i\ell_i \).

2. Let \( \Gamma_\pi \) be the closure of the set of all divisors in \( P^1 \) of type \( \pi \). If we identify the space of all divisors of degree \( N \) on \( P^1 \) with \( P^N \), then \( \Gamma_\pi \) is a subvariety of \( P^N \), where \( N = \sum_{i=1}^n i\ell_i \).

3. \( \dim \Gamma_\pi = \sum_{i=1}^n \ell_i \).

4. \( m(\pi) \) def = \( \prod_{i=1}^n \ell_i! \).

5. \( n(\pi) \) def = \( \frac{(\sum_{i=1}^n \ell_i)!}{\ell_1! \cdots \ell_n!} \).

Lemma 2.2. \( \deg \Gamma_\pi = m(\pi)n(\pi) \)

Proof  Let \( Q \in P^1 \), and \( N = \sum_{i=1}^n i\ell_i \). Let \( H_Q = \{ D \mid D \ni Q, \deg D = N \} \). Then \( H_Q \) is a hyperplane in \( P^N \). Then \( \Gamma_\pi \cap H_Q = \sum_{i=1}^n i\Gamma_{\pi,i} \), where \( \Gamma_{\pi,i} = \{ D \in \Gamma_\pi \mid D \ni iQ, D \not\ni (i + 1)Q \} \). Therefore, \( \Gamma_\pi \cap H_{Q_1} \cap \cdots \cap H_{Q_L} \), where \( L = \sum \ell_i \), has \( n(\pi) \) points (set theoretically) and each point has multiplicity \( m(\pi) \). Therefore, \( \deg \Gamma_\pi = n(\pi)m(\pi) \).

Corollary 2.3. Assume that \( L \) is a line in \( P^2 \). Fix a partition \( \pi = [a_1, ..., a_n] \). Let \( S(\pi) \) be the (locally closed) set of all smooth curves \( C \) of degree \( d \) in \( P^2 \) such that \( C \cap L \) is of type \( \pi \). Then \( \deg S(\pi) = n(\pi)m(\pi) \).
3. Recursion method

In this section, we want to describe the method of calculating the degree of $V(\delta, \lceil d \rceil)$. You can also consult the paper [R2] or [R3].

3.1. Degeneration of $\mathbb{P}^2$. Let $S$ be the blow up of $\mathbb{C} \times \mathbb{P}^2$ at a point $(0, p)$, and let $\pi' : S \to \mathbb{C} \times \mathbb{P}^2$ be the blowdown map and $\pi = \rho_2 \circ \pi' : S \to \mathbb{C}$ be the composition map. Then

$$\pi^{-1}(t) = \begin{cases} \mathbb{P}^2 & \text{if } t \neq 0 \\ \mathbb{P}^2 \cup \tilde{\mathbb{P}}^2 & \text{if } t = 0, \end{cases}$$

where $\tilde{\mathbb{P}}^2$ is the blow up of $\mathbb{P}^2$ at $p$. In the case $\pi^{-1}(0) = \mathbb{P}^2 \cup \tilde{\mathbb{P}}^2$, $E \overset{\text{def}}{=} \mathbb{P}^2 \cap \tilde{\mathbb{P}}^2$ called the axis is a line in $\mathbb{P}^2$, and an exceptional divisor in $\tilde{\mathbb{P}}^2$. This gives the degeneration of $\mathbb{P}^2$ to $\mathbb{P}^2 \cup \tilde{\mathbb{P}}^2$.

3.2. Degeneration of curve in $\mathbb{P}^2$. As $\mathbb{P}^2$ is degenerated to $\mathbb{P}^2 \cup \tilde{\mathbb{P}}^2$, a curve in $\mathbb{P}^2$ also degenerates to $C_1 \cup C_2$, where $C_1 \subset \mathbb{P}^2$ and $C_2 \subset \tilde{\mathbb{P}}^2$. The point is that the degenerated curve $C_1$ has smaller degree, therefore we can apply recursion and the other part $C_2$ has simple configuration, so it’s easy to count its dimension, degree etc. Clearly, $N(\delta, d) \overset{\text{def}}{=} \deg V(\delta, \lceil d \rceil)$ is the same as the number of plane curves with $\delta$ nodes and through $\left(\binom{d+2}{2} - 1 - \delta\right)$ generally given points. Let’s put $\left(\binom{d+1}{2} - 1 - \delta\right)$ points into $\mathbb{P}^2$ and $(d + 1)$ points into $\tilde{\mathbb{P}}^2$, i.e,

$$\begin{array}{c}
\begin{array}{ccc}
\mathbb{P}^2 & \sim \longrightarrow & \mathbb{P}^2 \cup \tilde{\mathbb{P}}^2 \\
\cup & \cup & \cup \\
C & \sim \longrightarrow & C_1 \cup C_2 \\
\cup & \cup & \cup \\
\left(\binom{d+2}{2} - 1 - \delta\right) \text{ pts} & \sim \longrightarrow & \left(\binom{d+1}{2} - 1 - \delta\right) \text{ pts} \quad (d + 1) \text{ pts},
\end{array}
\end{array}$$

and think of all possible kinds of degeneration of $\delta$ nodes. Then, $\deg C_1 = d - 1$, and $C_2$ is a smooth rational curve plus several rulings (possibly multiple) [R3]. After all, the number of all possible curves in $\mathbb{P}^2 \cup \tilde{\mathbb{P}}^2$ satisfying given conditions is $N(\delta, d)$.

3.3. Cogenus 3 case. (Table 1.) We fix the degree and the number of nodes to explain more clearly. The way to get a general formula is exactly the same, but much more computation. Let $d = 5$ (degree 5 curves) and $\delta = 3$ (3 nodes). There is a technical reason to assume
that \( d > \delta \). It makes computation easy. Then we have 11 points on \( \mathbb{P}^2 \) and 6 points on \( \mathbb{P}^2 \). The formula of \([V]\) gives \( N(3, 5) = 7915 \).

(1) Case A.

This case is easy. It’s just \( N(3, 4) = \deg \overline{V(\exists, \Delta)} = 675 \). Generally, the degree of this locus is \( N_{3,d-1} \)

(2) Case B.

\( C_1 \cap E = C_2 \cap E = [4] \). This means that \( C_1 \) and \( E \) meet in 4 distinct points. \( \deg C_1 = 4 \), but the condition of \( C_1 \) has 11 points + 2 nodes = 13 conditions, so one condition is needed. One of the four points on \( C_1 \cap E \) gives this condition. So the number of all curves through 12 points (given 11 points + one axis point) and having two nodes is \( N(2, 4) \) (generally, \( N(2, d-1) \)). And the others on \( E \cap C_1 \) give three conditions on \( C_2 \). \( C_2 \) has one node, so has one ruling. So the number of ways of choosing a ruling is \( 6 + 3 \) (generally \( (d+1) + (d-2) \)). So degree of Case B is \( N(2, d-1) \times (2d-1) \).

(3) Case C and Case D.

The method is similar as that of B.

| Case | M | \( N(C_1) \) | \( N(C_2) \) | \( \pi \) on \( E \) | \( d=5 \) | General Case |
|------|---|-------------|-------------|----------------|------------|--------------|
| A    | 1 | 3           | 0           | \([d-1]\)       | 675        | \( 1 \times N(3, d-1) \times 1 \) |
| B    | 1 | 2           | 1           | \([d-1]\)       | 2025       | \( 1 \times N(2, d-1) \times (2d-1) \) |
| C    | 1 | 1           | 2           | \([d-1]\)       | 756        | \( 1 \times N(1, d-1) \times (2d^2 - 5d + 3) \) |
| D    | 1 | 0           | 3           | \([d-1]\)       | 35         | \( 1 \times 1 \times (4d^3 - 24d^2 + 47d - 30)/3 \) |
| E    | 2 | 2           | 0           | \([d-3,1]\)     | 2020       | \( 2 \times N([d-3,1], 2, d-1) \times 1 \) |
| F    | 2 | 1           | 1           | \([d-3,1]\)     | 1316       | \( 2 \times (6d^3 - 42d^2 + 90d - 56) \times (2d-3) \) |
| G    | 2 | 0           | 2           | \([d-3,1]\)     | 24         | \( 2 \times 1 \times (4d^2 - 24d + 32) \) |
| H    | 3 | 1           | 0           | \([d-4,0,1]\)   | 405        | \( 3 \times N([d-4,0,1], 1, d-1) \times 1 \) |
| I    | 3 | 0           | 1           | \([d-4,0,1]\)   | 54         | \( 3 \times 3(d-4) \times (2d-4) \) |
| J    | 4 | 1           | 0           | \([d-5,2]\)     | 320        | \( 4 \times N([d-5,2], 1, d-1) \times 1 \) |
| K    | 4 | 0           | 1           | \([d-5,2]\)     | 0          | \( 4 \times 2(d-4)(d-5) \times (2d-5) \) |
| L    | 4 | 0           | 0           | \([d-5,0,0,1]\) | 16         | \( 4 \times 4(d-4) \times 1 \) |
| M    | 6 | 0           | 0           | \([d-6,1,1]\)   | 0          | \( 6 \times 6(d-4)(d-5) \times 1 \) |
| N    | 8 | 0           | 0           | \([d-7,3]\)     | 0          | \( 8 \times 4(d-4)(d-5)(d-6) \times 1 \) |
| Sum  |    |             |             | \([d-7,3]\)     | 7915       | *7           |

Table 1. (Calculation of \( N(3,5) \), and General Case \( N(3,d) \))
*1. M means multiplicity.
*2. N(C_1) means the number of nodes of C_1.
*3. N(C_2) means the number of nodes of C_2.
*4. N(\delta, d) = \text{deg } \mathcal{V}(\delta, |).
*5. N([\pi], \delta, d) = \text{degree of the closure of the set of all curves } C \text{ of degree } d \text{ having } \delta \text{ nodes and } C \cap L \text{ is of type } \pi \text{ to fixed line } L.
*6. In General Case, M \times A \times B means that M is the multiplicity, A is the degree of the locus of curves } C_1 \text{ in } \mathbb{P}^2 \text{ and B is the degree of the locus of curves } C_2 \text{ in } \mathbb{P}^2.
*7. N(3,d) = \frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + 423/2d^3 - 229d^2 - 829/2d + 525.
(4) Case E.

Since \( C_2 \) is nonsingular, each point on \( C_1 \cap E = \pi = [2,1] \) (generally, \([d-3,1]\)) gives a condition on \( C_2 \), therefore the degree of curves \( C_2 \) is just one. \( C_1 \) is the curve with two nodes and one tangency to \( E \). So \( C_1 \) has exactly 14 conditions (11 points + 2 nodes + 1 tangency). Therefore, the degree of Case E is the same as the degree of the locus of all curves with two nodes and one tangency condition to fixed line. The method of calculating this degree is similar to that for the Severi variety \( V(\varepsilon, \triangle) \), except when we degenerate \( P^2 \) to \( P^2 \cup \tilde{P}^2 \), we make the tangency on \( C_1 \) go to \( \tilde{P}^2 \), i.e,

\[
\begin{align*}
P^2 & \sim \rightarrow \quad P^2 \cup \tilde{P}^2 \\
C_1 & \sim \rightarrow \quad C_{11} \cup \quad C_{12} \\
11 \text{ pts} & \sim \rightarrow \quad 7 \text{ pts} \quad 4 \text{ pts} + 1 \text{ tangency to } E.
\end{align*}
\]

Table 2 is the table of Case E. The interesting subcategory is Case H1. This case doesn’t happen during calculation of the degree of the Severi variety \( V(\varepsilon, \lceil - \infty \rceil) \). This case occurs when the curve \( C_1 \) degenerates to \( C_{12} \) which contains a double ruling.

| Case | \( M \) | \( N(C_1) \) | \( N(C_2) \) | \( \pi \) on \( E \) | \( d=5 \) | General Case |
|------|------|------|------|------|------|-------------|
| A1   | 1    | 2    | 0    | \([d-2]\) | 126  | \( 1 \times N(2, d-2) \times 1 \times 2(d-2) \) |
| B1   | 1    | 1    | 1    | \([d-2]\) | 280  | \( 1 \times N(1, d-2) \times (2d-4) \times 2(d-3) \) |
| C1   | 1    | 0    | 2    | \([d-2]\) | 20   | \( 1 \times 1 \times (2d^2 - 11d + 15) \times 2(d-4) \) |
| D1   | 2    | 1    | 0    | \([d-4,1]\) | 432  | \( 2 \times N([d-4,1], 1, d-2) \times 1 \times 2(d-2) \) |
| E1   | 2    | 0    | 1    | \([d-4,1]\) | 64   | \( 2 \times 2(d-4) \times (2d-6) \times 2(d-3) \) |
| F1   | 3    | 0    | 0    | \([d-5,0,1]\) | 54   | \( 3 \times 3(d-4) \times 1 \times 2(d-2) \) |
| G1   | 4    | 0    | 0    | \([d-6,2]\) | 0    | \( 4 \times 2(d-4)(d-5) \times 1 \times 2(d-2) \) |
| H1   | 1    | 0    | 2    | \([d-4,2]\) | 10   | \( 2(d-1) + 2(d-4) \) |
| Sum  |      |      |      |       | 1010 | \( 9d^5 - 90d^4 + 300d^3 - 327d^2 - 76d + 190 \) |

(5) Case F.

Since \( C_1 \) has 13 conditions (11 points + 1 node + 1 tangent condition), one more condition is needed. The point on \( E \cap C_1 \) gives one more condition. Since the divisor \( E \cap C_1 \) has one multiple point, we have to divide two cases to fix one point of that divisor.

(a) We fix an ordinary point.

In this case, the condition of \( C_1 \) has 14 conditions. As in Case E, we do the whole thing again, we get the formula
\[6(d - 3)^3 + 8(d - 3)^2 + 2(d - 3) + 2(d - 4)(2d - 5) = (6d^3 - 42d^2 + 90d - 56).\] The number of ways of choosing a ruling is \((2d - 3)\), for the \((d + 1)\) points in \(\mathbb{P}^2\) plus the \((d - 4)\) points on axis \(C_1 \cap E\). (We can’t choose the fixed point on \(E\) which gives the condition on \(C_1\) and tangent point.) So in this case, the degree is \((6d^3 - 42d^2 + 90d - 56) \times (2d - 3)\).

(b) We fix the tangent point.

In this case, \(C_1\) has 14 conditions. (11 points + 1 node + 1 tangent condition at fixed point.) As in Case E, we do the whole thing again, we get \(3(d - 3)^2 + 4(d - 3) + (2d - 5) = (3d^2 - 12 + 10)\). Since the tangent point is fixed on \(C_1\), this point is not fixed on \(C_2\). This gives the degree of the locus of rational smooth curves of \(C_2\). This is two since this is the discriminant of the partition \([0,1]\]. And the number of ways of choosing a ruling is just \((d - 3)\), for the \((d - 3)\) points on \(C_1 \cap E\) (as in previous subcase, we can’t choose a tangent point. Also, we can’t choose given inside points because there are no ordinary fixed points on \(C_1 \cap E\)). So in this case, the degree is \((3d^2 - 12d + 10) \times 2(d - 3)\).

(6) Case G.

Since \(C_1\) has 12 conditions (11 points + 1 tangency on \(E\), two conditions are needed. Two points on \(C_1 \cap E\) give these conditions. But, as in Case F, the tangency condition gives us two ways to fix two points.

(a) We fix one ordinary point and one tangent point.

Since \(C_1\) has full conditions, such a curve exists and is unique. The counting of degree of the locus of curves \(C_2\) is a little tricky. If we choose both of the two rulings in \((d - 4)\) points on the axis (as in Case F, we can’t choose the fixed ordinary point and tangent point.), then the degree of the locus of rational smooth curves of \(C_2\) is 4 since this is the discriminant of the partition \([1,1]\], and if we choose one ruling in axis point and one ruling in \((d + 1)\) points in \(\mathbb{P}^2\), then the degree of locus of rational smooth curves is two (discriminant of \([0,1]\)). So in this case, the degree is \(4 \times \left(\binom{d-4}{2}\right) + 2(d + 1)(d - 4)\).

(b) We fix two ordinary points.

The degree of the locus of curves \(C_1\) is \(2(d - 4)\) (discriminant of \([d - 5, 1]\), and the number of ways of choosing a ruling is \(\binom{d-5}{2} + (d - 5)(d + 1) + \binom{d-5}{2}\). So in this case, the degree is \(2(d - 4) \times (2d^2 - 9d + 10)\).

(7) Case I, J, and K.
The method to calculate these cases is similar as those of Case E, F, and G.

(8) Case L.

$C_2$ has no node, so all points on $E \cap C_1$ give conditions on $C_2$, so just one such curve exists. The degree of the locus of curves $C_1$ is just the discriminant of the divisor of $C_1 \cap E$. By Corollary 2.3, this is $4(d - 4)$.

(9) Case M and N. The method is similar as that of Case L.

3.4. Cogenus 4. The way to calculate in the case of cogenus 4 is exactly the same as that of cogenus 3. There are just more cases, so the only thing to do is to be careful not to miss any. Case F involves calculating the degree of the variety of all curves with three nodes and one tangency to fixed line. This counting method is the same as the case E of cogenus 3. Table 3 and 4 are the calculations of low degrees. Below each Table, we give the general formula of the degree of these cases. Compared with Table 1, Table 4 has new cases (Cases O1, P1, and Q1). These Cases correspond to the cases having double rulings in $\mathbb{P}^2$. 
Table 3. (Calculation of N(4,5), and N(4,6))

| Case | Multiplicity | Nodes of $C_1$ | Nodes of $C_2$ | $\pi$ on E | $d = 5$ | $d = 6$ |
|------|--------------|----------------|----------------|-----------|---------|---------|
| A    | 1            | 4              | 0              | [d-1]     | 666     | 36975   |
| B    | 1            | 3              | 1              | [d-1]     | 6075    | 87065   |
| C    | 1            | 2              | 2              | [d-1]     | 6300    | 39690   |
| D    | 1            | 1              | 3              | [d-1]     | 945     | 4032    |
| E    | 1            | 0              | 4              | [d-1]     | 15      | 70      |
| F    | 2            | 3              | 0              | [d-3,1]   | 4728    | 99160   |
| G    | 2            | 2              | 1              | [d-3,1]   | 10440   | 90708   |
| H    | 2            | 1              | 2              | [d-3,1]   | 1920    | 12800   |
| I    | 2            | 0              | 3              | [d-3,1]   | 0       | 224     |
| J    | 3            | 2              | 0              | [d-4,0,1] | 2520    | 19170   |
| K    | 3            | 1              | 1              | [d-4,0,1] | 1350    | 7002    |
| L    | 3            | 0              | 2              | [d-4,0,1] | 0       | 252     |
| M    | 4            | 2              | 0              | [d-5,2]   | 1696    | 30240   |
| N    | 4            | 1              | 1              | [d-5,2]   | 0       | 5824    |
| O    | 4            | 0              | 2              | [d-5,2]   | 0       | 0       |
| P    | 4            | 1              | 0              | [d-5,0,0,1]| 320    | 1344    |
| Q    | 4            | 0              | 1              | [d-5,0,0,1]| 0      | 128     |
| R    | 5            | 0              | 0              | [d-6,0,0,0,1]| 0    | 25      |
| S    | 6            | 1              | 0              | [d-6,1,1] | 0       | 0       |
| T    | 6            | 0              | 1              | [d-6,1,1] | 0       | 0       |
| U    | 8            | 1              | 0              | [d-7,3]   | 0       | 0       |
| V    | 8            | 0              | 1              | [d-7,3]   | 0       | 0       |
| W    | 8            | 0              | 0              | [d-7,1,0,1]| 0      | 0       |
| X    | 9            | 0              | 0              | [d-7,0,2] | 0       | 0       |
| Y    | 12           | 0              | 0              | [d-8,2,1] | 0       | 0       |
| Z    | 16           | 0              | 0              | [d-9,4]   | 0       | 0       |
| Sum  |              |                |                |           | 36975   | 437517  |

The general formula for $d \geq 5$ is:

$$N(4,d) = \frac{27}{8}d^8 - 27d^7 + 1809/4d^6 - 642d^5 - 2529d^4 + 37881/8d^3 + 18057/4d^2 - 8865d + 88645$$
Table 4. (Calculation of $N([2,1],3,4)$, and $N([3,1],3,5)$)

| Case | Multiplicity | Nodes of $C_1$ | Nodes of $C_2$ | $\pi$ on $E$ | $d = 4$ | $d = 5$ |
|------|--------------|----------------|----------------|------------|---------|---------|
| A1   | 1            | 3              | 0              | [d-2]      | 90      | 5400    |
| B1   | 1            | 2              | 1              | [d-2]      | 504     | 10800   |
| C1   | 1            | 1              | 2              | [d-2]      | 240     | 2268    |
| D1   | 1            | 0              | 3              | [d-2]      | 0       | 40      |
| E1   | 2            | 2              | 0              | [d-4,1]    | 360     | 16160   |
| F1   | 2            | 1              | 1              | [d-4,1]    | 672     | 7968    |
| G1   | 2            | 0              | 2              | [d-4,1]    | 0       | 240     |
| H1   | 3            | 1              | 0              | [d-5,0,1]  | 378     | 3240    |
| I1   | 3            | 0              | 1              | [d-5,0,1]  | 0       | 324     |
| J1   | 4            | 1              | 0              | [d-6,2]    | 0       | 2560    |
| K1   | 4            | 0              | 1              | [d-6,2]    | 0       | 128     |
| L1   | 4            | 0              | 0              | [d-6,0,0,1]| 0       | 0       |
| M1   | 6            | 0              | 0              | [d-7,1,1]  | 0       | 0       |
| N1   | 8            | 0              | 0              | [d-8,3]    | 0       | 0       |
| O1   | 1            | 1              | 2              | [d-3,1]    | 96      | 344     |
| P1   | 1            | 0              | 3              | [d-3,1]    | 24      | 60      |
| Q1   | 2            | 0              | 2              | [d-5,2]    | 0       | 48      |
| Sum  |              |                |                |            | 2364    | 49580   |

Here, $N(\pi,3,d-1)$ is the degree of the locus of all plane curves $C$ of degree $d-1$ having 3 nodes and $C \cap L$ is of type $\pi$ to fixed line $L$.

The general formula for $d \geq 4$ is:

- $N([d-3,1],3,d-1) = 9d^7 - 126d^6 + 603d^5 - 891d^4 + 1118d^3 - 1116d^2 + 416d^3 + 2416$
- $N([d-5,2],2,d-1) = 9d^6 - 135d^5 + 750d^4 - 1785d^3 + 1249d^2 + 1188d - 1116$
- $N([d-4,0,1],2,d-1) = 27d^5 - 297d^2 + 549d^2 - 1341d^2 - 357d^2 + 495$
- $N([d-6,1,1],1,d-1) = 18d^6 - 234d^5 + 1038d^2 - 1734d^3 + 720$
- $N([d-5,0,0,1],1,d-1) = 12d^5 - 96d^2 + 220d - 120$
- $N([d-7,3],1,d-1) = 4d^5 - 76d^4 - 540d^3 - 1732d^2 + 2304d - 720$.

4. Degree of Severi polynomial

**Definition 3.** A Severi polynomial is a polynomial $N(\pi, \delta, d)$ which is the degree of the locus of curves $C$ in $\mathbb{P}^2$ with degree $d$ having $\delta$ nodes and $C \cap L$ is of type $\pi$ to fixed line $L$.

**Remark 4.1.** The fact that $N(\pi, \delta, d)$ is a polynomial in $d$ is obvious because of the recursion formula of Ran [R2].
Proposition 4.2. Fix $\delta$ and $\pi = [\ell_1, ..., \ell_n]$. Then
\[ \deg N(\pi, \delta, d) = 2\delta + \sum_{i=2}^{n} \ell_i. \]

Proof. We use induction on $d$. $N(d) = N(\pi, \delta, d)$ is the number of curves in $\mathbb{P}^2$ with degree $d$, having $\delta$ nodes, type $\pi$ conditions to fixed line and through given $n = \left(\binom{d+2}{2} - 1 - \delta - (\sum_{i=1}^{n} (i-1)\ell_i)\right)$ points.

As in the previous section, we degenerate $\mathbb{P}^2$ into $\mathbb{P}^2 \cup \tilde{\mathbb{P}}^2$, and put $n_1 = \left(\binom{d+1}{2} - 1 - \delta\right)$ points into $\mathbb{P}^2$ and $n_2 = (d + 1 - (\sum_{i=1}^{n} (i-1)\ell_i))$ points into $\tilde{\mathbb{P}}^2$, and degenerate the tangency conditions into $\tilde{\mathbb{P}}^2$, i.e,

\[
\begin{array}{ccc}
\mathbb{P}^2 & \sim \to & \mathbb{P}^2 \cup \tilde{\mathbb{P}}^2 \\
\cup & \cup & \cup \\
C & \sim \to & C_1 \cup C_2 \\
\cup & \cup & \cup \\
n \text{points} & \sim \to & n_1 \text{points} \quad n_2 \text{points} + \text{tangency conditions to} \ L.
\end{array}
\]

Then $N(\pi, \delta, d)$ is the sum of the Severi polynomials of all limit components. So, if we prove that the degree of each polynomial is less than or equal to $2\delta + \sum_{i=2}^{n} \ell_i$, we are done. Each polynomial of a limit component is the product of the polynomial of the upper part of that component (the set of the locus $C_1$) and the polynomial of the lower part (the set of the locus $C_2$). First, look at the degree of the polynomial of the lower part. This degree is the sum of the number of rulings, $\delta_2$, and the degree of the polynomial of the locus of the smooth curves with a divisor of type $\pi$ to fixed line, that is $\sum_{i=2}^{n} \ell_i$. Second, look at the degree of the polynomial of the upper part. $C_1$ has $\delta_1$ nodes and $C_1 \cap E$ is of type $\pi' = [\ell'_1, ..., \ell'_m]$ satisfying the equation
\[ \delta_1 + \sum_{i=1}^{m} (i-1)\ell'_i = \delta - \delta_2 \tag{2} \]

Since $C_1$ has degree $d-1$, by the induction, the polynomial of the upper part has degree at most $2\delta_1 + \sum_{i=2}^{m} \ell'_i$. So the degree of the polynomial of each component is less than or equal to $\delta_2 + \sum_{i=2}^{n} \ell_i + 2\delta_1 + \sum_{i=2}^{m} \ell'_i$. From the equation (2), we get
\[ \delta_2 + \sum_{i=2}^{n} \ell_i + 2\delta_1 + \sum_{i=2}^{m} \ell'_i \leq 2\delta + \sum_{i=2}^{n} \ell_i. \tag{3} \]

The equality holds just for the component which is the locus of curves such that $C_1$ has $\delta$ nodes and $C_2$ is a smooth rational curve.
Corollary 4.3. Let $a_{2\delta}^\delta (a_{2(\delta-1)}^{\delta-1})$ be the coefficient of the degree $2\delta(2(\delta - 1), \text{respectively})$ term of the Severi polynomial $N(\delta, d)$ ($N(\delta - 1, d)$, respectively). This is the leading coefficient of each polynomial by the Proposition 4.2. Then

$$a_{2\delta}^\delta \times 2\delta = a_{2(\delta-1)}^{\delta-1} \times 6, \text{ hence } a_{2\delta}^\delta = \frac{3^\delta}{\delta!}.$$  

Proof In the proof of Proposition 4.2, equality holds for just one limit component. The Severi polynomial of this component is $N(\delta, d - 1)$. Let’s find the limit components of degree one less than the degree of Severi polynomial. By equation (3), there exist exactly two such components. One such component, say $A_1$, consists of curves such that $C_1$ has $(\delta - 1)$ nodes and $C_2$ has one ruling, and another component, say $A_2$, consists of curves such that $C_1$ has $(\delta - 1)$ nodes, $C_1 \cap E$ is of type $[d - 3, 1]$ and $C_2$ is a smooth rational curve. In Table 1(Table 3), $A_1$ is the component of Case B (Case B, respectively) and $A_2$ is the component of Case E (Case F). The coefficient of the degree $(2\delta - 1)$ term of the polynomial of the component $A_1$ is $2a_{2(\delta-1)}^{\delta-1}$. The coefficient of the degree $(2\delta - 1)$ term of the polynomial of the component $A_2$ is $4a_{2(\delta-1)}^{\delta-1}$. So the sum is $6a_{2(\delta-1)}^{\delta-1}$. Therefore we get the following recursion formula

$$N(\delta, d) = N(\delta, d - 1) + 6a_{2(\delta-1)}^{\delta-1}d^{2\delta-1} + \text{lower degree terms}.$$ Integrating, we get $a_{2\delta}^\delta \times (2\delta) = 6 \times a_{2(\delta-1)}^{\delta-1}$. 

Remark 4.4. $N(7, d) = \frac{243}{560}d^4 + \text{lower degree terms}$. 

Corollary 4.5. Let $a_{2\delta-1}^\delta$ be the coefficient of the degree $(2\delta - 1)$ term of the polynomial $N(\delta, d)$. Then

$$a_{2\delta-1}^\delta = -(2\delta) \times a_{2\delta}^\delta, \text{ hence } a_{2\delta-1}^\delta = \frac{-2 \times 3^\delta}{(\delta - 1)!}.$$  

Proof The way to prove this corollary is the same as that of corollary 4.3. Let’s find the components with the polynomial of degree two less than the degree of Severi polynomial. By equation (3), there exist exactly three such components. First component, $B_1$, consists of curves such that $C_1$ has $(\delta - 2)$ nodes and $C_2$ has two rulings. Second component, $B_2$, consists of curves such that $C_1$ has $(\delta - 2)$ nodes, $C_1 \cap E$ is of type $[d - 3, 1]$ and $C_2$ has one ruling. Third one, $B_3$, consists of curves such that $C_1$ has $(\delta - 2)$ nodes, $C_1 \cap E$ is of type $[d - 5, 2]$ and $C_2$ is smooth. In Table 1(Table 3), $B_1$ is the component of Case C (Case C, 

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respectively) and $B_2$ is the component of Case F (Case G, respectively) and $B_3$ is the component of Case J (Case M, respectively). Calculation of the coefficients of the degree $(2\delta - 2)$ term of those polynomials give $2a_{2(\delta-2)}^\delta, 8a_{2(\delta-2)}^\delta, 8a_{2(\delta-2)}^\delta$ respectively. So the sum is $18a_{2(\delta-2)}^\delta$. The polynomial of the limit components $A_1$ and $A_2$ also have a term of this degree. The coefficient of this degree term of polynomial of $A_1$ is $-a_{2(\delta-1)}^\delta + 2(-2(\delta - 1)a_{2(\delta-1)}^\delta + a_{2(\delta-3)}^\delta)$. The coefficient of this degree term of polynomial of $A_2$ is $-8a_{2(\delta-1)}^\delta + 4(-4(\delta - 1)a_{2(\delta-1)}^\delta + a_{2(\delta-3)}^\delta) + 8a_{2(\delta-2)}^\delta + 16a_{2(\delta-2)}^\delta$. Using $a_{2\delta-3}^\delta = -2(\delta - 1)a_{2\delta-2}^\delta$ (by induction on $\delta$) and $a_{2\delta-1}^\delta \times 2(\delta - 1) = 6a_{2\delta-2}^\delta$ (by Corollary 4.3), we get that the coefficient of this degree term is $-9a_{2(\delta-1)}^\delta(2\delta - 1)$. So, we get the following formula,

\[ N(\delta, d) = N(\delta, d - 1) + 6a_{2(\delta-1)}^\delta d^{2\delta-1} - 9a_{2(\delta-1)}^\delta(\delta - 2)d^{2\delta-2} + \text{lower degree terms} \]

Integrating, we get

\[ N(\delta, d) = a_{2\delta}^\delta d^{2\delta} - (2\delta)a_{2\delta}^\delta d^{2\delta-1} + \text{lower degree terms}. \]

**Remark 4.6.** $N(7, d) = \frac{243}{560}d^{14} - \frac{243}{46}d^{13} + \text{lower degree terms}.$

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