On Multi-step BCFW Recursion Relations

Bo Feng\textsuperscript{ab}, Junjie Rao\textsuperscript{a}, Kang Zhou\textsuperscript{a}

\textsuperscript{a}Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, 310027, P. R. China
\textsuperscript{b}Center of Mathematical Science, Zhejiang University, Hangzhou, 310027, P. R. China

Abstract: In this paper, we extensively investigate the new algorithm known as the multi-step BCFW recursion relations. Many interesting mathematical properties are found and crucially, with these aspects understood, one can find a systematic process to complete the calculation after finite, definite steps and reach the correct answer, without recourse to any specific knowledge from field theories, except mass dimension and helicities. This process is constituted of the pole concentration and inconsistency elimination. Terms that survive inconsistency elimination cannot be determined by the new algorithm. They include polynomials and their generalizations, which are new mathematical objects to be explored. Afterwards, we apply this process to Standard Model plus gravity to illustrate both of its power and limitation. Having a systematic algorithm, we also tentatively discuss how to improve its efficiency by reducing its steps, which will be the next major topic.

Keywords: Amplitudes, Recursion Relation.
# Contents

1. Introduction  
2. Multi-step BCFW Recursion Relations  
   2.1 Extraction operators  
   2.2 Deformation generator algebra  
   2.3 Commutativity at integrand and integral levels  
3. Systematic Algorithm of Finite, Definite Steps  
   3.1 Pole concentration  
   3.2 Kinematic mass dimension  
   3.3 The master formula  
   3.4 Inconsistency elimination  
4. Applications in Standard Model plus Gravity  
   4.1 Two separated sectors  
   4.2 Simplified diagrammatic rules  
   4.3 Amplitudes and (pseudo) polynomials of $D = 0, 1, 2$  
5. Discussions  
   A. Independent Kinematic Variables  
   B. Examples of Pole Concentration  
      B.1 First systematic sequence  
      B.2 Second systematic sequence  
   C. (Pseudo) Polynomials and Saturated Fractions  
      C.1 Polynomials and pseudo polynomials  
      C.2 Saturated fractions
1. Introduction

In the past decade, the BCFW recursion relation [1, 2] had been a very efficient on-shell method to calculate tree-level scattering amplitudes. Pedagogical reviews on this topic can be found in [3, 4]. Still, it encounters certain difficulties when there exists no ‘good’ deformation [5, 6], i.e., the real amplitude does not vanish under the large $z$ limit, where $z$ is the deformation parameter. The recursion relation then fails to capture a residual part called the boundary term, which corresponds to the residue at infinity of the deformed amplitude.

Many related studies have been achieved including: introducing auxiliary fields to eliminate boundary terms [7, 8], analyzing Feynman diagrams to isolate boundary terms [9, 10, 11], expressing boundary terms in terms of roots of amplitudes [12, 13, 14], collecting factorization limits to interpolate boundary terms [15] and using other types of deformations [16].

Recently, a new algorithm named as the multi-step BCFW recursion relations [17] was established to tackle this problem universally and systematically. It considerably widens the category of quantum field theories of solvable tree amplitudes [18]. However, some common puzzles encountered in practice still lacks a formal study. One core question is: How to reach the correct answer within finite, definite steps, if an amplitude is solvable by the algorithm?

In this paper, we will explore multi-step BCFW recursion relations extensively, by investigating the algebra of BCFW deformation generators and commutativity of constant extractions. Next, we will seek for a universal approach to reach the answer and ensure that it is correct. This safety promise relies on very little knowledge of a particular QFT, except mass dimension and helicities, hence the algorithm is expected to be able to solve all massless tree amplitudes, with certain limitation addressed below.

It is well known that on-shell methods heavily rely on factorization properties of amplitudes, and the latter is a reflection of locality and unitarity. These properties are mathematically implemented on poles of amplitudes and their residues. For amplitudes that admit polynomials, no on-shell methods so far can fix this ambiguity. One can list all possible forms as basis, but to determine the coefficients will unfavorably call for more traditional means such as Feynman rules. In this work, we will clarify the applicable range of multi-step BCFW recursion relations, and explore all possible forms of polynomials and their generalized cousins called pseudo polynomials and saturated fractions. These two generalized objects can be fixed by other types of deformations, and they have many interesting mathematical aspects to be explored.

The paper is organized as follows. In section 2, we review the multi-step BCFW recursion relations and explore the commutativity of constant extractions. In section 3, we propose the systematic process to calculate amplitudes after finite, definite steps, and clarify its applicable range and limitation. In section 4, we apply this process to Standard Model plus gravity to demonstrate its workability.

2. Multi-step BCFW Recursion Relations

In this section, we briefly review the multi-step BCFW recursion relations, in the novel language of extraction operators. After that, the commutativity of constant extractions is investigated.
2.1 Extraction operators

For a general BCFW deformation \( \langle a_i | b_i \rangle \), namely

\[
\lambda_{a_i} \rightarrow \lambda_{a_i} - z_i \lambda_{b_i}, \quad \tilde{\lambda}_{b_i} \rightarrow \tilde{\lambda}_{b_i} + z_i \tilde{\lambda}_{a_i},
\]

let’s define two operations on an amplitude-like rational function \( R(\lambda_i, \tilde{\lambda}_i) \) via\(^1\)

\[
P_i[R] = -\sum_{\text{finite}} \int_{z_i} \frac{dz_i}{z_i} R(\lambda_{a_i} - z_i \lambda_{b_i}, \tilde{\lambda}_{b_i} + z_i \tilde{\lambda}_{a_i}),
\]

\[
C_i[R] = \int_{\infty} \frac{dz_i}{z_i} R(\lambda_{a_i} - z_i \lambda_{b_i}, \tilde{\lambda}_{b_i} + z_i \tilde{\lambda}_{a_i}),
\]

where \( P_i \) and \( C_i \) are the pole and constant ‘extraction operators’, which capture residues at finite locations except zero and infinity respectively. For a physical amplitude \( A \), \( P_i \) can capture only all or a part of its physical poles. But a general \( R \) such as \( P_i A \) or \( C_i A \) may also contain spurious poles, which is well known. Therefore the detectable poles at finite locations can be either physical or spurious.

By definition \( P_i + C_i = I \), where \( I \) is the identity operator. When we calculate an amplitude, starting by the 0th step, the amplitude is unknown, so is the \( C_0 \) operation. However, the \( P_0 \) operation represents exactly the BCFW recursion relation, hence we actually reconstruct this part by employing factorization properties, rather than manipulating the unknown amplitude. Conventionally, \( C_0 \) is called the boundary term with respect to \( P_0 \), which will be dissected into many parts to be determined. The dissection means, by expanding \( I \) for \((n + 1)\) times repeatedly, we have

\[
I = P_n + C_n P_{n-1} + \ldots + C_n C_{n-1} \cdots C_2 P_1 + C_n C_{n-1} \cdots C_2 C_1 P_0 + C_n C_{n-1} \cdots C_2 C_1 C_0,
\]

note that \( I \) acts on \( A \) implicitly. If the final boundary term \( C_n C_{n-1} \cdots C_2 C_1 C_0 \) vanishes, we have

\[
I = P_n + C_n P_{n-1} + \ldots + C_n C_{n-1} \cdots C_2 P_1 + C_n C_{n-1} \cdots C_2 C_1 P_0.
\]

This identity formally represents the ‘multi-step BCFW recursion relations’. Importantly, the workability of this multi-step approach relies on the existence of a series of deformations numbered by \( 0, 1, \ldots, n \) for which \( C_n C_{n-1} \cdots C_2 C_1 C_0 = 0 \), and the latter is the key condition we will mainly focus on.

The operators above have a general algebraic property, namely the projectivity:

\[
C_i C_i = C_i,
\]

To prove this, we first explicitly expand the deformed \( R \) as\(^2\)

\[
R(z_i) = \sum_k \frac{b_{0k} + b_{1k} z_i}{\left( a_{0k} + a_{1k} z_i + a_{2k} z_i^2 \right)^{d_k}} + c_0 + \sum_l c_l z_i^l,
\]

\(^1\)\( P_i \) and \( C_i \) used here are identical to \( P^2 \) and \( C^2 \) in the appendix of [17].

\(^2\)In practice, one can use the ‘Apart’ function in Mathematica to isolate pole and regular terms with respect to \( z \).
with \( d_k \geq 1 \). In the expansion, when \( a_{2k} \) vanishes, \( b_{1k} \) must also vanish, otherwise a linear recombination of the numerator can further lower \( d_k \) by one\(^3\). Now observe that performing the same deformation twice is equivalent to replacing \( z_i \) by \((z_i + z_i')\), as

\[
R(z_i, z_i') = R(z_i + z_i') = \sum_k \frac{b_{0k} + b_{1k}(z_i + z_i')}{a_{0k} + a_{1k}(z_i + z_i') + a_{2k}(z_i + z_i')^2} d_k + c_0 + \sum_l c_l(z_i + z_i')^l, \tag{2.7}
\]

then

\[
\oint_{\infty} \frac{dz_i'}{z_i'} \oint_{\infty} \frac{dz_i}{z_i} R(z_i, z_i') = \oint_{\infty} \frac{dz_i'}{z_i'} \left( c_0 + \sum_l c_l z_i^l \right) = c_0, \tag{2.8}
\]

hence \( C_i C_i R = C_i R = c_0 \). By using \( P_i = I - C_i \) it is trivial to find that

\[
P_i P_i = P_i, \quad C_i P_i = P_i C_i = 0. \tag{2.9}
\]

Besides projectivity, a more intricate property is the commutativity:

\[
C_i C_j = C_j C_i, \tag{2.10}
\]

which demands certain condition, as will be investigated shortly. If it holds, again with \( P_i = I - C_i \) one can find that

\[
P_i P_j = P_j P_i, \quad C_i P_j = P_j C_i. \tag{2.11}
\]

When all \( C \)'s are chosen to be commutative in expansion (2.4), each term is ‘orthogonal’ to the others. This orthogonality has a nice meaning: Each term contains non-overlapping pole terms, consequently one can capture all pole terms step by step without checking back and forth. While commutativity may simplify the calculation considerably, it is obviously not necessary for (2.4) to work.

One last digression is when we do practical calculations, it is convenient to use \( P_i + C_i = I \) to switch between \( P_i \) and \( C_i \), depending on which operation is easier. To check the equivalence between two visually different expressions, in appendix A we introduce a simple trick to solve all constraints and get a set of independent kinematic variables. This trick can uniquely fix the form of an expression, no matter by which means it is obtained (it is better to proceed it on computer programs).

### 2.2 Deformation generator algebra

Now we begin to explore the commutativity of \( C \)'s, which can be decomposed into the commutativity at integrand level and at integral level. The former is encoded in two successive deformations, and the latter is encoded in two successive contour integrals, which will use Laurent expansion in \( w = 1/z \). Before all of these, we need to first study the BCFW deformation generators and their algebra.

\(^3\)The \( z^2 \) term in the denominator can only originate from a spurious pole \( \langle i|K|j \rangle \) where \( K \) contains at least two external momenta other than \( i,j \), when it is deformed by \( \langle i|j \rangle \). All other physical poles can at most contribute terms linear in \( z \) under a BCFW deformation.
Let’s define the BCFW deformation generator with respect to $\langle i | j \rangle$ as

$$D_{\langle i | j \rangle} \equiv -\lambda_j^i \frac{\partial}{\partial \lambda_i^j} + \tilde{\lambda}_i^j \frac{\partial}{\partial \tilde{\lambda}_j^i}, \quad (2.12)$$

then the familiar BCFW deformation becomes

$$\exp(zD_{\langle i | j \rangle}) R(\lambda_i, \tilde{\lambda}_j) = R(\lambda_i - z\lambda_j, \tilde{\lambda}_j + z\tilde{\lambda}_i). \quad (2.13)$$

Although by default the spinorial partial derivatives treat all spinors as independent, we must also impose the momentum conservation constraint on physical amplitudes. Without doubt, this constraint will affect the independence of spinorial partial derivatives, but it will not affect the commutator algebra of $D_{\langle i | j \rangle}$. Below we will provide a simple argument.

Note that any $D_{\langle i | j \rangle}$ automatically annihilates the sum of all external momenta, i.e.,

$$D_{\langle i | j \rangle} \sum p = D_{\langle i | j \rangle}(\lambda_i \tilde{\lambda}_j + \lambda_j \tilde{\lambda}_i) = 0, \quad (2.14)$$

so we claim that momentum conservation is a trivial constraint. To get some intuition, one can consider a spherical surface, for which any rotation generator, say $L_{xy}$, annihilates the constraint

$$x^2 + y^2 + z^2 = r^2. \quad (2.15)$$

To parameterize one of the spherical symmetries explicitly, we can define an angle $\theta_{xy}$ via

$$L_{xy} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \theta_{xy}}, \quad (2.16)$$

then while $x, y$ are no longer independent on the sphere, $\theta_{xy}$ can be arbitrary, as this degree of freedom moves a point around on a subset of the spherical surface. From this viewpoint, the commutator algebra of $L_{xy}, L_{yz}, L_{zx}$ is obviously unchanged. More profoundly, it is these rotation generators that fully generate the spherical surface. Given a particular point in $\mathbb{R}^3$, rotation generators move it around to sweep over the entire surface of a fixed distance from the origin.

This picture can be exactly generalized to the case of BCFW deformation generators. We can define an ‘angle’ for each deformation in a complex spinorial sense, via

$$D_{\langle i | j \rangle} \equiv \frac{\partial}{\partial \theta_{\langle i | j \rangle}}, \quad (2.17)$$

then $\theta_{\langle i | j \rangle}$ parameterizes one of the symmetries that preserve momentum conservation, and hence momentum conservation will not change the commutator algebra of $D_{\langle i | j \rangle}$ at all. Instead, this constraint is fully generated by $2C_n^2 = n(n - 1)$ BCFW deformation generators. Given a particular point in $\mathbb{C}^{4n}$, namely the complex spinorial space $(\lambda_i, \tilde{\lambda}_i)$, BCFW deformation generators move it around to sweep over the entire codimension-4 surface of a fixed sum of external momenta. While physically, this sum is zero.
Since the commutator algebra is unchanged, we are free to treat all spinors as independent to derive the commutation relations. Imagine the 0th step of deformation is \( \langle i|j \rangle \), then the 1st step can be one of the four types as named below:

\[
\begin{align*}
\langle k|l \rangle &= \text{independent}, \\
\langle i|l \rangle / \langle k|j \rangle &= \text{straight descendent}, \\
\langle l|i \rangle / \langle j|k \rangle &= \text{skew descendent}, \\
\langle j|i \rangle &= \text{cross descendent}.
\end{align*}
\] (2.18)

The generators of first two types commute with that of \( \langle i|j \rangle \), i.e.,

\[
\begin{align*}
[D_{\langle i|j \rangle}, D_{\langle k|l \rangle}] &= 0, \\
[D_{\langle i|j \rangle}, D_{\langle i|l \rangle}] &= [D_{\langle i|j \rangle}, D_{\langle k|j \rangle}] = 0.
\end{align*}
\] (2.19)

For the last two types,

\[
\begin{align*}
2h_i &= -\lambda^\alpha_i \frac{\partial}{\partial \lambda^\alpha_i} + \tilde{\lambda}^\alpha_i \frac{\partial}{\partial \tilde{\lambda}^\alpha_i}, \\
2h_i, D_{\langle i|j \rangle} &= [D_{\langle i|j \rangle}, 2h_j] = D_{\langle i|j \rangle},
\end{align*}
\] (2.20)

where \( h_i \) is the helicity operator with respect to the \( i \)-th particle, for a function covariant under the little group (an amplitude does have this scaling property).

For the skew descendent case, use the Baker-Campbell-Hausdorff formula

\[
\exp X \exp Y = \exp(X + Y) \exp \left( \frac{1}{2} [X, Y] \right), \quad \text{for} \quad [X, [X, Y]] = [Y, [X, Y]] = 0,
\] (2.21)

and due to commutativity of the straight descend, we have

\[
\exp (z_0 D_{\langle i|j \rangle}), \exp (z_1 D_{\langle j|k \rangle}) = \exp (z_0 D_{\langle i|j \rangle} + z_1 D_{\langle j|k \rangle}) 2 \sinh \left( \frac{1}{2} z_0 z_1 D_{\langle i|k \rangle} \right).
\] (2.22)

Hence skew descendent deformations \( \langle i|j \rangle \) and \( \langle j|k \rangle \) commute if \( D_{\langle i|k \rangle} \) annihilates the amplitude, however, this is a too stringent condition which often trivializes the analysis. Therefore in general, skew descendent deformations do not commute.

### 2.3 Commutativity at integrand and integral levels

By applying the BCFW deformation generators, let’s perform a constant extraction on rational function \( R(\lambda_i, \tilde{\lambda}_i) \), given by

\[
CR = \oint_\infty \frac{dz}{z} R(z) = \oint_0 \frac{dw}{w} R \left( \frac{1}{w} \right),
\] (2.23)

where we have changed variable \( w \equiv 1/z \), so the infinity for \( z \) is the zero for \( w \). However, the residue at this zero is not a naive one. Recall (2.6) in terms of \( w \), it reads

\[
R \left( \frac{1}{w} \right) = \sum_k \frac{(b_{1k} + b_{0k} w) 2d_k - 1}{(a_{2k} + a_{1k} w + a_{0k} w^2)^d_k} + c_0 + \sum_l \frac{c_l}{w^l},
\] (2.24)
a naive substitution of $w = 0$ will cause divergence in the third term above. On the other hand, it is clear that after the expansion, the third term actually has no simple pole at $w = 0$, since there is already one $w$ in the denominator of the integrand. To remove this divergent term, before the contour integration we must Laurent expand $R(1/w)$ around $w = 0$, i.e., we need to first factor out a divergent factor $1/w^d$ with $d \geq 1$, leaving a finite fraction at $w = 0$, then Taylor expand it around $w = 0$. A simple example is

$$
\frac{b_0 + b_1 z + b_2 z^2}{a_0 + a_1 z} = \frac{1}{w} \frac{b_2 + b_1 w + b_0 w^2}{a_1 + a_0 w} = \frac{b_2}{a_1 w} \left( 1 + \frac{b_1}{b_2} w + \frac{b_0}{b_2} w^2 \right) \left( 1 + \frac{a_0}{a_1} w \right)^{-1}, \quad (2.25)
$$

then we can Taylor expand the function after $1/w$ around $w = 0$, and the contour integral will only pick up the constant part in this expression. Similarly, performing two successive constant extractions gives

$$
C_1 C_0 R = \oint_0 dw_1 \oint_0 dw_0 R \left( \frac{1}{w_1}, \frac{1}{w_0} \right), \quad (2.26)
$$

with

$$
R \left( \frac{1}{w_1}, \frac{1}{w_0} \right) = \exp \left( \frac{1}{w_1} D_1 \right) \exp \left( \frac{1}{w_0} D_0 \right) R. \quad (2.27)
$$

For independent and straight descendent cases, $[D_1, D_0] = 0$, so the order of deformations is irrelevant, and hence commutativity of these two types holds at integrand level\(^4\).

However, before performing the integral, double Laurent expansion of a fraction is a bit tricky. First, to properly factor out the overall factor in terms of $w_0$ and $w_1$, we need to ensure that in

$$
R \left( \frac{1}{w_1}, \frac{1}{w_0} \right) = \frac{1}{w_0^{a_0} w_1^{a_1}} \frac{P(w_0, w_1)}{\prod_i Q_i(w_0, w_1)}, \quad (2.28)
$$

both $P(w_0, w_1)$ and $Q_i(w_0, w_1)$ are irreducible polynomials, i.e., there is no common factor $w_0^{a_0} w_1^{a_1}$ of all monomials in each of them. Then, we find the expansions of a rational function (namely a fraction) in opposite orders may be different, which arises when one of the factors in the denominator contains no constant term. For example, take $g(w_1, w_0) = 1/(w_1 + w_0)$ and expand it around $w_0 = 0$ (it’s impossible to further expand around $w_1 = 0$), we have

$$
g(w_1, w_0) = \frac{1}{w_1} \left( 1 - \frac{w_0}{w_1} + \frac{w_0^2}{w_1^2} + \ldots \right), \quad (2.29)
$$

and for the reverse order,

$$
g(w_1, w_0) = \frac{1}{w_0} \left( 1 - \frac{w_1}{w_0} + \frac{w_1^2}{w_0^2} + \ldots \right), \quad (2.30)
$$

hence they are clearly different. In general, if one of the $Q_i$’s happens to satisfy $Q_i(0,0) = 0$, the double expansion depends on the order. Conversely, if all $Q_i$’s obey $Q_i(0,0) \neq 0$, the order of double expansion is irrelevant. Since the contour integral simply picks up the constant part in the expansion, commutativity of the denominator expansion is equivalent to commutativity at integral level.\(^4\)

\(^4\)It is possible that two constant extractions commute, even if they do not commute at integrand level, but we will not consider this trivial case here.
In practice, since it is clear that a detectable pole of merely one of the two successive deformations always contains a constant term after factoring out a proper factor, we should only focus on the overlap of two sets of detectable poles. But it’s impractical to trace each term at each step, for figuring out the existence of a constant term.

Combining all these above, we may draw the conclusion: commutativity of $C$ operators conditionally holds for the independent and straight descendent cases. Then these two types are considered as ‘good’, as they enjoy the orthogonal property (2.11). However, merely the good types are not sufficient to capture all physical pole terms, as will be explained in the end of appendix B. Hence we will not proceed further into commutativity of $C$ operators, but return to seek for the condition of $C_n C_{n-1} \cdots C_2 C_1 C_0 = 0$ such that the $(n + 1)$ steps can fully capture the amplitude.

Nevertheless, this investigation gives a crucial hint for the subsequent analysis: In the expansion, the coefficient of $z$ after a deformation will be become a new pole of the corresponding boundary term, with power one or higher. To be concrete, consider the example below

$$\frac{1}{a} \to \frac{1}{a + b z} = \frac{w}{(b + aw)} = \frac{w}{b} \left(1 + \frac{a}{b} w\right)^{-1} = \frac{w}{b} \left(1 - \frac{a}{b} w + \frac{a^2}{b^2} w^2 - \ldots\right),$$

where $b$ is the only source of poles after expansion.

3. Systematic Algorithm of Finite, Definite Steps

In this section, we propose the systematic process to capture the full amplitude after finite, definite steps of BCFW constant extractions. The condition for its calculation to be correctly completed is simply $C_n C_{n-1} \cdots C_2 C_1 C_0 = 0$. To achieve this, a form of all poles in the final boundary term is given, after a sequence of constant extractions, which is called the ‘pole concentration’. This sequence is designed for covering all situations and how to optimize it case by case is set aside temporally. Having the final form of all poles, merely using the information of mass dimension and helicities is sufficient to judge whether the final boundary vanishes.

3.1 Pole concentration

Now we use pole concentration to capture all poles regardless of they are physical or spurious by applying the logic of (2.31). Explicitly, each time we perform a BCFW constant extraction on an amplitude, one or more of its physical poles will be filtered out, and consequently each corresponding boundary term will contain one or more pole(s) mutated from the original physical pole(s).

For example, consider denominator $\langle 12 \rangle \langle 23 \rangle$ (the numerator is neglected for our purpose), under constant extraction $\langle 1|3 \rangle$,

$$\frac{1}{\langle 12 \rangle \langle 23 \rangle} \to \frac{1}{((12) - z \langle 32 \rangle) \langle 23 \rangle} \Rightarrow \frac{1}{\langle 23 \rangle^2},$$

where pole $\langle 12 \rangle$ has been replaced by $\langle 32 \rangle$. And crucially, under a next constant extraction, pole $\langle 23 \rangle^2$ is either unchanged or replaced by another pole as a whole. This means once two poles are stacked, they
are stacked forever. The same logic also works for anti-holomorphic poles. For a multi-particle pole, we first need to turn it into a product of holomorphic and anti-holomorphic poles, with a proper choice of deformation. As an example, under constant extraction \( \langle 1|4 \rangle \),

\[
\frac{1}{P_{123}^2} \rightarrow \frac{1}{P_{123}^2 + z\langle 4|2 + 3|1 \rangle} \Rightarrow \frac{1}{\langle 4|2 + 3|1 \rangle},
\]

(3.2)

next, under constant extraction \( \langle 2|1 \rangle \),

\[
\frac{1}{\langle 4|2 + 3|1 \rangle} \rightarrow \frac{1}{\langle 4|2 + 3|1 \rangle + z\langle 43|32 \rangle} \Rightarrow \frac{1}{\langle 43|32 \rangle},
\]

(3.3)

or under constant extraction \( \langle 2|5 \rangle \),

\[
\frac{1}{\langle 4|2 + 3|1 \rangle} \rightarrow \frac{1}{\langle 4|2 + 3|1 \rangle - z\langle 45|21 \rangle} \Rightarrow \frac{1}{\langle 45|21 \rangle},
\]

(3.4)

in either way we are again left with two-particle poles.

In general, one can first turn the multi-particle pole \( P^2 \) into \( \langle i_1|P|j_1 \rangle \), where \( P \) includes either \( i_1 \) or \( j_1 \). Note that \( p_{i_1} \) or \( p_{j_1} \) in \( P \) is already filtered out by \( \langle i_1 \rangle \) or \( \langle j_1 \rangle \). Next, one can continue to filter out more momenta in \( P \) by using \( \langle i_1|j_2 \rangle \) or \( \langle i_2|j_1 \rangle \), where \( P \) includes \( j_2 \) or \( i_2 \), until this pole is split into a product of two-particle poles. Alternatively one can split \( \langle i_1|P|j_1 \rangle \) directly by using \( \langle k|j_2 \rangle \) or \( \langle i_2|l \rangle \), where \( P \) includes \( j_2 \) or \( i_2 \) but not \( k \) or \( l \). Either way turns the pole into a product of one holomorphic and one anti-holomorphic poles in the end. Then for two-particle poles, once they are stacked, they must mutate as a whole afterwards. After finite steps, all poles can be encapsulated in only one holomorphic and one anti-holomorphic poles, with powers larger than one in general.

In appendix B, two example sequences of BCFW constant extractions are given to turn all poles of the final boundary term into a common denominator, given by\(^5\)

\[
\frac{(\text{polynomial})}{\langle i_1i_2 \rangle^m \langle i_3i_4 \rangle^\overline{m} \times \langle \text{remaining factor} \rangle},
\]

(3.5)

where \( i_1, i_2, i_3, i_4 \) are four different arbitrary particle labels. The remaining factor is a rational function which is dimensionless and helicity-neutral, see (3.9) for example. Note that we have not reduced the denominator against the numerator. At the first glance, the reason to reach this final denominator is that one can use one more deformation, say \( \langle i_1|i_4 \rangle \), to get the maximal large \( z \) suppress, since all poles after concentration are vulnerable to it. But in fact, there is a less obvious argument for eliminating the final boundary term without introducing one more step, as will be shown later.

Here, \( m/\overline{m} \) gets contribution from physical holomorphic/anti-holomorphic poles, and both of them get contributions from physical multi-particle poles. In general, \( m \) and \( \overline{m} \) need not to be equal, since not all possible poles are physical for a particular amplitude. To see the range of \( m, \overline{m} \), we will analyze all possible physical poles for various \( n \)'s. When \( n = 4 \), only a half of all two-particle poles can appear

\(^5\)The choice of sequence is not unique, and how to optimize it to shorten the steps is a valuable future problem.
in the amplitude, since they are doubly duplicated by momentum conservation. When \( n = 5 \), there are only two-particle poles, as three-particle poles are equivalent to them by momentum conservation. When \( n \geq 6 \), multi-particle poles arise. Their particle numbers range from 3 to \((n - 3)\), to avoid duplications of two-particle poles by momentum conservation. To further avoid duplications of themselves by momentum conservation, one can fix the pole momentum by demanding it to always include one pivot particle, and then the number of multi-particle poles is reduced by one half.

According to this counting, the maxima of \( m, \overline{m} \) are

\[
m_{\text{max}} = m_{\overline{m}} = C_n^2 + \frac{1}{2}(C_n^3 + \ldots + C_n^{n-3}) = 2^{n-1} - (n + 1),
\]

which nicely covers the special cases of \( n = 4 \) and \( n = 5 \).

However, there is one little subtlety in (3.5): For a given amplitude, while \( m, \overline{m} \) can be easily read off by analyzing all of its non-vanishing factorization limits, its final boundary term, in general, contains not only poles \( \langle i_1i_2 \rangle_m \langle i_3i_4 \rangle_{\overline{m}} \), but also the same poles of higher orders from the dimensionless and helicity-neutral remaining factor. This phenomenon also occurs in each intermediate step, for each corresponding intermediate boundary term. A simple example is the MHV amplitude \( A(1^{-}, 2^{-}, 3^{+}, 4^{+}) \), given by

\[
A = \frac{\langle 12 \rangle^3}{\langle 34 \rangle},
\]

and deformation \( \langle \bar{1}|3 \rangle \) turns it into (recall that \( z = 1/w \))

\[
A \rightarrow \frac{((12) - z(32))^3}{(23)(34)(41) - z(43)} = \frac{1}{w^2(23)(34) - \langle 43 \rangle - \langle 41 \rangle w} = \frac{1}{w^2(23)(34)} \langle 32 \rangle \left( 1 - \frac{\langle 32 \rangle}{\langle 43 \rangle} \right)^3 \left( 1 - \frac{\langle 41 \rangle}{\langle 32 \rangle} \right)^{-1},
\]

note that pole \( \langle 41 \rangle \) is turned into \( \langle 43 \rangle \), but its power can be larger than one. Explicitly, the corresponding boundary term is

\[
C_{\langle \bar{1}|3 \rangle} A = \frac{1}{(23)(34)(43)} \left( \frac{\langle 41 \rangle^2\langle 32 \rangle^3}{\langle 43 \rangle^2} - 3\frac{\langle 41 \rangle\langle 12 \rangle\langle 32 \rangle^2}{\langle 43 \rangle} + 3\langle 12 \rangle^2\langle 32 \rangle \right)
\]

\[
= \frac{\langle 12 \rangle^2\langle 32 \rangle}{(23)(34)^2} \left( \frac{\langle 41 \rangle^2\langle 32 \rangle^2}{\langle 43 \rangle^2\langle 12 \rangle^2} - 3\frac{\langle 41 \rangle\langle 32 \rangle}{\langle 43 \rangle\langle 12 \rangle} + 3 \right),
\]

where the term in parentheses is the remaining factor in (3.5). The advantage of packing up many pole terms into a dimensionless helicity-neutral factor is that, if we can show this representative factor cannot exist, all terms behind it including those with higher-power poles, must also be forbidden.

One digressive comment is that, so far we have found BCFW deformations to be the only type which admits a feasible pole concentration. A counterexample is, there is no straightforward pole concentration for Risager deformations [19]. We will not further explain the claim here, but it is not hard to confirm it. This is another specialty of BCFW deformations, in addition to that BCFW deformations automatically preserve (or generate) the momentum conservation constraint.
3.2 Kinematic mass dimension

To prepare for the later analysis, we need to understand some general information of amplitudes: mass dimension and helicities, with which the applicable range of multi-step BCFW recursion relations can be clarified.

First, for QFTs in 4-dimension, the mass dimension of an \(n\)-particle amplitude is \((4-n)\). We can use the LSZ reduction formula to prove this. Schematically, an \(n\)-particle amplitude \(A\) is defined via

\[
\prod^n (\int d^4x e^{ipx} \varepsilon \Delta) \langle \Phi_1 \ldots \Phi_n \rangle = \delta^4 \left( \sum p \right) A, \tag{3.10}
\]

where \(\langle \Phi_1 \ldots \Phi_n \rangle\) is the \(n\)-point function, \(\varepsilon\) and \(\Delta\) are the wave-function and kinematic operator for each field \(\Phi\). For a bosonic field, the mass dimensions of \(\varepsilon\), \(\Delta\) and \(\Phi\) are 0, 2 and 1 respectively, for a fermionic field, the mass dimensions of \(\varepsilon\), \(\Delta\) and \(\Phi\) are \(1/2\), 1 and \(3/2\) respectively. Hence the mass dimension of

\[
\int d^4x e^{ipx} \varepsilon \Delta \tag{3.11}
\]

is \(-1\). There are \(n\) such pieces, plus the momentum conservation delta function, the mass dimension of \(A\) is clearly \((4-n)\).

One special bosonic field is the graviton. By the perturbative definition \(g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}\), it should be dimensionless. To treat it as ordinary bosonic fields, we need to redefine it via

\[
g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu}, \tag{3.12}
\]

such that \(h_{\mu \nu}\) carries mass dimension 1 and \(\kappa\) carries \(-1\). Choosing \(\kappa\) to be \(\sqrt{8\pi G}\), the free field part of Einstein-Hilbert Lagrangian will have an analogous form as that of ordinary bosonic fields. Consequently, \(\kappa\) becomes the coupling constant of gravity.

One can also rediscover \(\kappa\) via the on-shell method. For gravity, three-particle amplitudes including at least one graviton are

\[
A(1^{-h}, 2^{+h}, 3^{-2}) = \kappa \begin{pmatrix} 12 \\ 23 \\ 31 \end{pmatrix}^{-2} \begin{pmatrix} 23 \\ 31 \end{pmatrix}^{2h+2} \begin{pmatrix} 31 \\ 23 \end{pmatrix}^{h+2},
\]

\[
A(1^{-h}, 2^{+h}, 3^{+2}) = \kappa \begin{pmatrix} 12 \\ 23 \end{pmatrix}^{-2} \begin{pmatrix} 23 \\ 31 \end{pmatrix}^{2h+2} \begin{pmatrix} 31 \\ 23 \end{pmatrix}^{-2h+2}, \tag{3.13}
\]

where \(h = 0, 1/2, 1, 2\), for all ‘realistic’ theories. No matter which value \(h\) takes, \(\kappa\) always carries mass dimension \(-1\), since the mass dimension of three-particle amplitudes is 1.

In general, we can reverse this logic and define the ‘kinematic mass dimension’ of an \(n\)-particle amplitude as

\[
D = 4 - n - \sum_i (D_c)_i, \tag{3.14}
\]

where \((D_c)_i\) is the mass dimension of the coupling constant for each vertex. From now on, we will focus on the kinematic part of an amplitude, which is constructed recursively.
For Standard Model\(^6\), all coupling constants are dimensionless so \(D = 4 - n\), which is a non-positive number for various \(n\)’s. For gravitational interactions \(D_c = -1\) and we will show that \(D = 2\).

When \(D < 0\), there is at least one irreducible denominator of the amplitude, which means a pole to be detected by BCFW deformations. Conversely, when \(D \geq 0\), the amplitude may admit some invulnerable terms to BCFW deformations which include polynomials and ‘pseudo polynomials’. The classification of these objects can be found in appendix C.

3.3 The master formula

By using mass dimension and helicities, let’s derive the master formula for subsequent discussions. After pole concentration, the final boundary term schematically reads\(^7\)

\[
\frac{1}{\langle 12 \rangle m [34]^m} \prod_{i=1}^{n} \langle i |^{\alpha_i} \prod_{i=1}^{n} [i |^{\beta_i}, \tag{3.15}
\]

where we have temporally taken \(i_1, i_2, i_3, i_4 = 1, 2, 3, 4\). The reason to use un-contracted spinors is that, this is more compact to capture the helicity information, and it can save the Schouten identity manipulations, as one can freely recombine them to get the desired spinorial products. Of course, the cost is that one needs to rule out all those illegitimate combinations. This treatment is similar to the method used in \([20]\). Now the helicity configuration enforces that

\[
\begin{align*}
-2h_1 + m &= \alpha_1 - \beta_1, \\
-2h_2 + m &= \alpha_2 - \beta_2, \\
2h_3 + m &= \beta_3 - \alpha_3, \\
2h_4 + m &= \beta_4 - \alpha_4, \\
2h_i &= \beta_i - \alpha_i, \quad (i = 5, \ldots, n)
\end{align*}
\tag{3.16}
\]

where \(m, m\) are known for a particular amplitude. Note that there are \(2n\) variables, with only \(n\) helicity constraints. We will fully exploit the \(n\) remaining degrees of freedom to derive the master formula.

The kinematic mass dimension of (3.15) is

\[
D' = -(m + m) + \frac{1}{2} \left( \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i \right), \tag{3.17}
\]

where obviously, \(\sum \alpha\) and \(\sum \beta\) must be both even to form spinorial products. Also, we have \(m + m \geq 1\) with \(m, m \geq 0\), and \(\alpha, \beta \geq 0\). For a legitimate final boundary term, \(D'\) equals to \(D\) defined in (3.14).\(^{6}\)

\(^{6}\)The \(\phi^3\) interaction is not included in Standard Model. But since its coupling constant is of mass dimension 1, it is in fact more benign for the subsequent analysis.

\(^{7}\)In general, the final boundary term’s numerator is a polynomial but we only focus on one term, as the identical analysis applies to all terms. The remaining factor in (3.5) is dropped from now on, since it is dimensionless and helicity-neutral.
When \( D \neq D' \) under all circumstances, correct dimension and helicities cannot be satisfied simultaneously, then the final boundary term is eliminated. One direct way to achieve this inconsistency is to show \( D'_{\text{min}} \) is larger than \( D \). First we need to figure out this minimum by eliminating one degree of freedom for each particle, and there are two variables \( \alpha \) and \( \beta \) to be chosen. For \( i = 5, \ldots, n \),

\[
\frac{1}{2} (\alpha_i + \beta_i) = -h_i + \beta_i = h_i + \alpha_i,
\]

(3.18)

when \( h_i \) is negative, \((-h_i + \beta_i)\) is guaranteed to be positive, similarly when \( h_i \) is non-negative, \((h_i + \alpha_i)\) is guaranteed to be non-negative. To manifest the non-negativity of \( D' \), our choice is

\[
\frac{1}{2} (\alpha_i + \beta_i) = |h_i| + \min(\alpha_i, \beta_i).
\]

(3.19)

Extending this logic for all particles, yields

\[
D' = -(m + \overline{m}) + \sum_{i=1,2} \left( |h_i - \frac{m}{2}| + \min(\alpha_i, \beta_i) \right) + \sum_{i=3,4} \left( |h_i + \frac{\overline{m}}{2}| + \min(\alpha_i, \beta_i) \right) + \sum_{i=5}^{n} (|h_i| + \min(\alpha_i, \beta_i)),
\]

(3.20)

which is the master formula and explicitly,

\[
\sum_{i=5}^{n} (|h_i| + \min(\alpha_i, \beta_i)) = \sum_{h<0} (-h_i + \beta_i) + \sum_{h \geq 0} (h_i + \alpha_i),
\]

(3.21)

which separates the sum into two parts according to the helicities. The final boundary term (3.15) now reads \((p_i = |i\rangle|\bar{i}\rangle\) is a helicity-neutral momentum with additional mass dimension 1)

\[
\frac{1}{\langle 12 \rangle^m [34] \overline{m}} \prod_{i=1,2} \left( |i\rangle^{2h_i} p_i^{\alpha_i} / |\bar{i}\rangle^{-2h_i + m} p_i^{\beta_i} \right) \prod_{i=3,4} \left( |i\rangle^{2h_i + \overline{m}} p_i^{\alpha_i} / |\bar{i}\rangle^{-2h_i - \overline{m}} p_i^{\beta_i} \right) \prod_{h<0} \langle i\rangle^{-2h_i} p_i^{\beta_i} \prod_{h \geq 0} \langle i\rangle^{2h_i} p_i^{\alpha_i},
\]

(3.22)

where / means one of two candidate expressions is chosen to manifest the non-negativity of \( D' \), as this choice also manifests the ‘extra neutral momenta’, in addition to the ‘net spinors’ that carry the helicity information. While the latter content is mandatory, the former is optional since it is brought in to fill the extra capacity of mass dimension. There is no unique choice of picking these extra \( \alpha \)‘s and \( \beta \)‘s as long as the total dimension is correct.

For \( i = 5, \ldots, n \), picking \(|h_i|\) trivially maximizes \( D' \). But for \( h_1, h_2, h_3, h_4 \), careful analysis is needed as \( m, \overline{m} \) are involved. Rewrite (3.20) as

\[
D' = -(m + \overline{m}) + T_{1234} + \sum_{i=1}^{n} |h_i| + \sum_{i=1}^{n} \min(\alpha_i, \beta_i),
\]

(3.23)

where

\[
T_{1234} = \sum_{i=1,2} \left( |h_i - \frac{m}{2}| - |h_i| \right) + \sum_{i=3,4} \left( |h_i + \frac{\overline{m}}{2}| - |h_i| \right),
\]

(3.24)
since \( m, \overline{m} \) and \( \sum |h| \) are fixed, we only need to manipulate \( T_{1234} \). It’s easy to check that

\[
\begin{align*}
|h_i - \frac{m}{2}| - |h_i| &= \begin{cases} 
\frac{m}{2}, & h_i < 0 \\
\frac{m}{2} - 2h_i, & 0 \leq h_i < \frac{m}{2} \\
-\frac{m}{2}, & \frac{m}{2} \leq h_i
\end{cases} \\
|h_i + \frac{\overline{m}}{2}| - |h_i| &= \begin{cases} 
-\frac{\overline{m}}{2}, & h_i < -\frac{\overline{m}}{2} \\
\frac{\overline{m}}{2} + 2h_i, & -\frac{\overline{m}}{2} \leq h_i < 0 \\
\frac{\overline{m}}{2}, & 0 \leq h_i
\end{cases}
\end{align*}
\] (3.25)

to maximize \( T_{1234} \), one must take \( h_1, h_2 \) to be two minimal helicities and \( h_3, h_4 \) to be two maximal ones\(^8\).

On the other hand, even if one chooses \( h_1, h_2, h_3, h_4 \) arbitrarily among all \( h_i \)'s, \( D'_{\text{min}} \) is no less than zero. To see this, neglecting other non-negative parts in \( D' \), let’s focus on the quantity

\[
\overline{T}_{1234} \equiv -(m + \overline{m}) + T_{1234},
\]

by its definition \( T_{1234} \) has a simple geometrical meaning: It is the sum of four distances from \( h_1, h_2 \) to line \( h = \frac{m}{2} \), and from \( h_3, h_4 \) to \( h = -\frac{\overline{m}}{2} \). It’s easy to find that its minimum is \((m + \overline{m})\), when four points are on one horizontal line and the line is within the region between \( h = \frac{m}{2} \) and \( h = -\frac{\overline{m}}{2} \). When this uniform line moves outside the region, \( T_{1234} \) increases by \( 4 \times \) (distance above or below). When this line is not uniform, one can always rearrange the four points to render \( T_{1234} \) increase. Therefore \( \overline{T}_{1234} \) is always non-negative, so is \( D'_{\text{min}} \).

Also note that

\[
\begin{align*}
-\frac{m}{2} + |h_i - \frac{m}{2}| - |h_i| &= 0 \Big/ (-2h_i) \Big/ (-m), \\
-\frac{\overline{m}}{2} + |h_i + \frac{\overline{m}}{2}| - |h_i| &= (-\overline{m}) \Big/ (2h_i) \Big/ 0,
\end{align*}
\] (3.27)

which all take integer values. According to (3.23), if \( \sum |h| \) is fractional, \( D' \) must be fractional.

### 3.4 Inconsistency elimination

We now use (3.20) to show that, all massless tree amplitudes except those admit (pseudo) polynomials of given mass dimension and helicities, shall be fully determined by multi-step BCFW recursion relations. In the following analysis, we pretend not to know any knowledge of QFT in the Lagrangian paradigm, except mass dimension and helicities. These are the only data needed to construct three-particle amplitudes, and recursions extend them to all higher-point cases.

First note that, after pole concentration, there is no need to find any further deformation to kill the final boundary term, because another choice will not change its pole form so one can always rearrange the entire series of pole concentration to reach the desired relabeling. Hence it must be the last step if chosen properly, and the direct way is to use the ‘inconsistency elimination’.

\(^8\)Now we should strictly use \( i_1, i_2, i_3, i_4 \) instead of \( 1, 2, 3, 4 \) to admit a possible relabeling, in order to implement the desired arrangement of pole concentration.
Within its own framework, if inconsistency elimination can exclude the final boundary term, the new algorithm will be a powerful and completely independent approach to calculate corresponding amplitudes. Otherwise, terms that survive it need to be identified, similarly as (pseudo) polynomials. In fact, we do discover a new type of object called the ‘saturated fraction’ by this way.

From the master formula (3.20), it is already known that $D'_{\text{min}} \geq 0$. Therefore if $D < 0$, $D < D'_{\text{min}}$ always holds. This tells us the nontrivial cases are of $D \geq 0$. When $D \geq D'_{\text{min}}$, we have the inconsistency criteria below to eliminate the final boundary term:

1. Fractional Dimension (FD): If $D' = \text{fractional}$. This arises when $\sum |h| = \text{fractional}$, but Lorentz invariance demands the dimension of (3.15) to be an integer, which implies that fermions must appear in pairs to be consistent.

2. Pair Mismatch (PM): If $\sum \alpha = \text{odd}$ and $\sum \beta = \text{odd}$, assume that FD is excluded already. In this case, (3.15) cannot be written as a fraction in terms of Lorentz invariant spinorial products, even though the dimension of (3.15) is an integer.

3. Spinor Excess (SE): If there exists an $i$, such that $\alpha_i > \sum_{j \neq i} \alpha_j$ or $\beta_i > \sum_{j \neq i} \beta_j$. In this case, spinorial contraction will force (3.15) to vanish, even though $\sum \alpha = \text{even}$ and $\sum \beta = \text{even}$.

Altogether, there are four layers of inconsistency criteria:

0. $D < D'_{\text{min}}$.
1. If $D \geq D'_{\text{min}}$, consider FD.
2. If $D \geq D'_{\text{min}}$, and FD is excluded, consider PM.
3. If $D \geq D'_{\text{min}}$, and both FD and PM are excluded, consider SE.

It’s obvious that these inconsistency criteria only mention general properties of field theories. Hence inconsistency elimination is theory independent in general, while in practice, knowing some theory dependent properties would help simplify discussions case by case. If the final boundary term can survive all four criteria, it must admit ‘saturated fractions’ (SF). Note that we have already set aside polynomials and pseudo polynomials, because they can be found without using the master formula (3.20). Altogether, there are three types of objects invulnerable to BCFW deformations:

1. Polynomials, such as $\langle 12 \rangle$.
2. Pseudo polynomials of $n = 4$, such as $[34]/\langle 12 \rangle$. When $\langle 12 \rangle \rightarrow 0$, we must also have $[34] \rightarrow 0$ since it is a BCFW deformation that is being used. Then the ratio $[34]/\langle 12 \rangle$ is finite like a polynomial.
3. Saturated fractions of $n \geq 5$, such as $[34][56]/\langle 12 \rangle$. When $\langle 12 \rangle \rightarrow 0$, this fraction is divergent.

Among these three objects, a polynomial is completely inert to BCFW constant extractions, in fact it is invulnerable to any type of deformation in on-shell methods. A pseudo polynomial is also completely
inert to BCFW constant extractions, while this requires momentum conservation. A saturated fraction is form-inert to BCFW constant extractions\(^9\), but with its particle labels rearranged. The last two objects are vulnerable to other types of deformations, such as Risager deformations \(^{[19]}\). Detailed exploration of all these three types is presented in appendix C.

So far, we have witnessed how the systematic process of multi-step BCFW recursion relations can be arranged for solving a particular amplitude. In summary, there are four universal steps:

1. Analyze all non-vanishing factorization limits to determine the amplitude’s common denominator, which is a product of all physical poles. This stage can be done almost purely diagrammatically.

2. Figure out the amplitude’s kinematic mass dimension, then combine this with its helicity configuration to determine all possible (pseudo) polynomials. If none of them arises, we assume the amplitude can be fully determined and proceed to the next step.

3. Choose four particles of two maximal and two minimal helicities to determine the denominator’s form of the final boundary term, and arrange a sequence of BCFW constant extractions to proceed pole concentration. The sequence must be able to capture all physical poles, such that each of them contributes to the final denominator via powers \(m, \bar{m}\). With this ensured, the sequence should be as short as possible. This optimization is an important future problem.

4. Use all four inconsistency criteria layer by layer to eliminate the final boundary term. If it fails, identify all possible saturated fractions. Then discuss whether these saturated fractions are legitimate, if not, clarify the argument to rule them out as mentioned in the end of appendix C. This delicate treatment to remove all dependence on spurious poles is another valuable future problem.

4. Applications in Standard Model plus Gravity

Having a general guide of multi-step BCFW recursion relations, naturally we would like to see how it applies to specific theories. As familiar examples, let’s first consider the realistic theories, \(i.e.,\) Standard Model plus gravity\(^{10}\).

For reader’s convenience, we rewrite the master formula below

\[
D' = -(m + \bar{m}) + \sum_{i=1,2} \left| h_i - \frac{m}{2} \right| + \sum_{i=3,4} \left| h_i + \frac{\bar{m}}{2} \right| + \sum_{i=5}^{n} |h_i| + \sum_{i=1}^{n} \min(\alpha_i, \beta_i),
\]

(4.1)

recall that one should take \(h_1, h_2\) to be two minimal helicities and \(h_3, h_4\) to be two maximal ones.

4.1 Two separated sectors

Let’s first consider Standard Model and gravity separately. From the previous section, it is known that for Standard Model, \(D = 4 - n\). On the other hand, since \(D'_{\text{min}}\) is no less than zero, when \(D \leq -1\), the final

\(^9\)Here we mean a pure saturated fraction. A mixed saturated fraction is a pure saturated fraction times a polynomial. Its transform under a constant extraction will be demonstrated in appendix C.

\(^{10}\)All other theories can be analyzed analogously. However, as the amplitude’s kinematic mass dimension goes up, more types of polynomials, pseudo polynomials and saturated fractions may arise, and one needs to identify them carefully.
boundary term must be eliminated. This directly tells that all Standard Model amplitudes of \( n \geq 5 \) are solvable, leaving amplitudes of \( n = 4 \) corresponding to \( D' = 0 \) to be further analyzed. As will be shown later, the \( n = 4 \) case admits (pseudo) polynomials 1 and \((34)/[12])^{\pm 1}.

For pure gravity, assume one of the Feynman diagrams of an \( n \)-particle amplitude contains \( v_m \) \( m \)-leg vertices and \( p \) internal propagators, it is clear that

\[
\sum_m m v_m - 2p = n, \quad \sum_m v_m = p + 1, \quad \Rightarrow \quad \sum_m (m - 2)v_m = n - 2,
\]

(4.2)

and each \( m \)-leg vertex brings in \((m - 2)\) \( \kappa \)'s, hence from (3.14) we have

\[
D = 4 - n - (-1)\sum_m (m - 2)v_m = 4 - n + (n - 2) = 2,
\]

(4.3)

in fact, \( D = 2 \) holds for all amplitudes with only gravitational interactions, as later shown by (4.6). Now we compare \( D \) with \( D' \). When \( n \geq 6 \), \( D'_{\min} = 4 > 2 \) so this case is completely solvable. When \( n = 5 \), from (4.1), and under the most conservative case \(-m/2 \leq -2 < 2 \leq m/2\), the all-plus helicity configuration is admitted (similar for all-minus) with the saturated fraction

\[
\frac{[34]^2[35]^2[45]^2}{\langle 12 \rangle^4}.
\]

(4.4)

When \( n = 4 \), the all-plus helicity configuration admits pseudo polynomials like

\[
\frac{[34]^4}{\langle 12 \rangle^4} \langle p_{xy} \rangle, \quad \frac{[34]^5}{\langle 12 \rangle^5} \langle 13 \rangle \langle 24 \rangle,
\]

(4.5)

where \( x, y \) are unspecified. And the all-but-one-minus case gives \( D'_{\min} = 4 > 2 \) already.

However, from three-particle amplitudes (3.13) (with \( h = 2 \)) and non-vanishing factorization limits, it is not hard to show that any amplitude’s helicity configuration of pure gravity must be between MHV and anti-MHV. For the MHV configuration, (4.1) gives \( D'_{\min} = 8 > 2 \). Therefore pure gravity is in fact completely solvable.

In general, for Standard Model plus gravity we have \( D \leq 2 \). Note that an amplitude which contains gravitational vertices only always obeys \( D = 2 \), regardless of how many or what kinds of external legs it owns. This specialty implies that one can arbitrarily attach more particles to a known amplitude without changing its mass dimension, via gravitational interaction.

### 4.2 Simplified diagrammatic rules

To simplify the general discussion of Standard Model plus gravity, we introduce the diagrammatic rules called ‘stretch and shrink’. The first example is the gauge interaction, as shown in Figure 1.

Fixing four external gauge bosons, this 4-point vertex can be stretched into two connected 3-point vertices, without changing the vertex’s mass dimension. This tricky equivalence holds at the level of mass dimension and helicities, which are the only information required for inconsistency elimination.
Figure 1: 4-point gauge vertex is ‘equivalent’ to two connected 3-point vertices. Wavy lines represent gauge bosons.

Figure 2: Stretch or shift rule of gravitational vertex. Bold lines represent Standard Model particles, and zigzag lines represent gravitons.

In other words, we have chosen a representative sub-diagram to encode the same information of mass dimension and helicities, and reduce the types of equivalent sub-diagrams in the analysis. Following this logic, all higher-point vertices in Standard Model plus gravity can be stretched into a number of connected 3-point vertices, except the special $\phi^4$ vertex. This simplified rule is notably advantageous in gravitational interaction.

As shown in Figure 2, gravitons can be shifted from any place to any place in a sub-diagram, without changing its mass dimension. Physically, this is because gravity is universal, gravitons can emit from any part of a system. Mathematically, this is because an $m$-point gravitational vertex carries coupling constant $\kappa^{m-2}$, where $\kappa$ carries mass dimension $-1$. This vertex can contain gravitons only, or it can be attached by Standard Model lines. Therefore, the $m$-point vertex can be stretched into $(m - 2)$ connected 3-point vertices, with the exception of $\phi^4$ vertex.

For convenience we define the ‘gravitational component’, as shown in Figure 3. All vertices within this component are gravitational, while its external legs can be either Standard Model particles or gravitons, or both. There is one special graviton which will attach to another component. A trivial case is that there is no vertex at all, so this special graviton becomes the only component.

Gravitational components also obey the simplified rules, and for convenience they are usually shrunk into one component, as shown in Figure 4. This pack-up can reduce many sub-diagrams of gravitational components to one sub-diagram. It’s free to attach or detach a gravitational component, since it will not change the mass dimension of the other component.

Summarizing the simplified diagrammatic rules, we are now left with the representative vertices only, as shown in Figure 5.
4.3 Amplitudes and (pseudo) polynomials of $D = 0, 1, 2$

In Standard Model plus gravity, the nontrivial cases are $D = 0, 1, 2$. By using the representative vertices in Figure 5, the following discussion is considerably shorten.

$D = 0$ case: First we consider $D = 0$, all possible amplitudes are listed diagrammatically in Figure 6. Similar to gravitational components, Standard Model components presented here only contain Standard Model vertices. Also note the Standard Model components attached by a single graviton and a nontrivial gravitational component, are listed separately for clarity.
Figure 6: Amplitudes of $D = 0$.

Figure 7: (Pseudo) polynomials of $D = 0$.

For $D'_{\text{min}} = 0$, polynomials and pseudo polynomials arise when all helicities are the same. Then the last three diagrams in Figure 6 are excluded, since a three-point Standard Model vertex can never have three same helicities. The second diagram is also excluded since a single graviton has helicity $\pm 2$.

Therefore, (pseudo) polynomials come from the first and third diagrams, as listed in Figure 7. The first three diagrams in Figure 7 admit polynomial 1, while the fourth diagram admits pseudo polynomial $((34)/(12))^{\pm 1}$, as Yukawa interaction requires the two fermions of its vertex to have the same helicities. The fourth diagram cannot be attached by a gravitational component, since fermions and gauge bosons must appear in pairs of opposite helicities when coupling with gravitons. Finally, since four gauge bosons must have the MHV configuration, $((34)/(12))^{\pm 2}$ is excluded.

Amplitudes of $D = 0$ diagrams other than those in Figure 7 are solvable. We see that it’s convenient to attach a gravitational component to a known diagram, since it does not change the mass dimension. This one-line attachment is equivalent to maximally separating gravitational and Standard Model vertices into two sub-diagrams, when building a single representative diagram.

$D = 1$ case: Continue this fashion for $D = 1$, all possible amplitudes are listed in Figure 8. Here, the 3-point Standard Model vertex can be one of the following four types: (a) 3-gauge interaction ($\pm 1, +1, -1$);
(b) gauge-fermion-fermion interaction ($\pm 1, +1/2, -1/2$); (c) gauge-scalar-scalar interaction ($\pm 1, 0, 0$); (d) scalar-fermion-fermion (Yukawa) interaction ($0, +1/2, -1/2$).

For the left diagram in Figure 8, when vertices of type (a), (b), (c) and (d) are attached by a single graviton, the four helicities are ($\pm 1, +1, -1, \pm 2$), ($\pm 1, +1/2, -1/2, \pm 2$), ($\pm 1, 0, 0, \pm 2$) and ($0, +1/2, -1/2, \pm 2$) respectively\footnote{Here $\pm 1$ and $\pm 2$ are independent, they do not necessarily have the same sign.}. Plugging the data in (4.1), corresponding $D'_{\text{min}}$’s are 3, 3, 3 and 2, which excludes all four cases. In one words, the left diagram is excluded simply due to the single graviton.

For the right diagram in Figure 8, when vertices of type (a), (b) and (c) are attached by a gravitational component, to consider the most conservative case, this component only contains external scalars, since higher-spin particles must appear in pairs of opposite helicities, which will not decrease $D'_{\text{min}}$. The helicities are ($\pm 1, +1, -1, 0, 0, \ldots$), ($\pm 1, +1/2, -1/2, 0, 0, \ldots$) and ($\pm 1, 0, 0, 0, 0, \ldots$) respectively, and $\ldots$ denotes more scalars besides the minimal five. Applying (4.1), corresponding $D'_{\text{min}}$’s are 3, 2 and 1, which excludes first two cases. However, the third case is also excluded even if its $D'_{\text{min}}$ is allowed. The argument is that no spinorial product can be formed by only $|1\rangle_2$ or $|1\rangle_2$, which is known as the Spinor Excess of inconsistency elimination. The only polynomial comes from the vertex of type (d), as given in Figure 9. This polynomial is $\langle 12 \rangle$ or $[12]$.\footnote{Note that these $D = 2$ polynomials can be of either $n = 4$ or $n \geq 5$ (for which all unspecified particles are scalars). For $n = 4$, there are dimensionless pseudo polynomials of the form $([34]/(12))^x$, which can be an additional factor of the polynomials above. This factor will lead to a global shift of all four helicities. But incidentally, there is no extra legitimate pseudo polynomial after adding it.}

**D = 2 case**: The last case is $D = 2$. The possible amplitude is given in Figure 10, and corresponding polynomials are listed in Figure 11. The first one is $P^2_{x'y'}$, where $x, y$ are two unspecified scalars. The second one is $\langle 1x \rangle [x2]$, with one pair of fermions of opposite helicities. The third one is $\langle 12 \rangle [34]$, with two pairs of fermions of opposite helicities. One may consider a fourth one, with one pair of gauge bosons of opposite helicities, which is allowed since its $D'_{\text{min}}$ is 2. But this case is also excluded, as no spinorial product can be formed by only $|1\rangle^2 |2\rangle^2$ or $|1\rangle^2 |2\rangle^2$.

Note that these $D = 2$ polynomials can be of either $n = 4$ or $n \geq 5$ (for which all unspecified particles are scalars). For $n = 4$, there are dimensionless pseudo polynomials of the form $([34]/(12))^x$, which can be an additional factor of the polynomials above. This factor will lead to a global shift of all four helicities. But incidentally, there is no extra legitimate pseudo polynomial after adding it.
Figure 9: Polynomials of $D = 1$.

Figure 10: Amplitudes of $D = 2$.

Figure 11: Polynomials of $D = 2$.

Last but not the least, we need to check the analysis above has covered all possible diagrams, by using a compact formula given by

$$ D = 2 - \sum_{i=1}^{s} (s_i - 2), \quad (4.6) $$

where $D$ is the kinematic mass dimension of a Standard Model 'skeleton', as shown in Figure 12. This
The skeleton amplitude is constituted of \( S \) Standard Model components connected by internal gravitons and \( s_i \geq 3 \) is the number of external legs for each component. As when \( s_i = 2 \), there is no Standard Model vertex. Each component reduces to a Standard Model line, which cannot be a part of the skeleton by its definition, but it can be the ‘flesh’ attached to it, namely a gravitational component.

The proof of this formula is simple. For \( S \) connected components, there are \((S-1)\) internal gravitons. Each internal graviton has two attached points, which bring in two \( \kappa \)'s. Hence from (3.14), we have

\[
D = 4 - \sum_{i=1}^{S} s_i - (-2)(S - 1),
\]

which is identical to (4.6) after a trivial rearrangement.

Having this skeleton, to build a general amplitude, more gravitational components (either nontrivial components or single gravitons) can be attached to it. By applying the simplified diagrammatic rules, they can be packed into one single gravitational component.

Now let’s consider \( D = 0 \), we have \( S = 1, s_1 = 4 \) and \( S = 2, s_1 = s_2 = 3 \). These two cases correspond to the first and fourth diagrams in Figure 6. They can be attached by gravitational components, which gives the rest four diagrams. For \( D = 1 \), we only have \( S = 1, s_1 = 3 \). This diagram cannot stand alone, since there is no on-shell massless 3-particle amplitude. Hence it must be accompanied by gravitational components, which gives the two diagrams in Figure 8. For \( D = 2 \), the amplitude contains only gravitational vertices, which corresponds to the diagram in Figure 10. Therefore all possible diagrams have been covered.
5. Discussions

In this work, we are mainly concerned with the workability of the new algorithm known as multi-step BCFW recursion relations. The key techniques of this approach are pole concentration and inconsistency elimination. Its applicable range is also clarified and we find three types of objects invulnerable to BCFW deformations: polynomials, pseudo polynomials and saturated fractions. While the last two objects can be determined by other types of deformations, how to deal with polynomials is probably beyond usual on-shell methods, and it may lead to important generalizations of the present approach. Moreover, when saturated fractions arise, we need to discuss whether they are legitimate, if not, how to find an argument to rule them out is another valuable topic. Again, we would to emphasize this systematic algorithm relies on general properties of field theories only, such as Lorentz invariance, locality and unitarity. The major information we have used are mass dimension and helicities.

Ensuring its workability, we try to further improve the efficiency of multi-step BCFW recursion relations by taking two sophisticated aspects into account, as listed below:

(a) Knowing the (final or intermediate) boundary term’s schematic form (in terms of un-contracted spinors), it is also very natural to seek for a deformation which renders the boundary term vanish under the large $z$ limit, other than employing inconsistency elimination. In practice, one can consider both ways at each step to shorten the sequence of pole concentration. More profoundly, inconsistency elimination is only an argument afterwards, as any boundary term that vanishes must be killed by a good deformation with respect to that step.

(b) In practice, it is often evident that there is no need to reach the final denominator (3.5). Merely a particular intermediate form is sufficient to complete the calculation correctly. This is due to the fact that pole concentration is for eliminating as many neutral momenta as possible in the denominator, and hence increasing the spinors in the numerator, in order to keep the helicities fixed.

Let’s illustrate these two aspects through three simple examples. The first one is the MHV amplitude $A(1^{-1}, 2^{+1}, 3^{-1}, 4^{+1}, \ldots, n^{+})$, where $1^{-}$ and $3^{-}$ are non-adjacent. Assuming all lower-point MHV amplitudes are known, non-vanishing factorization limits give the common denominator

$$\frac{1}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n-1, n \rangle \langle n1 \rangle},$$

(5.1)
on the other hand, the kinematic dimension of this amplitude is $(4 - n)$, as the gauge coupling constant is dimensionless. Then correct helicities require the numerator to be $|1\rangle^4 |3\rangle^4$ schematically, which uniquely fixes the amplitude as

$$\frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n-1, n \rangle \langle n1 \rangle},$$

(5.2)
from this example, we see the schematic form is a simple but powerful tool.

The second example is amplitude $A(1^{-1}, 2^{+1}, 3^{-1}, 4^{+1}, 5^{-2})$ in Einstein-Maxwell theory. Non-vanishing factorization limits give all physical poles as $[12][32][14][34][15][25][35][45]$. Since its kinematic dimension
is 2, correct helicities require the schematic form to be

\[
\frac{[1] \cdot [2] \cdot [3] \cdot [4] \cdot \prod^4 \langle \bullet \rangle \langle \bullet \rangle}{[12] \cdot [32] \cdot [14] \cdot [34] \cdot [15] \cdot [35] \cdot [45]},
\]

(5.3)

where \(\langle \bullet \rangle \langle \bullet \rangle\)'s are unspecified neutral momenta. At this stage, one can already check that

\[
\langle 1 | 5 \rangle \langle 2 | 5 \rangle \langle 3 | 5 \rangle \langle 4 | 5 \rangle
\]

(5.4)

are good deformations because each of them induces \(z^3\) in the denominator, while the numerator can at most contain \(z^2\) to avoid Spinor Excess, i.e., \(\prod^4 \langle \bullet \rangle\) in the numerator can at most contain two identical spinors to form non-vanishing spinorial products, which restricts \(\prod^2 \langle \bullet \rangle\) in the same way. For this case, to find a deformation with maximal large \(z\) suppress is clearly more straightforward than using inconsistency elimination after a number of deformations.

The third example is even more interesting. Consider amplitude \(A(1^{-1}, 2^{+1}, 3^{-1}, 4^{+1}, 5^{-1}, 6^{+1})\) again in Einstein-Maxwell theory, factorization limits, mass dimension and helicities together fix its schematic form to be

\[
\langle 1 | 2 \rangle \langle 3 | 2 \rangle \langle 4 | 2 \rangle \langle 5 | 2 \rangle \langle 6 | 2 \rangle \times \prod^4 \langle \bullet \rangle \langle \bullet \rangle
\]

(5.5)

we will show that two successive deformations, namely

\[
\langle 3 | 1 \rangle \rightarrow \langle 5 | 1 \rangle
\]

(5.6)

can already capture the full amplitude. First, after constant extraction \(\langle 3 | 1 \rangle\), similar trick of pole concentration gives its schematic form as

\[
\frac{1}{P^2_{12} P^2_{14} P^2_{16} P^2_{32} P^2_{34} P^2_{36} P^2_{54} P^2_{56}} \times \prod^2 \langle \bullet \rangle \langle \bullet \rangle
\]

(5.7)

and constant extraction \(\langle 5 | 1 \rangle\) turns it into

\[
\langle 1 | 20 \rangle \langle 2 | 10 \rangle \langle 3 | 10 \rangle \langle 4 | 10 \rangle \langle 5 | 4 \rangle \langle 6 | 2 \rangle \times \prod^2 \langle \bullet \rangle \langle \bullet \rangle
\]

(5.8)

then there is no need to proceed further, because in the numerator Spinor Excess already arises, as \(\prod^4 \langle \bullet \rangle\) can never saturate \(1^{20}\) to form non-vanishing spinorial products. By this way, two steps can already get the correct answer, while a blind pole concentration in general requires \(4(6 - 3) = 12\) steps. Therefore, it is not always necessary to reach the final denominator, when eliminating part of neutral momenta in the denominator enforces the numerator to contain sufficient identical spinors for triggering Spinor Excess.

But this is not the end of the story. When proceeding the calculation

\[
I = P_{5|1} C_{5|1} P_{3|1} C_{5|1},
\]

(5.9)
while it is just shown that $C_{[5|1]}C_{[3|1]} = 0$, incidentally we also find $C_{[5|1]}P_{[3|1]} = 0$. This means $⟨5|1⟩$ is a good deformation and hence one step is enough. Since particles $1^{-1}$, $3^{-1}$ and $5^{-1}$ are symmetric in the helicity configuration, $⟨3|1⟩$ is also a good deformation.

In general, there is a last good deformation corollary: After the $n$-th step, when $I = (\text{known terms})_n + C_n \cdots C_0$ is reached, we can further expand it by an $(n + 1)$-th step as

$$I = P_{n+1} + C_{n+1}(\text{known terms})_n + C_{n+1}C_n \cdots C_0,$$

(5.10)

assume the $(n + 1)$-th step is the last step for which $C_{n+1}C_n \cdots C_0 = 0$. And if

$$C_{n+1}(\text{known terms})_n = 0,$$

(5.11)

then the $(n + 1)$-th step is a good deformation. This corollary is powerful in practical calculations, since unnecessary steps can be saved if we incidentally encounter the condition above.

Back to the mainline, these two aspects (a) and (b) will be demonstrated more systematically, with more examples in our future work, with a possible joint use of the last good deformation corollary. The major goal is to improve the efficiency provided the workability is ensured. The exit of this maze is now located, and how to shorten the correct route is a complicated yet fascinating problem. Finally, we would like to highlight the power of simple analysis by mass dimension and helicities. These cheap information possibly lie in the core of the future study of efficiency.

Acknowledgement

The authors would like to thank Qingjun Jin and Rijun Huang for valuable discussions. JR is grateful to Qingjun Jin for correcting the errors of the early manuscript. This work is supported by Qiu-Shi funding and Chinese NSF funding under contracts No.11031005, No.11135006 and No.11125523.

A. Independent Kinematic Variables

In the calculation of amplitudes, it is common that visually different expressions in terms of spinorial products are actually equivalent. Although one can use a numerical method to check this equivalence, it is still favorable to know the analytic way.

For an $n$-particle amplitude, let’s start with all holomorphic spinorial products $⟨ij⟩$’s as below:

\[

g_{12} \quad g_{13} \quad g_{14} \quad g_{15} \quad \cdots \\
g_{23} \quad g_{24} \quad g_{25} \quad \cdots \\
g_{34} \quad g_{35} \quad \cdots \\
\vdots \quad \vdots \quad \vdots \quad \ddots \\
g_{1,n-1} \quad g_{2,n-1} \quad g_{3,n-1} \quad g_{4,n-1} \quad \cdots \\
g_{1,n} \quad g_{2,n} \quad g_{3,n} \quad g_{4,n} \quad \cdots \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots
\]

(A.1)
We can pick $|1\rangle, |2\rangle$ as two reference spinors, and for $i, j \neq 1, 2$ all independent Schouten identities can be solved via
\[
\langle ij \rangle = \frac{1}{\langle 12 \rangle} \langle 1i \rangle \langle 2j \rangle.
\]
(A.2)

There are $C_n^2 \langle ij \rangle$'s and $C_{n-2}^2$ Schouten identities, so there are $C_n^2 - C_{n-2}^2 = 2n - 3$ independent $\langle ij \rangle$'s.

We can do the same thing for anti-holomorphic spinorial products $[ij]$'s, and get $(4n - 6)$ independent kinematic variables as below:
\[
\begin{align*}
\langle 12 \rangle & \quad [12] \\
\langle 13 \rangle & \quad \langle 23 \rangle & \quad [13] & \quad [23] \\
\langle 14 \rangle & \quad \langle 24 \rangle & \quad [14] & \quad [24] \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
\langle 1, n-1 \rangle & \quad \langle 2, n-1 \rangle & \quad [1, n-1] & \quad [2, n-1] \\
\langle 1n \rangle & \quad \langle 2n \rangle & \quad [1n] & \quad [2n]
\end{align*}
\]
(A.3)

But the momentum conservation constraint has not been imposed so far. Adding this constraint, we can solve, for example, $[13], [14], [23]$ and $[24]$ via
\[
\begin{align*}
\langle 1 \rangle \sum P[1] = 0 : & \quad \langle 13 \rangle[13] + \langle 14 \rangle[14] = -\Sigma_{11} - \langle 12 \rangle[12], \\
\langle 2 \rangle \sum P[1] = 0 : & \quad \langle 23 \rangle[13] + \langle 24 \rangle[14] = -\Sigma_{21}, \\
\langle 1 \rangle \sum P[2] = 0 : & \quad \langle 13 \rangle[23] + \langle 14 \rangle[24] = -\Sigma_{12}, \\
\langle 2 \rangle \sum P[2] = 0 : & \quad \langle 23 \rangle[23] + \langle 24 \rangle[24] = -\Sigma_{22} - \langle 12 \rangle[12],
\end{align*}
\]
(A.4)

where $\Sigma_{ij} = \sum_{k=5}^{n} \langle ik \rangle[jk]$. Now there are $(4n - 10)$ independent kinematic variables left. The form of an expression in terms of these variables is uniquely fixed.

**B. Examples of Pole Concentration**

We have claimed that all poles of the final boundary term can be turned into a common denominator $\langle i_1 i_2 \rangle^m[i_3 i_4]$. Here, two systematic sequences are presented to achieve it.

**B.1 First systematic sequence**

For the first example, there are four series of BCFW constant extractions, namely I, II, III and IV:
\[
\begin{align*}
\text{I} : & \quad \langle 2|3 \rangle & \quad \langle 3|4 \rangle & \quad \ldots & \quad \langle n-2|n-1 \rangle & \quad \langle n-1|n \rangle \\
\text{II} : & \quad \langle 1|n-1 \rangle & \quad \langle 2|n-1 \rangle & \quad \ldots & \quad \langle n-3|n-1 \rangle & \quad \langle n-2|n-1 \rangle \\
\text{III} : & \quad \langle 3|2 \rangle & \quad \langle 4|3 \rangle & \quad \ldots & \quad \langle n-3|n-4 \rangle & \quad \langle n-2|n-3 \rangle \\
\text{IV} : & \quad \langle n-3|1 \rangle & \quad \langle n-3|2 \rangle & \quad \ldots & \quad \langle n-3|n-5 \rangle & \quad \langle n-3|n-4 \rangle
\end{align*}
\]
(B.1)
where III and IV and can be copied from I and II, by reducing \( n \) to \( (n - 2) \) and swapping the holomorphic and anti-holomorphic deformed spinors. This sequence has \( 4(n - 3) \) steps in total.

To see how this fully works, we assume that all physical poles occur (restrictions such as color order, must be disregarded). For convenience, let’s define sets \( H = \{1, \ldots, n\} \) and \( A = [1, \ldots, n] \) to denote all holomorphic and anti-holomorphic poles of \( n \) particles respectively.

Also, the set of all multi-particle poles is denoted by \( M \). To fit the analysis of pole concentration, we will classify all multi-particle poles according to which particle is absent, in a default order. Concretely, they are categorized as

\[
(p_2 + X_{23})^2 \quad (P_{23} + X_{24})^2 \quad (P_{24} + X_{25})^2 \quad \ldots \quad (P_{2,n-4} + X_{2,n-3})^2 \quad (P_{2,n-3} + X_{2,n-2})^2 \quad (P_{2,n-2} + X_{2,n-1})^2
\]

where \( P_{ij} = p_i + \ldots + p_j \), \( X_{ij} \) is a sum of external momenta without those from \( p_i, \ldots, p_j \) and \( i, \ldots, j \) is the default order. As the pole momentum includes at least three particles, \( X_{23} \) must at least include two particles, \( X_{24} \) must at least include one and \( X_{25} \) can be empty. Similarly, \( X_{2,n-3} \) must at most include two particles, \( X_{2,n-2} \) must at most include one and \( X_{2,n-1} \) must be empty. Analogous restriction works for all \( X_{2i} \)’s in between. In this list, \( p_2 \) is the pivot momentum which is always included, and \( p_3, \ldots, p_{n-1} \) becomes the absent momentum one by one.

Now for \( \langle 2|3 \rangle \) in series I, the affected two-particle poles are

\[
\langle 2\bullet \rangle \notin H \text{ ex } \langle 23 \rangle \quad \langle 3 \rangle \notin A \text{ ex } [23]
\]

where ‘ex’ means ‘except’, namely all \( \langle 2\bullet \rangle \) poles are filtered out except \( \langle 23 \rangle \), and all \( \langle 3 \rangle \) poles are filtered out except \( [23] \). The affected multi-particle pole in \( M \) is

\[
(p_2 + X_{23})^2 \rightarrow [2|X_{23}|3]
\]

since \( (p_2 + X_{23})^2 \) includes particle 2 but not 3, it is \( \langle 2|3 \rangle \) detectable, while all other multi-particle poles are not, hence they remain unchanged.

Then, for \( \langle 3|4 \rangle \) we have

\[
\langle 3\bullet \rangle \notin H \text{ ex } \langle 34 \rangle \quad \langle 23 \rangle \rightarrow \langle 24 \rangle \quad \langle 4 \rangle \notin A \quad \langle 3 \rangle \in A \text{ ex } [34]
\]

where \( \langle 23 \rangle \) turns into \( \langle 24 \rangle \) in \( H \) and all \( \langle 3 \rangle \) poles except \( [34] \) are revived, which gives a net effect that all \( \langle 4 \rangle \) poles are filtered out. Now the second multi-particle pole affected is

\[
(P_{23} + X_{24})^2 \rightarrow [3|p_2 + X_{24}|4]
\]

again, since \( (P_{23} + X_{24})^2 \) includes particle 3 but not 4, it is \( \langle 3|4 \rangle \) detectable, while all other multi-particle poles remain unchanged. But don’t forget the spurious pole from \( \langle 2|3 \rangle \), namely

\[
[2|X_{23}|3] \rightarrow [2|X_{24}|4] \text{ or } [2|x_{24}|x_{244}]
\]
here when \(X_{23}\) includes two particles, and one is 4 which is filtered out by \(|4\rangle\), we are left with one particle \(x_{24}\), where \(x_{ij}\) denotes one external momentum except that from \(p_i, \ldots, p_j\). A nice fact is that the split poles \(2 x_{24}\) and \(x_{23}4\) are not the two-particle poles already filtered out by \(\langle 2|3\rangle\) and \(\langle 3|4\rangle\), in other words, the next steps will take care of them, so there is no need to look back.

Continue this fashion, for \(\langle 4|5\rangle\) we have

\[
\langle 4\bullet \rangle \notin H \text{ ex } \langle 45 \rangle \quad \langle 34 \rangle \rightarrow \langle 35 \rangle \quad \langle 24 \rangle \rightarrow \langle 25 \rangle \quad (B.8)
\]

and

\[
[2|X_{24}|4\rangle \rightarrow [2|X_{25}|5\rangle \text{ or } [2x_{25}|x_{25}5\rangle
\]

\[
[3|p_2 + X_{24}|4\rangle \rightarrow [3|p_2 + X_{25}|5\rangle \text{ or } [32|25\rangle
\]

\[
(P_{24} + X_{25})^2 \rightarrow [4|P_{23} + X_{25}|5\rangle
\]

again the split two-particle poles will be taken care of by the next steps. Note the descendent poles from \((P_{24} + X_{25})^2, \ldots, (P_{2,n-3} + X_{2,n-2})^2, P_{2,n-2}^2\) will no longer produce split poles. After step \(\langle n - 2|n - 1\rangle\), all descendent poles from \(M\) except split poles are

\[
[2|X_{2,n-1}|n - 1\rangle \quad [3|p_2 + X_{2,n-1}|n - 1\rangle \quad [4|P_{23} + X_{2,n-1}|n - 1\rangle \ldots
\]

\[
[n - 4|P_{2,n-5} + X_{2,n-1}|n - 1\rangle \quad [n - 3|P_{2,n-4} + X_{2,n-1}|n - 1\rangle \quad [n - 2|P_{2,n-3}|n - 1\rangle
\]

and the last step \(\langle n - 1|n\rangle\) will turn them into (split poles are neglected)

\[
[3|P_{12}|n\rangle \quad [4|P_{13}|n\rangle \text{ or } [4|P_{23}|n\rangle \ldots
\]

\[
[n - 4|P_{1,n-5}|n\rangle \text{ or } [n - 4|P_{2,n-5}|n\rangle \quad [n - 3|P_{1,n-4}|n\rangle \text{ or } [n - 3|P_{2,n-4}|n\rangle \quad [n - 2|P_{2,n-3}|n\rangle
\]

For two-particle poles, after series I we have

\[
H = \{\langle \bullet n \rangle\}, \quad A = [1, \ldots, n - 1],
\]

so all poles in \(H\) include \(|n\rangle\) and any pole in \(A\) does not include \(|n\rangle\). Of course, all these poles can have powers larger than one.

After series II, it is easy to check that

\[
H = \{\langle n - 1, n \rangle\}, \quad A = [1, \ldots, n - 2].
\]

For spurious poles of the form \(|\bullet \bullet \bullet \bullet \rangle\) from \(M\), series II also nicely turns them into the poles in \(H\) and \(A\). To verify this, let’s single out the following series in II (note that \(\langle 1|n - 1\rangle\) and \(\langle 2|n - 1\rangle\) are set aside temporally):

\[
\langle 2|n - 1\rangle \ldots \langle 3|n - 1\rangle
\]
and it will be enough to take care of spurious poles \( \bullet \bullet \bullet \). Explicitly, we have
\[
\langle 2 | n - 1 \rangle : [3 | P_{12} | n] \quad \rightarrow \quad [32] \langle n - 1, n \rangle
\]
\[
\langle 3 | n - 1 \rangle : [4 | P_{13} | n] \quad \text{or} \quad [4 | P_{23} | n] \quad \rightarrow \quad [43] \langle n - 1, n \rangle
\]
\[
\vdots
\]
\[
\langle n - 4 | n - 1 \rangle : [n - 3 | P_{1,n-4} | n] \quad \text{or} \quad [n - 3 | P_{2,n-4} | n] \quad \rightarrow \quad [n - 3, n - 4] \langle n - 1, n \rangle
\]
\[
\langle n - 3 | n - 1 \rangle : [n - 2 | P_{2,n-3} | n] \quad \rightarrow \quad [n - 2, n - 3] \langle n - 1, n \rangle
\]
Therefore these spurious poles finally become parts of \( H \) and \( A \) after series I and II, we are only left with \( \{ \langle n - 1, n \rangle \} \) and \( \{1, \ldots, n - 2\} \). To concentrate them completely, series III and IV are needed, which are copied from I and II by replacing \( n \) by \( n - 2 \) and swapping \( \bullet \) and \( \bullet \) for all deformations.

After series III and IV, a trivial imitation gives
\[
H = \{ \langle n - 1, n \rangle \}, \quad A = \{ [n - 3, n - 2] \}.
\]
Note that \( H \) is in fact completely inert to series III and IV as only \( A \) is manipulated. Therefore we manage to turn all poles, regardless of two or more particles, physical of spurious, into a common denominator
\[
\frac{1}{\langle i_1 i_2 \rangle^m \langle i_3 i_4 \rangle^m}
\]
where \( i_1, i_2, i_3, i_4 \) are four different arbitrary particles after a trivial relabeling.

For \( n = 4 \), merely series I and II can turn all poles into such a form (in fact series III and IV do not exist). Consider the denominator \( P_{23}^2 P_{24}^2 P_{34}^2 \), as there are many equivalent choices related by momentum conservation, after
\[
\langle 2 | 3 \rangle \quad \langle 3 | 4 \rangle \quad \langle 1 | 3 \rangle \quad \langle 2 | 3 \rangle
\]
it becomes \( \langle 34 \rangle^3 \langle 12 \rangle^3 \). Again a trivial relabeling gives the general form \( \langle i_1 i_2 \rangle^3 \langle i_3 i_4 \rangle^3 \).

**B.2 Second systematic sequence**

As such a sequence is not unique, below we briefly present another example, given by
\[
\text{I : } \langle 1 | n \rangle \quad \langle 2 | n \rangle \quad \ldots \quad \langle n - 3 | n \rangle \quad \langle n - 2 | n \rangle
\]
\[
\text{II : } \langle n - 2 | 1 \rangle \quad \langle n - 2 | 2 \rangle \quad \ldots \quad \langle n - 2 | n - 4 \rangle \quad \langle n - 2 | n - 3 \rangle
\]
\[
\text{III : } \langle 1 | n - 1 \rangle \quad \langle 2 | n - 1 \rangle \quad \ldots \quad \langle n - 4 | n - 1 \rangle \quad \langle n - 3 | n - 1 \rangle
\]
\[
\text{IV : } \langle n - 3 | 1 \rangle \quad \langle n - 3 | 2 \rangle \quad \ldots \quad \langle n - 3 | n - 5 \rangle \quad \langle n - 3 | n - 4 \rangle
\]
which also has \( 4(n - 3) \) steps. After series I, we are left with two-particle poles
\[
H = \{ \langle n \bullet \rangle \}, \quad A = \{1, \ldots, n - 1\},
\]

\[\text{– 30 –}\]
and spurious poles (for multi-particle poles, $p_n$ is the pivot momentum which is always included)

$$[1|X_{n1}|n⟩ \quad [2|X_{n2}|n⟩ \quad \ldots \quad [n-4|X_{n,n-4}|n⟩ \quad [n-3|X_{n,n-3}|n⟩ \quad (B.21)$$

where $X_{ni}$ is a sum of external momenta without those from $p_n, p_1, \ldots, p_i$. Note that when $i = n-2$, the spurious pole $[n-2|X_{n,n-2}|n⟩$ is split into $[n-2, n-1]|n-1, n⟩$. After series II, we are left with

$$H = \{⟨n•⟩, |•⟩ \neq |n-2⟩\}, \quad A = \{[n-2, •], |•⟩ \neq |n⟩\}. \quad (B.22)$$

After series III, we are left with

$$H = \{⟨n,n-1⟩\}, \quad A = \{[n-2, •], |•⟩ \neq |n-1⟩, |n⟩\}. \quad (B.23)$$

Finally, after series IV, the final denominator reads also as $⟨n, n⟩^{n-3, n-2}$. For this sequence, in each series all deformations are straight descendent. However, two steps of different series can be skew descendent.

In general, not all physical poles can be detected by independent and straight descendent deformations alone. To calculate an $n$-particle amplitude by using only these two types of deformations, one can assign $a$ labels for $⟨i⟩$ and $(n-a)$ labels for $|j⟩$ in a deformation series $⟨i|j⟩$. Then two-particle poles $[i_1 i_2]$ with $i_1, i_2 \in I_a$ and $⟨j_1 j_2⟩$ with $j_1, j_2 \in I_{n-a}$ cannot be detected, where $I_a$ and $I_{n-a}$ denote the sets of $⟨i⟩$ and $|j⟩$ respectively.

C. (Pseudo) Polynomials and Saturated Fractions

This part gives the classification of all three types of objects that cannot be determined by multi-step BCFW recursion relations. They include (pseudo) polynomials and saturated fractions.

C.1 Polynomials and pseudo polynomials

Now, we list all (pseudo) polynomials that satisfy certain dimension and helicities, up to $D = 2$. This list can be similarly extended for $D \geq 3$.

First consider $n \geq 5$, as there is a tricky issue of $n = 4$. For dimension (or polynomial degree) $D = 0$, there is only one choice of polynomial, namely 1. The helicity configuration is simply

$$(0, 0, 0, 0, 0, \ldots) \quad (C.1)$$

where $\ldots$ denotes more scalars besides the minimal five.

For dimension $D = 1$, there are two choices: $⟨••⟩$ or $[••]$ and the helicity configuration is

$$\begin{pmatrix} ±\frac{1}{2}, ±\frac{1}{2}; 0, 0, 0, \ldots \end{pmatrix} \quad (C.2)$$
For dimension $D = 2$, there are three choices: \((\bullet\bullet)(\bullet\bullet), (\bullet\bullet)[\bullet\bullet]\) or \[[\bullet\bullet][\bullet\bullet]\]. One also needs to separate the cases for which one or two pair(s) of particle labels in the spinorial products are identical. Hence the helicity configuration can be

\[
\begin{align*}
&\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, 0, \ldots\right) \quad \left(\pm \frac{1}{2}, \pm \frac{1}{2}, 1, 0, 0, \ldots\right) \quad (\pm 1, \pm 1, 0, 0, 0, \ldots) \\
&\left(-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, 0, 0, \ldots\right) \quad \left(-\frac{1}{2}, +\frac{1}{2}, 0, 0, 0, \ldots\right) \quad (0, 0, 0, 0, 0, \ldots)
\end{align*}
\]  

(C.3)

When $n = 4$, there exist fraction-like pseudo polynomials. Explicitly, for the following identity

\[
\left(\frac{[34]}{\langle 12 \rangle}\right) = \left(\frac{[32]}{\langle 14 \rangle}\right) = \left(\frac{[42]}{\langle 13 \rangle}\right) = \left(\frac{[31]}{\langle 42 \rangle}\right) = \left(\frac{[12]}{\langle 34 \rangle}\right) = \left(\frac{[14]}{\langle 32 \rangle}\right)
\]  

(C.4)

one can check that, there is no effective BCFW deformation that can detect poles above, as the numerator vanishes simultaneously when the denominator reaches to zero, due to momentum conservation. Moreover, pseudo polynomials are in fact inert to BCFW deformations, for example under \langle 1|3\rangle,

\[
\left(\frac{[34]}{\langle 12 \rangle}\right) + z\left(\frac{[14]}{\langle 32 \rangle}\right) = \frac{[14]}{\langle 32 \rangle} + z = \frac{[34]}{\langle 12 \rangle},
\]

again due to momentum conservation. In general, a dimensionless pseudo polynomial takes the form

\[
\left(\frac{[34]}{\langle 12 \rangle}\right)^x,
\]

(C.6)

where $x = \pm 1, \pm 2, \pm 3, \pm 4$, as the spin is restricted within 2.

Since the dimensionless pseudo polynomial behaves like a numerical factor, polynomials of $n \geq 5$ can be multiplied by these factors to generate all possible pseudo polynomials. Hence, for $n = 4$ we have the following helicity configurations, corresponding to all (pseudo) polynomials:

\[
\begin{align*}
D = 0 : \quad & (y, y, y, y) \quad -2 \leq y \leq 2 \\
D = 1 : \quad & \left(\frac{1}{2} + y, \frac{1}{2} + y, y, y\right) \quad -2 \leq y \leq \frac{3}{2} \\
D = 2 : \quad & (y, y, y, y) \quad -2 \leq y \leq 2 \\
& \quad \left(1 + y, 1 + y, y, y\right) \quad -2 \leq y \leq 1 \\
& \quad \left(-\frac{1}{2} + y, \frac{1}{2} + y, y, y\right) \quad -\frac{3}{2} \leq y \leq \frac{3}{2}
\end{align*}
\]

(C.7)

where $y = 0, x/2$. Since (pseudo) polynomials of $n = 4$ are dimensionally equal, up to a global shift of all four helicities, for simplicity one can first select a set of representatives to analyze. As one example, the
representative helicity configuration can be

\[
D = 0 : \quad (0, 0, 0, 0)
\]
\[
D = 1 : \quad \left( \pm \frac{1}{2}, \pm \frac{1}{2}, 0, 0 \right)
\]
\[
D = 2 : \quad \begin{cases} 
(0, 0, 0, 0) \\
(\pm 1, \pm 1, 0, 0) \\
\left( -\frac{1}{2}, \frac{1}{2}, 0, 0 \right)
\end{cases}
\]  \hspace{1cm} (C.8)

then \( y \) must be shifted around to cover all possible cases in (C.7).

C.2 Saturated fractions

The SF is an irreducible fraction, but impossible to be killed by BCFW constant extractions. A minimally
\( z \)-power inducing constant extraction will only change its particle labels, but not its form. For example,

\[
\frac{[34][56]}{(12)}
\]  \hspace{1cm} (C.9)

is a pure SF, and constant extraction \( \langle 1|3 \rangle \) turns it into

\[
C_{\langle 1|3 \rangle} \frac{[34][56]}{(12)} = \frac{[14][56]}{(32)}.
\]  \hspace{1cm} (C.10)

For a pure SF, all effective BCFW deformations are minimally \( z \)-power inducing. Next consider

\[
\frac{[34]^2[56]}{(12)} = [34] \times \frac{[34][56]}{(12)},
\]  \hspace{1cm} (C.11)

which is a mixed SF, namely a pure SF times a polynomial. When we use \( \langle 1|3 \rangle \) again, it is not minimally
\( z \)-power inducing, \( i.e., \)

\[
\frac{[34]^2[56]}{(12)} \rightarrow \frac{([34] + z[14])^2[56]}{(12) - z(32)} \sim O(z),
\]  \hspace{1cm} (C.12)

for this mixed SF, one minimally \( z \)-power inducing deformation is \( \langle 1|5 \rangle \), for which the constant extraction
will give only one term with relabeling. But \( \langle 1|3 \rangle \) gives more than one term, as

\[
C_{\langle 1|3 \rangle} \frac{[34]^2[56]}{(12)} = -2[34] \times \frac{[14][56]}{(32)} - \frac{[14]^2(12)[56]}{(32)^2},
\]  \hspace{1cm} (C.13)

note that the first term above is also a mixed SF again with relabeling. In general, under a non-minimally
\( z \)-power inducing constant extraction, a mixed SF will transform into more than one term. These terms
include a special one, which is related to the original SF by relabeling.
To investigate SF’s more systematically, let’s analyze one special category: When $0 \leq h_1, h_2 < m/2$ and $0 \leq h_3, h_4$, the dependence on $m, \overline{m}$ dissolves in (3.20), then

$$D' = \sum_{i=1,2} (-h_i + \beta_i) + \sum_{i=3,4} (h_i + \alpha_i) + \sum_{h<0} (-h_i + \beta_i) + \sum_{h\geq0} (h_i + \alpha_i),$$  \hspace{1cm} (C.14)

and the corresponding final boundary term schematically reads

$$\frac{1}{\langle 12 \rangle^m \langle 34 \rangle^m} \prod_{i=1,2} \langle i \rangle^{-2h_i+m} p_i^\beta_i \prod_{i=3,4} \langle i \rangle^{2h_i+\overline{m}} p_i^\alpha_i \prod_{h<0} \langle i \rangle^{-2h_i} p_i^\beta_i \prod_{h\geq0} \langle i \rangle^{2h_i} p_i^\alpha_i.$$  \hspace{1cm} (C.15)

**1st type:** Let’s consider a first type of amplitudes, (1st type): $n$ necessarily unique, so we only present one example for each case)

For this type, FD arises when $\langle h, h, h, h, \ldots \rangle$ with $0 \leq h \leq 2$. We will explain why nontrivial SF’s only exist for $n \geq 5$ in the end. At this point, (C.14) becomes

$$D' = (n - 4)h + \beta_1 + \beta_2 + \sum_{i=3}^n \alpha_i,$$  \hspace{1cm} (C.16)

and the final boundary term schematically reads

$$\frac{1}{\langle 12 \rangle^{2h}} \langle 1 \rangle^{\beta_1} \langle 2 \rangle^{\beta_2} \langle 3 \rangle^{\alpha_3} \langle 4 \rangle^{\alpha_4} \ldots \langle n \rangle^{\alpha_n} \times [1]^{\beta_3}[2]^{\beta_4}[3]^{2h+\alpha_3}[4]^{2h+\alpha_4} \ldots [n]^{2h+\alpha_n},$$  \hspace{1cm} (C.17)

where we have maximally reduced $\langle 1 \rangle, \langle 2 \rangle$ in the fraction, hence in the numerator

$$(\text{num. of } \{\bullet\}) = \beta_1 + \beta_2 + \sum_{i=3}^n \alpha_i, \hspace{0.5cm} (\text{num. of } \{\bullet\}) = (n - 2)2h + \beta_1 + \beta_2 + \sum_{i=3}^n \alpha_i.$$  \hspace{1cm} (C.18)

For this type, FD arises when $n = \text{odd}$ and $h = 1/2, 3/2$ with $D \geq D'_{\min}$, PM arises when $D' - nh = \text{odd}$ with $D \geq D'_{\min}$ and $2nh = \text{even}$. From (C.16) and (C.17), it’s easy to check that:

When $D = 0$, there is no consistent choice. We do not consider (pseudo) polynomials here.

When $D = 1$, $n = 5$, $h = 1$ is SF, $h = 1/2$ is FD, $h = 0$ is PM.

When $D = 1$, $n = 6$, $h = 1/2$ is SF, $h = 0$ is PM.

When $D = 2$, $n = 5$, $h = 1$ is PM, $h = 1/2$ is FD.

When $D = 2$, $n = 6$, $h = 1$ is SF, $h = 1/2$ is PM.

When $D = 2$, $n = 7$, $h = 1/2$ is FD.

When $D = 2$, $n = 8$, $h = 1/2$ is SF.

For $D = 0, 1, 2$ all other cases are inconsistent since $D < D'_{\min}$. SF’s above are (the form of an SF is not necessarily unique, so we only present one example for each case)

$$\text{SF}(D = 1, n = 5, h = 1) = \frac{[34][35][45]}{(12)^2}, \hspace{0.5cm} \text{SF} \left( D = 1, n = 6, h = \frac{1}{2} \right) = \frac{[34][56]}{(12)},$$

$$\text{SF}(D = 2, n = 6, h = 1) = \left( \frac{[34][56]}{(12)} \right)^2, \hspace{0.5cm} \text{SF} \left( D = 2, n = 8, h = \frac{1}{2} \right) = \frac{[34][56][78]}{(12)}.$$  \hspace{1cm} (C.19)

---

12All amplitudes in this part are not necessarily physical, as the investigation is purely of mathematical interest.
2nd type: Similarly, a second type of amplitudes is \( (h, h, h + 1/2, h, h, \ldots) \) with \( 0 \leq h \leq 3/2 \), then

\[
D' = (n - 4)h + \frac{1}{2} + \beta_1 + \beta_2 + \sum_{i=3}^{n} \alpha_i. \tag{C.20}
\]

For this type, we find that:

- When \( D = 0 \), there is no consistent choice.
- When \( D = 1 \), \( n = 5, h = 1/2 \) is SF, \( h = 0 \) is FD.
- When \( D = 1 \), \( n = 6, 7, 8, \ldots \), \( h = 0 \) is FD.
- When \( D = 2 \), \( n = 5, h = 3/2 \) is SF, \( h = 1 \) is FD, \( h = 1/2 \) is PM, \( h = 0 \) is FD.
- When \( D = 2 \), \( n = 6, h = 3/2, 1, 1/2, 0 \) is FD.
- When \( D = 2 \), \( n = 7, h = 1/2 \) is SF, \( h = 0 \) is FD.
- When \( D = 2 \), \( n = 8, 9, 10, \ldots \), \( h = 0 \) is FD.

SF’s above are

\[
\text{SF} \left( D = 1, n = 5, h = \frac{1}{2} \right) = \frac{[34][35]}{\langle 12 \rangle},
\]

\[
\text{SF} \left( D = 2, n = 5, h = \frac{3}{2} \right) = \frac{[34]^2[35]^2[45]}{\langle 12 \rangle^3}, \quad \text{SF} \left( D = 2, n = 7, h = \frac{1}{2} \right) = \frac{[34][35][67]}{\langle 12 \rangle^3}. \tag{C.21}
\]

3rd type: A third type is \( (h, h, h + 1/2, h + 1/2, h, \ldots) \) with \( 0 \leq h \leq 3/2 \), then

\[
D' = (n - 4)h + 1 + \beta_1 + \beta_2 + \sum_{i=3}^{n} \alpha_i. \tag{C.22}
\]

For this type, we find that:

- When \( D = 0, 1 \), there is no consistent choice.
- When \( D = 2 \), \( n = 5, h = 1 \) is SF, \( h = 1/2 \) is FD, \( h = 0 \) is PM.
- When \( D = 2 \), \( n = 6, h = 1/2 \) is SF, \( h = 0 \) is PM.
- When \( D = 2 \), \( n = 7, 8, 9, \ldots, h = 0 \) is PM.

SF’s above are

\[
\text{SF} \left( D = 2, n = 5, h = 1 \right) = \frac{[34][35]^2[45]}{\langle 12 \rangle^2}, \quad \text{SF} \left( D = 2, n = 6, h = \frac{1}{2} \right) = \frac{[34][35][56]}{\langle 12 \rangle}. \tag{C.23}
\]

4th type: A fourth type is \( (h, h, h + 1, h, h, \ldots) \) with \( 0 \leq h \leq 1 \), then

\[
D' = (n - 4)h + 1 + \beta_1 + \beta_2 + \sum_{i=3}^{n} \alpha_i. \tag{C.24}
\]

For this type, we find that:

- When \( D = 0 \), there is no consistent choice.
- When \( D = 1 \), \( n = 5, 6, 7, \ldots, h = 0 \) is SE.
- When \( D = 2 \), \( n = 5, h = 1 \) is SF, \( h = 1/2 \) is FD, \( h = 0 \) is PM.
When $D = 2$, $n = 6$, $h = 1/2$ is SF, $h = 0$ is PM.
When $D = 2$, $n = 7, 8, 9, \ldots$, $h = 0$ is PM.

SF’s above are

\[
SF (D = 2, n = 5, h = 1) = \frac{[34]^2[35]^2}{\langle 12 \rangle^2}, \quad SF (D = 2, n = 6, h = \frac{1}{2}) = \frac{[34][35][36] \langle 12 \rangle}{\langle 12 \rangle^2}.
\] (C.25)

We will not continue to present a fifth type, as these toy examples have provided enough intuition for the generation of SF’s. Naturally SF’s are highly un-physical, but their existence in some artificial (effective) theories cannot be excluded. Sometimes even if an SF is admitted, we can use other arguments to exclude it, such as its denominator contains a spurious pole. Consider an SF whose denominator is $\langle 12 \rangle^2$, or $\langle 12 \rangle$ but the amplitude has vanishing factorization limit under $\langle 12 \rangle \to 0$, then one needs to delicately remove all dependence on the spurious pole $\langle 12 \rangle^2$ or $\langle 12 \rangle$ of the known terms in (2.4), if it appears.

For $n = 4$, an SF is equivalent to a pseudo polynomial. Consider

\[
\frac{[34]}{\langle 12 \rangle} = -\frac{[32]}{\langle 14 \rangle}.
\] (C.26)

under $\langle 12 \rangle$, the LHS is a polynomial while the RHS is an SF. Plus momentum conservation, for the RHS both its form and particle labels are preserved. For this reason, SF’s of $n = 4$ are trivially inert to BCFW constant extractions, therefore they are in fact pseudo polynomials.

References

[1] R. Britto, F. Cachazo and B. Feng, “New recursion relations for tree amplitudes of gluons,” Nucl. Phys. B 715, 499 (2005) [hep-th/0412308].
[2] R. Britto, F. Cachazo, B. Feng and E. Witten, “Direct proof of tree-level recursion relation in Yang-Mills theory,” Phys. Rev. Lett. 94, 181602 (2005) [hep-th/0501052].
[3] B. Feng and M. Luo, “An Introduction to On-shell Recursion Relations,” Front. Phys. 7, 533 (2012) [arXiv:1111.5759 [hep-th]].
[4] H. Elvang and Y. t. Huang, “Scattering Amplitudes,” arXiv:1308.1697 [hep-th].
[5] N. Arkani-Hamed and J. Kaplan, “On Tree Amplitudes in Gauge Theory and Gravity,” JHEP 0804, 076 (2008) [arXiv:0801.2385 [hep-th]].
[6] C. Cheung, “On-Shell Recursion Relations for Generic Theories,” JHEP 1003, 098 (2010) [arXiv:0808.0504 [hep-th]].
[7] P. Benincasa and F. Cachazo, “Consistency Conditions on the S-Matrix of Massless Particles,” arXiv:0705.4305 [hep-th].
[8] R. H. Boels, “No triangles on the moduli space of maximally supersymmetric gauge theory,” JHEP 1005, 046 (2010) [arXiv:1003.2989 [hep-th]].
[9] B. Feng, J. Wang, Y. Wang and Z. Zhang, “BCFW Recursion Relation with Nonzero Boundary Contribution,” JHEP 1001, 019 (2010) [arXiv:0911.0301 [hep-th]].

– 36 –
[10] B. Feng and C. Y. Liu, “A Note on the boundary contribution with bad deformation in gauge theory,” JHEP 1007, 093 (2010) [arXiv:1004.1282 [hep-th]].

[11] B. Feng and Z. Zhang, “Boundary Contributions Using Fermion Pair Deformation,” JHEP 1112, 057 (2011) [arXiv:1109.1887 [hep-th]].

[12] P. Benincasa and E. Conde, “On the Tree-Level Structure of Scattering Amplitudes of Massless Particles,” JHEP 1111, 074 (2011) [arXiv:1106.0166 [hep-th]].

[13] P. Benincasa and E. Conde, “Exploring the S-Matrix of Massless Particles,” Phys. Rev. D 86, 025007 (2012) [arXiv:1108.3078 [hep-th]].

[14] B. Feng, Y. Jia, H. Luo and M. Luo, “Roots of Amplitudes,” arXiv:1111.1547 [hep-th].

[15] K. Zhou and C. Qiao, “General tree-level amplitudes by factorization limits,” arXiv:1410.5042 [hep-th].

[16] C. Cheung, C. H. Shen and J. Trnka, “Simple Recursion Relations for General Field Theories,” arXiv:1502.05057 [hep-th].

[17] B. Feng, K. Zhou, C. Qiao and J. Rao, “Determination of Boundary Contributions in Recursion Relation,” JHEP 1503, 023 (2015) [arXiv:1411.0452 [hep-th]].

[18] Q. Jin and B. Feng, “Recursion Relation for Boundary Contribution,” arXiv:1412.8170 [hep-th].

[19] K. Risager, “A Direct proof of the CSW rules,” JHEP 0512, 003 (2005) [hep-th/0508206].

[20] D. A. McGady and L. Rodina, “Higher-spin massless S-matrices in four-dimensions,” Phys. Rev. D 90, no. 8, 084048 (2014) [arXiv:1311.2938 [hep-th]].