On new weakly periodic Gibbs measures of the Ising model on the Cayley tree of order \( \leq 5 \)

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Abstract. For the Ising model on the Cayley tree, we find new weakly periodic Gibbs measures corresponding to normal subgroups of index two in the group representation of the Cayley tree of order \( k \leq 5 \).

1. Introduction
One of the main problems for the Ising model is to describe all limiting Gibbs measures corresponding to this model. It is well known that for the Ising model, such measures form a nonempty, convex, and compact subset in the set of all probability measures. The problem of completely describing the element of this set is far from being completely solved. Some translation-invariant (see, e.g., [1],[8],[14]), periodic [3], [9], and continuum sets of non-periodic [8], [10] Gibbs measures for the Ising model on a Cayley tree have already been described.

To extend the set of periodic Gibbs measures corresponding to the Ising model, the notion of periodic measures was generalized to that of weakly periodic Gibbs measures in [6],[11]-[13]. In the mentioned paper under some conditions on parameters for the Ising model (on the Cayley tree of order \( \geq 6 \)), the existence of such new measures was proved. But the used methods did not allow us to find conditions for the existence of weakly periodic Gibbs measures. Therefore, main aim of this paper to study existence of weakly periodic (non-periodic) Gibbs measures for the mentioned model on the Cayley tree of order less than \( 5 \).

This paper is organized as follows. In Section 2 we give necessary definitions and formulate the problem. Section 3 is devoted to weakly periodic (non-periodic) Gibbs measures corresponding to the normal subgroups of index two in the group representation of the Cayley tree.

2. Definitions and the main problem.
Let \( \tau^k = (V, L), k \geq 1 \), be the Cayley tree of order \( k \), i.e. an infinity graph every vertex of which is incident to exactly \( k+1 \) edges. Here \( V \) is the set of all vertices, \( L \) is the set of all edges of the tree \( \tau^k \). It is known that \( \tau^k \) can be represented as \( G_k \), which is the free product of \( k+1 \) cyclic groups of the second order [4], [8].

For an arbitrary point \( x^0 \in V \) we set \( W_n = \{ x \in V | d(x^0, x) = n \} \), \( V_n = \bigcup_{m=0}^{n} W_m \), \( L_n = \{ < x, y > \in L | x, y \in V_n \} \), where \( d(x, y) \) is the distance between the vertices \( x \) and \( y \) in the Cayley tree, i.e. the number of edges in the shortest path joining the vertices \( x \) and \( y \).
Let $\Phi = \{-1, 1\}$ and let $\sigma \in \Omega = \Phi^V$ be a configuration, i.e. $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. Let $A \subset V$. We let $\Omega_A$ denote the space of configurations defined on the set $A$ and taking values in $\Phi$.

We consider the Hamiltonian of the Ising model:

$$H(\sigma) = -J \sum_{<x,y>\in L} \sigma(x)\sigma(y), \quad (2.1)$$

where $J \in R$, $\sigma(x) \in \Phi$ and $<x,y>$ are nearest neighbors.

Let $h_x \in R$, $x \in V$. For every $n$, we then define a measure $\mu_n$ on $\Omega_{V_n}$ setting

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\}, \quad (2.2)$$

where $\beta = \frac{1}{T}$ ($T$ is temperature, $T > 0$), $\sigma_n = \{\sigma(x), x \in V_n\} \in \Omega_{V_n}$, $Z_n^{-1}$ is the normalizing factor, and

$$H(\sigma_n) = -J \sum_{<x,y>\in L} \sigma(x)\sigma(y).$$

The compatibility condition for the measures $\mu_n(\sigma_n)$, $n \geq 1$, is

$$\sum_{\sigma^{(n)}} \mu_n(\sigma_{n-1}, \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}), \quad (2.3)$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

Let $\mu_n, n \geq 1$ be a sequence of measures on the sets $\Omega_{V_n}$ that satisfy compatibility condition (2.3). By the Kolmogorov theorem, we then have a unique limit measure $\mu$ on $\Omega_V = \Omega$ (called the limit Gibbs measure) such that

$$\mu(\sigma_n) = \mu_n(\sigma_n)$$

for every $n = 1, 2, ...$. It is known that measures (2.2) satisfies the condition (2.3) if and only if the set $h = \{h_x, x \in G_k\}$ of quantities satisfies the condition

$$h_x = \sum_{y \in S(x)} f(h_y, \theta), \quad (2.4)$$

where $S(x)$ is the set of direct successors (children) of the point $x \in V$ (see.[8], [1], [14]). Here, $f(x, \theta) = \arctanh(\theta \tanh x)$, $\theta = \tanh(J\beta)$, $\beta = \frac{1}{T}$.

Let $G_k/\hat{G}_k = \{H_1, ..., H_r\}$ be a factor group, where $\hat{G}_k$ is a normal subgroup of index $r \geq 1$.

**Definition 2.1.** A set $h = \{h_x, x \in G_k\}$ of quantities is called $\hat{G}_k$-periodic if $h_{xy} = h_x$, for all $x \in G_k$ and $y \in \hat{G}_k$.

For $x \in G_k$ we denote by $x_1$ the unique point of the set $\{y \in G_k : (x, y)\} \setminus S(x)$.

**Definition 2.2.** A set of quantities $h = \{h_x, x \in G_k\}$ is called $\hat{G}_k$-weakly periodic, if $h_x = h_{i_1}$, for any $x \in H_i, x_{i_1} \in H_j$.

We note that the weakly periodic $h$ coincides with an ordinary periodic one (see Definition 2.1) if the quantity $h_x$ is independent of $x_1$.

**Definition 2.3.** A Gibbs measure $\mu$ is said to be $\hat{G}_k$-(weakly) periodic if it corresponds to the $\hat{G}_k$-(weakly) periodic $h$. We call a $G_k$-periodic measure a translation-invariant measure.

In this paper, we study weakly periodic Gibbs measures and demonstrate that such measures exist for the Ising model on a Cayley tree of order $\leq 5$. 

2
3. Weakly periodic measures

The level of difficulty in the describing of weakly periodic Gibbs measures is related to the structure and index of the normal subgroup relative to which the periodicity condition is imposed. It is known (see Chapter 1 of [8]) that in the group \( G_k \), there is no normal subgroup of odd index different from one. Therefore, we consider normal subgroups of even indices. Here, we restrict ourself to the case of index two.

We describe \( \hat{G}_k \)-weakly periodic Gibbs measures for any normal subgroup \( \hat{G}_k \) of index two. We note (see Chapter 1 of [8]) that any normal subgroup of index two of the group \( G_k \) has the form

\[
H_A = \left\{ x \in G_k : \sum_{i \in A} \omega_x(a_i) = \text{even} \right\},
\]

where \( \emptyset \neq A \subseteq N_k = \{1, 2, \ldots, k + 1\} \), and \( \omega_x(a_i) \) is the number of letters \( a_i \) in a word \( x \in G_k \).

Let \( A \subseteq N_k \) and \( H_A \) be the corresponding normal subgroup of index two. We note that in the case \( |A| = k + 1 \), i.e., in the case \( A = N_k \), weak periodicity coincides with ordinary periodicity. Therefore, we consider \( A \subset N_k \) such that \( A \neq N_k \). Then, in view of (2.4), the \( H_A \)-weakly periodic set of \( h \) has the form

\[
h_x = \begin{cases} 
  h_1, & x \in H_A, x_\perp \in H_A, \\
  h_2, & x \in H_A, x_\perp \in G_k \backslash H_A, \\
  h_3, & x \in G_k \backslash H_A, x_\perp \in H_A, \\
  h_4, & x \in G_k \backslash H_A, x_\perp \in G_k \backslash H_A,
\end{cases}
\]

(3.1)

where \( h_i, i = 1, 2, 3, 4 \), satisfy the following equations:

\[
\begin{align*}
  h_1 &= |A|f(h_3, \theta) + (k - |A|)f(h_1, \theta), \\
  h_2 &= (|A| - 1)f(h_3, \theta) + (k + 1 - |A|)f(h_1, \theta), \\
  h_3 &= (|A| - 1)f(h_2, \theta) + (k + 1 - |A|)f(h_4, \theta), \\
  h_4 &= |A|f(h_2, \theta) + (k - |A|)f(h_4, \theta).
\end{align*}
\]

(3.2)

Consider operator \( W : R^4 \rightarrow R^4 \), defined by

\[
\begin{align*}
  h_1' &= |A|f(h_3, \theta) + (k - |A|)f(h_1, \theta) \\
  h_2' &= (|A| - 1)f(h_3, \theta) + (k + 1 - |A|)f(h_1, \theta) \\
  h_3' &= (|A| - 1)f(h_2, \theta) + (k + 1 - |A|)f(h_4, \theta) \\
  h_4' &= |A|f(h_2, \theta) + (k - |A|)f(h_4, \theta).
\end{align*}
\]

(3.3)

Note that the system of equations (3.2) is the equation \( h = W(h) \).

It is obvious that the following sets are invariant with respect to operator \( W \):

\[
I_1 = \{ h \in R^4 : h_1 = h_2 = h_3 = h_4 \}, \quad I_2 = \{ h \in R^4 : h_1 = h_4; h_2 = h_3 \},
\]

\[
I_3 = \{ h \in R^4 : h_1 = -h_4; h_2 = -h_3 \}.
\]

Set \( \alpha = \frac{1 + \theta}{1 + \theta} \).

**Theorem 3.1.** ([7]). The following assertions hold:

1) For the Ising model, all \( H_A \)-weakly periodic Gibbs measures on \( I_1 \) and \( I_2 \) are translation invariant.

2) For \( |A| = k \), all \( H_A \)-weakly periodic Gibbs measures are translation invariant.
3) For $|A| = 1$ and $k = 4$, there exists a critical value $\alpha_{cr}$ ($\approx 0.1569$) such that there exist five $H_A$-weakly periodic Gibbs measures $\mu_0, \mu_1^+, \mu_1^-, \mu_2^+, \mu_2^-$, for $0 < \alpha < \alpha_{cr}$, three $H_A$-weakly periodic Gibbs measures $\mu_0, \mu_1^+, \mu_1^-$, for $\alpha = \alpha_{cr}$, and only one $H_A$-weakly periodic Gibbs measure $\mu_0$ for $\alpha > \alpha_{cr}$.

4) For $|A| = 1$, $k \geq 6$, and $\theta \in (\theta_1, \theta_2)$, where $\theta_{1,2} = \frac{k-1+\sqrt{k^2-6k+1}}{2k}$, there exist three $H_A$-weakly periodic Gibbs measures $\mu_0$ and $\mu^+$, $\mu^-$ on $I_3$.

**Remark 3.2.** The measures $\mu^-, \mu^+, \mu_1^+, \mu_1^-$ are $H_A$-weakly periodic Gibbs measures for the Ising model. All the other measures constructed in Theorem 3.1 are translation invariant.

In parts 3) and 4) of Theorem 3.1 the cases $k = 4$ and $k \geq 6$ are considered (see [11]). Therefore remain the cases $k = 2, k = 3, k = 5$. In this paper we consider these cases.

Using the fact that

$$f(h, \theta) = \arctanh(\theta \tanh h) = \frac{1}{2} \ln \frac{(1 + \theta) e^{2h} + (1 - \theta)}{(1 - \theta) e^{2h} + (1 + \theta)},$$

and introducing the notation $z_i = e^{2h_i}$, $i = 1, 2, 3, 4$ we obtain the following system of equations instead of (3.3):

$$
\begin{align*}
z_1 &= \frac{z_4 + \alpha}{\alpha z_3 + 1} |A| \cdot \left( \frac{z_4 + \alpha}{\alpha z_3 + 1} \right)^{(k-|A|)} \\
z_2 &= \frac{z_3 + \alpha}{\alpha z_2 + 1} |A|^{-1} \cdot \left( \frac{z_3 + \alpha}{\alpha z_2 + 1} \right)^{(k+1-|A|)} \\
z_3 &= \frac{z_2 + \alpha}{\alpha z_2 + 1} |A|^{-1} \cdot \left( \frac{z_2 + \alpha}{\alpha z_2 + 1} \right)^{(k+1-|A|)} \\
z_4 &= \frac{z_1 + \alpha}{\alpha z_1 + 1} |A| \cdot \left( \frac{z_1 + \alpha}{\alpha z_1 + 1} \right)^{(k-|A|)}.
\end{align*}
$$

(3.4)

We consider the case $|A| = 1$. Introduce the notation $f(x) = \frac{x + \alpha}{\alpha x + 1}$. Then the system of equations (3.4) takes the form

$$
\begin{align*}
z_1 &= f(z_3) \cdot (f(z_1))^{k-1} \\
z_2 &= (f(z_1))^k \\
z_3 &= (f(z_4))^k \\
z_4 &= f(z_2) \cdot (f(z_4))^{k-1}.
\end{align*}
$$

(3.5)

**Theorem 3.3.** Let $|A| = 1$.

1) For $k = 2$ and $k = 3$ there is not any $H_A$-weakly periodic (non translation-invariant) Gibbs measures corresponding to a set of quantities from $I_3$.

2) Let $k = 5$. For the weakly periodic Gibbs measures corresponding to the set of quantities from $I_3$ there exists a critical value $\alpha_{cr}$ ($\approx 0.377$) such that there is not any $H_A$-weakly periodic (non translation-invariant) Gibbs measures for $\alpha > \alpha_{cr}$, there are two $H_A$-weakly periodic Gibbs measures for $\alpha = \alpha_{cr}$, and there are four $H_A$-weakly periodic (non translation-invariant) Gibbs measures for $0 < \alpha < \alpha_{cr}$.

**Proof.** System of equations (3.5) on the invariant set $I_3$ has the following form:

$$
\begin{align*}
z_1 &= f\left(\frac{1}{z_2}\right) \cdot (f(z_1))^{k-1} \\
z_2 &= (f(z_1))^k.
\end{align*}
$$

(3.6)

Thus the system (3.6) can be reduced to the equation

$$
z_1 = \left( \frac{z_1 + \alpha}{\alpha z_1 + 1} \right)^{k-1} \frac{\alpha (z_1 + 1) + (1 + \alpha z_1)^k}{(\alpha + z_1)^k + \alpha (1 + \alpha z_1)^k}.
$$

(3.7)
Denoting \( u = \frac{z_1 + z_2}{z_1 + 1} \) we reduce equation (3.7) to the equation
\[
\alpha^2 u^{2k} - \alpha u^{2k-1} + u^{k+1} - u^{k-1} + \alpha u - \alpha^2 = 0.
\]
Equation (3.8) is equivalent to
\[
(u^2 - 1)P_{2k-2}(u) = 0,
\]
where \( P_{2k-2}(u) \) is a symmetric polynomial of degree \( 2k - 2 \). It is well known that setting \( u + 1/u = \xi \), we can decrease the degree of the equation \( P_{2k-2}(u) = 0 \) twofold, i.e., reduce this equation to the equation \( P_{k-1}(\xi) = 0 \), where \( P_{k-1}(\xi) \) is a nonsymmetric polynomial of degree \( k - 1 \) in the general case. But for \( k \geq 6 \), the equation \( P_{k-1}(\xi) = 0 \) cannot be solved in radicals.

Now we prove the first part of Theorem 3.3. Let \( k = 3 \). Then the equation (3.9) has the following form
\[
(u^2 - 1)(\alpha^2 u^4 - \alpha u^3 + (\alpha^2 + 1)u^2 - \alpha u + \alpha^2) = 0.
\]
In this case one solution of the equation (3.10) is \( u = 1 \). Therefore we assume that \( u \neq 1 \). From (3.10) we have
\[
\alpha^2 u^4 - \alpha u^3 + (\alpha^2 + 1)u^2 - \alpha u + \alpha^2 = 0.
\]
Denoting \( \xi = u + \frac{1}{u} \), we obtain the equation
\[
\alpha \xi^2 - \alpha \xi + 1 - \alpha^2 = 0.
\]
It is easy to show that the equation for \( \xi > 2 \) does not have a solution, so the equation (3.10) has no solution other than \( u = 1 \).

By similar way one can show that for \( k = 2 \) the equation (3.8) has no solution other than \( u = 1 \).

We consider the case \( k = 5 \), where equation (3.9) has the form
\[
(u^2 - 1)\left(\alpha^2 u^8 - \alpha u^7 + \alpha^2 u^6 - \alpha u^5 + (\alpha^2 + 1)u^4 - \alpha u^3 + \alpha^2 u^2 - \alpha u + \alpha^2\right) = 0.
\]
From (3.11) we have \( u^2 - 1 = 0 \) or
\[
\alpha^2 u^8 - \alpha u^7 + \alpha^2 u^6 - \alpha u^5 + (\alpha^2 + 1)u^4 - \alpha u^3 + \alpha^2 u^2 - \alpha u + \alpha^2 = 0.
\]
Since \( u > 0 \), we have for equation (3.11), \( u = 1 \) is a solution. We assume that \( u \neq 1 \). Setting \( \xi = u + \frac{1}{u} > 2 \), from (3.12) we obtain the equation
\[
\alpha^2 \xi^4 - \alpha \xi^3 - 3\alpha^2 \xi^2 + 2\alpha \xi + \alpha^2 + 1 = 0.
\]
It is well known (see [5], p.28) that the number of positive roots of the polynomial (3.13) does not exceed the number of sign changes of the sequence of its coefficients. Therefore, the equation (3.13) has at most two positive solutions. From equation (3.13) we find the parameter \( \alpha \):
\[
\alpha_1 = \frac{\xi^3 - 2\xi + \sqrt{\xi^6 - 8\xi^4 + 16\xi^2 - 4}}{2(\xi^4 - 3\xi^2 + 1)} := \gamma_1(\xi),
\]
\[
\alpha_2 = \frac{\xi^3 - 2\xi - \sqrt{\xi^6 - 8\xi^4 + 16\xi^2 - 4}}{2(\xi^4 - 3\xi^2 + 1)} := \gamma_2(\xi).
\]
Denoting \( v = \xi^2 \) and \( \varphi(v) = v^3 - 8v^2 + 16v - 4 \). We consider \( \varphi'(v) = 3v^2 - 16v + 16 \). It is clear that \( \varphi'(v) > 0 \) for \( v > 4 \). On the other hand \( \varphi(4) < 0 \), \( \varphi(+\infty) > 0 \). It follows that \( \varphi(v) = 0 \) has a unique solution \( v_0 \) for \( v > 4 \). Therefore, the system of inequalities
\[
\begin{cases}
\xi^6 - 8\xi^4 + 16\xi^2 - 4 \geq 0 \\
\xi > 2,
\end{cases}
\]
true for \( \xi \in [\xi_0, +\infty) \), where \( \xi_0 = \sqrt{v_0}(\approx 2.214) \).

Note that \( \gamma_1(\xi_0) = \gamma_2(\xi_0) \).

One can check that

\[
\lim_{\xi \to +\infty} \alpha_i(\xi) = 0, \quad i = 1, 2.
\]

(3.16)

Note that if \( \xi \in [\xi_0, +\infty) \) then the equation \( \gamma_1^2(\xi) = 0 \) has a unique solution, which is \( \xi_1 \approx 2.3849 \), and also we get \( \gamma_1(\xi_0) \approx 0.311 \), \( \gamma_1(\xi_1) \approx 0.377 \).

Hence it is clear that the function \( \gamma_1(\xi) \) reaches its maximum in \( [\xi_0, +\infty) \) at \( \xi_1 \). Consequently, for \( \alpha \in (0, \gamma_1(\xi_0)) \cup \{\gamma_1(\xi_1)\} \) there exists a unique \( \xi > 2 \) satisfying the equation (3.14), for \( \alpha \in [\gamma_1(\xi_0), \gamma_1(\xi_1)) \) there exist two \( \xi > 2 \) satisfying the equation (3.14), moreover if \( \alpha > \gamma_1(\xi_1) \) then there is not \( \xi > 2 \) satisfying the equation (3.14).

Now consider the function \( \gamma_2(\xi) \). For \( \xi \in [\xi_0, +\infty) \) the equation \( \gamma_2^2(\xi) = 0 \) has not a solution. Since \( \gamma_2(\xi_0) \approx 0.311 \), by (3.16) we have that the function \( \gamma_2(\xi) \) is decreasing. Then we get following: for \( \alpha \in (0, \gamma_2(\xi_0)) \) there exists a unique \( \xi > 2 \) which satisfies the equation (3.15), for \( \alpha > \gamma_2(\xi_0) \) there is not \( \xi > 2 \) satisfying the equation (3.15).

Let \( n_\alpha \) be the number of solutions of the equation (3.13). Then \( n_\alpha \) has the following form

\[
n_\alpha = \begin{cases} 
2, & \text{if } \alpha \in (0, \alpha_{cr}) \\
1, & \text{if } \alpha = \alpha_{cr} \\
0, & \text{if } \alpha \in (\alpha_{cr}, +\infty) 
\end{cases}
\]

For \( \alpha \in (0, \alpha_{cr}) \) from \( u + \frac{1}{u} = \xi \) we get four solutions of the equation (3.12). In this case the equation (3.11) has five solutions. For \( \alpha = \alpha_{cr} \) from \( u + \frac{1}{u} = \xi \) we get that the equation (3.12) has two solutions. Consequently equation (3.11) has three solutions. In the case \( \alpha > \alpha_{cr} \) the equation (3.11) has a unique solution \( u = 1 \). Theorem is proved.

**Remark 3.4.** The new Gibbs measures described in Theorem 3.3 allow describing of a continuum set of non-periodic Gibbs measures that differ from the well-known ones.

**Remark 3.5.** In the paper [2] for \( k = 2, 3 \) the authors construct many ‘new’ Gibbs measures which are similar to the weakly periodic measures of the Ising model on the Cayley tree. Comparing their result with our Theorem 3.3 and with [7, Theorem 3] one can see that in the case \( k = 2, 3 \) we did not obtain \( H_{1^*} \)- weakly periodic measures under conditions of these theorems. If \( k = 3 \) and some conditions of Theorem 3.3 are not satisfied then we do not know about existence of any type of weakly periodic Gibbs measures.

**Aknowledgments.** The author thanks Professor U.A.Rozikov for useful discussions.

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