CRAMÉR-TYPE MODERATE DEVIATIONS UNDER LOCAL DEPENDENCE

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We establish Cramér-type moderate deviation theorems for sums of locally dependent random variables and combinatorial central limit theorems. Under some mild exponential moment conditions, optimal error bounds and convergence ranges are obtained. Our main results are more general or sharper than the existing results in the literature. The main results follows from a more general Cramér-type moderate deviation theorem for dependent random variables without any boundedness assumptions, which is of independent interest. The proofs couple Stein’s method with a recursive argument.

1. Introduction. Moderate deviations estimate the relative errors for distributional approximations. Since Cramér (1938) proved a moderate deviation result for tail probabilities of sums of independent random variables, Cramér-type moderate deviation theorems have been widely applied to estimate rare event probabilities. Specially, for independent and identically distributed (i.i.d.) random variables $X_1, \ldots, X_n$ with zero mean and unit variance satisfying that $\mathbb{E}e^{t|X_1|} \leq c$ for some $t_0 > 0$, it follows that
\[
\left| \frac{\mathbb{P}(W_n > x)}{1 - \Phi(x)} - 1 \right| \leq An^{-1/2}(1 + x^3) \quad \text{for} \quad 0 \leq x \leq an^{1/6},
\]
where $W_n = (X_1 + \cdots + X_n)/\sqrt{n}$, $\Phi(x)$ is the standard normal distribution function, and $A$ and $a$ are positive constants depending only on $t_0$ and $c$. We remark that the range $0 \leq x \leq an^{1/6}$ and the error term $n^{-1/2}(1 + x^3)$ are optimal for i.i.d. random variables. For other results on Cramér-type moderate deviations, we refer the reader to Linnik (1961) and Petrov (1975).

Moderate deviation theorems for independent random variables have been well studied in the literature. However, the data may not be independent in the era of big data. It is necessary to develop the corresponding limit theory for dependent random variables.

In this paper, we focus on Cramér-type moderate deviations for sum of locally dependent random variables (see Section 2) and combinatorial central limit theorems (see Section 3). A family of locally dependent random variables means that certain subset of the random variables are independent of those outside their respective neighborhoods, which is a generalization of $m$-dependence. Although absolute error bounds of normal approximation for sums of locally dependent random variables have been well studied in the literature (see, e.g., Baldi and Rinott, 1989; Baldi, Rinott and Stein, 1989; Rinott, 1994; Dembo and Rinott, 1996; Chen and Shao, 2004; Fang, 2019), few results for Cramér-type moderate deviation theorem for locally dependent random fields have been proved even when assuming that the random variables are bounded. Under certain dependence structures, Raič (2007) proved a

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large deviation result with some sophisticated assumptions, which, however, seems to be too restricted to apply to other applications. In Theorem 2.1, we provide a Cramér-type moderate deviation result under local dependence and some mild exponential moment conditions.

Combinatorial central limit theorem is the central limit theorem for a family of permutation statistics \( \sum_{i=1}^{n} X_{i,\pi(i)} \), where \( n \geq 1, X := \{ X_{i,j} : 1 \leq i, j \leq n \} \) is an \( n \times n \) array of random variables, and \( \pi \) is a uniform random permutation of \( \{1, 2, \ldots, n\} \), independent of \( X \). Absolute error bounds of normal approximation for \( \sum_{i=1}^{n} X_{i,\pi(i)} \) have also been well studied in the literature (see Hoeffding, 1951; Ho and Chen, 1978; Goldstein, 2005; Chen, Fang and Shao, 2013; Chen and Fang, 2015). For relative error bounds, Frolov (2019) obtained a moderate deviation result under some Bernstein-type conditions. However, he did not provide the error bound. In Theorem 3.1, we prove a Cramér-type moderate deviation result for combinatorial central limit theorems with best possible convergence rates and ranges.

Classical proofs of Cramér-type moderate deviations are based on the conjugate method and Fourier transforms, which perform well when dealing with independent random variables. Nevertheless, it is not easy to apply the Fourier transform without independence assumptions. Alternatively, Stein’s method is a powerful tool in dealing with dependent structures. Since introduced by Stein (1972), Stein’s method has been widely applied to prove optimal Berry–Esseen bounds and \( L_1 \) bounds with explicit constant factors for many distributional approximations (Chen, Goldstein and Shao, 2011; Chatterjee, 2014), and moreover, it turns out that Stein’s method can also be used to obtain moderate deviation theorems. For examples, Chen, Fang and Shao (2013) first applied Stein’s method to prove Cramér-type moderate deviation results for normal approximation via Stein identity, and recently, Shao, Zhang and Zhang (2021) further obtain a Cramér-type moderate deviation result for nonnormal approximations. In both papers, the authors made some boundedness assumptions about the random variables of interest. To relax boundedness assumptions, Zhang (2019) applied Stein’s method using the exchangeable pair approach to develop a Cramér-type moderate deviation result for unbounded case. However, Zhang (2019)’s result cannot be applied to deal with locally dependent random variables.

In order to prove Theorem 2.1 and Theorem 3.1, we consider the Stein identity approach of Stein’s method. Specifically, let \( W \) be a random variable, and assume that there exists a random function \( \hat{K}(u) \) and a random variable \( R \) such that for all absolutely continuous functions \( f \), the following identity holds:

\[
\mathbb{E}\{ W f(W) \} = \mathbb{E}\left\{ \int_{-\infty}^{\infty} f'(W + u) \hat{K}(u) du \right\} + \mathbb{E}\{ R f(W) \}.
\]

The equality (1.1) is called Stein identity (see Section 2.5 of Chen, Goldstein and Shao (2011)). Both \( L_1 \) bounds and Berry–Esseen bounds via Stein’s identity have been well studied in the literature, and we refer the readers to Chen, Goldstein and Shao (2011) for a detailed survey. Based on (1.1), Chen, Fang and Shao (2013) proved a Cramér-type moderate deviation theorem for \( W \) under the following conditions: there exists \( \delta_0, \delta_1, \delta_2 \) and \( \theta \) such that

\[
\hat{K}(u) = 0 \text{ for } |u| > \delta_0, \quad \mathbb{E}\{ \hat{K}_1 |W| \} - 1 \leq \delta_1 (1 + |W|),
\]

\[
\mathbb{E}\{ \hat{K}_1 |W| \} \leq \theta, \quad |\mathbb{E}\{ R |W| \}| \leq \delta_2 (1 + |W|).
\]

However, the conditions may be restricted to apply in some applications. First, the random function \( \hat{K}(u) \) is assumed to be positive and supported on a bounded interval \([-\delta_0, \delta_0]\), where the constant \( \delta_0 \) is of order \( O(n^{-1/2}) \) in some typical applications. Second, the conditional expectations may not be easy to calculate if we know few on the distribution of \( W \).

To improve Chen, Fang and Shao (2013)’s moderate deviation result, we establish a general Cramér-type moderate deviation result (Theorem 4.1) without assuming that the random
function $\tilde{K}(u)$ is positive and supported on a bounded interval, which may be of independent interest for other applications. There are several advantages of our result. First, optimal error bounds and optimal ranges are obtained for moderate deviations of locally dependent sums and combinatorial central limit theorems. Second, we relax the boundedness assumption and thus our general theorem can be applied to a much wider class of statistics.

The rest of this paper is organized as follows. We give the result for locally dependent random variables in Section 2. Moderate deviation for combinatorial central limit theorems are discussed in Section 3. Our general theorem is given in Section 4. We prove our general result in Section 5. Finally, the proofs of our results in Sections 2 and 3 are presented in Sections 6 and 7. Some supplementary materials are given in the appendix.

2. Moderate deviation for sums of locally dependent random variables. In this section, we prove a Cramér-type moderate deviation theorem for sums of locally dependent random variables.

We follow the notation in Chen and Shao (2004). Let $\mathcal{J}$ be an index set and let $\{X_i, i \in \mathcal{J}\}$ be a field of random variables with zero means and finite variances. Let $W = \sum_{i \in \mathcal{J}} X_i$ and assume that $\text{Var}(W) = 1$. For $A \subset \mathcal{J}$, write $X_A = \{X_i, i \in A\}$, $A^c = \{j \in \mathcal{J} : j \notin A\}$ and denote by $|A|$ the cardinality of $A$.

We now introduce the following local dependence conditions:

(LD1) For each $i \in \mathcal{J}$, there exists $A_i \subset \mathcal{J}$ such that $X_i$ is independent of $X_{A_i^c}$.

(LD2) For each $i \in \mathcal{J}$, there exists $B_i \subset \mathcal{J}$ such that $B_i \supseteq A_i$ and $X_{A_i}$ is independent of $X_{B_i^c}$.

We note that local dependence satisfying (LD1) and (LD2) is a generalization of $m$-dependence. These local dependence conditions were firstly introduced by Chen and Shao (2004), and we refer the reader to other types of local dependence structures in Baldi and Rinott (1989); Baldi, Rinott and Stein (1989); Rinott (1994); Dembo and Rinott (1996); Fang (2019). Absolute error bounds such as $L_1$ bounds and Berry–Esseen bounds for locally dependent random variables have also been well studied in the literature. For example, in Section 4.7 of Chen, Goldstein and Shao (2011), an $L_1$ bound was established under (LD1) and (LD2). Chen and Shao (2004) proved several sharp Berry–Esseen bounds under different local dependence conditions and some polynomial moment conditions. Although Cramér-type moderate deviations have been proved for $m$-dependent random variables (see, e.g., Heinrich (1982)), as far as we know, no Cramér-type moderate deviation results have been obtained for locally dependent random variables even for bounded cases.

Let $N_i = \{j \in \mathcal{J} : B_i \cap B_j \neq \emptyset\}$ and let $\kappa := \max_{i \in \mathcal{J}} |N_i|$. Let $n = |\mathcal{J}|$. Assume that there exist $a_n \geq 1$ and $b \geq 1$ such that for all $i \in \mathcal{J}$,

$$E\left\{\exp\left(a_n \sum_{j \in B_i} |X_j|\right)\right\} \leq b. \tag{2.1}$$

We have the following theorem.

**Theorem 2.1.** Under (LD1) and (LD2), and assume that (2.1) holds. Then

$$\left|\frac{P[W > z]}{1 - \Phi(z)} - 1\right| \leq C \delta_n (1 + z^3) \tag{2.2}$$

for $0 \leq z \leq c a_n^{1/3} \min\{1, \kappa^{-1/3}(1 + \theta_n)^{-2/3}\}$, where $C$ and $c$ are absolute constants and

$$\delta_n = \kappa^2 a_n^{-1}(1 + \theta_n^{-1}) \quad \text{and} \quad \theta_n = b^{1/2} n^{1/2} a_n^{-1}.$$
Remark. When $a_n$ is of order $O(n^{1/2})$ and $\kappa$ and $b$ are of order $O(1)$, we have $\theta_n = O(1)$ and $\delta_n = O(n^{-1/2})$. Therefore, the error bound in (2.2) is of order $(1 + z^3)/\sqrt{n}$ and the range is $0 \leq z \leq cn^{1/6}$. Specifically, for i.i.d. random variables $\xi_1, \ldots, \xi_n$ satisfying that $\mathbb{E}\xi_1 = 0$, $\text{Var}(\xi_1) = 1/n$ and $\mathbb{E}\sqrt{n}\xi_1 \leq b_0$ for some $b_0 > 0$, we have that (2.1) holds with $B_i = \{i\}$, $a_n = \sqrt{n}$ and $b = b_0$. Hence, Theorem 2.1 reduces to
\[
\left| \frac{\mathbb{P}(\sum_{i=1}^n \xi_i > z)}{1 - \Phi(z)} - 1 \right| \leq Cn^{-1/2}(1 + z^3) \quad \text{for } 0 \leq z \leq cn^{1/6},
\]
where $c, C$ are constants depending only on $b_0$. Thus, Theorem 2.1 is optimal in the sense that it provides optimal error bounds and ranges for sum of i.i.d. random variables.

Remark. We remark that there are some different dependence structures other than (LD1) and (LD2) in the literature, e.g., decomposable random variables, dependency graphs, and so on. For decomposable random variables, Raič (2007) proved a large deviation result with some sophisticated assumptions, which maybe too strict to apply to other applications.

To illustrate that our result gives optimal error bounds and ranges for other settings, we consider the following corollary for $m$-dependent random fields. Let $d \geq 1$ and let $\mathbb{Z}^d$ denote the $d$-dimensional space of positive integers. For any $i = (i_1, \ldots, i_d), \bar{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, we define the distance by $|i - \bar{j}| := \max_{1 \leq k \leq d}|i_k - j_k|$, and for $A, B \subset \mathbb{Z}^d$, we define the distance between $A$ and $B$ by $\rho(A, B) = \inf \{|i - j| : i \in A, j \in B\}$. Let $\mathcal{J}$ be a subset of $\mathbb{Z}^d$, and we say a field of random variables $\{X_i : i \in \mathcal{J}\}$ is an $m$-dependent random field if $\{X_i, i \in A\}$ and $\{X_{j}, j \in B\}$ are independent whenever $\rho(A, B) > m$ for any $A, B \subset \mathcal{J}$. If we choose $A_i = \{j \in \mathcal{J} : |i - j| \leq m\}$, $B_i = \{j \in \mathcal{J} : |i - j| \leq 2m\}$, then (LD1) and (LD2) are satisfied with $\kappa = (8m + 1)^d$, and Theorem 2.1 reduces to the following corollary.

Corollary 2.2. Let $\{X_i : i \in \mathcal{J}\}$ be an $m$-dependent random field on $\mathbb{Z}^d$ with $\mathbb{E}\{X_i\} = 0$, $W = \sum_{i \in \mathcal{J}} X_i$ and $\text{Var}(W) = 1$. If (2.1) is satisfied, then (2.2) holds with $\kappa = (8m + 1)^d$.

Remark. Under the conditions of Corollary 2.2 with $d = 1$, Corollary 4.1 of Heinrich (1982) reduces to the following result:
\[
\left| \frac{\mathbb{P}(\sum_{i=1}^n X_i > z)}{1 - \Phi(z)} - 1 \right| \leq Cn^{-3}(1 + z^3)
\]
for $0 \leq z \leq ca_n n^{-1/3}$ where $C$ and $c$ are constants depending only on $m$ and $b$. Since $\mathbb{E}W^2 = 1$, we have $m \sum_{i=1}^n \mathbb{E}X_i^2 > 1$. On the other hand, by (2.1), $\sup_{1 \leq i \leq n} \mathbb{E}X_i^2 \leq Cb_0 a_n^{-2}$. Thus, $n a_n^{-2} \geq C_0$, for some constant $C_0$ depend on $m$ and $d$. Up this constant $C_0$, Heinrich’s result is not better than ours and the moderate deviations for $m$-dependent random field on $\mathbb{Z}^d$ seems to be new.

3. Moderate deviation for combinatorial central limit theorems. Let $n \geq 1$, and let $\mathbf{X} := \{X_{ij} : 1 \leq i, j \leq n\}$ be an $n \times n$ array of independent random variables with $\mathbb{E}\{X_{ij}\} = a_{i,j}$ and $\text{Var}(X_{ij}) = c_{i,j}^2$. Moreover, assume that
\[
\sum_{i=1}^n a_{i,j} = 0 \quad \text{for all } 1 \leq j \leq n, \quad \sum_{j=1}^n a_{i,j} = 0 \quad \text{for all } 1 \leq i \leq n,
\]
and
\[
\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j}^2 + \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n c_{i,j}^2 = 1.
\]
Let $\mathcal{S}_n$ be the collection of all permutations over $[n] := \{1, 2, \ldots, n\}$ and let $\pi$ be a random permutation chosen uniformly from $\mathcal{S}_n$ independent of $X$. Let

$$W = \sum_{i=1}^{n} X_{i, \pi(i)}.$$  

(3.3)

Combinatorial central limit theorems for $\tilde{W} := \sum_{i=1}^{n} a_{i, \pi(i)}$, which is a special case of $W$, was firstly introduced by Hoeffding (1951). For the random variable $\tilde{W}$, Goldstein (2005) proved a Berry–Esseen theorem for $\tilde{W}$ by Stein’s method and zero–bias coupling, and Chen, Fang and Shao (2013) also gives the moderate deviation result of the normal approximation, where the convergence rate and range depend on $\max_{i,j} |a_{ij}|$.

Hu, Robinson and Wang (2007) proved a moderate deviation result for the simple random sample problem, which is an application of the combinatorial central limit theorems. The Berry–Esseen bounds of combinatorial central limit theorems for $W$ was firstly studied by Ho and Chen (1978), who proved an error bound using the concentration inequality approach, and Chen and Fang (2015) obtained a new error bound $451n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}[|X_{i,j}|^3]$ via exchangeable pair approach. Recently, Frolov (2019) gave a Cramér-type moderate deviation result for general combinatorial central limit theorems under some Bernstein type conditions, but the author did not provide the error bounds.

The following theorem provides a Cramér-type moderate deviation result for $W$.

**Theorem 3.1.** Assume that there exist $\alpha_n \geq 1$ and $b \geq 1$ such that

$$\max_{1 \leq i,j \leq n} \mathbb{E}\{\exp(\alpha_n |X_{i,j}|)\} \leq b.$$  

(3.4)

Then

$$\left| \frac{\mathbb{P}(W > z)}{1 - \Phi(z)} - 1 \right| \leq C\delta_n(1 + z^3),$$  

(3.5)

for $0 \leq z \leq c\alpha_n^{1/3} \min\{1, b^{-1}(\theta_n^{-1/2} + \theta_n)^{-1}\}$, where $C$ and $c$ are absolute constants, $\theta_n = n^{1/2} \alpha_n^{-1}$ and $\delta_n = b^2(\alpha_n^{-1} + n^{-1/2})(\theta_n^{-2} + \theta_n^6)$.

**Remark.** If $\max_{1 \leq i,j \leq n} |X_{i,j}|$ is of order $O(n^{-1/2})$, then we can choose $\alpha_n = O(n^{1/2})$ and $b = O(1)$, and (3.5) reduces to

$$\left| \frac{\mathbb{P}(W > z)}{1 - \Phi(z)} - 1 \right| \leq Cn^{-1/2}(1 + z^3)$$

for $0 \leq z \leq cn^{1/6}$ for some constants $c, C > 0$.

**Remark.** In Chen, Fang and Shao (2013), the authors proved a moderate deviation for the case where $X_{i,j} = a_{i,j}$ is nonrandom. Specially, our result recovers (4.1) in Chen, Fang and Shao (2013).

**4. A general theorem via Stein identity.** In this section, we proof a general theorem for dependent random variables, which will be used to prove Theorems 2.1 and 3.1. The theorem is based on Stein identity, and it is also of independent interest and can be applied to many other applications. Let $W$ be the random variable of interest satisfying the Stein identity (1.1)
with a random function $\hat{K}(u)$ and a random variable $R$. To give our general theorem, we first introduce the following notation. For $t \geq 0$ and for $u \in \mathbb{R}$, let

\begin{align}
(4.1) \quad & K(u) = \mathbb{E}\{\hat{K}(u)\}, \quad \hat{K}_1 = \int_{-\infty}^{\infty} \hat{K}(u) du, \\
(4.2) \quad & \hat{K}_{2,t} = \int_{-\infty}^{\infty} |u| e^{t|u|} |\hat{K}(u)| du, \\
(4.3) \quad & \hat{K}_{3,t} = \int_{|u| \leq 1} e^{2t|u|} (\hat{K}(u) - K(u))^2 du, \\
(4.4) \quad & \hat{K}_{4,t} = \int_{|u| \leq 1} |u| e^{2t|u|} (\hat{K}(u) - K(u))^2 du,
\end{align}

and

\begin{align}
(4.5) \quad & M_t = \int_{|u| \leq 1} e^{t|u|} |K(u)| du.
\end{align}

For any $\beta \geq 0$ and $t \geq 0$, let

\begin{align}
(4.6) \quad & \Psi_{\beta,t}(w) = \begin{cases} e^{tw} + 1 & \text{if } w \leq \beta, \\
2e^{t\beta} - e^{t(2\beta - w)} + 1 & \text{if } w > \beta.
\end{cases}
\end{align}

We remark that the function $\Psi_{\beta,t}$ is a smoothed version of the truncated exponential function, which plays an important role in relaxing the boundedness assumption in applications.

Our general result is based on the following conditions:

(A1) Assume that there exist constants $m_0 > 0$, $\rho > 0$ and $r_j \geq 0$, $\tau_j \geq 0$ for $j = 0, 1, \ldots, 4$ such that for all $\beta, t \in [0, m_0]$,

\begin{align}
(4.7) \quad & \mathbb{E}\{|R| \Psi_{\beta,t}(W)\} \leq r_0 (1 + t^{\tau_0}) \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
(4.8) \quad & \mathbb{E}\{|\mathbb{E}\{\hat{K}_1|W\} - 1| \Psi_{\beta,t}(W)\} \leq r_1 (1 + t^{\tau_1}) \mathbb{E}\{\Psi_{\beta,t}(W)\},
\end{align}

\begin{align}
(4.9) \quad & \mathbb{E}\{\hat{K}_{2,t} \Psi_{\beta,t}(W)\} \leq r_2 (1 + t^{\tau_2}) \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
(4.10) \quad & \mathbb{E}\{\hat{K}_{3,t} \Psi_{\beta,t}(W)\} \leq r_3 (1 + t^{\tau_3}) \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
(4.11) \quad & \mathbb{E}\{\hat{K}_{4,t} \Psi_{\beta,t}(W)\} \leq r_4 (1 + t^{\tau_4}) \mathbb{E}\{\Psi_{\beta,t}(W)\}
\end{align}

and

\begin{align}
(4.12) \quad & \sup_{0 \leq t \leq m_0} M_t \leq \rho.
\end{align}

We now state our general result.

**Theorem 4.1.** Under condition (A1). Let $\tau = \max\{\tau_0 + 1, \tau_1 + 2, \tau_2 + 3, \tau_3 + 1, \tau_4 + 1\}$ and let

\begin{align}
(4.13) \quad & z_0 = \min\{m_0, 0.02e^{-\tau/2}(r_0^{1/(\tau_0+1)} + r_1^{1/(\tau_1+2)} + r_2^{1/(\tau_2+3)})^{-1}\}.
\end{align}

We have

\begin{align}
(4.14) \quad & \left| \mathbb{P}[W > z] - \frac{1}{1 - \Phi(z)} \right| \leq \left( \frac{4}{\delta(m_0)} + C(150^\tau + \rho)e^{\tau^2/2} \right) \delta(z)
\end{align}
for $0 \leq z \leq z_0$, where $\Phi(z)$ is the standard normal distribution function and $C$ is an absolute constant, and

\begin{equation}
\delta(z) = r_0(1 + z^{\tau_0+1}) + r_1(1 + z^{\tau_1+2}) + r_2(1 + z^{\tau_2+3}) + r_3(1 + z^{\tau_3+1}) + r_4^{1/2}(1 + z^{\tau_4+1}).
\end{equation}

We give some remarks on our general result.

REMARK. Chen, Fang and Shao (2013) proved a moderate deviation for Stein identities under a boundedness assumption (1.2). On that basis, for $0 \leq t \leq \delta_0^{-1}$, it can be shown that (4.9) is satisfied with $r_2 = 3\theta\delta_0$. Moreover, for all $t, \beta \in (0, \delta_0^{-1})$, one can verify (see, e.g., (5.5) and (5.6) of Chen, Fang and Shao (2013)) that there exists a constant $C > 0$ depending only on $\theta$ such that

\[ \mathbb{E}\{W|\Psi_{\beta,t}(W)\} \leq C(1 + t) \mathbb{E}\{\Psi_{\beta,t}(W)\}. \]

Thus, we have that (4.7) and (4.8) are satisfied with $r_0 = C'\delta_2, r_1 = C'\delta_1$ and $\tau_0 = \tau_1 = 1$, where $C' > 0$ is an absolute constant. Therefore, by Theorem 4.1, we have that (4.14) holds with $m_0 = \delta_0^{-1}, \rho = \theta, \tau = 3, \tau_0 = \tau_1 = 1, \tau_2 = \tau_3 = \tau_4 = 0, r_0 = C'\delta_2, r_1 = C'\delta_1, r_2 = 3\theta\delta_0$ and

\[ r_3 = 8 \int_{|u| \leq \delta_0} \mathbb{E}\{(K(u) - K(u))^2\} du, \quad r_4 = 8\delta_0 \int_{|u| \leq \delta_0} \mathbb{E}\{(K(u) - K(u))^2\} du. \]

Note that this result involves two terms $r_3$ and $r_4^{1/2}$ that did not appear in Chen, Fang and Shao (2013). However, in many applications, both $r_3$ and $r_4^{1/2}$ have the same order as $\delta_0$. This shows that our result Theorem 4.1 covers Theorem 3.1 of Chen, Fang and Shao (2013) with the cost of two additional terms.

5. Proof of Theorem 4.1. In this section, we provide the proof of Theorem 4.1. Our proof is novel in two ways. On one hand, the proof of Theorem 4.1 is a combination of Stein’s method and a recursive method. The recursive method has been applied to obtain optimal Berry–Esseen bounds for both univariate and multivariate normal approximations; see Raic (2003); Raic (2019) and Chen, Röllin and Xia (2020) for examples. On the other hand, use a truncated exponential function to control tail probabilities. It is known that exponential-type tail probabilities play a crucial role in the proof of Cramér-type moderate deviations. In Chen, Fang and Shao (2013) and Shao, Zhang and Zhang (2021), the authors used exponential functions directly to prove upper bounds for such tail probabilities. In the present paper, a key observation is that the exponential function can be replaced by a smoothed truncated exponential function $\Psi_{\beta,t}$ (defined in (4.6) in Section 4) when proving exponential-type tail probabilities, and the function $\Psi_{\beta,t}$ plays an important role in relaxing the boundedness assumption when applying our general theorem.

This section is organized as follows. We first develop two preliminary lemmas, Lemmas 5.1 and 5.2, whose proofs are postponed to Section 5.3. In Lemma 5.1, we establish an upper bound of the ratio for the expectation of a smoothed indicator function, and we provide a Berry–Esseen bound under (1.1) in Lemma 5.2. The proof of Theorem 4.1 is given in Section 5.2, where we apply Lemmas 5.1 and 5.2 and a smoothing inequality.
5.1. **Preliminary lemmas.** We first introduce some notation. Let \( Z \sim N(0, 1) \), \( \phi(w) = (1/\sqrt{2\pi}) e^{-w^2/2} \) and \( \Phi(w) = \int_{-\infty}^{w} \phi(t) dt \). In what follows, we write \( Nh = \mathbb{E}\{h(Z)\} \) for any function \( h \). For any \( z \geq 0 \) and \( \varepsilon > 0 \), let
\[
\hat{h}_{z, \varepsilon}(w) = \begin{cases} 
1 & \text{if } w \leq z, \\
0 & \text{if } w > z + \varepsilon, \\
1 + e^{-1}(z - w) & \text{if } z < w \leq z + \varepsilon.
\end{cases}
\]

Let
\[
(5.1) \quad C_0 = \sup_{0 \leq z \leq z_0} \left\{ \frac{\mathbb{P}[W > z] - (1 - \Phi(z))}{\delta(z)(1 - \Phi(z))} \right\},
\]
where \( \delta(z) \) is given in (4.15). The following lemma gives a relative error for the test function \( \hat{h}_{z, \varepsilon} \).

**Lemma 5.1.** Assume that condition (A1) holds and \( z_0 \) in (4.13) satisfies \( z_0 \geq 8 \). Let \( z \) be a fixed real number satisfying \( 8 \leq z \leq z_0 \), and let \( \varepsilon := \varepsilon(z) = 40 e^{\tau/2} r_2 (1 + z^2) \). We have
\[
(5.2) \quad \frac{|\mathbb{E}\{\hat{h}_{z, \varepsilon}(W)\} - Nh_{z, \varepsilon}|}{\delta(z)(1 - \Phi(z))} \leq 0.75 \left(C_0 + \frac{1}{\delta(m_0)}\right) + (184 + 2\rho) e^{\tau/2} + (150 e^{\tau/2})^\tau.
\]

We also need to develop a Berry–Esseen bound to prove Theorem 4.1. The following lemma is a slight modification of Theorem 2.1 in Chen, Röllin and Xia (2020), because the Stein identity (1.1) in our paper involves an additional error term \( \mathbb{E}\{Rf(W)\} \) compared with that in Chen, Röllin and Xia (2020). The proof is given in Section 5.3, where we used a similar argument to the proof of Theorem 2.1 in Chen, Röllin and Xia (2020).

**Lemma 5.2.** Let \( W \) be a random variable satisfying \( \mathbb{E}W = 0 \) and \( \mathbb{E}W^2 = 1 \). Assume that (1.1) and (A1) hold. Then,
\[
\sup_{z \in \mathbb{R}} |\mathbb{P}(W \leq z) - \Phi(z)| \leq 4r_0 + 4r_1 + 28r_2 + 20r_3 + 13r_4^{1/2}.
\]

5.2. **Proof of Theorem 4.1.** It follows from Lemma 5.2 that
\[
(5.3) \quad \sup_{0 \leq z \leq 9} \left| \frac{\mathbb{P}(W \leq z) - \Phi(z)}{\delta(z)(1 - \Phi(z))} \right| \leq \frac{28}{1 - \Phi(9)}.
\]

It now suffices to prove (4.14) for the case \( 9 \leq z \leq z_0 \). When \( z_0 \geq 8 \), from (4.13), we have
\[
(5.4) \quad 0.02 e^{\tau/2} \min\{r_0^{-1/(\tau_0 + 1)}, r_1^{-1/(\tau_1 + 2)}, r_2^{-1/(\tau_2 + 3)}\} \geq 8,
\]
and one can verify that
\[
(5.5) \quad \max\{r_0, r_1, r_2\} \leq 0.02 e^{\tau/2}.
\]

Next, we use a smoothing inequality to prove the upper bound for the case \( 9 \leq z \leq z_0 \). Let \( \varepsilon := \varepsilon(z) = 40 e^{\tau/2} r_2 (1 + z^2) \). By the following well-known inequalities:
\[
(5.6) \quad \frac{1}{1 + x} \phi(x) \leq 1 - \Phi(x) \leq \frac{1}{x} \phi(x) \quad \text{for all } x \geq 1,
\]
we have
\[
(5.7) \quad \phi(z - \varepsilon) \leq e^{\varepsilon z} \phi(z) \leq e^{\varepsilon z} (1 + z)(1 - \Phi(z)).
\]
By (4.13), we have 
\[ z_0 \leq 0.02e^{-\tau/2}r_2^{-1/(r_2+3)} \], in other word
\[ (5.8) \quad r_2 \leq \left( \frac{0.02e^{-\tau/2}}{z_0} \right)^{r_2+3}. \]
It then follows that for \( z_0 \geq 8 \),
\[ z_0 \varepsilon \leq 40e^{\tau/2}r_2 z_0 \varepsilon + 40e^{\tau/2}r_2 z_0^{1+r_2} \]
\[ \leq 40e^{\tau/2} \left( \frac{0.02e^{-\tau/2}}{z_0} \right)^{r_2+3} z_0 + 40e^{\tau/2} \left( \frac{0.02e^{-\tau/2}}{z_0} \right)^{r_2+3} z_0^{1+r_2} \]
\[ (5.9) \quad \leq \frac{80e^{\tau/2}(0.02e^{-\tau/2})^3}{z_0^2} \leq 0.03, \]
where we use the fact that \( \tau \geq 3 \). Thus, it follows that
\[ (5.10) \quad e^{z_0 \varepsilon} \leq 1.05. \]
Also, note that for \( 8 \leq z \leq z_0 \),
\[ (5.11) \quad (1 + z^{r_2}) \leq 2z^{r_2} \leq \frac{1}{256}z^{r_2+3} \leq 0.004(1 + z^{r_2+3}), \]
and by (5.8),
\[ (5.12) \quad r_2(1 + z^{r_2+3}) \leq (0.02e^{-\tau/2})^{r_2+3} \frac{1 + z_0^{r_2+3}}{z_0^{r_2+3}} \leq 0.04e^{-\tau/2}. \]
Then, by (5.11) and (5.12), we have for \( 8 \leq z \leq z_0 \),
\[ \varepsilon(z) = 40e^{\tau/2}r_2(1 + z^{r_2}) \]
\[ (5.13) \quad \leq 0.16e^{\tau/2}r_2(1 + z^{r_2+3}) \]
\[ \leq 0.1. \]
Noting that \( 1 + z \leq 1.25z_0 \) for all \( 8 \leq z \leq z_0 \), by (5.7), (5.9) and (5.10) we have
\[ \Phi(z) - \Phi(z - \varepsilon) \leq \varepsilon \phi(z - \varepsilon) \]
\[ \leq \varepsilon e^{z\varepsilon}(1 + z)(1 - \Phi(z)) \]
\[ \leq 1.25z_0\varepsilon e^{z_0\varepsilon}(1 - \Phi(z)) \]
\[ \leq 0.05(1 - \Phi(z)), \]
and thus,
\[ (5.14) \quad (1 - \Phi(z - \varepsilon)) \leq (1 - \Phi(z)) + (\Phi(z) - \Phi(z - \varepsilon)) \leq 1.05(1 - \Phi(z)). \]
On the other hand, for \( 8 \leq z \leq z_0 \), we have
\[ (5.15) \quad (1 + z)(1 + z^{r_2}) \leq 1.25^2z^{r_2+1} \leq \frac{1.25^2}{64}z^{r_2+3} \leq 0.04(1 + z^{r_2+3}). \]
Then, by (5.10) and (5.15), we have
\[ \Phi(z) - \Phi(z - \varepsilon) \leq \varepsilon (1 + z)e^{z\varepsilon}(1 - \Phi(z)) \]
\[ (5.16) \quad \leq 40r_2e^{\tau/2}e^{z\varepsilon}(1 + z^{r_2})(1 + z)(1 - \Phi(z)) \]
\[ \leq 2r_2e^{\tau/2}(1 + z^{r_2+3})(1 - \Phi(z)), \]
and similarly,
\begin{equation}
\Phi(z + \varepsilon) - \Phi(z) \leq 2r_2 e^{r/2} (1 + z^{r^2 + 3})(1 - \Phi(z)).
\end{equation}

Recall that \(C_0\) is defined as in (5.1). By (5.2) and (5.17), we have for \(8 \leq z \leq z_0\),
\begin{equation}
P(W \leq z) - \Phi(z)
\end{equation}
\begin{equation}
\leq E\{h_{z,\varepsilon}(W) - Nh_{z,\varepsilon}\} + \Phi(z + \varepsilon) - \Phi(z)
\end{equation}
\begin{equation}
\leq (0.75(C_0 + \delta(m_0)^{-1}) + (186 + 2\rho)e^{r/2} + (150e^{r/2})^r)\delta(z)(1 - \Phi(z)).
\end{equation}

Let \(\varepsilon' = 40e^{r/2}r_2(1 + (z - \varepsilon)^{r^2})\). By (5.2) with replacing \(z\) and \(\varepsilon\) by \(z - \varepsilon\) and \(\varepsilon'\), respectively, and by (5.14), we have for \(9 \leq z \leq z_0\),
\begin{equation}
\left|E\{h_{z-\varepsilon',\varepsilon'}(W) - Nh_{z-\varepsilon',\varepsilon'}\}\right|
\end{equation}
\begin{equation}
\leq (0.8(C_0 + \delta(m_0)^{-1}) + (194 + 2.1\rho)e^{r/2} + 1.05(150e^{r/2})^r)\delta(z)(1 - \Phi(z)).
\end{equation}

Thus, by (5.16) and (5.19), we have for \(9 \leq z \leq z_0\),
\begin{equation}
P(W \leq z) - \Phi(z)
\end{equation}
\begin{equation}
\geq E\{h_{z-\varepsilon',\varepsilon'}(W)\} - Nh_{z-\varepsilon',\varepsilon'} - (\Phi(z) - \Phi(z - \varepsilon))
\end{equation}
\begin{equation}
\geq -(0.8(C_0 + \delta(m_0)^{-1}) + (196 + 2.1\rho)e^{r/2} + 1.05(150e^{r/2})^r)\delta(z)(1 - \Phi(z)).
\end{equation}

By (5.18) and (5.20), we have for \(9 \leq z \leq z_0\),
\begin{equation}
\left|P(W \leq z) - \Phi(z)\right|
\end{equation}
\begin{equation}
\leq (0.8(C_0 + \delta(m_0)^{-1}) + (196 + 2.1\rho)e^{r/2} + 1.05(150e^{r/2})^r)\delta(z)(1 - \Phi(z)).
\end{equation}

Moving \(\delta(z)(1 - \Phi(z))\) in (5.21) to the left-hand side (LHS) and taking the supremum over \(9 \leq z \leq z_0\), and by (5.3), we have
\begin{equation}
C_0 \leq (0.8(C_0 + \delta(m_0)^{-1}) + (530 + 2.1\rho)e^{r/2} + 1.05(150e^{r/2})^r) + \frac{28}{1 - \Phi(9)}.
\end{equation}

Solving the recursive inequality (5.22), we obtain
\[C_0 \leq 4\delta(m_0)^{-1} + C(150^r + \rho)e^{r/2},\]
which proves (4.14).

5.3. Proofs of Lemmas 5.1 and 5.2. In order to prove Lemma 5.1, we need to prove the following lemmas. Recall that \(\Psi_{\beta,t}(w)\) is defined as in (4.6). The first lemma gives an upper bound for the truncated exponential moment \(E\{\Psi_{\beta,t}(W)\}\).

**Lemma 5.3 (Exponential bound).** Assume that conditions (4.7)–(4.9) hold and \(z_0\) in (4.13) satisfies \(z_0 \geq 8\). We have
\begin{equation}
E\{\Psi_{\beta,t}(W)\} \leq 4e^{t^2/2} \text{ for } 0 \leq t \leq z_0,
\end{equation}
where \(\Psi_{\beta,t}(w)\) is defined in (4.6).
PROOF. Since $z_0 \geq 8$, we have (5.5) holds. Let
\[
\delta_1(t) = r_0(1 + t^{\gamma+1}) + r_1(1 + t^{\gamma+2}) + r_2(1 + t^{\gamma+3}) \quad \text{for } t \geq 0.
\]
Then, recalling that $\tau \geq 3$ and that $z_0$ is defined in (4.13), by a similar argument to that in (5.12), we have $\delta_1(z_0) \leq 0.1 e^{-\tau/2} \leq 0.1$. We prove a more general result as follows: for all $0 \leq t \leq m_0$,
\[
(5.24) \quad E\{\Psi_{\beta,t}(W)\} \leq 2e^{t^2/2+2\delta_1(t)},
\]
which, together with the fact that $\delta_1(z_0) \leq 0.1$, implies (5.23) immediately.

It now suffices to prove (5.24). For $t \geq 0$, let $h(t) = E\{\Psi_{\beta,t}(W)\}$. As $\Psi_{\beta,t}(w) \leq 2e^{t\beta} + 1$, then $h(t) < \infty$ for all $0 \leq t \leq m_0$. Write
\[
\Psi_{\beta,t}'(w) = \frac{\partial}{\partial w}\Psi_{\beta,t}(w), \quad \Psi_{\beta,t}''(w) = \frac{\partial^2}{\partial w^2}\Psi_{\beta,t}(w).
\]
By the definition of $\Psi_{\beta,t}(W)$,
\[
(5.25) \quad \frac{\partial}{\partial t}\Psi_{\beta,t}(w) = \begin{cases} \! w e^{tw} & \text{if } w \leq \beta, \\
\! 2\beta e^{\beta} - (2\beta - w)e^{t(2\beta - w)} & \text{if } w > \beta,
\end{cases}
\]
\[
(5.26) \quad \Psi_{\beta,t}'(w) = \begin{cases} \! t e^{tw} & \text{if } w \leq \beta, \\
\! t e^{(2\beta - w)} & \text{if } w > \beta,
\end{cases}
\]
and
\[
(5.27) \quad \Psi_{\beta,t}''(w) = \begin{cases} \! t^2 e^{tw} & \text{if } w \leq \beta, \\
\! -t^2 e^{(2\beta - w)} & \text{if } w > \beta.
\end{cases}
\]
Also,
\[
(5.28) \quad \frac{\partial}{\partial t}\Psi_{\beta,t}(w) \leq w(\Psi_{\beta,t}(w) - 1), \quad \Psi_{\beta,t}'(w) \leq t\Psi_{\beta,t}(w).
\]
By (5.25) and the first inequality of (5.28), it follows that
\[
(5.29) \quad h'(t) = E\left\{\frac{\partial}{\partial t}\Psi_{\beta,t}(W)\right\} \leq E\{W(\Psi_{\beta,t}(W) - 1)\}.
\]
By (1.1) and (5.28) and noting that $E W = 0$ and $|\Psi_{\beta,t}(w) - 1| \leq \Psi_{\beta,t}(w)$ for all $w \in \mathbb{R}$, we have
\[
(5.30) \quad E\{W(\Psi_{\beta,t}(W) - 1)\} = E\left\{\int_{-\infty}^{\infty} \Psi_{\beta,t}'(W + u)\hat{K}(u)du\right\} + E\{R(\Psi_{\beta,t}(W) - 1)\}
\]
\[
= E\left\{\int_{-\infty}^{\infty} \Psi_{\beta,t}'(W)\hat{K}(u)du\right\} + E\{R(\Psi_{\beta,t}(W) - 1)\}
\]
\[
+ E\left\{\int_{-\infty}^{\infty} (\Psi_{\beta,t}(W + u) - \Psi_{\beta,t}'(W))\hat{K}(u)du\right\}
\]
\[
\leq th(t) + tE\left\{|\hat{K}_{1}|W\right\} - 1|\Psi_{\beta,t}(W)\right\} + E\{|R\Psi_{\beta,t}(W)|\}
\]
\[
+ E\left\{\int_{-\infty}^{\infty} (\Psi_{\beta,t}(W + u) - \Psi_{\beta,t}'(W))\hat{K}(u)du\right\}.
\]
By (5.27), for all $w \in \mathbb{R}$, we have

$$
|\Psi'_{\beta,t}(w + u) - \Psi'_{\beta,t}(w)| \leq |u| \sup_{s \leq |u|} |\Psi''_{\beta,t}(w + s)|
$$

(5.31)

$$
\leq |u| t^2 \sup_{s \leq |u|} |\Psi''_{\beta,t}(w + s)|
\leq |u| t^2 e^{t|u|} |\Psi_{\beta,t}(w)|.
$$

By (4.2), (4.9) and (5.31), we have that the last term of (5.30) can be bounded by

$$
E \left\{ \int_{-\infty}^{\infty} (\Psi'_{\beta,t}(W + u) - \Psi'_{\beta,t}(W'))K(u)du \right\}
\leq t^2 \sup_{s \leq |u|} |\Psi''_{\beta,t}(w + s)|
\leq |u| t^2 e^{t|u|} |\Psi_{\beta,t}(w)|.
$$

By (5.7), (5.8), (5.30) and (5.32) into (5.29) yields

$$
h'(t) \leq t h(t) + r_0(1 + t^{\tau_0}) h(t) + r_1 t(1 + t^{\tau_1}) h(t) + r_2 t^2(1 + t^{\tau_2}) h(t).
$$

Solving the differential inequality yields

$$
h(t) \leq 2 \exp \left\{ \frac{t^2}{2} + r_0 \left( t + \frac{t^{\tau_0+1}}{\tau_0 + 1} \right) + r_1 \left( \frac{t^2}{2} + \frac{t^{\tau_1+2}}{\tau_1 + 2} \right) + r_2 \left( \frac{t^3}{3} + \frac{t^{\tau_2+3}}{\tau_2 + 3} \right) \right\}.
$$

(5.33)

Since $\tau_0, \tau_1, \tau_2 \geq 0$, by Young's inequality, we have

$$
\frac{t}{\tau_0 + 1} \leq 2(1 + t^{\tau_0+1}),
$$

(5.34)

$$
\frac{t^2}{2} + \frac{t^{\tau_1+2}}{\tau_1 + 2} \leq (1 + t^{\tau_1+2}),
$$

$$
\frac{t^3}{3} + \frac{t^{\tau_2+3}}{\tau_2 + 3} \leq (1 + t^{\tau_2+3}).
$$

Combining (5.33) and (5.34) yields (5.24), as desired.

The following lemma establish an error bound for differences between tail probabilities of $W$ and $Z$. This is one of the key observations in our proof, since we can give a bound between $\mathbb{P}(W > z)$ and $\mathbb{P}(Z > z)$ for $z$ slightly large than $z_0$, which is very important in the recursive argument.

**Lemma 5.4.** Assume that the conditions in Lemma 5.1 hold. For $8 \leq z \leq z_0$, $0 \leq \varepsilon \leq 2$, $|u| \leq 1$ and $u \wedge 0 \leq s \leq u \vee 0$, we have

$$
\mathbb{P}(W + s > z) - \mathbb{P}(Z + s > z) \leq 2e^{t/2}e^{z|u|} \delta(z)(1 - \Phi(z))(C_0 + c_0)
$$

(5.35)

and

$$
\mathbb{P}(W + s > z + \varepsilon) - \mathbb{P}(Z + s > z + \varepsilon) \leq 2e^{t/2}e^{z|u|+z\varepsilon} \delta(z)(1 - \Phi(z))(C_0 + c_0),
$$

(5.36)

where $C_0$ is defined as in (5.1), $\tau$ is as in Theorem 4.1, and $c_0 := 1/\delta(m_0) + (150e^{t/2})^\tau$.

**Proof of Lemma 5.4.** We first introduce some inequalities. For $z \geq 8$ and $0 \leq a \leq 3$, we have

$$
1 - \Phi(z - a) \leq \frac{1}{z - a} \phi(z - a) \leq \frac{e^{z\alpha} + 1}{z - a} \phi(z)
$$

(5.37)

$$
\leq \frac{(1 + z^2)e^{z\alpha}}{z(z - a)}(1 - \Phi(z)) \leq 2e^{z\alpha}(1 - \Phi(z)).
$$
Moreover, noting that
\[(1 + (z + 3)\ell) \leq 1.1(z + 3)\ell \leq 1.1 \times 1.375^\ell z^{\ell} \leq e^{\tau/2}(1 + z^{\ell}) \quad \text{for all } 1 \leq \ell \leq \tau \text{ and } z \geq 8,
\]
and by the definition of \(\delta(z)\) as in Theorem 4.1, we have
\[
\delta(z + a) \leq e^{\tau/2}\delta(z) \quad \text{for all } 0 \leq a \leq 3 \text{ and } z \geq 8.
\]
We first prove (5.35). To this end, we consider three cases.

(1). If \(s > 0\), then by the definition of \(C_0\) in (5.1), by (5.37) and noting that \(|s| \leq |u| \leq 1\), we have
\[
|\mathbb{P}[W + s > z] - \mathbb{P}[Z + s > z]| \leq C_0\delta(z - s)(1 - \Phi(z - s)) \leq 2C_0e^{z|u|}\delta(z)(1 - \Phi(z)).
\] (5.39)

(2). If \(s < 0\) and \(z - s \leq z_0\), by (5.38) and noting that \(|s| \leq 1\),
\[
|\mathbb{P}[W + s > z] - \mathbb{P}[Z + s > z]| \leq C_0\delta(z - s)(1 - \Phi(z - s)) \leq C_0e^{\tau/2}\delta(z)(1 - \Phi(z)).
\] (5.40)

(3). If \(s < 0\) but \(z - s > z_0\), it then follows that \(z_0 \geq z \geq 8\) and \(|z - z_0| \leq |s| \leq |u| \leq 1\).

By (5.1), (5.37) and (5.38),
\[
|\mathbb{P}[W + s > z] - \mathbb{P}[Z + s > z]| \leq \mathbb{P}[W > z_0] + \mathbb{P}[Z > z_0] \\
\leq (1 - \Phi(z_0)) + C_0\delta(z_0)(1 - \Phi(z_0)) + 1 - \Phi(z_0) \\
\leq 2e^{\tau/2}(1 - \Phi(z))(1 + e^{\tau/2}C_0\delta(z)).
\]

By (4.13), we have
\[
z_0 \geq 0.02e^{-\tau/2}\min\left\{50e^{\tau/2}m_0, \frac{1}{3}r_0^{-1/(\tau_0 + 1)}, \frac{1}{3}r_1^{-1/(\tau_1 + 2)}, \frac{1}{3}r_2^{-1/(\tau_2 + 3)}\right\}.
\]
Hence, by (4.15) and recalling that \(c_0 = 1/\delta(m_0) + (150e^{\tau/2})\), we have
\[
\frac{1}{\delta(z_0)} \leq c_0.
\]

By (5.38) and the fact that \(z \leq z_0 \leq z + 1\), we now have
\[
1 = \frac{\delta(z)}{\delta(z)} \leq e^{\tau/2}\delta(z)\frac{1}{\delta(z_0)} \leq c_0e^{\tau/2}\delta(z).
\]

Therefore, it follows that
\[
|\mathbb{P}[W + s > z] - \mathbb{P}[Z + s > z]| \leq 2(C_0 + c_0)e^{\tau/2}e^{z|u|}\delta(z)(1 - \Phi(z)).
\] (5.41)

Combining (5.39)–(5.41) yields (5.35). The inequality (5.36) can be shown similarly. \(\square\)

We next give the proof of Lemma 5.1, which couples Stein’s method with the recursive method. The proof includes two parts. First, based on Stein’s method, we split the numerator on the left hand side of (5.2) into several terms. Second, with the help of Lemmas 5.3 and 5.4, we bound these parts by recursive arguments.

**Proof of Lemma 5.1.** We first introduce some notation and inequalities. We fix \(8 \leq z \leq z_0\) in this proof. Because \(z_0 \geq 8\), we have that (5.5) holds. We also choose \(\beta := z_0\) in the function \(\Psi_{\beta,t}(w)\), and let \(\varepsilon = 40e^{\tau/2}r_2(1 + z^{\tau_2})\). By (5.10) and (5.13), we have \(e^{\beta\varepsilon} \leq 1.05\) and \(\varepsilon \leq 0.1\).
Now, consider the Stein equation

\[(5.42) \quad f'(w) - wf(w) = h_{z,\varepsilon}(w) - \mathbb{E}\{h_{z,\varepsilon}(Z)\},\]

and let \(f := f_{z,\varepsilon}\) be its solution. Let \(g(w) = wf(w)\) and let \(v(w) = (2\pi)^{-1/2} \int_0^w e^{-(s+\varepsilon-s)^2/2} ds\).

Recall that \(Z \sim N(0, 1)\), \(Nh_{z,\varepsilon} = \mathbb{E}h_{z,\varepsilon}(Z)\), \(\Phi(\cdot)\) is the standard normal probability density function and \(\Phi(\cdot)\) is the standard normal distribution function. It can be shown that (see, e.g., Lemma 5.3 of Chen and Shao (2004))

\[(5.43) \quad Nh_{z,\varepsilon} = \Phi(z) + \varepsilon v(1) = \Phi(z) + \int_z^{z+\varepsilon} \left(1 + \frac{z-s}{\varepsilon}\right) \phi(s) ds,\]

\[(5.44) \quad f(w) = \begin{cases} \frac{1 - \Phi(w)}{\phi(w)} Nh_{z,\varepsilon} - \frac{\varepsilon}{\phi(w)} v \left(1 + \frac{z-w}{\varepsilon}\right) & \text{if } z < w \leq z + \varepsilon, \\ \frac{1 - \Phi(w)}{\phi(w)} Nh_{z,\varepsilon} & \text{if } w > z + \varepsilon, \end{cases}\]

and

\[(5.45) \quad g'(w) = \begin{cases} \frac{(1 + w^2)\Phi(w)}{\phi(w)} + w \left(1 - Nh_{z,\varepsilon}\right) & \text{if } w \leq z, \\ \frac{(1 + w^2)(1 - \Phi(w))}{\phi(w)} - w Nh_{z,\varepsilon} \\ - \frac{\varepsilon(1 + w^2)}{\phi(w)} v \left(1 + \frac{z-w}{\varepsilon}\right) + \frac{w(z-w+\varepsilon)}{\varepsilon} & \text{if } z < w \leq z + \varepsilon, \\ \frac{(1 + w^2)(1 - \Phi(w))}{\phi(w)} - w Nh_{z,\varepsilon} & \text{if } w > z + \varepsilon. \end{cases}\]

Thus, by (1.1) and (5.42),

\[(5.46) \quad \left|\mathbb{E}\{h_{z,\varepsilon}(W)\} - Nh_{z,\varepsilon}\right| = \left|\mathbb{E}\{f'(W) - Wf(W)\}\right| = \left|\mathbb{E}\{f'(W)\} - \mathbb{E}\left\{\int_{-\infty}^{\infty} f'(W + u)K(u)du\right\} - \mathbb{E}\{Rf(W)\}\right| \leq |I_1| + |I_2| + |I_3|,\]

where

\[I_1 = \mathbb{E}\left\{\int_{-\infty}^{\infty} f'(W + u) - f'(W)K(u)du\right\}, \quad I_2 = \mathbb{E}\{f'(W)(1 - \hat{K})\}, \quad I_3 = \mathbb{E}\{Rf(W)\}.\]

For \(I_1\), by (5.42), we have

\[(5.47) \quad I_1 = I_{11} + I_{12} + I_{13} + I_{14},\]

where

\[I_{11} = \mathbb{E}\left\{\int_{-\infty}^{\infty} (g(W + u) - g(W))\hat{K}(u)du\right\},\]
Lemma 5.1

It now suffices to prove (5.51). Combining (5.52), (5.53), and (5.54), we have

\[ \int |u| \mathbb{E}\left\{ |u\tilde{K}(u)| \right\} du \leq 2r_2. \] (5.54)

Moreover,

\[ 1 - Nh_{z,\varepsilon} \leq 1 - \Phi(z). \] (5.55)
Thus,

\[
I_{111} \leq 2(1 - Nh_{z,e}) \mathbb{E} \left\{ \int_{-\infty}^{\infty} |u\hat{K}(u)| du \right\} \leq 2r_2(1 - \Phi(z)).
\]

(2) **Bound of** \( I_{112} \). Observe that

\[
0 \leq \frac{(1 + w^2)}{\phi(w)} + w \leq 3(1 + w^2)e^{w^2/2} \quad \text{for } 0 \leq w \leq z.
\]

For any \( 0 \leq a \leq b \leq z \) and for any \( u \land 0 \leq s \leq u \lor 0 \), we have

\[
\mathbb{E}\{(1 + (W + s)^2)e^{(W + s)^2/2} |\hat{K}(u)|1(a \leq W + s \leq b)\}
\]

\[
\leq (1 + b^2) \mathbb{E}\{|\hat{K}(u)|e^{(W + s)^2/2 - b(W + s) + b(W + s)}1(a \leq W + s \leq b)\}
\]

\[
\leq (1 + b^2)e^{a^2/2 - ab} \mathbb{E}\{|\hat{K}(u)|e^{b(W + s)}1(a \leq W + s \leq b)\}
\]

\[
\leq (1 + b^2)e^{-a^2/2}e^{-b^2/2} \mathbb{E}\{|\hat{K}(u)|e^{b|u|\Psi_{z_0,0}(W)}\}.
\]

Denote by \( \lfloor z \rfloor \) the greatest integer which is smaller than or equal to \( z \). Noting that for \( u \land 0 \leq s \leq u \lor 0 \), by (5.45), (5.55) and (5.57) and applying (5.58) with \( a = j - 1, b = j \) and \( a = \lfloor z \rfloor, b = z \), respectively, we have

\[
|\mathbb{E}\{g'(W + s)\hat{K}(u)1(0 \leq W + s \leq z)\}|
\]

\[
\leq 3(1 - \Phi(z)) \mathbb{E}\{(1 + (W + s)^2)e^{(W + s)^2/2} |\hat{K}(u)|1(0 \leq W + s \leq z)\}
\]

\[
\leq 3(1 - \Phi(z)) \sum_{j=1}^{\lfloor z \rfloor} \mathbb{E}\{(1 + (W + s)^2)e^{(W + s)^2/2} |\hat{K}(u)|1(j - 1 \leq W + s \leq j)\}
\]

\[
+ 3(1 - \Phi(z)) \mathbb{E}\{(1 + (W + s)^2)e^{(W + s)^2/2} |\hat{K}(u)|1(\lfloor z \rfloor \leq W + s \leq z)\}
\]

\[
\leq 3e^{1/2}(1 - \Phi(z)) \sum_{j=1}^{\lfloor z \rfloor} (1 + j^2)e^{-j^2/2} \mathbb{E}\{|\hat{K}(u)|e^{j|u|\Psi_{z_0,j}(W)}\}
\]

\[
+ 3e^{1/2}(1 - \Phi(z))(1 + z^2)e^{-z^2/2} \mathbb{E}\{|\hat{K}(u)|e^{z|u|\Psi_{z_0,z}(W)}\}.
\]

Thus, by the definition of \( I_{112} \), and by (4.9) and Lemma 5.3, we have for \( 0 \leq z \leq z_0 \),

\[
I_{112} \leq 5(1 - \Phi(z)) \sum_{j=1}^{\lfloor z \rfloor} (1 + j^2)e^{-j^2/2} \mathbb{E}\{\hat{K}_{2,j}\Psi_{z_0,j}(W)\}
\]

\[
+ 5(1 - \Phi(z))(1 + z^2)e^{-z^2/2} \mathbb{E}\{\hat{K}_{2,z}\Psi_{z_0,z}(W)\}
\]

\[
\leq 20r_2(1 - \Phi(z)) \left( \sum_{j=1}^{\lfloor z \rfloor} (1 + j^2)(1 + j^2) + (1 + z^2)(1 + z^2) \right).
\]
For all $\ell \geq 0$ and $z \geq 8$, it can be shown that

\[
\sum_{j=1}^{[z]} (1 + j^2)(1 + j^\ell) \leq \sum_{j=1}^{[z]} 1 + j^2 + j^\ell + j^{2+\ell} \leq z + \frac{z^3}{3} + \frac{z^2}{2} + \frac{z}{6} + \sum_{j=1}^{[z]} (j^\ell + j^{2+\ell}) \leq 0.51z^{\ell+3} + \sum_{j=1}^{[z]} (j^\ell + j^{2+\ell})
\]

and for any $m \geq 0$ and $n \geq 1$ we have

\[
\sum_{j=1}^{n} j^m \leq n^m + \sum_{j=1}^{n-1} \int_{j}^{j+1} x^m dx = n^m + \frac{1}{m+1} n^{m+1} \leq \left(\frac{1}{m+1} + \frac{1}{n}\right)n^{m+1}.
\]

Recalling that $z \geq 8$ and $\ell \geq 0$, by (5.60) with $n = [z]$ and $m = \ell$ or $\ell + 2$, we have

\[
\sum_{j=1}^{[z]} (1 + j^2)(1 + j^\ell) \leq 0.51z^{\ell+3} + \left(\frac{1}{\ell+1} + \frac{1}{8}\right)z^{\ell+1} + \left(\frac{1}{\ell + 1} + \frac{1}{8}\right)z^{\ell+3} 
\leq \left(0.51 + \frac{1}{8^2(\ell+1)} + \frac{1}{8^3} + \frac{1}{\ell + 3} + \frac{1}{8}\right)z^{\ell+3} \leq z^{\ell+3}.
\]

Then, for all $\ell \geq 0$ and $z \geq 8$,

\[
\sum_{j=1}^{[z]} (1 + j^2)(1 + j^\ell) \leq (1 + z)^{\ell+3},
\]

(5.62)

\[
(1 + z^2)(1 + z^\ell) \leq 2\left(1 + \frac{1}{64}\right)z^{\ell+2} \leq \frac{2.032}{8}z^{\ell+3} \leq 0.26(1 + z^{\ell+3}).
\]

Thus,

(5.63) $I_{112} \leq 26r_2(1 + z^{\ell+3})(1 - \Phi(z))$.

(3) Bound of $I_{113}$. According to Eqs. (4.5) and (4.6) in Chen and Shao (2004), we have $|f(w)| \leq 1$ and $|f'(w)| \leq 1$ for $w \in \mathbb{R}$. Thus, recalling that $g(w) = w f(w)$ and by the fact that $\varepsilon \leq 1$, we have

(5.64) $|g'(w)| \leq |f(w) + w f'(w)| \leq 1 + z + \varepsilon \leq 4(z + 1)$ if $z + \varepsilon \geq w \geq z$.

By (5.45) and (5.64) and the fact that

(5.65) $\left|\frac{(1 + w^2)(1 - \Phi(w))}{\phi(w)} - w\right| \leq 1$ for $w \geq 8$,

we have

(5.66) $|g'(w)| \leq 4(z + 1)$ if $w \geq z$.

For any $\ell \geq 0$ and $z \geq 8$, we have

(5.67) $(1 + z)^2(1 + z^\ell) \leq 2 \times 1.125^2z^{\ell+2} \leq 0.32(1 + z^{\ell+3})$. 
By (4.9), (5.6), (5.23), (5.66) and (5.67) and the Markov’s inequality,

\[
I_{113} \leq 4(1 + z) \mathbb{E} \left\{ \int_{-\infty}^{\infty} \int_{0, u}^{0, u} 1(W + s > z) |\hat{K}(u)| du \right\} \\
\leq 4(1 + z) \Psi_{z_0, z}(z)^{-1} \mathbb{E} \left\{ \int_{-\infty}^{\infty} |u| \hat{K}(u) |\Psi_{z_0, z}(W + |u|)| du \right\}
\]

(5.68)

\[
\leq 4(1 + z) e^{-z^2} \mathbb{E} \left\{ \int_{-\infty}^{\infty} e^{z|u|} |u| \hat{K}(u) |\Psi_{z_0, z}(W)| du \right\} \\
\leq 16(2\pi)^{1/2} r_2(1 + z)(1 + z^{2}) \phi(z) \\
\leq 40.2 r_2(1 + z)^2(1 + z^{2})(1 - \Phi(z)) \\
\leq 13 r_2(1 + z^{2}+3)(1 - \Phi(z)).
\]

Therefore, (5.48) follows from (5.56), (5.63) and (5.68).

(ii) Proof of (5.49). By the Markov inequality,

\[
|I_{12}| \leq \mathbb{E} \left\{ \int_{|u| > 1} 1(W + u > z) |\hat{K}(u)| du \right\} \\
+ \mathbb{E} \left\{ \int_{|u| > 1} 1(W > z) |\hat{K}(u)| du \right\} \\
\leq \mathbb{E} \left\{ \int_{-\infty}^{\infty} |u| e^{-z^2} \Psi_{z_0, z}(W + |u|) |\hat{K}(u)| du \right\} \\
+ \mathbb{E} \left\{ \int_{-\infty}^{\infty} |u| e^{-z^2} \Psi_{z_0, z}(W) |\hat{K}(u)| du \right\} \\
\leq 2 \mathbb{E} \left\{ \int_{-\infty}^{\infty} e^{-z^2} \Psi_{z_0, z}(W) |u| e^{z|u|} |\hat{K}(u)| du \right\} \\
\leq 2 e^{-z^2} \mathbb{E} \{ \hat{K}_{2, z} \Psi_{z_0, z}(W) \} \\
\leq 8 r_2(1 + z^{2}) e^{-z^2/2},
\]

where we used (4.9) and (5.23) in the last line. By (5.6), (5.10) and (5.15), we have

\[
|I_{12}| \leq 8(2\pi)^{1/2} r_2(1 + z^{2}) \phi(z) \\
\leq 21 r_2(1 + z)(1 + z^{2})(1 - \Phi(z)) \\
\leq r_2(1 + z^{2}+3)(1 - \Phi(z)),
\]

which proves (5.49).

(iii) Proof of (5.50). Observe that (see also (2.5) of Chen, Röllin and Xia (2020))

\[
|h_{z, \varepsilon}(w + u) - h_{z, \varepsilon}(w)| \leq \frac{1}{\varepsilon} \int_{u \wedge 0}^{u \vee 0} 1[z < w + s \leq z + \varepsilon] ds \\
\leq 1(z - u \vee 0 < w \leq z - u \wedge 0 + \varepsilon).
\]

(5.69)
Recall that $8 \leq z \leq z_0$, and by (5.69) and Fubini’s theorem,
\[ |I_{13}| \leq \int_{|u| \leq 1} \mathbb{E}\{|h_{z,\varepsilon}(W + u) - h_{z,\varepsilon}(W)|\} |K(u)| du \]
(5.70)
\[ \leq \frac{1}{\varepsilon} \int_{|u| \leq 1} \int_{u \wedge 0}^{u \vee 0} \mathbb{P}[z < W + s \leq z + \varepsilon]|K(u)| ds du \]
\[ \leq I_{131} + I_{132} + I_{133}, \]
where
\[ I_{131} := \int_{|u| \leq 1} (\Phi(z - 0 \wedge u + \varepsilon) - \Phi(z - 0 \vee u)) |K(u)| du, \]
(5.71)
\[ I_{132} := \frac{1}{\varepsilon} \int_{|u| \leq 1} \int_{u \wedge 0}^{u \vee 0} \{\mathbb{P}[W + s > z] - \mathbb{P}[Z + s > z]\} |K(u)| ds du, \]
\[ I_{133} := \frac{1}{\varepsilon} \int_{|u| \leq 1} \int_{u \wedge 0}^{u \vee 0} \{\mathbb{P}[W + s > z + \varepsilon] - \mathbb{P}[Z + s > z + \varepsilon]\} |K(u)| ds du. \]
One can easily verify that $(1 + z^{\tau_2})(1 + z^3)/(1 + z^{\tau_2 + 3})$ is a decreasing function for $z \geq 1$ and $\tau_2 \geq 0$, and
\[ \sup_{z \geq 2} \frac{(1 + z^{\tau_2})(1 + z^3)}{1 + z^{\tau_2 + 3}} \leq 2. \]
(5.72)
Thus, for $z \geq 8$,
\[ (1 + z) \leq 1.125z \leq 0.02(1 + z^3), \]
(5.73)
\[ (1 + z^{\tau_2})(1 + z^3) \leq 2(1 + z^{\tau_2 + 3}). \]
(5.74)
Moreover, if $\beta = 0$, then $1 \leq \Psi_{\beta,t}(w) \leq 3$ for all $w$ and $t$, and by (4.9) with $\beta = 0$, we have
\[ \int_{|u| \leq 1} e^{z|u|} u K_u(u) du \leq \mathbb{E}\{\hat{K}_{2,z}\} \leq 3r_2(1 + z^{\tau_2}). \]
(5.75)
For $I_{131}$, by (5.6), (5.7) and (5.73), for $z \geq 8$ and $|u| \leq 1$,
\[ \Phi(z - 0 \wedge u + \varepsilon) - \Phi(z - 0 \vee u) \leq (|u| + \varepsilon) \phi(z - 0 \vee u) \]
\[ \leq (|u| + \varepsilon) e^{z|u|} \phi(z) \]
(5.76)
\[ \leq (|u| + \varepsilon) e^{z|u|}(1 + z)(1 - \Phi(z)) \]
\[ \leq (0.02|u| + 0.02\varepsilon)e^{z|u|}(1 + z^3)(1 - \Phi(z)). \]
Then, recalling that $\varepsilon = 40e^{\tau_2/2}r_2(1 + z^{\tau_2})$, by (4.12), (5.74) and (5.75),
\[ I_{131} \leq 0.02(1 + z^3)(1 - \Phi(z)) \int_{|u| \leq 1} (|u| + \varepsilon) e^{z|u|} |K(u)| du \]
(5.77)
\[ \leq (0.06 + 0.8pe^{\tau_2/2})r_2(1 + z^3)(1 + z^{\tau_2})(1 - \Phi(z)) \]
\[ \leq (0.12 + 1.6pe^{\tau_2/2})r_2(1 + z^{\tau_2 + 3})(1 - \Phi(z)). \]
As for $I_{132}$, by Lemma 5.4 and (5.75) and recalling that $\varepsilon = 40e^{\tau/2}r_2(1 + z^2)$, we have for $8 \leq z \leq z_0$ and $|u| \leq 1$,
\begin{align}
I_{132} &\leq 2e^{\tau/2}e^{-1}(C_0 + c_0)\delta(z)(1 - \Phi(z)) \int_{|u| \leq 1} e^{z|u|} |uK(u)| du \\
&\leq 6e^{\tau/2}r_2e^{-1}(C_0 + c_0)\delta(z)(1 - \Phi(z))(1 + z^2) \\
&\leq 0.15(C_0 + c_0)\delta(z)(1 - \Phi(z)).
\end{align}
(5.78)

As for $I_{133}$, as $8 \leq z \leq z_0$ and $0 \leq \varepsilon \leq 1$, by Lemma 5.4 and (5.10) again, we have
\begin{align}
I_{133} &\leq 0.15e^{z\varepsilon}(C_0 + c_0)\delta(z)(1 - \Phi(z)) \\
&\leq 0.16(C_0 + c_0)\delta(z)(1 - \Phi(z)).
\end{align}
(5.79)

By (5.71) and (5.77)–(5.79), we have
\[ |I_{13}| \leq 0.31(C_0 + c_0)\delta(z)(1 - \Phi(z)) + (0.12 + 1.6pe^{\tau/2})r_2(1 + z^{\tau_1+3})(1 - \Phi(z)). \]

This proves (5.50).

(iv) **Proof of (5.51).** Without loss of generality, we assume that $r_4 > 0$. Without this assumption, the proof would be even easier. Note that by (5.69),
\[ |I_{14}| \leq \mathbb{E} \left\{ \int_{|u| \leq 1} \mathbf{1}\{z - 0 \vee u < W \leq z - 0 \wedge u + \varepsilon\} |\hat{K}(u) - K(u)| du \right\}. \]

Recall Young’s inequality
\[ ab \leq \frac{a^2}{2c} + \frac{cb^2}{2} \quad \text{for } a, b \geq 0 \text{ and } c > 0. \]

Applying Young’s inequality with $a = \mathbf{1}\{z - 0 \vee u < W \leq z - 0 \wedge u + \varepsilon\}$, $b = |\hat{K}(u) - K(u)| \mathbf{1}\{W > z - 0 \vee u\}$ and
\[ c = \frac{e^{z|u|}}{99e^{\tau/2}}(e^{\tau/2}(4.1C_0 + 4.1c_0 + 1.6) + r_4^{-1/2}|u|), \]
we have
\begin{align}
|I_{14}| &\leq \frac{1}{2} \int_{|u| \leq 1} c^{-1} \mathbb{P}[z - 0 \vee u < W \leq z - 0 \wedge u + \varepsilon] du \\
&\quad + \frac{1}{2} \mathbb{E} \left\{ \int_{|u| \leq 1} c(\hat{K}(u) - K(u))^2 \mathbf{1}\{W > z - 0 \vee u\} du \right\} \\
&:= I_{141} + I_{142}.
\end{align}
(5.80)

Using a similar argument to (5.76), and by (5.74) and the fact that $\tau_4 \geq 0$, we have
\begin{align}
\mathbb{P}[z - 0 \vee u < Z \leq z - 0 \wedge u + \varepsilon] &\leq |u|e^{z|u|}(1 + z^{\tau_4+1})(1 - \Phi(z)) + 0.8e^{\tau/2}r_2(1 + z^2)(1 + z^{\tau_4})(1 - \Phi(z)) \\
&\leq r_4^{-1/2}|u|e^{z|u|}\delta(z)(1 - \Phi(z)) + 1.6e^{\tau/2}\delta(z)(1 - \Phi(z)).
\end{align}
(5.81)
By (5.81) and Lemma 5.4, and noting that $e^{20\varepsilon} \leq 1.05$ in (5.10),
\begin{align*}
P[z - 0 \lor u < W \leq z - 0 \land u + \varepsilon] \\
&\leq P[z - 0 \lor u < Z \leq z - 0 \land u + \varepsilon] \\
&+ |P[W \geq z - 0 \lor u] - P[Z \geq z - 0 \lor u]| \\
&+ |P[W \geq z - 0 \land u + \varepsilon] - P[Z \geq z - 0 \land u + \varepsilon]| \\
&\leq (e^{r/2}(4.1C_0 + 4.1c_0 + 1.6) + r_4^{-1/2}|u|)e^{z|u|}\delta(z)(1 - \Phi(z)).
\end{align*}

(5.82) Then, we have
\begin{equation}
I_{141} \leq 99e^{r/2}\delta(z)(1 - \Phi(z)).
\end{equation}

Moreover, as $z \geq 8$, by the Markov inequality and by Lemma 5.3,
\begin{align*}
E\left\{ \int_{|u| \leq 1} |u|e^{z|u|}(\hat{K}(u) - K(u))^2 1(W > z - 0 \lor u)du \right\} \\
&\leq e^{z^2}E\left\{ \int_{|u| \leq 1} |u|e^{2z|u|}(\hat{K}(u) - K(u))^2 \Psi_{z_0,z}(W)du \right\} \\
&\leq r_4(1 + z^{r_4})e^{z^2}E\{\Psi_{z_0,z}(W)\} \\
&\leq 4(2\pi)^{1/2}r_4(1 + z^{r_4})(1 + z)(1 - \Phi(z)) \\
&\leq 21r_4(1 + z^{r_4 + 1})(1 - \Phi(z)),
\end{align*}

where in the last line we used the inequality that
\[(1 + z)(1 + z^{r_4}) \leq 2(1 + z^{r_4 + 1}) \quad \text{for } z \geq 8.
\]

Similarly,
\begin{align*}
E\left\{ \int_{|u| \leq 1} e^{z|u|}(\hat{K}(u) - K(u))^2 1(W > z - 0 \lor u)du \right\} \\
&\leq 4(2\pi)^{1/2}r_3(1 + z)(1 + z^{r_3})(1 - \Phi(z)) \\
&\leq 21r_3(1 + z^{r_3 + 1})(1 - \Phi(z)).
\end{align*}

Then, by (5.84) and (5.85), we have
\begin{align*}
I_{142} &\leq \frac{e^{r/2}(4.1C_0 + 4.1c_0 + 1.6)}{198e^{r/2}} \times 21r_4(1 + z^{r_4 + 1})(1 - \Phi(z)) \\
&+ \frac{21r_4^{1/2}}{198e^{r/2}}(1 + z^{r_4 + 1})(1 - \Phi(z)) \\
&\leq 0.44(C_0 + c_0)\delta(z)(1 - \Phi(z)) + r_3(1 + z^{r_3 + 1})(1 - \Phi(z)) \\
&+ r_4^{1/2}(1 + z^{r_4 + 1})(1 - \Phi(z)) \\
&\leq (0.44C_0 + 0.44c_0 + 1)\delta(z)(1 - \Phi(z)).
\end{align*}

Combining (5.83) and (5.86) yields (5.51).
(v) Proof of (5.52). Note that (see, e.g., p. 2010 of Chen and Shao (2004))

\begin{equation}
|f'(w)| \leq \begin{cases} 
1 - \Phi(z) & \text{if } w < 0, \\
\left( \frac{w\Phi(w)}{\phi(w)} + 1 \right) (1 - \Phi(z)) & \text{if } 0 \leq w \leq z, \\
1 & \text{otherwise.}
\end{cases}
\end{equation}

Observe that

\begin{align*}
|I_2| & \leq \mathbb{E}\{ |f'(W)(1 - \mathbb{E}\{ \hat{K}_1|W\})| 1(W \leq 0) \} \\
& \quad + \mathbb{E}\{ |f'(W)(1 - \mathbb{E}\{ \hat{K}_1|W\})| 1(0 < W \leq z) \} \\
& \quad + \mathbb{E}\{ |f'(W)(1 - \mathbb{E}\{ \hat{K}_1|W\})| 1(W > z) \} \\
& := I_{21} + I_{22} + I_{23}.
\end{align*}

For $I_{21}$, since $-1 \leq w\Phi(w)/\phi(w) \leq 0$ for $w \leq 0$, and by (4.8) and (5.87), we have for $z \geq 8$

\begin{equation}
I_{21} \leq (1 - \Phi(z)) \mathbb{E}\{ |\mathbb{E}\{ \hat{K}_1|W\} - 1| \}
\end{equation}

\begin{equation}
\leq 2r_1(1 - \Phi(z)) \\
\leq 0.1r_1(1 + z^{r_1+2})(1 - \Phi(z)).
\end{equation}

For $I_{22}$, by (4.8) and (5.87), we have

\begin{equation}
I_{22} \leq (2\pi)^{1/2}(1 - \Phi(z)) \mathbb{E}\{ (W^{W^{2/2}} + 1)\mathbb{E}\{ \hat{K}_1|W\} - 1 \} 1(0 < W \leq z) \\
\leq 2.6(1 - \Phi(z)) I_{24} + 5.2r_1(1 - \Phi(z)),
\end{equation}

where

\begin{align*}
I_{24} & := \mathbb{E}\{ W^{W^{2/2}}|\mathbb{E}\{ \hat{K}_1|W\} - 1 \} 1(0 < W \leq z) \} \\
& = \sum_{j=1}^{[z]} \mathbb{E}\{ W^{W^{2/2}}|\mathbb{E}\{ \hat{K}_1|W\} - 1 \} 1(j-1 < W \leq j) \\
& \quad + \mathbb{E}\{ W^{W^{2/2}}|\mathbb{E}\{ \hat{K}_1|W\} - 1 \} 1([z] < W \leq z) \\
& := I_{241} + I_{242}.
\end{align*}

For $I_{241}$, noting that

\begin{equation}
w^{2/2} - jw \leq (j-1)^2/2 - j(j-1) = -j^2/2 + 1/2 \quad \text{for } j-1 < w \leq j,
\end{equation}

we have for $z \geq 8$, by (4.8) and (5.23),

\begin{align*}
I_{241} & \leq \sum_{j=1}^{[z]} j \mathbb{E}\{ e^{W^{2/2-jW+jW}}|\mathbb{E}\{ \hat{K}_1|W\} - 1 \} 1(j-1 \leq W \leq j) \\
& \leq e^{1/2} \sum_{j=1}^{[z]} j e^{-j^2/2} \mathbb{E}\{ |\mathbb{E}\{ \hat{K}_1|W\} - 1| \Psi_{z_0,j}(W) \} \\
& \leq 4e^{1/2}r_1 \sum_{j=1}^{[z]} j(1 + j^{r_1}) \\
& \leq 13.2r_1(1 + z^{r_1+2}).
\end{align*}
The last inequality is similar to that in (5.62). Similarly, for \( z \geq 8 \),

\[
I_{242} \leq 4e^{1/2}r_1 z(1 + z^{\tau_1}) \leq 1.7r_1(1 + z^{\tau_1+2}).
\]  

By (5.90) and (5.91), we have

\[
I_{24} \leq 15r_1(1 + z^{\tau_1+2}) \quad \text{for } z \geq 8.
\]

By (5.89) and (5.92), we have for \( z \geq 8 \),

\[
I_{22} \leq 39r_1(1 + z^{\tau_1+2})(1 - \Phi(z)) + 5.2r_1(1 - \Phi(z))
\]

\[
\leq 40r_1(1 + z^{\tau_1+2})(1 - \Phi(z)).
\]

As for \( I_{23} \), by Lemma 5.3 and (4.8), (5.6) and (5.87) and recalling \( z \geq 8 \), we have

\[
I_{23} \leq \mathbb{E}\{|\mathbb{E}\{|\mathbb{E}\{K_1 | W\} - 1|1(W > z)\}| - 1|\Psi_{\sigma_0, z}(W)\}| 
\]

\[
\leq 4r_1(1 + z)(1 + z^{\tau_1})e^{-z^2/2}
\]

\[
\leq 10.1r_1(1 + z)(1 + z^{\tau_1})\phi(z)
\]

\[
\leq 10.1r_1(1 + z)^2(1 + z^{\tau_1})(1 - \Phi(z))
\]

\[
\leq 25.2r_1(1 + z^{\tau_1+2})(1 - \Phi(z)).
\]

By (5.88), (5.93) and (5.94), we have

\[
|I_2| \leq 66r_1(1 + z^{\tau_1+2})(1 - \Phi(z)).
\]

(vi) Proof of (5.53). It is known that \( 0 \leq f(w) \leq 1 \) (see p. 2010 of Chen and Shao (2004)). Note that by (5.44) and (5.55),

\[
|I_3| \leq I_{31} + I_{32} + I_{33},
\]

where

\[
I_{31} = (1 - \Phi(z)) \mathbb{E}\{|R| 1(W \leq 0)\},
\]

\[
I_{32} = \sqrt{2\pi}(1 - \Phi(z)) \mathbb{E}\{e^{W^2/2}|R| 1(0 < W \leq z)\},
\]

\[
I_{33} = \mathbb{E}\{|R| 1(W > z)\}.
\]

For \( I_{31} \), by (4.7) with \( t = 0 \), we have

\[
I_{31} \leq r_0(1 + z^{\tau_0+1})(1 - \Phi(z)) \quad \text{for } z \geq 8.
\]

For \( I_{32} \), similar to (5.93), we have

\[
I_{32} \leq 60r_0(1 + z^{\tau_0+1})(1 - \Phi(z)).
\]

For \( I_{33} \), similar to (5.94), we have

\[
I_{33} \leq 21r_0(1 + z^{\tau_0+1})(1 - \Phi(z)).
\]

Combining the foregoing inequalities, we have

\[
|I_3| \leq 82r_0(1 + z^{\tau_0+1})(1 - \Phi(z)).
\]

This proves (5.53). \( \square \)
Now, we prove Lemma 5.2.

**Proof of Lemma 5.2.** In this proof, we develop a Berry–Esseen bound using the idea in Chen, Röllin and Xia (2020). Let

\[ \gamma := \sup_{z \in \mathbb{R}} | \mathbb{P}(W \leq z) - \Phi(z) |, \]

and let \( \varepsilon = \gamma / 2 \). Consider the Stein equation \((5.42)\), and denote by \( f_{z,\varepsilon} \) the solution to \((5.42)\), which is given in \((5.44)\). Let \( g_{z,\varepsilon}(w) = w f_{z,\varepsilon}(w) \). By Chen and Shao (2004), we have

\[ 0 \leq f_{z,\varepsilon} \leq 1, \quad |f_{z,\varepsilon}'| \leq 1. \]

Note that

\[ \gamma \leq \sup_{z \in \mathbb{R}} | \mathbb{E}\{h_{z,\varepsilon}(W)\} - Nh_{z,\varepsilon}| + 0.4\varepsilon. \]

Now, we bound the first term on the right hand side of \((5.97)\). By \((1.1)\) and \((5.42)\), we have

\[
\begin{align*}
\mathbb{E}\{h_{z,\varepsilon}(W)\} - Nh_{z,\varepsilon} &= \mathbb{E}\{f_{z,\varepsilon}'(W)\} - \mathbb{E}\{W f_{z,\varepsilon}(W)\} \\
&= \mathbb{E}\{f_{z,\varepsilon}'(W)\} - \mathbb{E}\{R f_{z,\varepsilon}(W)\} - \mathbb{E}\left\{ \int_{-\infty}^{\infty} f_{z,\varepsilon}'(W + u) \hat{K}(u) du \right\} \\
&= J_1 + J_2 + J_3 + J_4 + J_5,
\end{align*}
\]

where

\[
\begin{align*}
J_1 &= \mathbb{E}\{f_{z,\varepsilon}'(W)(1 - \hat{K}_1)\}, \\
J_2 &= -\mathbb{E}\{R f_{z,\varepsilon}(W)\}, \\
J_3 &= -\mathbb{E}\left\{ \int_{|u| > 1} (f_{z,\varepsilon}'(W + u) - f_{z,\varepsilon}'(W)) \hat{K}(u) du \right\}, \\
J_4 &= -\mathbb{E}\left\{ \int_{|u| \leq 1} (f_{z,\varepsilon}'(W + u) - f_{z,\varepsilon}'(W)) \hat{K}(u) du \right\}, \\
J_5 &= -\mathbb{E}\left\{ \int_{|u| \leq 1} (f_{z,\varepsilon}'(W + u) - f_{z,\varepsilon}'(W)) (\hat{K}(u) - K(u)) du \right\}.
\end{align*}
\]

By \((4.7)-(4.11)\) with \( t = 0 \) and noting \( 1 \leq \Psi_{\beta,0}(w) \leq 2 \), we have

\[ \mathbb{E}|R| \leq 2r_0, \quad \mathbb{E}|\mathbb{E}\{\hat{K}_1|W\} - 1| \leq 2r_1, \]

\[ \mathbb{E}\hat{K}_{2,0} \leq 2r_2, \quad \mathbb{E}\hat{K}_{3,0} \leq 2r_3, \quad \mathbb{E}\hat{K}_{4,0} \leq 2r_4. \]

By Eqs. (2.12), (2.15) and (2.16) of Chen, Röllin and Xia (2020), we have with \( a = 0.18 \),

\[ |J_4| \leq 4r_2 + \frac{4\gamma + 0.8\varepsilon}{\varepsilon} r_2, \]

\[ |J_5| \leq a\gamma + 0.2a\varepsilon + \frac{2r_3}{a} + (2a + 0.4/a)r_4^{1/2} + 5r_4^{1/2}. \]

Combining \((5.97), (5.98), (5.100)\) and \((5.101)\) gives

\[ \gamma \leq 0.4\gamma + 2r_0 + 2r_1 + 16.8r_2 + 11.2r_3 + 7.6r_4^{1/2}. \]

Solving the recursive inequality yields the desired result. \( \square \)
6. Proof of Theorem 2.1. We apply Theorem 4.1 to prove Theorem 2.1. To this end, we first construct Stein identity. Then, we prove a preliminary lemma which help us to prove Theorem 2.1. Denote by $C, C_1, C_2, \ldots$ absolute constants, which may take different values in different places. For $a, b \in \mathbb{R}$, let $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

For each $i \in J$, let $Y_i = \sum_{j \in A_i} X_j$. Further, define
\begin{equation}
\hat{K}_i(u) = X_i\{1(-Y_i \leq u < 0) - 1(0 \leq u \leq -Y_i)\}, 
\hat{K}(u) = \sum_{i \in J} \hat{K}_i(u),
\end{equation}
and define
\begin{equation}
K_i(u) = \mathbb{E}\{\hat{K}_i(u)\}, \quad K(u) = \sum_{i \in J} K_i(u).
\end{equation}
Note that $\mathbb{E}X_i = 0$ and $X_i$ and $W - Y_i$ are independent, thus $\mathbb{E}\{X_i f(W - Y_i)\} = 0$. Therefore,
\begin{align*}
\mathbb{E}\{W f(W)\} &= \sum_{i \in J} \mathbb{E}\{X_i(f(W) - f(W - Y_i))\} \\
&= \mathbb{E}\int_{-\infty}^{\infty} f'(w + t) \hat{K}(t) dt.
\end{align*}
Hence, it follows that (1.1) holds with $R = 0$ and $\hat{K}(t)$ defined as in (6.1).

6.1. A preliminary lemma. Let $\hat{K}_1, \hat{K}_{2,t}, \hat{K}_{3,t}, \hat{K}_{4,t}$ and $M_t$ be as in (4.1)–(4.5) with $\hat{K}(u)$ in (6.1) and $h(t) = \mathbb{E}\{\Psi_{\beta,t}(W)\}$. The following lemma provides the upper bounds of the terms in Condition (A1), whose proof is put in Appendix A.

Lemma 6.1. Under (LD1) and (LD2), let $m_0 = (a_n^{1/3} / 4) \wedge (a_n / 16)$. For $0 \leq t, \beta \leq m_0$, we have
\begin{align}
\mathbb{E}\{|\mathbb{E}\{\hat{K}_1|W\} - 1|\Psi_{\beta,t}(W)\} &\leq C_1(bkna_{n}^{-3} + b^{1/2}\kappa^{1/2}n^{1/2}a_{n}^{-2})(1 + t) \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
\mathbb{E}\{\hat{K}_{2,t}\Psi_{\beta,t}(W)\} &\leq C_2bna_{n}^{-3} \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
\mathbb{E}\{\hat{K}_{3,t}\Psi_{\beta,t}(W)\} &\leq C_3(\kappa bna_{n}^{-3} + \kappa^2 b^2 n^2 a_n^{-5})(1 + t^2) \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
\mathbb{E}\{\hat{K}_{4,t}\Psi_{\beta,t}(W)\} &\leq C_4(\kappa bna_{n}^{-4} + \kappa^2 b^2 n^2 a_n^{-6})(1 + t^2) \mathbb{E}\{\Psi_{\beta,t}(W)\},
\end{align}
and
\begin{equation}
\sup_{0 \leq t \leq m_0} M_t \leq C_5 bna_n^{-2}.
\end{equation}

6.2. Proof of Theorem 2.1. We apply Theorem 4.1 to prove Theorem 2.1. Recalling that $\theta_n = b^{1/2}n^{1/2}a_n^{-1}$, by Lemma 6.1, we have that condition (A1) is satisfied with $r_0 = \tau_0 = 0$ and
\begin{align}
r_1 &= C_1\kappa \theta_n(\theta_n + 1)a_n^{-1}, \quad \tau_1 = 1, \\
r_2 &= C_2\theta_n^2 a_n^{-1}, \quad \tau_2 = 0, \\
r_3 &= C_3(\kappa \theta_n^2 + 1)a_n^{-1}, \quad \tau_3 = 2, \\
r_4 &= C_4(\kappa \theta_n^2 + 1)a_n^{-2}, \quad \tau_4 = 2, \\
\rho &= C_5 \theta_n^2,
\end{align}
where $C_1, C_2, C_3, C_4$ and $C_5$ are absolute constants. Recalling the definition of $\delta(t)$ in (4.15), and that $m_0 = (a_n^{1/3}/4) \wedge (a_n/16)$, we have

$$
\delta(m_0) \geq (r_0 + r_1 + r_2 + r_3 + r_4^{1/2})(1 + m_0^3)
\geq C\theta_n\kappa(1 + \theta_n + \kappa\theta_n^3)(a_n^{-1} + a_n^2 \wedge 1) \geq C\theta_n\kappa.
$$

Combining (6.10) and (6.11) and noting that $\kappa \geq 1$, we can see that the right hand side of (4.14) is less than

$$
C\left(\frac{1}{\kappa\theta_n} + \theta_n^2 + 1\right)\delta(z) \leq 2C\left(\frac{1}{\theta_n} + \theta_n^2\right)(r_1 + r_2 + r_3 + r_4^{1/2})(1 + z^3)
\leq C'\delta_n(1 + z^3),
$$

where $\delta_n = \kappa^3a_n^{-1}(1 + \theta_n^6)$, $C'$ is an absolute constant and we use the fact that $x^2 + 1/x > 1$ for $x > 0$ in the first inequality. On the other hand, by (6.8) and (6.9), we have

$$
r_1^{1/3} + r_2^{1/3} \leq C\kappa^{-1/3}a_n^{-1/3}(1 + \theta_n)^{2/3}.
$$

Applying Theorem 4.1 and by (6.12) and (6.13), we obtain the desired result.

7. Proof of Theorem 3.1. In this section, we use the exchangeable pair to construct Stein identity (1.1). For any $k \geq 1$ and $k$-fold index $i \in \mathbb{N}^k$, we denote by $i_j$ its $j$-th element. Let $[n]_k := \{i = (i_1, \ldots, i_k) \in \mathbb{N}^k : 1 \leq i_1 \neq \ldots \neq i_k \leq n\}$ be a class of $k$-fold indices. Let $I := (I_1, I_2)$ be chosen uniformly from $[n]_2$ and be independent of $\pi$ and $X$, and let $W' = W - X_{i_1,\pi(i_1)} - X_{i_2,\pi(i_2)} + X_{I_1,\pi(i_2)} + X_{I_2,\pi(i_1)}$. Then, it follows that $(W, W')$ is an exchangeable pair. Moreover, we have

$$
\mathbb{E}\{W - W'|X, \pi\} = \frac{1}{n(n-1)} \sum_{i \in [n]_2} \mathbb{E}\{X_{i_1,\pi(i_1)} + X_{i_2,\pi(i_2)} - X_{i_1,\pi(i_2)} - X_{i_2,\pi(i_1)}|X, \pi\}
= \frac{2}{n-1}(W - R),
$$

where

$$
R = -\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} X_{i,j}.
$$

By exchangeability, with $\lambda = 2/(n-1)$ and $\Delta = W - W'$, we have

$$
0 = \mathbb{E}\{(W - W')(f(W) + f(W'))\} = 2\mathbb{E}\{\Delta f(W)\} - \mathbb{E}\{\Delta (f(W) - f(W - \Delta))\}
= 2\lambda \mathbb{E}\{(W - R)f(W)\} - \mathbb{E}\{\Delta (f(W) - f(W - \Delta))\}.
$$

Rearranging (7.2) yields

$$
\mathbb{E}\{W f(W)\} = \mathbb{E}\int_{-\infty}^{\infty} f'(W + u)\tilde{K}(u)du + \mathbb{E}\{R f(W)\},
$$

where

$$
\tilde{K}(u) = \frac{1}{2\lambda} \mathbb{E}\{\Delta (1(-\Delta \leq u \leq 0) - 1(0 < u \leq -\Delta))|X, \pi\}
= \frac{1}{4n} \sum_{i \in [n]_2} D_{i,\pi(i)}(1(-D_{i,\pi(i)} \leq u \leq 0) - 1(0 < u \leq -D_{i,\pi(i)}))
$$

in (7.3).
and $D_{i,j} = X_{i_1,j_1} + X_{i_2,j_2} - X_{i_1,j_2} - X_{i_2,j_1}$ for any $i = (i_1, i_2)$ and $j = (j_1, j_2)$. Therefore, the condition (1.1) is satisfied.

In what follows, denote by $C, C_1, C_2, \ldots$ absolute constants, which may take different values in different places.

7.1. A preliminary lemma. The following lemma will be useful in the proof of Theorem 3.1, and the proof of this lemma is put in Appendix B. Let $\hat{K}_1, \hat{K}_2, \hat{K}_{3,t}, \hat{K}_{4,t}$ and $M_t$ be as in (4.1)–(4.5) with $\hat{K}(u)$ defined as in (7.3) and $h(t) = \mathbb{E}\Psi_{\beta,t}(W)$.

**Lemma 7.1.** For $n \geq 4$ and $0 \leq t, \beta \leq \alpha_n^{1/3}/64$, we have
\begin{align*}
(7.4) & \quad \mathbb{E}\{||R|\Psi_{\beta,t}(W)\} \leq C_0 bC_n^{-1} \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
(7.5) & \quad \mathbb{E}\{\hat{K}_1 - 1|\Psi_{\beta,t}(W)\} \leq C_1 b(n\alpha_n^{-3} + n^{1/2} \alpha_n^{-2} + n^{-1/2}) \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
(7.6) & \quad \mathbb{E}\{\hat{K}_2|\Psi_{\beta,t}(W)\} \leq C_2 b\alpha_n^{-3} \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
(7.7) & \quad \mathbb{E}\{\hat{K}_3|\Psi_{\beta,t}(W)\} \leq C_3 b^2(n\alpha_n^{-3} + n^2 \alpha_n^{-5})(1 + t^2) \mathbb{E}\{\Psi_{\beta,t}(W)\}, \\
(7.8) & \quad \mathbb{E}\{\hat{K}_4|\Psi_{\beta,t}(W)\} \leq C_4 b^2(n\alpha_n^{-4} + n^2 \alpha_n^{-6})(1 + t^2) \mathbb{E}\{\Psi_{\beta,t}(W)\}
\end{align*}

and
\begin{align*}
(7.9) & \quad \sup_{0 \leq t \leq m_0} M_t \leq C_5 b\alpha_n^{-2}.
\end{align*}

7.2. Proof of Theorem 3.1. We apply Theorem 4.1 to prove Theorem 3.1. Recalling that $\theta_n = n^{1/2} \alpha_n^{-1}$, by Lemma 7.1, we have condition (A1) is satisfied with $m_0 = \alpha_n^{1/3}/64$,
\begin{align*}
(7.10) & \quad r_0 = C_0 b\alpha_n^{-1}, \quad \tau_0 = 0, \quad r_1 = C_1 b((\theta_n^2 + \theta_n)\alpha_n^{-1} + n^{-1/2}), \quad \tau_1 = 0, \quad r_2 = C_2 b\theta_n^2\alpha_n^{-1}, \quad \tau_2 = 0, \\
& \quad r_3 = 2C_3 b^2(\theta_n^2 + 1)^2\alpha_n^{-1}, \quad \tau_3 = 2, \quad r_4 = 2C_4 b^2(\theta_n^2 + 1)^2\alpha_n^{-2}, \quad \tau_4 = 2, \quad \rho = C_5 b\theta_n^2,
\end{align*}
where $C_1, C_2, C_3, C_4$ and $C_5$ are absolute constants. Recalling the definition of $\delta(t)$ in (4.15), and noting that $m_0 = \alpha_n^{1/3}/64$, we have
\begin{align*}
(7.11) & \quad \delta(m_0) \geq r_2(1 + m_0^3) \geq r_2m_0^3 \geq C\theta_n^2.
\end{align*}
Combining (7.10) and (7.11), we have that the right hand side of (4.14) is less than
\begin{align*}
(7.12) & \quad C\left(\frac{1}{\theta_n^2} + \theta_n^2 + 1\right)\delta(z) \leq 2C\left(\frac{1}{\theta_n^2} + \theta_n^2\right)(r_0 + r_1 + r_2 + r_3 + r_4^{1/2})(1 + z^3) \leq C\delta_n(1 + z^3),
\end{align*}
where $\delta_n = (\alpha_n^{-1} + n^{-1/2})(\theta_n^{-2} + \theta_n^6)$, and we used the fact that $x^2 + 1/x^2 > 1$ for $x > 0$ in the first inequality. On the other hand, by (7.10), we have
\begin{align*}
(7.13) & \quad r_0^{1/(\tau_0+1)} \leq Cb\alpha_n^{-1}, \\
& \quad r_1^{1/(\tau_1+2)} \leq Cb^{1/2}(\theta_n + 1)(1 + \theta_n^{-1/2})\alpha_n^{-1/2}, \\
& \quad r_2^{1/(\tau_2+3)} \leq Cb^{1/3}\theta_n^{2/3}\alpha_n^{-1/3}.
\end{align*}
By (7.13),
\begin{align*}
(7.14) & \quad r_0^{1/(\tau_0+1)} + r_1^{1/(\tau_1+2)} + r_2^{1/(\tau_2+3)} \leq Cb(1 + \theta_n)(1 + \theta_n^{-1/2})\alpha_n^{-1/3}.
\end{align*}
Applying Theorem 4.1, and by (7.12) and (7.14), we obtain the desired result. $\square$
Throughout this section, we follow the notation and settings in Section 6. We write \( h(t) = \mathbb{E}\{\Psi_{\beta,t}(W)\} \) and for any \( i \), let \( V_i = \sum_{i\in B_t} X_i \), \( T_i = \sum_{i\in B_t} |X_i| \) and \( W_i = W - V_i \). In what follows, we give two general lemmas, which will be used in the proof of Lemma 6.1. The following lemmas give us some technical inequalities.

**Lemma A.1.** Under (LD1) and (LD2), let \( \zeta_i = \zeta(X_{A_i}) \geq 0 \) be a function of \( X_{A_i} \). Then, for \( 0 \leq t \leq m_0 \),

(A.1) \[ \mathbb{E}\left\{ \zeta_i e^{3T_i} \Psi_{\beta,t}(W_i) \right\} \leq 81 h^{1/4} h(t) \mathbb{E}\left\{ \zeta_0 e^{3a_i T_i/8} \right\}, \]

(A.2) \[ \mathbb{E}\left\{ \zeta_i T_i^2 e^{3T_i} \Psi_{\beta,t}(W_i) \right\} \leq C h^{1/4} h(t) \tau^{-1} \mathbb{E}\left\{ \zeta_i e^{3a_i T_i/4} \right\} + C h^2 \tau a_n^{-4} h(t), \]

where \( C > 0 \) is an absolute constant and \( \tau > 0 \) is any positive number.

**Proof of Lemma A.1.** By Hölder’s inequality, for any random variables \( U_1, U_2, U_3 \geq 0 \), we have

(A.3) \[ \mathbb{E}\left\{ U_1 U_2 U_3 \right\} \leq \left( \mathbb{E}\left\{ U_1 \right\}^{1+\varepsilon} / \varepsilon \right)^{1/(1+\varepsilon)} \left( \mathbb{E}\left\{ U_2 \right\}^{1+\varepsilon} / \varepsilon \right)^{1/(1+\varepsilon)} \left( \mathbb{E}\left\{ U_3 \right\}^{1+\varepsilon} / \varepsilon \right)^{1/(1+\varepsilon)}, \]

where \( \varepsilon = 16 m_0 / a_n \). Then

(A.4) \[ 0 < \varepsilon \leq 1, \quad \varepsilon m_0^2 \leq 1/3 \quad \text{and} \quad (1 + \varepsilon) m_0 / \varepsilon \leq a_n / 8. \]

Applying (A.3) with \( U_1 = \zeta_i, U_2 = e^{T_i} \) and \( U_3 = \Psi_{\beta,t}(W_i) \), and by (A.4), we have

\[ \mathbb{E}\left\{ \zeta_i \Psi_{\beta,t}(W_i) e^{3T_i} \right\} \]

\[ \leq \left( \mathbb{E}\left\{ \zeta_i e^{3(1+\varepsilon) T_i / \varepsilon} \right\} \right)^{\varepsilon / (1+\varepsilon)} \left( \mathbb{E}\left\{ \zeta_i \Psi_{\beta,t}(W_i)^{1+\varepsilon} \right\} \right)^{1/(1+\varepsilon)} \]

(A.5) \[ = \left( \mathbb{E}\left\{ \zeta_i e^{3(1+\varepsilon) T_i / \varepsilon} \right\} \right)^{\varepsilon / (1+\varepsilon)} \left( \mathbb{E}\left\{ \zeta_i \right\} \right)^{1/(1+\varepsilon)} \left( \mathbb{E}\left\{ \Psi_{\beta,t}(W_i)^{1+\varepsilon} \right\} \right)^{1/(1+\varepsilon)} \]

\[ \leq \left( \mathbb{E}\left\{ \zeta_i e^{3a_i T_i / 8} \right\} \right) \left( \mathbb{E}\left\{ \Psi_{\beta,t}(W_i)^{1+\varepsilon} \right\} \right)^{1/(1+\varepsilon)}, \]

where the equality in the third line follows from the fact that \( W_i \) is independent of \( \zeta_i \) and the last inequality follows from (A.4) and \( \Psi_{\beta,t} \geq 1 \). Recalling the definition of \( \Psi_{\beta,t}(w) \) in (4.6), we have for any \( u \) and \( v \),

(A.6) \[ \Psi_{\beta,t}(u + v) \leq \Psi_{\beta,t}(u) e^{l[v]}. \]

By (A.6) and Hölder’s inequality,

(A.7) \[ \mathbb{E}\left\{ \Psi_{\beta,t}(W_i)^{1+\varepsilon} \right\} \leq \mathbb{E}\left\{ \Psi_{\beta,t}(W)^{1+\varepsilon} e^{(1+\varepsilon) T_i} \right\} \leq H_1 \times H_2, \]

where

\[ H_1 = \mathbb{E}\left\{ \Psi_{\beta,t}(W)^{(1+\varepsilon)^2} \right\}, \quad H_2 = \left( \mathbb{E}\left\{ e^{(1+\varepsilon)^2 T_i / \varepsilon} \right\} \right)^{\varepsilon / (1+\varepsilon)}. \]

Recalling the definition of \( \Psi_{\beta,t}(w) \), we have

\[ \Psi_{\beta,t}(w) \leq 2 e^{m_0^2} + 1 \leq 3 e^{m_0^2} \quad \text{for} \ 0 \leq \beta, t \leq m_0, \]

which further implies that

(A.8) \[ \Psi_{\beta,t}(w)^{(1+\varepsilon)^2} \leq (3 e^{m_0^2})^{2e+\varepsilon^2} \Psi_{\beta,t}(w). \]
By (A.4) and (A.8), we have
\[ H_1 \leq 27e^{m_0^2(2\varepsilon^2 + \varepsilon^2)}h(t) \leq 27e^{3m_0^2\varepsilon}h(t) \leq 81h(t). \]

For \( H_2 \), by (A.4) and by Hölder’s inequality again, we have
\[ H_2 \leq \mathbb{E}e^{a_nT_i/4} \leq b^{1/4}. \]

Combining (A.5), (A.7), (A.9) and (A.10) yields (A.1).

We now prove (A.2). Expanding the square term of the left hand side of (A.2), we have for all \( \tau > 0 \),
\begin{equation}
\mathbb{E}\{\zeta_i^2e^{3T_i}\Psi_{\beta,t}(W_i)\} = \sum_{j \in B_i} \sum_{k \in B_i} \mathbb{E}\{\zeta_i|X_jX_k|e^{3T_i}\Psi_{\beta,t}(W_i)\}
\leq \kappa \sum_{j \in B_i} \mathbb{E}\{\zeta_iX_j^2e^{3T_i}\Psi_{\beta,t}(W_i)\}
\leq \frac{\kappa^2}{2\tau} \mathbb{E}\{\zeta_i^2e^{6T_i}\Psi_{\beta,t}(W_i)\} + \frac{\kappa^2T}{2} \sum_{j \in B_i} \mathbb{E}\{X_j^4\Psi_{\beta,t}(W_i)\}.
\end{equation}

For the first term of the right hand side of (A.11), by (A.1) with replacing \( \zeta_i \) by \( \zeta_i^2 \) and \( 3T_i \) by \( 6T_i \), we obtain
\[ \mathbb{E}\{\zeta_i^2e^{6T_i}\Psi_{\beta,t}(W_i)\} \leq 81b^{1/4}h(t) \mathbb{E}\{\zeta_i^2e^{3a_nT_i/4}\}. \]

For the second term of the right hand side of (A.11), by (A.6), we have for any \( j \in B_i \), with \( W_{ij} = W - \sum_{k \in B_i \cup B_j} X_k \),
\[ \mathbb{E}\{X_j^4\Psi_{\beta,t}(W_i)\} \leq \mathbb{E}\{X_j^4e^{T_i}\Psi_{\beta,t}(W_{ij})\}. \]

Similar to (A.1), we obtain
\[ \mathbb{E}\{X_j^4e^{T_i}\Psi_{\beta,t}(W_{ij})\} \leq 81b^{1/2}h(t) \mathbb{E}\{X_j^4e^{3a_nT_j/8}\}. \]

Observing that \( |X_j| \leq T_j \), we have the expectation term of the right hand side of (A.13) can be bounded by
\[ \mathbb{E}\{X_j^4e^{T_i}\} \leq \alpha_n^{-4} \mathbb{E}\{(a_nT_j)^4e^{3a_nT_j/8}\} \leq C\alpha_n^{-4} \mathbb{E}e^{a_nT_j/2} \leq C\alpha_n^{-4}b^{1/2}h(t). \]

Substituting (A.12)–(A.14) into (A.11) yields (A.2).

**Lemma A.2.** Under (LD1) and (LD2), for each \( i \in \mathcal{J} \), let \( \xi_i = \xi(X_{A_i}) \) be a function of \( X_{A_i} \), satisfying that \( \mathbb{E}\xi_i = 0 \). Let \( S = \sum_{i \in \mathcal{J}} \xi_i \). For \( 0 \leq t, \beta \leq m_0 \) and any positive number \( \tau \), we have
\begin{equation}
\mathbb{E}\{S^2\Psi_{\beta,t}(W)\} \leq 81b^{1/4}h(t) \sum_{i \in \mathcal{J}} \mathbb{E}\{\xi_i^2e^{3a_nT_i/8}\}
\end{equation}
\[ + Cb\kappa^2h(t) \sum_{i \in \mathcal{J}} \sum_{j \in \mathcal{J}\setminus N_i} \mathbb{E}|\xi_j|(e^{T_j}\Psi_{\beta,t}(W)) + \tau a_n^{-4}. \]

**Proof of Lemma A.2.** Expanding the left hand side of (A.15) yields
\begin{equation}
\mathbb{E}\{S^2\Psi_{\beta,t}(W)\} := I_1 + I_2,
\end{equation}
where \[ I_1 = \sum_{i \in J} \sum_{j \in N_i} \mathbb{E}\{\xi_i \xi_j \Psi_{\beta,t}(W)\}, \quad I_2 = \sum_{i \in J} \sum_{j \in J \setminus N_i} \mathbb{E}\{\xi_i \xi_j \Psi_{\beta,t}(W)\}. \]

We now give the bounds of \( I_1 \) and \( I_2 \) separately. Observe that

\[
\{(i, j) : i \in J, j \in N_i\} = \{(i, j) : B_i \cap B_j \neq \emptyset\} = \{(i, j) : j \in J, i \in N_j\}.
\]

Recall that \( W_i = W - \sum_{j \in B_i} X_j \). For \( I_1 \), we have

\[
I_1 \leq \frac{1}{2} \sum_{i \in J} \sum_{j \in N_i} \mathbb{E}\{(\xi_i^2 + \xi_j^2) \Psi_{\beta,t}(W)\}
\]

\[
= \frac{1}{2} \sum_{i \in J} \sum_{j \in N_i} \mathbb{E}\{\xi_i^2 \Psi_{\beta,t}(W)\} + \frac{1}{2} \sum_{i \in J} \sum_{j \in N_i} \mathbb{E}\{\xi_j^2 \Psi_{\beta,t}(W)\}
\]

\[
= \frac{1}{2} \sum_{i \in J} \sum_{j \in N_i} \mathbb{E}\{\xi_i^2 \Psi_{\beta,t}(W)\} + \frac{1}{2} \sum_{j \in J} \sum_{i \in N_j} \mathbb{E}\{\xi_j^2 \Psi_{\beta,t}(W)\}
\]

\[
= \sum_{i \in J} \sum_{j \in N_i} \mathbb{E}\{\xi_i^2 \Psi_{\beta,t}(W)\} \leq \kappa \sum_{i \in J} \mathbb{E}\{\xi_i^2 \Psi_{\beta,t}(W_i) e^{T_i}\},
\]

where we used (A.17), (A.6) and \(|N_i| \leq \kappa\) in the last line. By Lemma A.1 with \( \zeta_i = \xi_i^2 \), we obtain

\[
\mathbb{E}\{\xi_i^2 \Psi_{\beta,t}(W_i) e^{T_i}\} \leq 81 b^{1/4} h(t) \mathbb{E}\{\xi_i^2 e^{3a T_i / 8}\}.
\]

Substituting (A.19) into (A.18), we have

\[
I_1 \leq 81 b^{1/4} k h(t) \sum_{i \in J} \mathbb{E}\{\xi_i^2 e^{3a T_i / 8}\}.
\]

For \( i, j \in J \), let \( V_{ij} = \sum_{k \in B_i \cup B_j} X_k \), \( W_{ij} = W - V_{ij}, T_{ij} = \sum_{k \in B_i \cup B_j} |X_k| \). It is easy to see that \( |T_{ij}| \leq T_i + T_j \). In order bound \( I_2 \), for any \( i \in J \) and \( j \notin N_i \), by Taylor’s expansion, we have

\[
\mathbb{E}\{\xi_i \xi_j \Psi_{\beta,t}(W)\}
\]

\[
= \mathbb{E}\{\xi_i \xi_j \Psi_{\beta,t}(W_i)\} + \mathbb{E}\{\xi_i \xi_j V_i \Psi_{\beta,t}'(W_i)\} + \int_0^1 \mathbb{E}\{\xi_i \xi_j V_i^2 \Psi_{\beta,t}''(W_i + uV_i)\}(1 - u)du
\]

\[
= \mathbb{E}\{\xi_i \xi_j \Psi_{\beta,t}(W_i)\} + \mathbb{E}\{\xi_i \xi_j V_i \Psi_{\beta,t}'(W_{ij})\}
\]

\[
+ \int_0^1 \mathbb{E}\{\xi_i \xi_j V_i (V_{ij} - V_i) \Psi_{\beta,t}'(W_{ij} + u(W_i - W_{ij}))\}(1 - u)du
\]

\[
+ \int_0^1 \mathbb{E}\{\xi_i \xi_j V_i^2 \Psi_{\beta,t}''(W_i + uV_i)\}du.
\]

If \( j \notin N_i \), then \( \xi_i \) is independent of \( (\xi_j, W_i) \) and \( \xi_j \) is independent of \( (\xi_i, V_i, W_{ij}) \). Recalling that \( \mathbb{E}\xi_i = \mathbb{E}\xi_j = 0 \), we have \( \mathbb{E}\{\xi_i \xi_j \Psi_{\beta,t}(W_i)\} = \mathbb{E}\{\xi_i \xi_j V_i \Psi_{\beta,t}'(W_{ij})\} = \mathbb{E}\{\xi_i \xi_j V_i^2 \Psi_{\beta,t}''(W_{ij})\} = 0 \). By (5.27) and by the monotonicity of \( \Psi_{\beta,t}(\cdot) \), we have for any \( 0 \leq u \leq 1 \),

\[
|\mathbb{E}\{\xi_i \xi_j V_i (V_{ij} - V_i) \Psi_{\beta,t}'(W_{ij} + u(W_i - W_{ij}))\}|
\]

\[
\leq \ell^2 \mathbb{E}\{(|\xi_i \xi_j V_i (V_{ij} - V_i)| \Psi_{\beta,t}(W_{ij} + u(W_i - W_{ij}))\}
\]
\begin{align*}
&\leq t^2 \sum_{l \in B_i} \sum_{m \in B_j \setminus B_i} \mathbb{E}\{|\xi_i \xi_j X_l X_m | (\Psi_{\beta,t}(W_i) + \Psi_{\beta,t}(W_{ij}))\} \\
&\leq 2t^2 \sum_{l \in B_i} \sum_{m \in B_j \setminus B_i} \mathbb{E}\{|\xi_i \xi_j X_l X_m | \Psi_{\beta,t}(W_{ij}) e^{t(T_i + T_j)}\},
\end{align*}

and similarly,
\[ |\mathbb{E}\{\xi_i \xi_j V_i^2 \Psi_{\beta,t}^\prime(W_i + uV_i)\}| \leq 2t^2 \sum_{l \in B_i} \sum_{m \in B_i} \mathbb{E}\{|\xi_i \xi_j X_l X_m | \Psi_{\beta,t}(W_{ij}) e^{t(T_i + T_j)}\}.\]

Observe that
\[(T_i + T_j)^2 e^{t(T_i + T_j)} \leq 4(T_i^2 e^{2tT_i} + T_j^2 e^{2tT_j}).\]

Hence, it follows that
\begin{align*}
I_2 &\leq 2t^2 \sum_{i \in J} \sum_{j \in J \setminus N_i} \sum_{l \in B_i} \sum_{m \in B_i \cup B_j} \mathbb{E}\{|\xi_i \xi_j X_l X_m | \Psi_{\beta,t}(W_{ij}) e^{t(T_i + T_j)}\} \\
&\leq 2t^2 \sum_{i \in J} \sum_{j \in J \setminus N_i} \mathbb{E}\{|\xi_i \xi_j (T_i + T_j)^2 e^{t(T_i + T_j)} \Psi_{\beta,t}(W_{ij})\} \\
&\leq 8t^2 \sum_{i \in J} \sum_{j \in J \setminus N_i} \mathbb{E}\{|\xi_i \xi_j |(T_i^2 e^{2tT_i} + T_j^2 e^{2tT_j}) \Psi_{\beta,t}(W_{ij})\} \\
&= 16t^2 \sum_{i \in J} \sum_{j \in J \setminus N_i} \mathbb{E}\{|\xi_i \xi_j |T_i^2 e^{2tT_i} \Psi_{\beta,t}(W_{ij})\},
\end{align*}

where we used (A.17) in the last line. If \(j \in J \setminus N_i\), then \(\xi_j\) is independent of \((\xi_i, T_i, W_{ij})\).

Therefore, by (A.2) in Lemma A.1, we obtain
\begin{align*}
I_2 &\leq 16t^2 \sum_{i \in J} \sum_{j \in J \setminus N_i} \mathbb{E}|\xi_j| \mathbb{E}\{|\xi_i |T_i^2 e^{2tT_i} \Psi_{\beta,t}(W_{ij})\} \\
&\leq 16t^2 \sum_{i \in J} \sum_{j \in J \setminus N_i} \mathbb{E}|\xi_j| \mathbb{E}\{|\xi_i |T_i^2 e^{3tT_i} \Psi_{\beta,t}(W_{ij})\} \\
&\leq Cb\kappa t^2 h(t) \sum_{i \in J} \sum_{j \in J \setminus N_i} \mathbb{E}|\xi_j| (\tau^{-1} \mathbb{E}\{\xi_i^2 e^{3\alpha T_i/4}\} + \tau^2 \alpha^{-4}).
\end{align*}

Combining (A.20) and (A.21), we complete the proof. \(\Box\)

Based on Lemmas A.1 and A.2, we are now ready to give the proof of Lemma 6.1.

**Proof of Lemma 6.1.** Recall that \(\hat{K}(u), K(u), \hat{K}_i(u)\) and \(K_i(u)\) are defined as in (6.1) and (6.2), and
\begin{equation}
\hat{K}_1 = \int_{-\infty}^{\infty} \hat{K}(t) dt = \sum_{i \in J} X_i Y_i, \quad \hat{K}_{2,t} = \sum_{i \in J} \int_{-\infty}^{\infty} |u| e^{t|u|} \hat{K}_1(u) du.
\end{equation}

Since \(\mathbb{E}W^2 = 1\), it follows that \(\mathbb{E}\hat{K}_1 = 1\). For (6.3), by Jensen’s and Hölder’s inequalities, we have
\begin{align}
\mathbb{E}\{|\mathbb{E}[\hat{K}_1 W] - \mathbb{E}\hat{K}_1 |\Psi_{\beta,t}(W)\| \} &\leq \mathbb{E}\{|\hat{K}_1 - \mathbb{E}\hat{K}_1 |\Psi_{\beta,t}(W)\| \} \\
&\leq h(t)^{1/2} (\mathbb{E}\{|\hat{K}_1 - \mathbb{E}\hat{K}_1 |^2 \Psi_{\beta,t}(W)\|)^{1/2},
\end{align}
Recall that $\hat{K}_1 - \hat{\mathbb{E}} K_1 = \sum_{i \in J} (X_i Y_i - \mathbb{E} X_i Y_i)$. Then, applying Lemma A.2 with $\xi_i = X_i Y_i - \mathbb{E} \{X_i Y_i\}$ and $\tau = b^{1/2}$, we have

\[
\mathbb{E} \{ |\hat{K}_1 - \mathbb{E} \hat{K}_1|^2 \Psi_{\beta, t}(W) \} \leq G_1 + G_2,
\]

where

\[
G_1 = 81 b^{1/4} \kappa h(t) \sum_{i \in J} \mathbb{E} \{ e^{\xi_i^2 T_i / 8} \},
\]

\[
G_2 = C b^{1/2} T_i^2 h(t) \sum_{i \in J} \sum_{j \in J \setminus N_i} \mathbb{E} |\xi_j| \left( b^{-1/2} \mathbb{E} \{ e^{\xi_i T_i / 4} \} + b^{1/2} a_n^{-1} \right).
\]

Recalling that $T_i = \sum_{j \in B_i} |X_j|$, we have $|\xi_i| \leq T_i^2 + \mathbb{E} T_i^2$, and thus, for $0 < s \leq 3/4$,

\[
\mathbb{E} \{ (X_i Y_i - \mathbb{E} X_i Y_i)^2 e^{\xi_i T_i} \} \leq a_n^{-4} \mathbb{E} \{ \mathbb{E} e^{a_n T_i} \}^{s+1/4} \leq C a_n^{-4} e^{a_n T_i / 2} \leq C b^{1/2} a_n^{-2}.
\]

Substituting (A.25) and (A.26) into (A.24) gives

\[
\mathbb{E} \{ |\hat{K}_1 - 1|^2 \Psi_{\beta, t}(W) \} \leq C (b \kappa n a_n^{-4} + b^{1/2} \kappa^2 n^2 a_n^{-6}) h(t) \left( 1 + t^2 \right),
\]

which proves (6.3) together with (A.23).

We next prove (6.4). Recalling that $\hat{K}_i(u)$ is defined as in (6.1), by (A.6) and applying Lemma A.1 with $\xi_i = |\hat{K}_i(u)|$, we have

\[
| \mathbb{E} \{ \hat{K}_i(u) \Psi_{\beta, t}(W) \} | \leq \mathbb{E} \{ \hat{K}_i(u) |e^{T_i \Psi_{\beta, t}(W)}| \} \leq 81 b^{1/4} h(t) \mathbb{E} \{ |\hat{K}_i(u)| e^{3a_n T_i / 8} \}.
\]

Thus, by (A.22) and recalling that $t \leq a_n / 16$ and $|X_i| \leq T_i, |Y_i| \leq T_i$, we have

\[
\mathbb{E} \{ \hat{K}_2, t \Psi_{\beta, t}(W) \} \leq 81 b^{1/4} h(t) \sum_{i \in J} \mathbb{E} \{ |X_i^2 Y_i| e^{a_n T_i / 2} \} \leq 81 b^{1/4} h(t) \sum_{i \in J} \mathbb{E} \{ T_i^3 e^{a_n T_i / 2} \} \leq C b n a_n^{-3} h(t).
\]

We now move to prove (6.5) and (6.6) together. By definition,

\[
\mathbb{E} \{ \hat{K}_3, t \Psi_{\beta, t}(W) \} \leq \int_{|u| \leq 1} e^{2u |u|} \mathbb{E} \{ (\hat{K}(u) - \hat{K}(u))^2 \Psi_{\beta, t}(W) \} \, du
\]

and

\[
\mathbb{E} \{ \hat{K}_4, t \Psi_{\beta, t}(W) \} \leq \int_{|u| \leq 1} |u| e^{2u |u|} \mathbb{E} \{ (\hat{K}(u) - \hat{K}(u))^2 \Psi_{\beta, t}(W) \} \, du.
\]

For fixed $u$, applying Lemma A.2 with $\xi_i = \hat{K}_i(u) - \hat{K}_i(u)$ and $\tau = b^{1/2} a_n$, we have

\[
\mathbb{E} \{ (\hat{K}(u) - \hat{K}(u))^2 \Psi_{\beta, t}(W) \} = H_1(u) + H_2(u),
\]
where
\[
H_1(u) = 81b^{1/4}h(t) \sum_{i \in \mathcal{J}} \mathbb{E}\{(\hat{K}_i(u) - K_i(u))^2 e^{3a_n T_i/8}\},
\]
\[
H_2(u) = Cb^{3/2} t^2 h(t) \sum_{i \in \mathcal{J}} \sum_{j \in \mathcal{J} \setminus N_i} \mathbb{E}|\hat{K}_j(u) - K_j(u)| \left( b^{-1/2}a_n^{-1} \mathbb{E}\{(\hat{K}_i(u) - K_i(u))^2 e^{3a_n T_i/4}\} + b^{1/2}a_n^{-3} \right).
\]
For \(H_1(u)\), recalling that \(|X_i| \leq T_i, |Y_i| \leq T_i\) and \(t \leq a_n/16\),
\[
\int_{-\infty}^{\infty} e^{2|u|} \mathbb{E}\{(\hat{K}_i(u))^2 e^{3a_n T_i/8}\} du \leq \mathbb{E}\{|X_i Y_i| e^{a_n T_i/2}\}
\]
(A.30)
\[
\leq Ca_n^{-3} \mathbb{E}\{(a_n T_i)^3 e^{a_n T_i/2}\}
\]
\[
\leq Cb^{3/4}a_n^{-3},
\]
and similarly,
\[
\int_{-\infty}^{\infty} e^{2|u|} \mathbb{E}\{K_i(u)^2 e^{3a_n T_i/8}\} du \leq Cb^{3/4}a_n^{-3}.
\]
(A.31)
For \(H_2(u)\), note that \(|\hat{K}_i(u)|^2 \leq |X_i|^2\), and we have
\[
\mathbb{E}\{|\hat{K}_i(u)|^2 e^{3a_n/4}\} \leq \mathbb{E}\{|X_i|^2 e^{3a_n/4}\} \leq \mathbb{E}\{T_i^2 e^{3a_n/4}\} \leq Cb a_n^{-2}
\]
and
\[
\mathbb{E}\{|K_i(u)|^2 e^{3a_n/4}\} \leq Cb a_n^{-2}.
\]
Then,
\[
H_2(u) \leq Cb^{3/4} \kappa^2 t^2 a_n^{-3} h(t) \sum_{i \in \mathcal{J}} \sum_{j \in \mathcal{J} \setminus N_i} \mathbb{E}|\hat{K}_j(u) - K_j(u)|.
\]
(A.32)
Similar to (A.30) and (A.31),
\[
\int_{-\infty}^{\infty} e^{2|u|} \mathbb{E}|\hat{K}_j(u) - K_j(u)| du \leq \mathbb{E}\{|X_i Y_i| + \mathbb{E}|X_i Y_i|\} e^{2T_i} \leq Cb^{1/2}a_n^{-2}.
\]
(A.33)
Substituting (A.29)–(A.33) into (A.27) gives (6.5). The inequality (6.6) can be shown similarly.

It now remains to prove (6.7). By definition,
\[
\sup_{0 \leq t \leq m_0} M_t = \int_{|u| \leq 1} e^{m_0|u|} |K(u)| du
\]
\[
\leq \sum_{i \in \mathcal{J}} \mathbb{E}\left\{ \int_{|u| \leq 1} |\hat{K}_i(u)| e^{m_0|u|} du \right\}
\]
\[
\leq 2 \sum_{i \in \mathcal{J}} \mathbb{E}\left\{ |X_i Y_i| e^{m_0|Y_i|} \right\} \leq 2a_n^{-2} \sum_{i \in \mathcal{J}} \mathbb{E}\left\{ a_n^2 T_i^2 e^{m_0 T_i} \right\}
\]
\[
\leq Cna_n^{-2}b.
\]
This completes the proof.

\[\square\]

**APPENDIX B: PROOF OF LEMMA 7.1**

This section includes three subsections. In Appendix B.1, we prove Lemma 7.1. Before that, we give some preliminary lemmas, whose proofs are given in Appendices B.2 and B.3.
Lemma 7.1 can be shown similarly. For any (B.3)

For the first term in the RHS of (B.3). Since (B.4)

Let \( \sigma \) be a uniform permutation on \([n-m]\) which is independent of \( X \) and let \( S = \sum_{i \in [n-m]} X_i, \sigma(i) \). For \( k = 1, 2 \), and for any \( i, j \in [n-m] \), let \( \zeta_{ij} := \zeta(X_{i,j}) \) be a positive function of \( X_{i,j} \). We have

\[
\mathbb{E}\{\zeta_{i,\sigma(i)} \Psi_{\beta,t}(S)\} \leq 4b^{1/16} \mathbb{E}\Psi_{\beta,t}(S) \max_{v \in [n-m]} \mathbb{E}\{\zeta_{i,v} e^{tT_{i,v}}\},
\]

(B.1)

\[
\mathbb{E}\{\zeta_{\sigma^{-1}(j),j} \Psi_{\beta,t}(S)\} \leq 4b^{1/16} \mathbb{E}\Psi_{\beta,t}(S) \max_{v \in [n-m]} \mathbb{E}\{\zeta_{i,v} e^{tT_{i,v}}\},
\]

(B.2)

\[
\mathbb{E}\{\zeta_{\sigma^{-1}(j),\sigma(i)} \Psi_{\beta,t}(S)\} \leq 4b^{1/8} \mathbb{E}\Psi_{\beta,t}(S) \max_{u,v \in [n-m]} \mathbb{E}\{\zeta_{u,v} e^{t(T_{i,v} + T_{u,j})}\}.
\]

(B.3)

**Proof.** We only prove (B.1), because (B.2) and (B.3) can be shown similarly. For any \( i \in [n-m] \), let \( S^{(i)} = \sum_{i' \in A(i)} X_{i', \sigma(i')}. \) By definition, we have

\[
\mathbb{E}\{\zeta_{i,\sigma(i)} \Psi_{\beta,t}(S)\} = \frac{1}{(n-m)_k} \sum_{j \in [n-m]} \mathbb{E}\{\zeta_{i,j} \Psi_{\beta,t}(S)\} | \sigma(i) = j\}
\]

(B.4)

Since \( X_{i,j} \) is conditionally independent of \( S^{(i)} \), given the event that \( \sigma(i) = j \), and \( X_{i,j} \) is independent of \( \sigma \), the last conditional expectation in (B.4) can be rewritten as

\[
\mathbb{E}\{\zeta_{i,j} e^{tT_{i,j}} \Psi_{\beta,t}(S^{(i)}) | \sigma(i) = j\} = \mathbb{E}\{\zeta_{i,j} e^{tT_{i,j}}\} \mathbb{E}\{\Psi_{\beta,t}(S^{(i)}) | \sigma(i) = j\}.
\]

For the second term on the RHS of (B.5), note that \( \Psi_{\beta,t}(w + x) \leq e^{t|x|} \Psi_{\beta,t}(w) \) and \(|S - S^{(i)}| \leq T_{i,j}\) given \( \sigma(i) = j \). With \( \varepsilon = \alpha_n^{-2/3} \), by Hölder’s inequality, and noting that \( T_{i,j} \) is independent of \( \sigma \), we have

\[
\mathbb{E}\{\Psi_{\beta,t}(S^{(i)}) | \sigma(i) = j\} \leq \mathbb{E}\{e^{tT_{i,j}} \Psi_{\beta,t}(S) | \sigma(i) = j\} \leq (\mathbb{E}\{e^{(1+\varepsilon)tT_{i,j}}\})^{\varepsilon/(1+\varepsilon)} (\mathbb{E}\{\Psi_{\beta,t}^{1+\varepsilon}(S)\})^{1/(1+\varepsilon)}.
\]

(B.6)

Noting that \( 0 < t \leq \beta \leq \alpha_n^{1/3}/64 \), we obtain

\[
0 < \varepsilon < 1, \quad (1+\varepsilon)t/\varepsilon \leq 2t/\varepsilon \leq \alpha_n/32, \quad \varepsilon\beta t \leq 0.001.
\]

For the first term in the RHS of (B.6), because \( k = 1, 2 \) and \( i, j \in [n-m] \), by (3.4), we obtain

\[
\mathbb{E}\{e^{(1+\varepsilon)tT_{i,j}}\} \leq \mathbb{E}e^{\alpha_n T_{i,j}/32} \leq \max_{i,j \in [n-m]} \mathbb{E}e^{\alpha_n |X_{i,j}|} \leq b^{1/16}.
\]

(B.8)

Noting that \( \Psi_{\beta,t}(w) \leq 2e^{t\beta} + 1 \leq 3e^{t\beta} \), we have

\[
\Psi_{\beta,t}(w)^{1+\varepsilon} \leq (3e^{t\beta})^{\varepsilon} \Psi_{\beta,t}(w) \leq 4 \Psi_{\beta,t}(w).
\]
Thus,
\begin{align} 
\mathbb{E}\left\{ \Psi_{\beta,t}(S)^{1+\varepsilon} \mid \sigma(i) = j \right\} & \leq 4 \mathbb{E}\left\{ \Psi_{\beta,t}(S) \right\} \mathbb{E}\left\{ \Psi_{\beta,t}(S) \mid \sigma(i) = j \right\}.
\end{align}
Combining (B.4)–(B.6), (B.8) and (B.9), we have
\begin{align*}
\mathbb{E}\left\{ \zeta_{i,j} e^{T_{i,j}} \Psi_{\beta,t}(S(i)) \mid \sigma(i) = j \right\} & \leq 4b^{1/16} \max_{v \in [n-m]_k} \mathbb{E}\left\{ \zeta_{i,v} e^{T_{i,v}} \right\} \mathbb{E}\left\{ \Psi_{\beta,t}(S) \mid \sigma(i) = j \right\}.
\end{align*}
Now, taking average over \( j \in [n - m]_k \) yields (B.1). Using a similar argument, we obtain (B.2) and (B.3).

**Lemma B.2.** For \( i, j \), let \( \xi_{i,j} \) := \( \xi(X_{i,j}) \) be a function of \( X_{i,j} \) such that \( \mathbb{E}\xi_{i,\pi(i)} = 0 \). For any \( i \in [n]_2 \) and \( j' \in [n]_2 \), where \( [n]_2 := \{(k, l) \in [n]_2 : k, l \in [n] \setminus A(i)\} \), we have
\begin{align*}
& \mathbb{E}\left\{ \xi_{i,\pi(i)} \xi_{i',\pi(i')} \Psi_{\beta,t}(W) \right\} \\
& \leq Cb(1 + t^2) h(t) n^{-4} \sum_{j' \in [n]_2} \left( \mathbb{E}\left\{ |\xi_{i,j} T^2_{i,j} e^{2t |T_{i,j}|} | \right\} \mathbb{E}\left\{ |\xi_{i',j'}| \right\} + \mathbb{E}\left\{ |\xi_{i',j'} T^2_{i',j'} e^{2t |T_{i',j'}|} | \right\} \mathbb{E}\left\{ |\xi_{i,j}| \right\} \right) \\
& \quad + Cb(1 + t^2) h(t) n^{-4} \sum_{j' \in [n]_2} \left( \alpha_n^{-2} + n^{-1} + 1(E_{j,j'}) \right) \mathbb{E}\left\{ |\xi_{i,j}| \right\} \mathbb{E}\left\{ |\xi_{i',j'}| \right\},
\end{align*}
where \( E_{j,j'} = \{ A(j) \cap A(j') \neq \emptyset \} \).

Recalling that \( D_{i,j} \) is defined in (7.3), we give the following lemmas.

**Lemma B.3.** For \( i, j \in [n]_2 \), let \( g_{i,j}(u) = D_{i,j} \left( (1 - D_{i,j}) \mathbb{1}(u \leq 0) - \mathbb{1}(0 < u \leq -D_{i,j}) \right) \) and \( \bar{g}_{i,j}(u) = g_{i,j}(u) - \mathbb{E}g_{i,\pi(i)}(u) \). We have
\begin{align} 
\left| \int_{|u| \leq 1} |u|^v e^{2t |u|} \mathbb{E}\left\{ \left( \sum_{i \in [n]_2} \bar{g}_{i,\pi(i)}(u) \right)^2 \Psi_{\beta,t}(W) \right\} du \right| 
\leq Cb^2 \left( n^2 \alpha_n^{5-v} + n \alpha_n^{-3-v} \right)(1 + t^2) h(t), \quad \text{for } v = 0, 1.
\end{align}

**Lemma B.4.** For \( i, j \in [n]_2 \), let \( H_{i,j}(u) = D_{i,j}^2 \) and \( \bar{H}_{i,j}(u) = H_{i,j}(u) - \mathbb{E}H_{i,\pi(i)}(u) \). We have
\begin{align} 
\mathbb{E}\left\{ \left( \sum_{i \in [n]_2} \bar{H}_{i,\pi(i)} \right)^2 \Psi_{\beta,t}(W) \right\} & \leq Cb^2 \left( n^4 \alpha_n^{-6} + n^3 \alpha_n^{-4} \right)(1 + t^2) h(t).
\end{align}

The proofs of Lemmas B.2 and B.3 are given in Appendix B. The proof of Lemma B.4 is similar to that of Lemma B.3 and thus we omit the details.

We are now ready to give the proof of Lemma 7.1.

**Proof of Lemma 7.1.** We prove (7.4)–(7.9) one by one.

(i). Proof of (7.4). Let \( \bar{X}_{i,j} = X_{i,j} - a_{i,j} \). By Hölder’s inequality, we have
\begin{align} 
\mathbb{E}\left\{ |R| \Psi_{\beta,t}(W) \right\} & \leq h^{1/2}(t) \left( \mathbb{E}\left\{ R^2 \Psi_{\beta,t}(W) \right\} \right)^{1/2}.
\end{align}
By (3.1) and (7.1), we have
\[
\mathbb{E}\{R^2\Psi_{\beta,t}(W)\} = \frac{1}{n^2} \mathbb{E}\left\{ \left| \sum_{i=1}^{n} \sum_{j=1}^{n} X_{i,j} \right|^2 \Psi_{\beta,t}(W) \right\}
\]
\begin{equation}
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i' \geq 1} \sum_{j' = 1}^{n} \mathbb{E}\{X_{i,j} X_{i',j'} \Psi_{\beta,t}(W)\}.
\end{equation}

(B.13)

Now, for fixed \(i, j, i', j' \in [n]\),
\[
\mathbb{E}\{X_{i,j} X_{i',j'} \Psi_{\beta,t}(W)\} = \mathbb{E}\{X_{i,j} X_{i',j'} \Psi_{\beta,t}(W), \pi(i) = j, \pi(i') = j' \} + \mathbb{E}\{X_{i,j} X_{i',j'} \Psi_{\beta,t}(W), \pi(i) = j, \pi(i') \neq j' \} + \mathbb{E}\{X_{i,j} X_{i',j'} \Psi_{\beta,t}(W), \pi(i) \neq j, \pi(i') = j' \} + \mathbb{E}\{X_{i,j} X_{i',j'} \Psi_{\beta,t}(W), \pi(i) \neq j, \pi(i') \neq j' \}.
\]

For the first term, with \(W^{(i,i')} = W - \sum_{k \in \{i,i'\}} X_{k,\pi(k)}\), let consider the corresponding conditional expectation,
\[
\mathbb{E}\{X_{i,j} X_{i',j'} \Psi_{\beta,t}(W) | \pi(i) = j, \pi(i') = j' \} \leq \frac{1}{2} \mathbb{E}\{(X_{i,j}^2 + X_{i',j'}^2) \Psi_{\beta,t}(W) | \pi(i) = j, \pi(i') = j' \}
\]
\[
\leq \frac{1}{2} \mathbb{E}\{(X_{i,j}^2 + X_{i',j'}^2)e^{t(|X_{i,j}| + |X_{i',j'}|)} \Psi_{\beta,t}(W^{(i,i')}) | \pi(i) = j, \pi(i') = j' \}
\]
\[
= \frac{1}{2} \mathbb{E}\{(X_{i,j}^2 + X_{i',j'}^2)e^{t(|X_{i,j}| + |X_{i',j'}|)} \} \mathbb{E}\{\Psi_{\beta,t}(W^{(i,i')}) | \pi(i) = j, \pi(i') = j' \},
\]
where in the last line we used the fact that \((X_{i,j}, X_{i',j'})\) and \(W^{(i,i')}\) are conditionally independent given \(\pi(i) = j\) and \(\pi(i') = j'\). Recalling that that \(t \leq \alpha_n^{1/3}/4 \leq \alpha_n/4\) and by (3.4), we have
\begin{equation}
\mathbb{E}\{X_{i,j} e^{t|X_{i,j}|}\} \leq C \alpha_n^{-2} \mathbb{E}\{|\alpha_n X_{i,j}|^2 e^{\alpha_n |X_{i,j}|/4}\}
\leq C \alpha_n^{-2} \mathbb{E}\{e^{\alpha_n |X_{i,j}|/2}\} \leq Cb^{1/2} \alpha_n^{-2}.
\end{equation}

(B.14)

Choosing \(\varepsilon = \alpha_n^{-2/3}\) and according to (3.4) and (B.7), we have
\[
\mathbb{E}\{e^{(1+\varepsilon)t(|X_{i,j}| + |X_{i',j'}|)/\varepsilon}\} \leq \mathbb{E}\{e^{\alpha_n (|X_{i,j}| + |X_{i',j'}|)/2}\} \leq C b,
\]
and
\[
\mathbb{E}\{\Psi_{\beta,t}(W^{(i,i')}) | \pi(i) = j, \pi(i') = j' \}
\leq \mathbb{E}\{e^{t|W - W^{(i,j)}|} \Psi_{\beta,t}(W) | \pi(i) = j, \pi(i') = j' \}
\]
\[
\leq \mathbb{E}\{e^{(1+\varepsilon)(|X_{i,j}| + |X_{i',j'}|)/\varepsilon}\} \frac{1}{1+\varepsilon} \mathbb{E}\{\Psi_{\beta,t}(W) | \pi(i) = j, \pi(i') = j' \}
\]
\[
\leq C b^{1/2} \mathbb{E}\{\Psi_{\beta,t}(W) | \pi(i) = j, \pi(i') = j' \}.
\]

Therefore, we have
\begin{equation}
\mathbb{E}\{X_{i,j} X_{i',j'} \Psi_{\beta,t}(W) | \pi(i) = j, \pi(i') = j' \}
\leq C b \alpha_n^{-2} \mathbb{E}\{\Psi_{\beta,t}(W) | \pi(i) = j, \pi(i') = j' \}.
\end{equation}

(B.15)
Moreover, noting that \( X_{i',j'} \) is independent of \( (X_{i,j}, W) \) given \( \pi(i) = j \) and \( \pi(i') \neq j' \), we have

\[
\mathbb{E}\{X_{i,j}X_{i',j'}\Psi_{\beta,t}(W)|\pi(i) = j, \pi(i') \neq j'\} = \mathbb{E}\{X_{i',j'}\} \mathbb{E}\{X_{i,j}\Psi_{\beta,t}(W)|\pi(i) = j, \pi(i') \neq j'\} = 0.
\]

Similarly,

\[
\mathbb{E}\{X_{i,j}X_{i',j'}\Psi_{\beta,t}(W)|\pi(i) \neq j, \pi(i') = j'\} = 0.
\]

Furthermore, if \( \pi(i) \neq j \) and \( \pi(i') \neq j' \), we have

\[
\mathbb{E}\{X_{i,j}X_{i',j'}\Psi_{\beta,t}(W)|\pi(i) \neq j, \pi(i') \neq j'\}
= \mathbb{E}\{X_{i,j}X_{i',j'}\} \mathbb{E}\{\Psi_{\beta,t}(W)|\pi(i) \neq j, \pi(i') \neq j'\}
= \begin{cases} 
\mathbb{E}\{\hat{X}_{i,j}^2\} \mathbb{E}\{\Psi_{\beta,t}(W)|\pi(i) \neq j, \pi(i') \neq j'\} & \text{if } i = i' \text{ and } j = j', \\
0 & \text{otherwise.}
\end{cases}
\]

By \((\text{B.14})\), we obtain

\[
\mathbb{E}\{|\hat{X}_{i,j}X_{i',j'}\Psi_{\beta,t}(W)|\pi(i) \neq j, \pi(i') \neq j'\|
\leq Cb\alpha_n^{-2} \mathbb{E}\{\Psi_{\beta,t}(W)|\pi(i) \neq j, \pi(i') \neq j'\} 1((i,j) = (i',j')).
\]

Substituting \((\text{B.15})\)–\((\text{B.18})\) to \((\text{B.13})\) and using \((\text{B.12})\) yields \((\text{7.4})\).

\(\text{(ii). Proof of (7.5).}\) Recalling the definition of \(\hat{K}_1\) in \((\text{4.1})\) and \(\hat{K}(u)\) in \((\text{7.3})\), we have

\[
\hat{K}_1 = \frac{1}{4n^2} \sum_{i \in [n]^2} D_{i,\pi(i)}^2.
\]

By \((\text{3.2})\) and \((\text{B.19})\), one can verify (see, e.g., Eq. \((\text{3.10})\) in Chen and Fang (2015)) that 
\(\mathbb{E}K_1 - 1\) \(\leq 2/\sqrt{n}\). Thus,

\[
\mathbb{E}\{|\hat{K}_1 - 1|\Psi_{\beta,t}(W)\} \leq \mathbb{E}\{|\hat{K}_1 - \mathbb{E}\hat{K}_1|\Psi_{\beta,t}(W)\} + \frac{2}{\sqrt{n}} \mathbb{E}\{\Psi_{\beta,t}(W)\}.
\]

For the first term of the R.H.S. of \((\text{B.20})\), recalling that \(\bar{h}(t) = \mathbb{E}\Psi_{\beta,t}(W)\), by Hölder’s inequality, we have

\[
\mathbb{E}\{|\hat{K}_1 - \mathbb{E}\hat{K}_1|\Psi_{\beta,t}(W)\}
\leq \frac{\bar{h}^{1/2}(t)}{4n} \left( \mathbb{E}\left\{ \left( \sum_{i \in [n]^2} (D_{i,\pi(i)}^2 - \mathbb{E}D_{i,\pi(i)}^2) \right)^2 \Psi_{\beta,t}(W) \right\} \right)^{1/2}.
\]

Applying Lemma \(\text{B.4}\) to the expectation in the R.H.S of \((\text{B.21})\), we obtain

\[
\mathbb{E}\{|\hat{K}_1 - \mathbb{E}\hat{K}_1|\Psi_{\beta,t}(W)\} \leq Cb(n\alpha_n^{-3} + n^{1/2}\alpha_n^{-2})\bar{h}(t).
\]

Combining \((\text{B.20})\) and \((\text{B.22})\) yields \((\text{7.5})\).

\(\text{(iii). Proof of (7.6).}\) Recalling \(\hat{K}(u)\) as in \((\text{7.3})\), we have

\[
\mathbb{E}\{\hat{K}_{2,t}\Psi_{\beta,t}(W)\} = \int_{-\infty}^{\infty} |u|e^{t|u|} \mathbb{E}\{\hat{K}(u)\Psi_{\beta,t}(W)\} du
\leq \frac{1}{4n^2} \sum_{i \in [n]^2} \mathbb{E}\{|D_{i,\pi(i)}|\beta |\mathbb{E}D_{i,\pi(i)}|\Psi_{\beta,t}(W)\}.
\]
Then, applying Lemma B.1 with \( k = 2, m = 0, \sigma = \pi, S = W, \) and \( \zeta_{i,j}(u) = \left| D_{i,j} \right|^3 e^{t|D_{i,j}|}\), we have for any \( i \in [n]_2, \)

\[
\mathbb{E}\{|D_{i,\pi(i)}|^3 e^{t|D_{i,\pi(i)}|}\Psi_{\beta,t}(W)\} \\
\leq C b^{1/8} h(t) \max_{j \in [n]_2} \mathbb{E}\{|D_{i,j}|^3 e^{t|D_{i,j}|+tT_{i,j}}\} \\
\leq C b^{1/8} h(t) \alpha_n^{-3} \max_{j_1,j_2 \in [n]} \sum_{i \in \{1, \ldots, n\}} \sum_{j \in \{j_1, j_2\}} \mathbb{E}\{\left| \alpha_n X_{i,j} \right|^3 e^{6t|X_{i,j}|}\} \\
\leq C b \alpha_n^{-3} h(t),
\]

where the last inequality follows from (3.4). Therefore, by (B.23) and (B.24), we have

\[
\mathbb{E}\{\hat{K}_{2,t} \Psi_{\beta,t}(W)\} \leq C b n \alpha_n^{-3}.
\]

(iv). Proofs of (7.7) and (7.8). Recall the definitions in (4.3), (4.4) and (7.3), and we have

\[
\mathbb{E}\{\hat{K}_{3,t} \Psi_{\beta,t}(W)\} = \int_{|u| \leq 1} e^{2t|u|} \mathbb{E}\{(\hat{K}(u) - K(u))^2 \Psi_{\beta,t}(W)\} du.
\]

By Lemma B.3 with \( v = 0 \), we complete the proof of (7.7). By Lemma B.3 with \( v = 1 \), the inequality (7.8) follows similarly.

(v). Proofs of (7.9). Recalling the definition of \( \hat{K}(u) \) in (7.3), by Fubini’s theorem we have

\[
\sup_{0 \leq t \leq \alpha_n^{1/3}/64} M_t \leq \frac{1}{n} \sum_{i \in [n]_2} \mathbb{E}\{e^{\alpha_n |D_{i,\pi(i)}|/64} |D_{i,\pi(i)}|^2\} \leq C b n \alpha_n^{-2}
\]

where the last inequality follows from the similar argument in (B.24).

**B.2. Some useful lemmas.** In order to prove Lemmas B.2 and B.3, we need to show some preliminary lemmas. Recall that \( S_n \) is the collection of all permutations over \( [n] \).

**Lemma B.5.** For \( n \geq 4, m = 0, 1, 2, \) let \( S \) and \( \sigma \) be defined as in Lemma B.1. For any \( i, j \in [n-m], \) we have

\[
|\mathbb{E}\{X_{i,\sigma(i)} \Psi'_{\beta,t}(S)\}| \leq C(n^{-1/2} \alpha_n^{-1} + \alpha_n^{-2}) b^{1/2} (1 + t^2) \mathbb{E}\Psi_{\beta,t}(S),
\]

\[
|\mathbb{E}\{X_{\sigma^{-1}(j),\sigma(i)} \Psi'_{\beta,t}(S)\}| \leq C(n^{-1/2} \alpha_n^{-1} + \alpha_n^{-2}) b^{1/2} (1 + t^2) \mathbb{E}\Psi_{\beta,t}(S),
\]

\[
|\mathbb{E}\{X_{\sigma^{-1}(j),\sigma(i)} \Psi'_{\beta,t}(S)\}| \leq C(n^{-1/2} \alpha_n^{-1} + \alpha_n^{-2}) b^{1/2} (1 + t^2) \mathbb{E}\Psi_{\beta,t}(S).
\]

**Proof of Lemma B.5.** We only prove (B.27), because (B.28) and (B.29) can be shown similarly. Note that

\[
\mathbb{E}\{X_{i,\sigma(i)} \Psi'_{\beta,t}(S)\} = \frac{1}{n-m} \sum_{j=1}^{n-m} \mathbb{E}\{X_{i,j} \Psi'_{\beta,t}(S)\} |\sigma(i) = j\}
\]

\[
= \frac{1}{n-m} \sum_{j=1}^{n-m} \mathbb{E}\{X_{i,j} \Psi'_{\beta,t}(S)\} |\sigma(i) = j\} \\
+ \frac{1}{n-m} \sum_{j=1}^{n-m} \mathbb{E}\{X_{i,j} (\Psi'_{\beta,t}(S) - \Psi'_{\beta,t}(S^{(i)}))\} |\sigma(i) = j\}
\]

\[
:= I_1 + I_2.
\]
Denote by $\tau_{i,j}$ the transposition of $i$ and $j$, and define
\[
\sigma_{i,j} = \begin{cases} 
\sigma & \text{if } \sigma(i) = j, \\
\sigma \circ \tau_{i,\sigma^{-1}(j)} & \text{if } \sigma(i) \neq j.
\end{cases}
\]
Then $\sigma_{i,j}(i) = j$. For any given distinct $k_1, \ldots, k_{n-m-1} \in [n-m] \setminus \{i\}$ and $l_1, \ldots, l_{n-m-1} \in [n-m] \setminus \{j\}$, denote by $A$ the event that $\{\sigma_{i,j}(k_u) = l_u, u = 1, \ldots, n-m-1\}$. Then,
\[
\mathbb{P}(A) = \mathbb{P}(A, \sigma(i) = j) + \sum_{u=1}^{n-m-1} \mathbb{P}(A, \sigma(i) = l_u, \sigma(k_u) = j)
\]
\[
= \frac{1}{(n-m)!} + (n-m-1) \frac{1}{(n-m)!} = \frac{1}{(n-m-1)!}.
\]
On the other hand, we have
\[
\mathbb{P}(\sigma(k_u) = l_u, u = 1, \ldots, n-m-1|\sigma(i) = j) = \frac{1}{(n-m-1)!}.
\]
This proves that $\mathcal{L}(\sigma_{i,j}) = \mathcal{L}(\sigma|\sigma(i) = j)$. Moreover, with $S^{(i)} = S - X_{i,\sigma(i)}$, $S_{i,j} = \sum_{i' = 1}^{n} X_{i',\sigma_{i,j}(i')}$, and let $S_{i,j}^{(i)} = S_{i,j} - X_{i,j}$, it follows that
\[
\mathcal{L}(S_{i,j}^{(i)}) = \mathcal{L}(S^{(i)}|\sigma(i) = j).
\]
Noting that $X_{i,j}$ is independent of $\Psi_{\beta,t}^{(S^{(i)})}$ conditional on $\sigma(i) = j$, and recalling that $\mathbb{E}X_{i,j} = a_{i,j}$, we have
\[
\mathbb{E}\{X_{i,j} \Psi_{\beta,t}^{(S^{(i)})}|\sigma(i) = j\} = a_{i,j} \mathbb{E}\{\Psi_{\beta,t}^{(S^{(i)})}|\sigma(i) = j\} = a_{i,j} \mathbb{E}\{\Psi_{\beta,t}^{(S_{i,j}^{(i)})}\}.
\]
Therefore, recalling that $\sum_{j \in [n]} a_{i,j} = 0$, by assumption (3.1), we obtain
\[
I_1 = \frac{\mathbb{E}\Psi_{\beta,t}^{(S)}}{n-m} \sum_{j \in [n-m]} a_{i,j} + \frac{1}{n-m} \sum_{j \in [n-m]} a_{i,j} \left(\mathbb{E}\Psi_{\beta,t}^{(S_{i,j}^{(i)})} - \mathbb{E}\Psi_{\beta,t}^{(S)}\right)
\]
\[
(B.31)
= I_{11} + I_{12},
\]
where
\[
I_{11} = -\frac{\mathbb{E}\Psi_{\beta,t}^{(S)}}{n-m} \sum_{j \in [n]\setminus[n-m]} a_{i,j},
\]
\[
I_{12} = \frac{1}{n-m} \sum_{j \in [n-m]} a_{i,j} \left(\mathbb{E}\Psi_{\beta,t}^{(S_{i,j}^{(i)})} - \mathbb{E}\Psi_{\beta,t}^{(S)}\right).
\]
For $I_{11}$, by (3.4) and Jensen’s inequality,
\[
\max_{i,j} |a_{i,j}| \leq \max_{i,j} \mathbb{E}|X_{i,j}| \leq \alpha_n^{-1} \max_{i,j} \mathbb{E}\{|\alpha_n X_{i,j}|\}
\]
\[
(B.32)
\leq \alpha_n^{-1} \max_{i,j} \log \mathbb{E}e^{\alpha_n X_{i,j}} \leq \alpha_n^{-1} \log b.
\]
Thus, by (5.28), noting that $0 \leq m \leq 2$ and $n - m \geq n/2$, we have
\[
|I_{11}| \leq \frac{t m \alpha_n^{-1} \log b}{n-m} \mathbb{E}\Psi_{\beta,t}^{(S)} \leq C t n^{-1} \alpha_n^{-1} \log b \mathbb{E}\Psi_{\beta,t}^{(S)}.
\]
\[B.33\]
For $I_{12}$, note that
\[
S - S^{(i)}_{t,j} = (X_{t,\sigma(i)} + X_{\sigma^{-1}(j),j} - X_{\sigma^{-1}(j),\sigma(i)}) 1(\sigma(i) \neq j) + X_{t,\sigma(i)} 1(\sigma(i) = j)
\]
\[
\leq |X_{t,\sigma(i)}| + |X_{\sigma^{-1}(j),j}| + |X_{\sigma^{-1}(j),\sigma(i)}|.
\]

Moreover,
\[
|\Psi_{\beta,t}(S^{(i)}_{t,j}) - \Psi_{\beta,t}(S)|
\leq t|S - S^{(i)}_{t,j}|e^{t|S - S^{(i)}_{t,j}|} \Psi_{\beta,t}(S)
\]
\[
\leq 3t\Psi_{\beta,t}(S) \left(|X_{t,\sigma(i)}|e^{3t|X_{t,\sigma(i)}|} + |X_{\sigma^{-1}(j),j}|e^{3t|X_{\sigma^{-1}(j),j}|} + |X_{\sigma^{-1}(j),\sigma(i)}|e^{3t|X_{\sigma^{-1}(j),\sigma(i)}|}\right).
\]

Applying Lemma B.1 with $k = 1$ and $\zeta_{i,j} = |X_{i,j}|e^{2t|X_{i,j}|}$, and noting that $t \leq \alpha_n/64$, we have
\[
\mathbb{E}|\Psi_{\beta,t}(S^{(i)}_{t,j}) - \Psi_{\beta,t}(S)| \leq 36b^{1/8}t \mathbb{E}\Psi_{\beta,t}(S) \max_{i,j \in [n]} \mathbb{E}\{|X_{i,j}|e^{4t|X_{i,j}|}\}
\]
\[
\leq 36\alpha_n^{-1}b^{1/8}t \mathbb{E}\Psi_{\beta,t}(S) \max_{i,j \in [n]} \mathbb{E}\{|\alpha_n X_{i,j}|e^{\alpha_n|X_{i,j}|/16}\}
\]
\[
\leq C\alpha_n^{-1}b^{1/8}t \mathbb{E}\Psi_{\beta,t}(S) \max_{i,j \in [n]} \mathbb{E}\{e^{\alpha_n|X_{i,j}|/8}\}
\]
\[
\leq C\alpha_n^{-1}b^{1/4}t \mathbb{E}\Psi_{\beta,t}(S).
\]

By (B.32) and (B.34), we obtain
\[
|I_{12}| \leq C\alpha_n^{-2}t(b^{1/4} \log b) \mathbb{E}\Psi_{\beta,t}(S).
\]
Combining (B.33) and (B.35) yields
\[
|I_1| \leq Cb^{1/2}(n^{-1}\alpha_n^{-1} + \alpha_n^{-2})t \mathbb{E}\Psi_{\beta,t}(S).
\]

For $I_2$, observing that
\[
\mathbb{E}\{|X_{i,j}|(\Psi_{\beta,t}(S) - \Psi_{\beta,t}(S^{(i)}))\} = t^2 \mathbb{E}\{|X_{i,j}|^2 e^{t|X_{i,j}|}\} \Psi_{\beta,t}(S),
\]
we have
\[
|I_2| \leq t^2 \mathbb{E}\{|X_{i,j}|^2 e^{t|X_{i,j}|}\} \Psi_{\beta,t}(S).
\]
Applying Lemma B.1 with $k = 1$ and $\zeta_{i,j} = |X_{i,j}|^2 e^{2t|X_{i,j}|}$, we have
\[
|I_2| \leq Ct^2b^{1/8}h(t) \max_{i,j \in [n]} \mathbb{E}\{|X_{i,j}|^2 e^{2t|X_{i,j}|}\}
\]
\[
\leq Ct^2b^{1/4} \alpha_n^{-2} \mathbb{E}\Psi_{\beta,t}(S).
\]
Combining (B.36) and (B.37) yields (B.27). 

Recall that $\tau_{i,j}$ is the transposition of $i$ and $j$. For $n \geq 4$, $m = 0,1,2$, and any permutation $\sigma \in S_{n-m}$, define the transform
\[
P_{i,j,\sigma} = \begin{cases} 
\sigma & \text{if } \sigma(i) = \sigma(j), \\
\sigma \circ \tau_{\sigma^{-1}(j_1),i_1} & \text{if } \sigma(i_1) \neq j_1 \text{ and } \sigma(i_2) = \sigma(j_2), \\
\sigma \circ \tau_{\sigma^{-1}(j_2),i_2} & \text{if } \sigma(i_1) = j_1 \text{ and } \sigma(i_2) \neq \sigma(j_2), \\
\sigma \circ \tau_{\sigma^{-1}(j_1),i_1} \circ \tau_{\sigma^{-1}(j_2),i_2} \circ \tau_{i_1,i_2} & \text{if } \sigma(i_1) \neq j_1 \text{ and } \sigma(i_2) \neq \sigma(j_2).
\end{cases}
\]
The transformation \((B.38)\) was constructed by Goldstein (2005), and further applied by Chen and Fang (2015) to prove a Berry–Esseen bound for combinatorial central limit theorems. In the following lemmas, we use this transformation to calculate the conditional expectations of functions of \(W\) given \(\pi(i_1) = j_1\) and \(\pi(i_2) = j_2\).

**Lemma B.6.** Let \(S\) and \(\sigma\) be defined as in Lemma B.1. For any \(i = (i_1, i_2) \in [n - m]^2\), \(j = (j_1, j_2) \in [n - m]^2\) and \(1 \leq p, q \leq 2\), we have

\[
(B.39) \quad \mathbb{E}\{X_{\sigma^{-1}(j_q), \sigma(i_p)} | \Psi_{\beta,t}(S) \mathbf{1}(\sigma(i_1) = j_1 \text{ or } \sigma(i_2) = j_2)\} \leq Cb\alpha_n^{-1} \mathbb{E}\Psi_{\beta,t}(S).
\]

**Proof of Lemma B.6.** Let \(\Gamma_1 = \{\sigma(i_1) = j_1 \text{ or } \sigma(i_2) = j_2\}\) and \(\Gamma_{u,v} = \{\sigma(u) = j_q, \sigma(v) = j_v\}\). By the law of total expectation, for any \(1 \leq p, q \leq 2\), we have

\[
(B.40) \quad \mathbb{E}\{X_{\sigma^{-1}(j_q), \sigma(i_p)} | \Psi_{\beta,t}(S) \mathbf{1}(\Gamma_1)\} = \sum_{u,v \in [n-m]} \mathbb{E}\{X_{u,v} | \Psi_{\beta,t}(S) \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\} \mathbb{P}(\Gamma_{u,v}),
\]

where we used \((5.28)\) in the last line. Since \((X_{u,v}, X_{u,j_q})\) is independent of \((S^{(u)}, \pi)\), we have

\[
(B.41) \quad \mathbb{E}\{X_{u,v} | \epsilon^{1+\epsilon}X_{u,j_q} | \Psi_{\beta,t}(S^{(u)}) \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\} \leq \mathbb{E}\{X_{u,v} | \epsilon^{1+\epsilon}X_{u,j_q} | \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\} \leq Cb^{1/4} t\alpha_n^{-1} \mathbb{E}\{\Psi_{\beta,t}(S^{(u)}) \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\}.
\]

By Hölder’s inequality, we have

\[
(B.42) \quad \mathbb{E}\{\Psi_{\beta,t}(S^{(u)}) \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\} \leq \mathbb{E}\{\epsilon^{1+\epsilon}X_{u,j_q} | \Psi_{\beta,t}(S) \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\} \leq \left(\mathbb{E}\{\epsilon^{1+\epsilon}X_{u,j_q} | \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\}\right)^{\epsilon/(1+\epsilon)} \left(\mathbb{E}\{\Psi_{\beta,t}^{1+\epsilon}(S) | \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\}\right)^{1/(1+\epsilon)}.
\]

By the property of conditional expectation and the fact that \(X\) is independent of \(\sigma\), we have the right hand side of \((B.42)\) is equal to

\[
(B.43) \quad \left(\mathbb{E}\{\epsilon^{1+\epsilon}X_{u,j_q} | \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\}\right)^{\epsilon/(1+\epsilon)} \mathbb{P}(\Gamma_1 | B_{\alpha})^{\epsilon/(1+\epsilon)} \left(\mathbb{E}\{\Psi_{\beta,t}^{1+\epsilon}(S) | \mathbf{1}(\Gamma_1 \cap \Gamma_{u,v})\}\right)^{1/(1+\epsilon)} \mathbb{P}(\Gamma_1 \cap \Gamma_{u,v})^{1/(1+\epsilon)} \leq Cb^{1/64} \mathbb{E}\{\Psi_{\beta,t}^{1+\epsilon}(S) | \Gamma_1 \cap \Gamma_{u,v}\} \mathbb{P}(\Gamma_1 \cap \Gamma_{u,v}),
\]

where the inequality follows from \((B.8)\) and \((B.9)\) and the fact that \(\Psi_{\beta,t} \geq 1\). By the property of conditional expectation,

\[
(B.44) \quad \mathbb{E}\{\Psi_{\beta,t}^{1+\epsilon}(S) | \Gamma_1 \cap \Gamma_{u,v}\} \mathbb{P}(\Gamma_1 \cap \Gamma_{u,v}) = \mathbb{E}\{\Psi_{\beta,t}(S) \mathbf{1}(\Gamma_1) | \Gamma_{u,v}\}.
\]

Combining \((B.40)-(B.44)\), we have

\[
(B.45) \quad \mathbb{E}\{X_{\sigma^{-1}(j_q), \sigma(i_p)} | \Psi_{\beta,t}(S) \mathbf{1}(\Gamma_1)\} \leq Cb^{1/4} t\alpha_n^{-1} \mathbb{E}\{\Psi_{\beta,t}(S) \mathbf{1}(\Gamma_1)\}.
\]
For the expectation term on the right hand side of (B.45),
\[ E\{\Psi_{\beta,t}(S) \, 1(\Gamma_1)\} \]
\[ = \sum_{v_1, v_2 \in [n-m]} E\{\Psi_{\beta,t}(S) \, 1(\Gamma_1) | \sigma(i_1) = v_1, \sigma(i_2) = v_2\} \times \mathbb{P}(\sigma(i_1) = v_1, \sigma(i_2) = v_2) \]
\[ = \sum_{v_1, v_2 \in [n-m]} 1(v_1 = j_1 \text{ or } v_2 = j_2) E\{\Psi_{\beta,t}(S) | \sigma(i_1) = v_1, \sigma(i_2) = v_2\} \times \mathbb{P}(\sigma(i_1) = v_1, \sigma(i_2) = v_2). \quad (B.46) \]

For \( i = (i_1, i_2), \ v = (v_1, v_2), \) let \( \sigma_{i,v} = \mathcal{P}_{i,v} \sigma \) and \( S_{i,v} = \sum_{r=1}^{n-m} X_{r,\sigma_{i,v}(r)} \). By (3.14) of Chen and Fang (2015) (see also Lemma 4.5 of Chen, Goldstein and Shao (2011)), we have
\[ E\{\Psi_{\beta,t}(S) | \sigma(i_1) = v_1, \sigma(i_2) = v_2\} = E\{\Psi_{\beta,t}(S_{i,v})\}. \quad (B.47) \]

Moreover, by the construction of \( S_{i,v} \), it follows that
\[ |S - S_{i,v}| \leq |X_{i_1,v_1} + |X_{i_2,v_2}| + \sum_{i \in \{i_1, i_2\}} \sum_{v \in \{v_1, v_2\}} |X_{\sigma^{-1}(v), \sigma(i)}| \]
\[ + \sum_{i \in \{i_1, i_2\}} |X_{i, \sigma(i)}| + \sum_{v \in \{v_1, v_2\}} |X_{\sigma^{-1}(v), v}|. \quad (B.48) \]

By (A.6) and Hölder’s inequality we have
\[ E\{\Psi_{\beta,t}(S_{i,v})\} \leq E\{e^{t|S - S_{i,v}|} \Psi_{\beta,t}(S)\} \]
\[ \leq (E\{e^{(1+\varepsilon)|S - S_{i,v}|} \})^{1/(1+\varepsilon)} E\{\Psi_{\beta,t}(S)^{1+\varepsilon}\}. \quad (B.49) \]

By the similar argument to (B.8) and (B.9) again, we obtain
\[ E\{\Psi_{\beta,t}(S_{i,v})\} \leq Cb^{1/2} E\{\Psi_{\beta,t}(S)\}. \quad (B.50) \]

Combining (B.46), (B.47) and (B.49), we obtain
\[ E\{\Psi_{\beta,t}(S) \, 1(\Gamma_1)\} \]
\[ \leq Cb^{1/2} E\{\Psi_{\beta,t}(S)\} \sum_{v_1, v_2 \in [n-m]} 1(v_1 = j_1 \text{ or } v_2 = j_2) \mathbb{P}(\sigma(i_1) = v_1, \sigma(i_2) = v_2) \]
\[ \leq Cb^{1/2} n^{-1} E\{\Psi_{\beta,t}(S)\}. \quad (B.51) \]

By (B.45) and (B.51), we complete the proof. \( \square \)

The following lemma, whose proof is based on Lemmas B.5 and B.6, plays an important role in the proof of Lemma B.2.

**Lemma B.7.** Let \( \pi, X \) and \( W \) be defined as in Theorem 3.1, and recall that \( \mathcal{P}_{1,j} \) is defined as in (B.38). For \( i, j \in [n] \), \( i' \in [n] \) and \( j' \in [n] \), let \( \mathcal{I} = (i, i'), \mathcal{J} = (j, j') \), and
\[ \pi_{i,j} = \mathcal{P}_{1,j} \pi, \quad \pi_{i',j'} = \mathcal{P}_{i,j} \pi_{i,j}, \quad W^{(\mathcal{I})}_{\mathcal{J}} = \sum_{i' \in [n] \setminus \{i_1, i_2, i_1', i_2'\}} X_{i', \pi_{i',j'} (i')} \]

Then,
\[ |E\{\Psi_{\beta,t}(W^{(\mathcal{I})}_{\mathcal{J}})\} - h(t)| \leq Cb^2 (n^{-1} + \alpha_n^{-1})(1 + t^2)h(t). \quad (B.52) \]
PROOF. Recall that $h(t) = \mathbb{E}\Psi_{\beta,t}(W)$. To bound the difference between $\mathbb{E}\Psi_{\beta,t}(W_{i,j})$ and $\mathbb{E}\Psi_{\beta,t}(W)$, we consider the following three steps. In the first step, we construct an auxiliary random variable $S^{(i,j)}_V$ that is close to $W$ and has the same distribution as $W_{i,j}$. In the rest, we apply Taylor’s expansion to calculate the difference of the expectations.

**Step 1. Constructing $S^{(i,j)}_V$.** Note that $\pi$ is a random permutation chosen uniformly from $S_n$, and it follows from Eq. (3.14) of Chen and Fang (2015) (see also Lemma 4.5 of Chen, Goldstein and Shao (2011)) that

$$
\mathcal{L}(\pi_{i,j}) = \mathcal{L}(\pi|\pi(i) = j).
$$

Write $W_{i,j} = \sum_{i' \in [n]} X_{i',\pi_{i,j}(i')}$ and $W^{(i)}_{i,j} = \sum_{i' \notin A(i)} X_{i',\pi_{i,j}(i')}$, and it follows from (B.53) that $\mathcal{L}(W_{i,j}) = \mathcal{L}(W|\pi(i) = j)$ and $\mathcal{L}(W^{(i)}_{i,j}) = \mathcal{L}(W^{(i)}|\pi(i) = j)$. To calculate $\mathbb{E}\Psi_{\beta,t}(W_{i,j})$, we introduce an auxiliary permutation $\sigma$ as follows. Let $\sigma$ be a uniform permutation from $[n] \setminus \{i_1, i_2\}$ to $[n] \setminus \{j_1, j_2\}$, independent of everything else, and let

$$
S^{(i)} = \sum_{i' \notin A(i)} X_{i',\sigma(i')}.
$$

It also follows from Lemma 4.5 of Chen, Goldstein and Shao (2011) that

$$
\mathcal{L}(W^{(i)}_{i,j}) = \mathcal{L}(S^{(i)}).
$$

Moreover, noting that $\{i_1, i_2\} \cap \{i', j'\} = \emptyset$, using (B.53) twice implies that

$$
\mathcal{L}(\pi_{i,j}) = \mathcal{L}(\pi|\pi(i) = j).
$$

Recalling (B.38), we define

$$
\sigma_{V,j'} = \mathcal{P}_{V,j'}, \quad S^{(i)}_{V,j'} = \sum_{i' \in [n] \setminus \{i\}} X_{i',\sigma_{V,j'}(i')} \quad \text{and} \quad S^{(i,j')}_{V,j'} = \sum_{i' \in [n] \setminus \{i\} \cup \{j\}} X_{i',\sigma_{V,j'}(i')}.
$$

Then, it follows by definition that $\mathcal{L}(W_{i,j}^{(i,j)}) = \mathcal{L}(S^{(i,j')}_{V,j'})$.

**Step 2. Bounding $|\mathbb{E}\Psi_{\beta,t}(S^{(i,j)}_V) - h(t)|$.** We first bound $|\mathbb{E}\Psi_{\beta,t}(S^{(i,j')}_{V,j'}) - \mathbb{E}\Psi_{\beta,t}(S^{(i)})|$, and the bound of $|\mathbb{E}\Psi_{\beta,t}(S^{(i)}) - h(t)|$ can be obtained similarly. By Taylor’s expansion,

$$
\Psi_{\beta,t}(w) = \Psi_{\beta,t}(w_0) + (w - w_0)\Psi'_{\beta,t}(w_0) + \frac{1}{2}(w - w_0)^2 \mathbb{E}\{\Psi''_{\beta,t}(w_0 + U(w - w_0))(1 - U)\},
$$

where $U$ is a uniform random variable over the interval $[0,1]$ independent of all others. Applying Taylor’s expansion (B.57) with $w_0 = S^{(i)}$ and $w = S^{(i,j')}_{V,j'}$, we have

$$
\mathbb{E}\{\Psi_{\beta,t}(S^{(i,j')}_{V,j'})\} = \mathbb{E}\{\Psi_{\beta,t}(S^{(i)})\} + \mathbb{E}\{ \left(S^{(i,j')}_{V,j'} - S^{(i)}\right)\Psi'_{\beta,t}(S^{(i)})\}
$$

$$
+ \mathbb{E}\{ \left(S^{(i,j')}_{V,j'} - S^{(i)}\right)^2\Psi''_{\beta,t}(S^{(i)}) + U(S^{(i,j')}_{V,j'} - S^{(i)})\}(1 - U)\}.
$$
Denote by $B_{i,j}$ the event that $\{\sigma(i_1) = j_1$ or $\sigma(i_2) = j_2\}$ and denote by $B_{i,j}^c$ the complement of $B_{i,j}$. For the second term of (B.58), by the construction of $\mathcal{P}_{Y,Y'}$, we have
\[
\left| \mathbb{E}\{ (S_{Y,Y'}^{(i,j)} - S^{(i)}) \Psi_{\beta,t}(S^{(i)}) \} \right| \\
\leq \left| \mathbb{E}\{ X_{\sigma^{-1}(j_1),\sigma(i_1)} \Psi_{\beta,t}(S^{(i)}) 1(B_{Y,Y'}) \} \right| \\
+ \left| \mathbb{E}\{ X_{\sigma^{-1}(j_2),\sigma(i_2)} \Psi_{\beta,t}(S^{(i)}) 1(B_{Y,Y'}) \} \right| \\
+ \left| \mathbb{E}\{ X_{\sigma^{-1}(j_1'),\sigma(i_1')} \Psi_{\beta,t}(S^{(i)}) 1(B_{Y,Y'}) \} \right| \\
+ \left| \mathbb{E}\{ X_{\sigma^{-1}(j_2'),\sigma(i_2')} \Psi_{\beta,t}(S^{(i)}) 1(B_{Y,Y'}) \} \right| \\
+ \sum_{j' \in \{j_1',j_2'\}} \left| \mathbb{E}\{ X_{j',\sigma(i')} \Psi_{\beta,t}(S^{(i)}) \} \right| \right. \\
\left. + \sum_{j' \in \{j_1',j_2'\}} \left| \mathbb{E}\{ X_{\sigma^{-1}(j_1),\sigma(i_1)} \Psi_{\beta,t}(S^{(i)}) \} \right| + \left| \mathbb{E}\{ X_{\sigma^{-1}(j_2),\sigma(i_2)} \Psi_{\beta,t}(S^{(i)}) \} \right| \right).
\]
Applying Lemmas B.5 and B.6 with $S = S^{(i)}$ under a relabeling of indices, we obtain
(B.59) \[
\left| \mathbb{E}\{ (S_{Y,Y'}^{(i,j)} - S^{(i)}) \Psi_{\beta,t}(S^{(i)}) \} \right| \leq C(n^{-1}\alpha_n^{-1} + \alpha_n^{-2})b(1 + t^2) \mathbb{E}\Psi_{\beta,t}(S^{(i)}).
\]
For the third term of (B.59), we have
(B.60) \[
\left| \mathbb{E}\{ (S_{Y,Y'}^{(i,j)} - S^{(i)})^2 \Psi_{\beta,t}(S^{(i)}) + U(S_{Y,Y'}^{(i,j)} - S^{(i)})(1 - U) \} \right| \\
\leq 2t^2 \mathbb{E}\{ (S_{Y,Y'}^{(i,j)} - S^{(i)})^2 e^{L((S_{Y,Y'}^{(i,j)} - S^{(i)}) - \Psi_{\beta,t}(S^{(i)}))} \} \\
\leq 24t^2 \left\{ \sum_{k \in \{i_1,i_2,i_1',i_2'\}} \sum_{\ell \in \{j_1,j_2,j_1',j_2'\}} \mathbb{E}\{ X_{\sigma^{-1}(k),\sigma(\ell)}^2 e^{24t|X_{\sigma^{-1}(k),\sigma(\ell)}|} \Psi_{\beta,t}(S^{(i)}) \} \\
+ \sum_{\ell \in \{j_1,j_2,j_1',j_2'\}} \mathbb{E}\{ X_{\sigma^{-1}(\ell),\sigma(\ell)}^2 e^{24t|X_{\sigma^{-1}(\ell),\sigma(\ell)}|} \Psi_{\beta,t}(S^{(i)}) \} \right\}.
\]
Applying Lemma B.1 with $S = S^{(i)}$ and $\zeta_{i,j} = |X_{i,j}|^2 e^{24t|X_{i,j}|}$ under a relabeling of indices, and recalling that $t \leq \alpha_n/64$, we obtain
\[
\mathbb{E}\{ X_{\sigma^{-1}(k),\sigma(\ell)}^2 e^{24t|X_{\sigma^{-1}(k),\sigma(\ell)}|} \Psi_{\beta,t}(S^{(i)}) \} \\
\leq 4b^{1/8} \mathbb{E}\Psi_{\beta,t}(S^{(i)}) \max_{u,v \in [n]} \mathbb{E}\{ |X_{u,v}|^2 e^{24t|X_{u,v}| + t(|X_{k,v}| + |X_{u,v}|)} \} \\
\leq Cb^{1/8} \alpha_n^{-2} \mathbb{E}\Psi_{\beta,t}(S^{(i)}) \max_{u,v \in [n]} \mathbb{E}\{ (1 + |\alpha_n X_{u,v}|^2) e^{13\alpha_n |X_{u,v}|/32} \} \\
\leq Cb^{1/8} \alpha_n^{-2} \mathbb{E}\Psi_{\beta,t}(S^{(i)}) \max_{u,v \in [n]} \mathbb{E}\{ e^{\alpha_n |X_{u,v}|/2} \} \\
\leq Cb\alpha_n^{-2} \mathbb{E}\Psi_{\beta,t}(S^{(i)}),
\]
where we used (3.4) in the last line. Similarly, we obtain
\[
\mathbb{E}\{X_{i_1\pi(i)}^2 e^{2t|X_{i_1\pi(i)}|}\psi_{\beta,t}(S^{(i)})\} \leq Cb\alpha_n^{-2} \mathbb{E}\psi_{\beta,t}(S^{(i)}),
\]
\[
\mathbb{E}\{X_{j_1\pi^{-1}(j_1)}^2 e^{2t|X_{j_1\pi^{-1}(j_1)}|}\psi_{\beta,t}(S^{(i)})\} \leq Cb\alpha_n^{-2} \mathbb{E}\psi_{\beta,t}(S^{(i)}).
\]
Hence, we have
\[
\text{R.H.S. of (B.60)} \leq Cb\alpha_n^{-2}(1+t^2) \mathbb{E}\psi_{\beta,t}(S^{(i)}).
\]
Together with (B.58) and (B.59), we have
\[(B.61) \quad |\mathbb{E}\{\psi_{\beta,t}(W^{(\mathcal{I})}_{\mathcal{I},\mathcal{J}})\} - \mathbb{E}\psi_{\beta,t}(S^{(i)})| \leq Cb(n^{-1} + \alpha_n^{-2})(1+t^2) \mathbb{E}\psi_{\beta,t}(S^{(i)}).
\]
A similar argument yields
\[(B.62) \quad |\mathbb{E}\psi_{\beta,t}(S^{(i)}) - h(t)| \leq Cb(n^{-1} + \alpha_n^{-2})(1+t^2)h(t).
\]
Then applying Lemma B.1 to the last expectation of (B.61) and similar to (B.60), we have
\[
(B.63) \quad \mathbb{E}\psi_{\beta,t}(S^{(i)}) = \mathbb{E}\psi_{\beta,t}(W^{(i)}) \leq \mathbb{E}\{\psi_{\beta,t}(W)e^{t|W^{(i)}|}\} \leq Cb\alpha_n(t).
\]
Combining (B.61)–(B.63) and recalling that \( t \leq \alpha_n/64 \), we obtain
\[
|\mathbb{E}\{\psi_{\beta,t}(W^{(\mathcal{I})}_{\mathcal{I},\mathcal{J}})\} - h(t)| \leq Cb^2(n^{-1} + \alpha_n^{-2})(1+t^2)h(t).
\]
This completes the proof. \( \square \)

**B.3. Proofs of Lemma B.2 and B.3.** First, we apply Lemmas B.1 and B.7 to prove Lemma B.2.

**Proof of Lemma B.2.** Applying the Taylor expansion (B.57) to \( \psi_{\beta,t}(W) \) yields
\[(B.64) \quad \mathbb{E}\{\xi_{i_1\pi(i)}\xi_{i'\pi(i')}\psi_{\beta,t}(W)\} = \frac{1}{(n)_4}\sum_{j\in[n/2]}\sum_{j'\in[n/2]}^{j}\mathbb{E}\{\xi_{i_1\pi(i)}\xi_{i'\pi(i')}\psi_{\beta,t}(W)|\pi(\mathcal{I}) = \mathcal{J}\}
\]
\[= Q_1 + Q_2 + Q_3,
\]
where
\[Q_1 = \frac{1}{(n)_4}\sum_{j\in[n/2]}\sum_{j'\in[n/2]}^{j}\mathbb{E}\{\xi_{i_1j}\xi_{i'j'}\psi_{\beta,t}(W^{(\mathcal{I})})|\pi(\mathcal{I}) = \mathcal{J}\},
\]
\[Q_2 = \frac{1}{(n)_4}\sum_{j\in[n/2]}\sum_{j'\in[n/2]}^{j}\mathbb{E}\{\xi_{i_1j}\xi_{i'j'}v_{\mathcal{I},\mathcal{J}}\psi_{\beta,t}(W^{(\mathcal{I})})|\pi(\mathcal{I}) = \mathcal{J}\},
\]
\[Q_3 = \frac{1}{(n)_4}\sum_{j\in[n/2]}\sum_{j'\in[n/2]}^{j}\mathbb{E}\{\xi_{i_1j}\xi_{i'j'}v_{\mathcal{I},\mathcal{J}}^2\psi_{\beta,t}(W^{(\mathcal{I})} + U(W - W^{(\mathcal{I})}))|\pi(\mathcal{I}) = \mathcal{J}\},
\]
and where \( \mathcal{I} = (i_1, i_2, i'_1, i'_2) \), \( \mathcal{J} = (j_1, j_2, j'_1, j'_2) \), \( W^{(\mathcal{I})} = \sum_{i\in[n]\setminus\mathcal{I}}X_{i\pi(i)} \), \( V_{\mathcal{I},\mathcal{J}} = X_{i_1,j_1} + X_{i_2,j_2} + X_{i'_1,j'_1} + X_{i'_2,j'_2} \) and \( U \) is a uniform random variable on \([0,1]\) independent of \( X \) and \( \pi \).
For $Q_1$, as $(\xi_{i,j}, \xi_{i',j'})$ is conditionally independent of $W^{(Z)}$ given $\pi(I) = J$, and as $(\xi_{i,j}, \xi_{i',j'})$ is also independent of $\pi$, we have

$$\mathbb{E}\{\xi_{i,j}\xi_{i',j'} \mid \beta, t(W^{(Z)}) \mid \pi(I) = J\} = \mathbb{E}\{\xi_{i,j}\xi_{i',j'} \mid \mathbb{E}\{\Psi_{\beta, t}(W^{(Z)}) \mid \pi(I) = J\}. \tag{B.65}$$

Let $W_{I,J}$ be defined as in Lemma B.7. By (B.66), we have

$$\mathbb{E}\{\Psi_{\beta, t}(W^{(Z)}) \mid \pi(I) = J\} = \mathbb{E}\{\Psi_{\beta, t}(W_{I,J})\} = h(t) + |\mathbb{E}\{\Psi_{\beta, t}(W^{(Z)}) \mid \pi(I) = J\} - h(t)|.$$

Taking average over $j \in [n]_2, j' \in [n]_2^{(j)}$ on both sides of (B.65) and applying Lemma B.7 gives

$$|Q_1| \leq \frac{h(t)}{(n)_4} \sum_{j \in [n]_2} \sum_{j' \in [n]_2^{(j)}} \left| \mathbb{E}\xi_{i,j} \mathbb{E}\xi_{i',j'} \right|$$

$$+ \frac{1}{(n)_4} \sum_{j \in [n]_2} \sum_{j' \in [n]_2^{(j)}} \left| \mathbb{E}|\xi_{i,j}| \mathbb{E}|\xi_{i',j'}|| \mathbb{E}\{\Psi_{\beta, t}(W_{I,J})\} - h(t)\right| \tag{B.66}$$

For the first term of the right hand side of (B.66), since $\mathbb{E}\xi_{i',\pi(I')} = 0$, it follows that $\sum_{j' \in [n]_2} \mathbb{E}\xi_{i',j'} = 0$. Therefore,

$$\sum_{j' \in [n]_2} \mathbb{E}\xi_{i',j'} = \sum_{j' \in [n]_2} \mathbb{E}\xi_{i',j'} \leq \sum_{j' \in [n]_2} \mathbb{1}(E_{j,j'}) \mathbb{E}\xi_{i',j'}, \tag{B.67}$$

where $E_{j,j'} = \{A(j) \cap A(j') \neq \emptyset\}$. Hence, we have

$$|Q_1| \leq Cb^2(n-1 + \alpha_n^2)(1 + t^2)h(t) \sum_{j' \in [n]_2} (1(E_{j,j'}) \mathbb{1}(E_{j,j'}) + \alpha_n^{-2} + n^{-1})(\mathbb{E}|\xi_{i,j}| \mathbb{E}|\xi_{i',j'}|). \tag{B.68}$$

We now consider $Q_2$. Since $\xi_{i,j}\xi_{i',j'}V_{I,J}$ is independent of $W^{(Z)}$ conditional on $\pi(I) = J$, we have

$$Q_2 = \frac{1}{(n)_4} \sum_{j \in [n]_2} \sum_{j' \in [n]_2^{(j)}} \mathbb{E}\{\xi_{i,j}\xi_{i',j'} \mid V_{I,J}\} \mathbb{E}\{\Psi_{\beta, t}(W^{(Z)}) \mid \pi(I) = J\}$$

$$= \frac{1}{(n)_4} \sum_{j \in [n]_2} \sum_{j' \in [n]_2^{(j)}} \mathbb{E}\{\xi_{i,j}\xi_{i',j'} \mid V_{I,J}\} \mathbb{E}\{\Psi_{\beta, t}(W_{I,J})\} \tag{B.69}$$

$$= Q_{2,1} + Q_{2,2} + Q_{2,3},$$

where

$$Q_{2,1} = \frac{1}{(n)_4} \sum_{j \in [n]_2} \sum_{j' \in [n]_2^{(j)}} \mathbb{E}\{\xi_{i,j}\xi_{i',j'} \mid V_{I,J}\} \mathbb{E}\{\Psi_{\beta, t}(W)\}.$$
\[ Q_{2.2} = \frac{1}{(n)_4} \sum_{j \in [n]_2} \sum_{j' \in [n]_2} \mathbb{E}\{ \xi_{i,j} \xi_{i',j'} V_{i,j'} \} \mathbb{E}\{ \Psi'_{\beta,t}(W) \}, \]
\[ Q_{2.3} = \frac{1}{(n)_4} \sum_{j \in [n]_2} \sum_{j' \in [n]_2} \mathbb{E}\{ \xi_{i,j} \xi_{i',j'} V_{i,j'} \} \mathbb{E}\{ \Psi''_{\beta,t}(W + U(W_{i,j}^{(Z)} - W))(W_{i,j}^{(Z)} - W) \}, \]
and \( V_{i,j} = X_{i_1,j_1} + X_{i_2,j_2} + V_{i',j'} = X_{i_1,j_1} + X_{i_2,j_2} \). Similar to (B.67), we have
\[ |Q_{2.1}| \leq Cn^{-4}t(h(t) \sum_{j,j' \in [n]_2} 1(E_{j,j'}) \mathbb{E}\{ \xi_{i,j} V_{i,j} \} \mathbb{E}\{ \xi_{i',j'} \}). \]
(B.70)
\[ |Q_{2.2}| \leq Cn^{-4}t(h(t) \sum_{j,j' \in [n]_2} 1(E_{j,j'}) \mathbb{E}\{ \xi_{i,j} \} \mathbb{E}\{ \xi_{i',j'} V_{i,j'} \}). \]

We next consider \( Q_{2.3} \). By (5.27) and using a similar argument for (B.60),
\[ \mathbb{E}\{ \Psi''_{\beta,t}(W + U(W_{i,j}^{(Z)} - W))(W_{i,j}^{(Z)} - W) \} \]
\[ \leq t^2 \mathbb{E}\{ \Psi_{\beta,t}(W)e^{t(W_{i,j}^{(Z)} - W)}|W_{i,j}^{(Z)} - W| \} \]
\[ \leq Cb\alpha^{-1}(1 + t^2)h(t). \]
Therefore,
\[ |Q_{2.3}| \leq Cb(1 + t^2)h(t)n^{-4}\alpha^{-1}_n \left( \sum_{j,j' \in [n]_2} |\mathbb{E}\{ \xi_{i,j} V_{i,j} \} \mathbb{E}\{ \xi_{i',j'} \}| + \sum_{j,j' \in [n]_2} |\mathbb{E}\{ \xi_{i,j} \} \mathbb{E}\{ \xi_{i',j'} V_{i,j'} \}| \right). \]
(B.72)

By (B.69), (B.70) and (B.72), we have
\[ |Q_2| \leq Cb(1 + t^2)h(t)n^{-4} \sum_{j,j' \in [n]_2} (\alpha^{-1}_n + 1(E_{j,j'})) \left( |\mathbb{E}\{ \xi_{i,j} V_{i,j} \} \mathbb{E}\{ \xi_{i',j'} \}| + |\mathbb{E}\{ \xi_{i,j} V_{i,j'} \} \mathbb{E}\{ \xi_{i',j} \}| \right) \]
\[ \leq Cb(1 + t^2)h(t)n^{-4} \sum_{j,j' \in [n]_2} \left( \mathbb{E}\{ |\xi_{i,j} \xi_{i,j'}| V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)} - W_{i,j}^{(Z)}) \} \mathbb{E}\{ \xi_{i',j'} |V_{i,j}^2 \} \right) \right) \]
\[ + Cb(1 + t^2)h(t)n^{-4} \sum_{j,j' \in [n]_2} (\alpha^{-2}_n + 1(E_{j,j'})) \left( |\mathbb{E}\{ \xi_{i,j} \} \mathbb{E}\{ \xi_{i',j'} |V_{i,j}^2 \} \right) \right), \]
where the second inequality follows from Cauchy’s inequality and the fact that \( |V_{i,j}| \leq T_{i,j} \) for any \( i,j \). Finally, for \( Q_3 \), by (5.27) again, and noting that \( V_{i,j} = V_{i,j} + V_{i',j'} \), we have
\[ |\mathbb{E}\{ \xi_{i,j} \xi_{i,j'} V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)} - W_{i,j}^{(Z)})(1 - U)|\pi(i) = j; \pi(i') = j' \} \]
\[ \leq t^2 \mathbb{E}\{ |\xi_{i,j} \xi_{i,j'} V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)}) \} e^{t|V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)})|} \}
\[ = t^2 \mathbb{E}\{ |\xi_{i,j} \xi_{i,j'} V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)}) \} e^{t|V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)})|} \}
\[ = t^2 \mathbb{E}\{ |\xi_{i,j} \xi_{i,j'} V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)}) \} e^{t|V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)})|} \}
\[ \leq 2t^2 \mathbb{E}\{ |\xi_{i,j} \xi_{i,j'} |T_{i,j}^2 e^{2t|V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)})|} + T_{i,j}^2 e^{2t|V_{i,j}^2 \Psi_{\beta,t}(W_{i,j}^{(Z)})|} \} \}
\[ \leq Cb(t) \}
\[ \mathbb{E}\{ \Psi_{\beta,t}(W) e^{t|W_{i,j}^{(Z)} - W|} \} \leq Cb h(t). \]
By (B.74) and (B.75), we have

\[ |Q_3| \leq Cbn^{-4}t^2 h(t) \sum_{j' \in [n]_2} \left( \mathbb{E}\{\xi_{1,j}T_1^2 e^{2t|T_1,j|}\} \mathbb{E}|\xi_{1,j'}| + \mathbb{E}\{|\xi_{1,j}T_1^2 e^{2t|T_1,j'|}|\} \mathbb{E}|\xi_{1,j}| \right). \]

Combining (B.64), (B.68), (B.73) and (B.76) yields the desired result. \( \square \)

We finish our paper by proving Lemma B.3, which is based on Lemmas B.1 and B.2.

**Proof of Lemma B.3.** First, for \( v = 0 \), expanding the square term in (B.10) yields

\[ \mathbb{E}\left\{ \left( \sum_{i \in [n]_2} \tilde{g}_{i,\pi(i)}(u) \right)^2 \Psi_{\beta,t}(W) \right\} = H_1(u) + H_2(u), \]

where

\[ H_1(u) = \frac{1}{16n^2} \sum_{i \in [n]_2} \sum_{i' \in [n]_2 \setminus [n]_2^{(i)}} \mathbb{E}\{\tilde{g}_{i,\pi(i)}(u)\tilde{g}_{i',\pi(i')} (u) \Psi_{\beta,t}(W)\}, \]

\[ H_2(u) = \frac{1}{16n^2} \sum_{i \in [n]_2} \sum_{i' \in [n]_2^{(i)}} \mathbb{E}\{\tilde{g}_{i,\pi(i)}(u)\tilde{g}_{i',\pi(i')} (u) \Psi_{\beta,t}(W)\}. \]

For \( H_1(u) \), by Young’s inequality,

\[ H_1(u) \leq \frac{1}{16n^2} \sum_{i \in [n]_2} \sum_{i' \in [n]_2 \setminus [n]_2^{(i)}} \mathbb{E}\{(\tilde{g}_{i,\pi(i)}^2(u) + \tilde{g}_{i',\pi(i')}^2(u)) \Psi_{\beta,t}(W)\} \]

\[ \leq Cn^{-1} \sum_{i \in [n]_2} \mathbb{E}\{\tilde{g}_{i,\pi(i)}^2(u) \Psi_{\beta,t}(W)\}. \]

Taking integration on both sides of (B.78) implies

\[ \int_{|u| \leq 1} e^{2t|u|} \mathbb{E}\{\tilde{g}_{i,\pi(i)}^2(u) \Psi_{\beta,t}(W)\} du \]

\[ \leq Cn^{-1} \mathbb{E}\{\Psi_{\beta,t}(W) \left(|D_{1,\pi(i)}|^3 e^{2t|D_{1,\pi(i)}|} + \mathbb{E}|D_{1,\pi(i)}|^3 e^{2t|D_{1,\pi(i)}|}\right)\}. \]

Applying Lemma B.1 with \( k = 2, m = 0, \sigma = \pi \) and \( \zeta_{i,j} = |D_{i,j}|^3 e^{2t|D_{i,j}|} \), we have

\[ \mathbb{E}\{|D_{1,\pi(i)}|^3 e^{2t|D_{1,\pi(i)}|} \Psi_{\beta,t}(W)\} \]

\[ \leq Cb \mathbb{E}\{\Psi_{\beta,t}(W)\} \max_{v \in [n]_2} \mathbb{E}\{|D_{1,v}|^3 e^{2t|D_{1,v}|} e^{t \sum_{r=1}^2 X_{i_{r,v}}}|\} \]

\[ \leq Cb^{1/16} \mathbb{E}\{\Psi_{\beta,t}(W)\} \max_{v \in [n]_2} \mathbb{E}\{|\alpha_n D_{1,v}|^3 e^{2t|D_{1,v}|} e^{t \sum_{r=1}^2 X_{i_{r,v}}}|\} \]

\[ \leq Cb\alpha_n^{-3} \mathbb{E}\{\Psi_{\beta,t}(W)\}, \]

and similarly,

\[ \mathbb{E}|D_{1,\pi(i)}|^3 e^{2t|D_{1,\pi(i)}|} \leq Cb\alpha_n^{-3}. \]

By (B.78)–(B.81), we have

\[ \int_{|u| \leq 1} e^{2t|u|} H_1(u) du \leq Cb\alpha_n^{-3} \mathbb{E}\{\Psi_{\beta,t}(W)\}. \]
For $H_2(u)$, by Lemma B.2, with $\xi_{ij} = g_{ij}(u)$, we have any $i \in [n]_2, i' \in [n]_2^{(i)}$, and $j, j' \in [n]_2$, (B.83)

$$\mathbb{E}\{\tilde{g}_{i,\pi(i)}(u)\tilde{g}_{i',\pi(i')}(u)\Psi_{\beta,i}(W)\}$$

$$\leq Cb(1 + t^2)h(t)n^{-4} \sum_{j,j' \in [n]_2} \mathbb{E}\{|\tilde{g}_{ij}(u)|T_{ij}^2e^{2|T_{ij}|}\} \mathbb{E}\{|\tilde{g}_{ij'}(u)|\}$$

$$+ Cb(1 + t^2)h(t)n^{-4} \sum_{j,j' \in [n]_2} \mathbb{E}\{|\tilde{g}_{ij'}(u)|T_{ij'}^2e^{2|T_{ij'}|}\} \mathbb{E}\{|\tilde{g}_{ij}(u)|\}$$

$$+ Cb(1 + t^2)h(t)n^{-4} \sum_{j,j' \in [n]_2} (\alpha_n^{-2} + n^{-1} + 1(E_{j,j'})) \mathbb{E}\{|\tilde{g}_{ij}(u)|\} \mathbb{E}\{|\tilde{g}_{ij'}(u)|\}$$

$$:= H_{21}(u) + H_{22}(u) + H_{23}(u).$$

First we consider $H_{21}(u)$. Since $\mathbb{E}\{|\tilde{g}_{ij}(u)|\} \leq 2 \mathbb{E}|D_{ij}|$, by Fubini’s theorem, we have for any $i \in [n]_2, i' \in [n]_2^{(i)}$, and $j, j' \in [n]_2$,

$$\left| \int_{|u| \leq 1} e^{2t|u|} \mathbb{E}\{|\tilde{g}_{ij}(u)|T_{ij}^2e^{2|T_{ij}|}\} \mathbb{E}\{|\tilde{g}_{ij'}(u)|\} du \right|$$

(B.84)

$$\leq 2 \int_{|u| \leq 1} e^{2t|u|} \mathbb{E}\{|\tilde{g}_{ij}(u)|T_{ij}^2e^{2|T_{ij}|}\} \mathbb{E}\{|D_{ij'}|\} du$$

$$\leq 2 \mathbb{E}\left\{ |D_{ij}|^2e^{2|D_{ij}|} + \mathbb{E}\{|D_{i,\pi(i)}|^2e^{2|D_{i,\pi(i)}|}\} T_{ij}^2e^{2|T_{ij}|} \right\} \mathbb{E}\{|D_{ij'}|\}$$

where use the same notation as in Lemma B.2. Then, by (3.4) and Young’s inequality, we have

$$\mathbb{E}\left\{ |D_{ij}|^2e^{2|D_{ij}|}T_{ij}^2e^{2|T_{ij}|} \right\} \mathbb{E}\{|D_{ij'}|\}$$

(B.85)

$$\leq C\alpha_n^{-5} \mathbb{E}\{ |\alpha_n^{-1}D_{ij}|^2e^{2|D_{ij}|}T_{ij}^2e^{2|T_{ij}|} \} \mathbb{E}\{ \alpha_n^{-1}|D_{ij'}| \}$$

$$\leq Cb\alpha_n^{-5}.$$

Similar to (B.85),

(B.86)

$$\mathbb{E}\left\{ |D_{ij}|^2e^{2|D_{ij}|}T_{ij}^2e^{2|T_{ij}|} \right\} \mathbb{E}\{|D_{ij'}|\} \leq Cb\alpha_n^{-5}.$$

Using (B.84)–(B.86), we have

(B.87)

$$\left| \int_{|u| \leq 1} e^{2t|u|} \mathbb{E}\{|\tilde{g}_{ij}(u)|T_{ij}^2e^{2|T_{ij}|}\} \mathbb{E}\{|\tilde{g}_{ij'}(u)|\} du \right| \leq Cb\alpha_n^{-5}.$$

Furthermore, we have,

(B.88)

$$\left| \int_{|u| \leq 1} e^{2t|u|} H_{21}(u) du \right| \leq Cb(1 + t^2)\alpha_n^{-5}h(t)$$

Moreover, by the same argument, we have

$$\left| \int_{|u| \leq 1} e^{2t|u|} H_{22}(u) du \right| \leq Cb(1 + t^2)\alpha_n^{-5}h(t)$$

(B.89)

$$\left| \int_{|u| \leq 1} e^{2t|u|} H_{23}(u) du \right| \leq Cb(1 + t^2)(\alpha_n^{-5} + n\alpha_n^{-3})h(t)$$
By (B.88) and (B.89), we have we have

\[ \int_{|u| \leq 1} e^{2t|u|} H_2(u) du \leq C \gamma^2 \left( n^2 \alpha_n^{-2} + n \alpha_n^{-3} \right) \left( 1 + t^2 \right) \mathbb{E}[\Psi_{\beta,t}(W)]. \]

By (B.82) and (B.90), we complete the proof (B.10) for \( v = 0 \). The inequality (B.10) for the case \( v = 1 \) can be shown similarly.

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