On Regularity and Mass Concentration Phenomena for the Sign Uncertainty Principle

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Received: 24 March 2020 / Accepted: 5 September 2020 / Published online: 19 September 2020
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Abstract

The sign uncertainty principle of Bourgain et al. asserts that if a function \( f : \mathbb{R}^d \to \mathbb{R} \) and its Fourier transform \( \hat{f} \) are nonpositive at the origin and not identically zero, then they cannot both be nonnegative outside an arbitrarily small neighborhood of the origin. In this article, we establish some equivalent formulations of the sign uncertainty principle, and in particular prove that minimizing sequences exist within the Schwartz class when \( d = 1 \). We further address a complementary sign uncertainty principle, and show that corresponding near-minimizers concentrate a universal proportion of their positive mass near the origin in all dimensions.

Keywords  Sign uncertainty principle · Fourier transform · Schwartz class · Bandlimited function · Mass concentration

Mathematics Subject Classification  42A85 · 42B10 · 46A11

1 Introduction

Motivated by a problem in the theory of zeta functions over algebraic number fields, Bourgain et al. [1] investigated the class of functions \( \mathcal{A}_+(d) \), defined as follows. Given \( d \geq 1 \), a function \( f : \mathbb{R}^d \to \mathbb{R} \) is said to be eventually nonnegative if \( f(x) \geq 0 \) for
all sufficiently large $|x|$. Normalize the Fourier transform,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i \langle x, \xi \rangle} \, dx,$$

where $\langle \cdot, \cdot \rangle$ represents the usual inner product in $\mathbb{R}^d$. Let $A_+(d)$ denote the set of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ which satisfy the following conditions:

- $f \in L^1(\mathbb{R}^d)$, $\hat{f} \in L^1(\mathbb{R}^d)$, and $\hat{f}$ is real-valued (i.e., $f$ is even);
- $f$ is eventually nonnegative while $\hat{f}(0) \leq 0$;
- $\hat{f}$ is eventually nonnegative while $f(0) \leq 0$.

Note that any function $f \in A_+(d)$ is uniformly continuous. Consider the quantity

$$r(f) := \inf\{r > 0 : f(x) \geq 0 \text{ if } |x| \geq r\},$$

which corresponds to the radius of the last sign change of $f$. The product $r(f)r(\hat{f})$ is unchanged if we replace $f$ with $x \mapsto f(\lambda x)$ for some $\lambda > 0$, and thus becomes a natural object to consider. One of the initial observations in [1] is that the quantity

$$A_+(d) := \inf_{f \in A_+(d) \setminus \{0\}} \sqrt{r(f)r(\hat{f})}$$

is uniformly bounded from below away from zero. In fact, the following two-sided inequality is established in [1, §3]:

$$\frac{1}{\sqrt{2\pi e}} \leq \liminf_{d \to \infty} \frac{A_+(d)}{\sqrt{d}} \leq \limsup_{d \to \infty} \frac{A_+(d)}{\sqrt{d}} \leq \frac{1}{\sqrt{2\pi}}. \tag{1.3}$$

In particular, the radii $r(f), r(\hat{f})$ of the last sign change of $f, \hat{f}$, respectively, cannot both be made arbitrarily small, unless $f \in A_+(d)$ is identically zero. Consequently, the aforementioned results can be regarded as manifestations of a sign uncertainty principle.

The sign uncertainty principle of Bourgain et al. inspired a number of subsequent works [4,6,7]; see also [2,8]. Gonçalves et al. [7] proved that radial minimizers for (1.2) exist. More precisely, in each dimension $d \geq 1$, they showed that there exists a radial function $f \in A_+(d)$, satisfying $\hat{f} = f$, $f(0) = 0$, and $r(f) = A_+(d)$, and that such a minimizer must necessarily vanish at infinitely many radii greater than $A_+(d)$. The precise shape of minimizers remained a mystery in all dimensions $d \geq 1$, until Cohn and Gonçalves [4] exhibited an explicit minimizer in twelve dimensions. In particular, they relied on the following ingredients in order to show that $A_+(12) = \sqrt{2}$:

- A Poisson-type summation formula for radial Schwartz functions $f : \mathbb{R}^{12} \rightarrow \mathbb{C}$ based on the modular form $E_6$;
- An explicit construction via a remarkable integral transform discovered by Viazovska [16] which turns modular forms into radial eigenfunctions of the Fourier transform.
The first ingredient leads to the lower bound $A_+(12) \geq \sqrt{2}$. The second ingredient produces the explicit minimizer, and in particular leads to the upper bound $A_+(12) \leq \sqrt{2}$. Moreover, the minimizer is shown to belong to the Schwartz class $S(\mathbb{R}^{12})$, and to be a $+1$ eigenfunction of the Fourier transform; see [4, Fig. 1] for the corresponding plots. It then becomes natural to consider a complementary problem, associated to $-1$ eigenfunctions of the Fourier transform, which we now describe.

Let $A_-(d)$ denote the set of integrable functions $f: \mathbb{R}^d \to \mathbb{R}$, with integrable, real-valued Fourier transform $\hat{f}$, such that $\hat{f}(0) \leq 0$ while $f$ is eventually nonnegative, and $f(0) \geq 0$ while $-\hat{f}$ is eventually nonnegative. Define the quantity

$$A_-(d) := \inf_{f \in A_-(d) \setminus \{0\}} \sqrt{r(f)r(\hat{f})}. \quad (1.4)$$

Then [4, Theorem 1.4] guarantees that the chain of inequalities (1.3) still holds if $A_+(d)$ is replaced by $A_-(d)$. It further ensures the existence of radial minimizers for (1.4) which are $-1$ eigenfunctions of the Fourier transform, and necessarily have infinitely many zeros after the last sign change. Logan [12] has solved the optimization problem (1.4) in the one-dimensional case $d = 1$. Problem (1.4) turns out to be closely related to the sphere packing problem and, in light of the recent breakthroughs [5,16], it has also been solved in dimensions $d \in \{8, 24\}$; see [3, Prop. 7.1] and [4, §1]. In particular, $A_-(1) = 1$, $A_-(8) = \sqrt{2}$, and $A_-(24) = 2$. Moreover, if $d \in \{8, 24\}$, then minimizers for (1.4) belong to the Schwartz class, and modulo symmetries are unique within this class. However, no such refined regularity properties can be asserted a priori in any other dimension.

### 1.1 Main Results

In this article, we investigate regularity properties and mass concentration phenomena exhibited by minimizing sequences of (1.2) and (1.4).

We use the letter $s$ to denote a sign from $\{+,-\}$, and shall sometimes identify the signs $\{+,-\}$ with the integers $\{+1,-1\}$. A function $f: \mathbb{R}^d \to \mathbb{R}$ is said to be bandlimited if the support of its distributional Fourier transform $\hat{f}$ is compact, denoted $\text{supp}(\hat{f}) \subset \mathbb{R}^d$. For $s \in \{+,-\}$, define the quantities

$$A_s^B(d) := \inf_{f \in A_s(d) \setminus \{0\}, \text{supp}(\hat{f}) \subset \mathbb{R}^d} \sqrt{r(f)r(\hat{f})}; \quad A_s^S(d) := \inf_{f \in A_s(d) \cap S(\mathbb{R}^d) \setminus \{0\}} \sqrt{r(f)r(\hat{f})},$$

where the infima are taken over nonzero functions in $A_s(d)$ which are bandlimited and belong to the Schwartz space, $S(\mathbb{R}^d)$, respectively. It is then natural to wonder about the relationship between the sharp constants $A_s^B(d)$, $A_s^S(d)$, and $A_s^S(d)$.

The identities $A_-(8) = A_S^S(8)$, $A_-(24) = A_S^S(24)$, and $A_+(12) = A_S^S(12)$ are known, simply because the corresponding minimizers in $S(\mathbb{R}^d)$ have been explicitly constructed. If $(s,d) = (-,-,1)$, then uniqueness of minimizers (modulo symmetries) in $S(\mathbb{R})$ fails, but from knowledge of the corresponding minimizers one can likewise...
infer that $A_{-}(1) = A_{S}^{1}(1)$. According to [1, Théorème 3.2], the two-sided inequality

$$A_{s}(d) \leq A_{S}^{s}(d) \leq 2A_{s}(d)$$

(1.5)

holds when $s = +1$, and the argument presented there can be easily adapted to the case $s = -1$. As remarked in [1], it is not at all clear that the first inequality in (1.5) should be an equality.\(^1\) Our first main result settles this question for $(s, d) = (+, 1)$, where all three aforementioned sign uncertainty principles are seen to be equivalent.

**Theorem 1** $A_{+}(1) = A_{B}^{+}(1) = A_{S}^{+}(1)$.

For other combinations of signs and dimensions, barely anything is known about regularity properties of near-minimizers for $A_{s}(d)$, and the following conjecture remains open in its full generality.

**Conjecture 1** For any $s \in \{+, -\}$ and $d \geq 1$, it holds that $A_{s}(d) = A_{S}^{s}(d) = A_{S}^{s}(d)$.

The proof of Theorem 1 builds upon a few observations from [1]: Given $f \in A_{+}(d)$, there exists $x_{0} \in \mathbb{R}^{d}$ satisfying $|x_{0}| \leq r(f)$, such that $f(x_{0}) < 0$. Convolving with the sum of Dirac measures $\delta_{x_{0}} + \delta_{-x_{0}} + 2\delta_{0}$ yields a new function $g \in A_{+}(d)$, satisfying $g(0) < 0$ and $r(g) \leq 2r(f)$. This provides additional room for a further convolution with an appropriate smooth function, and ultimately enables selected minimizing sequences to be found within a smoother function space. To improve on this in the one-dimensional setting, we show that any minimizer of (1.2) is strictly negative on a punctured neighborhood of the origin; in particular, the point $x_{0}$ can be taken arbitrarily close to the origin if $d = 1$.

The previous paragraph and the explicit minimizer found in [4] together imply the existence of minimizers which are nonpositive in a neighborhood of the origin in dimensions $d \in \{1, 12\}$. In fact, the minimizer in twelve dimensions is nonpositive on the open ball $B_{\sqrt{2}}^{12} \subseteq \mathbb{R}^{12}$ centered at the origin of radius $\sqrt{2} = A_{+}(12)$, which makes it tempting to conjecture that such a property holds in arbitrary dimensions. The numerical examples from [4,6,7] provide further evidence for this possibility; see [6, Fig. 1] for the plot of a numerical approximation of a minimizer for $A_{+}(1)$.

An adaptation of the proof of Theorem 1 yields the following improvement over (1.5) in higher dimensions $d > 1$: There exist constants $\delta_{d} \in (0, 1)$, satisfying $\sqrt{d}(2\pi e)^{d/2}\delta_{d} \to 1$, as $d \to \infty$, such that

$$A_{S}^{s}(d) \leq (2 - \delta_{d})A_{s}(d).$$

(1.6)

In fact, if $d > 1$, then we are able to identify small, but not arbitrarily small, values of $|x_{0}|$ for which a given minimizer $f$ satisfies $f(x_{0}) < 0$. This is the main reason why our methods do not seem sufficient to establish Conjecture 1 for the Schwartz class if $(s, d) \neq (+, 1)$. For further details, see Sect. 2.3.

Even though our current techniques do not seem fit to establish any nontrivial equivalence for the $-1$ sign uncertainty principle, one might still hope to identify other

\(^{1}\) [1, p. 1218]: “Il n’est point évident que la borne $A$ définie par $A = \inf A(f)$, quand on impose de surcroît à $f$ d’appartenir à $\mathcal{S}$, coïncide avec celle définie pour $f$ parcourant $L^{1}$.”
regularity properties exhibited by near-minimizers. As previously mentioned, problem (1.4) has been solved if $d \in \{1, 8, 24\}$. Moreover, if $d \in \{8, 24\}$, then the minimizer is unique modulo symmetries, belongs to the Schwartz class, and is nonnegative in a neighborhood of the origin. If $d = 1$, then uniqueness fails, but the nonnegative property still holds in all known examples; see Fig. 1. In particular, one may expect every suitable function which is sufficiently close to a minimizer to concentrate a universal proportion of its positive mass on the smallest ball centered at the origin that contains all of its negative mass. Our second main result confirms these heuristics in all dimensions. Before stating it, recall (as shown in [1,4]) that we can restrict attention to radial functions. More precisely, the quantity $A_s(d)$ coincides with the minimal value of $r(g)$ in the following optimization problem.

**Fourier Eigenvalue Linear Programming Problem (s-FELPP)**

Let $s \in \{+, −\}$. Minimize $r(g)$ over all radial $g : \mathbb{R}^d \to \mathbb{R}$ such that

- $g \in L^1(\mathbb{R}^d) \setminus \{0\}$ and $\hat{g} = sg$;
- $g(0) = 0$ and $g$ is eventually nonnegative.

Given $r > 0$, let $B_r^d \subseteq \mathbb{R}^d$ denote the open ball of radius $r$ centered at the origin, and $g_+ := \max\{g, 0\}$.

**Theorem 2** Given $d \geq 1$, there exist constants $\varepsilon_d, \sigma_d > 0$, such that

$$\int_{B^d_{r(g)}} g_+ \geq \sigma_d \|g\|_{L^1(\mathbb{R}^d)},$$

whenever $g \in A_-(d)$ is a radial function such that $\hat{g} = -g$, $g(0) = 0$, and satisfies $|r(g) - A_-(d)| \leq \varepsilon_d$. 

Fig. 1 Plot of $g := \hat{f} - f$, where $f(x) = (1 - |x|)_+$, $\hat{f}(\xi) = \frac{\sin^2(\pi\xi)}{(\pi \xi)^2}$. The function $g$ is a minimizer of the $-1$-FELPP when $d = 1$. 

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We conclude the Introduction with a remark connecting bandlimited functions, Poisson summation, and reconstruction formulae. According to Theorem 1, \( A_+^B(1) = A_+^B(1) \). From this and the Poisson summation formula, a nontrivial conclusion can be withdrawn as in [11,12]. We record it in the following result, as it may prove useful in further investigations surrounding the Fourier Eigenvalue Linear Programming Problem.

**Proposition 1** Let \( f \in A_+(1) \) be a nonzero, bandlimited function, such that \( \text{supp}(\hat{f}) \subseteq [-\frac{1}{2}, \frac{1}{2}] \). Then \( r(f) \geq 1 \). Moreover, \( r(f) = 1 \) if and only if there exists \( \alpha > 0 \), such that

\[
\hat{f}(x) = \alpha \frac{\sin^2(\pi x - 1)}{x^2 - 1}.
\]

**1.2 Outline**

We prove Theorem 1 in Sect. 2, Theorem 2 in Sect. 3, and Proposition 1 in Sect. 4. The arguments rely on several lemmata which we choose to formulate in general dimensions whenever possible, and prove in Sect. 5. We dedicate the final Sect. 6 to a connection with the sign uncertainty principle for Fourier series on the torus, recently established in [6].

**1.3 Notation**

Let \( \mathbb{N} := \{1, 2, \ldots\} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Given \( f : \mathbb{R}^d \to \mathbb{R} \), let \( f_+ := \max\{f, 0\} \), and \( f_- := \max\{-f, 0\} \). In this way, \( f_+ \) and \( f_- \) are nonnegative functions which are never positive at the same point, and satisfy \( f = f_+ - f_- \) and \( |f| = f_+ + f_- \). Given a set \( E \subseteq \mathbb{R}^d \), its indicator function is denoted by \( \mathbb{1}_E \), and its Lebesgue measure by \( |E| \). Given \( r > 0 \), we continue to let \( B_r^d \subseteq \mathbb{R}^d \) denote the open ball of radius \( r \) centered at the origin.

**2 Proof of Theorem 1**

**2.1 Proof of \( A_+(1) = A_+^B(1) \)**

It suffices to show that \( A_+(1) \geq A_+^B(1) \), and we proceed in four steps.  

**Step 1.** Let \( f \in A_+(1) \) be a minimizer for (1.2) satisfying \( \hat{f} = f \), \( f(0) = 0 \). Then there exists \( \varepsilon > 0 \), such that \( f(x) < 0 \), for every \( x \in (-\varepsilon, \varepsilon) \setminus \{0\} \).

The proof of Step 1 hinges on two distinct observations, the first of which is inspired by work of Logan [11,12] on various extremal problems concerning the behavior of positive-definite bandlimited functions. The following result holds for any sign \( s \in \{+, -\} \) in arbitrary dimensions \( d \geq 1 \), and should be compared with [11, Lemma], where a one-dimensional variant of the same result is proved.
Lemma 1 Given $s \in \{+, -\}$, $d \geq 1$, let $f \in A_s(d)$ be radial, such that $\widehat{f} = sf$, $f(0) = 0$. Suppose that there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^d \setminus \{0\}$, such that $x_n \to 0$, as $n \to \infty$, and $sf(x_n) \geq 0$, for all $n$. Then $\int_{\mathbb{R}^d} |y|^2 |f(y)| \, dy < +\infty$, and

$$\int_{\mathbb{R}^d} |y|^2 f(y) \, dy \leq 0. \quad (2.1)$$

The second observation is that a chain of rearrangement-type inequalities can be set up in such a way as to contradict estimate (2.1). A similar approach already proved fruitful in [7, §4]. To implement it in the present context, let $f \in A_+(1)$ be a minimizer for (1.2), which we suppose to be $L^1$-normalized, $\|f\|_{L^1} = 1$, and to satisfy $\widehat{f} = f$, $f(0) = 0$; see [7, §3]. If $f$ fails to be strictly negative on any punctured neighborhood of the origin, then the hypotheses of Lemma 1 for $s = +1$ are verified. Writing $f = f_+ - f_-$, we then have that

$$\int_0^\infty y^2 f(y) \, dy = \int_0^{r(f)} y^2 f_+(y) \, dy - \int_0^{r(f)} y^2 f_-(y) \, dy + \int_{r(f)}^\infty y^2 f(y) \, dy \leq 0. \quad (2.2)$$

Setting $\sigma(f) := \|f_+\|_{L^1(0, r(f))}$, and appealing to [7, Lemma 12], we can bound each of the integrals on the right-hand side of (2.2) as follows:

$$\int_0^{r(f)} y^2 f_+(y) \, dy \geq \int_0^{\sigma(f)} y^2 \, dy; \quad (2.3)$$

$$\int_0^{r(f)} y^2 f_-(y) \, dy \leq \int_{r(f) - \frac{1}{4}}^{r(f)} y^2 \, dy; \quad (2.4)$$

$$\int_{r(f)}^\infty y^2 f(y) \, dy \geq \int_{r(f)}^{r(f) + \frac{1}{4} - \sigma(f)} y^2 \, dy. \quad (2.5)$$

To see why this is the case, note that $\|f\|_{L^\infty} \leq 1$, and that

$$\int_{\mathbb{R}} f_+ + \int_{\mathbb{R}} f_- = \int_{\mathbb{R}} |f| = 1;$$

$$\int_{\mathbb{R}} f_+ - \int_{\mathbb{R}} f_- = \int_{\mathbb{R}} f = \widehat{f}(0) = 0,$$

and thus $\int_{\mathbb{R}} f_+ = \int_{\mathbb{R}} f_- = \frac{1}{2}$, moreover, the functions $f$, $f_\pm$ are even, and so their masses are equally spread over the positive and negative half-lines. Estimates (2.2)–(2.5) immediately imply that

$$\frac{\sigma(f)^3}{3} - \frac{r(f)^3 - (r(f) - \frac{1}{4})^3}{3} + \frac{(r(f) + \frac{1}{4} - \sigma(f))^3 - \sigma(f)^3}{3} \leq 0,$$

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which can be equivalently rewritten as

$$\left( r(f) + \frac{1}{4} \right) \sigma(f) \left( r(f) + \frac{1}{4} - \sigma(f) \right) \geq \frac{r(f)}{8}. \quad (2.6)$$

We seek for a sufficiently good upper bound for \( \sigma(f) \) which will then force the desired contradiction. With this purpose in mind, observe that the left-hand side of (2.6) defines an increasing function of \( \sigma(f) \), provided \( \sigma(f) < \frac{1}{4} \). The next result provides a slight improvement over [7, Lemma 14] in terms of the admissible radius \( r(f) \).

**Lemma 2** Let \( f \in \mathcal{A}_+(1) \) be such that \( \| f \|_{L^1} = 1, \hat{f} = f, f(0) = 0, \) and \( r(f) \in \left[ \frac{1}{4}, \frac{1}{\sqrt{2}} \right] \). Then the following inequality holds:

$$f_+(x) \leq \frac{1}{2} + \frac{\sin(2\pi (r(f) - \frac{1}{4})x) - \sin(2\pi r(f)x)}{\pi x}, \quad (2.7)$$

for every \( x \in [0, r(f)] \).

With the two observations in place, we may now finish the proof of Step 1. Since \( \| f \|_{L^\infty} \leq \| \hat{f} \|_{L^1} = \| f \|_{L^1} = 1 \) and \( \| f_- \|_{L^1(0,r(f))} = \frac{1}{4} \), the following superlevel set estimate holds:

$$|\{ x \in [0, r(f)] : f(x) \geq 0 \}| \leq r(f) - \frac{1}{4}. \quad (2.8)$$

Since \( f \) is a minimizer for (1.2), and \( \hat{f} = f \), it follows from [7, Theorem 2] that \( 0.45 \leq r(f) \leq 0.595 \); in particular, \( r(f) \in \left[ \frac{1}{4}, \frac{1}{\sqrt{2}} \right] \). As a consequence, we may appeal to (2.8) in order to estimate

$$\sigma(f) = \int_{[0, r(f)]} f_+ \leq \sup_{J \subseteq [0,r(f)]} \int_{|J|=r(f)-\frac{1}{4}} f_+ \leq \int_{\frac{1}{4}}^{r(f)} \left( \frac{1}{2} + \frac{\sin(2\pi (r(f) - \frac{1}{4})x) - \sin(2\pi r(f)x)}{\pi x} \right) dx := \Phi(r(f)), \quad (2.9)$$

where from the first to the second line we invoked Lemma 2 and [7, Lemma 11], using the fact that the integrand on the right-hand side defines an increasing function of \( x \). We are thus reduced to a straightforward analysis of the function \( \Phi \). It is easy to check that \( \Phi(r(f)) \leq \Phi(0.595) < 0.121 \), and therefore the chain of inequalities (2.9) implies \( \sigma(f) < 0.121 \). But if \( \sigma(f) < 0.121 \) and \( 0.45 \leq r(f) \leq 0.595 \), then (2.6) does not hold. This yields the desired contradiction, and concludes the proof of Step 1.

**Step 2.** There exists a minimizing sequence \( \{ f_n \}_{n \in \mathbb{N}} \subseteq \mathcal{A}_+(1) \) for (1.2), such that the following conditions hold, for every \( n \):

(a) \( f_n(0) < 0 \);
(b) \( f_n(x) > 0 \), for every \( |x| > r(f_n) \);
(c) \( \hat{f}_n = f_n \);
(d) \( x^2 f_n(x) \geq c_n \), for some \( c_n > 0 \) and all sufficiently large \( |x| \).

Let \( f \in \mathcal{A}_+(1) \) be a minimizer for +1-FELPP. By Step 1, there must exist a sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq (0, \infty) \) such that \( x_n \to 0 \), as \( n \to \infty \), and \( f(x_n) < 0 \), for every \( n \). Define an associated sequence \( \{T_n\}_{n \in \mathbb{N}} \) of tempered distributions via

\[
T_n := \delta_{x_n} + \delta_{-x_n} + 2\delta_0.
\]

Setting \( g_n := T_n \ast f \), one easily checks that \( \{g_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}_+(1) \) is a minimizing sequence for (1.2). Indeed, the quantity \( g_n(x) = f(x - x_n) + f(x + x_n) + 2f(x) \)

is nonnegative if \( x \geq r(f) + x_n \), and satisfies \( g_n(0) = 2f(x_n) < 0 \) (in particular, \( g_n \) does not vanish identically). On the other hand,

\[
\hat{T}_n(\xi) = 2\cos(2\pi x_n \xi) + 2 \geq 0,
\]

and so \( \hat{g}_n(0) = 4\hat{f}(0) \leq 0 \). By the scaling argument detailed in [7, §3.3], we lose no generality in assuming that \( r(\hat{g}_n) = r(\hat{f}_n) \). In this case, \( \{g_n + \hat{g}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}_+(1) \) is a minimizing sequence for (1.2) satisfying conditions (a) and (c). Condition (b) can be achieved by further adding a suitable Gaussian function: Setting

\[
h_n := g_n + \hat{g}_n - \frac{g_n(0) + \hat{g}_n(0)}{2} \exp(-\pi \cdot 2),
\]

one again checks that \( \{h_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}_+(1) \) is a minimizing sequence for (1.2) which satisfies conditions (a)–(c). In order to further ensure condition (d), we make use of the following simple observation.

**Lemma 3** Given \( d \geq 1 \), there exists a function \( \eta \in \mathcal{A}_+(d) \), such that \( \hat{\eta} = \eta \), \( \eta(0) < 0 \), and \( |x|^{d+1} \eta(x) \geq 1 \), for all sufficiently large values of \( |x| \).

Let \( \eta \in \mathcal{A}_+(1) \) be given by Lemma 3, and pick the smallest \( r_0 > 0 \) such that \( x^2 \eta(x) \geq 1 \), for every \( |x| \geq r_0 \). Given \( n \in \mathbb{N} \), set \( \beta_n = 1 \) whenever \( r(h_n) \geq r_0 \). Otherwise, given \( \delta > 0 \) which is sufficiently small so that \( r(h_n) + \frac{\delta}{n} < r_0 \), set \( \beta_n = \beta_n(\delta) > 0 \) in such a way that

\[
h_n(x) + \beta_n \eta(x) > 0, \text{ for every } x \in \left[r(h_n) + \frac{\delta}{n}, r_0\right].
\]

That such a choice of \( \beta_n \) is possible follows from the fact that each function \( h_n \) is eventually (strictly) positive. Then the sequence \( \{f_n := h_n + \beta_n \eta\}_{n \in \mathbb{N}} \subseteq \mathcal{A}_+(1) \) is minimizing for (1.2), and satisfies properties (a), (c), and (d). Letting \( \delta \to 0^+ \), we ensure that condition (b) is fulfilled as well. This concludes the verification of Step 2.

**Step 3.** Let \( f \in \mathcal{A}_+(1) \) be such that \( \|f\|_{L_1} = 1 \), \( \hat{f} = f \), \( f(0) < 0 \). Suppose that there exist constants \( c, R > 0 \), such that
(i) $f(x) > 0$, for every $|x| > r(f)$;
(ii) $x^2 f(x) \geq c$, for every $|x| \geq R$.

Then, given $\varepsilon > 0$, there exists a bandlimited function $g \in A_+(1)$, such that $g(x) > 0$, for every $|x| \geq r(f) + \varepsilon$.

Fix a nonnegative, even, compactly supported, smooth function $\psi \in C^\infty_0(\mathbb{R})$, such that $\|\psi\|_{L^2} = 1$. Set $\varphi := \psi \ast \psi$ and $\varphi_\delta(x) := \varphi(\delta x)$. As $\delta \to 0^+$, the family $\{\varphi_\delta\}_{\delta > 0}$ constitutes an approximation to the identity. Therefore the bandlimited function $g_\delta := f \ast \varphi_\delta$ should provide a good approximation for $f$, for small enough values of $\delta$. We turn to the details.

Let $f, c, R$ be as above, and let $\varepsilon > 0$ be arbitrary but given. We show that $\delta = \delta(f, c, R, \varepsilon)$ can be chosen sufficiently small, so that $g = g_\delta$ belongs to $A_+(1)$, and satisfies $g(x) > 0$, for every $|x| \geq r(f) + \varepsilon$. Let $R_1 \geq R$ be such that

$$|\hat{\varphi}(\xi)| \leq 10^{-6} c|\xi|^{-3}, \text{ for every } |\xi| \geq R_1. \quad (2.10)$$

This is certainly possible since $\hat{\varphi}$ is rapidly decreasing. By letting $\delta \to 0^+$, the difference $f - g_\delta$ can be made uniformly close to 0 in the interval $[r(f) + \varepsilon, 2R_1]$. Thus it suffices to consider $|x| > 2R_1$, in which case the following chain of inequalities holds, as long as $\delta > 0$ is sufficiently small:

$$g_\delta(x) = \int_{-\infty}^{\infty} f(x - \xi) \hat{\varphi}_\delta(\xi) \, d\xi \geq \int_{-\delta^{1/2}}^{\delta^{1/2}} f(x - \xi) \hat{\varphi}_\delta(\xi) \, d\xi - \int_{x-r(f)}^{x+r(f)} \hat{\varphi}_\delta(\xi) \, d\xi$$

$$\geq \frac{c}{4\delta^2} \left[ \int_{-\delta^{1/2}}^{\delta^{1/2}} \hat{\varphi}_\delta(\xi) \, d\xi - \int_{x-r(f)}^{x+r(f)} \hat{\varphi}_\delta(\xi) \, d\xi \right] \geq \frac{c}{8\delta^2} - \int_{x-r(f)}^{x+r(f)} \hat{\varphi}_\delta(\xi) \, d\xi. \quad (2.11)$$

In the first inequality, we used hypothesis (i) to ensure that $f(x - \xi) \geq 0$ unless $\xi \in [x - r(f), x + r(f)]$, and the fact that $\|f\|_{L^\infty} \leq 1$; the second inequality holds for sufficiently small $\delta > 0$, in view of hypothesis (ii); and the third inequality holds for sufficiently small $\delta > 0$ since $\{\hat{\varphi}_\delta\}_{\delta > 0}$ is an approximation to the identity. Invoking (2.10), we have that

$$0 \leq \hat{\varphi}_\delta(\xi) = \delta^{-1} \hat{\varphi}(\delta^{-1} \xi) \leq \frac{1}{\delta} \frac{c}{10^6 |\delta^{-1} \xi|^3} = \frac{c\delta^2}{10^6 |\xi|^3},$$

for every $|\xi| \geq \delta R_1$. Since $|x| > 2R_1$, it follows that $|x - r(f)| \geq \delta R_1$, and therefore the last integral on the right-hand side of (2.11) can be estimated as follows:

$$\int_{x-r(f)}^{x+r(f)} \hat{\varphi}_\delta(\xi) \, d\xi \leq \frac{c\delta^2}{10^6} \int_{x-r(f)}^{x+r(f)} \frac{d\xi}{|\xi|^3} \leq \frac{c/2}{10^6} \frac{\delta^2}{(x - r(f))^2} \leq \frac{2c}{10^6} \frac{\delta^2}{x^2} \leq \frac{c}{10^4 x^2}, \quad (2.12)$$

provided $2\delta^2 < 10^2$. Estimates (2.11), (2.12) together imply that $g_\delta(x) > 0$, for every $|x| > 2R_1$, as long as $\delta > 0$ is sufficiently small. Finally, one easily checks that $g_\delta \in A_+(1)$, provided $\delta > 0$ is sufficiently small. For instance, $g_\delta(0) \leq 0$ since $f(0) < 0$ (the strict inequality is crucial here) and $f$ is continuous. Step 3 follows.
Let $d$ and define the corresponding optimal constants $A_{+}(1)$ for (1.2), satisfying $\|f_n\|_1 = 1$, $f_n(0) < 0$, $f_n(x) > 0$ if $|x| \geq r(f_n)$, $\tilde{f}_n = f_n$, and $x^2 f_n(x) \geq c_n$ if $|x|$ is sufficiently large. The existence of such a sequence follows from Step 2. Running the proof of Step 3 on each $f_n$ individually, we find that there exists a bandlimited function $g_n = f_n * \tilde{\phi}_\delta \in A_{+}(1)$, such that $r(g_n) \leq r(f_n) + \frac{1}{n}$. Since $\widehat{g_n} = \widehat{f_n} \psi_\delta$ has pointwise the same sign as $\widehat{f_n}$, we may let $n \to \infty$ and conclude that $\{g_n\}_{n \in \mathbb{N}} \subseteq A_{+}(1)$ is a minimizing sequence for (1.2) consisting of bandlimited functions, as desired.

2.2 Proof of $A_{+}(1) = A_{+}^S(1)$

Following [1, §1], consider the restricted classes

\[
\tilde{A}_{+}(d) := \{ f \in A_{+}(d) : \hat{f} = f, f(0) < 0 \}; \\
\tilde{A}_{+}^S(d) := \{ f \in A_{+}(d) \cap S(\mathbb{R}^d) : \hat{f} = f, f(0) < 0 \},
\]

and define the corresponding optimal constants

\[
\tilde{A}_{+}(d) := \inf_{f \in \tilde{A}_{+}(d) \setminus \{0\}} \sqrt{r(f) r(\hat{f})}; \\
\tilde{A}_{+}^S(d) := \inf_{f \in \tilde{A}_{+}^S(d) \setminus \{0\}} \sqrt{r(f) r(\hat{f})}.
\]

Our next result reveals that these constants coincide in all dimensions.

**Proposition 2** Let $d \geq 1$. Then $\tilde{A}_{+}(d) = \tilde{A}_{+}^S(d)$.

**Proof of Proposition 2** It suffices to show that $\tilde{A}_{+}(d) \geq \tilde{A}_{+}^S(d)$. With this goal in mind, let $f \in \tilde{A}_{+}(d)$. Given $\delta > 0$, consider a nonnegative, radial, compactly supported, smooth function $\psi_\delta \in C_0^\infty(\mathbb{R}^d)$, such that $\text{supp}(\psi_\delta) \subseteq B^d_\delta$; further assume $\psi_\delta$ to be $L^1$-normalized, $\|\psi_\delta\|_{L^1} = 1$. Define $\varphi_\delta := \psi_\delta * \psi_\delta$, $g := f * \varphi_\delta$, and

\[
h := \widehat{g} * \varphi_\delta + g \tilde{\varphi}_\delta.
\]  

(2.13)

The following lemma lists the key properties of the function $h$.

**Lemma 4** Let $f \in \tilde{A}_{+}(d)$. Given $\varepsilon > 0$, there exists $\delta > 0$, such that the function $h$ defined in (2.13) above satisfies the following properties:

(a) $h \in S(\mathbb{R}^d)$, $\hat{h} = h$, $h(0) < 0$;
(b) $r(h) \leq r(f) + \varepsilon$.

Part (a) of Lemma 4 implies that $h \in \tilde{A}_{+}^S(d)$, and the inequality $\tilde{A}_{+}(d) \geq \tilde{A}_{+}^S(d)$ then follows from part (b) of Lemma 4 by letting $\varepsilon \to 0^+$.

The desired identity, $A_{+}(1) = A_{+}^S(1)$, follows from the chain of inequalities

\[
A_{+}(1) \leq \tilde{A}_{+}^S(1) \leq \tilde{A}_{+}^S(1) \leq A_{+}(1) \leq A_{+}(1).
\]  

(2.14)

The first two inequalities in (2.14) are trivial, and the third one follows from Proposition 2. Our next result addresses the last inequality in (2.14).
Proposition 3 \( \tilde{A}_+(1) \leq A_+(1) \).

While the argument in Proposition 2 could be adapted to handle the present case as well, we offer a different proof which does not hinge on Lemma 4 and could prove of independent interest.

**Proof of Proposition 3** Let \( f \in A_+(1) \) be a minimizer for (1.2) satisfying \( \tilde{f} = f \), \( f(0) = 0 \); in particular, \( r(f) = \tilde{A}_+(1) \). Given \( \varepsilon > 0 \), we can invoke Step 1 of Sect. 2.1 in order to ensure that there exists \( x_0 \in (-\varepsilon, \varepsilon) \), such that \( f(x_0) < 0 \). Similarly to Step 2 of Sect. 2.1, consider the measure \( T := \delta_{x_0} + \delta_{-x_0} + 2\delta_0 \), which is positive-definite in the sense that

\[
\tilde{T}(\xi) = 2\cos(2\pi x_0 \xi) + 2 \geq 0, \quad (2.15)
\]

for every \( \xi \), and define \( g := T * f \). Since

\[
g(x) = f(x - x_0) + f(x + x_0) + 2f(x), \quad (2.16)
\]

it follows that \( g \) is real-valued, integrable, and even. Moreover, \( g(0) = 2f(x_0) < 0 \). Since \( |x_0| \leq \varepsilon \), it follows from (2.16) that

\[
g(x) \geq 0, \text{ if } |x| \geq r(f) + \varepsilon. \quad (2.17)
\]

In light of (2.15), we have that \( \tilde{g}(\xi) = \tilde{T}(\xi)\tilde{f}(\xi) \geq 0 \) if and only if \( \tilde{f}(\xi) \geq 0 \). In particular,

\[
\tilde{g}(\xi) \geq 0, \text{ if } |\xi| \geq r(f). \quad (2.18)
\]

Moreover, \( \tilde{g}(0) = 4f(0) = 0 \). Consider the dilation \( h := g(\cdot) \), where \( \lambda := (1 + \varepsilon/r(f))^{1/2} \). Then \( h(0) = g(0) < 0 \), \( \tilde{h}(0) = \lambda^{-1} \tilde{g}(0) = 0 \), and the functions \( h, \tilde{h} \) are nonnegative on the interval \( ([r(f)^2 + \varepsilon r(f)]^{1/2}, \infty) \). Real-valuedness, integrability, and evenness of \( h, \tilde{h} \) follow from those of \( g \). Therefore \( h + \tilde{h} \in \tilde{A}_+(1) \), whence \( \tilde{A}_+(1) \leq (r(f)^2 + \varepsilon r(f))^{1/2} \). Letting \( \varepsilon \to 0^+ \), it follows that \( \tilde{A}_+(1) \leq r(f) = \tilde{A}_+(1) \). \( \square \)

**2.3 A Short Proof of the Higher Dimensional Improvement (1.6)**

Let \( f \in A_+(d) \) be a nonzero function, for which the product \( r(f)r(\tilde{f}) \) is sufficiently close to \( \tilde{A}_+(d)^2 \). By the usual reductions, we may assume that the function \( f \) is radial, and satisfies \( f = \tilde{f} \), \( f(0) = 0 \), and \( \|f\|_{L^1} = 1 \). In particular,

\[
|[x \in B^d_{r(f)}: f(x) < 0]| \geq \int_{B^d_{r(f)}} f_- = \frac{1}{2}.
\]

This implies the following superlevel set estimate:

\[
|[x \in B^d_{r(f)}: f(x) \geq 0]| \leq v_d r(f)^d - \frac{1}{2}, \quad (2.19)
\]
where, as usual, \( \nu_d = 2d^{-1} \pi^{\frac{d}{2}} \Gamma(\frac{d}{2})^{-1} \) is the Lebesgue measure of the unit ball in \( \mathbb{R}^d \).

Suppose that \( r > 0 \) is such that \( \tilde{f}(x) \geq 0 \), for every \( x \in B^d_r \). Trivially, \( r < r(f) \).

In order to improve on this, we simply note that (2.19) implies \( r^d \leq r(f)^d - (2\nu_d)^{-1} \).

Thus there exists \( x_0 \in \mathbb{R}^d \), satisfying

\[
|x_0| \leq \left( r(f)^d - (2\nu_d)^{-1} \right)^{\frac{1}{d}}, \text{ and } f(x_0) < 0.
\]

The rest of the argument follows very similar ideas to those in Sect. 2.2. In particular, we obtain the following inequality:

\[
\hat{A}^S_+(d) \leq \left( 1 + \left( 1 - \frac{1}{2\nu_d \hat{A}_+(d)^d} \right)^{\frac{1}{d}} \right) \hat{A}_+(d).
\]  

(2.20)

Setting \( \theta_d := (2\nu_d)^{-1} \hat{A}_+(d)^{-d} \), we estimate

\[
1 - (1 - \theta_d)^{\frac{1}{d}} = \frac{1}{d} \int_{1-\theta_d}^1 t^{\frac{1}{d} - 1} \, dt \geq \frac{\theta_d}{d}.
\]

Stirling’s formula and (1.3) together imply that \( \theta_d \geq C \sqrt{d} (2\pi e)^{-d/2} \), for some absolute constant \( C > 0 \). Plugging this back into (2.20) yields (1.6).

### 3 Proof of Theorem 2

The proof of Theorem 2 proceeds by contradiction. We start by establishing the following claim. If Theorem 2 does not hold, then there exists a radial minimizer \( f \in \hat{A}_-(d) \) of (1.4), satisfying \( \hat{f} = -f \), \( f(0) = 0 \), such that \( f(x) \leq 0 \) for all \( x \in B^d_0 \).

To see why this is necessarily the case, start by observing that, if Theorem 2 does not hold, then there exists a minimizing sequence \( \{ f_n \}_{n \in \mathbb{N}} \subseteq \hat{A}_-(d) \) for (1.4) consisting of radial functions, satisfying \( \hat{f}_n = -f_n \), \( f_n(0) = 0 \), \( \| f_n \|_{L^1} = 1 \), \( |r(f_n) - \hat{A}_-(d)| \leq \frac{1}{n} \), and

\[
\int_{B^d_r(f_n)} (f_n)_+ \leq \frac{1}{n}.
\]

(3.1)

No generality is lost in assuming that the sequence \( \{ r(f_n) \}_{n \in \mathbb{N}} \) is strictly decreasing. By Jaming’s higher dimensional version of Nazarov’s uncertainty principle [9], there exists a constant \( K_d > 0 \) such that, for every \( n \in \mathbb{N} \),

\[
\int_{B^d_r(f_n)} f_n \leq -K_d.
\]

(3.2)

In fact, this amounts to a straightforward modification of [7, Lemma 23], as was already observed in [4, §3.2]. Since \( \hat{f}_n = -f_n \) and \( \| f_n \|_{L^1} = 1 \), it follows that

\[
\| f_n \|^2_{L^2} \leq \| f_n \|_{L^1} \| f_n \|_{L^{\infty}} \leq \| f_n \|_{L^1} \| \hat{f}_n \|_{L^1} = \| f_n \|^2_{L^1} = 1.
\]
The Banach–Alaoglu Theorem then implies, possibly after extraction of a subsequence, that the sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges weakly to some function \( f \in L^2 \). Since \( \mathcal{A}_-(d) \) is convex, we can then apply Mazur’s Lemma to produce a sequence \( \{g_n\}_{n \in \mathbb{N}} \) which converges strongly to \( f \) in \( L^2 \), with each \( g_n \) being a finite convex combination of elements from \( \{f_m\}_{m \geq n} \). For each \( n \in \mathbb{N} \), \( g_n \) is radial, \( \hat{g}_n = -g_n \), \( g_n(0) = 0 \), \( \|g_n\|_{L^2} \leq 1 \), and, possibly after passing to a further subsequence, the following additional properties hold:

\[
|r(g_n) - \mathcal{A}_-(d)| \leq \frac{1}{n}; \tag{3.3}
\]
\[
\int_{B^d_{r(g_n)}} (g_n)^+ \leq \frac{1}{n}; \tag{3.4}
\]
\[
\int_{B^d_{r(g_n)}} g_n \leq -\frac{K_d}{2}; \tag{3.5}
\]
\[
g_n \to f \text{ almost everywhere, as } n \to \infty. \tag{3.6}
\]

Property (3.3) follows from the fact that the sequence \( \{r(f_n)\}_{n \in \mathbb{N}} \) is decreasing, together with each \( g_n \) being a convex combination of elements from the tail sequence \( \{f_m\}_{m \geq n} \). Properties (3.4), (3.5) follow from the latter observation, together with the elementary inequality \( (x + y)^+ \leq x^+ + y^+ \), valid for every \( x, y \in \mathbb{R} \), and estimates (3.1), (3.2). Property (3.6) follows at once by extracting a further subsequence. In particular, from (3.5) it follows that

\[
\int_{B^d_{r(f)}} f \leq -\frac{K_d}{4},
\]

and therefore \( f \) does not vanish identically. Moreover, (3.4) implies that

\[
\int_{B^d_{r(f)}} f^+ = 0,
\]

and hence \( f \leq 0 \) in \( B^d_{r(f)} \). To conclude the proof of the claim, we still have to check that \( f \) satisfies all required properties. From (3.3) it follows that \( r(f) = \mathcal{A}_-(d) \). From (3.6) it follows that \( f \) is radial. That \( \hat{f} = -f \) follows at once from weak convergence. Setting \( r_1 := r(f_1) = \sup_{n \in \mathbb{N}} r(f_n) \), we may apply Fatou’s Lemma to the sequence \( \{g_n + \mathbb{1}_{B_{r_1}^d}\}_{n \in \mathbb{N}} \) (consisting of nonnegative functions which converge almost everywhere to \( f + \mathbb{1}_{B_{r_1}^d} \)) and deduce that \( f \in L^1 \), and that

\[
\hat{f}(0) = \int_{\mathbb{R}^d} f \leq \liminf_{n \to \infty} \int_{\mathbb{R}^d} g_n = 0.
\]

In particular, \( f(0) = -\hat{f}(0) \geq 0 \). Since \( f \leq 0 \) in \( B^d_{r(f)} \), it follows that \( f(0) = 0 \). This concludes the verification of the claim.
We will need a higher dimensional version of the rearrangement inequalities used in Sect. 2.1. The following elementary result from [10, §1.14] suffices for our application.

**Lemma 5** (Bathtub Principle, [10]) Let \( h : \mathbb{R}^d \to \mathbb{R} \) be a measurable function, such that \( |\{x \in \mathbb{R}^d : h(x) < t\}| \) is finite for all \( t \in \mathbb{R} \). Let the number \( G > 0 \) be given, and define a class of measurable functions on \( \mathbb{R}^d \) by

\[
C_G = \left\{ g : \mathbb{R}^d \to [0, 1] : \int_{\mathbb{R}^d} g = G \right\}.
\]

Then the minimization problem

\[
I = \inf_{g \in C_G} \int_{\mathbb{R}^d} gh
\]

is solved by \( g(x) = \mathbb{1}_{[h < s]}(x) + c \mathbb{1}_{[h = s]}(x) \), where

\[
s = \sup\{t \in \mathbb{R} : |\{x \in \mathbb{R}^d : h(x) < t\}| \leq G\} ; \quad c|x \in \mathbb{R}^d : h(x) = s| = G - |\{x \in \mathbb{R}^d : h(x) < s\}|.
\]

The minimizer is unique if \( G = |\{x \in \mathbb{R}^d : h(x) < s\}| \) or if \( G = |\{x \in \mathbb{R}^d : h(x) \leq s\}| \).

If Theorem 2 does not hold, then we have already verified the existence of a radial minimizer \( f \in A_-(d) \) of (1.4), satisfying \( \hat{f} = -f \), \( f(0) = 0 \), such that \( f \leq 0 \) on \( B^d_{r(f)} \). Normalizing \( \|f\|_{L^1} = 1 \), so that \( \|f\|_{L^\infty} \leq 1 \), we then have that

\[
-\int_{B^d_{r(f)}} f = \int_{\mathbb{R}^d \setminus B^d_{r(f)}} f = \frac{1}{2} . \tag{3.8}
\]

Invoking Lemma 1 with \( s = -1 \), and then Lemma 5 with \( h(y) = |y|^2 \), we have that

\[
0 \geq \int_{\mathbb{R}^d} |y|^2 f(y) \, dy = \int_{\mathbb{R}^d \setminus B^d_{r(f)}} |y|^2 f(y) \, dy + \int_{B^d_{r(f)}} |y|^2 f(y) \, dy \geq \int_{B^d_{r(f)}} |y|^2 \, dy - \int_{B^d_{r(f)} \setminus B^d_{2r(f)}} |y|^2 \, dy = \frac{v_d}{1 + 2/d} (t^{d+2} + s^{d+2} - 2r(f)^{d+2}) . \tag{3.11}
\]

Here, \( v_d := 2d-1 \pi^{d/2} \Gamma^{d/2} -1 \) denotes the Lebesgue measure of the unit ball \( B^d_1 \subseteq \mathbb{R}^d \) and, in light of (3.8), the radii \( s < t \) in (3.10) are defined in such a way as to ensure that \( |B^d_{r(f)} \setminus B^d_s| = |B^d_1 \setminus B^d_{r(f)}| = \frac{1}{2} \). Equivalently,

\[
s = \left( r(f)^d - \frac{1}{2v_d} \right)^{1/d} , \quad t = \left( r(f)^d + \frac{1}{2v_d} \right)^{1/d} .
\]

\( \square \) Springer
The strict convexity of the function \( x \mapsto |x|^{1+\frac{2}{d}} \) implies that
\[
t^d + 2 + s^{d+2} > 2 \left( \frac{t^d + s^d}{d} \right)^{\frac{d+2}{d}} = 2r(f)^{d+2},
\]
and hence (3.11) defines a positive quantity. The chain of inequalities (3.9)–(3.11) then leads to the desired contradiction. This concludes the proof of Theorem 2.

4 Proof of Proposition 1

Let \( f \in \mathcal{A}_+(1) \setminus \{0\} \) be a bandlimited function, such that \( \text{supp}(\hat{f}) \subseteq \left[ -\frac{1}{2}, \frac{1}{2} \right] \). With the Fourier transform normalized as in (1.1), the Poisson summation formula implies
\[
\sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} \hat{f}(k).
\]
Dilating by a parameter \( \alpha > 0 \), and translating by an arbitrary \( \beta \in \mathbb{R} \), yields
\[
\alpha \sum_{n \in \mathbb{Z}} f(\alpha n + \beta) = \sum_{k \in \mathbb{Z}} \hat{f}(\frac{k}{\alpha}) \exp \left( 2\pi i \frac{k}{\alpha} \beta \right). \tag{4.1}
\]
The function \( \hat{f} \) is continuous, and so \( \hat{f}(\pm \frac{1}{2}) = 0 \). Thus the right-hand side of (4.1) equals \( \hat{f}(0) \) provided \( \alpha \leq 2 \), and so
\[
\sum_{n \in \mathbb{Z}} f(\alpha n + \beta) = \alpha^{-1} \hat{f}(0) \leq 0, \tag{4.2}
\]
for every \((\alpha, \beta) \in (0, 2] \times \mathbb{R}\). Aiming at a contradiction, suppose \( r(f) < 1 \). Given \( \alpha \in (0, 2] \) so that \( r(f) \leq \frac{\alpha}{2} \leq 1 \), set \( \beta = \frac{\alpha}{2} \). From (4.2), it follows that
\[
\sum_{n \in \mathbb{Z}} f((2n + 1)\beta) \leq 0, \tag{4.3}
\]
for every \( \beta \in [r(f), 1] \). But \( f(x) \geq 0 \), for every \( |x| \geq r(f) \), and so (4.3) implies
\[
f(x) = 0, \text{ for every } x \in [r(f), 1].
\]
Being the Fourier transform of a compactly supported function, the function \( f \) extends to an entire function on the whole complex plane, and as such it cannot vanish on a nondegenerate interval without being identically zero. This shows that \( r(f) \geq 1 \).

Now, suppose \( r(f) = 1 \). Then replicating the argument from the previous paragraph with \( \beta = \frac{\alpha}{2} = 1 \) yields
\[
f(2n + 1) = 0, \text{ for every } n \in \mathbb{Z}. \tag{4.4}
\]
Since the function \( f \) does not change signs at the zeros \( \pm 3, \pm 5, \ldots \), we also have that
\[
f'(2n + 1) = 0, \quad \text{for every } n \in \mathbb{Z} \setminus \{-1, 0\}.
\] (4.5)

Set \( g(x) := f(2x + 1) \). Then \( g \in L^2(\mathbb{R}) \), and \( \hat{g} \) is supported on \([-1, 1]\). By the Paley–Wiener–Schwarz Theorem [14], the function \( g \) coincides with the restriction to \( \mathbb{R} \) of a complex-valued entire function of exponential type \( 2\pi \). It follows from Vaaler’s interpolation formula [15, Theorem 9] that
\[
g(x) = \left( \frac{\sin \pi x}{\pi} \right)^2 \left( \sum_{m \in \mathbb{Z}} \frac{g(m)}{(x - m)^2} + \sum_{n \in \mathbb{Z}} \frac{g'(n)}{x - n} \right),
\]
where the expression on the right-hand side converges uniformly on compact subsets of the real line. Conditions (4.4), (4.5) then translate into
\[
g(x) = \left( \frac{\sin \pi x}{\pi} \right)^2 \left( \frac{g'(-1)}{x + 1} + \frac{g'(0)}{x} \right),
\]
which can be rewritten in terms of the original function \( f \) as
\[
f(2x + 1) = \left( \frac{\sin \pi x}{\pi} \right)^2 \left( \frac{2f'(-1)}{x + 1} + \frac{2f'(1)}{x} \right).
\]

Since \( f \) is even, \( f'(-1) = -f'(1) \). Consequently,
\[
f(2x + 1) = 2f'(1) \left( \frac{\sin \pi x}{\pi} \right)^2 \frac{1}{x(x + 1)}.
\]

A change of variables yields
\[
f(x) = \frac{8f'(1) \sin^2(\pi \frac{x-1}{2})}{\pi^2 x^2 - 1},
\] (4.6)

which belongs to the class \( \mathcal{A}_+(1) \) if and only if \( f'(1) \geq 0 \). For the converse direction, note that the Fourier transform of the function \( f \) given by (4.6) is easy to calculate:
\[
\hat{f}(\xi) = \frac{4f'(1)}{\pi} (1_{[-\frac{1}{2}, 0]} - 1_{[0, \frac{1}{2}]}) (\xi) \sin(2\pi \xi).
\]

In particular, this shows that \( f \) is a bandlimited function with Fourier support \([-\frac{1}{2}, \frac{1}{2}]\); see Fig. 2. It is easy to check that \( f \in \mathcal{A}_+(1) \), and that \( r(f) = 1 \). This concludes the proof of Proposition 1.
5 Proofs of Lemmata

Proof of Lemma 1 Let $s \in \{+,-\}$ and $f \in \mathcal{A}_-(d)$ be as in the statement of the lemma. Since $\hat{f} = sf$, $f$ is even, and $sf(x_n) \geq 0 = f(0)$, we have that

$$\frac{f(0) - sf(x_n)}{|x_n|^2} = \int_{B_r^d(f)} \frac{1 - \cos(2\pi \langle x_n, y \rangle)}{|x_n|^2} f(y) \, dy + \int_{\mathbb{R}^d \setminus B_r^d(f)} \frac{1 - \cos(2\pi \langle x_n, y \rangle)}{|x_n|^2} f(y) \, dy \leq 0.$$

(5.1)

Uniformly on compact subsets of the real line,

$$\lim_{t \to 0} \frac{1 - \cos(2\pi tu)}{t^2} = 2\pi^2 u^2.$$

(5.2)

It follows that the first summand on the right-hand side of (5.1) is bounded independently of the sequence $\{x_n\}_{n \in \mathbb{N}}$, and tends to a finite limit, as $n \to \infty$.

Let $e_k \in \mathbb{R}^d$ denote the $k$-th coordinate unit vector, and write $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. Since $f$ is radial, we lose no generality in assuming that $x_n = \lambda_n e_k$, for some $\lambda_n > 0$ and $1 \leq k \leq d$. Then $|x_n| = \lambda_n$ and $\langle x_n, y \rangle = \lambda_n y_k$. Given $R \geq r(f)$, the second summand on the right-hand side of (5.1) (whose integrand is nonnegative on
the region of integration) can be bounded from below as follows:

\[
\int_{\mathbb{R}^d \setminus B_{r(f)}^d} \frac{1 - \cos(2\pi (x_n, y))}{|x_n|^2} f(y) \, dy \geq \int_{B_R^d \setminus B_{r(f)}^d} \frac{1 - \cos(2\pi (x_n, y))}{|x_n|^2} f(y) \, dy.
\]

It then follows from (5.1), (5.2) that

\[
\sup_{R \geq r(f)} \int_{B_R^d \setminus B_{r(f)}^d} y_k^2 f(y) \, dy < +\infty,
\]

whence \(\int_{\mathbb{R}^d} |y|^2 |f(y)| \, dy < +\infty\). Finally, (2.1) follows from noting that

\[
\int_{\mathbb{R}^d} y_k^2 f(y) \, dy = \frac{1}{2\pi^2} \lim_{n \to \infty} \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi (x_n, y))}{|x_n|^2} f(y) \, dy \leq 0,
\]

and summing in \(k\).

**Proof of Lemma 2** Reasoning as in the proof of [1, Théorème 1.1], we have that

\[
f_+(x) \leq 2 \int_0^\infty f_-(y)(1 - \cos(2\pi xy)) \, dy. \tag{5.3}
\]

If \(x \in [0, r(f)]\), then \(y \mapsto 1 - \cos(2\pi xy)\) defines a nonnegative, increasing function of \(y\) on the interval \([0, r(f)]\), provided \(r(f) \leq \frac{1}{\sqrt{2}}\). Since \(\text{supp}(f_-) \subseteq [-r(f), r(f)]\), and \(\|f_-\|_{L^1(0,r(f))} = \frac{1}{4} \leq r(f)\), by [7, Lemma 11] it then follows that

\[
\int_0^\infty f_-(y)(1 - \cos(2\pi xy)) \, dy \leq \int_{r(f) - \frac{1}{4}}^{r(f)} (1 - \cos(2\pi xy)) \, dy = \frac{1}{4} + \frac{\sin(2\pi r(f) - \frac{1}{4})x - \sin(2\pi r(f)x)}{2\pi x}. \tag{5.4}
\]

Estimate (2.7) follows at once from (5.3), (5.4), and the lemma is proved.

**Proof of Lemma 3** Let \(\mathbb{1}_{B_1^d}\) denote the indicator function of the unit ball \(B_1^d \subset \mathbb{R}^d\), and define the convolutions \(\chi := \mathbb{1}_{B_1^d} * \mathbb{1}_{B_1^d}\) and \(\varphi := \chi \hat{\chi}\). Explicitly,

\[
\chi(x) = (2 - |x|)_+, \quad \hat{\chi}(\xi) = \left(\frac{J_{d/2}(2\pi |\xi|)}{|\xi|^{d/2}}\right)^2,
\]

where \(J_{d/2}\) denotes the Bessel function of the first kind. One easily checks that \(\varphi, \hat{\varphi} \in L^1\). Moreover, the function \(\varphi\) is bandlimited, and standard decay properties of the Bessel functions imply the existence of a constant \(A > 0\), such that

\[
\varphi(x) = \int_{B_2^d} (2 - |t|) \frac{J_{d/2}^2(2\pi |x - t|)}{|x - t|^d} \, dt \geq A|x|^{-(d+1)},
\]
provided $|x|$ is sufficiently large. Let $\psi := \varphi + \hat{\varphi}$. Then $\hat{\psi} = \psi$, and $|x|^{d+1}\psi(x) \geq A$, for all sufficiently large values of $|x|$. This is still not the desired function since $\psi(0) > 0$, but one can simply add an appropriate Gaussian function. For instance, the function $x \mapsto \eta(x) := A - \frac{1}{2} \psi(x) - \frac{2}{2\pi} \psi(0) \exp(-\pi |x|^2) \in A_+(d)$ satisfies all the required properties.

Proof of Lemma 4  If $\delta > 0$ is small enough, then
\[ g(0) = \int_{B_{2\delta}^d} f \varphi_\delta < 0, \tag{5.5} \]
since the function $f$ is continuous and $f(0) < 0$. Moreover, $\hat{g}(0) = \hat{f}(0)(\hat{\psi}_\delta(0))^2 < 0$. Further note that $g(x) \geq 0$, provided $f \geq 0$ on the ball $x + B_{2\delta}^d$, and so it suffices to take $\delta \leq \frac{\varepsilon}{2}$ in order to ensure that $r(g) \leq r(f) + \varepsilon$. With these preliminary observations in place, we are now ready for the proof of the lemma.

For part (a), one easily checks that $h$ is a Schwartz function which coincides with its own Fourier transform. Moreover,
\[ h(0) = (\hat{g} * \varphi_\delta)(0) + g(0)(\hat{\psi}_\delta(0))^2, \]
where both summands are negative. For the first summand, note that $(\hat{g} * \varphi_\delta)(0) = \int_{B_{2\delta}^d} \hat{g}(\hat{\varphi}_\delta) = \hat{f}(0)(\hat{\psi}_\delta(0))^2$ is negative if $\delta > 0$ is small enough, since the function $\hat{g}$ is continuous and $\hat{g}(0) < 0$. For the second summand, this is clear in light of (5.5) and the real-valuedness of $\hat{\psi}_\delta$.

For part (b), we seek to verify that $h(x) \geq 0$, if $|x| \geq r(f) + \varepsilon$, which will follow from
\[ (\hat{g} * \varphi_\delta)(x) \geq 0 \text{ and } g(x)\hat{\varphi}_\delta(x) \geq 0, \text{ if } |x| \geq r(f) + \varepsilon. \]
The lower bound for $g\hat{\varphi}_\delta$ follows immediately from $r(g) \leq r(f) + \varepsilon$, whereas $\hat{f} = f$ implies
\[ \hat{g} * \varphi_\delta = (\hat{f} * \hat{\varphi}_\delta) * \varphi_\delta = (f * \hat{\varphi}_\delta) * \varphi_\delta. \]
But $r(f * \hat{\varphi}_\delta) \leq r(f)$ since $\hat{\varphi}_\delta = (\hat{\psi}_\delta)^2 \geq 0$, and so the lower bound for $\hat{g} * \varphi_\delta$ follows as before. This concludes the verification of part (b), and the proof of the lemma.

6 From Continuous to Discrete Uncertainty

Very recently, the authors [6] established new and very general sign uncertainty principles which apply to certain classes of bounded linear operators and metric measure spaces. As a corollary, we obtained a sign uncertainty principle for Fourier series on the $d$-torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, which we proceed to describe.

Given $s \in \{+, -\}$ and $d \geq 1$, let $\mathcal{P}_s(\mathbb{T}^d)$ denote the class of continuous, even functions $g : \mathbb{T}^d \to \mathbb{R}$, such that $\hat{g} \in \ell^1, \hat{g}(0) \leq 0$, and $s\hat{g}$ is eventually nonnegative.
while \( sg(0) \leq 0 \). Given \( g \in \mathcal{P}_s(\mathbb{T}^d) \), define the quantities\(^2\)

\[
\begin{align*}
  r(g; \mathbb{T}^d) &: = \inf \{ r > 0 : g(x) \geq 0 \text{ if } \sqrt{d} \geq |x|_2 \geq r \}; \\
  k_s(\hat{g}) &: = \min \{ k \geq 1 : s\hat{g}(n) \geq 0 \text{ if } |n|_2 \geq k \}.
\end{align*}
\]

(here, \(| \cdot |_2\) denotes the Euclidean norm) together with the optimal constant

\[
\mathbb{P}_s(\mathbb{T}^d) := \inf_{g \in \mathcal{P}_s(\mathbb{T}^d) \setminus \{0\}} \sqrt{r(g; \mathbb{T}^d) k_s(\hat{g})}. \tag{6.1}
\]

A straightforward application of [6, Theorem 1.2] reveals that

\[
\mathbb{P}_s(\mathbb{T}^d) \geq (1 + o(1)) \sqrt{\frac{d}{2\pi e}}.
\]

Identity \( A_+^s(1) = A_+^B(1) \) from Theorem 1 leads to the following connection between the continuous and discrete versions of the one-dimensional +1 sign uncertainty principle.

**Proposition 4** Given \( s \in \{+, -\} \) and \( d \geq 1 \), it holds that

\[
\mathbb{P}_s(\mathbb{T}^d) \leq A_+^B(d).
\]

In particular,

\[
\mathbb{P}_+(\mathbb{T}^1) \leq A_+^B(1).
\]

**Proof** Let \( f \in \mathcal{A}_s(d) \) be nonzero and bandlimited. Assume that \( \text{supp}(\hat{f}) \subseteq [-a, a]^d \), for some \( a > 0 \). Let \( f_\lambda(x) := f(\lambda x) \), in which case \( \hat{f}_\lambda(\xi) = \lambda^{-d} \hat{f}(\xi/\lambda) \) and \( \text{supp}(\hat{f}_\lambda) \subseteq [-a\lambda, a\lambda]^d \). Note that, as long as \( 2a\lambda < 1 \), \( \hat{f}_\lambda \) can be seen as a function on \( \mathbb{T}^d \cong [-\frac{1}{2}, \frac{1}{2}]^d \). Set \( \lambda = \frac{r(f)}{n} \), where \( n \in \mathbb{N} \) is sufficiently large so that \( 2a\lambda < 1 \), and define \( g(\xi) := s\hat{f}_\lambda(\xi) \). By the higher dimensional version of a result of Plancherel and Pólya [13],

\[
\sum_{k \in \mathbb{Z}^d} |\hat{g}(k)| = \sum_{k \in \mathbb{Z}^d} |f(\lambda k)| < \infty, \tag{6.2}
\]

whence \( g \in \mathcal{P}_s(\mathbb{T}^d) \setminus \{0\} \). Moreover, \( r(g; \mathbb{T}^d) = r(\hat{f}_\lambda) = \frac{r(f)}{n} r(\hat{f}) \), and \( k_s(\hat{g}) \leq n \). Therefore

\[
\mathbb{P}_s(\mathbb{T}^d)^2 \leq r(g; \mathbb{T}^d) k_s(\hat{g}) \leq r(f) r(\hat{f}).
\]

Since \( f \in \mathcal{A}_s(d) \) was an arbitrary nonzero bandlimited function, this finishes the proof of the proposition. \( \square \)

\(^2\) Trivially, \( r(g; \mathbb{T}^d) \leq \frac{\sqrt{d}}{2} \) for all continuous functions \( g : \mathbb{T}^d \to \mathbb{R} \).
Acknowledgements  F.G. acknowledges support from the Deutsche Forschungsgemeinschaft through the Collaborative Research Center 1060. D.O.S. is supported by the EPSRC New Investigator Award “Sharp Fourier Restriction Theory”, Grant No. EP/T001364/1. J.P.G.R. acknowledges financial support from the Deutscher Akademischer Austauschdienst. The authors are indebted to the anonymous referees for careful reading and valuable suggestions.

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