Self-Stabilization Through the Lens of Game Theory

Krzysztof R. Apt\textsuperscript{1,2} and Ehsan Shoja\textsuperscript{3}

\textsuperscript{1} CWI, Amsterdam, The Netherlands
\textsuperscript{2} MIMUW, University of Warsaw, Warsaw, Poland
\textsuperscript{3} Sharif University of Technology, Tehran, Iran

Abstract. In 1974 E.W. Dijkstra introduced the seminal concept of self-stabilization that turned out to be one of the main approaches to fault-tolerant computing. We show here how his three solutions can be formalized and reasoned about using the concepts of game theory. We also determine the precise number of steps needed to reach self-stabilization in his first solution.

1 Introduction

In 1974 Edsger W. Dijkstra introduced in a two-page article \cite{Dijkstra1974} the notion of self-stabilization. The paper was completely ignored until 1983, when Leslie Lamport stressed its importance in his invited talk at the ACM Symposium on Principles of Distributed Computing (PODC), published a year later as \cite{Lamport1984}. Things have changed since then. According to Google Scholar Dijkstra’s paper has been by now cited more than 2300 times. It became one of the main approaches to fault-tolerant computing. An early survey was published in 1993 as \cite{Manohar1993}, while the research on the subject until 2000 was summarized in the book \cite{Dijkstra2002}. In 2002 Dijkstra’s paper won the PODC influential paper award (renamed in 2003 to Dijkstra Prize). The literature on the subject initiated by it continues to grow. There are annual Self-Stabilizing Systems Workshops, the 18th edition of which took part in 2016.

The idea proposed by Dijkstra is very simple. Consider a distributed system viewed as a network of machines. Each machine has a local state and can change it autonomously by inspecting its local state and the local states of its neighbours. Some global states are identified as legitimate. A distributed system is called self-stabilizing if it satisfies the following three properties (the terminology is from \cite{Dijkstra1974}):

- closure: starting from an arbitrary global state, the system is guaranteed to reach a legitimate state,
- stability: once a legitimate state is reached, the system remains in it forever,
- fairness: in every infinite sequence of moves every machine is selected infinitely often.
Dijkstra proposed in [10] three solutions to self-stabilization in which, respectively, four and three state machines were used, where \( n \) is the number of machines. The proofs were provided respectively in [7] (republished as [11]), [8] and [9] (republished with small modifications as [12]). In his solutions a legitimate state is identified with the one in which exactly one machine can change its state.

In this paper we show how Dijkstra’s solutions to self-stabilization can be naturally formulated using the standard concepts of strategic games, notably the concept of an improvement path. Also we show how one can reason about them using game-theoretic terms. We focus on Dijkstra’s first solution but the same approach can be adopted to other solutions.

The connections between self-stabilization and game theory were noticed before. We discuss the relevant references in the final section. The analysis of the original Dijkstra’s solutions using game theory is to our knowledge new.

This paper connects two unrelated areas, each of which has developed its own well-established notation and terminology. To avoid possible confusion, let us clarify that in what follows \( S_i \) denotes a set of strategies of a player in a strategic game, while the letter \( S \) denotes a variable in a solution to the self-stabilization problem. Further, the notion of a state in the self-stabilization refers to the range of a variable and not to an assignment of values to all variables, as is customary in the area of program semantics.

2 Preliminaries

A strategic game \( G = (S_1, \ldots, S_n, p_1, \ldots, p_n) \) for \( n > 1 \) players consists of a non-empty set \( S_i \) of strategies and a payoff function \( p_i : S_1 \times \cdots \times S_n \to \mathbb{R} \), for each player \( i \). We denote \( S_1 \times \cdots \times S_n \) by \( S \), call each element \( s \in S \) a joint strategy and abbreviate the sequence \((s_j)_{j \neq i}\) to \( s_{-i} \). Occasionally we write \((s_i, s_{-i})\) instead of \( s \). We call a strategy \( s_i \) of player \( i \) a best response to a joint strategy \( s_{-i} \) of his opponents if for all \( s'_i \in S_i \), \( p_i(s'_i, s_{-i}) \geq p_i(s_i, s_{-i}) \). A joint strategy \( s \) is called a Nash equilibrium if each \( s_i \) is a best response to \( s_{-i} \). (In the literature these equilibria are often called pure Nash equilibria to distinguish them from Nash equilibria in mixed strategies. The latter ones have no use in this paper.)

Further, we call a strategy \( s'_i \) of player \( i \) a better response given a joint strategy \( s \) if \( p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}) \). We call \( s \to s' \) an improvement step (abbreviated to a step) if \( s' = (s'_i, s_{-i}) \) for some better response \( s'_i \) of player \( i \) given \( s \). So \( p_i(s') > p_i(s) \).

An improvement path is a maximal sequence

\[
s^1 \to s^2 \to \ldots \to s^k \to \ldots
\]

such that each \( s^i \to s^{i+1} \) is an improvement step.

In the next section we consider specific strategic games on directed graphs. Fix a finite directed graph \( G \). We say that a node \( j \) is a neighbour of the node \( i \) in \( G \) if there is an edge \( j \to i \) in \( G \). Let \( N_i \) denote the set of all neighbours of
node \(i\) in the graph \(G\). We now consider a strategic game in which each player is a node in \(G\). Fix a non-empty set of strategies \(C\) that we call \textit{colours}.

We divide the players in two categories: those who play a coordination game and those who play an anti-coordination game. More specifically,

- the players are the nodes of \(G\),
- the set of strategies of player (node) \(i\) is a set of colours \(A(i)\) such that \(A(i) \subseteq C\),
- if the player plays the coordination game, then his payoff function is defined by
  \[
  p_i(s) = |\{ j \in N_i \mid s_i = s_j \}|
  \]
- if the player plays the anti-coordination game, then his payoff function is defined by
  \[
  p_i(s) = |\{ j \in N_i \mid s_i \neq s_j \}|
  \]

So each node simultaneously chooses a colour and the payoff to the player who plays the \textit{coordination game} is the number of its neighbours that chose its colour, while the payoff to the player who plays the \textit{anti-coordination game} is the number of its neighbours that chose a different colour.

The games on directed graphs in which all players were playing the coordination game were studied in [4]. Corresponding games on undirected graphs were considered in [2] and on weighted undirected graphs in [25]. In turn, the games in which some players played the coordination game while other players played the anti-coordination game were studied (in a more general context of weighted hypergraphs) in [28]. If the underlying (weighted) graph is undirected the game always has a Nash equilibrium, which is not the case if the graph is directed. The absence of Nash equilibria is crucial in the context of this paper.

We now move on to the subject of this paper and introduce the following concepts concerning improvement paths.

\textbf{Definition 1.} \textit{Fix a strategic game.}

- A joint strategy is \textbf{legitimate} if exactly one player does not play a best response in it.
- An improvement path ensures
  - \textbf{closure} if some joint strategy in it is legitimate,
  - \textbf{stability} if the successors of the legitimate joint strategies in it are legitimate,
  - \textbf{fairness} if every player is selected in it infinitely often,
  - \textbf{self-stabilization (in \(k\) steps)} if every player is selected in it infinitely often and from a certain point (after \(k\) steps) each joint strategy in it is legitimate.

- A game admits \textbf{closure/stability/fairness} if it is ensured by every improvement path in it.
- A game admits \textbf{self-stabilization (in \(k\) steps)} if it is ensured by every improvement path in it (in \(k\) steps).
For a more refined analysis we shall need the concept of a scheduler.

Definition 2.

– A scheduler is a function $f$ that given a joint strategy $s$ that is not a Nash equilibrium and a player $i$ who does not hold in $s$ a best response selects a strategy $f(s, i)$ for $i$ that is a better response given $s$.

– Consider a scheduler $f$. An improvement path

$$s^1 \rightarrow s^2 \rightarrow \ldots \rightarrow s^k \rightarrow \ldots,$$

is generated by $f$ if for each $k \geq 1$, if $s^k$ is not a Nash equilibrium, then for some $i \in \{1, \ldots, n\}$, $s^{k+1} = (f(s^k, i), s^k_{-i})$.

– A scheduler $f$ ensures self-stabilization (in $k$ steps) if every improvement path generated by it ensures self-stabilization (in $k$ steps).

So a game admits self-stabilization (in $k$ steps) if every scheduler ensures self-stabilization (in $k$ steps). Schedulers in the context of strategic games were extensively considered in [3], though they selected a player and not his strategy. The ones used here correspond in the terminology of [3] to the state-based schedulers.

3 Dijkstra’s first solution

We start by recalling the first solution to the self-stabilization problem given in [10]. We assume a directed ring of $n$ machines, each having a local variable and a program. The variables assume the values from the set $\{0, \ldots, k-1\}$, where $k \geq n$ and $\oplus$ stands for addition modulo $k$. Each program consists of a single rule of the form

$$P \rightarrow A$$

where $P$ is a condition, called a privilege, on the local variables of the machine and its predecessor in the ring, and $A$ is an assignment to the local variable. The variable of a considered machine is denoted by $S$ and the variable of its predecessor by $L$.

The program for machine 1 is given by the rule

$$L = S \rightarrow S := S \oplus 1$$

and for the other machines by the rule

$$L \neq S \rightarrow S := L.$$
– exactly one privilege is true,
– this property remains true forever.

Moreover, every machine is selected in this sequence infinitely often.

In the terminology introduced in the Introduction the above system of machines is self-stabilizing.

We can model the above solution by means of the following strategic game \( G \) on a directed ring involving \( n \) players:

– each player has the same set \( C \) of strategies (called colours), where \( |C| \geq 2 \),
– exactly one player plays the anti-coordination game on the ring,
– all other players play the coordination game on the ring.

To fix notation we assume that it is player 1 who plays the anti-coordination game. So the payoff functions are simply:

\[
p_1(s) := \begin{cases} 
0 & \text{if } s_1 = s_n \\
1 & \text{otherwise} 
\end{cases}
\]

and for \( i \neq 1 \)

\[
p_i(s) := \begin{cases} 
0 & \text{if } s_i \neq s_{i-1} \\
1 & \text{otherwise} 
\end{cases}
\]

We arrange the colours in \( C \) in a cyclic order and given a colour \( c \) we denote its successor in this order by \( c' \). The following result provides a game-theoretic account of the above solution to the self-stabilization problem.

**Theorem 1.** Consider the game \( G \). Suppose that \( n \geq 3 \) and \( |C| \geq n \). Let \( f \) be a scheduler such that

\[f(s, 1) = s_1'.\]

Then \( f \) ensures self-stabilization in \( G \).

Thus the only restriction on the scheduler \( f \) is that for player 1 it selects the next colour in the cyclic order on \( C \) (as \( s_1' \) denotes the successor of \( s_1 \)).

**Proof.** There is a 1-1 correspondence between the maximal sequences of moves of the machines in Dijkstra’s solution and the improvement paths generated by the schedulers satisfying the stated condition. \( \square \)

We shall return to the above result in Section 6. It is useful to point out why we did not incorporate the specific choice of the strategies into the payoff functions and used a scheduler instead. This alternative would call for selecting \( \{0, \ldots, k - 1\} \) as the set of strategies for each player and using the following payoff function for player 1, where \( \oplus \) stands for addition modulo \( k \):

\[
p_1(s) := \begin{cases} 
0 & \text{if } s_1 \neq s_n \oplus 1 \\
1 & \text{otherwise} 
\end{cases}
\]
However, the resulting game would then correspond to a setup in which the program for machine 1 is

\[ S \neq L \oplus 1 \rightarrow S := L \oplus 1. \]

Moreover, the resulting game does not admit self-stabilization (and a fortiori the resulting programs for the machines do not form a solution for self-stabilization). Indeed, assume three players and \( k = 3 \), so that the strategies of the players are 0, 1, 2. Then the following infinite improvement path does not ensure closure:

\[
(200 \rightarrow 220 \rightarrow 120 \rightarrow 122 \rightarrow 112 \rightarrow 012 \rightarrow 011 \rightarrow 001 \rightarrow 201 \rightarrow)^*,
\]

where each joint strategy is displayed as a string of three numbers from \{0, 1, 2\} and "*" stands for the infinite repetition of the exhibited prefix of an improvement path.

4 Dijkstra’s three-state solution

Next we discuss Dijkstra’s three-state solution to the self-stabilization problem. We follow here the presentation he gave in [12], where he provided a particularly elegant correctness proof.

There are \( n \) machines arranged in an undirected ring, the first one called the bottom machine, the last one called the top machine, and the other machines called normal.

The condition of each rule is now on the local variables of the machine and its two neighbours. The variable of a considered machine is denoted by \( S \), of its left neighbour by \( L \) and of its right neighbour by \( R \). All variables range over the set \{0, 1, 2\} and \( \oplus \) stands for addition modulo 3.

The program for the bottom machine is given by the rule

\[ S \oplus 1 = R \rightarrow S := S \oplus 2, \]

for each normal machine by the rule

\[ L = S \oplus 1 \vee S \oplus 1 = R \rightarrow S := S \oplus 1, \]

and for the top machine by the rule

\[ L = R \wedge S \neq R \oplus 1 \rightarrow S := R \oplus 1. \]

Dijkstra proved that the above system of machines is self-stabilizing.

This solution can be represented and reasoned about using strategic games, though these games are not anymore coordination or anti-coordination games. First note that, in contrast to the case of Dijkstra’s first solution, this solution cannot be modeled using strategic games with 0/1 payoffs. To see it assume \( n = 3 \) and consider the global state of the system described by \( (2, 1, 0) \). Then
the privilege of machine 2 is true, since \( L = S \oplus 1 \), as \( 2 = 1 \oplus 1 \). After machine 2 is selected the global state changes to \((2, 2, 0)\). In this state the privilege of machine 2 is again true, since \( S \oplus 1 = R \), as \( 2 \oplus 1 = 0 \). So in the improvement path of the corresponding strategic game player 2 can be selected twice in succession. This can be modelled only using at least three payoff values.

To capture such a possibility we need to analyze when a machine can be selected twice in succession. This can happen when successively \( L = S \oplus 1 \) and \( S \oplus 1 = R \) are true or successively \( S \oplus 1 = R \) and \( L = S \oplus 1 \) are true. Taking into account the action of the assignment \( S := S \oplus 1 \) the first possibility means that initially \( L = S \oplus 1 \land S \oplus 2 = R \) is true and the second possibility that initially \( S \oplus 1 = R \land L = S \oplus 2 \) is true. These two options can be rewritten as \( S \oplus 1 \in \{L, R\} \land S \oplus 2 \in \{L, R\} \).

To complete this analysis note that a machine can be selected only once in succession, when initially \( L = S \oplus 1 \land S \oplus 2 \neq R \) is true or \( S \oplus 1 = R \land L \neq S \oplus 2 \) is true, which can be rewritten as \( S \oplus 1 \in \{L, R\} \land S \oplus 2 \notin \{L, R\} \).

Translating it into a game-theoretic notation that uses indices we are brought into the following strategic game \( G \) for \( n \) players. Each player has \( \{0, 1, 2\} \) as the set of strategies. The payoff functions are defined as follows, where we assume that player 1 corresponds to the bottom machine and player \( n \) to the top machine:

\[
p_1(s) := \begin{cases} 0 & \text{if } s_1 \oplus 1 = s_2 \\ 1 & \text{otherwise} \end{cases}
\]

for \( 1 < i < n \)

\[
p_i(s) := \begin{cases} 0 & \text{if } s_i \oplus 1 \in \{s_{i-1}, s_{i+1}\} \land s_i \oplus 2 \in \{s_{i-1}, s_{i+1}\} \\ 1 & \text{if } s_i \oplus 1 \in \{s_{i-1}, s_{i+1}\} \land s_i \oplus 2 \notin \{s_{i-1}, s_{i+1}\} \\ 2 & \text{otherwise} \end{cases}
\]

\[
p_n(s) := \begin{cases} 0 & \text{if } s_1 = s_{n-1} \land s_n \neq s_1 \oplus 1 \\ 1 & \text{otherwise} \end{cases}
\]

Dijkstra’s result concerning the above system of three-state machines is captured by the following theorem.

**Theorem 2.** Consider the above game \( G \). Suppose that \( n \geq 3 \). Let \( f \) be a scheduler such that

\[
\begin{align*}
f(s, 1) &= s_1 \oplus 2, \\
f(s, i) &= s_i \oplus 1, \text{ where } 1 < i < n, \\
f(s, n) &= s_1 \oplus 1.
\end{align*}
\]

Then \( f \) ensures self-stabilization in \( G \).

**Proof.** Every maximal sequence of moves of the machines in Dijkstra’s three-state solution corresponds to an improvement path generated by a scheduler satisfying the stated conditions. Conversely, every improvement path generated
by a scheduler satisfying the stated conditions corresponds to a maximal sequence of moves of the machines in Dijkstra’s three-state solution with each improvement step that results for a player $i$ in the payoff increase by 2 mapped to two consecutive moves of machine $i$. 

\[ \square \]

5 A four-state solution

Finally, we consider a four-state solution. Instead of Dijkstra’s solution that uses two Boolean variables per machine we consider a modified solution due to [16] that uses per machine a single variable that can take four values. We assume the set up and terminology of the previous section, with the following differences.

The variable of machine 1 now ranges over \{1, 3\}, of machine $n$ over \{0, 2\}, and all other variables range over \{0, 1, 2, 3\}. Further, $\oplus$ stands now for addition modulo 4.

The program for the bottom machine is given by the rule

$$S \oplus 1 = R \rightarrow S := S \oplus 2,$$

for each normal machine by the rule

$$L = S \oplus 1 \lor S \oplus 1 = R \rightarrow S := S \oplus 1,$$

and for the top machine by the rule

$$L = S \oplus 1 \rightarrow S := S \oplus 2.$$

Following the considerations of the previous section this solution can be modeled by the following strategic game $G$ for $n$ players. The sets of strategies are as follows: for player 1: \{1, 3\}, for player $n$: \{0, 2\}, and for all other players: \{0, 1, 2, 3\}.

The payoff functions are defined as follows, where we assume that player 1 corresponds to the bottom machine and player $n$ to the top machine:

$$p_1(s) := \begin{cases} 0 & \text{if } s_1 \oplus 1 = s_2 \\ 1 & \text{otherwise} \end{cases}$$

for $1 < i < n$

$$p_i(s) := \begin{cases} 0 & \text{if } s_i \oplus 1 \in \{s_i-1, s_i+1\} \land s_i \oplus 2 \in \{s_i-1, s_i+1\} \\ 1 & \text{if } s_i \oplus 1 \in \{s_i-1, s_i+1\} \land s_i \oplus 2 \not\in \{s_i-1, s_i+1\} \\ 2 & \text{otherwise} \end{cases}$$

$$p_n(s) := \begin{cases} 0 & \text{if } s_n \oplus 1 = s_{n-1} \\ 1 & \text{otherwise} \end{cases}$$

The reason for using three values in the payoff functions $p_i$, where $1 < i < n$, is as in the previous section. The corresponding result concerning self-stabilization of the above system of four-state machines is now captured by the following game-theoretic theorem.
Theorem 3. Consider the above game $G$. Suppose that $n \geq 3$. Let $f$ be a scheduler such that
\[ f(s, 1) = s_1 \oplus 2, \]
\[ f(s, i) = s_i \oplus 1, \text{ where } 1 < i < n, \]
\[ f(s, n) = s_n \oplus 2. \]
Then $f$ ensures self-stabilization in $G$.

Proof. The same as the proof of Theorem 2. \qed

6 A game-theoretic analysis of the first solution

We now analyze in detail the strategic game $G$ introduced in Section 3 with the aim of proving a stronger result about the first solution to self-stabilization. We begin with the following observation.

Note 1. The game $G$ admits no Nash equilibria.

Proof. Suppose otherwise. Let $s$ be a Nash equilibrium of $G$. Then every player $i \neq 1$ holds in $s$ the colour of its predecessor. Hence all players hold in $s$ the same colour, in particular players 1 and $n$. But then player 1 does not hold in $s$ a best response, which yields a contradiction. \qed

Corollary 1. The game $G$ admits stability.

Proof. Suppose $s \to s'$ is an improvement step in the game $G$ and that $s$ is legitimate. Then by the definition of the game either $s'$ is legitimate or it is a Nash equilibrium. So the claim follows by Note 1. \qed

We shall use below the following observation.

Note 2. Consider a coordination game on a chain of $n$ players in which each player has the same set of strategies. Then all improvement paths in this game are of length $\leq \frac{n(n-1)}{2}$. Further, improvement paths of length $\frac{n(n-1)}{2}$ exist.

Proof. Suppose the chain is $1 \to 2 \to \ldots \to n$. Consider an improvement path $\xi$. Each player $i$ can adopt in $\xi$ at most $i-1$ colours, namely the strategies held by his predecessors in the chain. So each player $i$ can be involved in at most $i-1$ improvement steps. Consequently the length of $\xi$ is bound by $\sum_{i=1}^{n} (i-1) = \frac{n(n-1)}{2}$.

To establish the second claim take an initial joint strategy $s$ in which all colours differ. Then the required number of steps is achieved by scheduling the players in the ‘rightmost first’ order, so
\[ (n), (n - 1, n), (n - 2, n - 1, n), \ldots, (2, 3, \ldots, n), \]
where to increase readability we separated the consecutive phases using brackets. \qed
Theorem 4. The game $G$ admits fairness.

Proof. Consider an improvement path $\xi$. We first prove that player 1 is infinitely often selected in $\xi$. Suppose otherwise. By Note 1 $\xi$ is infinite, so from some moment on player 1 is never selected in the infinite suffix $\phi$ of $\xi$. Break the ring by removing the link between players $n$ and 1 and consider the resulting coordination game on the chain $1 \rightarrow 2 \rightarrow \ldots \rightarrow n$. Then $\phi$ is an infinite improvement path in this game, which contradicts Note 2.

Note now that if some player $i$ is finitely often selected in $\xi$, then so is its successor. Together with the above conclusion this implies successively that players $n, n-1, \ldots, 2$ are infinitely often selected in $\xi$. $\square$

So to prove that $G$ admits self-stabilization we only need to check that it admits closure. However, this holds only for games with two or three players. In fact, we have the following result.

Theorem 5. Consider the game $G$.

(i) If $n = 2$ then $G$ admits self-stabilization in 0 steps.
(ii) If $n = 3$ then $G$ admits self-stabilization in 2 steps.
(iii) If $n > 3$ then $G$ does not admit self-stabilization.

Proof. For simplicity we view each joint strategy as a string over the set of colours that we denote by the initial letters of the alphabet. Different letters stand for different colours.

(i) In this case every joint strategy is legitimate.

(ii) For brevity we say that a joint strategy $s$ is an $i$-strategy, where $0 \leq i \leq 2$, if exactly $i$ players hold in $s$ a best response. The only 0-strategy is of the form $aba$. We reach from it in one step a 1-strategy $cba$ (assuming $|C| > 2$) or a 2-strategy $bba, aaa$ or $abb$.

So consider now an arbitrary 1-strategy. If it is player 1 who plays the best response, then $s$ is of the form $acb$ (so in this case $|C| > 2$). Then the only possible improvement steps are $acb \rightarrow aab$ or $acb \rightarrow acc$. In both cases we reach a 2-strategy in one step.

If it is player 2 who plays the best response, then $s$ is of the form $aaa$ or $aab$, which contradicts the fact that $s$ is a 1-strategy. Finally, if it is player 3 who plays the best response, then $s$ is of the form $baa$ or $aaa$, which also contradicts the fact that $s$ is a 1-strategy.

We conclude that a legitimate joint strategy is always reached in at most 2 steps.

(iii) Assume that $n > 3$. Then the following infinite improvement path does not ensure closure:

$$
(bba^{n-4}ab \rightarrow aba^{n-4}ab \rightarrow aba^{n-4}aa \rightarrow \ast abb^{n-4}ba \rightarrow aab^{n-4}ba \rightarrow bab^{n-4}ba \rightarrow baa^{n-4}ab \rightarrow \ast),
$$

where each inner $\ast$ stands for an appropriate sequence of $n-4$ improvement steps, while the outer $\ast$ stands for the infinite repetition of the exhibited prefix of an improvement path. $\square$
The above result explains the need for a scheduler. As before we assume a cyclic order on the set of colours and denote the successor of colour $c$ by $c'$. The following result improves upon Theorem 1. The differences are discussed after the proof.

**Theorem 6.** Consider the game $G$. Suppose that $n \geq 3$ and $|C| \geq n - 1$. Let $f$ be a scheduler such that

$$f(s, 1) = s'_1.$$  

Then $f$ ensures self-stabilization in $G$ in $\frac{1}{2}(3n + 1)(n - 2)$ steps.

**Proof.** We split the proof in two parts. The slightly unusual naming of joint strategies in Part 1 will become clear in Part 2.

**Part 1: self-stabilization.**

Consider an improvement path $\xi$ generated by the scheduler $f$ that starts in a joint strategy $s$. Call a joint strategy **lean** if the players $2, \ldots, n$ hold in it at most $n - 2$ different colours. We now establish a number of claims about $\xi$.

**Claim 1.** A lean joint strategy appears in $\xi$.

**Proof.** By Theorem 4 eventually some player $i \in \{3, \ldots, n\}$ is selected in $\xi$. The resulting joint strategy becomes then lean. $\square$

Let $s''$ be the first lean joint strategy in $\xi$. Call a colour **fresh** in $\xi$ if it is not held in $s''$ by any player $i \neq 1$. Fresh colours exist since $|C| \geq n - 1$. Let $c$ be the first fresh colour that follows, in the cyclic order on $C$, the colours that are held in $s''$ by players $i \neq 1$.

**Claim 2.** Player 1 eventually introduces in $\xi$ the colour $c$.

**Proof.** By the definition of the scheduler and Theorem 4. $\square$

**Claim 3.** Player 1 eventually introduces in $\xi$ the successor $c'$ of the colour $c$.

**Proof.** By the definition of the scheduler and Theorem 4. $\square$

Consider now the joint strategies $s^1$ and $s^5$ resulting from the steps described in Claims 2 and 3. Let

$$s^4 \rightarrow s^5$$

be the last step of the segment $s^1 \rightarrow s^5$ of $\xi$. So $s^1_1 = s^4_1 = s^5_n = c$ and $s^5_1 = c'$. 

Take now a joint strategy $s^6$ from the segment $s^1 \rightarrow s^5$, different from $s^1$ and $s^5$. In $s^6$ player 1 is not selected. Moreover, by the definition of the game, each better response of a player different than 1 is the colour of his predecessor. So only player 1 can introduce in $\xi$ colour $c$.

This implies by induction that each time some player $i$ switches in $s^6$ to the colour $c$, all players $1, \ldots, i - 1$ hold in $s^b$ the colour $c$. So the only possibility that player $n$ holds the colour $c$ in $s^4$ is that all players hold in $s^4$ the colour $c$. Informally, the colour $c$ ‘travelled the whole ring’. So $s^4$ is a legitimate joint strategy. Hence by Corollary 1 and Theorem 4 the scheduler $f$ ensures self-stabilization.
Part 2: computing the bound.

Recall that $s''$ is the first lean joint strategy in $\xi$. Let $s'$ be the first joint strategy in the segment $s \rightarrow^* s''$ of $\xi$ such that in the segment $s' \rightarrow^* s''$ player 1 is not selected. We first determine the maximum number of steps in the prefix $s \rightarrow^* s'$ of $\xi$. Since $s''$ is the first lean joint strategy in $\xi$, in the prefix $s \rightarrow^* s'$ only players 1 and 2 are selected. Moreover, by the choice of $s'$ the last step in this prefix involves player 1. Further, player 1 can be selected the second time only after player $n$ has been selected and no player can be selected twice in succession. These constraints leave only two possible schedulings that yield $s \rightarrow^* s'$, namely 1 and 2, 1.

However, the prefix $s \rightarrow^* s'$ cannot have 2 steps. Indeed, otherwise it would have the form

$$(c_1, c_2, \ldots, c_n) \rightarrow (c_1, c_1, c_3, \ldots, c_n) \rightarrow (c'_n, c_1, c_3, \ldots, c_n),$$

where $c_1 = c_n$. So $(c_1, c_1, c_3, \ldots, c_n)$ is lean, which contradicts the choice of $s''$ as the first lean joint strategy in $\xi$. Consequently the prefix $s \rightarrow^* s'$ can have at most 1 step.

Let now $\xi'$ be the suffix of $\xi$ that starts in $s'$. We now determine the number of steps in $\xi'$ that yield self-stabilization. We can assume that it takes in $\xi$ at least three steps to reach $s^5$, as otherwise the bound holds. Consider the last three steps in $\xi$ that lead to $s^5$:

$s^2 \rightarrow s^3 \rightarrow s^4 \rightarrow s^5$.

We noticed already that in $s^4$ all players hold the colour $c$. Also, the last $n$ steps in $\xi$ that lead to $s^4$ consist of switching to the colour $c$. Hence $s^2$ is of the form $(c, \ldots, c, a, b)$, where $a \neq c$ and $b \neq c$.

**Case 1** $a = b$.

Then $s^2$ is legitimate. We first compute the number of steps in the prefix $\chi$ of $\xi'$ leading from $s'$ to $s^4$. Consider some player $i$. In $\chi$ he can be involved in two types of steps:

- in which he switches to a colour held in $s'$ by one his predecessors $1, \ldots, i-1$,
- in which he switches to a colour introduced in $\chi$ by player 1 (to identify such steps in $\chi$ we can ‘mark’ such colours in some way).

The first possibility leads to at most $i - 1$ steps, while the second one to at most $n - 2$ steps since starting from the lean joint strategy $s''$ (and hence from $s'$) player 1 can change his colour in $\chi$ at most $n - 2$ times. This means that the total number of steps in $\chi$ is at most

$$\sum_{i=1}^{n} (i - 1 + n - 2) = \frac{n(n-1)}{2} + n(n-2).$$

Deducting 2 for the steps $s^2 \rightarrow s^3 \rightarrow s^4$ we get the bound $\frac{n(n-1)}{2} + n(n-2) - 2$ on the number of steps in $\xi'$ that yield self-stabilization.
Case 2 $a \neq b$.

Then $s^2$ is not legitimate but $s^3$ is, so we need to compute the number of steps in $\xi'$ leading from $s'$ to $s^3$. To this end we modify $\xi'$ to another improvement path $\psi$ by replacing the step $s^2 \rightarrow s^3$ by

$$s^2 \rightarrow (c, \ldots, c, a, a) \rightarrow s^3$$

and apply the reasoning from Case 1 to $\psi$. This yields the above bound on the number of steps in $\psi$ needed to reach $(c, \ldots, c, a, a)$ and hence the same bound on the number of steps in $\xi'$ leading from $s'$ to $s^3$.

We noticed already that the prefix $s \rightarrow s'$ can have at most 1 step, so we conclude that $\xi$ ensures self-stabilization in $\frac{n(n-1)}{2} + n(n-2) - 2 + 1 = \frac{1}{2}(3n+1)(n-2)$ steps. \hfill $\Box$

The original bound of [10] on the number of colours was $|C| \geq n$. The authors of [15] noticed that it can be lowered to $|C| \geq n - 1$ and that it is optimal in the sense that for $|C| = n - 2$ the claim of the theorem does not hold. The latter observation was established by noting that starting from the joint strategy

$$c_2 c_1 c_{n-2} \ldots c_2 c_1$$

the counterclockwise scheduling of the players combined with the selecting of the colours in the assumed cyclic order by player 1 generates an infinite improvement path which does not yield self-stabilization. The fact that self-stabilization can be reached in $O(n^2)$ steps when $|C| \geq n$ was established in [22]. Finally, Theorem 5 shows that the use of a scheduler in Theorem 6 is necessary.

Next, we show that $\frac{1}{2}(3n+1)(n-2)$ is also a lower bound.

Example 1. Consider the game $G$ for $n$ players with $|C| \geq n - 1$. Assume the cyclic order $c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_{n-1} \rightarrow \ldots$ on $C$. So if $|C| = n - 1$, then $c'_{n-1} = c_1$ and otherwise $c'_{n-1} = c_n$.

Then the following prefix of an improvement path is generated by every scheduler mentioned in Theorem 6 and ends in a legitimate joint strategy:

$$c_1 c_{n-1} c_{n-2} \ldots c_1 \rightarrow c_2 c_{n-1} c_{n-2} \ldots c_2 \rightarrow \ldots \rightarrow c_{n-1} c_{n-2} c_1 c_2 \ldots c_{n-2} c_1 \rightarrow c_{n-1} c_{n-2} c_1 c_2 \ldots c_{n-2} c_1 \rightarrow \cdots \rightarrow c_{n-1} c_{n-2} c_1 c_2 \ldots c_{n-2} c_1 \rightarrow c_{n-1} c_{n-2} c_1 c_2 \ldots c_{n-2} c_1 c_{n-1} c_{n-2} \rightarrow c'_{n-1} c_{n-2} c_1 c_2 \ldots c_{n-2} c_1 c_{n-1} c_{n-2}.$$
legitimate joint strategy $c'_{n-1}c_{n-1} \ldots c_{n-1}c_{n-1}$, so ‘too early’. Therefore we modify this scheduling to

$$(n), (n-1, n), (n-2, n-1, n), \ldots, (3, 4, \ldots, n-1), (2, 3, \ldots n-2, n, n-1, n).$$

This way we ensure that the legitimate joint strategy is reached only after $$\frac{n(n-1)}{2} - 2$$ steps. Alternatively, we could use the scheduling

$$(n, n-1, \ldots, 2), (n, n-1, \ldots, 3), \ldots, (n, n-1, n-2), (n, n-1), (n),$$
$$(n, n-1, \ldots, 2), (n, n-1, \ldots, 3), \ldots, (n, n-1, n-2), (n).$$

The first and the last two lines consist in total of $$1 + \frac{n(n-1)}{2} - 2$$, so $$\frac{(n+1)(n-2)}{2}$$ steps, while each of the remaining $$n-2$$ lines consists of $$n$$ steps. Therefore the total number of steps to reach $c'_{n-1}c'_{n-1} \ldots c'_{n-1}c_{n-1}$ equals $$\frac{(n+1)(n-2)}{2} + n(n-2) = \frac{1}{2}(3n+1)(n-2)$$. Note that no other listed joint strategy is legitimate.

\[\Box\]

7 Related work and discussion

Starting from [19], a paper that relates secret sharing and multiparty communication protocols to game theory, a growing literature keeps revealing rich connections between game theory and distributed computing. For a short overview of the early connections see Section 4 of [18].

Let us mention a couple of more recent examples. The authors of [1] provide a game-theoretic analysis of the leader election algorithms on a number of networks for both the synchronous case and the asynchronous case. In turn, [14] provides a framework in which the processes and the environment of a distributed system are viewed as players in an extensive game, in which implementations are interpreted as strategies with an implementation being correct if the corresponding strategy is winning.

To discuss the papers about connections between game theory and self-stabilization note first that we followed here the original Dijkstra’s definition of a legitimate global state as the one in which exactly one machine can change its state. If we view a legitimate global state as the one in which no machine can change its state and drop the fairness assumption then we enter the area of self-stabilizing algorithms. An early example of such an algorithm is the one introduced in [27] that computes a maximal independent set (MIS).

Probably the first paper that noted the connection between the self-stabilizing algorithms and game theory is [6], where the notion of a selfish stabilization is introduced. The authors attached to each node of a graph a cost function (a customary alternative to the payoff functions in the definition of strategic games) to derive a simple self-stabilizing algorithm that constructs a spanning tree in a final state corresponding to a Nash equilibrium of the underlying strategic game. In turn, the authors of [20] related self-stabilization to uncoupled dynamics, a procedure used in game theory to reach a Nash equilibrium in situations when players do not know each others’ payoff functions.
Recently, the authors of [29] observed that self-stabilizing algorithms that compute a maximal weighted independent set (MWIS) and MIS can be analyzed using game-theoretic tools. To relate this work to ours recall that in our setup we defined a legitimate joint strategy as the one in which exactly one player does not play a best response. Consider now an alternative definition that equates the legitimate joint strategy with a Nash equilibrium. We need now to recall the following definition due to [24]. We say that a strategic game has the finite improvement property (FIP) if every improvement path is finite.

The authors of [29] found that the self-stabilizing algorithms that compute a MWIS and a MIS correspond to natural strategic games on graphs that have the FIP. The computations of such an algorithm then correspond to the (necessarily finite) improvement paths in the corresponding game. They also noticed that if a game on a graph has the FIP then after an appropriate translation to a distributed system a self-stabilizing algorithm is obtained. Indeed, the FIP ensures the closure property, while the stability is immediate. These observations also clarify the set up of the just discussed papers [6] and [20].

We conclude this discussion of relations between self-stabilization and game theory by the following remark. The author of [17] introduced the concept of a weak self-stabilization which guarantees that a distributed system reaches a legitimate state only by some (and thus not necessarily all) sequence of moves. This concept can be easily incorporated into our framework by stipulating that a game admits weak self-stabilization if from every initial joint strategy some improvement path ensures self-stabilization. Schedulers that ensure self-stabilization obviously establish weak self-stabilization. This property naturally corresponds to the class of weakly acyclic games introduced in [30] and [24]. They are defined by the following weakening of the FIP: a game is weakly acyclic if for every initial joint strategy there exists a finite improvement path that starts in it. For a thorough analysis of weakly acyclic games see [3] from which we adopted the concept of a scheduler.

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