The Connection Between Density Matrix Method, Supersymmetric Quantum Mechanics and Lewis-Riesenfeld Invariant Theory

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This paper is concerned with the connection between density matrix method, supersymmetric quantum mechanics and Lewis-Riesenfeld invariant theory. It is shown that these three formulations share the common mathematical structure: specifically, all of them have the invariant operators which satisfy the Liouville-Von Neumann equation and the solutions to the time-dependent Schrödinger equation and/or Schrödinger eigenvalue equation can be constructed in terms of the eigenstates of the invariants.

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I. LEWIS-RIESENFELD INVARIANT THEORY

Lewis-Riesenfeld invariant theory can be applied to the solutions of time-dependent Schrödinger equation. In order to illustrate the Lewis-Riesenfeld invariant theory [1] easily, we consider a one-dimensional system whose Hamiltonian $H(t)$ is time-dependent. According to the Lewis-Riesenfeld invariant theory [1], a Hermitian operator $I(t)$ is called invariant if it satisfies the following invariant equation, i.e., the Liouville-Von Neumann equation (in the unit $\hbar = 1$)

$$\frac{\partial I(t)}{\partial t} + \frac{1}{i}[I(t), H(t)] = 0. \tag{1.1}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$I(t)|\lambda_n, t\rangle = \lambda_n |\lambda_n, t\rangle, \tag{1.2}$$

where

$$\frac{\partial \lambda_n}{\partial t} = 0. \tag{1.3}$$

The time-dependent Schrödinger equation (in the unit $\hbar = 1$) for the system is

$$i\frac{\partial |\Psi(t)\rangle_s}{\partial t} = H(t)|\Psi(t)\rangle_s. \tag{1.4}$$

In terms of the Lewis-Riesenfeld invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of Eq.(1.4) differs from the eigenfunction $|\lambda_n, t\rangle$ of the invariant $I(t)$ only by a phase factor $\exp[\frac{1}{i}\phi_n(t)]$, then the general solution of the Schrödinger equation (1.4) can be written as

$$|\Psi(t)\rangle_s = \sum_n C_n \exp[\frac{1}{i}\phi_n(t)] |\lambda_n, t\rangle, \tag{1.5}$$

where

$$\phi_n(t) = \int_0^t \langle \lambda_n, t' | H(t') - i\frac{\partial}{\partial t'} |\lambda_n, t'\rangle \, dt', \tag{1.6}$$

$$C_n = \langle \lambda_n, t = 0 |\Psi(t = 0)\rangle_s.$$
$|\lambda_n, t\rangle = \exp\left[\frac{i}{\hbar}\phi_n(t)\right]|\lambda_n, t\rangle \ (n = 1, 2, \cdots)$ are said to form a complete set of the solutions of Eq.(1.4). The statement outlined above is the basic content of the Lewis-Riesenfeld invariant theory.

A time-dependent unitary transformation operator can be constructed to transform $I(t)$ into a time-independent invariant $I_V \equiv V(t)I(t)V(t)$ [2,3] with

$$I_V |\lambda_n\rangle = \lambda_n |\lambda_n\rangle,$$

$$|\lambda_n\rangle = V(t)|\lambda_n, t\rangle. \tag{1.7}$$

Under the unitary transformation $V(t)$, the Hamiltonian $H(t)$ is correspondingly changed into $H_V(t)$

$$H_V(t) = V(t)H(t)V(t) - V(t)i\frac{\partial V(t)}{\partial t}. \tag{1.9}$$

In accordance with this unitary transformation method [2], it is very easy to verify that the particular solution $|\lambda_n, t\rangle_{s\theta}$ of the time-dependent Schrödinger equation associated with $H_V(t)$

$$i\frac{\partial |\lambda_n, t\rangle_{s\theta}}{\partial t} = H_V(t)|\lambda_n, t\rangle_{s\theta} \tag{1.10}$$

is different from the eigenfunction $|\lambda_n\rangle$ of $I_V$ only by the same phase factor $\exp[\frac{1}{\hbar}\phi_n(t)]$ as that in Eq.(1.5), i.e.,

$$|\lambda_n, t\rangle_{s\theta} = \exp[\frac{1}{\hbar}\phi_n(t)]|\lambda_n\rangle. \tag{1.11}$$

Substitution of $|\lambda_n, t\rangle_{s\theta}$ of Eq.(1.10) into Eq.(1.11) yields

$$\dot{\phi}(t)|\lambda_n\rangle = H_V(t)|\lambda_n\rangle, \tag{1.12}$$

which means that $H_V(t)$ differs from $I_V(t)$ only by a time-dependent multiplying c-number factor. It can be seen from Eq.(1.12) that the particular solution of Eq.(1.10) can be easily obtained by calculating the phase from Eq.(1.12). Thus, one is led to the conclusion that if the $V(t)$, $I_V$, $H_V(t)$ and the eigenfunction $|\lambda_n\rangle$ of $I_V$ have been found, the problem of solving the complicated time-dependent Schrödinger equation (1.4) reduces to that of solving the much simplified equation (1.10).

II. DENSITY MATRIX METHOD

Density matrix method has many helpful applications to the semiclassical theory of laser [4].

The density matrix operator is defined to be $\rho(t) = |\Psi(t)\rangle \langle \Psi(t)|$, the wavefunction $|\Psi(t)\rangle$ of which agrees with the time-dependent Schrödinger equation. It follows that the density operator satisfies the following time-evolution equation

$$\frac{\partial \rho(t)}{\partial t} + \frac{1}{i} [\rho(t), H(t)] = 0, \tag{2.1}$$

which is just the Liouville-Von Neumann equation (1.1). This, therefore, means that the density operator $\rho(t)$ is the Lewis-Riesenfeld invariant.

In what follows we will discuss the density matrix method that solves the time-dependent Schrödinger equation governing the semiclassical laser process. As an illustrative example, we consider a two-level atomic system ($|a\rangle$ and $|b\rangle$) interacting with the classical Maxwellian electromagnetic fields. The time-dependent wavefunction $|\Psi(t)\rangle$ can be expressed in terms of $|a\rangle$ and $|b\rangle$, namely,

$$|\Psi(t)\rangle = c_a(t)|a\rangle + c_b(t)|b\rangle, \tag{2.2}$$

and the density matrix can be rewritten as

$$\rho(t) = \begin{pmatrix} c_a & \ast \\ \ast & c_b^* \end{pmatrix} = \begin{pmatrix} c_ac_b^* & c_ac_b^* \\ c_bc_a^* & c_bc_b^* \end{pmatrix} \tag{2.3}$$

in the representation of $|a\rangle$ and $|b\rangle$. 2
The Hamiltonian of two-level atomic system takes the form
\[ H = H_0 + V, \]
where the atomic free Hamiltonian \( H_0 \) and the interaction Hamiltonian \( V \) are respectively of the form
\[ H_0 = \begin{pmatrix} \omega_a & 0 \\ 0 & \omega_b \end{pmatrix}, \quad V = \begin{pmatrix} 0 & V_{ab} \\ V_{ba} & 0 \end{pmatrix}. \]

Substitution of (2.3)-(2.5) into (2.1) yields
\[ \rho_{aa} = -i V_{ab} \rho_{ba} + c.c., \quad \rho_{bb} = i V_{ab} \rho_{ba} + c.c., \quad \rho_{ab} = -i (\omega_a - \omega_b) \rho_{ab} + i V_{ab} (\rho_{aa} - \rho_{bb}) \]
with \( \rho_{aa} = c_a c_a^\dagger, \rho_{ab} = c_a c_b^\dagger, \rho_{ba} = c_b c_a^\dagger \) and \( \rho_{bb} = c_b c_b^\dagger \). So the resolution of the Schrödinger equation can be ascribed to the solution of the Eq.(2.6) (i.e., the Liouville-Von Neumann equation (1.1)), which is similar to the case in Lewis-Riesenfeld invariant theory, where we should obtain the expression for the invariant \( I(t) \) via the Liouville-Von Neumann equation and solve the eigenvalue equation (1.2) of \( I(t) \).

### III. SUPERSYMMETRIC QUANTUM MECHANICS

Supersymmetric quantum mechanics is applicable to the energy eigenvalue problems of one-dimensional potential walls.

The sequence of ideas of supersymmetric quantum mechanics is to obtain the eigenstates of the following two Hamiltonians
\[ H_\pm = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_\pm(x) \]
via the eigenstates of \( H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \), where \( V_\pm(x) \) agrees with the following two equations:
\[ \frac{1}{2}[V_+(x) + V_-(x)] = W(x)^2, \quad \frac{1}{2}[V_+(x) - V_-(x)] = \frac{\hbar}{\sqrt{2m}} W'(x) \]
with \( W'(x) \) denoting the derivative of \( W(x) \) with respect to \( x \), and \( W(x) \) is so defined that the ground state \( \psi_0(x) \) of \( H \) is
\[ \psi_0(x) = \exp \left[ -\frac{\sqrt{2m}}{\hbar} \int^x dx W(x) \right]. \]

Define the lowering and raising operators \( A \) and \( A^\dagger \) as follows
\[ A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \]
and therefore we have \( H_- = A^\dagger A \) and \( H_+ = AA^\dagger \) and the commuting relation \( [A, A^\dagger] = \frac{2\hbar}{\sqrt{2m}} W'(x) \).

By using the properties of lowering and raising operators \( A \) and \( A^\dagger \), we can obtain the eigenstates of \( H_\pm \), which are constructed in terms of the eigenstates of \( H \).

It is readily verified that here the Hamiltonian \( H \) serves as the invariant that satisfies the Liouville-Von Neumann equation. Since here \( H \) is time-independent, we should calculate the commuting relation \( [H_+, H] \) only. According to Eqs.(3.2), we obtain \( V_+ = W(x)^2 + \frac{\hbar^2}{2m} W'(x) \) and consequently
\[ V_+ = \frac{\hbar^2}{2m} \frac{d^2 \psi_0(x)}{dx^2}. \]

In accordance with the eigenvalue equation \( [-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)] \psi_0(x) = \varepsilon_0 \psi_0(x) \), we get \( V_+(x) = V(x) - \varepsilon_0 \). So
\[ [H_+, H] = [V_+(x) - V(x), -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}] = 0, \]
because of \( V_+(x) - V(x) = -\varepsilon_0 \). This, therefore, means that the Hamiltonian \( H \) is an invariant.

For the detailed applications of supersymmetric quantum mechanics to the energy eigenvalue problems of one-dimensional potential walls and related problems, Readers may be referred to the references [5–8].
IV. CONCLUDING REMARKS

In general, the Lewis-Riesenfeld invariant theory is applicable to the multi-level atomic quantum system interacting with the second-quantized electromagnetic fields [9,10], while the density matrix method is appropriate to treat the multi-level atomic quantum system interacting with the classical Maxwellian electromagnetic fields (waves) [4]. The former formulation can also be applied to the motion of a charged spinning particle in a classical magnetic fieldcite [11] or of a photon moving in a sufficiently perfect curved fibercite [9].

In the present paper we show that the above three formulations share the common mathematical structure: specifically, all of them have the invariant operators which satisfies the Liouville-Von Neumann equation and the solutions to the time-dependent Schrödinger equation and/or Schrödinger eigenvalue equation can be constructed in terms of the eigenstates of the invariants.

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[1] Lewis H and Riesenfeld W B 1969 J. Math. Phys. 10 1458
[2] Gao X C, Xu J B and Qian T Z 1991 Phys. Rev. A 44 7016
[3] Gao X C, Xu J B and Qian T Z 1991 Phys. Lett. A 152 449; Gao X C, Fu J, Xu J B, Zou X B, 1999 Phys. Rev. A 59 55
[4] Li F L Advanced Laser Physics (Chin. Sci. Tech. Uni. Press, Hefei China, 1992) Chap. 3-4
[5] Cooper F and Freedman B 1983 Ann. Phys. 146 262
[6] Gendenshtein L 1983 JEPT Lett. 38 356
[7] Sukumar C V 1987 J. Phys. A 20 2461
[8] Dutt R, Khare A and Sukhatme U P 1988 Am. J. Phys. 56 163
[9] Shen J Q and Zhu H Y 2003 Ann. Phys.(Leipzig) 12 131
[10] Shen J Q, Zhu H Y and Mao H 2002 J. Phys. Soc. Jpn. 71 1440
[11] Shen J Q, Zhu H Y, Shi S L and Li J 2002 Phys. Scr. 65 465