On the largest critical value of $T_n^{(k)}$

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Abstract

We study the quantity $\tau_{n,k} := \frac{|T_n^{(k)}(\omega_{n,k})|}{T_n^{(k)}(1)}$, where $T_n$ is the Chebyshev polynomial of degree $n$, and $\omega_{n,k}$ is the rightmost zero of $T_n^{(k+1)}$.

Since the absolute values of the local maxima of $T_n^{(k)}$ increase monotonically towards the end-points of $[-1, 1]$, the value $\tau_{n,k}$ shows how small is the largest critical value of $T_n^{(k)}$ relative to its global maximum $T_n^{(k)}(1)$.

In this paper, we improve and extend earlier estimates by Erdős–Szegő, Eriksson and Nikolov in several directions. Firstly, we show that the sequence $\{\tau_{n,k}\}_{n=k+2}^{\infty}$ is monotonically decreasing in $n$, hence derive several sharp estimates, in particular

$$
\tau_{n,k} \leq \begin{cases} 
\tau_{k+4,k} & = \frac{3}{2k+3}, & n \geq k + 4, \\
\tau_{k+6,k} & = \frac{1}{2k+3} \left( \frac{2}{1} \right)^2 \beta_k, & n \geq k + 6,
\end{cases}
$$

where $\beta_k < \frac{2}{\sqrt{10}} \approx 1.032$.

We also obtain an upper bound which is uniform in $n$ and $k$, and that implies in particular

$$
\tau_{n,k} \approx \left( \frac{2}{e} \right)^k, \quad n \geq k^{3/2}; \quad \tau_{n,n-m} \approx \left( \frac{m}{2} \right)^{m/2} n^{-m/2}, \quad \tau_{n,n/2} \approx \left( \frac{4}{\sqrt{27}} \right)^{n/2}.
$$

Finally, we derive the exact asymptotic formulae for the quantities

$$
\tau_k^* := \lim_{n \to \infty} \tau_{n,k} \quad \text{and} \quad \tau_m^* := \lim_{n \to \infty} n^{m/2} \tau_{n,n-m},
$$

which show that our upper bounds for $\tau_{n,k}$ and $\tau_{n,n-m}$ are asymptotically correct with respect to the exponential terms given above.

1 Introduction and statement of the results

We study the quantity

$$
\tau_{n,k} := \frac{|T_n^{(k)}(\omega_k)|}{T_n^{(k)}(1)},
$$

where $T_n$ is the Chebyshev polynomial of degree $n$, and $\omega_k$ is the rightmost zero of $T_n^{(k+1)}$.

Since the absolute values of the local maxima of $T_n^{(k)}$ increase monotonically towards the end-points of $[-1, 1]$, the value $\tau_{n,k}$ shows how small is the largest critical value of $T_n^{(k)}$ relative to its global maximum $T_n^{(k)}(1)$ (see Figure 1).

This value is useful in several applications which include some Markov-type inequalities [7], [14], [18], the Landau–Kolmogorov inequalities for intermediate derivatives [8], [18], where the
estimates of $f^{(k)}$ on a subinterval slightly smaller than $[-1, 1]$ are needed, and also in studying extreme zeros of ultraspherical polynomials.

Let us mention previous results. For the first derivative ($k = 1$), Erdős–Szegő [7] showed that
\[ \tau_{3,1} = \frac{1}{3}, \quad \tau_{4,1} = \frac{1}{3}(2^{1/2}) \]
proved that
\[ \tau_{n,1} \leq \frac{1}{4}, \quad n \geq 5. \quad (1.1) \]
For arbitrary $k \geq 1$, Eriksson [8] and Nikolov [14] independently showed that
\[ \tau_{n,k} \leq \frac{1}{2k+1}, \quad n \geq k+2, \quad (1.2) \]
with a better estimate when $n$ is large relative to $k$,
\[ \tau_{n,k} \leq \frac{1}{2k+1} \cdot \frac{8}{2k+7}, \quad n \gtrsim k^{3/2}. \quad (1.3) \]
(The exact condition in [8], [14] was $\omega_{n,k} \geq 1 - \frac{8}{2k+7}$, which implies the above inequality between $n$ and $k$ via the upper estimate $\omega_{n,k} < 1 - \frac{k^2}{2\pi^2}$.)

In this paper, motivated by the applications mentioned above, we refine and extend inequalities (1.1)–(1.3) in several directions.

1) Our first observation is a monotone behaviour of the value $\tau_{n,k}$ with respect to $n$.

\[ \text{Theorem 1.1} \quad \text{For a fixed } k \in \mathbb{N}, \text{the values } \tau_{n,k} \text{ decrease monotonically in } n, \text{i.e.,} \]
\[ \tau_{n+1,k} < \tau_{n,k} < \cdots < \tau_{k+3,k} < \tau_{k+2,k}. \quad (1.4) \]
In particular, for any fixed $k \in \mathbb{N}$ and any $m \geq 2$, we have
\[ \tau_{n,k} \leq \tau_{k+m,k}, \quad n \geq k + m. \quad (1.5) \]

In fact, such a monotone decrease of the relative values of the local extrema takes place for the ultraspherical polynomials $P_n^{(\lambda)}$ with any parameter $\lambda > 0$. This remarkable result is due to Szász [19] and, for reader’s convenience and to keep the paper self-contained, we state it as Theorem 2.1 and give a short proof.

2) Our next result is several sharp estimates for $\tau_{n,k}$ which follow from (1.5). Namely, since $\tau_{k+m}^{(k)}$ is a symmetric polynomial of degree $m$, for small $m = 2.6$ we compute the value of its largest extremum, hence $\tau_{k+m,k}$, explicitly and then use (1.5).
Theorem 1.2 We have

\[
\tau_{n,k} \leq \begin{cases} 
\tau_{k+2,k} = \frac{1}{2k+1}, & n \geq k + 2, \\
\tau_{k+3,k} = \frac{1}{2k+1} \left( \frac{2}{k+2} \right)^{1/2}, & n \geq k + 3, \\
\tau_{k+4,k} = \frac{1}{2k+1} \left( \frac{3}{k+3} \right), & n \geq k + 4.
\end{cases}
\tag{1.6}
\]

These estimates contain earlier results (1.1)-(1.2) as particular cases, and the last inequality in (1.6) improves (1.3) by the factor of \(\frac{3}{4}\) and removes the unnecessary restrictions on \(n\) and \(k\).

The next pair of estimates strengthens (1.6). It also shows that, although the nice pattern for \(\tau_{k+m,k}\) in (1.6) is no longer true for \(m \geq 5\), an approximate behaviour \(\tau_{k+m,k} \approx \left( \frac{m}{k+m} \right)^{m/2}\) is very much suggestive.

Theorem 1.3 We have

\[
\tau_{n,k} \leq \begin{cases} 
\tau_{k+5,k} = \frac{1}{2k+1} \left( \frac{4}{k+4} \right)^{3/2} \alpha_k, & n \geq k + 5, \\
\tau_{k+6,k} = \frac{1}{2k+1} \left( \frac{5}{k+5} \right)^{2} \beta_k, & n \geq k + 6,
\end{cases}
\tag{1.7}
\]

where the values \(\alpha_k, \beta_k\) increase monotonically to the following limits,

\[
\alpha_k < \alpha_* = \frac{\sqrt{3(3 + \sqrt{6})}}{4} = 1.0108.., \quad \beta_k < \beta_* = \frac{2 + \sqrt{10}}{5} = 1.0325..
\]

Let us note that, because of monotonicity of \(\tau_{n,k}\), for any fixed moderate \(k\) and a moderate \(n_0\), one can compute numerically the value \(\tau_{n_0,k}\), thus getting for particular \(k\) the estimate

\[
\tau_{n,k} \leq \tau_{n_0,k}, \quad n \geq n_0.
\]

which would be better than those in (1.6) and (1.7).

3) An approximate behaviour \(\tau_{k+m,k} \approx \left( \frac{m}{k+m} \right)^{m/2}\) in (1.6)-(1.7) suggests that when \(m\) is fixed and \(k\) grows, then \(\tau_{n,n-m} = \tau_{k+m,k}\) is of a polynomial decay in \(n\), i.e.,

\[
\tau_{n,n-m} = O\left(n^{-m/2}\right) \quad (n \to \infty),
\]

while when \(k\) is fixed and \(n\) grows, we have an exponential estimate in \(k\),

\[
\tau_{n,k} = O\left(e^{-\gamma k}\right) \quad (n \to \infty).
\]

We prove that such a behaviour is indeed the case by establishing first the upper bounds for \(\tau_{n,k}\) which are uniform in \(n\) and \(k\), and then considering different relations between \(n\) and \(k\).
Theorem 1.4 For every $n, k \in \mathbb{N}$ with $n \geq k + 2$, we have
\[
\tau_{n,k}^2 \leq \frac{1}{2} \left( 1 + \frac{k}{n} \right) \left( \frac{n}{k} \right)^{2k} \left( \frac{n + k}{n - k} \right)^{-1}
\leq c_1^2 k^{3/2} \left( 1 - \frac{k^2}{n^2} \right)^{1/2} \frac{(2n)^{2k} (n - k)^{n-k}}{(n + k)^{n+k}}, \quad c_1^2 = \frac{e^2}{2\sqrt{\pi}}.
\] (1.8)

As a consequence of Theorem 1.4 we obtain the following statement.

Theorem 1.5 We have the following estimates:

(i) if $k \in \mathbb{N}$ is fixed and $n$ grows, then
\[
\tau_{n,k} \leq c_2 \left( \frac{2}{e} \right)^{-2} k^{1/4} (1 - k^2/n^2)^{-k/2},
\] (1.10)

in particular
\[
\tau_{n,k} < c_2 \left( \frac{2}{e} \right)^{-2} k^{1/4}, \quad n \geq k^{3/2};
\] (1.11)

(ii) if $n - k = m \in \mathbb{N}$ is fixed and $n$ grows, then
\[
\tau_{n,n-m} \leq c_3 m^{1/4} \left( \frac{m}{2} \right)^{m/2} n^{-m/2};
\] (1.12)

(iii) if $k = \lfloor \lambda n \rfloor \in \mathbb{N}$, where $\lambda \in (0, 1)$, and $n$ grows, then we have an exponential decay
\[
\tau_{n,\lambda n} \leq c_4 n^{1/4} \rho_{\lambda}^{n/2}, \quad \rho_{\lambda} < 1,
\] in particular
\[
\tau_{n,n/2} < c_1 n^{1/4} \left( \frac{4}{\sqrt{27}} \right)^{n/2}.
\]

We can reformulate Theorem 1.5 in the form which shows, for a fixed $k$ and growing $n$, the rate of decrease of the values $\tau_{n,k}$ in (1.4).

Corollary 1.6 We have
\[
\tau_{n,k} \lesssim \begin{cases} 
 k^{-m/2}, & n \geq k + m, \\
 \left( \frac{4}{\sqrt{27}} \right)^k, & n \geq 2k, \\
 \left( \frac{2}{e} \right)^k, & n \geq k^{3/2}.
\end{cases}
\] (1.13)

Remark 1.7 Exponential estimate (1.11) becomes superior to the polynomial estimates (1.7) only when $k \geq 10$.

4) Finally, we establish the asymptotics of the values of $\lim_{n \to \infty} \tau_{n,k}$ and $\lim_{n \to \infty} \tau_{n,n-m}$ which shows that the upper bounds in (1.11) and (1.12) are asymptotically correct with respect to the exponential terms therein.

Theorem 1.8 We have
\[
\tau_k := \lim_{n \to \infty} \tau_{n,k} = C_0 \left( \frac{2}{e} \right)^k \rho_k^{n/2}, \quad \rho_k = a_0 k^{1/3} k^{-1/6} (1 + O(k^{-1/3})),
\] (1.14)
\[
\tau_m := \lim_{n \to \infty} n^{m/2} \tau_{n,n-m} = C_1 \left( \frac{e m}{2} \right)^{m/2} \rho_m^{m/2}, \quad \rho_m = a_1 m^{1/3} m^{-1/6} (1 + O(m^{-1/3})),
\] (1.15)

where the pairs of constants
\[
a_0 = 1.8557..., \quad C_0 = 1.1966.. \quad \text{and} \quad a_1 = 2.3381..., \quad C_1 = 1.0660...
\]
can be explicitly represented in terms of the Airy function.
2 Monotonicity of the sequence \( \{\tau_{n,k}\}_{n \geq k+2} \)

Here, we prove that

\[
\mu_{i,n} := \frac{P_n^{(\lambda)}(y_{i,n})}{P_n^{(\lambda)}(1)},
\]

the relative values of the ordered local extrema of the ultraspherical polynomials \( P_n^{(\lambda)} \) with parameter \( \lambda \) decay monotonically with respect to \( n \) for any \( \lambda > 0 \). This includes Theorem 1.1 as a particular case since \( T_n^{(k)} \), the \( k \)-th derivative of the Chebyshev polynomials of degree \( n \), coincide up to a factor with \( P_n^{(\lambda)} \) where \( \lambda = k \).

We start with recalling some known facts about the ultraspherical polynomials (for more details, see [20] Chapter 4.7).

For \( \lambda > -\frac{1}{2} \), \( \{P_n^{(\lambda)}\}_{n \in \mathbb{N}_0} \) stands for the sequence of ultraspherical polynomials, which are orthogonal on \([-1, 1]\) with respect to the weight function \( w_\lambda(x) = (1-x^2)^{\lambda-1/2} \), with the standard normalization

\[
P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}, \quad \lambda \neq 0.
\]

The Chebyshev polynomials of the first and the second kind and the Legendre polynomials are particular cases of ultraspherical polynomials, they correspond up to a factor to the values \( \lambda = 0, 1 \) and \( \frac{1}{2} \), respectively. Moreover, due to the properties

\[
T_n^{(k)} = c_{n,k} P_n^{(\lambda)}(x), \quad \lambda = k, \quad k = 1, \ldots, n.
\]

We will work with the re-normalised ultraspherical polynomials

\[
p_n^{(\lambda)}(x) := P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1),
\]

so that \( p_n^{(\lambda)}(1) = 1 \). It is clear that the absolute values of the local extrema of \( p_n^{(\lambda)} \) are equal to the relative values of the local extrema of \( P_n^{(\lambda)} \) compared to \( P_n^{(\lambda)}(1) \).

Theorem 1.1 is a consequence of the following general statement.

**Theorem 2.1** Let \( y_{1,n}^{(\lambda)} > y_{2,n}^{(\lambda)} > \cdots > y_{n-1,n}^{(\lambda)} \) be the zeros of the ultraspherical polynomial \( p_{n-1}^{(\lambda+1)} \), i.e., the abscissae of the local extrema of \( p_{n-1}^{(\lambda+1)} \), in the reverse order. Set \( y_{n,n}^{(\lambda)} := -1 \), and denote

\[
\mu_{i,n}^{(\lambda)} := |p_n^{(\lambda)}(y_{i,n}^{(\lambda)})|, \quad i = 1, \ldots, n.
\]

1) If \( \lambda > 0 \), then

\[
\mu_{i,n+1}^{(\lambda)} < \mu_{i,n}^{(\lambda)} \quad \text{for} \quad i = 1, 2, \ldots, n.
\]
2) If \(-\frac{1}{2} < \lambda < 0\), then inequalities (2.3) hold with the opposite sign.

3) If \(\lambda = 0\), then \(p_n^{(\lambda)} = T_n\) and we have equalities in (2.3) as all local extrema of \(T_n\) and \(T_{n+1}\) are of the absolute value 1.)

**Proof.** We omit index \(\lambda\), so set \(p_n := p_n^{(\lambda)}\), and we will use the next two identities which readily follow from [20 eqn.(4.7.28)]:

\[
p_n(x) = -\frac{1}{n + 2\lambda} x p_n'(x) + \frac{1}{n + 1} p_{n+1}'(x),
\]

\[
p_{n+1}(x) = -\frac{1}{n + 2\lambda} p'_n(x) + \frac{1}{n + 1} x p_{n+1}'(x).
\]

From those we deduce that

\[
p_n(x)^2 - p_{n+1}(x)^2 = (1 - x^2)\left[\frac{1}{(n + 1)^2} p_{n+1}'(x)^2 - \frac{1}{(n + 2\lambda)^2} p_n'(x)^2\right],
\]

and we rearrange his equality as follows,

\[
f(x) := p_n(x)^2 + \frac{1 - x^2}{(n + 2\lambda)^2} p_n'(x)^2 p_{n+1}(x)^2 + \frac{1 - x^2}{(n + 1)^2} p_{n+1}'(x)^2.
\]

Clearly, \(f\) is a polynomial of degree 2\(n\) which interpolates both \(p_n\) and \(p_{n+1}\) at the points of their local maxima in \([-1, 1]\). Moreover, \(f'\) vanishes at the zeros of both \(p_n\) and \(p_{n+1}\), therefore, with some constant \(c_n^{(\lambda)}\),

\[
f'(x) = c_n^{(\lambda)} p_n'(x)p_{n+1}'(x).
\]

Next, we determine the sign of \(c_n^{(\lambda)}\). Let \(a_n, a_{n+1}\) be the leading coefficients of \(p_n\) and \(p_{n+1}\), respectively, and note that, since \(p_n(1) = p_{n+1}(1) = 1\), we have \(a_n, a_{n+1} > 0\). Then, equating the leading coefficients of \(f'\) in representations (2.5) and (2.6) respectively, we obtain

\[
2n a_n^2 \left(1 - \frac{n^2}{(n + 2\lambda)^2}\right) = c_n^{(\lambda)} n(n + 1) a_n a_{n+1},
\]

whence

\[
c_n^{(\lambda)} = \frac{1}{n + 1} a_n \frac{4\lambda(2n + 2\lambda)}{(n + 2\lambda)^2} \Rightarrow \text{sign } c_n^{(\lambda)} = \text{sign } \lambda.
\]

Thus, (2.6) becomes

\[
f'(x) = c \lambda p_n'(x)p_{n+1}'(x), \quad c = c_n, \lambda > 0.
\]

Now we can prove Theorem 2.1. Let \(\lambda > 0\). Then from (2.7) and the interlacing of zeros of \(p_n\) and \(p_{n+1}\) we conclude that

\[
f'(x) < 0, \quad x \in (y_{i,n}^{(\lambda)}, y_{i+1,n}^{(\lambda)}), \quad i = 1, \ldots, n,
\]

i.e., \(f\) is monotonically decreasing on each interval \((y_{i,n}^{(\lambda)}, y_{i+1,n}^{(\lambda)})\). From (2.5), we have

\[
f(y_{i,n}^{(\lambda)}) = p_n(y_{i,n}^{(\lambda)})^2 = |\mu_{i,n}^{(\lambda)}|^2,
\]

\[
f(y_{i,n+1}^{(\lambda)}) = p_{n+1}(y_{i,n+1}^{(\lambda)})^2 = |\mu_{i,n+1}^{(\lambda)}|^2.
\]

therefore

\[
|\mu_{i,n+1}^{(\lambda)}| < |\mu_{i,n}^{(\lambda)}|, \quad i = 1, \ldots, n.
\]

Clearly, if \(\lambda < 0\), then the sign is reversed. \(\square\)
Proof of Theorem 1.1 By (2.1) and (2.2), we have
\[ \frac{T_n^{(k)}(x)}{T_n^{(k)}(1)} = \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)} = p_{n-k}^{(\lambda)}(x), \quad \lambda = k \geq 1. \]
Hence, \(\tau_{n,k} = \mu_{1,n-k}\), and then Theorem 2.1 yields
\[ \tau_{n,k} = \mu_{1,n-k} > \mu_{1,n+1-k} = \tau_{n+1,k}. \]

Theorem 1.1 is proved.

3 Proof of Theorems 1.2-1.3

By Theorem 1.1 for any fixed \(m\), the value \(\tau_{k+m,k}\) gives an upper bound for all \(\tau_{n,k}\), namely
\[ \tau_{n,k} \leq \tau_{k+m,k}, \quad n \geq k + m, \]
so here we determine the latter values directly for \(m = 2.6\).

We will need the expansion formula for the \(n\)-th Chebyshev polynomial,
\[ T_n(x) = \frac{n}{2} \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{(n-i-1)!}{i!(n-2i)!} (2x)^{n-2i} \]
\[ = 2^{n-1} x^n - 2^{n-3} nx^{n-2} + 2^{n-6} n(n-3) x^{n-4} - \frac{1}{3} 2^{n-8} n(n-4)(n-5) x^{n-6} + \ldots \]
From this we compute expression for \(T_n^{(n-6)}\) in (3.7), and then differentiate it to find all further derivatives \(T_n^{(n-m)}\) for \(m = 5,2\).

We will denote the point of the rightmost local extrema of \(T_n^{(k)}\) by \(x_\ast\), i.e., \(x_\ast := \omega_{n,k}\). Since \(T_n^{(k+1)}(x_\ast) = 0\) then, for the value of \(T_n^{(k)}(x_\ast)\) we will also use simplifications arising from the formula
\[ T_n^{(k)}(x_\ast) = T_n^{(k)}(x_\ast) - c_{n,k} x_\ast T_n^{(k+1)}(x_\ast) \]
where we choose the constant \(c_{n,k}\) to cancel high degree monomials.

1) The case \(k = n - 2\) (or equivalently \(n = k + 2\)). We have
\[ T_n^{(n-2)}(x) = c^{-1} \left[ 2(n-1)x^2 - 1 \right], \]
whence \(x_\ast = 0\) and
\[ \tau_{n,n-2} = \frac{T_n^{(n-2)}(x_\ast)}{T_n^{(n-2)}(1)} = \frac{1}{2n-3} \Rightarrow \tau_{k+2,k} = \frac{1}{2k+1}. \]

2) The case \(k = n - 3\) (or equivalently \(n = k + 3\)). We obtain
\[ T_n^{(n-3)}(x) = c^{-1} \left[ 2(n-1)x^3 - 3x \right], \]
hence \(c T_n^{(n-3)}(1) = 2n - 5\). From (3.3), we find \(x_\ast^2 = \frac{1}{2(n-1)}\) and
\[ c T_n^{(n-3)}(x_\ast) = -2x_\ast = -\frac{2}{\sqrt{2(n-1)}}. \]
Respectively,
\[
\tau_{n,n-3} = \frac{|T^{(n-3)}_{n}(x_*)|}{T^{(n-3)}_{n}(1)} = \frac{1}{2n-5} \sqrt{\frac{2}{n-1}} \Rightarrow \tau_{k+3,k} = \frac{1}{2k+1} \sqrt{\frac{2}{k+2}}.
\]

3) The case \( k = n - 4 \) (or equivalently \( n = k + 4 \)). We have
\[
T^{(n-4)}_{n}(x) = c^{-1} \left[ 4(n-1)(n-2)x^4 - 12(n-2)x^2 + 3 \right].
\] (3.5)
hence
\[
cT^{(n-4)}_{n}(1) = 4n^2 - 24n + 35 = (2n-5)(2n-7).
\]
From (3.4), we find \( x^2_* = \frac{3}{2(n-1)} \) and
\[
cT^{(n-4)}_{n}(x_*) = -6(n-2)x^2_* + 3 = \frac{3}{n-1} \left[ -3(n-2) + (n-1) \right] = -\frac{3(2n-5)}{n-1}.
\]
Respectively,
\[
\tau_{n,n-4} = \frac{|T^{(n-4)}_{n}(x_*)|}{T^{(n-4)}_{n}(1)} = \frac{1}{2n-7} \frac{3}{n-1} \Rightarrow \tau_{k+4,k} = \frac{1}{2k+1} \frac{3}{k+3}.
\]
The cases 1)-3) prove estimates (1.6), hence Theorem 1.2. \( \square \)

4) The case \( k = n - 5 \) (or equivalently \( n = k + 5 \)). We have
\[
T^{(n-5)}_{n}(x) = c^{-1} \left[ 4(n-1)(n-2)x^5 - 20(n-2)x^3 + 15x \right].
\] (3.6)
hence
\[
cT^{(n-5)}_{n}(1) = 4n^2 - 32n + 63 = (2n-9)(2n-7).
\]
From (3.5), we find
\[
x^2_* = \frac{3(n-2) + \sqrt{9(n-2)^2 - 3(n-1)(n-2)}}{2(n-1)(n-2)} = \frac{1}{n-1} \frac{3 + \sqrt{6-t}}{2}, \quad t := t_k = \frac{3}{n-2} = \frac{3}{k+3}.
\]
and
\[
cT^{(n-5)}_{n}(x_*) = -4x_* \left[ 2(n-2)x^2_* - 3 \right].
\]
After simplifications we obtain
\[
\tau_{n,n-5} = \frac{|T^{(n-5)}_{n}(x_*)|}{T^{(n-5)}_{n}(1)} = \frac{1}{2n-9} \frac{8}{(n-1)^{3/2}} \alpha_k \Rightarrow \tau_{k+5,k} = \frac{1}{2k+1} \frac{4^{3/2}}{(k+4)^{3/2}} \alpha_k,
\]
where
\[
\alpha_k := \frac{1}{2\sqrt{2}} \frac{\sqrt{3 + \sqrt{6-t}}}{2-t} (\sqrt{6-t} - t) = \frac{1}{2\sqrt{2}} \frac{(y + 3)^{3/2}}{y + 2} =: f(y), \quad y := \sqrt{6-t}.
\]
The function \( f \) is increasing for \( y > 0 \), hence
\[
t_k > t_{k+1} \Rightarrow y_k < y_{k+1} \Rightarrow \alpha_k < \alpha_{k+1} < \alpha_\star
\]
where

\[
\alpha_* = \lim_{t \to 0} \alpha_k = \frac{1}{2\sqrt{2}} \sqrt{\frac{3 + \sqrt{6}}{2}} \sqrt{\frac{3(3 + \sqrt{6})}{4}}.
\]

5) The case \( k = n - 6 \) (or equivalently \( n = k + 6 \)). From (3.2), we have

\[
T_n^{(n-6)}(x) = c^{-1} \left[ 8(n-1)(n-2)(n-3)x^6 - 60(n-2)(n-3)x^4 + 90(n-3)x^2 - 15 \right], \tag{3.7}
\]

hence

\[
cT_n^{(n-6)}(1) = 8n^3 - 108n^2 + 478n - 693 = (2n-11)(2n-9)(2n-7).
\]

From (3.6), we find

\[
x_*^2 = \frac{5(n-2)}{2(n-1)(n-2)} + \frac{\sqrt{25(n-2)^2 - 15(n-1)(n-2)}}{2(n-1)(n-2)}
\]

\[
= \frac{1}{n-1} \left( \frac{5 + \sqrt{10 - 3t}}{2} \right), \quad t := t_k = \frac{5}{n-2} = \frac{5}{k+4}
\]

and

\[
cT_n^{(n-6)}(x_*) = -4(2n-7)x_*^2 \left[ 2(n-2)x_*^2 - 5 \right].
\]

After simplifications, we obtain

\[
\tau_{n,n-6} = \frac{T_n^{(n-6)}(x_*)}{T_n^{(n-6)}(1)} = \frac{1}{2n-11} \frac{5^2}{(n-1)^2} \beta_k \quad \Rightarrow \quad \tau_{k+6,k} = \frac{1}{2k+1} \frac{5^2}{(k+5)^2} \beta_k,
\]

where

\[
\beta_k := \frac{2}{5^2} \left( \frac{5 + \sqrt{10 - 3t} - t}{2} \right) = \frac{2}{5^2} \left( y + 5 \right) = g(y), \quad y := \sqrt{10 - 3t}.
\]

The function \( g \) is increasing for \( y > 1 \), hence

\[
t_k > t_{k+1} \quad \Rightarrow \quad y_k < y_{k+1} \quad \Rightarrow \quad \beta_k < \beta_{k+1} < \beta_*,
\]

where

\[
\beta_* = \lim_{t \to 0} \beta_k = \frac{2}{5^2} \frac{5 + \sqrt{10}}{2} = \frac{2 + \sqrt{10}}{5}.
\]

The cases 4)-5) prove estimates (4.7), hence Theorem 1.3 \( \square \)

4 \ Estimates based on the Duffin–Shaeffer majorant

In this section, we prove Theorem 1.4. Our proof is based on the upper bound \( \tau_{n,k} < \delta_{n,k} \) which uses the so-called Duffin–Shaeffer majorant.

**Definition 4.1** With \( T_n \) the Chebyshev polynomial of degree \( n \), and \( S_n(x) := \frac{1}{n} \sqrt{1-x^2} T_n'(x) \), we define the Duffin–Shaeffer majorant \( D_{n,k}(\cdot) \) as

\[
D_{n,k}(x) := \left\{ |T_n^{(k)}(x)|^2 + |S_n^{(k)}(x)|^2 \right\}^{1/2}, \quad x \in (-1,1).
\]

(4.1)

This majorant was introduced by Shaeffer–Duffin [16] who proved that, if \( p \) is a polynomial of degree not exceeding \( n \), then

\[
\|p\| \leq 1 \quad \Rightarrow \quad |p^{(k)}(x)| \leq D_{k,n}(x), \quad x \in (-1,1).
\]

(4.2)

which may be viewed as a generalization of the pointwise Bernstein inequality \( |p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|p\| \) to higher derivatives.
Lemma 4.2 The majorant $D_{n,k}$ has the following properties.

1. We have
   \[ |T^{(k)}_n(x)| \leq D_{n,k}(x) \quad \text{for all } x \in (-1, 1). \tag{4.3} \]

2. $D_{n,k}(x) = |T^{(k)}_n(x)|$ at zeros of $S^{(k)}_n$, in particular,
   \[ D_{n,k}(0) = |T^{(k)}_n(0)| \quad \text{if } n - k \text{ is even}. \tag{4.4} \]

3. The majorant $D_{n,k}(\cdot)$ is a strictly increasing function on $[0, 1]$.

4. We have the explicit formulae
   \[ \frac{1}{n^2} [D_{n,1}(x)]^2 = \frac{1}{1 - x^2} \text{ and} \]
   \[ \frac{1}{n^2} [D_{n,k}(x)]^2 = \sum_{m=0}^{k-1} \frac{b_{m,n}}{(1 - x^2)^{k+m}}, \quad k \geq 2, \tag{4.5} \]
   \[ \text{where} \]
   \[ b_{m,n} = c_{m,k} (n^2 - (m+1)^2) \cdots (n^2 - (k-1)^2), \tag{4.6} \]
   \[ c_{m,k} := \begin{cases} 1, & m = 0, \\ (k-1+m)(2m-1)!!^2, & m \geq 1. \end{cases} \tag{4.7} \]

Proof. Claim 1 and the first half of Claim 2 follow directly from Definition 4.1. Equality (4.4) is due to the fact that $T_n$ and $S_n$ are of different parity, so if $n - k$ is even, then $T^{(k)}_n$ is an even function and $S^{(k)}_n$ is an odd one, hence $S^{(k)}_n(0) = 0$. The third property was proved by Schaeffer–Duffin [16], and it also follows easily from the formulas (4.5)-(4.6) which were established by Shadrin [17]. \hfill \square

Here are few particular expressions for $D_{n,k}(\cdot)$.

\[ \frac{1}{n^2}[D_{n,1}(x)]^2 = \frac{1}{1 - x^2}, \]
\[ \frac{1}{n^2}[D_{n,2}(x)]^2 = \frac{(n^2 - 1)}{(1 - x^2)^2} + \frac{1}{(1 - x^2)^3}, \]
\[ \frac{1}{n^2}[D_{n,3}(x)]^2 = \frac{(n^2 - 1)(n^2 - 4)}{(1 - x^2)^3} + \frac{3(n^2 - 4)}{(1 - x^2)^4} + \frac{9}{(1 - x^2)^5}, \]
\[ \frac{1}{n^2}[D_{n,4}(x)]^2 = \frac{(n^2 - 1)(n^2 - 4)(n^2 - 9)}{(1 - x^2)^4} + \frac{6(n^2 - 4)(n^2 - 9)}{(1 - x^2)^5} + \frac{45(n^2 - 9)}{(1 - x^2)^6} + \frac{225}{(1 - x^2)^7}. \]

Lemma 4.3 Let $\omega_k := \omega_{n,k}$ be the rightmost zero of $T^{(k+1)}_n$. Then
   \[ \omega_k < x_k, \quad \text{where} \quad x_k := 1 - \frac{k^2}{n^2}. \tag{4.8} \]

Proof. The claim can be deduced from numerous upper bounds for the extreme zeros of ultraspherical polynomials. For instance, in [14] Nikolov proved that $\omega_k^2 \leq \frac{n^2 - (k+2)^2}{n^2}$, with some $\alpha_{n,k} > 0$, hence
   \[ \omega_k^2 \leq \frac{n^2 - (k+2)^2}{n^2} \leq \frac{n^2 - k^2}{n^2} = x_k. \tag{4.9} \]
From (4.3), monotonicity of $D_{n,k} \cdot$ and inequality (4.8), it follows immediately that

$$|T_n^{(k)}(\omega_k)| \leq D_{n,k}(\omega_k) < D_{n,k}(x_k),$$

hence the following statement.

**Proposition 4.4** We have

$$\tau_{n,k} < \delta_{n,k}, \quad \delta_{n,k} := \frac{D_{n,k}(x_k)}{T_n^{(k)}(1)}.$$

We proceed with estimates of $\delta_{n,k}$, using the explicit expression (4.5) for $D_{n,k}(\cdot)$.

**Lemma 4.5** We have

$$\tau_{n,k}^2 < \delta_{n,k}^2 = A_{n,k}B_{n,k},$$

where

$$A_{n,k} = \frac{(2k-1)!!}{k^{2k}} \sum_{m=0}^{k-1} c_{m,k} \frac{n^{2m}}{k^m (n^2 - 1^2) \cdots (n^2 - m^2)}, \quad (4.11)$$

$$B_{n,k} = \frac{n^{2k}}{n^2(n^2 - 1^2) \cdots (n^2 - (k-1)^2)}. \quad (4.12)$$

**Proof.** From (4.5) – (4.6), we obtain

$$[\delta_{n,k}]^2 = \frac{[D_{k,n}(x_k)]^2}{[T_n^{(k)}(1)]^2} = \frac{1}{[T_n^{(k)}(1)]^2} \sum_{m=0}^{k-1} \frac{c_{m,k}}{(1-x_k^2)^{m}(n^2 - (m+1)^2) \cdots (n^2 - (k-1)^2)}$$

$$= \frac{n^2(n^2-1^2) \cdots (n^2-(k-1)^2)}{[T_n^{(k)}(1)]^2(1-x_k^2)^k} \sum_{m=0}^{k-1} \frac{c_{m,k}}{(1-x_k^2)^m (n^2-1^2) \cdots (n^2-m^2)}$$

and substitution

$$\frac{1}{[T_n^{(k)}(1)]^2} = \frac{(2k-1)!!}{n^2(n^2 - 1^2) \cdots (n^2 - (k-1)^2)} \cdot \frac{1}{1-x_k^2} = \frac{k^2}{n^2},$$

gives (4.10) – (4.12) after a rearrangement.

**Remark 4.6** Whereas the value $\tau_{n,k}$ is defined only for $n \geq k + 2$, the values of $A_{n,k}$ and $B_{n,k}$ in (4.11) – (4.12) are well-defined for $n \geq k$. We will use this fact in the next lemma where the values $A_{k,k}$ and $B_{k,k}$ will be considered.

**Lemma 4.7** We have

$$\tau_{n,k}^2 < \delta_{n,k}^2 = A_{n,k}B_{n,k},$$

where

$$A_{n,k} \leq \frac{1}{2} \frac{(2k)!}{k^{2k}}, \quad B_{n,k} = \frac{n+k n^{2k}(n-k)!}{n(n+k)!}. \quad (4.14)$$

**Proof.** Expression for $B_{n,k}$ in (4.14) is just a rearrangement of (4.12).

As to the inequality for $A_{n,k}$ in (4.14), it is clear from (4.11) that $A_{n,k}$ decreases when $n$ grows, therefore

$$A_{n,k} \leq A_{k,k}, \quad n \geq k.$$
With \( n = k \), we have \( x_k = 1 - \frac{k^2}{n^2} = 0 \), and also \( D_{k,0}(0) = T_k^{(k)}(0) \) by (4.4), therefore

\[
A_{k,k} B_{k,k} = [\delta_{k,k}]^2 = \frac{[D_{k,k}(0)]^2}{[T_k^{(k)}(1)]^2} \quad \text{by (4.4), therefore}
\]

\[
A_{k,k} B_{k,k} = \frac{1}{B_{k,k}} = \frac{1}{2k} \frac{(2k)!}{k^{2k}},
\]

hence the result. \( \Box \)

**Remark 4.8** If we consider the first estimate in (4.9), namely

\[
\omega_k \leq x_k', \quad \text{where} \quad x_k'^2 = 1 - \frac{(k + 2)^2}{n^2}, \quad n \geq k + 2,
\]

then we obtain

\[
A_{n,k} \leq A_{k+2,k}' = \frac{(2k)!}{2k^{2k}} \gamma_k^2 = (k + 2) \left( \frac{k}{2k + 1} \right)^{2k}.
\]

i.e., we can improve the estimate (4.14) (and all subsequent estimates) by the factor of \( \gamma_k \) (or \( \gamma_k^2 \)). Note that

\[
\gamma_k \approx \frac{1}{\sqrt{2 \pi}} e^{\frac{1}{2}}.
\]

Now, we prove Theorem 4.9 which is the following statement.

**Theorem 4.9** For every \( n, k \in \mathbb{N} \) with \( n \geq k + 2 \), we have

\[
\tau_{n,k}^2 \leq \frac{1}{2} \left( 1 + \frac{k}{n} \right) \left( \frac{n}{k} \right)^{2k} \left( \frac{n + k}{n - k} \right)^{-1}
\]

\[
\leq c_1^2 k^{\frac{1}{2}} \left( 1 - \frac{k^2}{n^2} \right)^{\frac{1}{2}} \frac{(2n)^2 (n - k)^{n-k}}{(n + k)^{n+k}}, \quad c_1^2 = \frac{e^2}{2 \sqrt{\pi}}.
\]

**Proof.** The first part is just the estimate (4.13),

\[
\tau_{n,k}^2 < \delta_{n,k}^2 = A_{n,k} B_{n,k} < \frac{1 + k}{n} \frac{n + k}{k^{2k}} \frac{(n - k)! (2k)!}{(n + k)!}.
\]

To prove the second inequality we use the following version of Stirling’s formula

\[
\sqrt{2\pi} \left( \frac{N}{e} \right)^N \sqrt{N} < N! < e \left( \frac{N}{e} \right)^N \sqrt{N}.
\]

This gives

\[
\frac{1}{2} \frac{n + k}{n} \frac{n^{2k} (n - k)! (2k)!}{(n + k)!} \leq \frac{1 + k}{n} \frac{n^{2k} \sqrt{n - k \sqrt{2\pi}} \sqrt{n - k} (n - k)^{n-k}(2k)^{2k}}{(n + k)^{n+k}}
\]

\[
= \frac{e^2}{2 \sqrt{\pi}} \frac{n}{n + k} \frac{\sqrt{n - k \sqrt{2\pi}} (n - k)^{n-k}(2k)^{2k}}{(n + k)^{n+k}}
\]

\[
= c_1^2 k^{\frac{1}{2}} \left( 1 - \frac{k^2}{n^2} \right)^{\frac{1}{2}} \frac{(n - k)^{n-k}(2n)^{2k}}{(n + k)^{n+k}},
\]

and that finishes the proof. \( \Box \)
5 Proof of Theorem 1.5

We rewrite inequality (4.16) in a more convenient form
\[
\tau_{n,k} \leq c_1 k^{\frac{1}{2}} \left( 1 - \frac{k^2}{n^2} \right)^{\frac{1}{2}} \left( \frac{2n}{n+k} \right)^{n+k} \left( \frac{n-k}{2n} \right)^{n-k}, \quad c_1 = \frac{e}{2\sqrt{\pi}}. \tag{5.1}
\]

We will prove each part of Theorem 1.5 as a separate lemma.

Lemma 5.1 If \(k \in \mathbb{N}\) is fixed and \(n\) grows, then
\[
\tau_{n,k} \leq c_1 \left( \frac{2}{e} \right)^k \frac{k^{1/4}}{(1 - \frac{k^2}{n^2})^{k/4}}. \tag{5.2}
\]
in particular
\[
\tau_{n,k} < c_2 \left( \frac{2}{e} \right)^k k^{1/4}, \quad n \geq k^{3/2}. \tag{5.3}
\]

Proof. We write (5.1) in the form
\[
\tau_{n,k}^2 \leq c_1^2 L_1 L_2,
\]
where
\[
L_1 := k^{\frac{1}{2}} \left( 1 - \frac{k^2}{n^2} \right)^{1/2} < k^{1/2}, \tag{5.4}
\]
and
\[
L_2 := \left( \frac{2n}{n+k} \right)^{n+k} \left( \frac{n-k}{2n} \right)^{n-k} = 2^k \left( 1 - \frac{k}{n} \right)^{n-k} \left( 1 + \frac{k}{n} \right)^{n+k}. \]

We use the inequalities \((1 - \frac{1}{x})^{x-1/2} < \frac{1}{x}\) and \((1 + \frac{1}{x})^{x+1/2} > e\), where \(x > 1\), to derive
\[
\left( 1 - \frac{k}{n} \right)^{n-k} = \left( 1 - \frac{k}{n} \right)^{(\frac{x-1}{x})k} \left( 1 - \frac{k}{n} \right)^{-k/2} < \frac{1}{e^k} \left( 1 - \frac{k}{n} \right)^{-k/2},
\]
\[
\left( 1 + \frac{k}{n} \right)^{n+k} = \left( 1 + \frac{k}{n} \right)^{(\frac{x+1}{x})k} \left( 1 + \frac{k}{n} \right)^{k/2} > e^k \left( 1 + \frac{k}{n} \right)^{k/2}.
\]

Therefore
\[
L_2 < 2^{2k} \frac{1}{e^{2k}} \frac{1}{(1 - \frac{k^2}{n^2})^{k/2}},
\]
and that combined with (5.4) proves (5.2).

If \(n \geq k^{3/2}\) and \(k \geq 2\), then \((1 - \frac{k^2}{n^2})^{k/4} > (1 - \frac{1}{n})^{k/4} > 2^{-1/2}\), so (5.3) is valid with \(c_2 = 2^{1/2}c_1\).

If \(k = 1\), then \(n \geq 3\), and \((1 - \frac{k^2}{n^2})^{k/4} > 2^{-1/2}\) as well, and that proves (5.3) as well. \(\square\)

Lemma 5.2 If \(n - k = m\) is fixed and \(n\) grows, then
\[
\tau_{n,n-m} \leq c_3 m^{1/4} \left( \frac{me}{2} \right)^{m/2} n^{-m/2}. \tag{5.5}
\]

Proof. We consider the inequality (5.1)
\[
\tau_{n,k}^2 \leq c_1^2 k^{\frac{1}{2}} \left( 1 - \frac{k^2}{n^2} \right)^{1/2} \left( \frac{2n}{n+k} \right)^{n+k} \left( \frac{n-k}{2n} \right)^{n-k},
\]
and then estimate the factors using substitution \(n - k = m\) where appropriate. We have
\[
k^{\frac{1}{2}} \left( 1 - \frac{k^2}{n^2} \right)^{1/2} = (n-k)^{1/2} \left( \frac{k(n+k)}{n^2} \right)^{1/2} \leq 2^{1/2} m^{1/2},
\]
\[
\left( \frac{2n}{n+k} \right)^{n+k} = \left( 1 + \frac{n-k}{n+k} \right)^{n+k} = \left( 1 + \frac{m}{n+k} \right)^{n+k} < e^m,
\]
\[
\left( \frac{n-k}{2n} \right)^{n-k} = \left( \frac{m}{2n} \right)^m.
\]

Thus, (5.5) follows with \(c_3 = 2^{1/4}c_1\). \(\square\)
Lemma 5.3 If \( k = \lfloor \lambda n \rfloor \), where \( \lambda \in (0, 1) \), then as \( n \) grows, we have an exponential decay

\[
\tau_{n,k} \leq c_4 n^{1/4} \rho_{\lambda}^{n/2}, \quad \rho_{\lambda} < 1,
\]

in particular

\[
\tau_{n,n/2} \leq c_4 n^{1/4} \left( \frac{4}{\sqrt{27}} \right)^{n/2}.
\]

Proof. With \( k = \lfloor \lambda n \rfloor \), set \( \lambda' := \frac{k}{n} \), and note that

\[
\lambda n - 1 \leq k \leq \lambda n \implies \lambda - \frac{1}{n} \leq \lambda' \leq \lambda.
\]

Substitution \( k = \lambda' n \) in (5.1) gives

\[
\tau_{n,k} \leq c_4^2 k^2 \left( 1 - \frac{k^2}{n^2} \right)^{1/2} \left( \frac{2n}{n + k} \right)^{n+k} \left( \frac{n - k}{2n} \right)^n,
\]

where

\[
\rho_{\lambda'} = \left( \frac{2}{1 + \lambda'} \right)^{1+\lambda'} \left( \frac{1 - \lambda'}{2} \right)^{1-\lambda'} < 1, \quad \lambda' \in (0, 1).
\]

On using that \( g(x) := \ln \rho_x \) satisfies \( g'(x) > -1 \) for \( x \in (0, 1) \), we derive from (5.7) that

\[
\rho_{\lambda'} < e^{1/n} \rho_{\lambda},
\]

and that proves (5.6) with \( c_4 = e^{1/2n} c_1 \). If \( \lambda = \frac{1}{2} \) we obtain \( \rho_{1/2} = 2(\frac{1}{2})^{1/2}/(\frac{3}{2})^{3/2} = \frac{1}{\sqrt{27}} \). \( \square \)

6 The asymptotic formulas

In this section, we derive the asymptotic formulas (1.14)-(1.15) of Theorem 1.8.

1) We start with the asymptotic formula for

\[
\tau^*_{k} := \lim_{n \to \infty} \tau_{n,k}.
\]

For \( \alpha, \beta > -1 \), we denote by \( \{ P^{(\alpha,\beta)}_m \} \) the sequence of Jacobi polynomials which are orthogonal with respect to the weight \( w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta} \), with the standard normalization

\[
P^{(\alpha,\beta)}_m(1) = \binom{m + \alpha}{m}.
\]

Note that derivatives of the Chebyshev polynomials are related to Jacobi polynomials in the following way,

\[
T^{(k)}_n = c_{n,k} P^{(\nu,\nu)}_m, \quad m = n - k, \quad \nu = k - \frac{1}{2}.
\]

We will use the asymptotic property of Jacobi polynomials which is described in terms of Bessel functions (see [20, sect. 8.1]), namely the following equality from [21]

\[
\lim_{m \to \infty} m^{-\alpha} P^{(\alpha,\beta)}_m(y_{m,r}) = \left( \frac{j_{\alpha+1,r}}{2} \right)^{-\alpha} J_{\alpha}(j_{\alpha+1,r}),
\]

where \( y_{m,r} \) is the point of the \( r \)-th local extremum of \( P^{(\alpha,\beta)}_m \) counted in decreasing order and \( j_{\nu,r} \) is the \( r \)-th positive zero of the Bessel function \( J_{\nu} \).
Lemma 6.1 We have
\[ \tau_k^* = \Gamma(\nu + 1) \left( \frac{j_{\nu+1,1}}{2} \right)^{-\nu} |J_\nu(j_{\nu+1,1})|, \quad \nu = k - \frac{1}{2}. \] (6.4)

Proof. By (6.1)-(6.2), since \( \omega_{n,k} = y_{m,1} \), we have
\[ \tau_k^* = \lim_{m \to \infty} \frac{|P_m^{(\nu,\nu)}(y_{m,1})|}{|P_m^{(\nu,\nu)}(1)|} = \lim_{m \to \infty} \frac{m^{-\nu} |P_m^{(\nu,\nu)}(y_{m,1})|}{m^{-\nu} (m + \nu)} = \frac{L_1}{L_2}. \]

By (6.3),
\[ L_1 = \left( \frac{j_{\nu+1,1}}{2} \right)^{-\nu} |J_\nu(j_{\nu+1,1})|, \]

while for the denominator we use
\[ \left( \frac{m + \nu}{m} \right) = \frac{\Gamma(m + \nu + 1)}{\Gamma(m + 1) \Gamma(\nu + 1)}, \quad \lim_{m \to \infty} m^{-\nu} \frac{\Gamma(m + \nu + 1)}{\Gamma(m + 1)} = 1, \]
to obtain
\[ L_2 = 1/\Gamma(\nu + 1) \]
and that proves the lemma. \(\square\)

Lemma 6.2 ([12]) The first positive zero \( j_{\nu,1} \) of the Bessel function \( J_\nu \) obeys the following asymptotic expansion
\[ j_{\nu,1} = \nu + a\nu^{1/3} + \mathcal{O}(\nu^{-1/3}), \quad a = -i_1/2^{1/3} = 1.8557... \] (6.5)

where \( i_1 \) is the first negative zero of the Airy function \( \text{Ai}(x) \).

Lemma 6.3 We have
\[ J_\nu(j_{\nu+1,1}) = -\left( \frac{2}{\nu} \right)^{2/3} \text{Ai}^\prime(i_1) + \mathcal{O}(\nu^{-1}). \] (6.6)

Proof. We will need the asymptotic behavior of \( J_\nu(\nu x) \) for large (fixed) \( \nu \) and \( x \geq 1 \) (that is, around the first positive zero \( j_{\nu,1} \)), which is given by the following formula (see [15 Chapter 11] or [12]),
\[ J_\nu(\nu x) = \phi(z) \frac{\text{Ai}(\nu^{2/3} z) (1 + \mathcal{O}(\nu^{-2})) + \frac{\text{Ai}^\prime(\nu^{2/3} z)}{\nu^{4/3}} \left( B_0(z) + \mathcal{O}(\nu^{-2}) \right)}{\nu^{1/3}}, \] (6.7)

where \( 0 < B_0(z) \leq B_0(0) \) for \( z \leq 0 \) and
\[ z = -\left( \frac{3}{2} \sqrt{x^2 - 1} - \frac{3}{2} \sec^{-1}(x) \right)^{2/3}, \quad \phi(z) = \left( \frac{4z}{1 - x^2} \right)^{1/4}, \quad x \geq 1. \]

Let \( x = 1 + \delta \), where \( \delta = \mathcal{O}(\nu^{-2/3}) \) and \( \delta > 0 \). Then
\[ \sec^{-1}(x) = \arccos \left( \frac{1}{1 + \delta} \right) = \sqrt{2\delta} \left( 1 - \frac{5\delta}{12} + \mathcal{O}(\delta^2) \right), \]
whence we obtain for \( z \) and \( \phi(z) \)
\[ z = -2^{1/3} \nu (1 + \mathcal{O}(\delta)), \quad \phi(z) = 2^{1/3} + \mathcal{O}(\delta). \]

Substitution of these quantities in (6.7) yields
\[ J_\nu(\nu(1 + \delta)) = \left( \frac{2}{\nu} \right)^{1/3} \left( \text{Ai} \left( -\nu^{2/3} z^{2/3} \right) + \mathcal{O}(\nu^{-2/3}) \right) + \mathcal{O}(\nu^{-1}) \] (6.8)
From (6.5), we have

\[ j_{\nu + 1, 1} = \nu + 1 - \frac{i_1}{2^{1/3}} (\nu + 1)^{1/3} + O(\nu^{-1/3}) \]
\[ = \nu (1 + \delta_0), \quad \delta_0 = -\frac{i_1}{2^{1/3}} \nu^{-2/3} + \nu^{-1} + O(\nu^{-4/3}), \]

so putting this into (6.8), we conclude

\[ J_\nu(j_{\nu + 1, 1}) = \left(\frac{2}{\nu}\right)^{1/3} \left( \text{Ai}(i_1 - \frac{2}{\nu})^{1/3} + O(\nu^{-2/3}) + O(\nu^{-2/3}) \right) \]
\[ = \left(\frac{2}{\nu}\right)^{2/3} \text{Ai}'(i_1) + O(\nu^{-1}), \]

and that proves the lemma.

Proof of Theorem 1.8 part 1.14. With the substitution \( \nu = k - \frac{1}{2} \), we obtain

\[ |J_k - \frac{1}{2} (j_{\frac{k}{2}, 1})| \left(\frac{2}{k}\right)^{2/3} \left( |\text{Ai}'(i_1)| + O(k^{-1/3}) \right), \]
\[ j_{\frac{k}{2}, 1} = k + \frac{1}{2} + ak^{1/3} + O(k^{-1/3}), \]
\[ (j_{\frac{k}{2}, 1} - (k - \frac{1}{2})) = \left(\frac{2}{k}\right)^{k-1/2} e^{-1/2} e^{-ak^{1/3}} (1 + O(k^{-1/3})), \]
\[ \Gamma(k + \frac{1}{2}) = \frac{(2k)!}{4^k k! \sqrt{\pi}} = \left(\frac{2}{e}\right)^k \sqrt{\pi} (1 + O(k^{-1})), \]

and formula (6.4) gives

\[ \tau_k^* = C_0 \left(\frac{2}{e}\right)^k e^{-a_0 k^{1/3}} k^{-1/6} (1 + O(k^{-1/3})), \]

where

\[ C_0 = 4^{1/3} \sqrt{\pi} / e |\text{Ai}'(i_1)|, \quad a_0 = -i_1 / 2^{1/3}, \]

and that proves the first part of Theorem 1.8.

2) Next, we will prove the asymptotic formula for

\[ \tau_m^{**} := \lim_{n \to \infty} n^{m/2} \tau_{n,n-m}. \]

Note that, with \( m = n - k \) fixed, and provided that the limit exists, we have

\[ \tau_m^{**} := \lim_{k \to \infty} (k + m)^{m/2} \tau_{k+m,k} = \lim_{k \to \infty} k^{m/2} \tau_{k+m,k}. \]

We will use the relation

\[ T^{(k)}_{k+m} = c_{m,k} P_m^{(\lambda)}, \quad \lambda = k, \]

and the asymptotic properties of the ultraspherical polynomials \( P_m^{(\lambda)} \) expressed in terms of the Hermite polynomials \( H_m \) (see [20, eq. (5.6.3)])

\[ \lim_{\lambda \to \infty} \lambda^{-m/2} P_m^{(\lambda)} \left( \frac{x}{\sqrt{\lambda}} \right) = \frac{H_m(x)}{m!}. \quad (6.9) \]

Lemma 6.4. We have

\[ \tau_m^{**} = 2^{-m} |H_m(x'_m)|, \]

where \( x'_m \) is the point of the rightmost extremum of \( H_m \).
**Proof.** With \( \omega_{k+m,k} \) and \( x_m' \) being the points of the rightmost local extrema of \( T_{k+m}^{(k)} = c_{m,k}P_{m}^{(k)} \) and \( H_m \), respectively, it follows from (6.9) that, for a fixed \( m \), we have

\[
\tau_{k+m} = \frac{\left| P_{m}^{(k)}(\omega_{k+m,k}) \right|}{P_{m}^{(k)}(1)} \sim \frac{k^{m/2}|H_m(x_m')|}{m^{\left(m+2k-1\right)/m}} \sim 2^{-m}k^{-m/2}|H_m(x_m')|, \quad k \to \infty,
\]

and this implies

\[
\tau_{m}^{**} = \lim_{k \to \infty} k^{m/2}\tau_{k+m} = 2^{-m}|H_m(x_m')|.
\]

Lemma is proved. \( \square \)

**Proof of Theorem 1.8, part 1.15.** For approximation of \( H_m(x_m') \) we will use the formula of Plancherel - Rotach ([20, Theorem 8.22.9]). Actually, we need only the third part of this theorem, concerning the approximation of \( H_m \) around its turning point, where the behaviour of the polynomial changes from oscillatory to monotonically increasing. It states that if

\[
x = (2m+1)^{1/2} - 2^{-\frac{1}{3}}3^{-\frac{1}{3}}m^{-\frac{1}{3}}t, \quad t \in \mathbb{C},
\]

then

\[
e^{-x^2/2}H_m(x) = 3^{\frac{1}{2}}\pi^{-\frac{1}{4}}2^{\frac{m+1}{2}}(m!)^{\frac{1}{2}}m^{\frac{1}{4}}\left\{ A(t) + O(m^{-\frac{5}{6}}) \right\},
\]

where \( A(z) = 3^{-\frac{1}{2}}\pi \text{Ai}(-3^{-\frac{1}{2}}z) \) is the normalized Airy function. Moreover, the asymptotic formula (6.11) holds uniformly when \( t \in \mathbb{C} \) is bounded.

Let \( x_m \) be the largest zero of \( H_m \), then ([20, eq. (6.32.5)])

\[
x_m = (2m+1)^{1/2} - 2^{-\frac{1}{3}}3^{-\frac{1}{3}}m^{-\frac{1}{3}}i_1^* + O(m^{-5/6}), \quad m \geq 1,
\]

where \( i_1^* = -3^{1/3}i_1 \) is the first zero of \( A(z) \). Since \( H_m'(x) = 2mH_{m-1}(x) \), we have for \( m \geq 2 \)

\[
x_m' = x_{m-1} = (2m-1)^{1/2} - 2^{-\frac{1}{3}}3^{-\frac{1}{3}}m^{-\frac{1}{3}}i_1^* + O(m^{-5/6}) = x_m - (2m)^{-\frac{1}{4}} + O(m^{-5/6}),
\]

and we can put \( x_m' \) in the form (6.10), with \( t = t_m' \) where

\[
t_m' = i_1^* + 3^{\frac{1}{3}}m^{-\frac{1}{3}} + O(m^{-2/3}).
\]

Then formula (6.11) gives

\[
H_m(x_m') = e^{\frac{i}{2}(x_m')^2}3^{\frac{1}{2}}\pi^{-\frac{1}{4}}2^{\frac{m+1}{2}}(m!)^{\frac{1}{2}}m^{-\frac{1}{4}}\left\{ A(t_m') + O(m^{-\frac{5}{6}}) \right\}
\]

\[
= -\left(2em\right)^{\frac{m}{2}}e^{-|i_1|^{1/3}m^{-1/6}\sqrt{2\pi/e} \text{Ai}'(i_1)}\left(1 + O(m^{-1/3})\right).
\]

Finally, we obtain

\[
\tau_{m}^{**} = 2^{-m}|H_m(x_m')| = \left(\frac{em}{2}\right)^{m/2}e^{-a_1m^{1/3}}m^{-1/6}\left(1 + O(m^{-1/3})\right).
\]

where

\[
C_1 = \sqrt{\frac{2\pi}{e} \text{Ai}'(i_1)}, \quad a_1 = |i_1|.
\]

Theorem 1.8 is proved. \( \square \)
7 Remarks

1. As was mentioned in introduction, Theorem 2.1 is due to Szász [19]. The statement of Theorem 2.1 appears in [1, pp. 304–305] along with a proof of the Legendre case ($\lambda = 1/2$). The proof of this case originally was given by Szegő, who confirmed a conjecture made by J. Todd. Our proof follows the same approach. An alternative proof of Theorem 1.1 can be obtained using some results of Bojanov and Naidenov in [2].

2. To obtain better approximation of $\tau_{n,k}$ one needs more precise asymptotic formulae for ultraspherical and Hermite polynomials and bounds for their extreme zeros. In this connection we refer to [3, 4, 5, 6, 9, 11, 22].

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