A RISK-SHARING FRAMEWORK OF BILATERAL CONTRACTS

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Abstract. We introduce a two-agent problem which is inspired by price asymmetry arising from funding difference. When two parties have different funding rates, the two parties deduce different fair prices for derivative contracts even under the same pricing methodology and parameters. Thus, the two parties should enter the derivative contracts with a negotiated price, and we call the negotiation a risk-sharing problem. This framework defines the negotiation as a problem that maximizes the sum of utilities of the two parties. By the derived optimal price, we provide a theoretical analysis on how the price is determined between the two parties. As well as the price, the risk-sharing framework produces an optimal amount of collateral. The derived optimal collateral can be used for contracts between financial firms and non-financial firms. However, inter-dealers markets are governed by regulations. As recommended in Basel III, it is a convention in inter-dealer contracts to pledge the full amount of a close-out price as collateral. In this case, using the optimal collateral, we interpret conditions for the full margin requirement to be indeed optimal.

Key words. Bilateral contracts, risk-sharing, piece-wise concave utility, collateral

AMS subject classifications. 93E20, 91G20, 91G40

1. Introduction. In the aftermath of the financial crisis, it has become customary in recent years for trading desks to make several adjustments in derivative transactions for counterparty default risk, funding spreads, collateral, etc. For pricing derivatives with the collective adjustments, many methodologies have been developed by, e.g., [47, 23, 19, 50, 29]. However, it is known that the fair values derived by the developed methodologies are not fully recouped from counterparties. This can possibly due to inclusion of funding spread. For traders, if the increased funding costs are not compensated from the counterparty, it will be losses on the trades. However, considering the choices of funding in derivative prices is a violation of Modigliani–Miller (MM) theorem. For MM theorem to be valid, the absence of frictional financial distress costs is required; see [44, 49]. Therefore, considering funding cost/benefit may be justified by market frictions.

Even so, there still remain some puzzles related to the funding adjustment. First, when funding cost/benefit is accepted, it gives rise to asymmetry of theoretical prices between two contractors even under the same pricing methodology and parameters. The value fair to one party may not be fair to the counterparty since funding rates of the counterparty is different from those of the other party. Second, as asked by [36, 37], if funding cost should be really considered, possibly due to market frictions, why do banks buy Treasury bonds that return less than their funding costs?

Motivated by the related issues, we introduce a two-agent problem. Instead of using the individual fair values, two parties may enter a contract through negotiation by sharing costs, as briefly mentioned by [43]. We describe the negotiation problem as maximizing the sum of utilities of both parties and we call this a risk-sharing problem. Then, the risk-sharing framework theoretically analyzes how the price equilibrium is determined and we can interpret the questions on funding adjustment by using the derived price. The optimal price from the risk-sharing framework will depend on the risk aversion parameters and relative negotiation power between the two parties, but they are not observable in markets. Therefore, the importance of this study is on providing sound theoretical interpretations for the puzzles on funding difference.

The other part of the solution in the risk-sharing framework is collateral. In recent times, most OTC derivative contracts are collateralized. There are multiple procedures for the margin, but in

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bilateral contracts, it is general to post variation margin and initial margin. In our model, the focus is on variation margin which traces mark-to-market exposures. As stated in [13, p.15], “for variation margin, the full amount necessary to fully collateralise the mark-to-market exposure of the non-centrally cleared derivatives must be exchanged.” This full collateralization on variation margin has been settled as a market convention in inter-dealer transactions. On the other hand, there is no such convention between banks and sovereign or corporate clients. Indeed, it is partly or not collateralized for contracts between financial firms and non-financial firms. Therefore, the risk-sharing framework provides the optimal amount of variation margin for the contract between a financial firm and non-financial firm. Since inter-dealer contracts are governed by regulation in practice, we interpret the meaning behind the margin requirement.

The optimal collateral in our model is represented by a certain stochastic process. Thus, full variation margin may not be optimal in general. However, we do not conclude that the convention is unreasonable. Variation margin is posted on a daily or intra-day basis. If the amount was calculated by a complicated rule at each time, the amount would be unacceptable for some parties and this can be a possible cause of conflict. Hence, rather than coming to a sensitive conclusion, we analyze the situation for the margin requirement to be optimal. The market convention will turn out to be based on certain conditions on funding cost/benefit considered in derivative prices and hedging strategy taken by two parties. Especially, we will see later that the full margin requirement is related to the absence of market friction.

One mathematical difficulty to deal with the risk-sharing problem is that the amount by breach of contract is given by piece-wise concave functions. Mathematically similar problems were solved by [24, 15, 14, 51]. In [24], portfolio optimization problems were considered where the agent switches utilities. They used duality method that cannot be applied to our problem as we cannot impose a positive constraint for the portfolio. In [15, 14], the piece-wise concave property arose from different lending/borrowing rates and they solved the optimization problem by using HJB equations. In their problem, the associated HJB equations had a homothetic property. Moreover, with a mild assumption that the lending rate is smaller than the borrowing rate, the Hamiltonian became continuous in their cases. However, in our problem, we deal with two state processes taken by two utilities, so we cannot make use of a similar approach.

We circumvent the above difficulties by imposing some conditions on funding spread depending on choices of utilities. For the funding spread, we assume that the lending and borrowing rates are the same for each party. To be more precise, the two parties fund themselves on their own funding rate which may not be the same as OIS rate, but the lending and borrowing rates are the same. Moreover, the funding costs/benefits for delivering collateral of one or both parties will be ignored for characterization of the optimal solution. More precisely, we will examine two cases. First, we will consider two risk-averse agents whose funding rates for delivering margin are OIS rate. Second, we will also consider one risk-averse agent and one risk-neutral agent, and in this case, the funding rate of the risk-neutral agent does not need to be OIS rate. To streamline this paper, we mainly deal with the two risk-averse agents in main sections and report the second case in Appendix C.

This funding condition can be understood that the party is an entity which invests the capital without or with a small leverage, or the party can post collateral with secured funding. Even though the secured funding for variation margin is not so general, some realistic cases are discussed by [3]. In addition, this setup on funding spread can be partly justified by the results in [42] which showed that, in many classes of derivatives, hedgers do not need to switch funding state between lending and borrowing positions. In particular, it is guaranteed that if a hedger does not enter borrowing state and the lending rate is same as OIS rate, we can ignore the funding impacts.

In our model, we include default risk, funding spread, and collateral. We consider incomplete markets that hedgers cannot access to assets for hedging default risk such as bonds and CDSs. The reference filtration is generated by a Brownian motion. The mark-to-market exposure is calculated as clean price which is the classical risk-neutral price without default risks and funding spread. Moreover, for risk-averse agents, we consider exponential utilities. For simplicity, we consider non-
Appendix C

In section 2, the risk-sharing problem is introduced. We start from defining a filtration and making an assumption on default intensities in subsection 2.1. Before giving the details, we explain our motivation with a simple model in subsection 2.2. Then we describe cash-flow which are determined by dividends, margins, and close-out amount. Both parties entering the contract will have a portfolio given in subsection 2.3 depending on the cash-flow. The introduced risk-sharing problem is maximizing the sum of utilities of discounted portfolio values at termination of the contract. In subsection 2.4, the original form of the risk-sharing problem is reduced so that it is represented on the reference filtration. Then we define admissible sets more precisely with this reduced problem. We mainly deal with the reduced problem in this paper. In section 3, we define a dynamic version of the main problem, and optimal collateral is characterized. Then given the optimal collateral, we derive a condition to find an optimal price in section 4 and examples are given in section 5.

2. Modeling.

2.1. Mathematical Setup. We consider two parties entering a bilaterally cleared contract. We call the two parties “Agent A” and “Agent B”, respectively. In what follows, an index $A$ (resp. $B$) is used to stand for the Agent A (resp. Agent B). We consider a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with physical probability $\mathbb{P}$ and let $\mathbb{E}$ be the expectation under $\mathbb{P}$. For $i \in \{A, B\}$, we define non-negative random variables $\tau^i$ on $(\Omega, \mathcal{G}, \mathbb{P})$ such that $\mathbb{P}(\tau^i = 0) = 0$ and $\mathbb{P}(\tau^i > t) > 0$, for any $t \geq 0$, to represent default times of the agents. We let

$$\tau := \tau^A \land \tau^B, \quad \bar{\tau} := \tau \land T,$$

where $T > 0$ is the maturity of a certain derivative contract. We denote by $(W_t)_{t \geq 0}$ a $d$-dimensional standard Brownian motion under $\mathbb{P}$. The reference filtration $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$ is the usual natural filtration of $(W_t)_{t \geq 0}$, and the full filtration $\mathcal{G}$ is defined as

$$\mathcal{G} = (\mathcal{G}_t)_{t \geq 0} = (\mathcal{F}_t \vee \sigma(\tau^i \leq u : u \leq t, i \in \{A, B\}))_{t \geq 0}.$$

Then, we consider a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$. Note that for any $i \in \{A, B\}$, $\tau^i$ is a $\mathcal{G}$-stopping time but may fail to be an $\mathbb{F}$-stopping time. Unless stated, every process is a $(\mathbb{P}, \mathcal{G})$-semimartingale. As a convention, for any $\mathcal{G}$-progressively measurable process $(u_t)_{t \geq 0}$ and $(\mathbb{P}, \mathcal{G})$-semimartingale $(U_t)_{t \geq 0}$, $\int_0^t u_s dU_s = \int_0^{t \wedge \tau^i} u_s dU_s$, where the integral is well defined. In addition, for any $\mathcal{G}$-stopping time $\theta$ and process $(\xi_t)_{t \geq 0}$, we denote

$$\xi^\theta := \xi_{\wedge \theta},$$

and when $\xi_{\theta -}$ exists, denote $\Delta \xi^\theta := \xi^\theta - \xi_{\theta -}$. For $i \in \{A, B\}$, $t \geq 0$, we also let

$$G^i_t := \mathbb{P}(\tau^i > t | \mathcal{F}_t) \quad \text{and} \quad G_t := \mathbb{P}(\tau > t | \mathcal{F}_t).$$

The following assumption stands throughout this paper.

Assumption 2.1. (i) $(G_t)_{t \geq 0}$ is non-increasing and absolutely continuous with respect to Lebesgue measure.

(ii) For any $i \in \{A, B\}$, there exists a process $h^i$, defined as

$$h^i_t := \lim_{u \downarrow 0} \frac{\mathbb{P}(t < \tau^i \leq t + u, \tau > t | \mathcal{F}_t)}{\mathbb{P}(\tau > t | \mathcal{F}_t)},$$

and $(M^i_t)_{t \geq 0} := (\mathbf{1}_{\tau^i \leq t \wedge \tau} - \int_0^{t \wedge \tau} h^i_s ds)_{t \geq 0}$ is a $(\mathbb{P}, \mathcal{G})$-martingale.
We denote \( h := h^A + h^B \). By (i) in Assumption 2.1, there exists an \( \mathbb{F} \)-progressively measurable process \( (h^0_t)_{t \geq 0} \) such that
\[
 h^0_t = \lim_{u \downarrow 0} \frac{1}{u} \mathbb{P} \left( t < \tau \leq t + u | \mathcal{F}_t \right),
\]
and \((M_t)_{t \geq 0} := (1_{t \leq \tau} - \int_0^t h^0_s \, ds)_{t \geq 0}\) is also a \((\mathbb{P}, \mathbb{G})\)-martingale. When \( \tau^A \) and \( \tau^B \) are independent on \( \mathbb{F} \)
\[
 h^A := h - h^0 = 0.
\]
In general, it is not the case. Moreover, by (i) in Assumption 2.1 and \([27, \text{Corollary 3.4}]\), \( \tau \) avoids any \( \mathbb{F} \)-stopping time. In other words, for any \( \mathbb{F} \)-stopping time \( \tau^F \),
\[
 \mathbb{P}(\tau = \tau^F) = 0.
\]

Remark 2.2. It is worth discussing the meaning of the item (i) in Assumption 2.1 in view of both modeling and mathematical aspects. Without the assumption, \( G \) is only \((\mathbb{P}, \mathbb{G})\)-supermartingale, thus, by the Doob-Meyer decomposition, there exist a \((\mathbb{P}, \mathbb{G})\)-martingale \( \nu \) and non-increasing process \( \upsilon \) such that \( G = \nu + \upsilon \). Therefore, assuming \( G \) is non-increasing is equivalent to setting \( \nu = 0 \), i.e., we ignore some parts of random effects in default times for mathematical simplicity. In addition, the condition (i) is equivalent to the statement that for any \( \mathbb{F} \)-martingale \( \xi \), the stopped process \( \xi^T \) is a \( \mathbb{G} \)-martingale, see Proposition 3.4 in \([33]\). Thus, the condition is close to \((H)\)-hypothesis.

On this setup, we can reduce the full filtration using Lemma A.1 reported in Appendix A. For denoting the spaces of random variables and processes, we use standard notations which are given in Appendix B.

2.2. A Motivation. Before delving into the details, we will explain a motivation of our risk-sharing problem with a simple model. Let us consider two agents, \( A \) and \( B \) with constant funding rates \( R^A \) and \( R^B \). We moreover, consider a situation that the Agent \( A \) buys from the Agent \( B \) an uncollateralized bond of unit notional amount and maturity \( T \). If the two agents were able to fund by the (so-called) risk-free rate \( r \), the fair value of the two parties would be \( e^{-rT} \). However, when \( R^i \), \( i \in \{A, B\} \), are not equal to \( r \), the fair values of two parties are different to the parties, and the two parties would want to recoup their individual adjustments: \( (e^{-R^iT} - e^{-rT}) \). Given the asymmetry by funding difference, we want to model how the price is determined.

To this end, let \( p \) be the adjustment price “given to \( A \)” on top of the (clean) risk-neutral price \( e^{-rT} \), e.g., for the bond contract, \( A \) pays \( e^{-rT} + p \) to \( B \) at initiation of the contract. This money is invested in their funding accounts up to \( T \), and at the maturity, the Agent \( A \) will receive 1 dollar amount from the Agent \( B \). Therefore, the respective profit and loss of the two parties at \( T \) will be
\[
 V_T^{A,p} = (e^{-rT} + p)e^{R^AT} + 1, \quad V_T^{B,p} = (e^{-rT} - p)e^{R^BT} - 1.
\]
For the time being, we assume that the two parties both have the same preference of exponential utility as \( U(x) = -e^{-x} \). Then, we find an optimal adjustment price \( p^* \) to maximize the two parties’ aggregated utility of discounted P/L by their own funding rates, namely,
\[
 p^* = \arg \max_{p \in \mathbb{R}} \left[ U \left( -e^{-rT} + p + e^{-R^AT} \right) + \lambda U \left( e^{-rT} - p - e^{-R^BT} \right) \right],
\]
for some \( \lambda > 0 \). The parameter \( \lambda \) can be interpreted as a relative bargaining power of \( B \). By straightforward calculation, (2.2) becomes
\[
 p^* = -\frac{e^{-R^AT} + e^{-R^BT}}{2} + e^{-rT} - \frac{\ln(\lambda)}{2}.
\]
From (2.3), if the two agents have the same negotiation power, i.e., $\lambda = 1$, the optimal adjustment price $p^*$ is determined as the middle of the individual adjustments, i.e.,

\[
(2.4) \quad p^* = \frac{1}{2} \left( e^{-R^A_T} - e^{-r_T} + (e^{-R^B_T} - e^{-r_T}) \right).
\]

In addition, as $\lambda$ increases, the contract becomes more advantageous to the Agent B. In the case that the Agent B is a government, $\lambda$ can be large possibly due to tax benefits in buying treasury securities. However, it should be mentioned that $\lambda$ is generally not observable in markets, so the importance of our model mainly remains in theoretical analysis. In the following sections, we describe the agents’ P/L with more details in terms of hedging portfolios for entering a derivative contract.

Remark 2.3. A condition of funding transfer policy (FTP) that is beneficial to both parties was also discussed by [4]. It was shown that (2.4) is one of the choices satisfying their condition; see [4, Proposition 5.1]. However, there may be many choices of the FTP satisfying the condition, so instead, we investigate the prices which are the best to the parties.

2.3. Hedging Portfolio under Bilateral Contracts. In this section, under CVA, DVA, funding spread, and collateral, we define the two parties’ hedging portfolios for entering a contract. We mostly depict the hedging portfolio in view of $A$. Then the portfolio of $B$ can be derived by a similar way.

2.3.1. Dividend, Close-out Amount, and Collateral. We begin this section by explaining the cash-flow in bilateral contracts. Consider two agents who want to enter a bilateral contract which exchanges promised dividends. We denote the cumulative dividend process by $\mathcal{D}$. We assume that $\mathcal{D}$ is an $\mathcal{F}$-adapted càdlàg process and is independent of defaults. The value of $\mathcal{D}$ is determined by an $n$-dimensional $\mathcal{F}$-adapted (i.e., non-defaultable) underlying asset $S = (S^1, \ldots, S^n)$ that satisfies the following stochastic differential equation (SDE):

\[
(2.5) \quad dS^i_t = \mu^i S^i_t \, dt + (\sigma^i_t)^\top S^i_t \, dW_t, \quad 1 \leq i \leq n,
\]

where $\sigma^i \in \mathbb{R}^d$ and $\mu^i \in \mathbb{R}$ are $\mathcal{F}$-predictable. Moreover, we denote $\mu \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^{n \times d}$ such that $(\mu)_i = \mu^i$ and $\text{row}(\sigma)_i = \sigma^i$.

It is not assumed that $n = d$. In other words, the considered market may or may not be complete regardless of whether assets to hedge default risk, such as CDSs and bonds, are traded. In this paper, we consider markets with the absence of assets to hedge the default risk. We only assume that for all $t$, $\sigma_t$ is of full rank so that we can define the risk premium $\Lambda$ as a solution of

\[
(2.6) \quad \sigma \Lambda = (\mu - r \mathbf{1}),
\]

where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^d$ and $r$ is an $\mathcal{F}$-adapted process which represents overnight indexed swap (OIS) rate. We will later use $\Lambda$ for a pricing measure to define close-out amount. Recall that the existence of $\Lambda$ guarantees arbitrage-free condition in classical context. However, since the classical definition of arbitrage opportunity does not reflect adequately the hedger-specific nature of bilateral contracts, there have been many studies to redefine arbitrage opportunity properly in the context of bilateral contracts. The condition being developed is slightly different from paper to paper, but often absence of arbitrage opportunity is obtained with similar conditions to (2.6). See, for example, [12, Proposition 3.3]. For definitions of hedger-specific arbitrage opportunities, readers can refer to [12, 10, 6, 7, 8, 41, 40].

We set, as a convention, a positive value (resp. negative) of dividend process at a certain moment to mean that the Agent $A$ pays to (resp. is paid by) the Agent $B$. For example, if $A$ sells a put option on $S$ with the exercise price $\kappa$ and maturity $T$, then for any $t \geq 0$, $\mathcal{F}_t = \mathcal{F}_{\geq T} (\kappa - S_T)^+$. Note that the initial price exchanged at initiation of the contract is not a part of $\mathcal{D}$. We will include
the initial price in the hedging portfolios as their initial value. Because jumps of an \( \mathcal{F} \)-adapted càdlàg process are exhausted by \( \mathcal{F} \)-stopping times by [35, Theorem 4.21], \( \tau \) avoids any \( \mathcal{F} \)-stopping time (recall (2.1)), \( \mathcal{D} \) does not jump at default, i.e.,

\[
\Delta \mathcal{D}_\tau = 0, \text{ almost surely.}
\]

Let us turn to explain close-out amount and margin process. The obligation on dividend \( \mathcal{D} \) may not be fully honored at one party’s default. For the default risk, covenants of the close-out amount and collateral are documented in a Credit Support Annex\(^1\) before initiation of the contract. At the event of default, the dividend stream stops and CSA close-out amount should be settled. However, because of the default, the defaulting party would not be able to pay the full close-out amount. To mitigate the risk of losses at default, collateral is exchanged between the two parties. In bilateral contracts, \textit{variation margin} and \textit{initial margin} are posted in general, and the close-out amount is often determined as mark-to-market exposure.

In our model, only \textit{variation margin} is a part of the control variables of our stochastic control problem that will be introduced later. We exclude \textit{initial margin} for simplicity. As stated in [13, p.12], “the amount of \textit{variation margin} reflects the size of this current exposure,” and it is recommended that “the full amount necessary to fully collateralise the mark-to-market exposure of the non-centrally cleared derivatives must be exchanged” [13, p.15]. The meaning behind this regulation will be discussed later based on our model.

\textbf{Remark 2.4.} For general practices of \textit{initial margin}, readers can refer to [45, 22]. In addition, \textit{initial margin} causes associated BSDEs anticipated. For the numerical simulation of anticipated BSDEs, readers can refer to [1].

Now, we depict the close-out amount and \textit{variation margin} mathematically. One of the popular choices to calculate the market exposure is \textit{clean price} which is basically the classical risk-neutral price. As used in the classical pricing, we use the “so-called” risk-free rate. Note that it is a little out of context to call it the risk-free rate since arguments under bilateral contracts are from the reality that dealers cannot access to the risk-free rate, yet it is acknowledged that OIS rate is the best proxy for the so-called risk-free rate. Thus, in what follows, we use OIS rate for evaluating the \textit{clean price}, and denote it by \((r_t)\) for \( t \geq 0 \). We assume \((r_t)\) is \( \mathcal{F} \)-adapted and denote by \( B \) the money market account on \((r_t)\), namely

\[
B_t := \exp \left( \int_0^t r_s \, ds \right) \text{ for any } t \geq 0.
\]

We can find the pricing measure \( Q \) such that \( B^{-1} S^i, 1 \leq i \leq n, \text{ are } (\mathcal{Q}, \mathcal{F})\)-local martingales because \( \sigma \) is of full rank as in (2.6). Let \( e_t \) denote the new mark-to-market exposure at \( t \leq T \). We assume that the market exposure is calculated as \textit{clean price}:

\textbf{Assumption 2.5.} For any \( t \in [0, T] \), \( e_t = B_t \mathbb{E}^Q \left( \int_t^T B_s^{-1} \, d\mathcal{D}_s \big| \mathcal{F}_t \right) \).

By Assumption 2.5, we can derive some properties of \((e_t)\). We report the proof in Appendix D.

\textbf{Lemma 2.6.} (i) \( e_T = 0 \).
(ii) \( e_t = 1_{\tau \leq T} e_{\tau} \).
(iii) \( \frac{d}{dt} e_t = (r_t e_t + B_t Z_t \mathbb{A}_t) dt + B_t Z_t \, dW_t - d\mathcal{D}_t, \text{ for } t \in [0, T] \), for some \( Z \in \mathbb{H}_{T, loc}^{2, d} \).
(iv) \( e_{\tau^-} = e_{\tau} \text{ almost surely.} \)

\textbf{Remark 2.7.} Notice that \( Z \) is closely related with the delta risk of \( \hat{e}_t = B_t^{-1} e_t \). Indeed, if \( e \) is Malliavin differentiable and is a smooth function of \( S_t \), i.e., \( \hat{e}_t = \hat{e}(t, S_t) \), then by Clark-Ocone

\(^1\)A part of ISDA Master Agreement.
Lemma 2.6

We assume that \( L \) Agent B (resp. A) only when \( \tau \leq T \). Similarly, \( m_t \geq 0 \) (resp. \( m_t < 0 \)) means that \( A \) posts (resp. receives) the margin to \( B \) at time \( t \leq T \). We assume \( (m_t)_{t \geq 0} \) is an \( \mathbb{F} \)-adapted càdlàg process. Note that \( m \) is chosen to be \( \mathbb{F} \)-adapted for consistency in financial modeling. The collateral is required because we do not know the full information of default. Therefore, the amount of collateral is calculated only by available information \( \mathbb{F} \). The admissible set will be defined more precisely when the risk-sharing problem is introduced.

Once one party announces bankruptcy, the margin process stops. Therefore, at the default \( \tau \leq T \), the amount of collateral is \( m_{\tau-} \), but wealth which amounts to \( e_{\tau} + \Delta \mathcal{D}_{\tau} (= e_{\tau} \text{ a.s.}) \) should be transferred from \( A \) to \( B \). In addition, the loss by breach of the contract will be inflicted to the Agent B (resp. A) only when \( \tau = \tau^A \) and \( e_{\tau} \geq m_{\tau-} \) (resp. \( \tau = \tau^B \) and \( e_{\tau} < m_{\tau-} \)). Denoting the loss rate of the Agent A (resp. B) by \( L^A \) (resp. \( L^B \)), the amount by breach is

\[
1_{\tau = \tau^A} L^A(e_{\tau} - m_{\tau-})^+ - 1_{\tau = \tau^B} L^B(e_{\tau} - m_{\tau-})^-.
\]

We assume that \( L^i, \ i \in \{A, B\} \), are positive constant. Finally, we can define the full cash-flow \( \mathcal{C} \) as

\[
\mathcal{C}_t := 1_{\tau > t} \mathcal{D}_t + 1_{\tau \leq t} (\mathcal{D}_t + e_{\tau}) - 1_{\tau = \tau^A} L^A(e_{\tau} + \Delta \mathcal{D}_{\tau} - m_{\tau-})^+ + 1_{\tau = \tau^B \leq t} L^B(e_{\tau} + \Delta \mathcal{D}_{\tau} - m_{\tau-})^-.
\]

By (2.7) and the last item in Lemma 2.6, for any \( t \leq T \), almost surely

\[
\mathcal{C}_t = 1_{\tau > t} \mathcal{D}_t + 1_{\tau \leq t} (\mathcal{D}_t + e_{\tau}) - 1_{\tau = \tau^A \leq t} L^A(e_{\tau} - m_{\tau-})^+ + 1_{\tau = \tau^B \leq t} L^B(e_{\tau} - m_{\tau-})^-.
\]

In the next section, we define a self-financing portfolios to hedge against \( \mathcal{C} \) with more details. We construct the portfolio in view of the Agent A since the portfolio of the Agent B is in most ways similar. Before proceeding, we provide some remarks related to possible extensions of our model.

Remark 2.8.  
(i) One may argue that clean price is not an appropriate close-out amount since the Agent B’s default is not considered. However, taking default risk of the Agent B into the exposure may heavily penalize the surviving party, because the default event of one party can negatively affect the creditworthiness of the surviving party, especially when the defaulting member has an impact on systemic risk. For such discussion, readers can refer to [21].

(ii) In practice, variation margin is called on intra-day basis (say, two or three times per day). In this paper, we assume a continuous margin process for simplicity. One may want to model variation margin as a càdlàg step process to describe reality more precisely, cf. [20].

(iii) Underlying assets subject to defaults are beyond the scope of this paper. For modeling with emphasis on contagion risk, readers may want to refer to [38, 17, 18, 16].

(iv) In reality, it is hard to estimate exact default intensities. For example, dependence between the Agent B’s exposure and default probability is not negligible, which is sometimes called right/wrong way risk, but it is challenging to estimate the dependence from market quotation. Thus, such issues lead to robust pricing arguments. See [34, 9].

2.3.2. Self-Financing Hedging Portfolio. The funding sources of an agent can be an external funding provider, treasury department, repo markets, etc. After the financial crisis, such funding rates do not represent the risk-free rate (approximately OIS rate in recent times). We consider \( \mathbb{F} \)-adapted processes \((R^{i,m}_t)_{t \geq 0}, \ i \in \{A, B\}\), to represent the margin funding rates offered from margin lenders, and denote that for \( t \geq 0 \),

\[
B^{i,m}_t := \exp \left( \int_0^t R^{i,m}_s \, ds \right).
\]
Another cash-flow stream associated with margin process is remuneration from margin receivers. When variation margin is pledged by (resp. received by) the party A, the party B should remunerate (resp. should be remunerated by) the party A with respect to an interest rate. We let \( R^m \) and \( B^m \) denote the remuneration rate and account of the party A. Therefore, for each party, the net cost/benefit involved in posting the margin is determined by \( R_i^m - R^m, i \in \{A, B\} \), and we denote this spread by \( s^{i,m} \), i.e.,

\[
(2.10) \quad s^{i,m} := R_i^m - R^m, \quad i \in \{A, B\}.
\]

In general, rehypothecation is allowed for variation margin, in other words, the margin account can be used to maintain the hedging portfolio. Moreover, assume that the two parties should finance their operations by interest rates \( R^i \), \( i \in \{A, B\} \), for constructing the rest of the portfolios. We denote the associated funding account and spread by \( B^i \) and \( s^i \), \( i \in \{A, C\} \), respectively, i.e.,

\[
(2.11) \quad B^i_t := \exp \left( \int_0^t R^i_s \, ds \right), \\
(2.12) \quad s^i_t := R^i_t - r_t.
\]

Then, each party constructs their hedging portfolio using the above accounts and risky assets \( (B^i, B^{i,m}, B^m, S), i \in \{A, B\} \). Let \( \varphi^i := (\eta^i, \eta^{i,m}, \eta^m, \eta^{i,S}) \) denote the respective units of \( (B^i, B^{i,m}, B^m, S), i \in \{A, B\} \), in the hedging portfolios and we call \( \varphi^i \) the trading strategy of the party \( i \). We assume that \( \varphi^i, i \in \{A, B\}, \) are \( \mathbb{G}\)-predictable and use the convention that a positive unit of trading strategy means long position.

If the party A posts collateral which amounts to \( m_t \) at \( t \), she needs to deliver \( \eta^{A,m}_t \) shares of the account \( B^{A,m}_t \) from the margin lender. Then, the party A will have \( \eta^m_t \) shares of the margin account \( B^m_t \) to which the remuneration from the party B is accrued. Thus,

\[
(2.13) \quad \eta^m B^m = m, \\
(2.14) \quad \eta^{A,m} B^{A,m} + \eta^m B^m = 0, \\
(2.15) \quad \eta^{B,m} B^{B,m} - \eta^m B^m = 0.
\]

Remark 2.9. As in practice, we consider cash collateral which is rehypothecated. Sometimes it is possible that risky assets can be posted as collateral and the margin account is segregated, which
means that the account is not included in the hedging portfolio. When we consider a different convention, the mathematical structure of wealth process also become different. For the various conventions, readers may want to refer to [12, 28]. In addition, the amount of collateral may depend on the value of the hedging portfolio, which is sometimes called endogenous collateralization. [23] and [46] discussed endogenous collateralization by PDEs and BSDEs, respectively.

Now, we are in a position to define a self-financing portfolio.

Definition 2.10. If $V_t^A = V_t^A(\phi^A, C)$, $t \in [0, T]$, defined as

$$V_t^A = \eta_t^A B_t^A + \eta_t A^m B_t^A + \eta_t B^m_t + \eta_t^A S_t,$$

satisfies

$$V_t^A = V_0^A + \int_0^{t \land \bar{T}} \eta_s^A dB_s^A + \int_0^{t \land \bar{T}} \eta_s A^m dB_s^A + \int_0^{t \land \bar{T}} \eta_s B^m dB_s^m + \int_0^{t \land \bar{T}} \eta_s^A dS_s - \mathcal{C}_{t \land \bar{T}},$$

for any $t \in [0, T]$, then $V_t^A$ is called the self-financing portfolio of the Agent A.

Remark 2.11. Note that for $t > \bar{t}$, $V_t^A = V_\bar{t}^A$. In (2.17), $\mathcal{C}_{t \land \bar{T}} = \mathcal{C}_t$, for any $t \geq 0$, by the definition (2.8).

The self-financing portfolio of the Agent B is defined similarly. The difference is the direction of variation margin and $C$. Then, by (2.10), (2.13)-(2.17), we can see that self-financing portfolio processes of the Agent A and, similarly, the Agent B follow

$$dV_t^A = \left(R_t^A V_t^A - s_t A^m m_t + \eta_t^A S_t \mu_t - 1 R_t^A \right) dt + \eta_t B^m \sigma_t dW_t - d\mathcal{C}_t,$$

$$dV_t^B = \left(R_t^B V_t^B + s B^m m_t + \eta_t B^S \sigma_t - 1 R_t^B \right) dt + \eta_t B^S \sigma_t dW_t + d\mathcal{C}_t,$$

where $\circ$ is component-wise product. If we consider an agent who has a naked position against market risk, we set $\eta_t^A = 0$. Before examining whether (2.18) and (2.19) are well defined, we first want to introduce our target problem.

We find the best initial price and amount of variation margin to optimize the aggregated utilities of both parties. If there were no adjustment in pricing, the classical risk-neutral price $e_0$ should be exchanged at initiation of the contract. Let $p$ denote the amount paid to the Agent A on top of $e_0$. Therefore, initial price paid to the Agent A is $e_0 + p$. More precisely, denoting the initial endowment of each party by $\nu^A$ and $\nu^B$,

$$V_0^A = \nu^A + e_0 + p, \quad \text{and} \quad V_0^B = \nu^B - e_0 - p.$$

Thus, $V^i$ depends on the choice of $(p, m)$. For simplicity of notations, we often suppress $(p, m)$, e.g., $V^i = V^i p, m, i \in \{A, B\}$. Then, with an admissible set $A$, utilities $U_i \colon \mathbb{R} \to \mathbb{R}$, and $\lambda > 0$, we define the risk-sharing problem as follows:

$$p^*, m^* = \arg \max_{(p, m) \in A} \mathbb{E} \left[ U_A((B_t^A)^{-1} V_t A^p, m) + \lambda U_B((B_t^B)^{-1} V_t B^p, m) \right].$$

We will define $A$ more precisely in the following section. Note that hedging strategies are not control variables. In other words, we assume that two parties choose their strategies by their own methodologies not by the risk-sharing framework. It can be said that $\lambda$ is the relative bargaining power of the Agent B, or how much the Agent A wants to enter the contract. One can also think of $\lambda$ as the belief of how much funding spread should be acknowledged in derivative transactions.
As we set the conventions in subsection 2.3.2, \( p^* \) is the amount paid to the Agent A on top of \( c_0 \). This additional payment is necessary because of default risk and funding spread. If two parties price the contracts individually, the calculated prices may be different to each party because of different funding spread on this model. Therefore, when the contract is made with the initial price \( c_0 + p^* \), some parties should accept a cost. Thus, \( p^* \) can be seen as the cost that is agreed by the two parties to enter the contract, so we call \( p^* \) agreement-cost. We also call \( m^* \) optimal collateral (or margin), and \( (p^*, m^*) \) risk-sharing contract.

Before giving the detail, we first provide motivation about the discounting factors behind the choice of our model (2.20). In (2.20), the values of the portfolios were adjusted by discounting factors. The discounting factors are necessary for a fairness since the two agents have different funding rates. In general, the higher default risk is, the higher funding rate is. However, a hedging portfolio grows with respect to its funding rate (recall (2.18) and (2.19)). Therefore, without the discounting factors, we penalize a party under a healthier credit condition.

One may want to put the discounting factors outside of utilities as it is a typical choice in portfolio optimization literature. In this case, when the portfolio processes evolve forwardly, the effect of funding rates is mixed with risk aversion parameters in the utilities. However, the future value is purely discounted without consideration of risk aversions, so we would again end up with punishing or rewarding a certain party depending on risk aversions. An argument in the same context was discussed in [48].

For the utilities, we will investigate two cases:

\[
U_A(x) = x, \quad U_B(x) = -e^{-\gamma B x},
\]

\[
U_A(x) = -e^{-\gamma A x}, \quad U_B(x) = -e^{-\gamma B x},
\]

for some \( \gamma^i > 0 \). We choose the exponential utilities mainly for simplicity. To use a power utility, we need for \( V^i, i \in \{A, B\} \), to be lower bounded. To this end, boundedness condition should be imposed to \( (p, m) \), but this makes the exposition more complicated. Moreover, an explicit form of optimal collateral is not generally obtained under power utilities.

To solve the risk-sharing problem, we need different restrictions to funding spread depending on the choice of utilities for characterizing the optimal collateral. The restrictions are required mainly because the value functions w.r.t variation margin is not concave. More precisely, we will need that \( s^B,m = 0 \) in (2.21), and \( s^A,m = s^B,m = 0 \) in (2.22). The conditions on funding spread can be assumed not only when the capital structure of a party has small leverage but also when the party achieves secured funding for variation margin. This situation is not common, but some examples for the secured funding were discussed by [3].

There is another interpretation to keep the funding condition without loss of much generality, which is partly justified by a complete market argument. It was shown in complete market models that an agent can guarantee that they do not switch their position of funding state between lending and borrowing position, depending on the structure of the payoff. This binary nature of funding state is related to whether payoff functions are non-increasing or non-decreasing with respect to underlying assets. For the details, see Proposition 5.8 in [32] and refer to [42]. To streamline this paper, we deal with cases of (2.22), in the main sections. For the cases of risk-neutral agent, we report the analysis in Appendix C. Therefore, the following assumption stands throughout the following sections except Appendix C.

**Assumption 2.12.**

(i) \( s^A,m = s^B,m = 0 \),

(ii) \( U_A(x) = -e^{-\gamma A x} \) and \( U_B(x) = -e^{-\gamma B x} \).

In the next section, we represent (2.20) in a reduced form with a more precise definition of the admissible set. (2.20) is one type of principal-agent problems. This problem is often called the first best case in typical principal-agent context. In general, it is challenging to solve principal-agent problems because the solvability of involved equations, e.g., coupled FBSDEs, is not easy to obtain.
Since we also encounter a similar difficulty as well as non-concavity, we need to modify the dynamic version of our problem and impose some restrictions depending on the utilities. We will explain this point with more detail in section 3.

2.4. Reduction of Filtration. We start this section with introducing a long list of notations. The following notations are often used in this paper. For \( i \in \{A, B\}, t \in [0, T] \),

\[
\begin{align*}
(2.23) \quad \hat{V}_t^A &= (B_t^A)^{-1}(V_t^A - e_{t\wedge \tau}), \quad \hat{V}_t^B := (B_t^B)^{-1}(V_t^B + e_{t\wedge \tau}), \\
(2.24) \quad v_t &= (B_t^A)^{-1}e_t, \quad c_t := (B_t^A)^{-1}m_t, \\
(2.25) \quad \delta_t &= v_t - c_t, \quad K_t := B_t^A(B_t^B)^{-1}, \\
(2.26) \quad \pi_t^i := (B_t^i)^{-1}v_t^i S \circ S_i \sigma_t, \quad \Delta_t^i := B_t(B_t^i)^{-1}Z_t, \\
(2.27) \quad \bar{\phi}_t^A := \pi_t^A - \Delta_t^A, \quad \bar{\phi}_t^B := \pi_t^B + \Delta_t^B, \\
(2.28) \quad \sigma_i \Lambda_i^t := (\mu_t - R_t^i 1), \quad b_i^t := \Lambda_t^i - \Lambda_t, \\
(2.29) \quad \Theta_t(\delta) := 1_{\tau_A=t}L^A\delta^+ - 1_{\tau_B=t}L^B\delta^-.
\end{align*}
\]

We give some remarks on the above notations.

Remark 2.13. By (2.23), \( \hat{V}^i, i \in \{A, B\} \) are (discounted) adjustment processes. By (2.24), \( v \) is the discounted market exposure, and \( c \) is the discounted collateral, and by (2.25), \( \delta \) is the difference between the two processes. Note that \( \delta = v - c = 0 \) means full collateralization. By (2.26), \( \Delta^i, i \in \{A, B\} \), are the delta-risk of the market exposure adjusted by the funding rate of each party. By (2.27), \( \bar{\phi}^i, i \in \{A, B\} \), are the difference between the amount invested in the risky assets and delta risk of the clean price. i.e., \( \bar{\phi}^i \) can be seen as the hedging error. If the Agent B does not hedge the market risk, we have \( \bar{\phi}^B = \Delta^B \). Notice that if \( b^i = 0 \), then \( R^i = r \), by (2.28).

We will find the projections of \( \hat{V}^i \) onto \( \mathbb{F} \), then we will deal with the risk-sharing problem mainly with the reduced processes. For any \( i \in \{A, B\} \), we let \( \bar{\phi}^i \) denote the \( \mathbb{F} \)-predictable reduction of \( \tilde{\phi}^i \) until \( \tau \). Namely, \( \bar{\phi}^i, i \in \{A, B\} \), are \( \mathbb{F} \)-predictable and

\[
1_{t \leq \tau} \bar{\phi}_t^i = 1_{t \leq \tau} \tilde{\phi}_t^i.
\]

By Itô’s formula and (2.24), \( v \) satisfies, for \( t \in [0, T] \),

\[
dv_t = (-s^i_t v_t + \Delta_t^A \Lambda_t) dt + \Delta_t^A dW_t - (B_t^i)^{-1} \Delta_t^B dt.
\]

Note that \( v \) is exogenously given. Thus, if \( R^i, i \in \{A, B\} \), are independent with \( V^i \) and \( \hat{V}^i \) are well-defined, then \( V^i \) are also well defined by (2.23).

Theorem 2.14. Assume \( s^i, s^i_m, i \in \{A, B\} \), are bounded and

\[
\sum_{i \in \{A, B\}} \int_0^T \left( |\delta_t|^2 + |\Lambda_t|^2 + |\phi_t^i|^2 + |b_t^i|^2 + |\Delta_t^i|^2 \right) dt < \infty, \quad a.s.
\]

Then, the following processes \( v^A \) and \( v^B \), are well-defined:

\[
(2.30) \quad dv^A_t = (\phi_t^A \Lambda_t^A + \Delta_t^A b_t^A + s_t^A v_t) dt + \phi_t^A dW_t, \\
(2.31) \quad dv^B_t = (\phi_t^B \Lambda_t^B - \Delta_t^B b_t^B - s_t^B K_t v_t) dt + \phi_t^B dW_t.
\]

Moreover, assume that \( R^i, i \in \{A, B\} \), are independent with \( V^i \), and

\[
v_0^A = v^A + p, \\
v_0^B = v^B - p.
\]

Then \( v^i, i \in \{A, B\} \), are \( \mathbb{F} \)-optional reductions of \( \hat{V}^i \) until \( \tau \), i.e., \( v^i, i \in \{H, C\} \), are \( \mathbb{F} \)-optional and \( 1_{t \leq s} \hat{V}_s^i = 1_{t \leq s} v^i_t \), for any \( t \geq 0 \).
Proof. It is easy to check the first assertion. To check the second part, we apply Itô’s formula to $(B^A_t)^{-1}V^A_t$ and this yields

$$d((B^A_t)^{-1}V^A_t) = -R^A_t(B^A_t)^{-1}V^A_t \, dt + (B^A_t)^{-1} \, dV^A_t$$

$$= \mathbb{1}_{t \leq \bar{\tau}} \varphi^A_t \, dt + \mathbb{1}_{t \leq \bar{\tau}} \varphi^A_t \, dW_t - (B^A_t)^{-1} \, d\mathcal{C}_t.$$  

In addition, by (2.8) together with $v_\tau = v_{\tau-}$, a.s,

\begin{equation}
(B^A_t)^{-1} \, d\mathcal{C}_t = \mathbb{1}_{t \leq \bar{\tau}}(B^A_t)^{-1} \, d\mathcal{D}_t + d(\mathbb{1}_{t \leq \bar{\tau}})v_{\tau-} - d(\mathbb{1}_{t \leq \bar{\tau}})\Theta_{\tau}(\delta_{\tau-}).
\end{equation}

Then, by combining (2.32) and (2.33), we have

\begin{equation}
d\bar{V}^A_t = d((B^A_t)^{-1}(V^A_t) - v_{t\wedge \tau})
= \mathbb{1}_{t \leq \bar{\tau}}(s^A_t v_t + \bar{\phi}^A_t \Lambda^A_t + \Delta^A_t b^A_t) \, dt
+ \mathbb{1}_{t \leq \bar{\tau}} \bar{\phi}^A_t \, dW_t - \mathbb{1}_{t \geq \bar{\tau}}(B^A_t)^{-1} \, d\mathcal{D}_t - d(\mathbb{1}_{t \leq \bar{\tau}})(v_{\tau-} - \Theta_{\tau}(\delta_{\tau-})).
\end{equation}

It follows that

\begin{align*}
d(\mathbb{1}_{t \leq \bar{\tau}} V^A_t) &= \bar{V}^A_t \, d(\mathbb{1}_{t \leq \bar{\tau}}) + \mathbb{1}_{t \leq \bar{\tau}} \, d\bar{V}^A_t - \delta_\tau(dt) \Delta \bar{V}^A_t
= \bar{V}^A_t \, d(\mathbb{1}_{t \leq \bar{\tau}}) + \mathbb{1}_{t \leq \bar{\tau}}(s^A_t v_t + \bar{\phi}^A_t \Lambda^A_t + \Delta^A_t b^A_t) \, dt + \mathbb{1}_{t \leq \bar{\tau}} \bar{\phi}^A_t \, dW_t
- d(\mathbb{1}_{t \geq \bar{\tau}})(v_{\tau-} - \Theta_{\tau}(\delta_{\tau-})) - \delta_\tau(dt) \Delta \bar{V}^A_t
= \mathbb{1}_{t \leq \bar{\tau}} \, dV^A_t - d(\mathbb{1}_{t \geq \bar{\tau}}) \bar{V}^A_t - d(\mathbb{1}_{t \geq \bar{\tau}})(v_{\tau-} - \Theta_{\tau}(\delta_{\tau-})).
\end{align*}

Let $\mathcal{Y}_t := \mathbb{1}_{t \leq \bar{\tau}} v^A_t + \mathbb{1}_{t \geq \bar{\tau}}(v_{\tau-} - v_{\tau-} + \Theta_{\tau}(\delta_{\tau-}))$. Again, by Itô’s formula together with $v_\tau = v_{\tau-}$, a.s,

\begin{equation}
d\mathcal{Y}_t = \mathbb{1}_{t \leq \bar{\tau}} \, dv^A_t - d(\mathbb{1}_{t \leq \bar{\tau}})v^A_t \, d(\mathbb{1}_{t \leq \bar{\tau}})(v_{\tau-} - \Theta_{\tau}(\delta_{\tau-}))
= \mathbb{1}_{t \leq \bar{\tau}} \, dv^A_t - d(\mathbb{1}_{t \leq \bar{\tau}})(v_{\tau-} - \Theta_{\tau}(\delta_{\tau-})).
\end{equation}

Thus, if $\mathcal{Y}_0 = \bar{V}^A_0$, we obtain $\mathcal{Y}_t = \bar{V}^A_t$, for any $t \in [0, T]$. Moreover, $v^A_t$ is the $\mathbb{F}$-optional reduction of $\bar{V}^A_t$ and more precisely,

\begin{equation}
\bar{V}^A_t = \mathbb{1}_{t \leq \bar{\tau}} v^A_t + \mathbb{1}_{t \geq \bar{\tau}}(v_{\tau-} - v_{\tau-} + \Theta_{\tau}(\delta_{\tau-})).
\end{equation}

Similarly, we can attain that

\begin{equation}
\bar{V}^B_t = \mathbb{1}_{t \leq \bar{\tau}} v^B_t + \mathbb{1}_{t \geq \bar{\tau}}(v_{\tau-} + K_{\tau-}v_{\tau-} - K_{\tau-}\Theta_{\tau}(\delta_{\tau-})).
\end{equation}

Notice that control of $m$ is equivalent to that of $\delta$ since $\epsilon$ is given exogenously. Thus, we solve (2.20) with respect to the two state processes depending on $\delta$:

$$V^{i,p,m} = V^{i,p,\delta}.$$  

Moreover, we denote that $c^* := (B^A)^{-1}m^*$ and $\delta^* := v - c^*$.

Now, we are ready reduce the risk-sharing problem. Recall from (2.20) that our goal is to maximize the sum of utilities of discounted portfolios over all $(p, \delta) \in \mathcal{A}$:

\begin{equation}
E\left[U_A((B^A)^{-1}V^A_{p,\delta}) + \lambda U_B((B^B)^{-1}V^B_{p,\delta})\right].
\end{equation}
To this end, we will represent the two terms in (2.36) as reduced forms. Indeed, by Lemma A.1, where the integrability conditions hold

\[
\mathbb{E}\left[U_A\left((B^A_T)^{-1}V^{A,p,\delta}_T\right)\right] \\
= \mathbb{E}\left[U_A\left(1_{T<\tau}v^{A,p}_T + 1_{\tau \leq T}(v^{A,p}_T + \Theta^\delta_\tau)\right)\right] \\
= \mathbb{E}\left[G_TU_A(v^{A,p}_T) + \int_0^T G_t\left[h^A_tU_A\left(v^{A,p}_t + L^A\delta^+_t\right) + h^B_tU_A\left(v^{A,p}_t - L^B\delta^+_t\right)\right] dt\right],
\]

and

\[
\mathbb{E}\left[U_B\left((B^B_T)^{-1}V^{B,p,\delta}_T\right)\right] \\
= \mathbb{E}\left[U_B\left(1_{T<\tau}v^{B,p}_T + 1_{\tau \leq T}(v^{B,p}_T - K^\delta_\tau)\right)\right] \\
= \mathbb{E}\left[G_TU_B(v^{B,p}_T) + \int_0^T G_t\left[h^B_tU_B\left(v^{B,p}_t - L^B K^\delta_t\right) + h^A_tU_B\left(v^{B,p}_t + L^B K^\delta_t\right)\right] dt\right].
\]

We define \(g_t := 1_{\delta \geq 0}g^+_t + 1_{\delta < 0}g^-_t\), where

\[
g^+_t(v^A, v^B, \delta) := G_t\left[h^A_tU_A\left(v^A_t + L^A\delta + \lambda U_B\left(v^B_t - L^B K^\delta_t\right)\right) + h^B_tU_B\left(v^A_t + \lambda U_B\left(v^B_t\right)\right)\right] \\
g^-_t(v^A, v^B, \delta) := G_t\left[h^B_tU_B\left(v^A_t + L^B\delta + \lambda U_B\left(v^B_t - L^B K^\delta_t\right)\right) + h^A_tU_A\left(v^A_t + \lambda U_B\left(v^B_t\right)\right)\right].
\]

For the above reduction to be valid, we assume the following integrability condition:

\[
\sum_{i \in \{A, B\}} \left|U_i(v^i_t)\right| + \int_0^T \left|U_i(v^i_t)\right| dt < \infty,
\]

and we define the admissible set of collateral \(D\) for a given Borel set \(A \subseteq \mathbb{R}\) as follows:

**Definition 2.15.** \(\delta \in D\), if \(\delta \in \mathbb{H}^2_T\) and

(i) \(\delta \in A\), \(d\mathbb{P} \otimes dt\) \(-\) a.s,
(ii) \(\mathbb{E}\left[\int_0^T \left|g_t(v^{A,p}_t, v^{B,p}_t, \delta_t)\right| dt\right] < \infty\).

Then, the risk-sharing problem can be rewritten as

\[
\max_{(p, \delta) \in A}\mathbb{E}\left[G_TU_A(v^{A,p}_T) + \lambda G_TU_B(v^{B,p}_T) + \int_0^T g_t(v^{A,p}_t, v^{B,p}_t, \delta_t) dt\right],
\]

where \(A = \mathbb{R} \times D\) and \(v^i, i \in \{A, B\}\), are defined in (2.23)-(2.28), (2.30), and (2.31). Note that \(v^i\) does not depend on \(\delta\) since we assume \(s^{A,m} = s^{B,m} = 0\). When the Agent A is risk-neutral, we only need that \(s^{B,m} = 0\), and in this case \(D\) should be defined in a slightly different way. We will discuss this with more details in Appendix C.

**Remark 2.16.** (i) The risk-sharing framework can be thought of as a two-agent problem.

Since there is no party who can solely decide the contract, the mathematical structure of our risk-sharing problem is different from that of typical principal-agent problems. For example, for the Agent A, in both perspectives of funding impacts and loss given defaults, posting collateral to the Agent B is not beneficial. Say, we consider A as an agent, subject to B as a principal. Then, if we solve the agent problem first, e.g., as in [30], it always gives the trivial solution \(\delta^* = v - c^* = \infty\).
Remark 2.16, we need the sharing problem by using martingale optimality principle. Then we argue by verification that the agreement-cost by using martingale optimality principle. By MOP, Appendix C, then the agreement-cost $p^\ast$ will be found with the given optimal variation margin $\delta^\ast$. However, mainly because of the non-concave property of our problem addressed in Remark 2.16, we need to impose some restrictions to the funding spread in delivering collateral for the characterization. Recall that we consider two cases: a risk-neutral Agent A and risk-averse the Agent B investing their capital with a small leverage so that $s^{B,m} = 0$, and two risk averse parties with $s^{B,m} = s^{A,m} = 0$. As we mentioned, the mathematical analysis for a risk-neutral agent is deferred to Appendix C. In what follows, we first derive an optimal collateral for both cases, then we give financial interpretations later. The two most notable features are the weak dependence with default intensities and relationship with the full margin requirement. The discussion about the relationship between the optimal collateral and margin requirement is an important part of this paper. This funding condition is not necessary for finding $p^\ast$ if $\delta$ is not a control variable, e.g. $A = \{\delta_0\}$ for some $\delta_0 \in \mathbb{R}$.

We define a dynamic version of (2.38) and use martingale optimality principle (MOP) as in [39]. To this end, we define a set of controls which coincide with a given $\varepsilon \in \mathcal{D}$ up to a certain time $t \leq T$. We denote the set by $\mathcal{D}(t, \varepsilon)$, i.e., for $\varepsilon \in \mathcal{D}$, $\mathcal{D}(t, \varepsilon) := \{ \delta \in \mathcal{D} | \delta_{nt} = \varepsilon, \Delta t \}$. Now, define the dynamic version of (3.1) as

$$
J^\varepsilon_t(p) := \text{ess sup}_{\delta \in \mathcal{D}(t, \varepsilon)} \mathbb{E}\left[ G_T U_A(v^A_T, \delta) + \lambda G_T U_B(v^B_T, \delta) + \int_0^T g_s(v^A_s, v^B_s, \varepsilon_s) ds \mathbb{I}_F \right].
$$

Then we characterize the optimal collateral by using martingale optimality principle. By MOP, $(J^\varepsilon_0)_{0 \leq t \leq T}$ is chosen as a càdlàg version such that for any $\varepsilon \in \mathcal{D},$

$$
\left\{ J^\varepsilon_t + \int_0^t g_s(v^A_s, v^B_s, \varepsilon_s) ds \right\}_{0 \leq t \leq T}
$$

is a $(\mathbb{P}, \mathbb{F})$-supermartingale. Moreover, for the optimal collateral $\delta^\ast$ for $J_0,$

$$
\left\{ J^\ast_t + \int_0^t g_s(v^A_s, v^B_s, \delta^\ast_s) ds \right\}_{0 \leq t \leq T}
$$

is a $(\mathbb{P}, \mathbb{F})$-martingale. When the admissibility is guaranteed, a solution to (3.1) can be found by verification.

Before moving on, to represent the optimal collateral by one stochastic process, we define a process $(X_t)_{t \geq 0}$ such that

$$
J^\varepsilon_t := \text{ess sup}_{\delta \in \mathcal{D}(t, \varepsilon)} \mathbb{E}\left[ G_T U_A(v^A_T, \delta) + \lambda G_T U_B(v^B_T, \delta) + \int_0^T g_s(v^A_s, v^B_s, \varepsilon_s) ds \mathbb{I}_F \right].
$$

$$
U_A(X_t) := \frac{U_A(v^A_t)}{-U_B(v^B_t - v^B_0)} = -\exp \left[ -\gamma^A \left( v^A_t - \frac{\gamma^B}{\gamma^A} (v^B_t - v^B_0) \right) \right].
$$
Namely, $X_t = \nu_t^A - (\gamma B/\gamma A)(\nu_t^B - \nu_0^B)$. More precisely, $X$ is given by

\begin{equation}
X_t = \nu_0^A + \int_0^t \left[ s_t v_t + \phi_t^A \Lambda_t^A - \frac{\gamma B}{\gamma A} \phi_t^B \Lambda_t^B + \Delta_t^A b_t^A + \frac{\gamma B}{\gamma A} \Delta_t^B b_t^B \right] ds + \int_0^t \phi_s dW_s.
\end{equation}

Then, (3.1) will be represented w.r.t $X$, and where $\phi_t := \phi_t^A - (\gamma B/\gamma A)\phi_t^B$ and $s_t := s_t^A + (\gamma B/\gamma A) s_t^B K_t$.

**Theorem 3.1.** Assume that the integrability condition (2.37) hold. Define

\begin{equation}
\delta^*_t(p, X) := \arg \max_{\delta \in A} \{ U_A(X_t) \psi_t^A(\delta) + \lambda U_B(\nu_t^B - p) \},
\end{equation}

\begin{equation}
\psi_t^A(\delta) := -h_t^A U_A(L^A \delta^+) - h_t^B U_A(-L^B \delta^-),
\end{equation}

\begin{equation}
\psi_t^B(\delta) := -h_t^A U_B(-L^A K_t \delta^+) - h_t^B U_B(L^B K_t \delta^-).
\end{equation}

If $\delta^*(p, X) \in D$, then $\delta^*(p, X)$ is a solution of (3.1). Moreover,

\begin{equation}
J_0 = \mathbb{E} \left[ \beta_T \left[ U_A(X_T) + \lambda U_B(\nu_T^B - p) \right] \right],
\end{equation}

where $\beta_t := -G_t U_B(\nu_t^B - \nu_0^B)$ and

\begin{equation}
\hat{f}_t(p, x) := \beta_t \left[ U_A(x) \psi_t^A(\delta^*_t(p, x)) + \lambda U_B(\nu_t^B - p) \psi_t^B(\delta^*_t(p, x)) \right].
\end{equation}

**Proof.** For $\varepsilon \in D$, define $\xi^\varepsilon_t := J_t + \int_0^t g_s(\nu_s^A, \nu_s^B, \varepsilon_s) ds$. Notice that $J_t$ is independent of $\varepsilon \in D$ and by (3.6) and (3.7),

\[-[G_t U_B(\nu_t^B - \nu_0^B)]^{-1} \hat{g}_t(\nu_t^A, \nu_t^B, \varepsilon_t) = U_A(X_t) \psi_t^A(\varepsilon_t) + \lambda U_B(\nu_t^B) \psi_t^B(\varepsilon_t).
\]

Therefore, for any $\varepsilon \in D$, $\xi^\varepsilon - \xi^{\delta^*(p, X)}$ is a $(\mathbb{P}, \mathbb{F})$-supermartingale. Moreover, for any $\varepsilon \in D$

\begin{equation}
\mathbb{E}[\xi_T^{\varepsilon} - \xi_T^{\delta^*(p, X)}] \leq \mathbb{E}[\xi_0^{\varepsilon} - \xi_0^{\delta^*(p, X)}] = 0.
\end{equation}

Thus, (3.8) is obtained where the admissibility of $\delta^*(p, X)$ is guaranteed.

To find the explicit form of $\delta^*(p, X)$, we consider $A = \mathbb{R}$ and represent (3.5) as

\[\delta^*_t(p, x) := \arg \max_{\delta \in A} \left( \mathbb{1}_{\delta < 0} f^-(t, p, x, \delta) + \mathbb{1}_{\delta \geq 0} f^+(t, p, x, \delta) \right),\]

for some functions $f^-, f^+$. Then, $f^i, i \in \{-, +\}$ are continuously differentiable in $\delta$ and for any $(t, p, x)$, there exist $I^i_t(p, x)$ such that

\begin{equation}
\partial \delta f^i_t(t, p, x, I^i_t(p, x)) = 0.
\end{equation}

Then, $\delta^*(p, X)$ can be attained at $I^-, I^+$, and zero. We can easily see that

\begin{equation}
f^-(t, p, x, \delta) := h_t^B \left[ U_A(x + L^B \delta) + \lambda U_B(\nu - L^B K_t \delta) \right]
\end{equation}

\begin{equation}
+ h_t^A \left[ U_A(x + \lambda U_B(\nu - L^B K_t \delta) \right],
\end{equation}

\begin{equation}
f^+(t, p, x, \delta) := h_t^A \left[ U_A(x + L^A \delta) + \lambda U_B(\nu - L^A K_t \delta) \right]
\end{equation}

\begin{equation}
+ h_t^B \left[ U_A(x + \lambda U_B(\nu - L^A K_t \delta) \right].
\end{equation}
Therefore, we obtain that
\begin{equation}
I^{-}(p, x) := \frac{\gamma^B v^B - \gamma^B p - \gamma^A x - \ln \left( \lambda K + \gamma^B \right)}{L^B(\gamma^B K + \gamma^A)}, \tag{3.14}
\end{equation}
\begin{equation}
I^{+}(p, x) := \frac{\gamma^B v^B - \gamma^B p - \gamma^A x - \ln \left( \lambda K + \gamma^B \right)}{L^A(\gamma^B K + \gamma^A)}. \tag{3.15}
\end{equation}

The exact form of $\delta^*$ can be obtained by characterizing the region
\[ \left\{ \max_{\mathbb{R}} f^- > \max_{\mathbb{R}} f^+ \right\}. \]

The calculation of the region is a straightforward but tedious; see, e.g., [24]. We only obtain the exact form for a simple case which will be seen later. We complete Theorem 3.1 by the next lemma. The proof is reported in Appendix D.

**Lemma 3.2.** Let $A = \mathbb{R}$, and assume $(e_t)_{t \geq 0}, (Z_t)_{t \geq 0}, (\pi^i_t)_{t \geq 0}, i \in \{A, B\}$, are bounded. Then $\delta^*(p, X) \in D$ and (2.37) hold.

**Example 3.3.** Consider an Agent A who hedges delta-risk and an Agent B who does not hedge, i.e., $\pi^A = \Delta^A, \pi^B = 0$. They enter a bond contract that is paid by the Agent A, namely, $D = \mathbb{1}_{[\tau, \infty]}$. We assume that OIS rate $(r_t)_{t \geq 0}$ follows the next SDE:
\[ dr_t = k(\theta - r_t) \, dt + \rho \sqrt{r_t} \, dW_t^Q, \]
for some $k, \theta, \rho \in \mathbb{R}$ and a risk-neutral measure $Q$. Moreover, we assume that $h^i, i \in \{A, B\}$, are bounded and $s^B = s^{B,m} = 0$. Then, by Clark-Ocone formula, for $t \geq 0$,
\[ Z_t = -\rho \sqrt{r_t} A^2(t, T) B_t^{-1} e_t, \]
where
\[ e_t = A^1(t, T) e^{-r_t A^2(t, T)}, \]
\[ A^1(t, T) := \frac{2ae^{(a+k)(T-t)/2}}{2a + (a+k)(e^{a(T-t)} - 1)}, \]
\[ A^2(t, T) := \frac{2(e^{a(T-t)} - 1)}{2a + (a+k)(e^{a(T-t)} - 1)}, \]
\[ a := \sqrt{k^2 + 2 \rho^2}. \]

Since $r > 0$, $Z$ is bounded. Hence, all conditions in Lemma 3.2 are satisfied.

Now, we are ready to discuss the financial interpretation of the optimal collateral. In the next section, the financial meanings of (3.14)-(3.15) and the relationship with the margin requirement will be discussed.

### 3.1. Analysis of Collateral

In this section, we provide financial interpretations of the optimal collateral derived in the previous sections. Collateral is posted for default risk. In our model, there are two main components in default risk: intensities and loss rates. We first discuss a weak dependence between the optimal collateral and default intensities. We begin with giving the explicit form of $\delta^*$ in the following lemma. We report the proof in Appendix D.
LEMMA 3.4. Assume \( A = \mathbb{R} \). Then, \( \delta^*_i(p, x) \) is given by

\[
\delta^*_i(p, x) = (0 \lor I^*_i(p, x)) + (0 \land I^-_i(p, x)),
\]

where

\[
I^*_i(p, x) := -\gamma^A_X - \gamma^B_{\nu^B} \ln (\lambda K^B_{\gamma^B} + \gamma^A),
\]

\[
I^-_i(p, x) := -\gamma^A_X - \gamma^B_{\nu^B} \ln (\lambda K^B_{\gamma^B} + \gamma^A).
\]

Note from (3.16)-(3.18) that the optimal collateral depends only on the loss rates \( L^i \) not on default intensities \( h^i \), which is a rather natural consequence. Collateral is required for loss given default not for the default itself. Put differently, collateral is about how much loss would be inflicted at default and not about how likely default occurs. Recalling \( \delta^* = v - c^* \) and observing (3.17) and (3.18), the magnitude of the optimal variation margin \( c^* \) increases as \( L^i, i \in \{A, B\} \), increase.

We discuss the effect of loss rates with more details. By (3.16), when \( \delta^*(p, X) \leq 0, I^-(p, X) = \delta^*(p, X) \). In this case, as \( L^B \) increases, \( c^* = v - \delta^*(p, X) \) decrease because of the increased average loss of collateral posted to the Agent B. On the other hand, when \( \delta^*(p, X) \geq 0 \), the optimal collateral \( c^* = v - \delta^*(p, X) \), is independent of \( L^B \) and increases w.r.t \( L^A \). Again, this is because the high loss rate makes it risky for the Agent B to post collateral to the Agent A.

The relationships with \( p \) and \( \lambda \) are self-explanatory. If the contract starts from giving a high price \( p \), to the Agent A at initiation of the contract, the Agent A needs to post more collateral in return. Moreover, the higher \( \lambda \) is, i.e., the strong bargaining power the Agent B has, the more the Agent A should post more collateral.

In addition, recalling \( X_t = v^A_t - (\gamma^B / \gamma^A)(v^B_t - v^B_0) \), it seems that (3.16) suggests that optimal collateral ratio should be decided by the relative performance of each party. It is not obviously applicable in practice. However, we can use \( X \) to derive an interesting interpretation from the full margin requirement of Basel III, which will be discussed in the next section.

**Remark 3.5.** From (3.16)-(3.18), the major factor for collateral is the loss rate. In practice, loss rates are often chosen as 0.6 regardless of entities. Our model together with the practice on loss rates partly explains the margin requirement applied to all banks.

### 3.2. Analysis of the Full Margin Requirement

In this section, we interpret the meaning behind the inter-dealer market convention that is required by Basel III. It can be understood that the inter-dealer convention is \( \delta^*(p, X) = 0 \). By (3.14) and (3.15), the full margin convention requires that

\[
X_t + \frac{1}{\gamma^A} (s^A - s^B)t = \frac{\gamma^B}{\gamma^A}(\nu^B - p) - \frac{1}{\gamma^A} \ln \left( \frac{\lambda\gamma^B}{\gamma^A} \right), \quad d\mathcal{P} \otimes dt \quad \text{a.s.}
\]

Therefore, since \( \{X_t + (\gamma^A)^{-1} \int_0^t (s^A - s^B) \, ds\}_{t \geq 0} \) should be constant, it is necessary that \( \phi = 0 \). If two parties’ hedging strategies are chosen independently of each other, by (3.4), \( \phi = 0 \) may mean \( \phi^A = \phi^B = 0 \). It follows that \( \pi^A = \Delta^A \) and \( \pi^B = -\Delta^B \). In other words, the two parties should hedge the delta-risk of market exposure. In addition, together with (3.4), this constant condition implies that

\[
(s^A + \frac{\gamma^B}{\gamma^A} s^B K_t) v + \Delta^A b^A + \frac{\gamma^B}{\gamma^A} \Delta^B b^B + \frac{\gamma^B}{\gamma^A} s^A - s^B = 0, \quad d\mathcal{P} \otimes dt \quad \text{a.s.}
\]

For (3.20) to hold with arbitrary \( \Delta^A \) and \( \Delta^B \), we should have \( s^A = s^B = 0 \). Therefore, for the market convention to be optimal, the following two conditions are necessary:

- both agents hedge the delta-risk of clean price,
- funding spreads are not transferred to each party.
The second item seems like an expected result because the condition $\delta^* = 0$ inherently considers two parties whose earnings from the margin is symmetric. If one can make a profit or suffer a loss by margin process, $\delta^* = 0$ may not be optimal. A debate is still underway whether funding spread should be recouped from counterparties and how to handle the accounting; see, e.g., [36, 37, 25, 5, 2]. Indeed, in frictionless markets, choices of funding are separated with pricing as MM theorem properly applies. However, with frictional distress costs, shareholders’ decision can depend on the choices of funding. In such cases, the margin requirement is not optimal anymore. Therefore, the second condition on funding transfer can be understood that the margin requirement of Basel III inherently considers frictionless financial markets.

In the next section, we will derive a maximum principle of $p^*$ for (2.38). Mainly because of an issue from non-concavity, for finding the optimal pair $(p^*, \delta^*)$, we need either $s_{B,m} = s_{A,m} = 0$ or $A = \{\delta_0\}$, for some $\delta_0 \in \mathbb{R}$. The second condition means that the variation margin $c$ is fixed as a given process.

4. Optimal Initial Prices. Throughout this section, the conditions in Lemma 3.2 are assumed so that the admissibility is obtained. The next maximum principle for $p^*$ is basically a first order condition. First, we consider the case that $\delta$ is not a control variable, i.e. $A = \{\delta_0\}$, for some $\delta_0 \in \mathbb{R}$. In previous sections, we have assumed that $s_{A,m} = s_{B,m} = 0$. However, when $A$ is singleton, we do not need the condition on margin rate.

Theorem 4.1. Assume $A = \{\delta_0\}$, for some $\delta_0 \in \mathbb{R}$. Therefore, $\delta^* = \delta_0$. Let $X^* := X^{p^*, \delta^*}$, $f^* := \hat{f}(p^*, X^*)$, and for given $t \leq T$, define $Q_t \in \mathbb{R}^2$ as the set that $f_t^*(\cdot)$ is not differentiable. Assume

\[
\mathbb{E}
\left[
\beta_T U_A'(X^*_T) - \beta_T \lambda U_B'(\nu - p^*)
\right.
\]

\[
\left. + \int_0^T \mathbb{1}_{(p^*, X^*) \notin Q_t} (\partial_p \hat{f}_t(p^*, X^*_t) + \partial_x \hat{f}_t(p^*, X^*_t)) \, dt \right] = 0.
\]

Then $p^*$ is the optimal initial price.

Proof. Notice that $f_t^*(\cdot)$ is concave for any $t \in [0, T]$, so is differentiable a.e. The maximum principle (4.2), is basically a first order condition. We only need to check whether $(p^*, X^*)$ is not absorbed in $Q$. Since we assume $A = \{\delta_0\}$, for some $\delta_0 \in \mathbb{R}$, $\delta^*$ does not depend on $(p, X)$. We let, for any process $\varphi$, $\varphi^* := \varphi |_{p^*}$ and, for arbitrary $p \in \mathbb{R}$, $\Delta \varphi^* := \varphi^* - \varphi$. Then,

\[
\mathbb{E}[\Delta Y_t^*] = \mathbb{E}
\left[
\beta_T (U_A(X_T^*) - U_A(X_T^*)) + \beta_T \lambda (U_B(\nu - p) - U_B(\nu - p^*)) + \int_0^T \Delta f_t^* \, dt
\right]
\]

\[
\leq \mathbb{E}
\left[
\beta_T U_A'(X_T^*) \Delta X_T^* - \beta_T \lambda U_B'(\nu - p^*) \Delta p^* + \int_0^T \Delta f_t^* \, dt
\right]
\]

\[
\leq \mathbb{E}
\left[
\beta_T U_A'(X_T^*) \Delta X_T^* - \beta_T \lambda U_B'(\nu - p^*) \Delta p^*
\right]
\]

\[
+ \mathbb{E}
\left[
\int_0^T \mathbb{1}_{(p^*, X^*) \notin Q_t} (\partial_p \hat{f}_t(p^*, X^*_t) \Delta X_t^* + \partial_x \hat{f}_t(p^*, X^*_t) \Delta p^*) \, dt
\right].
\]

The last inequality is obtained by concavity of $f^*$ and (4.1). Notice that $\Delta X_t^* = \Delta p^*$, for any
is reduced to subsection 3.2, it was shown that delta-hedge of Theorem 4.1, we deal with examples in the next section. Thus, in this case, therefore, we obtain
\[ p \text{ does not fall in the non-differentiable set of } f, \text{ dP } \otimes \text{dt} - \text{a.s.} \]

\[ E \left[ \beta_T U_A'(X_T^*) - \beta_T U_B'(v^B - p^*) + \int_0^T I(p, X_t^*) \in Q_t \left( \partial_p \hat{f}_t(p^*, X_t^*) + \partial_x \hat{f}_t(p^*, X_t^*) \right) \text{ dt} \right] = 0. \]

When we control \((p, \delta)\) together, two conditions on the funding spread, \(s^{A,m} = s^{B,m} = 0\), are required. The proof is analogous to that of Theorem 4.1.

**Proposition 4.2.** Assume \(s^{A,m} = s^{B,m} = 0\). Let \(X^* := X^{p^*} \in Q_t \in \mathbb{R}^2\) as the set that \(f^*_i(\cdot)\) is not differentiable. Assume
\[ I_{(p^*, X^*)} = 0, \text{ dP } \otimes \text{dt} - \text{a.s.} \]
i.e., \((p^*, X^*)\) does not fall in the non-differentiable set of \(\hat{f}, \text{ dP } \otimes \text{dt} - \text{a.s.} \)

Then \((p^*, \delta^*)\) is the risk-sharing contract.

We deal with examples in the next section.

**5. Examples.** In subsection 3.2, it was shown that delta-hedge of clean price and the absence of market frictions are necessary for the full margin requirement to be optimal. We first derive the risk-sharing contract given the conditions.

**Example 5.1.** Assume \(s^{i,m} = s^i = \phi^i = 0, i \in \{A, B\}, \ A = \mathbb{R}\). Therefore, \(X^p = v^A + p\), and \(\beta = G\). We will check that \((p^*, \delta^*) = (\hat{p}, 0)\) where

\[ \hat{p} := \frac{\gamma^B v^B - \gamma^A v^A}{\gamma^B + \gamma^A} - \frac{1}{\gamma^B + \gamma^A} \ln \left( \frac{\lambda \gamma^B}{\gamma^A} \right). \]

By (3.17)-(3.18), we have \(I_+^+(X^p, p) = (L^A)^{-1}(\hat{p} - p)\), \(I_-^+(X^p, p) = (L^B)^{-1}(\hat{p} - p)\), where \(\hat{p}\) is defined as (5.1). Thus, by taking \(p = \hat{p}\), we recover the full margin convention: \(\delta^* = 0\). In addition,

\[ X^\hat{p} + L^A I^+ = v^A + \hat{p}, \]
\[ v^B - p + L^B I^- = v^B - \hat{p}. \]

Therefore, by (3.9), \(\hat{f}_t\) is differentiable at \((\hat{p}, X^\hat{p})\), i.e., for \(t \in [0, T]\), \(Q_t = 0\). Moreover, by straightforward calculation, \(\partial_p \hat{f}_t(p, x) = -\partial_x \hat{f}_t(p, x)\), and

\[ U_A'(v^A + \hat{p}) = \lambda U_B'(v^B - \hat{p}). \]

Therefore, we obtain \((p^*, \delta^*) = (\hat{p}, 0)\). Note that when \(\gamma^B = \gamma^A, v^A = v^B = 0, \lambda = 1\), then \(\hat{p} = 0\). Thus, in this case, \((p^*, \delta^*) = (0, 0)\).

If we take \(\gamma^B = \gamma^A = 1\) and \(v^B = v^A = 0\, (5.1)\) is reduced to \(\hat{p} = -\ln(\lambda)/2\). In particular, when the two parties have the same negotiation power, i.e., \(\lambda = 1\), we have \(\hat{p} = 0\). It can be said that \(\hat{p}\) represents the amount of adjustment by agents’ preference and negotiation power, which are non-observable information in markets. Since it is hard for both parties to agree on such parameters. In addition, it is notable that \(\hat{p}\) does not depend on \(h^i, L^i, i \in \{A, B\}\), because the price is mathematically derived from fully collateralized contracts. If one party does not hedge the delta-risk, we cannot have an explicit solution for \(\hat{p}\), so we discuss only the existence of \(p^*\) satisfying (4.2).
Example 5.2. Assume $s^i,m = 0$, and $\pi^A = \Delta^A$, $\pi^B = 0$, i.e., $\phi^A = 0$, $\phi^B = \Delta^B$. We consider constant default intensities and, without loss of generality, assume that $\gamma^A = \gamma^B$, $L^A = L^B = 0.5$, $\lambda = 1$, and $\nu^A = \nu^B = 0$. Therefore, for $t \in [0, T]$,

$$
X^p_t = p - \int_0^t \left[ \left( s^p_{s} + \frac{\gamma^B}{\gamma^A} s^B_{s} \right) v_{s} + \Delta^B (b^B_{s} - \Lambda^B_{s}) + \Delta^A b^A_{s} \right] ds - \int_0^t \Delta^B dW_s, \\
I^p_i(p, X^p_t) = -X^p_t - p, \quad i \in \{-, +\}.
$$

Since $\gamma^A = \gamma^B$, we denote $U := U_A = U_B$. By straightforward calculation, we can check that $Q_t = 0$, for $t \leq T$, and

$$
\partial_p \tilde{f}_t(p, X^p_t) + \partial_x \tilde{f}_t(p, X^p_t) = \left\{ \begin{array}{ll}
-\gamma^B \beta_s (U(X^p_t) - U(-p)), & -X^p_t - p \geq 0, \\
-\gamma^A \beta_t (U(X^p_t) - U(-p)), & -X^p_t - p < 0.
\end{array} \right.
$$

Recall that $X^p_t$ increases as $p$ increases. Therefore, both $\partial_p \tilde{f} + \partial_x \tilde{f}$ and $[U'(X_T) - U'(-p)]$ decrease w.r.t $p$. Moreover, both terms tend to $-\infty$ (resp. $-\infty$) as $p \to -\infty$ (resp. $p \to \infty$). Thus, there exists $p \in \mathbb{R}$ satisfying (4.2). Once $p^*$ is obtained, $\delta^*$ can be found as well, but in this case, $(p^*, \delta^*)$ may not be $(\hat{p}, 0)$, i.e., full collateralization may not be optimal.

6. Conclusion. In this paper, we introduced a new risk-sharing framework to understand how two parties enter bilateral contracts with the presence of entity-specific information such as default risk and funding spread. Based on our model, we can explain why banks buy Treasury bonds that return less than their funding rate. The analysis of the optimal collateral in the risk-sharing framework interprets the meaning behind the margin requirement in Basel III: two parties hedge delta risk of clean price and funding spread is not considered in derivative prices. Note that the full collateralization is really optimal in frictionless financial markets, which is an inherent assumption in Basel III. It is possible that this conclusion can change if we include gap risk, KVA, and hedging strategies are also control variables. We leave such analysis as a further research topic.

Appendix A. An Auxiliary Lemma. The next lemma is borrowed from [11] and often used in this paper.

**Lemma A.1.** Let $i \in \{A, B\}$.

(i) Let $U$ be an $\mathcal{F}_s$-measurable, integrable random variable for some $s \geq 0$. Then, for any $t \leq s$,

$$
\mathbb{E}(\mathbb{1}_{s \leq \tau} U | \mathcal{G}_t) = \mathbb{1}_{t \leq \tau} G^{-1}_t \mathbb{E}(G_s U | \mathcal{F}_t).
$$

(ii) Let $(U_t)_{t \geq 0}$ be a real-valued, $\mathbb{F}$-predictable process and $\mathbb{E}|U_\tau| < \infty$. Then,

$$
\mathbb{E}(\mathbb{1}_{\tau = \tau^\tau \leq \tau} U_\tau | \mathcal{G}_t) = \mathbb{1}_{t \leq \tau^\tau} G^{-1}_t \mathbb{E}\left( \int_t^T h^i_s G_s U_s ds | \mathcal{F}_t \right).
$$

Appendix B. Spaces of Random Variables and Stochastic Processes. In this paper, we denote spaces of random variables and stochastic processes as follows.

**Definition B.1.** Let $m \in \mathbb{N}$ and $p \geq 2$.

- $L^p_{\mathbb{F}}$: the set of all $\mathbb{F}_\cdot$-measurable random variables $\xi$, such that

$$
\|\xi\|_p := \mathbb{E}(|\xi|^p)^{\frac{1}{p}} < \infty.
$$

- $S^p_{\mathbb{F}}$: the set of all real valued, $\mathbb{F}$-adapted, càdlàg processes $(U_t)_{t \geq 0}$, such that

$$
\|U\|_{S^p_{\mathbb{F}}} := \mathbb{E}\left( \sup_{t \leq T} |U_t|^p \right)^{\frac{1}{p}} < \infty.
$$
• \( \mathbb{H}^{m}_{T} \): the set of all \( \mathbb{R}^{m} \)-valued, \( \mathbb{F} \)-predictable processes \( (U_{t})_{t \geq 0} \), such that
\[
||U||_{\mathbb{H}^{m}_{T}} := \mathbb{E}\left( \int_{0}^{T} |U_{t}|^{p} \, dt \right)^{\frac{1}{p}} < \infty.
\]

• \( \mathbb{H}^{m}_{T,loc} \): the set of all \( \mathbb{R}^{m} \)-valued, \( \mathbb{F} \)-predictable processes \( (U_{t})_{t \geq 0} \), such that
\[
\int_{0}^{T} |U_{t}|^{p} \, dt < \infty, \quad a.s.
\]

When \( d = 1 \), we denote \( \mathbb{H}^{p}_{T} := \mathbb{H}^{p,1}_{T} \) and \( \mathbb{H}^{p}_{T,loc} := \mathbb{H}^{p,1}_{T,loc} \).

**Appendix C. A Risk-Neutral Agent under Incremental Cash-flow.** In this section, we will derive an optimal collateral with a risk-neutral Agent A: \( U_{A}(x) = x \). As in previous sections, the Agent B is risk-averse as \( U_{B}(x) = -e^{-\gamma x} \). In this case, we can relax the assumption on margin funding rate of A. Then, we intend to derive similar arguments as in subsection 3.1 and subsection 3.2 with assuming

\[(C.1) \quad s^{A,m} > 0.\]

Now, to model the incremental cash-flow, assume that the bank has had contracts given by some endowed càdlàg \( \mathbb{F} \)-adapted processes \( (\mathcal{D}^{E}_{t}, e^{E}_{t}, m^{E}_{t}) \) before initiation of the new contract. If the two parties do not enter the new contract, the cash-flow remains as
\[
\mathcal{E}^{E}_{t} = \mathbb{1}_{t > \tau} \mathcal{D}^{E}_{t} + \mathbb{1}_{\tau \leq t} (\mathcal{D}^{E}_{\tau} + e^{E}_{\tau}) - \mathbb{1}_{\tau = \tau^{A} \leq t} L^{A}(e^{E}_{\tau} - m^{E}_{\tau})^{+} + \mathbb{1}_{\tau = \tau^{B} \leq t} L^{B}(e^{E}_{\tau} - m^{E}_{\tau})^{-}.
\]

On the other hand, with the new contract, the exposure and margin become \( (e^{E} + e) \) and \( (m^{E} + m) \), respectively. Therefore, with the new contract, the summed cash-flows are
\[
\mathcal{E}^{S}_{t} := \mathbb{1}_{t > \tau} (\mathcal{D}^{E}_{t} + \mathcal{D}_{t}) + \mathbb{1}_{\tau \leq t} (\mathcal{D}^{E}_{\tau} + \mathcal{D}_{\tau} + e^{E}_{\tau} + e_{\tau}) - \mathbb{1}_{\tau = \tau^{A} \leq t} L^{A}(e^{E}_{\tau} + e_{\tau} - m^{E}_{\tau} - m_{\tau}^{-})^{+} + \mathbb{1}_{\tau = \tau^{B} \leq t} L^{B}(e^{E}_{\tau} + e_{\tau} - m^{E}_{\tau} - m_{\tau}^{-})^{-}.
\]

Thus, the amount that should be dealt with by the Agent A is the increment from \( \mathcal{E}^{E} \) to \( \mathcal{E}^{S} \), namely for \( t \leq T \),
\[
\mathcal{E}_{t} := \mathcal{E}^{S}_{t} - \mathcal{E}^{E}_{t} = \mathcal{D}_{t} \vee (\tau^{A} - \tau^{B}) + \mathbb{1}_{\tau \leq t} \mathcal{E}_{\tau} - \mathbb{1}_{\tau = \tau^{A} \leq t} L^{A}(e^{\tau^{A} - m^{\tau^{A}} + e^{\tau} - m^{E}_{\tau} + e^{-} - m^{E}_{\tau}^{-})^{+} + \mathbb{1}_{\tau = \tau^{B} \leq t} L^{B}(e^{\tau^{A} + e^{-} - m^{E}_{\tau}^{-}}).
\]

Thus, we denote the amount of breach of the contract as
\[(C.2) \quad \Theta_{t}(\delta) = \mathbb{1}_{\tau = \tau^{A} = t} L^{A}(\delta^{A} + e^{A} + \delta^{A} - \delta^{A}^{-}) - \mathbb{1}_{\tau = \tau^{B} = t} L^{B}(\delta^{A} + e^{A} - \delta^{A}^{-}).\]

Moreover, by (C.1), the \( \mathbb{F} \)-reduction of \( V^{A} \), which is derived in Theorem 2.14, becomes slightly different as
\[
\begin{align*}
&d\bar{v}^{A}_{t} = (s^{A,m}_{t} \delta_{t} + \alpha^{A}_{t}) \, dt + \phi^{A}_{t} \, dW_{t} \\
&\alpha^{A}_{t} := s^{A,\Delta}_{t} \Delta^{A} + \delta^{A}_{t} \Delta^{B} \\
&s^{A,\Delta}_{t} := s^{A}_{t} - s^{A,m}_{t}.
\end{align*}
\]

Notice that \( v^{A} \) depends on \( \delta \) since we assumed that \( s^{A,m} > 0 \), which is the main mathematical difference from the main sections. We still assume that the Agent B can deliver the collateral
without any excessive cost/benefit, i.e., \( s^{B,m} = 0 \). In this case, \( v^B \) does not depend on \( \delta \). Because of the dependence between \( v^A \) and \( \delta \), we impose a slightly stronger condition for the admissible set of collateral \( D \):

\[
G_T U_A(v^A_t, \delta_t) + |\gamma G_T U_B(v^B_t)| + \int_0^T |g_t(v^A_{t-}, v^B_{t-}, \delta_{t-})| \, dt < \infty,
\]

where we now denote \( g_t := 1_{\delta_{t-}^+ < \delta_t^+} g_t^+ + 1_{\delta_{t-}^- < \delta_t^-} g_t^- \) and

\[
g^+_t(v^A, v^B, \delta) := G_t [h^A_t U_A(v^A + L^A(\delta - \delta^E_t - \epsilon)) + \lambda h^A_t U_B(v^B - L^A K_t(\delta - \delta^E_t - \epsilon)) + h^B_t (U_A(v^A + L^A(\delta^E_t + \epsilon)) + \lambda U_B(v^B - L^B K_t(\delta + \delta^E_t - \epsilon)) + h^A_t (U_A(v^A - L^A(\delta^E_t + \epsilon)) + \lambda U_B(v^B + L^A K_t(\delta^E_t + \epsilon)))].
\]

As in section 3, the first task is to characterize the optimal collateral by MOP. To this end, we slightly modify (3.1) by merging the one terminal condition \( G_T v^A_T \) into \( dt \)-integral term. Observe that Itô’s formula yields

\[
d(G_t v^A_t) = G_t [s^A_t \delta_t + \alpha^A_t - h^0_t v^A_t] \, dt + G_t v^A_t \, dW_t.
\]

If \( G^A, \delta^A \in \mathbb{H}_2^2 \), the Itô’s integral term is an \((\mathbb{P}, \mathbb{F})\)-local martingale. Thus,

\[
E[G_T v^A_T - (v^A + p)] = E\left[ \int_0^T G_t [s^A_t \delta_t + \alpha^A_t - h^0_t v^A_t] \, dt \right].
\]

Thus, (3.1) can be written as

\[
\max_{\delta \in D} E\left[ (v^A + p) + \lambda G_T U_B(v^B_T) \right] \]

\[
+ \int_0^T [g_t(v^A_{t-}, v^B_{t-}, \delta_{t-}) + G_t(s^A_{t-} \delta_t + \alpha^A_t - h^0_t v^A_{t-})] \, dt.
\]

Then, we define a dynamic version of (C.6) as:

\[
J^c_T(p) := \text{ess sup}_{\delta \in D(c,t)} E\left[ (v^A + p) + \lambda G_T U_B(v^B_T) \right] \]

\[
+ \int_c^t \left[ g_s(s^A_{s-} \delta_s + \alpha^A_s - h^0_s v^A_{s-}) + G_s(s^A_{s-} \delta_s + \alpha^A_s - h^0_s v^A_{s-}) \right] \, ds \bigg| \mathcal{F}_t.
\]

Then \( J^c_T \) is chosen as a càdlàg version such that for any \( \epsilon \in \mathcal{D}, \)

\[
\left\{ J^c_t + \int_0^c \left[ g_s(s^A_{s-} \delta^c_s + \alpha^A_s - h^0_s v^A_{s-}) + G_s(s^A_{s-} \delta^c_s + \alpha^A_s - h^0_s v^A_{s-}) \right] \, ds \right\}_{0 \leq t \leq T}
\]

is an \((\mathbb{P}, \mathbb{F})\)-supermartingale. Moreover, for the optimal collateral \( \delta^* \) for \( J_0 \),

\[
\left\{ J^* + \int_0^t \left[ g_s(s^A_{s-} \delta^*_s + \alpha^A_s - h^0_s v^A_{s-}) + G_s(s^A_{s-} \delta^*_s + \alpha^A_s - h^0_s v^A_{s-}) \right] \, ds \right\}_{0 \leq t \leq T}
\]

is an \((\mathbb{P}, \mathbb{F})\)-martingale. The detail is summarized in the following theorem. Before giving the theorem, we introduce two notations. We separate \( \delta \) from \( v^A \) by denoting \( \tilde{v}^A_t = dv^A_t - s_t \alpha^A \delta_t \, dt \), more precisely,

\[
\tilde{v}^A_t = \int_0^t \alpha^A s \, ds + \int_0^t \phi^A_s \, dW_s.
\]
Note that $\tilde{v}_0^A = 0$. In addition, we denote

$$I_t := \int_t^T G_s h^\Delta_s \, ds.$$  

The optimal collateral will later be represented by $(I_t)_{t \geq 0}$. This term appears in this section since $s^{A,m}$ can be positive. If we consider the cost of delivering collateral, then when to default becomes also important. However, the effect of default time can be still marginal. Recall the definition that $h^\Delta = h - h^B$. Therefore, $(I_t)_{t \geq 0}$ can be understood as a correcting term of collateral for the dependence of default times. When $\tau^A$ and $\tau^B$ are independent, we have $h = h^B$ and $I = 0$.

**Theorem C.1.** Assume that $G h^\Delta$ is deterministic. Define

$$\hat{\delta}_t (\hat{v}_t^A, v_t^B) := \arg \max_{\delta \in A} \left[ \hat{g}_t (\hat{v}_t^A, v_t^B, \delta) + I_t s_t^{A,m} \delta \right],$$

$$\hat{g}_t (v^A, v^B, \delta) := \mathbb{1}_{\delta + h^B \geq 0} v^A + \mathbb{1}_{\delta + h^B \geq 0} v^B, \quad \hat{g}_t^\Delta (v^A, v^B, \delta) := \hat{g}_t (v^A, v^B, \delta) + G_t \left( s_t^{A,m} \delta + \alpha_t^A - h^B v^A \right), \quad i \in \{-, +\}.$$

If $\hat{\delta}(v^A, v^B) \in \mathcal{D}$, then $(\hat{\delta}(\hat{v}_t^A, v_t^B))_{0 \leq t \leq T}$ is a solution of (3.1).

**Proof.** For $\varepsilon \in \mathcal{D}$, we define an $(\mathbb{P}, \mathbb{F})$-semimartingale $(Y_t)_{t \geq 0}$ as

$$Y_t := J_t^\varepsilon - \mathbb{E} \left[ \int_t^T G_u h^\Delta_u \left( \int_t^u s_v^{A,m} \varepsilon_v \, dv \right) \, du \bigg| \mathcal{F}_t \right],$$

$$= J_t^\varepsilon - \mathbb{E} \left[ \int_t^T \left( \int_s^T G_u h^\Delta_u \, du \right) s_v^{A,m} \varepsilon_v \, dv \bigg| \mathcal{F}_t \right],$$

$$= J_t^\varepsilon - \int_0^t I_t s_u^{A,m} \varepsilon_u \, du.$$

Notice that $Y$ does not depend on $\varepsilon$. We also define

$$\xi_t^\varepsilon := Y_t + \int_t^T I_t s_u^{A,m} \varepsilon_u \, du + \int_0^t \left[ g_u (v_u^{A,\varepsilon}, v_u^{B, \varepsilon}) + G_u (s_u^{A,m} \varepsilon_u + \alpha_u^A - h^B v_u^{A,\varepsilon}) \right] \, du.$$

To simplify (C.11), note that

$$g_t (v_t^{A,\varepsilon}, v_t^{B, \varepsilon}) + G_t (s_t^{A,m} \varepsilon_t + \alpha_t^A - h^B v_t^{A,\varepsilon})$$

$$= \left[ g_t (v_t^{A,\varepsilon}, v_t^{B, \varepsilon}) - G_t h_t v_t^{A,\varepsilon} \right] + G_t h_t v_t^{A,\varepsilon} - h^B v_t^{A,\varepsilon} + G_t (s_t^{A,m} \varepsilon_t + \alpha_t^A)$$

$$= \left[ g_t (v_t^{A,\varepsilon}, v_t^{B, \varepsilon}) - G_t h_t v_t^{A,\varepsilon} \right] + G_t h_t v_t^{A,\varepsilon} + G_t (s_t^{A,m} \varepsilon_t + \alpha_t^A).$$

In addition, by Fubini’s theorem,

$$\int_0^t G_u h^\Delta_u \left( \int_0^u s_v^{A,m} \varepsilon_v \, dv \right) \, du = \int_0^t s_v^{A,m} \varepsilon_v \left( \int_s^t G_u h^\Delta_u \, du \right) \, dv.$$  

Moreover,

$$\int_0^t G_u h^\Delta_u v_u^{A,\varepsilon} \, du = \int_0^t G_u h^\Delta_u \left[ v_u^{A,\varepsilon} - \int_0^u s_v^{A,m} \varepsilon_v \, dv \right] \, du + \int_0^t G_u h^\Delta_u \left( \int_0^u s_v^{A,m} \varepsilon_v \, dv \right) \, du$$

$$= \int_0^t G_u h^\Delta_u \hat{v}_u^A \, du + \int_0^t s_v^{A,m} \varepsilon_v \left( \int_s^t G_u h^\Delta_u \, du \right) \, dv.$$  

(C.12)
and for \( t \leq T \),

\[
C.11 \quad g_t(v_t^A, v_t^B, \varepsilon_t) - G_t h_t v_t^A + G_h_t \nu_t^A = g(\tilde{v}_t^A, v_t^B, \varepsilon_t).
\]

Therefore, (C.11) can be rewritten as

\[
Y_t + \int_0^t I_s s_t A,m \varepsilon_s ds + \int_0^t \left[ g_s(v_s^A, v_s^B, \varepsilon_s) + G_s(s_s A,m \varepsilon_s + \alpha_s^A - h_0 v_s^A) \right] ds \\
= Y_t + \int_0^t I_s s_t A,m \varepsilon_s ds + \int_0^t s_t A,m \varepsilon_s \left[ \int_s^t G_u h_u^A du \right] ds \\
+ \int_0^t \left( g_s(v_s^A, v_s^B, \varepsilon_s) - G_s h_s v_s^A + G_h_s \nu_s^A + G_s(s_s A,m \varepsilon_s + \alpha_s^A) \right) ds \\
= Y_t + \int_0^t I_s s_t A,m \varepsilon_s ds + \int_0^t \left( g_s(v_s^A, v_s^B, \varepsilon_s) + G_s(s_s A,m \varepsilon_s + \alpha_s^A - h_0 \nu_s^A) \right) ds \\
= Y_t + \int_0^t I_s s_t A,m \varepsilon_s ds + \int_0^t \tilde{g}_s(v_s^A, v_s^B, \varepsilon_s) ds.
\]

Then, since \( Y \) is independent of \( \varepsilon \in \mathcal{D} \), by the assumption of admissibility of \( \tilde{\delta}(\tilde{v}^A, v^B) \), for any \( \varepsilon \in \mathcal{D} \), we have that \( \xi_T = \xi_T(\tilde{v}^A, v^B) \) is a \((\mathcal{F}, \mathcal{F})\)-supermartingale. It follows that for any \( \varepsilon \in \mathcal{D} \),

\[
E[\xi_T - \xi_T(\tilde{v}^A, v^B)] \leq E[\xi_T - \xi_T(\tilde{v}^A, v^B)] = 0.
\]

Then, by the admissibility, \( \tilde{\delta}(\tilde{v}^A, v^B) \) is a solution of (3.1).

The last step is to show \( \tilde{\delta}(\tilde{v}^A, v^B) \) is admissible given some conditions. We consider \( A = \mathbb{R} \) and find the explicit form of \( \tilde{\delta}(\tilde{v}^A, v^B) \) for the case. Then, the integrability condition is easy to check. First, notice that \( \tilde{g}^+, \tilde{g}^- \) are continuously differentiable and strictly concave in \( \delta \). Thus, for any \( (t, v_t^A, v_t^B) \), there exists \( \tilde{I}^+_t(v_t^A, v_t^B), i \in \{-, +\} \) such that

\[
C.12 \quad \partial_\delta \tilde{g}^i_t(v_t^A, v_t^B, \tilde{I}_t(v_t^A, v_t^B)) + s_t A,m I_t = 0
\]

Then, it is easy to check that \( \tilde{\delta}(\tilde{v}^A, v^B) \) is attained at \( I^- \), \( I^+ \), and \(- \delta^E \). Observe the precise forms of \( \tilde{g}^i, i \in \{-, +\} \), are

\[
\tilde{g}_t^i(v_t^A, v_t^B, \delta) := G_t \left[ \delta_t A^A + (h_t B^B + s_t A,m) \delta + \alpha_t A^A + (h_t B^L B - h_t A^L A)(\delta^E)^+ \right] \\
+ \lambda h_t B^U B v_t B - h_t B \lambda K_t (\delta^E)^+ + \lambda U_t B^U B v_t B + h_t B \lambda K_t (\delta^E)^- + \lambda h_t B^U B v_t B - h_t B \lambda K_t (\delta^E)^- + \lambda h_t B^U B v_t B - h_t B \lambda K_t (\delta^E)^-
\]

Therefore, assuming \( h_t^L \lambda^+ > 0, i \in \{A, B\}, \lambda^+, i \in \{-, +\} \), can be explicitly represented as

\[
C.13 \quad \tilde{I}^-_t(v_t^A, v_t^B) = - (\delta^E)^+ + \frac{v_t B}{K_t B} + \frac{1}{\gamma B K_t B} \ln \left( \frac{G_t [h_t B^B B + s_t A,m^A] + s_t A,m I_t}{G_t \lambda B^B K_t h_t B^L B} \right),
\]

\[
C.14 \quad \tilde{I}^+_t(v_t^A, v_t^B) = (\delta^E)^- + \frac{v_t B}{K_t A} + \frac{1}{\gamma B K_t A} \ln \left( \frac{G_t [h_t A^L A + s_t A,m^B] + s_t A,m I_t}{G_t \lambda B^B K_t h_t A^L A} \right).
\]

Then Theorem C.1 is completed by the next lemma. The proof is similar to that of Lemma 3.2, so we omit it.

**Lemma C.2.** Let \( A = \mathbb{R} \). Assume that \( \delta^E, e, Z, \pi_i, i \in \{A, B\}, \) are bounded. Moreover, assume \( h_t^L \lambda^+ \in \mathbb{E}^2_L \) and

\[
C.15 \quad \frac{(G + I) A^A}{G h_t^L \lambda^+}, i \in \{A, B\},
\]

are bounded. Then \( \tilde{\delta}(\tilde{v}^A, v^B) \in \mathcal{D} \).
Different from (3.14) and (3.15), \( \hat{\theta}(\hat{v}^A, v^B) \) depends on \( h^A \) and \( h^B \). As mentioned, this dependence arises from the funding impact of \( s^A,m \), and by setting \( s^A,m = 0 \), (C.15)-(C.15) are reduced to

\[
\hat{I}^- (v^A, v^B) = - (\theta^E)^+ + \frac{v^B}{K_i L^B} - \frac{\ln (\lambda \gamma^B K_i)}{\gamma^B K_i L^B},
\]

\[
\hat{I}^+ (v^A, v^B) = (\theta^E)^- + \frac{v^B}{K_i L^A} - \frac{\ln (\lambda \gamma^B K_i)}{\gamma^B K_i L^A}.
\]

In addition, we can derive similar interpretations as in subsection 3.1. In what follows, we moreover, assume that all parameter are constant and the default times are independent on \( \mathbb{F} \), i.e., \( I^i = 0 \). The full collateral convention can be said that \( \delta^* = v - c^* = 0 \) and \( \delta^E = 0 \), \( d\mathbb{P} \otimes dt \)-a.s. Therefore, \( I^i = 0 \), for any \( t \leq T \). By (C.15) and (C.16), full collateralization requires that

\[
v^i - \frac{1}{\gamma^B} (s^A - s^B)t = \frac{1}{\gamma^B} \ln \left( \frac{\lambda \gamma^B h^i L^i}{h^i L^i + s^A} \right), \quad i \in \{A, B\}, \quad d\mathbb{P} \otimes dt - a.s.
\]

In particular, \((v^B - (s^A - s^B)t)/\gamma^B)_{t \geq 0}\) should be constant. Thus, (C.20) implies that \( \phi^B = \pi^B + \Delta^B = 0 \), i.e., delta-hedge, and

\[
-s^B K v + \phi^B \Lambda^B_B - \Delta^B b^B - \frac{s^A - s^B}{\gamma^B} = 0, \quad d\mathbb{P} \otimes dt - a.s.
\]

Consider a contract such that \( Z \neq 0 \), so necessarily \( v \neq 0 \) and \( \Delta^B \neq 0 \). Since (C.21) should hold for all contracts such that \( Z \neq 0 \), (C.21) implies that \( s^B = b^B = A - s^B = 0 \). Equivalently, by (2.12), (2.28), \( s^B = s^A = s^A,m = 0 \). Therefore, the margin requirement hinges on the assumption of absence of funding impacts and delta-hedge of the Agent B. No property of the Agent A’s hedging strategy was derived since we assumed the Agent A is risk-neutral.

**Appendix D. Proofs of Lemmas.**

**Proof of Lemma 2.6.** (i) is from the definition and (ii) is a directly obtained from (i). For (iii), notice that \( B_t^{-1} e_t + \int_0^t B_s^{-1} d\mathbb{D}_s \) is an \((\mathbb{Q}, \mathbb{F})\)-local martingale. Thus, by (local) martingale representation property, there exists \( Z \in \mathbb{H}_{T, loc}^2 \) such that for any \( t \geq 0 \),

\[
B_t^{-1} e_t + \int_0^t B_s^{-1} d\mathbb{D}_s = \int_0^t Z_s dW^Q_s,
\]

where \( W^Q \) is the Brownian motion under \( \mathbb{Q} \), i.e., \( W_t^Q = W_t + \int_0^t \lambda_s \, ds \). Therefore, \( (e_t)_{t \geq 0} \) follows the SDE:

\[
de_t = r_t e_t \, dt + B_t Z_t \, dW^Q_t - d\mathbb{D}_t
\]

\[
= (r_t e_t + B_t Z_t \lambda_t) \, dt + B_t Z_t \, dW^Q_t - d\mathbb{D}_t.
\]

By (iii), \( (e_t)_{t \geq 0} \) is an \( \mathbb{F} \)-adapted càdlàg process, but \( \tau \) avoids \( \mathbb{F} \)-stopping times. Thus, \( \Delta e_\tau = 0 \) almost surely, equivalently \( e_\tau^- = e_\tau^+ \) a.s. \( \square \)

**Proof of Lemma 3.2.** Let \( i \in \{A, B\} \) and \( \Psi(x) := e^{C^i x} \) for \( C \in \mathbb{R} \). It suffices to show that for any \( C \in \mathbb{R} \), \( \Psi(X), \Psi(v^i) \), are in \( S^2_C \). Note that \( v = (B^A)^{-1} e \) is bounded and, by (2.26) and (2.27), \( \Delta^i \) and \( \phi^i \) are also bounded. Denoting

\[
\alpha^A := \phi^A \lambda^A + \Delta^A b^A + s^A v
\]

\[
\alpha^B := \phi^B \lambda^B - \Delta^B b^B - K s^B v,
\]
we can write $dv^i_t = \alpha^i_t \, dt + \phi^i_t \, dW_t$. Applying Itô's formula to $\Psi(v^i)$,

\begin{equation}
(\text{D.2}) \quad d\Psi(v^i_t) = \left(C\alpha^i_t + (C\phi^i_t)^2/2\right)\Psi(v^i_t) \, dt + C\phi^i_t\Psi(v^i_t) \, dW_t.
\end{equation}

By the assumptions, the coefficients in (\text{D.2}) are uniformly Lipschitz continuous. Thus, there exists a unique solution of (\text{D.2}) such that

\[
\mathbb{E}\left[\sup_{t \leq T} |\Psi(v^i_t)|^2\right] < \infty.
\]

In particular, $U_i(v^i) \in \mathbb{S}^2_T$, so we obtain the integrability condition (2.37). It is similarly obtained that $\Psi(X) \in \mathbb{S}^2_T$. Let $(C_i)_{i \geq 0}$ be an arbitrary bounded deterministic process. Then, we also have $\exp(CX) \in \mathbb{S}^2_T$ and it follow that

\[
U_A \left( \frac{-\gamma^A}{\gamma^B K + \gamma^A} X \right) \in \mathbb{S}^2_T \quad \text{and} \quad U_B \left( \frac{K\gamma^A}{\gamma^B K + \gamma^A} X \right) \in \mathbb{S}^2_T.
\]

Thus, by (3.5)-(3.7), (3.14), and (3.15), we obtain $\delta^*(p, X) \in \mathcal{D}$. \hfill \(\Box\)

**Proof of Lemma 3.4.** Recall from (3.12) and (3.13) that

\[
\partial_{\delta} f^+(t, p, x, \delta) := h^A_t L^A \left[ -\gamma^A U_A(x + L^A \delta) + \lambda \gamma^B K_i U_B(\nu^B - p - L^B K_i \delta) \right],
\]

\[
\partial_{\delta} f^-(t, p, x, \delta) := h^B_t L^B \left[ -\gamma^A U_A(x + L^B \delta) + \lambda \gamma^B K_i U_B(\nu^B - p - L^B K_i \delta) \right].
\]

Let $I^i_t(p, x)$ denote the function such that $\partial_{\delta} f^i(t, p, x, I^i_t(p, x)) = 0$. Since $f^i$, $i \in \{-, +\}$, are concave w.r.t $\delta$, $I^i$ uniquely exists. Let us denote

\[
\tilde{f}^+(t, p, x) := \max_{\delta \leq 0} f^+(t, p, x, \delta),
\]

\[
\tilde{f}^-(t, p, x) := \max_{\delta \geq 0} f^-(t, p, x, \delta),
\]

\[
\tilde{f}(t, p, x) := \max_{\delta \in \mathbb{R}} f(t, p, x, \delta).
\]

Then it follows that

\[\tilde{f}(t, p, x) := f(t, p, x, \delta^*(t, p, x)), \quad \tilde{f}^+(t, p, x) = \tilde{f}^+(t, p, x) \mathbb{1}_{\tilde{f}^+(t, p, x) \geq \tilde{f}^-(t, p, x)} + \tilde{f}^-(t, p, x) \mathbb{1}_{\tilde{f}^+(t, p, x) \leq \tilde{f}^-(t, p, x)} \].

Thus, for finding $\delta^*$, we should characterize the region that $\tilde{f}^+(t, p, x) \geq \tilde{f}^-(t, p, x)$. The solutions $\delta^i$, $i \in \{-, +\}$, of $\tilde{f}^i$ can be found explicitly:

\[
\delta^+_i(p, x) = \begin{cases} 0, & 0 > \partial_{\delta} f^+(t, p, x, 0), \\ I^+_i(p, x), & 0 \leq \partial_{\delta} f^+(t, p, x, 0), \end{cases}
\]

\[
\delta^-_i(p, x) = \begin{cases} I^-_i(p, x), & \partial_{\delta} f^-(t, p, x, 0) \leq 0, \\ 0, & 0 < \partial_{\delta} f^-(t, p, x, 0). \end{cases}
\]

Moreover, notice that since $h^i \geq 0$ and $L^i \geq 0$, $\partial_{\delta} f^+(0) + \partial_{\delta} f^-(0) \geq 0$. The rest of the proof is merely a straightforward comparison of $\tilde{f}^i$ in each region. In what follows, we suppress $t, x, p$.

(I) Let $0 \leq \partial_{\delta} f^+(0) \land \partial_{\delta} f^-(0)$. In other words,

\[
\gamma^A x + \gamma^B p \leq \gamma^B \nu^B - \ln \left( \frac{\lambda K_i \gamma^B}{\gamma^A} \right).
\]
Thus, $I^i \geq 0$, $i \in \{-, +\}$, and $\delta^- = 0$. Moreover,

$$
\tilde{f}^- - \tilde{f}^+ = f^-(0) - f^+(I^+) = f^+(0) - f^+(I^-) \leq 0.
$$

Hence, $\delta^* = \delta^+ = I^+ \geq 0$.

(II) Let $0 > \delta f^+(0) \vee \delta f^-(0)$. Then, $\delta^+ = 0$ and $I^- \leq 0$. Therefore, by similar calculation, $\delta^* = \delta^- = I^- \leq 0$. \qed

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