Online Allocation with Two-sided Resource Constraints

Qixin Zhang∗†
School of Data Science
City University of Hong Kong
Kowloon, Hong Kong, China
Cainiao Network
Hang Zhou, China

Wenbing Ye∗
Cainiao Network
Hang Zhou, China

Zaiyi Chen∗
Cainiao Network
Hang Zhou, China

Haoyuan Hu
Cainiao Network
Hang Zhou, China

Enhong Chen
University of Science and Technology of China
Hefei, China

Yu Yang
School of Data Science
City University of Hong Kong
Kowloon, Hong Kong, China

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Abstract
Motivated by many interesting real-world applications in logistics and online advertising, we consider an online allocation problem subject to lower and upper resource constraints, where the requests arrive sequentially, sampled i.i.d. from an unknown distribution, and we need to promptly make a decision given limited resources and lower bounds requirements. First, with knowledge of the measure of feasibility, i.e., \( \alpha \), we propose a new algorithm that obtains \( 1 - O(\frac{1}{\alpha}) \) - competitive for the offline problems that know the entire requests ahead of time. Inspired by the previous study (Devanur et al., 2019), this algorithm adopts an innovative technique to dynamically update a threshold price vector for making decisions. Moreover, an optimization method to estimate the optimal measure of feasibility is proposed with theoretical guarantee at the end of this paper. Based on this method, if we tolerate slight violation of the lower bounds constraints with parameter \( \eta \), the proposed algorithm is naturally extended to the settings without strong feasible assumption, which cover the significantly unexplored infeasible scenarios.

Keywords: Online Assignment, Online Resource Allocation, Competitive Algorithm, Concentration Inequality

∗. Contributed equally.
†. Most of the work was done when Qixin Zhang acts as a research intern in Cainiao Network.
1. Introduction

Online resource allocation is a prominent paradigm for sequential decision making over multiple rounds subject to the resource constraints, increasingly attracting the wide attention of researchers and practitioners in theoretical computer science (Mehta et al., 2007; Devanur and Jain, 2012; Devanur et al., 2019), operations research (Agrawal et al., 2014; Li and Ye, 2021) and machine learning communities (Balseiro et al., 2020; Li et al., 2020). In these settings, the requests arrive online and offer a bid price; we need to serve every request via one of available channels, which consumes some amounts of different resources. The objective of decision maker is to maximize the cumulative revenue subject to the resource capacity constraints. Such problem frequently appears in many applications including online Adwords (Mehta et al., 2007), online combinatorial auctions (Chawla et al., 2010), online linear programming (Agrawal et al., 2014; Buchbinder and Naor, 2009), online routing (Buchbinder and Naor, 2006), online multi-leg flight seats and hotel rooms allocation (Talluri et al., 2004).

However, the aforementioned online resource allocation framework only considers the capacity (upper bound) constraints for resources. In practice, we usually encounter the lower bound resources constraints, i.e., decisions should at least consume some amounts of resources in the end. Such kind of requirements usually come from the contracts between decision makers and resource providers, e.g., advertising platform and advertisers respectively. Let us give several examples.

1. Guaranteed Advertising Delivery: In the online advertising scenario, the advertising publishers will sell the ad impressions in advance with promising to provide each advertiser an agree-on number of target impressions over a fixed time period, which is usually written in the contracts. Furthermore, the advertising platform also consider other constraints, such as advertisers’ budgets and impressions inventories, and simultaneously maximize multiple accumulative objectives regarding different interested parties, e.g., Gross Merchandise Volume (GMV) for ad providers and publisher’s revenue. We generally formulate this widely used guaranteed delivery advertising model as an online resource allocation problem with two-sided constraints (Zhang et al., 2020).

2. Fair Channel Constraints: We consider the online orders assignment in an e-commerce platform, where the platform allocates the orders/packages to different warehouses/logistics providers. Besides the resources used in packing and transportation, for some special channels, such as new/small-scale warehouses/logistics providers or those in developing areas, we usually trend to set a lower bound to their daily accepted orders/packages, according to the governmental regulations and contracts between e-commerce platforms and warehouses/logistics providers.

3. Timeliness Achievement Constraints: The delivery time for orders/packages is highly related with the customers’ shopping experience. Therefore, besides the previous fair constraints, the online shopping platforms also consider the time resource in assignment. For every order/package, the platforms usually use the timeline achievement rate $r_u \in [0, 1]$ to denote the probability of arriving at the destination in required $u$ days if we assign this order to one channel, which could be estimated via the historical data and features of order and channel/logistics providers. As we know, long delivery time will impair the consumer’s shopping experience, but reducing delivery time means increasing cost. In order to balance the customers’ shopping experience and transportation costs, the platforms always set a predefined lower threshold to the average time achievement ratio.

1.1 Related Works

Online allocation problems have been extensively studied in theoretical computer science and operations research communities. In this section, we overview the related literature.
When the incoming requests are adversarially chosen, there is a stream of literature that studies online allocation problems. Mehta et al. (2007) and Buchbinder et al. (2007) first study the AdWords problem and provide an algorithm that obtains a \( (1 - 1/e) \)-approximations to the offline optimal allocation, which is optimal under the adversarial input model. However, the adversarial assumption may be too pessimistic about the requests. In order to get more significant outcomes, Devanur and Jain (2012) propose the random permutation model, where an adversary first selects a sequence of requests which are then presented to the decision maker in random order. This model is more general than the stochastic i.i.d. setting in which requests are drawn independently and at random from an unknown distribution. Specifically, in this new stochastic model, Devanur and Hayes (2009) revisit the AdWords problem and present a dual training algorithm with two phases: a training phase in which data is used to estimate the dual variables by solving a linear program and an exploitation phase in which actions are taken using the estimated dual variables. Their algorithm can be shown to obtain \( 1 - o(1) \) competitive ratio, which is problem dependent. Feldman et al. (2010) showed that training-based algorithms could resolve more general linear online allocation problems with similar regret guarantees. Pushing these ideas one step further, Agrawal et al. (2014) consider primal and dual algorithm that dynamically update dual variables by periodically solving a linear program using the data collected so far. Meanwhile, Kesselheim et al. (2014) take the same policy with only considering renewing the primal variables. Recently, Devanur et al. (2019) take other innovative techniques to dynamically update the dual variables via some decreasing potential function derived from probability inequalities. These algorithms also obtain \( 1 - o(1) \) approximation guarantee under some mild assumptions. While the algorithms described above usually require solving large linear problems periodically, there is a recent line of work seeking simple algorithms that have no need of solving a large linear programming. Balseiro et al. (2020) studies a simple dual mirror descent algorithm for online allocation problems with concave reward functions and stochastic inputs, which attains \( O(\sqrt{KT}) \) regret, where \( K \) and \( T \) are the number of resources and requests respectively, i.e., updating dual variables via mirror descent algorithm and avoids solving large auxiliary linear programming. Simultaneously, Li et al. (2020) present a similar fast algorithm that updates the dual variable via projected gradient ascent in every round for linear rewards. Specially, all of these literature only consider the online allocation with capacity constraints.

1.2 Contributions

Our contribution can be summarized as follows:

1. To the best of our knowledge, this paper gives the first theoretical results to the online resource allocation problems with general two-sided constrains without assuming \( L_k = O(T) \), covering the models in many previous works (Mehta et al., 2007; Devanur et al., 2019; Lobos et al., 2021), with or without strong feasibility.

2. Under the gradually improved assumptions, we give three algorithms each of which returns a solution satisfying the two-sided constraints and achieving \( 1 - O(\alpha/\epsilon) \) competitive ratio with high probability, where \( \alpha \) is the measure of feasibility and the requirements of \( \epsilon \) is given by Theorem 4. Specially, under the knowledge of \( \alpha \), our result is the first one with theoretical guarantee under the competitive analysis framework. Meanwhile, from the Devanur et al. (2019), the performance of our algorithm is near-optimal, not only for \( T \) but also for \( K \).

3. To tackle the unknown parameter \( \alpha \), we also propose an optimization method to estimate it in the last section, guiding us to investigate the new settings without the strong feasible assumption. Throughout merging the estimate method into the previous framework, a new algorithm is proposed with a solution achieving \( 1 - O(\epsilon) \) competitive ratio. Due to objective regulations and contracts, we are unclear about the feasibility of the expected resource allocation problems and frequently encounter the infeasible settings when considering the lower bounds constraints.
Significantly, these new results provide a solution to these practically unexplored issues, if we allow violating the lower bounds constraints within parameter $\eta$.

2. Preliminaries and Assumptions

2.1 Two-sided Resource Allocation Framework

2.1.1 Offline Two-sided Resource Allocation

We consider the following framework for offline resource allocation problems. There are $J (= |\mathcal{J}|)$ different types of requests; each request $j \in \mathcal{J}$ could be served via some channel $i \in \mathcal{I}$, which will consume $a_{ijk}$ amount of resource $k \in \mathcal{K}$ and generate $w_{ij}$ amount of revenue where we use the $\mathcal{K}$ to denote the different $K$ resources. In the process of a $T$-round resource allocation, the function $R(t) : [T] \rightarrow \mathcal{J}$ to record the types of requests, i.e., the type of $t$-th request is $R(t)$. For each resource $k \in \mathcal{K}$, the $U_k(L_k)$ to denote the upper(lower) capacity requirement. The objective of the two-sided resource allocation is to maximize the revenue subject to the two-sided resource capacity constraints. The following is the offline linear programming for resource allocation which knows the entire sequence of requests ahead of time:

$$W_R = \max \sum_{i \in \mathcal{I}, j \in [T]} w_{iR(j)} x_{ij}$$

s.t. $L_k \leq \sum_{i \in \mathcal{I}, j \in [T]} a_{iR(j)k} x_{ij} \leq U_k, \forall k \in \mathcal{K}$

$$\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in [T]$$

$$x_{ij} \in \{0, 1\}, \forall i \in \mathcal{I}, j \in [T]$$

According to the linear programming 1, not picking any channel is feasible for resource allocation problems. Therefore, the $\mathcal{I}$, we assume, includes this option $\perp$, where $a_{\perp jk} = 0$ and $w_{\perp j} = 0$ for any $j \in \mathcal{J}$ and $k \in \mathcal{K}$.

2.1.2 Online Two-sided Resource Allocation

In practice, it’s impossible to know about the all incoming requests in advance. We now consider a online version of the two-sided resource allocation framework:

**Assumption 1** The requests arrive sequentially and are independently drawn from some unknown distribution $\mathcal{P} : \mathcal{J} \rightarrow [0, 1]$, where $\mathcal{P}(j)$ denotes the probability of the type $j \in \mathcal{J}$ and we set $p_j = \mathcal{P}(j)$ $\forall j \in \mathcal{J}$.

Under the Assumption 1 we could define the expected instance as follows.

$$W_E = \max \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j w_{ij} x_{ij}$$

s.t. $L_k \leq \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j a_{ijk} x_{ij} \leq U_k, \forall k \in \mathcal{K}$

$$\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in \mathcal{J}$$

$$x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in \mathcal{J}$$

If we take the same policy for every request $j \in \mathcal{J}$ in LP 1, it’s easy to verify that $E(\sum_{i \in \mathcal{I}, j \in [T]} a_{iR(j)k} x_{ij}) = \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j a_{ijk} x_{ij}$ and $E(\sum_{i \in \mathcal{I}, j \in [T]} w_{iR(j)} x_{ij}) = \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j w_{ij} x_{ij}$. Therefore, we could view...
the LP 2 as a relaxation version of the expectation of LP 1. Moreover, the $W_E$ is a upper bound of the expectation of $W_R$.

**Lemma 1** $W_E \geq E(W_R)$.

**Proof:**

| Sample Instance I | Sample Instance II |
|-------------------|-------------------|
| $\max_{i \in \mathcal{I}, j \in [T]} \sum w_{iR(j)} x_{ij}$ | $\max_{i \in \mathcal{I}, j \in [T]} \sum w_{iR(j)} x_{iR(j)}$
| $s.t. L_k \leq \sum_{i \in \mathcal{I}, j \in [T]} a_{iR(j)k} x_{ij} \leq U_k, \forall k \in \mathcal{K}$ | $s.t. L_k \leq \sum_{i \in \mathcal{I}, j \in [T]} a_{iR(j)k} x_{iR(j)} \leq U_k, \forall k \in \mathcal{K}$
| $\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in [T]$ | $\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in \mathcal{J}$
| $x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in [T]$ | $x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in \mathcal{J}$

We use the $W_{R_1}$ ($W_{R_2}$) to denote the optimal value of LP 1 (LP 1). Because the LP 1 is a relaxation version of LP 1, $W_{R_1} \geq W_R$. For any optimal solution $(x_{ij}^* \forall i \in \mathcal{I}, j \in [T])$ of LP 1, it’s easy to verify that the solution $x_{ij} = \frac{\sum_{R \in \mathcal{R}} x_{ij}^*}{\#\{R \in \mathcal{R} \mid R(t) = j\}} \forall i \in \mathcal{I}, j \in \mathcal{J}$ is feasible for LP 1 so that $W_{R_2} \geq W_{R_1}$. Moreover, the average of optimal solution for all possible LP 1 is a feasible solution for LP 2. Thus $W_E \geq E(W_{R_2}) \geq E(W_{R_1}) \geq E(W_R)$.

To demonstrate the performance of the proposed algorithms, we need to make some assumptions about the parameters.

**Assumption 2**

1. For every request $j \in \mathcal{J}$, the consumption of resources $0 \leq a_{ijk} < +\infty \forall i \in \mathcal{I}, k \in \mathcal{K}$ and the revenues $0 \leq w_{ij} < +\infty j \in \mathcal{K}$. Moreover, we set $\bar{a}_k = \max_{i \in \mathcal{I}, j \in \mathcal{J}} a_{ijk}$ and $\bar{w} = \max_{i \in \mathcal{I}, j \in \mathcal{J}} w_{ij}$.

2. We, in advance, know the lower and upper bound requirements regarding every resource $k \in \mathcal{K}$, i.e., $U_k$ and $L_k$ and the number of requests, i.e., $T$.

**Assumption 3** (Strong feasible condition)

There exist a constant $\alpha > 0$ making the following linear constraints feasible:

$$L_k + \alpha \bar{a}_k \leq \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T_{p_j} a_{ijk} x_{ij} \leq U_k, \forall k \in \mathcal{K}$$

$$\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in \mathcal{J}$$

$$x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in \mathcal{J}$$

where we call $\alpha$ the measure of feasibility.

**2.2 Concentration Inequalities**

In this section, we present some frequently used concentration inequalities in the theoretical analysis of the proposed algorithms.

**Lemma 2**
1. Suppose that $|X| \leq c$ and $E(X) = 0$. For any $t > 0$,
\[
E(\exp(tX)) \leq \exp(\frac{\sigma^2}{c^2}(e^{tc} - 1 - tc))
\]
where $\sigma^2 = \text{Var}(X)$.

2. If $X_1, X_2, \ldots, X_n$ are independent r.v., $P(|X_i| \leq c) = 1$ and $E(X_i) = \mu, \forall i = 1, \ldots, n$, then for any $\epsilon > 0$ the following inequality holds
\[
P\left(\left|\sum_{i=1}^{n} X_i - \frac{\mu}{n}\right| \geq \epsilon\right) \leq 2\exp\left(-\frac{n\epsilon^2}{2\sigma^2 + \frac{2}{3}c^2}\right)
\]
where $\sigma^2 = \sum_{i=1}^{n} \text{Var}(X_i)$.

3. Competitive Algorithms for Online Resource Allocation with Two-sided Constraints

In this section, we propose the online algorithms for resource allocation with two-sided constraints by progressively weakening the following assumptions.

1. With Known Distribution
2. With Known Optimal Value
3. Completely Unknown Distribution

For better exhibition, we use $E(\beta)$ to denote the following expected problem with lower bound $L_k + \beta T\bar{a}_k$ for every resource $k \in K$, i.e.,
\[
W_{\beta} = \max \sum_{i \in I, j \in J} Tp_{ij}w_{ij}x_{ij}
\]
\[
s.t. L_k + \beta T\bar{a}_k \leq \sum_{i \in I, j \in J} Tp_{ij}a_{ijk}x_{ij} \leq U_k, \forall k \in K
\]
\[
\sum_{i \in I} x_{ij} \leq 1, \forall j \in J
\]
\[
x_{ij} \geq 0, \forall i \in I, j \in J
\]
where the $W_{\beta}$ is the optimal value for problem $E(\beta)$. Meanwhile, we use $\{x(\beta)_{ij}^*, \forall i \in I, j \in J\}$ for the optimal solution for $E(\beta)$.

3.1 With Known Distribution

In this section, we assume we completely know the distribution $P$. Before going into the detail, we first give a high-level description of our algorithm and outcomes.

**High-level Overview:**

1. With the knowledge of $P$, we could directly solve the expected problem $E(\tau)$ and obtain the optimal solution $\{x(\tau)_{ij}^*, \forall i \in I, j \in J\}$, where deviation parameter $\tau = \frac{1}{2\sigma^2}$. The optimal solution $\{x(\tau)_{ij}^*, \forall i \in I, j \in J\}$ proposes some information or preference for allocation. For fixed request $j \in J$, if $x(\tau)_{i_1j}^* \geq x(\tau)_{i_2j}^*$, we trend to assign this type of requests to channel $i_1$ instead of $i_2$. Therefore, motivated via this intuition, we design the algorithm $\hat{P}$: assigning request $j \in J$ to channel $i \in I$ with probability $(1 - \epsilon)x_{ij}^*$. 


2. According to the algorithm $\tilde{P}$, if we use the random variable $X_k^P$ for resource $k$ consumed via one request sampled from distribution $P$ and r.v. $Y^P$ for the revenue, then it’s easy to observe that $(1-\epsilon)\frac{L_k}{\bar{a}_k} + \epsilon \bar{a}_k \leq \mathbb{E}(X_k^P) \leq (1-\epsilon)\frac{U_k}{\bar{a}_k}$ and $\mathbb{E}(Y^P) = (1-\epsilon)\frac{\bar{w}}{\bar{a}_k}$. Because the expectation is strictly included in $[\frac{L_k}{\bar{a}_k}, \frac{U_k}{\bar{a}_k}]$, we could prove that the algorithm $\tilde{P}$ will generate feasible solution for online resource problem with two-sided constraints (LP 1) via the Bernstein inequalities (Lemma 2). Moreover, this is the reason why we enlarge the lower bound $L_k$ with extra $\tau T \bar{a}_k$ and scale the solution with factor $1-\epsilon$. Meanwhile, we also verify that the accumulative revenue will be larger than $(1-2\epsilon)\bar{w}$ w.p. $1-\epsilon$ (in Theorem 2).

3. Previously, we have proved the accumulative revenue larger than $(1-2\epsilon)\bar{w}$ with high probability. In truth, we want to compare the revenue of algorithm $\tilde{P}$ with $\bar{w}$. That is, although lifting the lower resource constraints from $L_k$ to $L_k + \tau T \bar{a}_k$ will guarantee the feasibility of the algorithm $\tilde{P}$, this method also cause the change of baseline in analysing the accumulative revenues. Therefore, one hard challenge is how to derive the relationship between the $\bar{w}$ and $\bar{w}_{E}$. Under the strong feasible condition, we obtain $\bar{w} \geq (1-\frac{\alpha}{\alpha})\bar{w}_{E}$, where $\alpha$ is the measure of feasibility. Finally, we could prove the Theorem 1.

\begin{algorithm}[h]
\caption{Algorithm $\tilde{P}$}
\begin{algorithmic}
\State \textbf{Input:} $\tau = \frac{1-\epsilon}{1-\epsilon}$, $P$
\State \textbf{Output:} $\{x_{ij}\}_{i\in I, j\in J}$
\State $\{x(\tau)_{ij}\}_{i\in I, j\in J} = \text{argmax} E(\tau)$.
\State When $j$-th request comes, we assign this request to channel $i$ with probability $(1-\epsilon)x(\tau)_{ij}$. That is, If assigning the $j$-th request to channel $i$, we set $x_{ij} = 1$, otherwise $x_{ij} = 0$.
\end{algorithmic}
\end{algorithm}

**Theorem 1** Under the assumption 1-3, if $\forall \epsilon \geq 0$ and $\gamma = \max(\frac{\bar{a}_k}{U_k}, \frac{\bar{a}_k}{(1-\tau)\bar{a}_k-L_k}, \frac{\bar{w}}{\bar{a}_k}) = O(\frac{\epsilon^2}{\ln(\frac{1}{\epsilon})})$, the algorithm $\tilde{P}$ returns a solution satisfying the constraints and achieves an objective value $(1-\epsilon)W_{E}$ w.p. $1-\epsilon$.

3.1.1 Proof under the known distribution assumption

Like the previous description, we also use the random variable $X_{jk}^P$ to denote resource $k$ consumed via the $j$-th request, and r.v $Y_{j}^P$ for the revenue the $j$-th request brings, under the algorithm $\tilde{P}$. 

We first prove that, for every resource $k$, the consumed resource is below the capacity $U_k$ w.h.p..

\[
P\left(\sum_{j=1}^{T} X_{jk}^{\hat{P}} \geq U_k\right)
\]
\[
= P\left(\sum_{j=1}^{T} (X_{jk}^{\hat{P}} - E(X_{jk})) \geq U_k - T E(X_{jk}^{\hat{P}})\right)
\]
\[
\leq \exp\left(\frac{\left(U_k - T E(X_{jk}^{\hat{P}})\right)^2}{2 T \sigma^2 + \frac{2}{3} \bar{a}_k (U_k - T E(X_{jk}^{\hat{P}}))}\right)
\]
\[
= \exp\left(\frac{\left(U_k - T E(X_{jk}^{\hat{P}})\right)^2}{2 T \sigma^2 + \frac{2}{3} \bar{a}_k}\right)
\]
\[
\leq \exp\left(\frac{\epsilon^2}{2(1 - \frac{2}{3} \epsilon) \bar{a}_k}\right)
\]
\[
\leq \frac{\epsilon}{2 K + 1}
\]

where the first equality follows from $E(X_{jk}^{\hat{P}}) = E(X_{jk}^{\check{P}}) = \cdots = E(X_{jk}^{P})$; the first inequality from Lemma 2 and setting $\sigma^2 = Var(X_{jk}^{\hat{P}})$; in the second inequality, we could easily verify that $\sigma^2 \leq E((X_{jk}^{\hat{P}})^2) \leq \bar{a}_k E(X_{jk}^{\check{P}}) \leq \frac{(1 - \epsilon) \bar{a}_k U_k}{T}$, $U_k - T E(X_{jk}^{\hat{P}}) \geq \epsilon U_k$ and $\frac{T \sigma^2}{U_k - T E(X_{jk}^{\hat{P}})} \leq \bar{a}_k \frac{1}{\epsilon}$ so that

\[
- \frac{\left(U_k - T E(X_{jk}^{\hat{P}})\right)^2}{2 T \sigma^2 + \frac{2}{3} \bar{a}_k} \leq - \frac{\epsilon U_k}{2 \bar{a}_k \left(1 - \frac{2}{3} \epsilon\right) U_k} = - \frac{\epsilon^2}{2(1 - \frac{2}{3} \epsilon) U_k};
\]

the third inequality from $\gamma \geq \frac{\epsilon}{U_k}$; the final inequality from $\gamma = O\left(\frac{\epsilon}{\ln(\frac{1}{\epsilon})}\right)$.

Next, we verify that the algorithm $\hat{P}$ satisfies the lower resource bound with high probability.

\[
P\left(\sum_{j=1}^{T} X_{jk}^{\hat{P}} \leq L_k\right)
\]
\[
= P\left(\sum_{j=1}^{T} (E(X_{jk}^{\hat{P}}) - X_{jk}) \geq T E(X_{jk}^{\hat{P}}) - L_k\right)
\]
\[
\leq \exp\left(\frac{(T E(X_{jk}^{\hat{P}}) - L_k)^2}{2 T \sigma^2 + \frac{2}{3} \bar{a}_k (T E(X_{jk}^{\hat{P}}) - L_k))}\right)
\]
\[
= \exp\left(\frac{(T E(X_{jk}^{\hat{P}}) - L_k)^2}{2 T \sigma^2 + \frac{2}{3} \bar{a}_k}\right)
\]
\[
\leq \exp\left(\frac{\epsilon^2}{2(1 - \frac{2}{3} \epsilon) \bar{a}_k - L_k}\right)
\]
\[
\leq \frac{\epsilon}{2 K + 1}
\]

where the first inequality from Lemma 2 and setting $\sigma^2 = Var(X_{jk}^{\hat{P}})$; in the second inequality, we could easily verify that $\sigma^2 = Var(X_{jk}^{\hat{P}}) = Var(\bar{a}_k - X_{jk}^{\hat{P}}, X_{jk}^{\hat{P}}) \leq E((\bar{a}_k - X_{ij}^{\hat{P}})^2) \leq \bar{a}_k E(\bar{a}_k - X_{jk}) \leq \bar{a}_k \frac{1}{\epsilon}$.
\begin{align*}
\frac{(1-\epsilon)\bar{a}_k(T\bar{a}_k-L_k)}{\frac{1}{2}\bar{a}_k^2 + \frac{1}{4}a_k^2}, \quad T\mathbb{E}(X_{jk}) - L_k \geq \epsilon(T\bar{a}_k - L_k), \text{ and } \frac{T\sigma^2}{T\mathbb{E}(X_{jk}) - L_k} \leq \bar{a}_k \frac{1}{\epsilon}, \text{ so that } \frac{(T\mathbb{E}(X_{jk}) - L_k)}{2 T\mathbb{E}(X_{jk}) - L_k} \leq -\frac{\epsilon(T\bar{a}_k-L_k)}{2a_k \frac{1}{2}\bar{a}_k^2 + \frac{1}{4}a_k^2}; \text{ the final inequality from } \gamma = O\left(\frac{e^2}{\ln(\frac{1}{\epsilon})}\right).
\end{align*}

Therefore, from the previous outcomes, the consumed resource \( k \) satisfies our requirements, w.h.p.

Next, we investigate the revenue the algorithm \( \hat{P} \) brings.

\begin{align*}
P\sum_{j=1}^{T} Y_{j}^{\hat{P}} &\leq (1-2\epsilon)W_{\tau} \\
&= P\sum_{j=1}^{T} \left(\mathbb{E}(Y_{j}^{\hat{P}}) - Y_{j}^{\hat{P}}\right) \geq T\mathbb{E}(Y_{j}^{\hat{P}}) - (1-2\epsilon)W_{\tau} \]
&\leq \exp\left\{-\frac{(T\mathbb{E}(Y_{j}^{\hat{P}}) - (1-2\epsilon)W_{\tau})^2}{2T\sigma^2 + \frac{2}{3}w(T\mathbb{E}(Y_{j}^{\hat{P}}) - (1-2\epsilon)W_{\tau})}\right\} \tag{7}
&= \exp\left\{-\frac{(T\mathbb{E}(Y_{j}^{\hat{P}}) - (1-2\epsilon)W_{\tau})}{2T\mathbb{E}(Y_{j}^{\hat{P}}) - (1-2\epsilon)W_{\tau} + \frac{2}{3}w}\right\}
&\leq \exp\left\{-\frac{\epsilon^2}{2(1-\epsilon)w}\right\}
&\leq \frac{\epsilon}{2K+1}
\end{align*}

where the first equality follows from \( \mathbb{E}(Y_{j}^{\hat{P}}) = \mathbb{E}(Y_{2}^{\hat{P}}) = \cdots = \mathbb{E}(Y_{T}^{\hat{P}}) \); the first inequality from Lemma 2 and setting \( \sigma^2 = \text{Var}(Y_{j}^{\hat{P}}) \); in the second inequality, we could easily verify that \( \sigma^2 \leq \mathbb{E}((Y_{j}^{\hat{P}})^2) \leq w\mathbb{E}(Y_{j}^{\hat{P}}) \leq \frac{(1-\epsilon)\bar{a}_k W_{\tau}}{2} \) and \( E(Y_{j}^{\hat{P}}) = (1-\epsilon)W_{\tau} \) so that \( \frac{(T\mathbb{E}(Y_{j}^{\hat{P}}) - (1-2\epsilon)W_{\tau})}{T\sigma^2} \leq -\frac{\epsilon W_{\tau}}{2a_k \frac{1}{2}\bar{a}_k^2 + \frac{1}{4}a_k^2}; \) the final inequality from \( \gamma = O\left(\frac{e^2}{\ln(\frac{1}{\epsilon})}\right) \).

We summarize all outcomes in the following theorem 2

**Theorem 2** Under the assumption 1-3, if \( \epsilon \geq 0 \) and \( \gamma = O\left(\frac{e^2}{\ln(\frac{1}{\epsilon})}\right) \), the algorithm \( \hat{P} \) returns a solution satisfying the constraints and achieves an objective value \((1-2\epsilon)W_{\tau}\) w.p \( 1-\epsilon \).

**Proof:** From equation 5-7

\begin{align*}
P\sum_{j=1}^{T} Y_{j}^{\hat{P}} \leq (1-2\epsilon)W_{\tau} + \sum_{k \in K} P\left(\sum_{j=1}^{T} X_{jk}^{\hat{P}} \notin \{L_k, U_k\}\right) \leq (2K+1) \frac{\epsilon}{2K+1} \tag{8}
\end{align*}

where we set \( \gamma = \max\left(\frac{\bar{a}_k}{W_{\tau}}, \frac{\bar{a}_k}{\frac{1}{2}\bar{a}_k^2 + \frac{1}{4}a_k^2}\right) \) and \( \gamma = O\left(\frac{e^2}{\ln(\frac{1}{\epsilon})}\right) \).

From theorem 2, we have verified the cumulative revenue is larger than \((1-2\epsilon)W_{\tau}\), w.p. \( 1-\epsilon \). Thus, in order to compare the revenue with \( W_{E} \), we should derive the relationship between \( W_{\tau} \) and \( W_{E} \). However, due to the deviation \( \tau T\bar{a}_k \), it’s hard to directly obtain this relationship. We will tackle this problem in the next subsection.
3.1.2 SENSITIVE ANALYSIS

In order to guarantee the lower bound for every resource \( k \in \mathcal{K} \), we use the optimal solution for \( E(\tau) \) to design \( \tilde{P} \), instead of solution of \( E(0) \). However, although we could easily obtain the competitive ratio between the cumulative revenue and \( W_\tau \) under the algorithm \( \tilde{P} \), it’s hard to derive the ratio based on \( W_E \). Hence, in this subsection, we demonstrate the relationship between \( W_\tau \) and \( W_E \).

Before going into the detail, we first investigate the difference between problem \( E(\tau) \) and \( E(0) \). We only change the lower bound for every resource \( k \in \mathcal{K} \) with extra \( \tau T \hat{a}_k \). To investigate how these changes affect the optimal value, we consider the dual problems for \( E(\beta) \) and \( E(0) \).

The dual problem of the expected instance, i.e., problem \( E(0) \):

\[
\begin{align*}
\min & \, \sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j \\
\text{s.t.} & \, \sum_{k \in \mathcal{K}} (\alpha_k - \beta_k) T p_j a_{ijk} - T p_j w_{ij} + \rho_j \geq 0 \, \forall i \in \mathcal{I}, j \in \mathcal{J} \\
& \, \alpha_k, \beta_k, \rho_j \geq 0, k \in \mathcal{K}, j \in \mathcal{J}
\end{align*}
\]

The dual problem of \( E(\tau) \):

\[
\begin{align*}
\min & \, \sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k (L_k + \tau T \hat{a}_k) + \sum_{j \in \mathcal{J}} \rho_j \\
\text{s.t.} & \, \sum_{k \in \mathcal{K}} (\alpha_k - \beta_k) T p_j a_{ijk} - T p_j w_{ij} + \rho_j \geq 0 \, \forall i \in \mathcal{I}, j \in \mathcal{J} \\
& \, \alpha_k, \beta_k, \rho_j \geq 0, k \in \mathcal{K}, j \in \mathcal{J}
\end{align*}
\]

We can observe that LP 9 and LP 10 have the same feasible set. Moreover, \( \sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k (L_k + \tau T \hat{a}_k) + \sum_{j \in \mathcal{J}} \rho_j = \sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j - \sum_{k \in \mathcal{K}} \tau T \hat{a}_k \beta_k \). Therefore, if we know about some relationships between \( \sum_{k \in \mathcal{K}} \tau T \hat{a}_k \beta_k \) and \( \sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j \) under the dual constraints, we might obtain the ratio between \( W_\tau \) with \( W_E \).

Motivated by the factor-revealing Linear Programming method in theoretical computer science community (Jain et al., 2003), we propose the factor-revealing linear fractional programming for answering this question.

3.1.3 FACTOR REVEALING LINEAR FRACTIONAL PROGRAMMING

As we know, \( W_E \geq W_\tau \). To derive the competitive ratio based on \( W_E \), we urgently want to control \( W_\tau \) via \( W_E \), i.e., find a number \( c \in (0, 1) \) to making \( W_\tau \geq (1 - c)W_E \). How to derive this unknown parameter? Motivated by the dual LP (9) and LP (10), if \( \sum_{k \in \mathcal{K}} \frac{\epsilon}{1 - \tau} T \hat{a}_k \beta_k \leq c(\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j) \) for any dual feasible solution, we could obtain \( W_\tau \geq (1 - c)W_E \).

We translate this idea into mathematical terminology, i.e., the following linear fractional programming:

\[
\max \, \frac{\sum_{k \in \mathcal{K}} \frac{\epsilon}{1 - \tau} T \hat{a}_k \beta_k}{\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j}
\]

\[
\begin{align*}
\text{s.t.} & \, \sum_{k \in \mathcal{K}} (\alpha_k - \beta_k) T p_j a_{ijk} - T p_j w_{ij} + \rho_j \geq 0 \, \forall i \in \mathcal{I}, j \in \mathcal{J} \\
& \, \alpha_k, \beta_k, \rho_j \geq 0, k \in \mathcal{K}, j \in \mathcal{J}
\end{align*}
\]

Due to \( W_E \geq 0 \), the \( \sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j \geq W_E \geq 0 \) for any dual feasible solution. With this information, we transfer this linear fractional programming into linear programming.
(LP 12) by setting $\tilde{\alpha}_k = \frac{\sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} \rho_j}{\sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} \rho_j}$, and $z = \frac{\sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} \rho_j}{\sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} \rho_j}$.

$$\max \sum_{k \in K} \frac{\epsilon}{1 - \epsilon} T \tilde{\alpha}_k \tilde{\beta}_k$$

$$\text{s.t. } \sum_{k \in K} (\tilde{\alpha}_k - \tilde{\beta}_k) T p_j a_{ijk} - T p_j w_j z + \tilde{\rho}_j \geq 0 \forall i \mathcal{I}, j \in \mathcal{J}$$

$$\sum_{k \in K} \tilde{\alpha}_k U_k - \sum_{k \in K} \tilde{\beta}_k L_k + \sum_{j \in \mathcal{J}} \tilde{\rho}_j = 1$$

It’s easy to verify the equivalence between equation 11 and LP 12 and we investigate the dual problem of LP 12.

$$\min t$$

$$\text{s.t. } \sum_{i \in \mathcal{I}} d_{ij} \leq t \quad (13)$$

$$\sum_{i \in \mathcal{I}, j \in \mathcal{J}} d_{ij} T p_j a_{ijk} \leq t U_k$$

$$\sum_{i \in \mathcal{I}, j \in \mathcal{J}} d_{ij} T p_j a_{ijk} \geq t L_k + \frac{\epsilon}{1 - \epsilon} T \tilde{\alpha}_k$$

$$\forall d_{ij} \geq 0, t \in \mathbb{R}, \forall i \in \mathcal{I}, j \in \mathcal{J}$$

We reformulate LP 13 with the new variables $d_{ij} := tz_{ij}$.

$$t^* = \min t$$

$$\text{s.t. } t \left( \sum_{i \in \mathcal{I}} z_{ij} - 1 \right) \leq 0$$

$$t \left( \sum_{i \in \mathcal{I}, j \in \mathcal{J}} z_{ij} T p_j a_{ijk} - U_k \right) \leq 0$$

$$t \left( \sum_{i \in \mathcal{I}, j \in \mathcal{J}} z_{ij} T p_j a_{ijk} - L_k \right) \geq \frac{\epsilon}{1 - \epsilon} T \tilde{\alpha}_k$$

$$\forall z_{ij} \geq 0, t \in \mathbb{R}, \forall i \in \mathcal{I}, j \in \mathcal{J}$$

Therefore, according to strong feasible condition (assumption 3), we know there exist a point $x_{ij} \geq 0$ making $\sum_{i \in \mathcal{I}, j \in \mathcal{J}} x_{ij} T p_j a_{ijk} \leq U_k, \sum_{i \in \mathcal{I}} x_{ij} \leq 1,$ and $\sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j a_{ijk} x_{ij} \geq L_k + \alpha T a_k$. Therefore, the solution, $t = \frac{\pi}{\alpha}$ and $z_{ij} = x_{ij}, \forall i \in \mathcal{I}, j \in \mathcal{J}$, is a feasible point for equation 14. As a result, $t^* \leq \frac{\pi}{\alpha}$ so that, under the dual constraint, $\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k (L_k + \frac{\epsilon}{1 - \epsilon} T \tilde{\alpha}_k) + \sum_{j \in \mathcal{J}} \rho_j = \sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \beta_k \rho_j \geq (1 - \frac{\pi}{\alpha}) (\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j)$. Therefore, $W_{\tau} \geq (1 - \frac{\pi}{\alpha}) W_E$.

**Lemma 3** Under the strong feasible condition, i.e., assumption 3,

$$W_{\tau} \geq (1 - \frac{\tau}{\alpha}) W_E$$

With the Lemma 3 and Theorem 2, the cumulative revenue is larger than $(1 - 2\epsilon) W_{\tau} \geq (1 - 2\epsilon) (1 - \frac{\pi}{\alpha}) W_E \geq (1 - (2 + \frac{\pi}{\alpha}) \epsilon) W_E$. We finish the proof for Theorem 1.
3.2 With Known $W_\tau$

In this section, we only assume the knowledge of the optimal value $W_\tau$. Based on this assumption, we design an algorithm $A$ only using the $W_\tau$, which achieves at least $(1 - (2 + \frac{1}{\alpha}))W_E$ and satisfies the constraints requirements w.p. $1 - \epsilon$. Like the previous section, we first give a high-level overview for the techniques and the algorithms.

**High-level Overview:**

Similarly, for algorithm $A$, we also use the r.v $X^A_{jk}$ for resource $k$ consumed via the $j$-th request, and r.v $Y^A_j$ for the revenue. In general, the key to previous analysis is to bound the failure probability of three events: (1) $\sum_{j=1}^{T} X^A_{jk} \geq U_k, \forall k \in K$; (2) $\sum_{j=1}^{T} X^A_{jk} \leq L_k, \forall k \in K$; (3) $\sum_{j=1}^{T} Y^A_{jk} \leq (1 - 2\epsilon)W_\tau$. Like the previous analysis, we use the Bernstein inequalities to bound the failure probability of the algorithm $P$. As we know, the Bernstein inequalities are derived from the moment generating functions. Therefore, we first design some moment generating function $F(A_1A_2\ldots A_sP^{T-s})$ for controlling the probability of three bad events of the hybrid algorithm $A_1A_2\ldots A_sP^{T-s}$ which runs the algorithm $A_i$ for $i$-th request when $i \leq s$ and $P$ for next $T - s$ rounds. Surprisingly, the $F$ satisfies some decrease property for the special hybrid algorithm $A^sP^{T-s}$, i.e., $F(A^{s+1}P^{T-s-1}) \leq F(A^sP^{T-s})$ and $F(P^T) \leq \epsilon$. Hence, the $F(A^T) \leq F(A^{T-1}P) \leq \ldots \leq F(P^T) \leq \epsilon$.

Next, we will demonstrate how to find the satisfied moment generating function $F$. Before that, we first present the algorithm $A$ and the outcomes.

**Algorithm 2 Algorithm A**

**Input:** $\epsilon, L, U, \gamma, \tau$

**Output:** $(x_{ij})_{i \in K, j \in [T]}$

**Initialize** $\phi_{k,0} = \exp(\frac{\ln(1-\epsilon)}{\gamma}), \phi_{k,0} = \exp(-\frac{\ln(1-\epsilon)U_k}{\gamma}), \psi_0 = \exp(-\frac{(1 - 2\epsilon)\ln(1-\epsilon)}{\gamma})$

**for** $t = 1$ to $T$ **do**

When a request $j \in J$ coming, then use the following option $i^*$:

\[ i^* = \arg \min_i \{- \ln(1 - \epsilon) / U_i, \gamma \} \]

\[ a_{i^*j} = \exp(-\frac{\ln(1-\epsilon)}{\gamma} - a_{i^*j}) \]

\[ X^A_{i^*j} = x_{i^*j}, Y^A_{i^*j} = w_{i^*j} \]

Update $\phi_{i^*j} = \phi_{i^*j}\exp(-\frac{\ln(1-\epsilon)}{\gamma} - X^A_{i^*j})$ Update $\phi_{i^*j} = \phi_{i^*j}\exp(-\frac{\ln(1-\gamma)}{\gamma} - X^A_{i^*j})$

Update $\psi_t = \psi_t\exp(-\frac{\ln(1-\gamma)}{\gamma} - Y^A_{i^*j})$

**Theorem 3** Under the assumption $1.3$, if $\epsilon \geq 0$ and $\gamma = O(\frac{\epsilon^2}{\ln(\frac{1}{\alpha})})$, the algorithm $A$ returns a solution satisfying the constraints and achieves an objective value $(1 - (2 + \frac{1}{\alpha}))W_1$ w.p $1 - \epsilon$.

3.2.1 Proof under the knowledge of $W_\tau$

Firstly, we design some moment generating functions to control the failure probability for the hybrid algorithm $A_1A_2\ldots A_sP^{T-s}$.
For the first bad event,

\[ P \left( \sum_{j=1}^{s} X_{jk}^A + \sum_{j=s+1}^{T} X_{jk}^\tilde{P} \geq U_k \right) \]

\[ \leq \min_{t > 0} E \left( \exp \left( t \left( \sum_{j=1}^{s} X_{jk}^A + \sum_{j=s+1}^{T} X_{jk}^\tilde{P} - U_k \right) \right) \right) \]

\[ = \min_{t > 0} E \left( \exp \left( t \left( \sum_{j=1}^{s} X_{jk}^A - s T U_k \right) + \sum_{j=s+1}^{T} X_{jk}^\tilde{P} - \frac{T-s}{T} U_k \right) \right) \]

\[ = \min_{t > 0} \mathbb{E} (\Delta^s(t) \exp \left( T - \frac{s}{T} \frac{\sigma^2}{\bar{a}_k} \left( e^{\frac{t}{\bar{a}_k}} - 1 - \frac{t}{\bar{a}_k} \right) + \frac{-(T-s)\bar{t}U_k}{T} \right)) \]

\[ \leq \min_{t > 0} \mathbb{E} (\Delta^s(t) \exp \left( T - \frac{s}{T} \frac{\sigma^2}{\bar{a}_k} \left( e^{\frac{t}{\bar{a}_k}} - 1 - \frac{t}{1-\epsilon} \right) \right) \]

\[ \leq \min_{t > 0} \mathbb{E} (\Delta^s(t) \exp \left( -\frac{(1-\epsilon)(T-s)U_k}{\bar{a}_k} \right) \left( (1+\eta) \ln(1+\eta) - \eta \right)) \]

\[ \leq \min_{t > 0} \mathbb{E} (\Delta^s(t) \exp \left( -\frac{T-s}{T} \frac{\sigma^2}{\gamma(1-\frac{2}{3}\epsilon)} \right) \left( e^{\frac{t}{\bar{a}_k}} - 1 - \frac{t}{1-\epsilon} \right) \]

\[ \leq \mathbb{E} (\Delta^s(t) \exp \left( -\frac{T-s}{T} \frac{\sigma^2}{\gamma(1-\frac{2}{3}\epsilon)} \right) ) \]

where the first inequality follow from \( \exp (t(\sum_{j=1}^{s} X_{jk}^A + \sum_{j=s+1}^{T} X_{jk}^\tilde{P} - U_k)) \geq 0 \) when \( \sum_{j=1}^{s} X_{jk}^A + \sum_{j=s+1}^{T} X_{jk}^\tilde{P} \geq U_k \); in the second equality, we set \( \Delta^s(t) = \exp \left( t \left( \sum_{j=1}^{s} X_{jk}^A - \frac{s}{T} U_k \right) \right) \); the second inequality from Lemma 2 and \( T \mathbb{E}(X_{jk}^\tilde{P}) \leq (1-\epsilon)U_k \); the third inequality from \( \sigma^2 = \text{Var}(X_{jk}^\tilde{P}) \leq \frac{(1-\epsilon)\bar{a}_k U_k}{T} \); in the fourth inequality, we set \( t = \frac{-\ln(1-\epsilon)}{\bar{a}_k}, \eta = \frac{\epsilon}{1-\epsilon} \); then the fifth inequality from \( (1+\eta) \ln(1+\eta) - \eta \geq \frac{2\epsilon^2}{\gamma(1-\frac{2}{3}\epsilon)} \); the final inequality from the definition of \( \gamma \).
Next, we bound the second bad events via the moment generating function:

\[ P\left(\sum_{j=1}^{s} X_{A_j} + \sum_{j=s+1}^{T} X_{\tilde{P}_j} \leq L_k\right) \]

\[ \leq \min_{t>0} E\left(exp(t(L_k - s \sum_{j=1}^{s} X_{A_j} - T \sum_{j=s+1}^{n} X_{\tilde{P}_j})))\right) \]

\[ = \min_{t>0} E\left(exp(t\left(\frac{s}{T} L_k - \sum_{j=1}^{s} X_{A_j} + T\left(\frac{s-1}{T} L_k - \sum_{j=s+1}^{n} X_{\tilde{P}_j}\right)\right)\right) \]

\[ = \min_{t>0} E(\Delta_1(t)exp(t \sum_{j=s+1}^{T} (E(X_{A_j}) - X_{\tilde{P}_j} + \frac{T-s}{T}t(L_k - T E(X_{\tilde{P}_j})))) \]

\[ \leq \min_{t>0} E(\Delta_1(t)exp((n-s)\frac{\sigma^2}{\bar{a}_k^2} (e^{t\bar{a}_k} - 1 - t\bar{a}_k) - \frac{(T-s)e(T\bar{a}_k - L_k)}{T}t) \]

\[ \leq \min_{t>0} E(\Delta_1(t)exp((-1+\epsilon)(T-s)(T\bar{a}_k - L_k)(e^{t\bar{a}_k} - 1 - t\bar{a}_k - t\frac{\epsilon}{1+\epsilon}\bar{a}_k)) \]

\[ \leq E(\Delta_1^q(t_1) \leq (1+\epsilon)(T-s)(T\bar{a}_k - L_k)((1+\eta)\ln(1+\eta) - \eta)) \]

\[ \leq E(\Delta_1^q(t_2) \leq \frac{1+\epsilon}{\gamma U_k} \frac{e^2}{2\gamma(1-\frac{\epsilon}{2})} \]

where in the second equality, we set \( \Delta_1(t) = exp(t(\frac{s}{T} L_k - \sum_{j=1}^{s} X_{A_j})) \); the second inequality from Lemma 2 and \( T E(X_{\tilde{P}_j}) \geq (1-\epsilon)L_k + eT\bar{a}_k \); the third inequality from \( \sigma^2 = Var(X_{\tilde{P}_j}) = Var(\bar{a}_k - X_{\tilde{P}_j}) \leq \bar{a}_k E(\bar{a}_k - X_{\tilde{P}_j}) \leq \bar{a}_k (1+\epsilon)(T\bar{a}_k - L_k) \); in the fourth inequality, we set \( t = \frac{\ln(1+\eta)}{\bar{a}_k}, \eta = \frac{\epsilon}{1+\epsilon} \); then the fifth inequality from \( (1+\eta)\ln(1+\eta) - \eta \geq \frac{(\eta^2)}{2+2\eta} \); the final inequality from the definition of \( \gamma \).
Finally, we use the moment generating function to control the failure probability of the third bad event.

\[
P(\sum_{j=1}^{s} Y_j^A_j + \sum_{j=s+1}^{T} Y_j^\tilde{P} \leq (1 - 2\epsilon)W_r) \\
\leq \min_{t > 0} \mathbb{E}(\exp(t((1 - 2\epsilon)W_r - \sum_{j=1}^{s} Y_j^A_j - \sum_{j=s+1}^{T} Y_j^\tilde{P}))) \\
= \min_{t > 0} \mathbb{E}(\exp(t(\frac{s}{T}(1 - 2\epsilon)W_r - \sum_{j=1}^{s} Y_j^A_j) + t(\frac{T-s}{T}(1 - 2\epsilon)W_r - \sum_{j=s+1}^{T} Y_j^\tilde{P}))) \\
= \min_{t > 0} \mathbb{E}(\Delta_2^s(t)\exp(t \sum_{j=s+1}^{T} (\mathbb{E}(Y_j^\tilde{P}) - Y_j^\tilde{P}) + \frac{T-s}{T}t((1 - 2\epsilon)W_r - T\mathbb{E}(Y_j^\tilde{P})))) (18) \\
\leq \min_{t > 0} \mathbb{E}(\Delta_2^s(t)\exp((T - s)\frac{\sigma_1^2}{\bar{w}^2}(e^{t\bar{w}} - 1 - t\bar{w}) + \frac{-(T - s)t\bar{w}}{T}W_r)) \\
\leq \min_{t > 0} \mathbb{E}(\Delta_2^s(t)\exp\left(\frac{(1 - \epsilon)(T - s)W_r}{\bar{w}}(e^{t\bar{w}} - 1 - t\bar{w} - \frac{\epsilon}{1 - \epsilon}t\bar{w})\right)) \\
\leq \mathbb{E}(\Delta_2^s(\frac{-\ln(1 - \epsilon)}{\bar{w}})\exp\left(-\frac{(1 - \epsilon)(T - s)W_r}{\bar{w}}((1 + \eta)\ln(1 + \eta) - \eta))\right)) \\
\leq \mathbb{E}(\Delta_2^s(\frac{-\ln(1 - \epsilon)}{\gamma W_r})\exp\left(-\frac{T-s}{T}\frac{\epsilon^2}{2(1 - \frac{3}{4}\epsilon)\gamma}\right)) \\
\leq \mathbb{E}(\Delta_2^s(\frac{-\ln(1 - \epsilon)}{\gamma W_r})\exp\left(-\frac{T-s}{T}\frac{\epsilon^2}{2(1 - \frac{3}{4}\epsilon)\gamma}\right))
\]

where in the second equality, we set \( \Delta_2^s(t) = \exp(t((\frac{s}{T}(1 - 2\epsilon)W_r - \sum_{j=1}^{s} Y_j^A_j))) \); the second inequality from Lemma 2 and \( T\mathbb{E}(Y_j^\tilde{P}) = (1 - \epsilon)W_r \); the third inequality from \( \sigma_1^2 = \text{Var}(Y_j^\tilde{P}) \leq \bar{w}\mathbb{E}(Y_j^\tilde{P}) \leq \frac{(1 - \epsilon)\bar{w}W_r}{T} \); in the fourth inequality, we set \( t = -\frac{\ln(1 - \epsilon)}{\bar{w}}, \eta = \frac{\epsilon}{1 - \epsilon} \); then the fifth inequality from \( (1 + \eta)\ln(1 + \eta) - \eta \geq \frac{\eta^2}{2T + \eta} \); the final inequality from the definition of \( \gamma \).

With the inequalities 16-18, we set the \( \mathcal{F}(A_1A_2...A_s\tilde{P}^{T-s}) \) to the sum of upper bound for the three bad situations, i.e.,

\[
\mathcal{F}(A_1A_2...A_s\tilde{P}^{T-s}) = \mathbb{E}(\Delta_2^s(\frac{-\ln(1 - \epsilon)}{\gamma W_r})\exp\left(-\frac{T-s}{T}\frac{\epsilon^2}{2(1 - \frac{3}{4}\epsilon)\gamma}\right) + \sum_{k \in K}\Delta_1^s(\frac{-\ln(1 - \epsilon)}{\gamma U_k})\exp\left(-\frac{T-s}{T}\frac{\epsilon^2}{2(1 - \frac{3}{4}\epsilon)\gamma}\right)) \\
+ \sum_{k \in K}\Delta_1^s(\frac{-\ln(1 - \epsilon)}{\gamma U_k})\exp\left(-\frac{T-s}{T}\frac{\epsilon^2}{2(1 - \frac{3}{4}\epsilon)\gamma}\right))
\]

It’s easy to get \( \mathcal{F}(\tilde{P}^{T}) = (2K + 1)\exp\left(\frac{-\epsilon^2}{2(1 - \frac{3}{4}\epsilon)\gamma}\right) \leq \epsilon \). Next, we verify:

**Lemma 4** \( A = \arg\min_B \mathcal{F}(A_1^{s-1}B\tilde{P}^{T-s}) \)
Proof: From the description of algorithm A (in the algorithm 3.2), we will allocate the \( s \)-th request to the best channel \( i^* \), where

\[
i^* = \arg \min_{i \in \mathcal{K}} \sum_{k \in \mathcal{K}} \exp \left( -\frac{\ln(1 - \epsilon)}{\gamma U_k} \sum_{j=1}^{s-1} X_{jk}^A - \frac{s-1}{T} U_k + \frac{-\ln(1 - \epsilon)}{\gamma U_k} a_{isk} \right) + \sum_{k \in \mathcal{K}} \exp \left( -\frac{\ln(1 - \epsilon)}{\gamma U_k} (\frac{s}{T} L_k - \sum_{j=1}^{s-1} X_{jk}^A) + \frac{\ln(1 - \epsilon)}{\gamma L_k} a_{isk} \right) + \exp \left( -\frac{\ln(1 - \epsilon)}{\gamma W_r} (\frac{s}{T} (1 - 2\epsilon) W_r - \sum_{j=1}^{s-1} Y_{j}^A) - \frac{\ln(1 - \epsilon)}{\gamma W_r} w_{is} \right)
\]

(20)

where \( X_{jk}^A \) is the resource consumed via the algorithm A’s decision for \( j \)-th request, and \( Y_{j}^A \) for the revenue.

After multiplying the equation \( k \) with \( \exp \left( -\frac{T-s}{T} \frac{t^2}{2(1-\frac{t^2}{2})} \right) \) and taking expectation, it’s easy to get \( A = \arg \min_{i \in \mathcal{K}} F(A_i - 1B^T) \), according to equation 20.

With the Lemma 4, \( F(A^*P^T) = \min_{i \in \mathcal{K}} F(A_i - 1B^T) \leq F(A_i - 1P^T) \leq F(A_i - 1P^T) \leq \epsilon \). We finish the proof of theorem 3.

3.3 Completely Unknown Distribution

In this section, we consider the completely unknown distribution setting. Under this challenging assumption, we propose an algorithm \( A_1 \), which divides the incoming \( T \) requests into multiple stages. In each stage, instead of the exact optimal value \( W_\beta \), we only use an estimate of \( W(\beta) \), based on the requests of the past stages, to run a similar algorithm with the algorithm \( A_1 \), where \( \beta = \epsilon \) in the algorithm \( A_1 \) (However, we take \( \beta = \tau \) in the section 3.2). Before presenting the entire analysis of the \( A_1 \), we first give a high-level summaries.

High-level Overview:

1. The algorithm \( A_1 \) cut \( T \) requests into \( l \) stages, where \( \epsilon^2l = 1 \) and the \( \epsilon \) is the predefined error parameter. In each stage \( r \in \{0, 1, \ldots, l-1\} \), we first use the information of past requests to build an estimate \( Z_r \) of \( W_r \) and then run a similar algorithm with \( A \) to handle the next \( t_r \) requests via using the estimate \( Z_r \) instead of \( W_r \), where \( t_r = 2^r \epsilon T \). Specially, in stage 0, we use the first \( \epsilon T \) requests to estimate the \( W_r \) and then serve the following requests of stage 0. That is, none of the first \( \epsilon T \) requests are served in the algorithm \( A_1 \). For convenience, we also set \( t_{l-1} = \epsilon T \).

2. Next, we demonstrate the outcomes about the algorithm \( A_1 \). We set \( \delta = \frac{2\epsilon}{T}, \epsilon_{x,r} = \sqrt{\frac{2\ln(\frac{1}{\epsilon_{x,r}})}{t_r}} \).

\( \forall r \in \{-1, 0, 1, \ldots, l-1\} \) and \( \epsilon_{y,r} = \sqrt{\frac{2\ln(\frac{\epsilon_{x,r}}{\epsilon})}{t_r}} \), \( \forall r \in \{0, 1, \ldots, l-1\} \). In each stage \( r \), we first verify that \( 1 - (4 + \frac{2\epsilon}{T}) \epsilon_{x,r} \epsilon_{y,r} \leq Z_r \leq W_r \) w.p. \( 1 - 2\delta \). Then, like the previous analysis, we prove that the cumulative revenue is at least \( \frac{1}{\epsilon_{y,r}} (1 - \epsilon_{y,r}) \) and the consumed amount of every resource \( k \) is between \( \frac{1}{\epsilon_{y,r}} \frac{t_r(L_k + (1 - \epsilon_{y,r}) \gamma W_k)}{1 + \epsilon_{x,r}} \) and \( \frac{t_r L_k}{1 + \epsilon_{x,r}} \) with probability at least 1 - \( \delta \). Finally, with probability at least 1 - 3\( \delta \), the cumulative revenue is at least \( \sum_{i=0}^{t_r-1} \frac{1}{\epsilon_{y,r}} \frac{t_r(L_k + (1 - \epsilon_{y,r}) \gamma W_k)}{1 + \epsilon_{x,r}} \) and the cumulative consumed resource of \( A_1 \) \( k \in \mathcal{K} \) is between \( \sum_{i=0}^{t_r-1} \frac{t_r L_k}{1 + \epsilon_{x,r}} \) and \( \sum_{i=0}^{t_r-1} \frac{t_r L_k}{1 + \epsilon_{x,r}} \). Due to \( \gamma_1 = O(\frac{\epsilon^2}{\ln(\frac{1}{\epsilon})}) \), we could keep \( \sum_{i=0}^{t_r-1} \frac{t_r L_k}{1 + \epsilon_{x,r}} \leq U_k \sum_{i=0}^{t_r-1} \frac{t_r(L_k + (1 - \epsilon_{y,r}) \gamma W_k)}{1 + \epsilon_{x,r}} \) \( \geq L_k \) and \( \sum_{i=0}^{t_r-1} \frac{t_r L_k}{1 + \epsilon_{x,r}} \) \( \geq \) (1 - \( O(\frac{\epsilon^2}{\ln(\frac{1}{\epsilon})}) \))\( W_E \). This completes the high-level overview of the analysis of \( A_1 \).
Before going into the detail, we first present the algorithm \( A_1 \) and the outcomes.

**Algorithm 3 the algorithm \( A_1 \)**

**Input:** \( \epsilon, L_k, U_k, \gamma_1, \alpha \)

**Output:** \( \{x_{ij}\}_{i \in \mathcal{K}, j \in [T]} \)

Set \( l = \log_2(\frac{1}{\epsilon}) \)

Set \( t_r = \epsilon^2 T \) and \( t_{r-1} = \epsilon T \)

Set \( \delta = O\left(\frac{1}{lK}\right) \)

for \( r = 0 \) to \( l-1 \) do

Set

\[
W_r^{-1} = \max \sum_{i \in \mathcal{I}} \sum_{j=1}^{t_r} w_{iR(j)} x_{ij}
\]

s.t.

\[
\frac{t_r-1}{T}(L_k + \epsilon T \bar{a}_k) \leq \sum_{i \in \mathcal{I}} \sum_{j=1}^{t_r} a_{iR(j)} x_{iR(j)} \leq \frac{t_{r-1}}{T} U_k, \forall k \in \mathcal{K}
\]

\[
\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in [1 + t_{r-1}, t_r]
\]

\[
x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in [1 + t_{r-1}, t_r]
\]

Set \( Z_r \)

Set \( \epsilon_{r,r} = \sqrt{\frac{4T \gamma_1 \ln \left( \frac{4T}{\epsilon_{r,r}} \right)}{t_r}} \)

Set \( \epsilon_{y,r} = \sqrt{\frac{4T \ln \left( \frac{4T \gamma_1}{\epsilon_{y,r}} \right)}{t_r}} \)

Initialize \( \phi_{k,0} \)

Initialize \( \varphi_{k,0} \)

Initialize \( \psi_y \)

for \( t=1 \) to \( t_r \) do

If a request \( j \in \mathcal{K} \), then use the following option \( i^* \):

\[
i^* = \arg \min_i \left( \sum_{k \in \mathcal{K}} \phi_{k,t-1} \exp \left( \frac{\ln \left( 1 + \epsilon_{r,r} \right) \gamma_1}{U_k} a_{ijk} \right) + \sum_{k \in \mathcal{K}} \varphi_{k,t-1} \exp \left( \frac{\ln \left( 1 + \epsilon_{r,r} \right) \gamma_1}{U_k} (\bar{a}_k - a_{ijk}) \right) \right)
\]

\[
+ \psi_{t-1} \exp \left( \frac{\ln \left( 1 + \epsilon_{y,r} \right) \gamma_1}{\bar{w}} w_{ij} \right)
\]

\[
X^A_{ik} = a_{i^*,jk}, Y^A_{ik} = w_{i^*}
\]

Update \( \phi_{k,t} = \phi_{k,t-1} \exp \left[ \frac{\ln \left( 1 + \epsilon_{r,r} \right) \gamma_1}{U_k} (X^A_{ik} - \frac{\ln \left( 1 + \epsilon_{r,r} \right) \gamma_1}{U_k} a_{ijk}) \right] \)

Update \( \varphi_{k,t} = \varphi_{k,t-1} \exp \left[ \frac{\ln \left( 1 + \epsilon_{r,r} \right) \gamma_1}{U_k} (X^A_{ik} - \frac{\ln \left( 1 + \epsilon_{r,r} \right) \gamma_1}{U_k} (\bar{a}_k - a_{ijk})) \right] \)

Update \( \psi_{t} = \psi_{t-1} \exp \left[ \frac{\ln \left( 1 + \epsilon_{y,r} \right) \gamma_1}{\bar{w}} (Y^A_{ik} - \frac{\ln \left( 1 + \epsilon_{y,r} \right) \gamma_1}{w_{ij}}) \right] \)

Theorem 4 Under the assumption 1 3, if \( \tau_1 + \epsilon \leq \alpha \) and \( \gamma_1 = \max \left( \frac{\bar{a}_k}{U_k}, \frac{a_{i^*,jk}}{(1-\tau_1)\gamma_1 T \bar{a}_k - \epsilon \bar{w}}, \frac{\bar{w}}{W_{r+1}} \right) = O\left(\frac{1}{\ln \left( \frac{4T}{\epsilon_{r,r}} \right)} \right) \), the algorithm 3 returns a solution satisfying the constraints and achieves an objective value \( (1 - O\left(\frac{1}{\ln \left( \frac{4T}{\epsilon_{r,r}} \right)} \right)) W_E \) w.p \( 1 - \epsilon \), where the predefined parameter \( \epsilon \geq 0 \), \( \tau_1 = \frac{\sqrt{\epsilon}}{1 + \sqrt{\epsilon}} \), \( \delta = \frac{\epsilon}{\alpha l} \) and \( l = \log_2\left( \frac{1}{\epsilon} \right) \).

3.3.1 Concentration of \( Z_r \)

In the first step, we give the relationship between \( Z_r \) and \( W_r \).
Theorem 5 Under assumption 1-3, if $\tau_0 + \epsilon \leq \alpha$ and $\gamma_1 = \max(\bar{\alpha}, \frac{\tilde{\alpha}}{(1 - \tau_0)\tilde{L}_k - \tilde{a}_k}) = O(\frac{\epsilon^2}{\ln(\frac{1}{\epsilon})})$, with probability $1 - 2\delta$, we have

$$\frac{t_rW_r}{T}(1 - (2 + \frac{1}{\alpha})\epsilon\epsilon_{x,r}) \leq W^r \leq \frac{t_rW_r}{T}(1 + (2 + \frac{1}{\alpha})\epsilon\epsilon_{x,r})$$

where the predefined parameter $\epsilon \geq 0$, $\tau_0 = \frac{\sqrt{\delta}}{1 - \sqrt{\delta}}$, $\delta = \frac{\alpha}{\alpha - \epsilon}$ and $l = \log_2(\frac{1}{\epsilon})$.

Proof

RHS: We consider the dual of problem $E(\epsilon)$ as:

$$\min \sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k (L_k + \epsilon T\tilde{a}_k) + \sum_{j \in J} Tp_j \rho_j$$

s.t. $\sum_{k \in K} (\alpha_k - \beta_k) a_{ijk} - w_{ij} + \rho_j \geq 0 \forall i \in \mathcal{I}, j \in \mathcal{J}$

$$\alpha_k, \beta_k, \rho_j \geq 0, k \in K, j \in J$$

(23)

Note: The constraints of LP 23 is different from those in the section 3.1.2. However, if setting the dual variables $\rho_j := Tp_j \rho_j$ in section 3.1.2, we could easily obtain the same constraints of LP 23.

Due to the independent and identically distributed property, we could consider an equivalent definition of $W^r$.

$$W^r = \max \sum_{i \in \mathcal{I}, j \in [t_r]} w_{iR(j)} x_{ij}$$

s.t. $\frac{t_r}{T}(L_k + \epsilon T\tilde{a}_k) \leq \sum_{i \in \mathcal{I}, j \in [t_r]} a_{iR(j)k} x_{iR(j)} \leq \frac{t_r}{T} U_k, \forall k \in K$

$$\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in [t_r]$$

$$x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in [t_r]$$

(24)

where the realization $R$ maps every incoming request to their type.

Similarly, the dual of LP 24:

$$W^r = \min \sum_{k \in K} \alpha_k \frac{t_r}{T} U_k - \sum_{k \in K} \beta_k \frac{t_r}{T} (L_k + \epsilon T\tilde{a}_k) + \sum_{j \in [t_r]} \rho_{R(j)}$$

s.t. $\sum_{k \in K} (\alpha_k - \beta_k) a_{R(j)k} - w_{iR(j)} + \rho_{R(j)} \geq 0 \forall i \in \mathcal{I}, j \in [t_r]$.

$$\alpha_k, \beta_k, \rho_{R(j)} \geq 0, k \in K, j \in [t_r]$$

(25)

We denote the primal and dual optimal solution of $E(\epsilon)$ as $(\alpha_k^*, \beta_k^*, \rho_j^*)$ and $(x_{ij}^*)$. Meanwhile, we present the KKT conditions in the following:

$$\sum_{k \in K} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^* - w_{ij} x_{ij}^* + \rho_j^* x_{ij}^* = 0$$

$$\rho_j^* (\sum_{i \in \mathcal{I}} x_{ij}^* - 1) = 0$$

$$\alpha_k^* (\sum_{ij} Tp_j a_{ijk} x_{ij}^* - U_k) = 0$$

(26)

$$\beta_k^* (L_k + \epsilon T\tilde{a}_k - \sum_{ij} Tp_j a_{ijk} x_{ij}^*) = 0$$
Then, it’s easy to observe the constraints of LP 25 is a subset of those of LP 23 so that (α_k^*, β_k^*, ρ_k^*) is feasible for LP 25.

Hence,

\[
W' = \sum_{k \in K} \alpha_k^* \frac{t_r}{T} U_k - \sum_{k \in K} \beta_k^* \frac{t_r}{T}(L_k + cT\bar{a}_k) + \sum_{j \in [\tau]} \rho_{ik}^*(j)
\]

\[ \leq \sum_{k \in K} \alpha_k^* \left( \frac{t_r}{n} U_k - \sum_{j \in [\tau], i \in I} a_{iR(j)k} x_{ijk}^* \right) + \sum_{k \in K} \beta_k^* \left( \sum_{j \in [\tau], i \in I} a_{iR(j)k} x_{ijk}^* - \frac{t_r}{T}(L_k + cT\bar{a}_k) \right) \]

\[ + \sum_{j \in [\tau]} (\rho_{ik}^*(j) + \sum_{i \in I, k \in K} (\alpha_k^* - \beta_k^*) a_{iR(j)k} x_{ijk}^*) \]

We have divided the equation 26 into three parts. Next, we will derive the relationship between \( W' \) and \( W_r \) via controlling these three parts.

For part (1): From KKT conditions, if \( \sum_{i \in I, j \in J} T_{p_j} a_{ijk} x_{ij}^* \leq U_k \), then \( \alpha_k^* = 0 \). Thus, we only consider the resource \( k \) making \( \sum_{i \in I, j \in J} T_{p_j} a_{ijk} x_{ij}^* = U_k \). Due to the previous equality, it’s easy to observe \( E(\sum_{i \in I} a_{iR(j)k} x_{ijk}^*) = \frac{U_k}{T} \) \( \forall j \in [\tau] \). Thus, according to the Bernstein inequalities (Lemma 2), we have

\[
P\left( \sum_{j \in [\tau], i \in I} a_{iR(j)k} x_{ijk}^* \right) \leq \left( 1 - \epsilon_{x,r} \right) \frac{t_r}{T} U_k
\]

\[ \leq \exp\left(-\frac{\epsilon_{x,r}^2}{2(1 - \frac{t_r}{T})^2 n U_k^2} \right)
\]

where \( \sigma = \sqrt{\text{Var}(\sum_{i \in I, j \in J} a_{iR(j)k} x_{ijk}^*)} \).

Similarly, for part (2), we only consider the \( k \) making \( \sum_{i \in I, j \in J} T_{p_j} a_{ijk} x_{ij}^* = L_k + cT\bar{a}_k \). Thus, following similar proof of equation 28,

\[
P\left( \sum_{j \in [\tau], i \in I} (\bar{a}_k - a_{iR(j)k} x_{ijk}^*) \right) \leq \left( 1 - \epsilon_{x,r} \right) \frac{t_r}{T} (L_k - cT\bar{a}_k)
\]

\[ \leq \exp\left(-\frac{\epsilon_{x,r}^2}{2(1 - \frac{t_r}{T})^2 n U_k^2} \right)
\]

As for the final part (3), from KKT conditions, it’s easy to verify \( \sum_{i \in I, k \in K} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^* \leq \sum_{i \in I} w_{ij} x_{ij}^* + \rho_j^* (\sum_{i \in I} x_{ij}^*) = \sum_{i \in I, k \in K} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^* - \sum_{i \in I} w_{ij} x_{ij}^* + \rho_j^* = 0 \), so \( \sum_{i \in I, k \in K} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^* \leq \sum_{i \in I} w_{ij} x_{ij}^* \leq [0, \bar{w}] \). Moreover, \( E(\rho_{R(j)}^* + \sum_{i \in I, k \in K} (\alpha_k^* - \beta_k^*) a_{iR(j)k} x_{ijk}^*) = \frac{t_r}{T} W_r (1 + \epsilon_{x,r}) \). Therefore, following the same techniques, we have

\[
P\left( \sum_{j \in [\tau]} (\rho_{R(j)}^* + \sum_{i \in I, k \in K} (\alpha_k^* - \beta_k^*) a_{iR(j)k} x_{ijk}^*) \right) \geq \frac{t_r}{T} W_r (1 + \epsilon_{x,r}) \geq \exp\left(-\frac{\epsilon_{x,r}^2}{2(1 - \frac{t_r}{T})^2 n U_k^2} \right)
\]
Finally, when $\gamma_1 = O\left(\frac{e^2}{\log T}\right)$, with probability at least $1 - \delta$,

\[
\sum_{j \in [t], i \in \mathcal{I}} a_{iR(j)} x_{iR(j)}^* \geq (1 - \epsilon_{x,r}) \frac{t_r}{T} U_k
\]

\[
\sum_{j \in [t], i \in \mathcal{I}} \left(\bar{a}_k - a_{iR(j)} x_{iR(j)}^*\right) \geq (1 - \epsilon_{x,r}) \frac{t_r}{T} ((1 - \epsilon) T \bar{a}_k - L_k)
\]

\[
\sum_{j \in [t], i \in \mathcal{I}} (\rho_{iR(j)}^* + \sum_{i \in \mathcal{I}, k \in \mathcal{K}} (\bar{\alpha}_k^* - \beta_k^*) a_{iR(j)} x_{iR(j)}^*) \leq \frac{t_r}{T} W_{\epsilon}(1 + \epsilon_{x,r})
\]

Therefore, with probability at least $1 - \delta$,

\[
1 + 2 + 3
\]

\[
= \sum_{k \in \mathcal{K}} \alpha_k^* \frac{t_r}{T} U_k - \sum_{j \in [t], i \in \mathcal{I}} a_{iR(j)} x_{iR(j)}^* + \sum_{k \in \mathcal{K}} \beta_k^* \frac{t_r}{T} ((1 - \epsilon) T \bar{a}_k - L_k) - \sum_{j \in [t], i \in \mathcal{I}} (\bar{a}_k - a_{iR(j)} x_{iR(j)}^*) (1 - \epsilon_{x,r}) \frac{t_r}{T} U_k + \sum_{k \in \mathcal{K}} \beta_k^* \frac{t_r}{T} ((1 - \epsilon) T \bar{a}_k - L_k)) + (3)
\]

\[
\leq \epsilon_{x,r} \sum_{k \in \mathcal{K}} (\alpha_k^* \frac{t_r}{T} U_k + \beta_k^* \frac{t_r}{T} ((1 - \epsilon) T \bar{a}_k - L_k))) + (3)
\]

\[
\leq \epsilon_{x,r} \sum_{k \in \mathcal{K}} (\alpha_k^* \frac{t_r}{T} U_k + \beta_k^* \frac{t_r}{T} ((1 - \epsilon) T \bar{a}_k - L_k))) + \frac{t_r}{T} W_{\epsilon}(1 + \epsilon_{x,r})
\]

\[
\leq \epsilon_{x,r} \left(\frac{t_r}{T} W_{\epsilon} + \sum_{k \in \mathcal{K}} \beta_k^* t_r \bar{a}_k\right) + \frac{t_r}{T} W_{\epsilon}(1 + \epsilon_{x,r})
\]

\[
\leq (1 + \frac{1}{\alpha - \epsilon}) \epsilon_{x,r} \frac{t_r}{T} W_{\epsilon}
\]

where the first inequality from $\sum_{j \in [t], i \in \mathcal{I}} a_{iR(j)} x_{iR(j)}^* \geq (1 - \epsilon_{x,r}) \frac{t_r}{T} U_k$ and $\sum_{j \in [t], i \in \mathcal{I}} (\bar{a}_k - a_{iR(j)} x_{iR(j)}^*) \geq (1 - \epsilon_{x,r}) \frac{t_r}{T} ((1 - \epsilon) T \bar{a}_k - L_k)$; the second inequality from $\sum_{j \in [t], i \in \mathcal{I}} (\rho_{iR(j)}^* + \sum_{i \in \mathcal{I}, k \in \mathcal{K}} (\bar{\alpha}_k^* - \beta_k^*) a_{iR(j)} x_{iR(j)}^*) \leq \frac{t_r}{T} W_{\epsilon}(1 + \epsilon_{x,r})$; the third inequality from $W_{\epsilon} = \sum_{k \in \mathcal{K}} \alpha_k^* \frac{t_r}{T} U_k - \sum_{k \in \mathcal{K}} \beta_k^* ((L_k + \epsilon T \bar{a}_k) + \sum_{j \in \mathcal{J}} (T \bar{a}_j) \geq \sum_{k \in \mathcal{K}} \bar{\alpha}_k \frac{t_r}{T} U_k - \sum_{k \in \mathcal{K}} \beta_k^* (\epsilon T \bar{a}_k + L_k)$; the final inequality from the section 3.1.2, $\sum_{k \in \mathcal{K}} \beta_k^* T \bar{a}_k \leq \frac{1}{\alpha - \epsilon} W_{\epsilon}$.

**LHS:** In order to derive the lower bound of $W^r$, we consider the problem $E(\epsilon + \epsilon_{x,r})$. After that, we imitate the algorithm $P_i$ (algorithm 3.1) to design $P_i$ assigning request $j \in \mathcal{J}$ to channel $i \in \mathcal{I}$ with probability $(1 - \epsilon_{x,r}) x(\epsilon + \epsilon_{x,r}) r_j^*$. From the similar techniques in section 3.1, we could prove that, with probability $1 - \delta$,

\[
P\left(\sum_{j=1}^{t_r} X_{jk} \geq \frac{t_r}{T} U_k\right) \leq \exp\left\{-\frac{t_r \epsilon_{x,r}^2}{2(1 - \frac{\epsilon}{\alpha - \epsilon}) T \bar{a}_k / U_k}\right\}
\]

\[
P\left(\sum_{j=1}^{t_r} Y_{jk} \leq (1 - \frac{1}{\alpha - \epsilon}) \frac{t_r}{T} W_{\epsilon}\right) \leq \exp\left\{-\frac{t_r \epsilon_{x,r}^2}{2(1 - \frac{\epsilon}{\alpha - \epsilon}) T \bar{a}_k / W_{\epsilon}}\right\}
\]

\[
P\left(\sum_{j=1}^{t_r} X_{jk} \leq \frac{t_r}{T} L_k\right) \leq \exp\left\{-\frac{t_r \epsilon_{x,r}^2}{2(1 - \frac{\epsilon}{\alpha - \epsilon}) T (1 - \tau_1) T \bar{a}_k - L_k}\right\}
\]

where the second inequality from the truth that the problem $E(\epsilon)$ is satisfied with the strong feasible condition with the measure parameter $\alpha - \epsilon$ and $W_{\epsilon_{x,r}} \geq W_{\epsilon_{x,r} + \epsilon} \geq W_{\epsilon_{x,r} + \epsilon}$; the final inequality from $(1 - \frac{\epsilon}{\alpha - \epsilon}) T \bar{a}_k - L_k \geq (1 - \frac{\epsilon_{x,r}}{1 - \epsilon_{x,r}}) T \bar{a}_k - L_k \geq (1 - \epsilon - \tau_1) T \bar{a}_k - L_k$. 

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Finally,
\[
P((\sum_{j=1}^{t_r} Y_j^\beta_t) \leq (1 - (2 + \frac{1}{\alpha - \epsilon})\epsilon_{x,r}) \frac{t_r}{T} W_r) + \sum_{k \in K} P(\sum_{j=1}^{t_r} X_{jk}^\beta_t \notin \left[ \frac{t_r}{T} L_k, \frac{t_r}{T} U_k \right]) \leq (2K + 1) \exp\{-\frac{t_r \epsilon_{x,r}^2}{4T\gamma}\} \leq \delta. \tag{34}
\]

Therefore, we know \( W_r \geq (1 - (2 + \frac{1}{\alpha - \epsilon})\epsilon_{x,r}) \frac{t_r}{T} W_r \), w.p. 1 - \( \delta \). 

### 3.3.2 Proof without Knowledge of Distribution

Now, we prove that, at each stage \( r \), the algorithm \( A_1 \) return a solution whose cumulative revenue is at least \( \frac{t_r Z_r}{T}(1 - \epsilon_{y,r}) \). Meanwhile, the consumed amount of every resource \( k \) is between \( \frac{t_r}{T}(L_k + (\epsilon - \frac{t_r \epsilon_{x,r}}{L_k})T\bar{a}_k)(1 + \epsilon_{x,r}) \) and \( \frac{t_r}{T}(1 + \epsilon_{x,r}) \) with probability at least \( 1 - \delta \).

**First step:** We design the similar algorithm \( \hat{P}_2 \): allocating request \( j \) to channel \( i \) with \( x(\epsilon_{x,r})^\gamma \).

**Lemma 5** In the \( r \)-th stage, if \( \gamma = O\left(\frac{1}{\ln(\frac{1}{\delta})}\right) \), the algorithm \( \hat{P}_2 \) returns a solution satisfying the
\[
\sum_{j=1}^{t_r+1} Y_j \geq (1 - \epsilon_{y,r}) \frac{t_r}{T} Z_r, \quad \text{and} \quad \sum_{j=1}^{t_r+1} X_{jk} \in \left[ \frac{t_r}{T}((1 + \epsilon_{x,r})L_k + (\epsilon + \epsilon_{x,r}) \bar{a}_k), \frac{t_r}{T}(1 + \epsilon_{x,r})U_r \right]
\]

w.p \( 1 - 3\delta \).

**Proof** At first,
\[
P(\sum_{j=1}^{t_r+1} X_{jk}^\beta_t \geq (1 + \epsilon_{x,r}) \frac{t_r}{T} U_k) \leq \exp\{-\frac{t_r \epsilon_{x,r}^2}{2(1 + \frac{t_r \epsilon_{x,r}}{3}) T U_k}\} \leq \exp\{-\frac{t_r \epsilon_{x,r}^2}{2(1 + \frac{t_r \epsilon_{x,r}}{3}) \gamma}\} \leq \delta \frac{2K + 1}{2K + 1}
\]

For lower bound constraints,
\[
P(\sum_{j=1}^{t_r+1} (\bar{a}_k - X_{jk}^\beta_t) \geq (1 + \epsilon_{x,r}) \frac{t_r}{T}((1 - \epsilon)T\bar{a}_k - L_k)) \leq \exp\{-\frac{t_r \epsilon_{x,r}^2}{2(1 + \frac{t_r \epsilon_{x,r}}{3}) (1 - \epsilon)T\bar{a}_k - L_k}\} \leq \exp\{-\frac{t_r \epsilon_{x,r}^2}{2(1 + \frac{t_r \epsilon_{x,r}}{3}) \gamma}\} \leq \delta \frac{2K + 1}{2K + 1}
\]

where if \( \sum_{j=1}^{t_r+1} (\bar{a}_k - X_{jk}^\beta_t) \leq (1 + \epsilon_{x,r}) \frac{t_r}{T}((1 - \epsilon)T\bar{a}_k - L_k) \), we could obtain \( \sum_{j=1}^{t_r+1} X_{jk}^\beta_t \geq \frac{t_r}{T}(L_k + (\epsilon - \frac{t_r \epsilon_{x,r}}{1 + \epsilon_{x,r}})T\bar{a}_k)(1 + \epsilon_{x,r}) \).
For accumulative revenue in $r$-th stage, we take a slightly different techniques to derive :

\[
P( \sum_{j=t_{r+1}}^{t_r} Y_{j}^{T} \leq (1 - \epsilon_{y,r}) \frac{t_r}{T} Z_r)
\]

\[
= P( \sum_{j=t_{r+1}}^{t_r} \mathbb{E}(Y_{j}^{T}) - Y_{j}) \geq \frac{t_r}{T}(\mathbb{E}(Y_{j}^{T}) - (1 - \epsilon_{y,r}) Z_r))
\]

\[
\leq \exp(- \frac{(\mathbb{E}(Y_{j}^{T}) - (1 - \epsilon_{y,r}) Z_r)^2 t_r}{T(2T\sigma_1^2 + \frac{w}{3} \mathbb{E}(Y_{j}^{T}) - (1 - \epsilon_{y,r}) Z_r)})
\]

\[
= \exp(- \frac{t_r(\epsilon_{y,r} \mathbb{E}(Y_{j}))^2}{2T\mathbb{E}(Y_{j})w + \frac{2}{3} \epsilon_{y,r} w \mathbb{E}(Y_{j}))})
\]

\[
\leq \exp(- \frac{t_r \epsilon_{y,r}}{2(1 + \frac{w}{3})T \frac{w}{Z_r}})
\]

\[
\leq \frac{\delta}{2K + 1}
\]

where the second inequality follows $\sigma_1^2 = \text{Var}(Y_{j}^{T}) \leq w_{max} \mathbb{E}(Y_{j})$ and $\mathbb{E}(Y_{j}^{T}) - (1 - \epsilon_{y,r}) Z_r) \geq \epsilon_{y,r} T \mathbb{E}(Y_{j}^{T})$, the third inequality follows $Z_r \leq T \mathbb{E}(Y_{j})$, and the final from $\epsilon_{y,r} = \sqrt{\frac{4T \ln(K)}{Z_r t_r}}$.

Therefore, $P(\sum_{j=t_{r+1}}^{t_r} Y_{j}^{T} \leq (1 - \epsilon_{y,r}) \frac{t_r}{T} Z_r) + \sum_{k \in K} P(\sum_{j=t_{r+1}}^{t_r} X_{jk} \notin [\frac{t_r}{T}(1 + \epsilon_{x,r})L_k - (\epsilon(1 + \epsilon_{x,r}) - \epsilon_{x,r})T \bar{a}_k)] \frac{t_{r+1}}{T} U_r) \leq \epsilon. \quad \blacksquare$
Second Step: Imitating the methods in the section 3.2, we first derive the moment generating function for the event that the consumed resource \( k \in K \) is larger than \((1 + \epsilon_{x,r})\frac{T}{T} U_k\), i.e.,

\[
P(\sum_{j=1+\epsilon_r}^{s+\epsilon_r} X_{jk} + \sum_{j=s+\epsilon_r}^{t_r+\epsilon_r} X_{jk}^P \geq \frac{(1 + \epsilon_{x,r})T}{T} U_k)
\]

\[
\leq \min_{t>0} E(\exp(t(\sum_{j=1+\epsilon_r}^{s+\epsilon_r} X_{jk}^P + \sum_{j=s+\epsilon_r}^{t_r+\epsilon_r} X_{jk}^P - \frac{(1 + \epsilon_{x,r})T}{T} U_k)))
\]

\[
\leq \min_{t>0} E(\exp(t(\sum_{j=1+\epsilon_r}^{s+\epsilon_r} X_{jk}^P - \frac{(1 + \epsilon_{x,r})s U_k}{T} + t(\sum_{j=s+\epsilon_r}^{t_r+\epsilon_r} X_{jk}^P - \frac{(1 + \epsilon_{x,r})(T - s) U_k}{T})))
\]

\[
\leq \min_{t>0} E(\exp(t\sum_{j=1+\epsilon_r}^{t_r+\epsilon_r} (X_{jk}^P - E(X_{jk}^P))) + \frac{t_r - s}{T} T E(X_{jk}^P - (1 + \epsilon_{x,r})U_k)))
\]

\[
\leq \min_{t>0} E(\exp(t(\sum_{j=1+\epsilon_r}^{s+\epsilon_r} X_{jk}^P - \frac{(1 + \epsilon_{x,r})s U_k}{T} + t(\sum_{j=s+\epsilon_r}^{t_r+\epsilon_r} X_{jk}^P - \frac{(1 + \epsilon_{x,r})(T - s) U_k}{T})))
\]

\[
\leq \min_{t>0} E(\exp(t((1 + \epsilon_{x,r})s U_k - (1 + \epsilon_{x,r})s U_k)))
\]

where \( \Delta^s(t) = \exp(t(\sum_{j=1+\epsilon_r}^{s+\epsilon_r} X_{jk}^P - \frac{(1 + \epsilon_{x,r})s U_k}{T})). \)

For lower bound, we set \( Z_{jk}^P = \tilde{a}_k - X_{jk}^P \),

\[
P(\sum_{j=1+\epsilon_r}^{s+\epsilon_r} Z_{jk} + \sum_{j=s+\epsilon_r}^{t_r+\epsilon_r} Z_{jk}^P \geq \frac{(1 + \epsilon_{x,r})T \tilde{a}_k - L_k}{T})
\]

\[
\leq \min_{t>0} E(\exp(t(\sum_{j=1+\epsilon_r}^{s+\epsilon_r} Z_{jk}^P + \sum_{j=s+\epsilon_r}^{t_r+\epsilon_r} Z_{jk} - \frac{(1 + \epsilon_{x,r})(T \tilde{a}_k - L_k)}{T})))
\]

\[
\leq \min_{t>0} E(\exp(t(\sum_{j=1+\epsilon_r}^{s+\epsilon_r} Z_{jk}^P - \frac{(1 + \epsilon_{x,r})s (T \tilde{a}_k - L_k)}{T}) + t(\sum_{j=s+\epsilon_r}^{t_r+\epsilon_r} Z_{jk}^P - \frac{(1 + \epsilon_{x,r})(T - s) (T \tilde{a}_k - L_k)}{T})))
\]

\[
\leq \min_{t>0} E(\exp(t\sum_{j=1+\epsilon_r}^{t_r+\epsilon_r} (Z_{jk}^P - E(Z_{jk}^P))) + \frac{t_r - s}{T} T E(Z_{jk}^P - (1 + \epsilon_{x,r})(T \tilde{a}_k - L_k)))
\]

\[
\leq \min_{t>0} E(\exp(\frac{(t_r - s)(T \tilde{a}_k - L_k)}{T}((1 + \eta) \ln(1 + \eta) - \eta)))
\]

\[
\leq \min_{t>0} E(\exp(\frac{t_r - s}{T} 2(1 + \frac{a_{x,r}}{2})))
\]

\[
\leq \min_{t>0} E(\exp(\frac{t_r - s}{T} 4\gamma))
\]

(36)
where the third inequality follows from $\Delta_3^s(t) = \exp(t(\sum_{j=1+tr}^{s+tr} Z_{jk}^2 - (1+\epsilon_{x,r})s(1-\epsilon)T\bar{a}_k - L_k)))$; the fifth inequality from $T\mathbb{E}(Z_{jk}^2) \leq (1-\epsilon)T\bar{a}_k - L_k$ and $\sigma^2 = \text{Var}(Z_{jk}^2) \leq \bar{a}_k\mathbb{E}(Z_{jk}^2)$.

In final, for revenue, we have

\[
P(\sum_{j=1+tr}^{s+tr} Y_j + \sum_{j=s+1+tr}^{tr+1} Y_j^2 \leq (1-\epsilon_{y,r})\frac{tr}{T}Z_r)
\]

\[\leq \min_{t>0} \mathbb{E} \left( \exp(t(1-\epsilon_{y,r})\frac{tr}{T}Z_r - \sum_{j=1+tr}^{s+tr} Y_j^2 - \sum_{j=s+1+tr}^{tr+1} Y_j^2) \right)
\]

\[\leq \min_{t>0} \mathbb{E} \left( \exp(t(\frac{s}{T}(1-\epsilon_{y,r})Z_r - \sum_{j=1+tr}^{s+tr} Y_j^2) + t(\frac{tr-s}{T}(1-\epsilon_{y,r})Z_r - \sum_{j=s+1+tr}^{tr+1} Y_j^2)) \right)
\]

\[\leq \min_{t>0} \mathbb{E}(\Delta_3^s(t)\exp(t \sum_{j=s+1+tr}^{tr+1} (\mathbb{E}(Y_j^2) - Y_j^2) + \frac{tr-s}{T}(1-\epsilon_{y,r})Z_r - n\mathbb{E}(Y_j^2)))
\]

\[\leq \min_{t>0} \mathbb{E}(\Delta_3^s(t)\exp((t_r-s)\frac{\sigma_1^2}{\hat{w}}(e^{\hat{w}} - 1 - \hat{w}) - (t_r-s)\epsilon_{y,r}\mathbb{E}(Y_j^2)))
\]

\[\leq \mathbb{E}(\Delta_3^s(\frac{\ln(1+\epsilon_{y,r})}{\hat{w}})\exp(-\frac{(t_r-s)\mathbb{E}(Y_j^2)}{\hat{w}}((1+\eta)\ln(1+\eta) - \eta)))
\]

\[\leq \mathbb{E}(\Delta_3^s(\frac{\ln(1+\epsilon_{y,r})}{\hat{w}})\exp(-\frac{(t_r-s)\epsilon^2_{y,r}Z_r}{T\hat{w}}))
\]

where the third inequality follows from $\Delta_3^s(t) = \exp(t(\frac{s}{T}(1-\epsilon_{y,r})Z_r - \sum_{j=1+tr}^{s+tr} Y_j^2))$; the fourth inequality from $Z_r \leq T\mathbb{E}(Y_j^2)$; the fifth inequality from $\sigma_1^2 = \text{Var}(Y_j^2) \leq \hat{w}\mathbb{E}(Y_j^2)$.

With the equation 35 37, we set:

\[
\mathcal{F}_r(A_1 A_2 \ldots A_s \hat{\mathcal{P}}_2^{s-r}) = \mathbb{E}(\Delta_1^s(\frac{\ln(1+\epsilon_{x,r})}{\gamma U_k})\exp(-\frac{t_r-s\epsilon^2}{T^2 4\gamma}) + \Delta_1^s(\frac{\ln(1+\epsilon_{x,r})}{\gamma U_k})\exp(-\frac{t_r-s\epsilon^2}{T^2 4\gamma})
\]

\[+ \Delta_2^s(\frac{\ln(1+\epsilon_{y,r})}{\hat{w}})\exp(-\frac{(t_r-s)\epsilon^2_{y,r}Z_r}{T\hat{w}}))
\]

\[= \min_{t>0} \mathbb{E}(\Delta_3^s(\frac{\ln(1+\epsilon_{y,r})}{\hat{w}})\exp(-\frac{(t_r-s)\epsilon^2_{y,r}Z_r}{T\hat{w}}))
\]

It’s easy to get $\mathcal{F}_r(\hat{\mathcal{P}}_2^{sr}) \leq \delta$, if $\gamma_1 = O(\frac{\epsilon^2}{\ln(1+\epsilon_{y,r})})$. Using the same technique in Section 3.2, we also can derive that

Lemma 6 $A_1 = \arg \min_B \mathcal{F}_r(A_1^{s-1} B \hat{\mathcal{P}}_2^{s-r})$

Therefore, during the stage $r$, the algorithm $A_1$ return a solution satisfying:

\[
\sum_{j=t+tr}^{tr+1} Y_j \geq (1-\epsilon_{y,r})\frac{tr}{T}Z_r
\]

\[
\sum_{j=t+1+tr}^{t+1} X_{jk} \leq \frac{(1+\epsilon_{x,r})t_r}{T}U_r
\]

\[
\sum_{j=t+tr}^{t+1} (\bar{a}_k - X_{jk}) \leq \frac{(1+\epsilon_{x,r})t_r}{T}((1-\epsilon)T\bar{a}_k - L_k)
\]

with probability at least $1-\delta$. 

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Finally, with previous outcomes, the following result can be proved:

\[
\sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_{r+1}} X_{jk} \leq \sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_{r+1}} X_{jk} \leq \sum_{r=0}^{l-1} (1 + \epsilon_{x,r}) \frac{t_r}{T} U_r \leq U_k
\]

\[
(1 - \epsilon) Ta_k - \sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_{r+1}} X_{jk} = \sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_{r+1}} (\bar{a}_k - X_{jk}) \leq \sum_{r=0}^{l-1} (1 + \epsilon_{x,r}) \frac{t_r}{T} ((1 - \epsilon) Ta_k - L_k) \leq (1 - \epsilon) Ta_k - L_k
\]  
\[\text{(39)}\]

\[
\sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_{r+1}} Y_j \geq \sum_{r=0}^{l-1} (1 - \epsilon_{y,r}) \frac{Z_r t_r}{T} \geq \sum_{r=0}^{l-1} (1 - \epsilon_{y,r}) \frac{t_r (1 - (4 + \frac{1}{2 \alpha - r}) \epsilon_{x,r}^{-1})}{T} W_r
\]

\[
geq \sum_{r=0}^{l-1} (1 - \epsilon_{y,r}) \frac{t_r (1 - (4 + \frac{1}{2 \alpha - r}) \epsilon_{x,r}^{-1})}{T} W_r \geq (1 - O(\frac{\epsilon}{\alpha - \epsilon})) W_E
\]  
\[\text{(40)}\]

We finish the verification of theorem 4.

4. Discussion on the Strong Feasible Condition

In this section, we investigate the strong feasible condition, i.e., Assumption 3. Before going into the detail, we first explain why we need this assumption and what role this condition plays in the previous proofs. After that, we consider a practical question "If the problems are satisfied with the strong feasible condition, How to estimate the \(\alpha^m\)". On the other hand, how about the infeasible settings? If viewing the lower bounds as soft thresholds, whether we could find a variant of the algorithm \(A_1\), which minimally violates the lower bound constraints. We will answer these questions in the following section.

4.1 The Importance of Strong Feasible Condition

First, we characterize the settings without satisfying the strong feasible condition.

**Theorem 6** Under the Assumption 1-2, for the fixed \(\{L_k, U_k, \mathcal{P}\}\), if we could not find an \(\alpha > 0\) making the problem \(E(\alpha)\) in (4) feasible, then

1. the problem \(E(0)\) is infeasible.

2. Otherwise, for any feasible solution \(\{x_{ij}\}_{i \in \mathcal{I}, j \in \mathcal{J}}\) of \(E(0)\), there exist a number \(k \in \mathcal{K}\) making \(\sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j a_{ijk} x_{ij} = L_k\).

**Proof**

1. When LP 2 is infeasible, it’s easy to verify the infeasibility of equation 3 so that we could not find an \(\alpha > 0\).

2. When LP 2 is feasible, if there exist a solution \(\{x_{ij}\}_{i \in \mathcal{I}, j \in \mathcal{J}}\) making \(\sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j a_{ijk} x_{ij} > L_k\), \(\forall k \in \mathcal{K}\), we could set the \(\alpha_1 = \min_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j a_{ijk} x_{ij} - L_k > 0\). It’s easy to verify the feasibility of the problem \(E(\alpha_1)\). By contradiction, there will be some number \(k \in \mathcal{K}\) making \(\sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j a_{ijk} x_{ij} = L_k\).

Therefore, without the strong feasible condition, we have to deal with these two settings in the Theorem 6. It seems impossible to design efficient algorithms for the infeasible problem \(E(0)\).
As for the second setting, like the section 3.1, we also assume the knowledge of the optimal value $\{x(0)_{ij}\}_{i \in I, j \in J}$ of $E(0)$ and design the similar algorithm $\tilde{P}_0$ allocating the request $j \in J$ to channel $i \in I$ with probability $x(0)_{ij}$. However, due to the existence of a $k \in K$ making $\sum_{i \in I, j \in J} T p_j a_{ijk} x(0)_{ij} = L_k$, it’s impossible to control the probability of this bad event that $\sum_{j = 1}^{T} X_{jk} \leq L_k (= E(\sum_{j = 1}^{T} X_{jk}))$ without more information about the distribution $P$. For instance, where $X \sim U[0, 1]$, $P(X \leq E(X)) = \frac{1}{2}$.

Thus, with the strong feasible condition, we could enlarge the lower bound $L_k$ to design an algorithm keeping the final decisions feasible with high probability. Moreover, from the section 3.1.2, the strong feasible condition can build up connections between the optimal value $W_E$ and the optimal value of the problems enlarging the lower bound $L_k$. Finally, from the strong feasible condition, we also could control the dual sum $\sum_{k \in K} \beta_k T \bar{a}_k$ via the sum $\sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} \rho_j$, which is a key technique in verifying the guarantee of estimate of $W_\epsilon$ in section 3.3.1.

### 4.2 Estimate the measure of feasibility

In practice, we are not able to know $\alpha$ ahead. This parameter measures the feasibility of original problem, and plays a central role in the algorithm $A_1$. Moreover, as we demonstrate, the lower bound constraints usually come from the unintentional governmental regulations or contracts, which sometimes cause the infeasibility of the original problem. Thus, distinguishing the infeasible settings is in urgent need. Before present answers, we introduce the concept of the optimal measure of feasibility ($\alpha^*$), i.e.,

$$\alpha^* = \max \alpha$$

subject to

$$L_k + \alpha T \bar{a}_k \leq \sum_{i \in I, j \in J} T p_j a_{ijk} x_{ij} \leq U_k, \forall k \in K$$

$$\sum_{i \in I} x_{ij} \leq 1, \forall j \in J$$

$$x_{ij} \geq 0, \forall i \in I, j \in J$$

(41)

**Theorem 7** From the definition of $\alpha^*$, we have

1. When $\alpha^* \geq 0$, the problem $E(0)$ is feasible. Otherwise, it’s infeasible.

2. When $\alpha^* > 0$, the problem $E(\alpha)$ is feasible, where $0 < \alpha \leq \alpha^*$. Therefore, the problem $E(0)$ is satisfied with the strong feasible condition with parameter $\alpha$.

From Theorem 7, with the knowledge of $\alpha^*$, we could easily check the feasibility of $E(0)$. Meanwhile, we also obtain the parameter $\alpha$ if the optimal measure of feasibility $\alpha^* > 0$. Next, we demonstrate how to estimate the $\alpha^*$ via data (Algorithm 4).

**Theorem 8** Under the assumption 1-2, if $\gamma_2 = O(\frac{\epsilon^2}{\ln(\frac{1}{\delta})}) \leq \max(\frac{\bar{a}_k}{U_k}, \frac{\bar{a}_k}{(1-\alpha^*)T \bar{a}_k - L_k})$, Algorithm 4 with $t_r$ i.i.d. requests outputs $\hat{\alpha}_r$ such that

$$\hat{\alpha}_r \in [\alpha^* - \frac{\epsilon_{x,r}}{1 - \epsilon_{x,r}}, \alpha^* + \epsilon_{x,r}]$$

w.p. $1 - 2\delta$, where $\epsilon_{x,r} = \sqrt{\frac{4 \gamma_2 T \ln(\frac{1}{\delta})}{r}}$.

**Proof**
Algorithm 4: the algorithm $M$

**Input:** \{\(a_{ijk}\)\}_{i \in I, j \in J, k \in K}, \(R : [t_r] \rightarrow J\), \(0 \leq r \leq l - 1\) and \(\delta > 0\)

**Output:** \(\hat{\alpha}_r - \epsilon_{x,r}\)

Set \(\epsilon_{x,r} = \sqrt{4 \gamma^2 \ln(2)/t_r}\)

Set \(\hat{\alpha}_r = \max \alpha\)

\[
\begin{align*}
\text{s.t.} \; \frac{t_r}{T} (L_k - \alpha T \bar{a}_k) &\leq \sum_{i \in I, j \in [t_r]} a_{iR(j)k} x_{iR(j)} \leq \frac{t_r}{T} U_k, \forall k \in K \\
\sum_{i \in I} x_{iR(j)} &\leq 1, \forall j \in [t_r] \\
x_{iR(j)} &\geq 0, \forall i \in I, j \in [t_r]
\end{align*}
\]

**RHS:** The realization \(R : [t_r] \rightarrow J\) maps the incoming \(t_r\) requests into their types. Taking the same techniques in section 3.3.1. First, the dual of LP 41 is

\[
\begin{align*}
\min_{k \in K} \sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} T p_j \rho_j \\
\text{s.t.} \sum_{k \in K} (\alpha_k - \beta_k) a_{ijk} + \rho_j \geq 0 \forall i \in I, j \in J, k \in K
\end{align*}
\]

where we denote the primal and dual optimal solution of LP 41 and LP 43 as \((\alpha_k^*, \beta_k^*, \rho_j^*)\) and \((x_{ij}^*, \alpha^*)\) respectively.

According to the KKT conditions, we have that

\[
\begin{align*}
\sum_{k \in K} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^* + \rho_j^* x_{ij}^* &= 0 \\
\sum_{k \in K} T \bar{a}_k \beta_k^* &= 1 \\
\rho_j^* (\sum_{i \in I} x_{ij}^* - 1) &= 0 \\
\alpha_k^* (\sum_{ij} T p_j a_{ijk} x_{ij}^* - U_k) &= 0 \\
\beta_k^* (L_k + \alpha^* T \bar{a}_k - \sum_{ij} T p_j a_{ijk} x_{ij}^*) &= 0
\end{align*}
\]

Similarly, the dual of LP 42 is

\[
\begin{align*}
\min_{k \in K} \sum_{k \in K} \alpha_k t_r U_k - \sum_{k \in K} \beta_k t_r L_k + \sum_{j \in [t_r]} \rho_{R(j)} \\
\text{s.t.} \sum_{k \in K} (\alpha_k - \beta_k) a_{iR(j)k} + \rho_{R(j)} \geq 0 \forall i \in I, j \in [t_r] \\
\sum_{k \in K} t_r \bar{a}_k \beta_k &= 1 \\
\alpha_k, \beta_k, \rho_{R(j)} &\geq 0, k \in K, j \in [t_r]
\end{align*}
\]
Apparently, the solution \((\frac{T\alpha_k}{t_r}, \frac{T\beta_k}{t_r}, \frac{T\rho^*_k}{t_r})\) is feasible for LP 45. Then we have that

\[
\hat{\alpha}_r = \frac{T}{t_r} \left( \sum_{k \in K} \alpha^*_k \frac{t_r}{T} U_k - \sum_{k \in K} \beta^*_k \frac{t_r}{T} L_k + \sum_{j \in \{r\}} \rho^*_R(j) \right)
\]

\[
= \frac{T}{t_r} \left( \sum_{k \in K} \alpha^*_k \frac{t_r}{T} U_k - \sum_{j \in \{r\}, i \in I} a_{iR(j)k} x^*_{iR(j)} \right) + \sum_{k \in K} \beta^*_k \left( \sum_{j \in \{r\}, i \in I} a_{iR(j)k} x^*_{iR(j)} - \frac{t_r}{T} L_k \right)
\]

\[
+ \sum_{j \in \{r\}} \left( \rho^*_R(j) + \sum_{i \in I, k \in K} (\alpha^*_k - \beta^*_k) a_{iR(j)k} x^*_{iR(j)} \right)
\]

\[(46)\]

where the final equality follows from the KKT conditions \(\sum_{k \in K}(\alpha^*_k - \beta^*_k) a_{iR(j)k} x^*_{iR(j)} + \rho^*_j (\sum_{i \in I} x^*_ij - 1) = 0\), so that \(\sum_{i \in I, k \in K}(\alpha^*_k - \beta^*_k) a_{iR(j)k} x^*_{iR(j)} + \sum_{i \in I} \rho^*_j x^*_ij = \sum_{i \in I, k \in K}(\alpha^*_k - \beta^*_k) a_{iR(j)k} x^*_{iR(j)} + \rho^*_j = 0\).

For part 1, we only consider the resource \(k\) making \(\sum_{i \in I, j \in J} T p_j a_{ijk} x^*_{ij} = U_k\). By Lemma 2, it is easy to get that

\[
P(\sum_{j \in \{r\}, i \in I} a_{iR(j)k} x^*_{iR(j)}) \leq (1 - \epsilon_{x,r}) \frac{t_r}{T} U_k \leq \exp\left(-\frac{\frac{t_r}{T} \frac{\epsilon^2_{x,r}}{2} \alpha_k}{3}\right)
\]

\[(47)\]

where \(E(\sum_{i \in I} a_{iR(j)k} x^*_{iR(j)}) = \frac{t_r}{T}, \forall R(j)\).

For part 2, we only consider the \(k\) making \(\sum_{i \in I, j \in J} T p_j a_{ijk} x^*_{ij} = L_k + \alpha^* T \bar{a}_k\). We have that

\[
P(\sum_{j \in \{r\}, i \in I} (\bar{a}_k - a_{iR(j)k}) x^*_{iR(j}) \leq (1 - \epsilon_{x,r}) \frac{t_r}{T} ((1 - \alpha^* T \bar{a}_k - L_k)) \leq \exp\left(-\frac{\frac{t_r}{T} \frac{\epsilon^2_{x,r}}{2} \alpha_k}{3(1 - \alpha^*) (1 - \alpha^*) \frac{\epsilon_{x,r}}{T} \bar{a}_k - L_k}\right)
\]

\[(48)\]

Finally, when \(\gamma = O\left(\frac{x^2}{m(\frac{\epsilon}{\sqrt{T}})}\right)\), we have that

\[
\sum_{j \in \{r\}, i \in I} a_{iR(j)k} x^*_{iR(j)} \geq (1 - \epsilon_{x,r}) \frac{t_r}{T} U_k
\]

\[
\sum_{j \in \{r\}, i \in I} (\bar{a}_k - a_{iR(j)k}) x^*_{iR(j)} \geq (1 - \epsilon_{x,r}) \frac{t_r}{T} ((1 - \alpha^*) T \bar{a}_k - L_k)
\]

w.p. at least \(1 - \delta\).

For those \(k\) such that \(L_k + \alpha^* T \bar{a}_k < \sum_{i \in I, j \in J} T p_j a_{ijk} x^*_{ij} < U_k\), we know that they have no effect to \(\hat{\alpha}_r\) following the complementary slackness in (44). Therefore, with probability at least \(1 - \delta\)
we have that

\[(1 + 2)\]

\[
= \frac{T}{t_r} \sum_{k \in K} \alpha_k^* \left( t_r U_k - \sum_{j \in [t_r], i \in I} a_{iR(j)k} x^*_{iR(j)} \right) + \sum_{k \in K} \beta_k^* \left( \sum_{j \in [t_r], i \in I} a_{iR(j)k} x^*_{iR(j)} - \frac{t_r}{T} L_k \right)
\]

\[
= \frac{T}{t_r} \sum_{k \in K} \alpha_k^* \left( t_r U_k - \sum_{j \in [t_r], i \in I} a_{iR(j)k} x^*_{iR(j)} \right) + \sum_{k \in K} \beta_k^* \left( \frac{t_r}{T} (1 - \alpha^*) T \bar{a}_k - L_k \right) - \sum_{j \in [t_r], i \in I} (\bar{a}_k - a_{iR(j)k}) x^*_{iR(j)}
\]

\[+ \sum_{k \in K} \beta_k^* \alpha^* t_r \bar{a}_k\]

\[
\leq \frac{T}{t_r} (\epsilon_{x,r} \sum_{k \in K} \alpha_k^* \frac{t_r}{T} U_k + \epsilon_{x,r} \sum_{k \in K} \beta_k^* \frac{t_r}{T} ((1 - \alpha^*) T \bar{a}_k - L_k) + \sum_{k \in K} \beta_k^* \alpha^* t_r \bar{a}_k)
\]

\[
= \frac{T}{t_r} (\epsilon_{x,r} \sum_{k \in K} \alpha_k^* \frac{t_r}{T} U_k - \sum_{k \in K} \beta_k^* \frac{t_r}{T} L_k) + (\epsilon_{x,r} (1 - \alpha^*) + \alpha^*) \sum_{k \in K} \beta_k^* t_r \bar{a}_k
\]

\[
\leq \frac{T}{t_r} (\epsilon_{x,r} \alpha^* \frac{t_r}{T} + (\epsilon_{x,r} (1 - \alpha^*) + \alpha^*) \frac{t_r}{T})
\]

\[= \alpha^* + \epsilon_{x,r}
\]

where the first inequality from \(\sum_{j \in [t_r], i \in I} a_{iR(j)k} x^*_{iR(j)} \geq (1 - \epsilon_{x,r}) \frac{T}{t_r} U_k\) and \(\sum_{j \in [t_r], i \in I} (\bar{a}_k - a_{iR(j)k}) x^*_{iR(j)} \geq (1 - \epsilon_{x,r}) \frac{T}{t_r} (1 - \alpha^*) T \bar{a}_k - L_k\); the second inequality from \(\sum_{k \in K} \beta_k^* T \bar{a}_k = 1\) and \(\alpha^* = \sum_{k \in K} \alpha_k^* U_k - \sum_{k \in K} \beta_k^* L_k + \sum_{j \in [t_r]} T p_j \rho_j \geq \alpha_k^* U_k - \sum_{k \in K} \beta_k^* L_k\).

**LHS:** We design an algorithm \(\hat{P}_3\) by allocating request \(j\) to channel \(i\) with probability \((1 - \epsilon_{x,r})x^*_{ij}\), where \((x^*_{ij}, \alpha^*)\) is the optimal solution for LP 41. Following the very similar proofs in Section 3.1.1, we have that

\[
P(\sum_{j=1}^{t_r} X^\hat{P}_3_{jk} \geq \frac{t_r}{T} U_k) \leq \exp \left(-\frac{\epsilon_{x,r}^2}{2(1 - \frac{\delta}{2})^2 T^2 U_k} \right) \leq \delta
\]

\[
P(\sum_{j=1}^{t_r} X^\hat{P}_3_{jk} \leq \frac{t_r}{T} (L_k + (\alpha^* - \frac{\epsilon_{x,r}}{1 - \epsilon_{x,r}}) T \bar{a}_k)) \leq \exp \left(-\frac{\epsilon_{x,r}^2}{2(1 - \frac{\delta}{2})^2 T^2 U_k (1 - \alpha^*) T \bar{a}_k - L_k} \right) \leq \delta
\]

Thus w.p. at least \(1 - \delta\), we could find a solution \(\{(1 - \epsilon_{x,r})x^*_{iR(j)}\}\) whose consumed resource for each \(k\) is between \(\frac{T}{t_r} (L_k + (\alpha^* - \frac{\epsilon_{x,r}}{1 - \epsilon_{x,r}}) T \bar{a}_k)\) and \(\frac{T}{t_r} U_k\) via the algorithm \(\hat{P}_3\). According to the definition of \(\hat{\alpha}_r\) in LP 42, we have that \(\hat{\alpha}_r \geq \alpha^* - \frac{\epsilon_{x,r}}{1 - \epsilon_{x,r}}\).

In conclusion, we have that \(\alpha^* - \frac{\epsilon_{x,r}}{1 - \epsilon_{x,r}} - \epsilon_{x,r} \leq \hat{\alpha}_r - \epsilon_{x,r} \leq \alpha^*\), w.p. \(1 - 2\delta\).

Now \(\hat{\alpha}_r - \epsilon_{x,r}\) can be viewed as an good estimate for \(\alpha^*\), if we have enough data.

**4.3 Without the Knowledge of \(\alpha\).**

Based on the estimation of \(\alpha^*\) in previous subsection, we propose an algorithm \(A_2\) (Algorithm 5) without strong feasible assumption. Like the previous methods, we cut the all \(T\)-round requests into \(l\) stages. In each stage \(r\), we first estimate \(\hat{\alpha}_{t_{r-1}}\) of \(\alpha^*\) via the requests from previous stage. Then we consider the new expected problem \(E(\min(0, \hat{\alpha}_{t_{r-1}} - \epsilon_{x,r} - \eta))\) and run the same algorithm \(A_1\) (Algorithm 3) for this new problem, where \(\eta\) is the predefined violating parameter.
Algorithm 5 the algorithm $A_2$

**Input:** $t$, $L_k$, $U_k$, $\gamma$, $\eta$

**Output:** $\{x_{ij}\}_{i,j \in \mathcal{I}}$

Set $l = \log_2(\frac{t}{\epsilon})$

Set $t_r = 2^r T$ and $t_{r-1} = t T$

for $r = 0$ to $l-1$ do

Set

$$\hat{\alpha}_t = \max \alpha$$

subject to

$$\frac{t_r-1}{T} (L_k + \alpha T \bar{a}_k) \leq \sum_{i \in \mathcal{I}} \sum_{j=1+t_{r-1}}^{t_r} a_{iR(j)k} x_{iR(j)} \leq \frac{t_r-1}{T} U_k, \forall k \in \mathcal{K}$$

$$\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in [1 + t_{r-1}, t_r]$$

$$x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in [1 + t_{r-1}, t_r]$$

Set $\epsilon_{x,r} = \sqrt{\frac{4T \text{Var}(\epsilon)}{t_r}}$

$\eta_0 = \min(0, \hat{\alpha}_t - \epsilon_{x,r-1} - \eta)$

Set

$$W^{r-1} = \max \sum_{i \in \mathcal{I}} \sum_{j=1+t_{r-1}}^{t_r} w_{iR(j)} x_{ij}$$

subject to

$$\frac{t_r-1}{T} ((L_k + \eta_0 T \bar{a}_k) + \epsilon T \bar{a}_k) \leq \sum_{i \in \mathcal{I}} \sum_{j=1+t_{r-1}}^{t_r} a_{iR(j)k} x_{iR(j)} \leq \frac{t_r-1}{T} U_k, \forall k \in \mathcal{K}$$

$$\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in [1 + t_{r-1}, t_r]$$

$$x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in [1 + t_{r-1}, t_r]$$

Set $Z_r = \frac{W^{r-1}_{2w}}{(1+(2+\frac{T}{\gamma})\epsilon_{x,r-1})(t_r-1)}$

Set $\epsilon_{y,r} = \sqrt{\frac{4T \text{Var}(\epsilon)}{2w}}$

Initialize $\phi_{k,0} = \exp(-\frac{(1+\epsilon_{x,r}) \ln(1+\epsilon_{x,r})}{T \gamma} - \frac{(t_r-1) \epsilon_{x,r}}{T \gamma})$

Initialize $\varphi_{k,0} = \exp(-\frac{(1+\epsilon_{x,r}) \ln(1+\epsilon_{x,r})((1+\epsilon_{y,r} T \bar{a}_k - L_k - n_0 T \bar{a}_k) - \frac{(t_r-1) \epsilon_{y,r}}{T \gamma}}{T \gamma})$

Initialize $\psi_0 = \exp(-\frac{(1-\epsilon_{y,r}) \ln(1+\epsilon_{y,r}) Z_r}{T \gamma} - \frac{(t_r-1) Z_r \epsilon_{y,r}^2}{4T \gamma})$

for $r=1$ to $t_r$ do

If a request $j \in \mathcal{I}$, then use the following option $i^*$:

$$i^* = \arg \min_i \left\{ \sum_{k \in \mathcal{K}} \phi_{k,t-1} \exp(\frac{\ln(1+\epsilon_{x,r})}{U_k \gamma_3} a_{ijk}) + \sum_{k \in \mathcal{K}} \varphi_{k,t-1} \exp(\frac{\ln(1+\epsilon_{x,r})}{U_k \gamma_3} (\bar{a}_k - a_{ijk})) + \psi_{t-1} \exp(\frac{\ln(1+\epsilon_{y,r})}{w} w_{ij}) \right\}$$

$$X_{tk}^A = a_{i^*k}, Y_{tk}^A = w_{i^*j}$$

Update $\phi_{k,t} = \phi_{k,t-1} \exp(\frac{\ln(1+\epsilon_{x,r})}{U_k \gamma_3} (X_{tk}^A - \frac{\ln(1+\epsilon_{x,r})}{U_k \gamma_3} a_{ijk} + \frac{\epsilon_{y,r}^2}{4T \gamma}))$

Update $\varphi_{k,t} = \varphi_{k,t-1} \exp(\frac{\ln(1+\epsilon_{x,r})}{U_k \gamma_3} (X_{tk}^A - \frac{\ln(1+\epsilon_{x,r})}{U_k \gamma_3} (L_k + n_0 T \bar{a}_k) + (1+\epsilon_{x,r}) x_{ijk} T \bar{a}_k) - X_{tk}^A + \frac{\epsilon_{y,r}^2}{4T \gamma})$

Update $\psi_t = \psi_{t-1} \exp(\frac{\ln(1+\epsilon_{y,r})}{w} (1-\epsilon_{y,r}) Z_r - Y_{tk}^A) + \frac{\epsilon_{y,r}^2}{4T \gamma})$
**Theorem 9** Under the assumption 1-2, given \( \gamma_3 = O\left(\frac{\epsilon^2}{\ln(\frac{1}{\epsilon})}\right) \leq \max\left(\frac{\hat{a}_k}{T_{k}}, \left(1 - \min(0, \alpha^* - \eta)\right)T_{k} - L_k, W_{\min(0, \alpha^* - \eta)}\right) \) and \( \eta \geq \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} + \epsilon \), Algorithm 5 achieves an objective value \( (1 - O(\epsilon))W_{\min(0, \alpha^* - \eta)} \) and returns a solution satisfying the constraint \([L_k + (\min(0, \alpha^* - \sqrt{\epsilon} - \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} - \eta))T_{\hat{k}}, U_k]\) for every resource \( k \), w.p \( 1 - \epsilon \), where the predefined \( \epsilon > 0 \), \( \delta = \frac{\epsilon}{\mathcal{M}} \) and \( l = \log_2(\frac{1}{\epsilon}) \).

**Proof** In each stage \( r \), we serve \( t_r = 2^r \epsilon T \) requests, where \( 2^r \epsilon = 1 \) and \( r \in \{0, 1, \ldots, l - 1\} \). For \( r > 0 \), according to Theorem 8 we can estimate \( \hat{\alpha}_{r-1} \) by Algorithm 4 with \( \delta = \frac{\epsilon}{\mathcal{M}} \) and requests in stage \( r - 1 \), which gives that

\[
\alpha^* - \frac{\epsilon_{x,r-1}}{1 - \epsilon_{x,r-1}} \leq \hat{\alpha}_{r-1} \leq \alpha^* + \epsilon_{x,r-1}
\]

w.p. \( 1 - \frac{2\epsilon}{\mathcal{M}} \).

In stage \( r \), consider a new offline resource allocation problem

\[
\max \sum_{i \in \mathcal{I}} \sum_{j=1+t_r}^{t_{r+1}} w_{iR(j)}x_{iR(j)}
\]

s.t. \( t_r (L_k + \min(0, \hat{\alpha}_{t_{r-1}} - \epsilon_{x,r-1} - \eta)T_{\hat{k}}) \leq \sum_{i \in \mathcal{I}} \sum_{j=1+t_r}^{t_{r+1}} a_{iR(j)k}x_{iR(j)} \leq t_r U_k, \forall k \in \mathcal{K} \)

\[
\sum_{i \in \mathcal{I}} x_{iR(j)} \leq 1, \forall j \in [1 + t_r, t_{r+1}]
\]

\[
x_{iR(j)} \geq 0, \forall i \in \mathcal{I}, j \in [1 + t_r, t_{r+1}]
\]

When \( \alpha^* - \frac{\epsilon_{x,r-1}}{1 - \epsilon_{x,r-1}} \leq \hat{\alpha}_{r-1} \leq \alpha^* + \epsilon_{x,r-1} \), we could easily verify that the expected problem \( E(\min(0, \hat{\alpha}_{t_{r-1}} - \epsilon_{x,r-1} - \eta)) \) satisfies the strong feasible condition with parameter \( \eta > 0 \). Therefore, according to Theorem 4 and the analysis in section 3.3.2, we have that

\[
\sum_{i \in \mathcal{I}} \sum_{j=1+t_r}^{t_{r+1}} w_{iR(j)}x_{iR(j)} \geq (1 - \epsilon_{y,r}) \frac{t_r}{T} Z_r
\]

\[
\sum_{i \in \mathcal{I}} \sum_{j=1+t_r}^{t_{r+1}} a_{iR(j)k}x_{iR(j)} \leq \left(1 + \epsilon_{x,r}\right) \frac{t_r}{T} U_k
\]

\[
\sum_{j=t_{r+1}}^{t_{r+1}} \left(\hat{a}_k - \sum_{i \in \mathcal{I}} a_{iR(j)k}x_{iR(j)}\right) \leq \left(1 + \epsilon_{x,r}\right) \frac{t_r}{T} \left(\left(1 - \epsilon - \min(0, \hat{\alpha}_{t_{r-1}} - \epsilon_{x,r-1} - \eta)\right)T_{\hat{k}} - L_k\right)
\]

With the knowledge of \( \sqrt{\epsilon} \geq \epsilon_{x,r} \geq \epsilon_{x,r+1} \geq \epsilon \) and \( Z_r \geq 1 - (4 + \frac{2}{\eta - \epsilon}) \epsilon_{x,r-1} W_{\min(0, \hat{\alpha}_{t_{r-1}} - \epsilon_{x,r-1} - \eta)} \), we have that

\[
\sum_{i \in \mathcal{I}} \sum_{j=1+t_r}^{t_{r+1}} w_{iR(j)}x_{iR(j)} \geq (1 - \epsilon_{y,r}) \frac{t_r}{T} \left(1 - (4 + \frac{2}{\eta - \epsilon}) \epsilon_{x,r-1}\right) W_{\min(0, \hat{\alpha}_{t_{r-1}} - \epsilon_{x,r-1} - \eta)}
\]

\[
\geq (1 - \epsilon_{y,r}) \frac{t_r}{T} \left(1 - (4 + \frac{2}{\eta - \epsilon}) \epsilon_{x,r-1}\right) W_{\min(0, \alpha^* - \eta)}
\]

\[
\sum_{i \in \mathcal{I}} \sum_{j=1+t_r}^{t_{r+1}} a_{iR(j)k}x_{iR(j)} \leq \left(1 + \epsilon_{x,r}\right) \frac{t_r}{T} U_k
\]

\[
\sum_{j=t_{r+1}}^{t_{r+1}} \left(\hat{a}_k - \sum_{i \in \mathcal{I}} a_{iR(j)k}x_{iR(j)}\right) \leq \left(1 + \epsilon_{x,r}\right) \frac{t_r}{T} \left(\left(1 - \epsilon - \min(0, \alpha^* - \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} - \sqrt{\epsilon} - \eta)\right)T_{\hat{k}} - L_k\right)
\]
Moreover, it is easy to check

\[
\sum_{r=0}^{l-1} (1 - \epsilon_{y,r}) \frac{t_r}{T} (1 - (4 + \frac{2}{\eta - \epsilon}) \epsilon_{x,r-1}) W_{\min(0, \alpha^* - \eta)} \geq (1 - O(\epsilon)) W_{\min(0, \alpha^* - \eta)}
\]

\[
\sum_{r=0}^{l-1} (1 + \epsilon_{x,r}) \frac{t_r}{T} \geq U_k
\]

\[
\sum_{r=0}^{l-1} (1 + \epsilon_{x,r}) \frac{t_r}{T} \left( (1 - \epsilon - \min(0, \alpha^* - \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} - \sqrt{\epsilon} - \eta) T \tilde{a}_k - L_k) \right)
\]

\[
\leq (1 - \epsilon - \min(0, \alpha^* - \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} - \sqrt{\epsilon} - \eta)) T \tilde{a}_k - L_k
\]

Therefore, we finish the proof of Theorem 9.

A direct outcome from Theorem 9 is:

**Lemma 7** Given \( \gamma_3 = O\left(\frac{\epsilon^2}{\ln(\frac{K}{\epsilon})}\right) \),

1. if \( \alpha^* \geq \sqrt{\epsilon} + \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} + \eta \), the Algorithm 5 achieves an objective value \((1 - O(\epsilon)) W_E\) and returns a solution satisfying the constraint \([L_k, U_k]\) for every resource \(k\), w.p \(1 - \epsilon\).

2. if \( \alpha^* \leq 0\), the algorithm 5 achieves an objective value \((1 - O(\epsilon)) W_{\alpha^* - \eta}\) and returns a solution satisfying the constraint \([L_k + (\alpha^* - \sqrt{\epsilon} - \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} - \sqrt{\epsilon} - \eta)] T \tilde{a}_k, U_k]\) for every resource \(k\), w.p \(1 - \epsilon\).

From the Lemma 7 we know that when we are confident about the feasible condition, i.e., \( \alpha^* \geq \sqrt{\epsilon} + \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} + \eta \), the Algorithm 5 will return a solution satisfying the constraints and achieves an objective value \((1 - O(\epsilon)) W_E\) w.p \(1 - \epsilon\). As for the infeasible setting, under the permitted violating parameter \(\eta\), the Algorithm 5 achieves an objective value \((1 - O(\epsilon)) W_{\alpha^* - \eta}\) and violates the nearest feasible lower bound \(L_k + \alpha^* T \tilde{a}_k\) with extra \((\sqrt{\epsilon} + \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} + \eta) T \tilde{a}_k = (\eta + O(\epsilon)) T \tilde{a}_k\).

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