Research Article
Neural Network-Based Output Feedback Fault Tolerant Tracking Control for Nonlinear Systems with Unknown Control Directions

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In this study, an adaptive output feedback fault tolerant control (FTC) scheme is proposed for a class of multi-input and multioutput (MIMO) nonlinear systems with multiple constraints. The neural network (NN) is adopted to handle the unknown nonlinearity by means of its superior approximation capability. Based on it, the state observer is designed to estimate the unmeasured states, and the nonlinear disturbance observer is constructed to tackle the external disturbances. In addition, the Nussbaum function is utilized to cope with the actuator faults, which are coupled with the unknown control directions. Combining with the Lyapunov theory, a NN-based output feedback FTC law is developed for the MIMO nonlinear systems, and the boundedness of all closed-loop system error signals is proved. Simulation results on the unmanned helicopter are performed to demonstrate the effectiveness of the proposed controller.

1. Introduction

During these years, the control issue of multi-input and multioutput (MIMO) nonlinear systems has been drawing great attention due to their generalized and natural description for many practical applications, such as ocean vessel system, mobile robot system, and unmanned aerial vehicle system [1–4]. Since existing linear control methods do not match with the nonlinear systems, many effective nonlinear control approaches have been proposed and considerable results have been received. In [5–9] and the references therein, the adaptive backstepping control and sliding mode control schemes were, respectively, developed for a class of nonlinear systems to achieve acceptable control performance. Nevertheless, as the system structure becomes more and more complex, the unknown nonlinearity caused by nonlinear elements always appears and is worthy of further study.

Nowadays, in order to handle the unknown nonlinearity in the nonlinear systems, numerous control strategies have been presented, such as the neural network (NN) [10–13], extended state observer [14, 15], and fuzzy logic system [16–18]. In particular, the NNs are often combined with other control techniques to deal with the specific nonlinear systems due to their unique properties. In [10, 11], the adaptive backstepping-based neural control schemes were severally proposed to handle the unknown nonlinear functions for the helicopter and near space vehicle (NSV) systems. In [12], the NNs were combined with the model predictive control to overcome the so-called quadratic programming problem for a class of nonholonomic chained systems. In [13], a robust NN sliding mode control law was developed for a two-axis motion control system with unknown nonlinear terms and external disturbances. However, plenty of the above-mentioned literature are based on the available states, which are strict for some practical control systems. Actually, when the sensors malfunction, the signal usually cannot be accurately measured, and the state feedback approach is unreliable. Therefore, the high-quality controller needs to be further designed when the MIMO systems suffer from unmeasurable states.

The state observer is a common method to estimate the unmeasurable states in the academic community [19]. Meanwhile, fruitful and significant results can be found in
the recent literature [20–26]. In [20], an adaptive NN output feedback optimized controller was designed for the nonlinear systems with unknown dead zones. In [21, 22], the output feedback control problem for uncertain stochastic nonlinear systems was investigated by employing the state observer approach. In [23], an observer-based finite time controller was developed for a class of second-order nonlinear homogenous systems. Considering the objective existence of performance constraint, the output feedback control schemes were separately presented for the single-input and single-output and MIMO nonlinear systems in [24, 25]. In [26], a stable fuzzy output feedback controller was constructed for a class of nonlinear systems by means of the small-AAIN Approach. Nevertheless, it is worth noting that the control directions are usually unknown, and actuator faults occur frequently in many practical applications [27, 28]. Therefore, for the purpose of maintaining admissible system performance, more attentions and deeper considerations should be paid to the issues of unknown control directions and actuator faults.

At present, Nussbaum gain theory, which was first proposed by Nussbaum in 1983 [29], has become one of the most valid methods to cope with the unknown control directions. Meanwhile, existing achievements show that the combination of Nussbaum gain control technology with nonlinear control method attains favorable control performance in practice [30–33]. In [30], a distributed consensus controller for uncertain nonlinear systems with unknown control directions was designed to complete control task. In [31], the adaptive protocol was presented for nonlinear systems when both control directions and parameters were all unknown. In [32], a predefined performance-based adaptive controller was proposed for MIMO nonlinear systems in presence of unknown control directions and unknown hysteresis nonlinearities. In [33], an adaptive fuzzy tracking controller was designed to assure the stability of the stochastic nonlinear systems. However, the actuator faults are neglected in the above literature, whose occurrence will worsen the system performance and even lead to instability. Over the years, a quantity of attentions have been focused on fault tolerant control (FTC), and some effective control strategies have been implemented to faulty system. In [34], a robust adaptive sliding mode FTC scheme was proposed for coaxial helicopter to deal with actuator faults. In [35], a fuzzy adaptive nonlinear FTC strategy was presented for hypersonic vehicles with actuator stuck and loss of effectiveness faults. In [36], an adaptive decentralized FTC algorithm was developed for NSV attitude dynamics with actuator faults and control surface damage. Nonetheless, besides the negative factors mentioned above, the time-varying disturbances derived from the outside world also should be further considered.

The problem of robust control has a long history, and a number of disturbance rejection methods have been developed in past decades [15, 19, 37–40]. Especially, the nonlinear disturbance observer (NDO) has been extensively employed in practice since it does not depend on complete information of the disturbance model. In [15], the NDO-based composite fuzzy control issue was investigated when the uncertain nonlinear systems suffered from unknown dead zone. In [19], a NDO-based output feedback control scheme was designed for uncertain nonlinear systems with unknown nonlinearity and external disturbance. By combining the NDO and asymptotic tracking control techniques, a composed control approach was proposed for the spacecraft formation flying system under nonzero disturbances in [38]. However, when the unknown nonlinearity, unknown control directions, unmeasurable states, actuator faults, and external disturbances appear simultaneously, the control performance of the MIMO nonlinear system faces severe challenges, and it is of great significance to design high-quality control algorithms.

Motivated by above analysis, the radial basis function neural network (RBFNN), state observer, Nussbaum function, and NDO are combined with the backstepping technique to achieve satisfactory tracking control property. The main contributions can be summarized as follows:

1. Different from the traditional state feedback control [7], an output feedback controller is designed to tackle the unmeasured states and unknown nonlinearity by means of the RBFNN and state observer;
2. Compared with some direct adaptive fault estimation method using projection function [11], the presented Nussbaum-based FTC approach can overcome the singularity problem in a simpler way and reduce the complexity of controller design;
3. The developed output feedback FTC scheme can guarantee satisfactory tracking performance for the MIMO nonlinear systems under multiple negative effects.

The rest of this paper is organized as follows. Problem formulation and preparation knowledge are presented in Section 2. Section 3 derives the main results. Simulation studies on unmanned helicopter are carried out in Section 4. Section 5 draws the conclusion.

2. Problem Formulation and Preparation

Consider a class of MIMO nonlinear systems with actuator faults and external disturbances as follows:

$$\dot{X}_i = F_i(\Xi_i) + X_{i+1} + D_i(t), \quad i = 1, 2, \ldots, n - 1,$$

$$\dot{X}_n = F_n(\Xi_n) + B\theta u + D_n(t),$$

$$y = X_1,$$

where $X_i \in \mathbb{R}^n$ and $\Xi_i = [X_i^T, X_{i+1}^T, \ldots, X_n^T]^T \in \mathbb{R}^n$ with $i = 1, 2, \ldots, n$ being the system state vectors. $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ denote the input vector and output vector, respectively. $F_i(\Xi_i) \in \mathbb{R}^n$ defines an unknown smooth nonlinear function. $B = \text{diag}[B_1, B_2, \ldots, B_m]$ is the unknown nonzero constant control gain matrix. $\theta = \text{diag}[\theta_1, \theta_2, \ldots, \theta_m]$ refers to the actuator fault factor, which describes the unknown remaining control efficiency of $i^{th}$ actuator. $D_i(t)$ represents external time-varying disturbance. In this paper, it is assumed that only the system output $y$ is available for measurement.
Complexity

Remark 1. The considered MIMO nonlinear systems can be employed to describe many practical plants, such as ocean vessel system [2], unmanned helicopter system [4], and NSV system [11]. Moreover, it is worth mentioning that the control directions are unknown in many application requirements and usually coupled with actuator faults. Hence, for reflecting the system dynamics more practically, the unknown control directions and actuator faults are considered simultaneously in this work.

The control objective of this study is to design a robust adaptive NN output feedback FTC scheme, such that all error signals of the closed-loop system are convergent, and the output y can follow the desired trajectory $y_d$. To this end, the following definition, lemmas, and assumptions are introduced.

Definition 1 (see [33]). If a continuous function $H(\cdot)$ fulfills the performances as follows:

$$
\lim_{s \to +\infty} \sup_{\tau \in [0, T]} \frac{1}{s} \int_0^\tau H(\tau) d\tau = +\infty,
$$

$$
\lim_{s \to -\infty} \inf_{\tau \in [0, T]} \frac{1}{s} \int_0^\tau H(\tau) d\tau = -\infty.
$$

Then $H(\cdot)$ is called a Nussbaum function. At present, the functions $t^2 \cos(\tau)$, $e^t \cos(\tau)$, $t^2 \sin(\tau)$, and $e^t \sin(\pi t/2)$ have been proven to be the Nussbaum-type functions in [41, 42]. In this paper, $H(\tau) = t^2 \cos(\tau)$ is used.

Lemma 1 (see [33]). Let $V(t) \geq 0$ and $\tau_i(t)$ be smooth functions defined on $[0, t_f]$, and let $H(\tau_i)$ be smooth Nussbaum-type function. If the following inequality holds:

$$
V(t) \leq -b_0 V(t) + b_1 + \sum_{i=1}^n |\psi_i H(\tau_i) + 1| \tau_i,
$$

where $b_0$, $b_1$, and $\psi_i$ are suitable positive constants, then $V(t)$, $\tau_i(t)$ and $\sum_{i=1}^n |\psi_i H(\tau_i) + 1| \tau_i$ must be bounded on $[0, t_f]$.

Lemma 2 (see [10]). Owing to the powerful nonlinear approximation capability, RBFNN is frequently employed to approximate any unknown smooth nonlinear function $E(\gamma)$, which can be expressed as

$$
E(\gamma) = \tilde{\Phi}^T M(\gamma) + \sigma,
$$

where $\gamma \in \mathbb{R}^n$ is the input vector, $\tilde{\Phi} \in \mathbb{R}^{i \times n}$ is the weight matrix, $\sigma$ is the approximation error, and $M(\gamma) = [M_1(\gamma), M_2(\gamma), \ldots, M_j(\gamma)] \in \mathbb{R}^j$ is the basis function vector with $M_i(\gamma)$ being

$$
M_i(\gamma) = \exp \left( -\frac{(\gamma - c_i)^T (\gamma - c_i)}{\sigma_i} \right), \quad i = 1, 2, \ldots, j,
$$

where $c_i$ and $\sigma_i$ denote the center and width of the basis function, respectively.

Then, the RBFNN (4) can approach any unknown smooth function $E(\gamma)$ in the form of

$$
E(\gamma) = \Phi^*^T M(\gamma) + \sigma^*,
$$

where $\Phi^*$ is the optimal weight matrix, $\sigma^*$ is the approximation error satisfying $\|\sigma^*\| \leq \sigma$, and $\sigma$ is a positive constant.

Assumption 1 (see [7]). For the bounded desired trajectory $y_d$ and its derivatives $\dot{y}_d$ and $\ddot{y}_d$, there exists an unknown positive constant $t_0$ making $\Pi_0 = \{(y_d, \dot{y}_d, \ddot{y}_d) : \|y_d\|^2 + \|\dot{y}_d\|^2 + \|\ddot{y}_d\|^2 \leq t_0\}$ holds.

Assumption 2 (see [11]). The unknown external disturbance $d_i, i = 1, 2, \ldots, n$ is supposed to satisfy $\|D_i\| \leq D_i$ and $\|D_i\| \leq \zeta_i$, where $D_i$ and $\zeta_i$ are unknown positive constants.

Assumption 3 (see [11, 33]). The control gain $B_i$ is supposed to have the unknown but same sign with each other, and it satisfies $B_i \leq |B_i| \leq B_{\sigma}$, where $B_i > 0$ is the known constant. Moreover, the fault factor $\Theta_i$ is assumed to be unknown constant and satisfies $0 < \theta_m \leq \theta_i \leq 1$, where $\theta_m$ is the known lower bound.

Remark 2. For a practical system, the tracking mission should be realizable, and there should exist a feasible controller to achieve it, which means that assumption 1 is reasonable. In addition, assumption 2 is provided to illustrate that if the external disturbance is unbounded, it may result in that the system cannot provide enough energy to accomplish the specified object. Similarly, if the actuator loses too much effectiveness, the whole system may lose the FTC capacity. Therefore, it is reasonable for assumptions 1–3, and they have been extensively used in existing literature such as [4, 5, 7, 16–18, 40–42].

3. Control Design and Stability Analysis

In this section, a backstepping-based robust adaptive output feedback FTC control scheme will be developed for the MIMO nonlinear systems to deal with the unknown nonlinearity, unavailable states, unknown control directions, actuator faults, and external disturbances. The block diagram of the design thread is given in Figure 1.

3.1. Model Transformation and State Observer Design.

Since the control gain $B_i$ and fault factor $\Theta_i$ of the $i^{th}$ actuator are all unknown and coupled with the corresponding control input, it is difficult to design the controller directly. According to the properties of $B_i$ and $\Theta_i$, it can be obtained that $0 < B_i \Theta_i \leq |B_i| \leq B_{\sigma}$. By defining $h = B \Theta$, it is known that $h$ is an invertible matrix. In order to promote the control design, we introduce the new state variable $x_i = h^{-1} X_i$, the new smooth function $f_i(x_i) = h^{-1} f_i(X_i)$, and the new disturbance $d_i(t) = h^{-1} D_i(t)$. Then, the MIMO nonlinear system (1) can be transformed into the following system:

$$
\dot{x}_i = f_i(x_i) + x_{i+1} + d_i(t), i = 1, 2, \ldots, n - 1,
$$

$$
x_n = f_n(x_n) + u + d_n(t),
$$

$$
y_1 = x_1.
$$
where \( \bar{x}_i = [x_i^T, x_{i+1}^T, \ldots, x_{n}^T]^T \in \mathbb{R}^n \) and \( y_1 \in \mathbb{R}^n \) are the new system state vector and output vector.

**Remark 3.** Considering assumption 2 and assumption 3, we obtain that the new disturbance \( \hat{d}_i(t) \) and its first derivative are also bounded. That is, there exist unknown positive constants \( \bar{d}_i \) and \( \bar{q}_i \), such that \( \|\hat{d}_i\| \leq \bar{d}_i \) and \( \|\hat{d}_i\| \leq \bar{q}_i \). Moreover, we should note that the state variable \( x_i \) and the output vector \( y_1 \) of the new MIMO nonlinear system (7) are all unavailable.

Here, the following RBFNN is adopted to approximate the unknown nonlinear term \( L_i f_i(\bar{x}) \):

\[
L_i f_i(\bar{x}) = \Phi_i^T M_i(\bar{x}) + \sigma_i^*(\bar{x}), \quad i = 1, 2, \ldots, n, \tag{8}
\]

where \( L_i = L_i^T > 0 \) is the designed constant matrix, \( M_i(\bar{x}) \) is the basis function, and \( \Phi_i^* \) is the weight matrix.

Due to the unavailability of the system states \( x_i \), the above function approximations (8) are invalid. Hence, we use the following approximation to design the state estimator:

\[
L_i \tilde{f}_i(\tilde{x}) = \Phi_i^T M_i(\tilde{x}), \quad i = 1, 2, \ldots, n, \tag{9}
\]

where \( \tilde{x}_i = [\tilde{x}_i^T, \tilde{x}_{i+1}^T, \ldots, \tilde{x}_n^T]^T \) is the estimation of \( x_i \), and \( \Phi_i \) is the estimation of \( \Phi_i^* \).

It is obvious that the approximation \( L_i \tilde{f}_i(\tilde{x}) \) depends on the available \( \Phi_i \) and \( \tilde{x}_i \). In order to cope with the unknown system states in the MIMO nonlinear systems (7), the following state observer is designed [19]:

\[
\begin{align*}
\dot{\tilde{x}}_i &= x_{i+1} + f_i(\tilde{x}_i) - k_i \tilde{x}_i + \tilde{x}_i y + \tilde{d}_i, \quad i = 1, 2, \ldots, n - 1, \\
\dot{\tilde{x}}_n &= u + f_n(\tilde{x}_n) - k_n \tilde{x}_1 + \tilde{x}_n y + \tilde{d}_n,
\end{align*} \tag{10}
\]

where \( k_i = \text{diag}[k_{i1}, k_{i2}, \ldots, k_{in}] \) with \( k_{ij} > 0 \), \( \tilde{x}_i = \text{diag}[\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots, \tilde{x}_{in}] \) with \( \tilde{x}_{ij} > 0 \) \((j = 1, 2, \ldots, n)\), \( \tilde{d}_i \) is the estimation of \( d_i \). The corresponding updating laws with respect to \( \tilde{d}_i \) and \( \Phi_i \) will be given gradually along with the controller design process in the next subsection.

Define \( z_i = x_i - \tilde{x}_i \), \( \lambda_i = L_i^{-1} \sigma_i^*(\bar{x}) \), and \( \Phi_i = \Phi_i^* - \Phi_i \).

Considering (7–10) and differentiating \( z_i \) yield

\[
\begin{align*}
\dot{z}_i &= x_{i+1} + L_i^{-1}(\Phi_i^* M_i(\bar{x}) + \sigma_i^*(\bar{x})) + d_i - \tilde{x}_{i+1} \\
&= -L_i^{-1}\Phi_i^* M_i(\bar{x}) + k_i \tilde{x}_i - \tilde{x}_i y - \tilde{d}_i, \\
&= -k_i z_{i+1} + k_i x_{i+1} + \lambda_i - \tilde{x}_i y + L_i^{-1}\Phi_i^* M_i(\bar{x}) \\
&+ L_i^{-1}\Phi_i^* M_i(\bar{x}) - M_i(\bar{x}) - \varepsilon_i z_i + \tilde{\Delta}, \\
\dot{z}_n &= -k_n z_{i+1} + k_n x_{i+1} + \lambda_n - \tilde{x}_n y + L_n^{-1}\Phi_n^* M_n(\bar{x}) \\
&+ L_n^{-1}\Phi_n^* M_n(\bar{x}) - M_n(\bar{x}) - \varepsilon_n z_n + \tilde{\Delta}_n,
\end{align*} \tag{11, 12}
\]

where \( \varepsilon_i = \varepsilon_i^T > 0 \) is the designed diagonal matrix, and \( \tilde{\Delta}_i \) is the estimation error of the auxiliary variable \( \Delta_i \), which will be given in the following.

Then, the observation error dynamics (11) and (12) can be rewritten as

\[
\begin{align*}
\dot{\varepsilon}_i &= N \varepsilon_i + K x_i + L_i^{-1}\Phi_i^* Y(\bar{x}, \tilde{x}) + \lambda_i \\
&+ \tilde{\Delta} + L_i^{-1}\Phi_i^* M_i(\tilde{x}) - h y, \tag{13}
\end{align*}
\]

where

\[
N = \begin{bmatrix}
-k_1 & -\varepsilon_1 & 1 & \cdots & \cdots & \cdots \\
-k_2 & -\varepsilon_2 & I_2 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\
-k_n & -\varepsilon_n & I_{n-1} & \cdots & \cdots & -\varepsilon_n \\
\end{bmatrix}, \quad Z = [z_1^T, z_2^T, \ldots, z_n^T]^T, \\
\lambda = [\lambda_1^T, \lambda_2^T, \ldots, \lambda_n^T]^T, \\
K = [k_1^T, k_2^T, \ldots, k_n^T]^T, \\
L_i = \text{diag}[L_1^T, L_2^T, \ldots, L_n^T], \\
\Phi^* = \text{diag}[\Phi_1^*, \Phi_2^*, \ldots, \Phi_n^*], \\
\tilde{\Delta} = [\tilde{\Delta}_1^T, \tilde{\Delta}_2^T, \ldots, \tilde{\Delta}_n^T]^T, \\
h = [\tilde{x}_1^T, \tilde{x}_2^T, \ldots, \tilde{x}_n^T]^T, \\
M_i(\bar{x}) = [M_{i1}^T(\bar{x}), M_{i2}^T(\bar{x}), \ldots, M_{in}^T(\bar{x})]^T, \\
M_n(\bar{x}) = [M_{n1}^T(\bar{x}), M_{n2}^T(\bar{x}), \ldots, M_{nn}^T(\bar{x})]^T.
\]

Here, proper parameters \( k_i \) should be selected to ensure that \( N \) is Hurwitz. In other words, for a given matrix \( P = P^T > 0 \), there exists positive definite matrix \( \Lambda = \Lambda^T > 0 \) such that

\[
N^T P + PN \leq -\Lambda. \tag{14}
\]
Due to the unknown disturbance $d_i$, the following auxiliary variable $\Delta_i$ is introduced:

$$\Delta_i = d_i + \varepsilon_i x_i, \quad i = 1, 2, \ldots, n. \quad (15)$$

In light of (15), the NDOs are designed as

$$\begin{cases}
\tilde{d}_i = \Delta_i - \varepsilon_i x_i, & i = 1, 2, \ldots, n - 1, \\
\hat{\Delta}_i = \varepsilon_i (\tilde{f}_i(\hat{x}) + \varepsilon x_{i+1} - \hat{x}_i + \hat{\Delta}_i),
\end{cases} \quad (16)$$

$$\begin{cases}
\tilde{d}_n = \Delta_n - \varepsilon_n x_n, \\
\hat{\Delta}_n = \varepsilon_n (\tilde{f}_n(\hat{x}_n) + u - \varepsilon_n x_n + \hat{\Delta}_n). 
\end{cases} \quad (17)$$

Define $\Delta_i = \Delta_i - \hat{\Delta}_i$ and $\tilde{d}_i = d_i - \hat{d}_i$. Then, we have

$$\Delta_i = d_i + \varepsilon_i x_i - \tilde{d}_i - \varepsilon_i \hat{x}_i = \tilde{d}_i + \varepsilon_i z_i. \quad (18)$$

On the one hand, by invoking (15) and (16), we obtain

$$\begin{align}
\hat{d}_i &= \hat{d}_i + \varepsilon_i \hat{x}_i - \hat{\Delta}_i \\
&= \hat{d}_i + \varepsilon_i (L_i^{-1} \Phi_i^* Y_i(\hat{x}_i, \hat{x})) \\
&\quad + L_i^{-1} \tilde{F}_i^T M_i(\hat{x}_i) + \lambda_i + z_{i+1} - \Delta_i - \varepsilon_i z_i, \\
&i = 1, 2, \ldots, n - 1. \quad (19)
\end{align}$$

On the other hand, by invoking (15) and (17), we obtain

$$\begin{align}
\hat{\Delta}_n &= \hat{\Delta}_n - \varepsilon_n \hat{x}_n - \hat{\Delta}_n \\
&= \hat{\Delta}_n + \varepsilon_n (L_n^{-1} \Phi_n^* Y_n(\hat{x}_n, \hat{x}_n)) \\
&\quad + L_n^{-1} \tilde{F}_n^T M_n(\hat{x}_n) + \lambda_n - \Delta_n - \varepsilon_n z_n. \quad (20)
\end{align}$$

Consider the following Lyapunov function candidate:

$$V_0 = Z^T P Z + \frac{1}{2} \sum_{i=1}^{n} \Delta_i^T \Delta_i. \quad (21)$$

Invoking (13), (14), (19) and (20), one obtains

$$\begin{align}
V_0 &\leq -Z^T A Z + 2Z^T PK_{x_1} + 2Z^T P L + 2Z^T P \hat{\Delta}_n \\
&\quad + 2Z^T P L^{-1} \Phi^* Y(\hat{x}, \hat{x}) + 2Z^T P L^{-1} \Phi^* M(\hat{x}) \\
&\quad - 2Z^T P Y + \sum_{i=1}^{n} \Delta_i^T \Delta_i + \sum_{i=1}^{n} \Delta_i^T \Omega_i^* L_i^{-1} \Phi_i^* Y_i(\hat{x}_i, \hat{x}_i) \\
&\quad + \sum_{i=1}^{n} \Delta_i^T \varepsilon_i L_i^{-1} \Phi_i^* M_i(\hat{x}_i) + \sum_{i=1}^{n} \Delta_i^T \varepsilon_i \lambda_i \\
&\quad - \sum_{i=1}^{n-1} \Delta_i^T \varepsilon_i \Delta_i - \sum_{i=1}^{n-1} \Delta_i^T \varepsilon_i z_i + \sum_{i=1}^{n-1} \Delta_i^T \varepsilon_i z_{i+1}. \quad (22)
\end{align}$$

Based on Young’s inequality, the following inequalities can be obtained:

$$2Z^T P K_{x_1} \leq h_m Z^T P K^T P^T Z + \|e_1\|^2 + \zeta_0,$$

$$2Z^T P \lambda \leq Z^T P P^T Z + \sum_{i=1}^{n} \lambda_i^2,$$

$$2Z^T P L^{-1} \Phi^* Y(\hat{x}, \hat{x}) \leq Z^T P P^T Z + \sum_{i=1}^{n} \beta_i^2 \|L_i^{-1}\| \|\Phi_i^*\|^2,$$

$$2Z^T P L^{-1} \Phi^* M(\hat{x}) \leq r_1 Z^T P P^T Z + \frac{1}{r_1} \sum_{i=1}^{n} |Z_{i-1}^2| \|\Phi_i\|^2,$$

$$2Z^T P \Delta \leq Z^T P P^T Z + \sum_{i=1}^{n} \Delta_i^T \Delta_i,$$

$$-2Z^T P y \leq Z^T P h T P^T Z + \frac{1}{r_2} \sum_{i=1}^{n} |Z_{i-1}^2| \|\Phi_i\|^2.$$

Consider the following Lyapunov function candidate:

$$\sum_{i=1}^{n} \Delta_i^T \Delta_i + \frac{1}{2} \sum_{i=1}^{n} |Z_{i-1}^2| \|\Phi_i\|^2,$$

$$\sum_{i=1}^{n} \Delta_i^T \varepsilon_i \Delta_i + \sum_{i=1}^{n} \sum_{i=1}^{n-1} \Delta_i^T \varepsilon_i z_i + \sum_{i=1}^{n-1} \Delta_i^T \varepsilon_i z_{i+1}.$$

where $r_1$ and $r_2$ are the designed positive constants, $\|z\| \leq \varepsilon_0$, $h_m = 1/\beta_m^2$, $\varepsilon = \max (\|e_1\|, \|\lambda_i\| \leq \lambda_1)$, $|Y_i(\hat{x}_i, \hat{x}_i)| \leq \beta_i |M_i(\hat{x}_i)| \leq \delta_1$, $e_1$ is the tracking error, which will be given subsequently.

Substituting the above inequalities into (22) produces
\[
\dot{V}_0 \leq -Z^T \left( \Lambda - \Lambda_1 - \frac{\tau^2 + \tau' }{2} I_1 \right) Z + 2\|e_1\|^2
\]
\[
- \sum_{i=1}^n \Delta_i^T \left( e_i - \frac{7 + r_2 I_1}{2} \right) \Delta_i + R_0
\]
\[
+ \sum_{i=1}^n \left( \frac{1}{r_1} + \frac{\tau^2}{2r_2} \right) L_i^{-1} \| e_i \|^2 + 2\|e_i\|^2 + \sum_{i=1}^n (1 + 1/2\tau') \bar{\Delta}_i^T + 1/2 \sum_{i=1}^n \bar{\Delta}_i^T
\]
where \( \Lambda_1 = P \{ h_{\text{in}} K T + h h^T + (r_1 + 3) I_1 \} P^T \), \( R_0 = \sum_{i=1}^n (1 + 1/2\tau') \bar{\Delta}_i^T \).

### 3.2. Robust Adaptive Neural Output Feedback Control Design

**Step 1.** Define the following tracking errors:
\[
e_1 = y - y_d
\]
\[
\dot{e}_1 = \dot{y} = \dot{y}_d = \dot{h} + \dot{e}_2 = \dot{h} + z_2 + d_1 - \dot{y}_d
\]
(25)
where \( \dot{x}_d \) is the designed virtual control law.

Considering (7) and taking the time derivative of \( e_1 \) yield
\[
\dot{e}_1 = X_1 - \dot{\gamma}_d = h \dot{x}_1 - \dot{y}_d
\]
\[
= F_1(y) + h(x_2d + e_2 + z_2 + d_1) - \dot{y}_d
\]
(26)

Because of the unknown smooth function \( F_1(y) \), the following RBFNN is used to approximate it:
\[
F_1(y) = Q_F^{-1} (\Phi^*_F M_F(y) + \sigma_F^*(y)),
\]
(27)
where \( Q_F = Q_F^T > 0 \) is the designed matrix, \( M_F(y) \) is Gaussian function, and \( \Phi^*_F \) is the weight matrix.

Substituting (27) into (26) follows
\[
\dot{e}_1 = Q_F^{-1} \Phi^*_F M_F(y) + Q_F^{-1} \sigma_F^*(y) + h x_{2d} + h e_2
\]
\[
+ h z_2 + h d_1 - \dot{y}_d
\]
(28)

Consider the Lyapunov function candidate as
\[
\dot{V}_1 = V_0 + \frac{1}{2} e_i^T e_i + \frac{1}{2} \text{tr} \left( \Phi_i^T \Theta_i^T \Phi_i + \Theta_i^T \Phi_i^T \Phi_i \right)
\]
\[
\text{tr} \left( \Phi_i^T \Phi_i \right)
\]
(29)

where \( \Theta_i = \Theta_i^T > 0 \) and \( \Omega_i = \Omega_i^T > 0 \) are the designed matrixes, \( \Phi_i = \Phi_i^T - \Phi_i \).

Invoking (24) and (28), we obtain
\[
\dot{V}_1 \leq -Z^T \left( \Lambda - \Lambda_1 - \frac{\tau^2 + \tau'}{2} I_1 \right) Z + 2\|e_1\|^2 + R_0
\]
\[
+ e^T Q_i^{-1} \Phi^*_F^T M_F(y) - \sum_{i=1}^n \Delta_i^T \left( e_i - \frac{7 + r_2 I_1}{2} \right) \Delta_i
\]
\[
+ \frac{1}{2} \| e_i(\Phi_i^T \Theta_i^T \Phi_i) \| + \frac{\tau^2}{2r_2} \| e_i(\Phi_i^T \Theta_i^T \Phi_i) \|^2 + e^T Q_i^{-1} \sigma_F^*(y)
\]
(26)
\[
\leq -Z^T \left( \Lambda - \Lambda_1 - \frac{\tau^2 + \tau'}{2} I_1 - C \right) Z + e^T h x_{2d}
\]
\[
+ e^T Q_i^{-1} \Phi^*_F^T M_F(y) - \sum_{i=1}^n \Delta_i^T \left( e_i - \frac{7 + r_2 I_1}{2} \right) \Delta_i
\]
\[
+ \frac{1}{2} \| e_i(\Phi_i^T \Theta_i^T \Phi_i) \|^2 + \frac{1}{2} \| e_i(\Phi_i^T \Theta_i^T \Phi_i) \|^2 + \text{tr} \left( \Phi_i^T \Theta_i^T \Phi_i \right)
\]
(26)
where \( C = \text{diag} [0_{\text{non}}, 1/2 I_1, \ldots , 0_{\text{non}}] \), \( \| \sigma_F^* \| \leq \sigma_F \).

Design the virtual control function \( x_{2d} \) and parameter adaptive laws as
\[
x_{2d} = H(\tau) \mu_i
\]
(31)
\[
\hat{\tau}_i = e_i \mu_i, i = 1, 2, \ldots, n
\]
(32)
\[
\mu = q_i e_i - \dot{\gamma}_d + Q_i^{-1} \Phi^*_F M_F(y) - \dot{\gamma}_d
\]
(33)
\[
\dot{\Phi}_i = \Theta_i(M_i(\tilde{X}_1) e_i^{T} L_i^{-1} - \Theta_i \tilde{X}_1) \]
(34)
\[
\dot{\Phi}_F = \Omega_F(M_F(y) e_i^{T} Q_i^{-1} - \Theta_i \tilde{X}_1)
\]
(35)
where \( \tau = [\tau_1, \tau_2, \ldots, \tau_n]^T \), \( \mu = [\mu_1, \mu_2, \ldots, \mu_n]^T \), \( e_i \) is the \( i \)th element of \( e_i \), \( H(\tau) = \text{diag} (H(\tau_1), H(\tau_2), \ldots, H(\tau_n)) \),
\( H(\tau_i) \) is Nussbaum-type function, which is chosen as
\( H(\tau_i) = \tau_i^2 \cos(\tau_i) \), \( q_i = \text{diag}\{q_{i1}, q_{i2}, \ldots, q_{in}\} \), \( q_{il} \), \( \omega_1 \) and \( \omega_F \) are designed positive constants.

Considering (31)–(35), the following facts can be obtained:

\[
e^T_i h x_{2d} = \sum_{i=1}^{n} h_i H(\tau_i) \dot{\tau}_i = \sum_{i=1}^{n} [h_i H(\tau_i) + 1] \dot{\tau}_i - e^T_i \mu_e \tag{36}
\]

\[
e^T_i Q_F^{-1} \Phi F^T M_F(\dot{y}) - tr\left\{ \Phi F^T \Omega F^{-1} \dot{\Phi}_F \right\} = tr\left\{ \Phi F^T \Phi F \right\}
\leq -\frac{1}{2} \omega_2 \| \Phi F \|^2 + \frac{1}{2} \omega_F \| \Phi F \|^2,
\]

\[
-\text{tr}\left\{ \Phi F^T \Phi F^{-1} \dot{\Phi}_F \right\} = -\text{tr}\left\{ \Phi F^T M(F) e^T_i L_i^{-1} - \Phi F^T \omega F \dot{\Phi}_F \right\}
\leq e^T_i \epsilon_i - \left( \frac{1}{2} \omega_1 - \frac{1}{4} \omega_2 \right) \| L_i^{-1} \|^2 \| \Phi F \|^2 + \frac{1}{2} \omega_F \| \Phi F \|^2,
\]

where \( h = \text{diag}\{h_1, h_2, \ldots, h_n\} \) with \( h_i \) being the diagonal element.

Substituting (36)–(38) into (30) gives

\[
\dot{V}_1 \leq -Z^T A Z - e^T_i \epsilon_i e_i - \left( \frac{1}{2} \omega_1 - \frac{1}{4} \omega_2 \right) \| L_i^{-1} \|^2 \| \Phi F \|^2
- \frac{1}{2} \omega_F \| \Phi F \|^2 - \sum_{i=1}^{n} \Delta_i^T \left( e_i - \frac{7 + r_2}{2} I_1 \right) \Delta_i + e^T_i \epsilon_i e_i
- \Delta_i^T \left( e_i - \frac{8 + r_2}{2} I_1 \right) \Delta_i + \sum_{i=1}^{n} [h_i H(\tau_i) + 1] \dot{\tau}_i
+ \sum_{i=1}^{n} \left( \frac{1}{r_1} + \frac{r_2}{2} \right) \| L_i^{-1} \|^2 \| \Phi F \|^2 + R_1,
\]

where \( A = \Lambda - \Lambda_1 - \eta^2 + \eta \| \Phi F \|^2 / 2 I_1 - C - C_1, C_1 = \text{diag}\{1/2 I_1, 0_{n \times m}, \ldots, 0_{n \times m}\} \), \( \epsilon_i = q_i - 9 + 3 \delta_2^* \| \Phi F \|_F^2 + 1/2 \| \Phi F \|_F^2 + 1/2 \| \Phi F \|_F^2 + d_i \).

Step 2. Considering (10) and differentiating \( e_2 \) give

\[
e^2 = \dot{x}_3 + \dot{f}(\dot{x}_2) - k_2 \dot{x}_1 + \chi_2 y + d_2 - x_{3d}
= x_{3d} + e_3 + L_i^{-1} \Phi F^T M(F) \dot{\Phi}_F^T \dot{x}_{3d} - k_2 \dot{x}_1 + \chi_2 y
+ d_2 - x_{3d},
\]

In particular, the dynamic surface control technique is employed to avoid the repeated computation of \( \dot{x}_{3d} \) and achieve its available derivative. Let \( x_{3d} \) pass the first-order filter \( \eta_2 \) [7].

\[
h_2 \eta_2 + \eta_2 = x_{3d}, \eta_2(0) = x_{3d}(0),
\]

where \( h_1 = \text{diag}\{h_2, h_3, \ldots, h_{2n}\} > 0 \) is the time constant matrix of the filter.

By defining \( \omega_2 = \eta_2 - x_{3d} \), we have

\[
\omega_2 = \eta_2 - x_{3d}
= -q_2 e_2 + \eta_2 + k_2 \dot{x}_1 - \chi_2 y
- \bar{d}_2 - L_i^{-1} \Phi F^T M(F) \dot{x}_{3d}.
\]

where \( q_2 = \text{diag}\{q_{21}, q_{22}, \ldots, q_{2n}\}, q_{2i} \) is the designed positive constant.

The parameter adaptive law is proposed as

\[
\dot{\Phi}_F = \Theta_2 (M(F) e^T_i L_i^{-1} - \omega F \Phi F)
\]

where \( \Theta_2 = \Theta_2^T > 0 \) is the designed matrix, and \( \omega_2 > 0 \) is the designed constant.

Invoking (43), we obtain

\[
\dot{e}_2 = -q_2 e_2 + e_3 - h_1^{-1} \omega_2 + l_2(\bullet)
\]

where \( q_2 = \text{diag}\{q_{21}, q_{22}, \ldots, q_{2n}\}, q_{2i} \) is the designed positive constant.

Choose the following Lyapunov function candidate as

\[
V_2 = V_1 + e^T_2 e_2 + \frac{1}{2} tr\left\{ \Phi F^T \Theta_2^{-1} \Phi F \right\} + \frac{1}{2} \omega_2 \| \omega F \|^2,
\]

The time derivative of \( V_2 \) is

\[
\dot{V}_2 = \dot{V}_1 + e^T_2 \dot{e}_2 - tr\left\{ \Phi F^T \Theta_2^{-1} \dot{\Phi}_F \right\} + \omega^T_2 \dot{\omega}_2.
\]
Considering (44), we obtain
\[
-\text{tr}\left\{\Phi_T^T\Theta_T^{-1}\Phi_T\right\} = -\text{tr}\left\{\Phi_T^T M_3(\hat{x}_3) e_T L_2^{-1} - \Phi_T^T \omega_2 \Phi_T\right\}
\leq e_T^T e_T \left(\frac{1}{2} \omega_2 - \frac{\delta^2}{4} \|L_2^{-1}\|^2\right) \|\Phi_T\|^2 + \frac{1}{2} \omega_2 \|\Phi_T^*\|^2.
\]
(48)

Substituting (39), (45) and (48) into (47), we have
\[
\begin{align*}
\dot{V}_2 &\leq - Z^T A Z - \sum_{i=1}^n e_i e_i - \sum_{i=1}^n \left(\frac{1}{2} \omega_i - \frac{\delta_i^2}{4} \|L_i^{-1}\|^2\right) \|\Phi_i\|^2 \\
&\quad - \frac{1}{2} \omega_i \|\Phi_i\|^2 - \sum_{i=1}^n \bar{\Delta}_i^T \left(e_i - \frac{7 + r_2 I_1}{2}\right) \bar{\Delta}_i + e_i^T e_i \\
&\quad - \bar{\Delta}_i^T \left(e_i - \frac{8 + r_2 I_1}{2}\right) \bar{\Delta}_i + \sum_{i=1}^n \left[h_i H\left(\tau_i\right) + 1\right] \tau_i + R_2 \\
&\quad + \sum_{i=1}^n \left(\frac{1}{r_1} + \frac{\epsilon^2}{2r_2}\right) \|L_i^{-1}\|^2 \delta_i^2 \|\Phi_i\|^2 - \omega_i^2 (h_i^2 - 1) \omega_i,
\end{align*}
\]
(49)

where $I_i(\bullet)$ is a smooth function vector in regard to $\Pi_i(\bullet)$.
Since the set $\Pi_i(\bullet)$ is compact, the smooth function $I_i(\bullet)$ has a maximum $I_i$ on set $\Pi_i(\bullet)$ for the given initial conditions.
Considering (51), the virtual control law is developed as
\[
x_{(i+1)d} = - q_i e_i - e_{i+1} + \xi_i y + \hat{d}_i - L_i^{-1} \Phi_i^T M_3(\hat{x}_3),
\]
(53)

where $q_i = \text{diag}(q_{i1}, q_{i2}, \ldots, q_{in})$, $q_{ij}, j = 1, 2, \ldots, n$ is the designed positive constant.
The parameter adaptive law is designed as
\[
\dot{\hat{d}}_i = \Theta_i(\Pi_i(\hat{x}_3)) e_T L_i^{-1} - \omega_i \Phi_i,
\]
(54)

where $\Theta_i = \Theta_i^T > 0$ is the designed matrix, and $\omega_i > 0$ is the designed constant.
Invoking (53), we obtain
\[
\dot{e}_i = - q_i e_i - e_{i+1} + \xi_i y + \hat{d}_i(\bullet).
\]
(55)

Select the Lyapunov function candidate as
\[
V_i = V_{i+1} + \frac{1}{2} e_i^T e_i + \frac{1}{2} \text{tr}\left\{\Phi_i^T \Theta_i^{-1} \Phi_i\right\} + \frac{1}{2} \omega_i^2 \omega_i,
\]
(56)

By invoking (54), it outputs
\[
\begin{align*}
\dot{\omega}_i &= -h_i^{-1} \omega_i + \left(\frac{\partial x_{id}}{\partial x_{i-1}} - \frac{\partial x_{id}}{\partial x_{i-2}} - \frac{\partial x_{id}}{\partial x_{i-1}} - \frac{\partial x_{id}}{\partial y} \right) y - \frac{\partial x_{id}}{\partial \hat{d}_{i-1}} \hat{d}_{i-1} \\
&= -h_i^{-1} \omega_i + I_i(e_i, e_{i-1}, y, \hat{x}_{i-1}, \hat{\Phi}_{i-1}, \hat{d}_{i-1}),
\end{align*}
\]
(52)

where $R_2 = R_1 + 1/2 \omega_2 \|\Phi_T^*\|^2 + \hat{T}_2, \epsilon_2 = q_2 - 2 I - 1/2 \|h_i^2 - 1\|^2.

\bf{Step 3}. (3 \leq i \leq n - 1): Considering (10) and differentiating $e_i$ give
\[
\dot{e}_i = x_{(i+1)d} + e_{i+1} + L_i^{-1} \Phi_i^T M_3(\hat{x}_3) - k_1 \hat{x}_i
\]
\[ \quad + \hat{x}_i y + \hat{d}_i - \hat{x}_{id}. \]
(50)

Similar to Step 2, let $x_{id}$ pass the following first-order filter $h_i [7]:$
\[
h_i \hat{h}_i + \xi_i = x_{id}, \eta_i(0) = x_{id}(0),
\]
(51)

where $h_i = \text{diag}(h_{i1}, h_{i2}, \ldots, h_{in}) > 0$ is the time constant of the filter.
Define $\omega_i = \eta_i - x_{id}$. Differentiating $\omega_i$ yields
\[
-\text{tr}\left\{\Phi_i^T \Theta_i^{-1} \Phi_i\right\} = -\text{tr}\left\{\Phi_i^T M_3(\hat{x}_3) e_i^T L_i^{-1} - \Phi_i^T \omega_i \Phi_i\right\}
\leq e_i^T e_i \left(\frac{1}{2} \omega_i - \frac{\delta_i^2}{4} \|L_i^{-1}\|^2\right) \|\Phi_i\|^2 + \frac{1}{2} \omega_i \|\Phi_i^*\|^2.
\]
(57)

Differentiating (56), we obtain
\[
\dot{V}_i \leq - Z^T A Z - \sum_{i=1}^{n-1} e_i e_i - \sum_{i=1}^{n-1} \left(\frac{1}{2} \omega_i - \frac{\delta_i^2}{4} \|L_i^{-1}\|^2\right) \|\Phi_i\|^2 \\
- \frac{1}{2} \omega_i \|\Phi_i\|^2 - \sum_{i=2}^n \bar{\Delta}_i^T \left(e_i - \frac{7 + r_2 I_1}{2}\right) \bar{\Delta}_i + e_i^T e_i \\
- \bar{\Delta}_i^T \left(e_i - \frac{8 + r_2 I_1}{2}\right) \bar{\Delta}_i + \sum_{i=1}^n \left[h_i H\left(\tau_i\right) + 1\right] \tau_i \\
+ \sum_{i=1}^n \left(\frac{1}{r_1} + \frac{\epsilon^2}{2r_2}\right) \|L_i^{-1}\|^2 \delta_i^2 \|\Phi_i\|^2 + R_i \\
- \sum_{i=2}^{n-1} \omega_i^T (h_i^2 - 1) \omega_i,
\]
(58)
where \( R_i = R_{i-1} + 1/2\omega_i\|\Phi_i^*\|^2 + I_i^T \), \( \epsilon_i = q_i - 3/2I - 1/2\|h_i^{-1}\|^2I_1 \).

\[ R_i = R_{i-1} + 1/2\omega_i\|\Phi_i^*\|^2 + I_i^T \]

\[ \epsilon_i = q_i - 3/2I - 1/2\|h_i^{-1}\|^2I_1 \]

**Step 4.** Taking the derivative of \( e_n \) yields

\[
\dot{e}_n = u + L_n^T \Phi_n M_n(\bar{x}_n) - k_n x_n y + \bar{a}_n - x_{nd}.
\]

Similarly, let \( x_{nd} \) pass the following filter \( \eta_n \) [7]:

\[
h_n \dot{\eta}_n + \eta_n = x_{nd}, \eta_n(0) = x_{nd}(0).
\]

where \( h_n = \text{diag}[h_{n1}, h_{n2}, \ldots, h_{nm}] > 0 \) is the time constant of the filter.

Define \( \omega_n = \eta_n - x_{nd} \). Taking the derivative of \( \omega_n \) yields

\[
\dot{\omega}_n = -h_n^{-1} \omega_n - \frac{\partial x_{nd}}{\partial x_n} \dot{x}_n - \frac{\partial x_{nd}}{\partial y} \dot{y} - \frac{\partial x_{nd}}{\partial \bar{a}_n} \dot{\bar{a}}_n - \frac{\partial x_{nd}}{\partial \bar{a}_n} \dot{\bar{a}}_n
\]

\[ \dot{\Phi}_n = \Theta_n(M_n \bar{x}_n)(c_T L_n^{-1} \omega_n) \Phi_n, \]

where \( \rho_n \) is the designed constant.

Substituting (66) and (67) into (65), we have

\[
\dot{V}_n \leq -Z^T AZ - \sum_{i=1}^n e_i^T e_i - \sum_{i=1}^n \left( \frac{1}{2} \omega_i^T \frac{\delta^2}{4} \left( \frac{1}{2} \|L_i^{-1}\|^2 \right) \right) \Phi_i^2
\]

\[
-\frac{1}{2} \left( \delta^2 / 4 \right) \| \Phi_i^2 \|^2 - \sum_{i=2}^n \Delta_i^T \left( \epsilon_i - \frac{7 + r_2}{2} I_1 \right) \Delta_i + R_n
\]

\[
- \sum_{i=1}^n \left( \frac{1}{r_1} + \frac{r^2}{2r_2} \right) \left( \|L_i^{-1}\|^2 \right) \Phi_i^2
\]

\[
\sum_{i=1}^n \left( \frac{1}{2} \omega_i \left( h_i^{-1} - I_1 \right) \omega_i + \sum_{i=1}^n \left( h_i H(r_i) + 1 \right) \dot{r}_i \right)
\]

\[ \leq -Z^T AZ - \sum_{i=1}^n e_i^T e_i - \Delta_1^T \left( \epsilon_1 - \frac{8 + r_2}{2} I_1 \right) \Delta_1
\]

\[
- \frac{1}{2} \left( \delta^2 / 4 \right) \| \Phi_i^2 \|^2 - \sum_{i=2}^n \Delta_i^T \left( \epsilon_i - \frac{7 + r_2}{2} I_1 \right) \Delta_i + R_n
\]

\[
- \sum_{i=1}^n \left( \frac{1}{2} \omega_i \left( h_i^{-1} - I_1 \right) \omega_i + \sum_{i=1}^n \left( h_i H(r_i) + 1 \right) \dot{r}_i \right)
\]

\[ \sum_{i=1}^n \left( \frac{1}{2} \omega_i \left( h_i^{-1} - I_1 \right) \omega_i + \sum_{i=1}^n \left( h_i H(r_i) + 1 \right) \dot{r}_i \right)
\]

\[ \sum_{i=1}^n \left( \frac{1}{2} \omega_i \left( h_i^{-1} - I_1 \right) \omega_i + \sum_{i=1}^n \left( h_i H(r_i) + 1 \right) \dot{r}_i \right)
\]

where \( R_n = R_{n-1} + 1/2\omega_n \Phi_n^2 + I_i^T \), \( \epsilon_n = q_n - 3/2I - 1/2\|h_n^{-1}\|^2I_1 \).

3.3. Stability Analysis. Hereto, the following theorem is proposed to reveal the main results of this work.

**Theorem 1.** Consider a class of MIMO nonlinear systems (1) satisfying assumption 1-3. The state observer is designed as (10), and the NDOs are developed as (16-17), and the virtual control laws are proposed as (31), (43), and (53). Under the presented output feedback FTC scheme (62), all tracking error signals of the closed-loop system are uniformly ultimately bounded.

\[ \dot{V}_n \leq -sV_n + \sum_{i=1}^n \left[ h_i H(r_i) + 1 \right] \dot{r}_i, \]

where \( V_n \) and \( \dot{V}_n \) are given by
According to Lemma 1, it can be obtained that \( V_n \) and \( \sum_{i=1}^{n} [h_iH(\tau_i) + 1] \tau_i \) are bounded over \([0, t_f] \). By defining \( \Gamma_1 = \max_{t \in [0, t_f]} \sum_{i=1}^{n} [h_iH(\tau_i) + 1] \tau_i \), we obtain
\[
V_n \leq -sV_n + \Gamma,
\]
where \( \Gamma = \Gamma_0 + \Gamma_1 \).

Integrating (71) over \([0, t] \) yields
\[
V_n \leq \frac{\Gamma}{s} + \left( V_n(0) - \frac{\Gamma}{s} \right) e^{-st}.
\]

From (72) and the definition of \( V_n \), we obtain that \( V_n \) is exponentially convergent, i.e., \( \lim_{t \to \infty} V_n = \Gamma/s \). Hence, all the error signals of the closed-loop remain bounded. This concludes the proof.

Remark 4. In this paper, the actuator faults, which are coupled with the unknown control gains, are considered, and the Nussbaum function is utilized to eliminate their adverse effect. Compared with some direct adaptive method [11], the developed control algorithm can effectively avoid the problem of singularity and simplify the proof process. Compared with some existing observer-based output feedback control methods [18, 20, 43], the proposed robust FTC scheme can guarantee the closed-loop stability for the MIMO nonlinear system with multiple negative effects simultaneously, including unmeasured states, unknown nonlinearity, unknown control directions, actuator faults, and external disturbances.

Remark 5. In the control design, different design parameters will result in different system performance. Choosing larger \( \epsilon_i \) will increase the control gains and is beneficial for tracking results within bounds. However, excessive control gains require more control energy and may cause input saturation. Consequently, in order to achieve applicable control performance, the selection of corresponding design parameters should satisfy the following conditions and reach a compromise:
\[
A > 0, \epsilon_i > 0, 2 \epsilon_i - (8 + r_2) I_1 > 0, h_i^{-1} - I_1 > 0,
\]
\[
\omega_F > 0, \omega_i - \left( \frac{1}{2} + \frac{2}{r_1} + \frac{r_i^2}{r_2} \right) \delta_1^2 \| L_i^{-1} \| > 0.
\]

4. Simulation Examples

In this section, simulations studies on the unmanned helicopter position movement are given to illustrate the feasibility of the obtained results. The position dynamic model of unmanned helicopter with actuator faults and external disturbances can be described as [44]
\[
\dot{p} = v, \quad m \dot{v} = BB_F + mg\xi + D(t),
\]
where \( g \) is the gravitational acceleration, \( \xi = [0, 0, 1]^T \), and \( m \) defines the gross mass of the unmanned helicopter. \( P = [X, Y, Z]^T \) and \( V = [u, v, w]^T \) denote the position and velocity vectors, respectively. \( B = \text{diag} \{ B_1, B_2, B_3 \} \) is the unknown constant control gain matrix, \( \theta = \text{diag} \{ \theta_1, \theta_2, \theta_3 \} \), \( \delta \) is the remaining control efficiency, \( D(t) \) is external time-varying unknown disturbance, \( F_R = FR, F = [F_x, F_y, F_z]^T \) is the control input, and \( R \) refers to the rotation matrix whose definition can be found in [44].

In the simulation, we suppose that the unmanned helicopter is operated in constant attitude angles. Furthermore, the initial positions are assumed as \([X_0, Y_0, Z_0]^T = [1, 1, 0]^T \). Here, the unmanned helicopter is required to track the following desired trajectories:
The external time-varying disturbances are given by

\[
\begin{align*}
X_d &= 10 \cos\left(\frac{\pi}{10} t\right) m, \\
Y_d &= 10 \sin\left(\frac{\pi}{10} t\right) m, \\
Z_d &= -10(1 - e^{-3t}) m,
\end{align*}
\]  

(75)

Figure 2: Tracking errors \(e_{1t}\) without robust FTC.

Figure 3: Tracking trajectories \(y\) and \(y_d\) without robust FTC.

Figure 4: Contrastive results of tracking errors under proposed controller and adaptive FTC method [11].

Figure 5: Contrastive results of output \(y\) under the proposed controller and the adaptive FTC method [11].
In view of the analysis process in Section 3, the control object is to design control input vector $F$ to make the output $y$ track the desired trajectory $y_d$ as closely as possible. The basic parameters of the unmanned helicopter are selected as $m = 800$ kg, $g = 9.8$, $\phi = 10^\circ$, $\theta = 15^\circ$ and $\psi = 20^\circ$. The control gains and actuator faults are assumed as $B = \text{diag}[1.1, 1.2, 1.3]$ and $\theta = \text{diag}[0.8, 0.7, 0.6]$, respectively. The relevant designed parameters are selected as $q_1 = \text{diag}[5, 5, 5]$, $q_2 = \text{diag}[1000, 200, 500]$, $k_1 = \text{diag}[2, 2, 2]$, $k_2 = \text{diag}[1, 1, 1]$, $\chi_1 = \text{diag}[2, 2, 2]$, $\chi_2 = \text{diag}[1, 1, 1]$, $\omega_1 = 0.001$, $L_1 = \text{diag}[20, 20, 20]$.

First, the position tracking error $e_1 = y - y_d$ without robust FTC is shown in Figure 2. It can be noted that all tracking errors deviate from the origin severely. To reflect the movement of the helicopter more visually, Figure 3 is presented, where the magenta lines $Pd$ represent the desired trajectories, and the black lines $Ps$ are the actual outputs. From Figure 3, it can be seen that if there is no effective robust FTC scheme, satisfactory trajectory tracking performance cannot be obtained. Simulation results of Figures 2 and 3 reveal that the actuator faults, which are coupled with the unknown control directions and external disturbances, have a great negative influence on the control performance of the helicopter system. If these negative effects can not be tackled timely, they will lead to degradation of system performance and even catastrophic consequences.

In order to ensure the safe flight for the unmanned helicopter with unavailable states, unknown control directions, actuator faults, and external disturbances, a robust adaptive NN output feedback FTC scheme is developed, and the corresponding simulations are given by Figures 4–7.

![Figure 6: Input signals of the proposed controller.](image)

![Figure 7: The norm of the RBFNN weight matrix estimation $\hat{M}$.](image)

Figure 4 provides the contrastive results of tracking errors, where the blue lines $e_{ip}$ denote the tracking errors under the proposed controller, and the red lines $e_{in}$ denote these under adaptive FTC strategy proposed in [11]. According to Figure 4, it can be observed that both control schemes can guarantee that the tracking errors converge to a small boundary. However, compared with the adaptive FTC strategy proposed in [11], the developed control algorithm has faster convergence rate and smaller tracking error. Meanwhile, the three-dimensional graph of the output trajectory $y$ is provided in Figure 5, where the blue lines $Pp$ denote the system output under the proposed controller, and the red lines $Pn$ refer to these under adaptive FTC strategy proposed in [11]. From Figure 5, we can see that the presented control scheme can make the system output reach the desired trajectory faster, which reflects the same conclusion as Figure 4. The curve of designed fault tolerant controller is provided in Figure 6. Figure 6 manifests that the control command signals are convergent and can adjust dynamically to restrain the adverse impacts. Furthermore, the RBFNN weight matrix changes in reasonable boundary as time goes on, which is displayed in Figure 7.

5. Conclusion

In this paper, a NN-based adaptive output feedback FTC scheme has been proposed for uncertain MIMO nonlinear systems with unmeasured states, unknown control directions, actuator faults, and external disturbances. The RBFNN has been utilized to approximate the unknown nonlinear function, and the NDO has been constructed to estimate the external disturbances. Especially, the unknown actuator faults and control directions have been handled by the constructed Nussbaum function. The stability of the closed-loop system has been proved, and the performance of the presented controller has been confirmed through simulation results on unmanned helicopter position movement. In the future, the novel robust FTC strategy for the variable gain nonlinear systems with random faults and disturbances is worthy of further exploration, which are the other common difficulties in practice.
Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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