Solvability of the quaternionic Monge-Ampère equation on compact manifolds with a flat hyperKähler metric.

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Abstract
A quaternionic version of the Calabi problem was formulated in [6]. It conjectures a solvability of a quaternionic Monge-Ampère equation on a compact HKT manifold (HKT stays for HyperKähler with Tor-sion). In this paper this problem is solved under the extra assumption that the manifold admits a flat hyperKähler metric compatible with the underlying hypercomplex structure. The proof uses the continuity method and a priori estimates.

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0 Introduction.

In recent years there was suggested a quaternionic analogue of the classical real and complex Monge-Ampère equations. Thus in [3] the author has introduced quaternionic Monge-Ampère equation on the flat quaternionic space $\mathbb{H}^n$ and solved the Dirichlet problem for it under appropriate assumptions. Then M. Verbitsky and the author [5] have generalized the equation to the broader class of so called hypercomplex manifolds. They have also formulated a conjecture about existence of a solution of this quaternionic Monge-Ampère equation which is a quaternionic analogue of the well known Calabi problem for the complex Monge-Ampère equation. Moreover they proved a uniform a priori estimate for its solution (under some assumption) and uniqueness of a solution up to a constant. The goal of this paper is to solve the conjecture in the case of a compact hypercomplex manifold which admits a flat hyperKähler metric compatible with the underlying hypercomplex structure (see Theorem 0.10 below).

Recall that the original Calabi problem for the complex Monge-Ampère equation was formulated by him in 1954. It was eventually solved by Yau in 1976 [31]. Before this, Aubin [8] had made significant progress towards its proof. A real version of the Calabi problem was formulated and solved by Cheng and Yau [13].

Let us also mention that recently Harvey and Lawson [22] have extended the notion of (homogeneous) Monge-Ampère equation beyond real, complex, and quaternionic cases.

In order to formulate the main result precisely, let us recall the notions of hypercomplex and HKT-manifolds.

0.1 Definition. A hypercomplex manifold is a smooth manifold $M$ together with a triple $(I, J, K)$ of complex structures satisfying the usual quaternionic relations:

$$IJ = -JI = K.$$
(2) It follows that the dimension of a hypercomplex manifold $M$ is divisible by 4.

(3) Hypercomplex manifolds were explicitly introduced by Boyer [12].

Let $(M^{4n}, I, J, K)$ be a hypercomplex manifold. Let us denote by $\Lambda^p_q(I) (M)$ the vector bundle of differential forms of the type $(p, q)$ on the complex manifold $(M, I)$. By the abuse of notation we will also denote by the same symbol $\Lambda^p_q(I) (M)$ the space of $C^\infty$-sections of this bundle.

Let

$$\partial: \Lambda^p_q(I) (M) \rightarrow \Lambda^{p+1}_q(I) (M)$$

be the usual $\partial$-differential on differential forms on the complex manifold $(M, I)$.

Set

$$\partial J := J^{-1} \circ \overline{\partial} \circ J. \quad (0.2)$$

0.3 Claim (28). (1) $\Lambda^p_q(I) (M) \rightarrow \Lambda^q_p(I) (M)$.

(2) $\partial J: \Lambda^p_q(I) (M) \rightarrow \Lambda^{p+1}_q(I) (M)$.

(3) $\partial \partial J = -\partial J \partial$.

0.4 Definition (28). Let $k = 0, 1, \ldots, n$. A form $\omega \in \Lambda^{2k,0}(M)$ is called real if

$$\overline{J \circ \omega} = \omega.$$ We will denote the subspace of real $C^\infty$-smooth $(2k, 0)$-forms on $(M, I)$ by $\Lambda^{2k,0}_{I, \mathbb{R}}(M)$.

0.5 Lemma. Let $(M, I, J, K)$ be a hypercomplex manifold. Let $f: M \rightarrow \mathbb{R}$ be a smooth function. Then $\partial \partial_J f \in \Lambda^{2,0}_{I, \mathbb{R}}(M)$.

We call $\partial \partial_J h$ the quaternionic Hessian of $f$. In many respects it is analogous to the usual real and complex Hessians. It becomes particularly transparent on the flat space $\mathbb{H}^n$ where it can be written in coordinates; see the discussion in Section 2.

0.6 Definition. Let $\omega \in \Lambda^{2,0}_{I, \mathbb{R}}(M)$. Let us say that $\omega$ is non-negative (notation: $\omega \geq 0$) if

$$\omega(Y, Y \circ J) \geq 0.$$
for any (real) vector field $Y$ on the manifold $M$. The form $\omega$ is called strictly positive (notation: $\omega > 0$) if $\omega(Y, Y \circ J) > 0$ for any non-vanishing (real) vector field $Y$.

Equivalently, $\omega$ is non-negative (resp., strictly positive) if and only if $\omega(Z, \bar{Z} \circ J) \geq 0$ (resp., $> 0$) for any non-vanishing $(1, 0)$-vector field $Z$.

Let $g$ be a Riemannian metric on a hypercomplex manifold $M$. The metric $g$ is called quaternionic Hermitian (or hyperhermitian) if $g$ is invariant with respect to the group $SU(2) \subset \mathbb{H}$ of unitary quaternions, i.e. $g(X \cdot q, Y \cdot q) = g(X, Y)$ for any (real) vector fields $X, Y$ and any $q \in \mathbb{H}$ with $|q| = 1$.

Given a quaternionic Hermitian metric $g$ on a hypercomplex manifold $M$, consider the differential form

$$\Omega := \omega_J - \sqrt{-1} \omega_K$$

where $\omega_L(A, B) := g(A, B \circ L)$ for any $L \in \mathbb{H}$ with $L^2 = -1$, and any real vector fields $A, B$ on $M$. It is easy to see that $\Omega$ is a $(2, 0)$-form with respect to the complex structure $I$. Moreover $\Omega$ is real in the sense of Definition 0.6, thus $\Omega \in \Lambda^{2,0}_{I,\mathbb{R}}(M)$.

**0.7 Definition.** The metric $g$ on $M$ is called HKT-metric if

$$\partial \Omega = 0.$$ 

We call such a form $\Omega$, corresponding to an HKT-metric, an HKT-form.

**0.8 Remark.** HKT manifolds were introduced in the physical literature by Howe and Papadopoulos [23]. For the mathematical treatment see Grantcharov-Poon [20] and Verbitsky [28]. The original definition of HKT-metrics in [23] was different but equivalent to Definition 0.7; the latter was given in [20].

**0.9 Remark.** The classical hyperKähler metrics (i.e. Riemannian metrics with holonomy contained in the group $Sp(n)$) form a subclass of HKT-metrics. It is well known that a quaternionic Hermitian metric $g$ is hyperKähler if and only if the form $\Omega$ is closed, or equivalently $\partial \Omega = \bar{\partial} \Omega = 0$.

Now we can formulate the main result.

**0.10 Theorem.** Let $(M^{4n}, I, J, K)$ be a compact connected hypercomplex manifold with an HKT form $\Omega_0$. Let us assume in addition that it admits a flat hyperKähler metric compatible with the underlying hypercomplex

\[\text{In this paper all HKT-metrics, and consequently HKT-forms, are assumed to be infinitely smooth.}\]
structure. Let $f \in C^\infty(M)$ be a real valued function. Then there exists a unique constant $A$ such that the quaternionic Monge-Ampère equation

$$(\Omega_0 + \partial \partial_J \phi)^n = Ae^f \Omega_0^n$$

(0.3)

has a $C^\infty$-smooth solution.

0.11 Remark. (1) It was shown in [6] that solution $\phi$ is unique up to an additive constant.

(2) The constant $A$ is determined as follows. Let $\Omega$ be the HKT-form corresponding to the flat hyperKähler metric whose existence is assumed in the theorem. Then $A$ is found from the equation

$$\int_M Ae^f \cdot \Omega^n \wedge \bar{\Omega}^n = \int_M \Omega_0^n \wedge \bar{\Omega}^n.$$

(3) This theorem was conjectured by M. Verbitsky and the author in [6] in a more general form: without the assumption of existence a flat hyperKähler metric.

(4) The equation (0.3) is a non-linear second order elliptic differential equation. The ellipticity was shown in [6].

(5) Existence of a flat hyperKähler metric implies that the hypercomplex structure $(I, J, K)$ is locally flat, i.e. locally isomorphic to the standard flat space $\mathbb{H}^n$ of $n$-tuple of quaternions.

(6) This theorem can be stated in a slightly more refined form involving Hölder spaces rather than $C^\infty$, see Theorem 5.3 below.

(7) The obvious example of a hypercomplex manifold $M$ satisfying the assumptions of the theorem is a quaternionic torus: quotient of $\mathbb{H}^n$ by a lattice. However there are more examples coming from the Bieberbach classification of crystallographic groups (see e.g. [30]).

Notice that recently Verbitsky [29] has suggested a geometric interpretation of solutions of the equation (0.3) under appropriate assumptions on the right hand side.

The proof of the theorem uses the continuity method and a priori estimates. The standard elliptic regularity machinery, discussed in Section 5, implies that it suffices to prove a $C^{2,\alpha}$ a priori estimate for some $\alpha \in (0, 1)$. The $C^0$ estimate was obtained first in [6] under more general assumptions than in Theorem 0.10. Very recently Shelukhin and the author [4] have obtained a $C^0$ estimate by a different method and under different assumptions.
than in [6] which however are also satisfied in Theorem 0.10. The main point of this paper is to make two following steps: first to obtain $C^0$ estimate on a Laplacian of $\phi$ (Section 3), and then to deduce from it a $C^{2,\alpha}$ estimate (Section 4). The first step uses a modification of the well known Pogorelov’s method. This modification is not completely straightforward, and this is exactly the step where all the assumptions of the theorem are used, i.e. existence of a flat hyperKähler metric. The second step uses a quaternionic version of the Evans-Krylov method (see Section 4 for further references). It works under more general assumptions, namely on manifolds with locally flat hypercomplex structure (which may not admit a compatible hyperKähler metric).

In Section 1 we recall relevant definitions and facts from the quaternionic linear algebra. In Section 2 we recall few facts on HKT-manifolds. These two sections contain no new results, they are added for convenience of the reader only.

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1 Quaternionic linear algebra.

The standard theory of vector spaces, basis, and dimension works over any non-commutative field, e.g. $\mathbb{H}$, exactly like in the commutative case. The only remark is that one should distinguish between right and left vector spaces. The two cases are completely parallel. We will restrict to the case of right vector spaces, i.e. vectors are multiplied by scalars on the right.

However the theory of non-commutative determinants is quite different and deserves special discussion. We will need to remind the notion of Moore determinant on the class of quaternionic matrices called hyperhermitian\footnote{The Moore determinant was used in the original approach of [2] to define the quaternionic Monge-Ampère operator on the flat space $\mathbb{H}^n$. Later on, this operator was generalized in [5] to more general class of hypercomplex manifolds without using explicitly the Moore determinant. However this notion often still seems to be convenient while working on the flat space; in particular it will be used extensively in this paper.} They are analogues of real symmetric and complex hermitian matrices. The behavior of the Moore determinant of such matrices is analogous in many respects to the behavior of the usual determinant of real symmetric and com-
plex hermitian matrices. We believe that any general identity or inequality known for usual determinant of the real symmetric or complex hermitian matrices can be generalized to the Moore determinant of hyperhermitian matrices, though the proofs might be slightly more tricky. Here we review some of the relevant material. The discussion on determinants follows [2] where most of the proofs can be found. Another good reference to quaternionic determinants is [7]; for a relation of quaternionic determinants to a general theory [17] of non-commutative (quasi-) determinants see [16].

1.1 Definition. Let \( V \) be a right \( \mathbb{H} \)-vector space. A hyperhermitian semilinear form on \( V \) is a map \( a : V \times V \to \mathbb{H} \) satisfying the following properties:

(a) \( a \) is additive with respect to each argument;
(b) \( a(x \cdot y, q) = a(x, y) \cdot q \) for any \( x, y \in V \) and any \( q \in \mathbb{H} \);
(c) \( a(x, y) = a(y, x) \).

1.2 Remark. Hyperhermitian semi-linear forms on \( V \) are in bijective correspondence with real valued quadratic forms on the underlying real space \( \mathbb{R}V \) of \( V \)

\[ b : \mathbb{R}V \to \mathbb{R} \]

which are invariant under multiplication by the norm one quaternions, i.e. \( b(x \cdot q) = b(x) \) for any \( x \in V \) and any \( q \in \mathbb{H} \) with \( |q| = 1 \).

1.3 Example. Let \( V = \mathbb{H}^n \) be the standard coordinate space considered as right vector space over \( \mathbb{H} \). Fix a hyperhermitian \( n \times n \)-matrix \( (a_{ij})_{i,j=1}^n \), i.e. \( a_{ij} = \bar{a}_{ji} \), where \( \bar{q} \) denotes the usual quaternionic conjugation of \( q \in \mathbb{H} \). For \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) define

\[ A(x, y) = \sum_{i,j} \bar{x}_i a_{ij} y_j \]

(notice the order of the terms!). Then \( A \) defines hyperhermitian semilinear form on \( V \).

The set of all hyperhermitian \( n \times n \)-matrices will be denoted by \( \mathcal{H}_n \). Then \( \mathcal{H}_n \) a vector space over \( \mathbb{R} \).

In general one has the following standard claims.

1.4 Claim. Fix a basis in a finite dimensional right quaternionic vector space \( V \). Then there is a natural bijection between the space of hyperhermitian semilinear forms on \( V \) and the space \( \mathcal{H}_n \) of \( n \times n \)-hyperhermitian matrices.
This bijection is in fact described in previous Example 1.3.

**1.5 Claim.** Let $A$ be a matrix of the given hyperhermitian form in a given basis. Let $C$ be transition matrix from this basis to another one. Then the matrix $A'$ of the given form in the new basis is equal

$$A' = C^* A C,$$

where $(C^*)_{ij} = C_{ji}$.

**1.6 Remark.** Note that for any hyperhermitian matrix $A$ and for any matrix $C$ the matrix $C^* A C$ is also hyperhermitian. In particular the matrix $C^* C$ is always hyperhermitian.

**1.7 Definition.** A hyperhermitian semilinear form $a$ is called **positive definite** if $a(x,x) > 0$ for any non-zero vector $x$. Similarly $a$ is called **non-negative definite** if $a(x,x) \geq 0$ for any vector $x$.

Let us fix on our quaternionic right vector space $V$ a positive definite hyperhermitian form $(\cdot, \cdot)$. The space with fixed such a form will be called **hyperhermitian space**.

For any quaternionic linear operator $\phi : V \to V$ in hyperhermitian space one can define the adjoint operator $\phi^* : V \to V$ in the usual way, i.e. $(\phi x, y) = (x, \phi^* y)$ for any $x, y \in V$. Then if one fixes an orthonormal basis in the space $V$ then the operator $\phi$ is selfadjoint if and only if its matrix in this basis is hyperhermitian.

**1.8 Claim.** For any selfadjoint operator in a hyperhermitian space there exists an orthonormal basis such that its matrix in this basis is diagonal and real.

Now we are going to define the Moore determinant of hyperhermitian matrices. The definition below is different from the original one [26] but equivalent to it.

Any quaternionic matrix $A \in M_n(\mathbb{H})$ can be considered as a matrix of an $\mathbb{H}$-linear endomorphism of $\mathbb{H}^n$. Identifying $\mathbb{H}^n$ with $\mathbb{R}^{4n}$ in the standard way we get an $\mathbb{R}$-linear endomorphism of $\mathbb{R}^{4n}$. Its matrix in the standard basis will be denoted by $\mathbb{R} A$, and it is called the realization of $A$. Thus $\mathbb{R} A \in M_{4n}(\mathbb{R})$.

Let us consider the entries of $A$ as formal variables (each quaternionic entry corresponds to four commuting real variables). Then $\det(\mathbb{R} A)$ is a homogeneous polynomial of degree $4n$ in $n(2n - 1)$ real variables. Let us denote by $Id$ the identity matrix. One has the following result.
1.9 Theorem. There exists a polynomial \( P \) defined on the space \( \mathcal{H}_n \) of all hyperhermitian \( n \times n \)-matrices such that for any hyperhermitian \( n \times n \)-matrix \( A \) one has \( \det(\mathbb{R}A) = P^4(A) \) and \( P(\text{Id}) = 1 \). \( P \) is defined uniquely by these two properties. Furthermore \( P \) is homogeneous of degree \( n \) and has integer coefficients.

Thus for any hyperhermitian matrix \( A \) the value \( P(A) \) is a real number, and it is called the Moore determinant of the matrix \( A \). The explicit formula for the Moore determinant was given by Moore [26] (see also [7]). From now on the Moore determinant of a matrix \( A \) will be denoted by \( \det A \). This notation should not cause any confusion with the usual determinant of real or complex matrices due to part (i) of the next theorem.

1.10 Theorem. (i) The Moore determinant of any complex hermitian matrix considered as quaternionic hyperhermitian matrix is equal to its usual determinant.

(ii) For any hyperhermitian \( n \times n \)-matrix \( A \) and any matrix \( C \in M_n(\mathbb{H}) \) the Moore determinant satisfies

\[
\det(C^*AC) = \det A \cdot \det(C^*C).
\]

1.11 Example. (a) Let \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \) be a diagonal matrix with real \( \lambda_i \)'s. Then \( A \) is hyperhermitian and the Moore determinant \( \det A = \prod_{i=1}^n \lambda_i \).

(b) A general hyperhermitian \( 2 \times 2 \)-matrix \( A \) has the form

\[
A = \begin{bmatrix} a & q \\ \bar{q} & b \end{bmatrix},
\]

where \( a, b \in \mathbb{R}, q \in \mathbb{H} \). Then \( \det A = ab - \bar{q}q \).

1.12 Definition. A hyperhermitian \( n \times n \)-matrix \( A = (a_{ij}) \) is called positive (resp. non-negative) definite if for any non-zero vector \( \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \) one has

\[
\xi^*A\xi = \sum_{ij} \xi_i a_{ij} \xi_j > 0 \text{ (resp. } \geq 0).\]

1.13 Claim. Let \( A \) be a non-negative (resp. positive) definite hyperhermitian matrix. Then \( \det A \geq 0 \) (resp. \( \det A > 0 \)).
Moreover there is a version of the Sylvester criterion of positive definite-
ness of a hyperhermitian matrix. It is formulated in terms of the Moore
determinants and is completely analogous to the classical real and complex
results, see [2], Theorem 1.1.13.

Let us remind now the definition of the mixed determinant of hyperher-
mitian matrices in analogy with the case of real symmetric matrices [1].

1.14 Definition. Let $A_1, \ldots, A_n$ be hyperhermitian $n \times n$-matrices. Con-
sider the homogeneous polynomial in real variables $\lambda_1, \ldots, \lambda_n$ of degree $n$
equal to $\det(\lambda_1 A_1 + \cdots + \lambda_n A_n)$. The coefficient of the monomial $\lambda_1 \cdots \lambda_n$
divided by $n!$ is called the mixed determinant of the matrices $A_1, \ldots, A_n$, and
it is denoted by $\det(A_1, \ldots, A_n)$.

Note that the mixed determinant is symmetric with respect to all vari-
ables, and linear with respect to each of them, i.e.

\[
\det(\lambda A_1' + \mu A_1'', A_2, \ldots, A_n) = \lambda \cdot \det(A_1', A_2, \ldots, A_n) + \mu \cdot \det(A_1'', A_2, \ldots, A_n)
\]

for any real $\lambda, \mu$. Note also that $\det(A, \ldots, A) = \det A$.

1.15 Theorem. The mixed determinant of positive (resp. non-negative)
definite matrices is positive (resp. non-negative).

This theorem is proved in [2], Theorem 1.1.15(i). Moreover a version of
the A.D. Aleksandrov inequality for mixed determinants can be proven, see
[2], Theorem 1.1.15 and Corollary 1.1.16.

2 HKT manifolds.

In this section we recall few facts about HKT-manifolds in addition to those
stated in the introduction.

2.1 Definition ([5]). Let $(M, I, J, K)$ be a hypercomplex manifold. A $C^2$-
smooth function

\[
h : M \to \mathbb{R}
\]
is called quaternionic plurisubharmonic if $\partial \partial_J h$ is a non-negative section of $\Lambda^2_{I,\mathbb{R}}(M)$. $h$ is called strictly plurisubharmonic if $\partial \partial_J h$ is strictly positive at
every point.
2.2 Remark. The notion of quaternionic plurisubharmonicity can be generalized to continuous functions, see [5], Section 5. On the flat space $\mathbb{H}^n$ this notion was earlier defined even for upper semi-continuous functions in [2].

Let us discuss the relations of plurisubharmonic functions to the HKT-geometry. Let us denote by $S_H(M)$ the vector bundle over a hypercomplex manifold $M$ such that its fiber over a point $x \in M$ is equal to the space of hyperhermitian forms on the tangent space $T_x M$. Consider the map of vector bundles

$$t: \Lambda_{I,\mathbb{R}}^{2,0}(M) \to S_H(M)$$

(2.1)

defined by $t(\eta)(A) = \eta(A, A \circ J)$ for any (real) vector field $A$ on $M$. Then $t$ is an isomorphism of vector bundles (this was proved in [28]).

2.3 Theorem ([5], Prop. 1.14). (1) Let $f$ be an infinitely smooth strictly plurisubharmonic function on a hypercomplex manifold $(M, I, J, K)$. Then $t(\partial J f)$ is an HKT-metric.

(2) Conversely assume that $g$ is an HKT-metric. Then any point $x \in M$ has a neighborhood $U$ and an infinitely smooth strictly plurisubharmonic function $f$ on $U$ such that $g = t(\partial J f)$ in $U$. Equivalently $\Omega = \partial J f$, where $\Omega$ is the HKT-form corresponding to $g$ (as defined in the introduction).

In this paper we will often work with the flat hypercomplex manifold $\mathbb{H}^n$. In this case there is an equivalent way to rewrite the quaternionic Hessian and Monge-Ampère operator. Now we are going to describe them following the original approach of [2]. We also believe that in this language the analogies with the classical real and complex cases become more explicit.

We will write a quaternion $q \in \mathbb{H}$ in the standard form

$$q = t + x \cdot i + y \cdot j + z \cdot k,$$

where $t, x, y, z$ are real numbers, and $i, j, k$ satisfy the usual quaternionic relations

$$i^2 = j^2 = k^2 = -1, \ i j = -j i = k, \ j k = -k j = i, \ k i = -i k = j.$$

The Dirac (or Cauchy-Riemann) operator $\frac{\partial}{\partial q}$ is defined as follows. For any $\mathbb{H}$-valued function $F$

$$\frac{\partial}{\partial q} F := \frac{\partial F}{\partial t} + i \frac{\partial F}{\partial x} + j \frac{\partial F}{\partial y} + k \frac{\partial F}{\partial z}.$$
Let us also define the operator \( \partial_{\bar{q}} \):
\[
\partial_{\bar{q}} F := \partial_{\bar{q}} F = \frac{\partial F}{\partial t} - \frac{\partial F}{\partial x} i - \frac{\partial F}{\partial y} j - \frac{\partial F}{\partial z} k.
\]

In the case of several quaternionic variables, it is easy to see that the operators \( \partial_{q_i} \) and \( \partial_{\bar{q}_j} \) commute:
\[
\left[ \partial_{q_i}, \partial_{\bar{q}_j} \right] = 0. \tag{2.2}
\]

For any real valued functions \( f \) on the flat space \( \mathbb{H}^n \) the matrix \( \left( \frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} \right) \) is hyperhermitian; it corresponds exactly (up to a constant) to the quaternionic Hessian. More precisely, using the isomorphism \( t \) from (2.1) one has:
\[
t(\partial \partial_J f) = \kappa \left( \frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} \right), \tag{2.3}
\]
where \( \kappa > 0 \) is a normalizing constant, by Proposition 4.1 of [5]. The precise value of \( \kappa \) will not be important. In what follows it will be convenient to renormalize the isomorphism \( t \) to make this constant to be 1. We will denote by \( Hess_\mathbb{H} f \) the matrix in the right hand side of (2.3) (with \( \kappa = 1 \)).

It is not hard to show that a \( C^2 \) smooth function \( f \) on \( \mathbb{H}^n \) is plurisubharmonic if and only if the hyperhermitian matrix \( \left( \frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} \right) \) is non-negative definite everywhere (see [2], [5]).

**2.4 Proposition ([2]).** (i) Let \( f : \mathbb{H}^n \to \mathbb{H} \) be a smooth function. Then for any \( \mathbb{H} \)-linear transformation \( A \) of \( \mathbb{H}^n \) (as a right \( \mathbb{H} \)-vector space) one has the identities
\[
\left( \frac{\partial^2 f(Aq)}{\partial q_i \partial \bar{q}_j} \right) = A^* \left( \frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} (Aq) \right) A.
\]

(ii) If, in addition, \( f \) is real valued then for any \( \mathbb{H} \)-linear transformation \( A \) of \( \mathbb{H}^n \) and any quaternion \( a \) with \( |a| = 1 \)
\[
\left( \frac{\partial^2 f(A(q \cdot a))}{\partial q_i \partial \bar{q}_j} \right) = A^* \left( \frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} (A(q \cdot a)) \right) A.
\]

It remains to rewrite the Monge-Ampère operator \((\partial \partial_J f)^n\) in this language. Up to a positive normalizing constant which we ignore, the Monge-Ampère operator of a real valued function \( f \) is equal to the Moore determinant \( \det \left( \frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} \right) \).
3 Second order estimate.

The main result of this section is Theorem 3.7 below. It establishes a uniform estimate on the Laplacian of the solution of the Monge-Ampere equation (0.3). Let us introduce a bit more notation. To shorten the notation, it will be convenient to denote the quaternions 1, i, j, k by $e_0, e_1, e_2, e_3$ respectively. Furthermore the $p$-th coordinate of a quaternionic $n$-tuple $q = (q_1, \ldots, q_n)$ will be written as

$$q_p = \sum_{i=0}^{3} x_p^i e_p,$$

where $x_p^i \in \mathbb{R}$. The partial derivative of a function $F$ with respect to the real coordinate $x_p^i$ will be denoted by $F_{x_p^i}$.

First we prove the following elementary inequality.

3.1 Proposition. Let $u \in C^4$ be a strictly plurisubharmonic function such that at a given point $z$ its quaternionic Hessian $(u_{ij})$ is diagonal. Then at this point $z$ one has

$$\sum_{p=0}^{3} \sum_{i,k=1}^{n} \frac{|u_{kk}x_p^i|^2}{u_{ii}u_{kk}} \leq 2 \sum_{p=0}^{3} \sum_{i,k,l=1}^{n} \frac{|u_{kkx_p^i}|^2}{u_{ii}u_{kk}}.$$  \hspace{1cm} (3.1)

Proof. Let us fix now the indices $i, k$ and compare the summands containing this pair of indices in both sides of (3.1).

First consider the case $i = k$. In the left hand side we have

$$\sum_{p=0}^{3} \frac{|u_{kkx_p^i}|^2}{u_{ii}u_{kk}}.$$  \hspace{1cm} (3.2)

In the right hand side of (3.1) we have

$$2 \sum_{p=0}^{3} \sum_{l=1}^{n} \frac{|u_{kkx_p^i}|^2}{u_{ii}u_{kk}}.$$  \hspace{1cm} (3.3)

It is clear that (3.2) \leq (3.3).

Let us consider the case now $i \neq k$. The left hand side of (3.1) contains two summands with the pair $i, k$:

$$\frac{1}{u_{ii}u_{kk}} \sum_p (|u_{kkx_p^i}|^2 + |u_{ii}x_p^i|^2).$$  \hspace{1cm} (3.4)
The right hand side of (3.1) contains two summands with the pair $i, k$:

$$\frac{2}{u_{i\bar{i}}u_{k\bar{k}}} \sum_{p,l} (|u_{k\bar{i}x_p}|^2 + |u_{i\bar{k}x_p}|^2) = \frac{4}{u_{i\bar{i}}u_{k\bar{k}}} \sum_{p,l} |u_{k\bar{i}x_p}|^2. \quad (3.5)$$

To finish the proof of proposition, it suffices now to show that (3.4) $\leq$ (3.5), or explicitly after cancelling out the term $u_{i\bar{i}}u_{k\bar{k}}$ on both sides, it reduces to

$$\sum_{p,l} (|u_{k\bar{i}x_p}|^2 + |u_{i\bar{k}x_p}|^2) \leq 4 \sum_{p,l} |u_{k\bar{i}x_p}|^2. \quad (3.6)$$

In order to show such a general inequality it suffices to sum up in the right hand side over $l = i, k$. Thus (3.6) follows from

$$\sum_{p} (|u_{k\bar{i}x_p}|^2 + |u_{i\bar{k}x_p}|^2) \leq 4 \sum_{p} (|u_{k\bar{i}x_p}|^2 + |u_{k\bar{i}x_p}|^2). \quad (3.7)$$

In the last inequality we may separate summands containing derivatives $kki$ and $kii$. These two inequalities are completely symmetric and obtained one from the other by exchange $i$ by $k$. Thus it is enough to show

$$\sum_{p} |u_{k\bar{k}x_p}|^2 \leq 4 \sum_{p} |u_{k\bar{i}x_p}|^2. \quad (3.8)$$

Let us define two operators $\partial_k, \partial_{\bar{k}}$ acting on the space of quaternion valued functions:

$$\begin{align*}
\partial_k \Phi &:= \sum_{p=0}^{3} \frac{\partial \Phi}{\partial x_k^p} \bar{e}_p, \\
\partial_{\bar{k}} \Phi &:= \sum_{p=0}^{3} \frac{\partial \Phi}{\partial x_{\bar{k}}^p} e_p,
\end{align*}$$

where $\bar{e}_p$ denotes the quaternionic conjugate of the quaternionic unit $e_p$ (here $p = 0, \ldots, 3$).

Let $\Delta_k = \sum_{p=0}^{3} \frac{\partial^2}{(\partial x_k^p)^2}$ be the Laplacian with respect to the $k$-th quaternionic variable. Clearly $\Delta_k = \partial_k \partial_{\bar{k}} = \partial_{\bar{k}} \partial_k$. Also for real valued function $u$ $\Delta_k u = u_{kk}$.

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Let us take $\Phi = u = \sum_q e_q u_{x_q}$. Then (3.8) is rewritten

$$|\Delta_k \Phi|^2 \leq 4 \sum_p |\partial_k \Phi_{x_p}^b|^2$$

We have

$$\Delta_k \Phi = \partial_k \partial_k \Phi = \sum_q (\partial_k \Phi)_{x_q^b} e_q.$$  

Denote $\Psi := \partial_k \Phi$. Then (3.9) is rewritten:

$$|\sum_q \Psi_{x_q^b} e_q| \leq 2 \sqrt{\sum_q |\Psi_{x_q^b}|^2}$$

(3.10)

But indeed by the Cauchy-Schwarz

$$|\sum_{q=0}^3 \Psi_{x_q^b} e_q| \leq \sum_{q=0}^3 |\Psi_{x_q^b}| \leq \sqrt{4} \cdot \sqrt{\sum_{q=0}^3 |\Psi_{x_q^b}|^2}.$$  

Q.E.D.

For any smooth function $g$ we denote by $g_a$, $g_{ab}$ the first and second derivatives of $g$ with respect to coordinates with indices $a$ and $a, b$ respectively (thus $a, b$ could be $x_p^i, x_q^j$).

3.2 Proposition. Let $U$ be a smooth function with values in hyperhermitian invertible matrices. Let $\det U = F$.

Then

$$Tr(U^{-1}U_{ab}) = Tr(U^{-1}U_a U^{-1}U_b) + (\log F)_{ab}$$

The proof is by straightforward computation using the identity

$$(\det U)_a = \det U \cdot Tr(U^{-1}U_a).$$

We will need few more formulas. Let $U$ denote the quaternionic Hessian of a function $u \in C^4$. Let $G$ be a smooth function with values in positive definite hyperhermitian matrices. Define the Laplacian

$$\Delta u := Tr(G^{-1}U).$$

(3.11)
3.3 Proposition. Let $\gamma: \mathbb{R} \to \mathbb{R}$ be a smooth function. Then

$$[\gamma'(\Delta u)]_{ab} = \gamma''(\Delta u) \cdot \Delta u_{ab}$$

Proof. We have

$$[\gamma'(\Delta u)]_{ab} = \gamma'(\Delta u) \cdot (\Delta u)_a \cdot (\Delta u)_b + \gamma'(\Delta u) \cdot (\Delta u)_{ab}.$$ 

Next we have

$$(\Delta u)_a = (Tr(G^{-1}U))_a =$$

$$-Tr(G^{-1}G_aG^{-1}U) + Tr(G^{-1}U_a).$$

Also

$$(\Delta u)_{ab} = [-Tr(G^{-1}G_aG^{-1}U) + Tr(G^{-1}U_a)]_b =$$

$$Tr(G^{-1}G_aG^{-1}G_bG^{-1}U) + Tr(G^{-1}G_bG^{-1}G_aG^{-1}U)$$

$$-Tr(G^{-1}G_{ab}G^{-1}U)$$

$$-Tr(G^{-1}G_aG^{-1}U_b) - Tr(G^{-1}G_bG^{-1}U_a)$$

$$+ Tr(G^{-1}U_{ab}).$$

The proposition follows. Q.E.D.

Given a fixed plurisubharmonic function $u \in C^4$, let us define another Laplacian

$$\Delta' v := Tr(U^{-1}V)$$

(3.12)

where $U$ and $V$ are quaternionic Hessians of $u$ and $v$ respectively. Proposition 3.3 implies immediately
3.4 Proposition. Let \( u \in C^4 \) be a strictly plurisubharmonic function. Let us assume that \( G \) is a flat hyperKähler metric. Then choose (locally) coordinates such that \( G \equiv \text{Id} \). Then in these coordinates

\[
[\gamma(\Delta u)]_{ab} = \gamma''(\Delta u) \cdot Tr(U_a) \cdot Tr(U_b) + \gamma'(\Delta u) Tr(U_{ab}).
\]

If moreover at a point \( z \) the matrix \( U(z) = (u_{ij}(z)) \) is diagonal then at this point \( z \) one has

\[
\Delta'(\gamma(\Delta u)) = \gamma''(\Delta u) \sum_{i,p} \frac{1}{u_{ii}} (\text{Tr}U_{x_i}^p)^2 + \gamma'(\Delta u) \sum_{i,k} \frac{u_{ikk}}{u_{ii}}.
\]

3.5 Corollary. Let \( u \in C^4 \) be a strictly plurisubharmonic function. Let us assume that \( G \) is a flat metric. Let us fix a point \( z \). Then choose (locally) coordinates such that \( G \equiv \text{Id} \) in a neighborhood, and \( U := (u_{ij}) \) is diagonal at \( z \). Let \( F := \det U \) as previously. Then in these coordinates we have at the point \( z \)

\[
\Delta'(\gamma(\Delta u)) = \gamma''(\Delta u) \sum_{i,p} \frac{1}{u_{ii}} (\text{Tr}U_{x_i}^p)^2 + \gamma'(\Delta u) \left[ \sum_{i,l,n,p} \frac{|u_{iln}|^2}{u_{ii}u_{nn}} + \sum_i (\log F)_{ii} \right],
\]

where \( \Delta \) and \( \Delta' \) are defined by (3.11) and (3.12) respectively.

**Proof.** The second equality is just immediate substitution of matrix \( U = \text{diag}(u_{11}, \ldots, u_{nn}) \). Let us prove the first one. By Proposition 3.4 it suffices to show that

\[
\sum_{i,k} \frac{u_{ikk}}{u_{ii}} = \sum_{i,p} \text{Tr}((U^{-1}U_{x_i}^p)^2) + \Delta(\log F).
\]

The left hand side of the last equality is equal to \( \sum_{k,p} \text{Tr}(U^{-1}U_{x_kx_p}^k) \). But by Proposition 3.2 the last expression is equal to

\[
\sum_{k,p} \left( \text{Tr}(U^{-1}U_{x_k}^p U^{-1}U_{x_p}^k) + (\log F)_{x_kx_p} \right) = \sum_{k,p} \text{Tr}((U^{-1}U_{x_k}^p)^2) + \sum_k (\log F)_{kk}.
\]

Q.E.D.
3.6 Proposition. Let $u \in C^4$ be a strictly plurisubharmonic function. Denote $\det U = F$ as previously. Let $G$ be a locally flat quaternionic metric. Then

$$\Delta'(2\sqrt{\Delta u}) \geq (\Delta u)^{-1/2} \Delta(\log F),$$

where $\Delta$ and $\Delta'$ are defined by (3.11) and (3.12) respectively.

**Proof.** We prove it pointwise. Let us fix a point $z$. We can choose coordinates near $z$ such that $G \equiv \text{Id}$ and $U(z)$ is diagonal. Then clearly $\Delta h = \sum_i h_{i\bar{i}}$.

Let us take in Corollary 3.5 $\gamma(x) = 2\sqrt{x}$. Then

$$\Delta'(2\sqrt{\Delta u}) = (\Delta u)^{-1/2} \sum_i (\log F)_{i\bar{i}}$$

$$+ (\Delta u)^{-1/2} \left[ \sum_{i,l,n,p} \frac{|u_{i\bar{l}nx_p}|^2}{u_{i\bar{i}}u_{n\bar{n}}} - \frac{1}{2\Delta u} \sum_{i,p} \frac{1}{u_{i\bar{i}}} \left( \sum_k u_{k\bar{k}x_p} \right)^2 \right].$$

It remains to show that the expression in the square brackets is non-negative. We are using the Cauchy-Schwarz inequality

$$\frac{1}{2\Delta u} \sum_{i,p} \frac{1}{u_{i\bar{i}}} \left( \sum_k u_{k\bar{k}x_p} \right)^2 = \frac{1}{2\Delta u} \sum_{i,p} \frac{1}{u_{i\bar{i}}} \left( \sum_k \frac{\sqrt{u_{k\bar{k}}} u_{k\bar{k}x_p}}{\sqrt{u_{k\bar{k}}}} \right)^2 \leq

\frac{1}{2} \sum_{i,p} \frac{1}{u_{i\bar{i}}} \sum_k \frac{|u_{k\bar{k}x_p}|^2}{u_{k\bar{k}}} = \frac{1}{2} \sum_{i,k,p} \frac{|u_{k\bar{k}x_p}|^2}{u_{i\bar{i}}u_{k\bar{k}}}.$$

But by Proposition 3.1 the last expression does not exceed $\sum_{i,l,n,p} \frac{|u_{i\bar{l}nx_p}|^2}{u_{i\bar{i}}u_{n\bar{n}}}$. Q.E.D.

Now we return back to the Monge-Ampère equation.

3.7 Theorem. Let $M$ be a compact manifold with a locally flat hypercomplex structure. Let us assume in addition that $M$ admits a metric $G$ which is parallel with respect to the Obata connection. Let $G_0$ be another HKT-metric on $M$. Let $\phi : M \to \mathbb{R}$ be a $C^4$-smooth solution of the quaternionic Monge-Ampère equation

$$\det(G_0 + \text{Hess}_H \phi) = e^\phi \det G_0$$  \hspace{1cm} (3.13)

\footnote{Any such metric $G$ parallel with respect to the Obata connection is automatically hyperKähler. Hence equivalently one can say that $M$ admits a locally flat hyperKähler metric compatible with the hypercomplex structure.}
where \( f \) is a \( C^2 \)-smooth function. Then there exists a constant \( C \) depending on \( M, G, G_0 \), and \( \| f \|_{C^2(M)} \) such that
\[
\| \Delta_G \phi \|_{C^0} \leq C
\]
where \( \Delta_G : C^2(M, \mathbb{R}) \to \mathbb{R} \) is the globally defined operator which in flat local coordinates is equal \( \Delta_G h := \text{Tr}(G^{-1} \cdot \text{Hess}_H(h)) \).

**Proof.** Let us denote
\[
\Delta' h := \text{Tr}((G_0 + \text{Hess}_H\phi)^{-1} \cdot \text{Hess}_H(h)).
\]

Let \( \Omega \) and \( \Omega_0 \) be the HKT-forms corresponding to \( G \) and \( G_0 \) respectively. We may assume that the solution \( \phi \) satisfies
\[
\int_M \phi \cdot \Omega_0^n \wedge \bar{\Omega}_0^n = 0.
\]
Then by Corollary 5.7 in [6] (the uniform estimate), there exists a constant \( C_1 \) such that \( \| \phi \|_{C^0} \leq C_1 \). Let us consider the function
\[
T := 2\sqrt{\text{Tr}(G^{-1} \cdot (G_0 + \text{Hess}_H(\phi)))} - \phi.
\]
In order to prove the theorem, it suffices to show that this function is bounded from above. Let \( z \in M \) be a point of maximum of the function \( T \). Then
\[
\Delta' T(z) \leq 0. \tag{3.14}
\]
Let \( g_0 \in C^\infty \) be a local potential of the metric \( G_0 \). Then \( u := g_0 + \phi \in C^4 \) is a strictly plurisubharmonic function. Let \( U \) denote its quaternionic Hessian. In flat local coordinates around \( z \) we can rewrite the Monge-Ampère equation (3.13) as
\[
\det U = F \tag{3.15}
\]
where \( F \) is identified with \( e^f \det G_0 \). Also in this notation \( T = 2\sqrt{\Delta_G u} - \phi \). From this and (3.14) we get
\[
(\Delta'(2\sqrt{\Delta_G u} - \phi))(z) \leq 0. \tag{3.16}
\]
By (3.16) and Proposition 3.6 we obtain
\[0 \geq (\Delta_G u)^{-1/2} \sum_i (\log F)_{ii} - \Delta'\phi = (3.17)\]
\[ (\Delta_G u)^{-1/2} \sum_i (\log F)_{ii} - Tr(U^{-1} \cdot (U - G_0)) = (3.18)\]
\[ (\Delta_G u)^{-1/2} \sum_i (\log F)_{ii} - n + Tr(U^{-1} \cdot G_0). (3.19)\]

Let us choose coordinates near \( z \) so that \( G \equiv Id \) and \( U(z) \) is diagonal. Let \( C_2 := n \| \log F \|_{C^2} \). Then we get
\[0 \geq -C_2 \cdot (\Delta_G u)^{-1/2} - n + Tr(U^{-1} \cdot G_0) \] (3.20)
\[-C_2 \cdot (\sum_i u_{ii}(z))^{-1/2} - n + \sum_i \frac{(g_0)_{ii}(z)}{u_{ii}(z)}. (3.21)\]

Let \( C_3 > 0 \) be a constant, depending on \( M \) and \( G_0 \) only, such that for all \( i = 1, \ldots, n \) one has
\[(g_0)_{ii}(z) \geq C_3. \]

This and (3.20)-(3.21) imply
\[C_3 \cdot \sum_i \frac{1}{u_{ii}(z)} \leq C_2 \cdot (\sum_i u_{ii}(z))^{-1/2} + n. (3.22)\]

Since \( \prod_i u_{ii} = F \), the arithmetic-geometric mean inequality implies that
\[(\sum_i u_{ii})^{-1/2} \leq n^{-1/2} F^{-1/2n}. (3.23)\]

Let \( C_4 = C_2 \cdot n^{-1/2} \| F^{-1/2n} \|_{C^0} \). Hence we get
\[C_3 \cdot \sum_i \frac{1}{u_{ii}(z)} \leq C_4 + n. (3.24)\]

Obviously (3.21) implies that \( \sum_i \frac{1}{u_{ii}(z)} \leq C_5 \), and hence
\[u_{ii}(z) > C_6 > 0 \text{ for all } i.\]

Since \( \prod_i u_{ii} = F \) we obtain
\[\sum_i u_{ii}(z) \leq C_7.\]

Hence \( (\Delta_G u)(z) \leq C_7 \). This implies the proposition. Q.E.D.
3.8 Remark. If $u \in C^2(M)$ then estimates on $||u||_{C^0(M)}$ and $||\Delta_G u||_{C^0(M)}$, imply an estimate on $||u||_{C^{1.\alpha}(M)}$ for any $0 < \alpha < 1$ by Theorem 8.32 in [18].

3.9 Remark. If we replace $G$ by any HKT-metric $G_1$, then we can define similarly the operator $\Delta_{G_1} h := Tr(G_1^{-1} \cdot Hess_H h)$. By a simple linear algebra it is easy to show that an estimate on $||\Delta_G u||_{C^0(M)}$ is equivalent to an estimate on $||\Delta_{G_1} u||_{C^0(M)}$.

4 $C^{2,\alpha}$-estimate.

Let $(M^{4n}, I, J, K)$ be a compact hypercomplex manifold. Let $\Omega_0 \in \Lambda^{2,0}$ be an HKT-form. We are interested in the quaternionic Monge-Ampère equation

$$(\Omega_0 + \partial \bar{\partial} \phi)^n = e^f \Omega_0^n. \quad (4.1)$$

Let us denote

$$\Delta \phi = \frac{\partial \bar{\partial} \phi \wedge \Omega_0^{n-1}}{\Omega_0^n}.$$ 

Clearly $\Delta$ is a linear second order elliptic operator without free term (i.e. $\Delta(1) = 0$) and with infinitely smooth coefficients.

The main result of this section is as follows.

4.1 Theorem. Let $M^{4n}$ be a compact manifold with locally flat hypercomplex structure. There exist $\alpha \in (0, 1)$ and $C > 0$, both depending on $M, \Omega_0$, $||f||_{C^2(M)}, ||\phi||_{C^0(M)}, ||\Delta \phi||_{C^0(M)}$ only, such that

$$||\phi||_{C^{2,\alpha}(M)} \leq C.$$ 

Recall that by Theorem 2.3(2) locally $\Omega_0$ can be represented by a potential $\Omega_0 = \partial \bar{\partial} g_0$ where $g_0 \in C^\infty$ is quaternionic strictly plurisubharmonic. Since $M$ is locally flat, Theorem 4.1 follows from the following version on $\mathbb{H}^n$ applied to $u = g_0 + \phi$.

4.2 Theorem. Let $u \in C^4$ be a quaternionic psh function in an open subset $\mathcal{O} \subset \mathbb{H}^n$ satisfying

$$\det(u_{ij}) = e^f \quad (4.2)$$

with $f \in C^2$. Let $\mathcal{O}' \subset \mathcal{O}$ be a relatively compact open subset. Then there exist $\alpha \in (0, 1)$ depending on $n, ||u||_{C^0(\mathcal{O})}, ||\Delta u||_{C^0(\mathcal{O})}, ||f||_{C^2(\mathcal{O})}$ only and a constant $C > 0$ depending in addition on $dist(\mathcal{O}', \partial \mathcal{O})$ such that

$$||u||_{C^{2,\alpha}(\mathcal{O}')} \leq C.$$
The proof of this theorem is a quaternionic version of the Evans-Krylov method [14]-[15], [25]. The complex version of it was considered by Siu [27] and B\'ok\'cki [11]. Our exposition closely follows B\'ok\'cki [11]; perhaps only Lemma 4.6 below is somewhat novel.

For a unit vector $\zeta \in \mathbb{H}^n$ we denote by $\Delta_{\zeta}$ the Laplacian on any translate of the (right) quaternionic line spanned by $\zeta$. Also let us denote by $U$ the quaternionic Hessian $(u_{ij})$. Thus $U$ is a hyperhermitian positive definite $n \times n$ matrix.

**4.3 Lemma.** Assume that $u, f$ satisfy the assumptions of Theorem 4.2. Then pointwise we have

$$Tr(U^{-1} \cdot \Delta_{\zeta} U) \geq \Delta_{\zeta} f. \quad (4.3)$$

**Proof.** We may assume that $\zeta = (1, 0, \ldots, 0)$. Then $\Delta_{\zeta} = \sum_{p=0}^{3} \frac{\partial^2}{(\partial x_p^1)^2}$. It is enough to show that

$$Tr(U^{-1} \cdot U_{x_p^1 x_p^1}) \geq f_{x_p^1 x_p^1} \quad (4.4)$$

for any $p = 0, \ldots, 3$. Differentiating the equality

$$\log \det U = f$$

twice with respect to $x_p^1$, we obtain

$$Tr(U^{-1} \cdot U_{x_p^1 x_p^1}) = f_{x_p^1 x_p^1} + Tr(U^{-1} U_{x_p^1} U^{-1} U_{x_p^1}).$$

In order to prove the lemma, it suffices to show that $Tr(U^{-1} U_{x_p^1} U^{-1} U_{x_p^1}) \geq 0$. More generally let us show that if $A, B$ are hyperhermitian matrices and $A > 0$ then

$$Tr(A^{-1} BA^{-1} B) \geq 0.$$

Since $A > 0$ we can diagonalize $A, B$ simultaneously. More precisely we can find an invertible quaternionic matrix $T$ and a real diagonal matrix $D$ such that

$$A = T^* T, \quad B = T^* D T.$$

Then

$$Tr(A^{-1} BA^{-1} B) = Tr(T^{-1} D^2 T) = Tr(D^2) \geq 0.$$
Lemma is proved. Q.E.D.

Let us recall now the weak Harnack inequality (see [18], Theorem 8.18, or [21] Theorem 4.15). Below we normalize everywhere the Lebesgue measure on $\mathbb{R}^N$ by $\text{vol}(B_1) = 1$ where $B_1$ is Euclidean ball of unit radius. We also denote $D_i = \frac{\partial}{\partial x_i}$.

4.4 Theorem (weak Harnack inequality). Let $B_R \subset \mathbb{R}^N$ be a Euclidean ball of radius $R$. Let $(a_{ij})_{i,j=1}^N \in L^\infty(B_R) \cap C^1(B_R)$, $a_{ij} = a_{ji}$, satisfy uniform elliptic estimate

$$\lambda ||\xi||^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda ||\xi||^2, \text{ for all } \xi \in \mathbb{R}^N$$

with $\lambda, \Lambda > 0$. Let $v \in C^2(B_R)$ be a function satisfying

$$v \geq 0,$$

$$\sum_{i,j} D_j(a_{ij}D_i v) \leq \psi,$$

where $\psi \in L^\infty(B_R)$. Then for any $0 < \theta < \tau < 1$ we have

$$\inf_{B_{\theta R}} v + R||\psi||_{L^\infty(B_R)} \geq C \left( R^{-N} \int_{B_{\tau R}} v \right)$$

where the constant $C$ depends only on $\lambda, \Lambda, \theta, \tau, N$.

4.5 Remark. We will use Theorem 4.4 in the following weaker form. We will take $R = 4r, \theta = 1/4, \tau = 1/2$. Then we deduce

$$r^{-N} \int_{B_r} v \leq C'(\inf_{B_r} v + r) \quad (4.5)$$

where the constant $C'$ depends on $\lambda, \Lambda, ||\psi||_{L^\infty(B_R)}, N$ only.

For $U = (u_{ij})$ as above define the operator $D$ by

$$Dh = \det U \cdot Tr(U^{-1} \cdot \text{Hess}_U h). \quad (4.6)$$

First let us prove an algebraic lemma.
4.6 Lemma. The operator $\mathcal{D}$ defined by (4.6) can be written in the divergence form as in Theorem 4.4, namely

$$\mathcal{D}h = \sum_{st} D_s(a_{st} D_t h)$$

where $a_{st}$ is a $4n \times 4n$ real symmetric matrix with $C^2$-smooth coefficients, and $s,t$ in the sum run over all real variables $x^i_p$.

Before we prove the lemma, let us prove the following linear algebraic claim.

4.7 Claim. Let $A,B$ be $n \times n$ hyperhermitian matrices. Suppose that $A$ is invertible. Then

$$\det A \cdot \text{Tr}(A^{-1}B) = n \det(A[n-1], B). \quad (4.7)$$

Proof. Both sides of the equality are linear in $B$. Hence it suffices to prove the equality for $B > 0$. Then $A$ and $B$ can be diagonalized simultaneously, more precisely the exists an invertible quaternionic matrix $T$ and a real diagonal matrix $D$ such that

$$B = T^* T, \quad A = T^* DT.$$ 

Then the left hand side of (4.7) is equal to

$$\det(T^* DT) \cdot \text{Tr}(T^{-1}D^{-1}T) = \det(T^* T) \cdot \det(D \cdot \text{Tr}(D^{-1})).$$

On the other hand the right hand side is equal to

$$n \det((T^* DT)[n-1], T^* T) = n \det(T^* T) \cdot \det(D[n-1], I_n).$$

Hence it suffices to assume that $B = I_n$ and $A = D$ is real diagonal, i.e.

$$\det D \cdot \text{Tr}(D^{-1}) = n \det(D[n-1], I_n).$$

The last identity for real diagonal $D$ is obvious. Q.E.D.

Proof of Lemma 4.6. Let us consider on $\mathbb{H}^n$ the complex structure $I$.

By [5], Corollary 4.6, for appropriate choice of flat $I$-complex coordinates on $\mathbb{H}^n$ one has

$$(\partial \partial_I h)^n = \kappa_n \det(h_{ij}) \cdot dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{2n}$$
where \( \kappa_n \) is a normalizing constant depending on \( n \) only (its precise value will not be important in the argument below). Polarizing the last equality we obtain for any \( n \)-tuple of functions \( h_1, \ldots, h_n \)

\[
(\partial\partial_Jh_1) \wedge \cdots \wedge (\partial\partial_Jh_n) = \det((h_1)_{ij}, \ldots, (h_n)_{ij})\Theta,
\]

where we have introduced the notation \( \Theta := \kappa_n dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{2n} \in \Lambda^{2n,0}_I(\mathbb{H}^n) \) for brevity. Hence

\[
Dh = \det U \cdot Tr(U^{-1}(h_{ij})) \tag{4.9}
\]

\[
n \det(U[n-1], (h_{ij})) \overset{4.8}{=} n(\partial\partial_Ju)^{n-1} \wedge \partial\partial_Jh \Theta \tag{4.10}
\]

Let \( (a_{st}) \) be the \( 4n \times 4n \) be the real symmetric matrix defined to be the realization of the \( n \times n \) quaternionic hermitian matrix \( \det U \cdot U^{-1} \). Then it is easy to see that \( Dh = \sum_{s,t} a_{st} D_s D_t h \).

Clearly the statement of Lemma 4.6 is equivalent to

\[
\sum_s D_s a_{st} = 0 \text{ for any } t. \tag{4.12}
\]

In order to prove the last equality let us rewrite it in a more invariant way. Let \( \nabla \) denote the flat connection on the tangent bundle of \( \mathbb{H}^n =: M \). Let \( q: T^*M \otimes TM \to \mathbb{R} \) be the natural pairing. Let

\[
Q: Sym^2TM \otimes T^*M \to TM
\]

be the natural contraction map given by \( Q(x \otimes y \otimes \xi) = \xi(y)x \).

It is clear that the quaternionic Hessian \( U = (u_{ij}) \) belongs to the space \( B \) of quadratic forms on \( \mathbb{H}^n \) which are invariant under the (right) multiplication by norm one quaternions. Hence the matrix \( a := (a_{st}) \), which corresponds to \( \det U \cdot U^{-1} \), belongs to \( C := B^* \otimes L \) where \( L \) denotes the line to which the Moore determinant belongs (below we will identify \( L \) more explicitly).

In this notation (4.12) is equivalent to

\[
Q(\nabla a) = 0. \tag{4.13}
\]

Since \( a \) changes as an appropriate tensor under all translations on \( \mathbb{H}^n \) and all linear transformations from \( GL_n(\mathbb{H}) \cdot GL_1(\mathbb{H}) \), and since \( \nabla \) commutes
with such transformations, the equation (4.13) is invariant under the group
\( H^\ast \ast (GL_n(\mathbb{H}) \cdot GL_1(\mathbb{H})) \). Hence it suffices to check it at the point 0. This
is the first order differential equation. The 1-jet of \( a \) at 0 belongs to the
space \( \mathcal{C} \bigoplus C \otimes \mathbb{R}(\mathbb{H}^n)^\ast \). The differential operator \( a \mapsto Q(\nabla a) \) obviously does
not depend on the first component of \( a \) in this sum. Thus let us denote by
\( j(a) \) the second component of \( a \). The subspace of \( \mathcal{C} \otimes \mathbb{R}(\mathbb{H}^n)^\ast \) corresponding
to solutions of the equation (4.13) is a \( GL_n(\mathbb{H}) \cdot GL_1(\mathbb{H}) \)-invariant proper
subspace (clearly it is not equal to the whole space, and that it is non-zero will
be seen from the last part of the argument where we will construct non-zero
examples of solutions of this equation). Let us study the decomposition of
\( \mathcal{C} \otimes \mathbb{R}(\mathbb{H}^n)^\ast \) under the group \( GL_n(\mathbb{H}) \cdot GL_1(\mathbb{H}) \). Actually it will be convenient
to replace this group by \( GL_n(\mathbb{H}) \times GL_1(\mathbb{H}) \) which is mapped surjectively
onto it. Also it will be convenient to replace all spaces and groups by their
complexifications. We have
\[
\mathbb{H}^n \otimes \mathbb{R} C = V \otimes C W
\]
where \( V = \mathbb{C}^{2n}, W = \mathbb{C}^2 \). It is well known (and easy to see directly) that
\[
\mathcal{B} \otimes \mathbb{R} C = Sym^2 V^\ast \otimes \det W^\ast,
\]
where in the right hand side \( Sym, \wedge, \) and \( \otimes \) are taken over \( C \) (here and below
we will omit this subscript). It is easy to see that the complexified line \( L \)
where the Moore determinant lies is equal to \( L \otimes \mathbb{R} C = \det V^\ast \otimes (\det W^\ast)^\otimes n \).
Hence we obtain
\[
\begin{align*}
(\mathcal{C} \otimes \mathbb{R}(\mathbb{H}^n)^\ast) \otimes \mathbb{R} C &= \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 
\end{align*}
\]
where all the tensor products on the right hand side are over \( C \).

Next the complexification of the group \( GL_n(\mathbb{H}) \times GL_1(\mathbb{H}) \) is equal to the
product \( GL(V) \times GL(W) \) of complex linear groups, where \( GL(V) \) acts on \( V \)
in the standard way, and similarly for \( W \).

Obviously \( \det W^\ast \otimes W^\ast \) is an irreducible \( GL(W) \)-representation. However
the \( GL(V) \)-representation \( Sym^2 V \otimes V^\ast \) is a direct sum of two irreducible
non-isomorphic subspaces (this easily follows from the Schur-Weyl duality,
see e.g. [19], §9.1.1). Hence the representation of \( GL(V) \times GL(W) \) in the
space (4.14) is also a direct sum of two irreducible non-isomorphic subspaces.
The complexification of the map $Q$ tensored with the identity map of $L \otimes \mathbb{C}$, which we also will denote by $Q$, maps

$$Q: (\text{Sym}^2 V \otimes \det W) \otimes V^* \otimes W^* \otimes (\det V^* \otimes (\det W^*)^\otimes n) \to V \otimes W \otimes (\det V^* \otimes (\det W^*)^\otimes n).$$

It is equal to the tensor product of the obvious contraction maps

$$\text{Sym}^2 V \otimes V^* \to V,$$
$$\det W \otimes W^* = \det W \otimes W^* \to W.$$

Hence the kernel of $Q$ is an irreducible non-zero $GL(V) \times GL(W)$-representation which we will denote by $K$.

Thus to prove the lemma it remains to show that $j(a) \in K$. Recall that $a$ corresponds to the hyperhermitian matrix $\det(Hess_{\mathbb{H}}(u)) \cdot Hess_{\mathbb{H}}(u)$ where $u$ is a function. Due to (4.11) and some linear algebra, this expression can be identified with $(\partial \partial_j u)^{n-1}$ (up to a constant). But the last expression satisfies

$$\partial((\partial \partial_j u)^{n-1}) = \partial_j((\partial \partial_j u)^{n-1}) = 0.$$

These equations give a non-trivial restriction on $j(a)$ and imply that $j(a)$ belongs to a proper $GL(V) \times GL(W)$-invariant subspace for all functions $u$. Hence $j(a)$ belongs either always (i.e. for any $a$) to $K$ or always to the other irreducible summand of the space (4.13). Let us show that the first case takes place.

It suffices to give an example of a function $u$ such that the corresponding $a$ satisfies $Q(\nabla a) = 0$ and $j(a) \neq 0$. Let us write

$$\mathbb{H}^n = \mathbb{R}^n \oplus \mathbb{R}^n \cdot i \oplus \mathbb{R}^n \cdot j \oplus \mathbb{R}^n \cdot k.$$

We take an arbitrary (smooth) function $u$ which depends only on variables from the first summand and is independent from the variables from the last three summands. Then the quaternionic Hessian is $u$ equal to its usual real Hessian with respect to the first $n$ real coordinates:

$$Hess_{\mathbb{H}} u = \left( \frac{\partial^2 u}{\partial t_\alpha \partial t_\beta} \right) =: (u_{\alpha \beta})_{n \times n}.$$
Then set
\[ A = \det(u_{\alpha \beta}) \cdot (u_{\alpha \beta})^{-1}. \]
Then by definition \( a \) is equal to the realization of \( A \). Hence
\[
a = \begin{bmatrix}
A & 0 & 0 & 0 \\
0 & A & 0 & 0 \\
0 & 0 & A & 0 \\
0 & 0 & 0 & A
\end{bmatrix}_{4n \times 4n}.
\]
Then our equation \( Q(\nabla a) = 0 \) is equivalent to
\[
\frac{\partial}{\partial t_{\alpha}} u_{\alpha \beta} = 0,
\]
where we have summation with respect to repeated indices. More explicitly
the last equation can be rewritten
\[
\frac{\partial}{\partial t_{\alpha}} (\det U \cdot U^{\alpha \beta}) = 0
\]
where \( U = (u_{\alpha \beta}), U^{\alpha \beta} = (U^{-1})_{\alpha \beta} \). Next we have
\[
\frac{\partial}{\partial t_{\alpha}} \det U = \det U \cdot \text{Tr}(U^{-1} \frac{\partial U}{\partial t_{\alpha}}),
\frac{\partial U^{-1}}{\partial t_{\alpha}} = -U^{-1} \frac{\partial U}{\partial t_{\alpha}} U^{-1}.
\]
Hence, again with a summation over repeated indices, we get
\[
\frac{\partial}{\partial t_{\alpha}} (\det U \cdot U^{\alpha \beta}) = \det U \left( U^{\rho q} u_{\rho q} U^{\alpha \beta} - U^{\alpha \rho} u_{\rho q} U^{q \beta} \right) = 0.
\]
Hence \( j(a) \in \mathcal{K} \) for any \( u \) as above. It is easy to see that \( u \) can be chosen so that \( j(a) \neq 0 \). This implies the lemma. Q.E.D.

Next let us observe that \( D \) is uniformly elliptic with constants \( \lambda, \Lambda \) depending on \( ||f||_{C^{0}}(\Omega), ||\Delta \phi||_{C^{0}}(\Omega) \) only. We are going to apply the weak Harnack inequality to the operator \( D \) to the function
\[
v := \sup_{B_{4r}} \Delta_{\xi} u - \Delta_{\xi} u
\]
where $B_{4r} = B(z_0, 4r) \subset O$ is a ball with a center $z_0 \in \mathcal{O}$.

The function $v$ satisfies by Lemma 4.3

$$\mathcal{D}v \leq - e^f \Delta f$$

where we have used $\Delta_f (u_{ij}) = (\Delta_f u)_{ij}$.

Hence we can apply the weak Harnack inequality in the form given in Remark 4.5

$$r^{-4n} \int_{B_{r}} (\sup_{B_{4r}} \Delta u - \Delta u) \leq C (\sup_{B_{4r}} \Delta u - \sup_{B_{r}} \Delta u + r)$$

(4.17)

where $C$ depends on $||f||_{C^2(O)}, ||\Delta f||_{C^0(O)}, n$ only.

4.8 Lemma. Let $A, B$ be positive definite hyperhermitian matrices of size $n$. Then

$$\text{Tr}(A^{-1}(A - B)) \leq n(\det A)^{-1/n}((\det A)^{1/n} - (\det B)^{1/n})$$

Proof. $A$ and $B$ can be simultaneously diagonalized. Thus the inequality follows from the corresponding result in the real diagonal case when it is equivalent to the arithmetic-geometric mean inequality. Q.E.D.

Lemma 4.8 implies that for any point $x, y \in O$

$$e^{f(y)} \text{Tr} \left( U^{-1}(y)(U(y) - U(x)) \right) \leq$$

$$n e^{f(y)(1 - \frac{1}{n})}(e^{f(y)/n} - e^{f(x)/n}) \leq C_1 ||x - y||,$$

(4.18)

(4.19)

where $C_1$ depends on $||f||_{C^1(O)}$ and $n$.

Let $\zeta \in \mathbb{H}^n$ be a vector. If we write it as a column, it defines a rank one hyperhermitian matrix $\zeta \otimes \zeta^*$. Namely $(\zeta \otimes \zeta^*)_{ij} = \zeta_i \cdot \tilde{\zeta}_j$. Clearly the matrix $\zeta \otimes \zeta^*$ is non-negative definite. We will need the following linear algebraic lemma which is completely analogous to the real and complex cases (see [18], Lemma 17.13, for the real case, and for the complex case [27], p. 103, or [11], Lemma 5.3).

4.9 Lemma. Let us fix $0 < \lambda < \Lambda < \infty$. One can find a natural number $N$, unit vectors $\zeta_1, \ldots, \zeta_N \in \mathbb{H}^n$, and $0 < \lambda_s < \Lambda_s < \infty$ depending on $\lambda, \Lambda, n$
only such that every hyperhermitian \((n \times n)\)-matrix \(A\) whose spectrum lies in \([\lambda, \Lambda]\) can be written

\[
\sum_{k=1}^{N} \beta_k \zeta_k \otimes \zeta_k^* \quad \text{with} \quad \beta_k \in [\lambda^*, \Lambda^*].
\]

The vectors \(\zeta_1, \ldots, \zeta_N\) can be chosen to contain any orthonormal basis of \(\mathbb{H}^n\).

**Proof.** For convenience of the reader we outline the argument which is not novel. Let us denote by \(K\) the set of quaternionic hermitian matrices whose spectrum lies in \([\lambda, \Lambda]\). This is a compact subset of the interior of the cone of positive definite hyperhermitian matrices; we denote this open cone by \(C\). Then there exists a convex compact polytope \(P \subset C\) which contains a neighborhood of \(K\). Let \(Vert(P)\) denote the set of vertices of \(P\). Using a diagonalization, every matrix \(X \in C\) can be written in the form

\[
X = \sum_{i=1}^{n} \alpha_i(X) (\xi_i(X) \otimes \xi_i^*(X)), \quad \text{(4.20)}
\]

where \(\alpha_i(X) > 0\), and \(\xi_i(X) \in \mathbb{H}^n\) are unit vectors. Let us fix such a decomposition for any vertex of \(P\). Let us define a new finite subset \(S_1 \subset C\) consisting of rank one non-negative definite matrices as follows

\[
S_1 := \left\{ \left( \sum_j \alpha_j(X) \right) \cdot (\xi_i(X) \otimes \xi_i^*(X)) \mid X \in Vert(P), \alpha_j(X), \xi_i(X) \text{ satisfy (4.20)} \right\}.
\]

Let us add to \(S_1\) matrices of the form \(e_i \otimes e_i^*\), \(i = 1, \ldots, n\), where \(e_1, \ldots, e_n\) are an orthonormal basis of \(\mathbb{H}^n\). Let us denote by \(S\) the new set. It is clear that \(P \subset conv(S)\). Hence \(conv(S)\) contains a neighborhood of \(K\). Now the lemma follows from the following general fact which is left to the reader (and where one takes \(Q = conv(S)\)): Let \(K\) be a compact subset of \(\mathbb{R}^N\) which is contained in the interior of a compact convex polytope \(Q\). Then there exists \(\varepsilon > 0\) such that any point \(x \in K\) can be written as a convex combination of vertices of \(Q\):

\[
x = \sum_v \beta(v) v \quad \text{with} \quad \beta(v) > \varepsilon,
\]

where the sum runs over all vertices of \(Q\), \(\sum_v \beta(v) = 1\). Q.E.D.
Proof of Theorem 4.2. The eigenvalues of $U = (u_{ij})$ belong to $[\lambda_1, \Lambda_1]$ with $0 < \lambda_1 < \Lambda_1 < \infty$ depending on $||\Delta u||_{C^0(\mathcal{O})}, ||f||_{C^0(\mathcal{O})}$ only. Hence the eigenvalues of $e^fU^{-1}$ are in $[\lambda, \Lambda]$ with $0 < \lambda < \Lambda < \infty$ under control. By Lemma 4.9 there exist $N$, unit vectors $\zeta_1, \ldots, \zeta_N \in \mathbb{H}^n$, and $0 < \lambda^* < \Lambda^* < \infty$ such that for any $y \in \mathcal{O}$

$$e^f(y)U^{-1}(y) = \sum_{i=1}^{N} \beta_k(y) \zeta_k \otimes \zeta_k^*$$

with $\beta_k(y) \in [\lambda^*, \Lambda^*]$.

Observe also that for a unit vector $\zeta \in \mathbb{H}^n$

$$Tr((\zeta \otimes \zeta^*)(u_{ij})) = Tr(\zeta^*(u_{ij})\zeta) = \Delta_\zeta u.$$  

This and (4.18)-(4.19) imply

$$\sum_{k=1}^{N} \beta_k(y)(\Delta_\zeta u(y) - \Delta_\zeta u(x)) \leq C_1||x - y|| \text{ for } x, y \in \mathcal{O}. \quad (4.21)$$

Let $\mathcal{O}'$ be a compact neighborhood of $\mathcal{O}'$ in $\mathcal{O}$. Consider now a ball $B_{4r} = B(z_0, 4r) \subset \mathcal{O}$ with center $z_0 \in \mathcal{O}''$. Let us denote

$$M_{k,r} := \sup_{B_r} \Delta_\zeta u, \quad m_{k,r} := \inf_{B_r} \Delta_\zeta u,$$

$$\eta(r) := \sum_{k=1}^{N} (M_{k,r} - m_{k,r}).$$

We will show that for some $\alpha \in (0, 1), r_0 > 0, C > 0$ under control

$$\eta(r) \leq Cr^\alpha$$

for $0 < r < r_0$.

Since $\zeta_1, \ldots, \zeta_N$ can be chosen to contain an orthonormal basis of $\mathbb{H}^n$, this will imply an estimate on $||\Delta u||_{C^0(\mathcal{O}')}$, Then the Schauder estimates (18, Theorem 4.6) will imply an estimate on $||u||_{C^{2,\alpha}(\mathcal{O}')}$. The condition $\eta(r) \leq Cr^\alpha$ is equivalent to

$$\eta(r) \leq \delta \eta(4r) + r, \quad 0 < r < r_1$$

where $\delta \in (0, 1), r_1$ are under control (18, Lemma 8.23). Summing up (4.17) over $\zeta = \zeta_l$ with $l \neq k$ for any fixed $k$ we get

$$r^{-4n} \sum_{B_r} (M_{l,4r} - \Delta_\zeta u) \leq C_3(\eta(4r) - \eta(r) + r). \quad (4.22)$$
But by (4.21) we have for any \( x \in B_{4r}, y \in B_r \)
\[
\beta_k(y)(\Delta_{\zeta_k} u(y) - \Delta_{\zeta_k} u(x)) \leq C_1 ||x - y|| + \sum_{l \neq k} \beta_l(y)(\Delta_{\zeta_l} u(x) - \Delta_{\zeta_l} u(y)) \leq C_4 r + \Lambda_* \sum_{l \neq k} (M_{l,4r} - \Delta_{\zeta_l} u(y)).
\]

Optimizing in \( x \) we get
\[
\Delta_{\zeta_k} u(y) - m_{k,4r} \leq \frac{1}{\Lambda_*} \left( C_4 r + \Lambda_* \sum_{l \neq k} (M_{l,4r} - \Delta_{\zeta_l} u(y)) \right).
\]

Integrating the last inequality over \( y \in B_r \) and using (4.22) we obtain
\[
r^{-4n} \int_{B_r} (\Delta_{\zeta_k} u(y) - m_{k,4r}) \leq C_5(\eta(4r) - \eta(r) + r). \tag{4.23}
\]

Let us estimate the left hand side of (4.23) from below. Since we have normalized the Lebesgue measure on \( \mathbb{H}^n \) so that \( \text{vol}(B_1) = 1 \), we have
\[
r^{-4n} \int_{B_r} (\Delta_{\zeta_k} u(y) - m_{k,4r}) \geq -m_{k,4r} + M_{k,4r} - C(M_{k,4r} - M_{k,r} + r) = C(M_{k,r} - m_{k,r}) - (C - 1)(M_{k,4r} - m_{k,4r}) + C(m_{k,r} - m_{k,4r}) - Cr \geq C(M_{k,r} - m_{k,r}) - (C - 1)(M_{k,4r} - m_{k,4r}) - Cr.
\]

Substituting this back into (4.23) we obtain
\[
C(M_{k,r} - m_{k,r}) - (C - 1)(M_{k,4r} - m_{k,4r}) - Cr \leq C_5(\eta(4r) - \eta(r) + r).
\]

Summing this up over \( k \) we get
\[
C\eta(r) - (C - 1)\eta(4r) \leq C_6(\eta(4r) - \eta(r) + r).
\]

Hence
\[
\eta(r) \leq \frac{C + C_6 - 1}{C + C_6} \eta(4r) + r.
\]

Theorem 4.2 is proved. Q.E.D.
5 Proof of the main theorem.

Let us assume that our compact connected hypercomplex manifold $M$ with an HKT-form $\Omega_0$ admits a real (in the quaternionic sense) and strictly positive (in particular, nowhere vanishing) $(2n,0)$-form $\Theta \in C^\infty(M, \Lambda^{2n,0}_I \mathbb{R})$ which is $I$-holomorphic, i.e. $\bar{\partial} \Theta = 0$. Consequently one has

$$\bar{\partial} \Theta = \bar{\partial}_J \Theta = 0.$$

For any integer $k \geq 1$ and $\beta \in (0,1)$ let us define

$$U^{k,\beta} := \{ \phi \in C^{k,\beta}(M) | \Omega_0 + \partial \partial_J \phi > 0 \text{ and } \int_M \phi \cdot \Omega_0^n \wedge \bar{\Omega}_0^n = 0 \},$$
$$V^{k,\beta} := \{ \chi \in C^{k,\beta}(M) | \chi > 0 \text{ and } \int_M (\chi - 1) \cdot \Omega_0^n \wedge \bar{\Theta} = 0 \}.$$

Define

$$\mathcal{M}(\phi) := \frac{(\Omega_0 + \partial \partial_J \phi)^n}{\Omega_0^n}.$$

We claim that

$$\mathcal{M} : U^{k+2,\beta} \rightarrow V^{k,\beta}$$

is a continuous map. The continuity is obvious, the only thing to check is that for any $\phi \in C^{k+2,\beta}(M)$ one has

$$\int_M (\mathcal{M}(\phi) - 1) \Omega_0^n \wedge \bar{\Theta} = 0.$$

Indeed the left hand side of the last equality is equal to

$$\int((\Omega_0 + \partial \partial_J \phi)^n - \Omega_0^n) \wedge \bar{\Theta} = \int \partial \partial_J \phi \wedge \left( \sum_{k=0}^{n-1} (\partial \partial_J \phi)^k \wedge \Omega_0^{n-k-1} \right) \wedge \bar{\Theta} =$$

$$\int \phi \wedge \partial \partial_J \left( \sum_{k=0}^{n-1} (\partial \partial_J \phi)^k \wedge \Omega_0^{n-k-1} \right) \wedge \bar{\Theta} = 0,$$

where in the last equality we have used the Leibnitz rule, $\partial^2 = \partial_J^2 = 0$, $\partial \partial_J = -\partial_J \partial$, and $\partial \Omega_0 = \partial_J \Omega_0 = \partial \bar{\Theta} = \partial_J \bar{\Theta} = 0$.

Next notice that $U^{k+2,\beta}$ and $V^{k,\beta}$ are open subsets in Banach spaces with the induced Hölder norms.
5.1 Proposition. Let \((M^{4n}, I, J, K)\) be a compact connected hypercomplex manifold with an HKT-form \(\Omega_0\) and which admits a form \(\Theta\) as above. Let \(k \geq 1\) be an integer, and let \(\beta \in (0, 1)\). Then the map \(M : U^{k+2, \beta} \to V^{k, \beta}\) is locally a diffeomorphism of Banach spaces, and in particular its image is an open subset.

Proof. By the inverse function theorem for Banach spaces, it suffices to show that the differential of \(M\) at any \(\phi \in U^{k+2, \beta}\) is an isomorphism of tangent spaces. The tangent space to \(U^{k+2, \beta}\) at any point is

\[
\tilde{C}^{k+2, \beta}(M) := \{ \phi \in C^{k+2, \beta}(M) \mid \int_M \phi \cdot \Omega_0^n \wedge \bar{\Omega}_0^n = 0 \}.
\]

The tangent space to \(V^{k, \beta}\) at \(N(\phi)\) is

\[
\tilde{C}^{k, \beta}(M) := \{ \chi \in C^{k, \beta}(M) \mid \int_M \chi \cdot \Omega_0^n \wedge \bar{\Theta} = 0 \}.
\]

The differential of \(M\) at \(\phi\) is equal to

\[
DM_{\phi}(\psi) = n \frac{\partial \partial J \phi}{{\Omega}_0^n} \wedge \partial \partial J \psi.
\]

 Defined by this formula we consider \(DM_{\phi}\) as a map \(C^{k+2, \beta}(M) \to C^{k, \beta}(M)\) (without restricting to \(\tilde{C}\)). Then obviously \(DM_{\phi}\) is a linear second order differential elliptic operator. It has no free term (i.e. \(DM_{\phi}(1) = 0\)), and its coefficients belong to \(C^{k, \beta}\).

By the strong maximum principle (see e.g. [18], Theorem 8.19) and since \(M\) is connected, the kernel of \(DM_{\phi}\) consists only of constant functions; thus it is one dimensional. The image of \(DM_{\phi}\) is a closed subspace of \(C^{k, \beta}(M)\) by [24], Theorem 1.5.4, which is a version of the Schauder estimates.

Since the symbol of any second order differential operator of real valued functions is self-adjoint, the index of \(DM_{\phi}\) equals 0 (for operators with coefficients from Hölder spaces see the book [24], the proof of Theorem C3 in §5.6 and Theorem 1.5.4). Hence \(\text{codim} Im(DM_{\phi}) = 1\). But since \(Im(DM_{\phi}) \subset C^{k, \beta}(M)\), and since \(\dim C^{k, \beta}(M) / \tilde{C}^{k, \beta}(M) = 1\), it follows that \(Im(DM_{\phi}) = C^{k, \beta}(M)\). Q.E.D.

5.2 Proposition. Let \((M^{4n}, I, J, K)\) be a compact manifold with a locally flat hypercomplex structure which admits a flat hyperKähler metric compatible
with the hypercomplex structure. Let $\Omega_0$ be an HKT-form (not necessarily corresponding to the hyperKähler metric). Let $k \geq 2$ be an integer, $\beta \in (0, 1)$. Then the map

$$\mathcal{M}: \mathcal{U}^{k+2,\beta} \to \mathcal{V}^{k,\beta}$$

is a diffeomorphism of Banach manifolds, in particular it is onto.

**Proof.** $\mathcal{M}$ is one-to-one by the uniqueness of the solution (in [6], Corollary 4.10, the uniqueness was proven for $C^\infty$-solutions, but exactly the same standard proof, based on ellipticity and the strong maximum principle, works under the current assumptions on smoothness).

Now notice that the assumptions of Proposition 5.1 are satisfied. Indeed let $G$ be a locally flat hyperKähler metric. Let $\Omega$ be the corresponding HKT-form. Then $\Theta = \Omega^n$ satisfies the assumptions of Proposition 5.1.

Thus by Proposition 5.1 it suffices to show that $\mathcal{M}$ is onto. Since $\mathcal{V}^{k,\beta}$ is obviously connected (it is even convex), and since the image of $\mathcal{M}$ is open by Proposition 5.1, it suffices to show that the image of $\mathcal{M}$ is a closed subset of $\mathcal{V}^{k,\beta}$.

Let we have a sequence of point in the image $\mathcal{M}\phi_i \in \mathcal{V}^{k,\beta}$ where $\phi_i \in \mathcal{U}^{k+2,\beta}$. By Theorems 3.7, 4.1 and the zero order estimate in [6], Corollary 5.7 (see also [4], Theorem 2), there exist $\alpha \in (0, 1)$ and a constant $C$ both depending on $||f||_{C^2}$, $(M, I, J, K)$, $\Omega_0$, and the locally flat hyperKähler metric, such that $||\phi_i||_{C^{2,\alpha}} < C$ for $i \gg 1$. By the Arzelà-Ascoli theorem choosing a subsequence we may assume that $\phi_i \to \phi$ in $C^2(M)$. Clearly $\phi \in C^{2,\alpha}(M)$ and one has $\mathcal{M}(\phi) = e^f$. In other words one has

$$(\Omega_0 + \partial\bar{\partial}J\phi)^n = e^f \Omega_0^n. \quad (5.1)$$

Also clearly $\Omega_0 + \partial\bar{\partial}J\phi \geq 0$. But because of (5.1) the inequality is strict, i.e. $\Omega_0 + \partial\bar{\partial}J\phi > 0$, and the equation (5.1) is elliptic with $C^\infty$ coefficients on the left hand side and with $C^{k,\beta}$ on the right hand side. Hence by Lemma 17.16 from [18] $\phi \in C^{k+2,\beta}(M)$. Thus $\phi \in \mathcal{V}^{k+2,\beta}$ and $e^f = \mathcal{M}(\phi) \in \text{Im}(\mathcal{M})$. Q.E.D.

Finally let us state the main result of the paper which is an immediate consequence of Proposition 5.2.

**5.3 Theorem.** Let $(M^{kn}, I, J, K)$ be a compact locally flat hypercomplex manifold which admits a flat hyperKähler form (of class $C^\infty$) compatible
with the hypercomplex structure. Let \( \Omega_0 \) be an HKT-form on \( M \) of class \( C^\infty \) (which does not necessarily correspond to the above mentioned hyperKähler metric). Let \( k \geq 2 \) be an integer, \( \beta \in (0, 1) \). Let \( f \in C^{k,\beta}(M) \).

Then there is a unique constant \( A > 0 \) such that quaternionic Monge-Ampère equation

\[
(\Omega_0 + \partial J\phi)^n = Ae^f\Omega^n_0
\]

has a unique, up to a constant, \( C^2 \) smooth solution \( \phi \) which necessarily belongs to \( C^{k+2,\beta}(M) \). If \( f \) is \( C^\infty \) smooth, then any solution \( \phi \) is also \( C^\infty \) smooth.

5.4 Remark. The constant \( A \) in the theorem is defined by the following condition. Let \( \Omega \) be the HKT-form corresponding to the locally flat hyperKähler metric. Then \( A \) is found from the equality

\[
\int_M Ae^f \cdot \Omega^n_0 \wedge \bar{\Omega}^n = \int_M \Omega^n_0 \wedge \bar{\Omega}^n.
\]

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