Chapter 1

Baryon physics in a five-dimensional model of hadrons

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We review the procedure to calculate baryonic properties using a recently proposed five-dimensional approach to QCD. We show that this method gives predictions to baryon observables that agree reasonably well with the experimental data.

1.1. Introduction

In 1973 Gerard 'tHooft proposed, in a seminal article [1], a dual description for QCD. He showed that in the limit of large number of colors \( N_c \) strongly-interacting gauge theories could be described in terms of a weakly-interacting theory of mesons. It was later recognized [2] that, in this dual description, baryons appeared as solitons made of meson fields, as Skyrme had pointed out long before [3]. These solitonic states were therefore referred as skyrmions.

Skyrmions have been widely studied in the literature, with some phenomenological successes [4]. Nevertheless, since the full theory of QCD mesons is not known, these studies have been carried out in truncated low-energy models either incorporating only pions [2, 3] or few resonances [4]. It is unclear whether these approaches capture the physics needed to fully describe the baryons, since the stabilization of the baryon size is very sensitive to resonances around the GeV. In the original Skyrme model with only pions, for instance, the inverse skyrmion size \( \rho_s^{-1} \) equals the chiral perturbation theory cut-off \( \Lambda_{\chi PT} \sim 4\pi F_\pi \) (as it should be, since this is the only scale of the model), rendering baryon physics completely incalculable. Other examples are models with the \( \rho \)-meson [5] or the \( \omega \)-meson [6] which were shown to have a stable skyrmion solution. The inverse size, also in this case, is of order...
$m_{\rho} \sim \Lambda \chi_{PT}$, which is clearly not far from the mass of the next resonances. Including the latter could affect strongly the physics of the skyrmion, or even destabilize it.

In the last ten years the string/gauge duality [7–9] has allowed us to gain new insights into the problem of strongly-coupled gauge theories. This duality has been able to relate certain strongly-coupled gauge theories with string theories living in more than four dimensions. A crucial ingredient in these realizations is a (compact) warped extra dimension that plays the role of the energy scale in the strongly-coupled 4D theory. This has suggested that the QCD dual theory of mesons proposed by 'tHooft [1] must be a theory formulated in more than 4 dimensions.

Inspired by this duality, a five-dimensional field theory has been proposed in Refs. [10, 11] to describe the properties of mesons in QCD. This 5D theory has a cut-off scale $\Lambda_5$ which is above the lowest-resonance mass $m_{\rho}$. The gap among these two scales, which ensures calculability in the meson sector, is related to the number of colors $N_c$ of QCD. In the large $N_c$-limit, one has $\Lambda_5/m_{\rho} \rightarrow \infty$ and the 5D model describes a theory of infinite mesonic resonances, corresponding to the Kaluza-Klein (KK) spectrum. This 5D model has provided a quite accurate description of meson physics in terms of a very limited number of parameters.

Further studies, boosted by this success, have recently shown that the 5D model can also successfully describe baryon physics [12–14]. As Skyrme proposed [3], baryons must appear in this 5D theory as solitons. These 5D skyrmion-like solitons have been numerically obtained (see Fig. 1.1) and their properties have been studied. Their inverse size $\rho_s^{-1} \sim m_{\rho}$ have been found to be smaller than the cut-off scale $\Lambda_5$, showing then that, contrary to the 4D case, they can be consistently studied in 5D effective theories. Indeed, the expansion parameter which ensures calculability is provided by $1/(\rho_s \Lambda_5) \ll 1$.

![Fig. 1.1. Energy density, in the plane of the 4D radial and the extra fifth coordinate, of the skyrmion in a 5D model for QCD.](image)

In this article we will review the properties of baryons obtained in Refs. [12–14]
using the five-dimensional model of QCD of Refs. [10, 11, 15]. We will show how
the calculation of the static properties of the nucleons, such as masses, radii and
form factors, are performed, and will compare the predictions of the model with
experiments. As we will see, these predictions show a reasonably good agreement
with the data.

There have been alternatives studies to baryon physics using 5D models. Nev-
evertheless, these studies have encountered several problems. For example, the first
approaches [16] truncated the 5D theory and only considered the effects of the first
resonances. This leads to skyrmions whose size is of the order of the inverse of the
truncation scale, and therefore sensitive to the discarded heavier resonances. Later
studies [17–19] were performed within the Sakai-Sugimoto model [20]. It was shown,
however, that baryons are not calculable in this framework as their inverse size is
of the order of the string scale which corresponds to the cut-off of the theory [17].

1.2. A five-dimensional model for QCD mesons

The 5D model that we will consider to describe mesons in two massless flavor
QCD is the following. This is a $U(2)_L \times U(2)_R$ gauge theory with metric $ds^2 = a(z)^2 (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2)$, where $x^\mu$ represent the usual 4 coordinates and $z$, which runs in the interval $[z_{UV}, z_{IR}]$, denotes the extra dimension. We will work in AdS
where the warp factor $a(z)$ is

$$a(z) = \frac{z_{IR}}{z},$$

and $z_{UV} \to 0$ to be taken at the end of the calculations. In this limit $z_{IR}$ coincides
with the AdS curvature and the conformal length

$$L = \int_{z_{UV}}^{z_{IR}} dz.$$

The $U(2)_L$ and $U(2)_R$ gauge connections, denoted respectively by $L_M$ and $R_M
(M = \{\mu, 5\})$, are parametrized by $L_M = L_M^a \sigma_a/2 + \hat{L}_M 1/2$ and $R_M = R_M^a \sigma_a/2 + \hat{R}_M 1/2$, where $\sigma_a$ are the Pauli matrices. This chiral gauge symmetry is broken by
the conditions on the boundary at $z = z_{IR}$ (IR-boundary), which read

$$(L_\mu - R_\mu)|_{z = z_{IR}} = 0, \quad (L_{\mu 5} + R_{\mu 5})|_{z = z_{IR}} = 0,$$

where the 5D field strength is defined as $L_{MN} = \partial_M L_N - \partial_N L_M - i[L_M, L_N]$, and analogously for $R_{MN}$. At the other boundary, the UV one, we can consider
generalized Dirichlet conditions for all the fields:

$$L_\mu|_{z = z_{UV}} = l_\mu, \quad R_\mu|_{z = z_{UV}} = r_\mu.$$

The 4D fields $l_\mu$ and $r_\mu$ are arbitrary but fixed and they can be interpreted, as we
will now discuss, as external sources for the QCD global currents. We will eventually
be interested in taking the sources to vanish.
We can now, inspired by the “holographic” formulation of the AdS/CFT correspondence [7–9], try to interpret the above 5D model in terms of a 4D QCD-like theory, whose fields we will generically denote by $\Psi(x)$ and its action by $S_4$. This is a strongly coupled 4D theory that possesses an $U(2)_L \times U(2)_R$ global symmetry with associated Noether currents $j^\mu_{L,R}$. If the 4D theory were precisely massless QCD with two flavors, the currents would be given by the usual quark bilinear, 

\[ j^\mu_{L,R} = \bar{Q}^i \gamma^\mu Q^i_{L,R}. \]

Defining $Z[l_\mu, r_\mu]$ as the generating functional of current correlators, we state our correspondence as

\[
Z[l_\mu, r_\mu] \equiv \int \mathcal{D}\Psi \exp \left[ iS_4[\Psi] + i \int d^4x \text{Tr} (j^\mu_{L,L} l_\mu + j^\mu_{R,R} r_\mu) \right]
\]

where the 5D partition function depends on the sources $l_\mu, r_\mu$ through the UV-boundary conditions in Eq. (1.4).

Eq. (1.5) leads to the following implication. Under local chiral transformations, $Z$ receives a contribution from the $U(2)$ anomaly, which is known in QCD. This implies [9, 21, 22] that the 5D action must contain a Chern-Simons (CS) term

\[
S_{CS} = -\frac{N_c^2}{24\pi^2} \int [\omega_5(L) - \omega_5(R)],
\]

whose variation under 5D local transformations which does not reduce to the identity at the UV exactly reproduces the anomaly. The CS coefficient will be fixed to $N_c = 3$ when matching QCD. The CS 5-form, defining $A = -iA_M dx^M$, is

\[
\omega_5(A) = \text{Tr} \left[ A(dA)^2 + \frac{3}{2} A^3(dA) + \frac{3}{5} A^5 \right].
\]

When $A$ is the connection of an $U(2)$ group, as in our case, one can use the fact that $SU(2)$ is an anomaly-free group to write $\omega_5$ as

\[
\omega_5(A) = \frac{3}{2} \hat{A} \text{Tr} [F^2] + \frac{1}{4} \hat{A} \left( d\hat{A} \right)^2 + d\text{Tr} \left[ \hat{A}AF - \frac{1}{4} \hat{A}A^3 \right],
\]

where $A = A + \hat{A} \mathbf{1}/2$ and $A$ is the $SU(2)$ connection. The total derivative part of the above equation can be dropped, since it only adds to $S_{CS}$ an UV-boundary term for the sources.

The full 5D action will be given by $S_5 = S_g + S_{CS}$, where $S_g$ is made of locally gauge invariant terms. $S_g$ is also invariant under transformations which do not reduce to the identity at the UV-boundary, and for this reason it does not contribute

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*The 5D semiclassical expansion we perform in our model corresponds, as we will explain in the following, to the large-$N_c$ expansion. This is why we are ignoring the $U(1)-SU(N_c)^2$ anomaly of QCD, which is subleading at large-$N_c$. Being this anomaly responsible for the $\eta'$ mass, our model will contain a massless $\eta'$. 
to the anomalous variation of the partition function. Taking the operators of the lowest dimensionality, we have

$$S_9 = - \int d^4 x \int_{z_{UV}}^{z_{IR}} dz a(z) \frac{M_5}{2} \left\{ \text{Tr} [L_{MN}L^{MN}] + \frac{\alpha^2}{2} \hat{L}_{MN}\hat{L}^{MN} + \{L \leftrightarrow R\} \right\}. \quad (1.9)$$

We have imposed on the 5D theory invariance under the combined \{x \rightarrow -x, L \leftrightarrow R\}, where x denotes ordinary 3-space coordinates. This symmetry, under which $S_{CS}$ is also invariant, corresponds to the usual parity on the 4D side. We have normalized differently the kinetic term of the SU(2) and U(1) gauge bosons, since we do not have any symmetry reason to put them equal. In the large-$N_c$ limit of QCD, however, the Zweig’s rule leads to equal couplings (and masses) for the $\rho$ and $\omega$ mesons, implying $\alpha = 1$ in our 5D model. Since this well-known feature of large-$N_c$ QCD does not arise automatically in our 5D framework (as, for instance, the equality of the $\rho$ and $\omega$ masses does), we will keep $\alpha$ as a free parameter. The CS term, written in component notation, will be given by

$$S_{CS} = \frac{N_c}{16\pi^2} \int d^5 x \left\{ \frac{1}{4} \epsilon^{MNPQ} \hat{L}_M \text{Tr} [L_{NO}L_{PQ}] \\
+ \frac{1}{24} \epsilon^{MNPQ} \hat{L}_M \hat{L}_{NO} \hat{L}_{PQ} - \{L \leftrightarrow R\} \right\}. \quad (1.10)$$

The 5D theory defined above has only 3 independent parameters: $M_5$, $L$ and $\alpha$.

Let us make again use of Eq. (1.5) to determine the current operators through which the theory couples to the external EW bosons. These currents are obtained by varying Eq. (1.5) with respect to $l_\mu$ (exactly the same would be true for $r_\mu$) and then taking $l_\mu = r_\mu = 0$. The variation of the l.h.s. of Eq. (1.5) simply gives the current correlator of the 4D theory, while in the r.h.s. this corresponds to a variation of the UV-boundary conditions. The effect of this latter can be calculated in the following way. We perform a field redefinition $L_\mu \rightarrow L_\mu + \delta L_\mu$ where $\delta L_\mu(x,z)$ is chosen to respect the IR-boundary conditions and fulfill $\delta L_\mu(x,z_{UV}) = \delta l_\mu$. This redefinition removes the original variation of the UV-boundary conditions, but leads a new term in the 5D action, $\delta S_5$. One then has

$$i \int d^4 x \text{Tr} [\{j^a_L(x)\} \delta l_\mu(x)] = i \int DA_M DR_M \delta S_5 [L,R] \exp [iS_5 [L,R]], \quad (1.11)$$

where the 5D path integral is now performed by taking $l_\mu = r_\mu = 0$, i.e. normal Dirichlet conditions. The explicit value of $\delta S_5$ is given by

$$\delta S_5 = \int d^4 x \text{Tr} [J^a_L(x) \delta l_\mu(x)] + \int d^5 x (\text{EOM}) \cdot \delta L, \quad (1.12)$$

where $J_{LM} = J^a_L \sigma^a + \hat{J}_{L\mu} \mathbf{1}$ and

$$J^a_L = M_5 (a(z)L^a_\mu) \bigg|_{z=z_{UV}}, \quad \hat{J}_{L\mu} = \alpha^2 M_5 (a(z)\hat{L}_\mu) \bigg|_{z=z_{UV}}. \quad (1.13)$$

The last term of Eq. (1.12) corresponds to the 5D “bulk” part of the variation, which leads to the equations of motion (EOM). Remembering that the EOM always have
zero expectation value\[6\] we find that we can identify \(J_{\mu}^L\) of Eq. (1.13) with the current operator on the 5D side: \(\langle J_{\mu}^L \rangle_{4D} = \langle J_{\mu}^L \rangle_{5D}\). Notice that the CS term has not contributed to Eq. (1.12) due to the fact that each term in \(S_{CS}\) which contains a \(\partial_z\) derivative (and therefore could lead to a UV-boundary term) also contains \(L_{\mu}\) or \(R_{\mu}\) fields; these fields on the UV-boundary are the sources \(l_{\mu}\) and \(r_{\mu}\) that must be put to zero.

### 1.2.1. Meson Physics and Calculability

The phenomenological implications for the lightest mesons of 5D models like the one described above have been extensively studied in the literature. Let us briefly summarize the main results here. If rewritten in 4D terms, the theory contains massless Goldstone bosons that parametrize the \(U(2)_L \times U(2)_R / U(2)_V\) coset and describe the pion triplet and a massless \(\eta'\). The pion decay constant is given by

\[
F_\pi^2 = 2M_5 \left( \int \frac{dz}{a(z)} \right)^{-1} = \frac{4M_5}{L}.
\]

(1.14)

The massive spectrum consists of infinite towers of vector and axial-vector spin-one KK resonances. Among the vectors we have an isospin triplet, the \(\rho^{(n)}\), and a singlet \(\omega^{(n)}\). The axial-vectors are again a triplet \(a_1^{(n)}\) and a singlet \(f_1^{(n)}\). We want to interpret, as our terminology already suggests, the lightest states of each tower as the \(\rho(770)\), \(\omega(782)\), \(a_1(1260)\) and \(f_1(1285)\) resonances, respectively. The model predicts at leading order, i.e. at tree-level,

\[
m_\rho = m_\omega \approx \frac{3\pi}{4L}, \quad m_{a_1} = m_{f_1} \approx \frac{5\pi}{4L},
\]

(1.15)

compatibly with observations. The model also predicts the decay constants \(F_i\) and couplings \(g_i\) for the mesons as a function of \(M_5\), \(L\) and \(\alpha\) that can be found in Refs. [10–12, 15]; here we only notice, for later use, their scaling with the 5D coupling:

\[
F_i \sim \sqrt{M_5}, \quad g_i \sim \frac{1}{\sqrt{M_5}},
\]

(1.16)

while the masses, as shown above, do not depend on \(M_5\). In Table 1.1 we show a fit to 14 meson quantities. The best fit is obtained for the values of \(1/L = 343\) MeV, \(M_5L = 0.0165\) and \(\alpha = 0.94\) for the three parameters of our model. The minimum Root Mean Square Error (RMSE) corresponding to those values is found to be 11% and the relative deviation of each single prediction is below around 15%.

Concerning the choice of the meson observables, some remarks are in order. First of all, we are only considering the lowest state of each KK tower because we expect the masses and couplings of the heavier mesons to receive large quantum corrections. Our model is indeed, as we will explain below, an effective theory valid\[b\]We have actually shown this here; notice that \(\delta L_{\mu}\) was completely arbitrary in the bulk, but the variation of the functional integral can only depend on \(\delta l_{\mu} = \delta L_{\mu}(x, z_{UV})\).
Table 1.1. Global fit to mesonic physical quantities. Masses, decay constants and widths are given in MeV. Physical masses have been used in the kinematic factors of the partial decay widths.

|                | Experiment | AdS5 | Deviation |
|----------------|------------|------|-----------|
| \( m_\rho \)   | 775        | 824  | +6%       |
| \( m_{a_1} \)  | 1230       | 1347 | +10%      |
| \( m_\omega \) | 782        | 824  | +5%       |
| \( F_\rho \)   | 153        | 169  | +11%      |
| \( F_\omega/F_\rho \) | 0.88 | 0.94 | +7%       |
| \( F_\pi \)    | 87         | 88   | +1%       |
| \( g_{\rho\pi\pi} \) | 6.0  | 5.4  | −10%      |
| \( L_9 \)      | \(6.9 \times 10^{-3}\) | \(6.2 \times 10^{-3}\) | −10% |
| \( L_{10} \)   | \(-5.2 \times 10^{-3}\) | \(-6.2 \times 10^{-3}\) | −12% |
| \( \Gamma(\omega \to \pi \gamma) \) | 0.75 | 0.81 | +8%       |
| \( \Gamma(\omega \to 3\pi) \) | 7.5   | 6.7  | −11%      |
| \( \Gamma(\rho \to \pi \gamma) \) | 0.068 | 0.077 | +13%      |
| \( \Gamma(\omega \to \pi \mu \mu) \) | \(8.2 \times 10^{-4}\) | \(7.3 \times 10^{-4}\) | −10% |
| \( \Gamma(\omega \to \pi \varepsilon \varepsilon) \) | \(6.5 \times 10^{-3}\) | \(7.3 \times 10^{-3}\) | +12% |

up to a cut-off \( \Lambda_5 \sim 2 \text{ GeV} \) and our tree-level calculations only correspond to the leading term of an \( E/\Lambda_5 \) expansion. Apart from this restriction, we must include in our fit observables with an experimental accuracy better than 10%. This is because we want to neglect the experimental error in order to obtain an estimate of the accuracy of our theoretical predictions. Much more observables can be computed, once the best-fit value of the parameters are obtained, and several of them have already been considered in the literature. For instance, one can study the other low-energy constants of the chiral lagrangian, the physics of the \( f_1 \) resonance or the pseudo–scalar resonances which arise when the explicit breaking of the chiral symmetry is taken into account [11]. It would also be interesting to compute the \( a_1 \to \pi \gamma \) decay, which is absent in our model at tree-level and only proceeds via loop effects or higher-dimensional terms of our 5D effective lagrangian.\(^d\)

As discussed in the Introduction, the semiclassical expansion in the 5D model should correspond to the large-\( N_c \) expansion on the 4D side. The results presented above provide a confirmation of this interpretation: at large-\( N_c \) meson masses are expected to scale like \( N_c^0 \), while meson couplings and decay constants scale like \( g_i, 1/F_i \sim 1/\sqrt{N_c} \). \(^c\) These scalings agree with Eq. (1.15) and (1.16) if the parameters \( \alpha, L \) and \( M_5 \) are taken to scale like\(^d\)

\[
\alpha \sim N_c^0, \quad L \sim N_c^0, \quad M_5 \sim N_c. \quad (1.17)
\]

This leads us to define the adimensional \( N_c \)-invariant parameter

\[
\gamma = \frac{N_c}{16\pi^2 M_5 L \alpha}, \quad (1.18)
\]

\(^c\)Higher order contributions will also change our tree-level prediction \( L_9 + L_{10} = 0 \), which is again related with the absence of the \( a_1-\pi-\gamma \) vertex.

\(^d\)This scaling can also be obtained from the AdS/CFT correspondence.
whose experimental value is $\gamma = 1.23$ and will be useful later on. We will also show in the following that the assumed scaling of the 5D parameters leads to the correct $N_c$ scaling in the baryon sector as well.

Other descriptions of vector mesons in terms of massive vector fields, \textit{i.e.} models with Hidden Local Symmetry (HLS) \cite{23, 24} or two-form fields \cite{25}, also correctly reproduce the meson physical properties. Nevertheless, we believe that 5D models, as the one discussed here, present more advantages. First of all, they contain less parameters. In the models of Refs. \cite{23–25}, for example, the mass and the couplings of each meson are independent parameters; also anomalous processes, those involving an odd number of pions, depend on several operators with unknown coefficients which arise at the same order, while in our case they all arise from a single operator, the 5D CS term. Finally, Vector Meson Dominance is automatic in our scenario, while it needs to be imposed “by hand” in the case of HLS.

Moreover, and perhaps more importantly, 5D models are calculable effective field theory in which higher-dimensional operators are suppressed by the cut-off of the theory $\Lambda_5$. Calculations can be organized as an expansion in $E/\Lambda_5$, where $E$ is the typical scale of the process under consideration. Given that the cut-off is parametrically bigger than the mass of the lightest mesons, reliable calculations of masses and couplings can be performed.

Let us now use naive dimensional arguments to estimate the maximal value of our cut-off $\Lambda_5$. This is determined by the scale at which loops are of order of tree-level effects. Computing loop corrections to the $F^2$ operator, which arise from the $F^2$ term itself, one gets $\Lambda_5 \sim 24\pi^3 M_5$. Nevertheless, one gets a lower value for $\Lambda_5$ from the CS term due to the $N_c$ dependence of its coefficient. Indeed, at the one-loop level, the CS term gives a contribution of order $M_5$ to the $F^2$ operator for $\Lambda_5 \sim 24\pi^3 M_5/N_c^{2/3}$. Even though the cut-off scale lowers due to the presence of the CS term, we can still have, in the large-$N_c$ limit, a 5D weakly coupled theory where higher-dimensional operators are suppressed. The cut-off can be rewritten as

$$\Lambda_5 \sim \frac{3\pi N_c^{1/3}}{2\gamma \alpha L} \sim 2 \text{ GeV},$$

where we have used the best-fit value of our parameters.

The power of calculability of our 5D model makes it very suitable for studying baryon physics. Indeed, the typical size of the 5D skyrmion solution will be of order $\rho_s \sim 1/m_\rho$, guaranteeing that effects from higher-dimensional operators will be suppressed by $m_\rho/\Lambda_5 \sim 0.4$. This is therefore, we believe, the first fully consistent approach towards baryon physics.

\*\*It must be possible, generalizing what was done in Ref. \cite{26}, to rewrite our model as a 4D HLS with infinitely many $U(2)$ hidden symmetry groups. The comparison with HLS models that we perform in this section only applies, therefore, to the standard case of a finite number of hidden symmetries.
1.3. Baryons from 5D Skyrmions

1.3.1. 4D Skyrmions from 5D Solitons

Time-independent configurations of our 5D fields, which correspond to allowed initial ($t \to -\infty$) and final ($t \to +\infty$) states of the time evolution, are labeled by the topological charge

$$B = \frac{1}{32\pi^2} \int d^3x \int_{z_{UV}}^{z_{IR}} dz \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \text{Tr} \left[L^\hat{\mu} L^\hat{\nu} - R^\hat{\mu} R^\hat{\nu}\right],$$

(1.19)

where the indices $\hat{\mu}, \hat{\nu}, \ldots$ run over the 4 spatial coordinates, but they are raised with Euclidean metric. We will now show that $B$ can only assume integer values, which ensures that it cannot be changed by the time evolution. This makes $B$ a topologically conserved charge which we identify with the baryon number. In order to show this, and with the aim of making the relation with the skyrmion more precise, it is convenient to go to the axial gauge $L^5 = R^5 = 0$. The latter can be easily reached, starting from a generic gauge field configuration, by means of a Wilson-line transformation. In the axial gauge both boundary conditions Eqs. (1.3) and (1.4) (in which we take now $l = r = 0$) cannot be simultaneously satisfied. Let us then keep Eq. (1.3) but modify the UV-boundary condition to

$$\tilde{L}_i |_{z = z_{UV}} = i \frac{U(x)\partial_i U(x)^\dagger}{1}, \quad \tilde{R}_i |_{z = z_{UV}} = 0,$$

(1.20)

where $\tilde{L}_i$ and $\tilde{R}_i$ are the gauge fields in the axial gauge and $i$ runs over the 3 ordinary space coordinates. The field $U(x)$ in the equation above precisely corresponds to the Goldstone field in the 4D interpretation of the model [22]. Remembering that $F \wedge F = d\omega_3$, where $\omega_3$ is the third CS form, the 4D integral in Eq. (1.19) can be rewritten as an integral on the 3D boundary of the space:

$$B = \frac{1}{8\pi^2} \int_{3D} \omega_3(\tilde{L}) - \omega_3(\tilde{R}).$$

(1.21)

The contribution to $B$ coming from the IR-boundary vanishes as the $L$ and $R$ terms in Eq. (1.21) cancel each other due to Eq. (1.3). This is crucial for $B$ to be quantized and it is the reason why we have to choose the relative minus sign among the $L$ and $R$ instanton charges in the definition of $B$. At the $x^2 \to \infty$ boundary, the contribution to $B$ also vanishes since in the axial gauge $\partial_i A_i = 0$ (in order to have $F_{5i} = 0$). We are then left with the UV-boundary which we can topologically regard as the 3-sphere $S_3$. Therefore, we find

$$B = -\frac{1}{8\pi^2} \int_{UV} \omega_3 \left[\tilde{L}_i \left(= i \frac{U\partial_i U^\dagger}{1}\right)\right] = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{Tr} \left[U \partial_i U^\dagger U \partial_j U^\dagger U \partial_k U^\dagger\right] \in \mathbb{Z}.$$

(1.22)

The charge $B$ is equal to the Cartan-Maurer integral invariant for $SU(2)$ which is an integer.
In the next section we will discuss regular static solutions with nonzero $B$. If they exist, they cannot trivially correspond to a pure gauge configuration. Moreover, the particles associated to solitons with $B = \pm 1$ will be stable given that they have minimal charge. Eq. (1.22) also makes the relation with 4D skyrmions explicit: topologically non-trivial 5D configurations are those for which the corresponding pion matrix $U(x)$ is also non-trivial. The latter corresponds to a 4D skyrmion with baryon number $B$. In a general gauge, the skyrmion configuration $U(x)$ will be given by

$$
U(x) = P \left\{ \exp \left[ -i \int_{uv} \frac{dz'}{2\pi} R_5(x, z') \right] \right\} \cdot P \left\{ \exp \left[ i \int_{uv} \frac{dz'}{2\pi} L_5(x, z') \right] \right\},
$$

where $P$ indicates path ordering. From a 4D perspective, the 5D soliton that we are looking for can be considered to be a 4D skyrmion made of Goldstone bosons and the massive tower of KK gauge bosons.

1.3.2. The Static Solution

In order to obtain the static soliton solution of the 5D EOM of our theory it is crucial to specify a correct Ansatz, which is best constructed by exploiting the symmetries of our problem. Let us impose, first of all, our solution to be invariant under time-reversal $t \rightarrow -t$ combined with $\hat{L} \rightarrow -\hat{L}$ and $\hat{R} \rightarrow -\hat{R}$, under which also the CS term is invariant. This transformation reduces, in static configurations, to a sign change of the temporal component of $L$ and $R$ and of the spatial components of $\hat{L}$ and $\hat{R}$. We can therefore consistently put them to zero. We also use parity invariance ($\{L \leftrightarrow R, x \leftrightarrow -x\}$) to restrict to configurations for which $L_i(x, z, t) = -R_i(-x, z, t)$, $L_{5,0}(x, z, t) = R_{5,0}(-x, z, t)$ and analogously for $\hat{L}$, $\hat{R}$. We impose, finally, invariance under ”cylindrical” transformations [27], i.e. the simultaneous action of 3D space rotations $x_a \sigma^a \rightarrow \theta^a x_a \sigma^a \theta$, with $\theta \in SU(2)$, and vector $SU(2)$ global transformations $L, R \rightarrow \theta (L, R) \theta^\dagger$. An equivalent way to state the invariance is that a 3D rotation with $\theta$ acts on the solution exactly as an $SU(2)$ vector one in the opposite direction (i.e. with $\theta^\dagger$) would do. The resulting Ansatz for the static solution (which we denote by “barred” fields) is entirely specified 4 real 2D fields

$$
\begin{align*}
\bar{R}_j^a(x, z) &= A_1(r, z) \hat{x}_a \hat{x}_j + \frac{1}{r} \varepsilon_{a j k} \hat{x}_k - \frac{\phi(x)}{r} \varepsilon^{(x, y)} \Delta^{(y), aj}, \\
\bar{R}_5^a(x, z) &= A_2(r, z) \hat{x}^a, \\
\alpha \bar{R}_0(x, z) &= \frac{s(r, z)}{r},
\end{align*}
$$

(1.24)

where $r^2 = \sum_i x^i x^i$, $\hat{x}^i = x^i / r$, $\varepsilon^{(x, y)}$ is the antisymmetric tensor with $\varepsilon^{(1, 2)} = 1$ and the “doublet” tensors $\Delta^{(1, 2)}$ are

$$
\begin{align*}
\Delta^{(x), ab} &= \begin{bmatrix}
\varepsilon^{a b c} \hat{x}^c \\
2 x^a x^b - \delta^{a b}
\end{bmatrix},
\end{align*}

(1.25)
Substituting the Ansatz in the topological charge Eq. (1.19) we find

\[ B = \frac{1}{2\pi} \int_0^{\infty} dr \int_{z_{\text{UV}}}^{z_{\text{IR}}} dz \epsilon^{\hat{\mu}\hat{\nu}} \left[ \partial_{\hat{\mu}}(-i\phi^* D_{\hat{\nu}} \phi + h.c.) + A_{\hat{\mu} \hat{\nu}} \right], \quad (1.26) \]

where \( x^{\hat{\mu}} = \{r, z\} \), \( A_{\hat{\mu}} = \{A_1, A_2\} \), \( A_{\hat{\mu} \hat{\nu}} \) its field-strength, \( \phi = \phi_1 + i\phi_2 \) and the covariant derivative will be defined in Eq. (1.35). The charge can be written, as it should, as an integral over the 1D boundary of the 2D space. Finite-energy regular solutions with \( B = 1 \) which obey Eqs. (1.3) and (1.4) must respect the following boundary conditions:

\[
\begin{align*}
z = z_{\text{IR}} : & \quad \{ \phi_1 = 0, \phi_2 = 0, A_1 = 0, A_2 = 0, s = 0 \} \\
& \quad \{ \phi_1 = 0, \phi_2 = -1, A_1 = 0, s = 0 \}
\end{align*}
\]

(1.27)

and

\[
\begin{align*}
\{ \phi_1/r \rightarrow A_1, (1 + \phi_2)/r \rightarrow 0, A_2 = 0, s = 0 \} & \quad \{ \phi = -ie^{i\pi z/L}, A_2 = \frac{L}{r}, s = 0 \}
\end{align*}
\]

(1.28)

Solutions of the EOM with the required boundary conditions exist, and have been obtained numerically in Ref. [13] using the COMSOL package [28] (see Appendix for details). The 2D energy density of this solution is given in Fig. 1.1.

\subsection*{1.3.3. Zero-Mode Fluctuations}

Let us now consider time-dependent infinitesimal deformations of the static solutions. Among these, the zero-mode (i.e. zero frequency) fluctuations are particularly important as they will describe single-baryon states. Zero-modes can be defined as directions in the field space in which uniform and slow motion is permitted by the classical dynamics and they are associated with the global symmetries of the problem, which are in our case \( U(2)_V \) and 3-space rotations plus 3-space translations. The latter would describe baryons moving with uniform velocity and therefore can be ignored in the computation of static properties like the form factors. Of course, the global \( U(1)_V \) acts trivially on all our fields and the global \( SU(2)_V \) has the same effect as 3-space rotations on the static solution (1.24) because of the cylindrical symmetry. The space of static solutions which are of interest for us is therefore parametrized by 3 real coordinates –denoted as collective coordinates– which define an \( SU(2) \) matrix \( U \).

To construct zero-modes fluctuations we consider collective coordinates with general time dependence, i.e. we perform a global \( SU(2)_V \) transformation on the static solution

\[ R_{\hat{\mu}}(x, z; U) = U \tilde{R}_{\hat{\mu}}(x, z) U^\dagger, \quad \tilde{R}_0(x, z) = \tilde{R}_0(x, z), \quad (1.29) \]
but we allow $U = U(t)$ to depend on time. It is only for constant $U$ that Eq. (1.29) is a solution of the time-dependent EOM. For infinitesimal but non-zero rotational velocity

$$K = k_0 \sigma^a/2 = -iU^\dagger dU/dt,$$

Eq. (1.29) becomes an infinitesimal deformation of the static solution. Along the zero-mode direction uniform and slow motion is classically allowed, for this reason our fluctuations should fulfill the time-dependent EOM at linear order in $K$ provided that $dK/dt = 0$.

From the action (1.9) and (1.10) the following EOM are derived

$$\begin{align*}
D_\phi \left( a(z) R^0_\phi \right) &+ \frac{\gamma_0 L}{4} e^{\alpha_0 \overline{\phi} \phi} R_{\overline{\phi} \phi} \tilde{R}_{\overline{\phi} \phi} = 0 \\
\alpha \partial_\phi \left( a(z) \tilde{R}^0_\phi \right) &+ \frac{\gamma_0 L}{4} e^{\alpha_0 \overline{\phi} \phi} \left[ \text{Tr} \left( R_{\overline{\phi} \phi} R_{\overline{\phi} \phi} \right) + \frac{1}{2} \tilde{R}_{\overline{\phi} \phi} R_{\overline{\phi} \phi} \right] = 0 \\
D_\phi \left( a(z) \tilde{R}^0_\phi \right) &- a(z) D_0 \tilde{R}_0 \phi - \frac{\gamma_0 L}{2} e^{\alpha_0 \overline{\phi} \phi} \left[ R_{\overline{\phi} \phi} \tilde{R}_{\overline{\phi} \phi} + R_{\overline{\phi} \phi} \tilde{R}_{\overline{\phi} \phi} \right] = 0 \\
\alpha \partial_\phi \left( a(z) \tilde{R}^0_\phi \right) &- \alpha a(z) \partial_0 \tilde{R}_0 \phi - \gamma_0 L e^{\alpha_0 \overline{\phi} \phi} \left[ \text{Tr} \left( R_{\overline{\phi} \phi} R_{\overline{\phi} \phi} \right) + \frac{1}{2} \tilde{R}_{\overline{\phi} \phi} R_{\overline{\phi} \phi} \right] = 0
\end{align*}$$

(1.30)

We only need to specify the EOM for one chirality since we are considering, as explained in the previous section, a parity invariant Ansatz. We would like to find solutions of Eq. (1.30) for which $R_\phi$ and $\tilde{R}_0$ are of the form (1.29); it is easy to see that the time-dependence of $U$ in Eq. (1.29) acts as a source for the components $R_0$ and $\tilde{R}_0$, which therefore cannot be put to zero as in the static case. Notice that the same happens in the case of the 4D skyrmion [4], in which the temporal and spatial components of the $\rho$ and $\omega$ mesons are turned on in the rotating skyrmion solution. Also, it can be shown that Eq. (1.30) can be solved, to linear order in $K$ and for $dK/dt = 0$, by the Ansatz in Eq. (1.29) if the fields $R_0$ and $\tilde{R}_0$ are chosen to be linear in $K$. Even though $K$ must be constant for the EOM to be solved, it should be clear that this does not imply any constraint on the allowed form of the collective coordinate matrix $U(t)$ in Eq. (1.29), which can have an arbitrary dependence on time. What we actually want to do here is to find an appropriate functional dependence of the fields on $U(t)$ such that the time-dependent EOM would be solved if and only if the rotational velocity $K = -iU^\dagger dU/dt$ was constant.

In order to solve the time-dependent equations (1.30) we will consider a 2D Ansatz obtained by a generalization of the cylindrical symmetry of the static case. The Ansatz for $R_\phi$ and $\tilde{R}_0$ is specified by Eq. (1.29) in which the static fields are given by Eq. (1.24). Due to the cylindrical symmetry of the static solution the fields in Eq. (1.29) are invariant under 3D space rotations $x_a \sigma^a \rightarrow x_a \sigma^a \theta$ combined with vector $SU(2)$ global transformations $L, R \rightarrow \theta (L, R) \theta^\dagger$ if $U$ also transforms as $U \rightarrow \theta^\dagger U \theta$. We are therefore led to consider a generalized cylindrical symmetry under which $k_0$ also rotates as the space coordinates do. Compatible with this symmetry and with the fact that $R_0$ and $\tilde{R}_0$ must be linear in $K$ we write the
Ansatz as
\[ R_0(x, z; U) = U \mathcal{R}_0(x, z; K) U^\dagger + i U \partial_0 U^\dagger, \quad \widehat{R}_\mu(x, z; U) = \widehat{\mathcal{R}}_\mu(x, z; K), \] (1.31)
where
\[ \begin{align*}
\alpha \mathcal{R}_0(x, z; K) &= \chi(x)(r, z)k_0 \Delta, (x, ab) + v(r, z)(k \cdot \hat{x}) \hat{x}^a \\
\alpha \mathcal{R}_i(x, z; K) &= \frac{\rho(r, z)}{r} (k^i - (k \cdot \hat{x}) \hat{x}^i) + B_1(r, z)(k \cdot \hat{x}) \hat{x}^i + Q(r, z) e^{ibc} k_0 \hat{x}_c.
\end{align*} \]
\[ \alpha \mathcal{R}_5(x, z; K) = B_2(r, z)(k \cdot \hat{x}) \] (1.32)

It must be observed that our Ansatz has not fixed the 5D gauge freedom completely; its form is indeed preserved by chiral SU(2)$_{L,R}$ gauge transformations of the form $g_R = U(t) \cdot g \cdot U^\dagger(t)$ and $g_L = U(t) \cdot \hat{g} \cdot U^\dagger(t)$ with
\[ g = \exp[i \alpha(r, z) x^a \sigma_a/(2r)], \] (1.33)
under which the 2D fields $\phi(x)$ and $\chi(x)$ defined respectively in Eq. (1.24) and (1.22) transform as charged complex scalars. The fields $A_\mu$ transform as gauge fields. There is also a second residual $U(1)$ associated with chiral $U(1)_{L,R}$ 5D transformations of the form $\hat{g}_R = \hat{g}$ and $\hat{g}_L = \hat{g}^\dagger$ with
\[ \hat{g} = \exp \left[ i \beta(r, z) \frac{(k \cdot \hat{x})}{\alpha} \right]. \] (1.34)

Under this second residual $U(1)$ only $B_\mu = \{B_1, B_2\}$ and $\rho$ transform non trivially; $B_\mu$ is a gauge field and $\rho$ a Goldstone. In order to make manifest the residual gauge invariance of the observables we will compute we introduce gauge covariant derivatives for the $\phi, \chi$ and $\rho$ fields
\[ \begin{align*}
(D_\mu \phi)(x) &= \partial_\mu \phi(x) + \epsilon^{xy} A_\mu \phi(y) \\
(D_\mu \chi)(x) &= \partial_\mu \chi(x) + \epsilon^{xy} A_\mu \chi(y) \\
D_\mu \rho &= \partial_\mu \rho - B_\mu.
\end{align*} \] (1.35)

At this point it is straightforward to find the zero-mode solution. The EOM for the 2D fields can be obtained by plugging the Ansatz in Eq. (1.30), while the conditions at the IR and UV boundaries are derived from Eq. (1.3) and (1.4), respectively. The boundary conditions at $r = 0$ are obtained by imposing the regularity of the Ansatz, while those for $r \to \infty$ come from requiring the energy of the solution to be finite and $B = 1$. Also in this case, numerical solutions can be obtained with the methods discussed in the appendix. The reader not interested in detail can simply accept that a solution of Eq. (1.30) exists and is given by our Ansatz for some particular functional form of the 2D fields which we are able to determine numerically. In the rest of the paper the 2D fields will always denote this numerical solution of the 2D equations.
The collective coordinate matrix $U(t)$ will be associated with static baryons. The classical dynamics of the collective coordinates is obtained by plugging Eqs. (1.29) and (1.31) in the 5D action. One finds

$$S = \sum_z \text{Lagrangian of Collective Coordinates}$$

The collective coordinate matrix $U$ will be associated with static baryons. The classical dynamics of the collective coordinates is obtained by plugging Eqs. (1.29) and (1.31) in the 5D action. One finds $S = \int dt L$ where

$$L = -M + \frac{\lambda}{2} k_a k^a,$$  (1.36)

The mass $M$ and the moment of inertia $\lambda$ are given respectively by

$$M = 8\pi M_5 \frac{1}{3} \int_0^\infty dr \int_{z_{UV}} \frac{dz}{z_{UV}} \left\{ a(z) \left[ -(D_{\mu} \phi)^2 - r^2 \left( \partial_\mu Q \right)^2 - 2Q^2 - \frac{r^2}{4} B_{\mu \nu} B_{\mu \nu} \right] + r^2 (D_\mu \chi)^2 + \frac{r^2}{2} \left( \partial_\mu \nu \right)^2 + (\chi(x) \chi(x) + \phi^2) \left( 1 + \phi(x) \phi(x) \right) - 4v \phi(x) \chi(x) \chi(x) \right\},$$

and

$$\lambda = 16\pi M_5 \frac{1}{3} \int_0^\infty dr \int_{z_{UV}} \frac{dz}{z_{UV}} \left\{ a(z) \left[ -(D_{\mu} \phi)^2 - r^2 (\partial_\mu Q)^2 - 2Q^2 - \frac{r^2}{4} B_{\mu \nu} B_{\mu \nu} \right] + r^2 (D_\mu \chi)^2 + \frac{r^2}{2} \left( \partial_\mu \nu \right)^2 + (\chi(x) \chi(x) + \phi^2) \left( 1 + \phi(x) \phi(x) \right) - 4v \phi(x) \chi(x) \chi(x) \right\},$$

The numerical values of $M$ and $\lambda$ are easily computed, once the numerical solution for the 2D fields is known. Using the best-fit values of the parameters we find $M = 1132$ MeV and $1/\lambda = 227$ MeV.

Let us give some more detail on this theory. For now we proceed at the classical level and we will discuss the quantization in the next section. Our lagrangian can be rewritten as

$$L = -M + \lambda \text{Tr} \left[ \dot{U}^\dagger \dot{U} \right] = -M + 2\lambda \sum_i \dot{u}_i^2,$$  (1.39)

where we have parametized the collective coordinates matrix $U$ as $U = u_0 \mathbf{1} + i u_i \sigma^i$, with $\sum_i u_i^2 = 1$. The lagrangian (1.39) is the one of the classical spherical rigid rotor. The variables $\{u_0, u_i\}$ are restricted to the unitary sphere $S^3$, which is conveniently parametrized by the coordinates $q^a \equiv \{x, \phi_1 \phi_2\}$ --which run in the $x \in [-1, 1], \phi_1 \in [0, 2\pi]$ and $\phi_2 \in [0, 2\pi]$ domains-- as

$$u_1 + i u_2 \equiv z_1 = \sqrt{\frac{1 - x}{2}} e^{i \phi_1}, \quad u_0 + i u_3 \equiv z_2 = \sqrt{\frac{1 + x}{2}} e^{i \phi_2},$$  (1.40)

where we also introduced the two complex coordinates $z_{1,2}$. We can now rewrite the lagrangian as

$$L = -M + 2\lambda g_{\alpha \beta} q^\alpha q^\beta,$$  (1.41)
where \( g \) is the metric of \( S^3 \) which reads in our coordinates

\[
ds^2 = g_{\alpha\beta} dq^\alpha dq^\beta = \frac{1}{4} \frac{1}{1-x^2} dx^2 + \frac{1}{2} d\phi_1^2 + \frac{1}{2} d\phi_2^2.
\] (1.42)

The conjugate momenta are \( p_\alpha = \partial L/\partial \dot{q}^\alpha = 4\lambda g_{\alpha\beta} \dot{q}^\beta \) and therefore the classical Hamiltonian is

\[
H_c = M + \frac{1}{8\lambda} p_\alpha g^{\alpha\beta}(q)p_\beta.
\] (1.43)

It should be noted that the points \( U \) and \(-U\) in what we denoted as the space of collective coordinates actually describe the same field configuration (see Eq. (1.29,1.31)). The \( SU(2) = S_3 \) manifold we are considering is actually the universal covering of the collective coordinate space which is given by \( S_3/\mathbb{Z}_2 \). This will be relevant when we will discuss the quantization.

### 1.3.5. Skyrmion Quantization

We should now quantize the classical theory described above, by replacing as usual the classical momenta \( p_\alpha \) with the differential operator \(-i\partial/\partial q^\alpha\) acting on the wave functions \( f(q) \). Given that the metric depends on \( q \), however, there is an ambiguity in how to extract the quantum hamiltonian \( H_q \) from the classical one in Eq. (1.43). This ambiguity is resolved by requiring the quantum theory to have the same symmetries that the classical one had. At the classical level, we have an \( SO(4) \approx SU(2) \times SU(2) \) symmetry under \( U \rightarrow U \cdot \theta^\dagger \) and \( U \rightarrow g \cdot U \) with \( \theta, g \in SU(2) \). These correspond, respectively, to rotations in space and to isospin (i.e. global vector) transformations, as one can see from the Ansatz in Eqs. (1.29,1.31). This is because \( K \) is invariant under left multiplication by \( g \), and that the Ansatz is left unchanged by performing a rotation \( x_\alpha \sigma^a \rightarrow \theta^\dagger x_\alpha \sigma^a \theta \) and simultaneously sending \( U \rightarrow U \cdot \theta \). The spin and isospin operators must be given, in the quantum theory, by the generators of these transformations on the space of wave functions \( f(q) \) which are defined by

\[
[S^a,U] = U\sigma^a/(2), \quad [I^a,U] = -\sigma^a/(2)U.
\] (1.44)
After a straightforward calculation one finds

\[
\begin{align*}
S^3 &= -\frac{i}{2} (\partial_{\phi_1} + \partial_{\phi_2}) \\
S^+ &= \frac{1}{\sqrt{2}} e^{i(\phi_1 + \phi_2)} \left[ i \sqrt{1-x^2} \partial_x + \frac{1}{2} \sqrt{\frac{1+x}{1-x}} \partial_{\phi_1} - \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \partial_{\phi_2} \right] \\
S^- &= \frac{1}{\sqrt{2}} e^{-i(\phi_1 + \phi_2)} \left[ i \sqrt{1-x^2} \partial_x - \frac{1}{2} \sqrt{\frac{1+x}{1-x}} \partial_{\phi_1} + \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \partial_{\phi_2} \right] \\
I^3 &= -\frac{i}{2} (\partial_{\phi_1} - \partial_{\phi_2}) \\
I^+ &= -\frac{1}{\sqrt{2}} e^{i(\phi_1 - \phi_2)} \left[ i \sqrt{1-x^2} \partial_x + \frac{1}{2} \sqrt{\frac{1+x}{1-x}} \partial_{\phi_1} + \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \partial_{\phi_2} \right] \\
I^- &= -\frac{1}{\sqrt{2}} e^{-i(\phi_1 - \phi_2)} \left[ i \sqrt{1-x^2} \partial_x - \frac{1}{2} \sqrt{\frac{1+x}{1-x}} \partial_{\phi_1} - \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \partial_{\phi_2} \right]
\end{align*}
\]

(1.45)

where the raising/lowering combinations are \( S^\pm = (S^1 \pm iS^2)/\sqrt{2} \).

The operators in Eq. (1.45) should obey the Hermiticity conditions \((S^3)^\dagger = S^3\), \((S^+)^\dagger = S^-,\) and analogously for the isospin. In order for the Hermiticity conditions to hold we choose the scalar product to be

\[
\langle A|B \rangle = \int d^3q \sqrt{g} f_A^\dagger(q) f_B(q),
\]

(1.46)

where \( \sqrt{g} = 1/4 \) in our parametrization of \( S_3 \). The reason why this choice of the scalar product gives the correct Hermiticity conditions is that \( S^a \) and \( I^a \) (where \( a = 1, 2, 3 \)) can be written as \( X^a \partial_a \) with \( X^a \) Killing vectors of the appropriate \( S_3 \) isometries. The Killing equation \( \nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0 \) ensures the generators to be Hermitian with respect to the scalar product (1.46).

Knowing that the scalar product must be given by Eq. (1.46) greatly helps in guessing what the quantum Hamiltonian, which has to be Hermitian, should be. We can multiply and divide by \( \sqrt{g} \) the kinetic term of \( H_c \) and move one \( \sqrt{g} \) factor to the left of \( p_a \). Then we apply the quantization rules and find\[6\]

\[
H_q = M - \frac{1}{8\lambda} \sqrt{g} \nabla_\alpha \left( \sqrt{g} g^{\alpha\beta} \partial_\beta \right) = M - \frac{1}{8\lambda} \nabla_\alpha \nabla^\alpha,
\]

(1.47)

which is clearly Hermitian. We can immediately show that \( H_q \) commutes with spin and isospin, so that the quantum theory is really symmetric as required: a straightforward calculation gives indeed

\[
H_q = M + \frac{1}{2\lambda} S^2 = M + \frac{1}{2\lambda} I^2.
\]

(1.48)

It would not be difficult to solve the eigenvalue problem for the Hamiltonian (1.47), but in order to find the nucleon wave functions it is enough to note that the versor of \( n \)-dimensional Euclidean space provides the \( n \) representation of the

\[6\] The last equality holds because \( H_q \) is supposed to be acting on the wave functions, which are scalar functions.
SO(n) isometry group. In our case, \( n = 4 = (2, 2) \), which is exactly the spin/isospin representation in which nucleons live. It is immediately seen that \( z_1 \), as defined in Eq. (1.40), has \( S^3 = I^3 = 1/2 \). Acting with the lowering operators we easily find the wave functions

\[
|p \uparrow \rangle = \frac{1}{\pi} z_1, \quad |n \uparrow \rangle = \frac{i}{\pi} z_2, \quad |p \downarrow \rangle = -\frac{i}{\pi} z_2, \quad |n \downarrow \rangle = -\frac{1}{\pi} z_1,
\]

which are of course normalized with the scalar product (1.46). The mass of the nucleons is therefore \( E = M + 3/(8\lambda) \).

Notice that the nucleon wave functions are odd under \( U \rightarrow -U \), meaning that they are double-valued on the genuine collective coordinate space \( S_3/Z_2 \). This corresponds, following [29], to quantize the skyrmion as a fermion and explains how we could get spin-1/2 states after a seemingly bosonic quantization without violating spin-statistic.

Let us now summarize some useful identities which will be used in our calculation. First of all, it is not hard to check that, after the quantization is performed the rotational velocity becomes

\[
k^a = -i \text{Tr}\left[ U^\dagger \dot{U} \sigma^a \right] = \frac{1}{\lambda} S^a,
\]

and analogously

\[
i \text{Tr}\left[ \dot{U} U^\dagger \sigma^a \right] = \frac{1}{\lambda} I^a.
\]

Other identities which we will use in our calculations are

\[
\langle \text{Tr} \left[ U \sigma^b U^\dagger \sigma^a \right] = -\frac{8}{3} S^b I^a \rangle,
\]

\[
\langle \text{Tr} \left[ U \sigma^b \tilde{x}_b (k \cdot \tilde{x}) U^\dagger \sigma^a \right] = -\frac{2}{3\lambda} I^a \rangle,
\]

where the VEV symbols \( \langle ... \rangle \) mean that those are not operatorial identities, but they only hold when the operators act on the subspace of nucleon states. Notice that the second equation in (1.52) is implied by the first one if one also uses the commutation relation (1.44), Eq. (1.51) and the fact that, on nucleon states, \( \langle \{ S^a, S^1 \} = \delta^{a1}/2\).}

### 1.3.6. The Nucleon Form Factors

The nucleon form factors parametrize the matrix element of the currents on two nucleon states. For the isoscalar and isovector currents we have

\[
\langle N_f(p')|J^S_\mu(0)|N_i(p)\rangle = \bar{u}_f(p') \left[ F_1^S(q^2) \gamma^\mu + i F_2^S(q^2) \right] \frac{q^\nu}{2 |q|^2} q^\nu u_i(p),
\]

\[
\langle N_f(p')|J^V_\mu(0)|N_i(p)\rangle = \bar{u}_f(p') \left[ F_1^V(q^2) \gamma^\mu + i F_2^V(q^2) \right] \frac{q^\nu}{2 |q|^2} q^\nu (2 I^a) u_i(p). \tag{1.53}
\]
where the currents are defined as $J^a_V = J^a_R + J^a_L$ and $J_S = 1/3 \left( J_R + J_L \right)$ in terms of the chiral ones. In the equation above $q \equiv p' - p$ is the 4-momentum transfer, $N_i$ and $N_f$ are the initial and final nucleon states and $u_i(p)$, $u_f(p')$ their wave functions, $I^a = \sigma^a/2$ is the isospin generators and $\sigma^{\mu\nu} \equiv i/2 [\gamma^\mu, \gamma^\nu]$. For the axial current $J^a_A = J^a_R - J^a_L$ we have

$$\langle N_f(p')|J^a_A (0)|N_i(p)\rangle = \bar{u}_f(p')G_A(q^2) \left[ \gamma_\mu - \frac{2M_N}{q^2} q^\mu \right] \gamma^5 I^a u_f(p).$$

(1.54)

Exact axial and isospin symmetries, which hold in our model, have been assumed in the definitions above.

In our non-relativistic model the current correlators will be computed in the Breit frame in which the initial nucleon has 3-momentum $-\vec{q}/2$ and the final $+\vec{q}/2$ (i.e. $p^\mu = (E, -\vec{q}/2)$ and $p'^\mu = (E, \vec{q}/2)$, and $q^2 = -\vec{q}^2$, with $E = \sqrt{M^2_N + \vec{q}^2/4}$). Notice that the textbook definitions in Eqs. (1.53,1.54) involve nucleon states which are normalized with $\sqrt{2E}$; in order to divide with our non-relativistic normalization we have to divide all correlators by $2M_N$. The vector currents become

$$\langle N_f(\vec{q}/2)|J^a_V (0)|N_i(-\vec{q}/2)\rangle = G_V^S(q^2) \chi^i \gamma^\mu \gamma^5 I^a \chi_i,$n

(1.55)

where we defined

$$G_E^{S,V}(q^2) = F_1^{S,V}(q^2) + \frac{q^2}{4M_N^2} F_2^{S,V}(q^2), \quad G_M^{S,V}(q^2) = F_1^{S,V}(q^2) + F_2^{S,V}(q^2),$$

(1.56)

and used the definition $(\vec{S} \times \vec{q})^i \equiv \epsilon^{ijk} S_j q^k$. The nucleon spin/isospin vectors of state $\chi_{i,f}$ are normalized to $\chi_i^\dagger \chi_i = 1$. For the axial current we find

$$\langle N_f(\vec{q}/2)|J^a_A (0)|N_i(-\vec{q}/2)\rangle = \chi_f^i \frac{E}{M_N} G_A(q^2) 2S^i_T \tau^a_T \chi_i,$n

(1.57)

$$\langle N_f(\vec{q}/2)|J^0_A (0)|N_i(-\vec{q}/2)\rangle = 0$$

where $S_T \equiv \vec{S} - \vec{q} \cdot \vec{S} - \vec{q}$ is the transverse component of the spin operator.

It is straightforward to compute the matrix elements of the currents in position space on static nucleon states. Plugging the Ansatz (1.24,1.29,1.32,1.31) in the definition of the currents (1.13) and performing the quantization one obtains quantum mechanical operators acting on the nucleons. The matrix elements are easily computed using the results of sect. 3.1. We finally obtain the form factors by
taking the Fourier transform and comparing with Eqs. (1.55, 1.57). We have:

\[
G^S_E = -\frac{N_c}{6\pi\gamma L} \int dr \, j_0(qr) \left( a(z) \partial_z s \right)_{U V}
\]

\[
G^V_E = \frac{4\pi M_5}{3\lambda} \int dr \, r^2 j_0(qr) \left[ a(z) \left( \partial_z v - 2(D_z\chi)_{(2)} \right) \right]_{U V}
\]

\[
G^S_M = \frac{8\pi MN_5\alpha}{3\lambda} \int dr \, r^3 j_1(qr) \left( a(z) \partial_z Q \right)_{U V}
\]

\[
G^V_M = \frac{MN_c}{3\pi L\gamma\alpha} \int dr \, r^2 j_1(qr) \left( a(z) (D_z\phi)_{(2)} \right)_{U V}
\]

\[
G_A = \frac{N_c}{3\pi\alpha\gamma L} \int dr \, \left[ a(z) j_1(qr) \left( (D_z\phi)_{(1)} - r A_z \right) - a(z) (D_z\phi)_{(1)} j_0(qr) \right]_{U V}
\]

where \( j_n \) are spherical Bessel functions which arise because of the Fourier transform.

1.4. Properties of baryons: Results

In this section we will present our results. After discussing some qualitative features, such as the large-\( N_c \) scaling of the form factors and the divergences of the isovector radii due to exact chiral symmetry, we extrapolate to the physically relevant case of \( N_c = 3 \) and perform a quantitative comparison with the experimental data.

Consistently with our working hypothesis that the 5D model really describes large-\( N_c \) QCD we find a 30\% relative discrepancy.

**Large-\( N_c \) Scaling**

As explained in sect. 2.1, all the three parameters \( \alpha, \gamma \) and \( L \) of our 5D model should scale like \( N_c^0 \), Eq. (1.17), in order for the large-\( N_c \) scaling of meson couplings and masses to be correctly reproduced. This implies the following scaling for the baryon observables. First, we notice that the solitonic solution is independent of \( N_c \) given that \( M_5 \) factorizes out of the action and does not appear in the EOM. This implies that the radii of the soliton does not scale with \( N_c \), while the classical mass \( M \) and the moment of inertia \( \lambda \) scale like \( N_c \). Using this we can read the \( N_c \)-scaling of the electric and magnetic form factors from Eq. (1.58):

\[
G^S_E \sim N_c, \quad G^V_E \sim N_c^0, \quad \frac{G^S_M}{M_N} \sim N_c^0, \quad \frac{G^V_M}{M_N} \sim N_c.
\]

In large-\( N_c \) QCD the baryon masses scale like \( N_c \) [31], as in our model. The matrix elements of the currents on nucleon states are also expected to scale like \( N_c \), even though cancellations are possible [32]. The radii, therefore, must scale

---

It is quite intuitive that the form factors can be computed in this way. Given that solitons are infinitely heavy at small coupling, in the Breit frame they are almost static during the process of scattering with the current. To check this, however, we should perform the quantization of the collective coordinates associated with the center-of-mass motion, as it was done in [30] for the original 4D Skyrme model.
like $N^0_c$ as we find and, looking at the definition (1.55), $G_{E}^{S,V}$ and $G_{M}^{S,V}/M_N$ should both scale like $N_c$ up to cancellations. It is very simple to understand why, both in QCD and in our model, there must be a cancellation in $G_{E}^{V}$. Remembering that the temporal component of the current at zero momentum gives the conserved charge and looking at the definitions (1.55), one immediately obtains $G_{E}^{V}(0) = 1/2$ because the skyrmion, as the nucleon, is in the $1/2$ representation of isospin. This condition is respected by our model as it is implied by the EOM, and fulfilled to great accuracy ($0.1\%$) by the numerical solution. Similarly we find at zero momentum $G_{S}^{E}(0) = N_c/6$ as required for a bound-state made of $N_c$ quarks of $U(1)_V$ charge $1/6$ each (in our conventions). Also this condition is implied by the EOM and verified by the numerical solution.

Concerning the second cancellation, i.e. $G_{S}^{M}/M_N \sim N^0_c$, we are not able to prove that it must take place in large-$N_c$ QCD as it does in our model. We can, however, check that it occurs in the naive quark model, or better in its generalization for arbitrary odd $N_c = 2k + 1$ [33]. In this non-relativistic model the Nucleon wave function is made of $2k + 1$ quark states $q_i$, $2k$ of which are collected into $k$ bilinear spin/isospin singlets while the last one has free indices which give to the Nucleon its spin/isospin quantum numbers. Of course, the wave function is symmetrized in flavor and spin given that the color indices are contracted with the antisymmetric tensor and the spatial wave function is assumed to be symmetric. The current operator is the sum of the currents for the $2k + 1$ quarks, each of which will assume by symmetry the same form as in Eq. (1.55). If $S_{1,2}$ and $I_{1,2}$ represent the spin and isospin operators on the quarks $q_{1,2}$, the operators $S_1 + S_2$ and $I_1 + I_2$ will vanish on the singlet combination of the two quarks, but $S_1 I_1 + S_2 I_2$ will not. The $k$ singlets will therefore only contribute to $G_{E}^{S}, G_{M}^{V}$ and $G_A$, which will have the naive scaling, while for the others we find cancellations.

A detailed calculation can be found in [34] where, among other things, the proton and neutron magnetic moments and the axial coupling are computed in the naive quark model. The magnetic moments are related to the form factor at zero momentum as $\mu_V/\mu_N = G_{M}^{V}(0)$ and $\mu_S/\mu_N = G_{M}^{S}(0)$ where $\mu_N = 1/(2M_N)$ is the nuclear magneton and $2\mu_V = \mu_p - \mu_n$, $2\mu_S = \mu_p + \mu_n$. In accordance with the previous discussion, the results in the naive quark model are $2\mu_S = \mu_u + \mu_d$ and $2\mu_V = 2k/3(\mu_u - \mu_d)$, where $\mu_{u,d}$ are the quark magnetic moments, while for the axial coupling one finds $g_A = G_A(0) = 2k/3 + 1$ which scales like $N_c$ as expected:

$$g_A = \frac{N_c}{3} + \frac{2}{3}. \quad (1.60)$$

Notice that for $N_c = 3$ the subleading term in the $1/N_c$-expansion represents a $60\%$ correction. We have of course no reason to believe that such big corrections should persist in the true large-$N_c$ QCD; this remark simply suggests that “large” $1/N_c$ corrections to the form factors are not excluded.
Divergences in the Chiral Limit

It is well known that in QCD the isovector electric \( \langle r^2_{E,V} \rangle \) and magnetic \( \langle r^2_{M,V} \rangle \) radii which are proportional, respectively, to the \( q^2 \) derivative of \( G^E \) and \( G^M \) at zero momentum, diverge in the chiral limit [35]. In our model, as in the Skyrme model, divergences in the integrals of Eq. (1.58) which define the form factors are due, as in QCD, to the massless pions. If all the fields were massive, indeed, any solution to the EOM would fall down exponentially at large \( r \) while in the present case power-like behaviors can appear. These power-like terms in the large-\( r \) expansion of the solution can be derived analytically by performing a Taylor expansion of the fields around infinity \( (1/r = 0) \), substituting into the EOM and solving order by order in \( 1/r \). The exponentially suppressed part of the solution will never contribute to the expansion. This procedure allows us to determine the asymptotic expansion of the solution completely, up to an integration constant \( \beta \). Substituting the expansion into the definitions of the form factors (1.58) one gets

\[
\begin{align*}
G^E_S & \propto \beta^3 \int dr \frac{1}{r^7} j_0(qr) + \ldots \\
G^E_V & \propto \beta^2 \int dr \frac{1}{r^7} j_0(qr) + \ldots \\
G^M_S & \propto \beta^3 \int dr \frac{1}{r^5} j_1(qr) + \ldots \\
G^M_V & \propto \beta^2 \int dr \frac{1}{r^2} j_1(qr) + \ldots
\end{align*}
\]

(1.61)

All the form factors are finite for any \( q \), including \( q = 0 \). The electric and magnetic radii, however, are defined as

\[
\langle r^2_{E,M} \rangle = -\frac{6}{G^E_M(q^2 = 0)} \left. \frac{dG^E_M(q^2)}{dq^2} \right|_{q^2 = 0},
\]

(1.62)

and taking a \( q^2 \) derivative of Eqs. (1.61) makes one more power of \( r^2 \) appear in the integral. It is easy to see that the scalar radii are finite, while the vector ones are divergent. For the axial form factor \( G_A \) we find

\[
G_A \propto \int dr \left[ \left( \frac{3}{r} \beta - \frac{1}{r^5} \beta^3 \right) j_1(qr) qr + \left( -\frac{1}{r} \beta + \frac{5}{r^5} \beta^3 \right) j_0(qr) + \ldots \right].
\]

(1.63)

The integral in Eq. (1.63) is convergent for any \( q \neq 0 \) but, however, it is not uniformly convergent for \( q \to 0 \). The leading \( 1/r \) term in Eq. (1.63) is indeed given by \( I(q) = \beta \int_0^\infty dr \frac{1}{r} \frac{(3j_1(qr))/(qr) - j_0(qr))}{j_0(qr) + \ldots} \), which is independent of \( q \) and equal to \( \beta/3 \), while the argument of the integral vanishes for \( q \to 0 \) so that exchanging the limit and integral operations would give the wrong result \( I(0) = 0 \). To restore uniform convergence and obtain an analytic formula for \( g_A \) one can subtract the \( I(q) \) term from the expression in Eq. (1.58) for \( G_A \) and replace it with \( \beta/3 \). Rewriting the axial form factor in this way is also useful to establish that the axial radius, which seems divergent if looking at Eq. (1.63), is on the contrary finite.
The $I(q)$ term, indeed, does not contribute to the $q^2$ derivative and the ones which are left in Eq. (1.63) give a finite contribution.

We have found, compatibly with the QCD expectation, that all the form factors and radii are finite but the isovector ones. Notice that the structure of the divergences is completely determined by the asymptotic large-$r$ behaviour of the solution, and not by its detailed form (i.e., for instance, by the actual value of the integration constant $\beta$ which depends on the entire solution). Our model coincides, in the IR, with the Skyrme model, therefore the asymptotic behaviour of the current densities is expected to be the same in the two cases. This explains why we obtained the same divergences as in the Skyrme model.

**Pion Form Factor and Goldberger-Treiman relation**

It is of some interest to define and compute the pion-nucleon form factor which parametrizes the matrix element on Nucleon states of the pion field. In the Breit frame (for normalized nucleon states) it is

$$
\langle N_f (\vec{q}/2) | \pi^a(0) | N_i (\vec{q}/2) \rangle = - \frac{i}{2M_N q^2} G_{NN\pi}(q^2) \chi_i^\dagger (2S^i) q_i (2I^a) \chi_i ,
$$

(1.64)

where $\pi^a(x)$ is the normalized and “canonical” pion field operator. The field is canonical in the sense that its quadratic effective lagrangian only contains the canonical kinetic term $L_2 = 1/2(\partial \pi_a)^2$, or equivalently that its propagator is the canonical one, without a non-trivial form factor. With this definition, $G_{NN\pi}$ is the vertex form factor of the meson-exchange model for nucleon-nucleon interactions [36] and corresponds to an interaction

$$
L_{NN\pi} = i (G_{NN\pi}(\square) \pi_a) \nabla^\mu \gamma_5 (2I^a) N .
$$

(1.65)

On-shell, the form factor reduces to the pion-nucleon coupling constant, $G_{NN\pi}(0) = g_{NN\pi}$, whose experimental value is $g_{NN\pi} = 13.5 \pm 0.1$.

The pion field which matches the requirements above is given by the zero-mode of the KK decomposition. In the unitary gauge $\partial_z (a(z) A_5) = 0$, where $A_M \equiv (L_M - R_M)/2$, and for AdS$_5$ space, one has

$$
A_5^{(un)} (x, z) = \frac{1}{F_\pi a(z)} \pi^a(x) \sigma_a ,
$$

(1.66)

where $F_\pi$ is given in Eq. (1.13). Gauge-transforming back to the gauge in which our numerical solution is provided and using the Ansatz in Eqs. (1.24,1.29) we find the pion field

$$
\pi^a = - \frac{F_\pi}{2 \zeta_{UV}} \int_{\zeta_{UV}}^{z_{in}} dz A_2 (r, z) \tilde{2}^b \text{Tr} [ U \sigma_b U^\dagger \sigma^a ] .
$$

(1.67)

*Nucleon scattering, in our model, is a soliton scattering process and we have no reason to believe that it can be described by meson-exchange, i.e. that contact terms are suppressed. Therefore, we will not attempt any comparison of our form factor with the one used in meson-exchange models.*
Table 1.2. Prediction of the nucleon observables with the microscopic parameters fixed by a fit on the mesonic observables. The deviation from the empirical data is computed using the expression \((\text{th} - \text{exp})/\min(|\text{th}|, |\text{exp}|)\), where \(\text{th}\) and \(\text{exp}\) denote, respectively, the prediction of our model and the experimental result.

| Experiment | AdS5 | Deviation |
|------------|------|-----------|
| \(M_N\)  | 940 MeV | 1130 MeV | +20% |
| \(\mu_S\) | 0.44  | 0.34     | -30% |
| \(\mu_V\) | 2.35  | 1.79     | -31% |
| \(g_A\)  | 1.25  | 0.70     | -70% |
| \(\sqrt{\langle r_{E,S}^2 \rangle}\) | 0.79 fm | 0.88 fm | +11% |
| \(\sqrt{\langle r_{E,V}^2 \rangle}\) | 0.93 fm | ∞        |       |
| \(\sqrt{\langle r_{M,S}^2 \rangle}\) | 0.82 fm | 0.92 fm | +12% |
| \(\sqrt{\langle r_{M,V}^2 \rangle}\) | 0.87 fm | ∞        |       |
| \(\sqrt{\langle r_A^2 \rangle}\) | 0.68 fm | 0.76 fm | +12% |
| \(\mu_p/\mu_n\) | -1.461 | -1.459 | +0.1% |

Taking the matrix element of the above expression and comparing with Eq. (1.64) one obtains

\[
G_{NN\pi}(q^2) = -\frac{8\pi}{3} M_N F_{\pi} q \int_0^\infty dr j_1(qr) \int dz r^2 A_2(r, z). \tag{1.68}
\]

At \(q \to 0\) the form factor \(G_{NN\pi}\) is completely determined by the large-\(r\) behavior of the field \(A_2\), given by \(A_2 \to \beta/r^2\). We then find

\[
g_{NN\pi} = -\frac{32\pi}{3} M_N F_{\pi} \beta L^2. \tag{1.69}
\]

By using Eqs. (33,34) of Ref. [2] which show that also \(g_A\) is determined by the asymptotic behavior of the axial current, one finds

\[
g_A = -\frac{32\pi}{3} F_{\pi}^2 \beta L^2, \tag{1.70}
\]

that, together with Eq. (1.69), leads to the famous Goldberger-Treiman relation \(F_{\pi} g_{\pi NN} = M_N g_A\). This relation, which is a consequence of having exact chiral symmetry, has been numerically verified to 0.01%.

**Comparison with Experiments**

Let us now compare our results with real-world QCD. We therefore fix the number of colors \(N_c = 3\) and choose our microscopic parameters to be those that gave the best fit to the mesonic quantities: \(1/L \simeq 343\) MeV, \(M_{5L} \simeq 0.0165\) and \(\alpha \simeq 0.94\) \((\gamma \simeq 1.23)\). The numerical results of our analysis and the deviation with respect to the experimental data are reported in table 1.2. We find a fair agreement with the experiments, a 36% total RMSE which is compatible with the expected size of \(1/N_c\) corrections. The axial charge \(g_A\) is the one which shows the larger (80%)
deviation, and indeed removing this observable the RMSE decreases to 21%. We cannot exclude that, in a theory in which the naive expansion parameter is $1/3$, enhanced 80% corrections to few observables might appear at the next-to-leading order. Nevertheless, we think that this result could be very sensitive to the pion mass and therefore could be substantially improved in 5D models that incorporate explicit chiral breaking. The reason for this is that $g_A$ is strongly sensitive to the large-$r$ behavior of the solution (see the discussion following Eq. (1.63)) which is in turn heavily affected by the presence of the pion mass. Notice that a larger value, $g_A \approx 0.99$, is obtained in the “complete” model described in Ref. [4], a model with similar features to our 5D scenario and which includes a nonzero pion mass. This expectation, however, fails in the original Skyrme model, where the addition of the pion mass does not affect $g_A$ significantly [37] and one finds $g_A \approx 0.65$.

Table 1.2 also shows the proton-neutron magnetic moment ratio, $\mu_p/\mu_n$, which is in perfect agreement with the experimental value. This observable is the only one in the list that includes two orders of the $1/N_c$ expansion. Indeed, due to the scaling $\mu_V \sim N_c$ and $\mu_S \sim N_c^0$, we have $\mu_p/\mu_n = -(\mu_V + \mu_S)/(\mu_V - \mu_S) \simeq -1 - 2\mu_S/\mu_V$.

In figs. 1.2, 1.3 and 1.4 we compare the normalized nucleon form factors at $q^2 \neq 0$ with the dipole fit of the experimental data. The shape of the scalar and
axial form factors is of the dipole type, the discrepancy is mainly due to the error in the radii. The shape of vector form factors is of course not of the dipole type for small $q^2$, but this is due to the divergence of the derivative at $q^2 = 0$. Including the pion mass will for sure improve the situation given that it will render finite the slope at zero momentum; it would be interesting to see if the dipole shape of these form factors is recovered in the presence of the pion mass. We also plot in the left panel of fig. 1.4 the deviation of ratio of the proton and neutron magnetic form factors from the large $N_c$ value which is given, due to the different large-$N_c$ scaling of the isoscalar and isovector components, by $G_P(q)/G_N(q) = -1$. Not only we find that this quantity is quite well predicted, with an error $\lesssim 15\%$, but we also see that its shape, in agreement with observations, is nearly constant away from $q^2 = 0$. Also in this case corrections from the pion mass are expected to go in the right direction.

1.5. Conclusions and outlook

We have shown that five-dimensional models, used to describe meson properties of QCD, can also be considered to study baryon physics. Baryons appear in these theories as soliton of sizes of order $1/m_\rho$ stabilized by the presence of the CS term. We have reviewed the procedure to calculate the static properties of the nucleons that have shown to be in reasonable good agreement with the experimental data. This shows, once again, that 5D models provide an alternative and very promising tool to study properties of QCD in certain regimes.

There are further issues that deserve to be analyzed. The most urgent one is the inclusion of a nonzero pion mass. As we have pointed out above, this will be crucial to calculate the isovector radii and, maybe, improve the prediction for $g_A$. For this purpose we need to use a 5D model along the lines of Ref. [10, 11] where an explicit breaking of the chiral symmetry, corresponding to the quark masses, is introduced. We can also use this approach to study systems with high baryon densities, analyze
possible phase transitions or study the properties of nuclear matter.

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A.1. Numerical Methods

In this technical appendix we explain how the numerical determination of the soliton solution is performed.

Equations of Motion and Boundary Conditions

Let us first of all write down the EOM for the 2D fields which characterize our solution is performed. In this technical appendix we explain how the numerical determination of the soliton solution is performed.

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In order to solve numerically the EOM, they must be rewritten as a system of elliptic partial differential equations. This can be achieved by choosing a 2D Lorentz gauge condition for the residual $U(1)$ gauge fields

$$\partial^\mu A_\mu = 0, \quad \partial^\mu B_\mu = 0. \quad (A.3)$$

The equations for $A_\nu$ become

$$J_\nu = \partial^\mu (r^2 a A^\mu_\nu) = r^2 a \partial^\mu A^\mu_\nu + \partial_\nu (r^2 a) A_\nu$$

which is an elliptic equation and a similar result is obtained for $B_\mu$.

The gauge condition needs only to be imposed at the boundaries, while in the bulk one can just solve the “gauge-fixed” EOM treating the two gauge field components as independent. The fact that the currents are conserved, $\partial_\nu J^\nu = 0$, implies indeed an elliptic equation for $\partial^\mu A_\mu$ which has a unique solution once the boundary conditions are specified. If imposed on the boundary, therefore, the gauge conditions are maintained also in the bulk.

The IR and UV boundary conditions on the 2D fields follow from Eq. (1.3) and Eq. (1.4) and from the gauge choice in Eq. (A.3). They are given explicitly by

$$z = z_{\text{IR}} : \begin{cases} \phi_1 = 0 \\ \partial_2 \phi_2 = 0 \\ A_1 = 0 \\ \partial_2 A_2 = 0 \\ \partial_2 s = 0 \end{cases} \quad \quad \begin{cases} \chi_1 = 0 \\ \partial_2 \chi_2 = 0 \\ \partial_2 v = 0 \\ \partial_2 Q = 0 \end{cases} \quad \begin{cases} \rho = 0 \\ B_1 = 0 \\ \partial_2 B_2 = 0 \end{cases}, \quad (A.4)$$

and

$$z = z_{\text{UV}} : \begin{cases} \phi_1 = 0 \\ \phi_2 = -1 \\ A_1 = 0 \\ \partial_2 A_2 = 0 \\ s = 0 \end{cases} \quad \begin{cases} \chi_1 = 0 \\ \chi_2 = -1 \\ v = -1 \\ Q = 0 \end{cases} \quad \begin{cases} \rho = 0 \\ B_1 = 0 \\ \partial_2 B_2 = 0 \end{cases}. \quad (A.5)$$

The boundary conditions at $r = \infty$ have to ensure that the energy of the solution is finite; this means that the fields should approach a pure-gauge configuration. At the same time one has to require that the solution is non-trivial and its topological charge (Eq. (1.19)) is equal to one. We have

$$r = \infty : \begin{cases} \phi = -ie^{i\pi z/L} \\ \partial_1 A_1 = 0 \\ A_2 = \frac{\pi}{\tau} \\ s = 0 \end{cases} \quad \begin{cases} \chi = ie^{i\pi z/L} \\ v = -1 \\ Q = 0 \end{cases} \quad \begin{cases} \rho = 0 \\ \partial_1 B_1 = 0 \\ B_2 = 0 \end{cases}. \quad (A.6)$$

The $r = 0$ boundary of our domain requires an ad hoc treatment, given that the EOM become singular there. Of course this boundary is not a true boundary of our 5D space, but it represents some internal points. Thus we must require the 2D solution to give rise to regular 5D vector fields at $r = 0$ and we must also require
the gauge choice to be fulfilled. These conditions are
\[
\begin{align*}
& r = 0 : \\
& \begin{cases}
\phi_1/r \to A_1 \\
(1 + \phi_2)/r \to 0 \\
A_2 = 0 \\
\partial_1 A_1 = 0 \\
s = 0
\end{cases} \\
& \begin{cases}
\chi_1 = 0 \\
\chi_2 = -v \\
\partial_1 \chi_2 = 0 \\
Q = 0 \\
B_2 = 0
\end{cases} \\
& \begin{cases}
\rho/r \to B_1 \\
\partial_1 B_1 = 0
\end{cases}.
\end{align*}
\]

(A.7)

A.1.1. COMSOL Implementation

To obtain the numerical solution of the EOM we used the COMSOL 3.4 package [28], which permits to solve a generic system of differential elliptic equations by the finite elements method. A nice feature of this software is that it allows us to extend the domain up to boundaries where the EOM are singular (i.e. the \( r = 0 \) line), because it does not use the bulk equations on the boundaries, but, instead, it imposes the boundary conditions.

In order to improve the convergence of the program and the numerical accuracy, one is forced to perform a coordinate and a field redefinition. The former is needed to include the \( r = \infty \) boundary in the domain in which the numerical solution is computed. The advantage of this procedure is the fact that in this way one can correctly enforce the right behaviour of the fields at infinity by imposing the \( r = \infty \) boundary conditions. A convenient coordinate change is given by
\[
\begin{align*}
x = c \arctan \left( \frac{r}{c} \right),
\end{align*}
\]
where \( x \) is the new coordinate used in the program and \( c \) is an arbitrary constant. The domain in the \( x \) direction is now reduced to the interval \([0, c\pi/2]\). The parameter \( c \) has been introduced to improve the numerical convergence of the solution. A good choice for \( c \) is \( c \sim 10 \), which allows to have a reasonable domain for \( x \) and, at the same time, does not compress the solution towards \( x = 0 \).

A field redefinition is needed to impose the regularity conditions at \( r = 0 \) (Eq. (A.7)). For this purpose we use the rescaled fields
\[
\begin{align*}
\phi_1 &= x \psi_1 \\
\phi_2 &= -1 + x \psi_2 \\
\rho &= x \tau
\end{align*}
\]
With these redefinitions, in the new coordinates, the \( r = 0 \) boundary conditions read as
\[
\begin{align*}
& r = 0 : \\
& \begin{cases}
\psi_1 - A_1 = 0 \\
\psi_2 = 0 \\
A_2 = 0 \\
\partial_x A_1 = 0
\end{cases} \\
& \begin{cases}
\chi_1 = 0 \\
\partial_x \chi_2 = 0 \\
v = -\chi_2 \\
Q = 0
\end{cases} \\
& \begin{cases}
\tau - B_1 = 0 \\
\partial_x B_1 = 0 \\
B_2 = 0
\end{cases}.
\end{align*}
\]

(A.10)

In order to ensure the convergence of the program another modification is needed. As already discussed, to obtain a soliton solution with non-vanishing topological charge we have to impose non-trivial boundary conditions for the 2D fields
at $r = \infty$ (Eq. (A.6)). It turns out that if imposing such conditions the program is not able to reach a regular solution. This is so because the $r = \infty$ boundary is singular and imposing non-trivial (though gauge-equivalent to the trivial ones) boundary conditions at a singular point spoils the regularity of the numerical solution; the same would happen if the topological twist was located at $r = 0$. To fix this problem we have to perform a gauge transformation which reduces the $r = \infty$ conditions to trivial ones and preserves the ones at $r = 0$ at the cost of introducing a “twist” on the UV boundary. For this, we use a transformation of the residual $U(1)$ chiral gauge symmetry associated to $SU(2)_{L,R}$ (Eq. (1.33)) with

\[
\alpha(r, z) = (1 - z/L) f(r),
\]

(A.11)

where $f(r)$ can be an arbitrary function which respects the conditions

\[
\begin{align*}
&f(0) = 0 \quad \text{and} \quad f''(0) = 0, \\
&f(\infty) \to \pi \quad \text{and} \quad f''(\infty) \to 0.
\end{align*}
\]

(A.12)

For $c \sim 10$ it turns out that a good choice for $f(r)$ is $f(r) = 2 \arctan r$. The gauge-fixing condition for $A_{\bar{\mu}}$ is now modified as

\[
\partial_r A_1 + \partial_z A_2 - (1 - z/L)f''(r) = 0,
\]

(A.13)

the UV boundary conditions are given by

\[
\begin{align*}
&x\psi_1 = \sin f(r) \\
&(-1 + x\psi_2) = -\cos f(r) \\
&A_1 = f'(r) \\
&\partial_z A_2 = 0 \\
&s = 0
\end{align*}
\]

and the $r = \infty$ constraints are now trivial

\[
\begin{align*}
&\psi_1 = 0 \\
&(-1 + x\psi_2) = 1 \\
&\partial_z A_1 = 0 \\
&A_2 = 0 \\
&s = 0
\end{align*}
\]

(A.14)

whereas the $r = 0$ and the IR boundary conditions are left unchanged. Notice that in the new gauge the EOM for $A_{\bar{\mu}}$ are modified in accord to Eq. (A.13), however they are still in the form of elliptic equations.

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