ON DOUBLE COSETS IN FREE GROUPS

RITA GITIK AND ELIYAHU RIPS

Abstract. It is shown that for any finitely generated subgroups \( H \) and \( K \) of a free group \( F \), and for any \( g \in F \) the double coset \( HgK \) is closed in the profinite topology of \( F \).

A well-known theorem of M. Hall [2] states (in different language) that any finitely generated subgroup of a free group is closed in the profinite topology. We show that a slight modification of its proof [6] yields a stronger result:

**Theorem.** For any finitely generated subgroups \( H \) and \( K \) of a free group \( F \), and for any \( g \in F \) the double coset \( HgK \) is closed in the profinite topology of \( F \).

A free group \( F = \langle X \rangle \) can be viewed as the fundamental group of a wedge \( W \) of \(|X|\) oriented circles labeled by elements of \( X \). Subgroups of \( F \) correspond bijectively to based covering spaces of \( W \), and any covering space of \( W \) is a graph which inherits orientation and labeling of its edges from \( W \).

Let \( X_0 \subset X \) and let \( \Gamma \) be a subgraph of a covering of \( W \). An \( X_0 \)-component of \( \Gamma \) is a maximal connected subgraph of \( \Gamma \) with all its edges labeled by elements of \( X_0 \).

**Lemma.** Let \( \Gamma \) be a subgraph of a covering of \( W \) such that \( \Gamma \) has finitely many vertices. There exists an embedding of \( \Gamma \) in a covering \( \Gamma' \) of \( W \) such that \( \Gamma \) and \( \Gamma' \) have the same vertices, and for any \( X_0 \subset X \) distinct \( X_0 \)-components of \( \Gamma \) remain distinct in \( \Gamma' \).

**Proof.** We give an algorithm for constructing \( \Gamma' \) by adding edges to \( \Gamma \) in a unique way. For any vertex \( v \) of \( \Gamma \) and for any \( x \in X \) the number of edges labeled with \( x \) having an endpoint at \( v \) is either 0, 1 or 2. If the number is 0, we add an edge labeled with \( x \) with both endpoints at \( v \). If the number is 1 or 2, let \( p_x \) be the maximal path consisting of edges labeled only with \( x \) and with an endpoint at \( v \). If \( p_x \) has both endpoints at \( v \) we do nothing. Otherwise we add to \( \Gamma \) an edge labeled with \( x \) connecting the endpoints of \( p_x \). It is clear that the projection from \( \Gamma \) to \( W \) extends uniquely to a covering map from \( \Gamma' \) to \( W \).

**Remark.** Let \( H \) be a finitely generated subgroup of \( F \), and let \( f \in F \setminus H \). Let \( \Gamma \) be the minimal connected subgraph of the covering \( C \) of \( W \) corresponding to \( H \) which contains the core of \( C \) (cf. [6]) and the path \( p \) beginning at the basepoint \( v_0 \) of \( C \) whose projection in \( W \) represents \( f \). Embed \( \Gamma \) in a covering \( \Gamma' \) as in the lemma. Then as \( \Gamma \) has finitely many vertices, so does \( \Gamma' \), therefore the subgroup \( M \) of \( F \)

1991 Mathematics Subject Classification. 20F32, 20E05, 20E26.
corresponding to $\Gamma'$ has finite index in $F$. As $p$ is not a closed path in $\Gamma$, it remains not closed in $\Gamma'$, hence $f \notin M$. As $\Gamma$ is a subgraph of $\Gamma'$, $H$ is a subgroup of $M$, proving M. Hall’s theorem (cf. [6]).

**Proof of the theorem.** As $HgK = H(gKg^{-1})g$, it is enough to consider the case $g = 1$.

By an observation due to P.Kropholler, we can replace $F$ by a subgroup of finite index (cf. [3]), so we can assume that $K$ is a free factor of $F$, $F = K \ast L$. Let $X_1$ and $X_2$ be sets of free generators of $K$ and $L$ respectively, then $X = X_1 \cup X_2$ is a set of free generators of $F$. Let $f \in F, f \notin HK$. Our goal is to construct a subgroup $M$ of finite index in $F$ such that $Mf \cap HK = \emptyset$.

Let $\Gamma, \Gamma'$ and $M$ be as in the remark. As $f \notin HK$, the $X_1$-component of $v_0$ in $\Gamma$ does not contain the endpoint of $p$, therefore the lemma implies that $\Gamma'$ has the same property. But the condition $Mf \cap HK = \emptyset$ is equivalent to the condition that the endpoint of $p$ does not belong to the $X_1$-component of $v_0$, proving the theorem.

**Remarks.**

1) The first published proof of the theorem is due to G.A. Niblo [3], who found a much more simple and elegant argument than the original proof of the authors [1].

2) A more general result saying that for any finitely generated subgroups $H_1, \ldots, H_n$ of a free group $F$ the set $H_1 \cdots H_n$ is closed in the profinite topology on $F$ was obtained by L. Ribes and P.A. Zalesskii [4], and by K. Henckell, S.T. Margolis, J.E. Pin and J. Rhodes [5].

3) It is easy to construct a closed subset $B$ of $F$ and a finitely generated subgroup $K$ such that the product $BK$ is not closed in the profinite topology of $F$.

**References**

[1] R. Gitik and E. Rips, *On Separability Properties of Groups*, Int J. of Algebra and Computation 5 (1995), 703-717.
[2] M. Hall, Jr., *Coset Representations in Free Groups*, Trans. AMS 67 (1949), 431-451.
[3] G.A. Niblo, *Separability Properties of Free Groups and Surface Groups*, J. of Pure and Applied Algebra 78 (1992), 77-84.
[4] L. Ribes and P.A. Zalesskii, *On the Profinite Topology on a Free Group I*, Bull. LMS 25 (1993), 37-42.
[5] K. Henckell, S.T. Margolis, J.E. Pin and J. Rhodes, *Ash’s Type II Theorem*, Profinite Topology and Malcev Products I, Int J. of Algebra and Computation 1 (1991), 411-436.
[6] J.R. Stallings, *Topology of Finite Graphs*, Invent. Math. 71 (1983), 551-565.