Abstract

In recent years, SGD and its variants have become the standard tool to train Deep Neural Networks. In this paper, we focus on the recently proposed variant Lookahead, which improves upon SGD in a wide range of applications. Following this success, we study an extension of this algorithm, the Multilayer Lookahead optimizer, which recursively wraps Lookahead around itself. We prove the convergence of Multilayer Lookahead with two layers to a stationary point of smooth non-convex functions with $O\left(\frac{1}{\sqrt{T}}\right)$ rate. We also justify the improved generalization of both Lookahead over SGD, and of Multilayer Lookahead over Lookahead, by showing how they amplify the implicit regularization effect of SGD. We empirically verify our results and show that Multilayer Lookahead outperforms Lookahead on CIFAR-10 and CIFAR-100 classification tasks, and on GANs training on the MNIST dataset.

1 INTRODUCTION

Because of their simplicity, performance, and better generalization properties, stochastic variants of gradient descent have become the standard tool to train Deep Neural Networks. Experimental evidence [Keskar et al., 2016][Hoffer et al., 2017] shows that the use of smaller batch sizes further improves generalization, though it hinders the possibility of using parallelism. The development of new stochastic methods remains an active area of research. In particular, the Lookahead optimizer introduced by Zhang et al. (2019) has been shown to improve the performance of first-order stochastic methods for training deep neural networks. It also improves the convergence speed and the generalization of stochastic methods in numerous applications (Zhang et al., 2019; Chavdarova et al., 2020). In this paper, we further research around this optimizer.

Lookahead is a wrapper around any other optimizer, which is called the base, or inner optimizer. It stores two states of the parameters, namely fast and slow weights. At each round, it initializes fast weights to the current value of slow weights, then performs $k$ updates of the fast weights, using the inner optimizer, and assigns a new value of slow weights to be the convex combination of fast and slow weights with coefficients $\alpha$ and $1 - \alpha$. Parameter $\alpha$ is called the slow learning rate, or synchronization parameter. Lookahead achieves faster convergence than its inner optimizer for classification tasks such as LSTM language models, transformers (Zhang et al., 2019) and GANs (Chavdarova et al., 2020). Moreover, Lookahead is stable to suboptimal choice of hyperparameters (Zhang et al., 2019).

However, a sufficient theoretical explanation of the success of Lookahead is still missing. Zhang et al. (2019) claim that Lookahead can be considered as a method for variance reduction. They proved that Lookahead with SGD as the inner optimizer reduces the variance on the noisy quadratic loss function. Wang et al. (2020) consider Lookahead from the perspective of multi-agent methods of optimization and show its convergence (with SGD as the inner optimizer) to a stationary point for smooth non-convex functions with $O\left(\frac{1}{\sqrt{T}}\right)$ rate, which matches the rate of SGD. Even though their analysis gives the same constant for the asymptotically most significant terms of SGD and Lookahead, it produces an additional term for Lookahead, which makes its convergence guarantee slightly worse (Chavdarova et al., 2020). However, in practice, we observe the opposite result.

While Lookahead drastically improves the performance at the early stages of training, it usually produces only marginally better test loss after its conver-
Multilayer Lookahead: a Nested Version of Lookahead

Given the success of Lookahead in a wide range of applications, we go further and study a new extension, which we call the \textit{Multilayer Lookahead} optimizer. Succinctly, we can define it recursively as follows: Multilayer Lookahead with \( n \) layers is a Lookahead optimizer, whose inner optimizer is Multilayer Lookahead with \( n - 1 \) layers.\footnote{This algorithm for 2 layers was briefly discussed by Chavdarova et al. (2020).}

Our contributions are the following:

1) We introduce Multilayer Lookahead - an algorithm obtained by recursively iterating the Lookahead optimizer. We empirically verify that Multilayer Lookahead outperforms both its inner optimizer and Lookahead on image classification and GANs training tasks.

2) We extend the analysis of Wang et al. (2020) for Lookahead and prove the convergence of the Lookahead with two layers with SGD as the inner optimizer towards a stationary point with rate \( O\left(\frac{1}{\sqrt{T}}\right) \).

3) We prove that if the inner optimizer has a linear convergence rate, then both Lookahead and Multilayer Lookahead preserve the linear convergence rate (but, possibly, with a worse constant).

4) We propose a new explanation for the better generalization of Lookahead over SGD following the idea of implicit regularization and the backward analysis framework (Smith et al. 2021; Nichol et al. 2018).

5) We show that Multilayer Lookahead exhibits a behavior similar to the “super-convergence” phenomenon studied by Smith & Topin (2019), where near top performance is achieved after just a few epochs.

2 \textbf{METHOD}

First, we formally revisit the original Lookahead algorithm (Zhang et al. 2019) and then describe our extension of this method—Multilayer Lookahead.

Lookahead stores two parameters: fast weights \( x_{r,i} \) and slow weights \( y_r \). At each round \( r \), it starts by initializing the fast weights \( x_{r,0} \) to the current value of \( y_r \), then performs \( k \) updates of the fast weights using the inner optimizer, and obtains \( x_{r,k} \). Finally, it performs the \textit{synchronization step} which moves the slow weights towards the final value of the fast weights by taking their convex combination: \( y_{r+1} = (1-\alpha)y_r + \alpha x_{r,k} \).

The idea of using slow weights makes the trajectory of the optimizer smoother in a setting with a high variance of stochastic gradients. See the pseudo-code in Algorithm 1.

Algorithm 1 Lookahead

\textbf{Require:} Loss function \( f \); inner optimizer \( A \); initial point \( y_0 \); synchronization parameter \( \alpha \); length of the inner loop \( k \); number of rounds \( R \)

\begin{enumerate}
\item for \( r = 0, 1, \ldots, R - 1 \) do
\item \( x_{r,0} = y_r \)
\item for \( i = 0, 1, \ldots, k - 1 \) do
\item Sample mini-batch \( \xi_{r,i} \)
\item \( x_{r,i+1} = x_{r,i} - A(f_{\xi_{r,i}}, x_{r,i}) \)
\item end for
\item \( y_{r+1} = (1-\alpha)y_r + \alpha x_{r,k} \) \text{ \textit{\& synchronization}}
\item end for
\item end for
\end{enumerate}

Comparing to the base optimizer, Lookahead needs extra space to store slow weights, which is at most two times more space in total. It requires also computing one additional convex combination per round, which has a negligible impact on time complexity.

We proceed now to formally introduce the Multilayer Lookahead optimizer. The easiest way to define it is recursive: Lookahead with \( n \) layers is a Lookahead, whose inner optimizer is Lookahead with \( n - 1 \) layers. Thus, it stores \( n \) states of the parameters, has \( n \) synchronization parameters \( \alpha_1, \ldots, \alpha_n \), and \( n \) numbers of steps per update \( k_1, \ldots, k_n \), one per each layer. We provide a pseudo-code for Lookahead with two layers in Algorithm 2. Among all \( n \) weights, we will often refer to the \textit{outer weights} (weights at level 1; those that update the least frequently) and the \textit{inner weights} (weights at level \( n \); those that update the most frequently). As before, by \textit{round} we mean one update of the outer variable, and by \textit{iteration} - one update of the inner variable, possibly followed by the synchronization with the variables from lower layers. One update of the outer variable requires computing \( k_n \) updates of the variable from the level 2, that in turn requires \( k_{n-1}k_{n-2} \) updates of the variable from the level 3 and so on. Hence, one update of the outer variable requires \( k_n \ldots k_1 \) updates of the inner variable, or, equivalently, one round consists of \( k_n \ldots k_1 \) iterations. In the following, we will refer to \( R \) as the number of rounds, and \( T = Rk_1k_2 \ldots k_n \) as the total number of iterations.

Since the Lookahead with \( n \) layers needs to store \( n + 1 \) states of the parameters, it may increase the space complexity up to \( n + 1 \) times. However, the additional time cost is still insignificant. Indeed, it requires the same number of gradients evaluation as the inner optimizer. Moreover, during each round it requires \( 1 + k_n + k_nk_{n-1} + \ldots + k_nk_{n-1} \ldots k_2 < k_nk_{n-1} \ldots k_1 \) synchronizations (under the natural assumption that \( k_i \geq 2 \ \forall i \)). Therefore, it requires less than 1 synchronization per iteration in average, independently of the number of layers.
3 CONVERGENCE ANALYSIS

Wang et al. (2020) proved the convergence of Lookahead, which uses SGD as the base optimizer, to a stationary point for smooth, not necessarily convex functions $f$. Their analysis relies on considering Lookahead as a specific instance of local SGD (Stich, 2018; Koloskova et al., 2020; Woodworth et al., 2020), where two agents solve the composite minimization problem $\min_{x,y} f_1(x) + f_2(y)$, subject to consensus constraint $x = y$ at the end of each round. Specifically, the first agent updates the fast weights by optimizing function $f_1 = f$, and the second one updates the slow weights by optimizing function $f_2 = f_3 \equiv 0$. After each agent performs $k$ steps, they synchronize their weights by taking their convex combination.

Following the same idea, we prove similar result for Lookahead with two layers. Let’s represent 2-layers Lookahead as an instance of local SGD for a specific composite optimization problem. Consider problem of minimizing $f_1(x) + f_2(y) + \sum_{i=1}^3 f_i(z)$, where $f_1 = f$, $f_2 = f_3 \equiv 0$, subject to consensus constraints $x = y$, if $t \mod k_1 = 0$ (by : we denote “divisible by”), and $x = z$, if $t \mod k_2$ does not hold, $t$ is a number of local updates, or iterations. Denote by $x_t, y_t, z_t$ the values of the parameters and by $\gamma_t$ the learning rate at iteration $t$. Define parameters matrix $X_t = (x_t, y_t, z_t) \in \mathbb{R}^{d \times 3}$, matrix of stochastic gradients $G_t = (g(x_t, \xi_t), 0, 0) \in \mathbb{R}^{d \times 3}$, and synchronization matrix:

$$P_t = \begin{cases} I_3, & t \mod k_1 \\
\begin{pmatrix} \alpha_1 & 0 \\
\alpha_1 & 1 - \alpha_1 \\
0 & 0 \\
\alpha_1 & 1 - \alpha_1 \\
\alpha_1 & 1 - \alpha_1 \\
0 & 0 \\
1 - \alpha_1 & 1 - \alpha_1 \\
1 - \alpha_1 & 1 - \alpha_1 \\
1 - \alpha_1 & 1 - \alpha_1 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}, & t \mod k_1 = 0 \\
\end{cases}
$$

and define the update rule as

$$X_{t+1} = (X_t - \gamma_t G_t) P_t \quad (1)$$

with initial conditions $X_0 = (z_0, z_0, z_0)^T$. Then the sequence $(x_t, y_t, z_t, t \geq 0)$, produced by this instance of local SGD, coincides with the sequence $(x_{t,i,j}, y_{t,i,j}, z_{t,i,j}, t \geq 0)$, produced by 2-layers Lookahead applied to optimizing $f$, where $r, i, j$ are defined from $t$ using the equality $t = rk_1k_2 + ik_1 + j$, $0 \leq i \leq k_2 - 1$, $0 \leq j \leq k_1 - 1$. Indeed, these sequences start from the same initialization and have the same update rule. Thus, instead of directly proving the convergence of 2-layers Lookahead, we can show an equivalent statement for the considered local SGD instance.

Algorithm 2 Lookahead with two layers

**Require:** Loss function $f$; inner optimizer $A$; initial point $z_0$; synchronization parameters $\alpha_1, \alpha_2$; lengths of the inner loops $k_1, k_2$; number of rounds $R$

1. for $r = 0, 1, \ldots, R - 1$
2. $y_{r,0} = z_r$
3. for $i = 0, 1, \ldots, k_2 - 1$
4. $x_{r,i,0} = y_{r,i}$
5. for $j = 0, 1, \ldots, k_1 - 1$
6. Sample mini-batch $\xi_{r,i,j}$
7. $x_{r,i,j+1} = x_{r,i,j} - A(f_{r,i,j}, x_{r,i,j})$
8. $y_{r,i+1} = (1 - \alpha_1)y_{r,i} + \alpha_1 x_{r,i,k_1}$
9. $z_{r+1} = (1 - \alpha_2)z_r + \alpha_2 y_{r,k_2}$

Denote $a = (\alpha_1\alpha_2, (1 - \alpha_1)\alpha_2, 1 - \alpha_2)^T \in \mathbb{R}^3$. It can be easily verified that $P_t a = a$ holds for any $t$, that is, $a$ is an eigenvector of $P_t$ with eigenvalue 1 for any $t$. Thus, multiplying $[1]$ by $a$ from the right, we get:

$$X_{t+1} a = X_t a - \gamma_t G_t a = X_t a - \gamma_t \alpha_1 \alpha_2 g(x_t, \xi_t). \quad (2)$$

Let $\theta_t = X_t a = \alpha_1 \alpha_2 x_t + (1 - \alpha_1)\alpha_2 y_t + (1 - \alpha_2)z_t$. Using this notation, we can rewrite equation $[2]$ as:

$$\theta_{t+1} = \theta_t - \gamma_t \alpha_1 \alpha_2 g(x_t, \xi_t) \quad (3)$$

Intuitively, $\theta_t$ is a convex combination of $x_t, y_t$ and $z_t$ that evolves smoothly, regardless of the synchronizations. Consequently, $\theta_t$ coincides with $x_t, y_t$ and $z_t$ at the end of each round.

As in Wang et al. (2020), we assume that the function $f$ is $L$-smooth, i.e. satisfies:

$$\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|, \forall x, y \in \mathbb{R}^d \quad (4)$$

and that stochastic gradients are independent, unbiased, and have uniformly bounded variance:

$$\{g(x, \xi), x \in \mathbb{R}^d\} \text{ independent} \quad (5)$$

$$E_\xi [g(x, \xi) | x] = \nabla f(x) \quad (6)$$

$$E_\xi [|| g(x, \xi) - \nabla f(x) ||^2 | x] \leq \sigma^2 \quad (7)$$

We will need one more technical assumption, which arises from the proof:

$$\forall t \geq 0 : \quad 1 - \gamma_t \alpha_1 \alpha_2 L - 2 \gamma_t^2 L^2 k_1^2 \times \left( (1 - \alpha_1)^2 \alpha_2^2 + 2(1 - \alpha_2)^2 + 2 \sigma^2 (1 - \alpha_2)^2 k_2^2 \right) \geq 0. \quad (8)$$

Let us elaborate on the last assumption. It can be considered in the form $g(\gamma_t) \geq 0$, where $g(\gamma_t)$ is the
left side of the inequality as a function of learning rate \( \gamma_t \). Note that \( g \) is continuous and strictly decreasing (for non-negative \( \gamma_t \)). Moreover, \( g(0) = 1 \) satisfies the inequality. Thus, the solution set is always non-empty and has the form:

\[
\gamma_t \leq \gamma_* \quad \forall t \geq 0
\]  

where \( \gamma_* = g^{-1}(0) \) is implicitly defined from \( 8 \).

**Theorem 1.** Suppose that learning rate is kept constant within each round: \( \gamma_{rk_i,k_i} = \gamma_{rk_i,k_2} \), \( \forall r \geq 0 \), \( 0 \leq i \leq k_2 - 1 \), \( 0 \leq j \leq k_1 - 1 \), and satisfies the following conditions:

\[
\lim_{r \to \infty} \gamma_{rk_i,k_2} = 0, \quad \sum_{r=0}^{\infty} \gamma_{rk_i,k_2} = \infty \quad (10)
\]

Then under the assumptions \( 4 \) - \( 8 \), we have:

\[
\frac{1}{k_1k_2} \sum_{r=0}^{R-1} \frac{R}{k_1k_2} \sum_{r=0}^{R-1} \left( \gamma_{rk_i,k_2} \sum_{j=0}^{k_2-1} \sum_{i=0}^{k_1-1} \mathbb{E} \left[ \| \nabla f(\theta_{rk_i,k_2+j}) \|^2 \right] \right) \to 0, \quad r \to \infty
\]

i.e., the weighted average of the expected squared norms of the gradients approaches to 0.

Recall that \( T = Rk_1k_2 \) is the total number of iterations of 2-layers Lookahead.

**Theorem 2.** Suppose that the learning rate is kept constant, i.e. \( \gamma_t = \gamma \forall t \geq 0 \). Then under the assumptions \( 4 \) - \( 8 \), we have:

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla f(\theta_t) \|^2 \right] \leq \frac{2(f(\theta_0) - f_{inf})}{\gamma \alpha_1 \alpha_2 T} + \gamma \alpha_1 \alpha_2 L\sigma^2 + 2\gamma^2 L^2 \sigma^2 k_1 \times \\
\left( (1 - \alpha_1)^2 \alpha_2^2 + 2(1 - \alpha_2)^2 + \frac{4}{3} \alpha_1^2 (1 - \alpha_2)^2 k_2^2 \right)
\]  

(11)

where \( f_{inf} \) denotes the infimum of the objective function. Further, by setting \( \gamma = \gamma_* \sqrt{T} \), where \( \gamma_* \) is defined in \( 9 \), we get the following bound:

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla f(\theta_t) \|^2 \right] \leq \frac{2(f(\theta_0) - f_{inf}) + \gamma_*^2 \alpha_1^2 \alpha_2^2 L\sigma^2}{\gamma_* \alpha_1 \alpha_2 \sqrt{T}} + \Theta \left( \frac{1}{T} \right) = \Theta \left( \frac{1}{\sqrt{T}} \right)
\]  

(12)

For the proof of all statements in this section, see Appendix A.4

**Claim 1.** If we optimize the bound on the right side of \( 11 \) over \( \gamma \) precisely, respecting the constraint \( 8 \), then for large \( T \) the best bound will be achieved for \( \alpha_1 = \alpha_2 = 1 \), that is, when 2-layers Lookahead degenerates to SGD.

Thus, as in the analysis of Lookahead by Wang et al. (2020), our result, while provides some theoretical guarantees for 2-layers Lookahead, does not capture the improvement of this method over SGD.

**Corollary 1.** If \( T \geq \frac{2(f(\theta_0) - f_{inf})}{\alpha_1 \alpha_2 T L\sigma^2} \), then the learning rate \( \gamma = \frac{1}{\alpha_1 \alpha_2 \sqrt{T}} \sqrt{2(f(\theta_0) - f_{inf})} \) satisfies the constraint \( 9 \) and gives the improved bound:

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla f(\theta_t) \|^2 \right] \leq \frac{2\gamma \alpha_1 \alpha_2 \sqrt{T}}{\sqrt{2(f(\theta_0) - f_{inf})}} + \Theta \left( \frac{1}{T} \right) = \Theta \left( \frac{1}{\sqrt{T}} \right)
\]

Furthermore, the obtained constant for the asymptotically most significant term \( \frac{1}{\sqrt{T}} \) is the best possible, that can be deduced from Theorem 2.

Let us make a few observations about the result of Corollary 1. First, note that the optimal constant for the leading term \( \frac{2\gamma \alpha_1 \alpha_2 \sqrt{T}}{\sqrt{2(f(\theta_0) - f_{inf})}} \) does not depend on \( \alpha_1 \) and \( \alpha_2 \). Hence, our result provides almost the same optimal bound for SGD and 2-layers Lookahead, where the difference comes only through the lower-order \( \Theta \left( \frac{1}{T} \right) \) term.

Second, the proposed learning rate \( \gamma = \frac{1}{\alpha_1 \alpha_2 \sqrt{T}} \sqrt{2(f(\theta_0) - f_{inf})} = \Theta \left( \frac{1}{\alpha_1 \alpha_2 \sqrt{T}} \right) \) depends reciprocally on synchronization parameters \( \alpha_1 \) and \( \alpha_2 \).

Finally, Theorem 2 guarantees \( \frac{1}{\sqrt{T}} \) convergence rate only for a fixed number of iterations \( T \) which is known in advance (since the proposed learning rate \( \gamma = \Theta \left( \frac{1}{\sqrt{T}} \right) \) depends on \( T \)). To obtain \( \frac{1}{\sqrt{T}} \) rate for \( T \to \infty \), we can combine 2-layers Lookahead with restarts (see Theorem 3 in Appendix).

We highlight one more property of Multilayer Lookahead: if the inner optimizer has a linear convergence rate, then Multilayer Lookahead preserves it (possibly with a worse constant). For details, see Appendix A.3.

### 4 GENERALIZATION ANALYSIS

The study of optimizers is not limited to their convergence properties. In Machine Learning, our final aim is to perform well on the new, unseen data. Optimizer
parameters have a crucial effect on generalization. For instance, mini-batch SGD generalizes better than full-gradient descent, with smaller mini-batches contributing to even better generalization (Keskar et al., 2016; Hoffer et al., 2017). The large learning rate also plays a regularization role (Lewkowycz et al., 2020; Li et al., 2019). To theoretically analyze such effects, we need a tool that captures the generalization property. Smith et al. (2021) assumed that the samples are split into a fixed set of mini-batches so that each sample belongs to exactly one mini-batch, and only the order of mini-batches shuffles randomly. Thus, the average final iterate of SGD at the end of the epoch, averaged across all possible permutations of the mini-batches, follows the ODE for the modified flow:

$$\dot{f}_{SGD}(y) = f(y) + \frac{\gamma}{4m} \sum_{i=0}^{m-1} \|\nabla f_i(y)\|^2 + O(\gamma^2)$$  \hspace{1cm} (14)

or, equivalently:

$$\dot{f}_{SGD}(y) = f(y) + \frac{\gamma}{4} \|\nabla f(y)\|^2 + \frac{\gamma}{4m} \sum_{i=0}^{m-1} \|\nabla f_i(y) - \nabla f(y)\|^2 + O(\gamma^2)$$  \hspace{1cm} (15)

From (15) we see that SGD penalizes not only sharp regions, but also non-uniform regions (with a high variance of stochastic gradients). Smith et al. (2021) exploits this fact to explain the improved generalization of SGD over GD. In section Section 1.3, we will provide an alternative explanation of this phenomenon.

The analysis made by Smith et al. (2021) still works in more a general setting, where instead of fixing the set of mini-batches, one randomly shuffles the samples across the mini-batches at the beginning of every epoch so that each sample belongs to exactly one mini-batch. In this case, by replacing the sum over all mini-batches by the expectation with a corresponding scaling factor everywhere in the proof, we get the alternative form of (14):

$$\dot{f}_{SGD}(y) = f(y) + \frac{\gamma}{4} \mathbb{E}[\|\nabla f_0(y)\|^2] + O(\gamma^2)$$  \hspace{1cm} (16)

To construct the modified flow for GD and SGD, Smith et al. (2021) use a backward error analysis. Here, we briefly summarize their approach, and we will apply it to derive the modified flow for Lookahead in the next section.

Consider the ODE for the modified flow $\dot{y} = -\nabla f(y), y(0) = y_0$ in general form:

$$\dot{y} = \tilde{h}(y), y(0) = y_0$$  \hspace{1cm} (17)

Denote the iterations of an optimizer with a learning rate $\gamma$, which starts from the same initial point $y_0$, by $y_1, y_2, \ldots$. We aim to find such $\tilde{h}$ that the average final iterate of the epoch $\mathbb{E}[y_m]$ lies on the trajectory of $y$. Let us search for $\tilde{h}$ in the following form:

$$\tilde{h}(y) = h(y) + \varepsilon h_1(y) + \varepsilon^2 h_2(y) + \ldots$$  \hspace{1cm} (18)

Smith et al. (2021) established that for such $\tilde{h}$ the solution of (17) satisfies:

$$y(\varepsilon) = y_0 + \varepsilon h(y_0) + \varepsilon^2 \left( h_1(y_0) + \frac{1}{2} \nabla h(y_0) \dot{h}(y_0) \right) + O(\varepsilon^3)$$  \hspace{1cm} (19)

It remains to find $h$ and $h_1$ from the condition $y(\varepsilon) = \mathbb{E}[y_m]$ with $\varepsilon = \gamma m$. After this, we can reconstruct the modified flow $\tilde{f}$ from the condition $\dot{h} = -\nabla f$. For more details, see Smith et al. (2021).
4.2 Implicit Regularizer of Lookahead

In this section, we apply the backward error analysis to derive the modified flow of the Lookahead (LA).

Before moving to our main result, let us introduce the auxiliary functions:

\[ AN(y) = \mathbb{E}[\|\nabla f_0(y)\|^2] = \mathbb{E}[\|\nabla f_i(y)\|^2] \forall i \]
\[ AI(y) = \mathbb{E}[\langle \nabla f_0(y), \nabla f_i(y) \rangle] = \mathbb{E}[\langle \nabla f_i(y), \nabla f_j(y) \rangle] \forall i \neq j \]
\[ ANG(y) = \nabla AN(y) \]
\[ AIG(y) = \nabla AI(y) \]

Here, \( AN \) and \( AI \) stand for Average Norm squared and Averaged Inner product, while \( ANG \) and \( AIG \) stand for Average Norm squared Gradient and Averaged Inner product Gradient correspondingly. The expectation is taken across all possible splittings on the mini-batches. With these notations, we can rewrite the modified flow for SGD as follows:

\[ \tilde{f}_{SGD}(y) = f(y) + \frac{\gamma}{4} AN(y) + O(\gamma^2) \]

Let us also rewrite \[ f_{GD}(y) = f(y) + \frac{\gamma}{4} m AN(y) + \frac{m}{4} (m-1) AI(y) \]

Taking the expectation of both sides, we get:

\[ \| \nabla f(y) \|^2 = \frac{1}{m^2} \left( \sum_{i=0}^{m-1} \| \nabla f_i(y) \|^2 + \sum_{i \neq j} \langle \nabla f_i(y), \nabla f_j(y) \rangle \right) \]

Taking the expectation of both sides, we get:

\[ \| \nabla f(y) \|^2 = \frac{1}{m^2} \left( m AN(y) + m(m-1) AI(y) \right) = \frac{1}{m} \left( AN(y) + (m-1) AI(y) \right) \]

Thus, the formula for \( f_{GD}(y) \) becomes

\[ f_{GD}(y) = f(y) + \frac{\gamma}{4} \left( AN(y) + (m-1) AI(y) \right) = f(y) + \frac{\gamma}{4m} AN(y) + \frac{\gamma(m-1)}{4m} AI(y) \]

Now we are ready to present the modified flow of the Multilayer Lookahead optimizer.

**Theorem 3.** Consider \( n \)-layers Lookahead with parameters \( \{k_1, \ldots, k_n\}, \{\alpha_1, \ldots, \alpha_n\} \) and SGD with learning rate \( \gamma \) as the inner optimizer. Suppose that the number of mini-batches per epoch \( m \) is divisible by the number of iterations per one round \( k_1 \ldots k_n \).

Besides, assume that at each epoch the samples are randomly shuffled across the mini-batches such that each sample belongs to exactly one mini-batch. Then the final iteration of the epoch for the \( n \)-layers Lookahead, averaged across the randomness in splitting on the mini-batches, lies on the trajectory of the ODE for the following modified flow:

\[ \tilde{f}_{LA-n}(y) = f(y) + \frac{\gamma}{4} AN(y) - \frac{\gamma}{4} m AN(y) + \frac{\gamma}{4} m(m-1) AI(y) \]

Specifically, the solution of the ODE \( \dot{y} = -\nabla \tilde{f}_{LA-n}(y), y(0) = y_0 \) satisfies \( y(\alpha_1 \ldots \alpha_n \gamma m) = \mathbb{E}[y_r], \) where \( r = m/(k_1 \ldots k_n) \), so that \( y_r \) - output of the Lookahead after one epoch (or \( r \) rounds), starting from the point \( y_0 \).

For the proof, see Appendix A.2

Let us explicitly state the formulas for LA and LA-2:

\[ f_{LA}(y) = f(y) + \frac{\alpha_1}{4} AN(y) - \frac{\gamma}{4} (1 - \alpha)(k-1) AI(y) \]

\[ f_{LA-2}(y) = f(y) + \frac{\alpha_1 \alpha_2 \gamma}{4} AN(y) - \frac{\gamma}{4} ((1 - \alpha_1)(k_1 - 1) + (1 - \alpha_2)\alpha_1(k_2k_1 - 1)) AI(y) \]

4.3 Comparing the Implicit Regularizers for GD, SGD and LA

Let us summarize the modified flows for the considered optimizers, assuming that each optimizer uses its own learning rate \( \gamma_{GD}, \gamma_{SGD}, \) and \( \gamma_{LA} \) (we omit \( O(\gamma^2) \) term everywhere for simplicity):

\[ \tilde{f}_{GD}(y) = f(y) + \frac{\gamma_{GD}}{4m} AN(y) + \frac{\gamma_{GD}(m-1)}{4m} AI(y) \]

\[ \tilde{f}_{SGD}(y) = f(y) + \frac{\gamma_{SGD}}{4m} AN(y) \]

\[ \tilde{f}_{LA}(y) = f(y) + \frac{\alpha_{LA}}{4m} AN(y) - \frac{\gamma_{LA}}{4} (1 - \alpha_{LA}(k-1)) AI(y) \]

To ensure that all optimizers make the same progress during one epoch, the sum of step sizes during one epoch has to be the same: \( \gamma_{GD} = m\gamma_{SGD} = m\alpha_{LA} \) (in general, each layer of Lookahead shrinks the progress of the outer variable by a factor of \( \alpha_i \)). Then, the coefficient before \( AN(y) \) is equal for all three optimizers, and the difference comes only through the coefficient before \( AI(y) \): it decreases in order GD, SGD, LA. Notably, we empirically observe that generalization improves in the same order. Thus, we claim that the generalization improvement of SGD over GD and
of LA over SGD caused by the amplifying role of the implicit regularizer \(-AI(y) = -E[\langle \nabla f_0(y), \nabla f_1(y) \rangle]\). Intuitively, by minimizing \(-AI(y)\), we maximize the average inner product between the gradients on different mini-batches, which in turn minimizes the angle between these gradients. Thus, the regularizer \(-AI(y)\) makes stochastic gradients more aligned.

From the formula for \(\tilde{f}_{LA}(y)\), we see that by increasing \(k\), we also enhance the constant before \(-AI(y)\), thus, increase the regularization role of Lookahead. For \(\alpha\), we cannot conclude: by increasing \(\alpha\), we amplify \(AN(y)\) but suppress \(-AI(y)\) regularizer.

We now compare implicit regularizers of LA and LA-2:

\[
\tilde{f}_{LA}(y) = f(y) + \frac{\alpha \gamma_{LA}}{4} AN(y) - (1 - \alpha) \gamma_{LA}(k - 1) - AI(y)
\]

\[
\tilde{f}_{LA-2}(y) = f(y) + \frac{\alpha_1 \alpha_2 \gamma_{LA-2}}{4} AN(y) - \gamma_{LA-2}((1 - \alpha_1)(k_1 - 1) + (1 - \alpha_2)\alpha_1(k_2k_1 - 1))AI(y)
\]

Let us assume that their coefficients are related by \(\alpha_1 = \alpha, \ k = k_1\), and the optimizers make equal progress during one epoch. Then \(\alpha \gamma_{LA} = \alpha_1 \alpha_2 \gamma_{LA-2}\), \(0 < \alpha_2 < 1 \Rightarrow \gamma_{LA-2} > \gamma_{LA}\). Then the constants before \(AN(y)\) again coincide. For \(AI(y)\), even the first term \(\frac{\gamma_{LA-2}}{4}(1 - \alpha_1)(k_1 - 1)\) for LA-2 exceeds the whole constant for LA (since \(\gamma_{LA-2} > \gamma_{LA}\)). Thus, by adding the second layer of Lookahead, we reinforce \(-AI(y)\) regularizer further. The same intuition works for any number of layers.

Note that scaling the learning rate for Lookahead by a factor of \(\frac{1}{\alpha}\) and for 2-layers Lookahead by a factor of \(\frac{1}{\alpha_1 \alpha_2}\) also arises in Corollary \(1\) which provides an asymptotically optimal learning rate for a convergence bound given by Theorem \(2\). Besides, we will see in the experimental section that the optimal learning rate is often greater for Multilayer Lookahead than for SGD. Thus, the assumption that optimizers make the same progress is theoretically justified and not detached from the practice.

5 EXPERIMENTS

CIFAR-10 and CIFAR-100 datasets consist of \(32 \times 32\) color images of 10 and 100 different classes correspondingly, split into 50000 training samples and 10000 test samples. We trained ResNet-18 [He et al., 2016] model for 300 epochs with batch size = 50 and using learning rate decay by a factor of 10 after 150th and 250th epochs. We selected this batch size to ensure that each epoch ends after the synchronization step in Lookahead, which makes the training plots less noisy (as shown by Zhang et al. [2019], Lookahead achieves the highest train and test accuracy right after the synchronization step). As the inner optimizer, we used SGD with momentum = 0.9. We compared SGD with Lookahead and Multilayer Lookahead with up to four layers. Note that when using grid search, the number of possible configurations of \(\alpha = (\alpha_1, \ldots, \alpha_n)\) and \(k = (k_1, \ldots, k_n)\) increases exponentially with the number of layers. Thus, we tried a moderate range of values for \(\alpha\) and \(k\) and do not claim that the best parameters that we found are close to optimal. We empirically noticed that for any number of Lookahead layers the best parameter \(\alpha = (\alpha_1, \ldots, \alpha_n)\) is always achieved for a non-increasing sequence of \(\alpha_1\) (that is, when Multilayer Lookahead gives more preference to the weight from inner loops during the synchronization). This observation enables to significantly reduce the tuning time for \(\alpha\).

For CIFAR-10, we performed a grid search for SGD over learning rate \(lr \in \{0.05, 0.1, 0.2, 0.3, 0.5\}\) and weight decay parameter \(\lambda \in \{10^{-3}, 3 \cdot 10^{-4}, 10^{-4}\}\), and got \(lr = 0.1, \lambda = 3 \cdot 10^{-4}\) performing the

\[\text{Figure 1: Test Accuracy During the Training on (a) CIFAR-10 and (b) CIFAR-100. LA-n Stands for Multilayer Lookahead with n Layers}\]
Multilayer Lookahead: a Nested Version of Lookahead

We note that the performance of Multilayer Lookahead with the most layers achieves better anytime performance. The plots, in both cases, Multilayer Lookahead with SGD and Lookahead. Moreover, as we can see from search, Multilayer Lookahead often outperforms both SGD and Lookahead. Thus, in settings were learning rate decay could not be used, Multilayer Lookahead outperforms both SGD and Lookahead by a large margin.

For additional experiments training GANs on the MNIST dataset see Appendix B.2.

6 CONCLUSION

In this work, we present Multilayer Lookahead - a method obtained by recursively wrapping Lookahead around any other optimizer. It increases space complexity linearly in the number of layers. However, the time overhead is marginal: on average, it requires taking less than one additional convex combination per iteration. We prove the convergence of Lookahead with two layers to a stationary point for smooth non-convex functions with $\mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$ rate. Besides, we show that Multilayer Lookahead amplifies the implicit regularizer yielded by SGD. Next, we demonstrate that our optimizer outperforms both SGD and Lookahead on the classification task on CIFAR-10 and CIFAR-100 datasets.

Our experiments show a clear advantage over the use of SGD or Lookahead. While there is an extra cost in tuning the synchronization parameters, we have found the algorithm to be quite robust to the choice of these parameters. Indeed, having all synchronization parameters being equal leads to almost optimal results in all of our experiments.

We also want to highlight that as it can be seen in our plots for test accuracy, the use of Multilayer Lookahead leads to a similar phenomenon as the so called “super-convergence” by Smith & Topin (2019). That is, only a few epochs are needed to achieve an almost top accuracy. In particular, before the first learning rate decay, Multilayer Lookahead completely outperforms both SGD and Lookahead. This can be quite relevant when the amount of training data is not known in advance and a learning rate decay cannot be scheduled, e.g., in the streaming model.

Table 1: Results on CIFAR-10

| Opt | $lr$ | $\alpha$ | test accuracy |
|-----|------|-----------|---------------|
| SGD | 0.1  | -         | 0.9527 ± 0.0019 |
| LA-1| 0.1  | 0.5 ($k=10$) | 0.9541 ± 0.0009 |
| LA-2| 0.1  | (0.7, 0.7) | 0.9555 ± 0.0016 |
| LA-3| 0.1  | (0.6, 0.8, 0.8) | 0.9553 ± 0.0021 |
| LA-4| 0.1  | (0.6, 0.75, 0.85, 0.85) | 0.9554 ± 0.0011 |

You can see the performance of each optimizer using the best parameters found in Table 1 (CIFAR-10) and Table 2 (CIFAR-100), and the corresponding training plots in Figure 1 (the results were averaged across 3 runs). They show that even for a moderate grid search, Multilayer Lookahead often outperforms both SGD and Lookahead. Moreover, as we can see from the plots, in both cases, Multilayer Lookahead with the most layers achieves better anytime performance. We note that the performance of Multilayer Lookahead was robust towards the choice of parameters ($\alpha$, $k$) in the considered range.

Table 2: Results on CIFAR-100

| Opt | $lr$ | $\alpha$ | test accuracy |
|-----|------|-----------|---------------|
| SGD | 0.01 | -         | 0.7904 ± 0.0032 |
| LA-1| 0.05 | 0.3 ($k=5$) | 0.7910 ± 0.0010 |
| LA-2| 0.03 | (0.5, 0.8) | 0.7885 ± 0.0022 |
| LA-3| 0.03 | (0.6, 0.75, 0.75) | 0.7910 ± 0.0020 |
| LA-4| 0.03 | (0.6, 0.75, 0.8, 0.8) | 0.7923 ± 0.0020 |

For CIFAR-100, we found the optimal learning rate for Multilayer Lookahead to be larger than for SGD (Table 2). In this case, we clearly see the regularization effect of Multilayer Lookahead: comparing to SGD, it performs worse on training data, but better on test data (see Appendix B.1 for the corresponding plots). In contrast, for CIFAR-10, the optimal learning rate for SGD and Multilayer Lookahead turned out to be the same. In this case, Multilayer Lookahead improves both train and test accuracy. These observations are consistent with our findings in Section 4.3 where we established that Multilayer Lookahead with a larger learning rate amplifies the regularization effect of SGD. However, when the learning rates are equal, we cannot conclude. Even though Multilayer Lookahead still yields an additional $-AI(y)$ regularizer, it suppresses $AN(y)$ regularizer.

Super-convergence. In all of our experiments, we found that the network can be trained with Multilayer Lookahead to almost top accuracy orders of magnitude faster than with similar techniques. After just 30 epochs, we can reach 90% test accuracy for CIFAR-10 and 70% for CIFAR-100. This accuracies can be achieved before the first learning rate decay. Thus, in settings were learning rate decay could not be used, Multilayer Lookahead outperforms both SGD and Lookahead by a large margin.

For additional experiments training GANs on the MNIST dataset see Appendix B.2.
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A MISSING PROOFS

A.1 Proofs for Section 3

Here, we prove the theorems about the convergence of 2-layers Lookahead to a stationary point, presented in section 3. This appendix is organized as follows: we start by two auxiliary technical lemmas, then prove Lemma 6 (Main lemma), which contains the main part of the proof for both Theorem 1 and Theorem 2. Finally, we give a proof for the three announced theorems.

A.1.1 Auxiliary Lemmas

Lemma 4. Assumption (10) implies the following two statements:

1) \( \sum_{r=0}^{R-1} \gamma_{rk_1k_2} \rightarrow 0, R \rightarrow \infty \)
2) \( \sum_{r=0}^{R-1} \gamma_{rk_1k_2} \rightarrow 0, R \rightarrow \infty \)

Proof. Let us prove only the first statement, for the second one the proof is identical. Since the sequence \( (\gamma_{rk_1k_2}) \) convergence to 0, we have:

\[ \forall \varepsilon > 0 \exists S : \forall r \geq S \Rightarrow \gamma_{rk_1k_2} < \varepsilon. \]

Therefore,

\[ \sum_{r=0}^{R-1} \gamma_{rk_1k_2} \leq \sum_{r=0}^{S-1} \gamma_{rk_1k_2} + \sum_{r=S}^{R-1} \gamma_{rk_1k_2} \leq \sum_{r=0}^{S-1} \gamma_{rk_1k_2} + \varepsilon \sum_{r=0}^{R-1} \gamma_{rk_1k_2} \]

Now, since \( \sum_{r=0}^{S-1} \gamma_{rk_1k_2} \) is a finite sum, and \( \sum_{r=0}^{R-1} \gamma_{rk_1k_2} \rightarrow \infty \) we have:

\[ \exists R_0 : \forall R > R_0 : \sum_{r=0}^{R-1} \gamma_{rk_1k_2} > \frac{\sum_{r=0}^{S-1} \gamma_{rk_1k_2}}{\varepsilon} \Rightarrow \sum_{r=0}^{S-1} \gamma_{rk_1k_2} < \varepsilon \sum_{r=0}^{R-1} \gamma_{rk_1k_2} \]

Substituting this in the previous inequality, we have:

\[ \frac{\sum_{r=0}^{R-1} \gamma_{rk_1k_2}}{\sum_{r=0}^{R-1} \gamma_{rk_1k_2}} \leq \frac{\sum_{r=0}^{S-1} \gamma_{rk_1k_2} + \varepsilon \sum_{r=0}^{R-1} \gamma_{rk_1k_2}}{\sum_{r=0}^{R-1} \gamma_{rk_1k_2}} < \frac{\varepsilon \sum_{r=0}^{R-1} \gamma_{rk_1k_2} + \varepsilon \sum_{r=0}^{R-1} \gamma_{rk_1k_2}}{\sum_{r=0}^{R-1} \gamma_{rk_1k_2}} = 2 \varepsilon \quad \forall T > R_0 \]

which completes the proof.

Lemma 5. Suppose that stochastic gradients \( g(x_j, \xi_j) \) satisfy assumptions (5) - (7). Then for any indexes \( a \leq b \) we have:

\[ \mathbb{E} \left[ \left\| \sum_{j=a}^{b} g(x_j, \xi_j) \right\|^2 \right] \leq 2(b-a+1) \left( \sigma^2 + \sum_{j=a}^{b} \mathbb{E}[\| \nabla f(x_j) \|^2] \right) \]
Proof.

\[ E \left[ \left\| \sum_{j=a}^{b} g(x_j, \xi_j) \right\|^2 \right] = E \left[ \left\| \sum_{j=a}^{b} (g(x_j, \xi_j) - \nabla f(x_j)) + \sum_{j=a}^{b} \nabla f(x_j) \right\|^2 \right] \leq \]

\[ \leq 2 \left( E \left[ \left\| \sum_{j=a}^{b} (g(x_j, \xi_j) - \nabla f(x_j)) \right\|^2 \right] + E \left[ \left\| \sum_{j=a}^{b} \nabla f(x_j) \right\|^2 \right] \right) = \]

\[ = 2 \left( \sum_{j=a}^{b} E \left[ \left\| (g(x_j, \xi_j) - \nabla f(x_j)) \right\|^2 \right] + E \left[ \left\| \sum_{j=a}^{b} \nabla f(x_j) \right\|^2 \right] \right) \leq \]

\[ \leq 2 \left( (b - a + 1)\sigma^2 + (b - a + 1) \sum_{j=a}^{b} E[\|\nabla f(x_j)\|^2] \right) = \]

\[ = 2(b - a + 1) \left( \sigma^2 + \sum_{j=a}^{b} E[\|\nabla f(x_j)\|^2] \right) \]

In the first inequality, we used \[\|p + q\|^2 \leq 2\|p\|^2 + \|q\|^2\).

For the second equality, we used that for \(j > i\):

\[ E \left[ (g(x_j, \xi_j) - \nabla f(x_j), g(x_i, \xi_i) - \nabla f(x_i)) \right] = E_x \left[ E_{\xi_j} \left[ (g(x_j, \xi_j) - \nabla f(x_j), g(x_i, \xi_i) - \nabla f(x_i)) \right] \right] = \]

\[ = E_{\xi_i} \left[ E_{\xi_j} \left[ (g(x_j, \xi_j) - \nabla f(x_j), g(x_i, \xi_i) - \nabla f(x_i)) \right] \right] = 0 \]

For the second inequality, we used assumption \([7]\) and inequality \[\|\sum_{i=1}^{n} p_i\|^2 \leq n \sum_{i=1}^{n} \|p_i\|^2\].

\[ \square \]

A.1.2 Main Lemma

**Lemma 6** (Main lemma). Suppose that learning rate is kept constant within each round: \(\gamma_{r+k_1 k_2 + i k_1 + j} = \gamma_{r k_1 k_2}, \forall r \geq 0, 0 \leq i \leq k_2 - 1, 0 \leq j \leq k_1 - 1\). Then under the assumptions \([4]\) - \([8]\) we have:

\[
\frac{1}{k_1 k_2 S_R} \sum_{r=0}^{R-1} \left( \gamma_{r k_1 k_2} \sum_{k=0}^{k_1 k_2 - 1} E[\|\nabla f(\theta_{r k_1 k_2} + j)\|^2] \right) \leq \frac{2(f(\theta_0) - f_{\infty})}{\alpha_1 \alpha_2 k_1 k_2 S_R} + \frac{\alpha_1 \alpha_2 L \sigma^2}{S_R} \sum_{r=0}^{R-1} \sum_{k_{r k_1 k_2}}^2 + \frac{2L^2 \sigma^2 k_1^2}{S_R} \left( (1 - \alpha_1)^2 \alpha_2^2 + 2(1 - \alpha_2)^2 + \frac{4}{3} \alpha_1^2 (1 - \alpha_2)^2 k_2^2 \right) \sum_{r=0}^{R-1} \sum_{k_{r k_1 k_2}}^2 (30)
\]

where \(S_R = \sum_{r=0}^{R-1} \gamma_{r k_1 k_2}\) and \(f_{\infty}\) is the infimum of \(f\).

**Proof.** Combining \(L\)-smoothness of \(f\) with the formula \([3]\), we have:

\[
f(\theta_{t+1}) - f(\theta_t) \leq (\theta_{t+1} - \theta_t, \nabla f(\theta_t)) + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 = \]

\[
= -\gamma_t \alpha_1 \alpha_2 (\nabla f(\theta_t), g(x_t, \xi_t)) + \frac{\gamma_t^2 \alpha_1^2 \alpha_2^2 L}{2} \|g(x_t, \xi_t)\|^2 \]

As in \([Wang et al. 2020]\), to simplify notations we denote by \(E_x[\cdot]\) the conditional expectation \(E_x[\cdot|\mathcal{F}_t]\), where \(\mathcal{F}_t\) is the sigma algebra generated by the noise in stochastic gradients until iteration \(t\). Taking the conditional expectation of both
sides of (31), for the first term on RHS we have:

\[
E_t[\langle \nabla f(\theta_t), g(x_t, \xi_t) \rangle] = \langle \nabla f(\theta_t), \nabla f(x_t) \rangle = \frac{1}{2} \left( \|\nabla f(\theta_t)\|^2 + \|\nabla f(x_t)\|^2 - \|\nabla f(\theta_t) - \nabla f(x_t)\|^2 \right) \\
\geq \frac{1}{2} \left( \|\nabla f(\theta_t)\|^2 + \|\nabla f(x_t)\|^2 - L^2 \|\theta_t - x_t\|^2 \right) = \\
= \frac{1}{2} \left( \|\nabla f(\theta_t)\|^2 + \|\nabla f(x_t)\|^2 - L^2 \|\theta_t - x_t\|^2 \right)
\]

(32)

For the second term on RHS of (31):

\[
E_t[\|g(x_t, \xi_t)\|^2] = E_t[\|g(x_t, \xi_t) - \nabla f(x_t) + \nabla f(x_t)\|^2] = E_t[\|g(x_t, \xi_t) - \nabla f(x_t)\|^2 + \|\nabla f(x_t)\|^2] \leq \\
\sigma^2 + \|\nabla f(x_t)\|^2
\]

(33)

Plugging (32) and (33) back into (31) and taking the total expectation, we get

\[
E[f(\theta_{t+1})] - E[f(\theta_t)] \leq -\frac{\gamma_t \alpha_t \alpha_2}{2} E[\|\nabla f(\theta_t)\|^2] - \frac{\gamma_t \alpha_t \alpha_2}{2} (1 - \gamma_t \alpha_1 \alpha_2 L) E[\|\nabla f(x_t)\|^2] + \frac{\gamma_t^2 \alpha^2 \sigma^2}{2} + \\
+ \frac{\gamma_t \alpha_t \alpha_2 L^2}{2} E[\|1 - \alpha_1 \alpha_2 x_t - (1 - \alpha_1) \alpha_2 y_t - (1 - \alpha_2) z_t\|^2]
\]

(34)

Let us decompose \( t = r k_1 k_2 + i k_1 + j \), where \( 0 \leq i \leq k_2 - 1, 0 \leq j \leq k_1 - 1 \). For shortness, denote \( s = r k_1 k_2 + i k_1 \), so that \( t = s + j \). Summing (34) from \( j = 0 \) to \( j = k_1 - 1 \), and using that \( \gamma_{r k_1 k_2 + i k_1 + j} = \gamma_{r k_1 k_2} \), we get:

\[
E[f(\theta_{s+k_1})] - E[f(\theta_s)] \leq -\frac{\gamma_{r k_1 k_2} \alpha_t \alpha_2}{2} \sum_{j=0}^{k_1-1} E[\|\nabla f(\theta_{s+j})\|^2] - \frac{\gamma_{r k_1 k_2} \alpha_t \alpha_2}{2} (1 - \gamma_{r k_1 k_2} \alpha_1 \alpha_2 L) \sum_{j=0}^{k_1-1} E[\|\nabla f(x_{s+j})\|^2] + \\
+ \frac{\gamma_{r k_1 k_2}^2 \alpha^2 \sigma^2}{2} k_1 + \frac{\gamma_{r k_1 k_2} \alpha_t \alpha_2 L^2}{2} \sum_{j=0}^{k_1-1} E[\|1 - \alpha_1 \alpha_2 x_{s+j} - (1 - \alpha_1) \alpha_2 y_{s+j} - (1 - \alpha_2) z_{s+j}\|^2]
\]

(35)

Taking into account that \( y_{s+j} = y_s, z_{s+j} = z_{r k_1 k_2} \), we can bound the last term on RHS of (35) as follows:

\[
E[\|1 - \alpha_1 \alpha_2 x_{s+j} - (1 - \alpha_1) \alpha_2 y_{s+j} - (1 - \alpha_2) z_{s+j}\|^2] = E[\|1 - \alpha_1 \alpha_2 (x_{s+j} - y_s) + (1 - \alpha_2) (x_{s+j} - z_{r k_1 k_2})\|^2] \leq \\
\leq 2 \left( E[\|1 - \alpha_1 \alpha_2 (x_{s+j} - y_s)\|^2] + E[\|1 - \alpha_2 (x_{s+j} - z_{r k_1 k_2})\|^2] \right) = \\
= 2 (1 - \alpha_1)^2 \alpha_2^2 E[\|x_{s+j} - y_s\|^2] + 2 (1 - \alpha_2)^2 E[\|x_{s+j} - z_{r k_1 k_2}\|^2]
\]

(36)

Let us bound the first term on RHS of (36):

\[
E[\|x_{s+j} - y_s\|^2] = E[\|x_{s+j} - x_s\|^2] = \gamma_{r k_1 k_2}^2 \sum_{j'=0}^{j-1} E[\|g(x_{s+j'}, \xi_{s+j'})\|^2] \leq \text{using Lemma 5} \leq \\
\leq 2 \gamma_{r k_1 k_2}^2 \sum_{j'=0}^{j-1} E[\|\nabla f(x_{s+j'})\|^2] \leq 2 \gamma_{r k_1 k_2}^2 \sum_{j'=0}^{j-1} E[\|\nabla f(x_{s+j'})\|^2]
\]

(37)

Bounding the second term on RHS of (36):

\[
E[\|x_{s+j} - z_{r k_1 k_2}\|^2] = E[\|x_{r k_1 k_2 + i k_1 + j} - z_{r k_1 k_2}\|^2] = E[\|x_{r k_1 k_2 + i k_1 + j} - y_{r k_1 k_2 + i k_1} + y_{r k_1 k_2 + i k_1} - z_{r k_1 k_2}\|^2] \leq \\
\leq 2 \left( E[\|x_{r k_1 k_2 + i k_1 + j} - y_{r k_1 k_2 + i k_1}\|^2] + E[\|y_{r k_1 k_2 + i k_1} - z_{r k_1 k_2}\|^2] \right) = 2 \left( E[\|x_{s+j} - y_s\|^2] + E[\|y_{r k_1 k_2 + i k_1} - z_{r k_1 k_2}\|^2] \right)
\]

(38)
Finally, the RHS of (38) can be bounded by:

\[
\mathbb{E}[\|y_{r_kk_1+k_1} - z_{r_kk_2}\|^2] = \mathbb{E}[\|y_{r_kk_1+k_1} - y_{r_kk_2}\|^2] = \\
= \mathbb{E}\left[\sum_{i'=0}^{i-1} (y_{r_{k_1}+i'+1})_k - y_{r_{k_2}+i'+1})_k\right]^2 \leq \sum_{i'=0}^{i-1} \mathbb{E}[\|y_{r_kk_2 + (i'+1)} - y_{r_kk_2 + i'}\|^2]
\]

\[
= i \sum_{i'=0}^{i-1} \mathbb{E}\left[\alpha_1 (y_{r_kk_1+i'+1})_k - \gamma_{r_kk_2} \sum_{j=0}^{k_1-1} g(x_{r_kk_1+i'+1+j}) + (1 - \alpha_1) y_{r_kk_2+i'+1} - y_{r_kk_2+i'}\right]^2
\]

\[
= \alpha_1^2 \gamma_{r_kk_2} \sum_{i'=0}^{i-1} \mathbb{E}\left[\sum_{j=0}^{k_1-1} g(x_{r_kk_1+i'+1+j}) + (1 - \alpha_1) y_{r_kk_2+i'+1} - y_{r_kk_2+i'}\right]^2 \leq \text{using Lemma 5}
\]

\[
\leq \alpha_1^2 \gamma_{r_kk_2} \sum_{i'=0}^{i-1} 2k_1 \left(\sigma^2 + \sum_{j=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{r_kk_1+i'+1+j})\|^2]\right) = \\
= 2\alpha_1^2 \gamma_{r_kk_2} k_1 \sigma^2 + 2i_0^2 \gamma_{r_kk_2} k_1 \sum_{j=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{r_kk_2+j})\|^2] \leq \\
\leq 2\alpha_1^2 \gamma_{r_kk_2} k_1 \sigma^2 + 2i_0^2 \gamma_{r_kk_2} k_1 \sum_{j=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{r_kk_2+j})\|^2] (39)
\]

Finally, the RHS of (38) can be bounded by:

\[
\mathbb{E}[\|x_{s+j} - z_{r_kk_2}\|^2] \leq \\
\leq 2 \left(2\gamma_{r_kk_2} j \left(\sigma^2 + \sum_{j'=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{s+j'})\|^2]\right) + 2i_0^2 \gamma_{r_kk_2} k_1 \sigma^2 + 2i_0^2 \gamma_{r_kk_2} k_1 \sum_{j'=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{r_kk_2+j'})\|^2]\right) = \\
= 4\gamma_{r_kk_2} \left(j \left(\sigma^2 + \sum_{j'=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{s+j'})\|^2]\right) + i_0^2 \gamma_{r_kk_2} k_1 \sigma^2 + i_0^2 \gamma_{r_kk_2} k_1 \sum_{j'=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{r_kk_2+j'})\|^2]\right) (40)
\]

Now, using (37) and (40), we can bound the RHS of (36):

\[
\mathbb{E}[\|(1 - \alpha_1 \alpha_2)x_{s+j} - (1 - \alpha_1)\alpha_2 y_{s+j} - (1 - \alpha_2)z_{s+j}\|^2] \leq \\
\leq 2(1 - \alpha_1)^2 \alpha_2^2 \cdot 2\gamma_{r_kk_2} j \left(\sigma^2 + \sum_{j'=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{s+j'})\|^2]\right) + \\
\leq + 2(1 - \alpha_2)^2 \cdot 4\gamma_{r_kk_2} \left(j \left(\sigma^2 + \sum_{j'=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{s+j'})\|^2]\right) + i_0^2 \gamma_{r_kk_2} k_1 \sigma^2 + i_0^2 \gamma_{r_kk_2} k_1 \sum_{j'=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{r_kk_2+j'})\|^2]\right) = \\
= 4\gamma_{r_kk_2} j \left((1 - \alpha_1)^2 \alpha_2^2 j + 2(1 - \alpha_2)^2 j + 2(1 - \alpha_2)^2 \alpha_1^2 k_1\right) + \\
+ 4\gamma_{r_kk_2} \left((1 - \alpha_1)^2 \alpha_2^2 j + 2(1 - \alpha_2)^2 \sum_{j'=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{s+j'})\|^2]\right) = \\
= 8\gamma_{r_kk_2} \alpha_1^2 (1 - \alpha_2)^2 i_0 k_1 \sum_{j'=0}^{k_1-1} \mathbb{E}[\|\nabla f(x_{r_kk_2+j'})\|^2] (41)
\]

We have bounded the constituents of the last sum on RHS of (35). Summing from \(j = 0\) to \(j = k_1 - 1\), and using \(\sum_{j=0}^{k_1-1} j < \frac{k_1^2}{2}\), we get:
\[
\sum_{j=0}^{k_1-1} E[(1 - \alpha_1 \alpha_2)x_{s+j} - (1 - \alpha_1)\alpha_2y_{s+j} - (1 - \alpha_2)z_{s+j}]^2 \leq \\
4\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2 + \\
2\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2 + \\
8\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2
\]

Eventually, we can upper-bound the RHS of (35):

\[
\begin{align*}
\mathbb{E}[f(\theta_s + k_1)] - \mathbb{E}[f(\theta_s)] &\leq \mathbb{E}[\nabla f(\theta_{s+j})]^2 \leq \\
&- \frac{\gamma_{k_1k_2} \alpha_1 \alpha_2}{2} \left(1 - \gamma_{k_1k_2} \alpha_1 \alpha_2 L - 2\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2 + \\
&+ 2\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2 + \\
&+ 8\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2
\right)
\end{align*}
\]

Recall that \( s = rk_1k_2 + ik_1 \). Summing from \( i = 0 \) to \( i = k_2 - 1 \), and using \( \sum_{i=0}^{k_2-1} i < \frac{k_2^2}{2} \), \( \sum_{i=0}^{k_2-1} i^2 < \frac{k_3}{3} \), we get:

\[
\begin{align*}
\mathbb{E}[f(\theta_{s+1})] - \mathbb{E}[f(\theta_{s})] &\leq \mathbb{E}[\nabla f(\theta_{s+j})]^2 \leq \\
&- \frac{\gamma_{k_1k_2} \alpha_1 \alpha_2}{2} \left(1 - \gamma_{k_1k_2} \alpha_1 \alpha_2 L - 2\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2 + \\
&+ 2\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2 + \\
&+ 8\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2
\right)
\end{align*}
\]

By assumption \( \mathbb{E}[\nabla f(x_{s+j})]^2 \) is negative. Therefore, we can bound the RHS by excluding this term:

\[
\begin{align*}
\mathbb{E}[f(\theta_{s+1})] - \mathbb{E}[f(\theta_{s})] &\leq \mathbb{E}[\nabla f(\theta_{s+j})]^2 + \\
&+ \frac{\gamma_{k_1k_2} \alpha_1 \alpha_2}{2} \left(1 - \gamma_{k_1k_2} \alpha_1 \alpha_2 L - 2\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2 + \\
&+ 2\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2 + \\
&+ 8\sum_{j=0}^{k_1-1} E[\nabla f(x_{s+j})]^2
\right)
\end{align*}
\]
Summing from \( r = 0 \) to \( r = R - 1 \), we get:

\[
E[f(\theta_{rk_1k_2})] - f(\theta_0) \leq -\frac{\alpha_1 \alpha_2}{2} \sum_{r=0}^{R-1} (\gamma_{rk_1k_2}) \sum_{j'=0}^{k_1k_2-1} E[\|\nabla f(\theta_{rk_1k_2+j'})\|^2] + \frac{\alpha_2^2 L^2 \sigma^2 k_1 k_2^2}{2} \sum_{r=0}^{R-1} \gamma_{rk_1k_2} + \alpha_1 \alpha_2 L \sigma^2 k_1^2 \left( (1 - \alpha_1)^2 \alpha_2^2 + 2(1 - \alpha_2)^2 + \frac{4}{3} \omega_1^2 (1 - \alpha_2)^2 k_2^2 \right) \sum_{r=0}^{R-1} \gamma_{rk_1k_2}
\]

After rearranging and multiplying by \( \frac{2}{\alpha_1 \alpha_2 k_1 k_2 S_R} \), where \( S_R = \sum_{r=0}^{R-1} \gamma_{rk_1k_2} \), and using that \( E[f(\theta_{rk_1k_2})] \geq f_{inf} \), where \( f_{inf} \) is the infimum of \( f \), we obtain the statement of the lemma:

\[
\frac{1}{k_1 k_2 S_R} \sum_{r=0}^{R-1} \sum_{j'=0}^{k_1 k_2-1} E[\|\nabla f(\theta_{rk_1k_2+j'})\|^2] \leq \frac{2(f(\theta_0) - f_{inf})}{\alpha_1 \alpha_2 S_R} + \frac{\alpha_1 \alpha_2 L \sigma^2}{S_R} \sum_{r=0}^{R-1} \gamma_{rk_1k_2} + \frac{2L^2 \sigma^2 k_1}{S_R} \left( (1 - \alpha_1)^2 \alpha_2^2 + 2(1 - \alpha_2)^2 + \frac{4}{3} \omega_1^2 (1 - \alpha_2)^2 k_2^2 \right) \sum_{r=0}^{R-1} \gamma_{rk_1k_2}
\]

(45)

### A.1.3 Proofs of the Stated Results

Now, we have all the ingredients to prove the first two theorems.

**Proof of Theorem 1** Using assumption (40) and Lemma (4), we obtain that \( 1/S_R \to 0 \), \( \sum_{r=0}^{R-1} \gamma_{rk_1k_2}/S_R \to 0 \), \( \sum_{r=0}^{R-1} \gamma_{rk_1k_2}/S_R \to 0 \). Applying it to the statement of Lemma (6) we get that RHS of (30) approaches to 0, which yields the result of the Theorem.

**Proof of Theorem 2** To prove Theorem 2 we substitute \( \gamma_{rk_1k_2} = \gamma = \text{const} \) in (30) and after minor simplifications obtain the desired statement.

**Proof of Claim 1** When \( \alpha_1 = \alpha_2 = 1 \), the third term in the RHS of (11) disappears and the bound becomes:

\[
\frac{1}{T} \sum_{t=0}^{T-1} E[\|\nabla f(\theta_t)\|^2] \leq \frac{2(f(\theta_0) - f_{inf})}{\gamma T} + \gamma L \sigma^2
\]

Optimizing it over \( \gamma \), we get the optimal value \( \gamma = \frac{1}{T} \sqrt{\frac{2(f(\theta_0) - f_{inf})}{L \sigma^2}} \) and the bound takes the form:

\[
\frac{1}{T} \sum_{t=0}^{T-1} E[\|\nabla f(\theta_t)\|^2] \leq \frac{2 \sigma \sqrt{2L(f(\theta_0) - f_{inf})}}{\sqrt{T}}
\]

Note, that we need \( T \geq \frac{2(f(\theta_0) - f_{inf})}{L \sigma^2} \) to ensure the constraint (8).

For any other choice of \( \alpha_1 \) and \( \alpha_2 \), the third term in the RHS of (11) is positive. Thus,

\[
\frac{2(f(\theta_0) - f_{inf})}{\gamma_1 \alpha_2 T} + \gamma_1 \alpha_2 L \sigma^2 + 2 \gamma L \sigma^2 k_1 \left( (1 - \alpha_1)^2 \alpha_2^2 + 2(1 - \alpha_2)^2 + \frac{4}{3} \omega_1^2 (1 - \alpha_2)^2 k_2^2 \right) > \frac{2(f(\theta_0) - f_{inf})}{\gamma_1 \alpha_2 T} + \gamma_1 \alpha_2 L \sigma^2 \geq |\text{by Cauchy’s inequality}| \geq \frac{2 \sigma \sqrt{2L(f(\theta_0) - f_{inf})}}{\sqrt{T}}
\]

which completes the proof.
Proof of Corollary 7. For any choice of $\gamma$, we get:

$$
\frac{2(f(\theta_0) - f^{inf})}{\gamma \alpha_1 \alpha_2 T} + \gamma \alpha_1 \alpha_2 L \sigma^2 + 2 \gamma^2 L^2 \sigma^2 k_1 \left((1 - \alpha_1)^2 \alpha_2^2 + 2(1 - \alpha_2)^2 + \frac{4}{3} \alpha_1^2 (1 - \alpha_2) \sigma^2 \right) \geq \\
\geq \frac{2(f(\theta_0) - f^{inf})}{\gamma \alpha_1 \alpha_2 T} + \gamma \alpha_1 \alpha_2 L \sigma^2 \geq \text{by Cauchy's inequality} \geq \frac{2 \sigma \sqrt{2L(f(\theta_0) - f^{inf})}}{\sqrt{T}}
$$

so the constant before $\frac{1}{\sqrt{T}}$ indeed cannot be decreased. By substituting the proposed value of $\gamma$, which was found as the equality point of Cauchy's inequality, we get the declared result.

**Theorem 7.** Let us call the run to be an interval between two consecutive restarts. Consider 2-layers Lookahead with restarts defined as follows. For each $m \geq 0$, we start the run $m$ from the fixed initial point $z_0$ and perform $R_m = 2^m$ rounds of 2-layers Lookahead using learning rate $\gamma_m = \frac{\gamma}{\sqrt{m}} = \frac{\gamma}{m}$. As usual, denote by $R$ the total number of rounds and by $T = Rk_2 k_2$ the total number of iterations. Then, we have:

$$
\frac{1}{T} \sum_{t=0}^{T-1} E[\|\nabla f(\theta_t)\|^2] = O\left(\frac{1}{\sqrt{T}}\right), T \to +\infty
$$

**Proof.** Let fix the number of rounds $R$ and consider a number of run $M$ at which round $R$ occurs. Such $M$ satisfies $\sum_{m=0}^{M-1} 4^m < R \leq \sum_{m=0}^M 4^m$. Denote

$$
S_m = \sum_{t'=0}^{R_m k_1 k_2 - 1} E[\|\nabla f(\theta_{t'})\|^2]
$$

where $\{\theta_{t'}\}_{t'=0}^{R_m k_1 k_2 - 1}$ - sequence produced by 2-layers Lookahead, starting from the point $z_0$ and using learning rate $\gamma_m$. Then $\{\theta_{t'}\}_{t'=0}^{R_m k_1 k_2 - 1}$ coincides with the sequence produced by the $m$-th run of 2-layers Lookahead with restarts. By Theorem 2, we have:

$$
\frac{1}{k_1 k_2 R_m} S_m \leq \frac{b_0}{\sqrt{k_1 k_2 R_m}} + \frac{b_1}{k_1 k_2 R_m} = \frac{b_0}{2^m \sqrt{k_1 k_2}} + \frac{b_1}{4^m k_1 k_2}
$$

where $b_0$ and $b_1$ - some constants, defined from (12). Multiplying by $k_1 k_2 R_m = k_1 k_2 4^m$, we get:

$$
S_m \leq c_0 2^m + c_1
$$

where $c_0 = b_0 \sqrt{k_1 k_2}$, $c_1 = b_1$.

Now, using that $R \leq \sum_{m=0}^M 4^m = B$, we can estimate:

$$
\sum_{t=0}^{T-1} E[\|\nabla f(\theta_t)\|^2] = \sum_{t=0}^{R k_1 k_2 - 1} E[\|\nabla f(\theta_t)\|^2] \leq \sum_{t=0}^{B k_1 k_2 - 1} E[\|\nabla f(\theta_t)\|^2] = \sum_{m=0}^M S_m \leq \sum_{m=0}^M (c_0 2^m + c_1) = c_0 (2^{M+1} - 1) + c_1 (M + 1)
$$

Further, we have $R \geq 1 + \sum_{m=0}^{M-1} 4^m = 1 + 4^M - 1 > 4^M$. Thus, $12R > 4^M + 1 > 2^{M+1} < 2\sqrt{3R}$. Also, $R > 4^M > 4^M - 1 \Rightarrow M - 1 < \log_4(R)$. Substituting it in the last expression, we get:

$$
c_0 (2^{M+1} - 1) + c_1 (M + 1) < 2c_0 \sqrt{3R} + c_1 (\log_4(R) + 2)
$$
Finally, we obtain:

\[
\sum_{t=0}^{Rk_1k_2-1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] < 2c_0 \sqrt{3R} + c_1 (\log_4 (R) + 2)
\]

Dividing back by normalization constant, we get:

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] = \frac{1}{Rk_1k_2} \sum_{t=0}^{Rk_1k_2-1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] < \frac{2c_0 \sqrt{3R}}{Rk_1k_2} + \frac{c_1 (\log_4 (R) + 2)}{Rk_1k_2} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)
\]

A.2 Proof of Theorem 3

Denote by \(LA^{(n)}(y_0, \{r, k_n, \ldots, k_1\}, \{\alpha_n, \ldots, \alpha_1\})\) the output of the \(n\)-layers Lookahead with parameters \(k = (k_1, \ldots, k_n)\), \(\alpha = (\alpha_1, \ldots, \alpha_n)\), which starts from the point \(y_0\) and performs \(r\) rounds using SGD as the base optimizer.

Lemma 8. For \(y_r = LA^{(n)}(y_0, \{r, k_n, \ldots, k_1\}, \{\alpha_n, \ldots, \alpha_1\})\), we have:

\[
y_r - y_0 = -\gamma \alpha_n \ldots \alpha_1 k_n \ldots k_1 \sum_{i=0}^{r-1} \nabla f_i^{(k_n \ldots k_1)}(y_0) + \mathcal{O}(\gamma^2) \tag{46}
\]

Proof. Base: \(n = 0\) \((LA^{(n)}\) degenerates to SGD):

\[
y_r - y_0 = \sum_{i=0}^{r-1} (y_{i+1} - y_i) = -\gamma \sum_{i=0}^{r-1} \nabla f_i(y_i) = -\gamma \sum_{i=0}^{r-1} \nabla f_i(y_0) + \mathcal{O}(\gamma^2)
\]

Induction step: suppose the statement holds for \(LA^{(n)}\). Let us prove it for \(y_r = LA^{(n+1)}(y_0, \{r, k_n+1, \ldots, k_1\}, \{\alpha_n+1, \ldots, \alpha_1\})\).

\[
y_r - y_0 = \sum_{i=0}^{r-1} (y_{i+1} - y_i) = \sum_{i=0}^{r-1} (1 - \alpha_{n+1}) y_i + \alpha_{n+1} x_{i, k_{n+1}} - y_i = \sum_{i=0}^{r-1} \alpha_{n+1} (x_{i, k_{n+1}} - y_i) = \alpha_{n+1} \sum_{i=0}^{r-1} (x_{i, k_{n+1}} - x_{i, 0}) \tag{47}
\]

Now, using the induction hypothesis for \(x_{i, k_{n+1}} = LA^{(n)}(x_{i, 0}, \{k_{n+1}, k_n, \ldots, k_1\}, \{\alpha_n, \ldots, \alpha_1\})\), we can continue:

\[
y_r - y_0 = \alpha_{n+1} \sum_{i=0}^{r-1} (\gamma \alpha_n \ldots \alpha_1 k_n \ldots k_1 \sum_{j=0}^{k_{n+1}-1} \nabla f_j^{(k_n \ldots k_1)}(x_{i, 0}) + \mathcal{O}(\gamma^2)) = \]

\[
= -\gamma \alpha_{n+1} \ldots \alpha_1 k_n \ldots k_1 \sum_{i=0}^{r-1} \sum_{j=0}^{k_{n+1}-1} \nabla f_j^{(k_n \ldots k_1)}(y_0) + \mathcal{O}(\gamma^2) = \]

\[
= -\gamma \alpha_{n+1} \ldots \alpha_1 k_{n+1} \ldots k_1 \sum_{i=0}^{r-1} \nabla f_i^{(k_{n+1} \ldots k_1)}(y_0) + \mathcal{O}(\gamma^2) \tag{48}
\]

which finishes the proof.
Lemma 9. For \( y_r = LA^{(n)}(y_0, \{r, k_n, \ldots, k_1\}, \{\alpha_n, \ldots, \alpha_1\}) \), we have:

\[
E[y_r] = y_0 - \gamma \alpha_n \ldots \alpha_1 r k_n \ldots k_1 \nabla f(y_0) + \frac{\gamma^2 \alpha_n^2 \ldots \alpha_1^2 r^2 k_n^2 \ldots k_1^2}{4} \nabla \| \nabla f(y_0) \|^2 + \\
+ \frac{\gamma^2 \alpha_n (1 - \alpha_n) \alpha_{n-1}^2 \ldots \alpha_1^2 k_n^2 \ldots k_1^2}{4} \mathbb{E} \left[ \sum_{i=0}^{r-1} \nabla \| \nabla f_i^{(k_n \ldots k_1)}(y_0) \|^2 \right] + \\
+ \frac{\gamma^2 \alpha_n \alpha_{n-1} (1 - \alpha_{n-1}) \alpha_{n-2}^2 \ldots \alpha_1^2 k_{n-1}^2 \ldots k_1^2}{4} \mathbb{E} \left[ \sum_{i=0}^{r_{k_{n-1}-1}} \nabla \| \nabla f_i^{(k_{n-1} \ldots k_1)}(y_0) \|^2 \right] + \\
+ \ldots + \\
+ \frac{\gamma^2 \alpha_n \ldots \alpha_2 \alpha_1 (1 - \alpha_1) k_1^2}{4} \mathbb{E} \left[ \sum_{i=0}^{r_1} \nabla \| \nabla f_i^{(k_1)}(y_0) \|^2 \right] - \\
- \frac{\gamma^2 \alpha_n \ldots \alpha_1}{4} \mathbb{E} \left[ \sum_{i=0}^{r_{k_1-1}} \nabla \| \nabla f_i^{(k_1)}(y_0) \|^2 \right] + O(\gamma^3) \quad (49)
\]

Proof. Let us prove by induction by the number of layers \( n \).

Base \((n = 0)\). For the corresponding result for SGD, see formula (18) in [Smith et al. 2021].

Induction step \((n \Rightarrow n + 1)\).

\[
y_r - y_0 = \sum_{i=0}^{r-1} (y_{i+1} - y_i) = \sum_{i=0}^{r-1} \left( (1 - \alpha_{n+1}) y_i + \alpha_{n+1} x_{i,k_{n+1}} - y_i \right) = \sum_{i=0}^{r-1} \alpha_{n+1} (x_{i,k_{n+1}} - y_i) = \\
= \alpha_{n+1} \sum_{i=0}^{r-1} (x_{i,k_{n+1}} - x_{i,0}) \quad (50)
\]

Taking the expectation of both sides, we get:

\[
E[y_r] - y_0 = \alpha_{n+1} \sum_{i=0}^{r-1} (E[x_{i,k_{n+1}}] - E[x_{i,0}]) \quad (51)
\]

Recall, one epoch of \( LA^{(n+1)} \) contains \( r \) rounds, and thus \( r \) calls of \( LA^{(n)} \) for updating its inner variable \( x \). For each round \( i \), denote by \( E_n \) the expectation w.r.t. the randomness inside of this round. In other words, we consider the input \( x_{i,0} \) and the set of samples used during \( i \)-th round of \( LA^{(n+1)} \) to be fixed, and the expectation is taken across all possible shuffles of these samples across the mini-batches \( \{f_{j_{k_n \ldots k_1}}\}_{j=0}^{k_{n-1}-1} \).

By the induction hypothesis, applied to \( x_{i,k_{n+1}} = LA^{(n)}(x_{i,0}, \{k_{n+1}, \ldots, k_1\}, \{\alpha_n, \ldots, \alpha_1\}) \), for every \( i \in \{0, \ldots, r - 1\} \) we get:
Combining the sums over $E_0$ in all second-order terms, we can rewrite the last formula in the following way:

\begin{equation}
\begin{aligned}
\mathbb{E}_n [x_{i,k_{n+1}}] - x_{i,0} &= -\gamma \alpha_0 \ldots \alpha_1 k_{n+1} \ldots k_1 \nabla f_i^{(k_{n+1} \ldots k_1)}(x_{i,0}) + \\
&+ \frac{\gamma^2 \alpha_0^2 \ldots \alpha_1^2 k_{n+1}^2 \ldots k_1^2}{4} \nabla \| \nabla f_i^{(k_{n+1} \ldots k_1)}(x_{i,0}) \|^2 + \\
&+ \frac{\gamma^2 \alpha_0 (1 - \alpha_0) \alpha_1^2 k_{n+1}^2 \ldots k_1^2}{4} \mathbb{E}_n \left[ \sum_{j=0}^{k_{n+1}-1} \nabla \| \nabla f_{ik_{n+1}+j}^{(k_{n+1} \ldots k_1)}(x_{i,0}) \|^2 \right] + \\
&+ \frac{\gamma^2 \alpha_0 \alpha_1 (1 - \alpha_0) \alpha_1^2 k_{n+1}^2 \ldots k_1^2}{4} \mathbb{E}_n \left[ \sum_{j=0}^{k_{n+1}-1} \nabla \| \nabla f_{ik_{n+1}+j}^{(k_{n+1} \ldots k_1)}(x_{i,0}) \|^2 \right] + \\
&+ \ldots \ldots + \\
&+ \frac{\gamma^2 \alpha_0 \ldots \alpha_2 \alpha_2 (1 - \alpha_1) k_1^2}{4} \mathbb{E}_n \left[ \sum_{j=0}^{k_{n+1}-1} \nabla \| \nabla f_{ik_{n+1}+j}^{(k_{n+1} \ldots k_1)}(x_{i,0}) \|^2 \right] - \\
&- \frac{\gamma^2 \alpha_0 \ldots \alpha_1}{4} \mathbb{E}_n \left[ \sum_{j=0}^{k_{n+1}-1} \nabla \| \nabla f_{ik_{n+1}+j}^{(k_{n+1} \ldots k_1)}(x_{i,0}) \|^2 \right] + \mathcal{O}(\gamma^3)
\end{aligned}
\end{equation}

(52)

Summing up for $i = 0 \ldots r - 1$, multiplying by $\alpha_{n+1}$, taking the total expectation, and using the law of total expectation ($\mathbb{E}[\mathbb{E}_n[\ldots]] = \mathbb{E}[\ldots]$), we get:

\begin{equation}
\begin{aligned}
\mathbb{E} [y_r] - y_0 &= -\gamma \alpha_0 \ldots \alpha_1 k_{n+1} \ldots k_1 \mathbb{E} \left[ \sum_{i=0}^{r-1} \nabla f_i^{(k_{n+1} \ldots k_1)}(x_{i,0}) \right] + \\
&+ \frac{\gamma^2 \alpha_0 \ldots \alpha_1^2 k_{n+1}^2 \ldots k_1^2}{4} \mathbb{E} \left[ \sum_{i=0}^{r-1} \nabla \| \nabla f_i^{(k_{n+1} \ldots k_1)}(x_{i,0}) \|^2 \right] + \\
&+ \frac{\gamma^2 \alpha_0 \alpha_1 (1 - \alpha_0) \alpha_1^2 k_{n+1}^2 \ldots k_1^2}{4} \mathbb{E} \left[ \sum_{i=0}^{r-1} \sum_{j=0}^{k_{n+1}-1} \nabla \| \nabla f_{ik_{n+1}+j}^{(k_{n+1} \ldots k_1)}(x_{i,0}) \|^2 \right] + \\
&+ \frac{\gamma^2 \alpha_0 \ldots \alpha_2 \alpha_2 (1 - \alpha_1) k_1^2}{4} \mathbb{E} \left[ \sum_{i=0}^{r-1} \sum_{j=0}^{k_{n+1}-1} \nabla \| \nabla f_{ik_{n+1}+j}^{(k_{n+1} \ldots k_1)}(x_{i,0}) \|^2 \right] - \\
&- \frac{\gamma^2 \alpha_0 \ldots \alpha_1}{4} \mathbb{E} \left[ \sum_{i=0}^{r-1} \sum_{j=0}^{k_{n+1}-1} \nabla \| \nabla f_{ik_{n+1}+j}^{(k_{n+1} \ldots k_1)}(x_{i,0}) \|^2 \right] + \mathcal{O}(\gamma^3)
\end{aligned}
\end{equation}

(53)

Combining the sums over $i$ and $j$, and using that $x_{i,0} = y_0 + \mathcal{O}(\gamma)$, so that we can replace the function argument $x_{i,0}$ by $y_0$ in all second-order terms, we can rewrite the last formula in the following way:
\[ E[y_i] - y_0 = -\gamma \alpha_{n+1} \ldots \alpha_1 k_{n+1} \ldots k_1 E \left[ \sum_{i=0}^{r-1} \nabla f^{(k_{n+1} \ldots k_1)}(x_{i,0}) \right] + \]

\[ \frac{\gamma^2 \alpha_{n+1} \alpha_n^2 \ldots \alpha_2 k_{n+1}^2 \ldots k_1^2}{4} E \left[ \sum_{i=0}^{r-1} \nabla \| \nabla f^{(k_{n+1} \ldots k_1)}(y_0) \| \right]^2 + \]

\[ + \frac{\gamma^2 \alpha_{n+1} \alpha_n (1 - \alpha_n) \alpha_2^2 k_{n+1}^2 \ldots k_1^2}{4} E \left[ \sum_{i=0}^{r_{k_{n+1} \ldots k_1} - 1} \nabla \| \nabla f^{(k_{n} \ldots k_1)}(y_0) \| \right]^2 + \]

\[ + \ldots + \]

\[ \frac{\gamma^2 \alpha_{n+1} \ldots \alpha_2 \alpha_1 (1 - \alpha_2) k_1^2}{4} E \left[ \sum_{i=0}^{r_{k_{n+1} \ldots k_2 - 1}} \nabla \| \nabla f^{(k_1)}(y_0) \| \right]^2 - \]

\[ - \frac{\gamma^2 \alpha_{n+1} \ldots \alpha_1}{4} E \left[ \sum_{i=0}^{r_{k_{n+1} \ldots k_1 - 1}} \nabla \| \nabla f_i(y_0) \| \right]^2 + O(\gamma^3) \]  (54)

Now let us transform the term \( E \left[ \sum_{i=0}^{r-1} \nabla f_i^{(k_{n+1} \ldots k_1)}(x_{i,0}) \right] \) to get rid of the dependency on \( x_{i,0} \).

\[ \nabla f_i^{(k_{n+1} \ldots k_1)}(x_{i,0}) = \nabla f_i^{(k_{n+1} \ldots k_1)}(y_i) = \nabla f_i^{(k_{n+1} \ldots k_1)}(y_0) + \nabla^2 f_i^{(k_{n+1} \ldots k_1)}(y_0)(y_i - y_0) + O(\gamma^2) \]  (55)

Substituting \( r = i, n = n + 1 \) in (46) to express \( y_i - y_0 \), we can continue:

\[ \nabla f_i^{(k_{n+1} \ldots k_1)}(x_{i,0}) = \nabla f_i^{(k_{n+1} \ldots k_1)}(y_0) - \]

\[ -\gamma \alpha_{n+1} \ldots \alpha_1 k_{n+1} \ldots k_1 \sum_{j=0}^{i-1} \nabla^2 f_j^{(k_{n+1} \ldots k_1)}(y_0) \nabla f_j^{(k_{n+1} \ldots k_1)}(y_0) + O(\gamma^2) \]  (56)

Summing (56) for \( i = 0 \ldots r - 1 \) and taking the expectation, we get:

\[ E \left[ \sum_{i=0}^{r-1} \nabla f_i^{(k_{n+1} \ldots k_1)}(x_{i,0}) \right] = E \left[ \sum_{i=0}^{r-1} \nabla f_i^{(k_{n+1} \ldots k_1)}(y_0) \right] - \]

\[ -\gamma \alpha_{n+1} \ldots \alpha_1 k_{n+1} \ldots k_1 E \left[ \sum_{i=0}^{r-1} \sum_{j=0}^{i-1} \nabla^2 f_j^{(k_{n+1} \ldots k_1)}(y_0) \nabla f_j^{(k_{n+1} \ldots k_1)}(y_0) \right] + O(\gamma^2) \]  (57)

Let us simplify the right-hand side.

\[ E \left[ \sum_{i=0}^{r-1} \nabla f_i^{(k_{n+1} \ldots k_1)}(y_0) \right] = E[r \nabla f(y_0)] = r \nabla f(y_0) \]  (58)

\[ E \left[ \sum_{i=0}^{r-1} \sum_{j=0}^{i-1} \nabla^2 f_j^{(k_{n+1} \ldots k_1)}(y_0) \nabla f_j^{(k_{n+1} \ldots k_1)}(y_0) \right] = \frac{1}{2} E \left[ \sum_{0 \leq i \leq r} \nabla^2 f_i^{(k_{n+1} \ldots k_1)}(y_0) \nabla f_i^{(k_{n+1} \ldots k_1)}(y_0) \right] = \]

\[ \frac{1}{2} E \left[ \sum_{i=0}^{r-1} \nabla^2 f_i^{(k_{n+1} \ldots k_1)}(y_0) \sum_{j=0}^{r-1} \nabla f_j^{(k_{n+1} \ldots k_1)}(y_0) - \sum_{i=0}^{r-1} \nabla^2 f_i^{(k_{n+1} \ldots k_1)}(y_0) \nabla f_i^{(k_{n+1} \ldots k_1)}(y_0) \right] = \]

\[ \frac{1}{2} E \left[ r^2 \nabla^2 f(y_0) \nabla f(y_0) - \sum_{i=0}^{r-1} \nabla^2 f_i^{(k_{n+1} \ldots k_1)}(y_0) \nabla f_i^{(k_{n+1} \ldots k_1)}(y_0) \right] = \]

\[ \frac{r^2}{4} \nabla \| \nabla f(y_0) \|^2 - \frac{1}{4} E \left[ \sum_{i=0}^{r-1} \nabla \| \nabla f_i^{(k_{n+1} \ldots k_1)}(y_0) \|^2 \right] \]  (59)
Substituting (60) in (54), we get the desired formula for $E$:

$$
E \left[ \sum_{i=0}^{r-1} \nabla f_i(k_{n+1}\ldots k_1) \right] = r \nabla f(y_0) - \frac{\gamma \alpha_{n+1} \ldots \alpha_1 k_{n+1} \ldots k_1}{4} (\nabla \|\nabla f(y_0)\|^2) + 
\frac{\gamma \alpha_{n+1} \ldots \alpha_1 k_{n+1} \ldots k_1}{4} \left[ \sum_{i=0}^{r-1} (\nabla \|\nabla f_i(k_{n+1}\ldots k_1)(y_0)\|^2) \right]
$$

Finally, substituting (60) in (54) and combining similar terms, we get the desired formula for $E[y_r] - y_0$ for $(n + 1)$-layers Lookahead.

**Proof of Theorem 3**

As before, let $m = r k_n \ldots k_1$, where $r$ - number of rounds of $n$-layers Lookahead per epoch, and denote $\beta = \alpha_n \ldots \alpha_1$. Then, we can rewrite the formula (60) in more compact way:

$$
E[y_r] = y_0 - \gamma m \nabla f(y_0) + \frac{\gamma^2 \beta^2 m^2}{4} \nabla \|\nabla f(y_0)\|^2 + 
\sum_{p=1}^{n} \frac{\gamma^2 \alpha_n \ldots \alpha_p (1 - \alpha_p) k_{p-1}^2}{4} \sum_{i=0}^{r k_n \ldots k_{p-1} + 1} (\nabla \|\nabla f_i(k_{p-1}\ldots k_1)(y_0)\|^2) - 
\frac{\gamma^2 \beta}{4} \left[ \sum_{i=0}^{m - 1} (\nabla \|\nabla f_i(y_0)\|^2) \right] + O(\gamma^3)
$$

Let us express $E \left[ \sum_{i=0}^{r k_n \ldots k_{p+1} - 1} \nabla \|\nabla f_i(k_{p+1}\ldots k_1)(y_0)\|^2 \right]$ in terms of $ANG(y_0)$ and $AIG(y_0)$.

$$
E \left[ \sum_{i=0}^{r k_n \ldots k_{p+1} - 1} \nabla \|\nabla f_i(k_{p+1}\ldots k_1)(y_0)\|^2 \right] = r k_n \ldots k_{p+1} E \left[ \nabla \|\nabla f_0(k_{p+1}\ldots k_1)(y_0)\|^2 \right] = 
= r k_n \ldots k_{p+1} \frac{1}{k_p \ldots k_1} E \left[ \nabla \|\nabla f_j(y_0)\|^2 \right] = 
= r k_n \ldots k_{p+1} \frac{1}{k_p \ldots k_1} \left[ \sum_{j=0}^{k_{p-1} - 1} \nabla \|\nabla f_j(y_0)\|^2 + \sum_{0 \leq j \neq j' \leq k_{p+1} - 1} \nabla \langle f_j(y_0), f_{j'}(y_0) \rangle \right] = 
= r k_n \ldots k_{p+1} \frac{1}{k_p \ldots k_1} \left( ANG(y_0) + (k_p \ldots k_1 - 1)AIG(y_0) \right) = 
= r k_n \ldots k_{p+1} \frac{1}{k_p \ldots k_1} \left( ANG(y_0) + (k_p \ldots k_1 - 1)AIG(y_0) \right) = 
= \frac{r^2 \alpha_n \ldots \alpha_p}{m} \left( ANG(y_0) + (k_p \ldots k_1 - 1)AIG(y_0) \right)
$$

Now, combining expression (62) for $E \left[ \sum_{i=0}^{r k_n \ldots k_{p+1} - 1} \nabla \|\nabla f_i(k_{p+1}\ldots k_1)(y_0)\|^2 \right]$ with the coefficient before this term in (61), we obtain:

$$
\frac{\gamma^2 \alpha_n \ldots \alpha_p (1 - \alpha_p) k_{p-1}^2 k_p \ldots k_1^2}{4} \left[ \sum_{i=0}^{r k_n \ldots k_{p+1} - 1} \nabla \|\nabla f_i(k_{p+1}\ldots k_1)(y_0)\|^2 \right] = 
= \frac{\gamma^2 \alpha_n \ldots \alpha_p (1 - \alpha_p) k_{p-1}^2 k_p \ldots k_1^2}{4} \frac{r^2 k_n^2 k_{p+1}^2}{m} \left( ANG(y_0) + (k_p \ldots k_1 - 1)AIG(y_0) \right) = 
= \frac{\gamma^2 \beta m}{4} (1 - \alpha_p) \alpha_p \ldots \alpha_1 \left( ANG(y_0) + (k_p \ldots k_1 - 1)AIG(y_0) \right)
$$

(63)
For the last term in (61), directly by the definition of $\text{ANG}(\gamma)$ we get

$$\frac{\gamma^2 \beta}{4} E \left[ \sum_{t=0}^{m-1} \nabla \| \nabla f_t(y_0) \| ^2 \right] = \frac{\gamma^2 \beta m}{4} \text{ANG}(y_0) \quad (64)$$

Substituting (63) and (64) in (61), we have:

$$E[y_r] = y_0 - \gamma \beta m \nabla f(y_0) + \frac{\gamma^2 \beta^2 m^2}{4} \nabla \| \nabla f(y_0) \| ^2 +$$

$$\frac{\gamma^2 \beta m}{4} \sum_{p=1}^{n} (1 - \alpha_p) \alpha_{p-1} \ldots \alpha_1 (\text{ANG}(y_0) + (k_p \ldots k_1 - 1) \text{AIG}(y_0)) -$$

$$- \frac{\gamma^2 \beta m}{4} \text{ANG}(y_0) + O(\gamma^3) \quad (65)$$

Let us collect the coefficients before $\text{ANG}(y_0)$ and $\text{AIG}(y_0)$.

For $\text{ANG}(y_0)$: $-\frac{\gamma^2 \beta m}{4} + \frac{\gamma^2 \beta m}{4} \sum_{p=1}^{n} (1 - \alpha_p) \alpha_{p-1} \ldots \alpha_1 = -\frac{\gamma^2 \beta m}{4} + \frac{\gamma^2 \beta m}{4} \sum_{p=1}^{n} (\alpha_{p-1} \ldots \alpha_1 - \alpha_p \ldots \alpha_1) = -\frac{\gamma^2 \beta m}{4} + \frac{\gamma^2 \beta m}{4} (1 - \beta) = -\frac{\gamma^2 \beta m}{4} + \frac{\gamma^2 \beta m}{4} - \frac{\gamma^2 \beta^2 m}{4} = -\frac{\gamma^2 \beta m}{4}$.

For $\text{AIG}(y_0)$: $\frac{\gamma^2 \beta m}{4} \sum_{p=1}^{n} (1 - \alpha_p) \alpha_{p-1} \ldots \alpha_1 (k_p \ldots k_1 - 1)$.

Hence, (65) can be simplified in the following way:

$$E[y_r] = y_0 - \gamma \beta m \nabla f(y_0) + \frac{\gamma^2 \beta^2 m^2}{4} \nabla \| \nabla f(y_0) \| ^2 -$$

$$- \frac{\gamma^2 \beta^2 m}{4} \text{ANG}(y_0) + \frac{\gamma^2 \beta m}{4} \sum_{p=1}^{n} (1 - \alpha_p) \alpha_{p-1} \ldots \alpha_1 (k_p \ldots k_1 - 1) \text{AIG}(y_0) + O(\gamma^3) \quad (66)$$

Now, substituting $\varepsilon = \gamma \beta m$ in (19) (we instantly get $h(y) = -\nabla f(y)$ from equating the first order terms, so it was directly substituted in the formula below), we obtain that continuous solution of the ODE for the modified flow satisfies:

$$y(\gamma \beta m) = y_0 - \gamma \beta m \nabla f(y_0) + \gamma^2 \beta^2 m^2 \left( h_1(y_0) + \frac{1}{4} \nabla \| \nabla f(y_0) \| ^2 \right) + O(\gamma^3) \quad (67)$$

Equating the right-hand sides of (66) and (67), we can find $h_1(y)$:

$$h_1(y) = -\frac{1}{4m} \text{ANG}(y) + \frac{1}{4 \beta m} \sum_{p=1}^{n} (1 - \alpha_p) \alpha_{p-1} \ldots \alpha_1 (k_p \ldots k_1 - 1) \text{AIG}(y) \quad (68)$$

Thus, we recover the formula of the modified flow for $n$-layers Lookahead:

$$\tilde{h}_{LA-n}(y) = -\nabla f(y) + \gamma \beta m h_1(y) + O(\gamma^2) =$$

$$= -\nabla f(y) - \frac{\gamma \beta}{4} \text{ANG}(y) + \frac{\gamma}{4} \sum_{p=1}^{n} (1 - \alpha_p) \alpha_{p-1} \ldots \alpha_1 (k_p \ldots k_1 - 1) \text{AIG}(y) + O(\gamma^2) \quad (69)$$

Finally, from $\tilde{h}_{LA-n}(y) = -\nabla \tilde{f}_{LA-n}(y)$, we get:

$$\tilde{f}_{LA-n}(y) = f(y) + \frac{\gamma \beta}{4} \text{AN}(y) - \frac{\gamma}{4} \sum_{p=1}^{n} (1 - \alpha_p) \alpha_{p-1} \ldots \alpha_1 (k_p \ldots k_1 - 1) \text{AI}(y) + O(\gamma^2) \quad (70)$$

and unrolling $\beta = \alpha_n \ldots \alpha_1$, we get the desired result.
A.3 Multilayer Lookahead Preserves Linear Convergence Rate

In this section, we show that if the inner optimizer has a linear convergence rate (either in terms of distance to the optimal point or in terms of the function value), then Lookahead (with any number of layers) preserves linear convergence rate.

Recall the update rule of the Lookahead:

\[
\begin{align*}
  x_{t,i+1} &= x_{t,i} - \gamma g_{t,i}, \quad i = 0, \ldots, k - 1 \\
  y_{t+1} &= (1 - \alpha) y_t + \alpha x_{t,k} \quad (71)
\end{align*}
\]

In the following, we denote by \( f \) the function to be minimized, \( x^* \) its global minimum, and \( f^* \) its minimum value.

Claim 2. Suppose that the inner optimizer of Lookahead has a linear convergence rate in terms of the distance to the optimal point; that is, it gives a sequence of iterations \((x_t, t \geq 0)\), such that \( \|x_{t+1} - x^*\| \leq c \|x_t - x^*\|, \quad c < 1 \). Then Lookahead also has a linear convergence rate.

Proof. From the linear convergence speed of the inner optimizer, we have:

\[
\|x_{t,k} - x^*\| \leq c^k \|x_{t,0} - x^*\| = c^k \|y_t - x^*\|
\]

Hence,

\[
\|y_{t+1} - x^*\| \leq \|(1 - \alpha) y_t + \alpha x_{t,k} - x^*\| \leq (1 - \alpha) \|y_t - x^*\| + \alpha \|x_{t,k} - x^*\| \\
\leq (1 - \alpha) \|y_t - x^*\| + \alpha c^k \|y_t - x^*\| = (1 - \alpha + \alpha c^k) \|y_t - x^*\|
\]

Thus, we get:

\[
\|y_{t+1} - x^*\| \leq (1 - \alpha(1 - c^k)) \|y_t - x^*\|
\]

and \( (1 - \alpha(1 - c^k)) < 1 \) since \( c^k < 1 \).

Claim 3. Let \( f \) be a convex function and assume that the inner optimizer of Lookahead has a linear rate of convergence in terms of function value; that is, it gives a sequence of iterations \((x_t, t \geq 0)\), such that \( f(x_{t+1}) - f^* \leq c(f(x_t) - f(x^*)) \), \( c < 1 \). Then Lookahead also has a linear convergence rate.

Proof. From the linear convergence speed of the inner optimizer, we have:

\[
f(x_{t,k}) - f^* \leq c^k (f(x_{t,0}) - f^*) = c^k (f(y_t) - f^*)
\]

Hence,

\[
f(y_{t+1}) - f^* \leq f((1 - \alpha) y_t + \alpha x_{t,k}) - f^* \leq |\text{using convexity}| \leq (1 - \alpha) f(y_t) + \alpha f(x_{t,k}) - f^* = \\
= (1 - \alpha)(f(y_t) - f^*) + \alpha(f(x_{t,k}) - f^*) \leq (1 - \alpha + \alpha c^k)(f(y_t) - f^*)
\]

Thus, we get:

\[
f(y_{t+1}) - f^* \leq (1 - \alpha(1 - c^k))(f(y_t) - f^*)
\]

and \( (1 - \alpha(1 - c^k)) < 1 \) since \( c^k < 1 \).
Remark 1. Both claims remain valid when the inner optimizer is stochastic and the linear convergence holds in expectation. The proof for this case is identical, thus omitted.

Remark 2. Since we can consider Multilayer Lookahead with $n$ layers as a 1-layer Lookahead with $n - 1$ layers Lookahead as the inner optimizer, we can extend the result of both claims to the Multilayer Lookahead by induction over the number of layers. However, with each new layer, the constant of convergence degrades.

Remark 3. As an application of Claim [3], we get a linear convergence of Lookahead with gradient descent as the inner optimizer for smooth and strongly convex functions.

Both lemmas prove linear convergence of Lookahead with the same constant $1 - \alpha(1 - c^{k})$ (but in different senses). Since $c^{k} < 1$, the constant of convergence decreases as alpha increases, so it achieves the minimum value for $\alpha = 1$, that is when Lookahead degenerates to its base optimizer. We should have expected this because we made only minimal assumptions on the loss function and the inner optimizer.

B ADDITIONAL EXPERIMENTS

B.1 Additional details on classification on CIFAR-10 and CIFAR-100

Here, we present additional details for the Section 5.

B.2 Training GANs on MNIST

In this section, we present results of training GANs for digits generation on MNIST dataset. We chose Deep Convolutional GANs (DCGANs) model (Radford et al. 2015), where both Discriminator and Generator entirely consist of Convolutional (Transposed Convolutional) layers. You can find the architectures for Generator and Discriminator in Figure 4. The Generator takes a noise vector of $128 \times 1 \times 1$ size from the normal distribution as an input. We used the non-saturating loss function for GANs (Goodfellow et al. 2014) and Adam optimizer (Kingma & Ba 2014), which also served as the inner optimizer for Multilayer Lookahead. We fixed Adam parameters to be $\beta_{1} = 0.5$, $\beta_{2} = 0.999$, and tried learning rate in $\{0.0002, 0.001\}$. Such choice of parameters (with learning rate 0.0002) was suggested by (Radford et al. 2015) as the default choice for DCGANs. However, we expected the optimal learning rate for Multilayer Lookahead to be larger. We used the same number of steps $k = 5$ and synchronization parameter $\alpha = 0.5$ for each layer of the Multilayer Lookahead, and tried number of layers from 1 to 6.

For each optimizer, we trained our models on 100 epochs using batch size = 50. We found the learning rate $= 0.0002$ to be better for Adam and Multilayer Lookahead with 3 layers, and the learning rate $= 0.001$ to be better for the number of layers from 4 to 6. For comparing the models, we used the Inception Score metric (Salimans et al. 2016). In the report, we do not use another popular metric - Frechet Inception Distance, because we found this metric to be unstable for MNIST. Finally, we compared the optimizers with tuned learning rate in Figure 5. Our preliminary analysis shows that Multilayer Lookahead can significantly improve both Adam and Lookahead.
Figure 2: Additional Training Plots for CIFAR-10 Classification
Figure 3: Additional Training Plots for CIFAR-100 Classification

(a) Train Loss
(b) Test Loss
(c) Train Accuracy

Figure 4: Models Used for GANs

(a) Discriminator Architecture
(b) Generator Architecture
Figure 5: Comparing Adam and Multilayer Lookahead with Adam as base optimizer on training GANs on MNIST. For each optimizer, the best learning rate from \{0.0002, 0.001\} is chosen.