Fractals and Scars on a Compact Octagon

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A finite universe naturally supports chaotic classical motion. An ordered fractal emerges from the chaotic dynamics which we characterize in full for a compact 2-dimensional octagon. In the classical to quantum transition, the underlying fractal can persist in the form of scars, ridges of enhanced amplitude in the semiclassical wave function. Although the scarring is weak on the octagon, we suggest possible subtle implications of fractals and scars in a finite universe.

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Chaos is often avoided in fundamental physics and cosmology. A recent surge of interest in a finite universe has highlighted a well known chaotic system, a compact negatively curved space [1]. Most efforts in cosmology have struggled to circumvent the inherent chaos. Instead of avoiding chaos, we investigate the structure of the chaotic flows on a compact space. Since we are able to deduce analytically many of the chaotic properties, we concentrate on the compact 2d octagon and study the classical motion and the emergent fractal properties. Poincaré realized that the periodic orbits, though seemingly special, define the entire structure of a dynamical system. With the advent of chaos, periodic orbits grow exponentially in number as a function of their length. The invariant set of periodic trajectories packs itself into a fractal to accommodate the dense proliferation of orbits. We derive a very precise counting of the periodic orbits, determine the topological entropy, derive a recurrence relation for all of the fixed points, and analytically compute a spectrum of fractal dimensions. We then numerically confirm our analytic results by generating the fixed points and measuring the dimensions of the set.

The fractals may not be effaced in the classical to quantum transition. A phenomenon known as scarring of the quantum wave function has been witnessed in systems which are classically chaotic [2]. The scars refer to ridges along which the amplitude of the semiclassical wave function is enhanced. We argue that the scars can be interpreted as the smooth remnant of the underlying classical fractal. We relate the closed orbits on the 2d space to the occurrence of scars. The compact octagon is so thoroughly chaotically mixed that the scarring is minimal, consistent with the fractal properties as discussed. In a finite model of the universe, the faint scars could seed a weblike distribution of matter threaded across the space.

Geodesic motion on a compact 2d pseudosphere is completely chaotic. On the Poincaré unit disc the metric can be written

\[ ds^2 = -d\eta^2 + 4(1 - r^2)^{-2} \left( dr^2 + r^2 d\phi^2 \right). \]  

(1)

Geodesics are semi-circles which are orthogonal to the boundary at \( r = 1 \). We consider null geodesics for which the coordinates \((r, \phi)\) have the conjugate momenta

\[ (\Pi_r, \Pi_\phi) = 4(1 - r^2)^{-2}(\dot{r}, \dot{r}^2 \dot{\phi}). \]  

(2)

There are two conserved quantities. Since \( \Pi_\phi = 0 \), there is a conserved angular momentum

\[ L = 4(1 - r^2)^{-2} \dot{r}^2 \dot{\phi}. \]  

(3)

and the kinetic energy is conserved

\[ \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) 4(1 - r^2)^{-2} = \frac{1}{2}. \]  

(4)

Condition [3] is the same as the restriction that photons travel at unit speed.

We consider the simplest compact surface of negative curvature with genus 2. The fundamental domain is a symmetric regular octagon with the opposite edges identified in pairs as in fig. 1. The identifications are effected by a discrete subgroup of the Lorentz group. The horizontal identifying boost can be represented in coordinates \( x = r \cos \phi \) and \( y = r \sin \phi \) as

\[ g_0 = \begin{pmatrix} \cosh \tau_1/2 & \sinh \tau_1/2 \\ \sinh \tau_1/2 & \cosh \tau_1/2 \end{pmatrix} \]  

(5)

with \( \cosh \tau_1/2 = 1 + \sqrt{2} \), while the other four are related by a rotation of \( g_0 \) through the angle \( j \pi/4 \). The four generators are related to their respective inverses by a rotation through \( \pi \). We denote all the generators and their inverses compactly as \( g_j \) with \( j = -4, \ldots, 3 \) so that the \( g_0 g_{-4} g_{3} g_{-1} = g_{2} g_{-2} = g_{1} g_{-3} = 1 \). These generators obey the one relation

\[ g_0 g_{-3} g_{2} g_{-1} = g_{3} g_{-3} g_{1} g_{-2} g_{3} = 1. \]  

(6)

A geodesic is uniquely specified by two parameters \((\theta, L)\) where \( \theta \) is the angular coordinate of the geodesic as it intersects the boundary in the past. Balasz and Voros [3] developed a reentry map which takes initial values for \((\theta, L)\) through the appropriate edge and back into the fundamental domain. The map which iterates all values through edge 0 of fig. 1 is

\[ \tan(\theta_{n+1}/2) = e^{-\tau_1} \tan(\theta_n/2) \]  

(7)

\[ L_{n+1} = (\cosh \tau_1 + \cos \theta_n \sinh \tau_1)L_n - \sinh \tau_1 \sin \theta_n. \]

The reentry map for orbits exiting any of the 8 edges is effected by shifting \( \theta \rightarrow \theta + j \pi/4 \) for \( j = -4, \ldots, 3 \). The map is Hamiltonian and area preserving.

A trajectory is a point in the plane \((\theta, L)\) and those trajectories lying within the intersection of the two darkened separating curves of fig. 3 exit edge 0. A separating curve is

\[ L_0 = \frac{\sin(\theta - \pi/8)}{\cos(\theta - \pi/8) - \coth \tau_2} \]  

(8)
A word cannot begin and end on inverse letters: a repeated \(n\)-letter word of the form \(g_IM_j\) is equivalent to the repeated \((n-1)\)-letter word \(g_j...g_k\). An \(n\)-letter word can be built from every \(O(n-1)\) by attaching any of the letters in the alphabet except the one which is inverse to the end letter. So, there are \(7O(n-1)\) of these. More \(n\)-letter words can be built from every \(A(n-1)\) by attaching any of the letters in the alphabet except for the two letters which are either the inverse of the end letter or the inverse of the beginning letter. So, there are \(6B(n-1)\) of these. Although \((n-1)\)-letter words of the form \(g_i...g_i^{-1}\) do not contribute to \(N(n-1)\), they do contribute to \(N(n)\) for \(n \geq 3\) by adding any letter other than \(g_i\) or \(g_i^{-1}\). The number of words of order \((n-1)\) that begin and end on inverse letters is

\[
I(n-1) = \sum_{m=2}^{n-2} A(m).
\]

It follows that

\[
N(n) = 7O(n-1) + 6B(n-1) + 6I(n-1). \tag{11}
\]

Eqn. (11) can be simplified by noting that \(O(n) = N(n-1)\). We can combine these relations to reexpress \(I(n)\) for \(n \geq 3\) as

\[
N(n) = 7N(n-2) + 6N(n-1) - 6N(1). \tag{12}
\]

There is additional pruning at higher orders due to the one relation eqn. (10). However, this pruning is extremely sparse and affects the counting very little. The number in (10) is actually slightly smaller than \(N(n) = 7^n\) but tends toward this behaviour as \(n \to \infty\). To see this, let \(N \sim 6^n\) in the limit of large \(n\) and solve for \(b\) to obtain \(N \to 7^n\). The topological entropy is then,

\[
H_T = \lim_{n \to \infty} \frac{1}{n} \ln N = \ln 7.
\]

There is a spectrum of weighted dimensions which measures the distribution of frequencies and the distribution of the points in space. The spectrum is defined by covering the set with boxes of size \(\epsilon\) on a side and taking the limit as as \(\epsilon \to 0\),

\[
D_q = \frac{1}{q-1} \lim_{\epsilon \to 0} \frac{\ln \sum_{i=1}^N N(e)^q p_i}{\ln \epsilon}. \tag{13}
\]

with \(p_i\) the weight assigned to the \(i\)th box. If all closed orbits are equally weighted then \(p_i = 1/N(e)\) and the fractal is self-affine with the \(D_q\) equal. For a multifractal, some of the orbits are more popular than others and at least some of the \(D_q\) are unequal.

Following the reasoning of ref. [4] for the Baker’s map, we can use the areas of fig. 2 to analytically determine the spectrum of dimensions for the map. A given exit section of area \(A_i\) overlaps with the reentry strip into \(7\) overlap sections of area \(A_j\). The map then takes these \(7\) areas and lays them down in nearly vertical strips preserving the

\[
N(n) = O(n) + \Delta(n). \tag{9}
\]
area of each. The natural measure on the vertical strip is then the relative area \( f_j = A_j / A \). The vertical strips will have nearly the same height and differ only in relative width, so the relative width is also given by \( f_j \). There is a simple scaling symmetry: if any of the vertical strips were to be multiplied by the width of that strip, the features of the entire set are reproduced. This kind of reproduction on smaller and smaller scales is precisely what makes the set fractal. We can exploit the scaling to deduce the fractal dimension \( D \). In this vertical approximation the dimension is fractal. We can exploit the scaling to deduce the fractal dimension \( D \) [7]. In this vertical approximation the fractal is self-similar and fills an area. It implies \( \theta_\ast = -4\pi \) and \( L = 0 \). These correspond to the 4 short period orbits and their inverses shown in fig. [6].

The fixed points at any order will satisfy an equation of the form \( y = (Ay + B)/(Cy + D) \) and the periodic orbits are solutions to a quadratic equation. The fixed points are found by using the following recurrence relations:

\[
A^{(n)} = A^{(n-1)} + ab_{j(n)}^{-1} C^{(n-1)} \\
B^{(n)} = B^{(n-1)} + ab_{j(n)}^{-1} D^{(n-1)} \\
C^{(n)} = C^{(n-1)} + ab_{j(n)} A^{(n-1)} \\
D^{(n)} = D^{(n-1)} + ab_{j(n)} B^{(n-1)} .
\]

At order \( n = 5 \), there are \( \sum_{n=1}^{n=5} N(n) = 19,624 \) periodic points exactly as predicted by eqn. (12). These are shown in fig. [6].

Numerically we count the number of boxes needed to cover the set as \( \epsilon \to 0 \). We measure the box-counting dimension as 2, as predicted. To find the weighted dimensions we let \( p_i \) equal the number of fixed points in a box and it is important to note that we count each time a given fixed point occurs, so short orbits are more heavily weighted. We find the information dimension is \( D_1 = 1.4 \pm 0.1 \). The weight assigned to each box is sensitive to the number of windings around the space and therefore measures a different topological property than eqn. (10).

In the transition to quantum mechanics, remnants of chaos have been found in the form of smooth scars of enhanced amplitude along the classical periodic orbits [2]. We argue that the scars are the residue of the most frequently encountered orbits in the fractal set.

The semiclassical wave function can be constructed as a superposition of modes \( \Psi(x) = \sum_k \phi_k \psi_k(x) \) with \( \psi_k \) eigenvectors of the Laplace-Beltrami operator, \( \nabla^2 \psi_k = -k^2 \psi_k \). The intuition that the quantum system blurs
any chaotic features derives from two important conjectures. The first conjecture asserts that the eigenvalues are given by random matrix theory for systems which are classically completely chaotic. As a consequence, the amplitudes \( \hat{\phi}_g \) are drawn from a Gaussian random ensemble with a flat spectrum and should be thoroughly mixed and featureless. The second conjecture, due to Berry, implies that the eigenmodes will be concentrated in the regions of phase space which are visited by the classical trajectories given an infinite time. A completely chaotic region of phase space is covered by mixing trajectories and at first glance Berry’s conjecture seems to imply a uniform distribution.

While these arguments for uniformity are convincing, one might wonder what happened to the structure of the periodic orbits. It is no surprise that the periodic orbits, which play such a prominent role in the classical dynamics, should also play a defining role in the quantum system. Periodic loops have in fact been the backbone of a quantization prescription in quantum chaos \( \mathbb{G} \). Scars are the first evidence of quantum features related to these closed orbits. Heller discovered scars in the evolution of a wave-packet moving along a short orbit in a chaotic stadium \( \mathbb{G} \). He deduced that only certain eigenstates would contribute to the wave function as it traversed the orbit. Since the wave-packet was peaked about a single periodic orbit, the few contributing eigenstates must also show enhanced intensity along that orbit. The fewer the number of eigenstates which contribute to the wave-packet, the larger the intensity of the scar along that loop.

Rather than being inconsistent with Berry’s conjecture, the scar can be interpreted as a consequence of it. Scars occur in the regions of space where classical trajectories spend the most time. A typical trajectory will near a periodic orbit and then, due to the instability, will veer away onto a different periodic orbit. In this way, typical paths can be built from segments of periodic orbits. The most frequently visited periodic orbits tend to scatter the most substantially. For a fully chaotic system, the fractal set of closed orbits will fill a volume in phase space. This would lead to some uniformity in the distribution of classical orbits and hence their quantum counterparts, as Berry suggested. However, not all closed orbits are created equal. Some may be visited by typical aperiodic trajectories more frequently than others.

The scarring phenomenon has been investigated primarily in terms of the dynamics of a wave-packet for test particle motion. Since the periodic orbits are unstable, a particle will depart from that trajectory by a factor of \( \exp(-\lambda t) \) where \( \lambda \) is the positive Lyapunov exponent. The frequency \( \omega \) with which the wave-packet traces out the path must be sufficiently short, \( \lambda/\omega \leq 1 \), if the orbit is to be susceptible to scarring. For paths of length \( \ell \), the frequency is \( \omega = 2\pi/\ell \). A hyperbolic surface is specified by \( \lambda = 1 \) in curvature units and the scarring condition is simply \( \ell \leq 2\pi \). Only the shortest orbits scar. The short orbits are also the ones visited most frequently.
 filamentary and web-like.

Our own 3d universe does show vast voids and huge filaments in the distribution of large-scale structure. Galaxies would trace the scars and chart the shortest loops around a finite universe. If our universe is finite and negatively curved, the cobweb of luminous matter might be a residue of primordial quantum scars.

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