In this paper, we propose a secure multiparty protocol for the feature selection problem. Let \( D \) be the set of data, \( F \) the set of features, and \( C \) the set of classes, where the feature value \( x(F_i) \) and the class \( x(C) \) are given for each \( x \in D \) and \( F_i \in F \). For a triple \((D, F, C)\), the feature selection problem is to find a consistent and minimal subset \( F' \subseteq F \), where ‘consistent’ means that, for any \( x, y \in D, x(F_i) = y(F_i) \) for all \( F_i \in F' \), and ‘minimal’ means that any proper subset of \( F' \) is no longer consistent. The feature selection problem corresponds to finding a succinct description of \( D \), and has various applications in the field of artificial intelligence. In this study, we extend this problem to privacy-preserving computation model for multiple users. We propose the first algorithm for the privacy-preserving feature selection problem based on the fully homomorphic encryption. When parties \( A \) and \( B \) possess their own personal data \( D_A \) and \( D_B \), they jointly compute the feature selection problem for the entire data set \( D_A \cup D_B \) without revealing their privacy under the semi-honest model.

### 1 Introduction

#### 1.1 Motivation and related works

Feature selection is one of the typical problems in machine learning. For example, the human genome consists of 3.1 billion base pairs, of which at most a few dozen pairs are said to affect a particular disease. Feature selection extracts a set of features from this very sparse data that match a specific purpose, and the results are used by various machine learning algorithms. We will review the definition of the feature selection problem, its computational complexity, and the approximate solutions that have been proposed so far.

Feature selection is defined by a data set \( D \), a feature set \( F \), and a class set \( C \). Here, in particular, we consider feature \( F_i \in F \) and class \( C \) to be binary, but it is easy to extend the problem to multi-label. Thus, a feature \( F_i \) and a class for a data \( x \in D \) are denoted by \( x(F_i), x(C) \in \{0, 1\} \), and thus, when \( |F| = k \), the data \( x \) is associated with a binary vector of length \( k + 1 \).

Given a triple \((D, F, C)\), the goal of an algorithm is to extract a consistent and minimal \( F' \subseteq F \), where \( F' \) is consistent if, for any \( x, y \in D, x(F_i) = y(F_i) \) for all \( F_i \in F' \) implies \( x(C) = y(C) \) and a feature set \( F' \subseteq F \) is minimal if any proper subset of \( F' \) is no longer consistent.
To our knowledge, the most common method for finding features that characterize $C$ is to select features that show higher relevance in some statistical measure. The relevance of individual features can be estimated using statistical measures such as mutual information and Bayesian risk. For example, at the bottom row of Table 1 the mutual information score $I(F_1, C)$ of each feature $F_1$ to class labels is described. We can see that $F_1$ is more relevant than $F_5$, since $I(F_1, C) > I(F_5, C)$. Based on the mutual information score, $F_1$ and $F_2$ of Table 1 will be selected to explain $C$. However, looking into $D$ more closely, we understand that $F_1$ and $F_2$ cannot determine $C$ uniquely. In fact, we find $x_2$ and $x_5$ with $x_2(F_1) = x_5(F_1)$ and $x_2(F_2) = x_5(F_2)$ whose class labels are different. On the other hand, we can also find the fact that $F_4$ and $F_5$ uniquely determine $C$ by the formula $C = F_4 \oplus F_5$ while $I(F_4, C) = I(F_5, C) = 0$ holds. Therefore, the traditional method based on relevance scores of individual features misses the right answer.

This problem is well known as the problem of interacting features, which has been intensively studied in machine learning research. The literature describes a class of feature selection algorithms that can solve this problem, referred to as consistency-based feature selection [2][11][14][16][17]. CWC (Combination of Weakest Components) [16] is the most simplest one of these consistency-based feature selection algorithms. CWC is the simplest of such consistency-based feature selection algorithms, and even though CWC uses the most rigorous measure, it shows one of the best performances in terms of accuracy as well as computational speed compared to other methods [15].

### 1.2 Our contribution

We extend the feature selection problem to multi-users having their own private datasets and propose the first secure multi-party protocol to jointly compute the feature selection over the entire data without revealing their private information.

Our proposed method is a two-party protocol based on the fully homomorphic encryption. Given a public-key cryptosystem, let $E[m]$ be an integer $m$ encrypted with its public key; if $E[m + n]$ (the ciphertext of $m + n$) can be computed from $E[m]$ and $E[n]$ only with public information, in particular without decrypting $E[m]$ and $E[n]$, then $E$ is said to be additive homomorphic, and if $E[mn]$ can also be computed from $E[m]$ and $E[n]$ in addition, then $E$ is said to be fully homomorphic. Besides, when any plaintext $m$ can encrypt into any element of a set consisting of sufficiently many ciphertexts and for each execution of encryption, such a ciphertext is chosen probabilistically, $E$ is said to be probabilistic. For a cryptosystem, being probabilistic is required to satisfy the security of ciphertext indistinguishability: given $x_1, x_2$ and $E[x_i]$ when $i$ is secretly selected one of 1 and 2 uniformly at random, it is computationally impossible to guess $i$.

In the last two decades, various homomorphic encryptions have been proposed that satisfy those homomorphic properties. The first (probabilistic) additive homomorphic encryption was proposed by Paillier [12]. Somewhat homomorphic encryption that allows a sufficient number of additions and a restricted number of multiplications have also been proposed [3][5][6], and by using these cryptosystems, we can compute more difficult problems, such as the inner product.
of two vectors. The first fully homomorphic encryption with unlimited number of additions and multiplications was proposed by Gentry [9], and since then, useful libraries for the fully homomorphic encryption have been developed especially for bitwise operations and floating point operations.

TFHE [7][8] is known as a fastest fully homomorphic encryption specialized for bitwise operations. In this study, we use TFHE to design our algorithm for the multi-party protocols of feature selection problems. We assume that parties A and B have their own private data \( D_A \) and \( D_B \), and \( F = \{ F_1, \ldots, F_m \} \) and \( C = \{ 0, 1 \} \) are known. Under the assumption that the parties can use their respective TFHE, say \( E_A \) and \( E_B \), the goal of the parties is to jointly compute the result of CWC algorithm on the plain data \( D = D_A \cup D_B \), without revealing any other information about \( D_A \) and \( D_B \).

We summarize the results of this work in Table 2. baseline is a naive algorithm that simulates the original CWC [16] over the ciphertexts using TFHE operations. Given \( D, F, C \), the essential task of CWC is to sort \( F = \{ F_1, \ldots, F_k \} \) in an increasing order of their relevance to \( C \). Using the sorted \( F_i \in F \), CWC decides whether or not \( F_i \) should be selected for \( i = 1, \ldots, k \). The resulting features \( \{ F_{i_1}, \ldots, F_{i_t} \} \subseteq F \) are the output of CWC.

It is well-known that sorting, the main task in CWC, is one of most difficult problems in secure computation. So we propose an improvement of the baseline algorithm reducing the cost of sorting. We show the time and space complexities for both algorithms in Table 2. This significantly improved the time complexity while maintaining the space complexity. We also implemented the baseline algorithm and examined its running time for real data. As a result, it was confirmed that most of the time was spent on sorting. The implementation of an improved algorithm is a future work.

2 Preliminaries

2.1 CWC algorithm over plaintext

For the dataset \( D \) associated with \( F \) and \( C \), we generally assume that \( D \) contains no error, i.e., if \( x(F_i) = y(F_i) \) for all \( i, x(C) = y(C) \). When \( D \) contains such errors, these are removed beforehand, then as a result, \( D \) contains at most one \( x \in D \) with the same feature values.

We describe the original algorithm for finding a minimal consistent features in Algorithm 1. Given \( D \) with \( F_i \) and \( C = \{ 0, 1 \} \), a data \( x \in D \) of \( x(C) = 1 \) is called a positive data and \( y \in D \) of \( y(C) = 0 \) is called a negative data. Let \( n \) be the number of positive data and \( m = |D| - n \). Let \( F_p \) be the \( p \)-th positive data (\( 1 \leq p \leq n \)) and \( F_q \) the \( q \)-th negative data (\( 1 \leq q \leq m \)). Then, the bit string \( B_i \) of length \( nm \) is defined by: \( B_i[m(p-1)+q] = 0 \) if \( x_p(F_i) = x_q(F_i) \) and \( B_i[m(p-1)+q] = 1 \) otherwise. \( B_i[m(p-1)+q] = 0 \) means that \( F_i \) is not consistent with the pair \( (x_p, x_q) \) because \( x_p(F_i) = x_q(F_i) \) despite \( x_p(C) = x_q(C) \). Recall that \( F_i \) is said to be consistent only if \( x(F_i) = y(F_i) \) implies \( x(C) = y(C) \) for any \( x, y \in D \). Thus, \( |B_i| \) is defined to be the number of 1s in \( B_i \).

For a subset \( F' \subseteq F, F' \) is said to be consistent, if for any \( p \in [1, n] \) and \( q \in [1, m] \), there exists \( i \) such that \( F_i \in F' \) and \( B_i[m(p-1)+q] = 1 \) hold. Using this, CWC removes irrelevant features from \( F \) to construct a minimal consistent feature set\[1\].

Algorithm 1 The algorithm CWC for plaintext

1: Input: A dataset \( D \) associated with features \( F = \{ F_1, \ldots, F_k \} \) and class \( C = \{ 0, 1 \} \).
2: Output: A minimal consistent subset \( S \subseteq F \).
3: Sort \( F_1, \ldots, F_k \) in the incremental order of \( |B_i| \).
4: Let \( \pi \) be the sorted indices of \( \{ 1, \ldots, k \} \).
5: for \( i = 1, \ldots, k \) do
6: \hspace{1cm} if \( F \setminus \{ F_{\pi[i]} \} \) is consistent then
7: \hspace{2cm} update \( F \leftarrow F \setminus \{ F_{\pi[i]} \} \)
8: \hspace{1cm} end if
9: end for

In Table 3 we show an example of \( D \) and the corresponding \( B_i \). Let us consider the behavior of CWC on this example. All \( B_i (1 \leq i \leq 4) \) are computed as preprocessing. Then, the features are sorted by the order \( |B_2| = 5 \leq |B_4| = 5 \leq |B_3| = 6 \leq |B_1| = 8 \) and \( \pi = (2, 4, 3, 1) \). By the consistency order \( \pi \), CWC checks whether \( F_{\pi[4]} \) can be removed from the current \( F \). By the consistency measure, CWC removes \( F_2 \) and \( F_3 \) and the resulting \( \{ F_1, F_3 \} \) is the output. In fact, we can predict the class of \( x \) by the logical operation \( \bar{x}(F_1) \land x(F_3) \).

\[1\] Finding a smallest consistent feature set is clearly NP-hard due to an obvious reduction from the minimum set cover.
Table 3: An example dataset $D$ with $F = \{F_1, F_2, F_3, F_4\}$ and $C = \{0, 1\}$. For $n = 2$ and $m = 5$, the bit string $B_i$ is defined by the value of $x_p(F_i) = x_q(F_i)$. For example, $B_1 = (1, 1, 0, 1, 1, 1, 0, 1, 1)$ because $x_p(F_1) = x_q(F_1)$ only for the two pairs $(p, q) = (1, 4), (4, 5)$. Similarly, $B_2 = (1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0), B_3 = (0, 1, 1, 0, 1, 1, 1, 0, 1), B_4 = (0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1).

| $x_i \in D$ | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $C$ |
|-----------|-------|-------|-------|-------|-----|
| $x_1$     | 0     | 1     | 1     | 0     | 1   |
| $x_2$     | 1     | 0     | 1     | 0     | 0   |
| $x_3$     | 1     | 1     | 0     | 0     | 0   |
| $x_4$     | 0     | 1     | 0     | 1     | 0   |
| $x_5$     | 0     | 0     | 1     | 1     | 1   |
| $x_6$     | 1     | 0     | 1     | 0     | 0   |
| $x_7$     | 1     | 1     | 0     | 0     | 0   |

2.2 TFHE: a faster fully homomorphic encryption

For the privacy-preserving CWC, $D$ is entirely encrypted by a fully homomorphic encryption. We review the TFHE \([8]\), one of the fastest libraries allowing the bitwise operations of ciphertexts using $+$ or $\oplus$ and bitwise multiplication ($\cdot$). On TFHE, any integer is encrypted bitwise: For $\ell$-bit integer $m = (m_1, \ldots, m_\ell)$, we denote its bitwise encryption by $E[m] = (E[m_1], \ldots, E[m_\ell])$, for short. These operations are denoted by $E[x] \oplus E[y] \equiv E[x \oplus y]$ and $E[x] \cdot E[y] \equiv E[x \cdot y]$ for $x, y \in \{0, 1\}$ and the ciphertexts $E[x]$ and $E[y]$. An encrypted array is denoted in the same way. For example, when $x$ and $y$ are integers of length $\ell$ and $\ell'$, respectively, we abbreviate the bitwise encryption of a sequence of integers, e.g., $E(x, y)$ denotes the following ciphertext.

\[
E(x, y) = (E[x], E[y]) = ((E[x_1], \ldots, E[x_\ell]), (E[y_1], \ldots, E[y_\ell]))
\]

By the elementary operations $E[x] \oplus E[y]$ and $E[x] \cdot E[y]$, TFHE allows all arithmetic and logical operations. Here, we describe how to construct the adder and comparison operations. Let $x, y$ be $\ell$-bit integers and $x_i, y_i$ be the $i$-th bit of $x, y$ respectively. Let $c_i$ be the $i$-th carry-in bit and $s_i$ is the $i$-th bit of the sum $x + y$. Then, we can get $E[x + y]$ by the bitwise operations of ciphertexts using $s_i = x_i \oplus y_i \oplus c_i$ and $c_{i+1} = (x_i \oplus c_i) \cdot (y_i \oplus c_i) \oplus c_i$. Based on the adder, we can construct other operations like subtraction, multiplication, and division. For example, $E[x - y]$ is obtained by $E[x + (-y)]$, where $(-y)$ is the bit complement of $y$ obtained by $y_i \oplus 1$ for all $i$-th bit. On the other hand, we also review the comparison. We want to get $E[x < y]$ without decrypting $x$ and $y$ where $x < y = 1$ if $x < y$ and $x < y = 0$ otherwise. Here, we can get the bit $x < y$ as the most significant bit of $x + (-y)$ over ciphertexts. Similarly, we can compute the encrypted bit $E[x = y]$ for the equality test.

Adopting those operations of TFHE, we design a secure multi-party CWC. In this paper, we omit the details of TFHE (see e.g., \([7, 8]\)).

3 Algorithms

3.1 Baseline algorithm

We propose our baseline algorithm, which is a privacy-preserving version of CWC. In this subsection, we consider a two-party protocol, where a party $A$ has its private data and outsources the computation of CWC to another party $B$, but our baseline algorithm is easily extended to a multi-party protocol where more than two parties cooperate one another to select features with the joint data. During the computation, party $B$ should not gain other information than the number $n$ of positive data, the number $m$ of negative data and the number $k$ of features. Note that party $A$ can conceal the actual number of data by inserting dummy data and telling the inflated numbers $n$ and $m$ to $B$. The algorithm can distinguish dummy data by adding an extra bit indicating the data is a dummy if the bit is 1. For each class the values of features and dummy bit of data in the class are encrypted by public key of $A$ and sent to $B$.

The algorithm consists of three steps: Computing encrypted bit string $E(B_i)$, sorting $E(B_i)$'s and executing feature selection on $E(B_i)$'s.

Since all data in this subsection is encrypted by public key of $A$, we omit the description of the encryption function $E$ to simplify presentation.
3.1.1 Computing \( B_i \)

We can compute \( B_i[m(n-1)+q] \) by \( (x_p(F_1) \oplus x_q(F_1)) \vee x_p(d) \vee x_q(d) \), where \( x(d) \) represents the dummy bit for data \( x \). \( (x_p(F_1) \oplus x_q(F_1)) \) becomes 0 iff \( F_1 \) is inconsistent for the pair of \( x_p \) and \( x_q \). The part “\( \vee x_p(d) \vee x_q(d) \)” is added to make the whole value 1 when one of the data is a dummy. It takes \( O(kmn) \) time and space in total.

3.1.2 Sorting \( B \)’s

We can compute \( |B_i| \) in encrypted form by summing up values in \( B_i \) in \( O(mn \log(mn)) \) time (noting that each operation on integers of \( \log(mn) \) bits takes \( O(\log(mn)) \) time). Instead we can set an upper bound \( b_{\text{max}} \) of the bits used to store consistency measure to reduce the time complexity to \( O(mnb_{\text{max}}) \).

Then sorting \( B \)’s in the incremental order of consistency measures can be done using any sorting network in which comparison and swap are conducted in encrypted form without leaking the information about ordering of features. Note that, in this approach, the algorithm has to spend \( \Theta(mn + \log k) \) time to swap (or pretend to swap) two bit strings and original feature indices of \( \log k \) bits regardless that two features are actually swapped or not. Since this is the heaviest part in our baseline algorithm, we will show how to improve it. Using AKS sorting network \([1]\) of size \( O(k \log k) \), the total time for sorting \( B \)’s is \( O(mnb_{\text{max}} + (mn + b_{\text{max}} + \log k)k \log k) \).

In our experiments we use a more practical sorting network of Batcher’s odd-even mergesort \([4]\) of size \( O(k \log^2 k) \). Recently, a simple oblivious radix sort \([10]\) in \( O(k \log k) \) algorithm under the assumption that the bit length of each integer is constant.

3.1.3 Selecting features

Let \((F_{\pi(1)}, \ldots, F_{\pi(k)})\) be the sorted list of features. We first compute a sequence of bit strings \((Z_2, \ldots, Z_k)\) of length \( mn \) each such that \( Z_i[h] = \bigvee_{j=1}^{k} B_{\pi(j)}[hi] \) for any \( 2 \leq i \leq k \) and \( 1 \leq h \leq mn \), namely \( Z_i \) is the bit array storing cumulative or of each position \( h \) for \( B_{\pi(1)}, B_{\pi(i+1)}, \ldots, B_{\pi(k)} \). The computation takes \( O(kmn) \) time and space.

For feature selection, we simulate Algorithm \([1]\) on encrypted \( B \)’s and \( Z \)’s. In addition we use two 0-initialized bit arrays, \( R \) of length \( k \) and \( S \) of length \( mn \). \( R[i] \) is meant to store 1 if the \( i \)-th feature (in sorted order) is selected. \( S \) is used to keep track of the cumulative or for the bit strings of the currently selected features. Namely, \( S[h] \) is set to \( \bigvee_{\alpha=1}^{\ell} B_{\pi(j)}[hi] \) if \( \ell \) features \( \{F_{\pi(j_1)}, \ldots, F_{\pi(j_\ell)}\} \) have been selected at the moment.

Suppose that we are in the \( i \)-th iteration of the for loop of Algorithm \([1]\). Note that \( F \setminus \{F_{\pi(i)}\} \) is consistent iff \( \bigwedge_{h=1}^{mn} (Z_{i+1}[hi] \vee S[h]) \) is 1. Since we keep the \( i \)-th feature iff \( F \setminus \{F_{\pi(i)}\} \) is inconsistent, the algorithm sets \( R[i] = \neg \bigwedge_{h=1}^{mn} (Z_{i+1}[hi] \vee S[h]) \). After computing \( R[i] \), we can correctly update \( S \) by \( S[h] \leftarrow S[h] \lor (R[i] \land B_{\pi(i)}[hi]) \) for every \( 1 \leq h \leq mn \) in \( O(mn) \) time.

Since each feature is processed in \( O(mn) \) time, the total computational time is \( O(kmn) \).

3.1.4 Summing up analysis

The bottleneck of computational time is \( O(mnb_{\text{max}} + (mn + b_{\text{max}} + \log k)k \log k) \) of the sorting step. Since CWC works with any consistent measure, we do not have to use \( |B_i| \) in full accuracy, and thus, we assume that \( b_{\text{max}} \) is set to be a constant. Under the assumption, we obtain the following theorem.

**Theorem 1** We can securely simulate CWC in \( O(kmn \log k + k \log^2 k) \) time and \( O(kmn) \) space without revealing the private data of the parties under the assumption that TFHE is secure.

**Proof.** According to the discussion above, computing \( B_i \) for all features takes \( O(kmn) \) time and space, sorting features takes \( O(mnb_{\text{max}} + (mn + b_{\text{max}} + \log k)k \log k) \), and selecting features takes \( O(kmn) \) time. Finally, party \( B \) computes in \( O(k \log k) \) time an integer array \( P \) with \( P[h] = R[h] \cdot \pi[h] \), which stores the original indices of selected features. Party \( B \) randomly shuffles \( P \) and sends to party \( A \) as the result of CWC. Therefore, we can securely simulate CWC in \( O(kmn \log k + k \log^2 k) \) time and \( O(kmn) \) space. \( \square \)

3.2 Improvement of secure CWC

The task of sorting is a major bottleneck for CWC in secret computation presented above. The reason is that pointers cannot be moved over ciphertexts. For example, consider the case of secure integer sort. Let the variables \( x \) and \( y \) contain integers \( a \) and \( b \), respectively. Here, by performing the secure operation \( a < b \), the result is obtained as
Theorem 2 CWC. Finally, we obtain the following result.

Using the mix network, we propose the improved secure CWC (Algorithm 3) reducing the time complexity to \(O(nm)\) from \(O(mnk \log^2 k)\) in Algorithm 2. We can assume that we cannot know how they were shuffled by comparing the encrypted values. An example run of Algorithm 3 is illustrated in Fig. 1. As shown in this example, the party \(A\) sends the sorted \(F_i\) with random noise \(r_i\) to the mix network. Since the improved algorithm uses the mix network mechanism \([13]\) as a subroutine, we first give a brief overview of the mix network.

The purpose of a mix network is, given an encrypted sequence \((E[x_i], E[y_i])\), to obtain a random permutation \(\pi(E[x], E[y])\), where \(E[x]\) and \(E[y]\) are re-encrypted and shuffled. Recall that \(E\) is a probabilistic encryption. Thus, we cannot know how they were shuffled by comparing \(\pi(E[x], E[y])\) and the original \((E[x], E[y])\). Among two parties \(A\) and \(B\), the mix network can be realized using the public key encryption of \(A\) and \(B\). We show such a mix network in Algorithm 2. We can assume that \(A\) cannot know any information about the permutation without decryption.

Using the mix network, we propose the improved secure CWC (Algorithm 3) reducing the time complexity to \(O(mnk + k \log^2 k)\). An example run of Algorithm 3 is illustrated in Fig. 1. As shown in this example, the party \(A\) can securely sort \(k\) randomized features in \(O(k \log k)\) time and then swap each associated bit string of length \(mn\) in \(O(kmn)\) time. After this preprocessing, the parties obtain a minimal consistent features decrypting the output of CWC. Finally, we obtain the following result.

Theorem 2 Algorithm 3 can securely simulate CWC in \(O(kmn + k \log^2 k)\) time and \(O(kmn)\) space without revealing the private data of the parties under the assumption that TFHE is secure.

Proof. The party \(B\) shuffles \(F\) by a permutation \(\pi(F_1, \ldots, F_k) = (F_{i_1}, \ldots, F_{i_k})\). The parties communicate only in the step 5 and 6. \(B\) can decrypt any \(E_B[F_{i_1} + r_1]\), but due to the added noise, he cannot know anything about \(F_i\). On the other hand, \(A\) obtains the plaintext \(\pi[F_i]\), but he cannot compute \(\pi^{-1}\). Thus, the parties cannot get the rank of the original feature \(F_i\) from each party’s information alone. Therefore, the protocol of Algorithm 3 is as secure as TFHE. On the other hand, the time and space complexities are clear because the algorithm moves \(B_i\) of length \(mn\) at most \(O(k)\) times. It follows that the time complexity is reduced to \(O(kmn + k \log^2 k)\).

4 Experiments

We implemented our baseline algorithm in C++ using TFHE library for bitwise operations on fully homomorphic encryption. The experiments were conducted on the machine with Intel Core i7-6567U (3.30GHz) and 16GB RAM. In the following, \(m\) (resp. \(n\)) is the number of positive (resp. negative) data and \(k\) is the number of features.
Privacy-Preserving Multiparty Protocol for Feature Selection Problem

Figure 1: Improved secure CWC. (1): Parties $A$ and $B$ jointly compute $B_i$ and $||B_i||$ for each feature $F_i$. (2) and (3): $A$ obtains a shuffled data by the mix network between $A$ and $B$. (4): $A$ sorts $F_i$ by $||B_i||$. (5): $B$ receives the rank of $F_i$ with a random noise $r_i$ and returns their permutation to $A$. (6): $A$ moves the bit string $B_i$ according to the information from $B$. (7): $A$ and $B$ jointly compute the selected features by $\pi^{-1}$ and selected features over the renamed space.

Table 4 shows the time for computing $B_i$ in three different sizes of $mn$. As the theoretical time bound $O(mn)$ suggests, the time linearly increases to the size of $mn$. We note that $B_i$ can be computed independently from other $B_j$ with $i \neq j$, and thus, they can be computed in parallel.

Table 5 shows the time for computing $||B_i||$ while changing the size of $mn$ and upper bound $b_{\text{max}}$ of bits to store consistency measures. Since there are $O(mn)$ additions to a $b_{\text{max}}$-bits integer, the time complexity is $O(mnb_{\text{max}})$. We can observe that the time per addition linearly increases to $b_{\text{max}}$. Note that the computation of $||B_i||$ for all features can be conducted in parallel.

Table 6 shows the time for swapping a pair of data in the sorting procedure. Since the theoretical time complexity is $O(mn + b_{\text{max}} + \lceil \log k \rceil)$, the time is mostly dominated by $mn$.

Since the whole procedure of sorting takes a long time, we estimate it from the time for a single swap in Table 6. Table 7 shows the estimated total time for sorting $B_i$'s with OEM sort. Here we assume that all the swaps are conducted in serial (without utilizing parallelism of sorting network).

Table 8 shows the time for feature selection from sorted list of $B_i$'s. The results follow the theoretical time complexity $O(kmn)$.

Table 4: Time for computing $B_i$'s

| $mn$ | time [sec] |
|------|-------------|
| 100  | 6.04        |
| 500  | 30.06       |
| 1000 | 60.14       |

Table 5: Time for computing $||B_i||$

| $mn$ | $b_{\text{max}}$ | time [sec] | time per addition [sec] |
|------|------------------|-------------|------------------------|
| 100  | 7                | 27.624      | 0.28                   |
| 500  | 9                | 175.826     | 0.35                   |
| 1000 | 10               | 394.967     | 0.40                   |
Table 6: Time for swapping a pair of data

| $k$  | $mn$ | $b_{\max}$ | $[\log k]$ | time [sec] |
|------|------|------------|------------|-----------|
| 10   | 100  | 7          | 4          | 8.88      |
| 10   | 500  | 9          | 4          | 39.59     |
| 10   | 1000 | 10         | 4          | 78.05     |
| 50   | 100  | 7          | 6          | 9.05      |
| 50   | 500  | 9          | 6          | 39.73     |
| 50   | 1000 | 10         | 6          | 78.04     |
| 100  | 100  | 7          | 7          | 9.10      |
| 100  | 500  | 9          | 7          | 39.93     |
| 100  | 1000 | 10         | 7          | 77.94     |

Table 7: Estimated total time for sorting B’s with OEM sort.

| $k$  | $mn$ | # swaps | time [sec] |
|------|------|---------|-----------|
| 10   | 100  | 63      | 559.57    |
| 10   | 500  | 63      | 2494.23   |
| 10   | 1000 | 63      | 4917.40   |
| 50   | 100  | 543     | 4911.44   |
| 50   | 500  | 543     | 21573.39  |
| 50   | 1000 | 543     | 42376.26  |
| 100  | 100  | 1471    | 13386.10  |
| 100  | 500  | 1471    | 58732.62  |
| 100  | 1000 | 1471    | 114646.80 |

Table 8: Time for feature selection from sorted list of B_i’s

| $k$  | $mn$ | time [sec] |
|------|------|-----------|
| 10   | 100  | 111.96    |
| 10   | 500  | 558.17    |
| 10   | 1000 | 1114.24   |
| 50   | 100  | 589.35    |
| 50   | 500  | 2941.07   |
| 50   | 1000 | 5919.71   |
| 100  | 100  | 1179.06   |
| 100  | 500  | 5952.54   |
| 100  | 1000 | 11867.00  |

Table 9 summarizes the time for each step of our baseline algorithm under the assumption that the parallelism is not used. The table shows that the sorting part is the bottleneck.

As we can see from the experimental results (e.g. Table 9), most of the computational time of the baseline algorithm is spent on sorting. Thus, the implementation of the improved secure CWC is an important future work. Although we have implemented a two-party protocol, our algorithm including the improved secure CWC can be easily extended to general multiparty protocols.

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Table 9: Time for each step. Step 1: Computing $B_i$’s. Step 2: Sorting $B_i$’s. Step 3: feature selection.

| $k$  | $mn$ | Step 1 [sec] | Step 2 [sec] | Step 3 [sec] |
|------|------|--------------|--------------|--------------|
| 10   | 100  | 60.37        | 835.81       | 111.96       |
| 10   | 500  | 300.59       | 4252.49      | 558.17       |
| 10   | 1000 | 601.41       | 8867.07      | 1114.24      |
| 50   | 100  | 301.85       | 6292.64      | 589.35       |
| 50   | 500  | 1502.95      | 30364.69     | 2941.07      |
| 50   | 1000 | 3007.05      | 62124.61     | 5919.71      |
| 100  | 100  | 603.70       | 16148.50     | 1179.06      |
| 100  | 500  | 3005.90      | 76315.22     | 5952.54      |
| 100  | 1000 | 6014.10      | 154143.50    | 11867.00     |

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