NONTRIVIAL CLASSES IN $H^*(\text{Imb}(S^1,R^n))$ FROM NONTRIVALENT GRAPH COCYCLES

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Abstract. We construct nontrivial cohomology classes of the space $\text{Imb}(S^1,R^n)$ of imbeddings of the circle into $R^n$, by means of Feynman diagrams. More precisely, starting from a suitable linear combination of nontrivalent diagrams, we construct, for every even number $n \geq 4$, a de Rham cohomology class on $\text{Imb}(S^1,R^n)$. We prove nontriviality of these classes by evaluation on the dual cycles.

1. Introduction

In the recent years Quantum Field Theories have provided useful tools for addressing problems in Algebraic Topology. An example is the computation of the real cohomology of the spaces $\text{Imb}(S^1,R^n)$ of imbeddings of the circle into $R^n$ for $n \geq 3$. The works of Bar-Natan [1] and Kontsevich [7], have shown how the perturbative expansion of Chern-Simons Quantum Field Theory yields the so-called “Vassiliev invariants” of the space of ordinary knots. The precise statement is that there exists a complex of (trivalent) Feynman diagrams whose cohomology classes give rise to 0-cohomology classes of the space $\text{Imb}(S^1,R^3)$.

Chern-Simons is a Topological Quantum Field Theory which can be defined in three dimensions only and produces Feynman diagrams whose vertices have valence equal to three. There exist however other Quantum Field Theories that, while sharing the same topological properties, are defined over manifolds of any dimension and produce Feynman diagrams not necessarily trivalent. These Topological Quantum Field Theories are known as “BF Theories” and their properties are studied in details in [5] and [6].

In [4] it is shown how the perturbative expansion of BF Theories can define a complex of Feynman diagrams and a chain map into the de Rham complex of $\text{Imb}(S^1,R^n)$, for any $n > 3$. Moreover this map gives rise to an injective map in cohomology, when restricted to trivalent diagrams. The cohomology of this complex of Feynman diagrams is called “graph cohomology”.

A different approach to the problem of computing the cohomology of $\text{Imb}(S^1,R^n)$ has been considered by V. Vassiliev in [11]: there exists in fact a spectral sequence converging to the cohomology of $\text{Imb}(S^1,R^n)$ for every $n \geq 4$, whose $E_1$ term coincides with the graph cohomology (see also [10]). Thus, Vassiliev spectral sequence gives an “upper bound” to the cohomology of $\text{Imb}(S^1,R^n)$, since nontrivial cohomology classes of $\text{Imb}(S^1,R^n)$ must come from some element in $E_1$, while [4] gives a “lower bound”, since at least those elements corresponding to trivalent diagrams give rise to nontrivial elements in cohomology. In general it is unclear whether the collapsing always happens at the $E_1$ term, as conjectured by Kontsevich [7], or there are some “graph cocycles” which do not give rise to classes in $H^*(\text{Imb}(S^1,R^n))$. 
In this note we want to extend the results of [4] in the following direction:

**Theorem 1.** For every even \( n \geq 4 \), there exists a nontrivial class in \( H^{(n-3)/3+1}(\text{Imb}(S^1, \mathbb{R}^n)) \) associated to a nontrivalent graph cocycle.

A similar question has been posed by R. Bott [3] regarding the possibility of constructing 1-dimensional cocycles on the space \( \text{Imb}(S^1, \mathbb{R}^3) \) of ordinary knots. In this case, however, it is not even established the convergence of the spectral sequence. We also mention the work of D. Sinha [8] on the cohomology of the spaces of imbeddings of the real line into \( \mathbb{R}^n \). Using Goodwillie’s calculus, one can define a spectral sequence whose \( E_1 \)-term, as in Vassiliev’s case, is isomorphic to (a variant of) the graph cohomology. Recently P. Lambrechts and I. Volic have announced a proof of the collapse of Sinha spectral sequence at the \( E_1 \) term.

Here is the plan of the paper. We will first recall the definition of graph cohomology (Sections 2 and 3), and the construction of the map from graph cohomology to the de Rham cohomology of \( \text{Imb}(S^1, \mathbb{R}^n) \) (Section 4). Then, in Section 5 we will prove Theorem 1 by considering a nontrivial graph cocycle consisting of nontrivalent diagrams and mapping it to \( H^*(\text{Imb}(S^1, \mathbb{R}^n)) \). Nontriviality of this class will follow from an explicit evaluation on the dual cycle.

### 2. Graph cohomology

We first briefly recall some definitions given in [4]. By \( D^{k,m}_o \) and \( D^{k,m}_e \) we mean the real vector spaces generated by decorated diagrams of order \( k \) and degree \( m \), of odd and even type, respectively. The diagrams consist of an oriented circle and many edges joining vertices which may lie either on the circle (external vertices) or off the circle (internal vertices). We require all the vertices to be at least trivalent. If we denote by \( e \) the number of edges of a diagram, \( v_i \) the number of its internal vertices and \( v_e \) the number of its external vertices, then the order \( k \) of a diagram is \( e - v_i \) (minus the Euler characteristic) while the degree \( m \) is \( 2e - 3v_i - v_e \) (the deviation from being a trivalent diagram). We also define a chord diagram to be a decorated diagram whose vertices are all external.

The decoration in \( D^{k,m}_o \) is given by numbering the vertices (up to even cyclic permutations), numbering the internal vertices (up to even permutations) and orienting the edges (up to reversal). By convention, we number external vertices from 1 to \( v_i \) and internal vertices from \( v_i + 1 \) to \( v_i + v_e \). An extra decoration is needed on the edges connecting the same external vertex, namely an ordering of the two half-edges forming them. The decoration in \( D^{k,m}_e \) is given by numbering the external vertices (up to even cyclic permutations) and numbering the edges (up to even permutations). Finally, both in \( D^{k,m}_o \) and \( D^{k,m}_e \) we quotient by the subspace generated by all diagrams containing two edges joining the same pair of vertices and diagrams containing edges whose end-points are the same internal vertex.

The coboundary operators \( \delta_o : D^{k,m}_o \to D^{k,m+1}_o \) and \( \delta_e : D^{k,m}_e \to D^{k,m+1}_e \) are linear operators whose action on a diagram \( \Gamma \) is given by the signed sum of all the diagrams obtained from \( \Gamma \) by contracting, one at a time, all the regular edges and arcs of the diagrams. Here by arc we mean a piece of the oriented circle between two consecutive vertices, and by regular edge we mean an edge with at least one internal end-point. The signs are as follows: if we contract an arc or edge connecting the vertex \( i \) with the vertex \( j \), and oriented from \( i \) to \( j \), then the sign is \((-1)^j\) if
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If we contract an edge labelled by $\alpha$, then the sign is $(-1)^{\alpha+1+v_e}$, where $v_e$ is the number of external vertices of the diagram.

These complexes are called graph complexes, and the relevant cohomology groups $H^{k,m}(D_o)$ and $H^{k,m}(D_e)$ are know as graph cohomology. When we write $(D^{k,m}, \delta)$ (resp. $H^{k,m}(D)$), we mean either the odd or the even-type graph complex (resp. graph cohomology).

3. Graph homology

Graph homology is given by the dual vector space $(D^{k,m})^*$ and the boundary operator $\partial: (D^{k,m})^* \rightarrow (D^{k,m-1})^*$ defined as the adjoint to $\delta$. The homology groups are denoted by $H_{k,m}(D)$. Notice that since there is a preferred basis of $D^{k,m}$ given by the diagrams, we can identify $(D^{k,m})^*$ with $D^{k,m}$, and consider $\partial$ as an operator on $D^{k,m}$ that decollapses the vertices of the diagrams in all possible ways.

For instance, the boundary of a trivalent diagram is always zero, while the boundary of a diagram $\Gamma$ with all trivalent vertices except a quadrivalent external vertex is the sum of three trivalent diagrams as in figure 1 (odd case) and 2 (even case).

**Figure 1.** Boundary of a quadrivalent external vertex (odd case)

We claim that all classes of $H_{k,0}(D)$ can be represented by chord diagrams only. More precisely, $H_{k,0}(D_o)$ is isomorphic to the quotient of the space of chord diagrams $CD^{k,0}_o$ by the subspace generated by the diagrams of figure 3 and figure 4. Similarly, $H_{k,0}(D_e)$ is isomorphic to the quotient of the space of chord diagrams $CD^{k,0}_e$ by the subspace generated by the diagrams of figure 5 and figure 6. The proof of this fact in the odd case is given in [1], Thm. 6. The even case is completely analogous. The subspaces of figure 3 and figure 5 are know as the 4$T$ relations, while the subspaces of figure 4 and figure 6 are the 1$T$ relations.

Notice that the 4$T$ relations are the boundary of the sum (with proper sign) of two of the three nontrivalent diagrams considered in figure 4 and 5.
Figure 2. Boundary of a quadrivalent external vertex (even case)

Figure 3. 4T relations of odd type

Figure 4. 1T relation of odd type
We now want to recall the definition and main properties of the configuration space integrals \[\text{[2, 4]}\].

First, for any compact smooth manifold \(M\), we define the open configuration space \(C^0_q(M)\) of \(q\) points in \(M\) to be the space of the \(q\)-uples \((x_1, \ldots, x_q)\) \(\in M^q\) such that \(x_i \neq x_j\) for \(i \neq j\). Then, we consider the Axerlod–Singer–Fulton–MacPherson compactification \(C_q(M)\) of \(C^0_q(M)\), given by blowing-up all the diagonals \[\text{[9]}\]. These \(C_q(M)\) are smooth compact manifolds with corners. The space \(C_q(\mathbb{R}^n)\) is defined as the submanifold of \(C_{q+1}(\mathbb{S}^n)\) where the \((q + 1)\)st point is fixed to be the point \(\infty \in \mathbb{S}^n\).

If \(\gamma\) is an imbedding of \(S^1\) into \(\mathbb{R}^n\), then one can force some of the points in the configuration space to lay on \(\gamma\), thus obtaining the configuration space \(C^0_{r,s}(\mathbb{R}^n, \gamma)\) of \(r\) points on \(\gamma \subset \mathbb{R}^n\) and \(s\) points in \(\mathbb{R}^n\). We denote by \(C_{r,s}(\mathbb{R}^n, \gamma)\) its compactification. Putting all these spaces together, one gets a fiber bundle \(p: C_{r,s}(\mathbb{R}^n) \to \text{Imb}(S^1, \mathbb{R}^n)\), whose fibers are compact.

Next, one considers the maps
\[
\phi_{ij}: C^0_{r,s}(\mathbb{R}^n) \to S^{n-1} \quad (x_1, \ldots, x_{r+s}) \mapsto \frac{x_i - x_j}{|x_i - x_j|}
\]
and observes that they extend smoothly to maps \(\phi_{ij}: C_{r,s}(\mathbb{R}^n) \to S^{n-1}\). Let \(\omega^{n-1}\) be a symmetric top form on \(S^{n-1}\) which we assume to be concentrated around some fixed direction. Here “symmetric” means that \(\omega^{n-1}\) satisfies \(\alpha^* \omega^{n-1} = (-1)^n \omega^{n-1}\), where \(\alpha\) is the antipodal map on \(S^{n-1}\). The pull-back via \(\phi_{ij}\) of \(\omega^{n-1}\) is a smooth
(n−1)-form θij on Cr,s(Rn) called tautological form. Forms θi are defined as the pull-back of ω−1 via the map

\[ C_{r,s} \xrightarrow{\pi} C_{r,0} = C_r \times \text{Imb}(S^1, \mathbb{R}^n) \xrightarrow{pr \times id} S^1 \times \text{Imb}(S^1, \mathbb{R}^n) \xrightarrow{D} S^{n-1} \]

where \( C_r \) is a component of the compactified configuration space of \( r \) points on \( S^1 \), \( \pi \) forgets the \( s \) points outside the knot, \( pr_i \) is the projection on the \( i \)th point and \( D \) is the normalized derivative \( D(t, \psi) = \dot{\psi}(t)/|\dot{\psi}(t)| \).

We are now ready to define the maps \( I^{k,m} \). We consider a diagram \( \Gamma \in D_{k,m} \) and number all its edges and vertices (if they are not already numbered by their decoration). Then to the edge between the vertices \( i \) and \( j \) we associate the tautological form \( \theta_{ij} \), and take the wedge product over all the edges of \( \Gamma \). Finally we integrate (push-forward) this product along the map \( p: C_{r,s}(\mathbb{R}^n) \to \text{Imb}(S^1, \mathbb{R}^n) \), where \( r \) is the number of internal vertices of \( \Gamma \) and \( s \) the number of external vertices. Extending this map by linearity we obtain

\[ I^{k,m}: D_{k,m} \to \Omega^{(n-3)k+m}(\text{Imb}(S^1, \mathbb{R}^n)) \]

As shown in [4] these are cochain maps for every \( n > 3 \) and they induce injective maps in cohomology for \( m = 0 \).

One can wonder whether there exists an analogous construction for the dual theory, namely, whether one can define chain maps

\[ (D_{k,m}, \partial) \to (C_{(n-3)k+m}(\text{Imb}(S^1, \mathbb{R}^n)), \partial) \]

or, at least, maps in homology

\[ H_{k,m}(D) \to H_{(n-3)k+m}(\text{Imb}(S^1, \mathbb{R}^n)). \]

An answer can be given for the case \( m = 0 \) by associating to every trivalent chord diagram \( \Gamma \in \mathcal{CD}^{k,0} \) a cycle of \( \text{Imb}(S^1, \mathbb{R}^n) \), denoted by \( i_k(\Gamma) \), constructed as follows. Consider first an “imbedding” \( \psi_{\Gamma} \) of the diagram \( \Gamma \) in \( \mathbb{R}^n \), i.e., an immersion of the oriented circle of \( \Gamma \) into \( \mathbb{R}^n \) whose only singularities are transversal double points, and such that \( \psi_{\Gamma}(t_i) = \psi_{\Gamma}(t_j) \) if and only if \( t_i \) and \( t_j \) are the end-point of a chord in \( \Gamma \). We assume that the indices \( i \) and \( j \) are the same indices of the chord in \( \Gamma \) which has been contracted to form the singular point \( \psi_{\Gamma}(t_i) = \psi_{\Gamma}(t_j) \). Next, for every pair of points \( t_i, t_j \in S^1 \) (say, \( i < j \)) such that \( \psi_{\Gamma}(t_i) = \psi_{\Gamma}(t_j) \) and for every \( z \in S^{n-3} \), we consider the following loop in \( \mathbb{R}^n \):

\[ a^{i,j}(z)(t) = \begin{cases} 
0 & \text{if } t \notin [t_i - \epsilon, t_i + \epsilon], \\
\epsilon \delta \exp \left(1/[(t - t_i)^2 - \epsilon^2]\right) & \text{if } t \in [t_i - \epsilon, t_i + \epsilon], 
\end{cases} \]

with \( \epsilon, \delta > 0 \). By adding all the loops \( a^{i,j}(z) \) to the immersion \( \psi_{\Gamma} \) in correspondence with all the double points, we remove (blow-up) all the singularities and obtain a family of imbeddings which is parameterized by \( k \) points on \( S^{n-3} \), where \( k \) is the number of chords of \( \Gamma \). We look at this family of imbeddings as a \((n-3)k\)-cycle in \( \text{Imb}(S^1, \mathbb{R}^n) \). (We remark that the main result of [4] is that the cycle of imbeddings obtained from the chord diagram \( \Gamma \) is dual to the de Rham cocycle obtained from a trivalent graph cocycle containing \( \Gamma \), provided this graph cocycle exists). Finally we extend these maps \( i_k \) by linearity and we have:

**Proposition 2.** The maps

\[ i_k: \mathcal{CD}^{k,0} \to H_{(n-3)k}(\text{Imb}(S^1, \mathbb{R}^n)) \]
defined above, descend to maps on $H_{k,0}(D)$.

Proof. Since $H_{k,0}(D)$ is isomorphic to $CD^{k,0}$ modulo the $4T$ and $1T$ relations, we have to check that the linear combination of the diagrams of figure $3$, $4$, $5$, and $6$ are sent to trivial cycles.

Consider in fact the four cycles of imbeddings corresponding to the diagrams of a $4T$ relation. These cycles (or better, their images) are equal everywhere except near a small ball in $\mathbb{R}^n$, where they are described as follows: two strands of the imbedded circle meet in a double point $b$ (which is then blown-up) and a third strand is blown-up around one of the four (half) strands merging in $b$. Denote by $i_k(4T)$ the sum of these four cycles. Now, the chain of imbeddings whose boundary is $i_k(4T)$ is constructed by considering a chain in $\text{Imb}(S^1, \mathbb{R}^n)$ equal to the cycles in $i_k(4T)$, except that the third strand is lifted around $b$ along an $((n-3)+1)$-sphere centered in $b$, with $4$ holes corresponding to the four (half) strands merging in $b$ (see figure 7).

Notice that the orientation of the $((n-3)+1)$-chain is automatically fixed by the orientation of the strand we are lifting and the fact that we have fixed, once for all, an orientation for $\mathbb{R}^n$.

As for the $1T$ relations, one can notice that when we collapse and blow-up a “short chord” (i.e., a chord whose vertices are consecutive vertices on the oriented circle) we produce a cycle as in figure $8$.

If $n = 3$ we have the difference of two ordinary knots which are clearly isotopic, namely there exist a $1$-chain of imbeddings whose boundary is precisely the difference of these two knots. In higher dimensions, one can split the “blow-up” into two chains, $K_1$ and $K_2$. Suppose in fact that the two strands $l_1$ and $l_2$ meeting at the double point lie on a plane in $\mathbb{R}^n$; then set $K_1$ to be given by those imbeddings lying
on one side of the plane determined by $l_1$ and $l_2$, and $K_2$ those lying on the other side, in such a way that the cycle of imbeddings which “blows-up” the double point is $K_1 - K_2$. Therefore, the chains $K_1$ and $K_2$ play the role for $n > 3$ of the two ordinary knots, and it is not difficult to see that there exists a higher dimensional analogue to the 1-chain connecting the two ordinary knots, namely that there exists a $(n - 2)$-chain of embeddings whose boundaries are $K_1$ and $-K_2$. □

5. Proof of Theorem 1

In the quest for nontrivalent graph cocycles, one easily sees that there are no nontrivial elements in $H^{1,1}(D_o)$, $H^{2,1}(D_o)$, $H^{3,1}(D_o)$, $H^{1,1}(D_e)$ and $H^{2,1}(D_e)$, and that the first example of nontrivial nontrivalent graph cocycle is the generator of $H^{3,1}(D_e)$. This cocycle is represented in figure 9.

![Figure 9. Cocyloce of even type of order 3 and degree 1](image)

Theorem 1 is proved by showing that the image of this graph cocycle through $H^{3,1}(I) : H^{3,1}(D_e) \to H^{(n-3)3+1}(\text{Imb} (S^1, \mathbb{R}^n))$ is a nontrivial cohomology class of $\text{Imb} (S^1, \mathbb{R}^n)$. This will follow from the evaluation of this de Rham cocycle on the dual cycle, which we are now going to construct.

![Figure 10. Cycle of even type of order 3 and degree 1](image)

Let us first consider the linear combination of diagrams of figure 10. We notice that it can be seen as the difference between two chains of diagrams $D_1 - D_2$ (the $D_i$ are the chains of diagrams in square brackets) such that the boundary $\partial$ of each $D_i$ gives one of the 4T relations of figure 5. It is also immediate to check that $D_1 - D_2$ is indeed a cycle. Next we associate a $((n-3)3+1)$-chain of $\text{Imb} (S^1, \mathbb{R}^n)$ to $D_1$ and $D_2$ as follows: since $\partial D_i$ is a linear combination of trivalent diagrams, we can apply the map $i_3$ of equation (3). As shown in the proof of Proposition 2 the chain $i_3(D_1)$, resp. $i_3(D_2)$, is the boundary of a $((n-3)3+1)$-chain of $\text{Imb} (S^1, \mathbb{R}^n)$ which we denote by the symbol $i_3(D_i)$, resp. $i_3(D_2)$. Namely, we have:

$$i_3(D_i) = \partial(i_3(D_i)) \quad i = 1, 2$$

More specifically, what we have to do is to consider an “imbedding” of the first diagram of $D_1$, resp. $D_2$, i.e., an immersion of the oriented circle of this diagram into $\mathbb{R}^n$ which is an imbedding except for a double and a triple point obtained by
identifying the end-points of each chord. We get \( i_3(D_1) \) resp. \( i_3(D_2) \), by blowing-up these intersections according to the constructions of Section 4.

The difference \( i_3(D) = i_3(D_1) - i_3(D_2) \) of these two chains of imbeddings is the \((n - 3)3 + 1\)-cycle of \( \operatorname{Imb}(S^3, \mathbb{R}^n) \) that we associate to \( D = D_1 - D_2 \), and it is represented in figure 11. In other words, we have extended the map of equation (3) to \( H_3^{\text{im}}(D_e) \).

Let us define now

\[
\Xi_1 = \int_{C_{4,1}(\mathbb{R}^n)} \theta_{13} \theta_{45} \theta_{35} \theta_{25} \\
\Xi_2 = \int_{C_{5,0}(\mathbb{R}^n)} \theta_{13} \theta_{14} \theta_{25}
\]

so that \( \Xi = \Xi_1 + 2 \Xi_2 \) is the configuration space integral associated to the diagram of figure 9. The Theorem is proved by showing that the evaluation of \( \Xi \) on \( i_3(D_1) - i_3(D_2) \) is different from zero.

Recall that the tautological forms \( \theta_{ij} \) are constructed using a symmetric top form \( \omega^{n-1} \) on \( S^{n-1} \), concentrated around some fixed direction, say \((1, 0, \ldots, 0) \in S^{n-1} \subset \mathbb{R}^n\). This means that we are requiring \( \omega^{n-1} \) to have support near \((1, 0, \ldots, 0)\) and near \((-1, 0, \ldots, 0)\). Moreover we can always suppose that the images in \( \mathbb{R}^n \) of the chains \( i_3(D_1) \) and \( i_3(D_2) \) lay on a plane perpendicular to \((1, 0, \ldots, 0)\), except near the blow-ups.

Therefore the only contributions to the evaluation of \( \Xi_1 \) and \( \Xi_2 \) on \( i_3(D) \), arise from those parts of the configuration space integral in which the tautological forms
“points toward the direction $(1, 0, \ldots, 0)$”. This implies in particular that the evaluation of $\Xi_1$ on $i_3(D)$ is zero and that, in the evaluation of $\Xi_2$ on $i_3(D)$, the only nontrivial contributions arise from that part of the configuration space integral $\Xi_2$ in which $\theta_{25}$ is integrated near the blow-up of the double point and $\theta_{14}\theta_{13}$ is integrated near the blow-up of the triple point.

Moreover, it not difficult to see that the evaluation of $\Xi_2$ on $i_3(D)$ can be explicitly computed as twice the degree of the map

$$f: \quad C^{0,0}_5(\mathbb{R}^n) \rightarrow S^{n-1} \times S^{n-1} \times S^{n-1}$$

$$\quad (x_1, x_2, x_3, x_4, x_5) \mapsto \left( \frac{x_1-x_3}{|x_1-x_3|}, \frac{x_5-x_3}{|x_5-x_3|}, \frac{x_2-x_4}{|x_2-x_4|} \right).$$

restricted to a compact subset $K$ of $C^{0,0}_5(\mathbb{R}^n)$. This subset $K$ is given by the cartesian product of two 1-dimensional manifolds, two $(n-2)$-sphere and a $(n-1)$-manifold, imbedded in $\mathbb{R}^n$ as follows: each of the 1-dimensional manifolds is linked to a $(n-2)$-sphere, and the $(n-1)$-manifold wraps around one of the spheres.

For instance if we consider $x = ((1, 0, \ldots, 0), (1, 0, \ldots, 0), (1, 0, \ldots, 0))$, we can easily see that the point is regular and that the number of elements in the counter-image is one. In other words, the evaluation of $\Xi_2$ on $i_3(D)$ is different from zero, and this concludes the proof of Theorem 1.

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