Geodesic Witten diagrams with an external spinning field

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Abstract

We explore AdS/CFT correspondence between geodesic Witten diagrams and conformal blocks (conformal partial waves) with an external symmetric traceless tensor field. We derive an expression for the conformal partial wave with an external spin-1 field and show that this expression is equivalent to the amplitude of the geodesic Witten diagram. We also show the equivalence by using conformal Casimir equation in embedding formalism. Furthermore, we extend the construction of the amplitude of the geodesic Witten diagram to an external arbitrary symmetric traceless tensor field. We show our construction agrees with the known result of the conformal partial waves.
1 Introduction

Conformal blocks or conformal partial waves (CPW) are fundamental objects in conformal field theory (CFT) and finding their compact expression has long been a research subject of CFT (see, for example, [1, 2, 3, 4, 5, 6, 7, 8]). Because of recent progress in the conformal bootstrap program [9, 10], it has become necessary to obtain better expressions for numerical calculations. In particular, a formula for CPW of external operators with spin such as the stress tensor is important. It will enable us to apply the conformal bootstrap program to various areas, for example, critical phenomena and quantum gravity as the gravity dual of CFT.

From the viewpoint of AdS/CFT correspondence [11, 12, 13], the gravity dual of the conformal partial wave has not been well understood. However, the authors of [14] proposed the correspondence between CPW and geodesic Witten diagrams (GWD) up to normalization based on recent results in the AdS$_3$/CFT$_2$ correspondence of Virasoro conformal blocks [15, 16, 17, 18, 19, 20]. GWD are diagrams that represent the scattering process on AdS spacetime, such as the Witten diagram. The difference between GWD and the Witten diagram is the following. In the usual Witten diagram, the interactions are integrated over all points in the bulk. On the other hand, GWD interactions are restricted at geodesics between external operators. They showed the correspondence between the amplitude of GWD and CPW with four external scalar fields by direct computation and conformal Casimir equation. Moreover, they decomposed the Witten diagrams into GWD. The CPW expansion of the Witten diagrams has been also
discussed in [21, 22, 23, 24]. In [14], the authors considered the external scalar fields only. The generalization of their results to external fields in arbitrary representation could be useful for the conformal bootstrap program. There have been several developments since [14], for example, [25, 26, 27, 28, 29, 30, 31].

In this paper, we extend the correspondence in [14] to the case of scalar exchange GWD of an external field with spin and three scalar fields (Figure 1). Our work is a first step toward constructing GWD with external fields in arbitrary representation. We explicitly show the correspondence (20) between CPW and GWD with an external spin-1 field up to normalization. In order to construct the amplitude of GWD, we introduce the usual three point interaction, such as $A_{\mu}g^{\mu\nu}\partial\phi\partial\phi$. We also show that the amplitude of GWD satisfies the conformal Casimir equation. Moreover, we construct the amplitude of GWD with an external spin-$n$ field and find a three point interaction (52) for the construction. Our construction of GWD agrees with the known formula of CPW in [32].

This paper is organized as follows. In section 2, we review the correspondence between the scalar CPW and the scalar GWD in [14]. In section 3, we show the correspondence between CPW and GWD of an external spin-1 field and three external scalar fields with scalar exchange in Poincaré coordinates. In section 4, we construct the amplitude of GWD with an external spin-$n$ field. This amplitude agrees with the known results of CPW. We also discuss three point coupling in GWD. In section 5, we summarize the results and discuss future work. We note useful formulas for our calculation in appendix A and check the relation between the scalar three point function in CFT and the amplitude of the three point scalar GWD in appendix B. We show that GWD with an external spin-1 field satisfies the conformal Casimir equation in appendix C.
2 Review of conformal partial waves and geodesic Witten diagrams

In this section, we review AdS$_{d+1}$/CFT$_d$ correspondence between conformal partial waves and geodesic Witten diagrams in [14]. We focus on scalar CPW for later analysis.

In conformal field theories, four point functions of primary operators can be expanded by CPW $W_{\Delta,\ell}(x_i)$ (see, for example, [33, 34]),

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_{\ell} C_{12\ell} C_{34} W_{\Delta,\ell}(x_i),$$  

(1)

where $O$ is a primary operator with conformal dimension $\Delta$ and spin $\ell$, $C_{12\ell}$ and $C_{34}$ are the OPE coefficients. If $O_i$ are the scalar primary fields with conformal dimension $\Delta_i$, the conformal block $G_{\Delta,\ell}(u, v)$ is related to CPW $W_{\Delta,\ell}(x_i; \Delta_i)$,

$$W_{\Delta,\ell}(x_i; \Delta_i) = \left(\frac{x_{12}^2}{x_{14}^2}\right)^{\frac{1}{2} \Delta_{12}} \left(\frac{x_{13}^2}{x_{14}^2}\right)^{\frac{1}{2} \Delta_{13}} G_{\Delta,\ell}(u, v) \left(\frac{x_{12}^2}{x_{14}^2}\right)^{\frac{1}{2} (\Delta_1 + \Delta_2)} \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2} (\Delta_3 + \Delta_4)},$$

(2)

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$, $x_{ij} \equiv x_i - x_j$ and $u, v$ are conformal cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$  

(3)

Conformal blocks and CPW can be determined from conformal symmetry and they do not depend on the details of CFT.

As noted above, GWD was proposed as the gravity dual of CPW up to normalization. We can define the amplitude of GWD by integrating the bulk vertices over geodesics between external fields. For example, the amplitude of the scalar exchange GWD with four external scalar fields $W_{\Delta,0}(x_i; \Delta_i)$ (Figure 2) is defined as

$$W_{\Delta,0}(x_i; \Delta_i) \equiv \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' G_{b\delta}(y(\lambda), x_1; \Delta_1) G_{b\delta}(y(\lambda'), x_2; \Delta_2) G_{b\delta}(y(\lambda), y(\lambda'); \Delta)$$

$$\times G_{b\delta}(y(\lambda'), x_3; \Delta_3) G_{b\delta}(y(\lambda'), x_4; \Delta_4),$$

(4)

where $\lambda$ and $\lambda'$ are proper time coordinates of geodesics $\gamma_12$ and $\gamma_{34}$. The terms $\gamma_{ij}$ are the geodesics between boundary points $x_i$ and $x_j$, $y(\lambda)$ and $y(\lambda')$ are coordinates of $\gamma_12$ and $\gamma_{34}$. The bulk-boundary propagator and the bulk-bulk propagator on AdS spacetime are denoted by $G_{b\delta}$ and $G_{bb}$, respectively \[1^\dagger\] (see, for example, [35, 36]),

$$G_{b\delta}(y, x_i; \Delta_i) \equiv \left(\frac{u}{u^2 + |x - x_i|^2}\right)^{\Delta_i},$$

(5)

$$G_{bb}(y, y'; \Delta) \equiv \xi^2 F_1 \left(\frac{\Delta}{2} + \frac{\Delta + 1}{2}, \Delta + 1 - \frac{d}{2}; \xi^2\right),$$

(6)

$$\xi \equiv \frac{2u u'}{u^2 + u'^2 + |x - x'|^2}.$$  

(7)

\[1^\dagger\] For simplicity, we consider the symmetric traceless representation only and we suppress the index of spin in this section. Generally, the independent number of CPW is not one if we consider nonzero external spin.

\[2^\dagger\] Our normalization is the convention of [14].
Figure 2: Scalar exchange geodesic Witten diagram with four external scalar fields. Each line has the same meaning as the lines in Figure 1.

Now, we consider \((d+1)\)-dimensional Poincaré coordinates \(y^\mu = \{u, x^a\}\) and the metric is

\[
ds^2 = \frac{du^2 + dx^a dx^a}{u^2}.
\]  

(8)

The explicit form of \(y(\lambda)\) is

\[
\begin{align*}
u(\lambda) &= \frac{|x_1 - x_2|}{2 \cosh \lambda}, \\
x^a(\lambda) &= \frac{x_{1}^{a} + x_{2}^{a}}{2} - \frac{x_{1}^{a} - x_{2}^{a}}{2} \tanh \lambda,
\end{align*}
\]

(9)  

(10)

and the same is true of \(y(\lambda')\).

Surprisingly, (4) is the same form of a double integral representation for the scalar CPW in [4, 5]. Moreover, one can show that (4) satisfies the conformal Casimir equation by using embedding formalism. Therefore, we can conclude that the amplitude of GWD corresponds to CPW. Some readers may wonder why we use the exchange GWD rather than contact diagrams. This is because an equation of the bulk-bulk propagator in the scalar exchange GWD corresponds to the conformal Casimir equation and the contact GWD do not satisfy the conformal Casimir equation. In the next section, we will extend this result to the correspondence with an external spin-1 field.

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3 We fix the AdS radius as \(R_{AdS} = 1\).

4 In particular, [4]'s equation (32) corresponds to (4) \((d = 4)\) with change of variables,

\[
u = \frac{e^{-2\lambda}}{1 + e^{-2\lambda}}, \quad \lambda = \xi^{-1}.
\]
3 Direct proof of the correspondence with an external spin-1 field

In this section, we show the correspondence between conformal partial waves and geodesic Witten diagrams of an external spin-1 field and three scalar fields with scalar exchange up to normalization. We derive CPW with an external spin-1 field explicitly based on (4) and rewrite it in terms of the spin-1 propagator in AdS spacetime.

Let us introduce an integral
\[ \int d^dx O(x) \langle 0 \rangle \langle 0 | \tilde{O} (x) \]
and insert (12) into (1). This insertion becomes a projection into CPW,
\[ \int d^dx \langle O_1 (x_1) O_2 (x_2) O (x) \rangle \langle \tilde{O} (x) O_3 (x_3) O_4 (x_4) \rangle \propto W_{\Delta,0} (x_i) + K_O W_{d-\Delta,0} (x_i), \]
where \( K_O \) is a constant. This is because the l.h.s of (13) satisfies the conformal Casimir equation and the insertion of (12) does not change the transformation properties. \( W_{\Delta,0} (x_i) \) is the shadow CPW and its boundary condition is different from \( W_{\Delta,0} (x_i) \)'s boundary condition at \( x_{ij} \to 0 \). We can ignore \( K_O W_{d-\Delta,0} (x_i) \) by imposing the appropriate boundary condition.

In preparation for our calculation, we consider the relation between three point functions since we integrate a product of the three point functions in the shadow formalism. The forms of the three point functions in CFT are determined by conformal symmetry\(^5\)

\[ \langle O_1 (x_1) O_2 (x_2) O_3 (x_3) \rangle = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}, \tag{14} \]
\[ \langle J^a (x_1) O_2 (x_2) O_3 (x_3) \rangle = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \times \left( \frac{x_{12}^a}{|x_{12}|^2} - \frac{x_{13}^a}{|x_{13}|^2} \right), \tag{15} \]
where \( O_i (x_i) \) are scalar primary fields with conformal dimension \( \Delta_i \) and \( J^a (x_1) \) is a spin-1 primary field with conformal dimension \( \Delta_1 + 1 \). The relation between (14) and (15) is

\[ \left( \frac{\partial}{\partial x_{12}^a} + \frac{2\Delta_1 (x_{12})_a}{|x_{12}|^2} \right) \langle O_1 (x_1) O_2 (x_2) O_3 (x_3) \rangle = (\Delta_3 + \Delta_1 - \Delta_2) \langle J^a (x_1) O_2 (x_2) O_3 (x_3) \rangle. \tag{16} \]

For the remainder of the paper, we denote CPW of four external primary fields with conformal dimension \( \Delta_i \) and spin \( \ell_i \) by \( W_{\Delta,\ell}^{(\ell_1, \ell_2, \ell_3, \ell_4)} (x_i; \Delta_i) \). Here \( \Delta \) and \( \ell \) are the conformal dimension and spin.

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\(^5\)We ignore the OPE coefficients.

\(^6\)The independent number of CPW that we consider in this paper is one. This is because the degrees of freedom of the three point functions that we consider are one such as (14) and (15).
of an exchanging primary operator. Similarly, we denote the amplitude of GWD as $\mathcal{W}_{\Delta,\ell}^{(\ell_1,\ell_2,\ell_3,\ell_4)}(x_i; \Delta_i)$. By using (13) and (16), we get (up to normalization)

$$
\left( W^{(1,0,0,0)}_{\Delta,0}(x_i; \Delta_i) \right)_a = \int d^4x \langle \mathcal{J}_a(x_1) \mathcal{O}_2(x_2) \mathcal{O}(x) \rangle \langle \mathcal{O}(x) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle |_{BC}
$$

$$
= \left( \frac{\partial}{\partial x_1^a} + \frac{2\Delta(x_1) a}{|x_1|^2} \right) \int d^4x \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}(x) \rangle \langle \mathcal{O}(x) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle |_{BC}
$$

$$
= \left( \frac{\partial}{\partial x_1^a} + \frac{2\Delta(x_1) a}{|x_1|^2} \right) W^{(0,0,0,0)}_{\Delta,0}(x_i; \Delta_i)
$$

$$
= \left( \frac{\partial}{\partial x_1^a} + \frac{2\Delta(x_1) a}{|x_1|^2} \right) W^{(0,0,0,0)}_{\Delta,0}(x_i; \Delta_i),
$$

(17)

where $\tilde{\Delta}_i = \Delta_i + \delta_i$. Here $|_{BC}$ means imposing the appropriate boundary condition to ignore the shadow CPW and the explicit forms of the boundary conditions for CPW are

$$
\lim_{x_{12} \to 0} W^{(0,0,0,0)}_{\Delta,0}(x_i; \Delta_i) \to \frac{\text{constant}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta}},
$$

(18)

$$
\lim_{x_{12} \to 0} \left( W^{(1,0,0,0)}_{\Delta,0}(x_i; \tilde{\Delta}_i) \right)_a \to \frac{\text{constant}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta}} \times \frac{(x_{12})_a}{|x_{12}|^2}.
$$

(19)

Thus, we have obtained formula (17) of CPW with an external spin-1 field $W^{(1,0,0,0)}_{\Delta,0}(x_i; \tilde{\Delta}_i)$ in terms of (4).

However, the relationship between (17) and the spin-1 propagator in AdS spacetime is not clear. In order to make it manifest, we rewrite (17) in terms of the spin-1 propagator. In particular, we will show\footnote{We note that $\partial/\partial x_1^a$ acts on both $x_1$ and $y(\lambda)$.}

$$
\left( \frac{\partial}{\partial x_1^a} + 2\Delta \frac{(x_{12})_a}{|x_{12}|^2} \right) \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' G_{b\partial}(y(\lambda), x_1; \Delta_1) G_{b\partial}(y(\lambda), x_2; \Delta_2) G_{b\partial}(y(\lambda'), y; \Delta)
$$

$$
\times G_{b\partial}(y(\lambda'), x_3; \Delta_3) G_{b\partial}(y(\lambda'), x_4; \Delta_4)
$$

$$
= \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' \left( G^{1}_{b\partial}(y(\lambda), x_1; \Delta_1 + 1) \right)_a^{\mu} G_{b\partial}(y(\lambda), x_2; \Delta_2)
$$

$$
\times u(\lambda)^2 \frac{\partial}{\partial y^\mu(\lambda)} \left( G_{b\partial}(y(\lambda), y(\lambda'); \Delta) \right)
$$

$$
\times G_{b\partial}(y(\lambda'), x_3; \Delta_3) G_{b\partial}(y(\lambda'), x_4; \Delta_4),
$$

(20)

where $G^{1}_{b\partial}(y, x_1; \Delta_1 + 1)$ is the spin-1 bulk-boundary propagator (see, for example, [38, 39]),

$$
(G^{1}_{b\partial}(y, x_1; \Delta_1 + 1))_a^{\mu} \equiv \left( \frac{u}{u^2 + |x - x_1|^2} \right)^{\Delta_1} \left( \frac{\delta_a^\mu}{u^2 + |x - x_1|^2} - \frac{2 (y - x_1)_a (y - x_1)^\mu}{(u^2 + |x - x_1|^2)^2} \right).
$$

(21)

The l.h.s of (20) corresponds to the last line of (17). The r.h.s of (20) is a definition of the amplitude of GWD $W^{(1,0,0,0)}_{\Delta,0}(x_i; \tilde{\Delta}_i)$ with a three point interaction coefficient $u^2 \frac{\partial}{\partial y^\mu}$ that is the usual coupling such
as $A_\mu g^{\mu \nu} \partial_\nu \phi$. Therefore, (20) signifies the correspondence between CPW and GWD with an external spin-1 field.

In order to show (20), we deform

$$\frac{\partial}{\partial x_1^a} \int_{\infty}^{-\infty} d\lambda G_{b\delta}(y(\lambda), x_1; \Delta_1) G_{b\delta}(y(\lambda), x_2; \Delta_2) G_{bb}(y(\lambda), y(\lambda'); \Delta).$$

(22)

From (9), (10) and the definitions of the propagators, we find

$$\int_{\infty}^{-\infty} d\lambda G_{b\delta}(y(\lambda), x_1; \Delta_1) G_{b\delta}(y(\lambda), x_2; \Delta_2) G_{bb}(y(\lambda), y(\lambda'); \Delta)$$

$$= (\Delta_1 + \Delta_2) \frac{(x_{12})_a}{|x_{12}|^2} \int_{-\infty}^{\infty} d\lambda G_{b\delta}(y(\lambda), x_1; \Delta_1) G_{b\delta}(y(\lambda), x_2; \Delta_2) G_{bb}(y(\lambda), y(\lambda'); \Delta)$$

$$+ \int_{-\infty}^{\infty} d\lambda \left( G_{b\delta}^1(y(\lambda), x_1; \Delta_1 + 1) \right) \mu_a G_{b\delta}(y(\lambda), x_2; \Delta_2) u(\lambda)^2 \frac{\partial}{\partial y^\mu(\lambda)} \left( G_{bb}(y(\lambda), y(\lambda'); \Delta) \right)$$

$$- \frac{(x_{12})_a}{|x_{12}|^2} \int_{-\infty}^{\infty} d\lambda G_{b\delta}(y(\lambda), x_1; \Delta_1) G_{b\delta}(y(\lambda), x_2; \Delta_2) \partial y^\mu(\lambda) \frac{\partial}{\partial \lambda} G_{bb}(y(\lambda), y(\lambda'); \Delta).$$

(23)

After integration by parts of the last line of (23), we obtain

$$\int_{-\infty}^{\infty} d\lambda G_{b\delta}(y(\lambda), x_1; \Delta_1) G_{b\delta}(y(\lambda), x_2; \Delta_2) G_{bb}(y(\lambda), y(\lambda'); \Delta)$$

$$= -2\Delta_1 \frac{(x_{12})_a}{|x_{12}|^2} \int_{-\infty}^{\infty} d\lambda G_{b\delta}(y(\lambda), x_1; \Delta_1) G_{b\delta}(y(\lambda), x_2; \Delta_2) G_{bb}(y(\lambda), y(\lambda'); \Delta)$$

$$+ \int_{-\infty}^{\infty} d\lambda \left( G_{b\delta}^1(y(\lambda), x_1; \Delta_1 + 1) \right) \mu_a G_{b\delta}(y(\lambda), x_2; \Delta_2) u(\lambda)^2 \frac{\partial}{\partial y^\mu(\lambda)} \left( G_{bb}(y(\lambda), y(\lambda'); \Delta) \right).$$

(24)

Integrating (24) by $\Delta$ with $G_{b\delta}(y(\lambda'), x_3; \Delta_3) G_{b\delta}(y(\lambda'), x_4; \Delta_4)$, we obtain the final result (20). The formulas in appendix A are useful for the calculation.

Summarizing the above, we have explicitly shown the correspondence (20) between the conformal partial wave $W_{\Delta_0}^{(1,0,0,0)}(x_i; \Delta)$ and the amplitude of the geodesic Witten diagram $W_{\Delta_0}^{(1,0,0,0)}(x_i; \Delta_i)$ of an external spin-1 field and three scalar fields with scalar exchange. We can prove this correspondence by using conformal Casimir equation (see appendix C).

4 Generalization to an external spin-$n$ field

In this section, we extend the previous result to the geodesic Witten diagrams with an external spin-$n$ field. To see the equivalence between GWD and CPW, it is useful to employ so-called embedding

\[\text{Note that this interaction is one of the candidates. One can obtain the same tensor structure by using } A_\mu g^{\mu \nu} \nabla^2 \partial_\nu \phi, \text{ for example. Such three point interactions must be invariant under the isometry of AdS for the correspondence between CPW and GWD. In contrast to the Witten diagrams, forms of the three point interactions in GWD have no physical meaning because CPW does not depend on the dynamics.}\]

\[\text{We assume } |\Delta_1 - \Delta_2| < \Delta \text{ for integration by parts. This is the same condition for the convergence of the amplitude of GWD for the scalar three point function. We note the correspondence between the scalar three point function and GWD in appendix B.}\]
formalism. In section 4.1 we review the embedding formalism in order to note our notation. In section 4.2 we specify three point coupling in GWD and explicitly construct the amplitude of GWD with an external spin-$n$ field $\mathcal{W}_{(n,0,0)}^{(\Delta,0)}(x_i; \Delta_i)$. This expression agrees with the formula of CPW in [32].

4.1 Embedding space

It is a well-known fact that the conformal symmetry of $d$-dimensional CFT and the isometry of Euclidean AdS$_{d+1}$ are equivalent to $(d+2)$-dimensional Lorentz symmetry. By using this fact, we can describe CFT in $d$-dimension and a theory on AdS$_{d+1}$ space as a theory on $d+2$-dimensional embedding Minkowski spacetime. This formalism is called embedding formalism. (see, for example, [40, 41, 42, 43, 44, 45, 46, 32, 39].) Since the Lorentz transformation is linear, tensor structures in the embedding formalism become simple. From the above motivation, we review the embedding formalism. For more details about the embedding formalism in CFT, see [46, 32]. One can find the details of the embedding formalism for AdS in [39, 24].

Euclidean AdS$_{d+1}$ can be embedded into $(d+2)$-dimensional Minkowski spacetime $\mathbb{R}^{1,d+1}$ as

$$Y^2 = \eta_{AB}Y^AY^B = -1, \ Y^0 > 0,$$

where $Y^A$ denotes the coordinates of $\mathbb{R}^{1,d+1}$. We embed AdS$_{d+1}$ coordinates $y^\mu = \{u, x^a\}$ into $Y^A$ such that

$$Y^A \equiv (Y^+, Y^-, Y^a) = \frac{1}{u}(1, u^2 + x^2, x^a).$$

The conformal boundary of AdS, on which CFT lives, can be defined as the projective light cone

$$X^2 = 0, \ X^A \sim \lambda X^A \quad (\lambda \in \mathbb{R}).$$
We use the Poincaré section (Figure 3) for the $d$-dimensional flat space $\mathbb{R}^d$,

$$X^A \equiv (X^+, X^-, X^a) = (1, x^2, x^a). \quad (29)$$

Next, we embed the fields in CFT$_d$ and AdS$_{d+1}$ into the embedding space. Since the number of the fields in the embedding space is larger than that of the fields in CFT and AdS, we must impose the constraints for the fields in the embedding space. In particular, we impose a transverse condition to traceless symmetric tensors in both sides as

$$X_{A_1} T_{\partial}^{A_1 A_2 \cdots A_l} (X) = 0, \quad Y_{A_1} T_{b}^{A_1 A_2 \cdots A_l} (Y) = 0, \quad (30)$$

where $T_\partial$ is the tensor field in the boundary CFT and $T_b$ is in the bulk AdS. We further impose the condition to the primary field $T_{\partial}^{A_1 A_2 \cdots A_l} (X)$ as

$$T_{\partial}^{A_1 A_2 \cdots A_l} (\lambda X) = \lambda^{-\Delta} T_{\partial}^{A_1 A_2 \cdots A_l} (X). \quad (31)$$

When we consider the tensor fields, there is an efficient way to classify their tensor structures, which are called index-free notation, introduced in [46, 32, 39]. For this notation, we introduce auxiliary fields $Z$ for the boundary, and $W$ for the bulk to contract all indices:

$$T_\partial (X; Z) \equiv Z_{A_1} \cdots Z_{A_l} T_{\partial}^{A_1 A_2 \cdots A_l} (X), \quad T_b (Y; W) \equiv W_{A_1} \cdots W_{A_l} T_b^{A_1 A_2 \cdots A_l} (Y). \quad (32)$$

We can restrict $Z$ to $Z^2 = Z \cdot X = 0$ because these conditions do not lose the information of $T_\partial$. Similarly, we can restrict $W$ to $W^2 = W \cdot Y = 0$.

With index-free notation, we define the bulk-boundary propagators in embedding space [39] as

$$G_{\partial b}(X, Y; Z, W; \Delta) \equiv \frac{((-2X \cdot Y)(Z \cdot W) + 2(Z \cdot Y)(X \cdot W))^{J}}{(-2X \cdot Y)^{\Delta + J}}. \quad (33)$$

One can remove the auxiliary fields $Z$ and $W$ by using some differential operators introduced in [46, 32, 39]. Since the numerator of (33) is written as $J$ copies of the first-order polynomial of $Z$ (or $W$), to remove $Z$ and $W$ it is enough to use $\frac{\partial}{\partial Z}$ and $\frac{\partial}{\partial W}$ naively. One can also see that the bulk-bulk propagator in the embedding space is

$$G_{bb}(Y_1, Y_2; \Delta) \equiv \xi^{\Delta / 2} F_1 \left( \frac{\Delta}{2}, \frac{\Delta + 1}{2}, \Delta + 1 - \frac{d}{2}; \xi^2 \right), \quad (34)$$

$$\xi^{-1} \equiv -Y_1 \cdot Y_2. \quad (35)$$

Here (35) is the expression of (7) in the embedding formalism.

By employing the above ingredients, we can rewrite the amplitude of the scalar GWD (4) as following:

$$W_{(0,0,0,0)}^{(0,0,0)} (X_1; \Delta_1) = \int_{-\infty}^{\infty} d\lambda' \int_{-\infty}^{\infty} d\lambda G_{\partial \partial}^{00} (Y_1 (\lambda), X_1, \Delta_1) G_{\partial \partial}^{00} (Y_1 (\lambda), X_2, \Delta_2) G_{bb} (Y_1 (\lambda), Y_2 (\lambda'); \Delta)$$

$$\times G_{bb} (Y_2 (\lambda'), X_3; \Delta_3) G_{bb}^{00} (Y_2 (\lambda'), X_4; \Delta_4). \quad (36)$$
Here $Y_{1A}(\lambda)$, which is the point on the geodesic between the boundary points $X_1$ and $X_2$, can be written simply as

$$Y_{1A}(\lambda) = \frac{e^{-\lambda}X_{1A} + e^{\lambda}X_{2A}}{\sqrt{-2X_1 \cdot X_2}}.$$ (37)

Similarly, $Y_{2}(\lambda')$ can be written as

$$Y_{2A}(\lambda') = \frac{e^{-\lambda'}X_{3A} + e^{\lambda'}X_{4A}}{\sqrt{-2X_3 \cdot X_4}}.$$ (38)

On the other hand, the amplitude of GWD $W^{(1,0,0,0)}_{\Delta,0}(X_i; Z_1; \tilde{\Delta}_i)$ can be rewritten in the index-free notation as

$$W^{(1,0,0,0)}_{\Delta,0}(X_i; Z_1; \tilde{\Delta}_i) = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' \left\{ G^{0}_{b\bar{b}}(Y_1(\lambda), X_2; \Delta_1) G^{0}_{\bar{b}b}(Y_1(\lambda), X_2; \Delta_1) \right\} \times G^{0}_{b\bar{b}}(Y_2(\lambda'), X_3; \Delta_3) G^{0}_{\bar{b}b}(Y_2(\lambda'), X_4; \Delta_4).$$ (39)

Here we introduce a covariant derivative in the embedding AdS space [39] as

$$\nabla_A \equiv \frac{\partial}{\partial Y^A} + Y_A \left( Y \cdot \frac{\partial}{\partial Y} \right) + W_A \left( Y \cdot \frac{\partial}{\partial W} \right),$$ (40)

which also satisfies $Y^A \nabla_A = 0$. By using formulas in appendix A [40] one can check that (39) is equivalent to the r.h.s of (20).

One can translate the previous result (20) into

$$\left( Z_1 \cdot \frac{\partial}{\partial X_1} - 2\Delta_1 \frac{Z_1 \cdot X_2}{-2X_1 \cdot X_2} \right) W^{(1,0,0,0)}_{\Delta,0}(X_i; \tilde{\Delta}_i) = W^{(1,0,0,0)}_{\Delta,0}(X_i; Z_1; \tilde{\Delta}_i),$$ (41)

where $\tilde{\Delta}_i \equiv \Delta_i + \delta_i$. [41] represents the relation between the scalar GWD and GWD with an external spin-1 field in the embedding formalism. Of course, one can check this relation directly from (39). For the explicit computation to check [41] in the embedding formalism, appendix A [40] may be useful. For later convenience, we define

$$F^{(0,0,0)}_{\Delta,1,2,\Delta}(X_1, X_2, Y_2) \equiv \int_{-\infty}^{\infty} d\lambda G^{0}_{b\bar{b}}(Y_1, X_1; \Delta_1) G^{0}_{\bar{b}b}(Y_1, X_2; \Delta_1) G_{b\bar{b}}(Y_1, Y_2; \Delta),$$

$$F^{(n,0,0)}_{\Delta,1,2,\Delta}(X_1, X_2, Y_2; Z_1) \equiv \int_{-\infty}^{\infty} d\lambda G^{0}_{b\bar{b}}(Y_1, X_2; \Delta_2) \left\{ G^{n}_{b\bar{b}}(Y_1, X_1; Z_1, \nabla Y_1; \Delta_1) G_{b\bar{b}}(Y_1, Y_2; \Delta) \right\},$$

$$F^{(0,n,0)}_{\Delta,1,2,\Delta}(X_1, X_2, Y_2; Z_2) \equiv \int_{-\infty}^{\infty} d\lambda G^{0}_{b\bar{b}}(Y_1, X_1; \Delta_1) \left\{ G^{n}_{b\bar{b}}(Y_1, X_2; Z_2, \nabla Y_1; \Delta_2) G_{b\bar{b}}(Y_1, Y_2; \Delta) \right\}. \tag{42}$$

Here, $G_{b\bar{b}}(Y, X; Z, \nabla; \Delta)$ denotes

$$\frac{1}{(-2X \cdot Y)^{\Delta+n}} \left[ (-2X \cdot Y)(Z \cdot \nabla) + 2(Z \cdot Y)(X \cdot \nabla) \right] \cdots \left[ (-2X \cdot Y)(Z \cdot \nabla) + 2(Z \cdot Y)(X \cdot \nabla) \right]. \tag{43}$$
4.2 Explicit construction for an external spin-\(n\) field

Towards the generalization to arbitrary symmetric-traceless representation, we construct the amplitude of GWD with an external spin-\(n\) field. We also show that our construction of GWD agrees with the expression in [32]. Moreover, we discuss a corresponding three point interaction in the bulk.

Based on (39), it is straightforward to define GWD \(W_{\Delta,0}^{(n,0,0,0)}\) as

\[
W_{\Delta,0}^{(n,0,0,0)}(X_1; Z_1; \Delta_1) \\
= \int_{-\infty}^{\infty} d\lambda \left[ \int_{-\infty}^{\infty} d\lambda G_{bb}^0(Y_1(\lambda), X_1; \Delta_1) \right] G_{bb}^0(Y_2(\lambda), X_2; \Delta_2) \{ G_{bb}^0(Y_1(\lambda), X_1; \Delta_1) \}
\times G_{bb}^0(Y_2(\lambda'), X_2; \Delta_2) G_{bb}^0(Y_2(\lambda), X_4; \Delta_4)
\]

\[
= \int_{-\infty}^{\infty} d\lambda F_{(n,0,0)}^{(\Delta_1,\Delta_2,\Delta)}(X_1, X_2, Y_2; Z_1) G_{bb}^0(Y_2(\lambda'), X_3; \Delta_3) G_{bb}^0(Y_2(\lambda'), X_4; \Delta_4).
\]

In order to check that our construction agrees with the known results of CPW, it is better to understand the relation between \(W_{\Delta,0}^{(n,0,0,0)}\) and \(W_{\Delta,0}^{(n,0,0,0)}\). To this end, it is enough to study \(F_{(n,0,0)}^{(\Delta_1,\Delta_2,\Delta)}\).

It is known that CPW with symmetric-traceless tensors can be expressed by the scalar CPW with the differential operators such as (3.40) of [32]. They introduced the following differential operators\(^{[10]}\):

\[
D_{11} \equiv (X_1 \cdot X_2) \left( Z_1 \cdot \frac{\partial}{\partial X_2} \right) - (Z_1 \cdot X_2) \left( X_1 \cdot \frac{\partial}{\partial X_2} \right) - (Z_1 \cdot Z_2) \left( X_1 \cdot \frac{\partial}{\partial Z_2} \right) + (X_1 \cdot Z_2) \left( Z_1 \cdot \frac{\partial}{\partial Z_2} \right),
\]

\[
D_{22} \equiv (X_2 \cdot X_1) \left( Z_2 \cdot \frac{\partial}{\partial X_1} \right) - (Z_2 \cdot X_1) \left( X_2 \cdot \frac{\partial}{\partial X_1} \right) - (Z_2 \cdot Z_1) \left( X_2 \cdot \frac{\partial}{\partial Z_1} \right) + (X_2 \cdot Z_1) \left( Z_2 \cdot \frac{\partial}{\partial Z_1} \right),
\]

\[
D_{12} \equiv (X_1 \cdot X_2) \left( Z_1 \cdot \frac{\partial}{\partial X_2} \right) - (Z_1 \cdot X_2) \left( X_1 \cdot \frac{\partial}{\partial X_2} \right) + (Z_1 \cdot Z_2) \left( Z_1 \cdot \frac{\partial}{\partial Z_2} \right),
\]

\[
D_{21} \equiv (X_2 \cdot X_1) \left( Z_2 \cdot \frac{\partial}{\partial X_1} \right) - (Z_2 \cdot X_1) \left( Z_2 \cdot \frac{\partial}{\partial X_1} \right) + (Z_2 \cdot Z_1) \left( Z_2 \cdot \frac{\partial}{\partial Z_1} \right).
\]

By using these differential operators, one can show, for example,

\[
D_{11} F_{(\Delta_1+1,\Delta_2,\Delta)}^{(0,0,0)} = -\frac{1}{2} \int_{-\infty}^{\infty} d\lambda G_{bb}^0(Y_1, X_2; \Delta_2) \{ G_{bb}^1(Y_1, X_1; \Delta_1) \}
\times G_{bb}^0(Y_2(\lambda'), X_4; \Delta_4).
\]

One can repeat such manipulations and obtain

\[
(D_{11})^n F_{(\Delta_1+n,\Delta_2,\Delta)}^{(0,0,0)} = \left( -\frac{1}{2} \right)^n F_{(\Delta_1+n,\Delta_2,\Delta)}^{(n,0,0)},
\]

\[
(D_{22})^n F_{(\Delta_1,\Delta_2+n,\Delta)}^{(0,0,0)} = \left( -\frac{1}{2} \right)^n F_{(\Delta_1,\Delta_2+n,\Delta)}^{(0,0,0)},
\]

\[
(D_{12})^n F_{(\Delta_1,\Delta_2+n,\Delta)}^{(0,0,0)} = \left( -\frac{1}{2} \right)^n F_{(\Delta_1,\Delta_2+n,\Delta)}^{(n,0,0)},
\]

\[
(D_{21})^n F_{(\Delta_1+n,\Delta_2,\Delta)}^{(0,0,0)} = \left( -\frac{1}{2} \right)^n F_{(\Delta_1+n,\Delta_2,\Delta)}^{(0,n,0)}.
\]

\(^{[10]}\)The authors of [32] used \(P\) as the embedding space coordinates instead of \(X\).
Now the meaning of $D_{ij}$ is clear. Namely, the action of $D_{ij}$ increases the spin of the field at $X_i$ by one and decreases the scaling dimension of the field at $X_j$ by one. In particular, (48) and (49) agree with (3.40) of [32] in the case of CPW with an external spin-$n$ field and three external scalar fields. This result implies GWD can represent CPW with external tensor fields.

Finally, we discuss a three point interaction for $\mathcal{W}_{\Delta,0}^{(n,0,0,0)}$. One possible answer in the embedding space is

$$S_{int} = \int_{\text{AdS}} dY T_{\Delta_1}^{A_1 \cdots A_n} \phi_{\Delta_2} (\nabla_{A_1} \cdots \nabla_{A_n} \phi_{\Delta}).$$

(52)

The reason is as follows. After simple calculation, (44) becomes

$$\mathcal{W}_{\Delta,0}^{(n,0,0,0)}(X_i; Z_1; \Delta_i) = \int_{-\infty}^{\infty} d\lambda' \left[ \int_{-\infty}^{\infty} d\lambda G_{b \delta}^0(Y_1(\lambda), X_2; \Delta_2) \times G_{b \delta}^0(Y_1(\lambda), X_1; Z_1; \Delta_i) A_1 \cdots A_n \frac{\partial}{\partial Y_1^{A_1}} \cdots \frac{\partial}{\partial Y_1^{A_n}} G_{b b}(Y_1(\lambda), Y_2(\lambda'); \Delta) \right] \times G_{b \delta}^0(Y_2(\lambda'), X_3; \Delta_3) G_{b \delta}^0(Y_2(\lambda'), X_4; \Delta_4).$$

(53)

Here we define

$$G_{b \delta}^n(Y, X; Z, \Delta) A_1 \cdots A_n \equiv \frac{1}{n!} \frac{\partial}{\partial W^{A_1}} \cdots \frac{\partial}{\partial W^{A_n}} G_{b \delta}^0(Y, X; Z, W; \Delta).$$

(54)

From (53), we can choose a three point interaction to construct the amplitude of GWD $\mathcal{W}_{\Delta,0}^{(n,0,0,0)}$ as

$$S_{int} = \int_{\text{AdS}} dY T_{\Delta_1}^{A_1 \cdots A_n} \phi_{\Delta_2} \left( \frac{\partial}{\partial Y_1^{A_1}} \cdots \frac{\partial}{\partial Y_1^{A_n}} \phi_{\Delta} \right).$$

(55)

By virtue of the transverse condition (30) and the traceless condition of $T^{A_1 \cdots A_n}$, one can freely replace all of $\frac{\partial}{\partial Y}$ with $\nabla_A$; thus, we obtain the manifestly covariant expression (52). We stress that this three point interaction is not unique one for CPW with an external spinning field. We expect that any other three point interactions that are invariant under the isometry of AdS will give us the same tensor structure. This is because CPW in our case has unique tensor structure [32].

5 Summary and discussion

In this paper, we have explicitly constructed the amplitude of the scalar exchange geodesic Witten diagrams (44) that have an external field with spin and three external scalar fields. We have also found the three point interaction (52) in the bulk to construct our amplitude of GWD. Moreover, up to normalization, we have shown that these GWD are equivalent to the conformal partial waves that also have an external field with spin. There are two ways to prove this equivalence: comparing GWD with the known expression of CPW and checking that GWD satisfies the conformal Casimir equation. In the spin-1 case, we have proven the correspondence in both ways. In the spin-$n$ case, we have expressed our construction of GWD as scalar GWD with differential operators. We have confirmed that this expression is just the same one as CPW in [32].
We expect that our results can be generalized to the case of any symmetric-traceless fields. One can rewrite formula (3.40) of [32] in terms of the scalar GWD as

\[ W_{A,\ell}(\ell_1, \ell_2, \ell_3, \ell_4) = D_{12} D_{34} W_{A,\ell}^{(0,0,0,0)}, \] (56)

where \( D_{12} \) denotes the combination of \( D_{11}, D_{12}, D_{21}, D_{22} \) and \( H_{12} \) introduced in [32], and so dose \( D_{34} \).

In this paper, we have concentrated only on CPW whose tensor structure is unique. However, it is not true in the general cases. Since the three point functions in CFT have degrees of freedom of the tensor structure of more than one in general, the tensor structure of CPW can be different even if these have the same \( (\Delta_i, \ell_i) \) and \( (\Delta, \ell) \). This should also be true in the bulk picture; hence, we need an explicit dictionary between the tensor structures of CPW and three point interactions in the bulk.

We have studied only GWD with an external symmetric traceless tensor. However, CPW can contain a mixed-symmetry tensor structure in general. One can use index-free notation to construct such CPW by using Grassmann auxiliary fields introduced in [49, 50]. In our paper, we have rewritten \( W_{A,0}^{(n,0,0,0)} \) in terms of the scalar GWD \( W_{A,0}^{(0,0,0,0)} \) with differential operators acting on it. Thus \( W_{A,0}^{(0,0,0,0)} \) plays the role of a seed of GWD. The notion of a seed was introduced in the context of CPW in [51]. If one considers CPW with mixed-symmetry tensors, generally, the seed CPW include other CPW than the scalar CPW. Therefore it is important to find explicit forms of the seed GWD that correspond to such seed CPW.

One of the motivations to consider the bulk representation of CPW is for the new expression of the stress tensor CPW. This expression gives us the universal information about CFT and quantum gravity on AdS. For this purpose, we need to complete the above construction of GWD and we leave it for future work.

Recently, the authors of [52] have proposed a bulk dual of the OPE block (see also [53]). This bulk dual is an operator smeared over the subspace and invariant under isometry. If the points in the OPE block are spacelike, the bulk dual operator is smeared over the geodesics between these points. Since CPW can be written as the “two point function” of the OPE blocks, this proposal can explain why GWD corresponds to CPW. Their arguments are mainly based on the Lorentzian CFT; therefore, one can derive the Witten diagram representation of CPW even in the Lorentzian CFT. It is interesting to consider this correspondence with the spinning field.

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A Useful formulas for the calculations

In this appendix, we note useful formulas that make the calculations in the paper easier.
A.1 Formulas for section 3

These formulas are convenient for the proof of (20):

\[ u(\lambda)^2 + |x(\lambda) - x_1|^2 = \frac{|x_{12}|^2 e^\lambda}{2 \cosh \lambda}, \]  
(57)

\[ u(\lambda)^2 + |x(\lambda) - x_2|^2 = \frac{|x_{12}|^2 e^{-\lambda}}{2 \cosh \lambda}, \]  
(58)

\[ \frac{\partial x^b(\lambda)}{\partial x^a_1} = \frac{x_{12}^b}{2 \cosh \lambda}, \]  
(59)

\[ \frac{\partial x^a(\lambda)}{\partial \lambda} = -\frac{x_{12}^a}{2 \cosh^2 \lambda}, \]  
(60)

\[ \frac{\partial u(\lambda)}{\partial x^a_1} = -\frac{x_{12}u_a}{|x_{12}|^2} + \frac{x_{12}^a e^{-\lambda}}{|x_{12}| \cosh^2 \lambda}. \]  
(61)

A.2 Formulas for section 4

In our convention, the coordinate \( Y_1(\lambda) \) of the geodesic between the boundary points \( X_1 \) and \( X_2 \) is written as

\[ Y_{1A}(\lambda) = \frac{e^{-\lambda}X_{1A} + e^\lambda X_{2A}}{\sqrt{-2X_1 \cdot X_2}}. \]  
(62)

With \( Z_i \cdot X_i = X_i \cdot X_i = Z_i \cdot Z_i = 0 \) (\( i = 1, 2 \)), one can easily show

\[ (-2X_1 \cdot X_2) = (-2X_1 \cdot Y_1(\lambda))(-2X_2 \cdot Y_1(\lambda)), \]  
(63)

\[ (Z_1 \cdot Y_1(\lambda))(X_1 \cdot X_2) = (X_1 \cdot Y_1(\lambda))(Z_1 \cdot X_2), \]  
(64)

\[ (2Z_1 \cdot X_2) = (2Z_1 \cdot Y_1(\lambda))(-2Y_1(\lambda) \cdot X_2), \]  
(65)

\[ (2X_1 \cdot Z_2) = (2Z_2 \cdot Y_1(\lambda))(-2Y_1(\lambda) \cdot X_1). \]  
(66)

For integrating by parts, it is also useful to note that

\[ Z_1 \cdot Y_1(\lambda) = Z_1 \cdot \frac{dY_1}{d\lambda}, \]  
(67)

\[ Z_2 \cdot Y_1(\lambda) = -Z_2 \cdot \frac{dY_1}{d\lambda}. \]  

As for the scalar function \( f(Y_1(\lambda)) \), one can show

\[ X_1 \cdot \frac{\partial f(Y_1(\lambda))}{\partial X_1} = \frac{1}{(-2X_1 \cdot Y_1(\lambda))} (X_1 \cdot \nabla Y_1)f(Y_1(\lambda)), \]  
(68)

\[ Z_1 \cdot \frac{\partial f(Y_1(\lambda))}{\partial X_1} = \frac{1}{(-2X_1 \cdot Y_1(\lambda))} (Z_1 \cdot \nabla Y_1)f(Y_1(\lambda)). \]  
(69)

One can also show similar formulas with \( X_1 \to X_2 \) and \( Z_1 \to Z_2 \). To check these formulas, we use

\[ Y_{1A} \frac{\partial Y_{1A}}{\partial X_1} = 0, \]  

\( Y_{1A} \frac{\partial Y_{1A}}{\partial X_1} = 0, \) and

\[ \left( \frac{\partial Y_{1A}(\lambda)}{\partial X_1^B} \right) = \frac{1}{(-2X_1 \cdot Y_1(\lambda))} \left( \delta_A^B + \frac{Y_{1A}(\lambda)X_{1B}}{-2X_1 \cdot Y_1(\lambda)} \right), \]  
(70)

\[ \left( \frac{\partial Y_{1A}(\lambda)}{\partial X_2^B} \right) = \frac{1}{(-2X_2 \cdot Y_1(\lambda))} \left( \delta_A^B + \frac{Y_{1A}(\lambda)X_{2B}}{-2X_1 \cdot Y_1(\lambda)} \right). \]  
(71)
Formulas for the derivative with respect to $\lambda$ are
\[
\frac{d}{d\lambda} G_{bb}(Y_1(\lambda), Y_2; \Delta) = \frac{1}{X_1 \cdot Y_1(\lambda)} (X_1 \cdot \nabla Y_1) G_{bb}(Y_1(\lambda), Y_2; \Delta),
\]

\[
\frac{d}{d\lambda} (G_{a\eta}(Y_1(\lambda), X_1; \Delta_1) G_{b\rho}(Y_1(\lambda), X_2; \Delta_2)) = (\Delta_2 - \Delta_1) G_{a\eta}^0(Y_1(\lambda), X_1; \Delta_1) G_{b\rho}^0(Y_1(\lambda), X_2; \Delta_2).
\]

(72)

(73)

To see the equivalence of GWD between section 3 and section 4, it is useful to use the induced AdS metric
\[
G_{AB}(Y) \equiv \eta_{AB} + Y_A Y_B
\]

and the relation
\[
\eta^{AB} \nabla_B = G^{AB} \nabla_B.
\]

(74)

(75)

B The three point scalar geodesic Witten diagram

It is well known that the conformal three point function can be obtained from the three point Witten diagram integrated over all points in the bulk. In this appendix, we note that the three point scalar geodesic Witten diagram (Figure 4) also corresponds to the conformal three point function.

If we choose the geodesic between $X_1$ and $X_2$, the amplitude of the three point scalar geodesic Witten diagram $\mathcal{W}(X_1, X_2, X_3; \Delta_i)$ can be defined as
\[
\mathcal{W}(X_1, X_2, X_3; \Delta_i) \equiv \int_{-\infty}^{\infty} d\lambda \ G_{a\eta}^0(Y_1(\lambda), X_1; \Delta_1) G_{b\rho}^0(Y_1(\lambda), X_2; \Delta_2) G_{c\sigma}^0(Y_1(\lambda), X_3; \Delta_3) \\
= (-2X_1 \cdot X_2)^{-\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)} (-2X_1 \cdot X_3)^{-\Delta_3} \int_{-\infty}^{\infty} d\lambda \frac{e^{\lambda(\Delta_2-\Delta_1+\Delta_3)}}{(1 + ae^{2\lambda})^2},
\]

(76)

where we define $a \equiv \frac{-2X_2 \cdot X_3}{-2X_1 \cdot X_3}$. Changing the integral variable to $u = \frac{1}{1+ae^{2\lambda}}$, we obtain
\[
\mathcal{W}(X_1, X_2, X_3; \Delta_i) = \frac{1}{2} B \left( \frac{1}{2}(\Delta_3 + \Delta_2 - \Delta_1), \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1) \right) \\
\times \left( \frac{1}{-2X_1 \cdot X_2} \right)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)} \left( \frac{1}{-2X_2 \cdot X_3} \right)^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)}.
\]

(77)

Here $B(x, y)$ is the beta function $B(x, y) \equiv \int_0^1 du u^{x-1}(1 - u)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and this integral is convergent if $\Re x > 0, \Re y > 0$. For the convergence of the amplitude of GWD (77), we need the condition
\[
\Delta_3 > |\Delta_1 - \Delta_2|,
\]

(78)

which is the same condition in section 3. After substituting $(-2X_i \cdot X_j) = |x_{ij}|^2$ into (77), it is the same as (14) up to constant. In other words, the three point scalar GWD also provides the three point scalar correlation function in CFT. We note that (14) can be rewritten in terms of the three point GWD by replacing $G_{ab}(y(\lambda), y(\lambda'); \Delta)$ with $G_{\eta\rho}(y(\lambda), x_3; \Delta_3)$ in (24).
Figure 4: The three point scalar geodesic Witten diagram. The amplitude of this diagram also becomes the form of the scalar three point function in CFT as well as the Witten diagram.

C Proof by embedding formalism

In this appendix, we give a more transparent proof of the correspondence discussed in section 3 by using the embedding formalism and the conformal Casimir equation. The embedding formalism is reviewed in section 4.1.

C.1 Conformal Casimir equation

There is another way to show the correspondence between GWD $W^{(1,0,0,0)}_{\Delta,0}(x_i; \Delta_i)$ and CPW $W^{(1,0,0,0)}_{\Delta,0}(x_i; \Delta_i)$, namely, checking that GWD satisfies the conformal Casimir equation. This is because the conformal Casimir equation is the equation of which CPW is the solution. First, we derive the conformal Casimir equation for $W^{(1,0,0,0)}_{\Delta,0}(x_i; \Delta_i)$.

For convenience, we introduce the Lorentz generators $L_{AB}$ in $(d+2)$-dimension which are equivalent to the generators of conformal symmetry $SO(d+1,1)$ in $d$-dimension. Any local field $\mathcal{O}(x)$ in CFT is transformed under $L_{AB}$ as

$$[L_{AB}, \mathcal{O}(x)] = (L_x)_{AB}\mathcal{O}(x). \quad (79)$$

Here $L_x$ is the differential operator acting on fields at $x$ and $AB$ denotes the label of the generators. The explicit form of $(L_x)_{AB}$ depends on the conformal dimension $\Delta$ and spin $\ell$ of $\mathcal{O}(x)$, which are now suppressed. We will display $(L_x)_{AB}$ explicitly after introducing the embedding formalism.

In terms of the complete set for the conformal family, CPW $W^{(1,0,0,0)}_{\Delta,\ell}$ can be expressed as

$$W^{(1,0,0,0)}_{\Delta,0} = \frac{1}{C_{120}C_{34}} \sum_{\alpha} \langle 0 | J_1(x_1) \mathcal{O}_2(x_2) | \alpha \rangle \langle \alpha | \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) | 0 \rangle, \quad (80)$$

where $|\alpha\rangle$s denote the scalar primary state $|\mathcal{O}\rangle$ whose conformal dimension is $\Delta$ and its descendants. In order to derive the conformal Casimir equation for CPW, we define the quadratic Casimir $L^2 \equiv \frac{1}{2} L_{AB} L^{AB}$. The scalar primary state $|\mathcal{O}\rangle$ has the eigenvalue $C_2(\Delta, 0)$ of $L^2$ [6].

$$L^2 |\mathcal{O}\rangle = C_2(\Delta, 0) |\mathcal{O}\rangle = -\Delta(\Delta - d) |\mathcal{O}\rangle. \quad (81)$$
Since $L^2$ commutes with all generators $L_{AB}$, $L^2$ has the same eigenvalue in the descendants. Let us consider how $L^2$ acts on CPW. From [79] and the conformal invariance of the vacuum, we obtain

$$
(L_x^{(1)} + L_x^{(0)})_{AB} \langle 0 | J_1(x) O_2(x_2) | \alpha \rangle = - \langle 0 | J_1(x) O_2(x_2) L_{AB} | \alpha \rangle,
$$

where $\ell$ of $L_x^{(\ell)}$ implies spin of the operator on $x$. Using (82) twice, we get

$$
(L_x^{(1)} + L_x^{(0)})^2 \langle 0 | J_1(x) O_2(x_2) | \alpha \rangle = \langle 0 | J_1(x) O_2(x_2) L^2 | \alpha \rangle.
$$

Since all of $|\alpha\rangle$s have the same Casimir eigenvalue $C_2(\Delta, 0)$, we obtain the second-order differential equation

$$
(L_x^{(1)} + L_x^{(0)})^2 W_{\Delta,0}^{(1,0,0,0)} = C_2(\Delta, 0) W_{\Delta,0}^{(1,0,0,0)}.
$$

This equation is the so-called conformal Casimir equation for CPW. After taking the appropriate boundary condition, we will obtain a unique solution (up to constant). One can easily extend the above discussion to the case of generic external primary operators and intermediate states.

### C.2 Proof by the conformal Casimir equation

Based on the above preparation, we show that GWD $W_{\Delta,0}^{(1,0,0,0)}$ satisfies the conformal Casimir equation [84]. Any field $O$ that belongs to any spin representation on the projective light cone is transformed by the generators $L_{AB}$ as

$$
[L_{AB}, O(X; Z)] = (L_X^{(\ell_O)})_{AB} O(X; Z),
$$

where

$$
(L_X^{(\ell_O)})_{AB} \equiv X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} + S_{AB}^{(\ell_O)}.
$$

Here $S_{AB}^{(\ell_O)}$ depends on spin $\ell_O$ of $O(X; Z)$, for example,

$$
S_{AB}^{(0)} = 0, \quad (S_{AB}^{(1)})_{CD} = \eta_{AC} \eta_{BD} - \eta_{BC} \eta_{AD}.
$$

We define $S_{AB}^{(1)}$ as $(S_{AB}^{(1)})_{CD}$ in the index-free notation\footnote{We use $\frac{\partial}{\partial Z^A}$ instead of $D_A$ introduced in [46] because these are effectively the same operators when acting on first-order polynomials of $Z^A$.}

$$
S_{AB}^{(1)} \equiv Z_C (S_{AB}^{(1)})_{CD} \frac{\partial}{\partial Z^D} = Z_A \frac{\partial}{\partial Z^B} - Z_B \frac{\partial}{\partial Z^A}.
$$

Since the isometry group of AdS is also $SO(d + 1, 1)$, we can use the same operators for generators of isometry. For example, we can define a differential operator for isometry,

$$
(L_Y^{(0)})_{AB} \equiv Y_A \frac{\partial}{\partial Y^B} - Y_B \frac{\partial}{\partial Y^A}.
$$

It is enough to define $L_Y^{(0)}$ because we concentrate on scalar exchange in the bulk. We will use an important identity [47]

$$
-\frac{1}{2} (L_Y^{(0)})_{AB} (L_Y^{(0)})^{AB} f(Y) = \nabla^2_Y f(Y),
$$

\footnote{We use $\frac{\partial}{\partial Z^A}$ instead of $D_A$ introduced in [46] because these are effectively the same operators when acting on first-order polynomials of $Z^A$.}
where \( f(Y) \) is an arbitrary scalar function on AdS.

Let us check that GWD \( W_{\Delta,0}^{(1,0,0),0} \) satisfies the conformal Casimir equation. Our \( F_{\Delta_1,\Delta_2,\Delta}^{(1,0,0),0}(X_1, X_2, Y_2; Z_1) \) is manifestly invariant under \( SO(d + 1, 1) \) rotation. This means
\[
(L_{X_1}^{(1)} + L_{X_2}^{(0)} + L_{Y_2}^{(0)})_{AB} F_{\Delta_1,\Delta_2,\Delta}^{(1,0,0),0}(X_1, X_2, Y_2; Z_1) = 0.
\]
By using (90) and (91), we obtain a key identity
\[
-(L_{X_1}^{(1)} + L_{X_2}^{(0)})^2 F_{\Delta_1,\Delta_2,\Delta}^{(1,0,0),0} = \nabla^2_{Y_2} F_{\Delta_1,\Delta_2,\Delta}^{(1,0,0),0}.
\]
Since \( G_{\Delta\Delta}(Y_1, Y_2; \Delta) \) is an eigenfunction of \( \nabla^2_{Y_2} \) and its eigenvalue is \( \Delta(\Delta - d) \), we obtain
\[
-(L_{X_1}^{(1)} + L_{X_2}^{(0)})^2 W_{\Delta,0}^{(1,0,0),0} = \Delta(\Delta - d) W_{\Delta,0}^{(1,0,0),0},
\]
where we have assumed that the two geodesics do not intersect each other. This is just the same as the conformal Casimir equation for CPW. Thus we have shown that GWD \( W_{\Delta,0}^{(1,0,0),0} \) satisfies the conformal Casimir equation and GWD \( W_{\Delta,0}^{(1,0,0),0} \) is equivalent to CPW \( W_{\Delta,0}^{(1,0,0),0} \) in the embedding formalism.

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