Six-Dimensional Nearly Kähler Manifolds of Cohomogeneity One (II)

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Abstract: Let $M$ be a six dimensional manifold, endowed with a cohomogeneity one action of $G = SU_2 \times SU_2$, and $M_{\text{reg}} \subset M$ its subset of regular points. We show that $M_{\text{reg}}$ admits a smooth, 2-parameter family of $G$-invariant, non-isometric strict nearly Kähler structures and that a 1-parameter subfamily of such structures smoothly extends over a singular orbit of type $S^3$. This determines a new class of examples of nearly Kähler structures on $TS^3$.

1. Introduction

A Riemannian manifold $(M, g)$ is called nearly Kähler (NK) if it admits an almost complex structure $J$, such that $g$ is Hermitian and the Levi-Civita connection $\nabla$ satisfies $\left(\nabla_X J\right)X = 0$ for every vector field $X$. A NK structure $(g, J)$ is called strict if $\nabla_v J|_p \neq 0$ for every $p \in M$ and every $0 \neq v \in T_p M$ (see e.g. [13, 14, 22–24] for main properties). Significant examples are the so-called 3-symmetric spaces with their canonical almost complex structures. Recall also that the NK manifolds constitute one of the sixteen classes of Gray and Hervella’s classification of almost Hermitian manifolds and their canonical Hermitian connection $D$ has totally skew and $D$-parallel torsion ([1, 10, 15, 16, 23]).

According to Nagy’s structure theorem ([23, 24]), any complete strict NK manifold is finitely covered by a product of homogeneous 3-symmetric manifolds, twistor spaces of positive quaternion Kähler manifolds with their canonical NK structure and six dimensional strict NK manifolds. It is also known that, in six dimensions, the “strictness” condition is equivalent to the fact that the NK structure is not Kähler and that strict NK manifolds have a $SU_3$-structure with a parallel connection, are automatically Einstein and admit a nonzero Killing spinor, forcing the holonomy of the conical metric $dt^2 + r^2 g$ to be inside $G_2$ (see e.g. [1, 4, 10, 14, 17, 25]). These are the main mathematical motivations for investigating six dimensional strict NK manifolds. The compact manifolds of this kind are also of a very special interest in Theoretical Physics, since they belong to the
very few admissible cases in the context of compactifications with internal fluxes in heterotic string theories (the reader is referred to [8] and references therein). They are also related to other questions of interest in superstring theories (see e.g. [1, 4, 6, 10, 11, 25]).

Up to now, the only known examples of compact, six dimensional, strict NK manifolds are the six dimensional 3-symmetric spaces endowed with their natural NK structures, namely the standard sphere \( S^6 = G_2 / SU_3 \), the twistor spaces \( \mathbb{CP}^2 = Sp_2 / U(1) \times Sp_1 \) and \( F = SU_3 / U(1)^2 \), and the space \( S^3 \times S^3 = SU^3_2 / SU_2 \). This is actually the list of homogeneous strict NK manifolds in six dimensions ([3]).

In [26], we started the study of six dimensional strict NK manifolds \((M, g, J)\), admitting a compact Lie group \( G \) of automorphisms with a codimension one orbit. In these hypotheses, we proved that if \( M \) is compact:

a) \( G \) is semisimple and locally isomorphic to \( SU_3 \) or \( SU_2 \times SU_2 \);

b) if \( G \) is locally isomorphic to \( SU_3 \), then \((M, g)\) has constant sectional curvature (this is true also if \( M \) is not compact);

c) if \( G \) is locally isomorphic to \( SU_2 \times SU_2 \), then \( M \) is \( G \)-diffeomorphic to one of the following 3-symmetric spaces: the sphere \( S^6 = G_2 / SU_3 \), the projective space \( \mathbb{CP}^2 = Sp_2 / U(1) \times Sp_1 \) or \( S^3 \times S^3 = SU^3_2 / SU_2 \).

In this paper we focus on the six dimensional strict NK manifolds \((M, g, J)\) (not necessarily compact), on which \( G = SU_2 \times SU_2 \) acts by automorphisms with cohomogeneity one, i.e. with a codimension one orbit. In particular, we know that a principal isotropy subgroup is isomorphic to a one-dimensional torus \( T^1_{\text{diag}} \), which is diagonally embedded in \( G \) (see [26]).

We start from the following known fact: any strict NK structure \((g, J)\) on a given six dimensional, oriented manifold can be completely recovered by its Kähler form \( \omega = g(J\cdot, \cdot) \) and any non-degenerate 2-form \( \omega \), whose differential \( d\omega \) is stable in the sense of Hitchin ([19, 20]) and satisfying a suitable differential problem, is the Kähler form of a strict NK structure \((g, J)\) (see Theorem 2.2 for the complete statement). From this, we obtain that, locally, any \( G \)-invariant NK structure is completely determined by a smooth map \( f = (f_1, \ldots, f_3) : \mathbb{R} \times \mathbb{R}^5 \rightarrow \mathbb{R}^5 \), which solves a certain differential problem and which gives the components of the Kähler form \( \omega \) with respect to a special basis of \( G \)-invariant 2-forms along the points of a normal geodesic \( \gamma : \mathbb{R} \rightarrow M \). We then study the solvability of the differential problem on \( f \), first on an open set of \( G \)-principal points and then on a suitable tubular neighborhood of a singular orbit diffeomorphic to \( S^3 \). Our main results can be outlined as follows.

**Theorem 1.1.** Let \( G = SU_2 \times SU_2 \). Then:

1) There exists a 2-parameter family of non-isometric, non locally homogeneous, \( G \)-invariant strict NK structures on \( G / K \times \mathbb{R} \), with \( K = T^1_{\text{diag}} \).

2) There exists a family of non isometric, \( G \)-invariant strict NK structures on \( TS^3 \cong G \times SU_2 \mathbb{R}^3 \), smoothly parameterized by \( [0, +\infty[ \subset \mathbb{R} \). All such structures are non locally homogeneous, except precisely two of them, which are \( G \)-equivalent to those of suitable tubular neighborhoods of the singular \( G \)-orbit \( S \simeq S^3 \) in \( S^6 \) and \( S^3 \times S^3 \), respectively.

As an interesting consequence of (2), we have that the two locally homogeneous NK structures \((g_0, J_0)\) and \((g_1, J_1)\) on \( TS^3 \) are connected by a smooth family of non-locally homogeneous NK structures \((g_t, J_t), t \in [0, 1] \). The interesting problem to determine how many of these NK structures can be \( G \)-equivariantly completed remains unsolved.
Note that any such structure admitting a $G$-equivariant compactification would give a new NK structure on either $S^6$ or $S^3 \times S^3$.

Other interesting information is given by the proofs of the above results. In fact, the proof of (1) clearly shows that any smooth family of non-isometric $G$-invariant NK structures on $G/K \times \mathbb{R}$ can be described by at most two parameters. The proof of (2) shows that any $G$-invariant NK structure $(g, J)$ on $TS^3$ is isometric to one of the structures described above. We may therefore say that the isometric moduli space $\mathcal{M}$ of such NK structures on $TS^3$ can be identified with $\mathcal{M} = \mathbb{R}$.

The structure of the paper is as follows. In §2 and §3, we recall some known facts on stable 3-forms and six dimensional strict NK structures and use them to determine the differential problem that characterizes the map $f = (f_1, \ldots, f_5)$, associated with $G$-invariant strict NK structures. In §4, we show that, modulo a finite group of transformations, the map $f$ is an isometric invariant of any such NK structure and we give the explicit expressions for the $f$’s associated with locally homogeneous NK structures. In §5 and §6 we prove claim (1) and (2) of Theorem 1.1, respectively.

Given a semisimple Lie group $G$ (which, in almost all the paper, is always assumed to be $G = SU_2 \times SU_2$) and an action of $G$ on a manifold $M$, we denote by $g$ its Lie algebra, by $B$ the Cartan-Killing form of $g$ and we use symbols of the form $\hat{X}$ to indicate the vector fields on $M$ corresponding to elements $X \in g$.

2. Preliminaries

2.1. Stable 3-forms of oriented 6-dimensional vector spaces. Let $V$ be a 6-dimensional real vector space with a fixed orientation and consider the standard action on $\Lambda^3 V$ of the orientation preserving transformation in $G = GL^+(V)$. It is known (e.g. [20, 29]) that $\Lambda^3 V$ can be divided into the following disjoint $G$-invariant sets: a $G$-invariant hyper-surface $(\Lambda^3 V)_0 = \{ P = 0 \}$, given by the zero set of a suitable relative $G$-invariant irreducible polynomial $P$ of degree 4, and the complementary open sets

$$(\Lambda^3 V)_- = \{ P < 0 \}, \quad (\Lambda^3 V)_+ = \{ P > 0 \}.$$ 

These two sets are open $G$-orbits and their generic stabilizers have connected components conjugate to $SL_3(\mathbb{C})$ and $SL_3(\mathbb{R}) \times SL_3(\mathbb{R})$ respectively.

The polynomial $P$ is defined as follows. We consider the isomorphism $A : \Lambda^5 V^* \to \text{Hom}(V^*, \Lambda^6 V^*) \cong V \otimes \Lambda^6 V^*$ that is induced by the wedge product $\wedge : \Lambda^5 V^* \otimes V^* \to \Lambda^6 V^*$. We fix a non-zero element $\tau \in \Lambda^6 V^*$ and for every $\theta \in \Lambda^3 V^*$ we define $S_\theta \in \text{End}(V)$ as follows: given $v \in V$,

$$A(t_v \theta \wedge \theta) = S_\theta(v) \otimes \tau. \quad (2.1)$$

One can check that $S_\theta^2 = P(\theta)Id_V$ for some polynomial map $P : \Lambda^3 V^* \to \mathbb{R}$ and that $P$ is irreducible, of degree 4 and $SL(V)$-invariant (see e.g. [29], p. 80), which depends on the choice of volume form as follows: if $P$, $P'$ are determined using volume forms $\tau$, $\tau' = c \cdot \tau$, respectively, then

$$P' = \frac{1}{c^2} P. \quad (2.2)$$

By definitions, for any $\theta \in (\Lambda^3 V)_- = \{ P < 0 \}$, the endomorphism

$$J_\theta := \frac{1}{\sqrt{-P(\theta)}} S_\theta \quad (2.3)$$
is a complex structure. We call any 3-form $\theta \in (\Lambda^3 V)_-$ a stable 3-form and the corresponding $J_\theta$ the complex structure determined by $\theta$ (relative to the given orientation). We conclude pointing out that for any $\theta \in (\Lambda^3 V)_-$ the complex 3-form

$$\alpha = \frac{1}{2} \left( \theta + i J_\theta^* \theta \right)$$

is of type (3,0) (see e.g. Prop. 2 and formulae (8), (9) in [20]). In particular, $\iota (v + i J_\theta v) \alpha = 0$ for any $v \in V$ and hence, for any $v_1, v_2, v_3 \in V$,

$$\theta (J_\theta v_1, J_\theta v_2, J_\theta v_3) = \theta (J_\theta v_1, v_2, v_3).$$

(2.5)

2.2. Stable 3-forms and NK-structures. Let $M$ be a 6-dimensional oriented manifold.

**Definition 2.1.** A 3-form $\psi$ on $M$ is called stable if $\psi_x$ is stable for any $x \in M$. If $\psi$ is stable, we consider the almost complex structure $J_\psi$ on $M$ such that $(J_\psi)_x := J_{\psi_x}$ at every $x \in M$, where $J_{\psi_x}$ is defined as in (2.3).

The following is well-known (see e.g. [7,19,20,27,28]).

**Theorem 2.2.** Let $\omega \in \Lambda^2 T^* M$ and $\psi \in \Lambda^3 T^* M$ so that:

i) $\psi$ is stable, $\omega$ is $J_\psi$-invariant and $g = \omega (\cdot, J_\psi \cdot)$ is positively or negatively defined;

ii) there exists $\mu \in \mathbb{R}^+$ so that

$$\begin{cases}
  d \omega = 3 \psi \\
  d (J_\psi^* \psi) = -2 \mu \cdot \omega \wedge \omega.
\end{cases}$$

(2.6)

Then $(g, J_\psi)$ (or $(g, -J_\psi)$) is a strict NK structure on $M$ with scalar curvature $s = 30 \mu$.

Conversely, let $(g, J)$ be a strict NK structure on $M$ with (constant) scalar curvature $s$ and denote by $\omega = g(J \cdot, \cdot)$ and $\psi = \frac{1}{2} d \omega$. Then $\omega$ and $\psi$ satisfy (i) and (ii) with $\mu = \frac{s}{30}$ and $J = \pm J_\psi$.

**Remark 2.3.** In (i), the condition “$\omega$ is $J_\psi$-invariant” is indeed redundant, if the system (2.6) is satisfied. Namely, $3 \omega \wedge \psi = \frac{1}{2} d (\omega^2) = 0$.

**Remark 2.4.** Conditions (i) and (ii) were considered for the first time by Reyes Carrión in his PhD thesis.

2.3. NK-structures of cohomogeneity one for an $SU_2 \times SU_2$-action. We recall that the action of a compact connected Lie group $G$, acting almost effectively and isometrically on a Riemannian manifold $(M, g)$, is of cohomogeneity one if the generic orbits have codimension one. The points in such generic orbits (i.e. whose $G$-isotropy is, up to conjugation, minimal) are called regular and constitute an open and dense $G$-invariant subset $M_{\text{reg}} \subset M$.

If we denote by $\xi$ a unit vector field on $M_{\text{reg}}$, which is orthogonal to all $G$-orbits, its integral curves are geodesics that meet every $G$-orbit orthogonally. Moreover, any regular orbit $G \cdot p = G/K$ admits a tubular neighborhood which is $G$-equivariantly isometric to the product $]a, b[ \times G/K$, $]a, b[ \subset \mathbb{R}$, endowed with the metric

$$g = dt^2 + g_t,$$

(2.7)
where $g_t$ is a smooth family of $G$-invariant metrics on $G/K$ and the vector field $\partial/\partial t$ corresponds to $\xi$.

Let $(g, J)$ be an NK structure on $M$ and $G$ a compact connected Lie group acting isometrically with cohomogeneity one on $(M, g)$. Throughout the following we will always suppose that $G$ preserves the almost complex structure $J$. Note that this condition is automatically satisfied if the manifold is compact and the metric has not constant sectional curvature (see [22], Prop. 3.1).

Due to (2.7), any regular point of an NK manifold of cohomogeneity one admits a neighborhood which is locally identifiable with a Riemannian manifold of the form $(\omega, \psi)$, that satisfies (i) and (ii) of Theorem (2.2) and so that for every $X \in \mathfrak{g}$,

$$
 g(\xi, \xi) = 1, \quad g(\xi, \hat{X}) = 0. \quad (2.8)
$$

In the rest of the paper, we will be concerned with the strict NK-structures on 6-dimensional manifolds of cohomogeneity one w.r.t. $G = SU_2 \times SU_2$. By Lemma 3.3 in [26], which holds also when the manifold is not compact, we can suppose that, up to an automorphism of $\mathfrak{g}$, the Lie algebra $\mathfrak{f}$ of a regular isotropy subgroup $K$ is diagonally embedded into a Cartan subalgebra of $\mathfrak{g}$, i.e.

$$
 \mathfrak{f} = \mathbb{R} \cdot (H, H) \in su_2 + su_2 \quad (2.9)
$$

for some generator $H$ of a Cartan subalgebra $\mathfrak{h} \subset su_2$.

We conclude this section by fixing some notation and a particular basis for $su_2 + su_2$, which is particularly useful for our computations and will be constantly used in the rest of the paper.

- $\mathcal{B}$ always denotes the (negatively defined) Cartan-Killing form of $su_2$;
- $n = (\mathbb{R} \cdot H)\perp$ is the $\mathcal{B}$-orthogonal complement of $\mathfrak{h} = \mathbb{R} \cdot H$ in $su_2$;
- with no loss of generality, we always assume that $\text{ad}(H)|_n$ is a complex structure on $n$;
- $(E, V)$ is a basis for $n$ with $\mathcal{B}(E, E) = \mathcal{B}(V, V) = -1$ and $V = [H, E]$;
- the basis of $su_2 + su_2$ which we constantly consider is given by the elements

$$
 U := (H, H), \quad A := (H, -H), \\
 E_1 := (E, 0), \quad V_1 := (V, 0), \quad E_2 := (0, E), \quad V_2 := (0, V); \quad (2.10)
$$

- $\gamma$ is the curve $\gamma_t = (t, eK) \in ]a, b[ \times (SU_2 \times SU_2)/K$ and $\xi = \frac{\partial}{\partial t}$;
- for any point $\gamma_t$, we denote by $B_t$ the basis for $T_{\gamma_t}M$ equal to

$$
 B_t = (\xi, \hat{A}, \hat{E}_1, \hat{V}_1, \hat{E}_2, \hat{V}_2)_\gamma; \quad (2.11)
$$

we also denote by $B_t^* = (\xi^*, A^*, E_1^*, V_1^*, E_2^*, V_2^*)_\gamma$ the corresponding dual coframe in $T_{\gamma_t}^*M$. With no loss of generality, we will always assume that $M$ is oriented and that $B_t$ is in the prescribed orientation for any $t$.

3. The Equations

In this section, we want to determine the differential problem that characterizes the $G$-invariant pairs $(\omega, \psi)$, with $G = SU_2 \times SU_2$, on a manifold $M$ as in (2.9), corresponding to NK structures.
We first describe the $G$-invariant 2- and 3-forms on $M$. With the notation and assumptions of §2.3, any $G$-invariant $p$-form $\sigma$ is uniquely determined by the values of $\sigma|_{\gamma}$ on the tangent spaces $T_{\gamma}M$, $\gamma = (t,eK)$. Since the curve $\gamma([a,b]) \subseteq \text{Fix}(K)$, the form $\sigma$ is $G$-invariant if and only if $\sigma|_{\gamma}$ is $K$-invariant for any $t$.

Now, the tangent space $T_{\gamma}M$ decomposes into the following $K$-moduli
\[ T_{\gamma}M = (E_{1}|_{\gamma}, \tilde{V}_{1}|_{\gamma}) \oplus (E_{2}|_{\gamma}, \tilde{V}_{2}|_{\gamma}) \oplus \langle \xi_{\gamma}, \tilde{A}_{\gamma} \rangle \] (3.1)
where $K$ acts irreducibly on the first two and trivially on the last one.

Using this, we determine the spaces of $K$-invariant 2-forms of the tangent spaces $T_{\gamma}M$ and we obtain that the space of $G$-invariant 2-forms is generated over $C^\infty([a,b])$ by the five invariant forms $\omega^i, 1 \leq i \leq 5$, defined by
\[
\begin{align*}
\omega^1|_{\gamma} &= \xi^* \wedge A^*, \\
\omega^2|_{\gamma} &= E^*_1 \wedge V^*_1, \\
\omega^3|_{\gamma} &= E^*_2 \wedge V^*_2, \\
\omega^4|_{\gamma} &= E^*_1 \wedge E^*_2 + V^*_1 \wedge V^*_2, \\
\omega^5|_{\gamma} &= E^*_1 \wedge V^*_2 - V^*_1 \wedge E^*_2.
\end{align*}
\] (3.2)

In a similar way, we obtain that the space of $G$-invariant 3-forms on $M$ is generated over $C^\infty([a,b])$ by the eight invariant 3-forms $\psi^{ai}, a = 1, 2, i = 2, \ldots, 5$, defined by
\[
\begin{align*}
\psi^1|_{\gamma} := \xi^* \wedge \omega^i|_{\gamma}, \\
\psi^2|_{\gamma} := A^* \wedge \omega^i|_{\gamma}, 2 \leq i \leq 5.
\end{align*}
\] (3.3)

Due to this, the $G$-invariant strict NK structures on $M$ are in natural correspondence with the collections of real functions $f_i, p_{aj} \in C^\infty([a,b])$ such that the pairs
\[
\begin{align*}
&\omega = f_1 \omega^1, \\
&\psi = p_{aj} \psi^{aj}
\end{align*}
\] (3.3)
satisfy the conditions of Theorem 2.2 together with the constraints (2.8).

Let us now consider the following lemma.

**Lemma 3.1.** The condition $\psi = \frac{1}{3}d\omega$ on forms (3.3) is equivalent to the equations
\[
\begin{align*}
p_{22} &= p_{23} = 0, \\
p_{12} &= \frac{f'_2}{3} + \frac{f_1}{12}, \\
p_{13} &= \frac{f'_3}{3} - \frac{f_1}{12}, \\
p_{14} &= \frac{f'_4}{3}, \\
p_{15} &= \frac{f'_5}{3}, \\
p_{24} &= \frac{2}{3}p_{25}, \\
p_{25} &= -\frac{2}{3}f_4.
\end{align*}
\] (3.4)

**Proof.** Using the fact that the flow $\Phi^\xi$ of $\xi = \frac{\partial}{\partial t}$ commutes with the action of $G$ on $M$, it is immediate to realize that the $G$-invariant forms $\omega^i$ are also $\Phi^\xi$-invariant. By $G$- and $\Phi^\xi$-invariance and Koszul’s formula, it follows that for any $X, Y, Z \in \mathfrak{su}_2 + \mathfrak{su}_2$,
\[ d\omega^i(\xi, \hat{X}, \hat{Y}) = \omega^i(\xi, [\hat{X}, \hat{Y}]) \] (3.7)
and
\[ d\omega^i(\hat{X}, \hat{Y}, \hat{Z}) = \omega^j(\hat{X}, [\hat{Y}, \hat{Z}]) + \omega^j(\hat{Y}, [\hat{Z}, \hat{X}]) + \omega^j(\hat{Z}, [\hat{X}, \hat{Y}]). \] (3.8)

Using (3.7), (3.8) and the fact that $[E_i, V_i] = \frac{(-1)^{i+1}}{4}A(\text{mod } \mathbb{R})$ for $i = 1, 2$, we see that
\[
\begin{align*}
d\omega^1_{\gamma} &= \frac{1}{4} \xi^* \wedge (\omega^2 - \omega^3)|_{\gamma}, \\
d\omega^2_{\gamma} &= d\omega^3_{\gamma} = 0, \\
d\omega^4_{\gamma} &= -2(A^* \wedge \omega^5)|_{\gamma}, \\
d\omega^5_{\gamma} &= 2(A^* \wedge \omega^4)|_{\gamma}.
\end{align*}
\] (3.9)
From this and the $G$-equivariance, we have that the equality

$$\psi = p_{11} \xi^* \wedge \omega^j + p_{21} A^* \wedge \omega^j = \frac{1}{3} d \omega = \frac{1}{3} d(f_j \omega^j)$$

is equivalent to (3.4)–(3.5). □

The next lemma gives the conditions corresponding to the stability of $\psi$ and to condition (2.8).

**Lemma 3.2.** Given a pair (3.3) satisfying (3.4)–(3.6), we have that $\psi$ is stable, $\omega$ is $J_{\psi}$-invariant and $g = \omega(\cdot, J_{\psi} \cdot)$ satisfies (2.8) if and only if at all points of $|a, b|$ the following conditions hold

i) $f_1 < 0$ and $f_4$ and $f_5$ are of the form

$$f_4 = f_4 \cos \theta_o, \quad f_5 = f_4 \sin \theta_o$$

for a suitable function $0 < f_4 \in C^\infty(|a, b|)$ and some constant $\theta_o \in \mathbb{R}$;

ii)

$$4f_4^2 - \left( (f'_2 + \frac{f_1}{4}) \left( f'_3 - \frac{f_1}{4} \right) \right) (f_1)^2 = 0,$$

(iii)

$$\left( f'_2 + \frac{f_1}{4} \right) (f'_3 - \frac{f_1}{4}) > 0.$$  \hspace{1cm} (3.13)

If (i)–(iii) are satisfied, $J_\psi$ is represented in the basis (2.11) by a matrix of the form

$$J_\psi = \begin{pmatrix} K & 0 \\ 0 & L \end{pmatrix},$$

where, if we put $q = p_{14}^2 + p_{15}^2 - p_{12} p_{13},$

$$K = \begin{pmatrix} 0 & \sqrt{\frac{p_{24}^2 + p_{25}^2}{q}} \\ -\sqrt{\frac{q}{p_{24}^2 + p_{25}^2}} & 0 \end{pmatrix},$$

$$L = \frac{1}{\sqrt{q(p_{24}^2 + p_{25}^2)}} \begin{pmatrix} 0 & -p_{15} p_{24} + p_{14} p_{25} & p_{12} p_{24} & -p_{12} p_{25} & -p_{13} p_{25} & -p_{13} p_{24} & p_{15} p_{24} - p_{14} p_{25} \\ p_{14} p_{24} - p_{15} p_{25} & 0 & p_{13} p_{25} & 0 & p_{15} p_{24} - p_{14} p_{25} & 0 & -p_{12} p_{24} \\ -p_{12} p_{25} & p_{12} p_{24} & 0 & p_{13} p_{24} & p_{15} p_{24} - p_{14} p_{25} & 0 & p_{13} p_{25} \end{pmatrix}.$$

**Proof.** We fix a point $p = \gamma_o$ in the curve $\gamma$ and we consider the orientation of $T_p M$ given by $B_o$. From definitions, up to a factor, we get

$$S_{\psi_p}(\xi_p) = (p_{14} p_{24} + p_{15} p_{25}) \hat{\xi}_p + (p_{12} p_{13} - p_{14}^2 - p_{15}^2) \hat{A}_p,$$

$$S_{\psi_p}(\hat{A}_p) = (p_{24}^2 + p_{25}^2) \hat{\xi}_p - (p_{14} p_{24} + p_{15} p_{25}) \hat{A}_p.$$  \hspace{1cm} (3.16) (3.17)

Assume now that $\psi_p$ is stable (i.e. $P(\psi_p) < 0$) and that (2.8) holds. Since $S_{\psi_p} = \sqrt{-P(\psi_p)} J_{\psi_p}$ and $g_p(J_{\psi_p}(\xi_p), \xi_p) = 0$, we have

$$g(S_{\psi_p}(\xi_p), \xi_p) = p_{14} p_{24} + p_{15} p_{25} = 0 \Rightarrow f'_4 f_5 - f'_5 f_4 = 0. \hspace{1cm} (3.18)$$
Using (3.18) in (3.16) and (3.17), we get that
\[
P(\psi_p) = -(p_{24}^2 + p_{25}^2)(p_{14}^2 + p_{15}^2 - p_{12}p_{13}) - \frac{4}{81} \left( f_4^2 + f_5^2 \right) \left( (f_4')^2 + (f_5')^2 - \left( f_2' + \frac{f_1}{4} \right) \left( f_3' - \frac{f_1}{4} \right) \right) < 0. \tag{3.19}
\]

From this we see that \( f_4^2 + f_5^2 > 0 \). By (3.18) either \( \frac{f_4}{f_5} \) or \( \frac{f_5}{f_4} \) is constant (according to the case \( f_4 \neq 0 \) or \( f_5 \neq 0 \)). If we set \( f_4 := \sqrt{f_4^2 + f_5^2} \), we get that \( f_4 \) and \( f_5 \) are as in (3.11). Moreover, using (3.16), (2.8) and (3.19),
\[
f_1(t_0) = \omega_p(\xi, \widehat{A}) = -\frac{\sqrt{-P(\psi_p)}}{p_{14}^2 + p_{15}^2 - p_{12}p_{13}} < 0, \tag{3.20}
\]
and (i) follows. Then, (ii) follows from equality (3.20) using (i) and (3.5)–(3.6), while (iii) follows from (ii) using the fact that \( f_1 \) and \( f_4 \) do not vanish.

Conversely, it is immediate to check that (3.11)–(3.13) imply that \( \psi_p \) is stable and that (2.8) is satisfied.

Finally, assume (3.11)–(3.13). Through a direct but lengthy calculation, one can check that the complex structure \( J_\psi = \frac{1}{\sqrt{-P(\psi_p)}} S_\psi \) is of the form (3.14) at any point \( \gamma_1 \). Using this, one can also check that \( \omega \) is \( J_\psi \)-invariant. \( \square \)

**Theorem 3.3.** Let \((\omega, \psi)\) be as in (3.3). It satisfies (2.8) and all conditions of Theorem 2.2, but the positivity of \( g = \omega(\cdot, J_\psi \cdot) \), if and only if the \( p_{ij} \)'s are as in (3.4)–(3.6) and the functions \( f_i \)'s satisfy the following:

i) \( f_4 \) and \( f_5 \) are of the form \( f_4 = f_4 \cos \theta_0 \) and \( f_5 = f_4 \sin \theta_0 \) for some constant \( \theta_0 \in \mathbb{R} \) and a positive function \( 0 < f_4 \in C^\infty(]a, b[) \);

ii) \( f_1 < 0 \) and \( (f_3')^2 - \left( f_2' + \frac{f_1}{4} \right) \left( f_3' - \frac{f_1}{4} \right) > 0 \) at all points;

iii) \( f_1, f_2, f_3, f_4 \) satisfy the differential system
\[
\begin{align*}
(f_2' + \frac{1}{4} f_1) f_1' + 12 \mu f_1 f_2 &= 0, \\
(f_3' - \frac{1}{4} f_1) f_1' + 12 \mu f_1 f_3 &= 0, \\
(f_4 f_1)' - 4 \frac{f_4}{f_1} + 12 \mu f_1 f_4 &= 0, \\
f_1 (f_2' - f_3') + 12 \mu f_2 f_3 - \frac{f_1}{4} &= 0,
\end{align*}
\tag{3.21}
\]

Together with the algebraic condition
\[
4 f_4^2 - \left( (f_4')^2 - \left( f_2' + \frac{f_1}{4} \right) \left( f_3' - \frac{f_1}{4} \right) \right) (f_1)^2 \bigg|_{t=t_0} = 0 \tag{3.22}
\]

to be satisfied at some \( t_0 \in ]a, b[ \).

**Proof.** After Lemmas 3.1 and 3.2, it remains to consider the equation \( d \widehat{\psi} = -2 \mu \omega \wedge \omega \) with \( \widehat{\psi} = J_\psi \psi \).
By the $G$-invariance, $\hat{\psi} = J^*_\psi \psi$ is of the form $\hat{\psi} = \sum a_j \hat{p}_{aj} \psi^a$ for suitable $\hat{p}_{aj} \in C^\infty([a, b[). Using (2.5) and (3.14), we see that for $j = 2, \ldots, 5$,

$$\hat{p}_{1j} = \sqrt{\frac{p_{14}^2 + p_{15}^2 - p_{12}p_{13}}{p_{24}^2 + p_{25}^2}} p_{2j}, \quad \hat{p}_{2j} = -\sqrt{\frac{p_{14}^2 + p_{15}^2 - p_{12}p_{13}}{p_{24}^2 + p_{25}^2}} p_{1j}. \quad (3.23)$$

On the other hand, if we assume that (2.8) is satisfied, then (3.20) holds and therefore, using (3.4)–(3.6),

$$\hat{p}_{21} = \hat{p}_{11} = \hat{p}_{12} = p_{13} = 0, \quad (3.24)$$

$$\hat{p}_{22} = \frac{f_1}{3} \left( f'_2 + \frac{f_1}{4} \right), \quad \hat{p}_{23} = \frac{f_1}{3} \left( f'_3 - \frac{f_1}{4} \right), \quad (3.25)$$

$$\hat{p}_{24} = \frac{f_1 f'_4}{3}, \quad \hat{p}_{25} = \frac{-2 f_5}{f_1}, \quad \hat{p}_{14} = \frac{2 f_4}{3 f_1}, \quad \hat{p}_{15} = \frac{2 f_4}{3 f_1}. \quad (3.26)$$

We now compute $d\hat{\psi}$ along the curve $\gamma$. It is not difficult to see that

$$dA^* = \frac{1}{4}(\omega^3 - \omega^2), \quad (3.27)$$

and therefore, using (3.9), (3.10), (3.27) together with the fact that $\omega^i \wedge \omega^j = 0$ for $j > i = 2, 3$, we get that, at the points of $\gamma$,

$$d\hat{\psi} = \hat{p}_{22} \xi^* \wedge A^* \wedge \omega^2 + \hat{p}_{23} \xi^* \wedge A^* \wedge \omega^3 + (\hat{p}_{24} - 2\hat{p}_{15}) \xi^* \wedge A^* \wedge \omega^4$$

$$+ (\hat{p}_{25} + 2\hat{p}_{14}) \xi^* \wedge A^* \wedge \omega^5 + \frac{1}{4}(\hat{p}_{22} - \hat{p}_{23}) \xi^* \wedge \omega^2 \wedge \omega^3. \quad (3.28)$$

Now

$$\omega \wedge \omega|_\gamma = 2 \sum_{i=2}^5 f_1 f_i \xi^* \wedge A^* \wedge \omega^i|_\gamma + 2(f_2 f_3 - f_4 f_5) \omega^2 \wedge \omega^3|_\gamma \quad (3.29)$$

and therefore, comparing (3.28) and (3.29) and using (3.24), (3.24), (3.26), we see that the equation $d\hat{\psi} = -2\mu \omega \wedge \omega$ is equivalent to the system of equations (3.21).

Summing up, by Lemmas 3.1 and 3.2 and previous arguments, the conditions of Theorem 2.2 (with the only exception of the positivity of $g$) are equivalent to (i), (ii), (3.21) and (3.12). On the other hand, a straightforward check shows that whenever the $f_i$’s satisfy the first three equations of (3.21), then the derivative of (3.12) coincides with the derivative of the last equation of (3.21) multiplied by $-f_i(t)^2$. Hence, if we assume that $f_i(t) < 0$, for any solution of (3.21) the derivative of (3.12) is identically 0 and (3.12) is satisfied as soon as it is satisfied at just one point $t_o = [a, b[. \quad \square$

**Remark 3.4.** Given a NK structure $(g, J)$ on the manifold $M$, we may consider the corresponding Kähler form $\omega$ and the almost complex structure $\omega$ with $\psi = d\omega$. By Theorem 2.2 we have that $J = \pm J_\psi = J_{\pm \psi}$ and therefore the NK structure $(g, J)$ is uniquely associated to $(f_1, f_2, f_3, f_4, f_5)$ or to $(-f_1, -f_2, -f_3, -f_4, -f_5)$, where the $f_i$’s are the functions described in the theorem above.
4. Cohomogeneity One NK Structures that are Locally Homogeneous

4.1. Local and global homogeneity.

**Proposition 4.1.** Let \((M, g, J)\) be a 6-dimensional strict NK-manifold, admitting a cohomogeneity one action of \(SU_2 \times SU_2\) preserving the NK structure. Then \((M, g, J)\) is locally homogeneous if and only if it is locally equivalent to one of the following compact homogeneous NK spaces: (a) the standard sphere \(S^6 = G_2 / SU_3\); (b) the twistor space \(\mathbb{C}P^3 = Sp_2 / T^1 \times Sp_1\); (c) the homogeneous space \(S^3 \times S^3 = SU_2^3 / (SU_2)_{\text{diag}}\).

**Proof.** Given \(p_o \in M\), let \(g\) be the Lie algebra of the germs at \(p_o\) of the Killing vector fields of \((M, g)\) preserving \(J\), and denote by \(\mathfrak{t} \subset g\) the isotropy subalgebra at \(p_o\). By hypothesis

\[
\mathfrak{g} \supset su_2 + su_2, \quad \mathfrak{t} \supset \mathfrak{t} \cap (su_2 + su_2) = \mathbb{R}.
\]

We recall also that \(\mathfrak{t}\) is reductive and naturally embeddable into \(su_3\). It follows that it is isomorphic to \(\mathbb{R}, \mathbb{R}^2, su_2, \mathbb{R} + su_2\) or \(su_3\). We denote by \(G\) the simply connected Lie group with \(Lie(G) = \mathfrak{g}\) and by \(K \subset G\) the connected Lie subgroup with \(Lie(K) = \mathfrak{t}\).

We claim that \(K\) is closed in \(G\). Suppose not, so that \(\mathfrak{t} = Lie(\overline{K}) \supset \mathfrak{t}\). In this case \(\mathfrak{t}\) is an ideal of \(\mathfrak{t}\) and therefore, using the reductiveness of \(\mathfrak{t}\), there exists an \(\text{ad}(\mathfrak{t})\)-invariant decomposition \(\mathfrak{t} = \mathfrak{t} + p\) with \(p \neq \{0\}\) and \([\mathfrak{t}, p] = 0\).

Consider the reductive decomposition \(\mathfrak{g} = \mathfrak{t} + V\) of \(\mathfrak{g}\), with \(V \simeq \mathbb{C}^3\) and the adjoint action \(\text{ad}_V|_V \subset \mathfrak{su}(V)\). From the existence of the space \(p \neq \{0\}\) commuting with \(\mathfrak{t}\), we see that \(\mathfrak{t}\) is isomorphic to \(\mathbb{R}\) or to \(su_2\). Since \(K\) is not closed, the Lie algebra \(\mathfrak{t}\) is not semisimple (see e.g. [18] p.152), hence \(\mathfrak{t} \cong \mathbb{R}\) and therefore \(\mathfrak{t} = \mathfrak{t} \cap (su_2 + su_2)\). Hence \(K\) coincides with the isotropy subgroup of \(S = SU_2 \times SU_2\). In particular, \(K\) is closed in \(S\). Being semisimple, \(S\) is closed in \(G\) and \(K\) is closed in \(G\), too, contradicting \(K \subset \overline{K}\).

Being \(K \subset G\) closed, \((M, g)\) is locally isometric to an homogeneous NK manifold \((\overline{M} = G / K, \overline{g}, \overline{J})\) ([30,31]). Looking at the classification of 6-dimensional homogeneous NK manifolds ([3]) and using [26] Thm. 1.1, one can directly check that an homogeneous NK manifold admits a cohomogeneity one action of \(SU_2 \times SU_2\) if and only if it is one of those listed in the statement. \(\square\)

4.2. Locally homogeneous NK structures and their associated functions \(f_i\)’s.

4.2.1. The invariant NK structure of \(S^6 = G_2 / SU_3\). We identify the standard 6-dimensional sphere with the unit sphere \(S^6 \subset \text{Im} \mathbb{O} \simeq \mathbb{R}^7\) in the space of imaginary octonions, endowed with the standard metric \(g\) induced by the Euclidean product \(\langle x, y \rangle = \frac{1}{2}(x \bar{y} + y \bar{x})\). We recall that the algebra of octonions \(\mathbb{O}\) can be defined as the vector space \(\mathbb{H}^2 = \mathbb{H} \oplus (\mathbb{H} \cdot \epsilon)\), endowed with the product rule (see e.g. [5])

\[
(q_1 + q_2 \epsilon) \cdot (r_1 + r_2 \epsilon) := (q_1 r_1 - r_2^* q_2) + (r_2 q_1 + q_2 r_1^*) \epsilon.
\]

The sphere inherits a natural almost complex structure \(J\), which is defined as follows: for \(p \in S^6\) and \(v \in T_p S^6\), we define \(J_p(v) := \frac{1}{2}(p \cdot v - v \cdot p)\). It is well known that the pair \((g, J)\) is a \(G_2\)-invariant strict NK structure with scalar curvature \(s = 30\) ([12]).

Identifying unit quaternions with elements of \(SU_2\) in a standard way, we have the following action of \(G := SU_2 \times SU_2\) on \(\text{Im} \mathbb{O}\):

\[
(q_1, q_2) \cdot (a + b \cdot \epsilon) = (q_1 \cdot a \cdot q_1^* + (q_2 \cdot b \cdot q_1^*) \cdot \epsilon. \quad (4.1)
\]
This action provides an embedding of $G \subset G_2 = Aut(S^6, g, J)$ and the orbit space $S^6/G$ is one-dimensional with principal orbit diffeomorphic to $SU_2 \times SU_2/T^1_{\text{diag}}$, and two singular orbits

$$G \cdot i \simeq SU_2 \times SU_2/T^1 \times SU_2 = S^2, \quad G \cdot \varepsilon \simeq SU_2 \times SU_2/SU_2_{\text{diag}} = S^3.$$ 

The curve $\gamma_t = \cos t \cdot i + \sin t \cdot \varepsilon$ is a normal geodesic for this action, i.e. it is a geodesic orthogonal to all $G$-orbits. Moreover, a basis for $su_2 + su_2$, with the same properties of (2.10), is given by

$$U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ (4.2)

It is now easy to check that the Kähler form $\omega$ is given by $\omega = f_i \omega^i$, where the 2-forms $\omega_i$, $1 \leq i \leq 5$, are defined as in (3.2) and the functions $f_i$ are as follows:

$$f_1 = -\sin t, \quad f_2 = \frac{1}{8}(4 - 9 \sin^2 t) \cdot \cos t, \quad f_3 = -\frac{1}{8} \sin^2 t \cdot \cos t,$$

$$f_4 = 0, \quad f_5 = \frac{3}{8} \sin^2 t \cdot \cos t.$$

It is immediate to check that, for $t \in [0, \frac{\pi}{4}]$, these functions satisfy Theorem 3.3 (i)–(iii) with $\mu = \frac{s}{30} = 1$.

4.2.2. The invariant NK structure of $\mathbb{C}P^3 = Sp_2/T^1 \times Sp_1$. It is known that there exists an invariant strict NK structure $(g, J)$ on the twistor space $Sp_2/T^1 \times Sp_1 = \mathbb{C}P^3$, with scalar curvature $s = 60$, which can be described as follows. We consider the $\text{ad}(t_1 + sp_1)$-invariant decomposition $sp_2 = (t_1 + sp_1) + p_1 + p_2$, where $p_1 \simeq \mathbb{R}^2$ and $p_2 \simeq \mathbb{H}$. The module $p_1 + p_2$ identifies with the tangent space of $\mathbb{C}P^3$ at the origin $o := [T^1 \times Sp_1]$ and the metric $g$ can be described as the unique $\text{Ad}(T^1 \times Sp_1)$-invariant scalar product on $p_1 + p_2$ with the following properties: $g(p_1, p_2) = 0$, it induces on $p_2 = \mathbb{H}$ the standard Euclidean product $g(q_1, q_2) = Re(q_1^* \cdot q_2)$ and $g(W, W) = \frac{1}{2}$, where $W = \text{diag}(j, 0) \in p_2$ (see e.g. [32]). Finally, $J$ is defined as the unique $Sp_2$-invariant almost complex structure which corresponds to the multiplication by $-i$ on $p_1$ and by $i$ on $p_2$.

The subgroup $G = Sp_1 \times Sp_1 \subset Sp_2$ acts on $\mathbb{C}P^3$ with codimension one principal orbits $G$-equivalent to $SU_2 \times SU_2/T^1_{\text{diag}}$. A singular orbit is $G \cdot o$ and the curve $\gamma_t = \exp\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \cdot o$ is a normal geodesic for the action.

A basis for $su_2 + su_2 \subset sp_2$ as in (2.10) is given by the following matrices in $sp_2$:

$$U = \text{diag} \begin{pmatrix} i & i \\ i & -i \end{pmatrix}, \quad A = \text{diag} \begin{pmatrix} i & i \\ i & -i \end{pmatrix}, \quad E_1 = \text{diag} \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix},$$

$$V_1 = \text{diag} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \quad E_2 = \text{diag} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \quad V_2 = \text{diag} \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}.$$
As in the previous example, an easy computation shows that

\[ f_1 = \sin t \cdot \cos t, \quad f_2 = \frac{1}{16} (2 \sin^2 t - \cos^2 t) \cdot \cos^2 t, \]
\[ f_3 = \frac{1}{16} (2 \cos^2 t - \sin^2 t) \cdot \sin^2 t, \quad f_4 = 0, \quad f_5 = -\frac{3}{16} \sin^2 t \cdot \cos^2 t. \]

For \( t \in [0, \frac{\pi}{2}] \), the functions \(-f_i\)'s satisfy Theorem 3.3 (i)–(iii) with \( \mu = \frac{s}{30} = 2 \) (see Remark 3.4).

4.2.3. The invariant NK structure of \( S^3 \times S^3 = SU^2_3/(SU_2)_{\text{diag}} \). We recall that \( SU^2_3/(SU_2)_{\text{diag}} \) can be naturally identified with \( S^3 \times S^3 \) using the fact that \( L := SU^3_2 \) acts transitively on \( S^3 \times S^3 \cong SU_2 \times SU_2 \) by the map

\[ (g_1, g_2, g_3) \cdot (x_1, x_2) = (g_1 x_1 g_3^{-1}, g_2 x_2 g_3^{-1}) \]

with isotropy \( (SU_2)_{\text{diag}} \subset SU^3_2 \). We denote by \( h = \{ (X, X, X) : X \in su_2 \} \) the isotropy subalgebra of \( l \) and we consider the \( \text{ad}(h) \)-invariant decomposition \( l = h + m \), in which \( m = \{ (X_1, X_2, X_3) \in l, \sum_{i=1}^3 X_i = 0 \} \).

It is known that there exists an invariant NK structure \((g, J)\) on \( SU^2_3/(SU_2)_{\text{diag}} \), with scalar curvature \( s = 60 \), which is defined as follows. The almost complex structure \( J \) is (up to a sign) the unique invariant tensor which acts on \( m \) as follows:

\[ J(X_1, X_2, X_3) = \frac{1}{\sqrt{3}} (2X_3 + X_1, 2X_1 + X_2, 2X_2 + X_3). \]

The metric \( g \) is the invariant Riemannian metric which induces on \( m \) the inner product\( g := -\frac{1}{24}(B \oplus B \oplus B)|_{m \times m} \), where \( B \) is the Cartan-Killing form of \( su_2 \).

We consider the subgroup \( G \subset L \) given by \( G = \{ (g, h, g) \in L; g, h \in SU_2 \} \). Then \( G \cong SU_2 \times SU_2 \) acts on \( SU^3_2/(SU_2)_{\text{diag}} \) with cohomogeneity one; the following curve is easily checked to be a normal geodesic

\[ \gamma_t = \exp(tN) \cdot o, \quad N = \sqrt{6} \begin{pmatrix} i & 0 & 0 \\ i^2 & -i & 0 \\ 0 & 0 & i \end{pmatrix} \in m. \quad (4.3) \]

If we choose the same basis \( U, A, E_1, V_1, E_2, V_2 \) of \( su_2 + su_2 \) as in Sec. §4.2.1, we easily see that the corresponding functions \( f_i \) are given by

\[ f_1 = -\frac{\sqrt{3}}{3}, \quad f_2 = \frac{\sqrt{3}}{36} \sin(2\sqrt{6}t), \quad f_3 = 0, \]
\[ f_4 = 0, \quad f_5 = -\frac{\sqrt{3}}{36} \sin(\sqrt{6}t), \]

which satisfy Theorem 3.3 (i)–(iii) with \( \mu = \frac{s}{30} = 2 \) for \( t \in [0, \frac{\pi}{2\sqrt{6}}] \).
4.3. The functions $f_i$'s as isometric invariants.

**Proposition 4.2.** Let $(\omega = f_i \omega^i, \psi)$ and $(\tilde{\omega} = f_i \tilde{\omega}^i, \tilde{\psi})$ be pairs as in (3.4) satisfying Theorem 3.3 (i)-(iii) and associated with NK structures $(g, J)$ and $(\tilde{g}, \tilde{J})$, respectively, on $M = \sigma, b \times G / K$. Let also $f_4 = \sqrt{f_2^2 + f_5^2}$ and $\tilde{f}_4 = \sqrt{\tilde{f}_2^2 + \tilde{f}_5^2}$.

There exists a local isometry $\phi : \mathcal{U} \subset M \rightarrow M$ between $g$ and $\tilde{g}$ if and only if, on a suitable subinterval $I \subset [a, b]$ and modulo compositions with suitable shifts of parameters $t \rightarrow t + c$, the quadruple $(f_1, f_2, f_3, f_4)$ is $\tau(f_1, f_2, f_3, f_4)$, where $\tau$ belongs to the group of transformations generated by

\[ \tau_1(x_1(t), x_2(t), x_3(t), x_4(t)) = (-x_1(-t), x_2(-t), x_3(-t), x_4(-t)), \quad (4.4) \]
\[ \tau_2(x_1(t), x_2(t), x_3(t), x_4(t)) = (-x_1(t), -x_2(t), -x_3(t), x_4(t)), \quad (4.5) \]
\[ \tau_3(x_1(t), x_2(t), x_3(t), x_4(t)) = (x_1(t), -x_2(t), -x_3(t), x_4(t)). \quad (4.6) \]

**Proof.** Let $\phi : \mathcal{U} \rightarrow \mathcal{V}$ be an isometry, where $\mathcal{U}, \mathcal{V}$ are open subsets of $M$, with $\phi_* g = \tilde{g}$, hence $\phi_* J = \pm \tilde{J}$. We denote by $s$ and $\tilde{s}$ the Lie algebras of Killing vector fields on $(\mathcal{U}, g)$ and $(\mathcal{V}, \tilde{g})$ respectively, so that the map $\phi$ induces a Lie isomorphism $\phi_* : s \rightarrow \tilde{s}$.

First of all, we assume that $s \cong \tilde{s} \cong g = su_2 \oplus su_2$.

In this case, $\phi$ maps (locally) any $G$-orbit into a $G$-orbit and therefore $\phi_* (\xi) = \pm \xi$. Replacing $\phi$ by $h \circ \phi$ for a suitable $h \in G$ and up to a reparameterization $t \mapsto t + c$, we can suppose that $\phi(\gamma_t) = \gamma_{\pm t}$. We can always reduce the case $\phi(\gamma_t) = \gamma_t$, by possibly considering the map $\sigma : (t, gK) \mapsto (-t, gK)$ and the isometry $\phi' = \phi \circ \sigma$ between $\sigma_* g$ and $\tilde{g}$. Notice that the quadruple $(f_1', f_2', f_3', f_4')$ associated with $(\sigma_* g, \sigma_* J)$ is obtained from the quadruple of $(g, J)$ by the transformation (4.4).

If we now think of $\phi_*$ as an automorphism of the Lie algebra $g$, we see that $\phi_*(\mathfrak{k}) = \mathfrak{k}$ because $\phi$ preserves $\gamma$. It then follows that $\phi_*$ maps the centralizer of $\mathfrak{k}$ in $g$ onto itself, hence $\phi_*(A) = \pm A$.

**Case 1.** $\phi_*(A) = A$. If $n_i$ ($i = 1, 2$) denotes the linear span of $\{E_i, V_i\}$ in $g$, we see that $\phi_*$ either preserves or exchanges $n_1$ and $n_2$.

**Lemma 4.3.** If $\phi_*(n_i) = n_i$ for $i = 1, 2$, then the two quadruples coincide.

**Proof.** Put $\Phi_i = \phi_*|_{n_i} \in \text{End} (n_i)$ for $i = 1, 2$. Since $\phi_*(A) = A$, we have that $\Phi_i$ commutes with $\text{ad}(A)|_{n_i}$, hence det $\Phi_i = 1$. With respect to the basis $B_i = \{E_i, V_i\}$ of $n_i$, we can write the matrix of $\Phi_i$ as $\begin{pmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) \end{pmatrix}$ for some $\theta_i \in \mathbb{R}$, $i = 1, 2$. This implies that $\phi^*(\omega_j)|_{\gamma_t} = \omega_j|_{\gamma_t}$ for $j = 1, 2, 3$, while $\phi^* (\omega^4)|_{\gamma_t} = \cos(\theta_1 - \theta_2)\omega^4|_{\gamma_t}$ and $\phi^* (\omega^5)|_{\gamma_t} = \sin(\theta_1 - \theta_2)\omega^5|_{\gamma_t}$. Since $\phi^* (f_j \omega^j)|_{\gamma_t} = f_j \phi^* (\omega^j)|_{\gamma_t} = \tilde{f}_j \omega^j|_{\gamma_t}$, the claim follows. \(\Box\)

**Lemma 4.4.** If $\phi_*(n_1) = n_2$ and $\phi_*(n_2) = n_1$, then $(f_1, f_2, f_3, f_4) = \tau_3(f_1, f_2, f_3, f_4)$.

**Proof.** Let $\Phi = \begin{pmatrix} 0 & \Phi_1 \\ \Phi_2 & 0 \end{pmatrix}$ be the matrix of the endomorphism $\phi_*|_n$ w.r.t. the basis $B$ given by $(E_1, V_1, E_2, V_2)$, where $\Phi_i \in O(2)$ for $i = 1, 2$. Since $\phi_*$ commutes with $\text{ad}(A)$, we have that det $\Phi_i = -1$ for $i = 1, 2$. This implies that $\phi^* (\omega^1)|_{\gamma_t} = \omega^1|_{\gamma_t}$ and $\phi^* (\omega^2)|_{\gamma_t} = -\omega^2|_{\gamma_t}$, while $\phi^* (\omega^4)|_{\gamma_t} = (a\omega^4 + b\omega^5)|_{\gamma_t}$ and $\phi^* (\omega^5)|_{\gamma_t} = (-b\omega^4 + a\omega^5)|_{\gamma_t}$ for some constant $a, b$ with $a^2 + b^2 = 1$. Since $f_j \phi^* (\omega^j)|_{\gamma_t} = \tilde{f}_j \omega^j|_{\gamma_t}$, we get our claim. \(\Box\)
Case 2. \( \varphi_a(A) = -A \). We consider the automorphism \( \psi \) of \( G = SU_2 \times SU_2 \) given by \( \psi(g_1, g_2) = (g_2, g_1) \) and the corresponding diffeomorphism \( \hat{\psi} \) of \( G/K \) and of \( M \) given by \( \hat{\psi}(t, g)K = (t, \psi(g)K) \). Note that \( \varphi_a A = -A \) and that the NK structure \( (\hat{\psi}_a \hat{g}, \hat{\psi}_a \hat{J}) \) is represented by the quadruple \( \tau_2(f_1, f_2, f_3, f_4) \). By construction, the isometry \( \hat{\phi} := \hat{\psi} \circ \varphi \) satisfies the hypothesis of Case 1 and we get our claim.

The sufficiency can be dealt with in a similar way.

We now discuss the case where \( \mathfrak{s} \supseteq \mathfrak{g} \). By the results in [26] and Proposition 4.1, this occurs when the NK structure \( (g, J) \) and \( (\tilde{g}, \tilde{J}) \) are both locally homogeneous with \( \mathfrak{s} \simeq g_2, sp_2 \) or \( su_2 \). In all these cases, there exists a local isometry \( \psi : V \subset M \rightarrow V \subset M \) of \( (M, \tilde{g}) \), which preserves \( \tilde{J} \) and so that \( (\psi \circ \varphi)_a(g) = g \). Indeed, in the first two cases there is only one subgroup locally isomorphic to \( G \) in \( G_2 \) or \( Sp_2 \) up to conjugation. In the third case, two subgroups of \( L := SU_2^3 \), isomorphic to \( SU_2^3 \) and acting by cohomogeneity one on \( Q := SU_2^3/SU_2 \) are related by an outer automorphism \( \sigma \) of \( L \) interchanging two factors and any such \( \sigma \) induces an automorphism of the NK structure on \( Q \).

Hence, there exists a local equivalence \( \varphi \) between the NK structures \( (g, J) \) and \( (\tilde{g}, \tilde{J}) \) if and only if there exists a (possibly different) local equivalence of the two NK structures that, in addition, maps \( \mathfrak{g} \) into itself. From this, the conclusion follows from the first part of the proof. \( \Box \)

5. The Space of Solutions of the Differential System and Non-Homogeneous NK Structures

In this section, we study the space of solutions to the differential problem described in Theorem 3.3 (ii) and (iii). To this purpose, we consider the following change of variables. For any map \( (f_1, f_2, f_3, f_4) : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^4 \) with \( f_1 < 0 \), choose \( t_0 \in ]a, b[ \) and define

\[
\begin{align*}
  s(t) &:= \int_{t_0}^t \frac{1}{f_1(u)} \, du, & g(t) &:= \frac{1}{2} \int_{t_0}^t f_1(u) \, du.
\end{align*}
\]

Being \( s' = f_1 > 0 \), we may consider the inverse change of parameter \( t = s(t) \) and set

\[
\begin{align*}
  h_1(s) &:= g(t(s)), & h_2(s) &:= f_2(t(s)) + f_3(t(s)), \\
  h_3(s) &:= f_2(t(s)) - f_3(t(s)), & h_4(s) &:= 2 f_4(t(s)).
\end{align*}
\]

Using the fact that \( \frac{dt}{ds} = f_1(t(s)) \), one gets that \( h_1'(s) = \frac{1}{2} (f_1(t(s)))^2 \) and the system (3.21) turns out to be equivalent to a handier differential system on the \( h_1 \)'s. The results are given in the following proposition.

**Proposition 5.1.** The change of variables (5.1) gives a one-to-one correspondence between the maps \( (f_1, f_2, f_3, f_4) : [a, b] \rightarrow \mathbb{R}^4 \) satisfying (ii) and (iii) of Theorem 3.3 and the solutions \( (h_1, h_2, h_3, h_4) : [\alpha, \beta] \rightarrow \mathbb{R}^4 \) to the regular O.D.E. system:

\[
\begin{align*}
  h_1'' + \frac{2(h_1')^2h_3 + \frac{4}{\pi^2} h_4'}{h_2^2 - h_3^2 - h_4^2} &= 0, \\
  h_2' + 24 \mu h_1' h_2 &= 0, \\
  h_3' - \frac{2(h_1')^2h_3 + \frac{4}{\pi^2} h_4'}{h_2^2 - h_3^2 - h_4^2} + 24 \mu h_1' h_3 &= 0, \\
  h_4' + 24 \mu h_1' h_4 - 4 h_4 &= 0,
\end{align*}
\]
with initial conditions $h_i(0) := a_i$, $h'_i(0) := b_i$ satisfying the equations
\[
a_1 = 0, \quad \mathcal{I}(a_2, a_3, a_4, b_1, b_2, b_3, b_4) = 0_{\mathbb{R}^4},
\]
where the map $\mathcal{I} = (\mathcal{I}^1, \ldots, \mathcal{I}^4) : \mathbb{R}^7 \to \mathbb{R}^4$ is defined by
\[
\begin{align*}
\mathcal{I}^1 &= 12 \mu (a_2^2 - a_3^2 - a_4^2) + b_1 + b_3, \\
\mathcal{I}^2 &= 4a_4^2 + b_2^2 - b_3^2 - b_4^2 - b_1^2 - 2b_3b_1, \\
\mathcal{I}^3 &= a_2b_2 - a_3b_3 - a_4b_4 - a_3b_1, \\
\mathcal{I}^4 &= \frac{9\mu}{2} b_1 (a_2^2 - a_3^2 - a_4^2) + a_4^2,
\end{align*}
\]
and so that, for any $t$, the first derivative $b_j(t) = h'_j(t)$ satisfies the inequalities
\[
b_1 > 0, \quad b_2^2 - b_3^2 - b_4^2 - b_1^2 - 2b_1b_3 < 0.
\]

**Proof.** One can directly check that under the change of variables (5.1) the system (3.21) is equivalent to
\[
\begin{align*}
&h''_2 + 24\mu h'_2 h_2 = 0, \quad h'''_2 + h''_1 + 24\mu h'_1 h_3 = 0, \quad h''_4 + 24\mu h'_1 h_4 - 4h_4 = 0, \quad (5.6) \\
&h'_3 + h'_1 + 12\mu (h_2^2 - h_3^2 - h_4^2) = 0. \quad (5.7)
\end{align*}
\]
By the proof of Theorem 3.3, we also know that (3.22) is satisfied at $t_0$ if and only if it is satisfied for any $t$. So, by the same arguments, we have that the condition (3.22) is equivalent to the differential equation
\[
(h'_2)^2 - (h'_3)^2 - (h'_4)^2 - (h'_1)^2 - 2h'_1 h'_3 + 4 h'_3^2 = 0. \quad (5.8)
\]
Differentiating (5.7) and subtracting (5.6) 2, we get
\[
h_2h'_2 - h_3h'_3 - h_4h'_4 - h'_1h_3 = \frac{1}{2} (h_2^2 - h_3^2 - h_4^2)' - h'_1 h_3 = 0. \quad (5.9)
\]
Now, we differentiate (5.9) and replace the expressions for the second derivatives $h''_i$, $i = 2, 3, 4$, determined by (5.6), so to obtain the equation
\[
(h'_2)^2 - (h'_3)^2 - (h'_4)^2 - h'_1 h'_3 - 24\mu h'_1 (h_2^2 - h_3^2 - h_4^2) - 4h_4^2 = 0. \quad (5.10)
\]
Then, subtracting (5.8) from (5.10), we have
\[
(h'_1)^2 + h'_1 h'_3 - 8 h_4^2 - 24\mu h'_1 (h_2^2 - h_3^2 - h_4^2) = 0, \quad (5.11)
\]
and, using (5.7), we get
\[
h'_1 (h_2^2 - h_3^2 - h_4^2) + \frac{2}{9\mu} h_4^2 = 0. \quad (5.12)
\]
Finally, differentiating (5.12) and using (5.9) we obtain the differential equation
\[
h''_1 + \frac{2(h'_1)^2 h_3 + \frac{4}{9\mu} h'_1 h_4}{h_2^2 - h_3^2 - h_4^2} = 0, \quad (5.13)
\]
which together with (5.6) gives the regular system (5.2). From the way Eq. (5.13) has been derived, it is clear that Eqs. (5.7), (5.8), (5.9) and (5.12) are identically satisfied if and only if they are true at \( s = 0 \). They can therefore be considered as algebraic conditions on the initial conditions, to which one has to add the condition \( h_1(0) = 0 \) due to the definition of \( h_1 \). This gives (5.3). Since the inequalities (5.5) correspond to Theorem 3.3 (ii), the proposition follows. □

**Remark 5.2.** The proof of the previous theorem shows that the functions \( I_i \), \( 1 \leq i \leq 4 \), are first integrals of the system (5.2) and that the curve \( f = (f_1, f_2, f_3, f_4) \) satisfying (ii) and (iii) of Theorem 3.3 with \( \mu = 1 \) can be identified with a curve in the subset

\[
N := I^{-1}(0) \cap \{ b_1 > 0 \} \cap \{ b_2^2 - b_3^2 - b_4^2 < 0 \} \subset \mathbb{R}^7.
\]

Using Proposition 4.2 and the definitions of the functions \( h_i \)'s, one can check that two maximal solutions of (5.2) correspond to locally isometric structures \((g, J), (\bar{g}, \bar{J})\) (with \( g \) and \( \bar{g} \) possibly non positively defined) if and only if their images in \( N \) are equal up to a transformation in the group \( T \) of transformations of \( \mathbb{R}^7 \) generated by the elements

\[
\tau_1(a_2, \ldots, a_4, b_1, \ldots, b_4) = (-a_2, a_3, a_4, b_1, -b_2, b_3, b_4),
\]

\[
\tau_2(a_2, \ldots, a_4, b_1, \ldots, b_4) = (-a_2, -a_3, a_4, -b_1, -b_2, -b_3, b_4).
\]

In particular, we have that a solution of (5.2) corresponds to a locally homogeneous NK structure if and only if its trace in \( N \) is contained in one of the three curves, corresponding to the \( f_i \)'s described in §4.2.1–4.2.3, up to transformations in \( T \).

**Theorem 5.3.** There exists a 2-dimensional family of non-isometric, non locally homogeneous, strict NK structures on \( M = \{ \} - \varepsilon, \varepsilon[ \times G/K \) for a suitable \( \varepsilon > 0 \).

**Proof.** Consider the initial data for the differential problem (5.2)

\[
x_0 = (a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) = \frac{1}{36}(0, \sqrt{3}, \sqrt{3}, \sqrt{6}, 4, 0, 0, -2\sqrt{2}),
\]

whose solution \( H(t) \) corresponds to the homogeneous NK structure described in §4.2.3. The subset \( N \) defined in (5.14) can be easily shown to be a 3-dimensional smooth submanifold in a suitable neighborhood \( \mathcal{U} \) of \( x_0 \). We can also suppose that \( \mathcal{U} \cap \tau(\mathcal{U}) = \emptyset \) for any \( \tau \in T \). Since \( H(t) \) defines a positively defined metric \( g \), we can shrink \( \mathcal{U} \) so that any point \( x \in \mathcal{U} \setminus \text{Trace}(H) \) gives a solution corresponding to a non-locally homogeneous strict NK structure. By Remark 5.2, any 2-dimensional submanifold, transversal to the solutions in \( \mathcal{U} \), gives a 2-parameter family of initial data corresponding to non-equivalent NK structures. □

### 6. Cohomogeneity One NK Manifolds with One Singular Orbit \( S^3 \)

Let \( G = \text{SU}_2 \times \text{SU}_2 \) and consider its natural cohomogeneity one action on \( M := G \times_H V \), where the subgroup \( H \) is the diagonal subgroup \((\text{SU}_2)_{\text{diag}}\) and the vector space \( V \cong \mathbb{R}^3 \) is \( H \)-isomorphic to the Lie algebra \( \mathfrak{h} \) acted on by \( H \) via the standard adjoint representation. The manifold \( M \) is clearly diffeomorphic to \( T S^3 \cong S^3 \times \mathbb{R}^3 \) and it can be realized as a tubular neighborhood of the singular orbit \( S \cong S^3 \) in \( S^6 \) or \( S^3 \times S^3 \) endowed with the \( G \)-manifold structures described in §4.2.1 and §4.2.3.

In this section we classify (up to isometries) all \( G \)-invariant strict NK structures defined on \( M \).
Let $E_{\pm}, V_{\pm} \in \mathfrak{g}$ be defined by

$$E_{\pm} := E_1 \pm E_2, \quad V_{\pm} := V_1 \pm V_2,$$

and note that $\mathfrak{h} = \text{Span}(U, E_+, V_+)$, while $\mathfrak{m} := \text{Span}(A, E_-, V_-)$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. We also fix the curve $\gamma_t := [(e, tU)]$ in $M$ with $t \in \mathbb{R}$ and for any 2-form $\omega$ on $M$ we denote by $\omega_t = \omega|_{\gamma_t}$ its restriction to $\gamma_t$. We know that for any invariant 2-form $\omega$ on $M \setminus S$ the restriction $\omega_t$ is of the form $\omega_t = \sum_1^5 f_i \omega_t^i$, where the $\omega^i$’s are defined in (3.2).

**Proposition 6.1.** A $G$-invariant 2-form $\omega$ on $M \setminus S$ corresponding to $\omega_t = \sum_1^5 f_i \omega_t^i$ on $\gamma_t$, $t \neq 0$, admits a smooth extension on the whole $M$ if and only if all $f_i$’s extend smoothly at $t = 0$ and the following conditions are satisfied: denoting $\alpha_i := f_i(0), \beta_i := f_i'(0), 1 \leq i \leq 5$,

i) $f_1, f_4$ are even and $f_2, f_3, f_5$ are odd; in particular $\alpha_2 = \alpha_3 = \alpha_5 = 0$;

ii)

$$\beta_3 = \frac{1}{2} \alpha_1 + \beta_2, \quad \beta_5 = -\frac{1}{4} \alpha_1 - \beta_2, \quad \alpha_4 = 0. \tag{6.1}$$

Moreover if $\omega$ extends on $M$, we have that $(\omega_{\gamma_0})^3 \neq 0$ if and only if $\alpha_1 \neq 0$.

**Proof.** Let $t, x, y$ be the cartesian coordinates on $V$ determined by the basis \(\left(\frac{1}{\sqrt{2}} U, E_+, V_+\right)\) and notice that \(\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \hat{A}, \hat{E}_-, \hat{V}_-\}\) is a frame field in a neighborhood of $p_0 := \gamma_0$ with dual coframe \(\{dt, dx, dy, A^*, E^*_-, V^*_-\}\). We have that

$$\hat{E}_+|_{\gamma_t} = -\frac{t}{\sqrt{2}} \frac{\partial}{\partial y}|_{\gamma_t}, \quad \hat{V}_+|_{\gamma_t} = \frac{t}{\sqrt{2}} \frac{\partial}{\partial x}|_{\gamma_t},$$

and therefore (with the notations as in (3.2))

$$E_1^* = -\frac{\sqrt{2}}{t} dy + 2\hat{E}_-, \quad V_1^* = \frac{\sqrt{2}}{t} dx + 2\hat{V}_-,$$

$$E_2^* = -\frac{\sqrt{2}}{t} dy - 2\hat{E}_-, \quad V_2^* = \frac{\sqrt{2}}{t} dx - 2\hat{V}_-.$$

Using (3.2), one can find that

$$\omega_t = f_1 \ dt \wedge A^* + f_2 \left( E_+^* - \frac{\sqrt{2}}{t} dy \right) \wedge \left( V_-^* + \frac{\sqrt{2}}{t} dx \right) + \frac{2}{t} f_4 \left( -E_-^* \wedge dy + V_-^* \wedge dx \right) + \frac{4}{t^2} f_5 \left( -2E_-^* \wedge V_-^* \right)$$

$$+ f_3 \left( -E_-^* - \frac{\sqrt{2}}{t} dy \right) \wedge \left( -V_-^* + \frac{\sqrt{2}}{t} dx \right) + \frac{2\sqrt{2}}{t} f_4 \left( -E_-^* \wedge dy + V_-^* \wedge dx \right)$$

$$= 2 \frac{f_2 + f_3 + 2 f_5}{t^2} \ dx \wedge dy + (f_2 + f_3 - 2 f_5) \ E_-^* \wedge V_-^* + f_1 \ dt \wedge A^* + \sqrt{2} \frac{f_3 - f_2}{t} \ (dx \wedge E_-^* + dy \wedge V_-^*) + \frac{2\sqrt{2}}{t} f_4 \left( -dx \wedge V_-^* + dy \wedge E_-^* \right). \tag{6.2}$$
The restriction to $V\setminus\{0\}$ of the $G$-invariant form $\omega$ on $M \setminus S$ gives a $H$-equivariant map

$$\tilde{\omega} : V\setminus\{0\} \longrightarrow \Lambda^2(V^* + m^*) \cong V + m + V^* \otimes m^*$$

and $\tilde{\omega}$ smoothly extends to the whole $V$ if and only if each component $\tilde{\omega}^V$, $\tilde{\omega}^m$, $\tilde{\omega}^{V \otimes m^*}$ does.

We consider the component $\tilde{\omega}^V$. Under suitable identifications, we see that $\tilde{\omega}^V$ is a $SO_3$-equivariant map such that $\tilde{\omega}^V(t, 0, 0) = (\phi(t), 0, 0)$, where $\phi(t) := 2 \frac{f_2 + f_3 + 2f_5}{t^2}$ for $t \neq 0$. It is then easy to see that $\tilde{\omega}^V$ extends smoothly on the whole $V$ if and only if $\phi$ extends to a smooth odd function on $\mathbb{R}$. This means that $f_2 + f_3 + 2f_5$ extends as an odd function with

$$\beta_2 + \beta_3 + 2\beta_5 = 0.$$  \hspace{1cm} (6.3)

The condition on $\tilde{\omega}^m$ is similar and its extendability is equivalent to the extendability of $f_2 + f_3 - 2f_5$ as on odd function.

We now identify the $H$-modules $V$ and $m$ by means of the map $U \mapsto A$, $E_+ \mapsto E_-$, $V_+ \mapsto V_-$ and we further split $V^* \otimes m^* = S^2(V^*) \oplus V$ and $\tilde{\omega}^{V \otimes m^*} = \tilde{\omega}_1 + \tilde{\omega}_2$ accordingly. From (6.2) we have that the condition on $\tilde{\omega}_2$ is equivalent to say that $\frac{\tilde{\omega}_2}{t}$ extends as an odd function, i.e. $f_4$ extends evenly with $f_4(0) = 0$.

As for the extendability of $\tilde{\omega}_1$, we first write $\tilde{\omega}_{1t} := \tilde{\omega}_1{|}_{t^i}$ as

$$\tilde{\omega}_{1t} = \frac{1}{\sqrt{2}} \left( f_1 dt^2 + \frac{2(f_3 - f_2)}{t} (dx^2 + dy^2) \right).$$

From this and the invariance under the symmetry $t \mapsto -t$, we see that both $f_1$ and $\frac{f_3 - f_2}{t}$ must extend smoothly to even functions on the whole $\mathbb{R}$. This implies that $f_2 - f_3$ extends as an odd smooth function. By previous remarks we deduce that $f_2$, $f_3$ and $f_5$ extend as odd smooth functions.

We now determine $\tilde{\omega}_1$ explicitly at any point $p = (t, x, y) \in V$ with $x^2 + t^2 \neq 0$. If $\theta$, $\phi$ are defined as

$$\sin \theta = \frac{y}{\rho}, \quad \cos \theta = \frac{\sqrt{t^2 + x^2}}{\rho} \quad \text{where} \quad \rho = |p|, \quad \sin \phi = \frac{x}{\sqrt{t^2 + x^2}}, \quad \cos \phi = \frac{t}{\sqrt{t^2 + x^2}},$$

then the transformation

$$B := \begin{pmatrix} \cos \theta \cdot \cos \phi & \cos \theta \cdot \sin \phi & \sin \theta \\ -\sin \phi & \cos \phi & 0 \\ -\sin \theta \cdot \cos \phi & -\sin \theta \cdot \sin \phi & \cos \theta \end{pmatrix}$$

maps $p$ into $(\rho, 0, 0)$. Hence $\tilde{\omega}_1(p) = \tilde{\omega}_1(B^{-1}(\rho, 0, 0)) = B^{-1} \cdot \tilde{\omega}_1(\rho, 0, 0)$ is equal to

$$\tilde{\omega}_1(p) = \frac{1}{\sqrt{2}} \left[ f_1(\rho) (\cos \theta \cdot \cos \phi \ dt + \cos \theta \cdot \sin \phi \ dx + \sin \theta \ dy) \right]^2 + \lambda(\rho) \left( (\cos \phi \ dx + \sin \phi \ dy)^2 + (\cos \theta \cdot \sin \phi \ dt - \sin \theta \cdot \sin \phi \ dx + \cos \theta \ dy)^2 \right)$$

$$\begin{aligned} &\quad = \frac{1}{\sqrt{2}} \left( f_1(\rho) \frac{t^2}{\rho^2} + \lambda(\rho) \left( 1 - \frac{t^2}{\rho^2} \right) \right) dt^2 + \frac{1}{\sqrt{2}} \left( f_1(\rho) \frac{x^2}{\rho^2} + \lambda(\rho) \left( 1 - \frac{x^2}{\rho^2} \right) \right) dx^2 \\ &\quad \quad + \frac{1}{\sqrt{2}} \left( f_1(\rho) \frac{y^2}{\rho^2} + \lambda(\rho) \left( 1 - \frac{y^2}{\rho^2} \right) \right) dy^2 \end{aligned}$$
\[+\sqrt{\frac{t x}{\rho^2}} (f_1(\rho) - \lambda(\rho)) dt \otimes dx + \sqrt{\frac{t y}{\rho^2}} (f_1(\rho) - \lambda(\rho)) dt \otimes dy\]
\[+\sqrt{\frac{t y}{\rho^2}} (f_1(\rho) - \lambda(\rho)) dx \otimes dy,\]

where \(\lambda = \frac{2(f_3 - f_1)}{\rho}\). Since both \(f_1\) and \(\lambda\) extend as even smooth functions we see that \(\bar{\omega}_1\) extends smoothly if and only if

\[\alpha_1 = f_1(0) = \lambda(0) = 2(\beta_3 - \beta_2).\]  (6.4)

From this and (6.3), condition (ii) follows. Finally, from (i) and (ii), one has that \(\omega_0 = \alpha_1 (dt \wedge A^\ast + \frac{\sqrt{3}}{2}(dx \wedge E^\ast + dy \wedge V^\ast))\), from which the last assertion follows. \(\square\)

We now study the case when the two-form \(\omega\) is the Kähler form of a strict NK structure.

**Proposition 6.2.** Let \(\omega\) be a \(G\)-invariant two form on \(M\setminus S\) and \(\omega_t = \sum_i f_i \omega^i\) its restriction to \(\gamma_t\). Suppose that \(\omega\) is the Kähler form of a strict NK structure and that it extends smoothly to the whole \(M\). If \(f_1(0) \neq 0\), then

i) \(f_4 = 0\) and \(f_5'(0) = 0\);

ii) the 3-form \(d\omega\) is stable at all points of \(M\).

**Proof.** First of all, notice that if \(\omega\) extends smoothly, then the \(f_i\)’s satisfy the system (3.21) for any \(t \in \mathbb{R}\). Now, to prove i), recall that, by the proof of Lemma 3.2, one of the ratios \(\frac{f_4}{f_5}\) or \(\frac{f_5'}{f_4}\) is constant. Since \(f_4\) and \(f_5\) extend to an even and an odd function, respectively, it follows that either \(f_4 = 0\) or \(f_5 = 0\).

Assume that \(f_5 = 0\). Then, by Proposition 6.1, the even function \(f_4\) satisfies the differential equation (3.21)3 with \(f_4(0) = 0\) and \(f_4'(0) = 0\). This implies that \(f_4 \equiv 0\) contradicting the fact that \(f_4^2 + f_5^2 \neq 0\) by Theorem 3.3 (i). So \(f_4 = 0, f_5 \neq 0\) and, by the same argument, \(f_5'(0) \neq 0\).

For (ii), recall that by Proposition 6.1 the form \((\omega_t)^3\) is a volume form in \(T_{\gamma_t}M\) for every \(t\) and that the \(G\)-invariant 3-form \(d\omega\) is stable at all points of \(M\) if and only if \(P'(d\omega|_{\gamma_t}) < 0\) for any \(t\), where \(P'\) is the polynomial map on 3-forms, defined in §2.1 on the base of the volume form \(\tau_t = (\omega_t)^3\). This volume form can be expressed in terms of \(\tau_t = \omega_t^1 \wedge \omega_t^2 \wedge \omega_t^3\) by

\[\tau_t' = 6 f_1(f_2 f_3 - f_5^2) \tau_t.\]

Hence, from (2.2) and the expression (3.19) for the polynomial map \(P\), determined using the volume form \(\tau_t\), for \(t > 0\) we have

\[P'(d\omega|_{\gamma_t}) = -\frac{f_5^2}{9^3 f_1^2 (f_2 f_3 - f_5^2)^2} \cdot \left((f_5')^2 - \left(\frac{f_2'}{4} + \frac{f_1}{4}\right)\left(f_3' - \frac{f_1}{4}\right)\right).\]

Now, \(P'(d\omega|_{\gamma_t}) < 0\) for \(t \neq 0\) by (3.12) and therefore we only need to prove that \(P'(d\omega|_{\gamma_0}) < 0\). This follows from the fact that using (3.12) and Prop. 6.1,

\[P'(d\omega|_{\gamma_0}) = \lim_{t \to 0^+} P'(d\omega|_{\gamma_t}) = \lim_{t \to 0^+} \frac{4 f_5^4}{9^3 f_1^4 (f_2 f_3 - f_5^2)^2} \]
\[= -\frac{4(f_5'(0))^4}{9^3 f_1(0)^4 (f_2(0) f_3'(0) - (f_5'(0))^2)^2} = -\frac{45 (f_5'(0))^4}{9^3 f_1(0)^8} < 0.\] \(\square\)
Corollary 6.3. Let \( f_i, 1 \leq i \leq 5 \), be smooth functions on some interval \( ]-a, a[ \subseteq \mathbb{R} \) with

i) \( f_4 \equiv 0 \) and \( f_1 \) is even with \( f_1 < 0 \);
ii) \( f_2, f_3, f_5 \) are odd with \( f_3'(0) = \frac{1}{2} f_1(0) + f_2'(0) \) and \( f_5'(0) = -\frac{1}{3} f_1(0) - f_2'(0) \neq 0 \);
iii) the \( f_i \)'s satisfy the differential system (3.21) together with the algebraic condition (3.22).

Then there exists a tubular neighborhood \( T_\varepsilon = G \cdot \gamma ||0,\varepsilon|| \) of \( S, 0 < \varepsilon \leq a \), and a \( G \)-invariant strict NK structure \( (g, J) \) on \( T_\varepsilon \), whose Kähler form \( \omega \) is the \( G \)-invariant 2-form associated with \( \omega_t = \sum f_i \omega^i \) at the points \( \gamma_t \).

Proof. The initial conditions \( f_1(0) < 0 \) and \( f_5'(0) \neq 0 \) imply that the functions \( f_i \)'s satisfy (i) and (ii) of Theorem 3.3 on a suitable subinterval \( ]-\varepsilon, \varepsilon[ \). Therefore the two-form \( \omega \), corresponding to \( \omega_t = \sum f_i \omega^i, t \neq 0 \), defines a strict NK structure \( (g, J) \) (with \( g \) possibly not positive definite) on \( T_\varepsilon \setminus S \). By Propositions 6.1 and 6.2, \( \omega \) extends smoothly also at the singular \( G \)-orbit \( S \) and do is stable everywhere. Hence the corresponding NK structure \( (g, J) \) extends smoothly on the whole \( T_\varepsilon \).

It remains to show that, by possibly choosing a smaller \( \varepsilon > 0 \), the metric \( g \) is positive definite on \( T_\varepsilon \). Using the notations of the proof of Proposition 6.1, the tangent space at \( p_0 = \gamma_0 \) is identifiable with \( T_{p_0}T = V \oplus m \) as the \( H \)-module. Note that \( U = \gamma_0 \in V \) and that

\[
JU = \lim_{t \to 0} J(\gamma'_t) = \lim_{t \to 0} \frac{1}{f_1} A_{\gamma_t} = \frac{1}{\alpha_t} A \in m
\]

by (3.15) and (3.22). In particular, we have that \( JV \cap m \neq \emptyset \) and, being \( V \) and \( m \) irreducible \( H \)-modules, it follows that \( JV = m \). On the other hand, from (6.2), we see that \( \omega_{p_0}(V, V) = \omega_{p_0}(m, m) = 0 \) and hence that \( g_{p_0}(V, m) = 0 \).

Since \( g \) satisfies (2.8), we have that \( g_{p_0}(U, U) = \lim_{t \to 0} g(\gamma'_t, \gamma'_t) = 1 \). From this, \( H \)-invariance and \( J \)-Hermitianity, it follows that \( g_{p_0} \) is positive definite on \( V \times V \) and \( m \times m \), and hence on the whole tangent space \( T_{p_0}T_\varepsilon \). A value \( \varepsilon > 0 \), so that \( g \) is positive definite on \( T_\varepsilon \), can be now chosen by a simple continuity argument. \( \square \)

We now prove the main result of this section.

Theorem 6.4. There exists a one parameter family of non isometric, \( G \)-invariant strict NK structures on \( TS^3 \cong S^3 \times \mathbb{R}^3 \).

Proof. In order to find \( G \)-invariant strict NK structures on \( TS^3 \), we look for functions \( f_i \)'s satisfying (i)–(iii) of Corollary 6.3. Since we need solutions of (3.21) and (3.22) with \( f_1 \) nowhere vanishing, we may replace that system of differential equations and algebraic conditions by those obtained through the change of variables considered in §5, whose notations will be kept throughout the following. Setting \( \mu = 2 \) and using the change of variables (5.1) with \( t_0 = 0 \) and \( h_4(s) = 2 f_5(t(s)) \), we see that looking for functions \( f_i \)'s as in Corollary 6.3 is equivalent to looking for functions \( h_i, 1 \leq i \leq 4 \), on \( ]-\varepsilon, \varepsilon[ \) so that:

a) they are odd;
b) they satisfy (5.2) at any \( s \neq 0 \);
c) they satisfy the initial conditions (here \( b_i := h_i'(0) \))

\[
b_1 > 0, \quad b_3 = -b_1, \quad b_2 = -b_4 \neq 0.
\]

(6.5)
Indeed, the initial conditions in Corollary 6.3 (i), (ii) are equivalent to (6.5), which in turn imply conditions (5.4), since \( a_i = h_i(0) = 0 \) for any \( 1 \leq i \leq 4 \). Note also that condition \( b_2 \neq 0 \) can be replaced by \( b_2 < 0 \), since the transformation \( h_2 \rightarrow -h_2 \) (corresponding to the change \( (f_2, f_3) \rightarrow (-f_3, -f_2) \)) gives rise to equivalent NK structures (see Proposition 4.2, transformation (4.6)).

Since the system (5.2) is singular at \( s = 0 \), a necessary condition for the existence of solutions \( h_i \)'s satisfying (a)–(c) is that

\[
\lim_{s \to 0} \frac{2(h'_i)^2h_3 + \frac{2}{9}h_4h'_4}{s} = 2 \left( b_1^2 - \frac{1}{9}b_2^2 \right) = 0. \tag{6.6}
\]

This means that (6.5) can be replaced by the initial conditions

\[
b_2 = -3b_1 \sqrt{b_1}, \quad b_3 = -b_1, \quad b_4 = 3b_1 \sqrt{b_1}, \quad b_1 > 0, \tag{6.7}
\]

and that the whole problem corresponds to looking for smooth odd functions \( h_i \)'s satisfying (5.2) and (6.7) on an interval \( -\varepsilon, \varepsilon \).

For this, it is convenient to re-write the smooth odd functions \( h_i \)'s as follows:

\[
h_i = s p_i, \; i = 1, 2, 4, \quad h_3 = s(p_3 - p_1),
\]

where the \( p_i \)'s are some even functions. Then, (5.2) and (6.7) are equivalent to the following system on even functions \( p_i \)'s:

\[
\begin{align*}
p''_1 + \frac{2}{s^2} [p'_1 + \frac{18}{9} p_1 p'_1 (p_3 - p_1)^2 + p_4 p'_4]
& + \frac{2}{s^2} [p_1^2 (p_3 - p_1)^2 + p_2^2 (p_3 - p_1)^2 - p_4^2]
= 0, \\
p''_2 + \frac{2}{s^2} p'_2 + 48 p_1 p_2 + 48 s p_1 p_2 = 0, \\
p''_3 + \frac{2}{s^2} p'_3 + 48 p_1 (p_3 - p_1) + 48 s p_1 (p_3 - p_1) = 0, \\
p''_4 + \frac{2}{s^2} p'_4 + 4 p_4 (12 p_1 - 1) + 48 s p'_4 p_4 = 0,
\end{align*}
\tag{6.8}
\]

satisfying the initial conditions (here, \( c_i := p_i(0) \), while \( p'_i(0) = 0 \) by evenness)

\[
c_2 = -3c_1 \sqrt{c_1}, \quad c_3 = 0, \quad c_4 = 3c_1 \sqrt{c_1}, \quad c_1 > 0. \tag{6.9}
\]

If we now set \( \mathcal{P} : (-\varepsilon, \varepsilon) \to \mathbb{R}^4 \) with \( \mathcal{P} := (p_1, \ldots, p_4) \) and \( \mathcal{Q} := \mathcal{P}' \), we see that (6.8) and (6.9) are equivalent to a system of the form

\[
\begin{align*}
\mathcal{P}' &= \mathcal{Q}, \\
\mathcal{Q}' &= \frac{1}{s^2} A(\mathcal{P}) + \frac{1}{s} B(\mathcal{P}, \mathcal{Q}) + C(s, \mathcal{P}, \mathcal{Q})
\end{align*}
\tag{6.10}
\]

with initial conditions

\[
\mathcal{P}(0) = (c_1, -3c_1 \sqrt{c_1}, 0, 3c_1 \sqrt{c_1}), \quad \mathcal{Q}(0) = 0, \quad c_1 > 0, \tag{6.11}
\]

where \( A, B \) and \( C \) are smooth \( \mathbb{R}^4 \)-valued functions defined on suitable open neighborhoods of \( \mathcal{P}(0) \in \mathbb{R}^4, (\mathcal{P}(0), 0) \in \mathbb{R}^8 \) and \( (0, \mathcal{P}(0), 0) \in \mathbb{R}^9 \), respectively.

We claim that there exists a formal power series solution \( \widehat{\mathcal{P}} \) of (6.10) of the form

\[
\widehat{\mathcal{P}}(s) = \sum_{n=0}^{\infty} \frac{\mathcal{P}_{2n}}{(2n)!} s^{2n}, \quad \widehat{\mathcal{Q}}(s) = \sum_{n=1}^{\infty} \frac{\mathcal{P}_{2n}}{(2n - 1)!} s^{2n-1}. \tag{6.12}
\]
In fact, if \( \hat{\mathcal{P}}, \hat{\mathcal{Q}} \) are as in (6.12), for suitable \( A_{2n}, B_{2n+1}, C_{2n} \in \mathbb{R}^4 \), one has that
\[
A(\hat{\mathcal{P}}(s)) = \sum_{n=0}^{\infty} \frac{A_{2n}}{(2n)!} s^{2n}, \quad B(\hat{\mathcal{P}}(s), \hat{\mathcal{Q}}(s)) = \sum_{n=0}^{\infty} \frac{B_{2n+1}}{(2n + 1)!} s^{2n+1},
\]
\[
C(s, \hat{\mathcal{P}}(s), \hat{\mathcal{Q}}(s)) = \sum_{n=0}^{\infty} \frac{C_{2n}}{(2n)!} s^{2n},
\]
and \( \hat{\mathcal{P}} \) and \( \hat{\mathcal{Q}} \) are formal solutions of (6.10) if and only if, for any \( n \geq 0 \),
\[
P_{2n+2} = \frac{A_{2n+2}}{(2n+2)(2n+1)} + \frac{B_{2n+1}}{2n+1} + C_{2n}, \tag{6.13}
\]
Since
\[
A_{2n+2} = dA\big|_{\mathcal{P}(0)} \cdot P_{2n+2} \mod (\mathcal{P}_0, \mathcal{P}_2, \mathcal{P}_4, \ldots, \mathcal{P}_{2n}),
\]
\[
B_{2n+1} = \frac{d^{2n+1}}{ds^{2n+1}} B(\hat{\mathcal{P}}(s), \hat{\mathcal{Q}}(s)) = \frac{d^{2n}}{s^{2n}} \left( \frac{\partial B}{\partial \mathcal{P}} \cdot \hat{\mathcal{P}}'(s) + \frac{\partial B}{\partial \mathcal{Q}} \cdot \hat{\mathcal{P}}''(s) \right)
\]
\[
= \left. \frac{\partial B}{\partial \mathcal{Q}} \right|_{(\mathcal{P}(0),0)} \cdot P_{2n+2} \mod (\mathcal{P}_0, \mathcal{P}_2, \mathcal{P}_4, \ldots, \mathcal{P}_{2n}),
\]
Equation (6.13) can be re-written in the form
\[
P_{2n+2} = \frac{1}{(2n+2)(2n+1)} dA\big|_{\mathcal{P}(0)} \cdot P_{2n+2} + \frac{1}{2n+1} \left. \frac{\partial B}{\partial \mathcal{Q}} \right|_{(\mathcal{P}(0),0)} \cdot P_{2n+2} + D_{2n}, \tag{6.14}
\]
for a fixed function \( D_{2n} \) of \( \mathcal{P}_0, \mathcal{P}_2, \mathcal{P}_4, \ldots, \mathcal{P}_{2n} \). It follows that there exists a formal series solution of the form (6.12) if and only if for every \( n \),
\[
\mathcal{L}_{2n} \cdot P_{2n+2} = D_{2n},
\]
where \( \mathcal{L}_{2n} \) is the \( 4 \times 4 \)-matrix
\[
\mathcal{L}_{2n} = Id - \frac{1}{(2n+2)(2n+1)} dA\big|_{\mathcal{P}(0)} - \frac{1}{2n+1} \left. \frac{\partial B}{\partial \mathcal{Q}} \right|_{(\mathcal{P}(0),0)}.
\]
Since
\[
dA\big|_{\mathcal{P}(0)} = \begin{pmatrix} 6 & 0 & -2 & -\frac{4}{3^{\sqrt{3}}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \left. \frac{\partial B}{\partial \mathcal{Q}} \right|_{(\mathcal{P}(0),0)} = \begin{pmatrix} 6 & 0 & 0 & -\frac{2}{3^{\sqrt{3}}} \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},
\]
we have that \( \det(\mathcal{L}_{2n}) = \frac{(2n^2 - 3n - 8)(2n - 1)}{(2n + 1)^3} \neq 0 \) for every \( n \geq 0 \) and the sequence \( P_{2n+2} = \mathcal{L}_{2n}^{-1} \cdot D_{2n} \) uniquely determines a formal solution of the form (6.12).

By the Malgrange Theorem ([21], Thm. 7.1), the existence of the formal solution \( (\hat{\mathcal{P}}, \hat{\mathcal{Q}}) \) implies the existence of a smooth solution \( (\mathcal{P}, \mathcal{Q}) \) of (6.8), whose Taylor expansion at 0 coincides with \( (\hat{\mathcal{P}}, \hat{\mathcal{Q}}) \). This solution can be determined with \( \mathcal{P} \) even: In fact, given an arbitrary smooth solution \( \hat{\mathcal{P}} \) of (6.8) with Taylor expansion \( \hat{\mathcal{P}} \) at 0, the even
function \( \mathcal{P}(t) = \tilde{\mathcal{P}}(|t|) \) is smooth also at 0 (because \( \tilde{\mathcal{P}} \) has no odd degree monomials) and \( (\mathcal{P}, \mathcal{Q} = \mathcal{P}') \) is a solution of (6.10), as it is immediately checked.

A smooth even solution \( \mathcal{P} \) gives rise to a strict NK structure \((g, J)\) on \( T\mathbb{S}^3 \). Recall that \( g \) is Einstein and hence real analytic by [9]. Since the Killing vector field \( \tilde{A} \) is non-vanishing along the geodesic \( \gamma_1 \) and \( f_1(t) \) coincides (up to a multiple) with the norm \( ||\tilde{A}||_{\gamma_1} \) (see (3.15) and (3.20)), we have that \( f_1(t) \) is real analytic. By a direct inspection of the system (5.2), we also have that \( h_2, h_3, h_4 \) and the corresponding \( \mathcal{P} \) are real-analytic. From this we get that any NK structure on \( T\mathbb{S}^3 \) determined by the solutions \( \mathcal{P} \) are uniquely determined by the initial data and in particular, by the choice the parameter \( b_1 = c_1 > 0 \) in (6.9).

It remains to show that two NK structures \((g, J), (\tilde{g}, \tilde{J})\), corresponding to initial data \( b_1, \tilde{b}_1 > 0 \), are isometric if and only if \( b_1 = \tilde{b}_1 \).

First of all, we claim that \( b_1 \) corresponds to a locally homogenous structure if and only if \( b_1 = \frac{1}{2} \) or \( b_1 = \frac{1}{3} \). In fact, \((g, J)\) is locally homogeneous if and only if there is an isometric equivalence between an open neighborhood \( \mathcal{U} \) of \( p_o = \gamma_0 \) and an open subset \( \mathcal{U}' \) of one of the compact homogeneous NK spaces \( N \), described in Proposition 4.1. If \( g \) is properly identified with a subalgebra of \( \text{aut}(N) \), the isometric equivalence can be assumed to be \( g \)-equivariant. Notice that the only spaces \( N \), admitting a cohomogeneity one action of \( G \) with a singular orbit \( G/H \cong \mathbb{S}^3 \), are \( \mathbb{S}^6 \) and \( \mathbb{S}^3 \times \mathbb{S}^3 \). Now, the claim can be checked using the explicit descriptions of homogeneous NK structures in §4.2. Indeed, when \( N = \mathbb{S}^3 \times \mathbb{S}^3 \) the expressions in §4.2.3 immediately give that \( b_1 = \frac{1}{2} \), while in the case \( N = \mathbb{S}^6 \) more work is needed. In fact, one has to: a) rescale the metric considered in §4.2.1 in order to have \( \mu = \frac{s}{30} = 2 \); b) replace the geodesic \( \gamma \) with \( \tilde{\gamma}(t) = \gamma(-\sqrt{2}t + \frac{\pi}{2}) \), so that \( ||\tilde{\gamma}'|| = 1 \) and \( \tilde{\gamma}(0) \in S \); c) change \( J \) into \(-J\) so that the new functions \( \tilde{f}_1 \) meet all our assumptions on signs. They are \( \tilde{f}_1 = -\frac{1}{\sqrt{2}} \cos(\sqrt{2}t) \) and \( \tilde{f}_i(t) = -\frac{1}{\sqrt{2}} f_i(-\sqrt{2}t + \frac{\pi}{2}), i \geq 2, \) and give \( b_1 = \frac{1}{4} \).

Consider now initial data \( b_1, \tilde{b}_1 > 0 \), corresponding to locally isometric NK structures \((g, J), (\tilde{g}, \tilde{J})\). By the previous claim, we may assume that they are not locally homogeneous. In this case, if \( \phi : \mathcal{U} \subset T\mathbb{S}^3 \rightarrow \mathcal{U}' \subset T\mathbb{S}^3 \) is a local isometry between \( g \) and \( g' \) on a neighborhood \( \mathcal{U} \) of \( p_o = \gamma_0 \in S \cong \mathbb{S}^3 \), we have that \( \phi \) locally maps \( G \)-orbits into \( G \)-orbits. In particular, \( p_o \) is mapped into a point of the singular \( G \)-orbit \( S \) and, by composition with some element of \( G \), we may assume that \( \phi(p_o) = p_o \) and \( \phi(\gamma_t) = \gamma_t \) for every \( \gamma_t \in \mathcal{U} \). From this and Proposition 4.2, it follows that the quadruples \((f_1, f_2, f_3, f_4)\) and \((\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4)\), corresponding to \((g, J), (\tilde{g}, \tilde{J})\), coincide. This implies that \( b_1 = \tilde{b}_1 \), and the proof is concluded. \( \square \)

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