Segregation of large particles in dense granular flows: A granular Saffman effect?

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We report on the scaling between the lift force on a single large particle segregating in a monodisperse dense granular flow and a velocity lag of the particle. This scaling suggests a viscous-inertial origin for the lift force, similar to the Saffman lift force in (micro) fluids. These findings are relevant for modelling of particle-size segregation and our approach opens up a new avenue for the numerical study of granular flows. For dense granular flows this study indicates an interplay between polydispersity and viscous-inertial forces, striking new parallels with fluids and suspensions.

Size-polydispersity is intrinsic to non-equilibrium systems like granular materials [1]. It gives them the ability to size-segregate when agitated, a process which spatially separates different sized grains [2,3], but is different from phase separation in classical fluids. Particle-size segregation in dense granular flows [5,6] has been intensively studied [e.g. 7–23], but a fundamental question remains unanswered: why do large particles segregate?

It is generally understood that in dense granular flows both small and large particles are pushed away from high shear regions [8,9] or pulled by gravity [10,11]. The reason for the separation of the two species is that small particles move more effectively; they can carry proportionally more of the kinetic energy [13,14] and are also more likely to move into the gaps between larger particles; a process referred to as kinetic sieving [10,11]. Unfortunately, the concept of kinetic sieving breaks down when the large-particle concentration is very low.

Current models for size-segregation perform well when the small and large-particle concentrations are nearly equal [1,24,25]. When accounting for the effect of size-segregation asymmetry [26,27], models have been extended to more unequal concentrations, but they remain inaccurate in the limit of low large-particle concentrations. Extending models to this limit is critical because during segregation, and even after reaching a steady state, regions of low large-particle concentration occur and can persist throughout the flow [20,23,29]. Moreover, current models are either completely or partly phenomenological. Thus, to advance modelling, we should aim to understand the physical origin of segregation to derive the free-variables from their microscopic quantities. An important related issue is that current constitutive models for dense granular flows only work with an average particle size [33,34]. If we are to implement size-distributions in these models a better understanding of micro-scale effects between large and small particles seems crucial.

In contrast to particle-size segregation, particle migration in suspensions, in the limit of low concentrations, is generally well understood [e.g. 35,36]. Arguably this progress has been aided by the fact that the fluid forces acting on a particle can be calculated, which can not be said for granular media. This inspired us to treat the particles that surround an intruder as a continuum and attempt to understand the forces acting on a segregating particle based on the measured continuum fields.

Recently, Guillard et al. [39] measured for the first time the segregation lift force on a single large intruder particle in a mono-disperse granular flow by attaching the intruder to a spring perpendicular to the plane (see Fig. 1). They found scaling laws that linked the total upward force on the intruder to shear and pressure gradients. These scaling laws predict the direction of segregation of large particles in different flow configurations depending on whether a shear or pressure gradient has the strongest contribution. However, they do not shed light on the origin of the lift force.

In this letter we present new physical insights into the origin of the lift force on large intruders in three-dimensional mono-disperse dense granular flows. We do so, firstly, by taking a different approach to Guillard et al. [39] and determine the lift force $F_L$ by decomposing the total upward force on an intruder as $F_{tot} = F_L + F_b$, where $F_b$ is a generalized buoyancy force for dense granular media.

![FIG. 1. Schematic of the simulations: 3D mono-disperse granular flow down an incline, with angle $\theta = 22^\circ$. Only base and surface particles are shown. The flow contains three intruder particles that are held with springs around three different z-positions $z_j$ (intruder positions in the schematic are to scale), but move freely in the x-y plane.](image-url)
ular media. Secondly, we discover an unreported velocity lag of the intruder. A physical origin for the lift force is given by its scaling with the lag.

Methods.—We use MercuryDPM, based on discrete particle methods (MercuryDPM.org; [37, 38]), and investigate three-dimensional (3D) flows of mixtures of spherical dry frictional particles flowing down an incline of $\theta = 22^\circ$. All simulation parameters are non-dimensionalized such that the particle density is $\rho_p = 6/\pi$ and the gravitational acceleration is $g = 1$, with vertical component $g_z = \cos \theta$. The simulations are conducted in a box with dimensions $(x, y, z) = (30, 8.9, 40)$, with periodic walls in the $x$ and $y$ directions. The particles that make up the bulk of the flow have a diameter $d_b = 1$. The intruder diameter is varied between $d_i = 0.5$ and $3.2$. The rough base consists of particles of diameter $0.85$ and the flow height is $h = 32 \pm 0.5$.

We place three identical intruders in the flow at vertical positions $z_i = 5, 15$ and $23$ (see Fig. 1). Each intruder is attached to a spring [39], which applies a vertical force $F_{sp} = -k(z_i - z_f)$ proportional to the vertical distance between the intruder position $z_i$ and its corresponding $z_f$. Here $k = 20$ is the spring stiffness. We also simulate $k = \infty$ by fixing the intruder at $z_i = z_f$. Our findings are independent of $k$, so unless stated otherwise all data reported are for $k = 20$. We do not discuss the data for $z_i = 5$ because the intruder experiences boundary effects, likely due to layering near the bed, as reported in [40].

A linear spring-dashpot model [40, 41] with linear elastic and linear dissipative contributions is used for the normal forces between particles. The restitution coefficient for collisions $r_c = 0.1$ and the contact duration $t_c = 0.005$. This results in a different stiffness depending on the particle size. We verified that our findings are not the result of this difference in stiffness nor the dependence on $r_c$ and $t_c$. The friction coefficient for contacts between bulk particles $\mu_{bh}$ and between bulk and intruder particles $\mu_{bi}$ equals 0.5, unless otherwise stated.

Applying coarse-graining (CG) [40, 42, 43], after a steady state has been reached, we obtain time-averaged 3D continuum fields for $\nu$ the local solids fraction, and $\sigma$ the stress tensor, which satisfy the conservation laws. The CG-width $w$ is 1.0 [42]. We approximate the bulk solids fraction at the position of the intruder $\nu(x_i, y_i, z_i) = \nu_i = \frac{V_i}{V}$ using the ratio of the particle volume $V_i = \frac{4}{3} \pi (d_i/2)^3$ and the Voronoi volume $\tilde{V}_i$, which we obtain through 3D weighted Voronoi tessellation (math.lbl.gov/voro++-[14]). All error-bars correspond to a 95% confidence interval.

Results.—We find that the downstream velocity $v_{xi}$ of a large intruder experiences a lag $\lambda_x = \langle v_{xi}(t) - v_{xi}(z_i, t) \rangle$ with respect to the downstream velocity $v_a(z_i)$ of the bulk at height $z_i$. Figure 2(a) shows that a large intruder, $S > 1$, lags ($\lambda_x < 0$), while for a same sized intruder, $S = 1$, there is no lag, within the fluctuations. Figure 2(b) shows that the lag increases at higher positions in the flow. Fitting these data gives the following scaling for the lag: $\lambda_x = a(\frac{1}{S} - 1)/\sqrt{\lambda_H}$, where $\lambda_H$ is the hydrostatic pressure $P_H = \nu p g_z (h - z)$, with $\nu = 0.577$. Both data can be fitted with the same value for the coefficient $a$. Interestingly, but outside the scope of this study, for $S < 1$, when the intruder sinks, $\lambda_x$ flips sign and becomes a velocity raise (increase).

Pressure.—We look for the origin of the lag in the pressure $P = \text{Tr}(\sigma)/3$. Figure 3(a) shows the cross-section $P(x, 0, z)$ for different size ratios. For $S \leq 1$ the pressure is hydrostatic, i.e., $P \approx P_H$, which is expected for this type of flow [15]. For $S > 1$, $P$ changes and a high pressure region develops at the bottom-front side of the intruder. Pressure fluctuations of lower magnitude also appear around the intruder. This demonstrates that the presence of a large particle modifies the local pressure around it. Although it is known that pulling an object through a granular medium affects the local pressure [16, 17], here the intruder is not actively pulled but can freely flow in the $x$-$y$ plane.

In order to isolate the non-hydrostatic effects in the pressure we study $P_L = P - P_H$. Figure 3(b) shows that for $S \leq 1$ $P_L$ is zero, within the fluctuations, while $P_L$ increases for $S > 1$ and is characterized by positive regions (over-pressure) in the lower right and upper left quadrants, and negative regions in the lower left and upper right quadrants. It seems reasonable now to correlate the lift force and the velocity lag to this non-hydrostatic pressure.

Buoyancy.—Different definitions for granular buoyancy forces exist [e.g.39-48]. In order to determine the lift force $F_L$ on the intruder we derive a new definition of the buoyancy force $F_b$, resulting from $P_H$. Taking inspiration from [48] and using our approximation $\nu(x_i, y_i, z_i) = \nu_i$ for the solids fraction at the intruder position, we integrate $P_H$ over the surface $A_i$ of $V_i$. With the divergence

![FIG. 2. (a) The intruder lag $\lambda_x$, normalized by the average bulk velocity at $z_i$, as a function of $S$. (b) Lag as a function of depth $z_i$ for $S = 2.4$. The dashed lines in (a) and (b) are fits of $\lambda_x/v_a(z_i) = a(\frac{1}{S} - 1)/\sqrt{\lambda_H(v_a(z_i))}$ and $\lambda_x = a(\frac{1}{S} - 1)/\sqrt{\lambda_H}$, respectively, with $a = 0.023$.](image-url)
theorem we find:

$$F_b = \int \hat{A}_i \cdot \mathbf{n} \cdot \mathbf{e}_z d\hat{A}_i = \nu \rho_p g_z \int \hat{V}_i = \nu \rho_p g_z \hat{V}_i$$

(1)

Here \( \mathbf{n} \) is the normal outward vector to \( \hat{A}_i \) and \( \mathbf{e}_z \) is the upward unit vector. Substituting \( \hat{V}_i = \hat{V}_i / \nu_i \) we obtain:

$$F_b = \nu \rho_p g_z \hat{V}_i$$

(2)

Effectively this is a generalized Archimedes principle at the particle level defined through an effective density that is equal to mass divided by Voronoi volume. Figure 3(a) shows that the measured \( \nu_i \) strongly depends on \( S \) and is bigger than the bulk solids fraction \( \nu \) for \( S > 1 \). This means that a larger intruder occupies a larger fraction of its Voronoi volume. The important consequence of the ratio \( \nu / \nu_i \) in \( F_b \) is that for \( S > 1 \) the buoyancy force will be less than the gravity force \( F_g = \rho_p g_z \hat{V}_i \), as shown in Fig. 3(b) where \( F_b / F_g < 1 \). When \( S = 1 \), \( \nu \) equals \( \nu_i \), and the buoyancy force balances \( F_g \). In the limit of \( S \to \infty \), \( \nu_i \to 1 \) and thus \( F_b \) corresponds to the buoyancy force in a fluid with density \( \rho = \rho_p \). This generalized buoyancy force differs from the classical Archimedian buoyancy definition \( F_b = \nu \rho_p g_z \hat{V}_i \) in a granular fluid, which has two problems: it is independent of \( S \), and more critically, predicts that \( F_b < F_g \) if \( S = 1 \). If this were true, there must be an additional force to balance \( F_g \), but we measure no such force.

We can now determine the lift force \( F_L \) similar to the way we obtained \( F_L \), i.e., by subtracting \( F_b \) from the total upward force on the intruder: \( F_L = F_{tot} - F_b \). We determine \( F_{tot} = F_{vz} + F_{tz} \) using the measured vertical normal and tangential contact forces, \( F_{vz} \) and \( F_{tz} \), respectively. Fig. 3(b) shows that \( F_L / F_g \) is approximately zero for \( S = 1 \), increases rapidly for \( S > 1 \) and tends to a finite value above \( S = 2 \). The plot of \( (F_b + F_L) / F_g \) shows that there is an optimal size ratio for segregation, in agreement with experimental findings [7] and predictions [49].

Saffman Lift.—Now we investigate whether the lag and the lift force are correlated. Such a correlation is known to exist in fluids: The Saffman lift force is found to scale with the velocity lag of a particle in a fluid [50, 51]:

$$F_{Saffman} = 1.615 \sqrt{\eta \gamma} \rho \lambda_x d_i^2 \sgn(\dot{\gamma})$$

(3)

where \( \eta \) is the fluid viscosity, \( \dot{\gamma} = \partial_x \dot{v}_z(z_i) \) the shear-rate and \( \rho \) the fluid density. Saffman [50] derived this relation taking the fluid properties in the absence of the particle and considered the limit:

$$\frac{\rho \lambda_x d_i^2}{2\eta} \ll \left( \frac{\rho d_i^2}{4\eta} \right)^{0.5}$$

(4)

where the first term is the Reynolds number for the lag \( Re_\lambda \) and the second term is the shear-rate Reynolds number \( Re_\gamma \). This relation physically corresponds to the viscous and inertial forces in the fluid being of similar magnitude. The square root in Eqs. (3) and (4) follows from this assumption. Whether Eq. (4) is valid for dense granular flows in general remains to be seen, nonetheless it is valid for our system; we find \( Re_\lambda = O(10^{-3}) \) and \( Re_\gamma = O(10^{-1}) \), using \( \rho = \rho_p \), with \( \eta \) and \( \dot{\gamma} \) measured from CG-fields far away from the intruder.

In order to test if a Saffman-like relation exists between \( F_L \) and \( \lambda_x \) we define

$$F_L = b \sqrt{\mu P_H \rho_p \lambda_x d_i^2 \sgn(\dot{\gamma})}$$

(5)

analogous to Eq. (3). Here \( b \) a dimensionless coefficient that accounts for unknown dependencies, and \( \lambda_x = a(1/S-1)/\sqrt{P_H} \). The bulk properties are taken in the absence of an intruder, hence the bulk friction \( \mu = \eta \dot{\gamma} / P_H \). The pressure \( P_H \) drops out of Eq. (5), indicating that the
The lift force is independent of the depth. We verify that this is true (see Fig. 5), in agreement with [39]. Next, we fit Eq. (5) to $F_L$ in Fig. 4(b), using the value for $a$ obtained in Fig. 2 and find that it captures the trend in $F_L$. Using the same values for $a$ and $b$ we can fit Eq. (5) to the lift force measured as a function of depth, as shown in Fig. 6. This demonstrates that Eq. (5) is the correct scaling between the lift force, size ratio, pressure and lag at constant inclination angle. The fact that this scaling is Saffman-like suggests a viscous-inertial origin for the segregation of large particles in the limit of low large-particle concentrations.

To provide further support for the finding that the generalized buoyancy force does not support the weight of a large intruder ($S > 1$) we set the intruder-bulk friction $\mu_{bi}$ to zero and find that $F_L$ is reduced, as shown in Fig. 4(c). Critically, this leads to a large intruder sinking instead of rising. The measured total upward force $F_{tot} = F_{n_i} + F_{L_i}$ on the intruder is lower than $F_g$, which demonstrates that $F_L$ is less than $F_g$. Interestingly, since $F_L$ does not completely disappear it should have both a geometric and frictional origin. We verified that $P_L$ is reduced but does not disappear. Lower-friction particles were recently found to sink below higher-friction particles in mono-disperse granular mixtures [52].

Discussion.—We report that a single large particle in a dense granular flow is surrounded by a non-hydrostatic pressure field. This coincides with a velocity lag $\lambda_x$ and a lift force $F_L$, coupled through Eq. (5), causing the particle to rise against gravity. These findings suggest that the mechanism of squeeze expulsion [11]—which has been used to qualitatively explain the segregation of large particles in dense granular flows since 1988—is a granular equivalent of the Saffman effect; a viscous-inertial lift force [50, 51].

The decomposition of $F_{tot}$ into the lift force and the generalized buoyancy force is critical to the preceding analysis. Moreover, it provides a physical explanation for the sinking of very large intruders [53, 54], as well as for the optimal size ratio for segregation [7, 49] and the unexplained trend of $F_{tot}(S)$ in Fig. 6 of [39]. Namely, if we consider the limit of Eq. (5) at large size ratios, we see that the lag approaches a constant value, while the buoyancy force approaches a fluid buoyancy with density $\rho = \nu \rho_p$. Gravity will then outgrow the total upward force and the particle will sink.

A Saffman-like lift force indicates that viscous-inertial effects play a role in polydisperse dense granular flows. This warrants further investigation because of the relevance for our understanding of these flows. The fact that

![FIG. 4](image_url)

*FIG. 4. (a) Local intruder solids fraction $\nu_i$ versus $S$. Different (almost collapsing) symbols correspond to intruders with $\mu_{bi} = 0.5$, $\mu_{bi} = 0$, $z_i = 15$, $z_i = 23$, $k = 20$ and $k = \infty$. Solid line corresponds to $(\nu - 1)S^{1.2} + 1$. The schematic depicts the Voronoi volume $V_i$ (dotted octagon) of the intruder (dashed circle). (b) The forces $F_n$, $F_L$, and $F_L + F_b$, normalized by $F_g$, for $z_i = 23$. The solid red line is a fit of Eq. (5) with $a = 0.023$ and $b = 93$. (c) The forces $F_b$, $F_L$, $F_{n_i}$ and $F_{t_i}$, normalized by $F_g$, for $S = 2.4$, $\mu_{bi} = 0$ and 0.5, at $z_i = 15$."

![FIG. 5](image_url)

*FIG. 5. The total upward force $F_{tot}$ (red) and the lift force $F_L$ (blue open symbols) on an intruder, as a function of depth, for $S = 2.4$, normalized by the gravity force. The solid line is a fit of Eq. (5) using the functional form for the lag $\lambda_x = a(1/S - 1)/\sqrt{P_L}$, with $a = 0.0230$, while the dashed line uses the raw lag data. Both fits have the same value for $b = 93$. Near the bed a boundary effect occurs, likely due to layering [40]."
our system falls within the limits (Eq. (4)) for the fluid Saffman effect is reassuring. Interestingly, $F_{\text{Saffman}}$ is characterized by a square root dependence on the shear-rate $\dot{\gamma}$ (Eq. (3)), while this term drops out of Eq. (5). Determining the full dependency of the lift force on $\mu$, $\mu_0$ is as crucial as investigating the change in $F_L$ when the large-particle concentration increases.

The observed lag must originate from a horizontal drag force. Unfortunately, drag forces on a free-flowing object in granular media, in contrast to a dragged object, have received little attention [35]. We are currently investigating whether a Stokesian drag, found by Tripathi and Khakhar [35] for a heavy sinking mono-disperse intruder, also holds for rising large intruders. Obtaining the drag on free-flowing particles is important for the rheology of granular flows in general, but foremost because drag is a cornerstone of models for particle-size segregation in dense granular flows.

In order to unify Eq. (5) with the scaling laws found by Guillard et al. [39] and develop a multi-scale model for the segregation of large intruders in dense granular flows the lag will have to be expressed in terms of $\lambda_x = f(\partial P/\partial z, \partial \tau / \partial z)$, where $\tau$ is the shear stress. This is far from trivial and ongoing work. For a formal proof that the Saffman relation holds in granular fluids, the analytical derivation by Saffman could be repeated for a granular rheology.

Interestingly, our approach of treating the granular medium surrounding a particle as a continuum and obtaining the force laws can potentially be extended to every particle in a system, leading to new micro-inspired macroscopic models.

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