On eigenstate thermalization in the SYK chain model

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Abstract

Eigenstate thermalization hypothesis (ETH) explains how generic observables of individual isolated quantum systems in pure states can exhibit thermal behaviors. ETH ansatz usually holds true in quantum chaotic systems. In this paper, we examine a one dimensional lattice generalization of the Sachdev-Ye-Kitaev model with spatial local random interaction of Majorana fermions, the so-called SYK chain model. The model is maximally chaotic but its Rényi entanglement entropy study suggests that the model does not rapidly thermalize. We show, however, that for two conventional few-body operators, the ensemble averaged theory of the SYK chain model strictly satisfies ETH conditions. We also demonstrate that for every single realization of the ensemble, the operators loosely satisfy ETH, and consequently rapidly thermalize, albeit, with larger fluctuations. We comment on the difference between application of ETH for individual systems and ensemble averaged ones. We also use our results to comment on the implications for eigenstate correlation functions and a putative gravitational dual theory.

I. INTRODUCTION

Chaotic quantum many-body systems are usually expected to rapidly thermalize even when they are isolated from their surrounding environments. Eigenstate thermalization hypothesis (ETH) provides a basis for that [1–3]. It demonstrates that a closed quantum system that starts in a pure state, far from the equilibrium, can rapidly thermalize with respect to a collection of its operators if those operators satisfy a few conditions. In such a case, the expectation values of the operators evolve to thermal values and remain near those values with exponentially small fluctuations for most of the time. A recent study of Rényi entanglement entropy of a maximally chaotic quantum many-body system, however, suggests that the chaotic system under the study slowly thermalizes [4]. The system is a one-dimensional generalization of the original Sachdev-Ye-Kitaev (SYK) model, called the SYK chain model [5].

In this paper, we focus on the problem of thermalization of the SYK chain model from the perspective of the eigenstate thermalization hypothesis. The SYK chain model is a quenched disorder one-dimensional lattice model where each site of the lattice contains $N$ Majorana fermions with quartic random interactions [5]. The Majorana fermions of nearest neighboring sites interact with each other via 2-2 interactions. The model has interesting properties...
similar to the original SYK model \cite{6,7} such as local criticality, extensive zero temperature entropy, and maximal chaos. On top of that, the model shows unique properties that are absent in the zero-dimensional model of SYK such as diffusive energy transport and butterfly velocity for the propagation of chaos in space \cite{5}. From the condensed matter physics perspective, the model is a rare example of a strongly correlated interacting lattice model that is chaotic and at the same time solvable. It provides a fascinating platform for studying several properties of strongly correlated systems, such as thermalization, entanglement propagation, many-body localization, phase transition, dissipative transport, etc.

From the holographic duality perspective, the SYK chain model is expected to be dual to some kind of incoherent black hole \cite{4}. Since an isolated SYK chain model can only undergo unitary evolutions, understanding the thermalization of the model when isolated, which is the subject of the present paper, can shed light on the formation and evaporation of the putative black hole, and on the black hole information paradox \cite{8}.

We check, in this paper, the satisfaction of ETH conditions for two conventional generic simple few-body operators in the SYK chain model. We both check the case of single realizations of the disorder and the case of ensemble averages of realizations of the disorder. Through exact diagonalization, we demonstrate that single realizations of the disorder loosely satisfy ETH (see below), and ensemble averages of the disorder strictly satisfy ETH conditions. Therefore, we establish that the SYK chain model, and consequently its putative black hole dual, can rapidly thermalize through the mechanism of ETH. We comment on differences of our results with those of the Rényi entanglement entropy study \cite{4} in the discussion section.

ETH satisfaction for the original SYK \cite{18}, the complex SYK \cite{19}, and the supersymmetric SYK \cite{18} models have been established. Those models are zero dimensional models with all-to-all interactions. The interest about the role of the all-to-all interaction and spatial locality has already been raised \cite{18}. Our results shows that the presence of spatial locality and absence of all-to-all interaction have no determining role in the thermalization of the SYK chain model in the limits that we studied the model, which are the solvable limits of the model.

Furthermore, we establish in this paper, by studying off-diagonal matrix elements of the

\footnote{For recent developments on resolving the information paradox from another angle and challenges it faces see \cite{9,17}.}
operators, that the SYK chain model behaves like a random matrix theory (RMT) up to a certain energy scale and beyond that it shows non-RMT behaviors. We recognize the energy scale, by analogy with the theory of disordered conductors, as the Thouless energy \( E_T \) and confirm that the Thouless energy is directly proportional to a positive power of the inter-sites coupling strength in the model \( J_1 \). The more strongly the neighboring SYK sites in the chain are coupled together, the larger the energy scale for which the system exhibits RMT behavior.

The remainder of the paper is organized as follows. In section II we review eigenstate thermalization hypothesis and the SYK chain model. We define the term *loose satisfaction* of ETH and also discuss the Thouless energy and diffusion constant of the SYK chain model. In section III we numerically check ETH for three cases of coupling strengths of the SYK chain model, both for single realizations of the disorder and for ensemble averages of realizations of the disorder. We comment on the difference between application of ETH for single realizations and ensemble averaged ones. We further recognize the RMT behavior of the model for certain energy ranges and indicate the relation of its Thouless energy with the inter-sites coupling strength. In section IV we conclude the paper with a summary and discussion of implications of our results, future directions, and comparison with the study of Rényi entanglement entropy.

II. OVERVIEW

A. Eigenstate thermalization hypothesis

Eigenstate thermalization hypothesis (ETH) explains how isolated quantum systems in pure states can come to thermal equilibrium \[1,3\]. In this subsection, I review the hypothesis.

Assume that we have an isolated quantum system with finitely many degrees of freedom \( \infty > N \gg 1 \) and with non-degenerate discrete energy spectrum \( \{E_n\} \). We denote the Hamiltonian of the system by \( H \) and its energy eigenstates by \( \{|n\} \). For each pure state \( |\psi\rangle \) of the quantum system, the quantum average (or expectation value of) energy is

\[
\langle H \rangle_Q = \langle \psi | H | \psi \rangle
\]  \hspace{1cm} (1)
and the quantum energy uncertainty is
\[ \Delta H_Q = \left[ \langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2 \right]^{1/2}. \] (2)

ETH can make general statements about thermalization of pure states whose quantum energy uncertainty is much smaller than their quantum average energy
\[ \Delta H_Q \ll \langle H \rangle_Q. \] (3)

The smallness of \( \Delta H_Q \) with respect to \( \langle H \rangle_Q \) in (3) is sufficient to be of the form \( \Delta H_Q \sim \langle H \rangle_Q / N^\nu \) for some \( \nu > 0 \).

For later use, we assign to each pure state \(| \psi \rangle\) of an isolated quantum system an artificial temperature \( 1/\beta_\psi \) such that the expectation value of the Hamiltonian in pure state \(| \psi \rangle\) and in thermal state \( e^{-\beta_\psi H} \) are equal,
\[ \langle H \rangle_Q = \langle \psi | H | \psi \rangle = \frac{\text{Tr} \ e^{-\beta_\psi H} H}{\text{Tr} \ e^{-\beta_\psi H}} = \langle H \rangle_{\text{th}}. \] (4)

Eq. (4) defines \( \beta_\psi \). It sets the temperature such that the average energy of the system in the pure state \(| \psi \rangle\) and its corresponding thermal state \( e^{-\beta_\psi H} \) are equivalent.

Next, consider a finite set of generic observables \( \{ A^\alpha \}_{\alpha=1}^\Gamma \) whose total number \( \Gamma \) is much smaller than the dimension of the Hilbert space, i.e. \( \Gamma \ll \dim(H) \). Observables \( A^\alpha \) are generic in the sense that one can either imagine measuring them in the laboratory, or, as in recent literature, they are non-extensive. The latter means that it only involves few bodies in a many-body system. For example, the single-site number operator or the two-site hopping operator are non-extensive.

For any state \(| \psi \rangle\), the set \( \{ A^\alpha \}_{\alpha=1}^\Gamma \) are chosen such that the quantum average of the observables
\[ \langle A^\alpha \rangle_Q = \langle \psi | A^\alpha | \psi \rangle \] (5)
are far from their thermal averages at temperature \( 1/\beta_\psi \)
\[ \langle A^\alpha \rangle_{\text{th}} = \frac{\text{Tr} \ e^{-\beta_\psi H} A^\alpha}{\text{Tr} \ e^{-\beta_\psi H}}. \] (6)
Thus, as the system starts from state $|\psi\rangle$, the observables starts from non-equilibrium initial expectation values $\langle A^{\alpha}(0) \rangle_Q = \langle A^{\alpha} \rangle_Q \neq \langle A^{\alpha} \rangle_{th}$ (Fig. 1).

One can expand the pure state $|\psi\rangle$ in terms of energy eigenstates of the Hamiltonian

$$|\psi\rangle = \sum_n c_n |n\rangle. \quad (7)$$

If the system starts from state $|\psi\rangle$, it evolves to

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |n\rangle \quad (8)$$

at time $t$. The expectation values of the observables accordingly evolve to

$$\langle A^{\alpha}(t) \rangle_Q = \langle \psi(t)|A^{\alpha}|\psi(t)\rangle = \sum_{m,n} c_m^* c_n A^{\alpha}_{mn} e^{i(E_m - E_n)t/\hbar} \quad (9)$$

where

$$A^{\alpha}_{mn} = \langle m|A^{\alpha}|n\rangle \quad (10)$$
is the matrix element of the observable $A^{\alpha}$ in the energy eigenbasis.

Thus far, we have been carefully preparing the stage. We have now everything in place to express ETH.

ETH states that if in addition to all the above assumptions and conditions, the generic observables $\{A^{\alpha}\}_{\alpha=1}^{\Gamma}$ satisfy two more conditions, the expectation values of them $\langle A^{\alpha}(t) \rangle_Q$, which started from non-equilibrium values $\langle A^{\alpha}(0) \rangle_Q$, will reach their thermal equilibrium values $\langle A^{\alpha} \rangle_{th}$ and remain close to those for most of the times (Fig. 1).

Those two conditions of ETH are as follows: For all $n$, $m$, and $1 \leq \alpha \leq \Gamma$, ETH demands

$$A^{\alpha}_{nn} = A^{\alpha}(E_n), \quad (11)$$

$$|A^{\alpha}_{nm}| \sim e^{-S/2} \quad (12)$$

where $A^{\alpha}$ are smooth functions of energy and $S$ is the entropy of the quantum system which scales with the number of degrees of freedom $N$. ETH demands that the diagonal matrix elements of the generic observables in the energy basis $A^{\alpha}_{nn}$ lie on smooth curves of energy, $A^{\alpha}(E)$, and their off diagonal matrix elements be exponentially small by a factor of order
FIG. 1. Eigenstate Thermalization Hypothesis. If ETH conditions are satisfied for a set of observables \( \{A^\alpha\}_{\alpha=1}^{\Gamma} \), the observables will thermalize during unitary evolution of the pure state of the system. The observables are chosen such that their quantum expectation values over the pure state start from non-equilibrium values. ETH demonstrates that their expectation values \( \langle A^\alpha(t) \rangle_Q \) will reach their equilibrium values \( \langle A^\alpha \rangle_{th} \) and remain close to that for most of the times. Fluctuations around the equilibrium values are exponentially small for most of the times. But large fluctuations are allowed to rarely occur by ETH.

\[ e^{-S/2} \text{ or } e^{-N/2}. \] In such cases, ETH shows that the system reaches thermal equilibrium with respect to the set of observables \( \{A^\alpha\}_{\alpha=1}^{\Gamma} \). This means that each observable \( A^\alpha \) reaches its thermal average values \( \langle A^\alpha \rangle_{th} \) and remain close to that for most of the times. Fluctuations around the thermal values can rarely be large, but for most of the times they are exponentially small. 2

To prove, ETH demonstrates that infinite time average of the observables

\[ \overline{A^\alpha} = \lim_{T \to \infty} \frac{1}{T} \int_0^\infty \langle A^\alpha(t) \rangle \, dt \]  

(14)

2 In some literature, the conditions of ETH are expressed as satisfaction of the following equation

\[ A^\alpha_{mn} = A^\alpha(E_n)\delta_{mn} + e^{-S/2}f^\alpha(E,\omega)R_{mn} \]  

(13)

where \( \overline{E} = (E_n + E_m)/2 \), \( \omega = E_m - E_n \), \( f^\alpha \) is a smooth function, and \( R_{mn} \) is a Gaussian random variable with zero mean and unit variance. The only difference between conditions (11), (12) that were originally introduced as ETH by its founders [11] and condition (13), is the presence of random variable \( R_{mn} \) in the off-diagonal matrix elements. However, it is not necessary in general that the off-diagonal matrix elements be randomly distributed in order for ETH to be satisfied and for the closed system to reach thermal equilibrium. Quantized chaotic systems satisfy Eq. (13). Nonetheless, in general a specific type of distribution of the off diagonal matrix elements, here for example a Gaussian distribution, is not required for thermalization. For this reason, we neglect studying distributions of the off-diagonal elements in this work.
are approximately equal to the smooth function $A^\alpha(E_n)$, which is equal to the diagonal matrix elements by Eq. (11), and to the thermal and microcanonical average values of the observables

$$\bar{A}^\alpha \simeq A^\alpha(E_n) = A^\alpha_{nn} \simeq \langle A^\alpha \rangle_{th} \simeq \langle A^\alpha \rangle_{micro}. \tag{15}$$

The approximations are good up to $O(1/S)$ and $O(\Delta H^2)$ \cite{2, 3}.

Moreover, ETH shows that infinite time average of fluctuations around the thermal values are suppressed by the off digonal matrix elements as follows

$$\left| \langle A^\alpha(t) \rangle_Q - \bar{A}^\alpha \right|^2 = \sum_{m \neq n} |c_n|^2 |c_m|^2 |A^\alpha_{nm}|^2. \tag{16}$$

If the off-digonal matrix elements satisfy Eq. (12), the right hand side of the above equation will be exponentially small

$$\left| \langle A^\alpha(t) \rangle_Q - \bar{A}^\alpha \right|^2 \sim e^{-S}. \tag{17}$$

This is because $|c_n|^2$ and $|c_m|^2$ sum to unity.

Eq. (15) shows that the infinite time average values of the observables are equal to their thermal average values, and Eq. (17) guarantees that fluctuations around those thermal average values are exponentially small for most of the times.

This is ETH in its most strict sense. In some cases, however, all the conditions of ETH are not simultaneously satisfied. Nevertheless, one can still draw the general conclusions of ETH for thermalization of the observables. For example, in some systems, such as single realization of SYK chain model as we discuss in Sec. III B, Eq. (12) is not satisfied and the off diagonal matrix elements are larger than exponential. Nonetheless, since their maximum absolute values is still small, one can use Eq. (16) to obtain

$$\left| \langle A^\alpha(t) \rangle_Q - \bar{A}^\alpha \right|^2 \leq \max_{n \neq m} |A^\alpha_{nm}|^2. \tag{18}$$

and conclude that the fluctuations are still small, although not exponentially small.

In such cases we say ETH is loosely satisfied. As long as the off diagonal matrix elements are one or more orders of magnitude smaller than the diagonal elements we don’t get significant deviations from thermal average values in generic times and we can assume that thermalization is obtained in the system.
The other point about the conditions of ETH is for inequality (3) which requires the quantum fluctuation of energy be much smaller than quantum average energy. This inequality is used in the proof of ETH to demonstrate Eq. (15) for the diagonal matrix elements. If one can check satisfaction of Eq. (15) directly, one does not need to check the validity of inequality (3). This is what we do in Sec. III. We directly check Eq. (15) both for single realization of the SYK chain model and for the ensemble averaged theory and show our results there.

B. SYK Chain model

SYK chain model [5] is a generalization of the original SYK model [6, 7]. We first briefly overview the original SYK model and then describe the SYK chain model in this subsection.

The original SYK model is an ensemble averaged theory [6, 7]. Any single realization of the ensemble consists of \( N \) Majorana fermions with quartic random all-to-all interactions. The Hamiltonian of each realization is

\[
H = \sum_{1 \leq i < j < k < l \leq N} J_{ijkl} \chi_i \chi_j \chi_k \chi_l
\]

(19)

where the Majorana fermion operators satisfy the Clifford algebra

\[
\{\chi_i, \chi_j\} = \delta_{ij}
\]

(20)

and the couplings \( \{J_{ijkl}\} \) are independent random real numbers drawn from a Gaussian distribution with zero mean

\[
\overline{J_{ijkl}} = 0,
\]

(21)

and variance

\[
\overline{J_{ijkl}^2} = \frac{3! J^2}{N^3}.
\]

(22)

Here, \( J \) sets the average strength of the couplings.

The SYK model is solvable at large \( N \). It exhibits many interesting properties at strong coupling \( N \gg \beta J \gg 1 \) such as extensive zero temperature entropy, local criticality (power law correlation in time), and maximal chaos. In such limit, this averaged theory also shows
FIG. 2. SYK chain model. The model is made of $M$ sites with periodic boundary condition. Each site contains $N$ Majorana fermions with SYK interaction. Nearest neighboring sites interact via four fermion interaction with two from each site. (Graphic is adapted from [5])

holographic behavior. It demonstrates connections to nearly $AdS_2$ gravity. The model, however, is (0+1)-dimensional and has no spatial locality. As a result, some expected properties of higher dimensions such as diffusion and butterfly effect cannot be realized in the model.

A higher dimensional lattice generalization of the SYK with spatial locality has been recently proposed in [5]. In (1+1)-dimension the model is a chain of SYK sites or links and is called SYK chain model Fig. 2.

We focus on the SYK chain model in this paper. Similar to the SYK model, the SYK chain model is also an ensemble averaged theory. The Hamiltonian of a single realization of the model is given by

$$\begin{align*}
H &= \sum_{x=1}^{M} \sum_{1 \leq i < j < k < l \leq N} J_{ijkl,x} \chi_{i,x,\chi_{j,x,\chi_{k,x,\chi_{l,x}}} + 1} + 1 \\
&\quad + \sum_{x=1}^{M} \sum_{1 \leq i < j \leq N \atop 1 \leq k < l \leq N} J'_{ijkl,x} \chi_{i,x,\chi_{j,x,\chi_{k,x,\chi_{l,x}}} + 1} \\
\text{(23)}
\end{align*}$$

where $\{J_{ijkl,x}\}$ and $\{J'_{ijkl,x}\}$ are independent zero mean Gaussian random couplings with variances

$$\begin{align*}
\overline{J^2_{ijkl,x}} &= \frac{3! J_0^2}{N^3}, \\
\overline{J'^2_{ijkl,x}} &= \frac{J_1^2}{N^3}.
\text{(24)}
\end{align*}$$

Each site of the lattice, labeled by $x = 1, \cdots, M$, contains $N$ Majorana fermions with on-site SYK interaction. Nearest neighbor SYK sites also have 2-2 interactions (the second summation in Eq. (23)). The Majorana fermions satisfy anti-commutation relation,

$$\{\chi_{i,x}, \chi_{j,y}\} = \delta_{xy} \delta_{ij} \quad \text{(25)}$$
and periodic boundary condition $\chi_{i,1} = \chi_{i,M}$. One can describe local properties of the model in terms of an effective coupling constant

$$J = \sqrt{J_0^2 + J_1^2}. \quad (26)$$

The SYK chain model is solvable at large $N$. At strong coupling $N \gg \beta J \gg 1$ the model has the interesting properties of the original SYK model: maximal chaos, extensive zero temperature entropy, and local criticality. On top of that, due to its spatial locality, the model exhibits new features such as diffusion of energy and butterfly effect.

The model at strong coupling describes a strongly correlated diffusive metal. Also the relation $D = v_B^2/2\pi T$ between its temperature independent diffusion constant $D$ and its butterfly speed $v_B$ is consistent with proposals of incoherent metals. The SYK chain model, at strong coupling, is a rare example of strongly interacting chaotic lattice models that are tractable. Therefore, as also mentioned in Ref. [5], “it provides an interesting platform for studying various properties of strongly correlated systems, such as thermalization, entanglement propagation, dissipative transport, etc.”

Ref. [4] studied spread of Rényi entropy of the SYK chain model and commented on thermalization of the model. However, the eigenstate thermalization for the model has not yet been directly investigated. In the following sections we investigate the eigenstate thermalization of the SYK chain model.

The diffusion constant $D$ of the model is a function its couplings strengths:

$$D = \frac{2\pi J_1^2}{3\sqrt{2}J\alpha_K} \quad (27)$$

where $\alpha_K \approx 2.852$ [5]. The diffusion of energy in the system indicated by constant $D$ is occurred due to the dynamics of the collective mode of the time reparametrization field, recognized as light degrees of freedom in [4], which is the most important collective mode of the system in the low-energy-long-wavelength limit. Gu et. al. [5] find that single Majorana fermions do not propagate between sites but only collective modes made by pairs of fermions have spatial dynamics.

In study of off-diagonal matrix elements of observables in next section we recognize the
Thouless energy,

$$E_T = \frac{\hbar D}{L^2},$$

where \( L \) is the size of the system, and we investigate whether the Thouless energy agree with Eq. (27) or not.

### III. NUMERICAL CHECKS OF EIGENSTATE THERMALIZATION

In this section we numerically check the validity of the eigenstate thermalization hypothesis for the SYK chain model. We set up our model and set its parameters, and explain how one can construct the model numerically in terms of large matrices. Then we check ETH for single realization of the disorder and ensemble averages of 1000 realizations of the disorder.

#### A. Setup

We construct a SYK chain model with \( M = 4 \) SYK sites and put \( N = 6 \) Majorana fermions on each site. The total number of Majorana fermions in our system is \( NM = 24 \). We set \( J = \sqrt{5} \) and consider three different regimes for the intra-sites and inter-sites (nearest neighbor sites) coupling strengths:

(a) \( J_0 = 2, \ J_1 = 1 \)  Stronger intra-sites couplings

(b) \( J_0 = J_1 = \sqrt{5}/2 \)  Equally strong inter- and intra-sites couplings

(c) \( J_0 = 1, \ J_1 = 2 \)  Stronger inter-sites couplings

Notice that the SYK chain model in case (b) is still different from the original SYK model, because the couplings in the SYK chain model is not all-to-all as opposed to those in the original SYK model. The couplings in the SYK chain model are of two types: On-site couplings (characterized by \( J_0 \)) and between Majorana fermions of nearest neighboring sites couplings (characterized by \( J_1 \)). Thus, in no case we should expect to get directly the results of testing ETH in the SYK model.

To construct the matrix representation of the model, we first relabel the Majorana fermions by Greek indices as follows

$$\chi_{i,x} \rightarrow \chi_\alpha$$

(29)
where
\[ \alpha = i + (x - 1)N. \]  
(30)

The index \( \alpha \) runs from 1 to \( NM \), while \( 1 \leq i \leq N \) and \( 1 \leq x \leq M \). The Majorana fermions \( \chi_\alpha \) satisfy the Clifford algebra
\[ \{ \chi_\alpha, \chi_\beta \} = \delta_{\alpha \beta}. \]  
(31)

Since Majorana fermion operators are Hermitian \( \chi_\alpha = \chi_\alpha^\dagger \) we can build operators
\[
\begin{align*}
    c_\alpha &= \chi_{2\alpha} - i\chi_{2\alpha-1}, \\
    c_\alpha^\dagger &= \chi_{2\alpha} + i\chi_{2\alpha-1}, \quad \alpha = 1, \cdots, NM/2
\end{align*}
\]  
(32)

that satisfy the canonical anticommutation relation for complex fermions
\[ \{ c_\alpha, c_\beta \} = \{ c_\alpha^\dagger, c_\beta^\dagger \} = 0, \quad \{ c_\alpha, c_\beta^\dagger \} = \delta_{\alpha \beta}. \]  
(34)

To construct a representation, we pick a vacuum state \( |0\rangle \) that is annihilated by all the \( c_\alpha \)'s
\[ c_\alpha |0\rangle = 0 \]  
(35)

and create the basis of representation using \( c_\alpha^\dagger \)'s
\[ (c_1^\dagger)^{n_1} \cdots (c_L^\dagger)^{n_L} |0\rangle, \quad n_\alpha = 0, 1, \]  
(36)

where \( L = NM/2 \). This gives us \( 2^{NM/2} \) states. We can now explicitly write recursion relations for the matrix representation of \( \{ \chi_\alpha \} \) [20]

\[
\begin{align*}
    \chi^{(K)}_\beta &= \chi^{(K-1)}_\beta \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta = 1, \cdots, NM - 2, \\
    \chi^{(K)}_{NM-1} &= I_{2^{K-1}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
    \chi^{(K)}_{NM} &= I_{2^{K-1}} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\end{align*}
\]  
(37-39)
where $I_d$ is the $d \times d$ identity matrix, and $K$ indicates the level of recursion and is between $1 \leq K \leq NM/2$. We need to construct $\chi_{\alpha} = \chi_{\alpha}^{(NM/2)}$ for $\alpha = 1, \cdots, NM$. To this end, we start the recursion by setting the initial values of $\chi_{\alpha}^{(1)}$ to Pauli matrices $Y$ and $X$,

$$
\chi_{1}^{(1)} = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \chi_{2}^{(1)} = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

This gives us the entire set of $\{\chi_{\alpha}\}$. The matrices of $\chi_{\alpha}$’s constructed this way are $2^{NM/2} \times 2^{NM/2}$ matrices. For our case, $NM = 24$, those are $4096 \times 4096$ matrices.

Having all the $\{\chi_{\alpha}\}$ constructed, we can now retrieve the original Latin indices for Majorana fermions $\{\chi_{i,x}\}$ using relations

$$
i = \alpha \text{ mod } N, \quad x = \frac{\alpha - i}{N} + 1.
$$

By use of the original Latin indices we construct the Hamiltonian of the SYK chain model Eq. (23).

What we did above was to map every two Majorana fermions (labeled by Latin indices) onto one complex fermion (Labeled by Greek indices). In the language of complex fermions, we can say that there are $M$ sites and on each site there can be $N/2$ complex fermions ($N$ is even here and throughout the paper). Therefore, the indices of complex fermions $\alpha = 1, \cdots, N/2$ belong to site $x = 1$, the indices $\alpha = N/2 + 1, \cdots, N$ belong to site 2, the indices $\alpha = N + 1, \cdots, 3N/2$ belong to site 3, \cdots, and finally the indices of complex fermions $\alpha = \frac{N(M-1)}{2} + 1, \cdots, \frac{NM}{2}$ belong to site $x = M$. It is more convenient to express operators in terms of complex fermion operators as we do below

In following subsections we check ETH for the particle number operator and the hopping operator. Both of these operators are non-extensive and include one and two bodies respectively. The particle number operator can be constructed using the annihilation and creation operators

$$
\hat{n}_1 = c_{1}^{\dagger}c_1
$$

where we used complex fermions indices (the Greek indices) for annihilation and creation operators $c_{\alpha}, c_{\beta}^{\dagger}$. Physically, this means that we have mapped first two Majorana fermions

\footnote{In another picture one can map the complex fermions to qubits and regard fermion operators as gate operators that act on qubits.}
of the first SYK site \( \chi_{1,1}, \chi_{2,1} \) to one complex fermion and the number operator \( \hat{n}_1 \) counts
the number of that complex fermion which can be 0 or 1. The particle number operator
defined this way is rather a local operator and is only influenced by the first two Majorana
fermions of the first SYK site.

The hopping operator, on the other hand, can be defined as a non-local operator between
two different SYK sites. Using the complex fermion indices again we define a hopping
operator between site 1 and site 3 as follows:

\[
\hat{h}_{13} = \hat{c}_N^\dagger c_1 + c_1^\dagger \hat{c}_{N+1}
\]

(43)

In the language of complex fermions, this is a hopping operator between complex fermion
\( \alpha = 1 \) which belongs to site \( x = 1 \) and complex fermion \( \alpha = N + 1 \) which belongs to site
\( x = 3 \). In the language of Majorana fermions, the operator engage the first two Majorana
fermions of the first site \( \chi_{1,1}, \chi_{2,1} \) and the first two Majorana fermions of the third site
\( \chi_{2N+1,3}, \chi_{2N+2,3} \). Although the hopping operator is completely non-local, it is still non-
extensive as it only includes few Majorana fermions.

We check ETH for the particle number operator \( \hat{n}_1 \) of Eq. (42) and the hopping operator
\( \hat{h}_{13} \) of Eq. (43) below.

**B. Single realization of the disorder**

SYK chain model is an ensemble averaged theory. However, ETH is formulated for
individual systems, not an ensemble average of quantum systems. Thus, it is essential to
check whether ETH is satisfied for single realization of SYK chain model, before considering
the ensemble averaged theory.

If single realization of the disorder fails to satisfy ETH, it is not clear what conclusion one
can draw from satisfaction of ETH for the disorder averaged theory. This is because all the
steps of the proof of thermalization of the generic observables in the eigenstate thermalization
hypothesis rest on the assumption that the system of interest is an individual system with
a fixed Hamiltonian.

Fortunately, we found that for the operators we are studying in this paper, ETH is *loosely*
satisfied for single realizations of the disorder and one does not have to be worried about
FIG. 3. Matrix plots of $|O_{mn}| = |\langle m|O|n\rangle|$ for the particle number operator $O = \hat{n}_1$ for three different values of coupling strengths $J_0, J_1$. We choose a single realization of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions on each site. The horizontal and vertical axes show the energy eigenvalues, $E_m/J, E_n/J$, corresponding to the matrix elements $|O_{mn}|$. All three plots show agreement with ETH in its loose sense. The diagonal elements of the number operator matrices in the energy eigenbases fluctuate near the microcanonical ensemble average value $1/2$ while the off-diagonal ones are zero or small. The non-zero off-diagonal matrix elements fluctuate around the expected exponentially small value of ETH by about two orders of magnitude. The maximum absolute value of the off-diagonal matrix elements are about 0.16. This suggests that ETH is not satisfied in its strict sense but it is satisfied in its loose sense. For the purpose of presentation, the data are downsampled to $512 \times 512$. Top Left Panel: $J_0 = 1, J_1 = 2, J = \sqrt{5}$, Top Right Panel: $J_0 = 2, J_1 = 1, J = \sqrt{5}$, Bottom Panel: $J_0 = J_1 = \sqrt{5}/2, J = \sqrt{5}$.

contradiction between the results of the ensemble averaged theory and the single theory (See Sec. II A for the definition of the term: loose satisfaction of ETH).

We show in this subsection our results for the particle number operator $\hat{n}_1$ and the hopping operator $\hat{h}_{13}$ between first and third sites.
FIG. 4. Matrix plots of $|O_{mn}| = |\langle m | O | n \rangle|$ for the hopping operator between site 1 and 3, $O = \hat{h}_{13}$, for three different values of coupling strengths $J_0, J_1$. We construct single realizations of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions on each site. The horizontal and vertical axes show the energy eigenvalues, $E_m / J$, $E_n / J$, corresponding to the matrix elements $|O_{mn}|$. The diagonal elements of the hopping operator matrices in the energy eigenbases are strictly equal to the microcanonical ensemble average value zero while the off-diagonal ones are zero or small. The non-zero off-diagonal elements fluctuates around the expected exponential value by about two orders of magnitude. The maximum absolute value of the off-diagonal matrix elements are about 0.2. ETH in its strict sense is not satisfied. However, ETH is satisfied in its loose sense. For the purpose of presentation, the data are downsampled to $512 \times 512$. Top Left Panel: $J_0 = 1$, $J_1 = 2$, $J = \sqrt{5}$, Top Right Panel: $J_0 = 2$, $J_1 = 1$, $J = \sqrt{5}$, Bottom Panel: $J_0 = J_1 = \sqrt{5}/2$, $J = \sqrt{5}$.

Microcanonical average value of $\hat{n}_1$ is

$$\langle \hat{n}_1 \rangle_{\text{micro}} = 1/2. \quad (44)$$

ETH, from Eq. [15], expects that the diagonal matrix elements of $\hat{n}_1$ in the energy eigenbasis to be about this value.

For the off-diagonal matrix elements, ETH, in its strict sense, expects that they all be of
order $e^{-S/2}$, Eq. (12). However, ETH in its loose sense, as we defined in Sec. II A, expect that their maximum absolute values be sufficiently small.

For all three cases of couplings (a-c) introduced at the beginning of Sec. III A (i.e. $J_0 = 2, J_1 = 1$; $J_0 = J_1 = \sqrt{5}/2$; and $J_0 = 1, J_1 = 2$), we constructed single realizations of the SYK chain model and calculated absolute values of matrix elements of the number operator $\hat{n}_1$ in energy eigenbasis, namely $|\langle m|\hat{n}_1|n\rangle|$. We observed that the diagonal matrix elements fluctuate around the microcanonical average value $1/2$ of Eq. (44) while the off-diagonal matrix elements are small.

Fig. 3 shows this result in form of matrix plots. As one can see the diagonal elements have dark blue color (associated to 0.5 and numbers around that in the color bar) while the off-diagonal elements have light cyan color (associated to zero and numbers close to that). In fact, we observed that most of the off-diagonal matrix elements are zero to machine precision and only a small fraction of them are non-zero. We observed that most of the elements of the non-zero fraction are not as small as $e^{-S/2}$. So, ETH in its strict sense is not satisfied.

However, we observed that the non-zero fraction of the off diagonal matrix elements fluctuate around the expected value $e^{-N} \approx 0.002$ by only about two orders of magnitude. The maximum absolute value of the off-diagonal matrix elements was about 0.16 which is also sufficiently small. Therefore, we can state that ETH in its loose sense is satisfied. And $\langle \hat{n}_1(t) \rangle_Q$ will reach its thermal value $1/2$ over time with at most 32% fluctuations around that value for most of the times.

For the hopping operator, we observed similar results. We show our results in form of matrix plots in Fig. 4.

The on-diagonal matrix elements of the hopping operator are strictly zero on theoretical grounds,

$$\langle n|\hat{h}_{13}|n\rangle = 0.$$ (45)

One can easily obtain the above equation by noting that the operator is hermitian and the model is fermionic.

From Eq. (45) one can show that the microcanonical average value of the hopping operator $\hat{h}_{13}$ is also zero,

$$\left\langle \hat{h}_{13} \right\rangle_{\text{micro}} = 0.$$ (46)

We observed strict satisfaction of Eq. (45) in our numerical construction of single real-
izations of the SYK chain model for all three cases of the couplings (a-c). Since the micro-canonical average value, Eq. (46), is also zero, we found that the first condition of ETH about the equivalence of the on-diagonal matrix elements and the microcanonical average value is strictly satisfied for the hopping operator.

For the off-diagonal matrix elements of the hopping operator $\hat{h}_{13}$, we observed that a large fraction of them are zero. The non-zero fraction fluctuates around the expected exponential value $e^{-N} \approx 0.002$ by about two orders of magnitude. We found that the maximum absolute value of the off-diagonal matrix elements are about 0.2.

Since the off-diagonal matrix elements are not exponentially small, ETH is not satisfied in its strict sense. But, since they are sufficiently small, ETH is satisfied in its loose sense for the hopping operator $\hat{h}_{13}$.

The quantum expectation value of the hopping operator $\langle \hat{h}_{13}(t) \rangle_Q$ will reach its equilibrium zero value of Eq. (46) and remain close to that for most of the times. Fluctuations are expected to be at most about 0.2 for most of the times.

C. Ensemble average over the disorder: Diagonal elements

As we mentioned in the previous subsection, as far as we know, there is no theoretical ground for applying ETH to an ensemble averaged theory and it is not evident what implications such an application can have. However, since the observables of interest in this paper $\hat{n}_1$ and $\hat{h}_{13}$ satisfy ETH (in its loose sense) for single realization of the SYK chain model, based on the results of the preceding subsection, it is worth studying their relations to ETH for the ensemble averaged theory for possible implications.

We check satisfaction of Eq. (15) for the diagonal matrix elements of $\hat{n}_1$ in the ensemble averaged theory in this subsection. The diagonal matrix elements of the hopping operator $\hat{h}_{13}$ strictly satisfy Eq. (15) in single realizations of the disorder as we demonstrated in the preceding subsection. Therefore, they strictly satisfy Eq. (15) in the ensemble averaged theory as well.

We check satisfaction of Eq. (12) for the off-diagonal matrix elements of $\hat{n}_1$ and $\hat{h}_{13}$ in the ensemble averaged theory in the next subsection. Checking satisfaction of Eqs. (15) and (12) is what we mean here by satisfaction of ETH for the ensemble averaged theory.

To check ETH for an ensemble averaged theory, we follow the procedure employed in
FIG. 5. The absolute difference between the on-diagonal elements $O_{nn}$ of the particle number operator, $O = \hat{n}_1$, and its thermal value $O_{th} = \text{Tr}[\rho_{mc}O]/Z = 1/2$ as a function of energy eigenvalues $E_n/J$ for three different coupling strengths $J_0, J_1$. The ensemble average is taken over 1000 realizations of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions on each site. We observe visible dependence of the curves to the ratio $J_1/J_0$. Nevertheless, the smoothness of the curves and smallness of the difference between the diagonal elements and the thermal value show complete agreement with ETH for all three pairs of coupling strengths.

Refs. [18, 19]. The procedure is as follows: First, we partition the energy range into small intervals. Second, we compute diagonal matrix elements of some operator $O$ in energy eigenbasis, $O_{nn} = \langle n|O|n\rangle$, for several realizations of the disorder. Third, we take average over all $O_{nn}$'s whose corresponding energy eigenvalues $E_n$'s fall into the same energy interval. Fourth, we report the resultant quantity as the disorder averaged value of the diagonal matrix element of $O$ in that energy interval. Last, we compare the resultant quantity with the expectation of ETH. For all three cases of couplings (a-c) defined in Sec. III A, we constructed 1000 realizations of the SYK chain model and calculated absolute difference between the diagonal matrix elements of the number operator $O = \hat{n}_1$ in energy eigenbasis and the microcanonical average value of the operator $1/2$, namely $|O_{nn} - 1/2|$. Using the above procedure, we took average over all 1000 realizations for each case of couplings separately.
The result is shown in Fig. 5. As one can observe, ETH expectation is satisfied for the diagonal elements of the number operator. The deviations from the microcanonical average value is small, and, as a result, Eq. (15) is satisfied. Hence, ETH is satisfied for the diagonal matrix elements of the number operator \( \hat{n}_1 \) for all three cases of the couplings.

There are some interesting features in Fig. 5. For all three cases of the couplings, the deviation from the microcanonical average value is smaller at smaller absolute energies and becomes larger at larger absolute energies. This feature is also seen in the original SYK, supersymmetric SYK, and complex SYK models [18, 19]. The growth of the deviation from microcanonical value is here faster for the case of stronger intra-sites couplings \((J_0 = 2, J_1 = 1)\) and become slower as we go to the case of stronger inter-sites couplings \((J_0 = 1, J_1 = 2)\). This difference reveals different natures of the intra-site and inter-sites coupling terms in the Hamiltonian of the SYK chain model, Eq. (23). However, since the deviations are within the the domain \(O(1/N)\) allowed by ETH, they won’t affect satisfaction of ETH.

D. Ensemble average over the disorder: Off-diagonal elements

ETH expects the off-diagonal matrix elements of a generic observable \( \mathcal{O} \) in energy eigenbasis to be exponentially small,

\[
g_{\mathcal{O}}(\bar{E}, \omega) := \langle m | \mathcal{O} | n \rangle \sim e^{-S/2}. \tag{47}
\]

Here we denote the off-diagonal matrix elements of the operator \( \mathcal{O} \) by \( g_{\mathcal{O}} \) and classify that as a function of eigenenergy average \( \bar{E} = (E_m + E_n)/2 \) and eigenenergy difference \( \omega = E_m - E_n \). This allows us to study the chaotic and random matrix theory (RMT) like features of the off-diagonal matrix elements. In RMT, \( g_{\mathcal{O}}(\bar{E}, \omega) \) at fixed \( \bar{E} \) is a constant function of \( \omega \). Therefore, to compare the behavior of the SYK chain model with that of a random matrix ensemble, we fix \( \bar{E} \) and draw \( g_{\mathcal{O}}(\bar{E}, \omega) \) as a function of \( \omega \).

The part of the function \( g_{\mathcal{O}}(\bar{E}, \omega) \) that is almost constant with respect to \( \omega \) will show RMT behavior and the part that is a non-trivial function of \( \omega \) will exhibit non-RMT manner. The cross-over between RMT and non-RMT behaviors occurs at the characteristic energy \( E_T \). In local models with spatial structure this characteristic energy is often associated with
the diffusive Thouless energy. The SYK chain model has spatial structure (the model is one dimensional) and diffusion of collective modes are established for this model [5], so we can recognize $E_T$ as the Thouless energy of the model. Similar recognition is even made for the complex, original, and supersymmetric SYK models which are zero dimensional [18 19].

Here, the SYK chain model is a one dimensional model and its diffusion constant has been theoretically obtained, Eq. (27). Therefore, we take one further step and check whether the Thouless energy which is proportional to the diffusion constant, Eq. (28),

$$E_T \propto D$$

(48)

follows the dependence of the diffusion constant of the SYK chain model to the coupling strengths, Eqs. (27),

$$D = \frac{2\pi J^2}{3\sqrt{2}J_{\alpha K}}$$

(49)

or not.

Now, to construct $g_O(\bar{E},\omega)$ for the ensemble averaged theory of the SYK chain model, we follow a procedure similar to the one explained in Sec. III C. The procedure is as follows: First, we partition the domains of $\omega$ and $\bar{E}$ into small intervals with length $\delta\omega$ and $\delta\bar{E}$. Second, we compute off-diagonal matrix elements of the operator $O$ in energy eigenbasis, $g_O(\bar{E},\omega) = \langle m|O|n \rangle$, for several realizations of the disorder. Third, we take average over all $g_O(\bar{E},\omega)$’s whose corresponding $\omega$ and $\bar{E}$ fall into the same intervals ($\omega, \omega + \delta\omega$) and ($\bar{E}, \bar{E} + \delta\bar{E}$) respectively. Fourth, we report the resultant quantity $\bar{g}_O(\bar{E},\omega)$ as the disorder averaged value of the off-diagonal matrix elements of $O$ in those intervals of $\omega$ and $\bar{E}$. Last, we compare the resultant quantity with the expectation of ETH.

For all three cases of couplings ($J_0 = 2, J_1 = 1; J_0 = J_1 = \sqrt{5}/2$; and $J_0 = 1, J_1 = 2$), we constructed 1000 realizations of the SYK chain model and calculated absolute values of the off-diagonal matrix elements of the number operator $\hat{n}_1$ and the hopping operator $\hat{h}_{13}$ in energy eigenbasis. Using the above procedure, we took average over all 1000 realizations for each case of couplings separately. The results are shown in Figs. 6 and 7.

As observed in Figs. 6 and 7, ETH expectations are satisfied for the off-diagonal matrix elements of the number operator $\hat{n}_1$ and the hopping operator between sites one and three $\hat{h}_{13}$. The ensemble averaged values of the off-diagonal elements are of order $e^{-N} \approx 0.002$ for both of the operators. Thus, ETH is strictly satisfied for all three cases of the couplings.
FIG. 6. The absolute value of the off-diagonal elements $O_{mn}$ of the particle number operator, $O = \hat{n}_1$, as function of difference between the corresponding energy eigenvalues $\omega/J = (E_m - E_n)/J$ at a fixed average eigenenergy $(E_n + E_m)/2$ for three different coupling strengths $J_0$, $J_1$. The ensemble average is taken over 1000 realizations of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions on each site. We consider fixed average energy $(E_n + E_m)/2NM = .002 \pm .001$ and take a moving average in $\omega$ over 800 nearest neighbors of the matrix elements. The exponentially small values of the off-diagonal elements agree with ETH. Also, the curves show dependences on ratio $J_1/J$ for larger $\omega/J$ which indicates that the Thouless energy of the model and the range of its RMT like behavior is increases as $J_1/J$ is increased, in accordance with theoretical results for the diffusion constant of the model.

We also observer that the values of the Thouless energy $E_T$, determined by the cross-over between the constant dependence on $\omega$ (RMT behavior) and non-constant dependence (non-RMT behavior), increases as we increase $J_1$ from value 1 in the red curves to $\sqrt{5}/2$ in the green curves, to $J_1 = 2$ in the blue curves while we keep $J = \sqrt{5}$ fixed. Therefore, we find that the results for the diffusive Thouless energy $E_T$ of the SYK chain model are in agreement with the analytic results of Ref. [5] for the diffusion constant of the model, Eq. (49), and the definition of the Thouless energy Eqs. (28) or (48).

These results indicate that for stronger inter-sites couplings $J_1$ the range of $\omega$ for which the system behaves chaotically like RMT increases.
The absolute value of the off-diagonal elements $O_{mn}$ of the hopping operator between sites 1 and 3, $O = \hat{h}_{13}$, as function of difference between the corresponding eigenenergies $\omega/J = (E_m - E_n)/J$ at a fixed average eigenenergy $(E_n + E_m)/2$ for three different coupling strengths $J_0$, $J_1$. The ensemble average is taken over 1000 realizations of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions on each site. We consider fixed average energy $(E_n + E_m)/2NM = .002 \pm .001$ and take a moving average in $\omega$ over 800 nearest neighbors of the matrix elements. The exponentially small values of the off-diagonal elements agree with ETH. Also, the curves show dependences on ratio $J_1/J$ for larger $\omega/J$ which indicates that the Thouless energy of the model and the range of its RMT like behavior is increases as $J_1/J$ is increased, in accordance with theoretical results for the diffusion constant of the model.

**IV. DISCUSSION AND CONCLUSION**

In this paper we investigated satisfaction of eignestate thermalization hypothesis in the strongly correlated, non-local, (1+1) dimensional model of SYK chain. To verify ETH, we focused mainly on two conventional few-body operators, the local operator of single particle number that acts on one site of the chain and the non-local operator of hopping that acts on two non-neighboring sites of the chain. We investigated satisfaction of ETH for those operators both in single realization of the SYK chain model and in ensemble average of many realizations of the model. We conventionally took the ensemble averages over the spectrum.
of energy.

Through exact diagonalization, we found that the two operators loosely satisfy ETH in single realization of the SYK chain model and strictly satisfy ETH in the ensemble averaged theory of the model. The difference lay in the relative largeness of the off-diagonal matrix elements of the operators in single realizations from ETH expectations. Nevertheless, we found that those elements are yet small enough to allow thermalization, albeit with larger fluctuations around the thermal values. We also found that the Thouless energy, and consequently the diffusion constant, in the SYK chain model increases by increase of the inter-sites coupling strength, in agreement with analytical predictions [5].

If we can generalize our result of satisfaction of ETH from the single number operator and the hopping operator in the SYK chain model to all generic few-body non-extensive operators, then we can make some comments on the applications and implications of the findings of this paper. As a few and far between solvable example of strongly correlated chaotic lattice model, the SYK chain model have provided an excellent platform for studying entanglement propagation [4], thermalization, and other physical phenomena. Our results show that this strongly correlated solvable model can rapidly thermalize through the mechanism of ETH. Moreover, the spatial locality of the model and the absence of all-to-all interaction as opposed to those in complex SYK [19], Majorana SYK [18], and supersymmetric SYK [18], which are all zero dimensional models with all-to-all interactions, have no essential role in the thermalization of the model as long as we stay in solvable strong coupling regime of the SYK chain model, i.e. $N \gg \beta J \gg 1$.

From the perspective of holographic duality, the SYK chain model might be considered to be dual to some kind of incoherent black hole [5, 21]. The satisfaction of ETH in this context implies thermalization of black hole in the putative gravitational dual of the model. More precisely, satisfaction of ETH implies that the correlation functions in energy eigenstates look thermal. This further means that correlations in energy eigenstates can be well approximated by dual computations in the black hole geometry.

In comparison with the Rényi entanglement entropy study [4], our result shows that thermalization mechanism of ETH and consequently correlation functions are mainly controlled by the light degrees of freedom of the model, associated with time reparametrization field that gives rise to rapid thermalization. The heavy modes that [4] found by studying the Rényi entanglement entropy, which can induce slow thermalization, are not crucial for ETH
mechanism of thermalization and for correlation functions.

The SYK chain model is a (1+1) dimensional generalization of the SYK model. We established ETH for such a generalization in this paper. Gu et. al. [5] introduce further higher dimensional generalizations of the SYK model. It would be interesting to investigate ETH for those models and check whether ETH still holds as dimensions go further up, symmetries vary, and type of interactions changes.

Previous works on relation between ETH and SYK or SYK-like models [18, 19], and this work, investigated ETH for two conventional operators of single number and the hopping. They all also assumed that the satisfaction of ETH conditions for the ensemble averaged theory implies thermalization of the theory. It would be also interesting to explore whether further generic operators on those models satisfy ETH and whether thermalization of an ensemble averaged theory can be derived from satisfaction of ETH for those theories as in the case of individual theories. We leave these issues for future research.

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