Hybrid nanofluid flow and heat transfer past a permeable stretching/shrinking surface with a convective boundary condition

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Abstract. The steady flow and heat transfer past a permeable stretching/shrinking surface in a hybrid nanofluid with a convective boundary condition is studied. The governing equations of the problem are transformed to the similarity equations by using similarity transformation technique. The problem is solved numerically using the boundary value problem solver (bvp4c) in Matlab software. The plots of the skin friction coefficient and the local Nusselt number as well as the velocity and temperature profiles for selected parameters are presented. Results show that dual solutions exist for a certain range of the stretching/shrinking and suction parameters. The critical values of these parameters decrease with the increasing of the copper (Cu) nanoparticle volume fractions. It is found that the heat transfer rate for hybrid nanofluid is higher than that for nanofluid for the impermeable stretching surface. It is also found that the increasing of the copper (Cu) nanoparticle volume fractions enhances the skin friction coefficient and reduces the local Nusselt number for the shrinking surface. The rise in Biot number leads to the increment of the temperature at the surface and widen the thermal boundary layer for both branches. A temporal stability analysis is performed to determine the stability of the dual solutions in a long run, and it is revealed that only one of them is stable while the other is unstable.

1. Introduction
The study of fluid flow past a stretching sheet is very significant in the engineering and industrial processes such as wire drawing, hot rolling, extrusion, and metal spinning. Crane [1] was the first to study the steady flow over a stretching sheet and obtained the similarity solution. However, the study of the flow past a shrinking sheet was not gained much attention compared to that of the stretching case. Goldstein [2] stated that the flow of the shrinking sheet is basically a backward flow. In this regard, Wang [3] examined the liquid film behaviour over an unsteady stretching sheet and observed the growth of the uncommon flow owing to shrinking. The investigation of an exact solution of the Navier-Stokes equations over a shrinking sheet was conducted by Miklavčič and Wang [4]. They discovered that the vorticity within a boundary layer is not bounded, and a steady flow exists by applying an adequate suction at the boundary. As stated in Fang et al. [5], a shrinking sheet flow demonstrates different physical phenomena compared to the stretching sheet flow.
The temperature conditions such as constant surface temperature and heat flux are usually used in heat transfer analysis. In some situations, the transfer of heat to the surface depends on the temperature of the surface as what happens mostly in heat exchangers. Thus, the convective boundary condition should be considered. For instance, Aziz [6] studied the flow of Blasius over a flat plate subject to convective boundary condition and found that the increase in convective parameter tend to increase the surface temperature. Then, some authors considered the convective boundary condition to revisit the previous studies such as Ishak [7], Makinde and Aziz [8], Ishak et al. [9], Rahman et al. [10], and Mansur and Ishak [11].

The problem on heat transfer enhancement has earned much attention for the past few years. In this regard, thermal scientists have recommended adding nanosized metallic or non-metallic particles in the base fluid so that higher thermal conductivity can be achieved, as the thermal conductivity of nanoparticles is greater than that of the base fluid. The resulting mixture giving better physical and chemical properties referred as a nanofluid. The term nanofluid was first introduced by Choi and Eastman [12] in the year 1995. However, a new type of nanofluid is investigated to further improve the heat transfer rate which is called hybrid nanofluid. Hybrid nanofluid recognized as an advanced nanofluid which consists of two distinct nanoparticles, whereas regular nanofluid contains single nanoparticle dissolves in the base fluid. Hybrid nanofluid gives better thermophysical properties, consequently improve the heat transfer performance.

The boundary layer flow and heat transfer of hybrid nanofluid were carried out by several researchers for example, the flow of a hybrid nanofluid along a stretching surface by considering Cu-Al2O3 nanoparticles with and without magnetic effects was examined by Devi and Devi [13, 14]. They observed that hybrid nanofluid improves the heat transfer rate rather than regular nanofluid. Lately, many researchers studied the flow and heat transfer characteristics of hybrid nanofluid by considering various aspects in the literature such as Hayat and Nadeem [15], Yousefi et al. [16], Subhani and Nadeem [17], Ghadikolaei et al. [18], Usman et al. [19], Rostami et al. [20], and Waini et al. [21].

Upon reviewing the existing studies, we are motivated to investigate the behaviour of the hybrid nanofluid flow and heat transfer past a permeable stretching/shrinking surface subject to convective boundary condition by employing Tiwari and Das [22] nanofluid model. Here, copper (Cu) and alumina (Al2O3) nanoparticles are considered. Then, these nanoparticles are suspended in water to form hybrid nanofluid. The present numerical results are compared with the published data for results validation.

2. Mathematical formulation
A steady flow and heat transfer past a permeable stretching/shrinking surface of a hybrid nanofluid is considered. In Figure 1, the x- axis is measured along the plate and the y- axis is perpendicular to it where the plate is located at y = 0.

![Figure 1](image-url)
It is assumed that the surface is stretched/shrunk with the velocity \( u_w(x) = ax \) where \( a \) is a constant and the wall mass flux velocity is \( v_0 \). It is assumed that the bottom surface of the plate is heated by convection from a hot fluid of uniform temperature \( T_f \) which supplies a heat transfer coefficient \( h_f \). The surface temperature \( T_w \) is the result of a convective heating process which is characterized by the hot fluid and \( T_\infty \) is the ambient fluid temperature, thus, we have \( T_f > T_w > T_\infty \). Here, hybrid nanofluid is considered where the nanoparticles size is assumed to be uniform and the nanoparticles agglomeration effects on thermophysical properties is ignored due to the nanofluid is synthesized from the nanoparticles and the base fluid as a stable mixture.

Under these assumptions, the governing equations for continuity, momentum and energy of a hybrid nanofluid are written as follows (see Devi and Devi [14]; Tiwari and Das [22]):

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \quad (1) \\
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\mu_{hnf}}{\rho_{hnf}} \frac{\partial^2 u}{\partial y^2} \quad (2) \\
\frac{u}{\partial x} + v \frac{\partial T}{\partial y} &= \frac{k_{hnf}}{(\rho C_p)_{hnf}} \frac{\partial^2 T}{\partial y^2} \quad (3)
\end{align*}
\]

subject to:

\[
\begin{align*}
v &= v_0, \quad u = \lambda u_w(x), \quad -k_{hnf} \frac{\partial T}{\partial y} &= h_f(T_f - T) \quad \text{at} \quad y = 0 \\
&\quad u \rightarrow 0, \quad T \rightarrow T_\infty \quad \text{as} \quad y \rightarrow \infty \quad (4)
\end{align*}
\]

where \( u \) and \( v \) are the velocity components of the hybrid nanofluid along the \( x \)- and \( y \)-axes and \( T \) is the hybrid nanofluid temperature. Further, \( \rho_{hnf}, \mu_{hnf}, (\rho C_p)_{hnf}, \) and \( k_{hnf} \) represents the density, dynamic viscosity, heat capacity, and thermal conductivity of the hybrid nanofluid. Following Devi and Devi [14], and Oztop and Abu-Nada [23], the equations as given in Table 1 are employed in order to evaluate the thermophysical properties for nanofluid and hybrid nanofluid. Here, the subscripts \( f, n_f, hnf, n_1, \) and \( n_2 \) represents the fluid, nanofluid, hybrid nanofluid, Al_2O_3, and Cu nanoparticles solid components, respectively. Meanwhile, Table 2 provides the physical properties of the nanoparticles and the base fluid as in Oztop and Abu-Nada [23].

Following Devi and Devi [14], we are looking for a similarity solution of equations (1)-(4) by using the following similarity variables:

\[
\psi = x \sqrt{a \nu_f} f(\eta), \quad \theta(\eta) = (T - T_\infty)/(T_f - T_\infty), \quad \eta = y \sqrt{a / \nu_f} \quad (5)
\]

where the stream function denoted by \( \psi \) with \( u = \partial \psi / \partial y \) and \( v = -\partial \psi / \partial x \) so that the continuity equation (1) is satisfied identically. Therefore, we have:

\[
\begin{align*}
u = ax f'(\eta), \quad \nu &= -\sqrt{a \nu_f} f(\eta) \quad (6)
\end{align*}
\]

so that:

\[
\nu_0 = -\sqrt{a \nu_f} S \quad (7)
\]

where \( S = f'(0) \) is the mass flux parameter with \( S > 0 \) represents fluid suction while \( S < 0 \) represents fluid injection or removal, and \( \nu_f \) represents the base fluid kinematic viscosity.
Table 1. Nanofluid and hybrid nanofluid thermophysical properties

| Properties            | Nanofluid                              | Hybrid nanofluid                        |
|-----------------------|----------------------------------------|-----------------------------------------|
| Heat capacity         | \((\rho C_p)_nf = (1 - \varphi_1) (\rho C_p)_f + \varphi_1 (\rho C_p)_n1\) | \((\rho C_p)_{hnf} = (1 - \varphi_2) \left[ (1 - \varphi_1) (\rho C_p)_f + \varphi_1 (\rho C_p)_n1 \right] + \varphi_2 (\rho C_p)_{n2}\) |
| Density               | \(\rho_{nf} = (1 - \varphi_1) \rho_f + \varphi_1 \rho_{n1}\) | \(\rho_{hnf} = (1 - \varphi_2) [(1 - \varphi_1) \rho_f + \varphi_1 \rho_{n1}] + \varphi_2 \rho_{n2}\) |
| Dynamic viscosity     | \(\mu_{nf} = \frac{\mu_f}{(1 - \varphi_1)^{2.5}}\) | \(\mu_{hnf} = \frac{\mu_f}{(1 - \varphi_1)^{2.5} (1 - \varphi_2)^{2.5}}\) |
| Thermal conductivity  | \(k_{nf} = \frac{k_{n1} + 2k_f - 2\varphi_1(k_f - k_{n1})}{k_{n1} + 2k_f + \varphi_1(k_f - k_{n1})} \times (k_f)\) | \(k_{hnf} = \frac{k_{n2} + 2k_{nf} - 2\varphi_2(k_{nf} - k_{n2})}{k_{n2} + 2k_{nf} + \varphi_2(k_{nf} - k_{n2})} \times (k_{nf})\) where \(k_{nf} = \frac{k_{n1} + 2k_f - 2\varphi_1(k_f - k_{n1})}{k_{n1} + 2k_f + \varphi_1(k_f - k_{n1})} \times (k_f)\) |

Substituting (5) into equations (2) and (3), we obtain the following ordinary (similarity) differential equations:

\[
\frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f} f'''' + f f'' - f'^2 = 0
\]

\[
\frac{1}{\text{Pr} (\rho C_p)_{hnf}/(\rho C_p)_f} \theta'' + f \theta' = 0
\]

subject to:

\[
f(0) = S, \quad f'(0) = \lambda, \quad \theta'(0) = -\frac{k_f}{k_{hnf}} \text{Bi} \left(1 - \theta(0)\right)
\]

\[
f''(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty
\]

where prime denotes differentiation with respect to \(\eta\), \(\text{Pr} = \nu_f/\alpha_f\) represents the Prandtl number, \(\text{Bi} = \left(\nu_f/k_f\right)\sqrt{\nu_f/\alpha_f}\) represents the Biot number and \(\lambda\) represents the parameter of stretching/shrinking sheet with \(\lambda > 0\) corresponds to a stretching sheet, \(\lambda < 0\) to a shrinking sheet and \(\lambda = 0\) to a static sheet. Note that, \(\text{Bi} \to \infty\) refers to the constant wall temperature condition, \(\theta(0) = 1\). It is worth mentioning that by considering regular fluid (\(\varphi_1 = \varphi_2 = 0\)) and \(\text{Bi} \to \infty\), equations (8) and (9) reduce to equations (8) and (9) of the paper by Wang et al. [24].

The parameters of physical interest are the skin friction coefficient \(C_f\) and the local Nusselt number \(Nu_x\), which are defined as:

Table 2. Nanoparticles and fluid thermophysical properties

| Properties | Al_{2}O_{3} | Cu | Water |
|------------|-------------|----|-------|
| \(k (W/mK)\) | 40          | 400 | 0.613 |
| \(C_p(J/kgK)\) | 765         | 385 | 4179  |
| \(\rho (kg/m^3)\) | 3970        | 8933 | 997.1 |

Substituting (5) into equations (2) and (3), we obtain the following ordinary (similarity) differential equations:

\[
\frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f} f'''' + f f'' - f'^2 = 0
\]

\[
\frac{1}{\text{Pr} (\rho C_p)_{hnf}/(\rho C_p)_f} \theta'' + f \theta' = 0
\]

subject to:

\[
f(0) = S, \quad f'(0) = \lambda, \quad \theta'(0) = -\frac{k_f}{k_{hnf}} \text{Bi} \left(1 - \theta(0)\right)
\]

\[
f''(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty
\]

where prime denotes differentiation with respect to \(\eta\), \(\text{Pr} = \nu_f/\alpha_f\) represents the Prandtl number, \(\text{Bi} = \left(\nu_f/k_f\right)\sqrt{\nu_f/\alpha_f}\) represents the Biot number and \(\lambda\) represents the parameter of stretching/shrinking sheet with \(\lambda > 0\) corresponds to a stretching sheet, \(\lambda < 0\) to a shrinking sheet and \(\lambda = 0\) to a static sheet. Note that, \(\text{Bi} \to \infty\) refers to the constant wall temperature condition, \(\theta(0) = 1\). It is worth mentioning that by considering regular fluid (\(\varphi_1 = \varphi_2 = 0\)) and \(\text{Bi} \to \infty\), equations (8) and (9) reduce to equations (8) and (9) of the paper by Wang et al. [24].

The parameters of physical interest are the skin friction coefficient \(C_f\) and the local Nusselt number \(Nu_x\), which are defined as:
\[ C_f = \frac{\tau_w}{\rho_f u_{w0}^2}, \quad Nu_x = \frac{x q_w}{k_f (T_f - T_\infty)} \]  

(11)

where \( \tau_w \) is the surface shear stress and \( q_w \) is the heat flux from the stretching/shrinking surface, which are defined as:

\[ \tau_w = \mu_{hnf} \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k_{hnf} \left( \frac{\partial T}{\partial y} \right)_{y=0} \]  

(12)

Using (5), (11) and (12), one gets:

\[ Re_x^{1/2} C_f = \frac{\mu_{hnf}}{\mu_f} f''(0), \quad Re_x^{-1/2} Nu_x = -\frac{k_{hnf}}{k_f} \theta'(0) \]  

(13)

where the local Reynolds number is 

\[ Re_x = u_w(x) x / \nu_f \]

### 3. Stability analysis

The temporal stability analysis is conducted by considering the unsteady form of equations (2) and (3) where equation (1) remains unchanged. We use the variable of the dimensionless time, \( \tau = at \) to study the temporal stability of the flow as suggested by Merkin [25], and Weidman et al. [26]. Using variables in (5), we have:

\[ \tau = at, \quad \psi = x \sqrt{a/v_f} f(\eta, \tau), \quad \theta(\eta, \tau) = (T - T_\infty) / (T_f - T_\infty), \quad \eta = y \sqrt{a/v_f} \]  

(14)

Using (14), equations (8) and (9) can be written as:

\[ \frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f \eta^2} + f \frac{\partial f}{\partial \eta} \left( \frac{\partial f}{\partial \eta} \right)^2 - \frac{\partial^2 f}{\partial \eta \partial \tau} = 0 \]  

(15)

\[ \frac{1}{Pr \sqrt{\rho c_p hnf/\rho c_p f}} \frac{\partial^2 \theta}{\partial \eta^2} + f \frac{\partial \theta}{\partial \eta} - \frac{\partial \theta}{\partial \tau} = 0 \]  

(16)

subject to:

\[ f(0, \tau) = S, \quad \frac{\partial f}{\partial \eta} (0, \tau) = \lambda, \quad \frac{\partial \theta}{\partial \eta} (0, \tau) = -\frac{k_f}{k_{hnf}} Bi(1 - \theta(0, \tau)) \]

\[ \frac{\partial f}{\partial \eta} (\eta, \tau) \rightarrow 0, \quad \theta(\eta, \tau) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty \]  

(17)

Following Weidman et al. [26], to determine the stability of the solution \( f = f_0(\eta) \) and \( \theta = \theta_0(\eta) \) satisfying the boundary value problem (8)-(10), we write:

\[ f(\eta, \tau) = f_0(\eta) + e^{-\gamma \tau} F(\eta), \quad \theta(\eta, \tau) = \theta_0(\eta) + e^{-\gamma \tau} G(\eta) \]  

(18)

where the unknown eigenvalue denotes by \( \gamma \), while \( F(\eta) \) and \( G(\eta) \) are relatively small toward \( f_0(\eta) \) and \( \theta_0(\eta) \). Equation (18) is substitute into equations (15) to (17) to obtain the linear eigenvalue problems as follows:

\[ \frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f} F'''' + f_0 F''' + f_0'' F' - 2 f_0' F'' + \gamma F' = 0 \]  

(19)

\[ \frac{1}{Pr \sqrt{\rho c_p hnf/\rho c_p f}} G'''' + f_0 G''' + \theta'_0 F + \gamma G = 0 \]  

(20)
subject to:

\[ F(0) = 0, \quad F'(0) = 0, \quad G'(0) = \frac{k_f}{k_{hf}} \mathrm{Bi} \ G(0) \]
\[ F'(\eta) \to 0, \quad G(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \quad (21) \]

Here, the smallest eigenvalue \( \gamma \) determines the stability of the corresponding steady flow solution \( f_0(\eta) \) and \( \theta_0(\eta) \). Following Harris et al. [27], without loss of generality, we study the stability for the case of \( F''(0) = 1 \). By setting \( F''(0) = 1 \), we find the smallest eigenvalue \( \gamma \) in equations (19) and (20).

4. Results and discussion

The boundary value problem solver which is available in Matlab software called ‘bvp4c’ is employed to solve the Eqs. (8)-(10) numerically. The bvp4c solver employs the finite difference scheme and the 3-stage Labatto IIIa formula, where the initial guess and changes step size is supplied to obtain the required accuracy of the solution. Also, the appropriate thickness of the boundary layer, \( \eta_{\infty} \) must be selected relying on the parameters used to obtain the accurate solutions. The detail of this particular solver is clearly discussed in Shampine et al. [28]. To conduct this study, 0.1 solid volume fraction of \( \text{Al}_2\text{O}_3 \) (i.e. \( \phi_1 = 0.1 \)) is added to the base fluid as suggested by Devi and Devi [14]. Consequently, several solid volume fractions of \( \text{Cu} \) are added into the mixture in order to form \( \text{Cu-} \text{Al}_2\text{O}_3/\text{water hybrid nanofluid. As in Oztop and Abu-Nada [23], we are consider the Prandtl number, Pr = 6.2 which represents water as the base fluid, and will be fixed to obtained the numerical results.}

Table 3. Comparison of \(-\theta'(0)\) for regular fluid (\( \phi_1 = \phi_2 = 0 \)) with published data for different values of \( \text{Pr} \) when \( \lambda = 1, S = 0 \), and \( \text{Bi} \to \infty \). The present results demonstrate an excellent agreement to those obtained by Devi and Devi [14], Wang [24], and Gorla and Sidawi [29].

| Pr  | Gorla and Sidawi [29] | Wang [24] | Devi and Devi [14] | Present results |
|-----|----------------------|-----------|--------------------|----------------|
| 2   | 0.9114               | 0.9114    | 0.91135            | 0.911357       |
| 6.13|                      | 1.75968  |                    | 1.759682       |
| 7   | 1.8954               | 1.8954    | 1.89540            | 1.895400       |
| 20  | 3.3539               | 3.3539    | 3.35390            | 3.353893       |

The variation of \( \text{Re}_x^{1/2} C_f \) and \( \text{Re}_x^{-1/2} N_{ux} \) for selected parameters are presented in Figure 2 to 5. From these figures, the enhancement of \( \text{Re}_x^{1/2} C_f \) and reduction of \( \text{Re}_x^{-1/2} N_{ux} \) at the shrinking surface are observed with the increasing of \( \phi_2 \). For more specific, Figures 2 and 3 show the variation of \( \text{Re}_x^{1/2} C_f \) and \( \text{Re}_x^{-1/2} N_{ux} \) against \( \lambda \) for numerous values of \( \phi_2 \) when \( \text{Bi} = 1, S = 2.2, \phi_1 = 0.1, \) and \( \text{Pr} = 6.2 \). Then, Figures 4 and 5 describe the variation of \( \text{Re}_x^{1/2} C_f \) and \( \text{Re}_x^{-1/2} N_{ux} \) against \( S \) for various values of
φ₂ when Bi = 1, λ = −1, φ₁ = 0.1, and Pr = 6.2. It is seen that the solutions are not unique for a certain value of λ and S. No similarity solutions are obtained when λ < λc and S < Sc, because of the boundary layer separation occur from the surface. The increase of φ₂ tend to expand the solution domain with the critical values λc and Sc are slightly decreased. Note that, λc = −1.24658, −1.37508 and −1.47488 are the critical values for φ₂ = 0.01, 0.05 and 0.1, respectively. Meanwhile, it is found that for φ₂ = 0.01, the dual solutions exist when Sc ≥ 1.97037. For φ₂ = 0.05, Sc ≥ 1.87608 and for φ₂ = 0.1, Sc ≥ 1.81145.

Table 4. Values of \(Re_x^{1/2}C_f\) and \(Re_x^{-1/2}Nu_x\) for Cu/water (φ₁ = 0) and Cu–Al₂O₃/water (φ₁ = 0.1) with various values of φ₂ when Pr = 6.135, S = 0, λ = 1, and Bi → ∞

| φ₂   | Cu/water | Cu–Al₂O₃/water | Cu/water | Cu–Al₂O₃/water |
|------|----------|----------------|----------|----------------|
|      | Devi and Present results | Devi and Present results | Devi and Present results | Devi and Present results |
| 0.005| -1.026272| -1.026115      | -1.327098| 1.768118        |
| 0.02 | -1.104327| -1.104196      | -1.409490| 1.791406        |
| 0.04 | -1.208432| -1.208325      | -1.520894| 1.822984        |
| 0.06 | -1.313402| -1.313311      | -1.634279| 1.855179        |
| 0.1  | -1.528793| -1.869764      | 1.921657 | 2.141644        |

The profiles of \(f'(\eta)\) and \(θ(\eta)\) for numerous values of φ₂ are presented in Figures 6 and 7. Note that the rise in φ₂ led to the increment of the velocity for the upper branch but it is reduces for the lower branch, whereas the temperature increases for both branches as shown in Figures 6 and 7. Moreover, Figure 8 depicts the hybrid nanofluid temperature θ(η) for numerous values of Biot number Bi when λ = −1, φ₁ = 0.1, φ₂ = 0.1, S = 2.2, and Pr = 6.2. We observe that the increasing values of Bi expands the thermal boundary layer thickness for both branches. According to Aziz [6], Bi is directly proportional to \(h_f\) for fixed cold fluid properties and fixed freestream velocity. However, the thermal resistance is inversely proportional to \(h_f\) on the hot fluid side. Therefore, as Bi increases, the thermal resistance will decrease which lead to the increment of the surface temperature θ(η).
Figure 4. Effects of $\varphi_2$ on $Re_x^{1/2} C_f$ against $S$

Figure 5. Effects of $\varphi_2$ on $Re_x^{1/2} Nu_x$ against $S$

Figure 6. Effects of $\varphi_2$ on $f' (\eta)$

Figure 7. Effects of $\varphi_2$ on $\theta (\eta)$

Figure 8. Effects of $\text{Bi}$ on $\theta (\eta)$

Figure 9. Smallest eigenvalues $\gamma$ against $\lambda$
The smallest eigenvalues $\gamma$ against $\lambda$ when $\phi_1 = 0.1$, $\phi_2 = 0.1$, and $S = 2.2$ are plotted in Figure 9. Referring to Eq. (21), an initial decay of disturbance occurs for positive value of $\gamma$ as $\tau \to \infty$, means that the flow is in a stable mode. In the meantime, an initial growth of disturbance occurs for negative value of $\gamma$, means that the flow is in unstable mode. As $\lambda$ are approaching $\lambda_c$, it is observed that $\gamma$ tends to zero for both upper (stable) and lower (unstable) branches. This behaviour implies that the solutions are bifurcate at the critical values.

5. Conclusion
In this paper, the hybrid nanofluid flow and heat transfer past a permeable stretching/shrinking surface subject to convective boundary condition was studied. Results showed that the heat transfer rate for the hybrid nanofluid is higher than that for the regular nanofluid for the impermeable stretching surface. It was found that the increasing of $\phi_2$ led to enhance the values of $Re_x^{1/2}C_f$ and but reduce the values of $Re_x^{3/2}Nt_x$ for the shrinking surface. As $\phi_2$ increase, the range of $\lambda$ and $S$ for which the dual solutions exist are decreased. Also, the increment of $f'(\eta)$ and $\theta(\eta)$ inside the boundary layer for the upper branch, while $f'(\eta)$ decreases but $\theta(\eta)$ increases for the lower branch is observed as $\phi_2$ increase. The rise in Biot number led to the increment of $\theta(\eta)$ at the surface and widen the thermal boundary layer for both branches. It is discovered that the second solution is not practicable, while the first solution is stable and physically reliable in a long run from the temporal stability analysis.

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