A superconductor-insulator transition in a one-dimensional array of Josephson junctions

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We consider a one-dimensional Josephson junction array, in the regime where the junction charging energy is much greater than the charging energy of the superconducting islands. In this regime we critically reexamine the continuum limit description and establish the relationship between parameters of the array and the ones of the resulting sine-Gordon model. The later model is formulated in terms of quasi-charge. We argue that despite arguments to the contrary in the literature, such quasi-charge sine-Gordon description remains valid in the vicinity of the phase transition between the insulating and the superconducting phases. We also discuss the effects of random background charges, which are always present in experimental realizations of such arrays.

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I. INTRODUCTION

The standard description of Josephson junction arrays is given by the theory by Bradley and Doniach (BD) \cite{BD}. This theory predicts that a Josephson junction array can either be in superconducting or insulating regime, depending on the ratio of the Josephson to the charging energy of the superconducting grains. The two phases are separated by the Kosterlitz-Thouless transition.

The theory of Bradley and Doniach assumes that the gate voltage applied to the arrays is equal to zero. This implies that the electrostatic energy of the islands is minimized when their charge is equal to the integer number of the elementary charge of Cooper pair. More recent work of Glazman and Larkin (GL) \cite{GL} demonstrates that the gate voltage could be another important parameter of the Josephson junction arrays, as two additional phases of the array are possible when it is varied.

The BD transition was recently observed in one dimensional Josephson junction arrays \cite{BD_exp}. However, the results of Ref. \cite{BD_exp} show only qualitative agreement with the theories of Refs. \cite{BD} and \cite{GL}.

First of all, the BD theory completely neglects the capacitance of the Josephson junctions $C$ as opposed to the capacitance of the superconducting islands $C_0$. GL does take nonzero $C$ into account, but only as a small perturbation. In experiment, however, $C \gg C_0$. Not only does it mean that the junction capacitance terms are large, but it also leads to the long ranged Coulomb interactions between grains which are far away from each other, which is something both BD and GL neglect. As a result, the point in the parameter space where the transition happens is experimentally far from the theoretical point predicted by the BD theory.

It is therefore of some interest to develop a theory of the Josephson junction arrays which would take the condition $C \gg C_0$ into account. In fact, such theories were already proposed in the context of both tunnel \cite{tunnel} and Josephson \cite{Josephson} junction arrays. The main idea of this approach is to concentrate on the dynamics of charge of the islands as opposed to their superconducting phases. As a starting point of the theory, one takes the problem of single Josephson junction with nonzero capacitance $C$ treated in the full quantum mechanical fashion. The solution of this problem is a Josephson junction described by its quasicharge. For a system of connected junctions this description in the continuum limit yields the sine-Gordon Lagrangian describing dynamics of the quasicharge.

The sine-Gordon model in (1+1)-dimensions is one of the best studied models of field theory. This model admits a quantum phase transition: the excitation spectrum depends on the coupling constant and becomes gapless when this constant exceeds a certain critical value. In the underlying Josephson array this transition corresponds to the superconductor-insulator transition predicted by BD. We have to note, however, that according to the opinion widely circulating in the literature the quasicharge sine-Gordon description of the Josephson junction arrays loses its validity at large coupling constants and therefore cannot ascertain the existence of the transition \cite{GL}.

In this paper we critically examine the conditions of validity of the sine-Gordon description of Josephson junction arrays. We conclude that the quasi-charge sine-Gordon description remains self-consistent in the region of large couplings. Hence one can use it to get quantitative information about the array even in the vicinity of the transition.

We have to emphasize that the BD theory also describes the transition in terms of the sine-Gordon equation. However, being written in the limit $C \ll C_0$, the parameters of that equation are different from the one extracted from the sine-Gordon equation considered in this paper. We conclude therefore that while the universality class of the transition does not depend on whether $C$ is large or small, the details of the description do depend on it.

Secondly, contrary to Ref. \cite{GL}, the experiments failed to observe any additional phases of the array due to the change in gate voltage. Most likely, this is the result of the presence of random background charges in the realistic Josephson junction arrays, which lead to the voltage being randomized along the length of the array \cite{random}. An important outstanding question is thus the effect the random charges may have on the array and how...
their presence affects the transition.

The problem of Josephson junction arrays with random background charges is equivalent to the problem of randomly pinned charge density waves [8], which was extensively studied in the literature. The best understood case is that of classical pinned charge density waves, which corresponds to the array deeply in the insulating phase. In that regime, the random background charges lead to the AC conductance of the array [3] [11] [12]

$$\sigma(\omega) \propto \omega^4, \omega \ll \omega_p$$  

on the condition that the array length is much longer than the so-called Larkin length $l_p$ [12]. Here $\omega_p$ is the pinning frequency. Additionally, for a special value of the sine-Gordon coupling parameter $\beta^2 = 4\pi$ (see later in the text for a precise definition of $\beta$), the AC conductance is known to go as [13]

$$\sigma(\omega) \propto \omega^2 \ln \omega$$  

This can in principle be checked experimentally. In this paper we discuss a relationship between $\beta$, $l_p$, $\omega_p$ and parameters of the junction array.

Unfortunately, at other values of the coupling parameter $\beta$, the solution to the random background charge problem is not known. Therefore, we are not able at this time to discuss how the presence of the random background charges affects the superconductor-insulator transition.

The rest of this paper is organized as follows. In the next chapter we derive the quasicharge sine-Gordon equation. Then we discuss its applicability and show that it can indeed describe the transition superconductor-insulator transition. In the end we discuss how the random background charges affect the properties of the array.

II. DERIVATION OF THE MODEL

Let us consider a one-dimensional array of weakly coupled Josephson junctions (see Fig. 1). A single junction is described by the Hamiltonian

$$H = -\frac{E_c}{2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{2} E_J \cos(\phi)$$  

where $E_c = (2e)^2$ is the charging energy of the junction, $E_J$ is the Josephson energy and $\phi$ is the phase difference on the junction.

We work in the limit $E_J \gg E_c$. In this regime, the energy spectrum consists of narrow bands separated by gaps (see Ref. [14, 15]). The interband splitting frequency is estimated as

$$W = 2\sqrt{E_c E_J}$$  

We shall restrict our consideration to the lowest band. Therefore $W$ will serve as the ultraviolet cut-off in our effective theory. The energy eigenvalues in the lowest band are labelled by the quasicharge $q$:

$$E = E_b [1 - \cos(2\pi q)]$$  

where $E_b = 16E_J \left( \frac{E_c}{\pi^2 E_J} \right)^\frac{1}{2} \exp \left( -2\sqrt{\frac{E_J}{E_c}} \right)$.

The junction array consists of junctions separated by superconducting islands. Following [3], we treat the array in the adiabatic approximation, so that the $n$-th junction is characterized by a slowly varying function $q_n(t)$. The characteristic frequencies are supposed to be much smaller than the interband splitting $W$. The charge on the $n$-th junction is $2e \times q_n$. Since the total charge in the unit cell is zero, the charge on the wire connecting the two junctions is $2e(q_n - q_{n+1})$. The Coulomb energy of the wire is

$$E_{\text{Coulomb}} = \frac{(2e)^2}{2C_0} \sum_n (q_n - q_{n+1})^2 \approx \int dx \frac{(2e)^2}{2C_0} q_x^2$$  

where $a$ is the size of the island. The inductive energy of the junction is

$$E_{\text{Inductive}} = \frac{1}{2} \sum_n (2e)^2 L q_n^2,$$  

where $L$ is the inductance of the islands. The experiments conducted in Ref. [14] indicate that real arrays have a very considerable inductance ($L \gg \frac{L}{E_c}$), though its microscopic origin remains somewhat obscure.

Combined, Eqs. [16, 17] produce the Lagrangian describing the array,

$$\mathcal{L} = \int dx \left\{ \frac{(2e)^2}{2a} L q_x^2 - a \frac{(2e)^2}{2C_0} q_x^2 - \frac{E_b}{a} [1 - \cos(2\pi q)] \right\}$$  

By introducing a new variable $Q = 2\pi q/\beta$ the latter Lagrangian can be rewritten in the form of the sine-Gordon model:

$$\mathcal{L} = \hbar v_c^{-1} \int dx \left[ \frac{1}{2} \dot{Q}^2 - \frac{1}{2} v_c^2 Q_x^2 - \frac{m_0^2}{\beta^2} (1 - \cos(2Q)) \right]$$  

where

$$m_0^2 = \frac{\pi^2 E_b}{L c^2},$$  

$$v_c = \frac{a}{\sqrt{LC_0}}.$$  

FIG. 1: An equivalent scheme of an elementary cell of the array. Black dots denote the superconducting islands.
and
\[
\beta^2 = \frac{(2\pi)^2 \hbar v_c C_0}{(2e)^2 a} = \frac{(2\pi)^2 \hbar}{(2e)^2} \sqrt{\frac{C_0}{L}} = 2\pi R_0 \sqrt{\frac{C_0}{L}},
\]
(13)
where \(R_0 = \hbar/4e^2\) is the quantum of Cooper pair resistance.

The sine-Gordon equation \(\sigma\) goes through a quantum phase transition as \(\beta^2\) is tuned through \(8\pi\). Let us estimate whether this value of \(\beta\) can realistically be achieved. Assuming that the island has a form of a thin superconducting wire we can use for its impedance \(\sqrt{L/C_0}\) the formula derived by Likharev [15]:
\[
\sqrt{L/C_0} = \frac{4\pi}{c} \left[ 1 + \frac{2\pi \lambda^2}{A \ln(D/d)} \right]^{1/2}
\]
(14)
where \(\lambda\) is the penetration depth, \(A\) is the cross-section area of the wire, \(d\) is its thickness and \(D\) is the distance from the gate. Taking into account that \(\hbar c/e^2 \approx 137\) we get
\[
\beta^2 \approx \frac{4.3}{8\pi} \left[ 1 + \frac{2\pi \lambda^2}{A \ln(D/d)} \right]^{1/2}
\]
(15)
We see that \(\beta^2\) can be driven through the transition by changing \(\lambda\) which can be achieved by adding nonmagnetic impurities [17].

III. APPLICABILITY LIMITS OF THE SINE-GORDON MODEL

As we have mentioned above, the sine-Gordon model Eq. \(\sigma\) is one of the best studied models of one-dimensional field theory. Its spectrum and behavior of correlation functions dramatically depend on the value of coupling constant \(\beta^2\), because \(\beta^2\) controls the renormalization group (RG) dimension of the cosine term in Eq. \(\sigma\). At \(\beta^2 < 8\pi\), the cosine term is relevant, all excitations have spectral gaps and the system is an insulator. At \(\beta^2 < 4\pi\) the spectrum consists of electrically charged solitons and antisolitons and their neutral bound states, while at \(8\pi > \beta^2 > 4\pi\) the bound states disappear. At \(\beta^2 > 8\pi\), the cosine term is irrelevant and the spectrum becomes gapless. This is a superconducting regime; in the absence of disorder it allows ballistic transport through the system.

Let us critically examine conditions of validity of the sine-Gordon model [16]. It is based on the following assumptions:

- The harmonicity of the effective potential. The potential acquires cosine form [1] in the limit \(E_f \gg E_c\). The presence of higher harmonics will affect the integrability of the sine-Gordon model, but will not change the fact of the transition since these harmonics will become irrelevant even sooner than the primary one.

- The abiaticity. All characteristic energies of sine-Gordon model (in particular, the spectral gaps) must be much smaller than the interband splitting \(W\) of Eq. [4].

- The continuous approximation. The discrete array is replaced by the continuous one. This approximation requires that characteristic wave vectors are much smaller than \(1/a\) and the energy gaps are much smaller than \(\Lambda = \hbar v_c/a\).

- Absence of dissipation. It is assumed that islands are superconducting and there are no normal resistors in the scheme.

- We assume that all characteristic frequencies are much smaller than the plasma frequency in the islands so that electric charge spreads instantaneously through the entire island.

Let us look closer at the requirement of adiabaticity and validity of the continuous approximation. The spectrum of the sine-Gordon model consists of particle branches with the relativistic dispersion
\[
\omega^2 = v_f^2 k^2 + m_j^2 / \hbar^2,
\]
(16)
where the spectrum consists of solitons \(s\) and antisolitons \(\bar{s}\) with \(m_s = m_{\bar{s}} = M\) and (at \(\beta^2 < 4\pi\)) their bound states (breathers) with spectral gaps
\[
m_j = 2M \sin(\gamma j/2), \quad j = 1, \ldots, [\pi/\gamma], \quad \gamma = \frac{\pi \beta^2}{8\pi - \beta^2}
\]
(17)
The largest spectral gap is of order of \(M\). The self-consistency of the theory requires that its value must not exceed the cut-off. According to [16], we have
\[
M = \frac{2\Gamma(\gamma/2)}{\sqrt{\pi} \Gamma(1/2 + \gamma/2)} \left[ \frac{m_0^2 \Gamma(1 - \beta^2/8\pi)}{16\Lambda^2 \Gamma(1 + \beta^2/8\pi)} \right]^{1 - \beta^2/4\pi}
\]
(18)
The following inequalities must be fulfilled:
\[
M \ll \Lambda \ll W
\]
(19)
Inequality \(M, \Lambda \ll W\) is equivalent either to
\[
m_0 \ll \gamma \Lambda, \quad E_b \ll 1/e^2 L
\]
(20)
at small \(\beta^2\) or to
\[
m_0 \ll \Lambda, \quad E_b \ll \epsilon^2 / C_0
\]
(21)
at \(\beta^2 \approx 8\pi\). The validity of the continuous approximation requires that \(\Lambda \ll W\), that is
\[
\frac{\hbar^2}{C_0 L} \ll E_f E_c, \quad \text{or} \quad \beta^2 \ll WC_0/4e^2
\]
(22)
From the previous inequalities it follows that the only condition on the latter quantity is \(WC_0/\epsilon^2 \ll \left( E_f / E_c \right)^4 \exp[2\sqrt{E_f/E_c}] \) which still leaves room for \(\beta^2 \approx 8\pi\).

IV. PHASE TRANSITION

As we have mentioned above, one way to drive the system through the transition is by changing \(\beta^2\). In
fact, it is also possible to change $E_b$ or equivalently $m_0$, to go through the transition. This would correspond to the experimental set up [3, 9], where $E_j$ is lowered by applying the external magnetic field to the junctions in the SQUID geometry, leading to the increase of $E_b$ and $m_0$.

The Renormalization Group (RG) flow diagram of the sine-Gordon equation is shown on Fig. 2. If $\beta^2 < 8\pi$, the sine-Gordon equation is always in the massive regime. This corresponds to the insulating behavior of the array. However if $\beta^2 > 8\pi$, two regimes are possible. For sufficiently large $m_0$, the behavior is still massive. For $m_0$ smaller than a certain critical value $m_c \propto \Lambda (\beta^2 - 8\pi)$, the cosine term in Eq. (10) becomes irrelevant and can be neglected. The sine-Gordon equation becomes massless. This corresponds to the part of the diagram whose RG flow lines end on the $m_0 = 0$ axis. In this regime the charge density wave propagates ballistically.

![RG flow diagram](image)

**FIG. 2:** The RG flow diagram of the sine-Gordon equation. One can tune through the transition by either changing the value of $\beta$ or $m_0$.

One of the more experimentally relevant predictions of the theory for the ballistic regime follows from neglecting the cosine term in Eq. (10). Then the Josephson junction array can be described by the following very simple Lagrangian

$$\mathcal{L} = \frac{4\pi^2}{\beta^2} \hbar v_c^{-1} \int dx \left[ \frac{1}{2} \dot{Q}^2 - \frac{1}{2} v_c^2 Q_x^2 \right]$$

(23)

This is none other than the so-called Luttinger liquid with the Luttinger parameter $g = \frac{\beta^2}{4\pi}$. It is well known [18] that its conductance is given by

$$G = \frac{(2e)^2}{\hbar} g = \frac{\beta^2 (2e)^2}{4\pi \hbar}$$

(24)

As $\beta$ is lowered (or $m_0$ is increased, according to Fig 2), the conductance is decreased until it reaches its critical value (at $\beta^2 = 8\pi$),

$$G^* = 2 \frac{(2e)^2}{\hbar} = \frac{2}{R_Q}.$$  

(25)

If $\beta$ is decreased further, the array enters the regime of the Coulomb blockade and becomes an insulator.

**V. RANDOM BACKGROUND CHARGES**

In the presence of random background charges, the charging energy of the superconducting islands Eq. (11) gets modified,

$$E_{\text{Coulomb}} = \int dx \frac{(2e)^2}{2\hbar c} (q_x - V(x))^2,$$

(26)

where $V(x)$ is a time independent random function of the coordinate $x$ with short ranged correlations

$$\langle V(x)V(y) \rangle = V_0 \delta(x-x').$$

(27)

It is convenient to shift the variables $q(x) \rightarrow q(x) + \int_0^x dy V(y)$ to find the following Lagrangian for the Josephson junction array,

$$\mathcal{L} = \frac{\hbar}{2} \int dx \left[ \frac{1}{v_c} \dot{Q}^2 - v_c Q_x^2 - \frac{2m_0^2}{\beta^2} \{1 - \cos (\beta Q + \chi)\} \right],$$

(28)

where

$$\chi(x) = 2\pi \int_0^x dy V(y).$$

(29)

It is clear that $\chi(x)$ is a Brownian motion: $\sqrt{\langle \chi^2(x) \rangle} = 2\pi \sqrt{V_0}$. At distances of the order of $l \propto V_c^{-1}$ the phases of the cosine term in Eq. (28) become completely uncorrelated, so at length scale much bigger than that we can consider $\cos \{\chi(x)\}$ to be white noise in space. $l$ is the length at which the random background charge accumulates to be of the order of the Cooper pair charge.

Under these conditions, the problem defined in Eq. (28) becomes equivalent to the pinned charge density wave problem studied in a number of publications [12]. A particularly well understood regime is that of a Coulomb blockade, $\beta \ll 8\pi$. There the sine-Gordon Lagrangian Eq. (28) becomes purely classical, corresponding to the sine-Gordon equation

$$v_c^{-1} \partial_x^2 Q - v_c \partial_x^2 Q - \frac{m_0^2}{\beta} \sin \{Q + \chi\} = 0.$$  

(30)

If $Q_0(x)$ is a time independent solution to Eq. (30), then the small oscillations around that solution are described by

$$v_c \partial_x^2 \delta Q + m_0^2 \cos \{Q_0 + \chi\} \delta Q = v_c^{-1} \omega^2 \delta Q,$$

(31)

where $\delta Q$ is the amplitude of the oscillations and $\omega$ is their frequency.

It is well known in the literature [3, 10, 11] that this problem possesses a fundamental frequency called the pinning frequency $\omega_p$. At $\omega \gg \omega_p$, it is possible to neglect the cosine term in Eq. (31) to find that the oscillations $\delta Q$ are the plain waves with the wave vector $k$ and with the frequencies $\omega \propto v_c k$. At $\omega \ll \omega_p$, the oscillations are localized in space, and their spatial extent (localization length) $l_p$ is called the Larkin (or pinning) length. The pinning frequency is known to scale with $m_0$ as

$$\omega_p \propto m_0^{\frac{3}{2}}.$$  

(32)
while the Larkin length scales as

\[ l_p = \frac{1}{\omega_p} \propto m_0^{-\frac{1}{3}}. \]  

(33)

On the condition that the Larkin length is much smaller than the total length of the array, the conductivity of the array at a frequency \( \omega \) much smaller than the pinning frequency is given by

\[ \sigma(\omega) \propto \rho(\omega), \quad \rho(\omega) \propto \omega^4, \quad \omega \ll \omega_p. \]  

(34)

Here \( \rho(\omega) \propto \omega^4 \) is the probability that there exists a solution to Eq. (31) at a frequency \( \omega \) (or the density of states of Eq. 31). The formula Eq. (34) as well as Eq. (32) could be checked experimentally.

Finally, at a special value of \( \beta^2 = 4\pi \), the Eq. (28) can be solved using a completely different technique [13], to lead to the AC conductivity Eq. 2.

At values of \( \beta \) other than \( \beta^2 \ll 8\pi \) or \( \beta^2 = 4\pi \), the problem defined in Eq. (28) remains unsolved.

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