Representations of Gelfand-Graev type for the unitriangular group

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Abstract

We consider the analog of Gelfand-Graev representations of the unitriangular group. We obtain the decomposition into the sum of irreducible representations, prove that these representations are multiplicity free, calculate the Hecke algebra.

1 Introduction and main definitions

A representation of Gelfand-Graev is a representation of the group GL(n, \(F_q\)) (more generally, of the finite Chevalley group) induced from a nondegenerate character of its maximal unipotent subgroup. The main property of these representations that they are multiplicity free. These representations appeared first in the papers [1, 2]. It was proved that the algebra of G-endomorphisms (further referred as the Hecke algebra) of these representations is commutative; this is equivalent to the property of being multiplicity free. The basis of the Hecke algebra was constructed in the paper [2], for GL(n, \(F_q\)), and, later, for finite Chevalley groups, in [3, 4]. The Gelfand-Graev representations play an important role in the representation theory; many papers appear on properties of these representations and their generalizations.

In this paper, we study representations \(V(\lambda)\) of the unitriangular group; this representations are analogs of Gelfand-Graev ones. We shall prove that \(V(\lambda)\) are also multiplicity free. We shall give a complete description of all its irreducible components \(V_{S,a}(\lambda)\). Following the orbit method (see [10, 11]), we associate the coadjoint orbit \(\Omega_{S,a}(\lambda)\) with the irreducible component \(V_{S,a}(\lambda)\). In the paper, we find a canonical form \(\lambda_{S,a} \in \Omega_{S,a}(\lambda)\) and generators of the defining ideal of the orbit \(\Omega_{S,a}(\lambda)\). In a sequel of the paper, we find a basis of the Hecke algebra \(\mathcal{H}(\lambda)\). The main results are formulated in theorems 3.2, 3.4, 3.5, 4.1, 5.2.

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Decomposition of the representation $V(\lambda)$ into a sum of irreducible components admits the interpretation in terms of the theory of basic characters (basic representations) developed by C. André (see, for instance, [5, 6, 7, 8]). One can consider the representation $V(\lambda)$ as the induced representation from the basic (precisely, regular irreducible) representation of the unitriangular subgroup of size $n - 1$ [9]. The representation $V(\lambda)$ decomposes into the representations $V_S(\lambda)$ that in its turn decomposes into irreducible components $V_{S,a}(\lambda)$ (see (9)). Notice that the representations $V_S(\lambda)$ are basic or sums of the basic representations.

Firstly, we shall give a remark on the orbit method. The orbit method appeared in 1962 in the paper [10]. There was shown that there exists one to one correspondence between irreducible representations of a connected nilpotent Lie group and its coadjoint orbits. Later, in [12], it was proved that the orbit method is also true for any unipotent group over a finite field (see also [5, 13]). There are some requirements on the characteristic of the field; since the matrix exponent is used in the orbit method, the characteristic of the finite field have to be great enough. For the group $\text{UT}(n, \mathbb{F}_q)$, it is sufficient to put $\text{char} \mathbb{F}_q \geq n - 1$.

In this paper, the characteristic is arbitrary. The irreducible representations $V_{S,a}(\lambda)$ are induced from characters of associative polarizations (see definition 2.1). This enables to remove matrix exponents from the process of construction of representations [6, 15].

Let $\mathbb{F}_q$ be a finite field of $q$ elements. The unitriangular group $G = \text{UT}(n, \mathbb{F}_q)$ consists of all upper triangular matrices of size $n \times n$ with units on the diagonal and entries from the field $\mathbb{F}_q$. We consider further that $n > 2$. Denote by $\mathfrak{g}$ a subspace of all upper triangular matrices with zeros on the diagonal. It is obvious that $G = E + \mathfrak{g}$, where $E$ is the unity matrix. The subspace $\mathfrak{g}$ is an associative algebra with respect to the matrix multiplication and, therefore, a Lie algebra.

We shall give the following definition: a root is a pair $(i, j)$ of integers, where $1 \leq i < j \leq n$. The partial operation of addition is defined on the set of all roots

$$R = \{(i, j) : 1 \leq i < j \leq n\}$$

as follows: $(i, j) + (j, s) = (i, s)$.

The set of all roots $R$ decomposes into the subsets $R = R_+ \sqcup R_0 \sqcup R_-$, where

$$R_+ = \{(i, j) : i + j < n + 1\},$$

$$R_0 = \{(i, j) : i + j = n + 1\},$$

$$R_- = \{(i, j) : i + j > n + 1\}.$$
Matrix unities \( \{ E_\alpha : \alpha \in R \} \) generate a basis in the algebra \( g \). The algebra \( g \) is a direct sum of the subspaces \( g = g_+ \oplus g_0 \oplus g_- \), where
\[
g_\pm = \text{span}\{ E_\alpha : \alpha \in R_\pm \},
g_0 = \text{span}\{ E_\alpha : \alpha \in R_0 \}.
\]
The subspaces \( g_+ \), \( g_0 \), \( g_- \) are subalgebras of the associative algebra \( g \). Then
\[
G_\pm = E + g_\pm, \quad G_0 = E + g_0
\]
are subgroups in \( G \). Any root \( \alpha \in R \) determines the one-parameter subgroup
\[
\{ x_\alpha(t) = E + tE_\alpha, \ t \in \mathbb{F}_q \}.
\]

Introduce the following notations:
1) \( k = \lfloor \frac{n-1}{2} \rfloor \). Then \( n = 2k + 1 \), if \( n \) is odd, \( n = 2(k + 1) \), if \( n \) is even;
2) \( \varepsilon = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even}. \end{cases} \)

We say that a root from \( R_+ \) is a simple root, if it can’t be presented as a sum of two roots from \( R_+ \). The set of all simple roots is a union of two subsets \( \Pi_0 \cup \Pi \), where
\[
\Pi_0 = \{ (i, i+1) : 1 \leq i \leq k \},
\Pi = \{ (i, n-i) : 1 \leq i \leq k \}.
\]

Notice that
\[
\Pi_0 \cap \Pi = \begin{cases} \emptyset, & \text{if } n = 2(k + 1), \\ (k, k+1), & \text{if } n = 2k + 1. \end{cases}
\]

**Definition 1.1.** A character of an associative algebra \( a \) is a linear form on it that is zero on \( a^2 \).

Notice that any character of an associative algebra is a character of it as a Lie algebra. Any character of the associative algebra \( g_+ \) is uniquely determined by its values on \( \{ E_\alpha : \alpha \in \Pi_0 \cup \Pi \} \).

**Definition 1.2.** A character \( \lambda : g_+ \rightarrow \mathbb{F}_q \) is nondegenerate, if \( \lambda(E_\gamma) \neq 0 \) for any \( \gamma \in \Pi \setminus \Pi_0 \).

Fix some nontrivial complex character of the additive group of the field \( \mathbb{F}_q \) (i.e. homomorphism \( \mathbb{F}_q \rightarrow \mathbb{C}^* \)). We shall denote this character by \( e^x \), where \( x \in \mathbb{F}_q \).

Any character \( \lambda \) of the associative algebra \( g_+ \) determines a complex character (i.e. one-dimensional representation) \( \xi_\lambda \) of the group \( G_+ \) by the formula
\[
\xi_\lambda(1 + x) = e^{\lambda(x)}, \quad x \in g_+.
\]
Definition 1.3. The representation $V(\lambda) = \text{ind}(\xi_\lambda, G_+, G)$, where $\lambda$ is a non-degenerate character of $g_+$, is called a representation of Gelfand-Graev type.

Proposition 1.4. If $\lambda$, $\lambda'$ are nondegenerate characters of $g_+$ that coincide on $\{E_\alpha : \alpha \in \Pi\}$, then $V(\lambda) \cong V(\lambda')$.

Proof. Let $v_0 \in V(\lambda)$ be the generating vector of the induced representation, $g_+ v_0 = \xi_\lambda(g_+) v_0$. For any root $\alpha \in \Pi_0 \setminus \Pi$, there exists a unique root $\beta(\alpha) \in R_0$ such that the sum $\gamma(\alpha) = \alpha + \beta(\alpha)$ is determined and belongs to $\Pi \setminus \Pi_0$. Indeed, if $\alpha = (i, i + 1)$, then $\beta(\alpha) = (i + 1, n - i)$ and $\gamma(\alpha) = (i, n - i)$. The subspace, spanned by $E_{\beta(\alpha)}$, where $\alpha \in \Pi_0 \setminus \Pi$, is an associative subalgebra with zero multiplication. Therefore, for any two roots $\alpha, \alpha' \in \Pi_0 \setminus \Pi$, the elements of the corresponding one-parameter subgroups $x_{\beta(\alpha)}(t)$ and $x_{\beta(\alpha')}(t')$ commutes. Consider the element

$$g_0 = \prod x_{\beta(\alpha)}(t_\alpha) \in G,$$

where $\alpha$ is running through $\Pi_0 \setminus \Pi$, and $t_\alpha \in \mathbb{F}_q$ is a solution of the equation

$$\lambda(E_\alpha) + \lambda(E_{\gamma(\alpha)}) t_\alpha = \lambda'(E_\alpha).$$

Using the equalities

$$x_\alpha(s)x_{\beta(\alpha)}(t) = x_{\beta(\alpha)}(t)x_\alpha(s)x_{\gamma(\alpha)}(st),$$

we obtain $g_0 g_0 v_0 = \xi_{\lambda'}(g_+) g_0 v_0$. □

2  Associative polarizations

Let $a$ be an arbitrary nilpotent associative algebra over an arbitrary field $K$. Adjoin the unity element $E$ to the algebra $a$. Then $G = E + a$ is an unipotent group. The algebra $a$ is a Lie algebra with respect to the commutator $[x, y] = xy - yx$. Let $\lambda \in a^*$. Recall that a polarization for $\lambda$ is a Lie subalgebra $p$ of $a$ that is a maximal isotropic subspace with respect to the skew symmetric bilinear form $B_\lambda(x, y) = \lambda([x, y])$.

Definition 2.1. An associative polarization of $\lambda \in a^*$ is a polarization $p$ that obeys the following conditions

i) $p$ is an associative subalgebra of $a$,

ii) $\lambda(p^2) = 0$.

Is it true that any $\lambda \in a^*$ has an associative polarization? In general, the answer is negative.

Example 2.2. The associative algebra $a = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \right\}$ is commutative.
Then it is a commutative Lie algebra. Any linear form $\lambda$ on $a$ has a unique polarization, which coincides with $a$. If $\lambda(E_{13}) \neq 0$, then $a$ is not an associative polarization.

Suppose that $a$ is an associative nilpotent algebra over $\mathbb{F}_q$. If $p$ is an associative polarization for $\lambda \in a^*$, then formula (1) defines a one-dimensional complex representation $\xi_\lambda$ of the group $P = E + p$. Denote by $M(\lambda)$ the induced representation $\text{ind}(\xi_\lambda, P, G)$.

**Proposition 2.3.** Given $a$, $\lambda$, $p$ as above, then

1) the following formula for the character $\chi_\lambda$ of representation $M(\lambda)$ holds

$$\chi_\lambda(1 + x) = \frac{1}{\sqrt{|\Omega|}} \sum_{\mu \in \Omega(\lambda)} e^{\mu(x)}, \quad x \in a; \quad (3)$$

2) $\dim M(\lambda) = q^{\text{codim} p} = \sqrt{|\Omega|}$;

3) the representation $M(\lambda)$ doesn’t depend on a choice of associative polarization;

4) the representation $M(\lambda)$ is irreducible;

5) let two linear forms $\lambda$ and $\lambda'$ have associative polarizations; the representations $M(\lambda)$ and $M(\lambda')$ are equivalent if and only if $\lambda$ and $\lambda'$ lie in a common coadjoint orbit.

**Proof.** For any $\lambda \in a^*$, we denote by $a^\lambda = \{a \in a : \lambda([a, a]) = 0\}$ the stabilizer of $\lambda$ in the Lie algebra $a$. Obviously, the equality $\lambda((E + a)x) = \lambda(x(E + a))$ is equivalent to $\lambda(ax) = \lambda(xa)$. Hence, $E + a^\lambda$ coincides with the stabilizer $G^\lambda$ of linear form $\lambda$ in the group $G$. It implies

$$|\Omega| = \frac{|G|}{|G^\lambda|} = \frac{|g|}{|g^\lambda|} = q^{\dim g - \dim g^\lambda} = q^{\dim \Omega} \quad (4)$$

The proof may be finished similarly as in papers [6, 12, 15, 13].

**Remark.** The main result of the paper [16] implies that the formula (3) in general does not true for $\text{ut}(n, \mathbb{F}_q)$. Therefore, it is not true that every linear form on $\text{ut}(n, \mathbb{F}_q)$ has an associative polarization. Applying the classification of coadjoint orbits for the unitriangular Lie groups of lower sizes [14], one can prove the existence of associative polarization for $n \leq 7$.

### 3 Orbits and representations for $\lambda_{S,a}$

Let $\lambda$ be an nondegenerate character of $g_+$ as an associative algebra. In this section, we construct families of the linear forms $\lambda_{S,a}$ where $S$ is a subset of $\Pi$ and $a \in \Lambda_S$. For any $\lambda_{S,a}$, we construct an associative polarization $p_S$ and
corresponding irreducible representation \( V_{S,a}(\lambda) \). In what follows, we obtain a description of the coadjoint orbit \( \Omega_{S,a}(\lambda) \); we shall show that the representations \( \{ V_{S,a}(\lambda) \} \) are pairwise nonequivalent.

We shall treat any root \( \gamma = (i,j) \) as a box in empty \( n \times n \)-matrix. The number \( i \) is called the row of the root \( \gamma \), and \( j \), respectively, the column of the root \( \gamma \). We shall say that a root \( \gamma' = (i',j') \) lies on the left side (respectively, stronger on the left side) of the root \( \gamma \), if \( j' \leq j \) (respectively, \( j' < j \)). We define similarly relations of lying on the right side, over and lower.

Let \( S \) be an arbitrary subset of \( \Pi \).

**Notations.**

1) Denote by \( L^0_S \) a subset that consists of all roots \( \gamma = (i,j) \) obeying the following conditions:
   i) \( 1 \leq i \leq k \) and \( i + j \geq n + 1 \),
   ii) there is no roots of \( S \) in the \( i \)th row and \( j \)th column,
   iii) all roots of \( \Pi \) that are lying stronger over and stronger on the left side of \( \gamma \) belong to \( S \).
2) The subset \( L^0_S \) is empty for the odd \( n \). For \( n = 2(k + 1) \), the subset \( L^0_S \) consists of the single root \( \gamma = (k + 1, j) \) obeying conditions ii) and iii).
3) \( L^+_S \). A root \( \gamma \) of \( R_0 \cup R_- \) belongs to \( L^+_S \), if it belongs to the same column and lies stronger over some root of \( L^0_S \).
4) \( L^-_S \). A root \( \gamma' = (n - i, j) \) belongs to \( L^-_S \) if and only if \( \gamma = (i, j) \) belongs to \( L^+_S \).
5) \( L_S = L^+_S \cup L^0_S \cup L^0\cup L^-_S \).
6) \( R_S = R_+ \cup L_S \).
7) \( |S| = s, \ |R_+| = r_+, \ |R_0| = r_0 \).

Notice that \( |L^+_S| = |L^-_S| = |S| = s, \ |L^0_S| = k - s, \ |L^0\cup L^-_S| = \varepsilon, \ k + \varepsilon = r_0 \).

Consider the following ordering in the set all roots \( R \): we say that \( \beta \geq \alpha \), if \( \beta \) lies lower than \( \alpha \), or in the same row and on the left side from \( \alpha \). Order the subset \( R_+ \setminus S \) with respect to this ordering \( R_+ \setminus S = \{ \alpha_1 < \alpha_2 < \ldots < \alpha_{r_+ - s} \} \).

**Lemma 3.1.**

1) For any root \( \alpha_i \), there exists a unique root \( \beta_i \in R \setminus R_S \) such that \( \alpha_i + \beta_i \) belongs to \( S \cup L^0_S \).
2) For any \( 1 \leq i < j \leq r_+ - s \), the sum \( \alpha_i + \beta_j \) is either undefined, or is defined and belongs to \( (R_+ \setminus \{ S \cup \Pi_0 \}) \cup L^+_S \).

**Proof** Follows from the definitions. \( \square \)

Denote by \( I_S \) (resp. \( I^+_S, I^0_S, I^{00}_S \)) a subspace spanned by the system \( E_\gamma, \gamma \in L_S \) (resp. \( \gamma \in L^+_S, \alpha \in L^0_S \)). Obviously,

\[ I_S = I^+_S \oplus I^0_S \oplus I^{00}_S \oplus I^-_S. \]
The subspace $I_S$ is an associative subalgebra with zero multiplication. The subspace
\[ p_S = \text{span}\{E_\gamma : \gamma \in R_S\} = g_+ \oplus I_S \]
is also an associative subalgebra in $g$, and $P_S = E + p_S$ is a subgroup in $G$.

Consider the subset $\Lambda_S$ that consists of all functions
\[ a : L^0_S \sqcup L^0_S \sqcup L^-_S \to \mathbb{F}_q \]
such that $a(\gamma) \neq 0$ for any $\gamma \in L^0_S$. One may identify
\[ \Lambda_S \cong \mathbb{F}_q^{s+\varepsilon} \times (\mathbb{F}_q^*)^{k-s}. \]

The number of elements of $\Lambda_S$ equals to $q^{s+\varepsilon}(q-1)^{k-s}$.

For any $a \in \Lambda_S$, we define a linear form $\lambda_{S,a}$ on $g$ as follows:
1) $\lambda_{S,a}(E_\gamma) = \lambda(E_\gamma)$ for all $\gamma \in \Pi_0 \setminus \Pi$ and $\gamma \in S$,
2) $\lambda_{S,a}(E_\gamma) = a(\gamma)$ for all $\gamma \in L^0_S \sqcup L^0_S \sqcup L^-_S$,
3) $\lambda_{S,a}(E_\gamma) = 0$ for all other $E_\gamma \in g$.

Notice that the definition of $\Lambda_S$ implies
i) $\lambda_{S,a}(E_\gamma) \neq 0$ for all $\gamma \in L^0_S$;
ii) $\lambda_{S,a}(E_\gamma)$ may have an arbitrary values, when $\gamma \in L^0_S \sqcup L^-_S$.
iii) $\lambda_{S,a}(E_\gamma) = 0$ for all $\gamma$ of $R_+ \setminus \{S \sqcup \Pi_0\}$ and all $\gamma \in L^+_S$.

Easy to see that the linear form $\lambda_{S,a}$ is a character of the associative algebra $p_S$ in the sense of definition 1.1. Following formula (1), we define a complex character $\xi_{\lambda,S,a}$ of the subgroup $P_S = E + p_S$. We denote
\[ V_{S,a}(\lambda) = \text{ind}(\xi_{\lambda,S,a}, P_S, G). \]

**Theorem 3.2.** Let $\lambda$ be a nondegenerate character of the algebra $g_+$ (see definition 1.2). Then
1) the subalgebra $p_S$ is an associative polarization for the linear form $\lambda_{S,a}$,
2) every representation $V_{S,a}(\lambda)$ is irreducible.

**Proof.** By proposition 2.3, the statement 1) implies 2) 1). Let us prove 1).

The subspace $p_S$ is an associative subalgebra, and $\lambda_{S,a}(p^2_S) = 0$. It is sufficient to prove that $p_S$ is a maximal isotropic subspace for the skew symmetric bilinear form $\lambda_{S,a}([x,y])$. Suppose the contrary. Assume that there exists $x \in g \setminus p_S$ such that $\lambda_{S,a}([p_S,x]) = 0$. Then
\[ x = \sum_{j=1}^{r_{+}-s} b_j E_{\beta_j}. \]
Let $i$ be the smallest number obeying $b_i \neq 0$. Lemma 3.1 and definition of $\lambda_{S,a}$ imply that

$$\lambda_{S,a}([E_{\alpha_i}, E_{\beta_i}]) = c_i \neq 0, \quad \lambda_{S,a}([E_{\alpha_i}, E_{\beta_j}]) = 0$$

for all $j > i$.

Hence $\lambda_{S,a}([p_S, x]) = c_i b_i \neq 0$. This leads to contradiction. Therefore, $p_S$ is a maximal isotropic subspace. $\square$

Denote by $\Omega_{S,a}(\lambda)$ the orbit of $\lambda_{S,a} \in g^*$ with respect to the coadjoint representation of the group $G$.

Introduce the following notations: $\overline{\mathbb{F}}_q$ is an algebraic closure of the field $\mathbb{F}_q$, $\overline{g} = g \otimes \overline{\mathbb{F}}_q$, $\overline{\Omega}_{S,a}(\lambda)$ is the coadjoint orbit of $\lambda_{S,a}$ with respect to the group $\overline{G} = UT(n, \overline{\mathbb{F}}_q)$. The orbit $\overline{\Omega}_{S,a}(\lambda)$ is closed, since every orbit of a regular action of a nilpotent group on an arbitrary affine algebraic variety is closed ([17], 11.2.4). We shall find a system of generators of the defining ideal $I_{S,a}$ of the orbit $\overline{\Omega}_{S,a}(\lambda)$.

Notice that $\dim \overline{\Omega}_{S,a}(\lambda) = 2\text{codim } p_S = 2(r_+ - s)$. Respectively,

$$\text{codim } \overline{\Omega}_{S,a}(\lambda) = \dim g - \dim \overline{\Omega}_{S,a}(\lambda) = 2r_+ + r_0 - 2(r_+ - s) = r_0 + 2s.$$  

Notice that the number of roots in $S \sqcup L_S$ also equals to $r_0 + 2s$, since $|S| = |L_S^+| = |L_S^-| = s$, and $|L_0^S| = r_0 - s$. In what follows, we shall correspond some element of the symmetric algebra $S(\overline{g}) = \overline{\mathbb{F}}_q[\overline{g}]$ to any root $\gamma \in S \sqcup L_S$, and we shall prove that the constructed system of elements will generate the defining ideal $I_{S,a}$.

Denote by $X$ the upper triangular matrix with zeros on the diagonal and the following entries over the diagonal: any place $(i, j)$, where $1 \leq i < j \leq n$, is filled by the matrix unit $E_{ij}$. Every minor of the matrix $X$ is an element of the symmetric algebra $S(\overline{g})$, that is a polynomial on $\overline{g}^*$. Given a root $\gamma \in R$, we consider the system $S_\gamma$ that consists of the root $\gamma$ and also of all roots from $S \sqcup L_0^S$ lying strongly over and strongly on the right side from $\gamma$. Denote by $M_\gamma$ the minor of the matrix $X$ that has the systems of rows and columns just as $S_\gamma$ has.

Let $\gamma = (n - i, j) \in L_0^-$. Consider the characteristic matrix $X - \tau E$; cutting first $i$ columns and last $i$ rows, we obtain the minor $|X - \tau E|_i$. Then

$$|X - \tau E|_i = P_{\gamma,0}\tau^{n-2i} + P_{\gamma,1}\tau^{n-2i-1} + \ldots + P_{\gamma,n-2i}.$$  

Denote

$$F_\gamma = \begin{cases} M_\gamma, & \text{if } \gamma \in S \sqcup L_0^S \sqcup L_S^0 \sqcup L_0^{00}, \\ P_{\gamma,1}, & \text{if } \gamma \in L_0^- . \end{cases}$$
Denote by $F^0_\gamma$ a value of $F_\gamma$ at the point $\lambda_{S,a}$.

**Remark 3.3.** Notice that $F^0_\gamma = c\lambda_{S,a}(E_\gamma)$. Here $c$ equals to a product of values of $\lambda_{S,a}$ on some $E_\mu$, where $\mu < \lambda$ and $\mu$ belongs to $S \cup L^0_S$. Therefore, $c \neq 0$.

**Theorem 3.4.** The defining ideal $\mathcal{I}_{S,a}$ of the orbit $\Omega_{S,a}(\lambda)$ is generated by the algebraically independent system of polynomials

$$\{F_\gamma - F^0_\gamma : \gamma \in S \cup L_S\} \quad (5)$$

**Proof.** As above we order $R = \{\gamma_1 < \ldots < \gamma_N\}$, where $N = \frac{n(n-1)}{2}$, with respect to the ordering introduced above (before lemma 3.1). The ordering on the set of $R$ provides the ordering on the set of all matrix units

$$\{E_{i,j} : 1 \leq i < j \leq n\}.$$

The associative algebra $g$ has a chain of ideals

$$\mathfrak{g}_i = \langle E_{1,n} \rangle \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_N = g,$$

where $\mathfrak{g}_i$ is a span of all matrix units with numbers $\leq i$. Denote by $\mathcal{I}_i$ the ideal in $S(\mathfrak{g})$ generated by whose elements of (5) that has number $\leq i$. The last ideal $\mathcal{I}_N$ coincides with the ideal $\mathcal{I}$ generated by the system of generators (5).

It is not difficult to show that

$$F_{\gamma_i} = cE_{\gamma_i} + \Phi_{i-1} \bmod \mathcal{I}_{i-1}, \quad (6)$$

where $c$ is the constant as in remark above, $\Phi_{i-1}$ is some polynomial of $S(\mathfrak{g}_{i-1})$. Any $E_{\gamma_i}$ belongs to $\mathfrak{g}_i$, does not belong to $\mathfrak{g}_{i-1}$; using (6), we conclude that the system of generators (5) is algebraically independent, and the ideal $\mathcal{I}$ is prime. The number of generators of the set (5) equals to $r_0 + 2s$; this is equal to codimension of the orbit $\Omega_{S,a}(\lambda)$. Therefore, $\dim \text{Ann} \mathcal{I} = \dim \Omega_{S,a}(\lambda)$.

Obviously, the generators (5) annihilate at the point $\lambda_{S,a}$. To finish the proof, it is sufficient to show that the ideal $\mathcal{I}$ is invariant with respect to the adjoint representation of the group $G$. Using direct calculations, one can show that for any $1 \leq m \leq n - 1$ the element $(\text{ad} E_{m,m+1})F_{\gamma_i}$ belongs to the ideal $\mathcal{I}_{i-1}$. $\square$

**Theorem 3.5.** Linear forms $\lambda_{S,a}$ and $\lambda_{S',a'}$ lie in a common $\text{Ad}^*_G$-orbit if and only if they coincide.

**Proof.** It is obvious that, if $\lambda_{S,a} = \lambda_{S',a'}$, then they are lying in the common orbit. Let us prove the contrary.

Suppose that $\lambda_{S,a}$ and $\lambda_{S',a'}$ are lying in a common $\text{Ad}^*_G$-orbit. Then they are lying in a common $\text{Ad}^*$-orbit with respect to the group $\overline{G}$. The defining ideal
of the common orbit is generated by the system of polynomials \( \{ F_\gamma \} \). The values of every polynomial of \( \{ F_\gamma \} \) at the points \( \lambda_{S,a} \) and \( \lambda_{S',a'} \) coincide.

Suppose that \( S \neq S' \). Order the set \( \Pi \) in accordance with the number of row. Choose a number \( i \) such that

i) the subset of all roots of \( S \) with number of row \( < i \) coincides with the same subset of \( S' \);

ii) the root \( \gamma = (i, n - i) \) belongs to \( S \) and does not belong to \( S' \).

Since \( \gamma \in S \), there is some \( \gamma_* = (i, j_*) \in L_0^+ \) that lies strongly on the right side from \( \gamma \). In accordance with remark 3.3, a value \( F_{\gamma_*}^0 \) of the polynomial \( F_{\gamma_*} \) at the point \( \lambda_{S,a} \) equals to \( c \lambda_{S,a}(E_{\gamma_*}) \), where \( c \neq 0 \). Since \( \gamma_* \in L_0^+ \), we have \( \lambda_{S,a}(E_{\gamma_*}) = 0 \) and, therefore, \( F_{\gamma_*}^0 = 0 \).

On the other hand, since \( \gamma \notin S' \), we have \( \gamma_* \in L_0^{S'} \). Arguing similarly, we obtain \( F_{\gamma_*}^0 \neq 0 \). This leads to contradiction. Hence, \( S = S' \).

The equality \( S = S' \) implies \( L_0^S = L_0^{S'} \) and \( L_0^S = L_0^{S'} \). Calculating values \( F_{\gamma}^0 \) for \( \gamma \in L_0^S \), we conclude that \( a = a' \). \( \square \)

4 Decomposition into irreducible components

Theorem 4.1.

1) Every representation \( V_{S,a}(\lambda) \) can be realized as a subrepresentation of the representation \( V(\lambda) \).

2) Any representation of Gelfand-Graev type is decomposed into a sum of irreducible components

\[
V(\lambda) = \bigoplus V_{S,a}(\lambda),
\]

where \( S \subset \Pi \) and \( a \in \Lambda_S \). The multiplicity of every irreducible component of \( V(\lambda) \) is equal to one.

Proof. Let \( v \) be an eigenvector in \( V_{S,a}(\lambda) \) for \( G_+ = \{ 1 + x : x \in g_+ \} \); its eigenvalue has a form \( e^{\nu(x)} \), where \( \nu(x) \) is a character of the associative algebra \( g_+ \). Then \( \nu(x) \) is called an \( g_+ \)-weight, and corresponding eigenvector \( v \) is called an \( g_+ \)-weight vector.

To prove statement 1), it is sufficient to show that there exists nontrivial homomorphism \( V(\lambda) \rightarrow V_{S,a}(\lambda) \). This is equivalent to existence of nonzero \( g_+ \)-weight vector of weight \( \lambda \) in \( V_{S,a}(\lambda) \). The proof of this statement is similar to proof of proposition 1.4. Let \( f_0 \) be the generating vector of representation \( V_{S,a}(\lambda) \). For any root \( \alpha \in \Pi \setminus S \), there exists a unique root \( \gamma(\alpha) = (i, j_*) \in L_0^S \) that lies strongly on the right side from \( \alpha \). Then \( \gamma(\alpha) = \alpha + \beta(\alpha) \), where \( \beta = (n - i, j_*) \in R_- \). The subspace, spanned by the set \( E_{\beta(\alpha)}, \alpha \in \Pi \setminus S \), is a subalgebra with zero multiplication. Hence, for all \( \alpha, \alpha' \in \Pi_0 \setminus \Pi \), the
elements of one-parameter subgroups \( x_{\beta(\alpha)}(t) \) and \( x_{\beta(\alpha')}(t') \) commute. Recall that \( \lambda_{\alpha}(E_{\alpha}) = 0 \) and \( \lambda_{\alpha}(E_{\gamma(\alpha)}) = a(\gamma(\alpha)) \neq 0 \). Consider the element
\[
g_0 = \prod x_{\beta(\alpha)}(t_{\alpha}) \in G,
\]
where \( \alpha \) is running through \( \Pi \setminus S \), and \( t_{\alpha} \in \mathbb{F}_q \) is a solution of the equation
\[
\lambda_{\alpha}(E_{\gamma(\alpha)}) t_{\alpha} = \lambda(E_{\alpha}).
\]
The vector \( g_0 f_0 \) is a \( g+ \)-weighted vector with the weight \( \lambda \). This proves 1).

Taking into account theorems 3.2, 3.4 to prove the statement 2), it is sufficient to show that dimensions of representations of left and right hand sides of formula (7) coincide. Denote
\[
V_S(\lambda) = \bigoplus_{a \in \Lambda_S} V_{S,a}(\lambda).
\]
We have \( |\Lambda_S| = q^{r+s+\varepsilon} (q-1)^{k-s} \), \( \dim V_{S,a}(\lambda) = q^{r+s-\varepsilon} \), and
\[
\dim V_S(\lambda) = q^{r+s+\varepsilon} (q-1)^{k-s}.
\]
Then
\[
\dim \left( \bigoplus_{S \subset \Pi} V_S(\lambda) \right) = q^{r+s+\varepsilon} \sum_{s=0}^{k} C_s^k (q-1)^{k-s} = q^{r+s+k+\varepsilon} = q^{r+s+r_0} = \dim V(\lambda).
\]

\[\square\]

**Corollary 4.2.** The number of irreducible components in the representations of Gelfand-Graev type \( V(\lambda) \) for the group \( \text{UT}(n, \mathbb{F}_q) \) does not depend on choice of nondegenerate character \( \lambda \), and it equals to \( q^{\varepsilon}(2q-1)^k \).

**Proof.** The number of irreducible components of \( V(\lambda) \) is equal to
\[
\sum_{s=0}^{k} C_s^k |\Lambda_S| = \sum_{s=0}^{k} C_s^k q^{s+\varepsilon} (q-1)^{k-s} = q^\varepsilon \sum_{s=0}^{k} C_s^k q^s (q-1)^{k-s} = q^\varepsilon (2q-1)^k.
\]

5 The Hecke algebra

Let \( G \) be an arbitrary finite group, \( \mathcal{A}_G \) be its group algebra over \( \mathbb{C} \), \( H \) be a subgroup of \( G \), \( \xi \) be a character (one-dimensional representation) of the group \( H \). Denote by \( P_\xi \) the element \( \sum_{h \in H} \xi(h^{-1}) h \) in the group algebra \( \mathcal{A}_G \).

Let us realize the induced representation \( V = \text{ind}(\xi, H, G) \) in the space \( \mathcal{A}_G P_\xi \) by left multiplication. The Hecke algebra \( \mathcal{H}(V) \) (i.e. the the algebra of
all $G$-endomorphisms of $V$) is isomorphic to subalgebra $P_\zeta A_G P_\zeta$ with inverted multiplication. The algebra $P_\zeta A_G P_\zeta$ is spanned by the system $\{P_\zeta x P_\zeta : x \in G\}$. The following is well known (see \[4, \text{Lemma 84}\]).

1) The element $P_\zeta x P_\zeta$ is uniquely determined up to constant nonzero multiple determined by double class $H x H$. So the algebra $\mathcal{H}(V)$ is spanned by the elements $P_\zeta x P_\zeta$, where $x$ is running through a system of representatives of double classes $H x H$.

2) The elements $P_\zeta x P_\zeta$, where $x$ is running through a system of whose representations of double $(H, H)$ classes that satisfy $P_\zeta x P_\zeta \neq 0$, form a basis of $\mathcal{H}(V)$.

3) Define a character $x_\xi$ on $x H x^{-1}$ by the formula $x_\xi(y) = \xi(x^{-1} y x)$. The element $P_\zeta x P_\zeta$ is nonzero if and only if $\xi = x_\xi$ on $x H x^{-1} \cap H$. Summarizing 1)-3), we obtain the following proposition.

**Proposition 5.1.** Let $V = \text{ind}(\zeta, H, G)$. Then the system of elements $P_\zeta x P_\zeta$, where $x$ is running through a system of whose representatives of the double $(H, H)$ classes that satisfy $\xi = x_\xi$ on $x H x^{-1} \cap H$, form a basis of $\mathcal{H}(V)$.

Turn to representations of Gelfand-Graev type. We denote by $\mathcal{H}(\lambda)$ the Hecke algebra of the representation $V(\lambda)$. Since $V(\lambda)$ is multiplicity free, the Hecke is commutative. The dimension of Hecke algebra equals to the number of irreducible components of $V(\lambda)$, it is equal to $q^\varepsilon (2q - 1)^k$. Put $H = G_+$ and $\xi = \xi_\lambda$ (see formula (1.1)). Our goal is to construct a system of elements $\{x\}$ in $G$ such that $P_\zeta x P_\zeta$ is a basis in $\mathcal{H}(\lambda)$.

Given a subset $S \subset \Pi$, we define the subset $\Lambda'_S$ that consists of all vectors $(b_1, \ldots, b_{k+\varepsilon})$, where all $b_i \in \mathbb{F}_q$, and $b_i \neq 0$, if $(i, n - i) \in S$.

We construct the matrix $X_{S,b} = (x_{ij})$ as follows.

1) The matrix $X_{S,b}$ lies in $E + g_0 + g_-$, i.e. $x_{ii} = 1$ and $x_{ij} = 0$ for all $i > j$ and for whose pairs $i < j$ that obey $i + j < n + 1$.

2) First, we fill the last column of the matrix $X_{S,b}$. For any $1 \leq i \leq k + \varepsilon$ we put $x_{in} = b_i$. There is a unique root $(i, n - i) \in \Pi$ in each row $1 \leq i \leq k$. If $(i, n - i) \notin S$, then we put $x_{n-i,n} = 0$. If $(i, n - i) \in S$, then we put $x_{n-i,n} = b_i$. So $x_{i,n} = x_{n-i,n}$, if $1 \leq i \leq k$ and $(i, n - i) \in S$.

3) Now, we fill the other columns. If $(i, n - i) \in S$, then we put $x_{i+1,n-i} = \ldots = x_{n-i-1,n-i} = 0$. If $(i, n - s) \notin S$, then we put

$$
\begin{pmatrix}
  x_{i+1,n-i} \\
  \vdots \\
  x_{n-i-1,n-i}
\end{pmatrix}
= x_{i,n}
\begin{pmatrix}
  x_{i+1,n-i} \\
  \vdots \\
  x_{n-i-1,n-i}
\end{pmatrix}.
$$
Theorem 5.2. The system of elements
\[
\{P_\xi X_{S,b} P_\xi : S \subset \Pi, b \in \Lambda'_S\}
\]
is a basis of \(\mathcal{H}(V)\).

Proof. By direct calculations, we show that any \(x \in \{X_{S,b}\}\) obeys \(\xi = x\xi\) on \(xHx^{-1} \cap H\), where \(H = G_+\) and \(\xi = \xi_\lambda\). After, we prove that elements of \(\{X_{S,b}\}\) lie in different double classes. The number of elements of (10) is equal to
\[
\sum_{s=0}^{k} C^s_k (q - 1)^s q^{k+\varepsilon} = q^\varepsilon (2q - 1)^k,
\]
i.e. it is equal to \(\dim \mathcal{H}(V)\). By proposition 5.1, this concludes the proof. \(\Box\)

6 Calculations for small \(n\)

In all examples below, using the Killing form, we identify \(g^*\) with the space of lower triangular matrices with zeros on the diagonal.

Example 6.1. Case \(n = 3\). In this case,
\[
\begin{align*}
g &= \left\{ \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\}, & R_+ &= \{(1, 2)\}, & R_0 &= \{(1, 3)\}, & R_- &= \{(2, 3)\}, \\
g_+ &= \left\{ \begin{pmatrix} 0 & x_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, & g_0 &= \left\{ \begin{pmatrix} 0 & 0 & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, & g_- &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_{23} \end{pmatrix} \right\},
\end{align*}
\]
The set \(\Pi\) coincides with \(R_+ = \{(1, 2)\}\). Nondegenerate character \(\lambda\) on \(g_+\) is defined by one number \(\lambda(E_{12}) = c \neq 0\). There exist only two subsets in \(\Pi\): the empty set and \(\Pi\). Recall that, in the odd case, \(L_S^{00} = \emptyset\).

i) \(S = \emptyset\). Then \(L_+^S = L_-^S = \emptyset, L_0^S = \{(1, 3)\}, \quad \Lambda_S \cong F_q^* = \{a \in F_q : a \neq 0\}, \quad \lambda_{S,a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)
The coadjoint orbit \(\Omega_{S,a}(\lambda)\) is defined in
\[
g^* = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{31} & y_{32} & 0 \end{pmatrix} \right\} \quad \text{by equation } y_{31} = a.
\]
ii) $S = \{(1, 2)\}$. Then $L_S^+ = \{(1, 3)\}$, $L_S^- = \{(2, 3)\}$, $L_S^0 = \emptyset$, $\Lambda_S \cong \mathbb{F}_q = \{a \in \mathbb{F}_q\}$.

$$p_S = \begin{cases} \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix} \end{cases}, \quad \lambda_{S,a} = \begin{pmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ 0 & a & 0 \end{pmatrix}$$

The coadjoint orbit $\Omega_{S,a}(\lambda)$ is defined in $\mathfrak{g}^*$ by the equations $y_{31} = 0$, $y_{21} = c$, $y_{32} = a$. Any representation of Gelfand-Graev type $V(\lambda)$ decomposes into a sum of irreducible representations that correspond to mentioned above orbits. The number of irreducible components equals to $2q - 1$.

The matrices $X_{S,b}$ from theorem 5.2 has the form

$$\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } b \in \mathbb{F}_q^*, \quad \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } a \in \mathbb{F}_q.$$

**Example 6.2.** Case $n = 4$. In this case, $R_+ = \{(1, 2), (1, 3)\}, R_0 = \{(2, 3), (1, 4)\}, R_- = \{(2, 4), (3, 4)\}$.

$$\mathfrak{g} = \begin{cases} \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}, \quad \mathfrak{g}_+ = \begin{cases} \begin{pmatrix} 0 & x_{12} & x_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases},$$

$$\mathfrak{g}_0 = \begin{cases} \begin{pmatrix} 0 & 0 & 0 & x_{14} \\ 0 & 0 & x_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}, \quad \mathfrak{g}_- = \begin{cases} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}.$$
The coadjoint orbit \( \Omega_{S,a}(\lambda) \) if defined in 

\[
g^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 \\ y_{41} & y_{42} & y_{43} & 0 \end{pmatrix} \right\} \text{ by the equations } y_{41} = a_1, \quad \begin{vmatrix} y_{31} & y_{32} \\ y_{41} & y_{42} \end{vmatrix} = -a_1 a_2.
\]

ii) \( S = \{(1,3)\} \). Then \( L^+_S = \{(1,4)\}, \quad L^-_S = \{(3,4)\}, \quad L^0_S = \emptyset, \quad L^{00}_S = \{(2,4)\}, \quad \Lambda_S \cong \mathbb{F}_q^2 = \{(a_1, a_2) \in \mathbb{F}_q^2\}, \)

\[
p_S = \left\{ \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \quad \lambda_{S,a} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & a_1 & a_2 & 0 \end{pmatrix}.
\]

The coadjoint orbit \( \Omega_{S,a}(\lambda) \) is defined in \( g^* \) by the equations \( y_{41} = 0, \quad y_{42} = a_1, \quad y_{31} = c, \quad y_{42} y_{21} + y_{13} y_{31} = a_2 c. \)

The coadjoint orbit \( \Omega_{S,a}(\lambda) \) is defined in \( g^* \) by the equations \( y_{31} = 0, \quad y_{21} = c, \quad y_{32} = a. \) Any representation of Gelfand-Graev type \( V(\lambda) \) decomposes into a sum of irreducible representations that correspond to mentioned above orbits. The number of irreducible components equals to \( q(2q - 1) \).

The matrices \( X_{S,b} \) from theorem 5.2 has the form

\[
\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where } b \in \mathbb{F}_q^*, \quad a \in \mathbb{F}_q, \quad \begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & ba & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where } a, b \in \mathbb{F}_q.
\]

**Example 6.3.** Case \( n = 5 \). In that case \( R_+ = \{(1,2), (1,3), (1,4), (2,3)\}, \quad R_0 = \{(2,4), (1,5)\}, \quad R^- = \{(2,4), (3,4), (3,5), (4,5)\}, \)

\[
g = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & x_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad g_+ = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & 0 \\ 0 & 0 & x_{13} & x_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
g_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & x_{15} \\ 0 & 0 & 0 & x_{24} & 0 \\ 0 & 0 & 0 & 0 & x_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad g_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

The set \( \Pi \) coincides with \( \{(2,3), (1,4)\} \). A nondegenerate character \( \lambda \) on \( g_+ \) is defined by pair of numbers \( \lambda(E_{14}) = c_1 \neq 0, \lambda(E_{23}) = c_2 \) (the second number may by arbitrary). There are four subsets in \( \Pi \).
i) \( S = \emptyset \). Then \( L^+_S = L^-_S = \emptyset, \ L^0_S = \{(1, 5), (2, 4)\}, \ \Lambda_S \cong F_q^2 = \{(a_1, a_2) \in F_q^2 : a_1, a_2 \neq 0\}, \)

\[
p_S = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \lambda_{S,a} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ a_1 & 0 & 0 & 0 \end{pmatrix}.
\]

The coadjoint orbit \( \Omega_{S,a}(\lambda) \) is defined in

\[
g^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 \\ y_{41} & y_{42} & y_{43} & 0 \\ y_{51} & y_{52} & y_{53} & y_{54} \end{pmatrix}
\]

by the equations \( y_{51} = a_1, \ \begin{vmatrix} y_{41} & y_{42} \\ y_{51} & y_{52} \end{vmatrix} = -a_1 a_2 \).

ii) \( S = \{(1, 4)\}. \ Then \( L^+_S = \{(1, 5)\}, \ L^-_S = \{(4, 5)\}, \ L^0_S = \{(2, 5)\}, \ \Lambda_S \cong F_q^* \times F_q = \{(a_1, a_2) \in F_q^2 : a_1 \neq 0\}, \)

\[
p_S = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{45} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \lambda_{S,a} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 \end{pmatrix}.
\]

The coadjoint orbit \( \Omega_{S,a}(\lambda) \) is defined in \( g^* \) by the equations \( y_{51} = 0, \ y_{52} = a_1, \ y_{41} = c, \ y_{52} y_{21} + y_{53} y_{31} + y_{54} y_{41} = a_2 c_1 \).

iii) \( S = \{(2, 3)\}. \ Then \( L^+_S = \{(2, 4)\}, \ L^-_S = \{(3, 4)\}, \ L^0_S = \{(1, 5)\}, \ \Lambda_S \cong F_q^* \times F_q = \{(a_1, a_2) \in F_q^2 : a_1 \neq 0\}, \)

\[
p_S = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{45} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \lambda_{S,a} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_2 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 \end{pmatrix}.
\]

The coadjoint orbit \( \Omega_{S,a}(\lambda) \) is defined in \( g^* \) by the equations

\[
 y_{51} = a_1, \ \begin{vmatrix} y_{41} & y_{42} \\ y_{51} & y_{52} \end{vmatrix} = 0, \ \begin{vmatrix} y_{31} & y_{32} \\ y_{51} & y_{52} \end{vmatrix} = -a_1 c_2, \ \begin{vmatrix} y_{41} & y_{43} \\ y_{51} & y_{53} \end{vmatrix} = -a_1 a_2.
\]

iv) \( S = \{(1, 4), (2, 3)\}. \ Then \( L^+_S = \{(1, 5), (2, 5)\}, \ L^-_S = \{(3, 5), (4, 5)\}, \ L^0_S = \emptyset, \ \Lambda_S \cong F_q^2 = \{(a_1, a_2) : a_1, a_2 \in F_q\}, \)

\[
p_S = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{45} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \lambda_{S,a} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & 0 \end{pmatrix}.
\]

The coadjoint orbit \( \Omega_{S,a}(\lambda) \) is defined in \( g^* \) by the equations \( y_{51} = y_{52} = 0, \ y_{53} = a_1, \ y_{41} = c_1, \ y_{53} y_{31} + y_{54} y_{41} = a_2 c_1, \ \begin{vmatrix} y_{31} & y_{32} \\ y_{41} & y_{42} \end{vmatrix} = -c_1 c_2.\)
The coadjoint orbit $\Omega_{S,a}(\lambda)$ is defined in $g^*$ by the equations $y_{31} = 0$, $y_{21} = c$, $y_{32} = a$. Any representation of Gelfand-Graev type $V(\lambda)$ decomposes into a sum of irreducible representations that correspond to mentioned above orbits. The number of irreducible components equals to $(2q - 1)^2$. The matrices $X_{S,b}$ from theorem 5.2 has the form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & b_1 \\
0 & 1 & 0 & 0 & b_2 \\
0 & 0 & 1 & 0 & b_2 \\
0 & 0 & 0 & 1 & b_1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 & a_1 \\
0 & 1 & 0 & 0 & a_1b_1 \\
0 & 0 & 1 & 0 & b_1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 & a_1 \\
0 & 1 & 0 & 0 & a_1a_2 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 & a_1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 & a_1 \\
0 & 1 & 0 & 0 & a_2 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

where $b_1, b_2 \in \mathbb{F}_q^*$, $a_1, a_2 \in \mathbb{F}_q$.

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