Existence and Uniqueness Theorems for Generalized Set Differential Equations

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Abstract In this paper the concept of generalized differentiability for set-valued mappings proposed by A.V. Plotnikov, N.V. Skripnik is used. The generalized set-valued differential equations with generalized derivative are considered and the existence and uniqueness theorems are proved.

Keywords set-valued mapping, generalized derivative, existence and uniqueness theorems

1. Introduction

The concept of derivative for set-valued mapping was first entered by M. Hukuhara[1]. Then the problems of differentiability of fuzzy mappings were considered by T.F. Bridgland[2], J.N. Tyurin[3], H.T. Banks and M.Q. Jacobs[4], A.V. Plotnikov[5, 6], A.N. Vityuk[7], B. Bede and S.G. Gal[8], A.V. Plotnikov and N.V. Skripnik[9]. The properties of these derivatives were considered in[10-18].

F.S. de Blasi and F. Iervolino begun studying of set-valued differential equations (SDEs) in semilinear metric spaces[12,19-21]. Now it developed in the theory of SDEs as an independent discipline. The properties of solutions, the impulsive SDEs, control systems and asymptotic methods for SDEs were considered[5,6,9-11,16-24]. On the other hand, SDEs are useful in other areas of mathematics. For example, SDEs are used as an auxiliary tool to prove the existence results for differential inclusions. Also, one can employ SDEs in the investigation of fuzzy differential equations. Moreover, SDEs are a natural generalization of usual ordinary differential equations in finite (or infinite) dimensional Banach spaces[19].

In[9] a new concept of a derivative of a set-valued mapping that generalizes the concept of Hukuhara derivative was entered and a new type of a set-valued differential equation such that the diameter of its solution can whether increase or decrease (for example, to be periodic) was considered. In the ideological sense this definition of the derivative is close to the definitions proposed in[5,6,8].

In this paper the generalized set-valued differential equations with generalized derivative are considered and the existence and uniqueness theorems are proved.

2. The Generalized Derivative

Let $\text{conv}(\mathbb{R}^n)$ be a space of all nonempty convex closed sets of $\mathbb{R}^n$ with Hausdorff metric

$$h(A,B) = \min \{ r \geq 0 : A \subset B + S_r(0), B \subset A + S_r(0) \},$$

where $A,B \in \text{conv}(\mathbb{R}^n)$, $S_r(0) = \{ s \in \mathbb{R}^n : \| s \| \leq r \}$.

**Definition 1[1].** Let $X,Y \in \text{conv}(\mathbb{R}^n)$. A set $Z \in \text{conv}(\mathbb{R}^n)$ such that $X = Y + Z$ is called a Hukuhara difference of the sets $X$ and $Y$ and is denoted by $Y_{h}Z$.

From Rådström's Embedding Lemma[25] it follows that if this difference exists, then it is unique.

Let $I = [t^*,t] \subset \mathbb{R}$; $X : I \rightarrow \text{conv}(\mathbb{R}^n)$ be a set-valued mapping; $(t_0 - \Delta, t_0 + \Delta) \subset I$ be a $\Delta$-neighbourhood of a point $t_0 \in I$; $\Delta > 0$.

For any $t \in (t_0 - \Delta, t_0 + \Delta)$ consider the following Hukuhara differences if these differences exist.

$$X(t) = \frac{h}{h}X(t_0), \quad t \geq t_0 \quad (1)$$

$$X(t_0) = \frac{h}{h}X(t), \quad t \geq t_0 \quad (2)$$

$$X(t_0) = \frac{h}{h}X(t), \quad t \leq t_0 \quad (3)$$

$$X(t_0) = \frac{h}{h}X(t_0), \quad t \leq t_0 \quad (4)$$

The differences (1) and (2)[(3) and (4)] are called the right[left] differences. From the definition of the Hukuhara difference it follows that both one-sided differences exist only in the case when $X(t) = F + \{ f(t) \}$ for $t \in [t_0 + \Delta]$ or $t \in [t_0 - \Delta, t_0]$.

If all differences (1)-(4) exist then
In the theory of differential equations, there exists only one of the possible solutions if we consider the Hukuhara difference. Let us assume that there exist only two limits for a set-valued mapping defined on an interval. Then, for each of the above mentioned cases only one of the possible solutions exists.

### Definition 2

The set-valued mapping $X: I \to \text{conv}(R^n)$ is differentiable in the generalized sense at every point of the interval if it is differentiable in the generalized sense at every point of this interval.

### Definition 3

The set-valued mapping $X: I \to \text{conv}(R^n)$ is differentiable in the generalized sense almost everywhere on the interval if it is differentiable in the generalized sense almost everywhere on the interval.

### Theorem 1

Let a set-valued mapping $X: [t_0, T] \to \text{conv}(R^n)$ be such that for all $t \in [t_0, T]$, $X(t) = X(t_0) + \int_{t_0}^{t} G(s)ds$ or $X(t) = X(t_0) + \int_{t_0}^{t} G(s)ds$.

### 3. Generalized Differential Equations with the Generalized Derivative

First, consider a differential equation with the generalized derivative that is similar to a differential equation with the Hukuhara derivative, i.e.

$$ DX = F(t, X), \quad X(t_0) = X_0. \quad (9) $$

where $DX(t)$ is the generalized derivative of a set-valued mapping $X: [t_0, T] \to \text{conv}(R^n)$, $F: [t_0, T] \times \text{conv}(R^n) \to \text{conv}(R^n)$ is a set-valued mapping.

**Definition 4.** A set-valued mapping $X: [t_0, T] \to \text{conv}(R^n)$ is said to be solution of differential equation (9) if it is absolutely continuous and satisfies (9) almost everywhere on $[t_0, T]$.

**Remark 1.** Unlike the case of differential equations with Hukuhara derivative, if a differential equation with the generalized derivative (9) has a solution then there exists an infinite number of solutions irrespective of the conditions on the right-hand side of the equation.

**Example 1.** Consider the following differential equation with the generalized derivative

$$ DX = [-1,1], \quad X(0) = [-2,2]. \quad (10) $$

It is easy to check that the following set-valued mappings are the solutions of equation (10):

- $X_1(t) = [-2-t,2+t], \quad t \in [0,1]$
- $X_2(t) = [-2+t,2-t], \quad t \in [0,1]$
- $X_3(t) = \begin{cases} [-2-t,2+t], & t \in [0,0.5] \\ [-2.5+t,2.5-t], & t \in [0.5,1] \\ [-2+t,2-t], & t \in [0,0.25] \\ [-1.5-t,1.5+t], & t \in [0.25,0.5] \\ [-2.5+t,2.5-t], & t \in [0.5,1] \end{cases}$

Also it is possible to construct other solutions, thus only $X_1(t)$ will be the solution of the corresponding differential equation with the Hukuhara derivative

$$ DX = [-1,1], \quad X(0) = [-2,2] $$

and $X_1(t)$ and $X_2(t)$ are solutions of the differential equation with the generalized derivative (in the sense of [8]).

Therefore we will consider the other differential equation with the generalized derivative:

$$ DX(t) \overset{h}{\Phi} (\Phi(t))F(t, X(t)) = \Phi(\Phi(t))F_2(t, X(t)). \quad (11) $$

where $X(t_0) = X_0$,

- $\Phi(t) \in [1, \phi > 0], \quad 0, \quad \phi \leq 0$

**Definition 5.** A set-valued mapping $X: [t_0, T] \to \text{conv}(R^n)$ is called the solution of differential equation (11) if it is continuous and on any subinterval $[\tau_1, \tau_2] \subset [t_0, T]$, where function $\Phi(t)$ of constant signs, satisfies the integral equation

$$ X(t) + \int_{\tau_1}^{\tau_2} \Phi(\phi(s))F_1(s, X(s))ds = X(\tau_1) + \int_{\tau_1}^{\tau_2} \Phi(\phi(s))F_2(s, X(s))ds. $$

If on the interval $[\tau_1, \tau_2]$ the function $\phi(t) > 0$, then $X(t)$
satisfies the integral equation
\[ X(t) = X(t_{1}) + \int_{t_{1}}^{t} F_{2}(s, X(s))\,ds \]
for \( t \in [t_{1}, t_{2}] \) and \( \text{diam}X(t) \) increases.

If on the interval \([t_{1}, t_{2}]\) the function \( \phi(t) < 0 \), then we have
\[ X(t) = X(t_{1}) + \int_{t_{1}}^{t} F_{2}(s, X(s))\,ds = X(t_{2}), \]
i.e. \( X(t) = X(t_{1}) + \int_{t_{1}}^{t} F_{2}(s, X(s))\,ds \) and \( \text{diam}X(t) \) decreases.

If on the interval \([t_{1}, t_{2}]\) the function \( \phi(t) = 0 \), then we have \( X(t) = X(t_{1}) \).
So we can enter the other equivalent definition of a solution of equation (11).

**Definition 6.** A set-valued mapping \( X : [t_{1}, T] \rightarrow \text{conv}(R^{n}) \) is called the solution of differential equation (11) if it is absolutely continuous, satisfies (11) almost everywhere on \([t_{1}, T]\) and
\[ \text{diam}X(t) = \begin{cases}
  \text{increases if } \phi(t) > 0, \\
  \text{is constant if } \phi(t) = 0, \\
  \text{decreases if } \phi(t) < 0.
\end{cases} \]

**Example 2.** Consider the following differential equation with generalized derivative
\[ DX = \Phi(-\sin t)[-2, 4] = \Phi(\sin t)[1, 3], \quad X(0) = [2, 4]. \quad (12) \]
As \( \sin t > 0 \) for \( t \in (0, \pi) \) we have
\[ X(t) = [2, 4] + \int_{0}^{t} [1, 3]ds = [2, 4] + [t, 3t] = [2 + t, 4 + 3t] \]
for \( t \in [0, \pi] \).
So for \( t = \pi \) we get \( X(\pi) = [2 + \pi, 4 + 3\pi] \).
Further as \( \sin t < 0 \) for \( t \in (\pi, 2\pi) \) we have
\[ X(t) = [2 + \pi, 4 + 3\pi] + \int_{\pi}^{t} [-2, 4]ds = [2 + \pi, 4 + 3\pi] + \int_{\pi}^{t} [-2, 4]ds = [2 + \pi + 2t, 4 + 7\pi - 4t]. \]
So for \( t = \frac{1 + 4\pi}{3} < 2\pi \) we get \( X\left(\frac{1 + 4\pi}{3}\right) = \left[\frac{8 + 5\pi}{3}\right] \).

**Figure 1.** The graph of a solution of system (12)

It means that the solution exists only for \( t \in \left[0, 1 + \frac{4\pi}{3}\right] \) (see fig. 1).

**Example 3.** Consider the same differential equation with generalized derivative but with \( \phi(t) = -\sin(10t) \):
\[ DX = \Phi(\sin(10t))[-2, 4] = \Phi(-\sin(10t))[1, 3], \quad X(0) = [2, 4] \quad (13) \]
As \( -\sin 10t < 0 \) for \( t \in \left(0, \frac{\pi}{10}\right) \) we have
\[ X(t) = [2, 4] + \int_{0}^{t} [-2, 4] = [2 + 2t, 4 - 4t] \quad \text{for } t \in \left(0, \frac{\pi}{10}\right). \]
Further as \( -\sin 10t > 0 \) for \( t \in \left(\frac{\pi}{10}, \frac{3\pi}{5}\right) \) we get
\[ X(t) = \left[2 + \frac{\pi}{10}, 4 - \frac{2\pi}{5}\right] + \int_{\pi/10}^{t} [1, 3]ds = \left[2 + t, 4 - \frac{7\pi}{10} + 3t\right]. \]
Further as \( -\sin 10t < 0 \) for \( t \in \left(\frac{3\pi}{5}, \pi\right) \) we get
\[ X(t) = \left[2 + \frac{3\pi}{5}, 4 - \frac{3\pi}{10}\right] + \int_{3\pi/5}^{t} [-2, 4]ds = \left[2 - \frac{3\pi}{5} + 2t, 4 + \frac{7\pi}{10} - 4t\right]. \]
So for \( t = \frac{1}{3} + \frac{3\pi}{20} > \frac{3\pi}{10} \) we have \( X(\frac{1}{3} + \frac{3\pi}{20}) = \left[\frac{8}{3}, \frac{\pi}{10}\right]. \)

**Figure 2.** The graph of a solution of system (13)

It means that the solution exists only for \( t \in \left[0, 1 + \frac{3\pi}{20}\right] \) (see fig. 2).

**Remark 3.** It is obvious that the mappings \( F_{1}(t, X) \), \( F_{2}(t, X) \) define only on "how much" the mapping \( X(t) \) changes in case of its "decrease"(\( F_{1}(t, X) \)) or "increase"(\( F_{2}(t, X) \)) and function \( \phi(t) \) defines what will occur to \( X(t) \) ["decrease" or "increase"]. If \( \phi(t) = 0 \) irrespective of \( F_{1}(t, X(t)) \) and \( F_{2}(t, X(t)) \) the mapping \( X(t) \) will be constant.

**Example 4.** Consider the differential equation from Example 2 with \( \phi(t) = 0 \) for \( t \in [0, 2\pi] \). Then \( X(t) = [2, 4] \) for \( t \in [0, 2\pi] \).

**Remark 4.** If we take \( \phi(t) = -1 \) then we will have
\[ X(t) = [2, 4] + \int_{0}^{t} [-2, 4]ds = [2 + 2t, 4 - 4t]. \]
Then for \( t = \frac{1}{3} \) we get \( X\left(\frac{1}{3}\right) = \left[\frac{8}{3}\right] \). So the solution exists
we can guarantee the existence of solution for 
, where .

Then equation (11) is the ordinary differential equation with Hukuhara derivative 

for all 
and is not embedded for .

Therefore, using[17] we get that the equation (11) has a solution defined on , where satisfies the condition 

2) 

for . Then equation (11) is the ordinary differential equation with Hukuhara derivative 

for all .

According to Definition 5 consider the following integral equation

for and prove the existence of solution on the some interval .

As , then there exists such that the set is embedded in the set for all .

As , then is not embedded for . And, it is obviously, that can be found out from the equation

Therefore, for all the set is embedded in the set .

3b) As for all , then

Therefore, as

3c) Let us find such that and consider

3) Choose any natural . Sequentially on the intervals

Figure 3. The graph of a solution of system (12) for , where .

So for all we can guarantee the existence of solution of the differential equation on the interval .

Let be a space of all nonempty strictly convex closed sets of and all element of holds:

Theorem 2. Let the set-valued mappings , in the domain

where , and are summable on where is continuous and has the finite number of intervals where

Then there exists a solution of equation (11) defined on the interval , where satisfies the conditions

1) 

where

Proof. Let us consider some cases.

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3) Choose any natural . Sequentially on the intervals
Let us build the successive approximations of the solution

\[X^k(t) = X_0 \quad \text{for} \quad t_0 - \Delta \leq t \leq t_0, \]

\[X^k(t) = X_0 \int_{t_0}^{t} \frac{h}{a} F(s, X^k(s - \Delta))ds \quad \text{for} \quad t \in [t_0, t_0 + d] \]

(17)

By 3) \(X^k(t)\) is exist and \(X^k(t) \in C^p(R^a)\) for all \(k \in N\) and \(t \in [t_0, t_0 + d]\). Also by conditions i) and ii) of the theorem \(X^k(t)\) is continuous on \([t_0, t_0 + d]\) for all \(k \in N\). Besides \(h(X^k(t), X_0) = h\left(X_0, \int_{t_0}^{t} \frac{h}{a} F(s, X^k(s - \Delta))ds, X_0\right) \leq h\left(0, \int_{t_0}^{t} F(s, X^k(s - \Delta))ds, X_0\right) \leq \int_{t_0}^{a} h\left(F(s, X^k(s - \Delta)), 0\right)ds \leq \int_{t_0}^{a} m(s)ds \leq \phi(t_0 + d) + b.

Hence, it follows that the sequence of the set-valued mappings \(\{X^k(t)\}\) is uniformly bounded:

\[h(X^k(t), 0) \leq h(X_0, 0) + b.\]

Let us show that the set-valued mappings \(X^k(t)\) are equi-continuous. For any \(\alpha < \beta\) \(\alpha, \beta \in [t_0, t_0 + d]\) and any natural \(k\) the inequality holds

\[h(X^k(\alpha), X^k(\beta)) = h\left(X_0, \int_{t_0}^{a} F(s, X^k(s - \Delta))ds, X_0\right) \leq \int_{t_0}^{a} h\left(F(s, X^k(s - \Delta)), 0\right)ds \leq \int_{t_0}^{a} m(s)ds \leq \phi(\beta) - \phi(\alpha).

The function \(\phi(t)\) is absolutely continuous on \([t_0, t_0 + d]\) as the integral of the summable function with a variable top limit. Hence, for any \(\varepsilon > 0\) there exists \(\delta(\varepsilon) > 0\) such that for all \(\alpha, \beta\) such that \(0 < \beta - \alpha < \delta\) the inequality \(h(X^k(\alpha), X^k(\beta)) < \varepsilon\) is fair, the sequence \(\{X^k(t)\}\) is equi-continuous.

According to ASKOLI theorem[28] we can choose a uniformly converging subsequence of the sequence \(\{X^k(t)\}\).

Its limit is a continuous set-valued mapping that we will denote by \(X(t)\). As and the first summand is less than \(\varepsilon\) for \(\Delta \leq \frac{d}{k} < \delta\) in view of the equi-continuity of the set-valued mappings \(\{X^k(t)\}\), then along the chosen subsequence \(\{X^k(s - \Delta)\}\) converges to \(X(t)\). Owing to the theorem conditions in (15) it is possible to pass to the limit under the sign of the integral. We receive that the set-valued mapping \(X(t)\) satisfies equation (16) and \(X(t_0) = X_0\), i.e. \(X(t)\) is the solution of (15) on the interval \([t_0, t_0 + d]\).

In case when the function \(\phi(t)\) changes sign on the interval \([t_0, t_0 + a]\), the existence of the solution is proved combining cases 1)-3). The theorem is proved.

**Theorem 3.** Let the set-valued mappings \(F(t, X), F_2(t, X) : R \times C^p(R^a) \rightarrow C^p(R^a)\) in the domain

\[Q = \{(t, X) \in R \times C^p(R^a) : t \in [t_0, t_0 + a], h(X, X_0) \leq b\}

satisfy the conditions of Theorem 2 and satisfy the conditions

\[h(F_1(t, X), F_1(t, X)) \leq L_1 h(X', X'),
\[h(F_2(t, X), F_2(t, X)) \leq L_2 h(X', X')

for all \((t, X), (t, X) \in Q\).

Then there exists the unique solution of equation (11) defined on the interval \(t \in [t_0, t_0 + d]\).

The proof is similar to[17,24].

Finally we consider example for case \(C^p(R^a)\).

**Example 6.** Consider the following differential equation with generalized derivative

\[DX^h = \Phi(t - 1, X) = \Phi(t - 1, X) = \frac{1}{2} X, X(0) = S_1(0), (18)\]

where \(X : R \rightarrow C^p(R^a)\).

It is obvious that \(X(t) = \begin{cases} S_1(0), & t \leq 1, \\ S_{1+e}(0), & t > 1 \end{cases}\) is the solution of differential equation (18) (see fig. 4).

**Remark 5.** Also it is possible to prove the similar results if \(CC(R^a)\) be a space of all nonempty \(M\)- strongly convex closed sets of \(R^a\) and all element of \(R^a\) [29].

**Remark 6.** Let’s notice that we considered some continuous function \(\phi(t)\) but it is also possible to take \(\phi(t, X) = \text{diam}(X(t)) - c(t)\), for example, where \(c(t)\) - is the diameter of some etalon set-valued mapping.

## 4. Conclusions

In this paper the concept of generalized differentiability (proposed in[9]) for set-valued mappings is used. The new type of the set-valued differential equation – generalized set differential equations – is considered. The existence and uniqueness theorems for set-valued differential equations
with generalized derivative are proved.

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