

Wigner distribution functions for complex dynamical systems: a path integral approach

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Starting from Feynman’s Lagrangian description of quantum mechanics, we propose a method to construct explicitly the propagator for the Wigner distribution function of a single system. For general quadratic Lagrangians, only the classical phase space trajectory is found to contribute to the propagator. Inspired by Feynman’s and Vernon’s influence functional theory we extend the method to calculate the propagator for the reduced Wigner function of a system of interest coupled to an external system. Explicit expressions are obtained when the external system consists of a set of independent harmonic oscillators. As an example we calculate the propagator for the reduced Wigner function associated with the Caldeira-Legett model.

I. INTRODUCTION

To explain the dynamic behavior of a quantum mechanical system of interest one generally needs to extract the time dependence of the density matrix. Then, an arbitrary observable represented by a Hermitian operator can be obtained by taking the trace of the operator times the density matrix.

In a classical description, however, the density matrix emerges as a continuous, time dependent phase space distribution function whereas physical observables are identified with ordinary functions. Multiplying the latter with the phase space distribution function and integrating over phase space yields the commonly known recipe to calculate all relevant expectation values. Both approaches can be linked if one addresses the Wigner function[1] which is nothing but the Wigner–Weyl transform of the quantum mechanical density matrix. Inheriting from the density matrix all necessary information to calculate any observables of interest, the Wigner satisfies the Wigner-Liouville equation, the quantum mechanical analog of the classical Liouville equation.

Alternative to the Hamiltonian description of quantum mechanics one may adopt Feynman’s Lagrangian formulation of quantum mechanics[2]. This approach essentially relies on the calculation of the Green’s function or propagator appearing as a path integral, rather than on solving the dynamical equations equation for the density matrix. Moreover, also Feynman’s approach exhibits a direct link to classical mechanics through the explicit occurrence of the classical action in the path integral.

In this work, adopting exclusively Feynman’s Lagrangian formulation, we construct the path integral for the propagator of the single particle Wigner function in section II. In section III we extend the obtained result by computing also the path integral for the propagator of the reduced Wigner function of a system coupled to an external quantum system. This extension may be seen as the Wigner-Weyl formulation of the theory of influence functionals developed by Feynman and Vernon[3]. In section IV we derive the explicit Wigner influence functional for a particle coupled to a set of independent harmonic oscillators. Finally, as a non trivial example, we calculate the Wigner function propagator of a particle described by the Caldeira-Legett model[4].

II. SINGLE-PARTICLE PHASE SPACE PROPAGATOR

The well-known Wigner distribution function[1] provides a phase space description of quantum mechanics, and its dynamics is usually formulated as an inhomogeneous partial integro-differential equation. In earlier work[5,6] we investigated the relation between this equation of motion and classical phase space trajectories. However, inspired by Feynman’s Lagrangian description of quantum mechanics[2], in this paper we have explicitly constructed a propagator (i.e., a Green’s function) for the phase space distribution.

Consider first a single-particle system, for simplicity in the notations in one dimension, whose coordinates are denoted by \(x\). The quantum-mechanical amplitude for the system to go from position \(x_a\) at time \(t = t_a\) to position \(x_b\) at time...
It follows, by comparison with Eq. (II.5), that the density matrix at time \( t = t_b \) is given by
\[
K (x_b, t_b | x_a, t_a) = \int_{x(t_a) = x_a}^{x(t_b) = x_b} Dx(t) \exp \left( \frac{i}{\hbar} S [x(t)] \right),
\]
(II.1)

where \( S [x(t)] \) is the action of the system for a trajectory \( x(t) \). Let \( \{ |n \rangle \} \) denote a complete orthonormal set of states. The amplitude \( A(n_b, t_b | n_a, t_a) \) for being in state \( |n_b \rangle \) at time \( t_b \), if initially in state \( |n_a \rangle \) at time \( t_a \), is then given by
\[
A(n_b, t_b | n_a, t_a) = \int \langle n_b | x_b \rangle K (x_b, t_b | x_a, t_a) \langle x_a | n_a \rangle \text{d}x_a \text{d}x_b.
\]
(II.2)

The corresponding transition probability \( P(n_b, t_b | n_a, t_a) = |A(n_b, t_b | n_a, t_a)|^2 \) can thus be written as
\[
P(n_b, t_b | n_a, t_a) = \int \int \int \int \langle n_b' | x_b \rangle \langle n_b | x_b \rangle K^* (x_b', t_b | x_a', t_a) K (x_b, t_b | x_a, t_a) \left\{ \sum_n P(n, t_a | n_a) \langle n_a | n_a' \rangle \right\} \text{d}x_a \text{d}x_b \text{d}x_a' \text{d}x_b'.
\]
(II.3)

Therefore the total probability to be in state \( |n_b \rangle \) at \( t = t_b \) is the sum over all possible transitions from \( |n_a \rangle \) to \( |n_b \rangle \), weighted by the initial probability to be in a given state \( |n_a \rangle \):
\[
P(n_b, t_b) = \sum_n P(n, t_a | n_a) P(n_b, t_b | n_a, t_a),
\]
(II.4)

such that
\[
P(n_b, t_b) = \int \int \int \langle n_b' | x_b \rangle \langle n_b | x_b \rangle K^* (x_b', t_b | x_a', t_a) K (x_b, t_b | x_a, t_a) \left\{ \sum_n P(n, t_a | n_a) \langle n_a | n_a' \rangle \right\} \text{d}x_a \text{d}x_b \text{d}x_a' \text{d}x_b'.
\]
(II.5)

where the term between braces is identified as the initial density matrix \( \rho (x_a, x_a', t_a) \) of the system, since
\[
\rho (x, x', t) = \sum_n \langle x | n \rangle P(n, t) \langle n | x' \rangle.
\]
(II.6)

Because of the orthonormality of the states, the probability to be in a state \( |n_b \rangle \) at time \( t_b \) can also be extracted from the density matrix by
\[
P(n_b, t_b) = \int \langle n_b | x_b \rangle \langle n_b | x_b \rangle K (x_b, t_b | x_a, t_a) \text{d}x_b.
\]
(II.7)

It follows, by comparison with Eq. (II.5), that the density matrix at time \( t_b \) is given by
\[
\rho (x_b, x_b', t_b) = \int \int K (x_b, t_b | x_a, t_a) K^* (x_b', t_b | x_a', t_a) \rho (x_a, x_a', t_a) \text{d}x_a \text{d}x_a'.
\]
(II.8)

The density matrices can be expressed in terms of the Wigner distribution function by an inverse Weyl transform
\[
f (x, p, t) = \int \rho \left( x + \frac{\xi}{2}, x - \frac{\xi}{2}, t \right) e^{-ip\xi/\hbar} \frac{d\xi}{2\pi\hbar} \leftrightarrow \rho (x, x', t) = \int f \left( \frac{x + x'}{2}, p, t \right) e^{ip\xi/\hbar} d\xi dp.
\]
(II.9)

Defining the propagator \( K_w \) of the Wigner function as
\[
f (x_b, p_b, t_b) = \int \int K_w (x_b, p_b, t_b | x_a, p_a, t_a) f (x, p, t_a) \text{d}x_a \text{d}p_a,
\]
(II.10)

one may extract it from Eq. (II.11):
\[
K_w (x_b, p_b, t_b | x_a, p_a, t_a) = \int \int e^{-\frac{i}{\hbar} (p_b \xi_a - p_a \xi_b)} K \left( x_b + \frac{\xi_b}{2}, t_b | x_a + \frac{\xi_a}{2}, t_a \right) K^* \left( x_b - \frac{\xi_b}{2}, t_b | x_a - \frac{\xi_a}{2}, t_a \right) \frac{d\xi_b d\xi_a}{2\pi\hbar}
\]
(II.11)

\[
= \int \int \int \int \int \int \text{D}x \text{D}p \exp \left( \frac{i}{\hbar} (-p_b \xi_b + S \left[ x + \frac{\xi}{2} \right] - S \left[ x - \frac{\xi}{2} \right]) \right) \frac{d\xi_b d\xi_a}{2\pi\hbar}.
\]
(II.12)
Note that the explicit time dependence in the path variables $x(t)$ and $\xi(t)$ is omitted in order not to overload the notations.

For example, consider $x$ to be the coordinate of a (nonrelativistic) particle with mass $m$ in an external potential $V(x)$, described by the action

$$S[x] = \int_{t_a}^{t_b} \left( \frac{m}{2} \dot{x}^2 - V(x) \right) dt.$$ (II.13)

One then finds

$$S\left[x + \frac{\xi}{2}\right] - S\left[x - \frac{\xi}{2}\right] = \int_{t_a}^{t_b} \left( m\dot{x} - V \left( x + \frac{\xi}{2} \right) \right) dt.$$ 

After an integration by parts for the kinetic term one obtains the following expression for the propagator $K_w$:

$$K_w(x_b, p_b, t_b|x_a, p_a, t_a) = \frac{1}{2\pi\hbar} \int \int x(t_b) = x_b \quad x(t_a) = x_a \quad p(t_b) = m\dot{x}_b \quad p(t_a) = m\dot{x}_a \quad Dx \quad D\xi \exp \left( \frac{i}{\hbar} \int_{t_a}^{t_b} \left( m\dot{x} + V \left( x + \frac{\xi}{2} \right) - V \left( x - \frac{\xi}{2} \right) \right) dt \right) \frac{d\xi_d d\xi_a}{2\pi\hbar}.$$ (II.14)

Note that in previous work we obtained the same expression (Eq. (7) in [7]) for the propagator of the Wigner function by an infinitesimal time lapse expansion of the Wigner-Liouville equation. With this alternative derivation we have shown that the Wigner function can also be constructed directly from the Lagrangian form of quantum mechanics without the need of the usual canonical quantization procedure. This is important because some problems are much more complicated in Hamiltonian form, or a Hamiltonian description might not even exist. Furthermore, we have shown that the propagator (II.14) for the Wigner function is completely general, and not restricted to the phase space evolution of pure states.

It is clear that the double path integral (II.14) can only be solved analytically for a limited set of problems. As an example of such a problem consider a particle in a time dependent quadratic potential $V(x, t) = a(t) + b(t)x + c(t)x^2$. In that case the propagator can be explicitly evaluated:

$$K_w^{\text{quad}}(x_b, p_b, t_b|x_a, p_a, t_a) = \int x(t_b) = x_b \quad x(t_a) = x_a \quad p(t_b) = m\dot{x}_b \quad p(t_a) = m\dot{x}_a \quad Dx \quad D\xi \exp \left( \frac{-i}{\hbar} \int_{t_a}^{t_b} (m\dot{x} + b(t) + 2c(t)x) \xi dt \right) 2\pi\hbar$$

$$\propto \int x(t_b) = x_b \quad x(t_a) = x_a \quad p(t_b) = m\dot{x}_b \quad p(t_a) = m\dot{x}_a \quad Dx \quad |\delta (m\dot{x} + b(t) + 2c(t)x) |$$

$$= \delta (x_b - x_c(t_b|x_a, p_a, t_a)) \delta (p_b - p_{cl}(t_b|x_a, p_a, t_a)),$$ (II.15)

where $x_c(t_b|x_a, p_a, t_a)$ and $p_{cl}(t_b|x_a, p_a, t_a)$ are the position and momentum at time $t_b$ along the classical trajectory with initial position $x_a$ and momentum $p_a$ at time $t_a$. This confirms the well known result that the Wigner-Liouville equation can exactly be solved by the method of characteristics for all harmonic problems. It is the basic formula for most of the subsequent calculations. In the appendix a detailed derivation is given, based on the Feynman propagator for this type of potentials.

III. INFLUENCE FUNCTIONALS IN PHASE SPACE

In the previous section we derived the phase space propagator for a single-particle system. But many interesting systems consist of interacting subsystems. One of those subsystems is usually of particular interest. In this section
we generalize the previous phase space description with the help of influence functionals [3], allowing to describe the behavior of the subsystem of interest, coupled with an external (quantum) system, solely in terms of its own variables.

Consider a more complicated system that consists of two subsystems with coordinates \( x \) and \( u \). The subsystems are coupled by a potential \( V_I(x, u) \), which is incorporated in the action as \( S_I[x, u] \).

The extension of the path integral (III) thus takes the form

\[
K(x_b, u_b, t_b|x_a, u_a, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \int_{u(t_a)=u_a}^{u(t_b)=u_b} DxDu \exp \left( \frac{i}{\hbar} (S_x[x] + S_u[u] + S_I[x, u]) \right) \tag{III.1}
\]

where \( S_x[x] \) and \( S_u[u] \) are the non-interacting contributions of the subsystems \( x \) and \( u \) to the action. Again, for brevity, the path variables \( x \) and \( u \) are implicitly assumed to be time dependent. If \( \{\langle n\rangle\} \) denotes a complete orthonormal set of states for the \( x \) subsystem, and \( \{\langle j\rangle\} \) similarly for the \( u \) subsystem, the amplitude (II.2) can be generalized to

\[
A(n_b, j_b, t_b|n_a, j_a, t_a) = \int \int \int \int \langle n_b|x_b\rangle \langle j_b|u_b\rangle K(x_b, u_b, t_b|x_a, u_a, t_a) \langle x_a|n_a\rangle dx_a dx_b du_b du_a,
\tag{III.2}
\]

The conditional transition probability \( P(n_b, j_b, t_b|n_a, j_a, t_a) = |A(n_b, j_b, t_b|n_a, j_a, t_a)|^2 \) for subsystem \( x \) to go from state \( |n_a\rangle \) at \( t_a \) to state \( |n_b\rangle \) at \( t_b \), while subsystem \( u \) goes from \( |j_a\rangle \) to \( |j_b\rangle \) thus becomes

\[
P(n_b, j_b|n_a, j_a, t_a) = A(n_b, j_b, t_b|n_a, j_a, t_a) \langle n_b|x_b\rangle \langle j_b|u_b\rangle K(x_b, u_b, t_b|x_a, u_a, t_a) \langle x_a|n_a\rangle \times
\langle n_b|x_b\rangle \langle j_b|u_b\rangle K(x_b, u_b, t_b|x_a, u_a, t_a) \langle x_a|n_a\rangle dx_a dx_b du_b du_a dx'_a dx'_b du'_b du'_a.
\tag{III.3}
\]

From here we assume that only the description of subsystem \( x \) is physically relevant. In other words, one is interested in the probability \( P(n_b, t_b|n_a, t_a) \) of the transition of subsystem \( x \) from state \( |n_a\rangle \) to \( |n_b\rangle \). This can be found from the conditional probability (III.3) by summing over all final states \( |j_b\rangle \) and initial states \( |j_a\rangle \), weighted by the probability that subsystem \( u \) was initially in state \( |j_a\rangle \):

\[
P(n_b, t_b|n_a, t_a) = \sum_{j_a, j_b} P(j_a, t_a) P(n_b, j_b, t_b|n_a, j_a, t_a).
\tag{III.4}
\]

Substitution of the conditional probability (III.3) into Eq. (III.1) and regrouping terms leads to

\[
P(n_b, t_b|n_a, t_a) = \int \int \int \int \langle x_b'|n_b\rangle \langle n_a|x_b\rangle \langle x_a'|x_a\rangle \langle x_a|n_a\rangle \times
\int_{x(t_a)=x_a}^{x(t_b)=x_b} \int_{x'(t_a)=x_a'}^{x'(t_b)=x_b'} DxDx' \exp \left( \frac{i}{\hbar} (S_x[x] - S_x[x']) \right) F[x, x'] dx dxdx'dx',
\tag{III.5}
\]

The sum over the initial states \( |j_a\rangle \) is clearly identified as the initial density matrix of subsystem \( u \), whereas the closure relation ensures that the sum over all final states \( |j_b\rangle \) reduces to \( \delta(u_b - u'_b) \). Then, substitution of expression (III.1) for the propagators \( K(\cdots) \) and rearranging terms one ends up with the following expression for the required transition probability:

\[
P(n_b, t_b|n_a, t_a) = \int \int \int \langle x_b'|n_b\rangle \langle n_a|x_b\rangle \langle x_a'|x_a\rangle \langle x_a|n_a\rangle \times
\int_{x(t_a)=x_a}^{x(t_b)=x_b} \int_{x'(t_a)=x_a'}^{x'(t_b)=x_b'} DxDx' \exp \left( \frac{i}{\hbar} (S_x[x] - S_x[x']) \right) F(x, x') dx dxdx'dx',
\tag{III.6}
\]

with

\[
F[x, x'] = \int \int \int \rho(u_a, u'_a, t_a) \delta(u_b - u'_b) \left\{ \int_{u(t_a)=u_a}^{u(t_b)=u_b} D_u \exp \left( \frac{i}{\hbar} (S_u[u] + S_I[x, u]) \right) \right\} \times
\left\{ \int_{u'(t_a)=u'_a}^{u'(t_b)=u'_b} D_{u'} \exp \left( -\frac{i}{\hbar} (S_u[u'] + S_I[x', u']) \right) \right\} du_adu_b du'_adu'_b.
\tag{III.7}
\]
\( \mathcal{F} \left[ x, x' \right] \) contains a double path integral as indicated by the braces, and it can be regarded as an influence functional since it describes the full influence of subsystem \( u \) on subsystem \( x \).

The relation (II.9) readily allows to write the influence functional in terms of the initial Wigner distribution function of the \( u \) system. Expressed in the center-of-mass and relative coordinate system it becomes:

\[
\mathcal{F} \left[ x, x' \right] = \int \int \int \int f \left( u_a, p_a, t_a \right) e^{i(p_a \eta_a - p_b \eta_b)/\hbar} \times \\
\times \int_{u(t_a)=u_a}^{u(t_a)=u_b} \int_{\eta(t_a)=\eta_a}^{\eta(t_a)=\eta_b} D\eta D\eta \exp \left( \frac{i}{\hbar} \left( S_a \left[ u + \frac{q}{2} \right] + S_l \left[ x, u + \frac{q}{2} \right] \right) - S_a \left[ u - \frac{q}{2} \right] - S_l \left[ x', u - \frac{q}{2} \right] \right) \\
\times \frac{dp_a dp_b}{2\pi \hbar} du_a du_b d\eta_a d\eta_b, \quad (III.8)
\]

where a factor \( \delta(\eta_b) \) was replaced by its plane wave representation.

Proceeding along the lines (II.3–II.8) as in section II one obtains the time evolution of the reduced density matrix of subsystem \( x \):

\[
\rho \left( x_b, x'_b, t_b \right) = \int \int \left\{ \int_{x \left( t_a \right) = x_a}^{x \left( t_a \right) = x_b} \int_{x \left( t_a \right) = x_a}^{x \left( t_a \right) = x_a} Dx Dx' \exp \left( \frac{i}{\hbar} \left( S_x \left[ x \right] - S_x \left[ x' \right] \right) \right) \mathcal{F} \left[ x, x' \right] \right\} \rho \left( x_a, x'_a, t_a \right) dx_a dx'_a, \quad (III.9)
\]

which differs from (II.8) merely by the occurrence of the influence functional in the path integral.

Note that we have assumed the two subsystems to be initially independent so that the probability \( P \left( j_a, t_a \right) \) of finding \( u \) in state \( \left| j_a \right> \) is independent of the state of \( x \). This means that the initial total density matrix was supposed to be separable.

Some useful properties of the influence functional are listed below. The identity

\[
\mathcal{F}^{*} \left[ x, x' \right] = \mathcal{F} \left[ x', x \right], \quad (III.10)
\]

follows directly from Eq. (II.7) by interchanging \( x \) and \( x' \). Note that this property will later ensure that the propagator for the Wigner function is always a real quantity. Furthermore, if there are a number of statistically and dynamically independent subsystems \( u_j \) acting on \( x \) and if \( \mathcal{F}^j \left( x, x' \right) \) is the influence functional of the \( j \)th subsystem on \( x \), then the total influence function is the product of all the individual functionals \( \mathcal{F}^j \):

\[
\mathcal{F} \left[ x, x' \right] = \prod_{j=1}^{N} \mathcal{F}^j \left[ x, x' \right]. \quad (III.11)
\]

This property is a direct consequence of the total initial density emerging as a simple product when all \( u_j \) are statistically independent. If they are also dynamically independent, then each density matrix can propagate separately.

Finally, it is often convenient to write the influence functional in the form

\[
\mathcal{F} \left[ x, x' \right] = \exp \left( \frac{i}{\hbar} \Phi \left[ x, x' \right] \right) \quad (III.12)
\]

where \( \Phi \left[ x, x' \right] \) is called the influence phase. For independent subsystems, as in (II.11), the corresponding influence phases add. According to Eq. (III.10), the influence phase turns out to be antisymmetric under the exchange of \( x \) and \( x' \) if the phase is real, and symmetric if the phase is imaginary. More properties on influence functionals can be found in [3].

It follows from Eq. (II.5), and in analogy with the analysis presented in section II until Eq. (II.12), that the propagator for the reduced Wigner distribution function becomes

\[
K_w \left( x_b, p_b, t_b \mid x_a, p_a, t_a \right) = \int \int \int_{x \left( t_a \right) = x_a}^{x \left( t_a \right) = x_b} \int_{\xi \left( t_a \right) = \xi_a}^{\xi \left( t_a \right) = \xi_b} D\xi D\xi' \exp \left( \frac{i}{\hbar} \left( \begin{array}{c} -\rho \xi_{t_a}^b \\ S_x \left[ x + \frac{\xi}{2} \right] + S_l \left[ x, x + \frac{\xi}{2} \right] \\ -S_x \left[ x - \frac{\xi}{2} \right] \\ \end{array} \right) \right) \mathcal{F} \left[ x + \frac{\xi}{2} - \frac{x}{2}, x - \frac{x}{2} \right] \frac{d\xi_d d\xi}{2\pi \hbar}. \quad (III.13)
\]

If the action of the system \( x \) is of the form

\[
S_x \left[ x \right] = \int_{t_a}^{t_b} \left( \frac{m}{2} \dot{x}^2 - V(x) \right) dt, \quad (III.14)
\]
(i.e., without a magnetic field) the further analysis in section [III] until Eq. [III.14] simplifies this propagator:

\[ K_u \left( x_b, p_b, t_b \big| x_a, p_a, t_a \right) = \frac{1}{2\pi\hbar} \int_{p(t_b) = m\tilde{x}_b}^{x(t_b) = x_b} Dx \int D\xi F \left[ x + \frac{\xi}{2}, x - \frac{\xi}{2} \right] \times \exp \left( -\frac{i}{\hbar} \int_{t_a}^{t_b} \left( \dot{m}\dot{\xi} + \left( V \left( x + \frac{\xi}{2} \right) - V \left( x - \frac{\xi}{2} \right) \right) \right) dt \right). \] (III.15)

The crucial ingredient for further development is the influence functional \( F \left[ x, x' \right] \). Clearly representing the propagator of subsystem \( u \) under the influence of subsystem \( x \), the path integrals in (III.7) can be calculated analytically if \( S_u \left[ u \right] + S_I \left[ x, u \right] \) is quadratic in \( u \). Below the results for a bare harmonic action \( S_u \) with a linear coupling \( S_I \) will be discussed in some detail.

IV. HARMONIC SUBSYSTEMS WITH LINEAR COUPLING

A. A single oscillator

In this section we have reduced the subsystem \( u \) to a single harmonic oscillator with mass \( M \) and with potential energy \( V \left( u \right) = M\omega^2u^2/2 \), interacting with the subsystem \( x \) of interest. Taking the interaction energy to be \( u \cdot \gamma(x) \), with an arbitrary function \( \gamma(x) \) and a linear dependence on \( u \), we are left with \( S_I \left[ x, u \right] = -\int_{t_a}^{t_b} u\gamma(t) dt \), where we added a subscript \( t \) to \( \gamma \) to remember the time at which its path variable \( x \) should be evaluated. The total action of subsystem \( u \), to be used in the influence functional (III.8), is thus

\[ S_u \left[ u \right] + S_I \left[ x, u \right] = \int_{t_a}^{t_b} \left( \frac{M}{2}\dot{u}^2 - \frac{M}{2}\omega^2u^2 - u\gamma(t) \right) dt. \] (IV.1)

The argument in the exponent of Eq. (III.8) thus becomes linear in \( \eta \) and in \( u \):

\[
\mathcal{F} \left[ x, x' \right] = \iiint f \left( u_a, p_a, t_a \right) e^{i(p_a - p_u)\eta / \hbar} \int_{u(t_a) = u_a}^{u(t_b) = u_b} Du \exp \left( -\frac{i}{\hbar} \int_{t_a}^{t_b} u \left( \gamma(t) - \gamma(t') \right) dt \right) \times \int_{\eta(t_a) = \eta_a}^{\eta(t_b) = \eta_b} D\eta \exp \left( -\frac{i}{\hbar} \int_{t_a}^{t_b} \left( M\dot{\eta} - M\omega^2u\eta - \eta \frac{\gamma(t) + \gamma(t')}{2} \right) dt \right) \frac{dp_a dp_b d\eta_a d\eta_b}{2\pi\hbar}. \]

After an integration by parts of the kinetic term \( \int_{t_a}^{t_b} \dot{u} \eta dt = \dot{u}b \eta - \dot{u}_a \eta_a - \int_{t_a}^{t_b} \dddot{u} \eta dt \), and imposing \( M\dddot{u} = p_{a,b} \), the path integral over \( \eta \) becomes unconstrained:

\[
\mathcal{F} \left[ x, x' \right] = \iiint f \left( u_a, p_a, t_a \right) \int_{p(t_a) = M\dot{u}_a}^{p(t) = M\dot{u}_{a,b}} Dp_a \exp \left( -\frac{i}{\hbar} \int_{t_a}^{t_b} \left( \gamma(t) - \gamma(t') \right) u dt \right) \times \int D\eta \exp \left( \frac{i}{\hbar} \int_{t_a}^{t_b} \left( -M\dddot{u} - M\omega^2u + \frac{\gamma(t) + \gamma(t')}{2} \right) \eta dt \right) \frac{dp_a d\eta a d\eta b}{2\pi\hbar}. \] (IV.2)

The path integral over all \( \eta \) in the last line is of the form of the quadratic path integral (II.15), and restricts the phase space trajectories of \( u \)

\[ M \left( \dddot{u} + \omega^2u \right) + \frac{\gamma(t) + \gamma(t')}{2} = 0, \] (IV.3)

with the formal solution

\[ u \left( t \right) = u_a \cos \omega \left( t - t_a \right) + \frac{\dot{u}_a}{\omega} \sin \omega \left( t - t_a \right) - \int_{t_a}^{t} \frac{\gamma(s) + \gamma(t') \sin \omega \left( t - s \right)}{2M} ds. \]

The remaining path integral on the first line of (IV.2) imposes that the initial velocity is \( \dot{u}_a = p_a / M \). The trajectories in \( u \) are thus reduced to a single path, with the conditions \( p_{a,b} = M\dot{u}_{a,b} \) at the end points, which also eliminate the
integrations over $u_b$ and $p_b$. One thus readily arrives at

$$\mathcal{F}[x, x'] = \exp \left( \frac{i}{2\hbar M \omega} \int_{t_a}^{t_b} \left[ (\gamma_s(x) + \gamma_s(x')) (\gamma_t(x) - \gamma_t(x')) \sin \omega (t - s) \right] ds dt \right) \times
$$

$$\times \int \int f(u_a, p_a, t_a) \exp \left( -\frac{i}{\hbar} \int_{t_a}^{t_b} \left[ u_a \cos \omega (t - t_a) + \frac{p_a}{M \omega} \sin \omega (t - t_a) \right] \left( \gamma_t(x) - \gamma_t(x') \right) dt \right) dr_a dp_a. \quad (IV.4)$$

One might be concerned about the normalizing factors accompanying the several delta functions in the derivation, but this possible problem is resolved by considering the uncoupled limit $\gamma = 0$. An alternative derivation, using the explicit solution of the Feynman path integral [9] for the action $(S_0[u] + S_t[x,u])$, confirms this result.

Clearly the first line in expression (IV.4) is independent of the initial state of the harmonic oscillator. This term represents an effective interaction of system $x$ with itself. The second line is an expectation value which transfers all necessary information about the initial state of the harmonic $u$ system into $x$.

1. Example: initial wave packet

Despite the classical trajectories which govern its dynamics, the influence functional (IV.4) is intrinsically of quantum mechanical nature, because the initial Wigner distribution function $f(u_a, p_a, t_a)$ of the oscillator is bound to satisfy the uncertainty principle. A sharply defined initial distribution like $\delta(u_a - u_0)\delta(p_a - p_0)$ can not be of the form (II.9). However, a valid initial wave function could be a Gaussian wave packet

$$\Psi_G(u, t_a) = \frac{1}{\sqrt{\Delta \sqrt{2\pi}}} \exp \left( -\frac{(u - u_0)^2}{4\Delta^2} \right) e^{i p_0 u / \hbar}. \quad (IV.5)$$

From the corresponding density matrix $\rho_G(u, u', t) = \Psi_G^* (u', t) \Psi_G (u, t)$ one easily finds the Wigner distribution function (II.9) of this wave packet

$$f_G(u, p, t_a) = \frac{1}{\pi \hbar} \exp \left( -\frac{(u - u_0)^2}{2\Delta^2} - 2\Delta^2 \left( \frac{p - p_0}{\hbar} \right)^2 \right). \quad (IV.6)$$

Then, by performing the integrations in Eq. (IV.4) and rearranging some terms one finds the following influence phase (III.12):

$$\Phi_G(x, x') = \frac{1}{2M \omega} \int_{t_a}^{t_b} \int_{t_a}^{t_b} (\gamma_s(x) + \gamma_s(x')) (\gamma_t(x) - \gamma_t(x')) \sin \omega (t - s) ds dt$$

$$- u_0 \int_{t_a}^{t_b} (\gamma_t(x) - \gamma_t(x')) \cos \omega (t - t_a) dt - \frac{p_0}{M \omega} \int_{t_a}^{t_b} (\gamma_t(x) - \gamma_t(x')) \sin \omega (t - t_a) dt$$

$$+ \frac{i}{\hbar} \int_{t_a}^{t_b} \int_{t_a}^{t_b} (\gamma_s(x) - \gamma_s(x')) (\gamma_t(x) - \gamma_t(x')) \left( \frac{\Delta^2 + \frac{\hbar^2}{4\Delta^2 M \omega^2}}{\Delta^2 - \frac{\hbar^2}{4\Delta^2 M \omega^2}} \right) \cos \omega (s - t) ds dt \quad (IV.6)$$

The real part of the influence phase is the same as one would obtain from a (forbidden) initial Wigner distribution function $\delta(u_a - u_0)\delta(p_a - p_0)$. Besides the effective interacting of system $x$ with itself, the real part now contains an external driving potential which oscillates in time with a frequency $\omega$ and its spatial dependence is given by $\gamma(x)$. The magnitude of this driving potential depends on the initial average (vacuum) displacement of the oscillator $(u_0, p_0)$. The imaginary part of the influence phase results from the uncertainty on the initial position and momentum of $u$; its physical significance becomes more apparent in the next example. Note that some terms vanish under specific conditions, e.g., if the average initial position or momentum are zero. The last term becomes zero whenever $\Delta^2 = \frac{\hbar^2}{2M \omega^2}$.

This condition is satisfied if the oscillator was initially in an unsqueezed coherent state. In that case one finds the ground state or vacuum influence phase:

$$\Phi_{vac}[x, x'] = \frac{1}{2M \omega} \int_{t_a}^{t_b} \int_{t_a}^{t_b} (\gamma_s(x) + \gamma_s(x')) (\gamma_t(x) - \gamma_t(x')) \sin \omega (t - s) ds dt$$

$$+ \frac{i}{4M \omega} \int_{t_a}^{t_b} \int_{t_a}^{t_b} (\gamma_s(x) - \gamma_s(x')) (\gamma_t(x) - \gamma_t(x')) \cos \omega (s - t) ds dt. \quad (IV.7)$$
2. Example: thermal equilibrium

The case of thermal equilibrium at the start of course deserves some additional attention. The initial equilibrium Wigner function $\mathcal{F}$ of the harmonic oscillator $u$ is then given by

$$f_{eq}(u_a, p_a, t_a) = \frac{\tanh \frac{1}{2} \beta \hbar \omega}{\pi \hbar} \exp \left( - \frac{\tanh \frac{1}{2} \beta \hbar \omega}{\hbar \omega} \left( M \omega^2 u_a^2 + \frac{p_a^2}{M} \right) \right).$$  \hspace{1cm} (IV.8)

It follows from Eq. (IV.3) that the equilibrium influence phase $\Phi_{eq}[x, x']$ is

$$\Phi_{eq}[x, x'] = \frac{1}{2M \omega} \int_{t_a}^{t_b} \int_{t_a}^{t_s} (\gamma_s(x) + \gamma_s(x')) (\gamma_t(x) - \gamma_t(x')) \sin \omega (t - s) \, ds \, dt + \frac{i \coth \frac{1}{2} \beta \hbar \omega}{4M \omega} \int_{t_a}^{t_b} \int_{t_a}^{t_s} (\gamma_t(x) - \gamma_t(x')) (\gamma_u(x) - \gamma_u(x')) \cos \omega (t - s) \, ds \, dt \hspace{1cm} (IV.9)

While the real parts of $\Phi_{eq}$ and $\Phi_{vac}$ are the same, the imaginary part of $\Phi_{eq}$ has increased by a factor $\coth \left( \frac{\beta \hbar \omega}{2} \right)$ as compared to $\Phi_{vac}$, as a consequence of thermal broadening of the distribution. Note now that $\coth \left( \frac{\beta \hbar \omega}{2} \right) = [1 + n_B(\omega)] + n_B(\omega)$, with $n_B(\omega)$ the Bose-Einstein distribution with zero chemical potential. From a physics point of view one would therefore associate the imaginary part of $\Phi_{eq}$ with the emission and absorption of the quanta represented by the oscillator. When the temperature is zero system $x$ can only interact with the zero point fluctuations of $u$, allowing only losses through spontaneous emission. A more detailed discussion on this can be found in [3].

B. Many independent oscillators

The generalization of the result from one to many oscillators is trivial if the oscillators are independent. According to property (III.1) the total influence functional then becomes the product of all the individual influence functionals. Consider every oscillator $u_j$ to have a mass $M_j$, a frequency $\omega_j$ and consider the interaction energy with $x$ to be $u_j \cdot \gamma_j(x)$. Then we immediately arrive at the following expression for the total influence phase of a collection of $N$ oscillators

$$\mathcal{F}_N[x, x'] = \exp \left( \frac{i}{\hbar} \sum_{j=0}^{N} \frac{1}{2M_j \omega_j} \int_{t_a}^{t_b} \int_{t_a}^{t_s} (\gamma_j(x_s) + \gamma_j(x'_s)) (\gamma_j(x_t) - \gamma_j(x'_t)) \sin \omega_j (t - s) \, ds \, dt \right) \times \prod_{j=0}^{N} \int_{t_a}^{t_b} \int_{t_a}^{t_s} \left( u_j \cos \omega_j (t - t_a) + \frac{p_a}{M_j \omega_j} \sin \omega_j (t - t_a) \right) (\gamma_j(x_t) - \gamma_j(x'_t)) \, dt \, dp_a, \hspace{1cm} (IV.10)

where $f_j(u_a, p_a, t_a)$ represents the initial Wigner function of the $j$th oscillator. Let us consider a simple example now that can easily be generalized to more complicated situations.

1. Example: Equilibrium oscillators with bilinear coupling

If we assume the interaction energy to be bilinear in $\{u_j, x\}$, such that $u \cdot \gamma_j(x) = \gamma_j u \cdot x$, and if additionally all oscillators are initially in thermal equilibrium $\Phi_{eq}$, then the influence phase associated with influence functional $\Phi_{eq}$ is given by

$$\Phi_{eq,N}[x, x'] = \sum_{j=0}^{N} \frac{\gamma_j^2}{2M_j \omega_j} \int_{t_a}^{t_b} \int_{t_a}^{t_s} (x_s + x'_s) (x_t - x'_t) \sin \omega_j (t - s) \, ds \, dt \quad \text{and} \quad \Phi_{eq,N}[x, x'] = \sum_{j=0}^{N} \frac{\gamma_j^2}{4M_j \omega_j} \coth \frac{1}{2} \beta \hbar \omega_j \int_{t_a}^{t_b} \int_{t_a}^{t_s} (x_s - x'_s) (x_t - x'_t) \cos \omega_j (t - s) \, ds \, dt.$$
In the continuum limit, when \( N \to \infty \) while \( \omega_{j+1} - \omega_j \to 0 \), we can assume there is a distribution of oscillators, such that the relevant weight \( \Gamma(\omega) d\omega \) of the oscillators between \( \omega \) and \( \omega + d\omega \) is

\[
\Gamma(\omega) = \sum_{j=0}^{\infty} \frac{\gamma_j^2}{M_j} \delta(\omega - \omega_j).
\]

In this case the influence phase becomes

\[
\Phi_{eq,many}[x,x'] = \frac{1}{2} \int_{t_a}^{t_b} \int_{t_a}^{t_b} \left( \int_0^\infty \Gamma(\omega) \sin(\omega (t - s)) \, d\omega \right) (x_s + x'_s) (x_t - x'_t) \, ds \, dt + \frac{i}{\hbar} \int_{t_a}^{t_b} \int_{t_a}^{t_b} \left[ \int_0^\infty \left( \frac{\hbar \Gamma(\omega)}{4} \coth \frac{\beta \hbar \omega}{2} \right) \cos(\omega (t - s)) \, d\omega \right] (x_t - x'_t) (x_s - x'_s) \, ds \, dt.
\]

If \( x \) is the coordinate of an otherwise free particle with mass \( m \) we find the following propagator for its reduced Wigner function

\[
K_w(x_b,p_b|t_b|x_a,p_a,t_a) = \frac{1}{2\pi \hbar} \int_{x(t_a)=x_a}^{x(t_b)=x_b} \int_{p(t_a)=m \dot{x}_a}^{p(t_b)=m \dot{x}_b} D\xi D\bar{\xi} \exp \left( -\frac{1}{\hbar^2} \int_{t_a}^{t_b} R(t - s) \xi_t \cdot \bar{\xi}_s \, ds \, dt \right) \times \exp \left( -\frac{i}{\hbar} \int_{t_a}^{t_b} \left( m \ddot{x} - \int_{t_a}^{t} A(t - s) x_s \, ds \right) \cdot \xi_s \, dt \right),
\]

with \( A(t) \) and \( R(t) \) defined as:

\[
A(t) = \int_0^\infty \Gamma(\omega) \sin(\omega t) \, d\omega,
\]

\[
R(t) = \int_0^\infty \frac{h \Gamma(\omega)}{4} \coth \left( \frac{\beta \hbar \omega}{2} \right) \cos(\omega t) \, d\omega.
\]

2. Example: Caldeira-Legett model and thermalization

In the specific case that \( \Gamma(\omega) = 2\eta \omega \), as considered in detail by Caldeira and Leggett in [4], we obtain \( A(t) = -2\eta \delta'(t) \). If additionally the temperature \( T_0 \) is high enough such that we can approximate \( \left[ \frac{\hbar \omega}{2} \coth \left( \frac{\beta \hbar \omega}{2} \right) \right] \approx \frac{\hbar \omega}{2} + O(\beta) \), we find \( R(t) = \eta kT_b \delta(t - s) \). Therefore we arrive at the following expression for the propagator in the high temperature limit

\[
K_w(x_b,p_b|t_b|x_a,p_a,t_a) = \frac{1}{2\pi \hbar} \int_{x(t_a)=x_a}^{x(t_b)=x_b} \int_{p(t_a)=m \dot{x}_a}^{p(t_b)=m \dot{x}_b} D\xi D\bar{\xi} \exp \left( -\frac{\eta \hbar k T_b}{\hbar^2} \int_{t_a}^{t_b} \xi_t^2 \, dt \right) \exp \left( -\frac{i}{\hbar} \int_{t_a}^{t_b} \left( m \ddot{x} + \eta \dot{x} \right) \cdot \xi_s \, dt \right).
\]

It can easily be shown, for example by timeslicing the \( \xi \) path integral, that this propagator can be rewritten as a single path integral:

\[
K_w(x_b,p_b|t_b|x_a,p_a,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \int_{p(t_a)=m \dot{x}_a}^{p(t_b)=m \dot{x}_b} D\xi \exp \left( -\frac{1}{4\eta k T_b} \int_{t_a}^{t_b} (m \ddot{x} + \eta \dot{x})^2 \, dt \right).
\]

If we are not interested in the real space motion of \( x \), for example because the initial distribution of the particle is homogeneous in space, but only in the marginal propagator \( K_w(p_b,t_b|p_a,t_a) \) to go from \( p_a \) to \( p_b \), then we get

\[
K_w(p_b,t_b|p_a,t_a) = \int_{p(t_a)=p_a}^{p(t_b)=p_b} Dp \exp \left( -\frac{1}{2\eta k T_b} \int_{t_a}^{t_b} \frac{\left( \dot{p} \right)^2}{2} + \frac{\left( \eta \dot{p} \right)^2}{2} + \frac{p^2}{2m} \, dt \right).
\]

\(^1\) under some debateably assumptions about the limits in the integral.
An integration by parts shows that the term in $\dot{p}p$ only contributes at the boundaries:

$$K_w(p_b, t_b| p_a, t_a) = \exp \left( -\frac{p_b^2 - p_a^2}{4mkT_b} \right) \int_{p(t_a) = p_a}^{p(t_b) = p_b} \mathcal{D}p \exp \left( -\frac{1}{2\eta kT_b} \int_{t_a}^{t_b} \left( \frac{\dot{p}^2}{2} + \frac{\eta^2}{m} \frac{p^2}{2} \right) dt \right),$$

The remaining path integral is a simple Gaussian Feynman path integral. This path integral can be solved with standard techniques which yield the following expression for the reduced momentum space Wigner propagator:

$$K_w(p_b, t_b| p_a, t_a) = \frac{1}{\sqrt{2\pi mkT_b}} \exp \left( -\frac{1}{2\eta kT_b} \left( \frac{p_b - p_a}{1 - \exp \left(-\frac{\eta m}{m} (t_b - t_a) \right)} \right)^2 \right).$$

The maximal transition probability is, as expected, attained along the solution of the classical equation of motion $\dot{p} + \frac{\eta}{m} p = 0$, i.e. when $p_b = p_a \exp \left(-\frac{\eta}{m} (t_b - t_a) \right)$. The variance of the propagator is given by

$$\sigma^2 = mkT_b \left( 1 - \exp \left( -\frac{2\eta}{m} (t_b - t_a) \right) \right).$$

The time evolution of the effective temperature $T_e$ of the particle $x$ is therefore given by

$$T_e(t_b - t_a) = T_b \left( 1 - \exp \left( -\frac{2\eta}{m} (t_b - t_a) \right) \right),$$

such that the system thermalizes at a characteristic time $\tau = \frac{m}{2\eta}$. Finally consider the time $t_b - t_a \gg \tau$, such that the system is thermalized and $T_e \approx T_b$, then the propagator becomes

$$\lim_{(t_b-t_a) \to \infty} K_w(p_b, t_b| p_a, t_a) = \frac{1}{\sqrt{2\pi mkT_b}} \exp \left( -\frac{1}{kT_b} \frac{p_b^2}{2m} \right).$$

This means that the reduced Wigner function of the particle thermalizes into a Maxwell-Boltzmann distribution, regardless its initial Wigner function.

V. CONCLUSION

In conclusion we have used Feynman’s Lagrangian description of quantum mechanics to express the propagator of the Wigner function as a path integral. Propagating Wigner functions rather than wave functions has a double advantage. First of all one can directly propagate uncertain initial configurations in time instead of pure states only. Secondly the attained Wigner function propagator becomes a delta function in the classical trajectory for all harmonic problems and subproblems. In contrast to our previous derivation in [7] based on the Wigner-Liouville equation, the present treatment does not rely on a canonical quantization procedure of the Hamiltonian. This Lagrangian formulation, with the help of influence functionals, allows for a transparent description of two interacting subsystems, and to find the reduced Wigner function propagator of one of the two subsystems. In the last section we have generalized this result to a system interacting with many other, mutually independent, subsystems. As an example we considered in more detail the model by Caldeira and Leggett, and showed how to calculate the reduced Wigner function propagator for it, using the techniques explained in this manuscript.
Appendix A: Wigner propagator for quadratic potentials

In this appendix the propagator \([15]\) of the Wigner function for quadratic potentials of the form
\[ V_{\text{quad}}(x, t) = a(t) + b(t)x + c(t)x^2 \]  
(A.1)
is derived from the well known \([11]\) Feynman propagator \(K_{\text{quad}}(x_b, t_b|x_a, t_a)\) for this type of potentials:
\[ K_{\text{quad}}(x_b, t_b|x_a, t_a) = \sqrt{\frac{m}{2\pi i\hbar f(t_b, t_a)}} \exp \left( \frac{i}{\hbar} S_{\text{quad}}(x_b, t_b|x_a, t_a) \right), \]  
(A.2)
and
\[ S_{\text{quad}}(x_b, t_b|x_a, t_a) = \int_{t_a}^{t_b} \left( \frac{m}{2} x^2 - a(t) - b(t)x - c(t)x^2 \right) dt, \]  
(A.3)

where \(S_{\text{quad}}(x_b, t_b|x_a, t_a)\) is the action of the system along a classical trajectory from \((x_a, t_a)\) to \((x_b, t_b)\), to be determined from the classical equation of motion
\[ m\ddot{x} + 2c(t)x = -b(t) \text{ with } x(t_a) = x_a, \quad x(t_b) = x_b. \]  
(A.4)
The function \(f(t_a, t_b)\) only depends on the initial and final time, and is independent of the positions and momenta. It is the solution of the differential equation
\[ \left( m \frac{d^2}{dt^2} + 2c(t) \right) f(t_a, t_b) = 0 \text{ with } \frac{f(t_a, t_b)}{t} = 1. \]  
(A.5)

For general time dependence of \(c(t)\), these differential equations rarely have a solution in closed form, but it is sure that two linearly independent solutions, say \(x_1(t)\) and \(x_2(t)\), of the homogeneous equations exist:
\[ m\ddot{x}_{1,2} + 2c(t)x_{1,2} = 0. \]  
(A.6)

Their Wronskian \(\frac{dx_1(t)}{dt}x_2(t) - \frac{dx_2(t)}{dt}x_1(t)\) is independent of \(t\), because \([A.0]\) reveals that its time derivative is zero
\[ \frac{dx_1(t)}{dt}x_2(t) - \frac{dx_2(t)}{dt}x_1(t) = W \text{ independent of } t. \]  
(A.7)

Since \(f(t_a, t_b)\) is also a solution of the homogeneous differential equation \([A.0]\), it is a linear combination of \(x_1(t)\) and \(x_2(t)\). Taking the boundary conditions into account it becomes
\[ f(t_a, t_b) = \frac{\hbar(t_a, t_b)}{W} \text{ with } h(s, t) = x_1(s)x_2(t) - x_2(s)x_1(t). \]  
(A.8)

If one imposes that the solution of the homogeneous differential equation \([A.0]\) exhausts the boundary conditions at \(t_a\) and \(t_b\), the trajectory \(x(t)\) is of the form
\[ x(t) = \frac{\hbar(t_a, t_b)}{h(t_a, t_b)}x_a + \frac{\hbar(t_a, t_b)}{h(t_a, t_b)}x_b + x_p(t), \]  
(A.9)
where the particular solution \(x_p(t)\) has to satisfy the boundary conditions \(x_p(t_a) = 0 = x_p(t_b)\). It is easily found by the variation of parameters method, with the result:
\[ x_p(t) = -\frac{\hbar(t_a, t_b)}{h(t_a, t_b)} \int_{t_a}^{t} \frac{b(s)h(t_a, s)}{mW} ds - \frac{\hbar(t_a, t_b)}{h(t_a, t_b)} \int_{t}^{t_b} \frac{b(s)h(s, t_b)}{mW} ds. \]  
(A.10)

It is fairly easy to calculate the initial and final velocities \(\dot{x}_{a,b}\), which are of particular relevance below:
\[ \dot{x}_a = \frac{1}{\hbar(t_a, t_b)} \left( x_a \frac{\partial h(t_a, t_b)}{\partial t_a} - x_b W + \int_{t_a}^{t_b} \frac{b(s)}{m} h(s, t_b) ds \right), \]  
(A.11)
\[ \dot{x}_b = \frac{1}{\hbar(t_a, t_b)} \left( x_b W + x_b \frac{\partial h(t_a, t_b)}{\partial t_b} - \int_{t_a}^{t_b} \frac{b(s)}{m} h(t_a, s) ds \right). \]  
(A.12)
Applying an integration by parts $\int \dot{x}^2 dt = \dot{x} - \int x \ddot{x} dt$ in the kinetic contribution to the classical action, it can be rewritten as

$$S_{\text{quad}}(x_b, t_b | x_a, t_a) = \frac{m}{2} \left( [\dot{x}]_{t=t_b}^{t=t_a} - \int_{t_a}^{t_b} b(t) \frac{\dot{x}(t)}{m} dt \right)$$

$$= \frac{m}{h(t_a, t_b)} \left( x_b \frac{d}{dt_a} \frac{\partial h(t_a, t_b)}{dt_a} - x_a \frac{d}{dt_a} \frac{\partial h(t_a, t_b)}{dt_a} \right) + W x_a x_b - x_b \int_{t_a}^{t_b} b(s) \frac{h(s, t_b)}{m} ds - x_a \int_{t_a}^{t_b} b(s) \frac{h(s, t_a)}{m} ds - \frac{1}{2} \int_{t_a}^{t_b} b(s) x_p(s) ds,$$  \hspace{1cm} (A.13)

where the boundary velocities and the homogenous contribution $x(t)$ have been filled out.

In the propagator (II.12) for harmonic interactions of the form (A.1), only terms linear in $\xi_{a,b}$ survive in the exponent, and one is left with

$$K_{\text{quad}}(x_b, p_b, t_b | x_a, p_a, t_a) = \frac{mW}{(2\pi \hbar)^2 h(t_b, t_a)} \times$$

$$\times \int \exp \left( \frac{i}{\hbar} \xi_b \left( -p_b + \frac{m}{h(t_a, t_b)} \left( x_a W + x_b \frac{dh(t_a, t_b)}{dt_b} - \int_{t_a}^{t_b} b(s) \frac{h(s, t_b)}{m} ds \right) \right) \right) d\xi_b$$

$$\times \int \exp \left( \frac{i}{\hbar} \xi_a \left( p_a - \frac{m}{h(t_a, t_b)} \left( x_a \frac{dh(t_a, t_b)}{dt_a} - x_b W + \int_{t_a}^{t_b} b(s) \frac{h(s, t_b)}{m} ds \right) \right) \right) d\xi_a,$$  \hspace{1cm} (A.14)

where the remaining integrals are $\delta$ functions. Taking the results for the boundary velocities into account, one thus readily finds

$$K_{\text{quad}}(x_b, p_b, t_b | x_a, p_a, t_a) = \delta(x_b - x_{\text{cl}}(t_b | x_a, p_a, t_a)) \delta(p_b - p_{\text{cl}}(t_b | x_a, p_a, t_a)),$$  \hspace{1cm} (A.15)

which is the desired result (II.15).

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