DIMENSION-FREE ESTIMATES FOR THE DISCRETE SPHERICAL MAXIMAL FUNCTIONS

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Abstract. We prove that the discrete spherical maximal functions (in the spirit of Magyar, Stein and Wainger) corresponding to the Euclidean spheres in $\mathbb{Z}^d$ with dyadic radii have $L^p(\mathbb{Z}^d)$ bounds for all $p \in [2, \infty)$ independent of the dimensions $d > 5$. An important part of our argument is the asymptotic formula in the Waring problem for the squares with a dimension-free multiplicative error term. By considering new approximating multipliers we will show how to absorb an exponential in dimension (like $C^d$ for some $C > 1$) growth in norms arising from the sampling principle of Magyar, Stein and Wainger, and ultimately deduce dimension-free estimates for the discrete spherical maximal functions.

1. Introduction

1.1. Motivations and statement of the main results. For $t > 0$ let $E_t := \{ y \in \mathbb{R}^d : t^{-1}y \in E \}$ denote the dilate of a set $E \subseteq \mathbb{R}^d$. If $I \subseteq \mathbb{R}_+ := (0, \infty)$ is a non-empty index set such that $E_t \cap \mathbb{Z}^d \neq \emptyset$ for every $t \in I$, then for every $x \in \mathbb{Z}^d$ we define an averaging operator by

$$\mathcal{M}_E^x f(x) := \frac{1}{|E_t \cap \mathbb{Z}^d|} \sum_{y \in E_t \cap \mathbb{Z}^d} f(x - y), \quad f \in \ell^1(\mathbb{Z}^d).$$

For $\emptyset \neq I \subseteq \mathbb{R}_+$ and $E \subseteq \mathbb{R}^d$ as above, and $p \in [1, \infty]$ let $0 < C(p, I, E) \leq \infty$ be the smallest constant in the following maximal inequality

$$\left\| \sup_{t \in I} \| \mathcal{M}_E^x f \|_{L^p(\mathbb{Z}^d)} \right\|_{L^p(\mathbb{Z}^d)} \leq C(p, I, E) \| f \|_{L^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d).$$

Note that $C(\infty, I, E) = 1$, since $\mathcal{M}_E^x$ is an averaging operator, and $C(p, I_1, E) \leq C(p, I_2, E)$ if $I_1 \subseteq I_2$.

In this paper we are mainly concerned with the discrete averaging operators corresponding to the Euclidean unit spheres $S := S^{d-1} := \{ x \in \mathbb{R}^d : |x| = 1 \}$ in $\mathbb{R}^d$, where $S_t := S^{d-1}_t := \{ x \in \mathbb{R}^d : |x| = t \}$. For $x \in \mathbb{Z}^d$ and $t \in \sqrt{N} := \{ r \in (0, \infty) : r^2 \in \mathbb{N} \}$ we shall denote the discrete spherical average by

$$\mathcal{A}_E^x f(x) := \mathcal{M}_{S}^x f(x) := \frac{1}{|S_t \cap \mathbb{Z}^d|} \sum_{y \in S_t \cap \mathbb{Z}^d} f(x - y), \quad f \in \ell^1(\mathbb{Z}^d).$$

We shall also use the convention that $S_0 := \{ 0 \}$, and $\mathcal{A}_E^x(f) := f(x)$.

Maximal inequalities corresponding to the discrete averaging operators $\mathcal{M}_E^x$ were extensively investigated by Magyar [27], and Magyar, Stein and Wainger [28]. In the latter work a complete result was established, which asserts that $C(p, \sqrt{N}, S^{d-1}) < \infty$ if and only if $p > \frac{d}{d-1}$ and $d \geq 5$. The restricted weak-type endpoint result was also proved for $\sup_{t \in \sqrt{N}} |\mathcal{A}_E^x f|$ by Ionescu [21].

The main purpose of this work is to understand the asymptotic behavior of the best constants in maximal inequalities corresponding to spherical averages $\mathcal{M}_E^x$ as $d \to \infty$. Namely, we prove dimension-free estimates for the dyadic maximal function corresponding to $\mathcal{A}_E^x$, where the time parameter $t$ runs over the dyadic set $I = D_{\geq 1} = \{ 2^n : n \in \mathbb{N}_0 \}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{ 0 \}$. Our main result can be formulated as follows:

**Theorem 1.1.** For every $p \in [2, \infty]$ there exists a constant $C_p > 0$ such that

$$\sup_{t \geq 1} C(p, D_{\geq 1}, S^{d-1}) \leq C_p. \quad (1.2)$$

Theorem 1.1 is motivated by a question of Eli Stein from the mid 1990’s about the dimension-free estimates for the discrete Hardy–Littlewood maximal functions corresponding to the Euclidean balls, which using our notation can be stated as follows:

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**Stein’s question.** Let $B^2 := B^2(d) := \{ x \in \mathbb{R}^d : |x| \leq 1 \}$ be the Euclidean unit ball centered at the origin. Is it true that there is a constant $C > 0$ such that

$$
\sup_{d \in \mathbb{N}} C(2, \mathbb{R}^+ \cup B^2(d)) = \sup_{d \in \mathbb{N}} C(2, \sqrt{\mathbb{N}}, B^2(d)) \leq C? \tag{1.3}
$$

We now give some remarks about Theorem 1.1 and Stein’s question.

1. In fact, Theorem 1.1 is a purely $\ell^2(\mathbb{Z}^d)$ result, and by interpolation with $\ell^\infty(\mathbb{Z}^d)$ it suffices to prove that there is a constant $C > 0$ such that

$$
\sup_{d \geq 5} C(2, \mathbb{D}_{\geq 1}, S^{d-1}) \leq C. \tag{1.4}
$$

2. Recently, the first and the third author in a collaboration with Bourgain and Stein [8] made a first step towards establishing (1.3) and proved that there is a constant $C > 0$ such that

$$
\sup_{d \in \mathbb{N}} C(2, \mathbb{D}_{\geq 1}, B^2(d)) \leq C. \tag{1.5}
$$

This is a dyadic version of Stein’s question, which gives some evidence that (1.3) might be true.

3. An initial goal of [8] was motivated by the desire to establish (1.3) by noting a simple inequality

$$
\sup_{0 \leq t \leq T} |\mathcal{M}_t^B f(x)| \leq \sup_{0 \leq t \leq T} |A_t^B f(x)|, \quad T > 0, \quad x \in \mathbb{Z}^d, \quad f \in \ell^1(\mathbb{Z}^d), \tag{1.6}
$$

where $A_t^B f(x) = f(x)$. Since $S_0 = \{0\}$ we easily obtain (1.6) thanks to the identity

$$
\mathcal{M}_t^B f(x) = \frac{1}{|B_t^2(d) \cap \mathbb{Z}^d|} \sum_{\lambda \in \mathbb{N}^d, \lambda \leq t^2} |\mathcal{S}_{\sqrt{\lambda}} \cap \mathbb{Z}^d| A_t^B f(x), \quad f \in \ell^1(\mathbb{Z}^d),
$$

which is a consequence of the disjoint decomposition

$$
B_t^2(d) \cap \mathbb{Z}^d = \bigcup_{\lambda \in \mathbb{N}^d, \lambda \leq t^2} \mathcal{S}_{\sqrt{\lambda}} \cap \mathbb{Z}^d, \quad t > 0. \tag{1.7}
$$

Taking into account (1.6) it is easy to see that if one could find a constant $C > 0$ such that

$$
\sup_{d \geq 5} C(2, \sqrt{\mathbb{N}}, S^{d-1}) \leq C, \tag{1.8}
$$

then (1.3) would imply a positive answer to Stein’s question. Inequality (1.3) is the primary motivation behind this project. Unfortunately, the authors of [8] were unable to prove (1.3). In this paper we returned to this problem with a number of new ideas and in Theorem 1.1 (or more precisely in (1.3)) we establish a dyadic variant of (1.8). Appealing to the following estimate

$$
\| \sup_{t \in \sqrt{\mathbb{N}}} |A_t^B f| \|_{\ell^2(\mathbb{Z}^d)} \leq \| \sup_{t \in \mathbb{D}_{\geq 1}} |A_t^B f| \|_{\ell^2(\mathbb{Z}^d)} + \left( \sum_{n \in \mathbb{N}^d} \left( \sup_{t \in \sqrt{\mathbb{N}} \cap \mathcal{S}_n} |A_t^B f - A_2^B f|^2 \right)^{1/2} \right)_{\|_{\ell^2(\mathbb{Z}^d)}} \tag{1.9}
$$

one sees that inequality (1.3) can be thought of as the first step towards proving (1.8), since the first norm in (1.9) is bounded thanks to (1.4). Now the proof of (1.8) is reduced to bounding the square function in (1.8), which in turn may be reduced to understanding the difference

$$
\| A_t^B f - A_2^B f \|_{\ell^2(\mathbb{Z}^d)}. \tag{1.10}
$$

It is expected that (1.10) should be controlled from above by a constant (independent of $d$) multiple of $\lambda^{-3}$. A similar problem is apparent in [8] while $\mathcal{M}_t^B$ is studied in the context of inequality (1.3), and arguing as above the matter is reduced to understanding (1.10) with $\mathcal{M}_t^B$ in place of $A_t^B$. Interestingly, due to (1.6), this question is also related to controlling (1.10).

Although our methods have severe limitations and nothing can be said about (1.10) at the moment, we believe that (1.6) is the correct idea that motivates this work and it will be helpful in establishing inequality (1.3) in Stein’s question.

4. The idea of using (1.6) goes back to Stein [34], see also Stein and Strömberg [35], where the dimension-free estimates for the continuous Hardy–Littlewood maximal functions corresponding to the Euclidean balls were proved, see (1.14) below. We shall describe this method in a moment, as it may give an alternative way to approach (1.8).
5. In the proof of Theorem 1.1 we recover the asymptotic formula for the number of lattice points in the spheres $S^{d-1}_{\sqrt{t}}$ for $d \geq 5$, see [22] Theorem 20.2, p. 456 and [30] Theorem 5.7, p. 146. In fact, in Theorem 5.1 we improved qualitatively and quantitatively the asymptotic formula in the classical Waring problem for the squares and obtained the multiplicative error term that satisfies certain uniformities with respect to radii and dimensions. This part of our paper may be of independent interest. More precisely, see (5.17), we prove that there exists a constant $C > 0$ independent of the dimension such that

$$|S^{d-1}_{\sqrt{x}} \cap \mathbb{Z}^d| = \frac{\pi^{d/2}}{\Gamma(d/2)} \lambda^{d/2-1} \mathcal{S}_d(\lambda) \left(1 + o(1)\right) \quad \text{as} \quad \lambda \geq Cd^d \quad \text{and} \quad d \to \infty,$$  \hspace{1cm} (1.11)

and the singular series $\mathcal{S}_d(\lambda)$ given by (5.15) satisfies $\frac{1}{2} \leq \mathcal{S}_d(\lambda) \leq \frac{5}{2}$ for all $d \geq 16$ and $\lambda \in \mathbb{N}$. We have not found in the existing literature anything about the uniformities with respect to radii and dimensions in the context of the asymptotic formula (1.11). However, we believe it is a very natural problem, which is interesting in its own right, and might have been studied in the past. Another natural question of great interest arises whether a similar formula holds in (1.11) if $1 \leq \lambda \leq Cd^d$ and $d \to \infty$. Our method does not work in this regime. We hope to investigate this problem in future work, and also in the context of (1.10) as well as Stein’s question.

6. We conclude with a brief mention that our method allows us to verify (1.8) for large scales, see Remark 5.1. Namely, there are universal constants $C, C_3 > 0$ such that

$$\sup_{d \geq 5} C(2, \sqrt{t} \cap (C_3 d^{3/2}, \infty), S^{d-1}) \leq C.$$

However, if $1 \leq \lambda \leq C_3^2 d^d$ then (in view of (1.9)) inequality (1.8) is reduced to investigating (1.10), for which a preliminary step is to estimate

$$||S^{d-1}_{\sqrt{x} \cap \mathbb{Z}^d} - |S^{d-1}_{\sqrt{x}} \cap \mathbb{Z}^d||.$$

(1.12)

At this moment it is not clear whether it is possible to gain any power of $\lambda^{-1}$ in (1.10).

During the work on article [8] the authors had even been thinking about counterexamples for (1.5), since essentially at the same time it was shown [3] Theorem 2] that the dimension-free phenomenon may fail in the discrete setup. Surprisingly, one can prove that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for certain ellipsoids $E(d) \subset \mathbb{R}^d$ and all $d \in \mathbb{N}$ one has $C_p(\mathbb{R}_+, E(d)) \geq C_p(\log d)^{1/p}$. This stands in sharp contrast to the situation that we know from the continuous setup [1] [5] [6] [7] [23] [34] [35]. We also refer to the survey articles [17] and [10] for more exhaustive exposition of the dimension-free phenomena in the continuous setting. On the other hand, for the cubes $B^{\infty}(d) := [-1, 1]^d$ it was also shown in [9] Theorem 3] that for every $p \in (3/2, \infty]$ there is a constant $C_p > 0$ such that $\sup_{d \in \mathbb{N}} C(p, \mathbb{R}_+, B^{\infty}(d)) \leq C_p$. For $p \in (1, 3/2]$ it still remains open whether $\sup_{d \in \mathbb{N}} C(p, \mathbb{R}_+, B^{\infty}(d))$ is finite. However, one can prove that for all $p \in [1, \infty]$ there is a constant $C_p > 0$ such that $\sup_{d \geq 1} C(p, \mathbb{R}_+, B^{\infty}(d)) \leq C_p$. All these circumstances were the turning point, which forced the authors of [8] to change the above-described strategy in 3. and find a different way to prove (1.5). We adapt the scheme of the proof and some of these strategies from [8] to prove Theorem 1.1 more precisely (1.3). Although the methods developed in [8] are important in this paper, there are some novel ideas in our approach that we now highlight:

1. As opposed to the situation considered in [8], here we use a variant of the Hardy–Littlewood circle method to analyze the Fourier multipliers (1.20) corresponding to the spherical averages $A_1^d$. This is a consequence of a more singular nature of averages $A_1^d$, which is noticeable in the fact that the family of spheres $(S^{d-1}_{t_1})_{t \in \mathbb{R}_+}$ fails to be nested in contrast to the family of balls $(B^2_1(d))_{t \in \mathbb{R}_+}$, i.e. if $t_1 < t_2$ then $S^{d-1}_{t_1} \cap S^{d-1}_{t_2} = \emptyset$, whereas $B^2_1(d) \subset B^2_1(d)$. As a matter of fact, we follow the ideas of Magyar, Stein and Wainger [28] and adjust their approach to the dimension-free problem. This is a delicate process described in Theorem 3.1 where it was essential to keep track carefully of how the constants arising in the error terms of the underlying circle method depend on $d$. As a result we have obtained (1.11) with a multiplicative dimension-free error term, which allows us to circumvent the absence of a dilatation structure on $\mathbb{Z}^d$. While working on [8] the authors were not able to detect the correct relationship between radii $\lambda$ and dimensions $d$ that would guarantee (1.11). This was one of the obstacles why the strategy described in 3. had been given up in [8].

2. An important tool of the Magyar, Stein and Wainger paper [28] is the sampling principle, see [28] Corollary 2.1] (or Proposition 5.2 below), which is a general abstract theorem that allows
one to compare $L^p(\mathbb{R}^d)$ boundedness of certain convolution operators on $\mathbb{R}^d$ with $\ell^p(\mathbb{Z}^d)$ boundedness of analogous operators on $\mathbb{Z}^d$. Although very useful in many discrete problems, a literal application of the sampling principle, as in [25], to our problem makes no sense, since it produces an exponential in dimension (like $C^d$) growth in norm which is prohibited, see Proposition 4.2. This also prevented the authors of [8] to use the spherical maximal function to prove [1, 3]. In the remarks after [28, Proposition 2.1] the authors ask whether the constant in [28, Corollary 2.1] can be taken to be independent of $d$ or even $C = 1$. To the best of our knowledge, if the multiplier in the sampling principle [28] takes values in the space of bounded linear operators between two finite-dimensional Banach spaces $B_1$ and $B_2$, the question about the dimension-free $\ell^p(\mathbb{Z}^d; B_1) \to \ell^p(\mathbb{Z}^d; B_2)$ estimates in the sampling principle [28] for all $p \in [1, \infty]$ is still open. However, if $B_1$ and $B_2$ are finite-dimensional Hilbert spaces Kovrizehkin [26], and also recently Tao gave independently negative answers to this question as long as $p$ is sufficiently close to 1 or $\infty$. The answer probably is negative for all $p \neq 2$, but at this moment it is open. In the Hilbert space setup if $p = 2$ the $C^d$ factor from the sampling principle in [28] may be deleted by a simple application of the Plancherel theorem. In our case the situation is more complicated since we work with non-Hilbert spaces. Even for $p = 2$ it is suspected that the constant in the sampling principle [28] (in the non-Hilbert setting) is no longer 1, and in fact depends exponentially on the dimension, but we do not have a proof of this. Working on the current project, along the way, we come across some perhaps unexpected property of approximating multipliers, which essentially permits us to eat the exponential growth in dimension from the sampling principle. Specifically, we modified the approximating multiplier from [28] by considering new multipliers $4.1$ and $4.2$, which produce acceptable error terms $4.3$, and absorb the exponential growth arising in the sampling principle, see Theorem 4.2. We hope that our approach will be also useful when the dimension-free estimates will be discussed for $p \neq 2$.

3. Finally, in Lemma 4.1 and Lemma 4.2 we provided estimates of the Fourier transform corresponding to the continuous spherical measure on $S^{d-1}$ that lead to the dimension-free bounds in Lemma 1.3 and Lemma 1.6. Although the Fourier transform estimates of the surface measures received considerable attention over the years, much to our surprise it seems that no extensions delivered in Lemma 1.1 and Lemma 1.2 have appeared in the literature. The estimates from Lemma 1.2 are actually very much in the spirit of Bourgain’s result [4, eq. (10), (11), (12), p. 1473], where the estimates of multipliers associated with the Hardy–Littlewood averages over convex symmetric bodies in $\mathbb{R}^d$ are provided in terms of the corresponding isotropic constants. Interestingly, in contrast to Bourgain’s proof [1] the proofs of our results are based on elementary manipulations of the Bessel functions, (like change the contour of integration, see Lemma 1.1) and uniform estimates of the Bessel functions that lead to the conclusion of Lemma 1.2. Lemma 1.2 plays an essential role in the estimates of multipliers $5.1$ and $5.2$. This is a new ingredient which was not apparent in [28].

Since the Magyar, Stein and Wainger paper [28] is critical in this paper and has had a profound impact on the discrete harmonic analysis we conclude with a brief mention about the current state of the art in the related areas. Kesler, Lacey and Mena [23, 24] started to develop sparse estimates in the context of discrete spherical averages. It is a very successful line of research, which significantly enhanced the field of discrete harmonic analysis. In [24] conjecturally sharp sparse bounds for $\sup_{\lambda \in \mathbb{N}} |A^\lambda f|$ were proved, and used to give a new proof of Magyar, Stein and Wainger theorem [28] as well as the endpoint result of lonescu [21] in a fairly unified way. Another interesting line of research has been initiated by Hughes [20], who asked about the bounds for $\mathcal{C}(p, \mathbb{L}, S^{d-1})$, where $\mathbb{L} \subset \mathbb{N}$ is a lacunary set. Hughes also observed (even though the Magyar–Stein–Wainger theorem [28] is sharp) that it also makes sense to study $\mathcal{C}(p, 2N + 1, S^{d-1})$ for $d = 4$ upon restricting the radii $\lambda$ to odd integers. Specifically, Hughes [20] constructed a very sparse set $\mathbb{L} \subset \mathbb{N}$ of radii such that $\mathcal{C}(p, \mathbb{L}, S^{d-1}) \leq 4 \mathcal{C}(p, \mathbb{N}, S^{d-1})$ for $\frac{4}{d-1} \leq p \leq 2$ and $d \geq 4$. The latter result was recently extended by Kesler, Lacey and Mena [23], where it was shown that $\mathcal{C}(p, \mathbb{L}, S^{d-1}) \leq 4 \mathcal{C}(p, \mathbb{N}, S^{d-1}) \leq 4 \mathcal{C}(p, \mathbb{N}, S^{d-1})$ for $\frac{d-1}{d+1} < p \leq 2 \mathcal{C}(p, \mathbb{N}, S^{d-1})$ and $d \geq 5$. The case $d = 4$ was recently established by Anderson and Madrid [2] for $\frac{4}{d+1} < p \leq 2$ and all lacunary sequences $\mathbb{L} \subset \mathbb{N} \setminus \mathbb{N}$. Cook and Hughes [16] studied similar problems in the context of Birch forms, and specifically recovered the main result from [23]. They also illustrated in [16] that for any $1 < p < \frac{d+1}{d-1}$ there exists a set of lacunary radii $\mathbb{L} \subset \mathbb{N}$ such that $\mathcal{C}(p, \mathbb{L}, S^{d-1}) = \infty$. This negative result remains also true [16] for averages over more general forms in the spirit of Birch. This is a remarkable phenomenon that exhibit some peculiar features in the discrete world. Finally, it is worth noting that the negative result
from [15] does not exclude positive results for \(1 < p < \frac{d}{d-1}\). Namely, Cook showed that \(C(p, l, S^{d-1}) < \infty\) for all \(1 < p \leq \infty\) and \(d \geq 5\) by constructing a very sparse sequence of radii [15], he also showed a similar phenomenon [14] for the averages associated to a certain class of homogeneous algebraic hypersurfaces.

1.2. Dimension-free estimates in the continuous setting. We now make a link between the strategy described above with the ideas of the proof of dimension-free estimates for the Hardy–Littlewood maximal functions associated with the continuous averaging operators over the Euclidean balls in \(\mathbb{R}^d\).

For every \(t > 0\) and \(x \in \mathbb{R}^d\) we define the continuous Hardy–Littlewood averaging operator by

\[
M_t^B f(x) := \frac{1}{|B_t|} \int_{B_t^d} f(x - y) dy, \quad f \in L^1_{\text{loc}}(\mathbb{R}^d).
\]

For \(\emptyset \neq I \subseteq \mathbb{R}_+\) and \(p \in [1, \infty]\) let \(0 < C(p, I, B^2(d)) \leq \infty\) be the smallest constant in the following maximal inequality

\[
\| \sup_{t \in I} |M_t^B f| \|_{L^p(\mathbb{R}^d)} \leq C(p, I, B^2(d)) \| f \|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d).
\]  

(1.13)

Using a standard covering argument for \(p = 1\) and interpolation with \(p = \infty\) it is not hard to see that \(C(p, \mathbb{R}_+, B^2(d)) < \infty\) for every \(p \in (1, \infty]\), since \(C(\infty, \mathbb{R}_+, B^2(d)) = 1\). Stein [33] (see also Stein and Strömberg [35]) proved that there exists a constant \(C_p > 0\) depending only on \(p \in (1, \infty]\) such that

\[
\sup_{d \in \mathbb{N}} C(p, \mathbb{R}_+, B^2(d)) \leq C_p.
\]  

(1.14)

The key idea from [33] to establish (1.14) is to use the spherical averaging operator, defined for any \(t > 0\) and \(x \in \mathbb{R}^d\) by

\[
A_t^d f(x) := \int_{S^{d-1}} f(x - t\theta) d\mu^d(\theta), \quad f \in C_c(\mathbb{R}^d),
\]

where \(\mu^d\) denotes the normalized surface measure on \(S^{d-1}\), see [23] below. We now see that the spherical operator \(A_t^d\) from [14] is a discrete analogue of the operator \(A_t^d\). Let \(C(p, I, S^{d-1})\) be the best constant in inequality (1.13) with \(A_t^d\) in place of \(M_t^B\). It is very well known from the results of Stein [32] for all \(d \geq 3\), and Bourgain [3] for \(d = 2\) that \(C(p, \mathbb{R}_+, S^{d-1}) < \infty\) if and only if \(\frac{d}{d-1} < p \leq \infty\). We also know that \(C(p, \mathbb{D}, S^{d-1}) < \infty\) for all \(1 < p \leq \infty\) as it was shown by Calderón [11] and independently by Coifman and Weiss [15]. Using polar coordinates one easily sees that

\[
\sup_{t > 0} |M_t^B f(x)| \leq \sup_{t > 0} |A_t^d f(x)|.
\]  

(1.15)

Now the method of rotations enables one to view high-dimensional spheres as an average of rotated low-dimensional ones, and consequently one can conclude that for every \(d \geq 2\) and \(p > \frac{d}{d-1}\) we have

\[
C(p, \mathbb{R}_+, S^{d}) \leq C(p, \mathbb{R}_+, S^{d-1}) < \infty.
\]  

(1.16)

Hence, the sequence \((C(p, \mathbb{R}_+, S^{d-1}))_{d \in \mathbb{N}}\) is non-increasing, and in particular bounded, in \(d > \frac{p}{p-1}\). Therefore, to prove (1.14) we fix \(p \in (1, \infty]\) and pick the smallest integer \(d_0 \in \mathbb{N}\) such that \(d_0 > \frac{p}{p-1}\). We can assume, without loss of generality, that \(d > d_0\), then by (1.15), (1.16) and Stein’s result we conclude

\[
C(p, \mathbb{R}_+, B^2(d)) \leq C(p, \mathbb{R}_+, S^{d-1}) \leq C(p, \mathbb{R}_+, S^{d_0-1}) < \infty,
\]

and (1.14) follows.

Thinking about Stein’s question (1.3) the authors of [3] tried to adapt the ideas of the proof of (1.14) to the discrete setting. Although in [14] there is a discrete analogue of (1.14) it is completely unclear whether there is a discrete analogue of (1.16). More precisely, it is interesting to know whether for every \(d \geq 5\) and \(p > \frac{d}{d-1}\) the following is true

\[
C(p, \mathbb{R}_+, S^{d}) \leq C(p, \mathbb{R}_+, S^{d-1}).
\]  

(1.17)

The estimate (1.17) is not easy even for \(p = 2\), mainly due to the lack of the dilatation structure and the method of rotation on \(\mathbb{Z}^d\), which were essential to establish (1.14) in [23][33]. A natural remedy for the first of these obstacles is the asymptotic formula with a dimension-free multiplicative error term as in (1.17), whereas for the second one are symmetries of \(S_\sqrt{T} \cap \mathbb{Z}^d\):

(a) If \((x_1, \ldots, x_d) \in S_\sqrt{T} \cap \mathbb{Z}^d\), then \((\varepsilon_1 x_1, \ldots, \varepsilon_d x_d) \in S_\sqrt{T} \cap \mathbb{Z}^d\) for any \((\varepsilon_1, \ldots, \varepsilon_d) \in \{-1,1\}^d\).

(b) If \((x_1, \ldots, x_d) \in S_\sqrt{T} \cap \mathbb{Z}^d\), then \((x_{\tau(1)}, \ldots, x_{\tau(d)}) \in S_\sqrt{T} \cap \mathbb{Z}^d\) for any permutation \(\tau \in \text{Sym}(d)\).
These kind of symmetries were also strongly exploited in [8] for the discrete Euclidean balls.

Let us emphasize that if we could prove (1.14) then it would imply (1.15), and moreover, in view of the Magyar, Stein and Wainger theorem [28] we would be able to give an affirmative answer to Stein’s question even for all $p \in (1, \infty)$ in place of 2 in (1.15), which would be a genuine discrete analogue of inequality (1.13).

Finally, we remark that the method from [34, 35] is limited to the Euclidean balls. The case of general convex symmetric bodies $G \subset \mathbb{R}^d$ (which means that $G$ is a convex, compact subset of $\mathbb{R}^d$ which is symmetric and has a non-empty interior), requires a different approach. Let $C(p, G)$ be the best constant in (1.16), where $B^2$ is replaced with a general convex symmetric bodies $G \subset \mathbb{R}^d$.

Stein’s work [34] gave rise to the famous conjecture, which asserts that for every $p \in (1, \infty]$ there is a constant $C_p > 0$ such that

$$\sup_{d \in \mathbb{N}} \sup_{G \subset \mathbb{R}^d} C(p, R^d, G) \leq C_p,$$

where $\mathbb{B}(d)$ is the set of all convex symmetric bodies in $\mathbb{R}^d$. This problem has been studied for four decades by several authors. We now briefly list the current state of the art concerning (1.18) as well as its relations to dimension-free phenomena in the discrete setting.

1. Bourgain [4, 5], and independently Carbery [12], proved (1.18) for $p \in (3/2, \infty]$. They also showed a dyadic variant of (1.18) for $p \in (1, \infty)$ with $D := \{2^n : n \in \mathbb{Z}\}$ in place of $\mathbb{R}^d$. Although inequality (1.15) for $p \in (1,3/2]$ remains still open, the case of $q$-balls is quite well understood.

2. For the $q$-balls $B^q(d)$ (see definition (2.2)) we know that for every $p \in (1, \infty]$ and for every $q \in [1, \infty]$ there is a constant $C_{p,q} > 0$ such that

$$\sup_{d \in \mathbb{N}} C(p, \mathbb{R}^d, B^q(d)) \leq C_{p,q}.$$ 

This was established by Müller in [25] (for $q \in [1, \infty]$) and by Bourgain in [6] (for cubes $q = \infty$).

3. Dimension-free estimates in the discrete setting were initiated by the first and the third authors in a collaboration with Kosz and Plewa [25] proved that for every $G \in \mathbb{B}(d)$ one has

$$C(p, \mathbb{R}^d, G) \leq C(p, \mathbb{R}^d, G), \quad \text{for all } p \in [1, \infty],$$

where the case $p = 1$ corresponds to the optimal constants in the weak type $(1,1)$ inequalities respectively in $\mathbb{R}^d$ and $\mathbb{Z}^d$. It was also shown in [25] that $C(1, \mathbb{R}^d, B^\infty(d)) = C(1, \mathbb{R}^d, B^\infty(d))$, which in view of Aldaz’s result [11] yields $C(1, \mathbb{R}^d, B^\infty(d)) \rightarrow_{d \to \infty} \infty$. Thus the boundedness of $\sup_{d \in \mathbb{N}} C(p, \mathbb{R}^d, B^\infty(d))$ for $p \in (1,3/2]$ cannot be deduced by interpolation from [9, Theorem 3] for $p \in (3/2, \infty]$. Inequality (1.19) exhibits a well known phenomenon in harmonic analysis, which states that it is harder to establish bounds for discrete operators than the bounds for their continuous counterparts, and this is the best what we could prove in this generality.

1.3. Overview of the methods and proofs. We now give an overview of the proofs of our main results. The proof of Theorem 1.1 is similar in spirit to the proof of inequality (1.15) (we refer to [8] for the details), but with several new difficulties arising that require new ideas to overcome. The proof of Theorem 1.2 will consist of three steps. Specifically, we shall prove the following dimension-free estimates, which will result in inequality (1.2).

**Theorem 1.2.** There exist absolute constants $C, C_0, C_1, C_2, C_3 > 0$ such that the following is true.

1. The large-scale estimate holds

$$\sup_{d \geq 5} C(2, D_{C_1, \infty}, S^{d-1}) \leq C,$$

where $D_{C_1, \infty} := \{t \in D_{\geq 1} : t \geq C_2 t^{d/2}\}$.

2. The intermediate-scale estimate holds

$$\sup_{d \geq 5} C(2, D_{C_1, C_2}, S^{d-1}) \leq C,$$

where $D_{C_1, C_2} := \{t \in D_{\geq 1} : C_1 t^{d/2} \leq t \leq C_2 t^{d/2}\}$.

3. The small-scale estimate holds

$$\sup_{d \geq 5} C(2, D_{C_0}, S^{d-1}) \leq C,$$

where $D_{C_0} := \{t \in D_{\geq 1} : 1 \leq t \leq C_0 t^{d/2}\}$. 
Since we are working with the dyadic numbers the exact values of $C_0, C_1, C_2, C_3$ will never play a role as long as they are absolute constants. Indeed, if we establish (1.20), (1.21) and (1.22) for some constants $C, C_0, C_1, C_2, C_3 > 0$, then (1.20), (1.21) and (1.22) remain true with any other set of constants $C', C'_0, C'_1, C'_2, C'_3 > 0$ in place of $C_0, C_1, C_2, C_3$ and some constant $C' > 0$, that may depend on $C_0, C_1, C_2, C_3$, in place of $C$. Taking into account this remark we may always adjust $C_0 = C_1$ and $C_2 = C_3$ in Theorem 1.2 which immediately yields Theorem 1.3. In what follows, the implied constants will be always allowed to depend on $C_0, C_1, C_2, C_3$ and we shall be mainly concerned with proving Theorem 1.2.

If we restrict ourselves to small dimensions $5 \leq d < 16$, then Theorem 1.2 holds by 28. Therefore throughout the proof of Theorem 1.2 it suffices to assume that $d \geq 16$.

The proof of Theorem 1.2 uses a variety of Fourier methods to understand the multipliers $m_\ell$ (see 3.3) corresponding to the averages $A_\ell^d$, with $\ell = \lambda \in \mathbb{N}$. In Section 8 that handles the large-scale case (1.20), when $t \geq C_3 d^{5/2}$, we follow the ideas of Magyar, Stein and Wainger 28 and use a variant of the Hardy–Littlewood circle method to establish the asymptotic formula for $m_\ell$ for $t \geq C_3 d^{5/2}$, see Theorem 8.1. In the proof of Theorem 5.1 we had to keep track carefully of how the constants in all error terms depend on the dimension. Only an exponential growth of the form $C d$ is admissible, which consequently determines of how large the constant $C_3$ must be. This is the novelty of this paper, which can be thought of as a dimension-free variant of the circle method. To the best of our knowledge this aspect has not been discussed in the literature. An important consequence of Theorem 8.1 is the asymptotic formula (1.11) with a dimension-free multiplicative error term, which permits us to overcome the problem with the absence of dilatation structure on $\mathbb{Z}^d$. Theorem 5.1 is a key building block of Proposition 5.1 where the approximating multipliers $5.1$ and (5.2) are defined. This is the place, where our approach diverges from the method presented in 25. By introducing decomposition (5.3) in Proposition 5.1 and manipulating the parameter $n$ in the definition of the second multiplier (5.2) we were able to absorb (using the decay from the Gauss sums (5.20)) the exponential dimension dependence produced by the sampling principle of Magyar, Stein and Wainger 28 Corollary 2.1, see Theorem 5.2. Although the maximal function corresponding to the second approximating multiplier (5.2) is handled in Theorem 5.2 some price must be paid, since the sampling principle cannot be used to bound the maximal function corresponding to the first approximating multiplier (5.1). Fortunately the latter maximal function is estimated in Theorem 5.1. Here is the place, where the estimates from Lemma 4.2 (in the spirit of Bourgain’s isotropic-constant-estimates from 3) of the Fourier transform of continuous spherical measures enter into play. The estimates from Lemma 4.1 and Lemma 4.2 may be new and of independent interest. We hope that these observations will be useful in extending dimension-free estimate (1.4) for $p \neq 2$. It is noteworthy that the large-scale case for the Hardy–Littlewood maximal function from 8 is a simple consequence of the comparison principle from 9 Theorem 1], since the averages $M d^{2^d}$ are defined over the nested family of discrete Euclidean balls $B_2^{d}(d) \cap \mathbb{Z}^d$. This contrasts sharply with the above-described situation of the maximal spherical averages, which being more singular required a dimension-free variant of the circle method that we developed to circumvent various difficulties with the lack of nestedness for the spheres. We finally remark that Theorem 5.1 is also critical in the proof of Proposition 5.4 which in turn plays a pivotal role in the intermediate-scale estimate (1.24).

In Section 6 we handle the intermediate-scale case (1.24), when $C_1 d^{1/2} \leq t \leq C_2 d^{1/2}$. Here we mainly follow the ideas developed in 8 Section 2. However, there are subtle adjustments necessary to fit the arguments to the new situation. The multiplier (5.3) has an additional symmetry

$$m_\ell(\xi + (1/2, \ldots, 1/2)) = (-1)^2 m_\ell(\xi), \quad \xi \in \mathbb{T}^d, \quad t \in \sqrt{N},$$

(1.23) which causes new complications. Proposition 6.3 and Proposition 6.4 reveal the importance of property (1.23), which is reflected by the appearance of various quantities depending on $\xi + (1/2, \ldots, 1/2)$ in 4.1.1 and (6.1). The proofs of Proposition 6.3 and Proposition 6.4 exploit the symmetries of spheres $S_t \cap \mathbb{Z}^d$ described in (a) and (b). The invariance of the spheres $S_t \cap \mathbb{Z}^d$ under the permutation group (b) brought into play probabilistic tools, which are especially important in the proof of inequality (6.7) from Proposition 6.1 as well as in Lemma 8.0. There are three results: Lemma 6.4, Lemma 6.4 and Lemma 6.5 necessary for the proof of inequality (6.1) to work. Lemma 6.3 asserts that an essential amount of mass of the sphere $S_t \cap \mathbb{Z}^d$ is concentrated in the region where the coordinates are large. Its proof can be reduced to the corresponding result for balls from 8 upon proving Lemma 5.4 which gives a simple comparison between numbers of lattice points in balls and spheres. Lemma 6.4, as in 8, leads us to the so-called decrease dimension trick in Lemma 6.5. The decrease dimension trick resemble to some extent the method of rotations 34. 35 from the continuous setting that enables one to view high-dimensional spheres as an average of rotated low-dimensional ones. Here we use Lemma 6.5 to view the original
multiplier $m_l$ as an average of new multipliers $m_i^{(r)}$ (see (6.12)) associated with spheres $S_l^{r-1} \cap \mathbb{Z}^r$ in lower dimensional spaces whose radii satisfy the relation $l \geq C_0 r^{3/2}$ from the previous case with respect to the new dimensions $r \in \mathbb{N}$ such that $r \leq d$. Now the machinery from the previous section can be used. Specifically, Proposition 4.4 can be applied to the $m_l^{(r)}$, and this with the aid of a variant of the convexity lemma established in [8] (see Lemma 3.7) yields Proposition 6.1.

Finally, in Section 7 we handle the small-scale case (1.22), when $1 \leq t \leq C_0 d^{1/2}$. We proceed much the same way as in [8] Section 3]. We find suitable approximating multipliers and reduce the matters to the square function estimates using Proposition 7.1. Then we establish Lemma 7.2, which states that an decay of the multipliers is closely related to the Krawtchouk polynomial (7.6), see [19]. Using a uniform bound for the Krawtchouk polynomials (see Property 3.3 in Theorem 7.1) we are able to deduce in Proposition 7.3 a decay of the multipliers $m_l$ at infinity. Proposition 7.3 is the main result of Section 7 and its proof follows very closely the proof of corresponding result for the balls from [8] Section 3], therefore we refer to [8] for more details.

2. Notation

Here we set out some basic notation that will be used throughout the paper. The letter $d \in \mathbb{N}$ is the dimension for the dimension throughout this paper.

2.1. Basic notation. The sets $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ have standard meaning.

1. We use $\mathbb{N} := \{1, 2, \ldots \}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ to denote the sets of positive integers and non-negative integers, respectively. We also set $\mathbb{N}_N := \{1, 2, \ldots , N\}$ for any $N \in \mathbb{N}$.

2. Let $\mathbb{D} := \{2^n : n \in \mathbb{Z}\}$ denote the set of all dyadic numbers in $\mathbb{R}_+ := (0, \infty)$ and the set of all dyadic numbers in $\mathbb{N}$ will be denoted by $\mathbb{D}_N := \{2^n : n \in \mathbb{N}_N\}$.

3. For any $x \in \mathbb{R}$ we will use the floor function $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$, the fractional part $\{x\} := x - [x]$ and the distance to the nearest integer $\|x\| := \text{dist}(x, \mathbb{Z})$.

4. For further reference observe that for $\eta \in \mathbb{T}$ one has $2\|\eta\| \leq |\sin(\pi \eta)| \leq \|\eta\|$, since $|\sin(\pi \eta)| = |\sin(\pi \|\eta\|)|$ and $2\|\eta\| \leq |\sin(\pi \|\eta\|)| \leq \pi \|\eta\|$, for $0 \leq \|\eta\| \leq 1/2$. (2.1)

5. We use $1_A$ to denote the indicator function of a set $A$. If $S$ is a statement we write $1_S$ to denote its indicator, equal to 1 if $S$ is true and 0 if $S$ is false. For instance $1_A(x) = 1_{x \in A}$.

6. For $d \in \mathbb{N}$ let $\text{Sym}(d)$ be the permutation group on $\mathbb{N}_d$. We know that $|\text{Sym}(d)| = d!$. For $A \subseteq \text{Sym}(d)$ let $\mathbb{P}[A] := |A|/d!$ be the uniform distribution on the symmetry group $\text{Sym}(d)$. The expectation $\mathbb{E}$ will be always taken with respect to the uniform distribution $\mathbb{P}$ on the symmetry group $\text{Sym}(d)$.

2.2. Asymptotic notation and magnitudes.

1. The letters $C, c, C_0, C_1, \ldots > 0$ will always denote absolute constants which do not depend on the dimension, however their values may vary from occurrence to occurrence.

2. For two nonnegative quantities $A, B$ we write $A \lesssim B$ ($A \gtrsim B$) if there is an absolute constant $C_0 > 0$ (which possibly depends on $\delta > 0$) such that $A \leq C_0 B$ ($A \geq C_0 B$). We will write $A \approx B$ when $A \lesssim B$ and $A \gtrsim B$ hold simultaneously. We will omit the subscript $\delta$ if irrelevant.

3. For two nonnegative quantities $A, B$ we will also use the convention that $A \lesssim B$ ($A \gtrsim B$) to say that there is an absolute constant $C > 0$ such that $A \leq C B$ ($A \geq C B$). If $A \lesssim B$ and $A \gtrsim B$ hold simultaneously, then we write $A \approx B$.

4. For a function $f : X \to \mathbb{C}$ and positive-valued function $g : X \to (0, \infty)$, we write $f = O(g)$ if there exists a constant $C > 0$ such that $|f(x)| \leq C g(x)$ for all $x \in X$. We will also write $f = \Theta(g)$ if the implicit constant depends on $\delta$. For two functions $f, g : X \to \mathbb{C}$ such that $g(x) \neq 0$ for all $x \in X$ we write $f = o(g)$ if $\lim_{x \to \infty} f(x)/g(x) = 0$.

2.3. Euclidean spaces. Denote by $1$ the vector $(1, \ldots , 1) \in \mathbb{R}^d$.

1. The Euclidean space $\mathbb{R}^d$ is endowed with the standard inner product $x \cdot \xi := \langle x, \xi \rangle := \sum_{k=1}^d x_k \xi_k$.
for every $x = (x_1, \ldots, x_d)$ and $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, and the corresponding Euclidean norm is denoted by $|x| := |x|_2 = \sqrt{x \cdot x}$ for any $x \in \mathbb{R}^d$.

2. We will write $\tau \circ x = (x_{\tau(1)}, \ldots, x_{\tau(d)})$ for every $x \in \mathbb{R}^d$ and $\tau \in \text{Sym}(d)$.

3. We will identify the $d$-dimensional torus $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ with the unit cube $Q := [-1/2, 1/2]^d$.

4. For $x \in \mathbb{R}^d$ let $[x]$ be defined as the unique vector in $\mathbb{Z}^d$ such that $x - [x] \in [-1/2, 1/2)^d$. In particular note that for $\xi \in Q$ we have $|\xi| = 0$.

5. For $x \in \mathbb{R}^d$ we will write $||x||^2 := \sum_{j=1}^d |x_j|^2$, where $||x|| = \text{dist}(x, \mathcal{Z})$ for $x \in \mathbb{N}_d$. Note that for $\xi \in Q$ the norm $|\xi|$ coincides with the Euclidean norm $|\xi|$.

6. We define

$$B^d := B^d(d) := \left\{ x \in \mathbb{R}^d : |x|_q := \left( \sum_{1 \leq k \leq d} |x_k|^q \right)^{1/q} \leq 1 \right\} \text{ for } q \in [1, \infty),$$

$$B^\infty := B^\infty(d) := \{ x \in \mathbb{R}^d : |x|_\infty := \max_{1 \leq k \leq d} |x_k| \leq 1 \}. \quad (2.2)$$

7. Let $S_t := S^d_{d-1} := \{ x \in \mathbb{R}^d : |x| = t \}$ denote the sphere with radius $t > 0$ centered at the origin. If $t = 1$ we abbreviate $S^d_{d-1}$ to $S^d$.

8. The symbol $\sigma^d$ denotes the canonical surface measure on the unit sphere $S^{d-1}$, and let

$$\mu^d := \frac{1}{\sigma^d(S^{d-1})}\sigma^d \quad (2.3)$$

be its normalization.

2.4. $L^p$ spaces. $(X, \mathcal{B}(X), \nu)$ denotes a measure space $X$ with $\sigma$-algebra $\mathcal{B}(X)$ and $\sigma$-finite measure $\nu$.

1. The set of $\nu$-measurable complex-valued functions defined on $X$ will be denoted by $L^0(X)$.

2. The set of functions in $L^0(X)$ whose modulus is integrable with $p$-th power is denoted by $L^p(X)$.

3. Let $B$ be a Banach space and let $L^0(X; B)$ denote the space of all $B$-measurable functions. In our applications we can restrict our attention to the case when the underlying Banach space is finite dimensional. Our estimates will be independent of the Banach spaces in question and limiting arguments will encompass the results in the desired generality.

4. For $p \in (0, \infty]$ we define the $L^p$ spaces of $B$-valued functions

$$L^p(X; B) := \{ F \in L^0(X; B) : \| F \|_{L^p(X; B)} = \| |F| B\|_{L^p(X)} < \infty \}.$$  

5. In our case we will usually have $X = \mathbb{R}^d$ or $X = \mathbb{T}^d$ equipped with the Lebesgue measure, and $X = \mathbb{Z}^d$ endowed with the counting measure. If $X$ is endowed with counting measure we will abbreviate $L^p(X)$ to $\ell^p(X)$, and $L^p(X; B)$ to $\ell^p(X; B)$.

2.5. Fourier transforms.

1. The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ will be denoted by

$$\mathcal{F} f(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx \text{ for any } \xi \in \mathbb{R}^d.$$

More generally, if $\nu$ is a complex Borel measure on $\mathbb{R}^d$ then the Fourier transform of $\nu$ is defined by

$$\mathcal{F} \nu(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} d\nu(x) \text{ for any } \xi \in \mathbb{R}^d.$$

If $f \in \ell^1(\mathbb{Z}^d)$ we define the discrete Fourier transform by setting

$$\hat{f}(\xi) := \sum_{x \in \mathbb{Z}^d} f(x) e^{-2\pi i \xi \cdot x} \text{ for any } \xi \in \mathbb{T}^d,$$

To simplify notation we will denote by $\mathcal{F}^{-1}$ the inverse Fourier transform on $\mathbb{R}^d$ or the inverse Fourier transform (Fourier coefficient) on the torus $\mathbb{T}^d$. It will cause no confusions since their meaning will be always clear from the context.
3. Asymptotic formulae and dimension-free Waring problem

In what follows we are only interested in $t > 0$ such that $\lambda = t^2 \in \mathbb{N}$. Throughout the paper $t$ and $\lambda$ are always related by the equation

$$t = \sqrt{\lambda}.$$

The following notation will be used

$$w_\lambda(x) := 1_{S_{\sqrt{\lambda}} \cap \mathbb{Z}^d}(x), \quad x \in \mathbb{Z}^d, \quad \lambda \in \mathbb{N}.$$  

The Fourier transform of $w_\lambda$ is given by

$$\hat{w}_\lambda(\xi) = \sum_{x \in S_{\sqrt{\lambda}} \cap \mathbb{Z}^d} e^{2\pi i \xi \cdot x}$$ \hspace{1cm} (3.1)

for any $\xi \in T^d \equiv [-1/2, 1/2]^d$. Note that

$$\hat{w}_\lambda(0) = |\{x \in \mathbb{Z}^d : |x|^2 = \lambda\}| = |S_{\sqrt{\lambda}} \cap \mathbb{Z}^d|.$$ \hspace{1cm} (3.2)

The spherical averaging operator $A^d_t$ given by (1.1) is a convolution operator with the kernel

$$K_t(\xi) := \frac{1}{|S_t \cap \mathbb{Z}^d|} 1_{S_t \cap \mathbb{Z}^d}(x) = \frac{1}{|S_{\sqrt{\lambda}} \cap \mathbb{Z}^d|} w_\lambda(x), \quad x \in \mathbb{Z}^d.$$  

The corresponding multipliers are normalized exponential sums given by

$$m_t(\xi) := K_t(\xi) = \frac{1}{|S_t \cap \mathbb{Z}^d|} \sum_{x \in S_t \cap \mathbb{Z}^d} e^{2\pi i \xi \cdot x} = \frac{1}{|S_{\sqrt{\lambda}} \cap \mathbb{Z}^d|} \hat{w}_\lambda(\xi), \quad \xi \in \mathbb{T}^d.$$ \hspace{1cm} (3.3)

Note that the kernel $K_t$ as well as its multiplier $m_t(\xi)$ are invariant under permutations, namely, for any $\tau \in \text{Sym}(d)$, one has

$$K_t(\tau \circ x) = K_t(x), \quad \text{and} \quad m_t(\tau \circ \xi) = m_t(\xi).$$ 

These invariance properties will be crucial in our further arguments. For further reference we note that

$$m_t(\xi + 1/2) = (-1)^3 m_t(\xi), \quad \xi \in \mathbb{T}^d.$$ \hspace{1cm} (3.4)

The proof of (3.4) is based on the identity $\sum_{i=1}^d x_i \equiv \sum_{i=1}^d x_i^2 \equiv \lambda \pmod{2}$ for $x \in S_t \cap \mathbb{Z}^d$. Consequently, for $x \in S_t \cap \mathbb{Z}^d$ we have

$$e^{2\pi i \xi \cdot x} = e^{-\pi |x|^2} e^{2\pi i (\xi + 1/2) \cdot x} = (-1)^3 e^{2\pi i (\xi + 1/2) \cdot x},$$

and (3.4) follows.

3.1. Asymptotic formula for $\hat{w}_\lambda(\xi)$. We shall employ the circle method of Hardy and Littlewood to derive an asymptotic formula for $\hat{w}_\lambda(\xi)$. As a consequence we recover the asymptotic formula in the classical Waring problem for the squares for all $d \geq 5$, since $\hat{w}_\lambda(0) = |S_{\sqrt{\lambda}} \cap \mathbb{Z}^d|$. In fact, in our asymptotic formula, we obtain certain uniformities with respect to radii $\lambda \geq Cd^2$ (for an absolute constant $C > 0$) and dimensions $d \geq 5$. This refinement of the Waring problem, as we shall see later, is an essential novelty of this paper and will be important in our further arguments. Before we formulate the main result of this section we need to set necessary notation and terminology.

Let

$$\delta := 1/\lambda > 0.$$  

Then we write

$$e^{-2\pi \lambda} w_\lambda(x) = e^{-2\pi} \int_0^1 e^{2\pi i |x|^2 - \lambda \alpha} d\alpha = \int_0^1 e^{2\pi i |x|^2 (\alpha + i\delta)} e^{-2\pi i \lambda \alpha} d\alpha, \quad x \in \mathbb{Z}^d,$$

and consequently we obtain

$$\hat{w}_\lambda(\xi) = e^{2\pi} \int_0^1 s(\alpha, \delta, \xi) e^{-2\pi i \lambda \alpha} d\alpha,$$

where $s(\alpha, \delta, \xi)$ is an absolutely convergent series given by

$$s(\alpha, \delta, \xi) := \sum_{x \in \mathbb{Z}^d} e^{2\pi i x \cdot \xi} e^{2\pi i |x|^2 (\alpha + i\delta)}.$$
Let
\[ N := \lfloor \sqrt{t} \rfloor = \lfloor t \rfloor \] (3.5)
and consider corresponding Farey sequence
\[ H_N := \left\{ \frac{p}{q} \in \mathbb{Q} : 0 \leq p \leq q \leq N, (p, q) = 1 \right\}. \]

Now we make a Farey dissection at level \( N \) of the unit interval
\[ [0, 1) = \bigcup_{p/q \in H_N} V_N(p/q), \]
where the sets \( V_N(p/q) \) are pairwise disjoint intervals, and
\[ \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{p}{q} \right| < \frac{1}{2Nq} \right\} \subseteq V_N\left( \frac{p}{q} \right) \subseteq \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{p}{q} \right| < \frac{1}{Nq} \right\}. \]

Further we define
\[ \tilde{V}_N\left( \frac{p}{q} \right) := V_N\left( \frac{p}{q} \right) - \frac{p}{q}, \quad \text{for } \frac{p}{q} \in H_N. \] (3.6)

Then we see that
\[ \tilde{V}_N\left( \frac{p}{q} \right) = \left\{ \alpha \in \mathbb{R} : -\beta(N, p/q) \leq \alpha < \frac{\gamma(N, p/q)}{Nq} \right\}, \]
where \( \beta(N, p/q) \) and \( \gamma(N, p/q) \) both belong to the interval \([1/2, 1]\); here and later on we identify 0 with 1.

Using these sets we decompose our multiplier
\[ \hat{w}_\alpha(x) = e^{2\pi i x^T \alpha} \sum_{p/q \in H_N} \int_{V_N(p/q)} s(\alpha + p/q, \delta, \xi) e^{-2\pi i \lambda(\alpha + p/q)} d\alpha. \] (3.7)

Note that
\[ s(\alpha + p/q, \delta, \xi) = \sum_{x \in \mathbb{Z}^d} e^{2\pi i |x|^2 p/q + 2\pi i \xi} h_{\alpha, \delta}(x), \]
where
\[ h_{\alpha, \delta}(x) := e^{2\pi i |x|^2 (\alpha + i\delta)}, \quad x \in \mathbb{R}^d. \]

For further reference note that
\[ F(h_{\alpha, \delta})(y) = \left( \frac{1}{2(\delta - i\alpha) \sqrt{\pi}} \right)^{d/2} e^{-\frac{\pi |x + i\alpha|^2}{\sqrt{\pi \delta^2}}}, \quad y \in \mathbb{R}^d. \] (3.8)

Now summing over the reminders modulo \( q \) we may write
\[ s(\alpha + p/q, \delta, \xi) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i |n|^2 p/q} \sum_{m \in \mathbb{Z}^d} e^{2\pi i (qm + n) \xi} h_{\alpha, \delta}(qm + n). \] (3.9)

By the Poisson summation formula applied to the inner sum in (3.9) we obtain
\[ \sum_{n \in \mathbb{Z}^d} e^{2\pi i (qm + n) \xi} h_{\alpha, \delta}(qm + n) = \sum_{x \in \mathbb{Z}^d} e^{2\pi i x \cdot n/q} \delta^{-d} F(h_{\alpha, \delta})(x/q - \xi). \]

Therefore, coming back to (3.7) we conclude
\[ \hat{w}_\alpha(\xi) = e^{2\pi i \xi^T \alpha} \sum_{p/q \in H_N} \int_{\tilde{V}_N(p/q)} \sum_{x \in \mathbb{Z}^d} \delta^{-d} \sum_{n \in \mathbb{Z}^d} e^{2\pi i |n|^2 p/q + 2\pi i \xi} F(h_{\alpha, \delta})(x/q - \xi) e^{-2\pi i \lambda(\alpha + p/q)} d\alpha. \]

For \((p, q) = 1\) and \( x \in \mathbb{Z}^d \) let \( G(p/q; x) \) be the \( d \)-dimensional Gaussian sum
\[ G(p/q; x) := \delta^{-d} \sum_{n \in \mathbb{Z}^d} e^{2\pi i |n|^2 p/q + 2\pi i \xi} \sum_{x \in \mathbb{Z}^d} G(p/q; x) F(h_{\alpha, \delta})(x/q - \xi) e^{-2\pi i \lambda(\alpha + p/q)} d\alpha. \]

Using (3.10) we now write
\[ \hat{w}_\alpha(\xi) = e^{2\pi i \lambda p/q} \sum_{p/q \in H_N} \int_{\tilde{V}_N(p/q)} \sum_{x \in \mathbb{Z}^d} G(p/q; x) F(h_{\alpha, \delta})(x/q - \xi) e^{-2\pi i \lambda \alpha} d\alpha. \] (3.11)

If \( \xi = 0 \) then (3.11) yields the formula for the number of lattice points in \( S_t \), see (3.12).
Theorem 3.1. Let \( \lambda \geq 0 \) and satisfies the estimate
\[
\psi(\xi) := \prod_{j=1}^{d} \varphi(\xi_j), \quad \xi \in \mathbb{R}^d.
\] (3.12)

We have almost prepared the ground to formulate the main result of this section. Let us define
\[
M_1(\lambda, \xi, n) := \frac{1}{2} \lambda^{d/2-1} \sum_{p/q \in H_N \atop q < n} e^{-2\pi i \lambda p/q} G(p/q; y/\xi q - \xi) F_{\sigma}(\sqrt{\lambda}\|q\xi - \xi\|),
\] (3.13)
and
\[
M_2(\lambda, \xi, n) := \frac{1}{2} \lambda^{d/2-1} \sum_{p/q \in H_N} \sum_{x \in \mathbb{Z}^d} e^{-2\pi i \lambda p/q} G(p/q; x) \psi(q\xi - x) F_{\sigma}(\sqrt{\lambda}\|x/\xi - \xi\|).
\] (3.14)

If \( n > N = \lfloor \sqrt{\lambda} \rfloor \), then \( M_2(\lambda, \xi, n) = 0 \) and analogously \( M_1(\lambda, \xi, 1) = 0 \). We now state our main result of this section, which can be thought of as a dimension-free variant of the solution to the Waring problem. This phenomenon is exhibited by the asymptotic formula stated in \( \Box \), where the multiplicative error term has been obtained in the classical Waring problem for the squares as long as \( \lambda \geq C d^3 \) and \( d \to \infty \).

**Theorem 3.1.** There exists an absolute constant \( C > 0 \) such that for all integers \( d \geq 5 \) and \( \lambda > 0 \) satisfying \( \lambda \geq C d^3 \), and for all \( n \in \mathbb{N}_{N+1} \) with \( N = \lfloor \sqrt{\lambda} \rfloor \) and \( \xi \in \mathbb{T}^d \) we have
\[
|\hat{w}_\lambda(\xi) - M_1(\lambda, \xi, n) - M_2(\lambda, \xi, n)| \lesssim \lambda^d (d\lambda)^{d/4}.
\] (3.15)

If \( n = N + 1 \), then \( 3.16 \) gives the following estimate
\[
|\hat{w}_\lambda(\xi) - M_1(\lambda, \xi, N + 1)| \lesssim \lambda^d (d\lambda)^{d/4}.
\] (3.16)

Taking \( \xi = 0 \) in \( 3.16 \), for \( d \geq 16 \) one has
\[
|S_{\sqrt{\lambda}} \cap \mathbb{Z}^d| = \frac{\pi^{d/2}}{\Gamma(d/2)} \lambda^{d/2-1} \mathcal{S}_d(\lambda) \big(1 + o(1)\big) \quad \text{as} \quad \lambda \geq C d^3 \text{ and } d \to \infty,
\] (3.17)

where the singular series \( \mathcal{S}_d(\lambda) \) is given by
\[
\mathcal{S}_d(\lambda) := \sum_{q=1}^{\infty} \sum_{1 \leq p \leq q \atop (p,q) = 1} e^{-2\pi i \lambda p/q} G(p/q; 0)
\] (3.18)

and satisfies the estimate \( \frac{1}{2} \leq \mathcal{S}_d(\lambda) \leq \frac{3}{2} \) independently of the dimension \( d \geq 16 \) and \( \lambda \in \mathbb{N} \). Finally, the asymptotic formula from \( 3.17 \) also ensures that
\[
|S_{\sqrt{\lambda}} \cap \mathbb{Z}^d| \sim \frac{\pi^{d/2}}{\Gamma(d/2)} \lambda^{d/2-1} \mathcal{S}(\sqrt{\lambda}) \sim \mathcal{S}(S^{d-1}) \lambda^{d/2-1}, \quad \lambda \geq C d^3, \quad d \geq 16.
\] (3.19)

Before we turn to the proof of Theorem 3.1 we need several auxiliary lemmas.

**Lemma 3.1.** For any integers \( p, q \in \mathbb{Z} \) such that \( 0 \leq p \leq q \) and \( (p, q) = 1 \) one has
\[
|G(p/q; x)| \leq (2/q)^{d/2}, \quad x \in \mathbb{Z}^d.
\] (3.20)

Moreover,
\[
\sum_{n \in \mathbb{Z}^d} |G(p/q; n)|^2 = 1.
\] (3.21)

**Proof.** The proof of Lemma 3.1 is based on simple calculations. Thanks to the product structure of the Gauss sums we may assume, without loss of generality, that \( d = 1 \).

**Lemma 3.2.** For each \( T_0 > 0 \) there is a constant \( C(T_0) > 0 \) such that
\[
\sum_{x \in \mathbb{Z} \setminus \{0\}} e^{-T|x+y|^2} \leq C(T_0)^d e^{-T/4},
\]
unformly in \( T \geq T_0 \) and \( y \in \mathbb{Q} \).
Lemma 3.3. Let \( C \) be the constant in \( C(T_0) := e^{T_0/4} \sum_{x \in \mathbb{Z}^d} e^{-T_0|x|^2/4}. \) This completes the proof of Lemma 3.2.

Lemma 3.3. Let \( d \geq 5 \) and define
\[
I_\lambda(\xi) := e^{2\pi \int_R e^{-2\pi i \alpha} \left( \frac{1}{2(1 - i\beta)} \right)^{d/2} \exp \left(-\frac{\pi \lambda|\xi|^2}{2(1 - i\beta)} \right) d\alpha}, \quad \lambda > 0, \quad \xi \in \mathbb{R}^d.
\]
Then one has
\[
I_\lambda(\xi) = \frac{1}{2} \mathcal{F}(\sigma^d)(\sqrt{\lambda}\xi), \quad \lambda > 0, \quad \xi \in \mathbb{R}^d.
\]
Proof. The proof of Lemma 3.3 can be found in \([28, \text{Lemma 6.1}]\). □

We now prove Theorem 3.7. Appealing to \((3.8)\), changing the variable \( \alpha = \delta \beta \) and using the fact that \( \delta = 1/\lambda \) we see that
\[
e^{2\pi \int_R \mathcal{F}(h_{\alpha,\delta})(y) e^{-2\pi i \lambda \alpha} \, d\alpha} = \delta^{d/2 - 1} \lambda^{d/2} e^{-2\pi i \beta} \frac{1}{2(1 - i\beta)} \int_R \exp \left(-\frac{\pi \lambda|\xi|^2}{2(1 - i\beta)} \right) d\alpha,
\]
where in the last equality we have used Lemma 3.3.

Recalling the expression \((3.11)\) for \( \hat{w}_\lambda(\xi) \) and using \((3.22)\) with \( y = \|q\xi\|/q - \xi \) and \( y = x/q - \xi \) we may write
\[
\hat{w}_\lambda(\xi) = (M_1 - E_1 + E_2) + (M_2 - E_3 + E_4),
\]
where \( M_1 := M_1(\lambda, \xi, n) \) and \( M_2 := M_2(\lambda, \xi, n) \) are defined respectively in \((3.13)\) and \((3.14)\) and \( E_j := E_j(\lambda, \xi, n) \), \( 1 \leq j \leq 4 \), are defined by setting
\[
E_1 := e^{2\pi \int_{\mathbb{R}} \mathcal{F}(h_{\alpha,\delta})(y) \left( \frac{1}{2(1 - i\beta)} \right)^{d/2} \exp \left(-\frac{\pi \lambda|\xi|^2}{2(1 - i\beta)} \right) d\alpha},
\]
and
\[
E_2 := e^{2\pi \int_{\mathbb{R}} \mathcal{F}(h_{\alpha,\delta})(y) \left( \frac{1}{2(1 - i\beta)} \right)^{d/2} \exp \left(-\frac{\pi \lambda|\xi|^2}{2(1 - i\beta)} \right) d\alpha},
\]
with the sets \( \hat{V}_N(p/q) \) defined in \((3.9)\). Now, in view of \((3.22)\), the proof of Theorem 3.7 will be completed once we prove that
\[
|E_j| \lesssim (d\lambda)^{d/4}, \quad \text{for} \quad j = 1, 2, 3, 4,
\]
and establish the asymptotic formulae from \((3.17)\) and \((3.19)\). The details will be split into five steps.
Step 1. We now prove (3.24) for \( j = 1 \). By (3.20) and the fact that \((\hat{V}_N(p/q))^c \subseteq [-1/(2Nq), 1/(2Nq))^c\) we obtain

\[
|E_1| \lesssim^d \sum_{1 \leq q \leq N} q^{1-d/2} \int_{|\alpha| \geq (2Nq)^{-1}} |\mathcal{F}(f_{\alpha})([q\xi]/q - \xi)| \, d\alpha.
\]

By (3.5) we have \(|\mathcal{F}(f_{\alpha})([q\xi]/q - \xi)| \leq 2^{-d/2}(\delta^2 + \alpha^2)^{-d/4}\), and since \(N = \lfloor \sqrt{\lambda} \rfloor\), we conclude

\[
|E_1| \lesssim^d \sum_{1 \leq q \leq N} q^{1-d/2} \int_{|\alpha| \geq (2Nq)^{-1}} |\alpha|^{-d/2} \, d\alpha \lesssim^d \sum_{1 \leq q \leq N} q^{1-d/2}(Nq)^{d/2-1} \leq \lambda^{d/4}.
\]

Step 2. We now prove (3.24) for \( j = 2 \). Using (3.20) and (3.8) we obtain

\[
|E_2| \lesssim^d \sum_{1 \leq q \leq N} q^d \left( \sum_{|\alpha| \leq 1/(Nq)} \sum_{x \in \mathbb{Z}^d \setminus \{q\xi\}} (q^2(\delta^2 + \alpha^2))^{-d/4} \exp\left( \frac{-\pi \delta}{2(\delta^2 + \alpha^2)} |x/q - \xi|^2 \right) \right) \, d\alpha.
\]

Since \(N^2 \simeq \lambda\) and \(\alpha^2 q^2 N^2 \leq 1\) we see that

\[
\frac{\pi \delta}{2(\delta^2 + \alpha^2)q^2} = \frac{\pi}{2\lambda + 2\alpha^2 q^2} \geq T_0,
\]

where \(T_0 > 0\) is a universal constant. Therefore, by Lemma 3.2 with \(y = \|q\xi\| - q\xi \in Q\) we have

\[
|E_2| \lesssim^d \sum_{1 \leq q \leq N} q^d \int_{|\alpha| \leq 1/(Nq)} (q^2(\delta^2 + \alpha^2))^{-d/4} \exp\left( \frac{-\pi \delta}{2(\delta^2 + \alpha^2)} |x/q - \xi|^2 \right) \, d\alpha.
\]

Using the inequality \(e^{-x} \leq (d/4)^{d/4}e^{-d/4}x^{-d/4}\) with \(x = \frac{\pi \delta}{2(\delta^2 + \alpha^2)q^2}\), and recalling \(\delta = 1/\lambda\), we are led to

\[
|E_2| \lesssim^d d^{d/4} \sum_{1 \leq q \leq N} q \int_{|\alpha| \leq 1/(Nq)} \delta^{-d/4} \, d\alpha \lesssim^d (d\lambda)^{d/4}.
\]

Step 3. We now prove (3.24) for \( j = 3 \). By definition of \(\psi\) the term \(\psi(q^2 \xi - x)\) is non-zero for at most one \(x \in \mathbb{Z}^d\); moreover, \(\|\psi\|_{L^\infty(\mathbb{R}^d)} \leq 1\). Hence, (3.20) and (3.8) imply

\[
|E_3| \lesssim^d \sum_{1 \leq q \leq N} q \int_{|\alpha| \geq 1/(2Nq)} \sup_{x \in \mathbb{Z}^d} \left( q^{-d/2}(\delta^2 + \alpha^2)^{-d/4} \exp\left( \frac{-\pi \delta}{2(\delta^2 + \alpha^2)} |x/q - \xi|^2 \right) \right) \, d\alpha
\]

\[
\lesssim^d \sum_{1 \leq q \leq N} q^{1-d/2} \int_{|\alpha| \geq 1/(2Nq)} |\alpha|^{-d/2} \, d\alpha \lesssim^d \sum_{1 \leq q \leq N} q^{1-d/2}(Nq)^{d/2-1} \leq^d \lambda^{d/4}.
\]

Step 4. It remains to verify (3.24) for \( j = 4 \). Note that, by definition of \(\psi\) if \(1 - \psi(q^2 \xi - x) \neq 0\), then \(|x - q\xi|_\infty > 1/8\). Hence, (3.20) and (3.8) show that

\[
|E_4| \lesssim^d \sum_{1 \leq q \leq N} q \int_{|\alpha| \leq 1/(Nq)} \sum_{x \in \mathbb{Z}^d} (q^2(\delta^2 + \alpha^2))^{-d/4} \exp\left( \frac{-\pi \delta}{2(\delta^2 + \alpha^2)} |x/q - \xi|^2 \right) \, d\alpha
\]

\[
= \sum_{1 \leq q \leq N} q \int_{|\alpha| \leq 1/(Nq)} \sum_{x \in \mathbb{Z}^d} (q^2(\delta^2 + \alpha^2))^{-d/4} \chi_{\{|x - q\xi|_\infty > 1/8\}} g(x - q\xi) \, d\alpha,
\]

where

\[
g(y) := \exp\left( \frac{-\pi \delta}{2q^2(\delta^2 + \alpha^2)} |y|^2 \right), \quad y \in \mathbb{R}^d.
\]

Since \(N^2 \simeq \lambda\) and \(\alpha^2 q^2 N^2 \leq 1\), by (3.25) we may apply Lemma 3.2 with \(y = \|q\xi\| - q\xi \in Q\) to conclude

\[
|E_4| \lesssim^d \sum_{1 \leq q \leq N} q \int_{|\alpha| \leq 1/(Nq)} (q^2(\delta^2 + \alpha^2))^{-d/4}
\]

\[
\times \left( \chi_{\{|\|q\xi\| - q\xi|_\infty > 1/8\}} \exp\left( \frac{-\pi \delta}{2q^2(\delta^2 + \alpha^2)} \|q\xi\| - q\xi|^2 \right) + \exp\left( \frac{-\pi \delta}{8q^2(\delta^2 + \alpha^2)} \right) \right) \, d\alpha
\]

\[
\lesssim^d \sum_{1 \leq q \leq N} q \int_{|\alpha| \leq 1/(Nq)} (q^2(\delta^2 + \alpha^2))^{-d/4} \exp\left( \frac{-\pi \delta}{128q^2(\delta^2 + \alpha^2)} \right) \, d\alpha.
\]
Now, using the inequality $e^{-x} \leq (d/4)^{d/4}e^{-d/4}x^{-d/4}$ with $x = \frac{\pi d}{128(d^2+\alpha^2)q^2}$ we are thus led to

$$|E_d| \lesssim d^{d/4} \sum_{1 \leq p \leq N} q \int_{|\alpha| \leq 1/(Nq)} \delta^{-d/4}d\alpha \lesssim d^d (d\lambda)^{d/4}.$$ 

This also completes the proof of inequality (3.15) as well as (3.16).

**Step 5.** We now establish asymptotic formula in (3.17). For this purpose we observe that (3.16) with $\xi = 0$ and (3.2) yields

$$\left| |S_\pi \cap \mathbb{Z}^d| - M_1(\lambda,0,N+1) \right| \lesssim d (d\lambda)^{d/4}. \quad (3.26)$$

Combining $\mathcal{F} \sigma(0) = \sigma(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ with (3.13) we obtain

$$M_1(\lambda,0,N+1) = \frac{\pi^{d/2}}{\Gamma(d/2)} d^{d-1} \mathcal{S}_d(\lambda;N),$$

where

$$\mathcal{S}_d(\lambda;P) := \sum_{p=1}^P \sum_{1 \leq p \leq q \ (p,q)=1} e^{-2\pi i \lambda p/q} G(p/q;0), \quad P \in \mathbb{N}.$$ 

For $d \geq 16$ it is also not difficult to see that

$$\frac{1}{2} \leq \mathcal{S}_d(\lambda) \leq \frac{3}{2}, \quad (3.27)$$

since

$$\mathcal{S}_d(\lambda) = 1 + \sum_{q=3}^\infty \sum_{1 \leq p \leq q \ (p,q)=1} e^{-2\pi i \lambda p/q} G(p/q;0),$$

and by (3.20) for $d \geq 16$ we have

$$\sum_{q=3}^\infty \sum_{1 \leq p \leq q \ (p,q)=1} e^{-2\pi i \lambda p/q} G(p/q;0) \leq \sum_{q=3}^\infty q^{-1-d/2} q^{d/2} \leq 2^{d/2} 3^{1-d/2} + 2^{d/2} \int_3^\infty y^{-d/2+1} dy \leq 2^{d/2} \cdot \frac{1}{d/2 - 2} \leq 1/2.$$ 

In a similar way, recalling that $N \approx \lambda^{1/2}$ and using (3.20), we obtain

$$|\mathcal{S}_d(\lambda) - \mathcal{S}_d(\lambda;N)| \leq 2^{d/2} \sum_{q=N+1}^\infty q^{-d/2+1} \lesssim d^{-d/4+1}. \quad (3.28)$$

Combining (3.20) and (3.28) we conclude

$$\left| |S_\pi \cap \mathbb{Z}^d| - \frac{\pi^{d/2}}{\Gamma(d/2)} d^{d/2-1} \mathcal{S}_d(\lambda) \right| \lesssim d (d\lambda)^{d/4} + \frac{1}{\Gamma(d/2)} \lambda^{d/4} \lesssim d (d\lambda)^{d/4}. \quad (3.29)$$

Moreover, since $\Gamma(d/2) \sim d^{d/2}$ we see that

$$(d\lambda)^{d/4} \lesssim d \lambda^{-d/4+1} \lambda^{d/4} \lesssim \frac{\pi^{d/2}}{\Gamma(d/2)} \lambda^{d/2-1} = d^{\beta} \left( \frac{\beta}{\lambda} \right)^{d/4-1} \frac{\pi^{d/2}}{\Gamma(d/2)} \lambda^{d/2-1}. \quad (3.30)$$

Therefore, (3.27), (3.29) and (3.30) ensure, for some universal constant $C_0 > 0$, that

$$|S_\pi \cap \mathbb{Z}^d| = \frac{\pi^{d/2}}{\Gamma(d/2)} \lambda^{d/2-1} \mathcal{S}_d(\lambda) \left( 1 + O(C_0^d \lambda^{-1}) \right). \quad (3.31)$$

which for a large absolute constant $C > 0$ and for all integers $d \geq 16$ and $\lambda > 0$ obeying $\lambda \geq C d^3$ implies (3.17). Asymptotic (3.31) and (3.27) also imply (3.19) and the proof of Theorem 3.1 is completed. $\square$
3.2. Lattice points for balls and spheres. We shall also need a comparison between numbers of lattice points in balls and spheres.

Lemma 3.4. Let \( d \geq 5 \). Then for all \( t > 0 \) such that \( \lambda = t^2 \in \mathbb{N} \) we have
\[
|B_t^2(d-4) \cap \mathbb{Z}^{d-4}| \leq |S_t^{d-1} \cap \mathbb{Z}^d|,
\]
(3.32)
Consequently, for such \( t \) we have
\[
|S_t^{d-1} \cap \mathbb{Z}^d| \leq |B_t^2(d) \cap \mathbb{Z}^d| \leq (2t+1)^4 |S_t^{d-1} \cap \mathbb{Z}^d|.
\]
(3.33)

Proof. By the Lagrange four squares theorem for each \( 0 \leq j \leq \lambda \) we find integers \( y_j, d-3, y_j, d-2, y_j, d-1, y_j, d \) such that
\[
\sum_{i=d-3}^d y_{j,i}^2 = \lambda - j.
\]
Then for each \( x = (x_1, \ldots, x_{d-4}) \in B_t^2(d-4) \cap \mathbb{Z}^{d-4} \) we define \( \Phi: B_t^2(d-4) \cap \mathbb{Z}^{d-4} \rightarrow S_t^{d-1} \cap \mathbb{Z}^d \) by setting
\[
\Phi(x) := (x_1, \ldots, x_{d-4}, y_j, d-3, y_j, d-2, y_j, d-1, y_j, d),
\]
if \( (x_1, \ldots, x_{d-4}) \in S_t^{d-5} \setminus \mathbb{Z}^{d-4} \), with \( 0 \leq j \leq \lambda \). Since \( B_t^2(d-4) \cap \mathbb{Z}^{d-4} \) decomposes as the disjoint union \( \bigcup_{j=0}^{\lambda} S_t^{d-5} \setminus \mathbb{Z}^{d-4} \) the function \( \Phi \) is an injection from \( B_t^2(d-4) \cap \mathbb{Z}^{d-4} \) to \( S_t^{d-1} \cap \mathbb{Z}^d \). Thus, we have proved (3.32). It remains to justify the second inequality in (3.33). Observe that
\[
B_t^2(d) \cap \mathbb{Z}^d \subseteq (B_t^2(d-4) \cap \mathbb{Z}^{d-4}) \times \sqrt{r} [t, t] + \mathbb{Z}^d,
\]
and consequently \( |B_t^2(d) \cap \mathbb{Z}^d| \leq (2t+1)^4 |B_t^2(d-4) \cap \mathbb{Z}^{d-4}| \), which by (3.32) gives (3.33). \( \square \)

4. General estimates for certain Fourier multipliers

In this section we gather general estimates for Fourier multipliers in the continuous and discrete setup, which will be used later on in the paper. We first provide dimension-free estimates of the Fourier transform corresponding to the spherical measure in \( \mathbb{R}^d \).

4.1. Fourier transform estimates for the continuous spherical measures. For \( r \geq 2 \), let \( \mu^r \) denote the normalized spherical surface measure on \( S^{r-1} \) as in (2.3). We shall abbreviate \( \mu^r \) to \( \mu \) if \( r = d \).

Dimension-free estimates of the Fourier transforms corresponding to the spherical measures are provided in Lemma 4.1 and Lemma 4.2. These two results may be of independent interest.

Lemma 4.1. There exists a constant \( c > 0 \) such that for all \( r \geq 2 \) and \( \eta \in \mathbb{R}^r \) we have
\[
|\mathcal{F} \mu^r(\eta)| \lesssim e^{-2\pi |\eta| / \sqrt{r}} + e^{-cr},
\]
where the implicit constant is independent of \( r \) and \( \eta \).

Proof. The lemma is obvious when the dimension \( r \) is small. Thus, from now on we only focus on sufficiently large \( r \in \mathbb{N} \). Throughout the proof we abbreviate \( \mu^r \) to \( \mu \), \( \sigma^r \) to \( \sigma \), and \( S^{r-1} \) to \( S \). It is well known, for instance by [18] Appendix B.4, that
\[
\mathcal{F} \sigma(\eta) = \frac{2\pi}{|\eta|^{r/2 - 1}} J_{r/2 - 1}(2\pi |\eta|), \quad \eta \in \mathbb{R}^r,
\]
(4.1)
where \( J_{\nu} \) is the Bessel function of the first kind, which can be written, for \( \Re \nu > -1/2 \) and \( u > 0 \) as
\[
J_{\nu}(u) := \frac{u^\nu}{2^\nu \Gamma(\nu + 1/2) \sqrt{\pi}} \int_{-1}^1 e^{isu (1 - s^2)^{\nu-1/2}} ds.
\]
(4.2)
Therefore, using (4.1) together with (4.1) and (4.2) we have
\[
\mathcal{F} \mu(\eta) = \frac{2\pi^{-r/2}}{\sigma(S) \Gamma(\frac{r}{2} + 1/2)} \int_{-1}^1 e^{-2\pi |\eta| s (1 - s^2)^{r/4}} ds = \frac{\Gamma(r/2)}{\pi^{1/2} \Gamma(\frac{r}{2} + 1/2)} \int_{-1}^1 e^{-2\pi |\eta| s (1 - s^2)^{\frac{1}{\sqrt{r}}}} ds.
\]
By Stirling’s formula we deduce that
\[
|\mathcal{F} \mu(\eta)| \lesssim \sqrt{r} \int_{-1}^1 e^{-2\pi |\eta| s (1 - s^2)^{\frac{1}{\sqrt{r}}}} ds = \int_{-\sqrt{r}}^{\sqrt{r}} e^{-2\pi |\eta| s / \sqrt{r} (1 - s^2 / r)^{\frac{1}{\sqrt{r}}}} ds.
\]
(4.3)
Setting
\[ M(u) := \int_{-\sqrt{r}}^{\sqrt{r}} e^{-2\pi i u s} (1 - s^2/r)^{r-3} \, ds, \quad u \in \mathbb{R}, \]
and noting that \( M(u) = M(|u|) \), \( u \in \mathbb{R} \), we see that the proof of Lemma 4.1 reduces to showing that for sufficiently large \( r \in \mathbb{N} \) we have
\[ |M(u)| \leq e^{-2\pi u} + e^{-cr}, \quad u \geq 0. \tag{4.4} \]
We now establish (4.4). We first note
\[ |M(u)| \leq \left| \int_{r^{1/2}/100 \leq |s| \leq r^{1/2}} e^{-2\pi i u s} \left( 1 - \frac{s^2}{r} \right)^{r-3} \, ds \right| + \left| \int_{|s| \leq r^{1/2}/100} e^{-2\pi i u s} \left( 1 - \frac{s^2}{r} \right)^{r-3} \, ds \right|. \]
Observe that
\[ \left| \int_{r^{1/2}/100 \leq |s| \leq r^{1/2}} e^{-2\pi i u s} \left( 1 - \frac{s^2}{r} \right)^{r-3} \, ds \right| \leq 2r^{1/2}(1 - 10^{-4})^{r-3} \lesssim e^{-cr}, \]
since \( 1 - \frac{s^2}{r} \leq 1 - \frac{1}{10^4} \) for \( |s| \geq r^{1/2}/100 \). The proof of (4.4) will be completed if we show that
\[ \left| \int_{|s| \leq r^{1/2}/100} e^{-2\pi i u s} \left( 1 - \frac{s^2}{r} \right)^{r-3} \, ds \right| \lesssim e^{-2\pi u} + e^{-cr}. \tag{4.5} \]
To prove (4.5) we will change the contour of integration. Namely, let \( \gamma := \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \gamma_3 \) be the rectangle with the parametrization
\[
\begin{align*}
\gamma_0(s) &:= s & &\text{for } s \in [-r^{1/2}/100, r^{1/2}/100], \\
\gamma_1(s) &:= -is + \frac{r^{1/2}}{100} & &\text{for } s \in [0, 1], \\
\gamma_2(s) &:= -s - i & &\text{for } s \in [-r^{1/2}/100, r^{1/2}/100], \\
\gamma_3(s) &:= -i(1 - s) - \frac{r^{1/2}}{100} & &\text{for } s \in [0, 1].
\end{align*}
\]
The function \( z \mapsto e^{-2\pi i u (1 - s^2/r)^{r-3}} \) is holomorphic in \( \{z \in \mathbb{C} : |z| < \sqrt{r}/10 \} \) so the Cauchy integral theorem implies
\[
\left| \int_{|s| \leq r^{1/2}/100} e^{-2\pi i u s} \left( 1 - \frac{s^2}{r} \right)^{r-3} \, ds \right| \leq \sum_{j \in \{1, 3\}} \left| \int_0^1 e^{-2\pi i \gamma_j(s)u} \left( 1 - \frac{\gamma_j(s)^2}{r} \right)^{r-3} \gamma_j'(s) \, ds \right| + \left| \int_{|s| \leq r^{1/2}/100} e^{2\pi i (s+i)u} \left( 1 - \frac{(s+i)^2}{r} \right)^{r-3} \, ds \right|. \tag{4.6}
\]
Observe now that
\[ \sum_{j \in \{1, 3\}} \left| \int_0^1 e^{-2\pi i \gamma_j(s)u} \left( 1 - \frac{\gamma_j(s)^2}{r} \right)^{r-3} \gamma_j'(s) \right| \, ds \leq \sum_{j \in \{1, 3\}} \int_0^1 \left| 1 - \frac{\gamma_j(s)^2}{r} \right|^{r-3} \, ds \lesssim e^{-cr}, \tag{4.7} \]
since for \( s \in [0, 1] \) and sufficiently large \( r \in \mathbb{N} \) we have
\[ \sum_{j \in \{1, 3\}} \left| 1 - \frac{\gamma_j(s)^2}{r} \right|^{r-3} \leq 2 \left( 1 - \frac{1}{10^4} \right)^{50r^{1/2}} \lesssim 2 \left( 1 - \frac{1}{10^5} \right)^{50r^{1/2}} \lesssim e^{-cr}. \]
We also have \( e^{2\pi i (s+i)u} = e^{-2\pi u} e^{2\pi i u} \). Thus it suffices to prove that for sufficiently large \( r \in \mathbb{N} \) we have
\[ \left| \int_{|s| \leq r^{1/2}/100} e^{2\pi i u s} \left( 1 - \frac{(s+i)^2}{r} \right)^{r-3} \, ds \right| \lesssim 1, \quad u \geq 0. \tag{4.8} \]
Then (4.6) combined with (4.7) and (4.8) yields (4.5). We now observe that
\[
1 - \frac{(s+i)^2}{r} \leq \begin{cases} 
1 + \frac{2\pi}{r}, & \text{if } |s| \leq 5, \\
1 - \frac{2\pi}{r}, & \text{if } 5 < |s| \leq \frac{4}{100},
\end{cases}
\]
and consequently
\[
\left| \int_{|s| \leq r^{1/2}/100} e^{2\pi is} \frac{1}{r} \frac{(s+i)^2}{r} ds \right| \leq 1 + \int_{|s| \leq r^{1/2}/100} \left( 1 - \frac{s^2}{2r} \right) \frac{2\pi}{r} ds
\]
\[
\leq 1 + \frac{r^{1/2}}{1} \int_{-1}^{1} (1 - s^2)^{-1/2}/2ds
\]
\[
\leq 1 + \mathcal{F} \mu(0)
\]
\[
\simeq 1,
\]
where in the penultimate inequality we used (4.3) with \(\eta = 0\). This completes the proof of (4.6), hence, also the proof of (4.4). Therefore, the proof of Lemma 4.1 is completed. \(\square\)

**Lemma 4.2.** For the normalized spherical measure \(\mu := \mu^d\) on \(S^{d-1}\) as in (2.3) with \(d \geq 2\), one has
\[
|\mathcal{F} \mu(\xi) - 1| \leq 2\pi^2 (|\xi|/\sqrt{d})^2, \quad \xi \in \mathbb{R}^d,
\]
and
\[
|\mathcal{F} \mu(\xi)| \leq (|\xi|/\sqrt{d})^{-1/2}, \quad \xi \in \mathbb{R}^d,
\]
where the implicit constant is independent of \(d\) and \(\xi\).

**Proof.** Using symmetry we see that
\[
\mathcal{F} \mu(\xi) = \int_{S^{d-1}} e^{-2\pi i x \cdot \xi} d\mu(x) = \int_{S^{d-1}} \prod_{j=1}^{d} \cos(2\pi x_j \xi_j) d\mu(x).
\]
For any sequence \((a_j : j \in \mathbb{N}_d) \subseteq \mathbb{C}\) and \((b_j : j \in \mathbb{N}_d) \subseteq \mathbb{C}\), if \(\sup_{j \in \mathbb{N}_d} |a_j| \leq 1\) and \(\sup_{j \in \mathbb{N}_d} |b_j| \leq 1\) then
\[
\left| \prod_{j=1}^{d} a_j - \prod_{j=1}^{d} b_j \right| \leq \sum_{j=1}^{d} |a_j - b_j|.
\]
Since \(\cos(2x) = 1 - 2\sin^2 x\), hence, (4.11) and inequality (2.1) imply, for \(\xi \in \mathbb{R}^d\), that
\[
|\mathcal{F} \mu(\xi) - 1| \leq 2 \int_{S^{d-1}} \sum_{j=1}^{d} \sin^2(\pi x_j \xi_j) d\mu(x)
\]
\[
\leq 2\pi^2 \sum_{j=1}^{d} \xi_j^2 \int_{S^{d-1}} x_j^2 d\mu(x).
\]
This proves (4.9), since \(\int_{S^{d-1}} x_j^2 d\mu(x) = d^{-1} \int_{S^{d-1}} |x|^2 d\mu(x) \leq d^{-1}\) for any \(j \in \mathbb{N}_d\).

It remains to justify (4.10). Take \(\xi \in \mathbb{R}^d\) and assume first that \(|\xi| \leq d\sqrt{d}\). Then \(d \geq |\xi|/\sqrt{d}\) and thus Lemma 4.1 with \(r = d\) easily gives (4.10). Therefore, from now on we assume that \(|\xi| > d\sqrt{d}\). Recalling (4.4) (with \(r = d\)) we write
\[
\mathcal{F} \mu(\xi) = \frac{2\pi}{\sigma(S^{d-1})|\xi|^{d/2}} J_{d/2-1}(2\pi|\xi|), \quad \xi \in \mathbb{R}^d,
\]
with \(J_{d/2-1}\) being the Bessel function (4.2). By (3.1) eq. 10.14.1 we have
\[
|J_{d/2-1}(y)| \leq 1, \quad y \in \mathbb{R}.
\]
Inequality (4.12) and the formula \(\sigma(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\) imply the first bound below
\[
|\mathcal{F} \mu(\xi)| \leq \min \left\{ \frac{\pi \Gamma(d/2)}{\pi^{d/2} |\xi|^{d/2-1}}, \frac{C_d}{|\xi|^{(d-1)/2}} \right\},
\]
whereas the second estimate (with some \(C_d > 0\)) follows from well known asymptotics of the Bessel function \(|J_{d/2-1}(2\pi|\xi|)| \lesssim_d |\xi|^{-1/2}\), see e.g. (3.1) eq. 10.17.3. If \(2 \leq d \leq 4\) we apply the second bound to
Proof. We first prove (4.13). Exploiting the symmetries of Proposition 4.4. have to proceed differently. This will be illustrated in the next section. Therefore, using (4.11) and the formula \( \cos(2\theta) \), which gives

\[
|F(\xi)| \lesssim \frac{d^{3/2-1/2}}{(\pi d)^{3/2}|\xi|^{3/2-1/2}} \lesssim \frac{1}{|\xi|^{1/2}},
\]

whenever \( |\xi| > d^{3/2} \). This proves (4.10) and completes the proof of Lemma 4.2.

4.2. Fourier transform estimates for the discrete spherical measures. The proofs of inequalities (4.20), (4.21) and (4.22) in Theorem 1.2 will use Proposition 4.3, which provides estimates of the multiplier \( m_2(\xi) \) at the origin. On the other hand, they will appeal respectively to Proposition 4.3, Proposition 4.4 and Proposition 4.1, which provide estimates of the multiplier \( m_2(\xi) \) at infinity. All these estimates will be described in terms of a proportionality constant

\[
\kappa(d, \lambda) := \left( \frac{\lambda}{d} \right)^{1/2} = \frac{t}{\sqrt{d}},
\]

This quantity is just a reparametrization of the proportionality constant from the Euclidean ball case \( \mathbb{S} \), where the radius \( N \) is replaced with \( \sqrt{\lambda} \).

Proposition 4.3. Let \( d, \lambda \in \mathbb{N} \) be such that \( d \geq 2 \) and denote \( t = \sqrt{\lambda} \). Then for every \( \xi \in \mathbb{S}^d \) we have

\[
|m_2(\xi) - 1| \leq 2\pi^2 \kappa(d, \lambda)^2 \|\xi\|^2.
\]

Additionally, we have

\[
|m_2(\xi) - (-1)^\lambda| \leq 2\pi^2 \kappa(d, \lambda)^2 \|\xi + 1/2\|^2.
\]

Proof. We first prove (4.13). Exploiting the symmetries of \( S_t \cap \mathbb{Z}^d \) we have

\[
m_2(\xi) = \frac{1}{|S_t \cap \mathbb{Z}^d|} \sum_{x \in S_t \cap \mathbb{Z}^d} \prod_{j=1}^d \cos(2\pi x_j \xi_j).
\]

Therefore, using (4.11) and the formula \( \cos(2x) = 1 - 2\sin^2 x \), we obtain

\[
|m_2(\xi) - 1| \leq \frac{2}{|S_t \cap \mathbb{Z}^d|} \sum_{x \in S_t \cap \mathbb{Z}^d} \sum_{j=1}^d \sin^2(\pi x_j \xi_j).
\]

Observe that \( |\sin(\pi xy)| \leq |x| |\sin(\pi y)| \) for every \( x \in \mathbb{Z} \) and \( y \in \mathbb{R} \), and also for every \( i, j \in \mathbb{N}_d \) one has

\[
\sum_{x \in S_t \cap \mathbb{Z}^d} x_i^2 = \sum_{x \in S_t \cap \mathbb{Z}^d} x_j^2 = \frac{1}{d} \sum_{x \in S_t \cap \mathbb{Z}^d} |x|^2 = \kappa(d, \lambda)^2 |S_t \cap \mathbb{Z}^d|.
\]

Thus, taking into account these observations and changing the order of summations we obtain

\[
|m_2(\xi) - 1| \leq \frac{2}{|S_t \cap \mathbb{Z}^d|} \sum_{j=1}^d \sin^2(\pi \xi_j) \frac{1}{d} \sum_{x \in S_t \cap \mathbb{Z}^d} |x|^2 \leq 2\pi^2 \kappa(d, \lambda)^2 \|\xi\|^2,
\]

where in the last line we have used (2.1). Now (4.13) is justified.

It remains to prove (4.14). Using (4.3) and (4.13) we see that

\[
|m_2(\xi) - (-1)^\lambda| = |m_2(\xi + 1/2) - 1| \leq 2\pi^2 \kappa(d, \lambda)^2 \|\xi + 1/2\|^2,
\]

and the proof of (4.14) is completed.

Proposition 4.4. There exist universal constants \( C, c > 0 \) such that for \( d \geq 16, \lambda \geq Cd^3 \), and \( \xi \in \mathbb{S}^d \) one has

\[
|m_2(\xi)| \lesssim e^{-2\pi \kappa(d, \lambda)\|\xi\|} + e^{-2\pi \kappa(d, \lambda)\|\xi + 1/2\|} + \lambda^{-2} + e^{-cd}.
\]
Proof. Recalling that \( N = \lfloor \sqrt{d} \rfloor \), \( t = \sqrt{d} \) and invoking (3.3), (3.10), and (3.19) from Theorem 3.1 (together with \( \Gamma(d/2) \approx d^{d/2} \)), we obtain
\[
\left| m_r(x) - M_1(\lambda, \xi, N + 1) \right| \lesssim d \left( \frac{d \lambda}{|S_r \cap \mathbb{Z}^d|} \right)^{d/4} \lesssim d \left( \frac{d \lambda}{d^{d/4-1}} \right)^{d/4-3} \lesssim \frac{1}{\lambda},
\]
if \( \lambda \geq C d^3 \) for large enough \( C > 0 \).

By (3.19) and definition (3.13) of \( M_1(\lambda, \xi, N + 1) \) it suffices to estimate
\[
R_1 = R_1(d, \lambda, \xi) := \sum_{p/q \in H_N, q \in [1,2]} e^{-2\pi \iota q/p} G(p/q; \|q\xi\|/\xi),
\]
and
\[
R_2 = R_2(d, \lambda, \xi) := \sum_{p/q \in H_N, q \geq 3} e^{-2\pi \iota q/p} G(p/q; \|q\xi\|/\xi).
\]

We first handle \( R_2 \), which is easier. We use \( \sup_{y \in \mathbb{R}^d} |F_\mu(y)| \leq 1 \) and (3.20) to obtain
\[
|R_2| \lesssim \sum_{q \geq 3} q \cdot \left( \frac{q}{d^2} \right)^{d/2} = 2d/2 \sum_{q \geq 3} q^{-d/2+1} = 3 \cdot \left( \frac{2}{3} \right)^{d/2} + 2d/2 \sum_{q \geq 4} q^{-d/2+1}
\]
\[
\lesssim (2/3)^{d/2} + \frac{2d}{2} \int_3^\infty y^{-d/2+1} dy \lesssim (2/3)^{d/2} \lesssim \frac{e^{-d/5}}{d/2}.
\]

Now it remains to estimate \( R_1 \). We may assume that \( \xi \in Q \).

Clearly, if \( p/q \in H_N \) and \( q \in [1,2] \), then \( p/q \in \{1/1, 1/2\} \) since we identify 0/1 with 1/1. By definition (4.10) for \( x \in \mathbb{Z}^d \) we have \( G(1/1; x) = 1 \) and
\[
G(1/2; x) = 2^{-d} \prod_{j=1}^d (1 + e^{\pi \iota (x_j + 1)}) = \prod_{j=1}^d I_{2z+1}(x_j) = I_{(2z+1)^d}(x).
\]
Therefore, we have
\[
R_1 = F_\mu(\sqrt{d}[\xi - \xi]) + e^{-\pi \iota \lambda} I_{(2z+1)^d}([2\xi]) F_\mu(\sqrt{d}[\xi]/\xi - \xi) = F_\mu(\sqrt{d}[\xi]) + e^{-\pi \iota \lambda} I_{(2z+1)^d}([2\xi]) F_\mu(\sqrt{d}[\xi]/\xi - \xi),
\]
where in the second equality we have used that \( F_\mu \) is even and \( \xi = 0 \) for \( \xi \in [-1/2, 1/2]^d \). Note that
(i) if \( \xi_j \in [-1/4, 1/4] \) then \( 2\xi_j \in [-1/2, 1/2] \), and hence, \( [2\xi] = 0 \) and \( [2\xi]/\xi - \xi = 0 \),
(ii) if \( \xi_j \in [-1/2, -1/4] \) then \( 2\xi_j \in [-1, -1/2] \), and hence, \( [2\xi] = -1 \) and \( [2\xi]/\xi - \xi = 1/2 - \xi_j \),
(iii) if \( \xi_j \in [1/4, 1/2] \) then \( 2\xi_j \in [1/2, 1] \), and hence, \( [2\xi] = 1 \) and \( [2\xi]/\xi - \xi = 1/2 - \xi_j \).

Consequently, we see that
\[
I_{(2z+1)^d}([2\xi]) = I_{(-1/2, -1/4) \cup (1/4, 1/2)^d}(\xi),
\]
and
\[
\|2\xi\|/\xi - \xi = \|\xi + 1/2\| \quad \text{for} \quad \|\xi\| \in (2z + 1)^d.
\]
Using (4.10) and Lemma 4.1 with \( r = d \) we obtain
\[
|R_1| \lesssim \exp \left( -2\pi \sqrt{d}[\xi]/\sqrt{d} \right) + I_{(2z+1)^d}([2\xi]) \exp \left( -2\pi \sqrt{d}[2\xi]/2 - \xi/\sqrt{d} \right) + e^{-cd}
\]
\[
\lesssim e^{-2\pi \iota \lambda d[\xi]} + e^{-2\pi \iota \lambda \|\xi+1/2\|} + e^{-cd},
\]
where \( c > 0 \) is the universal constant from Lemma 4.1. This completes the proof of Proposition 4.4.

4.3. Symmetric diffusion semigroups. Large part of our estimates will rely on dimension-free bounds for symmetric diffusion semigroups. Namely, for every \( t > 0 \) let \( P_t \) be the semigroup with the multiplier
\[
p_t(\xi) := e^{-\pi \sum_{i=1}^d \sin^2(\pi \xi_i)} \quad \text{for} \quad \xi \in \mathbb{T}^d.
\]
It is very well known that for every \( p \in (1, \infty) \) there is \( C_p > 0 \) independent of \( d \in \mathbb{N} \) such that for every \( f \in L^p(\mathbb{Z}^d) \) we have
\[
\left\| \sup_{t > 0} |P_t f| \right\|_{L^p(\mathbb{Z}^d)} \leq C_p \|f\|_{L^p(\mathbb{Z}^d)}.
\]
We refer to [33] and to [9] Section 4.1 for more details.
We close this section by giving a simple application of inequality 1.18 and Lemma 4.2.

**Lemma 4.5.** Let \( d \geq 2 \) and for \( t > 0 \) define
\[
a_t(\xi) := F\mu(t(\xi - \|\xi\|)), \quad \xi \in \mathbb{R}^d.
\]
Then, for all \( f \in L^2(\mathbb{Z}^d) \) one has
\[
\left\| \sup_{t \in \mathbb{D}} |F^{-1}(a_t \hat{f})| \right\|_{L^2(\mathbb{Z}^d)} \lesssim \|f\|_{L^2(\mathbb{Z}^d)},
\]
where the implicit constant is independent of the dimension.

**Proof.** The multiplier \( a_t \) is 1-periodic in each coordinate \( \xi_j \) for \( j \in \mathbb{N}_d \) thus it is well defined as a function on \( \mathbb{T}^d \). Moreover, for \( \xi \in Q \) we have \( \xi - \|\xi\| = \xi \) and \( |\xi - \|\xi\|| = \|\xi\| \). Therefore, recalling that \( \kappa(d, \lambda) = td^{-1/2} \) and using Lemma 4.2 we obtain, for \( \xi \in \mathbb{T}^d \) and \( t > 0 \) the estimates
\[
|a_t(\xi) - 1| \lesssim (\kappa(d, \lambda)\|\xi\|)^2, \quad \text{and} \quad |a_t(\xi)| \lesssim (\kappa(d, \lambda)\|\xi\|)^{-1/2}.
\]
Now, (1.20) implies that
\[
|p_{\xi, d^{-1}}(\xi) - a_t(\xi)| \lesssim \min \{ (\kappa(d, \lambda)\|\xi\|)^2, (\kappa(d, \lambda)\|\xi\|)^{-1/2} \}, \quad \xi \in \mathbb{T}^d,
\]
where \( p_{\xi, d^{-1}} = p_{\kappa(d, \lambda)\xi} \) is the multiplier from (4.17) corresponding to the semigroup operator \( P_{\kappa(d, \lambda)p} \).

Hence, using (1.18) and (4.21) we obtain
\[
\left\| \sup_{t \in \mathbb{D}} |F^{-1}(a_t \hat{f})| \right\|_{L^2(\mathbb{Z}^d)} \leq \left( \sup_{t > 0} |p_{\xi, d^{-1}} f| \right)_{L^2(\mathbb{Z}^d)} + \left( \sum_{t \in \mathbb{D}} \left| F^{-1} \left( (p_{\xi, d^{-1}} - a_t) \hat{f} \right)^2 \right| \right)_{L^2(\mathbb{Z}^d)} \lesssim \|f\|_{L^2(\mathbb{Z}^d)}.
\]
This completes the proof of Lemma 4.5. \( \square \)

We have also a continuous analogue of Lemma 4.5.

**Lemma 4.6.** Let \( d \geq 2 \). Then, for all \( f \in \mathbb{L}^2(\mathbb{R}^d) \) one has
\[
\left\| \sup_{t \in \mathbb{D}} |F^{-1}(F\mu(t \cdot)\hat{f})| \right\|_{\mathbb{L}^2(\mathbb{R}^d)} \lesssim \|f\|_{\mathbb{L}^2(\mathbb{R}^d)},
\]
where the implicit constant is independent of the dimension.

**Proof.** The proof of Lemma 4.5 goes much the same way as the proof of the previous lemma; the only change is the use of the heat semigroup on \( \mathbb{R}^d \) in place of the semigroup \( P_t \). We omit the details. \( \square \)

5. **Proof of Theorem 1.2: Large-scale estimate 1.20**

The goal of this section is to prove inequality 1.20 in Theorem 1.2. In order to do this we shall need a number of estimates and expansions for the multiplier \( \hat{w}_\lambda \) given by 3.41. For \( q \geq 1 \), \( (p, q) = 1 \), \( t > 0 \), and \( \xi \in \mathbb{T}^d \) we denote
\[
a_{t, p/q}(\xi) := \frac{\lambda^{d/2-1}}{2|S_t \cap \mathbb{Z}^d|} e^{-2\pi i p q / q \|\xi\|} G(p, q; \|\xi\|) \hat{f}\sigma \left( \sqrt{X}(\|\xi\|/q - \xi) \right).
\]
Next, for \( 1 \leq n \leq N + 1 \), \( t > 0 \) and \( \xi \in \mathbb{T}^d \) we let
\[
b_{t, n}(\xi) := \frac{\lambda^{d/2-1}}{2|S_t \cap \mathbb{Z}^d|} \sum_{p/q \in H_N} \sum_{x \in \mathbb{Z}^d} e^{-2\pi i p q / q \|x\|} \psi(\|x\| - x) \hat{f}\sigma \left( \sqrt{X}(x/q - \xi) \right),
\]
recall that \( N = [\sqrt{X}] \), see 3.50. Theorem 3.61 immediately gives a decomposition of \( m_t \) in terms of the above multipliers.

**Proposition 5.1.** There exists a universal constant \( C > 0 \) such that for all integers \( d \geq 16 \) and \( \lambda > 0 \) satisfying \( \lambda \geq Cd^3 \), and all integers \( 1 \leq n \leq N + 1 \) (with \( N = [\sqrt{X}] \)), we have the decomposition
\[
m_t(\xi) = \sum_{p/q \in H_N} a_{t, p/q}(\xi) + b_{t, n}(\xi) + E_{t, n}(\xi),
\]
where the error term \( E_{t, n}(\xi) \) satisfies
\[
|E_{t, n}(\xi)| \lesssim \frac{d^{3/4}}{\lambda^{d/4-1}},
\]
uniformly in \( \xi \in \mathbb{T}^d \), \( 1 \leq n \leq N + 1 \), \( \lambda \geq Cd^3 \) and \( d \geq 16 \).
Proof. The first sum in (5.3) and \( b_{t,n}(\xi) \) correspond to the normalized multipliers from (5.13) and (5.14), respectively. Thus (5.19) from Theorem 3.1 applies giving

\[
|E_{t,n}(\xi)| \lesssim \|f\|_{L^2(\mathbb{Z}^d)},
\]

which in turn combined with (3.10) and (3.20) yields (5.4).

\[\Box\]

**Theorem 5.1.** There exists a universal constant \( C > 0 \) such that for all \( d \geq 16 \) and all integers \( 1 \leq p \leq q \) so that \( (p,q) = 1 \), one has

\[
\| \sup_{t \in \mathbb{D}, n} |F^{-1}(a_{t,p/q} f)| \|_{L^2(\mathbb{Z}^d)} \lesssim \|f\|_{L^2(\mathbb{Z}^d)},
\]

where the implicit constant is independent of \( p, q, \) and the dimension \( d \).

Proof. Fix \( q \in \mathbb{N} \) and for \( w \in \mathbb{N}_q^d \) we define the sets \( T_w := \{ \xi \in \mathbb{T}^d : \langle \xi \rangle \equiv w \ (\text{mod} \ q) \} \). Then any \( f \in L^2(\mathbb{Z}^d) \) admits the decomposition

\[
f = \sum_{w \in \mathbb{N}_q^d} f_w, \quad \text{with} \quad \hat{f}_w := \hat{f} \cdot \mathbb{1}_{T_w}.
\]

For further use we note that the functions \( \hat{f}_w \) have pairwise disjoint supports. For \( w \in \mathbb{N}_q^d \) one has

\[
\xi - \langle \xi \rangle \equiv \langle w \rangle, \quad \xi \in T_w.
\]

Indeed, writing \( \langle w \rangle = w + kq \) for some \( k \in \mathbb{Z}^d \), we see that

\[
\xi - \langle w \rangle = k + \langle \xi \rangle - \langle w \rangle.
\]

We obtain \( \langle \xi - \langle w \rangle \rangle = k \), since

\[
\langle \xi - \langle w \rangle \rangle = \langle w \rangle - \langle \xi \rangle \in \left[ \frac{-1}{2q}, \frac{1}{2q} \right] \subseteq Q.
\]

Thus, (5.7) justifies (5.6). By (3.19) from Theorem 3.1 (with \( C > 0 \) large enough) we have

\[
\lambda_t^{d/2-1} \leq \frac{1}{\sigma(S)} \quad t \in \mathbb{D},
\]

By definition (5.1) and decomposition (5.5) we estimate

\[
\| \sup_{t \in \mathbb{D}, n} |F^{-1}(a_{t,p/q} f)| \|_{L^2(\mathbb{Z}^d)} \lesssim \sum_{w \in \mathbb{N}_q^d} \|G(p/q; w)\| \sup_{t \in \mathbb{D}, n} \|F^{-1}(F_t \langle \xi \rangle \hat{f}_w)\|_{L^2(\mathbb{Z}^d)}.
\]

Now by (5.6) and Lemma 4.9 we obtain

\[
\| \sup_{t \in \mathbb{D}, n} |F^{-1}(F_t \langle \xi \rangle \hat{f}_w)\|_{L^2(\mathbb{Z}^d)} = \| \sup_{t \in \mathbb{D}, n} |F^{-1}(F_t \langle \xi - \langle w \rangle \rangle \hat{f}_w)\|_{L^2(\mathbb{Z}^d)} \leq \| \hat{f}_w \|_{L^2(\mathbb{Z}^d)}.
\]

Finally the Cauchy–Schwarz inequality together with (3.21), (3.25) and Plancherel’s theorem we conclude

\[
\| \sup_{t \in \mathbb{D}, n} |F^{-1}(a_{t,p/q} f)| \|_{L^2(\mathbb{Z}^d)} \lesssim \left( \sum_{w \in \mathbb{N}_q^d} \|G(p/q; w)\|^2 \right)^{1/2} \left( \sum_{w \in \mathbb{N}_q^d} \| \hat{f}_w \|_{L^2(\mathbb{Z}^d)} \right)^{1/2} = \|f\|_{L^2(\mathbb{Z}^d)}.
\]

The proof of Theorem 5.1 is thus completed. \( \Box \)

We now pass to our second main ingredient necessary to complete the proof of inequality (1.20) in Theorem 1.2.

**Theorem 5.2.** There exist universal constants \( C > 0 \) and \( n_0 \in \mathbb{N} \) such that for all \( d \geq 16 \) one has

\[
\| \sup_{t \in \mathbb{D}, n} |F^{-1}(b_{t,n_0} f)| \|_{L^2(\mathbb{Z}^d)} \lesssim \|f\|_{L^2(\mathbb{Z}^d)},
\]

where the implicit constant is independent of the dimension.
The proof of Theorem 5.2 will take up the bulk of this section. It will be based on a reduction to the sampling principle of Magyar, Stein and Wainger [28]. We now have to set the necessary notation. Assume that \( q \in \mathbb{N} \) and let \( B \) be a finite dimensional Banach space. Let \( m \in L^\infty(\mathbb{R}^d; B) \) be a function supported in \( q^{-1}Q \) and define
\[
m^q \text{per}(\xi) := \sum_{x \in \mathbb{Z}^d} m(\xi - x/q), \quad \xi \in \mathbb{R}^d. \tag{5.8}
\]
Then \( m^q \text{per} \) is \( 1/q \) periodic in each coordinate; in particular it may be regarded as a function on \( \mathbb{T}^d \). Consider the multiplier operators \( T \) and \( T^q \text{dis} \) given by
\[
\mathcal{F}(Tf)(\xi) = m(\xi)\mathcal{F}(f)(\xi), \quad f \in L^2(\mathbb{R}^d), \quad \xi \in \mathbb{R}^d \tag{5.9}
\]
and
\[
\tilde{T}^q \text{dis}f(\xi) = m^q \text{per}(\xi)\hat{f}(\xi), \quad f \in \ell^2(\mathbb{Z}^d), \quad \xi \in \mathbb{T}^d. \tag{5.10}
\]
We shall need two transference results from [28].

**Proposition 5.2** ([28] Corollary 2.1 with \( B_1 = C \) and \( B_2 = B \). Let \( 1 \leq p \leq \infty \) and \( q \in \mathbb{N} \). Then, there exists a universal constant \( C > 0 \) such that
\[
\|T^q \text{dis}\|_{\ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d; B)} \leq C^d\|T\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d; B)}.
\]
(5.11)
The constant \( C \) is independent of \( d, p, q \) and the finite dimensional Banach space \( B \).

**Remark.** The estimate (5.11) is not explicitly stated in [28]. However, it can be easily deduced by a careful inspection of the proof of [28] Corollary 2.1. We now present an argument which absorbs the exponential growth in (5.11) by choosing a large integer \( n_0 \in \mathbb{N} \) in Theorem 5.2.

Further we consider the multiplier
\[
m(\xi) = \sum_{x \in \mathbb{Z}^d} \gamma_x \Phi(\xi - x/q), \quad \xi \in \mathbb{R}^d,
\]
satisfying
a) \( \Phi \in C_c^\infty(q^{-1}Q) \) and as a function on \( Q \) it has the Fourier expansion
\[
\Phi(\xi) = \sum_{x \in \mathbb{Z}^d} \varphi_x e^{-2\pi i x \cdot \xi}, \quad \xi \in Q,
\]
with \( \sum_{x \in \mathbb{Z}^d} |\varphi_x| \leq A \).

b) \( (\gamma_x)_{x \in \mathbb{Z}^d} \) is a \( q\mathbb{Z}^d \) periodic sequence; i.e., \( \gamma_x = \gamma_{x'} \) if \( x - x' \in q\mathbb{Z}^d \).

Using a) and b) and Plancherel’s theorem we conclude that
\[
\|\mathcal{F}^{-1}(m\hat{f})\|_{\ell^2(\mathbb{Z}^d)} \leq A \sup_{x \in \mathbb{Z}^d} |\gamma_x| \|f\|_{\ell^2(\mathbb{Z}^d)}.
\]
(5.12)

We now are ready to prove Theorem 5.2.

**Proof of Theorem 5.2** By the monotone convergence theorem it suffices to show that for \( C > 0 \) and \( n_0 \in \mathbb{N} \) large enough the uniform estimate
\[
\sup_{J \in \mathbb{N}} \sup_{t \in \mathbb{D}_{C, n_0}[0, 1]} \|\mathcal{F}^{-1}(b_{t, n_0} \hat{f})\|_{\ell^2(\mathbb{Z}^d)} \lesssim \|f\|_{\ell^2(\mathbb{Z}^d)}, \tag{5.13}
\]
holds true. From now on \( J \in \mathbb{N} \) is fixed. We also fix a function \( \varphi' \in C_c^\infty((-1/2, 1/2]) \) which is equal to \( 1 \) on \([-1/4, 1/4]\), and satisfies \( \|\varphi'\|_{L^\infty(\mathbb{R})} \leq 1 \); and denote
\[
\psi' := \prod_{j=1}^d \varphi'(\xi_j), \quad \xi \in \mathbb{R}^d.
\]
Then clearly, \( \psi \psi' = \psi \), where \( \psi \) is the function defined by (5.12). We set
\[
H_\infty := \bigcup_{N \in \mathbb{N}} H_N = \{p/q \in \mathbb{Q} : 1 \leq p \leq q, (p, q) = 1\},
\]
recall that we identify \( 0/1 \) with \( 1/1 \). For every \( p/q \in H_\infty \) we also define two operators \( U^q : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d; B) \), where \( B := L^\infty(N_{\mathbb{Z}^d} \cap \mathbb{D}^d) \), and \( V^{p/q} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d; B) \) by setting
\[
U^q f := \left( \mathcal{F}^{-1} \left( \sum_{x \in \mathbb{Z}^d} \psi_q(\xi - x/q)\mathcal{F}(\mu(\sqrt{\lambda}(\xi - x/q))\hat{f}) \right) \right)_{\lambda \in \Lambda_{N_{\mathbb{Z}^d}}}, \quad f \in \ell^2(\mathbb{Z}^d),
\]
(5.14)
where
\[ \psi_q(\xi) = \psi(q \xi), \quad \psi'_q(\xi) = \psi'(q \xi), \quad \xi \in \mathbb{R}^d, \]
and
\[ V^{p/q} f := F^{-1} \left( \sum_{x \in \mathbb{Z}^d} G(p/q; x) \psi'_q(\xi - x/q) \hat{f} \right), \quad f \in l^2(\mathbb{Z}^d). \]

Since \( t \in D_{C, \infty} \) we have \( t \geq C \delta^{3/2} \) so that \( N = \lceil \sqrt{X} \rceil \geq C \delta^{3/2} - 1 \). Recalling (5.2), using (5.19) and the fact that \( \psi q \psi'_q = \psi_q \)
we see that
\[ \| \sup_{t \in D_{C, \infty} \cap [0,1]} |F^{-1}(b_{t,n_0} \hat{f})| \|_{l^2(\mathbb{Z}^d)} \lesssim \sum_{q \geq n_0} \| U^q V^{p/q} f \|_{l^2(\mathbb{Z}^d; B)}, \tag{5.15} \]
uniformly in \( 1 \leq n_0 \leq C \delta^{3/2} \). We specify \( n_0 \) later.

Now, observe that the proof of Theorem 5.10 will be completed once we show that there are universal constants \( D_1, D_2 > 0 \) such that
\[ \| U^q f \|_{l^2(\mathbb{Z}^d; B)} \leq D_1 \| f \|_{l^2(\mathbb{Z}^d)}, \tag{5.16} \]
\[ \| V^{p/q} f \|_{l^2(\mathbb{Z}^d)} \leq D_2 q^{-d/2} \| f \|_{l^2(\mathbb{Z}^d)}, \tag{5.17} \]
uniformly in \( p/q \in H_\infty \) and \( d \geq 16 \). Indeed, using (5.15) and assuming momentarily that (5.16) and (5.17) hold and taking \( n_0 := \lceil (D_1D_2)^{10} \rceil + 1 \) we obtain
\[ \| \sup_{t \in D_{C, \infty} \cap [0,1]} |F^{-1}(b_{t,n_0} \hat{f})| \|_{l^2(\mathbb{Z}^d)} \lesssim \| f \|_{l^2(\mathbb{Z}^d)} \sum_{q \geq n_0} (D_1D_2)^d q^{-d/2+1} \]
\[ \lesssim \| f \|_{l^2(\mathbb{Z}^d)} \sum_{q \geq n_0} q^{-2d/5+1} \left( \frac{(D_1D_2)^{10}}{q} \right)^{d/10} \lesssim \| f \|_{l^2(\mathbb{Z}^d)}, \]
as long as \( d \geq 16 \). Therefore we reduced the proof of Theorem 5.2 to showing (5.16) and (5.17).

**Proof of estimate (5.16).** Considering \((m^q_p(x))_{\lambda \in N, j_0 \in \mathbb{Z}^d} \in B = L^\infty(N_j \cap \mathbb{Z}^d)\) from (5.8) with
\[ m(\xi) := (\psi_q(\xi) \mathcal{F}_\mu(\sqrt{x} \xi))_{\lambda \in N, j_0 \in \mathbb{Z}^d}, \quad \xi \in \mathbb{R}^d \]
it is easy to see, from definition (5.14), that \( U^q \) is equal to \( T_{\text{dis}}^d \) defined in (5.16). Since \( \text{supp} \ m \subseteq q^{-1}Q \)
and \( m \) belongs to \( L^\infty(\mathbb{R}^d; B) \) and satisfies \( \| m \|_{L^\infty(\mathbb{R}^d)} \leq 1 \), hence, Proposition 5.2 applies and we obtain
\[ \| U^q \|_{l^2(\mathbb{Z}^d; \rightarrow l^2(\mathbb{Z}^d; B))} \lesssim \| T \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}, \]
where \( T \) is defined by (5.20). The latter operator norm, using Lemma 4.8 (see also 3.3 [55]), is bounded and for \( f \in L^2(\mathbb{R}^d) \) we have
\[ \| Tf \|_{L^2(\mathbb{R}^d; B)} \lesssim \| F^{-1}(\psi_q F f) \|_{L^2(\mathbb{R}^d)} \leq \| \psi_q \|_{L^\infty(\mathbb{R}^d)} \| f \|_{L^2(\mathbb{R}^d)} \leq \| f \|_{L^2(\mathbb{R}^d)}. \]
Thus (5.16) follows with a universal constant \( D_1 > 0 \).

**Proof of estimate (5.17).** Here we apply inequality (5.12) with \( \gamma_x := G(p/q; x) \) and \( \Phi := \psi'_q \). Note that \( \Phi \) satisfies condition a); here we shall use the fact that \( \psi'_q \in C_c^\infty(Q) \). Indeed, \( \Phi \in C_c^\infty(q^{-1}Q) \) and by Fourier inversion theorem, we have \( \varphi_x = \int_Q \psi'(q \xi) e^{2\pi i x \xi} \xi \ d\xi = q^{-d} \mathcal{F}(\psi')(x/q) \). Since \( \mathcal{F}(\psi') \) is a tensor product of Schwartz functions, we thus obtain \( \sum_{x \in \mathbb{Z}^d} |\varphi_x| \lesssim 1 \). Therefore, \( \Phi \) satisfies condition a) of inequality (5.12) with a constant \( A^d \) in place of \( A \), where \( A \) is a universal constant depending only on \( \psi' \). Moreover, \( G(p/q; x) \) is \( q \mathbb{Z}^d \) periodic, and \( \text{supp}_{x \in \mathbb{Z}^d} G(p/q; x) \leq (2/q)^{d/2} \), by (3.20). Therefore, an application of inequality (5.12) is justified and leads to (5.17) with \( D_2 = \sqrt{2A} \).}

**5.1. All together: proof of inequality (1.20) in Theorem 1.2.** Let \( C_3 > 0 \) be a universal constant so large that the conclusions of Proposition 5.10 in Theorem 5.11 and Theorem 5.2 are satisfied. Take \( n_0 \in \mathbb{N} \) large enough so that the conclusion of Theorem 5.2 holds. Then, using Proposition 5.11 together with Theorem 5.1 and Theorem 5.2 and Plancherel's theorem we see that
\[ \| \sup_{t \in B_{C_3}} |A_t f| \|_{L^2(Z^d)} \lesssim \| f \|_{L^2(Z^d)} + \sum_{t \in B_{C_3}} \| F^{-1}(E_{t,0} f) \|_{L^2(Z^d)} \]
\[ \lesssim \left( 1 + \sum_{\lambda \geq C_3^2d} A^d \frac{d^{d/4}}{\lambda^{d/4-1}} \right) \| f \|_{L^2(Z^d)} \]
\[ \lesssim \left( 1 + \sum_{\lambda \geq C_3^2d^3} \frac{d^d}{\lambda^2} \left( \frac{d^d A^4}{\lambda} \right)^{d/4-3} \right) \| f \|_{L^2(Z^d)}, \]

where \( A > 1 \) is a universal constant and \( E_{t,0} \) is the error term satisfying (5.1). Since \( d \geq 16 \) taking in (5.18) a constant \( C_3 > 0 \) such that \( C_3^2 \geq 2A^4 \) we obtain
\[ \| \sup_{t \in B_{C_3}} |A_t f| \|_{L^2(Z^d)} \lesssim \left( 1 + \sum_{\lambda \geq C_3^2d^3} \lambda^{-2} \right) \| f \|_{L^2(Z^d)} \lesssim \| f \|_{L^2(Z^d)}. \]

This completes the proof of (5.20). \( \square \)

**Remark 5.1.** We note that it is possible to show a slightly stronger result than (1.20). Namely, there are universal constants \( C, C_3 > 0 \) such that
\[ \sup_{d \geq 5} C(2, \sqrt{\pi} \cap (C_3 d^{3/2}, \infty), S^{d-1}) \leq C. \]

To prove (5.19) we need a refinement of (1.19), where \( \sup_{t \in \mathbb{Z}} \) is replaced by \( \sup_{t > 0} \). For this purpose we have to prove the following estimate
\[ \| \langle \eta, \mathcal{F}(\mu^r)(\eta) \rangle \| \lesssim 1, \quad r \geq 2, \quad \eta \in \mathbb{R}^r, \]

where the implicit constant is independent of \( r \) and \( \eta \in \mathbb{R}^r \). Proceeding as in the proof of Lemma 4.1 we may deduce (5.20). We omit the details.

### 6. Proof of Theorem 1.2 Intermediate-scale estimate 1.21

This section is devoted to the proof of inequality (1.21) in Theorem 1.2. The main estimate for Fourier transforms in this regime is the following proposition.

**Proposition 6.1.** Let \( 100d \leq \lambda \leq d^3 \). Then for all \( \xi \in \mathbb{T}^d \) we have
\[ |m_1(\xi)| \lesssim (\kappa(d, \lambda)|\xi||)^{-1} + (\kappa(d, \lambda)|\xi + 1/2|^(-1) + \kappa(d, \lambda)^{-4}. \]

Assuming for a moment Proposition 6.1 we may proceed as follows. We let
\[ V_{\xi} := \{ i \in N_d : \cos(2\pi \xi_i) < 0 \} = \{ i \in N_d : 1/4 < \| \xi_i \| \leq 1/2 \} \text{ for } \xi \in \mathbb{T}^d. \]

We will study the maximal functions associated with the following multipliers
\[ p_{\lambda}^1(\xi) := e^{-\kappa(d, \lambda)^2 \sum_{i=1}^d \sin^2(\pi \xi_i)} \quad \text{if } |V_{\xi}| \leq d/2, \]
\[ p_{\lambda}^2(\xi) := (-1)^d e^{-\kappa(d, \lambda)^2 \sum_{i=1}^d \cos^2(\pi \xi_i)} \quad \text{if } |V_{\xi}| > d/2. \]

The sets \( V_{\xi} \) and the multipliers \( p_{\lambda}^1 \) and \( p_{\lambda}^2 \) will be also used in the small scales case. Propositions 4.3 and 6.1 imply the following estimates.

**Proposition 6.2.** Let \( d, \lambda \in \mathbb{N} \) be such that \( 100d \leq \lambda \leq d^3 \). Then for every \( \xi \in \mathbb{T}^d \) we have the following bounds:

1. if \( |V_{\xi}| \leq d/2 \), then
\[ |m_1(\xi) - p_{\lambda}^1(\xi)| \lesssim \min \{ (\kappa(d, \lambda)|\xi||)^{-1}, \kappa(d, \lambda)|\xi| \} + \kappa(d, \lambda)^{-1}; \]

2. if \( |V_{\xi}| > d/2 \), then
\[ |m_1(\xi) - p_{\lambda}^2(\xi)| \lesssim \min \{ (\kappa(d, \lambda)|\xi + 1/2||)^{-1}, \kappa(d, \lambda)|\xi + 1/2| \} + \kappa(d, \lambda)^{-1}. \]

**Proof.** Note that \( \sum_{i=1}^d \sin^2(\pi \xi_i) \simeq |\xi|^2 \) and \( \sum_{i=1}^d \cos^2(\pi \xi_i) \simeq |\xi + 1/2|^2 \). We consider two cases.

If \( |V_{\xi}| \leq d/2 \), then \( |\xi + 1/2| \gtrsim \sqrt{d} \geq 1 \). Therefore using Proposition 6.1 we obtain
\[ |m_1(\xi) - p_{\lambda}^1(\xi)| \lesssim (\kappa(d, \lambda)|\xi||)^{-1} + (\kappa(d, \lambda)|\xi + 1/2||)^{-1} + \kappa(d, \lambda)^{-4} \lesssim (\kappa(d, \lambda)|\xi||)^{-1} + \kappa(d, \lambda)^{-1}. \]
Moreover, combining \( |m_t(\xi) - p^1_t(\xi)| \leq |m_t(\xi) - 1| + |1 - p^1_t(\xi)| \) with (4.13) we see that
\[
|m_t(\xi) - p^1_t(\xi)| \lesssim \kappa(d, \lambda)\|\xi\|.
\]

Consider now \( |V_t| \geq d/2 \). In this case \( \|\xi\| \gtrsim \sqrt{d} \geq 1 \). Thus, using Proposition 6.1 we have
\[
|m_t(\xi) - p^2_t(\xi)| \lesssim (\kappa(d, \lambda)\|\xi\|)^{-1} + (\kappa(d, \lambda)\|\xi + 1/2\|)^{-1} + \kappa(d, \lambda)^{-4} \lesssim (\kappa(d, \lambda)\|\xi + 1/2\|)^{-1} + \kappa(d, \lambda)^{-1}.
\]
Next, \( |m_t(\xi) - p^2_t(\xi)| \leq |m_t(\xi) - (\lambda^{-1})^2| + |(\lambda^{-1})^2 - p^2_t(\xi)| \) together with (4.14) gives
\[
|m_t(\xi) - p^2_t(\xi)| \lesssim \kappa(d, \lambda)\|\xi + 1/2\|.
\]
Hence, (6.5) and (6.6) are proved as claimed. \( \square \)

Having proved Proposition 6.2 we can now establish estimate (6.21) in Theorem 7.2. Here we shall need the symmetric diffusion semigroup given by (4.17).

Proof of inequality (6.21) in Theorem 7.2. Let \( f \in \ell^2(\mathbb{Z}^d) \) and we write \( f = f_1 + f_2 \), where \( f_1(\xi) := \hat{f}(\xi)\mathbf{1}_{\{0 \leq t \leq d/2\}}(\xi) \). Then
\[
\left\| \sup_{t \in B_{c_1, c_2}} |F^{-1}(m_t\hat{f})| \right\|_{\ell^2(\mathbb{Z}^d)} \leq \sum_{i=1}^2 \left\| \sup_{t \in B_{c_1, c_2}} |F^{-1}(p^1_t\hat{f}_i)| \right\|_{\ell^2(\mathbb{Z}^d)}
+ \sum_{i=1}^2 \left\| \left( \sum_{t \in B_{c_1, c_2}} |F^{-1}(m_t - p^1_t\hat{f}_i)| \right)^2 \right\|_{\ell^2(\mathbb{Z}^d)}^{1/2}.
\]

The usual square function argument permits therefore to reduce the problem to controlling the maximal functions associated with the multipliers \( p^1_t \) and \( p^2_t \). Indeed, taking \( C_1 = 10 \) and \( C_2 = 1 \), Plancherel's theorem and Proposition 6.2 imply
\[
\sum_{i=1}^2 \left\| \left( \sum_{t \in B_{c_1, c_2}} |F^{-1}(m_t - p^1_t\hat{f}_i)| \right)^2 \right\|_{\ell^2(\mathbb{Z}^d)}^{1/2} \lesssim \|f_1\|_{\ell^2(\mathbb{Z}^d)} + \|f_2\|_{\ell^2(\mathbb{Z}^d)} \leq 2\|f\|_{\ell^2(\mathbb{Z}^d)}.
\]

Thus we only have to bound the maximal functions corresponding to the multipliers (6.3) and (6.4). Indeed, since \( p^1_t(\xi) = p^{(d, \lambda)^2}(\xi) \), then by (4.13) we obtain
\[
\left\| \sup_{t > 0} |F^{-1}(p^1_t\hat{f}_1)| \right\|_{\ell^2(\mathbb{Z}^d)} \lesssim \|f_1\|_{\ell^2(\mathbb{Z}^d)}.
\]

It is also not difficult to see that
\[
\left\| \sup_{t > 0} |F^{-1}(p^1_t\hat{f}_2)| \right\|_{\ell^2(\mathbb{Z}^d)} \lesssim \|f_2\|_{\ell^2(\mathbb{Z}^d)}.
\]

Indeed, since \( p^1_t(\xi) = (\lambda^{-1})^2p^{(d, \lambda)^2}(\xi - 1/2) \), letting \( \hat{F}_2(\xi) := \hat{f}_2(\xi + 1/2) \) we obtain
\[
\sup_{t > 0} |F^{-1}(p^1_t\hat{f}_2)| = \sup_{t > 0} |F^{-1}(p_t\hat{F}_2)|.
\]

Therefore using (4.13) and the fact that \( \|F_2\|_{\ell^2(\mathbb{Z}^d)} = \|f_2\|_{\ell^2(\mathbb{Z}^d)} \leq \|f\|_{\ell^2(\mathbb{Z}^d)} \) we get (6.8). This finishes the proof of inequality (1.21) in Theorem 1.2. \( \square \)

6.1. Some preparatory estimates. What is left is to prove Proposition 6.3. The key idea will be to use the dimension reduction trick from [8]. The proof of Proposition 6.3 will require several lemmas, which are similar to those proved in [8] Section 2. Their proofs will mostly rely on Lemma 5.4 and the methods developed in [8] Section 2.

Lemma 6.3. Let \( d \geq 10 \). Given \( \varepsilon_1, \varepsilon_2 \in (0, 1] \) we define for every \( \lambda \in \mathbb{N} \) the set
\[
E = \left\{ x \in S_t \cap \mathbb{Z}^d : \{i \in \mathbb{N}_d : |x_i| \geq \varepsilon_2\kappa(d, \lambda)\} \leq \varepsilon_1d \right\}.
\]
If \( \varepsilon_1, \varepsilon_2 \in (0, 1/10] \) and \( d \geq \kappa(d, \lambda) \geq 10 \), then we have
\[
|E| \lesssim e^{-\frac{d}{2}|S_t \cap \mathbb{Z}^d|}.
\]
Proof. Considering
\[ \hat{E} = \{ x \in B_2^d(d) \cap \mathbb{Z}^d : |\{ i \in \mathbb{N}_d : |x_i| \geq \varepsilon \kappa(d, \lambda) \}| \leq \varepsilon_1 d \} \]
and using [8] Lemma 2.4 (notice that this result is still valid if we allow \( N \in \sqrt{n} \) there) we see that
\[ |\hat{E}| \lesssim e^{-\frac{1}{8}|B^2_2(d) \cap \mathbb{Z}^d|}. \tag{6.9} \]
Clearly \( E \subseteq \hat{E} \), thus inequality \((6.9)\) and Lemma \([8, \text{Lemma 2.3}]\) show that
\[ |E| \leq |\hat{E}| \lesssim e^{-\frac{1}{8}|B^2_2(d) \cap \mathbb{Z}^d|} \leq e^{-\frac{1}{8}(2t + 1)^4|\mathcal{S}_t \cap \mathbb{Z}^d|} \lesssim e^{-\frac{1}{8}|\mathcal{S}_t \cap \mathbb{Z}^d|}, \]
where in the last inequality used that \( t = \sqrt{n} \kappa(d, \lambda) \leq d^{1/2} \). This completes the proof of the lemma. \( \square \)

Having proved Lemma 6.3 we can now perform the dimension-reduction trick as in [8, Section 2.3]. First we need to justify an analogue of [8, Lemma 2.7]. Here a concentration inequality for the hypergeometric distribution (see [8, Lemma 2.5]) will be important.

**Lemma 6.4.** Let \( d \geq 10 \) and \( \lambda \in \mathbb{N} \). For \( \varepsilon \in (0, 1/50] \) and \( r \) an integer between 1 and \( d \) we define
\[ E := \{ x \in \mathcal{S}_t \cap \mathbb{Z}^d : \sum_{i=1}^r x_i^2 < \varepsilon^3 \kappa(d, \lambda)^2 r \} \tag{6.10} \]
If \( d \geq \kappa(d, \lambda) \geq 10 \) then we have
\[ |E| \lesssim e^{-\frac{1}{8}|\mathcal{S}_t \cap \mathbb{Z}^d|}. \]

**Proof.** Let \( \delta_1 \in (0, 1/10] \) be such that \( \delta_1 = 5\varepsilon \) and define \( I_x := \{ i \in \mathbb{N}_d : |x_i| \geq \varepsilon \kappa(d, \lambda) \} \). We have \( E \subseteq E_1 \cup E_2 \), where
\[ E_1 := \{ x \in \mathcal{S}_t \cap \mathbb{Z}^d : \sum_{i \in I_x \cap \mathbb{N}_r} x_i^2 < \varepsilon^3 \kappa(d, \lambda)^2 r \text{ and } |I_x| \geq \delta_1 d \}, \]
\[ E_2 := \{ x \in \mathcal{S}_t \cap \mathbb{Z}^d : |I_x| < \delta_1 d \}. \]

By Lemma 6.3 (with \( \varepsilon_1 = \delta_1 \) and \( \varepsilon_2 = \varepsilon \)) we have \( |E_2| \lesssim e^{-\frac{1}{8}|\mathcal{S}_t \cap \mathbb{Z}^d|} \).

To estimate \( E_1 \) we note that \( I_{\tau^{-1} \circ x} = \tau(I_x) \) for any \( \tau \in \text{Sym}(d) \) and \( x \in \mathcal{S}_t \cap \mathbb{Z}^d \). Therefore we have
\[
|E_1| = \sum_{x \in \mathcal{S}_t \cap \mathbb{Z}^d} \frac{1}{d^t} \sum_{\tau \in \text{Sym}(d)} \mathbb{I}_{E_1}(\tau^{-1} \circ x)
= \sum_{x \in \mathcal{S}_t \cap \mathbb{Z}^d} \mathbb{P}[\{ \tau \in \text{Sym}(d) : \sum_{i \in \tau(I_x) \cap \mathbb{N}_r} x_i^2 < \varepsilon^3 \kappa(d, \lambda)^2 r \text{ and } |I_x| \geq \delta_1 d \}]
\leq \sum_{x \in \mathcal{S}_t \cap \mathbb{Z}^d} \mathbb{P}[\{ \tau \in \text{Sym}(d) : |\tau(I_x) \cap \mathbb{N}_r| \leq \varepsilon r \} \mathbb{I}_{|I_x| \geq \delta_1 d}
\leq 2e^{-\frac{1}{8}|\mathcal{S}_t \cap \mathbb{Z}^d|},
\]
where the last bound is a consequence of [8, Lemma 2.5] (with \( I = I_x \), \( J = \mathbb{N}_r \), \( \delta_2 = \varepsilon = \frac{d}{8} \)). \( \square \)

Lemma 6.4 below will play an essential role in the proof of Proposition 6.1.

**Lemma 6.5.** For \( d \in \mathbb{N} \), \( \lambda = \ell^2 \in \mathbb{N} \) and \( \varepsilon \in (0, 1/50] \), if \( 10 \leq \kappa(d, \lambda) \leq d \), then for every \( 4 \leq r \leq d \) and \( \xi \in \mathbb{T}^d \) we have
\[ |m_{\ell}(\xi)| \lesssim \sup_{\varepsilon \kappa(d, \lambda)^2 r \leq \xi \leq \lambda} |m^{(r)}(\xi_1, \ldots, \xi_r)| + e^{-\frac{1}{8}}, \tag{6.11} \]
where
\[ m^{(r)}(\eta) = \frac{1}{|S_t^{r-1} \cap \mathbb{Z}^r|} \sum_{x \in S_t^{r-1} \cap \mathbb{Z}^r} e^{2\pi i x \eta}, \quad \eta \in \mathbb{T}^r, \quad r \geq 4, \tag{6.12} \]
is the lower dimensional multiplier with \( r \in \mathbb{N} \) and \( t > 0 \).
Proof. The case \( r = d \) is trivial so we assume that \( r < d \). We identify \( \mathbb{R}^d = \mathbb{R}^r \times \mathbb{R}^{d-r} \) and \( \mathbb{T}^d = \mathbb{T}^r \times \mathbb{T}^{d-r} \) and we will write \( \mathbb{R}^d \ni x = (x^1, x^2) \in \mathbb{R}^r \times \mathbb{R}^{d-r} \) and \( \mathbb{T}^d \ni \xi = (\xi^1, \xi^2) \in \mathbb{T}^r \times \mathbb{T}^{d-r} \), respectively. Note that Lemma 6.4 gives the disjoint decomposition

\[
S_1 \cap \mathbb{Z}^d = E \cup \bigcup_{\lambda \geq \varepsilon \sqrt{d}, \varepsilon (d, \lambda)^2 r} (S_{\sqrt{r-1}} \cap \mathbb{Z}^r) \times (S_{\sqrt{d-r-1}} \times \mathbb{Z}^{d-r}),
\]

(6.13)

with \( E \) being defined in (6.10). Invoking (6.13) we immediately obtain (6.11). □

The utility of the above lemma comes from the fact that, once \( r \) is appropriately chosen, Lemma 6.5 allows us to apply Proposition 4.4 for the lower-dimensional multipliers \( m^{(r)}(\sqrt{\xi_1, \ldots, \xi_r}) \).

Lemma 6.6. Let \( C > 0 \) be a universal constant for which the conclusion of Proposition 4.4 holds and take \( \delta \in (0, 1/2) \) and \( \varepsilon \in (0, 1/50] \). Let \( d, \lambda \in \mathbb{N} \) be such that \( \kappa (d, \lambda) \leq d \) and take \( r \) an integer such that \( 1 \leq r \leq d \) and

\[
\max \{21, \varepsilon^{\frac{3d}{2}} \kappa (d, \lambda) \delta C^{-d/2}/4 \} \leq r \leq \max \{21, \varepsilon^{\frac{3d}{2}} \kappa (d, \lambda) \delta C^{-d/2}/2 \}.
\]

(6.14)

Then, for every \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{T}^d \) we have

\[
|m_t (\xi)| \lesssim_{d, \varepsilon} \kappa (d, \lambda) r^{-4} + \varepsilon^{-2} \varepsilon^{3/2} (\kappa (d, \lambda) ||\eta||) + \varepsilon^{-2} \varepsilon^{3/2} (\kappa (d, \lambda) ||\eta|| + 1/2),
\]

(6.15)

where \( \eta = (\xi_1, \ldots, \xi_r) \).

Proof. If \( \kappa (d, \lambda) \leq \varepsilon^{-\frac{1}{2}} \), then there is nothing to do, since the implied constant in question is allowed to depend on \( \delta \) and \( \varepsilon \). We will assume that \( \kappa (d, \lambda) \geq \varepsilon^{-\frac{1}{2}} \), which ensures that \( \kappa (d, \lambda) \geq 10 \). For such \( \kappa (d, \lambda) \) if \( \varepsilon^{\frac{3d}{2}} \kappa (d, \lambda) \delta C^{-d/2}/4 \leq 21 \), then \( \kappa (d, \lambda) \simeq_{\varepsilon, \delta} 1 \), and (6.15) is again obvious. Therefore, without loss of generality we assume that \( \kappa (d, \lambda) \geq 10 \) and that

\[
21 \leq \varepsilon^{\frac{3d}{2}} \kappa (d, \lambda) \delta C^{-d/2}/4 \leq \varepsilon^{\frac{3d}{2}} \kappa (d, \lambda) \delta C^{-d/2}/2.
\]

(6.16)

In view of Lemma 6.5 and (6.14) we have

\[
|m_t (\xi)| \lesssim_{d, \varepsilon} \sup_{\varepsilon^{3d}(d, \lambda)^2 r \leq ||\xi|| \leq \lambda} |m^{(r)}(\eta)| + \kappa (d, \lambda) r^{-4},
\]

(6.17)

where \( \eta = (\xi_1, \ldots, \xi_r) \). By (6.16) we have \( \kappa (d, \lambda) \geq \sqrt{C} \varepsilon^{-1/2} \), hence \( \varepsilon^{3d}(d, \lambda)^2 r \geq Cr^{-1+2d} \geq Cr^3 \). Thus, for \( l \geq \varepsilon^{3d}(d, \lambda)^2 r \) we are allowed to apply Proposition 4.4 in dimension \( r \) to each of the multipliers \( m^{(r)}(\eta) \). In view of (6.17) we thus have

\[
|m_t (\xi)| \lesssim_{d, \varepsilon} \sup_{\varepsilon^{3d}(d, \lambda)^2 r \leq ||\xi|| \leq \lambda} \left( e^{-2\varepsilon^{3d}(d, \lambda)^2 r ||\eta||} + e^{-2\varepsilon^{3d}(d, \lambda)^2 r ||\eta|| + 1/2} + l^{-2} + e^{-cr} + \kappa (d, \lambda) r^{-4} \right).
\]

(6.18)

Recalling (6.10) and noting that for \( l \geq \varepsilon^{3d}(d, \lambda)^2 r \) we have \( \kappa (r, l) \geq \varepsilon^{3/2} \kappa (d, \lambda) \) and \( l \gtrsim \varepsilon \kappa (d, \lambda) r^3 \), we see that the above inequality leads to (6.15) as desired. □

6.2. All together: proof of Proposition 6.1. We have prepared all necessary tools to prove Proposition 6.1. We shall be working under the assumptions of Lemma 6.5 with \( \delta = 2/7 \) and \( \varepsilon = 1/50 \). While proving (6.14) we may assume that \( \kappa (d, \lambda) \geq \max \{10, \varepsilon^{1/5} 50^{1/2} \sqrt{C} \} \), where \( C \) is a constant from Lemma 6.6, as in the other case the inequality is obvious. Then the assumption (6.14) on the integer \( r \) becomes

\[
21 \leq 50 \varepsilon^{\frac{3d}{2}} \kappa (d, \lambda) \delta C^{-d/2}/4 \leq 50 \varepsilon^{\frac{3d}{2}} \kappa (d, \lambda) \delta C^{-d/2}/2 \leq d;
\]

(6.18)

the last inequality is a consequence of the assumption \( \lambda \leq d^3 \) which forces \( \kappa (d, \lambda) \leq d \).

We shall also require a variant of the convexity inequality from [8, Lemma 2.6]. The difference between Lemma 6.7 and [8, Lemma 2.6] lies in the fact that we do not require \( u_1, \ldots, u_d \) to be monotone.

Lemma 6.7 (cf. [8, Lemma 2.6]). Assume that we have a sequence \( (u_j : j \in \mathbb{N}) \) with \( 0 \leq u_j \leq (1-\delta_0)/2 \) for some \( \delta_0 \in (0, 1) \). Suppose that \( I \subseteq \mathbb{N} \) satisfies \( \delta_1 \leq |I| \leq d \) for some \( \delta_1 \leq (0, 1] \). Then for every \( J = \{d_0, d \} \cap \mathbb{Z} \) with \( 0 \leq d_0 \leq d \) we have

\[
\mathbb{E} \left[ \exp \left( - \sum_{j \in r(I) \cap J} u_j \right) \right] \leq 3 \exp \left( - \frac{\delta_0 \delta_1}{20} \sum_{j \in J} u_j \right).
\]
Proof. Denote \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_d) \) with \( \tilde{u}_j = u_j \) for \( j \in J \) and \( \tilde{u}_j = 0 \) for \( j \in \mathbb{N}_d \setminus J \). Let \( v = (v_1, \ldots, v_d) \) be a non-increasing rearrangement of \( \tilde{u} \). Since \( \tilde{u} = \tilde{\tau} \circ v \) for some \( \tilde{\tau} \in \text{Sym}(d) \) we see that

\[
E\left[ \exp \left( - \sum_{j \in \tilde{\tau}(I) \cap J} u_j \right) \right] = E\left[ \exp \left( - \sum_{j = \tilde{\tau}(I)} \tilde{u}_j \right) \right] = E\left[ \exp \left( - \sum_{j \in \tilde{\tau}(I)} u_j \right) \right].
\]

Clearly \( 0 \leq v_d \leq \ldots \leq v_1 \leq (1 - \delta_0)/2 \), hence, applying \cite{[8]} Lemma 2.6] with \( J = \mathbb{N}_d \) we obtain

\[
E\left[ \exp \left( - \sum_{j \in \tilde{\tau}(I) \cap J} u_j \right) \right] \leq 3 \exp \left( - \frac{\delta_0 \delta_1}{20} \sum_{j \in J} v_j \right) = 3 \exp \left( - \frac{\delta_0 \delta_1}{20} \sum_{j \in J} u_j \right).
\]

and the proof is completed. \( \square \)

Proof of Proposition 6.3.

Letting

\[
r_0 := \left[ \frac{50}{\delta_0^2} \kappa(d, \lambda)^4 C^{-\delta/2}/4 \right] + 1,
\]

we see that any \( r \in [r_0, 2r_0] \) still satisfies \cite{[6.18]}. Fix \( \xi \in \mathbb{T}^d \). Clearly, there exist \( \tau, \theta \in \text{Sym}(d) \) such that

\[
\|\xi_{\tau(1)}\| \geq \|\xi_{\tau(2)}\| \geq \ldots \geq \|\xi_{\tau(d)}\| \quad \text{and} \quad \|\xi_{\theta(1)} + 1/2\| \geq \|\xi_{\theta(2)} + 1/2\| \geq \ldots \geq \|\xi_{\theta(d)} + 1/2\|.
\]

Let

\[
I := \tau(\mathbb{N}_{r_0}) \cup \theta(\mathbb{N}_{r_0}), \quad r := |I|
\]

so that \( r_0 \leq r \leq 2r_0 \). Since both sides of \cite{6.11] are invariant under the permutation group \text{Sym}(d) \) without loss of generality we may assume that \( I = \mathbb{N}_r \). Thus Lemma 6.3 gives

\[
|m_r(\xi)| \lesssim_{d, \lambda} \kappa(d, \lambda)^{1/4} + e^{-2\pi \kappa^{3/2}(d, \lambda)|\eta|} + e^{-2\pi \kappa^{3/2}(d, \lambda)|\eta + 1/2|},
\]

(6.19)

where \( \eta = (\xi_1, \ldots, \xi_r) \). Note that \( \mathbb{N}_r \) contains at least \( r_0 \) largest numbers among the elements of both \( \{\|\xi_j\| : j \in \mathbb{N}_d\} \) and \( \{\|\xi_j + 1/2\| : j \in \mathbb{N}_d\} \), which means that

\[
\left| \left\{ j \in \mathbb{N}_r : \|\xi_j\| \geq \max_{i \in \mathbb{N}_{r_0}\setminus \mathbb{N}_r} \|\xi_i\| \right\} \right| \geq r_0,
\]

\[
\left| \left\{ j \in \mathbb{N}_r : \|\xi_j + 1/2\| \geq \max_{i \in \mathbb{N}_{r_0}\setminus \mathbb{N}_r} \|\xi_i + 1/2\| \right\} \right| \geq r_0.
\]

(6.20)

In the proof we shall need a modification of \( V_\xi \) from \cite{6.2} given by

\[
V_\eta := \{ i \in \mathbb{N}_r : \cos(2\pi \eta_i) < 0 \} = \{ i \in \mathbb{N}_r : 1/4 < |\eta_i| \leq 1/2 \} \quad \text{for} \quad \eta \in \mathbb{T}^r.
\]

We will consider two cases: either \( |V_\eta| \leq r/2 \) or \( |V_\eta| > r/2 \).

Firstly assume that \( |V_\eta| \leq r/2 \). Here our goal is to prove that

\[
|m_r(\xi)| \lesssim \kappa(d, \lambda)^{1/4} + (\kappa(d, \lambda)|\xi|)^{-1},
\]

(6.21)

which clearly implies \cite{6.11]. If \( |V_\eta| \leq r/2 \) then at least \( r/2 \) numbers among the elements \( \{\|\eta_j\| : j \in \mathbb{N}_r\} \) are smaller than \( 1/4 \), so that \( \|\eta + 1/2\| \geq 2^{-5/2}r_0^{1/2} \). In this case \cite{6.19] implies

\[
|m_r(\xi)| \lesssim \kappa(d, \lambda)^{1/4} + \exp(-2\pi \kappa^{3/2}(d, \lambda)|\eta|).
\]

(6.22)

Suppose for a moment that \( |\eta|^2 \geq \frac{1}{4} \|\xi\|^2 \). Then \cite{6.22] implies \cite{6.24]. Thus we can assume that

\[
\|\xi_1\|^2 + \ldots + \|\xi_r\|^2 \leq \frac{1}{4} \|\xi\|^2.
\]

(6.23)

Let \( \varepsilon_1 = 1/10 \) and assume first that

\[
\|\xi_j\| \leq \frac{\varepsilon_1^{1/2}}{10 \kappa(d, \lambda)} \quad \text{for all} \quad r < j \leq d.
\]

(6.24)

We will use \cite{4.15] and the Cauchy–Schwarz inequality we have

\[
|m_r(\xi)|^2 \leq \frac{1}{|S_r \cap \mathbb{Z}^d|} \sum_{x \in S_r \cap \mathbb{Z}^d} \exp \left( - \sum_{j=r+1}^d \sin^2(2\pi x_j \xi_j) \right).
\]

For \( x \in S_r \cap \mathbb{Z}^d \) we define

\[
I_x := \{ i \in \mathbb{N}_d : \varepsilon \kappa(d, \lambda) \leq |x_i| \leq 2\varepsilon_1^{-1/2} \kappa(d, \lambda) \},
\]

\[
E := \{ x \in S_r \cap \mathbb{Z}^d : |I_x| \geq \varepsilon_1 d/2 \}.
\]
Then by Lemma 6.3 (with $\varepsilon_2 = \varepsilon$) we obtain $|E^c| \lesssim e^{-\frac{\varepsilon}{2}|S| \cap \mathbb{Z}^d}$. Therefore,
\[
|m_\varepsilon(\xi)|^2 \lesssim \frac{1}{|S| \cap \mathbb{Z}^d} \sum_{x \in E} \exp \left( - \sum_{j \in I_r \cap J_r} \sin^2(2\pi x_j \xi_j) \right) + e^{-\frac{\varepsilon}{2}}.
\]
(6.25)

where $J_r = N \setminus N_r$. From (6.20) and the definition of $I_x$ we see that
\[
\sin^2(2\pi x_j \xi_j) \geq 4\|2\pi x_j \xi_j\|^2 \geq 16\varepsilon^2 \kappa(d, \lambda)^2 \|\xi_j\|^2
\]
for $j \in I$. Consequently, we obtain
\[
\frac{1}{|S| \cap \mathbb{Z}^d} \sum_{x \in E} \exp \left( - \sum_{j \in I_r \cap J_r} \sin^2(2\pi x_j \xi_j) \right) \leq \frac{1}{|S| \cap \mathbb{Z}^d} \sum_{x \in E} \exp \left( - 16\varepsilon^2 \kappa(d, \lambda)^2 \sum_{j \in I_r \cap J_r} \|\xi_j\|^2 \right),
\]
(6.26)

where $c > 0$ is a universal constant. In order to obtain the last inequality in (6.26) we use the fact that
\[
\gamma(E) = E
\]
and $I_{\gamma^{-1} \xi} = I_x$ for every $\gamma \in \text{Sym}(d)$ and $x \in S$. Using Lemma 6.7 with $\delta_1 = \varepsilon_1/2$, $d_0 = r$, $I = I_x$ and $d_0 = 3/5$, to conclude that
\[
E \left[ \exp \left( - 16\varepsilon^2 \kappa(d, \lambda)^2 \sum_{j \in I_x \cap J_x} \|\xi_j\|^2 \right) \right] \lesssim \exp \left( - c \kappa(d, \lambda)^2 \sum_{j=r+1}^d \|\xi_j\|^2 \right),
\]
for a universal constant $c' > 0$ and for all $x \in E$. This proves (6.20) since by (6.23) we obtain
\[
\exp \left( - c \kappa(d, \lambda)^2 \sum_{j=r+1}^d \|\xi_j\|^2 \right) \leq \exp \left( - \frac{3c' \kappa(d, \lambda)^2}{4} \sum_{j=1}^d \|\xi_j\|^2 \right).
\]

Coming back to (6.24) we have thus proved that
\[
|m_\varepsilon(\xi)| \lesssim \kappa(d, \lambda)^{-4} + e^{-d/22} \lesssim \kappa(d, \lambda)^{-4} + (\kappa(d, \lambda)^2 \|\xi\|)^{-1}.
\]

Therefore, (6.21) is true under the assumption (6.24).

Assume now that (6.24) does not hold, i.e. that for some $j \in N_d \setminus N_r$ we have $\|\xi_j\| \geq \frac{\varepsilon_1/200}{\kappa(d, \lambda)}$. Then, using (6.20) we see that
\[
\|\eta\|^2 \geq \frac{\varepsilon_1 r_0}{100s(d, \lambda)} \geq \frac{\varepsilon_1 r}{200s(d, \lambda)^2}.
\]

Thus, invoking (6.22) and recalling that $\varepsilon_1 = 1/10$ we obtain
\[
|m_\varepsilon(\xi)| \lesssim \kappa(d, \lambda)^{-4} + \exp(-2\pi 5^{-3/2} \kappa(d, \lambda)\|\eta\|) \leq \kappa(d, \lambda)^{-4} + \exp(-\pi 5^{-3/2} 10^{-9/2} \sqrt{r}) \lesssim \kappa(d, \lambda)^{-4},
\]

since $r \approx \kappa(d, \lambda)^4$. This completes the proof of (6.21).

It remains to consider the case $|V_0| > r/2$. We prove that
\[
|m_\varepsilon(\xi)| \lesssim \kappa(d, \lambda)^{-4} + (\kappa(d, \lambda)^2 \|\xi + 1/2\|)^{-1}.
\]
(6.27)

The above bound clearly implies (6.21). Actually, (6.27) may be easily deduced from the previous case (6.21). Indeed, by (6.3) we have $|m_\varepsilon(\xi)| = |m_\varepsilon(\xi + 1/2)|$. Now, for the vector $\xi := \xi + 1/2$ we have $\tilde{\eta} = \eta + 1/2$ and $|V_0| \leq r/2$. Moreover, (6.20) is still satisfied with $\xi$ in place of $\xi$. Therefore, we can repeat the argument used in the proof of (6.21) with $\xi$ replacing $\xi$. This leads to (6.27) as desired.

The proof of Proposition 6.1 is thus completed. \hfill \Box

7. Proof of Theorem 1.2: small-scale estimate (1.22)

This section is devoted to the proof of inequality (1.22) in Theorem 1.2. We shall proceed in much the same way as in the corresponding case for the discrete Euclidean balls from [8]. Therefore, we only briefly point out the main differences. The strategy of the proof of inequality (1.22) is similar to the proof of inequality (1.21). The approximating multiplies $\beta^1_\lambda(\xi)$ and $\beta^2_\lambda(\xi)$ (see (6.3), (6.4)) depend on the size of the set $V_0$ defined in (6.22). Proposition 7.1, which is the main results of this section, is a variant of Proposition 6.1 adjusted to the small scales.

**Proposition 7.1.** Let $d \geq 5$ and assume that $\kappa(d, \lambda) \leq 1/5$. Then for every $\lambda \in \mathbb{N}$ and $\xi \in \mathbb{T}^d$ we have the following bounds with the constant $c \in (0, 1)$ as in (6.1). Namely,
1. if $|V_2| \leq d/2$, then
\[
|m_t(\xi) - p^2_t(\xi)| \lesssim \min \left\{ e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^{d} \sin^2(\pi \xi_i)} \right\},
\] (7.1)
2. if $|V_2| \geq d/2$, then
\[
|m_t(\xi) - p^2_t(\xi)| \lesssim \min \left\{ e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^{d} \cos^2(\pi \xi_i)} \right\}.
\] (7.2)

Assuming momentarily Proposition 7.1 we can now deduce inequality (1.22) in Theorem 1.2. The argument is essentially the same as in the proof of inequality (1.21). The only difference is the use of Proposition 7.1 in place of Proposition 6.2 however, we give it for the sake of completeness.

**Proof.** Let $f \in L^2(\mathbb{Z}^d)$ and we write $f = f_1 + f_2$, where $f_1(\xi) := \hat{f}(\xi) \mathbb{I}_{[0, T^d]}(\xi \in d/2)^{(\xi)}$. Then
\[
\left\| \sup_{t \in T_0} |\mathcal{F}^{-1}(m_t)\|_{L^2(\mathbb{Z}^d)} \right\|_{L^2(\mathbb{Z}^d)} \leq \sum_{i=1}^{d} \left( \sum_{t \in \mathbb{N}_0} \left| \mathcal{F}^{-1}(p^2_t \hat{f}_i) \right| \right)^{1/2} \left\| \mathcal{F}^{-1}(m_t - p^2_t \hat{f}_i) \right\|_{L^2(\mathbb{Z}^d)}^{1/2}.
\]
The usual square function argument permits to reduce the problem to bounding the maximal functions associated with the multipliers $p^1_t$ and $p^2_t$. Taking $C_0 = 1/5$, Plancherel's theorem and Proposition 7.1 we obtain
\[
\sum_{i=1}^{d} \left( \sum_{t \in \mathbb{N}_0} \left| \mathcal{F}^{-1}(m_t - p^1_t \hat{f}_i) \right| \right)^{1/2} \left\| \mathcal{F}^{-1}(m_t) \right\|_{L^2(\mathbb{Z}^d)} \lesssim \|f_1\|_{L^2(\mathbb{Z}^d)} + \|f_2\|_{L^2(\mathbb{Z}^d)} \leq 2\|f\|_{L^2(\mathbb{Z}^d)}.
\]
It remains to bound the maximal functions corresponding to the multipliers $p^1_t$ and $p^2_t$. This was already done in (6.1) and (6.8), and is a simple consequence of (1.21). Hence, the proof of inequality (1.22) in Theorem 1.2 is completed. \(\square\)

The rest of Section 7 is devoted to the proof of Proposition 7.1.

### 7.1. Some preparatory estimates

We shall need a version of [3] Lemma 3.2] with discrete spheres in place of discrete balls.

**Lemma 7.2.** For every $d, \lambda \in \mathbb{N}$, $d \geq 5$, if $\kappa(d, \lambda) \leq 1/5$ and $\lambda \geq k \geq 2^9$ max$(\lambda, \kappa(d, \lambda)\lambda)$, then
\[
|\{x \in S_t \cap \mathbb{Z}^d : |i \in \mathbb{N}_0 : x_i = \pm 1| \leq \lambda - k\} | \leq (2t + 1)^2 2^{-k+1}|S_t \cap \mathbb{Z}^d|.
\] (7.3)

**Proof.** We define $A := \{x \in \mathbb{Z}^d : |i \in \mathbb{N}_0 : x_i = \pm 1| \leq \lambda - k\}$. Then, by [3] Lemma 3.2] (second inequality below) and Lemma 5.3 (last inequality below) we obtain
\[
|A \cap S_t \cap \mathbb{Z}^d| \leq |A \cap B_t^2(d) \cap \mathbb{Z}^d| \leq 2^{-k+1}B_t^2(d) \cap \mathbb{Z}^d| \leq 2^{-k+1}(2t + 1)^4|S_t \cap \mathbb{Z}^d|,
\]
and the proof of (7.3) is completed. \(\square\)

Lemma 7.2 will be essential in our next result.

**Proposition 7.3.** Let $d \geq 5$ and assume that $\kappa(d, \lambda) \leq 1/5$. Then, for every $\lambda \in \mathbb{N}$ and $\xi \in \mathbb{T}^d$ we have the following estimate
\[
|m_t(\xi)| \lesssim e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^{d} \sin^2(\pi \xi_i)} + e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^{d} \cos^2(\pi \xi_i)},
\] (7.4)
where $c \in (0, 1)$ is the absolute constant from (7.7).

**Proof.** Note that when $\lambda \leq 2^{10}$ then $\kappa(d, \lambda)^2 = \lambda/d \lesssim d^{-1}$ and (7.4) is obvious. Thus in what follows we assume that $\lambda \geq 2^{10}$. For any $x \in \mathbb{Z}^d$ we define the sets
\[
I_x := \{i \in \mathbb{N}_d : x_i = \pm 1\} \quad \text{and} \quad E := \{x \in S_t \cap \mathbb{Z}^d : |I_x| > \lambda/2\}.
\]
Since $\lambda = \kappa(d, \lambda)^2d$, using Lemma 7.2 with $k = \lambda - |\lambda/2|$, we see that
\[
|E^c| \leq (2t + 1)^4 2^{-\lambda/2+2}|S_t \cap \mathbb{Z}^d| \lesssim 2^{-3\lambda/8}|S_t \cap \mathbb{Z}^d| \leq e^{-\frac{c(d,\lambda)^2d}{100}}|S_t \cap \mathbb{Z}^d|.
\]
In view of these estimates it now suffices to show that

\[
\frac{1}{|S_t \cap \mathbb{Z}^d|} \sum_{x \in S_t \cap \mathbb{Z}^d} \prod_{j=1}^d \cos(2\pi x_j \xi_j) \lesssim e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^d \sin^2(\pi \xi_i)} + e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^d \cos^2(\pi \xi_i)}. \tag{7.5}
\]

As in [8, Section 3] the proof of (7.5) will rely on the properties of the Krawtchouk polynomials. For the convenience of the reader we recall their definitions and basic properties. For every \(n \in \mathbb{N}_0\) and integers \(x, k \in [0, n]\) we define the \(k\)-th Krawtchouk polynomial

\[
k_k^{(n)}(x) := \frac{1}{\binom{n}{k}} \sum_{j=0}^k (-1)^j \binom{x}{j} \binom{n-x}{k-j}. \tag{7.6}
\]

We gather properties of Krawtchouk polynomials required to establish (7.5).

**Theorem 7.1.** For every \(n \in \mathbb{N}_0\) and integers \(x, k \in [0, n]\) we have

1. Symmetry: \(k_k^{(n)}(x) = k_k^{(n)}(k)\).
2. Reflection symmetry: \(k_k^{(n)}(n-x) = (-1)^k k_k^{(n)}(x)\).
3. A uniform bound: there exists a constant \(c \in (0, 1)\) such that for all \(n \in \mathbb{N}_0\) the following inequality

\[
|k_k^{(n)}(x)| \leq e^{-\frac{4d}{100} c^2} \tag{7.7}
\]

holds for all integers \(0 \leq x, k \leq n/2\).

The proof of Theorem 7.1 can be found in [19], see also the references therein. It turns out that the left-hand side of (7.5) is essentially the Krawtchouk polynomial (with appropriate parameters \(k, n\)). In order to see this reduction one has to repeat the proof of inequality [8, Section 3, (3.20)] with \(S_t\) in place of \(B_N\). Once this is done we can easily deduce (7.5) with \(c \in (0, 1)\) as in (7.7).

### 7.2. All together.

We are now ready to prove Proposition 7.7.

**Proof of Proposition 7.7.** Firstly, we assume that \(|V_{\xi}| \leq d/2\), then (7.4) implies that

\[
|m_t(\xi)| \lesssim e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^d \sin^2(\pi \xi_i)},
\]

since \(\sum_{i=1}^d \cos^2(\pi \xi_i) \geq d/4 \geq 1/4 \sum_{i=1}^d \sin^2(\pi \xi_i)\). Thus

\[
|m_t(\xi) - e^{-\kappa(d,\lambda)^2 \sum_{i=1}^d \sin^2(\pi \xi_i)}| \lesssim e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^d \sin^2(\pi \xi_i)}. \tag{7.8}
\]

On the other hand, using (7.8) from Proposition 7.3 we obtain

\[
|m_t(\xi) - e^{-\kappa(d,\lambda)^2 \sum_{i=1}^d \sin^2(\pi \xi_i)}| \lesssim \kappa(d,\lambda)^2 \sum_{i=1}^d \sin^2(\pi \xi_i). \tag{7.9}
\]

We now see that (7.8) and (7.9) imply (7.1).

Secondly, we assume that \(|V_{\xi}| \geq d/2\), then (7.7) implies that

\[
|m_t(\xi)| \lesssim e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^d \cos^2(\pi \xi_i)},
\]

since \(\sum_{i=1}^d \sin^2(\pi \xi_i) \geq d/4 \geq 1/4 \sum_{i=1}^d \cos^2(\pi \xi_i)\). Thus

\[
|m_t(\xi) - (-1)^k e^{-\kappa(d,\lambda)^2 \sum_{i=1}^d \cos^2(\pi \xi_i)}| \lesssim e^{-\frac{c(d,\lambda)^2}{100} \sum_{i=1}^d \cos^2(\pi \xi_i)}. \tag{7.10}
\]

On the other hand, (7.10) Proposition 4.3 implies

\[
|m_t(\xi) - (-1)^k e^{-\kappa(d,\lambda)^2 \sum_{i=1}^d \cos^2(\pi \xi_i)}| \lesssim \kappa(d,\lambda)^2 \sum_{i=1}^d \cos^2(\pi \xi_i). \tag{7.11}
\]

We now see that (7.10) and (7.11) imply (7.2). This completes the proof of the proposition.

\[\square\]
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