1D Hubbard model elementary objects scattering

J. M. P. Carmelo\textsuperscript{a,b,c} and P. D. Sacramento\textsuperscript{b,d}

\textsuperscript{a}Center of Physics, University of Minho, Campus Gualtar, P-4710-057 Braga, Portugal
\textsuperscript{b}Beijing Computational Science Research Center, Beijing 100084, China
\textsuperscript{c}Institut f"ur Theoretische Physik III, Universit"at Stuttgart, D-70550 Stuttgart, Germany and
\textsuperscript{d}CFIF, Instituto Superior Técnico, TU Lisbon, P-1049-001 Lisboa, Portugal

(Dated: 30 August 2013)

In terms of electron processes, the 1D Hubbard model is a nonperturbative problem. That renders the description in terms of electron scattering of the microscopic processes that control the model properties a very difficult task. In this paper we study the corresponding scattering processes of the elementary objects whose occupancy configurations generate the energy eigenstates from the electron vacuum. Due to the related occurrence of an infinite set of conservation laws associated with the model integrability, such objects are found to undergo only zero-momentum forward-scattering collisions. The description of the model dynamical properties in terms of such elementary objects scattering events then drastically simplifies. The corresponding 1D Hubbard model scattering theory refers to arbitrary values of the densities and finite repulsive interaction \( U > 0 \). Each ground-state - excited-state transition is associated with a well defined set of elementary zero-momentum forward-scattering events. The elementary-object scatterers dressed \( S \) matrix is expressed as a \textit{commutative} product of \( S \) matrices, each corresponding to a two-object scattering event. This commutative factorization is stronger than the factorization associated with Yang-Baxter equation for the original spin-1/2 electron bare \( S \) matrix. The power-law singularities exponents in the finite-energy correlation-functions of the metallic phases of a wide class of 1D integrable and non-integrable systems are momentum dependent. In the present exactly solvable model such an exponent momentum dependence is controlled by the phase shifts and corresponding dressed \( S \) matrix considered in this paper.

PACS numbers: 03.65.Ca, 71.10.Pm, 71.27.+a, 03.65.Nk

I. INTRODUCTION

On a one-dimensional (1D) lattice, the Hubbard model \cite{1,2} for correlated electrons with effective on-site repulsion \( U \) and nearest-neighbor transfer integral \( t \) is a non-perturbative many-electron problem. Fortunately, it is solvable by Bethe-ansatz (BA) \cite{3,4}. In the low-energy limit, the model behavior is described by a two-component Luttinger-Tomonaga-liquid theory \cite{7,8,9}. Within that limit, the BA solution may be combined with bosonization \cite{7,8} or the model conformal invariance \cite{10,11} to evaluate the asymptotics of correlation functions.

A pseudofermion dynamical theory (PDT) \cite{12,13,14} for the 1D Hubbard model, which is a generalization to arbitrary \( U/t > 0 \) values of the \( U/t \gg 1 \) method of Refs. \cite{16,17}, has been used to calculate finite-energy spectral and correlation functions. More recently, alternative methods valid for both integrable and non-integrable 1D correlated problems have reached similar results \cite{18,19,20,21,22,23}. (While here \textit{finite energy} means an energy larger than the typical linear Luttinger-Tomonaga-liquid theory excitation energy, in this paper finite energy often refers to both low energy and high energy in the sense that there are no restrictions to its values.)

In the case of the 1D Hubbard model, all the above methods rely in part on the combination of the BA solution with suitable invariances, symmetries, and/or numerical computations. Which microscopic elementary processes are behind both the low-energy and finite-energy spectral and correlation function behaviors obtained by such techniques is an issue that is not well understood and thus deserves further investigations. It is an extremely complex issue in terms of the non-perturbative many-electron microscopic processes. On the other hand, it is shown in this paper that it drastically simplifies in terms of microscopic processes of the elementary objects whose occupancy configurations generate exact energy eigenstates from the electron vacuum. The operator algebra associated with the pseudoparticle representation of Refs. \cite{26,27} has in a recent paper, Ref. \cite{28}, been uniquely related to the electron creation and annihilation operators. The “bridge” between the electrons and the elementary objects of such a representation has been found to refer to rotated electrons. Those are generated from the electrons by a unitary transformation performed by the BA solution. The studies of this paper focus on the properties of the related scatterers and scattering centers, which are some of the elementary objects of the rotated-electron related representation of Refs. \cite{26,28}. We find that such a representation leads to a uniquely defined choice of scattering states basis. For \( U/t > 0 \) the states of that basis are found to be in one-to-one correspondence to the excited energy eigenstates of ground states with arbitrary values of the electronic density \( n \) and spin density \( m \).

The present studies are an extension of the preliminary results on the PDT related pseudofermion scattering mechanisms presented in short form in Ref. \cite{29}. That preliminary study has identified correctly the pseudofermion
dressed phase shifts and $S$ matrices that control the PDT one- and two-electron spectral-weight distributions [12–13]. However, it lacked many aspects of the pseudofermion scattering theory introduced in this paper needed both to justify its validity and to clarify the physics behind it. It did not include figures illustrating the dependence of the important two-pseudofermion phase shifts that control the PDT spectral weight distributions on the scatterer and scattering center momenta and on-site repulsion ratio $U/t$. Furthermore, the relation of the theory scatterers and scattering centers to the rotated electrons that are generated from the electrons by a unitary transformation performed by the BA solution was not discussed. Hence the effects of the pseudofermion transformation laws under the electron - rotated-electron unitary transformation were not investigated. The operator that counts the number of rotated electrons singly occupied sites is for $U/t > 0$ the generator of the hidden $U(1)$ symmetry in the model global $[SU(2) \times SU(2) \times U(1)]/Z_2$ symmetry more recently found in Ref. [30]. The preliminary studies of Ref. [29] have not accounted for the interplay of that symmetry with the model scattering properties. Moreover, important issues such as a clear definition of the one-pseudofermion “in” and “out” asymptote pseudofermion scattering states was lacking. The relation of the latter states to the corresponding many-pseudofermion “in” and “out” states was neither given.

In this paper all such issues are addressed and clarified.

From the relation of the PDT to the scattering theory studied in the following, one confirms that the corresponding exotic scattering centers of the 1D Hubbard model are observed by angle-resolved photoelectron spectroscopy in quasi-1D materials [31–33]. The model or its extensions describe as well the effects of correlations in semiconductor - metal transitions of doped quasi-1D materials [34] and are of interest for systems of ultra-cold atoms on 1D optical lattices [35].

In Ref. [28] it is shown that 1D Hubbard model elementary-object representations other than those used in the studies of this paper refer to alternative sets of degenerate energy eigenstates that span well-defined model reduced subspaces. This applies for instance to the holon and spinon representations of Refs. [35] [36]. The corresponding holon and spinon scattering theories of Ref. [36] and Refs. [37, 38], respectively, are actually different. However, both such theories use the ground state whose excited states span the subspaces wherein such theories are defined as the holon and spinon vacuum. Alike in the elementary-object scattering theory studied in this paper, for such holon and spinon scattering theories the elementary objects created under the transitions to the excited states play the role of scattering centers. Consistent with the ground state playing the role of vacuum of the latter theories elementary objects, the scatterers are as well holons and spinons created under such transitions. Hence the holon and spinon scattering theories only account for the phase shifts of such scatterers. As a result, the theories of Ref. [36] and Refs. [37, 38] do not account for most phase shifts found in this paper to control the one- and two-electron spectral-weight distributions. Indeed most of the latter phase shifts are of scatterers that pre-existed in the ground state of the excited state under consideration.

The relation of the elementary-object representations used in the studies of this paper to the holons and spinons of Refs. [34–38] is an issue that has been clarified [28]. That relation is consistent with the major advantage of the scattering theory considered in this paper relative to the holon and spinon scattering theories referring indeed to the explicit description of the microscopic processes that control the model dynamical and spectral properties. Such an advantage follows from the occupancy configurations of the elementary objects of the former theory generating the excited energy eigenstate from the electron vacuum rather then from a ground state. Therefore, the pseudofermion scattering theory accounts for both the phase shifts of the scatterers that pre-exist in the ground state and those that are created under the transitions from it to the excited states. Furthermore, the pseudofermion scattering theory refers to excited states of ground states with arbitrary electronic density $n$ and spin density $m$. On the other hand, the scattering theory of Ref. [36] and that of Refs. [37, 38] applies to densities $n \in [0, 1]; m = 0$ and $n = 1; m = 0$, respectively.

Due to the nonperturbative character of the many-electron problem, the microscopic elementary processes studied in this paper are in terms of electron scattering very involved. The simplicity of the elementary-objects scattering events studied in this paper stems both from the occupancy configurations of such elementary objects generating exact energy eigenstates and from the related occurrence of an infinite number of conservation laws [39–41]. Those are associated with the 1D Hubbard model integrability [3] [4]. The elementary-object representation used in our investigations [26, 28] naturally emerges from the interplay of the model global $[SU(2) \times SU(2) \times U(1)]/Z_2$ symmetry [30] with its exact BA solution [28]. Consistent, such conservation laws are explicit in their scattering theory, in that the scatterers and scattering centers are only allowed to undergo zero-momentum forward scattering events.

The pseudofermion scattering theory goes beyond the BA solution in that it accounts for both excited states inside and outside that solution subspace, which is spanned by the Bethe states. For Bethe states it is meant the energy eigenstates inside the BA solution subspace. Those can be chosen to be either lowest-weight states (LWSs) or highest-weight states (HWSs) of the $\eta$-spin and spin $SU(2)$ algebras algebras [12] in the model global $[SU(2) \times SU(2) \times U(1)]/Z_2$ symmetry [30]. The $\eta$-spin (and spin) and $\eta$-spin projection (and spin projection) of the energy eigenstates are denoted by $S_\eta$ and $S_\alpha^\eta$ (and $S_\alpha$ and $S_\alpha^s$), respectively. The $S_\alpha$ and $S_\alpha^s$ values of the LWSs and HWSs are such that $S_\alpha = -S_\alpha^s$ and $S_\alpha = S_\alpha^s$, respectively. Here $\alpha = \eta$ for $\eta$-spin and $\alpha = s$ for spin. In this paper we use the BA solution LWS
representation.

The paper is organized as follows: The model and the elementary objects arising from the rotated electrons as defined in Refs. [26, 28] are the topics addressed in Section II. In Section III it is found that a pseudofermion scattering theory naturally emerges from such elementary objects by means of a unitary transformation. It slightly shifts the pseudoparticle discrete momentum values, which renders the corresponding pseudofermion spectrum without energy interaction terms. The corresponding pseudofermion dressed phase shifts are studied in Section IV. The effects of the pseudofermion transformation laws on their scattering properties and the relation of the theory dressed phase shifts to the spectral weights of the PDT are issues also addressed in that section. Finally, the concluding remarks are presented in Section V. Complementary useful information is given in four appendices: Appendix A provides further information on the related pseudoparticle representation. The integral equations that define the two-pseudofermion phase shifts are given in Appendix B. In that Appendix some other phase-shift related issues are also addressed. Appendix C discusses the extension of the pseudofermion scattering theory to excited states of ground states with densities \( n = 1 \) and/or \( m = 0 \). Finally, in Appendix D information of the interaction and densities dependence of energy scales useful for the studies of this paper is reported.

II. THE MODEL AND THE ELEMENTARY OBJECTS EMERGING FROM THE ELECTRON - ROTATED-ELECTRON UNITARY TRANSFORMATION

The Hubbard model Hamiltonian, under periodic boundary conditions, on a 1D lattice with a site number \( N_a \gg 1 \) very large and even and in a chemical potential \( \mu \) and magnetic field \( H \) is given by,

\[
\hat{H} = \hat{H}_{\text{symm}} - \sum_{j=\eta,s} \mu_\alpha \hat{S}_\alpha^{z3},
\]

where

\[
\hat{H}_{\text{symm}} = t [\hat{T} + 4n \hat{V}_D]; \quad u = U/4t, \quad \hat{S}_\eta^{z3} = -\frac{1}{2} [\hat{N}_a - \hat{N}] ; \quad \hat{S}_s^{z3} = -\frac{1}{2} [\hat{N}_\uparrow - \hat{N}_\downarrow],
\]

\[
\hat{T} = - \sum_{\sigma=\uparrow,\downarrow} \sum_{j=1}^{N_a} \left[ c_{j,\sigma}^{\dagger} c_{j+1,\sigma} + c_{j+1,\sigma}^{\dagger} c_{j,\sigma} \right] ; \quad \hat{V}_D = \sum_{j=1}^{N_a} (\hat{n}_{j,\uparrow} - 1/2) (\hat{n}_{j,\downarrow} - 1/2).
\]

Here \( \hat{T} \) is the kinetic-energy operator in units of \( t \), \( \hat{V}_D \) the electron on-site repulsion operator in units of \( U, u = U/4t \) is the electron on-site interaction in units of \( 4t \), which is often used in this paper, \( \mu_t = 2\mu, \mu_s = 2\mu_B H, \mu_B \) is the Bohr magneton, and \( \hat{S}_\eta^{z3} \) and \( \hat{S}_s^{z3} \) are the diagonal generators of the \( \eta \)-spin and spin \( SU(2) \) symmetry algebras \([33, 34]\), respectively. The operator \( c_{j,\sigma}^{\dagger} \) (and \( c_{j,\sigma} \)) that appears in the above equations creates (and annihilates) a spin-projection \( \sigma \) electron at lattice site \( j = 1, ..., N_a \). The operator \( \hat{n}_{j,\sigma} = c_{j,\sigma}^{\dagger} c_{j,\sigma} \) counts the number of spin-projection \( \sigma \) electrons at such a lattice site. The electronic number operators read \( \hat{N} = \sum_{\sigma=\uparrow,\downarrow} \hat{N}_\sigma \) and \( \hat{N}_\sigma = \sum_{j=1}^{N_a} \hat{n}_{j,\sigma} \). The momentum operator is given by \( \hat{P} = \sum_{\sigma=\uparrow,\downarrow} \sum_{k} \hat{n}_\sigma (k) k \), where the spin-projection \( \sigma \) momentum distribution operator reads \( \hat{n}_\sigma (k) = c_{k,\sigma}^{\dagger} c_{k,\sigma} \) and the operator \( c_{k,\sigma}^{\dagger} \) (and \( c_{k,\sigma} \)) creates (and annihilates) a spin-projection \( \sigma \) electron of momentum \( k \).

Throughout this paper we use in general units of both Planck constant \( \hbar \) and lattice constant \( a \) one. We denote the lattice length by \( L = N_a a = N_a \). The LWSs have electronic densities \( n = N/L \) and spin densities \( m = [N_{\uparrow} - N_{\downarrow}] / L \) whose ranges obey the inequalities \( 0 \leq n \leq 1 \) and \( 0 \leq m \leq n \), respectively. The description of the states corresponding to densities such that \( 0 \leq n \leq 1 ; 1 \leq n \leq 2 \) and \( -n \leq m \leq n ; -(2-n) \leq m \leq (2-n) \), respectively, is achieved by application onto the LWSs of off-diagonal generators of the \( \eta \)-spin and spin \( SU(2) \) symmetry algebras.

The two global \( SU(2) \) symmetries of the Hubbard model on a bipartite lattice, including the present 1D lattice, have been known for a long time \([33, 34]\). The studies of Ref. [15] revealed that the model global symmetry was at least \( SO(4) = [SU(2) \otimes SU(2)] / Z_2 \). The recent investigations of Ref. [30] found that for finite on-site interaction values it is larger and given by \( [SO(4) \otimes U(1)] / Z_2 = SO(3) \otimes SO(3) \otimes U(1) \). That global symmetry may be rewritten as \( [SU(2) \times SU(2) \times U(1)] / Z_2^3 \). It stems from the local gauge \( SU(2) \times SU(2) \times U(1) \) symmetry of the Hubbard model on a bipartite lattice with vanishing transfer integral, \( t = 0 \). For finite \( U \) and \( t \) values the latter local symmetry becomes a group of permissible unitary transformations. The corresponding local \( U(1) \) canonical transformation is not the ordinary gauge \( U(1) \) subgroup of electromagnetism. It is rather a “nonlinear” transformation \([16]\).

The BA solution accounts for the quantum number occupancy configurations that generate the representations of the \( c \) hidden \( U(1) \) symmetry algebra in \( [SU(2) \otimes SU(2) \otimes U(1)] / Z_2^3 = [SO(4) \otimes U(1)] / Z_2 \) beyond \( SO(4) \) \([28]\). The energy eigenstates outside the BA solution subspace have exactly the same \( c \) hidden \( U(1) \) symmetry algebra representations.
as the Bethe states from which they are generated by the off-diagonal $\eta$-spin and spin operator algebras. This is why there is no contradiction whatsoever between the global $[SU(2) \otimes SU(2) \otimes U(1)]/Z_2^2 = [SO(4) \otimes U(1)]/Z_4$ symmetry found in Ref. 30 for the Hubbard model on any bipartite lattice and the results of Ref. 42, concerning the counting of the $4N_a$ energy eigenstates of the Hubbard model on the bipartite 1D lattice. The studies of that reference have not explicitly considered the model $c$ hidden $U(1)$ symmetry algebra beyond $SO(4)$, yet have used the BA solution, which includes the quantum-number occupancy configurations that generate the representations of the $c$ hidden $U(1)$ symmetry algebra [28].

In the following we shortly report how the elementary objects used in our studies emerge from the electrons. Here and in Appendix A1 $|l_r, l_{\eta s}, u\rangle$ denotes the model energy eigenstates whose LWSs of both $SU(2)$ symmetry algebras are the Bethe states, $|l_r, l_{\eta s}, u\rangle$. General non-LWS energy eigenstates are generated from the Bethe states as given in Eq. (A1) of that Appendix. The state labels $l_{\eta s}$ and $l_r$ stand for the set of numbers $[S_\eta, S_s, M_0^{\eta s}, M_s, -1/2, M_s, -1/2]$ and all remaining quantum numbers, respectively, needed to uniquely define an energy eigenstate whereas the Bethe-state label $p_{\eta s}^s$ refers to the specific $l_{\eta s}$ values $[S_\eta, S_s, 0, 0]$.

Within the $u \rightarrow \infty$ limit, the energy eigenstates $|l_r, l_{\eta s}, u\rangle$ adiabatically correspond to states $|l_r, l_{\eta s}, \infty\rangle$, which refer to one of the many choices of $u \rightarrow \infty$ energy eigenstates. In this paper we consider such $u \rightarrow \infty$ energy eigenstates. They can be generated from the electron vacuum $|0_{\text{elec}}\rangle$ by application onto it of a uniquely defined operator $\hat{G}$, i.e. $|l_r, l_{\eta s}, \infty\rangle = \hat{G} |0_{\text{elec}}\rangle$. The complete set of $4N_a$ energy eigenstates, $\{ |l_r, l_{\eta s}, \infty\rangle \}$, uniquely defines an electron-rotated-electron unitary transformation performed by the BA solution. (It is uniquely defined in Ref. 28 in terms of its $4N_a \times 4N_a$ matrix elements between energy eigenstates.) The rotated electrons have been contracted inherently to rotated-electron single and double occupancies being good quantum numbers for $u > 0$.

The elementary objects considered in Ref. 26,27 naturally emerge from the degrees of freedom separation of the rotated-electron occupancy configurations that generate the energy eigenstates from the electron vacuum.28 Such degrees of freedom correspond to the state representations of the $c$ hidden $U(1)$ symmetry, $\eta$-spin $SU(2)$ symmetry, and spin $SU(2)$ symmetry algebras in the model global $[SU(2) \otimes SU(2) \otimes U(1)]/Z_2^2$ symmetry algebra. The first step of such elementary objects emergence is the electron-rotated-electron unitary transformation performed by the BA, which is directly related to that symmetry. Indeed, the generator of the global $c$ hidden $U(1)$ symmetry beyond $SO(4)$ is the number of rotated-electron singly occupied sites operator $[28,30]$,

$$2\hat{S}_c \equiv \hat{V} \hat{G} \hat{V} \hat{G} \hat{V} = \sum_{j=1}^{N_a} \hat{s}_{j,c} : \hat{s}_{j,c} = \sum_{\sigma=\uparrow, \downarrow} \hat{n}_{j,\sigma} \left( 1 - \hat{n}_{j,-\sigma} \right).$$

Here $2\hat{S}_c$ and the related operator $\hat{D}$ count the number of electron singly-occupied sites and doubly-occupied sites and read,

$$2\hat{S}_c = \sum_{j=1}^{N_a} \hat{s}_{j,c} ; \quad \hat{D} = (\hat{N} - 2\hat{S}_c)/2 ; \quad \hat{s}_{j,c} = \sum_{\sigma=\uparrow, \downarrow} \hat{n}_{j,\sigma} \left( 1 - \hat{n}_{j,-\sigma} \right),$$

respectively. The eigenvalues $2\hat{S}_c = 0, 1, \ldots$ of the operator $2\hat{S}_c$, Eq. (4), are thus the number of rotated-electron singly occupied sites [30].

It is convenient to express the local generator $\hat{s}_{j,c}$ of the Hamiltonian electron-interaction term gauge $U(1)$ symmetry in $SU(2) \otimes SU(2) \otimes U(1)$, which is given in Eq. (4), and the alternative local generator $\hat{s}_{j,c}$, as follows,

$$\hat{s}_{j,c} = \hat{n}_{j,c} = \hat{f}_{j,c}^\dagger \hat{f}_{j,c} ; \quad \hat{s}_{j,c} = (1 - \hat{n}_{j,c}) = \hat{f}_{j,c} \hat{f}_{j,c}^\dagger , \quad j = 1, \ldots, N_a .$$

Here $\hat{f}_{j,c}$ and $\hat{f}_{j,c}$ stand for the following creation and annihilation operators, respectively, which obey a fermionic algebra,

$$\hat{f}_{j,c}^\dagger = c_{j,\sigma}^\dagger (1 - \hat{n}_{j,\sigma}) + (-1)^j c_{j,\sigma} \hat{n}_{j,\sigma} ; \quad \hat{f}_{j,c} = c_{j,\sigma} (1 - \hat{n}_{j,\sigma}) + (-1)^j c_{j,\sigma} \hat{n}_{j,\sigma} , \quad j = 1, \ldots, N_a .$$
In terms of the operators obtained from the electron rotation of those given in Eq. (7),

\[ f_{j,c} = \hat{V}^\dagger \hat{f}_{j,c} \hat{V} = \hat{e}_{j,\uparrow}^\dagger (1 - \hat{n}_{j,\downarrow}) + (-1)^j \hat{e}_{j,\downarrow}^\dagger \hat{n}_{j,\uparrow}, \]
\[ f_{j,c} = \hat{V}^\dagger \hat{f}_{j,c} \hat{V} = \hat{e}_{j,\uparrow}^\dagger (1 - \hat{n}_{j,\downarrow}) + (-1)^j \hat{e}_{j,\downarrow}^\dagger \hat{n}_{j,\uparrow}, \quad j = 1, \ldots, N_a, \]  

(8)

the two alternative generators of the global \( U(1) \) symmetry have the following very simple expressions,

\[ 2\hat{S}_c = \sum_{j=1}^{N_a} n_{j,c} = \sum_{j=1}^{N_a} f_{j,c}^\dagger f_{j,c}; \quad 2\hat{S}_c^h = \sum_{j=1}^{N_a} (1 - n_{j,c}) = \sum_{j=1}^{N_a} f_{j,c}^\dagger f_{j,c}. \]  

(9)

Here,

\[ n_{j,c} = f_{j,c}^\dagger f_{j,c}. \]  

(10)

Hence the operators, Eq. (5), have been obtained from electron rotation of those given in Eq. (1) inherently to be associated with the \( c \) hidden \( U(1) \) symmetry degrees of freedom of the rotated electrons. In this paper we use in general a notation within which \( \tilde{O} \) stands for the operator that in terms of rotated-electron creation and annihilation operators, Eq. (A10) of Appendix A. On the other hand, here we use the operator representation of Refs. [26, 28] within terms of electron creation and annihilation operators, respectively. However, for simplicity, no upper index is used onto the electron-rotated \( c \) pseudoparticle operators \( f_{j,c}^\dagger \) and \( f_{j,c} \), Eq. (8), and \( n_{j,c} \), Eq. (10).

One may introduce corresponding momentum-dependent operators,

\[ f_{q_j,c}^l = (f_{q_j,c})^\dagger = \frac{1}{\sqrt{N_a}} \sum_{j'=1}^{N_a} \cos q_j q_{j'} f_{j,c}^\dagger, \quad j = 1, \ldots, N_a. \]  

(11)

where the discrete momentum values \( q_j = (2\pi/N_a) I_j^c \), such that \( q_{j+1} - q_j = 2\pi/N_a \), are defined in Eqs. (A7) and (A10) of Appendix A.

The unitary operator \( \hat{V} \) in Eq. (8) is chosen to be that associated with the electron-rotated-electron unitary transformation performed by the BA solution. For that specific choice of unitary operator, the operators \( f_{q_j,c}^l \) and \( f_{q_j,c} \), Eq. (11), create and annihilate the \( c \) pseudoparticles previously considered in Refs. [26, 27]. In such references the \( c \) pseudoparticles emerged from an empirical association with the BA quantum numbers \( q_j = (2\pi/N_a) I_j^c \), Eqs. (A7) and (A10) of Appendix A. On the other hand, here we use the operator representation of Refs. [26, 28] within which their operators emerge naturally from the rotated-electron \( c \) hidden \( U(1) \) symmetry degrees of freedom [28].

The six generators of the global \( \eta \)-spin and spin \( SU(2) \) symmetry algebras commute with the electron-rotated-electron unitary operator [28, 30]. Hence they have the same expressions when expressed in terms of electron and rotated-electron, respectively, creation and annihilation operators, so that the following equality holds,

\[ \hat{S}_a = \hat{S}_a^l = \sum_{j=1}^{N_a} \hat{s}_j^l, \quad \alpha = \eta, s, \quad l = \pm, x_3. \]  

(12)

The electron-rotated \( \alpha = \eta, s \) local operators \( \hat{s}_j^l, \alpha \) read,

\[ \hat{s}_{j,\eta}^l = (1 - n_{j,c}) \hat{q}_j^\dagger; \quad \hat{s}_{j,s}^l = n_{j,c} \hat{q}_j^\dagger, \quad l = \pm, x_3, \]  

(13)

where \( n_{j,c} \) is the \( c \) pseudoparticle local density operator, Eq. (10). Moreover, \( \hat{q}_j^\dagger = \hat{q}_j^x + i \hat{q}_j^y \) and \( \hat{q}_j^z \), where \( x_1, x_2, x_3 \) denotes the Cartesian coordinates, are the following \( \eta s \) quasi-spin operators,

\[ \hat{q}_j^\dagger = (\hat{e}_{j,\uparrow}^\dagger + (-1)^j \hat{e}_{j,\downarrow}^\dagger) \hat{c}_{j,\dagger}; \quad \hat{q}_j = (\hat{q}_j^\dagger)^\dagger; \quad \hat{q}_j^z = (\hat{n}_{j,\downarrow} - 1/2). \]  

(14)

The six local operators in Eq. (13) are generated by electron rotating the remaining six generators besides \( \hat{s}_{j,c} \) of the Hamiltonian electron-interaction term local gauge \( SU(2) \otimes SU(2) \otimes U(1) \) symmetry.

As mentioned above, the local operators \( f_{j,c}^\dagger \) and \( f_{j,c} \), Eq. (8), refer to the \( c \) hidden \( U(1) \) symmetry degrees of freedom of the rotated electrons. Similarly, the three local operators \( \hat{s}_{j,s} \) and three local operators \( \hat{s}_{j,\eta}^l \) where \( l = \pm, x_3 \), Eqs. (13) and (14), correspond to the rotated-electron spin \( SU(2) \) symmetry and \( \eta \)-spin \( SU(2) \) symmetry degrees of freedom, respectively. The latter operators are associated with the spin-1/2 spinons and \( \eta \)-spin-1/2 \( \eta \)-spinons, respectively, as defined within the elementary-object representation of Refs. [26, 28]. Equations (5), (8), (13), and
together with the unitary operator $\hat{V}$ 4$^{N_a}$ × 4$^{N_a}$ matrix elements between energy eigenstates given in Ref.

28 define the $c$ pseudoparticle, $\eta$-spinon, and spinons operators in terms of the electron creation and annihilation operators.

The $\eta s$ quasi-spin operators $\hat{q}_j^l$, Eq. (14), may be rewritten as,

$$\hat{q}_j^l = \hat{s}_j^{l,\eta} + \hat{s}_j^{l,\eta'}, \ l = \pm, x_3 .$$

(15)

This reveals that the three local spinon operators $\hat{s}_j^{l}$, and the three local $\eta$-spinon operators $\hat{s}_j^{l,\eta}$ are particular cases of the general $SU(2)$ local $\eta s$ quasi-spin operators $\hat{q}_j^l$. The latter operators refer to all lattice sites and are associated with a general quasi-spin $SU(2)$ symmetry. On the other hand, the three local spinon operators $\hat{s}_j^{l}$, and three local $\eta$-spinon operators $\hat{s}_j^{l,\eta}$ are associated with the spin and $\eta$-spin $SU(2)$ symmetry algebra representations, respectively. Such two sets of three local operators are defined in two independent sets of sites: (i) The 2$S_c$ spin-up and spin-down rotated-electron singly occupied sites and (ii) the 2$S_c^0$ rotated-electron doubly-occupied and unoccupied sites, respectively. The $c$ pseudoparticle and $\eta s$ pseudoparticle hole local density operators $(1-n_{j,c})$ and $n_{j,c}$ in the expressions of the operators $\hat{s}_j^{l,\eta}$ and $\hat{s}_j^{l,s}$, Eq. (13), play the role of projectors onto such two sets of lattice-site rotated-electron occupancies, respectively.

The electron - rotated-electron transformation associated with the operator $\hat{V}$ is unitary. Hence the rotated-electron operators $\hat{c}_{j,\sigma}$, and $\hat{c}_{j,\sigma'}$, Eq. (9), have the same anticommutation relations as the corresponding electron operators $\hat{c}_{j,\sigma}$ and $\hat{c}_{j,\sigma'}$, respectively. It follows that straightforward manipulations based on Eqs. (8) and (14) lead to the following algebra for the $c$ pseudoparticle operators,

$$\{f_{j,c}^\dagger, f_{j',c}\} = \delta_{j,j'}; \ \{f_{j,c}^\dagger, f_{j',c}\} = \{f_{j,c}, f_{j',c}\} = 0,$$

(16)

and the $c$ pseudoparticle operators and the local $\eta s$ quasi-spin operators,

$$[f_{j,c}^\dagger, \hat{q}_j^l] = [\hat{q}_j^l, f_{j',c}^\dagger] = [f_{j,c}^\dagger, \hat{s}_j^{l,\eta}] = [\hat{s}_j^{l,\eta'}, f_{j',c}^\dagger] = 0, \ l = \pm, x_3, \ \alpha = \eta, s .$$

(17)

Alike the local $c$ pseudoparticle operators in Eq. (16), the corresponding momentum dependent operators, Eq. (11), obey an anticommuting algebra,

$$\{f_{\alpha,\eta}^\dagger, f_{\eta',\alpha'}\} = \delta_{\alpha,\alpha'}^\prime; \ \{f_{\alpha,\eta}^\dagger, f_{\eta',\alpha'}\} = \{f_{\alpha,\eta}, f_{\eta',\alpha'}\} = 0 .$$

(18)

The first of such relations is valid provided that the two operators in the anticommutators act onto subspaces whose BA quantum numbers $I_j^n$ in the band discrete momentum values $\{q_j\} = \{2\pi/N_a I_j^n\}$, Eqs. (A7) and (A10) of Appendix A are both integers or half-odd integers.

From the use of Eqs. (13) and (14) one confirms that the $SU(2)$ algebra obeyed by electron-rotated local quasi-spin operators $\hat{q}_j^l$, where $l = x_3, \pm$, and corresponding $\eta$-spin ($\alpha = \eta$) and spin ($\alpha = s$) operators $\hat{s}_j^{l,\alpha}$ is the usual one,

$$[\hat{q}_j^l, \hat{q}_j^{l'}] = \delta_{j,j'} 2 \hat{s}_j^{x_3}; \ [\hat{q}_j^l, \hat{q}_j^{l'}] = \mp \delta_{j,j'} \hat{q}_j^{l'} ,$$

(19)

and

$$[\hat{s}_j^{l,\alpha}, \hat{s}_j^{l,\alpha'}] = \delta_{j,j'} \delta_{\alpha,\alpha'} 2 \hat{s}_j^{x_3}; \ [\hat{s}_j^{l,\alpha}, \hat{s}_j^{l,\alpha'}] = \mp \delta_{j,j'} \delta_{\alpha,\alpha'} \hat{s}_j^{l,\alpha} , \ \alpha, \alpha' = \eta, s ,$$

(20)

respectively. Moreover, one has that $[\hat{q}_j^l, \hat{q}_j^{l'}] = 0$ and $[\hat{s}_j^{l,\alpha}, \hat{s}_j^{l,\alpha'}] = 0$ where $l = 0, \pm$ and $\alpha, \alpha' = \eta, s$.

The $c$ pseudoparticle and $\eta s$ quasi-spin operator algebras refer to the whole Hilbert space. On the other hand, those of the $\eta$-spinon and spinon operators correspond to well-defined subspaces. Those are spanned by states whose number 2$S_c$ of rotated-electron singly occupied sites is fixed. This assures that the value of the corresponding $\eta$-spinon number, $M_\eta = [N_a - 2S_c]$, and spinon number, $M_c = 2S_c$, is fixed as well.

Moreover, the studies of the Ref. 28 have confirmed that the $\nu\nu$ pseudoparticles (and $s\nu$ pseudoparticles) considered in Refs. 26, 27 are $\eta$-spin-neutral (and spin-neutral) composite objects containing $\nu = 1, \ldots, \infty$ pairs of $\eta$-spin-1/2 anti-bound $\eta$-spinons with opposite $\eta$-spin projection (and of spin-1/2 bound spinons with opposite spin projection). The $c$ and $\alpha\nu$ pseudoparticles where $\alpha = \eta, s$ of Refs. 26, 28 are an extension to the whole Hilbert space of those of Refs. 47, 48 for the $c$ and $s1$ pseudo particle two-component subspace. An additional well-defined number of $\eta$-spinons (and spinons) remain invariant under the electron - rotated-electron unitary transformation performed by the BA solution considered below. Since they remain unbound, they are called unbound $\pm 1/2$ $\eta$-spinons (and unbound $\pm 1/2$ spinons). The values of the numbers $M_{\eta, s, \pm 1/2}^n$ of unbound $\pm 1/2$ $\eta$-spinons and $M_{\eta, s, \pm 1/2}^n$ of unbound $\pm 1/2$ spinons
are fully controlled by the \( \eta \)-spin \( S_\eta \) and \( \eta \)-spin projection \( S_\eta^z \) and spin \( S_s \) and spin projection \( S_s^z \), respectively, of the subspace or state under consideration as follows,

\[
M^\alpha_{\alpha'} = [M^\alpha_{\alpha'-1/2} + M^\alpha_{\alpha'+1/2}] = \pm S_\alpha \; ; \quad M^\alpha_{\alpha,\pm 1/2} = [S_\alpha \mp S_\alpha^z], \quad \alpha = \eta, s.
\]  

(21)

Thus the \( \eta \)-spin \( S_\eta \), \( \eta \)-spin projection \( S_\eta^z \), spin \( S_s \), and spin projection \( S_s^z \) of an energy eigenstate are fully determined by the \( \eta \)-unbound spinons and unbound spinons occupancies. For Bethe states with finite spin \( S_s \) and/or \( \eta \)-spin \( S_\eta \) all unbound spinons and/or unbound \( \eta \)-spinons have spin up and \( \eta \)-spin up, respectively.

One can introduce \( \alpha \nu \) pseudoparticle operators \( f^\dagger_{\eta,\alpha \nu} \), labeled by the discrete momentum values \( q_j \), Eqs. (A7) and (A10) of Appendix A, such that \( j = 1, \ldots, N_{\alpha \nu} \). Here \( \alpha = \eta, s \) and \( \nu = 1, \ldots, \infty \). Those are the conjugate variables of the \( \alpha \nu \) effective lattice real-space coordinates of site index \( j = 1, \ldots, N_{\alpha \nu} \) defined in Ref. 28. For subspaces for which the ratio \( N_{\alpha \nu}/N_a \) involving the number \( N_{\alpha \nu} \) of sites of the \( \alpha \nu \) effective lattice, Eqs. (A8) and (A9) of Appendix A is finite, such operators are given by,

\[
f^\dagger_{\eta,\alpha \nu} = (f_{\eta,\alpha \nu})^\dagger = \frac{1}{\sqrt{N_{\alpha \nu}}} \sum_{j'=1}^{N_{\alpha \nu}} e^{+iq_j \alpha \nu j'} f^\dagger_{j',\alpha \nu}, \quad j = 1, \ldots, N_{\alpha \nu}.
\]  

(22)

Such local operators have anticommuting relations,

\[
\{f^\dagger_{\eta,\alpha \nu}, f_{\eta',\alpha \nu'} \} = \{f^\dagger_{\eta,\alpha \nu}, f^\dagger_{\eta',\alpha \nu'} \} = 0.
\]  

(23)

Again, the first of such relations is valid provided that the two operators in the anticommutators act onto subspaces whose BA quantum numbers \( T_{\alpha \nu} \) in the \( \alpha \nu \) band discrete momentum values \( \{q_j \} = \{[2\pi/N_a] T_{\alpha \nu}\} \), Eqs. (A7) and (A10) of Appendix A are both integers or half-odd integers.

The transformation laws of the elementary objects that emerge from the rotated electrons under the electron -rotated-electron unitary transformation play an important role in the identification of the scatterers and scattering centers of the theory studied in this paper. The latter are found in the following to be the PDT \( c \) pseudofermions and \( \alpha \nu \) pseudoparticles [12–13]. Except for a slight shift of their discrete momentum values, which renders such objects without energy interaction terms, they have exactly the same properties as the corresponding \( c \) pseudoparticles and \( \alpha \nu \) pseudoparticles, respectively, from which they are generated.

### III. THE PSEUDOFERMION SCATTERING THEORY

Scattering theories of BA solvable models involve dressed \( S \) matrices [49–51]. In this section we introduce the pseudoparticle - pseudofermion unitary transformation and corresponding pseudofermion and pseudofermion-hole dressed \( S \) matrices. The \( \eta \)-spin 1/2 unbound \( \eta \)-spinons and spin 1/2 unbound spinons are found to be scattering-less elementary objects as far as their internal \( SU(2) \) degrees of freedom is concerned. On the other hand, the theory scatterers and scattering centers have no internal degrees of freedom such as spin or \( \eta \)-spin. Hence their scattering is associated with fully diagonal dressed \( S \) matrices. Consistent, our analysis of the problem follows the standard quantum non-relativistic scattering theory of spin-less particles [52]. For simplicity and without loss in generality, in this section we consider in general densities in the ranges \( n \in [0,1] \) and \( m \in \{0, n\} \). In Appendix C the pseudofermion scattering theory is extended to PSs of ground states with densities \( n = 1 \) and/or \( m = 0 \).

#### A. The 1D Hubbard model in the pseudofermion subspace: The pseudofermion canonical momentum, associated functionals, and exotic pseudofermion algebra

In the remaining of this paper we use a label \( \beta \) that refers both to the \( \beta = c \) band and \( \beta = \alpha \nu \) band excitation branches. Here \( \alpha = \eta \) and \( \alpha = s \) correspond to \( \eta \)-spin and spin and \( \nu = 1, \ldots, \infty \) to the number of \( \eta \)-spinon and spinon pairs, respectively. The pseudofermion scattering theory studied in this paper refers to the 1D Hubbard model in the pseudofermion subspace (PS). Such a subspace is spanned by a given ground state with arbitrary values of the electronic density \( n \) and spin density \( m \) and all excited energy eigenstates whose generation from it involve changes in the occupancy configurations of a finite number of \( \beta \) pseudoparticles, unbound spinons, and unbound \( \eta \)-spinons.

For the excited states belonging to the PS, the following ratios then vanish as \( N_a \rightarrow \infty \): \( \delta N_\beta / N_a \rightarrow 0 \), \( \delta M^\alpha_{s\eta} / N_a \rightarrow 0 \), and \( \delta M^\alpha_{\eta \eta} / N_a \rightarrow 0 \). Here \( \delta N_\beta, \delta M^\alpha_{s\eta}, \) and \( \delta M^\alpha_{\eta \eta} \) denote the deviations in the numbers \( N_\beta, M^\alpha_{s\eta} = 2S_s \), and \( M^\alpha_{\eta \eta} = 2S_\eta \) of \( \beta \) pseudoparticles, unbound spinons, and unbound \( \eta \)-spinons, respectively, under the ground-state - excited-state transitions. (The numbers of unbound \( \eta \)-spinons and unbound spinons are defined in Eq. (21).) We
emphasize that within the definition of the PS there are no restrictions on the value of the excitation energy and excitation momentum.

For the energy eigenstates that span the PS, the rapidity functionals defined by the thermodynamic BA equations, Eqs. (A1)–(A6) of Appendix A have the following exact property,

\[ \Lambda_\alpha(q_j) = \sin k_c(q_j) = k_0(q_j) \; ; \; \Lambda_\beta(q_j) = k_0(q_j) \; , \; j = 1, ..., N_a , \]

The set of discrete numbers \( \bar{q}_j = \bar{q}(q_j) \) where \( j = 1, ..., N_{a\beta} \) in the arguments of the functions \( k_0(q_j) \) for \( \beta = c \) and \( \Lambda_0(q_j) \) for \( \beta = \alpha \nu \) appearing in Eq. (24) play a central role in the pseudofermion scattering theory. They are the \( \beta \) pseudofermion discrete canonical-momentum values. Their spacing is to first order in \( 1/N_a \),

\[
\bar{q}_{j+1} - \bar{q}_j = \frac{2\pi}{N_a} + \text{h.o.},
\]

where h.o. stands for terms of second order in \( 1/N_a \). Such pseudofermions have been previously used in the PDT studies \([12,13]\). A \( \beta \) pseudofermion carries discrete canonical momentum \( \bar{q}_j = \bar{q}(q_j) \) whereas the corresponding \( \beta \) pseudoparticle carries the discrete momentum \( q_j \). Often in the remaining of this paper we call \( q_j \) bare momentum, to distinguish it from the corresponding canonical momentum, \( \bar{q}_j = \bar{q}(q_j) \).

For the \( \beta = \nu \eta \) (and \( \beta = \nu \eta \neq s_1 \)) branches, the relation \( \Lambda_\beta(q_j) = \Lambda_0(q_j) \), Eq. (24), is valid provided that the hole concentration \( (1 - n) \) (and spin density \( m \)) is finite. For a \( S_n = 0; n = 1 \) (and \( S_n = 0; m = 0 \)) ground state, the number of \( \nu \eta \) (and \( \nu \eta \neq s_1 \)) band discrete momentum values \( N_{\nu \eta} \) (and \( N_{\nu \eta} \)), Eqs. (A8) and (A9) of Appendix A vanish, \( N_{\nu \eta} = 0 \) (and \( N_{\nu \eta} = 0 \)). Therefore, the corresponding pseudofermion branch does not exist. Hence the ground-state rapidity \( \Lambda_0(q_j) \) appearing in Eq. (24) is undefined. For the excited states of such a \( S_n = 0; n = 1 \) (and \( S_n = 0; m = 0 \)) ground state the value of \( \Lambda_\beta(q_j) \) for \( \beta = \nu \eta \) (and \( \beta = \alpha \nu \neq s_1 \)) is an issue addressed in Appendix C.

By use of the expressions provided in Eq. (24) in the thermodynamic BA equations, Eqs. (A1)–(A6) of Appendix A one uniquely finds that, to leading order in \( 1/N_a \), the discrete canonical-momentum values have the following form,

\[
\bar{q}_j = \bar{q}(q_j) = q_j + \frac{Q_{\beta}^\Phi(q_j)}{N_a} = \frac{2\pi}{N_a} I_j^\beta + \frac{Q_{\beta}^\Phi(q_j)}{N_a} , \; \beta = c, \alpha \nu , \; j = 1, ..., N_{\alpha \beta} .
\]

Here \( N_{\alpha \beta} \) is given in Eqs. (A8) and (A9) of that Appendix for both \( \beta = c \) and \( \beta = \alpha \nu \). The \( \beta \) band discrete momentum value \( q_j \) in Eq. (26) can be written as \( q_j = [2\pi/N_a] I_j^\beta \), where \( I_j^\beta \) with \( j = 1, ..., N_{\alpha \beta} \) are BA quantum numbers. As given in Eqs. (A7) and (A10) of Appendix A those can have either integer or half-odd integer values.

The relation, Eq. (26), uniquely defines a one-to-one correspondence between the two sets \( \{ q_j \} \) and \( \{ \bar{q}_j \} \). That correspondence defines the \( \beta \) pseudoparticle - \( \beta \) pseudofermion unitary transformation. The corresponding \( \beta \) pseudoparticle - \( \beta \) pseudofermion unitary operator,

\[
\hat{S}_\beta^\Phi = \sum_{\beta} \frac{1}{2} \sum_{j=1}^{N_{\alpha \beta}} f_{\beta,\gamma}^{\Phi} (q_j) / N_{\alpha \beta} \hat{f}_{\beta,\gamma} ; \; \left( \hat{S}_\beta^\Phi \right)^\dagger = \sum_{\beta} \frac{1}{2} \sum_{j=1}^{N_{\alpha \beta}} f_{\beta,\gamma}^{\Phi} (q_j) / N_{\alpha \beta} \hat{f}_{\beta,\gamma}^\dagger.
\]

is such that,

\[
\hat{f}_{\beta,\gamma} = f_{\beta,\gamma}^{\Phi} / N_{\alpha \beta} = \left( \hat{S}_\beta^\Phi \right)^\dagger \hat{S}_\beta^\Phi .
\]

Here \( f_{\beta,\gamma}^{\Phi} \) and \( f_{\beta,\gamma} \) are the \( \beta \) pseudoparticle operators, Eqs. (11) and (22). The corresponding \( \beta \) pseudofermion operators are denoted by \( f_{\beta,\gamma}^{\Phi} \) and \( f_{\beta,\gamma} \), respectively. Except for the slightly shifted discrete canonical momentum values, Eq. (26), which below is shown to render the \( \beta \) pseudofermion spectrum without energy interaction terms, the \( \beta \) pseudofermions have the same properties as the corresponding \( \beta \) pseudoparticles.

The quantity \( Q_{\beta}^\Phi(q_j) \) in Eqs. (20), (27), and (28) plays a key role in the pseudofermion scattering theory. It reads \([12]\),

\[
Q_{\beta}^\Phi(q_j) = \sum_{\beta} \sum_{j' = 1}^{N_{\alpha \beta}} 2\pi \Phi_{\beta',\beta}(q_j, q_{j'}) \delta N_{\beta'}(q_{j'}) .
\]
The $\beta'$ band momentum distribution function deviation $\delta N_{\beta'}(q_{j'})$ appearing here is defined in Eq. (A19) of Appendix A. The elementary-object representation used in the studies of this paper refers to the limit of a very large system, $N_a \gg 1$. Thus we often approximate the discrete bare momentum values $q_j$, such that $|q_{j+1} - q_j| = 2\pi N_a$, by a continuum variable $q$. The corresponding general $\beta$ band bare momentum distribution function deviation then reads,

$$
\delta N_{\beta}(q) = \frac{2\pi}{N_a} \sum_{\nu=1}^{N_{\beta}^p} \delta(q - q_{\nu}) - \frac{2\pi}{N_a} \sum_{h=1}^{N_{\beta}^s} \delta(q - q_{h}), \quad \text{if } \delta G_{\beta} \text{ even for } \beta = c, s1,
$$

$$
= \frac{2\pi}{N_a} \sum_{\nu=1}^{N_{\beta}^p} \delta(q - q_{\nu}) - \frac{2\pi}{N_a} \sum_{h=1}^{N_{\beta}^s} \delta(q - q_{h}) \pm \frac{\pi}{N_a} \sum_{j=\pm 1} j \delta(q - j q_{f_{\beta}}), \quad \text{if } \delta G_{\beta} \text{ odd for } \beta = c, s1,
$$

$$
\delta N_{\beta}(q) = \frac{2\pi}{N_a} \sum_{\nu=1}^{N_{\beta}^p} \delta(q - q_{\nu}), \quad \text{for } \beta = \alpha \nu \neq s1,
$$

(30)

Here and throughout this paper, $\delta(x)$ denotes the usual Dirac delta-function distribution. Since there is a one-to-one correspondence between the canonical momentum $\hat{q}(q)$ and the bare-momentum $q$, we may refer to the bare momentum $q$ of a pseudofermion. By that it is meant the bare momentum $q$ corresponding to the pseudofermion canonical momentum $\hat{q}(q)$. For instance, $q_1, ..., q_{N_{\beta}^p}$ are in Eq. (30) the bare momentum values of the $N_{\beta}^p$ pseudofermions added under the transition to the excited state and $q_1, ..., q_{N_{\beta}^s}$ those of the $N_{\beta}^s$ pseudofermion holes added under that transition. $\delta G_{\beta}$ stands in that equation for the deviation in the value of the number,

$$
G_{\beta} = \delta_{\beta,c} \sum_{\nu=1}^{N_{\beta}^c} B_{\alpha} + \delta_{\beta,s1}[N_{c} + N_{s1}]; \quad B_{\alpha} = \sum_{\nu=1}^{\infty} N_{\alpha \nu}, \quad \alpha = \eta, s.
$$

(31)

The additional term in the Eq. (30) expression corresponding to the deviation $\delta G_{\beta}$ being an odd integer number rather than an even integer number results from the corresponding $\beta = c, s1$ band momentum shift, $\pm \pi/N_a$.

Moreover, the quantity $\pi \Phi_{\beta,\beta'}(q_j, q_{j'})$ in Eq. (29) is a function of both the bare-momentum values $q_j$ and $q_{j'}$, given by,

$$
\pi \Phi_{\beta,\beta'}(q_j, q_{j'}) = \pi \Phi_{\beta,\beta'} \left( \frac{\Lambda_{\eta \nu}^0(q_j)}{u}, \frac{\Lambda_{\eta \nu}^0(q_{j'})}{u} \right).
$$

(32)

It is found below that $\pi \Phi_{\beta,\beta'}(q_j, q_{j'})$ (or $-\pi \Phi_{\beta,\beta'}(q_j, q_{j'})$) is an elementary two-pseudofermion phase shift. It is such that $q$ is the bare-momentum value of a $\beta$ pseudofermion or $\beta$ pseudofermion hole scattered by a $\beta'$ pseudofermion [or $\beta'$ pseudofermion hole] of bare-momentum $q'$ created under a ground-state - excited-state transition.

The function $\pi \Phi_{\beta,\beta'}(r, r')$, Eq. (32), is the unique solution of the integral equations, Eqs. (B11) - (B15) of Appendix B. The ground-state rapidity functions $\Lambda_{\nu \nu}^0(q_j)$, where $\Lambda_{\nu \nu}^0(q_j) = \sin k_{\nu}^0(q_j)$ for $\beta = c$, are defined in terms of their inverse functions in Eq. (A15) of that Appendix. We recall that for $S_a = 0; n = 1$ (and $S_a = 0; m = 0$) ground states there are no $\beta = \eta \nu$ (and $\beta = sv \neq s1$) bare-momentum bands. Indeed, Eqs. (A8) and (A9) of Appendix A lead to $N_{\alpha \nu} = 0$ (and $N_{a \nu} = 0$) for such states. As reported in Appendix C, then the ground-state rapidity functions $\Lambda_{\nu \nu}^0(q)$ (and $\Lambda_{\nu \nu}^0(q)$) must be replaced by those of the excited state, $\Lambda_{\alpha \nu}(q)$ (and $\Lambda_{\alpha \nu}(q)$), respectively. It is found below that the functions, Eq. (32), are phase shifts originated by well-defined ground-state - excited-state transitions.

In the $u \to \infty$ limit, the ground-state rapidity momentum function $k_{\nu}^0(q_j)$, Eq. (A15) of Appendix A is given by $k_{\nu}^0(q_j) = q_j$. Hence, according to Eq. (24), for all PSS it reads, $k_{c}(q_j) = k_{\nu}^0(q_j) = \tilde{q}_j$. The $u \to \infty$ spinless fermions of Refs. [16, 17] have been constructed inherently to carry the rapidity momentum $k_j = k_c(q_j)$. Since $k_c(q_j) = \tilde{q}_j$ as $u \to \infty$, such spinless fermions are nothing but the $c$ pseudofermions as defined here for $u \to \infty$. The spinless fermions have creation and annihilation operators $\hat{b}_{k_j}$ and $\hat{b}_{k_j}$, respectively, which have the same expression in terms of electron creation and annihilation operators as the corresponding $u > 0 c$ pseudofermion operators $\hat{f}_{q_j,c}$ and $\hat{f}_{q_j,c}$ in terms of rotated-electron creation and annihilation operators, so that,

$$
f_{q_j,c} = \hat{V}^\dagger b_{k_j} \hat{V}; \quad f_{q_j,c} = \hat{V}^\dagger b_{k_j} \hat{V}.
$$

(33)

Such relations hold except for unimportant phase factors provided that $\hat{V}$ is the electron - rotated-electron unitary operator associated with the specific unitary transformation performed by the BA solution.
An important functional related to that defined in Eq. (29) reads,
\[ Q_{\beta}(q_j) = Q_{\beta}^0 + Q_{\beta}^q(q_j). \] (34)

The related quantity, \( Q_{\beta}^0/N_\alpha \), is the shift in the \( \beta = c, \alpha \nu \) band discrete bare-momentum value \( q_j \) under the ground-state - excited-state transition. Its values are,
\[ Q_{c}^0 = 0; \quad \sum_{\beta=\eta, s}^\infty \sum_{\nu=1}^\infty \delta N_{\alpha \nu} \text{ even}; \quad Q_{c}^0 = \pm \pi; \quad \sum_{\beta=\eta, s}^\infty \sum_{\nu=1}^\infty \delta N_{\alpha \nu} \text{ odd}; \]
\[ Q_{\alpha \nu}^0 = 0; \quad \delta N_{c} + \delta N_{\alpha \nu} \text{ even}; \quad Q_{\alpha \nu}^0 = \pm \pi; \quad \delta N_{c} + \delta N_{\alpha \nu} \text{ odd}; \quad \alpha = \eta, s. \] (35)

When under such a transition the BA quantum numbers \( I^\beta_j \) in \( q_j = [2\pi / N_\alpha] I^\beta_j \), Eqs. (A7) and (A10) of Appendix A change from integers (or half-odd integers) to half-odd integers (or integers), a finite shift, \( \delta N_{\alpha \nu}/N_\alpha = \pm \pi / N_\alpha \) occurs.

Analysis of the discrete canonical-momentum value expression, Eqs. (26) and (29), reveals that for the ground state the canonical momentum \( \bar{q}_j \) and corresponding bare momentum \( q_j \) have the same value, \( \bar{q}_j = q_j \). Hence the ground-state limiting \( \beta \) canonical momenta and \( c \) and \( s^1 \) Fermi canonical momenta equal the corresponding bare momenta. However, for the excited states the \( c \) and \( s^1 \) Fermi canonical momentum values are shifted. The corresponding deviations play an important role in the spectral properties, as discussed below in Section IV D.

The bare-momentum distribution-function deviations second-order energy spectrum, Eq. (A18) of Appendix A can be exactly expressed in terms of the corresponding \( \beta \) pseudofermion canonical-momentum distribution-function deviations as follows,
\[ \delta E_{\epsilon, \eta, s} = \sum_{\beta=c, s^1} \sum_{j=1}^{N_{\alpha \beta}} \varepsilon_{\beta}(q_j) \delta N_{\beta}(q_j) + \sum_{\beta \nu \neq c, s^1} \sum_{j=1}^{N_{\alpha \nu}} \varepsilon_{\alpha \nu}(q_j) \delta N_{\alpha \nu}(q_j) + 2|\mu|D_r + 2\mu_B |H| S_r. \] (36)

The deviations second-order terms of the equivalent pseudoparticle energy spectrum, Eq. (A18) of Appendix A refer to zero-momentum forward-scattering interactions. The unitary transformation relations, Eqs. (26) and (29), have absorbed such second-order energy terms into the pseudofermion canonical momentum. Consistent, the pseudoparticle \( f \) functions, Eq. (A24) of Appendix A and momentum-shift functional, Eq. (29), contain the same two-pseudofermion phase shifts.

Importantly, the \( \beta \)-pseudofermion - \( \beta \)-pseudofermion unitary transformation maps a quantum problem in terms of pseudoparticles with residual zero-momentum forward-scattering energy interaction terms, the second-order terms in Eq. (A18), into a non-interacting pseudofermion problem. Indeed, the equivalent pseudofermion energy spectrum, Eq. (36), has no interaction energy terms, so that the pseudofermions are not energy entangled. Such a lack of pseudofermion energy entanglement plays a key role in the PDT. As shortly discussed below in Section IV D it allows the one- and two-electron spectral functions to be expressed as convolutions of \( c \) and \( s^1 \) pseudofermion spectral functions [12, 13].

The energy dispersions \( \varepsilon_{\beta}(q_j) \) in the energy spectrum, Eq. (36), have exactly the same dependence on \( q_j \) as those defined by Eqs. (A21) and (A22) of Appendix A on \( q_j \). The limiting analytical behaviors of such energy dispersions are provided in Appendix D. Moreover, the finite-energy part of the energy spectrum, Eq. (A18) of Appendix A is expressed in Eq. (36) in terms of the numbers \( D_r \) and \( S_r \), Eq. (A20) of that Appendix. \( D_r \) is the number of rotated-electron doubly occupied sites. \( S_r \) is that of those spin-down rotated-electron singly occupied sites whose spins are not contained into two-spinon \( s^1 \) pseudofermions. The numbers \( D_r \) and \( S_r \) vanish for all ground states. Hence their deviations are given by \( \delta D_r = D_r \) and \( \delta S_r = S_r \).

That for \( N_\alpha \gg 1 \) and up to contributions of \( 1/N_\alpha \) order the \( c \) and \( s^1 \) pseudofermions have no residual energy interactions is a result of major physical importance [12, 13]. Hence in the following we confirm that such elementary objects have indeed no residual energy interactions. The contributions to the PDT one- and two-electron spectral-weight distributions that are more involved to be accounted for are those of the \( c \) and \( s^1 \) pseudofermions processes. Indeed, LWS ground states are not populated by \( \alpha \nu \neq s^1 \) pseudofermions, unbound \(-1/2 \) \( \eta \)-spinons, and unbound \(-1/2 \) spinons. That much simplifies accounting for the contribution of such elementary-objects creation under the transitions to the excited states to the corresponding one- and two-electron spectral weights [12, 13]. Hence for simplicity and without loss in generality, in the following we consider PSs for which \( M_{\eta - 1/2}^{\alpha \nu} = M_{\epsilon - 1/2}^{\alpha \nu} = 0 \) in Eq. (24) and \( \delta N_{\beta}(q_j) = 0 \) for the \( \beta \neq c, s^1 \) branches in Eq. (A18) of Appendix A.

Our analysis refers to the limit of a very large system, \( N_\alpha \gg 1 \). Hence we approximate both the \( \beta = c, s^1 \) band discrete bare momentum values, \( q_j \), and the corresponding discrete canonical momentum values, \( \bar{q}_j \), by continuum variables, \( q \) and \( \bar{q} = q + Q_{\beta}^q(q)/N_\alpha \), respectively. The excited-state \( \beta = c, s^1 \) deviations \( \delta N_{\beta}(q) \) have the general form,
Eq. (30). For the PSs under consideration, the energy spectrum, Eq. (A18) of Appendix A, becomes in terms of continuum momentum variables 

\[
\delta E_{1,\beta} = \frac{N_a}{2\pi} \sum_{\beta=c,s1} \int_{-q_\beta}^{q_\beta} dq \varepsilon_\beta(q) \delta N_\beta(q) 
+ \frac{1}{N_a} \left( \frac{N_a}{2\pi} \right)^2 \sum_{\beta=c,s1} \sum_{\beta'=-c,s} \int_{-q_\beta}^{q_\beta} dq \int_{-q_{\beta'}}^{q_{\beta'}} dq' \frac{1}{2} f_{\beta,\beta'}(q,q') \delta N_\beta(q) \delta N_{\beta'}(q').
\]  

(37)

The energy dispersions \(\varepsilon_\beta(q)\) and \(f\) functions appearing here are those defined in Eqs. (A24)-(A26) and Eq. (A24) of Appendix A, respectively. Interestingly, such \(f\) functions involve only the \(\beta\) group velocities \(v_\beta(q)\) and \(v_\beta q_F(q_\beta)\), Eq. (A25) of Appendix A and the two-pseudofermion phase shifts, Eq. (32), of the scattering theory considered below. For low-energy excitations, both the first-order and second-order energy terms, Eq. (57), contribute to the finite-size spectrum of conformal-field theory \([14]\). The corresponding finite-size corrections are of \(1/N_a\) order \([10, 11, 14]\).

The energy spectrum, Eq. (37), can be rewritten in terms of the functional \(Q^0_\beta(q)\), Eq. (29), as follows \([54]\),

\[
\delta E_{1,\beta} = \frac{N_a}{2\pi} \sum_{\beta=c,s1} \int_{-q_\beta}^{q_\beta} dq \varepsilon_\beta(q) \delta N_\beta(q) 
+ \sum_{\beta=c,s1} \left[ \frac{1}{2\pi} \int_{-q_\beta}^{q_\beta} dq v_\beta(q) Q^0_\beta(q) \delta N_\beta(q) + \frac{v_\beta}{2\pi N_a} \sum_{\nu \pm 1} \left( Q^\nu_\beta(\nu q_F) \right)^2 \right].
\]  

(38)

We now confirm that the expression of this energy functional in terms of the \(\beta = c, s1\) pseudofermion canonical momenta \(\bar{q}\), Eq. (20), indeed simplifies to,

\[
\delta E_{1,\beta} = \frac{N_a}{2\pi} \sum_{\beta=c,s1} \int_{-q_\beta}^{q_\beta} dq \varepsilon_\beta(q) \delta N_\beta(q).
\]  

(39)

To first order in \(1/N_a\), the deviation \(\delta N_\beta(\bar{q})\) in Eq. (59) accounts for two types of contributions. The first contribution type results from the shift, \(q \rightarrow q + Q^0_\beta(q)/N_a\), in the arguments of the deviation \(\delta N_\beta(q)\) \(\delta\)-functions, Eq. (30). It is accounted for by considering that \(\varepsilon_\beta(\bar{q}) = \varepsilon_\beta(q)\) and \(\delta N_\beta(\bar{q}) = \delta N_\beta(q + Q^0_\beta(q)/N_a)\) in Eq. (30). This leads to the energy terms \(\frac{N_a}{2\pi} \sum_{\beta=c,s1} \int_{-q_\beta}^{q_\beta} dq \varepsilon_\beta(q) \delta N_\beta(q)\) and \(\frac{1}{2\pi} \sum_{\beta=c,s1} \int_{-q_\beta}^{q_\beta} dq v_\beta(q) Q^0_\beta(q) \delta N_\beta(q)\) in Eq. (38). To first order in \(1/N_a\), exactly the same energy terms are obtained if one uses instead \(\varepsilon_\beta(q) = \varepsilon_\beta(q + Q^0_\beta(q)/N_a)\) and \(\delta N_\beta(q) = \delta N_\beta(q)\) in Eq. (39). This latter choice is the most convenient for adding to \(\delta N_\beta(q)\) the second contribution.

The first contribution type refers to small changes in the momentum of the original \(\beta\) pseudoparticle creation and annihilation processes described by the deviations, Eq. (30). On the other hand, the second contribution to the \(\beta\) pseudofermion canonical-momentum distribution function deviation \(\delta N_\beta(\bar{q})\) involves the quantity \(\nu Q^\nu_\beta(\nu q_F)/N_a = (\nu \delta q^\nu_F - \delta q^\nu_F)\). Here \(\delta q^\nu_F\) and \(\delta q^\nu_F\) are the excited-state deviations of the \(\beta\) pseudofermion and \(\beta\) pseudoparticle right (\(\nu = 1\)) and left (\(\nu = -1\) Fermi points, which read,

\[
\delta q^\nu_F = \frac{Q^\nu_F(\nu q_F)}{N_a} ; \quad \delta q^\nu_F = \frac{2\pi}{N_a} \delta N^F_{\beta,\nu}, \quad \beta = c, s1, \quad \nu = \pm 1,
\]  

(40)

respectively. In this equation, \(\delta N^F_{\beta,\nu} = [\delta N^0_{\beta,\nu} + \nu Q^0_F/2\pi]\) denotes the deviation in the number of \(\beta = c, s1\) pseudofermions at the right (\(\nu = +1\)) and left (\(\nu = -1\)) \(\beta = c, s1\) Fermi points, under the ground-state - excited-state transition. Such a deviation includes the effects from the possible shifts, \(\pm Q^0_F/N_a\) and \(\pm Q^0_F/2\pi\), of the bare discrete momentum values \(q_j\), Eq. (A7) of Appendix A and corresponding BA quantum numbers \(l_j^F\), Eq. (A10) of that Appendix, respectively. (The quantity \(Q^0_F\) is that given in Eq. (35) for \(\beta = c, s1\)). On the other hand, the deviation \(\delta N^0_{\beta,\nu}\) does not account for the effects from such shifts. Its value follows only from the net change in the \(\beta = c, s1\) pseudofermion occupations at the corresponding \(\nu = \pm 1\) Fermi points.

The second term, \(Q^\nu_F(\nu q_F)/N_a\), in the deviation \(\delta q^\nu_F\), Eq. (40), is that associated with the second contribution type to \(\delta N_\beta(\bar{q})\), occurs in that expression even when \(\delta q^\nu_F = 0\). In that case there is no change in the \(\nu = \pm 1\) \(\beta = c, s1\) Fermi-point occupations under the ground-state - excited-state transition. To first order in \(1/N_a\), the two types of contribution are accounted for provided that one uses the following expressions for \(\varepsilon_\beta(q)\) and \(\delta N_\beta(q)\) in the
energy functional, Eq. (33),

\[ \varepsilon_{\beta}(q) = \varepsilon(\beta, q + Q_{\beta}(q)/N_a) = \varepsilon(\beta) + v_{\beta}(q) \frac{Q_{\beta}^2(q)}{N_a} \]

\[ \delta N_{\beta}(q) = \delta N_{\beta}(q) + \frac{Q_{\beta}^2(q)}{N_a} \delta(q - \epsilon q_{F\beta}). \]  

(41)

Consistent with our above discussion, the extra term \( \frac{Q_{\beta}^2(q)}{N_a} \delta(q - \epsilon q_{F\beta}) \) in the \( \delta N_{\beta}(q) \) expression accounts for the second contribution type. The use on the right-hand side of Eq. (39) of the expressions, Eq. (41), readily leads to the full energy expression, Eq. (38). It is of second order in the \( \beta \) pseudofermion momentum distribution function deviations \( \delta N_{\beta}(q) \). (We recall that \( \varepsilon(\beta, q_{F\beta}) = 0 \), so that the second contribution type leads indeed only to the energy term \( \sum_{\beta=c,s} \sum_{q=-1}^{1} (Q_{\beta}^2(q_{F\beta}))^2 \) in Eq. (38).)

On the one hand, within the \( \beta \) pseudofermion representation the quantity \( Q_{\beta}^2(q)/N_a \) (and \( Q_{\beta}^2(q)/N_a \)) in the argument of the rapidity function \( k_\beta^0(q + Q_{\beta}^2(q)/N_a) \) (and \( \Lambda_\beta^0(q + Q_{\beta}^2(q)/N_a) \)) is behind the \( \beta \) pseudofermion residual interactions. This is confirmed by the form of the second-order energy terms in the first expression of Eq. (38). Those would vanish if \( Q_{\beta}^2(q)/N_a = 0 \). On the other hand, within the \( \beta \) pseudofermion representation that quantity is rather incorporated in the canonical momentum. Consistent, we have just confirmed that the \( \beta = c, s \) pseudofermions are not energy entangled: The momentum shift \( Q_{\beta}^2(q)/N_a \) in the \( \beta \) pseudofermion canonical momentum \( q = q + Q_{\beta}^2(q)/N_a \), Eq. (39), exactly cancels the second-order energy terms in Eq. (38). That within the \( \beta \) pseudofermion representation the quantity \( Q_{\beta}^2(q) \) is incorporated in the canonical momentum has consequences though in the exotic algebra obeyed by the \( \beta \) pseudofermion creation and annihilation operators. Consider a \( \beta \) pseudofermion of canonical momentum \( \bar{q} \) and a \( \beta' \) pseudofermion of canonical momentum \( \bar{q}' \). Here \( \bar{q} \) and \( \bar{q}' = \bar{q} \) correspond to an excited energy eigenstate \( \beta \) band and the ground state \( \beta' \) band, respectively. The band canonical momentum having the form \( q = q + Q_{\beta}^2(q)/N_a \) implies that the effective anticommutators involving the creation and/or annihilation operators of these two pseudofermions have the general form [12],

\[ \{ f_{\bar{q},\beta}, f_{\bar{q}',\beta'} \} = \delta_{\beta,\beta'} \frac{1}{N_a} e^{-i(\bar{q} - \bar{q}')/2} e^{iQ_{\beta}(q)/2} \sin \left( \frac{Q_{\beta}(q)/2}{\sin((\bar{q} - \bar{q}')/2)} \right), \]

(42)

and \( \{ f_{\bar{q},\beta}, f_{\bar{q}',\beta} \} = \{ f_{\bar{q},\beta}, f_{\bar{q}',\beta'} \} = 0 \). Here \( Q_{\beta}(q_j) = Q_{\beta}^0 + Q_{\beta}^0(q_j) \) is the functional, Eq. (34), whose value is specific to the excited energy eigenstate under consideration. The related quantity, \( Q_{\beta}(q_j)/N_a = [Q_{\beta}^0 + Q_{\beta}^0(q_j)]/N_a \), is the overall shift in the discrete canonical-momentum value that results from the ground-state - excited-state transition.

Note that for \( \beta = \beta' \) the unitarity of the pseudofermion - pseudofermion transition preserves the pseudofermion operator algebra provided that the canonical momentum values \( \bar{q} \) and \( \bar{q}' \) correspond to the same excited state for \( \beta \) band. The exotic form of the anticommutator in Eq. (42) follows from for \( \beta' = \beta' \) the canonical momenta \( \bar{q} \) and \( \bar{q}' \) corresponding rather to the excited state \( \beta \) band and the ground state \( \beta \) band, respectively. Such an exotic \( \beta \) pseudofermion algebra plays an important role in the one- and two-electron finite-energy spectral weight distributions [12-14]. Consistent with the form of the anticommutator in Eq. (42) for \( \beta = \beta' \) and as discussed below in Section IV.D the functional \( Q_{\beta}(q) \) controls the quantum overlaps associated with such spectral weight distributions.

The effective character for \( \beta = \beta' \) of the anticommutator in Eq. (42) results from it involving two operators acting onto subspaces with different \( \beta \) band discrete canonical momentum values. Such subspaces are those of the ground state and excited state, respectively. The corresponding shake-up effects makes it to be an effective anticommutator. Indeed, the standard operator commutators involve operators acting onto the same Hilbert space. However, the effective anticommutators of such a form are physically meaningful. They control the orthogonal-catastrophe quantum overlaps associated with the PDT one- and two-electron spectral-weight distributions [12-14].

B. The ground-state - virtual-state transition

The momentum value of an unbound ±1/2 \( \eta \)-spinon (and a unbound ±1/2 spinon) internal degrees of freedom is \( q_{\eta}=1/2 = 0 \) or \( q_{\eta}=-1/2 = \pi \) (and \( q_{\eta}=\pm 1/2 = 0 \)) for all energy eigenstates with finite occupancy of such objects. That such momentum values remain unchanged under the ground-state - excited-state transitions reveals that as far as their internal degrees of freedom is concerned such objects are not scatterers. Neither do the \( \beta \) pseudofermions scatter off on them, so that they are not scattering centers. Hence the scattering processes described in the following involve only the \( \beta \) pseudofermions.

Each transition from the ground state to a PS excited energy eigenstate can be divided into three steps. The first process is a scatter-less finite-energy and finite-momentum excitation that transforms the ground state onto
a well defined virtual state. For the \( \alpha \nu \) branches, that excitation can involve a change in the number of discrete bare-momentum values. Following Eqs. (A8) and (A9) of Appendix A, those are given by,

\[
\delta N_{a_1} = \delta N_c - \delta N_{s_1} - 2 \sum_{\nu=2}^{\infty} \delta N_{sv} ; \quad \delta N_{a\nu} = \delta M_{a\nu}^n + 2 \sum_{\nu'=\nu+1}^{\infty} (\nu' - \nu) \delta N_{a\nu'} , \quad \alpha \nu \neq s 1 .
\]

For the ground state, the numbers of discrete bare-momentum values read,

\[
N_{a_1}^0 = N_\uparrow ; \quad N_{a\nu}^0 = (N_\uparrow - N_\downarrow) , \quad s\nu \neq s 1 ; \quad N_{a\nu}^0 = (N_a - N) ,
\]

and \( N_{a\nu}^0 = N_{a\nu} \) is given by \( N_{a\nu} = N_a \) for the whole Hilbert space. The \( \beta \neq c, s 1 \) branches have no finite pseudofermion occupancy in the ground state. In spite of that, for the present densities ranges one can define the discrete momentum number values \( N_{a\beta} = N_{a\beta}^0 \) of the corresponding unoccupied bands. For the \( \beta \neq c, s 1 \) branches, those are the numbers \( N_{a\nu}^0 \) and \( N_{a\nu}^0 \) given in Eq. (43). Thus, for \( \beta \neq c, s 1 \) branches with finite pseudofermion occupancy in the virtual state the discrete bare-momentum shifts, Eq. (35), and deviations, Eq. (43), are relative to the values of such unoccupied bands.

In addition and following the change in the number of discrete bare-momentum values, this excitation also involves the pseudofermion creation and annihilation processes and pseudofermion particle-hole processes that generate the PS excited states. The excitation momentum of the corresponding ground-state - virtual-state transition reads,

\[
\delta P_{\eta, s \nu}^0 = \sum_{j=1}^{N_a} q_j \delta N_c(q_j) + \sum_{\nu=1}^{N_{a\nu}} q_j \delta N_{sv}(q_j) + \sum_{\nu=1}^{N_{a\nu}} \sum_{j=1}^{N_{a\nu}} |\pi - q_j| N_{sv}(q_j) + \pi M_{\eta, -1/2} , \quad M_{\eta, -1/2} = M_{\eta, 1/2} + \sum_{\nu=1}^{\infty} \nu N_{sv} ; \quad \delta N_{sv}(q_j) = N_{sv}(q_j) \text{ for } s\nu \neq s 1 .
\]

Here \( M_{\eta, 1/2} \) is the \( \eta \)-spin-projection \(-1/2\) unbound \( \eta \)-spinon number, Eq. (21).

The momentum spectrum is of first order in the \( \beta \) momentum distribution function deviations. Thus it is convenient to express it in terms of the corresponding occupancies of the \( \beta \) band bare momentum values \( q_j \), as given here, rather than of those of the \( \beta \) band canonical momentum values \( \tilde{q}_j = \tilde{q}(q_j) \). On the other hand, the excitation energy is provided in Eq. (30).

In this first scatter-less step, the pseudofermions acquire the excitation energy needed for the second and third steps. The second step may give rise to a momentum contribution to be added to that given in Eq. (45).

### C. Pseudofermion scattering processes, dressed S matrices, and phase shifts

In order to study the second and third processes of the ground-state - excited-state transition, it is useful to express the many-pseudofermion states and operators in terms of one-pseudofermion states and operators, respectively. The PS energy and momentum eigenstates can be written as direct products of states. Those are generated by the occupancy configurations of each of the \( \alpha \nu \) branches with finite pseudofermion occupancy. Moreover, the many-pseudofermion states generated by occupancy configurations of each \( \alpha \nu \) branch can be expressed as a direct product of \( N_{a\beta} \), one-pseudofermion states. Each of the latter states refers to one discrete bare-momentum value \( q_j \), where \( j = 1, ..., N_{a\beta} \).

The Hamiltonian of the quantum problem described by the 1D Hubbard model in the PS, whose energy spectrum is for the \( \beta \) pseudofermion representation given in Eq. (30), has within that representation a uniquely defined expression of the general form,

\[
: \hat{H} := \sum_{\beta} \sum_{j=1}^{N_{a\beta}} \hat{H}_{\beta, \tilde{q}_j} + \sum_{\alpha} \hat{H}_{\alpha} .
\]

Here we have denoted the ground-state normal ordered Hamiltonian by : \( \hat{H} ; \), \( \hat{H}_{\beta, \tilde{q}_j} \) is the one-pseudofermion Hamiltonian associated with excited-state \( \beta \) pseudofermion or \( \beta \) pseudofermion hole of canonical momentum \( \tilde{q}_j \), and \( \hat{H}_{\alpha} \) refers to the unbound \( \eta \)-spinons \( (\alpha = \eta) \) and unbound spinons \( (\alpha = s) \) whose \( SU(2) \) internal degrees of freedom are scatter-less.
For each many-pseudofermion PS virtual state reached under the first step of the transition from the ground state to the excited energy eigenstate, the number of Hamiltonian terms, $H_{\beta,j}$, equals that of one-pseudofermion states of the virtual state. This number reads,

$$N_{\alpha} + N_{s,\ell} + \sum_{\beta \neq \ell, s} \theta(|\delta N_{\beta}|) N_{\alpha_{\beta}}. \quad (47)$$

Here $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x = 0$. The numbers $N_{\alpha} = N_{c} + N_{c}^{h} = N_{a}, N_{s,\ell} = N_{s_{1}} + N_{s_{1}^{h}}$, and $N_{\alpha_{\beta}} = N_{\beta} + N_{\beta}^{h}$ refer to the virtual state and corresponding excited energy eigenstate under consideration. The pseudofermion-hole number, $N_{\beta}^{h}$, is provided in Eq. (A9) of Appendix A.

The second scatter-less process generates the “in” state. The one-pseudofermion states belonging to the many-pseudofermion “in” state are the “in” asymptote states of the pseudofermion scattering theory. The unitary operator $S_{\beta}^{0}$, whose conjugate operator $S_{\beta}^{0 \dagger}$ generates the virtual-state “in”-state transition is of the form,

$$\hat{S}^{0}_{\beta} = \prod_{j} s^{0}_{\beta} ; \quad f^{\dagger}_{q}, Q_{\beta}^{0} / N_{\alpha_{\beta}} = (\hat{S}^{0}_{\beta})^{\dagger} f^{\dagger}_{q}, \hat{S}^{0}_{\beta}, \quad (48)$$

Here $f^{\dagger}_{q}, \beta$ and $f_{q}, \beta$ are the $\beta$ pseudoparticle operators, Eqs. (11) and (22), and the $Q_{\beta}^{0} / N_{\alpha}$ shift Hermitian operator $\hat{G}_{\beta}$ is within a continuum bare-momentum representation given by,

$$\hat{G}_{\beta} = -i \sum_{q' \neq q} \left[ \frac{\partial}{\partial q'} f^{\dagger}_{q', \beta} \right] f_{q', \beta}. \quad (49)$$

For simplicity and without loss in generality, consider that the many-pseudofermion “in” state of the virtual state is a Bethe state. The virtual state can then be written as a $\beta$ pseudoparticle Slater determinant,

$$\vert \text{virt}, l_{r}, j_{\text{ra}}, u \rangle = \prod_{\beta} \prod_{j=1}^{N_{\alpha_{\beta}}} f_{q}, \beta \vert 0_{\text{elec}} \rangle, \quad (50)$$

where alike in Section II $\vert 0_{\text{elec}} \rangle$ stands for the electron vacuum. That vacuum remains invariant under the application of the unitary operator $\left(\hat{S}^{0}_{\beta}\right)^{\dagger}$. Combination of that property with the operator $f^{\dagger}_{q}, \beta$ transformation law under that unitary operator, Eq. (48), one finds that,

$$\vert \text{in}, l_{r}, j_{\text{ra}}, u \rangle = \prod_{\beta} \left(\hat{S}^{0}_{\beta}\right)^{\dagger} \vert \text{virt}, l_{r}, j_{\text{ra}}, u \rangle = \prod_{\beta} \prod_{j=1}^{N_{\alpha_{\beta}}} f^{\dagger}_{q}, Q_{\beta}^{0} / N_{\alpha_{\beta}} \vert 0_{\text{elec}} \rangle. \quad (51)$$

Hence application of the unitary operator $\left(\hat{S}^{0}\right)^{\dagger} = \prod_{\beta} \left(\hat{S}^{0}_{\beta}\right)^{\dagger}$ onto the many-pseudofermion virtual state gives rise to the “in”-state. Under that process, the virtual state one-pseudofermion states discrete bare-momentum values $q_{j}$ are shifted to the excited-state discrete bare-momentum value $q_{j} + Q_{\beta}^{0} / N_{\alpha}$, where $Q_{\beta}^{0}$ is given in Eq. (35). (Note that $Q_{\beta}^{0}$ may vanish, as given in that equation.)

This second step may add a finite momentum to that given in Eq. (48) such that the total excitation momentum reads,

$$\delta P_{\ell_{1}, \text{ra}} = \delta P_{\ell_{1}, \text{ra}}^{0} + \epsilon 2kF \frac{Q_{\ell_{1}}^{0}}{\pi} + i' kF \frac{Q_{\ell_{1}}^{0}}{\pi}, \quad \ell, \ell' = \pm 1 \quad (52)$$

Finally, the third step consists of a set of two-pseudofermion scattering events. It corresponds to the “in”-state “out”-state transition, where the latter state is the FS excited energy eigenstate under consideration. The generator of that transition is the conjugate of the following unitary operator,

$$\hat{S}^{\Phi} = \prod_{\beta} \hat{S}^{\Phi}_{\beta}, \quad (53)$$
where $\hat{S}_\beta^\Phi$ is the $\beta$ pseudoparticle - $\beta$ pseudofermion unitary operator, Eq. (27).

The one-pseudofermion states belonging to the many-pseudofermion “out” state are the “out” asymptote pseudofermion scattering states. Application of the unitary operator $(\hat{S}_\beta^\Phi)\dagger = \prod_\beta (\hat{S}_\beta^\Phi)^\dagger$, Eq. (27), onto the many-pseudofermion “in” state, shifts its one-pseudofermion states discrete bare-momentum values $q_j + Q_\beta^\Phi/N_a$ to the “out”-state discrete canonical-momentum values $q_j + Q_\beta(q_j)/N_a$. It follows that the generator of the overall virtual-state - “out”-state transition is the unitary operator $(\hat{S}_T)\dagger$ whose conjugate reads,

$$
\hat{S}_T = \hat{S}_\beta^\Phi \hat{S}_\beta^0 = \prod_\beta \hat{S}_\beta; \quad f_{q_j + Q_\beta(q_j)/N_a;\beta}^\dagger = (\hat{S}_\beta^\dagger)^{\dagger} f_{q_j;\beta}^{\dagger} \hat{S}_\beta,
$$

$$
\hat{S}_\beta = e^{\sum_{j=1}^{N_{a\beta}} f_{q_j + Q_\beta(q_j)/N_a;\beta}^\dagger f_{q_j;\beta}^\dagger} \hat{S}_\beta^\dagger = e^{\sum_{j=1}^{N_{a\beta}} f_{q_j - Q_\beta(q_j)/N_a;\beta} f_{q_j;\beta}}.
$$

Thus application of the unitary operator $(\hat{S}_T)\dagger = \prod_\beta (\hat{S}_\beta)\dagger$ onto the corresponding many-pseudofermion virtual state, shifts its one-pseudofermion states discrete bare-momentum values $q_j$ directly into the “out”-state discrete canonical-momentum values $q_j + Q_\beta(q_j)/N_a$.

$$
|\text{out}, l_r, l_{\eta_s}, u\rangle = \prod_\beta (\hat{S}_\beta)\dagger |\text{virt}, l_r, l_{\eta_s}, u\rangle = \prod_\beta \prod_{j=1}^{N_a} f_{q_j + Q_\beta(q_j);\beta}^\dagger |0_{\text{elec}}\rangle = \prod_\beta \prod_{j=1}^{N_a} f_{q_j;\beta}^\dagger |0_{\text{elec}}\rangle.
$$

The “in” state and “out” state are different representations of the same PS excited energy eigenstate. Specifically, they refer to the alternative $\beta$ pseudoparticle and $\beta$ pseudofermion representations of that state. Consistent, the canonical-momentum shift $Q_\beta^\Phi(q_j)/N_a$ does not contribute to the physical momentum, Eq. (52), and the “in” state and “out” state can be shown to differ by a mere overall phase factor,

$$
|\text{out}, l_r, l_{\eta_s}, u\rangle = S_T^\Phi |\text{in}, l_r, l_{\eta_s}, u\rangle,
$$

$$
S_T^\Phi = e^{i\delta_T}; \quad \delta_T = \sum_\beta \sum_{j=1}^{N_{a\beta}} Q_{\beta}^\Phi(q_j)/2.
$$

That the one-pseudofermion states of the many-pseudofermion “in” state and “out” state are the “in” and “out” asymptote pseudofermion scattering states, respectively, implies that the one-pseudofermion Hamiltonian $\hat{H}_{\beta,q_j}$ plays the role of the unperturbed Hamiltonian $\hat{H}_0$ of the spin-less one-particle nonrelativistic scattering theory [22]. It follows that the matrix elements between one-pseudofermion states of $\hat{S}_\beta^\Phi$ are diagonal. Therefore, these operators are fully defined by the set of their eigenvalues of such states. The same applies to the generator $\hat{S}_\beta^\Phi$, Eq. (53). The matrix elements of that generator between many-pseudofermion “in” states are also diagonal and thus it is fully defined by the set of its eigenvalues of such states.

The unitarity of the operators $S_T^\Phi$ and $\hat{S}_\beta$ implies that each of their eigenvalues has modulus one. It can thus be written as the exponent of a purely imaginary number. In the case of a $\beta$ one-pseudofermion state of bare momentum $q_j$, such eigenvalues are given by,

$$
S_{\beta}^\Phi(q_j) = e^{iQ_{\beta}^\Phi(q_j)} = \prod_{\beta'} \prod_{j'=1}^{N_{a\beta'}} S_{\beta,\beta'}(q_j, q_{j'}); \quad j = 1, \ldots, N_{a\beta},
$$

$$
S_{\beta}(q_j) = e^{iQ_{\beta}(q_j)} = e^{iQ_{\beta}^\Phi(q_j)} \prod_{\beta'} \prod_{j'=1}^{N_{a\beta'}} S_{\beta,\beta'}(q_{j'}, q_j); \quad j = 1, \ldots, N_{a\beta},
$$

respectively. Here $Q_{\beta}^\Phi(q_j)$ and $Q_{\beta}(q_j)$ are the functionals, Eq. (29) and Eq. (44), respectively. By use of the functional $Q_{\beta}^\Phi(q_j)$ expression, Eq. (29), we find that the quantity $S_{\beta,\beta'}(q_j, q_{j'})$ in Eq. (57) reads,

$$
S_{\beta,\beta'}(q_j, q_{j'}) = e^{\pm i2\pi \frac{\phi_{\beta,\beta'}(q_j, q_{j'})}{\delta N_{\beta'}(q_{j'})}},
$$

where the functions $\pi \phi_{\beta,\beta'}(q_j, q_{j'})$ are given in Eq. (32).
The effects produced by a ground-state - excited-state transition beyond the ground-state - virtual-state transition occupancy configuration changes are those of interest for the scattering theory. Except for the latter changes, the only effect of, under such a transition, moving the $\beta$ pseudofermion or $\beta'$ pseudofermion hole of virtual-state canonical-momentum $q_j = q_j$ once around the length $L$ lattice ring is that its wave function acquires the overall phase factor $S_{,\beta}(q_j)$, Eq. (57). This property is consistent with the lack of interaction energy terms in the spectrum of the $\beta$ pseudofermions, Eq. (56). The procedure of moving the $\beta$ pseudofermion or $\beta'$ pseudofermion hole once around the length $L$ lattice ring refers to a method to derive the corresponding dressed $S$ matrix. It is precisely the overall phase factor $S_{,\beta}(q_j)$, Eq. (57), acquired by its wave function.

The phase factor $S_{,\beta}(q_j)$, Eq. (55), in the wave function of the $\beta$ pseudofermion or $\beta'$ pseudofermion hole of bare-momentum $q_j$ results from an elementary two-pseudofermion zero-momentum forward-scattering event. Its scattering center is a $\beta'$ pseudofermion ($\delta N_{,\beta'}(q_j') = 1$) or $\beta'$ pseudofermion hole ($\delta N_{,\beta'}(q_j') = -1$) created under the ground-state - excited-state transition. The third step of that transition involves a well-defined set of elementary two-pseudofermion scattering events. Under those, all $\beta$ pseudofermions and $\beta'$ pseudofermion holes of bare-momentum $q_j + Q_{,\beta}^0/N_a$ of the “in” state play the role of scatterers. This leads to the overall scattering phase factor $S_{,\beta}^0(q_j)$ in their wave function, Eq. (57). On the other hand, the $\beta'$ pseudofermions or $\beta'$ pseudofermion holes of bare momentum $q_j + Q_{,\beta'}^0/N_a$ created under the ground-state - “in” state transition play the role of scattering centers. This is confirmed by noting that $S_{,\beta'}(q_j, q_j') = 1$ for $\delta N_{,\beta'}(q_j') = 0$. (The lattice elementary objects play as well the role of scatterers.) Thus, out of the scatterers whose number equals that of the one-pseudofermion states, Eq. (57), the scattering centers are only those whose bare-momentum distribution-function deviation is finite.

That the many-pseudofermion “in” and “out” states, which are a direct product of one-pseudofermion “in” and “out” asymptote pseudofermion scattering states, respectively, are PS excited energy eigenstates is behind the validity of the pseudofermion scattering theory. Indeed, the validity of any scattering theory requires that the excited states associated with the asymptotic one-particle scattering states have a well-defined energy. For the present quantum problem, this requirement is fulfilled provided that the excited states associated with the one-particle scattering states are model energy eigenstates. Such a requirement is obeyed by all the excited states associated with the one-particle scattering states of the pseudofermion scattering theory.

The following properties play an important role in the pseudofermion scattering theory:

1. The elementary two-pseudofermion scattering processes associated with the phase factors, Eq. (55), conserve the total energy and total momentum. This stems from the occurrence of an infinite number of conservation laws (41), which are associated with the model integrability (31) being explicit in the present pseudofermion scattering theory. As a result, its scatterers and scattering centers only undergo zero-momentum forward scattering.

2. That the elementary two-pseudofermion scattering processes are of zero-momentum forward-scattering type also implies that they conserve the individual “in” asymptote $\beta$ pseudofermion bare-momentum value $q_j + Q_{,\beta}^0/N_a$ and energy. (The additional canonical-momentum scattering phase-shift term, $Q_{,\beta}^0(q_j)/N_a$, does not contribute to the physical momentum, Eq. (52).)

3. These processes also conserve the $\beta$ branch, usually called channel in the scattering language (52).

4. The scattering amplitude does not connect quantum objects with different $\eta$ spin or spin. (All such objects have no internal degrees of freedom such as $\eta$-spin or spin.)

5. For each $\beta$ pseudofermion or $\beta'$ pseudofermion hole of virtual-state bare-momentum $q_j$, the dressed $S$ matrix associated with the ground-state - excited-state transition is simply the phase factor $S_{,\beta}(q_j)$, Eq. (57).

The one-$\beta$-pseudofermion or one-$\beta'$-pseudofermion hole phase factor $S_{,\beta}^0(q_j)$ of the present 1D quantum problem corresponds to the usual one-particle phase factor $s_l(E)$ of similar three-dimensional quantum problems. The latter depends on the energy $E$ and angular-momentum quantum numbers $l$ and $m$. (See, for example, Eq. (6.9) of Ref. (52).) In the present 1D case, the energy $E$ and the quantum numbers $l$ and $m$ are replaced by the bare-momentum $q_j$ in the phase factor $S_{,\beta}^0(q_j)$. The $\beta$ pseudofermion or $\beta'$ pseudofermion hole energy is uniquely defined by the absolute bare-momentum value $|q_j|$. In 1D the sign of $q_j$ corresponds to the three-dimensional angular-momentum quantum numbers. Another difference is that $s_l(E)$ is associated with a single scattering event. Here, $S_{,\beta}^0(q_j)$ results in general from several scattering events. Each of such events corresponds to a well defined factor $S_{,\beta'}(q_j, q_j')$, Eq. (55), in the $S_{,\beta}^0(q_j)$ expression, Eq. (57). There are as many of such factors as $\beta'$ pseudofermion and $\beta'$ pseudofermion hole scattering centers created under the transition to the virtual state and corresponding excited energy eigenstate under consideration. The factor 2 in the phase factor of Eq. (6.9) of Ref. (52) corresponds to the phase-shift definition of the standard nonrelativistic scattering theory for spin-less particles. We use in general here such a definition. As
discussed below in Section \[\text{IV.A}\] it introduces the overall scattering phase shift \(\delta_\beta^\delta(q_j) = Q_\beta^\delta(q_j)/2\) and overall phase shift \(\delta_\beta(q_j) = Q_\beta(q_j)/2\). However, if instead we insert a factor 1, we would have an overall scattering phase shift \(Q_\beta^\delta(q_j)\) and overall phase shift \(Q_\beta(q_j)\). (That is the phase-shift definition used in Refs. \[36\] and \[38\].)

The factorization of the BA bare \(S\) matrix for the original spin-1/2 electrons is associated with the so-called Yang-Baxter equation (YBE) \[39\]. On the other hand, the factorization of the \(\beta\) pseudofermion or \(\beta\) pseudofermion-hole dressed \(S\) matrix \(S_\beta(q_j)\), Eq. \[57\], in terms of the elementary two-pseudofermion \(S\) matrices \(S_{\beta,\beta}(q_j, q_j')\), Eq. \[68\], is \textit{commutative}. Such commutativity is stronger than the symmetry associated with the YBE. This results from the \(c, c\), \(\eta\nu\), and \(\nu\nu\) pseudofermions and \(c\) and \(s\) pseudofermion holes, which are the scatterers and scattering centers, having no internal degrees of freedom such as \(\eta\)-spin or spin.

The BA bare \(S\) matrix refers to spin-1/2 electrons. Therefore its form reflects the spin-1/2 \(SU(2)\) symmetry of its scatterers and scattering centers. That of the \(\beta\) pseudofermions refers to neutral particles. This is in spite of the \(\eta\)-spinons and spinons carrying \(\eta\)-spin 1/2 and spin 1/2, respectively. On the other hand, the \(M^{\text{bo}}\) anti-bound \(\eta\)-spinons (and \(M^{\text{bo}}\) bound spinons) have \(\eta\)-spin 1/2 (and spin 1/2) but are anti-bound (and bound) within composite neutral objects, which play the role of scatterers and scattering centers. On the other hand, the \(\eta\)-spin (and spin) internal \(SU(2)\) degrees of freedom of the \(\eta\)-spin-1/2 unbound \(\eta\)-spinons and spin-1/2 unbound spinons have a scattering-less character.

Indeed, as mentioned above, the momentum values of the \(\eta\)-spin-1/2 unbound \(\eta\)-spinons and spin-1/2 unbound spinons remain unchanged under the transition from an “in” state to the corresponding “out” state (excited energy eigenstate). This confirms that as far as their \(SU(2)\) internal degrees of freedom is concerned such objects are not scatterers. Neither do the \(\eta\) pseudofermions scatter off on them, so that they are not scattering centers. That is consistent with the unbound \(\eta\)-spinons (and unbound spinons) playing the passive role of unoccupied sites of both the \(\eta\)-spin (and spin) effective lattice and the corresponding \(\eta\nu\) (and \(\nu\nu\)) effective lattices.

The \(\beta = c, \alpha\nu\) effective lattice elementary-object motion can alternatively be described in terms of \(\beta\) pseudofermions or \(\beta\) pseudofermion holes. Both the \(\beta\) pseudofermions and \(\beta\) pseudofermion holes are scatterers of the pseudofermion scattering theory. When created under transitions to excited states, they play as well the role of scattering centers. Under transitions to excited states, only \(c\) pseudofermion holes and \(s\) pseudofermion holes are created. They play a major role in the transport of charge and spin, respectively \[28\]. According to Eq. \[\text{A9}\] of Appendix \(\text{A}\), the numbers of \(c\) pseudofermion holes and \(s\) pseudofermion holes can be written as,

\[
N^c_\eta = M^\text{un}_\eta = M^\text{un} + \sum_{\nu=1}^\infty 2\nu N^\nu \eta ; \quad N^s_\nu = M^\text{un} + \sum_{\nu=1}^\infty 2(\nu - 1)N^\nu \nu ,
\]

(59)

respectively. The studies of Ref. \[28\] reveal that, out of the \(N^c_\eta\) \(c\) pseudofermion holes (and \(N^s_\nu\) \(s\) pseudofermion holes), Eq. \[59\], the \(M^\text{un}\) unbound +1/2 \(\eta\)-spinons (and \(M_s\) unbound +1/2 spinons) use \(M^\text{un}\) \(c\) pseudofermion holes (and \(M_s\) \(s\) pseudofermion holes) as hosts to couple to \(\eta\)-spin (and spin) proves. As justified in that reference, they are \textit{host shadows} of the corresponding \(M^\text{un}\) unbound +1/2 \(\eta\)-spinons (and \(M_s\) unbound +1/2 spinons.) Indeed, they carry their \(x_3\)-component \(\eta\)-spin (and spin) \(U(1)\) currents. Such unbound \(\eta\)-spinon and unbound spinon degrees of freedom are associated with the \(\eta\)-spin and spin \(U(1)\) symmetry algebra state representations within the \(\eta\)-spin and spin \(SU(2)\) symmetry algebras, respectively.

On the other hand, the remaining \(\sum_{\nu=1}^\infty 2\nu N^\nu \) \(c\) pseudofermion holes (and \(\sum_{\nu=1}^\infty 2(\nu - 1)N^\nu \) \(s\) pseudofermion holes), out of the \(N^c_\eta\) \(c\) pseudofermion holes (and \(N^s_\nu\) \(s\) pseudofermion holes), Eq. \[59\], are found in Ref. \[28\] to be neutral shadows of the excited-states \(N^\nu \eta\nu\) \(\eta\nu\) pseudofermions (and \(N^\nu \nu\nu\) pseudofermions with \(\nu > 1\) spinon pairs.) By neutral shadows it is meant that the virtual elementary \(\eta\)-spin (and spin) currents carried by the remaining \(\sum_{\nu=1}^\infty 2\nu N^\nu \) \(c\) pseudofermion holes (and \(\sum_{\nu=1}^\infty 2(\nu - 1)N^\nu \) \(s\) pseudofermion holes) are exactly cancelled by those of the corresponding \(N^\nu \eta\nu\) \(\eta\nu\) pseudofermions (and \(N^\nu \nu\nu\) pseudofermions with \(\nu > 1\) spin on pairs.) That is consistent with a \(2\nu\eta\)-spin pseudofermion (and \(2\nu\)-spinon \(\nu\nu\) pseudofermion) being a \(\eta\)-spin-neutral (and spin-neutral) composite object.

The dressed \(S\) matrix commutative factorization is that consistent with the form of the \(\beta\) pseudofermion canonical-momentum-occupancy configurations that describe the PS excited energy eigenstates. Such canonical-momentum occupancy configurations are superpositions of local occupancy configurations with the same number of \(\beta\) pseudofermions. Those elementary objects have different positions in each configuration. Hence the number of \(\beta\) pseudofermions belonging to \(\beta\) branches with finite occupancy in the virtual state is the same for all occupancy configurations. The relative position of these quantum objects is different in each such a configuration. Consider that under a specific ground-state - excited-energy-eigenstate transition a \(\beta\) pseudofermion or \(\beta\) pseudofermion hole moves once around the lattice ring. It then scatters the same \(\beta' = c, \alpha\nu\) pseudofermion or \(\beta' = c, s\) pseudofermion-hole scattering centers, but in different order for different occupancy configurations. However, it is required that the phase factor \(e^{i\delta_\beta^\delta}\) acquired by the \(\beta\) pseudofermion or \(\beta\) pseudofermion hole should be the same, independently of that order. This is consistent with the commutativity of the dressed \(S\)-matrix factors \(S_{\delta,\beta}(q_j, q_j')\) in the overall dressed \(S\) matrix.
FIG. 1: The elementary two-pseudofermion phase shift $\pi \Phi_{c,c}(q, q')$ in units of $\pi$ as a function of $q$ and $q'$ for $n = 0.59$, $m = 0$, and (a) $U/t \rightarrow 0$, (b) $U/t = 0.3$, (c) $U/t = 4.9$, and (d) $U/t = 100$. The scatterer and scattering-center bare-momentum values $q$ and $q'$, respectively, correspond to the right and left axis of the figures, respectively.

$S_\beta(q_j)$, Eq. (57). Such a commutativity follows from the elementary $S$ matrices $S_{\beta,\beta'}(q_j, q_{j'})$, Eq. (58), being simple phase factors, rather than matrices of dimension larger than one.

IV. THE PSEUDOFERMION PHASE SHIFTS

In this section we study the $\beta$ pseudofermion phase shifts associated with the dressed $S$ matrix introduced above in Section III. The effects of the pseudofermion transformation laws under the electron - rotated-electron unitary transformation on such phase shifts and corresponding scattering properties and the relation of those to the PDT are issues also addressed in this section.
A. Phase-shift definition

As above, our analysis refers to periodic boundary conditions and \( N_a \gg 1 \). The \( c \) effective lattice considered in Ref. \[28\] equals the original lattice. On the other hand, the spacing,

\[
a_{\alpha\nu} = \frac{L}{N_{a_{\alpha\nu}}} = \frac{N_a}{N_{a_{\alpha\nu}}} a, \quad N_{a_{\alpha\nu}} \geq 1,
\]

of the remaining \( \beta = \alpha\nu \) effective lattices also considered in that reference is for \( n \neq 1 \) and \( m \neq 0 \) larger than that of the original lattice. According to its definition, Eq. \[60\], the corresponding length, \( L = N_a a = N_{a_{\beta}} a_{\beta} \), equals that the latter lattice. (In general we use in this paper units of lattice constant \( a \), for which \( N_a = L \).)

Depending on the asymptote coordinate choice, there are two possible definitions of the \( \beta \) pseudofermion or \( \beta \) pseudofermion hole phase shifts. Both are associated with the dressed \( S \) matrix \( S_{\beta}(q) \), Eq. \[57\]. Indeed, the uniquely defined quantity is that dressed \( S \) matrix. The two choices of asymptote coordinates for the \( \beta \) pseudofermion or \( \beta \) pseudofermion hole correspond to \( x \in [-L/2, +L/2] \) and \( x \in [0, +L] \), respectively.

If when moving around the lattice ring, the \( \beta \) pseudofermion or \( \beta \) pseudofermion hole departs from the point \( x = -L/2 \) and arrives to \( x = L/2 \), one finds that,

\[
\lim_{x \to L/2} \bar{q} x = q L/2 + Q_{\beta}/2 + Q^\Phi_{\beta}(q)/2 = q L/2 + \delta_{\beta}(q),
\]

where

\[
\delta_{\beta}(q) = Q_{\beta}(q)/2 = Q^0_{\beta}/2 + Q^\Phi_{\beta}(q)/2.
\]

For this asymptote coordinate choice, \( \delta_{\beta}(q) \) is the overall \( \beta \) pseudofermion or \( \beta \) pseudofermion-hole phase shift. Moreover, from analysis of Eqs. \[29\] and \[42\] it follows that \( \pi \Phi_{\beta',\beta''}(q_j, q_j') \) is an elementary two-pseudofermion phase shift. This phase-shift definition corresponds to that used in standard quantum non-relativistic scattering theory \[52\]. It is such that the dressed \( S \) matrix \( S_{\beta}(q_j) \), Eq. \[57\], can be expressed as,

\[
S_{\beta}(q_j) = e^{i2\delta_{\beta}(q_j)}, \quad j = 1, ..., N_{a_{\beta}}.
\]

The factor 2 appearing here in the elementary argument corresponds to the usual form of the \( S \) matrix for that theory. This phase-shift definition is consistent with an exact Theorem due to Fumi \[53\]. Within that definition, Eq. \[62\], \( Q^0_{\beta}/2 = 0, \pi/2 \) corresponds to the scatter-less term \( -l \pi/2 \) of the three-dimensional partial-wave problem of orbital angular momentum \( I \) \[52, 53\]. Although the orbital angular momentum vanishes in 1D, the scatter-less phase shift, Eq. \[63\], plays a similar role.

In this paper we follow the definition of the standard quantum non-relativistic scattering theory and choose the overall \( \beta \) pseudofermion phase shift definition, \( Q_{\beta}(q)/2 \), associated with Eq. \[61\]. On the other hand, if when moving around the lattice ring the \( \beta \) pseudofermion (or \( \beta \) pseudofermion hole) departs from the point \( x = 0 \) and arrives to \( x = L \), one finds that,

\[
\lim_{x \to L} \bar{q} x = q L + Q^0_{\beta} + Q^\Phi_{\beta}(q) = q L + Q_{\beta}(q),
\]

where \( q \) refers to the ground state. For this asymptote coordinate choice, \( Q_{\beta}(q) \) is the overall \( \beta \) pseudofermion or \( \beta \) pseudofermion hole phase shift and \( 2\pi \Phi_{\beta',\beta''}(q_j, q_j') \) is an elementary two-pseudofermion phase shift.

The overall pseudofermion phase-shift choice, \( Q_{\beta}(q) = Q^0_{\beta} + Q^\Phi_{\beta}(q) \), is associated with the asymptote condition, Eq. \[64\]. It corresponds to a generalization of the conventional phase-shift definitions previously used in some of the BA literature. Examples are the holon-spinon scattering theories of Refs. \[37, 38\] and \[36\], respectively. (All the discussions and analysis presented below in this paper also apply to the phase-shift definition, \( Q_{\beta}(q) = Q^0_{\beta} + Q^\Phi_{\beta}(q) \), provided that the \( \beta \) phase shifts, \( \delta_{\beta}(q) = Q_{\beta}(q)/2 \), are multiplied by two.)

B. The two-pseudofermion phase shifts: Bare-momentum dependence and Levinson’s Theorem

The bare-momentum two-pseudofermion phase shifts, \( \pi \Phi_{\beta',\beta''}(q, q') \), are related to the rapidity two-pseudofermion phase shifts, \( \pi \Phi_{\beta',\beta''}(r, r') \), by Eq. \[32\]. The latter phase shifts are defined by the integral equations, Eqs. \[131\]-\[134\] of Appendix \[C\] Corresponding simplified \( m = 0 \) two-pseudofermion phase-shift equations, closed-form expressions valid at densities \( n = 1 \) and \( m = 0 \), and \( m = 0 \) analytical expressions valid for \( U/t \to 0 \) are provided in Appendix \[C\]
FIG. 2: The elementary two-pseudofermion phase shift $\pi \Phi_{c,\beta}(q, q')$ in units of $\pi$ as a function of $q$ and $q'$ for the same densities and $U/t$ values as in Fig. 1. Alike in that figure, the scatterer and scattering-center bare-momentum values $q$ and $q'$, respectively, correspond to the right and left axis, respectively.

The two-pseudofermion phase shifts, $\pi \Phi_{\beta,\beta'}(q, q')$, control the expression of the $f$ functions, Eq. (A24) of Appendix A. Moreover, analysis of the expressions, Eqs. (A21) and (A22) of that Appendix, reveals that also the $\beta$ energy dispersions involve such phase shifts. Specifically, according to Eq. (A21) of Appendix A, the energy associated with the creation of one $\beta$ pseudofermion has a contribution from phase shifts felt by all ground-state $c$ Fermi sea $c$ pseudofermions, as a result of its creation. Consistent, the $\int_{-Q}^{Q} dk$ integration in such expressions may be rewritten in terms of a corresponding bare-momentum integration, $\int_{-q_{F}}^{q_{F}} dq' = \int_{-2k_{F}}^{2k_{F}} dq'$. Alternatively, it may be written as a sum, $\sum_{j'=1}^{N_{a}} N_{c}^{0}(q_{j'})$. Here $N_{c}^{0}(q_{j'}) = 0$ and $N_{c}^{0}(q_{j'}) = 1$ for $|q_{j'}| > 2k_{F}$ and $|q_{j'}| < 2k_{F}$, respectively. Under such a variable transformation, $\Phi_{\alpha,\beta}(q, q) = \Phi_{c,\beta}(q_{j'}, q_{j})$, Eq. (32). In it the $c$ pseudofermion and the $\beta$ pseudofermion are the scatterer and the scattering center, respectively. (Here we have used the notation $\Phi_{c,\beta}(q_{j'}, q_{j})$ rather than our usual notation $\Phi_{c,\beta}(q_{j}, q_{j'})$. That choice is that consistent with the argument of the corresponding energy dispersion $\epsilon_{\beta}^{0}(q_{j})$ in Eq. (A21) of Appendix A being denoted by $q_{j}$.)

In Figs. 4-6 some of the two-pseudofermion phase shifts, Eq. (32), in the expressions of the overall phase shifts $Q_{\beta}(q_{j})/2$, Eqs. (29) and (34), associated with the excited states that mostly contribute to the PDT one- and two-electron spectral weights are plotted. They are plotted as a function of the scatterer and scattering-center bare momenta $q$ and $q'$, respectively, for electronic density $n = 0.59$, spin density $m \to 0$, and the $U/t$ values $U/t \to 0$ and
FIG. 3: The elementary two-pseudofermion phase shift \( \pi \Phi_{s1,c}(q, q') \) in units of \( \pi \) as a function of \( q \) and \( q' \) for the same densities and \( U/t \) values as in Fig. 1. Alike in that figure, the scatterer and scattering-center bare-momentum values \( q \) and \( q' \), respectively, correspond to the right and left axis, respectively.

\( U/t = 0.3, 4.9, 100 \). The analytical expressions of the \( m = 0 \) two-pseudofermion phase shifts plotted in Figs. 10 for \( U/t \to 0 \) are given in Eq. (C25)-(C30) of Appendix C. The electronic density \( n = 0.59 \) and the \( U/t = 4.9 \) value are those used in Refs. [15, 32, 33] for the description of the TCNQ photoemission dispersions observed in the quasi-1D organic compound TTF-TCNQ (tetrathiafulvalene tetracyanoquinodimethane).

The two-pseudofermion phase shift \( \pi \Phi_{c,n1}(q, q') \), plotted in Fig. 5 in units of \( \pi \), has values in the domain \( \Phi_{c,n1}(q, q') \in [-1, 1] \). Within the standard quantum non-relativistic scattering theory phase shift definition given in Eq. (61), this corresponds to the phase-shift range \( \pi \Phi_{c,n1}(q, q') \in [-\pi, \pi] \). Note that the width of this domain is \( 2\pi \), whereas the definition of Eq. (64) would lead to a domain of width \( 4\pi \).

The phase shifts of Refs. [53, 54] correspond to a particular case of the general elementary two-pseudofermion phase shifts considered here. If one considers densities in the ranges \( n \in [0, 1] \) and \( m \in [0, n] \) and the \( c \) and \( s1 \) excitation branches two-component PSs spanned by energy eigenstates such that \( N_\beta = 0 \) for the \( \beta \neq c, s1 \) branches and \( M_\alpha^{\text{un}} = -1/2 = 0 \) for \( \alpha = \eta, s \), the general integral equations, Eqs. (B1)-(B15) of Appendix B, reduce to the integral equations, Eqs. (20)-(23) of Ref. [53] and Eqs. (32)-(38) of Ref. [54].

As shown in the following, there are no \( \beta \) pseudofermion bound states. This ensures that the corresponding phase shifts \( \pi \Phi_{\beta, \beta'}(q, q') \), which are associated with the elementary two-pseudofermion scattering events, obey Levinson's Theorem [56]. Such a theorem states that when in the reference frame of the scattering center the momentum of the scatterer tends to zero, the phase shift is given by \( \pi N_b \). Here \( N_b \) is the number of bound states. In that frame, the
phase shift $\pi \Phi_{\beta,\beta'}(q, q')$ reads $\pi \Phi_{\beta,\beta'}(q - q', 0)$.

The two requirements, of absence of $\beta$ pseudofermion bound states and validity of Levinson’s Theorem, are fulfilled provided that the two-pseudofermion phase shifts of the present theory obey the following equation,

$$\lim_{q - q' \to 0} \pi \Phi_{\beta,\beta'}(q - q', 0) = 0.$$  

(65)

To check whether this equation is obeyed, after a straightforward algebra involving the integral equations, Eqs. (B1)-(B15) of Appendix B, we have found that $\pi \Phi_{\beta,\beta'}(r, r') = -\pi \Phi_{\beta,\beta'}(-r, -r')$. This result combined with the use of Eq. (32) and the odd character of the ground-state rapidity functions, $\Lambda^0_{\beta}(q) = -\Lambda^0_{\beta}(-q)$, then leads to,

$$\pi \Phi_{\beta,\beta'}(q, q') = -\pi \Phi_{\beta,\beta'}(-q, -q').$$

(66)

This latter symmetry implies that $\pi \Phi_{\beta,\beta'}(q - q', 0)$ is an odd function of $q - q'$. This confirms both the validity of Levinson’s Theorem, Eq. (65), and the absence of $\beta$ pseudofermion bound states.
FIG. 5: The elementary two-pseudofermion phase shift \( \pi \Phi_{\beta,\eta}(q, q') \) in units of \( \pi \) as a function of \( q \) and \( q' \) for the same densities and \( U/t \) values as in Fig. 1. Alike in that figure, the scatterer and scattering-center bare-momentum values \( q \) and \( q' \), respectively, correspond to the right and left axis, respectively.

C. Effects of the pseudofermion transformation laws on the scattering properties

The \( \beta \) pseudofermions have the same transformation laws under the electron - rotated-electron unitary transformation as the corresponding \( \beta \) pseudoparticles. For the latter objects, such transformation laws and corresponding invariances are studied in Ref. [28]. Here the issue of the effects of the pseudofermion transformation laws under the electron - rotated-electron unitary transformation on their scattering properties is addressed. Our analysis focuses on the pseudofermions that are invariant under that transformation in the cases of PSs whose ground states have densities in the ranges \( n \in [0, 1] \) and \( m \in [0, n] \). The same problem is addressed in Appendix C for PSs of ground states with densities \( n = 1 \) and/or \( m = 0 \). As discussed below, the effects of the pseudofermion invariance under the electron - rotated-electron unitary transformation are found to be different for these two PS types.

The \( \beta \) pseudofermions are not in general invariant under the electron - rotated-electron unitary transformation. The exception is as the canonical momentum \( \vec{q} \) of the composite \( \beta = \alpha \nu \) pseudofermions approaches its limiting values, \( \vec{q} \to \vec{q}(\pm q_{\alpha \nu}) \). Alike the corresponding \( \alpha \nu \) pseudoparticles [28], they are invariant under the electron - rotated-electron unitary transformation provided that the corresponding group velocity vanishes, \( v_{\alpha \nu}(\pm q_{\alpha \nu}) = 0 \).

In contrast to the band limiting momentum values of usual band particles and Fermi-liquid quasi-particles, the \( \alpha \nu \) band bare-momentum limiting values \( \pm q_{\alpha \nu} \), Eq. (A12) of Appendix A, of the composite \( \alpha \nu \) pseudoparticles can change due to shake-up effects. Those are caused by the ground-state - excited-state transitions. Specifically
\[\pm \delta q_{av} = \pm \pi \delta N_{a_\nu}/N_a\] so that such an exotic behavior occurs when the deviation \(\delta N_{a_\nu}\), Eq. \ref{eq:deltaN}, generated by the ground-state - excited-state transition is finite.

Interestingly, it is shown in Appendix \[B\] that the functional \(\tilde{q} = \tilde{q}(q)\), Eq. \ref{eq:tildeq}, is such that the ground-state limiting \(\alpha\nu\) band canonical-momentum values remain those of the ground state, \(\bar{q}\), state limiting ground-state - excited-state transition is finite. Hence the limiting \(\alpha\nu\) band canonical-momentum values remain those of the ground state, \(\bar{q}\), as \(v_{av}(\pm q_{av}) = 0\). Here \(q_{av}\) is given in Eq. \ref{eq:qav} of Appendix \[A\]. The condition \(v_{av}(\pm q_{av}) = 0\) is met for all \(\alpha\nu \neq s1\) branches in all \(a\nu\) bands. For the \(s1\) branch it is met only for \(a\nu\) whose ground state has a small or vanishing number, \(N_{s1}\), of \(s1\) pseudofermions. Hence, provided that \(N_{s1}/N_a \to 0\) as \(N_a \to \infty\).

At the limiting momentum values, \(\bar{q}(\pm q_{av}) = \pm q_{av}\), the energies of the \(\alpha\nu\) pseudofermions equal those of the corresponding \(\alpha\nu\) pseudoparticles and read \[28\] \[
\varepsilon_{\nu\nu}(\pm q_{av}) = \nu(\varepsilon_{n,-1/2} + \varepsilon_{n,1/2} + 1 - \delta_{n,1})2\nu|\mu| + \delta_{n,1}2\nu\mu^0; \quad \varepsilon_{\nu\nu}(\pm q_{av}) = 0,
\varepsilon_{\nu\nu}(\pm q_{av}) = \nu(\varepsilon_{s,-1/2} + \varepsilon_{s,1/2}) = 2\nu\mu B |H|; \quad \varepsilon_{\nu\nu}(\pm q_{av}) = 0.
\] (67)

The energy scale \(2\mu^0\) is the \(n = 1\) Mott-Hubbard gap, whose limiting behaviors are given in Eq. \ref{eq:mu0} of Appendix \[B\]. For the densities and \(\nu\) dependences of the chemical potential \(\mu\) and magnetic energy scale \(2\nu\mu B H\) appearing here, see Eqs. \ref{eq:mu}-\ref{eq:H} of that Appendix. Furthermore, \(\varepsilon_{n,\pm1/2}\) and \(\varepsilon_{s,\pm1/2}\) denote in Eq. \ref{eq:energy} the following energies of the unbound \(\eta\)-spinons and spinons \[28\] 
\[
\varepsilon_{n,\pm1/2} = 2|\mu|; \quad \varepsilon_{s,\pm1/2} = 2\mu B |H|; \quad \varepsilon_{s,\mp1/2} = 0, \quad \sgn\{1-n\} = \mp 1,
\] (68)
respectively.

As found in Ref. \[28\], \(\varepsilon_{\nu\nu}(\pm q_{av}) = 0\) (and \(\varepsilon_{\nu\nu}(\pm q_{av}) = 0\)) are in Eq. \ref{eq:energy} the \(\nu\nu\) pseudofermion anti-binding energy (and \(\nu\nu\) pseudofermion binding energy.) Indeed, the \(2\nu = 2, 4, ..., \infty\) \(\eta\)-spinons (and spinons) that are part of one composite \(\nu\nu\) pseudofermion (and one composite \(\nu\nu\) pseudofermion) lose their anti-bound (and bound) character as \(\bar{q} \to \pm q_{av}\) (and as \(\bar{q} \to \pm q_{av}\)). That property plus the vanishing of such composite elementary objects group velocity are directly related to the invariance of such elementary objects under the electron - rotated-electron unitary transformation.

**Active scatterers** are those whose overall phase shifts generated by the ground-state - excited-state transitions lead to a shift of the corresponding canonical-momentum values. Thus, the \(\alpha\nu\) pseudofermions of limiting canonical momentum \(\bar{q} = \pm q_{av}\) and vanishing velocity \(v_{av}(\pm q_{av}) = 0\) are not active scatterers. Next we investigate the properties of such \(\alpha\nu\) pseudofermions as scattering centers. By use of the integral equations, Eqs. \ref{eq:beta} of Appendix \[B\] and Eqs. \ref{eq:beta1}, \ref{eq:beta2} of Ref. \[14\], we find after some algebra that for \(\alpha\nu' \neq s1\) branches the two-pseudofermion phase shift \(\pi \Phi_{\beta,\alpha,\nu'}(q, t q_{av})\) may be expressed as follows,

\[
\pi \Phi_{\beta,\alpha,\nu'}(q, t q_{av}) = \frac{i\pi}{2} \left[\delta_{\beta,\nu'} - \delta_{\beta,\alpha,\nu'}(-\delta_{\nu,\nu'} + \nu + \nu' - |\nu - \nu'|)\right] + \frac{i}{2} \sum_{t'=\pm1} \left[\pi \Phi_{\beta,\nu}(q, t' k_F) - \delta_{\nu,\nu'} 2\pi \Phi_{\beta,\nu}(q, t' k_F)\right], \quad t = \pm 1.
\] (69)

For \(\alpha\nu' = s1\), this expression is valid only in the limit of vanishing velocity, \(v_{s1}(\pm q_{s1}) = 0\), within which \(2\pi \Phi_{\beta,s1}(q, t q_{av}) = \mp \pi k_F |H|\) is felt by the \(\beta\) pseudofermion or \(\beta\) pseudofermion hole active scatterer whose \(\beta\) band has finite pseudofermion occupancy for the excited state under consideration. \(\nu = \alpha\nu'\), then the values of \(q\) are such that \(|q| < q_{av}\). Otherwise they can have any value and thus correspond to any excited-state active \(\beta\) scatterer.

The form of the two-pseudofermion phase-shift expression, Eq. \ref{eq:phi}, reveals that, except for the constant phase-shift terms, creation of one \(\eta\nu'\) pseudofermion at canonical momentum \(q_{\nu'} = t(\pi - 2k_F)\) is felt by the \(\beta\) pseudofermion or \(\beta\) pseudofermion-hole active scatterer as a \(c\) pseudofermion Fermi-points current excitation. Such an excitation is associated with a shift, \(\pi t/N_a\), of both \(c\) bare-momentum Fermi points. Thus, such a scatterer effectively feels it is scattered by \(c\) Fermi-point current shifts, rather than by the \(\eta\nu'\) pseudofermion created at canonical momentum \(q_{\nu'} = t(\pi - 2k_F)\) under the transition to the excited state.

Similarly, Eq. \ref{eq:phi} demonstrates that, again, except for the constant phase-shift terms, creation of one \(s\nu' \neq s1\) pseudofermion at canonical momentum \(q_{\nu'} = t(k_F - k_F)\) is felt by a \(\beta\) pseudofermion or \(\beta\) pseudofermion-hole active scatterer as a \(c\) and \(s1\) pseudofermion Fermi-points current excitation. It corresponds to a shift, \(\pi t/N_a\), of both \(c\) bare-momentum Fermi points and a shift, \(\mp 2\pi/N_a\), of both \(s1\) bare-momentum Fermi points. Such a scatterer...
effectively feels it is scattered by $c$ and $s1$ Fermi-point current shifts, rather than than the $sv' \neq s1$ pseudofermion created at canonical momentum $q_B' = i [kF_1 - kF_2]$. The same applies to creation of one $s1$ pseudofermion at canonical momentum $q_B = 2kF$ as $kF_1 \rightarrow 2kF$, $kF_2 \rightarrow 0$, and thus $v_{s1}(i q_B) = 0$.

Active scattering centers are those that contribute to the scattering phase shift, Eq. (29). In spite of the exotic properties reported above, a vanishing-velocity $\alpha\nu$ pseudofermion created under a transition to an excited state at a limiting canonical momentum is an active scattering center. On the other hand, small-momentum and low-energy $c$ and $s1$ pseudofermion particle-hole processes in the vicinity of the corresponding $\beta = c, s1$ Fermi points, called elementary processes (C) below in Section IV D do not generate active scattering centers. Within the latter processes, the phase shifts generated by the $\beta = c$, $s1$ pseudofermion particle-like excitations exactly cancel those produced by the $\beta = c$, $s1$ pseudofermion hole-like excitations.

Above the effects of the pseudofermion transformation laws under the electron - rotated-electron unitary transformation were studied for PSs of $S_\alpha > 0$ ground states. Such effects are of a different type in the case of a $\alpha\nu \neq s1$ pseudofermion created onto a $S_\alpha = 0$ ground state. It is a simple exercise to show that such an elementary object energy obeys the equality, Eq. (0). Consistent with its invariance under the electron - rotated-electron unitary transformation, for $u > 0$ creation of one $\eta\nu$ pseudofermion (and one $sv' \neq s1$ pseudofermion) onto a $S_\alpha = 0$ (and $S_\nu = 0$) ground state leads to a change $\nu = 1, 2, ...$ in the number of lattice sites doubly occupied both by electrons and rotated electrons (and singly occupied both by spin-down electrons and spin-down rotated electrons). The $2\nu = 2, 4, 6, ...$ neutrons (and $2\nu' = 2, 4, 6, ...$ spindons) of that vanishing-velocity $\eta\nu$ pseudofermion (and vanishing-velocity $sv'$ pseudofermion) are not energetically anti-bound (and bound). In spite of their energetic non-anti-binding (and unbinding) character, such $2\nu$ neutrons (and $2\nu'$ spinons) remain in a collective $\eta$-spin-singlet (and spin-singlet) configuration.

The corresponding $\alpha\nu \neq s1$ momentum band does not exist for a $S_\alpha > 0$ ground state. Hence one has that $Q_{\nu\nu}(0)/N_\alpha = Q_{\alpha\nu}(0)/N_\nu$ for an excited state whose generation from that state involves creation of a single $\alpha\nu \neq s1$ pseudofermion. Moreover, the invariance under the electron - rotated-electron transformation of that $\alpha\nu \neq s1$ pseudofermion implies that it is not an active scatterer. For the above excited state, a necessary condition for such an object not being an active scatterer is that it is not a scatterer. The $\alpha\nu \neq s1$ bare-momentum band of the corresponding “in” state has a single value at $q = \pm q_{\alpha\nu} = 0$. Thus it is required that the canonical-momentum band of the “out” state has also a single value at $\bar{q} = \pm q_{\alpha\nu} = 0$. This implies that $Q_{\nu\nu}(0)/N_\alpha = Q_{\alpha\nu}(0)/N_\nu = 0$, and thus that the $\alpha\nu \neq s1$ pseudoparticle remains invariant under the pseudoparticle - pseudofermion unitary transformation. This is why the corresponding $\alpha\nu \neq s1$ pseudofermion is not a scatterer.

On the one hand, the following two-pseudofermion phase shifts vanish for the PS excited states of a $S_\eta = 0$ (and $S_\nu = 0$) ground state: $\pi \Phi_{s1,\eta\nu}(q, 0) = \pi \Phi_{s1,\nu\nu}(q, 0') = 0$ (and $\pi \Phi_{c,\eta\nu'}(q, 0) = \pi \Phi_{c,\nu\nu'}(q, 0') = 0$ for $\nu' > 1$). On the other hand, both the effective phase shift $\pi \Phi_{c,\nu\nu'}(c, q')$ (and $\pi \Phi_{c,\nu\nu'}(c, q')$ for $\nu' > 1$, $c = 2, 4, 6, ...$) of Appendix C and the two-pseudofermion phase shift $\pi \Phi_{c,\nu\nu'}(c, q')$ (and $\pi \Phi_{s1,\nu\nu'}(c, q')$ for $\nu' > 1$) of a $c$ (and $s1$) scatterer with bare momentum $q$ originated from its collision with the $\eta\nu$ (and $sv' \neq s1$) pseudofermion scattering center, respectively, Eq. (0) of that Appendix, have an interesting property: In addition to the $c$ (and $s1$) scattering-center bare momentum $q'$ or scatterer bare-momentum $q$, the two-pseudofermion phase shifts $\pi \Phi_{c,\eta\nu}(c, q')$ (and $\pi \Phi_{c,\eta\nu}(c, q')$ (and $\pi \Phi_{s1,\nu\nu'}(c, q')$ for $\nu' > 1$) are a function of the set of $2\nu = 2, 4, 6, ...$ (and $2\nu' = 2, 4, 6, ...$) bare-momentum values $\{q_\nu\}$ of the corresponding $2\nu$ (and $2\nu' = 2, 4, 6, ...$) neutral-shadow $c$ (and $s1$) pseudofermion holes considered in Section III C.

The requirement that $Q_{\eta\nu}(0)/N_\alpha = 0$ is shown in Appendix C to impose a specific form to the corresponding two-pseudofermion phase shifts $\pi \Phi_{c,\alpha\beta'}(0, q')$. Since the $\alpha\nu \neq s1$ pseudofermion is not a scatterer, $\pi \Phi_{c,\alpha\beta'}(0, q')$ is an effective virtual phase shift with no requirement to obey Levinson’s Theorem. The consequences of the above symmetries physically appear within the interplay of the $\eta\nu$ (and $sv' \neq s1$) pseudofermion with its $2\nu = 2, 4, 6, ...$ (and $2\nu' - 1 = 2, 4, 6, ...$) neutral-shadow $c$ (and $s1$) pseudofermion holes. As discussed in Section III C, $2\nu = 2, 4, 6, ...$ (and $2\nu' - 1 = 2, 4, 6, ...$) neutral-shadow $c$ (and $s1$) pseudofermion holes emerge under a transition to an excited state for each $\eta\nu$ (and $sv' \neq s1$) pseudofermion created under it. The set of $2\nu = 2, 4, 6, ...$ (and $2\nu' - 1 = 2, 4, 6, ...$) virtual two-pseudofermion phase shifts of the $\eta\nu$ (and $sv' \neq s1$) pseudofermion due to its collisions with the corresponding $2\nu = 2, 4, 6, ...$ (and $2\nu' - 1 = 2, 4, 6, ...$) neutral-shadow $c$ (and $s1$) pseudofermion-hole scattering centers also exactly cancel each other, so that $Q_{\eta\nu}(0) = 0$ (and $Q_{sv'}(0) = 0$).

D. Role of the pseudofermion scattering phase shifts in the model dynamical and spectral properties

In this section we revisit some of the spectral weights of the PDT of Refs. 12–15. Our aim is to specify which pseudofermion microscopic processes considered in the studies of this paper contribute to and control such
spectral weights. Note though that the weights given in the following have been previously derived in Refs. 12 13.

An important property specific to the pseudofermion scattering theory is that the phase shift \( \delta_{\beta}\left(q_{j}\right) = Q_{\beta}\left(q_{j}\right)/2 = Q_{\beta}^{0}/2 + Q_{\beta}^{1}\left(q_{j}\right)/2 \), Eqs. (41) and (22), refers to all \( j = 1, ..., N_{a} \) \( \beta \)-band scatterers of the excited state. This includes both those created under the transition from the ground state to the excited state and the scatterers that pre-existed in the ground state. That phase-shift scattering term, \( Q_{\beta}^{0}\left(q_{j}\right)/2 \), is expressed in Eq. (29) in terms of a suitable superposition of two-pseudofermion phase shifts, \( \pi \Phi_{\beta,\beta'}\left(q_{j}, q_{j'}\right) \). Such a superposition is uniquely defined for each excited state. The non-scattering term, \( Q_{\beta}^{1}/2 \), in the phase shift \( \delta_{\beta}\left(q_{j}\right) = Q_{\beta}\left(q_{j}\right)/2 \), Eqs. (41) and (22), results from the BA quantum numbers shifts, Eq. (35).

That the phase shift \( \delta_{\beta}\left(q_{j}\right) = Q_{\beta}\left(q_{j}\right)/2 \) refers to all \( j = 1, ..., N_{a} \) \( \beta \)-band scatterers of the excited state is a property that does not hold for the holon and spinon scattering theories of Refs. 36–38. Their phase shifts refer only to the scatterers created under the transition from the ground state to the excited states. Those scatterers play as well the role of scattering centers. As discussed in the following, this prevents such theories of describing most of the elementary processes that control the model dynamical and spectral properties.

The \( N_{a} \) \( \beta \)-band scatterers are the \( N_{\beta} \) \( \beta \) pseudofermions and \( N_{\beta}^{b} \) \( \beta \) pseudofermion holes that populate the excited state. Here \( N_{a,\beta} = \left[N_{\beta} + N_{\beta}^{b}\right] \), Eq. (A8) of Appendix A. For ground states with densities in the ranges \( n \in [0, 1] \) and \( m \in [0, n] \) such numbers are for instance for the \( \beta = c, s \) 1 bands of the PS excited states given by \( N_{c} = \left[N + \delta N_{c}\right] \), \( N_{c}^{b} = \left[N_{c} - N - \delta N_{c}\right] \), \( N_{s} = \left[N_{s} + \delta N_{s}\right] \), and \( N_{s}^{b} = \left[N_{s} - N_{s} + \delta N_{s}^{b}\right] \) where \( \delta N_{c}/N_{c} \to 0 \), \( \delta N_{s}/N_{s} \to 0 \), and \( \delta N_{s}^{b}/N_{s} \to 0 \) as \( N_{a} \to \infty \).

Thus the pseudofermion scattering theory provides the phase shifts of a macroscopic number \( N_{a} \) of \( \beta \) pseudofermion holes that populate the excited state. Here \( \bar{c}_{\beta,\beta'}(\tau) = 1 \) and \( \bar{c}_{\beta,\beta'}(\tau) = 0 \) as \( N_{a} \to \infty \). In Appendix B, the \( \beta \)-band scatterers acquire a phase shift \( \pi \Phi_{\beta,\beta'}\left(q_{j}, q_{j'}\right) \) due to their interaction with each created \( \beta' \) scattering center. This is confirmed by the form of the scattering phase shift \( Q_{\beta}^{0}\left(q_{j}/2 \right), \) Eq. (29).

Both the pseudofermion scattering theory studied in this paper and the related PDT 12 13 account for the effects of the changes in the \( \beta \) pseudofermion occupancy configurations that occur under the ground-state - excited-state transitions. Within the PDT, the elementary processes that generate the PS excited energy eigenstates from the ground state have three types:

(A) Finite-energy and finite-momentum elementary \( \beta = c, s \) 1 pseudofermion (and \( \beta = c, s \) 1 pseudofermion, unbound \( \eta \)-spinon, and unbound spinon) processes involving creation or annihilation (and creation) of one or a finite number of pseudofermions (and pseudofermions, unbound \( \eta \)-spinons, and unbound \( \eta \)-spinons) with canonical momentum values \( q_{j} \neq \pm q_{F,\beta} \) (and canonical momentum values \( q_{j} \neq \pm q_{3} \) and momentum values \( q_{3,1/2} = \pi \) and \( q_{3,-1/2} = 0 \), respectively);

(B) Zero-energy and finite-momentum processes that change the number of \( \beta = c, s \) 1 pseudofermions at the \( \tau = +1 \) right and \( \tau = -1 \) left \( \beta = c, s \) 1 Fermi points.

(C) Low-energy and small-momentum elementary \( \beta = c, s \) 1 pseudofermion particle-hole processes in the vicinity of the \( \tau = +1 \) right and \( \tau = -1 \) left \( \beta = c, s \) 1 Fermi points, \( \beta = c, s \) 1 pseudofermion momentum occupancy configurations generated by the above elementary processes (A) and (B).

According to the PDT, the elementary processes (A), (B), and (C) lead to qualitatively different contributions to the spectral-weight distributions. The PDT studies of Refs. 12 13 considered that creation of \( \eta \) pseudofermions and \( \nu \) \( \eta \) pseudofermions at the limiting bare-momentum values is felt by both the \( c \) and \( s \) 1 scatterers as effective \( c \) scattering centers and effective \( c \) and \( s \) 1 scattering centers, respectively. However, such studies applied only to \( \beta = c \) scatterers and \( \beta = s \) 1 scatterers of bare-momentum value \( q = \pm q_{F,\beta} \). Hence the general two-pseudofermion expression, Eq. (68), generalizes that result to all active \( \beta = c, \nu \) scatterers of arbitrary bare momentum \( q \). In Appendix B, the separate contributions of the PDT processes (A) and (B) to the overall scattering phase shift \( Q_{\beta}^{0}\left(q_{j}/2 \right), \) Eq. (29), are specified.

The pseudoparticles have zero-momentum forward-scattering energy interaction terms. Those are the \( f \)-function terms in Eqs. (A18) of Appendix A. The PDT relies on the corresponding pseudofermions having no energy interac-
tions for $u > 0$, as given in Eq. (56). This allows the expressions of the $u > 0$ one- and two-electron spectral functions in terms of a sum of terms, each of which is a convolution of a $c$ pseudofermion and a $s$1 pseudofermion spectral function. The latter spectral functions account for the effects of the Anderson’s orthogonality catastrophes [57]. Those result from the discrete canonical-momentum overall overlaps, $Q_\beta(q_j)/N_\alpha$. Such effects play a major role in the one- and two-electron matrix elements quantum overlaps that control the corresponding spectral-weight distributions [12, 13]. They emerge in the $c$ and $s$1 pseudofermion spectral functions through the exotic anticommutation relations, Eq. (42), which involve the overall phase-shift functional $Q_\beta(q_j)$, Eq. (51). Such spectral functions have the general form [12, 13],

$$B_{QB}(k, \omega) = \frac{N_\alpha}{2\pi} \sum_{m_\beta, +1} \frac{A^{(0, 0)}_\beta}{A^{(0, 0)}_\beta a_\beta(m_{\beta, +1}, m_{\beta, -1})} \times \delta\left(\omega - \frac{2\pi}{N_\alpha} \sum_{i=\pm} m_{\beta, i}\right) \delta\left(k - \frac{2\pi}{N_\alpha} \sum_{i=\pm} \nu_{\beta, i}\right), \quad \beta = c, s1. \quad (71)$$

Here the lowest peak weight $A^{(0, 0)}_\beta$ is that associated with the transition from the ground state to an excited state generated by the processes (A) and (B). The relative weights $a_\beta(m_{\beta, +1}, m_{\beta, -1})$ are generated by the additional processes (C). Such processes occur in the linear part of the $\beta = c, s1$ energy dispersions, which corresponds to the vicinity of the $\beta = c, s1$ Fermi points. Thus their energy spectra involve the $\beta = c, s1$ Fermi velocities, $v_\beta$, Eq. (A20) of Appendix A. The number of elementary pseudofermion - pseudofermion-hole processes (C) of momentum $\pm 2\pi/N_\alpha$ in the vicinity of the $\beta, c$ Fermi points is denoted in the summations of Eq. (71) by $m_{\beta, a}, = 1, 2, 3, \ldots$.

After a suitable algebra that involves the effective pseudofermion anticommutators, Eq. (42), one finds that the lowest peak weight in Eq. (71) reads [12, 13],

$$A^{(0, 0)}_\beta = \left(\frac{1}{N_{a\beta}}\right)^{2N_{\beta}} \prod_{j=1}^{N_{a\beta}} \left[1 + \sin^2\left(\frac{N_{\beta}(q_j)[Q_{\beta}(q_j) - \pi] + \pi}{2}\right) \prod_{j=1}^{N_{a\beta}} \left[1 + \sin\left(\frac{\pi j}{N_{a\beta}}\right)\right]^{2N_{a\beta} - |j|} \times \prod_{i=1}^{N_{a\beta}} \prod_{j=1}^{N_{a\beta}} \theta(j - \bar{i}) \sin^2\left(\frac{N_{\beta}(q_j)N_{\beta}(q_i)[Q_{\beta}(q_j) - Q_{\beta}(q_i) + 2\pi(j - i) - \pi N_{a\beta}] + \pi N_{a\beta}}{2N_{a\beta}}\right) \times \prod_{i=1}^{N_{a\beta}} \prod_{j=1}^{N_{a\beta}} \sin^2\left(\frac{N_{\beta}(q_j)N_{\beta}(q_i)[\pi(j - i) + \frac{Q_\beta(q_j)}{N_{a\beta}} + \frac{Q_\beta(q_i)}{N_{a\beta}}]}{\pi N_{a\beta}}\right), \quad \beta = c, s1. \quad (72)$$

The number of $\beta$ band discrete momentum values, $N_{a, \beta}$, that of $\beta$ pseudofermions, $N_{\beta} = \sum_{j=1}^{N_{a, \beta}} N_{\beta}(q_j)$, and the corresponding $\beta$ band momentum distribution function, $N_{\beta}(q_j)$, in this expression are those of the excited state generated by the processes (A) and (B). On the other hand, $Q_{\beta}(q_j)$ is the present scattering theory phase-shift functional, Eqs. (29), (34), and (37). The deviations in that functional expression are those generated by the corresponding ground-state - excited state transition.

Furthermore, the general expression of the relative weights $a_\beta(m_{\beta, +1}, m_{\beta, -1})$ also appearing in Eq. (71), which result from transitions to the tower of excited energy eigenstates generated by the processes (C), reads [12, 13],

$$a_\beta(m_{\beta, +1}, m_{\beta, -1}) = \prod_{i=\pm} a_{\beta, \nu}(m_{\beta, \nu})\left[1 + \mathcal{O}\left(\frac{\ln N_\alpha}{N_\alpha}\right)\right], \quad \beta = c, s1. \quad (73)$$

Here the relative weight $a_{\beta, \nu}(m_{\beta, \nu})$ is given by,

$$a_{\beta, \nu}(m_{\beta, \nu}) = \prod_{j=1}^{m_{\beta, \nu}} \frac{(2\Delta_\beta' + j - 1)}{j} = \frac{\Gamma(m_{\beta, \nu} + 2\Delta_\beta')}{\Gamma(m_{\beta, \nu} + 1) \Gamma(2\Delta_\beta')}, \quad \beta = c, s1, \quad \nu = \pm 1, \quad (74)$$

where $\Gamma(x)$ is the usual gamma function. It follows from Eq. (74) that,

$$a_{\beta, \nu}(1) = 2\Delta_\beta' = [\delta_{\nu} q_\beta/(2\pi/N_\alpha)]^2, \quad \beta = c, s1, \quad \nu = \pm 1. \quad (75)$$

The four $\beta = c, s1$ and $\nu = \pm 1$ functionals, $2\Delta_\beta'$, in this equation play a major role in the PDT. They are fully controlled by the corresponding four excited-states right ($\nu = +1$) and left ($\nu = -1$) $\beta = c, s1$ canonical-momentum
Fermi-point deviations \( \delta q_{\beta} \), Eq. (10). It follows from that equation that in units of \( 2\pi/N_{a} \) those four \( \beta = c, s1 \) and \( \iota = \pm 1 \) deviations read \( \delta q'_{\beta} / (2\pi/N_{a}) = i\delta N_{\beta,1}^{F} + Q_{\beta}^{F}(i\epsilon\beta) / 2\pi \). Thus the functionals \( 2\Delta_{\beta}^{J} \) can be written as,

\[
2\Delta_{\beta}^{J} = \left( i\delta N_{\beta,1}^{0,F} + \frac{Q_{\beta}(i\epsilon\beta)}{2\pi} \right)^{2} = \left( \frac{i\delta N_{\beta,1}^{0,F} - i\log S_{\beta}(i\epsilon\beta)}{2\pi} \right)^{2}, \quad \beta = c, s1, \quad \iota = \pm 1. \tag{76}
\]

The deviations \( \delta N_{\beta,1}^{0,F} \) and \( \delta N_{\beta,1}^{F} \) in the number of \( \beta = c, s1 \) pseudofermions at the right (\( \iota = +1 \)) and left (\( \iota = +1 \)) \( \beta = c, s1 \) Fermi points account and do not account, respectively, for the possible shift of the BA quantum numbers \( I_{β}^{β} \), Eqs. (A7) and (A10) of Appendix A. Here \( Q_{\beta}(q_{β})/2 \) stands again for the pseudofermion scattering theory phase-shift functional, Eqs. (29) and (34), and \( S_{β}(q) = e^{i\Phi_{β}(q)} \), Eq. (57), is the corresponding \( β \) pseudofermion dressed \( S \) matrix. In the case of Eq. (76), such a phase shift and \( S \) matrix refer to scatterers at the four \( β = c, s1 \) Fermi points, \( q_{β} = i\epsilon\beta \). Consistent with Eq. (76), the log branch in Eq. (76) is \( \log S_{\beta}(i\epsilon\beta) = i\epsilon\beta \).

According to Eq. (76), the functional, Eq. (76), is the relative weight of the \( α, β \) pseudofermion spectral function \( m_{β,α} = 1 \) peak. Moreover,

\[
a_{β}(1, 0) = 2\Delta_{β}^{J+1}; \quad a_{β}(0, 1) = 2\Delta_{β}^{J-1}, \quad \beta = c, s1. \tag{77}
\]

The relative weight, Eq. (74), has the following asymptotic behavior,

\[
a_{β,α}(m_{β,α}) \approx \frac{1}{(2\Delta_{β}^{J})^{2}} (m_{β,α})^{2\Delta_{β}^{J-1}}; \quad 2\Delta_{β}^{J} \neq 0; \quad \beta = c, s1, \quad \iota = \pm 1. \tag{78}
\]

The \( \beta = c, s1 \) pseudofermion spectral function lowest peak weight \( A_{β}^{(0,0)} \), Eq. (72), involves products \( \prod_{\iota=1}^{N_{a}} \) over all \( N_{a,b} = [N_{b} + N_{b}^{β}]_{β} \) corresponding excited-state \( \beta = c, s1 \) band scatterers. That refers both to those that pre-exist in the ground state and are created under the transition to the excited state. This confirms that the 1D Hubbard model dynamical and spectral properties are within the pseudofermion scattering theory controlled by microscopic processes that have contributions from all such \( \beta = c, s1 \) scatterers. In contrast, the holon and spinon scattering theories of Refs. [37, 38] consider only the few phase shifts associated with scatterers created under the transition to the excited state. Thus they do not account for most of the microscopic processes that contribute to the important lowest peak weights \( A_{β}^{(0,0)} \).

Such holon and spinon scattering theories neither include most of the phase shifts that for \( n \neq 1 \) appear in the expressions of the exponents in the dynamical and spectral function expressions. Indeed, within the PDT such exponents are linear combinations of the four \( \beta = c, s1 \) and \( \iota = \pm 1 \) phase-shift related functionals, \( 2\Delta_{β}^{J} \), Eq. (76).

Specifically, the exponents \( \zeta_{β}(k) \) and \( \zeta \) that control important line-shape singularities in the vicinity of well-defined \( (k, \omega) \) plane \( \beta = c, s1 \) branch lines and points have the general form [12, 13, 15],

\[
\zeta_{β}(k) = -1 + \sum_{\beta'=c,s1,\iota'=\pm 1} 2\Delta_{β'}^{J}(q)|_{q=c0[k-P_{0}]}, \quad \beta = c, s1. \tag{79}
\]

\[
\zeta = -2 + \sum_{\beta'=c,s1,\iota'=\pm 1} 2\Delta_{β'}^{J}; \quad \omega \neq \pm v_{β}(k-P_{0}), \quad \beta = c, s1, \tag{79}
\]

\[
\zeta = -1 - 2\Delta_{β}^{J} + \sum_{\beta'=c,s1,\iota'=\pm 1} 2\Delta_{β'}^{J}; \quad \omega \approx \pm v_{β}(k-P_{0}), \quad \beta = c, s1, \tag{79}
\]

respectively. Here \( c_{0} = +1 \) and \( c_{0} = -1 \) refer to \( \beta = c, s1 \) pseudofermion creation and annihilation, respectively. The momentum \( q \) dependence of \( 2\Delta_{β'}^{J}(q) \) in the \( \zeta_{β}(k) \) expression stems for a \( \beta = c, s1 \) branch line involving a phase-shift contribution, \( c_{0} 2\pi \Phi_{β,β}(i\epsilon\beta, q) \). It emerges within the corresponding scattering phase shift \( Q_{β}^{F}(i\epsilon\beta, q) \), Eq. (29).

The bare momentum \( q \) corresponds to the canonical momentum \( \vec{q} = \vec{q}(q) \) of the \( \beta = c, s1 \) pseudofermion branch-line created or annihilated under the transition to the excited state. The \( \beta = c, s1 \) energy dispersion \( ε_{β}(q) \), Eq. (A22) of Appendix A, is such an elementary object that determines the \( (k, \omega) \) plane shape of the corresponding \( \beta = c, s1 \) pseudofermion branch-line [12, 15]. Moreover, \( P_{0} \) stands in Eq. (79) for the following momentum, which may vanish,

\[
P_{0} = \pi M_{\eta,−1/2} + 2k_{F} δ2I_{F} + k_{F1} δ2J_{s1}^{F} + (π - 2k_{F})2J_{F}. \tag{80}
\]

Here \( 2J_{F} = δN_{β,1}^{F} - δN_{β,−1}^{F} \) for \( \beta = c, s1 \) and \( 2J_{F}^{L} = N_{p,−1}^{L} - N_{p,−1}^{L} \) where \( N_{p,−1}^{L} \) stands for the number of \( pν \) pseudofermions created at the \( pν \) band limiting momentum values \( a_{pν} = 0(k-P_{0}) \), Eq. (A13) of Appendix A in the second expression of Eq. (B20) of Appendix B for \( β = pλ \).
FIG. 6: The elementary two-pseudofermion phase shift $\pi \Phi_{s1,1}(q,q')$ in units of $\pi$ as a function of $q$ and $q'$ for the same densities and $U/t$ values as in Fig. 1. Alike in that figure, the scatterer and scattering-center bare-momentum values $q$ and $q'$, respectively, correspond to the right and left axis, respectively.

Such $c$ and $s1$ branch lines occur, for example, in the one-electron spectral functions $^{[15]}$. Under the processes (A) and (B) that generate the one-electron excited states, one $c$ pseudofermion or one $c$ pseudofermion hole scattering center and one $s1$ pseudfermion hole scattering center are created. On the other hand, the four scatterers with phase shifts $Q_c(\pm q_{Fc})/2$ and $Q_{s1}(\pm q_{Fs1})/2$ in the expression, Eq. (70), of the four functionals $2\Delta_\pm^c$ and $2\Delta_\pm^{s1}$, respectively, refer to four scatterers: two $c$ scatterers and two $s1$ scatterers. Such functionals control the corresponding branch-line momentum-dependent exponent $\zeta_\beta(k)$, Eq. (79). This applies applies as well to the exponents in the low-energy spectral and correlation functions in the vicinity of a $(k, \omega)$ plane point, $(k, \omega) = (P_0, 0)$. Those also involve the four $\beta = c, s1$ and $\iota = \pm 1$ phase shifts $Q_\beta(q_{F\beta})/2$ $^{[14]}$. In this second case, such functionals control the exponent $\zeta$, Eq. (79).

In the case of a one-electron branch line, one of the $c$ and $s1$ scattering centers is created at a momentum away from the corresponding $c$ or $s1$ Fermi points and the other at one of such points. On the other hand, in the case of the low-energy line shape in the vicinity of a $(k, \omega)$ plane point, $(k, \omega) = (P_0, 0)$, both $c$ and $s1$ scattering centers are created at two of the corresponding Fermi points. It then follows that for a one-electron branch line (and line shape in the vicinity of a $(k, \omega)$ plane point) out of the four $\beta = c, s1$ and $\iota = \pm 1$ scatterers with phase shifts $Q_\beta(q_{F\beta})/2$, three (and two) pre-exist in the ground state.

Concerning the excitation associated with the one-electron branch line (and line shape in the vicinity of a $(k, \omega)$ plane point), the scattering theory of the holon-spinon representation of Ref. $^{[36]}$ only provides the phase shift of one
scatterer (and two scatterers), out of the four \( \beta = c, s_1 \) and \( \nu = \pm 1 \) phase shifts \( Q_\beta(\nu q_F)/2 \). It is that (and those) of the scatterer (and two scatterers) that is (and are) created at \( \beta = c, s_1 \) Fermi points under the transition to the excited state. Hence that theory does not contain the other three (and two) phase shifts that control the branch lines (and line shape in the vicinity of \((k, \omega)\) plane points) exponents \( \zeta_\beta(k) \) (and \( \zeta \)). The same limitations occur in the case of the holon-spinon representation scattering theory of Refs. \[37\] \[38\] which in addition is valid only at half filling. Such results confirm the unsuitability of the two holon-spinon representations scattering theories of Refs. \[36\] \[37, 38\] to describe the model dynamical and spectral properties. That follows from for them the ground states being mere holon and spinon vacua, without elementary-object occupancies.

For further useful information on the relation of the \( \beta \) pseudofermion representation to the holon-spinon representations of Refs. \[30\] \[37\] \[38\], see Ref. \[28\]. Such representations are shown in that paper to refer to alternative sets of degenerate energy eigenstates that span well-defined model reduced subspaces.

The whole one-electron \((k, \omega)\) plane spectral-weight distribution is controlled only by the \( \beta \) pseudofermion scattering theory general phase shifts \( Q_\beta(q_j)/2 \), Eqs. \[29\] and \[34\]. The corresponding dynamical theory state summations are in general a very complex numerical problem. In the \( u \rightarrow \infty \) limit such phase shifts become independent of \( q_j \). That much simplifies the computation of the one-electron spectral functions over the whole \((k, \omega)\) plane. They are plotted in Fig. 1 of Ref. \[10\] for electronic density \( n = 0.5 \) and spin density \( m = 0 \).

Numerical studies on spectral functions as those of Ref. \[24\], which do not rely directly on a specific elementary-object representation, implicitly account for the microscopic processes and mechanisms of the pseudofermion scattering theory and PDT, respectively.

V. CONCLUDING REMARKS

In this paper it has been shown that a set of 1D Hubbard model natural scatterers emerges from the rotated-electron occupancy configurations degrees of freedom separation, as defined in Refs. \[26\] \[28\]. Such scatterers are the elementary objects previously used within the PDT studies of the one- and two-electron spectral-weight distributions \[12\] \[13\]: The \( c \) pseudofermions, the spin-neutral composite \( 2\nu \)-spinon \( sv \) pseudofermions, and the \( \eta \)-spin-neutral composite \( 2\nu-\eta \)-spinon \( yv \) pseudofermions. Those of such elementary objects created under a transition to an excited energy eigenstate play as well the role of the theory scattering centers. On the other hand, the \( \eta \)-spin-1/2 unbound \( \eta \)-spinos and spin-1/2 unbound spinons, respectively, are scattering-less elementary objects as far as their \( \eta \)-spin and spin, respectively, \( SU(2) \) symmetry internal degrees of freedom is concerned. Hence all the \( \beta = c, \alpha \nu \) pseudofermion scattering theory dressed \( S \) matrices have dimension one.

The theory scatterers and scattering centers that naturally emerge from rotated-electron configurations refer to elementary objects whose occupancy configurations generate exact energy eigenstates. That is why such occupancy configurations have been constructed inherently to account for an infinite number of conservation laws \[39\] \[41\], which are associated with the model integrability \[3\] \[6\]. As a result, the theory scatterers and scattering centers undergo only zero-momentum forward scattering events. This is in contrast to the underlying very involved non-perturbative many-electron processes.

There is a \( \beta \) pseudofermion scattering theory for each ground state and corresponding PS. The theory is valid for PSs of ground states with arbitrary values of the electronic density \( n \) and spin density \( m \). Within each of such large subspaces, the \( c \) pseudofermions, \( sv \) pseudofermions, and \( yv \) pseudofermions emerge from the \( c \) pseudoparticles, \( sv \) pseudoparticles, and \( yv \) pseudoparticles, respectively. This occurs through a well-defined unitary transformation. Under it, the \( \beta \) pseudofermions discrete momentum values are slightly shifted. This leads to the corresponding \( \beta \) pseudofermions discrete canonical momentum values, which are phase-shift dependent. Importantly, that renders the \( \beta \) pseudofermion spectrum without energy interaction terms. This property drastically simplifies the expression of electron spectral functions in terms of pseudofermion spectral functions. Otherwise, the \( \beta \) pseudoparticles and \( \beta \) pseudofermions have the same properties.

Concerning the elementary objects that participate in the theory scattering events, for densities \( n \neq 1 \) and \( m \neq 0 \) the ground states are populated only by \( c \) and \( s_1 \) pseudofermions and \( c \) and \( s_1 \) pseudofermion holes. (The \( c \) and \( s_1 \) momentum bands of \( S_\eta = 0; n = 1 \) and \( S_\eta = 0; m = 0 \) ground states are full, so that such states have no \( c \) and \( s_1 \) pseudofermion holes.) For all densities, ground states are not populated by \( \alpha \nu \neq s_1 \) pseudofermions. Therefore, only in the case of the \( c \) and \( s_1 \) bands do the discrete canonical-momentum overall shifts, \( Q_\beta(q_j)/Na \), lead to Andersons orthogonality catastrophes \[57\]. Those play a major role in the one- and two-electron matrix elements quantum overlaps that control the corresponding spectral-weight distributions \[12\] \[15\].

The “in” and “out” states of the pseudofermion scattering theory are exact excited energy eigenstates. They can be written as a simple direct product of “in” asymptote and “out” asymptote one-pseudofermion scattering states, respectively. Such a property combined with the also simple form of the \( \beta \) pseudofermion and \( \beta \) pseudofermion hole dressed \( S \) matrices is behind the suitability of the present representation to the describe the finite-energy spectral
and correlation properties of the model metallic phase. As discussed in Section [V D] the pseudofermion microscopic scattering processes control the PDT one- and two-electron spectral-weight distributions [12, 15]. This applies to finite values of the on-site repulsion, at both low and finite excitation energy.

The relation of the elementary objects of the representation used in the studies of this paper to the holons and spinons of the scattering theories of Ref. [30] and Refs. [37, 38], respectively, is an issue that has been clarified in Ref. [28]. As discussed in Section [V D] the major advantage of the theory considered in this paper relative to such holon and spinon scattering theories refers to the explicit description of the microscopic processes that control the model dynamical and spectral properties. Such an advantage follows from the present theory accounting for both the phase shifts of the scatterers that pre-exist in the ground state and those that are created under the transitions to the excited states. In contrast, the holon and spinon scattering theories account only for the few phase shifts of their scatterers, which are those created under the transitions to the excited states. That is consistent with the holon and spinon vacuum os such theories being the ground state, whose structure and occupancies are not accounted for.

Several properties predicted by the 1D Hubbard model have been observed in low-dimensional complex materials [34]. The investigations presented in Refs. [15, 31–33] confirm that the PDT describes successfully the unusual finite-energy spectral features observed by angle-resolved photoelectron spectroscopy in quasi-1D organic metals. Combination of such investigations results with those of this paper confirms that the scattering centers of the pseudofermion scattering theory are observed in such materials.

Acknowledgments

We thank N. Andrei, S.-J. Gu, H. Q. Lin, A. Moreno, A. Muramatsu, and K. Penc for illuminating discussions and the hospitality and support of the Beijing Computational Science Research Center. J. M. P. C. thanks the hospitality and support of the Beijing Computational Science Research Center. We thank N. Andrei, S.-J. Gu, H. Q. Lin, A. Moreno, A. Muramatsu, and K. Penc for illuminating discussions and the hospitality and support of the Beijing Computational Science Research Center. J. M. P. C. thanks the hospitality and support of the Beijing Computational Science Research Center.

Appendix A: Useful information on the pseudoparticle representation

Here some general results useful for the studies of the paper are presented. Our BA representation refers to LWSs of both the spin and \(\eta\)-spin \(SU(2)\) algebras. For such Bethe states the numbers, \(M^{\alpha \eta}_{\eta,0}\) and \(M^{\alpha \eta}_{\eta,1}\), Eq. (21), of \(\eta\)-spin projection \(-1/2\) unbound \(\eta\)-spinons and spin projection \(-1/2\) unbound spinons, respectively, vanish. Moreover, we use the notation, \(|l_r, l_{\eta s}, u\rangle\), for the energy eigenstates. Within it, the Bethe states are written as \(|l_r, l^0_{\eta s}, u\rangle\). A non-LWS energy eigenstate \(|l_r, l_{\eta s}, u\rangle\) is generated from the corresponding Bethe state \(|l_r, l^0_{\eta s}, u\rangle\) as follows,

\[ |l_r, l_{\eta s}, u\rangle = \prod_{\alpha=\eta,s} \left[ \frac{1}{\sqrt{C_\alpha}} (\hat{S}^{+\alpha}_{\alpha})^{M^{\alpha \eta}_{\eta,0}} (\hat{S}^{+\alpha}_{\alpha})^{M^{\alpha \eta}_{\eta,1}} \right] |l_r, l^0_{\eta s}, u\rangle. \]  

(A1)

Here,

\[ C_\alpha = \langle l_r, l^0_{\eta s}, u\rangle (\hat{S}^{+\alpha}_{\alpha})^{M^{\alpha \eta}_{\eta,0}} (\hat{S}^{+\alpha}_{\alpha})^{M^{\alpha \eta}_{\eta,1}} |l_r, l^0_{\eta s}, u\rangle \]

\[ = \prod_{j=1}^{M^{\alpha \eta}_{\alpha,1}} \left[ M^{\alpha \eta}_{\alpha,0} + 1 - j' \right], \quad M^{\alpha \eta}_{\alpha,0} = 1, \ldots, M^{\alpha \eta}_{\alpha,1}, \alpha = \eta, s, \]  

(A2)

are normalization constants, \(\hat{S}^+\) and \(\hat{S}^-\) are the \(\eta\)-spin (\(\alpha = \eta\)) and spin (\(\alpha = s\)) off-diagonals generators, Eq. (12) for \(i = \pm\), and \(\alpha = \eta, s\), \(M^{\alpha \eta}_{\alpha,0} = 2\alpha\) denotes the total number of unbound \(\eta\)-spinons (\(\alpha = \eta\)) and unbound spinons (\(\alpha = s\)), and \(l_{\eta s}\) and \(l^0_{\eta s}\) stand for the set of numbers \([S_\eta, S_\eta, M^{\alpha \eta}_{\eta,0}, M^{\alpha \eta}_{\eta,1}]\) and \([S_\eta, S_s, 0, 0]\), respectively. Within our notation, \(l^0_{\eta s}\) refers to limiting values of the general label \(l_{\eta s}\) such that \(M^{\alpha \eta}_{\eta,1} = M^{\alpha \eta}_{\eta,0} = 0\), which are those suitable to the Bethe states. The label \(\eta s\) in \(l_{\eta s}\) and \(l^0_{\eta s}\) symbolizes the two \(SU(2)\) symmetries associated with the numbers \(S_\eta, S_\eta\) and \(S_s, S_s\), respectively, in \([S_\eta, S_s, M^{\alpha \eta}_{\eta,0}, M^{\alpha \eta}_{\eta,1}]\). Moreover, \(l_r\) stands for all remaining quantum numbers beyond \(l_{\eta s}\) needed to uniquely define an energy eigenstate \(|l_r, l_{\eta s}, u\rangle\). This includes the eigenvalue \(2S_c\) of the hidden \(U(1)\) symmetry generator.
The energy eigenvalues of the Bethe states $|l_\alpha, l^0_\nu, u\rangle$ are given by,

$$E_{l_\alpha, l^0_\nu} = E_{\text{symm}} + \mu M_{\eta, +1/2}^\text{un} + \mu_B H M_{\eta, +1/2}^\text{un},$$

$$E_{\text{symm}} = -2t \sum_{i=1}^{N_a} [N_c(q_i) \cos k(q_i) + u] - u/2 + 4t \sum_{i=1}^{N_a} N_{\nu\nu}(q_i) \left[ \Re \left\{ (1 - (\Lambda_{\nu\nu}(q_i) + i\nu u)^2 \right\} - \nu u \right]. \quad (A3)$$

For the non-LWSs, the unbound $\eta$-spinon number $M_{\eta, +1/2}^\text{un}$ and unbound spinon number $M_{\eta, -1/2}^\text{un}$ appearing in the first expression given here are replaced by $[M_{\eta, +1/2}^\text{un} - M_{\eta, -1/2}^\text{un}]$ and $[M_{\eta, +1/2}^\text{un} - M_{\eta, -1/2}^\text{un}]$, respectively.

The energy spectrum, Eq. (A3), involves the rapidity functions $k(q_i)$ and $\Lambda_{\nu\nu}(q_i)$. Those are related to the rapidity function $\Lambda_{\nu\nu}(q_i)$ through a set of coupled thermodynamic BA equations. For the Bethe states, such rapidity functions are for each energy eigenstate obtained by solution of those equations. In functional form, they read,

$$k_{\alpha\nu}(q_j) = q_j - 2 \frac{N_a \sum_{i=1}^{N_a} N_{\nu\nu}(q_{j'}) \arctan \left( \frac{\sin k_{\alpha\nu}(q_{j'})}{\nu u} \right)}{N_a \sum_{i=1}^{N_a} N_{\nu\nu}(q_{j'})}, \quad j = 1, \ldots, N_a, \quad \alpha = \eta, s. \quad (A4)$$

and

$$k_{\alpha\nu}(q_j) = q_j - 2 \frac{N_a \sum_{i=1}^{N_a} N_{\nu\nu}(q_{j'}) \arctan \left( \frac{\Lambda_{\nu\nu}(q_{j'}) - \sin k_{\alpha\nu}(q_{j'})}{\nu u} \right)}{N_a \sum_{i=1}^{N_a} N_{\nu\nu}(q_{j'})}, \quad j = 1, \ldots, N_a, \quad \alpha = \eta, s. \quad (A5)$$

Here

$$k_{\alpha\nu}(q_j) = 2 \Re \{ \arcsin(\Lambda_{\nu\nu}(q_j) + i\nu u) \}, \quad \nu = 1, \ldots, \infty, \quad j = 1, \ldots, N_{a\nu}, \quad \alpha = \eta, s. \quad (A6)$$

is the $\alpha\nu$ rapidity-momentum functional, which vanishes for $\alpha = s$. The function $\Theta_{\nu\nu'}$ is given in Eq. (B16) of Appendix B.

The discrete momentum values $q_j$ appearing in Eqs. (A3)-(A6) are given by,

$$q_j = \frac{2\pi}{N_a} I^c_j, \quad j = 1, \ldots, N_{ac}, \quad q_j = \frac{2\pi}{N_a} I^\alpha_j, \quad j = 1, \ldots, N_{a\alpha} \quad (A7)$$

Here the number $N_{ac}$ of $c$ band discrete momentum values and that $N_{a\alpha}$ of $\alpha\nu$ band discrete momentum values read,

$$N_{ac} = N_c + N^h_c = N_a, \quad N_{a\alpha} = N_{\alpha\nu} + N^h_{\alpha\nu}, \quad (A8)$$

where $N_c = 2S_c$ and $N_{\alpha\nu}$ denote the number of occupied $c$ and $\alpha\nu$ band discrete momentum values, respectively. Those of unoccupied values are given by,

$$N^h_c = 2S^h_c = [N_a - 2S_c],$$

$$N^h_{\alpha\nu} = [2S_a + 2 \sum_{\nu' = \nu + 1}^{\infty} (\nu - \nu')(N_{\alpha\nu})] = [M^\text{un}_\alpha + 2 \sum_{\nu' = \nu + 1}^{\infty} (\nu - \nu')(N_{\alpha\nu})], \quad \alpha = \eta, s. \quad (A9)$$

Furthermore, the BA quantum numbers $I^c_j$ and $\{I^\alpha_j\}$ in Eq. (A7) are either integers or half-odd integers. For simplicity, we often denote such numbers by $I^\beta_j$ where $\beta = c, \alpha\nu$, $\alpha = \eta, s$, and $\nu = 1, \ldots, \infty$. Their values are uniquely determined by the following boundary conditions,

$$I^\beta_j = 0, \pm 1, \pm 2, \ldots \text{ for } G_\beta \text{ even},$$

$$= \pm 1/2, \pm 3/2, \pm 5/2, \ldots \text{ for } G_\beta \text{ odd}, \quad (A10)$$
where

\[ G_\beta = \delta_{\beta,c} \sum_{\alpha=\eta,s} \sum_{\nu=1}^\infty N_{a\nu} + \delta_{\beta,\alpha\nu} [2S_c + N_{\alpha\nu} - 1], \quad \alpha = \eta, s. \tag{A11} \]

Within the \( N_a \to \infty \) limit that the thermodynamic BA equations, Eqs. (A4)-(A6), refer to it is often convenient to replace the \( \beta \) pseudoparticle discrete momentum values \( q_j \), such that \( q_{j+1} - q_j = 2\pi/N_a \), by corresponding continuous momentum variables \( q \). Those belong to domains \( q \in [-q_\beta, +q_\beta] \), whose limiting absolute values \( q_\beta \) read,

\[ q_c = \pi, \quad q_{c\nu} = \frac{\pi}{N_a} (N_{a\nu} - 1) \text{ for } N_{a\nu} \text{ odd}, \]

\[ = \frac{\pi}{N_a} N_{a\nu} \text{ for } N_{a\nu} \text{ even.} \tag{A12} \]

For the \( \beta = \alpha\nu \) branches the \( \beta \) discrete-momentum values distribution is symmetrical and bound by the momentum values \( \pm q_{a\nu} \), with \( q_{a\nu} \) given in Eq. (A12). On the other hand, if one accounts for corrections of order \( 1/N_a \), the \( c \) band \( q \) range becomes \( q \in [q_c^-, q_c^+] \) where \( q_c^\pm = \pm \pi q_{a\nu} \) plus \( 1/N_a \) corrections, as given in Eqs. (B.15)-(B.17) of Ref. [26].

For the present ground states with densities \( n \in [0,1] \) and \( m \in [0,n] \), the limiting momentum values, Eq. (A12), have simple expressions,

\[ q_c = \pi; \quad q_{s1} = k_{F1}; \quad q_{\nu} = [k_{F1} - k_{F\nu}], \quad \nu > 1; \quad q_{\eta\nu} = [\pi - 2k_{F\nu}]. \tag{A13} \]

Here we have again ignored corrections of order \( 1/N_a \) order.

The pseudofermion scattering theory studied in this paper refers to transitions from ground states to PS excited energy eigenstates. Ground states are described by compact \( c \) and \( s1 \) pseudofermion finite occupancies. Those correspond to bare-momentum ranges \( q \in [-q_{\beta}\eta, +q_{\beta}\eta] \) where \( \beta = c, s1 \). For densities in the ranges \( n \in [0,1] \) and \( m \in [0,n] \), the ground-state unbound \( +1/2 \) \( \eta \)-spinon and unbound \( +1/2 \) spinon numbers are \( M_{\eta,1/2}^n = M_{\eta}^n = N_{c} = N_a - N \) and \( M_{\eta,1/2}^m = M_{s}^m = N_{s1} = N_t - N_s \), respectively. On the other hand, such a ground state is not populated by unbound \( -1/2 \) \( \eta \)-spinon and unbound \( -1/2 \) spinon, \( M_{\eta,-1/2}^n = M_{s,-1/2}^m = 0 \). Moreover, the \( \alpha\nu \neq s1 \) branches have vanishing ground-state occupation. For such a ground state the \( c \) and \( s1 \) Fermi momenta are given by,

\[ q_{Fc} = 2k_F; \quad q_{Fs1} = k_{F1}. \tag{A14} \]

Here we have again ignored \( 1/N_a \) order corrections, which are provided in Eqs. (C.4)-(C.11) of Ref. [26].

We denote by \( k_{c}^0(q_j) \) and \( \Lambda_{c}^0(q_j) \) the specific rapidity-function solutions of the thermodynamic integral equations, Eqs. (A3)-(A6), that refer to a ground state. Those play an important role both in the PDT of Refs. [12-14] and pseudofermion scattering theory studied in this paper. Upon suitable manipulations of such thermodynamic BA equations, the ground-state functions \( k_{c}^0(q_j) \) and \( \Lambda_{c}^0(q_j) \) may be defined in terms of their inverse functions, \( q_j = q_j(\Lambda_{c,\beta}^0) \), as follows,

\[ q_j = F_{c}^0(\Lambda_{c}^0(q_j)) + (1)^{\beta,\eta,\nu} \int_{-Q}^{Q} dk \Phi_{c,\beta} \left( \sin k / u, \frac{\Lambda_{c}^0(q_j)}{u} \right), \quad j = 1, ..., \text{intified in terms of their inverse functions}, \]

\[ F_{c}^0(\Lambda_{c}^0(q_j)) = \text{arcsin}(\Lambda_{c}^0(q_j)) = k_{c}^0(q_j), \]

\[ F_{\alpha\nu}^0(\Lambda_{\alpha\nu}^0(q_j)) = \delta_{\alpha,\eta} \text{Re} \left[ \text{arcsin}(\Lambda_{\eta\nu}^0(q_j) + i\nu u) \right]. \tag{A15} \]

Here \( N_{\alpha\beta} = N_a \) is given in Eq. (A8) and (A9) and the two-pseudofermion phase shifts \( \Phi_{c,\beta}(r,r') \) are defined in Eqs. (B7)-(B9) of Appendix B. Moreover, the parameter \( Q \) and the related parameters \( B, r_c^0, \) and \( r_s^0 \) in other quantities of this paper may be expressed in terms of the ground-state rapidity functions \( k_c^0(q) \) and \( \Lambda_c^0(q) \) at the corresponding \( c \) and \( s1 \) Fermi points, Eq. (A14), respectively, as follows,

\[ Q = k_{c}^0(2k_{F}) ; \quad B = \Lambda_{s1}^0(k_{F1}) ; \quad r_c^0 = \frac{\sin Q}{u} ; \quad r_s^0 = \frac{B}{u}. \tag{A16} \]

For the \( S_n = 0; n = 1 \) and \( S_s = 0; m = 0 \) absolute ground state the \( \beta \neq c, s1 \) bands do not exist. On the other hand, for it the equations given in Eq. (A15) have an analytical solution for the \( \beta = c, s1 \) branches in terms of the
inverse of the rapidity functions $k_c^0(q_j)$ (such that $\Lambda_c^0(q_j) = \sin k_c^0(q_j)$) and $\Lambda_{s1}^0(q_j)$.

$$q_j = k_c^0(q_j) + 2 \int_0^\infty d\omega \frac{\sin(\omega \sin k_c^0(q_j))}{\omega (1 + e^{2\omega u})} J_0(\omega), \quad j = 1, ..., N_a,$$

$$q_j = \int_0^\infty d\omega \frac{\sin(\omega \Lambda_{s1}^0(q_j))}{\omega \cosh(\omega u)} J_0(\omega), \quad j = 1, ..., N_{s1}.$$

(A17)

Here $J_0(\omega)$ is a Bessel function.

The $\beta$ pseudofermion energy functional, Eq. (26), second-order in the $\beta$ band canonical-momentum-distribution function deviations terms exactly vanish. That is not so for the corresponding $\beta$ pseudoparticle energy functional, which to second order in such deviations reads,

$$\delta E_{\beta,\alpha} = \sum_{\beta} \sum_{j=1}^{N_{\beta}} \varepsilon_{\beta}(q_j) \delta N_{\beta}(q_j) + 2|\mu| M_{\eta, 1/2}^{un} + 2\mu_B |H| M_{s, 1/2}^{un},$$

$$+ \frac{1}{N_a} \sum_{\beta} \sum_{\beta'} \sum_{j=1}^{N_{\beta}} \sum_{j'=1}^{N_{\beta'}} \frac{1}{2} f_{\beta,\beta'}(q_j, q_{j'}) \delta N_{\beta}(q_j) \delta N_{\beta'}(q_{j'}).$$

(A18)

$\delta N_{\beta}(q_j) = N_{\beta}(q_j) - N_{\beta}^0(q_j), \quad j = 1, ..., N_{\beta}, \quad \beta = c, \alpha \nu,$

(A19)

where $N_{\beta}(q_j)$ and $N_{\beta}^0(q_j)$ are the corresponding excited-state and ground-state $\beta$ pseudoparticle momentum-distribution functions, respectively. Furthermore, $M_{\eta, 1/2}^{un}$ and $M_{s, 1/2}^{un}$ are in Eq. (A18) the numbers of $\eta$-spin projection $\pm 1/2$ $\eta$-spinons and spin projection $\pm 1/2$ spinons, respectively, Eq. (21).

Related important conserving numbers for the finite-energy physics are those of rotated-electron doubly occupied sites, $D_r$, and spin-down rotated-electron singly occupied sites whose rotated electrons are not associated with $s1$ pseudoparticles, $S_r$. They read,

$$D_r \equiv M_{n, 1/2} = [M_{\eta, 1/2}^{un} + M_{\eta, 1/2}^{bo} / 2] = [M_{\eta, 1/2}^{un} + \sum_{\nu=1}^{\infty} \nu N_{\eta \nu}],$$

$$S_r \equiv [M_{s, 1/2} - N_{s1}] = [M_{s, 1/2}^{un} + M_{s, 1/2}^{bo} / 2 - N_{s1}] = [M_{s, 1/2}^{un} + \sum_{\nu=2}^{\infty} \nu N_{\alpha \nu}],$$

(A20)

respectively.

The energy dispersions $\varepsilon_{\beta}^0(q_j)$ in Eq. (A18) can be expressed in terms of the two-pseudofermion phase shifts $\Phi_{c,\beta}(r, r')$, Eqs. (B7)-(B9) of Appendix B and of the ground-state rapidity functions $\Lambda_{\beta}^0(q_j)$ (with $\Lambda_{c}^0(q_j) \equiv \sin k_c^0(q_j)$ for $\beta = c$), which are defined by their inverse functions in Eq. (A15), as follows,

$$\varepsilon_{\beta}^0(q_j) = E_{\beta}^0(q_j) + 2t \int_{-Q}^Q dk \Phi_{c,\beta} \left( \frac{\sin k}{u}, \frac{4t\Lambda_{\beta}^0(q_j)}{U} \right) \sin k, \quad j = 1, ..., N_{a\beta},$$

$$E_c^0(q_j) = \frac{U}{2} - 2t \cos k_c^0(q_j),$$

$$E_{\alpha \nu}^0(q_j) = \delta_{\alpha \eta} \left\{ -\nu U + 4t \text{Re} \left[ \sqrt{1 - (\Lambda_{\eta \nu}^0(q_j) + i\nu u)^2} \right] \right\}. $$

(A21)
The related energy dispersions $\varepsilon_\beta(q_\beta)$ also appearing in Eq. (A18) are given by,
\[
\varepsilon_c(q_\beta) = \varepsilon^0_c(q_\beta) - \varepsilon^0_c(q_F c) = \varepsilon^0_c(q_\beta) + (1 - \delta_{n,1}) |\mu| + \delta_{n,1} \mu^0 - \mu_B |H|,
\]
\[
\varepsilon_{s1}(q_\beta) = \varepsilon^0_{s1}(q_\beta) - \varepsilon^0_{s1}(q_F s1) = \varepsilon^0_{s1}(q_\beta) + 2\mu_B |H|,
\]
\[
\varepsilon_{s\nu}(q_\beta) = \varepsilon^0_{s\nu}(q_\beta) + 2\nu\mu_B |H|,
\]
\[
\varepsilon_{s\eta\nu}(q_\beta) = \varepsilon^0_{s\eta\nu}(q_\beta) + (1 - \delta_{n,1}) 2\nu |\mu| + \delta_{n,1} 2\nu \mu^0,
\]  
(A22)

where $2\mu^0$ is the $n = 1$ Mott-Hubbard gap. Its limiting behaviors are provided in Eq. (D5) of Appendix D.

The zero-energy level of the energy bands in Eq. (A13) is such that,
\[
\varepsilon_c(\pm 2k_F) = \varepsilon_{s1}(\pm 2k_{F1}) = 0,
\]
\[
\varepsilon^0_{s\nu}(\pm [\pi - 2k_F]) = \varepsilon^0_{s\eta\nu}(\pm [k_{F1} - k_{F1}]) = 0.
\]  
(A23)

For the $\beta = \alpha \nu \neq s1$ and $\beta = c, s1$ bands it refers to the limiting momenta, Eq. (A13), and $\beta = c, s1$ Fermi momenta, Eq. (A14), respectively.

The $f$ function in the second-order terms of the $\beta$ pseudopotential energy functional, Eqs. (A18), reads
\[
f_{\beta,\beta'}(q_\beta, q_{\beta'}) = v_{\beta}(q_\beta) 2\pi \Phi_{\beta,\beta'}(q_\beta, q_{\beta'}) + v_{\beta}(q_{\beta'}) 2\pi \Phi_{\beta',\beta}(q_{\beta'}, q_\beta)
\]
\[
+ \frac{1}{2\pi} \sum_{\beta'' = c, s1} \sum_{\nu, \nu'} v_{\beta''} 2\pi \Phi_{\beta''\nu,\beta''}(q_{\nu'}, q_{\beta}) 2\pi \Phi_{\beta''\nu,\beta''}(q_{\beta'}, q_{\nu'}). 
\]  
(A24)

Within the continuum momentum representation, the group velocities $v_{\beta}(q_\beta)$ appearing here are given by,
\[
v_{\beta}(q) = \frac{\partial \varepsilon_\beta(q)}{\partial q} = \frac{\partial \varepsilon^0_\beta(q)}{\partial q}, \quad \beta = c, \eta \nu, s \nu, \quad \nu = 1, ..., \infty,
\]
\[
v_{\beta} \equiv v_{\beta}(q_{\beta}), \quad \beta = c, s1. 
\]  
(A25)

In Appendix D limiting behaviors of the energy dispersions, Eqs. (A21) and (A22), along with other useful energy scales are given.

Appendix B: Two-pseudofermion phase shifts

In this Appendix we first derive the integral equations that define the two-pseudofermion phase shifts $\pi \Phi_{\beta,\beta'}(r, r')$ on the right-hand side of Eq. (32) for densities in the ranges $n \in [0, 1]$ and $m \in [0, n]$. To achieve that goal, we solve the thermodynamic BA equations, Eqs. (A1)-(A6) of Appendix A up to first order in the deviations $\delta N_\beta(q_\beta)$, Eq. (A19) of that Appendix. Although the two-pseudofermion phase-shift integral equations given below refer to densities $n \in [0, 1]$ and $m \in [0, n]$, within the limit $n \to 1$ (and $m \to 0$) they provide correct $n = 1$ (and $m = 0$) results. Those are given in Appendix C.

Moreover, in this Appendix the two-pseudofermion phase-shift integral equations are used to show that the functional $\tilde{q}_\beta = \tilde{q}(q_\beta)$, Eq. (29), is such that the ground-state limiting $\alpha \nu$ band canonical-momentum values $\tilde{q}(\pm q_{\alpha \nu})$ are not shifted by the ground-state - excited-state transitions provided that $v_{\alpha \nu}(\pm q_{\alpha \nu}) = 0$. Finally, we consider the separate contributions of the PDT processes (A) and (B) defined in Section IV D to the overall scattering phase shift $\Phi_{\beta}(q)/2$, Eq. (29).

A first group of two-pseudofermion phase shifts obey integral equations by their own,
\[
\pi \Phi_{s1,c}(r, r') = -\arctan(r - r') + \int_{-r'_2}^{r'_2} dr'' G(r, r'') \pi \Phi_{s1,c}(r'', r'), 
\]
(B1)

\[
\pi \Phi_{s1,\eta \nu}(r, r') = -\frac{1}{\pi} \int_{-r'_2}^{r'_2} dr'' \frac{\arctan \left( \frac{r' - r''}{r'} \right)}{1 + (r - r'')^2} + \int_{-r'_2}^{r'_2} dr'' G(r, r'') \pi \Phi_{s1,\eta \nu}(r'', r'), 
\]
(B2)

and
\[
\pi \Phi_{s1,s \nu}(r, r') = \delta_{s, \nu} \arctan \left( \frac{r - r'}{2} \right) + (1 - \delta_{s, \nu}) \left\{ \arctan \left( \frac{r - r'}{\nu - 1} \right) + \arctan \left( \frac{r - r'}{\nu + 1} \right) \right\}
\]
\[
- \frac{1}{\pi} \int_{-r'_2}^{r'_2} dr'' \frac{\arctan \left( \frac{r'' - r'}{2} \right)}{1 + (r - r'')^2} + \int_{-r'_2}^{r'_2} dr'' G(r, r'') \pi \Phi_{s1,s1}(r'', r'). 
\]  
(B3)
The parameters \( r_C^0 \) and \( r_s^0 \) appearing in such equations are given in Eq. (A16) of Appendix A. The kernel \( G(r, r') \) reads,

\[
G(r, r') = -\frac{1}{2\pi} \left[ \frac{1}{1 + ((r - r')/2)^2} \right] \left[ 1 - \frac{1}{2} \left( t(r) + t(r') + \frac{l(r) - l(r')}{r - r'} \right) \right].
\] (B4)

Here

\[
t(r) = \frac{1}{\pi} \left[ \arctan(r + r_C^0) - \arctan(r - r_C^0) \right],
\] (B5)

and

\[
l(r) = \frac{1}{\pi} \left[ \ln(1 + (r + r_C^0)^2) - \ln(1 + (r - r_C^0)^2) \right].
\] (B6)

The kernel, Eqs. (B1) - (B6), was first introduced in Ref. [54] within the c and \( s \equiv s_1 \) pseudo-particle two-component PS considered in that reference.

A second group of two-pseudofermion phase shifts are expressed in terms of the basic functions, Eqs. (B1)-(B3), and of the phase shifts, Eqs. (B7)-(B9), as follows,

\[
\pi \Phi_{c,c}(r, r') = \frac{1}{\pi} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{s_1,c}(r'', r')}{1 + (r - r'')^2},
\] (B7)

\[
\pi \Phi_{c,s\nu}(r, r') = -\arctan \left( \frac{r - r'}{\nu} \right) + \frac{1}{\pi} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{s_1,s\nu}(r'', r')}{1 + (r - r'')^2},
\] (B8)

and

\[
\pi \Phi_{c,s\nu'}(r, r') = -\arctan \left( \frac{r - r'}{\nu} \right) + \frac{1}{\pi} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{s_1,s\nu'}(r'', r')}{1 + (r - r'')^2}.
\] (B9)

Finally, the remaining two-pseudofermion phase shifts can be expressed either only in terms of the functions, Eqs. (B7)-(B9),

\[
\pi \Phi_{s\nu,c}(r, r') = \arctan \left( \frac{r - r'}{\nu} \right) - \frac{1}{\pi} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{s_1,c}(r'', r')}{\nu[1 + (r - r'')^2]},
\] (B10)

\[
\pi \Phi_{s\nu,s\nu'}(r, r') = \frac{1}{2} \Theta_{\nu,\nu'}(r - r') - \frac{1}{\pi} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{s_1,s\nu'}(r'', r')}{\nu[1 + (r - r'')^2]},
\] (B11)

\[
\pi \Phi_{s\nu,s\nu'}(r, r') = -\frac{1}{\pi} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{c,s\nu'}(r'', r')}{\nu[1 + (r - r'')^2]},
\] (B12)

or both in terms of the basic functions, Eqs. (B1)-(B3), and of the phase shifts, Eqs. (B7)-(B9),

\[
\pi \Phi_{s\nu,c}(r, r') = -\arctan \left( \frac{r - r'}{\nu} \right) + \frac{1}{\pi} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{c,c}(r'', r')}{\nu[1 + (r - r'')^2]} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{s_1,c}(r'', r')}{2\pi}; \quad \nu > 1,
\] (B13)

\[
\pi \Phi_{s\nu,s\nu'}(r, r') = \frac{1}{\pi} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{s\nu,s\nu'}(r'', r')}{\nu[1 + (r - r'')^2]} - \frac{1}{2\pi} \Theta_{\nu,\nu'}(r - r''); \quad \nu > 1,
\] (B14)

\[
\pi \Phi_{s\nu,s\nu'}(r, r') = \frac{1}{2} \Theta_{\nu,\nu'}(r - r') + \frac{1}{\pi} \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{c,s\nu'}(r'', r')}{\nu[1 + (r - r'')^2]} - \int_{-r_C^0}^{r_C^0} dr'' \frac{\pi \Phi_{s_1,s\nu'}(r'', r')}{2\pi}. \quad \nu > 1.
\] (B15)
In the above two-pseudofermion phase shift expressions, \( \Theta_{\nu,\nu'}(x) \) is the function,
\[
\Theta_{\nu,\nu'}(x) = \delta_{\nu,\nu'} \left\{ 2 \arctan \left( \frac{x}{2l} \right) + \sum_{l=1}^{\nu-1} 4 \arctan \left( \frac{x}{2l} \right) \right\} + \left( 1 - \delta_{\nu,\nu'} \right) \left\{ 2 \arctan \left( \frac{x}{|\nu - \nu'|} \right) + \sum_{l=1}^{\nu - 1 - |\nu - \nu'|} 4 \arctan \left( \frac{x}{|\nu - \nu'| + 2l} \right) \right\},
\tag{B16}
\]
and \( \Theta_{\nu,\nu'}^{(1)}(x) \) is its derivative,
\[
\Theta_{\nu,\nu'}^{(1)}(x) = \frac{d\Theta_{\nu,\nu'}(x)}{dx} = \delta_{\nu,\nu'} \left\{ \frac{1}{\nu \left[ 1 + \left( \frac{x}{\nu} \right)^2 \right]} + \sum_{l=1}^{\nu-1} \frac{2}{l \left[ 1 + \left( \frac{x}{l} \right)^2 \right]} \right\} + \left( 1 - \delta_{\nu,\nu'} \right) \left\{ \frac{2}{|\nu - \nu'| \left[ 1 + \left( \frac{x}{|\nu - \nu'|} \right)^2 \right]} + \sum_{l=1}^{\nu - 1 - |\nu - \nu'|} \frac{4}{\left( |\nu - \nu'| + 2l \right) \left[ 1 + \left( \frac{x}{|\nu - \nu'| + 2l} \right)^2 \right]} \right\}.
\tag{B17}
\]

Next it is shown that the ground-state limiting \( \alpha \nu \) band canonical-momentum values \( \nu^0(q_{\alpha\nu}) \) are not shifted by the ground-state - excited state transitions provided that \( v_{\alpha\nu}(\pm q_{\alpha\nu}) = 0 \). From the use of the integral equations, Eqs. (B10)-(B15), for the two-pseudofermion phase shifts \( \pi \Phi_{\alpha\nu,\beta'}(q,q') \) involving \( \alpha \nu \neq s1 \) scatterers, it is found that,
\[
\pi \Phi_{\alpha\nu,\beta'}(\nu q_{\alpha\nu}, q) = \frac{\nu \pi}{2} \left[ \delta_{\beta',c,\nu} - \delta_{\alpha,s} \right], \quad \nu = \pm 1. \tag{B18}
\]
If \( \beta' = \alpha' \), this two-pseudofermion phase-shift expression is valid provided that \( q \neq \nu q_{\alpha\nu} \). From the use of the integral equations, Eqs. (B11)-(B13), for the two-pseudofermion phase shifts \( \pi \Phi_{s1,\beta'}(q,q') \) we find that in the \( r_s^0 \rightarrow 0 \) limit the two-pseudofermion phase shifts \( \pi \Phi_{s1,\beta'}(\nu q_{s1}, q) \) are as well given by Eq. (B18). Such a \( r_s^0 \rightarrow 0 \) limit is equivalent to the limit, \( N_{s1}/N_a \rightarrow 0 \) as \( N_a \rightarrow \infty \), within which the condition \( v_s(\nu q_{s1}) = 0 \) is fulfilled.

The use of Eq. (B18) in the scattering phase-shift expression, Eq. (29), leads to,
\[
\frac{Q_{\alpha\nu}(\nu q_{\alpha\nu})}{2} = -\frac{\nu \pi}{2} \left[ \delta_{\alpha,\nu} - \delta_{\alpha,s} \right] \delta_{N_\nu} - \sum_{\nu' = 1}^{\infty} (\nu + \nu' - |\nu - \nu'|) \delta_{N_{\nu'}}; \quad \nu = \pm 1. \tag{B19}
\]
Comparison of this expression with that of \( \nu \delta q_{\alpha\nu} = \nu \pi \delta N_{\alpha\nu} / N_a \) where \( N_{\alpha\nu} = [N_{\alpha\nu} + N_{\alpha\nu}^h] \) and \( N_{\alpha\nu}^h \) is provided in Eq. (A9) of Appendix A confirms that \( Q_{\alpha\nu}(\nu q_{\alpha\nu})/N_a = -\nu \delta q_{\alpha\nu} \). (For the \( s1 \) pseudofermions such results are valid again in the above limit within which \( v_s(\nu q_{s1}) = 0 \).)

As mentioned above, for a \( \beta' = \alpha' \) pseudofermion scattering center the two-pseudofermion phase shift expression, Eq. (B18), does not apply at \( q = \nu q_{\alpha\nu} \). However, from the form \( \Lambda_{\alpha\nu}(q) = \Lambda_{\alpha\nu}^0(q_{\alpha\nu}) \) of the PS excited energy-eigenstates rapidity function, Eq. (24), one confirms that the relation \( Q_{\alpha\nu}(\nu q_{\alpha\nu})/N_a = \nu \delta q_{\alpha\nu} \) is valid for all PS excited states provided that \( v_{\alpha\nu}(\nu q_{\alpha\nu}) = 0 \). Indeed, Eq. (B18) reveals that the rapidity functions of the excited energy eigenstates of a given ground state equal those of the latter state with in the argument of such functions the ground-state bare momentum replaced by the excited-state canonical momentum. This property implies that the corresponding bare-momentum and canonical-momentum bands have precisely the same momentum width. Hence one has that \( Q_{\alpha\nu}(\nu q_{\alpha\nu})/N_a = -\nu \delta q_{\alpha\nu} \) for all PS excited states provided that \( v_{\alpha\nu}(\nu q_{\alpha\nu}) = 0 \).

The PDT processes (A) and (B) considered in Section IV D lead to separate contributions to the overall scattering phase shift \( Q_{\beta'}(q)/2 \). In the following we consider these two separate contributions. The part of the \( \beta' \) bare-momentum distribution-function deviation generated by \( \beta' \) scattering centers can be written as \( \delta N^{NF}_{\beta'}(q') + \delta N^{NF}_{\beta'}(q' + \delta N_{s1}^F(q')) \delta_{\nu_{s1}(q',a)} \) for the \( \beta' = c, s1 \) branches and \( \delta N^{NF}_{\beta'}(q') + \delta N^{NF}_{\beta'}(q') \) for the \( \beta' = \alpha' \not\equiv s1 \) branches. Here \( \delta N^{NF}_{\beta'}(q') \) is generated by the processes called elementary processes (A) in Section IV D. Those create and annihilate (and create) \( \beta' = c, s1 \) pseudofermions (and \( \beta' = \alpha' \not\equiv s1 \) pseudofermions) away from the \( \beta' = c, s1 \) Fermi points and \( s1 \) limiting values \( \pm q_{s1} = \pm 2k_F \) when \( k_F \rightarrow 2k_F \) and \( k_F \rightarrow 0 \) (and from the limiting values \( \pm q_{\alpha'\nu} \)). On the other hand, \( \delta N^{NF}_{\beta'}(q') \) and \( \delta N^{NF}_{\beta'}(q') \) are generated by the processes called elementary processes (B) in that section. Those involve the creation and annihilation (and creation) \( \beta' = c, s1 \) pseudofermions (and \( \beta' = \alpha' \not\equiv s1 \) pseudofermions of vanishing velocity) at the \( \beta' = c, s1 \) Fermi points (and at the limiting momentum values \( \pm q_{\alpha'\nu} \)).

The general expressions of the deviations \( \delta N^{NF}_{\beta'}(q') \) and \( \delta N^{NF}_{\beta'}(q') \) associated with scattering centers generated by
processes \((B)\) read,
\[
\delta N_{β}^{F}(q') = \sum_{i=±1} \left[ \frac{δN_{β}^{F}}{2} + i \, δJ_{β}^{F} \right] \delta q_{i} q_{Fβ'} , \quad β' = c, s1 ;
\]
\[
\delta N_{β}^{L}(q') = \sum_{i=±1} \left[ \frac{δN_{β}^{L}}{2} + i \, δJ_{β}^{L} \right] \delta q_{i} q_{β'} , \quad β' = α' ν' , \tag{B20}
\]
respectively. (For the \(β' = s1\) branch, the deviation \(δN_{β}^{F}(q')\) applies only to the vanishing-velocity \(s1\) pseudofermion scattering centers with \(s1\) limiting values \(±q_{s1} = ±2k_F\). Those correspond to the \(k_{F1} → 2k_F\) and \(k_{F1} → 0\) limit.)

The deviation numbers \(δN_{β}^{F}\) (and \(δN_{β}^{L}\)) are in Eq. \((B20)\) such that \(δN_{β}^{F} = δN_{β}^{F} + δN_{β}^{NF}\) (and \(δN_{β}^{L} = δN_{β}^{L} + δN_{β}^{NF}\)). They can be expressed as \(δN_{β}^{F} = δN_{β}^{F,+1} + δN_{β}^{F,-1}\) (and \(δN_{β}^{L} = δN_{β}^{L,+1} + δN_{β}^{L,-1}\)). Here \(δN_{β}^{β,±1}\) is the deviation in the number of \(β' = c, s1\) pseudofermions at the right (+1) and left (-1) \(β' = c, s1\) Fermi point (and \(δN_{β}^{F,+1}\) is the deviation in the number of \(β' = α' ν'\) pseudofermions created at \(i q_{β'}\) with \(i = ±1\)). The deviation current numbers read \(2δJ_{β}^{F} = [δN_{β}^{F,+1} - δN_{β}^{F,-1}]\) for \(β' = c, s1\) pseudofermions (and \(2δJ_{β}^{L} = [δN_{β}^{L,+1} - δN_{β}^{L,-1}]\) for vanishing-velocity \(β' = α' ν'\) pseudofermions. For the \(β' = α' ν' \neq s1\) pseudofermions the deviations \(δN_{β}^{β} = δN_{β}^{F} + δN_{β}^{NF}\) equal the corresponding numbers, \(N_{β}^{β} = N_{β}^{F} + N_{β}^{NF}\). Indeed the \(β' = α' ν' \neq s1\) pseudofermion occupancy vanishes for the ground states.

From the linearity in the deviations of the overall scattering phase shift, Eq. \((29)\), one can write \(Q_{β}^{⊕}(q)/2 = [Q_{β}^{NF}(q)/2 + Q_{β}^{F}(q)/2\). Also the part of the total momentum deviation, Eq. \((15)\), associated with the elementary processes \((A)\) and \((B)\) can be written as \(δP_{NF} + δP_{F}\). After some algebra involving the use of Eqs. \((29)\), \((15)\), \((59)\), and \((B20)\), we reach the following expressions for such quantities,
\[
\frac{Q_{β}^{NF}(q)}{2} = \sum_{β'} \sum_{q'} \pi Φ_{β,ν}(q, q') \delta N_{β}^{NF}(q') , \tag{B21}
\]
\[
\frac{Q_{β}^{NF}(q)}{2} = \sum_{β' = c, s1} \sum_{q'} \pi Φ_{β,ν}(q, q') \delta N_{β}^{ν}(q') , \tag{B22}
\]
\[
δP_{NF} = \sum_{β' = c, s1} \sum_{q'} q' \delta N_{β}^{NF}(q') + \sum_{q'} \sum_{q''}[π - q'] δN_{q''}^{NF}(q') ;
\]
\[
δP_{F} = \pi [M_{q,-1/2} + \sum_{ν' = 1}^{∞} ν' N_{ν'}^{L}] \tag{B23}
\]
In the above expressions we used that \(δJ_{β}^{F} = J_{β}^{F}\) for the \(β' = α' ν' \neq s1\) scattering centers.

The general expression of the phase shift \(Q_{β}^{NF}(q)/2\), Eq. \((B22)\), is valid for all active \(β\) scatterers, as defined in Section \(IVC\). In the \(δP_{F}\) expression, Eq. \((B23)\), we have included the contribution from the unbound \(-1/2\) \(ν\)-spins. (The momentum contributions from the unbound \(+1/2\) \(ν\)-spins and unbound \(±1/2\) spinons vanish.) Note that the current contributions to the momentum spectrum \(δP_{F}\), Eq. \((B23)\), which multiply \(4k_F\) and \(2k_{F1}\), are identical to the current contributions to the scattering phase shift, Eq. \((B22)\), which multiply the phase shifts \(π Φ_{β,ν}(q, ν' 2k_F)\) and \(π Φ_{β,β1}(q, ν' k_{F1})\), respectively.
Appendix C: Pseudofermion scattering theory for densities \( n = 1 \) and/or \( m = 0 \)

The general pseudofermion scattering theory studied in this paper also applies to PSs of \( S_n = 0; n = 1 \) and/or \( S_n = 0; m = 0 \) ground states, provided that the specific issues addressed here are accounted for.

The general two-pseudofermion phase-shift \( \pi \Phi_{\beta,\beta'}(q_j, q_{j'}) \) expression, Eq. (33), applies to ground states with densities in the ranges \( n \in [0, 1] \) and \( m \in [0, n] \). On the other hand, for excited states belonging to PSs of \( S_n = 0; n = 1 \) (and \( S_n = 0; m = 0 \)) ground states and \( \beta = \eta \nu \) scatterers and/or \( \beta' = \eta \nu' \) scattering centers (and \( \beta = \eta \nu \neq s1 \) scatterers and/or \( \beta' = s \nu' \neq s1 \) scattering centers) the rapidity function \( \Lambda_0^0(q_j) \) and/or \( \Lambda_0^0(q_{j'}) \) in the argument of the phase-shift \( \pi \Phi_{\beta,\beta'}(r, r') \), Eq. (32), must be replaced by a corresponding excited-state functional \( \Lambda_0(q_j) \) and/or \( \Lambda_0(q_{j'}) \), respectively. Fortunately, however, the \( n \to 1 \) (and \( m \to 0 \)) limit of the above equations defining the two-pseudofermion phase-shifts \( \pi \Phi_{\beta,\beta'}(r, r') \) defines as well such phase shifts for PSs of an \( S_n = 0; n = 1 \) (and \( S_n = 0; m = 0 \)) ground state.

We start by using such a property in the case of PSs of an \( m = 0 \) ground state with electronic density in the range \( n \in [0, 1] \). (The correct phase-shift results for electronic density \( n = 1 \) are reached by taking the \( n \to 1 \) limit in the following equations within which \( r_0 \to 0 \).) The rapidity two-pseudofermion phase shifts \( \pi \Phi_{\beta,\beta'}(r, r') \) are defined by the integral equations, Eqs. (B1)-(B15) of Appendix B. By Fourier transforming such equations after accounting for that \( B \to \infty \) and thus \( r_0^2 = 4tB/U \to \infty \) for \( u \) finite within the \( m \to 0 \) limit, we arrive to the following equations valid for \( m = 0 \) and \( u \) finite,

\[
\pi \Phi_{c,c}(r, r') = -B(r - r') + \int_{-r_0}^{r_0} dr'' \ A(r - r'') \pi \Phi_{c,c}(r'', r'), \tag{C1}
\]

\[
\pi \Phi_{c,s1}(r, r') = -\frac{1}{2} \arctan \left( \frac{\pi}{2} \frac{r - r'}{2} \right) + \int_{-r_0}^{r_0} dr'' \ A(r - r'') \pi \Phi_{c,s1}(r'', r'), \tag{C2}
\]

\[
\pi \Phi_{s1,c}(r, r') = -\frac{1}{2} \arctan \left( \frac{\pi}{2} \frac{r - r'}{2} \right) + \frac{1}{4} \int_{-r_0}^{r_0} dr'' \ \frac{\pi \Phi_{c,s1}(r'', r')}{\cosh \left( \frac{\pi}{2} \frac{r - r''}{2} \right)}; \quad r \neq \pm \infty
\]

\[
= -\frac{\text{sgn}(r)\pi}{2\sqrt{2}}; \quad r = \pm \infty, \tag{C3}
\]

\[
\pi \Phi_{s1,s1}(r, r') = B(r - r') + \frac{1}{4} \int_{-r_0}^{r_0} dr'' \ \frac{\pi \Phi_{c,s1}(r'', r')}{\cosh \left( \frac{\pi}{2} \frac{r - r''}{2} \right)}; \quad r \neq \pm \infty
\]

\[
= \frac{\text{sgn}(r)\pi}{2\sqrt{2}}; \quad r = \pm \infty, \quad r' \neq r
\]

\[
= \left[ \text{sgn}(r) \right] \left( \frac{3}{2\sqrt{2}} - 1 \right) \pi; \quad r = r' = \pm \infty, \tag{C4}
\]

\[
\pi \Phi_{c,\nu}(r, r') = -\arctan \left( \frac{r - r'}{\nu} \right) + \int_{-r_0}^{r_0} dr'' \ A(r - r'') \pi \Phi_{c,\nu}(r'', r'), \tag{C5}
\]

\[
\pi \Phi_{c,\nu}(r, r') = 0; \quad \nu > 1, \tag{C6}
\]

\[
\pi \Phi_{s1,\nu}(r, r') = \frac{1}{4} \int_{-r_0}^{r_0} dr'' \ \frac{\pi \Phi_{c,\nu}(r'', r')}{\cosh \left( \frac{\pi}{2} \frac{r - r''}{2} \right)}; \tag{C7}
\]

\[
\pi \Phi_{s1,\nu}(r, r') = \arctan \left( \frac{r - r'}{\nu} \right); \quad r \neq \pm \infty; \quad \nu > 1,
\]

\[
= \frac{\pi}{\sqrt{2}}; \quad r = \pm \infty; \quad \nu > 1, \tag{C8}
\]
The two-pseudofermion phase shifts \( \Phi_{\nu,\nu'}(r, r') \), \( \Phi_{\nu',\nu}(r, r') \), and \( \Phi_{\nu,\nu'}(r, r') \) remain being given by Eqs. (B10), (B12) of Appendix B.

In the above expressions, the function \( \Theta_{\nu,\nu'}(x) \) is defined in Eq. (B10) of that Appendix,

\[
B(r) = \int_0^\infty d\omega \frac{\sin(\omega r)}{\omega(1 + e^{2\omega})} = \frac{i}{\pi} \ln \frac{\Gamma\left(\frac{1}{4} + i\frac{r-r'}{4}\right) \Gamma\left(1 - i\frac{r-r'}{4}\right)}{\Gamma\left(\frac{1}{4} - i\frac{r-r'}{4}\right) \Gamma\left(1 + i\frac{r-r'}{4}\right)},
\]

and

\[
A(r) = \frac{1}{\pi} \frac{dB(r)}{dr} = \frac{1}{\pi} \int_0^\infty d\omega \frac{\cos(\omega r)}{1 + e^{2\omega}},
\]

where \( \Gamma(x) \) is the usual \( \Gamma \) function. The four equations, Eqs. (C1) with \( s_1 = s, r = x \), and \( r_0 = x_0 \), are equivalent to Eqs. (A9)-(A12) of Ref. [54]. (For the phase shifts, Eqs. (C3) and (C4), this equality refers to values of \( r \) such that \( r \neq \infty \).)

The two-pseudofermion phase-shift expressions defined here for \( m = 0 \) are those of some of the two-pseudofermion phase shifts plotted in Figs. 1B. Specifically, the two-pseudofermion phase shifts \( \Phi_{c,c}(q, q'), \Phi_{s1,s1}(q, q'), \Phi_{s1,c}(q, q'), \Phi_{c,s1}(q, q'), \Phi_{s1,s1}(q, q') \), and \( \Phi_{c,s1}(q, q') \), Eq. (32), are plotted in such figures in units of \( \pi \) for \( n = 0.59, m = 0 \), and several \( U/t \) values.

As mentioned above, by taking the \( n \rightarrow 1 \) and \( r_0 \rightarrow 0 \) limits the phase-shift equations, Eqs. (C6) and (C8)-(C11), apply to PS excited states generated by transitions from a \( S_n = 0; n = 1 \) and \( S_n = 0; m = 0 \) ground state. Such a procedure leads to,

\[
\pi \Phi_{c,c}(r, r') = -B(r - r') = -\frac{i}{2} \ln \frac{\Gamma\left(\frac{1}{4} + i\frac{(r-r')}{4}\right) \Gamma\left(1 - i\frac{(r-r')}{4}\right)}{\Gamma\left(\frac{1}{4} - i\frac{(r-r')}{4}\right) \Gamma\left(1 + i\frac{(r-r')}{4}\right)},
\]

\[
\pi \Phi_{s1,c}(r, r') = -\frac{1}{2} \arctan\left(\sinh\left(\frac{\pi}{2}(r - r')\right)\right) = \frac{i}{2} \ln \left(-i \frac{1 + ie^{\pi(1-r')}}{1 - ie^{\pi(r-r')}}, \quad r \neq \pm \infty \right)
\]

\[
\pi \Phi_{s1,s1}(r, r') = -\frac{1}{2} \arctan\left(\sinh\left(\frac{\pi}{2}(r - r')\right)\right) = \frac{i}{2} \ln \left(-i \frac{1 + ie^{\pi(r-r')}}{1 - ie^{\pi(r-r')}}, \quad r \neq r' \neq \pm \infty \right)
\]

\[
\pi \Phi_{c,s1}(r, r') = -\arctan\left(\frac{r-r'}{\nu}\right), \quad \pi \Phi_{s1,\eta}(r, r') = \pi \Phi_{c,\eta'}(r, r') = 0, \quad \nu' > 1,
\]
For a latter equation the following closed-form expressions for the ground-state rapidity functions \( s_\nu \neq 0 \) ground state with electronic density in the range \( n \in [0,1] \) and \( m = 0 \). The form of the general two-pseudofermion phase-shift expression, Eq. (32), reveals that the evaluation of such \( u \to 0 \) analytical expressions requires that of the ground-state rapidity functions \( \Lambda_\beta^0(q) \) for \( \beta = c, s1, \eta \nu \) where \( \nu = 1, \ldots, \infty \). (For \( m = 0 \) ground state there are no \( su \neq s1 \) bands so that the corresponding rapidity functions \( \Lambda_\beta^0(q) \) are undefined for \( \nu > 1 \).

The \( \beta = c, s1, \eta \nu \) ground-state rapidity functions \( \Lambda_\beta^0(q) \) are in Eq. (A15) defined in terms of their inverse functions. For a \( m = 0 \) ground state with electronic density in the range \( n \in [0,1] \) we find in the \( u \to 0 \) limit from use of the latter equation the following closed-form expressions for the ground-state rapidity functions \( k_c^0(q) \), \( \Lambda_c^0(q) \), \( \Lambda_{su}^0(q) \), and \( \Lambda_{s1}^0(q) \),

\[
\begin{align*}
k_c^0(q) &= \frac{q}{2}, \quad |q| \leq 2k_F, \\
&= \text{sgn}(q) |q| - k_F|, \quad 2k_F \leq |q| < \pi \\
&= \text{sgn}(q) \pi, \quad |q| = \pi, \quad u \to 0, \\
\Lambda_c^0(q) &= \sin\left(\frac{q}{2}\right), \quad |q| \leq 2k_F, \\
&= \text{sgn}(q) \sin\left(|q| - k_F\right), \quad 2k_F \leq |q| < \pi, \\
&= 0, \quad |q| = \pi, \quad u \to 0, \\
\Lambda_{su}^0(q) &= \text{sgn}(q) \sin\left(\frac{|q| + \pi n}{2}\right), \quad 0 < |q| < (\pi - 2k_F), \\
&= 0, \quad q = 0, \\
&= \pm \infty, \quad q = \pm (\pi - 2k_F), \quad u \to 0, \\
\Lambda_{s1}^0(q) &= \sin(q), \quad |q| < k_F, \\
&= \pm \infty, \quad q = \pm k_F, \quad u \to 0,
\end{align*}
\]

and

\[
\begin{align*}
\pi \Phi_{s1,c}(q, q') &= -\text{sgn}\left(\sin k_c^0(q) - \sin k_c^0(q')\right) \frac{\pi}{C_c(q)} + \delta_{q1,2kF} \delta_{q,q'} |\text{sgn}(q)| \left(\frac{3}{2\sqrt{2}} - 1\right) \pi, \\
\pi \Phi_{c,s1}(q, q') &= -\text{sgn}\left(\sin k_c^0(q) - c_{s1}(q') \sin(q')\right) \frac{\pi}{C_c(q)}, \\
\pi \Phi_{s1,c}(q, q') &= -\text{sgn}\left(\sin(q) - \sin k_c^0(q')\right) \frac{\pi}{2}, \quad q \neq \pm k_F \\
&= -\frac{\text{sgn}(q) \pi}{2\sqrt{2}}, \quad q = \pm k_F, \quad u \to 0 \\
&= 0; \quad q = \pm k_F, \quad u = 0,
\end{align*}
\]
\[ \pi \Phi_{\alpha \nu, 1}(q, q') = 0, \quad q \neq \pm k_F \]
\[ = \frac{\text{sgn}(q)\pi}{2\sqrt{2}} \left[ 1 + \delta_{q, q'} 2(1 - \sqrt{2}) \right], \quad q = \pm k_F, \quad u \to 0 \]
\[ = 0, \quad q = \pm k_F, \quad u = 0, \]
(C28)

\[ \pi \Phi_{c, \nu 1}(q, q') = -\text{sgn} \left( \sin k^0_c(q) - c_{\nu 1}(q') \sin k^0_{\nu 1}(q') \right) \frac{2\pi}{C_c(q)}, \]  
(C29)
and

\[ \pi \Phi_{s, \nu 1}(q, q') = -\theta(k_F - |q|) \text{sgn} \left( \sin q - \sin k^0_{\nu 1}(q') \right) \frac{\pi}{2}. \]  
(C30)

Here the sign function is such that \( \text{sgn}(0) = 0 \) and the \( \theta(x) \) function reads \( \theta(x) = 1 \) for \( x > 0 \) and \( \theta(x) = 0 \) for \( x \leq 0 \), alike that of Eq. (17). Hence, \( \pi \Phi_{s, \nu 1}(\pm k_F, q') = 0 \). Moreover, in the above equations, \( k^0_c(q) = \lim_{u \to 0} k^0_c(q) \) where the \( u \to 0 \) value of \( k^0_c(q) \) is given in Eq. (C21).

\[ k^0_{\nu 1}(q) = \frac{q}{2} + \text{sgn}(q) k_F, \quad 0 < |q| \leq |\pi - 2k_F| \]
\[ = 0, \quad q = 0, \]  
(C31)

\[ C_c(q) = 2 \left[ \theta(2k_F - |q|) + \sqrt{2} \delta_{q, 2k_F} + 2 \theta(|q| - 2k_F) \right], \]  
(C32)

and

\[ c_{\nu 1}(q) = 1, \quad |q| < k_F, \quad c_{\nu 1}(q) = \infty, \quad q = \pm k_F \]
\[ c_{\nu 1}(q) = 1, \quad |q| < |\pi - 2k_F|, \quad c_{\nu 1}(q) = \infty, \quad q = \pm |\pi - 2k_F|. \]  
(C33)

Finally, we address the issue of why in spite of the lack of ground-state \( \eta \nu \) (and \( s \nu \neq s 1 \)) pseudofermion bands, the pseudofermion scattering theory can be generalized to the PSs of \( S_\eta = 0; n = 1 \) (and \( S_\nu = 0; m = 0 \)) ground states. The point is that the “in” asymptote one-pseudofermion scattering states do not contribute to the direct-product expression of the ground state but rather to that of the “in” state, as defined in Section III C. We start by considering

The point is that the “in” asymptote one-pseudofermion scattering states do not contribute to the direct-product expression of the ground state but rather to that of the “in” state, as defined in Section III C. We start by considering

...
pseudofermion and one $sv'$ $\neq s1$ pseudofermion, (b) one $\eta\nu$ pseudofermion, and (c) one $sv' \neq s1$ pseudofermion. Consistent with Eq. (39) and as discussed in Section IIIA the excited energy eigenstates (a) and (b) have $2\nu$ $c$ pseudofermion holes associated with the $\eta\nu$ pseudofermion and the excited states (a) and (c) have $2(\nu'-1)$ $s1$ pseudofermion holes associated with the $sv' \neq s1$ state.

The $S_\eta=0$; $S_r=0$; $2S_c=N_a$ absolute ground state is described by full $c$ and $s1$ pseudofermion bands whose $c$ and $s1$ Fermi momenta read $q_{Fe}=2kF_c=\pi$ and $q_{Fs1}=k_{F1}=q_{s1}=k_{F1}=\pi/2$, respectively. Thus, from the use of Eq. (40) of Appendix A we find that $\delta N_c=-2\delta N_c^h=-2\nu$, $\delta N_{s1}=-\nu+\nu'$, $\delta N_{s1}^h=2(\nu'-1)$ for the excited states (a), $\delta N_c=-2\nu$, $\delta N_{s1}=-\nu$, $\delta N_{s1}^h=0$ for the excited states (b), and $\delta N_c=-\delta N_c^h=0$, $\delta N_{s1}=-\nu'$, $\delta N_{s1}^h=2(\nu'-1)$ for the excited states (c).

It follows from Eqs. (C9), (C18), (C19), and (C20) that $\pi \Phi_{c,sv'}(q,0)=\pi \Phi_{\eta\nu,sv'}(q,0)=\pi \Phi_{sv',\eta\nu}(q,0)=0$ for the $sv' \neq s1$ and $\eta\nu'$ scattering centers of the PS excited states of the $S_\eta=0$; $S_r=0$; $2S_c=N_a$ absolute ground state. On the other hand, according to Eqs. (C9), (C10), and (C20), within the simultaneous $m \rightarrow 0$ and $n \rightarrow 1$ limits the two-pseudofermion phase shifts that contribute to the phase shifts $Q_{\eta\nu}(0)/2$ and $Q_{sv'}(0)/2$ of such a PS excited states simplify to $\pi \Phi_{\eta\nu,sv'}(r',r)=\pi \Phi_{sv',\eta\nu}(r',r)=\pi \Phi_{sv',s\nu}(r',r)=0$, $\pi \Phi_{\eta\nu,c}(r',r)=\arctan(r'/r)$ and $\pi \Phi_{sv',s\nu}(r',r)=\arctan(r'/r)$ for $sv' \neq s1$. It follows then from the use of Eq. (29) that for the excited energy eigenstates (a)-(c) the equation $Q_{\eta\nu}(0)/2=0$ and/or $Q_{sv'}(0)/2=0$ leads to,

$$
\frac{Q_{\eta\nu}(0)}{2} = \sum_{h=1}^{2\nu} \pi \Phi_{\eta\nu,c}(0,q_h) = \sum_{h=1}^{2\nu} \arctan \left( \frac{\Lambda_{\eta\nu}(0) - \Lambda_{\eta\nu}^0(q_h)}{\nu u} \right) = 0,
$$

$$
\frac{Q_{sv'}(0)}{2} = \sum_{h=1}^{2(\nu'-1)} \pi \Phi_{sv',s\nu}(0,q_h) = \sum_{h=1}^{2(\nu'-1)} \arctan \left( \frac{\Lambda_{sv'}(0) - \Lambda_{sv'}^0(q_h)}{(\nu'-1) u} \right) = 0, \quad sv' \neq s1. \quad (C34)
$$

Here the first and second equations refer to the $c$ branch of both the states (a) and (b) and to the $s1$ branch of both the states (a) and (c), respectively. In these equations, the set of $2\nu=2,4,6,...$ values $\{q_h\}$ and of $2(\nu'-1)=2,4,6,...$ values $\{q_h\}$ correspond to the above-mentioned excited-energy-eigenstate $c$ pseudofermion holes and $s1$ pseudofermion holes, respectively. As discussed in Section IIIA the physics revealed by these results is that addition of the $2\nu=2,4,6,...$ effective phase shifts of the $\alpha\nu \neq s1$ pseudofermion that result from its collisions with its $2\nu=2,4,6,...$ neutral-shadow scattering centers, as defined in Section IIIA exactly cancel each other, so that $Q_{\eta\nu}(0)=0$.

The absolute ground-state rapidity functions, $\Lambda^0_\eta(q) = \sin k_\eta(q)$ and $\Lambda^0_{s1}(q)$, are defined in terms of their inverse functions in Eq. (D12) of Appendix D. The form of the inequalities, Eq. (C34), reveals that the corresponding solutions, $\Lambda_{\eta\nu}(0) = \Lambda_{\eta\nu}(0,\{q_h\})$ and/or $\Lambda_{sv'}(0) = \Lambda_{sv'}(0,\{q_h\})$, are functions of the above sets of bare-momentum values $\{q_h\}$ and $\{q_h\}$, respectively.

We have obtained here Eq. (C34) from the $Q_{\eta\nu}(0)/2=0$ condition imposed by the invariance under the electron - rotated-electron unitary transformation of the $sv' \neq s1$ pseudofermion. For $\alpha\nu \neq \eta\nu$ and $\alpha\nu = sv' \neq s1$ that equation is obeyed by the functions $\Lambda_{\eta\nu}(0) = \Lambda_{\eta\nu}(0,\{q_h\})$ and $\Lambda_{sv'}(0) = \Lambda_{sv'}(0,\{q_h\})$ for the rapidities $\Lambda_{\eta\nu}(0)$ and $\Lambda_{sv'}(0)$, respectively. For the above excited states, the solution of the thermodynamic BA equations, Eqs. (A3)-(A6) of Appendix A leads exactly to the same functions, which thus again obey Eq. (C34). This confirms that for such excited states the exact solution of the BA equations is equivalent to imposing the symmetry requirement $Q_{\eta\nu}(0)/2=0$ and $Q_{sv'}(0)/2=0$. The latter is associated with the invariance under the electron - rotated-electron unitary transformation non-scatterer character of the $\eta\nu$ pseudofermion and/or $sv'$ pseudofermion under consideration, respectively. Hence the BA solution accounts for that symmetry.

The above functions $\Lambda_{\eta\nu}(0) = \Lambda_{\eta\nu}(0,\{q_h\})$ and $\Lambda_{sv'}(0) = \Lambda_{sv'}(0,\{q_h\})$ are to be used in the following expressions,

$$
\pi \Phi_{\eta\nu,c}(0,q') = \pi \Phi_{sv',c}(\Lambda_{\eta\nu}(0,\{q_h\}), \Lambda_{\eta\nu}^0(q')) = \arctan \left( \frac{\Lambda_{\eta\nu}(0,\{q_h\}) - \Lambda_{\eta\nu}^0(q')}{\nu u} \right);
$$

$$
\pi \Phi_{sv',s\nu}(0,q') = \pi \Phi_{sv',s\nu}(\Lambda_{sv'}(0,\{q_h\}), \Lambda_{sv'}^0(q')) = \arctan \left( \frac{\Lambda_{sv'}(0,\{q_h\}) - \Lambda_{sv'}^0(q')}{(\nu'-1) u} \right). \quad (C35)
$$

They assure that $Q_{\eta\nu}(0)/2=0$ and/or $Q_{sv'}(0)/2=0$. The simplest case corresponds to $\nu=1$ and/or $\nu'=2$. In that case, the solution of Eq. (C34) leads to $\Lambda_{q1}(0,q_1,q_2) = [\Lambda_{q1}^0(q_1) + \Lambda_{q1}^0(q_2)]/2$ and/or $\Lambda_{s2}(0,q_1',q_2') = [\Lambda_{s2}^0(q_1') + \Lambda_{s2}^0(q_2')]/2$, respectively. Hence, the symmetry requirement that the $\eta\nu$ pseudofermion (and $sv' \neq s1$ pseudo-fermion) considered here is not a scatterer implies that the corresponding rapidity function $\Lambda_{\eta\nu}(0) = \Lambda_{\eta\nu}(0,\{q_h\})$ (and $\Lambda_{sv'}(0) = \Lambda_{sv'}(0,\{q_h\})$) does not in general vanish. Rather, it is the unique solution of the first (and second) equation in Eq. (C34).
On the other hand, combination of this result with the two-pseudofermion phase shift expressions, Eqs. (C18) and (C19), reveals that the c scatterer two-pseudofermion phase shift, \( \pi \Phi_{c,\nu}(q, 0) \), and the s1 scatterer two-pseudofermion phase shift, \( \pi \Phi_{s1,\nu'}(q, 0) \), are such that,

\[
\pi \Phi_{c,\nu}(q, 0) = \pi \tilde{\Phi}_{c,\nu}(\frac{\Lambda_0(q)}{u}, \frac{\Lambda_{\nu}(0, \{ q_h \})}{u}) = -\arctan\left(\frac{\Lambda_0(q) - \Lambda_{\nu}(0, \{ q_h \})}{\nu u}\right), \\
\pi \Phi_{s1,\nu'}(q, 0) = \pi \tilde{\Phi}_{s1,\nu'}(\frac{\Lambda_0(q)}{u}, \frac{\Lambda_{\nu'}(0, \{ q'_h \})}{u}) = \arctan\left(\frac{\Lambda_0(q) - \Lambda_{\nu'}(0, \{ q'_h \})}{(\nu - 1) u}\right), \quad q \neq \pm k_F
\]

(C36)

In addition to the c or s1 scatterer bare-momentum q, the two-pseudofermion phase shifts provided in Eq. (C36) are functions of the set of \( 2\nu = 2, 4, 6, \ldots \) bare-momentum values \( \{ q_h \} \) or \( 2(\nu' - 1) = 2, 4, 6, \ldots \) bare-momentum values \( \{ q'_h \} \) of the \( 2\nu = 2, 4, 6, \ldots \) neutral-shadow c pseudofermion-hole scattering centers or \( 2(\nu' - 1) = 2, 4, 6, \ldots \) neutral-shadow s1 pseudofermion-hole scattering centers, respectively. As discussed in Section [11C], the latter emerge under the ground-state - excited-state transition so that their virtual \( \alpha \) elementary currents plus those of the \( \alpha' \neq s1 \) pseudofermion exactly cancel.

This confirms that as a result of the creation of one \( \eta \nu \) pseudofermion (and one \( \nu' \neq s1 \) pseudofermion), the c (and s1) scatterers acquire the phase shift \( \pi \Phi_{c,\nu}(q, 0) \) (and \( \pi \Phi_{s1,\nu'}(q, 0) \)). Its value is fully controlled by the \( 2\nu = 2, 4, 6, \ldots \) (and \( 2(\nu' - 1) = 2, 4, 6, \ldots \)) bare-momentum values of the corresponding \( 2\nu = 2, 4, 6, \ldots \) (and \( 2(\nu' - 1) = 2, 4, 6, \ldots \)) neutral-shadow c (and s1) pseudofermion-hole scattering centers. Thus, through the \( \{ q_h \} \) (and \( \{ q'_h \} \)) momentum dependence of the two-pseudofermion phase shift \( \pi \Phi_{c,\nu}(q, 0) \) (and \( \pi \Phi_{s1,\nu'}(q, 0) \)), the c (and s1) scatterers feel the \( \eta \nu \) pseudofermion (and the \( \nu' \neq s1 \) pseudofermion) created under the transition to the excited state as \( 2\nu = 2, 4, 6, \ldots \) corresponding neutral-shadow c effective scattering centers (and \( 2(\nu' - 1) = 2, 4, 6, \ldots \) corresponding neutral-shadow s1 effective scattering centers).

Similar results are obtained for PS excited states of \( S_n = 0; n = 1 \) and/or \( S_s = 0; m = 0 \) ground states with occupancy of a larger finite number of \( \beta \) pseudofermions belonging to several \( \beta = \alpha \nu \neq s1 \) branches. In that general case, the number of equations defining the set of rapidities \( \{ \Lambda_{\alpha\nu} \} \) is in general larger than above. Moreover, each of these equations is more involved than the two equations, Eq. (C34). And the equations that result from the \( Q_{\eta\nu}^F(0)/2 = 0 \) and \( Q_{\eta\nu}^F(0)/2 = 0 \) symmetry requirements involve additional phase shifts \( \pi \Phi_{\nu',\nu'}(r', r') \) and \( \pi \Phi_{\eta\nu,\nu'}(r, r') \), Eq. (B13) of Appendix B and Eq. (C20), respectively. The corresponding \( \beta = \alpha \nu \neq s1 \) pseudofermions are also invariant under both the electron - rotated-electron unitary transformation and pseudoparticle - pseudofermion unitary transformation.

Finally, the expression for \( Q_{\eta\nu}^F(0)/2 \) (and \( Q_{\eta\nu}^F(0)/2 \)), Eq. (C34), is also valid for the type of excited states of \( S_n = 0; n = 1 \) ground states with spin density \( m \neq 0 \) (and of \( S_s = 0; m = 0 \) ground states with electronic density \( n \neq 1 \)) with the following occupancies: no unbound \( \eta \)-spinons and no unbound spinons, finite pseudofermion occupancy in the c and s1 bands, plus one \( \eta \nu \) pseudofermion (and one \( \nu' \neq s1 \) pseudofermion). Indeed, within the \( n \to 1 \) limit for \( m \neq 0 \) (and the \( m \to 0 \) limit for \( n \neq 1 \)) the two-pseudofermion phase shifts that contribute to \( Q_{\eta\nu}^F(0)/2 \) (and \( Q_{\eta\nu}^F(0)/2 \)) also simplify to \( \pi \Phi_{\eta\nu,\nu'}(r, r') = 0 \) and \( \pi \Phi_{\eta\nu,\nu'}(r, r') = \text{arctan}\left(\frac{\sqrt{r'}}{r'}\right) \) (and to \( \pi \Phi_{\eta\nu,\nu'}(r', r') = \pi \Phi_{\eta\nu,\nu'}(r', r') = 0 \) and \( \pi \Phi_{\eta\nu,\nu'}(r', r) = \text{arctan}\left(\frac{\sqrt{r'}}{r'}\right) \) for \( \nu' \neq s1 \)). Similarly, the expressions for \( \pi \Phi_{\eta\nu,\nu}(0, q') \) and \( \pi \Phi_{\eta\nu,\nu}(q, 0) \) (and \( \pi \Phi_{\eta\nu,\nu}(0, q') \) and \( \pi \Phi_{\eta\nu,\nu}(q, 0) \)), Eqs. (C35) and (C36), respectively, are valid as well for the above type of excited states of \( S_n = 0; n = 1 \) ground states with spin density \( m \neq 0 \) (and excited states of \( S_s = 0; m = 0 \) ground states with electronic density \( n \neq 1 \)).

**Appendix D: Useful energy scales**

In this Appendix we introduce several energy scales that are extracted from the BA solution and play an important role in the studies of this paper.

For electronic densities \( n \neq 1 \), the values of the chemical potential \( \mu = \mu(n) \) and magnetic-field energy \( 2\mu_B H = 2\mu_B H(m) \) are fully controlled by the energy dispersions of the \( \beta = c, s1 \) bands, Eq. (A21) of Appendix A. They read

\[
\mu = \text{sgn}\{(1 - n)\} \left[ \epsilon_0^c(q_Fc) + \frac{1}{2} \epsilon_0^{s1}(q_{Fs1}) \right]; \quad 2\mu_B H = \text{sgn}\{m\} \epsilon_0^{s1}(q_{Fs1}), \quad n \neq 1.
\]

(D1)

The expressions given here are valid for the whole range of densities \( n \neq 1 \) and \( m \).
The corresponding chemical-potential dependence on the hole concentration \( x = (1 - n) \) is such that,

\[
\mu(x) = -\mu(-x) ; \quad \mu(x) \in [\mu^0, \mu^1] , \quad x \in [-1, 0] ; \quad \mu \in [-\mu^0, \mu^0] , \quad x = 0 ,
\]

\[
\mp \mu^1 = \lim_{x \to \pm 1} \mu(x) ; \quad \mp \mu^0 = \lim_{x \to 0} \mu(x) , \quad \mu^1 > \mu^0 . \tag{D2}
\]

The energy scales \( \mu^0 \) and \( \mu^1 \) are defined in the following.

The finiteness for \( u > 0 \) of the half-filling Mott-Hubbard gap, \( 2\mu^0 \), implies that the chemical potential curve \( \mu = \mu(x) \) has a well-defined discontinuity at \( n = 1 \). For \( u \gg 1 \) and \( m = 0 \) the corresponding energy scale \( 2\mu = 2\mu(x) \) is of the form,

\[
2\mu(x) = -\text{sgn}(x)[U - 4t \cos(\pi x)] , \quad x \in [-1, 0] \text{ and } x \in [0, 1] ,
\]

\[
\in [-U - 4t, U - 4t] , \quad x = 0 . \tag{D3}
\]

For \( u > 0 \) and \( n = 1 \) the Mott-Hubbard gap remains finite for all spin densities \( m \in [-1, 1] \). It is an even function of \( m \). For instance, at \( m = 0 \) and \( m = -1, 1 \) it is given by [3],

\[
2\mu^0 = U - 4t + 8t \int_0^\infty d\omega \frac{J_1(\omega)}{\omega(1 + e^{\pm 2iu})} = \frac{16t^2}{U} \int_1^\infty d\omega \frac{\sqrt{\omega^2 - 1}}{\sinh(\frac{2\pi\omega}{U})} , \quad m = 0 ,
\]

\[
= \sqrt{(4t)^2 + U^2 - 4t} , \quad m = -1,1 , \tag{D4}
\]

respectively, where \( J_1(\omega) \) is a Bessel function. For \( u \ll 1 \) and \( u \gg 1 \), this energy scale behaves as,

\[
2\mu^0 \approx \frac{8}{\pi} \sqrt{U} e^{-2\pi(\frac{1}{2})}, \quad m = 0 ; \quad 2\mu^0 \approx \frac{U^2}{8t} , \quad m = -1,1 , \quad u \ll 1 ,
\]

\[
2\mu^0 \approx |U - 4t| , \quad m \in [-1,1] , \quad u \gg 1 . \tag{D5}
\]

The energy scale \( 2\mu^1 \) associated with the minimum \(-\mu^1\) and maximum \( \mu^1 \) chemical-potential values is for all \( m \) magnitudes given by,

\[
2\mu^1 = |U + 4t| . \tag{D6}
\]

On the other hand, the magnetic energy scale \( 2\mu_B H \), Eq. (D1), dependence on the spin density \( m \) is such that,

\[
2\mu_B H(m) = -2\mu_B H(-m) ; \quad 2\mu_B H(m) \in [0, 2\mu_B H_c] , \quad m \in [-1 - |x|, 0] ,
\]

\[
2\mu_B H(0) = 0 ; \quad \mp 2\mu_B H = \lim_{m \to \pm 1 - |x|} 2\mu_B H(m) , \quad x = (1 - n) . \tag{D7}
\]

A closed-form expression for the dependence on \( U, t, \) and density \( n \) of the energy scale \( 2\mu_B H_c \) associated with the critical field \( H_c \) can be derived from the general \( 2\mu_B H \) expression, Eq. (D1) [58]. Since \( 2\mu_B H_c \) is a even function of the hole concentration \( x = (1 - n) \), we expressed it in terms of it ,

\[
2\mu_B H_c = \frac{1}{2} \sqrt{(4t)^2 + U^2} \left[ 1 + \frac{2}{\pi} \arccot \left( \frac{\sqrt{(4t)^2 + U^2}}{U} \tan(\pi|x|) \right) \right]
\]

\[
- U (1 - |x|) - 4t \cos(\pi x) \arctan(\frac{4t \sin(\pi |x|)}{U}) , \quad x = (1 - n) . \tag{D8}
\]

At \( u = 0 \) and for \( u \gg 1 \) its limiting behaviors are,

\[
2\mu_B H_c = \frac{4t \sin^2(\frac{\pi x}{2})}{U} , \quad u = 0 , \quad x = (1 - n) ,
\]

\[
\approx \frac{8(1 - |x|)t^2}{U} \left[ 1 + \frac{\sin(2\pi |x|)}{2\pi(1 - |x|)} \right] , \quad u \gg 1 , \quad x = (1 - n) . \tag{D9}
\]

As a function of \( n \), it has for instance the following values,

\[
2\mu_B H_c = 0 , \quad n = 0, 2 ,
\]

\[
= \frac{1}{2} \left[ \sqrt{(4t)^2 + U^2} - U \right] , \quad n = \frac{1}{2} \cdot \frac{3}{2} ,
\]

\[
= \sqrt{(4t)^2 + U^2} - U , \quad n = 1 . \tag{D10}
\]
Other energy scales involved in the studies of this paper are the \(c\) and \(\alpha \nu\) pseudoparticle energy dispersions, Eqs. (A21) and (A22) of Appendix A. Both the momentum widths \(2\pi (1 - n)\) and \(2\pi n\) of the \(\eta \nu\) and \(s \nu\) \(\neq s 1\) momentum bands, respectively, and their energy-dispersion bandwidths, \(|\varepsilon^0_{c, \nu}(q, u) - \varepsilon^0_{\nu, \nu}(0)|\), where \(\alpha \nu = \eta \nu\) and \(\alpha \nu = s \nu \neq s 1\), vanish in the \(n \to 1, m \to 0\) limit. Consistent, for the \(S_y = 0; n = 1; S_z = 0; m = 0; 2S_c = N_v\) absolute ground state the corresponding \(\alpha \nu \neq s 1\) pseudoparticle energy dispersions do not exist. On the other hand, the \(c\) pseudoparticle energy dispersions \(\varepsilon^0_c(q)\) and \(\varepsilon_c(q)\) and \(s 1\) pseudoparticle energy dispersions \(\varepsilon^0_{s 1}(q)\) and \(\varepsilon_{s 1}(q)\) have closed-form expressions, which read \[54\],

\[
\varepsilon^0_c(q) = \frac{-U}{2} - 2t \cos k^0_c(q) - 4t \int_0^\infty \frac{d\omega}{\omega(1 + e^{u/2\omega})} J_1(\omega), \quad q \in [-\pi, \pi],
\]

\[
\varepsilon_c(q) = \varepsilon^0_c(q) + \mu^0, \quad q \in [-\pi, \pi],
\]

\[
\varepsilon_{s 1}(q) = \varepsilon^0_{s 1}(q) = -2t \int_0^\infty \frac{d\omega}{\omega} \frac{\cos(\omega \Lambda^0_{s 1}(q))}{\omega \cosh(\omega u)} J_1(\omega), \quad q \in [-\pi/2, \pi/2].
\]

The absolute ground state rapidity functions \(k^0_c(q)\) (such that \(\Lambda^0_c(q) = \sin k^0_c(q)\)) and \(\Lambda^0_{s 1}(q)\) are defined in terms of their inverse functions as follows \[54\],

\[
q = k^0_c(q) + 2 \int_0^\infty \frac{d\omega}{\omega} \frac{\sin(\omega \sin k^0_c(q))}{\omega(1 + e^{u/2\omega})} J_0(\omega), \quad q \in [-\pi, \pi],
\]

\[
q = \int_0^\infty \frac{d\omega}{\omega} \frac{\sin(\omega \Lambda^0_{s 1}(q))}{\omega \cosh(\omega u)} J_0(\omega), \quad q \in [-\pi/2, \pi/2].
\]

On the other hand, for \(m \to 0\) and \(n \in [0, 1]\) all \(s \nu \neq s 1\) bands momentum and energy bandwidths vanish. Consistent, for a \(S_z; m = 0\) ground state the corresponding \(s \nu \neq s 1\) pseudoparticle energy dispersions, Eqs. (A21) and (A22) of Appendix A do not exist. For densities \(n \in [0, 1]\) and \(m = 0\) the \(c\) pseudoparticle, \(s 1\) pseudoparticle, and \(\eta \nu\) pseudoparticle energy dispersions \(\varepsilon_c(q), \varepsilon_{s 1}(q),\) and \(\varepsilon^0_{\nu}(q)\), respectively, defined in such equations have the following limiting behaviors for \(u \to 0\) and \(u \gg 1\),

\[
\varepsilon_c(q) = -4t \left[ \cos \left( \frac{q}{2} \right) - \cos \left( \frac{\pi n}{2} \right) \right], \quad |q| \leq 2k_F = \pi n, \quad u \to 0,
\]

\[
= -2t \left[ \cos \left( \frac{q}{2} - \frac{\pi n}{2} \right) - \cos \left( \frac{\pi n}{2} \right) \right], \quad 2k_F = \pi n \leq |q| \leq \pi, \quad u \to 0,
\]

\[
= -2t \left[ \cos(q) - \cos(\pi n) \right] - \frac{8nt^2}{U} \ln(2) \left[ \sin^2(q) - \sin^2(\pi n) \right], \quad |q| \leq \pi, \quad u \gg 1,
\]

\[
\varepsilon_{s 1}(q) = \varepsilon^0_{s 1}(q) = -2t \left[ \cos(q) - \cos \left( \frac{\pi n}{2} \right) \right], \quad |q| \leq k_F = \pi n/2, \quad u \to 0,
\]

\[
= -2\pi nt^2 \left[ 1 - \frac{\sin(\pi n)}{2\pi n} \right] \cos \left( \frac{q}{n} \right), \quad |q| \leq k_F = \pi n/2, \quad u \gg 1,
\]

\[
\varepsilon^0_{\nu}(q) = 4t \cos \left( \frac{|q| + \pi n}{2} \right), \quad |q| \leq (\pi - 2k_F) = \pi (1 - n), \quad u \to 0,
\]

\[
= \frac{8t^2(1 - n)}{U \nu} \left[ 1 - \frac{\sin(\pi n)}{2\pi (1 - n)} \right] \cos^2 \left( \frac{q}{2(1 - n)} \right), \quad |q| \leq (\pi - 2k_F) = \pi (1 - n), \quad u \gg 1.
\]

For the excited states of a fully polarized ground state, which for electronic densities \(n \in [0, 1]\) corresponds to \(m \to n\), the energy dispersions \(\varepsilon^0_c(q)\) and \(\varepsilon_c(q)\), \(\varepsilon^0_{\nu}(q)\) and \(\varepsilon_{\nu}(q)\), and \(\varepsilon^0_{\nu}(q)\) and \(\varepsilon_{\nu}(q)\), Eqs. (A21) and (A22) of
Appendix A have closed-form expressions,

\[ \varepsilon^0_c(q) = -\frac{U}{2} - 2t \cos q, \quad q \in [-\pi, \pi], \]
\[ \varepsilon_c(q) = -2t[\cos q - \cos(\pi n)], \quad q \in [-\pi, \pi], \]
\[ \varepsilon^0_{sv}(q) = -\frac{2t}{\pi} \int_{-\pi}^{\pi} dk \sin k \arctan \left( \frac{\sin k - \Lambda^0_{sv}(q)}{\nu u} \right), \quad q \in [-2k_F, 2k_F] = [-\pi n, \pi n], \]
\[ \varepsilon_{sv}(q) = \varepsilon^0_{sv}(q) + W_{sv}, \quad q \in [-2k_F, 2k_F] = [-\pi n, \pi n], \]
\[ \varepsilon^0_{\eta\nu}(q) = -\nu U + 4t \text{Re} \left[ \sqrt{1 - (\Lambda^0_{\eta\nu}(q)) + i\nu u} \right] \]
\[ = -\frac{2t}{\pi} \int_{-\pi}^{\pi} dk \sin k \arctan \left( \frac{\sin k - \Lambda^0_{\eta\nu}(q)}{\nu u} \right), \quad q \in [-(\pi - 2k_F), (\pi - 2k_F)] = [-\pi(1-n), \pi(1-n)], \]
\[ \varepsilon_{\eta\nu}(q) = \varepsilon^0_{\eta\nu}(q) + W_{\eta\nu}, \quad q \in [-(\pi - 2k_F), (\pi - 2k_F)] = [-\pi(1-n), \pi(1-n)]. \] (D16)

In this limit, \( k^0_n = q \), whereas the rapidity functions \( \Lambda^0_{\alpha\nu}(q) \) appearing in the above expressions are defined by their inversion functions,

\[ q = 2\text{Re}[\arcsin(\Lambda^0_{\eta\nu}(q_j) + i\nu u)] \]
\[ + \frac{1}{\pi} \int_{-\pi}^{\pi} dk \arctan \left( \frac{\sin k - \Lambda^0_{\eta\nu}(q)}{\nu u} \right), \quad q \in [-(\pi - 2k_F), (\pi - 2k_F)] = [-\pi(1-n), \pi(1-n)], \]
\[ q = -\frac{1}{\pi} \int_{-\pi}^{\pi} dk \arctan \left( \frac{\sin k - \Lambda^0_{\eta\nu}(q)}{\nu u} \right), \quad q \in [-2k_F, 2k_F] = [-\pi n, \pi n]. \] (D17)

Moreover, for the electronic density range \( n \in [0, 1] \) and spin density \( m \to n \) the \( sv \) and \( \eta\nu \) energy dispersion bandwidths appearing in some of the expressions provided in Eq. (D16) are given by,

\[ W_{sv} = \frac{1}{2} \sqrt{(4t)^2 + (\nu U)^2} \left[ 1 - \frac{2}{\pi} \arccot \left( \frac{\sqrt{(4t)^2 + (\nu U)^2}}{\nu U} \tan(\pi n) \right) \right] \]
\[ - \nu U - \frac{4t}{\pi} \cos(\pi n) \arctan \left( \frac{4t \sin(\pi n)}{\nu U} \right), \quad n \in [0, 1], \quad m \to n, \] (D18)

and

\[ W_{\eta\nu} = \frac{1}{2} \sqrt{(4t)^2 + (\nu U)^2} \left[ 1 + \frac{2}{\pi} \arccot \left( \frac{\sqrt{(4t)^2 + (\nu U)^2}}{\nu U} \tan(\pi n) \right) \right] \]
\[ - \nu U (1-n) + \frac{4t}{\pi} \cos(\pi n) \arctan \left( \frac{4t \sin(\pi n)}{\nu U} \right), \quad n \in [0, 1], \quad m \to n, \] (D19)

respectively. The energy bandwidths \( W_{sv} \) and \( W_{\eta\nu} \) given here are an increasing and decreasing function of \( n \), respectively. For instance, for \( n \to 0, \ n = 1/2, \) and \( n = 1 \) they read,

\[ W_{sv} = 0, \quad n \to 0, \quad m \to n, \]
\[ = \frac{1}{2} \left[ \sqrt{(4t)^2 + (\nu U)^2} - U \right], \quad n = \frac{1}{2}, \quad m \to n, \]
\[ = \sqrt{(4t)^2 + (\nu U)^2} - \nu U, \quad n = 1, \quad m \to n, \] (D20)

and

\[ W_{\eta\nu} = \sqrt{(4t)^2 + (\nu U)^2} - \nu U, \quad n \to 0, \quad m \to n, \]
\[ = \frac{1}{2} \left[ \sqrt{(4t)^2 + (\nu U)^2} - U \right], \quad n = \frac{1}{2}, \quad m \to n, \]
\[ = 0, \quad n = 1, \quad m \to n, \] (D21)

respectively.
For spin density \( m = 0 \) and electronic density \( n = 1 \) all energy bandwidths \( W_{\alpha \nu} \) vanish except that of the \( s1 \) pseudoparticle energy dispersion, which reads,

\[
W_{s1} = 2t \int_0^\infty d\omega \frac{J_1(\omega)}{\omega \cosh(\omega u)}, \quad n = 1, \quad m = 0. \tag{D22}
\]

Finally, for \( u \to 0 \) and \( u \gg 1 \), electronic densities \( n \in [0, 1] \), and spin densities \( m = 0 \) and \( m \to n \) the \( s\nu \) and \( \eta \nu \) dispersion energy bandwidths have the following limiting behaviors,

\[
W_{s\nu} = \begin{cases} 
\delta_{\nu, 1} \left\{ 2t \left[ 1 - \cos \left( \frac{\pi}{2} n \right) \right] \right\}, & m = 0, \quad u \to 0, \\
4t \sin^2 \left( \frac{\pi n}{2} \right) = 2\mu_B H_c, & m \to n, \quad u \to 0, \\
\delta_{\nu, 1} \left\{ \frac{2\pi nt^2}{\nu U} \left[ 1 - \sin \left( \frac{2\pi n}{2\pi n} \right) \right] \right\} = \delta_{\nu, 1} \frac{\pi}{4} (2\mu_B H_c), & m = 0, \quad u \gg 1, \\
8n t^2 \left[ 1 - \sin \left( \frac{2\pi n}{2\pi n} \right) \right] = \frac{1}{\nu} 2\mu_B H_c, & m \to n, \quad u \gg 1,
\end{cases}
\]

and

\[
W_{\eta\nu} = \begin{cases} 
4t \cos \left( \frac{\pi}{2} n \right) = 2|\mu|, & m = 0, \quad u \to 0, \\
4t \left[ 1 - \sin^2 \left( \frac{\pi n}{2} \right) \right] = 2|\mu|, & m \to n, \quad u \to 0, \\
8(1 - n) t^2 \left[ 1 - \sin \left( \frac{2\pi (1 - n)}{2\pi (1 - n)} \right) \right], & m = 0, \quad u \gg 1, \\
8n t^2 \sin \left( \frac{2\pi n}{2\pi n} \right) = \frac{1}{\nu} 2\mu_B H_c, & m \to n, \quad u \gg 1,
\end{cases}
\]

respectively.

[1] M. C. Gutzwiller, Phys. Rev. Lett. 10 (1963) 159.
[2] J. Hubbard, Proc. Roy. Soc. (London) A 276 (1963) 238.
[3] Elliott H. Lieb, F. Y. Wu, Phys. Rev. Lett. 20 (1968) 1445;
Elliott H. Lieb, F. Y. Wu, Physica A 321 (2003) 1.
[4] Minoru Takahashi, Progr. Theor. Phys 47 (1972) 69.
[5] F. Woynarovich, J. Phys. C 15 (1982) 85;
F. Woynarovich, J. Phys. C 15 (1982) 97.
[6] P. B. Ramos, M. J. Martins, J. Phys. A 30 (1997) L195;
M. J. Martins, P. B. Ramos, Nucl. Phys. B 522 (1998) 413.
[7] H. J. Schulz, Phys. Rev. Lett. 64 (1990) 2831.
[8] J. Voit, Rep. Prog. Phys. 58 (1995) 977.
[9] K.-V. Pham, M. Gabay, P. Lederer, Phys. Rev. B 61 (2000) 16 397.
[10] F. Woynarovich, J. Phys. A 22 (1989) 4243.
[11] H. Frahm, V. E. Korepin, Phys. Rev. B 42 (1990) 10553.
[12] J. M. P. Carmelo, K. Penc, D. Bozi, Nucl. Phys. B 725 (2005) 421;
J. M. P. Carmelo, K. Penc, D. Bozi, Nucl. Phys. B 737 (2006) 351, Erratum.
[13] J. M. P. Carmelo, K. Penc, Eur. Phys. J. B 51 (2006) 477.
[14] J. M. P. Carmelo, L. M. Martelo, K. Penc, Nucl. Phys. B 737 (2006) 237.
[15] J. M. P. Carmelo, D. Bozi, K. Penc, J. Phys.: Cond. Mat. 20 (2008) 415103;
D. Bozi, J. M. P. Carmelo, K. Penc, P. D. Sacramento, J. Phys.: Cond. Mat. 20 (2008) 022205.
[16] Karlo Penc, Karen Hallberg, Frédéric Mila, Hiroyuki Shiba, Phys. Rev. Lett. 77 (1996) 1390.
[17] Karlo Penc, Karen Hallberg, Frédéric Mila, Hiroyuki Shiba, Phys. Rev. B 55 (1997) 15 475.
[18] A. Imambekov, L. I. Glazman, Science 323 (2009) 228;
A. Imambekov, L. I. Glazman, Phys. Rev. Lett. 100 (2008) 206805.
[19] A. Imambekov, L. I. Glazman, Phys. Rev. Lett. 102 (2009) 126405.
[20] Thomas L. Schmidt, Adilet Imambekov, Leonid I. Glazman, Phys. Rev. Lett. 116 (2010) 116403;
Thomas L. Schmidt, Adilet Imambekov, Leonid I. Glazman, Phys. Rev. B, 82 (2010) 245104;
A. Shashi, L. I. Glazman, J.-S. Caux, A. Imambekov, Phys. Rev. B, 84 (2011) 045408.
