Solitary wave of the Schrödinger lattice system with nonlinear hopping

Ming Cheng

College of Mathematics, Jilin University, Changchun 130012, P.R. China

Abstract
This paper is concerned with the nonlinear Schrödinger lattice with nonlinear hopping. Via variation approach and the Nehari manifold argument, we obtain two types of solution: periodic ground state and localized ground state. Moreover, we consider the convergence of periodic solutions to the solitary wave.

1. Introduction
In the last decades, a great deal of attention has been paid to study the existence of solitary wave for the lattice systems\[9, 10, 11\]. They play a role in lots of physical models, such as nonlinear waves in crystals and arrays of coupled optical waveguides. The discrete nonlinear Schrödinger lattice is one of the most famous models in mathematics and physics. The existence and properties of discrete breathers(periodic in time and spatially localized) in discrete nonlinear Schrödinger lattice have been considered in a number of studies\[13, 14\].

In the present paper, we consider a variant of the discrete nonlinear Schrödinger lattice as follows:

\[i\dot{\psi}_l + (\Delta_d \psi)_l + \alpha \psi_l \sum_{j=1}^{d}(T_j \psi)_l + \beta |\psi_l|^{2\sigma} \psi_l = 0, \quad l \in \mathbb{Z}^d,\]

(1.1)
where $\alpha, \beta \in \mathbb{R}$, $(\Delta d \psi)_t = \sum_{m \in N_d} \psi_m - \psi_t$ and the nonlinear operator $T$ is defined by
\[
(T_j \psi)_t = |\psi_{l_1,\ldots,l_{j-1},l_j+1,l_j,l_{j+1},\ldots,l_d}|^2 + |\psi_{l_1,\ldots,l_{j-1},l_j-1,l_{j+1},\ldots,l_d}|^2.
\]

Here, $N_d$ denotes the set of the nearest neighbors of the point $l \in \mathbb{Z}^d$.

Note that for $\alpha = 0, \beta \neq 0$, it recovers the classical nonlinear Schrödinger lattice. For $\alpha \neq 0, \beta \neq 0$, it denotes the Schrödinger lattice with nonlinear hopping.

There has been a lot of interests in this equation as the modeling of waveguide arrays. Also, nonlinear hopping terms appear from Klein-Gordon and Fermi-Pasta-Ulam chains of anharmonic oscillators coupled with anharmonic inter-site potentials, or mixed FPU/KG chains. N. I. Karachalios et al. discuss the energy thresholds in the setting of DNLS lattice with nonlinear hopping terms by using fixed point method. The numerical results have also been obtained in their paper [1].

Our aim is to investigate the existence of nontrivial solitary wave for the infinite dimensional lattice (1.1). Here, we only consider the case of one dimension. i.e., $d = 1$. The case of $d > 1$ is similar. It notes that for classical nonlinear Schrödinger lattice, Weinstein discusses a connection among the dimensionality, the degree of the nonlinearity and the existence of the excitation threshold. They prove that if the degree of the nonlinearity $\sigma$ satisfies $\sigma \geq \frac{2}{d}$ where $d$ is the dimension, then there exists a ground state for the total power is greater than the excitation threshold and there is no ground state for the total power is less than the excitation threshold. However, we get that the power of solitary wave always has a lower bound for the equation (1.1) with $\sigma \geq 1$.

The paper is organized as follows. In Section 2, we firstly consider the $k$-periodic problem. Note that the dimension in space variable is finite. We obtain the nontrivial periodic solution by Nehari manifolds argument [6]. The existence of solitary wave is more complex. In Section 3, we follow the idea of [2, 3, 4, 5] to obtain the solitary wave. The key point is to show the norms of periodic ground state are bounded. It is based on the concentration compactness. In Section 4, we concern the convergence of periodic ground states to a solitary ground state.
2. Periodic solution

In this paper, we consider the following equation:

\[ i \dot{\psi}_l + (\Delta_d \psi)_l + \alpha \psi_l (|\psi_{l+1}|^2 + |\psi_{l-1}|^2) + \beta |\psi_l|^{2\sigma} \psi_l = 0, \quad l \in \mathbb{Z}, \quad (2.2) \]

where \( \sigma \geq 1 \).

To obtain breather, we seek the solution:

\[ \psi_l = e^{-i\omega t} u_l. \]

The equation of \( u_l \) is

\[ \omega u_l + (\Delta_d u)_l + \alpha u_l (|u_{l+1}|^2 + |u_{l-1}|^2) + \beta |u_l|^{2\sigma} u_l = 0, \quad l \in \mathbb{Z}. \quad (2.3) \]

Actually, we give the proofs only in the focusing case with \( \alpha, \beta > 0 \) and \( \omega < 0 \). For the defocusing case with \( \alpha, \beta < 0 \) and \( \omega > 4 \), the argument is similar. Here, we omit the details.

In this section, we prove the existence of \( k \)-periodic solution which satisfies

\[ u_{l+k} = u_l, \quad \text{for} \quad l \in \mathbb{Z}, \]

where \( k > 2 \) is an integer.

Let

\[ P_k = \left\{ l \in \mathbb{Z} | \left\lfloor \frac{k}{2} \right\rfloor \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor - 1 \right\}. \]

Consider the Banach space \( \ell^p_k \) with norm:

\[ \|u\|_{\ell^p_k} = \sum_{l \in P_k} |u_l|^p. \]

We mention that

\[ \|u\|_{\ell^q_k} \leq \|u\|_{\ell^p_k}, \quad 1 \leq p \leq q \leq \infty. \]

Denote that \( \langle \cdot, \cdot \rangle_k \) is natural inner product in \( \ell^2_k \).

Define the functional

\[ J_k(u) = \langle -\Delta_d u, u \rangle_k - \omega \langle u, u \rangle_k - \alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_l|^{2\sigma+2}, \]

\[ \text{for} \quad l \in \mathbb{Z}. \]
and Nehari manifold
\[ N_k = \left\{ u \in l^2_k | I_k(u) = \langle -\triangle_d u, u \rangle_k - \omega \langle u, u \rangle_k - \alpha \sum_{l \in P_k} |u_l|^2 (|u_{l+1}|^2 + |u_{l-1}|^2) \right\} \]

Then, the minimizer of the constrained variational problem:
\[ m_k = \inf_{u \in N_k} \{ J_k(u) \} \]

is the nontrivial periodic solution of (2.3). We mention that the minimizer is called a periodic ground state.

Note that \[ 0 \leq \langle \triangle_d u, u \rangle_k \leq 4||u||^2_{l^2_k}, \quad \text{for} \quad u \in l^2_k. \]

We want to obtain the periodic solution with prescribed frequency \( \omega < 0 \). With the Nehari manifold approach, we have the following result.

**Theorem 2.1.** Assume that the frequency \( \omega < 0 \) and \( \alpha, \beta > 0 \). There exists a positive \( k \)-periodic ground state \( u^k \) for the equation (2.3).

**Lemma 2.1.** Under the assumptions of Theorem 2.1. The Nehari manifold \( N_k \) is nonempty.

**Proof.** For \( t \geq 0 \) and \( u \neq 0 \), define
\[ \rho(t) = I_k(\sqrt{t}u) = t(\langle -\triangle_d u, u \rangle_k - \omega \langle u, u \rangle_k) - 2t^2 \alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - t^{\sigma+1} \beta \sum_{l \in P_k} |u_l|^{2\sigma+2}. \]

Then,
\[ \rho'(t) = \langle -\triangle_d u, u \rangle_k - \omega \langle u, u \rangle_k - 4t\alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - (\sigma + 1)t^\sigma \beta \sum_{l \in P_k} |u_l|^{2\sigma+2}. \]

There holds that \( \rho'(t) > 0 \) for \( t > 0 \) small enough.

Observe that
\[ \rho''(t) = -4\alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - (\sigma + 1)\sigma t^{\sigma-1} \beta \sum_{l \in P_k} |u_l|^{2\sigma+2} < 0. \]

Therefore, \( \rho(t) \) admits a unique zero point \( t^* \in (0, +\infty) \). This implies \( \sqrt{t^*}u \in N_k \). It completes the proof. \( \square \)
Lemma 2.2. Under the assumptions of Theorem 2.1, for $u \in N_k$, the function $J_k(\sqrt{tu})$ has a unique critical point at $t = 1$, which is a global maximum.

Proof. For $t > 0$ and $u \in N_k$, we get

\[ \theta(t) = J_k(\sqrt{tu}) \]

\[ = t \langle -\Delta_d u, u \rangle_k - \omega \langle u, u \rangle_k - t^2 \alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - \frac{t^{\sigma+1}\beta}{\sigma + 1} \sum_{l \in P_k} |u_l|^{2\sigma+2}. \]

Then,

\[ \theta'(t) = \langle -\Delta_d u, u \rangle_k - \omega \langle u, u \rangle_k - 2t\alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - t^\sigma \beta \sum_{l \in P_k} |u_l|^{2\sigma+2}. \]

It holds that $\theta'(t) > 0$ for $t > 0$ small enough.

Note that

\[ \theta''(t) = -2\alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - \sigma t^{\sigma-1} \beta \sum_{l \in P_k} |u_l|^{2\sigma+2} < 0. \]

We can see that $t = 1$ is the unique maximum point of $\theta(t)$. This implies the proof. \hfill \Box

Assume that $u^k$ is the $k$-periodic solution of (2.3), we have

\[ |\omega||u^k||^2_{P_k} \]

\[ \leq \langle -\Delta_d u^k, u^k \rangle_k - \omega \langle u^k, u^k \rangle_k \]

\[ = 2\alpha \sum_{l \in P_k} |u_l^k|^2 |u_{l+1}^k|^2 + \beta \sum_{l \in P_k} |u_l^k|^{2\sigma+2} \]

\[ \leq ||u^k||^2_{l^2} \left( \beta ||u^k||^{2\sigma}_{l^2} + 2\alpha ||u^k||^{2}_{l^2} \right). \]

Therefore

\[ ||u^k||_{l^2} \geq C_1 > 0. \]

where $C_1$ is the unique positive solution of equation:

\[ \beta x^{2\sigma} + 2\alpha x^2 + \omega = 0. \]

Observe that $C_1$ is independent of $k$.

Thus, we get a lower bound of the power of the periodic solutions.
**Theorem 2.2.** The power of the periodic solution must be greater than $C_1$.

**Lemma 2.3.** Under the assumptions of Theorem 2.1, $J_k(u)$ is bounded below for all $u \in N_k$.

**Proof.** Let $u \in N_k$. From the argument in (2.4) and (2.5), there exists $l_0 \in P_k$ and a positive constant $C_2$ such that

$$|u_{l_0}| > C_2 > 0.$$

Therefore,

$$J_k(u) = \alpha \sum_{l \in P_k} |u_l|^2 |u_{l-1}|^2 + \frac{\sigma \beta}{\sigma + 1} \sum_{l \in P_k} |u_l|^{2\sigma+2} > \frac{\sigma \beta}{\sigma + 1} C_2^{2\sigma+2}.$$

It completes the proof. \qed

**Lemma 2.4.** Under the assumptions of Theorem 2.1. Then, the minimizer of the constrained variational problem $m_k = \inf_{u \in N_k} \{J_k(u)\}$ could be attained.

**Proof.** Assume that $\{u^n\}$ is a minimizing sequence. We can see that there exists a constant $M > 0$ such that

$$\max J_k(u^n) \leq M.$$

Thus,

$$\|\omega\|_{l_{\infty}^k}^2 \leq \langle -\Delta u^n, u^n \rangle_k - \omega \langle u^n, u^n \rangle_k \leq M.$$

There holds that $\|u^n\|_{l_{\infty}^k}$ is bounded.

Note that $P_k$ is finite dimensional space. Passing to a subsequence, there exists $u^k$ such that $u^{n_j} \to u^k$ in $l_{\infty}^k$. Since the set $l_{\infty}^k$ is closed and the functional $J_k$ is continuous, we obtain that $u^k \in N_k$ and $J_k(u^k) = m_k$. \qed

By Lagrange multiplier method, there exists some constant $\lambda$ such that

$$\lambda \left( 2 \langle -\Delta u^k, v \rangle_k - 2\omega \langle u^k, v \rangle_k - 4\alpha \sum_{l \in P_k} u_l^k v_l (|u_{l+1}^k|^2 + |u_{l-1}^k|^2) - (2\sigma + 2) \beta \sum_{l \in P_k} |u_l^k|^{2\sigma} u_l^k v_l \right) + 2 \langle -\Delta u^k, v \rangle_k - 2\omega \langle u^k, v \rangle_k - 2\alpha \sum_{l \in P_k} u_l^k v_l (|u_{l+1}^k|^2 + |u_{l-1}^k|^2) - 2\beta \sum_{l \in P_k} |u_l^k|^{2\sigma} u_l^k v_l = 0.$$
Choose $v = u^k$. Note that $u^k \in \mathcal{N}_k$, there holds
\[
\lambda \left( -2\alpha \sum_{l \in \mathcal{P}_k} |u^k_l|^2 (|u^k_{l+1}|^2 + |u^k_{l-1}|^2) - \sigma \beta \sum_{l \in \mathcal{P}_k} |u^k_l|^{2\sigma + 2} \right) = 0.
\]

We have $\lambda = 0$. It implies that $u^k$ is a nontrivial solution of equation (2.2).

Now, we prove that $u^k$ is positive. Observe that
\[
\langle -\nabla_d |u|, |u| \rangle - \omega \langle |u|, |u| \rangle \leq \langle -\nabla_d u, u \rangle - \omega \langle u, u \rangle.
\]

Since that $u^k$ is the nontrivial solution. Then, there exists $t^{**} \in (0, 1]$ such that $\sqrt{t^{**}}|u^k| \in \mathcal{N}_k$. It is obvious that
\[
J_k(\sqrt{t^{**}}|u^k|) \leq m_k.
\]

Hence $J_k(\sqrt{t^{**}}|u^k|) = m_k$. We can assume that $u^k = \sqrt{t^{**}}|u^k|$.

Let $G(n, m)$ be the Green function of $-\nabla_d - \omega$. From [7], we have $G(n, m) > 0$ for $\omega < 0$. It obtains that
\[
u_n^k = \sum_{l \in \mathbb{Z}} G(n, l) \left( \alpha u^k_l (|u^k_{l+1}|^2 + |u^k_{l-1}|^2) + \beta |u^k_l|^{2\sigma + 2} u^k_l \right), \quad n \in \mathbb{Z}.
\]

Since that $u^k$ is nonnegative, there holds $u_n^k > 0$ for all $n \in \mathbb{Z}$. It completes the proof of Theorem 2.1.

3. Localized ground state

Here, we give some notations. Define the functional
\[
J(u) = \langle -\nabla_d u, u \rangle - \omega \langle u, u \rangle - \alpha \sum_{l \in \mathbb{Z}} |u_l|^2 |u_{l+1}|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in \mathbb{Z}} |u_l|^{2\sigma + 2},
\]

and Nehari manifold
\[
\mathcal{N} = \left\{ u \in l^2 | I(u) = \langle -\nabla_d u, u \rangle - \omega \langle u, u \rangle - \alpha \sum_{l \in \mathbb{Z}} |u_l|^2 (|u_{l+1}|^2 + |u_{l-1}|^2) \right.
\]
\[
- \beta \sum_{l \in \mathbb{Z}} |u_l|^{2\sigma + 2} = 0, u \neq 0 \left\}.
\]
where \( \langle \cdot, \cdot \rangle \) is natural inner product in \( l^2 \). Thus, we can see that the minimizer of the constrained variational problem:

\[
m = \inf_{u \in N} \{ J(u) \}
\]

is the nontrivial solitary wave of \((2.3)\). We call this minimizer a localized ground state. Similar with Lemma 2.1 and 2.2, the results are obtained by replacing \( J_k(u), I_k(u) \) to \( J(u), I(u) \).

In this section, to obtain the localized ground state \( u \) satisfying

\[
\lim_{l \to \infty} |u_l| = 0,
\]

we follow the idea of [2]. We want to pass to the limit as \( k \to \infty \). The key point is the following result.

**Lemma 3.1.** Under the assumptions of Theorem 2.1. Let \( u_k \) be the \( k \)-periodic solution. Therefore, the sequences \( m_k \) and \( ||u_k||^2_k \) are bounded.

**Proof.** First, we concern the sequences \( m_k \) are bounded. From the similar argument of Lemma 2.1, there holds that for any given \( u \in l^2 \), there exists \( t' \) such \( I(\sqrt{t'} u) < 0 \). Since the sequences with finite support are dense in \( l^2 \). Therefore, there exists \( \tilde{u} \) with finite support such that \( I(\tilde{u}) < 0 \). It obtains that there exists \( t'' \) such that \( I(\sqrt{t''} \tilde{u}) = 0 \). And \( m_k \leq J_k(\tilde{u}) = J(\sqrt{t''} \tilde{u}) \) is bounded.

Second, we prove that \( ||u^k||^2_k \) is uniformly bounded. Assume that \( ||u^k||^2_k \) is unbounded. Passing to a subsequence which is still denoted by itself, we have \( ||u^k||^2_k \to \infty \) for \( k \to \infty \). Let \( v^k = \frac{u^k}{||u^k||^2_k} \). One of the following should holds:

(i) \( v^k \) is vanishing, i.e. \( ||v^k||_\infty \to 0 \).

(ii) \( v^k \) is not vanishing. Passing to a subsequence which is still denoted by itself, there exists \( \delta > 0 \) and \( b^k \in \mathbb{Z} \) such that \( |v^k|_b > \delta \) for all \( k \).

Now, we rule out the case (i). There holds that

\[
0 = \frac{I_k(u^k)}{||u^k||^2_k} = \langle (-\Delta_d - \omega) v^k, v^k \rangle_k - 2\alpha \sum_{l \in P^k} |v^k_l|^2 |u^k_{l-1}|^2 - \beta \sum_{l \in P_k} |v^k_l|^2 |u^k_l|^{2\alpha}.
\]

Hence,

\[
|\omega| = |\omega||w^k||^2_k \leq \langle (-\Delta_d - \omega) v^k, v^k \rangle_k = 2\alpha \sum_{l \in P^k} |v^k_l|^2 |u^k_{l-1}|^2 + \beta \sum_{l \in P_k} |v^k_l|^2 |u^k_l|^{2\alpha}.
\]

(3.6)
Assume that
\[ A_k = \left\{ l \in P_k \middle| |u^k_l| < M_0 \right\}, \]
\[ B_k = P_k \setminus A_k, \]
where \( M_0 > 0 \) is a constant which is defined below.

Let \( M_0 \) be small enough such that
\[
2\alpha \sum_{l \in A_k} |v^k_l|^2 |u^k_{l-1}|^2 + \beta \sum_{l \in A_k} |v^k_l|^2 |u^k_l|^{2\sigma} < 2\alpha M_0^2 \sum_{l \in A_k} |v^k_{l+1}|^2 + \beta M_0^{2\sigma} \sum_{l \in A_k} |v^k_l|^2
\]
\[
\leq \frac{|\omega|}{2}.
\]

Combine with the equation (3.6), we have
\[
\frac{|\omega|}{2} \leq \liminf_{k \to \infty} 2\alpha \sum_{l \in B_k} |v^k_{l+1}|^2 |u^k_{l+1}|^2 + \beta \sum_{l \in B_k} |v^k_l|^2 |u^k_l|^{2\sigma}.
\] (3.7)

From the argument above, there exists a constant \( M' > 0 \) such that
\[
m_k = \sum_{l \in P_k} \alpha |u^k_l|^2 |u^k_{l+1}|^2 + \frac{\sigma \beta}{\sigma + 1} \sum_{l \in P_k} |u^k_l|^{2\sigma+2} < M'.
\]

Hence, \( \|u^k\|_{L^{2\sigma+2}} \) is uniformly bounded.

By Hölder’s inequality, we have
\[
\sum_n |v_n|^2 |u_n|^2 \leq \|u\|^{2\sigma+2}_{L^{2\sigma+2}} \|v\|_{L^{2\sigma+2}},
\]
\[
\sum_n |v_n|^2 |u_n|^{2\sigma} \leq \|u\|^{2\sigma}_{L^{2\sigma+2}} \|v\|_{L^{2\sigma+2}}^2
\]
and
\[
\|v\|_p \leq \|v\|^{\frac{p-2}{p}}_{L^\infty} \|v\|_{L^2}^{\frac{2}{p}}, \quad \text{for } p > 2.
\]
Since \(v^k\) is vanishing, we can see that
\[
\lim_{k \to \infty} ||v^k||_{l^p_k} = 0, \quad \text{for } p > 2.
\]
It concludes that
\[
\liminf_{k \to \infty} 2\alpha \sum_{B_k} |v_{i+1}^k|^2 |u_i^k|^2 + \beta \sum_{B_k} |v_i^k|^2 |u_i^k|^{2\sigma} \to 0, \quad \text{as } k \to \infty.
\]
It contradicts with (3.7).

Let’s rule out the non-vanishing case. By the discrete translation invariance, we can assume that \(b_k = 0\). Since \(||v^k||_{l^p_k} = 1\), there exists \(v = \{v_l\}\) such that \(v^k_l \to v_l \) for all \(l \in \mathbb{Z}\). It is obvious that \(v \in l^2\), \(||v||_{l^2} \leq 1\) and \(|v_0| \geq \delta\).

Since \(|v_0| \neq 0\), then \(|u_0^k| \to \infty\), as \(k \to \infty\). On the other hand, we have
\[
\frac{\sigma \beta}{\sigma + 1} |u_0^k|^{2\sigma + 2} \leq m_k \leq M'.
\]
It is a contradiction.

\[\blacksquare\]

**Theorem 3.1.** Assume that the frequency \(\omega < 0\) and \(\alpha, \beta > 0\). There exists a positive localized ground state \(u\) for the equation (2.3).

**Proof.** Let \(u^k \in l^2_k\) be a periodic ground state. From Lemma 3.1, the sequence \(||u^k||_{l^2_k}\) is bounded. Therefore, \(u^k\) is either vanishing or non-vanishing. In the case of vanishing, we have \(\lim_{k \to \infty} ||u^k||_{l^p_k} \to 0\), for \(p > 2\). There holds that
\[
|\omega|||u^k||_{l^2_k}^2 \leq \langle (-\Delta_d - \omega)u^k, u^k \rangle_k
= 2\alpha \sum_{i \in P_k} |u_i^k|^2 |u_{i-1}^k|^2 + \beta \sum_{i \in P_k} |u_i^k|^{2\sigma + 2}
\leq 2\alpha \sum_{i \in P_k} |u_i^k|^4 + \beta \sum_{i \in P_k} |u_i^k|^{2\sigma + 2} \to 0.
\]
It is a contradiction.

Thus, the sequence \(u^k\) is non-vanishing. By the discrete translation invariance, we assume that \(|u_0^k| \geq \delta > 0\). There exists \(u = \{u_l\}\) such that \(u_l^k \to u_l\) for all \(l \in \mathbb{Z}\). It is obvious that \(u \in l^2\) and \(u \neq 0\). Also, we obtains that \(u\) is a nontrival solution for (2.3) by point-wise limits. Now, we want to prove that \(u\) is a localized ground state.
Let $L$ be a positive integer such that
\[
\liminf_{k \to \infty} J_k(u^k) \geq \liminf_{k \to \infty} \alpha \sum_{-L \leq l \leq L} |u_l|^2 |u_{l-1}|^2 + \frac{\sigma \beta}{\sigma + 1} \sum_{-L \leq l \leq L} |u_l|^{2\sigma + 2} \\
\geq \alpha \sum_{-L \leq l \leq L} |u_l|^2 |u_{l-1}|^2 + \frac{\sigma \beta}{\sigma + 1} \sum_{-L \leq l \leq L} |u_l|^{2\sigma + 2}.
\]
Let $L \to \infty$, it obtains that
\[
\liminf_{k \to \infty} J_k(u^k) \geq J(u) \geq m, \\
\liminf_{k \to \infty} m_k \geq m. \tag{3.8}
\]
For any given $\epsilon > 0$, let $u' \in N$ such that
\[
J(u') = \alpha \sum_{l \in \mathbb{Z}} |u'_l|^2 |u'_{l-1}|^2 + \frac{\sigma \beta}{\sigma + 1} \sum_{l \in \mathbb{Z}} |u'_l|^{2\sigma + 2} < m + \epsilon.
\]
Choose $t_1 > 1$ such that
\[
J(t_1 u') < m + \epsilon, \quad I(t_1 u') < 0.
\]
From density argument, there exists a finite supported sequence $v = \{v_l\}$ sufficiently close to $t_1 u'$ in $l^2$ such that
\[
I(v) < 0 \quad \text{and} \quad \alpha \sum_{l \in \mathbb{Z}} |v_l|^2 |v_{l-1}|^2 + \frac{\sigma \beta}{\sigma + 1} \sum_{l \in \mathbb{Z}} |v_l|^{2\sigma + 2} < m + \epsilon.
\]
Thus, there exists $t_2 \in (0, 1)$ such that $t_2 v \in N$ and $J(t_2 v) < m + \epsilon$.

Choose $k$ large enough such that $P_k$ contains the support of $v$. Let $v^k \in l^2$ such that $v^k_l = t_2 v_l$ for $l \in P_k$. It concludes that
\[
I_k(v^k) = I(t_2 v), \\
J_k(v^k) = J(t_2 v) < m + \epsilon.
\]
It implies
\[
\limsup_{k \to \infty} m_k < m + \epsilon.
\]
Combining with (3.8), we have $\lim_{k \to \infty} m_k = m$. It completes the proof.

Remark 3.1. With the similar argument in (2.4), (2.5), the power of the localized ground state has a lower bound $C_1 > 0$. For more estimates, we refer to [1].
From the argument above, we can assume that \( u_{\mathbf{M}} > 0 \) is a ground state. We want to prove that \( \text{a translation} \) indeed, \( (2.3) \)

Proof. Let \( u^k \in l_k^2 \) be the periodic ground state and \( b_k \in \mathbb{Z} \). Now, we consider a translation

\[
u_{t}^{k} = u_{t+b_{k}}^k.
\]

From the argument above, we can assume that \( u_{t}^{k} \to u_{t} \) for all \( l \in \mathbb{Z} \) where \( u \) is a ground state. We want to prove that \( ||u^{k} - u||_{l_k^2} \) convergent to 0 as \( k \to \infty \). First, it concludes that

\[
J_k(u^k - u) \to 0, \quad I_k(u^k - u) \to 0, \quad \text{as} \quad k \to \infty.
\]

Indeed,

\[
J_k(u^k - u) = \langle -\Delta_d(u^k - u), (u^k - u) \rangle - \omega(\langle (u^k - u), (u^k - u) \rangle)
\]

\[
- \alpha \sum_{l \in P_k} |u_{t}^{k} - u_{l}^{k}|^2 |u_{l+1}^{k} - u_{l+1}^{k}|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_{l}^{k} - u_{l}|^{2\sigma + 2}
\]

\[
= J_k(u^k) - J_k(u) - 2 \langle -\Delta_d(u^k - u), u \rangle - 2 \omega(\langle (u^k - u), u \rangle)
\]

\[
- \alpha \sum_{l \in P_k} |u_{t}^{k} - u_{l}^{k}|^2 |u_{l+1}^{k} - u_{l+1}^{k}|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_{l}^{k} - u_{l}|^{2\sigma + 2}
\]

\[
+ \alpha \sum_{l \in P_k} |u_{t}^{k}|^2 |u_{l+1}^{k}|^2 + \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_{l}^{k}|^{2\sigma + 2}
\]

\[
- \alpha \sum_{l \in P_k} |u_{t}^{k}|^2 |u_{l+1}^{k}|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_{l}^{k}|^{2\sigma + 2}.
\]

Similar with the argument in [2], it obtains that \( J_k(u^k) \to J(u) = m \) and \(-2\langle -\Delta_d(u^k - u), u \rangle - 2 \omega(\langle (u^k - u), u \rangle) \to 0\), as \( k \to \infty \).

Since that \( ||u^{k}||_{l_k^2} \) and \( ||u||_{l_k^2} \) is bounded. For any given \( \epsilon > 0 \), there exists \( M > 0 \) such that \( \sum_{|l| \geq M} |u_{l}|^2 < \epsilon \). Therefore, we have

\[
\alpha \sum_{l \in P_k, |l| \geq M} |u_{l}^{k}|^2 |u_{l+1}^{k}|^2 - \alpha \sum_{l \in P_k, |l| \geq M} |u_{l}^{k} - u_{l}^{k}|^2 |u_{l+1}^{k} - u_{l+1}^{k}|^2
\]
\[ \leq \alpha \sum_{l \in P_k, l || M} (|u_t^{h_k}|^2 - |u_t^{h_k} - u_t|)|u_t + l| + \alpha \sum_{l \in P_k, l || M} |u_t^{h_k} - u_t|^2 (|u_t^{h_k}|^2 - |u_t^{h_k} - u_t + l|^2) \]

\[ \leq \alpha (2||u_t^{h_k}||_l ||u_t||_l^2 + ||u_t||_l^2) \sum_{l \in P_k, l || M} |u_t + l|^2 + \alpha (||u_t^{h_k}||_l^2 + ||u_t||_l^2) (2||u_t^{h_k}||_l^2 + ||u_t||_l^2) \left( \sum_{l \in P_k, l || M} |u_t + l|^2 \right) \]

\[ \leq M^2 ||u_t||_l^2 \]

for \( k \) large enough.

Also, we have

\[ \sum_{l \in P_k, l || M} \frac{\beta}{\sigma + 1} |u_t^{h_k} - u_t|^2 + \frac{\beta}{\sigma + 1} \sum_{l \in P_k, l || M} |u_t^{h_k} - u_t|^2 \]

\[ \leq M^2 ||u_t||_l^2 \]

for \( k \) large enough.

On the other hand, from the point limits, it concludes that

\[ - \alpha \sum_{l \in P_k, l || M} |u_t^{h_k} - u_t|^2 |u_t^{h_k} - u_t + l|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in P_k, l || M} |u_t^{h_k} - u_t|^2 \]

\[ + \alpha \sum_{l \in P_k, l || M} |u_t^{h_k} - u_t|^2 |u_t^{h_k} - u_t + l|^2 + \frac{\beta}{\sigma + 1} \sum_{l \in P_k, l || M} |u_t^{h_k} - u_t|^2 \]

\[ - \alpha \sum_{l \in P_k, l || M} |u_t^{h_k} - u_t|^2 |u_t^{h_k} - u_t + l|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in P_k, l || M} |u_t^{h_k} - u_t|^2 \]

for \( k \) large enough.

Combine with Hölder inequality, there holds \( J^k(u_t^{h_k} - u) \to 0 \).

With similar argument, we obtain \( I^k(u_t^{h_k} - u) \to 0 \). Therefore,

\[ J^k(u_t^{h_k} - u) - I^k(u_t^{h_k} - u) \]

\[ = \alpha \sum_{l \in P_k} |u_t^{h_k} - u| |u_t^{h_k} - u| |u_t^{h_k} - u| \]

\[ \geq \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_t^{h_k} - u| \]

Since \( |\cdot|_l \leq ||\cdot||_l^{2+2} \), we have \( ||u_t^{h_k} - u||_l \to 0 \). From Lemma 3.1 it is known that \( ||u_t^{h_k} - u||_l \to 0 \) for \( p > 2 \). Hence,

\[ |\omega||u_t^{h_k} - u||_l^2 \]

\[ \leq -\Delta_d(u_t^{h_k} - u), (u_t^{h_k} - u) - \omega((u_t^{h_k} - u), (u_t^{h_k} - u)) \]

13
\[
2\alpha \sum_{l \in P_k} |u_{i+1}^k - u_i|^2 |u_{i+1}^k - u_{i+1}|^2 + \beta \sum_{l \in P_k} |u_{i}^k - u_i|^{2\sigma + 2} \\
\leq 2\alpha ||u^k - u||_{l_k}^4 + \beta ||u^k - u||_{l_k}^{2\sigma + 2} \to 0.
\]

It completes the proof.

Acknowledgments.

References

[1] N. I. Karachalios, B. Sánchez-Rey, P. G. Kevrekidis and J. Cuevas, Breathers for the Discrete Nonlinear Schrödinger equation with nonlinear hopping

[2] A. Pankov and V. Rothos, Periodic and decaying solutions in discrete nonlinear Schrödinger with saturable nonlinearity, Proc. R. Soc. A (2008) 464, 3219-3236

[3] Pankov, A. 2005a Travelling waves and periodic oscillations in Fermi-Pasta-Ulam lattices. London, UK: Imperial College Press.

[4] Pankov, A. 2005b Periodic nonlinear Schrödinger equation with an application to photonic crystals. Milan J. Math. 73, 259-287.

[5] Pankov, A. 2006 Gap solitons in periodic discrete NLS equations. Nonlinearity 19, 27-40.

[6] Nehari, Z. 1960 On a class of nonlinear second order differential equations. Trans. Am. Math. Soc. 95, 101-123.

[7] Teschl, G. 2000 Jacobi operators and completely integrable nonlinear lattices. Providence, RI: American Mathematical Society.

[8] Weinstein, M. I., Excitation thresholds for nonlinear localized modes on lattices, Nonlinearity 12, 673 (1999).

[9] Johansson, M. and Aubry, S., Existence and stability of quasiperiodic breathers in the discrete nonlinear Schrödinger equation, Nonlinearity 10, 1151 (1997).
[10] Aubry, S., Breathers in nonlinear lattices: existence, linear stability and quantization, Physica D 103, 201 (1997).

[11] Zhang, G. P., Breather solutions of the discrete nonlinear schrödinger equations with unbounded potentials, J. Math. Phys 50(1), 013505 (2009).

[12] Lions, P. L., The concentration compactness principle in the calculus of variations I: The locally compact case, Ann. Inst. Henri Poincaré, Anal. Nonlinéaire 1, 223 (1984).

[13] Cuevas, J., Karachalios, N. I. and Palmero, F., Lower and upper estimates on the excitation threshold for breathers in discrete nonlinear Schrodinger lattices, J. Math. Phys. 50(11), 112705 (2009)

[14] Nikos I. Karachalios, Athanasios N. Yannacopoulos, Global existence and compact attractors for the discrete nonlinear Schrödinger equation, J. Differential Equations 217 (2005) 88-123