ON THE PROPAGATION OF REGULARITY OF SOLUTIONS OF THE KADOMTSEV-PETVIASHVILI (KPII) EQUATION

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ABSTRACT. We shall deduce some special regularity properties of solutions to the IVP associated to the KPII equation. Mainly, for datum $u_0 \in X_s(\mathbb{R}^2)$, $s > 2$, (see (1.2) below) whose restriction belongs to $H^m((x_0, \infty) \times \mathbb{R})$ for some $m \in \mathbb{Z}^+$, $m \geq 3$, and $x_0 \in \mathbb{R}$, we shall prove that the restriction of the corresponding solution $u(\cdot, t)$ belongs to $H^m((\beta, \infty) \times \mathbb{R})$ for any $\beta \in \mathbb{R}$ and any $t > 0$.

1. INTRODUCTION

We consider solutions of the initial value problem (IVP) associated to the Kadomtsev-Petviashvilli (KPII) equation,

$$\begin{aligned}
\partial_t u + \partial_x^3 u + \alpha \partial_x^{-1} \partial_y^2 u + u \partial_x u &= 0, \quad (x,y) \in \mathbb{R}^2, \quad t > 0, \quad \alpha = 1, \\
u(x,y,0) &= u_0(x,y),
\end{aligned}$$

(1.1)

the operator $\partial_x^{-1}$ is defined via the Fourier transform by

$$\hat{\partial_x^{-1}}f(\xi, \eta) = -\frac{i}{\xi} \hat{f}(\xi, \eta).$$

The KP equations (KPI ($\alpha = -1$) and KPII ($\alpha = 1$)) are models for the propagation of long, dispersive, weakly nonlinear waves which travel predominantly in the $x$ direction, with weak transverse effects. These equations were derived by Kadomtsev and Petviashvilli [10] as two-dimensional extensions of the Korteweg-de Vries equation (see (1.5) below). The KP equations have been studied extensively in the last few years in several aspects. For an interesting account of KP equations features and open problems we refer the reader to [12] (see also [14]).

Our main purpose in this paper is the study of smoothing properties of solutions of the IVP (1.1).

Before stating our result we briefly describe the development of the local well-posedness theory for the IVP (1.1). The first outcome regarding the local well-posedness of the IVP (1.1) was given by Ukai in [21] (see also [15], [9]) for initial data in $H^s(\mathbb{R}^2)$, $s \geq 3$. In [11] Bourgain proved local and...
global well-posedness of the IVP (1.1) in $L^2(\mathbb{T}^2)$ and $L^2(\mathbb{R}^2)$. Takaoka and Tzvetkov [18] and Isaza and Mejía [7] established local well-posedness for data in the anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$, $s_1 > -1/3$, $s_2 \geq 0$, where

$$H^{s_1,s_2}(\mathbb{R}^2) = \{ f \in \mathcal{S}'(\mathbb{R}^2) : \| f \|_{H^{s_1,s_2}} = \| \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \hat{f} \| < \infty \},$$

and $\langle \cdot \rangle^2 = 1 + | \cdot |^2$ (for previous results we refer [16], [19], [20]). Later Takaoka in [17] proved local well-posedness in the scaling anisotropic Sobolev space $H^{1/2,0}(\mathbb{R}^2)$ for any size data. They also obtained global well-posedness for small data in the homogeneous anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$, $s_1 > -1/2$, $s_2 = 0$, but imposing an additional low frequency condition in the initial data (i.e. $|D_x|^{1+\varepsilon}u_0 \in L^2(\mathbb{R}^2)$, for a suitable $\varepsilon > 0$). In [2] Hadac removed the latter condition on the initial data and showed local well-posedness for any data in $H^{s_1,s_2}(\mathbb{R}^2)$, $s_1 > -1/2$, $s_2 \geq 0$. Finally, Hadac, Herr and Koch obtained the local well-posedness in the scaling anisotropic Sobolev space $H^{-1/2,0}(\mathbb{R}^2)$ for any size data. They also obtained global well-posedness for small data in the homogeneous anisotropic Sobolev space $H^{-1/2,0}(\mathbb{R}^2)$ and local well-posedness in the same space for any size data. In the anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$ the best global result known for any size data was proved by Isaza and Mejía in [8] for $s_1 > -1/14$, $s_2 = 0$. We point out that the inverse scattering method provides global solution for the KPII equation only for small initial data (see [22]).

In our analysis we will use a result of Iorio and Nunes [4] regarding local well-posedness for the KP equations ($\alpha = \pm 1$ in (1.1)) and a general nonlinearity $\partial_x F(u)$ in Sobolev spaces $H^s(\mathbb{R}^2)$, $s > 2$. More precisely, we define

$$X_s = \{ f \in H^s(\mathbb{R}^2) : \partial_x^{-1} f \in H^s(\mathbb{R}^2) \}. \tag{1.2}$$

**Theorem A** ([4]). Let $u_0 \in X_s(\mathbb{R}^2)$, $s > 2$. There exist $T > 0$ and a unique $u = u(x,y,t)$ solution of the IVP (1.1) such that $u \in C([0,T];X_s)$. Moreover, the data-solution map is continuous in the $\| \cdot \|_s$-norm.

Our main result reads as follows:

**Theorem 1.1.** For $T > 0$, let $u$ be a solution in $[0,T]$ of equation (1.1) with initial data $u_0 \in X_s(\mathbb{R}^2)$, $s > 2$. Suppose that for an integer $n \geq 3$ and some $x_0 \in \mathbb{R}$, the restriction of $u_0$ to $(x_0,\infty) \times \mathbb{R}$ belongs to $H^n((x_0,\infty) \times \mathbb{R})$ and $\partial_x^{-1} \partial_y^3 u_0 \in L^2((x_0,\infty) \times \mathbb{R})$.

Then, for any $\nu > 0$ and $\varepsilon > 0$

$$\sup_{t \in [0,T]} \sum_{\alpha_1 + \alpha_2 \leq n} \int_{-\infty}^{\infty} \int_{x_0 + \varepsilon - \nu t}^{\infty} (\partial_x^{\alpha_1} \partial_y^{\alpha_2} u(x,y,t))^2 \, dx \, dy < \infty. \tag{1.3}$$

In particular, for all times $t \in (0,T]$ and for all $a \in \mathbb{R}$, $u(t) \in H^n((a,\infty) \times \mathbb{R})$.

**Remark 1.2.** We observe that the condition $\partial_x^{-1} \partial_y^3 u_0 \in L^2((x_0,\infty) \times \mathbb{R})$ is automatically fulfilled if $s \geq 3$. 
Remark 1.3. From our comments above and our proof of Theorem 1.1 it will be clear that the requirement \( u_0 \in X_s(\mathbb{R}^2) \) in Theorem 1.1 can be lowered.

As a direct consequence of Theorem 1.1 we can deduce

Corollary 1.4. Let \( u \in C(\mathbb{R} : X_s(\mathbb{R}^2)) \), \( s > 2 \), be a solution of the equation in (1.1) described in Theorem A. If there exist \( m \in \mathbb{Z}^+ \), \( m \geq 3 \), \( \hat{t} \in \mathbb{R} \), \( a \in \mathbb{R} \) such that

\[
  u(\cdot, \hat{t}) \notin H^m((a, \infty) \times \mathbb{R}),
\]

then for any \( t \in (-\infty, \hat{t}) \) and any \( \beta \in \mathbb{R} \)

\[
  u(\cdot, t) \notin H^m((\beta, \infty) \times \mathbb{R}).
\]

Next, one has that for appropriate class of data singularities of the corresponding solutions travel with infinite speed to the left in the \( x \)-variable as time evolves.

Corollary 1.5. Let \( u \in C(\mathbb{R} : X_s(\mathbb{R}^2)) \), \( s > 2 \), be a solution of the equation in (1.1) described in Theorem A. If there exist \( k, m \in \mathbb{Z}^+ \) with \( k \geq m \) and \( a, b \in \mathbb{R} \) with \( b < a \) such that

\[
  u_0 \in H^k((a, \infty) \times \mathbb{R}) \quad \text{but} \quad u_0 \notin H^m((b, \infty) \times \mathbb{R}), \tag{1.4}
\]

then for any \( t \in (0, \infty) \) and any \( v > 0 \) and \( \varepsilon > 0 \)

\[
  \sum_{\alpha_1 + \alpha_2 \leq k} \int_{-\infty}^{\infty} \int_{a + \varepsilon - vt}^{\infty} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} u(x, y, t)|^2 \, dx \, dy < \infty,
\]

and for any \( t \in (-\infty, 0) \) and \( \gamma \in \mathbb{R} \)

\[
  \sum_{\alpha_1 + \alpha_2 \leq m} \int_{-\infty}^{\infty} \int_{\gamma}^{\infty} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} u(x, y, t)|^2 \, dx \, dy = \infty.
\]

Remark 1.6.

(a) If in Corollary 1.5 in addition to (1.4) one assumes that

\[
  \sum_{\alpha_1 + \alpha_2 \leq k} \int_{-\infty}^{b} \int_{-\infty}^{\infty} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} u(x, y, t)|^2 \, dx \, dy < \infty,
\]

then by combining the results in this corollary with the group properties it follows that

\[
  \sum_{\alpha_1 + \alpha_2 \leq m} \int_{-\infty}^{\beta} \int_{-\infty}^{\infty} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} u(x, y, t)|^2 \, dx = \infty, \quad \text{for any} \ \beta \in \mathbb{R} \text{ and } t > 0.
\]
This shows that the regularity in the left hand side does not propagate forward in time.

(b) Notice that (1.3) tells us that the local regularity of the initial datum \( u_0 \) described in the statement of Theorem 1.1 propagates with infinite speed to its left in the x-variable as time evolves.

(c) In [5] we proved the corresponding result concerning the IVP for the k-generalized Korteweg-de Vries equation

\[
\begin{aligned}
\begin{cases}
\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, & x, t \in \mathbb{R}, \ k \in \mathbb{Z}^+,

u(x,0) = u_0(x).
\end{cases}
\end{aligned}
\]  

(1.5)

More precisely,

**Theorem B.** If \( u_0 \in H^{3/4^+}(\mathbb{R}) \) and for some \( l \in \mathbb{Z}^+ \), \( l \geq 1 \) and \( x_0 \in \mathbb{R} \)

\[
\| \partial_x^l u_0 \|^2_{L^2((x_0,\infty))} = \int_{x_0}^\infty |\partial_x^l u_0(x)|^2 dx < \infty,
\]  

(1.6)

then the solution of the IVP (1.5) provided by the local theory satisfies that for any \( v > 0 \) and \( \varepsilon > 0 \)

\[
\sup_{0 \leq t \leq T} \int_{x_0 + \varepsilon - vt}^\infty (\partial_x^j u)^2(x,t) dx < c,
\]  

(1.7)

for \( j = 0, 1, \ldots, l \) with \( c = c(l; \| u_0 \|_{3/4^+,2}; \| \partial_x^l u_0 \|_{L^2((x_0,\infty))}; v; \varepsilon; T) \).

In particular, for all \( t \in (0,T] \), the restriction of \( u(\cdot,t) \) to any interval \((x_1,\infty)\) belongs to \( H^l((x_1,\infty)) \).

Moreover, for any \( v > 0 \), \( \varepsilon > 0 \) and \( R > 0 \)

\[
\int_0^T \int_{x_0 + \varepsilon - vt}^{x_0 + R - vt} (\partial_x^{l+1} u)^2(x,t) dx dt < c,
\]  

(1.8)

with \( c = c(l; \| u_0 \|_{3/4^+,2}; \| \partial_x^l u_0 \|_{L^2((x_0,\infty))}; v; \varepsilon; R; T) \).

**Remark 1.7.** For solutions of the IVP associated to the Benjamin-Ono equation, that is,

\[
\begin{aligned}
\begin{cases}
\partial_t u - \mathcal{H} \partial_x^2 u + u \partial_x u = 0, & x \in \mathbb{R}, \ t > 0,

u(x,0) = u_0(x),
\end{cases}
\end{aligned}
\]  

(1.9)

where \( \mathcal{H} \) denotes the Hilbert transform, we also showed a similar property (see [6]).

**Remark 1.8.** In [5] we obtained the following result.

**Theorem C.** If \( u_0 \in H^{3/4^+}(\mathbb{R}) \) and for some \( n \in \mathbb{Z}^+ \), \( n \geq 1 \),

\[
\| x^{n/2} u_0 \|^2_{L^2((0,\infty))} = \int_0^\infty |x^n| |u_0(x)|^2 dx < \infty,
\]  

(1.10)
then the solution $u$ of the IVP (1.5) provided by the local theory satisfies that
\[
\sup_{0 \leq t \leq T} \int_0^\infty |x^n| |u(x,t)|^2 dx \leq c
\] (1.11)
with $c = c(n; \|u_0\|_{L^2((-\infty,0);T)}).$

Moreover, for any $\epsilon, \delta, R > 0, v \geq 0, m, j \in \mathbb{Z}^+, m + j \leq n, m \geq 1,$
\[
\sup_{\delta \leq t \leq T} \int_{\epsilon - vt}^\infty (\partial_x^m u)^2(x,t) x^j \, dx
+ \int_{\epsilon - vt}^T \int_{\epsilon - vt}^{R - vt} (\partial_x^{m+1} u)^2(x,t) x^{j-1} \, dx \, dt \leq c,
\] (1.12)
with $c = c(n; \|u_0\|_{L^2((-\infty,0);T)}; \delta; \epsilon; R; v).$

In [12] (p.783) Klein and Saut gave an example showing that initial data in the Schwartz class do not necessarily lead to solutions of the KPII equation in the Schwartz class. On the other hand, Levandovsky in [13] showed that for initial data $u_0$ satisfying
\[
\int_{\mathbb{R}^2} \left\{ u_0^2 + (\partial_x^3 u_0)^2 + (\partial_x^{-1} \partial_y u_0)^2 + x_+^L u_0^2 + x_+^{L'} (\partial_x u_0^2) \right\} \, dx \, dy < \infty
\] (1.13)
for all integer $L \geq 0,$ where $x_+ = \max\{0, x\},$ there exists a unique solution of the IVP (1.1) $u(t) \in C^\infty(\mathbb{R}^2)$ for $t \in (0,T)$.

We shall notice that solutions of the IVP (1.1) also share a smoothing property similar to the one proved by Kato ([11]) for solutions of the KdV equation (see [15]).

For the generalized KPII equation i.e.
\[
\partial_t u + \partial_x^3 u + \partial_x^{-1} \partial_y^2 u + u^p \partial_x u = 0 \quad p \in \mathbb{Z}^+, \quad p > 1,
\] (1.14)
it may be possible to obtain similar results as those in Theorem 1.1.

This paper is organized as follows. In Section 2, we introduce some tools that will be employed in the proof of Theorem 1.1. Section 3 will be devoted to the proof of Theorem 1.1.

2. PRELIMINARIES

Our argument of proof uses weighted energy estimates. In this case we will employ weights independent of the variable $y.$ More precisely, for each $\epsilon > 0$ and $b \geq 5\epsilon$ we define a function $\chi_{\epsilon,b} \in C^\infty(\mathbb{R})$ with $\chi_{\epsilon,b}'(x) \geq 0,$ and
\[
\chi_{\epsilon,b}(x) = \begin{cases} 
0, & x \leq \epsilon, \\
1, & x \geq b,
\end{cases}
\] (2.1)
which will be constructed as follows. Let \( \rho \in C_0^\infty(\mathbb{R}) \), \( \rho(x) \geq 0 \), even, with \( \text{supp } \rho \subseteq (-1, 1) \) and \( \int \rho(x)dx = 1 \) and define

\[
\nu_{\epsilon,b}(x) = \begin{cases} 
0, & x \leq 2\epsilon, \\
\frac{1}{b-3\epsilon}x - \frac{2\epsilon}{b-3\epsilon}, & x \in [2\epsilon, b-\epsilon), \\
1, & x \geq b-\epsilon,
\end{cases}
\]  
(2.2)

with

\[
\chi_{\epsilon,b}(x) = \rho_{\epsilon} * \nu_{\epsilon,b}(x)
\]  
(2.3)

where \( \rho_{\epsilon}(x) = \epsilon^{-1}\rho(x/\epsilon) \). Thus

\[
\text{supp } \chi_{\epsilon,b} \subseteq [\epsilon, \infty),
\]

\[
\text{supp } \chi'_{\epsilon,b}(x) \subseteq [\epsilon, b].
\]  
(2.4)

If \( x \in (3\epsilon, b-2\epsilon) \), then

\[
\chi'_{\epsilon,b}(x) \geq \frac{1}{b-3\epsilon}.
\]  
(2.5)

and for any \( x \in \mathbb{R} \)

\[
\chi'_{\epsilon,b}(x) \leq \frac{1}{b-3\epsilon}.
\]  
(2.6)

We will frequently use the following facts

\[
\chi_{\epsilon/5,b}(x) = 1, \quad \text{on supp } \chi_{\epsilon,b},
\]

\[
\chi''_{\epsilon,b}(x) \leq c\chi_{\epsilon/5,b+\epsilon}(x).
\]  
(2.7)

Throughout the article we will apply the following inequality of Gagliardo-Nirenberg’s type:

**Lemma 2.1.** Let \( f = f(x,y) \) be a function such that \( f\chi \in H^1(\mathbb{R}^2) \), where \( \chi = \chi(x) = \chi_{\epsilon,b} \) is as above. Then,

\[
\left( \int_{\mathbb{R}^2} f^4 \chi^2 \right)^{1/2} \leq c \int_{\mathbb{R}^2} f^2 \chi + c \int_{\mathbb{R}^2} (\partial_x f)^2 \chi + c \int_{\mathbb{R}^2} (\partial_y f)^2 \chi + c \int_{\mathbb{R}^2} f^2 \chi'.
\]  
(2.8)

**Proof.** It suffices to observe that

\[
f^2(x,y)\chi(x) \leq \int_{-\infty}^{+\infty} (2|f\partial_x f|\chi + f^2\chi') \, dx \quad \text{and}
\]

\[
f^2(x,y)\chi(x) \leq \int_{-\infty}^{+\infty} 2|f\partial_y f|\chi \, dy.
\]
Therefore
\[ \int f^4 \rho^2 \leq c \left( \int (|f \partial_x f| + f^2 \rho') \, dx \, dy \right) \left( \int |f \partial_y f| \rho \, dx \right). \]
In this way, (2.8) follows from Young’s inequality. \( \square \)

3. PROOF OF THEOREM 1.1

We begin by giving a brief sketch of the proof. By using a translation in \( x \) if necessary we may assume that \( x_0 = 0 \). For two integers \( \alpha_1, \alpha_2 \), with \( \alpha_1 \geq -1 \) and \( \alpha_2 \geq 0 \), let \( \alpha = (\alpha_1, \alpha_2) \), \( |\alpha| = \alpha_1 + \alpha_2 \) and \( \partial^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2} \). We apply \( \partial^\alpha \) to equation (1.1), multiply by 
\[ \partial^\alpha u \, \chi = \partial^\alpha u \, \chi_{e,b} (x + vt), \]
and integrate in \( \mathbb{R}^2 \). Formally assuming that we have enough regularity to apply integration by parts we obtain that
\[ \frac{1}{2} \frac{d}{dt} \left( \int \partial^\alpha u^2 \chi \, dx \, dy \right) \leq -\frac{1}{2} \int \partial^{\alpha-1} \chi' \, dx \, dy \int \partial^\alpha u \partial^\alpha u \, \chi \, dx \, dy \]
\[ = -\frac{1}{2} \int \partial^{\alpha-1} \chi' \, dx \, dy \left( \int \partial^\alpha u^2 \chi'' \, dx \, dy \right) \]
\[ + \frac{1}{2} \int \partial^\alpha u \partial^\alpha u \, \chi \, dx \, dy = C. \tag{3.1} \]

In order to write our expressions in a simple form we will use the following notation: for \( \alpha = (\alpha_1, \alpha_2) \),
\[ [\alpha_1, \alpha_2]_{e,b} \equiv [\alpha_1, \alpha_2] := \int \partial^\alpha u^2 \chi_{e,b} (x + vt) \, dx \, dy, \]
\[ [\alpha_1, \alpha_2]'_{e,b} \equiv [\alpha_1, \alpha_2]' = A_3 = \int \partial_x \partial^\alpha u^2 \chi_{e,b} ' (x + vt) \, dx \, dy, \tag{3.2} \]
\[ [\alpha_1, \alpha_2]''_{e,b} \equiv [\alpha_1, \alpha_2]'' = A_4 = \int \partial_x \partial_x \partial^\alpha u^2 \chi_{e,b} '' (x + vt) \, dx \, dy. \]

When \( n \geq 3 \), \( \chi_{e,b} (\cdot) u_0 \in H^n (\mathbb{R}^2) \) and \( \chi_{e,b} (\cdot) \partial_x^2 (\partial_x^{-1} \partial_y) u_0 \in L^2 (\mathbb{R}^2) \), we will use Gronwall’s lemma to show that
\[ \sup_{t \in [0,T]} [\alpha_1, \alpha_2] (t) \leq \sup_{t \in [0,T]} \int \partial^\alpha u^2 \chi_{e,b} (x + vt) \, dx \, dy \leq C \tag{3.3} \]
for all indices \( \alpha \) with \( 3 \leq |\alpha| \leq n \).

By induction we will suppose that (3.3) is proved for all cases with \( |\alpha| \leq n - 1 \) and we will refer to a case already proved as a former case.
For an index $\alpha$ with $|\alpha| = n$, our procedure will lead to verify that, as a consequence of a former case,

$$\int_0^T |A_1^\alpha(t)| dt \equiv c \int_0^T \int (\partial^\alpha u)^2 \chi' dxdy dt \leq C$$

and

$$\int_0^T |A_2^\alpha(t)| dt \equiv c \int_0^T \int (\partial^\alpha u)^2 \chi''' dxdy dt \leq C.$$  (3.4)

Notice that for $|\alpha| = 0, 1, 2$ with $\alpha_1 \geq 0$, inequalities (3.3) and (3.4) follow directly from the well-posedness of the IVP (1.1) with $u_0 \equiv u(0) \in H^{2+}(\mathbb{R}^2)$. Taking into account (3.4) and the fact that $A_3^\alpha \geq 0$ and $A_4^\alpha \geq 0$, we will restrict our attention to show that

$$|A_2^\alpha(t)| \equiv \int \partial^\alpha(u \partial_x u \partial^\alpha u \chi) dxdy \leq c \int (\partial^\alpha u)^2 \chi dxdy + g(t),$$  (3.5)

where $g \geq 0$ is a function with $\int_0^T g(t) dt \leq C$ (sometimes we will mix several cases together to obtain an inequality similar to (3.5)). We will continue denoting by $g$ a generic nonnegative integrable function on $[0, T]$.

Once (3.5) is obtained, Gronwall’s Lemma will give (3.3) for the case $\alpha$ under consideration. Also, from (3.1) to (3.5) it will follow that

$$\int_0^T [\alpha_1, \alpha_2]' dt \equiv \int_0^T A_3^\alpha(t) dt \equiv c \int_0^T \int (\partial_x \partial^\alpha u)^2 \chi' dxdy dt \leq C,$$  (3.6)

and

$$\int_0^T [\alpha_1, \alpha_2]'' dt \equiv \int_0^T A_4^\alpha(t) dt \equiv c \int_0^T \int (\partial_x^{-1} \partial_y \partial^\alpha u)^2 \chi' dxdy dt \leq C,$$  (3.7)

which guarantees for the case $(\alpha_1 + 1, \alpha_2)$ with $\alpha_1 \geq -1$ and the case $(\alpha_1 - 1, \alpha_2 + 1)$ with $\alpha_1 \geq 1$ that

$$\int_0^T |A_1^{(\alpha_1, \alpha_2)}| dt \equiv c \int_0^T \int (\partial_x \partial^\alpha u)^2 \chi' dxdy dt \leq C,$$  (3.8)

and

$$\int_0^T |A_1^{(\alpha_1 - 1, \alpha_2 + 1)}| dt \equiv c \int_0^T \int (\partial_x^{-1} \partial_y^{\alpha_1 - 1} \partial_y^{\alpha_2 + 1} u)^2 \chi' dxdy dt \leq C.$$  (3.9)

Since $|\chi''_{\epsilon, b}| \leq c \chi'/_{5, b+\epsilon}$, we will have that

$$\int_0^T |A_2^{(\alpha_1 + 1, \alpha_2)}| dt \leq C, \quad \text{and} \quad \int_0^T |A_2^{(\alpha_1 - 1, \alpha_2 + 1)}| dt \leq C.$$  (3.10)
In this way (3.3), (3.9), and (3.10) will give (3.4) for the cases \((\alpha_1 + 1, \alpha_2)\) and \((\alpha_1 - 1, \alpha_2 + 1)\).

We now begin the proof by considering the cases with \(|\alpha| = 2, \alpha_1 \geq 0\). Though the regularity of the solution provides (3.3) for these cases, we consider them in order to establish the local smoothing effects expressed in (3.6) and (3.7), which will be used in future cases.

**Case (2,0):**

With \(\alpha = (2,0)\), \(\partial^{\alpha} = \partial_x^2\) we estimate the cubic term \(A_5\) in (3.1). Using integration by parts and Sobolev’s embeddings,

\[
|A_5| = |\int \partial_x^2 (u \partial_x u) \partial_x^2 u \chi| = |\int 3\partial_x u (\partial_x^2 u)^2 \chi + u \partial_x^3 u \partial_x^2 u \chi|
\]

\[
= \frac{5}{2} \int \partial_x u (\partial_x^2 u)^2 \chi - \frac{1}{2} \int u (\partial_x^2 u)^2 \chi'
\]

\[
\leq c(\|\partial_x u\|_{L^\infty} + \|u\|_{L^\infty}) \|\partial_x^2 u\|_{L^2_{xy}} \leq c\|u\|^3_{C([0,T];H^{2+}(\mathbb{R}^2)}
\]

Besides,

\[
|A_1| + |A_2| \leq c \int (\partial_x^2 u)^2 \chi'' + c \int (\partial_x^2 u)^2 \chi''' \leq c\|u\|^2_{C([0,T];H^{2+}(\mathbb{R}^2)}
\]

Thus, by integrating (3.1) in \([0,T]\), and taking into account that the values of \([2,0]\) at \(t = 0\) and at \(t = T\) are bounded by \(c\|u\|^2_{C([0,T];H^{2+}(\mathbb{R}^2)}\), we obtain (3.6) and (3.7) for the case \((2,0)\), which, according to our notation (3.2), is

\[
\int_0^T ([2,0]' + [2,0]''') dt \leq C.
\]

Notice that this estimate provides (3.4) for the future case \(\alpha = (3,0)\).

**Case (1,1):**

With \(\alpha = (1,1)\) and \(\partial^{\alpha} = \partial_x \partial_y\), we apply integration by parts to obtain that

\[
|A_5| = |\int (2\partial_x \partial_y u \partial_x u + \partial_y u \partial_x^2 u + u \partial_x \partial_y u) \partial_x \partial_y u \chi|
\]

\[
= |\int \frac{3}{2} \partial_x u (\partial_x \partial_y u)^2 \chi - \frac{1}{2} u (\partial_x \partial_y u)^2 \chi' + \partial_y u \partial_x^2 u \partial_x \partial_y u \chi| \quad (3.12)
\]

\[
\leq \|u\|^3_{C([0,T];H^{2+}(\mathbb{R}^2)}
\]

and, proceeding as in the former case we have that

\[
\int_0^T ([1,1]' + [1,1]''') dt \leq C,
\]

which gives (3.4) for the case \(\alpha = (2,1)\).

**Case (0,2):**
The cubic term $A_5$ with $\alpha = (0, 2)$ in (3.1), is treated as the former cases to obtain that

$$|A_5| = |\int \frac{1}{2} \partial_x u (\partial^2_x u)^2 \chi - \frac{1}{2} u (\partial^2_x u)^2 \chi' + 2 \partial_x u \partial_x \partial_x u \partial^2_x u \chi| \leq C,$$

and from this estimate we then have that

$$\int_0^T ([0, 2] + [0, 2])^n d\tau = \int_0^T \int (\partial_x \partial^2_x u)^2 \chi' dxdy d\tau \leq C, \quad (3.14)$$

to be used in the case $\alpha = (1, 2)$.

For the estimates of order $|\alpha| = 3$ we will need to consider a single case with $\alpha_1 = -1$, namely the case (-1,3).

Case (-1,3):

For this case $\partial^\alpha = \partial_x^{-1} \partial^3_x$. From integration by parts and Young’s inequality it follows that

$$|A_5| = \frac{1}{2} |\int \partial_x^3 u^2 \partial_x^{-1} \partial^3_x u \chi| = \frac{1}{2} |\int (2u \partial_x^3 u^2 + 6 \partial_x u \partial^2_x u) \partial_x^{-1} \partial^3_x u \chi|$$

$$\leq \frac{1}{2} |\int \partial_x u (\partial_x^{-1} \partial^3_x u)^2 \chi - \int u (\partial_x^{-1} \partial^3_x u)^2 \chi' + \frac{1}{2} c||\partial_x u||_{L^\infty} \int |\partial_x^2 u||\partial_x^{-1} \partial^3_x u|\chi$$

$$\leq c||\partial_x u||_{L^\infty}[−1,3] + c||u||_{L^\infty}[0,2]^n + \frac{1}{2} c|−1,3|$$

$$\leq c + \frac{1}{2} c||\partial_x u||_{L^\infty}[−1,3] + g(t).$$

On the other hand, since $|\chi''''| \leq c\chi'/\sqrt{b+c}$, we see that in this case

$$|A_1 + A_2| \leq c \int (\partial_x^{-1} \partial^3_x u)^3 \chi'/\sqrt{b+c} d\tau \leq c [0,2]^n.$$

In this way, from the above estimates

$$\frac{d}{dt} [−1,3] \leq c[−1,3] + g(t),$$

which gives (3.3), (3.6), and (3.7) for this case.

We now turn to the cases with $|\alpha| = 3$. Thus we assume that $u_0$ satisfies the hypotheses in the statement of Theorem 1.1 with $n = 3$.

Case (3,0):

From integration by parts we see that

$$A_5 = \int \partial^3_x u (u \partial_x u) \partial^3_x u \chi = \int (4 \partial_x u \partial^2_x u + 3 \partial_x^2 u \partial^2_x u + u \partial^4_x u) \partial^3_x u \chi$$

$$= \frac{7}{2} \int \partial_x u (\partial^3_x u)^2 \chi - \frac{1}{2} \int u (\partial^3_x u)^2 \chi' - \frac{1}{2} \int (\partial^3_x u)^3 \chi' \equiv A_51 + A_52 + A_53.$$
By Sobolev embeddings

\[ |A_{51}| + |A_{52}| \leq (\| \partial_t u \|_{L^\infty} + \| u \|_{L^\infty}) \int (\partial_x^3 u)^2 \chi + c \int (\partial_x^3 u)^2 \chi' \]

\[ \leq c[3,0] + c[2,0]' \]  

(3.15)

The first term on the right hand side of (3.15) is the quantity to be estimated while the second term has finite integral in \([0,T]\) by (3.11). Now, from integration by parts and Young’s inequality

\[ |A_{53}| = |-2 \int \partial_t u \partial_x^2 u \partial_y^3 u \chi - \int \partial_t u (\partial_x^2 u)^2 \chi''| \]

\[ \leq c\| \partial_t u \|_{L^\infty}(\int (\partial_x^2 u)^2 \chi' + \int (\partial_x^2 u)^2 \chi' + \int (\partial_x^2 u)^2 |\chi''|) \]

\[ \leq c + c[2,0]' + c \]  

(3.16)

which is bounded after integration in \([0,T]\).

Since from the case (2,0), and inequalities (3.8) and (3.10) we have that \( |A_1| \) and \( |A_2| \) have finite integral in \([0,T]\), it follows that

\[ \frac{d}{dt} [3,0] \leq c[3,0] + g(t) \]

Therefore, as we have shown in the sketch of our proof, we obtain (3.3) (3.6), and (5.7) for the case (3,0). That is

\[ \sup_{t \in [0,T]} [3,0] \leq C \quad \text{and} \quad \int [3,0]' + [3,0]'' dt < \infty. \]

We will now turn to the cases (2,1), (1,2), and (0,3). As it will be seen, we need to consider these three cases together for the application of Gronwall’s lemma.

Case (2,1):

We have that

\[ |A_5| = \int \partial_x^2 \partial_y (u \partial_x u) \chi \]

\[ = \int (a_1 \partial_x^2 \partial_y u \partial_x u + a_2 \partial_x \partial_y u \partial_x^2 u + a_3 \partial_x u \partial_x^3 u + u \partial_x^3 \partial_y u \partial_x^2 \partial_y u) \chi \]

\[ \equiv A_{51} + A_{52} + A_{53} + A_{54} \]

We apply Young’s inequality and Sobolev embeddings to obtain that

\[ |A_{51} + A_{53}| \leq c \| \partial_t u \|_{L^\infty}[2,1] + c \| \partial_x u \|_{L^\infty}([3,0] + [2,1]) \]

\[ \leq c + c[2,1], \]  

(3.17)

since (3,0) is a former case and we have already seen that \([3,0] \leq c.\)
From integration by parts it follows that
\[ |A_{54}| = | - \frac{1}{2} \int \partial_x u (\partial_x^2 \partial_y u)^2 \chi - \frac{1}{2} \int u (\partial_x^2 \partial_y u)^2 \chi' | \leq c[2, 1] + c[1, 1]' . \] (3.18)

For \( A_{52} \), we integrate by parts to conclude that
\[ A_{52} = - \frac{a_2}{2} \int (\partial_x \partial_y u)^2 \partial_x^3 u \chi - \frac{a_2}{2} \int (\partial_x \partial_y u)^2 \partial_x^2 u \chi' \equiv A_{521} + A_{522} . \]

To estimate \( A_{521} \) we apply (2.8) and the facts that \( \chi_{b,e} = \chi_{b,e/5,e} \) and \( \chi_{e/5,e}^2 \leq \chi_{e/5,e} \) to conclude that
\[ |A_{521}| = c | \int (\partial_x \partial_y u)^2 \partial_x^3 u \chi \chi_{e/5,e} | \leq \left( \int (\partial_x \partial_y u)^4 \chi^2 \right)^{1/2} \left( \int (\partial_x^2 u)^2 \chi_{e/5,e}^2 \right)^{1/2} \leq c[3, 0]_{e/5,e} \left( \int (\partial_x \partial_y u)^2 \chi + (\partial_x^2 \partial_y u)^2 \chi + (\partial_x \partial_y u)^2 \chi' \right)^{1/2} \leq c([1, 1] + [2, 1] + [1, 2] + [0, 1]) \leq c + c[2, 1] + c[1, 2] + c[0, 1]' , \] (3.19)
since the cases (3,0) and (1,1) are former cases.

\( A_{522} \) can be treated in a similar manner to obtain that
\[ |A_{522}| \leq \left( \int (\partial_x \partial_y u)^4 (\chi')^2 \right)^{1/2} \left( \int (\partial_x^2 u)^2 \chi_{e/5,e}^2 \right)^{1/2} \leq c[2, 0]_{e/5,e} \left( \int (\partial_x \partial_y u)^2 \chi' + (\partial_x^2 \partial_y u)^2 \chi' + (\partial_x \partial_y u)^2 |\chi''| \right) \leq c([0, 1]' + [1, 1]' + [0, 2]' + [0, 1]'_{e/5,b+e}) , \] (3.20)
since
\[ |\chi''| = |\chi_{e,b}''| \leq c \chi_{e/5,b+e}' \]
and (2,0) is a former case.

On the other hand,
\[ |A_1| + |A_2| \leq c | \int (\partial_x^2 \partial_y u)^2 \chi' | + c | \int (\partial_x^2 \partial_y u)^2 \chi''' | \leq c[1, 1]' + [1, 1]'_{e/5,b+e} . \]

In this way, gathering the above estimates, and taking into account that the cases (0,1), (0,2), and (1,1) are former cases we conclude that
\[ \frac{d}{dt} [2, 1] \leq c[2, 1] + c[1, 2] + g(t) . \] (3.21)
Case (1,2):

\[ |A_5| = \int \partial_x \partial_y^2 (u \partial_x u) \partial_y^2 u \chi \]
\[ = \int (a_1 \partial_x \partial_y^2 u \partial_x u + a_2 \partial_x \partial_y^2 u \partial_y u + a_3 \partial_y^2 u \partial_x^2 u + a_4 \partial_y^2 u \partial_y u + u \partial_y^2 \partial_x^2 u) \partial_y^2 u \chi \]
\[ \equiv A_{51} + A_{52} + A_{53} + A_{54} + A_{55}. \]

Integrating by parts in the term \( A_{55} \), and proceeding as we did to obtain (3.17) and (3.18), we have that

\[ |A_{51} + A_{54} + A_{55}| \leq c[1,2] + c([1,2] + [2,1]) + c([1,2] + c[0,2]'). \]

Integration by parts with respect to \( y \) shows that \( A_{52} = 0 \).

For \( A_{53} \) we integrate by parts and apply (2.8) to conclude that

\[ |A_{53}| = \left| -\frac{a_3}{2} \int (\partial_y^2 u)^2 \partial_x^3 u \chi - \frac{a_3}{2} \int (\partial_y^2 u)^2 \partial_x^2 u \chi' \right| \]
\[ = \left| -\frac{a_3}{2} \int (\partial_y^2 u)^2 \chi \partial_x^3 u \chi_{e/5,\epsilon} + \frac{a_3}{2} \left( \int 2 \partial_y^2 u \partial_x \partial_y^2 u \partial_x \chi' + \int (\partial_y^2 u)^2 \partial_x \chi'' \right) \right| \]
\[ \leq c[3,0]^{1/2} \left( \int (\partial_y^2 u)^4 \chi^2 \right)^{1/2} + c \int |\partial_y^2 u \partial_x \partial_y^2 u| \chi' + c \int (\partial_y^2 u)^2 \chi'_{e/5,b+\epsilon} \]
\[ \leq c([0,2] + [1,2] + [0,3] + [1,1]'') + c([1,1]'' + [0,2]') + c[1,1]''_{e/5,b+\epsilon}. \]

Also,

\[ |A_1| + |A_2| \leq c \int (\partial_x \partial_y^2 u)^2 \chi'_{e/5,b+\epsilon} \leq c[0,2]'. \]

From the above estimates and taking into account that (0,2) and (1,1) are former cases we have that

\[ \frac{d}{dt}[1,2] \leq c[2,1] + c[1,2] + c[0,3] + g(t). \quad \text{(3.22)} \]

Case (0,3):

\[ A_5 = \int \partial_y^3 (u \partial_x u) \partial_y^3 u \chi \]
\[ = \int (\partial_y^3 u \partial_x u + 3 \partial_y^2 u \partial_y \partial_x u + 3 \partial_y u \partial_y^2 \partial_x u + u \partial_x \partial_y^3 u) \partial_y^3 u \chi \]
\[ \equiv A_{51} + A_{52} + A_{53} + A_{54}. \]

From Sobolev embeddings and Young’s inequality

\[ |A_{51} + A_{53}| \leq c[0,3] + c([1,2] + [0,3]). \]
Applying integration by parts we obtain
\[ |A_{52}| = | \frac{3}{2} \int \partial_\alpha \partial_\beta u(\partial_\gamma u)^2 \chi | = \frac{1}{2} \int (\partial_\gamma u)^3 \chi' \]
\[ = | \int \partial_\alpha u \partial_\beta u \partial_\gamma u \chi' | \leq c \int (\partial_\gamma u)^2 \chi' + c(\partial_\gamma u)^2 \chi' \]
\[ \leq c[-1, 3]' + c[1, 1]' . \]

For \( A_{54} \) we see that
\[ |A_{54}| = | - \frac{1}{2} \int \partial_\alpha u(\partial_\beta u)^2 \chi - \frac{1}{2} \int u(\partial_\gamma u)^2 \chi' | \]
\[ \leq c[0, 3] + c[-1, 3]' . \]

Also
\[ |A_1| + |A_2| \leq c \int (\partial_\gamma u)^2 \chi_{\tilde{e} / 5, b + \epsilon} \leq c[-1, 3]'_{\tilde{e} / 5, b + \epsilon} . \]

In this way we see that
\[ \frac{d}{dt}[0, 3] \leq c[0, 3] + c[1, 2]) + g(t) , \quad (3.23) \]

since the cases \((-1, 3)\) and \((1, 1)\) are former cases.

Hence, from \((3.21), (3.22), \text{and} (3.23)\) it follows that
\[ \frac{d}{dt}([2, 1] + [1, 2] + [0, 3]) \leq c([2, 1] + [1, 2] + [0, 3]) + g(t) , \]

which gives \((3.3), (3.6), \text{and} (3.7)\) for the three cases \((2, 1), (1, 2), \text{and} (0, 3)\) together.

For the cases with \(|\alpha| = 4\) we will see that the case \((4, 0)\) can be obtained independently of the other cases of the same order.

**Case (4,0):** For this case
\[ |A_5| = \int \partial_\alpha^4 u(\partial_\alpha u) \partial_\alpha^4 u \chi = \int (5 \partial_\alpha^4 u \partial_\alpha u + 10 \partial_\alpha^2 u \partial_\alpha^3 u + u \partial_\alpha^5 u) \partial_\alpha^4 u \chi | \]
\[ = | - \frac{5}{2} \int \partial_\alpha u(\partial_\alpha^4 u) \beta \chi - \frac{1}{2} \int u(\partial_\alpha^4 u)^2 \chi' + 10 \int \partial_\alpha^2 u \partial_\alpha^3 u \partial_\alpha^4 u \chi | \]
\[ \leq c[4, 0] + c[3, 0]' + | \int (\partial_\alpha^2 u)^2 (\partial_\alpha^3 u)^2 \chi | + c[4, 0] . \]

To estimate the last integral term we will use the notation \( \tilde{\chi} := \chi_{e / 5, b} \) and take into account that \( \chi_{e', b'} \leq c \chi_{e', b'} \) and \( (\tilde{\chi})^2_{e', b'} \leq c \tilde{\chi}_{e', b'} \) for \( 0 < e' < b' / 5 \). We will also apply the following Gagliardo-Nirenberg’s inequality:
\[ \int f^6 dx dy \leq c \int f^2 dx dy (\int ((\partial_\gamma f)^2 + (\partial_\delta f)^2) dx dy)^2 . \]
In this way,
\[
\left| \int (\partial_x^2 u)^2 (\partial_x^3 u)^2 \chi \right| = -\frac{1}{3} \int (\partial_x^2 u)^3 \partial_x^4 u \chi - \frac{1}{3} \int (\partial_x^2 u)^3 \partial_x^3 u \chi'
\]
\[
= \frac{1}{3} \left| \int (\partial_x^2 u)^3 \tilde{\chi} \partial_x^4 u \chi + \int (\partial_x^2 u)^3 \tilde{\chi} \partial_x^3 u \chi' \right|
\]
\[
\leq c \int (\partial_x^2 u \tilde{\chi})^6 + c[4,0] + c \int (\partial_x^2 u \tilde{\chi})^6 + c[2,0]'
\]
\[
\leq c \left( \int (\partial_x^2 u)^2 \tilde{\chi}^2 \right) \left( \int (\partial_x^2 u)^2 \tilde{\chi}^2 + (\partial_x^2 u)^2 (\tilde{\chi}')^2 + (\partial_x^2 u)^2 \tilde{\chi}^2 \right)^2
\]
\[
+ c[4,0] + c[2,0]'
\]
\[
\leq c + c[4,0] + c[2,0]'
\]
which together with (3.8) and (3.10) gives (3.3) for this case.

We will now consider the cases (3,1), (2,2), (1,3) and estimate them together.

**Case (3,1):**

\[
A_5 = \int \partial_x^3 \partial_y (u \partial_x u) \partial_x^4 \partial_y u \chi
\]
\[
= \int (a_1 \partial_x^3 \partial_y u \partial_x u + a_2 \partial_x^2 \partial_y \partial_x u \partial_x^2 u + a_3 \partial_x^2 \partial_y \partial_x u \partial_x u) \partial_x^3 \partial_y u \chi
\]
\[
+ \int (a_4 \partial_x \partial_y u \partial_x^4 u + u \partial_x \partial_y \partial_x \partial_x u) \partial_x^3 \partial_y u \chi
\]
\[
= A_{51} + A_{52} + A_{53} + A_{54} + A_{55}.
\]

Treating the last term in the former integral by integration by parts and proceeding as we did to obtain (3.17) and (3.18) we have that
\[
\left| A_{51} + A_{54} + A_{55} \right| \leq c[3,1] + c[4,0] + c[2,1]' \leq c + c[3,1] + c[2,1]'.
\]

For the remaining terms \( A_{52} \) and \( A_{53} \) we can use inequality (2.8) to obtain that
\[
\left| A_{52} + A_{53} \right| \leq c \left( \int (\partial_x^2 u \partial_x u)^4 \chi^2 \right)^{1/2} \left( (\partial_x^2 u)^4 \tilde{\chi}^2 \right)^{1/2}
\]
\[
+ c \left( (\partial_x^3 u)^2 \chi^2 \right)^{1/2} \left( (\partial_x \partial_x u)^4 \tilde{\chi}^2 \right)^{1/2} + c[3,1]
\]
\[
\leq c([2,1] + [3,1] + [2,2] + [1,1]'([2,0] + [3,0] + [2,1] + [1,0]')_{\varepsilon/5} + c([3,0] + [4,0] + [3,1] + [2,0]')([1,1] + [2,1] + [1,2] + [0,1]')_{\varepsilon/5}
\]
\[
+ c[3,1],
\]
where the subindex \((e/5,e)\) in the closing parenthesis means that all terms 
\([\cdot,\cdot]\) inside the parentheses are to be taken as 
\([\cdot,\cdot]_{e/5,e}\). Since

\[
([2,0] + [1,0]' + [1,1] + [0,1]')_{e/5,e} \leq c\|u\|_{C([0,T];H^2(\mathbb{R}^2))}
\]

and noticing the former cases in (3.24), we conclude that

\[
|A_{52} + A_{53}| \leq c + c([3,1] + [2,2] + [1,1]' + [4,0] + [2,0]')
\]

\[
\leq c + c([3,1] + [2,2] + [1,1]' + [2,0]'),
\]

and therefore, from the above estimates for this case,

\[
|A_5| \leq c[3,1] + c[2,2] + g(t).
\]  
(3.25)

Case (2,2):

We proceed as in case (3,1) to obtain analogous terms \(A_{51}\) to \(A_{56}\). We observe as before that

\[
|A_{51} + A_{55} + A_{56}| \leq c[2,2] + c[3,1] + c[1,2]',
\]

while for \(A_{52}, A_{53}\) and \(A_{54}\) we see that

\[
A_{52} + A_{53} + A_{54} = \int (a_2 \partial_x^2 \partial_y u \partial_x \partial_y u + a_3 \partial_x \partial_y^2 u \partial_x^2 u + a_4 \partial_y^2 u \partial_x^2 u) \partial_x^2 \partial_y u \chi,
\]

which can be treated by using inequality (2.8), as we did in the case (3,1), to conclude that

\[
|A_{52} + A_{53} + A_{54}| \leq c[2,2]
\]

\[
+ c([2,1] + [3,1] + [2,2] + [1,1]')([1,1] + [2,1] + [1,2] + [0,1]')_{e/5,e}
\]

\[
+ c([1,2] + [2,2] + [1,3] + [0,2]')([2,0] + [3,0] + [2,1] + [1,0]')_{e/5,e}
\]

\[
+ ([3,0] + [4,0] + [3,1] + [2,0]')([0,2] + [1,2] + [0,3] + [1,1]')_{e/5,e}
\]

\[
\leq c + c([3,1] + [2,2] + [1,1]' + [1,3] + [0,2]' + [4,0] + [2,0]').
\]

Thus, for the case (2,2),

\[
|A_5| \leq c[3,1] + c[2,2] + c[1,3] + g(t).
\]  
(3.26)

Case (1,3):

We see that

\[
A_5 = \int \partial_x \partial_y^3 (u \partial_x u) \partial_x \partial_y^3 u
\]

\[
= \int (a_1 \partial_x \partial_y^3 u \partial_x u + a_2 \partial_x \partial_y^2 u \partial_x^2 u + a_3 \partial_x \partial_y^2 u \partial_x \partial_y u + a_4 \partial_y^2 u \partial_x^2 u
\]

\[
+ a_5 \partial_y u \partial_x^2 \partial_y^3 u + a \partial_x^2 \partial_y^3 u) \chi \partial_x \partial_y^3 u
\]

\[
= A_{51} + \cdots + A_{56}.
\]
As before,
\[ |A_{51} + A_{55} + A_{56}| \leq c[1,3] + c[2,2] + c[0,3]' \] (3.27)
The terms \( A_{53} \) and \( A_{54} \) can be bounded using inequality (2.8) as it was done to obtain (3.24) above. In this case we have
\[ |A_{53} + A_{54}| \leq c[1,3] + c((1,2) + [2,2] + [1,3] + [0,2]'([1,0] + [2,0] + [1,2] + [0,1]'e/5,e + c([0,2] + [1,2] + [0,3] + [1,1]''\epsilon/5,e([2,1] + [3,1] + [2,2] + [1,1]'').
\]
Since
\[ ([0,1] + [1,1]'')\epsilon/5,e \leq c\|u\|^2_{C([0,T];H^2(\mathbb{R}^2))}, \]
it follows that
\[ |A_{53} + A_{54}| \leq c + c([2,2] + [1,3] + [0,2]' + [3,1] + [1,1]'). \] (3.28)

We now consider the term \( A_{52} \). For this term we will use the embedding \( W^{2,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \), (where \( W^{2,1} \) is the classical Sobolev space of \( L^1 \) functions having derivatives up to second order in \( L^1 \)). More precisely we will use the inequality
\[ \|f\|_{L^\infty(\mathbb{R}^2)} \leq c\|\partial_x \partial_t f\|_{L^1(\mathbb{R}^2)} \] (3.29).

Now,
\[ |A_{52}| = c\int \partial_x^3 u \partial_t^3 u \partial_x \partial_t^3 u \chi \leq c\int (\partial_x^3 u)^2 (\partial_t^3 u)^2 \chi \partial_t \chi + c[1,3] \] (3.30)
We estimate the \( L^\infty \) norm in (3.30) by using inequality (3.29) to conclude that
\[ \|(\partial_x^3 u)^2 \chi\|_{L^\infty,\chi} \leq c\int |\partial_x \partial_t [(\partial_x^3 u)^2 \chi]| \]
\[ = c\int |2\partial_x^2 \partial_t u \partial_x^3 u \chi + 2\partial_x^2 \partial_t u \partial_x^3 \partial_t u \chi + 2\partial_x^2 \partial_t u \partial_x^3 \partial_t u \chi'| \] (3.31)
\[ \leq c[2,1] + [3,0] + c[2,0] + c[3,1] + c[1,0]' + c[1,1]' \]
\[ \leq c[3,1] + g(t). \]
Therefore, from (3.27), (3.28), (3.30), and (3.31) it follows that
\[ |A_5| \leq c[3,1] + [2,2] + [1,3] + g(t). \] (3.32)

From the estimates obtained in (3.25), (3.26), and (3.32) for the cases (3,1), (2,2), and (3,3), we conclude that
\[ \frac{d}{dt}([3,1] + [2,2] + [1,3]) \leq c([3,1] + [2,2] + [1,3]) + g(t), \]
and thus we have (3.3) for the cases (3,1), (2,2), and (1,3).
Notice that we also have the option of treating the case (4,0) by using inequalities (2.8) or (3.29) to obtain an estimate for the four terms \([4,0], [3,1], [2,2],\) and \([1,3]\) together.

**Case (0,4):**

In this case

\[
A_5 = \int \partial_\gamma (u \partial_\delta u) \partial_\beta u \chi = \int \left( \partial_\gamma u \partial_\delta u + a_2 \partial_\gamma^3 u \partial_\delta \partial_\gamma u + a_3 \partial_\gamma^2 u \partial_\delta \partial_\gamma^2 u \\
+ a_4 \partial_\delta u \partial_\gamma \partial_\delta^3 u + u \partial_\delta \partial_\gamma^4 u \partial_\delta u \chi \right)
\]

\[
= A_{51} + \cdots + A_{55}.
\]

As usual, after applying integration by parts in \(A_{55}\) we find that

\[
|A_{51} + A_{54} + A_{55}| \leq c[0,4] + c[1,3] + c|\int u(\partial_\gamma^4 u)^2 \chi'|.
\]

Notice that the last term in the former expression is bounded by \(c[1,3]'\) which is also a case of order 4. However, we took precautions to avoid the appearance of the term \([0,4]\) in our bounds for the other cases of order 4. In particular, we achieve that in the case \((1,3)\) with the application of the embedding \(W^{2,1} \hookrightarrow L^\infty\) in (3.30) and (3.31).

In this way, since the case \((1,3)\) is already a former case, we conclude that

\[
|A_{51} + A_{54} + A_{55}| \leq c[0,4] + g(t).
\]

For the terms \(A_{52}\) and \(A_{53}\) we observe that they have the form \(\int \partial_\gamma u \partial_\beta u \partial_\gamma^4 u \chi\) with \(|\gamma|=3\) and \(\beta=2\) allowing us to apply inequality (2.8) as we did in the cases above. Thus, taking into account all former cases, we find that

\[
|A_{52} + A_{53}| \leq c[0,4] + g(t),
\]

and we can conclude that (3.3) is valid for this case.

**Cases with \(n \geq 5\):**

The cases with \(|\alpha|=n \geq 5\) are easier since, as we saw for \(n = 4\), we have enough regularity to estimate the terms with second order derivatives by means of the Sobolev embedding (3.29).

We group the cases \((n,0), (n-1,1), \cdots, (1,n-1)\), estimate the corresponding integrals \([n,0], [n-1,1], \cdots, [1,n-1]\) together in a single application of Gronwall’s lemma, and then consider the estimation of \([0,n]\) separately.

If \(\alpha = (\alpha_1, \alpha_2)\), with \(\alpha_1 + \alpha_2 = n\) and \(\alpha_1 \geq 1\), we observe that the expansion of \(A_5 = \int \partial_\alpha (u \partial_\delta u) \partial_\gamma u \chi\) consists of a sum of terms of the form

\[
A_\beta = \int \partial_\beta u \partial_\delta \partial_\gamma u \chi, \quad \text{with } |\beta| + |\gamma| = n.
\]
Proceeding as in the cases with $|\alpha| = 3, 4$ we first consider the terms with $|\beta| = 1$ or $|\gamma| = 0$, which can be treated by the Sobolev inequality $\|\partial_x u\|_{L^\infty} + |\partial_y u|_{L^\infty} \leq c$ and Young’s inequality. For the term with $|\beta| = 0$ ($|\gamma| = n$), $A_{(0,0)}$, we apply as before integration by parts to obtain the bound
\[ |A_{(0,0)}| \leq c\|\partial_x u\|_{L^\infty}[\alpha_1, \alpha_2] + c\|u\|_{L^\infty}[\alpha_1 - 1, \alpha_2]' . \tag{3.34} \]

The terms with $|\beta| = 2$ ($|\gamma| = n - 2$) or with $|\gamma| = 1$ ($|\beta| = n - 1$) can now be bounded, as we did in the case (1,3), by using the Sobolev embedding (3.29). Notice that in the estimates of these cases the term $[0,n]$ will not appear.

The intermediate terms with other combinations of $\beta$ and $\gamma$ will have $|\beta| \leq n - 2$ and $|\gamma| \leq n - 2$ and can be estimated by means of inequality (2.8) to give bounds which always come from former cases. In this way, adding all cases under consideration we have that
\[ \frac{d}{dt}[0,n] \leq [0,n] + g(t), \]
which gives (3.3) for these cases.

Now, we proceed to consider the case $(0,n)$ separately. Here, the estimation of the terms $A_\beta$ is carried out as in the former cases of order $n$. However, for $A_{(0,0)}$, instead of (3.34) we obtain,
\[ |A_{(0,0)}| \leq c\|\partial_x u\|_{L^\infty}[\alpha_1, \alpha_2] + c\|u\|_{L^\infty}[\alpha_1 - 1, \alpha_2]' \int (\partial_y u)^2 \chi' \leq c[\alpha_1, \alpha_2] + [1,n - 1]''. \]

Notice that the case $(1,n - 1)$ is of order $n$, but is already a former case. Therefore, taking into account that all cases $(n,0), \cdots (1,n - 1)$ are former cases we obtain the inequality
\[ \frac{d}{dt}[0,n] \leq [0,n] + g(t), \]
thus giving (3.3) for this case.

To justify the above formal computations we shall follow the following standard argument.

Consider data $u_0^\tau = \rho_\tau \ast u_0$ with $\rho \in C_0^\infty(\mathbb{R}^2)$, $\text{supp } \rho \subset B_1(0) = \{ z \in \mathbb{R}^2 : |z| < 1 \}$, $\rho \geq 0$, $\int_{\mathbb{R}^2} \rho(z) \, dz = 1$ and
\[ \rho_\tau(z) = \frac{1}{\tau^2} \rho\left( \frac{z}{\tau} \right), \quad \tau > 0. \]

For $\tau > 0$ consider the solutions $u^\tau$ of the IVP (1.1) with data $u_0^\tau$ where $(u^\tau)_{\tau > 0} \subset C([0,T] : H^\infty(\mathbb{R})).$
Using the continuous dependence of the solution upon the data we have that
\[
\sup_{t \in [0,T]} \| u^\tau(t) - u(t) \|_{s,2} \downarrow 0 \quad \text{as} \quad \tau \downarrow 0 \quad \text{for} \quad s > 2. \tag{3.35}
\]
Applying our argument to the smooth solutions \( u^\tau(\cdot, t) \) one gets that
\[
\sup_{[0,T]} \int_{\mathbb{R}^2} (\partial^\alpha u^\tau)^2 \chi_{\varepsilon,b}(x + vt) \, dx \, dy \leq c_0 \tag{3.36}
\]
for any \( \varepsilon > 0, \ b \geq 5\varepsilon, \ v > 0, \ c_0 = c_0(\varepsilon; b; v) > 0 \) but independent of \( \tau > 0 \).

Combining (3.35) and (3.36) and a weak compactness argument one gets that
\[
\sup_{[0,T]} \int_{\mathbb{R}^2} (\partial^\alpha u)^2 \chi_{\varepsilon,b}(x + vt) \, dx \, dy \leq c_0 \tag{3.37}
\]
which is the desired result.

This completes the proof of Theorem 1.1.

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