Research Article

Existence of Solutions for a Periodic Boundary Value Problem with Impulse and Fractional Derivative Dependence

Yaohong Li,1 Yongqing Wang,2 Donal O’Regan,3 and Jiafa Xu4

1School of Mathematics and Statistics, Suzhou University, Suzhou 234000, Anhui, China
2School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China
3School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, Galway, Ireland
4School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China

Correspondence should be addressed to Yongqing Wang; wyqing9801@163.com

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1. Introduction

This paper considers the existence of solutions of the following fractional-order impulsive periodic boundary value problem:

\[
\begin{aligned}
\dot{D}_q^a u(t) & = f(t, u(t), \dot{D}_q^a u(t)), \quad t \in J' \\
\Delta u(t_k) & = I_k(u(t_k)), \quad \Delta^\gamma D_q^a u(t_k), \quad k = 1, 2, \ldots, m, \\
\Delta^\gamma D_q^a u(t_k) & = J_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\
am(0) - b(1) & = 0, \quad a \Delta^\gamma D_q^a u(0) - b \Delta^\gamma D_q^a u(1) = 0,
\end{aligned}
\]

(1)

where \( \dot{D}_q^a \) and \( D_q^a \) represent the Caputo derivatives of orders \( q \) and \( \gamma \), and \( 1 < q < 2, 0 < \gamma < 1 \), and \( J = [0, 1], 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1, J' = J \setminus \{t_1, t_2, \ldots, t_m\} \). Here, \( f: J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) and \( I_k, J_k: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions. Now, \( \Delta u(t_k) = u(t_k^+ - u(t_k^-) \) and \( \Delta^\gamma D_q^a u(t_k) \) denote the right limit and the left limit of \( u(t) \) at the impulsive point \( t_k \). Also, \( \Delta^\gamma D_q^a u(t_k) = \dot{D}_q^a u(t_k^+) - \dot{D}_q^a u(t_k^-) \), where \( \dot{D}_q^a u(t_k^+) \) and \( \dot{D}_q^a u(t_k^-) \) denote the right limit and the left limit of \( \dot{D}_q^a u(t) \) at the impulsive point \( t_k \). If \( u(t_k^+) \) and \( \dot{D}_q^a u(t_k^-) \) exist, we let \( u(t_k^-) = u(t_k^+) \) and \( \dot{D}_q^a u(t_k^-) = \dot{D}_q^a u(t_k^+) \), where \( k = 1, 2, \ldots, m \). Also, \( a \) and \( b \) are two real constants with \( b > a > 0 \).

The theory of fractional differential equation has received a lot of attention because of its wide application in mathematical models (see [1–27] and the references therein). Fractional-order impulsive differential equations are a natural generalization of the case of nonimpulses and are used to describe sudden changes in their states, such as in optimal control, population dynamics, biological systems, financial systems, and mechanical systems with impact. We refer the reader to [28–36] and the references therein. In particular, Bai et al. [37] investigated a mixed boundary value problem of nonlinear impulsive fractional differential equation:

\[
\begin{aligned}
\dot{D}_q^a u(t) & = f(t, u(t)), \quad t \in J' \\
\Delta u(t_k) & = I_k(u(t_k)), \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\
u(0) + u(1) & = 0, \\
u'(0) + u'(1) & = 0,
\end{aligned}
\]

(2)

and some sufficient conditions on the existence and uniqueness of solutions for problem (2) are obtained under Lipschitz conditions. In [38], Zhang and Xu studied the following impulse periodic boundary value problem with the Caputo fractional derivative:
\[
\begin{aligned}
\Delta \nu(t_k) &= I_{k} \nu(t_k), \\
u'(t_k) &= I_{k} \nu(t_k), \\
u(0) - b \nu(1) &= 0, \\
u'(0) - b \nu'(1) &= 0,
\end{aligned}
\]  

(3)

using Green’s function in [36], and via the symmetry property of Green’s function and topological degree theory, the authors obtained the existence of positive solutions for (3) when the growth of \( f \) is superlinear and sublinear.

Inspired by the above research studies, in this paper, we consider fractional-order impulsive differential equations with generalized periodic boundary value conditions (1), where the nonlinear term, impulse terms, and periodic boundary conditions all depend on unknown functions and the lower-order fractional derivative of unknown functions. This is obviously more general and more widely applied, but it is also more complex and difficult to solve. Compared with (1), the nonlinear term, pulse term, and periodic boundary conditions of (3) are all independent of fractional derivatives, so it is a special form of (1). In this paper, we first give an equivalent integral form of solutions for problem (1) using some new Green’s functions. Next, we present some sufficient conditions for the existence of solutions for problem (1), where the nonlinear and impulse terms satisfy some nonlinear and linear growth conditions, which are different from the conditions in [36–38]. Finally, we present three examples to illustrate our main results.

2. Preliminaries and Lemmas

In this section, we only present some necessary definitions and lemmas about fractional calculus.

Definition 1 (see [39, 40]). The Riemann–Liouville fractional integral of order \( \alpha > 0 \) for a function \( f: (0, \infty) \rightarrow \mathbb{R} \) is defined as

\[
I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s) ds,
\]

where \( \Gamma(\cdot) \) is the Euler gamma function.

Definition 2 (see [39, 40]). The Caputo fractional derivative of order \( \alpha > 0 \) for a continuous and \( n \)-order differentiable function \( f: (0, \infty) \rightarrow \mathbb{R} \) is defined as

\[
^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{f^{(n)}(s)}{(t-s)^{\alpha}} ds,
\]

where \( \Gamma(\cdot) \) is the Euler gamma function and \( n = \lfloor \alpha \rfloor + 1 \), where \( \lfloor \alpha \rfloor \) is the smallest integer greater than or equal to \( \alpha \).

Lemma 1 (see [39, 40]). Let \( \alpha > 0 \). The differential equation

\[
^{C}D_{t}^{\alpha} u(t) = 0
\]

has a unique solution:

\[
u(t) = c_{0} + c_{1} t + \ldots + c_{n-1} t^{n-1},
\]

for some \( c_{i} \in \mathbb{R} \) (i = 0, 1, 2, \ldots, n - 1), where \( n = \lfloor \alpha \rfloor + 1 \).

Lemma 2 Let \( y \in C(J) \) and \( 1 < q < 2 \). The unique solution of the following periodic boundary value problem

\[
\begin{aligned}
^{C}D_{t}^{\alpha} u(t) &= y(t), \\
\Delta u(t_{k}) &= I_{k} \Delta u(t_{k}) = I_{k}, \\
u(0) - b \nu(1) &= 0, \\
u'(0) - b \nu'(1) &= 0,
\end{aligned}
\]

(7)

is expressed by

\[
u(t) = \sum_{i=1}^{m} K_{i} \int_{t_{i-1}}^{t_{i}} y(s) ds + \sum_{i=1}^{m} K_{i} \int_{t_{i-1}}^{t_{i}} y(s) ds + \sum_{i=1}^{m} K_{i} \int_{t_{i-1}}^{t_{i}} y(s) ds + \sum_{i=1}^{m} K_{i} \int_{t_{i-1}}^{t_{i}} y(s) ds,
\]

(8)

where

\[
K_{1}(t,s) = \begin{cases} 
\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{b(1-s)^{\alpha-1}}{(b-a)\Gamma(\alpha)} - \frac{\Gamma(2-\gamma)(1-s)^{\gamma-1}}{\Gamma(q-\gamma)} \left( t - \frac{b}{b-a} \right), & 0 \leq s \leq t \leq 1, \\
\frac{b(1-s)^{\alpha-1}}{(b-a)\Gamma(\alpha)} - \frac{\Gamma(2-\gamma)(1-s)^{\gamma-1}}{\Gamma(q-\gamma)} \left( t - \frac{b}{b-a} \right), & 0 \leq t \leq s \leq 1,
\end{cases}
\]

(9)

\[
K_{2}(t, t_{i}) = \begin{cases} 
a\Gamma(2-\gamma) & 0 < t_{i} < t \leq 1, i = 1, 2, \ldots, m, \\
(b-a)t_{i}^{\alpha-1} & 0 \leq t_{i} < t \leq 1, i = 1, 2, \ldots, m,
\end{cases}
\]

\[
K_{3}(t, t_{i}) = \begin{cases} 
\frac{\Gamma(2-\gamma)}{t_{i}^{\alpha-1}} \left( \frac{bt_{i}}{b-a} - t \right), & 0 \leq t_{i} < t \leq 1, i = 1, 2, \ldots, m,
\end{cases}
\]

\[
K_{4}(t, t_{i}) = \begin{cases} 
\frac{a}{a-b} & 0 < t_{i} < t \leq 1, i = 1, 2, \ldots, m, \\
\frac{b}{a-b} & 0 \leq t_{i} < t \leq 1, i = 1, 2, \ldots, m.
\end{cases}
\]
Furthermore,
\[ cD^\gamma_t u(t) = \int_0^t H_1(t, s) y(s)ds + \sum_{i=1}^{m} H_2(t, t_i) I_i, \quad (10) \]

where
\[ H_1(t, s) = \begin{cases} \frac{(t-s)^{\gamma-1}}{\Gamma(q-\gamma)} - \frac{(1-s)^{\gamma-1} t^{1-\gamma}}{\Gamma(q-\gamma)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\gamma-1} t^{1-\gamma}}{\Gamma(q-\gamma)}, & 0 \leq t \leq s \leq 1, \end{cases} \]
\[ H_2(t, t_i) = \begin{cases} 0, & 0 < t_i < t \leq 1, i = 1, 2, \ldots, m, \\ \left( \frac{t}{t_i} \right)^{1-\gamma}, & 0 \leq t \leq t_i < 1, i = 1, 2, \ldots, m. \end{cases} \quad (11) \]

**Proof.** Suppose \( u \) is a general solution of (7) on each interval \((t_k, t_{k+1}) (k = 0, 1, 2, \ldots, m)\). Then, using Lemma 1, (7) can be transformed into the following equivalent integral equation:
\[ u(t) = I_{0+}^\gamma y(t) - c_k - d_k t, \quad t \in (t_k, t_{k+1}], \quad (12) \]
where \( t_0 = 0 \) and \( t_{m+1} = 1 \). Also, we have
\[ cD^\gamma_t u(t) = I_{0+}^\gamma y(t) - \frac{d_k t^{1-\gamma}}{\Gamma(2-\gamma)} t \in (t_k, t_{k+1}], \quad (13) \]

From (12) and (13), according to (7), we obtain
\[ ac_0 - bc_m = bd_m - bI_{0+}^\gamma y(1), \quad (14) \]
\[ d_m = \Gamma(2-\gamma) I_{0+}^\gamma y(1). \quad (15) \]

Applying the right fractional-order impulsive condition of (7), we obtain
\[ d_{k-1} - d_k = \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} I_k, \quad (16) \]
\[ c_{k-1} - c_k = \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} I_k. \quad (17) \]

From (15) and (16), after a recursive calculation, we have
\[ d_0 = d_m + \Gamma(2-\gamma) \sum_{i=1}^{m} \frac{I_i}{t_i^{1-\gamma}} I_i = \Gamma(2-\gamma) I_{0+}^\gamma y(1) + \Gamma(2-\gamma) \sum_{i=1}^{m} \frac{I_i}{t_i^{1-\gamma}} I_i, \quad (18) \]

Similar to (18), we see that
\[ d_k = d_0 - \Gamma(2-\gamma) \sum_{i=1}^{k} \frac{I_i}{t_i^{1-\gamma}} I_i = \Gamma(2-\gamma) I_{0+}^\gamma y(1) + \Gamma(2-\gamma) \sum_{i=1}^{m} \frac{I_i}{t_i^{1-\gamma}} I_i. \quad (19) \]

From (13), (14), and (16), we have
\[ c_0 = \frac{b}{b-a} \left[ I_{0+}^\gamma y(1) - \Gamma(2-\gamma) I_{0+}^\gamma y(1) + \sum_{i=1}^{m} (I_i - \Gamma(2-\gamma) t_i^{1-\gamma} I_i) \right]. \quad (20) \]

From (17) and (20), after a recursive calculation, we have
\[ c_k = c_0 - \sum_{i=1}^{k} [I_i - \Gamma(2-\gamma) t_i^{1-\gamma} I_i] = \frac{b}{b-a} \left[ I_{0+}^\gamma y(1) - \Gamma(2-\gamma) I_{0+}^\gamma y(1) \right] \]
\[ + \frac{a}{b-a} \sum_{i=1}^{k} [I_i - \Gamma(2-\gamma) t_i^{1-\gamma} I_i] + \frac{b}{b-a} \sum_{i=1}^{m} [I_i - \Gamma(2-\gamma) t_i^{1-\gamma} I_i]. \quad (21) \]

For \( t \in I_0 = [t_0, t_1] \), substituting (18) and (20) into (12) and (13), we obtain
\[ u(t) = I_{0+}^\gamma y(t) - \frac{b}{b-a} I_{0+}^\gamma y(1) - \Gamma(2-\gamma) \left( t - \frac{b}{b-a} \right) I_{0+}^\gamma y(1) \]
\[ - \sum_{i=1}^{m} \Gamma(2-\gamma) \left( t - \frac{b t_i}{b-a} \right) I_i - \frac{b}{b-a} \sum_{i=1}^{m} I_i \]
\[ - \Gamma(2-\gamma) \left( t - \frac{b}{b-a} \right) \left( \frac{1}{t} \int_0^t \frac{(1-s)^{\gamma-1}}{\Gamma(q-\gamma)} y(s)ds + \sum_{i=1}^{m} \Gamma(2-\gamma) \left( \frac{b t_i}{b-a} - t \right) I_i - \frac{b}{b-a} \sum_{i=1}^{m} I_i \right) \]
where $K_1(t, s), K_2(t, t_i), K_3(t, t_i), H_1(t, s)$, and $H_2(t, t_i)$ are defined by (7) and (9).

For $J_k = [t_k, t_{k+1}], k = 1, 2, \ldots, m$, substituting (20) and (18) into (11) and (12), we have

\[
\begin{align*}
\frac{d^\gamma}{dt^\gamma} u(t) &= I_{0+}^\gamma y(t) - \left[ I_{0+}^\gamma y(1) + \sum_{i=1}^{m} \frac{J_i}{t_i^{\gamma-1}} \right] t^{\gamma-1} \\
&= \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(q)} y(s)ds - \frac{b}{b-a} \left[ \int_0^1 (1-s)^{\gamma-1} t^{\gamma-1} y(s)ds - \sum_{i=1}^{m} \frac{t}{t_i^{\gamma-1}} J_i \right] \\
&= \int_0^t H_1(t, s)y(s)ds + \sum_{i=1}^{m} H_2(t, t_i)I_i,
\end{align*}
\]

(22)

where $K_1(t, s), K_2(t, t_i), K_3(t, t_i), H_1(t, s)$, and $H_2(t, t_i)$ are defined as in (9) and (11) are continuous, and the following inequalities hold:

(i) $|K_1(t, s)| \leq (2b-a)(1-s)^{\gamma-1}/(b-a)I'_{(q)}(q) + ((2b-a)I'(2-\gamma)(1-s)^{\gamma-1}/(b-a)I'(q-y)), |H_1(t, s)| \leq (2(1-s)^{\gamma-1}/\Gamma(q-y)), t, s \in J_k$.

Lemma 3. Let $0 < a < b < +\infty$. Then, $K_1(t, s) + K_2(t, t_i)$ and $K_3(t, t_i)$ and $H_1(t, s)$ and $H_2(t, t_i)$ and $K_1(t, s) + K_2(t, t_i)$ and $K_3(t, t_i)$ and $H_1(t, s)$ and $H_2(t, t_i)$
\( |K_2(t,t)| \leq b \Gamma (2 - \gamma) / b - a, |H_2(t,t)| \leq 1, |K_3(t,t)| \leq b / b - a, \ t, t_i \in J \)

**Proof.** Directly observe that

\[
|K_1(t,s)| \leq \frac{(1 - s)^{\gamma - 1}}{\Gamma (q)} + \frac{b(1 - s)^{\gamma - 1}}{(b - a)\Gamma (q)} + \frac{\Gamma (2 - \gamma)(1 - s)^{\gamma - 1}}{\Gamma (q - \gamma)} \left( 1 + \frac{b}{b - a} \right),
\]

\[
|H_1(t,s)| \leq \frac{(1 - s)^{\gamma - 1}}{\Gamma (q - \gamma)} + \frac{(1 - s)^{\gamma - 1}t^{1 - \gamma}}{\Gamma (q - \gamma)} \leq \frac{(1 + t^{1 - \gamma})(1 - s)^{\gamma - 1}}{\Gamma (q - \gamma)},
\]

\[
|K_3(t,t_i)| \leq \frac{b}{b - a}, \quad t, t_i \in J,
\]

\[
|H_2(t,t_i)| \leq \left( \frac{t_i}{t} \right)^{1 - \gamma} \leq 1, \quad t, t_i \in J.
\]

**Lemma 4.** If the function \( f(t,u, D^\gamma u(t)) \) is continuous, then \( u \in E \) is a solution of (1) if and only if \( u \in E \) is a solution of the following integral equation:

\[
|u| = \sup_{t \in J} |u(t)| + \sup_{t \in J} |D^\gamma u(t)|.
\]

\[
u(t) = \int_0^1 K_1(t,s)f(s,u(s), D^\gamma u(s))ds + \sum_{i=1}^m K_2(t,t_i)J_i(u(t), D^\gamma u(t_i))
\]

\[
+ \sum_{i=1}^m K_3(t,t_i)I_i(u(t), D^\gamma u(t_i)).
\]

**Proof.** Assume that \( u \) satisfies (1). From Lemma 2, we see that \( u \) satisfies integral equation (26).

Conversely, assume that \( u \) satisfies integral equation (26). Applying Definition 2, by a direct fractional derivative computation, it follows that the solution given by (26) and (2) satisfies (1).

Define an operator \( T : E \rightarrow E \) as

\[
(Tu)(t) = \int_0^1 K_1(t,s)f(s,u(s), D^\gamma u(s))ds + \sum_{i=1}^m K_2(t,t_i)J_i(u(t), D^\gamma u(t_i))
\]

\[
+ \sum_{i=1}^m K_3(t,t_i)I_i(u(t), D^\gamma u(t_i)),
\]

\[
(\mathcal{D}^\gamma Tu)(t) = \int_0^1 H_1(t,s)f(s,u(s), D^\gamma u(s))ds + \sum_{i=1}^m H_2(t,t_i)J_i(u(t), D^\gamma u(t_i)).
\]
It is easy to prove that the function $u$ is a solution of (1) if and only if $u$ is a fixed point of the operator $T$.

For convenience, we list some hypotheses:

(B1) $0 < a < b < + \infty, 1 < q < 2, 0 < \gamma < 1$ with $q - \gamma > 1$

(B2) $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $I_k, J_k: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions \( \square \)

**Lemma 5.** Assume that (B1) and (B2) hold. Then, the operator $T: E \to E$ defined as in (27) is completely continuous.

**Proof.** We divide the proof into three steps. Set $\Omega_r = \{ u \in E, \| u \| \leq r \}$ for some $r > 0$. The steps are as follows:

(i) **Step 1.** $T$ is continuous from the continuity of the functions $K_1, K_2, K_3, H_1, H_2, f, I_k, J_k$.

(ii) **Step 2.** $T$ is uniformly bounded. Now, for $u \in \Omega_r$, we have $|f(t, u, \partial_t^q u)| \leq m_1, |I_k| \leq m_2, |J_k| \leq m_3$, where $m_i > 0, i = 1, 2, 3$.

In fact, for each $t \in I_k = [t_k, t_{k+1}], u \in \Omega_r, k = 0, 1, 2, \ldots, m$, from Lemma 3, we have

\[
|T u(t)| \leq \int_0^1 |K_1(t, s) f(s, u(s), \partial_t^q u(s))| \, ds + \sum_{i=1}^m |K_2(t, t_i) J_i(u(t_i), \partial_t^q u(t_i))| \\
+ \sum_{i=1}^m |K_3(t, t_i) I_i(u(t_i), \partial_t^q u(t_i))| \leq m_1 \int_0^1 |K_1(t, s)| \, ds + m_2 \sum_{i=1}^m |K_2(t, t_i)| + m_3 \sum_{i=1}^m |K_3(t, t_i)|,
\]

which and Lemma 4 imply that

\[
\|T u\| = \sup_{t \in I} |T u(t)| + \sup_{t \in I} |(\partial_t^q T u)(t)| \\
\leq m_1 \int_0^1 |K_1(t, s)| \, ds + m_2 \sum_{i=1}^m |K_2(t, t_i)| + m_3 \sum_{i=1}^m |K_3(t, t_i)| + m_1 \int_0^1 |H_1(t, s)| \, ds + m_2 \sum_{i=1}^m |H_2(t, t_i)| + m_3 \sum_{i=1}^m |J_k(t, t_i)|
\]

\[
\leq m_1 \int_0^1 \left( \frac{(2b-a)(1-s)^{\gamma-1}}{(b-a)\Gamma(q)} + \frac{(2b-a)\Gamma(2-\gamma)(1-s)^{\gamma-1}}{(b-a)\Gamma(q-\gamma)} + \frac{2(1-s)^{\gamma-1}}{\Gamma(q-\gamma)} \right) \, ds \\
+ \frac{b \Gamma(2-\gamma) + b - a) m_{m_2}}{b-a} + \frac{b m_{m_3}}{b-a} = m_1 \left[ \frac{(2b-a)}{(b-a)\Gamma(q+1)} + \frac{(2b-a)\Gamma(2-\gamma)}{(b-a)\Gamma(q-\gamma+1)} + \frac{2}{\Gamma(q-\gamma+1)} \right] \\
+ \frac{b \Gamma(2-\gamma) + b - a) m_{m_2}}{b-a} + \frac{b m_{m_3}}{b-a} = M.
\]
(iii) Step 3. $T$ is equicontinuous. For any $t_1, t_2 \in J_k, k = 0, 1, \ldots, m$, fixed $s \in J$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for $|t_1 - t_2| < \delta$, we have

\[
|K_1(t_1, s) - K_1(t_2, s)| < \frac{\varepsilon}{6m_1},
\]

\[
|K_2(t_1, t) - K_2(t_2, t)| < \frac{\varepsilon}{6m_2},
\]

\[
|K_3(t_1, t) - K_3(t_2, t)| < \frac{\varepsilon}{6m_3},
\]

(31)

\[
|H_1(t_1, s) - H_1(t_2, s)| < \frac{\varepsilon}{4m_1},
\]

\[
|H_2(t_1, t) - H_2(t_2, t)| < \frac{\varepsilon}{4m_2},
\]

\[
(Tu)(t_1) - (Tu)(t_2) = \int_0^1 (K_1(t_1, s) - K_1(t_2, s))f(s, u(s), \partial_j^\alpha D_1^\alpha u(s))ds
\]

\[
+ \sum_{i=1}^m (K_2(t_1, t) - K_2(t_2, t))J_i(u(t), \partial_j^\alpha D_1^\alpha u(t))
\]

\[
+ \sum_{i=1}^m (K_3(t_1, t) - K_3(t_2, t))J_i(u(t), \partial_j^\alpha D_1^\alpha u(t))
\]

\[
\leq m_1 \int_0^1 |K_1(t_1, s) - K_1(t_2, s)|ds + m_2m|K_2(t_1, t) - K_2(t_2, t)|
\]

\[
+ m_3m|K_3(t_1, t) - K_3(t_2, t)| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}
\]

\[
|\partial_j^\alpha D_1^\alpha (Tu)(t_1) - \partial_j^\alpha D_1^\alpha (Tu)(t_2)| = \int_0^1 (H_1(t_1, s) - H_1(t_2, s))f(s, u(s), \partial_j^\alpha D_1^\alpha u(s))ds
\]

\[
+ \sum_{i=1}^m (H_2(t_1, t) - H_2(t_2, t))J_i(u(t), \partial_j^\alpha D_1^\alpha u(t))
\]

\[
\leq m_1 \int_0^1 |H_1(t_1, s) - H_1(t_2, s)|ds + m_2m|H_2(t_1, t) - H_2(t_2, t)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}
\]

Thus,

\[
\|(Tu)(t_1) - (Tu)(t_2)\| < \varepsilon.
\]

which implies that $T(\Omega_\varepsilon)$ is equicontinuous on any subinterval $J_k, k = 0, 1, \ldots, m$. 

From the Arzela–Ascoli theorem, we deduce that $T: E \rightarrow E$ is completely continuous.

**Lemma 6** (Schauder fixed-point theorem, see [41, 42]). Let $X$ be a real Banach space, $C \subset X$ be a nonempty closed bounded and convex subset, and $F: C \rightarrow C$ be compact. Then, $T$ has at least one fixed point in $C$.

**Lemma 7** (Krasnoselskii fixed point theorem, see [41, 42]). Let $\Omega$ be a closed convex and nonempty subset of a Banach space $X$. Let $\Phi$ and $\Psi$ be the operators such that (i) $\Phi x + \Psi y \in \Omega$
whenever \( x, y \in \Omega \); (ii) \( \Phi : \Omega \to X \) is compact and continuous; and (iii) \( \Psi \) is a contraction mapping. Then, there exists an \( z \in \Omega \) such that \( z = \Phi z + \Psi z \).

**Lemma 8** (Banach’s fixed point theorem, see [43]). Let \( E \) be a Banach space, \( \Omega \subset E \) be closed, and \( F: \Omega \to \Omega \) be a strict contraction, i.e., \( |Fx - Fy| \leq k|x - y| \) for some \( k \in (0, 1) \) and all \( x, y \in \Omega \). Then, \( F \) has a unique fixed point in \( \Omega \).

### 3. Existence of the Solutions

For convenience, we give the following symbols:

\[
A_i = \int_0^1 \left[ \frac{(2b-a)(1-s)^{q-1}}{(b-a)\Gamma(q)} + \frac{(2b-a)\Gamma(2-\gamma)}{(b-a)\Gamma(q-\gamma)} + \frac{2(1-s)^{q-1}}{\Gamma(q-\gamma)} \right] a_i(s) ds,
\]

\[
B_i = \frac{[b\Gamma(2-\gamma) + b-a]mb_i}{b-a},
\]

\[
C_i = \frac{mbc_i}{b-a}, \quad i = 0, 1, 2.
\]

Now, we present our main theorems.

**Theorem 1** Assume that (B1) and (B2) hold, and the following hypotheses are satisfied:

\[
[f(t,u,v)] \leq a_0(t) + a_1(t)|u|^\lambda_1 + a_2(t)|v|^\lambda_2, \quad \forall t \in J, u, v \in \mathbb{R},
\]  

(C1) There exist three nonnegative functions \( a_0, a_1, a_2 \in L(J) \) and two constants \( \lambda_1, \lambda_2 \in (0, 1) \) such that

\[
|J_1(u,v)| \leq b_0 + b_1|u|^\mu_1 + b_2|v|^\mu_2,
\]

\[
|J_i(u,v)| \leq c_0 + c_1|u|^\nu_i + c_2|v|^\nu_i, \quad i = 1, 2, \ldots, m, \forall u, v \in \mathbb{R},
\]  

(C2) There exist eight positive constants \( b_1, b_2, c_1, c_2 \geq 0 \) and \( \mu_1, \mu_2, \nu_1, \nu_2 \in (0, 1) \) such that

Then, (1) has at least one solution in \( E \).

**Proof.** Let

\[
R_1 = \max \left\{ \frac{7(4A_0 + B_0 + C_0), (7A_1)^{11-\lambda_1}, (7A_2)^{11-\lambda_2}, (7B_1)^{11-\mu_1}, (7B_2)^{11-\mu_2}, (7C_1)^{11-\nu_1}, (7C_2)^{11-\nu_2}}{7(4A_0 + B_0 + C_0), (7A_1)^{11-\lambda_1}, (7A_2)^{11-\lambda_2}, (7B_1)^{11-\mu_1}, (7B_2)^{11-\mu_2}, (7C_1)^{11-\nu_1}, (7C_2)^{11-\nu_2}} \right\},
\]

\( \Omega_{R_1} = \{ u \in E: \|u\| \leq R_1 \} \).

Now, \( \Omega_{R_1} \) is a closed bounded convex subset of \( E \). For each \( u \in \Omega_{R_1} \), from (C1) and (C2), we have

\[
|Tu(t)| \leq \int_0^1 |K_1(t, s)f(s, u(s), \frac{\partial}{\partial t}u(s))| ds + \sum_{i=1}^m |K_2(t, t_i)J_i(u(t_i), \frac{\partial}{\partial t}u(t_i))| ds
\]

\[
+ \sum_{i=1}^m |K_3(t, t_i)J_i(u(t_i), \frac{\partial}{\partial t}u(t_i))| ds \leq \int_0^1 |K_1(t, s)| a_0(s) + a_1(s)|u(s)|^{\lambda_1} + a_2(s)|\frac{\partial}{\partial t}u(s)|^{\lambda_2} |ds
\]
Assume that (B1) and (B2) hold, and the following hypotheses are satisfied:

\[
\|T u\| = \sup_{t \in t} |u(t)| + \sup_{t \in t} \left| (C^i D^j T u)(t) \right|
\]

\[
\leq \int_0^1 \left( |K_1 (t, s) + H_1 (t, s)| \left( a_0 (s) + a_1 (s) \|u\|^{k_1} + a_2 (s) \|u\|^{k_2} \right) \right) ds
\]
\[
+ \sum_{i=1}^m \left| K_2 (t, t_i) \right| \left( b_0 + b_1 \|u\|^{p_1} + b_2 \|u\|^{p_2} \right)
\]
\[
+ \sum_{i=1}^m \left| K_3 (t, t_i) \right| \left( c_0 + c_1 \|u\|^{q_1} + c_2 \|u\|^{q_2} \right)
\]
\[
\leq A_0 + A_1 \|R_1\|^{k_1} + A_2 \|R_1\|^{k_2} + B_0 + B_1 \|R_1\|^{p_1} + B_2 \|R_1\|^{p_2}
\]
\[
+ C_0 + C_1 \|R_1\|^{q_1} + C_2 \|R_1\|^{q_2} \leq R_1,
\]

which implies that \( T(M_{R_1}) \subset M_{R_1} \).

From Lemmas 5 and 6, \( T \) has at least one fixed point in \( M_{R_1} \), so (1) has at least one solution in \( E \).

**Theorem 2.** Assume that (B1) and (B2) hold, and the following hypotheses are satisfied:

(C3) There exists a nonnegative function \( a_0 \in L (J) \), such that

\[
|I \left( u_1, v_1 \right) - I \left( u_2, v_2 \right)| \leq b_1 |u_1 - u_2| + b_2 |v_1 - v_2|,
\]
\[
|J \left( u_1, v_1 \right) - J \left( u_2, v_2 \right)| \leq c_1 |u_1 - u_2| + c_2 |v_1 - v_2|,
\]

\[
\forall u_1, u_2, v_1, v_2 \in \mathbb{R}, i = 1, 2, \ldots, m,
\]

If \( \Lambda = \sum_{i=1}^m (B_i + C_i) < 1/2 \), then (1) has at least one solution in \( E \).

**Proof.** We first define the operators. For \( u \in E \), let

\[
[f (t, u, v)] \leq a_0 (t), \quad \forall t \in J, u, v \in \mathbb{R}.
\]

(C4) There exist four positive constants \( b_1, b_2, c_1, c_2 \geq 0 \) such that

\[
\forall u_1, u_2, v_1, v_2 \in \mathbb{R}, i = 1, 2, \ldots, m.
\]
\[(\Phi u)(t) = \int_0^t K_1(t, s) f(s, u(s), \partial_t u(s)) \, ds,\]
\[(\Phi_1 u)(t) = \int_0^t H_1(t, s) f(s, u(s), \partial_t u(s)) \, ds,\]
\[(\Psi u)(t) = \sum_{i=1}^m K_2(t, t_i) I_i(u(t_i), \partial_t u(t_i)) + \sum_{i=1}^m K_3(t, t_i) I_i(u(t_i), \partial_t^2 u(t_i)),\]
\[(\Psi_1 u)(t) = \sum_{i=1}^m H_2(t, t_i) I_i(u(t_i), \partial_t^2 u(t_i)).\]  

Now,

\[(T u)(t) = (\Phi u)(t) + (\Psi u)(t),\]
\[(\partial_t^2 T u)(t) = \partial_t^2 (\Phi u)(t) + \partial_t^2 (\Psi u)(t) = (\Phi_1 u)(t) + (\Psi_1 u)(t).\]

Let
\[M_1 = \max_{1 \leq i \leq m} |I_i(0, 0)|,\]
\[M_2 = \max_{1 \leq i \leq m} |I_i(0, 0)|.\]  

Let
\[R_2 \geq \max\left\{2A_0, \frac{2\Theta}{1 - 2\lambda}\right\},\]
\[\Omega_{R_2} = \{u \in E: \|u\| \leq R_2\}.\]

\[\|\Psi(t) - \Psi(v)\| \leq \sum_{i=1}^m |K_2\| |I_i(t_i, \partial_t u(t_i)) - I_i(v(t_i), \partial_t v(t_i))| + \sum_{i=1}^m |K_3\| |I_i(t_i, \partial_t^2 u(t_i)) - I_i(v(t_i), \partial_t^2 v(t_i))|\]
\[\leq \sum_{i=1}^m |K_2\| |(c_1 |u(s) - v(s)| + c_2 |\partial_t u(s) - \partial_t v(s)|)| + \sum_{i=1}^m |K_3\| |(b_1 |u(s) - v(s)| + b_2 |\partial_t^2 u(s) - \partial_t^2 v(s)|)|\]
\[\leq \sum_{i=1}^m |K_2\| |(c_1 + c_2)\|u - v\| + \sum_{i=1}^m |K_3\| |(b_1 + b_2)\|u - v\|.\]  

\[\|\Psi_1(t) - \Psi_1(v)\| \leq \sum_{i=1}^m |H_2\| |I_i(t_i, \partial_t^2 u(t_i)) - I_i(v(t_i), \partial_t^2 v(t_i))| + \sum_{i=1}^m |H_3\| |I_i(t_i, \partial_t^2 u(t_i)) - I_i(v(t_i), \partial_t^2 v(t_i))|\]
\[\leq \sum_{i=1}^m |H_2\| |(c_1 |u(s) - v(s)| + c_2 |\partial_t^2 u(s) - \partial_t^2 v(s)|)| + \sum_{i=1}^m |H_3\| |(c_1 + c_2)\|u - v\|.\]  

Note that \(\Omega_{R_2}\) is a nonempty bounded closed convex subset of \(E\).

From Lemma 5, \(\Phi\) is completely continuous (i.e., condition (ii) of Lemma 7 is satisfied).

For any \(u, v \in \Omega_{R_2}\), from hypothesis (C4), we have
Therefore,
\[
\|\Psi u - \Psi v\| = \max_{t \in I} |\Psi u - \Psi v| + \max_{t \in I} |\Psi_1 u - \Psi_1 v|
\]
\[
\leq \sum_{i=1}^{m} \left[ |K_2 (t, t_i)| + |H_2 (t, t_i)| \right] (c_1 + c_2) \|u - v\|
\]
\[
+ \sum_{i=1}^{m} |K_3 (t, t_i)| (b_1 + b_2) \|u - v\|
\]
\[
\leq (C_1 + C_2 + B_1 + B_2) \|u - v\| = \Lambda \|u - v\|
\]
(47)

and since \( \Lambda < 1/2 \), \( \Psi \) is a contraction (so condition (iii) of Lemma 7 is satisfied).

For each \( u \in \Omega_{R_1} \) from hypothesis (C3), we have
\[
\|\Phi (t)\| \leq \int_0^1 |K_1 (t, s)| a_0 (s) ds \leq A_0 \leq \frac{R_2}{2}
\]
(48)

Consequently,
\[
\|\Phi u\| \leq \int_0^1 (|H_1 (t, s)| + |H_2 (t, s)|) a_0 (s) ds \leq A_0 \leq \frac{R_2}{2}
\]
(49)

For each \( v \in \Omega_{R_2} \), we have
\[
\|\Psi v\| \leq \|\Psi 0\| + \|\Psi v\| \leq \Lambda \|0\| + \Lambda \|v\| + \Theta \leq R_2
\]
(50)

where
\[
\Theta = m \left( b \Gamma (2 - \gamma) + b - a \right) \frac{M_2 + b}{b - a}
\]
(51)

Thus, for any \( u, v \in \Omega_{R_1} \), we obtain
\[
\|\Phi u + \Psi v\| \leq \|\Phi u\| + \|\Psi v\| \leq A_0 + \Lambda \|v\| + \Theta \leq R_2
\]
(52)

which implies that \( \Phi u + \Psi v \in \Omega_{R_1} \) (so condition (i) of Lemma 7 is satisfied).

In view of Lemma 7, there exists a \( u \in \Omega_{R_1} \) such that \( \Phi u + \Psi u = u \), so (1) has at least one solution in \( E \). \( \square \)

**Theorem 3.** Assume that (B1), (B2), and (C4) hold and the following hypothesis is satisfied:

(C5) There exist two nonnegative functions \( a_1, a_2 \in L (J) \) such that

\[
|f (t, u_1, v_1) - f (t, u_2, v_2)| \leq a_1 (t) |u_1 - u_2| + a_2 (t) |v_1 - v_2|, \quad \forall t \in J, u_1, u_2, v_1, v_2 \in \mathbb{R}
\]
(53)

If \( \Pi = (A_1 + A_2) + |b \Gamma (2 - \gamma) + b - a| (c_1 + c_2) + b(b_1 + b_2)/b - a < 1 \), then (1) has a unique solution in \( E \).

**Proof.** Choose

\[
M_0 = \max_{t \in J} f (t, 0, 0),
\]
\[
M_1 = \max_{1 \leq j \leq m} |I_j (0, 0)|,
\]
\[
M_2 = \max_{1 \leq j \leq m} |J_j (0, 0)|,
\]
\[
A' = \frac{(2b - a)}{(b - a)^2 (q + 1)} + \frac{(2b - a)^2 (q - \gamma)}{(b - a)^2 (q - \gamma + 1)} + \frac{2}{\Gamma (q + 1)}
\]
\[
B' = \frac{b \Gamma (2 - \gamma) M_2 + b M_1 + b - a}{b - a}
\]
(55)

where

\[
R_3 \geq \frac{1}{1 - \Pi} \left( A' M_0 + B' \right)
\]
(54)
First, we show that $T\Omega_{R_3} \subset \Omega_{R_3}$, where
\[ \Omega_{R_3} = \{ u \in E, \| u \| \leq R_3 \}. \]
For $u \in \Omega_{R_3}$, from hypotheses (C4) and (C5), we obtain

\[ |(Tu)(t)| \leq \int_0^1 |K_1(t, s)| \left[ |f(s, u(s), ^c D_t^s u(s)) - f(t, 0, 0)| + |f(t, 0, 0)| \right] ds \]
\[ + \sum_{i=1}^m |K_2(t, t_i)| \left[ |I_1(u(t_i), ^c D_t^t u(t_i)) - I_1(0, 0)| + |I_1(0, 0)| \right] \]
\[ + \sum_{i=1}^m |K_3(t, t_i)| \left[ |I_1(u(t_i), ^c D_t^t u(t_i)) - I_1(0, 0)| + |I_1(0, 0)| \right] \]
\[ \leq \int_0^1 |K_1(t, s)| \left[ (a_1(t) + a_2(t))\| u \| + M_0 \right] ds + \sum_{i=1}^m |K_2(t, t_i)| \left[ (c_1 + c_2)\| u \| + M_2 \right] \]
\[ + \sum_{i=1}^m |K_3(t, t_i)| \left[ (b_1 + b_2)\| u \| + M_1 \right], \quad (56) \]

\[ \|(^c D_t^s Tu)(t)\| \leq \int_0^1 |H_1(t, s)| \left[ |f(s, u(s), ^c D_t^s u(s)) - f(t, 0, 0)| + |f(t, 0, 0)| \right] ds \]
\[ + \sum_{i=1}^m |H_2(t, t_i)| \left[ |I_1(u(t_i), ^c D_t^t u(t_i)) - I_1(0, 0)| + |I_1(0, 0)| \right] \]
\[ \leq \int_0^1 |H_1(t, s)| \left[ (a_1(t) + a_2(t))\| u \| + M_0 \right] ds + \sum_{i=1}^m |H_2(t, t_i)| \left[ (c_1 + c_2)\| u \| + M_2 \right]. \]

Then,

\[ \|Tu\| \leq \int_0^1 \left[ |K_1(t, s)| + |H_1(t, s)| \right] \left[ (a_1(t) + a_2(t))\| u \| + M_0 \right] ds \]
\[ + \sum_{i=1}^m \left[ |K_2(t, t_i)| + |H_2(t, t_i)| \right] \left[ (c_1 + c_2)\| u \| + M_2 \right] \]
\[ + \sum_{i=1}^m |K_3(t, t_i)| \left[ (b_1 + b_2)\| u \| + M_1 \right] \leq \| u \| + A'M_0 + B' \leq R_3, \quad (57) \]

so $T\Omega_{R_3} \subset \Omega_{R_3}$.

Furthermore, from hypotheses (C4) and (C5), for all $u, v \in \Omega_{R_3}$, we have

\[ |(Tu)(t) - (Tv)(t)| \leq \int_0^1 |K_1(t, s)| \left| f(s, u(s), ^c D_t^s u(s)) - f(s, v(s), ^c D_t^s v(s)) \right| ds \]
\[ + \sum_{i=1}^m |K_2(t, t_i)| \left| I_1(u(t_i), ^c D_t^t u(t_i)) - I_1(v(t_i), ^c D_t^t v(t_i)) \right| \]
\[ + \sum_{i=1}^m |K_3(t, t_i)| \left| I_1(u(t_i), ^c D_t^t u(t_i)) - I_1(v(t_i), ^c D_t^t v(t_i)) \right|. \]
\[ \leq \int_0^1 |K_1(t,s)| (a_1(t) + a_2(t))\|u - v\| + \sum_{i=1}^m |K_2(t,t_i)| (c_1 + c_2)\|u - v\| \\
+ \sum_{i=1}^m |K_3(t,t_i)(b_1 + b_2)\|u - v\|,\]

\[ |(\mathcal{D}_t^\alpha Tu(t) - (\mathcal{D}_t^\alpha Tv(t))| \leq \int_0^1 |H_1(t,s)| |f(s,u(s),\mathcal{D}_t^\alpha u(s)) - f(s,v(s),\mathcal{D}_t^\alpha v(s))|\,ds \\
+ \sum_{i=1}^m |H_2(t,t_i)| |J_1(u(t_i),\mathcal{D}_t^\alpha u(t_i)) - J_1(v(t_i),\mathcal{D}_t^\alpha v(t_i))| \\
\leq \int_0^1 |H_1(t,s)| (a_1(t) + a_2(t))\|u - v\| + \sum_{i=1}^m |H_2(t,t_i)| (c_1 + c_2)\|u - v\|. \quad (58) \]

Thus,

\[ \Pi = \int_0^1 \left[ |K_1(t,s)| + |H_1(t,s)| \right] (a_1(t) + a_2(t))\|u - v\|\,ds \\
+ \sum_{i=1}^m \left[ |K_2(t,t_i)| + |H_2(t,t_i)| \right] (c_1 + c_2)\|u\| + \sum_{i=1}^m |K_3(t,t_i)(b_1 + b_2)\|u\| \]

where \( \Pi < 1 \), so \( T \) is a contraction. Lemma 8 guarantees that \( T \) has a unique fixed point in \( \Omega_{R_0} \), which is the unique solution of (1) in \( E \). This completes the proof. \( \square \)

4. Examples

In (1), let \( q = 1.25, \gamma = 0.15, a = 1, b = 2, t_1 = 0.5, \) and \( k = 1 \) and then, we obtain the following fractional-order impulsive differential equation:

\[
\begin{align*}
\mathcal{D}_t^{1.25} u(t) &= f\left(t, u(t), \mathcal{D}_t^{0.15} u(t)\right), \quad t \in (0, 1), t \neq 0.5, \\
\Delta u(0.5) &= I_1(u(0.5), \mathcal{D}_t^{0.15} u(0.5)), \\
\Delta \mathcal{D}_t^{0.15} u(0.5) &= I_1(u(0.5), \mathcal{D}_t^{0.15} u(0.5)), \\
\mathcal{D}_t^{0.15} u(0) &= 0, \\
\mathcal{D}_t^{0.15} u(0) - 2^\gamma \mathcal{D}_t^{0.15} u(1) &= 0.
\end{align*}
\]

(60)

By a direct observation, note that \( 0 < a < b < +\infty, 1 < q < 2, 0 < \gamma < 1 \) with \( q - \gamma > 1 \), so hypothesis (B1) is satisfied.

Example 1. In (60), let

\[ f\left(t, u(t), \mathcal{D}_t^{0.15} u(t)\right) = \frac{e^t}{50} + \frac{(1 - t)^2 (u(t))^{0.2}}{100} + \frac{e^{2t} (\mathcal{D}_t^{0.15} u(t))^{0.3}}{200}, \]

\[ \Delta u(0.5) = \sin \left( \frac{1 + 2(u(0.5))^{0.5} + 3(\mathcal{D}_t^{0.15} u(0.5))^{0.4}}{150} \right), \]

\[ \Delta \mathcal{D}_t^{0.25} u(0.5) = \sin \left( \frac{1 + 3(u(0.5))^{0.2} + 2(\mathcal{D}_t^{0.15} u(0.5))^{0.1}}{120} \right). \]

(61)
so hypothesis (B2) is satisfied. Set $^cD_t^{0.15}u(t) = v(t)$, and then, we obtain

\[
|f(t, u, v)| \leq \frac{e^t}{50} + \frac{(1-t)^2}{100} |u|^{0.2} + v^{0.3} = a_0(t) + a_1(t)|u|^{0.2} + a_2(t)|v|^{0.3},
\]

\[
|I_1(u, v)| \leq \frac{1}{150} + \frac{1}{75} |u|^{0.5} + \frac{1}{50} |v|^{0.4} = b_0 + b_1|u|^{0.5} + b_2|v|^{0.4},
\]

\[
|J_1(u, v)| \leq \frac{1}{120} + \frac{1}{40} |u|^{0.2} + \frac{1}{60} |v|^{0.1} = c_0 + c_1|u|^{0.2} + c_2|v|^{0.1},
\]

which implies that (C1) and (C2) are satisfied. Thus, all the hypotheses in Theorem 1 are satisfied, so (60) has at least one solution in $E$.

**Example 2.** In (60), let

\[
f(t, u, v) = \frac{(1-s)^2}{50} u(t) + ^cD_t^{0.15}u(t),
\]

\[
\Delta u(0.5) = \frac{1+2u(0.5) + 3^cD_t^{0.15}u(0.5)}{150},
\]

\[
\Delta^cD_t^{0.15}u(0.5) = \frac{1+3u(0.5) + 2^cD_t^{0.15}u(0.5)}{120},
\]

so hypothesis (B2) is satisfied. Set $^cD_t^{0.15}u(t) = v(t)$, and then, we obtain

\[
|f(t, u, v)| \leq \frac{(1-s)^2}{50} u(t),
\]

\[
|I_1(u_1, v_1) - I_1(u_2, v_2)| \leq \frac{1}{75} |u_1 - u_2| + \frac{1}{50} |v_1 - v_2|,
\]

\[
|J_1(u_1, v_1) - J_1(u_2, v_2)| \leq \frac{1}{40} |u_1 - u_2| + \frac{1}{60} |v_1 - v_2|,
\]

which implies that (C3) and (C4) are satisfied. Also, note that $\Lambda = 0.179707 < 0.5$. Then, all the hypotheses in Theorem 2 are satisfied, so (60) has at least one solution in $E$.

**Example 3.** In (60), let

\[
f(t, u, v) = \frac{(1-s)^2}{50} u(t) + \sqrt{(1-s)D_t^{0.15}u(t)},
\]

\[
\Delta u(0.5) = \frac{1+2u(0.5) + 3^cD_t^{0.15}u(0.5)}{150},
\]

\[
\Delta^cD_t^{0.15}u(0.5) = \frac{1+3u(0.5) + 2^cD_t^{0.15}u(0.5)}{120},
\]

so hypothesis (B2) is satisfied. Set $^cD_t^{0.15}u(t) = v(t)$, and then, we obtain

\[
|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{(1-s)^2}{100} |u_1 - u_2| + \frac{\sqrt{1-s}}{200} |v_1 - v_2| = a_3(t)|u_1 - u_2| + a_4(t)|v_1 - v_2|,
\]

\[
|I_1(u_1, v_1) - I_1(u_2, v_2)| \leq \frac{1}{75} |u_1 - u_2| + \frac{1}{50} |v_1 - v_2| = b_0|u_1 - u_2| + b_2|v_1 - v_2|,
\]

\[
|J_1(u_1, v_1) - J_1(u_2, v_2)| \leq \frac{1}{40} |u_1 - u_2| + \frac{1}{60} |v_1 - v_2| = c_1|u_1 - u_2| + c_2|v_1 - v_2|,
\]

which implies that (C1) and (C2) are satisfied. Thus, all the hypotheses in Theorem 1 are satisfied, so (60) has at least one solution in $E$. 

---

**Example 3.** In (60), let

\[
f(t, u, v) = \frac{(1-s)^2}{50} u(t) + \frac{\sqrt{1-s}}{200} D_t^{0.15}u(t),
\]

\[
\Delta u(0.5) = \frac{1+2u(0.5) + 3D_t^{0.15}u(0.5)}{150},
\]

\[
\Delta^cD_t^{0.15}u(0.5) = \frac{1+3u(0.5) + 2D_t^{0.15}u(0.5)}{120},
\]

so hypothesis (B2) is satisfied. Set $^cD_t^{0.15}u(t) = v(t)$, and then, we obtain

\[
|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{(1-s)^2}{100} |u_1 - u_2| + \frac{\sqrt{1-s}}{200} |v_1 - v_2| = a_3(t)|u_1 - u_2| + a_4(t)|v_1 - v_2|,
\]

\[
|I_1(u_1, v_1) - I_1(u_2, v_2)| \leq \frac{1}{75} |u_1 - u_2| + \frac{1}{50} |v_1 - v_2| = b_0|u_1 - u_2| + b_2|v_1 - v_2|,
\]

\[
|J_1(u_1, v_1) - J_1(u_2, v_2)| \leq \frac{1}{40} |u_1 - u_2| + \frac{1}{60} |v_1 - v_2| = c_1|u_1 - u_2| + c_2|v_1 - v_2|,
\]
which implies that (C4) and (C5) are satisfied. Note that $\Pi = 0.787135 < 1$. Then, all the hypotheses in Theorem 3 are satisfied, so (60) has a unique solution in $E$.

5. Conclusion

In this paper, we use fixed-point theorems to study fractional-order impulsive differential equation (1) with generalized periodic boundary value conditions. Very little is known on fractional-order impulsive differential equations with generalized periodic boundary value conditions where nonlinear terms and impulse terms depend on the unknown function and the lower-order fractional derivative of the unknown function. Our main results are obtained under some nonlinear and linear growth conditions corresponding to the relevant linear operators where the symmetry property of a Green’s function is not required, so our results generalize and improve works in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

The study was carried out in collaboration with all authors. All authors read and approved the final manuscript.

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