Eigenvectors of Baxter–Bazhanov–Stroganov \( \tau(2)(t_q) \) model with fixed-spin boundary conditions

N. Z. Iorgov\(^1\), V. N. Shadura\(^2\), Yu. V. Tykhyy\(^3\)

Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine

Abstract

The aim of this contribution is to give the explicit formulas for the eigenvectors of the transfer-matrix of Baxter–Bazhanov–Stroganov (BBS) model (\( N \)-state spin model) with fixed-spin boundary conditions. These formulas are obtained by a limiting procedure from the formulas for the eigenvectors of periodic BBS model. The latter formulas were derived in the framework of the Sklyanin’s method of separation of variables. In the case of fixed-spin boundaries the corresponding \( T-Q \) Baxter equations for the functions of separated variables are solved explicitly. As a particular case we obtain the eigenvectors of the Hamiltonian of Ising-like \( Z_N \) quantum chain model.

1 Introduction

During the last two decades, a considerable progress in application of the Separation of Variables (or functional Bethe ansatz) method to a broad class of integrable models of quantum chains and statistical physics has been achieved. This progress was initiated by the paper \([1]\) of Sklyanin who has proposed a recipe for a Separation of Variables in the case of quantum Toda chain where the algebraic Bethe ansatz fails. The idea is to write the eigenvectors of periodic problem as a linear combinations of the eigenvectors of an auxiliary problem (the auxiliary problem for the periodic Toda chain is the open Toda chain). The next step is to construct recursively the eigenvectors of the \( m \)-site auxiliary problem in terms of the eigenvectors of the \((m-1)\)-site auxiliary problem \([2,3]\).

In \([4]\), this program was realized for the inhomogeneous periodic Baxter–Bazhanov–Stroganov (BBS) model (or \( \tau(2)(t_q) \) model) \([5,6,7,8]\) defined in terms of cyclic \( L \)-operators \([9,7]\). It was shown that for every eigenvalue (found from the functional relations) of the transfer-matrix there corresponds an explicit formula for the eigenvector. So the problem of finding the eigenvectors reduces to the problem of finding the corresponding eigenvalues.

At special values of parameters, the periodic BBS model can be interpreted as a model with fixed-spin boundary conditions \([5]\). In this paper we derive the eigenvectors of this model from the eigenvectors of periodic BBS model by specializing corresponding parameters. In the case of fixed-spin boundary conditions, the structure of eigenvalues is simple, so the \( T-Q \) Baxter equations for the functions of separated variables can be solved explicitly in terms of cyclic function \( w_p(\gamma) \) \([10]\) which is a root of unity analog of the \( q \)-gamma function. It gives us formulas for the eigenvectors of the BBS model with fixed-spin boundary conditions.

\(^{1}\)iorgov@bitp.kiev.ua
\(^{2}\)shadura@bitp.kiev.ua
\(^{3}\)tykhyy@bitp.kiev.ua
At the end of the paper we show that the Hamiltonian of the Ising-like $\mathbb{Z}_N$ quantum chain model with fixed boundary spins found by Baxter \([3,11]\) can be obtained in the present framework by further specialization of parameters. It means, in particular, that we can find explicit formulas for the eigenvectors and the eigenvalues of the Hamiltonian. The formula for the eigenvalues coincides with Baxter’s one obtained from the functional relation for the transfer-matrix. Explicit formulas for the eigenvectors prove that the corresponding eigenvalues enter to the spectrum of the Hamiltonian with the multiplicity one.

2 Lattice and quantum chain formulations of BBS model

Following the notation of a recent paper of Baxter \([3]\), we define the BBS model as a statistical model of short-range interacting spins placed at the vertices of a rectangular lattice. We label the spin variables $s_{x,y}$ by a pair $(x, y)$ of integers: $x = 1, \ldots, n + 1$, and $y = 1, \ldots, m$. Each spin variable $s_{x,y}$ takes $N$ values ($N \geq 2$): $0, 1, \ldots, N - 1$. The model shall have $\mathbb{Z}_N$-symmetry and we may extend the range of the spins $s_{x,y}$ to all integers identifying two values if their difference is a multiple of $N$. The model has a chiral restriction on the values of vertically neighboring spins:

$$s_{x,y} - s_{x,y+1} = 0 \text{ or } 1 \mod N. \tag{1}$$

In the following we will consider the spin variables on two adjacent rows: $(k, l)$ and $(k, l+1)$, where $l$ is fixed and $k = 1, \ldots, n + 1$. Let us denote $s_{k,l} = \gamma_k$ and $s_{k,l+1} = \gamma'_k$. The model depends on the parameters $t_q$ and $a'_k, b'_k, c'_k, d'_k, a''_k, b''_k, c''_k, d''_k$, $k = 1, 2, \ldots, n + 1$. Each square plaquette of the lattice has the Boltzmann weight (see Fig\([\text{III}]\))

$$W_{\gamma}(\gamma_{k-1}, \gamma_k; \gamma'_{k-1}, \gamma'_k) = \sum_{m_{k-1}=0}^{1} \omega^{m_{k-1}(\gamma'_k-\gamma_{k-1})}(-\omega t_q)^{\gamma_k-\gamma'_k-m_{k-1}} \times \tag{2}$$

$$\times F_{k-1}(\gamma_{k-1} - \gamma'_{k-1}, m_{k-1}) F''_k(\gamma_k - \gamma'_k, m_{k-1}),$$

where $\omega = e^{2\pi i/N}$, and

$$F'_k(0,0) = 1, \quad F'_k(0,1) = -\omega t_q \frac{c'_k}{b'_k}, \quad F''_k(1,0) = \frac{d'_k}{b'_k}, \quad F''_k(1,1) = -\omega \frac{a''_k}{b''_k},$$

and expressions for $F''_k(\gamma_k - \gamma'_k, m_{k-1})$ are obtained from $F'_k(\gamma_k - \gamma'_k, m_{k})$ by substitutions: $a'_k, b'_k, c'_k, d'_k \to a''_k, b''_k, c''_k, d''_k$.

We consider the periodic boundary condition: $\gamma_{n+2} = \gamma_1$, $\gamma'_{n+2} = \gamma'_{1}$, where $n + 1$ is the number of sites on the lattice along the horizontal axis. The transfer-matrix of the periodic BBS model is $N^{n+1} \times N^{n+1}$ matrix with matrix elements

$$t_{n+1}(\gamma, \gamma') = \prod_{k=2}^{n+2} W_{\gamma}(\gamma_{k-1}, \gamma_k; \gamma'_{k-1}, \gamma'_k), \tag{3}$$

labelled by the sets of spin variables $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{n+1}\}$ and $\gamma' = \{\gamma'_1, \gamma'_2, \ldots, \gamma'_{n+1}\}$ of two neighbour rows.
The triangle with vertices marked by the spin variables $\gamma_{k-1}$, $\gamma'_{k-1}$, $m_{k-1}$ corresponds to the function $F'_{k}(\gamma_{k-1} - \gamma'_{k-1}, m_{k-1})$ in (2); the triangle $\gamma_{k}$, $\gamma'_{k}$, $m_{k-1}$ to $F''_{k}(\gamma_{k} - \gamma'_{k}, m_{k-1})$.

Considering $m_{k}$, $k = 1, \ldots, n + 1$, in (2) as auxiliary spin variables which take the two values 0 and 1, we can rewrite transfer-matrix (3) in a vertex formulation associating a statistical weight not to the plaquettes but to vertices each of them relating four spins: $m_{k-1}$, $m_{k}$, $\gamma_{k}$, $\gamma'_{k}$ (see Fig. 1). Then the weight associated with the $k$th vertex is

$$
\ell_{k}(t_{q}; m_{k-1}, m_{k}; \gamma_{k}, \gamma'_{k}) = \omega^{m_{k-1} \gamma_{k} - m_{k} \gamma'_{k}} (-\omega t_{q})^{m_{k} - \gamma_{k} - \gamma'_{k} - m_{k-1}} \times F''_{k}(\gamma_{k} - \gamma'_{k}, m_{k-1}) F'_{k}(\gamma_{k} - \gamma'_{k}, m_{k}). 
$$

and the transfer-matrix (3) can be rewritten as

$$
t_{n+1}(\gamma, \gamma') = \sum_{m_{1}, \ldots, m_{n+1}} \prod_{k=2}^{n+2} \ell_{k}(t_{q}; m_{k-1}, m_{k}; \gamma_{k}, \gamma'_{k}). 
$$

For our construction of the BBS model eigenvectors we will use a description of this model as a quantum chain model. To the each site $k$ of the quantum chain we associate the cyclic $L$-operator [9, 7] acting in a two-dimensional auxiliary space

$$
L_{k}(\lambda) = \begin{pmatrix}
1 + \lambda \varepsilon_{k}v_{k}, & \lambda u_{k}^{-1}(a_{k} - b_{k}v_{k}) \\
\lambda u_{k}(c_{k} - d_{k}v_{k}), & \lambda a_{k}c_{k} + v_{k}b_{k}d_{k}/\varepsilon_{k}
\end{pmatrix}, \quad k = 1, 2, \ldots, n + 1. 
$$

At each site $k$ we define ultra-local Weyl elements $u_{k}$ and $v_{k}$ obeying the commutation rules and normalization

$$
u_{j}u_{k} = u_{k}u_{j}, \quad \nu_{j}v_{k} = v_{k}v_{j}, \quad u_{j}v_{k} = \omega^{\delta_{j,k}}v_{k}u_{j}, \quad \omega = e^{2\pi i/N}, \quad u_{k}N = v_{k}N = 1. 
$$

In (6), $\lambda$ is the spectral parameter and we have five parameters $\varepsilon_{k}$, $a_{k}$, $b_{k}$, $c_{k}$, $d_{k}$ per site. At each site $k$ we define a $N$-dimensional linear space (quantum space) $V_{k}$ with the basis $|\gamma\rangle_{k}$, $\gamma \in \mathbb{Z}_{N}$ and natural scalar product $k\langle\gamma'\gamma\rangle_{k} = \delta_{\gamma',\gamma}$. In $V_{k}$ the Weyl elements $u_{k}$ and $v_{k}$ act as:

$$
u_{k} |\gamma\rangle_{k} = \omega^{\gamma} |\gamma\rangle_{k}, \quad v_{k} |\gamma\rangle_{k} = |\gamma + 1\rangle_{k}. 
$$

\[3\]
The correspondence between the lattice BBS model and its quantum chain analog is established through the relation
\[
\ell_k(t_q; m_{k-1}, m_k; \gamma_k, \gamma_k') = k \langle \gamma_k | L_k(\lambda) m_{k-1}, m_k | \gamma_k' \rangle_k
\]  
and the following connection between the parameters of these models
\[
\lambda = -\omega t_q, \quad \kappa_k = \frac{d'_k d''_k}{b'_k b''_k}, \quad a_k = \frac{c''_k}{b''_k}, \quad b_k = \omega \frac{d'_k d''_k d'_k}{b'_k b''_k}, \quad c_k = \frac{c'_k}{b'_k}, \quad d_k = \frac{d'_k d''_k}{b'_k b''_k}.
\]

We extend the action of the operators \( u_k, v_k \) to \( \mathcal{V}^{(n+1)} = \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_{n+1} \) defining this action to be trivial in all \( \mathcal{V}_s \) with \( s \neq k \). The monodromy matrix for the quantum chain with \( n + 1 \) sites is defined as
\[
T_{n+1}(\lambda) = L_1(\lambda) L_2(\lambda) \cdots L_{n+1}(\lambda) = \begin{pmatrix}
A_{n+1}(\lambda) & B_{n+1}(\lambda) \\
C_{n+1}(\lambda) & D_{n+1}(\lambda)
\end{pmatrix}.
\]

The transfer-matrix (5) is obtained taking the trace in the auxiliary space
\[
t_{n+1}(\lambda) = \text{tr} T_{n+1}(\lambda) = A_{n+1}(\lambda) + D_{n+1}(\lambda).
\]

This quantum chain is integrable because the \( L \)-operators (6) are intertwined by the twisted 6-vertex \( R \)-matrix
\[
R(\lambda, \nu) = \begin{pmatrix}
\lambda - \omega \nu & 0 & 0 & 0 \\
0 & \omega (\lambda - \nu) & \lambda (1 - \omega) & 0 \\
0 & \nu (1 - \omega) & \lambda - \nu & 0 \\
0 & 0 & 0 & \lambda - \omega \nu
\end{pmatrix},
\]

where \( L^{(1)}_k(\lambda) = L_k(\lambda) \otimes \mathbb{1}, L^{(2)}_k(\nu) = \mathbb{1} \otimes L_k(\nu) \). Relation (14) leads to \( [t_{n+1}(\lambda), t_{n+1}(\mu)] = 0 \). So \( t_{n+1}(\lambda) \) is a generating function for the commuting set of non-local and non-hermitian Hamiltonians of the model. It also follows from the intertwining relation (14) that \( [B_{n+1}(\lambda), B_{n+1}(\mu)] = 0 \) and therefore \( B_{n+1}(\lambda) \) is a generating function for another commuting set of operators.

The BBS model with fixed-spin boundary conditions can be obtained using periodic inhomogeneous BBS model with \( n + 1 \) sites (in each row if we consider the lattice formulation) if one fixes \( a'_{n+1} = d'_{n+1} = 0 \). In this case \( F'_{n+1}(\gamma_{n+1} - \gamma'_{n+1}, m) = 0 \) unless \( \gamma_{n+1} = \gamma'_{n+1} \). It means that \( t_{n+1}(\lambda) \) with such parameters and fixed \( \gamma_{n+1} \) can be interpreted as transfer-matrix of \( n \)-site BBS model with first and last \((n\text{th})\) spins interacting with fixed spin \( \gamma_{n+1} \) (fixed-spin boundary conditions).

Let us look what we have in \( L \)-operator formulation. Due to (10), the relations \( a'_{n+1} = d'_{n+1} = 0 \) give us \( \kappa_{n+1} = b_{n+1} = d_{n+1} = 0 \) and \( b_{n+1} d_{n+1} / \kappa_{n+1} = 0 \). So we get
\[
L_{n+1}(\lambda) = \begin{pmatrix}
1, & \lambda u_{n+1}^{-1} a_{n+1} \\
u_{n+1} c_{n+1}, & \lambda c_{n+1}^{-1} a_{n+1}
\end{pmatrix} = \begin{pmatrix}
1, & \lambda u_{n+1}^{-1} a_{n+1} \\
u_{n+1} c_{n+1}, & \lambda c_{n+1}^{-1} a_{n+1}
\end{pmatrix}.
\]

Since \( t_{n+1}(\lambda) \) commutes with \( u_{n+1} \) we may restrict the space of states of \((n + 1)\)-site problem to the states with fixed eigenvalue of \( u_{n+1} = \omega^{\gamma_{n+1}} \) in order to obtain \( n \)-site
problem. To consider different fixed spins on both boundaries we include one more $L$-operator $L_0(\lambda)$ like (15) but with operator $u_0$ and parameters $a_0$ and $c_0$. On the states with fixed eigenvalues $u_{n+1} = \omega^{\gamma_{n+1}}$ and $u_0 = \omega^{\gamma_0}$ we have

$$L_{n+1}(\lambda)L_0(\lambda) = (1 + \lambda \omega^{-\gamma_{n+1}} a_{n+1} c_0) \begin{pmatrix} 1, & \lambda a_0 \omega^{-\gamma_0} \\ c_{n+1} \omega^{\gamma_{n+1}}, & \lambda a_0 c_{n+1} \omega^{\gamma_{n+1} - \gamma_0} \end{pmatrix}.$$  

It means that we can imitate different boundary spins by one $L$-operator

$$L_{n+1}(\lambda) = \begin{pmatrix} 1, & \lambda \omega^{-\gamma_0} a_0 \\ u_{n+1} c_{n+1}, & \lambda a_0 c_{n+1} \omega^{\gamma_{n+1} - \gamma_0} \end{pmatrix},$$  

which is of the form (6) with special parameters. Thus in this paper we consider the monodromy matrix defined by (11) for the quantum chain with $n+1$ sites, where $L_{n+1}(\lambda)$ is given by (16). The corresponding transfer-matrix with fixed boundary spins is

$$t^B(\lambda) = \text{tr} T_{n+1}(\lambda) = \begin{pmatrix} 1, & \lambda \omega^{-\gamma_0} a_0 \\ 1, & \omega^{\gamma_{n+1}} c_{n+1} \end{pmatrix} L_1(\lambda) L_2(\lambda) \cdots L_n(\lambda),$$  

(17)

The problem of construction of the eigenvectors of the transfer-matrix $t^B(\lambda)$ will be solved as follows. First, we find the eigenvectors of $B_1(\lambda)$. Then the eigenvectors of $B_m(\lambda)$ are obtained recursively as linear combination of the eigenvectors of $B_{m-1}(\lambda)$. After that the eigenvectors for the transfer-matrix (17) are constructed as linear combinations of the eigenvectors of $B_{n+1}(\lambda)$ (the auxiliary model). The multi-variable coefficients of this decomposition admit the separation of variables and can be written as products of single-variable functions, each satisfying a Baxter difference equation which can be solved explicitly.

### 3 Eigenvalues and eigenvectors of $B_m(\lambda)$

#### 3.1 Eigenvalues of $B_m(\lambda)$

We start from the problem of construction of the eigenvectors of $B_m(\lambda)$. According to $[B_m(\lambda), B_m(\mu)] = 0$, we are looking for the eigenvectors of $B_m(\lambda)$ not depending on $\lambda$ with the eigenvalue being a polynomial in $\lambda$. Factorizing this polynomial we get

$$B_m(\lambda) \Psi_{\rho_m} = \lambda r_{m,0} \omega^{-\rho_{m,0}} \prod_{s=1}^{m-1} \left( \lambda + r_{m,s} \omega^{-\rho_{m,s}} \right) \Psi_{\rho_m},$$  

(18)

where $r_{m,s}, s = 0, \ldots, m-1$, is a set of constants and we shall use the phases

$$\rho_m = (\rho_{m,0}, \ldots, \rho_{m,m-1}) \in (\mathbb{Z}_N)^m$$  

(19)

as labels of the eigenvectors. These formulas are valid for all $m = 1, \ldots, n+1$, that is including the chain with boundary $L$-operator (16). As we shall see later, in the latter case not all the phases $\rho_{n+1}$ are possible but only those which satisfy the following restriction related to the boundary spin $\gamma_0$:

$$\hat{\rho}_{n+1} := \sum_{s=0}^{n} \rho_{n+1,s} = \gamma_0 + 1 \mod N.$$  

(20)
Let us define the “averaged” counterpart $O(\lambda^N)$ of a quantum cyclic operator $O(\lambda)$ using averaging procedure \cite{12}

$$O(\lambda^N) = \langle O \rangle(\lambda^N) = \prod_{s \in \mathbb{Z}_N} O(\omega^s \lambda)$$

(21)

and apply this procedure to the entries of the quantum $L$-operator \cite{3}. Denote the result by $L_k(\lambda^N)$

$$L_k(\lambda^N) = \left( \begin{array}{ccc}
1 - \epsilon \lambda^N & -\epsilon \lambda^N (a_k^N - b_k^N) \\
\epsilon \lambda^N (c_k^N - d_k^N) & \epsilon \lambda^N d_k^N / \lambda^N - \epsilon \lambda^N a_k^N c_k^N
\end{array} \right),$$

(22)

where $\epsilon = (-1)^N$, and call it as the “averaged” $L$-operator of the BBS model. In particular, the averaging of \cite{16} gives

$$L_{n+1}(\lambda^N) = \left( \begin{array}{cc}
1 & -\epsilon \lambda^N a_0^N \\
\epsilon \lambda^N a_0^N c_{n+1} & -\epsilon \lambda^N a_0^N c_{n+1}
\end{array} \right).$$

(23)

Accordingly, the averaged monodromy $T_m$ for the $m$-site chain is

$$T_m(\lambda^N) = L_1(\lambda^N) L_2(\lambda^N) \cdots L_m(\lambda^N) = \left( \begin{array}{cc}
A_m(\lambda^N) & B_m(\lambda^N) \\
C_m(\lambda^N) & D_m(\lambda^N)
\end{array} \right),$$

(24)

where the entries are polynomials of $\lambda^N$. By Proposition 1.5 from \cite{12} (see also \cite{13}), these polynomials coincide with averages $\langle A_m \rangle$, $\langle B_m \rangle$, $\langle C_m \rangle$ and $\langle D_m \rangle$ of the entries of \cite{11}. This proposition provides a tool for finding the $N$-th powers of the amplitudes $r_{m,s}$; applying (21) to (18) we obtain

$$B_m(\lambda^N) = (-\epsilon)^m \lambda^N r_{m,0} \prod_{s=1}^{m-1} (\lambda^N - \epsilon r_{m,s}^N).$$

(25)

This relation together with (22) and (24) allows to find $r_{m,s}^N$ in terms of the parameters $a_k^N$, $b_k^N$, $c_k^N$, $d_k^N$ and $\lambda^N$, $k = 1, \ldots, m$. The problem of finding the amplitudes $r_{m,s}$ is reduced to the problem of solving a $(m-1)$-th degree algebraic relation. As shown in \cite{4}, in the case of the homogeneous BBS chain model the problem is reduced to solving a quadratic equation only. The described procedure gives the amplitudes $r_{m,s}$ up to some roots of unity. In fact we can fix these phases arbitrarily because this leads just to relabeling of the eigenvectors. In what follows we suppose that we fixed some solution $\{r_{m,s}\}$ in terms of the parameters $a_k^N$, $b_k^N$, $c_k^N$, $d_k^N$ and $\lambda^N$. Again in the case $m = n + 1$ we can not define the phase of $r_{n+1,0}$ arbitrarily. We fix it by the relation

$$\tilde{r}_{n+1} := \prod_{s=0}^{n} r_{n+1,s} = \omega a_0.$$

(26)

Then the relations (20) and (26) provide correct coefficient $a_0 \omega^{-\tau_0}$ at $\lambda$ (the lowest term) in the eigenvalue of $B_{n+1}(\lambda)$.

### 3.2 One-site eigenvectors for the auxiliary problem

In BBS model a very important role is played by the cyclic function $w_p(\gamma)$ \cite{10} which depends on a $\mathbb{Z}_N$-variable $\gamma$ and on a point $p = (x, y)$ restricted to the Fermat curve $x^N + y^N = 1$. We define $w_p(\gamma)$ by the difference equation

$$\frac{w_p(\gamma)}{w_p(\gamma - 1)} = \frac{y}{1 - \omega^\gamma x}; \quad w_p(0) = 1; \quad \gamma \in \mathbb{Z}_N.$$
The Fermat curve restriction guarantees the cyclic property $w_p(\gamma + N) = w_p(\gamma)$. The function $w_p(\gamma)$ is a root of unity analog of the $q$-gamma function.

It is convenient to change the bases in the spaces $\mathcal{V}_k$. Instead of $|\gamma\rangle_k$, $\gamma \in \mathbb{Z}_N$, we will use the vectors

$$
\psi_{p_k}^{(k)} = \sum_{\gamma \in \mathbb{Z}_N} w_p(\gamma - \rho_k) |\gamma\rangle_k, \quad \rho_k \in \mathbb{Z}_N,
$$

which are eigenvectors of the upper off-diagonal matrix element of the operator $L_k(\lambda)$:

$$
\lambda u_k^{-1}(a_k - b_k v_k) \psi_{p_k}^{(k)} = \lambda r_k \omega^{-\rho_k} \psi_{p_k}^{(k)}.
$$

The coordinates of the Fermat curve points $p_k = (x_k, y_k)$ are defined as follows. Let us fix some value of $\rho$ have to identify $V$.

$$
3.3 \text{ Fermat curve points in the formulas for the eigenvectors of } B_m(\lambda)
$$

The formula for the eigenvectors of $B_m(\lambda)$ is defined in terms of $w_p(\gamma)$ function depending on four types of points on the Fermat curve $x^N + y^N = 1$:

$$
\tilde{p}_m = (\tilde{x}_m, \tilde{y}_m); \quad p_{m,s} = (x_{m,s}, y_{m,s}); \quad \tilde{p}_{m,s} = (\tilde{x}_{m,s}, \tilde{y}_{m,s}); \quad p_{m,s'} = (x_{m,s'}, y_{m,s'}). \quad (32)
$$

The coordinates of these points are expressed in the terms of amplitudes $r_{m,s}$, $m = 1, \ldots, n+1, s = 0, \ldots, m-1$ (defined as some solutions of equations (25), $m = 1, \ldots, n+1$) by

$$
x_{m,s'} = r_{m,s}/r_{m,s'}, \quad x_{m,s} = a_m \varkappa_m r_{m,s}/b_m, \quad \tilde{x}_{m,s} = d_m/(\varkappa_m c_m r_{m,s}), \quad s, s' \geq 1. \quad (33)
$$

In the construction of the eigenvectors of $B_m(\lambda)$ we need the points of type $p_{m,s}$ and $\tilde{p}_{m,s}$, $s \geq 1$, only for $m \leq n$. For all the other types of points $m$ runs up to $n + 1$. 

7
The values of $y_{m,s}^{m,s}$, $y_{m,s}$, $\tilde{y}_{m,s}$ are defined by the condition on $p_{m,s}^{m,s}$, $p_{m,s}$, $\tilde{p}_{m,s}$ to belong to the Fermat curve and the following relations (1 ≤ $l$ ≤ $m - 2$) on phases of $y$-coordinates:

$$
\tilde{r}_{m-1}r_{m,0}r_{m-1} = \frac{\prod_{s \neq l}^{m-2} y_{m-1,s}^{m-1,l} \prod_{l=1}^{m-1} y_{m-1,l}^{m-1,l} \prod_{s=1}^{m-3} y_{m-1,s}^{m-1,s}}{\prod_{s=1}^{m-1} y_{m-1,s}^{m-1,s}} = 1,
$$

(34)

where

$$
\tilde{r}_m = r_{m,0} r_{m,1} \cdots r_{m,m-1}.
$$

(35)

Practically we may define $y_{m,s}^{m,s}$, $y_{m,s}$, $\tilde{y}_{m,s}$ arbitrarily and the coordinates $\tilde{y}_{m,s}$ by (34). Then the points $\tilde{p}_{m,s}$, due to (25), will belong to the Fermat curve automatically [4].

The coordinates of the points $p_{m,0}$ and $\tilde{p}_m$, 1 ≤ $m$ ≤ $n$, are defined by

$$
x_{m,0} r_{m,0} = r_{m-1,0} a_{m,c_m}, \quad y_{m,0} r_{m,0} = x_1 x_2 \cdots x_{m-1} r_m,
$$

(36)

$$
\tilde{x}_m \tilde{r}_m = r_m, \quad \tilde{y}_m \tilde{r}_m = b_m d_m \tilde{r}_{m-1} / x_m.
$$

(37)

We use the same formulas for the case $m = n + 1$. Due to (26) and $r_{n+1} = a_0$, the solution of (37) is $\tilde{x}_{n+1} = \omega^{-1}$, $\tilde{y}_{n+1} = 0$. Correct limiting procedure gives $(w_{p_n}(\rho))^{-1} = \delta_{\rho,0}$. We define the coordinates of the Fermat curve point $p_{n+1,0}$ by

$$
x_{n+1,0} r_{n+1,0} = r_{n,0} a_0 c_{n+1}, \quad y_{n+1,0} r_{n+1,0} = x_1 x_2 \cdots x_n a_0.
$$

(38)

Note that from (36) we had to define $x_{n+1,0}$ by relation $x_{n+1,0} r_{n+1,0} = r_{n,0} a_0 c_{n+1} \omega^{\gamma_{n+1}-\gamma_0}$ but we want to avoid entering the values of boundary spins into the definition of Fermat curve points. Such change of definition of $x_{n+1,0}$ means that instead of $w(x_{n+1,0}, y_{n+1,0})(\rho)$ we have to use

$$
w(x_{n+1,0}, \omega^{\gamma_{n+1}-\gamma_0}, y_{n+1,0})(\rho) = \frac{w_{p_{n+1,0}}(\rho + \gamma_{n+1} - \gamma_0)}{w_{p_{n+1,0}}(\gamma_{n+1} - \gamma_0)}
$$

(39)

with point $p_{n+1,0}$ defined by (38). This formula allows to move the dependence on the values of boundary spins $\gamma_0$ and $\gamma_{n+1}$ to the argument of $w(x_{n+1,0}, y_{n+1,0})(\rho)$.

### 3.4 Recursive formulas for the eigenvectors of $B_m(\lambda)$

Recall from (31) that the vector $\Psi_{\rho_1,0} := \psi_{\rho_1,0}^{(1)} \in \mathcal{V}_1$ is eigenvector for $B_1(\lambda)$ and from (18), (19) that the eigenvectors $\Psi_{\rho_m}$ of $B_m(\lambda)$ were labeled by the vector $\rho_m = (\rho_{m,0}, \ldots, \rho_{m,m-1}) \in (\mathbb{Z}_N)^m$. Let us further define:

$$
\tilde{\rho}_m = \sum_{k=0}^{m-1} \rho_{m,k}; \quad \tilde{\rho}'_m = \sum_{k=1}^{m-1} \rho_{m,k}; \quad \rho'_m = (\rho_{m,1}, \ldots, \rho_{m,m-1}) \in (\mathbb{Z}_N)^{m-1}.
$$

(40)

The vector $\rho_{m,k}^{+k}$ denotes the vector $\rho_m$ in which $\rho_{m,k}$ is replaced by $\rho_{m,k} \pm 1$, i.e.

$$
\rho_{m,k}^{+k} = (\rho_{m,0}, \ldots, \rho_{m,k} \pm 1, \ldots, \rho_{m,m-1}), \quad k = 0, 1, \ldots, m - 1.
$$

The following Theorem proved in [4] gives a recursive procedure to obtain the eigenvectors $\Psi_{\rho_m} \in \mathcal{V}^{(m)}$, 2 ≤ $m$ ≤ $n$, of $B_m(\lambda)$ from the eigenvectors $\Psi_{\rho_{m-1}} \in \mathcal{V}^{(m-1)}$ of $B_{m-1}(\lambda)$ and single site vectors $\psi_{\rho_{m_0}}^{(m)} \in \mathcal{V}_m$ defined by (27). The recursion is starting from the already defined $\Psi_{\rho_{1,0}}$. The Theorem 1 is valid provided $r_m^{N_1} \neq 0$, the polynomials $B_m(\lambda^N)/\lambda^N$, $m = 2, \ldots, n$, have nonzero simple zeros and det $T_n(\epsilon r_m^{N_1}) \neq 0$ (cf. the definition of the $B$-representation in [12]).
Theorem 1 The vector

\[ \Psi_{\rho_m} = \sum_{\rho_{m-1} \in (\mathbb{Z}_N)^{m-1}} Q(\rho_{m-1}, \rho_m | \rho_m) \Psi_{\rho_{m-1}} \otimes \psi^{(m)}_{\rho_m} \]  

(41)

where

\[ Q(\rho_{m-1}, \rho_m | \rho_m) = \frac{\omega(\tilde{\rho}_{m-1} - \rho_{m-1,0})}{w_{\rho_{m-1}}(\rho_{m,0} - \rho_{m-1,0} - 1)w_{\rho_m}(\rho_m - \rho_{m-1})} \times \]
\[ \prod_{l=1}^{m-2} \prod_{k=1}^{m-1} w_{\rho_{m,k}}(\rho_{m-1,l} - \rho_{m,k}) \prod_{l=1}^{m-2} \prod_{(j \neq l)}^{m-1} w_{\rho_{m,j,l}}(\rho_{m-1,l} - \rho_{m,j}) \]

(42)

is eigenvector of \( B_m(\lambda) \):

\[ B_m(\lambda) \Psi_{\rho_m} = \lambda \rho_{m,0} \omega - \rho_{m,0} \prod_{k=1}^{m-1} (\lambda + r_{m,k} \omega - \rho_{m,k}) \Psi_{\rho_m}. \]

(43)

At the \( m - 1 \) zeros \( \lambda_{m,k} \) of the eigenvalue polynomial of \( B_m(\lambda) \)

\[ \lambda_{m,k} = -r_{m,k} \omega \rho_{m,k}, \quad k = 1, \ldots, m - 1, \]

(44)

the operators \( A_m \) and \( D_m \) act as shift operators for the \( k \)-th (and \( 0 \)-th for \( D_m \)) index of \( \Psi_{\rho_m} \):

\[ A_m(\lambda_{m,k}) \Psi_{\rho_m} = \varphi_k(\rho_m) \Psi_{\rho_m}^{+k}, \quad D_m(\lambda_{m,k}) \Psi_{\rho_m} = \varphi_k(\rho_m') \Psi_{\rho_m}^{-k}, \]

(45)

where

\[ \varphi_k(\rho_m') = -\frac{r_{m-1}}{r_m} \frac{\omega \rho_{m-1}}{\rho_{m,k}} F_m(\lambda_{m,k}/\omega) \prod_{s=1}^{m-2} y_{m-1,s}^{m,k}, \]

(46)

\[ \varphi_k(\rho_m') = -\frac{r_{m-1}}{r_m} \frac{\omega \rho_{m-1}}{\rho_{m,k}} \prod_{s=1}^{m-2} y_{m-1,s}^{m,k} F_s(\lambda_{m,k}), \]

(47)

\[ F_s(\lambda) = (b_s + \omega a_s \varphi_s \lambda) (\lambda c_s + d_s / \varphi_s). \]

(48)

The next theorem gives the formula for the eigenvectors of \( B_{n+1}(\lambda) \) in the space \( \mathcal{V}^{(n)} \) (which is restriction of the initial space of states \( \mathcal{V}^{(n+1)} \) to the subspace of fixed value \( \gamma_{n+1} \) of spin at \( (n+1) \)th site) in terms of the eigenvectors of \( B_n(\lambda) \). As was explained before, the components of \( \rho_{n+1} \) are not independent but satisfy (20). So we will label the eigenvectors \( \Psi_{\rho_{n+1}}^B \) of \( B_{n+1}(\lambda) \) by the set \( \rho_{n+1}' = (\rho_{n+1,1}, \ldots, \rho_{n+1,n}) \). Then \( \rho_{n+1,0} = \gamma_0 + 1 - \rho_{n+1}' \).

Theorem 2 The vector

\[ \Psi_{\rho_{n+1}}^B = \sum_{\rho_n \in (\mathbb{Z}_N)^n} Q^B(\rho_n | \rho_{n+1}) \Psi_{\rho_n} \]

(49)

where

\[ Q^B(\rho_n | \rho_{n+1}) = \frac{\omega(\gamma_{n+1} - \rho_{n+1})}{w_{\rho_{n+1,0}}(\gamma_{n+1} - \rho_{n+1,0} - 1)} \times \]
\[ \prod_{l=1}^{n-1} \prod_{k=1}^{n} w_{\rho_{n+1,k}}(\rho_{n+1,l} - \rho_{n+1,k}) \prod_{l=1}^{n-1} \prod_{(j \neq l)}^{n} w_{\rho_{n,j,l}}(\rho_{n,0} - \rho_{n,l}) \]

(50)
is eigenvector of $B_{n+1}(\lambda)$:

$$B_{n+1}(\lambda)\Psi^B_{\rho_{n+1}} = \lambda r_{n+1,0}\omega^{-\rho_{n+1,0}} \prod_{k=1}^{n} \left( \lambda + r_{n+1,k}\omega^{-\rho_{n+1,k}} \right) \Psi^B_{\rho_{n+1}}.$$  \hspace{1cm} (51)

At the $n$ zeros $\lambda_{n+1,k}$ of the eigenvalue polynomial of $B_{n+1}(\lambda)$

$$\lambda_{n+1,k} = -r_{n+1,k}\omega^{-\rho_{n+1,k}}, \quad k = 1, \ldots, n,$$  \hspace{1cm} (52)

the operators $A_{n+1}$ and $D_{n+1}$ act as:

$$A_{n+1} (\lambda_{n+1,k}) \Psi^B_{\rho_{n+1}} = 0, \quad D_{n+1} (\lambda_{n+1,k}) \Psi^B_{\rho_{n+1}} = \tilde{\varphi}(\rho_{n+1}) \Psi^B_{\rho_{n+1}}.$$  \hspace{1cm} (53)

**Proof.** The formula for the eigenvectors $\Psi^B_{\rho_{n+1}}$ of $B_{n+1}(\lambda)$ follows from the formulas (11) and (12) at $m = n+1$ after appropriate limiting procedure: the factor $w_{\rho_{n+1}}(\rho_{n+1} - \rho_{n+1} - 1)$ kills summation over $\rho_{n+1}$ (fixing $\rho_{n+1} = \gamma_0$) and becomes 1, the factor $w_{\rho_{n+1,0}}(\rho_{n+1,0} - \rho_{n,0} - 1)$ due to (59) becomes $w_{\rho_{n+1,0}}(\gamma_{n+1} - \tilde{\rho}_{n+1} - \rho_{n,0})$ up to an inessential constant multiplier $w_{\rho_{n+1,0}}(\gamma_{n+1} - \gamma_0)$. The action formula for $A_{n+1}$ follows from the identity $\lambda a_{\rho,\omega^{-\gamma_0}} A_{n+1}(\lambda) = B_{n+1}(\lambda)$. \hspace{1cm} □

4 **Eigenvalues and eigenvectors of the boundary transfer-matrix** $t^B(\lambda)$

After having determined in Theorem 2 the eigenvectors $\Psi^B_{\rho_{n+1}}$ of the auxiliary system, we are looking for the eigenvectors of $t^B(\lambda)$ from (17) as linear combinations of the eigenvectors $\Psi^B_{\rho_{n+1}}$.

From the expressions for the $L$-operators we can see that the eigenvalue of $t^B(\lambda)$ is

$$t^B(\lambda|\gamma_0, \gamma_{n+1}, \mathbf{E}) = E_0 + E_1\lambda + \cdots + E_n\lambda^n + E_{n+1}\lambda^{n+1},$$  \hspace{1cm} (54)

where $\mathbf{E} = \{E_1, \ldots, E_n\}$ and the values of $E_0$ and $E_{n+1}$ are

$$E_0 = 1, \quad E_{n+1} = \prod_{m=1}^{n} a_m c_m \cdot a_0 c_{n+1} \omega^{\gamma_{n+1} - \gamma_0}.$$  \hspace{1cm} (55)

The possible values of $\mathbf{E}$ can be found from the functional relations. We define the following polynomials depending on unknown set $\mathbf{E}$: $\tau(0)(\lambda) = 0$, $\tau(1)(\lambda) = 1$, $\tau(2)(\lambda) = t^B(\lambda)$ and recursively

$$\tau(j+1)(\lambda) = \tau(2)(\omega^{j-1}\lambda) \tau(j)(\lambda) - \omega^j z(\omega^{j-1}\lambda) \tau(j-1)(\lambda), \quad j = 2, 3, \ldots, N,$$  \hspace{1cm} (56)

where $z(\lambda) = \prod_{m=1}^{n+1} F_m(\lambda/\omega)$. Then the "truncation" identity

$$\tau(N+1)(\lambda) - \omega^N z(\lambda) \tau(N-1)(\omega\lambda) = A_{n+1}(\lambda^N) + D_{n+1}(\lambda^N)$$  \hspace{1cm} (57)

defines possible values of $\mathbf{E}$. Note, the polynomial $A_{n+1}(\lambda^N) + D_{n+1}(\lambda^N)$ corresponds to $\alpha_q + \bar{\alpha}_q$ in [5].
As was mentioned in [5], in the case of parameters considered in the present paper we have \( z(\lambda) = 0 \) (due to \( F_{n+1}(\lambda) = 0 \)) and therefore all the set of functional relations reduces to one relation (we use the averaging (21)):

\[
\langle t^B \rangle (\lambda^N) = \mathcal{A}_{n+1}(\lambda^N) + \mathcal{D}_{n+1}(\lambda^N) \tag{58}
\]

or, equivalently, \( \langle A_{n+1} + D_{n+1} \rangle = \langle A_{n+1} \rangle + \langle D_{n+1} \rangle \). At this point it is natural to conjecture the formula for the eigenvalues of transfer-matrix [5]:

\[
t^B(\lambda|\sigma_{n+1}) = \prod_{j=1}^{n+1} \left( \frac{\lambda}{s_j \omega^{-\sigma_j}} + 1 \right), \tag{59}
\]

where \( \sigma_{n+1} = (\sigma_1, \sigma_2, \ldots, \sigma_{n+1}) \in (\mathbb{Z}_N)^{n+1} \) and

\[
\sigma_1 + \sigma_2 + \cdots + \sigma_{n+1} = \gamma_{n+1} - \gamma_0 \mod N, \tag{60}
\]

The numbers \( s_j, j = 1, 2, \ldots, n + 1 \), satisfy two relations:

\[
\prod_{j=1}^{n+1} s_j^{-1} = \prod_{m=1}^{n} a_m a_0 a_{n+1}, \tag{61}
\]

\[
\prod_{j=1}^{n+1} \left( 1 - (-1)^N \frac{\lambda^N}{s_j^N} \right) = \mathcal{A}_{n+1}(\lambda^N) + \mathcal{D}_{n+1}(\lambda^N). \tag{62}
\]

The relation (62) follows from functional relation (58) and averaged (59). The relation (62) means that \((-1)^N s_j^N\) are roots of polynomial \( \mathcal{A}_{n+1}(\lambda^N) + \mathcal{D}_{n+1}(\lambda^N) \) with respect to \( \lambda^N \). Note, the relations (60) and (61) provide correct value of \(E_{n+1}\) in (55).

In the paper [5], Baxter conjectured that to each set \( \sigma_{n+1} \) satisfying (60) there corresponds the eigenvector \( \Phi_{\sigma_{n+1}} \) of transfer-matrix \( t^B(\lambda) \) with eigenvalue (59). The method of functional relations gives in principle the eigenvalues without information on their multiplicities (0, 1 or more). The method of separation of variables developed in [4] for BBS model gives the formulas for the eigenvectors if the eigenvalues are provided. In particular, more precise information on the multiplicities of the eigenvalues can be obtained.

According to [4], we are looking for \( \Phi_{\sigma_{n+1}} \) to be of the form

\[
\Phi_{\sigma_{n+1}} = \sum_{\rho'_{n+1}} Q^B(\rho'_{n+1}|\sigma_{n+1}) \Psi^B_{\rho'_{n+1}}, \tag{63}
\]

where

\[
Q^B(\rho'_{n+1}|\sigma_{n+1}) = \frac{\prod_{k=1}^{n} \tilde{q}_k(\rho_{n+1,k})}{\prod_{(m,m')=1}^{n} \omega_{n+1,m'}(\rho_{n+1,m} - \rho_{n+1,m'})}
\]

and the functions of separated variables \( \tilde{q}_k(\rho_{n+1,k}) \) satisfy the Baxter type difference equations, \( k = 1, 2, \ldots, n \):

\[
t^B(\lambda_{n+1,k}|\sigma_{n+1}) \tilde{q}_k(\rho_{n+1,k}) = \Delta^+_k(\lambda_k) \tilde{q}_k(\rho_{n+1,k} + 1) + \Delta^-_k(\omega \lambda_{n+1,k}) \tilde{q}_k(\rho_{n+1,k} - 1) \tag{64}
\]

with

\[
\Delta^+_k(\lambda) = \chi_k^{-1}(\lambda/\omega)^{-n} \prod_{m=1}^{n} F_m(\lambda/\omega); \quad \Delta^-_k(\lambda) = \chi_k(\lambda/\omega)^n F_{n+1}(\lambda/\omega); \tag{65}
\]
\[ \chi_k = \frac{r_{n+1,0} \tilde{r}_n}{a_0 \tilde{r}_n} \left( \prod_{m=1, m \neq k}^{n} \frac{y_{n+1,k}^{n+1,m}}{y_{n+1,m}} \right) \prod_{m=1}^{n-1} y_{n,m}^{n+1,k}. \]  

(66)

Since in the case of parameters considered in this paper we have \( F_{n+1}(\lambda) = 0 \) we get \( \Delta_k(\lambda) = 0 \), and equations (64) can be solved explicitly in terms of functions \( w_p(\gamma) \). The answer for \( Q^B(\rho_{n+1}' | \sigma_{n+1}) \) is

\[ Q^B(\rho_{n+1}' | \sigma_{n+1}) = \frac{\prod_{k=1}^{n} \prod_{j=1}^{n+1} w_{j,k}^{BB}(\sigma_j - \rho_{n+1,k})}{\prod_{m,m' = 1}^{n} w_{p_{m,k}^{BB}}(\rho_{n+1,m} - \rho_{n+1,m'})} \cdot \prod_{k=1}^{n} \prod_{m=1}^{n} w_{p_{m,k}^{BB}}(-\rho_{n+1,k} - 1), \]  

(67)

where the Fermat curve points \( p_{j,k}^{BB}, p_{m,k}^{B} \) and \( \tilde{p}_{m,k}^{B} \) are defined by

\[ x_{j,k}^{BB} = r_{n+1,k}/s_j, \quad x_{m,k}^{B} = b_m/\left( a_mz_m r_{n+1,k} \right), \quad x_{m,k}^{B} = r_{n+1,k} c_m z_m / d_m \]

and conditions on the discrete phases of \( y_{j,k}, y_{m,k}, \tilde{y}_{m,k} \):

\[ \prod_{m=1}^{n} \omega a_m d_m y_{m,k}^{BB} \tilde{y}_{m,k}^{BB} = \chi_k \prod_{j=1}^{n} y_{j,k}^{BB}, \quad k = 1, 2, \ldots, n. \]

The validity of these relations up to Nth root of unity is provided by (62). Straightforward calculation gives

\[ t^B(\lambda) \Phi_{\sigma_{n+1}} = t^B(\lambda | \sigma_{n+1}) \Phi_{\sigma_{n+1}}, \]

where the eigenvectors \( \Phi_{\sigma_{n+1}} \) and corresponding eigenvalues are defined by (63), (67), (59).

### 5 Ising-like \( \mathbb{Z}_N \)-model with fixed boundary spins

In this section we fix special values of parameters \( b_k = d_k = 0 \) of \( L \)-operators (6):

\[ L_k(\lambda) = \begin{pmatrix} 1 + \lambda z_k v_k, & \lambda u_k^{-1} a_k \\ u_k c_k, & \lambda a_k c_k \end{pmatrix}, \quad k = 1, 2, \ldots, n, \]

(68)

and consider the transfer-matrix (17) of the BBS model with fixed boundary spins. The coefficients of this transfer-matrix expansion in \( \lambda \) give a set of commuting operators. In the previous section we gave the formula for the common eigenvectors of this set. The coefficient at \( \lambda \) gives Hamiltonian of Ising-like \( \mathbb{Z}_N \) quantum chain model with fixed boundary spins [6, 11]

\[ H_1 = \sum_{k=1}^{n} z_k v_k + \sum_{k=0}^{n+1} a_k c_{k+1} u_k^{-1} u_{k+1}, \]

(69)

where \( u_0 = \omega^{\gamma_0}, u_{n+1} = \omega^{\gamma n+1} \) are fixed boundary spins. The corresponding eigenvalue

\[ E_1 = \sum_{j=1}^{n+1} s_j^{-1} \omega^{\sigma_j} \]

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follows from (59), the set $\sigma_{n+1}$ satisfy (60), and the amplitudes $s_j$ have to be found from (61) and (62). Using the explicit expressions for averaged $L$-operators (22) with $b_k = d_k = 0$ and (23) we can rewrite right-hand side of (62) as

$$A_{n+1}(\lambda^N) + D_{n+1}(\lambda^N) = \text{tr} \, T_{n+1}(\lambda^N) = \text{tr} \, \tilde{L}_1(\lambda^N) \cdots \tilde{L}_{n+1}(\lambda^N),$$

where

$$\tilde{L}_k(\lambda^N) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_k^{-N} & 0 \\ 0 & 0 & c_{k+1}^{-N} \end{pmatrix} L_k(\lambda^N) \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_k^{N} & 0 \\ 0 & 0 & c_{k+1}^{N} \end{pmatrix} = \begin{pmatrix} 1 - \epsilon \lambda^N \beta_{2k}^2 & -\epsilon \lambda^N \beta_{2k+1}^2 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

with $\epsilon = (-1)^N$, $\beta_{2k+1} = (a_k c_{k+1})^{N/2}$, $\beta_{2k} = \beta_k^{N/2}$. Finally we get

$$\text{tr} \, T_{n+1}(\lambda^N) = (1, -\epsilon \lambda^N \beta_1^2) \left\{ \prod_{k=1}^{n} \begin{pmatrix} 1 & -\epsilon \lambda^N \beta_{2k}^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\epsilon \lambda^N \beta_{2k+1}^2 \\ 1 & 0 \end{pmatrix} \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where the multipliers in the product over increasing $k$ are ordered from the left to the right. Rewriting it as determinant of $(2n+2) \times (2n+2)$ tri-diagonal matrix we get $s_1^N, \ldots, s_{n+1}^N$ from the eigenvalue problem for $(2n+2) \times (2n+2)$ bi-diagonal matrix

$$B = \begin{pmatrix} 0 & \beta_1 & 0 & \ldots & 0 & 0 \\ \beta_1 & 0 & \beta_2 & \ldots & 0 & 0 \\ 0 & \beta_2 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \beta_{2n+1} \\ 0 & 0 & 0 & \ldots & \beta_{2n+1} & 0 \end{pmatrix},$$

which has $2n+2$ eigenvalues $\pm s_j^{-N/2}$, $j = 1, 2, \ldots, n+1$. The method of finding eigenvalues of Hamiltonian (69) in terms of eigenvalues of $B$ was given in [11].

Another model with $A_n(\lambda)$ as generation function of the Hamiltonians has similar boundary conditions and was considered in [14]. A wide class of models with integrable boundary conditions can be obtained in the reflection equation approach with the use of the cyclic $L$-operators [6]. So it is interesting to construct the eigenvectors in these models. In particular case of the relativistic Toda chain at root of unity, the formulas for the eigenvectors were obtained in [15].

**Acknowledgements**

The authors are thankful to the organizers of the International Workshop “Classical and Quantum Integrable Systems” for their kind hospitality and excellent workshop. This work was partially supported by the grant INTAS-05-1000008-7865 and the Ukrainian State Foundation for Fundamental Research. The work of V.N.S. was also partially supported by the SCOPES-project IB7320-110848 of Swiss NSF.
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