A-geometrical approach to topological insulators with edge dislocations

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Abstract. A study of the propagation of electrons with varying spinor orientability is conducted using the coordinate transformation method. Topological insulators are characterized by an odd number of changes in the orientability of the spinors in the Brillouin zone. For defects the change in spinor orientability takes place for closed orbits in real space. Both the cases are characterized by nontrivial spin connections. Using this method we derive the form of the spin connections for topological defects in three-dimensional (3D) topological insulators. On the surface of a topological insulator, the presence of an edge dislocation gives rise to a spin connection controlled by torsion. We find that electrons propagate along 2D regions and confined circular contours. We compute for the edge dislocations the tunneling density of states. The edge dislocations violate parity symmetry, resulting in a current measured by the in-plane component of the spin on the surface. For a continuum distribution of edge dislocations with the Burger vector $B^{(2)}$ in the $y$-direction, we show that electron backscattering is absent for electrons with zero momentum in the $y$-direction.
1. Introduction

The propagation of electrons in solids is characterized by the topological properties of the electronic band spinors. Topological insulators [1–6, 8–14, 28] can be identified by an odd number of changes in the orientability [8] of the spinors in the Brillouin zone. As a result, nontrivial spin connections with a non-zero curvature characterized by the Chern numbers can be identified. In time-reversal invariant systems, one finds that for Kramer’s states the time-reversal operator \( T \) obeys \( T^2 = -1 \) and thus the second Chern number for four-dimensional (4D) space is given by \((-1)^v = -1\), where \( v \) is an odd number of orientability changes [24].

Real materials are imperfect and contain topological defects such as dislocations [15, 18, 35], disclinations [19, 20] and gauge fields induced by strain [21, 22]; therefore, a natural question is to formulate the physics of topological insulators in the presence of such defects [8]. These topological defects can be analyzed using the coordinate transformation method given in [26] which modifies the Hamiltonian for a topological insulator with a defect by the metric tensor and the spin connection [30–34].

The effect of strain field dislocations and disclinations has an important role in material science and can be studied using scanning tunneling microscopy (STM) and transmission electron spectroscopy (TEM). Therefore, we expect that the chiral excitations [29] will be sensitive to such defects.
In this paper, we will introduce the tangent space approach used in differential geometry [24, 33, 34] to study the propagation of electrons for a space-dependent coordinate [26]. We find that the continuum representation of the edge dislocation [26] generates a spin connection [30–32] which is controlled by the Burger vector.

Using this formulation, we obtain the form of the topological insulator in three dimensions which simplifies for the surface Hamiltonian (on the boundary). For the surface Hamiltonian, we find that the electronic excitations are confined to a 2D region and to a set of circular contours of radius $R_g(n)$.

The structure of this paper is as follows. In section 2, we introduce the geometrical method. In section 2.1, we present the geometrical method for the edge dislocations and strain fields. In section 2.2, we consider the effects of the strain fields on the 3D topological insulator (TI). The chiral model for the boundary surface is presented in section 3.1. Section 3.2 is devoted to the derivation of the metric tensor and spin connection for an edge dislocation [26]. In section 3.3, we identify the stable solutions. Section 3.4 is devoted to the stable 2D solutions $n = 0$ and section 3.5 is devoted to the stable solution for circular contours $n = \pm 1$. Section 4.1 is devoted to the computation of the tunneling density of states. In section 4.2, we present the results for the 2D region $n = 0$. Section 4.3 is devoted to a large number of dislocations. In section 4.4, we compute the tunneling density of states for the circular contours $n = \pm 1$. In section 5, we compute the current which is given by the in-plane spin component. In section 5.1, we show that this current is zero for a TI. In section 5.2, we show that in the presence of an edge dislocation the parity symmetry is violated, and a current, representing the in-plane spin component, is generated. In section 5.3, we consider the backscattering effect due to many dislocations. Section 6 presents the conclusions.

2. The geometrical method for dislocations and strain fields

2.1. General considerations

A perfect crystal is described by the lattice coordinates $\vec{r} = [x, y, z]$. For a crystal with a deformation, the coordinates $\vec{r}$ are replaced by $\vec{r} \rightarrow \vec{R} = \vec{r} + \vec{u} \equiv [X^1(\vec{r}), X^2(\vec{r}), X^3(\vec{r})]$, where $\vec{u}(\vec{r})$ is the local lattice deformation and $X^a, a = 1, 2, 3$, is the local coordinate that changes when we move from one point to another.

In a deformed crystal, we introduced a set of local vectors $e_a$ that are orthogonal to each other $(e_b, e_a) = \langle e^b | e_a \rangle = \delta^b_a$ and local coordinates $X^a, a = 1, 2, 3$. The unit vector $e_a$ can be represented in terms of a Cartesian fixed frame space with the coordinate basis $\partial_\mu, \mu = x, y, z$. In the fixed Cartesian frame, the coordinates are given by $x^\mu$. Using the Cartesian basis $\partial_\mu$ we expand the deformed medium in terms of the local tangent vector $e_a$: $e_a = e^\mu_a \partial_\mu$ (for the particular case when vectors $e_a$ are given by $e_a = \partial_a$, the transformation between the two basis is $e^\mu_a = \delta^\mu_a$). Any vector $\vec{X}$ (in the deformed space) can be represented in terms of the unit vectors $e_a$ or the $\partial_\mu$ (the tangent vectors in the Cartesian fixed coordinates space). The vector $\vec{X}$ can be represented in two different ways, $\vec{X} = X^a e_a = X^a \partial_\mu$ (when an index appears twice it is understood as a summation, $X^a e_a \equiv \sum_{a=1,2,3} X^a e_a$). The dual vector $e^a$ is a one form and can be expanded in terms of the one forms $dx^\mu$. We have $e^a = e^a_\mu dx^\mu$, where $e^a_\mu$ represents the matrix transformation $e^a = (\partial_\mu X^a) dx^\mu$. The scalar product of the components $e^b_\mu e^a_\nu = g_{\mu,\nu}$, $e^a_\mu e^b_\nu = \delta_{a,b}$ defines the metric tensors $g_{\mu,\nu}$ (in the Cartesian frame) and $\delta_{a,b}$ in the local medium frame.
2.2. Application to the topological insulators in three dimensions

The 3D electronic TI bands for Bi$_2$Se$_3$ and Bi$_2$Te$_3$ [14] can be represented using four projected states [16], (orbital = 1, 2) $\otimes$ |spin = $\uparrow, \downarrow$) (the Pauli matrix $\tau$ describes the orbital states and the Pauli matrix $\sigma$ describes the spin). The effective $h^{3D}$ Hamiltonian in the first quantized form is given by

$$H^{3D} = \hbar v_0 [k_x(\sigma_1 \otimes \tau_1) - k_z(\sigma_2 \otimes \tau_1) + \epsilon k_z(\sigma_3 \otimes \tau_1) + M(\tilde{k})(I \otimes \tau_3)].$$

The parameter $M(\tilde{k})$ determines whether the insulator is trivial or topological. For Bi$_2$Se$_3$ and Bi$_2$Te$_3$, the gap is inverted, namely $M(\tilde{k}) = -M_0 + B_1k_z^2 + B_2k_z^2$ with $M_0 > 0$, $B_1 > 0$, $B_2 > 0$, and therefore topological [11, 12, 16].

Using the metric tensor $g_{\mu,\nu}$ given by the coordinate transformation (the transformation between the two sets of coordinates—one without the dislocation and the other with the dislocation) $e_\mu^a e_\nu^b = g_{\mu,\nu}$ defines the Jacobian $\sqrt{G}$, where $G = \det[g_{\mu,\nu}]$. We find that the derivative for a spinor component $\Psi^{(\alpha)}(\vec{r})$, $\alpha = [1 = 1 \uparrow; 2 = 1 \downarrow; 3 = 2 \uparrow; 4 = 2 \downarrow]$, is replaced by the covariant derivative [31]:

$$\nabla_\mu \Psi^{(\alpha)}(\vec{r}) = \partial_\mu \Psi^{(\alpha)}(\vec{r}) + \frac{i}{8} \omega^{(a,b)}_{\mu} [\hat{\Gamma}_a, \hat{\Gamma}_b]_{\beta} \Psi^{(\beta)}(\vec{r}),$$

where $\hat{\Gamma}_a$, $a = 1, 2, 3, 4, 5$, are the matrices: $\hat{\Gamma}^1 = -\hat{\Gamma}^2 \equiv -(\sigma_2 \otimes \tau_1)$; $\hat{\Gamma}^2 = \Gamma^3 \equiv (\sigma_1 \otimes \tau_1)$; $\hat{\Gamma}^3 = \Gamma^3 \equiv (\sigma_3 \otimes \tau_1)$; $\hat{\Gamma}^4 = \Gamma^4 \equiv (I \otimes \tau_2)$; $\hat{\Gamma}^5 = \Gamma^5 \equiv (I \otimes \tau_3)$.

The spin connection $\omega^{a,b}_\mu$ determines the covariant derivative that is given in terms of the tangent vectors $e_\mu^a$: $e_\mu^a = \partial_\mu X^a(r)$; $a = 1, 2, 3$; $\mu = x, y, z$ (see appendix A).

$$\omega^{a,b}_\mu \equiv \frac{1}{2} e_v^{v,a} (\partial_\mu e_v^b - \partial_v e_\mu^b) - \frac{1}{2} e_v^{v,b} (\partial_\mu e_v^a - \partial_v e_\mu^a) - \frac{1}{2} e_v^{v,a} e_\sigma^{a,b} (\partial_\mu e_\sigma^c - \partial_\sigma e_\mu^c) e_\mu^c. \quad (3)$$

We not the asymmetry between $e_v^{v,a}$ and $e_a^{v,a}$: $e_v^{v,a} \equiv g_{v,v} e_\lambda^a$ and $e_a^{v,a} \equiv \delta_{a,b} e_\lambda^b$. As a result, the Hamiltonian in equation (1) in the second quantized form is replaced by

$$H^{(3D)} = \hbar v_0 \int d^3r \sqrt{G}[\hat{\Psi}^\dagger(\vec{r}) e_\mu^a \hat{\Gamma}_a (\vec{r}) (-i\nabla_\mu) - E_\xi (I \otimes I) + \hat{\Gamma}^5 (-M_0)] \Psi(\vec{r})$$

$$+ B_1 g^{\mu,\nu} (\nabla_\mu \Psi_1^\dagger(\vec{r}) \nabla_\nu \Psi_1(\vec{r})) - B_1 g^{\mu,\nu} (\nabla_\mu \Psi_2^\dagger(\vec{r}) \nabla_\nu \Psi_2(\vec{r}))],$$

where $e_\mu^a = \sum_a e_\mu^a \hat{\Gamma}_a \equiv \hat{\Gamma}_a (\vec{r})$, $[\hat{\Gamma}_\mu (\vec{r}) \hat{\Gamma}_v (\vec{r}) + \hat{\Gamma}_v (\vec{r}) \hat{\Gamma}_\mu (\vec{r})] = 2 g^{\mu,\nu} (\vec{r})$, $\det [g^{\mu,\nu} (\vec{r})] \equiv G$ and $\nabla_\mu$ is the covariant derivative given in terms of the spin connection given in equation (2): $\nabla_\mu \Psi^{(\alpha)}(\vec{r}) = \partial_\mu \Psi^{(\alpha)}(\vec{r}) + \frac{i}{8} \omega^{(a,b)}_{\mu} [\hat{\Gamma}_a, \hat{\Gamma}_b]_{\beta} \Psi^{(\beta)}(\vec{r})$.

2.3. The mechanical strain effect on $H^{(3D)}$

From the work [17], we learn that the effect of the strained field on Bi$_2$Se$_3$ is different from that on Bi$_2$Te$_3$. In Bi$_2$Se$_3$, the compressive strain decreases the Coulombic gap while increasing the inverted gap strength induced by the spin–orbit interaction. We will use the result in equation (4) to analyze the effect of strain. The strain field $\epsilon_{i,j}$ (symmetric in $i$, $j$) is related to the stress field $\sigma_{i,j}$ and the elastic stiffness Lame constants $\lambda$ and $\mu$: $\sigma_{i,j} = \lambda \delta_{i,j} \epsilon_{k,k} + 2 \mu \epsilon_{i,j}$. By applying a constant stress $\sigma_{i,j}$, one can determine the value of the constant strain field $\epsilon_{i,j}$ which is related to the tangent vectors $e^i = \delta_{i,j} + \epsilon_{i,j}$. In the present case, the spin connection and the Christofel tensor vanish. The metric tensor $g_{i,j}$ is given by: $g_{i,j} = \delta_{i,j} + 2 \epsilon_{i,j}$. Using this formulation, we
can investigate the effect of stress on Bi$_2$Se$_3$ at the $\Gamma$ point $\vec{k} = 0$. The TI Hamiltonian given in equation (4), $M(\vec{k}) = -M_0 + B_1k_x^2 + B_2(k_x^2 + k_y^2)$ with the inverted case $M_0 > 0$ [12]. The Hamiltonian in equation (4) is replaced by

$$H^{(3D\text{-strain})} = \hbar v_0 \int d^3r \sqrt{G}[\Psi^\dagger(\vec{r})[\hat{\Gamma}^a (\delta_{\mu,a} + e_{\mu,a})(-i\partial_\mu)] + \hat{\Gamma}^a (-M_0)$$

$$+ B(1 - 2e_{\mu,v})\partial_\mu \hat{\Gamma}^a \partial_v] \Psi(\vec{r})$$

$$\approx \hbar v_0 \int d^3r \sqrt{G}[\Psi^\dagger(\vec{r})[\hat{\Gamma}^a [\delta_{\mu,a}(1 + \langle \epsilon \rangle)(-i\partial_\mu)]$$

$$+ \hat{\Gamma}^4 (-M_0) + B(1 - 2\langle \epsilon \rangle)\partial_\mu \hat{\Gamma}^4 \partial_v] \Psi(\vec{r}).$$

(5)

In equation (5), we have used the average strain field $\langle \epsilon \rangle$, $\langle \epsilon \rangle \equiv \frac{\epsilon_{1,3} + \epsilon_{2,2} + \epsilon_{3,3}}{3}$. We replace the spinor field $\Psi(\vec{r})$ by $\Psi(\vec{r})\sqrt{(1 + \langle \epsilon \rangle)} \equiv \hat{\Psi}(\vec{r})$. As a result, we obtain

$$H^{(3D\text{-strain})} \approx \hbar v_0 \int d^3r \sqrt{G} \left[ \hat{\Psi}(\vec{r}) \hat{\Gamma}^\mu (-i\partial_\mu) + \hat{\Gamma}^4 (-M_0) + B \frac{1 - 2\langle \epsilon \rangle}{1 + \langle \epsilon \rangle} \partial_\mu \hat{\Gamma}^4 \partial_v \right] \hat{\Psi}(\vec{r}).$$

(6)

For the compressive case, $\langle \epsilon \rangle$ is negative, $\langle \epsilon \rangle \equiv -\langle |\epsilon| \rangle$. As a result, we observe that the inverted gap is enhanced: $\frac{|M_0|}{1 + |\epsilon|} > \frac{|M_0|}{1 - |\epsilon|}$. In the same way, we can show that the Coulomb interaction is reduced: we introduce the Hubbard Stratonovici field $a_0$ to describe the Coulomb interactions.

$$H^{\text{coul}} = \int d^3r \sqrt{G} \left[ I (-e) \cdot a_0 \Psi^\dagger(\vec{r}) \Psi(\vec{r}) + \frac{1 - 2e_{\mu,v}}{2} a_0 \partial_\mu \partial_v a_0 \right]$$

$$\approx \int d^3r \sqrt{G} \left[ I (-e) \cdot a_0 \Psi^\dagger(\vec{r}) \Psi(\vec{r}) + \frac{1 - 2\langle \epsilon \rangle}{2} a_0 \partial_\mu \partial_v a_0 \right].$$

(7)

Next we rescale $a_0 = \frac{A_0}{\sqrt{1 - 2\langle \epsilon \rangle}}$ and obtain

$$H^{\text{coul}} \approx \int d^3r \sqrt{G} \left[ I \frac{(-e)}{\sqrt{1 - 2\langle \epsilon \rangle}} A_0 \Psi^\dagger(\vec{r}) \Psi(\vec{r}) + A_0 \partial_\mu \partial_v A_0 \right].$$

(8)

We observe that for the compressive case the effective charge $e_{\text{eff}} \equiv \frac{(-e)}{\sqrt{1 - 2\langle \epsilon \rangle}} = \frac{(-e)}{\sqrt{1 + 2\langle \epsilon \rangle}}$ is reduced and therefore the Coulomb gap decreases, while at the same time the inverted gap increases, $\frac{|M_0|}{1 - |\epsilon|} > |M_0|$, in qualitative agreement with [17].

3. The chiral metal with an edge dislocation

3.1. Description of the chiral model

The low-energy Hamiltonian for the bulk 3D TI in the Bi$_2$Se$_3$ family was shown to behave on the boundary surface (the $x$, $y$-plane) as a 2D chiral metal [7].

$$H = \int d^2r \Psi^\dagger(\vec{r})[h^{\text{TI}} - \mu] \Psi(\vec{r})] \equiv \hbar v_F \int d^2r \Psi^\dagger(\vec{r})[i\sigma^1 \partial_y - i\sigma^2 \partial_x - \mu] \Psi(\vec{r}).$$

(9)

$h^{\text{TI}} = \hbar v_F[i\sigma^1 \partial_y - i\sigma^2 \partial_x]$ is the chiral Dirac Hamiltonian in the first quantized language. $v_F \approx 5 \times 10^5 \text{ m s}^{-1}$ is the Fermi velocity, $\sigma$ is the Pauli matrix describing the electron spin and $\mu$ is the chemical potential measured relative to the Dirac $\Gamma$ point. The Hamiltonian for the 2D surface $L \times L$ describes well excitations smaller than the bulk gap of the 3D TI at 0.3 eV. Moving...
away from the Γ point, the Fermi velocity becomes momentum dependent; therefore, we will introduce a momentum cut-off Λ to restrict the validity of the Dirac model. The chiral Dirac model in the Bloch representation takes the form \( h = \hbar v_F (\mathbf{K} \times \hat{\sigma}) \cdot \hat{\tau} \equiv \hbar v_F (-\sigma^1 k_y + \sigma^2 k_x) \). The eigen-spinors for this Hamiltonian are \( |u(\mathbf{K})\rangle = |u_1(\mathbf{K})\rangle, |u_{-1}(\mathbf{K})\rangle\rangle^{\top} = |\mathbf{K}\rangle \otimes [1, \mathbf{i} e^{i\chi(k_x, k_y)}]^\top \), where \( \chi(k_x, k_y) = \tan^{-1}(\frac{\xi}{\eta}) \) is the spinor phase and \( \epsilon = \hbar v_F \sqrt{k_x^2 + k_y^2} \) is the eigenvalue for particles. For holes we have the eigenvalue \( \epsilon = -\hbar v_F \sqrt{k_x^2 + k_y^2} \) and eigenvectors \( |v(\mathbf{K})\rangle = [\mathbf{v}_1(\mathbf{K})], [\mathbf{v}_{-1}(\mathbf{K})]\rangle^{\top} = |\mathbf{K}\rangle \otimes [-1, \mathbf{i} e^{i\chi(k_x, k_y)}]^\top \). The chirality operator is defined in terms of the chiral phase \( \chi(k_x, k_y) \):

\[
(\hat{\sigma} \times \frac{\mathbf{K}}{|\mathbf{K}|}) \cdot \hat{\tau} \equiv \sin[\chi(k_x, k_y)]\sigma^1 - \cos[\chi(k_x, k_y)]\sigma^2.
\] (10)

The chirality operator takes the eigenvalue—(counter-clockwise) for particles \( \sin(\chi(k_x, k_y))\sigma^1 - \cos(\chi(k_x, k_y))\sigma^2 \) \( |\mathbf{K}\rangle \otimes [1, \mathbf{i} e^{i\chi(k_x, k_y)}]^\top = -|\mathbf{K}\rangle \otimes [1, \mathbf{i} e^{i\chi(k_x, k_y)}]^\top \) and + (clockwise) for holes \( \sin(\chi(k_x, k_y))\sigma^1 - \cos(\chi(k_x, k_y))\sigma^2 \) \( |\mathbf{K}\rangle \otimes [-1, \mathbf{i} e^{i\chi(k_x, k_y)}]^\top = |\mathbf{K}\rangle \otimes [-1, \mathbf{i} e^{i\chi(k_x, k_y)}]^\top \).

### 3.2. The effect of edge dislocation on a two-dimensional chiral surface Hamiltonian

We use the notation \( x^a, \mu = x, y, \) and \( X^a, a = 1, 2, \) to describe the media with dislocations. For an edge dislocation in the \( x \)-direction, the Burger vector \( B^{(2)} \) is in the \( y \)-direction. The value of the Burger vector \( B^{(2)} \) is given by the shortest translation lattice vector in the \( y \)-direction. (For the TI Bi\(_2\)Se\(_3\), the length of the vector \( B^{(2)} \) is 5 times the interatomic distance.) Following [26], we introduce the coordinate transformation for an edge dislocation: \( \mathbf{r} = (x, y) \rightarrow \{X(\mathbf{r}) = x, Y(\mathbf{r}) = y + B^{(2)} \tan^{-1}(\frac{x}{\eta})\} \) with the core of the dislocation centered at \( \mathbf{r} = (0, 0) \). The matrix element fields \( e^a_\mu \) for the edge dislocation are given by

\[
e^a_\mu = \partial_\mu X^a(\mathbf{r}), \quad a = 1, 2, \quad \mu = x, y.
\] (11)

We express the Burger vector in terms of the partial derivatives with respect to the coordinates \( a = 1, 2 \) in the dislocation frame and \( \mu = x, y \) for the fixed Cartesian frame [26]:

\[
\partial_x e^2_\mu - \partial_y e^1_\mu = B^{(2)} \delta^2(\mathbf{r}).
\] (12)

Using the Stokes theorem, we replace the line integral \( \oint \int d^\mu \mathbf{e}^2_\mu(\mathbf{r}) \) by the surface integral \( \int \int d^\mu d^\nu [\partial_\mu e^2_\nu - \partial_\nu e^2_\mu] \). For a system with zero curvature and non-zero torsion \( T^{(2)}_{\mu,\nu} \), we find that the surface torsion tensor integral \( \int \int d^\mu d^\nu T^{(2)}_{\mu,\nu} \) is equal to \( \int \int d^\mu d^\nu [\partial_\mu e^2_\nu - \partial_\nu e^2_\mu] \), and therefore both integrals are equal to the Burger vector:

\[
\oint \int d^\mu \mathbf{e}^2_\mu(\mathbf{r}) = \int \int d^\mu d^\nu [\partial_\mu e^2_\nu - \partial_\nu e^2_\mu] = B^{(2)},
\] (13)

\[
\int \int d^\mu d^\nu T^{(2)}_{\mu,\nu} = \int \int d^\mu d^\nu [\partial_\mu e^2_\nu - \partial_\nu e^2_\mu] = B^{(2)}.
\]
where $dx^\mu \, dy^\nu$ represents the surface element. The tangent components $e_\mu^a$ can be expressed in terms of the Burger vector density $B^{(2)}/\delta^2(\vec{r})$ [26]:

$$e_x^2 = \left( \frac{B^{(2)}}{2\pi} \right) \frac{y}{(x^2 + y^2)}, \quad e_y^2 = 1 - \left( \frac{B^{(2)}}{2\pi} \right) \frac{x}{(x^2 + y^2)},$$

$$e_x^1 = 1, \quad e_y^1 = 0.$$  \hspace{1cm} (14)

Using the tangent components, we obtain the metric tensor $g_{\mu,\nu}$.

$$e_\mu^a e_\nu^b \equiv e_\mu^1 e_\nu^1 + e_\mu^2 e_\nu^2 = g_{\mu,\nu}(\vec{r}), \quad e_\mu^a e_\nu^b \equiv e_\mu^a e_\nu^b + e_\mu^b e_\nu^a = \delta_{a,b}.$$  \hspace{1cm} (15)

The inverse of the metric tensor $g_{\mu,\nu}(\vec{r})$ is the tensor $g^{\nu,\mu}(\vec{r})$ defined through the equation $g_{\mu,\nu}(\vec{r}) g^{\nu,\mu}(\vec{r}) = \delta_{\mu,\nu}$. Using the tangent vectors, we find to first order in the Burger vector (the restriction to first order is justified since the Burger vectors are much smaller in comparison with the sample size $L$, $B^{(i)}_i \ll 1$ for $i = 1, 2$) the metric tensor $g_{\mu,\nu}$ and the Jacobian transformation $\sqrt{G}$:

$$g_{x,x} = 1, \quad g_{x,y} = \frac{B^{(2)}}{2\pi} \frac{y}{x^2 + y^2}, \quad g_{y,y} = 1 - \frac{B^{(2)}}{2\pi} \frac{y}{x^2 + y^2}, \quad g_{y,x} = 0,$$

$$G = \text{det}[g_{\mu,\nu}] = 1 - \frac{B^{(2)}}{2\pi} \frac{y}{x^2 + y^2}.$$  \hspace{1cm} (16)

The inverse tensor is given by $g^{\nu,\mu} \approx 1$, $g^{x,x} = g^{x,y} = -\frac{B^{(2)}}{2\pi} \frac{y}{x^2 + y^2}$, $g^{y,y} = 1 + \frac{B^{(2)}}{\pi} \frac{x}{x^2 + y^2}$. Using the inverse tensor $g^{\nu,\mu}$, we obtain the inverse matrix $e^\mu_a$ which is given by

$$e_\mu^a = e_{a,b} g^{\nu,\mu} = (\delta_{a,b} e_\nu^b) g^{\nu,\mu} = e_\nu^a g^{\nu,\mu}.$$  \hspace{1cm} (17)

Using the components $e_\mu^a$, we compute the transformed Pauli matrices. The Hamiltonian in the absence of the edge dislocation is given by $h^{\text{TI}} = i\gamma^a \partial_a \equiv \sum_{a=1,2} i\gamma^a \partial_a$, where the Pauli matrices are given by $\gamma^1 = -\sigma^2$, $\gamma^2 = \sigma_1$, and $\gamma^3 = \sigma^3$. (We will use the convention that when an index appears twice we perform a summation over this index.) In the presence of the edge dislocation, the term $\gamma^a \partial_a$ must be expressed in terms of the Cartesian fixed coordinates $\mu = x, y$. As a result, the spinor $\Psi(\vec{r})$ transforms accordingly to the SU(2) transformation. If $\Psi(\vec{r})$ is the spinor for the deformed lattice, it can be related with the help of an SU(2) transformation to the spinor $\tilde{\Psi}(\vec{r})$ in the undeformed lattice: $\tilde{\Psi}(X, Y) = e^{-i \omega_{3,2} x \sigma^3} \Psi(x, y)$. Where $\delta\phi(x, y)$ is the rotation angle between the two sets of coordinates: $\delta\phi(x, y) = \tan^{-1}(\frac{Y}{X}) - \tan^{-1}(\frac{Y}{X})$. Using the rotation between the coordinates $X = x$ and $Y = y + \frac{B^{(2)}}{2\pi} \tan^{-1}(\frac{X}{Y})$ with the singularity at $x = y = 0$ gives us the derivative of the phase which is a delta function, $\partial_{\phi}\delta\phi(x, y) = -\partial_{\phi}\delta\phi(x, y) \propto \delta^2(x, y)$. Combining the transformation of the derivative with the SO(2) rotation in the plane, we obtain the form of the chiral Dirac equation in the Cartesian space (see appendix A) given in terms of the spin connection $\omega_{3,2}^{\mu,\nu}$ [24]:

$$i\gamma_\mu \partial_\nu \tilde{\Psi}(\vec{r}) = i\delta_{a,b} \gamma^b \partial_\nu \tilde{\Psi}(\vec{r}) = i\gamma^a e_\mu^a \left[ \partial_\mu + \frac{1}{2} [\gamma^b, \gamma^c] \omega_{3,2}^{\mu,\nu} \right] \Psi(\vec{r}).$$  \hspace{1cm} (18)

The Hamiltonian $h^{\text{TI}} \rightarrow h^{\text{edge}}$ is transformed to the dislocation edge Hamiltonian with the explicit form given by

$$h^{\text{edge}} = i\sigma^1 \partial_2 - i\sigma^2 \partial_1 = i\sigma^1 e_2^\mu \left[ \partial_\mu + \frac{1}{2} [\sigma^1, \sigma^2] \omega_{3,2}^{\mu,2} \right] - i\sigma^2 e_1^\mu \left[ \partial_\mu + \frac{1}{2} [\sigma^1, \sigma^2] \omega_{3,2}^{\mu,1} \right] = i(\sigma^1 e_2^\mu - \sigma^2 e_1^\mu) \left( \partial_\mu + \frac{1}{2} [\sigma^1, \sigma^2] \omega_{3,2}^{\mu,1} \right).$$  \hspace{1cm} (19)
To first order in the Burger vector we find \( \omega^1 \approx -\omega^2 = 0 \) and \( -\omega^3 = \omega^4 = -\frac{Z_0}{2} \delta^2(\vec{r}) \), see equations (A.12, A.13) in appendix A.

\[
h^\text{edge} \approx i\sigma^1 \left( \partial_\epsilon - \frac{i}{2} \sigma^3 B^{(2)} \delta^2(\vec{r}) \right) - i\sigma^2 \partial_\epsilon. \tag{20}
\]

In the second quantized form, the chiral Dirac Hamiltonian in the presence of an edge dislocations is given by

\[
H^\text{edge} \approx \int d^2 r \sqrt{G} \Psi^\dagger(\vec{r}) [h^\text{edge} - \mu] \Psi(\vec{r})
\]

\[
\equiv \hbar v_F \int d^2 r \sqrt{G} \Psi^\dagger(\vec{r}) \left[ i\sigma^1 \left( \partial_\epsilon - \frac{i}{2} \sigma^3 B^{(2)} \delta^2(\vec{r}) \right) - i\sigma^2 \partial_\epsilon - \mu \right] \Psi(\vec{r}), \tag{21}
\]

\( h^\text{edge} \) is the Hamiltonian in the first quantized language, \( \mu \) is the chemical potential and \( \Psi(\vec{r}) = [\Psi_\uparrow(\vec{r}), \Psi_\downarrow(\vec{r})]^T \) is the two-component spinor field.

### 3.3. The identification of the physical contours for the edge Hamiltonian \( h^\text{edge} \)

In order to identify the solutions, we will use the complex representation. The coordinates in the complex representation are given by \( z = \frac{1}{2}(x + iy), \quad \bar{z} = \frac{1}{2}(x - iy), \quad \partial_z = \partial_x - i\partial_y, \quad \partial_\bar{z} = \partial_x + i\partial_y. \) In this representation, the 2D delta function \( \delta^2(\vec{r}) \) is given by \( \delta^2(\vec{r}) \equiv \frac{1}{\pi} \delta(\frac{1}{z}) = \frac{1}{\pi} \delta_\epsilon(\frac{1}{z}) \) [36]. We will use the edge Hamiltonian \( h^\text{edge} \) and will compute the eigenfunctions \( u_\epsilon(z, \bar{z}) = [U_{\epsilon \uparrow}(z, \bar{z}), U_{\epsilon \downarrow}(z, \bar{z})]^T \) and \( v_\epsilon(z, \bar{z}) = [V_{\epsilon \uparrow}(z, \bar{z}), V_{\epsilon \downarrow}(z, \bar{z})]^T \). The eigenvalue equation is given by

\[
\epsilon U_{\epsilon \uparrow}(z, \bar{z}) = - \left[ \partial_z + \left( \frac{B^{(2)}}{2\pi} \right) \partial_z \left( \frac{1}{z} \right) \right] U_{\epsilon \downarrow}(z, \bar{z}),
\]

\[
\epsilon U_{\epsilon \downarrow}(z, \bar{z}) = \left[ \partial_{\bar{z}} + \left( \frac{B^{(2)}}{2\pi} \right) \partial_{\bar{z}} \left( \frac{1}{\bar{z}} \right) \right] U_{\epsilon \uparrow}(z, \bar{z}). \tag{22}
\]

The eigenfunctions \( u_\epsilon(z, \bar{z}) \) and \( v_\epsilon(z, \bar{z}) \) can be written with the help of a singular matrix \( M(z, \bar{z}) \) [27]:

\[
u_\epsilon(z, \bar{z}) = M(z, \bar{z}) \hat{F}_\epsilon(z, \bar{z}) \equiv \begin{bmatrix} e^{-\frac{B^{(2)}}{4\pi \bar{z}}} & 0 \\ 0 & e^{-\frac{B^{(2)}}{4\pi z}} \end{bmatrix} \begin{bmatrix} F_{\epsilon \uparrow}(z, \bar{z}) \\ F_{\epsilon \downarrow}(z, \bar{z}) \end{bmatrix}.
\]

(Note that \( F_\uparrow(z, \bar{z}) \) and \( F_\downarrow(z, \bar{z}) \) are the transformed eigenfunctions for \( \epsilon > 0 \) and \( \epsilon < 0 \), respectively.) In terms of the transformed spinors the eigenvalue equation \( h^\text{edge}(z, \bar{z}) u_\epsilon(z, \bar{z}) = \epsilon u_\epsilon(z, \bar{z}) \) and \( F_{\epsilon \downarrow}(z, \bar{z}) \) becomes

\[
\begin{bmatrix} F_{\epsilon \uparrow}(z, \bar{z}) \\ F_{\epsilon \downarrow}(z, \bar{z}) \end{bmatrix} = \begin{bmatrix} I(z, \bar{z}) & 0 \\ 0 & (I(z, \bar{z}))^* \end{bmatrix} \begin{bmatrix} -\partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{bmatrix} \begin{bmatrix} F_{\epsilon \uparrow}(z, \bar{z}) \\ F_{\epsilon \downarrow}(z, \bar{z}) \end{bmatrix},
\]

\(^1\) The Holomorphic representation of the delta function was brought to my attention by my colleague V P Nair.
where \( I(z, \bar{z}) = e^{-\frac{B(2)}{\pi}(z + \bar{z})} \equiv e^{\frac{2B(2)}{\pi}(\frac{y}{x^2+y^2})} \), \( I(z, \bar{z})^* = e^{-\frac{B(2)}{\pi}(z - \bar{z})} \), \( |I(z, \bar{z})| = 1 \). We search for zero modes \( \epsilon = 0 \) and find that
\[
\partial_z F_{\epsilon 1}(z, \bar{z}) = 0, \quad \partial_{\bar{z}} F_{\epsilon 1}(z, \bar{z}) = 0.
\] (23)

The solutions are given by the holomorphic representation \( F_{\epsilon = 01}(z, \bar{z}) = f_1(z) \) and the anti-holomorphic function \( F_{\epsilon = 01}(z, \bar{z}) = f_1(\bar{z}) \). The zero-mode eigenfunctions are given by
\[
u_e = 0, 1(z) = e^{-\frac{B(2)}{\pi}(\frac{1}{z})} f_1(z), \quad u_{\epsilon = 0, \downarrow}(\bar{z}) = e^{-\frac{B(2)}{\pi}(\frac{1}{\bar{z}})} f_1(\bar{z}).
\] (24)

Due to the presence of the essential singularity at \( z = 0 \), it is not possible to find analytic functions \( f_1(z) \) and \( f_1(\bar{z}) \) which vanish fast enough around \( z = 0 \) such that \( \int d^2z (\nu_e = 0, \lambda(z))^* u_{\epsilon = 0, \lambda(z)} < \infty \). Therefore, we conclude that a zero-mode solution does not exist. The only way to remedy the problem is to allow for states with finite energy.

In the next step, we look for finite energy states. We perform a coordinate transformation:
\[
z \rightarrow W[z, \bar{z}], \quad \bar{z} \rightarrow \bar{W}[z, \bar{z}].
\] (25)

We demand that the transformation is conformal and preserve the orientation. This restricts the transformations to holomorphic and anti-holomorphic functions [36]. This means that we have the conditions \( \partial_{\bar{z}} W[z, \bar{z}] = 0 \) and \( \partial_{z} \bar{W}[z, \bar{z}] = 0 \). As a result, we obtain \( W[z, \bar{z}] = W[z] \) and \( \bar{W}[z, \bar{z}] = \bar{W}[\bar{z}] \), which obey the eigenvalue equations
\[
\epsilon F_{\epsilon 1}(W, \bar{W}) = -\partial_W F_{\epsilon 1}(W, \bar{W}),
\] (26)
\[
\epsilon F_{\epsilon 1}(W, \bar{W}) = \partial_{\bar{W}} F_{\epsilon 1}(W, \bar{W}).
\]

This implies the conditions \( \frac{dW[z]}{dz} = (I(z, \bar{z}))^* \) and \( \frac{d\bar{W}[\bar{z}]}{d\bar{z}} = I(z, \bar{z}) \). Since \( I(z, \bar{z}) \) is neither holomorphic or anti-holomorphic and satisfies \( |I(z, \bar{z})| = 1 \), the only solutions for \( W[z] \) and \( \bar{W}[\bar{z}] \) must obey \( I(z, \bar{z}) = 1 \):
\[
I(z, \bar{z}) \equiv e^{2\pi B(2)\frac{(y)}{x^2+y^2}} = e^{i2\pi n}, \quad n = 0, \pm 1, \pm 2, \ldots
\] (27)

For \( I(z, \bar{z}) \neq 1 \), one obtains solutions that are unstable. The stable solutions will be given by a one-parameter \( s \) curve \((s) \) is the length of the curve \( \bar{r}(s) \equiv [x(s), y(s)] \) which obey the equation \( I(z, \bar{z}) = 1 \). The curve \( \bar{r}(s) \) allows us to define the tangent \( \bar{t}(s) \) and the normal vectors \( \bar{N}(s) \). This allows us to introduce a 2D region in the vicinity of the contour of \( \bar{r}(s) \) to \( \bar{R}(s, u) = \bar{r}(s) + u \bar{N}(s) \).

3.4. The wave function for the edge dislocation: the \( n = 0 \) contour

The condition \( I(z, \bar{z}) = e^{2\pi B(2)\frac{y}{x^2+y^2}} = 1 \) for \( n = 0 \) is satisfied for \( y = 0 \) and large values of \( y \) which obey \( 2 \frac{B(2)}{\pi} \left( \frac{y}{x^2+y^2} \right) \ll 1 \). The values of \( y \) that satisfy these conditions are restricted to \( I(z, \bar{z}) = e^{2\pi \frac{B(2)}{\pi} \frac{y}{x^2+y^2}} \approx 1 \). This condition is satisfied for values of \( y \) in the range
\[
2 \frac{B(2)}{\pi} \left( \frac{y}{x^2+y^2} \right) \lesssim \eta < \frac{\pi}{4} < 1.
\] (28)

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We introduce the radius $R_x = \frac{p^2}{2na}$ and find that the condition $I(z, z) \approx 1$ gives rise to condition for $y$. The solution is given by $x^2 + (y \pm \frac{2q}{d} R_x)^2 = (\frac{2q}{d} R_x)^2$. Therefore, for $|y| > |d| \geq (\frac{2q}{d})R_x > 2R_x$ we have $I \approx 1$, which corresponds to free particle eigenvalue equations.

\[ \epsilon F_{\uparrow}(x, y) = e^{\frac{B(1)}{\pi} \epsilon \xi} [-\partial_x + i\partial_y] F_{\downarrow}(x, y) \approx [-\partial_x + i\partial_y] F_{\downarrow}(x, y), \]

\[ \epsilon F_{\downarrow}(x, y) = e^{\frac{B(1)}{\pi} \epsilon \xi} [\partial_x + i\partial_y] F_{\uparrow}(x, y) \approx [\partial_x + i\partial_y] F_{\uparrow}(x, y). \]  

For $|y| > d$ the eigenfunctions are given by $U_{\epsilon, \uparrow}(x, y) = e^{-\frac{B(1)}{\pi} \epsilon \xi} F_{\epsilon, \uparrow}(x, y)$, $U_{\epsilon, \downarrow}(x, y) = e^{-\frac{B(2)}{\pi} \xi} F_{\epsilon, \downarrow}(x, y)$, where $F_{\epsilon, \uparrow}(x, y)$ and $F_{\epsilon, \downarrow}(x, y)$ are the eigenfunctions of equation (21). The envelope functions $e^{-\frac{B(2)}{\pi} \xi}$, $e^{-\frac{B(2)}{\pi} \xi}$, which multiply the wave functions $F_{\epsilon, \uparrow}(x, y)$, $F_{\epsilon, \downarrow}(x, y)$, impose vanishing boundary conditions for the eigenfunctions $U_{\epsilon, \downarrow}(x, y)$ and $U_{\epsilon, \uparrow}(x, y)$ at $y \rightarrow \pm \infty$. Therefore, we demand that the eigenfunctions $U_{\epsilon, \uparrow}(x, y)$, $U_{\epsilon, \downarrow}(x, y)$ should vanish at the boundaries $y = \pm L$. Since the multiplicative envelope functions for opposite spins are complex conjugate to each other, we have to make the choice that one of the spin components vanishes at one side and the other component at the opposite side. Two possible choices can be made:

\[ U_{\epsilon, \uparrow}(x, y = \frac{L}{2}) = e^{-\frac{B(2)}{\pi} \xi} F_{\epsilon, \uparrow}(x, \frac{L}{2}) = U_{\epsilon, \downarrow}(x, y = -\frac{L}{2}) \]

\[ U_{\epsilon, \downarrow}(x, y = -\frac{L}{2}) = e^{-\frac{B(2)}{\pi} \xi} F_{\epsilon, \downarrow}(x, -\frac{L}{2}) = 0 \]

or

\[ U_{\epsilon, \uparrow}(x, y = -\frac{L}{2}) = e^{-\frac{B(2)}{\pi} \xi} F_{\epsilon, \uparrow}(x, -\frac{L}{2}) = U_{\epsilon, \downarrow}(x, y = \frac{L}{2}) \]

\[ U_{\epsilon, \downarrow}(x, y = \frac{L}{2}) = e^{-\frac{B(2)}{\pi} \xi} F_{\epsilon, \downarrow}(x, \frac{L}{2}) = 0. \]

Making the first choice (both choices give the same eigenvalues and eigenfunction), we compute the eigenfunctions $F_{\epsilon, \uparrow}(x, y)$ and $F_{\epsilon, \downarrow}(x, y)$ for $|y| > d$ using the boundary conditions:

\[ F_{\epsilon, \uparrow}(x, y = \frac{L}{2}) = 0, \quad F_{\epsilon, \downarrow}(x, y = -\frac{L}{2}) = 0. \]  

Due to the fact that the solutions are restricted to $|y| > d$, no conditions need to be imposed at $x = y = 0$. In the present case, we consider a situation with a single dislocation. This is justified for a dilute concentration of dislocations typically separated by a distance $l \approx 10^{-6}$ m. (In principle, we need at least two dislocations in order to satisfy the condition that the sum of the Burger vectors is zero.) The eigenvalues are given by $\epsilon = \pm h\sqrt{p^2 + q^2}$. The value of $p$ is determined by the periodic boundary condition in the $x$-direction $p(m) = \frac{2\pi}{L}m = \frac{2\pi}{Na}m$, $m = 0, 1, \ldots, (N - 2), (N - 1)$ and $a$ is the lattice constant $a \approx \frac{2\pi}{N}$. The value of $q$ will be...
obtained from the vanishing boundary conditions at \( y = \pm \frac{L}{2} \). The eigenfunctions \( F_{n,\sigma}(x, y) \) will be obtained using the linear combination of the spinors introduced in section 3. In the Cartesian representation, we can build four spinors \( \Gamma_{p,q}(x, y) \), \( \Gamma_{p,-q}(x, y) \), \( \Gamma_{-p,q}(x, y) \), \( \Gamma_{-p,-q}(x, y) \) which are eigenstates of the chirality operator and are given by

\[
\Gamma_{p,q}(x, y) = e^{ipx} e^{iqy} \left( \frac{1}{i} e^{i\chi(p,q)} \right),
\]

\[
\Gamma_{p,-q}(x, y) = e^{ipx} e^{-iqy} \left( \frac{1}{i} e^{-i\chi(p,q)} \right),
\]

\[
\Gamma_{-p,q}(x, y) = e^{-ipx} e^{iqy} \left(-i e^{i\chi(p,q)} \right),
\]

\[
\Gamma_{-p,-q}(x, y) = e^{-ipx} e^{-iqy} \left(-i e^{-i\chi(p,q)} \right),
\]

where \( \tan[\chi(p, q)] = \frac{q}{p} \).

The TI Hamiltonian \( h_{TI}(x, y) = \hbar v_F [\sigma^1 \partial_y - i \sigma^2 \partial_x] \) is invariant under the symmetry operation \( x \rightarrow -x \) which is described by the transformation \( P_x \): \( P_x P_x^{-1} = -x; P_x \sigma^2 P_x^{-1} = -\sigma^2; P_x y P_x^{-1} = y; P_x \sigma^1 P_x^{-1} = \sigma^1. \)

The edge Hamiltonian \( h_{edge} \) contains in addition the term \( \sigma^2 \delta(\vec{r}) \), which changes sign under the symmetry operation \( P_i \). As a result the symmetry operation does not commute with the edge Hamiltonian \( [h_{edge}, P_i] \neq 0 \). This result demands that we construct two independent eigenfunctions \( F_{p>0,q}^{(n=0,R)}(x, y) \) (right-mover) for \( p > 0 \) and \( F_{p<0,q}^{(n=0,L)}(x, y) \) (left-mover) for \( p < 0 \).

Employing the boundary conditions given in equation (29), we obtain the amplitudes \( \frac{D(q)}{C(q)} = \frac{R(q)}{A(q)} \) and the discrete momenta \( q_+ \). Using the pair \( \Gamma_p(x, y), \Gamma_{-p,q}(x, y) \) for \( p > 0 \), we obtain

\[
F_{p>0,q}^{(n=0,R)}(x, y) = e^{ipx} e^{\frac{1}{2} \chi(p,q)} \left[ e^{i(q_+ y + \frac{1}{2} \chi(p,q_+))} + (-1)^k e^{-i(q_+ y + \frac{1}{2} \chi(p,q_+))} \right], \quad |y| > d,
\]

\[
F_{p>0,q}^{(n=0,L)}(x, y) = e^{ipx} e^{\frac{1}{2} \chi(p,q)} \left[ e^{i(q_+ y + \frac{1}{2} \chi(p,q_+))} + (-1)^k e^{-i(q_+ y + \frac{1}{2} \chi(p,q_+))} \right], \quad |y| > d.
\]

\[
q = q_+ = \frac{\pi}{L} k + \frac{1}{L} \tan^{-1} \left( \frac{q_+}{p} \right), \quad k = 1, 2, 3, \ldots, \quad \tan[\chi(p, q_+)] = \left( \frac{q_+}{p} \right),
\]

\[
\epsilon(p, q_+) = \pm \hbar v_F \sqrt{\left( \frac{2\pi}{L} m \right)^2 + q_+^2}.
\]

Similarly, for the second pair \( \Gamma_{-p,q}(x, y), \Gamma_{-p,-q}(x, y) \), \( p > 0 \), we obtain

\[
F_{p<0,q}^{(n=0,L)}(x, y) = e^{-ipx} e^{-\frac{1}{2} \chi(p,q,-)} \left[ e^{i(q_+ y + \frac{1}{2} \chi(p,q_-))} + (-1)^k e^{-i(q_+ y + \frac{1}{2} \chi(p,q_-))} \right], \quad |y| > d,
\]

\[
F_{p<0,q}^{(n=0,R)}(x, y) = -i e^{-ipx} e^{-\frac{1}{2} \chi(p,q,-)} \left[ e^{i(q_+ y - \frac{1}{2} \chi(p,q_-))} + (-1)^k e^{-i(q_+ y - \frac{1}{2} \chi(p,q_-))} \right], \quad |y| > d,
\]

\[
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\]
\[ q \equiv q_\mp = \frac{\pi}{L} k - \frac{1}{L} \tan^{-1}\left(\frac{q_-}{p}\right), \quad k = 1, 2, 3, \ldots, \quad \tan[\chi(p, q_-)] = \left(\frac{q_-}{p}\right), \]

\[ \epsilon(-p, q_-) = \pm \hbar v_F \left(\frac{2\pi}{L} m \right)^2 + q_\mp^2. \]  

(34)

For the state with zero momentum \( p = 0 \), we find that

\[ F^{(n=0,0)}_{\epsilon(p=0,q),\uparrow}(x, y) = 2 e^{\frac{\pi}{L} p} \cos\left(q y + \frac{\pi}{4}\right), \quad |y| > d, \]

\[ F^{(n=0,0)}_{\epsilon(p=0,q),\downarrow}(x, y) = 2 e^{\frac{-\pi}{L} p} \cos\left(q y - \frac{\pi}{4}\right), \quad |y| > d, \]  

(35)

\[ q = \frac{\pi}{2L} + \frac{\pi}{L} k, \quad k = 0, 1, 2, 3 \ldots, \]

\[ \epsilon(p = 0, q) = \pm \hbar v_F |q|. \]

The eigenfunctions for the dislocation problem for \( |y| > d \) will be given in terms of the envelope functions \( e^{-\frac{\pi}{2L}(\frac{1}{4\pi})}, e^{-\frac{\pi}{2L}(\frac{1}{4\pi})} (U_{\epsilon,\uparrow}^{(n=0,L)}(x, y) = e^{-\frac{\pi}{2L}(\frac{1}{4\pi})} F_{\epsilon,\uparrow}(x, y), U_{\epsilon,\downarrow}^{(n=0,L)}(x, y) = e^{-\frac{\pi}{2L}(\frac{1}{4\pi})} F_{\epsilon,\downarrow}(x, y)). \]

The explicit solutions are given by

\[ u_{\epsilon}^{(n=0,R)}(x, y) \equiv [U_{\epsilon,\uparrow}^{(n=0,R)}(x, y), U_{\epsilon,\downarrow}^{(n=0,R)}(x, y)]^T; \]

\[ u_{\epsilon}^{(n=0,L)}(x, y) \equiv [U_{\epsilon,\uparrow}^{(n=0,L)}(x, y), U_{\epsilon,\downarrow}^{(n=0,L)}(x, y)]^T. \]

The components of the spinor are given by

\[ U_{\uparrow}^{(n=0,R)}(x, y) \approx \frac{2\text{const}(B^{(2)})}{G^{\uparrow}(x, y) L} e^{-\frac{\pi}{2L}(\frac{1}{4\pi})} F^{(n=0,R)}_{\epsilon(p=0,q),\uparrow}(x, y), \]

\[ U_{\downarrow}^{(n=0,R)}(x, y) \approx \frac{2\text{const}(B^{(2)})}{G^{\uparrow}(x, y) L} e^{-\frac{\pi}{2L}(\frac{1}{4\pi})} F^{(n=0,R)}_{\epsilon(p=0,q),\downarrow}(x, y), \]

\[ U_{\uparrow}^{(n=0,L)}(x, y) \approx \frac{2\text{const}(B^{(2)})}{G^{\uparrow}(x, y) L} e^{-\frac{\pi}{2L}(\frac{1}{4\pi})} F^{(n=0,L)}_{\epsilon(p=0,L),\uparrow}(x, y), \]

\[ U_{\downarrow}^{(n=0,L)}(x, y) \approx \frac{2\text{const}(B^{(2)})}{G^{\uparrow}(x, y) L} e^{-\frac{\pi}{2L}(\frac{1}{4\pi})} F^{(n=0,L)}_{\epsilon(p=0,L),\downarrow}(x, y), \]  

(36)

\[ U_{\uparrow}^{(n=0,0)}(x, y) \approx e^{-\frac{\pi}{2L}(\frac{1}{4\pi})} F^{(n=0,0)}_{\epsilon(p=0,q),\uparrow}(x, y), \]

\[ U_{\downarrow}^{(n=0,0)}(x, y) \approx e^{-\frac{\pi}{2L}(\frac{1}{4\pi})} F^{(n=0,0)}_{\epsilon(p=0,q),\downarrow}(x, y), \]

where \( G(x, y) = 1 - \frac{B^{(2)} y}{2\pi \sqrt{2(x^2+y^2)}} \) is the Jacobian introduced by the edge dislocation. The eigenstates are normalized and obey \( \int dx \int dy \sqrt{G(x, y)}(U_{\sigma}^{(n=0,R)}(x, y))^*U_{\sigma'}^{(n=0,R)}(x, y) \approx \delta_{\sigma,\sigma'}, \quad \int dx \int dy \sqrt{G(x, y)}(U_{\sigma}^{(n=0,L)}(x, y))^*U_{\sigma'}^{(n=0,L)}(x, y) \approx \delta_{\sigma,\sigma'} \). The normalization factor \( \frac{2\text{const}(B^{(2)})}{L} \approx \frac{1}{2L} \) has a weak dependence on the Burger vector \( B^{(2)} \). This dependence is a consequence of the Jacobian \( \sqrt{G} \) which affects the normalization constant.

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Figure 1. The contours \((x(s))^2 + (y(s) - \frac{R_g}{n})^2 = \left(\frac{R_g}{n}\right)^2\) for \(n = \pm 1, \pm 2, \pm 3\) (in decreasing size), \(R_g(n) = \frac{R_g}{n}\). Note that \(n = 0\) corresponds to the equation \(y(s) = 0\) and \(|y| > d\) (see the text). The distance is measured in units of the Burger vector \(B\).

(37)

\[ (x(s))^2 + (y(s) \pm R_g(n))^2 = \left(\frac{R_g(n)}{|n|}\right)^2. \]

The centers of the contours are given by \([\bar{x}, \bar{y}] = [0, R_g(n)]\) for \(n \neq 0\). When \(n > 0\) the center of the contours has positive coordinates (upper contour) and for \(n < 0\) the center has negative coordinates (lower contour). Each contour is parameterized by the arc length \(0 \leq s < 2\pi \frac{R_g}{|n|}\), which is equivalent to \(0 \leq \varphi < 2\pi\). Each contour is parameterized by \(\bar{r}(s) = [x(s), R_g(n) + y(s)]\), where \(x(s) = R_g(n) \cos\left(\frac{s}{R_g(n)}\right)\) \(= R_g(n)\cos(\varphi)\) and \(y(s) = R_g(n) \sin\left(\frac{s}{R_g(n)}\right)\) \(= R_g(n)\sin(\varphi)\). We will extend this curve to a 2D strip with the coordinate \(u\) in the normal direction: for the curve \(\bar{r}(s) = [x(s), y(s)]\), we will use the tangent \(\bar{t}(s)\) and the
normal vector $\vec{N}(s)$. Therefore, the 2D region in the vicinity of the one-parameter curve $\vec{r}(s)$ is replaced by $\vec{r}(s) \rightarrow \vec{R}(s, u) = \vec{r}(s) + u\vec{N}(s)$.

\[
x(s, u) = R_g(n) \cos \left[ \frac{s}{R_g(n)} \right] + u \cos \left[ \frac{s}{R_g(n)} \right],
\]

\[
y(s, u) = R_g(n) \sin \left[ \frac{s}{R_g(n)} \right] + u \sin \left[ \frac{s}{R_g(n)} \right].
\]

(38)

We will restrict the width $|u|$ such that $\exp^{i2\pi n} \exp^{\pm i\eta} \approx 1$ where $\eta$ obeys $\eta < \frac{\pi}{4} < 1$, $|u| \leq \frac{R_g(n)}{l_{\pi n} - R_g(n)} \approx R_g(n) \left( \frac{\eta}{2\pi n} \right) < \frac{R_g(n)}{8\pi n}$. In these new coordinates, the Dirac equation is approximated for $|u| \leq R_g(n) \left( \frac{\eta}{2\pi n} \right) = \frac{D(n)}{2}$ by

\[
\epsilon F_{\epsilon\uparrow}(s, u) = -i (s, u) e^{-\frac{i}{s}(\epsilon)} \left[ \partial_u - \frac{i}{1 + u} \partial_q \right] F_{\epsilon\downarrow}(s, u) \approx -e^{-\frac{i}{s}(\epsilon)} [\partial_u - i\partial_q] F_{\epsilon\downarrow}(s, u),
\]

\[
\epsilon F_{\epsilon\downarrow}(s, u) = (i(s, u))^2 e^{\frac{i}{s}(\epsilon)} \left[ \partial_u + \frac{i}{1 + u} \partial_q \right] F_{\epsilon\uparrow}(s, u) \approx e^{\frac{i}{s}(\epsilon)} [\partial_u + i\partial_q] F_{\epsilon\uparrow}(s, u).
\]

(39)

The solution for the contour $n \neq 0$, $0 \leq s < 2\pi R_g(n); |u| \leq \frac{D(n)}{2}$.

The periodicity in $s$ allows us to represent the eigenfunctions in the form $F_{\epsilon\uparrow}(s, u) = \sum_{j=-\infty}^{\infty} \sum_{q} e^{ij(\frac{q}{\sqrt{n}})} e^{q u} F_{\epsilon\uparrow}(j, q)$ and $F_{\epsilon\downarrow}(s, u) = \sum_{j=-\infty}^{\infty} \sum_{q} e^{i(j+1)(\frac{q}{\sqrt{n}})} e^{q u} F_{\epsilon\downarrow}(j, q)$. We find that

\[
\epsilon F_{\epsilon}(\epsilon; j, q) = \left( iq + \frac{j}{R_g(n)} \right) F_{\epsilon}(\epsilon; j, q),
\]

\[
\epsilon F_{\epsilon}(\epsilon; j, q) = \left( iq + \frac{j+1}{R_g(n)} \right) F_{\epsilon}(\epsilon; j, q).
\]

(40)

The determinant of the two equations determines the relation between the eigenvalue $\epsilon$, the transverse momentum $Q(\epsilon)$ and the eigenfunctions $F_{\epsilon\uparrow}(j, q), F_{\epsilon\downarrow}(j, q)$. The eigenvalues are degenerate and obey $\epsilon(j = l; k) = \epsilon(j = -(l+1); k)$, where $l \geq 0$.

\[q = \frac{-i}{2R_g(n)} \pm Q(\epsilon) \quad \text{with} \quad Q(\epsilon) = \sqrt{\epsilon^2 - \left( \frac{l+\frac{1}{2}}{R_g(n)} \right)^2},\]

\[
F_{\epsilon}(l, q) = \left[ F_{\epsilon\uparrow}(l, q), F_{\epsilon\downarrow}(l, q) \right]^T \propto \left[ 1, e^{-i\kappa(\epsilon)l} \right]^T \quad \text{with} \quad \kappa(\epsilon, l) = \tan^{-1} \left( \frac{Q R_g(n)}{l+\frac{1}{2}} \right).
\]

(41)

The value of the transversal momentum $Q(\epsilon)$ will be determined from the boundary conditions at $\pm \frac{D(n)}{2}$. We will introduce a polar angle $\theta$ measured with respect to the Cartesian axes: the angle $0 < \varphi(n = 1) \leq 2\pi$ for the upper contour $n = 1$ centered at $[\bar{x} = 0, \bar{y} = R_g]$ is described.
by the polar coordinate $0 < \theta \leq \pi$ measured from the center of the Cartesian coordinate [0,0]. The lower contour centered at $[x = 0, y = -R_o]$ characterized by the angle $0 < \varphi(n = -1) \leq 2\pi$ is described by the polar angle $\theta$ restricted to $\pi < \theta \leq 2\pi$. We establish the correspondence between $\varphi(n = \pm 1)$ and $\theta$:

$$\varphi(n = 1) = 2\theta + \frac{3\pi}{2} \text{ for the upper contour } \quad n = 1, \quad 0 < \theta \leq \pi,$$

$$\varphi(n = -1) = 2\theta + \frac{3\pi}{2} + \pi \text{ for the lower contour } \quad n = -1, \quad 0 < \theta \leq \pi. \quad (42)$$

Following the discussion from the previous section, we will introduce the following boundary conditions:

$$F_{\epsilon\uparrow}^{(n=1)} \left( s, u = \frac{D}{2} \right) = 0, \quad F_{\epsilon\downarrow}^{(n=1)} \left( s, y = -\frac{D}{2} \right) = 0,$$

$$F_{\epsilon\uparrow}^{(n=-1)} \left( s, u = -\frac{D}{2} \right) = 0, \quad F_{\epsilon\downarrow}^{(n=-1)} \left( s, y = \frac{D}{2} \right) = 0. \quad (43)$$

$D(n = \pm 1) \equiv D.$

For the two contours $n = \pm 1$, we introduce eight spinors $\Gamma_{l, Q}^{(n=\pm 1)}(\varphi(n = \pm 1), u)$, $\Gamma_{l, -Q}^{(n=\pm 1)}(\varphi(n = \pm 1), u)$, $\Gamma_{l, -Q}^{(n=\pm 1)}(\varphi(n = \pm 1), u)$, $\Gamma_{l, -Q}^{(n=\pm 1)}(\varphi(n = \pm 1), u)$. Using this spinor, we will compute the eigenfunctions. For the case $n = 0$, we had only four spinors given in equation (30). The four spinors have been used to construct the eigenfunctions $F_{p > 0, q}^{(n=0, L)}(x, y)$ for $p > 0$ and $F_{p > 0, q}^{(n=0, R)}(x, y)$. Due to the fact that for each $n \neq 0$, we have two contours $n = \pm$ we have eight spinors that will be used to construct the eigenfunctions.

$$\Gamma_{l, Q}^{(n=\pm 1)}(\varphi(n = \pm 1), u) = e^{iL(\varphi(n=\pm 1))} e^{iQ u} \left( \begin{array}{c} 1 \\ e^{i(\varphi(n=\pm 1))} e^{-i(\varphi(n=\pm 1))} \end{array} \right),$$

$$\Gamma_{l, -Q}^{(n=\pm 1)}(\varphi(n = \pm 1), u) = e^{iL(\varphi(n=\pm 1))} e^{-iQ u} \left( \begin{array}{c} 1 \\ e^{i(\varphi(n=\pm 1))} e^{i(\varphi(n=\pm 1))} \end{array} \right), \quad (44)$$

$$\Gamma_{l, -Q}^{(n=\pm 1)}(\varphi(n = \pm 1), u) = e^{-iL(\varphi(n=\pm 1))} e^{iQ u} \left( \begin{array}{c} 1 \\ -e^{-i(\varphi(n=\pm 1))} e^{i(\varphi(n=\pm 1))} \end{array} \right),$$

$$\Gamma_{l, -Q}^{(n=\pm 1)}(\varphi(n = \pm 1), u) = e^{-iL(\varphi(n=\pm 1))} e^{-iQ u} \left( \begin{array}{c} 1 \\ -e^{-i(\varphi(n=\pm 1))} e^{-i(\varphi(n=\pm 1))} \end{array} \right).$$

Using the vanishing boundary condition given in equation (42), we construct for this case similar spinors to the one given in equation (31). In the present case, we have for each $n \neq 0$ two contours; therefore the number of spinors will be doubled. We find instead of the eigenfunction given in equation (33) two sets of eigenfunctions with momentum $Q_-$ (which replaces $q_-$, see (33)) and $Q_+$ (which replaces $q_+$, see (32)).
Using the boundary conditions given in equation (35), we determine the quantization conditions $Q_-, Q_+$ and the eigenfunctions for the $n = 1$ and $n = -1$ contours.

\[
Q_- = \frac{\pi}{D} k - \frac{1}{D} \tan^{-1}\left(\frac{Q_- R_x(1)}{l + \frac{1}{2}}\right), \quad k = 1, 2, 3 \ldots,
\]

\[
\tan[k(l, Q_-)] = \left(\frac{Q_- R_x(1)}{l + \frac{1}{2}}\right)^2
\]

\[
\epsilon(l, Q_-) = \pm \hbar \nu_{\nu} \sqrt{\frac{l + \frac{1}{2}}{R_y(1)}} + Q_-^2,
\]

\[
Q_+ = \frac{\pi}{D} k + \frac{1}{D} \tan^{-1}\left(\frac{Q_+ R_x(1)}{l + \frac{1}{2}}\right), \quad k = 1, 2, 3 \ldots,
\]

\[
\tan[k(l, Q_+)] = \left(\frac{Q_+ R_x(1)}{l + \frac{1}{2}}\right)^2
\]

\[
\epsilon(l, Q_+) = \pm \hbar \nu_{\nu} \sqrt{\frac{l + \frac{1}{2}}{R_y(n)}} + Q_+^2.
\]

(45)

Using the fact that the combined wave function on the contours $n = 1$ and $n = \pm 1$ must be finite, we obtain two sets of wave functions. We include the envelope function and obtain the wave function for $Q_-$ and $Q_+$: the envelope functions $e^{-\frac{\kappa(l)}{2} \frac{\pi}{2}}$, $e^{-\frac{\kappa(l)}{2} \frac{\pi}{2}}$ when projected to the contours take a complicated form. The envelope functions can be expressed in terms of the functions $\eta(u)$ and $\xi(\theta, u)$:

\[
\eta(u) = \frac{R_x(1)}{R_y(1) + u}, \quad \frac{|u|}{R_y(1)} < 1,
\]

\[
\xi(\theta, u) = \frac{-B^{(2)}(2\pi R_y(1) + u)((\sin[2\theta])^2 + (\eta(u) - \cos[2\theta])^2)}{2 \pi (R_y(1) + u) ((\sin[2\theta])^2 + (\eta(u) - \cos[2\theta])^2)}.
\]

(46)

We find for $Q_-:

\[
U_{e(l, Q_-)}(\theta, u) = G_{\pi l}^{-1}(\theta, u) \cdot [U_{e(l, Q_-)}^{(even, k)}(\theta, u) + U_{e(l, Q_-)}^{(odd, k)}(\theta, u)],
\]

\[
U_{e(l, Q_-)}^{(even, k)}(\theta, u) = 2i e^{\frac{\pi}{2} \kappa(l, Q_-)} \left[ e^{i(\theta, u) \sin[2\theta]} e^{-i\xi(\theta, u)(\eta(u) - \cos[2\theta])} e^{il(2\theta + \frac{\pi}{2})} \sin \left[ Q_- u + \frac{1}{2} \kappa(l, Q_-) \right] \right]
\]

\[
+ (-1)^j e^{i\xi(\theta, u) \sin[2\theta]} e^{-i\xi(\theta, u)(\eta(u) + \cos[2\theta])} e^{-il(2\theta + \frac{\pi}{2})} \sin \left[ Q_- u - \frac{1}{2} \kappa(l, Q_-) \right],
\]

\[
U_{e(l, Q_-)}^{(odd, k)}(\theta, u) = 2 e^{\frac{\pi}{2} \kappa(l, Q_-)} \left[ e^{i(\theta, u) \sin[2\theta]} e^{-i\xi(\theta, u)(\eta(u) - \cos[2\theta])} e^{il(2\theta + \frac{\pi}{2})} \cos \left[ Q_- u + \frac{1}{2} \kappa(l, Q_-) \right] \right]
\]

\[
+ (-1)^j e^{i\xi(\theta, u) \sin[2\theta]} e^{-i\xi(\theta, u)(\eta(u) + \cos[2\theta])} e^{-il(2\theta + \frac{\pi}{2})} \cos \left[ Q_- u - \frac{1}{2} \kappa(l, Q_-) \right],
\]

\[
U_{e(l, Q_-)}(\theta, u) = G_{\pi l}^{-1}(\theta, u) \cdot [U_{e(l, Q_-)}^{(even, k)}(\theta, u) + U_{e(l, Q_-)}^{(odd, k)}(\theta, u)],
\]

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At zero temperature, the STM tunneling current is described using the electronic surface states. Similarly, for \( \epsilon_{l} \) we obtain the wave function:

\[
U_{\epsilon_{l}(Q_{-})\downarrow}(\theta, u) = G^{\downarrow}(\theta, u) : U_{\epsilon_{l}(Q_{-})\uparrow}(\theta, u) ; U_{\epsilon_{l}(Q_{-})\downarrow}(\theta, u) ; U_{\epsilon_{l}(Q_{+})\downarrow}(\theta, u) ; U_{\epsilon_{l}(Q_{+})\uparrow}(\theta, u) ; U_{\epsilon_{l}(Q_{-})\downarrow}(\theta, u) ; U_{\epsilon_{l}(Q_{+})\uparrow}(\theta, u) \]

where \( G^{\downarrow}(\theta, u) \) is the Jacobian transformation induced by the metric tensor.

4. Computation of the scanning tunneling microscopy (STM) density of states

4.1. Description of the STM method

The STM tunneling current \( I \) is a function of the bias voltage \( V \), which gives spatial and spectroscopic information about the electronic surface states. At zero temperature, the
derivative of the current with respect to the bias voltage $V$ is given in terms of the single particle eigenvalues: $\epsilon(m, q_{-}) = \pm \hbar v_F \sqrt{(\frac{2\pi m}{\hbar})^2 + q_{-}^2}$, $\epsilon(m, q_{+}) = \pm \hbar v_F \sqrt{(\frac{2\pi m}{\hbar})^2 + q_{+}^2}$, $m = 0, 1, 2, 3, \ldots$, for contour $n = 0$. For the upper and lower circular contours $n = \pm 1$, we have: $\epsilon(l, Q_{-}) = \pm \hbar v_F \sqrt{\left(\frac{q_{-}}{R_s(1)}\right)^2 + Q_{-}^2}$, $\epsilon(l, Q_{+}) = \pm \hbar v_F \sqrt{\left(\frac{q_{+}}{R_s(1)}\right)^2 + Q_{+}^2}$, $l = 0, 1, 2, 3, \ldots$. The STM density of states is computed for a voltage $V$ between the STM tip and the sample. The tunneling current is a function of the bias voltage $V$ and the chemical potential $\mu > 0$ [23]:

$$\frac{dI}{dV} \propto D(E = eV; s, u) \equiv \sum_{n} D^{(n)}(E = eV; s, u)$$

$$= \sum_{\eta = \pm} \left[ \sum_{m} \sum_{q_{+} = q_{-}} \sum_{\sigma} \left| U^{(n = 0; m, q_{-})}_{\sigma}(x, y) \right|^2 \delta \left[ eV + \mu - \eta \hbar v_F \sqrt{\left(\frac{2\pi}{L} m\right)^2 + q_{-}^2} \right] \right]$$

$$+ \sum_{n = \pm 1} \sum_{l} \sum_{Q_{+} = Q_{-}} \sum_{\sigma} \left| U^{(n = \pm 1; l, Q_{-})}_{\sigma}(\theta, u) \right|^2 \delta \left[ eV + \mu - \eta \hbar v_F \sqrt{\left(\frac{l + \frac{1}{2}}{R_s(1)}\right)^2 + Q_{+}^2} \right]$$

$$= \left( \eta = + \right) \text{corresponds to electrons with energy } 0 < \epsilon \leq \mu \text{ and } \eta = - \text{ corresponds to electrons below the Dirac point } \epsilon < 0. \text{ In the rest of this paper, we will take the chemical potentials to be } \mu = 120 \text{ mV (this is the typical value for the TI). We will neglect the states with } \eta = - \text{ which correspond to particles below the Dirac cone. The density of states at the tunneling energy } eV \text{ is weighted by the probability density of the STM tip at the position } [x, y] \text{ for } n = 0. \text{ The contours for } n = \pm 1 \text{ will be parameterized in terms of the polar angle } \theta \text{ and the transverse coordinate } u. \text{ The proportionality factor } J \text{ for the tunneling probability (not shown in the equation) } \frac{dI}{dV} = J D(V; x, y) \text{ is a function of the distance between the tip and the sample. The notation } D^{(n)}(V; x, y) \text{ represents the tunneling density for the different contours.}$$

4.2. The tunneling density of states $D^{(n = 0)}(V; x, y)$ for $n = 0$

Summing up the single-particle states weighted with occupation probability $\left| U^{(n = 0; m, q_{-})}_{\sigma}(x, y) \right|^2$, we obtain a space-dependent density of states for the 2D boundary surface, $\frac{L}{2} \leq x \leq \frac{L}{2}$, and the coordinate $y$ is restricted to the regions $\frac{d}{2} < y \leq \frac{L}{2}$ and $\frac{L}{2} < y \leq \frac{d}{2}$. We will perform the computation at the thermodynamic limit; namely, we replace the discrete momentum $\frac{2\pi}{L} k$ by $Y = \frac{L}{N}$ and $\frac{2\pi}{L} m$ by $X = \frac{m}{N}$, where $N = \frac{L}{\frac{d}{2}}$. We find for the dimensionless momentum $\hat{q} = qa$ the equations: $\hat{q}_{+}(Y) = \pi Y \pm \frac{1}{N} \tan^{-1}\left[ \frac{\hat{q}_{+}(Y)}{2\pi X} \right]$ where $2\pi X = pa = \hat{p}$. As a result we obtain the following density of states $\frac{\partial \hat{q}_{+}}{\partial Y}$:

$$\frac{\partial \hat{q}_{+}}{\partial Y}^{-1} = \frac{1}{\pi} \frac{\hat{q}_{+}^2 + \hat{p}^2 - \frac{1}{N} \hat{p}}{\hat{q}_{+}^2 + \hat{p}^2},$$

$$\frac{\partial \hat{q}_{-}}{\partial Y}^{-1} = \frac{1}{\pi} \frac{\hat{q}_{-}^2 + \hat{p}^2 + \frac{1}{N} \hat{p}}{\hat{q}_{-}^2 + \hat{p}^2}.$$
Using these results, we compute the tunneling density of states in terms of the energy $\mu + eV$ measured with respect to the chemical potential $\mu$ and the transverse energy $\epsilon_\perp \equiv h v_F q_\perp$.

$$D^{(n=0)}(V; x, y) = \left(\frac{L}{h v_F}\right)^2 \left(\frac{B^{(2)}}{L}\right)^2 \frac{1}{4\sqrt{G(x, y)}} e^{-\frac{B^{(2)} v}{2\pi \sqrt{x^2 + y^2}}} \left[\int_0^{E_{\text{max}}} \frac{d\epsilon_\perp}{\sqrt{(\mu + eV)^2 - \epsilon_\perp^2}}\right] \times \left[\frac{1}{2} \left(1 + \frac{h v_F}{L(\mu + V)} \sqrt{1 - \left(\frac{\epsilon_\perp}{\mu + V}\right)^2}\right) + \frac{1}{2} \left(1 - \frac{h v_F}{L(\mu + V)} \sqrt{1 - \left(\frac{\epsilon_\perp}{\mu + V}\right)^2}\right)\right]$$

$$+ \frac{h v_F}{L} \left[H[\mu + V - \frac{h v_F}{2L}] - H[\mu + eV - E_{\text{max}}]\right] \cdot \left(\cos \left[\frac{(\mu + eV)}{h v_F} y - \frac{\pi}{4}\right]\right)^2$$

$$+ \left(\cos \left[\frac{(\mu + eV)}{h v_F} y - \frac{\pi}{4}\right]\right)^2 \right]\right]\right] \left(\frac{L}{h v_F}\right)^2 \left(\frac{B^{(2)}}{L}\right)^2 \frac{1}{4\sqrt{G(x, y)}} e^{-\frac{B^{(2)} v}{2\pi \sqrt{x^2 + y^2}}} \times \left[\frac{\pi}{2} (\mu + eV) + \frac{h v_F}{L} \left[H[\mu + V - \frac{h v_F}{2L}] - H[\mu + eV - E_{\text{max}}]\right]\right] \text{ for } |y| > d. \tag{51}$$

$H[\mu + eV - \frac{h v_F}{2L}]$ is the step function, which is one for $\mu + eV - \frac{h v_F}{2L} \geq 0$ and zero otherwise. $a = \frac{2\pi}{\Lambda}$ is the short-distance cut-off and $E_{\text{max}} = h v_F \Lambda < 0.3 \text{ eV}$ is the maximal energy that restricts the validity of the Dirac model. We observe in the second line that the asymmetry in the density of states $1 \pm \frac{h v_F}{2L} \sqrt{1 - \left(\frac{\epsilon_\perp}{\mu + V}\right)^2}$ cancels.

Equation (51) shows that the tunneling density of states is linear in the energy $\mu + eV$ (in the present case we have looked only for energies above the Dirac cone). For the chemical potential $\mu = 120 \text{ mV}$, the zero energy corresponds to the voltage $V = -120 \text{ mV}$. The tunneling density of states has a constant part at energies $\frac{h v_F}{2L} \approx 0.2 \text{ mV}$ for $-120 \text{ mV} < V < -119.8 \text{ mV}$. For $V > -119.8 \text{ mV}$ the density of states is proportional to $\mu + eV$.

In figure 2, we have plotted the tunneling density of states as a function of the coordinates $x$ and $y$. The shape of the plot is governed by the multiplicative factor $e^{-\frac{B^{(2)} v}{2\pi \sqrt{x^2 + y^2}}}$ which governs the solutions in equation (35). We observe that the density of state is maximal in the region $|y| < 10B^{(2)}$.

Figure 3 shows the dependence on the voltage $V$ and the coordinate $y$. We observe the linear increase in the tunneling density of states, which is maximal in the region $|y| < 10B^{(2)}$. 

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Figure 2. The tunneling density of states for $n = 0$, $\frac{dI}{dV} \propto D^{(n=0)}(\frac{x}{B}, \frac{y}{B}; \mu = 120 \text{ mV})$. The right corner represents the intersection of the $x$-coordinate, which runs from 30 (right corner) to $-30$, and the $y$-coordinate, which runs from $-30$ (right corner) to 30 in units of the Burger vector.

Figure 3. The tunneling density of states for $n = 0$ as a function of $y$ and $V \frac{dI}{dV} \propto D^{(n=0)}(\frac{x}{B}, \frac{y}{B}; \mu = 120 \text{ mV})$. The voltage range is $-120 \leq V \leq 50$ and the $y$-coordinate is in the range $-30 \leq \frac{y}{B} \leq 30$.

4.3. The tunneling density of states $D^{(n=0)}(V, x, y; \vec{r}, \ldots, \vec{r}_{2M})$ for $2M$ dislocations

For many dislocations which satisfy $\sum_{w=1}^{2M} B^{(2,w)} = 0$ (the sum of the Burger vectors is zero) with the core centered at $[x_w, y_w], w = 1, 2, \ldots, 2M$ the coordinate $\vec{r} = (x, y) \rightarrow [X(\vec{r}), Y(\vec{r})]$ is replaced by $[X(\vec{r}) = x, Y(\vec{r}) = y + \sum_w \frac{B^{(2,w)}}{2\pi} \tan^{-1}(\frac{y-y_w}{x-x_w})]$. Following the method used
we show the tunneling density of states for an even number of dislocations in the $y$-directions which have the core on the $y = 0$ axis ($\vec{r}_w = [x_w, y_w = 0]$, $w = 1, 2, 3, \ldots, 2M$). When the coordinate of the $w = 1, 2, 3, \ldots, 2M$ dislocations is replaced by a continuum variable $w$ which can be described by a domain wall model: $h_{\text{domain-wall}}(x, y) = h v_F [−i\sigma^1 \partial_y + i\sigma^2 \partial_x − \sigma^3 \kappa M(y)]$, where $M(y) = \text{sgn}[y] |M(y)|$ [25]. Using this model, we find that the tunneling density of states density $D^{\text{domain-wall}} (V; x, y)$ confined to $|y| < W$ (the width $W$ depends on the explicit form of the domain wall function $M(y)$ and strength $\kappa$) is given by: $D^{\text{domain-wall}} (V; x, y) \propto (\frac{v_F}{\hbar v_F})^2 e^{-2x \int_0^W dy' M(y')}$. This shows the
similarity between the result obtain from the domain-wall model and the large numbers of
dislocations given in equation (54).

4.4. The tunneling density of states $D^{(n=\pm 1)}(V; \theta, u)$ for the $n=\pm 1$ contours

Following the same procedure as that used for $n=0$ and using the eigenfunctions for $n=\pm 1$,
we find that

$$
D^{(n=\pm 1)}(V; \theta, u) \equiv D^{(n=\pm 1)}(\mu, V; \theta, u)_{\text{even}} + D^{(n=\pm 1)}(\mu, V; \theta, u)_{\text{odd}}.
$$

(55)

For the even $k'$s, we solve for the momentum $Q_+$ and $Q_-$ and find

$$
D^{(n=\pm 1)}(\mu, V; \theta, u)_{\text{even}} = \frac{(B^{(2)})^2}{2\pi R_\varepsilon(1) D(1) \sqrt{G(\theta, u)}} \sum_{Q_r=Q_+, Q_-} \sum_{l=0}^\infty \delta
\times \left[ eV + \mu - \hbar v_F \sqrt{\left( \frac{l + \frac{1}{2}}{R_\varepsilon(1)} \right)^2 + Q_r^2} \left( e^{-2\zeta(\theta, u) \sin[2\theta]} + e^{2\zeta(\theta, u) \sin[2\theta]} \right)
\times \left( \sin(Q_r u - \frac{1}{2} \kappa(l, Q_r))^2 + \sin(Q_r u + \frac{1}{2} \kappa(l, Q_r))^2 \right)
+ 2(-1)^l \sin(Q_r u + \frac{1}{2} \kappa(l, Q_r)) \sin(Q_r u - \frac{1}{2} \kappa(l, Q_r))
\times \left( \cos \left( l \left( \theta + \frac{3\pi}{2} \right) - \zeta(\theta, u)(-\eta(u) + \cos[2\theta]) \right) \right)
- \cos \left( (l + 1) \left( \theta + \frac{3\pi}{2} \right) + \zeta(\theta, u)(-\eta(u) + \cos[2\theta]) \right) \right];
$$

(56)

Similarly, for the odd $k'$s we find that

$$
D^{(n=\pm 1)}(\mu, V; \theta, u)_{\text{odd}} = \frac{(B^{(2)})^2}{2\pi R_\varepsilon(1) D(1) \sqrt{G(\theta, u)}} \sum_{Q_r=Q_+, Q_-} \sum_{l=0}^\infty \delta
\times \left[ (e^{-2\zeta(\theta, u) \sin[2\theta]} + e^{2\zeta(\theta, u) \sin[2\theta]}) \left( \cos \left( Q_r u - \frac{1}{2} \kappa(l, Q_r) \right) \right)^2
\times \left( \cos \left( Q_r u + \frac{1}{2} \kappa(l, Q_r) \right) \right)^2 \right)
+ 2(-1)^l \cos \left( Q_r u + \frac{1}{2} \kappa(l, Q_r) \right)
\times \cos \left( Q_r u - \frac{1}{2} \kappa(l, Q_r) \right) \left( \cos \left( l \left( \theta + \frac{3\pi}{2} \right) - \zeta(\theta, u)(-\eta(u) + \cos[2\theta]) \right) \right)
- \cos \left( (l + 1) \left( \theta + \frac{3\pi}{2} \right) + \zeta(\theta, u)(-\eta(u) + \cos[2\theta]) \right) \right].
$$

(57)

For the present case, the energy scale of the excitations is governed by the radius $R_\varepsilon(1)$ and
width $D$. The spectrum is discrete and we cannot replace it by a continuum density of states as
we did for the case $n=0$.
Figure 5. The discrete tunneling density of states for \( n = 1 \), as a function of the voltage \( VD^{(n=1)}(V; \theta = \frac{\pi}{2}, \frac{u}{B^{\frac{1}{2}}}, \mu = 120 \text{ mV}) \).

Figure 6. The tunneling density of states as a function of the polar angle \( \theta \) for a fixed energy. The periodicity in \( \theta \) is controlled by the discrete energy eigenvalues.

In figure 5 we show the tunneling density of states at a fixed polar angle \( \theta = \frac{\pi}{2} \) as a function of voltage \( V \). We observe that the density of states is dominated by high energy eigenvalues. These solutions are localized in energy. The range of the spectrum is above \( \mu + eV > 200 \text{ mV} \) which is well separated from the low-energy spectrum controlled by the \( n = 0 \) contour (which ranges from \(-120 \) to \(70 \text{ mV}\)).

Figure 6 shows the tunneling density of states as a function of the polar angle \( \theta \) for a fixed energy. The periodicity in \( \theta \) is controlled by the discrete energy eigenvalues.

In figure 7, we show the tunneling density of states at a fixed voltage \( V \) as a function of the polar angle \( 0 < \theta < \pi \) and width \( |u| < 0.1 \).

5. The charge current: the in-plane spin on the surface of the \( h^\text{TI} \) Hamiltonian

5.1. The current in the absence of the edge dislocation for \( h^\text{TI} \)

From the Hamiltonian given in equation (1) we compute the equation of motion for the velocity operator: \( \frac{dx}{dt} = \frac{i}{\hbar} [x, \hat{h}] = v_F \sigma^y \), \( \frac{dy}{dt} = \frac{i}{\hbar} [y, \hat{h}] = -v_F \sigma^x \). We multiply the velocity operator by
the charge \((-e)\) and identify the charge current operators: \(\hat{J}_x = (-e)v_F\sigma^2\), \(\hat{J}_y = (-e)(-v_F)\sigma^1\). This also represent the ‘real’ spin on the surface. Therefore, the charge current is a measure of the in-plane spin on the surface.

Integrating over the \(y\)-coordinate we obtain the current \(I_{x}^{T_{1}}\) in the \(x\)-direction. Using the eigenstates \(\Gamma_{p,q}(x, y)\) and \(\Gamma_{-p,q}(x, y)\) of the \(h^{T_{1}}\) Hamiltonian

\[
\Gamma_{p,q}(x, y) = e^{ipx} e^{iqy} \left( i e^{ix(p,q)} \right),
\]

\[
\Gamma_{-p,q}(x, y) = e^{-ipx} e^{iqy} \left( -ie^{-ix(p,q)} \right),
\]

we find \((\Gamma_{p,q}(x, y))(\sigma^2)(\Gamma_{p,q}(x, y)) = -(\Gamma_{-p,q}(x, y))(\sigma^2)(\Gamma_{-p,q}(x, y))\); therefore, we conclude that the current \(I_{x}^{T_{1}} = 0\) is zero.

5.2. The current for the ground state in the presence of an edge dislocation

We will compute the current in the presence of the edge dislocation. The current operator \(\hat{J}_{edge}^{x}(x, y)\) will be given in terms of the transformed currents. We find that the current density operator \(J_{x}^{edge}(x, y)\) is given by

\[
\hat{J}_{x}^{edge}(x, y) = (-e)v_F[\sigma^2 e_1^x - \sigma^1 e_2^x] = (-e)v_F\sigma^2 - (-e)v_F B^{(2)} \frac{y \sigma^1 + x \sigma^2}{x^2 + y^2} \approx (-e)v_F\sigma^2.
\]

We use the zero-order current operator \(\hat{J}_{edge}^{x}(x, y) \approx (-e)v_F\sigma^2\) to construct the second quantization form for the current density. The operator is defined with respect to the shifted ground state \(|\mu\rangle \equiv |\tilde{0}\rangle\) with the energy \(E = \epsilon - \mu\) measured with respect to the chemical potential and spinor field \(\Psi_{n=0}(x, y)\).

\[
J_{x}^{edge}(x, y) = \langle \mu | \Psi_{n=0}^{\dagger}(x, y) \hat{J}_{x}^{edge}(x, y) \Psi_{n=0}(x, y) |\mu\rangle.
\]
Using the spinor eigenfunction given in equation (35) and the second quantized form with the electron-like operators $\alpha_{E,R}$, $\alpha_{E,L}$ and hole-like $\beta_{E,R}$, $\beta_{E,L}$ we find that

$$\Psi_{n=0}(x, y; t) \approx \sum_{E>0} \left[ \alpha_{E,R} \left( \begin{array}{c} U_{n=0,R}^{(n=0,R)}(x, y) \\ U_{n=0,R}^{(n=0,R)}(x, y) \end{array} \right) \right] e^{-iEtf} + \beta_{E,R}^{\dagger} \left( \begin{array}{c} U_{n=0,R}^{(n=0,R)}(x, y) \\ U_{n=0,R}^{(n=0,R)}(x, y) \end{array} \right) e^{iEtf}$$

$$+ \alpha_{E,L} \left( \begin{array}{c} U_{n=0,L}^{(n=0,L)}(x, y) \\ U_{n=0,L}^{(n=0,L)}(x, y) \end{array} \right) \right] e^{-iEtf} + \beta_{E,L}^{\dagger} \left( \begin{array}{c} U_{n=0,L}^{(n=0,L)}(x, y) \\ U_{n=0,L}^{(n=0,L)}(x, y) \end{array} \right) e^{iEtf}. \tag{60}$$

The current is a sum of two terms computed with the eigen-spinor obtained in equation (35): $[U_{n=0,R}^{(n=0,R)}(x, y), U_{n=0,R}^{(n=0,R)}(x, y)]$ and $[U_{n=0,L}^{(n=0,L)}(x, y), U_{n=0,L}^{(n=0,L)}(x, y)]$. The energy $e_{\perp}$ thus dislocations, the density of states is asymmetric $1 + \frac{1}{\pi} \int_{L(x+y)}^{h_{V}} \sqrt{1 - \left( \frac{\xi - \mu}{\hbar V} \right)^{2}}$, resulting in a finite current. We integrate over the transversal direction $y$ and obtain the edge current $I_{x}^{n=0, \text{edge}}$.

$$I_{x}^{n=0, \text{edge}} = (-e) v_{F} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dy \langle \mu | J_{x}^{\text{edge}}(x, y) | \mu \rangle$$

$$= \frac{(-e) v_{F}}{4\pi} \left( \begin{array}{c} L \\ \hbar v_{F} \end{array} \right)^{2} \left( \frac{1}{L} \right)^{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dy \frac{e^{2\epsilon_{\perp}} \left( \epsilon_{\perp} \right)^{2}}{\sqrt{G(x, y)}} \int d\epsilon_{\parallel} \int d\epsilon_{\perp}$$

$$\times H \left[ \mu - \sqrt{\left( \epsilon_{\parallel} \right)^{2} + \left( \epsilon_{\perp} \right)^{2}} \right] \left( \frac{\hbar v_{F}}{L} \right) \cdot \frac{e_{\perp}}{\left( \epsilon_{\parallel} \right)^{2} + \left( \epsilon_{\perp} \right)^{2}}$$

$$= \frac{1}{4\pi} \left( \frac{-e v_{F}}{L} \right) \left( \frac{\mu}{\hbar v_{F}/L} \right) \left( \frac{B^{(2)}}{L} \right) \left( H \left[ \mu + eV - \frac{\hbar v_{F}}{L} \right] - H \left[ \mu + eV - E_{\text{max}} \right] \right); \tag{61}$$

$$f \left( \frac{B^{(2)}}{L} \right) \approx 6.22$$

$H[\mu - \sqrt{\left( \epsilon_{\parallel} \right)^{2} + \left( \epsilon_{\perp} \right)^{2}}]$ is the step function, which is one for $\sqrt{\left( \epsilon_{\parallel} \right)^{2} + \left( \epsilon_{\perp} \right)^{2}} \leq \mu$. The single-particle energies are $\epsilon_{\perp} = \hbar v_{F} q_{\perp}$ and $\epsilon_{\parallel} = \hbar v_{F} p$. For $L \approx 10^{-6}$ m, chemical potential $\mu = 120$ mV and $L_{\perp}^{2} \approx 100$, we find that the current $I_{x}^{n=0, \text{edge}}$ is in the range of mA.

To conclude, we have shown that the presence of an edge dislocation gives rise to a non-zero current, which is a manifestation of the in-plane component of the spin on the 2D surface. Therefore, a non-zero value $I_{x}^{n=0, \text{edge}} \neq 0$ will be an indication of the presence of edge dislocation. This effect might be measured using a coated tip with magnetic material used by the magnetic force microscopy technique.

5.3. The backscattering effect on the conduction band due to many dislocations

The edge Hamiltonian with many dislocations with Burger vectors $B^{(2)}$ in the $y$-direction is given by

$$h_{\text{edge-dislocations}} = i \sigma^{1} \partial_{y} - i \sigma^{2} \partial_{x} - \frac{i}{2} \sigma^{2} V_{\text{g}}, \tag{62}$$
where $V_{sc}$ is the scattering potential that is controlled by the distribution of dislocations

$$V_{sc}(\vec{r}) \equiv \sum_{u=1}^{2M} B^{(2,u)} \delta^2(\vec{r} - \vec{r}_w).$$

(To compute the current, we need to include an external potential which is not included in the Hamiltonian.) Using the field operator representation:

$$\Psi(\vec{r}) = \sum_{\vec{k}} \psi(\vec{k}) C(\vec{k}),$$

where $C(\vec{k}) = U^{(\pm)}(\vec{k})$ are the eigenfunctions in the absence of dislocations. We find

$$H = H_0 + H_{sc},$$

$$H_0 = \sum_{\vec{k}} \hbar v_F |k| [C^\dagger(\vec{k})C(\vec{k}) + B^\dagger(-\vec{k})B(-\vec{k})] - \mu \sum_{\vec{k}} [C^\dagger(\vec{k})C(\vec{k}) + B^\dagger(-\vec{k})B(-\vec{k})],$$

$$H_{sc} = \sum_{\vec{k}} \sum_{\vec{k}'} \hat{V}_\parallel(\vec{k}, \vec{k}') [C^\dagger(\vec{k})C(\vec{k}') + B^\dagger(-\vec{k})B(-\vec{k}')] + \hat{V}_\perp(\vec{k}, \vec{k}') [C^\dagger(\vec{k})B(\vec{k}') + B^\dagger(\vec{k})C(\vec{k}')] ,$$

$$\hat{V}_\parallel(\vec{k}, \vec{k}') = \frac{i}{4} \left[ e^{i\vec{k} \cdot \vec{r}} + e^{-i\vec{k} \cdot \vec{r}} \right] V_{sc}(\vec{k} - \vec{k}'),$$

$$\chi(\vec{k}) \equiv \chi(k_x, k_y) = \tan^{-1} \left( \frac{k_y}{k_x} \right).$$

For $\vec{k}' = -\vec{k}$, we find the backscattering matrix element for the conduction band:

$$\hat{V}_\parallel(\vec{k}, -\vec{k}) = -\frac{1}{2} \sin[\chi(\vec{k})] V_{sc}(2\vec{k}) = -\frac{1}{2} \frac{k_y}{k_x^2 + k_y^2} V_{sc}(2\vec{k}).$$

In agreement with the experimental situation where the chemical potential $\mu$ is large, we can study separately the effect of the many dislocations on the conduction band:

$$H \approx \sum_{\vec{k}} (\hbar v_F |k| - \mu) C^\dagger(\vec{k})C(\vec{k}) + \sum_{\vec{k}} \sum_{\vec{k}'} C^\dagger(\vec{k}) \hat{V}_\parallel(\vec{k}, \vec{k}') C(\vec{k}').$$

We observe that for $k_x = 0$ the backscattering is absent and allowed for other directions.

In the presence of a Fermi surface $\mu = \hbar v_F k_F$, for momenta $q_x, q_y \equiv k_y = 0$ (around the Fermi points $k_x = 0$ and $k_y = \pm k_F + q_x$), the propagation of the conduction electrons will be perfect.

This result is very different from a regular impurity scattering model $h_{\text{edge-impurity}} = i\sigma^1 \partial_x - i\sigma^2 \partial_y + V_{\text{imp}}(\vec{r})$, where $V_{\text{imp}}(\vec{r})$ the scalar scattering potential gives rise to an effective backscattering matrix element that vanishes for all the directions $\vec{k}$! (For edge dislocations in the x-direction with $B^{(1)}$, the backscattering is absent for $k_x = 0$.)

6. Conclusions

We have used the coordinate transformation method to investigate TI in the presence of deformations. We have computed the spin connection and the metric tensor for the 3D TI. This theory has been applied to the surface of a TI with an edge dislocation. We have shown that the tunneling density of states is confined to the 2D region $n = 0$ and to high-energy circular contours with $n = \pm 1$. The edge dislocations violate the parity symmetry. As a result a current that is a manifestation of in-plane spin orientation is generated. The in-plane spin orientation is a manifestation of the parity violation induced by the edge dislocation. The backscattering for conduction electrons vanishes for $k_y = 0$. We propose that scanning tunneling methods might be able to verify our prediction.
Appendix A

We assume that a 2D manifold with a mapping from the curved space $X^a$, $a = 1, 2$, to the local flat space $x^\mu$, $\mu = x, y$, exists. We introduce the tangent vector $[31] e^a_\mu(\vec{x}) = \frac{\partial X^a(\vec{x})}{\partial x^\mu}$, $\mu = x, y$, which satisfies the orthonormality relation $e^a_\mu(\vec{x}) e^b_\mu(\vec{x}) = \delta_{a,b}$ (here we use the convention that we sum over indices that appear twice). The metric tensor for the curved space is given in terms of the flat metric $\delta_{a,b}$ and the scalar product of the tangent vectors: $e^a_\mu(\vec{x}) e^b_\mu(\vec{x}) = g_{\mu,\nu}(\vec{x})$. The linear connection is determined by the Christoffel tensor $\Gamma^{\lambda}_{\mu,\nu}$:

$$\nabla_{\partial_x} \partial_y = -\Gamma^{\lambda}_{\mu,\nu} \partial_{\lambda}. \quad (A.1)$$

The Christoffel tensor is constructed from the metric tensor $g_{\mu,\nu}(\vec{x})$:

$$\Gamma^{\lambda}_{\mu,\nu} = -\frac{1}{2} \sum_{\tau=x,y} g^{\lambda,\tau}(\vec{x}) \left[ \partial_\nu g_{\tau,\tau}(\vec{x}) + \partial_\mu g_{\tau,\tau}(\vec{x}) - \partial_\tau g_{\mu,\nu}(\vec{x}) \right]. \quad (A.2)$$

Next, we introduce the vector field $\tilde{V} = V^a \partial_x = V^\mu \partial_\mu$, where $a = 1, 2$ are the components in the curved space and $\mu = x, y$ represents the coordinate in the fixed Cartesian frame. The covariant derivative of the vector field $V^a$ is determined by the spin connection $\omega^{a}_{\mu,b}$, which needs to be computed:

$$D_\mu V^a(\vec{x}) = \partial_\mu V^a(\vec{x}) + \omega^{a}_{\mu,b} V^b. \quad (A.3)$$

For a two-component spinor, we can identify the spin connection in the following way: the spinor in the curved space (generated by the dislocation) is represented by $\tilde{\Psi}(\vec{x})$ and in the Cartesian space it is given by $\Psi(\vec{x})$ [37]. The two-component spinor represents a chiral fermion which transforms under spatial rotation as a spin half-fermion:

$$\tilde{\Psi}(\vec{x}) = e^\frac{i}{2} \omega_{1,2} \sigma^3 \Psi(\vec{x}),$$

$$e^\frac{i}{2} \omega_{1,2} \sigma^3 \equiv e^\frac{i}{2} \omega_{a,b} \Sigma^{a,b} \equiv e^{\sum_{a=1,2} \sum_{b=1,2} \frac{1}{4} \omega_{a,b} \Sigma^{a,b}},$$

$$\omega_{a,b} \equiv -\omega_{b,a},$$

$$\Sigma^{a,b} \equiv \frac{1}{2} [\sigma^a, \sigma^b].$$

We have used the anti-symmetric property of the rotation matrix $\omega_{a,b} \equiv -\omega_{b,a}$, and the representation of the generator $\Sigma^{a,b}$ in terms of the Pauli matrices.

Therefore, for a two-component spinor we obtain the connection

$$D_\mu \Psi(\vec{x}) = \left( \partial_\mu + \frac{1}{4} \omega^{a}_{\mu} \Sigma_{a,b} \right) \Psi(\vec{x}) \equiv \left( \partial_\mu + \frac{1}{8} \omega^{a}_{\mu,\nu} \left[ \sigma_a, \sigma_b \right] \right) \Psi(\vec{x}). \quad (A.5)$$

Next we will compute the spin connection $\omega^{a}_{\mu,b}$ using the Christoffel tensor. In the physical coordinate basis $x^\mu$, the covariant derivative $D_\mu V^v(\vec{x})$ is determined by the Christoffel tensor:

$$D_\mu V^v(\vec{x}) = \partial_\mu V^v(\vec{x}) + \Gamma^{v}_{\mu,\nu} V^\lambda. \quad (A.6)$$

The relation between the spin connection and the linear connection can be obtained from the fact that the two covariant derivatives of the vector $\tilde{V}$ are equivalent:

$$D_\mu V^a = e^a_v D_\mu V^v. \quad (A.7)$$

Since we have the relation $V^a = e^a_v V^v$, it follows from the last equation that

$$D_\mu [e^a_v] = D_\mu \partial_v e^a = (D_\mu \partial_v) e^a + \partial_v (D_\mu e^a) = 0. \quad (A.8)$$
Using the definition of the Christoffel index and the differential geometry relation $\nabla_{\partial_i} \partial_j = -\Gamma^k_{\mu, i} \partial_k [31]$, we obtain the relation between the spin connection and the linear connection:

$$D_{\mu}[\omega^\mu_\nu] = \partial_\mu \omega^\rho_\nu - \Gamma^\rho_{\mu, i} \partial_i \omega^\rho_\nu + \omega^a_{\mu, i} \omega^{\mu a}_\nu \equiv 0. \quad (A.9)$$

Solving this equation, we obtain the spin connection given in terms of the Burger vector. We multiply from left equation (A.4) by the tangent vector $e^\mu_\nu$ and replace $\Gamma^\mu_{\mu, i}$, with the representation given in equation (63). We use the metric tensor relations $e^a_\mu(\vec{x})e^\mu_\nu(\vec{x}) = \delta_{\nu, b}$, $\omega^a_\mu(\vec{x})e^\mu_\nu(\vec{x}) = g_{\mu, \nu}(\vec{x})$, and find that $[31]

$$\omega^a_\mu = \frac{1}{2} e^{\nu, a}(\partial_\mu e^\nu_\nu - \partial_{\nu} e^\nu_\mu) - \frac{1}{2} e^{\nu, b}(\partial_\mu e^\nu_\nu - \partial_{\nu} e^\nu_\mu) - \frac{1}{2} e^{\rho, a} e^{\sigma, b}(\partial_\rho e_{\sigma, c} - \partial_{\sigma} e_{\rho, c}) e^c_\mu. \quad (A.10)$$

We note the asymmetry between $e^{\nu, a}$ and $e_{\alpha, \nu}$:

$$e^{\nu, a} \equiv g^{\nu, \lambda} e^\lambda_a \quad \text{and} \quad e_{\alpha, \nu} \equiv \delta_{\alpha, \nu} e^\nu_b.$$ 

For our case, we have a two-component spin connection $\omega^a_\mu$ and $\omega^b_\nu$:

$$\omega^a_\mu = \frac{1}{2} e^{v, a}(\partial_\mu e^v_v - \partial_v e^v_\mu) - \frac{1}{2} e^{\nu, b}(\partial_\mu e^\nu_\nu - \partial_{\nu} e^\nu_\mu) - \frac{1}{2} e^{\rho, a} e^{\sigma, b}(\partial_\rho e_{\sigma, c} - \partial_{\sigma} e_{\rho, c}) e^c_\mu,$$

$$\omega^b_\nu = \frac{1}{2} e^{v, a}(\partial_\nu e^v_v - \partial_v e^v_\nu) - \frac{1}{2} e^{\nu, b}(\partial_\nu e^\nu_\nu - \partial_{\nu} e^\nu_\nu) - \frac{1}{2} e^{\rho, a} e^{\sigma, b}(\partial_\rho e_{\sigma, c} - \partial_{\sigma} e_{\rho, c}) e^c_\nu. \quad (A.11)$$

These equations are further simplified with the help of equations (11)–(17) with $e^1_v = 0, e^1_v = 1$ and the Burger tensor $\partial_v e^2_x - \partial_x e^2_v = B^{(2)} \delta^2(\vec{r})$.

$$\omega^a_\mu = \frac{1}{2} g^{\nu, a}[\partial_\mu e^\nu_\nu - \partial_{\nu} e^\nu_\mu] - \frac{1}{2} g^{\rho, r} e^{\rho, a}_s e^s_\nu[\partial_\nu(\delta_{c, b} e^b_\rho) - \partial_\rho(\delta_{c, d} e^d_\nu)] e^c_\mu = \frac{1}{2} B^{(2)} \delta^2(\vec{r}) \left[ g^{v, x} e^1_\nu + g^{v, y} e^1_\nu - (g^{y, r} g^{v, x} - g^{x, r} g^{v, y}) e^1_\nu \right]$$

$$\approx \frac{1}{2} B^{(2)} \delta^2(\vec{r}) \left[ -\frac{B^{(2)}}{2\pi} \frac{y}{\nu^2 + x^2} - \left(1 - \frac{B^{(2)}}{2\pi} \frac{y}{\nu^2 + x^2} \right)^2 \right] \times \left( \frac{B^{(2)}}{2\pi} \frac{y}{\nu^2 + x^2} \right) \left(1 - \frac{B^{(2)}}{2\pi} \frac{x}{\nu^2 + y^2} \right)$$

$$\approx \frac{1}{2} B^{(2)} \delta^2(\vec{r}) \left[ -\frac{B^{(2)}}{2\pi} \frac{2y - x}{\nu^2 + x^2} \right] \quad (A.12)$$

and

$$\omega^b_\nu = \frac{1}{2} e^{v, a}(\partial_\nu e^v_v - \partial_v e^v_\nu) - \frac{1}{2} e^{\nu, b}(\partial_\nu e^\nu_\nu - \partial_{\nu} e^\nu_\nu) - \frac{1}{2} e^{\rho, a} e^{\sigma, b}(\partial_\rho e_{\sigma, c} - \partial_{\sigma} e_{\rho, c}) e^c_\nu = \frac{1}{2} g^{v, a} e^1_\nu[\partial_\nu e^\nu_v - \partial_v e^\nu_\nu] - \frac{1}{2} g^{\rho, r} e^1_\nu[\partial_v e^1_v - \partial_v e^1_v] - \frac{1}{2} g^{\rho, r} e^1_\nu g^{\rho, a} e^a_v[\partial_\nu e^c_\rho - \partial_\rho e^c_\nu] e^c_v = -\frac{B^{(2)}}{2} \delta^2(\vec{r}) g^{v, x} e^1_\nu - \frac{B^{(2)}}{2} \delta^2(\vec{r}) [g^{y, r} g^{v, x} - g^{x, r} g^{v, y}] e^1_\nu e^2_\nu \approx -\frac{B^{(2)}}{2} \delta^2(\vec{r}). \quad (A.13)$$

To first order for the Burger vector $B^{(2)}$ the spin connections are given by $\omega^a_\mu = -\omega^a_\mu \approx 0$ and $\omega^b_\nu = -\omega^b_\nu \approx -\frac{B^{(2)}}{2} \delta^2(\vec{r})$. 

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