Lovelock black holes in a string cloud background

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We present an exact static, spherically symmetric black hole solution to the third order Lovelock gravity with a string cloud background in seven dimensions for the special case when the second and third order Lovelock coefficients are related via \(\tilde{\alpha}_2^2 = 3\tilde{\alpha}_3^2\) (\(\equiv \tilde{\alpha}^2\)). Further, we examine thermodynamic properties of this black hole to obtain exact expressions for mass, temperature, entropy and also perform the thermodynamical stability analysis. We see that a string cloud background makes a profound influence on horizon structure, thermodynamic properties and the stability of black holes. Interestingly the entropy of the black hole is unaffected due to a string cloud background. However, the critical solution for thermodynamic stability is being affected by a string cloud background.

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I. INTRODUCTION

Black holes through quantum outcome indicate that they radiate due to the Hawking effect \cite{1}. In the absence of established theories of quantum gravity, black holes have become a main playground to divulge quantum gravity effects through their thermodynamics. Black holes have been used as theorists’ laboratories in many other relevant fields. Thermodynamic properties of black holes have been studied for many years, but established statistical explanations of black hole thermodynamics are still lacking. It shows that black holes also have the standard thermodynamic quantities, such as temperature, entropy, etc., and even possess abundant phase structures like the Hawking-Page phase transition \cite{2} and similar critical phenomena in ordinary thermodynamic systems.

Recent years witness the renewed interest towards the study of black hole solutions specially in modified theories of gravity \cite{3, 4}, as besides theoretical results, cosmological evidence, e.g. dark matter and dark energy, suggests possibility of changing the Einstein gravity. On the other hand, the Einstein gravity cannot be quantized (non-renormalizable), so it is believed that it is a low energy effective theory and could be modified with higher derivative terms at high energy \cite{5}. Modifications to the Einstein gravity theory, for instance the Lovelock theory \cite{3}, the \(f(R)\) gravity theory \cite{4}, etc. have been studied extensively. Those models in higher spacetime dimensions have very different features. For example, natures of stability in higher dimensions are quite different. Extending spacetime dimensionality in gravity theories has been one possible way to combine other interactions with gravity or often seems to be even required in many theories, e.g. Kaluza-Klein theory, a string theory, Brane world scenarios, etc. \cite{6}.

In these context, apart from the standard Einstein-Hilbert action, there also exist interesting theories of gravity in dimensions greater than four involving higher powers of the curvatures such that the field equations for the metric are at most in second-order. Among the higher curvature gravity Theories, the most extensively studied theory is the so-called Lovelock gravity \cite{3}, which naturally emerges when we wish to generalize the Einstein theory in higher dimensions by keeping all characteristics of usual general relativity excepting the linear dependence of the Riemann tensor. The Lovelock gravity is one of the most general second order theories in higher dimensional spacetime, which is free from ghosts. The Lovelock theory may play an important role in string theory where the low energy effective field theory of gravity contains higher curvature terms.

In this sense the Lovelock gravity \cite{3} is a natural extension to the Einstein gravity. It is constructed by sum of all the Euler densities of a \(2n\)-dimensional manifold. The Lagrangian is given by,

\[
\mathcal{L} = \sum_{n=0}^{t} \alpha_n \mathcal{L}_n,
\]

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where

\[ \mathcal{L}_n = \frac{1}{2^n} \delta_{\alpha_1 \beta_1 \ldots \alpha_n \beta_n} \prod_{r=1}^n R^{\alpha_r \beta_r}_{\mu_r \nu_r}. \]  

A spacetime dimension \( D \) can be written as \( D = 2t + 2 \) for even dimensions and \( D = 2t + 1 \) for odd dimensions. The nth order higher derivative terms \( \mathcal{L}_n \) becomes a surface term when \( D \leq 2n \). Non-trivial extra term contribute to equations of motion in higher dimensions but not in dimensions less than \( 2n \). Moreover, higher derivative terms can cancel a ghost term. For instance, the reference \([16]\) shows that the second order Lovelock (Gauss-Bonnet) terms cancel a ghost term. Boulware and Deser \([8]\) first found a static, spherically symmetric black hole solution with the Gauss-Bonnet corrections. Using the Gauss-Bonnet gravity static, spherically symmetric solutions are obtained later \([9, 10]\) with thermodynamical properties \([11]\). Static, spherically symmetric black hole solutions in the Lovelock gravity with general energy momentum tensors in any arbitrary dimension can be found in \([12]\) and also in references \([13, 14]\) and its thermodynamics in \([14]\). Further extensive studies on the Gauss-Bonnet black holes with a focus on thermodynamic properties have been found in \([15, 16]\). The special third order Lovelock gravity also received a significant attention, e.g., for a black solution and its thermodynamics in this theory with Born-Infeld source \([17]\). Also, topological properties of the general Lovelock black holes in the context of thermodynamics have been investigated \([18]\).

In this paper, we begin with finding static, spherically symmetric black hole solutions for a string cloud background for a specific case, i.e. \( \tilde{\alpha}_3^2 = 3\tilde{\alpha}_3 \) and examine thermodynamical properties in the third order Lovelock gravity. The solution in there can be utilized to calculate mass, temperature, entropy and heat capacity of black holes and explicitly study effects of a string cloud background. It turns out that the horizon and thermodynamic properties of Lovelock black hole in conjunction with background string could have some interesting features. It may be pointed out that gravity coupled to a cloud of strings may be very useful and important as the Universe can be considered as a collection of extended objects, like, one dimensional string \([19]\). The study of black holes in a cloud of strings model was initiated by Letelier \([19]\) modifying the Schwarzschild solutions for a cloud of strings as a source \([19]\), which was recently extended to the Gauss-Bonnet gravity \([20]\) and also to the Lovelock gravity \([21, 22]\). We show that a string cloud background makes a profound influence on horizon structure and thermodynamic quantities but entropy is not changed.

The paper is organized as follows: In Sec. II we begin examining the third order Lovelock action, which is a modification of the Einstein-Hilbert action, and also derive energy momentum tensors of a cloud of strings. The thermodynamics of a static black hole solution in this theory is explored in Sec. IV. Before that we find an exact static spherically symmetric black hole solution in Sec. III. The paper ends in Sec. V, which gives concluding remarks. We have used units that fix \( G = c = 1 \) and the metric signature, \((- , + , + , \cdots , +)\).

**II. LOVELOCK ACTION AND EQUATIONS OF MOTION**

The Lovelock theory is the most general theory of gravity that gives second order field equations in arbitrary dimensions. The recent interest in the Lovelock theory arose because its action appears as a low energy limit of a heterotic superstring theory. The simplest third order Lovelock action reads \([3]\):

\[ I_G = \frac{1}{2} \int_M dx^D \sqrt{-g} \left[ \mathcal{L}_1 + \alpha_2 \mathcal{L}_{GB} + \alpha_3 \mathcal{L}_{(3)} \right] + I_S, \]

where \( I_S \) is a matter action due to cloud of strings. The Einstein term \( \mathcal{L}_1 \) is \( R \), the second order Lovelock (Gauss-Bonnet) term \( \mathcal{L}_{GB} \) is

\[ \mathcal{L}_{GB} = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4R_{\mu\nu} R_{\mu\nu} + R^2, \]

and the third order Lovelock Lagrangian is

\[ \mathcal{L}_{(3)} = 2R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\rho\tau} R^{\rho\tau}_{\mu\nu} + 8R^{\mu\nu}_{\sigma\rho} R^{\sigma\kappa}_{\mu\nu} R^{\rho\tau}_{\kappa\tau} + 3R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\mu\nu} + 24R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\rho\tau} R^{\rho\tau}_{\mu\nu} + 16R^{\mu\nu}_{\sigma\rho} R_{\sigma\rho\tau} R^{\tau}_{\mu\nu} - 12RR_{\mu\nu} R_{\mu\nu} + R^2. \]

Here \( R \), \( R_{\mu\nu\gamma\delta} \) and \( R_{\mu\nu} \) are the Ricci scalar, the Riemann and the Ricci tensors, respectively. The coupling constants \( \alpha_2 \) and \( \alpha_3 \) in Eq. \([3]\) have dimensions, \([\text{length}]^{4-D}\) and \([\text{length}]^{6-D}\), respectively and will help us to explicitly bring
out changes in the general relativity equations. In the limits $(\alpha_2, \alpha_3) \to 0$, one recovers the Einstein-Hilbert action. Variation of the action with respect to the metric $g_{\mu\nu}$ yields modified field equations for the third order Lovelock gravity,

$$G^E_{\mu\nu} + \alpha_2 G^{GB}_{\mu\nu} + \alpha_3 G^{(3)}_{\mu\nu} = T_{\mu\nu},$$

where $G^E_{\mu\nu}$ is the Einstein tensor, while $G^{GB}_{\mu\nu}$ and $G^{(3)}_{\mu\nu}$ are given explicitly in [23], respectively as:

$$G^{GB}_{\mu\nu} = 2 \left( -R_{\mu\sigma\kappa\tau} R^{\kappa\sigma\tau}_{\nu} - 2R_{\mu\rho\nu\sigma} R^{\rho\sigma} - 2R_{\mu\rho} R^\rho_{\nu} + R R_{\mu\nu} \right) - \frac{1}{2} g_{\mu\nu} G_{GB},$$

and

$$G^{(3)}_{\mu\nu} = -3 \left( 4R^{\tau\rho\sigma\kappa} R_{\tau\rho\sigma\kappa\lambda\mu} R_{\nu\mu} - 8R^{\tau\rho} R_{\sigma\mu\lambda} R_{\lambda\mu} - 8R^{\tau\rho} R_{\sigma\mu\lambda} R_{\lambda\mu} - 8R^{\tau\rho} R_{\sigma\mu\lambda} R_{\lambda\mu} ight)$$

Next we turn attention to calculate energy-momentum tensor of a cloud of strings [19], for further details. The Nambu-Goto action of a string evolving in spacetime is given by

$$\mathcal{I}_S = \int_{\Sigma} \mathcal{L} \, d\lambda^0 \, d\lambda^1, \quad \mathcal{L} = m(\gamma)^{-1/2},$$

with the Lagrangian for a cloud of strings [19]:

$$\mathcal{L} = m \left[ -\frac{1}{2} \Sigma^{\mu\nu} \Sigma_{\mu\nu} \right]^{1/2}.$$  

The string worldsheet is associated with a bivector of the form

$$\Sigma^{\mu\nu} = \epsilon^{ab} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b},$$

where $\epsilon^{ab}$ is the two-dimensional Levi-Civita tensor and $\epsilon^{01} = -\epsilon^{10} = 1$. Here $m > 0$ is a constant such that each string and $\gamma$ is the determinant of an induced metric on the string world sheet $\Sigma$ given by

$$\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b}.$$  

($\lambda^0, \lambda^1$) with $\lambda^0$ and $\lambda^1$ which are a timelike and a spacelike parameter, respectively [24] is a parametrization of the world sheet $\Sigma$. Further, since $T^{\mu\nu} = \partial \mathcal{L} / \partial g^{\mu\nu}$, then the energy-momentum tensor for one string reads

$$T^{\mu\nu} = m \Sigma^{\mu\rho} \Sigma_{\rho\nu} / (\gamma)^{1/2}.$$  

Hence, a cloud of strings has the energy-momentum tensor

$$T^{\mu\nu} = \rho \Sigma^{\mu\sigma} \Sigma_{\sigma\nu} / (\gamma)^{1/2},$$

where $\rho$ is a proper density of a cloud of strings. The quantity $\rho (\gamma)^{-1/2}$ is a gauge-invariant density.
III. SPHERICALLY SYMMETRIC SOLUTION IN LOVELOCK GRAVITY

Here we wish to obtain static, spherically symmetric black hole solutions to Eq. (6) for the energy momentum tensors, Eq. (12), and investigate their horizons and thermodynamic properties. Hence, we assume the metric of the form:

\[
    ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_n^2,
\]

where \( \gamma_{ij} \) is a metric of a \((D - 2)\)-dimensional constant curvature space \( \kappa = 1, 0 \) or -1, representing spherical, flat and hyperbolic spaces, respectively. But, in this paper we shall confine ourselves to \( \kappa = 1 \). To find the metric function \( f(r) \), we should solve Eq. (14). Using this metric ansatz, an \( r - r \) component of the field equations of motion reduces to:

\[
    \left[ r^5 - 2\tilde{\alpha}_2 r^3 (f(r) - 1) + 3\tilde{\alpha}_3 r (f(r) - 1)^2 \right] f'(r) + (n - 1) r^4 (f(r) - 1) - (n - 3) \tilde{\alpha}_2 r^2 (f(r) - 1)^2 + (n - 5) \tilde{\alpha}_3 (f(r) - 1)^3 = \frac{2\rho}{\kappa_0 n} T_r^r,
\]

where a prime denotes a derivative with respect to \( r \). \( \tilde{\alpha}_2 \equiv (n - 1)(n - 2)\alpha_2 \) and \( \tilde{\alpha}_3 \equiv (n - 1)(n - 2)(n - 3)/(n - 4)\alpha_3 \). In general, the Eq. (14) has one real and two complex solutions. But it can also have three real solutions as well under appropriate conditions. We are seeking static, spherically symmetric real solutions, which restrict a density \( \rho \) and a bivector \( \Sigma_{\mu\nu} \) as a function of \( r \) only. Further, the only possible non-zero component of a bivector \( \Sigma \) is \( \Sigma^{tr} = -\Sigma^{rt} \). Thus \( T_t^t = T_r^r = -\rho \Sigma^{tr} \) and we obtain \( \partial_r(\sqrt{-g}T_t^t) = 0 \) which implies:

\[
    T_t^t = T_r^r = \frac{a}{r^n},
\]

for some real constant \( a \). Clearly the third order Lovelock gravity is non-trivial only for spacetime dimensions \( D \geq 7 \). Henceforth, to extract information from our analysis, we shall confine ourselves to \( D = 7 \) in which the Eq. (14) can be easily integrated and solution read

\[
    f(r) = 1 + \frac{\tilde{\alpha}_2}{3\tilde{\alpha}_3} r^2 + \frac{10^{1/3}}{3\alpha_3 A^{1/3}} r^6 + B^{1/3},
\]

where

\[
    A \equiv 5\tilde{\alpha}_2 (2\tilde{\alpha}_2 - 9\tilde{\alpha}_3) r^6 - 54\tilde{\alpha}_3^2 (ar + C_1) + 3\tilde{\alpha}_3 (3\Delta)^{1/2},
\]

and

\[
    B \equiv \frac{1}{3^3 \times 10^{1/3} \tilde{\alpha}_3} [5\tilde{\alpha}_2 (2\tilde{\alpha}_2 - 9\tilde{\alpha}_3) r^6 + 54\tilde{\alpha}_3^2 (ar - C_1) + 3\tilde{\alpha}_3 (3\Delta)^{1/2}],
\]

with

\[
    \Delta \equiv -25 (\tilde{\alpha}_2^2 - 4\tilde{\alpha}_3) r^{12} + 20\tilde{\alpha}_2 (2\tilde{\alpha}_2^2 - 9\tilde{\alpha}_3) r^6 (ar - C_1) + 108\tilde{\alpha}_3^2 (C_1 - ar)^2.
\]

Here we shall impose the condition \( \tilde{\alpha}_2^2 = 3\tilde{\alpha}_3 \) (\( \equiv \alpha^2 \)), which simplifies \( f(r) \) significantly with all coefficients intact and hence it is worthwhile to consider this case. This condition reduces \( f(r) \) to the following form

\[
    f(r) = 1 + \frac{r^2}{\alpha} + \frac{1}{21/3 \alpha} \left\{ -\alpha^3 g(r) + \alpha^2 \sqrt{-\alpha g(r)^2} \right\}^{1/3},
\]

where \( g(r) \) is given with the notation \( \omega \equiv \frac{2}{5} C_1 \),

\[
    g(r) = r^6 - 6\alpha r + 3\omega a.
\]

Note that the square root here should be defined including a sign of \(-\alpha g(r)\), i.e. \( \sqrt{-\alpha g(r)^2} = -\alpha g(r) \). One can confirm it by noticing that the solution has been obtained from solving a cubic equation. Eq. (14) can be rewritten as

\[
    -r^{-n+6} \frac{\partial}{\partial r} \left\{ r^{n-1} (r^2 F) + \tilde{\alpha}_2 r^{n-3} (r^2 F)^2 + \tilde{\alpha}_3 r^{n-5} (r^2 F)^3 \right\} = r^{-n+6} \frac{\partial}{\partial r} \int dr \frac{2r^n}{n} T_r^r(r),
\]

where \( F \) is a function of the form:

\[
    F = \frac{1}{2} \left( \frac{a_1}{r^4} + \frac{a_2}{r^2} + a_3 \right).
\]

The coefficients \( a_1, a_2, a_3 \) can be determined by solving the field equations.

Note that the above solution does not have a thermodynamic horizon only a surface singularity, but it provides a new perspective on the Lovelock gravity framework, which is of the utmost importance for understanding the structure of black holes in higher-dimensional spacetimes.
where \( F \equiv [1 - f(r)]/r^2 \). For \( n = 5 \) and \( \omega^2 = 3 \omega_3 = \alpha^2 \) this reduces to, with an integration constant \( \omega \),

\[
(\alpha F + 1)^3 = 1 - \frac{6\alpha\omega}{5r^5} + \frac{3\omega\omega}{r^6} = \frac{g}{r^6}
\]

Thus \( f(r) \) reads:

\[
f(r) = 1 + \frac{r^2}{\alpha} \left[ 1 - \left( 1 - \frac{6\alpha\omega}{5r^5} + \frac{3\omega\omega}{r^6} \right)^{1/3} \right].
\]

We consider only positive mass, i.e. \( \omega \geq 0 \). \( f(r) \) asymptotically behaves as \( \lim_{r \to \infty} f(r) = 1 \) as shown in FIG. 1 with various parameter values.

In fact, for \( D > 4 \) the Einstein gravity can be thought of as a particular case of the Lovelock gravity since the Einstein-Hilbert term is one of several terms that constitute the Lovelock action. Hence, for \( D > 4 \) and \( \alpha = 0 \), the higher dimensional Schwarzschild solution in a string cloud model reads [21]:

\[
f(r) = 1 - \frac{\omega}{r^{D-3}} + \frac{2a}{(D-2)r^{D-4}}.
\]

which goes to Eq. (26) for \( D = 7 \). Eq. (24), in the limit \( \alpha \to 0 \), again leads to

\[
f(r) = 1 - \frac{\omega}{r^{4}} + \frac{2}{3r^2}a + O(\alpha).
\]

which can also obtained from Eq. (25) when \( D = 7 \). With \( a = 0 \), \( f(r) \) in Eq. (24) can be identified with one in the reference [17] with \( \beta = 0 \), i.e., the vanishing Born-Infeld field. The solution can be exactly verified through the reference [12]. Since \( \omega \) represents mass it should be positive, \( \omega \geq 0 \).

Due to a fractional power on \( g(r) \) \( f(r) = 1 + \frac{r^2}{\alpha} - \frac{2}{3^\alpha} \) leads to a curvature singularity at \( r = r_* \), where \( g(r_*) = 0 \) as in the Gauss-Bonnet case [11]. Using the expression of the Ricci scalar \( R = -[(n^2 + 2)F + (n + 4)rF' + r^2F''] \) from [9], one sees \( \lim_{r \to r_*} R = \infty \). A horizon radius \( r_h \) is defined by \( f(r_h) = 0 \), i.e. \( 0 = (r_h^2 + \alpha)^3 - g(r_h) = D(r_h) \), which reduces to

\[
r_h^4 + \alpha r_h^2 = -\frac{2}{5}a r_h + \omega - \frac{1}{3}r_0^4.
\]

The solutions are intercepts between the curve \( l_1 = r^4 + \alpha r^2 \) and the straight line \( l_2 = -\frac{2}{5}a r + \omega - \frac{1}{3}\alpha^2 \). Let us first consider \( \alpha > 0 \) case. One can see from the form of \( f(r) = 1 + \frac{r^2}{\alpha} - \frac{\omega}{r^{2/3}} \) that horizons can exist only for \( \omega > 0 \), i.e. \( r_h > r_* \). For \( a \geq 0 \) there is only one horizon with the condition \( \omega - \alpha^2/3 \geq 0 \). In a particular situation that \( g \) has a negative minimum, i.e. \( g(r_m) < 0 \) where \( r_m = (\alpha a/5)^{1/3} \), a horizon \( r_h \) exists surrounded by a smaller \( r_* \) where \( g(r_*) = 0 \). For \( a < 0 \) \( g \) is a monotonically increasing function and \( g(0) > 0 \), so \( r_* \) does not exist and \( g \) is always positive for \( a > 0 \). There can exist at most two horizons under the condition \( (\omega - \alpha^2/3)_{\min} < \omega - \alpha^2/3 \leq 0 \), where \( (\omega - \alpha^2/3)_{\min} \) is a \( y \)-intercept of \( l_2 \) when \( l_1 \) and \( l_2 \) have only one intercept with a common slope. On the other hand, there exists only one horizon for \( \omega - \alpha^2/3 > 0 \) and \( \omega - \alpha^2/3 = (\omega - \alpha^2/3)_{\min} \) and no horizon for \( \omega - \alpha^2/3 < (\omega - \alpha^2/3)_{\min} \).

Next, consider \( \alpha < 0 \) case. It is useful to notice that a horizon cannot exists between \( r_0 = (-\alpha)^{1/2} \) and \( r_* \). This can be checked by \( D(r) > 0 \) for \( r_0 < r < r_* \) and \( D(r) < 0 \) for \( r_* < r < r_0 \). The maximum number of horizons is three for \( a > 0 \) and two for \( a < 0 \) for \( r < r_* \). Let us consider possibility to have two intercepts for \( r > r_0 \). First, the necessary conditions are \( a < 0 \) and \( \omega - r_0^4/3 \leq 2ar_0/5 \). The slopes of \( l_1 \) and \( l_2 \) are equal to \(-2a/5\) when they have one intercept. Thus \(-2a/5 > \partial l_1/\partial r|_{r=r_0}, i.e. -a \geq 5r_0^3 \). However, the condition \( \omega \leq 2ar_0/5 + r_0^4/3 \) conflicts with the positive mass condition \( \omega \geq 0 \), i.e. \( \omega \leq 2ar_0/5 + r_0^4/3 \leq -2r_0^4/5 - r_0^4/3 = -5r_0^4/3 \). Therefore, there can exist only one horizon in the region \( r > r_0 \) and the condition for the existence of one horizon is \( \omega > r_0^4/3 + 2ar_0/5 \). This condition is just \( g(r_0) < 0 \), which means \( r_* > r_0 \).

In this paper we are specially interested in the case that a singular point \( r = r_* \) is covered by a horizon. From now on we focus only on the case that one horizon exists for \( r > r_* \). Consider the case \( r_h = r_* = r_0 \). Define \( \omega_m \)

\[
\omega_m = \frac{1}{5}a r_0 + \frac{1}{15}r_0^4
\]

This gives a lower bound of mass if \( \omega_m \geq 0 \). If \( \omega_m < 0 \) the lower bound for \( \omega \) must be taken to be zero. Given \( (a, \alpha) \) if \( \omega_m < 0 \), which means \( a < -\frac{5}{6}(\omega)^{3/2} \), \( \omega = 0 \) gives a lower bound for a horizon, \( r_m = (\frac{5}{6}a \omega)^{1/5} \leq r_h \) from \( g(r) = 0 \), while when \( \omega_m \geq 0 \), the case \( \omega = \omega_m \) leads to \( r_h = r_0 = r_* \) at \( a = -\frac{5}{6}(\omega)^{3/2} \). Fig. 1 shows behaviors of the metric function \( f(r) \) with different values of parameters.
FIG. 1: Plots show a metric function $f(r)$ as a function of $r$ for different values of parameters.

FIG. 2: Plots show that for $a > 0$, $\alpha < 0$ and $\omega - \alpha^2/3 > 0$ there can exist maximum three horizons in the region $r < r_*$. 
FIG. 3: The plot illustrates that horizons can be obtained from intercepts between \( l_1 = r^4 - r_0^2 r^2 \) and \( l_2 = -\frac{2}{3} a r + \omega - \frac{1}{3} r_0^4 \).

IV. THERMODYNAMICS OF BLACK HOLES

In this section we present thermodynamic properties of the Lovelock black hole solution Eq. (24) based on the existence of a horizon for \( r > r_0 \). As we demonstrate in the following, like any other black holes it also has thermodynamic properties. The Arnowitt-Deser-Misner (ADM) mass is defined

\[
M = \frac{(D - 2) V_{D-2}}{16 \pi} \omega,
\]

where \( V_{D-2} = 2\pi^{(D-1)/2} \Gamma((D - 1)/2) \) is the area of a unit \((D - 2)\)-sphere. Thus the gravitational mass of the black hole is determined by \( f(r_h) = 0 \), which in terms of a horizon \( r_h \), from Eq. (24), reads

\[
M = \frac{1}{16} \pi^2 \left( 2 a r_h + \frac{5}{3} \alpha^2 + 5 r_h^4 + 5 \alpha r_h^2 \right).
\]

As \( a \to 0 \), Eq. (30) becomes

\[
M \to \frac{5}{16} \pi^2 \left( \frac{1}{3} \alpha^2 + r_h^4 + \alpha r_h^2 \right),
\]

which is found in [17] in the limits of the vanishing Born-Infeld electromagnetic field and cosmological constant, \((\beta, \Lambda) \to 0\). Furthermore, in the limits \((a, \alpha) \to 0\) it leads to \( M \to \frac{5}{16} \pi^2 r_h^4 \), which is mass for the Schwarzschild black hole in seven dimension [11]. Also, Eq. (30) can give the minimum mass value \( M_m \). To find a possible minimum let us consider,

\[
\frac{\partial M}{\partial r_h} = \frac{1}{16} \pi^2 (2a + 20r_h^3 + 10\alpha r_h) = 0.
\]

Eq. (31) can be viewed as intercepts of \( M'_1 = -2a \) and \( M'_2 = 20r_h^3 + 10\alpha r_h \). The latter has two zeros, \([0, (-\alpha/2)^{1/2}]\). For \( 0 \leq a \leq 20(-\alpha/6)^{3/2} \), Eq. (31) has two solutions. The greater one might give a possible minimum mass but it is smaller than \( r_0 = (-\alpha)^{1/2} \), so the minimum mass occurs at \( r = r_0 \).

\[
M_m = \frac{1}{16} \pi^2 \left[ 2a(-\alpha)^{1/2} + \frac{5}{3} \alpha^2 \right].
\]

For \( a > 20(-\alpha/6)^{3/2} \), \( M \) is an increasing function and \( M(r_0) > 0 \), so \( M_m \) is a minimum mass as well. This result verifies Eq. (28). Consider the case \( r_h = r_0 = r_* \). Because as \( \omega \) increased \( r_h \) and \( r_* \) are increased, as from the point \( r_h = r_0 = r_* \) the value of \( \omega \) decreases with fixed \( a \) and \( \alpha \) we do not have a horizon any more. Thus, we can see that
we cannot have less mass than $M_m$ as far as a horizon exists. For $a < 0$, Eq. (31) has one solution. For $-\frac{5}{6}r_0^3 < a < 0$, $\frac{\partial M}{\partial r}|_{r=r_0} > 0$ and $M(r_0) > 0$. In this case the minimum mass is the same as $M_m$. For $-5r_0^3 \leq a \leq -\frac{5}{6}r_0^3$, $\frac{\partial M}{\partial r}|_{r=r_0} > 0$ and $M(r_0) < 0$. For $a \leq -5r_0^3$, $\frac{\partial M}{\partial r}|_{r=r_0} < 0$ and $M(r_0) < 0$. In the latter two cases the minimum mass is neither $M(r_m)$ nor $M_m$ but just zero. For $-\frac{5}{6}r_0^3 \leq a \leq -\frac{5}{6}r_0^3$, $\frac{\partial M}{\partial r}|_{r=r_0} > 0$ and $M(r_0) < 0$. For $a \leq -\frac{5}{6}r_0^3$, $\frac{\partial M}{\partial r}|_{r=r_0} < 0$ and $M(r_0) < 0$. In the latter two cases the minimum mass is neither $M(r_m)$ nor $M_m$ but just zero. In the case $a < -\frac{5}{6}r_0^3$ there can be a situation that given $(a, \alpha)$, $r_h$ cannot be less than a certain minimum by changing $\omega$ since $\omega$ must be non-negative. As seen in the last section given $a < -\frac{5}{6}r_0^3$, $\alpha > 0$, there exists a minimum $r_h$ greater than $r_0$, $r_m = \left(\frac{6}{5}a\alpha\right)^{1/5}$. In Fig. 4 $M$ is plotted as a function of $r_h$ for different values of $(a, \alpha)$.

![Plots](image_url)

**FIG. 4:** Plots show mass $M$ as a function of a horizon radius for different values of parameters. The dotted vertical lines in (a) and (b) the dashed line in (b) correspond to $r = r_0$ and $r = r_m$, respectively.

The Hawking temperature associated with a black hole is calculated using $T_h = \frac{\kappa}{2\pi}$, where $\kappa$ is a surface gravity of a horizon. Hence, the temperature $T_h$ at the horizon can be calculated by the definition $T_h = f'(r_h)/4\pi$, which is simplified to

$$T_h = \frac{a + 10r_h^3 + 5\alpha r_h}{10\pi(\alpha + r_h^2)^2}. \quad (33)$$

Since we consider only $r_h > (-\alpha)^{1/2}$, a singular point in $T_h$ in the denominator can be avoided. From Eq. (31) a positivity of mass means $ar_h > -\frac{5}{6}\alpha^2 - \frac{5}{2}r_h^2 - \frac{5}{2}\alpha r_h^2$. This leads to the inequality, $r_h(a+10r_h^3+5\alpha r_h) > 5\left(\frac{3}{2}r_h^4 + \frac{1}{2}\alpha r_h^2 - \frac{1}{6}\alpha^2\right)$. One can easily see that the right hand side in the inequality is positive for $r > r_0$, i.e., $a + 10r_h^3 + 5\alpha r_h > 0$. Thus, $T > 0$ in this case. Therefore, whenever mass is positive, temperature is also positive. In the limit $a \to 0$, the
temperature, Eq. \(33\) goes to

\[ T_h \rightarrow \frac{2r_h^3 + \alpha r_h}{2\pi(\alpha + r_h^2)^{\frac{3}{2}}}, \]

which is found in \[17\] as \((\beta, \Lambda) \rightarrow 0\). In addition to it the limit \(\alpha \rightarrow \) further reduces it to the temperature for the Schwarzschild black hole, \(T_h \rightarrow 1/\pi r_h\). Fig. 5 plots show the Hawking temperature of the black holes for different values of parameters.

![Fig. 5](image-url)

**FIG. 5:** Plots show the Hawking temperature of the black holes, \(T_h\) as a function of a horizon radius for different values of parameters. The dotted vertical lines in (a) and (b) the dashed line in (b) correspond to \(r = r_0\) and \(r = r_m\), respectively.

For the Schwarzschild black hole in \(D\) dimension, the entropy of a black hole, \(S\) is given by \(S = A_h/4\) with the area of the horizon, \(A_h\), which is \(D - 2\) dimensional surface area of a sphere. However, the black hole is supposed to obey the first law of thermodynamics \(dM = TdS\). To calculate the entropy, the integral can be done with respect to \(r_h\),

\[
S = \int \frac{dM}{T} = \int T^{-1} \frac{\partial M}{\partial r_h} dr_h. \tag{34}
\]

Using the expression \(\frac{\partial M}{\partial r_h}\) in Eq. 31 leads to

\[
S = \int_0^{r_h} dr \frac{5}{4} \pi^3 (\alpha + r^2)^2 + \text{const.} = \frac{1}{12} \pi^3 \left[3r_h^5 + 10\alpha r_h^3 + 15\alpha^2 r_h\right] + \text{const.}. \tag{35}
\]
Here we calculate the entropy with integration from 0 to $r_h$. This is a usual way to define entropy in order to make entropy vanish when a horizon length becomes zero. Although a horizon length must be greater than $r_0$ it should not be a concern here because the difference is only an additive constant. The integrand in $S$ is positive, so $S \geq 0$ in any case. Also, it is worthwhile to notice that the entropy is independent of $\alpha$. A similar case happens in [17]. The entropy expression in [17] coincides with Eq. (35). Using the areas of spheres $A_n = 2\pi^{(n+1)/2}r^{n}/\Gamma[(n+1)/2]$ for an $n$ dimensional surface, $A_5 = \pi^3r^5$. Only the second term $+\pi^3r_h^5$ in Eq. (35) reflects the area law $S = A_h/4$ and the rest are usually considered as quantum corrections in higher dimension. Fig. 6 plots behaviors of the entropy in terms of a horizon radius for different values of $\alpha$.

The heat capacity is expressed as,

$$C = \frac{\partial M}{\partial T} = \frac{\partial M}{\partial r_h} \frac{\partial T}{\partial T_h}$$  \hspace{1cm} (36)

Using the expression $\frac{\partial M}{\partial r_h}$ in Eq. (31) and $\frac{\partial T_h}{\partial r_h}$,

$$\frac{\partial T_h}{\partial r_h} = \frac{5\alpha^2 - 4ar_h - 10r_h^4 + 15\alpha r_h^2}{10\pi (\alpha + r_h^2)^3},$$  \hspace{1cm} (37)

we get

$$C = \frac{5\pi^3 (\alpha + r_h^2)^3 [a + 5 (2r_h^3 + \alpha r_h)]}{4 [4ar_h + 5 (-\alpha^2 + 2r_h^4 - 3\alpha r_h^2)]}. \hspace{1cm} (38)$$

In the limit $a \to 0$, the heat capacity Eq. (38) goes to

$$C \to -\frac{5}{4} \pi^3 (\alpha + r_h^2)^3 (2r_h^3 + 3\alpha r_h),$$

which is found in [17] as $(\beta, \Lambda) \to 0$. We have just seen above that from the positive mass condition $a + 5 (2r_h^3 + \alpha r_h) > 0$. Using the same condition, we notice that in the denominator in Eq. (38), for $r_h > r_0$

$$4ar_h + 5 (-\alpha^2 + 2r_h^4 - 3\alpha r_h^2) > -\frac{25}{3} \alpha (r_h^2 + \frac{\alpha}{3}) > 0.$$  \hspace{1cm} (39)

Therefore, the heat capacity is always negative for $r > (-\alpha)^{1/2}$ with positive mass and hence the black hole is thermodynamically unstable in positive mass region. However, it does not necessarily mean that the black hole is unstable as the Schwarzschild black hole. The heat capacity of the black holes is plotted in Fig. 6 for various values of the parameters $(a, \alpha)$. It turns out that the parameters $(a, \alpha)$ influence the thermodynamic stability of black holes. There exist transition points at which a sign of heat capacity changes, i.e. boundaries between thermodynamically stable and unstable regions. Let us consider only a largest transition point $r_c$. For a fixed $\alpha$ and $a < -5r_0^3$, $r_c$ increases.
as $a$ decreases but $a \geq -5r_0^3r_c$ becomes a common point, $r_c = r_0$ due to the factor $(\alpha + r_h^2)^3$ and does not move as $a$ changes as can be seen in Eq. (38). For a fixed $a$ as $\alpha$ increases $r_c$ decreases. Around $r_h = r_c$ such black holes are thermodynamically stable for $r_h < r_c$ and unstable for $r_h > r_c$. Fig. 8 shows such characteristics. There exist some regions for $r_h > r_0$ where heat capacity is positive. However, one can check that points $r_{*m} = (\frac{4}{5}a\alpha)^{1/5}$ are always located beyond singular points and hence heat capacity is negative in the region of positive mass.

V. CONCLUSION

The Lovelock theory is a natural extension of the Einstein theory of general relativity to higher dimensions and it is of a great arena for research for theoretical physics. The Lovelock theory describes string inspired corrections of Einstein-Hilbert action and hence admits the general relativity as a particular case. In this paper, we have obtained exact static, spherically symmetric black hole solutions to the third order Lovelock gravity in a string cloud background in seven dimension with help of carefully choosing coefficients of the curvature correction terms, thereby generalizing the static, spherically symmetric black hole solutions for these theories. These solutions possess rich properties of black holes and in the limits go over to black holes in general relativity. If a horizon exists in (anti) de Sitter side the other must have one as well but only one horizon can exist in each space. We found the condition for the existence of a horizon. It turned out that the parameter $\alpha$, from higher order curvature terms, should be negative for the existence of a horizon. The horizon condition provides a lower bound for mass or a minimum horizon radius.
We then proceeded to find exact expressions for the thermodynamic quantities like the black hole mass, the Hawking temperature, entropy and heat capacity and in turn also analyzed the thermodynamic stability of black holes. In addition we explicitly brought out the effect of a string cloud background on black hole solutions and their thermodynamics. We found these quantities have simple expressions in terms of the horizon radius, which is quite complicated in most cases. In particular near \( r_h = r_c \) these black holes are thermodynamically stable with a positive heat capacity for the range \( r_h < r_c \) while for \( r_h > r_c \) the negativity of the heat capacity tells us that the black hole is thermodynamically unstable like the Schwarzschild black hole. The entropy does not obey horizon area formula. Interestingly the third order Lovelock black hole entropy has no correction from a string cloud background.

The possibility of a further generalization of these results in arbitrary dimensional Lovelock gravity is an interesting problem for future research.

\[ \text{FIG. 8: Plots show heat capacity around a transition point } r_c. \]

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