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Magnetic properties of four dimensional fermions

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Abstract. We investigate the Sakai-Sugimoto model at nonzero baryon chemical potential in a background magnetic field in the chiral symmetric phase. We find that a new form of baryonic matter shows up, and we investigate its properties. We find a generated axial current, a reduction in the amount of charge participating in dissipative interactions and a metamagnetic like phase transition at low temperature.

1. Introduction
Properties of strongly interacting fermions in 3 + 1 dimensions is a vastly important subject. Holographic representation of various systems of strongly interacting fermions can teach us about the possible phenomena one can expect. In particular an addition of a magnetic field can reveal all kinds of interesting properties (as it does in lower dimensions). The Sakai-Sugimoto model [1] is a system of strongly interacting fermions which seems to share quite a few phenomena with QCD. We explore here the behavior of this model at finite baryon density and magnetic field. Other works on parts of the subject include [2, 3, 4, 5, 6, 7]. In particular we look at the phase where chiral symmetry is unbroken, and find that baryonic charge can be carried by a new form of matter. Understanding its properties and possible interpretation is the goal of this note.

2. New form of baryonic matter
The Sakai-Sugimoto model consists of a probe D8-brane and a probe anti D8-brane in the background of N D4-branes compactified on a circle. The matter degrees of freedom are fermions in 3 + 1 dimensions in the fundamental representation of $SU(N)$, interacting with $SU(N)$ gauge fields propagating in one higher dimension. The properties of the fermions are encoded in the action of the probe branes. The baryon charge is the charge under the $U(1)$ gauge field living on the probe branes, and can be carried by strings ending on the D8 branes. These strings can start at the horizon and then they have the interpretation as being quarks, or can end on a D4-brane wrapping the $S^3$, in which case it is identified with a baryon. In the chiral symmetric phase, at zero magnetic field, all the baryonic charge is carried by the quarks.

1 Based on a talk given by G. Lifschytz.
Let us now turn on a magnetic field. The 8-brane DBI and CS actions in the deconfined background are given by [8],

\[ S_{DBI} = N \int_{u_T}^{\infty} du \frac{u^{5/2}}{\sqrt{1 - (a_0'(u))^2 + f(u)(a_1'(u))^2}} \left( 1 + \frac{h^2}{u^2} \right) \]

\[ S_{CS} = -N \int_{u_T}^{\infty} \left( \partial_2 a_3 a_0 V'(u) a_1 u' + \partial_2 a_3 a_0 V'(u) a_1 u' - a_3^2 \partial_2 a_0 a_1 a_0 + a_3 \partial_2 a_1 a_0 V' \right) , \]

where now \( f = 1 - (u_T^2/u^3) \), and where we have included both the 8-brane and anti-8-brane parts, with \( \bar{a}_0 = a_0 \) and \( \bar{a}_1 = -a_1 \). The boundary value of the axial field is now a parameter, rather than a field, in the gauge theory, which we set to zero, \( a_1(\infty) = 0 \). We modify the Chern-Simons action by throwing away boundary terms of the form

\[ \frac{1}{2} \partial_2 \left( a_3 a_1 a_0 \right) + \frac{1}{2} \partial_u \left( a_3 \partial_2 a_1 a_0 \right) \]

(2)

to obtain the correct five-dimensional currents, consistent with gauge invariance.

The integrated equations of motion are then given by

\[ \frac{\sqrt{u^5 + h^2 u^2 a_0'(u)}}{\sqrt{1 - (a_0'(u))^2 + f(u)(a_1'(u))^2}} = 3h a_1(u) + d \]

(3)

\[ \frac{\sqrt{u^5 + h^2 u^2 f(u)a_1'(u)}}{\sqrt{1 - (a_0'(u))^2 + f(u)(a_1'(u))^2}} = 3h a_0(u) + j_A - \frac{3}{2} h u \mu , \]

(4)

where \( d \) is the total baryon charge density and \( j_A \) is the axial current density. As can be seen from this equation there are two sources of baryonic charge. First there are strings stretching from the horizon to the D8-brane, being represented as a delta function source at the horizon, in the un-integrated equations of motion. Second there is the term \( 3h a_1(u) \) representing smeared D4-branes. The charge carried by these is \( -3ha_1(u_T) \).

We will now explore the properties of these objects.

2.1. Axial current

Looking at equation (4), there is an additional condition imposed by regularity at the horizon, \( a_0(u_T) = 0 \). Since \( f(u_T) = 0 \) as well, the consistency of the \( a_1 \) equation of motion (4) requires turning on a specific axial current density [8]

\[ j_A = \frac{3}{2} h \mu . \]

(5)

In terms of physical quantities, the axial current density is

\[ J_A = \frac{N_c}{4\pi^2} H_{\mu \nu} , \]

(6)

which agrees precisely with the result of [9], once we account for the different normalizations (the relative factor of \( N_c/2 \)). What carries this axial current. from the equation of motion it is clear that it is the term \( 3h a_0(u) \). This term represents the amount of axial current below some \( u \) however since \( a_0(u_T) = 0 \) it is clear that all the axial current comes from regions above the horizon. Thus it is associate with the smeared D4-branes living outside the horizon.
3. Transport properties

We will first consider orthogonal electric and magnetic fields and calculate the ohmic and Hall conductivities \[10\] employing the same method used by Karch and O’Bannon in the D3-D7 model \[11, 12\]. We consider the deconfined, chiral-symmetric phase with background magnetic field \(f_{23} = \partial_2 a_3 = h\) and electric field \(f_{02} = a_2 = e\). In order to include spacetime baryonic currents \(j_\mu\) (though we will call the baryon density \(d \equiv j_0\)), the spacetime components of the gauge fields will have \(u\)-dependent parts \(a_\mu(u)\) as well. Primes denote derivatives with respect to \(u\) and dots are derivatives with respect to \(t\).

We combine the actions of the 8-brane and the 8-\(\bar{\text{b}}\)-brane by considering the vector part of \(a_0\) and the axial part of \(a_1\), i.e. \(a_0^\text{D8} = a_0^\text{D8\bar{b}}\), \(a_1^\text{D8} = a_1^\text{D8\bar{b}}\). The DBI action with these background fields is

\[
S_{\text{DBI}} = N \int du u^{5/2} \sqrt{X}.
\] \hfill (7)

where

\[
X = 1 + \frac{h^2}{u^3} - \frac{e^2}{f(u)u^3} - a_0^2 f(u) + a_1^2 f(u) - \frac{e^2}{f(u)u^3} + \frac{e^2}{f(u)u^3} - \frac{eh}{u^3} \frac{a_0 a_3}{f(u)u^3} + \frac{2eh}{u^3} \frac{a_0 a_3}{f(u)u^3}.
\] \hfill (8)

The normalization constant is given by

\[
N = 2\Omega_4 T_{D8} R^5 = \frac{N_c}{6\pi^2} \frac{R^2}{(2\pi\alpha')^3}.
\] \hfill (9)

where the factor of 2 corresponds to the two halves of the embedding. The Chern-Simons action with appropriate addition to make it gauge invariant is

\[
S_{\text{CS}} + \Delta S = \frac{3N}{2} \int \left( a_{0[a_1]} + e a_{[1a_3]} \right). \hfill (10)
\]

The charge \(d\) and the currents \(j_i\) define tilde currents as follows,

\[
\tilde{d} = d + 3ha_1, \hfill (11)
\]

\[
\tilde{j}_1 = j_1 + 3h \left( a_0 - \frac{\mu}{2} \right) + 3ea_3, \hfill (12)
\]

\[
\tilde{j}_3 = j_3 - 3ea_1. \hfill (13)
\]

Solving the equation of motion and plugging back into the action, we find that the DBI part can be written as

\[
S_{\text{DBI}} = N \int du u^{5/2} \frac{1 + \frac{h^2}{u^3} - \frac{e^2}{f(u)u^3}}{\sqrt{Z}}.
\] \hfill (14)

where

\[
Z = \left( 1 + \frac{h^2}{u^3} - \frac{e^2}{f(u)u^3} \right) \left( u^5 + \tilde{a}^2 + \tilde{a}_2 + \frac{\tilde{j}_3}{f(u)} \right) - \left( \frac{hd}{u^3} + \frac{\tilde{e}_j}{f(u)} \right)^2 - \frac{\tilde{j}_1^2}{f(u)}. \hfill (15)
\]
For the action to be real, \( Z \) must clearly be non-negative. The only way this can happen is if all three terms in (15) have double zeros at the same value of \( u \), which we’ll call \( u^* \). For the first term, this means each factor must vanish at \( u^* \), so we get a total of four equations. These can be solved and one gets

\[
\begin{align*}
    u^3_s &= u^3_T \left( 1 + \frac{e^2}{h^2 + u^5_T} \right) \\
    j_3 &= \left( -\frac{h a_T}{u^3_T + h^2} + 3a_1T \right) e \\
    j_2 &= \frac{\sqrt{u^8_T + u^5_T h^2 + u^3_T d^2}}{u^3_T + h^2} e
\end{align*}
\]

and an axial current

\[
    j_1 = 3h \left( \frac{\mu}{2} - a_0^* \right) - 3ea_3^* .
\]

For small \( e \), \( a_3 = \mathcal{O}(e) \) and \( a_0^* = a_{0T} + \mathcal{O}(e^2) \). In addition, regularity at the horizon implies \( a_{0T} = 0 \), so we find

\[
    j_1 = \frac{3}{2} h \mu
\]

which matches the result found in the previous subsection.

4. Metamagnetic phase transition

We saw that as we turn on a magnetic field D4-branes come up from the horizon and are smeared over the D8-brane. Being above the horizon they do not participate is dissipative phenomena like longitudinal conductivity. It turns out that the process of the D4-brane moving up along the D8-brane is not smooth at low temperature but rather is a first order phase transition [13].

To solve the zero-temperature equations, one notices that dividing the two equations in (3, 4) gives

\[
    \frac{a_0'}{a_1'} = \frac{3ha_1 + d}{3ha_0}
\]

which can be integrated, using the boundary conditions \( a_1(\infty) = 0 \) and \( a_0(\infty) = \mu \). One then finds that

\[
(3ha_1 + d)^2 - (3ha_0)^2 = d^2 - 9h^2 \mu^2 .
\]

By defining a new coordinate

\[
z = \int_0^u \frac{3hdu}{\sqrt{u^8 + h^2 u^2 + d^2 - 9h^2 \mu^2}}
\]

The solution to the equations of motion with the conditions \( a_0(z = 0) = 0 \) and \( a_1(z = z_{\infty}) = 0 \) is then

\[
\begin{align*}
a_0 &= \frac{d \sinh z}{3h \cosh z_{\infty}} \\
a_1 &= \frac{d \cosh z}{3h \cosh z_{\infty}} - \frac{d}{3h} .
\end{align*}
\]
Figure 1. Three solutions for (a) the free energy $F$ and the (b) magnetization $M$ at $T = 0$ with $d = 1$. There is a first-order phase transition from the solid blue $z_{\infty} \sim h$ solution to the dashed red $z_{\infty} \to \infty$ solution at $h = 0.19$. The dotted grey solution, which corresponds to the third solution, connects the two stable solutions and is an unstable maximum of the free energy.

From these equations we see that

$$
\mu = \frac{d}{3h} \tanh z_{\infty}, \quad a_1(0) = \frac{d}{3h} \left( \frac{1}{\cosh z_{\infty}} - 1 \right),
$$

and thus,

$$
z_{\infty} = \int_0^\infty \frac{3hdu}{\sqrt{u^6 + h^2 u^2 + \frac{d^2}{\cosh^2 z_{\infty}}}}.
$$

To find the actual solution for $z_{\infty}(h,d)$, one needs to solve (26) numerically. We regularize the integral$^2$ by taking the lower limit of the integral defining $z$ to be $\epsilon$ and at the end sending $\epsilon \to 0$. For small $d$ and $h$ there are three solution for $z_{\infty}$, but for large enough $h$ only one remains which is $z_{\infty} \sim \log \epsilon \to \infty$. It should be noted that the appearance of three solution and the interesting phase structure which they imply is due to the use of the full DBI action rather than just the Yang-Mills approximation.

The free energy of the three solutions is shown in Fig. 1. At small $h$ the solution $z_{\infty} \sim h$ is preferred, but at some finite $h$ there is a jump to the branch parametrized by $z_{\infty} \to \infty$. This first-order phase transition is accompanied by jumps in both the magnetization and the chemical potential.

The preferred solution at large $h$ has

$$
\mu = \frac{d}{3h}, \quad F = \frac{d^2}{6h}.
$$

We also find that $a_0 = 0$ and $a_1 = -\frac{d}{3h}$ are constant all the way up to $u \to \infty$ where they jump to their asymptotic values $a_0 = \mu$ and $a_1 = 0$.

4.1. Non-zero temperature

We now turn to solving these equations in the more physical case of non-zero temperature. We will see that the properties seen for zero temperature persist. At high temperature there is

$^2$ Having to insert by hand a regulator is a particular feature of $T = 0$. At non-zero temperature, $u_T$ naturally cuts off integrals over $u$. 
only one solution which means a single phase. As the temperature is lowered, there is a critical temperature $T_c(d)$ below which one finds three solutions to the equations of motion for some range of $h$. One way to see this in the numerics is to plot $a_1(\infty)$ as a function of $a_1(uT)$. At high temperature the graph (at all values of $h$) cross the horizontal axis only once. However, at lower temperature there is a range of $h$ when there are three possible values of $a_1(uT)$ which gives $a_1(\infty) = 0$. Again, one of the solution is unstable and two are stable. There is a phase transition at some non-zero value of $h$. At a temperature slightly higher then the critical temperature both $\mu(h)$ and $M(h)$ develop what looks like a wiggle. As the temperature is lowered, we enter a regime where there is a phase transition; the chemical potential and the magnetization become discontinuous. At zero temperature, the jump of the chemical potential is such that the axial current $j_A = \frac{3}{2}h\mu$ attains its maximum value of $d/2$ and does not change any more as $h$ is varied. This implies that in the large $h$ phase the system is fully polarized. At non-zero temperature, $j_A$ jumps close to the maximum value and asymptotes to it at large $h$. Similarly if we had turned on an axial chemical potential rather then a vector chemical potential, there would be a jump in the vector current in the direction of the magnetic field.

To further describe the phase transition and to try to understand it better, we look at the value of the magnetic field at which the transition occurs as a function of the density and at fixed temperature. The results are plotted in Fig. 2(a) and closely fit

$$h \sim d^{2/3}.$$ 

We also find that even at non-zero temperature the free energy of the phase at large $h$ is well-approximated by

$$F \sim \frac{d^2}{6h}.$$ 

As was expected, the behavior here is similar to the zero-temperature case but with finite temperature corrections. One qualitative difference is in the location of the charges in the high $h$ phase. While almost all the charges in the form of smeared D4-branes, they are no longer all near the boundary at $u = \infty$ but are instead spread across all values of $u$.

In terms of the behavior as a function of the temperature, we find that for a given charge density $d$ above a certain temperature there will be no phase transition for any $h$; thus, the line of first-order phase transitions ends at a critical point above which there is a smooth cross-over. The critical line in the $T - d$ plane above which there is no phase transition is given in Fig. 2(b) and behaves at large $d$ as

$$T \sim \log d.$$ 

It appears that no matter how high the temperature is, one can find a large enough density for which a phase transition will occur at some strong magnetic field. Of course, at these parameters, for a given $L$, the preferred phase maybe the chiral-broken phase. However, as explained before, we can always take $L$ to be large enough that the chiral-symmetric phase is preferred.

5. Interpretation

To try to understand some of the physics of this transition, we explore the phase that is dominant at large $h$. At least some of its properties are similar to the properties of a free, massless fermion gas in four dimensions in its lowest Landau level.

A massless fermion in four dimensions with charge $e$ in a magnetic field $B$ has the following energy spectrum:

$$E = \sqrt{k^2 + 2eBn}$$ 

(31)
where $k_1$ is the momentum in the direction of the magnetic field and $n$ is a natural number labeling the Landau levels in the transverse directions. The degeneracy of each state is roughly $B$. If all fermions are in the lowest Landau level, the energy of the gas is given by

$$E_{tot} \sim B \int_0^{k_{max}} dk \, k = \frac{1}{2} B k_{max}^2,$$

(32)

and the density of fermions is

$$n_f \sim B k_{max},$$

(33)

giving a chemical potential and energy

$$\mu \sim \frac{n_f}{B}, \quad E \sim \frac{n_f^2}{B}.$$  

(34)

The lowest Landau level is not spin degenerate, thus when there’s a jump to the lowest Landau level, there is an increase in the magnetization. We can estimate when the system crosses from the lowest to the first Landau level; this occurs when it is energetically favorable to increase $n$ rather than add to the parallel momentum, meaning $k_{max}^2 \sim B$, which corresponds to

$$B \sim n_f^{2/3}. $$

(35)

Many of the properties of the large $h$ phase agree with those expected from free fermions in the lowest Landau level in four dimensions. The behavior of the free energy and chemical potential (27) match the free fermion result (34), and phase transition (28) resembles that of a transition into the lowest Landau level (35).

On the supergravity side, in the large $h$ phase almost all the baryonic charge$^3$ is represented by smeared D4-branes that crept up the D8-brane. Recently, there have been other indications that smeared D4-branes seem to be related to fermions in the lowest Landau level. For example, in [14] it was argued that in the confined, large $h$, pion-gradient phase the smeared D4-branes also behave as if they are free fermions and reproduced the thermodynamics of the lowest Landau level.

Furthermore, the smeared D4-branes holographically correspond to bound states of $N_c$ fermions, making up what looks like smeared baryons. This phenomena may be related to

$^3$ At $T = 0$ this will be all the baryonic charge.
the fact that in the lowest Landau level fermions are confined to move only in one spatial dimension, and thus any attractive potential will make them bind, even one which would not bind them in three spatial dimensions. Such an occurrence has been demonstrated to happen for a different situation related to the magnetically-induced chiral-symmetry breaking [15].

Because the Sakai-Sugimoto model provides a robust holographic model of QCD, it would be interesting to see if the phenomena described here could be found, for example, at RHIC in a strong magnetic field. We see the onset anomaly-driven behavior as soon as a magnetic field is turned on, but this may be an artifact of exactly massless quarks. In real QCD, a strong magnetic field may be required before any of these phenomena become manifest.

There are a few non-relativistic fermionic systems which resemble in some way the relativistic system discussed here. In particular, several condensed matter systems also exhibit first-order quantum phase transitions with discontinuous magnetizations. In the case of metamagnetism [16], the magnetization, which comes from spin ordering, jumps due to a large increase in the density of states near the Fermi surface. This effect sometimes occurs in paramagnetic materials with itinerant electrons in their outer shells. The magnetization, as a function of the external magnetic field, can have a large rise and sometimes a discontinuity due to a rapid transition of many fermion spins from anti-parallel to the magnetic field to parallel.

The phase transition we find in the SS model appears to have some resemblance to metamagnetism but also some differences. We start with a paramagnetic state and see a jump in the magnetization at a particular value of the magnetic field. However, after the jump, the magnetization decreases, while in materials exhibiting metamagnetism, the phase after the transition is ferromagnetic. In addition, while we find evidence the large-magnetic field phase in the SS model is associated with the lowest Landau level, metamagnetism, occur at more modest magnetic fields, and is not related to lowest Landau level physics.

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