Pseudo-spin canting transition in bilayer quantum Hall ferromagnets: a self-charging capacitor

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For sufficiently strong in-plane magnetic field a \( \nu_T = 1 \) bilayer quantum Hall pseudo-ferromagnet is expected to exhibit a soliton lattice. For sufficiently close layers and large in-plane field, we predict this incommensurate “planar” phase \( P_I \) to undergo a reentrant pseudo-spin canting transition to an incommensurate state \( C_I \), with a finite out-of-plane pseudo-magnetization component, corresponding to an interlayer charge imbalance in regions between solitons. At \( T > 0 \) the transition is in the 2d compressible Ising universality class, and at \( T = 0 \), the quantum transition is in heretofore unexplored universality class. The striking experimental signatures are the universal nonlinear charge-voltage and in-plane field relations, and the divergence of the differential bilayer capacitance at the transition, resulting in a bilayer capacitor that spontaneously charges itself, even in the absence of an applied interlayer voltage.

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There now exists considerable experimental [4] and theoretical [5] evidence for the bilayer phase coherent quantum Hall (QH) state at a total electron filling fraction \( \nu_T = 1 \), which is driven by Coulomb exchange interactions and is expected to survive even in the limit of vanishing interlayer tunneling, \( \Delta \). A number of complementary pictures of the state, include superfluidity of Chern-Simons composite boson [6], excitonic superfluid, and pseudo-ferromagnet [8].

In the latter picture, that we employ here, the \( z \) component \( m_z = (n_1 - n_2)2\pi\ell^2 \) (\( \ell = \sqrt{\hbar/eB} \) is the magnetic length) of the pseudo-spin magnetization unit vector \( \mathbf{m} = \mathbf{m}_z + m_x \mathbf{z} \), is the normalized interlayer imbalance in electron layer densities \( n_{1,2} \), while the azimuthal angle \( \phi = \phi_1 - \phi_2 \) of \( \mathbf{m}_z \) is the difference between the electron phases \( \phi_{1,2} \) in the two layers. Interlayer charging energy, \( 2\pi\ell^2\varepsilon_c \), explicitly breaks SU(2) pseudo-spin symmetry down to \( U(1) \times \mathbb{Z}_2 \), forcing \( \mathbf{m} \) to lie in the easy-xy-plane, defined by \( m_z = 0 \). The interlayer tunneling energy, acts like a pseudo-magnetic field directed along \( \mathbf{z} \), further explicitly breaking \( U(1) \) symmetry.

In this Letter, we predict and explore a striking, reentrant quantum phase transition from the charge-balanced “planar” (P) state into an interlayer charge-imbalanced “canted” (C) state. This novel state is characterized by the development of \( m_z \neq 0 \) and corresponds to the pseudo-magnetization \( \hat{m} \) spontaneously canting out of the easy-xy-plane. The transition is controlled by dimensionless parameters \( g \equiv 2\pi\ell^2\varepsilon_c/\Delta \) and \( b_\parallel \equiv B_\parallel/B_{C1} \), with a \( T = 0 \) phase boundary \( b_\parallel(g) \) illustrated in Fig.1, and \( B_{C1} \) a critical in-plane field for the commensurate-incommensurate (CI) transition. [9] For a range of parameters, we find that the transition can be continuous and for \( T = 0 \) is in a heretofore unexplored 2 + 1-d quantum compressible Ising universality class. At finite \( T \) this transition is the 2d compressible Ising universality class.

Two of many striking and experimentally testable consequences are (i) the universal nonlinear behavior of the interlayer charge imbalance \( q(V,B_\parallel) \) with gate voltage \( V \) and in-plane field \( B_\parallel \)

\[
q(V,B_\parallel) = |b_\parallel - b_c|^\beta q(b_\parallel - b_c)|^\chi)
\]

for \( b_\parallel > b_c(g) \), leading to a spontaneous interlayer charge imbalance \( q \), even in the absence of an applied interlayer voltage, \( V \to 0 \), i.e., the QH bilayer is a self-charging capacitor, and

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(iii) the concomitant divergence of the interlayer differential capacitance \( C = dq/dV \)

FIG. 1. Phase diagram illustrating a transition between the “planar incommensurate” \( P_I \) and “canted incommensurate” \( C_I \) phases of \( \nu_T = 1 \) QH bilayer, with insets showing cartoons of the two phases.
are universal. Both relations should be readily testable in a balanced gate geometry with the QH bilayer acting as a novel dielectric medium.

We also expect a universal suppression of the quantum Hall gap below the P-C transition, with $\delta \Delta_{QH}(V, B_\parallel) \propto -q(V, B_\parallel)\lambda^2$, accessible through activated behavior of the longitudinal resitivity. Similar predictions also hold for the finite $T$ transition, with $b_\parallel, b_\perp(g)$ replaced by $T, T_c(b_\parallel, g)$, and (for an incompressible soliton lattice) with exact critical exponents of the 2d Ising model.

The driving force for the P-C transition is the competition between the charging (hard axis anisotropy) energy density $E_{charge} = \frac{1}{2} \varepsilon_c m_z^2$, minimized by $m_z = 0$, and the exchange energy density $E_{exchange} = \frac{1}{2} \rho_s |\nabla \phi|^2$. Because $\rho_s \approx \rho_s^0 (1 - m_z^2)$ (at least in mean-field theory), the latter can be lowered for any planar twisted state by increasing $m_z$, via pseudo-spin canting out of the xy-plane, corresponding to the development of the interlayer charge imbalance. In the simplest $\Delta \to 0$ limit, for a uniformly twisted, $\phi = Qx$ staggered current carrying state, this canting instability will clearly take place for $Q > \sqrt{\varepsilon_c/\rho_s^0}$, corresponding to an antiferromagnetic "depairing" staggered (spin) current $J_s^z = \frac{\pi}{2} \rho_s |\nabla \phi| = \frac{\pi}{2} \sqrt{\rho_s \varepsilon_c} x$. In contrast, for a finite tunneling energy $\Delta$ there exist equilibrium twisted states, induced by an applied in-plane field $B_\parallel = B_\parallel \hat{y}$. Although the general driving mechanism is the same, as we discuss below, the details depend on whether the canting phase transition takes place out of the planar commensurate PC or planar incommensurate $P_C$ (soliton) state.

We begin our analysis by studying the classical energetics of the quantum Hall bilayer in the presence of an in-plane (physical) magnetic field $B_\parallel = B_\parallel \hat{y}$. The appropriate Hamiltonian, in the absence of quenched disorder, and for a fixed uniform value of $m_z$ is given by

$$\mathcal{H}_{cl} = \frac{\varepsilon_c}{2} m_z^2 + \frac{\rho_s}{2} |\nabla \phi|^2 - \frac{\Delta}{2 \pi T^2} \cos(\phi - Qx),$$

where $\Delta = \Delta_0 (1 - m_z^2)^{1/2}$ is the interlayer tunneling energy; $Q$ is the in-plane "magnetic wavevector" $Q = 2\pi B_\parallel d/\phi_0$ associated with $B_\parallel$, and $\lambda = \sqrt{\rho_s 2 \pi T^2 / \Delta}$ is a screening length for spatial deformations of $\phi(x)$. In the commensurate state, $B_\parallel < B_{C1} = \frac{2\phi_0}{(\pi^2 d\lambda)}$, the pseudo-spin $\mathbf{m}$ twists uniformly about the $z$-axis with $\phi(x) = Qx$. The corresponding $m_z$-dependent energy density is given by

$$\mathcal{E}_C[m_z] = \frac{8}{\pi z} \mathcal{A}_0 \left[ b_0^3 + (b_0^3 - b_0^2) m_z^2 - \frac{\pi^2}{8} (1 - m_z^2)^{1/2} \right],$$

where the first term is the planar (i.e., $m_z = 0$) exchange energy, the second consists of the interlayer charging energy, with $b_0 = \frac{\pi}{2} \sqrt{2 \pi T^2 / \Delta_0} = \frac{\pi}{2} \theta^{1/2}$ and the reduction of the exchange energy due to canting, and third, the tunneling energy. Simple analysis of $\mathcal{E}_C[m_z]$ predicts a continuous canting transition from $m_z = 0$ planar to $m_z \neq 0$ canted state, at $b_\parallel = b_\parallel^C(g) = \frac{\pi}{2} \sqrt{g + T}$. Given that it is $Z_2$ symmetry that is being broken, at $T = 0$ ($T \neq 0$) we generically expect such quantum (classical) $P_C-C_C$ transition to be in the well-studied 3d (2d) Ising universality class, and in the latter case characterized by exactly known exponents. In this scenario, this $P_C-C_C$ canting transition will be followed by the $C_C-C_I$ transition in a CI universality class, modified by long-range dipolar interactions and delayed by $m_z \neq 0$ to higher critical in-plane field $B_{CI} = B_0 (1 - m_z^2)^{-1/4}$.

However, because currently studied bilayer devices are characterized by large $g$, and therefore have $b_\parallel^C(g) > 1$, above results suggest that the system will first undergo a commensurate (planar)-incommensurate (planar) ($P_C-P_I$) transition at $b_\parallel = 1$ into a state characterized by a periodic array of 1d $y$-directed solitons. Consequently, we need to analyze the $P_I-C_I$ transition from within such incommensurate soliton state.

Although there has not been any direct experimental evidence, the CI transition picture was successfully used to interpret a precipitous drop in the QH gap, upon application of sufficiently strong $B_\parallel$. In addition to the intrinsic interest in the $P_I-C_I$ transition, the more easily detected canted-incommensurate (CI) state will also facilitate the detection and study of the soliton lattice.

The properties of 1d incommensurate soliton state are well documented in the literature. For $b_\parallel > 1$ ($Q > Q_{C1}$), the soliton chemical potential is negative and $y$-directed solitons enter the system at a density $n_s(b_\parallel)$ that continuously increases with $b_\parallel$. In this state, the phase $\phi_s(x)$ rotates uniformly between solitons, but slips by $2\pi$ over a soliton width $\lambda$, with $m_1(x) = 0$ failing to follow the winding rate $Q$ imposed by $B_\parallel$. Generically, solitons reduce the energy density relative to that of the commensurate state with $\mathcal{E}_I[m_z] = \mathcal{E}_C[m_z] - \delta \mathcal{E}_{\text{solitons}}[m_z]$.

Although, in the absence of fluctuations, exact expressions for $\phi_s(x)$ and the energy $\mathcal{E}_I$ (expressible in terms of elliptic integrals) are available, sufficiently close to the CI transition thermal fluctuations always qualitatively modify these $T = 0$ predictions. Consequently, there are at least three regimes above the CI transition. Asymptotically close to the transition, where the soliton array is sufficiently dilute, such that $n_s(b_\parallel) < n_T$, with $n_T$ determined by $n_T \mathbf{\lambda} = \frac{\pi}{2} e^{-1/(2 n_s \mathbf{\lambda}^2)} \approx 1/(2 \log \rho_s / \mathbf{T})$, thermal fluctuations-induced steric interaction, $T^2 \mathbf{\lambda}^2 / (8 \rho_s |x_1 - x_2|^4)$ dominates over the exponentially weak $T = 0$ interaction $\frac{\pi}{2} e^{-|x_1 - x_2| / \mathbf{\lambda}}$. Such enhanced soliton repulsion leads to significantly slower (than $T = 0$, $n_s(b_\parallel) \approx 2 \frac{\pi}{2} \log (b_\parallel - 1)^{-1}$ prediction),
power-law increase in soliton density, \( n_s(b_{||}) \sim \frac{\rho_s}{b_{||} - 1}^{1/2} \), and energy density given by

\[
\delta \mathcal{E}^{(T)}_{\text{solitons}}[m_z] \sim \frac{\Delta_0 \rho_0}{2\pi \ell^2 T} (1 - m_z^2)^{15/8} |b_{||} - (1 - m_z^2)^{-1/4}|^{3/2},
\]

(5)

Ignoring for simplicity a possible intermediate regime \( T \to 0 \) dilute regime, defined by \( n_T \lesssim n_s(b_{||}) \ll \lambda^{-1} \), for sufficiently high soliton density \( n_T \ll n_s(b_{||}) \ll \lambda^{-1} \), soliton interaction crosses over to contact interaction, with energy scale \( \rho_s \). In this dense regime \( n_s(b_{||}) \sim \lambda^{-1} |b_{||} - 1| \) and soliton energy density is given by

\[
\delta \mathcal{E}_{\text{solitons}}^{(dense)}[m_z] \sim \frac{\Delta_0}{2\pi \ell^2} (1 - m_z^2) |b_{||} - (1 - m_z^2)^{-1/4}|^2.
\]

(6)

Finally, in the super-dense limit, \( n_s(b_{||}) \gg \lambda^{-1} \), \( \phi_s(x) \) only shows periodic modulation with vanishing amplitude and the incommensurate energy density reduces to

\[
\delta \mathcal{E}_{\text{I}}^{(super-dense)}[m_z] \approx \frac{\varepsilon_c}{2} m_z^2 - \frac{\Delta_0 \pi}{128 \ell^2 b_{||}^2},
\]

(7)

with exchange and tunneling energy only leading to a vanishing, \( m_z \)-independent correction to the charging energy.

Putting these results together, standard analysis of \( \mathcal{E}_{\text{I}}[m_z] \) shows, that for sufficiently small \( g \) and sufficiently large \( b_{||} \) a continuous \( P_I - C_I \) transition can take place at \( b_{||}^{(I)}(g) \), determined by precisely which of the above regimes it falls into. Because the driving force (the exchange energy) behind the canting transition vanishes in the \( b_{||} \approx 1 \) limit, Eq. [4], we predict that \( P - C \) transition must be reentrant, as illustrated in Fig. [1].

So far we have restricted our analysis to a spatially uniform mean-field treatment. While this is appropriate for the commensurate state and the \( P_C - C_C \) transition, this clearly fails in the incommensurate state, where the soliton array breaks translational symmetry. A more detailed analysis shows that even within mean-field theory the canting instability is periodically modulated by \( (\partial_z \phi_s(x))^2 \), with canting \( (m_z \neq 0) \) confined to regions between solitons, where the twisting of \( \phi(x) \) and therefore the exchange energy cost is highest. One consequence of this is an upward shift of \( b_{||}^{(I)}(g) \), corresponding to the exchange energy density cost \( n_s \sqrt{\rho_s \varepsilon_c} \ell^2 \) associated with deformation of \( m_z(x) \) localized on solitons.

It is noteworthy, that, because \( \varepsilon_c \) vanishes while \( \rho_s \) saturates in the \( d \to 0 \) limit [8], in principle, for sufficiently small interlayer separation \( d \), the \( P - C \) transition must always take place. Whether it remains continuous, beyond our above mean-field theory analysis is a more difficult question. However, estimates of our model parameters from recent experiments [1], suggest that currently available bilayer devices have \( g \gg g_c \), and therefore should not display the \( P - C \) transition (see Fig. [2]).

Transcending these semi-microscopic, model-specific mean-field considerations, \( (\pm m_z) \mathbb{Z}_2 \) symmetry dictates the form of the effective classical Hamiltonian

\[
H = \int d^2 r \left[ \frac{\rho_s^2}{2} \left| \nabla m_z \right|^2 + \frac{\alpha_2}{2} m_z^2 + \alpha_4 m_z^4 - W \rho_s^2 \partial_z u \right] + \frac{c_g}{2} (\partial_y u)^2 + \frac{c_y}{2} (\partial_y u)^2
\]

(8)

valid near the finite temperature \( T_{P_I - C_C} \) transition and on scales longer than the soliton lattice spacing \( n_s^{-1} \). Given our discussion above, we expect the reduced temperature \( \alpha_2 \approx T/T_{P_I - C_I}(g) - 1 \), the compressional elastic constant \( c_g \) to vanish as \( b_{||} \to 1^+ \) (with precise form depending on the range of \( b_{||} \)), the tilt modulus \( c_y \approx \rho_s n_s / \lambda \), with both approaching \( \rho_s \) in the dense regime, and \( w \) a (pseudo)magneto-elastic coupling of the soliton lattice phonon \( u \) degree of freedom to the local electric dipole moment \( m_z \); because a decrease in the soliton density \( (\partial_z u > 0) \) increases the exchange energy of the twisted incommensurate state, we expect \( w > 0 \).

Standard analysis of \( H \) in Eq. (8) predicts that both \( \alpha_4 \) and \( w \) are relevant couplings for \( d < 4 \) and will therefore lead to non-mean-field critical behavior sufficiently close to the \( P_I - C_I \) transition. Preliminary renormalization group (RG) analysis in \( d = 4 - \epsilon \) dimensions suggests that the magneto-elastic coupling \( w \) can either drive the Ising transition first order or qualitatively modify it into a new \textit{scalar compressible} Ising model universality class. [14][15]

If latter scenario survives down to \( d = 2 \), critical behavior of the magnetization \( m_z \) and the associated magnetic susceptibility leads to the predictions of Eqs (8) for the corresponding interlayer charge imbalance \( q \) and the differential capacitance \( C \), with gate voltage \( V \) playing the role of the associated \( z \)-directed “magnetic field”.

We now turn to dynamics. Adapting the imaginary time action of the pseudo-ferromagnet [8] to the incommensurate soliton state, we find

\[
S = \int_0^{1/T} d\tau d^2 r \left[ i g m_z \partial_r u + \mathcal{H}[u, m_z] \right]
\]

(9)

where the dynamics originates from the Berry’s phase \( "\mathbb{P}\mathbb{P}" \) term, that encodes the ferromagnetic precessional dynamics, \( \gamma = g n_s / 2 \ell^2 \), and \( \alpha_2 \propto b_{||}^{(I)}(g) - b_{||} \) at \( T = 0 \).

The dynamics of interlayer charge and soliton lattice fluctuations contained in \( S \), Eq (8) can be probed through a linear response of interlayer charge imbalance \( \delta \mathbf{n}(k, \omega) \) to a time-dependent interlayer voltage \( V(k, \omega) \), applied in a balanced (i.e., keeping \( n_T = 1 \) capacitive geometry. The relevant response function is the dynamie dielectric constant \( \epsilon(k, \omega) = \epsilon(k, \omega) A/d \), both expressible in terms of the pseudo-spin linear susceptibility, \( \chi_{zz}(k, \omega) = -i \int dt d^2 r e^{-i \omega t - i k r} \langle \delta m_z^2(t, r), \delta m_z^2(0, 0) \rangle \).

3
It is straightforward to compute the $\epsilon(k, \omega)$ away from the $P_1 - C_1$ transition. For the $P_1$ phase we find

$$
\epsilon_P(k, \omega) = d \left( \frac{e}{4\pi^2\gamma} \right)^2 \frac{c_x k_x^2 + c_y k_y^2}{-\omega^2 + \omega P(k)^2},
$$

(10a)

$$
\omega_P(k) = \gamma^{-1}(c_x k_x^2 + c_y k_y^2)^{1/2}(\rho_s^2 k_x^2 + 2|\alpha_2|)^{1/2},
$$

(10b)

a result resembling that for the $\Delta = B_{||} = 0$ bilayer, here, with the translational Goldstone mode $u(r, t)$ playing the role analogous to the staggered $U(1)$ charge Goldstone phason mode $\phi(r, t)$. It is noteworthy that similar to recent theoretical predictions and experimental findings for interlayer tunneling, at long wavelengths the peak in the dielectric response $\epsilon(k, \omega)$ traces out the soliton lattice phonon dispersion $\omega_P(k)$.

Inside the $C_1$ phase we instead find

$$
\epsilon_C(k, \omega) = d \left( \frac{e}{4\pi^2\gamma} \right)^2 \frac{c_x k_x^2 + c_y k_y^2}{-\omega^2 + \omega C(k)^2},
$$

(11a)

$$
\omega_C(k) = \gamma^{-1}(c_x k_x^2 + c_y k_y^2)^{1/2}(\rho_s^2 k_x^2 + 2|\alpha_2|)^{1/2},
$$

(11b)

with $\epsilon(k \to 0, \omega \to 0)$ and therefore the associated capacitance (cf. Eq.3) diverging as the $P_1 - C_1$ transition is approached from above or below.

Near the $P_1 - C_1$ transition, the divergent correlation length $\lambda_m = \sqrt{\rho_s/|\alpha_2|}$ leads to breakdown of perturbation theory, and a full RG analysis of this quantum compressible Ising transition is in principle necessary. On general grounds, sufficiently close to $b_c(g)$, we expect $\epsilon(k, \omega)$ to display critical scaling

$$
\epsilon_{cr}(k, \omega) = k_y^{-2+\eta} \epsilon(k_x/k_x^{z_x}, \omega/k_y^{z_y}, k_y^{-z_y}),
$$

(12)

where $\nu$ and $z_{x,\omega}$ are the correlation length and anisotropy exponents, respectively. Preliminary RG analysis indicates that the upper critical dimension, below which the quartic ($\alpha_4$) and magneto-elastic ($\omega$) nonlinearities become qualitatively important is $d_{\text{crit}} = 2$. Hence, we expect mean-field description with $z_x = 1$, $z_\omega = 2$, $\eta = 0$, and $\nu = 1/2$ to accurately (up to logarithmic corrections) describe the $P_1 - C_1$ transition in these 2d devices.

Up to now we have ignored long-range part of the electrostatic dipolar interaction $\int_k U(k)|m_z(k)|^2$, with kernel $U(k) = 1/(1 - e^{-kd})/k$, and $\varepsilon_{d} = (\pi k d^2)^{2}/8$ is the dipolar energy per unit of length. As in real bulk magnets this interaction clearly favors a development of anti-aligned dipolar domains at the shortest possible length scales and therefore competes with the exchange energy $\mathbf{1}/2 \rho_s^2 |\nabla m_z|^2$, minimized by spatially homogeneous $m_z(r)$. In contrast to 3d systems, where domain size scales as $\sqrt{T}$, it is easy to show that in our 2d geometry, sufficiently below the $P_1 - C_1$ transition, the domain length is $d_\Omega \approx \xi_{\text{exch}}/\varepsilon_{d}$ ($\xi_{\text{exch}} \approx \rho_s^{2}/\lambda_m^{4}$) and therefore is a length that can in principle be tuned from bilayer thickness to a macroscopic length. Sufficiently close to the transition, the long-range dipolar interaction will always become important, and will lead to the canting instability in $m_z(r)$ to take place at a finite wavevector $k_d$, determined by $\rho_s^{2}\varepsilon_{d}^{2} = \pi/(3(k_d d))^{2} |1 - 1/k_d d^{2}e^{-k_d d}|$. Although the mean-field predictions of the thermodynamics at the transition should remain unchanged, the uniform ($k_s = 0$) pseudo-spin susceptibility, determining the capacitance $C$ will no longer diverge. For sufficiently large domains $\xi_d$, we expect fluctuations enhancement of $C$ near the transition, but with its divergence now cutoff by the dipolar domain size, and therefore with its peak scaling according to $k_d^{2}l_{E}^{(2-\eta)/z_{\omega}}$.

We have argued that in bilayer QH ferromagnets, for sufficiently strong field and close interlayer separation a quantum interlayer charging transition must take place. While arguments in favor of the transition and its striking properties are quite general, it is unlikely that our mean-field estimates of the phase boundary are quantitatively trustworthy. Microscopic calculations, exact diagonalization and quantum Monte Carlo studies are necessary to accurately determine the details of the phase diagram proposed here. We hope that the signatures of the $P - C$ transition studied here will stimulate experimental efforts to develop bilayer devices, where it can be observed.

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[1] J. P. Eisenstein et al., Phys. Rev. Lett. 68, 1383 (1992).
[2] S. Q. Murphy et al., Phys. Rev. Lett. 72, 782 (1994).
[3] I. B. Spielman et al., Phys. Rev. Lett. 84, 5808 (2000).
[4] I. B. Spielman et al., condmat/0012094.
[5] H. Fertig, Phys. Rev. B 40, 1087 (1989).
[6] A. H. MacDonald et al., Phys. Rev. Lett. 65, 775 (1990).
[7] X.-G. Wen, A. Zee, Phys. Rev. Lett. 69, 1811 (1992); Z. Ezawa, A. Iwazaki, Phys. Rev. B 48, 15189 (1993).
[8] K. Yang et al., Phys. Rev. Lett. 72, 732 (1994); K. Moon et al., Phys. Rev. B 51, 5138 (1995); K. Yang et al., ibid. 54, 11644 (1996).
[9] L. Balents and L. Radzihovsky, Phys. Rev. Lett. 86, 1825 (2001); A. Stern et al., ibid. 1829 (2001); M. Fogler and F. Wilczek et al., ibid. 1833 (2001).
[10] V. L. Pokrovsky, A. L. Talapov, Phys. Rev. Lett. 42, 65 (1979); M. E. Fisher, D. S. Fisher, Phys. Rev. B 25, 3192 (1982); C. B. Hanna et al., ibid. 63, 125305 (2001).
[11] J. Kyriakidis and L. Radzihovsky, condmat/0010329.
[12] Quantum fluctuations are not important in 2+1 dimensions, but quenched disorder is, sufficiently close to the CI transition.
[13] Such arguments for the commensurate state have also been made by C. Hanna, unpublished.
[14] L. Radzihovsky, unpublished.
[15] D. J. Bergman and B. I. Halperin, Phys. Rev. B 13, 2145 (1976).