Global unique solvability of the initial-boundary value problem for the equations of one-dimensional polytropic flows of viscous compressible multifluids

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Abstract
We consider the equations which describe polytropic one-dimensional flows of viscous compressible multifluids. We prove global existence and uniqueness of a solution to the initial-boundary value problem which corresponds to the flow in a bounded space domain.

Keywords: multifluid, viscous compressible flow, initial-boundary value problem, polytropic flow, global existence, uniqueness
1 Introduction

The paper is devoted to the analysis of the solvability of the equations of motion of multicomponent viscous compressible fluids (homogeneous mixtures of fluids, multifluids). Concerning the origin of the model and its physical interpretation, we refer the reader to [16]. An overview of the options for formulating the model and the known results can be found in [9] and [15]. Related multi-velocity models of multifluids are considered in [3], [7], [21] and [25]. As the first results on the well-posedness of the multidimensional equations of multifluids, we can refer to [4], [5] and [6]. Solvability for related models is shown in [18], [20], [22] and [23].

Weak solutions for multidimensional barotropic problems for the model considered in the paper are constructed in the steady version in [13] and [24] (polytropic case), and then in [11] and [14] (general case); in the unsteady version in [12] (polytropic case), and then in [17] (general case). Similar results for the heat-conductive model are obtained in [10]. For a number of
reasons, including the purpose of constructing more regular solutions, one-
dimensional formulations are of interest. It should be noted that the dimen-
sion with respect to the number of components (constituents) of a multifluid
is not logically and technically related to the number of the spatial variables,
and the interaction of the constituents via the viscous terms transforms the
system of differential equations governing the motion of a multifluid into a
system essentially different from the one-component system. Therefore, de-
spite the developed theory of one-dimensional flows of a viscous gas (see [8]
for example), the one-dimensional theory of multifluids remains rele-
vant.

The specificity of the paper is that we consider a variant of the model
with an average velocity in the transport operator.

2 Statement of the problem

Let us consider the system of equations governing motions of multicom-
ponent viscous compressible fluids without taking into account chemical re-
actions in the case of one spatial variable:

\[
\partial_t \rho_i + \partial_x (\rho_i v) = 0, \\
\rho_i (\partial_t u_i + v \partial_x u_i) = \partial_x P_i, \quad i = 1, \ldots, N.
\]

Here \(N \geq 2\) is the number of components, \(\rho_i\) is the density if the \(i\)-th con-
stituent, \(u_i\) is the velocity of the \(i\)-th component, \(v = \frac{1}{N} \sum_{i=1}^{N} u_i\) is the average
velocity of the multifluid, and \(P_i\) are the stresses. We accept generalized
Newton’s hypothesis

\[
P_i = -p + \sum_{j=1}^{N} \mu_{ij} \partial_x u_j,
\]

where \(p\) is the pressure, and the viscosity coefficients \(\{\mu_{ij}\}_{i,j=1}^{N}\) form the
symmetric matrix \(M\). Moreover, \(M > 0\), i. e. \((M \xi, \xi) \geq C_0(M) |\xi|^2\) for all
\(\xi \in \mathbb{R}^N\) with a constant \(C_0(M) > 0\).

The written equations together with the constitutive equation

\[
p = K \rho^\gamma, \quad \rho = \sum_{i=1}^{N} \rho_i, \quad K = \text{const} > 0, \quad \gamma = \text{const} > 1
\]

form a closed system

\[
\partial_t \rho_i + \partial_x (\rho_i v) = 0, \quad v = \frac{1}{N} \sum_{i=1}^{N} u_i,
\] (1)
\[ \rho_i (\partial_t u_i + v \partial_x u_i) + K \partial_x \rho^\gamma = \sum_{j=1}^{N} \mu_{ij} \partial_{xx} u_j, \quad i = 1, \ldots, N, \quad \rho = \sum_{i=1}^{N} \rho_i. \quad (2) \]

Let us consider this system in the rectangular \( Q_T \) (here and below \( Q_t = (0,1) \times (0,t) \)) with an arbitrary finite height \( T, 0 < T < \infty \), and endow this system with the following initial and boundary conditions (\( i = 1, \ldots, N \)):

\[ \rho_i|_{t=0} = \rho_{0i}(x), \quad u_i|_{t=0} = u_{0i}(x), \quad x \in [0,1], \quad \text{for all} \quad x \in (0,1), \quad (3) \]

\[ u_i|_{x=0} = u_i|_{x=1} = 0, \quad t \in [0,T]. \quad (4) \]

**Definition 1.** By a strong solution to the problem (1)–(4) we mean a collection of \( 2N \) functions \((\rho_1, \ldots, \rho_N, u_1, \ldots, u_N)\) such that the equations (1) and (2) are satisfied a.e. in \( Q_T \), the initial conditions (3) are satisfied for a.a. \( x \in (0,1) \), the boundary conditions (4) are satisfied for a.a. \( t \in (0,T) \), and the following inequalities and inclusions hold (\( i = 1, \ldots, N \))

\[ \rho_i > 0, \quad \rho_i \in L_\infty(0,T;W^1_2(0,1)), \quad \partial_t \rho_i \in L_\infty(0,T;L_2(0,1)), \]

\[ u_i \in L_\infty(0,T;W^1_2(0,1))^N \sqcap L_2(0,T;W^2_2(0,1)), \quad \partial_t u_i \in L_2(Q_T). \]

### 3 Main result

**Theorem 2.** Suppose that the initial data in (3) satisfy the conditions

\[ \rho_{0i} \in W^1_2(0,1), \quad \rho_{0i} > 0, \quad u_{0i} \in W^1_2(0,1), \quad i = 1, \ldots, N \quad (N \geq 2), \]

the symmetric viscosity matrix \( M \) is positive definite, and the polytropic index \( \gamma > 1 \) as well as the constants \( 0 < K, T < \infty \) are given. Then there exists a unique strong solution to the problem (1)–(4) in the sense of Definition 1.

Since the uniqueness and the corresponding local result are obtained in [19], the proof of Theorem 2 reduces to obtaining global a priori estimates, which is the main content of the paper.

### 4 Lagrangian coordinates

During the study of the problem (1)–(4), the parallel use of the Lagrangian coordinates is convenient. Let us accept \( y(x,t) = \int_{0}^{x} \rho(s,t) \, ds \) and \( t \)
as new independent variables. Then the system (1), (2) takes the form

$$\partial_t \rho_i + \rho \rho_i \partial_y v = 0, \quad v = \frac{1}{N} \sum_{i=1}^{N} u_i,$$

$$\frac{\rho_i}{\rho} \partial_t u_i + K \partial_y \rho \gamma = \sum_{j=1}^{N} \mu_{ij} \partial_y (\rho \partial_y u_j), \quad i = 1, \ldots, N, \quad \rho = \sum_{i=1}^{N} \rho_i,$$

the domain $Q_T$ is transformed into the rectangular $\Pi_T = (0, d) \times (0, T)$, where $d = \int_{0}^{1} \rho_0 \, dx > 0$, $\rho_0 = \sum_{i=1}^{N} \rho_{0i}$, and the initial and boundary conditions take the form $(i = 1, \ldots, N)$

$$\rho_i|_{t=0} = \tilde{\rho}_{0i}(y), \quad u_i|_{t=0} = \tilde{u}_{0i}(y), \quad y \in [0, d],$$

$$u_i|_{y=0} = u_i|_{y=d} = 0, \quad t \in [0, T].$$

5 Estimates of the concentrations

Let us consider a hypothetical solution $(\rho_1, \ldots, \rho_N, u_1, \ldots, u_N)$ to the problem (1)–(4) which possesses all necessary differential properties, and such that the densities $\rho_i, i = 1, \ldots, N$, are positive and bounded (see Definition 1).

First of all, we note that the summation of (5) with respect to $i = 1, \ldots, N$ gives

$$\partial_t \rho + \rho^2 \partial_y v = 0,$$

and hence

$$\partial_t \left( \frac{\rho_i}{\rho} \right) = 0, \quad i = 1, \ldots, N.$$

Hence, due to (7) we get the equalities

$$\frac{\rho_i(y, t)}{\rho(y, t)} = \frac{\tilde{\rho}_{0i}(y)}{\tilde{\rho}_0(y)} \quad \text{as} \quad (y, t) \in [0, d] \times [0, T]$$

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for all $i = 1, \ldots, N$, where $\tilde{\rho}_0 = \sum_{i=1}^{N} \tilde{\rho}_{0i}$. In the Eulerian coordinates $(x, t)$, the ratios $\frac{\rho_i}{\rho}$ satisfy the transport equations, and we only have the inequalities

$$\inf_{[0,1]} \frac{\rho_{0i}(x)}{\rho_0(x)} \leq \frac{\rho_i(x, t)}{\rho(x, t)} \leq \sup_{[0,1]} \frac{\rho_{0i}(x)}{\rho_0(x)} \leq 1 \quad \text{as} \quad (x, t) \in [0, 1] \times [0, T],$$

$$i = 1, \ldots, N. \quad (11)$$

6 First a priori estimates

Typically for the compressible Navier—Stokes theory, the energy inequality immediately entails the estimates for the kinetic energy, the rate of energy dissipation and the potential energy of the multifluid constituents, as we show below.

**Lemma 3.** Under the assumptions of Theorem 2, there exists a positive constant

$$C_1 \left( \left\{ \inf_{[0,1]} \frac{\rho_{0i}}{\rho_0} \right\}, \left\{ \|\sqrt{\rho_{0i}} u_{0i}\|_{L_2(0,1)} \right\}, \|\rho_0\|_{L_1(0,1)}, K, M, N, \gamma \right),$$

such that the following estimate holds

$$\sum_{i=1}^{N} \left( \|\sqrt{\rho} u_i\|_{L_\infty(0,T;L_2(0,1))} + \|\partial_x u_i\|_{L_2(Q_T)} \right) + \|\rho\|_{L_\infty(0,T;L_\gamma(0,1))} \leq C_1. \quad (12)$$

**Proof.** Let us multiply the equations (2) by $u_i$, integrate over $(0,1)$ and sum with respect to $i = 1, \ldots, N$. In view of (1), (11) and $M > 0$, the following relations hold

$$\sum_{i=1}^{N} \int_0^1 \left( \rho_i \partial_t u_i + \rho_i v \partial_x u_i \right) u_i \, dx = \frac{1}{2} \frac{d}{dt} \sum_{i=1}^{N} \int_0^1 \rho_i u_i^2 \, dx,$$

$$\sum_{i=1}^{N} \int_0^1 u_i K \partial_x \rho^\gamma \, dx = -KN \int_0^1 \rho^\gamma \partial_x v \, dx = \frac{KN}{\gamma - 1} \frac{d}{dt} \int_0^1 \rho^\gamma \, dx,$$

\[\text{Hereinafter, } C \text{ with indices denotes positive constants which depend on the initial data, physical constants and } T.\]
\[
\sum_{i,j=1}^{N} \mu_{ij} \int_{0}^{1} (\partial_{xx} u_j) u_i \, dx = - \sum_{i,j=1}^{N} \mu_{ij} \int_{0}^{1} (\partial_{x} u_i)(\partial_{x} u_j) \, dx \leq -C_0(M) \sum_{i=1}^{N} \int_{0}^{1} |\partial_{x} u_i|^2 \, dx, \quad (13)
\]

and hence we get the inequality
\[
\frac{d}{dt} \sum_{i=1}^{N} \int_{0}^{1} \left( \frac{1}{2} \rho_i u_i^2 + \frac{K}{\gamma - 1} \rho^\gamma \right) \, dx + C_0 \sum_{i=1}^{N} \int_{0}^{1} |\partial_{x} u_i|^2 \, dx \leq 0. \quad (14)
\]

We integrate \((14)\) over \((0, t)\), and using \((3)\), we obtain the bound
\[
\sum_{i=1}^{N} \int_{0}^{1} \left( \frac{1}{2} \rho_i u_i^2 + \frac{K}{\gamma - 1} \rho^\gamma \right) \, dx + C_0 \sum_{i=1}^{N} \int_{0}^{t} \int_{0}^{1} |\partial_{x} u_i|^2 \, dx \, d\tau \leq
\]
\[
\leq \sum_{i=1}^{N} \int_{0}^{1} \left( \frac{1}{2} \rho_0 u_0^2 + \frac{K}{\gamma - 1} \rho_0^\gamma \right) \, dx,
\]

which, due to \((11)\), implies the conclusion of Lemma 3.

**Remark 4.** In the Lagrangian coordinates, the bound \((12)\) takes the form
\[
\sum_{i=1}^{N} \left( \|u_i\|_{L_{\infty}(0,T;L_2(0,d))} + \|\sqrt{\rho} \partial_y u_i\|_{L_2(\Pi_T)} \right) + \|\rho\|_{L_{\infty}(0,T;L_{\gamma-1}(0,d))} \leq C_2(C_1, \gamma). \quad (15)
\]

**Remark 5.** In view of \((8)\), we obviously get the following inequality from \((12)\)
\[
\sum_{i=1}^{N} \|u_i\|_{L_2(0,T;L_{\infty}(0,1))} \leq C_1. \quad (16)
\]

### 7 Bound of the density from above

The crucial a priori estimates are the bounds of strict positiveness and boundedness of the densities of the multifluid constituents. First we prove the bound for the densities from above.
Lemma 6. There exists a constant $C_3 \left( C_1, \| \rho_0 \|_{L_\infty(0,1)}, \{\| \rho_0 u_0 \|_{L_1(0,1)} \}, K, M, N, T, d, \gamma \right)$ such that

$$\rho(x,t) \leq C_3 \quad \text{as} \quad (x,t) \in [0,1] \times [0,T]. \quad (17)$$

**Proof.** Let us rewrite the equations (2), using (1), in the form

$$\frac{1}{N} \sum_{j=1}^{N} \tilde{\mu}_{ij} \left( \partial_t (\rho_j u_j) + \partial_x (\rho_j v u_j) \right) + K \left( \sum_{j=1}^{N} \tilde{\mu}_{ij} \right) \partial_x \rho^\gamma = \frac{1}{N} \partial_{xx} u_i,$$

\[ i = 1, \ldots, N, \quad (18) \]

where $\tilde{\mu}_{ij}$ are the entries of the matrix $\tilde{M} = M^{-1} > 0$, and then sum (18) with respect to $i = 1, \ldots, N$, then we get

$$\partial_t V = \partial_x \left( \partial_x v - \tilde{K} \rho^\gamma - v V \right), \quad (19)$$

where $V = \frac{1}{N} \sum_{i,j=1}^{N} \tilde{\mu}_{ij} \rho_j u_j$, $\tilde{K} = \frac{K}{N} \sum_{i,j=1}^{N} \tilde{\mu}_{ij} > 0$. We denote

$$\alpha(x,t) = \int_{0}^{t} \left( \partial_x v - \tilde{K} \rho^\gamma - v V \right) d\tau + \int_{0}^{x} V_0 \, ds, \quad (20)$$

where $V_0(x) = V(x,0)$. In view of (12), we have

$$\| \partial_x \alpha \|_{L_\infty(0,T;L_1(0,1))} = \| V \|_{L_\infty(0,T;L_1(0,1))} \leq C_4(C_1, M, d),$$

\[ \sup_{0 \leq t \leq T} \int_{0}^{1} \alpha \, dx \leq T \max_{1 \leq i,j \leq N} |\tilde{\mu}_{ij}| \sum_{i=1}^{N} \sup_{0 \leq t \leq T} \int_{0}^{1} \rho u_i^2 \, dx + \tilde{K} T \sup_{0 \leq t \leq T} \int_{0}^{1} \rho^\gamma \, dx + \max_{1 \leq i,j \leq N} |\tilde{\mu}_{ij}| \sum_{i=1}^{N} \int_{0}^{1} \rho_0 |u_0| \, dx \leq C_5 \left( C_1, \{\| \rho_0 u_0 \|_{L_1(0,1)} \}, M, \tilde{K}, T, \gamma \right), \]

and hence, using Poincaré’s inequality, we get

$$\sup_{0 \leq t \leq T} \int_{0}^{1} |\alpha| \, dx \leq \sup_{0 \leq t \leq T} \int_{0}^{1} |\partial_x \alpha| \, dx + \sup_{0 \leq t \leq T} \int_{0}^{1} |\alpha| \, dx \leq C_6(C_4, C_5),$$

\[ \text{2The bound } C_6 \text{ depends on the size of the flow domain.} \]
and we arrive at the boundedness of \(\alpha\) in \(L_\infty(0, T; W_1^1(0, 1))\). Using this and the fact \(W_1^1(0, 1) \rightarrow L_\infty(0, 1)\), we obtain the estimate

\[
\|\alpha\|_{L_\infty(Q_T)} \leq C_7 (C_4, C_6).
\]

Let us note that, in view of (1), (19) and (20), the following relations hold

\[
d_t(\rho e^{\alpha}) = -\tilde{K} e^{\alpha} \rho^{\gamma+1} \leq 0, \quad \text{where} \quad d_t = \partial_t + v \partial_x,
\]

and hence

\[
\rho e^{\alpha} \leq \sup_{[0,1]} \rho_0 \exp \left( \int_0^1 |V_0| \, dx \right),
\]

so that we arrive at the conclusion of Lemma 6.

8 The bound for the derivative of the density

In order to obtain the bound for the densities from below, we first need to prove the boundedness of the first spatial derivative of the logarithm of the total density. Specifically, the following assertion holds.

**Lemma 7.** There exists a constant

\[
C_8 \left( C_1, C_2, C_3, \left\{ \|\tilde{u}_0\|_{L_2(0,d)} \right\}, \left\{ \left\| \frac{\tilde{\rho}_0}{\rho_0} \right\|_{W_2^1(0,d)} \right\}, \| (\ln \tilde{\rho}_0)' \|_{L_2(0,d)}, M, N \right)
\]

such that

\[
\| \partial_y \ln \rho \|_{L_\infty(0,T;L_2(0,d))} \leq C_8.
\] (21)

**Proof.** Let us use the equations in the form (5), (6). We rewrite the equations (6) as

\[
\frac{1}{N} \sum_{j=1}^N \tilde{\mu}_{ij} \frac{\rho_j}{\rho} \partial_t u_j + \frac{K}{N} \left( \sum_{j=1}^N \tilde{\mu}_{ij} \right) \partial_y \rho^\gamma = \frac{1}{N} \partial_y (\rho \partial_y u_i), \quad i = 1, \ldots, N,
\] (22)

and then sum (22) with respect to \(i = 1, \ldots, N\), then we get, using (10), the resulting relation

\[
\frac{1}{N} \sum_{i,j=1}^N \tilde{\mu}_{ij} \frac{\tilde{\rho}_0}{\rho_0} \partial_t u_j + \tilde{K} \partial_y \rho^\gamma = \partial_y (\rho \partial_y v).
\] (23)
We extract from (9) that
\[ \rho \partial_y v = -\partial_t \ln \rho \] (24)
and substitute this into (23), then we get
\[ \partial_y \ln \rho + \tilde{K} \partial_y \rho \gamma = -\frac{1}{N} \sum_{i,j=1}^N \tilde{\mu}_{ij} \rho_{0j} \partial_t u_j. \]

We multiply this equality by \( \partial_y \ln \rho =: w \) and integrate over \( y \in (0, d) \), then we obtain
\[ \frac{1}{2} \frac{d}{dt} \left( \int_0^d w^2 \, dy \right) + \tilde{K} \gamma \int_0^d \rho^\gamma w^2 \, dy = -\frac{1}{N} \sum_{i,j=1}^N \tilde{\mu}_{ij} \int_0^d \left( \frac{\rho_{0j} \partial_t u_j}{\rho_0} \right) w \, dy. \] (25)

Let us transform the right-hand side of (25) via the integration by parts and using (24):
\[ -\frac{1}{N} \sum_{i,j=1}^N \tilde{\mu}_{ij} \int_0^d \left( \frac{\rho_{0j} \partial_t u_j}{\rho_0} \right) w \, dy = -\frac{d}{dt} \left( \frac{1}{N} \sum_{i,j=1}^N \tilde{\mu}_{ij} \int_0^d \left( \frac{\rho_{0j} \partial_t u_j}{\rho_0} \right) w \, dy \right) + \]
\[ + \frac{1}{N} \sum_{i,j=1}^N \tilde{\mu}_{ij} \int_0^d \rho u_j (\partial_y v) \left( \frac{\rho_{0j}}{\rho_0} \right)' \, dy + \frac{1}{N} \sum_{i,j=1}^N \tilde{\mu}_{ij} \int_0^d \frac{\rho_{0j} \rho}{\rho_0} (\partial_y v)(\partial_y u_j) \, dy. \] (26)

Thus, after integration of (25) with respect to \( t \), taking into account (17) and (26), we get
\[ \| w \|^2_{L^2(0,d)} + 2 \tilde{K} \gamma \int_0^t \int_0^d \rho^\gamma w^2 \, dy \, d\tau \leq \]
\[ \leq \| w_0 \|^2_{L^2(0,d)} - \frac{2}{N} \sum_{i,j=1}^N \tilde{\mu}_{ij} \int_0^d \frac{\rho_{0j} \rho}{\rho_0} u_j w \, dy + \frac{2}{N} \sum_{i,j=1}^N \tilde{\mu}_{ij} \int_0^d \frac{\rho_{0j} \rho}{\rho_0} u_j w_0 \, dy + \]
\[ + \frac{2\sqrt{C_3}}{N} \sum_{i,j=1}^N \left\| \tilde{\mu}_{ij} \right\|_{L^2(0,d)} \left\| u_j \right\|_{L^\infty(0,d)} \left\| \sqrt{\rho} \partial_y v \right\|_{L^2(0,d)} \, d\tau + \]

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\[ + \frac{2}{N} \sum_{i,j=1}^{N} \left| \tilde{\mu}_{ij} \right| \sup_{[0,d]} \tilde{\rho}_{0j} \int_{0}^{t} \left\| \rho(\partial_y v)(\partial_y u_j) \right\|_{L_1(0,d)} \, d\tau, \]

where \( w_0 = (\ln \tilde{\rho}_0)' \). Using the estimates (15) and (16), we derive from this the inequality (21), concluding the proof of Lemma 7.

9 The estimate of the density from below

In this section, we finish obtaining the crucial estimates of strict positivity and boundedness of the densities via the following assertion.

Lemma 8. There exists a constant \( C_9(C_8, d) \) such that

\[ \rho(y, t) \geq C_9 \quad \text{as } (y, t) \in [0, d] \times [0, T]. \] (27)

**Proof.** The continuity equation for \( \rho \) immediately leads, for any \( t \in [0, T] \), to the existence of a point \( z(t) \in [0, d] \) such that

\[ \rho(z(t), t) = d. \] (28)

Hence, we can use the representation

\[ \ln \rho(y, t) = \ln \rho(z(t), t) + \int_{z(t)}^{y} \partial_s \ln \rho(s, t) \, ds, \]

from which, via Hölder’s inequality, and using (21) and (28), we get

\[ |\ln \rho(y, t)| \leq |\ln d| + \sqrt{d}\|w\|_{L_2(0,d)} \leq C_{10}(C_8, d). \]

This leads immediately to (27), and Lemma 8 is proved.

**Remark 9.** The equalities (10) and the estimates in Lemmas 6 and 8 imply that for all \( i = 1, \ldots, N \) we have

\[ C_{11} \leq \rho_i(y, t) \leq C_3 \quad \text{as } (y, t) \in [0, d] \times [0, T], \] (29)

where \( C_{11} = C_{11}\left( C_9, \left\{ \inf_{[0,d]} \frac{\tilde{\rho}_0(y)}{\rho_0(y)} \right\} \right) \).

10 Further bounds

Concluding the proof of Theorem 2, we obtain the estimates for the derivatives of the densities and velocities of the multifluid constituents.
Remark 10. From the estimates in Lemmas 6 and 7 and the formula (10), it follows that
\[ \left\| \partial_x \rho_i \right\|_{L_\infty(0,T;L_2(0,1))} \leq C_{12}, \quad i = 1, \ldots, N, \quad (30) \]
where
\[ C_{12} = C_{12} \left( C_3, C_8, \left\{ \left\| \left( \frac{\tilde{\rho}_0}{\rho_i} \right)' \right\|_{L_2(0,d)} \right\}, \left\{ \sup_{[0,d]} \frac{\tilde{\rho}_0}{\rho_0} \right\} \right). \]

Lemma 11. There exists
\[ C_{13} \left( C_1, C_3, C_{11}, C_{12}, \left\{ \| u'_0 \|_{L_2(0,1)} \right\}, K, M, N, T, \gamma \right) \]
such that
\[ \sum_{i=1}^{N} \left( \left\| \partial_x u_i \right\|_{L_\infty(0,T;L_2(0,1))} + \left\| \partial_{xx} u_i \right\|_{L_2(Q_T)} + \left\| \partial_t u_i \right\|_{L_2(Q_T)} \right) \leq C_{13}. \]

Proof. We derive from (30) that
\[ \left\| \partial_x \rho(t) \right\|_{L_2(0,1)} \leq C_{14} \left( C_{12}, N \right) \quad \forall t \in [0, T]. \quad (31) \]
Using the idea of [2], we square the momentum equations (2) and sum the result with respect to \( i = 1, \ldots, N \), then we get
\[ \sum_{i=1}^{N} \rho_i (\partial_t u_i)^2 + \sum_{i=1}^{N} \frac{1}{\rho_i} \left( \sum_{j=1}^{N} \mu_{ij} \partial_{xx} u_j \right)^2 - 2 \sum_{i=1}^{N} (\partial_t u_i) \left( \sum_{j=1}^{N} \mu_{ij} \partial_{xx} u_j \right) = \]
\[ = \sum_{i=1}^{N} \rho_i \left( K \frac{\partial_x \rho^\gamma}{\rho_i} + v \partial_x u_i \right)^2. \quad (32) \]
Let us introduce a function \( \beta(t) \) via the relation
\[ \beta(t) = \sum_{i,j=1}^{N} \mu_{ij} \int_{0}^{1} (\partial_x u_i)(\partial_x u_j) \, dx + \]
\[ + \sum_{i=1}^{N} \int_{0}^{t} \int_{0}^{1} \left( \rho_i (\partial_t u_i)^2 + \frac{1}{\rho_i} \left( \sum_{j=1}^{N} \mu_{ij} \partial_{xx} u_j \right)^2 \right) \, dx \, d\tau. \]
Then (32) and the inequalities (13), (17), (29) and (31) give the estimate\(^3\)
\[ \beta'(t) \leq C_{15} + C_{16} \left( \sum_{j=1}^{N} \| u_j \|_{L_\infty(0,1)}^2 \right) \left( \sum_{i,j=1}^{N} \mu_{ij} \int_{0}^{1} (\partial_x u_i)(\partial_x u_j) \, dx \right) \leq \]
\(^3\)Here the symmetry of the matrix \( M \) is used.
\[ \leq C_{15} + C_{16} \left( \sum_{j=1}^{N} \|u_j\|_{L_\infty(0,1)}^2 \right) \beta(t), \]

where \( C_{15} = C_{15}(C_3, C_11, C_14, K, N, \gamma) \), \( C_{16} = C_{16}(C_3, M) \), from which, via Gronwall’s lemma (see also (16)), it follows that

\[ \beta(t) \leq C_{17} \left( C_1, C_{15}, C_{16}, \{\|u_{0i}'\|_{L_2(0,1)}\}, M, T \right), \]

and we arrive at the conclusion of Lemma 11.

**Remark 12.** It follows immediately from the continuity equations (1) and the estimates in Lemmas 6 and 11 and Remark 10, that

\[ \|\partial_t \rho_i\|_{L_\infty(0,T;L_2(0,1))} \leq C_{18}(C_3, C_{12}, C_{13}), \quad i = 1, \ldots, N. \]

11 Proof of Theorem 2

Basing on the global a priori estimates proved in Sections 5–10, we can continue the local solution (obtained in Theorem 2 from [19]) into the entire \( Q_T \) (see, e. g., [1], P. 40, or [26], P. 20). The uniqueness of this solution is shown in [19]. Thus, Theorem 2 is proved.
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