TORIC SPACES AND FACE ENUMERATION ON SIMPLICIAL MANIFOLDS

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ABSTRACT. In this paper, we study the well-know $g$-conjecture for rational homology spheres in a topological way. To do this, we construct a class of topological spaces with torus actions, which can be viewed as topological generalizations of toric varieties. Along this way we prove that after doing stellar subdivision operations at some middle dimensional faces of an arbitrary rational homology sphere, the $g$-conjecture is valid. Furthermore, we give topological proofs of several fundamental algebraic results about Buchsbaum complexes and simplicial manifolds. In this process, we also get a few interesting results in toric topology.

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1. Introduction

Our motivating problem is the following celebrated $g$-conjecture in algebraic combinatorics, which was first proposed by McMullen for characterizing the face numbers of simplicial polytopes [20]. See the great survey article [33] about this conjecture by Swartz.

**Conjecture 1** ($g$-conjecture). The $g$-vector of a rational homology sphere is a $M$-vector.

To understand this conjecture, let us recall some notions.

For a $(d - 1)$-dimensional simplicial complex $\Delta$, the $f$-vector of $\Delta$ is

$$(f_0, f_1, \ldots, f_{d-1}),$$

where $f_i$ is the number of the $i$-dimensional faces of $\Delta$. Sometimes it is convenient to set $f_{-1} = 1$ corresponding to the empty set. The $h$-vector of $\Delta$ is the integer vector $(h_0, h_1, \ldots, h_d)$ defined from the equation

$$h_0 t^d + \cdots + h_{d-1} t + h_d = f_{-1}(t-1)^d + f_0(t-1)^{d-1} + \cdots + f_{d-1}.$$

The $g$-vector $(g_0, \ldots, g_{[d/2]})$ is defined to be $g_0 = 1$, $g_i = h_i - h_{i-1}$ for $1 \leq i \leq [d/2]$.

In order to define $M$-vectors we first introduce the pseudopowers. For any two positive integers $a$ and $i$ there is a unique way to write

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$
with \( a_i > a_{i-1} > \cdots > a_j \geq j \geq 1 \). Define the \( i \)th pseudopower of \( a \) as

\[
a^{(i)} = \binom{a_i + 1}{i+1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j+1}.
\]

For convenience we define \( 0^{(i)} = 0 \) for all \( i \). A sequence of integers \((k_0, k_1, k_2, \ldots)\) satisfies \( k_0 = 1 \) and \( 0 \leq k_{i+1} \leq k^{(i)}_i \) for \( i \geq 1 \) is called an \( M \)-sequence. Finite \( M \)-sequences are \( M \)-vectors. Its name comes from the following fundamental result of Macaulay.

**Theorem 2** (Macaulay [19], see [7, §4.2]). A sequence of integers \((k_0, k_1, k_2, \ldots)\) is an \( M \)-sequence if and only if there exists a connected commutative graded algebra \( A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots \) over a field \( k \) such that \( A \) is generated by its degree-one elements and \( \dim_k A_i = k_i \) for \( i \geq 0 \).

In 1980, by using results from algebraic geometry, Stanley gave a beautiful proof of Conjecture 1 for the case where \( \Delta \) is a polytopal sphere, i.e. the boundary complex of a simplicial polytope. Later, McMullen gave another proof of the same result without using algebraic geometry.

Let \( \Theta = (\theta_1, \ldots, \theta_d) \) be a l.s.o.p. (linear system of parameters) of the face ring \( \mathbb{Q}[\Delta] \) of a \((d-1)\)-complex \( \Delta \) (see §2.2 for the definitions). Stanley first noticed that when \( \Delta \) is Cohen-Macaulay (a class of simplicial complexes including simplicial spheres), the Hilbert function of \( \mathbb{Q}[\Delta]/\Theta \) is equal to the \( h \)-vector of \( \Delta \). So in the case of polytopal spheres, Conjecture 1 is an immediate consequence of Theorem 2 and the following theorem.

**Theorem 3** (Stanley [31], McMullen [21, 22]). If \( \Delta \) is the boundary of a simplicial \( d \)-polytope, then for a certain l.s.o.p. \( \Theta \) of \( \mathbb{Q}[\Delta] \), there exists a linear form \( \omega \in \mathbb{Q}[\Delta] \) such that the multiplication map

\[
\cdot \omega^{d-2i} : (\mathbb{Q}[\Delta]/\Theta)_i \to (\mathbb{Q}[\Delta]/\Theta)_{d-i}
\]

is an isomorphism for all \( i \leq d/2 \).

Stanley’s proof of the above theorem used deep results from algebraic geometry, in particular, the hard Lefschetz theorem for projective toric varieties. McMullen’s proof builds upon the notion of the polytope algebra, which may be thought of as a combinatorial model for the cohomology algebras of toric varieties.

Let \( \Delta \) be a rational homology \((d-1)\)-sphere. Then it satisfies the Dehn-Sommerville relations, i.e., \( h_i(\Delta) = h_{d-i}(\Delta) \) [18]. We say \( \Delta \) has Lefschetz property if there exists an l.s.o.p. \( \Theta \) for \( \mathbb{Q}[\Delta] \) and a linear form \( \omega \) satisfying the condition in Theorem 3. Apparently, the \( g \)-conjecture can be deduced from the following algebraic conjecture.

**Conjecture 4** (algebraic \( g \)-conjecture). Every rational homology sphere has Lefshetz property.
Recently, Adiprasito [1] announced a proof of conjecture 4, but his paper is too technical and difficult. Still, the first three sections of his paper are inspiring and readable.

A natural question, in the spirit of Stanley’s topological proof of Theorem 3, is

What is the topological spaces behind the g-conjecture for general simplicial spheres (or even rational homology spheres)?

In this paper, we answer this question by constructing a class of topological spaces with torus actions, as a generalization of toric varieties. It turns out that some algebraic properties of face rings can be explained by the topological properties of these toric spaces, such as the Dehn-Sommerville relations just correspond to the Poincaré duality of rational toric manifolds as we will show in §3.

We can deduce many interesting results from the well-behaved local topology of these toric spaces. For example, in §4 we prove that after doing stellar subdivision operations at some middle dimensional faces of an arbitrary rational homology sphere, the g-conjecture is valid (Corollary 4.9).

Another important research object in algebraic combinatorics is the class of Buchsbaum complexes, which includes homology manifolds. In §5, we calculate the rational cohomology of toric spaces associated to a Buchsbaum complex $\Delta$, especially when $\Delta$ is a rational homology manifold. This gives topological expositions for several fundamental algebraic results about Buchsbaum complexes (e.g. Proposition 5.8 and Theorem 5.9).

Our topological construction is inspired by Davis-Januszkiewicz’s [13] construction of toric manifolds over simple polytopes. Their pioneering work [13] is the beginning of a very recent field called toric topology. §2.4-2.8 are short introductions to the main two research objects in toric topology: moment-angle complexes and toric spaces by D-J construction.

2. Preliminaries

2.1. Notations and conventions. For an abstract simplicial complex $\Delta$, let $F_i(\Delta)$ be the set of $i$-dimensional faces (simplices) of $\Delta$. For convenience, we set $F_{-1} = \{\emptyset\}$. Unless otherwise stated, we assume $\Delta$ has $m$ vertices and identify $F_0(\Delta)$ with $[m] = \{1, \ldots, m\}$. By $\Delta^{m-1}$ we denote the simplex consisting of all subsets of $[m]$, and by $\partial \Delta^{m-1}$ the boundary complex of $\Delta^{m-1}$.

For a subset $J \subset [m]$, the full subcomplex $\Delta_J \subset \Delta$ is defined to be

$$\Delta_J = \{\sigma \in \Delta : \sigma \subset J\}.$$

A subset $I \subset [m]$ is a missing face of $\Delta$ if $I \not\subset \Delta$ but $J \in \Delta$ for all proper subsets $J \subset I$. 
The *link* and the *star* of a face $\sigma \in \Delta$ are the subcomplexes

\[ \text{lk}_\sigma \Delta = \{ \tau \in \Delta : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset \} ; \]
\[ \text{st}_\sigma \Delta = \{ \tau \in \Delta : \tau \cup \sigma \in \Delta \} . \]

The *join* of two simplicial complexes $\Delta$ and $\Delta'$, where the vertex set $F_0(\Delta)$ is disjoint from $F_0(\Delta')$, is the simplicial complex

\[ \Delta \ast \Delta' = \{ \sigma \cup \sigma' : \sigma \in \Delta, \sigma' \in \Delta' \} . \]

In particular, if $\Delta' = \Delta^0$ is a point, we say that $\Delta \ast \Delta'$ is the *cone over* $\Delta$, simply denoted $C\Delta$.

Let $\sigma \in \Delta$ be a nonempty face of $\Delta$. The *stellar subdivision* $\text{ss}_\sigma \Delta$ of $\Delta$ at $\sigma$ is obtained by replacing the star of $\sigma$ by the cone over its boundary:

\[ \text{ss}_\sigma \Delta = (\Delta \setminus \text{st}_\sigma \Delta) \cup (C(\partial \sigma \ast \text{lk}_\sigma \Delta)) . \]

If $\dim \sigma = 0$ then $\text{ss}_\sigma \Delta = \Delta$. Otherwise the complex $\text{ss}_\sigma \Delta$ acquires an additional vertex (the vertex of the cone). In this case, denote by $v_\sigma$ this new vertex.

If $k$ is a field, the *reduced Betti numbers* of $\Delta$ are $\tilde{\beta}_i(\Delta; k) := \dim_k \tilde{H}_i(\Delta; k)$.

A simplicial complex $\Delta$ is called a *triangulated manifold* (or simplicial manifold) if the geometric realization $|\Delta|$ is a topological manifold. More generally, a $d$-dimensional simplicial complex $\Delta$ is a *$k$-homology manifold* ($k$ is a field) if

\[ H_*(|\Delta|, |\Delta| - x; k) = \tilde{H}_*(S^d; k) \quad \text{for all } x \in |\Delta| , \]

or equivalently,

\[ H_*(\text{lk}_\sigma \Delta; k) = H_*(S^{d-|\sigma|}; k) \quad \text{for all } \emptyset \neq \sigma \in \Delta . \]

Especially, when $k = \mathbb{Q}$, it is also referred to as a *rational homology manifold*, and when $k = \mathbb{Z}$, it is simply called a *homology manifold*. The notions for manifold, such as orientable, closed, with boundary, etc., are similarly defined for $k$-homology manifold. For example, A pair $(\Delta, \partial \Delta)$ of simplicial complexes is a *$k$-homology $d$-manifold with boundary* if the following conditions hold:

- $\Delta - \partial \Delta$ is a $k$-homology $d$-manifold,
- $\partial \Delta$ is a $k$-homology $(d - 1)$-manifold, and
- for each $x \in |\partial \Delta|$, the homology groups $H_*(|\Delta|, |\Delta| - x; k)$ all vanish.

$\Delta$ is a *$k$-homology $d$-sphere* if it is a $k$-homology $d$-manifold with the same $k$-homology as $S^d$. Similarly, when $k = \mathbb{Q}$, it is also called a *rational homology sphere*, and for $k = \mathbb{Z}$, a *homology sphere*. (Remark: Usually, the terminology “homology sphere” means a manifold having the homology of a sphere. Here we take it in a more relaxed sense than its usual meaning.) The *$k$-homology ball* is defined similarly.
2.2. **Face rings and l.s.o.p.** For a commutative ring \( k \) with unit, let \( k[x_1, \ldots, x_m] \) be the polynomial algebra with one generator for each vertex in \( \Delta \). We make it a graded algebra by setting \( \deg x_i = 2 \). (This even grading is unusual for algebraists. The reason why we set \( \deg x_i = 2 \) rather than 1 is to make it agree with the grading of the cohomology of some toric spaces we constructed below.)

The *Stanley-Reisner ideal* of \( \Delta \) is

\[
I_\Delta := (x_{i_1} x_{i_2} \cdots x_{i_k} : \{i_1, i_2, \ldots, i_k\} \notin \Delta).
\]

The *Stanley-Reisner ring* (or *face ring*) of \( \Delta \) is the quotient

\[
k[\Delta] := k[x_1, \ldots, x_m] / I_\Delta.
\]

Since \( I_\Delta \) is a monomial ideal, the quotient ring \( k[\Delta] \) is graded by degree.

For a face \( \sigma = \{x_{i_1}, \ldots, x_{i_k}\} \in \mathcal{F}_{d-1}(\Delta) \), denote by \( x_\sigma = x_{i_1} \cdots x_{i_k} \in Q[\Delta] \) the face monomial corresponding to \( \sigma \).

Assuming \( k \) is a field, a set \( \Theta = \{\theta_1, \ldots, \theta_d\} \) consisting of \( d = \dim \Delta + 1 \) linear forms in \( k[\Delta] \) is called a *linear system of parameters* (l.s.o.p. for short), if \( k[\Delta]/\Theta \) is finite-dimensional as a vector space over \( k \); here \( \Theta := (\theta_1, \ldots, \theta_d) \) also denotes the ideal that the l.s.o.p generates. It can be shown that a linear sequence \( \theta_1, \ldots, \theta_d \) is an l.s.o.p if and only if the restriction \( \Theta_\sigma = r_\sigma(\Theta) \) to each face \( \sigma \in \Delta \) generates the polynomial algebra \( k[x_i : i \in \sigma] \); here \( r_\sigma : k[\Delta] \to k[x_i : i \in \sigma] \) is the projection homomorphism (see [7, Theorem 5.1.16]). For the case that \( k = \mathbb{Z} \), a linear sequence \( \theta_1, \ldots, \theta_d \) is referred to as an *integral l.s.o.p* if its reduction modulo \( p \) is an l.s.o.p. for \( \mathbb{Z}_p[\Delta] \) for any prime \( p \). Equivalently, \( \theta_1, \ldots, \theta_d \) is an integral l.s.o.p. if and only if the restriction \( \Theta_\sigma = r_\sigma(\Theta) \) to each simplex \( \sigma \in \Delta \) generates the polynomial algebra \( \mathbb{Z}[x_i : i \in \sigma] \).

**Remark 2.1.** If \( k \) is an infinite field, then there always exists an l.s.o.p for \( k[\Delta] \) by Noether normalization lemma, but if \( k \) is a finite field (or \( k = \mathbb{Z} \)) then an l.s.o.p. for \( k[\Delta] \) (or an integral l.s.o.p. for \( \mathbb{Z}[\Delta] \)) may fail to exist (cf. [10, Example 3.3.4]).

2.3. **Algebraic properties of face rings.** In this subsection we review some basic combinatorial and algebraic concepts used in the rest of our paper. Throughout this subsection, \( k \) is an infinite field of arbitrary characteristic.

Let \( \Delta \) be a simplicial complex of dimension \( d-1 \). The face ring \( k[\Delta] \) is a *Cohen-Macaulay ring* if for any l.s.o.p \( \Theta = \{\theta_1, \ldots, \theta_d\} \), \( k[\Delta]/\Theta \) is a free \( k[\theta_1, \cdots, \theta_d] \) module. In this case, \( \Delta \) is called a *Cohen-Macaulay complex over* \( k \).

Let \( A \) be a connected commutative graded \( k \)-algebra. The *socle* of \( A \) is the ideal

\[
\text{Soc}(A) = \{x \in A : A_+ \cdot x = 0\}.
\]

The face ring \( k[\Delta] \) is a *Gorenstein ring* if it is Cohen-Macaulay and for any l.s.o.p \( \Theta = \{\theta_1, \ldots, \theta_d\} \), \( \dim_k \text{Soc}(k[\Delta]/\Theta) = 1 \). In other words, \( k[\Delta] \) is a Poincaré duality
k-algebra. We call $\Delta$ **Gorenstein over** $k$ if its face ring $k[\Delta]$ is a Gorenstein ring. Further, $\Delta$ is called **Gorenstein** if $k[\Delta]$ is Gorenstein and $\Delta$ is not a cone, i.e., $\Delta \neq \Delta^0 \ast \Delta'$.

The face ring $k[\Delta]$ is said to be **Buchsbaum** if for every l.s.o.p $\{\theta_1, \ldots, \theta_d\}$ and all $1 \leq i \leq d$,

$$\{x \in k[\Delta]/(\theta_1, \ldots, \theta_{i-1}) : x\theta_i = 0\} = \text{Soc}(k[\Delta]/(\theta_1, \ldots, \theta_{i-1})).$$

Similarly, $\Delta$ is called **Buchsbaum over** $k$ in this case.

All these algebraic properties of face rings have combinatorial-topological characterisations as follows.

**Theorem 2.2.** Let $\Delta$ be a simplicial complex. Then

(a) (Reisner [29]) $\Delta$ is Cohen-Macaulay (over $k$) if and only if for all faces $\sigma \in \Delta$ (including $\sigma = \emptyset$) and $i < \dim \text{lk}_\sigma \Delta$, we have $\tilde{H}_i(\text{lk}_\sigma \Delta; k) = 0$.

(b) (Stanley [32], Theorem II.5.1) $\Delta$ is Gorenstein* (over $k$) if and only if it is a $k$-homology sphere.

(c) (Schenzel [30]) $\Delta$ is Buchsbaum (over $k$) if and only if it is pure and the link of each nonempty face is Cohen-Macaulay (over $k$).

Hence, every simplicial complex whose geometric realization is a $k$-homology manifold is Buchsbaum over $k$.

If $\Delta$ is Cohen-Macaulay, the following result of Stanley shows that the $h$-vector of $\Delta$ has a pure algebraic description.

**Theorem 2.3** (Stanley). Let $\Delta$ be a $(d-1)$-dimensional Cohen-Macaulay complex and let $\Theta = \{\theta_1, \ldots, \theta_d\}$ be an l.s.o.p. for $k[\Delta]$. Then

$$\dim_k(k[\Delta]/\Theta)_{2i} = h_i(\Delta), \quad \text{for all } 0 \leq i \leq d.$$
\{\theta_1, \ldots, \theta_d\}$ for $k[\Delta]$ and a linear form $\omega$ such that the multiplication maps
\[ \cdot \omega : (k[\Delta]/\Theta)_{2i} \to (k[\Delta]/\Theta)_{2i+2} \]
have full rank for all $i < d$, i.e. either injective or surjective. Such a linear form $\omega$ is called a weak Lefschetz element (WLE).

The WLP is closely related to the $g$-conjecture because of the following well known result (cf. [33]).

**Proposition 2.6.** Let $\Delta$ be a $k$-homology $(d-1)$-sphere. If $\Delta$ has the WLE over $k$ then $\Delta$ satisfies the $g$-conjecture; A linear form $\omega$ is a WLE if and only if the multiplication map $(k[\Delta]/\Theta)_{2[d/2]} \to (k[\Delta]/\Theta)_{2[d/2]+2}$ is surjective, or equivalently, the map $(k[\Delta]/\Theta)_{2[d/2]-2} \to (k[\Delta]/\Theta)_{2[d/2]}$ is injective.

We define a set of pairs $W(\Delta) \subset k^{f_0} \oplus k^{df_0}$ to be
\[ W(\Delta) = \{ (\omega, \Theta) : \Theta \text{ is an l.s.o.p. for } k[\Delta] \text{ and } \omega \text{ is a WLE} \}. \]
It is not hard to see that $W(\Delta)$ is a Zariski open set. So if $W(\Delta) \neq \emptyset$, it is open dense in $k^{f_0} \oplus k^{df_0}$. We will loosely use the term ‘generic choice’ of $\Theta$ or $\omega$ to mean that these elements are chosen from a non-empty Zariski open set, to be understood from the context.

If $k$ is an infinite field and $\omega$ is a WEL for $k[\Delta]/\Theta$, then the generic linear combination of $\omega$ and some other arbitrary one-forms $\omega_1, \ldots, \omega_k$ is also a WEL. This can be seen from the following elementary result in linear algebra theory.

**Lemma 2.7.** Suppose we are given $r \times s$ matrices $(r \leq s)$ $A_1, \ldots, A_j$ with entries in an infinite field $k$, and one of these matrices has rank $r$. Let $B_{b_1 \ldots b_j} = \sum_{i=1}^j b_i A_i$ ($b_i \in k$) be a linear combination of $A_1, \ldots, A_j$. Then the set
\[ X = \{ (b_1, \ldots, b_j) \in k^j : \text{rank } B_{b_1 \ldots b_j} = r \} \]
is a nonempty Zariski open subset in $k^j$.

**Proof.** Without loss of generality, we may assume $A_1, \ldots, A_k$ are square $r \times r$ matrices, and $|A_1| \neq 0$. Viewing $b_i$ as variables, then it is easily verified that the determinant $|B_{b_1 \ldots b_k}|$ is a nonzero homogeneous polynomial $f(b_1, \ldots, b_k)$ of degree $r$. The statement of this lemma follows immediately since $k$ is infinite. \( \square \)

### 2.4. Moment-angle complexes and manifolds

The moment-angle complexes first appeared in work of Davis and Januszkiewicz [13] and further studied in detail and named by Buchstaber and Panov [8]. They play a key role in the emerging field of toric topology, which has many connections with algebraic geometry, commutative algebra and combinatorics, etc.
Let $\Delta$ be a simplicial complex, and let $(D^2, S^1)$ denote the pair of a disk and its boundary circle. For each simplex $\sigma = \{i_1, \ldots, i_k\} \in \Delta$, set

$$B_\sigma = \{(z_1, \ldots, z_m) \in (D^2)^m : z_i \in S^1 \text{ when } i \notin \sigma\}.$$ 

The moment-angle complex associated to $\Delta$ is defined as

$$Z_\Delta := \bigcup_{\sigma \in \Delta} B_\sigma \subset (D^2)^m.$$ 

The standard coordinatewise action of the $m$-torus $T^m = \mathbb{R}^m/\mathbb{Z}^m$ on $(D^2)^m$ induces the canonical $T^m$-action on $Z_\Delta$.

We have a natural cellular decomposition of $Z_\Delta$ as follows. Consider the following decomposition of the disc $D^2$ into 3 cells: the 0-cell $e^0 = 1 \in D^2$; the 1-cell $e^1 = S^1 \setminus \{1\}$; the 2-cell $e^2 = D^2 \setminus S^1$. By taking product we obtain a cellular decomposition of $(D^2)^m$, and then $Z_\Delta$ embeds as a CW subcomplex in $(D^2)^m$.

Each cell of $Z_\Delta$ has the form

$$e_\sigma \times t_J = e^2_{i_1} \times \cdots \times e^2_{i_k} \times e^1_{j_1} \times \cdots \times e^1_{j_r},$$

where $\sigma = \{i_1, \ldots, i_k\} \in \Delta$, $J = \{j_1, \ldots, j_r\}$, $\sigma \cap J = \emptyset$. (We omit the 0-cell in the product.)

There is an alternative way to define $Z_\Delta$ in terms of the dual simple polyhedral complex $P_\Delta$, constructed in [13]. As a polyhedron, $P_\Delta$ is the cone over the barycentric subdivision $\Delta'$ of $\Delta$. Precisely, for each simplex $\sigma \in \Delta$ (including $\emptyset$), let $F_\sigma$ denote the geometric realization of the poset $\Delta_{\geq \sigma} = \{\tau \in \Delta : \tau \supseteq \sigma\}$. Hence, for $\sigma \neq \emptyset$, $F_\sigma$ is the subcomplex of $\Delta'$ consisting of all simplices of the form $\sigma = \sigma_0 < \sigma_1 < \cdots < \sigma_k$, and $F_\emptyset = P_\Delta$ is the cone on $\Delta'$. If $\sigma$ is a $(k-1)$-simplex, then we say that $F_\sigma$ is a face of codimension $k$.

The polyhedron $P_\Delta$ together with its decomposition into “faces” $\{F_\sigma\}_{\sigma \in \Delta}$ will be called a simple polyhedral complex. In particular, there are $m$ facets $F_i, \ldots, F_m$ of $P_\Delta$, in which $F_i$ is the geometric realization of the star of the $i$th vertex of $\Delta$ in $\Delta'$. Let $T_i = S^1$ be the coordinate circle subgroup of $T^m$. For each point $x \in P_\Delta$, define a subtorus

$$T(x) = \prod_{i : x \in F_i} T_i \subset T^m,$$

assuming that $T(x) = \{1\}$ if there are no facets containing $x$. Then define

$$Z_\Delta = P_\Delta \times T^m / \sim,$$  \hspace{1cm} (2.1)

where the equivalence relation $\sim$ is given by $(x, g) \sim (x', g')$ iff $x = x$ and $g^{-1}g' \in T(x)$. The action of $T^m$ on $P_\Delta \times T^m$ by the right translations descends to a $T^m$-action on $Z_\Delta$, and the orbit space of this action is just $P_\Delta$. These two definitions are equivalent and both have their own convenience in different situations (cf. [9, Chapter 6]).
Example 2.8. (i) Let $\Delta = \partial \Delta^{m-1}$ (the boundary of a simplex), then

$$
\mathcal{Z}_\Delta = (D^2 \times \cdots \times D^2 \times S^1) \cup (D^2 \times \cdots \times S^1 \times D^2) \cup \cdots
$$

$$
\cup (S^1 \times \cdots \times D^2 \times D^2) = \partial((D^2)^m) = S^{2m-1}.
$$

(ii) If $\Delta = \Delta_1 \ast \Delta_2$, then $\mathcal{Z}_\Delta = \mathcal{Z}_{\Delta_1} \times \mathcal{Z}_{\Delta_2}$.

$\mathcal{Z}_\Delta$ is a closed orientable topological manifold (resp. $k$-homology manifold) of dimension $m + d$ if and only if $\Delta$ is a homology $(d-1)$-sphere (resp. $k$-homology $(d-1)$-sphere). (see [12, §2.1]). In this case, we call $\mathcal{Z}_\Delta$ a moment-angle manifold (resp. $k$-homology moment-angle manifold). In particular, if $\Delta$ is a polytopal sphere, then $\mathcal{Z}_\Delta$ admits a smooth structure (see [10, Chapter 6]). In general, the smoothness of $\mathcal{Z}_\Delta$ is open.

2.5. Cohomology of moment-angle complexes. Throughout this subsection, $k$ is a commutative ring with unit.

For a simplicial complex $\Delta$, the Koszul complex of the face ring $k[\Delta]$ is defined as the differential $\mathbb{Z} \oplus \mathbb{N}^m$-graded algebra $(\Lambda[y_1, \ldots, y_m] \otimes k[\Delta], d)$, where $\Lambda[y_1, \ldots, y_m]$ is the exterior algebra on $m$ generators over $k$, and the multigrading and differential is given by

$$
mdeg y_i = (-1, 2e_i), \quad mdeg x_i = (0, 2e_i), \quad e_i \in \mathbb{N}^m \text{ is the } i\text{th unit vector};
$$

$$
dy_i = x_i, \quad dx_i = 0.
$$

It is known that $H^*(\Lambda[y_1, \ldots, y_m] \otimes k[\Delta], d) = \text{Tor}_{k[x_1, \ldots, x_m]}(k[\Delta], k)$. Then the Tor-algebra $\text{Tor}_{k[x_1, \ldots, x_m]}(k[\Delta], k)$ is canonically an $\mathbb{Z} \oplus \mathbb{N}^m$-graded algebra.

**Theorem 2.9** ([4],[10, Theorem 4.5.4]). The following isomorphism of algebras holds:

$$
H^*(\mathcal{Z}_\Delta; k) \cong \text{Tor}_{k[x_1, \ldots, x_m]}(k[\Delta], k),
$$

$$
H^p(\mathcal{Z}_\Delta; k) = \bigoplus_{i+2|J|=p} \text{Tor}_{k[x_1, \ldots, x_m]}^{-i,2J}(k[\Delta], k),
$$

where $J = (j_1, \ldots, j_m) \in \mathbb{N}^m$ and $|J| = j_1 + \cdots + j_m$.

We may view a subset $J \subset [m]$ as a $(0,1)$-vector in $\mathbb{N}^m$ whose $j$th coordinate is 1 if $j \in J$ and 1 otherwise. Then there is the following well known Hochster’s formula:

**Theorem 2.10** (Hochster [16], see also [10, Theorem 3.2.9]). For any subset $J \subset [m]$ we have

$$
\text{Tor}_{k[x_1, \ldots, x_m]}^{-i,2J}(k[\Delta], k) \cong \widetilde{H}^{-i-1}(\Delta_J; k),
$$

and $\text{Tor}_{k[x_1, \ldots, x_m]}^{-i,2J}(k[\Delta], k) = 0$ if $J$ is not a $(0,1)$-vector. We assume $\widetilde{H}^{-1}(\Delta_{\emptyset}; k) = k$ above.
So Tor\(_k\)\([x_1,\ldots,x_m]\)(k[\Delta], k) is isomorphic to \(\bigoplus_{J \subseteq [m]} \tilde{H}^*(\Delta_J; k)\) as \(k\)-modules, and this isomorphism endows the direct sum \(\bigoplus_{J \subseteq [m]} \tilde{H}^*(\Delta_J; k)\) with a \(k\)-algebra structure. On the other hand, Baskakov [3] directly defined a multiplication structure on \(\bigoplus_{J \subseteq [m]} \tilde{H}^*(\Delta_J; k)\) to make the isomorphism in the Hochster’s formula to be algebraic. Before describing this multiplication structure precisely, let us see some operations on the homology and cohomology of the full subcomplexes of \(\Delta\).

Let \(\tilde{C}^i(\Delta; k)\) (resp. \(\tilde{C}_i(\Delta; k)\)) denote the \(i\)th reduced simplicial cochain (resp. chain) group of \(\Delta\) with coefficients in \(k\). For an oriented (ordered) simplex \(\sigma = (i_1, \ldots, i_p) \in \Delta\), denote still by \(\sigma \in \tilde{C}^{p-1}(\Delta; k)\) the basis cochain corresponding to \(\sigma\); it takes value 1 on \(\sigma\) and vanishes on all other simplices. For simplicity we will omit the coefficient ring \(k\) from the notations throughout the rest of this subsection.

**Definition 2.11.** The **union product** in the simplicial cochains of full subcomplexes of \(\Delta\) is defined to be the \(k\)-bilinear operation

\[
\sqcup : \tilde{C}^{p-1}(\Delta_I) \otimes \tilde{C}^{q-1}(\Delta_J) \rightarrow \tilde{C}^{p+q-1}(\Delta_{I \cup J}), \quad p, q \geq 0,
\]

\[
\sigma \otimes \tau \mapsto \sigma \sqcup \tau
\]

in which \(\sigma \sqcup \tau\) is the juxtaposition of \(\sigma\) and \(\tau\) if \(I \cap J = \emptyset\) and \(\sigma \cup \tau\) is a simplex of \(\Delta_{I \cup J}\); zero otherwise.

Similarly, the **excision product** in the simplicial chains and cochains of full subcomplexes is defined by

\[
\sqcap : \tilde{C}_{p+q-1}(\Delta_I) \otimes \tilde{C}^{p-1}(\Delta_J) \rightarrow \tilde{C}_{q-1}(\Delta_{I \setminus J}), \quad p, q \geq 0.
\]

\[
\sigma \otimes \tau \mapsto \sigma \sqcap \tau
\]

Here \(\sigma \sqcap \tau = \varepsilon_{\sigma, \tau}(\sigma \setminus \tau)\) if \(J \subset I\), \(\tau \subset \sigma\) and \(\sigma \setminus \tau \subset I \setminus J\); zero otherwise, and \(\varepsilon_{\sigma, \tau}\) is the sign of the permutation sending \(\tau \sqcup (\sigma \setminus \tau)\) to \(\sigma\).

It is easily verified that the union product of cochains induces a union product of cohomology classes in the full subcomplexes of \(K\):

\[
\sqcup : \tilde{H}^{p-1}(\Delta) \otimes \tilde{H}^{q-1}(\Delta) \rightarrow \tilde{H}^{p+q-1}(\Delta), \quad p, q \geq 0.
\]  \hspace{1cm} (2.2)

Similarly, there is an induced excision product in homology and cohomology of the full subcomplexes of \(\Delta\). Union and excision product are related by the formula

\[
\psi(c \sqcap \phi) = (\phi \sqcup \psi)(c)
\]

for \(c \in \tilde{C}_{p+q-1}(\Delta_I)\), \(\phi \in \tilde{C}^{p-1}(\Delta_J)\) and \(\psi \in \tilde{C}^{q-1}(\Delta_{I \setminus J})\).

Intuitively, the union product (resp. excision product) is an analog of cup product (resp. cap product) in cohomology (resp. homology and cohomology) of a space. Actually, the union and excision product for \(\Delta\) do respectively induce the cup and cap product for \(\mathcal{Z}_\Delta\) (cf. [10, Chapter 4.5]).
Theorem 2.12 (Baskakov [3], see also [10, Theorem 4.5.8] or [15]). There is a ring isomorphism (up to a sign for each cohomology degree).

\[ H^*(Z_\Delta) \cong \bigoplus_{J \subset [m]} \tilde{H}^*(\Delta_J), \quad H^p(Z_\Delta) \cong \bigoplus_{J \subset [m]} \tilde{H}^{p-|J|-1}(\Delta_J), \]

where the ring structure on the right hand side is given by the union product \( \sqcup \) in (2.2).

Remark 2.13. There is an intrinsic topological exposition of Theorem 2.12. For a subset \( J \subset [m] \) and a \((k-1)\)-face \( \sigma \in \Delta_J \), we have a \((|J|+k)\)-cell \( e_\sigma \times t_{J\setminus \sigma} \) of \( Z_\Delta \). Let \( C^*(Z_\Delta) \) be the cellular cochain groups of \( Z_\Delta \). It has a basis of cochains \( e_\sigma^* t_{J\setminus \sigma} \) dual to the corresponding cells. Hence, the cup product in \( C^*(Z_\Delta) \) just corresponds to the dual cochains of the cartesian product of these cells. So it corresponds to the union product in the simplicial cochains of full subcomplexes of \( \Delta \). Similarly, we have the correspondence between the cap product for \( Z_\Delta \) and excision product for the full subcomplexes of \( \Delta \).

To conclude this subsection, we mention that the \( T^m \)-equivariant cohomology is considerably simpler than the ordinary cohomology of \( Z_\Delta \).

Theorem 2.14 ([13, Theorem 4.8]). The \( T^m \)-equivariant cohomology ring of the moment-angle complex \( Z_\Delta \) is isomorphic to the face ring of \( \Delta \):

\[ H^*_T(Z_\Delta) \cong k[\Delta]. \]

2.6. Quasitoric manifolds. In their pioneering work [13] Davis and Januszkiewicz suggested a topological generalisation of projective toric manifolds (nonsingular projective toric varieties), which became known as quasitoric manifolds. A quasitoric manifold is a \( 2d \)-dimensional manifold \( M \) with a locally standard action of \( T^d \) (that is, it locally looks like the standard coordinatewise action of \( T^d \) on \( \mathbb{C}^d \)) such that the quotient \( M/T^d \) can be identified with a simple \( d \)-polytope \( P \). Let us review this object as a guide to the further generalized spaces.

Let \( P \) be a simple \( d \)-polytope, \( \mathcal{F} = \{ F_1, \ldots, F_m \} \) the set of facets of \( P \). Given a map \( \lambda: \mathcal{F} \to \mathbb{Z}^d \), and write \( \lambda(F_i) \) in the standard basis of \( \mathbb{Z}^d \):

\[ \lambda(F_i) = \lambda_i = (\lambda_{i1}, \ldots, \lambda_{id})^T \in \mathbb{Z}^d, \quad 1 \leq i \leq m. \]

If the matrix

\[ A = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_{d1} & \cdots & \lambda_{dm} \end{pmatrix} \]

has the following property:

\[ \det(\lambda_{i1}, \ldots, \lambda_{id}) = \pm 1 \quad \text{whenever} \quad F_{i1} \cap \cdots \cap F_{id} \neq \emptyset \quad \text{in} \quad P, \quad (2.3) \]
then λ is called a characteristic function for P, and Λ is called a characteristic matrix.

Let \((P, \Lambda)\) be a characteristic pair consisting of a simple polytope \(P\) and its characteristic matrix \(\Lambda\). Denote by \(T_i = S^1\) the circle subgroup of \(T^d = \mathbb{R}^d/\mathbb{Z}^d\) corresponding to the subgroup \(\lambda_i \in \mathbb{Z}^d\). For each point \(x \in P\), define a subtorus

\[
T(x) = \prod_{i: x \in F_i} T_i \subset T^d.
\]

Then the quasitoric manifold \(M(P, \Lambda)\) is defined to be

\[
M(P, \Lambda) = P \times T^d / \sim,
\]

where the relation \(\sim\) is as in (2.1).

In particular, if \(P\) is a Delzant polytope (a simple \(d\)-polytope \(P \subset \mathbb{R}^d\) is called a Delzant polytope if for every vertex \(v \in P\) the normal vectors to the facets meeting at \(v\) can be chosen to form a basis of \(\mathbb{Z}^d\)), and the function \(\lambda\) is defined by the normal vectors of \(P\), then \(M(P, \Lambda)\) is a projective toric manifold (cf. [27]).

There is an equivalent way to define quasitoric manifolds from polytopal spheres. Let \(\Delta\) be a polytopal \((d - 1)\)-sphere. Then the simple polyhedral complex \(P_\Delta\) (see subsection 2.4) can be viewed as the dual simple polytope of the simplicial polytope which \(\Delta\) bounds. Suppose \(\lambda: \mathcal{F}_0(\Delta) \to \mathbb{Z}^d\), \(i \mapsto \lambda_i\) is a characteristic function, that is it satisfies the condition

\[
\det(\lambda_{i_1}, \ldots, \lambda_{i_d}) = \pm 1 \quad \text{whenever} \quad (i_1, \ldots, i_d) \in \Delta. \tag{2.4}
\]

By means of \(\lambda\), we get a \(T^d\)-space \(M(\Delta, \Lambda) := P_\Delta \times T^d / \sim\) as in the construction (2.1) of \(Z_\Delta\). Let \(M(P_\Delta, \Lambda)\) be the space constructed in the first way. It is obvious that \(M(P_\Delta, \Lambda) = M(\Delta, \Lambda)\). For notational consistency, we use the second construction \(M(\Delta, \Lambda)\) to denote a quasitoric manifold in the rest of this paper.

**Remark 2.15.**

(i) The condition (2.4) is equivalent to saying that the linear sequence \(\{\theta_i = \lambda_{i_1}x_1 + \cdots + \lambda_{i_d}x_d\}_{1 \leq i \leq d}\) is an integral l.s.o.p. for \(\mathbb{Z}[\Delta]\).

(ii) For every 2- or 3-dimensional simple polytope \(P\) there exists a quasitoric manifold over \(P\). (The 3-dimensional case is due to the Four Color Theorem.) But for \(n \geq 4\), there exist simple \(n\)-polytopes which do not arise as the base spaces of quasitoric manifolds, since the integral l.s.o.p. for a polytopal sphere \(\Delta\) may fail to exist when \(\dim \Delta \geq 3\).

Note that a characteristic matrix \(\Lambda\) for \(\Delta\) defines a map of lattices: \(\Lambda: \mathbb{Z}^m \to \mathbb{Z}^d\), \(e_i \mapsto \lambda_i\). Condition (2.4) implies that there is a short exact sequence

\[
0 \to \mathbb{Z}^{m-d} \to \mathbb{Z}^m \xrightarrow{\Lambda} \mathbb{Z}^d \to 0.
\]

The matrix \(\Lambda\) also induces an epimorphism of tori

\[
\exp \Lambda: T^m \to T^d, \quad T_i \mapsto \{e^{2\pi i \lambda_{i_1} t}, \ldots, e^{2\pi i \lambda_{i_d} t}\} \in T^d, \quad t \in \mathbb{R}\}.
\]
whose kernel we denote by $K_A$. Obviously, $K_A = T^{m-d}$. From the construction of moment-angle manifolds and quasitoric manifolds we can easily see the following relation between them.

**Proposition 2.16** ([10, Proposition 7.3.12]). The group $K_A = T^{m-d}$ acts freely and smoothly on $Z_\Delta$. There is a $T^d$-equivariant homeomorphism

$$Z_\Delta/K_A \cong M(\Delta, \Lambda).$$

The cohomology ring of a quasitoric manifold has a simple expression as follows.

**Theorem 2.17** (Davis-Januszkiewicz, [13]). Let $M(\Delta, \Lambda)$ be a quasitoric manifold constructed from a polytopal $(d-1)$-sphere $\Delta$ with $m$ vertices, $\Lambda = (\lambda_{ij})$ be the corresponding characteristic $d \times m$ matrix. Then the cohomology ring $H^*(M(\Delta, \Lambda); \mathbb{Z})$ is generated by the degree-two classes, and is given by

$$H^*(M(\Delta, \Lambda); \mathbb{Z}) = \mathbb{Z}[^\Delta] / \Theta,$$

where $\Theta$ is the ideal generated by the linear forms $\lambda_{i1}x_1 + \cdots + \lambda_{im}x_m$, $1 \leq i \leq m$.

### 2.7. Topological toric orbifolds and rational toric manifolds

In fact, the construction of quasitoric manifolds over simple polytopes can be generalized to cases where the base space is an arbitrary simple polyhedral complex (cf. [13, §2]).

Now we discuss such a generalization, and introduce the central topological spaces of this paper.

Let $\Delta$ be a simplicial complex of dimension $d-1$, $P_\Delta$ the simple polyhedral complex associated to $\Delta$. A map $\lambda : F_0(\Delta) \to \mathbb{Z}^d$, $i \mapsto \lambda_i = (\lambda_{i1}, \ldots, \lambda_{id})^T$ (here we require $\lambda_i$ to be primitive in $\mathbb{Z}^d$) is called a *generalized characteristic function* if the linear sequence $\{\theta_i = \lambda_{i1}x_1 + \cdots + \lambda_{im}x_m\}_{1 \leq i \leq d}$ is an l.s.o.p. for $\mathbb{Q}[\Delta]$. In this case, the $d \times m$ matrix $\Lambda = (\lambda_{ij})$ is called a *generalized characteristic matrix*. Note that the rational face ring $\mathbb{Q}[^\Delta]$ always admits an l.s.o.p.

For a *generalized characteristic pair* $(\Delta, \Lambda)$ consisting of a simplicial complex $\Delta$ and its generalized characteristic matrix $\Lambda$. We put $M(\Delta, \Lambda) = P_\Delta \times T^d / \sim$, where the equivalence relation is defined exactly as in the case of quasitoric manifold; as before, $M(\Delta, \Lambda)$ is a $T^d$-space over $P_\Delta$. We call $M(\Delta, \Lambda)$ a *toric space associated to $\Delta$ by D-J construction*.

**Terminology Convention.** To simplify terminologies, we will omit the word ‘generalized’ and the words ‘by D-J construction’ in the above definitions, since we always discuss such toric spaces in the rest of this paper.

As we have seen, the matrix $\Lambda$ defines a map of lattices: $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^d$, which can be extended to a exact sequence

$$0 \to \mathbb{Z}^{m-d} \to \mathbb{Z}^m \xrightarrow{\Lambda} \mathbb{Z}^d \to G \to 0.$$
Since \( \{\theta_1, \ldots, \theta_m\} \) is an l.s.o.p for \( \mathbb{Q}[\Delta] \), \( G \) is a finite group. Set \( N = \text{Im} \Lambda \). Then the above exact sequence splits into two short exact sequences:

\[
0 \rightarrow \mathbb{Z}^{m-d} \rightarrow \mathbb{Z}^m \xrightarrow{A} N \rightarrow 0,
\]

\[
0 \rightarrow N \rightarrow \mathbb{Z}^d \rightarrow G \rightarrow 0.
\]

We apply the functor \( \otimes_{\mathbb{Z}} S^1 \) \((S^1 \subset \mathbb{C})\) to them and we get

\[
0 \rightarrow \mathcal{T}_m \rightarrow T_m \rightarrow N \rightarrow 0,
\]

\[
0 \rightarrow \text{Tor}_1^\mathbb{Z}(G, S^1) = G \rightarrow T^d_N \rightarrow T^d \rightarrow 0 = G \otimes_{\mathbb{Z}} S^1.
\]

Thus the lattice map \( \Lambda \) induces an epimorphism of tori \( \exp \Lambda : T^m \rightarrow T^d \) with kernel \( K_\Lambda = T^{m-d} \times G \).

Recall that an action of a group on a topological space is almost free if all isotropy subgroups are finite. As in the case of quasitoric manifold, it can be shown that

**Proposition 2.18** (cf. [10, Theorem 4.8.5]). The group \( K_\Lambda = T^{m-d} \times G \) acts almost freely and properly on \( Z_\Delta \). There is a \( T^d \)-equivariant homeomorphism

\[
Z_\Delta/K_\Lambda \cong M(\Delta, \Lambda).
\]

In particular, if \( \Delta \) is a homology \((d-1)\)-sphere, then \( Z_\Delta \) is a closed, orientable manifold of dimension \( m + d \). So Proposition 2.18 implies that for a given characteristic matrix \( \Lambda, M(\Delta, \Lambda) \) is a closed, orientable \( 2d \)-dimensional orbifold with a \( T^d \)-action. The orientation of \( M(\Delta, \Lambda) \) can be defined as follows. First, give an orientation to \( X = Z_\Delta/T^{m-d} \) by choosing orientations of \( Z_\Delta \) and \( T^{m-d} \) respectively. Then since the \( G \)-action on \( X \) extends to a toral action, it preserves the orientation. Thus \( M(\Delta, \Lambda) = X/G \) inherits an orientation from \( X \). In this case, we call \( M(\Delta, \Lambda) \) a topological toric orbifold. Similarly, when \( \Delta \) is a rational homology sphere, \( Z_\Delta \) is a closed, orientable, rational homology \((m + d)\)-manifold. It follows that \( M(\Delta, \Lambda) \) is a closed, orientable, rational homology \( 2d \)-manifold, called a rational toric manifold. Furthermore, if \( \Delta \) is a rational homology ball, \( M(\Delta, \Lambda) \) is an orientable, rational homology \( 2d \)-manifold with boundary.

**Example 2.19.** Let \( \Delta = \partial \Delta^2 \), the boundary complex of a 2-simplex. Hence \( Z_\Delta = S^5 \) (Example 2.8 (i)). We define \( M(\Delta, \Lambda) \) in three cases with different \( \Lambda \).

(i) Take \( \Lambda \) to be

\[
\lambda_1 = (1, 0)^T, \quad \lambda_2 = (0, 1)^T, \quad \lambda_3 = (-1, -1)^T.
\]

In this case, \( M(\Delta, \Lambda) \) is the quotient space of \( S^5 \subset \mathbb{C}^3 \) under the diagonal \( S^1 \) action, so \( M(\Delta, \Lambda) = \mathbb{C}P^2 \) is a toric manifold.

(ii) Let \( a, b \neq 0 \) are relatively prime positive integers. Define \( \Lambda \) to be

\[
\lambda_1 = (1, 0)^T, \quad \lambda_2 = (0, 1)^T, \quad \lambda_3 = (-a, -b)^T.
\]
Then $M(\Delta, \Lambda)$ is the quotient space of $S^5 \subset \mathbb{C}^3$ under a twisted $S^1$ action:

$$M(\Delta, \Lambda) = \left\{ (z_1, z_2, z_3) \in S^5 \right\} / \sim, \quad (z_1, z_2, z_3) \sim (t^a z_1, t^b z_2, tz_3), \quad t \in S^1.$$  

This space is the so-called weighted projective space $\mathbb{C}P^2_{(a,b,1)}$ with weight $(a, b, 1)$ (cf. [17]).

(iii) Take $\Lambda$ to be

$$\lambda_1 = (1, -1)^T, \quad \lambda_2 = (1, 2)^T, \quad \lambda_3 = (-2, -1)^T.$$  

In this case, the kernel of $\Lambda : \mathbb{Z}^3 \to \mathbb{Z}^2$ is $\mathbb{Z} \cdot (1, 1, 1)$, and the cokernel of $\Lambda$ is $\mathbb{Z}^3$, which is generated by $\frac{1}{2}(\lambda_2 + 2\lambda_3)$. From Proposition 2.18 we know that $M(\Delta, \Lambda) = S^5 / (S^1 \times \mathbb{Z}^3) = \mathbb{C}P^2 / \mathbb{Z}^3$ with the following action of $\mathbb{Z}^3$

$$\varepsilon(z_1 : z_2 : z_3) = (z_1 : \varepsilon z_2 : \varepsilon^2 z_3),$$

where $\varepsilon$ is a primitive root of unity of degree 3. Such a space is known as a fake weighted projective space (cf. [11]).

Remark 2.20. Let $\Lambda = (\lambda_1, \ldots, \lambda_m)$ be a characteristic matrix for $\Delta$. Then we can defined a characteristic matrix $\Lambda'$ for the stellar subdivision $\Delta' = \text{ss}_\sigma \Delta$ ($\sigma = \{1, \ldots, k\} \in \Delta, \quad k \geq 2$) as follows. For any $(a_1, \ldots, a_k) \in \mathbb{Q}^k$ with $a_i \neq 0$ for $1 \leq i \leq k$, define

$$\Lambda' = (A \mid \lambda_m), \quad \lambda_{\nu\sigma} = a_1 \lambda_1 + \cdots + a_k \lambda_k.$$  

It is easy to check that $\Lambda'$ is a characteristic matrix for $\Delta'$.

Remark 2.21. For a field $k$ of positive characteristic, it seems that there is no corresponding $k$-toric space associated to a simplicial complex $\Delta$ unless $k = \mathbb{Z}_2$, in which case there is a $\mathbb{Z}_2$-toric space called a small cover associated to $\Delta$ if there is an l.s.o.p. for $\mathbb{Z}_2[\Delta]$ (cf. [13]). But we know that $\mathbb{Z}_2[\Delta]$ do not always have an l.s.o.p.

Since this article has a topological flavor, we focus on the case $k = \mathbb{Q}$ in the rest of the paper.

2.8. Cohomology of toric spaces associated to Cohen-Macaulay complexes. In this subsection, we assume $\Delta$ is a Cohen-Macaulay complex over $\mathbb{Q}$, i.e., for every l.s.o.p. $\{\theta_1, \ldots, \theta_d\}$ for $\mathbb{Q}[\Delta]$, the face ring $\mathbb{Q}[\Delta]$ is free as a $\mathbb{Q}[\theta_1, \ldots, \theta_d]$-module.

For a characteristic matrix $\Lambda$ for $\Delta$, let $T^{m-d} \subset T^m$ be the subtorus corresponding to the kernel $\mathbb{Z}^{m-d}$ of the lattice map $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^d$. First, let us consider the rational $T^{m-d}$-equivariant cohomology of $Z_\Delta$.

Proposition 2.22. The rational $T^{m-d}$-equivariant cohomology of $Z_\Delta$ is given by

$$H^*_\mathbb{T}^{m-d}(Z_\Delta; \mathbb{Q}) \cong \mathbb{Q}[\Delta]/\Theta,$$
where Θ is the ideal generated by the l.s.o.p. \[ \{ \theta_i = \lambda_{i1}x_1 + \cdots + \lambda_{im}x_m \}_{1 \leq i \leq d} \] corresponding to \( \Lambda = (\lambda_{ij}) \).

Before giving the proof, let us recall some notions about equivariant topology. For an \( i \)-torus \( (i > 0) \ T^i \), let \( T^i \rightarrow ET^i \rightarrow BT^i \) be the universal principal \( T^i \) bundle. Note that the universal principal \( S^1 \)-bundle is the infinite Hopf bundle \( S^\infty \rightarrow \mathbb{C}P^\infty \). So the classifying space \( BT^i \) of the \( i \)-torus \( T^i \) is the product \( (\mathbb{C}P^\infty)^i \) of \( i \) copies of \( \mathbb{C}P^\infty \), and the total space \( ET^i \) over \( BT^i \) can be identified with the \( i \)-fold product of the infinite-dimensional sphere \( S^\infty \). Let \( X \) be a \( T^i \)-space. Then the \( T^i \)-equivariant cohomology of \( X \) is isomorphic to the ordinary cohomology of the Borel construction \( ET^i \times_{T^i} X \). Here

\[ ET^i \times_{T^i} X := ET^i \times X/\sim, \]

where \((e, x) \sim (ge, gx)\) for any \( e \in ET^i, \ x \in X, \ g \in T^i \).

In addition, we may assume that the lattice map \( \Lambda : \mathbb{Z}^m \rightarrow \mathbb{Z}^d \) is onto. Indeed if this is not the case, suppose \( \text{Im} \ \Lambda = N \subset \mathbb{Z}^d \). (Remember that \( G = \mathbb{Z}^d/\mathbb{N} \) is a finite group.) Choose a basis of the lattice \( N \) and then we can get another characteristic matrix \( \Lambda' \) written in this basis. It is easy to see that there is a \( d \times d \) matrix \( B \in GL(d, \mathbb{Q}) \) such that \( \Lambda A = \Lambda' \). So the rational ideal generated by the l.s.o.p. \( \Theta' \) corresponding to \( \Lambda' \) is the same as \( \Theta \), and therefore \( \mathbb{Q}[\Delta]/\Theta = \mathbb{Q}[\Delta]/\Theta' \).

Proof of Proposition 2.22. The coefficient \( \mathbb{Q} \) will be implicit throughout the proof. Consider the principal \( T^d \)-bundle:

\[ ET^d \times (ET^{m-d} \times_{T^{m-d}} \mathbb{Z}_\Delta) \rightarrow ET^m \times_{T^m} \mathbb{Z}_\Delta. \]

The the Serre spectral sequence of this fibration has \( E_2 \)-term

\[ E_2^{pq} = H^p(ET^m \times_{T^m} \mathbb{Z}_\Delta; H^q(T^d)). \]

According to Theorem 2.14, \( E_2 = \mathbb{Q}[\Delta] \otimes \Lambda[v_1, \ldots, v_d] \), where \( \Lambda[v_1, \ldots, v_d] \) (deg \( v_i = 1 \)) is the exterior algebra over \( \mathbb{Q} \).

We assert that the differential \( d_2 \) of the \( E_2 \)-term sends \( v_i \) to \( \theta_i \in \mathbb{Q}[\Delta] \). To see this, we consider the bundle map:

\[
\begin{array}{ccc}
T^m & \longrightarrow & ET^d \times (ET^{m-d} \times \mathbb{Z}_\Delta) \\
\downarrow \exp \Lambda & & \downarrow \\
T^d & \longrightarrow & ET^d \times (ET^{m-d} \times_{T^{m-d}} \mathbb{Z}_\Delta)
\end{array}
\]

Theorem 2.14 shows that the Serre spectral sequence of the upper fibration has \( E_2 \)-term

\[ E_2 = \mathbb{Q}[\Delta] \otimes \Lambda[y_1, \ldots, y_m], \text{ deg } y_i = 1. \]

The homomorphism \((\exp \Lambda)^* : H^1(T^d) \rightarrow H^1(T^m)\) can be identified with the dual map of \( \Lambda : \mathbb{Z}^m \rightarrow \mathbb{Z}^d \). Hence \((\exp \Lambda)^*(v_i) = \lambda_{i1}y_1 + \cdots + \lambda_{im}y_m \). Since we have
$d_2(y_i) = x_i$ in the $E_2$-term of the Serre spectral sequence of the upper fibration (cf. Appendix A.2), the assertion is true.

The fact that $\mathbb{Q}[\Delta]$ is a free $\mathbb{Q}[\theta_1, \ldots, \theta_d]$-module implies that the Serre spectral sequence of the lower fibration collapses at the $E_3$-term: $E_3 = \mathbb{Q}[\Delta]/\Theta$. Notice that $ET^d \times (ET^{m-d} \times_{T^{m-d}} \mathbb{Z}_\Delta)$ is homotopy equivalent to $ET^{m-d} \times_{T^{m-d}} \mathbb{Z}_\Delta$. Hence

$$H^*_{T^{m-d}}(\mathbb{Z}_\Delta) = H^*(ET^{m-d} \times_{T^{m-d}} \mathbb{Z}_\Delta) \cong \mathbb{Q}[\Delta]/\Theta.$$ 

\[\square\]

In the Serre fibration $ET^{m-d} \times \mathbb{Z}_\Delta \to ET^{m-d} \times_{T^{m-d}} \mathbb{Z}_\Delta$, the projection onto the second factor of $ET^{m-d} \times \mathbb{Z}_\Delta$ descends to a projection $ET^{m-d} \times_{T^{m-d}} \mathbb{Z}_\Delta \to \mathbb{Z}_\Delta/T^{m-d}$, compose this with the quotient map $\mathbb{Z}_\Delta/T^{m-d} \to \mathbb{Z}_\Delta/K_A = M(\Delta, \Lambda)$ if necessary we get a map

$$p : ET^{m-d} \times_{T^{m-d}} \mathbb{Z}_\Delta \to M(\Delta, \Lambda).$$

**Theorem 2.23.** For any $(d-1)$-dimensional complex $\Delta$ (not necessarily being Cohen-Macaulay), we have the following ring isomorphism

$$p^* : H^*(M(\Delta, \Lambda); \mathbb{Q}) \cong H^*_{T^{m-d}}(\mathbb{Z}_\Delta; \mathbb{Q}),$$

which is induced by the quotient map $p$ above.

We include the proof of Theorem 2.23 in Appendix A.1 for the reader’s convenience.

**Corollary 2.24.** If $\Delta$ is Cohen-Macaulay, then we have a ring isomorphism

$$H^*(M(\Delta, \Lambda); \mathbb{Q}) \cong \mathbb{Q}[\Delta]/\Theta.$$ 

**Remark 2.25.** Although the integral cohomology of rational toric manifolds often has torsion, and the ring structure is subtle even in the simplest case of weighted projective spaces (see [17]), their rational cohomology has the same simple form as quaitoric manifolds.

As we have seen, every characteristic matrix for $\Delta$ defines an l.s.o.p. for $\mathbb{Q}[\Delta]$. Conversely, if $\Theta$ is an l.s.o.p. for $\mathbb{Q}[\Delta]$, then the associated $d \times m$ matrix $A = (a_1, \ldots, a_m)$ can be written as $A = (\frac{1}{p_1}, \ldots, \frac{1}{p_m})$ with $p_i \in \mathbb{Z}$, such that $A = (\lambda_1, \ldots, \lambda_m)$ is a characteristic matrix for $\Delta$. Let $\Theta_A$ be the l.s.o.p. corresponds to $A$. Then it is easy to see that $\mathbb{Q}[\Delta]/\Theta \to \mathbb{Q}[\Delta]/\Theta_A$, $x_i \mapsto p_i x_i$ is a ring isomorphism. So we will do not distinguish the ring $\mathbb{Q}[\Delta]/\Theta$ from the cohomology of $M(\Delta, \Lambda)$ for Cohen-Macaulay complex $\Delta$ because of Corollary 2.24.
3. Topology of Rational Toric Manifolds and Its Applications

Throughout this section, $\Delta$ is a Cohen-Macaulay complex of dimension $d - 1$. By $(\Delta, A)$ and $\Theta$, we denote a characteristic pair and the corresponding l.s.o.p. So $M(\Delta, A)$ is a 2$d$-dimensional toric space. The simplified notation $M_\Delta$ for $M(\Delta, A)$ will be also used whenever it creates no confusion.

3.1. Local topology of toric spaces. For $(\Delta, A)$ and a subset $S = \{i_1, \ldots, i_j\} \subset [m] = F_0(\Delta)$, let $A_S = (\lambda_{i_1}, \ldots, \lambda_{i_j})$ be the restriction $d \times j$ matrix, and let $\Theta_S = r_S(\Theta)$ be the image of $\Theta$ under the restriction map $r_S : \mathbb{Q}[\Delta] \to \mathbb{Q}[\Delta_S]$. For a $(k - 1)$-face $\sigma = \{i_1, \ldots, i_k\} \in \Delta$, setting $S_\sigma = F_0(st_\sigma \Delta) = \{i_1, \ldots, i_j\}$, then we get a $T^d$-space

$$M_\sigma = M(st_\sigma \Delta, A_{S_\sigma}) = \mathcal{Z}_{st_\sigma \Delta}/K_{A_{S_\sigma}}, \quad K_{A_{S_\sigma}} := \text{Ker} \exp A_{S_\sigma} : T^j \to T^d.$$ 

Since $st_\sigma \Delta$ is clearly Cohen-Macaulay, $H^*(M_\sigma; \mathbb{Q}) = \mathbb{Q}[st_\sigma \Delta]/\Theta_{S_\sigma}$.

On the other hand, let $T_{[m]\setminus S_\sigma}$ be the coordinate subtorus corresponding to the subset $[m] \setminus S_\sigma$. We get another $T^d$-space

$$\tilde{M}_\sigma = (\mathcal{Z}_{st_\sigma \Delta} \times T_{[m]\setminus S_\sigma})/K_A, \quad K_A := \text{Ker} \exp A : T^m \to T^d.$$ 

It is easy to see that $\tilde{M}_\sigma$ is the quotient space of $M_\sigma$ under a finite group $G \subset T^d$ action: $G \times M_\sigma \to M_\sigma$, so their rational cohomology rings are isomorphic (cf. Appendix A.1). Thus, the restriction map $r_S : \mathbb{Q}[\Delta]/\Theta \to \mathbb{Q}[\Delta_S]/\Theta_{S_\sigma}$ is induced by an inclusion $\psi_\sigma : \tilde{M}_\sigma \hookrightarrow M_\Delta$.

Now consider the subcomplex $lk_\sigma \Delta$. Reordering the vertices if necessary, there exists a matrix $A \in GL(d, \mathbb{Z})$ such that

$$A \cdot A_\sigma = \begin{pmatrix} U_\sigma \\ 0 \end{pmatrix} \quad \text{and} \quad A \cdot A_{S_\sigma} = \begin{pmatrix} U_\sigma \\ 0 \\ 0 \\ B \\ \Gamma \end{pmatrix},$$

where $U_\sigma$ is a full rank $k \times k$ upper triangle matrix. It is easily verified that the $(d - k) \times (j - k)$ matrix $\Gamma$ is a characteristic matrix for $lk_\sigma \Delta$. Thus, we can define a $(2d - 2k)$-dimensional toric space $N_\sigma$ associated to $lk_\sigma \Delta$ to be

$$N_\sigma = M(lk_\sigma \Delta, \Gamma) = \mathcal{Z}_{lk_\sigma \Delta}/K_\Gamma, \quad K_\Gamma := \text{Ker} \exp \Gamma : T^{j-k} \to T^{d-k}.$$ 

Viewing $\mathcal{Z}_{lk_\sigma \Delta}$ as a subspace of $\mathcal{Z}_{st_\sigma \Delta}$:

$$\mathcal{Z}_{lk_\sigma \Delta} = \{(x_1, \ldots, x_j) \in (D^2)^j : x_i = 0 \text{ for } i \in \sigma\} \subset \mathcal{Z}_{st_\sigma \Delta}.$$ 

Then $N_\sigma$ is a deformation retract of $M_\sigma$ induced by the deformation retraction from $\mathcal{Z}_{st_\sigma \Delta} = (D^2)^k \times \mathcal{Z}_{lk_\sigma \Delta}$ onto $\mathcal{Z}_{lk_\sigma \Delta}$, and $\pi_\sigma : M_\sigma \to N_\sigma$ is (rationally) an orientable $D^{2k}$-bundle.

In particular, if $\Delta$ is a rational homology $(d - 1)$-sphere (resp. rational homology $(d - 1)$-ball), $lk_\sigma \Delta$ is a rational homology $(d - k - 1)$-sphere (resp. rational homology $(d - k - 1)$-ball).
that $C$ is a rational toric $(2d-2k)$-manifold (resp. rational toric $(2d-2k)$-manifold with or without boundary). Let us look at an example.

**Example 3.1.** Let $\Delta$ be the boundary of a square with $\{1,3\}$ and $\{2,4\}$ as missing faces. So $\mathcal{Z}_\Delta = S^3 \times S^3$ (see Example 2.8). Define $\Lambda$ to be

$$\lambda_1 = (1,0)^T, \lambda_2 = (0,1)^T, \lambda_3 = (-1, -1)^T, \lambda_4 = (0, -1)^T.$$ 

Then $M_\Delta$ is the connected sum $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ (see [28]), where $\overline{\mathbb{C}P^2}$ is the projective space with the reversed orientation. The kernel subtorus $K_\Lambda = T^2$ corresponds to the sublattice

$$\mathbb{Z} \cdot (1, 1, 1, 0) \oplus \mathbb{Z} \cdot (0, 1, 0, 1).$$

It is not hard to verify that $N_i = S^2$ for all $1 \leq i \leq 4$; $M_i = \check{M}_i = D^2 \times S^2$ for $i = 1, 3$; and for $i = 2, 4$, $M_i = \check{M}_i$ is the total space of a $D^2$-bundle over $S^2$ such that the boundary of $M_i$ is the Hopf bundle:

$$S^1 \rightarrow S^3 \rightarrow S^2.$$

In the previous notations, we have a composition map

$$\rho_{\sigma} : N_{\sigma} \xrightarrow{\phi_{\sigma}} M_{\sigma} \xrightarrow{q_{\sigma}} \check{M}_{\sigma} \xrightarrow{\psi_{\sigma}} M_\Delta,$$

where $q_{\sigma}$ is the quotient map; $\phi_{\sigma}$ and $\psi_{\sigma}$ are inclusions, and there are induced ring isomorphisms

$$H^*(\check{M}_{\sigma}) \xrightarrow{\phi^*_{\sigma}} H^*(M_{\sigma}) \xrightarrow{\psi^*_{\sigma}} H^*(N_{\sigma}).$$

**3.2. Excision for rational toric manifolds with boundary.** In this subsection, we assume $\Delta$ is a rational homology ball with characteristic matrix $\Lambda$, so that $M_\Delta$ is a rational toric manifolds with boundary. The following lemma can be used to calculate the relative cohomology of the pair $(M_\Delta, \partial M_\Delta)$.

**Lemma 3.2** (Excision). Suppose a characteristic pair $(\Delta', \Lambda')$ satisfies that $\Delta'$ is a rational homology sphere of the same dimension as $\Delta$, $\Delta \subset \Delta'$ and $\Lambda$ is the restriction of $\Lambda'$ to $\Delta$. Let $D$ be the closure of $\Delta' - \Delta$, $\mathcal{D} = \mathcal{F}_0(D)$, and $r_D : \mathbb{Q}[\Delta']/\Theta' \rightarrow \mathbb{Q}[D]/\Theta'_D$ the restriction map. Then we have an isomorphism

$$H^*(M_\Delta, \partial M_\Delta; \mathbb{Q}) \cong \text{Ker } r_D.$$

**Proof.** Note that $D$ is a rational homology ball. Let $S = \mathcal{F}_0(\Delta') - \mathcal{F}_0(\Delta)$, $i = |S|$ and $\mathcal{U} = \mathcal{F}_0(\Delta') - \mathcal{F}_0(D)$, $j = |\mathcal{U}|$. We can define spaces

$$\check{M}_D = (\mathcal{Z}_D \times T^i_S)/K_{\Lambda'} \quad \text{and} \quad \check{M}_D = (\mathcal{Z}_D \times T^j_U)/K_{\Lambda'},$$

where $K_{\Lambda'} := \text{Ker exp } \Lambda'$.

As we showed in §3.1 that $M_\sigma$ and $\check{M}_\sigma$ are rational cohomology equivalent, in the same way, we have $H^*(M_\Delta, \partial M_\Delta; \mathbb{Q}) \cong H^*(\check{M}_\Delta, \partial \check{M}_\Delta; \mathbb{Q})$ and $H^*(M_D; \mathbb{Q}) \cong$
$H^*(\hat{M}_D; \Bbb{Q})$. Using Corollary 2.24 and five-lemma, we can readily deduce that $H^*(M_{\Delta}, \hat{M}_D; \Bbb{Q}) \cong \text{Ker} r_D$. Since $M_\Delta = \hat{M}_\Delta \cup \hat{M}_D$ and $\hat{M}_\Delta \cap \hat{M}_D = \partial \hat{M}_\Delta$, $H^*(M_{\Delta}, \hat{M}_D; \Bbb{Q}) \cong H^*(\hat{M}_\Delta, \partial \hat{M}_\Delta; \Bbb{Q})$ by excision. So the lemma is proved. \hfill \Box

**Corollary 3.3.** Let $I$ be the ideal of $\Bbb{Q}[\Delta]$ generated by $\{x_\sigma : \sigma \in \Delta - \partial \Delta\}$, $\Theta$ an l.s.o.p. for $\Bbb{Q}[\Delta]$ and $M_\Delta$ the corresponding rational toric manifold with boundary. Then we have an isomorphism

$$H^*(M_\Delta, \partial M_\Delta; \Bbb{Q}) \cong I/I\Theta.$$

**Proof.** Let $\Delta' = \Delta$, and $L = \Delta \cup_{\partial \Delta} \Delta'$ be the rational homology sphere obtained by gluing these two balls together, $I' = (x_\sigma : \sigma \in L - \Delta) \in \Bbb{Q}[L]$. Define an l.s.o.p. $\hat{\Theta}$ for $\Bbb{Q}[L]$ in a symmetrical way. Then we have a short exact sequence

$$0 \to I/(I \cap \hat{\Theta}) \to \Bbb{Q}[L]/\hat{\Theta} \to \Bbb{Q}[\Delta']/\Theta \to 0.$$

According to Lemma 3.2, $H^*(M_\Delta, \partial M_\Delta; \Bbb{Q}) \cong I/(I \cap \hat{\Theta})$. So it remains to prove that $I \cap \hat{\Theta} = I\Theta$.

First note that $I\Theta = I\hat{\Theta} \subset I \cap \hat{\Theta}$. For the inverse direction, suppose $f \in I \cap \hat{\Theta}$, then it can be written as $f = \sum p_i \hat{\theta}_i + \sum q_i \hat{\theta}_i + \sum r_i \hat{\theta}_i$, where $p_i \in I$, $q_i \in I'$ and $r_i \in \Bbb{Q}[\partial \Delta]$. Thus $\sum q_i \hat{\theta}_i + \sum r_i \hat{\theta}_i = f$, $p_i \hat{\theta}_i \in I \cap \hat{\Theta}$ since $f$, $p_i \hat{\theta}_i \in I \cap \hat{\Theta}$. It is easy to see that $I \cap I' = 0$. Hence if we write $\sum r_i \hat{\theta}_i = r + r' + r''$ with $r \in I$, $r' \in I'$ and $r'' \in \Bbb{Q}[\partial \Delta]$, then we have $r'' = 0$ and $r' = -\sum q_i \hat{\theta}_i$. By symmetry of both $L$ and $\hat{\Theta}$, we can deduce that $r = -\sum q_i \hat{\theta}_i$ with $q_i \in I$ to be the polynomial symmetric to $q_i$. It follows that $\sum q_i \hat{\theta}_i + \sum r_i \hat{\theta}_i = -\sum q_i \hat{\theta}_i \in I\Theta$, and so $f \in I\hat{\Theta}$ too. \hfill \Box

### 3.3. Poincaré duality of rational toric manifolds

In this subsection, we assume that $\Delta$ is a rational homology sphere (or ball) of dimension $d - 1$. So $M_\Delta$ is a rational toric 2d-manifold (resp. rational toric 2d-manifold with boundary), and therefore it should have (rational) Poincaré duality (resp. Lefschetz duality) property. After choosing an orientation of $M_\Delta$, denote by $[M_\Delta] \in H_{2d}(M_\Delta; \Bbb{Q})$ (resp. $H_{2d}(M_\Delta, \partial M_\Delta; \Bbb{Q})$) the fundamental class of $M_\Delta$ (resp. $(M_\Delta, \partial M_\Delta)$). The following lemma plays an important role in this paper.

**Lemma 3.4** (Poincaré duality). If $\Delta$ is a rational homology $(d - 1)$-sphere, then the map defined by

$$H^{2j}(M_\Delta; \Bbb{Q}) \xrightarrow{[M_\Delta]} H_{2d-2j}(M_\Delta; \Bbb{Q})$$

is an isomorphism for all $j$. Namely, the rational algebra $\Bbb{Q}[\Delta]/\Theta$ is a Poincaré duality algebra. Moreover, for any $(k - 1)$-face $\sigma = \{i_1, \ldots, i_k\} \in \Delta$, we have

$$\Bbb{Q} \cdot [M_\Delta] \cap x_\sigma = \Bbb{Q} \cdot (\rho_\sigma)_*([N_\sigma]),$$

where $[N_\sigma] \in H_{2d-2k}(N_\sigma; \Bbb{Q})$ is a rational fundamental class of $N_\sigma$, and $\rho_\sigma$ is defined by (3.2).
Proof. Since $M_\Delta$ is a rational homology manifold when $\Delta$ is a rational homology sphere, the first statement is obvious. For the second statement, let $e_\sigma = e_{i_1}^2 \times \cdots \times e_{i_k}^2$ be the 2k-cell of $Z_\Delta$ defined in Remark 2.13. Define the ‘orbit cell’ $\tilde{e}_\sigma$ to be the image of $e_\sigma$ in the orbit space $M_\Delta$ under the quotient map $Z_\Delta \to M_\Delta$.

(Remark: Actually, $\tilde{e}_\sigma$ may not be a cell in general, but a rational homology ball, which is homeomorphic to the quotient of $D^{2k}$ under a finite group $G \subset T^k$ action. However, a rational homology ball plays the same role as a cell in rational homology calculations. This is what we need.)

We claim that $x_\sigma \in \mathbb{Q}[\Delta]/\Theta$ is represented by the cocycle $\tilde{e}_\sigma^* \in C^{2k}(M_\Delta; \mathbb{Q})$ up to multiplication by an integer (see Appendix A.2 for a proof). Similarly, the fundamental class $[M_\Delta]$ is represented by the cycle

$$\pm \sum_{\tau \in F_{d-1}(\Delta)} \tilde{e}_\tau \in C_{2d}(M_\Delta; \mathbb{Q})$$

(up to multiplication by an integer for each term). Hence, the cap product $[M_\Delta] \smile x_\sigma$ is represented by

$$\pm \sum_{\tau \in F_{d-k-1}(\text{lk}_\sigma \Delta)} \tilde{e}_\tau, \text{ summing over all facets of } \text{lk}_\sigma \Delta.$$

(Compare with the relation between the cap product for $Z_\Delta$ and the excision product for full subcomplexes of $\Delta$.) But this is just a representative of a fundamental class of $N_\sigma$. □

Remark 3.5. The Poincaré duality of $\mathbb{Q}[\Delta]/\Theta$ can also be obtained in a purely algebraic way [32, I.12]. Lemma 3.4 provides a topological explanation of this algebraic phenomenon.

Similar to the Poincaré duality of rational toric manifolds, for rational toric manifolds with boundary we have

Lemma 3.6 (Lefschetz duality). If $\Delta$ is a rational homology $(d-1)$-ball, then the maps defined by

$$H^{2j}(M_\Delta, \partial M_\Delta; \mathbb{Q}) \xrightarrow{[M_\Delta]^\smile} H_{2d-2j}(M_\Delta; \mathbb{Q}), \text{ and}$$

$$H^{2j}(M_\Delta; \mathbb{Q}) \xrightarrow{[M_\Delta]^\smile} H_{2d-2j}(M_\Delta, \partial M_\Delta; \mathbb{Q})$$

are isomorphisms for all $j$. Moreover, for $x_\sigma \in I/I\Theta$ with $I = (x_\sigma : \sigma \in \Delta - \partial \Delta)$ (resp. $x_\sigma \in \mathbb{Q}[\Delta]$ ), $\mathbb{Q} : [M_\Delta] \smile x_\sigma = \mathbb{Q} : (\rho_\sigma)_*(\lbrack N_\sigma \rbrack)$, where $\lbrack N_\sigma \rbrack \in H_{2d-2k}(N_\sigma; \mathbb{Q})$ is a rational fundamental class of $N_\sigma$ (resp. $\lbrack N_\sigma, \partial N_\sigma \rbrack$).

3.4. Applications of the Poincaré duality lemma. Restricting attention to closed rational toric manifolds for simplicity, we assume $\Delta$ is a rational homology sphere throughout this subsection. As an application of Lemma 3.4, we have the following result which is an essential ingredient of this paper.
Proposition 3.7. For a face \( \sigma \in \Delta \), let \( \rho_{\sigma} : N_{\sigma} \to M_\Delta \) be the map defined in (3.2). Then for every \( 1 \leq i \leq d \), the map

\[
H^{2k}(M_\Delta; \mathbb{Q}) \oplus_{\sigma \in F_{i-1}(\Delta)} H^{2k}(N_{\sigma}; \mathbb{Q})
\]

is an injection for all \( k \leq d - i \).

Proof. It is equivalent to prove that

\[
\bigoplus_{\sigma \in F_{i-1}(\Delta)} H_{2k}(N_{\sigma}; \mathbb{Q}) \xrightarrow{\oplus (\rho_{\sigma})_*} H_{2k}(M_\Delta; \mathbb{Q})
\]

is a surjection for \( k \leq d - i \).

First we will show that for any \( 1 \leq i \leq d \), (3.4) holds for \( k = d - i \). As a consequence of Lemma 3.4, we have the following commutative diagram:

\[
\begin{array}{c}
\bigoplus_{\sigma \in F_{i-1}(\Delta)} \mathbb{Q} \cdot x_{\sigma} \xrightarrow{[M_\Delta]_\sim} \bigoplus_{\sigma \in F_{i-1}(\Delta)} H_{2d-2i}(N_{\sigma}; \mathbb{Q}) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
H^{2i}(M_\Delta; \mathbb{Q}) \xrightarrow{[M_\Delta]_\sim} H_{2d-2i}(M_\Delta; \mathbb{Q})
\end{array}
\]

(3.5)

Since the left vertical map is surjective (see [32, Lemma III.2.4]), so is the right vertical map.

Next for \( k < d - i \), notice that for each face pair \( \tau \supset \sigma \) with \( \dim \sigma = i - 1 \), \( \dim \tau = d - 1 - k \), there is a map \( \rho_{\tau|\sigma} : N_{\tau} \to N_{\sigma} \), and \( \rho_{\tau} \) factors through \( N_{\sigma} \) by this map. Hence, we have the following commutative diagram:

\[
\begin{array}{c}
\bigoplus_{\sigma \in F_{i-1}(\Delta)} \left( \bigoplus_{\tau \subseteq \sigma, \tau \in F_{d-1-k}(\Delta)} H_{2k}(N_{\tau}; \mathbb{Q}) \right) \xrightarrow{\bigoplus (\rho_{\tau|\sigma})_*} \bigoplus_{\sigma \in F_{i-1}(\Delta)} H_{2k}(N_{\sigma}; \mathbb{Q}) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\bigoplus (\rho_{\tau})_* \quad \quad \quad \bigoplus (\rho_{\sigma})_*
\end{array}
\]

(3.6)

We have already seen that the left vertical map is surjective, so the right vertical map is surjective too. Thus, (3.4) holds for all \( k \leq d - i \). \( \square \)
Remark 3.8. Proposition can be translated into a purely algebraic description, that is, for every $1 \leq i \leq d - 1$ and $k \leq d - i$, we have an injection

$$(\mathbb{Q}[\Delta]/\Theta)_{2k} \xrightarrow{\Theta r_{S_{\sigma}}} \bigoplus_{\sigma \in F_{i-1}(\Delta)} (\mathbb{Q}[\text{st}_{\sigma} \Delta]/\Theta_{S_{\sigma}})_{2k},$$

where $S_{\sigma} = F_0(\text{st}_{\sigma} \Delta)$ and $r_{S_{\sigma}}$ is the restriction map. For the special case that $i = 1$, this algebraic result is also obtained by Adiprasito [1, Lemma 3.4].

From the proof of Proposition 3.7, we can readily generalize it to

**Theorem 3.9.** If $\{x_{\sigma_1}, \ldots, x_{\sigma_{h_k}}\}$ is a basis for $(\mathbb{Q}[\Delta]/\Theta)_{2k}$, then the map

$$(\mathbb{Q}[\Delta]/\Theta)_{2j} \xrightarrow{\Theta r_{S_{\sigma_i}}} \bigoplus_{1 \leq i \leq h_k} (\mathbb{Q}[\text{st}_{\sigma_i} \Delta]/\Theta_{S_{\sigma_i}})_{2j},$$

is an injection for $j < d - k$ and an isomorphism for $j = d - k$.

**Proof.** The isomorphism comes from diagram (3.5), and the injection comes from diagram (3.6) and the following lemma.

**Lemma 3.10.** If $\{x_{\sigma_1}, \ldots, x_{\sigma_{h_k}}\}$ is a basis for $(\mathbb{Q}[\Delta]/\Theta)_{2k}$, then for each $n > k$, $(\mathbb{Q}[\Delta]/\Theta)_{2n}$ is spanned by the face monomials $\{x_{\tau} : \tau \in \bigcup_{i=1}^{h_k} \text{st}_{\sigma_i} \Delta\}$.

**Proof.** Since $(\mathbb{Q}[\Delta]/\Theta)_{2k+2} = (\mathbb{Q}[\Delta]/\Theta)_{2} \cdot (\mathbb{Q}[\Delta]/\Theta)_{2k}$, it is spanned by the monomials of the form $x_{\sigma_j}$. If $i \notin \sigma_j$ and $x_i x_{\sigma_j} \neq 0$, then $\tau = \{i\} \cup \sigma_j \in \text{st}_{\sigma_i} \Delta$, so assume $i \in \sigma_j$. Since $\theta_1, \ldots, \theta_d$ is an l.s.o.p for $\mathbb{Q}[\Delta]$, some linear combination of them has the form

$$\gamma = x_i + \sum_{x_l \notin \sigma_j} a_l x_l, \ a_l \in \mathbb{Q}.$$

Then in $\mathbb{Q}[\Delta]/\Theta$ we have $x_i x_{\sigma_j} = (x_i - \gamma)x_{\sigma_j}$, and so reduce to the case $x_i \notin \sigma_j$. This prove the case $n = k + 1$. Doing this inductively for $n = k + 2, \ldots, d$, we get the conclusion of the lemma.

Before proceeding further, let us define a combinatorial construction.

**Definition 3.11.** Let $\Delta$ be a pure simplicial complex of dimension $d - 1$. The first partially derived subdivision $D_1(\Delta)$ of $\Delta$ is defined to be the simplicial complex obtained from $\Delta$ by applying stellar subdivision operations at all facets of $\Delta$. (Using ‘partially’ because the term ‘derived subdivision’ usually means the barycentric subdivision.) Note that $D_1(\Delta)$ is well defined, that is it does not depend on the order of the stellar subdivision operations we perform. For convenience we define $D_0(\Delta) = \Delta$.

Recursively, define the $i$th partially derived subdivision $D_i(\Delta)$ of $\Delta$ to be the simplicial complex obtained from $D_{i-1}(\Delta)$ by applying stellar subdivisions on all
faces $\sigma \in F_{d-i}(\Delta)$. (It is easily verified that $F_{d-i}(\Delta) \subset F_{d-i}(D_{i-1}(\Delta))$ and $D_{i}(\Delta)$ is well defined.)

Recall that $v_{\sigma}$ denote the adding vertex in the stellar subdivision at the face $\sigma \in \Delta$. We have

$$F_{0}(D_{i}(\Delta)) = F_{0}(\Delta) \cup \{v_{\sigma} : \dim \sigma \geq d - i\}.$$ 

**Proposition 3.12.** Suppose $M_{D_{i}(\Delta)}$ is a rational toric manifold associated to $D_{i}(\Delta)$. Let $V_{i} = F_{0}(D_{i}(\Delta)) \setminus F_{0}(\Delta)$. Then for each $k < i$, we have an injection

$$H^{2k}(M_{D_{i}(\Delta)}; \mathbb{Q}) \oplus \rho_{v_{\sigma}}^{*} \rightarrow \bigoplus_{v_{\sigma} \in V_{i}} H^{2k}(N_{v_{\sigma}}; \mathbb{Q}).$$

Here $N_{v_{\sigma}}$ is the rational toric manifold associated to $\text{lk}_{v_{\sigma}}D_{i}(\Delta)$ defined in subsection 3.1.

**Proof.** As before, it is equivalent to show that

$$\bigoplus_{v_{\sigma} \in V_{i}} H^{2k}(N_{v_{\sigma}}; \mathbb{Q}) \oplus \rho_{v_{\sigma}}^{*} \rightarrow \bigoplus_{v_{\sigma} \in V_{i}} H^{2k}(M_{D_{i}(\Delta)}; \mathbb{Q})$$

is surjective for $k < i$.

As in the proof of Proposition 3.7, we have a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{v_{\sigma} \in V_{i}} (\bigoplus_{\tau \in F_{d-1-k}(D_{i}(\Delta))} H^{2k}(N_{\tau}; \mathbb{Q})) & \oplus \rho_{v_{\sigma}}^{*} & \bigoplus_{v_{\sigma} \in V_{i}} H^{2k}(N_{v_{\sigma}}; \mathbb{Q}) \\
\bigoplus \rho_{v_{\sigma}}^{*} & \downarrow & \bigoplus \rho_{v_{\sigma}}^{*} \\
H^{2k}(M_{D_{i}(\Delta)}; \mathbb{Q}) & \rightarrow & H^{2k}(M_{D_{i}(\Delta)}; \mathbb{Q})
\end{array}$$

By Proposition 3.7,

$$\bigoplus_{\tau \in F_{d-1-k}(D_{i}(\Delta))} H^{2k}(N_{\tau}; \mathbb{Q}) \oplus \rho_{v_{\sigma}}^{*} \rightarrow H^{2k}(M_{D_{i}(\Delta)}; \mathbb{Q})$$

is a surjection. Note that if $k < i$, then any face $\tau \in F_{d-1-k}(D_{i}(\Delta))$ must contain at least one vertex $v_{\sigma} \in V_{i}$. It follows that the left vertical map is surjective in the above diagram, then so is the right vertical map. \qed

For a subset $\mathcal{A} \subset F_{k}(\Delta)$, define $S(\Delta, \mathcal{A})$ to be the set of simplicial complexes obtained from $\Delta$ by a sequence of stellar subdivision operations at each face of $\mathcal{A}$. (In general, changing the order of stellar subdivision operations produces a different simplicial complex.)
Let $\Delta$ be a rational homology $(d - 1)$-sphere, $\Theta$ a generic l.s.o.p. for $\mathbb{Q}[\Delta]$ and suppose $\{x_{\sigma_1}, \ldots, x_{\sigma_h}\}$ is a basis of $((\mathbb{Q}[\Delta])/\Theta)_{2k}$. Set $\mathcal{A}_{k-1} = \{\sigma_1, \ldots, \sigma_h\} \subset F_{k-1}(\Delta)$. Then for any $\Delta' \in S(\Delta, \mathcal{A}_{k-1})$, we can give an l.s.o.p. $\Theta'$ for $\mathbb{Q}[\Delta']$ such that it is $\Theta$ when restricted to the vertices of $\Delta$ (cf. Remark 2.20). Let $V = F_0(\Delta') - F_0(\Delta) = \{v_{\sigma_1}, \ldots, v_{\sigma_h}\}$. Then $((\mathbb{Q}[\Delta'])/\Theta')_{2k}$ has a face monomial basis as follows.

Lemma 3.13. $((\mathbb{Q}[\Delta'])/\Theta')_{2k}$ has a basis of the form $\{x_{r_1}, \ldots, x_{r_s}\}$ such that $\tau_j \cap V \neq \emptyset$ for each $1 \leq j \leq s$.

Proof. Without loss of generality, we may assume $\Delta = \Delta_0$, $\Delta' = \Delta_h$, and $\Delta_i = ss_{\sigma_i} \Delta_{i-1}$ for $1 \leq i \leq h$. For notational simplicity, we use $\Theta$ to denote the l.s.o.p. for all $\mathbb{Q}[\Delta_i]$. Let $D_i (i \geq 1)$ be the closure of $\Delta_i - st_{v_{\sigma_i}} \Delta_i$. Then for each $1 \leq i \leq h$, we have a short exact sequence

$$0 \to J_i/(J_i \cap \Theta) \to \mathbb{Q}[\Delta_i]/\Theta \to \mathbb{Q}[D_i]/\Theta \to 0, \quad (3.7)$$

where $J_i$ is the ideal generated by the vertex $v_{\sigma_i}$.

From the short exact sequence

$$0 \to I/(I \cap \Theta) \to \mathbb{Q}[\Delta]/\Theta \to \mathbb{Q}[D_1]/\Theta \to 0, \quad I = (x_{\sigma_1}),$$

it is easy to see that $\{x_{\sigma_2}, \ldots, x_{\sigma_h}\}$ is a basis of $((\mathbb{Q}[D_1]/\Theta)_{2k}$. Thus the short exact sequence (3.7) implies that $((\mathbb{Q}[\Delta]/\Theta)_{2k}$ has a basis of the form $\{x_{r_1}, \ldots, x_{r_s}\} \cup \{x_{\sigma_2}, \ldots, x_{\sigma_h}\}$, where $v_{\sigma_i} \in \tau_j$ for all $1 \leq j \leq r$. Doing this inductively for $i = 2, 3, \ldots$, we see that $((\mathbb{Q}[\Delta']/\Theta')_{2k}$ has the desired basis. \hfill $\Box$

By using Lemma 3.13 and the same argument as in the proof of Proposition 3.7, we can get that:

Proposition 3.14. In the notation above, for any $\Delta' \in S(\Delta, \mathcal{A}_{k-1})$ and $i \leq d - k$, we have an injection

$$((\mathbb{Q}[\Delta'])/\Theta')_{2i} \to \bigoplus_{1 \leq j \leq h} ((\mathbb{Q}[st_{v_{\sigma_j}} \Delta']/\Theta')_{2i}.$$

4. WLP and subdivisions of rational homology spheres

Stellar subdivisions play an important role in piecewise-linear geometry. In this section, we investigate the problem that which stellar subdivisions of a rational homology sphere has WLP. (Babson-Nevo’s paper [2] is a good reference for the strong-Lefschetz property about this question.) We begin with a lemma which is needed later on.

Lemma 4.1. If a rational homology $(d - 1)$-sphere $\Delta$ has the form $\Delta = \partial \Delta^n * \Delta'$ with $n \geq \lfloor d/2 \rfloor$, then $\Delta$ has the WLP in the sense that for any l.s.o.p $\Theta$, there exists a WLE for $\mathbb{Q}[\Delta]/\Theta$. 
Proof. By Proposition 2.6, we only need to show that there exists a linear form \( \omega \in \mathbb{Q}[\Delta] \) such that the map

\[
\psi: (\mathbb{Q}[\Delta]/\Theta)_{2[d/2]} \to (\mathbb{Q}[\Delta]/\Theta)_{2[d/2]+2}
\]

is a surjection.

Choosing an arbitrary facet \( \sigma \in \partial \Delta \), we have \( \text{st}_\sigma \Delta = \sigma \ast \Delta' \). Consider the short exact sequence

\[
0 \to I \to \mathbb{Q}[\Delta] \to \mathbb{Q}[\text{st}_\sigma \Delta] \to 0.
\]  

(4.1)

Suppose \( F_0(\Delta) \setminus \sigma = \{i\} \), then it is easy to see that \( I = (x_i) \). If we quotient out by \( \Theta \) in (4.1), we obtain the short exact sequence

\[
0 \to I/(I \cap \Theta) \to \mathbb{Q}[\Delta]/\Theta \to \mathbb{Q}[\text{st}_\sigma \Delta]/\Theta_{S_\sigma} \to 0,
\]  

(4.2)

in which \( S_\sigma = F_0(\text{st}_\sigma \Delta) \).

Let \( k = \dim \Delta' + 1 \). Then \( k \leq [d/2] \), since \( n \geq [d/2] \) and \( n + k = d \) by assumption. As we have seen in subsection 3.1, \( \mathbb{Q}[\text{st}_\sigma \Delta]/\Theta_{S_\sigma} \) is isomorphic to the cohomology of the rational toric 2k-manifold \( N_\sigma \). Hence, we have \((\mathbb{Q}[\text{st}_\sigma \Delta]/\Theta_{S_\sigma})_{2l} = 0 \) for \( l \geq [d/2] + 1 \). It follows from (4.2) that

\[
(I/(I \cap \Theta))_{2l} = (\mathbb{Q}[\Delta]/\Theta)_{2l}, \quad \text{for } l \geq [d/2] + 1.
\]  

(4.3)

Let \( R \) be the image of the map \( \cdot x_i : \mathbb{Q}[\Delta]/\Theta \to \mathbb{Q}[\Delta]/\Theta \). Since \( I = (x_i) \), it follows that \( I/(I \cap \Theta) = R \). Combining this with (4.3) we get the surjection

\[
\psi: (\mathbb{Q}[\Delta]/\Theta)_{2[d/2]} \to (\mathbb{Q}[\Delta]/\Theta)_{2[d/2]+2}.
\]

The form \( \omega = x_i \) is what we need. \( \square \)

The WLP of a rational homology sphere and the one of its stellar subdivisions are related by the following algebraic result, which is proved initially by Böhm-Papadakis [5]. Here we give a simpler proof.

**Proposition 4.2.** Let \( \Delta \) be a rational homology \((d - 1)\)-sphere. For the stellar subdivision \( \Delta' = \text{ss}_\sigma \Delta \) at a face \( \sigma \in F_n(\Delta) \), we have:

(a) If \( \Delta \) has the WLP and \( n \geq d/2 \), then \( \Delta' \) has the WLP.

(b) If \( \Delta' \) has the WLP and \( n > d/2 \), then \( \Delta \) has the WLP.

Proof. (a) If \( \Delta \) has the WLP, then for a pair \((\omega_0, \Theta) \in \mathcal{W}(\Delta)\) we have a surjection

\[
\psi: (\mathbb{Q}[\Delta]/\Theta)_{2[d/2]} \to (\mathbb{Q}[\Delta]/\Theta)_{2[d/2]+2}.
\]

Since \( \dim \sigma \geq d/2 \), \( \dim \text{lk}_\sigma \Delta = d - \dim \sigma - 2 \leq [d/2] - 2 \). Therefore, as in the proof of Lemma 4.1, for the short exact sequence

\[
0 \to I/(I \cap \Theta) \to \mathbb{Q}[\Delta]/\Theta \to \mathbb{Q}[\text{st}_\sigma \Delta]/\Theta_{S_\sigma} \to 0,
\]
we have
\[(I/(I \cap \Theta))_{2l} = (Q[\Delta]/\Theta)_{2l}, \quad \text{for } l \geq [d/2].\]  
\tag{4.4}

So the map \((I/(I \cap \Theta))_{2[d/2]} \xrightarrow{\omega} (I/(I \cap \Theta))_{2[d/2]+2}\) is surjective too.

On the other hand, let \(\Theta'\) be a generic l.s.o.p. for \(Q[\Delta']\). Here we may assume \(\Theta\) is the restriction of \(\Theta'\) to \(x_1, \ldots, x_m\). Note that \(\text{lk}_{v_2} \Delta' = \partial \sigma \star \text{lk}_x \Delta\) is a \((d-2)\)-sphere, then by Lemma 4.1 and equation (3.3), for any \(i \in \sigma\), \(x_i\) is a WLE for \(Q[\text{st}_{v_2} \Delta']/\Theta'_{S_{v_2}}\), where \(S_{v_2} = F_0(\text{st}_{v_2} \Delta')\). Let \(\omega\) be a generic linear combination of \(\omega_0\) and \(x_i\). Now consider the following commutative diagram of exact sequences:
\[
\begin{array}{c}
0 \longrightarrow I'/(I' \cap \Theta') \longrightarrow Q[\Delta'/\Theta'] \longrightarrow Q[\text{st}_{v_2} \Delta']/\Theta'_{S_{v_2}} \longrightarrow 0 \\
\downarrow \omega \quad \quad \quad \quad \downarrow \omega \quad \quad \quad \quad \downarrow \omega \\
0 \longrightarrow I'/(I' \cap \Theta') \longrightarrow Q[\Delta'/\Theta'] \longrightarrow Q[\text{st}_{v_2} \Delta']/\Theta'_{S_{v_2}} \longrightarrow 0
\end{array}
\tag{4.5}
\]

(For notational convenience, we omit the degree subscripts in both rows, which are \(2[d/2]\) and \(2[d/2] + 2\) resp.) Thus, the right vertical map is a surjection. By Lemma 3.2, we have \(I'/(I' \cap \Theta') \cong I/(I \cap \Theta)\), so the left vertical map is a surjection too. The five-lemma then gives a surjection
\[
\omega : (Q[\Delta'/\Theta]_{2[d/2]} \rightarrow (Q[\Delta'/\Theta]_{2[d/2]+2}.
\]

Hence \(\Delta'\) has the WLP.

(b) We divide the proof into two cases, depending on the parity of \(d\). Consider first the case that \(d\) is odd. In this case, \(n > d/2\) is equivalent to \(n \geq [d/2] + 1\). Let the degree of the first and second rows in (4.5) are \(2[d/2]\) and \(2[d/2] + 2\) resp. Thus if \(\Delta'\) has WLP, then for a generic choice of \((\omega, \Theta) \in W(\Delta')\), the middle vertical map is an isomorphism by Poincaré duality, and so the left vertical map is an injection. Combining this with (4.4) and \(h_{[d/2]}(\Delta) = h_{[d/2]+1}(\Delta)\) we see that the map
\[
\omega : (Q[\Delta]/\Theta)_{2[d/2]} \rightarrow (Q[\Delta]/\Theta)_{2[d/2]+2}
\]
is an isomorphism. So \(\Delta\) has the WLP.

For the case that \(d\) is even, we have \(d/2 = [d/2]\). So \(n > d/2\) implies that \(\dim \text{lk}_x \Delta \leq [d/2] - 3\), and therefore by the same argument as before, we have
\[(I/(I \cap \Theta))_{2l} = (Q[\Delta]/\Theta)_{2l}, \quad \text{for } l \geq [d/2] - 1.\]  
\tag{4.6}

Note that when \(d\) is even, the WLP of \(\Delta\) is equivalent to the injectivity of
\[
\omega : (Q[\Delta]/\Theta)_{2[d/2]-2} \rightarrow (Q[\Delta]/\Theta)_{2[d/2]} \quad \text{for some } \omega.
\tag{4.7}
\]
This time let the degree of the first and second rows in (4.5) are \(2[d/2] - 2\) and \(2[d/2]\) resp. The WLP of \(\Delta'\) implies that for a generic choice of \((\omega, \Theta) \in W(\Delta')\), the middle vertical map is an injection. So the left vertical map is also an injection. It follows from Lemma 3.2 and (4.6) that the map in (4.7) is an injection too. So \(\Delta\) has the WLP.
Corollary 4.3. Let $\Delta$ be a rational homology $(d-1)$-sphere, $V_i = \mathcal{F}_0(\mathcal{D}_i(\Delta)) \setminus \mathcal{F}_0(\Delta)$. If $i \leq [d/2]$, then for every $v_\sigma \in V_i$, $\text{lk}_{v_\sigma} \mathcal{D}_i(\Delta)$ has the WLP.

Proof. It is easy to see that $\mathcal{F}_0(\Delta) = V_0 \subset V_1 \subset \cdots \subset V_i$. Let $k_\sigma = d - \dim \sigma$. Then $k_\sigma$ is the smallest number such that $v_\sigma \in V_{k_\sigma}$, and

$$\text{lk}_{v_\sigma} \mathcal{D}_{k_\sigma}(\Delta) = \partial \sigma \ast \text{lk}_{v_\sigma} \mathcal{D}_{k_\sigma-1}(\Delta)$$

is a rational homology $(d-2)$-sphere. So when $k_\sigma \leq i \leq [d/2]$, we have $\dim \sigma \geq [d/2] = [(d-1)/2]$, and then $\text{lk}_{v_\sigma} \mathcal{D}_{k_\sigma}(\Delta)$ has the WLP by Lemma 4.1.

When $i > k_\sigma$, $\text{lk}_{v_\sigma} \mathcal{D}_i(\Delta)$ is obtained from $\text{lk}_{v_\sigma} \mathcal{D}_{k_\sigma}(\Delta)$ by a sequence of stellar subdivisions at some faces of dimension greater than or equal to $d - i \geq [d/2] \geq (d-1)/2$. Hence we conclude our assertion with the help of Proposition 4.2 (a). \qed

Corollary 4.4. Let $\Delta$ be a rational homology $(d-1)$-sphere. Then for $i < d/2$, $\mathcal{D}_i(\Delta)$ has the WLP if and only if $\Delta$ has the WLP.

Proof. By definition, $\mathcal{D}_i(\Delta)$ is obtained from $\Delta$ by a sequence of stellar subdivisions at faces of dimension greater than or equal to $d - i > d/2$. Then the conclusion follows from Proposition 4.2. \qed

As an application of the previous results, we get the following theorem.

Theorem 4.5. For any rational homology $(d-1)$-sphere $\Delta$ and a generic l.s.o.p. $\Theta$ for $\mathbb{Q}[\mathcal{D}_k(\Delta)]$ with $k \leq [d/2]$, there exists a linear form $\omega \in \mathbb{Q}[\mathcal{D}_k(\Delta)]$ such that the map

$$\cdot : \mathbb{Q}[\mathcal{D}_k(\Delta)]/\Theta \xrightarrow{\Theta_{\mathcal{D}_k}} \bigoplus_{v_\sigma \in V_k} \mathbb{Q}[\text{st}_{v_\sigma} \mathcal{D}_k(\Delta)]/\Theta_{\mathcal{S}_{v_\sigma}},$$

where $V_k = \mathcal{F}_0(\mathcal{D}_k(\Delta)) \setminus \mathcal{F}_0(\Delta)$ and $\mathcal{S}_{v_\sigma} = \mathcal{F}_0(\text{st}_{v_\sigma} \mathcal{D}_k(\Delta))$. Since $k \leq [d/2]$, it follows by Corollary 4.3 and Lemma 2.7 that for a generic choice of linear form $\omega$, and for $i \leq [d/2]$, the right vertical map is injective in the following commutative diagram:

$$\begin{array}{ccc}
(\mathbb{Q}[\mathcal{D}_k(\Delta)]/\Theta)_{2i-2} & \xrightarrow{\Theta_{\mathcal{D}_k}} & \bigoplus_{v_\sigma \in V_k} (\mathbb{Q}[\text{st}_{v_\sigma} \mathcal{D}_k(\Delta)]/\Theta_{\mathcal{S}_{v_\sigma}})_{2i-2} \\
\downarrow \omega & & \downarrow \omega \\
(\mathbb{Q}[\mathcal{D}_k(\Delta)]/\Theta)_{2i} & \xrightarrow{\Theta_{\mathcal{D}_k}} & \bigoplus_{v_\sigma \in V_k} (\mathbb{Q}[\text{st}_{v_\sigma} \mathcal{D}_k(\Delta)]/\Theta_{\mathcal{S}_{v_\sigma}})_{2i}
\end{array}$$
Proposition 3.12 implies that the upper horizontal map is an injection for \( i \leq k \), so the left vertical map is also an injection for \( i \leq \min\{k, \lfloor d/2 \rfloor\} \). When \( d \) is even and \( k = d/2 \), this is an equivalent condition for \( \omega \) to be a WLE. \( \square \)

**Remark 4.6.** For an odd-dimensional rational homology \((d-1)\)-sphere \( \Delta \), Corollary 4.4 says that if \( D_{d/2-1}(\Delta) \) has the WLP then so does \( \Delta \). However, it seems that we can only get the WLP of \( D_{d/2}(\Delta) \) from Theorem 4.5.

Compare the last statement of Theorem 4.5 with the result in [23], which says that the barycentric subdivision of an odd-dimensional Cohen-Macaulay complex has the WLP.

**Corollary 4.7.** Let \( \Delta \) be a rational homology \((d-1)\)-sphere. Then the g-conjecture holds for \( D_{\lfloor d/2 \rfloor}(\Delta) \) and \( D_{\lceil d/2 \rceil}(\Delta) \).

**Theorem 4.8.** In the notation of Proposition 3.14, if \( k \geq \lceil \frac{d-1}{2} \rceil \), then for any \( \Delta' \in S(\Delta, \mathcal{A}_k) \), there exists a linear form \( \omega \in \mathbb{Q}[\Delta'] \) such that the map
\[
\cdot \omega : (\mathbb{Q}[\Delta']/\Theta')_{2i-2} \to (\mathbb{Q}[\Delta']/\Theta')_{2i}
\]
is an injection for \( i \leq \min\{d - k, \lfloor d/2 \rfloor\} \). Especially, if \( d \) is even, and \( k = d/2 \), then there is a WLE for \( \mathbb{Q}[\Delta']/\Theta' \).

**Proof.** By Proposition 3.14, we have an injection
\[
(\mathbb{Q}[\Delta']/\Theta')_{2i} \to \bigoplus_{1 \leq j \leq h_k} (\mathbb{Q}[\sigma_{v_j} \Delta']/\Theta')_{2i} \quad \text{for} \quad i \leq d - k - 1.
\]
Since \( \text{lk}_{v_{\sigma_j}} \Delta' \) is a \((d-2)\)-sphere, and has the form \( \partial \sigma_j * L \), it follows that if \( \dim \sigma_j = k \geq \lceil \frac{d-1}{2} \rceil \), there is a WLE for \( \mathbb{Q}[\sigma_{v_{\sigma_j}} \Delta']/\Theta' \) by Lemma 4.1. The same reasoning as in the proof of Theorem 4.5 gives the desired result. \( \square \)

**Corollary 4.9.** Let \( \Delta \) be a rational homology \((d-1)\)-sphere. Then the g-conjecture holds for any \( \Delta' \in S(\Delta, \mathcal{A}_{\lfloor d/2 \rfloor}) \) or \( S(\Delta, \mathcal{A}_{\lceil d/2 \rceil}) \).

5. Toric spaces associated to Buchsbaum complexes

In this section, we consider toric spaces associated to rational Buchsbaum complexes. This class of simplicial complexes plays a significant role in algebraic combinatorics and includes rational homology manifolds as a special subclass. We start with some general results about the cohomology properties of toric spaces by the D-J construction.
5.1. A decomposition of toric spaces. In this subsection, $\Delta$ is an arbitrary simplicial complex of dimension $d - 1$, and $M_\Delta = M(\Delta, A)$ is a toric space associated to $\Delta$. As we have seen in 2.7, $M_\Delta$ is the quotient space $C\Delta \times T^d / \sim$. So, $M_\Delta$ is the union of two spaces:

$$M_\Delta = (C\Delta \times T^d) \cup_{\Delta \times T^d} (I \times \Delta \times T^d / \sim), \quad (5.1)$$

where $I = [0, 1]$, with the relation $'\sim'$ defined on $\{0\} \times \Delta' \times T^d$ and the gluing identity map defined on $\{1\} \times \Delta \times T^d$. So we have a long exact sequence

$$\cdots \to H^*(M_\Delta, I \times \Delta \times T^d / \sim) \xrightarrow{j^*} H^*(M_\Delta) \xrightarrow{i^*} H^*(I \times \Delta \times T^d / \sim) \xrightarrow{\partial} H^{*+1}(M_\Delta, I \times \Delta \times T^d / \sim) \to \cdots \quad (5.2)$$

By excision and the fact that $(C\Delta \times T^d)/(\Delta \times T^d) = \Sigma\Delta \wedge T^d_+$, we have

$$H^*(M_\Delta, I \times \Delta \times T^d / \sim) \cong H^*(C\Delta \times T^d, \Delta \times T^d) \cong \tilde{H}^*(\Sigma\Delta) \otimes H^*(T^d).$$

Lemma 5.1. In the above notation, we have $\text{Im} j^* \cdot \tilde{H}^*(M_\Delta) = 0$

Proof. Consider another long exact sequence

$$\cdots \to H^*(M_\Delta, C\Delta \times T^d) \xrightarrow{f^*} H^*(M_\Delta) \xrightarrow{g^*} H^*(C\Delta \times T^d) \xrightarrow{\partial} H^{*+1}(M_\Delta, C\Delta \times T^d) \to \cdots$$

Since $T^d$ is contractible in $M_\Delta$, $H^k(M_\Delta) \xrightarrow{g^*} H^k(C\Delta \times T^d)$ is a zero map for $k > 0$. Exactness of the sequence then implies that $H^*(M_\Delta, C\Delta \times T^d) \xrightarrow{f^*} \tilde{H}^*(M_\Delta)$ is onto. On the other hand, from the definition of cup product, we can see that $j^*(\alpha) \sim f^*(\beta) = 0$ for any $\alpha \in H^*(M_\Delta, I \times \Delta \times T^d / \sim)$ and $\beta \in H^*(M_\Delta, C\Delta \times T^d)$. Thus, the lemma is proved. \qed

Lemma 5.2. For any nonempty face $\sigma \in \Delta$, the following composition is zero.

$$H^*(M_\Delta, I \times \Delta \times T^d / \sim) \xrightarrow{j^*} H^*(M_\Delta) \to H^*(M_\sigma),$$

where $M_\sigma \subset M_\Delta$ is the toric space defined in §3.1.

Proof. The composition in the lemma is the same as the composition

$$H^*(M_\Delta, I \times \Delta \times T^d / \sim) \to H^*(M_\sigma, I \times \text{st}_\sigma\Delta \times T^d / \sim) \to H^*(M_\sigma).$$

The middle relative cohomology is isomorphic to the reduced cohomology of

$$(C(\text{st}_\sigma\Delta) \times T^d)/(\text{st}_\sigma\Delta \times T^d) = \Sigma(\text{st}_\sigma\Delta) \wedge T^d_+ \simeq pt.$$ \(\text{The right homotopy equivalent follows from the fact that st}_\sigma\Delta \text{ is contractible. Hence this composition factors through a zero term, and so itself is zero.} \quad \square \)
5.2. Cohomology of toric spaces associated to Buchsbaum complexes. In this subsection, we will give topological proofs of several fundamental algebraic results about Buchsbaum complexes.

**Lemma 5.3.** Suppose $\Delta$ is a Buchsbaum complex. Let $\Delta \times T^d = \{0\} \times \Delta \times T^d$, and $\pi : \Delta \times T^d \to \Delta \times T^d / \sim$ the quotient map in the definition of $M_\Delta$. Then the following composition is zero for any $q > p \geq 0$.

$$H^{p+q}(\Delta \times T^d / \sim; \mathbb{Q}) \xrightarrow{\pi^*} H^{p+q}(\Delta \times T^d; \mathbb{Q}) \to H^p(\Delta; \mathbb{Q}) \otimes H^q(T^d; \mathbb{Q}).$$

Since the proof of Lemma 5.3 needs more complicated topological arguments, we put it in Appendix A.3.

**Corollary 5.4.** In the notations of the discussion preceding Lemma 5.1, if $\Delta$ is Buchsbaum, then the restriction of $j^*$ to the cohomology subgroup

$$\bigoplus_{q \geq p} \tilde{H}^p(\Sigma \Delta; \mathbb{Q}) \otimes H^q(T^d; \mathbb{Q}) \subset H^*(M_\Delta, I \times \Delta \times T^d / \sim; \mathbb{Q})$$

is an injection.

**Proof.** The statement of Lemma 5.3 is equivalent to saying that the cohomology subgroup $\bigoplus_{q \geq p} \tilde{H}^p(\Sigma \Delta; \mathbb{Q}) \otimes H^q(T^d; \mathbb{Q})$ is not in the image of the boundary homomorphisms $\partial$ in the sequence (5.2). So exactness of the sequence gives the result. $\square$

Recall the following fibration in the proof of Proposition 2.22

$$T^d \to E^{T^m} \times_{T^{m-d}} \mathcal{Z}_\Delta \to E^{T^m} \times T^m \mathcal{Z}_\Delta,$$

where $T^d$ is the torus associated to an l.s.o.p. $\Theta = \{\theta_1, \ldots, \theta_d\}$ for $\mathbb{Q}[\Delta]$. For the Serre spectral sequence of this fibration, the $E_2$-term, as we have seen in the proof of Proposition 2.22, is

$$E_2 = \mathbb{Q}[\Delta] \otimes \Lambda[v_1, \ldots, v_d], \quad d_2(v_i) = \theta_i.$$

For the $E_3$-term, we have the following result.

**Lemma 5.5.** If $\Delta$ is a Buchsbaum complex, then the $E_3$-term of the rational Serre spectral sequence of the above fibration is $E_3^{p,q} = 0$ if $p$ is odd, and

$$\dim E_3^{2p,q} = \begin{cases} \left(\begin{array}{c} d \\ p+q \end{array}\right) \tilde{\beta}_{p-1}(\Delta) & \text{for } q > 0, \\ h_p(\Delta) - \left(\begin{array}{c} d \\ p \end{array}\right) \sum_{i=1}^{p-1} (-1)^i \tilde{\beta}_{p-i-1}(\Delta) & \text{for } q = 0. \end{cases}$$

We will prove this lemma by using a double complex argument, whose calculation is due to Adiprasito [1, Lemma 3.4 and Proposition 3.9]. Note that the formula for $E_3^{2p,0}$ is just Schenzel’s formula (Theorem 2.4).
Proof of Lemma 5.5. Viewing \( \{ U_i := \text{st}_i \Delta \}_{i \in J(\Delta)} \) as an open cover of \( \Delta \), for a subset \( \sigma = \{ i_1, \ldots, i_k \} \subset [m] \), let \( U_\sigma \) denote the intersection \( U_{i_1} \cap \cdots \cap U_{i_k} \). It is easy to see that \( U_\sigma \neq \emptyset \) if and only if \( \sigma \in \Delta \), in which case \( U_\sigma = \text{st}_\sigma \Delta \). For each \( \text{st}_\sigma \Delta \), define a differential graded algebra

\[
\mathcal{L}^*(\text{st}_\sigma \Delta) := (\mathbb{Q}[\text{st}_\sigma \Delta] \otimes \Lambda[v_1, \ldots, v_d], d), \quad dv_i = \theta_i, \quad dx_i = 0; \quad \deg v_i = -1, \quad \deg x_i = 0.
\]

Particularly, when \( \sigma = \emptyset \), we briefly write \( \mathcal{L}^*(\Delta) \) as \( \mathcal{L}^* \). Note that \( \mathcal{L}^*(\text{st}_\sigma \Delta) \) as a \( \mathbb{Q}[m] \)-module has another even internal grading which is preserved by the differential. We will denote this grading by subscript.

Consider the double complex

\[
(\mathcal{R}^*, D) = \bigoplus_k \mathcal{R}^k = \bigoplus_k \bigoplus_{p-q=k} (\mathcal{R}^{p-q}, d, \delta),
\]

\[
\mathcal{R}^{p-q} = \bigoplus_{\sigma \in J_{p-1}(\Delta)} (\mathcal{L}^{-q}(\text{st}_\sigma \Delta), d), \quad D = \delta + (-1)^p d,
\]

where \( \delta \) is the Čech coboundary operator. There are two spectral sequences converging to the total cohomology \( H^*(\mathcal{R}^*, D) \). One spectral sequence starts with \( ^1E_1 = H_\delta \) and \( ^1E_2 = H_\delta H_\delta \), and another with \( ^2E_1 = H_\delta \) and \( ^2E_2 = H_\delta H_\delta \).

By Hochster’s theorem (unpublished, see [31, Stanley Theorem II.4.1]), we have

\[
^1E_1^{p-q} = \tilde{H}^{p-1}(\Delta; \mathbb{Q}) \otimes \Lambda^{-q}[v_1, \ldots, v_d].
\]

Since \( H^{p-1}(\Delta; \mathbb{Q}) \) has zero \( \mathbb{Q}[m] \)-module degree, \( ^1E_1 \) collapses at the \( E_1 \)-term. This implies that

\[
(H^{-k}(\mathcal{R}^*, D))_{2i} = \binom{d}{i} \tilde{H}^{i-k-1}(\Delta; \mathbb{Q}). \tag{5.3}
\]

Since \( \Delta \) is a Buchsbaum complex, Theorem 2.2 (c) implies that \( \text{st}_\sigma \Delta \) is Cohen-Macaulay for each \( \sigma \neq \emptyset \). Hence, for the second spectral sequence, we have

\[
^2E_1^{p-q} = 0 \quad \text{for } p, q > 0, \quad \text{and}
\]

\[
^2E_1^{0,0} = \bigoplus_{\sigma \in J_{p-1}(\Delta)} \mathbb{Q}[\text{st}_\sigma \Delta]/\Theta \quad \text{for } p > 0.
\]

It follows that this spectral sequence collapses at the \( E_2 \)-term, and so \( H^k(\mathcal{R}^*, D) = \bigoplus_{p-q=k} ^2E_2^{p-q} \). The \( E_1 \)-term also tells us that

\[
^2E_1^{0,-q} = ^2E_2^{0,-q} \quad \text{for } q > 0, \quad \text{and}
\]

\[
^2E_2^{0,-q} = H^{-q}(\mathcal{R}^*, D) \quad \text{for } q \geq 0. \tag{5.4}
\]

An easy calculation shows that the \( E_3 \)-term of the Serre spectral sequence in the lemma is just \( E_3^{2p,q} = (^2E_1^{0,-q})_{2p+2q} \). Combining this with formula (5.3) and (5.4), we get the desired dimension of \( E_3^{2p,q} \) for \( q > 0 \).
It remains to consider $E_3^{2p,0}$. Note that $\mathcal{L}_{2k}^*$ is a subcomplex of $\mathcal{L}^*$ since the internal grading is preserved by the differentials. Moreover, for each $k$ and $i$, $\mathcal{L}_{2k}^{-i}$ is a finite dimensional vector space over $\mathbb{Q}$, so we can calculate the Euler characteristic of $\mathcal{L}_{2k}^*$:

$$
\chi(\mathcal{L}_{2k}^*) = \sum_{0 \leq i \leq d} (-1)^i \dim_{\mathbb{Q}} \mathcal{L}_{2k}^{-i} = \sum_{0 \leq i \leq k} (-1)^i \dim_{\mathbb{Q}} \mathcal{L}_{2k}^{-i}.
$$

Recall the Hilbert series of $\mathbb{Q}[\Delta]$ is

$$
F(\mathbb{Q}[\Delta], \lambda) = \frac{h_0 + h_1 \lambda^2 + \cdots + h_d \lambda^{2d}}{(1 - \lambda^2)^d} = 1 + a_1 \lambda^2 + a_2 \lambda^4 + \cdots
$$

It follows that $\chi(\mathcal{L}_{2k}^*) = \sum_{i=0}^{k} (-1)^i d(k-i).$ A straightforward calculation shows that this number is equal to the coefficient of $\lambda^{2k}$ in the expansion of the polynomial

$$(1 - \lambda^2)^d(1 + a_1 \lambda^2 + a_2 \lambda^4 + \cdots),$$

which is just $h_k$.

On the other hand, we can also compute $\chi(\mathcal{L}_{2k}^*)$ in terms of the cohomology of $\mathcal{L}^*$, i.e.,

$$
\chi(\mathcal{L}_{2k}^*) = \sum_{0 \leq i \leq k} (-1)^i \dim_{\mathbb{Q}} H^{-i}(\mathcal{L}^*, d)_{2k}.
$$

Since $\chi(\mathcal{L}_{2k}^*) = h_k$ and $H^{-i}(\mathcal{L}^*, d)_{2k} = E_3^{2k-2i,i}$, we can immediately get the desired expression of $E_3^{2k,0}$ from the above calculation of $E_3^{2k,0}$.

Now we can give the cohomology of toric spaces associated to Buchsbaum complexes.

**Theorem 5.6.** Let $\Delta$ be a Buchsbaum complex, $\Theta$ an l.s.o.p. for $\mathbb{Q}[\Delta]$. Then for the associated toric space $M_\Delta$, we have

$$
H^k(M_\Delta; \mathbb{Q}) \cong \bigoplus_{2p+q=k} E_3^{p,q},
$$

where $E_3^{p,q}$ is given by Lemma 5.5. The ring structure of $H^k(M_\Delta; \mathbb{Q})$ is given by

$$
H^*(M_\Delta; \mathbb{Q}) \cong \mathcal{R} \oplus \mathbb{Q}[\Delta]/\Theta, \quad \mathcal{R}^k = \bigoplus_{q>0, 2p+q=k} (d_{p+q}) \check{H}^{p-1}(\Delta; \mathbb{Q}),
$$

where $\mathcal{R}$ has trivial multiplication structure.

**Proof.** According to Theorem 2.23 and lemma 5.5, it is equivalent to prove that $E_3 = E_\infty$ in the Serre spectral sequence of the fibration in Lemma 5.5.
For each vertex \( \{i\} \in [m] \), there is an inclusion \( X_i = Z_{st_i \Delta} \times T^{m-j_i} \subset Z_\Delta \), where \( j_i = \# \mathcal{F}_0(st_i \Delta) \), and we have a Serre fibration map:

\[
\begin{align*}
T^d \longrightarrow & \quad ET^m \times_{T^{m-d}} X_i \longrightarrow ET^m \times_{T^m} X_i \\
T^d \longrightarrow & \quad ET^m \times_{T^{m-d}} Z_\Delta \longrightarrow ET^m \times_{T^m} Z_\Delta \\
\end{align*}
\]

\( \varphi_i \) induces a morphism of cohomology \( \varphi_i^* : H^*(M_\Delta) \to H^*(M_i) \) (after an isomorphism given by Theorem 2.23). Since the Serre spectral sequence construction is functorial, \( \varphi_i \) also induces a morphism \( E_r \xrightarrow{\varphi_i^*} i_* E_r \) of Serre spectral sequences.

Let \( \Pi E_r \) be the spectral sequence defined in the proof of Lemma 5.5, \( \Phi_r = \bigoplus_{i=1}^m \varphi_i^* \). Then for the \( E_3 \)-term of the Serre spectral sequence, the map \( \Phi_3^{2p,q} \) is just the differential \( d_1 : \Pi (E_0^{0,-q})_{2p} \to \Pi (E_1^{1,-q})_{2p} \). So from (5.3) and (5.4), we have

\[
\begin{align*}
\text{Ker } \Phi_3^{2p,q} &= E_3^{2p,q} \quad \text{for } q > 0, \text{ and} \\
\text{Ker } \Phi_3^{2p,q} &= \Pi (E_2^{0,-q})_{2p+2q} = \left( \begin{array}{c} d \\ p+q \end{array} \right) \tilde{H}^{p-1}(\Delta; \mathbb{Q}) \quad \text{for } q \geq 0. \\
\end{align*}
\]

(5.5)

It follows that \( \dim \text{Ker } \Phi_3 = \sum_{p,q=0}^d \left( \begin{array}{c} d \\ p+q \end{array} \right) \tilde{\beta}_{p-1}(\Delta) \).

On the other hand, combining Lemma 5.2 and Corollary 5.4 with the fact that \( \Phi_\infty = \bigoplus_{i=1}^m \varphi_i^* \), we can get that

\[
\dim \text{Ker } \Phi_\infty \geq \dim \text{Im } j^* \geq \sum_{p,q=0}^d \left( \begin{array}{c} d \\ p+q \end{array} \right) \tilde{\beta}_{p-1}(\Delta). 
\]

However, since \( st_i \Delta \) is Cohen-Macaulay, \( i_* E_3 = i_* E_\infty \), which implies that

\[
\dim \text{Ker } \Phi_\infty \leq \dim \text{Ker } \Phi_3 = \sum_{p,q=0}^d \left( \begin{array}{c} d \\ p+q \end{array} \right) \tilde{\beta}_{p-1}(\Delta). 
\]

Hence these two inequalities shows that \( \text{Ker } \Phi_3 = \text{Ker } \Phi_\infty \), and therefore \( E_3^{2p,>0} \) survives to \( E_\infty \) by formula (5.5), but this already implies that the Serre spectral sequence collapses at the \( E_3 \)-term.

It remains to see the ring structure of \( H^*(M_\Delta; \mathbb{Q}) \). It is clear that \( E_3^{*,0} = \mathbb{Q}[\Delta]/\Theta \). So we can define \( R = E_3^{*,>0} \). Lemma 5.2 shows that \( \text{Ker } \Phi_3 \subset \text{Im } j^* \), and formula (5.5) shows that \( E_3^{*,>0} = \text{Ker } \Phi_3^{*,>0} \). Thus the cohomology ring formula follows immediately from Lemma 5.1.

\( \square \)

**Remark 5.7.** Note that the two inequalities in the proof of Theorem 5.6 also implies that \( \text{Ker } \Phi_3 = \text{Im } j^* \), where \( j^* \) is the map in the exact sequence (5.2).
From the proof of Theorem 5.6, we can get an interesting result about the socle of a Buchsbaum complex over $\mathbb{Q}$, which was initially obtained by Novik and Swartz [26] for any infinite field $k$.

**Proposition 5.8 ([26, Theorem 2.2]).** Let $\Delta$ be a Buchsbaum complex, $\Theta$ an l.s.o.p. for $\mathbb{Q}[\Delta]$. Then

$$\dim \text{Soc}(\mathbb{Q}[\Delta]/\Theta)_{2p} \geq \binom{d}{p} \tilde{\beta}_{p-1}(\Delta).$$

**Proof.** In formula (5.5), we have $\ker \Phi_{3}^{2p,0} = H^{0}(E_{2}^{0,0})_{2p} = \binom{d}{p} \tilde{H}^{p-1}(\Delta; \mathbb{Q})$. Since $\ker \Phi_{3} = \text{Im} j^{*}$, by using Lemma 5.1 and Theorem 5.6 we get that $\ker \Phi_{3}^{2p,0} \subset \text{Soc}(\mathbb{Q}[\Delta]/\Theta)_{2p}$, and so the inequality in the proposition holds. □

### 5.3. Toric spaces associated to rational homology manifolds.

In this subsection, $\Delta$ is a rational homology manifold without boundary. As we have shown, the toric space $M_{\Delta}$ is not a rational homology manifold unless $\Delta$ is a rational homology sphere. However, if we look at the local topology of $M_{\Delta}$ in the D-J construction, we can see that an open neighbourhood of a point $x \in M_{\Delta} - \{\ast\} \times T^{d}$, where $\ast$ is the cone point in $\mathcal{C}\Delta$, is the same as the case that $\Delta$ is a rational homology sphere. So $M_{\Delta} - \{\ast\} \times T^{d}$ is an open manifold, and the subspace $I \times \Delta \times T^{d} \sim M_{\Delta}$ (see (5.1)) is a compact manifold with boundary $\{1\} \times \Delta \times T^{d}$. Moreover, if $\Delta$ is $\mathbb{Q}$-orientable, then so is $I \times \Delta \times T^{d} / \sim$. The following result on its own is interesting in toric topology.

**Theorem 5.9.** Let $\Delta$ be a $(d-1)$-dimensional connected simplicial complex. For $j^{*}$ in (5.2), let $\mathcal{I} = \bigoplus_{k=1}^{d-1} (\text{Im} j^{*})_{2k}$. If $\Delta$ is an orientable rational homology manifold without boundary, then the quotient algebra

$$\mathcal{A} = H^{*}(M_{\Delta}; \mathbb{Q}) / \mathcal{I}$$

is a Poincaré duality algebra.

**Proof.** Since $\Delta$ is connected and orientable, it follows from Lemma 5.5 and Theorem 5.6 that the top cohomology group of $M_{\Delta}$ is $H^{2d}(M_{\Delta}) = \mathbb{Q}$. So it suffices to show that for any $0 \neq \alpha \in \mathcal{A}_{2k}$ with $k < d$, there exists $\beta \in \mathcal{A}_{2d-2k}$, such that $\beta \alpha \neq 0$.

Let $M_{1} = \mathcal{C}\Delta \times T^{d}$, $M_{2} = I \times \Delta \times T^{d} / \sim$ in the decomposition formula (5.1). Consider the following commutative diagram:

$$
\begin{align*}
H^{2d-2k}(M_{\Delta}, M_{1}) \otimes H^{2k}(M_{\Delta}) &\xrightarrow{i^{*}} H^{2d}(M_{\Delta}, M_{1}) \\
\cong i^{*} &\xrightarrow{i^{*}} \cong i^{*} \\
H^{2d-2k}(M_{2}, \partial M_{2}) \otimes H^{2k}(M_{2}) &\xrightarrow{i^{*}} H^{2d}(M_{2}, \partial M_{2})
\end{align*}
$$

(5.6)

The vertical isomorphisms come from excision.
Suppose $\alpha \in H^{2k}(M_\Delta)$ with $k < d$, such that its image is not zero in $A_{2k}$, i.e., $\alpha \notin \text{Im} j^*$, then by the exactness of (5.2), $0 \neq i^*(\alpha) \in H^{2k}(M_2)$. Since $M_2$ is an orientable manifold with boundary, the Lefschetz duality of $(M_2, \partial M_2)$ tells us that there exists $\beta \in H^{2d-2k}(M_2, \partial M_2)$ such that $\beta \sim i^*(\alpha) \neq 0$. Let $\beta' = (i^*)^{-1}(\beta)$. Diagram (5.6) shows that $\beta' \sim \alpha \neq 0$. Clearly the map $f^* : H^{2d}(M_\Delta, M_1) \to H^{2d}(M_\Delta)$ is an isomorphism. So $f^*(\beta') \sim \alpha = f^*(\beta' \sim \alpha) \neq 0$, and we get the desired element $f^*(\beta')$.

**Remark 5.10.** Suppose $[M_\Delta]$ is a generator of $H_{2d}(M_\Delta; \mathbb{Q}) \cong \mathbb{Q}$. Then for a face monomial $x_\sigma \in A$, we have

$$[M_\Delta] \cdot x_\sigma = \mathbb{Q} \cdot (\rho_\sigma)_*[\langle N_\sigma \rangle],$$

where $\langle N_\sigma \rangle$ is the rational fundamental class of $N_\sigma$. This can be proved in the same way as Lemma 3.4.

In §5.2 we have already seen that if $\Delta$ is a Buchsbaum complex, $\text{Ker } \Phi_3 = \text{Im } j^*$, $\mathcal{R} = \text{Ker } \Phi_3^{*>0}$ and $\text{Soc}(\mathbb{Q}[\Delta]/\Theta)_{2k} \supset \text{Ker } \Phi_3^{2k,0} = \{q\} \mathbb{H}_{k-1}(\Delta; \mathbb{Q})$. So Theorem 5.9 implies that if $\Delta$ is an orientable rational homology manifold, then

$$\dim \text{Soc}(\mathbb{Q}[\Delta]/\Theta)_{2k} = \binom{d}{k} \tilde{\beta}_{k-1}(\Delta),$$

$$\dim A_{2k} = h_k(\Delta) - \binom{d}{k} \sum_{i=0}^{k-1} (-1)^i \tilde{\beta}_{k-i-1}(\Delta), \quad \text{and}$$

$$A \cong \mathbb{Q}[\Delta]/(\Theta + I), \quad \text{where } I = \bigoplus_{k=1}^{d-1} \text{Soc}(\mathbb{Q}[\Delta]/\Theta)_{2k}.$$

So in fields of characteristic zero, Theorem 5.9 is a topological explanation of the following important result of Novik and Swartz:

**Theorem 5.11 ([25]).** Let $\Delta$ be a $(d-1)$-dimensional connected simplicial complex, $\mathbb{k}$ an infinite field. If $\Delta$ is an orientable $\mathbb{k}$-homology manifold without boundary, then for any l.s.o.p. $\Theta$ for $\mathbb{k}[\Delta]$.

$$\dim \text{Soc}(\mathbb{k}[\Delta]/\Theta)_{2i} = \binom{d}{i} \tilde{\beta}_{i-1}(\Delta; \mathbb{k}).$$

Moreover, let $I = \bigoplus_{i=1}^{d-1} \text{Soc}(\mathbb{k}[\Delta]/\Theta)_{2i}$, then $\mathbb{k}[\Delta]/(\Theta + I)$ is a Poincaré duality $\mathbb{k}$-algebra.

**Appendix A. Some topological facts about toric spaces**

A.1. **Proof of Theorem 2.23.** The cohomology with coefficients in $\mathbb{Q}$ will be implicit throughout the proof. First, we consider the case that $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^d$ is onto. In this case, $M(\Delta, \Lambda) = \mathcal{Z}_\Delta/T^{m-d}$, where $T^{m-d}$ is the kernel of the tori map $\exp \Lambda : T^m \to T^d$. Using the notation in subsection 2.4, we have a $T^m$-subspace
Let \( B_\sigma \subset \mathcal{Z}_\Delta \) for each \( \sigma \in \Delta \). By definition, if \( \dim \sigma = k - 1 \), \( B_\sigma = (D^2)^k \times T^{m-k}. \)

Then \( C_\sigma \cong T^{m-k} \) is a \( T^m \)-invariant subspace of \( B_\sigma \). Since \( C_\sigma \) is a deformation retract of \( B_\sigma \), \( C_\sigma/T^{m-d} = T^{d-k} \) is a deformation retract of \( B_\sigma/T^{m-d} \). It follows that the composition \( T^{m-d} \to B_\sigma \to B_\sigma/T^{m-d} \) induces an isomorphism

\[
H^*(B_\sigma) = H^*(B_\sigma/T^{m-d}) \otimes H^*(T^{m-d}).
\]

On the other hand, applying the Leray-Hirsch theorem to the fiber bundle \( T^{m-d} \to ET^m \times B_\sigma \to ET^m \times T^{m-d} B_\sigma \), we get that

\[
H^*(ET^m \times B_\sigma) = H^*(ET^m \times T^{m-d} B_\sigma) \otimes H^*(T^{m-d}).
\]

Hence we have an isomorphism

\[
p_\sigma^*: H^*(B_\sigma/T^{m-d}) \to H^*(ET^m \times T^{m-d} B_\sigma),
\]

which is induced by the restriction \( p_\sigma : ET^m \times T^{m-d} B_\sigma \to B_\sigma/T^{m-d} \) of \( p \).

Now we can get the desired cohomology isomorphism by double induction on the number of facets of \( \Delta \) and \( \dim \Delta \). We proceed the inductive argument by applying Mayer-Vietoris sequences, and the base of the induction is given above.

For the general case that \( M(\Delta, \Lambda) = \mathcal{Z}_\Delta/(T^{m-d} \times G) \), consider the quotient map \( \pi : \mathcal{Z}_\Delta/T^{m-d} \to M(\Delta, \Lambda) \). We only need to show that \( \pi^* : H^*(M(\Delta, \Lambda)) \to H^*(\mathcal{Z}_\Delta/T^{m-d}) \) is an isomorphism. Note that the \( G \)-action on \( \mathcal{Z}_\Delta/T^{m-d} \) extends to a toral action. Thus, \( H^*(\mathcal{Z}_\Delta/T^{m-d}) \) is fixed under the induced \( G \)-action. Recall the classical result for a finite \( G \)-action: \( G \times X \to X \), that is

\[
\pi^* : H^*(X/G; \mathbb{Q}) \to H^*(X; \mathbb{Q})^G := \{ x \in H^*(X; \mathbb{Q}) : gx = x \}
\]

is a ring isomorphism (see for example [6]). Then the theorem follows. \( \square \)

A.2. **Algebraic model for cellular cochains.** In this subsection, we prove the promised statement in the proof of Lemma 3.4. That is, for a \( (k-1) \)-face \( \sigma \in \Delta \), the monomial \( x_\sigma \in \mathbb{Q}[\Delta]/\Theta = H^*(M_\Delta; \mathbb{Q}) \) is, up to multiplication by an integer, represented by a cocycle \( \tilde{c}_\sigma \in C^{2k}(M_\Delta; \mathbb{Q}) \). First, we need to know the algebraic models for the cellular cochain algebras \( C^*(\mathcal{Z}_\Delta) \), \( C^*(ET^m) \), \( C^*(BT^m) \), \( C^*(ET^m \times T^m \mathcal{Z}_\Delta) \), etc.

Recall that in [10, §4.5], \( S^\infty \) is given a cell decomposition with one cell in each dimension; the boundary of an even cell is the closure of an odd cell, and the boundary of an odd cell is the 0-cell. Thus, the cellular cochain complex of \( S^\infty \) can be identified with the Koszul algebra

\[
\Lambda[y] \otimes \mathbb{Z}[u], \quad \deg y = 1, \quad \deg u = 2, \quad dy = u, \quad du = 0.
\]
Similarly, $\mathbb{C}P^{\infty}$ has a cell decomposition with one cell in each even dimension, and $C^{*}(\mathbb{C}P^{\infty})$ can be identified with the polynomial algebra $\mathbb{Z}[u]$. It follows that the cochain homomorphism $C^{*}(BT^{m}) \to C^{*}(ET^{m})$ induced by the universal principal $T^{m}$-bundle $ET^{m} \to BT^{m}$ has an algebraic model of the form

$$Z[u_{1}, \ldots, u_{m}] \to \Lambda[y_{1}, \ldots, y_{m}] \otimes Z[u_{1}, \ldots, u_{m}].$$

On the other hand, recall the Koszul complex $(\Lambda[y_{1}, \ldots, y_{m}] \otimes Z[\Delta], d)$ of the face ring defined in subsection 2.5. It is an algebraic model of $C^{*}(Z_{\Delta})$. Precisely, the cochain map

$$\Lambda[y_{1}, \ldots, y_{m}] \otimes Z[\Delta] \to C^{*}(Z_{\Delta}), \quad y_{i} \mapsto t_{i}^{*}; \quad x_{i} \mapsto e_{i}^{*}$$

induces an isomorphism in cohomology. It follows that the differential graded algebra

$$\mathcal{R} = Z[u_{1}, \ldots, u_{m}] \otimes \Lambda[y_{1}, \ldots, y_{m}] \otimes Z[\Delta], \quad dy_{i} = u_{i} - x_{i}$$

is an algebraic model of $C^{*}(ET^{m} \times T^{m} Z_{\Delta})$ (see for example [24, Theorem 12.6.1]). This has the consequence that for a $(k - 1)$-face $\sigma \in \Delta$, the monomial $x_{\sigma} \in Z[\Delta] = H^{*}(ET^{m} \times T^{m} Z_{\Delta})$ can be represented by the cocycle

$$(pt, e_{\sigma})^{*} \in C^{2k}(ET^{m} \times T^{m} Z_{\Delta}).$$

Now consider the fibration sequence

$$ET^{m-d} \times_{T^{m-d}} Z_{\Delta} \xrightarrow{i} ET^{m} \times_{T^{m}} Z_{\Delta} \xrightarrow{\pi} BT^{d}.$$ 

From the proof of Proposition 2.22 we know that the fiber inclusion map $i$ induces a homomorphism of rational cohomology $i^{*} : \mathbb{Q}[\Delta] \to \mathbb{Q}[\Delta]/\Theta$. Since $i^{*}( (pt, e_{\sigma})^{*} ) = (pt, e_{\sigma})^{*}$, $x_{\sigma} \in \mathbb{Q}[\Delta]/\Theta = H^{*}(ET^{m-d} \times_{T^{m-d}} Z_{\Delta}; \mathbb{Q})$ can also be represented by the cocycle

$$(pt, e_{\sigma})^{*} \in C^{2k}(ET^{m-d} \times_{T^{m-d}} Z_{\Delta}; \mathbb{Q}).$$

On the other hand, for the ‘orbit cell’ $\tilde{e}_{\sigma}$, the cellular cochain $\tilde{e}_{\sigma}^{*} \in C^{2k}(M_{\Delta})$ satisfies that $p^{*}(\tilde{e}_{\sigma}^{*}) = (pt, e_{\sigma})^{*}$ up to multiplication by an integer, where $p : ET^{m-d} \times_{T^{m-d}} Z_{\Delta} \to M_{\Delta}$ is the quotient map. Since $p^{*}$ is an isomorphism on rational cohomology by Theorem 2.23, the assertion follows.

### A.3. Proof of Lemma 5.3.

Let $\Delta'$ be the barycentric subdivision of $\Delta$. Recall that the relation ‘$\sim$’ in $\Delta \times T^{d}/ \sim$ is defined by means of the polyhedral decomposition

$$\Delta' = \bigcup_{i \in F_{0}(\Delta)} \text{st}_{i}\Delta'.$$

Thus, $\mathcal{U} = \{ U_{i} := \text{st}_{i}\Delta' \times T^{d}/ \sim \}_{i \in F_{0}(\Delta)}$ can be viewed as an ‘open’ cover of $\Delta \times T^{d}/ \sim$. For a subset $\sigma = \{i_{1}, \ldots, i_{k}\} \subset [m]$, let $\mathcal{U}_{\sigma}$ denote the intersection $\mathcal{U}_{i_{1}} \cap \cdots \cap \mathcal{U}_{i_{k}}$. There are some obvious facts:

(i) $\mathcal{U}_{\sigma} \neq \emptyset$ if and only if $\sigma \in \Delta$. 


For $\sigma \in \Delta$, $\mathcal{U}_\sigma$ is a toric space associated to the geometric realization of the poset $\Delta_{>\sigma} = \{ \tau \in K : \tau > \sigma \}$. This geometric realization, which we denote by $L_\sigma$, is a subcomplex of $\Delta'$ and combinatorially equivalent to $\text{lk}_\sigma \Delta$. Precisely,

$$\mathcal{U}_\sigma = F_\sigma \times T^{d-|\sigma|} / \sim,$$

where $F_\sigma \subset \Delta'$ is the geometric realization of the poset $\Delta_{>\sigma}$ as defined in §2.4, $T^{d-|\sigma|} = T^d / (T_{i_1} \times \cdots \times T_{i_k})$.

Now consider the Čech double complex

$$(K^*, d) = \bigoplus_k K^k = \bigoplus_{k} \bigoplus_{p+q=k} (K^{p,q}, \partial, \delta),$$

$$K^{p,q} = \bigoplus_{\sigma \in F_p(\Delta)} C^q(\mathcal{U}_\sigma, \partial; \mathbb{Q}), \quad d = \delta + (-1)^p \partial,$$

where $C^*(\mathcal{U}_\sigma, \partial; \mathbb{Q})$ is the rational cellular cochain complex of $\mathcal{U}_\sigma$ and $\delta$ is the Čech coboundary operator. There are two spectral sequences converging to the total cohomology $H^*(K^*, d)$. One spectral sequence starts with $E_1^q = H^q(\mathcal{U}_\sigma; \mathbb{Q})$, and another with $E_1^q = H^q$ and $E_2^q = H^q(\mathcal{U}_\sigma; \mathbb{Q})$. (The second one is also known as the Mayer-Vietoris spectral Sequence.)

Since $\mathcal{U}$ is an open cover,

$$E_1^q = E_1^{0,*} = C^*(\Delta \times T^d / \sim, \partial; \mathbb{Q}).$$

Hence, the first spectral sequence collapses at the $E_1$-term and therefore

$$H^*(K^*, d) \cong H^*(\Delta \times T^d / \sim; \mathbb{Q}).$$

For the second spectral sequence, we have $E_1^{p,q} = 0$ for $q$ odd, and

$$E_1^{p,2q} = \bigoplus_{\sigma \in F_p} H^{2q}(\mathcal{U}_\sigma; \mathbb{Q}).$$

This is because $\mathcal{U}_\sigma$ is a toric space associated to $L_\sigma$, which is a Cohen-Macaulay complex by the assumption that $\Delta$ is a Buchsbaum complex.

It follows from Appendix A.2 that a cohomology class of $H^{2q}(\mathcal{U}_\sigma; \mathbb{Q})$ can be represented by the cellular cochain $\sum k_\varrho e^{\varrho}_\sigma$, where $\varrho \in L_\sigma$ is a simplex of the form $\varrho = (\sigma_1 < \cdots < \sigma_q)$ with $\sigma_i \in \Delta_{>\sigma}$, $e^{\varrho}_\sigma$ is the ‘orbit cell’ corresponding to the cell $e_\varrho = e_{\sigma_1}^2 \times \cdots \times e_{\sigma_q}^2 \subset Z_{L_\sigma}$. An easy topological observation shows that $e_\varrho$ is the image of $(\sigma < \sigma_1 < \cdots < \sigma_q) \times T^q_\varrho$ under the quotient map $F_\sigma \times T^k \to Z_{L_\sigma}$, where $k = |F_0(L_\sigma)|$. Hence,

$$e_\varrho = \pi((\sigma < \sigma_1 < \cdots < \sigma_q) \times T^q_\varrho), \quad \text{where } \pi : F_\sigma \times T^{d-|\sigma|} \to \mathcal{U}_\sigma.$$
$\varrho \in F_{q-1}(L_\sigma)$ for some $\sigma \in F_p(\Delta)$, a diagram chasing in the double complex $K^{*,*}$ shows that $\alpha$ is represented by a cocycle $\beta \in C^{p+2q}(\Delta' \times T^d/\sim)$ of the form

$$\beta = \sum_{\tau \in F_{p+q}(\Delta')} k_\tau \tilde{c}_\tau,$$

where $\tilde{c}_\tau \subset \Delta' \times T^d/\sim$ has the form

$$\tilde{c}_\tau = \pi((\tau_1 < \cdots < \tau_p < \sigma < \sigma_1 \cdots < \sigma_q) \times T^d).$$

So $\pi^*(\beta) \in H^{p+q}(\Delta'; \mathbb{Q}) \otimes H^q(T^d; \mathbb{Q})$, and therefore lemma 5.3 holds.

\[\square\]

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