On Conformal Properties of the Dualized Sigma-Models

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Abstract

We have calculated the first-order $\beta$-functions for a $\sigma$-model (with dilaton) dualized with respect to an arbitrary Lie group that acts without isotropy. We find that non-abelian duality preserves conformal invariance for semi-simple groups, but in general there is an extra contribution to the $\beta$-function proportional to the trace of the structure constants, which cannot be absorbed into an additional dilaton shift. Two particular examples, a Bianchi V cosmological background and the $G \otimes G$ WZW model, are discussed.
1 Introduction

The subject of target space duality (also called “T-duality”) has been important in the study of string models, because of, for example, compactification issues and the inter-dependency of the model dynamics on different scales and for different manifolds. But historically, duality was performed with respect to an abelian isometry group.

During the last couple of years, non-abelian duality received significant attention. While formally rather similar to the abelian case, it poses several problems that have been successfully solved for abelian Lie groups, but become much more difficult for non-abelian ones. The inverse transformation, the transformation group, and the connection to the conformal properties of the $\sigma$-models are among those problems.

This paper provides the lacking explicit proof [12] that the target space duality spoils the conformal invariance only if the isometry group of the original conformal background has structure constants that are not traceless.

The outline of the paper is as follows. In section 2 we briefly summarize the formulae relevant to the subject of non-abelian duality transformations. For extensive reviews of both abelian and non-abelian duality, see [11, 12, 13]. The method that has been used to calculate the 1-loop $\beta$-functions is explained in section 4 and is followed by the explicit derivation valid for arbitrary duality groups (that act without isotropy). Following this, in section 5 we work out two examples of $\sigma$-models that underwent non-abelian duality transformation. The summary and discussion are in the section 6.

2 Non-Abelian Duality

Suppose that we are given the usual $\sigma$-model action, with a background that can be decomposed as follows ($\mathcal{E} = g + b$)\footnote{As explained in [1], this is the most general case without isotropy; this restriction is made purely to simplify the computations, but there is no reason to believe that our results are not general.}

$$
\mathcal{S}_\sigma = -\frac{1}{2} \int d^2 \sigma \left[ e^m(x^\mu) \mathcal{E}_{mn}(x^i) \tilde{e}^n(x^\mu) + e^m(x^\mu) \mathcal{E}_{mj}(x^i) \partial x^j \right.

+ \partial x^i \mathcal{E}_{in}(x^i) \tilde{e}^n(x^\mu) + \partial x^i \mathcal{E}_{ij}(x^i) \tilde{\partial} x^j - \frac{\sqrt{\gamma}}{4} R^{(2)} \Phi, \right]
$$

(2.1)
where \( e_m = e_m^\mu (x^\mu) \partial x^\mu \) and \( \bar{e}_m = e_m^\mu (x^\mu) \bar{\partial} x^\mu \) represent the coordinate frames that obey the Maurer-Cartan equation

\[
d e^m + \frac{1}{2} f_{pq}^m e^p \wedge e^q = 0
\] (2.2)

with \( f_{pq}^m \) being the structure constants of some Lie group \( G \). Equivalently, the generators of \( G \)

\[
[T_m, T_n] = f_{mn}^p T_p, \quad T_m = \theta^\mu_m (x^\mu) \partial_\mu, \quad \theta^\mu_m = (e^m_\mu)^{-1},
\] (2.3)

act as the Killing vectors on the background fields \( E \) and \( \Phi \) \([1, 7]\):

\[
\mathcal{L}_T g = 0, \quad \mathcal{L}_T b = d\varpi, \quad \mathcal{L}_T \Phi = 0,
\] (2.4)

where \( \varpi \) is some 1-form. When the action \( S_\sigma \) is invariant without total derivative terms, the \( G \)-isometry of Eq. (2.1) can be gauged by introducing covariant derivatives with the \( G \)-valued gauge fields \( v_\pm = v_\pm^m T_m \)

\[
\partial \rightarrow \nabla = \partial + v_+ \quad \text{and} \quad \bar{\partial} \rightarrow \bar{\nabla} = \bar{\partial} + v_-;
\] (2.5)

otherwise one needs to follow the procedure of \([10]\). Note that \( e_m \) by itself is a pure gauge:

\[
e_m^\mu (x^\mu) \partial x^\mu = \text{Tr} \left( T_m g^{-1} \partial g \right),
\] (2.6)

which means that one can pick the “unitary” gauge \([3]\)

\[
g = 1 \iff x^\mu = 0 \quad (\text{in suitable coordinates})
\] (2.7)

for the model (2.1), and reobtain the original \( S_\sigma \) by adding the following extra term to the action:

\[
S_\lambda = -\frac{1}{2} \int d^2 \sigma \left[ \lambda_m F^m \right], \quad \text{where} \quad F^m = \bar{\partial} v_+^m - \partial v_-^m - v_+^p f_{pq}^m v_-^q
\] (2.8)

The equations of motion \( \delta S_\lambda / \delta \lambda_m = 0 \) require the gauge field strengths \( F^m \) to vanish, thus constraining \( v_+^m \) to \( e^m \) and \( v_-^m \) to \( \bar{e}_m \).

To perform a duality transformation on the \( \sigma \)-model described by Eq. (2.1) and Eq. (2.8) means to promote the \( \lambda_m \)’s that were introduced as mere Lagrange multipliers to regular coordinates of the dual space-time by using the equations of motion for \( v_+^m, v_-^m \) to integrate \( v_{\pm}^m \) out of the action functional

\[
S_\sigma \left[ x^\mu = 0, x^i, v_{\pm}^m \right] + S_\lambda
\] (2.9)
This transformation gives rise to the dual action:

\[
\hat{S} = -\frac{1}{2} \int d^2\sigma \left\{ \left[ (\partial \lambda_m - \partial x^i \mathcal{E}_{im}) \left( \mathcal{E}_{qq} + \lambda_p f_{qp} \right)^{-1} \right]^{mn} (\partial \lambda_n + \mathcal{E}_{nj} \partial x^j) \right. \\
+ \mathcal{E}_{ij} \partial x^i \partial x^j - \frac{\sqrt{\gamma}}{4} R^{(2)} \left. \left[ \Phi + \ln \det (\mathcal{E}_{mn} + \lambda_p f_{mn}^p) \right] \right\} \tag{2.10}
\]

It can be identified as the \(\sigma\)-model action with the dual coordinates \(\hat{x}^m = \lambda_m\), \(\hat{x}^i = x^i\) and the dual dilaton field \(\hat{\Phi} = \Phi + \ln \det (\mathcal{E}_{mn}(\hat{x}^i) + \hat{x}^p f_{mn}^p)\).

For the rest of this paper, all the quantities arising from the dual \(\sigma\)-model \([2.10]\) will be labeled by “hats”.

### 3 Conformal Invariance and Sigma-Models

If one thinks of \(\sigma\)-models as strings propagating in the potential of some background fields (a space-time manifold and dilaton matter comprise a particularly popular choice), then consistency requires the \(\sigma\)-model to be conformally invariant not only on the classical, but also on the quantum level (for a review see e.g. \([1, 3]\)). The 1-loop \(\beta\)-functions are known to be:

\[
\beta^E_{\alpha \gamma} = R_{\alpha \gamma}^{(-)} - \nabla_\gamma \nabla_\alpha \Phi \tag{3.1}
\]

\[
\beta^\Phi = \frac{1}{3\alpha'} (c - c_{crit}) + \left[ (\nabla \Phi)^2 + 2 \nabla^2 \Phi - R^{(-)} - \frac{1}{6} H^2 \right], \tag{3.2}
\]

where \(H_{\alpha\beta\gamma}\) is the torsion tensor and \(\nabla\) and \(R^{(-)}\) are the covariant derivative and scalar curvature for the connection \(\omega^{(-)}\) defined by Eq. \([1.2]\). Note that \(\beta^E\) and \(\beta^\Phi\) are not independent, since, by using the Bianchi identities, one can derive that, provided \(\beta^E_{\alpha \gamma} = 0\),

\[
\nabla_\gamma \beta^\Phi = -2 \nabla^\alpha \beta^E_{\alpha \gamma} = 0 \tag{3.3}
\]

which means that \(\beta^\Phi\) is a \(c\)-number and can always be set equal to zero, once \(\beta^E_{\alpha \gamma}\) is known to vanish \([3]\). Thus one can concentrate one’s efforts on Eq. \([3.1]\).

### 4 Calculating the Beta-Functions

Probably the easiest way to calculate the dual \(\beta\)-functions is to use the Buscher’s method \([3]\): to consider non-abelian duality transformation of the
N=1 supersymmetric extension of the \( \sigma \)-model given by Eq. (2.1). We intend to benefit from the well-known fact that the component action of N=1 \( \sigma \)-models consists (after the elimination of the auxiliary fields) of purely bosonic part corresponding to Eq. (2.1), 2-fermion coupling to the bosonic connection and 4-fermion coupling to the bosonic Riemann tensor:

\[
S_{N=1} = \frac{1}{2} \int d^2 \sigma D_+ D_- \left[ \mathcal{E}_{\alpha \beta} (\phi^\alpha) D_+ \phi^\alpha \psi^\beta \right] \\
= \frac{1}{2} \int d^2 \sigma \left[ -\mathcal{E}_{\alpha \beta} (x^\alpha) \partial x^\alpha \tilde{\partial} x^\beta + \frac{1}{2} \psi^\alpha_+ \psi^\beta_+ \psi^\gamma_+ \psi^\delta_+ \mathcal{R}^{(-)}_{\alpha \beta \gamma \delta} \right. \\
+ \left. \psi^\alpha_+ (g_{\alpha \beta} \tilde{\partial} \psi^\beta + \psi^\beta_+ \tilde{\partial} x^\gamma \omega^\gamma_{\alpha \beta}) + \psi^\alpha_- (g_{\alpha \beta} \partial \psi^\beta - \psi^\beta_- \partial x^\gamma \omega^\gamma_{\alpha \beta}) \right]
\] (4.1)

where the connections in the cartesian coordinates are defined as

\[
\omega_{\alpha \beta}^{(\pm)} = \frac{1}{2} (g_{\alpha \beta} + g_{\alpha \gamma} \pm g_{\beta \gamma}) \mp \frac{1}{2} (b_{\alpha \beta} + b_{\alpha \gamma} \pm b_{\beta \gamma})
\] (4.2)

The following steps are involved in the procedure:

1. Write down the component action for the total extended N=1 action \( S = S_{\sigma, gauged} + S_\Lambda \).

2. Dualize \( S \) at the component level, with the unitary gauge (Eq. (2.7) and (4.13)) fixed.

3. Identify the dual connections and curvatures in terms of the original ones, using the equations of motion for the super gauge field.

These steps are equivalent to dualizing the superfield \( \Lambda \) as a whole, which produces the background fields identical to the ones of a bosonic model Eq. (2.10). This is the reason for the validity of the relations for the connections and curvatures that we obtain on the step 3.

We define our notation for the N=1 super Yang-Mills theory in 2 dimensions as follows (\( \partial \equiv \partial_+, \tilde{\partial} \equiv \partial_- \)):

\[
\begin{align*}
\{D_+, D_-\} &= 0 \\
\nabla_+ &= D_+ + \Gamma_+ \\
\Gamma_+ &= \Gamma^m_+ \mathbf{T}_m \quad \text{(see Eq. (2.3))} \\
\{\nabla_+, \nabla_-\} &= W \\
\{\nabla_+, \nabla_-\} &= 2\nabla_{\pm \pm} = 2(\partial_{\pm \pm} + \Gamma_{\pm \pm})
\end{align*}
\] (4.3)

\[
\begin{align*}
\{D_\pm, D_\pm\} &= 2\partial_{\pm \pm} \\
\nabla_\pm &= D_\pm + \Gamma_\pm \\
\Gamma_\pm &= \Gamma^m_\pm \mathbf{T}_m \\
\{\nabla_\pm, \nabla_\pm\} &= 2\nabla_{\pm \pm} = 2(\partial_{\pm \pm} + \Gamma_{\pm \pm})
\end{align*}
\] (4.4)
The covariant component fields are defined as:

\[ \phi^\alpha = x^\alpha \quad \nabla_\pm \phi^\alpha = \psi_\pm^\alpha \quad \left( \nabla_\pm \phi^i = D_\pm \phi^i \right) \]

\[ \Lambda_m = \lambda_m \quad \nabla_\pm \Lambda_m = \dot{\psi}_\pm^m \]

\[ \nabla_\pm W = G_\pm \quad \Gamma_{\pm\pm} = v_{\pm\pm} \quad \Gamma_+ = \zeta_+ \]

and the rules of covariant differentiation are

\[ \left( \nabla_{++} x \right)^\alpha = \partial_{++} x^\alpha + \delta_m^\alpha \theta_m^\mu \]

\[ \left( \nabla_{++} \psi \right)^\alpha = \partial_{++} \psi^\alpha + \delta_m^\alpha \theta_m^\mu \left( \partial_\mu \theta_\nu^m \right) e^\nu_\pm \psi_\pm^\rho \]

After removing the auxiliary fields \( W \), \( \nabla_+ \Lambda_m \) and \( \nabla_+ \nabla_\phi^\alpha \), the N=1 extended action

\[ S = S_\sigma + S_\Lambda = -\frac{1}{2} \int d^2 \sigma \left[ \mathcal{E}_{\alpha\beta} \nabla_+ \phi^\alpha \nabla_- \phi^\beta \right] - \frac{1}{2} \int d^2 \sigma \left[ \nabla_+ \nabla_- \left[ \Lambda_m W^m \right] \right] \]

becomes

\[ S_\sigma = -\frac{1}{2} \int d^2 \sigma \left[ \frac{1}{2} \psi_+^\alpha \psi_+^\beta \psi_-^\gamma \psi_-^\delta R^{(-)}_{\alpha\delta \gamma \beta} \right. \]

\[ \left. - \mathcal{E}_{\alpha\beta} \nabla_+ \phi^\alpha \nabla_- \phi^\beta + g_{\alpha\beta} \left( \psi_+^\alpha \nabla_- \psi_+^\beta + \psi_-^\alpha \nabla_+ \psi_-^\beta \right) \right] \]

\[ + \psi_+^\alpha \psi_-^\beta \nabla_- \nabla_+ \mathcal{E}^{(-)}_{\alpha\beta \gamma} + \mathcal{E}_{\alpha\beta} \left( -\psi_+^\alpha G_-^m \psi_-^\beta + G_+^m \psi_-^\beta \right) \]

\[ S_\Lambda = -\frac{1}{2} \int d^2 \sigma \left[ \dot{\psi}_+^m G_-^m - \dot{\psi}_-^m G_+^m \right] \]

\[ + \lambda_m \left( \partial_\mu^m - \partial_\nu^m - v_+^m J_{pq}^m v_-^q \right) + \lambda_m f_{pq}^m \left( G_+^q \psi_-^q - G_-^q \psi_+^q \right) \]

after we have discarded the total derivatives

\[ \frac{1}{2} \nabla_{++} \left( b_{\alpha\beta} \psi_+^\alpha \psi_-^\beta \right) = b_{\alpha\beta} \psi_+^\alpha \nabla_{++} \psi_-^\beta + \frac{1}{2} \psi_+^\alpha \psi_-^\beta b_{\alpha\beta} \nabla_{++} x^\gamma \]

The calculation will be significantly simplified in orthonormal frames, which we always can choose to have the following “triangular” form:

\[ E^M_\mu = E^M_{m}(x^i)e^m_\mu(x^\mu) \quad E^M_i = E^M_i(x^i) \]

\[ E^I_\mu = 0 \quad E^I_i = E^I_i(x^i) \]
Due to the above choice, the block form of $\Theta = E^{-1}$ is:

$$\Theta^\mu_M = (E^M_\mu)^{-1} \quad \Theta^i_M = 0 \quad (4.12)$$

$$\Theta^\mu_I = -\Theta^\mu_M E^M_i \Theta^i_I \quad \Theta^i_I = (E^I_i)^{-1}$$

After fixing the unitary gauge (see Eq. (2.7))

$$\phi^\mu = 0 \implies x^\mu = 0 \quad \text{and} \quad \zeta^m = e^m_\mu \psi^\mu_\pm \equiv \psi^m_\pm \quad (4.13)$$

$S = S_\sigma + S_\Lambda$ takes the following form:

$$S_\sigma = -\frac{1}{2} \int d^2 \sigma \left\{ \frac{1}{2} \Psi_+^A \Psi_+^B \Psi_-^C \Psi_-^D R^{(-)}_{ABCD} - v^p_+ E^P_\mu (1 + b)_{PQ} E^Q q v^q_- - v^p_+ E^P_\mu (1 + b)_{PB} E^B j \hat{\partial} x^j \right. $$

$$- \partial x^j E^A_i (1 + b)_{AQ} E^Q q v^q_- - \partial x^j E^A_i (1 + b)_{AB} E^B j \hat{\partial} x^j + \Psi_+^A \hat{\partial} \Psi_+^A + \Psi_+^A \Psi_-^B \left( E^C_i \partial x^j \omega^{(-)}_{A,BC} + E^Q q v^q_- \omega^{(-)}_{A,BQ} \right)$$

$$+ \Psi_-^A \hat{\partial} \Psi_-^A + \Psi_-^A \Psi_-^B \left( E^C_i \partial x^j \omega^{(+)}_{A,BC} + E^P p v^p_+ \omega^{(+)}_{A,BP} \right) + G^m E^M_m (1 + b)_{MB} \Psi_-^B + G^m E^m_+ (1 + b)_{AM} E^M m \right\}$$

$$S_\Lambda = -\frac{1}{2} \int d^2 \sigma \left( \hat{\psi}_m G^m_- - \hat{\psi}_m G^m_+ \right) \quad (4.15)$$

For our future convenience, we define these matrices:

$$L_{PQ} \equiv -\Theta^p_P \left[ \lambda_{m f_{pq}} \right] \Theta^q_Q$$

$$M^{PQ} \equiv \left[ ((1 + b)_{MM'} - L_{MM'})^{-1} \right]^{PQ}$$

$$T^{PQ} \equiv [1 - 2M]^{PQ} \quad \text{, note that (T)}^T = (T)^{-1}$$

The dual orthonormal frames are equal to:

$$\hat{E}^P_\mu = \Theta^p_Q M^{QP} \quad \hat{E}^P_i = -\left( E^I_i b_{IQ} + E^Q_i L_{Q'^{PQ}} \right) M^{QP} \quad \hat{E}^I_p = 0 \quad \hat{E}^I_i = E^I_i$$

with

$$\left( \partial_{\pm \pm} \hat{X} \right)^P = \hat{E}^P_p \partial_{\pm \pm} \lambda_p + \hat{E}^P_i \partial_{\pm \pm} x^i \quad \left( \partial_{\pm \pm} \hat{X} \right)^I = \left( \partial_{\pm \pm} X \right)^I \quad (4.18)$$

$$\hat{\Psi}_{\pm}^P = \hat{E}^P_\mu \hat{\psi}_{\pm \mu} + \hat{E}^P_i \psi_{\pm}^i \quad \hat{\Psi}_+^I = \Psi_+^I \quad (4.19)$$
The equations of motion for the gauge components \( v_{\pm \pm} \) and \( G_{\pm} \) yield
\[
\dot{\Psi}^Q_+ = \dot{\Psi}^Q_- \quad \text{and} \quad \dot{\Psi}^Q_- = T^{QR} \dot{\Psi}^-_R \tag{4.20}
\]
If we extend the definition of \( T \) onto all indices as following: \( T^{IJ} = 1 \) and \( T^{IQ} = T^{PQ} = 0 \), we can compactly write
\[
\dot{\Psi}^A_+ = \dot{\Psi}^A_- \quad \text{and} \quad \dot{\Psi}^A_- = T^{AB} \dot{\Psi}^-_B \tag{4.21}
\]
The contribution from the \( \nu \)-quadratic term equals
\[
v_{\mu}^P E_P^P \mathcal{M}^{-1}_{PQ} E^Q_{\nu} v_{\nu} = \left[ (\partial \dot{X})^P - \partial x^i E^P_i + \Psi^A_+ \Psi^B_- \omega_{ABQ} \right] \times
\]
\[
\mathcal{M}^{QP} \left[ -(\partial \dot{X})^R \mathcal{M}^{-1}_{RP} - \partial x^i E^R_j (1 - b + \mathcal{L})_{RP} + \Psi^C_+ \Psi^D_- \omega_{CD} \right] \tag{4.22}
\]
Finally, we can write down the dual component action:
\[
\dot{S} = -\frac{1}{2} \int d^2 \sigma \left\{ -(\partial \dot{X})^P \mathcal{M}^T_{PQ} (\partial \dot{X})^Q - (\partial \dot{X})^I (1 + b)_{IJ} (\partial \dot{X})^J + \right.
\]
\[
-(\partial \dot{X})^P \mathcal{M}^T_{PQ} \dot{\Theta}_j^i (\partial \dot{X})^J + (\partial \dot{X})^I \dot{\Theta}_j^i E^P_i \mathcal{M}^T_{PQ} (\partial \dot{X})^Q
\]
\[
-(\partial \dot{X})^I \dot{\Theta}_j^i \left( E^P_i \mathcal{L}_{PQ} E^Q_j + E^P_i b_{P,J} E^J_j + E^I b_{IQ} E^Q_j \right) \dot{\Theta}_j^i (\partial \dot{X})^J
\]
\[
+ \dot{\Psi}^A_+ \dot{\Psi}^A_- \dot{\Psi}^B_+ + \dot{\Psi}^A_+ \dot{\Psi}^B_- \left[ \dot{\omega}_{ABJ} (\partial \dot{X})^J + \dot{\omega}_{ABP} T^{PQ} (\partial \dot{X})^Q \right]
\]
\[
+ T^{AA'} \dot{\Psi}^A_+ \dot{\Psi}^A_- \dot{\Psi}^B_- \dot{\Psi}^B_+ T^{BB'} \dot{\omega}_{ABJ} (\partial \dot{X})^J
\]
\[
+ \frac{1}{2} \dot{\Psi}^A_+ \dot{\Psi}^B_- T^{CC'} \dot{\Psi}^A_- T^{DD'} \dot{\Psi}^B_+ \left[ \mathcal{R}^{(-)}_{ABCD} + \omega^{(-)}_{ABB} T^{PQ} \omega^{(+)}_{CDQ} \right] \right\}. \tag{4.23}
\]
It gives us the sought relations:
\[
\dot{\omega}_{ABJ}^{(-)} = \dot{\omega}_{ABJ}^{(-)} T^{CC}
\]
\[
\dot{\omega}_{ABJ}^{(+) \nu} = \omega_{ABJ}^{(+) \nu} T^{AA'} T^{BB'} T^{MPQ} \omega_{CDQ}^{(+)} \mathcal{R}_{ABCD} \tag{4.24}
\]
\[
\dot{R}^{(-)}_{ABCD} = \left[ \mathcal{R}^{(-)}_{ABCD} + \omega^{(-)}_{ABB} T^{PQ} \omega^{(+)}_{CDQ} \right] T^{CC'} T^{DD'} \tag{4.25}
\]

The contribution to the dual \( \beta \)-functions from the dilaton shift \( \dot{\Phi} - \Phi \) can be easily calculated if one observes that
\[
\ln \det \left( \mathcal{E}_{pq} + \lambda_m f^m_{pq} \right) = \ln \det \left( E^M_{\ M} \dot{\Theta}^P_M \right) \tag{4.27}
\]
Then, using the formula \( \ln \det = \text{Tr} \ln \), we can find that
\[
\dot{\Theta}^a_A \frac{\partial}{\partial \dot{x}^a} \ln \det \left( \mathcal{E}_{pq} + \lambda_m f^m_{pq} \right) = \left( \dot{\omega}_{M,AM}^{(-)} - \dot{\omega}_{M,AM}^{(-)} \right) - 2 f^m_{mn} \Theta^n_A \tag{4.28}
\]
Also, it’s easy to see that due to the relations Eq. (2.4) and Eq. (4.24),
\[ \hat{\nabla}_D \hat{\nabla}_B \Phi = [ \nabla_{D'} \nabla_B \Phi ] T^{D'D} \]  
(4.29)

Now, using the fact that
\[ \hat{\mathbf{R}}_{BD} = \left( \mathbf{R}_{BD} - \mathbf{R}_{QBQD} \right) T^{D'D} + \left( \omega_{I,BP} - \omega_{I,BP}' \right) \hat{\omega}_{I,PD} + \hat{R}_{QBQD} \]  
(4.30)
we arrive after some algebra to the central result of this work
\[ \hat{\beta}_{BD} = \beta_{BD}' T^{D'D} + \hat{\nabla}_D \left[ 2f^m_{mn} \Theta^n_B(\hat{x}^i) \right] \]  
(4.31)

Note that for non-semi-simple Lie groups with \( f^m_{mn} \neq 0 \), the right hand side of Eq. (4.31) can not in general be compensated by adding an extra piece to the dilaton field, unless for some particular case
\[ \Theta^n_B(\hat{x}^i) = \hat{\nabla}_B Y(\hat{x}^a) \]  
(4.32)

5 Applications

In this section we demonstrate two particular cases of non-abelian duality transformation that have been discussed in the literature.

First, the example of Bianchi V as a non-semi-simple subgroup of the \( SO(3,1) \) Lorentz group in 4 dimensions [2]. Second, the duality of the vector-gauged \( \mathcal{G} \otimes \mathcal{G} \) WZW model for arbitrary semi-simple group [1].

5.1 Non-Semi-Simple Group

In the paper [2], the authors pointed out one particular Bianchi-type non-semi-simple group, where conformal invariance of the dualized \( \sigma \)-model could not be achieved by any correction to the dilaton. They discussed the following initial background configuration:
\[ x^\mu = \{ x, y, z \} = \{ x^1, x^2, x^3 \} \quad x^i = \{ t \} = \{ x^4 \} \]
\[ f_{12}^2 = f_{13}^3 = 1 \quad \text{the rest is zero} \]
\[ g_{\alpha\beta} = \text{diag} \left( t^2, t^2 e^{-2x}, t^2 e^{-2x}, -1 \right) \quad b_{\alpha\beta} = \Phi = 0 \]  
(5.1.1)

We have explicitly checked that an attempt to resolve the equation
\[ \hat{\nabla}_D \left( \hat{\nabla}_B Y - 2f^m_{mn} \Theta^n_B \right) = 0 \]  
(5.1.2)
leads to the same unsatisfiable condition \( 1/t^2 = 0 \) that have been reported in [2].
5.2 $G \otimes G$ WZW Model

The vector-gauged $G \otimes G$ WZW Model for semi-simple $G$ was considered in [1, 3, 10] as an example of a model without isotropy:

$$ S_{G \otimes G} [g, x] = \frac{k}{2\pi} \int d^2 \sigma \text{Tr} \left[ \left( g^{-1} \partial g - x^{-1} \partial x \right) g^{-1} \bar{\partial} g \right] + k S_G^{WZ} [x] \quad (5.2.1) $$

The dual action in this case is

$$ \hat{S} = \frac{k}{2\pi} C_R \int d^2 \sigma \left[ \partial \lambda^m + (x^{-1} \partial x)^m \right] \left( 1 + \lambda_p f_p \right)^{-1}_{mn} \bar{\partial} \lambda^n + k S_G^{WZ} [x], \quad (5.2.2) $$

where in the group representation $R$, $\text{Tr}(T_R^m T_R^n) = C_R \delta_{mn}$ and the WZW action

$$ k S_G^{WZ} [x] = \frac{k}{2\pi} \left[ \frac{1}{2} \int d^2 \sigma \text{Tr} \left( x^{-1} \partial x \cdot x^{-1} \bar{\partial} x \right) + \Gamma^{WZ}_G [x] \right] $$

$$ = \frac{k}{4\pi} C_R \int d^2 \sigma \left[ (x^{-1} \partial x)^m \otimes (x^{-1} \bar{\partial} x)^m \right] $$

$$ + \left( b^{WZ}_G \right)_{mn} (x^{-1} \partial x)^m \wedge (x^{-1} \bar{\partial} x)^n $$

This calculation becomes relatively simple exercise in differential geometry techniques [14] after one chooses the frame

$$ E^M = \kappa_m d \hat{x}^m, \quad \text{where } \kappa \equiv (1 + \lambda_p f_p)^{-1} \text{ and } \hat{x}^m \equiv \lambda_m \quad (5.2.4) $$

$$ dE^M = \frac{1}{2} (1 + \kappa)^M_n f^q_{PQ} E^P \wedge E^Q \quad (5.2.5) $$

Specifics of this concrete problem prompt us not to go for the orthonormal frames. It’s easy to see that $\kappa = M^T$ (c.f. Eq. (4.17)). The values of dual connections:

$$ \hat{\omega}^{(-)}^I J P = (\kappa - 1)^I_q f^q_{IP} \quad \hat{\omega}^{(-)}^I N P = (2\kappa - 1)^N_q f^q_{IP} $$

$$ \hat{\omega}^{(-)}^M J P = -\kappa^I_q f^q_{MP} \quad \hat{\omega}^{(-)}^M N P = \left( \kappa^M_q f^q_{NP} - 2\kappa^N_q f^q_{MP} \right) $$

$$ \hat{\omega}^{(-)}^A_{BK} = 0 \quad (5.2.6) $$

agree with the values obtained by suitably modified (c.f. Eq. (5.2.4)) procedure, described above. The dual derivative of the dilaton shift equals

$$ \hat{Q}_M \ln \det (1 + \lambda_p f_p) = \kappa^P Q f^Q_{PM} \quad (5.2.7) $$

One can check, using the Jacobi identities and the relations

$$ 1 - \kappa = \kappa (\lambda_m f^m) = (\lambda_m f^m) \kappa \quad (5.2.8) $$

that the dual background is, indeed, 1-loop conformally invariant.
6 Conclusions and Discussions

We have provided the proof that non-abelian duality respects the conformal invariance of $\sigma$-models if the dualized isometry corresponds to a semi-simple Lie group. In the non-semi-simple case, an anomaly \[4, 11\] arises in the Jacobian of duality transformation and violates the conformal invariance. While formally our proof works only for the case without isotropy, we strongly believe that our result should be valid in general. We plan to address the isotropy case in the future.

Clearly, not all of the properties of abelian transformations can be recovered in the much more complex environment of non-abelian duality, which, unfortunately, is not sufficiently understood at the present time. Specifically, it is of great interest to investigate the nature of the symmetries in the space of Conformal Field Theories that are exhibited by the target-space duality. Also, it seems to be useful to classify non-semi-simple groups that due to some “accident” preserve the conformal invariance of the original $\sigma$-model, since they might have some physical significance \[2\].

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References

[1] A. Giveon, M. Roček “On Nonabelian Duality”, Nucl. Phys. B421 (1994) 173 or hep-th/9308154

[2] M. Gasperini, R. Ricci, G. Veneziano “A Problem with Non-Abelian Duality?”, Phys. Lett. B319 (1993) 438 or hep-th/9308112

[3] F. Quevedo “Abelian and Non-Abelian Dualities in String Backgrounds”, hep-th/9305055
X. de la Ossa, F. Quevedo “Duality Symmetries from Non-Abelian Isometries in String Theory”, Nucl. Phys. B403 (1993) 377 or hep-th/9210021

[4] E. Álvarez, L. Álvarez-Gaumè, Y. Lozano “On Non-Abelian Duality”, Nucl. Phys. B424 (1994) 155 or hep-th/9403155

[5] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer, G. Veneziano “Remarks on Non-Abelian Duality”, hep-th/9409011

[6] A. Giveon, M. Roček “Generalized Duality in Curved String-Backgrounds”, Nucl. Phys. B380 (1992) 128 or hep-th/9112070

[7] M. Roček, E. Verlinde “Duality, Quotients and Currents”, Nucl. Phys. B373 (1992) 630 or hep-th/9110053

[8] C. Callan, D. Friedan, E. Martinec, M. Perry “Strings in Background Fields”, Nucl. Phys. B262 (1985) 593

[9] T. Buscher “Studies of the Two-dimensional Nonlinear Sigma-model”, Ph. D. thesis (1988), unpublished
T. Buscher “Path-Integral Derivation of Quantum Duality In Nonlinear Sigma-Models”, Phys. Lett. B201 (1988) 466
T. Buscher “A Symmetry of the String Background Field Equations”, Phys. Lett. B194 (1987) 59

[10] K. Bardakci, E. Rabinovici, B. Säring “String Models with c < 1 components”, Nucl. Phys. B299 (1988) 151
K. Gawedzki, A. Kupianen “Coset Construction From Functional Integrals”, Nucl. Phys. B320 (1989) 625
C. Hull, B. Spence “The Gauged Nonlinear Sigma Model With Wess-Zumino Term”, Phys. Lett. B232 (1989) 204
C. Hull, B. Spence “The Geometry Of The Gauged Sigma Model With Wess-Zumino Term”, Nucl. Phys. B353 (1991) 379

[11] A. Giveon, M. Roček “Introduction to Duality”, hep-th/9406178, to appear in “Essays on Mirror Manifolds II”

[12] E. Álvarez, L. Álvarez-Gaumè, Y. Lozano “An Introduction to T-Duality in String Theory”, hep-th/9410327

[13] A. Giveon, M. Porrati, E. Rabinovici “Target Space Duality in String Theory”, Phys. Rep. 244 (1994) 77 or hep-th/9401139

[14] T. Eguchi, P. Gilkey, A. Hanson “Gravitation, Gauge Theories and Differential Geometry”, Phys. Rep. 66 (1980) 213