The properties of integral models for planar and axisymmetric unsteady jets

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This article reviews and builds upon recent progress that has been made in understanding the mathematical properties of integral models for unsteady turbulent jets. The focus is on models that describe the evolution of the volume flux and the momentum flux in a jet, whose source conditions are time dependent. A generalized approach that postpones making assumptions about the ‘internal’ properties of the flow, such as the radial dependence of the longitudinal velocity profile, turbulent transport and pressure, allows one to understand how the resulting integral equations are affected by model-specific assumptions. Whereas the assumptions invoked in previous unsteady jet models have resulted in a parabolic system of equations, generalized equations that are derived from first principles have a hyperbolic character and statistical stability that depends sensitively on assumptions that are normally invoked a priori. Unsteady axisymmetric jets with Gaussian velocity profiles have special properties, including a tendency to remain straight-sided (conical) and marginal stability in response to source perturbations. A distinct difference between planar jets and axisymmetric jets is that the mean energy flux, which plays a leading-order role in determining the unsteady dynamics of jets, is significantly lower in planar jets. We hypothesize that in order to maintain marginal stability the turbulence and pressure fields in planar jets adjust themselves, relative to axisymmetric jets, to compensate for the lower mean energy flux.

Keywords: turbulent jets; stability; integral models.

1. Introduction

A turbulent jet is the free-shear flow that is produced by a source of momentum in an otherwise quiescent environment. Unconfined jets have an approximately conical shape, the precise boundary of which is a contorted surface separating the irrotational (non-turbulent) environment from the rotational (turbulent) flow within the jet, as shown in Fig. 1(a). The bounding surface of the jet twists and folds and the jet mixes and entrains fluid from its surroundings, as described by Fig. 1(b). Examples of jets abound in fields as diverse as pollution dispersion, building ventilation, combustion, meteorology and oceanography. In applications, one typically needs to be able to predict the rate at which jets dilute and transport fluid. Such processes determine ventilation rates, the temperature structure in an enclosed space and the concentration and arrival time of contaminants.

Although the jets considered in this article are neutrally buoyant, many jets in the natural and built environment consist of fluid whose density differs from that of the surroundings, and are therefore buoyant. Such buoyant jets are typically referred to as turbulent plumes and can be elegantly and accurately described using classical plume theory, which deals with the evolution of bulk quantities associated with the flow. Turbulent jets are therefore a limiting case of the flows that can be described.
Fig. 1. Iso-regions of the passive scalar concentration in a direct numerical simulation of a steady turbulent jet $Re \approx 5000$ (see Craske & van Reeuwijk, 2015a, for details). Darker (red) regions indicate higher concentrations. The thick (blue) line denotes an enstrophy isosurface, demarcating rotational and irrotational parts of the flow. Window (b) illustrates the process of turbulent entrainment, the arrows denoting the instantaneous fluid velocity in the ambient. The color image is available in the online version of this article.

by classical plume theory, introductions to which can be found in Linden (2000), Kaye (2008) and Hunt & van den Bremer (2011). Classical plume theory grew from the similarity analysis of jets and plumes by Tollmien (1926), Schmidt (1941a,b) and, at a similar time in the Soviet Union, from the scaling arguments used to analyse thermal convection by Zeldovich (1937). Not until the model for axisymmetric plumes of Morton et al. (1956) did the theory acquire its present-day form and popularity as an operational model, followed by various extensions and generalizations that address the relative balance of buoyancy and inertia in the plume (Morton, 1959; Morton & Middleton, 1973; van den Bremer & Hunt, 2010). With the exception of the theory and experiments of Lee & Emmons (1961), in the context of classical plume theory planar plumes, generated from line sources of momentum and/or buoyancy, have received comparatively little attention (van den Bremer & Hunt, 2014).

As a canonical turbulent free-shear flow that can be usefully compared to mixing layers, wakes and axisymmetric jets (e.g. Bradbury, 1965; Heskestad, 1965; Gutmark & Wygnanski, 1976), planar jets have received considerable attention from experimentalists. In particular, Wygnanski & Fiedler (1969, 1970) and Gutmark & Wygnanski (1976) conducted experiments of an axisymmetric jet, a mixing layer and a planar jet, respectively. In this series of experiments the same conditional sampling technique was employed in each case to restrict measurements to the turbulent region of the flow and to facilitate a
fair comparison of the turbulence structure. The overall conclusion was that planar and axisymmetric jets have similar distributions of velocity, intermittency and turbulence intensity, adding credence to the hypothesis made by Bradbury (1965) that turbulent free-shear flows possess a universal structure.

The extension of classical plume theory with which the present article is concerned is the incorporation of statistical unsteadiness, which would arise if the source conditions of an otherwise steady plume were to vary in time. Examples of unsteady jets and plumes can be found in volcanic eruption columns, rapidly growing fires and time-dependent ventilation and heating in buildings. In the context of bulk plume models, the first extension to unsteady releases of buoyancy appears to have been the starting plume model of Turner (1962), followed by a more general model by Delichatsios (1979), the latter having introduced a system of partial differential equations to model arbitrary temporal variations in the source buoyancy flux. Alternative unsteady plume models were derived subsequently (Yu, 1990; Vul’fson & Borodin, 2001; Scase et al., 2006), each ostensibly based on slightly different assumptions about the profile of mean longitudinal (streamwise) velocity. Exposing the importance of these assumptions on the usefulness of the resulting models, Scase & Hewitt (2012) recently discovered that those proposed by Delichatsios (1979), Yu (1990) and Scase et al. (2006) were ill-posed. Craske & van Reeuwijk (2015a) subsequently demonstrated that the cause of the ill-posedness was the assumed form of the mean velocity profile and an alternative, generalized, formulation of the governing equations based on mean energy conservation was proposed, which ultimately led to the development of a well-posed Gaussian unsteady plume model (Craske & van Reeuwijk, 2016). The subsequent well-posed model of Woodhouse et al. (2016) also incorporates essential information about the shape of the velocity profile, but focuses on momentum conservation rather than energy conservation. From a mathematical modelling perspective, unsteady jets and plumes are fundamentally different to their steady-state counterparts and provide the opportunity to understand aspects of free-shear flows that are difficult to deduce from a steady state.

The aim of the present work is to review the recent progress that has been made in understanding the mathematical properties of axisymmetric unsteady jets, and to provide an extension to planar jets that might initiate and assist future research in the field. The theory and results we discuss pertaining to axisymmetric jets comprise a review of the author’s own work in collaboration with Maarten van Reeuwijk (Craske & van Reeuwijk, 2015a,b; van Reeuwijk & Craske, 2015; Craske & van Reeuwijk, 2016). The work we present on unsteady planar jets, and their relation to axisymmetric jets, is new. Where the emphasis of the previous work was the underlying physics and the development of a robust model, here the emphasis is on the mathematical properties of the governing equations and how the framework can be used to provide a deeper understanding of steady jets. We restrict our attention to jets for clarity, and because some of the most interesting aspects of unsteady plumes are inherited from jets.

The paper is organised as follows. Before deriving the governing equations for unsteady jets from first principles, we provide a brief preview of integral jet models in Section 2. Section 3.1 outlines the differences between planar and axisymmetric jets. In Section 3.2 we discuss local conservation equations for planar and axisymmetric jets, before deriving a system of generalized integral conservation equations in Section 3.3. In Section 3.4 we explain how radial profiles of the mean longitudinal velocity and turbulence affect the integral energy flux in jets. Section 4 consists of a systematic study of the structure of the governing integral equations. The analysis begins by considering volume conservation and the behaviour of the radius of the jet in Section 4.1, before revisiting classical unsteady similarity solutions from a generalized perspective in Section 4.2. The response of planar and axisymmetric jets to harmonic source perturbations is discussed in Section 4.3. In Sections 4.4–4.6 the properties of discontinuous solutions to the governing equations are considered. The results are discussed in Section 5 and conclusions and recommendations for future work are provided in Section 6.
2. Integral modelling of turbulent jets

Integral models of turbulent jets allow one to make predictions based on limited information about the details of a flow and are therefore useful in practical applications. The robustness of these integral models stems from conservation laws and the fact that fully developed jets and plumes are typically self-similar (see e.g. Rajaratnam, 1976; George, 1989; Barenblatt, 1996), which means that, relative to the local characteristic scales of the flow, their dynamics look ‘the same’ at any given longitudinal location. For self-similarity in the strictest sense, relatively far from a flow’s source, the characteristic scales necessarily obey power-law scalings with respect to the longitudinal coordinate (Barenblatt, 1996).

As we will discuss in Section 3, integration of the local equations of motion over a plane perpendicular to the mean direction of flow yields a system of differential equations in the remaining longitudinal coordinate and possibly time. In a steady state the resulting conservation equations for volume and momentum in a jet are

\[
\frac{dQ}{dz} = 2\alpha_0 \frac{M^{1/d}}{Q^{2/d-1}}, \quad \frac{dM}{dz} = 0, \quad (2.1a,b)
\]

respectively, where \(d = 1\) for planar jets and \(d = 2\) for axisymmetric jets. The volume flux in the jet is \(Q\), the momentum flux is \(M\) and \(z\) is the longitudinal coordinate. The coefficient \(\alpha_0\) is the entrainment coefficient: an assumed ratio between the induced irrotational flow in the ambient and the flow’s characteristic velocity (Taylor, 1945; Batchelor, 1954; Turner, 1986). Although \(\alpha_0\) is a constant for a given self-similar flow, its value is affected by several flow-specific properties such as buoyancy, source conditions and environmental conditions (see e.g. van Reeuwijk & Craske, 2015; George, 1989; Khorsandi et al., 2013, respectively), and has therefore been the subject of intense investigation.

A crucial feature of steady self-similar models of jets and plumes is that their structure is essentially independent of the assumptions one makes about the shape of the mean longitudinal velocity profile, turbulent transport and longitudinal pressure gradients (Craske, 2016). Such contributions constitute a fixed proportion of the leading-order contribution from the mean flow and, therefore, to within a constant of proportionality, do not affect the structure of the governing equations. Similarly, alternative definitions of \(\alpha_0\), and indeed assumptions about its value, have a superficial influence on the governing equations.

For unsteady jets the situation is different because the presence of temporal derivatives in the governing equations means that assumptions made about the internal structure of the flow play a non-trivial role (Craske & van Reeuwijk, 2015b). Specifically, in Section 3 we will see that the governing integral equations for volume and momentum conservation in unsteady jets are

\[
\frac{1}{\gamma} \frac{\partial}{\partial t} \left( \frac{Q^2}{M} \right) + \frac{\partial Q}{\partial z} = 2\alpha \frac{M^{1/d}}{Q^{2/d-1}}, \quad (2.2)
\]

\[
\frac{\partial Q}{\partial t} + \beta \frac{\partial M}{\partial z} = 0. \quad (2.3)
\]

The parameters \(\beta, \gamma\) and \(\alpha\), which describe the internal structure of the flow, play an independent role in determining the structure of (2.2)–(2.3). One therefore needs to appreciate the physics that the parameters represent and to understand how it manifests itself in the mathematical properties of the governing equations. Classical assumptions about the value of \(\gamma, \beta\) and \(\alpha\) in (2.2)–(2.3) result in the system being ill-posed (Scase & Hewitt, 2012), which suggests that the underlying physics might not be correctly represented. Hence, we are concerned with finding the physically correct values of \(\gamma, \beta\) and
\( \alpha \) and, conversely, understanding the extent to which their values can be inferred \textit{a priori} by inspecting the mathematical properties of (2.2)–(2.3).

3. The governing equations

3.1. Planar and axisymmetric jets

The jets considered in this paper are produced by either a line or point source of momentum, as shown in Fig. 2(a and b), respectively. In the case of a line source, we refer to the flow as a planar jet, due to the statistical symmetry in the span-wise \((\varphi)\) direction. In the case of a point source we refer to the flow as an axisymmetric jet, because the flow possesses a statistical symmetry in the azimuthal \((\varphi)\) direction. In both cases the flow can be regarded as statistically 2D. However, the way in which the flow is constrained in each case is very different: the flow per unit radian in the axisymmetric jet entrains fluid from a 2D space, whereas the flow per unit length in a planar jet entrains fluid from a 1D space. The source of the axisymmetric jet has compact support in two dimensions, whereas the source of the planar jet has compact support in just one dimension. We therefore use the variable \( d \) to distinguish between planar \((d = 1)\) and axisymmetric \((d = 2)\) jets. The geometrical difference between the two cases plays an important physical role, because it determines the rate at which quantities in the flow scale with respect to the longitudinal coordinate.

In practice, the assumption of axisymmetry or planar symmetry is an idealization, strictly valid as one of two limiting cases. As described by Fig. 3, whilst a series of identical point releases, spaced \( L \) units apart might be accurately modelled as axisymmetric jets in the near field \((z \ll L)\), in the far-field \((z \gg L)\) the array of jets would coalesce (for plumes see Cenedese & Linden, 2014), and it would be appropriate to model the flow as a planar jet, as shown in Fig. 3(a). Conversely, a line source of finite length \( L \) will, sufficiently far from the source, ultimately produce an axisymmetric jet, as shown in Fig. 3(b).

![Fig. 2](image)

**Fig. 2.** Schematic diagram and coordinate system for (a) a planar jet and (b) an axisymmetric jet, driven by a source of momentum flux \( M_s \).
3.2. Local conservation equations

In this section a system of integral equations are derived from the local equations of motion. To begin, consider the transport equation for longitudinal specific momentum (hereafter referred to as momentum) in an incompressible jet issuing from an infinitesimal line source \((d = 1)\) or point source \((d = 2)\), as shown in Fig. 2(a and b), respectively. We are interested in being able to predict and reason about the velocity field that one would expect to find in such a flow, not with the instantaneous fluctuations that one might observe in a single experiment. It is therefore appropriate to consider the Reynolds-averaged momentum equation

\[
\frac{\partial \overline{w}}{\partial t} + \frac{1}{r^{d-1}} \frac{\partial (r^{d-1} \overline{uw})}{\partial r} + \frac{\partial \overline{w}^2}{\partial z} + \frac{1}{r^{d-1}} \frac{\partial (r^{d-1} \overline{uw}')}{\partial r} + \frac{\partial \overline{w}'^2}{\partial z} = - \frac{\partial \overline{p}}{\partial z},
\]

(3.1)

where \(w\) is the velocity in the longitudinal \((z)\) direction, \(u\) is the velocity in the inhomogeneous cross-stream direction \((r)\) and \(p\) is the kinematic pressure, from which a hydrostatic component has been subtracted. The ensemble average of a quantity \(\chi\) is denoted \(\overline{\chi}\) and is defined such that \(\overline{\chi}' = 0\). In practice, an ensemble average over independent realizations of the flow can be supplemented with an average taken over the homogeneous \(\varphi\)-direction in the flow. It is assumed that the mean velocity in the \(\varphi\)-direction is equal to zero.

As described in Section 3.1, the geometry of the flow is characterized by \(d\), which corresponds physically to the dimension of a lateral cross section of the flow. Axisymmetric jets have a circular cross section, hence \(d = 2\), while planar jets, characterized by fluxes per unit length of their line source, have a cross section that is a line, hence \(d = 1\) (see Fig. 2). To obtain (3.1) it was assumed that the Reynolds number is sufficiently high, such that viscous terms make a negligible contribution in the governing equations. Multiplication of (3.1) by \(2\overline{w}\) and use of the continuity equation,

\[
\frac{1}{r^{d-1}} \frac{\partial (r^{d-1} \overline{u})}{\partial r} + \frac{\partial \overline{w}}{\partial z} = 0,
\]

(3.2)
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gives

\[
\frac{\partial \overline{w}^2}{\partial t} + \frac{1}{r^{d-1}} \frac{\partial (r^{d-1} \overline{w}^2)}{\partial r} + \frac{\partial \overline{w}^3}{\partial z} + \frac{2}{r^{d-1}} \frac{\partial (r^{d-1} \overline{w} \overline{w})}{\partial r} 
+ 2 \frac{\partial (\overline{w} \overline{w})}{\partial z} \overline{w} + \frac{2 \partial (\overline{w} \overline{w})}{\partial r} \overline{w} = \frac{2}{r^{d-1}} \frac{\partial \overline{w} \overline{w}}{\partial r} + 2 \frac{\partial \overline{w} \overline{w}}{\partial z} + 2 \overline{u} \overline{w} \frac{\partial \overline{w}}{\partial r},
\]

which describes the conservation of mechanical energy in the jet. While (3.2) imposes a diagnostic constraint on the flow field, (3.3) provides a prognostic equation. For this reason, (3.1) and (3.3) are a convenient starting point for the development of unsteady jet models. A similar approach was adopted in the analysis of steady plumes by Priestley & Ball (1955). It differs from the volume–momentum approach popularised by Morton et al. (1956), due to the assumptions that are made about the scaling of the Reynolds stress $\overline{u} \overline{w}$ (see van Reeuwijk & Craske, 2015, for details).

3.3. Integral conservation equations

Integration of equations (3.1) and (3.3) over a plane that is perpendicular to the $z$-coordinate (see Fig. 2) results in (Craske & van Reeuwijk 2015a)

\[
\frac{\partial Q}{\partial t} + \frac{\partial (\beta g M)}{\partial z} = 0,
\]

\[
\frac{\partial M}{\partial t} + \frac{\partial}{\partial z} \left( \gamma g \frac{M^2}{Q} \right) = \delta g \frac{M^{2+1/d}}{Q^{1+2/d}},
\]

which describe the bulk conservation of momentum and (mean) energy in the jet, respectively. The volume flux and momentum flux per unit length ($d = 1$) or per unit radian ($d = 2$) in the jet are

\[
Q \equiv 2 \int_0^\infty \overline{w} r^{d-1} \, dr, \quad M \equiv 2 \int_0^\infty \overline{w}^2 r^{d-1} \, dr,
\]

and $\beta_g$, $\gamma_g$ and $\delta_g$ are dimensionless profile coefficients:

\[
\beta_g \equiv \frac{2}{M} \int_0^\infty \left( \overline{w}^3 + \overline{w} \overline{w}^2 + \overline{p} \right) r^{d-1} \, dr,
\]

\[
\gamma_g \equiv \frac{2Q}{M^2} \int_0^\infty \left( \overline{w}^3 + 2\overline{w} \overline{w}^2 + 2\overline{p} \overline{w} \right) r^{d-1} \, dr,
\]

\[
\delta_g \equiv \frac{4Q^{1+2/d}}{M^{2+1/d}} \int_0^\infty \left( \overline{u} \overline{w} \frac{\partial \overline{w}}{\partial r} + \overline{w} \overline{w} \frac{\partial \overline{w}}{\partial z} + \overline{p} \frac{\partial \overline{w}}{\partial z} \right) r^{d-1} \, dr,
\]

which describe the total momentum flux, the gross mean energy flux and the mean-flow energy loss (turbulence production) in the jet, in a non-dimensional form. Under the assumption that the jet has an infinitesimal radial extent, it is normally argued (Craske & van Reeuwijk, 2015a; Tennekes & Lumley, 1972) that streamwise turbulent transport can be neglected, and that all but the first terms appearing in
the integrands of equations (3.7)–(3.9) can be neglected. Bearing in mind that the dimensionless integral of \( \overline{w}^2 \) is unity by definition, it is therefore convenient to define

\[
\gamma_m \equiv \frac{2Q}{M^2} \int_0^\infty \overline{w}^2 r^{d-1} dr, \quad \delta_m \equiv \frac{4Q^{1+2/d}}{M^{1+d}} \int_0^\infty \overline{u}' \overline{w} \frac{\partial \overline{w}}{\partial r} r^{d-1} dr, \tag{3.10a,b}
\]

as the leading-order contributions to the energy flux and the turbulence production, respectively. In deriving (3.4)–(3.5) we assumed that the required integrals actually exist, which for practical purposes amounts to assuming that \( \overline{w} = o(1/r^d) \) as \( r \to \infty \). In practice, one typically integrates to a finite value of \( r \) that ensures that the integrals capture ‘most’ of the jet (Craske & van Reeuwijk, 2015a). For further details regarding the limits of integration that are used in analysing jets and plumes, the reader is referred to Kotsovinos (1978).

In the unsteady jet equations (3.4)–(3.5), the dependent variable \( Q \) can be interpreted as a volume flux (the classical interpretation) or as the integral of specific momentum in the jet (non-classical). Similarly, \( M \) can be interpreted as a momentum flux (the classical interpretation) or as the integral of mean energy in the jet (non-classical). In the context of unsteady jets the non-classical interpretation is useful, because it is \( Q \) and \( M \) that appear as operands of the temporal derivatives in conservation equations for momentum (3.4) and energy (3.5), respectively.

Equations (3.4)–(3.5) constitute a closed system if the profile coefficients \( \gamma_g \), \( \beta_g \) and \( \delta_g \) can be expressed as constants or in terms of \( t, z, Q \) and/or \( M \). Here, rather than making assumptions about the profile coefficients, we investigate how they affect the mathematical properties of (3.4)–(3.5), and ask what this implies about the underlying physics.

### 3.4. Profile coefficients and self-similarity

The dimensionless profile coefficients that appear in (3.4)–(3.5) account for the ‘internal’ features of the flow. Integral equations are used to understand the leading-order behaviour of the system at the expense of introducing the profile coefficients as additional unknowns. When modelling jets, one hopes that a simple assumption about the profile coefficients will prove sufficient to capture the unknown dynamics of the system.

We will assume self-similarity of the flow and suppose that the mean longitudinal velocity profile has the form

\[
\frac{\overline{w}}{w_m(z,t)} = c_1 f(c_2 \eta), \tag{3.11}
\]

where \( \eta \equiv r/r_m(z,t) \) is a similarity variable, \( f \) is a function whose improper integral over \( r \geq 0 \) is bounded and \( c_1 \) and \( c_2 \) are scaling parameters. The variables \( r_m \) and \( w_m \) are characteristic length and velocity scales:

\[
\frac{r_m^d}{d} \equiv \frac{Q^2}{2M}, \quad \frac{w_m}{Q} \equiv \frac{M}{Q} \tag{3.12}
\]

and therefore \( Q = 2w_m r_m^d/d \) and \( M = 2w_m^2 r_m^d/d \). These characteristic scales can be given a notional interpretation as the velocity and radius (or half-width, for planar jets) associated with a uniform ‘top-hat’ velocity profile, respectively. If we also suppose that \( w' \) and \( p \) are (statistically) self-similar functions,
then it is clear that the profile coefficients do not depend on $z$. Given (3.11), our definitions of $w_m$, $r_m$ in (3.12) are consistent with $Q$ and $M$ when $c_1$ and $c_2$ are chosen such that

$$\frac{c_1^2}{dc_1} = \int_0^\infty f(\eta)\eta^{d-1}d\eta, \quad \frac{c_2^2}{dc_2} = \int_0^\infty f(\eta)^2\eta^{d-1}d\eta.$$ (3.13a,b)

The dimensionless profile coefficients are functionals (see (3.7)–(3.9)), that map spatially varying velocity profiles onto a single number. Among them, the mean dimensionless energy flux (3.10a) is particularly important, because it is a leading-order quantity, has been assigned a variety of different values by modellers in the past (see Craske & van Reeuwijk, 2015b, for details) and, as we will demonstrate in Section 4, has a profound influence on the properties of the governing integral equations. Simulations (Craske & van Reeuwijk, 2015a) and experiments (Wang & Law, 2002) of axisymmetric jets reveal that whilst $\gamma_g > \gamma_m$ and $\beta_g > 1$, the ratio $\gamma_g/\beta_g$ is approximately equal to $\gamma_m$ and has a value of around 4/3. It is therefore useful to consider and compare the value of $\gamma_m$ in planar and axisymmetric jets.

Using (3.8) the mean-flow component of the dimensionless energy flux is

$$\gamma_m[f|d] = \frac{\int_0^\infty f(\eta)^3\eta^{d-1}d\eta \int_0^\infty f(\eta)\eta^{d-1}d\eta}{\left(\int_0^\infty f(\eta)^2\eta^{d-1}d\eta\right)^2}.$$ (3.14)

Equation (3.14) provide a useful means of calculating $\gamma_m$ because it does not require knowledge of the scaling factors $c_1$ and $c_2$ to find $\gamma_m$. In other words, any profile shape described by $f$ can be inserted into (3.14), provided that the integrals exist. For Gaussian profiles, which are typically observed in experiments for both planar (Bradbury, 1965) and axisymmetric (Wang & Law, 2002) jets, $f(\eta) = \exp(-\eta^2)$ and (using (3.13a,b))

$$\bar{w} = \begin{cases} \sqrt{2}w_m\exp\left(-\frac{\pi}{2}\eta^2\right) & d = 1, \\ 2w_m\exp\left(-2\eta^2\right) & d = 2. \end{cases}$$ (3.15)

The use of (3.14) implies that

$$\gamma_m = \begin{cases} \sqrt{\frac{\pi}{3}} & d = 1, \\ \frac{4}{3} & d = 2. \end{cases}$$ (3.16)

An alternative means of determining $\gamma_m$ is to invoke an assumed relationship between turbulence and the mean flow and to solve the boundary layer equations exactly (see e.g. Rajaratnam, 1976). In particular, under the assumption of a uniform eddy viscosity, the mean velocity $\bar{w}$ in planar and axisymmetric jets takes the form $c_1\text{sech}^2(c_2\eta)$ and $c_1/(1+c_2^2\eta^2)^2$, respectively (Pope, 2000). The values of $\gamma_m$ corresponding to these profiles are displayed in Table 1. While the profiles are qualitatively similar in shape, their associated values of $\gamma_m$ differ considerably, ranging from 6/5 in a planar jet to 9/5 in an axisymmetric jet. The profile $c_1/(1+c_2^2\eta^2)^2$ is significantly more peaked that the Gaussian and therefore corresponds to a comparatively larger value of $\gamma_m$. 
Table 1 The mean dimensionless energy flux $\gamma_m$ in planar and axisymmetric jets for different velocity profiles. The functions $\text{sech}^2(\eta)$ and $1/(1 + \eta^2)^2$ correspond to the solutions of the boundary layer equations under the assumption of uniform eddy-viscosity (Pope, 2000), and are indicated with dashed lines in the final column, in comparison with the Gaussian profile, which is denoted with a solid line. The scaling parameters $c_1$ and $c_2$ are determined using (3.13)

| $d$ | $f(\eta)$ = | $\frac{\sqrt{4}}{3}$ | $\frac{6}{5}$ | $\frac{9}{5}$ |
|-----|---------------|----------------|--------------|--------------|
| 1   | $\exp(-\eta^2)$ | sech$^2(\eta)$ | $(1 + \eta^2)^{-2}$ | |
| 2   | $\frac{4}{3}$ | $-\frac{4}{3}$ | $\frac{9}{5}$ | |

The Cauchy–Schwarz inequality implies that $\gamma_m \geq 1$ when $\overline{w} \geq 0$, which corresponds physically to the fact that the rate at which mean energy is transported by a unidirectional mean flow is minimised when the velocity profile is uniform for $0 < r < r_m$ and zero elsewhere. Any deviation from uniformity in the velocity profile will increase the dimensionless energy flux. For a given profile, the dimensionless energy flux in axisymmetric jets is typically higher than it is in planar jets, because the largest velocities at the centre of the jet occupy a comparatively smaller proportion of the total lateral cross section. These properties can be seen be substituting a parameterised family of profiles into (3.14). Using

$$f(\eta|q) = \exp(-|\eta|^q),$$

as depicted in Fig. 4(a), (3.14) implies that

$$\gamma_m[f, d] = \frac{2^{2d/q}}{3^{d/q}},$$

which is illustrated in Fig. 4(b). Evidently there is a significant difference between $\gamma_m$ in planar and axisymmetric jets. Note that Gaussian profiles have the special property that $\gamma_m[f, d] = \gamma_m[f, 1]^d$.

The profile coefficient $\gamma_m$ depends quite sensitively on the behaviour of $f$ for large values of its argument, particularly when $d = 2$. This is an unfortunate property with regard to turbulence modelling, because at the edge of the jet, where the eddy-viscosity decays to zero and intermittency becomes a dominant feature (Pope, 2000), simple constant eddy-viscosity models yield predictions that are at odds with observations. The dependence of $\gamma_m$ on the velocity profile can be quantified by looking at the functional derivative of $\gamma_m$:

$$\frac{\delta \gamma_m}{\delta f} = \gamma_m \left( 3f(\eta)^2 \left( \int_0^\infty f(\eta)^3 \eta^{d-1} d\eta \right)^{-1} + \frac{dc_1}{c_1^d} - \frac{4dc_2^2 f(\eta)}{c_2^d} \right) \eta^{d-1}.$$ (3.19)
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Fig. 4. (a) Profile $f(\eta|q) = \exp(-|\eta|^q)$ for different values of $q$ and (b) the corresponding energy flux. The top-hat profile corresponds to the limit $q \to \infty$.

Fig. 5. The functional derivative of $\gamma_m$ with respect to $f$ evaluated at $f(\eta) = \exp(-\eta^2)$ for (a) planar jets and (b) axisymmetric jets.

Figure 5 plots $\delta \gamma_m / \delta f$ evaluated for a Gaussian profile $f(\eta) = \exp(-\eta^2)$. For relatively large values of $\eta$ the derivative of $\gamma_m$ with respect to the profile $f$ is either constant (planar jets), or depends linearly on $\eta$ (axisymmetric jets). Understanding how sensitively $\gamma_m$ depends on $f$ is useful for quantifying uncertainty when comparing integral models to results from simulations and for comparing different turbulence models.

The purpose of this section was to clarify the properties of the mean dimensionless energy flux $\gamma_m$ and to highlight the significant difference in $\gamma_m$ between planar and axisymmetric jets. The parameter $\gamma_m$ provides the dominant contribution to the gross dimensionless mean energy flux $\gamma_e$ and therefore warrants careful attention. In subsequent sections we will show that the value of $\gamma_m$ plays an important role in determining the dynamics of an unsteady jet. In particular the value $\gamma_m = 4/3$ corresponding to a Gaussian profile in an axisymmetric jet turns out to be special because it implies that an unsteady jet remains straight-sided and occupies a state of marginal stability.

4. Theory and analysis

In this section we study the structure of the governing equations and some of their solutions, asking what their mathematical properties imply about the physics of unsteady jets. Since we do not make an
assumption about the values of the dimensionless profile coefficients (3.7)–(3.9) *a priori*, we adopt a
generalized standpoint and find that some assumptions give rise to non-physical phenomena. On the
other hand, physically realistic values of the profile coefficients are a distinguished set that endow the
governing equations with special properties, which include the admission of solutions that are marginally
stable and straight-sided.

4.1. **Volume conservation**

The classical steady plume theory of Morton *et al.* (1956) focuses on volume conservation and momentum
conservation. It is therefore useful to derive a generalized volume conservation equation by combining
integral equations for momentum (3.4) and energy (3.5). Following Craske & van Reeuwijk (2015a),
the combination of (3.4) and (3.5) implies that

\[
\frac{1}{\gamma_g} \frac{\partial}{\partial t} \left( \frac{Q^2}{M} \right) + \frac{\partial Q}{\partial z} = 2\alpha \frac{M^{1/d}}{Q^{2/d-1}},
\]

(4.1)

where

\[
\alpha \equiv -\frac{\delta_g}{2\gamma_g} + \frac{Q^{2/d}}{M^{1+1/d}} \left( 1 - \frac{\beta_g}{\gamma_g} \right) \frac{\partial M}{\partial z}.
\]

(4.2)

Hence the entrainment coefficient \( \alpha \) can be expressed in terms of the profile coefficients appearing in
the momentum and energy conservation equations. The first term on the right hand side of (4.2) is equal
to the dimensionless production of turbulence (normalised by \( \gamma_g \)), while the second term relates to the
advective acceleration of the flow and is identically zero in the steady state, for which \( \partial_t M = 0 \). Indeed,
in the steady state, predictions from (4.1) and (4.2) are unaffected by the value of \( \beta_g \) and \( \gamma_g \). For further
details the reader is referred to Craske & van Reeuwijk (2015a) and van Reeuwijk & Craske (2015).

Equation (4.1) generalizes volume conservation equations that can be derived for unsteady jets by
using the continuity equation directly. The reason for this is that when integrating the continuity equation,
previous models (e.g. Delichatsios, 1979; Yu, 1990; Scase *et al.*, 2006) have had to make assumptions
about the shape of the velocity profile and the dependence that \( \alpha \) has on the flow. In the case of Scase
*et al.* (2006), an integral volume conservation equation was rigorously derived, but was restricted to
unsteady jets with top-hat velocity profiles, such that \( \gamma_g = 1 \) and \( \beta_g = 1 \). Obtaining (4.1) via integral
conservation equations for momentum and energy allows one to understand the physical origin of the
prefactor \( 1/\gamma_g \) and the dependence of \( \alpha \) on the ratio \( \gamma_g/\beta_g \), characterizing the magnitude of the mean
energy flux in the jet.

In the steady state (4.1) reduces to (2.1a), where \( \alpha_0 \equiv -\delta_g/(2\gamma_g) \) is the steady-state entrainment
coefficient (note that \( \partial_t M = 0 \) in (4.2) in the steady state); hence the solution to the steady jet equations
(2.1a,b) in the far field is

\[
Q = \left( \frac{4\alpha_0}{d} \right)^{d/2} \frac{M^{1/2}}{z}, \quad d = 1,
\]

\[
\left\{ \begin{array}{ll}
2\alpha_0 M^{1/2}z, & d = 2.
\end{array} \right.
\]

(4.3)

where the solution in the planar \( (d = 1) \) case is equivalent to the jet solution discussed in van den Bremer
& Hunt (2014).
Following Craske & van Reeuwijk (2015b), one can ask whether the characteristic area of the plume,

\[ \frac{2r_m^d}{d} \equiv \frac{Q^2}{M} = \left( \frac{4\alpha_0}{d\gamma_g} \right)^d, \]  

(4.4)
is affected by unsteady source conditions. Using (4.1), (4.2) and (4.3), we find that

\[ \frac{1}{\gamma_g} \frac{\partial}{\partial t} \left( \frac{Q^2}{M} \right) = \left( \frac{4\alpha_0}{d\gamma_g} \right)^{d/2} \left( \frac{3}{4} - \frac{\beta_g}{\gamma_g} \right) \frac{2}{M^{1/2}} \frac{\partial M}{\partial z}. \]  

(4.5)

Therefore, the area, or lateral extent, of planar and axisymmetric jets is invariant under temporal variation in source conditions if and only if \( \gamma_g/\beta_g = 4/3 \). Hence the value of \( \gamma_m \) provides a first indication of whether a jet’s area will be affected by unsteadiness. As identified by Craske & van Reeuwijk (2015b), Gaussian axisymmetric jets remain straight-sided, which contrasts with the behaviour of top-hat jets, whose area can change with respect to time. These properties also underpin the crucial differences between the models of Morton et al. (1956) and Priestley & Ball (1955), as discussed in Morton (1971) and van Reeuwijk & Craske (2015).

### 4.2. Time-similarity solutions

For source conditions that have a power-law dependence on time, Scase et al. (2006) demonstrated that the governing unsteady integral equations for top-hat velocity profiles admit similarity solutions (with respect to \( z \) and \( t \)). These solutions were subsequently generalized to arbitrary velocity and buoyancy profiles by Craske & van Reeuwijk (2016), which made evident how assumptions about the internal properties of plumes manifest themselves in the dynamics of unsteady plumes. In these similarity solutions the power-law dependence of the flow on the longitudinal coordinate \( z \) is independent of time. Here we restrict our analysis to jets, but extend the analysis of the axisymmetric case by Craske & van Reeuwijk (2015b) to the planar case.

Similarity solutions to the governing integral equations for momentum (3.4) and mean energy (3.5) have the form

\[ Q(z, t) = c_Q z^{1+d} t^{-1}, \quad M(z, t) = c_M z^{2+d} t^{-2}. \]  

(4.6a,b)

In the absence of additional dimensional parameters, the exponents of \( z \) and \( t \) are implied by dimensional analysis. Using the fact that \( \delta_g = -2\alpha_0/\gamma_g \), the coefficients are

\[ c_Q = \frac{(2\alpha_0)^d}{\beta_g(d+2)} \left( \frac{\beta_g}{\gamma_g} - (d + 3) \right)^{-d}, \]  

(4.7)

and

\[ c_M = \frac{c_Q}{\beta_g(d+2)}. \]  

(4.8)

Table 2 summarizes the properties of the similarity solutions, evaluated for planar and axisymmetric jets. As indicated in the final column, the characteristic velocity \( w_m \) does not depend on the profile
coefficients. However, the spreading-rate of the jet, corresponding to the penultimate column of Table 2, is affected by the ratio \( \gamma_g / \beta_g \).

If we neglect turbulent transport and pressure, and therefore assume that \( \gamma_g / \beta_g = \gamma_m \), then, using (3.16) and (4.4), we find that for Gaussian velocity profiles

\[
\frac{2r_m^d M_0}{Q_0^d d} = \begin{cases} \left(6\sqrt{3} - 8\right)^{-1} \approx 0.418 < 1 & d = 1, \\ 1 & d = 2, \end{cases}
\]

where the functions \( Q_0 \) and \( M_0 \) correspond to \( Q \) and \( M \) in a steady state. Equation (4.9) implies that the radius of the (Gaussian) planar jet is reduced relative to its steady-state value. For top-hat profiles \( \gamma_m = 1 \), and the predictions in Table 2 are in agreement with the reduction in jet radius that was predicted by Scase et al. (2006).

At a given longitudinal location, the behaviour of the radius of an unsteady jet depends on the relative balance of the integral momentum \( Q \) and the integral energy \( M \), because \( r_m^d \propto Q^2 / M \). Hence, the behaviour of the radius in an unsteady jet depends on the ratio of the energy flux to the momentum flux, which is equal to \( \gamma_g \gamma_m / \beta_g \). For the time-similarity solutions defined above, both the momentum flux \( \beta_g M \) and the energy flux \( \gamma_g M^2 / Q \) increase as \( z \) increases, the divergence resulting in the reduction of \( Q \) and \( M \) with respect to time. When \( \gamma_g / \beta_g = 4/3 \) these transport processes are balanced, and the jet radius retains its steady-state radius. When \( \gamma_g / \beta_g < 4/3 \), however, the divergence of the energy flux is relatively weak, which leads to a rise in \( M \) and therefore a relative reduction in \( r_m \).

An alternative perspective of the analysis presented above is to consider volume conservation, noting that the generalized entrainment coefficient (4.2) will not be equal to \( \alpha_0 \) in the case of the time-similarity solutions, because \( \partial_t M \neq 0 \). Using \( \delta_k = -2\gamma_k \alpha_0 \) and the solutions (4.6a,b), we find that

\[
\alpha = \alpha_0 \left( \frac{d + 1}{(2d + 4)\beta_g / \gamma_g - (3 + d)} \right) = \begin{cases} \alpha_0 \left( \frac{2}{3\sqrt{3} - 4} \right) \approx 1.67\alpha_0, & d = 1, \gamma_m = \sqrt{4/3}, \\ \frac{3\alpha_0}{3\alpha_0}, & d = 2, \gamma_m = 4/3. \end{cases}
\]
In the unsteady axisymmetric jet the entrainment coefficient increases to three times its steady state value, which ensures that the jet’s radius is not affected by unsteadiness. In the case of the planar jet, however, the entrainment coefficient increases to approximately $1.7\alpha_0$, which proves insufficient in preventing the jet from becoming narrower. These conclusions do not, however, account for the effects of turbulence and pressure, which are fields that may play an important role in determining the behaviour of planar jets in practice.

4.3. Perturbation analysis

In reality turbulent jets appear to have velocity profiles that are approximately Gaussian in form, rather than top-hat in form (see e.g. Wang & Law, 2002). It is nevertheless useful to reason about why top-hat velocity profiles are not found in nature and why a fully developed turbulent flow might choose to organize itself in a particular way. From a pragmatic viewpoint, the integration of top-hat unsteady jet models turns out to be extremely difficult; without regularization, Scase & Hewitt (2012) found that top-hat jet models are ill-posed, giving rise to the unbounded downstream growth of wave modes of arbitrarily small length. This behaviour prevents one from obtaining converged numerical approximations, because a refinement of the computational grid admits shorter, more rapidly growing, wave modes. It was subsequently demonstrated that such pathological behaviour is an artefact of the modelling assumptions (Craske & van Reeuwijk, 2015b). Physically realistic assumptions about the jet’s energy budget, as discussed in Section 3.3, lead to well-posed unsteady integral models.

Following the method used by Scase & Hewitt (2012), Craske & van Reeuwijk (2015b) determined the response of the generalized axisymmetric jet equations to harmonic perturbations applied to the source. In doing so they were able to deduce that velocity profiles for which $\gamma_g/\beta_g < 4/3$ result in an ill-posed system. Likewise, one supposes that a fully developed turbulent flow would not evolve into a state whose mean-flow statistics are unstable to infinitesimal perturbations. Such a viewpoint was proposed by Malkus (1956) in seeking marginally stable mean-flow solutions to turbulent channel flow, and led to the conjecture that the flow realized in practice is that which maximises total viscous dissipation (see Bertram, 2015, for further details). Analysis of a generalized system of integral equations for unsteady jets therefore tells us something useful about the steady state.

Here, we consider a harmonic perturbation to the steady mean flow of frequency $\sigma$, and generalize the perturbation analysis of Craske & van Reeuwijk (2015b) to include planar jets. To this end, it proves convenient to introduce the dimensionless variables

$$
\xi \equiv \frac{bz}{\beta_g w_m} = \frac{b\sigma}{\beta_g M_0^{1/2}} \left( \frac{2\alpha_0 b}{1 - b} \right)^{1/b-1} \varepsilon^{1/b}, \quad \tau \equiv \sigma t, \quad (4.11a, b)
$$

where $b \equiv 2/(d + 2)$ is a constant and $M_0$ is the underlying steady-state momentum flux. Following Scase & Hewitt (2012) and Craske & van Reeuwijk (2015b), we introduce a series expansion for $Q(\xi, \tau)$ and $M(\xi, \tau)$ in terms of a small parameter $\epsilon$:

$$
Q = Q_0 \left( 1 + \epsilon Q_1 + \epsilon^2 Q_2 + \ldots \right), \quad (4.12)
$$

$$
M = M_0 \left( 1 + \epsilon M_1 + \epsilon^2 M_2 + \ldots \right). \quad (4.13)
$$
The governing equation for the leading-order component of the perturbations $Q_1$ and $M_1$, can be expressed as

$$
\left( \frac{\partial}{\partial \tau} + \begin{bmatrix} 0 & 1 \\ -\gamma_s/\beta_s & -\gamma_s/\beta_s \end{bmatrix} \frac{\partial}{\partial \zeta} - \frac{b}{\zeta} \begin{bmatrix} 0 & -\gamma_s/2 \beta_s \\ -\gamma_s/\beta_s & -\gamma_s/2 \beta_s \end{bmatrix} \right) \begin{pmatrix} Q_1 \\ M_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.14)
$$

Excepting its dependence on $b$ and the more general parameter $\gamma_s/\beta_s$ in place of $\gamma_m$, (4.14) is identical to the linearized equation obtained by Craske & van Reeuwijk (2015b) for axisymmetric jets ($b = 1/2$).

Decomposing the dependent variables into Fourier modes, Craske & van Reeuwijk (2015b) assumed a solution of the form

$$
\begin{pmatrix} Q_1 \\ M_1 \end{pmatrix} = \begin{pmatrix} \hat{Q}_1(\zeta) \\ \hat{M}_1(\zeta) \end{pmatrix} \exp(i\tau). \quad (4.15)
$$

Substitution of (4.15) into (4.14) shows that

$$
\frac{d^2 \hat{M}_1}{d\zeta^2} + \left( 2i + \frac{b}{\zeta} \right) \frac{d \hat{M}_1}{d\zeta} + \left( \frac{ib}{2\zeta} - \frac{\beta_s}{\gamma_s} \right) \hat{M}_1 = 0. \quad (4.16)
$$

By transforming the dependent and independent variable according to $\hat{M}_* = \hat{M}_1 \exp(i\zeta(1 + \phi))$ and $\zeta_* = 2\phi i \zeta$, (4.16) can be expressed as

$$
\zeta_* \frac{d^2 \hat{M}_*}{d\zeta_*^2} + (b - \zeta_*) \frac{d \hat{M}_*}{d\zeta_*} - a \hat{M}_* = 0, \quad (4.17)
$$

where

$$
a \equiv b \left( \frac{1}{2} + \frac{1}{4\phi} \right), \quad \phi \equiv \sqrt{1 - \beta_s/\gamma_s}. \quad (4.18a,b)
$$

Equation (4.17) is a confluent hypergeometric equation, whose solutions are well known and documented extensively in Abramowitz & Stegun (1970, Section 13, p. 504). The limiting form of the solution for large $\zeta$ is (Abramowitz & Stegun, 1970, p. 508)

$$
\hat{M}_*(\zeta) \sim c_1 \exp \left[ i\zeta (\phi - 1) \right] \frac{\Gamma(b)}{\Gamma(a(\phi))} (2\phi i \zeta)^{a(\phi)-b}; \quad (4.19)
$$

hence the amplitude of the perturbations grow in the longitudinal (downstream) direction according to

$$
a(\phi) - b = \frac{2}{d + 2} \left( \frac{1}{4\phi} - \frac{1}{2} \right). \quad (4.20)
$$

In terms of the spatial coordinate $z$, noting that $\zeta \propto z^{d/2 + 1}$, it is perhaps surprising that (4.20) implies that the rate at which the perturbations grow in physical space is independent of $d$ and, therefore, not directly affected by the geometry of the flow. The growth rate is determined entirely by $\phi \equiv \sqrt{1 - \beta_s/\gamma_s}$, which depends on the relative magnitude of the energy flux and momentum flux in the jet. Hence the geometry
Fig. 6. The response to source perturbations of a planar jet with different ratios of the dimensionless energy flux to momentum flux $\gamma_g/\beta_g$. The continuous lines correspond to an exact solution of (4.17), while the dashed lines (only visible for $\zeta^{2/3} < 3$) correspond to the asymptotic solution (4.19) and the thick line to the modulus of (4.19). Note that $\zeta^{2/3} \propto z$ in a planar jet.

of the flow influences the growth rate indirectly via $\gamma_g/\beta_g$. When $1 \leq \gamma_g/\beta_g < 4/3$ the perturbations grow with respect to $\zeta$ and when $4/3 < \gamma_g/\beta_g$ the perturbations decay. As demonstrated by Craske & van Reeuwijk (2015b), top-hat velocity profiles correspond to the limit $\gamma_g/\beta_g \to 1$ and result in exponential growth. In contrast, the limit $\gamma_g/\beta_g \to 4/3$, which is equivalent to the mean velocity of an axisymmetric jet having a Gaussian profile, corresponds to a single harmonic of constant amplitude; hence profiles for which $\gamma_g/\beta_g = 4/3$ are marginally stable. The amplitude of the perturbed momentum flux in a planar jet is plotted for various values of $\gamma_g/\beta_g$ in Fig. 6.

In the light of observations that $\gamma_g/\beta_g = 4/3$ in axisymmetric jets (Craske & van Reeuwijk, 2015a), it would appear that from an integral perspective, turbulence organizes itself to produce a mean field that is marginally stable to perturbations, as described by Malkus (1956) for channel flow. However, if one were to assume that $\gamma_g/\beta_g < 4/3$ in planar jets, due to the fact that $\gamma_m = \sqrt{4/3} < 4/3$, based on the mean flow, then one would conclude that the planar jet is statistically unstable. One is therefore led to speculate whether subtle differences between the turbulence field in planar compared with axisymmetric jets compensate for the lower mean energy flux in the former, to ensure that $\gamma_g/\beta_g = 4/3$.

4.4. Characteristic curves

While the previous section described how the shape of the mean velocity profile in a turbulent jet determines the flow’s response to infinitesimal perturbations, in this section we consider the characteristic curves of the system and examine the differences between stable and unstable flow configurations in more detail. Using the momentum–energy formulation, we express the governing equation for unsteady jets as

$$
\frac{\partial}{\partial t} \left( \frac{Q}{M} \right) + \left[ \begin{array}{cc} 0 & \frac{\beta_g}{2\gamma_g M Q} \\ -\gamma_g M^2 \frac{\beta_g}{Q^2} & 2\gamma_g \frac{M}{Q} \end{array} \right] \frac{\partial}{\partial z} \left( \frac{Q}{M} \right) = \left( 0 \right) \frac{M^{2+1/d}}{Q^{1+2/d}}. \tag{4.21}
$$
Table 3 The eigenvalues $\lambda_1$ and $\lambda_2$ associated with planar and axisymmetric jets with Gaussian velocity profiles, under the assumption that $\gamma_g = \gamma_m$ and $\beta_g = 1$.

| $d$ | $\bar{w}/w_m$ | $\gamma_m$ | $\phi$ | $\lambda_1$ | $\lambda_2$ |
|-----|---------------|------------|--------|-------------|-------------|
| 1   | $\sqrt{2} \exp\left( -\frac{\pi r^2}{2 r_m^2} \right)$ | $\frac{4}{3}$ | $\sqrt[3]{\frac{1}{3}}$ | $1 + \frac{\sqrt{3}}{3}$ | |
| 2   | $2 \exp\left( -2 \frac{r^2}{r_m^2} \right)$ | $\frac{4}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | 2 |

Values of $\lambda$ satisfying

$$\begin{bmatrix} -\lambda & \frac{\beta_g Q}{M} \\ -\gamma_g \frac{M}{Q} & 2\gamma_g - \lambda \end{bmatrix} = 0,$$

(4.22)

Correspond to the eigenvalues, or the dimensionless velocity, associated with characteristic curves; hence

$$\frac{\lambda}{\beta_g} = \frac{\gamma_g}{\beta_g} (1 \pm \phi).$$

(4.23)

In general, when $\gamma_g/\beta_g \neq 1$ the system is hyperbolic, comprising two independent families of characteristic curves with eigenvalues $\lambda_1$ and $\lambda_2 > \lambda_1$. The special case for which $\gamma_g/\beta_g = 1$ is degenerate because the eigenvalues, and therefore the characteristic curves, coincide. In that case the eigenvectors of the system are no longer linearly independent and the system becomes parabolic (see e.g. Whitham, 1974), as described in the context of unsteady plumes by Scase et al. (2009, Appendix A). Physically, the degeneracy requires that the dimensionless mean energy flux is equal to the dimensionless momentum flux, so that $\gamma_g/\beta_g = 1$. For this to be the case, the turbulent transport and pressure work in the jet would need to be zero and the mean velocity profile would need to have a top-hat form. Experiments (Wang & Law, 2002) and simulations (Craske & van Reeuwijk, 2015a) of spatially developing jets, in addition to the stability analysis described in the previous section, suggest that the integral behaviour of real jets is hyperbolic. Under the assumption that $\gamma_g = \gamma_m$ and $\beta_g = 1$, the eigenvalues corresponding to $\gamma_m = \sqrt{4/3}$ (planar jets) and $\gamma_m = 4/3$ (axisymmetric jets) are provided in Table 3.

Along characteristic curves, the derivatives of two quasi-invariant quantities can be decoupled:

$$\frac{dY_i}{dr} = \delta_i \frac{M_0^{2+1/d}}{Q^{1+2/d + \gamma_g/\gamma}}.$$  

(4.24)
Following a step change in the source momentum flux, a slow ($\lambda_1$) and fast ($\lambda_2$) characteristic curve bound a region $S$, as shown in Fig. 7, in which the dynamics of the jet differ from those associated with the steady state. Either side of the characteristic curves, in regions $A$ (after the step change) and $B$ (before the step change), the system is unaffected by the unsteady source conditions and assumes a steady-state behaviour.

4.5. Nonlinear behaviour

For problems of practical significance one hopes to understand the effect of finite changes in source conditions on a jet’s behaviour. In this and the subsequent section we therefore consider nonlinear effects by assuming that the jet’s source momentum flux is subjected to a finite step change. The analysis allows one to predict the propagation speed of finite disturbances in the jet and (see Section 4.6) to deduce information about entrainment and turbulence production across finite step changes in the flow.

To simplify the analysis we consider the homogeneous system, $dY_j/dt = 0$, along characteristic curves. We therefore neglect the turbulence production term that appears on the right hand side of (4.24). Although turbulence production plays a crucial role in turbulent jets, studying the homogeneous system is worthwhile because it allows one to determine the role played by the momentum and energy flux in the jet in a simplified setting.

We start by reviewing the analysis of Craske & van Reeuwijk (2015b), before discussing the solutions and extending the analysis to deduce information about turbulence production in Section 4.6. If a discontinuity is imposed at $(z, t) = (0, 0)$ then $Q$ and $M$ will be constant in the region $S$ (see Fig. 7), bounded by fast and slow characteristic curves that emanate from $(0, 0)$. Therefore, the value of the invariants, and hence the values $Q_S$ and $M_S$, of $Q$ and $M$ in $S$, respectively, can be determined by tracing fast and slow characteristic curves to points outside $S$ at which the solution is known:

$$\frac{M_S}{(Q_S)^{\gamma g/\lambda_1}} = \frac{M_B}{(Q_B)^{\gamma g/\lambda_1}}, \quad \frac{M_S}{(Q_S)^{\gamma g/\lambda_2}} = \frac{M_A}{(Q_A)^{\gamma g/\lambda_2}}. \quad (4.26a, b)$$
Here $M_X$ and $Q_X$ denote the values of $M$ and $Q$ in a given region $X$, as indicated in Fig. 7. The solution of these equations is

$$
\frac{M_S}{M_B} = \left( \frac{M_A}{M_B} \right)^{1/2} \left( \frac{Q_A}{Q_B} \right)^{1/2}, \quad \frac{Q_S}{Q_B} = \left( \frac{M_A}{M_B} \right)^{1/2} \left( \frac{Q_A}{Q_B} \right)^{1/2}.
$$

(4.27a, b)

Without loss of generality, hereafter we will assume that $M_B = 1$. In other words, quantities such as $M_S$ and $M_A$ should be understood as the momentum flux in regions $S$ and $A$ relative to that in $B$. Starting jets therefore correspond to the limit $M_A \to \infty$. Noting that $Q_A/Q_B = Q_A = (M_A/M_B)^{1/2} = M_A^{1/2}$, because the radius is constant either side of the region $S$, we find that

$$
M_S = M_A^{1/4\phi+1/2}, \quad Q_S = M_A^{1/4\phi-\phi/2+1/4};
$$

(4.28a, b)

hence

$$
\frac{w_S^m}{w_m^m} = \frac{M_S}{Q_S} = M_A^{\phi/2+1/4}, \quad \frac{r_S^m}{r_m^m} = \frac{Q_S}{M_S^2} = M_A^{(\phi/4)-\phi/2}.
$$

(4.29a, b)

If the total energy flux and momentum flux are such that $\gamma_f/\beta_f = 4/3$, then $\phi = 1/2$ and region $S$ is indistinguishable from region $A$. Along the fastest characteristic, the velocity $w_m$ undergoes a step-change from $w_m^A$ to $w_m^B$. For a positive step change $w_m^A > w_m^B$, the fastest characteristic is a compression wave, while for a negative step change $w_m^A < w_m^B$, the fastest characteristic is a rarefaction wave. At a compression wave, faster moving fluid precedes slower fluid and it is necessary to impose a conservation relation across the discontinuity to determine a unique speed of the wave, which will be discussed in the next section.

When $\gamma_f/\beta_f < 4/3$, we find that $\phi < 1/2$ and (4.29a) predicts two compression waves ($w_m^A > w_m^B > w_m^S$) or two rarefaction waves ($w_m^A < w_m^S < w_m^B$), depending on the nature of the unsteady forcing at the source. If we were to neglect turbulent transport and assume that $\gamma_f/\beta_f = \gamma_m = \sqrt{4/3}$ for a planar jet with a Gaussian velocity profile (recall from Section 3.3 that $\gamma_f = \gamma_m$ and $\beta_f = 1$ if we neglect turbulent transport), then

$$
\log \left( \frac{w_m^S}{w_m^m} \right) = \sqrt{\frac{3}{16}} \log M_A \approx 0.433 \log M_A,
$$

(4.30)

$$
\log \left( \frac{r_m^S}{r_m^m} \right) = \frac{1}{4} \left( \frac{3 - 2\sqrt{3}}{1 - \sqrt{3}} \right) \log M_A \approx 0.158 \log M_A.
$$

(4.31)

The extent to which the planar jet departs from its steady-state behaviour in $S$ therefore depends on the magnitude of the step change, i.e. on $M_A$. If the source momentum flux were to double ($M_A = 2$), the predictions imply that the radius of the planar jet in $S$ would increase by a factor of $2^{0.158} \approx 1.116$ and the velocity in $S$ would increase by a factor of $2^{0.433} \approx 1.350 < \sqrt{2}$, where $M_A^{1/2} = \sqrt{2}$ would be the velocity in region $A$. To discern significant changes in the radius or velocity of the jet in region $S$, one would therefore need to impose a relatively large step change.

---

Note that the fluid we are modelling is strictly incompressible. The notion of a compression wave here refers to a feature of the hyperbolic integral equations rather than the local behaviour of the flow.
The arrangement and type of characteristic curves arising from arbitrary source conditions is illustrated in Fig. 8 for three values of the ratio $\gamma_g/\beta_g$. In Fig. 8(a) $\gamma_g/\beta_g = \sqrt{4/3}$, which corresponds to a planar jet with a Gaussian velocity profile and no transport by turbulence. As described above, this system comprises either two rarefaction waves when $M_A < 1$ or two compression waves when $M_A > 1$. Figure 8(c) corresponds to a case for which $\gamma_g/\beta_g > 4/3$ and shows that the system comprises a compression wave preceding a rarefaction wave when $M_A < 1$ and a rarefaction wave preceding a compression wave when $M_A > 1$. The case for which $\gamma_g/\beta_g = 4/3$ is a bifurcation point, at which the slower characteristic is effectively invisible and the faster characteristic is either a rarefaction wave ($M_A < 1$) or a compression wave ($M_A > 1$). The precise determination of the location of the compression waves is shown in Fig. 8 and relies on solving Rankine–Hugoniot jump conditions, as discussed in the next section.

4.6. Jump conditions

Along characteristic curves the governing equations admit discontinuities. In spite of these discontinuities, it is possible to obtain weak solutions to the equations. However, to find a unique solution to the given problem, it is necessary to incorporate an additional constraint. Physically, the constraint corresponds to a conservation law across the discontinuity and is commonly referred to as a ‘jump condition’.

In this section we impose momentum conservation across discontinuities in the solution to deduce the speed of travelling disturbances. This is useful from a practical point of view because it allows one to predict the time taken for contaminants to reach a particular location, for example. By imposing momentum conservation we relinquish energy conservation across discontinuities, and find that a finite amount of the energy associated with the mean flow is converted into turbulence kinetic energy. The amount of energy that is converted depends on the ratio of the dimensionless energy flux to the dimensionless...
momentum flux $\gamma_e/\beta_e$. For the value $\gamma_e = \beta_e = 4/3$ associated with marginal stability we find that turbulence production is minimized among stable flow configurations.

In the case of a compression wave, the characteristic velocity undergoes a step reduction (in $z$) across the characteristic curve. We would therefore like to know the velocity, lying somewhere between that occurring upstream and downstream of the discontinuity, that corresponds to the propagation speed of the characteristic curve. To this end, consider a single step change in the variable $X$ of magnitude $[X]_2 \equiv X_S - X_B$, propagating at an unknown velocity $\lambda M_A/Q_A$ along the fastest characteristic. Since the flow is unbounded and not subjected to any forces, we assume that it conserves momentum in an integral sense.

Assuming that the solution is continuously differentiable either side of the jump across the fastest characteristic (associated with $\lambda_2$), momentum conservation (3.4) in the region containing the step change is satisfied if

$$\int_{z^*}^{z_S} \frac{\partial Q}{\partial t} dz + \int_{z_B}^{z^*} \frac{\partial Q}{\partial t} dz - \beta_e [M]_2 = 0, \quad (4.32)$$

where $z_B > z^* > z_S$ and $[M]_2 \equiv M_S - M_B = M_S - 1$. Letting $z_S \to z^*$ from below and $z_B \to z^*$ from above,

$$\lambda_2 = \frac{Q_A}{M_A} \frac{dz^*}{dt} = \beta_e \frac{Q_A [M]_2}{M_A [Q]_2} = \frac{\beta_e}{M_A^{1/2}} \left( \frac{M_S - 1}{Q_S - 1} \right). \quad (4.33)$$

Similarly, letting $[M]_1 \equiv M_A - M_S$, momentum conservation across the slower characteristic implies that

$$\lambda_1 = \frac{Q_A}{M_A} \frac{dz^*}{dt} = \beta_e \frac{Q_A [M]_1}{M_A [Q]_1} = \frac{\beta_e}{M_A^{1/2}} \left( \frac{M_A - M_S}{M_A^{1/2} - Q_S} \right), \quad (4.34)$$

where we have utilized the fact that $Q_A = M_A^{1/2}$. For further details regarding the determination of jump conditions, we refer the reader to Whitham (1974).

Substitution for $Q_A$ from (4.28b) results in

$$\lambda_1 = \frac{\beta_e}{M_A^{1/2}} \left( \frac{M_A - M_A^{1/(4\phi)+(1/2)}}{M_A^{1/2} - M_A^{1/(4\phi)-\phi/2+1/4}} \right), \quad \lambda_2 = \frac{\beta_e}{M_A^{1/2}} \left( \frac{M_A^{1/(4\phi)+(1/2)} - 1}{M_A^{1/(4\phi)-\phi/2+1/4} - 1} \right). \quad (4.35)$$

Noteworthy is the fact that when $\beta_e = 1$ and $\phi = 1/2$, the latter corresponding to an axisymmetric jet with a Gaussian velocity profile,

$$\lambda_2 = \frac{1}{M_A^{1/2}} \left( \frac{M_A - 1}{M_A^{1/2} - 1} \right), \quad (4.36)$$

and when $M_A \to 1$, $\lambda_2 \to 2$, as predicted by (4.23). Moreover, when $M_A \to \infty$, $\lambda_2 \to 1$ which implies that a starting jet propagates at exactly the characteristic velocity associated with $M_A$, a prediction that is broadly in agreement with the theory and observations made by Turner (1964) and, notably, the theory for planar jets developed by Ruban & Vonatsos (2008). In contrast, when $\phi < 1/2$, $\lambda_2/\lambda_1 \to 1$ as
$M_A \to \infty$ and the two shocks coincide. However, in that case, relative to the velocity $w_m^A$, the velocity of the shocks approaches zero (i.e. both $\lambda_1 \to 0$ and $\lambda_2 \to 0$) and is accompanied by an increase in the area of the jet between the characteristic curves. Conversely, if $\phi > 1/2$, the velocity of the fastest characteristic relative to $w_m^A$ increases without bound. This behaviour is inconsistent with experimental observations and theoretical predictions and suggests that real jets, both planar and axisymmetric, might adjust themselves to occupy the special state for which $\gamma_g/\beta_g = 4/3$, as speculated at the end of Section 4.3 and shown in Fig. 8(b).

Having imposed the physically motivated constraint of momentum conservation over discontinuities in the flow, one can determine the extent to which energy is lost to turbulence across discontinuities. Figure 9 illustrates the behaviour of the flow following a step increase in the source momentum flux for planar jets (a) and axisymmetric jets (b), under the assumption that $\gamma_g/\beta_g = \gamma_m = \sqrt{4/3}$ and $\gamma_g/\beta_g = \gamma_m = 4/3$, respectively. In the absence of internal and boundary forces the jet conserves momentum across the shocks but does not conserve mean-flow energy; energy is lost by the mean flow and converted into turbulence kinetic energy. Moreover, since the volume flux is assumed to be constant in regions $A$, $S$ and $B$ (see Fig. 7) for the homogeneous problem, there must exist either positive or negative entrainment into the jet at the characteristic curve to balance the step change in $Q$, as illustrated schematically in Fig. 9.

To quantify the turbulence production along each characteristic, we integrate the integral energy equation (3.5) over the discontinuity, whence

$$
\lim_{\gamma_g z^* = \gamma_m z} \int_{\gamma_g z^*}^{\gamma_m z} \frac{\partial M}{\partial t} \, dz + \lim_{\gamma_g z^* = \gamma_m z} \int_{\gamma_g z^*}^{\gamma_m z} \frac{\partial M}{\partial t} \, dz - \gamma_g \left[ \frac{M^2}{Q} \right]_j = \Delta_j \gamma_g \left[ \frac{M^2}{Q} \right]_j, \quad (4.37)
$$

where $j = 1, 2$, and $\Delta_j < 0$ accounts for the energy lost to turbulence relative to the mean energy flux in the jet. Consequently we find that

$$
\lambda_j = \frac{Q_A}{M_A} \frac{dz^*}{dr} = \gamma_g (1 + \Delta_j) \frac{Q_A}{M_A} \left[ \frac{M^2}{Q} \right]_j, \quad j = 1, 2. \quad (4.38)
$$
Fig. 10. The discrete quantity of turbulence production $\Delta_1$ and $\Delta_2$ relative to the energy flux in the jet. (a) The production $\Delta_1$ across the slower characteristic and (b) the production $\Delta_2$ across the faster characteristic. The arrow points in the direction of increasing $M_A = 10^1, 10^2, 10^3, 10^4$ and the dashed line denotes the limiting case $M_A \to \infty$. The circles denote points on the curves for which $\gamma_g/\beta_g = 4/3$.

With (4.33) and (4.34), (4.38) can be solved for $\Delta_1$ and $\Delta_2$, to indicate the quantity of energy lost to turbulence over the step change (cf. turbulence production in a hydraulic jump):

$$
\Delta_1 = \frac{(M_1^{1/4\phi+1/2} - M_A)^2}{(M_A^{1/4\phi+1/2} - M_1^{1/4\phi+1/2}) (M_A^{1/4\phi-\phi/2+1/4} - M_1^{1/4\phi-\phi/2+1/4})} \left( \frac{\beta_g}{\gamma_g} \right) - 1, \quad (4.39)
$$

$$
\Delta_2 = \frac{(1 - M_A^{1/4\phi+1/2})^2}{(1 - M_A^{1/4\phi+1/2})(1 - M_A^{1/4\phi-\phi/2+1/4})} \left( \frac{\beta_g}{\gamma_g} \right) - 1. \quad (4.40)
$$

In general, the turbulence production associated with each characteristic in a starting jet ($M_A \to \infty$) is

$$
\lim_{M_A \to \infty} \Delta_1 = \frac{\beta_g}{\gamma_g} - 1, \quad \phi < 1, \quad (4.41)
$$

$$
\lim_{M_A \to \infty} \Delta_2 = \frac{\beta_g}{\gamma_g} - 1, \quad (4.42)
$$

the first case being restricted to $\phi \leq 1/2$, because when $\phi \geq 1/2$ the slower characteristic is not a compression wave. Figure 10(a and b) display how the discrete turbulence production is affected by the magnitude of the positive step change in $M_A > 1$ imposed at the source and the relative balance of energy flux and momentum flux in the jet. In comparison with the balanced state for which $\gamma_g/\beta_g = 4/3$, when $1 < \gamma_g/\beta_g < 4/3$, the jet produces less turbulence across the fastest characteristic, while $4/3 < \gamma_g/\beta_g$ implies greater turbulence production across the fastest characteristic. In the case for which $\gamma_g/\beta_g < 4/3$ additional turbulence production occurs across the compression wave that emerges on the slower characteristic. However, in the limit $\gamma_g/\beta_g \to 1$ the turbulence production across each characteristic is equal to zero.

In the context of the conclusions that were made about stability in Section 4.3 and Fig. 10(b), the results of this section reveal that the case for which $\gamma_g/\beta_g = 4/3$ minimizes turbulence production across
the jump among stable velocity profiles ($\gamma_g/\beta_g \geq 4/3$). That the flow would seek the most energetically favourable stable configuration is intuitively appealing and raises questions regarding the value of $\gamma_g/\beta_g$ in planar jets.

5. Discussion

As discussed in Section 3.4, the mean dimensionless energy flux $\gamma_m$ in planar jets is significantly less than it is in axisymmetric jets, due to their geometrical differences. In determining the behaviour of unsteady jets the mean dimensionless energy flux $\gamma_m$ plays a leading role, because it dominates the ratio $\gamma_g/\beta_g$ and therefore quantifies the magnitude of the gross mean energy flux (inclusive of turbulent transport) relative to the gross momentum flux. Calculation of $\gamma_m$ for a Gaussian profile in an axisymmetric jet reveals that $\gamma_m = 4/3$, which coincides with the canonical state for which the jet is marginally stable to perturbations applied to the mean flow at the source. In contrast, in planar jets we find that the dimensionless mean energy flux $\gamma_m = \sqrt{4/3}$ for a Gaussian profile is lower than it is in axisymmetric jets, and one is led to believe that planar jets are unstable to source perturbations and have a horizontal extent that increases following a step increase in their source momentum flux, as depicted in Fig. 9(a).

Inviting future experimental or numerical investigations, we hypothesize that in practice $\gamma_g/\beta_g = 4/3$ in both planar and axisymmetric jets, and that planar jets do remain straight sided and are marginally stable to source perturbations. This hypothesis implies that the difference between $\gamma_m = \sqrt{4/3}$ and $\gamma_m = 4/3$, in planar jets and axisymmetric jets, respectively, is accounted for by a difference in the organization of turbulence and pressure fields. An alternative hypothesis is that while $\gamma_g/\beta_g < 4/3$ in planar jets in the steady state, large perturbations from the steady state result in the profile coefficients adjusting themselves in a way that inhibits large deviations from straight-sidedness and guarantees stability of the mean flow. In either case, the observation that $\gamma_m$ is different in planar and axisymmetric jets suggests that there is a fundamental difference between planar and axisymmetric jets. However, Gutmark & Wygnanski (1976) and Pope (2000) note that the profile shapes and the magnitude of the Reynolds stresses are comparable in planar jets compared with axisymmetric jets. To investigate this subtle matter further, our recommendation is that experiments or simulations of unsteady planar jets should be undertaken, alongside a detailed analysis of their statistical stability.

The tendency of an axisymmetric jet to develop until a balanced, statistically stable state is reached is evident in Fig. 11, which plots a state space described by the quantities $\beta_g$, $\gamma_g$ and $\delta_g$. The points on the figure were obtained by evaluating the corresponding profile coefficients using the results of a direct numerical simulation of the Navier–Stokes equations (van Reeuwijk et al., 2016). Close to the source, the jet has a top hat velocity profile and $\beta_g = 1$, $\gamma_g = 1$ and $\delta_g = 0$. Further away from the source, the jet evolves towards a state in which $\gamma_g/\beta_g = 4/3$, to within a remarkably close agreement, and $\delta_g = -2\alpha_0\gamma_g$. We label states for which $\gamma_g/\beta_g < 1$ as being non-physical because in those cases the governing integral equations (3.4)–(3.5) are elliptic (recall from Section 3.4 that $\gamma_m < 1$ is only possible if $\overline{w}$ is negative for some $r$, which means that the correspondence between $\gamma_m[f/d]$, as defined in (3.14), and the Cauchy–Schwarz inequality is no longer valid). Complementary simulations of a planar jet would provide its state space trajectory and confirm whether, like axisymmetric jets, planar jets also tend towards a distinguished state in which $\gamma_g/\beta_g = 4/3$.

6. Conclusions

The turbulent jet has a special place in fluid mechanics, due to its many applications and ability to provide generic insights into turbulence and mixing. Amongst the wide variety of turbulent jets that one might
encounter in practical situations, those issuing from line sources (the planar jet) and point source (the axisymmetric jet) emerge as canonical cases. Over the past century relatively simple integral descriptions of these flows have provided practitioners with robust estimations and allowed theoreticians to develop elegant mathematical representations that capture their leading-order properties. An understanding of turbulent jets is, furthermore, a prerequisite for the understanding of turbulent plumes, whose governing integral equations contain additional terms arising from buoyancy.

Statistically unsteady jets are significantly more difficult to model than their steady-state counterparts, because their integral behaviour depends sensitively on the assumptions that are made about the underlying radial dependence of velocity, turbulence and pressure. In contrast, such assumptions do not play an active role in the dynamics of steady-state jets and a wide variety of different assumptions yield steady-state models that are essentially indistinguishable. Consequently, in order to develop accurate unsteady jet models, it is essential that integral models are derived from first principles, rather than as extrapolations of the classical steady-state models.

Mathematics plays a crucial role in the understanding and development of unsteady jet models, allowing one to probe a large parameter space and determine the consequences of the different assumptions that can be invoked to close the governing integral equations, for both planar and axisymmetric jets. These consequences were discussed at length in Section 4 of this article and include how the assumed velocity profile determines the extent to which an unsteady jet deviates from straight-sided behaviour, its stability in response to harmonic source perturbations and the structure of its characteristic curves. In the light of the growing availability of detailed velocity measurements in jets, it is remarkable that their integral properties still provide essential insights and is testimony to the value of soluble mathematical models.

Unsteady jets are not only worth studying in their own right, for their many applications in industrial and environmental fluid mechanics, but because they provide an improved understanding of the behaviour of steady-state jets. Just as it is difficult to examine the properties of a point in space without an appreciation of the space in which it is embedded, it is difficult to understand why a steady-state jet might ‘choose’ a particular state without considering its unsteady behaviour. It this article we demonstrated that the mean energy flux in planar jets compared with axisymmetric jets is very different, and that, in the
absence of turbulence, this implies significant differences in their unsteady behaviour. Yet, consideration of the statistical stability of unsteady jets and the evolution of nonlinear jumps in the flow suggests that the turbulence and pressure fields in each case might organize themselves in a way that compensates for these differences. Numerical simulations or experiments of unsteady planar jets would therefore provide a worthwhile complement to the data that currently exists for unsteady axisymmetric jets and plumes.

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