The Discrete Gaussian model, II.
Infinite-volume scaling limit at high temperature

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Abstract

The Discrete Gaussian model is the lattice Gaussian free field conditioned to be integer-valued. In two dimensions, at sufficiently high temperature, we show that the scaling limit of the infinite-volume gradient Gibbs state with zero mean is a multiple of the Gaussian free field.

This article is the second in a series on the Discrete Gaussian model, extending the methods of the first paper by the analysis of general external fields (rather than macroscopic test functions on the torus). As a byproduct, we also obtain a scaling limit for mesoscopic test functions on the torus.

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1 Introduction and main results

This is the second article in a series on the Discrete Gaussian model, which builds on the foundation provided by the first paper [8]. The Discrete Gaussian model is the Gaussian free field conditioned to be integer-valued. Its two-dimensional version is a model for a crystal interface (in 2+1 dimensions) undergoing a roughening transition, see [16, Section 6] for a textbook treatment. We refer to our first paper [8] for a more extensive introduction and discussion of the literature.

1.1. Discrete Gaussian model in infinite volume. In our first paper [8], we studied the scaling limit of the Discrete Gaussian model for macroscopic test functions on the torus. In the present article, we derive the scaling limit of its infinite-volume gradient Gibbs state, as well as the scaling limit for mesoscopic test functions on the torus, which is a byproduct of the proof of the infinite-volume result. These scaling limit results are the objects of Theorems 1.1 and 1.2 below.

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The infinite-volume limit of the two-dimensional Discrete Gaussian model will be taken through weak limits with periodic boundary conditions, cf. (1.3), and we permit a general finite-range interaction \( J \) in the definition of the model. To be precise, let \( J \subset \mathbb{Z}^d \setminus \{0\} \) be finite and symmetric under reflections and lattice rotations, and define the associated normalised range-\( J \) Laplacian \( \Delta_J \) by

\[
(\Delta_J f)(x) = \frac{1}{|J|} \sum_{y \in J} (f(x + y) - f(x)),
\]

for \( f : \mathbb{Z}^d \to \mathbb{R} \), where \(|J|\) denotes the number of elements of \( J \). Acting on test functions having mean zero, \((-\Delta_J)^{-1}\) has kernel

\[
(-\Delta_J)^{-1}(x,y) \sim -\frac{1}{2\pi v_J^2} \log |x - y|, \quad \text{as } |x - y| \to \infty,
\]

where \( v_J^2 = \frac{1}{2|J|} \sum_{x \in J} x^2 \). (1.2)

We now introduce the relevant finite-volume states. Let \( \Lambda_N \) be a two-dimensional discrete torus of side length \( L^N \) for integers \( L > 1, N \geq 1 \), and fix an origin \( 0 \in \Lambda_N \). Given the above step distribution \( J \), the Discrete Gaussian model on \( \Lambda_N \) at temperature \( \beta \in (0,\infty) \) has expectation, for any \( F : (2\pi \mathbb{Z})^{\Lambda_N} \to \mathbb{R} \) with \( F(\sigma) = F(\sigma + c) \) for any constant \( c \in 2\pi \mathbb{Z} \) and such that the following series converges, defined by

\[
\langle F \rangle_{J,\beta}^{\Lambda_N} \propto \sum_{\sigma \in \Omega^{\Lambda_N}} e^{-\frac{1}{\beta}(\sigma \cdot \Delta_J \sigma)} F(\sigma) = \sum_{\sigma \in \Omega^{\Lambda_N}} e^{-\frac{1}{\beta} \sum_{x-y \in J} (\sigma_x - \sigma_y)^2} F(\sigma)
\]

where the sum over \( x - y \in J \) counts every undirected edge \( \{x, y\} \) twice and

\[
\Omega^{\Lambda_N} = \{ \sigma \in (2\pi \mathbb{Z})^{\Lambda_N} : \sigma_x = 0 \}.
\]

Note that, as in our first paper [8], the factors of \( 2\pi \) in the spacing of the integers in \( 2\pi \mathbb{Z} \) are convenient (but could be absorbed by rescaling \( \beta \)), and, to relate better to the Coulomb gas literature (cf. references below), we use \( \frac{1}{\beta} \) rather than \( \beta \) to denote the inverse temperature of the Discrete Gaussian model. Equivalent to considering \( \sigma \) modulo constants, one can consider the gradient field \( \eta = (\eta_e)_{e \in E} \) where \( E \) are the directed nearest-neighbour edges of \( \mathbb{Z}^d \) and \( \eta_e = \sigma_x - \sigma_y \) when \( e = (x,y) \). Known correlation inequalities imply that, for any integer \( L > 1 \) and any finite-range interaction \( J \), the weak limit of \( \langle \cdot \rangle_{J,\beta}^{\Lambda_N} \) as \( N \to \infty \) exists (modulo constants or as a gradient field), see Appendix A. For concreteness, we define the infinite-volume limit in terms tori of side lengths \( 2^N \), i.e., when \( \Lambda_N \) has side length \( 2^N \),

\[
\langle \cdot \rangle_{J,\beta}^{2^N} := \lim_{N \to \infty} \langle \cdot \rangle_{J,\beta}^{\Lambda_N}.
\]

This limit \( \langle \cdot \rangle_{J,\beta}^{2^N} \) is a translation-invariant gradient Gibbs measure and every ergodic measure \( \langle \cdot \rangle \) in its extremal decomposition has zero mean, i.e., \( \langle \eta_e \rangle = 0 \) for all \( e \in E \), also see Appendix A. For \( J = J_{un} \) the usual nearest-neighbour interaction, \( \langle \cdot \rangle_{J,\beta}^{2^N} \) is the unique ergodic gradient Gibbs measure with zero mean on account of Theorem 9.1.1 in [49]. For general \( J \), such a characterisation has not been proved.

As is well-known (see refs. below for an overview over the existing literature on the subject), in the Discrete Gaussian model, the discreteness of the spins is responsible for a phase transition between a rough (or delocalised) high-temperature phase and an ordered (or localised) low-temperature phase. Our results apply to large temperatures \( \beta \). In contrast, in the regime of small \( \beta \), a Peierls expansion yields that the Discrete Gaussian field is localised (or ‘smooth’), e.g., there actually exists an (ordinary nongradient) Gibbs measure \( \langle \cdot \rangle_{J,\beta}^{2^N} \) satisfying

\[
\langle \sigma_x \sigma_y \rangle_{J,\beta}^{2^N} - \langle \sigma_x \rangle_{J,\beta}^{2^N} \langle \sigma_y \rangle_{J,\beta}^{2^N} \leq C e^{-c|x-y|}, \quad \text{for all } x, y \text{ and } \beta < c;
\]

see also [12][46] for very precise results on the extremal behaviour in this regime.
defined above is a multiple of the Gaussian free field on \( \mathbb{R}^2 \) when \( \beta \) is large. To state this precisely, given \( f \in C_c^\infty(\mathbb{R}^2) \) with \( \int f(x) \, dx = 0 \), let \( f_\epsilon : \mathbb{Z}^2 \to \mathbb{R} \) satisfy \( \sum_{x \in \mathbb{Z}^2} f_\epsilon(x) = 0 \) and, with \( d = 2 \),

\[
\max_{0 \leq k \leq 2, x \in \mathbb{Z}^2} |(\varepsilon^{-1} \nabla)^k f_\epsilon(x)| \leq C_f \varepsilon^{1 + d/2}, \quad \text{supp } f_\epsilon \subset [-C_f \varepsilon^{-1}, C_f \varepsilon^{-1}]^d,
\]

\[
\max_{x \in \mathbb{Z}^2} |(\varepsilon^{-1} \nabla)^k f_\epsilon(x) - f(\varepsilon x)| \to 0,
\]

where \( C_f \) is a constant and \( \nabla \) is the vector of discrete gradients on \( \mathbb{Z}^2 \), see Section 1.4. For example, if \( f = \nabla g \) for some \( g \in C_c^\infty(\mathbb{R}^2) \) and \( i \in \{1, 2\} \) then one can take \( f_i(x) = \varepsilon^{d/2}(g(\varepsilon x + \varepsilon \xi_i) - g(\varepsilon x)) \). Thus the following scaling limit in particular implies that of the gradient field \( \nabla \sigma \).

We use the notation \( (u, v)_{\mathbb{Z}^2} = \sum_{x \in \mathbb{Z}^2} u(x)v(x) \) for \( u, v : \mathbb{Z}^2 \to \mathbb{R} \) square summable, \( (f, g)_{\mathbb{R}^2} = \int_{\mathbb{R}^2} f(x)g(x) \, dx \) for \( f, g : \mathbb{R}^2 \to \mathbb{R} \) square integrable, and \( \Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \) is the Laplacian on \( \mathbb{R}^2 \).

**Theorem 1.1.** Let \( J \subset \mathbb{Z}^2 \setminus \{0\} \) be any finite-range step distribution that is invariant under lattice rotations and reflections and includes the nearest-neighbour edges. Then there exists \( \beta_0(J) \in (0, \infty) \) such that for the infinite-volume Discrete Gaussian Model \( \langle \cdot \rangle_{\mathbb{Z}^2}^{J,\beta} \) at temperature \( \beta \geq \beta_0(J) \), there is \( \beta_{\text{eff}}(J, \beta) = \beta + O_J(\varepsilon^{-d}) \in (0, \infty) \) with a universal constant \( c > 0 \) (independent of \( J \)) such that for any \( f \in C_c^\infty(\mathbb{R}^2) \) with \( \int f \, dx = 0 \) and \( f_\epsilon \) as in (1.7), as \( \varepsilon \to 0 \),

\[
\log \langle e^{(f_\epsilon, \sigma)_{\mathbb{Z}^2}} \rangle_{J,\beta} \to \frac{\beta_{\text{eff}}(J, \beta)}{2\nu^2}(f, (-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}.
\]  

Theorem 1.1 superficially resembles [8, Theorem 1.1], but we emphasise that we are now considering the infinite-volume state; correspondingly the covariance on the right-hand side is now \( (-\Delta_{\mathbb{R}^2})^{-1} \) instead of \( (-\Delta_{\mathbb{T}^2})^{-1} \). The comparison below [8, Theorem 1.1] with previous results for the Discrete Gaussian model however also applies to the infinite-volume version, i.e., to Theorem 1.1 of this paper.

Theorem 1.1 can be seen as an analogue for the Discrete Gaussian model (with \( \beta \geq \beta_0(J) \)) of the Naddaf–Spencer theorem [15] which applies to strictly convex smooth gradient models. In our first paper [8] we discuss many further references concerning such models and concerning discrete height functions, and we refer to [8] for a more detailed discussion and only list here the most relevant references. For the Discrete Gaussian and XY models, we of course mention the fundamental work of Fröhlich–Spencer [31,32] as well as the more recent articles [2,35,36,43–45,50,51]. For smooth gradient models, there is a very comprehensive picture including stochastic dynamics [33–34,38] and recent developments include [8,6,9,20,22,41,47,49,52]. For the smooth but nonconvex gradient models we refer to [10,11,18,19] and in particular [13] and [4] which use the renormalisation group approach. For other discrete height functions, recent works include [17,26,27,39,42]. Our first paper (and therefore this paper as well) relies in important ways on ideas developed in [14,23,25,28,29].

As a byproduct of the proof of Theorem 1.1 we also obtain the following mesoscopic scaling limit for the Discrete Gaussian model on the torus. (Effective error bounds also follow from the proof.)

**Theorem 1.2.** Under the same assumptions as in Theorem 1.1, there exists \( L = L(J) \) such that for the Discrete Gaussian model on the torus \( \Lambda_N \) of side length \( L^N \), for any \( f \in C_c^\infty(\mathbb{R}^2) \) with \( \int f \, dx = 0 \), \( f_\epsilon \) as in (1.7), and any sequence \( \varepsilon_N > 0 \) such that \( \varepsilon_N \to 0 \) as \( N \to \infty \) while \( \varepsilon_N L^N \to \infty \),

\[
\log \langle e^{(f_\epsilon, \sigma)_{\Lambda_N}} \rangle_{J,\beta} \to \frac{\beta_{\text{eff}}(J, \beta)}{2\nu^2}(f, (-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}, \quad \text{as } N \to \infty.
\]  

Note that the assumption \( \varepsilon_N L^N \to \infty \) is necessary. Indeed, if \( \varepsilon_N \ll L^{-N} \) then the support of \( f_\epsilon \) is not a subset of \( \Lambda_N \). Moreover, if \( \varepsilon_N = L^{-N} \) the limit would correspond to the macroscopic
scaling limit considered in [8, Theorem 1.1] which is different from the right-hand side above (given in terms of \((-\Delta_{\mathbb{Z}^d})^{-1}\) rather than \((-\Delta_{\mathbb{R}^d})^{-1}\).

For some of the related open questions, we refer to our discussion in [8, Section 1.3], but mention in addition that a characterisation of the gradient Gibbs measures with finite range \(J\) as in [19, Theorem 9.1.1] for the nearest-neighbour case would be interesting.

1.3. Outline of the paper. This paper relies heavily on our first article on the Discrete Gaussian model [8], and in particular we use the set-up and notation from Section 2 and Sections 4–6 of that paper. Even though we included some reminders below, we will often refer to [8] to avoid repetitiveness.

The proofs of Theorems 1.1 and 1.2 proceed by decomposing the external field from the moment-generating function into contributions from all scales, with each contribution smooth at the respective scale. This is set up in Section 2. Then, the main technical contribution of the present paper compared to [8] is an extension of the renormalisation group map, originally defined in [8, Section 7], to allow for a scale-dependent external field. This is carried out in Section 5 after technical preparation in the preceding sections.

Different methods to extend a renormalisation group flow by observables for pointwise correlation functions in similar setups to ours were considered in [7, 15, 24, 29]. These approaches do not allow to derive the infinite-volume scaling limit as in our main result, and we expect that the approach we develop here could have applications to other models.

1.4. Notation. We use the notation \(|a| \leq O(|b|)\) or \(a = O(b)\) to denote \(|a| \leq C|b|\) for an absolute constant \(C > 0\) and \(a \sim b\) to denote that \(\lim a/b = 1\) (where the limit is clear from the context). We stress that all constants appearing below are uniform in \(\beta\) unless explicitly stated.

Throughout the paper, the dimension will be \(d = 2\), but we sometimes write \(d\) to emphasise the source of the constant 2. Let \(e_1, \ldots, e_d\) be the basis of unit vectors with nonnegative components spanning \(\mathbb{Z}^d\) or the local coordinates of \(\Lambda\), and set \(\hat{e} = \{\pm e_1, \ldots, \pm e_d\}\). For a function \(f : \mathbb{Z}^d \to \mathbb{C}\) or \(f : \Lambda_N \to \mathbb{C}\), we write \(\nabla^\mu f(x) = f(x + \mu) - f(x)\) for \(\mu \in \hat{e}\). For any multi-index \(\alpha \in \{\pm 1, \ldots, \pm d\}^n\) with \(n = |\alpha| \geq 1\), we write \(\nabla^n f = \nabla^{\alpha_1} \cdots \nabla^{\alpha_n} f\). The vector of \(n\)-th order discrete partial derivatives are denoted by

\[
\nabla^n f(x) = (\nabla^{\mu_1} \cdots \nabla^{\mu_n} f(x) : \mu_k \in \hat{e} \text{ for all } k) ,
\]

and we write \(|\nabla^n f(x)|\) for the maximum over all of its components. \(\Delta\) without subscript denotes the unnormalised nearest-neighbour Laplacian,

\[
\Delta f(x) = \sum_{\mu \in \hat{e}} (f(x + \mu) - f(x)) = \sum_{\mu \in \hat{e}} \nabla^\mu f(x) = \frac{1}{2} \sum_{\mu \in \hat{e}} \nabla^\mu \nabla^{-\mu} f(x) ,
\]

whereas \(\Delta_J\) denotes the normalised Laplacian (1.1) with finite-range step distribution \(J\).

2 Scale-dependent external fields

In this section, after briefly reviewing some aspects from the setup of our first paper [8], we proceed to describe how the proofs of the above theorems follow by amending the renormalisation group flow constructed in [8] by suitable external fields \(u = (u_j)\), which start to appear at a characteristic scale \(j = j_f\) in the renormalisation. We then proceed, assuming these fields \(u\) to have a negligible overall effect, as expressed in Theorem 2.3 below, to conclude the proofs of Theorems 1.1 and 1.2. The remaining sections will be geared towards the proof of Theorem 2.3 which appears in Section 6.

2.1. Multiscale decomposition of the field. We first briefly review a few key aspects from the setup of our previous paper [8], which will prevail here. As in Section II we denote by \(\Lambda_N\) the
discrete torus of side length $L^N$ and we will later impose that $L$ is sufficiently large, see the discussion at the end of Section 3.4 for details; the infinite volume limit will then correspond to the limit $N \to \infty$. As explained in [8] Section 2, it is convenient to work with the mass-regularised Discrete Gaussian model \( \langle \cdot \rangle_{\beta,m^2} \) and take $m^2 \downarrow 0$ in the end. This is the probability measure \( \langle \cdot \rangle_{\beta,m^2} \equiv \langle \cdot \rangle_{\beta,m^2}^{\Lambda_N} \) obtained by replacing $-\Delta_J$ by $-\Delta_J + m^2$ in (1.3) and $\Omega^\Lambda_N$ by $Z_{\beta}^{\Lambda_N}$ where $Z_{\beta} = 2\pi \beta^{-1/2} \mathbb{Z}$, i.e., dropping the constraint $\sigma_0 = 0$. By [8] Lemma 2.1, then

$$
\langle F(\sigma) \rangle_{\beta} = \lim_{m^2 \downarrow 0} \langle F(\sigma) \rangle_{\beta,m^2},
$$

for any $F$ as appearing above (1.3) (and in particular for the choice $F(\sigma) = e^{(f,\sigma)}$ for any $f : \Lambda_N \to \mathbb{R}$).

The renormalisation group analysis will involve a decomposition of the covariance

$$
C(s,m^2) \overset{\text{def}}{=} (C(m^2)^{-1} - s\Delta)^{-1}, \quad \text{with} \quad C(m^2) = (-\Delta_J + m^2)^{-1} - \gamma \text{id},
$$

where the inverses are interpreted on $\mathbb{R}^{\Lambda_N}$ and $\Delta$ is the (unnormalised) nearest-neighbour Laplacian on $\Lambda_N$, and $\gamma$ and $s$ are parameters with $\gamma \in (0,\frac{1}{4})$ and $|s|$ tacitly assumed sufficiently small so that $C(m^2)^{-1} - s\Delta$ is positive definite. As in [8] (1.1)), and without loss of generality, we work from here on under the standing assumptions that $|s| \lesssim \varepsilon_s \theta_J$ (by which (2.2) is well-defined) and that, for an arbitrary constant $C > 0$, we have $\theta_J \geq C^{-1}$ and $\nu_J \geq C^{-1} \rho_J$, where $\theta_J$ and $\rho_J$ refer to the range and spectral characteristics of $J$, defined in [8] (3.3), (3.5), and $\varepsilon_s$ is the numerical constant appearing in [8] Proposition 3.4. The last two conditions hold for any fixed $J$ as in the theorems. (The use of the constant $C$ will yield uniform estimates over families of $J$ as above, see [8] Remark 1.2). We do not state these in our main theorems above, but still introduce $C$ to follow the same setup as in [8].

Under these assumptions, it follows that for suitable choice of $\gamma \in (0,\frac{1}{3})$, which we henceforth regard as fixed, one can decompose $C(s,m^2)$ from (2.2) as in [8] Section 4 (see in particular (4.4) therein) to obtain, for all $m^2 > 0$ (and $|s| \lesssim \varepsilon_s \theta_J$),

$$
C(s,m^2) = \Gamma_1(s,m^2) + \cdots + \Gamma_{N-1}(s,m^2) + \Gamma_N(s,m^2) + t_N(s,m^2)Q_N.
$$

The right-hand side is a sum over positive (semi-)definite (covariance) matrices indexed by $\Lambda_N$. The matrix $Q_N$ has all entries equal to $1/|\Lambda_N| = L^{-dN}$ and $t_N(s,m^2)$ is a scalar satisfying [8] (3.16)], in particular, diverging like $m^{-2}$ as $m^2 \downarrow 0$. The covariances $\Gamma_{j+1}$ and $\Gamma_j^{\Lambda_N}$ in (2.3) refer to those defined in [8] (4.2), (4.3)]. They correspond to a decomposition over scales $L^j$ of the covariance $C(s,m^2)$. By construction, the matrices $\Gamma_j$ have range $\frac{1}{L^j} L^j$ and their key analytical features are summarised in [8] Lemma 4.1. We will frequently use the following notation. For $f : \Lambda_N \to \mathbb{R}$, we define (with a slight abuse of notation) $\Gamma_j(f) = \Gamma_j * f$ where $(\Gamma_j * f)(x) = \sum_y \Gamma_j(x-y) f(y)$ with $\Gamma_j(x) = \Gamma_j(0,x)$, cf. [8] below (3.8)].

This completes the introduction of our setup. We observe that in fact, the parameter $s$ in (2.2), which implements the renormalisation of the temperature of the model, can be fixed from the start in the present paper as $s = s_0(J, \beta)$ with the latter as defined in [8] Proposition 8.1; we will return to this later.

In what follows, we write $E_\Gamma$ denotes the expectation of a Gaussian field $\zeta$ with covariance $\Gamma$. We will frequently write $E$ for $E_{\Gamma_{j+1}}$ when $j = 1, \ldots, N-2$ and $E$ for $E_{\Gamma_N^{\Lambda_N}}$ when $j = N-1$, whenever the scale $j$ is clear from the context. Since $\Gamma_N^{\Lambda_N}$ satisfies exactly the same upper bounds as $\Gamma_j$ with $j = N$, we will usually not distinguish between the cases $j + 1 < N$ and $j + 1 = N$. Generally, $j$ without further specification is allowed to take values $j = 1, \ldots, N - 1$.

2.2. Strategy. Contrary to the macroscopic torus scaling limit in [8], in which all the scales $j < N$ appearing in (2.3) were treated equally, we will have to distinguish in what follows a characteristic scale $j_f$ at which a given test function $f$ starts to induce a ‘perturbation,’ cf. [2.9]
below, which manifests itself as a shift (or translation) of the corresponding Gaussian field (at the same scale). This is because the infinite volume limit $N \to \infty$ in Theorem 1.1 is decoupled from the characteristic scale $j_f$, whereas [8] simply takes $j_f = N$. The induced perturbation influences the renormalisation group flow in all the larger scales $j \geq j_f$. The technical difficulties arising in this paper are due to these changes. Fortunately, it will turn out that the infinite chain of perturbations will only impact the analysis on a bounded region by the compact support condition on the external field (see Lemma 2.2 for example).

Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a finitely supported test function with $\sum_x f(x) = 0$. Let $j_f$ be the smallest integer ($\geq 1$) such that the support of $f$ and $\Delta f$ is contained in $[0, \frac{1}{2}L^j]^2$ up to a spatial translation. If $f : \Lambda_N \to \mathbb{R}$ then $j_f$ is defined similarly by identifying $\Lambda_N$ with $((0, L^N) \cap \mathbb{Z})^2 \subset \mathbb{Z}^2$, whence $j_f \leq N$. We call $j_f$ the smoothness scale of $f$ and will frequently assume that

$$\|f\|_{\ell^\infty(\mathbb{Z}^2)} \leq cL^{-2j_f}, \quad (2.4)$$

where $c$ will be an $L$-dependent small constant fixed below Lemma 2.2. The interpretation of $j_f$ as a smoothness scale becomes clear when we focus on lattice functions scaled like $f_\varepsilon$ given by (1.7). Indeed, each $\varepsilon^{-2} f_\varepsilon(\varepsilon^{-1}x)$ is an approximation of a smooth function, thus $j_{f_\varepsilon}$ is the scale where $f_\varepsilon$ becomes smooth: $L^{-j_{f_\varepsilon}} \approx \varepsilon^{-1}$.

The macroscopic scaling limit considered in [8] corresponds to $j_f = N$, but now we are interested in $j_f \ll N$. The analysis of the macroscopic scaling limit proceeds through a translation of the field by $\gamma f + C(s, m^2)(f + s\gamma \Delta f)$ at scale $N$, with $C(s, m^2)$ as given by (2.2). The term $\gamma f$ and the difference between $f$ and $f + s\gamma \Delta f$ will be insignificant and result from the preliminary renormalisation group step in [8] Section 2.3, which integrates out the i.i.d. field with variance $\gamma$, cf. (2.2), thus transforming the original discrete field into a smooth periodic potential (integrated with respect to a Gaussian measure). In view of (2.3), we now rewrite $C(s, m^2)$ as

$$C(s, m^2) = \Gamma_{\leq j_f}(s, m^2) + \sum_{j = j_f + 1}^{N-1} \Gamma_j(s, m^2) + \Gamma_{N}(s, m^2) + t_N(s, m^2)Q_N, \quad (2.5)$$

where, with hopefully obvious notation, $\Gamma_{\leq j} = \sum_{1 \leq k \leq j} \Gamma_k$. Our starting point in this paper for the proofs of Theorems 1.1 and 1.2 is also a translation, but at the smoothness scale $j_f$ rather than the macroscopic scale $N$, and by $\gamma f + \Gamma_{\leq j_f}(f + s\gamma \Delta f)$, see Lemma 2.1 below. An observation (made precise by Lemma 2.2 below) is that $\gamma f + \Gamma_{\leq j_f}(f + s\gamma \Delta f)$ is smooth at scale $j_f$ because $f$ is, while on the other hand, $\Gamma_k(f + s\gamma \Delta f)$ is smooth for $k > j_f$ because of the smoothing properties of the covariance $\Gamma_k$. We will show that this allows to implement translations iteratively for all scales $k \geq j_f$, with small errors accumulating from each scale $k$ starting from $k = j_f$ and that as $j_f \to \infty$ the sum of these errors is governed by the contribution from the scale $j_f$ and tends to 0 as $j_f \to \infty$.

2.3. Scale-dependent external fields. To formulate the above strategy more precisely, first recall (as mentioned above) that the parameter $s$ is fixed as $s = s_\beta^0(J, \beta)$ from the start of this paper. Further let $s_0 = s = s_\beta^0(J, \beta)$, and define (as in [8] (2.25))

$$Z_0(\varphi) = e^{\frac{s_0}{2}(\varphi, -\Delta \varphi) + \sum_{s \in \Lambda_N} \tilde{U}(\varphi_s)}, \quad \varphi \in \mathbb{R}^{\Lambda_N}, \quad (2.6)$$

with the function $\tilde{U}$ given by [8] (2.15), which is a $2\pi\beta^{-1/2}$ periodic function of a single real variable. The next lemma is a slight reformulation of [8] Lemma 2.3. For its statement let $\tilde{C}(s, m^2)$ be given as in [8] (2.26), i.e.,

$$\tilde{C}(s, m^2) = \gamma(1 + s\gamma \Delta) + (1 + s\gamma \Delta)C(s, m^2)(1 + s\gamma \Delta), \quad (2.7)$$

and recall the covariance decomposition (2.5).
Lemma 2.1. For all $\beta > 0$, $\gamma \in (0, \frac{1}{2})$, $m^2 \in (0, 1]$, $|s| = |s_0|$ small, one has for any $f \in \mathbb{R}^\Lambda_N$ such that $\sum_x f(x) = 0$, 

\[
\langle e(f, \sigma)^{\Lambda_N}_{\beta, m^2} \rangle \propto e^{\frac{1}{2}(f, \tilde{C}(s, m^2)f)} E_{C(s, m^2)}[Z_0(\varphi + \sum_{j<j} u_j)],
\]

where the expectation acts on $\varphi$ and 

\[
u_j = \begin{cases} 
0 & (j < j) \\
\gamma f + \Gamma_{<j}(f + s \gamma \Delta f) & (j = j) \\
\Gamma_j(f + s \gamma \Delta f) & (N > j > j) \\
\Gamma_N^A(f + s \gamma \Delta f) & (j = N).
\end{cases}
\]

Proof. By [8, Lemma 2.3],

\[
\sum_{\sigma \in Z_\beta^{\Lambda_N}} e^{-\frac{1}{2}(\sigma, -\Delta_j + m^2)\sigma} e^{(f, \sigma)} \propto e^{\frac{1}{2}(f, \tilde{C}(s, m^2)f)} E_{C(s, m^2)}[Z_0(\varphi + Af)],
\]

with

\[
A = (1 + s \gamma \Delta)^{-1} \tilde{C}(s, m^2) = \gamma + C(s, m^2)(1 + s \gamma \Delta).
\]

The statement follows by applying the decomposition (2.5) of $C(s, m^2)$ which gives

\[
Af = \sum_{j \in \mathbb{Z}} u_j + t_N Q_N(f + s \gamma \Delta f) = \sum_{j \in \mathbb{Z}} u_j,
\]

where the last equality follows because $\sum_x f(x) = 0$, and hence $Q_N f = Q_N \Delta f = 0$.

The renormalisation group flow constructed in [8], which we now sometimes refer to as the bulk renormalisation group flow, is in terms of the recursion (cf. [8, (7.3)])

\[
Z_{j+1}(\varphi') = E_{\Gamma_{j+1}} Z_j(\varphi' + \zeta), \quad \varphi' \in \mathbb{R}^{\Lambda_N},
\]

where here and below, $E_{\Gamma_{j+1}}$ is the Gaussian expectation with covariance $\Gamma_{j+1}$ which always acts on the field $\zeta$. To incorporate the scale-dependent external fields $u = (u_j)$ we now define $Z_0(u, \varphi) = Z_0(\varphi)$ and

\[
Z_{j+1}(u, \varphi') = E_{\Gamma_{j+1}} Z_j(u, \varphi' + \zeta + u_j), \quad \varphi' \in \mathbb{R}^{\Lambda_N},
\]

with $\Gamma_N^A$ instead of $\Gamma_{j+1}$ when $j + 1 = N$. Finally set

\[
\tilde{Z}_N(u, \varphi') = E_{t_N Q_N} Z_N(u, \varphi' + \zeta + u_N), \quad \varphi' \in \mathbb{R}^{\Lambda_N}.
\]

Together, (2.13), (2.15) and (2.3) imply in particular that the expectation appearing on the right-hand side of (2.8) can be recast as (with $E_{C(s, m^2)}$ acting on $\varphi$)

\[
E_{C(s, m^2)}[Z_0(\varphi + \sum_{j=j}^N u_j)] = \tilde{Z}_N(u, 0).
\]

Our analysis of the $\tilde{Z}_j(u, \varphi')$ relies on the property that the external fields $u_j$ are smooth on scale $j$ for all $j$, as demonstrated by the next lemma. Here assume that $j f$ in (2.9) is the smoothness scale of $f$, i.e., the smallest integer such that supp $f$ is contained in a block of side length $\frac{j}{4} L^j$. By definition, a block of size $L$ is any set of the form $x + ([0, L) \cap \mathbb{Z})^2$ for some $x \in L^\mathbb{Z}^2$. Let $\|u_j\|_2 = \|u_j\|_{C^2_0(\mathbb{Z}^2)} = \max_{n=0,1,2} \|\nabla^m u_j\|_{C^0(\mathbb{Z}^2)}$, cf. [8, (5.10)]. In the sequel we often tacitly view a function $f$ with domain $\Lambda_N$ (such as $u_j$) as defined on $\mathbb{Z}^2$ by identifying $\Lambda_N$ with $[0, L^N)^2$ and extending $f$ to have value 0 outside this set.
Lemma 2.2. There exists an $L$-independent constant $C > 0$ such that the following holds: for all $f: \mathbb{Z}^d \to \mathbb{R}$ satisfying $\sum f = 0$ and such that $f$ and $\Delta f$ have support in a block of side length $\frac{1}{4} L^{j_f}$, the functions $u_j$ defined by \((2.9)\) have support in blocks of side lengths $\frac{3}{4} L^j$ for $j \leq N - 1$ and
\[
\|u_j\|_{C^2} \leq CL^{2j+2}\|f\|_{L^\infty(\mathbb{Z}^d)}, \quad j \leq N. \tag{2.17}
\]

In particular, if \((2.4)\) holds with $c \leq (CL^2)^{-1}$, then $\sup_j \|u_j\|_{C^2} \leq 1$. From here on, we fix (any) such value of $c$; this choice is implicit when referring to \((2.4)\) in the sequel.

Proof. Let $g = f + s\gamma \Delta f$ and note that by assumption $g$ has support in a block of side length $\frac{1}{4} L^{j_f}$. Also, $\|g\|_{L^\infty} \leq (1 + 2|s|\gamma)\|f\|_{L^\infty} \leq C\|f\|_{L^\infty}$ since $\|\Delta f\|_{L^\infty} \leq 8\|f\|_{L^\infty}$ for any $j$. We may identify $\Gamma_j$ with its convolution kernel, i.e., $\Gamma_j \equiv \Gamma_j * g$. Then $\Gamma_j$ is supported in a block of side length $\frac{1}{4} L^j$ and satisfies $\|\nabla^\alpha \Gamma_j\|_{L^\infty} \leq CL^2$ for $|\alpha| \leq 2$ where $\nabla^\alpha \equiv \sum_{j=1}^L \nabla^{\alpha_j}$, see \cite{8} Corollary 4.1, thus
\[
\|\nabla^\alpha \Gamma_j g\|_{L^\infty} \leq L^{2j}\|\nabla^\alpha \Gamma_j\|_{L^\infty}\|g\|_{L^\infty} \leq CL^{2j+2}\|f\|_{L^\infty}. \tag{2.18}
\]
Thus the desired statement holds if $j < N$.

The same estimates hold when $j = N$, i.e., with $\Gamma_j$ replaced by $\Gamma_N^{\Lambda^\alpha}$ which satisfies analogous bounds, see \cite{8} Corollary 4.1. This completes the proof of the bound \((2.17)\).

The statement about the support of the $u_j$ follows immediately from the assumption that the support of $f$ and $g$ have diameter $\frac{1}{4} L^{j_f} \leq \frac{1}{4} L^j$ for all $j \geq j_f$ and that $\Gamma_N$ has range $\frac{1}{4} L^j$. \hfill \Box

2.4. Conclusion of the argument. In Section 3 we will show the following theorem from which the proof of Theorem 1.1 can be completed similarly as the torus result in \cite{8} Section 9. The theorem is stated under somewhat more general condition on the sequence $(u_j) \equiv (u_j \in \mathbb{R}^{\Lambda_j})$ of given external fields that are uniformly bounded and supported on a single block in the sense that:

(A_u) There exists $j_u$ such that $u_j = 0$ for $j < j_u$, $\|u_j\|_{C^2} \leq 1$ for each $j \leq N$, and $u_j$ is supported on the unique $B_0 \in B_j$ such that $0 \in B_0$ and $d(\partial B_0, \text{supp}(u_j)) > 4$.

For the same reason that $j_f$ was called a smoothness scale of $f$, we call $j_u$ the smoothness scale (of $u = (u_j)$). Note that, by translation invariance of the Discrete Gaussian model on the torus $\Lambda_N$, we may assume that $f$ is centred with respect to the block decomposition; that is, $\text{supp}(f)$ and $\text{supp}(\Delta f)$ are contained in the box $m + [0, \frac{1}{4} L_{j_f}]^2$, where $m$ is one of the lattice points closest to the center of some block $B \in B_j$ for all $j_f \leq j \leq N$. In particular, then, by Lemma 2.2, for all scales $j \leq N$, there is a block $B \in B_j$ such that whenever $L \geq C$, $N_0(\text{supp}(u_j)) \subset B$ where $N_0(X)$ denotes the set of points with $\ell^1$-distance at most $k$ from the set $X$. Thus the condition on the support of $u_j$ is not stronger than the condition on the support of $f$.

Theorem 2.3. Let $J$ be a finite-range step distribution as in the statements of Theorems 1.1 and 1.2. There are $\beta_0(J) \in (0, \infty)$, a (large) integer $L = L(J)$ (which can be chosen dyadic), and a constant $\alpha > 0$ such that if $u = (u_j)$ satisfies (A_u) there is $C > 0$ such that for $\beta \geq \beta_0(J)$ and $N > j_u$,
\[
\left| \tilde{Z}_N(u,0) - \frac{\tilde{Z}_N(0,0)}{\tilde{Z}_N(u,0)} \right| \leq CL^{-\alpha j_u}. \tag{2.19}
\]

Assuming Theorem 2.3 to hold, and in view of Lemma 2.1 the proofs of Theorems 1.1 and 1.2 are readily completed by means of the following elementary lemma, as explained below. This lemma is the infinite-volume analogue of \cite{8} Lemma 9.2; we postpone its proof to the end of this section and first give the details for the proof of Theorems 1.1 and 1.2. In what follows, for $N > j_f$, we tacitly identify $f_c$ with the corresponding function having domain on the torus $\Lambda_N$ by identifying $\text{supp}(f_c)$ with a suitable subset of the torus $\Lambda_N$. We write $C_{\Lambda_N} \equiv C$ for the covariance matrix defined in \((2.7)\) to stress the dependence on the underlying torus $\Lambda_N$. 

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Lemma 2.4. Let \( f \in C_c^\infty(\mathbb{R}^2) \) with \( \int f \, dx = 0 \) and \( f_\varepsilon \) be as in (1.7). Then
\[
\lim_{\varepsilon \to 0} \lim_{N \to \infty} \sum_{m^2 \in k_0} (f_\varepsilon, C_\Lambda_N(s, m^2) f_\varepsilon) = \frac{1}{v_j^2 + s} (f, (-\Delta_{\mathbb{R}^2})^{-1} f)_{\mathbb{R}^2},
\]
and the statement also holds if the two leftmost limits are replaced by \( N \to \infty \) with \( \varepsilon = \varepsilon_N \to 0 \) while \( \varepsilon_N L^N \to \infty \).

Proof of Theorems 1.1 and 1.2. Our proof proceeds as the following. We will first prove our main
Lemma 2.4. While
\[
\sum_{n=0}^{k} \frac{T^n}{n!} |(f_\varepsilon, \sigma)|^n = e^{T|(f_\varepsilon, \sigma)|} - \sum_{n=0}^{k} \frac{T^n}{n!} |(f_\varepsilon, \sigma)|^n = \frac{e^{\theta T|(f_\varepsilon, \sigma)|}}{(k+1)!} \leq e^{T(f_\varepsilon, \sigma) + e^{-T(f_\varepsilon, \sigma)}} (k+1)!
\]
for each \( n \in \mathbb{N} \). Also, for any \( T \geq 0 \), by the Taylor’s theorem (for the second equality), there exists \( \theta \in [0, 1] \) such that
\[
\sum_{n>k} \frac{T^n}{n!} |(f_\varepsilon, \sigma)|^n = e^{T|(f_\varepsilon, \sigma)|} - \sum_{n=0}^{k} \frac{T^n}{n!} |(f_\varepsilon, \sigma)|^n = \frac{e^{\theta T|(f_\varepsilon, \sigma)|}}{(k+1)!} \leq e^{T(f_\varepsilon, \sigma) + e^{-T(f_\varepsilon, \sigma)}} (k+1)!.
\]
But by \(30\) (see also \(43\) Proposition 1.2), we have the Gaussian domination
\[
\langle \epsilon^{(g, \sigma)} \rangle_{J, \beta} \leq \epsilon^q \| \epsilon^{(-\Delta)} \|^q_g
\]
for any \(g : \Lambda_N \to \mathbb{R}\) with \(\sum g = 0\), so we obtain
\[
\left( \sum_{n=k}^{\infty} \frac{\tau^n}{n!} (f, \sigma)^n \right)_{J, \beta} \leq \frac{2}{(k+1)!} \epsilon^{\frac{2}{\epsilon} T^2 \langle f, \epsilon \rangle_{J, \beta}}
\]
upon letting \(T = |\text{Re}(\tau)|\). In other words, \(\sum_{n=0}^{\infty} \frac{\tau^n}{n!} (f, \sigma)^n \to \langle \epsilon^{(f, \sigma)} \rangle_{J, \beta} \) as \(k \to \infty\), uniformly in \(\epsilon\) and \(N\), proving
\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \left( \sum_{n=0}^{\infty} \frac{\tau^n}{n!} (f, \sigma)^n \right)_{J, \beta} = \lim_{k \to \infty} \lim_{\epsilon \to 0, N \to \infty} \left( \sum_{n=0}^{k} \frac{\tau^n}{n!} (f, \sigma)^n \right)_{J, \beta}
\]
But by (2.26), the latter is \(\epsilon^2 \frac{\beta, \sigma (J, \beta)}{2 \pi^2} (f, (\Delta^2)^{-1} f)_{2, 2}\), completing the proof of Theorem 1.1. The extension for Theorem 1.2 is done analogously.

**Proof of Lemma 2.2.** In what follows, given \(f : \mathbb{Z}^2 \to \mathbb{R}\), we denote by \(\hat{f}\) its Fourier transform, defined as in \(8\) (3.19). By definition of \(\hat{C}(s, m^2)\) and since \(\hat{f}(0) = 0\), one has
\[
\lim_{N \to \infty} \lim_{m \to \infty} \left( \sum_{n=0}^{\infty} \frac{\tau^n}{n!} (f, \sigma)^n \right)_{J, \beta} = \lim_{k \to \infty} \lim_{\epsilon \to 0, N \to \infty} \left( \sum_{n=0}^{k} \frac{\tau^n}{n!} (f, \sigma)^n \right)_{J, \beta}
\]
where \(\lambda(p)\) is the Fourier multiplier of the (unnormalised) discrete Laplacian \(\Delta\) and \(\lambda_J(p)\) that of the (normalised) range-\(J\) Laplacian \(\Delta_J\), see [8] Section 3.2. By [8] Lemma 3.6,
\[
\lim_{\epsilon \to 0} \epsilon^{-2} \lambda(p) = |p|^2, \quad \lim_{\epsilon \to 0} \epsilon^{-2} \lambda_J(p) = v^2_J |p|^2
\]
and the fraction in the integrand in (2.31) is bounded by \(C|p|^{-2}\) uniformly in \(\epsilon\) and \(p \in [-\pi/\epsilon, \pi/\epsilon]^2\). Moreover, as we now argue, (1.7) implies that \(\hat{f}(\epsilon, p) \to \hat{f}(p)\) as \(\epsilon \to 0\) for each \(p \in \mathbb{R}^2\) and that \(|\hat{f}(\epsilon, p)| \leq C|p|(1 + |p|)^{-3}\). To see this in detail, we start from
\[
\hat{f}(\epsilon, p) = \sum_{y \in \mathbb{Z}^2} f(y/\epsilon) e^{-iy \cdot p}.
\]
For \(|\hat{f}(p) - \hat{f}(\epsilon, p)| \to 0\) pointwise, use \(f \in C_c^\infty(\mathbb{R}^2)\) and the last condition in (1.7) to see that, with \([\cdot]\) denoting the integer part,
\[
|\hat{f}(p) - \hat{f}(\epsilon, p)| \leq \int_{\mathbb{R}^2} |f(y)(e^{-iy \cdot p} - e^{-iy \cdot [y/\epsilon] \cdot p})| dy + \int_{\mathbb{R}^2} |f(y) - \epsilon^{-2} f\langle [y/\epsilon] \rangle| dy \to 0.
\]
To see the bound on \(\hat{f}(\epsilon, p)\), use summation by parts to write
\[
\lambda(p)|\hat{f}(p)| = |\hat{\Delta} f\langle p \rangle| = \left| \sum_{x \in \mathbb{Z}^2} e^{-iy \cdot x} \Delta f\langle x \rangle \right| \leq \|\Delta f\|_{\ell^1(\mathbb{Z}^2)}.
\]
By (1.7),
\[
\|\Delta f\|_{\ell^1(\mathbb{Z}^2)} \leq C f^2(\epsilon^{-1} + 1)^2 \|\Delta f\|_{\ell^\infty(\mathbb{Z}^2)} \leq 2 C f^2 \|\epsilon^{-1} \nabla\|^2 f\|_{\ell^\infty(\mathbb{Z}^2)} \leq 2 C f^2 \|\epsilon\|_{\ell^\infty(\mathbb{Z}^2)}^2,
\]
and by [8] Lemma 3.6, we have that \(\frac{1}{\epsilon^2 |p|^2} \lambda(p) \geq \frac{1}{\epsilon^2 |p|^2}\). Thus it follows that \(|\hat{f}(\epsilon, p)| \leq C|p|^{-2}\). On the other hand, since \(\sum f\langle \epsilon \rangle = 0\) and \(\|f\|_{\ell^\infty} \leq C f^2\), also
\[
|\hat{f}(\epsilon, p)| = \left| \sum_{y \in \mathbb{Z}^2} f\langle y/\epsilon \rangle (e^{-iy \cdot p} - 1) \right| \leq \|f\|_{\ell^\infty} \sum_{y \in \mathbb{Z}^2 : |y| \leq C_f} |y \cdot p| \leq C C f^2 |p|.
\]
and therefore $|\hat{f}_s(\varepsilon p)| \leq C|p|(1 + |p|)^{-3}$ when combined with $|\hat{f}_s(\varepsilon p)| \leq C|p|^{-2}$.

Finally, using the convergence in Fourier space and that the integrand is dominated by $C|p|^{-2} \times |p|(1 + |p|)^{-3})^2 \leq C(1 + |p|)^{-6}$ which is integrable over $\mathbb{R}^2$, the Dominated convergence theorem implies

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \int_{\mathbb{R}^2} \frac{1}{v_j^2 + s} |p|^{-2} |\hat{f}(p)|^2 dp = \frac{1}{v_j^2 + s} (f, (\Delta_{\mathbb{R}^2})^{-1} f)$$

as claimed.

3 Norms and contraction estimates

We now prepare the ground for the proof of Theorem 2.3, which will essentially follow by suitably extending the RG flow developed in [8]. This extension is designed to accommodate the external field $u$. In the present section, we discuss the necessary amendments to the norms introduced in [8] Section 5 required to carry this out, as well as the resulting contraction estimates, cf. [8], Section 6.

3.1. Norms and regulators without external field. We recall some essential elements of [8]. Given $\Lambda_N$, the discrete two-dimensional torus of side lengths $L^N$ and a distinguished point $0 \in \Lambda_N$, let $\pi_N : \mathbb{Z}^2 \to \Lambda_N$ be the canonical projection with $\pi_N(0) = 0$. Then for each $j = 0, \cdots, N$, $B_j$ (j-blocks) will be the sets of the form $\pi_N([-L, L] \cap \mathbb{Z}^2 + n L^j)$ for $n \in \mathbb{Z}^2$, $P_j$ (j-scale polymers) are any subsets (not necessarily connected) of $\Lambda_N$ that can be obtained as the union of $j$-blocks. For various notions related to $P_j$, see [8] Section 4. Functions $F(X, \varphi)$ smooth in $\varphi$ that only depend on $\varphi|_{X^*}$ for each $X \in P_j$ are called polymer activities at scale $j$, see [8], Section 5; here $X^*$ refers to the small-set neighborhood of $X$, see [8], Section 4.1.

In (2.13), $Z_j$ will always be parametrised as

$$Z_j(\varphi) = e^{-E_j[\Lambda_N]} \sum_{X \in P_j(\Lambda_N)} e^{U_j(\Lambda_N \setminus X, \varphi)} K_j(X, \varphi) \quad (3.1)$$

$$U_j(X, \varphi) = \frac{1}{2} s_j |\nabla \varphi|^2_X + \sum_{q \geq 1} L^{-2j} s_j q \sum_{x \in X} \cos(\sqrt{\beta q} \varphi(x)) = \frac{1}{2} s_j |\nabla \varphi|^2_X + W_j(X, \varphi), \quad (3.2)$$

with $U_j(\emptyset) = 0$ and initial conditions $s_0 \in \mathbb{R}$ given, $E_0 = 0, K_0(X, \varphi) = 1_{X=\emptyset}$ and $z_0 = (z_q^{(j)})_{q \geq 0}$ given, where the latter refer to the Fourier coefficients of the periodic potential $\tilde{U}$ in (2.13), see [8] (2.18). The coordinates $U_j$ and $K_j$ are polymer activities, and in [8] Sections 5 and 7, they are controlled using the norms $\|\cdot\|_{\Omega_j^{(j)}}$ and $\|\cdot\|_{\Omega_N^{(j)}}$. The latter norm needs an extension in the current work, so it will be reviewed in some detail here. It is defined in terms of positive parameters $r, a, L, \kappa_L, c_2, c_4, c_6, h$, which will essentially be fixed as in [8] in Section 3.3 below. The definition of the norms involves the regulator $G_j$, which is a weight defined for $X \in P_j$ and $\varphi \in \mathbb{R}^{\Lambda_N}$ by

$$G_j(X, \varphi) = \exp \left( \kappa_L \left( \|\nabla \varphi\|^2_{L^2(X)} + c_2 \|\nabla \varphi\|^2_{L^2(\partial X)} + \sum_{B \in B_j(X)} \|\nabla \varphi\|^2_{L^\infty(B^*)} \right) \right), \quad (3.3)$$

where $B_j(X)$ is the set of $j$-blocks constituting $X$, $\partial X$ denotes the inner $\ell^1$-vertex boundary of $X$, and with the relevant $L^p$-norms as introduced in [8] Definition 5.2. The semi-norms and norms on polymer activities are then given by (cf. [8] Definition 5.4)

$$\|D^n F(X, \varphi)\|_{n, T_j(X, \varphi)} = \sup_{\varphi \in \mathbb{R}^{\Lambda_N}} \left\{ D^n F(X, \varphi)(f_1, \cdots, f_n) : \|f_k\|_{C^2_{j^*}(X^*)} \leq 1 \forall k \right\} \quad (3.4)$$

$$\|F(X, \cdot)\|_{h, T_j(X)} = \sup_{\varphi \in \mathbb{R}^{\Lambda_N}} G_j(X, \varphi)^{-1} \sum_{n=0}^{\infty} \frac{h^n}{n!} \|D^n F(X, \varphi)\|_{n, T_j(X, \varphi)} \quad (3.5)$$

$$\|F\|_{\Omega_N^{(j)}} = \|F\|_{h, T_j} = \sup_{X \in P_j} A^{[X]_{j^*}} \|F(X, \cdot)\|_{h, T_j(X)} \quad (3.6)$$
We will also need the following somewhat more technical properties of the norms and regulators. For \( X \in \mathcal{P}_j \) and \( \varphi \in \mathbb{R}^\Lambda_N \), recall the definition \( w_j(X, \varphi)^2 = \sum_{B \in B_j(X)} \max_{a=1,2} \| \nabla_j^a \varphi \|_{L^\infty(B^*)}^2 \) and then that of the strong regulators

\[
\exp \left( c_w \kappa_L w_j(X, \varphi)^2 \right), \quad g_j(X, \varphi) = \exp \left( c_4 \kappa_L \sum_{a=0,1,2} W_j(X, \nabla_j^a \varphi)^2 \right),
\]

(3.7)

where \( W_j(X, \nabla_j^a \varphi)^2 = \sum_{B \in B_j(X)} \| \nabla_j^a \varphi \|_{L^\infty(B^*)}^2 \). For sharp integrability estimates, we subdecomposed in [8] Section 4.3 each scale \( j \) into \( M \) fractional scales \( j + s \) with \( s = 0, \ldots, 1 - 1/M \) when \( L = \ell^M \) with \( \ell \) an integer. Each covariance \( \Gamma_{j+1} \) from the finite-range decomposition (2.5) has the corresponding subdecomposition

\[
\Gamma_{j+1} = \Gamma_{j,j+1/M} + \cdots + \Gamma_{j+(M-1)/M,j+1}.
\]

The regulators \( G_{j+s} \) and the strong regulators \( g_{j+s} \) are also defined on these fine scales, see [8 (5.15)] and analogously for \( G_{j+s} \). The crucial property of \( G_{j+s} \) and \( g_{j+s} \) is stated in the next lemma, which is an extension of [8, Lemma 5.13] and proved in Appendix B. The fields \( X, \varphi, \xi \) appearing in the next lemma will correspond in practice to shifts induced by the external fields.

Lemma 3.1. For \( X \in \mathcal{P}_{j+s} \) and \( \varphi, \xi, \xi_B \in \mathbb{R}^\Lambda_N \) for each \( B \in B_{j+s}(X) \), define

\[
\log G_{j+s}(X, \varphi, \xi, \xi_B)_{B \in B_{j+s}(X)} = \kappa_L \| \nabla_{j+s}(\varphi + \xi) \|_{L^2(B_{j+s}(X))}^2 + 2 \kappa_L c_2 \| \nabla_{j+s}(\varphi + \xi) \|_{L^2(B_{j+s}(\partial X))}^2 + \kappa_L \| \nabla_{j+s}(\varphi + \xi_B) \|_{L^\infty(B^*)}^2.
\]

(3.9)

Assume \( 0 \leq j < N \), \( L = \ell^M \). For any choice of \( c_2 \) small enough compared to 1, there exist \( c_4 = c_4(c_2) \) and an integer \( \ell_0 = \ell_0(c_1, c_2) \) (both large), such that for all \( \ell \geq \ell_0 \), \( M \geq 1 \), \( s \in \{0, \frac{1}{M}, \ldots, 1 - \frac{1}{M}\} \) and \( \kappa_L > 0 \), for \( X \in \mathcal{P}_{j+s} \),

\[
G_{j+s}(X, \varphi, \xi, \xi_B)_{B \in B_{j+s}(X)} \leq \max_{a \in \{0\} \cup B_{j+s}(X)} g_{j+s}(X_{s+M-1}, \xi_a)G_{j+s+M-1}(X_{s+M-1}, \varphi).
\]

(3.10)

and \( X_{s+M-1} \) is the smallest \((j+s+M-1)\)-polymer containing \( X \) (see [8, Section 4.3]).

3.2. Norms and regulators with external field. To incorporate the effect of the scale-dependent external fields, we need an extension of the norms and regulators that take the external field into account. The following definition introduces modified regulators that effectively control the polymer activities perturbed by the external fields \((u_j)\).

Definition 3.2. Given \((u_j)_{j} \) satisfying \( [\mathcal{A}_u] \) define the \( \Psi \)-regulators (cf. [KM])

\[
G_j^\Psi(X, \varphi; u_j) = \sup_{t \in [0,1]} \exp \left( \kappa_L \| \nabla_j(\varphi + tu_j) \|_{L^2(X)}^2 + c_2 \kappa_L \| \nabla_j(\varphi + tu_j) \|_{L^2(\partial X)}^2 + \kappa_L W_j^\Psi(X, \varphi; u_j)^2 \right)
\]

(3.11)

where

\[
W_j^\Psi(X, \nabla_j^a \varphi; u_j)^2 = \sum_{B \in B_j(X)} \max_{t \in [0,1]} \| \nabla_j^a(\varphi + tBu_j) \|_{L^\infty(B^*)}^2.
\]

(3.12)

The dependence on \( u_j \) will often be hidden.

Remark 3.3. The main motivation for \( G_j^\Psi \) is to have \( \sup_{t \in [0,1]} G_j(X, \varphi + tu_j) \leq G_j^\Psi(X, \varphi) \) and hence

\[
\| K(X, \varphi + tu_j) \|_{h,T_j(X,\varphi)} \leq \| K(X) \|_{h,T_j(X)G_j^\Psi(X, \varphi)}, \quad t \in [0,1].
\]

(3.13)
Note that we could not use \( \sup_{t \in [0,1]} G_j(X, \varphi + tu_j) \) for \( G_j^\Psi \) because this definition does not factorise into connected components, i.e.,

\[
\sup_{t \in [0,1]} G_j(X \cup Y, \varphi + tu_j) \neq \sup_{t_1, t_2 \in [0,1]} G_j(X, \varphi + t_1 u_j) G_j(Y, \varphi + t_2 u_j) \tag{3.14}
\]

if \( X \not\subset Y = \emptyset \) but \( X^* \cap Y^* \neq \emptyset \). This is why we introduced the \( W^\Psi_j \).

Also note that since \( \|\nabla_j u_j\|_{L^2_j(X)} \|\nabla_{j_u} u_j\|_{L^2_j(\partial X)} \), \( W_j(X, \nabla^2 j u_j) \) are each bounded by some multiple of \( \|u_j\|_{C^j_2}^2 \), in particular, there exists finite \( C > 0 \), independent of \( X \), such that, under \([A_\Psi]\)

\[
G_j^\Psi(X, 0; u_j) \leq C \tag{3.15}
\]

The following are the key properties of \( G_j^\Psi \) (cf. the properties of \( G_j \) in \[8\] Section \[3\]).

**Proposition 3.4.** Let \((u_j)\) satisfy \([A_\Psi]\). Then there exists \( C_\Psi > 0 \) such that for \( L \) as in the assumption of Lemma \[3.4\] and sufficiently small \( c_2 \), \( c_w > 0 \), \((G_j^\Psi)_{j \geq 0} \equiv (G_j^\Psi(\cdot; u_j))_{j \geq 0} \) satisfies for each \((X, \varphi) \in \mathcal{P}_j \times \mathbb{R}^{AN} \), \( j \geq 0 \),

1. \( G_j^\Psi(X, \varphi) \geq G_j(X, \varphi) \),
2. \( G_j^\Psi(X, \varphi) = \prod_{Y \in \text{Comp}_j(X)} G_j^\Psi(Y, \varphi) \),
3. \( e^{c_w \kappa L w_j(X, \varphi + tu_j)^2} G_j^\Psi(Y, \varphi) \leq C_\Psi G_j^\Psi(X \cup Y, \varphi) \) if \( X \cap Y = \emptyset \) and \( t \in [0,1] \),
4. \( E[G_j^\Psi(X, \varphi + \zeta)] \leq C_\Psi 2^{\|X\|_j} G_{j+1}(X, \varphi') \) for all \( \varphi' \in \mathbb{R}^{AN} \).

**Proof.** By definition of \( G_j^\Psi \), properties (1) and (2) are clear. For (3), first observe from the definition of \( w_j(X, \varphi) \) (see above \([3.7]\)) that for each \( t' \in [0,1] \) and some geometric constant \( C > 0 \),

\[
w_j(X, \varphi + tu_j)^2 \leq 2 \sum_{B \in B_j(X)} \left( \|\nabla_j(\varphi + t'u_j)\|_{L^\infty(B')}^2 + (t-t')^2 \|\nabla_j u_j\|_{L^\infty(B')}^2 \right) + \|\nabla^2_j(\varphi + tu_j)\|_{L^\infty(B')}^2 \tag{3.16}
\]

\[
\leq 2 \sum_{B \in B_j(X)} \left( \|\nabla_j(\varphi + tu_j)\|_{L^\infty(B')}^2 + \|\nabla^2_j(\varphi + tu_j)\|_{L^\infty(B')}^2 \right) + C \|u_j\|_{C^j_2}^2.
\tag{3.17}
\]

We then note that for any \( B \in B_j(X), \) \( x_0 \in B \) and \( x \in B' \), there is another constant \( C > 0 \) such that

\[
|\nabla^\mu_j \varphi'(x)| \leq |\nabla^\mu_j \varphi(x_0)| + C \|\nabla^\mu_j \varphi\|_{L^\infty(B')} \tag{3.18}
\]

for all \( \mu \in \hat{\ell} \) (for example, cf. \[8\] \((A.37)\)), applied to \( f = \nabla_j^\mu \varphi \) and recall that \( x_0 \) and \( x \) belong to some small set \( X \), whence \( |X| \leq C \) and hence \( \|\nabla_j^\mu \varphi\|_{L^\infty(B')} \leq 2 \max_{\mu \in \ell} |\nabla^\mu \varphi(x_0)|^2 + 2C^2 \|\nabla^2_j \varphi\|_{L^\infty(B')} \). Summing over all \( x_0 \in B \), this implies

\[
\|\nabla_j \varphi\|_{L^\infty(B')} \leq 2 \|\nabla_j \varphi\|_{L^2_j(B)}^2 + 2C^2 \|\nabla^2_j \varphi\|_{L^\infty(B')}^2. \tag{3.19}
\]

Plugging this into \((3.17)\), we get

\[
c_w w_j(X, \varphi + tu_j)^2 \leq C c_w \|u_j\|_{C^j_2}^2 + \frac{1}{2} \|\nabla_j(\varphi + tu_j)\|_{L^2_j(X)}^2 + \frac{1}{2} W_j^\Psi(X, \nabla^2_j \varphi)^2 \tag{3.20}
\]
for \(c_w\) sufficiently small. The discrepancy between the left- and right-hand sides of item (3) of the statement of the proposition due to the boundary term of \(G_j\) can be treated by the discrete Sobolev trace theorem [8, Corollary A.2], which shows that there is \(C > 0\) such that

\[
c_2 \|\nabla_j (\varphi + t'u_j)\|_{L^2_j(\partial Y \setminus \partial (X \cup Y))}^2 \leq C_2 \left( \|\nabla_j (\varphi + t'u_j)\|_{L^2_j(Y)}^2 + \|\nabla_j (\varphi + t'u_j)\|_{L^2_j(X \cup Y)}^2 \right),
\]

(3.21)

so if \(c_2\) is sufficiently small so that \(C_2 \leq \frac{1}{2}\), then this together with (3.20) gives

\[
c_w u_j (X, \varphi + tu_j)^2 + \|\nabla_j (\varphi + t'u_j)\|_{L^2_j(Y)}^2 + c_2 \|\nabla_j (\varphi + t'u_j)\|_{L^2_j(X \cup Y)}^2 + W_{\Psi}^j (X \cup Y, \nabla_j^2 \varphi)^2 \leq C c_w \|u_j\|_{C^2_j}^2 + \|\nabla_j (\varphi + t'u_j)\|_{L^2_j(Y)}^2 + \|\nabla_j (\varphi + t'u_j)\|_{L^2_j(X \cup Y)}^2 + W_{\Psi}^j (X \cup Y, \nabla_j^2 \varphi)^2.
\]

(3.22)

After taking supremum over \(t'\), it follows that (3) holds for any \(C_{\Psi} \geq \exp(C_{c_w} K_L \|u_j\|_{C^2_j}^2)\), and \(C_{\Psi}\) can be chosen independent of \(j\) because of (A.2).

For (4), we may assume that \(j \leq N - 2\), since \(\Gamma^{A_N}_N\) satisfies the same estimates as \(\Gamma_N\). We use the regulator decomposition: by Lemma 3.1,

\[
G_j^\Psi (X, \varphi' + \zeta; u_j) \leq \prod_{k=1}^{M} \sup_{t \in [0,1]} g_{j+\frac{k-1}{M}} (X_{k/M}, \xi_k + 1_ktu_j) G_{j+1} (\overline{X}, \varphi')
\]

(3.23)

whenever \(\zeta = \sum_k \xi_k\) and \(X_{k/M}\) is the smallest polymer in \(P_{j+k/M}\) containing \(X \in P_j\) and \(\overline{X} = X_1\).

Using the covariance subdecomposition (3.5), we may decompose \(\xi \sim \mathcal{N}(0, \Gamma_{j+1})\) as the sum of independent \(\xi_k \sim \mathcal{N}(0, \Gamma_{j+k/M,j+(k+1)/M})\). Then each \(\mathbb{E}^{t_k} [g_{j+(k-1)/M}(X_{k/M}, \xi_k + 1_ktu_j)]\) are bounded using [8] Lemma 5.12. For \(k = 1\), we have from the definition of \(g_j\) that

\[
g_j(X_{M-1}, \xi_1 + tu_j) \leq g_j(X_{M-1}, \xi_1)^2 g_j(X_{M-1}, u_j)^2 \leq g_j(X_{M-1}, \xi_1)^2 e^{c \kappa_l \|u_j\|^2_{C^2_j}}
\]

(3.24)

for some \(c > 0\). Also for any \(k \in \{1, \ldots, M\}\), [8] Lemma 5.12 gives

\[
\mathbb{E}^{t_k} [g_{j+(k-1)/M}(X_{k/M}, \xi_k)] \leq 2|\xi_j|/M
\]

(3.25)

with the choice of \(L\) and \(\ell\) as in Lemma 3.1 (cf. [8] Appendix A.2). Therefore

\[
\mathbb{E} [G_j^\Psi (X, \varphi' + \zeta)] \leq e^{c \kappa_l \|u_j\|^2_{C^2_j}} 2|\xi_j|/M G_{j+1} (\overline{X}, \varphi')
\]

(3.26)

which implies the claim with the same choice of \(C_{\Psi}\) as in (3).

\[
\square
\]

Next we define a norm corresponding to the \(\Psi\)-regulators. This norm is defined in the same way as the \(\|\cdot\|_{h,T_j}\)-norm in (3.6) except that there is, apart from the use of \(G_j^\Psi\) instead of \(G_j\), also a change of the parameter \(A\) (large-set regulator) from \(A\) to \(A/2\). This is to compensate a combinatorial factor coming from reblocking in the next section, which will not significantly affect the resulting estimates.

**Definition 3.5.** Define, for \(\Psi_j : P_j \times \mathbb{R}^{A_N} \to \mathbb{R}\) such that \(\Psi_j(X) = \prod_{Y \in \text{Comp}_j(X)} \Psi_j(Y)\),

\[
\|\Psi_j(X)\|_{h,T_j^\Psi(X)} = \sup_{\varphi} G_j^\Psi (X, \varphi)^{-1} \|\Psi_j(X, \varphi)\|_{h,T_j(X, \varphi)}
\]

(3.27)

\[
\|\Psi_j\|_{h,T_j^\Psi} = \sup_{X \in P_j} (A/2)^{|X_j|} \|\Psi_j(X)\|_{h,T_j^\Psi(X)}.
\]

(3.28)
3.3. Contraction estimates. This short section can be regarded as an extension of [8 Section 6], but some results are now generalized to apply to the norm \( \|\cdot\|_{h,T_j^q}(X) \). In the following we write
\[
G_j^*(X,\varphi) = \begin{cases} 
G_j(X,\varphi) & \text{if } * = 0 \\
G_j^q(X,\varphi) & \text{if } * = \Psi.
\end{cases} 
\] (3.29)
Note that by applying Proposition 3.4 to both \( u_j \) and \( u_j' \equiv 0 \) (which also satisfies \([A_0]\)), one obtains that
\[
\mathbb{E}[G_j^*(X,\varphi' + \zeta)] \leq C \Psi 2^{|X|/|G_{j+1}(X,\varphi')}, \quad \text{for both } * \in \{0, \Psi\}. 
\] (3.30)
We also use the notation \( \|\cdot\|_{h,T_j^q}(X) \) for either \( \|\cdot\|_{h,T_j}(X) \) or \( \|\cdot\|_{h,T_j^q}(X) \) for either \( \|\cdot\|_{h,T_j} \) or \( \|\cdot\|_{h,T_j^q} \) when \( * = 0 \) or \( \Psi \), respectively.

Below, we refer to \( 2\pi/\sqrt{3} \)-periodic polymer activities to be the functions \( F(X,\varphi) \) such that \( t \mapsto F(X,\varphi + t) \) is \( 2\pi/\sqrt{3} \)-periodic, see [8 Definition 6.1]. Then its charge-\( q \) part is defined by the Fourier expansion
\[
F(X,\varphi + t) = \sum_{q \in \mathbb{Z}} e^{i\sqrt{\beta}qt} \hat{F}_q(X,\varphi), \quad t \in \mathbb{R},
\] (3.31)
and \( F \) is called neutral if \( F = \hat{F}_0 \). Recall that the norm \( \|\cdot\|_{h,T_j^q}(X) \) in (3.29) depends implicitly on a choice of \( u = (u_j)_j \) and the notion of small sets \( S_j \) at scale \( j \) from [8 Section 4.1].

Proposition 3.6. Let \( X \in S_j \), and let \( F \) be a \( 2\pi/\sqrt{3} \)-periodic polymer activity such that \( \|F\|_{h,T_j^q}(X) < \infty \) where \( * \in \{0, \Psi\} \). Let \( (u_j)_j \) satisfy \([A_0]\). Then for some \( C > 0 \) and \( L \geq L_0 \), the following hold.

- If \( F \) has charge \( q \) with \( |q| \geq 1 \), then for all \( \varphi' \in \mathbb{R}^\Lambda_N \),
  \[
  \|\mathbb{E}[F(X,\varphi' + \zeta)]_{h,T_{j+1}(X,\varphi')} \leq Ce^{\sqrt{\beta}q|\varphi'|} e^{-(|q|-1/2)^2 R_j+1(0)} \|F(X)\|_{h,T_j^q}(X)G_{j+1}(X,\varphi').
  \] (3.32)

- If \( F \) is neutral, then for all \( \varphi' \in \mathbb{R}^\Lambda_N \),
  \[
  \|\mathbb{E}[F(X,\varphi' + \zeta) - F(X,\zeta)]_{h,T_{j+1}(X,\varphi')} \leq CL^{-1}(\log L)^{1/2} \|F(X)\|_{h,T_j^q}(X)G_{j+1}(X,\varphi').
  \] (3.33)

Proof. The first item, (3.32) for \( * = 0 \) is just [8] Lemma 6.13.

For \( * = \Psi \), it suffices to argue that the conclusion of [8 Lemma 6.12] continues to hold under the modified assumption that \( \|F\|_{h,T_j^q}(X) < \infty \). Indeed with this at hand, the proof of (3.32) proceeds exactly as that of [8 Lemma 6.13], except that one invokes (3.30) above rather than [8 Proposition 5.9] towards the end of that proof. As to why the identity [8 (6.43)] still holds, one simply observes upon inspecting its proof that an analogue of the argument in [8 (6.50) - (6.52)] involving \( \|F(X)\|_{h,T_j^q}(X) \) still applies when combining (3.29) (which generalises [8 Lemma 5.13]) with [8 (5.36)].

To see the second point, we proceed similarly as in [8 Lemma 6.17]: writing \( \text{Rem}_0 \mathbb{E}[F](X,\varphi') = \mathbb{E}[F(X,\varphi' + \zeta) - F(X,\zeta)] \), Taylor’s theorem and neutrality of \( F \) give
\[
(\text{Rem}_0 \mathbb{E}[F])(X,\varphi') = \int_0^1 dt \left( 1 - t \right) D\text{Rem}_0 \mathbb{E}[F(X,\zeta + t\varphi')] (\delta\varphi'),
\] (3.34)
where \( \delta\varphi'(x) = \varphi'(x) - \varphi'(x_0) \) for a fixed point \( x_0 \in X \). But since \( D\text{Rem}_0 \mathbb{E}[F(X,\varphi')] = \mathbb{E}DF(X,\varphi' + \varphi') \), the left-hand side of (3.34) is bounded in absolute value by
\[
\frac{1}{h} \int_0^1 dt \left( 1 - t \right) \|\mathbb{E}[DF(X,\varphi' + t\varphi')]_{h,T_{j+1}(X,\varphi')} \| \delta\varphi' \|_{C_{j+1}^2(X^*)} \leq \frac{1}{h} C_d L^{-1} \int_0^1 dt \left( 1 - t \right) \|F(X)\|_{h,T_j^q(X)} \mathbb{E}[G_j^q(X,\varphi' + \zeta)] \| \delta\varphi' \|_{C_{j+1}^2(X^*)},
\] (3.35)
applying [8] (6.31) in the second line. Moreover, $\mathbb{E}[G^*_n(X, t\varphi' + \zeta)] \leq C_g 2^n |G_{j+1}(\overline{X}, t\varphi')|$ for both $* \in \{0, \Psi\}$, as follows readily from (3.30), and by [8] (6.100) (applied with $n = 2$),

$$
\mathbb{E}[G^*_n(X, t\varphi' + \zeta)] \|\delta \varphi'\|_{C^2_{j+1}(X^*)} \leq C (\log L)^{1/2} G_{j+1}(\overline{X}, \varphi').
$$

(3.36)

On the other hand, for $n \geq 1$, $D^n \text{Rem}_0 = D^n$ and thus by [8] (6.31), we immediately get

$$
|D^n(\text{Rem}_0 \mathbb{E} F)(X, \varphi)(f_1, \ldots, f_n)| \leq (C_g L)^{-n} \|D^n \mathbb{E} F(X, \varphi' + \zeta))\|_{h,T_j(X,\varphi')} \prod_{k=1}^n \|f_k\|_{C^2_{j+1}(X^*)}
$$

(3.37)

for some constant $C_g > 0$. We obtain (3.33) from (3.35), (3.37) by summing $\frac{h}{n} \|D^n \text{Rem}_0 \mathbb{E} F(X, \varphi')\|_{h,T_j(X,\varphi')}$ over $n \geq 0$.

Finally, we recall the definition of the reblocking operator from [8] Definition 6.19], defined for a $j$-scale polymer activity $F$ by

$$
SF(X) = \sum_{Y \in \mathcal{P}^\epsilon_j \cap \overline{X}} F(Y), \quad X \in \mathcal{P}^\epsilon_{j+1}
$$

(3.38)

and extended to disconnected $Z \in \mathcal{P}_{j+1}$ by $SF(Z) = \prod_{X \in \text{Comp}_{j+1}(Z)} SF(X)$. The following lemma extends the reblocking estimate from [8] Proposition 6.20. The only difference is that the bound on the right-hand side also holds for the weaker norm $\|\cdot\|_{h,T_j^\varphi}$.

**Proposition 3.7.** There exists a geometric constant $\eta > 0$ and $\varepsilon_{rb} := A^{-\eta}$ such that the following holds. Let $F$ be a polymer activity supported on large sets and satisfy $\|F\|_{h,T_j^\varphi} \leq \varepsilon_{rb}$. Then for any $L \geq 5$, $(A/2)^n \geq L(2eL^2)^{1+\eta}$, $X \in \mathcal{P}_{j+1}$ and $* \in \{0, \Psi\}$,

$$
\|\mathbb{E}[F(X, \cdot + \zeta)]\|_{h,T_j(X^*)} \leq (L^{-1} A^{-1}) |X|^{j+1} \|F\|_{h,T_j^\varphi}.
$$

(3.39)

**Proof.** The case $* = 0$ is exactly [8] Proposition 6.20. The case $* = \Psi$ is obtained by following the same proof, but $A$ is replaced by $A/2$ in view of the definition of $\|\cdot\|_{h,T_j^\varphi}$, see (3.28).

3.4. Choice of parameters. Finally, we explain how the parameters in the norms above are chosen.

First of all, the parameters $\kappa_L$, $c_2$, $c_4$, $c_w$ are chosen as in [8] Section 5) (see the end of Section 5.1 and Remark 5.11 therein), except that we impose the extra conditions resulting from the assumptions of Lemma 3.1 and Proposition 3.4. These do not contradict the conditions from [8] Section 5) as they only impose further smallness conditions on $c_w$, $c_2$, $c_4$.

Next, given a finite-range step distribution $J$, we fix an additional parameter $r \in (0, 1]$ such that (with $C = \sqrt{2c_b c_f^{-1}}$, an absolute constant from [8] Lemma 7.4, cf. also [8] (7.6) and Lemma 6.11) regarding the choices of $c_f$ and $c_h$, respectively)

$$
Cr \leq \rho_j^2,
$$

and we always impose the condition (with $C = 2 \max\{c_f^{-2}, c_f^{-1}\}$, also an absolute constant from [8] Lemma 7.4)

$$
\beta \geq C.
$$

(3.40)

(3.41)

The parameter $h$ is then chosen as in [8] Definition 7.2 as $h = \max\{\epsilon_f^{1/2}, rc_b \rho_j^{-2} \sqrt{\beta}, \rho_j^{-1}\}$. Finally, we will assume that $L \geq L_0$ and $A \geq A_0(L)$ with $L_0$ and $A_0(L)$ chosen to satisfy the assumptions of [8] Theorem 7.7 as well as of those of Lemma 3.1, Proposition 3.6 and Proposition 3.7 above. Moreover, we will always tacitly assume from here on that $L$ is $\ell$-adic, i.e., of the form $L = \ell^M$ for some integer $M \geq 1$, where $\ell := \min\{2^n : 2^n \geq \ell_0\}$ is the smallest dyadic integer larger than $\ell_0$ (with $\ell_0$ as supplied by Lemma 3.1 now fixed since $c_2$ is). This ensures that i) Lemma 3.1 is always in force and ii) eventually, (1.5) can be used (since $L$ is automatically dyadic). Later in Sections 5 and 6 further lower bound conditions on $L$ and $A$ will be imposed, which are consistent with our standing assumptions $L \geq L_0$ and $A \geq A_0(L)$. 

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4 Reblocking the external field

We will use a renormalisation group analysis in Section 3 to study the flow of the partition functions defined by (2.14). Ideally, we would like to write the renormalisation group maps in identical form as those of [8]. Section 7, but the introduction of the external field \( u_j \) breaks the algebraic form of \( U_j \) (see (3.2)) and the symmetry of the system that we used to define the localisation operators \( \text{Loc} \) in [8]. Thus, we will first reduce the problem caused by the external field to a setting where the form of \( U_j \) stays the same as in the original renormalisation group steps and then bound the perturbation created by this operation. This is achieved by the following proposition and lemma. For \( X \in \mathcal{P}_j \), recall that \( \mathcal{P}_j(X) \) denotes the set of all \( j \)-polymers \( Y \) such that \( Y \subseteq X \).

**Definition 4.1.** Given \( u_j \in \mathbb{R}^{A \times N} \) and scale-\( j \) polymer activities \( K_j \) and \( U_j \), define for \( X \in \mathcal{P}_j \),

\[
\mathcal{F}_\Psi[u_j, U_j, K_j; j](X, \varphi) = -K_j(X, \varphi) + \sum_{Y \in \mathcal{P}_j(X)} (e^{U_j(\varphi+u_j)} - e^{U_j(\varphi)}) Y K_j(Y, \varphi + u_j) \tag{4.1}
\]

where

\[
(e^{U_j(\varphi+u_j)} - e^{U_j(\varphi)}) Z \overset{\text{def.}}{=} \prod_{B \in \mathcal{B}_j(Z)} (e^{U_j(B, \varphi + u_j)} - e^{U_j(B, \varphi)}), \text{ for } Z \in \mathcal{P}_j \tag{4.2}
\]

and \( \mathcal{F}_\Psi[u_j, U_j, K_j; j](Z, \varphi) = \prod_{X \in \text{Comp}_j(Z)} \mathcal{F}_\Psi[u_j, U_j, K_j; j](X, \varphi) \) for general \( Z \in \mathcal{P}_j \).

The dependence of \( \mathcal{F}_\Psi \) on the scale \( j \) will often be omitted when it is clear from the context. The following is a purely algebraic statement. Note in particular that the assumptions on \( U_j, K_j \) appearing below will be satisfied by the choices in (3.1), (3.2).

**Proposition 4.2.** Assume that for some scale-\( j \) polymer activities \( K_j \) and \( U_j \),

\[
Z_j(\varphi) = e^{-E_j|A_N|} \sum_{X \in \mathcal{P}_j} e^{U_j(A \setminus X, \varphi)} K_j(X, \varphi), \tag{4.3}
\]

and that \( U_j \) is additive over blocks, i.e., \( U_j(X \cup Y) = U_j(X) + U_j(Y) \) for all \( X \cap Y = \emptyset, X, Y \in \mathcal{P}_j \). Let \( \Psi_j = \mathcal{F}_\Psi[u_j, U_j, K_j; j] \). Then

\[
Z_j(\varphi + u_j) = e^{-E_j|A_N|} \sum_{X \in \mathcal{P}_j} e^{U_j(A \setminus X, \varphi)} \prod_{Z \in \text{Comp}_j(X)} (K_j + \Psi_j)(Z, \varphi). \tag{4.4}
\]

If \( K_j, U_j \) are \( 2\pi/\sqrt{3} \)-periodic, then so is \( \Psi_j \). If \( u_j \) satisfies \([A_u]\) then \( \Psi_j(X) = 0 \) whenever \( B_0^c \cap X = \emptyset \).

**Proof.** This is a result of a simple reblocking argument. Using the assumption

\[
Z_j(\varphi + u_j) = e^{-E_j|A_N|} \sum_{X \in \mathcal{P}_j} e^{U_j(X, \varphi + u_j)} K_j(A \setminus X, \varphi + u_j), \tag{4.5}
\]

by making the substitution

\[
e^{U_j(X, \varphi + u_j)} = \prod_{B \in \mathcal{B}_j(X)} e^{U_j(B, \varphi + u_j)} = \sum_{Y \in \mathcal{P}_j(X)} (e^{U_j(\varphi + u_j)} - e^{U_j(\varphi)}) Y e^{U_j(X \setminus Y, \varphi)} \tag{4.6}
\]

we immediately obtain that

\[
Z_j(\varphi + u_j) = e^{-E_j|A_N|} \sum_{Y \in \mathcal{P}_j} e^{U_j(Y, \varphi)} \sum_{X' \in \mathcal{P}_j(A \setminus Y)} (e^{U_j(\varphi + u_j)} - e^{U_j(\varphi)}) X' K_j(A \setminus (Y \cup X'), \varphi + u_j). \tag{4.7}
\]

Then we arrive at (4.4) after factoring the above expression into connected components of \( A \setminus Y \).

The asserted periodicity of \( \Psi_j \) is plainly inherited from \( K_j, U_j \) and the last remark is a consequence of the fact that \( K_j(X, \varphi + u_j) = K_j(X, \varphi) \) for \( 0 \in X^* \) and \( u_j \) satisfying \([A_u]\). \( \square \)
For the next estimates, recall the definition of the space $\Omega_j^{\mathbb{U}}$ from \cite{8} Definition 7.1 and of $\Omega_j^K$ from \cite{8} Definition 7.2. In particular, the parameters these spaces and their norms depend on are always assumed to satisfy the conditions specified in Section 5.4.

**Lemma 4.3.** Suppose $(u_j)_j$ satisfies $A_n$. Given $U_j$ in form \eqref{1.22} and $K_j$ a $2\pi/\sqrt{3}$-periodic polymer activity, let $\Psi_j = \mathcal{F}[u_j, U_j, K_j]$. Then there exist $C > 0$ and $\varepsilon_\Psi > 0$ such that, whenever $\|\omega_j\|_{\Omega_j} := \max\{\|U_j\|_{\Omega_j^{\mathbb{U}}}, \|K_j\|_{\Omega_j^K}\} \leq \varepsilon_\Psi$,

1. $\|\Psi_j\|_{h,T_j} \leq C \|\omega_j\|_{\Omega_j}$;

2. for $X \in S_j$, $\|\mathbb{E}[\Psi_j(X, \varphi' + \zeta) - \hat{\Psi}_j(X, \varphi')]\|_{h,T_j+1} \leq A^{-|X_j|} \alpha_{\text{Loc}} \Psi_j \|\Psi_j\|_{h,T_j} G_j+1(\varphi, \varphi')$

where $\alpha_{\text{Loc}} = C \log L + C \min\{1, \sum_{q \geq 1} e^{\sqrt{3}q(1/2)\varepsilon_\beta T_j(0)}\}$ and $\hat{\Psi}_j$ is the charge-0 term of $\Psi_j$.

**Proof.** To prove (1), we first notice that by \cite{8} Lemma 7.4 (whose assumptions are satisfied by the assumptions of this lemma) and \eqref{3.13}, for $t \in \{0, 1\}$,

$$
\|U_j(B, \varphi + tu_j)\|_{h,T_j(B, \varphi)} \leq CA^{-1} \|U_j\|_{\Omega_j^{\mathbb{U}}} w_j(B, \varphi + tu_j)^2, \quad B \in B_j 
$$

$$
\|K_j(X, \varphi + tu_j)\|_{h,T_j(X, \varphi)} \leq A^{-|X_j|} \|K_j\|_{\Omega_j^K} G_j^\Psi(X, \varphi), \quad X \in \mathcal{P}_j.
$$

Also, using $\|F - 1\|_{h,T_j(B, \varphi)} \leq \|F\|_{h,T_j(B, \varphi)} e^{\|F\|_{h,T_j(B, \varphi)}}$,

$$
\|U_j(B, \varphi + tu_j) - U_j(B, \varphi)\|_{h,T_j(B, \varphi)} \leq CA^{-1} \|U_j\|_{\Omega_j^{\mathbb{U}}} \max_{t \in \{0, 1\}} w_j(B, \varphi + tu_j) e^{CA^{-1} \|U_j\|_{\Omega_j^{\mathbb{U}}} \max_{t \in \{0, 1\}} w_j(B, \varphi + tu_j)} \tag{4.10}
$$

Using the submultiplicativity of the $\|\cdot\|_{h,T_j(B, \varphi)}$-norm to bound the powers of $e^{U_j(B, \varphi + u_j)} - e^{U_j(B, \varphi)}$ and Proposition \ref{3.3} (3), it follows that

$$
\frac{\|\Psi_j(X, \varphi)\|_{h,T_j(X, \varphi)}}{G_j^\Psi(X, \varphi)} \leq \sum_{Y \in \mathcal{P}_j(X)} \frac{C}{|X_j|} \Psi_j(Y) A^{-|X_j|} \leq C' \|\omega_j\|_{\Omega_j} (A/2)^{-|X_j|} \tag{4.11}
$$

whenever $\|\omega_j\|_{\Omega_j}$ is sufficiently small. This proves (1). To show (2), take $X \in S_j$ and recall that $\Psi_j$ is $2\pi/\sqrt{3}$-periodic to decompose

$$
\Psi_j(X, \varphi) = \sum_{q \in \mathbb{Z}} \hat{\Psi}_{j,q}(X, \varphi) \tag{4.12}
$$

where $\hat{\Psi}_{j,q}$ is the charge-$q$ term of $\Psi_j$. Then apply \eqref{3.32} to bound $\mathbb{E}[\Psi_{j,q}(X, \varphi + \zeta)]$ for $|q| \geq 1$ and \eqref{3.33} to bound $\mathbb{E}[\hat{\Psi}_{j,0}(X, \varphi + \zeta) - \hat{\Psi}_{j,0}(X, \varphi)]$. \hfill $\square$

**5 The renormalisation group map with external field**

To prove the infinite-volume scaling limit we need an extended version of the renormalisation group maps that admits an external field at every scale. In this section we extend the (bulk) renormalisation group map from \cite{8} Section 7 to allow for such an external field. The starting point is the generalisation of the parametrisation of the partition function from \cite{8} (7.4) to take into account a local perturbation. In accordance with \eqref{1.4}, partition functions will now be parametrised as

$$
Z_j(\varphi, \Psi_j; (\Psi_k)_{k < j}|\Lambda_N) = e^{-E_j|\Lambda_N| + e_j} \sum_{X \in \mathcal{P}_j(\Lambda_N)} \prod_{Y \in \text{Comp}_j(X)} (K_j(Y, \varphi; (\Psi_k)_{k < j}) + \Psi_j(Y, \varphi)) \tag{5.1}
$$
and where $e_j$ is a scalar coupling constant (like $E_j$), but originating from a bounded number of blocks near the origin. Then the renormalisation group flow corresponding to

$$Z_{j+1}(\phi', 0; (\Psi_k)_{k\leq j}|\Lambda_N) = E Z_j(\phi' + \zeta, \Psi_j; (\Psi_k)_{k\leq j}|\Lambda_N), \quad (j < N - 1),$$

(5.2)

will be considered. Here recall that $E = E_{\Gamma,j+1}$ for $j \leq N - 1$ and $E = E_{\Gamma,N}$ for the last step $j = N - 1$.

5.1. Renormalisation group flow without external field. When $\Psi_k \equiv 0$ for each $k < j$, then we will just denote $K_j(\cdot; (\Psi_k)_{k\leq j})$ by $K_j(\cdot; 0)$; this corresponds to the setting of $[8]$. Here we briefly recall the main estimates for the renormalisation group map in this setting from $[8]$ Sections 7 and 8. This maps acts on the coupling constants $E_j \in \mathbb{R}$, $U_j$ of the form (5.2), and $K_j(\cdot; 0)$ from $[8]$ Sections 7 and 8. In particular, $U_j$ can be identified with its coupling constants $s_j$ and

$$z_j = (z_j^{(q)})_{q \geq 1}.$$

Also, we use the abbreviations $\omega_j = (U_j,K_j)$ and $\|\omega_j\|_{\Omega_j} = \max\{|U_j|_{\Omega_j}, |K_j|_{\Omega_j}^\beta\}$, where norms are still as in $[8]$ Definitions (7.1)–(7.2) with the parameters they depend on always assumed to satisfy the conditions of Section 5.4.

The following theorem puts together $[8]$ Theorems 7.6 and 7.7 for $j + 1 \leq N$ with its analogue $[8]$ Proposition 9.1 for the last step $j + 1 = N$.

Theorem 5.1. Fix a finite-range step distribution $J$ as in Theorem 1.1. There exist $\varepsilon_{nl}(\beta, A, L)$ such that the following holds for $\|\omega_j\|_{\Omega_j} \leq \varepsilon_{nl}$. For all $N$ and $0 \leq j \leq N - 1$, there is a map

$$\Phi_{\Lambda_j}^{\Delta_j + 1} = (E_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}): (E_j, s_j, z_j, K_j(\cdot; 0)) \mapsto (E_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}(\cdot; 0)),$$

(5.3)

such that (5.1), (5.2) hold with $e_j \equiv 0$ and $\Psi_k \equiv 0$, and $Z_0$ given by (2.6). The maps $E_{j+1} - E_j$, $s_{j+1}$, $z_{j+1}$, and $K_{j+1}$ are functions of $\omega_j$ satisfying

$$|s_{j+1}(\omega_j) - s_j| \leq C A^{-1} |\omega_j|_{\Omega_j},$$

(5.4)

$$|\langle E_{j+1} - E_j \rangle(\omega_j) + s_j \nabla(e_{j+1}^z - e_{j+1}^s)_{\Gamma_{j+1}(0)}| \leq C A^{-1} L^{-2j} |\omega_j|_{\Omega_j},$$

(5.5)

$$z_{j+1}(\omega_j) = L^2 e^{-\frac{1}{2} \beta q^2 T_{j+1}(0)}$$

(5.6)

for some $C > 0$ and there exists $\varepsilon_{nl} > 0$ such that whenever $|\omega_j|_{\Omega_j} \leq \varepsilon_{nl}$, $K_{j+1}$ is continuously (Fréchet-)differentiable and admits a decomposition $K_{j+1} = L_{j+1} + M_{j+1}$ satisfying the estimates

$$||L_{j+1}(\omega_j)||_{\Omega_j} \leq C_1 L^2 \alpha_{\text{Loc}} |\omega_j|_{\Omega_j},$$

(5.7)

$$||M_{j+1}(\omega_j)||_{\Omega_j} \leq C_2(\beta, A, L)|\omega_j|_{\Omega_j},$$

(5.8)

$$||D M_{j+1}(\omega_j)||_{\Omega_j} \leq C_2(\beta, A, L)|\omega_j|_{\Omega_j},$$

(5.9)

for some $C_1, C_2(\beta, A, L) > 0$, where $L_{j+1}$ is linear in $\omega_j$ and

$$\alpha_{\text{Loc}} = CL^{-3}(\log L)^{3/2} + C \min \left\{ 1, \sum_{q \geq 1} e^{\sqrt{\beta q} e^{-(q-1/2)r}} \right\}.$$

(5.10)

The next theorem concerns the existence of initial conditions independent of $N$ such that the renormalisation group flow exists for all $N$, i.e., that for all $N \geq 1$ and all $j \leq N - 1$,

$$(E_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}) = \Phi_{\Lambda_j}^{\Delta_j + 1}(E_j, s_j, z_j, K_j)$$

(5.11)

such that $||(U_j, K_j)||_{\Omega_j} < \varepsilon_{nl}$ for each $j \leq N$. With $j \leq N - 1$ instead of $j \leq N$, the theorem is exactly $[8]$ Proposition 8.1, and the bounds (5.12) for $j = N$ follows from the bounds with $j = N - 1$ by a single application of Theorem 5.1.
The extended version of the K-definition 5.3. The coordinate tion 5.1 and the definition of (bulk) renormalisation group coordinates incorporate a perturbation \( \Psi \). First, recall the definition of polymer activities from [8, Definition 5.2. Extended coordinates. We next define the extended renormalisation group coordinates that incorporate any \( \alpha > 0 \) such that \( \alpha \) is even and invariant under the lattice symmetries. For pairs of polymer activities, define

\[
\| \hat{K}_j \|_{\Omega_j^R} = \max \{ \| K_j(\cdot;0) \|_{h,T_j}, \| \hat{K}_j(\cdot;(\Psi_k)_{k<j}) \|_{h,T_j} \},
\]

(5.13)

Let \( \Omega_j^R \) be the Banach space (cf. [8, Appendix B]) of such pairs where the maximum is finite.

We also need a new definition of the product space of \( (U_j, \hat{K}_j, \Psi_j) \) as follows.

Definition 5.4 (Extended coordinates). Define the normed space of polymer activity perturbations based at the origin by

\[
\Omega_j^\Psi = \{ \Psi_j \text{ is } 2\pi/\sqrt{\beta} \text{-periodic} : \| \Psi_j \|_{h,T_j^\Psi} < \infty, \Psi_j(X) = 0 \text{ if } 0 \notin X^* \}
\]

(5.14)
equipped with the norm \( \| \cdot \|_{\Omega_j^\Psi} = \| \cdot \|_{h,T_j^\Psi} \). Also let \( \Omega_j^\Psi = \Omega_j^T \times \Omega_j^R \times \Omega_j^\Psi \), i.e.,

\[
\Omega_j^\Psi = \{ \omega_j \in (U_j, \hat{K}_j, \Psi_j) : \| \omega_j \|_{\Omega_j^\Psi} < +\infty \}, \quad \| \omega_j \|_{\Omega_j^\Psi} = \max \{ \| U_j \|_{\Omega_j^T}, \| \hat{K}_j \|_{\Omega_j^R}, \| \Psi_j \|_{\Omega_j^\Psi} \}.
\]

(5.15)

Given \( \varepsilon, C > 0 \) also define \( Y_j \equiv Y_j(\varepsilon, C) \subseteq \Omega_j^\Psi \) be the closed subset defined by the conditions

1. \( K_j(X, \varphi;0) = K_j(X, \varphi; (\Psi_k)_{k<j}) \) if \( 0 \notin X^* \);
2. \( \| \Psi_j \|_{\Omega_j^\Psi} \leq C \max \{ \| U_j \|_{\Omega_j^T}, \| \hat{K}_j \|_{\Omega_j^R} \} \) and \( \| \omega_j \|_{\Omega_j^\Psi} \leq \varepsilon \);
3. For \( X \in S_j \),

\[
\| \mathbb{E}[\Psi_j(X, \cdot + \zeta) - \hat{\Psi}_j(X, \zeta)] \|_{h,T_j+1(X)} \leq C \alpha_\Psi A^{-|X|/\alpha_{\text{Loc}}} \| \omega_j \|_{\Omega_j^\Psi}
\]

(5.16)

with \( \alpha_\Psi \) as defined below Lemma 4.3 (2).

In particular, if we define \( \Psi_j = F_\Psi[U_j, K_j(\cdot; (\Psi_k)_{k<j}), U_j; j] \) for given \( U_j, \hat{K}_j \), then, if their assumptions are satisfied, Proposition 4.2 and Lemma 4.3 imply \( (U_j, K_j(\cdot;0), K_j(\cdot; (\Psi_k)_{k<j}), \Psi_j) \in Y_j(\varepsilon, C) \) for some \( \varepsilon, C > 0 \) whenever \( \| U_j \|_{\Omega_j^T}, \| \hat{K}_j \|_{\Omega_j^R} \leq \varepsilon \).

\[20\]
5.3. Definition of the extended renormalisation group map. We will now introduce the extended renormalisation group map with the extra coordinates \( K_j(\cdot; \Phi_k)_{k<j} \) and \( \Phi_j \), which we denote by

\[
\Phi_{j+1} : (E_j, e_j, s_j, z_j, \tilde{K}_j, \Psi_j) \mapsto (E_{j+1}, e_{j+1}, s_{j+1}, z_{j+1}, \tilde{K}_{j+1}, 0)
\]  

\( \Phi \) (cf. [8, (7.12)] for \( \Phi_{j+1} \) which we now call the bulk part of the renormalisation group map); here the \( e_j \) are scalar coupling constants taking the role for the perturbation due to the external field that the \( E_j \) have for the bulk part of the renormalisation group map. In analogy with \( \Phi_{j+1} \), we will also denote the components of the map \( \Phi_{j+1} \) by \((\check{E}_{j+1}, e_{j+1}, \check{U}_{j+1}, K^0_{j+1}, K^\Psi_{j+1}) \) and require that

\[
(\check{e}_{j+1} - \check{E}_{j+1}[\Lambda])(E_j, e_j, \cdot) = (\check{e}_{j+1} - \check{E}_{j+1}[\Lambda])(0, 0, \cdot) + e_j - E_j[\Lambda].
\]  

The last condition can be imposed because the scalar prefactor \( e^{-E_j[\Lambda]+e_j} \) appearing in \( Z_j \) (see (5.11)) is mapped to the corresponding quantity at scale \( j+1 \) and hence does not contribute to the dynamics, see the discussion below [8, (7.12)] for the bulk case. Moreover, when we write \( \check{e}_{j+1}, \check{E}_{j+1} \) without \( e_j, E_j \) as their arguments, they are just \( \check{e}_{j+1}(0, 0, \cdot) \) and \( \check{E}_{j+1}(0, 0, \cdot) \) respectively.

We are thinking of \( \Phi_{j+1} \) as \( \Phi_{j+1} \) with a perturbation, which entails that the \( \check{E}_{j+1}, \check{U}_{j+1} \) and \( K^0_{j+1} \) will be given as in [8, Section 7], i.e., by Definitions 7.8 and 7.9 in that paper respectively. The other coordinates \( e_{j+1} \) and \( K^\Psi_{j+1} \) are defined explicitly as follows. The definition of \( K^\Psi_{j+1} \) is almost the same as that of \( K_{j+1} \) except for the perturbed activity \( \Psi_j \) and the one-point energy \( \check{e}_{j+1} \) arising from it.

Definition 5.5. For \( 0 \leq j \leq N - 1 \), let \( \zeta \) be the centred Gaussian random variable with covariance \( \Gamma_{j+1} \) if \( j \leq N - 2 \) and \( \Gamma_N \) if \( j = N - 1 \). Then for each \( Y \in \mathcal{P}_j \), define the map \((K_j, \Psi_j) \mapsto \check{e}_{j+1}(K_j, \Psi_j) \) by

\[
\check{e}_{j+1}(Y, \check{K}_j, \Psi_j) = \sum_{B \in B(Y : \check{K}_j \cap B_0)} \sum_{Z \in \mathcal{S}_j} \frac{1}{|Z \cap B_0|} \mathbb{E}[\check{\Psi}_{j,0}(Z, \zeta) + \check{K}_{j,0}(Z, \zeta; (\Phi_k)_{k<j}) - \check{K}_{j,0}(Z, \zeta; 0)]
\]  

(5.19)

where we recall that \( B_0 \) is the unique \( j \)-block such that \( 0 \in B_0 \) and let

\[
\check{e}_{j+1}(\check{K}_j, \Psi_j) = \check{e}_{j+1}(\Lambda_N, \check{K}_j, \Psi_j).
\]  

(5.20)

The map \((U_j, K_j, \Psi_j) \mapsto K^\Psi_{j+1} \) is defined by

\[
K^\Psi_{j+1}(U_j, \check{K}_j, \Psi_j, X) = \sum_{X_0, X_1, Z_0(B_{Z''})} e^{\check{e}_{j+1}[T] - \check{e}_{j+1}[T]} e^{U_{j+1}(X \setminus T)}
\]

\[
\times \mathbb{E}
\]

\[
\left[(e^{U_j} - e^{-\check{e}_{j+1}[B]} + \check{e}_{j+1}[B])e^{U_{j+1}}, X_0(\check{K}_j - \check{E}_j K_j)[X_1] \right]
\]

\[
\prod_{Z'' \in \text{Comp}_{j+1}(Z)} J^\Psi_j(B_{Z''}, Z''),
\]  

(5.21)

where the polymer powers follow the convention [8, (7.23), (7.24)], the summation \( * \) is running over disjoint \((j + 1)\)-polymers \( X_0, X_1, Z \) such that \( X_1 \neq Z, B_{Z''} \in B_{j+1}(Z'') \) for each \( Z'' \in \)
and note that

\[ Q_j^\Psi(D, Y, \varphi') = 1_{\bar{\mathcal{S}}_j}(\mathcal{L}c_{Y, D} \mathbb{E}[K_j(Y, \varphi' + \zeta; 0)]) \]

\[ + \frac{1_{D \subset B_j(B_0 \cap Y)} D \in B_j(Y)}{|Y \cap B_0^*|} \mathbb{E}[\hat{\Psi}_j, 0(Y, \zeta) + \hat{K}_j, 0(Y, \zeta; (\Psi_k)_{k < j}) - \hat{K}_j, 0(Y, \zeta; 0)) \]

\[ J_j^\Psi(B, X, \varphi') = 1_{B \in B_j(X)} \sum_{D \in B_j(Y)} \sum_{Y \in \mathcal{S}_j} Q_j^\Psi(D, Y, \varphi')(1_{\mathcal{Y} = X} - 1_{B = X}). \]

for \( D \in B_j, B \in B_j(X), Y \in \mathcal{P}_j \) and \( X \in \mathcal{P}_{j+1} \).

Note that each \((j + 1)\)-block \( B_{Z''} \) appearing in the summation defining \( K_j^\Psi \) is such that \( Z'' \in B_{Z''}^* \) since \( J_j^\Psi(B_{Z''}, Z'', \varphi') \) vanishes whenever \( Z'' \notin \mathcal{S}_j^+ \).

In the remainder of the argument, we will focus on the case \( j \leq N - 2 \), and hence \( \zeta \sim \mathcal{N}(0, \Gamma_{j+1}) \). The argument is identical for the case \( j \leq N - 1 \) because \( \Gamma_{N} \) satisfies the same estimates as \( \Gamma_{N} \).

The next theorem is the extension of Theorem 7.3 with essentially the same proof; see Appendix C for the proof. It shows that \( Z_{j+1} \) defined by the map \( \mathfrak{T}_{j+1} \) is indeed the desired partition function of scale \( j + 1 \).

**Theorem 5.6.** Let \( Z_j(\varphi, \Psi_j; (\Psi_k)_{k < j}|\Lambda) \) and \( Z_{j+1}(\varphi', 0; (\Psi_k)_{k < j}|\Lambda) \) be defined by (5.1) with coordinates \( (E_j, e_j, U_j, \tilde{K}_j, \Psi_j) \) and \( (\tilde{E}_{j+1}, e_{j+1}, U_{j+1}, \tilde{K}_{j+1}, 0) = \tilde{\mathfrak{T}}_{j+1} (E_j, e_j, U_j, \tilde{K}_j, \Psi_j) \) respectively. Then they satisfy (5.2) (and (5.18) holds).

### 5.4. Estimates for the extended renormalisation group map.

Since we have already established estimates on the bulk components \( \mathcal{E}_{j+1}, U_{j+1}, \) and \( \mathcal{K}^0 = \mathcal{K}_j(\cdot; 0) \) of the renormalisation group map in Theorems 7.6 and 7.7, we only need additional estimates for \( \mathcal{E}_{j+1} \) and \( \mathcal{K}_{j+1} \). Since we will not need a stable manifold theorem to tune parameters, a cruder control of these suffices.

**Theorem 5.7.** Let \( (u_j) \) satisfy (5.20) and the parameters be as in Section 3.2. If \( (U_j, \tilde{K}_j, \Psi_j) \in \mathcal{Y}_j(\epsilon, C_{\Psi}) \), for some \( \epsilon > 0 \) and \( C_{\Psi} \) as given by Proposition 5.4,

\[ |\epsilon_j^+(B, \Psi_j, \tilde{K}_j)| \leq CC_{\Psi} A^{-1} \| \omega_j \|_{\mathcal{H}_j}, \quad B \in B_j. \]

**Proof.** Let \( X \in \mathcal{S}_j \) be such that \( 0 \in X^* \) and \( B \in B_j(X) \). By (4.12) and the definition of \( \| \cdot \|_{\mathcal{H}_j} = \| \cdot \|_{h, T_j^\Psi} \), see (3.27)-(3.28),

\[ \| \mathbb{E}[\hat{\Psi}_j, 0(X, \zeta)] \| \leq (A/2)^{-|X|_{1j}} \| \Psi_j \|_{\mathcal{H}_j} \mathbb{E}[G_j^\Psi(X, \zeta)] \]

and by the assumption \( (U_j, \tilde{K}_j, \Psi_j) \in \mathcal{Y}_j \), we also have \( \| \Psi_j \|_{\mathcal{H}_j} \leq C_{\Psi} \| \omega_j \|_{\mathcal{H}_j} \). Similarly,

\[ \| \mathbb{E}[\hat{K}_j, 0(X, \zeta; (\Psi_k)_{k < j}) - \hat{K}_j, 0(X, \zeta; 0)] \| \leq 2 A^{-|X|_{1j}} \| \tilde{K}_j \|_{\mathcal{H}_j} \mathbb{E}[G_j(X, \zeta)] \]

and note that \( \| \tilde{K}_j \|_{\mathcal{H}_j} \leq \| \omega_j \|_{\mathcal{H}_j} \) by definition, see (5.15). By Proposition 3.4 and since \( |X|_{1j} \leq 4 \) for \( X \in \mathcal{S}_j \), we have that

\[ \mathbb{E}[G_j(X, \zeta)] \leq \mathbb{E}[G_j^\Psi(X, \zeta)] \leq C_{\Psi} 2^{|X|_{1j}} \leq 16 C_{\Psi}. \]

Hence, by definition of \( \epsilon_j^+ \) in (5.19), we obtain (5.26).
Theorem 5.8 (Estimate for remainder coordinate). Let 0 ≤ j ≤ N − 1 and the parameters be as in Section 7.4. Further assume (A_0) to hold and let C_Ψ be given by Proposition 3.4. Then the map K_j+1(U_j, K_j, Ψ_j) admits a decomposition

\[ K_j+1(U_j, K_j, Ψ_j) = L_j+1(K_j, Ψ_j) + M_j+1(U_j, K_j, Ψ_j) \]  

(5.30)
such that the following estimates hold: the map L_j+1 is linear in (K_j, Ψ_j) and there exist L_0, A_0(L), ε_n ≡ ε_n(β, A, L, C_Ψ) > 0 (only polynomially small in its arguments), C_1 > 0 independent of A and L and C_2 = C_2(β, A, L, C_Ψ) > 0 (only polynomially large in its arguments) such that for L ≥ L_0, A ≥ A_0(L), \( Ψ_j = (U_j, K_j, Ψ_j) ∈ Y_j(ε_n, C_Ψ) \),

\[ \|L_j+1(K_j, Ψ_j)\|_{Ω_j^+} ≤ C_1C_Ψ (L^2 α_{Loc} \|K_j(·; 0)\|_{Ω_j^X} + α_{Loc} \|Ψ_j\|_{Ω_j^X}), \]  

(5.31)

with \( α_{Loc} \) from Lemma 4.3 and \( M_j+1(U_j, K_j, Ψ_j) \) is continuously Fréchet-differentiable with

\[ \|M_j+1(Ψ_j)\|_{Ω_j^+} ≤ C_2(β, A, L, C_Ψ) \|Ψ_j\|^2 \]  

(5.32)

\[ \|D M_j+1(Ψ_j)\|_{Ω_j^+} ≤ C_2(β, A, L, C_Ψ) \|Ψ_j\| \]  

(5.33)

5.5. Proof of Theorem 5.8: bound of linear part. We first introduce \( L_j+1 \). Proceeding as in \( 8 \), Section 7.4, we may write the terms linear in \( U_j, K_j \) from (5.21) by keeping only the terms in (5.21) with

\[ \#(X_0, X_1, Z) := |X_0|_{j+1} + |Comp_{j+1}(X_1)| + |Comp_{j+1}(Z)| ≤ 1 \]  

(5.34)

and replacing exponentials by their linear approximations. This linearisation process is identical to that of \( 8 \) Section 7.4. For \( X ∈ P_{j+1}^c \), this gives

\[ L_j^Ψ(X, φ') = \sum_{Y, Y = X} 1_{Y ∈ P_j^c} \left( [K_j(Y, ζ + φ'; (Ψ_k)_{k ≤ j−1}) + Ψ_j(Y, ζ + φ')] − 1_{Y ∈ S_j} \sum_{D ∈ B_j(Y)} Q^Ψ(D, Y, φ') \right) \]

(5.35)

\[ + \sum_{D ∈ B_j(Y)} [E[U_j(D, ζ + φ')] + E_j+1|D| − c_{j+1}(D) − U_{j+1}(D, φ') + \sum_{Y ∈ S_j} Q^Ψ(D, Y, φ')] \]

\[ =: L_{j+1}^{(1)}(K_j)(X, φ') + L_{j+1}^{(2)}(Ψ_j)(X, φ') + L_{j+1}^{(3)}(K_j)(X, φ') \]

(5.35)

where, using the choice of \( U_{j+1} \) and \( c_{j+1} \), see \( 8 \), (7.21) and (5.19), respectively, we set

\[ L_{j+1}^{(1)}(K_j)(X, φ') = \sum_{Y, Y = X} 1_{Y ∈ P_j^c} [E[K_j(Y, ζ + φ'; 0) − 1_{Y ∈ S_j} E[Loc Y K_{j, q=0}(Y, φ' + ζ; 0)]], \]

(5.36)

\[ L_{j+1}^{(2)}(Ψ_j)(X, φ') = \sum_{Y, Y = X} 1_{Y ∈ P_j^c} [E[Ψ_j(Y, φ' + ζ)] − 1_{Y ∈ S_j} Ψ_{j, 0}(Y, ζ)], \]

(5.37)

\[ L_{j+1}^{(3)}(K_j)(X, φ') = \sum_{Y, Y = X} 1_{Y ∈ P_j^c} E[D_j(Y, φ' + ζ) − 1_{Y ∈ S_j} D_{j, 0}(Y, ζ)] \]

(5.38)

and

\[ D_j(Y, φ) := K_j(Y, φ; (Ψ_k)_{k ≤ j}) − K_j(Y, φ; 0). \]  

(5.39)

In fact, the \( L_{j+1} \) in Theorem 5.1 (see \( 8 \) Section 7.4) is identical to \( L_{j+1}^{(1)} \), i.e.,

\[ L_{j+1}(K_j(·; 0)) = L_{j+1}^{(1)}(K_j) \]

(5.40)

and also \( L_{j+1}^Ψ \) is a function of \( (K_j, Ψ_j) \), not depending on \( U_j \).
Proof of (5.31) of Theorem 5.8. We will show that the bound (5.31) holds for any choice of \( \varepsilon_{nl} \), where the latter refers to the (bulk) value supplied by Theorem 5.1, see above (5.7). Thus, let \( \omega_j = (U_j, \tilde{K}_j, \Psi_j) \in \mathcal{Y}_j(\varepsilon_{nl}, C\Psi) \). By (5.7) and (5.30), we already know that

\[
\| L_{j+1}^{(1)}(\tilde{K}_j)\|_{\Omega_{j+1}^{\varepsilon}} \leq C_1 L^2 \alpha_{\text{Loc}} \| K_j(\cdot; 0)\|_{\tilde{\Omega}_j^{\varepsilon}}. \tag{5.41}
\]

The estimate on \( L_{j+1}^{(2)} \) follows from the decomposition

\[
L_{j+1}^{(2)}(X, \varphi') = \sum_{Y \in \mathcal{S}_j, 0 \in Y^*} \mathbb{E}[\Psi_j(Y, \varphi' + \zeta) - \hat{\Psi}_{j,0}(Y, \zeta)] + \mathbb{S}(1_{Y \notin \mathcal{S}_j} \mathbb{E}[\Psi_j(\cdot, \cdot + \zeta)])(X, \varphi'). \tag{5.42}
\]

The summation is running over \( Y^* \ni 0 \) now because of the assumption that \( \Psi_j(Y, \varphi) = 0 \) if \( 0 \notin Y^* \) (which is a part of the assumption \( (\tilde{K}_j, \Psi_j) \in \mathcal{Y}_j(\varepsilon_{nl}, C\Psi) \)). Then the first term is bounded by \( C A^{-|X|_{j+1}} \alpha_{\text{Loc}} \| \omega_j \|_{\Omega_j^{\varepsilon}} \) because of the assumption \( (\tilde{K}_j, \Psi_j) \in \mathcal{Y}_j(\varepsilon, C\Psi) \) and (5.16) implied by it (here we also used that \( |Y|_{j+1} \leq |Y_j| \)). The second term is bounded using Proposition 3.7 with \( \# = \Psi \) with \( L \) and \( A = A(L) \) sufficiently large:

\[
\| \mathbb{S}[1_{Y \notin \mathcal{S}_j} \mathbb{E}[\Psi_j(\cdot, \cdot + \zeta)]] \|_{h, T_{j+1}^{(1)}(X)} \leq (L^{-1} A^{-1})^{|X|_{j+1}} \| \Psi_j \|_{\Omega_j^\Psi} \leq C \alpha_{\text{Loc}} A^{-|X|_{j+1}} \| \Psi_j \|_{\Omega_j^\Psi}. \tag{5.43}
\]

Finally, we bound

\[
L_{j+1}^{(3)}(X, \varphi') = \sum_{Y \in \mathcal{S}_j, 0 \in Y^*} \mathbb{E}[D_j(Y, \varphi' + \zeta) - \hat{D}_{j,0}(Y, \zeta)] + \mathbb{S}(1_{Y \notin \mathcal{S}_j} \mathbb{E}[D_j(\cdot, \cdot + \zeta)])(X, \varphi'). \tag{5.44}
\]

Again, the assumption \( D_j(Y, \zeta + \varphi') = 0 \) for \( Y^* \ni 0 \) (which, as above, is a part of the assumption \( (U_j, \tilde{K}_j, \Psi_j) \in \mathcal{Y}_j(\varepsilon_{nl}, C\Psi) \)) effectively restricts the sum in the first term to \( Y^* \ni 0 \), then Proposition 3.6 with case \( \# = 0 \) applies to give the bound \( C A^{-|X|_{j+1}} \alpha_{\text{Loc}} \| \tilde{K}_j \|_{\Omega_j^{\varepsilon}} \). For the second term, Proposition 3.7 with \( \# = 0 \) gives the bound same bound with the same choice of \( L \) and \( A \) as above.

\[ \square \]

5.6. Proof of Theorem 5.8: bound of non-linear part. Analogously as in [8] Section 7.5, the non-linear part \( M_{j+1}^{\Psi} = K_{j+1}^{\Psi} - L_{j+1}^{\Psi} \) (with \( L_{j+1}^{\Psi} \) as defined by the first line of (5.34)) can be decomposed into four parts,

\[
M_{j+1}^{\Psi}(U_j, \tilde{K}_j, X, \varphi') = \sum_{k=1}^{4} M_{j+1}^{(k)}(\Psi_j^k(\omega_j), X, \varphi'). \tag{5.45}
\]
with

\[ M_{j+1}^{\psi}(\mathcal{R}_j^\psi(\omega_j), X) = \sum_{X_0,X_1,Z,(B_{Z''})} e^{E_j+1} X_0 (K_j^\psi - C^\psi K_j^\psi) |X_1| \prod_{Z'' \in \text{Comp}_{j+1}(Z)} J_j^\psi(B_{Z''}, Z'') \]

(5.46)

\[ M_{j+1}^{\psi,2}(\mathcal{R}_j^\psi(\omega_j), X) = \sum_{X_0,X_1,Z,(B_{Z''})} e^{E_j+1} X_0 (K_j^\psi - C^\psi K_j^\psi) |X_1| \prod_{Z'' \in \text{Comp}_{j+1}(Z)} J_j^\psi(B_{Z''}, Z'') \]

(5.47)

\[ M_{j+1}^{\psi,3}(\mathcal{R}_j^\psi(\omega_j), X) = \sum_{X_0=X} \mathbb{E}\left[ \left( e^{U_j} - e^{U_{j+1}} - U_j + U_{j+1}^\psi \right) X_0 \right] \]

(5.48)

\[ M_{j+1}^{\psi,4}(\mathcal{R}_j^\psi(\omega_j), X) = \mathbb{E}\left[ \sum_{Y \in P_j} e^{U_j(Y)} (K_j(\cdot; (\Psi_k)_{k<j})) + \Psi_j)(X-Y) - S(K_j + \Psi_j)(X) \right] \]

(5.49)

where \( \mathcal{R}_j^\psi \equiv \mathcal{R}_j^\psi(\omega_j) \) is short for the collection

\[ (E_j+1)X - \epsilon_j+1(X), U_j, U_{j+1}^\psi, K_j(\cdot; (\Psi_k)_{k<j}) + \Psi_j, K_j^\psi, E^\psi K_j, J_j^\psi(\omega_j), \]

(5.50)

we consider \( X \mapsto E_j+1|X| - \epsilon_j+11_{0 \in X} \) as a polymer activity,

\[ U_{j+1}(X, \varphi') := -E_j+1|X| + \epsilon_j+1(X) + U_j+1(X, \varphi'), \]

(5.51)

and the rest of the notations are those of Definition 5.5. Also notice that \( U_{j+1} \) is used in place of \( U_{j+1} \) to simplify notations. These look somewhat complicated, but in view of [8] Lemma 7.12, it is actually sufficient to check some regularity properties of terms appearing in each \( M_{j+1}^{\psi,(k)} \) to show the differentiability of \( M_{j+1} \) along with the desired estimates (5.32) and (5.33). We now proceed to supply the necessary details. Our discussion follows closely the line of arguments yielding [8] Lemmas 7.11 and 7.12. We first gather the estimates that will lead to a suitable analogue of [8] Lemma 7.11. This is the object of the next lemma.

**Lemma 5.9.** Under the assumptions of Theorem 7.8, for any \( \delta > 0 \), there exists \( \varepsilon = \varepsilon(\delta, \beta, L, C_\psi) > 0 \) such that for \( \omega_j \in \mathcal{Y}_j(\varepsilon, C_\psi) \), \( B \in B_{j+1}, \), \( k \in \{0,1,2\}, \)

\[ \|\mathcal{U}(B, \varphi)\|_{h,T_j(\varphi,B)} \leq C(\delta, \beta, L, C_\psi)(1 + \delta \kappa w_{j}^2(B, \varphi))\|\omega_j\|_{T_j}, \]

(5.52)

\[ \|e^{\mathcal{U}(B, \varphi)} - \sum_{m=0}^{k} \frac{1}{m!}(\mathcal{U}(B, \varphi))^m\|_{h,T_j(\varphi,B)} \leq C(\delta, \beta, L, C_\psi)e^{\delta \kappa w_{j}^2(B, \varphi)}\|\omega_j\|_{T_j}^{k+1}, \]

(5.53)

where \( \mathcal{U} \) is either \( U_j \) or \( U_{j+1}^\psi \). The same inequalities hold with \( \mathcal{U}(B) \) and \( C(\delta, \beta, L, C_\psi) \) replaced by \( E_j+1|B| - \epsilon_j+1(B) \) and \( C(\beta, L, C_\psi) \), respectively, and \( \delta \) set to 0.

**Proof.** For \( \mathcal{U} = U_j \) or \( E_j+1|B| \), the asserted bounds are then an immediate consequence of [8] Lemma 7.14. For the remaining choices of \( \mathcal{U} \), recall the definition and the bound on \( U_{j+1} \) provided by [8] (7.39) and [8] (7.77) that for \( B \in B_{j+1}, \)

\[ \|U_{j+1}(B, \varphi)\|_{h,T_j(\varphi,B)} \leq C(\delta, \beta, L)(1 + \delta \kappa w_{j}^2(B, \varphi))\|\omega_j\|_{T_j}. \]

(5.54)

Also by Theorem 5.7 we have

\[ |\epsilon_j+1(B, \omega_j)| \leq CC_\psi A^{-1}\|\omega_j\|_{T_j}, \]

(5.55)
and since $\overline{U}_j^\varphi(X, \varphi) = \varphi_{j+1}(X) + \overline{U}_{j+1}(X, \varphi')$ by (5.51), we have

$$\|\overline{U}_{j+1}^\varphi(B, \varphi)\|_{h, T_j(B, \varphi)} \leq C(\delta, \beta, L, C_\varphi)(1 + \delta c_w \kappa L w_j(B, \varphi^2))\|\varphi\|_{\Omega_j},$$

(5.56) showing (5.52). For the second inequality, assume $\varepsilon \leq 1/C(\delta, \beta, L, C_\varphi)$ and $\|\varphi\|_{\Omega_j} \leq \varepsilon$, then the submultiplicativity of norm and (5.56) shows

$$\|e^{\overline{U}_{j+1}^\varphi}\|_{h, T_j(B, \varphi)} \leq e^{\overline{U}_{j+1}^\varphi\|_{h, T_j(B, \varphi)} \leq C(\delta, \beta, L, C_\varphi)e^{\delta c_w \kappa L w_j(B, \varphi^2)}.$$  

(5.57)

Then (5.56) and (5.57) shows (5.53).

We now state the analogue of [8, Lemma 7.11] in the present context.

**Lemma 5.10.** Under assumptions of Theorem 5.8, there exist $\varepsilon \equiv \varepsilon(\beta, L) > 0$, $\eta > 0$, $C \equiv C(c_w, \beta, L, C_\varphi)$ and $C_\varphi \equiv C(\varphi, \beta, L)$ such that

$$\|D e^{U_j(B, \varphi)}\|_{h, T_j(B, \varphi)} \leq C e^{c_w \kappa L w_j(B, \varphi^2)}$$

(5.58)

$$\|D^2 e^{U_j(B, \varphi)}\|_{h, T_j(B, \varphi)} \leq C e^{c_w \kappa L w_j(B, \varphi^2)}$$

(5.59)

$$\|D J^\varphi_j(B, Z, \varphi)\|_{h, T_j(B, \varphi)} \leq C A^{-1} e^{c_w \kappa L w_j(B, \varphi^2)}$$

(5.60)

$$\|D K^\varphi_j(Z, \varphi)\|_{h, T_j(Z, \varphi)} \leq C A A^{-1+(1+\eta)} |Z|_{j+1} e^{c_w \kappa L w_j(Z, \varphi^2)}$$

(5.61)

for $B \in B_{j+1}$, $Z \in P_{j+1}$ whenever $\omega_j \in \mathcal{Y}_j(\varepsilon(L), C_\varphi)$ and $\Omega_j$ is either $U_j$ or $\overline{U}_{j+1}$ or $\varepsilon_{j+1}B - \varepsilon'_{j+1}(B)$. In the final case, $e^{c_w \kappa L w_j(B, \varphi^2)}$ can be omitted.

**Proof.** The proof is mostly the same as that of [8, Lemma 7.11]. The bounds (5.58) and (5.59) are consequences Lemma 5.9; cf. the discussion around [8, (7.16)–(7.17)]. The bound (5.62) follows directly from (5.60) (cf. [8, Lemma 7.15]), which in turn follows from a bound on $\|D Q^\varphi_j\|_{h, T_j(Y, \varphi)}$ (namely, (5.66) below). To obtain this bound, notice that for $D \in B_J$, $Y \in S_j$,

$$Q^\varphi_j(D, Y, \varphi') = Q_j(D, Y, \varphi') + 1_{Y \in S_j} \int_{D \subset Y \cap B_0} \mathbb{E}[\hat{\Psi}_{j,0}(Y, \zeta) + \hat{D}_{j,0}(Y, \zeta)]$$

(5.63)

where $D_j(Y, \zeta) = K_j(Y, \zeta; (\Psi_k)_{k<j}) - K_j(Y, \zeta; 0)$ and $Q_j$ is defined by [8, (7.26)]. But [8, (7.53)] already bounds $Q_j(D, Y, \varphi')$, so we actually only have to bound $\mathbb{E}[\hat{\Psi}_{j,0}(Y, \zeta) + \hat{D}_{j,0}(Y, \zeta)]$. But

$$\|\mathbb{E}[\hat{\Psi}_{j,0}(Y, \zeta)]\|_{h, T_j(Y, \varphi')} \leq C(A/2)^{-|Y_j|} \|\Psi_j\|_{O^\varphi_j} \mathbb{E}[G^\varphi_j(Y, \zeta)] \leq C_\varphi C(A/4)^{-|Y_j|} \|\Psi_j\|_{O^\varphi_j},$$

(5.64)

$$\|\mathbb{E}[\hat{D}_{j,0}(Y, \zeta)]\|_{h, T_j(Y, \varphi')} \leq C A^{-|Y_j|} \|\bar{K}_j\|_{O^\varphi_j} \mathbb{E}[G_j(Y, \zeta)] \leq C(A/2)^{-|Y_j|} \|\bar{K}_j\|_{O^\varphi_j}$$

(5.65)

so it follows that $Q^\varphi_j$ is differentiable with

$$\|D Q^\varphi_j(D, Y, \varphi')\|_{h, T_j(Y, \varphi')} \leq C A^{-|Y_j|} e^{c_w \kappa L w_j(D, \varphi')}.$$  

(5.66)

For (5.61), notice that if we write $\mathcal{F}$ for the function

$$\mathcal{F}(U_j, K_j) = \mathcal{K}_j := \sum_{Y \in P_j} e^{U_j(Y)} K_j(Y),$$

(5.67)

it follows that $\mathcal{K}_j = \mathcal{F}(U_j, K_j; (\Psi_k)_{k<j}) + \Psi_j)$. So by inspecting the proof of [8, Lemma 7.16], one sees that $D \mathcal{K}_j^\varphi$ satisfies exactly the same bound as $D \mathcal{K}_j$ (see [8, (7.28), (7.61)] for its definition and bound), only with $A$ replaced by $A/2$, i.e.,

$$G^\varphi_j(Z, \varphi)^{-1}\|D \mathcal{K}_j^\varphi(Z, \varphi)\|_{h, T_j(Z, \varphi)} \leq C A(A/2)^{-(1+\eta)} |Z|_{j+1}.$$  

(5.68)

But for $A$ large enough, this is less than $C A A^{-(1+\eta')}|Z|_{j+1}$ for some $\eta' \in (0, \eta)$ as needed.
Proof of continuous differentiability of $\mathcal{M}^\Psi_{j+1}$ and \((5.32), (5.33))$. For $j \leq N - 2$, [8, Lemma 7.12] implies that the bounds on $\bar{\mathcal{M}}^\Psi_{j+1}(\omega_j) = (\mathcal{E}_{j+1}, U_j, \overline{\mathcal{K}}^\Psi_{j+1}, K_j(\cdot; (\Psi_k)_{k<j}) + \Psi_j, \overline{\mathcal{R}}^\Psi_{j+1}, \mathcal{E}^\Psi_{j+1}, J^\Psi_{j+1})$ provided by Lemma 5.9 and Lemma 5.10 are sufficient to prove the differentiability and bounds on $\mathcal{M}^\Psi_{j+1}(\omega_j)$, $k \in \{1, 2, 3, 4\}$. In fact, (6.61) now imposes bound in terms of $G^\Psi_j$ instead of $G_j$ but this does not affect the proof because [8, Lemma 7.12] uses the properties of $G_j$ that (1) $e^{\omega_u \cdot \varepsilon_j(X)^2} G_j(Y) \leq G_j(X \cup Y)$ if $X \cap Y = \emptyset$, (2) $G_j(X) = \prod_{\text{comps}(X)} G_j(X)$ and (3) $E[G_j(X, \varphi') \leq 2|\mathcal{G}_j(X, \varphi')]$. But the same properties are verified on account of Proposition 3.4 while the constant $C_\Psi$ only contributes as a multiplicative factor in each estimate.

For $j = N - 1$, all of the arguments of Sections 5.5–5.6 continue to apply as $\Gamma_N$ satisfies exactly the same bounds as required for $\Gamma_j$ when $j = N$.

6 Proof of Theorem 2.3

In Section 5, we defined the extended renormalisation map $\Phi_{j+1}$ corresponding to the finite torus $\Lambda_N$. In this section, we analyse the limit (as $N \to \infty$) of the final renormalisation group coordinates $(E_N, e_N, U_N, K_N, \Psi_N)_{N \geq 0}$ obtained by the iteration of the renormalisation group map up to scale $N$, with initial conditions provided by Theorem 5.2. This limit is not exactly as the same as the limit $j \to \infty$ of the local infinite volume limit; in the former limit the size of the torus $\Lambda_N$ is also varying as $N \to \infty$. For this reason, we temporarily write the dependence on $\Lambda_N$ of the coordinates explicitly in the following theorem and the corollary, e.g., the coordinates will be denoted $(E^\Lambda_N, e^\Lambda_N, U^\Lambda_N, K^\Lambda_N, \Psi^\Lambda_N)$ and the renormalisation group map will be denoted $\Phi^\Lambda_{j+1}$ and $\overline{\Phi}^\Lambda_{j+1}$ for the bulk and the extended flows, respectively.

Theorem 6.1. Let $J$ be any finite-range step distribution as in Theorem 1.1, choose the parameters as in Section 3.4, assume that $\beta \geq \beta_0$ as in Theorem 5.2, and let $(E^\Lambda_N, U^\Lambda_N, K^\Lambda_N)_{N \geq 0}$ be the (bulk) renormalisation group map on $\Lambda_N$ as in Theorem 5.2 i.e.,

$$(E^\Lambda_N, U^\Lambda_N, K^\Lambda_{j+1}(\cdot; 0)) = \Phi^\Lambda_{j+1}(E^\Lambda_j, U^\Lambda_j, K^\Lambda_j(\cdot; 0)), \quad 0 \leq j \leq N - 1.$$ (6.1)

Assume that $(u_j)_{j \geq 0}$ satisfies $[\mathcal{A}_u]$ and define $(e_j)_{0 \leq j \leq N}, (\Psi^\Lambda_j)_{0 \leq j \leq N}, (K^\Lambda_j(\cdot; (\Psi^\Lambda_k)_{k<j}))_{0 \leq j \leq N}$ inductively by

$$\Psi^\Lambda_j = \mathcal{F}_\Psi[u_j, U^\Lambda_j, K^\Lambda_j(\cdot; (\Psi^\Lambda_k)_{k<j}); j] \quad (6.2)$$
$$K^\Lambda_{j+1}(\cdot; (\Psi^\Lambda_k)_{k<j}) = K^\Lambda_{j+1}(U^\Lambda_{j+1}, K^\Lambda_{j+1}, \Psi^\Lambda_{j+1}) \quad (6.3)$$
$$e^\Lambda_j = e^\Lambda_{j+1} + e^\Lambda_{j+1}(K^\Lambda_{j+1}, \Psi^\Lambda_{j+1}) \quad (6.4)$$

with initial conditions $K^\Lambda_0(X) = 1_{X=0}$ and $e^\Lambda_0 = 0$. Then there exists $C > 0$ such that for all $N \geq 1$ and $0 \leq j \leq N$, if $L$ and $j_u$ are large enough, then

$$\max\{\|\hat{K}^\Lambda_j\|_{\Omega^\Lambda_j}, \|\hat{\Psi}^\Lambda_j\|_{\Omega^\Lambda_j}\} \leq C L^{-\alpha j}, \quad (6.5)$$

with decay factor $\alpha \equiv \alpha(\beta, J) > 0$ as in Theorem 6.2.

Proof. The asserted exponential decay in $j$ (uniform in $N$) is almost immediate from Theorems 5.1, 5.2, and 5.8 as we now explain. Throughout the remainder of the proof, we drop the superscripts $N$ and $\Lambda_N$. All the following estimates hold uniformly in $N$. By Theorem 5.2, it has already been shown that $\omega_j \in \mathcal{Y}_j(\varepsilon_n, C_\Psi)$ and $\|(U_j, K_j(\cdot; 0))\|_{\Omega_j} \leq C L^{-\alpha_j}$ for all $j \leq N$. We will now argue that there is $C' > 0$ such that, for all $j$, both

$$\|K_j(\cdot; (\Psi_k)_{k<j}) - K_j(\cdot; 0)\|_{\Omega^\Lambda_j} \leq C' L^{-\alpha_j} \quad (6.6)$$
$$\|\Psi_j\|_{\Omega^\Lambda_j} \leq C_\Psi(C + C') L^{-\alpha_j} \quad (6.7)$$
hold, where \( C \) refers to the constant in the bound \( \|(U_j, K_j(\cdot; 0))\|_{\Omega_j} \leq CL^{-\alpha_j} \). The claim then immediately follows by combining these two estimates with (6.12). We now show these two bounds by induction. For \( j \leq j_a \) there is nothing to prove, as \( \Psi_j \equiv 0 \) and \( K_j(\cdot; (\Psi_k)_{k<j}) \equiv K_j(\cdot; 0) \). Now assume (6.6) and (6.7) hold for some \( j \in [j_a, N] \). If \( j_a \) is sufficiently large, then these bounds and Lemma 4.3 imply that \( \omega_j \) falls into the admissible range of Theorem 5.8 i.e., \( (U^A_j, K^A_j, \Psi^A_j) \in \mathcal{Y}_j(\bar{\varepsilon}, C_\Psi) \). Then (5.31) and linearity of \( L^j_{\Psi \omega_j} \) give for \( \omega_j = (U_j, \bar{K}_j, \Psi_j) \in \mathcal{Y}_j(\varepsilon, C_\Psi) \) (with \( \varepsilon \equiv \bar{\varepsilon}_n \))

\[
\| L^j_{\Psi \omega_j} (\cdot; 0) - L^j_{\Psi \omega_j} (\cdot; (\Psi_k)_{k<j}, 0) \|_{\Omega_{j+1}^K} \leq C_1 C_\Psi \alpha_{\text{Loc}} \| K^A_j (\cdot; (\Psi_k)_{k<j}) - K^A_j (\cdot; 0) \|_{\Omega_j^K} \tag{6.8}
\]

and (5.33) gives

\[
\| M^j_{\Psi \omega_j} (\cdot; 0) - M^j_{\Psi \omega_j} (\cdot; (\Psi_k)_{k<j}, 0) \|_{\Omega_{j+1}^K} \leq C_2 \| K^A_j (\cdot; (\Psi_k)_{k<j}) - K^A_j (\cdot; 0) \|_{\Omega_j^K}. \tag{6.9}
\]

Here \( ((\Psi_k)_{k<j}, 0) \) refers to \( (\Psi_k')_{k<j} \) with \( \Psi_k' = \Psi_k \) for \( k < j \) and \( \Psi_j' = 0 \). For \( \varepsilon \) sufficiently small in \( \alpha_{\text{Loc}}^\Psi \) and \( (C_2 (\beta, A, L))^{-1}, (6.6), (6.8) \) and (6.9) imply

\[
\| K^A_{j+1} (\cdot; 0) - K^A_{j+1} (\cdot; (\Psi_k)_{k<j}, 0) \|_{\Omega_{j+1}^K} \leq 2C_1 C_\Psi C' \alpha_{\text{Loc}} L^{-\alpha_j} \tag{6.10}
\]

Similar arguments gives

\[
\| K^A_{j+1} (\cdot; (\Psi_k)_{k<j}) - K^A_{j+1} (\cdot; (\Psi_k)_{k<j}, 0) \|_{\Omega_{j+1}^K} \leq 2C_1 C_\Psi \alpha_{\text{Loc}} \| \Psi_j \|_{\Omega_j^\Psi} \tag{6.11}
\]

Together with (6.7), these inequalities imply

\[
\| K^A_{j+1} (\cdot; (\Psi_k)_{k<j}) - K^A_{j+1} (\cdot; 0) \|_{\Omega_{j+1}^K} \leq C'' L^\alpha \alpha_{\text{Loc}} L^{-\alpha(j+1)}. \tag{6.12}
\]

To proceed, we need the fact that \( L^\alpha \leq C (L^2 \alpha_{\text{Loc}})^{-1} \) for some \( C > 0 \), see the last remark of Theorem 5.2. Also since \( \alpha_{\text{Loc}}^\Psi = (\log L)^{-1} O(L^2 \alpha_{\text{Loc}}) \), we now have \( L^\alpha \alpha_{\text{Loc}} \leq C/\log L \) and therefore

\[
\| K^A_{j+1} (\cdot; (\Psi_k)_{k<j}) - K^A_{j+1} (\cdot; 0) \|_{\Omega_{j+1}^K} \leq \frac{C'}{\log L} L^{-\alpha(j+1)} \tag{6.13}
\]

which completes the induction step for (6.3) after choosing \( C' \log L \geq C''' \). To obtain (6.7) at scale \( j+1 \), one now uses that \( \|(U_{j+1}, K_{j+1}(\cdot; 0))\|_{\Omega_j} \leq C L^{-\alpha(j+1)} \) by Theorem 5.2 and the fact that \( \| K_{j+1} (\cdot; (\Psi_k)_{k<j}) \|_{\Omega_{j+1}^K} \leq (C + C') L^{-\alpha(j+1)} \) which follows by combining with the newly proved (6.6) at scale \( j+1 \), along with the fact that \( \| \Psi_{j+1} \|_{\Omega_{j+1}^\Psi} \leq \| \Psi \|_{\Omega_{j+1}^\Psi} \) which is uniformly bounded in \( N \) by Lemma 4.3.

\[ \square \]

**Corollary 6.2.** Under the assumptions of Theorem 6.7,

\[
\| e_{\alpha^N}^\Psi \| \leq O \left( \sum_{j \geq j_a} \| \tilde{K}^A_j \|_{\Omega_j^{\Psi}} \right) \leq O(L^{-\alpha j_a}) \tag{6.14}
\]

for \( j_a \) from \( (A_u) \) uniformly in \( N \).

**Proof.** We start from the the explicit expression \( e_{\alpha}^A_n = \sum_{j<n-1} \epsilon_{j+1} (\tilde{K}^A_j, \Psi^A_j) \). To see that the sum actually only starts from \( j = j_a \), note that, by construction, \( \Psi_k \equiv 0 \) for \( k < j_a \), and hence \( K^A_j (\cdot; (\Psi_k)_{k<j}) = K^A_j (\cdot; 0) \) for \( j \leq j_a \) which implies that \( \epsilon_{j+1} = 0 \) by its definition. (5.19). Hence \( |e_{\alpha}^A_n| \leq C \sum_{j_a \leq j \leq N-1} \| \tilde{K}^A_j \|_{\Omega_j^{\Psi}} \) and the sum is uniformly bounded in \( N \) because \( \| \tilde{K}^A_j \|_{\Omega_j^{\Psi}} = O(L^{-\alpha j_a}) \) uniformly in \( N \).

\[ \square \]
Theorem 2.3 is almost direct from the above two results.

Proof of Theorem 2.3 We first note that Lemma 2.2 implies that \((u_j)_{j \geq 0}\) defined by (2.10) satisfies \((A_n)\) with some \(j_n = j_f\), and so Theorem 6.1 and Corollary 6.2 may be used. We then assume that \(\beta \geq \beta_0(J)\) with \(\beta_0(J)\) as supplied by Theorem 5.2; pick \(L = L(J)\) large enough (and of the form specified in Section 5.3) such that the conclusions Theorem 6.1 hold and set \(A(J) = A_0(L)\) for this choice of \(L\).

For a constant field \(\zeta\), we have \(\nabla \zeta = 0\) and \(G_N^W(X, \zeta) = G_N^W(X, 0)\), so, with \(W_N\) denoting the non-gradient term (involving the cosines) in (3.2) with \(j = N\),
\[
e^{E_N|\Lambda_N| - e_N} Z_N(u, \zeta + u_N) = e^{\frac{1}{2}N |\nabla(\zeta + u_N)|^2 + W_N(\Lambda_N, \zeta + u_N)} + K_N(\Lambda_N, \zeta + u_N; (\Psi_k)_{k<N})
\]
\[
= e^{\frac{1}{2}N |\nabla \zeta|^2 + W_N(\Lambda_N; \zeta)} + K_N(\Lambda_N, \zeta; (\Psi_k)_{k<N}) + \Psi_N(\Lambda_N, \zeta)
\]
\[
= 1 + O(||W_N||_{\Omega_N^W} + ||K_N(\cdot; (\Psi_k)_{k<N})||_{\Omega_N^W}) G_N^W(\Lambda_N, 0)\] (6.15)
whenever \(||W_N||_{\Omega_N^W} \leq 1\) and we have used \(\Psi_N = \mathcal{F}_W[u_N, U_N, K_N(\cdot; (\Psi_k)_{k<N}); N]\) and Proposition 4.2 for the second equality. Also Lemma 4.3 bounds \(\Psi_N\) in terms of \(K_N(\cdot; (\Psi_k)_{k<N})\) in the third equality. Then by (2.15) and (6.15),
\[
\tilde{Z}_N(u, 0) = \mathbb{E}_{t_N} Z_N(u, \zeta + u_N)
\]
\[
= e^{-E_N|\Lambda_N| + e_N} 1 + O(||W_N||_{\Omega_N^W} + ||K_N(\cdot; (\Psi_k)_{k<N})||_{\Omega_N^W})\] (6.16)
For \(||W_N||_{\Omega_N^W} + ||K_N(\cdot; 0)||_{\Omega_N^W}\) sufficiently small, it follows that
\[
\frac{\tilde{Z}_N(u, 0)}{Z_N(0, 0)} = \exp(e_N) \frac{1 + O(||W_N||_{\Omega_N^W} + ||K_N(\cdot; (\Psi_k)_{k<N})||_{\Omega_N^W})}{1 + O(||W_N||_{\Omega_N^W} + ||K_N(\cdot; 0)||_{\Omega_N^W})} \] (6.17)
But \(||W_N||_{\Omega_N^W} \leq CL^{-\alpha N}\) by Theorem 5.2; \(||K_N||_{\Omega_N^W} \leq C_1 L^{-\alpha N}\) by Theorem 6.1 and \(|e_N| \leq C_2 L^{-\alpha J}\) by Corollary 6.2. This implies the desired conclusion.

A Existence of infinite-volume limit

We recall the Fröhlich–Park–Ginibre inequalities: Let \(\Lambda\) be finite, let \(C\) be a positive definite matrix, and let \(\langle \cdot \rangle_C\) be the expectation of the associated (generalised) Discrete Gaussian model:
\[
\langle F \rangle_C \propto \sum_{\sigma} e^{-\frac{1}{2} \langle \sigma, C^{-1} \sigma \rangle} F(\sigma).
\] (A.1)

By taking limits, the definition of \(\langle \cdot \rangle_C\) can also be extended to \(C\) positive semidefinite. The finite volume states \(\langle \cdot \rangle^\Lambda_{J,\beta}\) given by (1.3) then correspond to \(C = \beta(-\Delta J)^{-1}\) when \(\sigma\) is identified up to constants (as we do), see also [8, Lemma 2.1]. The results of [30] Section 3 (see also [A3 Proposition 1.2]) then imply that for \(f: \Lambda \to \mathbb{R}\) with \(\sum f = 0\):
\[
\langle e^{if(\sigma)} \rangle^\Lambda_{J,\beta} \leq e^{\frac{1}{2}(f, -\Delta J)^{-1} f},
\] (A.2)
\[
\langle (f, \sigma)^2 \rangle^\Lambda_{J,\beta} \leq (f, -\Delta J)^{-1} f.
\] (A.3)
Moreover, [30] Corollary 3.2 (1)] implies that
\[
\langle e^{if(\sigma)} \rangle^C_1 \leq \langle e^{if(\sigma)} \rangle^C_2 \quad \text{if} \  C_2 \leq C_1.
\] (A.4)

Proposition A.1. Let \(L > 1\) be an integer. For any finite-range step distribution \(J\) and any sequence of discrete tori \(\Lambda_N\) with side lengths \(L^N\), with \(N \in \mathbb{N}\), the measures \(\langle \cdot \rangle^\Lambda_{J,\beta}\) converge weakly as \(N \to \infty\) (when the field is identified up to constants). For any \(f: \mathbb{Z}^d \to \mathbb{R}\) with compact support and \(\sum f = 0\), one also has \(\langle e^{if(\sigma)} \rangle^\Lambda_{J,\beta} \to \langle e^{if(\sigma)} \rangle\) where \(\langle \cdot \rangle = \lim_{N \to \infty} \langle \cdot \rangle^\Lambda_{J,\beta}\) is the weak limit.
Proof. We consider the Laplacian $-\Delta^N$ as an operator on $\ell^2(\mathbb{Z}^d)$ with domain

$$D(-\Delta^N) = \{ f \in \ell^2(\mathbb{Z}^d) : f(0) = 0, f(x) = f(x + L^N y) \text{ for any } y \in \mathbb{Z}^d \}.$$  \hfill (A.5)

Then clearly $D(-\Delta^N) \subset D(-\Delta^{N+1})$ and $-\Delta^N = -\Delta^{N+1}$ on $D(-\Delta^N)$. This implies $-\Delta^N \geq -\Delta^{N+1}$ and hence $(-\Delta^N)^{-1} \leq (-\Delta^{N+1})^{-1}$. From (A.2), it follows that for any $f : \mathbb{Z}^d \to \mathbb{R}$ compactly supported and with $\sum f = 0$, $S_N(f) = \langle e^{i(f,\phi)} \rangle_{J,\beta}^N$ is increasing in $N$. In particular, since also $S_N(f) \leq 1$, the limit $S(f) = \lim_{N \to \infty} S_N(f)$ exists. To show $S(f)$ is the characteristic function of a probability measure on $(2\pi \mathbb{Z})^{2^d}/\text{constants}$ to which $\langle \cdot \rangle_{J,\beta}^N$ converges weakly, we will apply Minlos’ theorem. To this end, we consider $(2\pi \mathbb{Z})^{2^d}/\text{constants}$ as a topological vector space with the topology defined by the condition that $\varphi_k \to \varphi$ in $(2\pi \mathbb{Z})^{2^d}/\text{constants}$ if $\langle \varphi_k, g \rangle \to \langle \varphi, g \rangle$ for all compactly supported $g : \mathbb{Z}^d \to \mathbb{R}$ with $\sum g = 0$. In particular, $(2\pi \mathbb{Z})^{2^d}/\text{constants}$ is the dual of a nuclear space. To apply Minlos’ theorem we need to check that $S$ is continuous in this topology. But this is immediate from the correlation inequality (A.3) which implies that for any $g : \mathbb{Z}^d \to \mathbb{R}$ with compact support and $\sum g = 0$,

$$|S(f + g) - S(f)| = \lim_{N \to \infty} |S_N(f + g) - S_N(f)| \leq \lim_{N \to \infty} (g, (-\Delta^N)^{-1} g) = (g, (-\Delta)^{-1} g), \quad (A.6)$$

from which the continuity is clear.

The final statement about the convergence of $\langle e^{i(f,\phi)} \rangle_{J,\beta}^N$ follows from the weak convergence and (A.2) which implies that the random variables $e^{i(f,\phi)}$ are uniformly integrable. \hfill \Box

It is also standard, see [37] and analogous extensions to the gradient Gibbs setting as in [33,34], that any limit as in the previous proposition is translation invariant and satisfies the gradient Gibbs property. Moreover, the limit satisfies the analogous correlation inequalities.

Proposition A.2. The measure $\langle \cdot \rangle_{J,\beta}^{2^d}$ has tilt 0, i.e., for each gradient Gibbs state in the ergodic decomposition of $\langle \cdot \rangle_{J,\beta}^{2^d}$ the gradient field has mean 0.

Proof. The proof is analogous to that of [34, Theorem 3.2]. The correlation decay can be replaced by the following application of the Riemann–Lebesgue lemma. For $g : \mathbb{Z}^d \to \mathbb{R}$ with compact support, where now $\nabla \sigma : \mathbb{Z}^d \to \mathbb{R}^d$ denotes the vector of discrete forward derivatives, (A.3) implies

$$\langle (g, \nabla \sigma)^2 \rangle_{J,\beta}^{2^d} \leq C \int_{[-\pi,\pi]^d} \frac{|\hat{g}(p) \cdot p|^2}{|p|^2} dp. \quad (A.7)$$

Thus the distributional Fourier transform of $\langle \nabla \sigma, \sigma(0) \nabla \sigma(x) \rangle$ is integrable in the Fourier variable. From this, the Riemann-Lebesgue lemma implies that

$$\langle \nabla \sigma, \sigma(0) \nabla \sigma(x) \rangle_{J,\beta}^{2^d} \to 0 \quad (|x| \to \infty). \quad (A.8)$$

In particular, for every $i = 1, \ldots, d$, with $Q_R = [-R, R]^2 \cap \mathbb{Z}^2$,

$$\left\langle \left( \liminf_{R \to \infty} \frac{1}{|Q_R|} \sum_{x \in Q_R} \nabla \sigma, \sigma(x) \right)^2 \right\rangle_{J,\beta}^{2^d} \leq \liminf_{R \to \infty} \frac{1}{|Q_R|^2} \sum_{x,y \in Q_R} |\langle \nabla \sigma, \sigma(x) \nabla \sigma, \sigma(y) \rangle_{J,\beta}^{2^d}| = 0. \quad (A.9)$$

This implies that every measure $\mu$ in the ergodic decomposition of $\langle \cdot \rangle_{J,\beta}^{2^d}$ has mean 0 for $\nabla \sigma$ (see e.g. [34, Theorem 3.2] for a similar argument): indeed, for any such $\mu$, by (A.9) and ergodicity, one deduces that $|Q_R|^{-1} \sum_{x \in Q_R} \nabla \sigma, \sigma(x)$ converges $\mu$-a.s. and that the limit vanishes, whence $E_{\mu}[\nabla \sigma, \sigma(x)] = 0$. \hfill \Box
B Properties of the regulator with external field

Proof of Lemma [4]. In the proof, the notation

\[ W_{j+s}(X, \nabla_j^k \varphi)^2 = \sum_{B \in B_{j+s}(X)} \| \nabla_{j+s}^a \varphi \|_{L^\infty(B^*)}^2 \]  

(B.1)

will be used. For brevity, \( s + M^{-1} \) will be denoted \( s' \) and \( X_{s'} \) will be denoted \( X' \). We will bound each term appearing in \( \log G_{j+s}(X, \varphi + \xi_o) \). First, \( \| \nabla \varphi \|_{L^2(X)}^2 \) will be isolated from \( \| \nabla (\varphi + \xi_o) \|_{L^2(X)}^2 \). Let \( B \in B_{j+s}(X) \) and without loss of generality, let \( B, l_i (i = 1, 2, 3, 4) \) be as above but \( B = [1, L^{j+s}]^2 \). Then by discrete integration by parts,

\[ \sum_{x \in B} \nabla^\varepsilon_1 \varphi(x) \nabla^\varepsilon_1 \xi_o(x) = - \sum_{x \in 1} \xi_o(x) \nabla^\varepsilon_1 \varphi(x) - \sum_{x \in 1} \xi_o(x + e_1) \nabla^\varepsilon_1 \varphi(x) + \sum_{x \in B} \xi_o(x) \nabla^\varepsilon_1 \nabla^\varepsilon_1 \varphi(x). \]  

(B.2)

Hence in particular, summing this over each direction \( \pm e_1, \pm e_2 \), \( B \in B_{j+s}(X) \), and using the AM-GM inequality,

\[ t(\nabla \varphi, \nabla \xi_o)X \leq \tau t \| \xi_o \|_{L^2(X)}^2 + \tau^{-1} t \| \nabla_j^s \varphi \|_{L^2(X)}^2 + \tau \| \xi_o \|_{L^2(X)}^2 + \tau^{-1} \| \nabla_j^s \varphi \|_{L^2(X)}^2 \]

\[ \leq 2 \tau W_{j+s}(X, \xi_o) + \tau^{-1} (\| \nabla_j^s \varphi \|_{L^2(X)}^2 + W_{j+s}(X, \nabla^2 \varphi)^2) \]  

for any \( \tau > 0 \), and hence

\[ \| \nabla_j^s (\varphi + \xi_o) \|_{L^2(X)}^2 \leq \| \nabla_j^s \varphi \|_{L^2(X)}^2 + \| \nabla_j^s \xi_o \|_{L^2(X)}^2 \]

\[ + 2 \tau W_{j+s}(X, \xi_o) + \tau^{-1} (\| \nabla_j^s \varphi \|_{L^2(X)}^2 + W_{j+s}(X, \nabla^2 \varphi)^2) \]  

(B.3)

Next, we will use rather trivial bound on the other two terms of \( \log G_{j+s} \):

\[ \| \nabla_j^s (\varphi + \xi_o) \|_{L^2(X)}^2 \leq 2 \| \nabla_j^s \varphi \|_{L^2(X)}^2 + 2 W_{j+s}(X, \nabla_j^s \xi_o)^2 \]  

(B.4)

\[ \| \nabla_j^2 (\varphi + \xi_B) \|_{L^\infty(B^*)}^2 \leq 2 \| \nabla_j^2 \varphi \|_{L^\infty(B^*)}^2 + 2 \| \nabla_j^2 \xi_B \|_{L^\infty(B^*)}^2 \]  

(B.5)

By (B.3), (B.4), (B.6) and setting \( c_4 = \max\{2c_1, 2c_2, 2c_2\} \),

\[ \frac{1}{k_L} \log G_{j+s}(X, \varphi, \xi_o, (\xi_B)B) \leq c_1 \| \nabla_j^s \varphi \|_{L^2_j(X)}^2 + (2c_2 + c_1 \tau^{-1}) \| \nabla_j^s \varphi \|_{L^2_j(X)}^2 \]

\[ + 2c_1 (1 + \tau^{-1}) W_{j+s}(X, \nabla^2 \varphi) + \frac{1}{k_L} \log \max_{a \in \alpha} g_{j+s}(X, \xi_a). \]  

(B.7)

Now by repeated application of the discrete Sobolev trace theorem [8 (A.4)],

\[ \| \nabla_j^s \varphi \|_{L^2_j(X)}^2 \leq \| \nabla_j^s \varphi \|_{L^2_j(X')}^2 + 10 \| \nabla_j^s \varphi \|_{L^2_j(X' \setminus X)}^2 + 10 W_{j+s}(\nabla^2_j \varphi, X' \setminus X) \]  

(B.8)

hence by choosing \( \tau = c_1 c_2^{-1} \) and \( 30 c_2 \leq c_1 \),

\[ \log(G_{j+s}(X, \varphi, \xi, (\xi_B)B)/\max_a g_{j+s}(X, \xi_a)) \]

\[ \leq c_1 \| \nabla_j^s \varphi \|_{L^2_j(X')}^2 + 3 c_2 \| \nabla_j^s \varphi \|_{L^2_j(X')}^2 + 2 c_1 (1 + \tau^{-1}) W_{j+s}(\nabla^2_j \varphi, X') \]

\[ \leq c_1 \| \nabla_j^s \varphi \|_{L^2_j(X')}^2 + 3 \ell^{-1} c_2 \| \nabla_j^s \varphi \|_{L^2_j(X')}^2 + 2 \ell^{-2} c_1 (1 + \tau^{-1}) W_{j+s}(\nabla^2_j \varphi, X'). \]  

(B.9)

Hence the conclusion follows upon taking \( \ell \) large enough.
C Reblocking and fluctuation integral

Proof of Theorem 5.6. Throughout the proof, we write

$$\varphi = \varphi' + \zeta$$

with $\zeta \sim \Gamma_{j+1}$ and $\varphi', \zeta$ independent, and the fluctuation integral $E$ acts on the variable $\zeta$. As explained in [8, below (7.12)], we may assume that $\mathcal{E}_j = 0$ and $e_j = 0$. The first step is the reblocking

$$Z_j(\varphi, \Psi; (\Psi_k)_{k<j}) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Lambda \setminus X)} \mathcal{K}_j(X; (\Psi_k)_{k<j}) + \Psi_j(X)) = \sum_{X \in \mathcal{P}_{j+1}} e^{U_j(\Lambda \setminus X)} \mathcal{K}^\Psi_j(X)$$

(C.2)

where $\mathcal{K}^\Psi_j$ is defined in (5.22). In the next step, $e^{-E_{j+1}|B|+10e_{j+1}+U_{j+1}}$ replaces $U_j$ using the identity

$$e^{U_j(\Lambda \setminus X', \varphi)} = \prod_{B \in \mathcal{P}_{j+1}(\Lambda \setminus X')} \left( (e^{U_j(B|\varphi)} - e^{-E_{j+1}|B|+10e_{j+1}+U_{j+1}(B, \varphi')} + e^{-E_{j+1}|B|+10e_{j+1}+U_{j+1}(B, \varphi') \sum_{Y \in \mathcal{P}_{j+1}(\Lambda \setminus X')} (e^{U_j(\varphi) - e^{-E_{j+1}|B|+10e_{j+1}+U_{j+1}(\varphi'))})^Y \right)$$

(C.3)

and similarly $\mathcal{K}^\Psi_j - \mathcal{E}^\Psi_j \mathcal{K}_j$ replaces $\mathcal{K}^\Psi_j$ (recall $\mathcal{E}^\Psi_j \mathcal{K}_j$ from (5.22)) using the identity

$$\mathcal{K}^\Psi_j(X', \varphi) = \prod_{Z \in \text{Comp}_{j+1}(X')} \left( \mathcal{E}^\Psi_j \mathcal{K}_j(Z', \varphi') + (\mathcal{K}^\Psi_j(Z', \varphi) - \mathcal{E}^\Psi_j \mathcal{K}_j(\varphi')) \right)$$

(C.4)

Using the specific form of $\mathcal{E}^\Psi_j \mathcal{K}_j$ given by (5.22) the last right-hand side can be rewritten as

$$\mathcal{E}^\Psi_j \mathcal{K}_j(\varphi')^{[Z]} = \sum_{(B_{Z''})_{Z''}} \prod_{Z''} J^\Psi_j(B, Z'')$$

(C.5)

where the last sum $(B_{Z''})_{Z''}$ runs over collections of blocks $B_{Z''} \in B_{j+1}(Z'')$ and $Z'' \in \text{Comp}_{j+1}(Z)$. Rewriting $X'' = X' \cup Y$, the expectation $E Z_j$ can now be written as

$$Z_j(\varphi', 0; (\Psi_k)_{k<j}) = E Z_j(\varphi' + \zeta, \Psi; (\Psi_k)_{k<j}) = e^{-E_{j+1}|A|} \mathbb{E} \left[ \sum_{X'' \in \mathcal{P}_{j+1}} e^{1_{0 \in \Lambda \setminus X''} e_{j+1} + U_{j+1}(\Lambda \setminus X'')} e^{E_{j+1}|X''|} \times \sum_{X' \subset X''} (e^{U_j} - e^{-E_{j+1}|B|+10e_{j+1}+U_{j+1}}) X'' \setminus X' \times \sum_{Z \subset X'} (\mathcal{K}^\Psi_j - \mathcal{E}^\Psi_j \mathcal{K}_j)^{[X' \setminus Z]} \prod_{(B_{Z''})_{Z'' \in \text{Comp}_{j+1}(Z)}} J^\Psi_j(B, Z'')) \right].$$

(C.6)

The final result is obtained after taking $e^{E_{j+1}}$ out and another renormalisation: we write $X_0 = X'' \setminus X'$, $X_1 = X' \setminus Z$, $T = X_0 \cup X_1 \cup Z = X''$ and define for $X = \cup_{Z''} B^*_{Z''} \cup X_0 \cup X_1$,

$$K_{j+1}(X, \varphi'; (\Psi_k)_{k<j}) = \sum_{X_0, X_1, Z(B_{Z''})} \mathbb{E} \left[ (e^{U_j} - e^{-E_{j+1}|B|+10e_{j+1}+U_{j+1}}) X_0 (\mathcal{K}^\Psi_j - \mathcal{E}^\Psi_j \mathcal{K}_j)^{[X_1]} \prod_{Z'' \in \text{Comp}_{j+1}(Z)} J^\Psi_j(B, Z'') \right].$$

(C.7)
Note that only $T \subset X$ contribute because, by definition of $\mathcal{E}^\Psi K_j$, the whole expression vanishes when $Z \not\in S_{j+1}$. Therefore

$$Z_{j+1}(\varphi', 0; (\Psi_k)_{k \leq j}) = e^{-E_{j+1}|\Lambda|+\epsilon_{j+1}} \sum_{Z \in \mathcal{P}_{j+1}} e^{U_{j+1}(\Lambda \setminus Z, \varphi')} K_{j+1}(Z, \varphi'; (\Psi_k)_{k \leq j})$$

(C.8)

which is the desired form. The factorisation property of $K_{j+1}(\cdot; (\Psi_k)_{k \leq j})$ is inherited from that of $e^{U_j}$, $e^{U_{j+1}}$ and $K_j$. \hfill \Box

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