Post-Newtonian effects of Dirac particle in curved spacetime - II: the electron g-2 in the Earth’s gravity.

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The general relativistic effects to the anomalous magnetic moment of the electron $g_e$-2 in the Earth’s gravitational field have been examined. The magnetic moment of electrons to be measured on the Earth’s surface is evaluated as $\mu_m^{\text{eff}} \simeq (1 + 3\phi/c^2) \mu_m$ on the basis of the Dirac equation containing the post-Newtonian effects of the general relativity for fermions moving in the Earth’s gravitational field. This implies that the anomalous magnetic moment of $10^{-9}$ appears in addition to the radiative corrections in the quantum field theory. This may seem contradictory with the fact of the 12th digit agreement between the experimental value measured on the ground level $g_e(\text{EXP})$ and the theoretical value calculated in the flat spacetime $g_e(\text{SM})$. In this paper, we show that the apparent contradiction can be explained consistently with the framework of the general relativity.

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The electron magnetic moment $\mu_m$ equals to the Bohr magneton $\mu_B = e/2m$ in the Dirac theory and the electron g-factor is exactly $g_e = 2$\textsuperscript{1}. This is an important consequence obtained by applying the Dirac equation to a free fermion with the charge of $e$ and the spin of $1/2$ minimally coupled to the electromagnetic field. However, the $g_e$ of real electrons deviates from 2 by about 0.002 according to radiative corrections in the quantum field theory. The deviation from 2 is referred to as the anomalous magnetic moment $a_e$, which is defined as

$$a_e \equiv \frac{\mu_m}{\mu_B} - 1 = \frac{g_e}{2} - 1 \quad (= 0.001...) \quad (1)$$

The theoretical value of the anomalous magnetic moment calculated up to higher order radiative corrections within the standard model of elementary particles (SM) [1, 3] and the

\textsuperscript{1}In this paper we use unit $\hbar=c=1$. 

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experimental value obtained in precise measurements [2–4] are in agreement up to the 12th digit

\[ a_{e}(SM) - a_{e}(EXP) = 0.91 \text{ (0.82)} \times 10^{-12}, \tag{2} \]

as shown in Table 1.

The excellent agreement supports the accuracy of the standard model. Here we raise a question if the effect of the Earth’s gravity is properly treated in the extremely precise comparison. It should be noted that the effect of the gravitational field does not refer to quantum gravitational effect mediated by gravitons at the Planck scale but to effects of the spacetime curvature within the classical mechanics on the basis of the general relativity at low energy that appear in the Earth’s gravitational field. Such effects can be evaluated as the post-Newtonian effects and their typical scale is \(|\phi|/c^2 \approx 10^{-9}\) on the Earth’s surface. The experimental values of the electron anomalous magnetic moment have been obtained by observing the behavior of electrons on the Earth’s surface [2–4], while theoretical values are analytically and numerically calculated in the vacuum of the flat spacetime [1, 3]. Thus there exists a definite difference in the existence of the Earth’s gravitational fields.

Table 1  Comparison of the theoretical value and the experimental value of the anomalous magnetic moment of electron.

|                | anomalous magnetic moment \((a_e \equiv g_e/2 - 1)\) | Ref.               |
|----------------|-----------------------------------------------------|-------------------|
| Experiment     | \(a_e(\text{EXP}) = 1\ 159\ 652\ 180.73\ (0.28) \times 10^{-12}\) [2–4] |
| Theory         | \(a_e(\text{SM}) = 1\ 159\ 652\ 181.643\ (0.764) \times 10^{-12}\) [1, 3] |
| (Agreement)    | \(g_e(\text{EXP})/2 = 1.001\ 159\ 652\ 180\ 73\ ...\) |
|                | \(g_e(\text{SM})/2 = 1.001\ 159\ 652\ 181\ 643\ ...\) |

Here we simply estimate the electron anomalous magnetic moment in a curved spacetime within the combination of the quantum mechanics and the general relativity. A more precise derivation of the electron anomalous magnetic moment based on the quantum mechanics and the general relativity will be given in a separate paper [6].

The motion of free neutral particle in the Earth’s gravitational field can be described with the covariant equation of motion (Klein-Gordon equation) defined as

\[ g^{\mu\nu}p_\mu p_\nu - m^2 c^2 = 0. \tag{3} \]

The equation of motion of electrons with the charge \(e\) and the spin \(1/2\) should be obtained by substituting the momentum as \(p \rightarrow \sigma \cdot (p - eA)\) analogously to the Pauli’s method to implement the spin \(1/2\) into Schrödinger equation. The time component \(p_0 (= \mathcal{H})\) is obtained
as

\[ p_0 = N^i p_i + \frac{N m}{\epsilon} \left( 1 + \epsilon^2 \frac{\gamma^{ij} p_i p_j}{2m^2} - \epsilon^4 \frac{(\gamma^{ij} p_i p_j)^2}{8m^4} \right) + O(\epsilon^4), \]  

(4)

up to the post-Newtonian order \( O(\epsilon^2) \), where \( \epsilon \equiv 1/c \). \( N, N^i, \gamma^{ij} \) are the lapse function, shift function and 3-dimensional metric in the (3+1) differential form defined as

\[ g_{\mu\nu} = \begin{pmatrix} N^2 - N^i N_i & -N_j \\ -N_i & -\gamma^{ij} \end{pmatrix}. \]  

(5)

We adopt the Schwarzschild metric to describe the Earth’s gravitational field assuming that the Earth is a sphere with the mass of \( M \) and the radius of \( R \), the gravitational potential at the distance of \( r \) from the Earth’s center is \( \phi = -GM/r \) for \( r \geq R \). The post-Newtonian approximation is practically sufficient to describe the Earth’s gravitational field since it is as weak as \( \phi/c^2 \ll 1 \).

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \epsilon^{-2} \left( 1 + \epsilon^2 2 \phi + \epsilon^4 \phi^2 \right) dt^2 - \left( 1 - \epsilon^2 2 \phi \right) \left( dx^2 + dy^2 + dz^2 \right) + O(\epsilon^4) \]  

(6)

This metric uniquely determines the lapse function, shift function and the 3-dimensional metric as

\[ N = \epsilon^{-1} \left( 1 + \epsilon^2 \phi + \epsilon^4 \phi^2 \right), \]

\[ N^i = 0, \]

\[ \gamma^{ij} = (1 + \epsilon^2 \phi) \delta^{ij} \quad (i, j = 1, 2, 3) \]  

(7)

Substituting \( N, N^i \) and \( \gamma^{ij} \) into Eq. (4) and replacing the momentum with \( p \to \sigma \cdot (p - eA) \), the Hamiltonian of electron containing the effects of the general relativity becomes, after some algebra to rearrange, as

\[ \mathcal{H} \quad (\equiv p_0) \]

\[ = m\epsilon^{-2} + \frac{(p - eA)^2}{2m} - \mu_m \cdot B + m\phi 

+ \epsilon^2 \left( \frac{m\phi^2}{2} - \frac{(p - eA)^4}{8m^3} - 3\phi \mu_m \cdot B + \frac{3}{2m} (p - eA) \cdot \left( \phi (p - eA) \right) \right) 

- \frac{\nabla^2 (\mu_m \cdot B)}{4m^2} - \frac{(\mu_m \cdot B)^2}{2m} + \frac{1}{2m^2} (p - eA) \cdot \left( (\mu_m \cdot B) (p - eA) \right) \right) 

+ O(\epsilon^4), \]  

(8)

where \( \mu_m = \mu_B \sigma \) is the magnetic moment and \( \mu_B = e/2m \) is the Bohr magneton. The magnetic moment \( \mu_m \) in the flat spacetime can be defined as the operator which satisfies
the Heisenberg’s equation
\[ \frac{dS}{dt} = \frac{1}{i} \left[ S, \mathcal{H} \right] = \mu_m \times B. \] (9)

The effective magnetic moment \( \mu_m^{\text{eff}} = \mu_m^{\text{eff}}(\phi) \) can be derived by calculating the Hamiltonian of electrons in the curved spacetime under the Earth’s gravitational field as
\[ \mu_m^{\text{eff}}(\phi) \simeq (1 + 3e^2 \phi) \mu_m, \] (10)

which shows there appears an anomalous magnetic moment to deviate the \( g_e \)-factor from 2 as the result of the gravitational effects in the general relativity additionally to the radiative corrections in the quantum field theory. Consequently, when we define \( a_e^{\text{conv}} = a_e^{\text{conv}}(\phi) \) as the effective value of the anomalous magnetic moment of electrons on the ground level according to Eq. (1), the anomalous magnetic moment to be measured on the Earth’s surface should be
\[ |a_e^{\text{conv}}(\phi)| = \left| \frac{\mu_m^{\text{eff}}}{\mu_B} - 1 \right| \simeq 3e^2|\phi| \sim 2 \times 10^{-9} \] (11)

and a gravitationally induced anomaly appears at the ninth digit in \( g_e \)-factor. This anomaly seems contradictory to the 12th digits agreement between the experimental value measured on the ground level and the theoretical value calculated in the flat spacetime (see Table 1).

Below, we examine the interpretation of the experimental result of the Penning trap experiment [2, 3].

The cyclotron frequency \( \Omega_c \) and the spin precession frequency \( \Omega_s \) of an electron moving in an uniform magnetic field are measured in the Penning trap experiment [2, 3]. The anomalous magnetic moment of electrons \( \mu_m \) is regarded equal to the ratio of the anomalous spin precession frequency \( \Omega_s \equiv \Omega_s - \Omega_c \) to the cyclotron frequency \( \Omega_c \) as
\[ a_e^{\text{conv}}(\text{EXP}) = \frac{\Omega_s}{\Omega_c} = \frac{\Omega_s}{\Omega_c} - 1. \] (12)

The \( a_e^{\text{conv}}(\text{EXP}) \) has been treated as the electron anomalous magnetic moment in the conventional analysis without the consideration of gravitational effect. However, this experiment was carried out in the Earth’s gravitational field. Thus the experimentally measured frequencies in Eq. (12) are the effective frequencies in a curved spacetime. Hereafter, we derive the effective values of the cyclotron frequency \( \Omega_c^{\text{eff}} \) and the spin precession frequency \( \Omega_s^{\text{eff}} \) in a curved spacetime.

The general relativity requires that the translational motion of a free particle with the electric charge of \( e \) is described by the covariant equation of
\[ \frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = \frac{e}{m} F^{\mu\nu} u_\nu, \] (13)

where \( \tau \) is the proper time, \( u^\mu \) the four velocity vector \( u^\mu = (u^0, \mathbf{u}) \), \( F^{\mu\nu} \) the electromagnetic tensor and \( \Gamma^\mu_{\nu\lambda} \) the Christoffel symbol. Assuming the Earth’s gravitational field can be described with the Schwarzschild metric shown in Eq. (6), the equation of translational
motion of a free electron can be written as
\[ \frac{d\beta}{dt} = \left( 1 + (2\gamma^2 + 1)e^2 \phi \right) \frac{e}{\gamma m} \left( E + \beta \times B - (1 - 4\gamma^2 e^2 \phi) \beta (\beta \cdot E) \right) \]
\[ - \epsilon(1 + \beta^2) \nabla \phi + \epsilon(4\beta \cdot \nabla \phi) \beta + O(\epsilon^4), \]
(14)
where \( \beta = v/c, \gamma = 1/\sqrt{1 - \beta^2} \) and
\[ u^0 = \frac{dt}{d\tau} = 1/\sqrt{1 + 2\epsilon^2 \phi + 2\epsilon^4 \phi^2} - (1 - 2\epsilon^2 \phi) \beta^2 \]
\[ = \frac{1}{\gamma} \left( 1 + (2\gamma^2 - 1)e^2 \phi + O(\epsilon^4) \right) \]
(15)
is used. In the flat spacetime where \( \phi = 0 \), this equation results in the equation of motion of charged particles in the special relativity
\[ \frac{d\beta}{dt} = \frac{e}{\gamma m} \left( E + \beta \times B - (E \cdot \beta) \beta \right). \]
(16)
In addition, at the limit of \( |\beta| = |v|/c \ll 1 \), it leads to the Lorentz equation of motion
\[ \frac{d(mv)}{dt} = e(E + v \times B). \]
(17)
In general, this equation can be a spiral motion which is the combination of the translational acceleration and the rotational motion. The cyclotron frequency is the measure of the rotational motion. In other words, the cyclotron frequency corresponds to the frequency of the rotational motion of the constant-norm vector parallel to the velocity. We define the unit vector parallel to the three-dimensional velocity \( \beta \) as
\[ \beta \equiv \beta \hat{\beta}, \quad \hat{\beta} \cdot \hat{\beta} = \beta \cdot \beta = 1. \]
(18)
Here we use \( \beta \cdot \nabla \phi = 0 \) and assume the effects of the gravitational gradient is negligibly small, we obtain the equation of translational motion of electron as
\[ \frac{d\beta}{dt} = \hat{\beta} \times \left( 1 + (2\gamma^2 + 1)e^2 \phi \right) \frac{e}{m} \left( B - \gamma / (\gamma^2 - 1) (\beta \times E) \right). \]
(19)
In the measurement of the electron g\( -2 \) experiment (i.e., Penning trap method), the electron velocity is sufficiently small as \( \beta \ll 1 \) and the effective value of the electron’s cyclotron frequency is given as
\[ \Omega_{c_{\text{eff}}} = (1 + 3e^2 \phi) \Omega_c \]
(20)
where the \( \Omega_c \) is the electron’s cyclotron frequency in the flat spacetime defined as
\[ \Omega_c \equiv - (e/m) \left( B / \gamma - \gamma / (\gamma^2 - 1) (\beta \times E) \right). \]
We can interpret that the \( \Omega_{c_{\text{eff}}} (\phi) \) is used as the experimentally measured cyclotron frequency.

Here we analyze the behavior of an electron spin in the Earth’s gravitational field. The time evolution of an electron spin in the flat spacetime is described by the equation
\[ \frac{dS}{dt} = \mu_m \times B = \Omega_s \times S, \]
(21)
where \( \Omega_s = -(g_e/2)(e/m)B \). Rewriting this equation with four vector, we obtain
\[ \frac{dS^\mu}{d\tau} = \frac{g_e}{2m} \left( F^{\mu \nu} S_\nu + \frac{1}{c^2} u^\mu (S_\lambda F^{\lambda \nu} u_\nu) \right) - \frac{1}{c^2} u^\mu (S_\lambda \frac{du^\lambda}{d\tau}), \]
(22)
which is known as the BMT equation [5]. In the same manner as the Eq. (13), employing the covariant derivative into the BMT equation, we obtain the covariant equation of spin
motion in the general relativity as
\[
\frac{DS^\mu}{d\tau} = \frac{r_c e}{2m} \left( F^{\mu\nu} S_{\nu} + \frac{1}{c^2} u^\mu (S_\lambda F^{\lambda\nu} u_\nu) \right) - \frac{1}{c^2} u^\mu (S_\lambda \frac{Du^\lambda}{d\tau}).
\] (23)

Here we substitute the Schwarzschild metric defined by Eq. (6) into the covariant BMT equation. In the same way as we have derived the cyclotron frequency based on the equation of translational motion of electrons, we consider the case where the gradient of the gravitational potential \( \nabla \phi \) is negligibly small such as the case where the motion is limited on a horizontal plane. We define the four spin vector as \( s^\mu = (S^0, S^i) \) and the spatial component of the generalized BMT equation Eq. (23) up to the post-Newtonian order \( O(\epsilon^2) \) can be written as
\[
\frac{dS}{dt} = S \times \left( 1 + \epsilon^2 \phi (2\gamma^2 + 1) \right) \frac{g e}{2m} B.
\] (24)

We can take \( \beta \ll 1 \) in the measurement of electron \( g_e - 2 \) (Penning trap method), which implies that the effective value of the electron spin precession frequency is given as
\[
\Omega_s^{\text{eff}} = (1 + 3\epsilon^2 \phi) \Omega_s.
\] (25)

The experimental value of the anomalous magnetic moment obtained in the Penning trap method, denoted as \( a^{\text{eff}}_e \) below, should be compared with the ratio of the kinematically calculated effective values \( \Omega_c^{\text{eff}} \) and \( \Omega_s^{\text{eff}} \). Substituting the effective values \( \Omega_c^{\text{eff}} \) and \( \Omega_s^{\text{eff}} \) into Eq. (12), we obtain
\[
a^{\text{eff}}_e (\text{EXP}) = \frac{\Omega_s^{\text{eff}}}{\Omega_c^{\text{eff}}} - 1 = \frac{(1 + 3\epsilon^2 \phi) \Omega_s}{(1 + 3\epsilon^2 \phi) \Omega_c} - 1
\]
\[
= \frac{\Omega_s}{\Omega_c} - 1
\]
\[
= a^{\text{conv}}_e (\text{EXP}).
\] (26)

Although both effective values \( \Omega_c^{\text{eff}} \) and \( \Omega_s^{\text{eff}} \) are different from those in the flat spacetime, gravitational effects are canceled in their ratio and the \( a^{\text{eff}}_e (\text{EXP}) \) coincides with \( a^{\text{conv}}_e (\text{EXP}) \). It is reasonable to regard that the high precision agreement between the experimental and theoretical values is confirming the cancellation of gravitational effects. It also suggests that, if we take the ratio of the conventionally defined \( a_e \) in Eq. (1) to the Bohr magneton, the “true” anomalous magnetic moment \( a^{\text{eff}}_e (\text{EXP}) \) deviates from the “conventional” value \( a^{\text{conv}}_e (\text{EXP}) \) as
\[
a^{\text{eff}}_e (\text{EXP}) \equiv \frac{g_e^{\text{eff}}}{2} - 1 = \frac{\mu_m^{\text{eff}}}{\mu_B} - 1 = \frac{\Omega_s^{\text{eff}}}{\Omega_c} - 1
\]
\[
= (1 + 3\epsilon^2 \phi) \frac{\Omega_s}{\Omega_c} - 1
\]
\[
\neq \frac{\Omega_s}{\Omega_c} - 1 = a^{\text{conv}}_e (\text{EXP}).
\] (27)

Since the Earth’s gravity induces an anomaly even for \( g_e = 2 \), in which radiative corrections are not involved, the comparison between experimental value measured in the curved spacetime
and the theoretical value calculated in the flat spacetime results in the difference of

$$|a_{\text{eff}}^{\text{(EXP)}} - a_{\text{e(SM)}}| \simeq 3e^2 |\phi| \sim 2.1 \times 10^{-9}. \quad (28)$$

Here we introduce the redefinition of the effective Bohr magneton including the general relativistic effects of the Earth’s gravitational field as

$$\mu_{B}^{\text{eff}} \equiv \mu_{B}^{\text{eff}}(\phi) \simeq (1 + 3e^2 \phi) \mu_B. \quad (29)$$

In this case, we can treat the $a_{\text{eff}}^{\text{(EXP)}}$ as the invariant quantity independent of the gravitational field if we cancel the general relativistic effects by redefining the anomalous magnetic moment using the ratio of the effective magnetic moment to the effective Bohr magneton as

$$a_{\text{e}}^{\text{conv}}(\text{EXP}) = \frac{\mu_{m}^{\text{eff}}}{\mu_{B}^{\text{eff}}} - 1 = \frac{\Omega_{c}^{\text{eff}}}{\Omega_{e}^{\text{eff}}} - 1 = a_{\text{e}}^{\text{conv}}(\text{EXP}), \quad (30)$$

instead of its ratio to the Bohr magneton in the flat spacetime. The redefinition restores the validity of the test of the standard model of elementary particles via the comparison of the theoretical value $a_{\text{e(SM)}}$ and the experimental value $a_{\text{eff}}^{\text{(EXP)}}$.

In this paper, we discussed the case of classical and semi-classical electrons. Further discussion on the basis of the Dirac equation in curved space will be published in a separate paper [6].

The anomalous magnetic moment of muons was measured in the storage ring at much higher energy $\gamma=29.3$ and the gravitationally induced effects appears in the different manner from the electron case, which suggests a new type of corrections in the comparison of the theoretical and experimental values. The muon case is to be published in another separate paper [7].

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