A new (in)finite dimensional algebra for quantum integrable models

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Abstract

A new (in)finite dimensional algebra which is a fundamental dynamical symmetry of a large class of (continuum or lattice) quantum integrable models is introduced and studied in details. Finite dimensional representations are constructed and mutually commuting quantities - which ensure the integrability of the system - are written in terms of the fundamental generators of the new algebra. Relation with the deformed Dolan-Grady integrable structure recently discovered by one of the authors and Terwilliger’s tridiagonal algebras is described. Remarkably, this (in)finite dimensional algebra is a “q−deformed” analogue of the original Onsager’s algebra arising in the planar Ising model. Consequently, it provides a new and alternative algebraic framework for studying massive, as well as conformal, quantum integrable models.

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1 Introduction

Two-dimensional completely integrable models (continuum or lattice) are characterized by the existence of an (in)finite set of mutually commuting conserved quantities. Such property allows to solve the model exactly without considering approximation schemes. Since the exact solution of the planar Ising model by Onsager in 1944 [1], several methods have been proposed to analyse in a nonperturbative way integrable models. For instance, the factorized scattering theory (based on Yang-Baxter/star-triangle relations), quantum group symmetry, Bethe ansatz techniques and conformal field theory framework are constantly applied to derive exact results (such as the exact $S$–matrix, VEVs, forms factors,...) in quantum integrable models. For systems with an (in)finite number of degrees of freedom, integrability takes its roots in the existence of an (in)finite dimensional symmetry. For instance, in the context of (massless) conformal field theory, the infinite dimensional Virasoro algebra with fundamental generators $L_n$ satisfying

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0},$$

where $c$ denotes the central charge, actually gives a powerful algebraic approach to critical statistical systems and corresponding field theories [2]. Indeed, exact results like correlation functions (which were previously not accessible using standard techniques like renormalization group methods) can be derived in a systematic manner using the properties of the Virasoro algebra. Later on, other types of infinite dimensional conformal symmetries including for instance supersymmetry [3], parafermionic symmetry [4], $W$–algebras [5] and current algebras [6] have been considered, extending [1]. Similarly, they found several applications in a large class of critical statistical systems or conformal field theories with enlarged symmetries.

In the context of integrable massive quantum field theory or lattice systems, hidden symmetries associated with quantum groups [7][8] or deformed Virasoro algebra [9] (in the vicinity of critical points) have been introduced...
in order to derive exact results in these models. However, an infinite dimensional algebra characterizing the dynamical symmetry of a large class of quantum integrable massive models has not yet been found. The only known example in this direction is the Onsager’s algebra with generators \( A_k, G_l \) satisfying

\[
[A_k, A_l] = 4G_{k-l}, \quad [G_l, A_k] = 2A_{l+k} - 2A_{l+k}, \quad [G_k, G_l] = 0
\]

for any integers \( k, l \). This infinite dimensional Lie algebra was originally introduced in [11] in order to solve the planar Ising model in zero magnetic field. Although it played a crucial role in the original solution of the Ising model, this algebra only arises in a few other quantum integrable models (XY, superintegrable chiral Potts [10] and generalizations [11]). In these models, all conserved quantities \( I_{2k+1} \) can be simply expressed in terms of the fundamental generators \( A_k \) as

\[
I_{2k+1} = \frac{\kappa}{2}(A_k + A_{-k}) + \frac{\kappa^*}{2}(A_{k+1} + A_{-k+1}) ,
\]

for \( k \geq 0 \) where \( \kappa, \kappa^* \) are arbitrary parameters. Also, note that in the 80s the Onsager’s algebra was shown to be closely related with the integrable structure discovered by Dolan and Grady in [12]. Despite of its nice properties, the Onsager’s algebra remained an interesting curiosity in the last sixty years.

Clearly, identifying the underlying (in)finite dimensional symmetry in quantum integrable systems is a fundamental and important problem that we wish to adress in this paper. Indeed, we construct explicitly and study in details an (in)finite dimensional algebraic structure with fundamental generators \( W_{-k}, W_{k+1}, G_{k+1}, \tilde{G}_{k+1} \) satisfying

\[
[W_0, W_{k+1}] = [W_{-k}, W_1] = \frac{1}{(q^{1/2} + q^{-1/2})} (\tilde{G}_{k+1} - G_{k+1}),
\]

\[
[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = \rho W_{-k-1} - \rho W_{k+1},
\]

\[
[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = \rho W_{k+2} - \rho W_{-k},
\]

\[
[W_0, W_{-k}] = 0, \quad [W_1, W_{k+1}] = 0 ,
\]

and

\[
[G_{k+1}, G_{l+1}] = 0, \quad [\tilde{G}_{k+1}, \tilde{G}_{l+1}] = 0, \quad [\tilde{G}_{k+1}, G_{l+1}] + [G_{k+1}, \tilde{G}_{l+1}] = 0 ,
\]

with fixed scalar \( \rho \) and \( k, l \in \mathbb{N} \). Finite dimensional representations are obtained, and examples of quantum integrable systems (XXZ spin chain, Sine-Gordon and Liouville field theories) which enjoy this symmetry are given. More generally, our framework opens the possibility of analyzing a large class of quantum integrable models from a new point of view, in the spirit of Onsager’s approach [11]. The paper is organized as follows. In Section 2, the fundamental relations [4] are derived using its relation with a class of quadratic algebra, namely the reflection equation. Indeed, finite dimensional representations of its fundamental generators are shown to be generated from general solutions of the reflection equation. In particular, the closure of the algebra is ensured by the existence of a set of linear relations among the generators. Explicit expressions of mutually commuting quantities in terms of \( W_{-k}, W_{k+1}, G_{k+1}, \tilde{G}_{k+1} \), generalizing [3] are obtained. Also, we argue that similarly to the undeformed case, the integrable structure generated from our “\( q \)-deformed” Onsager’s algebra coincides with the deformed Dolan-Grady integrable structure recently discovered in [13] [14]. In particular, we exhibit the correspondence in the simplest cases. In the last Section, we give some examples of quantum integrable systems which enjoy this (in)finite dimensional symmetry.

## 2 Structure of the algebra

In the last thirty years, one of the most important progress in the approach of quantum integrable systems has been based on the star-triangle relations which originated in [11] [15] and led to the Yang-Baxter equations, the theory of quantum groups as well as the quantum inverse scattering method. Although the star-triangle relations
and the Onsager’s algebra \([2]\) first appeared in the work of Onsager, to our knowledge, a direct and explicit link between both structure has never been found or even noticed. Based on the recent results in \([13, 14]\), in this Section we will exhibit such a link, relating a class of quadratic algebras - the reflection equation sometimes called the boundary Yang-Baxter equation - and a new finite dimensional algebra with deformation parameter \(q\) generalizing the Onsager’s one.\(^3\). This link will allow us to derive the defining relations \([4]\) for the new algebra, as well as mutually commuting operators written in the basis of its fundamental generators.

### 2.1 Fundamental generators and recursion relations

Following \([13, 14]\), let us consider the quadratic algebra (reflection equation) which was first introduced by Cherednik in \([16]\):

\[
R(u/v) (K(u) \otimes \mathbb{I}) R(uv) (\mathbb{I} \otimes K(v)) = (\mathbb{I} \otimes K(v)) R(uv) (K(u) \otimes \mathbb{I}) R(u/v) .
\]

This equation arises, for instance, in the context of quantum integrable systems with boundaries \([17]\). We report the reader to the literature on the subject for more details. For our purpose, we restrict our attention to the trigonometric \(R\)–matrix \(R(u)\) which solves the Yang-Baxter equation. In the spin\(-\frac{1}{2}\) representation of \(U_q^{1/2}(sl_2)\), it reads

\[
R(u) = \sum_{i,j \in \{0, 3, \pm\}} \omega_{ij}(u) \sigma_i \otimes \sigma_j ,
\]

where

\[
\begin{align*}
\omega_{00}(u) &= \frac{1}{2}(q^{1/2} + 1)(u - q^{-1/2}u^{-1}) , \\
\omega_{33}(u) &= \frac{1}{2}(q^{1/2} - 1)(u + q^{-1/2}u^{-1}) , \\
\omega_{+-}(u) &= \omega_{-+}(u) = q^{1/2} - q^{-1/2} ,
\end{align*}
\]

and \(\sigma_j\) are Pauli matrices, \(\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2\). Suppose that one knows an “initial” two-dimensional matrix solution \(K^{(0)}(u)\) of \((6)\), then a family of solutions to \((6)\) can be easily obtained using the so-called “dressing” procedure \([17]\). Indeed, consider the fundamental solution (called \(L\)–operator) \(L(u)\) of the quantum Yang-Baxter algebra

\[
R(u/v)(L(u) \otimes L(v)) = (L(v) \otimes L(u))R(u/v) .
\]

In the basis \(\{S_\pm, s_3\}\) of the quantum algebra \(U_q^{1/2}(sl_2)\) with defining relations \([s_3, S_\pm] = \pm S_\pm\) and \([S_+, S_-] = (q^{s_3} - q^{-s_3})/(q^{1/2} - q^{-1/2})\), the \(L\)–operator takes the simple form:

\[
L(u) = \begin{pmatrix}
qu q^{s_3/2} - u^{-1}q^{-s_3/2} & (q^{1/2} - q^{-1/2})S_- \\
(q^{1/2} - q^{-1/2})S_+ & uq q^{s_3/2} - u^{-1}q^{-s_3/2}
\end{pmatrix}.
\]

From the results of \([17]\), it follows, for any parameter \(v \in \mathbb{C}\), that

\[
K^{(N)}(u) \equiv L_N(uv) \cdots L_1(uv) K^{(0)}(u) L_1(uv^{-1}) \cdots L_N(uv^{-1})
\]

acting on the quantum space \(V^{(N)} \equiv \bigotimes_{j=1}^N V_j\) also solves \((7)\). Due to the choice \([3]\), it is clear that this general “dressed” solution is a two-dimensional matrix in the auxiliary space with operator entries. Then, in full generality we decide to write it as

\[
K^{(N)}(u) = \sum_{j \in \{0, 3, \pm\}} \sigma_j \otimes \Omega_j^{(N)}(u) ,
\]

where \(\Omega_j^{(N)}(u)\) are rather complicated operators acting on the quantum space \(V^{(N)}\). Our main objective is now to write these operators in a more convenient form. In the following, we choose the trivial solution of \([3]\) to be \(K^{(0)}(u) \equiv (\sigma_+ + \sigma_-)/(q^{1/2} - q^{-1/2})\). Note that, according to \([3, 9]\), the operators \(\Omega_j^{(N)}(u)\) are combinations of Laurent polynomials of degree \(-2N \leq d \leq 2N\) in the spectral parameter \(u\) and operators acting solely on \(\bigotimes_{j=1}^N U_q^{1/2}(sl_2)\).

\(^3\)As we will see later on, defining relations for the Onsager’s algebra \([2]\) are recovered setting \(q = 1\).
2.1.1 Case $N = 1$

For simplicity, let us start by considering $N = 1$ in \cite{13}. Note that this special case was first studied in details in \cite{13}, having some interesting applications in the context of quasi-exactly solvable systems (Azbel-Hofstadter one in particular). This simplest “dressed” solution takes the form \cite{10} for $N = 1$ where the operators $\Omega_{j}^{(1)}(u)$ are easily written as \cite{13, 14}:

\[
\begin{align*}
\Omega_{0}^{(1)}(u) + \Omega_{3}^{(1)}(u) &= uq^{1/2}W_{0}^{(1)} - u^{-1}q^{-1/2}W_{1}^{(1)}, \\
\Omega_{0}^{(1)}(u) - \Omega_{3}^{(1)}(u) &= uq^{1/2}W_{1}^{(1)} - u^{-1}q^{-1/2}W_{0}^{(1)}, \\
\Omega_{+}^{(1)}(u) &= \frac{q^{1/2}u^{2} + q^{-1/2}u^{-2}}{c_{0}(q^{1/2} - q^{-1/2})} + \frac{G_{1}^{(1)}(1)}{q^{1/2} + q^{-1/2}} + \omega_{0}^{(1)}, \\
\Omega_{-}^{(1)}(u) &= \frac{q^{1/2}u^{2} + q^{-1/2}u^{-2}}{(q^{1/2} - q^{-1/2})} + \frac{c_{0}G_{1}^{(1)}(1)}{q^{1/2} + q^{-1/2}} + c_{0}\omega_{0}^{(1)},
\end{align*}
\]

where the generators $W_{0}^{(1)}, W_{1}^{(1)}, G_{1}^{(1)}, \tilde{G}_{1}^{(1)}$ have been introduced. As shown in \cite{13}, the generators $G_{1}^{(1)}, \tilde{G}_{1}^{(1)}$ are given by $G_{1}^{(1)} = [W_{1}^{(1)}, W_{0}^{(1)}], G_{1}^{(1)} = [W_{0}^{(1)}, W_{1}^{(1)}]$ where the $q$–commutator

\[
[X, Y]_{q} = q^{1/2}XY - q^{-1/2}YX
\]

has been introduced. In the basis of $U_{q^{1/2}}(sl_{2})$, they admit the following representations:

\[
\begin{align*}
W_{0}^{(1)} &= \frac{1}{c_{0}}vq^{1/4}S_{+}q_{s^{3/2}} + v^{-1}q^{-1/4}S_{-}q^{-s^{3/2}}, \\
W_{1}^{(1)} &= \frac{1}{c_{0}}v^{-1}q^{-1/4}S_{+}q_{s^{3/2}} + vq^{1/4}S_{-}q^{-s^{3/2}}, \\
G_{1}^{(1)} &= \frac{(q^{1/2} + q^{-1/2})}{c_{0}(q^{1/2} - q^{-1/2})} \left[ \frac{(v^{2} + v^{-2})}{(q^{1/2} + q^{-1/2})} W_{0}^{(j)} - (v^{2}q^{s_{3}} + v^{-2}q^{-s_{3}}) \right] + (q - q^{-1})S_{z}, \\
\tilde{G}_{1}^{(1)} &= \frac{(q^{1/2} + q^{-1/2})}{c_{0}(q^{1/2} - q^{-1/2})} \left[ \frac{(v^{2} + v^{-2})}{(q^{1/2} + q^{-1/2})} W_{0}^{(j)} - (v^{2}q^{-s_{3}} + v^{-2}q^{s_{3}}) \right] + \frac{1}{c_{0}}(q - q^{-1})S_{z}^{2}.
\end{align*}
\]

The remaining constant term $\omega_{0}^{(1)}$ in \cite{11} can be either obtained directly from \cite{13}, or follows by plugging \cite{10} with the structure \cite{11} and \cite{12} in \cite{13}. This last requirement imposes strong constraints on the generators, leading to the so-called Askey-Wilson relations \cite{19}. We report the reader to \cite{13} for details. In any case, one finds

\[
\omega_{0}^{(1)} = -\frac{(v^{2} + v^{-2})}{c_{0}(q - q^{-1})} W_{0}^{(j)},
\]

where the Casimir operator eigenvalue of $U_{q^{1/2}}(sl_{2})$ reads $W_{0}^{(j)} = q^{j+1/2} + q^{-j-1/2}$.

2.1.2 Case $N = 2$

For $N = 2$, the calculations are more involved but the procedure being straightforward, we do not need to report the detailed analysis. Part of the results below can be found in \cite{14}, but written in a slightly different form. In this case, the solution of the reflection equation \cite{13} is given by \cite{10} for $N = 2$ with (see also \cite{14})

\[
\begin{align*}
\Omega_{0}^{(2)}(u) + \Omega_{3}^{(2)}(u) &= uq^{1/2}(P_{0}^{(2)}(u)W_{0}^{(2)} + P_{1}^{(2)}(u)W_{1}^{(2)}) - u^{-1}q^{-1/2}(P_{0}^{(2)}(u)W_{1}^{(2)} + P_{1}^{(2)}(u)W_{2}^{(2)}), \\
\Omega_{0}^{(2)}(u) - \Omega_{3}^{(2)}(u) &= uq^{1/2}(P_{0}^{(2)}(u)W_{1}^{(2)} + P_{1}^{(2)}(u)W_{2}^{(2)}) - u^{-1}q^{-1/2}(P_{0}^{(2)}(u)W_{0}^{(2)} + P_{1}^{(2)}(u)W_{2}^{(2)}), \\
\Omega_{+}^{(2)}(u) &= \frac{(q^{1/2}u^{2} + q^{-1/2}u^{-2})}{c_{0}(q^{1/2} - q^{-1/2})} P_{0}^{(2)}(u) + \frac{1}{q^{1/2} + q^{-1/2}}(P_{0}^{(2)}(u)G_{1}^{(2)} + P_{1}^{(2)}(u)G_{2}^{(2)}) + \omega_{0}^{(2)}, \\
\Omega_{-}^{(2)}(u) &= \frac{(q^{1/2}u^{2} + q^{-1/2}u^{-2})}{(q^{1/2} - q^{-1/2})} P_{0}^{(2)}(u) + \frac{c_{0}}{q^{1/2} + q^{-1/2}}(P_{0}^{(2)}(u)\tilde{G}_{1}^{(2)} + P_{1}^{(2)}(u)\tilde{G}_{2}^{(2)}) + c_{0}\omega_{0}^{(2)},
\end{align*}
\]
where the generators $W_{-k}^{(2)}, W_{k+1}^{(2)}, G_{k+1}^{(2)}, \tilde{G}_{k+1}^{(2)}$ for $k \in \{0, 1\}$ have been introduced for further convenience. Their finite dimensional representations in the tensor product basis of $U_{q^{1/2}}(sl_2) \otimes U_{q^{1/2}}(sl_2)$ can be obtained as before, and coincide exactly with the ones corresponding to the general case $N$. So, we refer the reader to the next section for explicit expressions. Let us however mention that the exact expressions for the Laurent polynomials $P_{-k}^{(2)}(u)$ with $k \in \{0, 1\}$ are given by

\[
P_{0}^{(2)}(u) = q^{1/2}u^2 + q^{-1/2}u^{-2} + c_0(q^{1/2} - q^{-1/2})\omega_0^{(1)} - \frac{(v^2 + v^{-2})}{(q^{1/2} + q^{-1/2})}w_0^{(j)},
\]

\[
P_{-1}^{(2)}(u) = q^{1/2} + q^{-1/2}.
\]

Finally, the constant term in (14) reads

\[
\omega_0^{(2)} = \frac{(v^2 + v^{-2})}{(q^{1/2} + q^{-1/2})}w_0^{(j)}\omega_0^{(1)}.
\]

2.1.3 General case $N$

For general values of $N$, we want to find an explicit expression for $K^{(N)}(u)$ such that the dependance on the spectral parameter $u$ in the operators $\Omega_j^{(N)}(u)$ is disentangled, similarly to \[14\]. Based on previous results for $N = 1$ and $N = 2$, we propose the following ansatz in \[10\] for $K^{(N)}(u)$:

\[
\Omega_0^{(N)}(u) + \Omega_3^{(N)}(u) = uq^{1/2} \sum_{k=0}^{N-1} P_{-k}^{(N)}(u)W_{-k}^{(N)} - u^{-1/2}q^{1/2} \sum_{k=0}^{N-1} P_{-k}^{(N)}(u)W_{k+1}^{(N)},
\]

\[
\Omega_0^{(N)}(u) - \Omega_3^{(N)}(u) = uq^{1/2} \sum_{k=0}^{N-1} P_{-k}^{(N)}(u)W_{k+1}^{(N)} - u^{-1/2}q^{1/2} \sum_{k=0}^{N-1} P_{-k}^{(N)}(u)W_{-k}^{(N)},
\]

\[
\Omega_+^{(N)}(u) = \frac{q^{1/2}u^2 + q^{-1/2}u^{-2}}{c_0(q^{1/2} - q^{-1/2})}P_0^{(N)}(u) + \frac{1}{q^{1/2} + q^{-1/2}} \sum_{k=0}^{N-1} P_{-k}^{(N)}(u)G_{k+1}^{(N)} + \omega_0^{(N)},
\]

\[
\Omega_-^{(N)}(u) = \frac{q^{1/2}u^2 + q^{-1/2}u^{-2}}{q^{1/2} - q^{-1/2}}P_0^{(N)}(u) + c_0 \frac{1}{q^{1/2} + q^{-1/2}} \sum_{k=0}^{N-1} P_{-k}^{(N)}(u)G_{k+1}^{(N)} + c_0\omega_0^{(N)},
\]

where $P_{-k}^{(N)}(u)$ are Laurent polynomials to be determined. As explained above, according to the analysis of \[17\], one knows that

\[
K^{(N+1)}(u) \equiv L_{N+1}(uv)K^{(N)}(u)L_{N+1}(uv^{-1})
\]

is also a solution of \[5\] provided $L(u)$ obeys \[6\]. For the ansatz \[14\] to be correct for all $N$, we thus need to show that $K^{(N+1)}(u)$ keeps the form \[17\] with $N \rightarrow N + 1$. To do that, we proceed as follows. First, let us assume \[14\] given $N$ fixed. Then, we have to find an explicit relation between the tensor product of the “old” basis $W_{-k}^{(N)}, W_{k+1}^{(N)}, G_{k+1}^{(N)}, \tilde{G}_{k+1}^{(N)}$ with $U_{q^{1/2}}(sl_2)$ and the “new” one $W_{-k}^{(N+1)}, W_{k+1}^{(N+1)}, G_{k+1}^{(N+1)}, \tilde{G}_{k+1}^{(N+1)}$. Such relation can be obtained, however being rather complicated we report the reader to Appendix A for the explicit recursion relations \[11\] in the $(N+1)$–tensor product basis of $U_{q^{1/2}}(sl_2)$. At the same time, the following recursion relations for the Laurent polynomials:

\[
P_0^{(N+1)}(u) = \left( q^{1/2}u^2 + q^{-1/2}u^{-2} - \frac{(v^2 + v^{-2})}{(q^{1/2} + q^{-1/2})}w_0^{(j)} \right) P_0^{(N)}(u) + c_0(q^{1/2} - q^{-1/2})w_0^{(N)},
\]

\[
P_{-k}^{(N+1)}(u) = \left( q^{1/2} + q^{-1/2} \right) P_{-k+1}^{(N)}(u) - P_{-k}^{(N)}(u) \frac{(v^2 + v^{-2})}{(q^{1/2} + q^{-1/2})}w_0^{(j)} \text{ for } k \in \{1, ..., N - 1\},
\]

\[
P_{-N}^{(N+1)}(u) = q^{1/2} + q^{-1/2} P_{-N+1}^{(N)}(u)
\]

(19)
appear naturally. In addition, the constant term transforms as
\[ \omega_0^{(N+1)} = -\frac{(u^2 + v^{-2})}{(q^{1/2} + q^{-1/2})} w_0^{(N)} \omega_0^{(N)}. \] (20)

Remarkably, in the basis \( \Omega_j^{(N+1)}(u) \) the operators \( \Omega_j^{(N+1)}(u) \) can be drastically simplified. Indeed, using the ansatz [17] and the recursion relations for the polynomials \( P_k^{(N)}(u) \) as written above, these operators reduce to
\[
\begin{align*}
\Omega_0^{(N+1)}(u) + \Omega_3^{(N+1)}(u) &= (\Omega_0^{(N)}(u) + \Omega_3^{(N)}(u))|_{N \to N+1} + u q^{1/2} q^{s_3} \otimes \Delta_1^{(N+1)} - u^{-1} q^{-1/2} q^{-s_3} \otimes \Delta_2^{(N+1)}, \\
\Omega_0^{(N+1)}(u) - \Omega_3^{(N+1)}(u) &= (\Omega_0^{(N)}(u) - \Omega_3^{(N)}(u))|_{N \to N+1} + u q^{1/2} q^{-s_3} \otimes \Delta_2^{(N+1)} - u^{-1} q^{-1/2} q^{s_3} \otimes \Delta_1^{(N+1)}, \\
\Omega_+^{(N+1)}(u) &= \Omega_+^{(N)}(u)|_{N \to N+1} + (q^{1/2} - q^{-1/2}) \Gamma_+^{(N+1)}, \\
\Omega_-^{(N+1)}(u) &= \Omega_-^{(N)}(u)|_{N \to N+1} + (q^{1/2} - q^{-1/2}) \Gamma_-^{(N+1)},
\end{align*}
\]

where we denote
\[
\begin{align*}
\Gamma_+^{(N+1)} &= v q^{-1/4} S_- q^{s_3/2} \otimes \Delta_1^{(N+1)} + v^{-1} q^{1/4} S_- q^{-s_3/2} \otimes \Delta_2^{(N+1)} + \frac{1}{(q - q^{-1})} II \otimes \Delta_3^{(N+1)}, \\
\Gamma_-^{(N+1)} &= v^{-1} q^{1/4} S_+ q^{s_3/2} \otimes \Delta_1^{(N+1)} + v q^{-1/4} S_+ q^{-s_3/2} \otimes \Delta_2^{(N+1)} + \frac{c_0}{(q - q^{-1})} II \otimes \Delta_4^{(N+1)}.
\end{align*}
\]

It is clear that the extra (unwanted) operators \( \Delta_j^{(N+1)} \) with \( j \in \{1, ..., 4\} \) (written below) must be vanishing in order for the ansatz [17] to apply in the case \( N+1 \). To show that, let us focus for instance on \( \Delta_1^{(N+1)} \). According to previous analysis, one has
\[
\Delta_1^{(N+1)} = -c_0 (q^{1/2} - q^{-1/2}) \omega_0^{(N)} W_0^{(N)} - P_{-k+1}^{(N)}(u) W_{-N}^{(N)} - \sum_{k=1}^{N-1} \left( (q^{1/2} u^2 + q^{-1/2} u^{-2}) P_{-k}^{(N)}(u) - (q^{1/2} + q^{-1/2}) P_{-k+1}^{(N)}(u) \right) W_{-N}^{(N)}.
\]

In other words, the fundamental generators \( W_{-k}^{(N)}, W_{k+1}^{(N)}, G_{k+1}^{(N)}, \hat{G}_{k+1}^{(N)} \) must satisfy non-trivial linear relations. Furthermore, for consistency reasons the relations between them must be independent of the spectral parameter \( u \). It is then useful to notice that the Laurent polynomials [22] can be written as
\[
P_{-k+1}^{(N)}(u) = -\frac{1}{(q^{1/2} + q^{-1/2})} \sum_{n=k-1}^{N-1} \left( \frac{q^{1/2} u^2 + q^{-1/2} u^{-2}}{q^{1/2} + q^{-1/2}} \right) n-k+1 C_{-n}^{(N)}
\]

for \( 1 \leq k \leq N \), together with the initial condition \( P_{1}^{(1)}(u) \equiv 1 \). Here the coefficient \( C_{-k}^{(N)} \) are found to satisfy the recursion relations
\[
\begin{align*}
C_{0}^{(N)} &= -(q - q^{-1}) c_0 \omega_0^{(N-1)} - \frac{(u^2 + v^{-2})}{(q^{1/2} + q^{-1/2})} w_0^{(j)} C_{0}^{(N-1)}, \\
C_{-k}^{(N)} &= (q^{1/2} + q^{-1/2}) C_{-k+1}^{(N)} - \frac{(u^2 + v^{-2})}{(q^{1/2} + q^{-1/2})} w_0^{(j)} C_{-k}^{(N-1)} \quad \text{for} \quad 1 \leq k \leq N - 2, \\
C_{-N}^{(N)} &= (q^{1/2} + q^{-1/2}) C_{-N+2}^{(N)}
\end{align*}
\]

with \( C_{0}^{(1)} = -(q^{1/2} + q^{-1/2}) \). In particular, using [28] we immediately deduce for \( k \in \{1, ..., N - 1\} \)
\[
(q^{1/2} u^2 + q^{-1/2} u^{-2}) P_{-k}^{(N)}(u) - (q^{1/2} + q^{-1/2}) P_{-k+1}^{(N)}(u) = C_{-k}^{(N)}.
\]

Note that relations analogous to [22] also hold for \( \Delta_j^{(N+1)} \) with \( j \in \{2, 3, 4\} \) substituting \( W_{-k}^{(N)} \) by \( W_{k+1}^{(N)}, G_{k+1}^{(N)} \) or \( \hat{G}_{k+1}^{(N)} \), respectively. Then, as explained above, \( K^{(N+1)}(u) \) will have the structure [17] provided \( \Delta_j^{(N+1)} = 0 \) for
$j \in \{1, \ldots, 4\}$ are satisfied for all values of $N$. After replacing (25) in (22), we finally get the following (spectral parameter independent) linear relations among the fundamental generators

$$
c_0(q^{1/2} - q^{-1/2})\omega_0^{(N)} W_0^{(N)} - \sum_{k=1}^{N} C_{-k+1}^{(N)} W_{-k}^{(N)} = 0 ,
$$

$$
c_0(q^{1/2} - q^{-1/2})\omega_0^{(N)} W_1^{(N)} - \sum_{k=1}^{N} C_{k+1}^{(N)} W_{k+1}^{(N)} = 0 ,
$$

$$
c_0(q^{1/2} - q^{-1/2})\omega_0^{(N)} G_1^{(N)} - \sum_{k=1}^{N} C_{-k+1}^{(N)} G_{k+1}^{(N)} = 0 ,
$$

$$
c_0(q^{1/2} - q^{-1/2})\omega_0^{(N)} \tilde{G}_1^{(N)} - \sum_{k=1}^{N} C_{-k+1}^{(N)} \tilde{G}_{k+1}^{(N)} = 0
$$

(26)

with (24), (13) and

$$
C_{-k+1}^{(N)} = (q^{1/2} - q^{-1/2})^k (-1)^{N-k+1} \frac{(v^2 + v^{-2})^N}{(q^{1/2} + q^{-1/2})^N} \frac{1}{(N-k)!} \frac{N!}{(N-k)!(N-k)!}.
$$

for $k \in \{1, \ldots, N\}$. Using the representation (24), we have checked explicitly that these linear relations are satisfied for all $N$. For simplicity, details are reported in Appendix B. The terms $\Delta_j^{(N+1)}$ vanishing in (21), it follows that $K^{(N+1)}(u)$ is given by (10), with (17) using the substitution $N \rightarrow N+1$. This being true for $N = 1$ and $N = 2$ as shown in previous sections, we conclude that the general solutions of the reflection equations can be written as (10) with (17) for all values of $N$, where the algebraic structure is now encoded in the fundamental generators $W_{-k}^{(N)}, W_{k+1}^{(N)}, G_{k+1}^{(N)}, \tilde{G}_{k+1}^{(N)}$.

### 2.2 Integrable structure and generating function

Applied to quantum integrable systems on the lattice, the generalized quantum inverse scattering approach provides a powerful method in order to derive in a systematic way a family of independent mutually commuting quantities. For instance, as shown in (17) one can introduce the functional

$$
t^{(N)}(u) = tr_0 \{ K_+(u)K^{(N)}(u) \}
$$

(27)

with (10) and $K(u)$ which solves the “dual” reflection equation. Here $tr_0$ denotes the trace over the two-dimensional auxiliary space. Then, it is proven in (17) that (27) satisfies

$$
[t^{(N)}(u), t^{(N)}(v)] = 0 \quad \text{for all} \quad u, v \in \mathbb{C} ,
$$

i.e. $t^{(N)}(u)$ constitutes the generating function for the mutually commuting operators. For instance, let us plug the $c-$number solution of the “dual” reflection equation (20, 21)

$$
K_+(u) = \left( \begin{array}{cc}
\kappa^+ (q^{1/2} - q^{-1/2})(qu^2 - q^{-1/2}u^{-2}) & \kappa_-(q^{1/2} + q^{-1/2})(qu^2 - q^{-1/2}u^{-2})/c_0 \\
\kappa_+ (q^{1/2} - q^{-1/2})(qu^2 - q^{-1/2}u^{-2}) & \kappa_-(q^{1/2} + q^{-1/2})(qu^2 - q^{-1/2}u^{-2})/c_0
\end{array} \right)
$$

(29)

with $\kappa^0 \equiv \kappa^-$, $\kappa_\pm$ arbitrary complex parameters and (10) with (17) in (21). Immediately, we derive

$$
t^{(N)}(u) = \sum_{k=0}^{N-1} (qu^2 - q^{-1/2}u^{-2}) P_k^{(N)}(u) \mathcal{T}_2^{(N)} + \mathcal{F}(u) \mathbb{II},
$$

(30)

with (26) and

$$
\mathcal{F}(u) = \frac{(q^{1/2} + q^{-1/2})(qu^2 - q^{-1/2}u^{-2})}{c_0} \left( \frac{(q^{1/2}u^2 + q^{-1/2}u^{-2})}{(q^{1/2} - q^{-1/2})} P_0^{(N)}(u) + c_0 \omega_0^{(N)} \right) (\kappa_+ + \kappa_-) .
$$

(31)

The “dual” reflection equation follows from (3) by changing $u \rightarrow q^{-1/2}u^{-1}$, $v \rightarrow q^{-1/2}v^{-1}$ and $K(u)$ in its transpose.
Note that the function $\mathcal{F}(u)$ is obviously not important from an algebraic point of view. Here, we have introduced the operators $\mathcal{I}^{(N)}_{2k+1}$ which can be written in terms of the fundamental generators as

$$
\mathcal{I}^{(N)}_{2k+1} = \kappa_0 W^{(N)}_{-k} + \kappa_+ G^{(N)}_{k+1} + \kappa_- \tilde{G}^{(N)}_{k+1},
$$

for $k \in \{0, 1, ..., N-1\}$. In particular, the property (28) leads to

$$
[\mathcal{I}^{(N)}_{2k+1}, \mathcal{I}^{(N)}_{2l+1}] = 0 \quad \text{for all} \quad k, l \in \{0, ..., N-1\}.
$$

These latter commutation relations impose strong constraints on the fundamental generators. Indeed, plugging (31) in (32) one gets the commutation relations

$$
[W^{(N)}_{-k}, W^{(N)}_{-l}] = 0, \quad [W^{(N)}_{k+1}, W^{(N)}_{l+1}] = 0, \quad [W^{(N)}_{+1}, W^{(N)}_{-1}] = 0,
$$

$$
[G^{(N)}_{k+1}, G^{(N)}_{l+1}] = 0, \quad [G^{(N)}_{k+1}, \tilde{G}^{(N)}_{l+1}] = 0, \quad [G^{(N)}_{k+1}, \tilde{G}^{(N)}_{l+1}] = 0,
$$

$$
[W^{(N)}_{k+1}, \tilde{G}^{(N)}_{l+1}] = 0, \quad [W^{(N)}_{k+1}, \tilde{G}^{(N)}_{l+1}] = 0, \quad [W^{(N)}_{k+1}, \tilde{G}^{(N)}_{l+1}] = 0.
$$

A few remarks can now be done. First, although from the beginning we have considered tensor product representations of $U_{q^{1/2}}(s_{l_2})$, the form of (31) essentially relies on the structure (10) with (17). Then, classifying all possible finite, infinite dimensional or cyclic tensor product representations different from (61) is an interesting open question. Secondly, it should be stressed that given $N$, the relations (20) and their generalizations (see Appendix B (54)-(57)) are responsible of the truncation of the integrable hierarchy, i.e. any quantity $\mathcal{I}^{(N)}_{2k+1}$ for $k \geq N$ cannot be written in terms of all $\mathcal{I}^{(N)}_{2k+1}$ with $k \leq N-1$. It follows that given $N$, there are only $N$-independent mutually commuting fundamental quantities. To conclude, let us mention that for the special case $N = 1$ (and $k = 0$) it is easy to check that (31) coincides exactly with the result of [18].

2.3 Fundamental $q$–deformed commutation relations

We are now interested in the algebraic structure associated with the fundamental generators $W^{(N)}_{-k}, W^{(N)}_{k+1}, G^{(N)}_{k+1}, \tilde{G}^{(N)}_{k+1}$. The form of (17) and the representations (61) being determined by the quadratic algebra (5), this equation fixes all fundamental relations among the generators. To extract these relations, we proceed as follows. First, from (5) and (6), one has

$$
\pi^{(1/2)}[L(u)] = R(u),
$$

where $\pi^{(1/2)}$ denotes the spin–$1/2$ representation of $U_{q^{1/2}}(s_{l_2})$. Then, plugging $K^{(N)}(u)$ in (6), this latter equation can be written as

$$
(\pi^{(1/2)} \times id^{(N)})[L_{N+1}(uv^{-1})K^{(N)}(u)L_{N+1}(uv)](I \otimes K^{(N)}(v)) = (I \otimes K^{(N)}(v))(\pi^{(1/2)} \times id^{(N)})[L_{N+1}(uv)K^{(N)}(u)L_{N+1}(uv^{-1})].
$$

Being satisfied for any value of the spectral parameter $u$, following [13] we can consider its asymptotic expansion for $u \to \infty$. Replacing (10) with (17) in (35), one finds that the leading equation is trivially satisfied. However, the (next) two subleading ones read

$$
(\pi^{(1/2)} \times id^{(N)})[W^{(N+1)}_{0}]|_{u \to v^{-1}}K^{(N)}(v) = (\pi^{(1/2)} \times id^{(N)})[W^{(N+1)}_{0}]K^{(N)}(v),
$$

$$
(\pi^{(1/2)} \times id^{(N)})[W^{(N+1)}_{1}]|_{u \to v^{-1}}K^{(N)}(v) = (\pi^{(1/2)} \times id^{(N)})[W^{(N+1)}_{1}]K^{(N)}(v).
$$

Using the recursion relations (61) for the finite dimensional representations of the fundamental generators and $\pi^{(1/2)}[S_\pm] = \sigma_\pm$ and $\pi^{(1/2)}[S_3] = \sigma_3/2$, one has

$$
\pi^{(1/2)}[W^{(N+1)}_{0}] = \left(\begin{array}{cc}
q^{1/2}W^{(N)}_{0} & v/c_0
\end{array}\right), \quad \pi^{(1/2)}[W^{(N+1)}_{1}] = \left(\begin{array}{cc}
q^{-1/2}W^{(N)}_{1} & v^{-1}/c_0
\end{array}\right).
$$
It is now easy to simplify the intertwining relations (36). After some calculations, the constraints (39) appear naturally with the use of (31). These latter relations being satisfied (see Appendix B for details), omitting the index $N$ we end up with the defining relations (40) for all $k \in \{0, \ldots, N - 1\}$ provided one identifies
\[\rho = \frac{(q^{1/2} + q^{-1/2})^2}{c_0}.\] (37)

Note that some of the relations (40) already appear in (31). Actually, it is easy to give an alternative derivation of the $q$–deformed commutation relations (40) using the representation (61). Indeed, for any $N$ the constraints (39) are satisfied. Let us consider $N \rightarrow N+1$ in these constraints, and replace the generators $W^{(N+1)}$, $G^{(N+1)}$, $\tilde{G}^{(N+1)}$, $k \in \{0, \ldots, N\}$ by their finite dimensional representations (31). After some straightforward calculations, the $q$–deformed relations (40) arise explicitly. Consequently, this shows perfect consistency between the approach associated with the symmetries underlying (35), and the properties of the integrable structure (31), i.e. (31).

It follows that all elements $W^{(N)}$, $G^{(N)}$, $\tilde{G}^{(N)}$ for $k \in \{0, \ldots, N - 1\}$ are generated from $W^{(N)}$, $W^{(N)}$ using the recursion relations (40).

2.4 Relation with tridiagonal algebras and deformed Dolan-Grady hierarchy

The integrable structure (20) is known to be identical (22) with the Dolan-Grady construction introduced and studied in (12), but corresponding to a different notation. This latter structure was found to apply to the class of Hamiltonian of the form
\[\mathcal{H} = \kappa A_0 + \kappa^* A_1.\] (38)

As shown in (12), the integrability condition of the related models (Ising, XY,...) relies on the existence of two (necessary and sufficient) conditions, the “Dolan-Grady relations”, defined by
\[\{A_0, [A_0, [A_0, A_1]]\} = 16\{A_0, A_1\} \quad \text{and} \quad \{A_1, [A_1, [A_1, A_0]]\} = 16\{A_1, A_0\}.\] (39)

All higher mutually commuting quantities beyond (36) can be written solely in terms of the fundamental operators $A_0, A_1$. Note that provided $A_0, A_1$ satisfy (39), the whole Onsager’s algebra (2) is generated. Also, $A_0, A_1$ as well as the other generators of the Onsager’s algebra can be expressed in the basis of the loop algebra $sl_2$ (23, 24). We refer the reader to these works for details.

Surprisingly, a “$q$–deformed” analogue of the Dolan-Grady relations (39) recently appeared in the context of $P$– and $Q$–polynomial association schemes (26, 27, 28):
\[\{A, [A, [A, A^*]]_q\} = \rho [A, A^*] \quad \text{and} \quad \{A^*, [A^*, [A^*, A]]_q\} = \rho [A^*, A].\] (40)

By (28, Definition 3.9) the tridiagonal algebra $T$ is the associative algebra with unity generated by two symbols $A, A^*$ subject to the relations (10). We call $A, A^*$ the standard generators. Here $q$ is a deformation parameter (usually assumed to be not a root of unity) and $\rho$ is a fixed scalar. Let $V$ denote a finite dimensional irreducible module for $T$. Then the pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ is said to be a tridiagonal (TD) pair (22, Definition 1.1), which complete classification remains an open problem. The subset of TD pairs such that $A, A^*$ have eigenspaces of dimension one is called Leonard pairs, classified in (29). In particular, Leonard pairs satisfy (for details, see (30)) the so-called Askey-Wilson (AW) relations first introduced by Zhedanov in (19).

Other examples of TD pairs can be found in (31): for $\rho = 0$ in which case (10) reduce to $q$–Serre relations; for $q = 1$ and $\rho = 16$ which leads to the Dolan-Grady relations (39). The more general situation $\rho \neq 0$, $q \neq 1$ was recently considered in details in (13, 14). There, it was found that TD pairs $A, A^*$ admit a realization in terms of the quantum affine Kac-Moody algebra $U_{q^{1/2}}(\hat{sl}_2)$. This algebra is generated by $Q_\pm, Q_{\pm}$ and $H$ subjects to
\begin{align*}
q^{-1/2} Q_{\pm} Q_{\pm} - q^{1/2} Q_{\pm} Q_{\pm} &= 0, \\
q^{1/2} Q_{\pm} Q_{\pm} - q^{-1/2} Q_{\pm} Q_{\pm} &= \frac{q^{2H} - 1}{q^{1/2} - q^{-1/2}}, \\
q^H Q_{\pm} &= q^{\pm r} Q_{\pm} q^H, \quad q^H Q_{\pm} = q^{\pm r} Q_{\pm} q^H
\end{align*}
(41)
together with the \(q\)-Serre relations

\[
\begin{align*}
Q_\pm^3 + Q_\mp - (1 + q + q^{-1})Q_\pm^2 Q_\mp - (1 + q + q^{-1})Q_\pm Q_\mp^2 - Q_\pm Q_\mp^3 &= 0, \\
\overline{Q}_\pm^3 + Q_\mp - (1 + q + q^{-1})\overline{Q}_\pm^2 Q_\mp - (1 + q + q^{-1})\overline{Q}_\pm Q_\mp^2 - \overline{Q}_\pm Q_\mp^3 &= 0.
\end{align*}
\]

Also, the Hopf algebraic structure of \(U_{q^1/2}(\widehat{sL_2})\) is ensured by the coproduct \(\Delta : U_{q^1/2}(\widehat{sL_2}) \to U_{q^1/2}(\widehat{sL_2}) \times U_{q^1/2}(\widehat{sL_2})\) associated with \(\mathfrak{h}\) acting on the fundamental generators as

\[
\begin{align*}
\Delta(Q_\pm) &= Q_\pm \otimes I + q^{\pm H} \otimes Q_\pm, \\
\Delta(\overline{Q}_\pm) &= \overline{Q}_\pm \otimes I + q^{\mp H} \otimes \overline{Q}_\pm, \\
\Delta(q^H) &= q^H \otimes q^H.
\end{align*}
\]

More generally, one defines the \(N\)-coproduct \(\Delta^{(N)} : U_{q^1/2}(\widehat{sL_2}) \to U_{q^1/2}(\widehat{sL_2}) \otimes \cdots \otimes U_{q^1/2}(\widehat{sL_2})\) as

\[
\Delta^{(N)} \equiv (id \times \cdots \times id \times \Delta) \circ \Delta^{(N-1)}
\]

for \(N \geq 3\) with \(\Delta^{(3)} \equiv \Delta, \Delta^{(1)} \equiv id\). Note that the opposite \(N\)-coproduct \(\Delta^{(N)}\) is similarly defined with \(\Delta' \equiv \sigma \circ \Delta\), where the permutation map \(\sigma(x \otimes y) = y \otimes x\) for all \(x, y \in U_{q^1/2}(\widehat{sL_2})\) is used. As noticed in [13], it is not difficult to show that some linear combinations of \(U_{q^1/2}(\widehat{sL_2})\) generators satisfy \(\mathfrak{h}\). More generally, using the homorphism property of \(\mathfrak{h}\), it is straightforward to check that

\[
\begin{align*}
\mathfrak{A} &\equiv \Delta^{(N)}\left(\frac{1}{c_0}Q_+ + \overline{Q}_- + \epsilon_+ q^H\right) \quad \text{and} \quad \mathfrak{A}^* \equiv \Delta^{(N)}\left(\frac{1}{c_0}Q_- + \overline{Q}_+ + \epsilon_- q^{-H}\right)
\end{align*}
\]

defines a family of TD pairs (i.e. \(\mathfrak{h}\) is satisfied) for arbitrary parameters \(\epsilon_{\pm}\) and the identification \(\mathcal{R}\). Note that although the special case \(\epsilon_{\pm} = 0\) is considered in most of this paper, as pointed out in [13], the algebraic structure remains the same for \(\epsilon_{\pm} \neq 0\). The only difference arises in the exact expressions for the \(c\)–number coefficients in the linear relations \(\mathfrak{h}\).

A \(q\)-deformed analogue of the Dolan-Grady integrable structure [12] was proposed in [13] [14], also based on the properties of the quadratic algebra \(\mathfrak{h}\). In the basis of \(\mathfrak{A}, \mathfrak{A}^*\), the two first charges simply read

\[
\begin{align*}
I_1 &= \kappa \mathfrak{A} + \kappa^* \mathfrak{A}^*, \\
I_3 &= \kappa\left(\frac{c_0}{(q^{1/2} + q^{-1/2})^2}[[\mathfrak{A}, \mathfrak{A}^*]_q, \mathfrak{A}^*]_q + \mathfrak{A}^*\right) + \kappa^*\left(\frac{c_0}{(q^{1/2} + q^{-1/2})^2}[[\mathfrak{A}^*, \mathfrak{A}^*]_q, \mathfrak{A}]_q + \mathfrak{A}\right),
\end{align*}
\]

which are mutually commuting in virtue of \(\mathfrak{h}\). Comparison between the results of previous Sections and the ones of [13] can be done easily, and allow us to find the explicit expression of the fundamental generators \(W_{sL_2}^{(N)}, \mathfrak{G}_1^{(N)}, \mathfrak{G}_{k+1}^{(N)}\) in terms of \(\mathfrak{A}, \mathfrak{A}^*\) satisfying \(\mathfrak{h}\) in the simplest cases \(N = 1\) and \(N = 2\). Furthermore, it provides an alternative check of the fundamental \(q\)-deformed relations \(\mathfrak{h}\), the linear relations \(\mathfrak{h}\) as well as their generalizations \(\mathfrak{h}\) [13] [14].

### 2.4.1 Case \(N = 1\)

Assuming that the entries \(\Omega^{(1)}(u)\) are combinations of Laurent polynomials of degree \(-2 \leq d \leq 2\) in \(u\) and operators, it was shown in [13] that any solution \(K^{(1)}(u)\) takes the form \(\mathfrak{h}\) for \(N = 1\) and the identification

\[
\begin{align*}
W_0^{(1)} &= \mathfrak{A}, \\
W_1^{(1)} &= \mathfrak{A}^*, \\
\mathfrak{G}_1^{(1)} &= [[\mathfrak{A}, \mathfrak{A}^*]_q, \mathfrak{A}^*]_q, \\
\mathfrak{G}_{k+1}^{(1)} &= [[\mathfrak{A}^*, \mathfrak{A}]_q, \mathfrak{A}]_q.
\end{align*}
\]

It should be stressed that this does \textit{not} require the relation \(\mathfrak{h}\) to be satisfied, and only relies on the degree of the Laurent polynomials in \(\Omega^{(1)}(u)\). Nevertheless, the fact that \(K^{(1)}(u)\) satisfies \(\mathfrak{h}\) imposes strong constraints on the generators \(\mathfrak{A}, \mathfrak{A}^*\): plugging \(\mathfrak{h}\) for \(N = 1\) with \(\mathfrak{h}\) and using \(\mathfrak{h}\), one obtains the so-called Askey-Wilson relations [19]

\[
\begin{align*}
A^2A^* + A^*A^2 - (q + q^{-1})AA^* &= \frac{(q^{1/2} + q^{-1/2})^2}{c_0}A^* - \frac{(v^2 + v^{-2})}{c_0}w_0^{(j)}A, \\
A^*A^2 + AA^* - (q + q^{-1})A^*A^* &= \frac{(q^{1/2} + q^{-1/2})^2}{c_0}A - \frac{(v^2 + v^{-2})}{c_0}w_0^{(j)}A^*.
\end{align*}
\]
In particular, it is easy to check these relations for the representation (12). Having in mind the \(q\)-deformed relations (11) and the corresponding hierarchy (31) for \(\kappa_\pm = 0\), it is natural to propose from (11) the following identification:

\[
W^{(1)}_{-1} = \frac{c_0}{(q^{1/2} + q^{-1/2})^2} [A, \{A^*, A\}_q] + A^* ,
\]

\[
W^{(1)}_0 = \frac{c_0}{(q^{1/2} + q^{-1/2})^2} [A, \{A^*, A\}_q]_q + A^* ,
\]

\[
W^{(1)}_1 = \frac{c_0}{(q^{1/2} + q^{-1/2})^2} [\{A^*, A\}_q, A^*]_q + A .
\]  

(47)

This notation introduced in (16), together with (45) leads to a set of very simple linear relations among the fundamental generators. These relations are actually responsible of the truncation of the hierarchy (44) for \(N = 1\), i.e. one finds that \(I_3\) is proportional to \(I_1\). More generally, any higher charge \(I_{2k+1}\) for \(k \geq 1\) can be expressed in terms of the first one (11). Based on this truncation which occurs at \(N = 1\) and the defining relations (11), one finds a slightly generalized version of the first two equations in (46) for the special case \(N = 1\):

\[
c_0(q^{1/2} - q^{-1/2})w^{(1)}_0 W^{(1)}_{-l} - c_0(q^{1/2} - q^{-1/2})w^{(1)}_0 W^{(1)}_{-l-1} = 0 \quad \text{and} \quad c_0(q^{1/2} - q^{-1/2})w^{(1)}_0 W^{(1)}_{l+1} - c_0(q^{1/2} - q^{-1/2})w^{(1)}_0 W^{(1)}_{l+2} = 0
\]

(48)

for any \(l \geq 0\). In particular, the AW relations correspond to \(l = 0\).

2.4.2 Case \(N = 2\)

Based on the results of (11), we proceed similarly. Assuming now that the entries \(\Omega_j^{(1)}(u)\) are combinations of Laurent polynomials of degree \(-d \leq d \leq 4\) in \(u\) and operators, the solution \(K^{(2)}(u)\) takes the form (11) for \(N = 2\) with the identification

\[
W^{(2)}_0 = A , \quad W^{(2)}_1 = \frac{c_0}{(q^{1/2} + q^{-1/2})^2} [A, \{A^*, A\}_q]_q + A^* ,
\]

\[
W^{(2)}_2 = \frac{c_0}{(q^{1/2} + q^{-1/2})^2} [\{A^*, A\}_q, A^*]_q + A .
\]

\[
G^{(2)}_1 = [A^*, A]_q , \quad G^{(2)}_2 = \alpha_1 [A^*, A]_q^2 + \alpha_2 [A^*, A]_q^2 + \alpha_3 [A^*, A]_q^2 + \frac{(q^{1/2} - q^{-1/2})(A^2 + A^*)}{(q + q^{-1})} ,
\]

\[
\tilde{G}^{(2)}_1 = [A, A^*]_q , \quad \tilde{G}^{(2)}_2 = \alpha_1 [A^*, A]_q^2 + \alpha_2 [A^*, A]_q^2 + \alpha_3 [A, A^*]_q^2 + \frac{(q^{1/2} - q^{-1/2})(A^2 + A^*)}{(q + q^{-1})} ,
\]

where

\[
\alpha_0 = \frac{2}{c_0(q^{1/2} - q^{-1/2})} \left( \frac{u^{(j)}_0}{(q^{1/2} + q^{-1/2})^2} - 1 \right) \left( 1 - \frac{u^{(j)}_0}{(q + q^{-1})} \right) ,
\]

\[
\alpha_1 = - \frac{c_0(q^{1/2} - q^{-1/2})}{(q - q^{-1})(q^{1/2} + q^{-1/2})} , \quad \alpha_2 = \frac{c_0(q + q^{-1})}{(q - q^{-1})(q^{1/2} + q^{-1/2})} , \quad \alpha_3 = - \frac{c_0(q^{1/2} + q^{-1/2})}{(q^2 - q^{-2})} .
\]

It is important to notice that the expressions of the fundamental generators \(W^{(2)}_0, W^{(2)}_1, G^{(2)}_1, \tilde{G}^{(2)}_1\) in terms of \(A, A^*\) remains unchanged compared to the case \(N = 1\). In addition, the proposal (46) is confirmed. Plugging \(K^{(2)}(u)\) given by (11) with (14) in (15) and using (46), we obtain the \(N = 2\) generalization of the AW relations (16):

\[
[A, G^{(2)}_2]_q = \frac{2(u^2 + v^2)}{(q^{1/2} + q^{-1/2})^2} u^{(j)}_0 [A, [A^*, A]_q] + [A^*, [A^*, A]_q]_{q^{-1}} + \frac{2(u^2 + v^2)}{c_0} w^{(j)}_0 A^* + \frac{(q^{1/2} + q^{-1/2})^2}{c_0} \left( \frac{u^2 + v^2}{(q^{1/2} + q^{-1/2})^2} w^{(j)}_0 \right) A ,
\]

(50)

\[
[G^{(2)}_2, A^*]_q = \frac{2(u^2 + v^2)}{(q^{1/2} + q^{-1/2})^2} u^{(j)}_0 [A^*, [A, A^*]_q] + [A, [A, A^*]_q]_{q^{-1}} + \frac{2(u^2 + v^2)}{c_0} w^{(j)}_0 A^* + \frac{(q^{1/2} + q^{-1/2})^2}{c_0} \left( \frac{u^2 + v^2}{(q^{1/2} + q^{-1/2})^2} w^{(j)}_0 \right) A .
\]
Notice that \( [G_2^{(2)}_q, A]_q = [A, G_2^{(2)}_q]_q \) and \( [G_2^{(2)}_q, A^*]_q = [A^*, G_2^{(2)}_q]_q \), so that no other relations than (50) are obtained. Based on the algebraic structure (4), as before it is then natural to propose

\[
\begin{align*}
W_2^{(2)} & = \frac{c_0}{(q^{1/2} + q^{-1/2})^2} [A, G_2^{(2)}]_q + W_2^{(2)} , \\
W_3^{(2)} & = \frac{c_0}{(q^{1/2} + q^{-1/2})^2} [G_2^{(2)}, A^*]_q + W_3^{(2)} ,
\end{align*}
\]

(51)

where the explicit expression of \( W_2^{(2)}, W_3^{(2)}, G_2^{(2)} \) in terms of \( A, A^* \) are used in the r.h.s of (51). Identifying in the constraint (50), one finds immediately the linear relations (26) for \( N \geq 2 \). For \( N = 2 \), using the same argument as before about the truncation of the hierarchy, it follows that only \( \mathcal{I}_1 \) and \( \mathcal{I}_3 \) are independent for \( N = 2 \). All higher charges being linear combinations of them, we deduce

\[
\begin{align*}
c_0(q^{1/2} - q^{-1/2})\omega_0^{(2)} W_{-l}^{(2)} - c_0^{(2)} W_{-l-1}^{(2)} - c_0^{(2)} W_{-l-2}^{(2)} &= 0 , \\
c_0(q^{1/2} - q^{-1/2})\omega_0^{(2)} W_{l+1}^{(2)} - c_0^{(2)} W_{l+2}^{(2)} - c_0^{(2)} W_{l+3}^{(2)} &= 0
\end{align*}
\]

(52)

for any \( l \geq 0 \). For \( l = 0 \), the relations (50) are recovered.

### 2.4.3 General case \( N \)

For more general values of \( N \), although technically difficult it is well expected that the fundamental generators can be written solely in terms of \( A, A^* \). Indeed, for \( q = 1 \) the deformed Dolan-Grady integrable structure must reduce to the undeformed one, so that operators in the two hierarchies are in one-to-one correspondence. This goes beyond the scope of this paper, so we do not pursue the analysis for \( N > 2 \).

We now want to focus our attention on the construction of linear relations generalizing (26), similarly to (48) and (49). First, it is an exercise to show using (48) and (4) that the relation

\[
c_0(q^{1/2} - q^{-1/2})\omega_0^{(2)} G_{t+1}^{(1)} - c_0^{(1)} G_{t+2}^{(1)} = 0
\]

(53)

is satisfied, and similarly for \( \tilde{G}_{t+1}^{(1)} \). Indeed, this is in agreement with the argument based on the truncation of the hierarchy (31) for the more general case \( \kappa_\pm \neq 0 \). Any charge \( \mathcal{I}_{2k}^{(N)} \) for \( k \geq N \) being expressed as a linear combination of all charges \( \mathcal{I}_{2k}^{(N)} \) for \( k \leq N - 1 \), and the parameters \( \kappa, \kappa^*, \kappa_\pm \) being independent, we propose for general values of \( N \) the following relations generalizing (26)

\[
\begin{align*}
c_0(q^{1/2} - q^{-1/2})\omega_0^{(N)} W_{-l}^{(N)} - \sum_{k=1}^{N} c^{(N)}_{-k+l} W_{-k-l}^{(N)} &= 0 , \\
c_0(q^{1/2} - q^{-1/2})\omega_0^{(N)} W_{l+1}^{(N)} - \sum_{k=1}^{N} c^{(N)}_{-k+l} W_{k+l+1}^{(N)} &= 0 , \\
c_0(q^{1/2} - q^{-1/2})\omega_0^{(N)} G_{t+1}^{(N)} - \sum_{k=1}^{N} c^{(N)}_{-k+l} G_{k+l+1}^{(N)} &= 0 , \\
c_0(q^{1/2} - q^{-1/2})\omega_0^{(N)} \tilde{G}_{t+1}^{(N)} - \sum_{k=1}^{N} c^{(N)}_{-k+l} \tilde{G}_{k+l+1}^{(N)} &= 0
\end{align*}
\]

(54–57)

for any \( l \geq 0 \). For \( N = 1 \) and \( N = 2 \), these relations were obtained above. We have checked explicitly that these relations also hold for general values of \( N \), using the explicit finite dimensional representations of the generators. We report the reader to Appendix B for details. As a consistency check, let us mention that the relations (56), (57) actually follow from (53) and (54), using the q-deformed commutation relations (20).
3 Concluding remarks

The Onsager’s algebra is known to be generated from two elements $A_0, A_1$ satisfying the Dolan-Grady relations \[ A_0 A_1 = \frac{1}{2} [A_1, A_0] \quad \text{and} \quad [A_0, A_1] = \frac{1}{4} [A_0, A_0]. \] (58)

The relations (59) are actually sufficient to reconstruct the Onsager’s algebra (2). Furthermore, for finite dimensional representations the spectral properties of (38) (as well as arbitrary combinations of $A_k, G_l$) are known to be encoded in the closure of the algebra \[ \{23, 24\} \] which reads (for some coefficients $\alpha_k$)

\[ \sum \alpha_k A_{k-l} = 0, \quad \sum \alpha_k G_{k-l} = 0. \] (59)

In this paper, based on the link between the quadratic algebra $[23]$ and the deformed Dolan-Grady integrable structure recently discovered in $[13, 14]$, we have found that the algebra $[2]$ introduced by Onsager in $[1]$ admits a $q-$deformed infinite dimensional analogue $[3]$ with fundamental generators $W_{-k}, W_{k+1}, G_{k+1}, \tilde{G}_{k+1}$ with $k \in \mathbb{N}$. Similarly to the Onsager’s algebra, the integrable structure follows from the “$q$–deformed” Dolan-Grady relations (i.e. tridiagonal algebra) \[ \{40\} \] with $A \rightarrow W_0, A^* \rightarrow W_1$ and the “$q$–deformed” recursion relations in \[ \{41\} \]. It should be stressed that this new algebra possesses either finite or infinite dimensional representations: Finite dimensional representations \[ \{61\} \] have been obtained, in which case the generators satisfy a set of linear relations generalizing the Askey-Wilson ones \[ \{54\", \ldots \} \]. On the other hand, in the limit $N \rightarrow \infty$ vertex operators representations can be used \[ \{13\} \].

This new symmetry ensures the existence of an (in)finite number (associated with the (in)finite parameter $N$) of mutually commuting quantities given by \[ \{51\} \]. For the special (undeformed) case $q = 1$, the defining relations \[ \{49, 41\} \] coincide exactly with the ones considered in \[ \{12, 23\} \]. Indeed, simple comparison between \[ \{49, 41\} \] and \[ \{61\} \] gives the exact relation between our generators and the ones in \[ \{11\} \]:

\[ W_{-k}|_{q=1} = (A_k + A_{-k})/2, \quad W_{k+1}|_{q=1} = (A_{k+1} + A_{-k+1})/2, \quad G_{k+1}|_{q=1} = -\tilde{G}_{k+1}|_{q=1} = 4G_{k+1}. \] (60)

Also, the linear relations \[ \{20\} \] as well as their generalizations \[ \{53\", \ldots \} \] reduce to the ones proposed in \[ \{23\} \].

The most interesting problem now is to analyze quantum integrable models with this new mathematical framework, in order to extract any nonperturbative information. In this direction, identifying the models with such underlying symmetry is obviously the first thing to be done. In particular, it should be stressed that $[41]$ is closely related with $U_{q^{1/2}}(sl_2)$ with generators $Q_\pm, \overline{Q}_\pm \[ \{13, 14\] \]. Then, it is well expected that the characteristics of the model are encoded in the parameters $N, q, c_0, v$ whereas the fundamental generators should correspond to some observables. Below, we give various examples of quantum integrable models (lattice, massive, boundary or conformal) which enjoy the symmetry $[41]$.

• **XXZ open spin chain with general boundary conditions:** The fundamental generators $W_0^{(N)}, W_1^{(N)}$ are related \[ \{6\} \] with the nonlocal conserved charges obtained in \[ \{32\} \] using the method proposed in \[ \{38, 31\} \], and $N$ corresponds to the number of sites. The deformation parameter $q$ characterizes the anisotropy $\Delta = (q^{1/2} + q^{-1/2})/2$ of the model, whereas $\kappa, \kappa_\pm (c_0)$ are non-diagonal left (right) boundary conditions, respectively. Also, $v = 1$. Note that for more general right boundary conditions associated with extra parameters $\epsilon_\pm$, the algebraic structure remains essentially unchanged. It follows that the underlying fundamental symmetry of the XXZ open spin chain with general boundary conditions is the $q-$deformed Onsager’s algebra $[41]$ with representations $[61]$. This symmetry, sometimes called “boundary quantum group algebra”, is in one-to-one correspondence with the

\[ ^5 \text{Of course, the linear relations } [53", \ldots \} \text{ need to be clarified in this case.} \]

\[ ^6 \text{Note that the nonlocal charges derived in } [32] \text{ correspond to certain integrable boundary conditions, not the most general ones.} \]
tridiagonal algebra \([20, 27, 28]\) as shown in \([14]\). Based on previous analysis, it follows that the transfer matrix can be simply written as \([30]\). Details will be reported elsewhere.

- **Sine-Gordon quantum field theory:** In the bulk, the sine-Gordon model is known to possess nonlocal conserved charges \([7]\), usually denoted \(Q_\pm, \overline{Q}_\pm\), generating a \(U_{q/2}(sl_2)\) symmetry. Actually, \(W_0^{(N)}, W_1^{(N)}\) for \(N \to \infty\) admit a vertex operator representation in one-to-one correspondence with linear combinations of these charges \([13]\) and parametrized by \(c_0\) (arbitrary). The deformation parameter \(q\) and parameter \(v\) are easily related with the coupling constant \(\beta^2\) and the rapidity of the fundamental particles (soliton/antisoliton), respectively. In case of a non-dynamical \([21]\) or a dynamical boundary \([35]\) the model still remains integrable, but the symmetry is restricted. The corresponding nonlocal conserved charges have been constructed in \([33, 34]\), and generate an example of tridiagonal algebra \([13, 14]\). For these boundary integrable models, one has the identification \(N \to \infty\) and \(c_0 = 1\).

- **Liouville quantum field theory:** In this conformal limit of the sine-Gordon model, it is easy to check that either \(Q_+, \overline{Q}_-\) or \(Q_-, \overline{Q}_+\) are conserved (commuting with the stress-energy tensor). Then, an infinite number of conserved quantities are obtained from \([31]\) for \(N \to \infty\) and some vanishing parameters \(\kappa, \kappa^*, \kappa_\pm\). It follows that the Liouville field theory (as well as its boundary counterpart) enjoys the symmetry generated by a subalgebra of \([4]\).

- **Quasi-exactly solvable systems, Bethe ansatz and \(q\)–difference equations:** For general values of \(N \neq 1\), the spectral problem associated with \([31]\) leads to a system of partial \(q\)-difference equations that clearly needs further investigation. For the special case \(N = 1\), one obtains a second-order \(q\)-difference equation which has been considered in details in \([13, 33]\), and lead to Bethe equations. Interestingly, for the limit \(q \to 1\) it becomes the Heun (or similarly the Pöschl-Teller) equation. Furthermore, at this special value of the deformation parameter the Onsager algebra \([2]\) exhibited and studied in the context of quasi-exactly solvable systems and nonlinear holomorphic supersymmetry \([37]\) is recovered. Then, we expect our construction will provide a new (algebraic) approach to the surprising relation between conformal field theory and differential equations pointed out in \([38]\), as well as its massive counterpart.

Related problems will be considered elsewhere.

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**Appendix A: Tensor product representations of the fundamental generators**

\[
W_0^{(N+1)} = \frac{1}{c_0}vq^{1/4}S_+q^{s_3/2} \otimes I + v^{-1}q^{-1/4}S_-q^{-s_3/2} \otimes I + q^{s_3} \otimes W_0^{(N)},
\]
\[
W_1^{(N+1)} = \frac{1}{c_0}v^{-1}q^{-1/4}S_+q^{-s_3/2} \otimes I + vq^{1/4}S_-q^{-s_3/2} \otimes I + q^{-s_3} \otimes W_1^{(N)},
\]
\[
G_1^{(N+1)} = (q-q^{-1})S_-^2 \otimes I - \frac{(q^{1/2} + q^{-1/2})}{c_0(q^{1/2} - q^{-1/2})}(v^2q^{s_3} + v^{-2}q^{-s_3}) \otimes I + I \otimes G_1^{(N)}
\]
\[
+ (q-q^{-1}) \left( v^{-1}q^{-1/4}S_-q^{-s_3/2} \otimes W_0^{(N)} + vq^{1/4}S_-q^{-s_3/2} \otimes W_1^{(N)} \right) + \frac{(v^2 + v^{-2})w_0^{(j)}}{c_0(q^{1/2} - q^{-1/2})} \otimes I,
\]
\[
\tilde{G}_1^{(N+1)} = \frac{(q-q^{-1})}{c_0}S_+^2 \otimes I - \frac{(q^{1/2} + q^{-1/2})}{c_0(q^{1/2} - q^{-1/2})}(v^2q^{-s_3} + v^{-2}q^{s_3}) \otimes I + I \otimes \tilde{G}_1^{(N)}
\]
\[
+ \frac{(q-q^{-1})}{c_0} \left( v^{-1}q^{1/4}S_+q^{s_3/2} \otimes W_0^{(N)} + vq^{-1/4}S_+q^{s_3/2} \otimes W_1^{(N)} \right) + \frac{(v^2 + v^{-2})w_0^{(j)}}{c_0(q^{1/2} - q^{-1/2})} \otimes I,
\]
\[ W^{(N+1)}_{-k-1} = \frac{(w^{(j)}_0 - (q^{1/2} - q^{-1/2})q^{s_3})}{(q^{1/2} - q^{-1/2})^3} W^{(N)}_{k+1} - \frac{(v^2 + v^{-2})}{(q^{1/2} - q^{-1/2})^2} \otimes W^{(N+1)}_{-k} + \frac{(v^2 + v^{-2})w^{(j)}_0}{(q^{1/2} - q^{-1/2})^2} W^{(N+1)}_{-k} \]

\[ W^{(N+1)}_{k+2} = \frac{(w^{(j)}_0 - (q^{1/2} - q^{-1/2})q^{s_3})}{(q^{1/2} - q^{-1/2})^3} W^{(N)}_{k+1} - \frac{(v^2 + v^{-2})}{(q^{1/2} - q^{-1/2})^2} \otimes W^{(N+1)}_{k+1} + \frac{(v^2 + v^{-2})w^{(j)}_0}{(q^{1/2} - q^{-1/2})^2} W^{(N+1)}_{k+1} \]

\[ G^{(N+1)}_{k+2} = \frac{c_0 (q^{1/2} - q^{-1/2})^2}{N+1} S_+ \otimes G^{(N)}_{k+1} - \frac{1}{(q^{1/2} - q^{-1/2})} \left( v^2 q^{s_3} + v^{-2} q^{s_3} \right) \otimes G^{(N)}_{k+1} + \frac{1}{(q^{1/2} - q^{-1/2})^2} \otimes G^{(N)}_{k+2} \]

\[ \tilde{G}^{(N+1)}_{k+2} = \frac{(q^{1/2} - q^{-1/2})^2}{c_0 (q^{1/2} - q^{-1/2})^3} S_+ \otimes \tilde{G}^{(N)}_{k+1} - \frac{1}{(q^{1/2} - q^{-1/2})} \left( v^2 q^{s_3} + v^{-2} q^{s_3} \right) \otimes \tilde{G}^{(N)}_{k+1} + \frac{1}{(q^{1/2} - q^{-1/2})^2} \otimes \tilde{G}^{(N)}_{k+2} \]

for \( k \in \{0, 1, \ldots, N-1\} \).

**Appendix B: Generalized linear relations**

The purpose of this Appendix is to show that the linear relations \( 20, 26, 33 \), as well as their generalizations \( 59, 60, 67 \), are satisfied for all values of \( N \). For \( N \) fixed, let us first assume that \( W^{(N)}_{-k}, W^{(N)}_{k+1}, G^{(N)}_{k+1} \) and \( \tilde{G}^{(N)}_{k+1} \) with \( k \in \{0, \ldots, N\} \) satisfy \( 33 - 37 \). Then, a straightforward calculation based on the finite dimensional representations \( 61 \) shows that

\[
\sum_{k=1}^{N+1} C^{(N+1)}_{-k-1} W^{(N+1)}_{-k-1} - c_0 (q^{1/2} - q^{-1/2}) w^{(N+1)}_{0} W^{(N+1)}_{-l} = \frac{w^{(j)}_0}{(q^{1/2} - q^{-1/2})^3} \sum_{k=1}^{N+1} \beta^{(N+1)}_{-k-1} W^{(N)}_{k+1} - \frac{v^2 + v^{-2}}{q^{1/2} - q^{-1/2}} \sum_{k=1}^{N+1} \beta^{(N+1)}_{-k-1} W^{(N+1)}_{-k-1} - \frac{(q^{1/2} - q^{-1/2})}{(q^{1/2} - q^{-1/2})^3} \left( v q^{1/2} S_+ \right) \sum_{k=1}^{N+1} \beta^{(N+1)}_{-k-1} G^{(N)}_{k+1} + c_0 v - q^{1/2} S_+ w^{(N+1)}_{0} \sum_{k=1}^{N+1} \beta^{(N+1)}_{-k-1} G^{(N)}_{k+1} \]

\[+ q^{s_3} \sum_{k=1}^{N+1} \beta^{(N+1)}_{-k-1} W^{(N)}_{-k-1} \]

\[+ \frac{(v^2 + v^{-2})w^{(j)}_0}{(q^{1/2} - q^{-1/2})^3} \beta^{(N+1)}_{-k-1} W^{(N+1)}_{-l} - c_0 (q^{1/2} - q^{-1/2}) w^{(N+1)}_{0} W^{(N+1)}_{-l}, \]

where \( l \geq 0 \) and

\[ \beta^{(N+1)}_{-k-1} = \sum_{m=k}^{N+1} \left( \frac{(v^2 + v^{-2})w^{(j)}_0}{(q^{1/2} - q^{-1/2})^3} \right)^{m-k} C^{(N+1)}_{-k-1}. \]
Using (24), it is easy to notice that
\[ \beta^{(N+1)}_{-(k-1)} = (q^{1/2} + q^{-1/2})C^{(N)}_{-k+2} \quad \text{for} \ 2 \leq k \leq N + 1 \quad \text{and} \quad \beta^{(N+1)}_0 = -c_0(q - q^{-1})\omega^{(N)}_0. \]
Replacing these expressions and (20) in (62), all terms vanish in virtue of (54)-(57). Consequently,
\[ \sum_{k=1}^{N+1} C^{(N+1)}_{-k+1} W^{(N+1)}_{-k} - c_0(q^{1/2} - q^{-1/2})\omega^{(N+1)}_0 W^{(N+1)}_{-l} = 0, \quad (63) \]
provided (54)- (57) for \( N \) fixed. Similar relations also holds for \( W^{(N+1)}_{k+1}, G^{(N+1)}_{k+1} \) and \( \tilde{G}^{(N+1)}_{k+1} \) with \( k \in \{0, ..., N+1\} \). Now, we proceed by recursion:

- **General case \( N \) and \( l = 0 \):** the linear relations (26) are satisfied for \( N = 1 \) and \( N = 2 \), either corresponding to the Askey-Wilson relations (for \( N = 1 \)), or its \( N = 2 \) generalization (50). According to (63), it follows that (26) are satisfied for all values of \( N \).

- **General case \( N \) and arbitrary \( l \geq 0 \):** Due to (26), (10) admits the representation (17) and the mutually commuting operators take the form (31). For \( N = 1 \) and \( N = 2 \), we obtained (53) for arbitrary \( l \geq 0 \). Due to (63), (54) is indeed satisfied for all values of \( N \). Clearly, similar analysis can be done (see (53)) for \( W^{(N)}_{k+1}, G^{(N)}_{k+1} \) and \( \tilde{G}^{(N)}_{k+1} \) with \( k \in \{0, ..., N\} \). Then, we conclude that (54)- (57) are satisfied for all values of \( N \).

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