COMPLEX INTERPOLATION OF ORLICZ SEQUENCE SPACES AND ITS HIGHER ORDER ROCHBERG SPACES

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Abstract. We show that if \((\ell_{\phi_0}, \ell_{\phi_1})\) is a couple of suitable Orlicz sequence spaces then the corresponding Rochberg derived spaces of all orders associated to the complex interpolation method are Fenchel-Orlicz spaces. In particular, the induced twisted sums have the \((C[0, 1], C)\)-extension property.

1. Introduction

The present paper is a spiritual continuation of [1], where the authors deal with extensions of Orlicz sequence spaces. An extension of a Banach space \(Y\) is a quasi-Banach space \(X\) such that \(Y\) is isomorphic to a subspace of \(X\) and the respective quotient is also isomorphic to \(Y\). Androulakis, Cazaku and Kalton showed how Fenchel-Orlicz spaces (a natural generalization of Orlicz spaces to higher dimensions) may be used to obtain extensions of Orlicz sequence spaces with nontrivial type.

More generally, given Banach spaces \(Y\) and \(Z\), a twisted sum of \(Z\) and \(Y\) (the order is important) is a quasi-Banach space \(X\) such that \(Y\) is isomorphic to a subspace of \(X\) and \(X/Y\) is isomorphic to \(Z\). We may represent that in terms of a short exact sequence

\[
0 \to Y \to X \to Z \to 0
\]

The most famous case is when \(Y = Z = \ell_2\). If \(X \neq \ell_2\) then \(X\) is called a twisted Hilbert space. The first example of twisted Hilbert space was given by Enflo, Lindenstrauss and Pisier [6], followed some years later by the example of Kalton and Peck [9].

One can go the extra mile and twist twisted sums; in [11] Rochberg presented his derived spaces associated to the complex method of interpolation. Such derived spaces give us families of twisted sums. Take, for example, \(Z_2^{(1)} = \ell_2\) and \(Z_2^{(2)} = Z_2\), the Kalton-Peck space. In [3], Cabello Sánchez, Castillo and Kalton used Rochberg’s construction to obtain a family of Banach spaces \((Z_2^{(n)})_{n \geq 3}\) that fit nicely into short exact sequences

\[
0 \to Z_2^{(m)} \to Z_2^{(m+n)} \to Z_2^{(n)} \to 0
\]

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Using those spaces, they showed that being a twisted Hilbert space is not a 3-space property, since $Z_2^{(4)}$ is a Banach space which contains an isomorphic copy of $Z_2$, the respective quotient is isomorphic to $Z_2$, but $Z_2^{(4)}$ is not a twisted Hilbert space. We call the spaces $Z_2^{(n)}$ higher order extensions of $\ell_2$.

Our aim in this paper is to give a description and to study the properties of the higher order extensions of Orlicz sequence spaces obtained by complex interpolation. We prove that those spaces are Fenchel-Orlicz spaces through an adaptation of the arguments of Androulakis, Cazaku and Kalton in [1].

The structure of Fenchel-Orlicz space is behind the astonishing fact that if $T : \ell_2 \to C[0,1]$ is any operator then $T$ admits an extension to the Kalton-Peck space $Z_2$. We are able to show that our higher order extensions share the same property (considering complex scalars).

The structure of the paper is as follows: Section 2 contains background on Fenchel-Orlicz spaces, complex interpolation and the twisted sums it generates. In Sections 3 and 4 we show how to obtain quasi-Young functions from complex interpolation of Orlicz sequence spaces, and that the derived spaces induced by the interpolation process agree with the Fenchel-Orlicz spaces generated by those quasi-Young functions. In Section 5 we show that our twisted sums satisfy the aforementioned property of extension of operators with image in $C([0,1], C)$, and in Section 6 we conclude with a remark on how the results in [1] may be seen in the context of complex interpolation.

2. Background

2.1. Fenchel-Orlicz spaces. The standard reference for this topic is the work of Turett [12]. Usually Young functions are defined for real vector spaces, but we will need to consider their complex version here. A Young function $\phi : \mathbb{C}^n \to [0, \infty)$ is a convex function such that $\phi(0) = 0$, $\lim_{t \to \infty} \phi(tx) = \infty$ for every $x \in \mathbb{C}^n \setminus \{0\}$, and $\phi(e^{it}x) = \phi(x)$ for every $x \in \mathbb{C}^n$, $s \in \mathbb{R}$. We define the Fenchel-Orlicz space

$$\ell_\phi = \{(x^k) \in (\mathbb{C}^n)^\mathbb{N} : \exists \rho > 0 \text{ for which } \sum_{k=1}^{\infty} \phi\left(\frac{x^k_1}{\rho}, \cdots, \frac{x^k_n}{\rho}\right) < \infty\}$$

endowed with the complete norm

$$\|(x^k)\|_\phi = \inf\{\rho > 0 : \sum_{k=1}^{\infty} \phi\left(\frac{x^k_1}{\rho}, \cdots, \frac{x^k_n}{\rho}\right) \leq 1\}$$

When $n = 1$ we have the definition of an Orlicz space, so we call $\phi$ an Orlicz function. If $\phi(t) > 0$ for $t > 0$ we say that $\phi$ is nondegenerate. We are particularly interested in this case because $\phi_{|[0,\infty)}$ is strictly increasing and therefore has an inverse. When we write $\phi^{-1}$ we will always be referring to this inverse. We also have another important particularity of Orlicz functions: it is enough to define $\phi$ on $[0, \infty)$, since for $x \in \mathbb{C}$ we have $\varphi(x) = \varphi(|x|)$.

A Young function $\phi$ on $\mathbb{C}^n$ satisfies the $\Delta_2$ condition (or is said to be in the class $\Delta_2$) if there is a constant $M > 0$ such that $\phi(2x) \leq M\phi(x)$ for every $x \in \mathbb{C}^n$. The following lemma is an easy exercise.

**Lemma 2.1.** Let $\phi$ be an Orlicz function in the class $\Delta_2$.

1. There is $c \geq 0$ such that for every $x, y \in \mathbb{C}$

$$\phi(x + y) \leq c(\phi(x) + \phi(y))$$

2. If $x, y \in \mathbb{C}^n$, then

$$\phi(x + y) \leq \phi(x) + \phi(y)$$

3. If $x, y \in \mathbb{C}^n$ and $\phi(2x) \leq M\phi(x)$, then $\phi(x + y) \leq 2M\phi(x) + 2M\phi(y)$.
In particular, if \( \phi \) is strictly increasing then
\[
\phi^{-1}(s) + \phi^{-1}(t) \leq \phi^{-1}(c(s + t))
\]
for every \( s, t \in [0, \infty) \).

(2) If \( a > 0 \) then there is \( D_a > 0 \) such that
\[
\phi(ax) \leq D_a \phi(x)
\]
for every \( x \in \mathbb{C} \).

In the definition of twisted sum the middle space of the short exact sequence is a quasi-Banach space. Accordingly, we will need to work with quasi-Young functions: we say that \( \phi : \mathbb{C}^n \to [0, \infty) \) is quasi-Young if \( \phi(0) = 0, \lim_{t \to \infty} \phi(tx) = \infty \) for every \( x \in \mathbb{C}^n \setminus \{0\} \), \( \phi(e^{is}.x) = \phi(x) \) for every \( x \in \mathbb{C}^n \), \( s \in \mathbb{R} \), and there is a constant \( C > 0 \) such that
\[
\phi(tx + (1-t)y) \leq C(t\phi(x) + (1-t)\phi(y))
\]
for every \( x, y \in \mathbb{C}^n \), \( t \in [0, 1] \). We say that \( C \) is a quasi-convexity constant for \( \phi \).

If we define \( \ell_\phi \) and \( \| \cdot \|_\phi \) as above then \( \| \cdot \| \) is a quasinorm and \( \ell_\phi \) is a quasi-Banach space. Actually, we may easily adapt the argument of [1, Proposition 2.1] to complex variables and show that \( \| \cdot \|_\phi \) is equivalent to a norm.

2.2. Quasilinear maps. The standard way to build a twisted sum of \( Z \) and \( Y \) is through quasilinear maps [9]. Let \( W \) be a vector space containing \( Y \). We say that \( F : Z \to W \) is quasilinear from \( Z \) into \( Y \) if it is homogeneous and there is a constant \( Q > 0 \) such that
\[
\|F(z_1 + z_2) - F(z_1) - F(z_2)\|_Y \leq Q(\|z_1\|_Z + \|z_2\|_Z)
\]
for every \( z_1, z_2 \in Z \). Let
\[
Y \oplus_F Z = \{(w, z) : z \in Z, w - Fz \in Y\} \subset W \times Y
\]
be endowed with the complete quasinorm \( \|(w, z)\| = \|w - Fz\|_Y + \|z\|_Z \). Then \( Y \oplus_F Z \) is a twisted sum of \( Z \) and \( Y \), since the map \( y \mapsto (y, 0) \) from \( Y \) into \( Y \oplus_F Z \) defines an isometry and the respective quotient is isometric to \( Z \). For example, the Kalton-Peck space is defined through a quasilinear map \( K_2 : \ell_2 \to \ell_\infty \) from \( \ell_2 \) into \( \ell_2 \) such that
\[
K_2(x) = \sum x_n \log \frac{|x_n|}{\|x\|_2} e_n
\]
for every \( x \in \ell_2 \) of finite support.

2.3. Complex interpolation. If the standard way to build twisted sums is through quasilinear maps, complex interpolation is a established tool to build quasilinear maps. Let \( V \) be a Hausdorff topological vector space, and let \( \overline{X} = (X_0, X_1) \) be a couple of Banach spaces for which there are continuous injections \( i_j : X_j \to V, j = 0, 1 \). We call such a couple compatible. We can always suppose that \( V \) is a Banach space. Indeed, consider the sum space
\[
\Sigma(\overline{X}) = \{i_0(x_0) + i_1(x_1) : x_0 \in X_0, x_1 \in X_1\}
\]
with the complete norm.

\[ ||x|| = \inf \{ ||x_0||_{X_0} + ||x_1||_{X_1} : x = i_0(x_0) + i_1(x_1) \} \]

We may then replace \( i_j \) by the inclusion map \( X_j \subset \Sigma(X) \), \( j = 0, 1 \).

Let \( S = \{ z \in \mathbb{C} : 0 \leq Re(z) \leq 1 \} \), and let \( \mathcal{F}(\overline{X}) \) be the space of all bounded continuous functions \( f : S \to \Sigma(X) \) which are analytic on \( S^0 \) and such that the functions \( t \mapsto f(j + it) \) are continuous and bounded from \( \mathbb{R} \) into \( X_j \), \( j = 0, 1 \). The space \( \mathcal{F}(\overline{X}) \) is a Banach space with the norm

\[ ||f|| = \sup_{j=0,1} \sup_{r \in \mathbb{R}} ||f(j + it)||_{X_j} \]

For \( \theta \in (0, 1) \), let \( X_\theta = \{ f(\theta) : f \in \mathcal{F}(\overline{X}) \} \) endowed with the quotient norm

\[ ||x||_\theta = \inf \{ ||f|| : f \in \mathcal{F}(\overline{X}), f(\theta) = x \} \]

Then \( X_\theta \) is an interpolation space with respect to \( (X_0, X_1) \). The classical example is \( (\ell_\infty, \ell_1)_\theta = \ell_{p_\theta} \), where \( \frac{1}{p_\theta} = \theta \). For more information on interpolation, see [2].

### 2.4. Extensions induced by complex interpolation.

Quite surprisingly, a construction of Rochberg [11] yields higher order extensions of the interpolation space. Here we present the Rochberg spaces from the point of view of quasilinear maps [3]. If \( f \) is a function on some complex domain with values in a Banach space, we let \( \hat{f}[j; z] = \frac{p^{n}[j]}{p^j} \) be its \( j \)-th Taylor coefficient at \( z \). Let \( d^1X_\theta = X_\theta \), and suppose the space \( d^nX_\theta \) has already been defined. There is a homogeneous function \( B^n_\theta : d^nX_\theta \to \mathcal{F}(\overline{X}) \) such that

1. \( B^n_\theta[j; \theta] = x_j \) for every \( \theta = (x_{n-1}, \cdots, x_0) \in d^nX_\theta \) and every \( 0 < j < n - 1 \);
2. There is a constant \( C_n > 0 \) independent of \( x \) in \( d^nX_\theta \) such that \( ||B^n_\theta(x)|| \leq C_n||x|| \).

Let \( \Omega^n_\theta : d^nX_\theta \to \Sigma(\overline{X}) \) be defined by \( \Omega^n_\theta(x) = \hat{B}^n_\theta[n; \theta] \). Then \( \Omega^n_\theta \) is quasilinear from \( d^nX_\theta \) into \( X_\theta \) and we can define the derived space

\[ d^{n+1}X_\theta = X_\theta \oplus_{\Omega^n_\theta} d^nX_\theta = \{ (w, x) : x \in d^nX_\theta, w - \Omega^n_\theta(x) \in X_\theta \} \subset \Sigma(\overline{X}) \times d^nX_\theta \]

endowed with the quasinorm presented above. We notice that the spaces \( d^nX_\theta \) are independent of the choice of maps \( B^n_\theta \) satisfying (1) and (2), up to equivalence of quasinorms. Also, the quasinorm on \( d^nX_\theta \) is always equivalent to a norm.

These spaces form higher order extensions of the space \( X_\theta \). Again, the classical example comes from \( (\ell_\infty, \ell_1) \): the derived spaces at \( \theta = \frac{1}{2} \) are the spaces \( Z^{(1)}_\theta \) of the introduction. By higher order extensions we mean that we have short exact sequences

\[ 0 \to d^mX_\theta \to d^{m+n}X_\theta \to d^nX_\theta \to 0 \]

where the inclusion map is \( x \mapsto (x, 0) \) and the quotient map is \( (x, y) \mapsto y \).

The following fact will be important: once we have \( B^n_\theta \) it is possible to define \( B^{n+1}_\theta \) inductively. Indeed, let \( \varphi : S \to \overline{S} \) be a conformal map such that \( \varphi(\theta) = 0 \). Suppose \( B^n_\theta(x) \) is defined for every \( x \in d^nX_\theta \) and let \( x = (x_n, \cdots, x_0) \in d^{n+1}X_\theta \). In particular, \( x_n - \Omega^n_\theta(x_{n-1}, \cdots, x_0) \in X_\theta \). We can take

\[ B^{n+1}_\theta(x)(z) = B^n_\theta(x_{n-1}, \cdots, x_0)(z) + \frac{n!}{k_n^2}B^n_\theta(x_n - \Omega^n_\theta(x_{n-1}, \cdots, x_0))(z) \]
where \( k_n \) is a numerical constant depending only on the derivative of \( \varphi \) at \( \theta \) which ensures that
\[
\bar{B}_\theta^{n+1}[n; \theta] = x_n.
\]

### 3. Obtaining quasi-Young functions from complex interpolation

From now on, unless otherwise stated, we suppose our Orlicz functions satisfy the \( \Delta_2 \) condition. It is a classical result that if \( \phi_0, \phi_1 : \mathbb{C} \to [0, \infty) \) are nondegenerate Orlicz functions then for every \( \theta \in (0, 1) \) the function \( \phi_\theta : [0, \infty) \to [0, \infty) \) given by \( \phi_\theta^{-1} = (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^\theta \) defines an Orlicz function, and we have

\[
(\ell_\phi, \ell_\psi)_\theta = \ell_{\phi_\theta}
\]

with equivalence of norms (see [7]; see also [5, 4] for generalizations). Notice that everything works equally well if we let \( \ell_\phi = \ell_\infty \) or \( c_0 \) and take \( \phi_\theta^{-1} \equiv 1 \).

For \( x \in \ell_\phi \) we can take

\[
B_\theta^1(x)(z) = \|x\|_{\ell_\phi} \sum_{n=1}^{\infty} \phi_0^{-1}(\phi_\theta(|x_n|/\|x\|_{\ell_\phi}))^{1-\theta} \phi_1^{-1}(\phi_\theta(|x_n|/\|x\|_{\ell_\phi}))^{\theta} sgn(x_n)e_n
\]

where \( sgn(z) = \frac{z}{|z|} \) if \( z \in \mathbb{C} \setminus \{0\} \) and \( sgn(0) = 0 \). Therefore

\[
\Omega_\theta^1(x) = \sum_{n=1}^{\infty} \log \frac{\phi_0^{-1}(\phi_\theta(|x_n|/\|x\|_{\ell_\phi}))}{\phi_1^{-1}(\phi_\theta(|x_n|/\|x\|_{\ell_\phi}))} x_n e_n
\]

This together with the definition of the maps \( B_\theta^n \), will inspire our definition of Young functions. Since we must deal with one coordinate at a time we cannot use the norm of the full vector to build a Young function, so we will consider the coordinates of \( B_\theta^1(x) \) when \( \|x\|_{\ell_\phi} = 1 \). From now on we fix \( \theta \in (0, 1) \).

**Definition 3.1.** For \( x \in \mathbb{C} \) let \( g_x : \mathbb{S} \to \mathbb{C} \) be given by

\[
g_x(z) = \phi_0^{-1}(\phi_\theta(|x|))^{1-\theta} \phi_1^{-1}(\phi_\theta(|x|))^{\theta} sgn(x)
\]

Let \( n \geq 2 \) and suppose that \( g_x \) has been defined for every \( x \in \mathbb{C}^{n-1} \). Let \( x = (x_{n-1}, \ldots, x_0) \in \mathbb{C}^n \) and define \( g_x : \mathbb{S} \to \mathbb{C} \) by

\[
g_x(z) = g(x_{n-2}, \ldots, x_0)(z) + \frac{\varphi^{n-1}(z)}{\kappa_n} g_n(x_{n-1} - \tilde{g}(x_{n-2}, \ldots, x_0)(n-1; \theta))(z)
\]

where \( \kappa_n \) is from the definition of \( B_\theta^n \).

Notice that \( \tilde{g}_x[j; \theta] = x_j \) for \( 0 \leq j \leq n-1 \). Now we define our quasi-Young functions:

**Definition 3.2.** Let \( \phi_{\theta,1} = \phi_\theta \). For \( n \geq 2 \), let \( \phi_{\theta,n} : \mathbb{C}^n \to [0, \infty) \) be given by

\[
\phi_{\theta,n}(x_{n-1}, \ldots, x_0) = \phi_{\theta,n-1}(x_{n-2}, \ldots, x_0) + \phi_\theta(x_{n-1} - \tilde{g}(x_{n-2}, \ldots, x_0)(n-1; \theta))
\]

Of course, we must prove that those are indeed quasi-Young functions. We will need the following lemma.
Lemma 3.3. For every \( n \geq 1 \) there are constants \( \alpha_n, \beta_n \) such that for every \( x = (x_{n-1}, \cdots, x_0) \in \mathbb{C}^n \) we have
\[
|g_s(it)| \leq \alpha_n \phi_0^{-1}(\beta_n \phi_{0,n}(x))
\]
and
\[
|g_s(1+it)| \leq \alpha_n \phi_1^{-1}(\beta_n \phi_{0,n}(x))
\]

Proof. We prove it by induction. The case \( n = 1 \) is straightforward, so suppose the result is true for \( n - 1 \) and let \( c \) and \( D_n \) be the constants of Lemma 2.1 for \( \phi_0 \). We have:
\[
|g_s(it)| = \left| g_{(s_{n-2}, \ldots, s_0)}(it) + \frac{\varphi^{n-1}(it)}{k_n} g_{(s_{n-2}, \ldots, s_0, s_{n-1})}(it) \right|
\]
\[
\leq \alpha_{n-1} \phi_0^{-1}(\beta_{n-1} \phi_{0,n-1}(x_{n-2}, \ldots, x_0)) + \frac{\alpha_{n-1}}{|k_n|} \phi_0^{-1}(\beta_{n-1} \phi_{0,n-1}(n!(x_{n-1} - \hat{g}_{(s_{n-2}, \ldots, s_0)}(n-1; \theta))))
\]
\[
\leq \alpha_{n-1} \phi_0^{-1}(\beta_{n-1} \phi_{0,n-1}(x_{n-2}, \ldots, x_0)) + \frac{\alpha_{n-1}}{|k_n|} \phi_0^{-1}(\beta_{n-1} D_n \phi_{0,n-1}(x_{n-1} - \hat{g}_{(s_{n-2}, \ldots, s_0)}(n-1; \theta)))
\]
\[
\leq \alpha_{n-1} \max\{1, |k_n|^{-1}\} \phi_0^{-1}(c \beta_{n-1} D_n \phi_{0,n}(x))
\]
The proof for \( 1 + it \) is similar. \( \square \)

Theorem 3.4. For every \( n \geq 1 \) the function \( \phi_{0,n} \) is quasi-Young.

Proof. We will prove it by induction. The case \( n = 1 \) is clear, so suppose that \( \phi_{0,n-1} \) is quasi-Young with quasi-convexity constant \( C_{n-1} \). Notice that \( \phi_{0,n}(e^{is}x) = \phi_{0,n}(x) \),
\[
\lim_{t \to \infty} \phi_{0,n}(tx) = \infty
\]
for every \( x \neq 0 \in \mathbb{C}^n \), \( s \in \mathbb{R} \), and \( \phi_{0,n}(0) = 0 \). It remains to prove the quasi-convexity of \( \phi_{0,n} \). It is easy to see that this reduces to estimating
\[
\phi_0 \left( \sum_{j} (t_j g_{x^j})[n-1; \theta] - \hat{g}_{\sum_{j} t_j x^j}[n-1; \theta] \right)
\]
for \( t_j \in [0,1], t_0 + t_1 = 1 \), and \( x^j \in \mathbb{C}^{n-1}, j = 0, 1 \). Let \( h_1 : \mathbb{S} \to \mathbb{C} \) be given by
\[
h_1(z) = \sum_{j} (t_j g_{x^j}(z)) - g_{\sum_{j} t_j x^j}(z)
\]
Notice that \( h_1[j; \theta] = 0 \), for \( j = 0, \ldots, n-2 \). This means that the function \( h_2(z) = \frac{h_1(z)}{(z-\theta)^{n-1}} \) is bounded on \( \mathbb{S} \), analytic on \( \mathbb{S}^o \), and what we want to estimate is precisely \( \phi_0(h_2(\theta)) \).

Let \( d = d(\theta, \partial \mathbb{S}) \). By Lemma 3.3 we have:
\[
|h_2(it)| \leq \frac{1}{d^{n-1}} \left( \sum_{j} (t_j|g_{x^j}(it))| + |g_{\sum_{j} t_j x^j}(it)) \right)
\]
\[
\leq \frac{1}{d^{n-1}} \left( \sum_{j} (t_j \alpha_{n-1} \phi_0^{-1}(\beta_{n-1} \phi_{0,n-1}(x^j))) + \frac{\alpha_{n-1}}{d^{n-1}} \phi_0^{-1}(\beta_{n-1} \phi_{0,n-1}(\sum_{j} t_j x^j)) \right)
\]
\[
\leq \frac{\alpha_{n-1}}{d^{n-1}} \phi_0^{-1}(\beta_{n-1} \phi_{0,n-1}(\sum_{j} t_j x^j)) + \phi_0^{-1}(\beta_{n-1} C_{n-1} \sum_{j} t_j \phi_{0,n-1}(x^j))
\]
\[
\leq \frac{2\alpha_{n-1}}{d^{n-1}} \phi_0^{-1}(\beta_{n-1} C_{n-1} \sum_{j} t_j \phi_{0,n-1}(x^j))
\]
We get a similar estimate for $1 + it$, substituting $\phi_0$ by $\phi_1$. So, by the three-lines lemma,

$$|h_2(\theta)| \leq \frac{2\alpha_n-1}{d_n} \phi_0^{-1}(\beta_n C_{n-1} \sum t_j \phi_{0,n-1}(x^j))^{1-\theta} \phi_1^{-1}(\beta_n C_{n-1} \sum t_j \phi_{0,n-1}(x^j))^{\theta}$$

$$= \frac{2\alpha_n-1}{d_n} \phi_0^{-1}(\beta_n C_{n-1} \sum t_j \phi_{0,n-1}(x^j))$$

Applying $\phi_0$ and letting $D_{\phi_{0,n-1}}$ be the constant of Lemma 2.1 for $\phi_0$, we have

$$\phi_0(h_2(\theta)) \leq \phi_0(\frac{2\alpha_n-1}{d_n} \phi_0^{-1}(\beta_n C_{n-1} \sum t_j \phi_{0,n-1}(x^j)))$$

$$\leq D_{\phi_{0,n-1}} \beta_n C_{n-1} \sum t_j \phi_{0,n-1}(x^j)$$

It follows that $\phi_{0,n}$ is quasi-convex.

\[\square\]

**Observation**: The technique of the previous proof may also be used to show that each $\phi_{0,n}$ satisfies the $\Delta_2$ condition. Indeed, one may check that it is enough to estimate

$$\phi_0(2g_x [n-1; \theta] - g_x [n-1; \theta]),$$

for $x \in \mathbb{C}^{n-1}$, which may done by taking $h_1 = 2g_x - g_{2x}$.

4. Derived spaces are Fenchel-Orlicz spaces

Our goal now is to prove that if $\phi_0$ and $\phi_1$ are nondegenerate Orlicz functions satisfying the $\Delta_2$ condition (or if we allow $\ell_{\phi_0} = \ell_{\phi_1} = \ell_{c_0}$) and we let $\mathcal{X} = (\ell_{\phi_0}, \ell_{\phi_1})$ then the derived space $d^\phi \mathcal{X}$ is isomorphic to $\ell_{\phi_0}$. We will use the following result:

**Proposition 4.1** ([1], Proposition 3.2). Let $X$ be a sequence space which is complete under the quasi-norms $\| \cdot \|_1$ and $\| \cdot \|_2$, and suppose that the coordinate functionals are continuous in each norm. Then $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent.

**Lemma 4.2**. Let $n \geq 1$ and $(x_{n-1}, \cdots, x_0) \in \ell_{\phi_{n-1}}$. Then the function $h_{(x_{n-1}, \cdots, x_0)}$ defined by

$$h_{(x_{n-1}, \cdots, x_0)}(z) = \sum g_{(x_{n-1}, \cdots, x_0)}(z) e_k$$

is in $\mathcal{F}(\mathcal{X})$.

**Proof**. We will prove it by induction in $n$. The base case is $n = 1$, so we have $x_0 \in \ell_{\phi_0}$. We must divide in two cases now:

- **Case 1**: $\phi_0$ and $\phi_1$ are nondegenerate

  For $n_1 \leq n_2$ define

  $$h_{x_0}^{n_1,n_2}(z) = \sum_{k=n_1}^{n_2} g_{x_0}(k)(z)e_k = \sum_{k=n_1}^{n_2} \phi_0^{-1}(\phi_0(|x_0(k)|))^{1-\theta} \phi_1^{-1}(\phi_0(|x_0(k)|))^{\theta} sgn(x_0(k)) e_k$$

  Then

  $$\|h_{x_0}^{n_1,n_2}\|_{\mathcal{F}(\mathcal{X})} = \max\left\{ \left\| \sum_{k=n_1}^{n_2} \phi_0^{-1}(\phi_0(|x_0(k)|)) e_k \right\|_{\ell_{\phi_0}}, \left\| \sum_{k=n_1}^{n_2} \phi_1^{-1}(\phi_0(|x_0(k)|)) e_k \right\|_{\ell_{\phi_1}} \right\}$$
Notice that \( \sum_{k=1}^{\infty} \phi_0^{-1}(\phi_0([x_0(k)]))e_k \in \ell_{\phi_0} \) and \( \sum_{k=1}^{\infty} \phi_0^{-1}(\phi_0([x_0(k)]))e_k \in \ell_{\phi_1} \), so that \( \lim_{n \to \infty} \|h_{x_0}^{n,n'}\|_{\mathcal{F}(X)} = 0 \), and therefore \( h_{x_0} \) is the limit of the Cauchy sequence \( (h_{x_0}^{n,n'})_n \) in \( \mathcal{F}(X) \).

Case 2: \( \phi_0 \) is degenerate

In this case \( \ell_{\phi_0} = \ell_{\infty} = \Sigma(X) \) with equivalence of norms. We have

\[
\begin{align*}
&h_{x_0}(z) = \sum_{k=1}^{\infty} \phi_0^{-1}(\phi_0([x_0(k)])) e_k \\
&\text{If } y \in \ell_1 \text{ then } (y, h(z)) = \sum_{k=1}^{\infty} y(k) \phi_0^{-1}(\phi_0([x_0(k)])) e_k \text{ is absolutely convergent and the convergence is uniform on } z. \text{ Therefore } (y, h(z)) \text{ is continuous and bounded on } \mathbb{S} \text{ and analytic on } \mathbb{S}^0. \text{ Since } y \in \ell_1 \text{ was arbitrary, this implies that } h_{x_0} \text{ is continuous and bounded on } \mathbb{S} \text{ and analytic on } \mathbb{S}^0 \text{ as a function with values in } \ell_{\infty}. \text{ In particular, } t \mapsto h_{x_0}(t) \in \ell_{\infty} \text{ is continuous and bounded, and the argument of the previous case may be used to show that so is } t \mapsto h_{x_0}(1 + it) \in \ell_{\phi_1}. \text{ This shows that } h_{x_0} \in \mathcal{F}(\overline{X}).
\end{align*}
\]

Now, by induction, it is enough to prove that for \( n \geq 2 \) if \( (x_{n-1}, \ldots, x_0) \in \ell_{\phi_{0,n}} \) then \( (x_{n-1}(k) - \hat{g}(x_{n-2}(k), \ldots, x_0(k))|n - 1; \theta))e_k \in \ell_{\phi_0} \). But this is a direct consequence of the definition of \( \phi_{0,n} \).

\[\square\]

Proposition 4.3. If \( n \geq 1 \) then \( d^nX_0 = \ell_{\phi_{0,n}} \) as sets.

Proof. Again, the proof is by induction. The base case is the classical equality \( \ell_{\phi_0} = (\ell_{\phi_0}, \ell_{\phi_1})_\theta \). Suppose the result is true for \( n - 1 \). We begin by proving the inclusion \( \ell_{\phi_{0,n}} \subset d^nX_0 \). Let \( (x_{n-1}, \ldots, x_0) \in \ell_{\phi_{0,n}} \). This implies that \( (x_{n-2}, \ldots, x_0) \in \ell_{\phi_{0,n-1}} = d^{n-1}X_0 \). Let

\[
\begin{align*}
\Psi(x_{n-2}, \ldots, x_0) = (\hat{g}(x_{n-2}(k), \ldots, x_0(k)) [n - 1; \theta])e_k \in \mathbb{C}^{2^j}.
\end{align*}
\]

Then \( x_{n-1} - \Psi(x_{n-2}, \ldots, x_0) \in \ell_{\phi_0} \). We must show that \( x_{n-1} - \Omega_{\theta}^{n-1}(x_{n-2}, \ldots, x_0) \in \ell_{\phi_0} \), so, it is enough to show that \( \Omega_{\theta}^{n-1}(x_{n-2}, \ldots, x_0) - \Psi(x_{n-2}, \ldots, x_0) \in \ell_{\phi_0} \). To see that, let

\[
\begin{align*}
&h_{(x_{n-2}, \ldots, x_0)}(z) = \sum_{k=1}^{\infty} g(x_{n-2}(k), \ldots, x_0(k)) \left( \sum_{j=0}^{n-2} z_j \right) e_k.
\end{align*}
\]

Then \( h \in \mathcal{F}(\overline{X}) \) by Lemma 4.2 and the function \( l = h_{(x_{n-2}, \ldots, x_0)} - B_\theta(x_{n-2}, \ldots, x_0) \) is such that \( \hat{l}[j; \theta] = 0 \), \( j = 0, \ldots, n - 2 \). It follows that \( \hat{l}[n - 1; \theta] \in \ell_{\phi_0} \) and therefore \( \Omega_{\theta}^{n-1}(x_{n-2}, \ldots, x_0) - \Psi(x_{n-2}, \ldots, x_0) \in \ell_{\phi_0} \).

To prove the reverse inclusion, if \( (x_{n-1}, \ldots, x_0) \in d^nX_0 \) then \( (x_{n-2}, \ldots, x_0) \in d^{n-1}X_0 \) and \( x_{n-1} - \Omega_{\theta}^{n-1}(x_{n-2}, \ldots, x_0) \in \ell_{\phi_0} \). So, by the previous calculation, we have \( x_{n-1} - \Psi(x_{n-2}, \ldots, x_0) \in \ell_{\phi_0} \), and therefore \( (x_{n-1}, \ldots, x_0) \in \ell_{\phi_{0,n}} \).

\[\square\]

Proposition 4.4. If \( n \geq 1 \) then the coordinate functionals on \( \ell_{\phi_{0,n}} \) are continuous.

Proof. One may check that \( \phi_{0,n} \) is continuous, so if \( \|(x_{n-1}, \ldots, x_0)\|_{\ell_{\phi_{0,n}}} \leq 1 \) then

\[
\sum_{k=1}^{\infty} \phi_{0,n}(x_{n-1}(k), \ldots, x_0(k)) \leq 1.
\]

In particular, \( \phi_{0,n}(x_{n-1}(k), \ldots, x_0(k)) \leq 1 \) for every \( k \). By the continuity of \( \phi_{0,n} \) the set \( \phi_{0,n}^{-1}[0, 1] \) is bounded, which implies that the coordinate functionals are bounded.

\[\square\]

Proposition 4.5. If \( n \geq 1 \) then the coordinate functionals on \( d^nX_0 \) are continuous.
Theorem 5.1. For every $n$ space we will accordingly deal with the $C$ extension property if every operator

Recall that $\|B_0^n(x_{n-1}, ..., x_0)\| \leq C_n$. Now, for each $n \in \mathbb{N}$ the map $\delta_{\theta,n} : \mathcal{F}(\overline{X}) \to \Sigma(\overline{X})$ given by $\delta_{\theta,n}(f) = f^{(n)}(\theta)$ is continuous, so there is $M > 0$ independent of $x$ such that

$$\|\Omega^n_0(x_{n-1}, ..., x_0)\|_{\Sigma(\overline{X})} \leq M$$

It is clear that the coordinate functionals on $\Sigma(\overline{X})$ are continuous too, so there is $N > 0$ independent of $x$ (and of $k$) such that

$$\|\Omega^n_{\phi}(x_{n-1}, ..., x_0)(k)\| \leq N$$

Now the result follows from $|x_{n-1}(k)| \leq |x_{n-1}(k) - \Omega^n_{\phi}(x_{n-1}, ..., x_0)(k)| + |\Omega^n_{\phi}(x_{n-1}, ..., x_0)(k)|$.

All this implies

Theorem 4.6. For each $n \geq 1$ the identity $d^nX_\phi = \ell_{\phi,n}$ is an isomorphism.

5. The $C[0,1]$-extension property

If $X$ is a real Banach space and $Y$ is a subspace of $X$, we say that the pair $(Y,X)$ has the $C[0,1]$-extension property if every operator $T : Y \to C[0,1]$ admits an extension to $X$. If $X$ is a complex Banach space we will accordingly deal with the $C([0,1],\mathbb{C})$-extension property, which is defined analogously.

A combination of [1, Theorem 4.1] and the results of [8] show that if $\phi$ is a real Young function in the class $\Delta_2$ then $(Y,\ell_\phi)$ has the $C[0,1]$-extension property for every subspace $Y$ of $\ell_\phi$.

The goal of this section is to prove the following result:

Theorem 5.1. For every $n \leq m$ the pair $(\ell_{\phi,n}, \ell_{\phi,n})$ has the $C([0,1],\mathbb{C})$-extension property.

Recall that if $X$ is a real Banach space then on the complexification $X \oplus_\mathbb{C} X$ we put the norm

$$\|(x,y)\| = \sup_{\theta \in [0,2\pi]} \|\cos(\theta)x + \sin(\theta)y\|$$

The following result is clear:

Proposition 5.2. Let $(Y,X)$ be a pair of real Banach spaces with the $C[0,1]$-extension property. Then $(Y \oplus_\mathbb{C} Y, X \oplus_\mathbb{C} X)$ has the $C([0,1],\mathbb{C})$-extension property.

For a complex Young function $\phi$ we will let $\ell_\phi(\mathbb{R})$ be the real sequence space $\ell_{\phi,\mathbb{R}}$.

Lemma 5.3. For every $\theta \in (0,1)$ there is a conformal map $\varphi : \mathbb{S} \to \mathbb{D}$ such that $\varphi(t) \in \mathbb{R}$ for every $t \in (0,1)$ and $\varphi(\theta) = 0$. In particular, $\varphi^{(n)}(t) \in \mathbb{R}$ for every $n \geq 1$ and $t \in (0,1)$, and therefore we may take $k_n$ real.

Proof. Consider the conformal map $\chi : \mathbb{S} \to \mathbb{D}$ given by

$$\chi(t) = \frac{e^{i\pi t} - e^{i\pi \theta}}{e^{i\pi \theta} - e^{-i\pi \theta}}$$
We have \( \chi_\varphi(\theta) = 0 \). Also, one may check that if we write \( \chi_\varphi(z) = (f_1(z), f_2(z)) \in \mathbb{R}^2 \) then the ratio \( \frac{f_2(\theta)}{f_1(\theta)} \) is constant for \( \theta \in (0, 1) \), which means that we may obtain \( \varphi \) as in the enunciate by multiplying \( \chi_\varphi \) by a modulus one constant. The remark about \( k_a \) follows from noticing that it is defined in terms of the derivative of \( \varphi \) at \( \theta \).

\( \square \)

**Lemma 5.4.** For every \( n \geq 1 \) there is a constant \( a_n \) such that \( \phi_{\theta,n}(x) \leq a_n \phi_{\theta,n}(x + iy) \) for every \( x, y \in \mathbb{R}^n \).

**Proof.**

For the base case we may take \( a_1 = 1 \). By induction, it is enough to show that there is a constant \( c_n \) such that

\[
\phi_\theta(x_{n-1} - \hat{g}(x_{n-2}, \ldots , x_0)[n-1; \theta]) \leq c_n \phi_{\theta,n}(x_{n-1} + iy_{n-1}, \ldots , x_0 + iy_0)
\]

for every \( x, y \in \mathbb{R}^n \).

Let \( c \) be the constant of Lemma 2.1 for \( \phi_\theta \). We have that \( \phi_\theta(x_{n-1} - \hat{g}(x_{n-2}, \ldots , x_0)[n-1; \theta]) \) is bounded by

\[
\phi_\theta(x_{n-1} - \hat{g}(x_{n-2}, \ldots , x_0)[n-1; \theta] + iy_{n-1} - Im \hat{g}(x_{n-2} + iy_{n-2}, \ldots , x_0 + iy_0)[n-1; \theta])
\]

which in turn is smaller or equal to \( c \) times

\[
\phi_\theta(x_{n-1} + iy_{n-1} - \hat{g}(x_{n-2} + iy_{n-2}, \ldots , x_0 + iy_0)[n-1; \theta])
\]

and

\[
\phi_\theta(Re \hat{g}(x_{n-2} + iy_{n-2}, \ldots , x_0 + iy_0)[n-1; \theta] - \hat{g}(x_{n-2}, \ldots , x_0)[n-1; \theta])
\]

Now it is enough to bound \( \phi_\theta(Re \hat{g}(x_{n-2} + iy_{n-2}, \ldots , x_0 + iy_0)[n-1; \theta] - \hat{g}(x_{n-2}, \ldots , x_0)[n-1; \theta]) \) in terms of \( \phi_{\theta,n-1}(x_{n-2} + iy_{n-2}, \ldots , x_0 + iy_0) \). So let \( h_1 = g_{x_{n-2} + iy_{n-2}, \ldots , x_0 + iy_0} - g_{x_{n-2}, \ldots , x_0} - ig_{y_{n-2}, \ldots , y_0} \) and proceed as in the proof of Theorem 3.4.

\( \square \)

Now, Theorem 5.1 is simply a consequence of the following:

**Corollary 5.5.**

For every \( n \geq 1 \) there is a constant \( a_n \) such that \( \|x\|_{\ell_{\phi,n}} \leq a_n \|x + iy\|_{\ell_{\phi,n}} \) for every \( x, y \in (\mathbb{R}^n)^\mathbb{N} \). In particular, \( \ell_{\phi,n} \) is \( \mathbb{R} \)-isomorphic to the complexification \( \ell_{\phi,n}(\mathbb{R}) \oplus \mathbb{C} \ell_{\phi,n}(\mathbb{R}) \).

From our results, the Johnson-Szankowski twisted Hilbert spaces \( Z(JS) \) of [10] are Fenchel-Orlicz spaces up to a renorming, and \( (\ell_2, Z(JS)) \) has the \( C([0, 1], \mathbb{C}) \)-extension property. The Johnson-Szankowski twisted Hilbert spaces are examples of HAPpy spaces which are not asymptotically Hilbertian.

### 6. Final Remark

In [1], given an Orlicz sequence space \( \ell_\phi \) with nontrivial type, the authors use Lipschitz functions to build extensions of \( \ell_\phi \). If we take the identity as Lipschitz function and \( \ell_\phi \) is \( p \)-convex and \( q \)-concave for nontrivial \( p \) and \( q \), one may check that the extension of \( \ell_\phi \) obtained in [1] corresponds to the one induced by the couple \((\ell_{\infty}, (\ell_\phi)_p)\) at \( \frac{1}{p} \), where \( X_p \) is the \( p \)-concavification of \( X \). It follows that we automatically get higher order extensions of \( \ell_\phi \) in that case. It would be interesting to obtain higher order extensions when using other functions.
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