Thermophysical Model for Realistic Surface Layers on Airless Small Bodies: Applied to Study the Spin Orientation and Surface Dust Properties of (24) Themis from WISE/NEOWISE Multiepoch Thermal Light Curves

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Abstract
This work proposes a thermophysical model for realistic surface layers on airless small bodies (RSTPM) for the use of interpreting their multiepoch thermal light curves (e.g., WISE/NEOWISE). RSTPM considers the real orbital cycle, rotation cycle, rough surface, temperature-dependent thermal parameters, as well as contributions of sunlight reflection to observations. It is thus able to produce a precise temperature distribution and thermal emission of airless small bodies regarding the variations on orbital timescales. Details of the physics, mathematics, and numerical algorithms of RSTPM are presented. When used to interpret multiepoch thermal light curves by WISE/NEOWISE, RSTPM can give constraints on the spin orientation and surface physical properties, such as the mean thermal inertia or the mean size of dust grains, the roughness fraction, and the albedo via a radiometric procedure.

As an application example, we apply this model to the main-belt object (24) Themis, the largest object of the Themis family, which is believed to be the source region of many main-belt comets. We find multiepoch (2010, 2014–2018) observations of Themis by WISE/NEOWISE, yielding 18 thermal light curves. By fitting these data with RSTPM, the best-fit spin orientation of Themis is derived to be \( \lambda = 137^\circ, \beta = 59^\circ \) in ecliptic coordinates, and the mean radius of dust grains on the surface is estimated to be \( b = 140^{+500}_{-14}(6 \sim 640) \) \( \mu m \), indicating that the surface thermal inertia varies from \( \sim 3 \text{J m}^{-2} \text{s}^{-0.5} \text{K}^{-1} \) to \( \sim 60 \text{J m}^{-2} \text{s}^{-0.5} \text{K}^{-1} \) due to seasonal temperature variation. A more detailed analysis found that the thermal light curves of Themis show a weak feature that depends on the rotation phase, which is indicative of heterogeneous thermal properties or imperfections of the light-curve inversion shape model.

Unified Astronomy Thesaurus concepts: Infrared photometry (792); Small Solar System bodies (1469); Asteroid surfaces (2209); Computational methods (1965)

1. Introduction

In the solar system, there exist numerous small bodies that are believed to be small planetesimals that did not grow large enough to become planets. Thus some small bodies, especially low-albedo asteroids (e.g., C types) and fresh comets, should contain primitive materials remaining from the formation of the solar system, and hence can provide important clues to the composition of the solar nebula in which planets formed, thus improve our understanding of the origin of solar system.

The temperature distributions of the surface and subsurface layers on small bodies are crucial for the study about thermophysical properties of their surface materials. To obtain the surface and subsurface temperature distributions, we would need a so-called “surface thermophysical model,” which aims to simulate these temperature distributions on the basis of the realistic physical conditions, including the orbital motion, rotation state, shape topography, surface roughness, and surface thermophysical parameters (e.g., thermal inertia).

The first-generation of thermophysical models (TPMs) of small bodies, typically those of Spencer (1990), Lagerros (1996a, 1996b, 1997, 1998), and Delbo (2004), are mainly designed for the so-called “radiometric method,” which aims to interpret the disk-integrated thermal emission observations of asteroids. Due to the limitation of the spatial and time resolution of astronomical instruments at that time, very limited thermal infrared observations could be obtained for only a few asteroids of interest, therefore we were only able to estimate the global mean thermal inertias and mean roughnesses of these asteroids. However, the appearance of roughness causes the surface to emit in a non-Lambertian way, which in turn causes more flux to be observed at low solar phase angles. This effect is known as the “thermal infrared beaming effect” (Lagerros 1998), which leads to a somewhat similar effect as a surface with low thermal inertia. Thus the parameters of thermal inertia and roughness have an inevitable degeneracy in the radiometry procedure. To remove the degeneracy of thermal inertia and roughness, we would need thermal infrared observations at multiple solar phase angles, namely observations at multiple epochs.

During the past 30 years, thanks to numerous new thermal infrared data of small bodies from space telescopes such as IRAS, Spitzer, AKARI, and WISE/NEOWISE, and high-precision in situ thermal infrared imaging from space missions of small bodies such as Rosetta, Hayabusa2, and OSIRIS-REx, the requirements of observations at multiple epochs have been well met. Particularly the WISE/NEOWISE mission has obtained multiepoch thermal light curves of many small bodies. Now with these observations, we are able to remove the degeneracy of thermal inertia and roughness, and simultaneously obtain constraints of their values with the radiometric model. Hence, thermophysical modeling of small
bodies has made extraordinary progress in recent years. A review of previous TPMs of small bodies can be found in Delbo et al. (2015), and many updated versions of TPMs have been proposed to be applied to specific cases (Rozitis & Green 2011; Davidsson & Rickman 2014; Hanuš et al. 2015). Rozitis & Green (2011) made progress by modeling the thermal emission of a rough surface considering the shadowing effect, scattering of sunlight, and self-heating within the rough region. Davidsson & Rickman (2014) made improvements by considering 3D heat conduction and roughness on spatial scales smaller than the thermal skin depth. Hanuš et al. (2015) further introduced a varied-shape TPM scheme, where the asteroid shape and pole uncertainties are taken into account.

With these updated TPMs, the “thermal infrared beaming effect” on airless bodies (e.g., the Moon, small satellites, asteroids, and even comets) can be well explained, and thus the degeneracy of thermal inertia and roughness can be well resolved with the radiometric procedure. However, problems still remain when we have to explain multiepoch thermal infrared data from WISE/NEOWISE with the current TPMs:

First, for some small bodies, especially main-belt asteroids, W1-band and W2-band observations of WISE/NEOWISE could contain a significant amount of reflected sunlight. Sunlight scattering is related to the surface albedo and roughness, which also have an important influence on the surface thermal emission. How to model the sunlight reflection and thermal emission of the surface with a unified geometry and physical parameters becomes an important problem.

Second, if a target small body has a high orbital eccentricity or an obliquity close to 90°, it would show a significantly different temperature at different epochs, such as (3200) Phaethon in Yu et al. (2019) and (349) Dembowska in Yu et al. (2017). Since thermal parameters including specific heat capacity $c$ and thermal conductivity $\kappa$ are strong functions of temperature, the value of the thermal inertia defined as $\Gamma = \sqrt{\rho c \kappa}$ ($\rho$ means density) is certainly a strong function of temperature as well. Consequently, such small bodies can have significantly different thermal inertias at different epochs (see, e.g., Rozitis et al. 2018). In such cases, a mean thermal inertia derived by the radiometric procedure may not well represent the thermophysical properties of the surface materials.

If there is a dust mantle (regolith layer) on the surface, the specific heat capacity of the dust mantle can be expressed as a function of temperature and material type, while the thermal conductivity can be described as a function of the temperature, of the porosity of the dust mantle, of the density, and of the mean size of the dust grains (Gundlach & Blum 2013). The dust-mantle porosity and dust-grain density for small bodies with the same spectral type would not be expected to differ very much (Britt et al. 2002), whereas the mean grain size may be obviously different for various small bodies even if they have similar spectral type. In addition, the mean size of the dust grains would be nearly unchanged at each observation epoch. Therefore, for small bodies that are covered by dust mantle, it may be more appropriate to use the mean size of the dust grains rather than the mean thermal inertia as the free parameter in a radiometric procedure.

In this paper, we propose a thermophysical model for realistic surface layers on airless small bodies (RSTPM). The model considers not only the real shape and rough surface, but also the real orbital cycle, the rotational cycle, and even temperature-dependent thermal parameters in the thermal simulation process, as well as the contribution of sunlight reflection into the infrared radiometric procedure. In comparison to previous models, RSTPM differs in three aspects, including (1) a different mathematical technique that is used to solve the influence of the surface roughness on the energy balance equation of the surface boundary; (2) with the aim of removing the degeneracy of thermal inertia and roughness by interpreting multiepoch thermal light curves, the variation of the thermal parameters due to the temperature variation caused by the orbital cycle and the rotation cycle is taken into consideration; and (3) a combination model of simultaneously computing the thermal emission and the sunlight reflection under the same surface topography is proposed to fit infrared data in case of the data that contain significant sunlight reflection. The structure of the paper is as follows: Section 2 presents the details of the physics and mathematics of RSTPM. Section 3 gives the description of the temperature-dependent thermal parameters. Section 4 describes the radiometric procedure. In Section 5, the application of RSTPM to interpret multiyear thermal light curves of (24) Themis by WISE/NEOWISE is presented. Finally, Section 6 gives an open discussion and brief conclusion of this model.

2. Model Description
2.1. Thermal Diffusion

For any small body in space, we could imagine that following its rotation as well as orbital movement, the temperature distribution $T(t, r)$ all over the small body would vary with time, which is dominated by the energy conservation law,

$$\frac{\partial U}{\partial t} + \nabla \cdot q = \sum Q_s,$$

where $U = U[T(t, r)]$ is the density of the internal energy, $t$ represents time, $r$ means the position vector, $q$ is the heat flux, and $Q_s$ represents the possible energy production source, such as energy released by the decay of $^{26}$Al.

Generally, the so-called specific heat capacity $c(T)$, defined to be the amount of heat required to raise the temperature of the unit mass substance by one degree, is introduced as

$$c(T) = \frac{1}{\rho} \frac{\partial U}{\partial T},$$

so that the first term in Equation (1) can be rewritten as

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial T} \frac{\partial T}{\partial t} = \rho c_v(T) \frac{\partial T}{\partial t} \text{ or } \rho c_p(T) \frac{\partial T}{\partial t},$$

in consideration of whether the system is under constant volume ($c_v(T)$) or constant pressure ($c_p(T)$).

For small bodies, the thermal process generally happens under constant pressure. Thus the specific heat capacity at constant pressure $c_p(T)$ should be adopted. In the case of an airless small body, generally no mass transfer happens, thus the density $\rho$ should be constant; and without an internal heat source, the item $\sum Q_s = 0$ can be ignored. Then the energy conservation Equation (1) can be rewritten as the general
thermal diffusion equation,
\[
\rho c_p(T) \frac{\partial T}{\partial t} = -\nabla \cdot q. \tag{4}
\]

We can describe the shape of a small body with a polyhedron composed of N triangle facets, thus the small body could be divided into numerous tiny voxels in such a way that each voxel could be marked by two numbers \((m, n)\), where \(m\) means the radial direction toward the surface facet \(m\), and \(n\) means the \(n\)th voxel below the facet. For each voxel, integrating the two sides of Equation (4) leads to the following equation:
\[
\int \rho c_p(T) \frac{\partial T}{\partial t} \, dV = -\int \nabla \cdot q \, dV = \oint q \cdot (-dS). \tag{5}
\]
If the voxel \((m, n)\) is small enough, its temperature can be assumed to be constant within the voxel space, then Equation (5) can be discretized as follows:
\[
\rho c_p(T) \frac{\partial T}{\partial t} V_{mn} = \sum_{\alpha} q_{\alpha} \cdot \vec{S}_{\alpha}
= q_{\perp} \cdot \vec{S}_{\perp} + q_{\parallel} \cdot \vec{S}_{\parallel} + \sum_{\alpha} q_{\parallel\alpha} \cdot \vec{S}_{\parallel\alpha}, \tag{6}
\]
where \(V_{mn}\) is the volume of the voxel \((m, n)\), \(\alpha\) stands for a possible voxel adjoined to voxel \((m, n)\), \(\vec{S}_{\alpha}\) represents the cross-section area-vector between the two voxels, \(\perp\) stands for radial conduction, and \(\parallel\) stands for lateral conduction.

The heat flow between two voxels is the result of the spatial gradient of the temperature,
\[
q = -\kappa(T) \nabla T, \tag{7}
\]
where \(\kappa(T)\) is the so-called thermal conductivity. Thus the component of the heat flux between voxel \(\alpha\) and voxel \((m, n)\) and the corresponding cross-section area-vector could be expressed as
\[
q_{\alpha} = \kappa(T_{\alpha-mn}) \frac{T_{\alpha} - T_{mn}}{\delta h_{\alpha}} l_{\alpha-mn} \vec{S}_{\alpha} = S_{\alpha} n_{\alpha-mn}, \tag{8}
\]
respectively, where \(l_{\alpha-mn}\) is the unit direction vector from voxel \(\alpha\) to voxel \((m, n)\), \(n_{\alpha-mn}\) represents the unit normal vector, and \(\delta h_{\alpha}\) means the average distance between voxel \(\alpha\) and voxel \((m, n)\).

We assume that the typical size of a facet in the shape model we used is \(l_{\text{facet}}\), and the typical thermal penetration depth (generally called “thermal skin depth”) is \(l_{\text{st}}\). Then we can make the following approximations:
\[
\delta h_{\perp} \sim l_{\text{st}}, \quad S_{\perp} \sim l_{\text{facet}}^2 \delta h_{\perp} \sim l_{\text{facet}}, \quad S_{\parallel} \sim l_{\text{st}}^2, \quad \frac{S_{\parallel}}{\delta h_{\parallel}} \sim \left( \frac{l_{\text{st}}}{l_{\text{facet}}} \right)^3 \frac{S_{\parallel}}{\delta h_{\parallel}}. \tag{9}
\]
For typical small bodies, \(l_{\text{facet}}(\sim 10\text{m})\) is far more larger than \(l_{\text{st}}(\sim 10^{-2}\text{m})\), and
\[
\frac{l_{\text{st}}}{l_{\text{facet}}} \sim 10^{-3}, \quad \frac{S_{\parallel}}{h_{\parallel}} \sim 10^{-9} \frac{S_{\parallel}}{h_{\parallel}},
\]
thus the lateral conduction could be sufficiently small to be ignored.

On the other hand, the radial conduction items could be further simplified via the following approximations:
\[
S_{\parallel} \sim S_{\perp} \sim l_{\text{st}}, \quad V_{mn} \sim 6h_{S_{\parallel}},
q_{\perp} \cdot \vec{S}_{\perp} \sim \kappa(T_{m,n-1}) \frac{T_{mn} - T_{mn}}{\delta h} S_{\perp},
q_{\parallel} \cdot \vec{S}_{\parallel} \sim \kappa(T_{m,n+1}) \frac{T_{mn+1} - T_{mn} \delta h}{S_{\perp}} S_{\parallel}.
\]
Then Equation (6) could be simplified to be a 1D heat conduction equation as
\[
\rho c_p(T) \frac{\delta T}{\delta t} \approx \frac{q_{\perp} \cdot \vec{S}_{\perp} + q_{\parallel} \cdot \vec{S}_{\parallel}}{V_{mn}} \approx \frac{\delta}{\delta h} \left[ \kappa(T) \frac{\delta T}{\delta h} \right]. \tag{10}
\]
which would be precise enough to model the temperature distribution of the surface layers of small bodies. But if \(l_{\text{st}}\) is comparable to \(l_{\text{facet}}\), a 3D thermal diffusion model would be necessary.

### 2.2. Boundary Conditions

For a voxel at the very surface of an airless small body in space, the heat flow around this voxel contains not only conduction between adjoining voxels, but also absorbed incident solar flux and escaped thermal emission, thus the conservation of energy gives the surface boundary condition as
\[
\frac{\delta U}{\delta t} = Q_{\text{absorbed}} - Q_{\text{emitted}} + Q_{\text{conduction}} = 0, \tag{11}
\]
if a quasiequilibrium state is considered.

However, for some small bodies, especially irregular-shape asteroids (e.g., Eros and Toutatis), the effects from topography and roughness are significant. They need to be considered in the surface boundary condition.

Topography and roughness both indicate the fluctuation of height on the surface. Their difference lies in the spatial scale, where topography refers to a relatively large spatial scale (> tens of meters), which only causes shadow, while roughness refers to a macroscopic small spatial scale (meters), which can cause not only shadow, but also multiple scattering of sunlight and self-heating by self-thermal emission.

Therefore, for a small body, if its 3D shape model is composed of \(\sim 2000\) surface elements, the typical size of a facet element \(l_{\text{facet}}\) can be treated as the spatial scale of the topography, while the spatial scale of roughness, say \(l_{\text{R}}\), should be much smaller than \(l_{\text{facet}}\), but far larger than the thermal skin depth \(l_{\text{st}}\) (\(l_{\text{st}} \ll l_{\text{R}} \ll l_{\text{facet}}\)).

If the topography around facet \(m\) is significant, probably causing it to be shaded by other facets, then no sunlight will shine into this facet.

If the facet \(m\) is a significant rough surface, we can further divide it into several subfacets, where each subfacet \(i\) is treated as a smooth Lambertian surface. Then several significant effects would arise within the subfacets, such as:

Some subfacets may be shaded, causing the temperature of these subfacets to be lower than that of other unlit subfacets. If the small body is observed at zero phase angle, all observed subfacets would be the hotter ones, but when the
is the unit normal vector of subfacet \( L \).

\( L = \) is the \( \sigma \) of the smooth Lambertian surface, and \( \) averaged thermal emissivity over the entire emission spectrum can be approximated by

\[
\varepsilon = \frac{E_i T_i}{L \sigma}
\]

in which \( n_i \) is the unit normal vector of subfacet \( i \), and \( n_{i,\sigma} \) is the unit vector pointing in the opposite direction of the sunlight, \( L \) is the integrated solar flux at the distance of the asteroid, which can be approximated by

\[
L = \frac{L_{\odot}}{d_{\odot}^2},
\]

where \( L_{\odot} \) is the solar constant, about 1361.5Wm\(^{-2}\), and \( d_{\odot} \) is the heliocentric distance in au. As a result, the effective amount of light energy reflected out from the whole rough surface can be reduced, leading to an effective Bond albedo \( A_{\text{eff},\text{L}} \) smaller than the \( A_\text{L} \) of the smooth Lambertian surface.

The subfacets can absorb thermal radiation \( R_{th} \) from surrounding visible facets. As a result, the colder subfacets can be heated by the hotter ones. This effect is known as the “self-heating effect.” So the net thermal flux that escapes to space from a subfacet \( i \) can be written as

\[
E_{th}(i) = \varepsilon \sigma T_i^4 - R_{th}(i),
\]

where \( T_i \) represents the temperature of subfacet \( i \), \( \varepsilon \) is the averaged thermal emissivity over the entire emission spectrum of the smooth Lambertian surface, and \( \sigma \) is the Stefan-Boltzmann constant. As a result, the net fraction of thermal emission energy that escapes out to space from the whole rough facet would be reduced.

These effects together contribute to the so-called “thermal infrared beaming effect” (Lagerros 1998)—the observed disk-integrated flux would significantly decrease when the observation phase angle deviates from zero (Figure 1).

Therefore considering both topography and roughness, the surface boundary condition (11) for a subfacet on rough surface can be expressed as

\[
(1 - A_{\text{L}})L(i) + R_{th}(i) - \varepsilon \sigma T_i^4 + \kappa(T) \frac{\delta T}{\delta h} \bigg|_{h=0} = 0.
\]

On the other hand, when there are no internal heat sources in the small body, and the small body is large enough so that periodical variation of temperature happens only in the very thin surface layers during the timescale we consider, an isothermal core would form in the internal region. Thus the radial temperature gradient below a certain depth would vanish, which gives the internal boundary condition as

\[
\frac{\delta T}{\delta h} \bigg|_{h \to \infty} \to 0.
\]

2.3. Roughness Representation

In order to quantitatively consider the influence of roughness, we should establish a roughness model to mathematically represent it in the surface boundary condition. A good and widely used way to model the surface roughness (Spencer 1990; Lagerros 1998; Rozitis & Green 2011) is to express it by a fractional coverage of macroscopic bowl-shaped craters, symbolized by \( f_c \) \((0 < f_c < 1)\), whereas the remaining fraction, \( 1 - f_c \), represents a smooth flat Lambertian surface. The configuration of the used macroscopic crater can be described by the depth-to-diameter ratio \( \xi = h/D_{\text{rim}} \geq 0 \).

The macroscopic crater can be divided into several subfacets as shown in Figure 2, where each subfacet is the same as the smooth flat Lambertian surface. Then we could use the so-called “root mean square (rms) slope” \( \theta_{\text{rms}} \) to measure the degree of surface roughness. The rms slope is defined as...
where $\theta_i$ is the angular slope of facet $i$ to the local horizontal surface, and $a_i$ is the area of facet $i$. Generally, for a macroscopic spherical crater defined by $\xi$, its rms slope can be uniquely calculated, like $\theta_{\text{rms}}(\xi)$. Then for a rough surface indicated by a pair of $(f_i, \xi)$, the RMS slope can be evaluated as

$$\theta_{\text{RMS}}(f_i, \xi) = \sqrt{f_i \theta_{\text{rms}}(\xi)}.$$  

(18)

Therefore, different pairs of $(f_i, \xi)$ can generate surfaces with the same RMS slope, which, together with the thermal inertia, dominate the beaming effect (Rozitis & Green 2011). So in general cases of radiometric applications, we can fix the parameter $\xi$ to be constant while treating the roughness fraction $f_i$ as the free parameter so as to generate surfaces with a different RMS slope. Typically, for a hemispherical crater defined by $\xi = 0.5$, the rms slope is $\theta_{\text{rms}} = 50^\circ$, while, when $f_i = 0.0\sim 1.0$ is considered, the surface RMS slope $\theta_{\text{RMS}}$ is in the range $0\sim 50^\circ$, decided by the roughness fraction $f_i$.

2.3.1. Sunlight Multiscattering

We consider a subfacet $i$ on the rough surface represented by the macroscopic crater shown in Figure 2. If the total light flux incident onto the facet is $L(i)$, then the total light leaving the facet can be expressed as $A_B L(i)$, and the scattering light from surrounding visible facets can be estimated via

$$L_{\text{scat}}(i) = \sum_{j \neq i} f(i, j) A_B L(j),$$  

(19)

where $f(i, j)$ is the so-called “view factor” (Lagerros 1998), meaning the fraction of radiative energy received by facet $i$ of the total radiative energy leaving from facet $j$, assuming a Lambertian energy distribution. Therefore $f(i, j)$ can be expressed as

$$f(i, j) = \begin{cases} 0, & j = i \smallskip \vspace{0.5cm} \frac{a_i a_j}{\pi d_{ij}^4} \cos \theta_i \cos \theta_j, & j \neq i \end{cases}$$  

(20)

where $a_{ij}$ stands for the fraction of area that is visible to each other, $a_i$ is the area of facet $j$, $\theta_i$ is the incidence angle on facet $i$, $\theta_j$ is the emission angle from facet $j$, and finally, $d_{ij}$ is the distance between facet $i$ and $j$. Moreover, for a bowl-shaped crater as shown in Figure 3, we have

$$\cos \theta_i = \cos \theta_j = \frac{0.5d_{ij}}{R},$$

thus the view factors can be simplified to be

$$f(i, j) = \begin{cases} 0, & j = i \smallskip \vspace{0.5cm} \frac{a_i}{4\pi R^2}, & j \neq i \end{cases}$$  

(21)

where $R$ is the radius of the sphere that contains the crater.

![Figure 3. The geometry of a spherical crater (ξ = h/Drms), which is one part of a sphere with radius R. Point O is the center of the sphere. The rim of the crater has a diameter Drms, and the depth is h.](image)

On the other hand, when combining Equations (12) and (19), we can obtain

$$L(i) = L_s v_i \psi_i + \sum_{j \neq i} f(i, j) A_B L(j) \iff L_s v_i \psi_i = L(i) - \sum_{j \neq i} f(i, j) A_B L(j) = \sum_{j} A(i, j) L(j),$$  

(22)

where the coefficient matrix $A(i, j)$ is defined as

$$A(i, j) = \begin{cases} 1, & j = i \smallskip \vspace{0.5cm} -A_B f(i, j), & j \neq i \end{cases}$$  

(23)

being a symmetric matrix. Actually, Equation (22) forms a system of linear equations,

$$S = AL,$$

where the vector $S$ has elements $L_s v_i \psi_i$, namely $S(i) = L_s v_i \psi_i$. The linear equations can be easily solved by the Gauss elimination method, giving

$$L = A^{-1}S,$$

where $A^{-1}$ stands for the inverse matrix of $A$. Therefore the actually incident sunlight flux $L(i)$ onto the subfacet $i$ can be obtained to be

$$L(i) = \sum_{j} A^{-1}(i, j) S(j) = L_s \sum_{j} A^{-1}(i, j) v_j \psi_j.$$  

(24)

2.3.2. Thermal Self-heating

When we assume that the subfacet $i$ has a temperature $T_i$ and assume Lambertian emission, then the subfacet $i$ can absorb the integrated thermal radiation,

$$R_{th}(i) = (1 - A_{th}) \sum_{j \neq i} f(i, j) \varepsilon \sigma T_j^4$$  

(25)

from the surrounding visible subfacets, where $A_{th}$ is the average albedo over the entire thermal emission spectrum, and can be related to $\varepsilon$ as $1 - A_{th} = \varepsilon$ according to the Kirchhoff law.
Therefore the surface condition for a subfacet on a rough surface can be further expressed as

\[
(1 - A_B) L_x \sum_j A^{-1}(i, j) v_j \psi_j + (1 - A_{th}) \sum_{j \neq i} f(i, j) \varepsilon \sigma T_j^4 - \varepsilon \sigma T_i^4 + \kappa(T) \frac{\delta T}{\delta h} \bigg|_{h=0} = 0. 
\]  \hspace{1cm} (26)

2.3.3. Average Boundary Condition

If we integrate Equation (24) for all subfacets, the total effective absorbed energy by the macroscopic crater can be approximated as

\[
(1 - A_{\text{e,B}}) L_x \psi a_{\text{rim}} = \sum_j (1 - A_B) L(i) a_i = (1 - A_B) L_x \sum_j A^{-1}(i, j) v_j \psi_j a_i,
\]

where \(A_{\text{e,B}}\) stands for the effective Bond albedo of the crater, and the effective sunlit area of the crater equals

\[
\psi a_{\text{rim}} = \sum_j v_j \psi_j a_j.
\]

Thus we can obtain

\[
A_{\text{e,B}} = 1 - (1 - A_B) \frac{\sum_j \sum_i A^{-1}(i, j) v_j \psi_j a_i}{\sum_j v_j \psi_j a_j} = 1 - (1 - A_B) \frac{\sum_j \sum_i A^{-1}(j, i) v_j \psi_j a_j}{\sum_j v_j \psi_j a_j} \approx 1 - (1 - A_B) \left( \sum_j A^{-1}(j, i) \right),
\]

when the areas \(a_i\) of the subfacets are nearly the same. The symbol \(\langle \rangle_i\) represents the operation of averaging as

\[
\left\langle \sum_i A^{-1}(i, j) \right\rangle_j = \frac{1}{n} \sum_{j} \sum_i A^{-1}(i, j),
\]

in which \(n\) is the dimension of matrix \(A\) and \(A^{-1}\).

According to \(AA^{-1} = I\) and \(A^T = A\), we have

\[
\left\langle \sum_i A(i, j) \right\rangle_j \left\langle \sum_i A^{-1}(j, i) \right\rangle_j = 1.
\]

In addition, according to Equation (23),

\[
\left\langle \sum_i A(i, j) \right\rangle_j = \frac{1}{n} \sum_{j} \sum_i A(i, j) = 1 - A_B \frac{1}{n} \sum_{i} f(i, j),
\]

and

\[
\frac{1}{n} \sum_{j} \sum_i f(i, j) = \frac{1}{n} \sum_{j} \sum_i \frac{a_j}{4\pi R^2} \approx \frac{1}{4\pi R^2} \sum_i a_j = \frac{2\pi R h}{4\pi R^2} = \frac{4\xi^2}{1 + 4\xi^2}.
\]

Therefore

\[
\begin{align*}
\left\langle \sum_i A^{-1}(j, i) \right\rangle_j &= \frac{1}{1 - A_B} \left( \sum_i A(i, j) \right) = \frac{1}{1 - A_B} \frac{4\xi^2}{1 + 4\xi^2}, \\
A_{\text{e,B}} &\approx 1 - (1 - A_B) \left( \sum_i A^{-1}(j, i) \right) \\
&\approx 1 - \frac{(1 - A_B)}{1 - A_B} \frac{4\xi^2}{1 + 4\xi^2} = \frac{A_B}{1 + 4\xi^2(1 - A_B)}. \hspace{1cm} (31)
\end{align*}
\]

and finally, the averaged effective Bond albedo of a rough surface defined by \((f_r, \xi)\) can be evaluated as

\[
A_{\text{eff,B}} = (1 - f_r) A_B + f_r \frac{A_B}{1 + 4\xi^2(1 - A_B)}. \hspace{1cm} (32)
\]

On the other hand, from the point of view of energy conservation, we could define an effective temperature \(T\) and effective thermal emissivity \(\varepsilon_e\) for the macroscopic crater via

\[
T^4 = \frac{1}{a_{\text{rim}}} \sum_i T_i^4 a_i, \hspace{1cm} \varepsilon_e = \frac{E_c}{\sigma T^4 a_{\text{rim}}}, \hspace{1cm} (33)
\]

where \(E_c\) is the net thermal emission energy escaping from the whole crater,

\[
E_c = \sum_i E_{\text{th}}(i) a_i = \sum_i (\varepsilon \sigma T_i^4 - R_{\text{th}}) a_i. \hspace{1cm} (34)
\]

Then the effective thermal emissivity \(\varepsilon_e\) can be derived to be

\[
\varepsilon_e = \left( 1 - \frac{\sum_i a_i (1 - A_{\text{th}}) \frac{a_j}{4\pi R^2} \sigma T_j^4}{\sum_i \sigma T_i^4 a_i} \right) \varepsilon \\
\approx \left( 1 - \frac{(1 - A_{\text{th}}) \sum_i a_i}{4\pi R^2} \right) \varepsilon \\
= \frac{1 + 4\xi^2 A_{\text{th}}}{1 + 4\xi^2} \varepsilon, \hspace{1cm} (35)
\]

and the averaged effective thermal emissivity of a rough surface defined by \((f_r, \xi)\) can be evaluated as

\[
\varepsilon_{\text{eff}} = (1 - f_r) \varepsilon + f_r \frac{1 + 4\xi^2 A_{\text{th}}}{1 + 4\xi^2} \varepsilon. \hspace{1cm} (36)
\]

In a view of energy conservation for the whole rough facet, we therefore integrate the boundary condition Equation (15) for all subfacets and use the averaged effective Bond albedo \(A_{\text{eff,B}}\) (Equation (32)) as well as the averaged effective thermal emissivity \(\varepsilon_{\text{eff}}\) (Equation (36)). Then we could obtain the average boundary condition of the rough surface as

\[
(1 - A_{\text{eff,B}}) L_x \psi - \varepsilon_{\text{eff}} \sigma T^4 + \kappa(T) \frac{\delta T}{\delta h} \bigg|_{h=0} = 0, \hspace{1cm} (37)
\]

which can be applied in the cases for which we do not need to distinguish the temperature difference within the rough facet.
2.4. Standard Transformation

In order to simplify the solution of the above equations, it is useful to introduce a standard transformation as follows:

\[ u = \frac{T}{T_*}, \tau = \frac{t}{t_*}, x = \frac{h}{h_*}, \]  

(38)

where \( t_* \) represents the typical timescale of the thermal variation, \( h_* \) represents the spatial scale of thermal diffusion, and \( T_* \) represents the typical temperature of the system during the thermal process.

Then, the 1D heat conduction Equation (10) can be transformed as

\[ f_c(u) \frac{\delta u}{\delta T} = \frac{\delta}{\delta x} \left[ f_c(u) \left( \frac{\alpha_u T}{h_*^2} \right) \frac{\delta u}{\delta x} \right], \]  

(39)

where

\[ \alpha_u = \frac{\kappa_u}{\rho c_p u}, \quad f_c(u) = \frac{\kappa(T)}{\kappa_u}, \quad f_c(u) = \frac{c(T)}{c_u}, \]

in which \( \kappa_u \) and \( c_u \) are the reference constant thermal conductivity and heat capacity, introduced via

\[ \kappa_u = \kappa(T_*), \quad c_u = c(T_*). \]

The thermal state in the surface layers can change cyclically due to the variation of solar insolation caused by some kind of short-term cycle of motion, for example, a diurnal cycle or seasonal cycle. For such thermal cycles, generally we choose

\[ t_* = \frac{1}{\omega}, \]

where \( \omega \) is the rotational angular frequency or orbital angular frequency, and use the so-called thermal skin depth \( l_{sk*} \) as \( h_* \)

\[ h_* = l_{sk*} = \sqrt{\alpha_u/T_*} = \frac{\kappa_u}{\rho c_p \omega}. \]  

(40)

And then the 1D heat conduction Equation (39) can be simplified as

\[ f_c(u) \frac{\delta u}{\delta T} = \frac{\delta}{\delta x} \left[ f_c(u) \frac{\delta u}{\delta x} \right]. \]  

(41)

In realistic cases, both rotation and orbital motion can cause temperature variation. The rotation effect is more important in low-latitude regions, especially in equatorial regions, but there is no such effect in polar regions. However, if the small body has significant orbital eccentricity and axial tilt, then a seasonal thermal variation would become important, especially for high-latitude regions, and even dominate the temperature variation in polar regions.

2.4.1. Diurnal Cycle

For the diurnal cycle, it is convenient to use the subsolar temperature \( T_{ss} \) as \( T_* \)

\[ T_* = T_{ss} = \left[ \frac{(1 - A_{eff,B})L_{ss}}{\varepsilon \sigma} \right]^{1/4}, \]  

(42)

then the boundary condition Equation (26) of the rough surface can be converted into

\[ \left( \frac{1 - A_B}{1 - A_{eff,B}} \right) \sum_j A^{-1}(i, j) \psi_j + \sum_{j \neq i} \varepsilon f(i, j) u_j^i = u_i^i - \Phi_u f_u(u) \frac{\delta u}{\delta x} \bigg|_{x=0}. \]  

(43)

where

\[ \Phi_u = \frac{\Gamma_* \sqrt{\omega}}{\varepsilon \sigma T_*^3}, \]  

(44)

is the characteristic thermal parameter, in which

\[ \Gamma_* = \sqrt{\rho c_p \kappa_u} \]

is the corresponding characteristic thermal inertia.

2.4.2. Seasonal Cycle

For the seasonal cycle, if the orbital period is not that long in comparison to the rotation period, it is easy to simulate the temperature variation considering the orbital motion and rotation simultaneously until the surface temperature is affected by cyclic variation.

For the case in which the orbital period is far longer than the rotation period, we use the subsolar temperature at perihelion to define \( T_* \)

\[ T_* = \left[ \frac{(1 - A_{eff,B})L_{ss}}{\varepsilon \sigma} \right]^{1/4}, \]

where \( L_{ss} \) is the incident solar flux at perihelion distance \( d_{sp} \). And the surface boundary Equation (43) for each facet can be rotationally averaged at each orbital location via

\[ \left( \frac{d_{sp}}{d_{ss}} \right)^2 \left( \frac{1 - A_B}{1 - A_{eff,B}} \right) \sum_j A^{-1}(i, j) \psi_j + \sum_{j \neq i} \varepsilon f(i, j) \bar{u}_j^i = \bar{u}_i^i - \Phi_u f_u(u) \frac{\delta u}{\delta x} \bigg|_{x=0}. \]  

(45)

where the diurnal-averaged solar insolation

\[ \left( \sum_j A^{-1}(i, j) \psi_j \right) \]

is used, so as to compute the diurnal-averaged temperature \( \bar{u}_i \) by simulating the seasonal variation of this diurnal-averaged temperature for several orbital periods until it reaches a stable state.

Then, for a particular orbital position, the calculated diurnal-averaged surface temperature is used as an initial temperature to simulate the diurnal cycle of the temperature of each facet for several rotation periods. In this way, we can obtain a more precise temperature distribution, and hence more precise thermal emission in consideration of both diurnal and seasonal effects.

2.5. Numerical Algorithm

The partial differential Equation (41) and its boundary conditions Equation (43) or Equation (45) are very complicated
coupling equations, for which it is difficult to obtain analytic solutions, thus here we attempt to solve it with a numerical algorithm.

When we wish to numerically solve a time-dependent partial differential equation, like Equation (41), which would reach a cyclic stable solution, it is suitable to use a time-marching algorithm until the final period-stabilized solution. Let us use \( \delta \tau \) as the time step, \( \delta x \) as the spatial step, and \( u_{ij} \) to represent the current temperature of depth \( j \) below facet \( i \), while \( u_{ij}^n \) represents its temperature at the next time step. Then with the Crank-Nicholson scheme, we can obtain the differential forms of Equation (41) as

\[
\begin{align*}
    u_{ij}^n - u_{ij} &= f_{ij} - f_{ij-1}(u_{ij-1}^n - u_{ij}^n) + f_{ij}(u_{ij+1}^n - u_{ij}^n) \\
    &+ f_{ij-1}(u_{ij-1} - u_{ij}) + f_{ij}(u_{ij+1} - u_{ij}) \\
\end{align*}
\]  

(46)

where

\[
    f_{ij} = \frac{1}{2} f_{i,j}(u_{ij}) = \frac{1}{2} \tilde{\kappa}(u_{ij}) \frac{c_k}{\kappa} \delta \tau,
\]

(47)

in which

\[
    \tilde{\kappa}(u_{ij}) = \frac{\kappa(u_{ij}) V_{ij} + \kappa(u_{ij+1}) V_{ij+1}}{V_{ij} + V_{ij+1}},
\]

(48)

and \( V_{ij} \) stands for the volume of voxel \( (i,j) \). By putting the items of the next step \( u^n \) to the left side and the items of the current step \( u \) to the right side, Equation (46) can be further rewritten as

\[
    -f_{ij-1}u_{ij-1}^n + (1 + f_{ij-1} + f_{ij})u_{ij}^n - f_{ij}u_{ij+1}^n = f_{ij-1}u_{ij-1} + (1 - f_{ij-1} - f_{ij})u_{ij} + f_{ij}u_{ij+1}.
\]

(49)

For the surface boundary condition Equation (43) in the diurnal cycle, to ensure the stability as well as efficiency of the calculation, we could use a semi-implicit scheme, making it convert into

\[
    (u_{ij}^n)^4 + p_1 u_{ij}^n = p_2,
\]

(50)

in which

\[
    p_1 = \Phi \delta_T h(u_{ij}) \frac{1}{\delta x},
\]

\[
    p_2 = \Phi \delta_T h(u_{ij}) \frac{u_{ij}}{\delta x} + \left( \frac{1 - A_n}{1 - A_{eff,B}} \right) \sum_j A^{-1}(i,j) \nu_y \psi_j
    + \sum_{j=i}^j c_f(i,j) u_{ij}^2.
\]

The surface boundary condition Equation (45) in a seasonal cycle can also be transformed in a similar way as above.

In the process of numerical calculation, we can first calculate the parameters \( p_1 \) and \( p_2 \) with the temperatures and solar insolation at the current time step, and then obtain the surface temperature of the next time step,

\[
    u_{ij}^n = BC(p_1, p_2),
\]

(51)

where \( BC \) is named for the function to solve the nonlinear Equation (50). On the other hand, the internal boundary condition can simply be expressed as

\[
    u_{ij}^n - u_{ij,n-1} = 0.
\]

(52)

Then Equations (49), (51), and (52) could be combined into a special tridiagonal system of equations as

\[
\begin{bmatrix}
    b_1 & c_1 & 0 & \cdots & 0 \\
    a_2 & b_2 & c_2 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{j-1} & b_{j-1} & c_{j-1} & \cdots & a_N \\
    0 & a_N & b_N & \cdots & d_N
\end{bmatrix}
\begin{bmatrix}
    u_{ij1}^n \\
    u_{ij2}^n \\
    \vdots \\
    u_{ijN}^n \\
    u_{iN+1}^n
\end{bmatrix}
= \begin{bmatrix}
    d_1 \\
    d_2 \\
    \vdots \\
    d_N
\end{bmatrix}
\]

(53)

where

\[
    a_j = \begin{cases} 
    0, & j = 1 \ 
    -f_{ij-1}, & j = 2, N - 1 \ 
    1, & j = N
    \end{cases}
\]

\[
    b_j = \begin{cases} 
    1, & j = 1 \\
    1 + f_{ij-1} + f_{ij}, & j = 2, N - 1 \\
    -1, & j = N
    \end{cases}
\]

\[
    c_j = \begin{cases} 
    0, & j = 1 \\
    -f_{ij+1}, & j = 2, N - 1 \\
    0, & j = N
    \end{cases}
\]

\[
    d_j = \begin{cases} 
    BC(p_1, p_2), & j = 1 \\
    f_{ij-1}u_{ij-1} + (1 - f_{ij-1} - f_{ij})u_{ij} + f_{ij}u_{ij+1}, & j = 2, N - 1 \\
    0, & j = N
    \end{cases}
\]

Equation (53) is still a nonlinear equation because the coefficients contain the temperature-dependent function \( f_{ij} \). To solve this kind of nonlinear heat conduction equation, we could use the so-called “predictor-corrector” method, which is a two-step iterative procedure. The advance from temperature \( u \) at the current time \( \tau \) to the temperature \( u^n \) at the next time \( \tau + \delta \tau \) is implemented through the temperature \( u^{n-1/2} \) at an intermediate time step \( \tau + \delta \tau/2 \).

First, in the predictor step, we can obtain the intermediate temperature \( u^{n-1/2} \) from the current temperature \( u \) by taking the coefficient \( f_{ij}(u) \) with the current temperature \( u \).

Second, in the corrector step, the coefficients \( f_{ij}(u^{n-1/2}) \) are derived by the temperature \( u^{n+1/2} \) at time step \( \tau + \delta \tau/2 \), so as to obtain \( u^n \) from \( u \).

Both the predictor and corrector step need the solution of the linear tridiagonal system of equations, which already has known solution algorithms.

3. Thermal Parameters

3.1. Specific Heat Capacity

The specific heat capacity of simple materials as a function of temperature can be described by the Debye theory when the temperature is not very high (lower than the so-called Debye characteristic temperature). But for materials on a planetary surface or minerals, the relation between specific heat capacity and temperature actually deviates from the Debye theory. Nevertheless, the positive correlation that specific heat capacity...
increases with the increase in temperature is similar to the Debye formula. Hence, the Debye formula can serve as an approximation for the temperature dependence of the specific heat capacity of minerals or planetary surface materials.

Considering that the Debye formula of specific heat capacity is a time-consuming integration formula, here we present a simplified formula to approximate the positive correlation between specific heat capacity and temperature based on the Debye formula:

\[
c_{v}(T) = \frac{3k_B m_a}{m} \left\{ a \left( \frac{T}{T_0} \right)^3 \left[ b \left( \frac{T}{T_0} \right)^2 + c \left( \frac{T}{T_0} \right) + 4 \right] \right\},
\]

where the coefficients

\[a \approx 39.09, \quad b \approx 14.46, \quad c \approx 3.304,\]

\(k_B\) is the Boltzmann’s constant, \(m_a = \rho / n\) is the average atom mass of the material, and \(T_0\) is the “characteristic temperature” of the material that would be measured by experiments.

While we have no exact information about the surface materials on small bodies, we may find their most spectrum-resembling chondrites to estimate their heat capacities. The average atom mass \(m_a\) of several chondrites is listed in Table 1.

| Chondrite Type | \(m_a\) (1.66053873 \times 10^{-27}\) kg |
|---------------|----------------------------------------|
| CI            | 21.5515                               |
| CM            | 22.7817                                |
| H             | 24.9229                                |
| L             | 23.6574                                |
| LL            | 23.4194                                |
| EH            | 25.9846                                |
| EL            | 25.1344                                |

In principle, the characteristic temperature \(T_D\) of each chondrite can be obtained by fitting experimental data of specific heat capacities with Equation (54). By comparing Equation (54) with the experimental data of Macke et al. (2016) and Opeil et al. (2020), we suggest a constant

\[T_D \approx 700\ K\]

for various chondrites.

When the temperature becomes very high, the Debye theory is no longer suitable. Waples & Waples (2004) presented an empirical formula for the specific heat capacities of minerals and nonporous rocks at high temperature, where the relative heat capacity is given as

\[
N_{c_p}(T) = 0.716 + 1.72 \times 10^{-3}(T - 273.4) - 2.13 \times 10^{-6}(T - 273.4)^2 + 8.95 \times 10^{-10}(T - 273.4)^3,
\]

and the specific heat capacity is then estimated via

\[
c_p(T) = c_p(T_0) \frac{N_{c_p}(T)}{N_{c_p}(T_0)}.
\]

Then the variation in specific heat capacity on a very large temperature scale can be expressed as

\[
c_p(T) = \begin{cases} 
  c_p(300) \frac{N_{c_p}(T)}{N_{c_p}(300)}, & T \geq 300\ K, \\
  c_p(T_0) \frac{N_{c_p}(T)}{N_{c_p}(T_0)}, & T < 300\ K.
\end{cases}
\]

Here in Figure 4, we present the specific heat capacities of CI, CM, H, L, and LL chondrites obtained by the above methods.

### 3.2. Thermal Conductivity

For small bodies covered by a dust mantle, the thermal conductivity \(\kappa\) of the dust mantle can be related to the temperature \(T\), the mean radius \(b\) of grains, and the dust mantle porosity \(\phi\) via the model of Gundlach & Blum (2013),

\[
\kappa(T, b, \phi) = \kappa_{\text{solid}} \left( \frac{9\pi}{4} \frac{1 - \mu^2}{E} \frac{\gamma(T)}{b} \right)^{1/3} \chi_f e^{\delta(1-\phi)} + 8\sigma e T^3 \frac{e^{1-\phi}}{1 - \phi} b.
\]

The details of Equation (58) are described in Gundlach & Blum (2013).

Based on Equation (58), in Figure 5, we show how the thermal conductivities change with temperature for C-type and S-type asteroids, assuming a mean grain radius \(b = 0.5\ mm\) and a porosity \(\phi = 0.5\).

### 3.3. Thermal Inertia

As shown in Figures 4 and 5, both the specific heat capacities and thermal conductivities are strong functions of temperature, thus the so-called thermal inertia

\[
\Gamma = \sqrt{\rho c_p K},
\]

is also strongly temperature dependent (see Figure 6).

According to the surface boundary condition Equation (43), the thermal parameter that relates to thermal inertia decides how the surface temperature changes. The thermal inertia, as indicated by its name, generally implies the ability to maintain the thermal state. Thus, the larger the thermal inertia, the slower the temperature variation, and this would probably cause a significant thermal-delay effect that would generate an asymmetric temperature distribution between the sunrise side and the sunset side.

On the other hand, due to the strong positive correlation between thermal inertia and temperature, the rise in temperature could increase the thermal inertia, and thus slow down the temperature increase and enhance the thermal-delay effect, while inversely, the decline in temperature would reduce the thermal inertia, and thus promote the temperature decrease. As a result, the temperature distribution between the sunrise and sunset sides would tend to be even more asymmetric due to the temperature dependence of the thermal parameter.

In the following subsections, we use RSTPM to show how the temperature dependence of the thermal inertia enhances the
asymmetry of the temperature distribution in the diurnal cycle of small bodies. As examples, we consider S-type asteroids at 1 au from the Sun and the physical parameters listed in Table 2.

### 3.3.1. Mean Thermal Inertia versus Mean Grain Radius

As a comparison to the realistic case, first we have to define a constant mean thermal inertia, which ignores the temperature dependence. Therefore we need a mean temperature $\bar{T}$, and the mean thermal inertia is defined as

$$\bar{\Gamma} = c_T \bar{T}.$$  

For the test asteroids with the above conditions, the diurnal mean temperature on the equator can be estimated to be

$$\bar{T} = \left(1 - A_B\right)L_\odot \frac{1}{\varepsilon \sigma \pi} \approx 300 \text{ K}.$$  

Then the mean grain radius $\bar{b}$ would be the main parameter that decides the mean thermal inertia $\bar{\Gamma}$ as shown in Figure 7, which shows that a larger grain radius generally means a higher thermal inertia.

### 3.3.2. Enhanced Asymmetric Temperature Distribution

In Figure 8 we show the diurnal temperature variation obtained by two models with different thermal parameters. The solid curves (in red, green, and blue) are obtained by the RSTPM considering temperature-dependent thermal parameters in three cases of the mean grain radius

$$\bar{b} = 0.1, 1, 10 \text{ mm},$$  

respectively, while the dotted curves with the same colors are obtained by commonly used thermophysical models (CTPM for short) that ignore the temperature dependence, using the

![Figure 4. Model results of specific heat capacities of CI, CM, H, L, and LL chondrites.](image)

![Figure 5. Temperature-dependent thermal conductivities of C-type and S-type asteroids, assuming a mean grain radius $b = 0.5$ mm and a porosity $\phi = 0.5$.](image)

![Figure 6. Temperature-dependent thermal inertias of C-type (CM) and S-type (LL) asteroids, assuming a mean grain radius $b = 0.5$ mm, a surface porosity $\phi = 0.5$, and a grain density $3100 \text{ kgm}^{-3}$ for C-type but $3700 \text{ kgm}^{-3}$ for S-type asteroids.](image)

| Properties                  | Value    |
|-----------------------------|----------|
| Heliocentric distance       | 1 au     |
| Rotation obliquity          | $0^\circ$|
| Rotation period $P_R$       | 10 hr    |
| Roughness fraction $f_r$    | 0.0      |
| Bond albedo $A_B$           | 0.04     |
| Thermal emissivity $\varepsilon$ | 0.9      |
| Resembled chondrite         | LL       |
| Material density $\rho_m$   | 3700 kgm$^{-3}$ |
| Surface porosity $\phi$    | 0.5      |

Table 2: Assumed Parameters for the Test Asteroids

![Figure 7. The relation of the mean thermal inertia and the mean grain radius under the condition given in Table 2.](image)
corresponding mean thermal inertia

$$\tilde{\Gamma} = 63, 150, 466 \text{ Jm}^{-2} \text{s}^{-0.5} \text{ K}^{-1},$$

respectively.

First, when we compare the curves with different colors, we can see that higher thermal inertia tends to generate a stronger thermal-delay effect or asymmetric temperature distribution.

Second, when we compare the solid curves and the corresponding dotted curves with the same color, the temperature distribution tends to be more asymmetric if the thermal parameters’ temperature dependence is taken into consideration. This effect can lead to a temperature difference as large as \(\sim 10 \text{ K}\), which can hence induce a variation of the thermal emission within

$$\sim \left( \frac{300 + 10}{300} \right)^4 \sim 14\%.$$ 

Finally, the asymmetry enhancement tends to be more significant for cases with higher thermal inertia (blue curves).

4. Thermal Infrared Radiometry

The infrared radiation from a small body is related to its size as well as temperature distribution, which is decided by the surface thermophysical properties (e.g., albedo, roughness, and thermal inertia). Thus these properties could be well determined by fitting the measurements of its infrared radiation with the surface thermophysical model. This procedure is known as the so-called “thermal infrared radiometry.” However, for any body except the Sun in the solar system, the observed infrared radiation is the integration of its own thermal emission and the reflected sunlight, especially radiation at a wavelength <5 \(\mu\) m, containing a significant fraction of sunlight (e.g., the W1 and W2 bands of WISE/NEOWISE, see Figure 21 in Section 5.3.1). Thus both thermal emission and sunlight reflection should be taken into account in the radiometric model.

4.1. Disk-integrated Thermal Emission

On the basis of the previously described roughness model, we treat the facet \(i\) of the shape model and the subfacet \(j\) in the crater on facet \(i\) both as smooth Lambertian surfaces, and hence thermal radiation from them can be approximated by Lambertian radiation. So for a given observation phase angle \(\alpha\) and distance \(\Delta\), the thermal emission from facet \(i\) and subfacet \(ij\) that can be observed by the telescope will be \(\epsilon(\lambda)\pi B_{\lambda i} f_i\) and \(\epsilon(\lambda)\pi B_{\lambda ij} f_{ij}\), where \(\epsilon(\lambda)\) is the monochromatic emissivity at wavelength \(\lambda\), \(f_i\) and \(f_{ij}\) are the view factors of facet \(i\) and subfacet \(ij\) relative to the telescope, and \(B_{\lambda i}\) and \(B_{\lambda ij}\) are the Planck intensity function on a temperature \(T_i\) and \(T_{ij}\),

$$B(\lambda, T_i) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda kT_i}\right) - 1}.$$ (61)

The so-called view factor \(f_i\) is defined as

$$f_i = \frac{v_i a_i \bar{n}_i \cdot \bar{n}_{\text{obs}}}{\pi \Delta^2},$$ (62)

where \(a_i\) and \(\bar{n}_i\) are the area and normal vector of facet \(i\), \(\bar{n}_{\text{obs}}\) is the unit vector of the telescope direction in the body-fixed coordinate system, and \(v_i = 1\) indicates that facet \(i\) is observable from the telescope, otherwise \(v_i = 0\).

With the temperature distribution \(T_i\) and \(T_{ij}\) computed from the above numerical method, the observable thermal emission \(F_{\text{obs}}\) of the entire small body can be expressed as the integration of thermal emission from both the smooth and rough surface,

$$F_{\text{obs}}(\lambda) = (1 - f_i) \sum_{i=1}^{N} \epsilon(\lambda)\pi B_{\lambda i} f_i + f_i \sum_{i=1}^{N} \sum_{j=1}^{M} \epsilon(\lambda)\pi B_{\lambda ij} f_{ij}.$$ (63)

4.2. Disk-integrated Sunlight Reflection

While it could be a good approximation to calculate the absorption of solar energy by assuming a Lambertian surface, such approximation will not be good enough if we care about the exact reflected sunlight. Actually, the sunlight reflection by a realistic planetary surface deviates largely from the ideal Lambertian reflection due to multiple effects, including asymmetric scattering by irregular-shape dust particles and the beaming effect induced by macroscopic roughness (far larger than the dust particle size). The macroscopic-roughness model can be similar to the thermal roughness model above. To consider asymmetric scattering, we use the combined Lambert–Lommel–Seeliger law that introduces a correction coefficient \(C_L\) to the Lambertian reflection as

$$C_L(\psi_i, \psi_{\alpha i}, \alpha, w_i) = f(\alpha) \left( w_i + \frac{1}{\psi_i + \psi_{\alpha i}} \right),$$ (64)

where \(\psi_i\) and \(\psi_{\alpha i}\) are the cosines of the incident angle and emergence angle on facet \(i\), respectively, \(\alpha\) is the solar phase angle, and \(f(\alpha)\) is the phase correction function, according to Kaasalainen et al. (2001),

$$f(\alpha) \sim 0.5 \exp(-\alpha/0.1) - 0.5\alpha + 1.$$ 

Parameter \(w_i\) represents the weight of the Lambertian term in the scattering law, so we call it “scattering weight-factor.” To ensure 0 \(\leq C_L \leq 1\), the scattering weight-factor \(w_i\) is required to
be $0 \leq w_i \leq 0.5$. The value $w_i$ can be determined by doing optimization fitting to observations of sunlight reflection.

Hence, for a given epoch with a certain observation phase angle $\alpha$ and distance $\Delta$, the reflection of sunlight from facet $i$ and subfacet $ij$ can be expressed as

$$F_{\text{fl},i}(\lambda) = \pi B(\lambda, 5778) \frac{R_{\text{sun}}^2}{r_{\text{helio}}} \cdot A_\text{h} \cdot \psi_i \cdot f_i \cdot C_L, \quad (65)$$

$$F_{\text{fl},ij}(\lambda) = \pi B(\lambda, 5778) \frac{R_{\text{sun}}^2}{r_{\text{helio}}} \cdot A_\text{h} \cdot \psi_{ij} \cdot f_{ij} \cdot C_L, \quad (66)$$

where $\psi_i$ and $\psi_{ij}$ are the cosine values of the solar altitudes, $R_{\text{sun}}$ is the radius of the Sun, $r_{\text{helio}}$ nearly equals to the heliocentric distance of the asteroid, $B(\lambda, T)$ is the Planck intensity function, and $A_\text{h} = A(\lambda)$ is the albedo at wavelength $\lambda$.

Then the total reflected sunlight that can be observed by the telescope is the integration of reflection from all observable facets,

$$F_{\text{fl}}(\lambda) = \left(1 - f_i\right) \sum_{i}^{N} F_{\text{fl},i} + f_i \sum_{i}^{N} \sum_{j}^{M} F_{\text{fl},ij}. \quad (67)$$

And the total radiation flux that can be observed by the telescope at wavelength $\lambda$ would be the sum of thermal emission and sunlight reflection,

$$F_{\text{modtot}}(\lambda) = F_{\text{fl}}(\lambda) + F_{\text{th}}(\lambda). \quad (68)$$

### 4.3. Thermal Infrared Beaming Effect

As mentioned in Section 2.2, the appearance of roughness could lead to the so-called thermal infrared beaming effect, which causes the observed disk-integrated emission flux to decrease significantly when the observation phase angle deviates from zero. Here we use the RSTPM to investigate the beaming effect. As an example, a testing asteroid with the parameters listed in Table 2 and two cases of the roughness fraction $f_i = 0$ and 1.0 are investigated.

In Figure 9 we show the modeled 12 $\mu$m observation flux in an equatorial view but at various solar phase angles. The observation distance is fixed at 1 au, but two roughness fractions $f_i = 0$ and 1.0 are taken into account.

First, the difference between the dotted curve and the solid curve with the same color clearly reveals the mentioned beaming effect—more emission at zero solar phase angle.

Second, the beaming effect tends to be more significant for the cases of low thermal inertia (smaller grain size). The observed flux at zero solar phase can increase up to about 30% (compare the dotted and solid red curves) due to the beaming effect generated by a totally rough surface (roughness fraction $f_i = 1.0$).

Third, both the decrease in thermal inertia and the increase in roughness can lead to the tendency that more photons are emitted at zero solar phase angle. These two effects can be degenerate for observations at insufficient solar phase angles. In order to reduce the degeneracy of the surface thermal inertia and roughness from thermal infrared observations, observations at various solar phase angles are therefore necessary.

### 4.4. Radiometric Procedure

In order to reproduce the disk-integrated infrared observation of a small body with the RSTPM, we need its 3D shape model, rotation parameters (rotation period and rotation axis orientation), effective diameter $D_{\text{eff}}$, Bond albedo $A_B$, emissivity, and the thermophysical parameters $\rho$, $c_p$, and $\kappa$.

#### 4.4.1. Shape Model and Spin Orientation

The 3D shape model and spin state can be constructed by the light-curve inversion method developed by Kaasalainen & Torppa (2001) if we have observed enough light curves, or by inversion of radar Delay-Doppler images (Ostro et al. 2002). Since observations of radar delay-Doppler images are more difficult than light-curve observations, the light-curve inversion method has become the most commonly used method to derive shape and spin state of most small bodies. However, even for a light-curve observation, it is difficult to obtain enough light curves to constrain a unique spin orientation for many small bodies. It is fortunate that WISE/NEOWISE has observed very many thermal light curves of many small bodies, among which spin orientation still has different solutions. Therefore, by interpreting the thermal light curves of WISE/NEOWISE, it is possible to constrain the spin orientation to a unique solution.

#### 4.4.2. Size and Albedo

According to Fowler & Chillemi (1992), an asteroid’s effective diameter $D_{\text{eff}}$, defined by the diameter of a sphere with the same area as that of the shape model, can be related to its geometric albedo $p_i$ and absolute visual magnitude $H_i$ via

$$D_{\text{eff}} = \frac{1329 \times 10^{-H_i/5}}{\sqrt{p_i}} \text{ (km).} \quad (69)$$

In addition, the geometric albedo $p_i$ is related to the effective Bond albedo $A_{\text{eff},B}$ by

$$A_{\text{eff},B} = p_i q_{\text{ph}}, \quad (70)$$

where $q_{\text{ph}}$ is the phase integral, which can be approximated by

$$q_{\text{ph}} = 0.290 + 0.684G, \quad (71)$$

in which $G$ is the slope parameter in the $H$, $G$ magnitude system of Bowell et al. (1989), which can be obtained by photometric observation.
4.4.3. Roughness Fraction

On the other hand, the asteroid’s effective Bond albedo is the averaged result of both the albedo of the smooth and rough surface, which can be expressed as the following relationship according to Equation (32) \( (\xi = 0.5\) for a hemispherical crater): 

\[
A_{\text{eff,B}} = (1 - f_r)A_B + f_r \frac{A_B}{2 - A_B},
\]  

(72)

where \( A_B \) is the Bond albedo of a smooth Lambertian surface. Thus an input roughness fraction \( f_r \) and geometric albedo \( p_r \) can lead to a unique Bond albedo \( A_B \) and effective diameter \( D_{\text{eff}} \) to be used to fit the observations.

4.4.4. Thermal Emissivity

We can use the Bond albedo \( A_B \) to approximate the reflectance \( A_0(\lambda) \) at an observation wavelength \( \lambda \), so as to calculate the reflected sunlight at wavelength \( \lambda \). On the other hand, according to Kirchhoff’s law, the monochromatic emissivity \( \epsilon(\lambda) \) at wavelength \( \lambda \) can be approximately related to \( A_0(\lambda) \) via 

\[
\epsilon(\lambda) = 1 - A_0(\lambda),
\]

thus enabling the computation for the thermal emission at wavelength \( \lambda \). Under this approximation, the size, albedo, and emissivity are related to each other, thus becoming one free parameter in the fitting procedure.

4.4.5. Mean Thermal Inertia

For the thermophysical parameters, as a first approximation, we may ignore their temperature dependence and assume a mean thermal inertia of the whole surface to obtain a corresponding mean thermal parameter,

\[
\bar{\Phi} = \frac{\bar{c} p}{\varepsilon \sigma T_e^3},
\]

(73)

then we are able to obtain the surface temperature distribution to fit the thermal infrared observations.

Thus we actually have three free parameters: the mean thermal inertia, roughness fraction, and geometric albedo (or effective diameter), which would be extensively investigated in the fitting process. We use the so-called reduced \( \chi^2_r \), which is defined as

\[
\chi^2_r = \frac{1}{n - 3} \sum_{i=1}^{n} \left[ \frac{F_{\text{model}}(\lambda_{i}, p_r, f_r, \bar{\Phi}) - F_{\text{obs}}(\lambda_{i})}{\sigma_{\lambda_{i}}} \right]^2,
\]

(74)

to assess the fitting degree of model results with respect to the observations. The input parameters that give the minimum \( \chi^2_r \) could be treated as the most possible values of these parameters.

4.4.6. Mean Grain Radius

Considering that the surface temperature could differ largely in different regions and that thermal inertia is strongly temperature dependent, the obtained mean thermal inertia from the above radiometry process may not well reveal the physical condition of the surface materials. Thus it is necessary to remove the temperature effect so as to find the more basic properties that are not affected by temperature.

As mentioned in the section above, if we know the taxonomic type of a small body from spectral observation, we could find its most spectrum-resembling chondrite to estimate the specific heat capacity \( \varepsilon_c(T) \) at various temperatures.

Moreover, its surface mass density can also be estimated via

\[
\rho = (1 - \phi) \rho_m,
\]

(75)

where the material density \( \rho_m \) can be approximated as the density of the corresponding chondrite, such as (Opic et al. 2010)

\[
\rho_m = 3110 \text{ kg m}^{-3} \text{ for C-type,}
\]

\[
\rho_m = 3700 \text{ kg m}^{-3} \text{ for S-type,}
\]

and the porosity \( \phi \) can have values from 0.4 to 0.6 for the surface layers of airless bodies.

Finally, if the surface is covered by a dust mantle, the thermal conductivity can be related to the mean grain radius \( b \) as in Equation (58), meaning that \( b \) would be the main important free parameter that decides the realistic thermal inertia.

Then we can investigate the three free parameters, that is, the mean grain radius, the roughness fraction, and the geometric albedo (or effective diameter) in the fitting process. Still, we use the so-called reduced \( \chi^2_r \), which is defined as

\[
\chi^2_r = \frac{1}{n - 3} \sum_{i=1}^{n} \left[ \frac{F_{\text{model}}(\lambda_{i}, p_r, f_r, \bar{\Phi}) - F_{\text{obs}}(\lambda_{i})}{\sigma_{\lambda_{i}}} \right]^2,
\]

(76)

to assess the fitting degree of the model results with respect to the observations. The input parameters that give the minimum \( \chi^2_r \) could be treated as the most possible values of these parameters.

5. Application Example

As an example, we apply RSTPM to study the main-belt object (24) Themis, which has been believed to be the parent body of most of the currently known main-belt comets (MBCs), and hence should have a dust mantle on its surface. MBCs are so small that observations of them are difficult to obtain, whereas Themis is bright enough to be observed at both optical and infrared bands. So using thermal infrared observations to study the surface dust properties would be easier for Themis, the results of which can serve as a reference for the surface dust properties of MBCs regarding their possible connections.

Although light-curve observations of Themis have been obtained to derive its spin orientation together with shape model by the light-curve inversion method, a unique solution to the spin orientation is lacking so far. Currently, four different solutions to the spin orientation and shape model of Themis exist, as shown in Table 3 and Figure 10. Fortunately, WISE/NEOWISE has obtained multiepoch thermal light curves of Themis, thus enabling us to determine which of the four shape models is the best by using the RSTPM to fit these thermal light curves.
The four shape models of (24) Themis from the Database of Asteroid Models from Inversion Techniques. The shape models are shown in the same on-sky orientation.

![Shape 1](image1.png) ![Shape 2](image2.png) ![Shape 3](image3.png) ![Shape 4](image4.png)

Figure 10. The four shape models of (24) Themis from the Database of Asteroid Models from Inversion Techniques. The shape models are shown in the same on-sky orientation.

Table 3
Light-curve Inversion Shape Models of (24) Themis (Hanuš et al. 2016; Viikinkoski et al. 2017)

| Spin Orientation | Number of Vertices | Number of Facets |
|------------------|--------------------|------------------|
| shape 1 (331, 52) | 1018               | 511              |
| shape 2 (137, 59)| 1018               | 511              |
| shape 3 (139, 71)| 800                | 402              |
| shape 4 (329, 70)| 800                | 402              |

Notes. The shape models can be obtained from the database of asteroid models from inversion techniques.

5.1. WISE/NEOWISE Observation

The Wide-field Infrared Survey Explorer (WISE) mission has mapped the entire sky in four bands at 3.4 (W1), 4.6 (W2), 12 (W3), and 22 (W2) μm with resolutions from 6″1 to 12″. All four bands were imaged simultaneously, and the exposure times were 7.7 s in 3.4 and 4.6 μm and 8.8 s in 12 and 22 μm. The four-band survey started on 2010 January 7, and ended on 2010 August 6 after the outer cryogen tank was exhausted, which prevented the W4 channel from being used to obtain survey data from that time on. The W3 channel continued operation until 2010 September 29, when the inner cryogen reserve was exhausted, while the W1 and W2 channels kept working until the telescope was set into hibernation on 2011 February 1. The two-band survey was then resumed on 2013 December 13 (known as NEOWISE), and is still in service. It has obtained nearly six years of observations.

We found multiyear observations of Themis from WISE archive (see the website of the NASA/IPAC Infrared Science Archive [http://irsa.ipac.caltech.edu/]). The magnitude data are converted into flux with color corrections (W1: 2.0577, W2: 1.3448, W3: 1.0006, and W4: 0.9833), and all the derived monochromatic flux densities are set with an associated uncertainty of ±10% (Wright et al. 2010; Mainzer et al. 2011). The flux data are summarized in Tables 4, 5, and 6.

5.2. Input Parameters

In order to interpret these multiyear observations, RSTPM needs several input parameters, including the observation geometry, the shape model, the spin orientation, the rotation phase ϕr, the scattering weight-factor wfi, the geometric albedo pfl, the roughness fraction ffl, and the mean grain radius βfl.

The observation geometry at the time of each observation can easily be obtained according to the orbit of Themis and WISE. The spin orientation together with the shape model has four different choices, as listed in Table 3, which makes the rotation phase of each observation unclear as well. Hence the spin orientation would be the first parameter that needs to be investigated by the fitting procedure.

The scattering weight-factor wfi is crucial in the fitting procedure, as the W1- and W2-band observations contain a significant amount of sunlight reflection. Since this parameter is an artificial factor used to interpret the sunlight reflection, its physical significance is not that clear, thus we only need a scattering weight-factor wfi that could achieve a best-fitting degree to the observations.

The other parameters, including the geometric albedo pfl, the roughness fraction ffl, and the mean grain radius βfl, are all free parameters that would be determined by an optimization of fitting process.

5.3. Fitting with Rotationally Averaged Flux: Seasonal Effect

As mentioned above, rotation phases at different observation epochs are unclear due to the uncertainties of the spin orientation, thus at the first step, we choose the rotationally averaged model flux $F_{\text{model}}$ to fit the observations, by which the diurnal effect is eliminated, whereas the seasonal effect is highlighted to investigate the influence of the spin orientation.

5.3.1. Best-fit Spin Orientation

The available WISE/NEOWISE observations of Themis cover nearly eight different epochs, as shown in Figure 11. Therefore, these observations of the infrared flux can show seasonal variation, making it possible for us to investigate the probable spin orientation, roughness fraction, and thermal parameters.

By fitting observations with the rotationally averaged model flux generated by the RSTPM under the input of a different spin orientation and physical parameters, the best-fit results are selected and summarized in Table 7.

According to Table 7, in the case of shape 2, the minimum reduced χ² is much smaller than that obtained in the other three cases, indicating that shape 2 with spin orientation $(λ = 137°, \beta = 59°)$ should be the best solution to the spin orientation and shape model of Themis. This result is consistent with the recent work of O’Rourke et al. (2020), which also concluded $(λ = 137°, \beta = 59°)$ to be the best-fit spin orientation by TPM fitting to the Subaru/COMICS observations of Themis, indicating that this best solution of the spin orientation is not accidental.

On the other hand, for the case of shape 2, we obtain a best-fit scattering weight-factor $w_{fi} = 0.32$. Then, in the following
sections, we use shape 2 and \( w_t = 0.32 \) to further study the geometric albedo \( p_v \), the roughness fraction \( f_r \), the mean grain radius \( \bar{b} \), and the thermal inertia \( \Gamma \) of Themis.

### 5.3.2. Results of \( p_v, f_r, \) and \( \bar{b} \)

By fixing the shape model with the spin orientation (\( \lambda = 137^\circ, \beta = 59^\circ \)) and scattering weight-factor \( w_t = 0.32 \), we then fit the observations by scanning the roughness fraction \( f_r \) in the range of 0–1 and the mean grain radius \( \bar{b} \) in the range of 1~1000 \( \mu \text{m} \). With each pair of \((f_r, \bar{b})\), a best-fit geometric albedo \( p_v \), together with an effective diameter \( D_{\text{eff}} \) is found to compute the reduced \( \chi^2 \). The results are presented in Figure 12 as a contour of \( \chi^2(f_r, \bar{b}) \).

According to Figure 12, a well-constrained 1\( \sigma \)-level limit is derived for the roughness fraction \( f_r \) and the mean grain radius \( \bar{b} \), giving \( f_r = 0.4 \pm 0.15 \) (corresponding to an rms slope 31 \pm 6), and \( \bar{b} = 140^{+100}_{-50} \mu \text{m} \), respectively. The 3\( \sigma \)-level constraint for the roughness fraction is derived as \( f_r = 0.4^{+0.3}_{-0.4} \) (corresponding to an rms slope 31^{+11}_{-31})), whereas for the mean grain radius, a relatively wide 3\( \sigma \)-level limit is obtained as \( \bar{b} = 140^{+50}_{-140}(6 \sim 640) \mu \text{m} \).

According to the above-derived 1\( \sigma \) and 3\( \sigma \) ranges of the roughness fraction and the mean grain radius, the corresponding geometric albedo \( p_v \) and \( \chi^2 \) are selected, leading to the \( p_v \sim \chi^2 \) relation as shown in Figure 13. In this way, we obtain the 1\( \sigma \) and 3\( \sigma \)-level limits of the geometric albedo as \( p_v = 0.064^{+0.005}_{-0.003} \) and \( p_v = 0.064^{+0.006}_{-0.011} \), respectively, and consequently, the effective diameter of Themis can be derived to be \( D_{\text{eff}} = 201.6^{+8.4}_{-10.5} \text{ km (1}\sigma) \) and \( D_{\text{eff}} = 201.6^{+10.9}_{-11.5} \text{ km (3}\sigma) \) in consideration of the absolute visual magnitude \( H_v = 7.08 \) and slope parameter \( G = 0.19 \) (Harris et al. 1989).

Our result of the geometric albedo \( p_v = 0.064^{+0.005}_{-0.003} \) of Themis agrees with the result \( p_v = 0.07 \pm 0.01 \) of O’Rourke et al. (2020) in the range 0.06–0.072, although our result tends to be smaller, which consequently leads to a larger estimation of the effective diameter \( D_{\text{eff}} = 201.6^{+10.9}_{-11.5} \text{ km in comparison to the result } D_{\text{eff}} = 192^{+10}_{-10} \text{ km derived by O’Rourke et al. (2020). Differences including data inputs and modeling procedures between our work and O’Rourke et al. (2020) may both contribute to the slight differences of the geometric albedo and effective diameter.}

To verify the reliability of the outcomes derived by the above fitting procedure, we employ the observation-to-model ratio to examine how the model results match the observations at various observation wavelengths and geometries (Figure 14) because these factors are the basic variables of the observations.

The upper panel of Figure 14 shows the observation-to-model ratios at each observation wavelength, where the ratios are evenly distributed around 1.0 without significant wavelength-dependent features, indicating that the surface emissivity or albedo of Themis do not show a significant wavelength dependence, thus the combination model of the surface thermal emission and sunlight reflection is good enough to interpret WISE/NEOWISE observations of Themis.

The lower panel of Figure 14 presents the observation-to-model ratios at different solar phase angles, where the ratios are also uniformly distributed around 1.0, showing no distinct phase-angle-dependent features, indicating that the thermal infrared beaming effect of Themis is well resolved by our model, and hence the expectation of removing the degeneracy between thermal parameters and roughness by multiepoch data is well realized. Therefore it should be safe to claim that the above fitting procedure and derived results are reliable.

### 5.3.3. Seasonal Variation of Thermal Inertia

As illustrated in Section 3.3, thermal inertia is a strong function of temperature. Now with the above-derived profile of the mean grain radius, we can evaluate the change in surface thermal inertia of Themis due to the influence of the seasonal temperature variation.

In Figure 15, a map of the surface temperature of Themis is plotted as a function of local latitude and orbital mean anomaly. Each temperature has been averaged over one rotational period. We can clearly see that the temperature at each local latitude can reach maximum (summer) or minimum (winter) at different orbital positions as a result of seasonal effects. The temperature at the poles can vary from \( \sim 34 \) to \( \sim 206 \) K.

The significant temperature variation caused by seasonal effects can influence the thermal inertia of the surface materials. With Equations (59), (58), and (57), the variation of the surface thermal inertia can be evaluated, as shown in Figure 16. When we consider the 3\( \sigma \)-level results of the mean grain radius \( \bar{b} \), the surface thermal inertia of Themis may vary from a minimum profile of \( \sim 3 \text{ J m}^{-2} \text{s}^{-0.5} \text{K}^{-1} \) to a maximum value of \( \sim 60 \text{ J m}^{-2} \text{s}^{-0.5} \text{K}^{-1} \). Moreover, Figure 16 shows that in

### Table 4

| UT         | 3.4 \( \mu \text{m} \) (mJy) | 4.6 \( \mu \text{m} \) (mJy) | 12 \( \mu \text{m} \) (Jy) | 22 \( \mu \text{m} \) (Jy) | MA (\( \circ \)) | \( \epsilon_{\text{hi}} \) (au) | \( \Delta_{\text{obs}} \) (au) | \( \alpha \) (\( \circ \)) |
|------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------|-----------------------------|-----------------------------|-----------------------------|
| 2010-04-21 00:45 | 3.59 ± 0.36 | 9.81 ± 0.98 | 4.95 ± 0.49 | 13.22 ± 1.32 | 131.584 | 3.421 | 3.278 | 17.082 |
| 2010-04-21 13:27 | 3.98 ± 0.40 | 10.50 ± 1.05 | 5.02 ± 0.50 | 15.05 ± 1.51 | 131.673 | 3.421 | 3.272 | 17.083 |
| 2010-04-21 15:02 | 3.65 ± 0.36 | 10.95 ± 1.10 | 6.20 ± 0.62 | 15.99 ± 1.60 | 131.687 | 3.421 | 3.270 | 17.083 |
| 2010-04-21 16:37 | 3.56 ± 0.36 | 10.57 ± 1.06 | 6.05 ± 0.61 | 16.28 ± 1.63 | 131.702 | 3.421 | 3.269 | 17.083 |
| 2010-04-21 18:13 | 3.67 ± 0.37 | 10.45 ± 1.05 | 5.29 ± 0.53 | 15.62 ± 1.56 | 131.710 | 3.421 | 3.269 | 17.083 |
| 2010-04-21 19:48 | 3.69 ± 0.37 | 11.13 ± 1.11 | 6.25 ± 0.62 | 17.08 ± 1.71 | 131.724 | 3.421 | 3.268 | 17.083 |
| 2010-04-21 21:23 | 3.66 ± 0.37 | 10.25 ± 1.03 | 5.73 ± 0.57 | 15.50 ± 1.55 | 131.732 | 3.421 | 3.267 | 17.083 |
| 2010-04-22 00:54 | 3.92 ± 0.39 | 10.81 ± 1.08 | 5.99 ± 0.60 | 16.03 ± 1.60 | 131.761 | 3.421 | 3.265 | 17.083 |
| 2010-04-22 03:44 | 3.98 ± 0.40 | 12.70 ± 1.27 | 6.30 ± 0.63 | 17.28 ± 1.73 | 131.783 | 3.422 | 3.263 | 17.083 |

Note. The epoch marked in red is used as the reference epoch to generate thermal light curves.
Table 5
Mid-infrared Observations of (24) Themis: NEOWISE (2014–2016)

| UT              | Flux (mJy) | MA (°) | $\rho_{\text{eff}}$ (au) | $\Delta_{\text{obs}}$ (au) | $\alpha$ (°) |
|-----------------|------------|--------|--------------------------|----------------------------|--------------|
| 3.4 μm          | 4.6 μm     |        |                          |                            |              |
| 2014-06-10 20:42| 7.65 ± 0.76| 38.43 ± 3.84 | 39.354 | 2.854 | 2.633 | -20.826 |
| 2014-06-10 20:43| 7.80 ± 0.78| 40.10 ± 4.01 | 39.354 | 2.854 | 2.633 | -20.826 |
| 2014-06-10 23:52| 7.48 ± 0.75| 36.84 ± 3.68 | 39.376 | 2.854 | 2.635 | -20.826 |
| 2014-06-11 03:01| 7.47 ± 0.75| 36.10 ± 3.61 | 39.398 | 2.854 | 2.636 | -20.827 |
| 2014-06-11 06:11| 7.54 ± 0.75| 41.18 ± 4.12 | 39.421 | 2.854 | 2.638 | -20.828 |
| 2014-06-11 07:46| 7.25 ± 0.73| 34.22 ± 3.42 | 39.435 | 2.854 | 2.639 | -20.828 |
| 2014-06-11 09:20| 7.21 ± 0.72| 36.07 ± 3.61 | 39.443 | 2.854 | 2.640 | -20.829 |
| 2014-06-11 10:55| 7.39 ± 0.74| 35.15 ± 3.51 | 39.458 | 2.854 | 2.641 | -20.828 |

comparison to the uncertainties of the mean grain radius, the influence of the temperature variation on the thermal inertia is more significant.

5.3.4. Average Thermal Inertia

Despite the fact that thermal inertia is temperature dependent, most relevant works ignore this temperature dependency and only estimate the average thermal inertias. For comparison with these existing results, we estimate the seasonal average thermal inertia of Themis by using Equations (59), (58), and (57) with inputs of the derived mean grain radius and seasonal averaged temperature of Themis. The seasonal average temperature would be a function of local latitude, $\tilde{T}(\tilde{\theta})$, and can be estimated as

$$ (1 - A_{\text{eff,B}}) L_{\tilde{\theta}}(\tilde{\theta}) = \varepsilon \sigma \tilde{T}(\tilde{\theta})^4, $$

where $A_{\text{eff,B}}$ is the Bond albedo, $\varepsilon \sim 0.9$ is the average thermal emissivity, and $L_{\tilde{\theta}}(\tilde{\theta})$ is the seasonal average incoming solar flux on each latitude. The results are presented in Figure 17.

From Figure 17, we see that the average thermal inertia of Themis would be within $26 \sim 32 \text{Jm}^{-2} \text{s}^{0.5} \text{K}^{-1}$. As a comparison, our result of the average thermal inertia is well consistent...
with that of O’Rourke et al. (2020), who estimated Themis to have a mean thermal inertia of $\sim20^{+25}_{-10}\text{Jm}^{-2}\text{s}^{-0.5}\text{K}^{-1}$.

5.4. Fitting with Thermal Light Curves: Diurnal Effect

Since the WISE/NEOWISE data at different epochs do not perfectly cover an entire rotation period and have been observed at various solar phase angles, it is not proper to directly use them to generate thermal light curves. However, the orbital and rotational parameters of Themis are well known. Thus, in principle, we could derive the rotational phase of each observation data with respect to a defined local body-fixed coordinate system if we know the observed rotational phase at a particular epoch. These data could then be used to create thermal light curves.

The 3D shape model is used to define the local body-fixed coordinate system, where the $z$-axis is chosen to be the rotation axis, and “zero” rotational phase is chosen to be the “equatorial view ($0^\circ$)”, as shown in Figure 18. Moreover, if we define the viewing angle of one observation with respect to the body-fixed coordinate system to be $(\varphi, \theta)$, where $\varphi$ stands for the local longitude, and $\theta$ means the local latitude, then the rotational phase $\phi$ of this observation can be related to the local longitude $\varphi$ via

$$\phi = 1 - \frac{\varphi}{2\pi}. \quad (78)$$

If selecting a reference epoch and assuming that the rotational phase at this epoch is $\phi_0$, then all the rotational phases of other data could be derived in consideration of the observation time and geometry. Furthermore, for some particular epoch, the thermal light curves can be derived for each band by correcting the observed flux at various epochs into one rotation period at this epoch, where the correction is implemented via

$$F_{i,\text{corr}} = F_i \left( \frac{r_{i,\text{helio}}}{r_{0,\text{helio}}} \right)^2 \left( \frac{\Delta \phi_{\text{obs}}}{\Delta \phi_{\text{obs},0}} \right)^2, \quad (79)$$
Figure 11. The small magenta circles represent the orbital positions of (24) Themis at the time of each observation, covering eight different epochs, making it possible to resolve the spin orientation, roughness fraction, and thermal parameters by considering the seasonal variation of the observed infrared flux by WISE/NEOWISE.

Figure 12. The contour of $\chi^2(f_r, \bar{b})$, which is obtained by fitting the observations with two free parameters: the roughness fraction $f_r$ and the mean grain radius $\bar{b}$. The cyan region stands for the 1σ level constraint, and the red region represents the 3σ-level constraint.

Table 7
Best-fit Results from Fitting Observations with the Rotationally Averaged Model Flux

| Best-fitting Parameters | Minimum $\chi^2$ |
|-------------------------|-----------------|
| $w_f$ | $p_v$ | $f_r$ (μm) | $\bar{b}$ (μm) | $\chi^2$ |
| shape 1 | 0.37 | 0.067 | 0.45 | 150 | 0.383 |
| shape 2 | 0.32 | 0.064 | 0.40 | 140 | 0.291 |
| shape 3 | 0.50 | 0.058 | 0.50 | 140 | 0.982 |
| shape 4 | 0.50 | 0.060 | 0.50 | 140 | 0.842 |

$w_f$: scattering weight-factor.
$f_r$: roughness fraction; $\bar{b}$: mean grain radius.

Figure 13. $p_v \sim \chi^2_{\text{reduced}}$. Profiles fit to the observations in consideration of the derived 1σ and 3σ ranges of the roughness fraction and the mean grain radius.

Figure 14. The observation-to-model ratios as a function of wavelength (upper panel) and solar phase angle (under panel) for the case of the best-fit parameters.

Figure 15. Seasonal variation of the diurnal-averaged surface temperature as a function of local latitude. The so-called local latitude is defined as the complementary angle of the angle between the local normal vector and the rotation axis.
rotational phase $\zeta_{ph}$. But to correct for the flux, in order to reduce the flux errors caused by correction of Equation (79), we select eight separate reference epochs so as to use data that are close to each reference epoch (data within three days) to generate the thermal light curves. These eight reference epochs are marked in red in Tables 4, 5, and 6. Then for each of the reference epochs, the theoretical thermal light curves are simulated by the RSTPM to fit the above-generated observations of thermal light curves. The best-fit results are plotted in Figures 19 and 20.

Now questions arise what we can learn by fitting with thermal light curves. Since both irregular shape and surface heterogeneity can contribute to the rotational variation in flux in light curves, we can evaluate whether the surface is heterogeneous along the longitude by fitting with thermal light curves. To realize this purpose, we can investigate whether the ratios of observation/model have features that depend on the rotation phase.

Note that the fraction of sunlight reflection in each band observation of WISE/NEOWISE is significantly different. To show these differences, the best-fit parameters are input to RSTPM to estimate the fraction of sunlight reflection in each band observation at each epoch. The results are shown in Figure 21, where we can see that the fraction of reflection is almost zero at bands W4 and W3 ($<10^{-4}$), but comes to be non-negligible at band W2 ($\sim 10\% - 30\%$), and becomes dominating at band W1 (reaches up to $\sim 99\%$), so the observation-to-model ratios for bands W4 and W3 actually represent the deviation of the real thermal emissivity relative to the model-input emissivity $\approx 0.9$, while the ratios of observation/model for band W1 stand for the deviation of the real geometric albedo from the best-fit value $\approx 0.064$.

Therefore the ratios of observation/model for each band are separately plotted as a function of the rotation phase in Figure 22. From Figure 22, we see that the observation-to-model ratios for band W1 are evenly distributed around 1.0 at all rotation phases, showing no rotation-phase-dependent feature, hence indicating that the light-curve inversion shape model fits the W1-band data pretty well and the surface albedo of Themis does not significantly vary with longitude. However, for bands W4, W3, and W2, the observation-to-model ratios show a weak rotation-phase-dependent feature, as shown by the dashed red curves in Figure 22, where the variation trends of the three dashed red curves are similar to each other but are different from the trend of W1, indicating that surface materials at different longitude of Themis may have heterogeneous thermal properties if the shape model imperfection is small. However, the light-curve inversion shape model is only an ideal shape that achieves optimum fitting to the visible light curves. It may still be different from the real shape, and hence may not be able to produce the thermal light-curve very well. Therefore the possibility that the trends in the W2, W3, and W4 thermal light curves are caused by shape model imperfections cannot be removed.

6. Discussion and Conclusion

Thermophysical modeling is the basis of thermal infrared radiometry, which is the main method for measuring the thermophysical properties of surface materials on small bodies. So developing advanced TPMs have always been the direction of work in this field. For airless small bodies, the thermal state of the surface layers is influenced not only by the surface
thermophysical parameters, but also by geometric effects such as the roughness or topography, and state of motion, including rotation and orbital motion. The complexity of the problem lies in that these effects’ influence on the thermal state are coupled together, causing the surface thermal parameters and roughness to be inevitable degenerate in the radiometry procedure, hence multiepoch observations are necessary to remove the degeneracy of the thermal inertia and roughness in the radiometric procedure. For the use of interpreting multiepoch thermal light curves (e.g., WISE/NEOWISE), we therefore propose this thermophysical model for realistic surface layers on airless small bodies, the RSTPM, which simultaneously considers the real orbital cycle, the rotation cycle, a rough surface, temperature-dependent thermal parameters, as well as contributions of sunlight reflection to observation.

When we aim to interpret multiepoch observations, the temperature dependence of the thermal inertia becomes non-negligible if seasonal effect can cause significant temperature

Figure 19. Best-fit results to the thermal light curves of WISE/NEOWISE at bands W4, W3, and W2.

Figure 20. Best-fit results to the light curves of WISE/NEOWISE at band W1.
variation. As shown in Figure 16, the effect of seasonal temperature variations on thermal inertia can be more significant than the influence from the uncertainties of the mean dust-grain size, if there is a dust mantle on the surface, and the thermal conductivity of the dust mantle can be related to the temperature, the porosity of the dust mantle, and the mean dust-grain radius as in Equation (58). For such small bodies that are covered by a dust mantle, the mean dust-grain size of the dust mantle is therefore more suitable to be used as the free parameter to be determined from the radiometric procedure, which then provides a way to study the physical properties of dust on the surface of small bodies.

Of course, there are also small bodies that do not have a dust mantle on the surface. For example, in situ observations of (162173) Ryugu by Hayabusa2 show that most of Ryugu’s surface is covered by porous boulders (Okada et al. 2020). For such small bodies, there is even no dust mantle on the surface, hence there is no need to study the dust properties, and thermal inertia cannot be modeled as a function of the mean dust-grain size, whereas the RSTPM can be still used to study the mean thermal inertia of the surface, or the macroscopic porosity of the surface if the thermal inertia can be described as a function of temperature and porosity.

A dust mantle is more likely to appear on large main-belt objects, so WISE/NEOWISE becomes a versatile archive for studying the dust properties of these bodies. However, as shown in Figure 21, the W1-band observation is actually dominated by sunlight reflection, and even the W2-band

Figure 21. The fraction of sunlight reflection in the observed flux of Themis for each band of WISE/NEOWISE.

Figure 22. Ratios of observation/model as a function of rotation phase for each band.
observation contains non-negligible $\sim 10\% – 30\%$ sunlight reflection. Thus, a precise combination model of thermal emission and sunlight reflection is extremely necessary for interpreting the multiepoch thermal light curves of WISE/NEOWISE. The RSTPM performs very well in simultaneously simulating the thermal emission and sunlight reflection, as demonstrated by its successful application to (24) Themis. But it should be noted that the “scattering weight-factor” $w_f$ used in the reflection model (Equation (64)) is an artificial factor that does not have a clear physical significance, and this may need further examination by more observations and researches.

Nevertheless, the successful application of the RSTPM makes it possible for us to claim that this model is reliable, and is highly capable of deriving the physical properties of small bodies by interpreting the four-band WISE/NEOWISE observations obtained at multiple epochs. We thus propose that if it is used to fit rotationally averaged observations of multiple epochs, the RSTPM can study the spin orientation as well as the surface properties, including the geometric albedo, the roughness, and the mean thermal inertia or mean dust-grain size; and if it is used to fit thermal light curves, the RSTPM can investigate whether the surface materials at different longitudes are heterogeneous in terms of their thermophysical properties.

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