A Maximum Edge-Weight Clique Extraction Algorithm Based on Branch-and-Bound

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Abstract
The maximum edge-weight clique problem is to find a clique whose sum of edge-weight is the maximum for a given edge-weighted undirected graph. The problem is NP-hard and some branch-and-bound algorithms have been proposed. In this paper, we propose a new exact algorithm based on branch-and-bound. It assigns edge-weights to vertices and calculates upper bounds using vertex coloring. By some computational experiments, we confirmed our algorithm is faster than previous algorithms.

1 Introduction
For a simple undirected graph $G = (V, E)$, a vertex subset $C \subseteq V$ is called a clique if any pair of vertices in $C$ are adjacent. Given a simple undirected graph $G = (V, E)$, the maximum clique problem (MCP) is to find the clique of maximum cardinality. MCP has lots of practical applications: bioinformatics [2], coding theory [10, 4], economics [3] and more. MCP is known to be NP-hard [12], and the decision version is one of the Karp’s 21 NP-complete problems [15]. Since it has theoretical importance, there have been a number of studies in decades [31].

Given a simple undirected graph $G = (V, E)$ and non-negative weight $w(\cdot, \cdot)$ for each edge, the maximum edge-weight clique problem (MEWCP) is to find the clique of maximum weight. Obviously, MEWCP is a generalization of MCP. Because of edge-weights, MEWCP has practical applications that cannot be handled by MCP: pattern recognition [19], protein side-chain packing [11, 6], market basket analysis [7], communication analysis [9] and so on.

To obtain exact solutions of MEWCP, there are two approaches in previous works. One approach is formulating MEWCP into mathematical programming

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and solve it by existing solvers. Formulations based on integer programming (IP) \cite{13} and mixed integer programming (MIP) \cite{24} were proposed.

The other approach is based on branch-and-bound. Branch-and-bound algorithms recursively divide subproblems into smaller subproblems to search optimal solutions. For each subproblem, it calculates upper bounds of the weight of feasible solutions and prunes unnecessary subproblems that have no possibility to improve the incumbent (current best solution). Variety of algorithms adopt different strategies in branching strategy, search strategy and pruning rule. The branching strategy is how to divide a given problem into subproblems. The search strategy is the order in which subproblems are explored. The pruning rule is how to calculate upper bounds to prune unnecessary subproblems. A survey of branch-and-bound is shown in \cite{18}. For MEWCP, CBQ proposed in \cite{14} uses quadratic relaxation to obtain upper bounds. Our previous algorithm EWCLIQUE is also based on the branch-and-bound \cite{24}. EWCLIQUE decomposes edge-weights of each subproblem into three components, and calculates an upper bound for each of them.

In this paper, we propose a new branch-and-bound algorithm MECQ for MEWCP. For each subproblem, our algorithm assigns weights of edges to vertices. To obtain upper bounds using the assigned vertex weights, our algorithm calculates vertex coloring that is a procedure to divide the vertex set into a collection of independent sets. By some computational experiments, we confirm our algorithm is faster than previous methods.

The remainder of this paper is organized as follows. Our algorithm MECQ is described in Section 2. The results of computational experiments are in Section 3. We conclude the paper in Section 4.

2 Our algorithm MECQ

The proposed algorithm MECQ is based on the branch-and-bound. Hereafter let \( P(C, S) \) be a subproblem of MEWCP, where \( C \) is a constructed clique and \( S \) is a set of candidate vertices to be added to \( C \). Note that \( C \subseteq N(v) \) must be satisfied for any element \( v \in S \), where \( N(v) \) is the set of adjacent vertices of \( v \). \( P(\emptyset, V) \) corresponds to the instance of MEWCP. In this section, we first describe pruning rules of our algorithm. Next, we show the branching strategy that divides \( P(C, S) \) into subproblems, and the search strategy to determine the order of subproblems to be solved.

2.1 Pruning Rules

First, we describe an upper bound calculation for graphs where both vertices and edges are weighted. Then, we show that an upper bound of \( P(C, S) \) can be calculated in the same way.

For a graph \( G = (V, E) \), let \( w(v) \) and \( w(u, v) \) denote the weight of vertex \( v \) and the weight of edge \( (u, v) \), respectively. Hereafter we define \( w(u, v) = 0 \) for any \( (u, v) \notin E \) for simplicity. Let \( G(S) \) be a subgraph of \( G \) induced by a
2.1.1 Upper bound of vertex-and-edge-weighted graph

Vertex coloring is to color vertices such that no adjacent vertices have the same color. A vertex set of each color forms an independent set. The smallest number of colors needed to color a graph $G$ is called chromatic number $\chi(G)$. Let $\omega(G)$ be the clique number (the number of vertices in a maximum clique). Since at most one vertex can be included in a clique from each independent set, $\chi(G)$ is an upper bound of $\omega(G)$. Therefore heuristic vertex coloring is often used to obtain upper bounds for MCP \[27, 29\]. For the maximum weight clique problem (MWCP), the sum of the maximum vertex weight of each independent set is used as an upper bound \[16, 25\]. To calculate upper bounds of the MEWCP, we consider assigning edge weights to incident vertices. Let $\tau(v)$ be the index of the independent set including vertex $v$. Namely, $\tau(v) = i$ for all $v \in I_i$. Let $\sigma[v]$ be the total weight assigned to the vertex $v$ as follows:

$$\sigma[v] = w(v) + \sum_{i<\tau(v)} \max\{w(u, v) \mid u \in I_i \cap N(v)\}. \quad (1)$$

An example $G_{ex}$ of a vertex-and-edge-weighted graph is shown in Figure 1. Numbers in parentheses are the vertex weights. For $G_{ex}$, Figure 2 shows the
assignment of independent set indices $\tau(\cdot)$ and the weight $\sigma[\cdot]$ of equation 1. At most one vertex of $I_1 \cap N(v_5) = \{v_1, v_6\}$ can be included in a clique since $I_1$ is an independent set. Therefore we assign only the larger weight of edges $(v_1, v_5)$ and $(v_5, v_6)$ to $v_5$ for upper bound calculation. We can ignore smaller weights to tighten upper bounds.

The following lemma shows that an upper bound of the MEWCP can be calculated by using vertex coloring.

**Lemma 1.** For a clique $C$ in a vertex-and-edge-weighted graph, the following inequality holds where $k = \max \{\tau(v) \mid v \in V\}$:

$$W(C) \leq \sum_{i=1}^{k} \max \{\sigma[v] \mid v \in I_i\}. \quad (2)$$

**Proof.** Since at most one vertex in $I_i$ can be included in $C$, $|C \cap I_i| \leq 1$ holds. From the definition, $C \subseteq N(v)$ for all $v \in C$. Therefore following inequality is obtained:

$$W(C) = \sum_{v \in C} w(v) + \sum_{(u,v) \in E(C)} w(u,v) \quad (3)$$

$$= \sum_{v \in C} w(v) + \sum_{v \in C} \sum_{1 < \tau(v) \leq \tau} \sum_{u \in C \cap I_i} w(u,v) \quad (4)$$

$$\leq \sum_{v \in C} w(v) + \sum_{v \in C} \sum_{1 < \tau(v) \leq \tau} \max \{w(u,v) \mid u \in N(v) \cap I_i\} \quad (5)$$

$$= \sum_{v \in C} \sigma[v] \quad (6)$$

$$= \sum_{i=1}^{k} \sum_{v \in C \cap I_i} \sigma[v] \quad (7)$$

$$\leq \sum_{i=1}^{k} \max \{\sigma[v] \mid v \in I_i\} \quad (8)$$

\[\square\]

Our algorithm uses Lemma 1 to obtain upper bounds. In the example $G_{ex}$, the clique of maximum weight is $\{v_4, v_5, v_6\}$ and its weight is 35. Using $\sigma[\cdot]$ of Figure 2 an upper bound can be calculated by Lemma 1 as follows:

$$W(C) \leq \max \{\sigma[v_1], \sigma[v_3], \sigma[v_6]\} + \max \{\sigma[v_2], \sigma[v_4]\} + \max \{\sigma[v_5]\} \quad (9)$$

$$= 36 \quad (10)$$
### 2.1.2 Upper bound calculation for MEWCP

Let $F$ be any feasible solution of a subproblem $P(C, S)$ of MEWCP. $F$ is a union of $C$ and a subset of $S$. $W(F)$ can be calculated as follows:

$$W(F) = W(C) + \sum_{u \in C} \sum_{v \in S \cap F} w(u, v) + W(S \cap F). \quad (11)$$

In the branch-and-bound, our algorithm calculates upper bounds of $W(F)$ to prune unnecessary subproblems. Since the term $W(C)$ is obviously obtained in branching steps, we have to calculate an upper bound of $\sum_{u \in C} \sum_{v \in S \cap F} w(u, v) + W(S \cap F)$.

To obtain vertex-and-edge-weighted graphs, our algorithm assigns edge-weights of $\{(u, v) \mid u \in C, v \in S \cap F\}$ to vertices $v \in S \cap F$. Let $w_p(C, v)$ be the total edge-weight assigned to $v \in S \cap F$ as follows:

$$w_p(C, v) = \sum_{u \in C} w(u, v). \quad (12)$$

Hereafter $w_p(v)$ denotes $w_p(C, v)$ when $C$ can be obviously identified. Using $w_p(v)$ and equation (11) following equation holds:

$$W(F) = W(C) + \sum_{v \in S \cap F} w_p(v) + W(S \cap F). \quad (13)$$

Note that the assigned weight $w_p(\cdot)$ and equation (13) is originally proposed in our previous work [24].

For the vertex induced subgraph $G(S)$ of $P(C, S)$, let $G(C, S)$ be the subgraph that can be obtained by assigning the weight $w_p(v)$ to each vertex in $G(S)$. The proposed algorithm uses the vertex-and-edge-weighed graph $G(C, S)$ to calculate an upper bound of $W(F)$. Equation (13) indicates that the sum of $W(C)$ and an upper bound of clique weight in $G(C, S)$ is an upper bound of $W(F)$. Hence the proposed algorithm calculates an upper bound of $W(F)$ for $P(C, S)$ as follows:

1. Obtain the vertex-and-edge-weighted graph $G(C, S)$ using $w_p(\cdot)$ of equation (12).
2. Divide $S$ into mutually disjoint independent sets $I_1, I_2, \ldots, I_k$ by vertex coloring.
3. Calculate $\sigma[\cdot]$ for each vertex in $G(C, S)$ using equation (11).
4. Calculate an upper bound of $W(F)$ using Lemma (1).

### 2.2 Branch-and-bound

Algorithm (1) shows the main part of the proposed algorithm. The inputs are a graph $G = (V, E)$, edge-weights $w(\cdot, \cdot)$ and an initial solution $C_{\text{initial}}$. It searches
Algorithm 1 MECQ

INPUT: $G = (V, E)$, $w(\cdot, \cdot)$, $C_{initial}$

OUTPUT: a maximum edge-weight clique $C_{max}$

GLOBAL VARIABLES: $C_{max}$

1: $C_{max} \leftarrow C_{initial}$
2: EXPAND($\emptyset, V$)
3: return $C_{max}$

Algorithm 2 Solving a subproblem

INPUT: a subproblem $P(C, S)$

OUTPUT: Update $C_{max}$ to a better clique if it exists.

GLOBAL VARIABLES: $C_{max}$

1: procedure EXPAND($C, S$)
2: if $S = \emptyset$ then
3: if $W(C) > W(C_{max})$ then
4: $C_{max} \leftarrow C$
5: end if
6: return
7: end if
8: $\Pi, upper[\cdot] \leftarrow$ CALC_SEQ_AND_UB($C, S$)
9: for each $p_i$ in order of $\Pi$ do
10: if $W(C) + upper[p_i] > W(C_{max})$ then
11: EXPAND($C \cup \{p_i\}, (S \setminus \{p_j \mid j < i\}) \cap N(p_i)$)
12: end if
13: end for
14: end procedure

for solutions by the recursive procedure EXPAND. Our algorithm accepts a feasible solution $C_{initial}$ as an initial incumbent. Although our algorithm works when $C_{initial}$ is empty, given non-empty $C_{initial}$, our algorithm can use it as a lower bound and can efficiently prune subproblems in some cases.

Algorithm 2 shows the recursive procedure EXPAND to update the best solution so far. When $S$ is empty, it is the base case that updates the optimal solution $C_{max}$ (lines from 2 to 7). Otherwise, at line 8 the function CALC_SEQ_AND_UB returns a sequence $\Pi = [p_1, p_2, \ldots, p_{|S|}]$ of vertices in $S$ and an array $upper[\cdot]$ of upper bounds using vertex coloring (described in 2.2.1). In the loop of lines from 9 to 13 in Algorithm 2 it recursively searches solutions at line 11. The branching strategy, pruning rules and search strategy of our algorithm are as follows:

Branching Strategy

For each $p_i$ of $\Pi$, our algorithm generates a child subproblem $P(C \cup \{p_i\}, (S \setminus \{p_j \mid j < i\}) \cap N(p_i))$. Excepting the order of vertices in $\Pi$, this strategy is same as previous algorithm EWCLIQUE [24] and is widely
used in branch-and-bound algorithms of MCP and MWCP [22, 28, 26, 11].

**Pruning Rules**

For each \( P(C \cup \{p_i\}, (S \setminus \{p_j \mid j < i\}) \cap N(p_i)) \), an upper bound based on equation 2 is stored in the array \( \text{upper}[p_i] \).

**Search Strategy**

In order of \( \Pi = [p_1, p_2, \ldots, p_{|S|}] \), our algorithm searches \( P(C \cup \{p_i\}, (S \setminus \{p_j \mid j < i\}) \cap N(p_i)) \). Since \( \Pi = [p_1, p_2, \ldots, p_{|S|}] \) is ordered in non-increasing of \( \text{upper}[\cdot] \) (described in 2.2.1), this strategy is to find cliques of large weight early.

### 2.2.1 Subroutine CALC_SEQ_AND_UB

Algorithm 3 shows the function CALC_SEQ_AND_UB. It receives a subproblem \( P(C, S) \) and returns a sequence \( \Pi = [p_1, p_2, \ldots, p_{|S|}] \) of vertices in \( S \) and an array \( \text{upper}[\cdot] \) of upper bounds. The array \( \text{upper}[p_i] \) contains an upper bound of \( P(C \cup \{p_i\}, (S \setminus \{p_j \mid j < i\}) \cap N(p_i)) \). It is used at line 10 of Algorithm 2.

\( \Pi \) is ordered in non-increasing of \( \text{upper}[\cdot] \) and is used in branching strategy and search strategy.

Here we describe the detail of Algorithm 3. At line 3, it initializes \( \sigma[\cdot] \) to \( w_{\rho}(\cdot) \). Each iteration of the while loop from line 7 to 22, it increments \( k \) and constructs a maximal independent set \( I_k \), appends the vertices in \( I_k \) to \( \Pi \), and updates \( \sigma[\cdot] \). The loop terminates when all vertices are added to \( \Pi \). In the loop of lines from 11 to 18, it constructs a maximal independent set. In line 15, it appends the vertices in \( I_k \) to the head of \( \Pi \) in order of assignment to independent sets. In the maximal independent set construction, \( X \) is the set of candidate vertices to be added to the independent set. At line 12, our algorithm picks vertices from \( X \) in non-decreasing order of \( \sigma[\cdot] \). This makes \( \Pi \) non-increasing order of upper bounds. At line 20, our algorithm updates \( \sigma[\cdot] \) for vertices that are not added to any independent set and are adjacent to vertices in the constructed independent set.

### 3 Computational experiments

We implemented our algorithm MECQ in C++ to compare with previous algorithms. In the experiments, our algorithm received an initial solution \( C_{\text{initial}} \) calculated by phased local search (PLS) [20]. PLS is a heuristic based on local search. To avoid to be trapped into local optimums, it switches three phases that have different search policies. The one iteration of PLS consists of 50 searches of random phase, 50 searches of penalty phase and 100 searches of degree phase. Our algorithm used PLS with 10 iterations and used the best solution found by the PLS as an initial solution.
Algorithm 3 Calculate a vertex sequence and upper bounds

**INPUT:** a subproblem $P(C, S)$

**OUTPUT:** a vertex sequence $\Pi$ and an array $upper[\cdot]$

1: procedure `CALC_SEQ_AND_UB(C, S)`
2: 
3: for $v \in S$ do
4: \hspace{1em} $\sigma[v] \leftarrow w_{\rho}(v)$
5: end for
6: $S' \leftarrow S$ \hspace{1em} $\triangleright$ uncolored vertex set
7: $k \leftarrow 0$ \hspace{1em} $\triangleright$ number of independent sets
8: while $S' \neq \emptyset$ do
9: \hspace{1em} $k \leftarrow k + 1$
10: \hspace{1em} $I_k \leftarrow \emptyset$
11: \hspace{1em} $X \leftarrow S'$ \hspace{1em} $\triangleright$ candidate vertex set to add to $I_k$
12: \hspace{1em} while $X \neq \emptyset$ do
13: \hspace{2em} $v \leftarrow$ a vertex of minimum $\sigma[\cdot]$ in $X$
14: \hspace{2em} $upper[v] \leftarrow \sigma[v] + \sum_{i<k} \max \{\sigma[u] \mid u \in I_i\}$
15: \hspace{2em} $I_k \leftarrow I_k \cup \{v\}$ \hspace{1em} $\triangleright \tau(v) = k$
16: \hspace{2em} Append $v$ to the head of $\Pi$.
17: \hspace{2em} $X \leftarrow X \setminus N(v)$
18: \hspace{2em} $S' \leftarrow S' \setminus \{v\}$
19: end while
20: for $v \in S'$ do
21: \hspace{2em} $\sigma[v] \leftarrow \sigma[v] + \max \{w(u, v) \mid u \in N(v) \cap I_k\}$
22: end for
23: \hspace{2em} $\sigma[v] \leftarrow \sigma[v] + \max \{w(u, v) \mid u \in N(v) \cap I_k\}$
24: end procedure

3.1 Random graphs

We generated uniform random graphs. Edge-weights were uniform random integer values from 1 to 10. The compared algorithms are EWCLIQUE [24] and mathematical programming formulations of MIP proposed in [23]. We used the C++ implementation of EWCLIQUE that was used in our previous work [24]. For the formulations of MIP, we used the mathematical programming solver IBM CPLEX 12.5.

The compiler is g++ 5.4.0 with optimization option -O2. The OS is Linux 4.4.0. The CPU is Intel@CoreTM i7-6700 CPU 3.40 GHz. RAM is 16GB. Note that CPLEX is a multi-thread solver based on branch-and-cut, and our algorithm is a single-thread solver based on branch-and-bound.

Table 1 shows the CPU time for random graphs. The symbol $\epsilon$ shows that the CPU time is less than 0.01 sec. The column LB shows the weight of initial solutions given by PLS. For all conditions, our algorithm MECQ obtained optimal solutions in a shorter time than previous methods. For the random graphs, the initial solution given by PLS does not improve performance.
Table 1: CPU time for random graphs [sec]

| | \(V\) | \(d\) | weight | optimal | MECCQ + PLS | MECCQ | Total | MECCQ without PLS | \([23]\) | \([24]\) |
|---|---|---|---|---|---|---|---|---|---|---|
| 300 | 0.1 | 60.7 | 60.7 | 60.7 | 0.01 | 60.7 | 0.01 | 91.54 |
| 350 | 0.1 | 64.8 | 64.8 | 64.8 | 0.01 | 64.8 | 0.01 | 90.96 |
| 15000 | 0.1 | 174.7 | 148.3 | 0.73 | 408.56 | 409.29 | 402.75 | 460.90 | >1000 |
| 250 | 0.2 | 97.3 | 97.3 | 97.3 | 0.02 | 97.3 | 0.02 | 64.26 |
| 280 | 0.2 | 102.4 | 102.4 | 102.4 | 0.02 | 102.4 | 0.02 | 119.47 |
| 5500 | 0.2 | 254.8 | 212.2 | 0.37 | 319.27 | 319.64 | 319.93 | 440.29 | >1000 |
| 200 | 0.3 | 150.0 | 150.0 | 150.0 | 0.02 | 150.0 | 0.02 | 97.16 |
| 250 | 0.3 | 332.8 | 291.1 | 0.26 | 227.70 | 227.96 | 232.99 | 459.05 | >1000 |
| 160 | 0.4 | 185.5 | 185.5 | 185.5 | 0.02 | 185.5 | 0.02 | 97.16 |
| 200 | 0.4 | 224.0 | 224.0 | 224.0 | 0.03 | 224.0 | 0.03 | 57.54 |
| 1400 | 0.4 | 444.3 | 406.8 | 0.21 | 293.71 | 293.92 | 295.09 | 758.22 | >1000 |
| 140 | 0.5 | 272.7 | 272.7 | 272.7 | 0.01 | 272.7 | 0.01 | 21.98 |
| 170 | 0.5 | 300.6 | 300.6 | 300.6 | 0.03 | 300.6 | 0.03 | 21.98 |
| 750 | 0.5 | 560.3 | 546.5 | 0.15 | 164.32 | 164.47 | 164.76 | 603.91 | >1000 |
| 120 | 0.6 | 399.0 | 399.0 | 399.0 | 0.01 | 399.0 | 0.01 | 18.10 |
| 150 | 0.6 | 424.6 | 424.6 | 424.6 | 0.03 | 424.6 | 0.03 | 28.57 |
| 450 | 0.6 | 754.2 | 745.9 | 0.03 | 125.43 | 125.46 | 125.59 | 716.48 | >1000 |
| 100 | 0.7 | 583.5 | 583.5 | 583.5 | 0.01 | 583.5 | 0.01 | 52.72 |
| 110 | 0.7 | 607.1 | 607.1 | 607.1 | 0.01 | 607.1 | 0.01 | 31.23 |
| 270 | 0.7 | 1049.7 | 1049.7 | 1049.7 | 0.02 | 1049.7 | 0.02 | 62.78 |
| 80 | 0.8 | 879.0 | 879.0 | 879.0 | 0.01 | 879.0 | 0.01 | 7.28 |
| 90 | 0.8 | 978.0 | 978.0 | 978.0 | 0.01 | 978.0 | 0.01 | 21.51 |
| 170 | 0.8 | 1580.2 | 1580.2 | 1580.2 | 0.01 | 1580.2 | 0.01 | 485.50 | >1000 |
| 110 | 0.9 | 2666.4 | 2666.4 | 2666.4 | 0.02 | 2666.4 | 0.02 | 21.51 |

Table 2 shows the number of recursive iterations of MECQ and EWCLIQUE. The value of the time [\(\mu s\)] is calculated by CPU time per iteration. The value of the iteration ratio is the ratio of iterations of MECQ and EWCLIQUE. From the result, we confirm that although the computation time of upper bounds of MECQ is longer than EWCLIQUE, the iterations of MECQ is less than our previous algorithm EWCLIQUE. The difference of CPU time can be explained by this. One reason is MECQ calculates upper bounds of equation 13 at once. EWCLIQUE calculates upper bounds in two steps and calculates the sum of two upper bounds.

### 3.2 DIMACS benchmarks

DIMACS is a set of benchmarks for MCP [30]. We used them as benchmarks of MEWCP by giving weights to edges in the same way as [24, 20, 13, 14]. For each edge \((v_i, v_j)\), we gave the weight \(w(v_i, v_j) = (i+j) \mod 200 + 1\).

For the DIMACS benchmarks, the results of computational experiments for previous methods are shown in [14, 13]. Hence we also compared our algorithm with the branch-and-bound algorithm CBQ [14] and mathematical programming formulations proposed in [13]. We quote the results shown in [14, 13] to our
Table 2: Iterations for random graphs

| | iterations | iterations | time [µs] | iterations | time [µs] | iterations | time [µs] |
|---|---|---|---|---|---|---|---|
| MEQ | with PLS | without PLS | | EWCLIQUE | | | |
| | | | | | | | |
| V | d | | | | | | |
| 300 | 0.1 | 312.6 | 357.2 | ε | 1835.1 | ε | 19.46% |
| 350 | 0.1 | 443.4 | 514.5 | ε | 2721.0 | ε | 18.91% |
| 15000 | 0.1 | 56824211.6 | 56842802.3 | 7.09 | 702255007.1 | 0.66 | 8.09% |
| 250 | 0.2 | 974.5 | 1043.8 | ε | 6412.8 | ε | 16.28% |
| 280 | 0.2 | 1424.7 | 1515.2 | ε | 9862.9 | ε | 15.36% |
| 5500 | 0.2 | 75843118.8 | 75882002.3 | 4.22 | 1537843711.0 | 0.29 | 4.93% |
| 200 | 0.3 | 1547.8 | 1644.2 | ε | 14169.8 | ε | 11.60% |
| 250 | 0.3 | 3449.5 | 3896.9 | 2.57 | 34844.2 | 0.29 | 11.18% |
| 2500 | 0.3 | 65558818.2 | 65673225.6 | 3.55 | 1824084938.8 | 0.25 | 3.60% |
| 160 | 0.4 | 2740.6 | 2895.5 | 3.45 | 31813.3 | 0.31 | 9.10% |
| 200 | 0.4 | 5062.9 | 5868.7 | 1.70 | 70701.9 | 0.28 | 8.30% |
| 1400 | 0.4 | 73127999.6 | 73361911.2 | 4.02 | 2993314273.1 | 0.25 | 2.45% |
| 140 | 0.5 | 4511.5 | 5200.1 | 1.92 | 88105.7 | 0.23 | 5.90% |
| 170 | 0.5 | 11351.4 | 11829.1 | 2.54 | 224608.1 | 0.27 | 5.27% |
| 750 | 0.5 | 37702959.2 | 38847817.4 | 4.24 | 2342210511.1 | 0.26 | 1.66% |
| 120 | 0.6 | 8166.3 | 8804.9 | 2.27 | 208388.4 | 0.24 | 1.42% |
| 130 | 0.6 | 11338.8 | 13157.6 | 2.28 | 288494.1 | 0.24 | 4.56% |
| 450 | 0.6 | 27505132.9 | 2849833.9 | 4.34 | 2725764995.6 | 0.26 | 1.04% |
| 100 | 0.7 | 12737.1 | 13792.0 | 2.90 | 437892.1 | 0.25 | 3.15% |
| 110 | 0.7 | 24203.6 | 26032.4 | 2.30 | 950029.7 | 0.25 | 2.74% |
| 270 | 0.7 | 13547235.5 | 14004499.2 | 4.30 | 2141882035.0 | 0.28 | 0.69% |
| 80 | 0.8 | 17659.5 | 20083.0 | 2.49 | 617626.3 | 0.26 | 3.25% |
| 90 | 0.8 | 37616.7 | 45288.8 | 2.65 | 1578193.4 | 0.28 | 2.87% |
| 170 | 0.8 | 79478247.6 | 87159995.9 | 4.26 | 1510657832.0 | 0.32 | 0.58% |
| 70 | 0.9 | 30957.5 | 37263.2 | 2.95 | 2355972.7 | 0.26 | 1.58% |
| 80 | 0.9 | 102852.5 | 111808.0 | 3.33 | 9974393.5 | 0.29 | 1.11% |
| 110 | 0.9 | 5009165.9 | 5402681.4 | 4.31 | 1951189872.0 | 0.30 | 0.28% |

result tables. The CPU used in [14] is Intel® Core™ i7 2.90 GHz. The CPU used in [13] is Intel® Core™ i7 3.40 GHz.

Table 3 shows the CPU time for DIMACS. Table 4 shows the number of recursive iterations of MEQ and EWCLIQUE. Except for hamming8-2 and san200.9.1, our algorithm MEQ obtained optimal solutions in a shorter time than others. For hamming8-2 and san200.9.1, MEQ has usable performance. Only the MEQ with PLS solved all instances in the table in 1000 sec. Although the initial solutions given by PLS did not improve performance in random graphs, they worked well in DIMACS. Especially for benchmark families gen and san, it reduced a lot of computation time.
Table 3: CPU time for DIMACS (sec)

| Algorithm | $|I|$ | 100 | 400 | 200 | 500 | 1000 | 10000 |
|-----------|-----|-----|-----|-----|-----|------|-------|
| Brockhaus | 200 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 |
| hamming-2-2 | 64 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| keller4-2-2 | 171 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 |
| MANN-2 | 45 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| gb-1000-2-2 | 1000 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 |
| gb-1000-5-5 | 1000 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 |
| ggen200 | 200 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| hamming-2-2 | 64 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| keller4-2-2 | 171 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 |
| MANN-2 | 45 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| gb-1000-2-2 | 1000 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 |
| gb-1000-5-5 | 1000 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 |

11
| graph | $|V|$ | $d$ | MECQ with PLS iterations | MECQ without PLS iterations | time [µs] | EWCLIQUE time [µs] | EWCLIQUE ratio |
|-------|-----|-----|-------------------------|---------------------------|--------|------------------|----------|
| brock200.1 | 200 | 0.75 | 607449 | 6103600 | 3.95 | 132864116 | 0.25 | 0.46% |
| brock200.2 | 200 | 0.50 | 14073 | 19906 | 3.01 | 345371 | 0.29 | 5.76% |
| brock200.6 | 200 | 0.61 | 114928 | 130560 | 3.22 | 4282305 | 0.30 | 3.05% |
| brock200.61 | 200 | 0.66 | 287037 | 310735 | 3.77 | 1381425 | 0.35 | 2.25% |
| C125.9 | 125 | 0.90 | 4329351 | 4551897 | 5.45 | | | |
| c-fat200-1 | 200 | 0.08 | 28 | 38 | $\epsilon$ | 632 | $\epsilon$ | 6.01% |
| c-fat200-2 | 200 | 0.16 | 97 | 107 | $\epsilon$ | 6780 | $\epsilon$ | 1.58% |
| c-fat200-3 | 200 | 0.43 | 113 | 141 | $\epsilon$ | 13819445 | 0.54 | 0.00% |
| c-fat500-10 | 500 | 0.37 | 3853 | 3947 | 60.81 | 4282305 | 0.30 | 3.05% |
| c-fat500-2 | 500 | 0.07 | 92 | 126 | $\epsilon$ | 4679 | $\epsilon$ | 2.69% |
| c-fat500-5 | 500 | 0.19 | 324 | 404 | 24.75 | 1227023 | 0.35 | 0.03% |
| DSJC500 | 500 | 1.00 | 2419493 | 2494606 | 4.01 | 200152687 | 0.22 | 1.25% |
| gen200.95 | 200 | 0.90 | 13443980 | | | | |
| hamming-6-2 | 6 | 0.90 | 32 | 48 | $\epsilon$ | 896 | $\epsilon$ | 5.36% |
| hamming-6-4 | 6 | 0.35 | 265 | 265 | $\epsilon$ | 340 | $\epsilon$ | 77.94% |
| hamming-8-2 | 256 | 0.97 | 479056 | 479125 | 42.45 | 65731 | 0.29 | 5.76% |
| hamming-8-4 | 256 | 0.64 | 86587 | 86767 | 6.20 | 2475100 | 0.59 | 3.58% |
| johnson-16-2-4 | 120 | 0.76 | 309697 | 309697 | 0.58 | 1905154 | 0.13 | 16.26% |
| johnson8-2-4 | 70 | 0.77 | 79 | 79 | $\epsilon$ | 150 | $\epsilon$ | 52.67% |
| johnson8-4-4 | 70 | 0.77 | 354 | 361 | $\epsilon$ | 3953 | $\epsilon$ | 9.13% |
| keller4 | 171 | 0.65 | 61141 | 61170 | 3.32 | 2158496 | 0.32 | 1.25% |
| MANNa9 | 45 | 0.93 | 35116 | 35128 | 0.57 | 116041 | 0.17 | 30.27% |
| p-hat1000-1 | 1000 | 0.24 | 562124 | 591829 | 3.33 | 9800185 | 0.30 | 5.98% |
| p-hat1500-1 | 1500 | 0.25 | 4552944 | 4565892 | 4.30 | 106284583 | 0.31 | 4.30% |
| p-hat1300-1 | 300 | 0.24 | 3975 | 4221 | 2.37 | 50151 | 0.20 | 8.42% |
| p-hat1300-2 | 300 | 0.49 | 876123 | 1053858 | 6.60 | 13446327 | 0.32 | 0.78% |
| p-hat1500-1 | 500 | 0.25 | 27485 | 27601 | 2.90 | 408371 | 0.28 | 5.89% |
| p-hat700-1 | 700 | 0.25 | 110426 | 113403 | 3.35 | 167857 | 0.31 | 6.76% |
| san1000 | 1000 | 0.50 | 345909 | 383525 | 50.74 | | | |
| san200.7-1 | 200 | 0.70 | 6694 | 425248 | 4.59 | 38749894 | 0.14 | 0.11% |
| san200.7-2 | 200 | 0.70 | 335623 | 680897 | 5.90 | 4872878 | 0.37 | 1.40% |
| san200.9-1 | 200 | 0.90 | 1637449 | | | | |
| san200.9-2 | 200 | 0.90 | 4463399 | 25206475 | 72.79 | 30316916 | 2.75 | 1.72% |
| san400.7-1 | 400 | 0.50 | 11065 | 68967 | 9.13 | 43132933 | 1.40 | 0.16% |
| san400.7-2 | 400 | 0.70 | 547682 | 53869639 | 166.74 | | | |
| san400.7-3 | 400 | 0.70 | 2841349 | 57665379 | 64.93 | | | |
| san400.7-4 | 400 | 0.70 | 20591310 | 39873392 | 32.99 | | | |
| san200.7 | 200 | 0.70 | 1045157 | 1196523 | 3.63 | 55871960 | 0.33 | 2.14% |
4 Conclusion

We proposed a branch-and-bound algorithm MECQ for MEWCP. Our algorithm calculates upper bounds using vertex coloring. In the vertex coloring procedure, our algorithm assigns edge weights to vertices to calculate upper bounds. By some computational experiments, we confirmed our algorithm is faster than previous ones.

Although modern techniques are proposed for MCP [17, 21], they cannot be directly applied to MEWCP because of edge weights. To apply such techniques to MEWCP, modifying them is a future work.

Recently, quantum annealer is studied to solve NP-hard problems including MCP [8, 3]. Quantum annealer can solve the quadratic unconstrained binary optimization (QUBO) problem. Since quantum annealer solvers are heuristic, efficient exact solvers are required to evaluate them. QUBO can be formulated as MEWCP by the vertex-and-edge-weighted complete graphs where negative weight is allowed. Hence handling negative weight is one future work.

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