Scaling topological charge
in the $CP^3$ spin model

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Abstract
The $CP^3$ spin model is simulated at large correlation lengths in two dimensions. An overrelaxation algorithm is employed which yields reduced critical slowing down with dynamical exponents $z \approx 1$. We compare our results with recent multigrid data on the massgap $m$ and the spin susceptibility and confirm the absence of asymptotic scaling. As a new result we find scaling for the universal topological susceptibility with values extrapolating to $\chi_t/m^2 = 0.156(2)$ in the continuum limit.
1 Introduction

With the discovery of Monte Carlo simulation algorithms with reduced critical slowing down it has become possible to nonperturbatively study a number of two dimensional lattice field theories close to the continuum limit. As the ratio between physical length scales and the cutoff length reaches values of $O(100)$, these high precision simulations obtain a completely new quality compared with what could be done before. Cluster algorithms \[1\] are particularly efficient, but so far their success has remained restricted to $O(n)$ invariant vector models as far as continuous field manifolds are concerned. Recently multigrid techniques \[2\] succeeded in extending the class of models accessible to such high precision study to include the $CP^3$ and $SU(3) \times SU(3)$ systems with standard action \[3\]. In parallel with these nonlocal methods overrelaxation (OR) techniques have been developed for practical applications to spin systems \[4, 5\]. Although OR is expected to only reduce critical slowing down to dynamical exponents $z \approx 1$, thus producing a gain in efficiency proportional to the correlation length, it already makes a large difference in practice. Being a local algorithm it can be implemented very efficiently, with no overhead, and it is actually superior to multigrid for many simulations at intermediate correlation lengths of $O(10)$. In this letter we present OR results on the $CP^3$ model. The purpose is on the one hand to offer an independent check on results obtained with the new multigrid technique in \[3\] and on the other hand to supplement these data by values of the topological susceptibility, a particularly interesting observable in $CP^n$ models\[6\]. In sect.2 we define the model and present the OR algorithm used. The physical results extracted for the $CP^3$ model are discussed in sect.3.

2 $CP^n$ spin model and overrelaxation

The spins of $CP^{n-1}$ models live on the sites $x$ of a hypercubic $L \times L$ lattice and consist of rays in complex $n$ dimensional space. They can be labelled by unit vectors $z(x)$ entering into the standard action only through 1-dimensional

\[1\] During our numerical work we received a copy of \[3\] where \[3\] is refined by higher statistics, and the topological charge is studied, too. The results are now in slight conflict with ours, as we discuss in sect.3.
projectors

\[ P(x) = z(x)z^\dagger(x). \] (1)

The partition function then reads

\[ Z = \int \prod_x d^n z(x) d^n z^\dagger(x) \delta(|z(x)|^2 - 1) \exp \left\{ 2\beta \sum_{x\mu} \text{tr}[P(x)P(x + \hat{\mu})] \right\}, \] (2)

where \( \mu = 0, 1 \) and \( \hat{\mu} \) are unit vectors in the corresponding lattice directions. The fundamental correlation function is given by

\[ C(x) = \langle \text{tr}[P(x)P(0)] \rangle - 1/n, \] (3)

and the magnetic susceptibility \( \chi \) is constructed by summing the argument of \( C \) over all sites. We employ periodic boundary conditions, and the mass \( m \) or correlation length \( \xi = m^{-1} \) are defined through the temporal decay of \( C \) at zero spatial momentum [we use lattice units putting the lattice spacing \( a = 1 \)].

Topological quantities in the \( CP^{n-1} \) model are associated with the “composite” \( U(1) \) gauge field

\[ U_\mu(x) = \frac{z^\dagger(x) \cdot z(x + \hat{\mu})}{|z^\dagger(x) \cdot z(x + \hat{\mu})|}. \] (4)

As one forms plaquette fields \( U_p \) from it and \( F_p \) with

\[ U_p = e^{iF_p}, \quad -\pi < F_p \leq \pi, \] (5)

the topological charge

\[ Q = \frac{1}{2\pi} \sum_p F_p \] (6)

is integer, since the product over all \( U_p \) is unity on the torus. While the average of \( Q \) vanishes for symmetry reasons (which we monitored), the topological susceptibility

\[ \chi_t = \langle Q^2 \rangle / L^2 \] (7)

is expected to be a nontrivial physical quantity of length dimension \(-2\).

Local updates of a \( CP^{n-1} \) spin \( z \) at some site \( x \) have to sample the local Boltzmann factor

\[ p(z) \propto \exp(z^\dagger M(x)z), \] (8)
where the hermitean $n \times n$ matrix $M$ is given in terms of the nearest neighbors,

$$M(x) = 2\beta \sum_{|y-x|=1} P(y).$$

The decisive component in our realization of OR are microcanonical moves of individual spins which leave (8) unchanged. As has already been observed in [7], this can be achieved, if a normalized eigenvector $\psi$ of $M(x)$ is known, by reflecting

$$z \rightarrow -z + 2\psi\psi^\dagger \cdot z.$$  

In the $CP^1$ model, which coincides with the $O(3)$ model, the standard microcanonical reflections are reproduced for either choice of the two possible eigenvectors of $M$. For the case $n = 4$, which we study here, we use the eigenvector belonging to the largest eigenvalue, as that one is easy to obtain.

In the practical realization of hybrid OR we perform $N$ microcanonical sweeps with checkerboard ordering followed by one standard ergodic Metropolis sweep. In the latter, we move the $z$ spins with $U(2)$ rotations of all possible pairs of components of $z$. The eigenvectors of the $M(x)$ are simply constructed approximately by multiplying $M$ a few times on a start vector preconditioned to the sum of the nearest neighbor $z(y)$ spins. Then the reflected spin (10) is first considered as an update proposal, which is followed by an accept/reject step taking into account small changes in energy due to the approximation. With four multiplications we find for all our simulated $\beta$ values between 96% and 99.7% acceptance (growing with $\beta$), which makes this step practically microcanonical. Note that in this way the remaining small error in the eigenvectors does not lead to any systematic bias in the simulation.

The resulting integrated autocorrelation times as functions of the correlation length $\xi$ are summarized in Fig.1. Errors are displayed but mostly fall within the symbols. We employed the tuning strategy of taking the number of microcanonical sweeps $N$ roughly proportional to $\xi$. This was found to be optimal both in an extensive numerical study of the $O(3)$ model and in a rigorous analysis of the Gaussian free field model [8]. More precisely, in the order of growing $\xi$, we took $N = 1, 3, 6, 9, 12, 25$. All $\tau_{int}$ refer to single sweeps, and we did not distinguish OR and Metropolis here. In fact, close to the continuum limit, only the microcanonical OR steps, which are faster, are important. The dynamical exponents $z$ shown in Fig.1 result from fits to the
four largest $\xi$ values. While the zero momentum quantities $\chi$ and $\chi_t$ show the expected $z \approx 1$, the value for the energy seems to be even smaller. As $E$ can be written with a lattice derivative and couples to all nonzero momenta, this behavior cannot be excluded for an integrated autocorrelation time even if the slowest modes evolve with $z \approx 1$. The analogous situation for free fields is discussed in [8]. The topological charge was found to have very short autocorrelations, which coincides with observations in [6, 9].

With the results on exponents $z$ in [3], it is clear that for asymptotically large correlation lengths multigrid becomes superior to OR. Based on CPU times kindly communicated by the authors and on our own, we estimate the crossover in efficiency to occur at $\xi \approx 20-30$. In [4] a version of hybrid
OR was tested which makes microcanonical moves in the same embedded $U(1)$ spin models as are used for multigrid. The crossover occurs at similar $\xi$, and exponents $z$ close to unity are reached. It is interesting that the restriction of microcanonical moves to a small submanifold of spin space seems to be of no disadvantage. This opens the possibility of very simple exactly microcanonical updates in complex field spaces like $SU(3)$.

Let us finally mention that the data shown here represent a total of 166 hours on one XMP processor, of which 2/3 went into the largest lattice.

3 Results for the $CP^3$ model

Apart from [3, 6] a number of other studies of the $CP^3$ model have been attempted recently, among them [4, 5]. They both used cluster algorithms that turned out not to be really efficient enough to obtain high precision data at large correlation length. The reason for the choice of $CP^3$ was in most cases the expectation [10] that pathologies plague lattice definitions of the topological charge for smaller $n$, at least when the standard action and lattice are used.

Results from our runs on $L^2$ lattices are now summarized in the table:

| $\beta$ | $L$  | $\chi$   | $\chi_t \times 10^3$ | $\xi_1$    | $\xi_2$    |
|---------|------|----------|----------------------|------------|------------|
| 2.3     | 20   | 11.09(3) | 11.11(4)             | 2.579(5)   | 2.613(13)  |
| 2.5     | 32   | 26.17(5) | 5.20(2)              | 4.442(7)   | 4.488(16)  |
| 2.7     | 64   | 79.4(2)  | 1.731(8)             | 8.800(21)  | 8.845(46)  |
| 2.8     | 96   | 145.9(5) | 0.893(5)             | 12.640(36) | 12.784(84) |
| 2.9     | 128  | 274.7(1.5)| 0.435(4)            | 18.49(9)   | 18.79(22)  |
| 3.1     | 256  | 942.6(7.5)| 0.103(1)            | 38.37(25)  | 39.32(59)  |

All 1σ errors quoted have been determined by a jackknife binning procedure with various bin lengths and typically a few hundred effectively independent bins. This has been compared with the directly summed autocorrelation functions for $E, \chi, \chi_t$. The two columns for the correlation length $\xi_{1,2}$ require some discussion. We stored all estimates of the two point function at time separations $t = 0, \Delta, 2\Delta, \ldots$ with $\Delta = L/32$ or $\Delta = 1$ for $L = 20$. From two successive such values, time dependent effective masses were determined.
The masses whose time separations selfconsistently embrace $j\xi$ are quoted as $\xi_j$. The table shows that $\xi_1$, for our statistical accuracy, is systematically shorter than $\xi_2$. This is the expected signature of higher mass states in the channel probed by (3). Inspection of $\xi_3$ suggests that, with the present level of errors, $\xi_2$ is an acceptable estimate for the inverse massgap. The ratio $\xi_3/\xi_2$ is consistent with unity at the 1$\sigma$ level, although the mean exceeds unity for all but the last line in the table. We face here the recurring dilemma of having to compromise between systematic and statistical errors when estimating energies. As a consequence, it seems hard to exclude that under a worst case scenario, with two nearby levels in the probed channel, we still underestimate systematic errors. Comparing with cluster algorithms at this point, we see that OR (and multigrid) is inferior not only in speed, but also the lack of improved estimators is a severe disadvantage that is costly to overcome with just statistics. Finite size effects, on the other hand, are expected to be irrelevant with $mL \approx 8$, and this has been tested to some degree in [3]. Note that with the above procedure of extracting masses, and for the chosen $\beta$ values, all scales ($L, \xi, \Delta$) of the various lattices are roughly in the same proportion to one another. Therefore, neglecting scaling violations due to the finite lattice spacing, the ratio $\xi_2/\xi_1$ should be universal to a good approximation. Our data are consistent with this. If we assume it, on the other hand, we can determine the ratio to be about 1.01 from the smaller lattices, and apply this correction to $\xi_1$ on the largest two lattices. This gives a smaller error than taking $\xi_2$, and we feel that the neglect of scaling violations for this 1% correction is no problem. We use these mass estimates in the following analysis.

In [3] asymptotic scaling was found to be violated in a way similar to or even worse than for the $O(n)$ models [11]. We confirm this and plot in Fig.2 the ratio of the mass and the perturbative lattice scale in the two loop approximation

$$\Lambda^{(2)}_L = (\pi \beta)^{1/2} e^{-\pi \beta}. \tag{11}$$

Asymptotic scaling requires it to tend to a constant as $\beta^{-1} \to 0$ with a speed linear in $\beta^{-1}$. Clearly, no sign of this is visible in the (small) range over which we are able to vary $\beta^{-1}$. We also consider a modified bare coupling as discussed in [11]. From the perturbative expansion of the energy

$$E = \langle \text{tr}[P(x)P(x+\hat{\mu})] \rangle = 1 - \frac{3}{4} \beta^{-1} - \frac{3}{16} \beta^{-2} + O(\beta^{-3}) \tag{12}$$
we find that

$$\beta_E = \frac{3}{4(1 - E)}$$

(13)
is a correctly normalized alternative inverse bare (= short distance) coupling. Now $\Lambda^{(2)}_L$ can be reexpanded in $\beta_{E}^{-1}$, and the series is truncated to two loops in this parameter, whose values are taken from the measured energies. This leads to the other data points in Fig.2. They clearly vary less steeply, just like in the $O(n)$ case, but the behavior is not monotonic, and no really convincing extrapolation to $\beta_{E}^{-1} = 0$ is feasible either. The difference between the two sets of data is best taken as evidence of how far away we are from asymptotic behavior in the bare coupling. In [9] a different lattice regularization of the
Figure 3: Scaling behavior of the topological susceptibility in physical units

$CP^3$ model has been employed, and asymptotic scaling seems to be violated less drastically. It would be very interesting to develop OR or multigrid for this action and investigate universality for this case.

The situation with regard to scaling is rather different for the dimensionless product of physical quantities $\chi_4\xi^2$. In Fig.3 it is plotted against $1/\xi^2$, the squared lattice spacing in physical units. We see that data from the four largest lattices fit with a straight line and extrapolate to 0.156(2). The intrinsic length scale for the scaling violations seems to be about three lattice spacings, and for such correlation lengths their expansion in powers of the lattice spacing ceases to be useful. The breakaway is rather abrupt. While for $\xi = 8.8$ the leading term alone works very well, each of the smaller lattices needs a new power of $\xi^{-2}$ to be included to get an acceptable fit. We wish
to remark that an asymptotic expansion in powers $\xi^{-2}$ (neglecting the variation of logarithms) is suggested by renormalized perturbation theory \[12\] and by an effective action picture \[13\]. From the point of view of statistical mechanics, an arbitrary power of $\xi^{-1}$ would seem natural. We found that the four leftmost points in Fig.3 can indeed be extrapolated with anything between $\xi^{-1}$ to $\xi^{-4}$. As our data get very close to the continuum in Fig.3 (compare this with Fig.2!), the extrapolated values are not too different. If one wants to include these uncertainties in a systematic error one could still quote $\chi_t/m^2 = 0.16(1)$ in the continuum.

In \[6\] $\chi_t\xi^2$ is found to fall again at $\beta = 3.1, 3.3$ after rising and agreeing with our data for smaller $\beta$ values. Since we do not have data for $\beta = 3.3$, the only real discrepancy in the measured data hinges upon a smaller value beyond errors for $\xi$ at $\beta = 3.1$ in \[6\]. As mentioned in \[6\] the authors extract correlation lengths by fitting over a range of separations starting at $\xi$. We feel that there may also be a problem with higher states and propose a simple analysis with effective masses. In any case this discrepancy should be resolved in the future, if necessary, with new long runs.

Let us finally compare with results from the large $n$ expansion \[10\]. To leading order one expects

$$\chi_t\xi^2 = \frac{3}{4\pi n} = 0.05968 \ldots \text{ for } n = 4,$$  \hspace{1cm} (14)

from which we differ by a factor of about 2.6. First $1/n$ corrections have been computed in \[14\]. The authors find

$$\chi_t\tilde{\xi}^2 = \frac{1}{2\pi n} \left(1 - \frac{0.38}{n}\right) = 0.03979(1 - 0.095) \text{ for } n = 4,$$  \hspace{1cm} (15)

where $\tilde{\xi}$ is a second moment definition of the correlation length. The authors argue that, although $\tilde{\xi}^2 = 2\xi^2/3$ holds at $n = \infty$, the two lengths differ only by a few % for small $n$ with the connection being nonanalytic in $1/n$. This leads to an even larger discrepancy between our result and large $n$ despite an already small leading correction for $n = 4$. In summary, no quantitative agreement with the $1/n$ expansion at $n = 4$ is found.

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