Rate-Constrained Shaping Codes for Finite-State Channels with Cost

Yi Liu Member, IEEE, Yonglong Li Member, IEEE, Pengfei Huang Member, IEEE, Paul H. Siegel Life Fellow, IEEE

Abstract—Shaping codes are used to generate code sequences in which the symbols obey a prescribed probability distribution. They arise naturally in the context of source coding for noiseless channels with unequal symbol costs. Recently, shaping codes have been proposed to extend the lifetime of flash memory and reduce DNA synthesis time. In this paper, we study a general class of shaping codes for noiseless finite-state channels with cost and i.i.d. sources. We establish a relationship between the code rate and minimum average symbol cost. We then determine the rate that minimizes the average cost per source symbol (total cost). An equivalence is established between codes minimizing average symbol cost and codes minimizing total cost, and a separation theorem is proved, showing that optimal shaping can be achieved by a concatenation of optimal compression and optimal shaping for a uniform i.i.d. source.

I. INTRODUCTION

Shaping codes are used to encode information for use on channels with symbol costs under an average cost constraint. They find application in data transmission with a power constraint, where constellation shaping is achieved by addressing into a suitably designed multidimensional constellation or, equivalently, by incorporating, either explicitly or implicitly, some form of non-equiprobable signaling. More recently, shaping codes have been proposed for use in data storage applications: coding for flash memory to reduce device wear [17], and coding for efficient DNA synthesis in DNA-based storage [14]. Motivated by these applications, [18] investigated information-theoretic properties and design of rate-constrained fixed-to-variable length shaping codes for memoryless noiseless channels with cost and general i.i.d. sources. In this paper, we extend the results in [18] to rate-constrained shaping codes for finite-state noiseless channels with cost and general i.i.d. sources.

Finite-state noiseless channels with cost trace their conceptual origins to Shannon’s 1948 paper that launched the study of information theory [23]. In that paper, Shannon considered the problem of transmitting information over a telegraph channel. The telegraph channel is a finite-state graph and the channel symbols – dots and dashes – have different time durations, which can be interpreted as integer transmission costs. Shannon defined the combinatorial capacity of this channel and gave an explicit formula. He also determined the symbol probabilities that maximize the entropy per unit cost, and showed the equivalence of this probabilistic definition of capacity to the combinatorial capacity. In [4], this result was then generalized to arbitrary non-negative symbol costs. In [11], a new proof technique for deriving the combinatorial capacity was introduced for non-integer costs and another proof of the equivalence of combinatorial and probabilistic definitions of capacity was given. In [2] and [3], a generating function approach was used to extend the equivalence to a larger class of constrained systems.

We refer to the problem of designing codes that achieve the capacity, i.e., that maximize the information rate per unit cost, or, equivalently, that minimize the cost per information bit, as the type-II coding problem. Several researchers have considered this problem. In [4], modified Shannon-Fano codes, based on matching the probability of source and codeword sequences, were introduced, and they were shown to be asymptotically optimal. A similar idea was used in [22], where an arithmetic coding technique was introduced. Several works extend coding algorithms for memoryless channels to finite-state channels. In [2], a finite-state graph was transformed to its memoryless representation $M$ and a normalized geometric Huffman code was used to design a asymptotically capacity achieving code on $M$. In [8], the author extended the dynamic programming algorithm introduced in [7] to finite-state channels. The proposed algorithm finds locally optimal codes for each starting state, but the algorithm does not guarantee global optimality. In [6], an iterative algorithm that can find globally optimal codes was proposed.

The concepts of combinatorial capacity and probabilistic capacity can be generalized to the setting where there is a constraint on the average cost per transmitted channel symbol. The probabilistic capacity was determined in [20] and [9], where the entropy-maximizing stationary Markov chain satisfying the average cost constraint was found. The relationship between cost-constrained combinatorial capacity and probabilistic capacity was also addressed in [10]. The equivalence of the two definitions of cost-constrained capacity was proved in [25], and an alternative proof was recently given in [15], where methods of analytic combinatorics in several variables were used to directly evaluate the cost-constrained combinatorial capacity.

We refer to the problem of designing codes that achieve the cost-constrained capacity as the type-I coding problem. This problem has also been addressed by several authors. In [10], an asymptotically optimal block code was introduced by considering codewords that start and end at the same state. In [12], the authors construct fixed-to-fixed length and variable-to-fixed length codes based on state-splitting methods [1] for magnetic recording and constellation shaping applications. Other constructions can be found in [13], [24] and [26].

In this paper, we address the problem of designing shaping codes for noiseless finite-state channels with cost and general i.i.d. sources. We systematically study the fundamental properties of these codes from the perspective of symbol distribution, average cost, and entropy rate using the theory of finite-state word-valued sources. We derive fundamental bounds relating these quantities and establish an equivalence between optimal type-I and type-II shaping codes. A generalization of Varn code [28] is shown to provide an asymptotically optimal type-II shaping code for uniform i.i.d. sources. Finally, we prove separation theorems showing that optimal shaping for a general i.i.d. source can be achieved by a concatenation of optimal lossless compression with an optimal shaping code for a uniform i.i.d. source.
In Section II, we define finite-state channels with cost and review the combinatorial and probabilistic capacities associated with the type-I and type-II coding problems. In Section III, we define finite-state variable length shaping codes for channels with cost and characterize properties of the codeword process using the theory of finite-state word-valued sources. In Section IV, we analyze shaping codes for a fixed code rate, which we call type-I shaping codes. We develop a theoretical bound on the trade-off between the rate — or more precisely, the corresponding expansion factor — and the average cost of a type-I shaping code. We then study shaping codes that minimize average cost per source symbol (total cost). We refer to this class of shaping codes as type-II shaping codes. We derive the relationship between the code expansion factor and the total cost and determine the optimal expansion factor. In Section V, we consider the problem of designing optimal shaping codes. We prove an equivalence theorem showing that both type-I and type-II shaping codes can be realized using a type-II shaping code for a channel with modified edge cost. Using a generalization of Varn coding [28], we propose an asymptotically optimal type-II shaping code on this modified channel for a uniform i.i.d. source. We then extend our construction to arbitrary i.i.d. sources by introducing a separation theorem, which states that optimal shaping can be achieved by a concatenation of lossless compression and optimal shaping for a uniform i.i.d. source.

Due to space constraints, we must omit many detailed proofs, which can be found in [16]. However, we remark that several new proof techniques are required to extend the results on block shaping codes for memoryless channels in [18] to the corresponding results on finite-state shaping codes for finite-state channels in this paper.

II. Noiseless Finite-State Costly Channel

Let \( \mathcal{H} = (\mathcal{V}, \mathcal{E}) \) be an irreducible finite directed graph, with vertices \( \mathcal{V} \) and edges \( \mathcal{E} \). A finite-state costly channel is a noiseless channel with cost associated with \( \mathcal{H} \), where each edge \( e \in \mathcal{E} \) is assigned a non-negative cost \( w(e) \geq 0 \). We assume that between any pair of vertices \( (v_i, v_j) \in \mathcal{V} \times \mathcal{V} \), there is at most one edge. If not, we can always convert it to another graph that satisfies this condition by state splitting [19]. An example of such a channel is given in Example 1.

**Example 1.** In SLC NAND flash memory, cells are arranged in a grid and programming a cell affects its neighbors. One example of this phenomenon is inter-cell interference (ICI) [27]. Cells have two states: programmed, corresponding to bit 1, and erased, corresponding to bit 0. Due to ICI, programming a cell will damage its neighbors cells. Each length-3 sequence has a cost associated with the damage to the middle bit, as shown in Table I. We can convert this table into a directed graph with vertices \( \mathcal{V} = \{00, 01, 10, 11\} \), as shown in Fig. 1.

| \( e \) | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| \( w(e) \) | 1 | 2 | 4 | 4 | 2 | 3 | 4 | 4 |

**A. Channel capacity with average cost constraint**

Given a length-\( n \) edge sequence \( e^n \), the cost of this sequence is defined as \( W(e^n) = \sum_{i=1}^{n} w(e_i) \), and the average cost of this sequence is defined as \( A(e^n) = \frac{1}{n} W(e^n) \). If \( K_n(W) \) is the number of sequences of length-\( n \) with average cost less than or equal to \( W \), then the combinatorial capacity for a given average cost constraint [25], or cost-constrained capacity, is

\[
C_{I,\text{comb}}(W) \overset{\Delta}{=} \lim_{n \to \infty} \sup \frac{1}{n} \log_2 [K_n(W)].
\]

We also refer to this definition as type-I combinatorial capacity.

Let \( \mathcal{E} \) be a stationary Markov process with entropy rate \( H(\mathcal{E}) \) and average cost \( A(\mathcal{E}) \). The probabilistic capacity for a given average cost constraint \( W \), or cost-constrained probabilistic capacity, is

\[
C_{I,\text{prob}}(W) \overset{\Delta}{=} \sup_{A(\mathcal{E}) \leq W} H(\mathcal{E}).
\]

The maxentropic Markov chain for a given \( W \) was derived in [9] and [20]. The result relies on the one-step cost-enumerator matrix \( D(S) \), where \( S \geq 0 \), with entries

\[
d_{ij}(s) = \begin{cases} 2^{-Sw(e_{ij})} & \text{for edge } e_{ij} \text{ between } (v_i, v_j) \\ 0 & \text{if edge } e_{ij} \text{ doesn’t exist.} \end{cases}
\]

Denote by \( \lambda(S) \) its Perron root and by vectors \( E_L = [\rho_i] \) and \( E_R = [\rho_j] \) the corresponding left and right eigenvectors such that \( E_L E_R = 1 \). Given an average cost constraint \( W(S) \) the maxentropic Markov chain has transition probabilities

\[
P_{ij}(S) = \frac{1}{\lambda(S)} \rho_j 2^{-Sw(e_{ij})} \rho_j \rho_i
\]

such that

\[
W(S) = \frac{1}{\lambda(S)} \sum_{ij} P_{ij} 2^{-Sw(e_{ij})} \rho_j \rho_i,
\]

and the type-I probabilistic capacity of this channel is

\[
C_{I,\text{prob}}(W(S)) = \log_2 \lambda(S) + SW(S).
\]

It was shown in [25], [15] that \( C_{I,\text{comb}}(W) = C_{I,\text{prob}}(W) \).

**B. Channel capacity without cost constraint**

Denote by \( K(W) \) the number of distinct sequences \( e^n \) with cost equal to \( W \). The combinatorial capacity, or the type-II combinatorial capacity, of this channel is defined as

\[
C_{II,\text{comb}} \overset{\Delta}{=} \lim_{W \to \infty} \sup \frac{1}{W} \log_2 K(W).
\]

Similarly, the type-II probabilistic capacity of this channel is defined as

\[
C_{II,\text{prob}} \overset{\Delta}{=} \sup_{W} \frac{H(\mathcal{E})}{A(\mathcal{E})}.
\]

In [11], it was proved that the transition probabilities of the maxentropic Markov process are \( P_{ij}(S_0) \), where \( S_0 \) satisfies \( \lambda(S_0) = 1 \). It was also proved that

\[
C_{II,\text{comb}} = C_{II,\text{prob}} = S_0.
\]

In [2] and [3], the equivalence between \( C_{II,\text{comb}} \) and \( C_{II,\text{prob}} \) was extended to a larger class of constrained systems.
III. FINITE-STATE VARIABLE-LENGTH CODES: A WORD-VALUED SOURCE APPROACH

A. Finite-State Variable-Length Codes

Let $X = X_1X_2\ldots$, where $X_i \sim X$ for all $i$, be an i.i.d. source with alphabet $X = \{x_1, \ldots, x_q\}$. We denote by $P_i$ the probability of symbol $x_i$ and assume that $P_1 \geq P_2 \geq \ldots \geq P_q$. Let $|X|$ denote the size of the alphabet and $P(x^*)$ denote the probability of any finite sequence $x^*$. A finite-state variable-length code on graph $H$ is a mapping $\phi : V \times X^q \rightarrow E^*$. For simplicity and without loss of generality, we assume $q = 1$ and denote $\phi(v_i, x_j)$ as $y_{i,j}$. The encoding process associated with graph $H$ is defined as follows.

Given an encoded sequence $e_1e_2\ldots$, the probability of sequence $e_n^q$ is $Q(e_n^q)$, and the number of occurrences of $e$ in sequence $y_{i,j}$ is $N_e(y_{i,j})$. We can similarly prove that

$$E(N_e) = \sum_{y_{i,j}} N_e(y_{i,j}) \pi_G(v_i)P(x_i).$$

B. Finite-State Word-Valued Source

Introduced in [21], a word-valued source is a discrete random process that is formed by sequentially encoding the symbols of an i.i.d. random process $X$ into corresponding codewords over an alphabet $E$. In this paper, the mapping is a function of both input symbols and the starting state. We refer to the process $E$, formed by an i.i.d. source process and mapping $\phi$, as a finite-state word-valued source.

Given an encoded sequence $e_1e_2\ldots$, the probability of sequence $e_n^q$ is $Q(e_n^q)$, and the number of occurrences of $e$ is $N_e(e_n^q)$. The following properties of the process $E$ are of interest.

- The asymptotic symbol occurrence probability

$$\hat{P}_e = \lim_{n \to \infty} \frac{1}{n}E(N_e(e_n^q)).$$

- The asymptotic average cost

$$A(\phi) = \lim_{n \to \infty} \frac{1}{n}E(W(e_n^q)).$$

- The entropy rate

$$H(E) = \lim_{n \to \infty} \frac{1}{n}H(e_n^q).$$

We can prove the following lemma.

**Lemma 2.** For a finite-state code $\phi : V \times X^q \rightarrow E^*$ associated with graph $F$ and stationary distribution $\{\pi_F(y_{i,j}) = \pi_G(v_i)P(x_j)\}$ such that $E(N_e) < \infty$ for all $e \in E$ and $E(L) < \infty$, the asymptotic probability of occurrence $\hat{P}_e$ is given by

$$\hat{P}_e = E(N_e) \frac{1}{E(L)}.$$  

The asymptotic average cost $A(\phi)$ is

$$A(\phi) = \sum_{e \in E} \hat{P}_e w(e).$$

**Lemma 3.** For a finite-state code $\phi : V \times X^q \rightarrow E^*$ associated with graph $F$ such that $H(X) < \infty$ and $E(L) < \infty$, the entropy rate of the codeword process is

$$H(E) = \frac{qH(X)}{E(L)} = \frac{H(X)}{f}.$$  

Here $f = E(L)/q$ is the expansion factor of the mapping $\phi$.  

**Lemma 1.** The graph $F$ is irreducible. The encoding process is a Markov process associated with $F$ with transition probabilities

$$P(y_{i,j} | y_{i,j-1}) = P(x_i).$$

The stationary distribution of states $\{y_{i,j}\}$ is

$$\pi_F(y_{i,j}) = \pi_G(v_i)P(x_i).$$
C. Asymptotic normalized KL-divergence

Similar to the definition of \( \hat{P}_v \), the asymptotic probability of occurrence of state \( v \in V \) is defined as

\[
\hat{P}_v = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(N_v(E^n)) ,
\]

and we can prove that

\[
\hat{P}_{e_{ij}} = \sum_j \hat{P}_{e_{ij}}.
\]

Consider a finite-order Markov process \( \hat{E} \) associated with graph \( \mathcal{H} \) and transition probabilities

\[
t_H = \frac{\hat{P}_{e_{ij}}}{\hat{P}_{e_{ij}}} .
\]

Denote by \( \hat{P}(e^n_{ij}) \) the probability of a length-\( n \) sequence generated by this process. To measure the difference between \( \hat{E} \) and \( E \), we define the asymptotic normalized KL-divergence as

\[
\lim_{n \to \infty} \frac{1}{n} D(E^n || \hat{E}^n) = \lim_{n \to \infty} \sum_{e^n_{ij} \in E^n} \frac{Q(e^n_{ij})}{\hat{P}(e^n_{ij})} \log_2 \frac{\hat{P}(e^n_{ij})}{Q(e^n_{ij})} .
\]

The relationship between processes \( E \) and \( \hat{E} \) is summarized in the following lemma.

**Theorem 5.** If a cost-0 cycle does not exist, the minimum total cost of a type-II shaping code \( \phi : V \times X^q \to \hat{E}^* \) is given by

\[
T_{\text{min}} = f^* \sum_i \hat{P}^{\ast}_{e_{ij}} w(e_{ij}) = \frac{H(X)}{S^*} ,
\]

where \( \hat{P}^{\ast}_{e_{ij}} = \frac{\hat{P}_{e_{ij}}}{\hat{P}_{e_{ij}}} \) and \( S^* \) is a constant such that \( \lambda(S^*) = 1 \). and \( E_L = [\hat{P}_{e_{ij}} / \rho_i] \) and \( E_R = [\rho_i] \) are the corresponding eigenvectors such that \( E_L E_R = 1 \) and \( S \) is the constant such that

\[
H(\hat{E}) = -\sum_{ij} \hat{P}_{e_{ij}} \log_2 \frac{\hat{P}_{e_{ij}}}{\hat{P}_{e_{ij}}} = H(X) / f .
\]

On a cost-uniform graph, the average cost for any shaping code is a constant \( -\alpha \).

**Remark 2.** When the minimum average cost is achieved, we have \( H(\hat{E}) = H(E) \). As shown in Remark 1, the codeword sequence approximates a finite-order stationary Markov process with transition probabilities \( \{ \hat{P}_{e_{ij}} / \hat{P}_{e_{ij}} \} \).

Using Theorem 5, we study shaping codes that minimize average cost per source symbol, which are closely related to the type-II channel capacity introduced in Section II-B. The total cost of a shaping code is

\[
T(\phi) = \lim_{n \to \infty} e^{-(\lambda(X^n))} = f \sum_{ij} \hat{P}_{e_{ij}} w(e_{ij}) .
\]

We refer to the problem of minimizing the total cost as the type-II shaping problem. Shaping codes that achieve the minimum total cost are referred to as optimal type-II shaping codes. The corresponding optimization problem is as follows.

minimize \( f \sum_i \hat{P}^{\ast}_{e_{ij}} w(e_{ij}) \)

subject to \( H(\hat{E}) \geq H(E) = \frac{H(X)}{f} \)

We have the following theorem that determines the minimum achievable total cost of a shaping code.

**Theorem 6.** If a cost-0 cycle does not exist, the minimum total cost of a type-II shaping code \( \phi : V \times X^q \to \hat{E}^* \) is given by

\[
T_{\text{min}} = f^* \sum_i \hat{P}^{\ast}_{e_{ij}} w(e_{ij}) = \frac{H(X)}{S^*} ,
\]

where \( \hat{P}^{\ast}_{e_{ij}} = \frac{\hat{P}_{e_{ij}}}{\hat{P}_{e_{ij}}} \) and \( \hat{P}^{\ast}_{e_{ij}} \) is a constant such that \( \lambda(S^*) = 1 \), and \( E_L = [\hat{P}_{e_{ij}} / \rho_i] \) and \( E_R = [\rho_i] \) are the corresponding eigenvectors such that \( E_L E_R = 1 \). The corresponding expansion factor \( f^* \) is

\[
f^* = \frac{H(X)}{\sum_i \hat{P}^{\ast}_{e_{ij}}} \frac{\sum_i \hat{P}^{\ast}_{e_{ij}} w(e_{ij})}{S^*} .
\]

If there is a cost-0 cycle in \( \mathcal{H} \), the total cost is a decreasing function of \( f \).

V. OPTIMAL SHAPING CODE DESIGN

In this section, we consider the problem of designing optimal type-I and type-II shaping codes.

A. Equivalence Theorem

We consider the channel with modified edge costs

\[
w^{\prime}(e_{ij}) = -\log_2 \frac{\hat{P}^{\ast}_{e_{ij}}}{\hat{P}^{\ast}_{e_{ij}}} ,
\]

where \( \hat{P}^{\ast}_{e_{ij}} \) and \( \hat{P}^{\ast}_{e_{ij}} \) are given in Theorem 6. It is easy to check that the optimal type-II shaping codes on this channel are also optimal on the original channel, in the sense that the symbol occurrence probabilities \( \{ \hat{P}_{e_{ij}} \} \) are identical on both channels.

We can prove the following lemma.

**Lemma 7.** Given a noiseless finite-state costly channel with edge costs \( \{ w(e_{ij}) \} \). If there is a shaping code \( \phi : V \times X^q \to \hat{E}^* \) such that

\[
\sum_{ij} \hat{P}_{e_{ij}} w(e_{ij}) - H(X) < \delta ,
\]

then

\[
A_{\text{min}}(f) = \sum_{ij} \hat{P}_{e_{ij}} w(e_{ij}) - H(X) = \frac{H(X)}{f} .
\]

Observe that the resulting shaping code is optimal.
where \( w'(e_{ij}) = -\log_2(\hat{p}_{e_{ij}}/\hat{p}_{v_{ij}}) = S^*w(e_{ij}) + \log_2 \rho_i - \log_2 \rho_j \), for some \( \delta > 0 \), then the total cost of this code satisfies
\[
f' \sum_{ij} \hat{p}_{e_{ij}} w'(e_{ij}) - H(X) < \delta.
\]
(38)

The next two theorems establish the equivalence between type-I and type-II shaping codes.

**Theorem 8.** Given a noiseless finite-state costly channel with edge costs \( \{w(e_{ij})\} \). For any \( \gamma, \eta > 0 \), there exists a \( \delta > 0 \) such that if there exists a shaping code \( \phi : V \times \mathcal{X}^q \rightarrow \mathcal{E}^* \) with expansion factor \( f' \) such that
\[
f' \sum_{ij} \hat{p}_{e_{ij}} w'(e_{ij}) - H(X) < \delta,
\]
where \( w'(e_{ij}) = -\log_2(\hat{p}_{e_{ij}}/\hat{p}_{v_{ij}}) = S^*w(e_{ij}) + \log_2 \rho_i - \log_2 \rho_j \), then the average cost of this code is upper bounded by
\[
\sum_{ij} \hat{p}_{e_{ij}} w(e_{ij}) - \frac{H(X)}{S'} < \gamma
\]
and the expansion factor of this code \( f' \) satisfies \( |f' - f| < \eta \).

**Theorem 9.** Given a noiseless finite-state costly channel that does not contain a cost-0 cycle. Denote by \( S^* \) the constant such that \( \lambda(S^*) = 1 \) and by \( f^* \) the expansion factor of an optimal type-II shaping code. For any \( \gamma > 0 \), there exist \( \delta, \eta > 0 \) such that if a shaping code \( \phi : V \times \mathcal{X}^q \rightarrow \mathcal{E}^* \) with expansion factor \( f' \) satisfies
\[
\sum_{ij} \hat{p}_{e_{ij}} w(e_{ij}) - A_{\min}(f') < \delta, \quad |f' - f^*| < \eta,
\]
then the total cost of this code satisfies
\[
f' \sum_{ij} \hat{p}_{e_{ij}} w(e_{ij}) - \frac{H(X)}{S^*} < \gamma.
\]
(42)

**B. Generalized Varn Code**

We now describe an asymptotically optimal type-II shaping code for uniform i.i.d. sources based on a generalization of Varn coding [28]. Given a uniform i.i.d. input source \( \mathcal{X} \), a generalized Varn Code on the noiseless finite-state costly channel is a collection of tree-based variable-length mappings, \( \phi : V \times \mathcal{X}^q \rightarrow \mathcal{E}^* \). Denote by \( \mathcal{V}_k \) the set of codewords starting from state \( v_k \), namely
\[
\mathcal{V}_k = \{ \phi(v_k, x^q) | x^q \in \mathcal{X}^q \}.
\]
(43)

Codewords in \( \mathcal{V}_k \) are generated according to the following steps.
- Set state \( v_k \in \mathcal{V} \) as the root of the tree.
- Expand the root node. The edge costs \( \{w'(e_{ij})\} \) are the modified costs defined in Lemma 7. The cost of a leaf node is the cost of the path from root node to the leaf node.
- Expand the leaf node that has the lowest cost.
- Repeat the previous steps until the total number of leaf nodes \( M \geq |\mathcal{X}^q| \). Delete the leaf nodes that have the largest cost until the number of leaf nodes equals to \( |\mathcal{X}^q| \). Each path from the root node \( v_k \) to a leaf node represents one codeword in \( \mathcal{V}_k \).

The following lemma gives an upper bound on the total cost of a generalized Varn code.

**Lemma 10.** The total cost of a generalized Varn code \( \phi : V \times \mathcal{X}^q \rightarrow \mathcal{E}^* \) is upper bounded by
\[
T(\phi) \leq \frac{\log_2 M}{q} + \frac{\max_i \{w'(e_{ij})\}}{q} \xrightarrow{q \to \infty} \log_2 |\mathcal{X}|.
\]
(44)

**Remark 3.** By extending some leaf nodes to states that are not visited by the original code, we can make graph \( \mathcal{G}_0 \) a complete graph. Then we can choose any state as the starting state. This operation only adds a constant to the cost of a codeword and therefore does not affect the asymptotic performance of the generalized Varn code.

**Example 2.** For the channel introduced in Example 1, the optimal symbol distributions that minimize the total cost are shown in Table II. Based on the distribution, we can design a generalized Varn code on the channel with modified edge costs shown in Table III. The total cost as a function of codebook size is shown in Fig. 3.

**TABLE II: Probabilities for SLC flash channel that minimize total cost.**

| \( u \) | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| \( P_u \) | 0.4318 | 0.1323 | 0.1135 | 0.0593 | 0.1323 | 0.0405 | 0.0593 | 0.0310 |

**TABLE III: Modified cost for flash memory channel.**

| \( u \) | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| \( C(u) \) | 0.3805 | 2.0923 | 0.6068 | 1.5423 | 0.3855 | 2.0923 | 0.6068 | 1.5423 |

**Fig. 3:** The total cost of a generalized Varn code on the SLC flash channel.

**C. Separation Theorem**

We now present a separation theorem for shaping codes. It states that the minimum total cost can be achieved by a concatenation of optimal lossless compression with an optimal shaping code for a uniform i.i.d. source.

**Theorem 11.** Given an i.i.d. source \( \mathcal{X} \) and a noiseless finite-state costly channel with edge costs \( \{w(e_{ij})\} \), the minimum total cost can be achieved by a concatenation of an optimal lossless compression code with a binary optimal type-II shaping code for a uniform i.i.d. source.

**Theorem 12.** Given the i.i.d. source \( \mathcal{X} \), the noiseless finite-state costly channel with edge costs \( \{w(e_{ij})\} \), and the expansion factor \( f \), the minimum average cost can be achieved by a concatenation of an optimal lossless compression code with a binary optimal type-I shaping code for uniform i.i.d. source and expansion factor \( f' = \frac{H(X)}{S'} \).

By Theorem 9, the optimal type-I shaping code for uniform i.i.d. source in Theorem 12 can be replaced by a suitable optimal type-II shaping code for uniform i.i.d. source.
REFERENCES

[1] R. Adler, D. Coppersmith, and M. Hassner, “Algorithms for sliding block
codes,” IEEE Trans. Inf. Theory, vol. IT-29, no. 1, pp. 5–22, Jan. 1983.

[2] G. Böcherer, “Capacity-Achieving Probabilistic Shaping for Noisy and
Noiseless Channels”, Ph.D. dissertation, RWTH Aachen University, 2012.

[3] G. Böcherer, R. Mathar, V. C. da Rocha Jr., and C. Pimentel, “On the
capacity of constrained systems,” in Proc. Int. ITG Conf. Source Channel
Coding (SCC), 2010.

[4] I. Csiszár, “Simple proofs of some theorems on noiseless channels”, Inf.
Contr., vol. 14, pp. 285–298, 1969.

[5] R. Durrett, Probability: Theory and Examples, 3rd ed. Belmont, CA:
Duxbury, 2004.

[6] R. Fujita, K. Iwata, and H. Yamamoto, “An Iterative Algorithm to
Optimize the Average Performance of Markov Chains with Finite States,”
in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Paris, France, 2019, pp.1902
- 1906.

[7] M. J. Golin and G. Rote, “A dynamic programming algorithm for
constructing optimal prefix-free codes with unequal letter costs,” IEEE
Trans. Inf. Theory, vol. 44, no. 5, pp. 1770–1781, Sep. 1998.

[8] K. Iwata and T. Koyama, “A prefix-free coding for finite-state noiseless
channels with small coding delay,” in Proc. 2010 Int. Symp. Inf. Theory
& its Applications, Taichung, Taiwan, Oct. 2010, pp. 473–477.

[9] J. Justesen and T. Høholdt, “Maxentropic Markov chains,” IEEE Trans.
Inf. Theory, vol. IT-30, no. 4, pp. 665–667, Jul. 1984.

[10] R. Karabed, D. L. Neuhoff, A. Khayrallah, The Capacity of Costly
Noiseless Channels, Research report, IBM Research Division, 1988.

[11] A. Khandekar, R. J. McEliece, and E. Rodemich, “The Discrete Noiseless
Channel Revisited,” in Proc. 1999 Int. Symp. Communication Theory and
Applications, pp. 115-137, 1999.

[12] A. S. Khayrallah and D. L. Neuhoff, “Coding for channels with cost
constraints,” IEEE Trans. Inf. Theory, vol. 42, pp. 854-867, May 1996.

[13] V. Y. Krachkovsky, R. Karabed, S. Yang and B. A. Wilson, “On
modulation coding for channels with cost constraints”, in Proc. IEEE
Int. Symp. Inf. Theory, Honolulu, HI, Ju.-Jul. 2014, pp. 421–425.

[14] A. Lenz, Y. Liu, C. Rashtchian, P. H. Siegel, A. Wächter-Zeh, and E.
Yaakobi, “Coding for efficient DNA synthesis,” in Proc. IEEE Int. Symp.
Inf. Theory, Los Angeles, CA, Jun. 2020, pp. 2885-2890.

[15] A. Lenz, S. Melczer, C. Rashtchian, and P. H. Siegel, “Multivariate
Analytic Combinatorics for Cost Constrained Channels and Subsequence
Enumeration”, arXiv:2111.06105 [cs.IT], Nov. 2021.

[16] Y. Liu, “Coding Techniques to Extend the Lifetime of Flash Mem-
ories”, Ph.D. dissertation, University of California, San Diego, 2020.
https://escholarship.org/uc/item/43k8v2hz

[17] Y. Liu and P. Huang, “Rate-constrained shaping codes for structured sources”, IEEE Trans. Inf. Theory, vol. 66, no. 8, pp. 5261–5281, Aug. 2020.

[18] B. H. Marcus, R.M. Roth, and P.H. Siegel, An Introduction to Cod-
ing for Constrained Systems, Lecture Notes, 2001, available online at:
ronny.cswp.cs.technion.ac.il/wp-content/uploads/sites/54/2016/05/chapters1-9.pdf

[19] R. J. McEliece and E. R. Rodemich, “A maximum entropy Markov chain”, in Proc. 17th Conf. Inf. Sciences and Systems, Johns Hopkins University, Mar. 1983, pp. 245-248.

[20] M. Nishiara and H. Morita, “On the AEP of word-valued sources,” IEEE Trans. Inf. Theory, vol. 46, no. 3, pp. 1116–1120, May 2000.

[21] S. A. Savari and R. G. Gallager, “Arithmetic coding for finite-state noiseless
channels,” IEEE Trans. Inf. Theory, vol. 40, no. 1, pp. 100–107, Jan. 1994.

[22] C. E. Shannon, “A mathematical theory of communication, Part I, Part II,” Bell Syst. Tech. J. vol. 27, pp. 379–423, 1948.

[23] J. B. Soriaga and P. H. Siegel, “On distribution shaping codes for partial-
response channels,” in Proc. 41st Annual Allerton Conference on Communication, Control, and Computing, (Monticello, IL, USA), pp. 468-477, October 2003.

[24] J. B. Soriaga and P. H. Siegel “On the design of finite-state shaping encoders
for partial-response channels,” in Proc. of the 2006 Inf. Theory and Application
Workshop (ITA2006), San Diego, CA, USA, Feb. 2006.

[25] J. B. Soriaga and P. H. Siegel, “Near-capacity coding systems for partial-
response channels,” in Proc. IEEE Int. Symp. Inf. Theory, Chicago, IL, USA, p. 267, June 2004.

[26] V. Taranalli, H. Uchikawa and P. H. Siegel, “Error analysis and inter-cell interference mitigation in multi-level cell flash memories,” in Proc. IEEE Int. Conf. Commun.
(iCC), London, Jun. 2015 , pp. 271-276.

[27] B. Varn, “Optimal variable length codes (arbitrary symbol cost and equal code word probability),” Inform. Contr., vol. 19, pp. 289–301, 1971.