Energy bounds of sign-changing solutions to Yamabe equations on manifolds with boundary

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Abstract

We study the Yamabe equation in the Euclidean half-space. We prove that any sign-changing solution has at least twice the energy of a standard bubble. Moreover, a sharper energy lower bound of the sign-changing solution set is also established via the method of moving planes. This bound increases the energy range for which Palais-Smale sequences of related variational problem has a non-trivial weak limit.

1 Introduction

In this paper, we study the following semi-linear equation with Neumann-type boundary condition in the \(n\)-dimensional Euclidean half-space \(\mathbb{R}^n_+ := \{(x_1, ..., x_n) \in \mathbb{R}^n | x_n \geq 0\}, n \geq 3:\)

\[
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^n_+ \\
\frac{\partial u}{\partial \nu} + |u|^\frac{4}{n-2} u = 0 & \text{on } \partial \mathbb{R}^n_+,
\end{cases}
\]

where \(\nu\) is the inward-pointing normal vector. We are interested in the set of sign-changing solutions to equation (1.1).

Equation (1.1) is the boundary version of the classical Yamabe problem, in the scalar-flat case. Known as the generalization of the famous uniformization theorem for Riemann surfaces, the Yamabe problem concerns the existence of conformal metrics with constant scalar curvature on a given smooth compact Riemannian manifold \((M, g)\) without boundary. If one writes the conformal change in the form \(\bar{g} = u^{\frac{4}{n-2}}g\) for some smooth positive function \(u\), the problem is equivalent to proving the existence of positive solutions to the following equation:

\[
\Delta_g u - \frac{n - 2}{4(n - 1)} R_g u + K|u|^\frac{4}{n-2} u = 0 \text{ in } M,
\]

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where $\Delta_g := \text{div}_g \nabla$ is the Laplace-Beltrami operator, $R_g$ denotes the scalar curvature of $(M, g)$ and $K$ here corresponds to the prescribed constant scalar curvature of the conformal metric $\tilde{g}$. Existence of positive solutions to (1.2) has been obtained by the combined work of Aubin [8], Schoen [35], Trudinger [37] and Yamabe [39]. The study of multiplicity of positive solutions has also drawn wide attention ever since it was brought up by Schoen in his Stanford lectures in 1988. He asked if the set of positive solutions to (1.2) is compact in the sense that it is bounded in the $C^{2,\alpha}$-topology for some $0 < \alpha < 1$. This problem has been studied extensively by many mathematicians and was finally solved by Khuri, Marques and Schoen in [27], where they showed that the set is compact if the dimension of the manifold satisfies $3 \leq n \leq 24$. On the other hand, lack of compactness was proved by Brendle [9] and Brendle and Marques [11] when $n \geq 25$.

Less is known about the set of sign-changing solutions to equation (1.2). In a classical paper by Ding [18], he established the existence of infinitely many sign-changing solutions to (1.2) in the Euclidean space, or equivalently, on the standard sphere, with unbounded energy. Using a variation of the method of moving planes, Weth [38] obtained a sharper energy lower bound of the set of sign-changing solutions. For further references on the existence and multiplicity of sign-changing solutions to the Yamabe equation on general Riemannian manifolds, see for example Ammann and Humbert [7], Clapp and Fernández [14], Clapp, Saldaña and Szulkin [15], del Pino, Musso, Pacard and Pistoia [16, 17], Fernández and Petean [23], Henry [26], Musso and Wei [33], Petean [34] and the references within.

In the case where $(M, g)$ has a non-empty boundary, Escobar [20] proposed and studied the following boundary version of the Yamabe problem in the scalar-flat case. Given a smooth compact Riemannian manifold with boundary $\partial M$, is there a conformal metric with zero scalar curvature and constant boundary mean curvature? If we write similarly the conformal change as $\tilde{g} = u^{4n/(n-2)}g$, then the problem is equivalent to proving the existence of positive solutions to the following equation:

$$\begin{cases}
\Delta_g u - \frac{n-2}{4(n-1)} R_g u = 0 & \text{in } M \\
\frac{\partial u}{\partial v_g} - \frac{n-2}{2} h_g u + K|u|^{2/(n-2)} u = 0 & \text{on } \partial M,
\end{cases}$$

(1.3)

where $v_g$ is the inward-pointing normal vector, $h_g$ denotes the boundary mean curvature, and $K$ now corresponds to the prescribed constant mean curvature with respect to $\tilde{g}$. Regularity of solutions to (1.3) was obtained by Cherrier [13]. Existence results were established by Almaraz [2], Brendle and Chen [10], Escobar [20], Marques [30, 31] and Mayer and Ndiaye [32]. For compactness and non-compactness results, see for example Almaraz [3, 4], Almaraz, Queiroz and Wang [6], Felli and Ahmedou [21, 22], Ghimenti and Micheletti [24] and Kim, Musso and Wei [28].

Equation (1.1) is called critical due to the lack of compactness of the corresponding Sobolev trace embedding. As a result, the traditional variational method cannot be applied directly here to prove the existence of solutions.
Solutions to (1.1) are critical points of the functional

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 - \frac{1}{2} \int_{\partial \mathbb{R}^n} |u|^2, \]

where \( 2^* := \frac{2(n-1)}{n-2} \) denotes the critical power of the following Sobolev trace inequality:

\[ \int_{\mathbb{R}^n} |\nabla u|^2 \geq S_n \left( \int_{\partial \mathbb{R}^n} |u|^{2^*} \right)^{\frac{2}{2^*}}. \]  

(1.5)

Here \( S_n \) is the Sobolev best constant.

Define \( D^{1,2}(\mathbb{R}^n) \) to be the completion of smooth functions with compact support with respect to the norm

\[ \|u\|_D = \left( \int_{\mathbb{R}^n} |\nabla u|^2 \right)^{\frac{1}{2}}. \]

Throughout the paper, we use the term standard bubbles to denote the following functions:

\[ U(\epsilon, y; x, t) := \frac{(n-2)\epsilon}{(\epsilon + t)^2 + |x - y|^2} \] for \( x \in \mathbb{R}^{n-1}, t \geq 0, \) \( \epsilon > 0, \) \( y \in \mathbb{R}^{n-1}. \) We also write

\[ B := \{ U(\epsilon, y; x, t) : \epsilon > 0, y \in \mathbb{R}^{n-1} \} \]

as the set of all standard bubbles. We sometimes use \( U(\epsilon, y) \) to denote \( U(\epsilon, y; x, t) \) when there is no possible confusion. It is well known that \( B \) is the set of all the positive solutions to (1.1). Moreover, \( B \) is also the set of least energy critical points of \( I \) with \( I(U(\epsilon, y)) = \frac{1}{2} \int_{\mathbb{R}^n} S_n \left( \int_{\partial \mathbb{R}^n} |u|^{2^*} \right)^{\frac{2}{2^*}} \) for any \( \epsilon > 0, y \in \mathbb{R}^{n-1}. \) Equality holds in (1.5) if and only if \( u \) takes the form of standard bubbles (1.6) up to a constant multiple. See for example Escobar [19].

Via the extension method developed by Caffarelli and Silvestre [12], the existence of a solution \( u \) to (1.1) is equivalent to the existence of a solution \( \bar{u} \) to the following fractional equation:

\[ (-\Delta)^{\frac{1}{2}} \bar{u} = |\bar{u}|^{2^* - 2} \bar{u} \text{ in } \mathbb{R}^{n-1}, \]  

(1.7)

where \( \bar{u} = u|_{\partial \mathbb{R}^n}. \) Just recently, using a similar method as in Ding [18], Abreu, Barbosa and Ramirez [1] established the existence of sign-changing solutions to (1.7) with unbounded energy. In particular, they proved the following:

**Theorem 1.1** ([1]). Equation (1.7) has an unbounded sequence of sign-changing solutions \( \{\bar{u}_k\}_{k \in \mathbb{N}} \subset D^{1,2}(\mathbb{R}^{n-1}) \) when \( n \geq 4. \)
Here, the space \( D^{s,2}(\mathbb{R}^n) \), \( 0 < s < 1 \), is defined to be the completion of smooth functions with compact support with respect to the norm
\[
\|u\|_s := \left( \int_{\mathbb{R}^n} u(-\Delta)^s u \right)^{\frac{1}{2}}.
\]

For more details on the fractional Laplacian operator and related function spaces, see Section 2. Using the extension method, it is easy to see that an unbounded sequence of solutions to (1.7) in \( D^{1,2}(\mathbb{R}^n) \) corresponds to an unbounded sequence of solutions to (1.1) in \( D^{1,2}(\mathbb{R}^n) \).

As a corollary of Theorem 1.1, we can obtain the following existence result of sign-changing solutions to equation (1.1):

**Corollary 1.2.** There exists a sequence of sign-changing solutions \( \{u_k\}_{k \in \mathbb{N}} \subset D^{1,2}(\mathbb{R}^n) \) of (1.1) such that \( \lim_{k \to +\infty} I(u_k) \to +\infty \) when \( n \geq 4 \).

Corollary 1.2 establishes the existence of sign-changing solutions of different energy levels. As we have already mentioned, all positive solutions of (1.1), i.e., all standard bubbles, have the same level of energy \( \frac{1}{n-1} S_n^{n-1} \) and they are the set of least energy solutions. The following proposition states that any sign-changing solution to (1.1) has at least twice the energy of the standard bubbles:

**Proposition 1.3.** Every sign-changing solution \( u \in D^{1,2}(\mathbb{R}^n) \) of (1.1) satisfies \( I(u) > \frac{1}{n-1} S_n^{n-1} \) where \( S_n \) is the Sobolev best constant defined in (1.5).

A natural question is whether the energy lower bound in Proposition 1.3 is sharp. Our next result is inspired by [38] and gives a negative answer to that.

**Theorem 1.4.** There exists \( \gamma > 0 \) such that \( I(u) \geq \frac{1}{n-1} S_n^{n-1} + \gamma \) for any sign-changing solution \( u \in D^{1,2}(\mathbb{R}^n) \) of (1.1).

As an application of Theorem 1.4 let us consider the following equation in a bounded domain \( D \subset \mathbb{R}^n \) with smooth boundary \( \partial D \):

\[
\begin{cases}
\Delta u = 0 & \text{in } D \\
\frac{\partial u}{\partial \nu} + \lambda u + |u|^{2^* - 2} u = 0 & \text{on } \partial D,
\end{cases}
\]  

where \( \lambda \in \mathbb{R} \) and \( \nu \) points inwards. The corresponding functional is
\[
I_{\lambda,D}(u) = \frac{1}{2} \int_D |\nabla u|^2 - \frac{\lambda}{2} \int_D u^2 - \frac{1}{2^*} \int_{\partial D} |u|^{2^*}.
\]

A slight modification of [5, Theorem 1.3] to include sign-changing functions produces the following Struwe-type compactness result:

**Theorem 1.5 ([5]).** Suppose a sequence \( \{u_k\}_{k \in \mathbb{N}} \subset H^1(D) \) is such that \( \{I_{\lambda,D}(u_k)\} \) is bounded and \( \nabla I_{\lambda,D}(u_k) \to 0 \) as \( k \to +\infty \).
Then there exist \( m \in \{0, 1, 2, \ldots\} \), a solution \( u^0 \in H^1(D) \) of (1.8), \( m \) non-trivial solutions \( u^{(i)} \in D^{1,2}(\mathbb{R}^n) \) of (1.4), sequences \( \{R_k^{(i)}\}_n \geq 0 \subseteq \mathbb{N} \) and sequences \( \{x_k^{(i)}\}_n \subseteq \partial D \), \( 1 \leq j \leq m \) such that the whole satisfies the following conditions for \( 1 \leq j \leq m \), possibly after taking a subsequence:

(i) \( R_k^{(i)} \to +\infty \) as \( k \to +\infty \);

(ii) \( x_k^{(i)} \) converges as \( k \to +\infty \);

(iii) \( \left\| u_k - u^0 - \sum_{j=1}^{m} u_k^{(j)} \right\|_{H^1(D)} \to 0 \) as \( k \to +\infty \), where

\[
 u_k^{(j)}(x) = (R_k^{(j)})^\frac{n}{2n-1} u^{(j)}(R_k^{(j)}(x - x_k^{(j)})) .
\]

Moreover,

\[
 I_{\lambda,D}(u_k) - I_{\lambda,D}(u^0) - \sum_{j=1}^{m} I(u_k^{(j)}) \to 0 \quad \text{as} \quad k \to +\infty .
\]

Hence, Theorem 1.5 together with Theorem 1.4 gives the following:

**Corollary 1.6.** Let \( \Lambda := \min\{\gamma, \frac{1}{2(n-1)S_{n-1}}\} \), where \( \gamma \) is obtained in Theorem 1.4 and \( S_n \) is the Sobolev best constant defined as before. If \( \{u_k\}_k \subseteq H^1(D) \) is a sequence with

\[
 \nabla I_{\lambda,D}(u_k) \to 0,
\]

and

\[
 I_{\lambda,D}(u_k) \to c \in \left( 0, \frac{1}{n-1} S_{n-1} + \Lambda \right) \setminus \left\{ \frac{1}{2(n-1)} S_{n-1} - \frac{1}{n-1} S_{n-1} \right\},
\]

as \( k \to +\infty \), then a subsequence of \( \{u_k\}_k \) has a non-trivial weak limit.

**Remark 1.7.** We would like to add that Corollary 1.6 can be generalized to equations similar to (1.3) on compact Riemannian manifolds with boundary.

For the proof of Theorem 1.4 we use a certain variation of the moving planes method. This method was originated from classical papers of Serrin [36] and Gidas, Ni and Nirenberg [25]. It was usually applied to obtain certain symmetry properties of solutions. In our paper, we use this method to rule out the possibility of sign-changing solutions consisting of two bubbles of opposite signs and consequently establish a sharper lower bound of the energy.

The paper is organized as follows. In Section 2, we present some basics about equation (1.1) and the related fractional operator and function spaces. Some elementary lemmas involving the convergence and transformation of certain functions will also be obtained in this section. In Section 3, we first prove Proposition 1.3 using a direct variational argument. The proof of Theorem 1.4 is given in the rest of that section via the method of moving planes.

### 2 Preliminaries

Throughout the paper, we denote \( u^+ := \max\{u, 0\} \) and \( u^- := \max\{-u, 0\} \) for any \( u \in D^{1,2}(\mathbb{R}^n) \) so that \( u = u^+ - u^- \). Define \( C^1_b(\mathbb{R}^n) \) to be the space of all
bounded functions in $C^1(\mathbb{R}^n_+)$ with bounded gradient, endowed with the norm $\|u\|_{C^1(\mathbb{R}^n_+)} = \|u\|_{L^\infty(\mathbb{R}^n_+)} + \|\nabla u\|_{L^\infty(\mathbb{R}^n_+)}. $

We also define $\Gamma$ to be the set of all finite compositions of the following transformations on $u \in D^{1,2}(\mathbb{R}^n_+)$: the translations, rotations, rescalings and Kelvin transformation, respectively

$$(y * u)(x, t) := u(x - y, t), \ y \in \mathbb{R}^{n-1},$$

$$(A * u)(x, t) := u(A^{-1}x, t), \ A \in O(n - 1),$$

$$(\varepsilon * u)(x, t) := \varepsilon^{-\frac{n-2}{2}} u \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right), \ \varepsilon > 0,$$

$$(K * u)(x, t) := |(x, t)|^{2-n} u \left( \frac{x}{|x, t|^{\frac{2-n}{2}}}, \frac{t}{|x, t|^{\frac{2-n}{2}}} \right),$$

for any $x \in \mathbb{R}^{n-1}$ and $t \geq 0$. Here $O(n - 1)$ is the orthogonal group.

It is easy to see that the functional (1.4) is invariant under any transformation of $\Gamma$ and any $T \in \Gamma$ maps solutions of (1.1) to solutions of (1.1). The following identities are easily verified:

$$(y' * U)(\varepsilon, y; x, t) = U(\varepsilon, y + y'; x, t), \ y' \in \mathbb{R}^{n-1},$$

$$(A * U)(\varepsilon, y; x, t) = U(\varepsilon, Ay; x, t), \ A \in O(n - 1),$$

$$(\varepsilon' * U)(\varepsilon, y; x, t) = U(\varepsilon' \varepsilon, \varepsilon' y; x, t), \ \varepsilon' > 0,$$

$$(K * U)(\varepsilon, y; x, t) = U \left( \frac{\varepsilon}{\varepsilon^2 + |y|^2}, \frac{y}{\varepsilon^2 + |y|^2}; x, t \right),$$

where $U(\varepsilon, y; x, t)$ are the standard bubbles defined in (1.6).

The fractional Laplacian in $\mathbb{R}^n_+$ is a non-local pseudo-differential operator of the form:

$$(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n_+} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy$$

$$= C_{n,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n_+ \setminus B_{r}(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,$$

where $0 < s < 1$, $C_{n,s}$ is a positive constant depending only on $s$ and $n$ and $P.V.$ stands for the Cauchy principal value. In order for the integral to make sense, we require $u \in L^{2s} \cap C_{loc}^{1,1}(\mathbb{R}^n)$ where

$$L^{2s}(\mathbb{R}^n) = \{ u \in L^1_{loc}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} < +\infty \}.$$

Let's start by proving some elementary lemmas that we need in the next section.
Lemma 2.1. Suppose \{u_k\}_{k \in \mathbb{N}} \subset D^{1,2}(\mathbb{R}^n_+) is a sequence of solutions to (1.1) with \(\sup_{k \in \mathbb{N}} \|u_k\|_D < +\infty\) and

\[
\lim_{k \to +\infty} \int_{B_{r_k}(x_k) \cap \partial \mathbb{R}^n_+} |u_k|^2 = 0,
\]

for any \(x_k \in \partial \mathbb{R}^n_+, \) and \(r_k > 0\) with \(r_k \to 0.\) Then we have \(\sup_{k \in \mathbb{N}} \|u_k\|_{C^1(\mathbb{R}^n_+)} < +\infty.\)

Proof. In view of \(L^p\)-estimates and Sobolev embedding theories, it is enough to prove that \{u_k\}_{k \in \mathbb{N}} \subset D^{1,2}(\mathbb{R}^n_+) is bounded in \(L^\infty(\mathbb{R}^n_+).\) Let us assume by contradiction that there exists a sequence of points \(x_k \subset \partial \mathbb{R}^n_+\) such that

\[
s_k := \|u_k\|_{L^\infty(\mathbb{R}^n_+)} = |u_k(x_k)| \to +\infty.
\]

Let \(v_k(x) := s_k^{-1} u_k \left( \frac{x}{s_k} - x_k \right).\) By the scaling invariance of equation (1.1) we see that \{v_k\}_{k \in \mathbb{N}} is a sequence of solutions to (1.1) with \(\|v_k\|_{L^\infty(\mathbb{R}^n_+)} = |v_k(0)| = 1\) for all \(k.\) Therefore again in view of \(L^p\)-estimates and Sobolev embedding theories we know that \{v_k\}_{k \in \mathbb{N}} is bounded in \(C^1_0(\mathbb{R}^n_+).\) Let’s assume that up to a subsequence, still denoted as \(v_k, v_k \to v\) in \(C(\bar{B}_1(0) \cap \mathbb{R}^n_+).\) Let \(r_k := s_k^{-1} \to 0\) as \(k \to +\infty,\) we obtain using \(\|v_k\|_{C^1(\mathbb{R}^n_+)} = |v_k(0)| = 1:\)

\[
\int_{B_{r_k}(x_k) \cap \partial \mathbb{R}^n_+} |u_k|^2 = \int_{B_1(0) \cap \mathbb{R}^n_+} |v_k|^2 \to \int_{B_1(0) \cap \mathbb{R}^n_+} |v|^2 > 0,
\]

which is a contradiction to (2.2). This completes the proof. \(\square\)

Now let us use Lemma 2.1 to obtain the following convergence result:

Lemma 2.2. Let \{u_k\}_{k \in \mathbb{N}} \subset D^{1,2}(\mathbb{R}^n_+) be a sequence of solutions to (1.1) such that \(\|u_k - \tilde{w}_k\|_D \to 0\) for some sequence \{\tilde{w}_k\}_{k \in \mathbb{N}} \subset D^{1,2}(\mathbb{R}^n_+) \cap C^1_0(\mathbb{R}^n_+)\) which is bounded in \(C^1_0(\mathbb{R}^n_+).\) Then \(\|u_k - \tilde{w}_k\|_{L^\infty(\mathbb{R}^n_+)} \to 0\) as \(k \to +\infty.\)

Proof. Observe that for any sequence \(x_k \subset \partial \mathbb{R}^n_+, r_k > 0, r_k \to 0\) we have by Sobolev inequality (1.5):

\[
\left( \int_{B_{r_k}(x_k) \cap \partial \mathbb{R}^n_+} |u_k|^2 \right)^{1/2} \leq \|u_k - \tilde{w}_k\|_{L^\infty(\partial \mathbb{R}^n_+)} + \left( \int_{B_{r_k}(x_k) \cap \partial \mathbb{R}^n_+} |\tilde{w}_k|^2 \right)^{1/2} \\
\leq \|u_k - \tilde{w}_k\|_{L^\infty(\partial \mathbb{R}^n_+)} + o(1)\|\tilde{w}_k\|_{L^\infty(\partial \mathbb{R}^n_+)} \\
\to 0.
\]

Thus it follows from Lemma 2.1 that \{u_k\}_{k \in \mathbb{N}} is bounded in \(C^1_0(\mathbb{R}^n_+).\) Assume by contradiction that for a subsequence, there exist points \(z_k \subset \mathbb{R}^n_+\) such that

\[
\liminf_{k \to +\infty} |u_k(z_k) - \tilde{w}_k(z_k)| > 0.
\]

Define \(\bar{u}_k(x) = u_k(x + z_k)\) and \(\bar{w}_k(x) = \tilde{w}_k(x + z_k),\) then it is easy to see that \{\bar{u}_k\}_{k \in \mathbb{N}} and \{\bar{w}_k\}_{k \in \mathbb{N}} are uniformly bounded in \(C^1_0(\mathbb{R}^n_+).\) Moreover, \(\|\bar{u}_k - \bar{w}_k\|_D \to 0\) in \(\mathbb{R}^n_+.\)
We may assume without loss of generality that $\bar{u}_k \to u$ in $C_{loc}(\mathbb{R}^n)$ and $\bar{w}_k \to w$ in $C_{loc}(\mathbb{R}^n)$. Thus we have by (2.3):

$$|u(0) - w(0)| > 0.$$ 

Therefore, we have that

$$\lim_{k \to +\infty} \int_{B(1) \cap \mathbb{R}^n} |\bar{u}_k(x) - \bar{w}_k(x)| = \int_{B(1) \cap \mathbb{R}^n} |u(x) - w(x)| > 0.$$ 

This contradicts the fact that $||\bar{u}_k - \bar{w}_k||_{L^1} \to 0$ in $\mathbb{R}^n$, hence $\bar{u}_k - \bar{w}_k \to 0$ in $L^1_{loc}(\mathbb{R}^n)$, therefore completing the proof of the lemma.

The next lemma establishes certain transformation properties of the standard bubbles $U(\varepsilon, y)$.

**Lemma 2.3.** Let $\varepsilon, \varepsilon' > 0$ and $y, y' \in \mathbb{R}^{n-1}$. Then there exists a transformation $T \in \Gamma$ such that

$$TU(\varepsilon, y) = U(1, z),$$

$$TU(\varepsilon', y') = U(1, -z),$$

where $z = (z_1, 0, \ldots, 0) \in \mathbb{R}^{n-1}$ for some $z_1 > 0$.

**Proof.** We divide the proof into two cases:

Case 1: $\varepsilon = \varepsilon'$. After rescaling we may assume that $\varepsilon = \varepsilon' = 1$. Then we can find suitable translation and rotation that transform $y$ and $y'$ into $z = (z_1, 0, \ldots, 0)$ and $-z = (-z_1, 0, \ldots, 0)$ for some $z_1 > 0$. Thus there exists a transformation $T$ in $\Gamma$ as required.

Case 2: $\varepsilon \neq \varepsilon'$. Using proper rescaling, translation and rotation applied to both $U(\varepsilon, y)$ and $U(\varepsilon', y')$, we may assume that $(\varepsilon, y) = (1, 0)$, $\varepsilon' > 0$ and $y' = (y'_1, 0, \ldots, 0)$ for some $y'_1 \in \mathbb{R}$. Apply a transformation of the form $K \circ \bar{y}$ where $K$ is the Kelvin transformation and $\bar{y} = (\bar{y}_1, 0, \ldots, 0)$ for some $\bar{y}_1 \in \mathbb{R}$, and use the invariance properties of transformations in Section 2 to obtain

$$(K \circ \bar{y}) \ast U(1, 0) = U\left(\frac{1}{1 + \bar{y}^2_1}, \frac{\bar{y}}{1 + \bar{y}^2_1}\right),$$

$$(K \circ \bar{y}) \ast U(\varepsilon', y') = U\left(\frac{\varepsilon'}{(\varepsilon')^2 + (\bar{y}_1 + y'_1)^2}, \frac{\bar{y} + y'_1}{(\varepsilon')^2 + (\bar{y}_1 + y'_1)^2}\right).$$

It is elementary to see that

$$\frac{1}{1 + \bar{y}^2_1} = \frac{\varepsilon'}{(\varepsilon')^2 + (\bar{y}_1 + y'_1)^2}$$

has a real solution $\bar{y}_1 \in \mathbb{R}$. Let $\bar{y}_1$ be such a solution and set $\frac{1}{1 + \bar{y}^2_1} = \varepsilon_1 > 0$. Then from above we know that there exists a transformation $T \in \Gamma$ such that

$$TU(\varepsilon, y) = U(\varepsilon_1, \theta),$$

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\[ TU(\varepsilon', y') = U(\varepsilon_1, \theta'), \]

for some \( \theta, \theta' \in \mathbb{R}^{n-1} \). Now we can proceed as in Case 1 to complete our proof.

\[ \Box \]

### 3 Energy lower bound

We first prove Proposition 1.3 which states that any sign-changing solution to (1.1) has at least twice the energy of the standard bubbles.

**Proof of Proposition 1.3.** Let \( u \) be a sign-changing solution to (1.1). Testing \( u^\pm \) to the equation and using Sobolev trace inequality (1.5), we get

\[
0 = \int_{\mathbb{R}^n_+} (-\Delta u) u^\pm = \int_{\mathbb{R}^n_+} |\nabla u^\pm|^2 - \int_{\partial \mathbb{R}^n_+} |u^\pm|^{2^*'} \\
\geq (1 - S_n^{2^*} \|u^\pm\|_{L^{2^*} (\partial \mathbb{R}^n_+)}^2) \|u^\pm\|_{L^{2^*'} (\partial \mathbb{R}^n_+)}^2,
\]

where \( 2^* = \frac{2(n-1)}{n-2} \) is the critical exponent of the Sobolev trace embedding defined as before. In particular, \( \|u^\pm\|_{L^{2^*} (\partial \mathbb{R}^n_+)} \neq 0 \) and \( \|u^\pm\|_{L^{2^*'} (\partial \mathbb{R}^n_+)} > S_n^{n-1} \), we have

\[
I(u^\pm) = \frac{1}{2(n-1)} \|u^\pm\|_{L^{2^*} (\partial \mathbb{R}^n_+)}^2 \geq \frac{1}{2(n-1)} S_n^{n-1}.
\]

We claim that it must hold:

\[
I(u^\pm) > \frac{1}{2(n-1)} S_n^{n-1}.
\]

If this is not the case, let us assume that

\[
I(u^+) = \frac{1}{2(n-1)} S_n^{n-1},
\]

then \( u^+ \) must be a solution to equation (1.1). It follows from maximum principle that \( u^+ > 0 \) which contradicts the fact that \( u \) is sign-changing. Therefore we must have

\[
I(u) = I(u^+) + I(u^-) > \frac{1}{n-1} S_n^{n-1}.
\]

This ends the proof of Proposition 1.3. \( \Box \)

In the rest of this section, we will prove the sharper energy bound of Theorem 1.4. Let us first introduce some notations. Denote

\[
H_s := \{ x \in \mathbb{R}^n_+ : x_1 > s, \ s \in \mathbb{R} \},
\]
On the other hand, by condition 2 we obtain the following three conditions hold:

\[ L := 2^{-\frac{\pi}{2}} \left( \frac{n-2}{n} S_n \right)^{n-1}, \]  

where \( S_n \) is the Sobolev best constant as before.

The proof of Theorem 1.4 is based on the following two propositions. First, we use a variation of the moving planes method to prove the Proposition 3.1 which establishes the non-existence of sign-changing solutions to (1.1) of a special shape.

**Proposition 3.1.** Let \( H := H_0 \) be as defined before and \( \Omega \) be a non-empty compact subset of \( \partial^s H \). Then there is no sign-changing solution \( u \in D^{1,2}(\mathbb{R}^n) \) to (1.1) such that the following three conditions hold:

1. \( u(x) > u(r_0(x)) \) for \( x \in \Omega \),
2. \( \int_{\partial^s H \setminus (\partial \Omega \cup \partial \Omega)} |u|^{2^*} < L \),
3. \( \inf_{\partial^s H} u > \inf_{\partial^s H \setminus \partial \Omega} u \).

**Proof.** Let us prove by contradiction. Assume that there exists a sign-changing solution \( u \) to equation (1.1) that satisfies conditions 1-3. Consider \( w_s := u - u \circ r_s \), where \( r_s \) is the reflection about the hyperplane \( \{x_1 = s\} \). A direct calculation gives that \( w_s \) satisfies:

\[
\begin{cases}
\Delta w_s = 0 & \text{in } \mathbb{R}^n \\
\frac{\partial w_s}{\partial v} + V_s w_s = 0 & \text{on } \partial \mathbb{R}^n,
\end{cases}
\]  

where \( 0 \leq V_s \leq (2^* - 1)(|u| + |u \circ r_s|)^{2^*-2} \).

Writing \( w_0^- = \max(-w_0, 0) \), we have in particular that

\[
\begin{cases}
\Delta w_0^- = 0 & \text{in } \mathbb{R}^n_+ \\
\frac{\partial w_0^-}{\partial v} + V_0 w_0^- = 0 & \text{on } \partial \mathbb{R}^n_+.
\end{cases}
\]

Since \( w_0^- = 0 \) on \( \partial^s H \) and in a neighbourhood of \( \Omega \), an integration by parts gives

\[
\|w_0^\|_{L^2}^2 = \int_{\partial^s H \setminus \Omega} V_0(w_0^-)^2 \leq \|V_0\|_{L^{-1}(\partial^s H \setminus \Omega)} \|w_0^-\|_{L^{2^*}(\partial^s H \setminus \Omega)}.
\]

On the other hand, by condition 2 we obtain

\[
\int_{\partial^s H \setminus \Omega} |V_0|^{n-1} \leq (2^* - 1)^{n-1} \int_{\partial^s H \setminus \Omega} (|u| + |u \circ r_0|)^{2^*} \\
\leq (2^* - 1)^{n-1} 2^{2^*-1-1} \int_{\partial \mathbb{R}^n_+ \setminus (\partial \Omega \cup \partial \Omega)} |u|^{2^*} \\
< (2^* - 1)^{n-1} 2^{2^*-1-1} L.
\]
So, by the Sobolev trace inequality (3.3),
\[
S_n\|w^*_0\|^2_{L^2(\partial H; \Omega)} \leq \|V_0\|_{L^2(\partial H; \Omega)}\|w^*_0\|^2_{L^2(\partial H; \Omega)} < (2^* - 1)^{n-2}L^2\|w^*_0\|^2_{L^2(\partial H; \Omega)}.
\]
Since \((2^* - 1)^{n-2}L^2 = S_n\) by the definition of \(L\), we get \(w^*_0 \equiv 0\) on \(\partial' H \setminus \Omega\). In particular, \(w_0 \equiv 0\) in \(\partial' H\). Set
\[
\bar{s} = \sup\{s \in \mathbb{R} : w_s \equiv 0\text{ in } \partial' H_s\} > 0.
\]
Since \(u > 0\) somewhere on \(\partial R^*_n\) and \(u \to 0\) as \(|x| \to +\infty\), \(w_s < 0\) for some \(x\) in \(\partial' H_s\) for large \(s\). Thus \(\bar{s} < +\infty\). By continuity \(w_{\bar{s}} \equiv 0\) in \(\partial' H_{\bar{s}}\). Since \(w_{\bar{s}} = 0\) on \(\partial' H_{\bar{s}}\), it follows from the strong maximum principle that either \(w_{\bar{s}} \equiv 0\) or \(w_{\bar{s}} > 0\) in \(\partial' H_{\bar{s}}\). But \(w_{\bar{s}} \equiv 0\) is impossible because condition 3 implies that
\[
\inf_{\partial' H_{\bar{s}}} u \geq \inf_{\partial R^*_n} u > \inf_{\partial R^*_n \setminus \partial' R^*_n} u \geq \inf_{\partial R^*_n \setminus \partial' H_{\bar{s}}} u.
\]
Thus we have \(w_{\bar{s}} > 0\) on \(\partial' H_{\bar{s}}\). Choose a sufficiently large compact set \(\Omega_1 \subset \partial' H_{\bar{s}}\) such that
\[
\int_{\partial R^*_n \setminus (\Omega_1 \cup \Omega_1)} |u|^{2^*} < L.
\]
We can also choose \(s_1 > \bar{s}\) close to \(\bar{s}\) such that \(\Omega_1 \subset \partial' H_{s_1}\), \(w_{s_1} > 0\) in \(\Omega_1\) and
\[
\int_{\partial R^*_n \setminus (\Omega_1 \cup \Omega_1)} |u|^{2^*} < L.
\]
Using an inequality similar to (3.3) and arguing as before we conclude that \(w_{s_1} \equiv 0\) on \(\partial' H_{s_1}\). This contradicts the definition of \(\bar{s}\). Thus we have finished the proof. \(\Box\)

The second part of the proof of Theorem 1.4 is the following approximation result:

**Proposition 3.2.** Let \(|u_k|_{k \in \mathbb{N}}\) be a sequence of sign-changing solutions of (1.1) such that \(I(u_k) \to \frac{1}{n-1}S^{n-1}\). Then there exists a sequence of transformations \(|T_{k}|_{k \in \mathbb{N}}\) in \(\Gamma\) and positive numbers \(|z_{k,1}|_{k \in \mathbb{N}}\) such that, up to a subsequence, \(z_{k,1} \to +\infty\),
\[
\|T_k u_k - U(1, z_k) + U(1, -z_k)\|_D \to 0,
\]
and
\[
\|T_k u_k - U(1, z_k) + U(1, -z_k)\|_{L^\infty(R^*_n)} \to 0,
\]
where \(z_k = (z_{k,1}, 0, \ldots, 0) \in \mathbb{R}^{n-1}\).

**Proof.** For all \(k\) we test \(u_k^+\) to equation (1.1) to obtain
\[
I(u_k^+) = \frac{1}{2(n-1)}\|u_k^+\|_D^2 = \frac{1}{2(n-1)}\|u_k^+\|_{L^2(\partial R^*_n)}^2 \geq \frac{1}{2(n-1)}S^{n-1}.
\]
By the assumption of the proposition we obtain
\[ I(u_k) = I(u_k^+) + I(u_k^-) \to \frac{1}{n-1} s_n^{n-1}. \]
So we must have
\[ I(u_k^\pm) \to \frac{1}{2(n-1)} s_n^{n-1}. \]
Again by equation (3.4) we have
\[ \frac{\|u_k^\pm\|_{D}^2}{\|u_k^\pm\|_{L^2(\mathbb{R}^n)}} = \left(2(n-1)I(u_k^\pm)\right)^{\frac{1}{2}} \to S_n. \]
Thus it follows from classical results by Escobar [19] and Lions [29] that there exist \( \varepsilon_k^{(1)}, \varepsilon_k^{(2)} > 0, y_k^{(1)}, y_k^{(2)} \in \mathbb{R}^{n-1} \) such that
\[
\|u_k^\pm - U(\varepsilon_k^{(1)}, y_k^{(1)})\|_D \to 0, \\
\|u_k^\pm - U(\varepsilon_k^{(2)}, y_k^{(2)})\|_D \to 0.
\]
As a result,
\[
\|u_k - U(\varepsilon_k^{(1)}, y_k^{(1)}) + U(\varepsilon_k^{(2)}, y_k^{(2)})\|_D \to 0.
\]
From Lemma 2.3 there exists a transformation \( T_k \in \Gamma \) such that \( T_k U(\varepsilon_k^{(1)}, y_k^{(1)}) = U(1, z_k) \) and \( T_k U(\varepsilon_k^{(2)}, y_k^{(2)}) = U(1, -z_k) \) for some \( z_k = (z_{1,k}, 0, \cdots, 0) \) where \( z_{1,k} > 0 \). Therefore
\[
\|T_k u_k - U(1, z_k) + U(1, -z_k)\|_D \to 0.
\]
Using Lemma 2.2 we have
\[
\|T_k u_k - U(1, z_k) + U(1, -z_k)\|_{L^\infty(\mathbb{R}^n)} \to 0.
\]
Now let’s prove that \( z_{1,k} \to +\infty \). Suppose not, then up to a subsequence we assume \( z_{1,k} \to z_1 \geq 0 \). If \( z_1 = 0 \), then \( T_k u_k \to 0 \) in \( D^{1,2}(\mathbb{R}^n) \) which is impossible since \( I(T_k u_k) \to \frac{1}{n-1} s_n^{n-1} \). If \( z_1 > 0 \), then \( u = U(1, z_1) - U(1, -z_1) \) is a sign-changing solution to equation (1.1) with \( I(u) = \frac{1}{n-1} s_n^{n-1} \). This contradicts Proposition 1.3. Therefore we must have \( z_{1,k} \to +\infty \). \( \square \)

Now we are ready to prove Theorem 1.4

**Proof of Theorem 1.4** Assume by contradiction that there exists a sequence of solutions \( \{u_k\}_{k \in \mathbb{N}} \subset D^{1,2}(\mathbb{R}^n) \) such that \( I(u_k) \to \frac{1}{n-1} s_n^{n-1} \). Then by Proposition 3.2 there exists a sequence of transformations \( T_k \in \Gamma \) such that
\[
\|T_k u_k - U(1, z_k) + U(1, -z_k)\|_D \to 0,
\]
and
\[
\|T_k u_k - U(1, z_k) + U(1, -z_k)\|_{L^\infty(\mathbb{R}^n)} \to 0, \tag{3.5}
\]

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where \( z_k = (z_{k,1}, 0, \cdots, 0) \in \mathbb{R}^{n-1}, \ z_{k,1} \to +\infty \). Let \( v_k = T_k u_k \). Then using the invariance properties of equation (1.1) we know \( \{v_k\}_{k \in \mathbb{N}} \) are sign-changing solutions to (1.1) as well. Choose \( R > 0 \) large enough such that

\[
\left( \int_{\partial \mathbb{R}^n \setminus B_R(0)} U(1,0)^2 \right)^{\frac{1}{2}} < \frac{L^*}{2},
\]

where \( L \) is defined in (3.1). Since \( \|v_k - U(1,z_k) + U(1,-z_k)\|_{L^2(\partial \mathbb{R}^n)} \to 0 \) and \( z_{k,1} \to +\infty \), by Sobolev inequality (1.5), we know

\[
\int_{\partial \mathbb{R}^n \setminus B_R(z_k) \cup \mathbb{R}_R(z_k)} |v_k|^2 < L,
\]

for \( k \) large enough. Moreover, by (3.5) we have \( v_k > 0 \) in \( B_R(z_k) \cap \partial \mathbb{R}^n \), \( v_k < 0 \) in \( r_0(B_R(z_k)) \cap \partial \mathbb{R}^n \) and \( \inf_{\partial \mathbb{R}^n} v_k > \inf_{\partial \mathbb{R}^n \setminus \partial' H} v_k \) for \( k \) large enough. However it follows from Proposition 3.1 that such solutions \( \{v_k\}_{k \in \mathbb{N}} \) of (1.1) do not exist. This gives a contradiction and finishes the proof of Theorem 1.4.

\[\Box\]

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