Quantum fields in Bianchi type I spacetimes. The Kasner metric.

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Vacuum polarization of the quantized massive fields in Bianchi type I spacetime is investigated from the point of view of the adiabatic approximation and the Schwinger-DeWitt method. It is shown that both approaches give the same results that can be used in construction of the trace of the stress-energy tensor of the conformally coupled fields. The stress-energy tensor is calculated in the Bianchi type I spacetime and the back reaction of the quantized fields upon the Kasner geometry is studied. A special emphasis is put on the problem of isotropization, studied with the aid of the directional Hubble parameters. Similarities with the quantum corrected interior of the Schwarzschild black hole is briefly discussed.

PACS numbers: 04.62.+v, 04.70.-s

I. INTRODUCTION

In this paper we shall consider quantized massive fields in homogeneous anisotropic cosmological Bianchi type I models described by the line element

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2,$$

(1)

where a, b and c, the directional scale factors, are functions of time. A special emphasis will be put on the Kasner spacetime [1][3]

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2,$$

(2)

where the parameters $p_1, p_2$ and $p_3$ satisfy

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$

(3)

These conditions define a Kasner plane and a Kasner sphere (see Fig[1]). The Kasner metric is a solution of the vacuum Einstein field equations, and, because of its simplicity, it is also a solution of the equations of the quadratic gravity. The Kasner conditions exclude the possibility that all three exponents are equal, however, there are configurations for which two of them are the same. We shall call these configurations degenerate. The Kasner solution with $p_1 = -1/3$ and $p_2 = p_3 = 2/3$
has rotational symmetry. The choice of \( p_i \) in the form \((1, 0, 0)\) defines the flat Kasner metric. In the nondegenerate case one can order the parameters \( p_i \) as follows

\[
-\frac{1}{3} < p_1 < 0 < p_2 < \frac{2}{3} < p_3 < 1
\]  

(4)

and the comoving volume element expands in two directions and compresses in one. In what follows however, we shall take \( p_1 \) as an independent parameter and express all remaining quantities in terms of it. It can be done easily since

\[
p_3 = 1 - p_1 - p_2
\]

(5)

and

\[
p_2 = \frac{1}{2} \left( 1 - p_1 - \sqrt{1 + 2p_1 - 3p_1^2} \right)
\]

(6)

for the lower branch, and

\[
p_2 = \frac{1}{2} \left( 1 - p_1 + \sqrt{1 + 2p_1 - 3p_1^2} \right)
\]

(7)

for the upper branch. This terminology is self-explanatory if we consider the parameter \( p_2 \) as a function of \( p_1 \) (see Fig. 2). (For different parametrizations see Ref. [4]). There is an interesting relation between the Kasner metric and the metric that describes the closest vicinity of the Schwarzschild central singularity. Indeed, it can be shown that the Schwarzschild interior approaches the Kasner metric with, say, \( p_1 = -1/3 \) and \( p_2 = p_3 = 2/3 \) as \( r \) goes to 0 (see e.g. Ref. [5–7]).

The physical content of the quantum field theory in curved background is encoded in the regularized stress-energy tensor, \( \langle T^a_b \rangle \), and to certain extend in the field fluctuation, \( \langle \phi^2 \rangle \). In this formulation the spacetime is treated classically whereas the fields propagating on it are quantized. Even with this simplified approach the semiclassical theory should be able to describe quite a number of interesting phenomena, such as vacuum polarization, particle creation and the influence of the quantized field upon the background geometry [8–11]. Moreover, one expects that the results obtained within the framework of the quantum field theory in curved background remain accurate as long as the quantum gravity effects are negligible.

Ideally, the stress-energy tensor should depend functionally on a general metric or at least on a wide class of metrics and be related to the non-local one-loop effective action, \( W_R \), in a standard way. Unfortunately, such calculations are very hard (if not impossible) in practice. Indeed, the solutions of the field equations are not expressible in terms of the known special functions, the formal products of the operator valued distributions have to be regularized and the (perturbative)
FIG. 1: The Kasner sphere and the Kasner plane in the parameter space.

FIG. 2: Two branches of the allowable parameters in the \((p_1, p_2)\) space. The parameter \(p_3\) can be obtained from Eq. 5. The branch points represent degenerate configurations \((-1/3, 2/3, 2/3)\) and \((1, 0, 0)\).
series are divergent. All this makes the exact analytical calculations practically impossible and to circumvent these problems one is forced either to refer to the numerical methods or to make use of some approximations.

In this note we shall follow the latter approach and make use of the local one-loop effective action constructed within the framework of the Schwinger-DeWitt approximation \[ [12] [14], \] (see also Ref. [15]). In cosmology, however, there is another powerful approach to the problem, namely the adiabatic approximation \[ [16] [27] \] and closely related $n$-wave method \[ [15] [28] [30]. \] For the Robertson-Walker spacetime it has been shown that regardless of the chosen method, the results of the calculations are identical. It has been demonstrated that the Schwinger-DeWitt approach and the adiabatic method give precisely the same result in this context for \( 4 \leq D \leq 8 \). (See \[ [31] [33]. \]) Building on this we expect that a similar correspondence also appears in the anisotropic homogeneous cosmologies. Although we do not attempt to perform the full calculations of the stress-energy tensor within the framework of the adiabatic approach and show that the both methods yield the same result, here we will solve somewhat simpler problem and demonstrate that this equality holds for the vacuum polarization, \( \langle \phi^2 \rangle \), of the massive scalar field with arbitrary curvature coupling in the Bianchi type I cosmology. Specifically, it will be shown that the leading and the next-to-leading term of the approximate vacuum polarization calculated within the framework of the adiabatic approximation are precisely the same as the analogous terms calculated with the aid of the Schwinger-DeWitt approach. Interestingly, using our vacuum polarization results, we will be able to calculate the trace of the stress-energy tensor of the conformally coupled massive scalar field.

The paper is organized as follows. In section II we construct the leading and the next-to-leading term of the approximate vacuum polarization of the massive scalar field in the Bianchi type I spacetime. In section III the stress-energy tensor of the scalar, spinor and vector fields in the Kasner spacetime is calculated and discussed, whereas in Sec. IV we study the back reaction of the quantized fields upon the background geometry. To the best of our knowledge the results of Sections II-IV are essentially new. Throughout the paper the natural units are chosen and we follow the Misner, Thorne and Wheeler conventions.

II. VACUUM POLARIZATION

In this section we will be concerned with the neutral massive scalar field

\[
- \square \phi + (m^2 + \xi R) \phi = 0,
\]  
(8)
with the arbitrary curvature coupling, $\xi$, in the anisotropic Bianchi type I spacetime. Our main task is to construct the vacuum polarization.

### A. Adiabatic approximation

To simplify calculations we will introduce a new time coordinate

$$\eta = \int^t V^{-1/3} dt',$$

where $V = abc$, and redefine the field putting

$$f = V^{1/3} \phi. \tag{10}$$

The solution of the transformed equation can be written in the form

$$f = \frac{1}{(2\pi)^3/2} \int d^3 k \left[ A_k f_k(\eta) e^{i k x} + A_k^\dagger f_k^*(\eta) e^{-i k x} \right], \tag{11}$$

where $f_k(\eta)$ satisfies

$$f''_k(\eta) + (\Omega^2 + Q + Q_1) f_k(\eta) = 0 \tag{12}$$

with

$$\Omega^2 = V^{2/3} \left( m^2 + \frac{k_1^2}{a^2} + \frac{k_2^2}{b^2} + \frac{k_3^2}{c^2} \right), \tag{13}$$

$$Q = \frac{1}{3} \left( \frac{1}{3} - \xi \right) \left[ \left( \frac{a'}{a} - \frac{b'}{b} \right)^2 + \left( \frac{a'}{a} - \frac{c'}{c} \right)^2 + \left( \frac{b'}{b} - \frac{c'}{c} \right)^2 \right] \tag{14}$$

and

$$Q_1 = 2 \left( \xi - \frac{1}{6} \right) \left( \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} \right). \tag{15}$$

The $A_k$ obey the standard commutation relations

$$[A_k, A_{k'}] = [A_k^\dagger, A_{k'}^\dagger] = 0 \tag{16}$$

and

$$[A_k, A_{k'}^\dagger] = \delta(k - k'), \tag{17}$$

provided the functions $f_k$ satisfy the Wronskian condition

$$f_k(\eta)f_k'(\eta) - f_k'(\eta)f_k(\eta) = i. \tag{18}$$
The ground state is defined by the relation

\[ A_k |0 \rangle = 0 \]  \hspace{1cm} (19)

and the formal (divergent) expression for the vacuum polarization has the following simple form

\[ \langle \phi^2 \rangle = \frac{1}{(2\pi)^3 V^{2/3}} \int d^3k |f_k|^2. \]  \hspace{1cm} (20)

Now, we demand that the positive frequency functions \( f_k \) can be written in the form

\[ f_k = \frac{1}{(2V^{1/3}W)^{1/2}} \exp \left( -i \int^{\eta} V^{1/3}(s) W(s) ds \right), \]  \hspace{1cm} (21)

where \( W(\eta) \) satisfies the following differential equation

\[ \Omega^2 + Q + Q_1 - V^{2/3}W^2 + \frac{7V'^2}{36V^2} + \frac{V''W'}{6VW} + \frac{3W'^2}{4W^2} - \frac{V''}{6V} - \frac{W''}{2W} = 0. \]  \hspace{1cm} (22)

The solution of this equation can be constructed iteratively assuming that the functions \( W(\eta) \) (we have omitted the subscript \( k \)) can be expanded as

\[ W = \omega_0 + \omega_2 + \omega_4 + ... \]  \hspace{1cm} (23)

with the zeroth-order solution given by

\[ \omega_0 = \left( m^2 + \frac{k_1^2}{a^2} + \frac{k_2^2}{b^2} + \frac{k_3^2}{c^2} \right)^{1/2}. \]  \hspace{1cm} (24)

Now, in order to make our calculations more systematic and transparent, we shall introduce a dimensionless parameter \( \varepsilon \) that will help to determine the adiabatic order of the complicated expressions:

\[ \frac{d}{d\eta} \rightarrow \varepsilon \frac{d}{d\eta} \quad \text{and} \quad W = \sum_{j=0}^{\infty} \varepsilon^{2j} \omega_{2j}, \]  \hspace{1cm} (25)

and, after the substitution of (23) into (22), we collect the terms with the like powers of \( \varepsilon \). It should be noted that \( \omega_2 \) is of the second adiabatic order, \( \omega_4 \) is of the fourth adiabatic order, and so forth. Solving the chain of the algebraic equations of ascending complexity and substituting the thus obtained \( W \) into the formal expression for the vacuum polarization (20) one obtains

\[ \langle \phi^2 \rangle = \frac{1}{2(2\pi)^2 V} \int \frac{d^3k}{\omega_0} - \frac{\varepsilon^2}{2(2\pi)^2 V} \int \frac{d^3k}{\omega_0} \frac{\omega_2}{\omega_0^2} + \frac{\varepsilon^4}{2(2\pi)^2 V} \int \frac{d^3k}{\omega_0} \left( \frac{\omega_2^2}{\omega_0^2} - \frac{\omega_4}{\omega_0^2} \right) \]

\[ - \frac{\varepsilon^6}{2(2\pi)^2 V} \int \frac{d^3k}{\omega_0} \left( \frac{\omega_4^3}{\omega_0^4} - 2\frac{\omega_2\omega_4}{\omega_0^3} + \frac{\omega_6}{\omega_0^4} \right) + ... \]  \hspace{1cm} (26)

The first integral is divergent whereas the second one contains the terms that are divergent. Starting from the third term in the right hand side of the formal expression for the vacuum polarization
the integrals are finite and their computation in the anisotropic case presents no problems. Now, following the standard prescription we subtract from the formal the first two terms, i.e., we subtract all the terms of a given adiabatic order if at least one of them is divergent.

Thus far our analysis has been formal. Now, let us investigate when the adopted method can give well defined functions $\omega_n$. In order to make our analysis more precise let us introduce the directional Hubble parameters

$$H_a = \frac{a'}{a V^{1/3}}, \quad H_b = \frac{b'}{b V^{1/3}} \quad \text{and} \quad H_c = \frac{c'}{c V^{1/3}}$$

and observe that $\omega_0$ is much smaller than $\omega_2$ provided $H_i/m \ll 1$ where $i = a, b, c$. Moreover, in this regime one has

$$\frac{a^{(n)}}{m^naV^{n/3}} \sim \left(\frac{H_a}{m}\right)^n,$$

where $a^{(n)}$ denotes $n$-th derivative of $a(\eta)$. Since similar relations for the remaining directional scale factors $b$ and $c$ hold, the magnitude of the terms at the $n$-th adiabatic order are therefore equal to

$$\left(\frac{H_a}{m}\right)^\alpha \left(\frac{H_b}{m}\right)^\beta \left(\frac{H_c}{m}\right)^\gamma,$$

with $\alpha + \beta + \gamma = n$, where $\alpha, \beta, \gamma$ are nonnegative integers. One expects that in this regime the particle creation can be safely ignored.

Although the $\varepsilon^4$-term (i.e., the leading term) looks innocent it leads to the quite complicated final result. Indeed, integration over angles yields 723 terms of the type

$$F(a, b, c; \xi) \int \frac{dp p^2}{(m^2 + p^2)^r},$$

where

$$p^2 = \frac{k_1^2}{a^2} + \frac{k_2^2}{b^2} + \frac{k_3^2}{c^2}$$

and the functions $F(a, b, c; \xi)$ are constructed from $a, b, c$ and their derivatives. The number of derivatives in each term at each adiabatic order is constant. Finally, after integration over $p$ one obtains 135 terms. Similarly, the next-to-leading term (i.e. the sixth adiabatic order) consists of 761 terms.

As the vacuum polarization of the massive scalar field with the arbitrary coupling in a general Bianchi type I spacetime is rather complicated we will confine ourselves to the minimal and
conformal couplings only. Integrating over \( p \) we find

\[
\langle \phi^2 \rangle^\xi=1/6 = \langle \phi^2 \rangle^\xi=0 + \Delta,
\]

where

\[
\kappa \langle \phi^2 \rangle^\xi=0 = -\frac{2a^{(4)}}{15a} + \frac{2a^{(3)}b'}{45ab} + \frac{2a^{(3)}c'}{45ac} + \frac{13a''b''}{45ab} + \frac{8a''b'c'}{45abc} - \frac{15ab^2}{15ac^2} + \frac{9a^2}{135abc^2} - \frac{135a^3b}{135a^2c} + \frac{14a^3c'}{135a^3c} - \frac{10a^4}{27a^4} + \frac{4a^{(3)}a'}{9a^2} + \frac{4a'a''c'}{45a^2c} - \frac{16a^2a''}{15a^3} + \frac{4b'b''c'}{45b^2c} + \text{cycl},
\]

\[
\kappa \Delta = \frac{a^{(4)}}{9a} - \frac{a^{(3)}b'}{27ab} - \frac{a^{(3)}c'}{27ac} - \frac{8a''b''}{27ab} + \frac{8a''b'c'}{27abc} - \frac{2a''b'^2}{27ab^2} + \frac{2a''c'^2}{27ac^2} - \frac{10a''}{27a^2} + \frac{2a''b'c'}{27a^2bc} + \frac{8a''b'^2}{81a^3b} + \frac{8a''c'^2}{27a^2b^2} + \frac{8a^3c'}{81a^3c} - \frac{25a^4}{81a^4} - \frac{10a^{(3)}a'}{27a^2} - \frac{a'a''c'}{9a^2c} + \frac{8a^2a''}{9a^3} - \frac{b'b''c'}{9b^2c} + \text{cycl},
\]

\( \kappa = 32\pi^2m^2V^{4/3} \) and \( \text{cycl} \) denotes the terms that should be added after performing cyclic transformations \( \{a(\eta), b(\eta), c(\eta)\} \rightarrow \{b(\eta), c(\eta), a(\eta)\} \rightarrow \{c(\eta), a(\eta), b(\eta)\} \). The next-to-leading term is too complicated to be presented here. Finally observe that the thus constructed vacuum polarization can easily be expressed as a function of \( t \) by a simple transformation of the time coordinate.

Now, let us return to the Kasner spacetime and calculate the first two terms of the vacuum polarization. Because of the simplicity of the metric the result is independent of the coupling constant and reads

\[
\langle \phi^2 \rangle = \frac{1}{180\pi^2m^2t^4} \left( p_1^2 - p_3^3 \right) + \frac{1}{16\pi^2m^4t^6} \left( -\frac{8}{35}p_1^2 + \frac{8}{35}p_1^3 + \frac{1}{315}p_1^4 - \frac{2}{315}p_1^5 + \frac{1}{315}p_1^6 \right).
\]

As we shall see the second term in the right-hand-side of the above equation (multiplied by \( m^2 \)) is precisely the the main approximation of the trace of the stress-energy tensor of the conformally coupled massive field taken with the minus sign.

**B. Schwinger-DeWitt approximation**

As is well known the regularized vacuum polarization of the quantized massive scalar field in a large mass limit can be constructed within the framework of the Schwinger-DeWitt method.

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1 The general results can be obtained on request from the author.
Subtracting the terms that are divergent in the coincidence limit of the Schwinger-DeWitt approximation to the Green function, the \( \langle \phi^2 \rangle \) can be written as a series

\[
\langle \phi^2 \rangle = \frac{1}{16\pi^2m^2} \sum_{k=2} a_k \frac{1}{(m^2)^{k-1}(k-2)!},
\]

(36)

where \( a_k \) are the coincidence limit of the Hadamard-DeWitt coefficients. The usual criterion for the validity of the approximation is that the Compton length associated with the massive fields is smaller that the characteristic radius of curvature of the spacetime. Note, that one can safely use the conditions \( H_i/m \ll 1 \) in this regard. The first two coefficients have the form

\[
a_2 = \frac{1}{180} R_{abcd} R^{abcd} - \frac{1}{180} R_{ab} R^{ab} + \frac{1}{6} \left( \frac{1}{5} - \xi \right) R^a_{;a} + \frac{1}{2} \left( \frac{1}{6} - \xi \right)^2 R^2,
\]

(37)

and

\[
a_3 = \frac{1}{7!} b_3^{(0)} + \frac{1}{360} b_3^{(\xi)},
\]

(38)

where

\[
b_3^{(0)} = \frac{35}{9} R^3 + 17 R^a_{;a} R^{;a} - 2 R_{abc} R^{abc} - 4 R_{abc} R^{acb} + 9 R_{abc} e R^{abcd:e} - 8 R_{abc} c R^{ab} - \frac{14}{3} R_{ab} R^{ab} + 24 R_{abc} R^{b} R^{ac} - \frac{208}{9} R_{ab} R_a R^a R^{bc} + \frac{64}{9} R_{ab} R_{cd} R^{abcd} + \frac{16}{3} R_{ab} R_{cde} a R^{bcde} + \frac{80}{9} R_{ab} R_{ef} a e R^{bedf} + \frac{14}{3} R R_{abcd} R^{abcd} + 28 R a R a + 18 R_{ab} R_{a} a b + 12 R e R_{abcd} + \frac{44}{9} R_{abcd} R_{ef} ab R^{edef} \]

(39)

and

\[
b_3^{(\xi)} = -5 R^3 \xi + 30 R^3 \xi^2 - 60 R^3 \xi^3 - 12 R a R^{a} + 30 \xi^2 R_{a} R^{a} - 22 R \xi R_{a} a
\]

\[-6 R_{ab} R_{;a} a b - 4 \xi R_{ab} R^{ab} + 2 R \xi R_{ab} R^{ab} - 2 R \xi R_{abcd} R^{abcd} + 60 R \xi^2 R_{a} a.
\]

(40)

It can be shown that calculating Hadamard-DeWitt coefficients \( a_2 \) and \( a_3 \) for the anisotropic Bianchi type I spacetime described by the line element

\[
ds^2 = -V^{2/3} d\eta^2 + a^2(\eta) dx^2 + b^2(\eta) dy^2 + c^2(\eta) dz^2,
\]

(41)

one obtains for the vacuum polarization precisely the same results as in the adiabatic method. The main approximation \( a_2/16\pi^2m^2 \) equals the fourth-order adiabatic term and \( a_3/16\pi^2m^4 \) is the same as the sixth-order adiabatic term. This one-to-one correspondence should hold for the higher-order terms.
C. Trace of the stress-energy tensor

At first sight it seems that the calculations reported in this section have nothing to do with the stress-energy tensor. However, for the conformally coupled fields there is an interesting relation between the trace of the stress-energy tensor and the vacuum polarization [35]. Indeed, provided \( \xi = 1/6 \) one has

\[
T^a_a = \frac{a_2}{16\pi^2} - m^2 \langle \phi^2 \rangle.
\]  

(42)

Since the leading behavior of the vacuum polarization is proportional to the trace anomaly term, i.e., \( a_2/(4\pi)^2 \), the main approximation of the trace of the stress-energy tensor is simply the next-to-leading term of the vacuum polarization taken with the minus sign. Of course it equals also the minus sixth-order term calculated within the framework of the adiabatic approximation. In the next section we shall demonstrate, among other things, that the trace of the stress-energy tensor calculated from the one-loop effective action is given precisely by (42).

III. STRESS-ENERGY TENSOR OF QUANTIZED MASSIVE FIELDS

The one-loop effective action of the quantized fields in curved spacetime is nonlocal and describes both particle creation and the vacuum polarization. However, when the mass of the field is sufficiently large, the creation of the real particles is suppressed and the effective action becomes local and is determined by the geometry. To be more precise consider a test field of the mass \( m \) and the associated Compton length \( \lambda_C \) in a spacetime with the characteristic radius of curvature \( L \). One expects that if \( \lambda_C/L \ll 1 \) the vacuum polarization part of the effective action dominate, making the expansion in inverse powers of \( m^2 \) possible. Suppose that these assumptions are satisfied, then the effective action is given by the Schwinger-DeWitt expansion [13, 36–38]

\[
W_R = \frac{1}{32\pi^2} \sum_{n=3}^{\infty} \frac{(n-3)!}{(m^2)^{n-2}} \int d^4x \sqrt{|g|} \text{Tr} a_n, \quad (43)
\]

where \( a_n \) are the coincidence limit of the Hadamard-DeWitt coefficients and \( \text{Tr} \) is a supertrace operator. Inspections of Eq. (43) shows that the main approximation requires knowledge of the fourth Hadamard-DeWitt coefficients \( a_3 \). Their exact form is known for the vector, spinor and scalar fields, satisfying respectively

\[
(\delta^a_b \Box - \nabla_b \nabla^a - R^a_b - \delta^a_b m^2)\phi^{(1)} = 0,
\]

(44)

\[
(\gamma^a \nabla_a + m)\phi^{(1/2)} = 0
\]

(45)
and Eq. (8), where $\gamma^a$ are the Dirac matrices. In the main approximation, the effective action of the quantized scalar, spinor and vector fields, after discarding total divergences and expressing the final result in the basis of the curvature invariants, can be written as \cite{38}

$$W^{(1)}_{\text{ren}} = \frac{1}{192\pi^2 m^2} \int d^4 x g^{1/2} \left( \alpha_1 R \Box R + \alpha_2 R_{ab} \Box R^{ab} + \alpha_3 R^3 + \alpha_4 R \nabla R^{ab} 
+ \alpha_5 R_{abcd} R^{abcd} + \alpha_6 R_a \nabla R_b + \alpha_7 R_{ab} \nabla R^{ab} + \alpha_8 R_{ab} \nabla R^{ab} \nabla R^{abcd} 
+ \alpha_9 R_{ab} R^{ab} \nabla R_{cd} \nabla R^{cd} + \alpha_{10} R_{ab} \nabla R^{ab} \nabla R_{cd} \nabla R^{cd} \right)$$

$$= \frac{1}{192\pi^2 m^2} \int d^4 x g^{1/2} \sum_i \alpha_i I_i,$$

(46)

where the numerical coefficients $\alpha_i$ are given in a Table I. The (renormalized) stress-energy tensor

| $s = 0$ | $s = 1/2$ | $s = 1$ |
|-------|--------|--------|
| $\alpha_1$ | $\frac{1}{2} \xi^2 - \frac{1}{5} \xi + \frac{1}{50}$ | $-\frac{3}{250}$ | $-\frac{27}{250}$ |
| $\alpha_2$ | $\frac{1}{140}$ | $\frac{1}{28}$ | $\frac{9}{140}$ |
| $\alpha_3$ | $(\frac{1}{6} - \xi)^3$ | $\frac{1}{864}$ | $\frac{5}{72}$ |
| $\alpha_4$ | $-\frac{1}{30} (\frac{1}{6} - \xi)$ | $-\frac{1}{150}$ | $\frac{31}{60}$ |
| $\alpha_5$ | $\frac{1}{30} (\frac{1}{6} - \xi)$ | $\frac{7}{1430}$ | $\frac{1}{10}$ |
| $\alpha_6$ | $-\frac{1}{540}$ | $-\frac{25}{720}$ | $\frac{52}{63}$ |
| $\alpha_7$ | $\frac{2}{315}$ | $\frac{47}{1260}$ | $-\frac{19}{105}$ |
| $\alpha_8$ | $\frac{1}{1260}$ | $\frac{19}{1260}$ | $\frac{61}{140}$ |
| $\alpha_9$ | $\frac{17}{1560}$ | $\frac{29}{1560}$ | $\frac{67}{2320}$ |
| $\alpha_{10}$ | $-\frac{1}{270}$ | $-\frac{1}{108}$ | $\frac{1}{18}$ |

TABLE I: The coefficients $\alpha_i^{(s)}$ for the massive scalar with arbitrary curvature coupling $\xi$, spinor, and vector field

can be calculated form the standard relation

$$T^{ab} = \frac{2}{\sqrt{g}} \frac{\delta W^{(1)}_{\text{ren}}}{\delta g_{ab}}$$

(47)

and consists of the purely geometric terms constructed from the Riemann tensor, its covariant derivatives and contractions. Each $I_i$, after variations with respect to the metric tensor, leads to the covariantly conserved quantity. The type of the quantum field enters through the spin-dependent coefficients $\alpha_i$. The resulting stress-energy tensor is a linear combination of almost 100 local geometric terms. (Their actual number depends on the simplification strategies and identities satisfied by the Riemann tensor used during the calculation). The general formulas describing the stress-energy tensor have been given in Refs. \cite{39, 40}.  

Sometimes it is more efficient to adopt a less ambitious approach and instead of using general formulas of Refs. [40] focus on the effective action for a given line element and make use of the Lagrange-Euler equations. Below we shall briefly discuss how it can be achieved for the Kasner metric. First, observe that we can solve a more general problem and calculate components of the stress-energy tensor of the quantized fields in a general Bianchi type I spacetime with 
\[ g_{00} = -f(t). \]
(Henceforth, for calculational convenience, we slightly abuse our notation and put \( g_{11} = a(t), \) \( g_{22} = b(t) \) and \( g_{33} = c(t) \)). Since the effective action is invariant under the cyclic transformation
\[
\{a(t), b(t), c(t)\} \rightarrow \{b(t), c(t), a(t)\} \rightarrow \{c(t), a(t), b(t)\}
\]
it suffices to calculate \( T^0_0 \) and \( T^1_1 \). The remaining components can be obtained using this chain of transformation in \( T^1_1 \). Indeed, under the action of the cyclic transformation (48) one has
\[
T^1_1 \rightarrow T^2_2 \rightarrow T^3_3.
\]
For the first spatial component the Lagrange-Euler equations give
\[
T^1_1 = \frac{a}{96\pi^2 m^2 \sqrt{-g}} \left[ \frac{\partial L}{\partial a} + \sum_{k=1}^{n} (-1)^{k+1} \frac{d^k}{dt^k} \left( \frac{\partial L}{\partial a^{(k)}} \right) \right],
\]
where \( L \) is the Lagrange function density, \( n \) is the maximal order of derivatives of the function \( a(t) \) and \( a^{(k)} = \frac{d^k a}{dt^k} \). The component \( T^0_0 \) can be constructed in a similar way.

One can also start with the simplified line element with \( f(t) = 1 \) and using (50) calculate only \( T^1_1 \). The remaining spatial components can be obtained making use of the cyclic transformation whereas the time component of the stress-energy tensor can be constructed from the \( \nabla_a T^a_b = 0 \), which in the case at hand reduces to
\[
T^0_0 \left( \frac{\dot{a}}{2a} + \frac{\dot{b}}{2a} + \frac{\dot{b}}{2a} \right) - T^1_1 \frac{\dot{a}}{2a} - T^2_2 \frac{\dot{b}}{2b} - T^3_3 \frac{\dot{c}}{2a} + T^0_0 = 0.
\]
The integration constant should be put to zero at the end of the calculations. Since the Ricci tensor (and Ricci scalar) vanish for the Kasner metric the calculations can be substantially simplified from the very beginning. For example, the nonvanishing terms in \( \delta I_5 / \delta g_{ab} \) and \( \delta I_8 / \delta g_{ab} \) come solely from the variations of the Ricci tensor. Regardless of the choice of method the obtained results must be, of course, the same.

The stress-energy tensor of the quantized massive fields in the general Bianchi type I spacetime is very complicated and for obvious reasons it will be not presented here. We only remark that the number of terms in \( T^0_0 \) with the coefficients \( c_i \) unspecified is 1431 whereas the number of terms in the each spatial component of the stress-energy tensor is 1709. For \( a(t) = b(t) = c(t) \) the
result reduces to the well-known $T^b_a$ obtained in the spatially flat Robertson-Walker spacetime. Although the general stress-energy tensor in the Bianchi type I spacetime is quite complicated, the final result for the Kasner metric is not. Indeed, because of spatial homogeneity there are massive simplifications and the resulting $T^b_a$ consists of small number of terms and has a simple form that can schematically be written as follows:

$$T^{(i)b}_a = \frac{1}{96\pi^2 m^2 t^6} \text{diag}[T^{(i)}_0, T^{(i)}_1, T^{(i)}_2, T^{(i)}_3]_a$$  \hspace{1cm} (52)$$

where each $T^{(i)}_k$ is a sixth-order polynomial of $p_1$ and $i$ refers either to the upper branch or the lower branch in the parameter space. Since there is no danger of confusion the spin index is omitted.

### A. Massive scalar fields

First let us consider the quantized massive scalar field. Making the substitution \{a(t), b(t), c(t)\} $\to$ \{t^{2p_1}, t^{2p_2}, t^{2p_3}\} in the general formulas in the Bianchi type I spacetime, after some algebra, one obtains (for the lower \(l\) and upper \(u\) branches)

$$T^{(u)0}_0 = T^{(l)0}_0 = \frac{1}{96\pi^2 m^2 t^6} \left[ \frac{p_1^6}{105} + \frac{2p_1^5}{35} - \frac{11p_1^4}{21} - \frac{152p_1^3}{105} + \frac{40p_1^2}{21} + \left( \frac{32p_1}{15} + \frac{32p_1^3}{5} - \frac{128p_1^2}{15} \right) \xi \right],$$  \hspace{1cm} (53)$$

$$T^{(u)1}_1 = T^{(l)1}_1 = \frac{1}{96\pi^2 m^2 t^6} \left[ \frac{p_1^6}{105} + \frac{2p_1^5}{35} - \frac{11p_1^4}{21} + \frac{152p_1^3}{105} + \frac{40p_1^2}{21} + \left( \frac{32p_1}{15} + \frac{32p_1^3}{5} - \frac{128p_1^2}{15} \right) \xi \right],$$  \hspace{1cm} (54)$$

$$T^{(u)2}_2 = T^{(l)3}_3 = \frac{1}{96\pi^2 m^2 t^6} \left[ \frac{p_1^6}{105} - \frac{2p_1^5}{35} - \frac{4\beta p_1^4}{105} + \frac{29p_1^3}{105} + \frac{16\beta p_1^2}{105} - \frac{244p_1^3}{105} \right]$$

$$+ \frac{4\beta p_1^2}{21} + \frac{44p_1^2}{21} + \frac{32p_1^3}{3} - \frac{16\beta p_1^2}{5} - \frac{48p_1^2}{5},$$  \hspace{1cm} (55)$$

and

$$T^{(u)3}_3 = T^{(l)2}_2 = \frac{1}{96\pi^2 m^2 t^6} \left[ \frac{p_1^6}{105} - \frac{2p_1^5}{35} + \frac{4\beta p_1^4}{105} + \frac{29p_1^3}{105} + \frac{16\beta p_1^2}{105} - \frac{244p_1^3}{105} \right]$$

$$- \frac{4\beta p_1^2}{21} + \frac{44p_1^2}{21} + \frac{32p_1^3}{3} + \frac{16\beta p_1^2}{5} - \frac{48p_1^2}{5},$$  \hspace{1cm} (56)$$

where $\beta = \sqrt{1 + 2p_1 - 3p_1^2}$.

Inspection of the above formulas reveals some interesting general features: (i) for any allowable $p_1$ both $T^{0}_0$ and $T^{1}_1$ does not depend on the branch, (ii) the differences appear only for the remaining spatial components and they are related to the change of the sign of $\beta$, (iii) despite the dependence of
the stress-energy tensor on the branch there are only four independent components as $T_2^{(u)2} = T_3^{(l)3}$ and $T_3^{(u)3} = T_2^{(l)2}$, that is in concord with the symmetries of the background geometry, (iv) only $I_5, I_8, I_9$ and $I_{10}$ contribute to the final result, and, consequently, (v) the stress-energy tensor depends linearly on $\xi$. One expects, that except (v) the features (i)-(iv) are independent of the spin of the quantized field. The results of the calculations for the massive scalar field that are plotted in Figs. 3-6 reveal quite complicated (oscillatory) behavior of the $T_i$ on the $(p_1, \xi)$-space. Here we shall focus on $\xi = 0$ (the minimal coupling) and $\xi = 1/6$ (the conformal coupling), i.e., we restrict ourselves to its physical values. First observe that the components of the stress-energy tensor are either negative or positive, and they vanish for the degenerate configurations for which one of the $p_i$ equals 1. The basic properties are listed in the tables [II] and [III].

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\xi = 0$ & $T_0^0 \leq 0$ & $T_1^1 \geq 0$ & $T_2^2 \geq 0$ & $T_3^3 \geq 0$ \\
\hline
$\xi = 1/6$ & $T_0^0 \leq 0$ & $T_1^1 \geq 0$ & $T_2^2 \geq 0$ & $T_3^3 \geq 0$ \\
\hline
\end{tabular}
\caption{The sign of the components of the stress-energy tensor of the massive scalar field in the Kasner spacetime. The calculations have been carried out for the lower branch.}
\end{table}

As the functions $T_i$ have a simple structure

$$T_i(p_1) = p_1^2(p_1 - 1)W_3(p_1), \quad (57)$$

where $W_3(p_1)$ is a third-order polynomial, the first local extremum of the stress-energy tensor is always at $p_1 = 0$, whereas location of the second one (on the lower branch) is tabulated in Table [III]. For the upper branch the results for $T_2^2$ and $T_3^3$ should be interchanged. It should be noted that for the conformal coupling the second extremum is always at $p_1 = 2/3$, i.e., for the degenerate configuration.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
 & $T_0^0$ & $T_1^1$ & $T_2^2$ & $T_3^3$ \\
\hline
$\xi = 0$ & $p_1 = 2/3$ & $p_1 = 0.6785$ & $p_1 = 2/3$ & $p_1 = 0.6547$ \\
$\xi = 1/6$ & $p_1 = 2/3$ & $p_1 = 2/3$ & $p_1 = 2/3$ & $p_1 = 2/3$ \\
\hline
\end{tabular}
\caption{The extrema of the components of the stress-energy tensor of the massive scalar field. The calculations have been carried out for the lower branch.}
\end{table}
FIG. 3: $T_1 = 96\pi^2 m^2 t^6 T_0^0$ plotted as a function of $p_1$ and $\xi$.

FIG. 4: $T_1 = 96\pi^2 m^2 t^6 T_1^1$ plotted as a function of $p_1$ and $\xi$. 
FIG. 5: $T_2 = 96\pi^2 m^2 t^6 T_2^2$ plotted as a function of $p_1$ and $\xi$.

FIG. 6: $T_3 = 96\pi^2 m^2 t^6 T_3^3$ plotted as a function of $p_1$ and $\xi$. 
B. Massive spinor and vector fields

Similar calculations can be carried out for the massive fields of higher spin. First, let us consider the spinor field. The stress-energy tensor in the Kasner spacetime has a simple form

\[ T^{(u)0}_0 = \frac{1}{96\pi^2 m^2 t^6} \left( \frac{2p_1^6}{21} - \frac{4p_1^5}{21} + \frac{2p_1^4}{21} + \frac{16p_1^3}{105} - \frac{16p_1^2}{105} \right), \tag{58} \]

\[ T^{(u)1}_1 = \frac{1}{96\pi^2 m^2 t^6} \left( -\frac{2p_1^6}{105} - \frac{4p_1^5}{35} - \frac{2p_1^4}{105} - \frac{32p_1^3}{105} + \frac{16p_1^2}{35} \right), \tag{59} \]

\[ T^{(u)2}_2 = \frac{1}{96\pi^2 m^2 t^6} \left( -\frac{2p_1^6}{105} + \frac{4p_1^5}{35} + \frac{8\beta p_1^4}{105} - \frac{2p_1^4}{35} - \frac{8\beta p_1^3}{35} - \frac{24p_1^3}{105} + \frac{16\beta p_1^2}{105} + \frac{64p_1^2}{105} \right), \tag{60} \]

and

\[ T^{(u)3}_3 = \frac{1}{96\pi^2 m^2 t^6} \left( -\frac{2p_1^6}{105} + \frac{4p_1^5}{35} - \frac{8\beta p_1^4}{105} - \frac{2p_1^4}{105} - \frac{8\beta p_1^3}{35} - \frac{24p_1^3}{105} - \frac{16\beta p_1^2}{105} + \frac{64p_1^2}{105} \right). \tag{61} \]

Similarly for the vector field one obtains

\[ T^{(u)0}_0 = \frac{1}{96\pi^2 m^2 t^6} \left( -\frac{p_1^6}{7} + \frac{2p_1^5}{7} - \frac{p_1^4}{7} - \frac{16p_1^3}{21} + \frac{16p_1^2}{21} \right), \tag{62} \]

\[ T^{(u)1}_1 = \frac{1}{96\pi^2 m^2 t^6} \left( \frac{p_1^6}{35} + \frac{6p_1^5}{35} + \frac{52p_1^4}{105} + \frac{72p_1^3}{35} - \frac{296p_1^2}{105} \right), \tag{63} \]

\[ T^{(u)2}_2 = \frac{1}{96\pi^2 m^2 t^6} \left( \frac{p_1^6}{35} - \frac{6p_1^5}{35} - \frac{4\beta p_1^4}{35} - \frac{52p_1^3}{21} - \frac{64\beta p_1^2}{105} + \frac{388p_1^2}{105} - \frac{52\beta p_1^1}{105} - \frac{116p_1^1}{35} \right), \tag{64} \]

and

\[ T^{(u)3}_3 = \frac{1}{96\pi^2 m^2 t^6} \left( \frac{p_1^6}{35} - \frac{6p_1^5}{35} + \frac{4\beta p_1^4}{35} - \frac{52p_1^3}{21} - \frac{64\beta p_1^2}{105} + \frac{388p_1^2}{105} + \frac{52\beta p_1^1}{105} - \frac{116p_1^1}{35} \right). \tag{65} \]

The results for the lower branch can be obtained, as before, form the conditions \( T^{(l)0}_0 = T^{(u)0}_0 \), \( T^{(l)1}_1 = T^{(u)1}_1 \), \( T^{(l)2}_2 = T^{(u)2}_2 \) and \( T^{(l)3}_3 = T^{(u)3}_3 \). The components of \( T^b_a \) are still of the form given by Eq. (52) and the functions \( T_i \) are plotted in Figs. 7 and 8. A comparison of the results shows that the components of the stress-energy tensor change their sign with a change of spin and vanish for the degenerate configurations of the type \((1, 0, 0)\). The energy density \( \rho = -T^{(u)0}_0 \) is nonnegative for the spinor field whereas it is negative (or zero) for the vector field. Moreover, the quantum
FIG. 7: The functions $T_0$ (dashed curve), $T_1$ (dotted curve), $T_2$ (dot-dashed curve) and $T_3$ (solid curve) plotted as a function of $p_1$. $T_i$ are defined as $T_i = 96\pi^2 m^2 \ell^6 T_i^{(i)}$ (no summation) and $s = 1/2$.

| $s = 1/2$ | $T_0^0$ | $T_1^1$ | $T_2^2$ | $T_3^3$ |
|-----------|---------|---------|---------|---------|
| $p_1 = 2/3$ | $0.7067$ | $p_1 = 2/3$ | $0.6250$ | $p_1 = 0.6438$ |
| $s = 1$   | $p_1 = 2/3$ | $0.6890$ | $p_1 = 2/3$ | $0.6438$ |

TABLE IV: The extrema of the components of the stress-energy tensor of the massive spinor and vector fields. The calculations have been carried out for the lower branch.
FIG. 8: The functions $T_0$ (dashed curve), $T_1$ (dotted curve), $T_2$ (dot-dashed curve) and $T_3$ (solid curve) plotted as a function of $p_1$. $T_i$ are defined as $T_i = 96\pi^2 m^2 T^i (\text{no summation})$ and $s = 1$.

where $T_{a}^{(cl)b}$ is the classical part of the total stress-energy tensor. The resulting system of differential equations has to be solved self-consistently for the quantum-corrected metric. To simplify our discussion we shall assume that the cosmological constant and the coupling parameters $k_1$ and $k_2$ in the quadratic part of the total action functional

$$
\int d^4 x \sqrt{g} \left( k_1 R_{ab} R^{ab} + k_2 R^2 \right) \quad (67)
$$

vanish after the renormalization. Unfortunately, because of the technical complexity of the problem it is practically impossible to find the solution of the equations without referring to approximations or numerics. The exact self-consistent solutions exist only for simple geometries with a high degree of symmetry. Moreover, for the stress-energy tensor obtained from the effective action (46) there is a real danger that some classes of solution of the semiclassical equations would be non-physical. It is because of the appearance of the higher-order derivatives in the equations. Because of that our strategy (that is in concord with the philosophy of the effective lagrangians) is as follows. Since the modifications of the classical spacetime caused by the quantum effects are expected to be small, the natural approach to the problem is to solve the semiclassical equations perturbatively. If both the classical and the quantum parts of the total stress-energy tensor depend functionally on the metric, the equations to be solved have the following form

$$
G_{ab}[g] = 8\pi \left( T_{a}^{(cl)b}[g] + \varepsilon T_{a}^{(q)b}[g] \right),
$$

(68)
with
\[ g_{ab} = g_{ab}^{(0)} + \varepsilon \Delta g_{ab}, \]  

(69)

where \( \Delta g_{ab} \) is a first-order correction to the metric and to keep control of the order of terms in complicated series expansions, we have introduced once again the dimensionless parameter \( \varepsilon \).

Focusing on the first two term of the expansion one has
\[ G_{ab} = G_{ab}^{(0)} + \varepsilon \Delta G_{ab}. \]  

(70)

Of course, one expects that the quantized fields acting upon the classical Kasner spacetime deform it, i.e., the quantum corrected metric is still of the Bianchi type I type, but it is not the Kasner metric any more (See however Ref. [41]). Having this in mind we assume that each metric potential \( a(t), b(t) \) and \( c(t) \) can be expand as the classical background plus a correction. Since we are interested in the corrections to the classical vacuum solution we put \( T_{a}^{(cl)b} = 0 \), and, consequently, the metric, with a little prescience, can be expanded as
\[ a(t) = t^{2p_1} (1 + \varepsilon \psi_1(t)), \]  

(71)
\[ b(t) = t^{2p_2} (1 + \varepsilon \psi_2(t)), \]  

(72)
\[ c(t) = t^{2p_3} (1 + \varepsilon \psi_3(t)). \]  

(73)

Now, expanding the semiclassical Einstein field equations in the powers of \( \varepsilon \) and retaining the first two terms in the Einstein tensor, one has
\[ G_0^0 = -\frac{1}{t^2} (p_1 p_2 + p_1 p_3 + p_2 p_3) - \frac{\varepsilon}{2t} (p_2 + p_3) \psi'_1 - \frac{\varepsilon}{2t} (p_1 + p_3) \psi'_2 - \frac{\varepsilon}{2t} (p_1 + p_2) \psi'_3, \]  

(74)
\[ G_1^1 = \frac{1}{t^2} (p_2 - p_2^2 + p_3 - p_3^2 - p_2 p_3) - \frac{\varepsilon}{2t} (2p_2 + p_3) \psi'_2 - \frac{\varepsilon}{2t} (p_2 + 2p_3) \psi'_3 - \frac{\varepsilon}{2} \psi''_2 - \frac{\varepsilon}{2} \psi''_3, \]  

(75)
\[ G_2^2 = \frac{1}{t^2} (p_1 - p_1^2 + p_3 - p_3^2 - p_1 p_3) - \frac{\varepsilon}{2t} (2p_1 + p_3) \psi'_1 - \frac{\varepsilon}{2t} (p_1 + 2p_3) \psi'_3 - \frac{\varepsilon}{2} \psi''_1 - \frac{\varepsilon}{2} \psi''_3, \]  

(76)
\[ G_3^3 = \frac{1}{t^2} (p_1 - p_1^2 + p_2 - p_2^2 - p_1 p_2) - \frac{\varepsilon}{2t} (2p_1 + p_2) \psi'_1 - \frac{\varepsilon}{2t} (p_1 + 2p_2) \psi'_2 - \frac{\varepsilon}{2} \psi''_1 - \frac{\varepsilon}{2} \psi''_2. \]  

(77)

The solution of the zeroth-order equations is the Kasner metric, whereas the system of first the order equations
\[ \Delta G_{ab} = 8\pi T_{a}^{b}, \]  

(78)
where $\Delta G^b_a$ is given by the linear in $\varepsilon$ part of $G^b_a$, is more complicated. However, before going further it is worthwhile to briefly discuss our general strategy. Following Ref. [42] let us assume that for $t < t_0$ ($t_0 \gg t_{Pl}$) the stress-energy of the quantum fields vanishes. The modes with the frequencies satisfying $\tilde{\omega}_k(t_0) > t_0^{-1}$ are in the adiabatic regime for $t > t_0$ and the creation of the particles is exponentially damped. On the other hand, for the modes satisfying $\tilde{\omega}_k(t_0) < t_0^{-1}$ we have creation, although it can be made small taking sufficiently massive fields. In what follows the particle creation will be ignored.

Now return to the first order equations and analyze the degenerate configuration $(-1/3, 2/3, 2/3)$. The equations to be solved are of the form

$$\Delta G^a_a = \frac{1}{12\pi m^2} T^{(i)}_a \quad \text{(no summation over } a).$$

(79)

The general solution $(\psi_1(t), \psi_2(t))$ depends on three integration constants. The fourth one must be equated to zero on the account of the covariant conservation of the stress-energy tensor. Since $T^a_b(t) = 0$ for $t \leq t_0$ we have the Kasner metric tensor and its derivative (a left-hand derivative at $t_0$) in that region. Consequently one is left with a simple solution

$$\psi_1(t) = \left(\frac{1}{15} T_1 - \frac{1}{10} T_2\right) \frac{1}{t^4}$$

(80)

and

$$\psi_2(t) = -\frac{T_1}{12t^4}$$

(81)

with $\psi_2(t) = \psi_3(t)$. On the other hand, for a general configuration one has

$$\psi_i = \frac{B_i}{t^4},$$

(82)

where $B_i$ for the upper branch have the form

$$B_1 = \frac{T_1 (3p_1^2 - 2p_1 + 15)}{2880\pi m^2} + \frac{T_2 [3(\beta - 9) - 3p_1^2 + (3\beta - 10)p_1]}{5760\pi m^2} - \frac{T_3 [3(\beta + 9) + 3p_1^2 + (3\beta + 10)p_1]}{5760\pi m^2},$$

(83)

$$B_2 = \frac{T_1 [p_1(14 - 3p_1 + 3\beta) - 5(7 + \beta)]}{5760\pi m^2} + \frac{T_2 [31 + p_1(2 - 3p_1 - 3\beta) + \beta]}{5760\pi m^2} + \frac{T_3 [p_1(3p_1 - 2) - 4(4 + \beta)]}{2880\pi m^2},$$

(84)

and

$$B_3 = \frac{T_1 [p_1(14 - 3p_1 - 3\beta) + 5(\beta - 7)]}{5760\pi m^2} - \frac{T_2 [p_1(3p_1 - 2) + 4(\beta - 4)]}{2880\pi m^2} - \frac{T_3 [31 - \beta + p_1(2 - 3p_1 + 3\beta)]}{5760\pi m^2}. $$

(85)
For a given spin, $T_i$ are defined as in Eqs. (52). The results for the lower branch can be obtained by putting $\beta \rightarrow -\beta$ and taking $T_i$ appropriate for that branch. With a little effort one can check that the functions $\psi_1, \psi_2$ and $\psi_3$ satisfy Eq (74) and this completes the solution of the first order semiclassical equations.

Now we try to answer the natural question if the quantum effects dampen or strengthen the anisotropy [42]. As its natural measure let us take the ratios of the directional Hubble parameters of the quantum-corrected spacetime. To the first order in $\varepsilon$ one has

\begin{align*}
    H_{ab} &= \frac{H_a}{H_b} = \frac{p_1}{p_2} + \frac{\varepsilon t}{2} \left( \frac{\dot{\psi}_1}{p_2} - \frac{p_1 \dot{\psi}_2}{p_2^2} \right), \\
    H_{bc} &= \frac{H_b}{H_c} = \frac{p_2}{p_3} + \frac{\varepsilon t}{2} \left( \frac{\dot{\psi}_2}{p_3} - \frac{p_2 \dot{\psi}_3}{p_3^2} \right), \\
    H_{ca} &= \frac{H_c}{H_a} = \frac{p_3}{p_1} + \frac{\varepsilon t}{2} \left( \frac{\dot{\psi}_3}{p_1} - \frac{p_3 \dot{\psi}_1}{p_1^2} \right).
\end{align*}

As the result has a general structure $H_{ij} = H_{ij}^{(0)} + \delta H_{ij}$ we shall call $H_{ij}^{(0)}$ the classical part and $\delta H_{ij}$ its correction. First, consider the zeroth-order effects: if the $H_{ij}^{(0)}$ is positive the spacetime is expanding or contracting in the both spacetime directions, moreover, if $H_{ij}^{(0)} = 1$ then the evolution is isotropic. On the other hand, if the sign is negative then the spacetime is expanding in one direction and contracting in the other. From this one sees that the influence of the quantum fields depends not only on the relative signs of the classical Hubble parameters and their corrections, but also if $H_{ij}^{(0)}$ is bigger or smaller than 1.

Before we discuss the general case let us analyze the degenerate configuration $(-1/3, 2/3, 2/3)$. For the massive scalar field the sign of the perturbation $\delta H_{ab}$ depends the coupling constant $\xi$. Indeed, when $\xi < 47/216$ the perturbation is positive and the vacuum polarization isotropizes background spacetime. It should be noted that both minimally and conformally coupled fields make the background spacetime more isotropic. Moreover, it is precisely the same inequality that should hold for the coupling constant of the massive scalar field to make the interior of the Schwarzschild black hole more isotropic [5]. It becomes even more interesting when we realize that for the Schwarzschild black hole the degenerate Kasner metric is approached asymptotically only in the closest vicinity of the singularity. For the spinor field $\delta H_{ab}$ is always positive whereas for the vector fields it is always negative. Once again a similar behavior is observed for the quantum corrected interior of the Schwarzschild spacetime.
Now, let us return to the general case. We shall analyze the influence of the minimally and conformally coupled massive scalar fields on the anisotropy. Here we describe only the minimally coupled fields since a similar qualitative behavior of $H_{ij}^{(0)}$ and $\delta H_{ij}$ can be observed for the conformal coupling. On the lower branch (excluding configurations of the type $(0, 0, 1)$) the ratio $H_{ab}^{(0)}$ is always negative, $H_{bc}^{(0)}$ is positive for $p_1 < 0$ and negative for $p_1 > 0$, and finally $H_{ca}^{(0)}$ is negative for $p_1 < 0$ and positive for $p_1 > 0$. On the other hand, $\delta H_{ab}$ is positive for $p_1 < 0$ and negative for $p_1 > 0$. Further, $\delta H_{bc}$ is always positive, whereas $\delta H_{ca}$ is negative for $p_1 < 2/3$ and positive for $p_1 > 2/3$.

A similar analysis carried out for the upper branch shows that $H_{ab}^{(0)}$ is negative for $p_1 < 0$ and positive for $p_1 > 0$, $H_{bc}^{(0)}$ is positive for $p_1 < 0$ and negative for $p_1 > 0$, and $H_{ca}^{(0)}$ is always negative. The quantum correction $\delta H_{ab}$ is positive for $p_1 < 2/3$ and negative for $p_1 > 2/3$, $\delta H_{bc}$ is always negative, and, finally, $\delta H_{ca}$ is negative for $p_1 < 0$ and positive for $p_1 > 0$.

All this can be stated succinctly in the following way: roughly speaking, for the upper branch, the quantum effects tend to increase anisotropy in $(x, y)$-directions for $p_1 < 2/3$ and decrease for $p_1 > 2/3$. The anisotropy is always decreased by the vacuum polarization in $(y, z)$-directions and in $(x, z)$-directions the anisotropy is strengthened for $p_1 < 0$ and damped for $p_1 > 0$. On the other hand, for the lower branch the behavior of $\delta H_{12}$ is qualitatively similar to $\delta H_{23}$ on the upper branch, whereas $\delta H_{23}$ is qualitatively similar to $\delta H_{12}$. The qualitative behavior of $H_{31}$ is identical on both branches. Finally observe that for the corrections generated by the spinor and vector fields one has a similar equivalence. More specifically, analysis of $\delta H_{12}$ for the massive vectors shows that the anisotropy always increases, whereas that of $\delta H_{23}$ increases for $p_1 < 0$ and decreases for $p_1 > 0$. $\delta H_{31}$ leads to decreasing anisotropy for $p_1 < 0$ and to increasing for $p_1 > 0$. The appropriate results for the massive spinors field are opposite, i.e., ‘increase’ should be replaced by ‘decrease’ and vice-versa.

V. FINAL REMARKS

In this paper we have calculated the vacuum polarization, $\langle \phi^2 \rangle$, of the massive scalar field in the Bianchi type I spacetime within the framework of the Schwinger-DeWitt method and the adiabatic approximation. It has been demonstrated that both methods yield the same result. We expect that a similar equality will hold for the stress-energy tensors. Although we have verified this only for the trace of the stress-energy tensor of the conformally coupled scalar field, we believe that the demonstration of this equality in a general case is conceptually easy but quite involved computationally. Building on this we have calculated the stress-energy tensor of the scalar, spinor
and vector fields in the Bianchi type I spacetime making use the Schwinger-DeWitt one-loop effective action and checked the influence of the quantized fields upon the Kasner spacetime. The special emphasis has been put on the problem of isotropization of the background geometry. It should be emphasized once again that being local the Schwinger-DeWitt technique does not take particle creation into account. It is therefore possible that the actual influence of the quantized fields, e.g., calculated numerically, will be more pronounced [5]. On the other hand however, we expect that if the conditions \( H_i/m < 1 \) hold our results should provide a reasonable approximation. Finally observe that the semiclassical Einstein equations with the right hand side given by the stress-energy tensor of the quantized fields constructed from the one-loop effective action (46) may be treated as the theory with higher curvature terms. Theories of this type are currently actively investigated (see e.g. Refs. [43–45] and references therein).

Acknowledgments

The author would like to thank Darek Tryniecki for discussions.

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