Cosmological Consequences of Slow-Moving Bubbles in First-Order Phase Transitions

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(May 1, 2018)

In cosmological first-order phase transitions, the progress of true-vacuum bubbles is expected to be significantly retarded by the interaction between the bubble wall and the hot plasma. We examine the evolution and collision of slow-moving true-vacuum bubbles. Our lattice simulations indicate that phase oscillations, predicted and observed in systems with a local symmetry and with a global symmetry where the bubbles move at speeds less than the speed of light, do not occur inside collisions of slow-moving local-symmetry bubbles. We observe almost instantaneous phase equilibration which would lead to a decrease in the expected initial defect density, or possibly prevent defects from forming at all. We illustrate our findings with an example of defect formation suppressed in slow-moving bubbles. Slow-moving bubble walls also prevent the formation of 'extra defects', and in the presence of plasma conductivity may lead to an increase in the magnitude of any primordial magnetic field formed.

PACS numbers: 98.80.Cq, 11.27.+d, 64.60.Qb, 64.60.-i

I. INTRODUCTION

According to standard cosmology, the early universe is expected to have undergone a series of symmetry-breaking phase transitions as it expanded and cooled, at which topological defects may have formed [1]. Phase transitions are labelled first- or second-order, according to whether the position of the vacuum state in field space changes discontinuously or continuously as the critical temperature is crossed. A first-order phase transition proceeds by bubble nucleation and expansion. When at least \((4-n)\) of these bubbles collide (for \(n = 0, 1, 2\)), an \(n\)-dimensional topological defect may form in the region between them.

In recent years there has been considerable interest in the formation of defects in first-order phase transitions, in particular the validity of the so-called geodesic rule. The geodesic rule, first stated by Kibble [2], predicts that after a two-bubble collision the phase of the scalar field interpolates continuously between the values in each bubble, along the shortest path in field space. Early analysis [3] confirmed the geodesic rule for defect formation in both global and local theories, albeit using a planar approximation and neglecting the effect of the surrounding plasma. In later work, the finite conductivity of the plasma was considered [4] for local theories, as was the effect of slow-moving (i.e. speeds less than the speed of light) bubble walls in theories with a global symmetry [5]. These analyses confirm defect formation in first-order phase transitions, but make conflicting claims about the number of defects actually formed. In this paper we investigate this issue. Unlike previous work, we include the effect of slow-moving bubble walls in both the global and local cases. We use our results to make qualitative comparisons between defect densities formed in global and local theories, and by slow-moving and fast-moving bubble walls. As well as the consequences for defect formation, we also consider the implications of slow-moving walls for the formation of primordial magnetic fields at a first-order phase transition.

We take as our model the simplest spontaneously-broken gauge symmetry: the Abelian Higgs model, which has a local \(U(1)\) symmetry, with Lagrangian

\[
\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\Phi^\dagger \Phi), \tag{1}
\]

where \(D_\mu \Phi = \partial_\mu \Phi - ie A_\mu \Phi\) and \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\). The detailed form of the effective potential \(V(|\Phi|)\) will depend upon the particular particle-physics model being considered, but in order to be able to study the generic features of a first-order phase transition we shall take \(V\), following Ferrera and Melff [5], to be

\[
V(\Phi) = \lambda \left(\frac{\Phi^2}{2} - (|\Phi| - \eta)^2 - \frac{\varepsilon}{3} |\Phi|^3\right), \tag{2}
\]

where in a realistic model, \(\varepsilon = \varepsilon(T) \propto (T_c - T)\). \(V\) has a local minimum false-vacuum state at \(\Phi = 0\) which is invariant under the \(U(1)\) symmetry, and global minima true-vacuum states on the circle \(|\Phi| = \rho \exp(\eta/4) (3 + \varepsilon + \sqrt{1 + 6 \varepsilon + \varepsilon^2})\) which possess no symmetry. The dimensionless parameter \(\varepsilon\) is responsible for lifting the degeneracy between the two sets of minima – the greater \(\varepsilon\), the greater the potential difference between the false- and true-vacuum states, and hence the faster the bubbles will accelerate. By making the field and coordinate transformations

\[
\Phi \rightarrow \phi = \eta \Phi \tag{3}
\]

\[
x \rightarrow x' = \frac{x}{\sqrt{\lambda \eta}} \tag{4}
\]

\[
t \rightarrow t' = \frac{t}{\sqrt{\lambda \eta}} \tag{5}
\]

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it is possible to set $\lambda$ and $\eta$ to unity, so that the potential is parametrized only by $\epsilon$, and hereafter we shall use these transformed variables.

The bubble nucleation rate per unit time per unit volume is given by the ‘bounce’ solution of the Euclidean field theory. Ignoring quantum fluctuations, the phase $\theta$ is constant within each bubble, and uncorrelated between spatially-separated bubbles. Any non-zero gauge fields in the nucleation configuration will make a contribution to the action and hence the nucleation of bubbles with non-zero gauge fields is exponentially suppressed. When three or more bubbles collide, a phase-winding of $2\pi n$ can occur around a point, which by continuity must then be at $\Phi = 0$. In three spatial dimensions, this topologically-stable region of high-energy false vacuum is string-like – a cosmic string.

The formation and evolution of cosmic strings have been studied in great detail. Cosmic strings have been evoked as, amongst other things, possible seeds for cosmic structure formation, sources of cosmic rays, gravitational radiation and baryogenesis (see, e.g. [6]). In order to be able to assess the significance of cosmic strings in the evolution of the early universe, it is important to be able to estimate the initial defect density accurately. This depends on how the phases between two or more bubbles interpolate after collision. In particular, although strings are in general formed when three or more bubbles collide, a simultaneous three-bubble collision is unlikely – one would expect in general two-bubble collisions, with a third, or fourth bubble colliding some finite time later. If the phase inside a two-bubble collision is able to equilibrate quickly, and before a third bubble arrives, there may be a strong suppression of the initial string density. The effect of phase equilibration on the initial defect density was first investigated by Melfo and Perivolaropoulos [4]. They found a decrease of less than 10%, in models which possess a global symmetry and with bubbles moving at the speed of light.

The above description of defect formation, however, ignores any effect that the hot-plasma background may have on the evolution of the Higgs field, which may be significant in the early universe. Real-time simulations and analytic calculations for the (Standard Model) electroweak phase transition predicted that the bubble wall would reach a terminal velocity $v_{ter} \sim 0.1c$. The reason for this is simple: outside the bubble, where the $(SU(2) \times U(1))$ symmetry remains unbroken, all fields coupled to the Higgs are massless, acquiring their mass from the vacuum expectation value of the Higgs in the spontaneously-broken symmetry phase inside the bubble. Particles outside the bubble without enough energy to become massive inside bounce off of the bubble wall, retarding its progress through the plasma. The faster the bubble is moving, the greater the momentum transfer in each collision, and hence the stronger the retarding force. Thus a force proportional to the bubble-wall velocity appears in the effective equations of motion.

Ferrera and Melfo [4] studied bubble collisions in such an environment, for theories which possessed a global symmetry, and found that decaying phase oscillations occur inside a two-bubble collision, leading to a suppression of the defect formation rate [4]. Kibble and Vilenkin [6] studied phase dynamics in collisions of undamped bubbles in models with a local symmetry, and found, analytically, a different kind of decaying phase oscillation. When the finite conductivity of the plasma was included, these oscillations were found not to occur. However, Kibble and Vilenkin did not consider the behaviour of the phase after collisions of bubbles moving at speeds slower than the speed of light. Moreover, because of the symmetry assumptions made in their calculations, their results cannot be simply extrapolated to the slower-moving case.

The behaviour of the phase inside bubble collisions in local theories where the bubbles move at the speed of light, and in global theories with slow-moving bubbles has been considered. However, the most realistic scenario cosmologically – a gauge-theory phase transition where the bubbles are slowed significantly by the plasma (as might be expected at the electroweak- or GUT-scales) – has not been studied. This paper presents the results of our investigations into what happens in theories with a local symmetry, where the bubbles are moving at terminal velocities less than the speed of light. Our 3 + 1-dimensional simulations indicate that, for slow-moving bubbles, phase oscillations of either of the types described in [3] or [4] do not occur, before the effect of the plasma conductivity is even considered. We therefore expect that (a) fewer defects would form in a phase transition where the ‘Higgs’ field is coupled to a gauge field than in a global-symmetry phase transition, and (b) in local theories, fewer defects would form in slow-moving bubbles than fast-moving ones.

We should note in passing that we have ignored the effect of the expansion of the universe in our work. This is a good approximation for phase transitions which take place at late times, like the electroweak phase transition. At phase transitions which occur earlier, however, the Hubble expansion may have a significant effect on bubble and phase dynamics. This topic deserves consideration on its own, and work is currently in progress [11].

In the following section, we describe the effects of a slow-moving bubble wall on phase dynamics inside bubbles collisions in theories which possess a global symmetry. In section III, we discuss phase equilibration in theories with a local symmetry and present our new results in the case of slowly-moving local-symmetry bubbles. Our conclusions are supported by examples of defect formation suppressed in slow-moving bubbles. We show that the ‘extra defects’ found in [4] do not occur in heavily-damped environments. In section IV we discuss the formation of a primordial magnetic field. We show that the presence of the plasma conductivity results in a larger magnetic field for fast-moving bubbles. For slow-moving bubbles, the plasma conductivity stops the field dispersing. A larger magnetic field could also result in this case. A discussion of our results and conclusion are presented
in section V.

II. GLOBAL SYMMETRY

If the gauge coupling $\epsilon$ is set to zero, we have a theory with a global $U(1)$ symmetry

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) - V(\Phi^\dagger \Phi).$$

(6)

By writing $\Phi = \rho e^{i\theta}$, the equations of motion for the modulus $\rho$ and phase $\theta$ of the Higgs field are

$$\ddot{\rho} - \rho'' - (\partial_\mu \theta)^2 \rho = -\frac{\partial V}{\partial \rho}$$

(7)

$$\partial^\mu [\rho^2 \partial_\mu \theta] = 0.$$  

(8)

If the potential difference between the true- and false-vacuum states is much smaller than the height of the barrier separating them, the field equations may be solved using the ‘thin-wall’ approximation [8], by setting $\varepsilon = 0$. For our potential [2], this yields

$$|\Phi| = \eta \left[1 + \tanh \left(\frac{\sqrt{\lambda} \eta}{\rho} (s - R_0)\right)\right],$$

(9)

where $s^2 = x^2 - t^2$ and $R_0$ is the bubble radius on nucleation.

Note that $\theta = \text{constant}$ trivially satisfies the phase equation (8), and so if the phase is initially constant within each bubble, as we shall assume, there are no phase dynamics until the bubbles collide.

As described in the introduction however, we would like to investigate the behaviour of the phase in collisions of slow-moving bubbles. For a given theory, by considering the Boltzmann equations for scattering off of the Higgs field, it is possible to calculate the terminal velocity of the bubble wall [8]. Since we are not concerned here with the parameters of a specific particle-physics model, we choose instead to use a single damping parameter $\Gamma$ to model the interaction of the Higgs with the plasma. In the introduction we claimed that the plasma would introduce a term proportional to the bubble-wall velocity into the equations of motion. Since the phase $\theta$ of the Higgs field is not affected by the effects described, we assume that the plasma couples only to the modulus $\rho$. We then have effective equations of motion

$$\ddot{\rho} - \rho'' + \Gamma \dot{\rho} - (\partial_\mu \theta)^2 \rho = -\frac{\partial V}{\partial \rho}$$

(10)

$$\partial^\mu [\rho^2 \partial_\mu \theta] = 0.$$  

(11)

A damping term of this form has been used by several authors [8], [14], [13], and has also been derived from the stress-energy of the Higgs, assuming a coupling to the plasma [14]. Heckler [14] estimates $\Gamma \sim \frac{\rho_e T_c}{\hbar}$ for the electroweak phase transition, by comparing the energy generated by the frictional damping with the pressure on the wall due to the damping.

The effect of this damping term is that instead of accelerating up to the speed of light, the bubble walls reach a terminal velocity $v_{\text{ter}} < c$. By making the ansatz $\rho = \rho [x - x_0(t)]$, the terminal velocity can be calculated [6] by integrating the equation of motion for $\rho$

$$v_{\text{ter}} = \frac{\Delta V}{\Gamma} \int \rho^2 dx,$$

(12)

where $\Delta V$ is the difference in potential energy between the true- and false-vacuum states. Assuming that the wall has a Lorentz-contracted, moving profile of the form [8]

$$\rho = \rho_{tv} \left[1 + \tanh \left(\sqrt{\lambda} \rho_{tv} \gamma \left(r - v_{\text{ter}} t - R_0\right)\right)\right],$$

(13)

the integral in the denominator of (12) can be evaluated. Expanding $\gamma = (1 - v_{\text{ter}}^2)^{-1/2}$, we obtain

$$v_{\text{ter}} = \frac{A}{\sqrt{A^2 + 1}}$$

(14)

where $A = 6\Delta V/\sqrt{\lambda} \rho_{tv}^3$. We have simulated the evolution of bubbles in such a dissipative environment in 1+1-dimensions. Taking a static profile of the form (11) as the initial conditions, the terminal velocity of the bubble was calculated for a range of values of friction parameter $\Gamma$. The accuracy of formula (14), compared with terminal velocities calculated directly from simulations can be seen in Figure 3. This is a very useful result, as from it we can dial the input value of $\Gamma$ to produce the value of $v_{\text{ter}}$ corresponding to the particular particle-physics model we are interested in. Heckler [4], and Ferrera and Melfo [8] obtained a result like (12), and Haas [6] found the best-fit equation $v_{\text{ter}} = A + (1 - A)/\left(1 + B / r^{1.62}\right)$ from Langevin-equation simulations. However, equation (14), we believe, holds for all of the cases above, provided that $\varepsilon$ is small enough for the ‘thin wall’ approximation to hold, and is more useful when performing simulations. For example, it could be applied to the electroweak phase transition in the supersymmetric case, if the terminal velocity of the bubble wall were calculated.

Ferrera and Melfo [8] described how, in the context of a theory with a global symmetry, slow-moving bubble walls lead to phase oscillations. When two bubbles collide, the walls merge. Across the plane (in 3 spatial dimensions) of intersection, there exists a phase gradient $\Delta \theta$, the phase wave propagates into each bubble from the centre – see Figures 3 (b) and 3 (b). As the Goldstone boson is massless, and undamped, this wave travels at the speed of light. If the phase difference between the bubbles is $\Delta \theta$, the phase wave will carry a phase difference $+\Delta \theta/2$ into one of the bubbles, and $-\Delta \theta/2$ into the other, equilibrating the phase. If the bubble wall is moving at a terminal velocity $v_{\text{ter}} < c$, the
wave will catch up with the bubble wall, and rebound – the returning wave will now ‘flip’ the original phase profile. Thus phase oscillations occur inside the merged bubbles. Given three or more spatially-separated bubbles whose distribution of phases one would expect to generate a vortex on collision, a vortex, an anti-vortex, or none at all may form, depending on the profile of the phase inside the two bubbles at the moment of collision of the third – an example of how phase dynamics can affect the defect-formation process. The oscillations are damped, because the bubble walls continue to expand, increasing the volume over which the finite-energy wave must sweep, thus diluting the phase difference carried by the wave. Thus the converse of the above statement is not true – an initial distribution of phases which one would not expect to form a defect, will not produce one as a result of phase oscillations. Statistical simulations in two dimensions \([10]\) have shown that this leads to a suppression in the defect-formation rate – the slower the bubble walls, the fewer defects are formed per nucleated bubble.

III. LOCAL SYMMETRY

A. Phase dynamics inside two-bubble collisions

Including the gauge fields in our model, the field equations become

\[
\rho - \rho'' - (\partial_{\mu} \theta - eA_{\mu})^2 \rho = -\frac{\partial V}{\partial \rho} \tag{15}
\]

\[
\partial^{\mu} [\rho^2 (\partial_{\mu} \theta - eA_{\mu})] = 0 \tag{16}
\]

\[
\dot{A}_{\nu} - A_{\nu}'' - \partial_{\nu} (\partial \cdot A) = -2e\rho^2 \partial_{\nu} \theta. \tag{17}
\]

Since we now have a local \(U(1)\) symmetry, the phase \(\theta\) can be arbitrarily re-defined at any point in time by a gauge transformation, and so we need a gauge-invariant notion of phase. We define, following Kibble and Vilenkin \([11]\), the gauge-invariant phase difference between two points \(A\) and \(B\)

\[
\Delta\theta = \int_{A}^{B} dx \, (\partial_{i} - ieA_{i}), \tag{18}
\]

where \(i = 1, 2, 3\) and the integral is taken, for simplicity, along the straight line joining \(A\) and \(B\).

For bubbles which move at approximately the speed of light, it is possible to greatly simplify the field equations. If we consider a two-bubble collision, in a frame where the bubbles are nucleated simultaneously, by assuming that the bubbles instantly propagate at the speed of light, it is possible to impose \(SO(1,2)\) Lorentz symmetry on the field equations. Thus the fields are functions of \(z\) and \(\tau^2 = t^2 - x^2 - y^2\) only. With this assumption, and a step-function ansatz for the phase \(\theta\) at the time of collision, Kibble and Vilenkin \([11]\) solved the field equations for \(\Delta\theta\)

\[
\Delta\theta = \frac{2R}{L} \theta_0 \left( \cos e\eta (t - R) + \frac{1}{enR} \sin e\eta (t - R) \right), \tag{19}
\]

where \(2\theta_0\) is the initial phase difference between the spatially-separated bubbles and \(R\) is their radius on collision at \(t = 0\).

Equation \((19)\) describes decaying phase oscillations, the time scale of equilibration determined by the initial phase difference and radius of the colliding bubbles, the frequency of oscillation by the gauge-boson mass. These oscillations, and the accuracy of this formula for small initial phase differences, have recently been confirmed in simulations \([11]\).

However, the assumption that the bubbles move at, or close to the speed of light, does not appear to be realistic \([9]\). In this case, the symmetry assumptions made in \([11]\) are no longer valid. Moreover, it is not possible to replace the coordinate \(\tau^2 = t^2 - x^2 - y^2\) by the obvious choice \(\tau^2 = (v_{ter}t)^2 - x^2 - y^2\), since only the the modulus of the Higgs field, the bubble wall, is constrained in this way – the phase and gauge fields are still free to propagate causally.

In order to investigate whether phase oscillations – which occur in the global theory with slow-moving bubbles, and in the local theory with fast-moving bubbles – still occur in the local theory when the bubbles expand slower than the speed of light, we include the dissipation term \(\Gamma \dot{\rho}\) into the equation for the modulus of the Higgs field \(\rho\), without coupling it to the phase or the gauge fields. This is motivated in the same manner as described in the global case. A term proportional to \(\dot{\rho}\) is, of course, \(U(1)\) gauge-invariant.

Since there is no longer an obvious simplification of the equations of motion which might lead to an analytic solution, we turn to computer simulations. The equations

\[
\dot{\rho} - \rho'' - (\partial_{\mu} \theta - eA_{\mu})^2 \rho = -\frac{\partial V}{\partial \rho},
\]

\[
\partial^{\mu} [\rho^2 (\partial_{\mu} \theta - eA_{\mu})] = 0,
\]

\[
\dot{A}_{\nu} - A_{\nu}'' - \partial_{\nu} (\partial \cdot A) = -2e\rho^2 \partial_{\nu} \theta.
\]
of motion were discretized in the gauge-invariant way described in [18], choosing the temporal gauge $A_0 = 0$ in order to make the time evolution trivial. We used a lattice of size $200^3$ and a lattice spacing $a = 0.5$ – tests were performed on lattices with spacing down to $a = 0.1$ giving no qualitatively-different results. The time evolution was performed using a fourth-order Runge-Kutta algorithm. We took as initial conditions a static profile of the form (9) for $\rho = |\Phi|$, for two bubbles of radius $R = 5$, with phases $\theta = 0$ and $\theta = 2\pi/3$, centred at $(\pm 8, 0, 0)$. We choose to ignore any primordial magnetic field and, since the nucleation process is not expected to generate non-zero gauge fields (see Introduction), set all the gauge fields to zero initially.

The results of the simulations are displayed in Figures 1, 2 and 3. For the sake of clarity and to aid comparison between the different cases, we have chosen to present our results in terms of the evolution with time of the gauge-invariant phase difference $\Delta \theta$. We evaluated $\Delta \theta$ between the centres of the two bubbles, though the qualitative behaviour was found not to change when it was calculated between different points.

Figure 4 (a) shows the behaviour of the gauge-invariant phase difference for bubbles moving at the speed of light – the decaying oscillations calculated by Kibble and Vilenkin in the local case. In the global case, $\epsilon = 0$, we find that the phase does equilibrate, but on a much longer time-scale. Thus we would expect that for fast-moving bubbles, fewer defects are formed in local theories than global ones, since in order to form a defect a phase difference inside the two merged bubbles must be present when a third bubble collides.

In Figure 4 (b) we plot $\Delta \theta$ for slower-moving bubbles. For $\epsilon = 0$, we confirm in $3 + 1$-dimensions the decaying phase oscillations described by Ferrera and Melfo [19] and observed by them in $2 + 1$-dimensions. These oscillations are killed by adding in gauge fields – for a fixed bubble-wall velocity, the stronger the gauge coupling, the less time the gauge-invariant phase difference is non-zero, and hence the less likely a third collision will occur in time for a defect to form. Thus we would expect a lower defect-formation rate in local theories with slower-moving bubble walls.

Figure 5 illustrates our findings – it shows a cross-section through a non-simultaneous three-bubble collision, after all three bubbles have merged. In each case, the bubbles of initial radius $R = 5$, centred at $(\pm 8, 0, -10)$ and $(0, 0, 10)$, were given phases $\theta = -\pi/2, 0$ and $2\pi/3$. For identical initial conditions, we see that in the fast-moving case a vortex is formed, but when the bubbles are slowed down, the phase difference between the two bubbles has equilibrated by the time the third bubble collides, and no defect is formed.

In any cosmological phase transition where the bubble wall is significantly slowed down, we may also expect the plasma to have non-zero conductivity, which will affect the evolution of the fields and so needs to be considered in any attempt at a realistic model. We have simulated the effects of the finite-conductivity of the plasma, by adding on to the right-hand side of the gauge field equations (17) a conduction current $j^c$, whose spatial part is given by

$$j_c = \sigma E.$$  \hfill (20)

The corresponding charge density $\rho_c$ is fixed by the continuity relation $\partial_\mu j^c_\mu = 0$. For large values of the conductivity, it has been shown [17] that the oscillations in the gauge-invariant phase difference, which took place
FIG. 3. Damped bubbles bubbles merging: a) with no phase difference, and b) with phase difference $\Delta \theta = \pi/3$. In both cases $\varepsilon = 0.1$. Note in case b) the phase wave propagating inside the bubbles at the speed of light after collision, faster than the bubble walls. This phase wave will catch up with the outer wall of the bubble (not shown here), and rebound, producing phase oscillations.

in fast-moving bubbles with $\sigma = 0$, are exponentially damped.

Figure 6 (a) shows the evolution of the gauge-invariant phase difference in this case, where the walls are moving at the speed of light, for three different values of the conductivity $\sigma$. We confirm that, as $\sigma$ increases, the phase oscillations are more heavily suppressed, with practically no oscillations occurring for $\sigma \gtrsim 0.5$. In Figure 6 (b), we present the results of our simulations for slow-moving bubbles. For $\sigma = 0$, we have the case considered above – heavily suppressed oscillations. Increasing $\sigma$ merely serves to increase the suppression of phase oscillations: no new effect is observed.

Whereas it is already known that for bubbles moving at the speed of light, phase oscillations can be killed by a high conductivity $\sigma$, it is clear from our work that in slower-moving bubbles, the same effect can be obtained by a much lower value of $\sigma$.

### B. Extra defect production

An interesting consequence of slower-moving bubbles concerns the issue of ‘extra’ defect production at a two-bubble collision. Hawking, Moss and Stewart [15] first described (by energy considerations) how two true-vacuum bubbles travelling at nearly the speed of light would ‘pass through each other’, leading to the temporary restoration of the spontaneously-broken symmetry in a region between the two bubbles. This is illustrated in Figure 2 – the two bubbles collide, and bounce off of (or pass through) each other, producing a region of $\Phi = 0$ false vacuum inside the merged bubbles, which decays via oscillations of the bubble walls into the true-vacuum state. The size of the symmetry-restored region, and the time taken to decay completely to the true vacuum depends on the initial phase difference between the two bubbles, and the asymmetry parameter $\varepsilon$. Copeland and Saffin [12] showed how this could lead to the formation of ‘extra’ – in the sense that a defect would not be expected from the initial distribution of phases – flux-tube vortices in a gauge theory, around these regions of temporarily restored symmetry, and hence to an increase in the initial defect density after a phase transition.

Our simulations show – see Figure 6 (b) – that dissipation prevents this bouncing, or passing-through, of the bubbles. The excess energy, which would cause the symmetry restoration, is dissipated away by the plasma, and the bubbles simply merge. Thus there is no symmetry-restored region around which a non-zero winding of the phase can occur, and so no ‘extra’ defects would be formed.

### IV. MAGNETIC FIELDS

Another consequence of first-order phase transitions which may be cosmologically significant is the generation of primordial magnetic fields. Galaxies are observed to have magnetic fields $B_{\text{gal}} \sim 10^{-6} G$, coherent over large scales. Given a small initial seed field a dynamo mechanism, powered by the differential rotation of the galaxy in combination with the small-scale turbulent motion of the ionized gas, could generate the observed galactic fields. Many mechanisms for producing such a seed field have been proposed, one being bubble collisions at a first-order phase transition (see e.g. [20] and references therein). It had been believed that it was not possible to generate a seed field of sufficient magnitude at the electroweak phase...
transition for the dynamo mechanism to explain galactic magnetic fields as large as $10^{-6} \text{G}$. However, a recent paper by Davis, Lilley and Törnkvist [21] showed how, in a universe with a low matter density and in particular a positive cosmological constant, a dynamo mechanism may be able to generate observed galactic magnetic fields from a much smaller seed field. As a consequence, electroweak-scale magnetic fields may be viable primordial seed fields, and it is of interest to consider the effect of slow-moving bubble walls and finite plasma conductivity on the generation of magnetic fields.

If the gauge fields are set to zero initially, it can be seen from the equations of motion (15), (16) and (17), that non-zero gauge fields can only be generated where there exist spatial phase gradients, that is after the collision of two or more bubbles. After the collision of two bubbles, a loop of magnetic flux is generated around the circle of intersection [4]. The amount of flux generated is given by the integral of the gauge field $A$ around any loop which passes outside the bubbles. This is the same for all bubbles, regardless of size or speed

$$\int A_i dx^i = \frac{2\theta_0}{e},$$

and our simulations confirm this. When a third bubble collides, the fluxes combine, and if there is a phase winding of $2\pi$ around the centre, one flux quantum $2\pi/e$ will be trapped.

If there is no plasma conductivity, the magnetic flux generated is free to propagate at the speed of light away from the bubble collision. If the bubbles are expanding at the speed of light, then the fields can disperse no further outwards than the intersection of the bubbles. Copeland, Saffin and Törnkvist [22], [17] demonstrated how in this case two tubes of flux are produced — a ‘primary flux tube’ at the intersection of the colliding bubbles and a smaller, ‘secondary’ peak of opposite direction following behind it — see Figure 6 (a). Our simulations show that when non-zero conductivity is included, this secondary peak does not occur, and all the magnetic flux is concentrated into the primary peak, which is consequently larger — Figure 6 (b). We might expect that in this case, a larger magnetic field would form as all the flux generated by the two-bubble collision is aligned.

If, however, the bubbles are moving at speeds less than the speed of light, the flux is able to disperse into the plasma. Unless the bubble nucleation rate is extremely high (when a third bubble might be expected to collide quickly after the initial collision), no magnetic field will be able to form. For slow-moving bubbles, large-enough conductivity prevents the flux from dispersing, freezing it in to the plasma. Figure 6 (c) shows the magnetic field strength formed after a collision of two slow-moving bubbles, for $\sigma = 5$ (in reference [4] an estimate of $\sigma = T/e^2$ is given. The temperature at a phase transition is typically $T \sim \eta$, and so $\sigma = 5$ may be realistic). It can be seen that for this value of $\sigma$, a magnetic field does form, but it is spread through the inside of the bubble, rather than being concentrated in one or two narrow peaks at the wall, as seen in the fast-moving case. The height of the peak is lower by an order of magnitude — this is finite, rather than infinite conductivity and so some flux still escapes.

We note in passing here that provided that the plasma dynamics which slow down the bubble walls do not affect the bubble nucleation rate, the average number of bubbles nucleated per unit volume by the time the phase transition is completed will increase as the bubble wall velocity decreases. That is the average bubble radius on collision, the correlation length of the Higgs field $\xi$, decreases as the wall velocity decreases. Since the amount of flux generated at each collision is independent of the bubble radius, slow-moving bubbles will generate more flux, and hence a larger magnetic field when coarse-
grained over many bubble radii.

V. CONCLUSION

In this paper, we have examined the behaviour of colliding true-vacuum bubbles at a first-order cosmological phase transition. In the Abelian Higgs model, strings may form at the collision of three or more bubbles, but since a simultaneous three-bubble collision is very unlikely, the dynamics of the phase inside two-bubble collisions is crucial — if phase differences between two bubbles can be equilibrated quickly, and before the arrival of a third bubble, a topological defect will not form.

The most relevant phase transitions to cosmology involve gauge fields coupled to the symmetry-breaking field. In such phase transitions, the speed of the bubble walls will be considerably less than the speed of light, and yet the phase dynamics of slow-moving bubbles in a gauge field have not been considered previously. We have thus paid particular attention to the evolution of the phase inside collisions of bubbles moving at speeds much lower than the speed of light, in a $U(1)$ gauge theory.

In the simplest model, with no gauge fields and where the bubble walls accelerate up to the speed of light, the phase difference between two points is found to equilibrate. In models with a global symmetry where the bubble walls move slowly, and models with a local symmetry where the walls move at the speed of light, decaying phase oscillations have been observed. We find that in a $U(1)$ gauge theory, with slow-moving bubble walls, these oscillations are suppressed. On collision of two bubbles, instantaneous phase equilibration is observed. This would lead to a decrease in the initial expected defect density compared to the other cases. We have illustrated our claims by demonstrating an example of the suppression of defect formation in a local theory, due to nontrivial phase dynamics.

When two bubbles collide and merge, there will exist phase gradients across the intersection, a potential difference. In local theories it is necessary to define a gauge-invariant notion of the phase difference, which involves the gauge fields. Thus in local theories, the phase difference may equilibrate through the generation of gauge fields — there is in effect an ‘extra channel’ for the decay of the potential difference created on collision. This explains why we would expect fewer defects in local theories than in global ones. In a local theory, the phase difference between two bubbles is observed to equilibrate more quickly in slower-moving bubbles than in bubbles moving at the speed of light. This is due to the fact in slow-moving bubbles the rate of generation of phase gradients is lower, yet the gauge fields are not restricted to propagate at the speed of the bubble wall and are thus able to equilibrate the phase difference more rapidly.

If phase equilibration and hence the suppression of defect formation is aided by the coupling of gauge fields to the Higgs, it is interesting to ask whether another scalar field $\chi$ could have the same effect. In order to be able to dissipate the potential energy in the phase gradient such a field would need to couple to the phase of the Higgs, but also preserve the $U(1)$ symmetry of the Lagrangian. This can only be achieved (with terms at most quadratic in $\Phi$ and its derivatives) by Higgs couplings proportional to $\partial_{\mu}\chi[\Phi^\dagger \partial_{\mu}\Phi - (\partial_{\mu}\Phi^\dagger)\Phi] = \partial_{\mu}\chi[2i\rho^2\partial_{\mu}\theta]$, but in this case $\chi$ is effectively a gauge field [17]. It is possible that fermion couplings to the Higgs field would aid phase equilibration, but unfortunately this cannot be simulated easily.

We predict that fewer defects will form in gauge theories than global-symmetry theories since the phase difference is non-zero for less time after collision in gauge theories. In fact, if the bubble nucleation rate is low...
enough, it might be possible to effectively rule out the
formation of defects, solely on phase-dynamical grounds,
though percolation could presumably still be achieved.
This is a very interesting prospect, which could have sig-
nificant implications for cosmology – it may be possible
for example to circumvent the monopole problem without
needing inflation if defect formation is dynamically
suppressed in this way.

It has also been seen how it is unlikely that ‘extra
defects’, caused by bubbles bouncing off of each other
on collision, will be formed in cosmological phase transi-
tions, since the bubbles are retarded sufficiently by the
plasma for no such bouncing to occur. First-order phase
transitions can also generate a primordial magnetic field,
which may seed the galactic dynamo and hence be re-
sponsible for the galactic magnetic fields observed today.

A simple qualitative analysis suggests that in fast-moving
bubble walls, high conductivity (as would be expected
in the early universe) would lead to the generation of
a larger magnetic field. Where the bubble walls move
slower, we have demonstrated that a magnetic field can
form if the plasma has non-zero conductivity. In this
case, the smaller average bubble radius on collision may
cause more flux to be generated, producing a larger mag-
netic field. This may be significant in helping to beat the
lower-bound required by the dynamo model in order to
produce observed fields.

We have shown qualitatively how we expect the defect-
formation probability to be decreased by phase equilibra-
tion in two-bubble collisions. It would be interesting to
perform a statistical simulation of the type done in [10],
but for gauge-theory phase transitions, to see quantita-
tively how the defect density is affected by the terminal
velocity of the walls or the introduction of gauge fields.
We note that the argument given in the Introduction for
nucleating bubbles with zero gauge fields does not apply
if there already exists a primordial magnetic field before
bubble nucleation. In this case, it is not at all clear what
the preferred nucleation field configuration would be, and
we believe that a study of bubble nucleation in the pres-
ence of a magnetic field would be worth while. It is also
of some interest to consider the effect on phase dynamics
and defect formation of the Hubble expansion, since this
acts as a dissipation term on the phase as well as on the
bubble walls. We conclude, though, with a summary of
our findings.

In gauge theories, more defects are formed by fast-
moving bubble walls than by slower ones. In global the-
ories, the same is true.

For fast-moving bubble walls, more defects are formed
in global theories than in local ones. For slow-moving
bubble walls, the same is true.

VI. ACKNOWLEDGEMENTS

We would like to thank T.Kibble, P.Saffin, D.Steer and
especially O. Törnkvist for helpful comments and con-
versations. Computer facilities were provided by the UK
National Cosmology Supercomputing Centre in coopera-
tion with Silicon Graphics/Cray Research, supported by
HEFCE and PPARC. This work was supported in part
by PPARC and an ESF network grant. Support for M.L.
was provided by a PPARC studentship and Fitzwilliam
College, Cambridge.

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FIG. 7. Magnetic field strength along radius of collision of two bubbles. In figures (a) and (b) the bubbles are expanding at the speed of light, and in figure (c) the bubbles are slow-moving, with $\Gamma = 2$. On the left, conductivity $\sigma = 0$, in the centre $\sigma = 0.5$ and on the right $\sigma = 5$. The bubbles, of initial radius $R_0 = 5$ centred at $(\pm 8, 0, 0)$, were given phases $\theta = 0$ and $\theta = 2\pi/3$. The figures show the field strength at $t = 35$ for the fast-moving and $t = 52.5$ for the slow-moving bubbles. In the fast-moving case, with zero conductivity, a secondary peak of opposite sign follows inside the primary peak, as described in [7]. For non-zero conductivity, this secondary peak does not occur – all the flux is concentrated in the primary peak at the edge of the bubbles. In the slow-moving case, a smaller magnetic field forms which is spread over the bubble interior, rather than being concentrated in a peak at the edge.

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