Iterates of Markov operators and their limits

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Abstract
It is well known that iterates of quasi-compact operators converge towards a spectral projection, whereas the explicit construction of the limiting operator is in general hard to obtain. Here, we show a simple method to explicitly construct this projection operator, provided that the fixed points of the operator and its adjoint are known which is often the case for operators used in approximation theory.

We use an approach related to Riesz-Schauder and Fredholm theory to analyze the iterates of operators on general Banach spaces, while our main result remains applicable without specific knowledge on the underlying framework. Applications for Markov operators on the space of continuous functions $C(X)$ are provided, where $X$ is a compact Hausdorff space.

Keywords: Iterates of Markov operator, Quasi-compactness, spectral theory

The behaviour of the iterates of Markov operators has been studied extensively in modern ergodic theory, while in general the limiting operator is not explicitly given. A comprehensive overview on limit theorems for quasi-compact Markov operators can be found in Hennion and Hervé [8]. In this article, we construct the limit of the iterates of quasi-compact operators that satisfy a spectral condition. It will be shown under which conditions the limit exists and how the limiting projection operator can be explicitly constructed using the inverse of a Gram matrix. The explicit knowledge of the limiting operator is of interest in many applications.

This research is motivated by studying general Markov operators on the space of continuous functions $C(X)$, where $X$ is a compact Hausdorff space. Lotz [16] has already shown uniform ergodic theorems for Markov operators on $C(X)$. For specific classes of operators, the limiting operator has been provided as shown for instance by Kelisky and Rivlin [12], Karlin and Ziegler [10] and Gavrea and Ivan [6, 7]. Recently, Altomare [2] has shown a different approach using the concept of Choquet-boundaries and results from Korovkin-type approximation theory. Altomare et al. [1] have shown an application where they discussed differential operators associated with Markov operators, where also the knowledge of limit of the iterates is significant. Another application has been shown in the field of approximation theory, where the iterates can be used to prove lower estimates for Markov operators with sufficient smooth range, see Nagler et al. [17].

It is worthwhile to mention that in most methods the limiting operator has to be known apriori. Here, we show an elegant extension to general Banach spaces for quasi-compact Markov operators. This extension provides a very general framework to explicitly construct the limiting operator with a simple method without prior knowledge of this operator.

After an introductory example, we introduce briefly our notation and recall the most important results that are necessary to prove our results. All of these results are well-known and can be found, e.g., in the classical books of Ruston [20], Rudin [19], and Heuser [9]. In the next section, we discuss how the complemented subspace for some finite-dimensional eigenspace of an operator can be expressed in terms of the corresponding projection. We will start using the standard coordinate map to show the principle of our approach. Using a generalized version of the coordinate map we show conditions when the coordinate map on some eigenspace can be expressed in terms of a basis for this eigenspace and a basis of the corresponding eigenspace of the
adjoint operator. These results are used to prove the limiting behaviour of the iterates of quasi-compact Markov operators.

1. An introductory example

We now demonstrate the simplicity of our result in a short example on \( C([0, 1]) \), the space of continuous functions on the interval \([0, 1]\). Thereby, let \( n \) be a positive integer and suppose that \( \{x_j\}_{j=1}^n \) form a partition of \([0, 1]\), i.e. \( 0 = t_1 < t_2 < \cdots < t_n = 1 \). We consider the positive finite-rank operator \( T : C([0, 1]) \to C([0, 1]) \), defined for \( f \in C([0, 1]) \) by

\[
T f = \sum_{k=1}^n f(t_k) p_k, 
\]

where \( p_1, \ldots, p_n \in C([0, 1]) \) are positive functions that form a partition of unity, i.e., \( \sum_{k=1}^n p_k(t) = 1 \) for all \( t \in [0, 1] \). It is easy to see that in this case \( T1 = 1 \) and \( \|T\|_{op} = r(T) = 1 \), where \( r(T) \) is the spectral radius of \( T \). Besides, we assume that

\[
\sum_{k=1}^n t_k p_k(t) = t, \quad t \in [0, 1],
\]

i.e., \( Tf = f \) holds whenever \( f \) is a linear function. From that it follows already that \( p_1(0) = p_n(1) = 1 \) and \( T \) interpolates at 0 and 1, as

\[
T f(0) = \sum_{k=1}^n f(t_k) p_k(0) = f(t_1) = f(0), \quad T f(1) = \sum_{k=1}^n f(t_k) p_k(1) = f(t_n) = f(1).
\]

The introduced operator is a Markov operator, as it is a positive contraction and \( T1 = 1 \) holds. Two fixed points for \( T^* \) are given due to the interpolation at 0 and 1. If \( \delta_0, \delta_1 \) denote the continuous functionals that evaluate continuous functions at 0 and 1 respectively, then \( \delta_0(T f) = \delta_0(f) \) and \( \delta_1(T f) = \delta_1(f) \) holds for all \( f \in C([0, 1]) \).

In the following, we want to answer the question whether the limit of the iterates \( T^m \) for \( m \to \infty \) exists and if so to which operator the iterates converge. In Nagler [18] it has been shown that the partition of unity property, which is here equivalent to the ability to reproduce constant functions, guarantees that \( \sigma(T) \subset B(0, 1) \cup \{1\} \). To apply our main result, we have to specify the fixed point spaces of \( T \) and its adjoint \( T^* \). Using the partition of unity property of \( T \) and the ability of \( T \) to reproduce linear functions as well as the ability to interpolate at the endpoints of the interval \([0, 1]\), we derive the fixed-point spaces

\[
\ker(T - I) = \{ f \in C([0, 1]) : T f = f \} = \text{span}(1, x),
\]

\[
\ker(T^* - I) = \{ \alpha^* \in C([0, 1])^* : \alpha^*(T f) = \alpha^*(f) \text{ for all } f \in C([0, 1]) \} = \text{span}(\delta_0, \delta_1).
\]

Then we consider the Gram matrix

\[
G := \begin{pmatrix} \delta_0(1) & \delta_0(x) \\ \delta_1(1) & \delta_1(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},
\]

where the functionals of \( \ker(T^* - I) \) operate on the fixpoints of \( T \). Indeed, this matrix is invertible with

\[
A := G^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},
\]

and we are able to use the coefficients \( a_{11} = 1, a_{12} = 0, a_{11} = -1, a_{12} = 1 \) to conclude by Theorem 6 that

\[
\lim_{m \to \infty} \|T^m - P\|_{op} = 0,
\]
where the finite-rank projection \( P : C([0, 1]) \to \ker(T - I) \) is defined for \( f \in C([0, 1]) \) by
\[
P f = (a_{11} \delta_0(f) + a_{12} \delta_1(f)) \cdot 1 + (a_{21} \delta_1(f) + a_{22} \delta_0(f)) \cdot x
= \delta_0(f) \cdot 1 + \delta_1(f) - \delta_0(f) \cdot x = f(0) + (f(1) - f(0)) x.
\]
The iterates converge to the linear interpolation operator that interpolates at the endpoints of \([0, 1]\). In this example we demonstrated the underlying framework for finite-rank operators that reproduce constant and linear functions. Operators of this kind are, e.g., the Bernstein and the Schoenberg operator that are often used in CAGD and approximation theory. However, the convergence is guaranteed for all quasi-compact linear functions. Operators of this kind are, e.g., the Bernstein and the Schoenberg operator that are often used in CAGD and approximation theory. However, the convergence is guaranteed for all quasi-compact Markov operators. Note that the following implications hold:
\[
\text{finite-rank} \implies \text{compact} \implies \text{Riesz} \implies \text{quasi-compact}
\]

2. Notation

For the convenience of the reader this section provides not only the used notation throughout this article but also a compact overview over the most important facts that are used later. All results in this chapter can be found in the comprehensive books of Heuser [9], Rudin [19] and Ruston [20].

The general setting considers \( X \) as a complex Banach space equipped with a norm \( \| \cdot \|_X \). If the used norm is unambiguous we will just use the abbreviated version \( \| \cdot \| \). Note that the results shown here are also applicable on real Banach spaces using a standard complexification scheme as outlined, e.g, in Ruston [20, pp. 7–16].

The Banach algebra of bounded linear operators on \( X \) is denoted by \( \mathcal{L}(X) \) equipped with the usual operator norm \( \| \cdot \|_{\text{op}} \). The identity operator on \( X \) is \( I \in \mathcal{L}(X) \). The corresponding topological dual space \( \mathcal{L}(X, \mathbb{C}) \) is denoted by \( (X^*, \| \cdot \|_{X^*}) \). The range and null space of \( T \in \mathcal{L}(X) \) is denoted by \( \text{ran}(T) \) and \( \ker(T) \), respectively. The closure of \( M \subset X \) is denoted by \( \overline{M} \). We denote the space of all compact operators from \( X \) to \( Y \) by \( \mathcal{K}(X,Y) \).

2.1. Annihilators

For \( M \subset X \) and \( \Lambda \subset X^* \), we denote by \( M^\perp \) the annihilator of \( M \), i.e.
\[
M^\perp := \{ x^* \in X^* : x^*(x) = 0 \text{ for every } x \in M \} \subset X^*,
\]
and by \( \Lambda_\perp \) the pre-annihilator of the set \( \Lambda \), i.e.
\[
\Lambda_\perp = \{ x \in X : x^*(x) = 0 \text{ for every } x^* \in \Lambda \} \subset X.
\]
Recall that if \( X \) and \( Y \) are Banach spaces and \( T \in \mathcal{L}(X,Y) \), then
\[
\ker(T^*) = T(X)^\perp, \quad \ker(T) = T^*(Y^*)_\perp,
\]
where \( T^* \) denotes the adjoint of \( T \).

2.2. Fredholm, Weyl and Browder operators

An operator \( T \in \mathcal{L}(X) \) is said to have finite ascent if there exists \( k \in \mathbb{N} \) such that \( \ker(T^k) = \ker(T^{k+1}) \). The smallest integer with this property is the ascent of \( T \) and will be denoted by \( \text{asc}(T) \). Accordingly, \( T \) has finite descent if there exist \( k \in \mathbb{N} \) such that \( \text{ran}(T^k) = \text{ran}(T^{k+1}) \) and we denote by \( \text{dsc}(T) \) the smallest integer with this property and call this number the descent of \( T \). Recall, that
\[
\text{asc}(T) \leq m < \infty \quad \text{iff} \quad \ker(T^n) \cap \text{ran}(T^m) = \{0\}
\]
holds, where \( n > 0 \) is arbitrary. If both, the ascent and the descent, are finite, then they are equal. In this case, the operator \( T \) is said to have finite chain length \( p \) and yields a direct sum decomposition in the following way:
\[
X = \text{ran}(T^p) \oplus \ker(T^p).
\]
The concept of the ascent and descent. Recall that if asc(\(T\)) and dsc(\(T\)) are finite, and by induction, we will denote all such linear operators \(T \in \Phi(X)\) with finite ascent, \(p = asc(T) < \infty\), as operators where

\[
dim(ker(T^*)) < \dim(ker(T)) < \infty.
\]

We will denote all such linear operators \(T\) that have a Fredholm index less or equal than zero by the set \(\Phi_-(X)\). A bounded operator \(T \in \mathcal{L}(X)\) is said to be a Weyl operator if \(T \in \Phi(X)\) with index 0. The class of all Weyl operators on \(X\) will be denoted by \(\mathcal{W}(X)\).

A bounded operator \(T \in \mathcal{L}(X)\) is said to be a Browder operator if it is a Fredholm operator with finite chain length. We will denote the sets of all Browder operators on \(X\) by \(\mathcal{W}_p(X)\). Each Browder operator \(T\) is in fact a Weyl operator, as in that case \(asc(T) = dsc(T) < \infty\) and \(\alpha(T) = \beta(T) < \infty\) holds. A comparison between both classes is shown in Table 1. Note that due to the finite chain-length \(p\) of a Browder operator \(T\), the following properties hold:

1. \(dim(ker(T)) = dim(ker(T^*)) < \infty\),
2. \(X = ker(T^p) \oplus ran(T^p)\).

In this article, we are interested in operators \(T \in \mathcal{L}(X)\) where \((T - \lambda I)\) is a Browder operator. In that case, we construct an explicit projection for the space decomposition shown in the second item.

### 2.3. Spectral projections

We denote by \(r(T)\) the resolvent set of \(T \in \mathcal{L}(X)\). The resolvent of \(T\) corresponding to \(\lambda \in \mathbb{C}\) will be denoted by \(R(T, \lambda) := (T - \lambda I)^{-1}\). The spectrum of \(T\) is denoted by \(\sigma(T)\), the spectral radius by \(r(T)\).

Using functional calculus, it is well known that spectral projections exactly provide the space decomposition discussed previously. The spectral projection associated to a spectral set \(\sigma\) is given by

\[
P_\sigma := \frac{1}{2\pi i} \int_{\Gamma_\sigma} R(T, \lambda) d\lambda, \tag{5}
\]

where \(\Gamma_\sigma\) is a simple, closed integration path oriented counterclockwise that lies in the resolvent set \(\rho(T)\) and encloses \(\sigma\). Recall, that \(\lambda\) is a pole of the resolvent of \(T\) if and only if \(T - \lambda I\) has positive finite chain length \(p\) which also is the order of the pole. In this case \(\lambda \in \sigma_p(T)\), i.e., \(\lambda\) is an eigenvalue of \(T\). The spectral projector \(P_{\{\lambda\}}\) corresponding to \(\{\lambda\}\) satisfies

\[
\text{ran}(P_{\{\lambda\}}) = ker(T - \lambda I)^p \quad \text{and} \quad ker(P_{\{\lambda\}}) = ran(T - \lambda I)^p. \tag{6}
\]

If furthermore \(T - \lambda I\) is a Fredholm operator, i.e. a Browder operator, then \(\lambda\) is always an isolated eigenvalue of \(T\) and the associated spectral projection is finite-dimensional.

Note that the computation of the spectral projection using the formula provided in (5) is in general hard to calculate. In the next section, we will consider operators \(T\) where \(T - \lambda I\) is a Browder operator and explicitly construct the corresponding spectral projection \(P_{\{\lambda\}}\).
3. Invariant subspaces of linear operators

The aim of this section is to show how to construct a projection $P$ onto a generalized eigenspace of a bounded linear operator $T$ defined on a complex Banach space $X$ corresponding to an eigenvalue $\lambda \in \mathbb{C}$. To this end, we consider an operator $T$ such that $T - \lambda I$ is a Browder operator with ascent $\text{asc}(T - \lambda I) = p$. In this case, the projection has the property $\ker(P) = \text{ran}(T - \lambda I)^p$ which gives us generically the following space decomposition:

$$X = \ker(T - \lambda I)^p \oplus \text{ran}(T - \lambda I)^p = \text{ran}(P) \oplus \ker(P).$$

We provide a simple criterion under which assumptions this space decomposition is possible. Before we will look at a finite-dimensional generalized eigenspace of an operator $T \in \mathcal{L}(X)$, we will construct the projection on an arbitrary finite-dimensional subspace $M$ of a vector space $X$. On $M$ we introduce the classical coordinate map defined by a basis of $M$ and the corresponding dual basis of the dual space $M^*$. By the extension theorems of Hahn-Banach the coordinate map gives us a continuous projection of $X$ onto $M$. In the sequel, we will discuss conditions on the functionals that can be chosen in the coordinate map to build a dual basis. Finally, we apply the results to the generalized eigenspaces of a bounded linear operator $T$ on a Banach space $X$ and its adjoint $T^*$ corresponding to an eigenvalue $\lambda \in \mathbb{C}$. A necessary condition on the operator $T - \lambda I$ is being Fredholm with non-positive index. If in addition $T - \lambda I$ is a Browder operator, i.e., the index is zero and its chain length is finite, then the projection yields the previously mentioned direct sum decomposition of $X$.

Note that this space decomposition is already well known, see [4], provided $T - \lambda I$ has positive finite chain length. In contrast to existing literature we prove it using an explicitly constructed finite-rank projection $P$. This method uses in fact the restriction that $T - \lambda I$ has to be Weyl operator, i.e., a Fredholm operator of index zero, to guarantee that the corresponding generalized eigenspaces of $T$ and $T^*$ have finite dimension. This direct construction of the projection $P$ provides an alternative way to calculate the spectral projection corresponding to the eigenvalue $\lambda$ as by (5).

3.1. Dual basis and the coordinate map

Let $X$ be a normed vector space over the complex numbers and let $M \subset X$ be a closed subspace with $0 < \dim(M) < \infty$. In the sequel, we denote its dimension by $n = \dim(M)$. Moreover, let $\{e_1, \ldots, e_n\}$ be a basis for $M$. Then every $x \in M$ has a unique representation

$$x = \sum_{i=1}^{n} a_i^*(x)e_i,$$

where $\{a_1^*, \ldots, a_n^*\}$ are appropriate continuous linear functionals on $M$. By definition, each $u_i$ can also be represented by (7) which yields the characterization

$$a_i^*(e_k) = \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases}$$

for all $i, k \in \{1, \ldots, n\}$. In analogy to the construction of the frame operator on Hilbert spaces [4], we define a synthesis operator $\Phi : \mathbb{C}^n \rightarrow M$ by

$$\Phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i e_i.$$  

The adjoint of this operator $\Phi^* : M \rightarrow \mathbb{C}^n$ yields the analysis operator

$$\Phi^*(x) = \begin{pmatrix} a_1^*(x) \\ \vdots \\ a_n^*(x) \end{pmatrix}, \quad x \in X.$$
Combining both operators we can represent the coordinate map (7) by the composition \( \Phi \Phi^* : M \to M \),

\[ \Phi \Phi^*(x) = \sum_{i=1}^{n} a_i^*(x) e_i = x. \]

Note that according to (8) the matrix \( \Phi^* \Phi \in \mathbb{C}^{n \times n} \) is the identity on \( \mathbb{C}^n \):

\[
\Phi^* \Phi = \begin{pmatrix}
    a_1^*(e_1) & \cdots & a_n^*(e_1) \\
    \vdots & \ddots & \vdots \\
    a_1^*(e_n) & \cdots & a_n^*(e_n)
\end{pmatrix} = \begin{pmatrix}
    1 & 0 & \cdots & 0 \\
    0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1
\end{pmatrix} = I_n.
\]

Accordingly, the basis \( \{a_1^*, \ldots, a_n^*\} \subset M^* \) is said to be the dual basis for \( \{e_1, \ldots, e_n\} \subset M \). Applying the Theorem of Hahn-Banach, the coordinate map can be extended to the whole vector space \( X \).

**Lemma 1.** The operator \( \Phi \Phi^* : M \to M \) can be extended to a projection of the space \( X \) onto the closed set \( M \) and is bounded by

\[
\| (\Phi \Phi^*) (x) \| \leq \| x \| \sum_{i=1}^{n} \| a_i^* \|. 
\]

The matrix \( (\Phi^* \Phi)_{ij} \in \mathbb{C}^{n \times n} \) is invertible and the coordinate map \( \Phi \Phi^* \mid \) \( M \) which is restricted on \( M \) yields an isomorphism. The space \( X \) can be decomposed into

\[ X = M \oplus \ker(\Phi \Phi^*). \]

**Proof.** The continuous functionals \( a_i^* \) can be extended by the classical Hahn-Banach Theorem to \( X^* \) with the same properties as on \( M \). We denote the resulting extensions again as \( a_i^* \in X^* \). Therefore, \( \Phi \Phi^* : X \to M \) and

\[ (\Phi \Phi^*)(x) = \sum_{i=1}^{n} a_i^*(x) e_i = x \quad \text{for all } x \in M. \]

Moreover, the operator is bounded on \( X \) since for \( x \in X \) we have

\[
\| (\Phi \Phi^*)(x) \| = \left\| \sum_{i=1}^{n} a_i^*(x) e_i \right\| \leq \sum_{i=1}^{n} \| a_i^*(x) \| \| e_i \| \leq \| x \| \sum_{i=1}^{n} \| a_i^* \| ,
\]

where we used that \( \| e_i \| = 1 \) and the fact that \( \| a_i^*(x) \| \leq \| a_i^* \| \| x \| \). Clearly, \( (\Phi^* \Phi)^{-1} = I_n \). It yields also a projection, because for every \( x \in M \) we obtain \( (\Phi \Phi^*)(x) = x \) and therefore, \( (\Phi \Phi^*)^2 = (\Phi \Phi^*) \). As the operator \( \Phi \Phi^* \in \mathcal{L}(X) \) is a bounded projection onto the closed space \( M \), we obtain canonically the space decomposition \( X = M \oplus \ker(\Phi \Phi^*) \). \( \square \)
Figure 2: Commutative diagram showing the projection $\Phi A \Phi^* : X \rightarrow M$. Here the matrix $A$ is either the Moore-Penrose inverse of the matrix $\Phi^* \Phi$ or its inverse.

The key property to notice here is that $(\Phi^* \Phi)_{ij}$ is an invertible matrix and that $\Phi \Phi^*$ is a projection onto $M$. The commutative diagram shown in Figure 1 illustrates the behaviour of $\Phi$ and $\Phi^*$.

In the following, we show which functionals can be chosen instead of the dual basis such that $\Phi \Phi^*$ is still a projection where the analysis operator $\Phi^*$ now contains the new functionals. The next section shows that the matrix $\Phi^* \Phi$ must have full column rank.

### 3.2. Complemented subspaces and projections

We consider now the following problem. Given a set of linear functionals $\Lambda \subset X^*$, we ask whether it is possible to construct a projection onto the closed finite dimensional subspace $M \subset X$ with functionals chosen only from the set $\Lambda$. We give a characterization in the next theorem. As in the previous section, we consider a finite-dimensional subspace $M$ of $X$. Additionally, let $\Lambda \subset X^*$ be a finite-dimensional subspace of $X^*$. Let us denote by $\{e_1, \ldots, e_n\}$ and $\{e_1^*, \ldots, e_m^*\}$ a basis of $M$ and $\Lambda$, respectively. The synthesis operator $\Phi : C^n \rightarrow X$ is constructed as in (9), whereas the analysis operator $\Phi^* : X \rightarrow C^n$ is this time not defined as the adjoint of $\Phi$ but uses the basis functionals of $\Lambda$:

$$\Phi^*(x) := \begin{pmatrix} e_1^*(x) \\ \vdots \\ e_m^*(x) \end{pmatrix}, \quad x \in X.$$  

Let us assume that $\dim(\Lambda) \geq \dim(M)$ holds. Then we will show in the next theorem that again $\Phi \Phi^*$ yields a projection operator onto $M$ provided that $\Phi^* \Phi$ has full column rank.

**Theorem 1.** Let $\Lambda \subset X^*$ with $0 < \dim(M) \leq \dim(\Lambda) < \infty$ and let $n = \dim(M)$, $m = \dim(\Lambda)$. Then the operator $P \in \mathcal{L}(X)$ defined for $A = (a_{ij}) \in C^{n \times m}$ by

$$P x = \Phi A \Phi^* (x) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} e_j^*(x) e_i, \quad x \in X,$$

yields a projection onto $M$ if and only if the matrix

$$G := (\Phi^* \Phi) = \begin{pmatrix} e_1^*(e_1) & \cdots & e_1^*(e_n) \\ \vdots & \ddots & \vdots \\ e_m^*(e_1) & \cdots & e_m^*(e_n) \end{pmatrix} \in C^{m \times n}$$

has full column rank $n$. In this case, the matrix $A$ is determined by the Moore-Penrose inverse of $G$,

$$A = (G^T G)^{-1} G^T.$$

**Proof.** Let us first assume that for the matrix $G \in C^{m \times n}$ exists the Moore-Penrose inverse

$$G_{\text{left}}^{-1} = (G^T G)^{-1} G^T \in C^{n \times m},$$
and let \( A = G_{\text{ref}}^{-1} \). According to its definition we have

\[
A \cdot G = G_{\text{ref}}^{-1} \cdot G = I_n.  \tag{13}
\]

Now, we prove that \( P \), defined for \( x \in X \) as in (11), is a projection onto \( M \). To this end, we will show that \( P(x) = x \) holds for all \( x \in M \) by considering the basis of \( M \). Thus, we only have to prove \( P(e_k) = e_k \) for all \( k \in \{1, \ldots, n\} \). The direct calculation of \( P(e_k) \) yields

\[
P(e_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} e_j^\ast(e_k) e_i = \sum_{i=1}^{n} \left[ \sum_{j=1}^{m} a_{ij} e_j^\ast(e_k) \right] e_i.
\]

Applying (13) yields \( \sum_{j=1}^{m} a_{ij} e_j^\ast(e_k) = \delta_{ki} \). Therefore,

\[
P(e_k) = \sum_{i=1}^{n} p_i \delta_{ki} = e_k
\]

holds and we obtain \( P(X) = M \). Furthermore, \( P^2 = P \) on \( X \) as \( \{e_1, \ldots, e_n\} \) forms a basis for \( M \). Finally, we show the reverse direction. To this end, let us assume that \( P \) is a projection onto \( M \), i.e., \( P(X) = M \) and \( P^2 = P \) holds. Then \( P(e_k) = e_k \) must hold for any \( k \in \{1, \ldots, n\} \), as \( e_i \in M \). We calculate

\[
P(e_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} e_j^\ast(e_k) e_i = \sum_{i=1}^{n} \left[ \sum_{j=1}^{m} a_{ij} e_j^\ast(e_k) \right] e_i.
\]

This yields necessary the requirement \( \sum_{j=1}^{m} a_{ij} e_j^\ast(e_k) = \delta_{ik} \) for all \( k \in \{1, \ldots, n\} \). Therefore, we derive the matrix equation \( A \cdot G = I_n \) with the unknown coefficient matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times m} \). In fact, this equation has a solution if and only if the matrix \( G \) has a Moore-Penrose inverse \( G_{\text{ref}}^{-1} \), which concludes the proof.

Next, we will provide an upper bound of the projection operator \( P = \Phi A \Phi^\ast \) by the 1-norm of \( A \).

**Lemma 2.** Under the assumption of Theorem 1, the projection operator \( P \) defined by (11) has finite-rank and is bounded by

\[
\|Px\| \leq \|x\| \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|, \quad x \in X.
\]

**Proof.** Clearly, \( P \) is a finite rank operator. Let \( x \in X \). For arbitrary \( i \in \{1, \ldots, n\} \) we obtain

\[
\left| \sum_{j=1}^{m} a_{ij} e_j^\ast(x) \right| \leq \|x\| \sum_{j=1}^{m} |a_{ij}|
\]

because the dual basis is normalized, i.e., \( \|e_j^\ast\| = 1 \). Using the same argument for the basis of \( M \) we get

\[
\|Px\| = \|\Phi A \Phi^\ast x\| = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} e_j^\ast(x) e_i \leq \|x\| \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|.
\]

\( \square \)
3.3. Invariant subspaces and projections

In the following we will consider a linear operator $T$ defined on a complex Banach space $X$. As in the preceding sections we are interested in the construction of a projection onto a finite-dimensional subspace of $X$. Here, we choose $M$ as a generalized eigenspace of $T$ corresponding to an eigenvalue $\lambda \in \sigma_p(T)$. It will be shown that the set of functionals is exactly given by the corresponding generalized eigenspace of the adjoint $T^*$.

Accordingly, given some integer $p > 0$, we consider now the following two subspaces

$$M^p_\lambda = \ker(T - \lambda I)^p = \{x \in X : (T - \lambda I)^p x = 0\} \subset X, \quad (14)$$

$$\Lambda^p_\lambda = \ker(T^* - \lambda I)^p = \{x^* \in X^* : (T^* - \lambda I)^p x^* = 0\} \subset X^*. \quad (15)$$

Note that due to the fact that $\ker(T^* - \lambda I)^p = \ran((T - \lambda I)^p)^\perp$ holds by (2) the set $\Lambda^p_\lambda$ can also be determined as

$$\Lambda^p_\lambda = \{x^* \in X^* : x^* ((T - \lambda I)^p x) = 0 \text{ for all } x \in X\}. \quad (16)$$

To assure that both spaces (14) and (15) are finite-dimensional and that the dimension of the functionals $\Lambda^p_\lambda$ is greater than the dimension of $M^p_\lambda$, we assume in the following that $(T - \lambda I)^p$ is a Fredholm operator with negative index, i.e., $\ind(T - \lambda I)^p \leq 0$. Then we have by definition

$$n = \dim(M^p_\lambda) \leq \dim(\Lambda^p_\lambda) = m$$

and we can consider w.l.o.g. normalized bases of $M^p_\lambda$ and $\Lambda^p_\lambda$:

$$M^p_\lambda = \operatorname{span}\{e_1, \ldots, e_n\} \text{ and } \Lambda^p_\lambda = \operatorname{span}\{e^*_1, \ldots, e^*_n\} \quad (17)$$

such that $\|e_i\|_X = 1$ and $\|e^*_i\|_{X^*} = 1$. If we additionally suppose we have the following finite chain of inclusions

$$\ker(T - \lambda I) \subset \ker(T - \lambda I)^2 \subset \cdots \subset \ker(T - \lambda I)^p = \ker(T - \lambda I)^{p+1} \subset \cdots,$$

then the ascent of $T - \lambda I$ is specified as $p := \asc(T - \lambda I) < \infty$. Corresponding to the eigenvalue $\lambda \in \sigma_p(T)$, the set $M^p_\lambda$ contains all of the generalized eigenvectors of the operator $T$ and the set $\Lambda^p_\lambda$ contains all the dual generalized eigenvectors. More precisely, the set $\Lambda^p_\lambda$ contains all the generalized eigenvectors of the adjoint operator $T^*$ to the eigenvalue $\lambda$.

**Remark 1.** Note that the assumption on $T$ are not very restrictive. As shown in the end of the last chapter, every compact operator satisfies all of the conditions. Moreover, quasi-compact operators satisfy these condition in the case where $\lambda = 1$ is chosen. Especially, every operator where $T - \lambda I$ is a Brouwer operator fulfills these conditions, see the definition and comments in subsection 2.3.

Next, we will show how to construct a projection $P$ onto $\ker(T - \lambda I)^p$ to obtain the space decomposition

$$\ker(T - \lambda I)^p \oplus \ker(P)$$

such that $\ker(P) = \ran(T - \lambda I)^p$ holds. Note that in this case $\ran(T - \lambda I)^p$ is closed as it is the null space of the projection $P$.

First, we provide an equivalent characterization of the restrictions on $T$ to have finite chain length of the generalized eigenspaces of $T$ provided that $T - \lambda I$ is a Fredholm operator with $\ind(T - \lambda I) \leq 0$ to assure that the generalized eigenspaces of $T$ and $T^*$ are finite-dimensional. The next lemma shows that the ascent can be characterized by the column rank of the Gramian matrix constructed using the matrix (12). In the following, we will denote by $\Phi_-(X)$ all Fredholm operators defined on the Banach space $X$ that have an index less or equal to zero.

**Lemma 3.** Let $T \in \mathcal{L}(X)$ and $\lambda \in \sigma_p(T)$ such that $T - \lambda I \in \Phi_-(X)$. Then $T - \lambda I$ has finite ascent $p$, i.e.,

$$p = \asc(T - \lambda I) = p < \infty,$$

if and only if the Gramian matrix

$$G := (\Phi^*\Phi) = \begin{pmatrix}
  e^*_1(e_1) & \cdots & e^*_1(e_n) \\
  \vdots & \ddots & \vdots \\
  e^*_m(e_1) & \cdots & e^*_m(e_n)
\end{pmatrix} \in \mathbb{C}^{m \times n}
$$

has full column rank.
Proof. Suppose that $T - \lambda I$ is Fredholm operator with non-positive index. Then $(T - \lambda I)^p$ is also Fredholm with non-positive index $p \cdot \text{ind}(T - \lambda I)$. This follows by the index theorem [8, Thm. 23.1], as

$$\text{ind}((T - \lambda I) \cdots (T - \lambda I)) = \sum_{i=1}^{p} \text{ind}(T - \lambda I) = p \cdot \text{ind}(T - \lambda I).$$

Therefore, $\text{ran}(T - \lambda I)^p$ is closed [8, Prop. 24.3] and

$$n = \alpha((T - \lambda I)^p) = \dim(\ker(T - \lambda I)^p) \leq \dim(\ker(T^* - \lambda I)^p) = \beta((T - \lambda I)^p) = m.$$

Note that $(\Lambda^p_A)^\perp = (\text{ran}((T - \lambda I)^p)^\perp)^\perp = \text{ran}(T - \lambda I)^p = \text{ran}(T - \lambda I)^p$.

Let us now assume that $T - \lambda I$ has ascent $p$. In order to show that the columns of $G = \Phi^*\Phi$ are linearly independent, we choose $c = (c_1, \ldots, c_n)^T \in \mathbb{C}^n$ such that

$$\sum_{i=1}^{n} c_i e_j^*(e_i) = 0$$

for all $j \in \{1, \ldots, m\}$. Then we derive that $e_j^*(\sum_{i=0}^{n} c_i e_i) = 0$ for all $j \in \{1, \ldots, m\}$. Therefore, we can conclude with (3) that $\ker(e_j^*) = (\Lambda^p_A)^\perp = \text{ran}(T - \lambda I)^p$.

As $T - \lambda I$ has finite ascent $p$ we can conclude with [8] that $\ker(T - \lambda I)^p \cap \text{ran}(T - \lambda I)^p = \{0\}$ holds. As by definition also $\sum_{i=1}^{n} c_i e_i \in \ker(T - \lambda I)^p$ holds we derive that $\sum_{i=1}^{n} c_i e_i = 0$. From the linear independence of $\{e_1, \ldots, e_n\}$ it follows that $c_1 = \cdots = c_n = 0$. Therefore, the matrix $\Phi^*\Phi$ has full rank, as the columns are linearly independent.

To show that the converse is also true let us suppose that the matrix $G$ has full column rank. Hence, if $\sum_{i=1}^{n} c_i e_i \in \ker(T - \lambda I)^p$ holds it follows that every coefficient $c_i = 0$ for all $i \in \{1, \ldots, n\}$. Suppose now that $x \in \ker(T - \lambda I)^p \cap \text{ran}(T - \lambda I)^p$. Then $x$ can be written as linear combination $x = \sum_{i=1}^{n} c_i e_i$ for some coefficients $c_i \in \mathbb{C}$. As $\text{ran}(T - \lambda I)^p = (\Lambda^p_A)^\perp$, we obtain for all $j \in \{1, \ldots, m\}$ that

$$0 = e_j^* \left( \sum_{i=1}^{n} c_i e_i \right) = \sum_{i=1}^{n} c_i e_j^*(e_i).$$

We conclude that $c_i = 0$ for all $i \in \{1, \ldots, n\}$ as the matrix $G$ has full column rank. Finally, we have $x = 0$. Therefore, $\ker(T - \lambda I)^p \cap \text{ran}(T - \lambda I)^p = \{0\}$. By Equation 3 this is equivalent to the statement that the ascent of $T - \lambda I$ is $p$ and the proof is complete.

As the Gramian matrix has full column rank, we can construct a projection operator onto $\ker(T - \lambda I)$ according to Theorem 1. Consequently, as in the last section, we consider the finite-rank operator $P \in K(X)$ defined for $x \in X$ by

$$Px = (\Phi A^\Phi^*) (x) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} e_j^*(x) e_i,$$

where $e_i \in M_A$, $e_j^* \in \Lambda_A$ are the normalized bases and $A = (a_{ij}) \in \mathbb{C}^{n \times m}$. This time, the functionals $e_j^*$ are explicitly chosen as basis of $\ker(T^* - \lambda I)^p$ where the coefficients $a_{ij}$ serve as parameter. In this setting Theorem 1 yields a projection operator that projects onto the generalized eigenspace $M_A^p$ and provides a space decomposition of $X$ into $X = M_A^p \oplus \ker(P)$.

Corollary 1. Let $T \in \mathcal{L}(X)$ and $\lambda \in \sigma_p(T)$ such that $T - \lambda I \in \Phi_-(X)$ with ascent $p \in \mathbb{N}$. Then the linear operator $P \in K(X)$ defined for $x \in X$ as

$$Px = \Phi A^\Phi^*(x),$$

where $A$ is the Moore-Penrose inverse of $(\Phi^* \Phi)$, yields a continuous projection onto $M_A^p \subset X$, where $\text{ran}(P) = M_A^p = \ker(T - \lambda I)^p$ is a closed subspace.
Proof. This is a direct consequence of Lemma 1 and Lemma 3.

Note that in the current setting, we obtain a projection $P$ where $\text{ran}(P) = \ker(T - \lambda I)^p$ is a $T$-invariant subspace. Accordingly, we have the space decomposition

$$X = \text{ran}(P) \oplus \ker(P) = \ker(T - \lambda I)^p \oplus \ker(P).$$

In the following we are interested when also $\ker(P)$ is invariant with respect to the operator $T$. Then we can decompose the operator $T$ into

$$T = \begin{pmatrix} J & 0 \\ 0 & S \end{pmatrix} \in \mathcal{L}(\ker(T - \lambda I)^p \oplus \ker(P)),$$

where $J$ is the Jordan normal form of $T$ on the generalized eigenspace $\ker(T - \lambda I)^p$ and $S \in \mathcal{L}(\ker(P))$ is equal to the operator $T$ restricted to $\ker(P)$.

Remark 2. Even though we write the operator decomposition in matrix notation, we don’t assume the Banach space $X$ to be separable. The matrix form is only used to simplify notation as $\ker(T - \lambda I)$ is always finite-dimensional. In this case, $J$ is given according to some basis, whereas $S$ is not necessarily defined by a countable dense set in $X$.

Furthermore, we are not only interested whether $\ker(P)$ is invariant with respect to $T$, we also want to know under which conditions on $T$ the relation $\ker(P) = \text{ran}(T - \lambda I)^p$ holds. It turns out that this is exactly the case when the $T - \lambda I$ is a Browder operator, i.e., the operator $T - \lambda I$ has Fredholm index 0 and finite chain length $p$. We will discuss this particular case in the following. First, we show in the next lemma that the Fredholm index 0 leads to the invertibility of the Gramian matrix $\Phi^* \Phi$. Finally, we will prove that in this case $\ker(P) = \text{ran}(T - \lambda I)^p$ holds. We will conclude this section with an overview over related results.

Lemma 4. Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$ such that $T - \lambda I \in \Phi^-(X)$. Then $T - \lambda I$ is a Browder operator if and only if the matrix $G = \Phi^* \Phi$ is invertible.

Proof. If $G$ is invertible, then $T - \lambda I$ has finite ascent $p$ by Lemma 3 and $G$ is necessarily a square matrix, thus $\text{ind}(T - \lambda I)^p = 0$ as

$$n = \alpha((T - \lambda I)^p) = \beta((T - \lambda I)^p) = m,$$

using the definition of the nullity $\alpha((T - \lambda I)^p) = \dim \ker(T - \lambda I)^p$ and the deficiency $\beta((T - \lambda I)^p) = \dim \ker(T^* - \lambda I)^p$. As $\alpha(T - \lambda I) = \beta(T - \lambda I)$ and $\text{asc}(T - \lambda I) = p < \infty$, we can conclude by [3, Prop. 38.5 and 38.6] that also the descent of $T - \lambda I$ is finite. Therefore, $T - \lambda I \in \mathcal{W}_B(X)$, i.e., $T - \lambda I$ is a Browder operator with ascent $p$.

Assume to the contrary that $T - \lambda I$ is a Browder operator. Then $T - \lambda I$ has finite ascent $p$ and $\text{ind}(T - \lambda I) = 0$ by definition. As $\text{ind}(T - \lambda I) = 0$ the matrix $G = \Phi^* \Phi$ is a $n \times n$-matrix as $n = \alpha((T - \lambda I)^p) = \beta((T - \lambda I)^p)$ using the same argument as in (19). As we have the conditions $\text{ind}(T - \lambda I) = 0$ and $\text{asc}(T - \lambda I) = p$ we can apply Lemma 3 to conclude that the matrix $G$ has full rank and, thus, is invertible as a square matrix.

Next, we will prove that the null space of the projection $P$ is given by $\ker(P) = \text{ran}(T - \lambda I)^p$ provided that $T - \lambda I$ is a Fredholm operators with index 0 having finite chain length $p$, i.e., $T - \lambda I$ is a Browder operator. Note that the invertibility of $\Phi^* \Phi$ is already sufficient for this result.

Theorem 2 (Space decomposition). Let $T \in \mathcal{L}(X)$ and $\lambda \in \sigma_p(T)$ such that $T - \lambda I$ is a Browder operator with ascent $p$. Then

$$X = \ker(T - \lambda I)^p \oplus \text{ran}(T - \lambda I)^p,$$

where $\text{ran}(\Phi(\Phi^* \Phi)^{-1} \Phi^*) = \ker(T - \lambda I)^p$ and $\ker(\Phi(\Phi^* \Phi)^{-1} \Phi^*) = \text{ran}(T - \lambda I)^p$. 

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Proof. Let \( n = \dim(\ker(T - \lambda I)^p) = \dim(\ker(T^* - \lambda I)^p) < \infty \). As \((T - \lambda I)^p\) is a Fredholm operator, \(\text{ran}(T - \lambda I)^p\) is closed. We already have shown that \(\text{ran}(P) = \ker(T - \lambda I)^p\). In order to show \(\ker(P) = \text{ran}(T - \lambda I)^p\) let \(x \in \ker(P)\). Then we have

\[
0 = Px = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}e_j^*(x)e_i. \tag{20}
\]

As \(\{e_1, \ldots, e_n\}\) form a basis for \(\ker(T - \lambda I)^p\) by (14) and (17), relation (20) can only hold if

\[
\sum_{j=1}^{n} a_{ij}e_j^*(x) = 0
\]

for every \(i \in \{1, \ldots, n\}\). Using that \(A = (\Phi^*\Phi)^{-1}\) is invertible by Lemma 3 we obtain that \(e_j^*(x) = 0\) for all \(j \in \{1, \ldots, m\}\). Then it is easy to see that

\[
x \in (\Lambda^p_{\perp})_\perp = (\text{ran}((T - \lambda I)^p)\perp)_\perp = \text{ran}(T - \lambda I)^p,
\]

because \(\text{ran}(T - \lambda I)^p\) is closed.

Now let \(y \in \text{ran}(T - \lambda I)^p\). Accordingly, there is \(x \in X\) with \((T - \lambda I)^p x = y\). In this case also \(y \in \ker(P)\) holds, because

\[
Py = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}e_j^*((T - \lambda I)^p x)e_i = 0.
\]

In the last step we used that \(e_j^* \in \text{ran}((T - \lambda I)^p)\perp\). Finally, we obtain the space decomposition

\[
X = \text{ran}(P) \oplus \ker(P) = \ker(T - \lambda I)^p \oplus \text{ran}(T - \lambda I)^p,
\]

where \(\text{ran}(P) = \ker(T - \lambda I)^p\) and \(\ker(P) = \text{ran}(T - \lambda I)^p\). \(\square\)

We conclude this section with a theorem that gathers all the results we have shown for a bounded operator \(T\) with eigenvalue \(\lambda \in \sigma_p(T)\), where \(T - \lambda I\) is a Weyl operator, i.e., a Fredholm operator with zero index. Note once more that this restriction is important for our setting where the generalized eigenspaces have to be finite-dimensional.

**Theorem 3** (Characterization of the Browder operator \(T - \lambda I\)).

Let \(T \in \mathcal{L}(X)\) and \(\lambda \in \sigma_p(T)\) such that \(T - \lambda I \in \mathcal{W}(X)\). Then the following statements are equivalent:

1. \(T - \lambda I\) is a Browder operator, \(T - \lambda I \in \mathcal{W}_B(X)\),
2. the operator \(T - \lambda I\) has finite chain length, i.e., \(\text{asc}(T - \lambda I) = \text{dsc}(T - \lambda I) < \infty\),
3. the space \(X\) can be decomposed into \(X = \ker(T - \lambda I)^p \oplus \text{ran}(T - \lambda I)^p\),
4. the \(n \times n\) matrix \(G := (\Phi^*\Phi)\),

\[
G = \begin{pmatrix}
e_{1}^*(e_{1}) & \cdots & e_{n}^*(e_{n}) \\
\vdots & \ddots & \vdots \\
e_{1}^*(e_{1}) & \cdots & e_{n}^*(e_{n})
\end{pmatrix} \in \mathbb{C}^{n \times n}
\]

is invertible, where \(n = \dim \ker(T - \lambda I)^p = \dim \ker(T^* - \lambda I)^p\),
5. the operator \(P : X \to \ker(T - \lambda I)^p\) defined by

\[
P x = \Phi A \Phi^* (x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}e_j^* (x)e_i, \quad x \in X,
\]

yields a projection onto \(\ker(T - \lambda I)^p\), where \(A = (a_{ij}) := G^{-1}\).

The main contribution of this article is the invertibility of the Gram matrix and the construction of the projection operator as stated in the last item. We have shown an explicit construction of the projection operator \(P\) by the inverse of the Gramian matrix for the space decomposition in the third item. In the next section, we will apply these results to uniform ergodic theorems.
4. Application: Uniform ergodic theorems

We conclude this article by showing a relation between the theory developed in the last sections and uniform ergodic theorems. Since \[22\] has shown that if \(T\) is a contraction on a Banach space \(X\) then the Cesàro mean
\[
a_n(T) := n^{-1} \sum_{k=0}^{n-1} T^k
\]
converge strongly for \(n \to \infty\) if and only if the fixed points of \(T^*\) separate the fixed points of \(T^*\).

We show here that for a contraction \(T\) where \(T - I\) is a Weyl operator, i.e., a Fredholm operator of index 0, the this fixed point separation property is equivalent to the property that \(T - I\) has ascent one. This states in particular that \(T - I\) is in fact a Browder operator.

**Theorem 4.** Let \(T \in \mathcal{L}(X)\) such that \(\|T\|_{\text{op}} \leq 1\) and \(T - I \in \mathcal{W}(X)\). Then \(\ker(T - I)\) separates the points of \(\ker(T^* - I)\) if and only if the matrix
\[
G = \begin{pmatrix} e_1^*(e_1) & \cdots & e_n^*(e_n) \\ \vdots & \ddots & \vdots \\ e_n^*(e_1) & \cdots & e_n^*(e_n) \end{pmatrix} \in \mathbb{C}^{n \times n}
\]
is invertible, where \(n = \dim(T - \lambda I) = \dim(T^* - \lambda I)\).

**Proof.** We show first that if the fixed points of \(T\) separate the fixed points of \(T^*\) then the matrix \(G\) is invertible. To this end, let us assume to the contrary that the matrix \(G\) is not invertible. We will show that in this case \(\ker(T - I)\) does not separate \(\ker(T^* - I)\). If \(G\) is not invertible, then the rows of \(G\) are not linearly independent. Hence, we can assume there are \(c_1, \ldots, c_n \in \mathbb{C}\) such that
\[
\sum_{j=1}^{n} c_je_j^*(e_i) = 0, \quad \text{for all } i \in \{1, \ldots, n\},
\]
where there exists at least one coefficient with \(c_k \neq 0\). Then
\[
e_k^*(x) = \sum_{j \neq k} \left(\frac{c_j}{c_k}\right) e_j^*(x)
\]
for all \(x \in \ker(T - I)\) as \(e_1, \ldots, e_n\) form a basis. We conclude that \(\ker(T - I)\) does not separate \(\ker(T^* - I)\).

We prove next by contradiction that if \(G\) is invertible then \(\ker(T - I)\) separates \(\ker(T^* - I)\). To this end, assume that the fixed points of \(T\) do not separate the fixed points of \(T^*\). Then there are \(x_1^* \neq x_2^* \in \ker(T^* - I)\) such that for all \(x \in \ker(T - I)\)
\[
x_1^*(x) = x_2^*(x).
\]
Let \(x_1^* = \sum_{j=1}^{n} c_je_j^*\) and \(x_2^* = \sum_{j=1}^{n} b_je_j^*\). Then as well
\[
x_1^*(e_i) - x_2^*(e_i) = \sum_{j=1}^{n} (c_j - b_j)e_j^*(e_i) = 0
\]
holds for all \(i \in \{1, \ldots, n\}\). As \(c_i \neq b_i\) for at least one \(i \in \{1, \ldots, n\}\) the rows of \(G\) are linearly dependent and \(G\) is not invertible. \(\square\)

Finally, we extend our results of Theorem 3 with the result of the previous theorem.

**Corollary 2.** Let \(T \in \mathcal{L}(X)\) with \(\|T\|_{\text{op}} \leq 1\) such that \(T - I \in \mathcal{W}(X)\). Then the following statements are equivalent:

1. \(T - I\) has chain length one, i.e., \(\text{asc}(T - I) = \text{dsc}(T - I) = 1\),
2. $X = \ker(T - I) \oplus \ran(T - I)$,
3. $T - I \in \mathcal{W}_B(X)$,
4. $G$ is invertible,
5. $P = \Phi G^{-1} \Phi^*$ yields a projection onto $\ker(T - I)$,
6. The Cesáro means $n^{-1} \sum_{k=0}^{n-1} T^k$ converge in the strong operator topology towards $P$ for $n \to \infty$.

The last item follows in particular by work of [3, Thm. 3.16 on p. 215].

5. Iterates of quasi-compact Markov operators

Using the preceding results, we consider now the limit of the iterates of an operator $T \in \mathcal{L}(X)$ that has a non-trivial fixed point space. We will first introduce the concept of quasi-compact operators in a proper way and relate the quasi-compactness to the essential spectrum and the Browder essential spectrum. Note that if the iterates converge to a finite-rank operator, then this operator is quasi-compact by definition. Results for the peripheral spectrum of quasi-compact operators are given with corollaries for the case where the operator is positive. Finally, we will state different limit theorems for quasi-compact operators.

5.1. Quasi-compact operators and the peripheral spectrum

For Banach spaces there are several ways to define the essential spectrum for a bounded linear operator $T \in \mathcal{L}(X)$. If one considers the essential spectrum as the largest subset of the spectrum which remains invariant under compact perturbations one obtains the following definition of $\sigma_{\text{ess}}(T)$,

$$\sigma_{\text{ess}}(T) := \{ \lambda \in \mathbb{C} : T - \lambda I \not\in \mathcal{W}(X) \},$$

which also often said to be the essential Weyl spectrum, see Schechter [21]. However, this definition of the spectrum does not contain the limit points of the spectrum. If all these accumulation points are added, then one comes to the definition of Browder [2, 107], where a spectral value $\lambda \in \mathbb{C}$ is in the essential spectrum, if at least one of the following conditions hold:

1. $\ran(T - \lambda I)$ is not closed in $X$,
2. $\lambda$ is a limit point of the spectrum $\sigma(T)$,
3. $\bigcup_{k \in \mathbb{N}} \ker(T - \lambda I)^k$ is infinite dimensional.

This is indeed equivalent to the essential Browder spectrum,

$$\sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \not\in \mathcal{W}_B(X) \}.$$

The advantage of using $\sigma_{\text{ess}}$ is the perturbation invariance, while the advantage of the Browder spectrum $\sigma_b(T)$ is that $\sigma(T) \setminus \sigma_b(T)$ is a countable set. Summing up these facts, we have the relation

$$\sigma_{\text{ess}}(T) \subseteq \sigma_b(T) = \sigma_{\text{ess}}(T) \cup \text{acc } \sigma(T) \subset \sigma(T),$$

where $\text{acc } \sigma(T)$ denotes all the limit points of $\sigma(T)$. Nevertheless, the essential spectral radius is in both definition of the essential spectrum equal, i.e., all spectral limit points are on the boundary of $\sigma_{\text{ess}}(T)$.

Now, suppose $T \in \mathcal{L}(X)$ is a quasi-compact operator, i.e., the essential spectral radius is less than one. From this it follows that every spectral value $\lambda \in \sigma(T)$ with modulus larger than the essential spectral radius is an isolated eigenvalue and the operator $T - \lambda I$ is a Browder operator. Therefore, there always exists an eigenvalue $\lambda \in \sigma(T)$ with modulus equal to the spectral radius $r(T)$. Moreover, there are only finitely many eigenvalues on the peripheral spectrum. The next lemma gives a characterization.

**Lemma 5.** Let $T \in \mathcal{L}(X)$ be a quasi-compact operator with $r(T) \geq 1$. Then, there is at least one eigenvalue $\lambda$ with $|\lambda| = r(T)$. Besides, every spectral value $\lambda \in \sigma(T)$ with $|\lambda| > r_{\text{ess}}(T)$ is an isolated eigenvalue of $T$ and $T - \lambda I$ is a Browder operator. There are only finitely many eigenvalues on the peripheral spectrum of $T$.
Proof. By the definition of quasi-compactness, we have \( r_{ess}(T) < 1 \) and all of the spectral values outside with modulus larger than \( r_{ess}(T) \) are isolated. As already discussed above, \( T - \lambda I \) is a Browder operator. If \( \lambda \notin \sigma_p(T) \), then by Theorem 1 in [14], \( \lambda \) is a pole of the resolvent of finite rank. Applying Heuser [6, Proposition 50.3], we derive that \( \text{asc}(\lambda I) \) is positive and hence, \( \lambda \) is an eigenvalue of \( T \). As all the cluster points of the spectrum are on the boundary of \( \sigma_{ess}(T) \), there is an eigenvalue \( \lambda \in \sigma(T) \) with \( |\lambda| = 1 \). Finally, there are only finitely many on the peripheral spectrum as otherwise there would be an accumulation point outside of the essential Browder spectrum.

We now show that for the eigenvalues \( \lambda \) of quasi-compact operators \( T \in \mathcal{L}(X) \) lying on the peripheral spectrum, i.e., eigenvalues \( \lambda \) with modulus \( r(T) \), the associated Browder operator \( T - \lambda I \) has ascent one, whereas the dimension of the associated eigenspace can be arbitrary but finite. This result has already been shown in a similar setting by Hennion and Hervé [8, Proposition V.1] and is stated here with less assumptions on the operator \( T \).

**Lemma 6.** Let \( T \in \mathcal{L}(X) \) be a quasi-compact operator with \( r(T) \geq 1 \) such that \( \sup_{n \in \mathbb{N}} r(T)^{-n} \|T^n\|_{op} < \infty \). Then for every peripheral eigenvalue \( \lambda \in \sigma_{per}(T) \) the associated Browder operator \( T - \lambda I \) has ascent one.

**Proof.** Note that the existence of a peripheral eigenvalue has been shown in the lemma above. We consider now the eigenvalue \( \lambda \in \sigma_{p}(T) \) with \( |\lambda| = r(T) \). Let \( x \in \ker(T - \lambda I)^2 \). Then we can represent \( T^n x \) for all positive integers \( n \) by

\[
T^n x = (\lambda I + (T - \lambda I))^n x = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} (T - \lambda I)^k x = \lambda^n x - n\lambda^{n-1}(T - \lambda I)x.
\]

We will show now that \( (T - \lambda I)x = 0 \). To this end, using that \( |\lambda| = r(T) \) and that there exists of \( B > 0 \) such that \( r(T)^{-m} \|T^m\|_{op} < B \) for all positive integers \( m \), we calculate:

\[
\|n\lambda^{n-1}(T - \lambda I)x\| = \|\lambda^n x - T^n x\| \leq \|\lambda^n\| \|x\| + \|T^n x\| \leq r(T)^n \|x\| + \|T^n\|_{op} \|x\| \leq r(T)^n (1 + B) \|x\|.
\]

It is now easy to see that \( \|(T - \lambda I)x\| \leq \frac{r(T)(1+B)}{n} \|x\| \) and we conclude that \( x \in \ker(T - \lambda I) \) as \( n \) was arbitrary.

In the following, we consider the case when \( r(T) = \|T\|_{op} \). Operators of this kind are said to be normaloid and have been discussed in Heuser [3, Chapter 54]. We obtain the following corollary:

**Corollary 3.** Let \( T \in \mathcal{L}(X) \) be a quasi-compact operator with \( r(T) = \|T\|_{op} \). The exists at least one eigenvalue with modulus \( r(T) \). Furthermore, for every peripheral eigenvalue \( T - \lambda I \) is a Browder operator with ascent one.

**Proof.** For all positive integers \( n \) the inequality \( r(T) \leq \|T^n\|_{op}^{1/n} \leq \|T\|_{op} \) holds. Therefore, \( r(T)^{-n} \|T^n\|_{op} \leq 1 \) for all \( n \). The result follows by [Lemma 5] and [Lemma 6].

If we \( X \) is a Banach lattice and \( T \) is positive even stronger results can be made. According to Lotz [15], Krein and Rutman [13] have first shown that every positive compact operator on a Banach lattice with \( r(T) = \|T\|_{op} \) has a cyclic peripheral spectrum. This result has been generalized in Lotz [15, Theorem 4.10], where the peripheral spectrum of a positive operator \( T \in \mathcal{L}(X) \) on a Banach lattice \( X \) is cyclic if \( r(T) \) is a pole of the resolvent. Furthermore, Lotz [15] concluded that the peripheral spectrum of every positive compact operator is cyclic.

The next corollary sums up these results for positive quasi-compact operators.

**Corollary 4.** Let \( T \in \mathcal{L}(X) \) be a positive quasi-compact operator with \( r(T) = \|T\|_{op} = 1 \). Then \( 1 \in \sigma(T) \), i.e., \( T - I \) is a Browder operator of ascent one. Furthermore, the peripheral spectrum is cyclic and consists only of roots of unity.
Proof. It has been shown by Lotz \cite{13} that \( r(T) \in \sigma(T) \) if \( T \) is positive. In the case where \( T \) is a quasi-compact positive operator with \( r(T) = 1 \), \( T \) has real eigenvalue one and \( T - I \) is a Browder operator with ascent one.

The peripheral spectrum contains only finitely many eigenvalues of \( T \) and \( 1 \in \sigma_{\text{per}}(T) \). As the peripheral spectrum is cyclic, see the above mentioned result in \cite{13}, we conclude that \( \sigma_{\text{per}}(T) \) can only contain roots of unity.

If \( T \in \mathcal{L}(X) \) is a positive quasi-compact operator on a Banach lattice \( X \) with \( \|T\|_{op} = r(T) = 1 \). Then of course \( \sigma(T) \subset B(0,1) \) and using the preceding results we obtain that 1 is an isolated eigenvalue of \( T \) and the peripheral spectrum is cyclic. Let us denote by the positive integer \( l \) the number of spectral values in the peripheral spectrum. There are now two cases to discuss separately:

1. \( l = 1 \): then \( \sigma_{\text{per}}(T) = \{1\} \), otherwise
2. \( \sigma_{\text{per}}(T) = \{\rho^k \mid k \in \{1, \ldots, l\}\} \),

where \( \rho_k \) are the \( l \)-th roots of unity.

The first case has already been characterized by Katznelson and Tzafriri \cite{11}, who have shown that for every linear operator \( T \) on a Banach space \( X \) with \( \|T\|_{op} \leq 1 \) the limit

\[
\lim_{n \to \infty} \|T^{n+1} - T^n\|_{op} = \lim_{n \to \infty} \|T^n(T - I)\|_{op} = 0
\]

holds if (and only if) \( \sigma_{\text{per}}(T) \subset \{1\} \).

5.2. Operators with ascent one

The last results have shown that if \( \lambda \) is a peripheral eigenvalue of quasi-compact operator \( T \), the operator \( T - \lambda I \) has always ascent one. In this case the spaces \( M^1 \) and \( \Lambda^1 \) contain only eigenvectors of \( T \) and \( T^* \) respectively. They can now be represented by

\[
M^1 = \ker(T - \lambda I) = \{x \in X : Tx = \lambda x\}, \quad \Lambda^1 = \ker(T^* - \lambda I) = \{x^* \in X^* : x^*(Tx) = x^*(\lambda x) \text{ for all } x \in X\}.
\]

Let us denote by \( n = \dim(M^1) = \dim(\Lambda^1) \). The result of Theorem 3 yields a projection \( P : X \to M^1 \) onto the eigenspace space associated with \( \lambda \). Recall that the Gram matrix consisting of only of dual eigenvectors acting on the eigenvectors of \( T \),

\[
G := (\Phi^*\Phi) = \begin{pmatrix} e_1^*(e_1) & \cdots & e_1^*(e_n) \\ \vdots & \ddots & \vdots \\ e_m^*(e_1) & \cdots & e_m^*(e_n) \end{pmatrix} \in \mathbb{C}^{n \times n},
\]

is invertible. Setting \( A = (a_{ij}) = G^{-1} \), the projection has the form

\[
Px = \Phi(\Phi^*\Phi)^{-1}\Phi^*(x) = \sum_{i=1}^n \sum_{j=1}^m a_{ij} e_j^*(x)e_i, \quad \text{for } x \in X.
\]

The next lemma gives a characterization of this projection.

**Theorem 5.** Let \( T \in \mathcal{L}(X) \) and \( \lambda \in \sigma_p(T) \) such that \( T - \lambda I \) is a Browder operator. Then the following two statements are equivalent:

1. \( T - \lambda I \) has ascent one,
2. there exists a projection \( P \in \mathcal{K}(X) \) such that \( TP = PT = \lambda P \),
Proof. Suppose that the first statement holds. Then we obtain a projection \( P \in \mathcal{K}(X) \) from [Theorem 3] We have the property \( T \circ P = \lambda P \), because for \( x \in X \) we obtain
\[
(T \circ P)(x) = T(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} e_{j}^*(x)e_{i}) = \sum_{i=1}^{m} a_{ij} e_{j}^*(x)T(e_{i}) = \lambda P(x),
\]
as \( T e_{i} = \lambda e_{i} \) by \([21]\). Similarly, we obtain \( P \circ T = \lambda P \). Namely, for \( x \in X \) using \( e_{j}^*(Tx) = e_{j}^*(\lambda x) \) by \([22]\) it holds that
\[
(P \circ T)(x) = \sum_{i=1}^{m} a_{ij} e_{j}^*(T x)e_{i} = \sum_{i=1}^{m} a_{ij} e_{j}^*(\lambda x)e_{i} = \lambda P(x).
\]

Now we show that if there exists a projection \( P \in \mathcal{K}(X) \) with \( TP = PT = \lambda P \), then \( T - \lambda I \) has ascent one. As \( \ker(T - \lambda I) \subset \ker(T - \lambda I)^2 \) and \( \text{ran}(P) = \ker(T - \lambda I) \), it is enough to show that \( \ker(T - \lambda I)^2 \subset \text{ran}(P) \). Suppose \( x \in \ker(T - \lambda I)^2 \). Then \( (T - \lambda I)^2 x = 0 \) and \( (T - \lambda I)x \in \ker(T - \lambda I) = \text{ran}(P) \). Therefore, there is \( y \in \text{ran}(P) \) such that \( Py = (T - \lambda I)x \). Then
\[
y = Py = P^2 y = P(T - \lambda I) x = PT x - \lambda Px = \lambda Px - \lambda Px = 0.
\]
Thus, \( y = 0 \) and we obtain using \( 0 = y = Py = (T - \lambda I)x \) the final result, namely \( x \in \ker(T - \lambda I) = \text{ran}(P) \).

Lemma 7. Let \( T \in \mathcal{L}(X) \) and \( \lambda \in \sigma_{p}(T) \). Suppose there exists a projection \( P \in \mathcal{K}(X) \) such that \( TP = PT = \lambda P \). Then
\[
(T - T P)^n = T^n - T^n P = T^n - \lambda^n P
\]
holds for all \( n \in \mathbb{N} \).

Proof. We will use the fact that if \( P \) is a projection then \( I - P \) is a projection as well. Also note that \( I - P \) commutes with \( T \), as
\[
(I - P)T = T - TP = T - PT = T(I - P).
\]
Now we derive the result with the following steps:
\[
(T - T P)^n = (T - T P)^n = (T(I - P))^n = T^n(I - P)^n
\]
\[
= T^n(I - P) = T^n - T^n P = T^n - \lambda^n P.
\]

5.3. The limit of the iterates of quasi-compact operators

We assume in the following that \( \|T\|_{op} = 1 \) and \( r(T) = 1 \). First, we will restrict us to the fixed point space of a quasi-compact operator \( T \in \mathcal{L}(X) \) and assume that \( \sigma(T) \subset B(0,1) \cup \{1\} \), i.e., \( 1 \) is the only peripheral eigenvalue of \( T \). In this case, if \( T - I \in \mathcal{W}_{B}(X) \) has ascent one and the iterates will converge to the projection \( P \). Later we will consider the case where the peripheral spectrum is cyclic.

In the proof of our main result we will need the following lemma. As it is more convenient, we omit the proof here but prove it at the end of this section. The lemma states that isolated spectral values can be removed by the projection operator on the corresponding generalized eigenspace.

Lemma 8. Let \( T \in \mathcal{L}(X) \) and \( \lambda \in \sigma_{p}(T) \) such that \( T - \lambda I \in \mathcal{W}_{B}(X) \) with ascent \( p \). Let \( P \) be denote the projection onto \( \ker(T - \lambda I)^p \), defined by [Theorem 3]. Then \( \lambda \) is an isolated spectral value and \( \lambda \notin \sigma(T - TP) \).

Now we can show the following result:

Theorem 6. Let \( T \in \mathcal{L}(X) \) with \( r(T) = \|T\|_{op} = 1 \) satisfying the spectral condition \( \sigma(T) \subset B(0,1) \cup \{1\} \). Then \( T \) is quasi-compact if and only if
\[
\lim_{m \to \infty} \|T^m - P\|_{op} = 0,
\]
where \( P \in \mathcal{K}(X) \) is a finite-rank projection with \( TP = PT = P \).
Proof. Clearly, if the iterates $T^n$ converge to a finite-rank operator, then $T$ is a quasi-compact operator. Now let $T$ be quasi-compact with $\sigma(T) \subset B(0,1) \cup \{1\}$, then $r(T) = 1$ and 1 is an isolated peripheral eigenvalue. Thus, $T - I$ is Browder with ascent one. We now prove the limit of the iterates. By Theorem 2 the space $X$ has the decomposition

$$X = \ker(T - I) \oplus \text{ran}(T - I).$$

Therefore, we can decompose the operator $T$ into

$$T = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} \in \mathcal{L}(\ker(T - I) \oplus \text{ran}(T - I)),$$

with $S \in \mathcal{L}(\text{ran}(T - I))$. Then

$$T - P = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix},$$

where $P$ is the projection operator defined by Theorem 3. Using Lemma 8 we derive that $1 \notin \sigma(S) \subset B(0,1) \cup \{1\}$, and hence, $\sigma(S) \subset B(0,1)$. Therefore, the spectral radius of $S$ is strictly smaller than 1 and thus, the iterates $S^m$ converge to 0 in the operator norm as $m$ tends to infinity. Finally, applying Lemma 7 we obtain the final result

$$\lim_{m \to \infty} \|T^m - P\|_{op} = \lim_{m \to \infty} \|(T - P)^m\|_{op} \lim_{m \to \infty} \|S^m\|_{op} = 0.$$ 

The iterates $T^n$ converge in the strong topology to the operator $P$, the projection onto the fixpoint space of $T$. $\square$

**Corollary 5** (Convergence Rate). Let $T \in \mathcal{L}(X)$ be a quasi-compact operator with $r(T) = \|T\|_{op} = 1$ satisfying the spectral condition $\sigma(T) \subset B(0,1) \cup \{1\}$. Define

$$\gamma := \sup \{ |\gamma| : \gamma \in \sigma(T) \setminus \{1\} \}.$$ 

Then there exists a constant $1 \leq C \leq \gamma^{-1}$, such that for all $m \in \mathbb{N}$

$$\|T^m - P\|_{op} \leq C \cdot \gamma^m,$$

where $P \in \mathcal{K}(X)$ is the operator defined by Theorem 3.

**Proof.** According to the proof of Theorem 6 we decompose

$$T = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} \in \mathcal{L}(\ker(T - I) \oplus \text{ran}(T - I)).$$

Furthermore, we have that $\sigma(S) \subset B(0,1)$ and therefore we obtain $r(S) = \gamma < 1$. As $r(S) = \lim_{m \to \infty} \|S^m\|^{1/m}$, we obtain that there exists a constant $1 \leq C \leq \gamma^{-1}$ such that

$$\|S^m\| \leq C \cdot \gamma^m$$

for every $m \in \mathbb{N}$. $\square$

If a sequences of operators with the spectrum contained in $B(0,1) \cup \{1\}$ share the same fixpoints spaces, the following limit theorem holds.

**Corollary 6.** Let $T_n \in \mathcal{L}(X)$ be a sequence of continuous linear operators with $\sigma(T_n) \subset B(0,1) \cup \{1\}$ such that $T_n - \lambda I \in \mathcal{W}_B(X)$ with ascent one. Furthermore, assume that $\ker(T_n - I)$ and $\ker(T^*_n - I)$ are equal for all $n \in \mathbb{N}$. If $(k_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ is a strictly increasing sequence of positive integers, then

$$\lim_{n \to \infty} \|T_{kn}^* - P\|_{op} = 0,$$

where $P \in \mathcal{K}(X)$ is the operator defined by Theorem 3.
Proof. Follows directly by applying Corollary 5. We derive for \( n \in \mathbb{N} \) that

\[
\left\| T_n^k - P \right\| = \left\| S_n^k \right\| \leq C \gamma_n^k
\]

where

\[
\gamma_n := \sup \{ |\gamma| : \gamma \in \sigma(T_n) \setminus \{1\} \}.
\]

As 1 is an isolated eigenvalue, \( \gamma_n < 1 \) for all \( n \in \mathbb{N} \) and therefore, \( \left\| T_n^k - P \right\| \to 0 \) if \( n \) tends to infinity.

Finally we discuss the case, when the peripheral spectrum is cyclic.

**Theorem 7.** Let \( T \in \mathcal{L}(X) \) be a quasi-compact operator with \( r(T) = \|T\|_{op} = 1 \) with a non-trivial fixed point space. Furthermore, we assume the peripheral spectrum to be finite and cyclic. Then there exists \( k \in \mathbb{N} \) such that

\[
\lim_{m \to \infty} \|T^{km} - P\|_{op} = 0,
\]

where \( P \in \mathcal{K}(X) \) is the operator defined by Theorem 3 for applied to the operator \( T^k \).

**Proof.** As the peripheral spectrum is finite and cyclic and 1 \( \in \sigma_{per}(T) \), the spectrum contains only roots of unity. Let us denote by \( l \) the number of spectral values contained in the spectrum. Then

\[
\sigma_{per}(T) = \{ \rho_k^l : k \in \{1, \ldots, l\} \},
\]

where \( \rho_l \) is the \( l \)-th root of unity. By the spectral mapping theorem for the point spectrum, see e.g., Rudin [19, Theorem 10.33] we conclude that for the integer \( k := l(l-1) \cdots 2 \cdot 1 \) the peripheral spectrum of \( T^k \) contains only the eigenvalue 1. As \( T^k \) is also quasi-compact, we can derive the result by Theorem 6 applied to \( T^k \).

It is easy to see that quasi-compact Markov operators always satisfy the conditions of Theorem 7. For that case, we derive the corollary:

**Corollary 7.** Let \( T \in \mathcal{L}(X) \) be a quasi-compact Markov operator with \( \|T\|_{op} = 1 \). Then there exists \( k \in \mathbb{N} \) such that

\[
\lim_{m \to \infty} \|T^{km} - P\|_{op} = 0,
\]

where \( P \in \mathcal{K}(X) \) is the operator defined by Theorem 3 for applied to the operator \( T^k \).

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