The Power Graph of a Torsion-Free Group Determines the Directed Power Graph

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Abstract
The directed power graph \( \vec{G}(G) \) of a group \( G \) is the simple digraph with vertex set \( G \) such that \( x \rightarrow y \) if \( y \) is a power of \( x \). The power graph of \( G \), denoted with \( G(G) \), is the underlying simple graph.

In this paper, for groups \( G \) and \( H \), the following is proved. If every quasicyclic subgroup \( C_p^\infty \) of \( G \) has trivial intersection with every cyclic subgroup \( K \) of \( G \) such that \( K \not\leq C_p^\infty \), then \( G(G) \cong H(G) \) implies \( \vec{G}(G) \cong \vec{G}(H) \). Consequently, any two torsion-free groups having isomorphic power graphs have isomorphic directed power graphs.

1 Introduction
The directed power graph of a group, which was introduced by Kelarev and Quinn [9], is the simple directed graph whose vertices are all elements of the group, and in which \( x \rightarrow y \) if \( y \) is a power of \( x \). The power graph of a group is the underlying simple graph, and it was first studied by Chakraborty, Ghosh and Sen [7]. The power graph has been subject of many papers, including [1,3–6,8,10–17]. In these papers, combinatorial and algebraic properties of the power graph have received considerable attention, as well as the relation between a group and its power graph. For more details the survey [2] is recommended.

Cameron [3] proved that two finite groups that have isomorphic power graphs also have isomorphic directed power graphs. Cameron, Guerra and Jurina [5] proved that, for torsion-free groups \( G \) and \( H \) of nilpotency class 2, \( G(G) \cong H(G) \) implies \( \vec{G}(G) \cong \vec{G}(H) \). The authors in [5] also asked whether this is also true when at least one of the groups is torsion-free of nilpotency class 2. In [17] was given the affirmative answer to their question. This paper deals further with this problem. Here the result from [17] is extended by proving that any two torsion-free groups having isomorphic power graphs have isomorphic directed power graphs as well. What is more, \( G(G) \cong H(G) \) implies \( \vec{G}(G) \cong \vec{G}(H) \) whenever every quasicyclic subgroup \( C_p^\infty \) of \( G \) has trivial intersection with every cyclic subgroup of \( G \) not being contained in \( C_p^\infty \).

As mentioned before, great deal of combinatorial properties of the power graph have been studied by many authors, although such investigations are not a part of this paper. Aalipour et al. [1] showed that the power graph of every group of bounded exponent is perfect. They showed that the clique number of the power graph of any group is at most countable, and they posed the question whether the power graph of every group has at most countable chromatic number. Shitov [15] gave the affirmative answer to that question by proving that all power-associative groupoids, i.e. groupoids whose only one-generated subgroupoids are semigroups, have power graphs of at most countable chromatic numbers. Even though his result was combinatorial, it is one of his
2 Basic Notions and Notations

Graph $\Gamma$ is a structure $(V(\Gamma), E(\Gamma))$, or shortly $(V, E)$, where $V$ is a set, and where $E$ is a set of two-element subset of $V$. Elements of $V$ are called vertices, and elements of $E$ are called edges of the graph $\Gamma$. Two edges $x$ and $y$ of $\Gamma$ are adjacent if $\{x, y\} \in E$, and we write that fact with $x \sim y$, or shortly with $x \sim y$. A graph $\Delta$ is a subgraph of $\Gamma$ if the vertex set and the edge set of $\Delta$ are subsets of $V(\Gamma)$ and $E(\Gamma)$, respectively. We say that $\Delta$ is an induced subgraph of $\Gamma$ if $V(\Delta) \subseteq V(\Gamma)$ and if, for any $x, y \in V(\Delta)$, $x \sim y$ if and only if $x \sim \Delta y$. In this case we also say that $\Delta$ is the subgraph of $\Gamma$ induced by the set $V(\Delta)$, and the subgraph of $\Gamma$ induced by a set of vertices $X \subseteq V(\Gamma)$ we shall denote by $\Gamma[X]$. The complement of a graph $\Gamma$ is the graph $\bar{\Gamma}$ with the same vertex set as $\Gamma$ such that $x \sim y \bar{\Gamma}$ if and only if $x \not\sim y$.

Directed graph, or digraph, $\vec{\Gamma}$ is a structure $(V(\vec{\Gamma}), E(\vec{\Gamma}))$, or shortly $(\vec{V}, E)$, where $V$ is a set, whose elements are called vertices of $\vec{\Gamma}$, and where $E$ is a set of ordered pairs of different vertices of $\vec{\Gamma}$. If $(x, y) \in E$, then we shall say that $x$ is a direct predecessor of $y$, and that $y$ is a direct successor of $x$, and we will write that fact with $x \rightarrow y$, or more shortly with $x \rightarrow y$.

Closed neighborhood of a vertex $x$ of a graph $\Gamma$ is the set $\overline{\text{N}}_\Gamma(x) = \{y \mid y \sim x \text{ or } y = x\}$, and we may shortly denote it with $\overline{\text{N}}(x)$. If two vertices $x$ and $y$ of $\Gamma$ have the same closed neighborhood, then we shall write that fact with $x \equiv y$, or simply with $x \equiv y$. A path in a graph $\Gamma$ is a sequence of different vertices $x_0, x_1, x_2, \ldots, x_n$ such that $x_{i-1} \sim x_i$ for all $i \in \{1, 2, \ldots, n\}$, and the length of this path is $n$. If, for every two vertices $x$ and $y$ of $\Gamma$, there is a path in $\Gamma$ connecting $x$ and $y$, i.e., in which $x = x_0$ and $y = x_n$, then we say that the graph $\Gamma$ is connected. A connected component of a graph $\Gamma$ is any maximal connected induced subgraph of $\Gamma$. Distance between vertices $x$ and $y$ in a connected graph $\Gamma$, denoted with $d(x, y)$ is the minimal length of a path connecting $x$ and $y$. The maximal distance between two vertices of a connected graph is called the diameter of the graph $\Gamma$, and it is denoted with $\text{diam}(\Gamma)$. A clique of graph $\Gamma$ is a set of its vertices which induces a complete subgraph of $\Gamma$, where a graph is complete if it has no pair of different non-adjacent vertices.

All through this paper, algebraic structures such as groups we shall denote with bold capitals, and their universes we will denote with respective regular capital letters. For elements $x$ and $y$ of a group $G$ we shall write $x \approx_G y$, or simply $x \approx y$, if $\langle x \rangle = \langle y \rangle$, where $\langle x \rangle$ denotes the subgroup generated by $x$. With $\circ(x)$ we shall denote the order of the element $x$ of a group. We shall say that a subgroup $H$ of $G$ is intersection-free if $H \cap K$ is trivial for all cyclic subgroups $K$ of $G$ such that $K \not\subseteq H$. In this paper we deal with the power graph and the directed power graph of a group, and now we introduce the definitions of these graphs.

**Definition 2.1** The directed power graph of a group $G$ is the digraph $\vec{G}(G)$ whose vertex set is $G$, and in which there is a directed edge from $x$ to $y$, $x \neq y$, if there exists $n \in \mathbb{Z}$ such that $y = x^n$. If there is a directed edge from $x$ to $y$ in $\vec{G}(G)$, we shall write that fact with $x \rightarrow_G y$, or shortly with $x \rightarrow y$. Observations that affected some of the important proofs from this paper and from [17].
The power graph of a group $G$ is the graph $\mathcal{G}(G)$ whose vertex set is $G$, and whose vertices $x$ and $y$, $x \neq y$, are adjacent if there exists $n \in \mathbb{Z}$ such that $y = x^n$ or $x = y^n$. If $x$ and $y$ are adjacent in $\mathcal{G}(G)$, we shall write that fact with $x \sim_G y$, or shortly with $x \sim y$.

Throughout this paper, instead of dealing with the power graph of a group as defined above, it will be more convenient to state our arguments for the power graph defined as in the following definition. To avoid any ambiguity, the power graph defined as in the ensuing definition we will call the $Z^n$-power graph.

Definition 2.2 The directed $Z^n$-power graph of a group $G$ is the digraph $\tilde{\mathcal{G}}^\pm(G)$ whose vertex set is $G$, and in which there is a directed edge from $x$ to $y$, $x \neq y$, if there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $y = x^n$. If there is a directed edge from $x$ to $y$ in $\mathcal{G}^\pm(G)$, we shall write that fact with $x \xrightarrow{\pm} y$, or shortly with $x \xrightarrow{\pm} y$.

The $Z^n$-power graph of a group $G$ is the graph $\mathcal{G}^\pm(G)$ whose vertex set is $G$, and in which $x$ and $y$, $x \neq y$, are adjacent if there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $y = x^n$ or $x = y^n$. If $x$ and $y$ are adjacent in $\mathcal{G}^\pm(G)$, we shall write that fact with $x \sim_G y$, or shortly with $x \sim y$.

If elements $x$ and $y$ of a group $G$ have the same closed neighborhood in $\mathcal{G}^\pm(G)$, we will write that fact with $x \equiv_G y$, or simply with $x \equiv y$. By the following theorem, which was proved in [17], the results about the $Z^n$-power graph of a group also apply for the power graph.

Theorem 2.3 ([17, Theorem 5]) Let $G$ and $H$ be groups. Then $\mathcal{G}(G) \cong \mathcal{G}(H)$ if and only if $\mathcal{G}^\pm(G) \cong \mathcal{G}^\pm(H)$.

3 The Power Graph and the Directed Power Graph of a Group

In this section we prove that, if no quasicyclic subgroup of a group is intersection-free, then its power graph determines the directed power graph of the group. Consequently, the power graph of any torsion-free group determines the directed power graph of a group. By Theorem 2.3, the power graph and the $Z^n$-power graph determine each other up to isomorphism. Therefore, it is justified to provide all proofs in this section in the case of the $Z^n$-power graph and the directed $Z^n$-power graph instead of in the case of the power graph and the directed power graph.

For a group $G$, $G_{<\infty}$ shall denote the set of all elements of $G$ of finite order, $G_{\infty}$ the set of all elements of infinite order, and, for a prime $p$, with $G_p$ we will denote the set of all elements whose orders are powers of $p$. No element of infinite order of a group $G$ is adjacent in the graph $\mathcal{G}(G)$ to an element of finite order, and the identity element is adjacent to all non-identity element of finite order. Therefore, $G_{<\infty}$ induces a connected component of $\mathcal{G}(G)$ (and $\tilde{\mathcal{G}}(G)$). The subgraph of $\mathcal{G}(G)$ or $\tilde{\mathcal{G}}(G)$ induced by $G_{<\infty}$ we shall call the finite-order component of $\mathcal{G}(G)$ or $\tilde{\mathcal{G}}(G)$, respectively. Similarly, the subgraph of $\mathcal{G}(G)$ or $\tilde{\mathcal{G}}(G)$ induced by $G_{\infty}$ shall be called the infinite-order component of $\mathcal{G}(G)$ or $\tilde{\mathcal{G}}(G)$, respectively. Notice that, while the
finite-order component of the $Z^{\pm}$-power graph is also its connected component, this may not be the case with the infinite-order component.

By the following lemma, for any group $G$, an isomorphism from $G^{\pm}(G)$ to $G^{\pm}(H)$ maps the finite-order component of $G^{\pm}(G)$ onto the finite-order component of $G^{\pm}(H)$.

**Lemma 3.1** Let $G$ and $H$ be groups, and let $\varphi : G \to H$ be an isomorphism from $G^{\pm}(G)$ to $G^{\pm}(H)$. Then $\varphi(G_{<\infty}) = H_{<\infty}$.

**Proof.** Let $D$ induce a connected component of $G^{\pm}(H)$ containing elements of infinite order. Then, if $y \in D$, the set $\bigcup_{i \in \mathbb{Z} \setminus \{0\}} y^{2^i}$ is a clique which is a union of $\equiv_H$-classes of cardinality 2. On the other hand, the finite-order component of $G^{\pm}(G)$ does not contain such a clique. Therefore, any isomorphism from $G^{\pm}(G)$ to $G^{\pm}(H)$ maps $G_{<\infty}$ onto $H_{<\infty}$, which proves the lemma. $\blacksquare$

The previous lemma justifies us to split the proof of the main result of this paper into two subsection, one in which we deal with isomorphisms between the infinite-order components, and the other in which we deal with isomorphisms between the finite-order components of power graphs of the two groups.

### 3.1 The Isomorphism between the Infinite-Order Components

In this subsection it is proved that, if two groups $G$ and $H$ have isomorphic $Z^{\pm}$-power graphs, then infinite-order components of $G^{\pm}(G)$ and $G^{\pm}(H)$ are isomorphic too. Naturally, this also proves that any two torsion-free groups having isomorphic $Z^{\pm}$-power graphs also have isomorphic directed $Z^{\pm}$-power graphs.

The next lemma tells us an important relationship between any two elements of infinite order belonging to a same connected component of the $Z^{\pm}$-power graph of a group.

**Lemma 3.2** Let $G$ be a group. Then, for any $x$ and $y$ belonging to the same connected component of the infinite-order component of $G^{\pm}(G)$, subgroups $\langle x \rangle$ and $\langle y \rangle$ have non-trivial intersection.

**Proof.** Let $\Gamma = G^{\pm}(G)$, and let $C \subseteq G_{\infty}$ induce a connected component of $\Gamma$. Let us prove first that $\text{diam}(\Gamma[C]) = 2$. Let $x, y \in C$ be such that $d_\Gamma(x, y) > 2$. Therefore, in the path of minimal length from $x$ to $y$, there are consecutive elements $a$, $b$ and $c$ such that $a \xrightarrow{\leq} b$ and $b \xrightarrow{\leq} c$. Then, $\langle a \rangle \cap \langle c \rangle = \langle b^n \rangle$ for some $n \in \mathbb{N}$. Thus, we can make a shorter path from $x$ to $y$ by replacing vertices $a$ and $b$ or vertices $b$ and $c$ by vertex $b^n$. This way we get a shorter path from $x$ to $y$, which is a contradiction.

Finally, because $\text{diam}(\Gamma[C]) = 2$, it follows that $\langle x \rangle \cap \langle y \rangle$ is non-trivial for any $x$ and $y$ belonging to the same connected component of the infinite-order component of $G^{\pm}(G)$. $\blacksquare$
For an element $x$ of a group $G$, let us define sets $I_G(x)$, $O_G(x)$ and $M_G(x)$ as follows:

$$I_G(x) = \{ y \in V \setminus \{x^{-1}\} \mid y \nrightarrow_G x \},$$

$$O_G(x) = \{ y \in V \setminus \{x^{-1}\} \mid x \nrightarrow_G y \}$$

and

$$M_G(x) = I_G(x) \cup O_G(x).$$

Sometimes we may shortly denote them with $I(x)$, $O(x)$ and $M(x)$, respectively. Further, for a group $G$ and its $Z^\pm$-power graph $\Gamma^\pm = \mathcal{G}^\pm(G)$, with $\overline{\mathcal{G}}_G(x)$ and $\overline{\mathcal{M}}_G(x)$ we shall denote $\overline{\Gamma^\pm}[O(x)]$ and $\overline{\Gamma^\pm}[M(x)]$, respectively. Note that, for an element $x$ of infinite order of any group, one can recognize the element $x^{-1}$ as the only vertex which has the same closed neighborhood in $\mathcal{G}^\pm(G)$ as the vertex $x$.

The following lemma has been proved by Cameron, Guerra and Jurina \cite{CGJ}. Although in the original paper it was proved in the case of an element of a torsion-free groups, it is proved in the analogous way in the case of an element of infinite order of any group.

**Lemma 3.3** (\cite[Lemma 3.3]{CGJ}) Let $G$ be a group, and let $x$ be an element of infinite order of group $G$. Then $\overline{\mathcal{O}}_G(x)$ is a connected component of $\overline{\mathcal{M}}_G(x)$.

To prove the main result of this paper, we shall start by Proposition 3.4, which deals with certain connected components of infinite-order component of the $Z^\pm$-power graph where it is possible to reconstruct the directions of arcs of the directed $Z^\pm$-power graph.

We shall say that a graph is **almost connected** if it is a disjoint union of a connected graph and two copies of the trivial graph $K_1$. For a group $G$, suppose that $C$ induces a connected component of $\mathcal{G}^\pm(G)$ containing elements of infinite order, and let $x$ and $y$ be elements of $C$ non-adjacent in $\mathcal{G}^\pm(G)$. Then $\overline{\mathcal{O}}(x) \cap \overline{\mathcal{O}}(y)$ is an almost connected graph. Namely, by Lemma 3.2, $\langle x \rangle \cap \langle y \rangle$ is an infinite cyclic subgroup of $G$. Therefore, $\langle x \rangle \cap \langle y \rangle = \langle z \rangle$ for some $z \in C$, and $O(x) \cap O(y) = \langle z \rangle \setminus \{e\}$. Now we see that $z$ and $z^{-1}$ are isolated vertices of $\overline{\mathcal{O}}(x) \cap \overline{\mathcal{O}}(y)$. Also, $O(x) \cap O(y) \setminus \{z, z^{-1}\}$ induces a connected subgraph of $\overline{\mathcal{O}}(x) \cap \overline{\mathcal{O}}(y)$ because, for any $n, m \in \mathbb{Z} \setminus \{-1, 0, 1\}$, there is $k > 1$ relatively prime to both $n$ and $m$, and, therefore, vertices $z^n$ and $z^m$ are connected in $\overline{\mathcal{O}}(x) \cap \overline{\mathcal{O}}(y)$ with the path $z^n \xrightarrow{\pm k} z^k \xrightarrow{\pm k} z^m$. This observation will be useful in proofs of Proposition 3.4 and Proposition 3.6.

**Proposition 3.4** Let $G$ and $H$ be groups. Let $\varphi : G \to H$ be an isomorphism from $\mathcal{G}^\pm(G)$ to $\mathcal{G}^\pm(H)$, and let $C \subseteq G$ induce a connected component of $\mathcal{G}^\pm(G)$ which contains elements of infinite order. If there are $x, y \in C$, $x \nleftrightarrow_G y$, such that $\overline{\mathcal{M}}_G(x) \cap \overline{\mathcal{M}}_G(y)$ is an almost connected graph, then $\varphi|_C$ is an isomorphism from $(\mathcal{G}^\pm(G))[C]$ to $(\mathcal{G}^\pm(H))[\varphi(C)]$.

**Proof.** Let us denote $\varphi(C)$ with $D$. By Lemma 3.1, $D$ contains elements of infinite order. By Lemma 3.2, then $\emptyset = O_G(x) \cap O_G(y) \subseteq M_G(x) \cap M_G(y)$. Further, $I_G(x) \cap M_G(y) = I_G(y) \cap M_G(x) = \emptyset$, because otherwise, by Lemma 3.3, $\overline{\mathcal{M}}_G(x) \cap \overline{\mathcal{M}}_G(y)$ would not be almost connected. Therefore, $M_G(x) \cap M_G(y) = O_G(x) \cap O_G(y)$. Similarly, $M_H(\varphi(x)) \cap M_H(\varphi(y)) = O_H(\varphi(x)) \cap O_H(\varphi(y))$. 


Suppose that \( u \nless_G v \) for some \( u, v \in C \). If \( v = u^{-1} \), then \( u \equiv_G v \). This would imply that \( \varphi(u) \equiv_H \varphi(v) \), and that \( \varphi(v) = (\varphi(u))^{-1} \). So suppose that \( v \neq u^{-1} \). Let us prove that \( \varphi(u) \nless_H \varphi(v) \). Because \( v \in O_G(u) \) and by Lemma 3.2, the connected component of \( M_G(u) \) which contains \( v \) has infinite intersection with \( M_G(x) \cap M_G(y) \). It follows that the connected component of \( M_H(\varphi(u)) \) which contains \( \varphi(v) \) has infinite intersection with \( M_H(\varphi(x)) \cap M_H(\varphi(y)) \), which implies that \( \varphi(u) \nless_H \varphi(v) \). It is analogously proved that \( \varphi(u) \nless_H \varphi(v) \) implies \( u \nless_G v \). Therefore, the mapping \( \varphi|_C \) is an isomorphism from \( \langle \tilde{G}^\pm(G) \rangle[C] \) to \( \langle \tilde{G}^\pm(H) \rangle[\varphi(C)] \).

Now we shall deal with the rest of the connected components of the infinite-order component of the \( Z^\pm \)-power graph of a group. The following theorem, which was proved in [17], will serve as a useful tool here.

**Theorem 3.5** ([17, Theorem 21]) Let \( G \) be a torsion-free group of nilpotency class 2, and let \( G \) be a group such that \( \tilde{G}^\pm(G) \cong \tilde{G}^\pm(H) \). Then \( \tilde{G}^\pm(G) \cong \tilde{G}^\pm(H) \).

**Proposition 3.6** Let \( G \) and \( H \) be groups. Let \( \varphi : G \to H \) be an isomorphism from \( \tilde{G}^\pm(G) \) to \( \tilde{G}^\pm(H) \), and let \( C \subseteq G \) induce a connected component of \( \tilde{G}^\pm(G) \) which contains elements of infinite order. If \( M_G(x) \cap M_G(y) \) is an almost connected graph for no pair of elements \( x, y \in G \) such that \( x \nless_G y \), then \( \langle \tilde{G}^\pm(G) \rangle[C] \cong \langle \tilde{G}^\pm(H) \rangle[\varphi(C)] \).

**Proof.** Let \( D \) denote \( \varphi(C) \), which, by Lemma 3.1, contains elements of infinite order. Let \( x, y \in C \), and suppose that \( x \nless_G y \). Because \( M_G(x) \cap M_G(y) \) is not almost connected, and because \( O(x) \cap I(y) \neq \emptyset \) or \( I(x) \cap O(y) \neq \emptyset \) would imply \( x \nless_G y \), it follows that \( I_G(x) \cap I_G(y) \neq \emptyset \). Therefore, there is \( z \in C \) such that \( z \nless_G x \) and \( z \nless_G y \), i.e. \( x, y \in \langle z \rangle \). It follows that \( \langle x, y \rangle \subseteq \langle z \rangle \subseteq C \cup \{ e \} \). Also, if \( x \nless_G y \), then \( \langle x, y \rangle = \langle y \rangle \subseteq C \cup \{ e \} \) or \( \langle x, y \rangle = \langle x \rangle \subseteq C \cup \{ e \} \). Thus, \( C \cup \{ e \} \) is the universe of a locally cyclic torsion-free subgroup \( \tilde{C} \) of the group \( G \). Similarly, \( D \cup \{ e \} \) is the universe of a locally cyclic torsion-free subgroup \( \tilde{D} \) of the group \( H \). It follows that \( \tilde{G}^\pm(C) \cong \tilde{G}^\pm(D) \). Then, by Theorem 3.5 and because \( \tilde{C} \) and \( \tilde{D} \) are abelian, \( \tilde{G}^\pm(C) \cong \tilde{G}^\pm(D) \). Therefore, \( \langle \tilde{G}^\pm(G) \rangle[C] \cong \langle \tilde{G}^\pm(H) \rangle[D] \), which finishes our proof.

Now, with Proposition 3.4 and Proposition 3.6 on our hands, we are able to prove the main theorem of this subsection.

**Theorem 3.7** Let \( G \) and \( H \) be groups whose \( Z^\pm \)-power graphs have isomorphic infinite-order components. Then the directed \( Z^\pm \)-power graphs of \( G \) and \( H \) have isomorphic infinite-order components too.

**Proof.** Let \( C \) be a connected component of the infinite-order component of \( \tilde{G}^\pm(G) \). If there are some elements \( x, y \in C \) non-adjacent in \( \tilde{G}^\pm(G) \) for which \( M_G(x) \cap M_G(y) \) is almost connected, then, by Proposition 3.4, follows that \( \langle \tilde{G}^\pm(G) \rangle[C] \cong \langle \tilde{G}^\pm(G) \rangle[\varphi(C)] \). If \( M_G(x) \cap M_G(y) \) is almost connected for no pair of different elements \( x \) and \( y \) from \( C \) such that \( x \nless_G y \), then, by
Proposition 3.6, we get \((\overline{G}^\pm(G))[C] \cong (\overline{G}^\pm(G))[\varphi(C)]\). From this follows that the directed \(Z^\pm\)-power graphs of groups \(G\) and \(H\) have isomorphic infinite-order components.

\[\text{Corollary 3.8} \quad \text{Let } G \text{ be a torsion-free group, and } H \text{ be a group such that } G^\pm(G) \cong G^\pm(H). \text{ Then } \overline{G}^\pm(G) \cong \overline{G}^\pm(H).\]

Proof. By Lemma 3.1, \(G^\pm(G)\) and \(G^\pm(H)\) have isomorphic infinite-order components and isomorphic finite-order components. Because \(G\) is torsion-free, the finite-order component of \(G^\pm(G)\) has only one vertex, and so the same holds for \(G^\pm(H)\). Therefore, to prove that \(G^\pm(G) \cong G^\pm(H)\), it is sufficient to show that \(\overline{G}^\pm(G)\) and \(\overline{G}^\pm(H)\) have isomorphic infinite-order components. But, by Theorem 3.7, \(\overline{G}^\pm(G)\) and \(\overline{G}^\pm(H)\) do have isomorphic infinite-order components. Thus, the corollary has been proved.

Now, the subsequent statement follows directly by Theorem 2.3.

\[\text{Corollary 3.9} \quad \text{Let } G \text{ be a torsion-free group, and } H \text{ be a group such that } G(G) \cong G(H). \text{ Then } \overline{G}(G) \cong \overline{G}(H).\]

### 3.2 The Isomorphism between the Finite-Order Components

In this subsection we give the proof that, if two groups have isomorphic \(Z^\pm\)-power graphs, and if at least one of them does not contain any intersection-free quasicyclic subgroup, then the finite-order components of their directed \(Z^\pm\)-power graphs are also isomorphic. Proofs from this subsection rely on the ideas presented in [3] by Peter Cameron, where he showed that the power graph of a finite group \(G\) determines the directed power graph. There he noticed that it is possible to determine the direction of arcs between vertices from different \(\equiv_G\)-classes. He also observed that, although it may be impossible to determine directions of all arcs within a single \(\equiv_G\)-class, it is possible to determine the induced subgraph of \(G^\pm(G)\) by that \(\equiv_G\)-class up to isomorphism. The difference here is that the set of all elements of finite order of a group may not be finite, and it may not even be a universe of a subgroup of the group.

The following proposition is a generalization of [3, Proposition 4]. Notice that the finite-order component of the \(Z^\pm\)-power graph of a group has at least one vertex adjacent to all other vertices. Therefore, for a group \(G\), the set of all vertices of the finite-order component \(\Phi\) of \(G^\pm(G)\) adjacent to all other vertices of \(\Phi\) we shall call the center of \(\Phi\), and we will denote it with \(\text{Cen}(\Phi)\).

Proposition 3.10 Let \(G\) be a group such that \(|\text{Cen}(\Phi)| > 1\), where \(\Phi\) is the finite-order component of \(G^\pm(G)\), and let \(S = \text{Cen}(\Phi)\). Then one of the following holds:

- **Proposition 3.10** Let \(G\) be a group such that \(|\text{Cen}(\Phi)| > 1\), where \(\Phi\) is the finite-order component of \(G^\pm(G)\), and let \(S = \text{Cen}(\Phi)\). Then one of the following holds:
1. $G_{<\infty}$ is a Prüfer group. In this case $S = G_{<\infty}$, and $S$ is infinite.

2. $G_{<\infty}$ is a cyclic group of prime power order. In this case $S = G_{<\infty}$, and $S$ is finite.

3. $G_{<\infty}$ is a cyclic group whose order is the product of two different prime numbers. In this case $|S| \geq \frac{1}{2}|G_{<\infty}|$, and the set $G_{<\infty} \setminus S$ induces a disconnected subgraph of $\Phi$.

4. $G_{<\infty}$ is a cyclic group whose order is divisible by at least two different prime numbers, but whose order is not the product of two different primes. In this case the set $G_{<\infty} \setminus S$ induces a connected subgraph of $\Phi$.

5. There is a prime number $p$ such that the order of every element from $G_{<\infty}$ is a power of $p$, but $\langle G_{<\infty} \rangle$ is not a cyclic, nor a Prüfer group. In this case $|S| < \frac{1}{2}|G_{<\infty}|$, and the set $G_{<\infty} \setminus S$ induces a disconnected subgraph of $\Phi$.

Proof. Let $P$ be the set of all primes $p$ such that $G_{<\infty}$ contains an element of order $p$. In this proof, by the exponent of a subset $X$ of $G$ we shall mean the least $k \in \mathbb{N}$ such that $x^k = e$ for all $x \in X$.

Suppose first that the set $P$ contains only one prime. If $G_{<\infty}$ is a cyclic group, then $S = G_{<\infty}$, and $S$ is finite. If $G_{<\infty}$ is a Prüfer group, then $S = G_{<\infty}$ and $S$ is infinite. Suppose further that $G_{<\infty}$ is not the universe of a cyclic subgroup of $G$, nor it is the universe of a subgroup of $G$ isomorphic to a Prüfer group. Then $\langle G_{<\infty} \rangle$ is not a cyclic group, nor a Prüfer group, because $G_{<\infty}$ already contains all elements of finite order of $G$. Let us show that there is an element of $S$ of maximal order. Let $x \in S$ and $y \in G_{<\infty} \setminus S$. Then $y \rightarrow x$, because $\langle x \rangle \subseteq S$. Therefore, if $S$ had no element of maximal order, then there would be no element $y$ such that $y \rightarrow x$ for all $x \in S$, i.e. $S = G_{<\infty}$. This is a contradiction with the fact that $\langle G_{<\infty} \rangle$ is not isomorphic to a Prüfer group. Thus, there is an element $x \in S$ of maximal order, and $\langle x \rangle = S$. Let $o(x) = p^k$ for some $k \in \mathbb{N}$. Then there are $y, z \in G_{<\infty} \setminus S$ of order $p^{k+1}$ such that $y \neq z$. Then $\Phi[G_{<\infty} \setminus S](y)$ and $\Phi[G_{<\infty} \setminus S](z)$ are different connected components of $\Phi[G_{<\infty} \setminus S]$, and their cardinalities are at least $(p-1)p^k$. Therefore, $|S| < \frac{1}{2}|G_{<\infty}|$.

Suppose now that $|P| > 1$. Let $x \in S \setminus \{e\}$. For every $i \leq n$, $o(x)$ is divisible by $p_i$, because otherwise $x$ would not be adjacent to any element of order $p_i$. Therefore, $P$ is finite, and the exponent of $G_{<\infty}$ is $p_1^{k_1}p_2^{k_2} \cdots p_m^{k_m}$, for some $m > 1$ and for some primes $p_1, p_2, \ldots, p_m$. Further, for any $i \leq m$, there is an element $y_i \in G_{<\infty}$ of order $p_i^{k_i}$. Because $x \rightarrow y_i$, then $p_i^{k_i} \mid o(x)$. Therefore, $o(x)$ is equal to the exponent of $G_{<\infty}$, and $G_{<\infty} = \langle x \rangle$ because $x \in S$. Now, if $o(x)$ is not a product of two different prime numbers, then the graph $\Phi[G_{<\infty} \setminus S]$ is connected. However, if $o(x) = pq$, for different primes $p$ and $q$, then $\Phi[G_{<\infty} \setminus S]$ is disconnected and

$$|S| = (p-1)(q-1) + 1 = \frac{pq + pq - 2p - 2q + 4}{2} = \frac{pq}{2} + \frac{(p-2)(q-2)}{2} \geq \frac{pq}{2} = \frac{|G|}{2}.$$

This proves the proposition. □
Let us show now that, if we knew all $\approx$-classes and their relations in the $Z^\pm$-power graph, then, for two adjacent $\approx$-classes of elements of finite order, it would be possible to determine which one of them contains elements of greater order. The following fact was used in [3], although it was not given as a separate lemma there.

**Lemma 3.11** Let $G$ be a group, and let $x, y \in G_{<\infty}$, $x \neq y$. Then $x \xrightarrow{\pm} y$ if and only if at least one of the following holds:

1. $x \overset{p^\pm}{\sim} y$ and $|[y]_\approx| < |[x]_\approx|$;
2. $x \overset{p^\pm}{\sim} y$, $|[y]_\approx| = |[x]_\approx|$, and $x \overset{p^\pm}{\sim} z$ for some $z \in G_{<\infty}$ such that $[z]_\approx = \{z\}$, and $\mathcal{N}(z) \neq G$;
3. $x \approx y$.

**Proof.** It is known that, for any $n, m \in \mathbb{N}$, $n \mid m$ implies that $\varphi(n) \mid \varphi(m)$, where $\varphi$ denotes Euler totient function. What is more, $n \mid m$ implies $\varphi(n) < \varphi(m)$ unless $m = 2n$ for an odd number $n$, or unless $n = m$. But when $m = 2n$ for an odd number $n$, then, if an element $x$ of order $n$ is adjacent to an element $y$ of order $m$, the element $y$ is adjacent to an element $z$ of order 2, while $x$ is adjacent to no such element. Note that, beside the identity element, elements of order 2 are the only ones contained in one-element $\approx$-classes. Therefore, $x \xrightarrow{\pm} y$ if and only if one of the three conditions is fulfilled.

The above lemma will be useful for us, but from the $Z^\pm$-power graph we do not see $\approx$-classes. The following four lemmas will, with the help of Lemma 3.11, enable us to determine directions of arches of the directed $Z^\pm$-power graph between different $\equiv$-classes, and to determine directions of arches within $\equiv$-classes up to isomorphism. The following lemma is a generalization of [3, Proposition 5], and it is one of the essential facts for the proof of the main result of this subsection.

**Lemma 3.12** Let $G$ be a group such that $|\text{Cen}(\Phi)| = 1$, where $\Phi$ is the finite-order component of $G^\pm(G)$. Then every $\equiv$-class $C$ of $\Phi$ is one of the following forms:

1. $C$ is an $\approx$-class. Such an $\equiv$-class we shall call a simple $\equiv$-class.
2. $C = \{x \in \langle y \rangle \mid o(x) \geq p^s\}$, where $p$ is a prime number, $y$ is an element of order $p^r$ for some $r \in \mathbb{N}$, and where $s \in \mathbb{N}$ satisfies $r > s > 0$. In this case $C$ is a union of $r - s + 1 \approx$-classes, and we shall say that such an $\equiv$-class is a complex $\equiv$-class.
3. $C = \bigcup_{k \geq s} [x_k]_\equiv$ for some $s \geq 1$, where, for some prime $p$, each $x_k$ is an element of order $p^s$, and where $x_k \in \langle x_{k+1} \rangle$, for all $k \geq s$. Such an $\equiv$-class we shall call an infinitely complex $\equiv$-class.

**Proof.** It is easily seen that every $\equiv$-class is a union of $\approx$-classes. Also, if all elements of an $\equiv$-class have the same order, then that $\equiv$-class is also an $\approx$-class, i.e. it is a simple $\equiv$-class.
Let \( C \) be an \( \equiv \)-class, and suppose that \( C \) contains elements \( x \) and \( y \) of different orders. Let us prove that the \( \equiv \)-class \( C \) is complex or infinitely complex. Without loss of generality, let \( o(x) < o(y) \). Then \( o(x) \) is a divisor of \( o(y) \) because \( y \overset{\equiv}{\rightarrow} x \). Let us show that \( o(y) \) is a power of a prime number. If that was not the case, then there would be different prime numbers \( q_1 \) and \( q_2 \) such that \( q_1 | \frac{o(y)}{o(x)} \) and \( q_2 | o(x) \). Then there is an element of order \( \frac{o(x)y}{o(x)} \) which is adjacent to \( y \) and not to \( x \), which is a contradiction. Therefore, the order of \( y \) is a power of a prime number.

Now we know that there is a prime number \( p \) such that \( C \) contains only elements whose orders are powers of \( p \). Because \( |\mathrm{Cen}(\Phi)| = 1 \), \( C \) does not contain \( e \), i.e. it does not contain an element of order \( p^0 \). Further, if \( x \) and \( y \), such that \( o(x) < o(y) \), belong to the same \( \equiv \)-class \( C \), and if \( z \) is an element such that \( \langle x \rangle \leq \langle z \rangle \leq \langle y \rangle \), then \( N(y) \subseteq N(z) \subseteq N(x) \), because \( x, y \) and \( z \) have prime power orders. This implies that \( N(z) = N(x) \), i.e. \( z \equiv x \). Now, if \( C \) contains an element of maximal order, then \( C \) is a complex \( \equiv \)-class. Otherwise, \( C \) is an infinitely complex \( \equiv \)-class.

In the above lemma we introduced the notions of simple, complex and infinitely complex \( \equiv \)-classes. Although Lemma 3.12 deals with the case when the center of the finite-order component contains only the identity element of the group, we will use those terms when dealing with the finite-order component of any group.

**Lemma 3.13** Let \( G \) be a group such that \( |\mathrm{Cen}(\Phi)| = 1 \), where \( \Phi \) is the finite-order component of \( G^{\pm}(G) \), and let \( x_0, y_0 \in G_{\infty} \). If \( x_0 \not\equiv y_0 \) and \( x_0 \overset{\equiv}{\rightarrow} y_0 \), then \( x \overset{\equiv}{\rightarrow} y \) for all \( x \) and \( y \) such that \( x \equiv x_0 \) and \( y \equiv y_0 \).

**Proof.** Suppose that \([x_0]_\equiv\) is an infinitely complex \( \equiv \)-class. Then there is no element \( z \in G \setminus [x_0]_\equiv \) such that \( z \overset{\equiv}{\rightarrow} x_0 \). Therefore, for any \( x \in [x_0]_\equiv \), and for any element \( y \) such that \( y \overset{\equiv}{\rightarrow} x \) and \( y \not\equiv x \), follows \( x \overset{\equiv}{\rightarrow} y \). Now it only remains to prove the lemma in the case when neither of \( \equiv \)-classes \([x]_\equiv\) and \([y]_\equiv\) is infinitely complex. Suppose further that none of \([x]_\equiv\) and \([y]_\equiv\) is an infinitely complex \( \equiv \)-class.

It suffices to show that \( x \equiv x_0 \), \( y \equiv y_0 \), \( x_0 \not\equiv y_0 \) and \( x_0 \overset{\equiv}{\rightarrow} y_0 \) implies \( x \overset{\equiv}{\rightarrow} y_0 \) and \( x_0 \overset{\equiv}{\rightarrow} y_0 \), for any \( x, y, y_0 \in G \). Suppose that \( x \equiv x_0 \), \( x_0 \not\equiv y_0 \) and \( x_0 \overset{\equiv}{\rightarrow} y_0 \). If \([x_0]_\equiv\) is a simple \( \equiv \)-class, then it is easily seen that the implication holds, so suppose that \([x_0]_\equiv\) is a complex \( \equiv \)-class. Suppose that \( x \overset{\equiv}{\rightarrow} y_0 \). Then \( y_0 \overset{\equiv}{\rightarrow} x \), which implies \( \langle x \rangle \leq \langle y_0 \rangle \leq \langle x_0 \rangle \) and that the orders of \( x_0 \), \( x \) and \( y_0 \) are powers of a prime number. Therefore, \( N(x_0) \subseteq N(y_0) \subseteq N(x) \), and thus \( x_0 \equiv y_0 \), which is a contradiction. This proves that \( x \overset{\equiv}{\rightarrow} y_0 \). It is proved similarly that \( y \equiv y_0 \), \( x_0 \not\equiv y_0 \) and \( x_0 \overset{\equiv}{\rightarrow} y_0 \) implies \( x_0 \overset{\equiv}{\rightarrow} y \). Thus, the lemma has been proved.

The following lemma was proved by Cameron [3]. Although he did not state it as a separate proposition, it was one of the essential steps in his proof that the power graph of a finite group determines the directed power graph. It is one of the crucial facts for this subsection too.

In the remainder of this subsection, for a set \( S \subseteq G_{\infty} \), with \( \hat{S} \) we shall denote
the set 
\[ \hat{S} = \hat{N}(\overline{N}(S)), \]
where \( \overline{N}(S) = \bigcap_{x \in S} \overline{N}(x) \).

**Lemma 3.14** Let \( G \) be a group such that \( |\text{Cen}(\Phi)| = 1 \), where \( \Phi \) is the finite-order component of \( \mathcal{G}^+ (G) \). Let \( C \) be a complex \( \equiv \)-class. Then the following holds:

1. \(|\hat{C}| = p^r \) and \(|\hat{C}| - |C| = p^{s-1} \) for some \( r, s \in \mathbb{N} \) such that \( r > s > 0 \);
2. \( C \) is adjacent to no mutually non-adjacent \( \equiv \)-classes \( D \) and \( E \) such that \(|D|, |E| \leq |C|\).

Further, \( p^r \) and \( p^s \) are the maximum and the minimum order of an element of \( C \), respectively.

If \( C \) is a simple \( \equiv \)-class, then at least one of the above statements is not satisfied.

**Proof.** Suppose first that \( C \) is a complex \( \equiv \)-class, and let \( y \) be an element of \( C \) of maximal order. Let \( o(y) = p^r \) for a prime \( p \). Let us prove that \( \hat{C} = \langle y \rangle \).

Because \( y \) has prime power order, \( \langle y \rangle \subseteq \hat{C} \). Suppose now that there is an element \( z \in \hat{C} \setminus \langle y \rangle \). Because \( \hat{C} \subseteq \overline{N}(y) \), we get that \( \langle y \rangle \not\subseteq \langle z \rangle \). If \( z \) was not of prime power order, then \( C \) would be a simple \( \equiv \)-class. Also, if \( z \) is of prime power order, that \( \overline{N}(z) \subseteq \overline{N}(y) \) because \( \langle y \rangle \leq \langle z \rangle \). We also have \( \overline{N}(y) \subseteq \overline{N}(z) \) because \( z \in \hat{C} \), and therefore \( y \equiv z \), which is a contradiction. Now \( \hat{C} = \langle y \rangle \) implies that the first condition is fulfilled, and that \( p^r = |C| \). Also, if \( p^{s-1} \) is the maximal order of an element of \( \hat{C} \setminus C \), and if \( z \) is an element of \( \hat{C} \setminus C \) of order \( p^{s-1} \), then \( \langle z \rangle = \hat{C} \setminus C \). This implies that \( |C| - |\hat{C}| = p^{r-1} \).

Let us prove that, for the complex \( \equiv \)-class \( C \), the second condition is fulfilled too. If \( y \xrightarrow{\delta} z \), then \( |[z]_{\equiv}| < |C| \), but any such \( \equiv \)-class is adjacent to all other \( \equiv \)-classes adjacent to \( C \). Now it is sufficient to show that all \( \equiv \)-classes \( [z]_{\equiv} \) adjacent to \( C \), such that \( z \xrightarrow{\delta} y \), have greater cardinality than \( |C| \). If \( z \xrightarrow{\delta} y \), then

\[ |C| < p^r \leq p^r (p - 1) = \varphi(p^{r+1}) \leq [z]_{\equiv}, \]

where \( \varphi \) is Euler totient function. Therefore, the second condition is fulfilled too.

Let us prove now that a simple \( \equiv \)-class does not fulfill at least one of the two conditions. Let \( C \) be a simple \( \equiv \)-class such that the order of its element are divisible by at least two different primes \( p \) and \( q \). Then there are classes \( D \) and \( E \) which contain elements of orders \( p \) and \( q \), respectively. Therefore, \( C \) does not fulfill the second condition.

Suppose now that the elements of \( C \) are of prime power order. Obviously, if \( C = \{e\} \), then the first condition is not satisfied, so suppose further that there are \( k \in \mathbb{N} \) and a prime \( p \) such that all elements of \( C \) are have order \( p^k \). Let \( y \in C \), and suppose that \( C \) satisfies the first condition. Then

\[ |\hat{C}| = p^r < p^r + (p^r - 2p^{s-1}) = 2(p^r - p^{s-1}) = 2|C|. \]

Therefore, \( \hat{C} \) does not contain any element of order greater than \( p^k \). Now, in a similar way like in the first paragraph of this proof, it can be showed that \( \hat{C} = \langle y \rangle \), and therefore \( C \) does not fulfill the first condition. This proves the lemma. \( \square \)
Lemma 3.15 Let $G$ be a group such that $|\text{Cen}(\Phi)| = 1$, where $\Phi$ is the finite-order component of $\mathcal{G}^\pm(G)$. Let $C$ be an infinitely-complex $\equiv$-class. Then $|C| - |C| = p^s - 1$, for a prime $p$ and for some $s \in \mathbb{N}$, and $p^s$ is the minimum order of an element of $C$.

Proof. Because $C$ is an infinitely complex $\equiv$-class, there is a prime $p$ such that orders of all elements of $C$ are powers of $p$. Let $p^s$ be the minimum order of an element of $C$. Because $|\text{Cen}(\Phi)| = 1$, $C$ does not contain the identity element of the group, and, therefore, $s > 0$.

Because $x \overset{\lambda}{\rightarrow} y$ for no elements $y \in C$ and $x \in G_{<\infty} \setminus C$, the set $\overline{N}(C)$ is the universe of a quasicyclic subgroup $C_{p^\infty}$ of $G$. Further, because $\mathcal{G}^\pm(C_{p^\infty})$ is a complete graph, and because $\overline{N}(\overline{N}(C)) \subseteq \overline{N}(C)$, it follows that $\hat{C} = \overline{N}(C)$. Therefore, $|\hat{C} - |C| = p^s - 1$. This proves the lemma.

Before heading over to prove the main theorem of this subsection, we have just one more proposition to prove. Proposition 3.16 covers the case when there is a prime number $p$ such that the orders of all elements of finite order of the group are powers of $p$.

Proposition 3.16 Let $p$ be a prime number, let $G$ and $H$ be groups in which orders of all elements of finite order are powers of $p$, and let $G$ have no intersection-free quasicyclic subgroup. Let $\mathcal{G}^\pm(G)$ and $\mathcal{G}^\pm(H)$ have isomorphic finite-order components. Then $\hat{\mathcal{G}}^\pm(G)$ and $\hat{\mathcal{G}}^\pm(H)$ have isomorphic finite-order components too.

Proof. Because $G$ contains no intersection-free quasicyclic subgroup, $G$ is not a Prüfer group, and so neither is $H$. It further follows that $\mathcal{G}^\pm(G)$ has no infinitely complex $\equiv_G$-class $C$ such that $|\hat{C} \setminus C| = 1$. Therefore, $\mathcal{G}^\pm(H)$ also does not contain such infinitely complex $\equiv_H$-classes, which implies that $H$ also does not contain any intersection-free quasicyclic subgroup.

Let us denote the finite-order components of $\mathcal{G}^\pm(G)$ and $\mathcal{G}^\pm(H)$ with $\Phi$ and $\Psi$, respectively. Let $\varphi : G_{<\infty} \to H_{<\infty}$ be an isomorphism from $\Phi$ to $\Psi$. For a finite $\equiv$-class $C$ contained in $G_{<\infty}$ or $H_{<\infty}$, let us show that the maximal order of an element from $C$ is equal to $|\hat{C}|$. Let $c$ be an element of $C$ of maximal order. Then $\langle c \rangle \subseteq \hat{C}$. Suppose that there is an element $d \in \hat{C} \setminus \langle c \rangle$. Then the order of $d$ is greater than $\text{o}(c)$, and $\langle c \rangle < \langle d \rangle$, which implies $\overline{N}(d) \subseteq \overline{N}(c)$. Also, the fact that $d \in \overline{N}(\overline{N}(c))$ implies $\overline{N}(c) \subseteq \overline{N}(d)$, and thus $c \equiv d$, which is a contradiction. Therefore, the maximal order of an element of $C$ is $|\hat{C}|$. Also, for an infinitely complex $\equiv$-class $C$, $|C| = \aleph_0 = \sup\{\text{o}(x) \mid x \in C\}$.

Now, for $x, y \in G_{<\infty}$ such that $x \not\equiv_G y$ and $x \overset{p^s_G}{\rightarrow} y$, $x \not\equiv_G y$ implies $|\overline{N}_G(\overline{N}_G(y))| < |\overline{N}_G(\overline{N}_G(x))|$. Because $\varphi$ is an isomorphism from $\Phi$ to $\Psi$, $|\overline{N}_H(\overline{N}_H(\varphi(y)))| < |\overline{N}_H(\overline{N}_H(\varphi(x)))|$, and $\varphi(x) \overset{p^s_H}{\rightarrow} \varphi(y)$, and thus, by Lemma 3.13, $\varphi(x) \overset{p^s_H}{\rightarrow} \varphi(y)$. Now it remains to prove that $\varphi$ is determines isomorphisms between pairs of $\equiv$-classes of $\mathcal{G}^\pm(G)$ and $\mathcal{G}^\pm(H)$.

Let $C$ be an $\equiv_G$-class, and let $D = \varphi(C)$. Suppose that $C$ is finite. Then $D$ is a finite $\equiv_H$-class too. If $C$ contains the identity element, then $C = \text{Cen}(\Phi)$ and $D = \text{Cen}(\Psi)$. In this case $C$ and $D$ are cyclic subgroups of the same order, which implies that $(\hat{\mathcal{G}}^\pm(G))[C] \cong (\hat{\mathcal{G}}^\pm(H))[\varphi(C)]$. So suppose that $C$ does not contain the identity element. Then there is an element $z$ from $\langle C \rangle \setminus C$ of maximal
order. Let us denote \(|(C)| = |\hat{C}|\) with \(p^r\), and let us denote \(|(z)| = |\overline{N}_G(\overline{N}_G(z))|\) with \(p^{s-1}\). Then \(C\) contains elements of orders \(p^s, p^{s+1}, \ldots, p^r\). In the same way we conclude that \(D\) also contains elements of order \(p^s, p^{s+1}, \ldots, p^r\). Also, notice that \(\Phi[C]\) and \(\Psi[D]\) are complete subgraphs. Therefore, \((\overline{G}^±(G))[C] \cong (\overline{G}^±(H))[\varphi(C)]\) for any finite \(\equiv_G\)-class \(C\) of \(\Phi\).

Suppose now that \(C\) is an infinitely-complex \(\equiv_G\)-class. Then \(|\hat{C} \setminus C| = p^{s-1}\) for some \(s > 1\). Because \(\varphi\) is an isomorphism from \(\overline{G}^±(G)\) to \(\overline{G}^±(H)\), \(|\hat{D} \setminus D| = p^{s-1}\). It follows that both \(C\) and \(D\) are infinitely complex \(\equiv\)-classes containing elements of orders \(p^s, p^{s+1}, p^{s+2}, \ldots\), which implies \((\overline{G}^±(G))[C] \cong (\overline{G}^±(H))[\varphi(C)]\). This proves that \(\overline{G}^±(G)\) and \(\overline{G}^±(H)\) have isomorphic finite-order components. \(\square\)

**Theorem 3.17** Let \(G\) and \(H\) be groups, and \(G\) have no intersection-free quasicyclic subgroup. If \(Z^±\)-power graphs of \(G\) and \(H\) have isomorphic finite-order components, then their directed power graphs have isomorphic finite-order components too.

**Proof.** Let \(G\) and \(H\) be groups, and \(G\) have no intersection-free quasicyclic subgroup. Let us denote the finite-order components of \(\overline{G}^±(G)\) and \(\overline{G}^±(H)\) with \(\Phi\) and \(\Psi\), respectively. Let \(\Phi \cong \Psi\), and let \(|\text{Cen}(\Phi)| > 1\). Then \(|\text{Cen}(\Psi)| > 1\) too. Now, by Proposition 3.10, \(G_{<\infty}\) and \(H_{<\infty}\) are either both cyclic groups of the same order, or there is a prime \(p\) such that the order of every element of \(G_{<\infty}\) and \(H_{<\infty}\) is a power of \(p\). In the first case, trivially, the \(Z^±\)-power graphs of \(G\) and \(H\) have isomorphic finite-order components, while in the second case that follows by Proposition 3.16.

Suppose further that \(\Phi \cong \Psi\) and \(|\text{Cen}(\Phi)| = 1\). Then \(|\text{Cen}(\Psi)| = 1\) too. Let \(\psi : G \to H\) be an isomorphism from \(\Phi\) to \(\Psi\). Notice that if \(C\) is an \(\equiv_G\)-class, then \(\psi(C)\) is also an \(\equiv_H\)-class. Also, by Lemma 3.14, \(C\) and \(\psi(C)\) are either both simple \(\equiv\)-classes, or they are both complex \(\equiv\)-classes, or they are both infinitely-complex \(\equiv\)-classes. For sets \(X \subseteq G_{<\infty}\) and \(Y \subseteq H_{<\infty}\), we shall say that they are corresponding if there is an \(\equiv_G\)-class \(C\) such that \(X \subseteq C\) and \(Y \subseteq \psi(C)\).

Just like in the proof of Proposition 3.16, from the fact that \(G\) does not have any intersection-free quasicyclic subgroup, we conclude that \(H\) has no intersection-free quasicyclic subgroup too. Therefore, by Lemma 3.15, for any infinitely complex \(\equiv\)-class of \(\Phi\) or \(\Psi\) it is possible to determine all orders of elements contained in \(C\). Also, by Lemma 3.14, for every complex \(\equiv\)-class contained in \(G_{<\infty}\) or \(H_{<\infty}\) one can determine the orders of elements contained in this \(\equiv\)-class. Therefore, each complex or infinitely complex \(\equiv_G\)-class contains elements of same orders as its corresponding \(\equiv_H\)-class. In the remainder of this proof, when mentioning an \(\approx\)-class or an \(\equiv\)-class, we assume that those are classes which contain elements of finite order. We conclude that there is a bijection \(\vartheta : G_{<\infty} \to H_{<\infty}\) which maps every \(\approx_G\)-class onto the corresponding \(\approx_H\)-class. Notice that, by Lemma 3.13, all \(\equiv\)-classes contained in a same complex or infinitely complex \(\equiv\)-class \(C\) relate in the same way in the directed \(Z^±\)-power graph to any other \(\equiv\)-class outside the \(\equiv\)-class \(C\). Finally, \(\vartheta\) is an isomorphism from \((\overline{G}(G))[G_{<\infty}]\) to \((\overline{G}(H))[H_{<\infty}]\), because, by Lemma 3.11, for two adjacent \(\approx\)-classes one can tell which one contains elements of greater order. This proves the theorem. \(\square\)
3.3 Putting the Pieces Together

Now we are ready to prove the main result of this paper.

**Theorem 3.18** Let $G$ and $H$ be groups such that $\mathcal{G}^\pm(G) \cong \mathcal{G}^\pm(H)$. If $G$ has no intersection-free quasicyclic subgroup, then $\mathcal{G}^\pm(G) \cong \mathcal{G}^\pm(H)$.

**Proof.** By Lemma 3.1, $\mathcal{G}^\pm(G)$ and $\mathcal{G}^\pm(H)$ have isomorphic infinite-order components and isomorphic finite-order component. Then, by Theorem 3.7 and Theorem 3.17, $\mathcal{G}^\pm(G)$ and $\mathcal{G}^\pm(H)$ have isomorphic infinite-order components and isomorphic finite-order components. Therefore, groups $G$ and $H$ have isomorphic directed $\mathbb{Z}^\pm$-power graphs.

By Theorem 2.3, the ensuing corollary follows directly from Theorem 3.18.

**Corollary 3.19** Let $G$ and $H$ be groups such that $\mathcal{G}(G) \cong \mathcal{G}(H)$. If $G$ has no intersection-free quasicyclic subgroup, then $\tilde{G}(G) \cong \tilde{G}(H)$.

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