A REMARKABLE CONTRACTION OF SEMISIMPLE LIE ALGEBRAS

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INTRODUCTION

The ground field \( \mathbb{F} \) is algebraically closed and \( \text{char} \mathbb{F} = 0 \). Let \( G \) be a connected semisimple algebraic group of rank \( l \) with Lie algebra \( \mathfrak{g} \). Recently, E. Feigin introduced a very interesting contraction of \( \mathfrak{g} \) [1]. His motivation came from some problems in Representation Theory [3], and making use of this contraction he also studied certain degenerations of flag varieties [2]. Our goal is to elaborate on invariant-theoretic properties of these contractions of semisimple Lie algebras.

Fix a triangular decomposition \( \mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^- \), where \( \mathfrak{t} \) is a Cartan subalgebra. Then \( \mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t} \) is the fixed Borel subalgebra of \( \mathfrak{g} \). The corresponding subgroups of \( G \) are \( B, U, T \). Using the vector space isomorphism \( \mathfrak{g}/\mathfrak{b} \cong \mathfrak{u}^- \), we regard \( \mathfrak{u}^- \) as a \( B \)-module. If \( \mathfrak{b} \in \mathfrak{b} \) and \( \eta \in \mathfrak{u}^- \), then \( \mathfrak{b}, \eta \mapsto b \circ \eta \) stands for the corresponding representation of \( \mathfrak{b} \). That is, if \( p_- : \mathfrak{g} \to \mathfrak{u}^- \) is the projection with kernel \( \mathfrak{b} \), then \( b \circ \eta = p_-([b, \eta]) \).

Following [1, Sect. 2], consider the semi-direct product \( \mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{g}/\mathfrak{b})^a = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a \), where the superscript ‘a’ means that the \( \mathfrak{b} \)-module \( \mathfrak{u}^- \) is regarded as an abelian ideal in \( \mathfrak{q} \). We may (and will) identify the vector spaces \( \mathfrak{g} \) and \( \mathfrak{q} \) using the decomposition \( \mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}^- \). If \( (b, \eta), (b', \eta') \in \mathfrak{q} \), then the Lie bracket in \( \mathfrak{q} \) is given by

\[
[(b, \eta), (b', \eta')] = ([b, b'], b \circ \eta' - b' \circ \eta).
\]

The corresponding connected algebraic group is \( Q = B \ltimes N \), where \( N = \exp((\mathfrak{u}^-)^a) \) is an abelian normal unipotent subgroup of \( Q \). The exponential map \( \exp : (\mathfrak{u}^-)^a \to N \) is an isomorphism of varieties, and elements of \( Q \) are written as product \( s \cdot \exp(\eta) \), where \( s \in B \) and \( \eta \in \mathfrak{u}^- \). If \( (s, \eta) \mapsto s \cdot \eta \) is the representation of \( B \) in \( \mathfrak{u}^- \), then the adjoint representation of \( Q \) is given by

\[
\text{Ad}_Q(s \cdot \exp(\eta))(b, \eta') = (\text{Ad}(s)b, s \cdot (\eta' - b \circ \eta)).
\]

In this note, we explicitly construct certain polynomials that generate the algebras of invariants \( \mathbb{F}[\mathfrak{q}]^Q \) and \( \mathbb{F}[\mathfrak{q}^*]^Q \), and thereby prove that these two algebras are free. Furthermore, we also show that these polynomials generate the corresponding fields of invariants, \( \mathbb{F}(\mathfrak{q})^Q \) and \( \mathbb{F}(\mathfrak{q}^*)^Q \), and that \( \mathbb{F}[\mathfrak{q}] \) is a free \( \mathbb{F}[\mathfrak{q}]^Q \)-module and \( \mathbb{F}[\mathfrak{q}^*] \) is a free \( \mathbb{F}[\mathfrak{q}^*]^Q \)-module. The last assertion implies that the enveloping algebra of \( \mathfrak{q}, \mathcal{U}(\mathfrak{q}) \), is a free module

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over its centre. The Lie algebra \( q \) is a an Inönü-Wigner contraction of \( g \) (see [13, Ch. 7 § 2.5]), and we also discuss the corresponding relationship between the invariants of \( G \) and \( Q \).

Certain classes of non-reductive algebraic Lie algebras \( q \) such that \( F[q]^Q \) is a polynomial ring have been studied before. They include the centralisers of nilpotent elements in \( sl_{l+1} \) and \( sp_{2l} \) [9], \( \mathbb{Z}_2 \)-contractions of \( g \) [8], and the truncated seaweed (biparabolic) subalgebras of \( sl_{l+1} \) and \( sp_{2l} \) [5]. Our result enlarges this interesting family of Lie algebras.

Let \( q^* \) denote the set of regular elements of \( q^\ast \), i.e., \( x \in q^*_{\text{reg}} \) if and only if \( \dim Q.x \) is maximal. For many problems related to coadjoint representations, it is vital to have that \( \text{codim } (q^* \setminus q^*_{\text{reg}}) \geq 2 \) [8, 9]. However, we prove that if \( g \) is simple and not of type \( A_l \), then \( q^* \setminus q^*_{\text{reg}} \) contains a divisor.

**Notation.**

- the centraliser in \( g \) of \( x \in g \) is denoted by \( g^x \).
- \( \kappa \) is the Killing form on \( g \).
- \( g_{\text{reg}} \) is the set of regular elements of \( g \), i.e., \( x \in g_{\text{reg}} \) if and only if \( \dim g^x = l \).
- If \( X \) is an irreducible variety, then \( F[X] \) is the algebra of regular functions and \( F(X) \) is the field of rational functions on \( X \). If \( X \) is acted upon by an algebraic group \( A \), then \( F[X]^A \) and \( F(X)^A \) denote the subsets of respective \( A \)-invariant functions.
- If \( F[X]^A \) is finitely generated, then \( X/A := \text{Spec } (F[X]^A) \) and \( \pi : X \to X/A \) is determined by the inclusion \( F[X]^A \hookrightarrow F[X] \). If \( F[X]^A \) is graded polynomial, then the elements of any set of algebraically independent homogeneous generators will be referred to as basic invariants.
- \( S^i(V) \) is the \( i \)-th symmetric power of the vector space \( V \) and \( S(V) = \oplus_{i \geq 0} S^i(V) \).

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1. **On adjoint and coadjoint invariants of Inönü-Wigner contractions**

The algebra \( q = b \ltimes (u^-)^a \) is an Inönü-Wigner contraction of \( g \). For this reason, we recall the relevant setting and then describe a general procedure for constructing adjoint and coadjoint invariants of Inönü-Wigner contractions. The \( \mathbb{Z}_2 \)-contractions of \( g \) (considered in [7, 8]) are special cases of Inönü-Wigner contractions, and for them such a procedure is exposed in [8, Prop. 3.1]. However, a more general situation considered here requires another proof.

For a while, we assume that \( G \) is any connected algebraic group. Let \( H \) be an arbitrary connected subgroup of \( G \) and let \( m \) be a complementary subspace to \( \mathfrak{h} = \text{Lie } H \). Using the vector space isomorphism \( g/\mathfrak{h} \simeq m \), we regard \( m \) as \( H \)-module. Consider the invertible
linear map \( c_t : g \to g, t \in F \setminus \{0\} \), such that \( c_t(h + m) = h + t m \) \((h \in \mathfrak{h}, m \in m)\) and define the Lie algebra multiplication \([\; , \; ](t)\) on the vector space \( g \) by the rule

\[
[x, y]_t := c_t^{-1}([c_t(x), c_t(y)]), \quad x, y \in g.
\]

Write \( g(t) \) for the corresponding Lie algebra. The operator \( (c_t)^{-1} = c_t : g \to g(t) \) yields an isomorphism between the Lie algebras \( g = g(1) \) and \( g(t) \), hence all algebras \( g(t) \) are isomorphic. It is easily seen that \( \lim_{t \to 0} g(t) \simeq \mathfrak{h} \ltimes (\mathfrak{g}/\mathfrak{h})^a = \mathfrak{h} \ltimes m^a \).

The resulting Lie algebra \( \mathfrak{k} := \mathfrak{h} \ltimes m^a \) is called an Inönü-Wigner contraction of \( g \), cf. Example 7 in [13, Chapter 7, §2]. The corresponding connected algebraic group is \( K = H \ltimes \exp(m^a) \). We identify the vector spaces \( g \) and \( \mathfrak{k} \) using the decomposition \( g = \mathfrak{h} \oplus m \).

**Remark.** For \( g \) semisimple, the contraction \( g \sim b \ltimes u^- \) is presented in a more lengthy way, using structure constants, in [1, Remark 2.3].

### 1.1. To construct invariants of the coadjoint representation of \( \mathfrak{k} \), we proceed as follows. Let \( f \in S(g) = F[g^*] \) be a homogeneous polynomial of degree \( n \). Using the decomposition \( g = \mathfrak{h} \oplus m \), we consider the bi-homogeneous components of \( f \):

\[
f = \sum_{a \subseteq b} f^{(n-i,i)},
\]

where \( f^{(n-i,i)} \in S^{n-i}(\mathfrak{h}) \otimes S^i(m) \subset S^n(g) \), and both \( f^{(n-a,a)} \) and \( f^{(n-b,b)} \) are assumed to be nonzero. In particular, \( f^{(n-b,b)} \) is the bi-homogeneous component having the maximal degree relative to \( m \). Since \( g(t) \) and \( \mathfrak{k} \) are just the same vector spaces, we also can regard each \( f^{(n-i,i)} \) as an element of \( S_n(g(t)) \) or \( S_n(\mathfrak{k}) \).

**Theorem 1.1.** If \( f \in S^n(g)^G = F[g^*]^G_{\mathfrak{h}} \), then \( f^{(n-b,b)} \in S^n(\mathfrak{k})^K = F[\mathfrak{k}^*]^K \).

**Proof.** The isomorphism of Lie algebras \( c_t^{-1} : g \to g(t) \) implies that \( \sum_{a \subseteq b} t^{-i} f^{(n-i,i)} \in S(g(t))^{G(t)} \) for all \( t \neq 0 \). It is harmless to replace the last expression with the \( G(t) \)-invariant \( f(t) := \sum_{a \subseteq b} t^{-i} f^{(n-i,i)} \). Consider the line \( \langle f(t) \rangle \) in the projective space \( \mathbb{P}(S^n(g)) \). It is a standard fact on \( F^\times \)-actions in projective spaces that \( \lim_{t \to 0} \langle f(t) \rangle = \langle f^{(n-b,b)} \rangle \). Hence \( f^{(n-b,b)} \) is \( K \)-invariant. \( \square \)

Let us say that \( f^* := f^{(n-b,b)} \) is the **highest component** of \( f \in F[g^*]^G_{\mathfrak{h}} \) (with respect to the contraction \( g \sim \mathfrak{g} \)). Set \( \mathcal{L}^*(F[g^*]^G) = \{ f^* \mid f \in F[g^*]^G \text{ is homogeneous} \} \). Clearly, it is a graded subalgebra of \( F[\mathfrak{k}^*]^K \).

Invariants of the adjoint representation of \( \mathfrak{k} \) can be constructed in a similar way. Set \( m^* := \mathfrak{h}^\perp \), the annihilator of \( \mathfrak{h} \) in \( g^* \). Likewise, \( \mathfrak{h}^* = m^\perp \). Then \( g^* = m^* \oplus \mathfrak{h}^* \), and the adjoint operator \( c_t^* : g^* \to g^* \) is given by \( c_t^*(m^* + h^*) = t^{-1}m^* + h^* \) \((m^* \in m^*, h^* \in \mathfrak{h}^*)\).

Having identified \( g^* \) and \( \mathfrak{k}^* \), we can play the same game with homogeneous elements of \( S(g^*) = F[g] \). If \( \tilde{f} \in S^n(g) \), then \( \tilde{f}^{(i,n-i)} \) denotes its bi-homogeneous component that belong to \( S^i(m^*) \otimes S^{n-i}(\mathfrak{h}^*) \). The resulting assertion is the following:
Theorem 1.2. For $f \in S^n(g^*)^G$, let $\hat{f}^{(a,n-a)}$ be the bi-homogeneous component with minimal $a$, i.e., having the maximal degree relative to $h^* = m^\perp$. Then $\hat{f}^{(a,n-a)} \in S^n(t^*)^K$.

Likewise, we write $\hat{f}^\bullet := \hat{f}^{(a,n-a)}$ and consider the algebra of highest components, $L^\bullet(F[g]^G)$, which is a graded subalgebra of $F[t]^K$.

Lemma 1.3. The graded algebras $F[g^*]^G$ and $L^\bullet(F[g]^G)$ have the same Poincaré series, i.e., $\dim F[g^*]^G = \dim L^\bullet(F[g]^G)$ for all $n \in \mathbb{N}$; and likewise for $F[g]^G$ and $L^\bullet(F[g]^G)$.

Proof. It easily seen that, for each $n$, there is a basis $\{p_i\}$ for $F[g^*]^G_n$ such that $\{p_i^\bullet\}$ is a basis for $L^\bullet(F[g]^G)_n$. \hfill \Box

It is not always the case that $L^\bullet(F[g]^G) = F[t]^K$ or $L^\bullet(F[g]^G) = F[t]^K$. For instance, we will see below that, for $g$ semisimple and $q = b \ltimes (u^-)^\vee$, such an equality holds only for the invariants of the coadjoint representation. By the very construction, the algebras $L^\bullet(F[g]^G)$ and $L^\bullet(F[g]^G)$ are bi-graded. Moreover, it follows from [8, Theorem 2.7] that the algebras $F[t^*]^K$ and $F[t]^K$ are always bi-graded.

1.2. If $g$ is semisimple, then we may identify $g$ and $g^*$ (and hence $S(g)$ and $S(g^*)$) using the Killing form $\kappa$. If $h$ is also reductive, then $\kappa$ is non-degenerated on $h$ and one can take $m$ to be the orthocomplement of $h$ with respect to $\kappa$. Then $h^\perp \simeq m$ and the decompositions of $g$ and $g^*$ considered in the general setting of Inönü-Wigner contractions coincide. Moreover, we can also identify the vector spaces $t$ and $t^*$. However, to obtain invariants of the adjoint and coadjoint representations of $q$, one has to take the bi-homogeneous components of maximal degree with respect to different summands in the sum $g = h \oplus m$.

In this situation, Theorems 1.1 and 1.2 admit the following simultaneous formulation:

Suppose that $f \in F[g]^G_n \simeq S(g)^G_n$ and $f = \sum_{a \leq i < b} f^{(n-i,i)}$ is the bi-homogeneous decomposition relative to the sum $g = h \oplus m$. (That is, $\deg_h f^{(n-i,i)} = n - i$, etc.) Then, upon identifications of vector spaces $g, t$, and $t^*$, we have $f^{(n-a,a)} \in F[t]^K$ and $f^{(n-b,b)} \in F[t^*]^K$.

Such a phenomenon was already observed in the case of $\mathbb{Z}_2$-contractions of semisimple Lie algebras, i.e., if $h$ is the fixed-point subalgebra of an involution, see [8, Prop. 3.1].

2. INVARIANTS OF THE ADJOINT REPRESENTATION OF $Q$

In this section, we describe the algebra of invariants of the adjoint representation of $Q$.

To prove that a certain set of invariants generates the whole algebra of invariants, we use the following lemma of Igusa.

Lemma 2.1 (Igusa). Let $A$ be an algebraic group acting regularly on an irreducible affine variety $X$. Suppose that $S$ is an integrally closed finitely generated subalgebra of $F[X]^A$ and the morphism $\pi : X \to \text{Spec} S := Y$ has the properties:
(i) the fibres of $\pi$ over a dense open subset of $Y$ contain a dense $A$-orbit;

(ii) $\text{Im } \pi$ contains an open subset $\Omega$ of $Y$ such that $\text{codim } (Y \setminus \Omega) \geq 2$.

Then $S = \mathbb{F}[X]^A$. In particular, the algebra of $A$-invariants is finitely generated.

See e.g. [7, Lemma 6.1] for the proof.

Remark 2.2. The proof given in [7] shows that the above condition (i) can be replaced with the condition that $S \subset \mathbb{F}[X]^A$ generates the field $\mathbb{F}(X)^A$. (In fact, it is not hard to prove that (i) holds if and only if $S$ separates $A$-orbits in a dense open subset of $X$ if and only if $S$ generates $\mathbb{F}(X)^A$.)

Lemma 2.3. If $t \in \mathfrak{t}$ is regular and $u \in \mathfrak{u}$ is arbitrary, then (i) $t + u$ and $t$ belong to the same $\text{Ad } U$-orbit; (ii) $(t + u) \circ u^- = u^-.$

Proof. (i) Clearly, $(\text{Ad } U)t \subset t + u$ for all $t \in \mathfrak{t}$. If $t$ is regular, then $\dim(\text{Ad } U)t = \dim u$. It is also known that the orbits of a unipotent group acting on an affine variety are closed. Hence $(\text{Ad } U)t = t + u$.

(ii) This is obvious if $u = 0$. In general, this follows from (i). $\square$

Theorem 2.4. We have $\mathbb{F}[q]^Q \simeq \mathbb{F}[t]$, and the quotient morphism $\pi_Q : q \rightarrow \mathfrak{t}$ is given by $(t + u, \eta) \mapsto t$.

Proof. Clearly, $\mathbb{F}[q]^Q = (\mathbb{F}[q]^N)^B$. We prove that 1) $\mathbb{F}[q]^N \simeq \mathbb{F}[b]$ and 2) $\mathbb{F}[b]^B \simeq \mathbb{F}[t]$.

1) Consider the projection $\pi_N : q \rightarrow q/ (u^-)^a \simeq b$. Clearly, $N$ acts trivially on $q/ (u^-)^a$ and $\pi_N$ is a surjective $N$-equivariant morphism. Hence $\mathbb{F}[b] \subset \mathbb{F}[q]^N$. By Lemma 2.1, the equality $\mathbb{F}[b] = \mathbb{F}[q]^N$ will follow from the fact that generic fibres of $\pi_N$ are $N$-orbits.

If $t \in \mathfrak{t}$ is regular and $u \in \mathfrak{u}$ is arbitrary, then $b = t + u$ is a regular semisimple element of $\mathfrak{g}$. By (0.2) with $s = 1$, we have

$$\text{Ad}_Q(N)(b, \eta) = (b, \eta + b \circ u^-).$$

It then follows from Lemma 2.3 that $\text{Ad}_Q(N)(b, \eta) = (b, u^-)$. On the other hand, $\pi_N^{-1}(b) = (b, u^-)$, i.e., $\pi_N^{-1}$ is a single $N$-orbit whenever $b$ is regular semisimple.

2) Consider the projection $\pi_B : b \rightarrow b/ u \simeq t$. Clearly, $B$ acts trivially on $b/ u$ and $\pi_B$ is a surjective $B$-equivariant morphism. Hence $\mathbb{F}[t] \subset \mathbb{F}[b]^B$. By Lemma 2.1, the equality $\mathbb{F}[t] = \mathbb{F}[b]^B$ will follow from the fact that generic fibres of $\pi_B$ are $B$-orbits. Again, it follows from Lemma 2.3 that if $t \in \mathfrak{t}$ is regular, then $(\text{Ad } B)t = t + u = \pi_B^{-1}(t)$. $\square$

Remark 2.5. Theorem 2.4 can be proved in a less informative way. Notice that $[q, q] = \mathfrak{u} \ltimes (u^-)^a$ and therefore $\mathbb{F}[t] \subset \mathbb{F}[q]^Q$. Let $x \in \mathfrak{t}$ be regular semisimple. Then $q^x \simeq \mathfrak{g}^x = t$, since $\mathfrak{g}$ and $q$ are isomorphic as $T$-modules. The fibres of the morphism $\pi_Q : q \rightarrow \mathfrak{t}$, defined in Theorem 2.4, are linear spaces of dimension $\dim q - \dim \mathfrak{t} = \dim(\text{Ad } Q)x$. Hence a
generic fibre contains a dense $Q$-orbit and Lemma 2.1 applies. We also see that the algebra $\mathbb{F}[t]$ separates generic $Q$-orbits in $q$ and therefore $\mathbb{F}(q)^Q = \mathbb{F}(t)$.

Comparing with the adjoint representation of $\mathfrak{g}$, we see that, for $q$, the algebra of invariants remains polynomial, but the degrees of basic invariants drastically decrease! All the basic invariants in $\mathbb{F}[q]^Q$ are of degree 1. This clearly means that here $L^*(\mathbb{F}[\mathfrak{g}]^G) \subsetneq \mathbb{F}[q]^Q$.

3. Invariants of the coadjoint representation of $Q$

In this section, we describe the algebra of invariants of the coadjoint representation of $Q$. The coadjoint representation is much more interesting since $\mathbb{F}[q^*] = S(q)$ is a Poisson algebra, $S(q)^Q$ is the centre of this Poisson algebra, and $S(q)$ is related to the enveloping algebra of $q$ via the Poincaré-Birkhoff-Witt theorem.

Since $q$ is isomorphic to $\mathfrak{b} \oplus \mathfrak{g}/\mathfrak{b} \simeq \mathfrak{b} \oplus \mathfrak{u}^-$ as vector space, the dual vector space $q^*$ is isomorphic to $(\mathfrak{g}/\mathfrak{b})^* \oplus \mathfrak{b}^*$. Using $\kappa$, we identify $\mathfrak{b}^*$ with $\mathfrak{b}^- := \mathfrak{t} \oplus \mathfrak{u}^-$ and $(\mathfrak{g}/\mathfrak{b})^*$ with $\mathfrak{u}$. To stress that $q^*$ is regarded as a $Q$-module and $\mathfrak{b}^-$ appears to be a $Q$-stable subspace, we write $q^* = \mathfrak{u} \ltimes \mathfrak{b}^-$. If $(b, \eta) \in q$ and $(u, \xi) \in q^*$, i.e., $u \in \mathfrak{u}$ and $\xi \in \mathfrak{b}^-$, then the coadjoint representation of $q$ is given by the formula:

$$
(b, \eta) \star (u, \xi) = ([b, u], \phi(u, \eta) + b \ast \xi).
$$

Here $(b, \xi) \mapsto b \ast \xi$ is the coadjoint representation of $\mathfrak{b}$, and

$$
\phi : \mathfrak{u} \times \mathfrak{u}^- \simeq \mathfrak{u} \times \mathfrak{u}^* \xrightarrow{\psi} \mathfrak{b}^* \simeq \mathfrak{b}^-,
$$

where $\psi$ is the moment map associated with the $\mathfrak{b}$-module $\mathfrak{u}$. Upon our identifications, the mapping $\phi$ is directly defined by $\kappa(b, \phi(u, \eta)) := \kappa([b, u], \eta) = -\kappa(u, b \circ \eta)$.

Recall some well-known properties of the $B$-module $\mathfrak{u}$:

- If $\bar{e} \in \mathfrak{u}$ is regular nilpotent, then $\mathfrak{g}\bar{e} \subset \mathfrak{u}$ and hence $(\text{Ad } B)\bar{e}$ is dense in $\mathfrak{u}$.
- For any $e \in \mathfrak{u}$, each irreducible component of $(\text{Ad } G)e \cap \mathfrak{u}$ has dimension $\frac{1}{2} \dim(\text{Ad } G)e$ [12, 4.3.11].

Let $\text{Mor}_G(\mathfrak{g}, \mathfrak{g})$ denote the $\mathbb{F}[\mathfrak{g}]^G$-module of polynomial $G$-equivariant morphisms $F : \mathfrak{g} \rightarrow \mathfrak{g}$. By work of Kostant [6], $\text{Mor}_G(\mathfrak{g}, \mathfrak{g})$ is a free graded $\mathbb{F}[\mathfrak{g}]^G$-module of rank $l$. It was noticed by Th. Vust [14, Char. III, §2] (see also [10]) that a homogeneous basis of this module is obtained as follows. Let $f_1, \ldots, f_l$ be homogeneous algebraically independent generators of $\mathbb{F}[\mathfrak{g}]^G$. Each differential $df_i$ determines a polynomial $G$-equivariant morphism (covariant) from $\mathfrak{g}$ to $\mathfrak{g}^*$. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ via $\kappa$ yields a homogeneous covariant (or, vector field) $F_i = \text{grad} f_i : \mathfrak{g} \rightarrow \mathfrak{g}$. Then $F_1, \ldots, F_l$ form a homogeneous basis for $\text{Mor}_G(\mathfrak{g}, \mathfrak{g})$. If $\deg f_i = d_i$, then $\deg F_i = d_i - 1 =: m_i$. It is customary to say that $\{m_1, \ldots, m_l\}$ are the exponents of (the Weyl group of) $\mathfrak{g}$. Recall that if $\mathfrak{g}$ is simple and $m_1 \leq \ldots \leq m_l$, then $m_1 = 1$, $m_2 \geq 2$, and $m_i + m_{i+1}$ is the Coxeter number of $\mathfrak{g}$. 
The covariants $F_i$ have the following properties:

(i) $F_i(x) \in g^r$ for all $i \in \{1, 2, \ldots, l\}$ and $x \in g$;

(ii) The vectors $F_1(x), \ldots, F_l(x) \in g$ are linearly independent if and only if $x \in g_{\text{reg}}$ [6, Theorem 9].

It follows that $(F_1(x), \ldots, F_l(x))$ is a basis for $g^r$ if and only if $x \in g_{\text{reg}}$.

Lemma 3.1. If $x \in b$, then $F_i(x) \in b$. If $y \in u$, then $F_i(y) \in u$.

Proof. If $x \in b \cap g_{\text{reg}}$, then $g^r \subset b$. Hence $F_i(x) \in g^r \subset b$. Since $b \cap g_{\text{reg}}$ is open and dense in $b$, the assertion follows.

If $y \in u \cap g_{\text{reg}}$, i.e., $y$ is regular nilpotent, then $g^y \subset u$ [6]. The rest is the same. □

Consequently, letting $P_i := F_i|_u$, we obtain the covariants $P_1, \ldots, P_l \in \text{Mor}_B(u, u)$. Actually, we consider the $P_i$'s as $B$-equivariant morphisms $P_i : u \rightarrow u \subset b$. Using these covariants, we define polynomials $\widehat{P}_i \in \mathbb{F}[q^*] = \mathbb{F}[u \ltimes b^-]$ by the formula

\begin{equation}
\widehat{P}_i(u, \xi) = \varkappa(P_i(u), \xi), \quad i = 1, \ldots, l,
\end{equation}

where $u \in u$ and $\xi \in b^-$. 

Lemma 3.2. We have $\widehat{P}_i \in \mathbb{F}[q^*]^Q$.

Proof. Since $Q = B \ltimes N$, it suffices to verify that $\widehat{P}_i$ is both $B$- and $N$-invariant.

1) $\widehat{P}_i$ is $B$-invariant, since $P_i$ is $B$-equivariant.

2) For polynomials obtained from covariants $P_i$ as in (3.2), the invariance with respect to the commutative unipotent group $N$ is equivalent to that $[P_i(u), u] = 0$. Indeed, for $\eta \in u^-$, the coadjoint action of $\exp(\eta) \in N$ is given by $\exp(\eta)^*(u, \xi) = (u, \xi + \phi(u, \eta))$. Then

\[\widehat{P}_i(\exp(\eta) \cdot (u, \xi)) = \varkappa(P_i(u), \xi + \phi(u, \eta)) = \varkappa(P_i(u), \xi) + \varkappa(P_i(u), \phi(u, \eta)) = \widehat{P}_i(u, \xi) + \varkappa([P_i(u), u], \eta).\]

Hence $\widehat{P}_i(\exp(\eta) \cdot (u, \xi)) = \widehat{P}_i(u, \xi)$ for all $\eta$ if and only if $[P_i(u), u] = 0$. In our case, this follows from the corresponding property of $F_i$. □

Remark. We prove below that $\widehat{P}_i$ is the highest components of $f_i \in \mathbb{F}[g^*]^G$. In view of Theorem 1.1, this also implies that $\widehat{P}_i$ is $Q$-invariant.

Theorem 3.3. The algebra $\mathbb{F}[q^*]^Q$ is freely generated by $\widehat{P}_1, \ldots, \widehat{P}_l$, and $\mathbb{F}(q^*)^Q$ is the fraction field of $\mathbb{F}[q^*]^Q$.

Proof. Consider the morphism

\[\pi : q^* = u \ltimes b^- \rightarrow \mathbb{A}^l,\]
given by $\pi(u, \xi) = (\hat{P}_1(u, \xi), \ldots, \hat{P}_l(u, \xi))$. As in Section 2, to prove that $\pi$ is the quotient by $Q$, we are going to apply Lemma 2.1 to $\pi$.

If $e \in u$ is regular, then $P_i(e), \ldots, P_l(e)$ are linearly independent and form a basis for $g^e = u^e$. Therefore, (3.2) implies that $\pi$ is onto, and condition (ii) in Lemma 2.1 is satisfied.

Let us prove that $F(q^*)^Q = F(\hat{P}_1, \ldots, \hat{P}_l)$. Consider the morphism

$$\hat{\pi} : q^* \to (q^*/b^-) \times A^l = u \times A^l$$

defined by $\hat{\pi}(u, \xi) = (u, \hat{P}_1(u, \xi), \ldots, \hat{P}_l(u, \xi))$. If $e \in u \cap g_{\text{reg}}$, then Eq. (3.2) shows that $\hat{\pi}^{-1}(e, a)$ is an affine subspace of $q^*$ for any $a \in A^l$, and $\dim \hat{\pi}^{-1}(e, a) = \dim b - l = \dim u$. As in the proof of Theorem 2.4, this implies that $\hat{\pi}^{-1}(e, a)$ is a sole $N$-orbit. Thus, the coordinate functions on $u$ and $\hat{P}_1, \ldots, \hat{P}_l$ separate generic $N$-orbits of maximal dimension. By the Rosenlicht theorem, this implies that all these functions generate the field of $N$-invariants on $q^*$, i.e., $F(q^*)^N = F(u)(\hat{P}_1, \ldots, \hat{P}_l)$. Since $B$ has an open orbit in $u$, we have $F(u)^B = F$. Hence

$$F(q^*)^Q = (F(u)(\hat{P}_1, \ldots, \hat{P}_l))^B = F(\hat{P}_1, \ldots, \hat{P}_l).$$

In view of Remark 2.2, this is sufficient for using Lemma 2.1, and we conclude that $\hat{P}_1, \ldots, \hat{P}_l$ generate the algebra of $Q$-invariants on $q^*$. □

**Remark 3.4.** Although we have proved that $F(q^*)^N = F(u)(\hat{P}_1, \ldots, \hat{P}_l)$, it is not true that $F[q^*]^N = F[u][\hat{P}_1, \ldots, \hat{P}_l]$. The reason is that the morphism $\hat{\pi}$ defined in the previous proof does not satisfy condition (ii) of Lemma 2.1. That is, the closure of the complement of $\Im \hat{\pi}$ contains a divisor. One can prove that if $D = u \setminus (\text{Ad} B) \hat{e} = u \setminus (u \cap g_{\text{reg}})$, then this divisor is $D \times A^l$. Actually, we can explicitly point out a function in $F[q^*]^N \setminus F[u][\hat{P}_1, \ldots, \hat{P}_l]$. Let $v$ be a non-zero vector in the one-dimensional space $b^u$. We can regard $v$ as a linear function on $b^-$ and hence on $q^*$. Making use of Eq. (0.1), it is not hard to check that the subalgebra $(u^-)^a \subset q$ commutes with $v$, i.e., $v$ is a required $N$-invariant in the symmetric algebra $S(q)$.

Recall that, for an algebraic group $A$ with Lie algebra $a$, the *index* of $a$, $\text{ind} a$, equals $\text{trdeg} F(a^*)^A$. It is also true that the index cannot decrease under contractions, hence $\text{ind} q \geq \text{ind} g = l$. The above description of the field of $Q$-invariants implies that

**Corollary 3.5.** $\text{ind} q = l$.

**Theorem 3.6.** The polynomial ring $F[q^*]$ is a free $F[q^*]^Q$-module.

**Proof.** Since it is already known that $F[q^*]^Q$ is a polynomial algebra (of Krull dimension $l$), it suffices to prove that the quotient morphism $\pi : q^* \to q^*/Q \simeq A^l$ is equidimensional [11, Prop. 17.29]. This, in turn, will follow from the fact that the null-cone $\mathcal{N} = \pi^{-1}(\pi(0))$ is of dimension $\dim q - l$. To estimate the dimension of $\mathcal{N}$, consider the projection $p : \mathcal{N} \to u$ and partition $u$ into finitely many *orbital varieties*, i.e., the irreducible components
of \((\text{Ad } G)e_i \cap u\), where \(\{e_i\}\) runs over a finite set of representatives of all nilpotent \(G\)-orbits. Let \(Z_i\) be an irreducible component of \((\text{Ad } G)e_i \cap u\). Since \(\pi = (\bar{P}_1, \ldots, \bar{P}_t)\), Eq. (3.2) shows that
\[
\dim p^{-1}(Z_i) = \dim Z_i + \dim b - \dim \text{span}\{P_1(e_i), \ldots, P_l(e_i)\}.
\]
As \(\dim Z_i = \frac{1}{2} \dim (\text{Ad } G)e_i\), the condition that \(\dim p^{-1}(Z_i) \leq \dim q - l\) can easily be transformed into
\[
\dim g^e_i + 2 \dim \text{span}\{P_1(e_i), \ldots, P_l(e_i)\} \geq 3l.
\]
(3.3)
Recall that \(P_1, \ldots, P_l\) are just the restrictions to \(u\) of basic covariants \(F_1, \ldots, F_i\), and \(F_j = \text{grad } f_j\). Consequently, \(\dim \text{span}\{P_1(e_i), \ldots, P_l(e_i)\}\) equals the rank of the differential at \(e\) of the quotient morphism \(\pi_{g, G} : g \to g//G\). Therefore, (3.3) is precisely the inequality proved in [7, Theorem 10.6].

**Corollary 3.7.** The enveloping algebra \(\mathcal{U}(q)\) is a free module over its centre \(Z(q)\).

**Proof.** This is a standard consequence of the fact that \(\mathbb{F}[q^*] = S(q)\) is a free module over \(S(q)^Q\), \(S(q)^Q\) is the centre of the Poisson algebra \(S(q)\), and \(\text{gr } Z(q) = S(q)^Q\), cf. [6, Theorem 21], [4, Theorem 3.3].

**Remark 3.8.** By Theorem 3.6, the irreducible components of all fibres of \(\pi : q^* \to q^*/Q \simeq \mathbb{A}^l\) are of dimension \(\dim q - l\). However, unlike the case of the (co)adjoint representation of \(g\), the zero fibre of \(\pi\) is highly reducible. For, if \(\dim g^e_i + 2 \dim \text{span}\{P_1(e_i), \ldots, P_l(e_i)\} = 3l\), then every irreducible component of \((\text{Ad } G)e_i \cap u\) gives rise to an irreducible component of \(\pi^{-1}(\pi(0))\). A complete classification of nilpotent elements of \(g\) satisfying this equality is contained in [7, §10].

**Theorem 3.9.** We have \(L^*(S(g)^G) = S(q)^Q\). The polynomials \(\tilde{P}_1, \ldots, \tilde{P}_t \in \mathbb{F}[q^*]^Q = S(q)^Q\) are the highest components of \(f_1, \ldots, f_i \in S(g)^G\) in the sense of Subsection 1.1.

**Proof.**
1) Since \(\deg \tilde{P}_i = \deg f_i\) for all \(i\), it follows from Lemma 1.3 and Theorem 3.3 that \(L^*(S(g)^G)\) and \(S(q)^Q\) have the same Poincaré series. Hence these algebras coincide.

2) Recall that \(\deg f_i = d_i = m_i + 1\). According to Theorem 1.1, we have to take the decomposition \(g = b \oplus u^-\) and pick the bi-homogeneous component of \(f_i\) of maximal degree with respect to \(u^-\).

If the component \(f_i^{(0,d_i)} \in S^h(u^-)\) were non-trivial, then it would be a \(Q\)-invariant in \(S(q)\) and in particular a \(B\)-invariant (Theorem 1.1). Recall that if we work in \(q\), then \(u^- \simeq g/b\) as \(B\)-module. Since \(S(g/b) \simeq \mathbb{F}[u] \text{ and } \mathbb{F}[u]^B = \mathbb{F}\), we get a contradiction. Hence \(f_i^{(0,d_i)} = 0\).

Then next possible component is \(f_i^{(1,m_i)} \in b \otimes S^{m_i}(u^-)\). Using the identifications \(b^* \simeq b^-\) and \(u^* \simeq u^-\), we have \(f_i^{(1,m_i)} \in \mathbb{F}[b^-]_1 \otimes \mathbb{F}[u]_{m_i}\). That is, if considered as a function on \(g = b^- \oplus u\), it can be written as \(f_i^{(1,m_i)}(\xi, u) = \kappa(\bar{P}_i(u), \xi)\) for some morphism \(\bar{P}_i : u \to b\).
of degree \( m_i \). As we have already proved that \( f_i^{(0,d_i)} = 0 \), \( \bar{P}_i(u) \) is nothing but the value of \( \text{grad} f_i \) at \( u \). Hence \( \bar{P}_i = P_i \), and we are done. \( \square \)

4. FURTHER PROPERTIES OF THE COADJOINT REPRESENTATION

4.1. For the classical Lie algebras, the basic covariants \( F_i : g \to g \) (and hence \( P_i \)) have a simple description:

- if \( x \in \mathfrak{s}l_{l+1} \), then \( F_i(x) = x^i \), \( i = 1, 2, \ldots, l \);
- if \( x \in \mathfrak{sp}_{2l} \) or \( \mathfrak{so}_{2l+1} \), then \( F_i(x) = x^{2i-1} \), \( i = 1, 2, \ldots, l \);
- if \( x \in \mathfrak{so}_{2l} \), then \( F_i(x) = x^{2l-1} \), \( i = 1, 2, \ldots, l - 1 \). The covariant \( F_i \) that is related to the pfaffian is described as follows. Let \( x \) be a skew-symmetric matrix of order \( 2l \). For \( i \neq j \), let \( x_{ij} \) be the skew-symmetric sub-matrix of order \( 2l - 2 \) obtained by deleting \( i \)th and \( j \)th row and column. Set \( a_{ij} = \text{Pf}(x_{ij}) \) if \( i \neq j \), and \( a_{ii} = 0 \). Then \( F_i(x) = (a_{ij})_{i,j=1}^{2l} \). Clearly, \( \text{deg} F_i = l - 1 \), as required.

Results of Sections 2 and 3 explicitly yield the bi-degrees of basic invariants for \( q = b \times (u^-)^a \). For \( \mathbb{F}[q]^Q \), all the basic invariants have bi-degrees \((1, 0)\). For \( \mathbb{F}[q^a]^Q \), the basic invariants have bi-degrees \((m_i, 1)\), i.e., \( \bar{P}_i \in S^{m_i}(u^-) \otimes b \). In particular, for the coadjoint representation, the total degrees of the basic \( Q \)-invariants remain the same as for \( G \).

4.2. Hereafter we assume that \( g \) is simple and the basic invariants \( f_1, \ldots, f_l \in \mathbb{F}[g]^G \) are numbered such that \( d_i \leq d_{i+1} \). Then \( d_l = h \) is the Coxeter number of \( g \). We show that the corresponding \( Q \)-invariant \( \bar{P}_l \) has a rather simple form. In fact, it appears to be a monomial.

Let \( \Delta \) be the root system of \((g, t)\) and \( \Delta^+ \) the subset of positive roots corresponding to \( u \). Then \( \Pi = \{\alpha_1, \ldots, \alpha_l\} \) (resp. \( \theta \)) is the set of simple roots (resp. the highest root) in \( \Delta^+ \). Then \( \theta = \sum_{i=1}^l a_i \alpha_i \) and \( \sum_{i=1}^l a_i = h - 1 \). For any \( \gamma \in \Delta \), \( g_\gamma \) denotes the corresponding root subspace, and we fix a nonzero vector \( e_\gamma \in g_\gamma \).

**Lemma 4.1.** Up to a scalar multiple, we have \( \bar{P}_l = e_{-\alpha_1} \cdots e_{-\alpha_l} e_\theta \in S(q)^Q \).

**Proof.** Recall that \( q = b \oplus u^- \) as vector space, and here \( e_\theta \in b \) and \( e_{-\alpha_i} \in u^- \). By the very construction, \( \bar{P} := e_{-\alpha_1} \cdots e_{-\alpha_l} e_\theta \) is a \( T \)-invariant in \( S(q) \). Then, using Eq. (0.1), one readily verifies that \( \bar{P} \) is both \( U \)-invariant and \( N \)-invariant. Hence \( \bar{P} \) is a polynomial in \( \bar{P}_1, \ldots, \bar{P}_l \). Since \( \text{bi-deg} \bar{P} = (h - 1, 1) \) and \( m_i < m_l \) for \( i < l \), the subspace of bi-degree \((m_l, 1) = (h-1, 1)\) in \( S(q)^Q \) is one-dimensional and spanned by \( \bar{P}_l \). Hence the assertion. \( \square \)

Since \( \dim q = \dim g \), \( \text{ind} q = \text{ind} g \), and the (total) degrees of the basic invariants of the coadjoint representations for \( G \) and \( Q \) coincide, we have the equality

\[
\sum_{i=1}^l \deg \bar{P}_i = \frac{\dim q + \text{ind} q}{2},
\]
which is very useful in the study of the coadjoint representation, see e.g. [8, Theorem 1.2]. Unfortunately, \( q \) does not always possess another important ingredient, the so-called \( \text{codim}-2 \) property. Recall that \( x \in q^* \) is said to be regular if \( \dim Q_\cdot x \) is maximal. The set of all regular elements is denoted by \( q^*_\text{reg} \). It is an open subset of \( q^* \), and we say that \( q \) has the \( \text{codim}-2 \) property if \( \text{codim} \{ q^* \setminus q^*_\text{reg} \} \geq 2 \).

**Theorem 4.2.** The algebra \( q \) does not have the \( \text{codim}-2 \) property if \( g \) is not of type \( A_l \).

**Proof.** Suppose that \( q \) has the \( \text{codim}-2 \) property. Since (4.1) is satisfied, it follows from [8, Theorem 1.2] that the differentials \( (d\hat{P}_i)_x, \) \( i = 1, \ldots, l \), are linearly independent if and only if \( x \in q^*_\text{reg} \). In particular, any divisor \( D \subset q^* \) contains a point where the differentials of \( \hat{P}_1, \ldots, \hat{P}_l \) are linearly independent.

On the other hand, Lemma 4.1 shows that if \( a_i \geq 2 \) for some \( i \), then \( d\hat{P}_i \) vanishes at the hyperplane \( \{ e_{-\alpha_i} = 0 \} \), where \( e_{-\alpha_i} \) is regarded as a linear function on \( u \) and hence on \( q^* \). Thus, \( q \) cannot have the \( \text{codim}-2 \) property unless \( a_i = 1 \) for all \( i \), i.e., \( g \) is of type \( A_l \). \( \square \)

To prove the converse of this theorem, we need some preparations. For \( a_i \in \Pi \), let \( u_i \subset u \) denote the kernel of the linear form \( u \mapsto \kappa(e_{-\alpha_i}, u) \). By [6], \( u \setminus u \cap g_\text{reg} = \cup_i u_i \). Set

\[
\mathcal{Y} = \mathcal{Y}(q^*) = \{ x \in q^* \mid (d\hat{P}_i)_x, \ldots, (d\hat{P}_i)_x \text{ are linearly independent} \}.
\]

**Proposition 4.3.** If \( g = sl_{l+1} \), then \( \text{codim} \{ q^* \setminus \mathcal{Y} \} \geq 2 \).

**Proof.** Let \( a = (e, \xi) \) and \( a' = (e', \xi') \) be typical elements of \( q^* \), where \( e, e' \in u \) and \( \xi, \xi' \in b^- \). According to formulae of Subsection 4.1, \( \hat{P}_i(e, \xi) = \kappa(e^i, \xi) \). Recall that \( (d\hat{P}_i)_a \in q \) and \( \langle (d\hat{P}_i)_a, a' \rangle \) is the coefficient of \( t \) in the expansion of \( \hat{P}_i(a + ta') \). Consequently,

\[
\langle (d\hat{P}_i)_a, a' \rangle = \kappa(e^i, \xi') + \kappa(\sum_{k+m=i-1} e^k e^m, \xi).
\]

The vector \( (d\hat{P}_i)_a \) has the \( b^- \) and \( u^- \) components, and this equality shows that:

- the \( b^- \) component of \( (d\hat{P}_i)_a \) equals \( e^i \);
- the \( u^- \) component of \( (d\hat{P}_i)_a \), say \( (d\hat{P}_i)_a \{ u^- \} \), is determined by the equation

\[
\kappa((d\hat{P}_i)_a \{ u^- \}, e') = \kappa(\sum_{k+m=i-1} e^k e^m, \xi).
\]

Let \( O^\text{reg} \) and \( O^\text{sub} \) denote the regular and subregular nilpotent orbits in \( sl_{l+1} \), respectively. Then \( \overline{O^\text{sub} \cap u} = \cup_j u_j \). If \( e \in O^\text{reg} \cap u \), then the \( b^- \) components of \( (d\hat{P}_i)_{(e, \xi)}, i = 1, \ldots, l \), are linearly independent, regardless of \( \xi \). Hence \( (O^\text{reg} \cap u) \times b^- \subset \mathcal{Y} \).

If \( e \in O^\text{sub} \cap u \), then the \( b^- \) components of \( (d\hat{P}_i)_{(e, \xi)}, i = 1, \ldots, l - 1 \), are still linearly independent for any \( \xi \), but \( e^l = 0 \). However, if \( e \) is sufficiently general, then the \( u^- \) component of \( (d\hat{P}_i)_{(e, \xi)} \) appears to be nonzero for all \( \xi \) that belong to a dense open subset of \( b^- \). More precisely, suppose that \( e \in u_j \) and \( \kappa(e, e_{-\alpha_i}) \neq 0 \) for \( i \neq j \). Taking \( e' = e_{\alpha_j} \), one readily computes that

\[
\sum_{k+m=i-1} e^k e^m = e^{j-1} e_{\alpha_i} e^{l-j} \text{ is a nonzero multiple of } e_\theta.
\]

Hence, one can take any \( \xi \) such that \( \kappa(\xi, e_\theta) \neq 0 \).
Thus, there is a dense open subset \( \Omega \subset \bigcup_i u_i \times b^- \) such that \( \Omega \subset Y \), and the assertion follows. \( \square \)

It turns out that Proposition 4.3 together with (4.1) is sufficient to conclude that for \( g = \mathfrak{sl}_{l+1} \), \( q \) has the codim-2 property. This follows from the following general assertion:

**Theorem 4.4.** Let \( R \) be a connected algebraic group with Lie algebra \( \mathfrak{r} \). Suppose that (a) \( \text{ind } \mathfrak{r} = m \), (b) \( \mathbb{F}[\mathfrak{r}^*]^R = \mathbb{F}[p_1, \ldots, p_m] \) is a graded polynomial algebra, and (c) \( \sum_{i=1}^m \deg p_i = (\dim \mathfrak{r} + \text{ind } \mathfrak{r})/2 \). Then the following conditions are equivalent:

1) \( \text{codim } (\mathfrak{r}^* \setminus \mathfrak{r}^*_\text{reg}) \geq 2 \);

2) \( \text{codim } (\mathfrak{r}^* \setminus \mathcal{Y}(\mathfrak{r}^*)) \geq 2 \), where \( \mathcal{Y}(\mathfrak{r}^*) \) is defined as in (4.2) via the \( p_i \)'s.

If these conditions are satisfied, then actually \( \mathfrak{r}^*_\text{reg} = \mathcal{Y}(\mathfrak{r}^*) \).

**Proof.** The implication 1) \( \Rightarrow \) 2) is already proved in [8, Theorem 1.2]. To prove the converse, one can slightly adjust the proof given in [8], see also the proof of Theorem 1.2 in [9]. Set \( n = \dim \mathfrak{r} \). Let \( T(\mathfrak{r}^*) \) denote the tangent bundle of \( \mathfrak{r}^* \). The main part of that proof consists in a construction of two homogeneous polynomial sections of \( \wedge^{n-m}T(\mathfrak{r}^*) \), denoted \( \mathfrak{V}_1 \) and \( \mathfrak{V}_2 \). Write \( (\mathfrak{V}_i)_x \) for the value of \( \mathfrak{V}_i \) at \( x \in \mathfrak{r}^* \). These sections have the following properties:

- There exist \( F_1, F_2 \in \mathbb{F}[\mathfrak{r}^*] \) such that \( F_1 \mathfrak{V}_1 = F_2 \mathfrak{V}_2 \);

- \( (\mathfrak{V}_1)_x \neq 0 \) if and only if \( x \in \mathfrak{r}^*_\text{reg} \);

- \( (\mathfrak{V}_2)_x \neq 0 \) if and only if \( x \in \mathcal{Y}(\mathfrak{r}^*) \);

- \( \deg \mathfrak{V}_1 = (n - m)/2 \) and \( \deg \mathfrak{V}_2 = \sum_i (\deg p_i - 1) \).

This only requires assumptions a) and b). If c) is also satisfied, then \( \deg \mathfrak{V}_1 = \deg \mathfrak{V}_2 \). Therefore either of conditions 1), 2) implies the other. \( \square \)

Since \( q = b \times u^- \) does not have the codim-2 property if \( g \) is not of type \( A_l \), we cannot immediately conclude that in all cases \( x \in q^*_\text{reg} \) if and only if \( (d\hat{P}_1)_x, \ldots, (d\hat{P}_l)_x \) are linearly independent. Nevertheless, the fact that \( \hat{P}_1, \ldots, \hat{P}_l \) are the highest components of the basic \( G \)-invariants \( f_1, \ldots, f_l \) allows to circumvent this difficulty. It can be shown in general (see [15]) that the coadjoint representation \( (Q : q^*) \) has the following property:

**Claim 4.5.** For \( x \in q^* \) the following conditions are equivalent:

- The orbit \( Q \cdot x \) is of maximal dimension, which is \( \dim q - l \) in this situation;

- The differentials \( (d\hat{P}_i)_x, i = 1, \ldots, l \), are linearly independent.

In case of the coadjoint representation of \( g \), this is a result of Kostant [6, Theorem 9].
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