Chow groups of smooth varieties fibred by quadrics

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Abstract

Let \( f : X \to B \) be a proper flat dominant morphism between two smooth quasi-projective complex varieties \( X \) and \( B \). Assume that there exists an integer \( l \) such that all closed fibres \( X_b \) of \( f \) satisfy \( CH_0(X_b) = CH_1(X_b) = \ldots = CH_l(X_b) = \mathbb{Q} \). Then we prove an analogue of the projective bundle formula for \( CH_i(X) \) for \( i \leq l \). When \( B \) is a surface, \( X \) is projective and \( l = \lfloor \frac{\dim X - 3}{2} \rfloor \), this makes it possible to construct a Chow-Künneth decomposition for \( X \) that satisfies Murre’s conjectures. For instance we prove Murre’s conjectures for complex smooth projective varieties \( X \) fibred over a surface (via a flat morphism) by quadrics, or by complete intersections of dimension 4 of bidegree \((2, 2)\).

Introduction

Let \( X \) be a smooth projective complex variety of dimension \( d_X \). We write \( H_i(X) \) for the rational homology group \( H_i(X, \mathbb{Q}) \), this group is isomorphic to \( H^{2i}(X, \mathbb{Q}) \). The group \( CH_i(X) \) denotes the rational Chow group of \( i \)-cycles on \( X \) modulo rational equivalence.

This article is concerned with Jacob Murre’s Chow-Künneth decomposition problem for smooth projective varieties. In [14], Murre conjectured the following.

(A) \( X \) has a Chow-Künneth decomposition \( \{ \pi_0, \ldots, \pi_{2d} \} : \) There exist mutually orthogonal idempotents \( \pi_0, \ldots, \pi_{2d} \in CH_{d_X}(X \times X) \) adding to the identity such that \( (\pi_i)_*H_i(X) = H_i(X) \) for all \( i \).

(B) \( \pi_0, \ldots, \pi_{2l-1}, \pi_{d+l+1}, \ldots, \pi_{2d} \) act trivially on \( CH_l(X) \) for all \( l \).

(C) \( F^iCH_i(X) := \ker (\pi_{2l}) \cap \ldots \cap \ker (\pi_{2l+i-1}) \) doesn’t depend on the choice of the \( \pi_j \)’s. Here the \( \pi_j \)’s are acting on \( CH_l(X) \).

(D) \( F^1CH_1(X) = CH_1(X)_{\text{hom}} \).

A variety \( X \) that satisfies conjectures (A), (B) and (D) is said to have a Murre decomposition. If moreover the Chow-Künneth decomposition of conjecture (A) can be chosen so that \( \pi_i = t^{2d-i} \in CH_{d_X}(X \times X) \), then \( X \) is said to have a self-dual Murre decomposition. The relevance of Murre’s conjectures were demonstrated by Jannsen who proved [8] that these are true for all smooth projective varieties if and only if Bloch and Beilinson’s conjecture is true for all smooth projective varieties.
Here we are mainly interested in families of quadric hypersurfaces, although some of the results can be stated in more generality. Our strategy for constructing Chow-Künneth projectors consists in first computing the Chow groups of the total space $X$. In [15], we already proved

**Theorem 1** (Theorem 3.4 in [15]). Let $f : X \rightarrow B$ be a complex projective dominant morphism onto a complex quasi-projective variety $B$ of dimension $d_B$. Assume that there is an integer $l$ such that $CH_i(X_b) = \mathbb{Q}$ for all $i \leq l$ and all closed points $b \in B$. Then $CH_i(X)$ has niveau $\leq d_B$, i.e. it is supported in dimension $i + d_B$, for all $i \leq l$.

Examples for which the theorem above applies are given by varieties fibred by complete intersections of very low degree. For instance, if $Q$ is a quadric hypersurface, then we know that $CH_i(Q) = \mathbb{Q}$ for all $i < \frac{\dim Q}{2}$. The above theorem then makes it possible to establish some of the conjectures on algebraic cycles for smooth projective varieties fibred by quadrics:

**Theorem 2** (Theorem 4.2 in [15]). Let $X$ be a smooth projective complex variety fibred by quadric hypersurfaces over a smooth projective variety $B$. Then

- if $\dim B \leq 1$, $X$ is Kimura finite-dimensional [10] and satisfies Murre’s conjectures;
- if $\dim B \leq 2$, $X$ satisfies Grothendieck’s standard conjectures;
- if $\dim B \leq 3$, $X$ satisfies the Hodge conjecture.

Since smooth projective surfaces have a Murre decomposition [13], it is natural to seek for a Murre decomposition for smooth projective varieties fibred by quadrics over a surface. It turns out that, when $f : X \rightarrow B$ is a complex projective flat morphism from a smooth variety $X$ to a smooth quasi-projective $B$ whose closed fibres are quadrics, it is possible to compute explicitly most Chow groups of $X$ in terms of the Chow groups of $B$. Precisely, in section 1, we prove an analogue of the projective bundle formula for Chow groups:

**Theorem 3.** [Corollary to Theorem 1.5] Let $f : X \rightarrow B$ be a projective flat dominant morphism from a smooth quasi-projective complex variety $X$ to a smooth quasi-projective complex variety $B$ of dimension $d_B$. Let $l \geq 0$ be an integer. Assume that

$$CH_{l-i}(X_b) = \mathbb{Q}$$

for all $0 \leq i \leq \min(l, d_B)$ and for all closed points $b$ of $B$. Then $CH_l(X)$ is isomorphic to $\bigoplus_{i=0}^{d_X-d_B} CH_{l-i}(B)$ via the action of correspondences.

When the closed fibres of $f$ are quadrics, we thus obtain that $CH_l(X)$ is isomorphic to $\bigoplus_{i=0}^{d_X-d_B} CH_{l-i}(B)$ in a strong sense for all $l \leq \frac{d_X-d_B-1}{2}$. When furthermore $B$ is a surface, theorem 3 is the prerequisite for constructing idempotents in $CH_{d_X}(X \times X)$. In section 2, we carry out the construction of a Chow-Künneth decomposition for $X$ fibred by quadrics over a surface and prove
Theorem 4 (Corollary to Theorem 2.1). Let $X$ be a smooth projective complex variety which is the total space of a flat family of quadrics over a smooth projective curve or surface. Then $X$ has a self-dual Murre decomposition which satisfies the motivic Lefschetz conjecture.

This theorem generalises a previous result of del Angel and Müller-Stach [2] where a Murre decomposition was constructed for threefolds fibred by conics over a surface.

The motivic Lefschetz conjecture stipulates, for $\{\pi_i, 0 \leq i \leq 2d_X\}$ a Chow-Künneth decomposition for $X$, that the morphisms of Chow motives $(X, \pi_i \hom^0, d_X - i)$ induced by intersecting $d_X - i$ times with a hyperplane section are isomorphisms for all $0 \leq i \leq d_X$. The motivic Lefschetz conjecture follows from a combination of Kimura’s finite-dimensionality conjecture with the Lefschetz standard conjecture. It should be noted that, in order to prove theorem 2.1, no reference to Kimura’s finite-dimensionality property is used. Furthermore, it doesn’t seem possible to prove theorem 2.1 by using the approach of Gordon-Hanamura-Murre [5], as in loc. cit. $f$ is assumed to be smooth away from a finite number of points and to have a relative Chow-Künneth decomposition. Here we only assume $f$ to be flat and in the proof of theorem 2.1 we don’t consider the existence of a relative Chow-Künneth decomposition for $f$.

Notations. Chow groups are always meant with rational coefficients. The group $\text{CH}_i(X)$ is the $\mathbb{Q}$-vectorspace with basis the $i$-dimensional irreducible subschemes of $X$ modulo rational equivalence.

In section 2, motives are defined in a covariant setting and the notations are those of [16]. Briefly, a Chow motive $M$ is a triple $(X, p, n)$ where $X$ is a variety of pure dimension $d$, $p \in \text{CH}_d(X \times X)$ is an idempotent ($p \circ p = p$) and $n$ is an integer. The motive of $X$ is denoted $\mathcal{H}(X)$ and by definition is the motive $(X, \Delta_X, 0)$ where $\Delta_X$ is the class in $\text{CH}_{d_X}(X \times X)$ of the diagonal in $X \times X$. A morphism between two motives $(X, p, n)$ and $(Y, q, m)$ is a correspondence in $q \circ \text{CH}_{d+\Delta_X}(X \times Y) \circ p$. If $f : X \to Y$ is a morphism, $\Gamma_f \in \text{CH}_d(X \times Y)$ is the class of the graph of $f$. By definition we have $\text{CH}_i(X, p, n) = p_*\text{CH}_i(X)$ and $H_i(X, p, n) = p_\ast H_{i-2n}(X)$, where we write $H_i(X) := H^{2d-i}(X(C), \mathbb{Q})$ for singular homology.

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1 Chow groups for varieties fibred by varieties with Chow groups generated by hyperplane sections

We establish a formula that is analogous to the projective bundle formula for Chow groups.
For $X$ a projective variety, $h : CH_f(X) \to CH_{f-1}(X)$ denotes the intersection with a hyperplane section of $X$. It is well-defined [20, Cor. 21.10]. When $X$ is also smooth, consider a smooth linear hyperplane section $\iota : H \hookrightarrow X$. Let’s write $\Delta_H$ for the diagonal inside $H \times H$. The map $h$ is then induced by the correspondence $(\iota \times \iota)_\ast[\Delta_H] \in CH_{d_X-1}(X \times X)$ that we also denote $h$.

**Lemma 1.1.** Let $f : X \to B$ be a projective dominant morphism between two smooth quasi-projective varieties of respective dimension $d_X$ and $d_B$. Then there exists a non-zero integer $n$ such that $f_\ast h^{d_X-d_B} f^* : CH_f(B) \to CH_0(B)$ is multiplication by $n$. If moreover $B$ is projective, then

$$\Gamma_f \circ h^{d_X-d_B} \circ \Gamma_f = n \cdot \Delta_B \in CH_{d_B}(B \times B).$$

**Proof.** This follows from the projection formula applied to $(f \circ \iota)_\ast(f \circ \iota)^*$ and, when $B$ is projective, from Manin’s identity principle. See [12, Example 1 p. 450].

**Lemma 1.2.** Let $f : X \to B$ be a projective dominant morphism between two smooth quasi-projective varieties. Then $f_\ast h^{d_X-d_B} f^* : CH_f(B) \to CH_0(B)$ is the zero map for all $l < d_X - d_B$. If moreover $B$ is projective, then $\Gamma_f \circ h^l \circ \Gamma_f = 0 \in CH_{d_X-l}(B \times B)$ for all $l < d_X - d_B$.

**Proof.** Let’s first assume that $B$ is projective. Let $\iota : H^l \hookrightarrow X$ be a smooth linear section of $X$ of codimension $l$ that dominates $B$ and let $h^l$ be the class of $(\iota \times \iota)(\Delta_{H^l}) \in X \times X$. By definition we have $\Gamma_f \circ h^l \circ \Gamma_f = (p_{1,4})_\ast(p_{1,2}^\ast \Gamma_f \cap p_{2,3}^\ast h^l \cap p_{3,4}^\ast \Gamma_f)$, where $p_{i,j}$ denotes projection from $B \times X \times X \times B$ to the $(i,j)$-th factor. These projections are flat morphisms, therefore by flat pullback we have $p_{1,2}^\ast \Gamma_f = [\Gamma_f \times X \times B]$, $p_{2,3}^\ast h^l = [B \times \Delta_{H^l} \times B]$ and $p_{3,4}^\ast \Gamma_f = [B \times X \times \Gamma_f]$. It is easy to see that the closed subschemes $\Gamma_f \times X \times B$, $B \times \Delta_{H^l} \times B$ and $B \times X \times \Gamma_f$ of $B \times X \times X \times B$ intersect properly. Their intersection is given by $\{(f(h), h, h, f(h)) : h \in H\} \subset B \times X \times X \times B$. Since $f$ is projective, this is a closed subset of dimension $d_X - l$ and its image under the projection $p_{1,4}$ has dimension $d_B$, which is strictly less than $d_X - l$ by the assumption made on $l$. The projection $p_{1,4}$ is a proper map and hence by proper pushforward we get that

$$(p_{1,4})_\ast\{(f(h), h, h, f(h)) : h \in H^l\} = 0.$$

When $B$ is only assumed to be quasi-projective, the arguments above can be adapted by using refined intersections as in [4, Remark 16.1] and by noticing that the graph $\Gamma_f$ seen as a subscheme of $X \times B$ is proper over $X$ and over $B$ and that $H \times H$ is proper over $X$ via the two projections.

**Proposition 1.3.** Let $f : X \to B$ be a projective dominant morphism from a smooth quasi-projective variety $X$ of dimension $d_X$ to a smooth quasi-projective variety $B$ of dimension $d_B$. Then the map

$$(*) \quad \bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^* : \bigoplus_{i=0}^{d_X-d_B} CH_{d_X-i}(B) \to CH_i(X)$$

is injective.
Proof. Thanks to lemma 1.1 and to lemma 1.2, we have that

\[ f_\ast \circ h^i \circ f^\ast : CH_t(B) \to CH_{t+d_X-d_B-i}(B) \]

is multiplication by a non-zero integer if \( i = d_X - d_B \) and is zero if \( i < d_X - d_B \).

Consider the map \( \bigoplus_{i=0}^{d_X-d_B} f_\ast \circ h^i : CH_t(Y) \to \bigoplus_{j=0}^{d_X-d_B} CH_{t-j}(B) \). In order to prove the injectivity of \( \bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^\ast \), it suffices to show that the composite

\[
\left( \bigoplus_{j=0}^{d_X-d_B} f_\ast \circ h^j \right) \circ \left( \bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^\ast \right) : \bigoplus_{i=0}^{d_X-d_B} CH_{t-i}(B) \to CH_t(X) \to \bigoplus_{j=0}^{d_X-d_B} CH_{t-j}(B)
\]

is an isomorphism. Indeed it follows from lemma 1.1 and from lemma 1.2 that this composite map can be represented by an upper triangular matrix whose diagonal entries’ action on \( CH_{t-i}(B) \) is given by multiplication by \( n \) for some \( n \neq 0 \).

\[ \square \]

Proposition 1.4. Let \( f : X \to B \) be a flat dominant morphism from a quasi-projective variety \( X \) of dimension \( d_X \) to a quasi-projective variety \( B \) of dimension \( d_B \). Let \( l \geq 0 \) be an integer. Assume that

\[ CH_{t-i}(X_{\eta_B}) = Q \]

for all \( 0 \leq i \leq \min(l, d_B) \) and for all closed irreducible subschemes \( B_i \) of \( B \) of dimension \( i \), where \( \eta_B \) is the generic point of \( B \).

Then the map

\[ (\ast) \quad \bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^\ast : \bigoplus_{i=0}^{d_X-d_B} CH_{t-i}(B) \to CH_t(X) \]

is surjective.

\[ \text{Proof.} \quad \text{The case when } d_B = 0 \text{ is obvious. Let’s proceed by induction on } d_B. \text{ We have the localization exact sequence} \]

\[ \bigoplus_{D \in B^1} CH_t(X_D) \to CH_t(X) \to CH_{t-d_B}(X_{\eta_B}) \to 0, \]

where the direct sum is taken over all irreducible divisors of \( B \). If \( l \geq d_B \), let \( Y \) be a closed subscheme of \( X \) obtained as the scheme-theoretic intersection of \( d_X - l \) hyperplanes in general position. Then, by Bertini, for a suitable choice of hyperplanes, \( Y \) is irreducible, has dimension \( l \) and is such that \( f|_Y : Y \to B \) is dominant. The restriction map \( CH_t(X) \to CH_{t-d_B}(X_{\eta_B}) \) is by definition the direct limit of the flat pullback maps \( CH_t(X) \to CH_t(X_U) \) taken over all open subsets \( U \) of \( B \). Therefore \( CH_t(X) \to CH_{t-d_B}(X_{\eta_B}) \) sends the class of \( Y \) to the class of \( Y_{\eta_B} \) inside \( CH_{t-d_B}(X_{\eta_B}) \). But then this class is non-zero because \( Y_{\eta_B} \) is irreducible. Moreover, if \( [B] \) denotes the class of \( B \) in \( CH_{d_B}(B) \), then the class of \( Y \) is equal to \( h^{d_X-l} \circ f^\ast [B] \) in \( CH_t(X) \). Therefore, the composite map

\[ CH_{d_B}(B) h^{d_X-l} \circ f^\ast \to CH_t(X) \to CH_{t-d_B}(X_{\eta_B}) \]
is surjective.

Consider now the fibre square

\[
\begin{array}{ccc}
X_D & \xrightarrow{j'_D} & X \\
\downarrow f_D & & \downarrow f \\
D & \xrightarrow{j_D} & B.
\end{array}
\]

Then \( f_D : X_D \to D \) is flat and its fibres above points of \( D \) satisfy the assumptions of the theorem. Therefore, by the inductive assumption, we have a surjective map

\[
\bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^*_D : \bigoplus_{i=0}^{d_X-d_B} CH_{l-i}(D) \to CH_l(X_D).
\]

Furthermore, since \( f \) is flat and \( j_D \) is proper, we have the formula [4, 1.7]

\[
j'_D \ast h^{d_X-d_B-i} \circ f^*_D = h^{d_X-d_B-i} \circ f^* \circ j_D^* : CH_{l-i}(D) \to CH_l(X).
\]

Therefore, the image of (*) contains the image of

\[
\bigoplus_{D \in B^i} \bigoplus_{i=0}^{d_X-d_B} j'_D \ast h^{d_X-d_B-i} \circ f^*_D : \bigoplus_{i=0}^{d_X-d_B} CH_{l-i}(D) \to CH_l(X).
\]

Altogether, this implies that the map (*) is surjective. \( \square \)

We can now gather the statements and proofs of the two previous propositions into the following.

**Theorem 1.5.** Let \( f : X \to B \) be a flat projective dominant morphism from a smooth quasi-projective variety \( X \) of dimension \( d_X \) to a smooth quasi-projective variety \( B \) of dimension \( d_B \). Let \( l \geq 0 \) be an integer. Assume that

\[
CH_{l-i}(X_{\eta_{B_i}}) = \mathbb{Q}
\]

for all \( 0 \leq i \leq \min(l, d_B) \) and for all closed irreducible subschemes \( B_i \) of \( B \) of dimension \( i \), where \( \eta_{B_i} \) is the generic point of \( B_i \).

Then the map

\[
(*) \quad \bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^* : \bigoplus_{i=0}^{d_X-d_B} CH_{l-i}(B) \to CH_l(X)
\]

is an isomorphism. Moreover the map

\[
\bigoplus_{i=0}^{d_X-d_B} f^* \circ h^i : CH_l(X) \to \bigoplus_{i=0}^{d_X-d_B} CH_{l-i}(B)
\]

is also an isomorphism. \( \square \)
**Proposition 1.6.** Let \( f : X \to B \) be a morphism of complex varieties with \( B \) irreducible and let \( F \) be the geometric generic fibre of \( f \). Then there is a subset \( U \subseteq B(C) \) which is a countable intersection of nonempty Zariski open subsets such that for each point \( p \in U \), \( CH_i(X_p) \) is isomorphic to \( CH_i(F) \) for all \( i \).

**Proof.** Cf. [15, Proposition 3.2].

We then have the following corollaries to theorem 1.5.

**Corollary 1.7.** Let \( f : X \to B \) be a projective flat dominant morphism from a smooth quasi-projective complex variety \( X \) to a smooth quasi-projective complex variety \( B \) of dimension \( d_B \). Let \( l \geq 0 \) be an integer. Assume that

\[
CH_{l-i}(X_b) = \mathbb{Q}
\]

for all \( 0 \leq i \leq \min(l, d_B) \) and for all closed points \( b \) of \( B \).

Then the conclusion of theorem 1.5 holds.

**Proof.** Let \( B_i \) be an irreducible closed subscheme of \( B \) of dimension \( i \) and let \( f|_{B_i} : X|_{B_i} \to B_i \) be the restriction of \( f \) to \( B_i \). If \( CH_{l-i}(X_b) = \mathbb{Q} \) for all closed points \( b \in B \), then proposition 1.6 applied to \( f|_{B_i} \) implies that \( CH_{l-i}(X_{\eta_{B_i}}) = \mathbb{Q} \). Here \( \eta_{B_i} \) denotes a geometric generic point of \( B_i \). But then it is well-known [4, Ex. 1.7.6] that for a scheme \( X \) over a field \( k \), the pull-back map \( CH_*(X) \to CH_*(X_k) \) is injective. We are thus reduced to the statement of theorem 1.5.

**Corollary 1.8.** Let \( f : X \to B \) be a flat dominant morphism from a smooth projective complex variety \( X \) to a smooth projective complex variety \( B \) whose closed fibres are quadric hypersurfaces. Then the conclusion of theorem 1.5 holds for any \( l \leq \left\lfloor \frac{d_X - 3}{2} \right\rfloor \).

**Proof.** It is well-known (see e.g. [3]) that, for a quadric hypersurface \( Q \), \( CH_i(Q) = \mathbb{Q} \) for all \( i \leq \left\lfloor \frac{\dim Q - 1}{2} \right\rfloor \). Corollary 1.7 thus applies.

### 2 Murre’s conjectures for total spaces of flat families of quadric hypersurfaces over a surface

The main result of this section is the following.

**Theorem 2.1.** Let \( f : X \to S \) be a flat dominant morphism from a smooth projective complex variety \( X \) to a smooth projective complex surface \( S \) whose closed fibres \( X_s \) satisfy \( CH_l(X_s) = \mathbb{Q} \) for all \( l \leq \frac{d_X - 3}{2} \). Then \( X \) has a self-dual Murre decomposition and \( X \) satisfies the motivic Lefschetz conjecture.

**Corollary 2.2.** Let \( f : X \to S \) be a flat dominant morphism from a smooth projective complex variety \( X \) to a smooth projective complex surface \( S \) whose closed fibres are either quadric hypersurfaces or complete intersections of dimension 4 and bidegree \((2, 2)\). Then \( X \) has a self-dual Murre decomposition and \( X \) satisfies the motivic Lefschetz conjecture.
Proof. The Chow groups of a quadric hypersurface $Q$ satisfy $CH_i(Q) = \mathbb{Q}$ for all $i \leq \frac{\dim Q - 1}{2}$ and the Chow groups of a complete intersection $X_{2,2}$ of dimension 4 and bidegree $(2,2)$ satisfy $CH_0(X_{2,2}) = CH_1(X_{2,2}) = \mathbb{Q}$. This is for example proved in [3].

Before we proceed to a proof of theorem 2.1, we consider the case when $f : X \rightarrow S$ is a smooth quadric fibration.

2.1 The case of smooth families

In this subsection we are given $f : X \rightarrow B$ a smooth surjective morphism between smooth projective varieties with fibres being quadric hypersurfaces. In this case there are several ways to compute the Chow motive of $X$ in terms of the Chow motive of $B$. Since we are going to prove a more general statement we only give some indication on proofs.

Smooth families with a relative Chow-Künneth decomposition. Recall that quadric hypersurfaces are cellular varieties and that smooth quadric hypersurfaces are homogeneous varieties. Assume first that $X$ has the structure of a relative cellular variety over $B$. Then Köck [11] proved that $X$ has a relative Chow-Künneth decomposition over $B$ in the sense of [5]. If $f$ is only assumed to be smooth, then Iyer [6] showed that $f$ is étale locally trivial and deduced that $f$ has a relative Chow-Künneth decomposition.

By using the technique of Gordon-Hanamura-Murre [5], it is then possible to prove that

$$h(X) \simeq \bigoplus_{l=0}^{d_X - d_B} h(B)(l) \text{ for } d_X - d_B \text{ odd, and }$$

$$h(X) \simeq \bigoplus_{l=0}^{d_X - d_B} h(B)(l) \oplus h(B)\left(\frac{d_X - d_B}{2}\right) \text{ for } d_X - d_B \text{ even.}$$

Actually, in the case when $X$ has the structure of a relative cellular variety over $B$, this follows immediately from Manin’s identity principle.

When $d_X - d_B$ is odd we develop below an approach that bypasses the use of the fact that the smooth family $f : X \rightarrow B$ is étale locally trivial. This approach is the starting point towards extending the above result for smooth families to flat families.

Smooth families of odd relative dimension. If $Q$ is a smooth projective odd-dimensional quadric, we have that $CH_l(Q) = \mathbb{Q}$ for all $0 \leq l \leq \dim Q$ so that $Q$ has the same Chow groups (with rational coefficients) as the projective space of dimension $\dim Q$. Thus when $f$ has odd relative dimension, the situation is very similar to the case of projective bundles: Corollary 1.7 gives isomorphisms for all $0 \leq l \leq d_X$

$$\bigoplus_{i=0}^{d_X - d_B} h^{d_X - d_B - i} \circ f^* : \bigoplus_{i=0}^{d_X - d_B} CH_{l-i}(B) \longrightarrow CH_l(X).$$

If $H \hookrightarrow X$ is a smooth hyperplane section of $X$, then for $Y$ a smooth projective variety $H \times Y \hookrightarrow X \times Y$ is also a smooth hyperplane section of $X \times Y$. Therefore the isomorphism
above induces a similar isomorphism for the smooth map \( f \times \text{id}_Y : X \times Y \to B \times Y \) and Manin’s identity principle applies to give an isomorphism of Chow motives

\[
h(X) \simeq \bigoplus_{l=0}^{d_X-d_B} h(B)(l).
\]

There is yet another way of proceeding and this will be the path we will follow to prove the case of flat families over a surface (in which case the arguments above do not suffice as the Chow groups of singular quadrics are not all equal to \( \mathbb{Q} \)). We only give a sketch and point out where the difficulty is. Thanks to lemma 1.1 we can define idempotents \( \pi_0, \ldots, \pi_{d_X-d_B} \in CH_{d_X}(X \times X) \) as

\[
\pi_l := \frac{1}{n} \cdot h^{d_X-d_B-l} \circ \Gamma_f \circ \Gamma_f \circ h^l.
\]

Lemma 1.1 shows that these idempotents satisfy \( (X, \pi_l) \simeq h(B)(l) \) and lemma 1.2 shows that \( \pi_l \circ \pi_{l'} = 0 \) for all \( l' > l \). Moreover theorem 1.5 shows that \( CH_i(X) = \sum_l (\pi_l)_* CH_i(X) \) for all \( i \).

Now we have the following non-commutative Gram-Schmidt process [18, lemma 2.12].

**Lemma 2.3.** Let \( V \) be a \( \mathbb{Q} \)-algebra and let \( k \) be a positive integer. Let \( \pi_0, \ldots, \pi_n \) be idempotents in \( V \) such that \( \pi_i \circ \pi_j = 0 \) whenever \( i - j < k \) and \( i \neq j \). Then the endomorphisms

\[
p_l := (1 - \frac{1}{2} \pi_n) \circ \cdots \circ (1 - \frac{1}{2} \pi_{l+1}) \circ \pi_l \circ (1 - \frac{1}{2} \pi_{l-1}) \circ \cdots \circ (1 - \frac{1}{2} \pi_0)
\]

define idempotents such that \( p_l \circ p_j = 0 \) whenever \( i - j < k + 1 \) and \( i \neq j \).

**Proposition 2.4.** Let \( X \) be a smooth projective variety of dimension \( d \). Let \( \pi_0, \ldots, \pi_s \in CH_d(X \times X) \) be idempotents such that \( \pi_l \circ \pi_{l'} = 0 \) for all \( l' > l \), then the non-commutative Gram-Schmidt process of lemma 2.3 gives mutually orthogonal idempotents \( \{p_l\}_{l \in \{0, \ldots, s\}} \) such that we have isomorphisms of Chow motives \( (X, \pi_l) \simeq (X, p_l) \) for all \( l \).

**Proof.** In order to produce mutually orthogonal idempotents, it is enough to apply lemma 2.3 \( (l-1) \)-times. It is then enough to check the isomorphisms of Chow motives after each application of the process of lemma 2.3. Such isomorphisms are simply given by the correspondences \( p_l \circ \pi_l \); the inverse of \( p_l \circ \pi_l \) is \( \pi_l \circ p_l \) as can be readily checked.

This way we get mutually orthogonal idempotents \( p_0, \ldots, p_{d_X-d_B} \) such that \( (X, p_l) \simeq h(B)(l) \). In order to conclude it would be nice to know that \( CH_*(X) = CH_*(X, \sum p_l) \). In that case \( (X, \Delta_X - \sum p_l) \) is a Chow motive with trivial Chow groups which implies that \( \Delta_X = \sum p_l \) and hence \( h(X) \simeq \bigoplus_{l=0}^{d_X-d_B} h(B)(l) \). However, it is not clear how to prove from here that \( CH_*(X) = CH_*(X, \sum p_l) \) and this explains why the proof of the next section might seem convoluted.
2.2 Proof of theorem 2.1

We now assume that $f : X \to S$ is a flat dominant morphism defined over $\mathbb{C}$ from a smooth projective variety $X$ to a smooth projective surface $S$ whose closed fibres $X_s$ satisfy $CH_l(X_s) = \mathbb{Q}$ for all $l \leq \frac{d_X-3}{2}$.

The general strategy for proving theorem 2.1 consists in exhibiting some idempotents modulo rational equivalence with a prescribed action on Chow groups or cohomology groups and then to turn them into an orthogonal family. We do this step by step. Here’s a rough outline. At each step we check that the idempotents form an ordered family which is “semi-orthogonal” in the sense that $P_j \circ P_i = 0$ for $j > i$. This makes it possible to run the non-commutative Gram-Schmidt process of lemma 2.3 to get an orthogonal family of idempotents. Such an orthonormalising process does not affect the action of the idempotents on cohomology. At each step we need to keep track of the action of the idempotents on the Chow groups of $X$. For this purpose we check at each step that $P_j \circ P_i$ acts trivially on $CH_l(X)$ for all $l$ and all $j > i$.

First we construct idempotents $\pi^l_{2i}$ that factor through surfaces and “not through curves”.

Then for $l \leq \left\lfloor \frac{d_X-3}{2} \right\rfloor$ we construct idempotents $p^{alg}_{2l}$ and $p_{2l+1}$. We check that those idempotents act the way we want them to on the Chow groups of $X$. We deduce that they act as wanted on the cohomology of $X$.

We then define $p^l_{d_X}$ if $d_X$ is odd and mutually orthogonal idempotents $p^{alg}_{d_X}$ and $p_{d_X-1}$ if $d_X$ is even. We use a different construction than the one before as here we check directly that they act the way we want on the cohomology of $X$.

Finally we define $p_{2l} := p^{alg}_{2l} + p^l_{2l}$ for $2l < d_X$, and $p_l := t p_{2d_X-1}$ for $l > d_X$.

**Step 1.** Let $\pi^r_{2,S} \in CH_2(S \times S)$ be an idempotent with the following properties. Its homology class is the orthogonal projector on the orthogonal complement of $H_{1,1}(X) \cap H_2(X, \mathbb{Q})$ inside $H_2(X, \mathbb{Q})$ with respect to the choice of a polarisation on $S$. It acts trivially on $CH_1(S)$ and on $CH_2(S)$ and
\[
(\pi^r_{2,S})_* CH_0(S) = \text{Ker } (\text{alb}_S : CH_0(S)_{\text{hom}} \to \text{Alb}_S(k)).
\]

Such an idempotent exists, see [9].

**Step 2.** From lemma 1.1, let $n$ be the non-zero integer such that $\Gamma_f \circ h^{d_X-d_S} \circ \Gamma_f = n \cdot \Delta_S \in CH_2(S \times S)$. We set for $2i \neq d_X$,
\[
\pi^r_{2i} := \frac{1}{n} \cdot h^{d_X-d_S-i+1} \circ \Gamma_f \circ \pi^r_{2i} \circ \Gamma_f \circ h^{i-1} \in CH_{d_X}(X \times X).
\]

It is understood that $h^l = 0$ for $l < 0$. Because the correspondence $h$ is self-dual (that is $h = \tau h$) we see that
\[
\pi^r_{2d_X-2i} = \tau \pi^r_{2i}.
\]

It is expected that the correspondence $\pi^r_{2i}$ induces the projector on the orthogonal complement of $H_{i,i}(X) \cap H_{2i}(X, \mathbb{Q})$ inside $H_{2i}(X, \mathbb{Q})$. This will become apparent at the end of step 6.
Step 3. Orthogonality relations among the $\pi_{2i}^{tr}$.

**Proposition 2.5.** The $\pi_{2i}^{tr}$’s satisfy the following identities:

- $\pi_{2i}^{tr} \circ \pi_{2i}^{tr} = \pi_{2i}^{tr}$ for $2i \neq d_X$,
- $\pi_{2i}^{tr} \circ \pi_{2j}^{tr} = 0$ for all $i < j$ with $2i, 2j \neq d_X$.

**Proof.** By definition of the $\pi_{2i}^{tr}$’s we have

$$\pi_{2i}^{tr} \circ \pi_{2j}^{tr} = \frac{1}{n^2} \cdot h^{d_X-d_s-i+1} \circ t \Gamma_f \circ \pi_{2i}^{tr,S} \circ \pi_{2j}^{tr,S} \circ \Gamma_f \circ h^{-1}.$$ If $i = j$, then lemma 1.1 gives $\pi_{2i}^{tr} \circ \pi_{2i}^{tr} = \pi_{2i}^{tr}$. If $i < j$, then lemma 1.2 ensures that $\Gamma_f \circ h^{d_X-d_s+i-j} \circ t \Gamma_f = 0$ so that $\pi_{2i}^{tr} \circ \pi_{2j}^{tr} = 0$. \hfill $\Box$

Because we will need to keep track of the action of the idempotents on the Chow groups of $X$ after orthonormalising the family $\{\pi_{2i}^{tr} : 2i \neq d_X\}$, we state the following.

**Proposition 2.6.** The correspondence $\pi_{2i}^{tr} \circ \pi_{2i}^{tr}$ acts trivially on $CH_s(X)$ for all $i \neq j$.

**Proof.** The correspondence $\pi_{2i}^{tr}$ factors through $\pi_{2i}^{tr,S}$ and hence, thanks to step 1, $\pi_{2i}^{tr}$ acts trivially on $CH_j(X)$ for $j \neq i - 1$. \hfill $\Box$

**Step 4.** Orthonormalising the $\pi_{2i}^{tr}$. By proposition 2.4, after having applied lemma 2.3 a finite number of times to the set of idempotents $\{\pi_{2i}^{tr} : 2i \neq d_X\}$, we get a set of mutually orthogonal idempotents $\{p_{2i}^{tr} : 2i \neq d_X\}$ such that the Chow motives $(X, \pi_{2i}^{tr})$ and $(X, p_{2i}^{tr})$ are isomorphic for all $i$ with $2i \neq d_X$.

**Proposition 2.7.** Let $2i \neq d_X$. The action of $p_{2i}^{tr}$ on $CH_i(X)$ coincides with the action of $\pi_{2i}^{tr}$ for all $l$.

**Proof.** Considering the formula of lemma 2.3 that defines the idempotents $p_{2i}^{tr}$ inductively from the the idempotents $\pi_{2i}^{tr}$, this follows from proposition 2.6. \hfill $\Box$

**Step 5.** Let’s define the following idempotent

$$Q := \Delta_X - \sum_{2i \neq d_X} p_{2i}^{tr} \in CH_{d_X}(X \times X).$$

**Definition 2.8.** Let $(X, P)$ be a Chow motive. The subgroup of $CH_i(X, P)$ consisting of algebraically trivial cycles is denoted $CH_i(X, P)_{\text{alg}}$. This subgroup can be shown to coincide with the image of the map $P_* : CH_i(X)_{\text{alg}} \rightarrow CH_i(X)_{\text{alg}}$. It is said to be representable if there exist a curve $C$ and a correspondence $\alpha \in \text{Hom}(\mathfrak{h}_1(C)(i), (X, P))$ such that the induced map $\alpha_* : CH_0(C)_{\text{alg}} \rightarrow CH_i(X, P)_{\text{alg}}$ is surjective.

**Proposition 2.9.** The group $CH_i(X, Q)_{\text{alg}}$ of $l$-cycles modulo rational equivalence which are algebraically equivalent to zero is representable for all $l \leq \lfloor \frac{d_X - 3}{2} \rfloor$. 

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Proof. By proposition 2.7, the action of $Q$ on $CH_1(X)$ coincides with the action of $Q' := \Delta_X - \sum_{2i \neq d_X} \pi_{2i}$. Consider the map

$$
\Phi := \bigoplus_{i=1}^{l} h^{d_X - d_{S-i}} \circ f^* \circ (\Delta_S - \pi^{2tr}_{S}) = \bigoplus_{i=l-2}^{l} CH_{l-1}(S) \rightarrow CH_l(X).
$$

By corollary 1.7, the map $\Psi := \bigoplus_{i=1}^{l} f^* \circ h^i : CH_l(X) \rightarrow \bigoplus_{i=l-2}^{l} CH_{l-1}(S)$ is an isomorphism. Moreover, we have $Q' = \Phi \circ \Psi$ so that $\text{Im}(\Phi) = (Q')_* CH_l(X)$. We can then conclude that $Q_* CH_l(X)_{\text{alg}}$ is representable because $(\Delta_S - \pi^{2tr}_{S})_* CH_k(S)_{\text{alg}}$ is representable for all $k$.

**Step 6. The idempotents $p_{2i}^{alg}$ and $p_{2i+1}$.** We first construct idempotents $p_{2i}^{alg}$ and $p_{2i+1}$ for $l \leq \lfloor \frac{d_X - 3}{2} \rfloor$ that act appropriately on Chow groups. Then we construct idempotents $p_{2i-2}^{alg}$ and $p_{2i-1}$ if $d_X$ even and an idempotent $p_{2i-1}^{alg}$ if $d_X$ odd that act appropriately on homology.

In order to define $p_{2i}^{alg}$ and $p_{2i+1}$ for $l \leq \lfloor \frac{d_X - 3}{2} \rfloor$ we use the construction of [18, §1]. Let’s recall it. By Jannsen’s theorem [7], the category of motives for numerical equivalence is abelian semi-simple. Therefore we can construct idempotents modulo numerical equivalence $\overline{p}_{2i}$ and $\overline{p}_{2i+1}$ such that

$$(X, \overline{p}_{2i}) = \sum \text{Im}(\overline{\mathbf{f}}(l) \rightarrow (X, \overline{Q})) \text{ and } (X, \overline{p}_{2i+1}) = \sum \text{Im}(\overline{\mathbf{f}}(C)(l) \rightarrow (X, \overline{Q}))$$

Here the first sum runs over all morphisms $\overline{\mathbf{f}}(l) \rightarrow (X, \overline{Q})$ and the second sum runs over all curves $C$ and all morphisms $\overline{\mathbf{f}}(C)(l) \rightarrow (X, \overline{Q})$. We then see that there is an integer $n$ such that $(X, \overline{p}_{2i})$ is isomorphic to $\overline{\mathbf{f}}(\mathbb{P}^n)$ and a curve $C$ such that $(X, \overline{p}_{2i+1})$ is isomorphic to a direct summand of $\overline{\mathbf{f}}(C)(l)$. Because $\text{End}(\overline{\mathbf{f}}(\mathbb{P}^n)) = \text{End}(\mathbb{F}^{\oplus n})$ and $\text{End}(\overline{\mathbf{f}}(C)) = \text{End}(\mathbf{f}(C))$, we can lift the idempotents $\overline{p}_{2i}$ and $\overline{p}_{2i+1}$ to idempotents $p_{2i}$ and $p_{2i+1}$ modulo rational equivalence which are orthogonal to $\Delta_X - Q$ and such that $(X, p_{2i})$ is isomorphic to $\mathbb{F}(l)^{\oplus n}$ and $(X, p_{2i+1})$ is isomorphic to a direct summand of $\mathbf{f}(C)(l)$. If we construct these idempotents one after the other and replace $Q$ by $Q$ minus the last constructed idempotent at each step, we see that in addition to being orthogonal to $\Delta_X - Q$, the idempotents $\{p_{2i}^{alg}, p_{2i+1} \mid l \leq \lfloor \frac{d_X - 3}{2} \rfloor\}$ can be constructed so as to form a family of mutually orthogonal idempotents.

Now we check that these idempotents act the way we want on the Chow groups of $X$.

**Lemma 2.10** (lemma 3.3. in [19]). Let $P \in CH_{d_X}(X \times X)$ be an idempotent and assume that $CH_0(X, P)_{\text{alg}}$ is representable. Assume also that for all curves $C$ and all correspondences $\alpha \in \text{Hom}(\mathbf{f}(C), (X, P))$ we have that $\alpha$ is numerically trivial. Then $CH_0(X, P) = 0$.

**Proof.** Let $C$ be a curve and let $\gamma \in CH_1(C \times X)$ be a correspondence such that $\gamma_* CH_0(C)_{\text{alg}} = P_* CH_0(X)_{\text{alg}}$. Let then $\alpha := P \circ \gamma \circ \pi_1^C \in \text{Hom}(\mathbf{f}(C), (X, P))$. By [17,
Th. 3.6], which follows a decomposition of the diagonal argument à la Bloch-Srinivas [1], we get that \( P = P_1 + P_2 \) where \( P_2 \) is supported on \( D \times X \) for some divisor \( D \) in \( X \) and \( P_1 = \alpha \circ \beta \) for some \( \beta \in \text{Hom}((X, P), \mathfrak{h}_1(C)) \). By Chow’s moving lemma \( P_2 \) acts trivially on \( CH_0(X) \) so that \( P_1 = \alpha \circ \beta \) acts as the identity on \( P_1CH_0(X) \). By assumption, \( \pi = 0 \) and thus \( \beta \circ \alpha = 0 \). Because \( \text{End}(\mathfrak{h}_1(C)) = \text{End}(\overline{\mathfrak{h}}_1(C)) \), we get that \( \beta \circ \alpha = 0 \). It follows that \( P_1CH_0(X) = 0 \). \( \square \)

**Lemma 2.11.** Let \( P \in CH_d(X \times X) \) be an idempotent and assume that \( CH_0(X, P) \) is a finite-dimensional \( \mathbf{Q} \)-vector space. Assume also that for all correspondences \( \alpha \in \text{Hom}(\mathbf{1}, (X, P)) \) we have that \( \alpha \) is numerically trivial. Then \( CH_0(X, P) = 0 \).

**Proof.** The lemma can be proved along the same lines as lemma 2.10. \( \square \)

From lemmas 2.10 and 2.11, we get

**Proposition 2.12.** Let \( P \in CH_d(X \times X) \) be an idempotent and assume that \( CH_0(X, P)_{\text{alg}} \) is representable. Assume that \( (X, P) \) has no direct summand isomorphic to \( \mathbf{1} \) or to a direct summand of the \( \overline{\mathfrak{h}}_1 \) of a curve. Then \( CH_0(X, P) = 0 \). \( \square \)

The next lemma was mentioned to me by Bruno Kahn.

**Lemma 2.13.** Let \( P \in CH_{d_X}(X \times X) \) be an idempotent and assume that \( CH_0(X, P) = 0 \). Then there exists a smooth projective variety \( Y \) of dimension \( d_X - 1 \) and an idempotent \( P' \in CH_{d_X-1}(Y \times Y) \) such that \( (X, P) \simeq (Y, P', 1) \).

**Proof.** For a proof, see [16, Theorem 2.1]. \( \square \)

Let

\[
P := \Delta_X - \sum (p_{2l}^{\text{alg}} + p_{2l+1} + p_{2l+2}^{\text{tr}})
\]

where the sum is taken over all \( l \leq \lfloor \frac{d_X-3}{2} \rfloor \). We are finally in a position to prove the crucial

**Proposition 2.14.** For all \( l \leq \lfloor \frac{d_X-3}{2} \rfloor \) we have \( CH_l(X) = (p_{2l}^{\text{alg}} + p_{2l+1} + p_{2l+2}^{\text{tr}})_* CH_l(X) \).

**Proof.** The idempotent \( p_{2l+2}^{\text{tr}} \) acts trivially on \( CH_l(X) \) for all \( l' \neq l \) and so does \( p_{2l+2}^{\text{tr}} \) by proposition 2.7 (or more simply because \( p_{2l+2}^{\text{tr}} \) factors through \( \pi_{2l+2}^{\text{tr}} \) by the formula of lemma 2.3). By construction, the idempotents \( p_{2l}^{\text{alg}} \) and \( p_{2l+1} \) also act trivially on \( CH_l(X) \) for all \( l' \neq l \). Therefore it suffices to prove that \( P_lCH_l(X) = 0 \) for \( l \leq \lfloor \frac{d_X-3}{2} \rfloor \). The case \( l = 0 \) is proposition 2.12. By proposition 2.13, we get that \( (X, P) \) is isomorphic to \( (Y, P', 1) \) for some smooth projective \( Y \) and some idempotent \( P' \in CH_{\dim Y}(Y \times Y) \). We can then apply proposition 2.12 to \( (Y, P') \) and we obtain \( CH_l(X, P) = 0 \). An easy induction concludes the proof. \( \square \)

Proposition 2.14 yields that the Chow motive \((X, P)\) has trivial Chow groups in degrees less than \( \lfloor \frac{d_X-3}{2} \rfloor \).

It follows from lemma 2.13 that there exist a smooth projective variety \( Y \) and an idempotent \( q \in CH_{\dim Y}(Y \times Y) \) such that \((X, P)\) is isomorphic to \((Y, q, \lfloor \frac{d_X-1}{2} \rfloor)\). Let
\( \alpha \in \text{Hom}( (X, P), (Y, q_1, \lfloor \frac{d_X - 1}{2} \rfloor) ) \) denote such an isomorphism and let \( \beta \) be its inverse. In [16, §3], orthogonal idempotents \( q_0 \) and \( q_1 \in \text{End}( (Y, q) ) \) with the following properties are constructed:

- \( (q_0)_* H_s(Y) = q_* H_0(Y) \) and \( (q_1)_* H_s(Y) = q_* H_1(Y) \).
- The Chow motive \( (Y, q_0) \) is isomorphic to a direct sum of Chow motives of points.
- The Chow motive \( (Y, q_1) \) is isomorphic to a direct summand of the Chow motive of a curve.

Let’s then define the idempotent \( p_{d_X - 1}^{\text{alg}} := \beta \circ q_0 \circ \alpha \) if \( d_X \) is odd and mutually orthogonal idempotents \( p_{d_X - 2}^{\text{alg}} := \beta \circ q_0 \circ \alpha \) and \( p_{d_X - 1} := \beta \circ q_1 \circ \alpha \) if \( d_X \) is even.

By construction the idempotents \( p_{2l}^{\text{alg}} \) and \( p_{2l}^{tr} \) are mutually orthogonal for all \( 0 < l < \frac{\dim X}{2} \). Let’s thus define the idempotent

\[
p_{2l} := p_{2l}^{\text{alg}} + p_{2l}^{tr}.
\]

We also set \( p_0 = p_0^{alg} \). We have thus now at our disposal a set \( \{ p_l \}_{0 \leq l < d_X} \) of mutually orthogonal idempotents. Modulo homological equivalence, these define the Künneth projectors:

**Proposition 2.15.** The mutually orthogonal idempotents \( \{ p_l \}_{0 \leq l < d_X} \) satisfy

\[
(p_l)_* H_s(X) = H_l(X).
\]

**Proof.** For weight reasons we immediately see that \( (p_l)_* H_s(X) \subseteq H_l(X) \) for all \( l < d_X \). By proposition 2.14, we have that \( CH_l(X, \Delta_X - \sum \nu < d_X p_\nu) = 0 \) for \( l \leq \lfloor \frac{d_X - 2}{2} \rfloor \). As in the discussion above, lemma 2.13 then shows that there exists \( Y \) and an idempotent \( q \in CH_{\dim Y}(Y \times Y) \) such that \( (X, \Delta_X - \sum \nu < d_X p_\nu) \) is isomorphic to \( (Y, q, \lfloor \frac{d_X - 1}{2} \rfloor) \). Clearly \( H_l(Y, q, \lfloor \frac{d_X - 1}{2} \rfloor) = 0 \) for \( l < 2 \lfloor \frac{d_X - 1}{2} \rfloor \) so that \( (p_l)_* H_s(X) = H_l(X) \) for \( l < 2 \lfloor \frac{d_X - 1}{2} \rfloor \). It then follows from the definitions of \( p_{d_X - 1}^{\text{alg}} \) when \( d_X \) is odd and of \( p_{d_X - 2}^{\text{alg}} \) and \( p_{d_X - 1}^{tr} \) when \( d_X \) is even that \( (p_l)_* H_s(X) = H_l(X) \) for the remaining \( l \)’s that is for \( 2 \lfloor \frac{d_X - 1}{2} \rfloor \leq l < d_X \).

By Poincaré duality we then have for \( l < d_X \)

\[
(_l p_l)_* H_s(X) = H_{2d_X - l}(X).
\]

We are thus led to set for \( l > d_X \)

\[
p_l := _l p_{2d_X - l}.
\]

**Step 7. More orthogonality relations.**

**Lemma 2.16.** Let \( V \) and \( W \) be two smooth projective varieties and let \( \gamma \in CH^0(V \times W) \) be a correspondence such that \( \gamma_* \) acts trivially on zero-cycles. Then \( \gamma = 0 \).
Proof. We can assume that $V$ and $W$ are both connected. The cycle $\gamma$ is equal to $a \cdot [V \times W]$ for some $a \in \mathbb{Q}$. Let $z$ be a zero-cycle on $V$. Then $\gamma \cdot z = a \cdot \deg z \cdot [W]$. This immediately implies $a = 0$. \hfill \Box

The following lemma will be used in the proof of lemma 2.29.

Lemma 2.17. Let $\gamma \in CH^1(V \times W)$ be a correspondence such that both $\gamma_*$ and $\gamma^*$ act trivially on zero-cycles. Then $\gamma = 0$.

Proof. We can assume $V$ and $W$ are connected. We have Pic($V \times W$) = Pic($V$) × Pic($W$). The cycle $\gamma$ is thus equal to $D_1 \times [W] \oplus [V] \times D_2$ for some divisors $D_1 \in CH^1(V)$ and $D_2 \in CH^1(W)$. Let $z$ be a zero-cycle on $V$. Then $\gamma_* z = \deg z \cdot D_2$. This immediately implies $D_2 = 0$. Likewise, if $z \in CH_0(W)$, $\gamma^* z = 0$ implies $D_1 = 0$. We have thus proved that $\gamma = 0$. \hfill \Box

Proposition 2.18. Let $C$ and $C'$ be smooth projective curves and let $S$ be a smooth projective surface together with an idempotent $\pi_{tr,S}^2$ as in Step 1. Then

- $\text{Hom}(\pi_1(C)(1), \pi_1(C')) = 0$ for $l > 0$,
- $\text{Hom}((S, \pi_{tr,S}^2), \pi_1(C)) = 0$ for $l > 0$.

Proof. The result is trivial for dimension reasons if $l > 1$. Let’s thus consider the case $l = 1$. If $\gamma$ is a morphism that belongs to Hom($\pi_1(C)(1), \pi_1(C')$) (resp. Hom($((S, \pi_{tr,S}^2), \pi_1(C))$), then $\gamma$ is an element of $CH^0(C \times C')$ (resp. $CH^0(S \times C)$) such that $\gamma_*$ acts trivially on zero-cycles. By lemma 2.16 we get $\gamma = 0$. \hfill \Box

Proposition 2.19. Let $\{p_i : i \neq d_X\}$ be the idempotents constructed in Step 6. For all $0 \leq i, j < d_X$ we have the following relations.

- $p_i \circ p_j = 0$ for $i \neq j$,
- $p_i \circ t p_j = 0$.

Proof. The first point is clear by construction of the $p_i$’s for $0 \leq i < d_X$. Concerning the second point, we already know from Step 4 that $p_{2i}^{ttr} \circ t p_{2j}^{ttr} = 0$. Here is what is left to prove.

- $p_{2i}^{alg} \circ t p_j = 0$ for $0 \leq 2i, j < \dim X$. This follows immediately for dimension reasons and from the fact that $p_{2i}^{alg}$ factors through a zero-dimensional variety.
- $p_{2i+1} \circ t p_{2j+1} = 0$ for $0 \leq 2i + 1, 2j + 1 < d_X$. The correspondence $p_{2i+1} \circ t p_{2j+1}$ factors through a correspondence $\gamma \in \text{Hom}(\pi_1(C_i)(d_X - 2i - 1), \pi_1(C_i))$ for some curve $C_i$. By proposition 2.18, the group Hom($\pi_1(C_i)(d_X - i - j - 1), \pi_1(C_i)$) is zero for $d_X - i - j - 1 > 0$ and hence $p_{2i+1} \circ t p_{2j+1} = 0$.
- $p_{2i+1} \circ t p_{2j}^{tr} = 0$ for $0 \leq 2i + 1, 2j < d_X$. The correspondence $p_{2i+1} \circ t p_{2j}^{tr}$ factors through a correspondence $\gamma \in \text{Hom}((S, \pi_{tr,S}^2, d_X - i - j - 1), \pi_1(C_i))$ for some curve $C_i$. By proposition 2.18, the group Hom($((S, \pi_{tr,S}^2, d_X - i - j - 1), \pi_1(C_i))$ is zero for $d_X - i - j - 1 > 0$ and hence $p_{2i+1} \circ t p_{2j}^{tr} = 0$. \hfill \Box
Step 8. Orthonormalising the $p_i$’s. By proposition 2.19, the set of idempotents \( \{p_l : l \neq d_X\} \) is such that $p_l \circ p_{l'} = 0$ for $l < l'$ and $l, l' \neq d_X$. Therefore, we can apply proposition 2.4 to get a new set of mutually orthogonal idempotents, that we denote \( \{\Pi_l : l \neq d_X\} \). We then set \[
abla_{d_X} := \Delta_X - \sum_{l \neq d_X} \Pi_l.
\]

Step 9. The $\Pi_l$’s define a self-dual Chow-Künneth decomposition for $X$. We are now in a position to state the following.

Proposition 2.20. The set \( \{\Pi_l : 0 \leq l \leq 2d_X\} \) defines a Chow-Künneth decomposition for $X$ that enjoys the following properties:

- Self-duality, i.e. $\Pi_l = ^t\Pi_{2d_X-1}$ for all $l$.
- $(X, \Pi_{2l}, -l + 1)$ is isomorphic to a direct summand of the motive of a surface for $2l \neq d_X$.
- $(X, \Pi_{2l+1}, -l)$ is isomorphic to a direct summand of the motive of a curve for $2l + 1 \neq d_X$.

Proof. It is easy to see from the fact that $p_l = ^t p_{2d_X-1}$ for all $l \neq d_X$ and from the formula of lemma 2.3 that $\Pi_l = ^t \Pi_{2d_X-1}$ for all $l$. That the decomposition \( \{\Pi_l : 0 \leq l \leq 2d_X\} \) does indeed induce a Künneth decomposition, i.e. that $(\Pi_l)_* H_*(X) = H_*(X)$, follows for $l \neq d_X$ from the isomorphisms $(X, p_l) \simeq (X, \Pi_l)$ of proposition 2.4 and from proposition 2.15. It is then obvious that $(\Pi_{d_X})_* H_*(X) = H_{d_X}(X)$.

Concerning the last two points, this follows again from the fact that $(X, \Pi_l)$ is isomorphic to $(X, p_l)$ for all $l \neq d_X$ by proposition 2.4, and from the construction of $p_l$ carried out in Steps 4 and 6.

Step 10. On the middle idempotent $\Pi_{d_X}$. Here we characterise the support of the idempotent $\Pi_{d_X}$. It is an essential step towards proving Murre’s conjectures for $X$. Let’s start by showing that the $\Pi_l$’s act the same way as the $p_l$’s on Chow groups for $i \neq d_X$, i.e. we show that the action on Chow groups is not altered by the non-commutative Gram-Schmidt process. For this purpose we need the following.

Proposition 2.21. The correspondence $^t p_j \circ p_i$ acts trivially on $CH_*(X)$ for all $i, j < d_X$.

Proof. The idempotents $p_{2i}^{alg}$ (resp. $p_{2i}^{tr}$, $p_{2i+1}$) factor through $h(P_l)(i)$ (resp. $(S, \pi_2)^{tr,S}$, $i-1$), $h_1(C_j)(i)$ for some variety $P_l$ (resp. $S$, $C_j$) of dimension $0$ (resp. 2, 1). For dimension reasons we thus actually have $^t p_j \circ p_i = 0$ for $|2d_X - i - j| > 3$. By construction, we also have $^t p_{2i}^{alg} \circ p_{2i}^{tr} = 0$. Here are the remaining cases:

- $^t p_{d_X-1} \circ p_{d_X-1}$ acts trivially on $CH_*(X)$. Indeed, when $d_X$ is even, then $^t p_{d_X-1} \circ p_{d_X-1}$ factors through a morphism $\gamma \in \text{Hom}(h_1(C), h_1(C)(1))$ that clearly acts trivially on $CH_*(h_1(C))$. When $d_X$ is odd, there are two cases that need be treated. First $^t p_{d_X-1}^{alg} \circ p_{d_X-1}^{alg} = 0$ because it factors through a morphism $\gamma \in \text{Hom}(h(P), h(P)(1))$ for some
zero-dimensional variety $P$. Secondly, $t_{\Pi_{d_X}^{-1}}^i p_{d_X-1} \circ p_{d_X-1}$ acts trivially on $CH_*(X)$ because it factors through a morphism $\gamma \in \text{Hom}((S, \pi_2^{tr,S}), \mathfrak{h}(P)(2))$ for some zero-dimensional variety $P$ and hence $\gamma$ is seen to act trivially on $CH_*(S, \pi_2^{tr,S})$.

- $t_{\Pi_{d_X}^{-2}} p_{d_X-1}$ acts trivially on $CH_*(X)$. If $d_X$ is even, then $t_{\Pi_{d_X}^{-2}} p_{d_X-1}$ factors through a morphism $\gamma \in \text{Hom}(\mathfrak{h}_1(C), (S, \pi_2^{S}, 1))$ that clearly acts trivially on $CH_*(S, \pi_2^{S})(\mathfrak{h}_1(C))$. If $d_X$ is odd, then $t_{\Pi_{d_X}^{-2}} p_{d_X-1}$ factors through a morphism $\gamma \in \text{Hom}((S, \pi_2^{S}), \mathfrak{h}_1(C)(2))$ that clearly acts trivially on $CH_*(S, \pi_2^{S})$.

- $t_{\Pi_{d_X}^{-1}} p_{d_X-2}$ acts trivially on $CH_*(X)$. The proof is similar to the previous case and is left to the reader.

**Proposition 2.22.** The correspondence $p_j \circ p_i$ acts trivially on $CH_*(X)$ for all $i \neq j$ with $i, j \neq d_X$.

**Proof.** Recall that for $i \neq d_X$, we have $p_i = t_{\Pi_{d_X}^{-1}} p_{d_X-1}$. Then the proposition follows from a combination of propositions 2.19 and 2.21.

**Proposition 2.23.** Let $i \neq d_X$. The action of $\Pi_i$ on $CH_*(X)$ coincides with the action of $p_i$ for all $l$, i.e. for all $x \in CH_*(X)$ we have $(\Pi_i)_* x = (p_i)_* x$.

**Proof.** This can be read off the formula of lemma 2.3 using proposition 2.22.

**Proposition 2.24.** For all $l \leq \left\lfloor \frac{d_X-3}{2} \right\rfloor$ we have $CH_*(X) = (\Pi_{2l+2})_* CH_*(X)$.

**Proof.** This follows immediately from propositions 2.14 and 2.23.

We now give the two main propositions concerning the middle Chow-Künneth idempotent $\Pi_{d_X}$.

**Proposition 2.25.** If $X$ is odd-dimensional, then there is a curve $C$ such that $(X, \Pi_{d_X})$ is isomorphic to a direct summand of $\mathfrak{h}(C)(\frac{d_X-1}{2})$.

**Proof.** By proposition 2.24 we have $CH_l(X, \Pi_{d_X}) = 0$ for all $l \leq \frac{d_X-3}{2}$. Applying $\frac{d_X-1}{2}$ times lemma 2.13, we get a smooth projective variety $Z$ of dimension $\frac{d_X-1}{2}$ and an idempotent $q$ such that $(X, \Pi_{d_X}) \simeq (Z, q, \frac{d_X-1}{2})$. By proposition 2.20, we have $\Pi_{d_X} = \Pi_{d_X}$. Therefore, by duality, we get $(X, \Pi_{d_X}) \simeq (Z, \ell q)$. Thus $CH_l(Z, \ell q) = 0$ for all $l \leq \frac{d_X-1}{2}$. Applying $\frac{d_X-1}{2}$ times lemma 2.13 to $(Z, \ell q)$, we get a curve $C$ such that $(Z, \ell q)$ is isomorphic to a direct summand of $\mathfrak{h}(C)(\frac{d_X-1}{2})$. Dualizing, we see that $(Z, q)$ is isomorphic to a direct summand of $\mathfrak{h}(C)$. This finishes the proof.

**Proposition 2.26.** If $X$ is even-dimensional, then there is a surface $S$ such that $(X, \Pi_{d_X})$ is isomorphic to a direct summand of $\mathfrak{h}(S)(\frac{d_X-2}{2})$.

**Proof.** The proof follows the exact same pattern as the proof of the proposition 2.25.

Step 11. The motivic Lefschetz conjecture for $X$. 

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Lemma 2.16 gives $\gamma \in CH$.

Proof. In the first case, $\pi_{2i}^{tr} \circ h^{d-2i} \circ t \pi_{2i}^{tr} \in Hom((X, t \pi_{2i}^{tr}), (X, \pi_{2i}^{tr}, d - 2i))$ are isomorphisms for $2i < d$.

Proposition 2.27. The morphisms $\pi_{2i}^{tr} \circ h^{d-2i} \circ t \pi_{2i}^{tr} \in Hom((X, t \pi_{2i}^{tr}), (X, \pi_{2i}^{tr}, d - 2i))$ are isomorphisms for $2i < d$.

Proof. We claim that

$$\frac{1}{n} \cdot t \pi_{2i}^{tr} \circ h^{i-1} \circ t \Gamma \circ \Gamma \circ h^{i-1} \circ \pi_{2i}^{tr}$$

is the inverse of $\pi_{2i}^{tr} \circ h^{d-2i} \circ t \pi_{2i}^{tr}$. Indeed this follows from the formula defining the idempotents $\pi_{2i}^{tr}$ and from lemma 1.1.

By proposition 2.4 the motives $(X, p_{2i})$ and $(X, \Pi_{2i})$ are isomorphic for all $2i \neq d_X$. Let's thus consider the orthogonal decomposition $\Pi_{2i} = \Pi_{2i}^{alg} + \Pi_{2i}^{tr}$ arising from the latter isomorphism and from the decomposition $p_{2i} = p_{2i}^{alg} + p_{2i}^{tr}$.

Proposition 2.28. The morphisms

$$\Pi_{2i}^{alg} \circ h^{d-2i} \circ t \Pi_{2i}^{alg} \in Hom((X, t \Pi_{2i}^{alg}), (X, \Pi_{2i}^{alg}, d - 2i))$$

for $2i < d_X$ and

$$\Pi_{2i+1} \circ h^{d-2i-1} \circ t \Pi_{2i+1} \in Hom((X, t \Pi_{2i+1}), (X, \Pi_{2i+1}, d - 2i - 1))$$

for $2i + 1 < d_X$ are isomorphisms of Chow motives.

Proof. By the hard Lefschetz theorem $h_{d-i}^*: H^i(X) \to H_i(X)$ is an isomorphism for all $i \leq d$. In particular, $h_{d-i}^*: H_*(X, t \Pi_{2i}) \to H_*(X, \Pi_{2i})$ is an isomorphism for all $i \leq d$. The isomorphism of proposition 2.27 together with the hard Lefschetz theorem in degree $2i$ implies that $h_{d-2i}^*: H_*(X, t \Pi_{2i}^{alg}) \to H_*(X, \Pi_{2i}^{alg})$ is an isomorphism. We thus see that the morphisms of the proposition induce isomorphisms on homology. We can now conclude with [16, Propositions 5.1 & 5.2] by saying that $(X, \Pi_{2i}^{alg}, -i)$ is isomorphic to the motive of a zero-dimensional variety and that $(X, \Pi_{2i+1}, -i)$ is isomorphic to a direct summand of the $h_1$ of a curve.

Lemma 2.29. Let $\alpha \in CH_{2i}(X \times X)$. Then

- $\pi_{2j}^{tr} \circ \alpha \circ t \pi_{2j}^{tr} = 0$ for $j < i$.
- $\pi_{2j+1} \circ \alpha \circ t \pi_{2j}^{tr} = 0$ for $j < i$.
- $\pi_{2j}^{alg} \circ \alpha \circ t \pi_{2i}^{tr} = 0$ for $j \leq i$.

Proof. In the first case, $\pi_{2j}^{tr} \circ \alpha \circ t \pi_{2j}^{tr}$ factors through a correspondence $\gamma \in CH_{2i-j}(S \times S)$ such that $\gamma_z = \gamma^* z = 0$ for all $z \in CH_0(S)$. If $i > j + 2$ then clearly $\gamma = 0$. If $i = j + 2$, lemma 2.16 gives $\gamma = 0$. If $i = j + 1$, then lemma 2.17 gives $\gamma = 0$.

In the second case, there is a curve $C$ such that $\pi_{2j+1} \circ \alpha \circ t \pi_{2j}^{tr}$ factors through a correspondence $\gamma \in CH_{1+i-j}(S \times C)$ such that $\gamma_z = 0$ for all $z \in CH_0(S)$ and $\gamma^* z' = 0$ for all $z' \in CH_0(C)$. This implies $\gamma = 0$ by lemmas 2.16 and 2.17.

Finally, in the last case, there exists a zero-dimensional $P$ such that $\pi_{2j}^{alg} \circ \alpha \circ t \pi_{2j}^{tr}$ factors through a correspondence $\gamma \in CH_{1+i-j}(S \times P)$ such that $\gamma_z = 0$ for all $z \in CH_0(S)$ and $\gamma^* z' = 0$ for all $z' \in CH_0(P)$. We conclude as in the previous cases.
By proposition 2.28, in order to prove the motivic Lefschetz conjecture for X, it is enough to show that \( \Pi_{2i}^{al} \circ h^{d-2i} \circ \pi_i^{tr} = 0 \) and that \( \Pi_{2i}^{tr} \circ h^{d-2i} \circ \Pi_{2i}^{tr} \) is an isomorphism. The first point follows immediately from lemma 2.29 and from the formula of lemma 2.3 defining the \( \Pi_i \)'s in terms of the \( p_i \)'s. Concerning the second point, we know by proposition 2.27 that \( \Pi_{2i}^{tr} \circ \pi_{2i}^{tr} \circ h^{d-2i} \circ \pi_{2i}^{tr} \circ \Pi_{2i}^{tr} \) is an isomorphism with inverse \( \frac{1}{\pi} \Pi_{2i}^{tr} \circ \pi_{2i}^{tr} \circ h^{d-2i} \circ \pi_{2i}^{tr} \circ \Pi_{2i}^{tr} \). We can therefore conclude that \( \Pi_{2i}^{tr} \circ h^{d-2i} \circ \Pi_{2i}^{tr} \) is an isomorphism if we can show the equality

\[
\Pi_{2i}^{tr} \circ h^{d-2i} \circ \Pi_{2i}^{tr} = \Pi_{2i}^{tr} \circ \pi_{2i}^{tr} \circ h^{d-2i} \circ \pi_{2i}^{tr} \circ \Pi_{2i}^{tr},
\]

Having a close look at the non-commutative Gram-Schmidt process of lemma 2.3 we see that this reduces to the identities proved in lemma 2.29.

The motivic Lefschetz conjecture for X is thus established. \( \square \)

**Remark 2.30.** The morphism \( \Pi_i \circ h^{d-i} \circ \Pi_i \in \text{Hom}((X, \pi_i^{tr}), (X, \Pi_i, d - i)) \) is an isomorphism for any choice of a polarisation \( h \). In order to see this, we only need to check that \( \pi_{2i}^{tr} \circ h' \circ \pi_i^{tr} \in \text{Hom}((X, \pi_i^{tr}), (X, \pi_{2i}^{tr}, d - 2i)) \) is an isomorphism for all polarisations \( h' \). This follows from the fact, which is analogous to lemma 1.1, that for any choice of polarisations \( h_1, \ldots, h_{d - d_S} \), there exists a non-zero integer \( m \) such that \( \Gamma_f \circ h_1 \circ \ldots \circ h_{d - d_S} \circ \Gamma_f = m \cdot \Delta_S \in CH_{d_S}(S \times S) \).

**Step 12. Murre’s conjectures for X.** Thanks to propositions 2.20, 2.25 and 2.26, the following proposition settles Murre’s conjectures (B) and (D) for X.

**Proposition 2.31.** Let X be a smooth projective variety of dimension \( d \). Suppose X has a Chow-Künneth decomposition \( \{\Pi_i\}_{0 \leq i \leq 2d} \) such that, for all \( i \),

- \( \Pi_{2i} \) factors through a surface, i.e. there is a surface \( S_i \) such that \( (X, \Pi_{2i}) \) is a direct summand of \( b(S_i)(i - 1) \).
- \( \Pi_{2i+1} \) factors through a curve, i.e. there is a curve \( C_i \) such that \( (X, \Pi_{2i+1}) \) is a direct summand of \( b_1(C_i)(i) \).

Then homological and algebraic equivalence agree on X, X satisfies Murre’s conjectures (A), (B) and (D), and the filtration does not depend on the choice of a Chow-Künneth decomposition as above.

*Proof. See [16, Proposition 6.4]. \( \square \)*

**Remark 2.32.** Let X be a smooth projective variety defined over a subfield \( k \) of \( \mathbb{C} \). Assume that there is a flat dominant morphism \( f : X \to S \) to a smooth projective surface \( S \) defined over \( k \) such that for all field extensions \( K/k \) and all points \( \text{Spec} K \to S \) the fibre \( X_K \) is a quadric hypersurface. Then the conclusion of theorem 2.1 holds for X, i.e. X has a self-dual Murre decomposition which satisfies the motivic Lefschetz conjecture.
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