A FAKE SMOOTH $\mathbb{C}P^2\#\mathbb{R}P^4$

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ABSTRACT. We show that the manifold $\ast\mathbb{R}P^4\#\ast\mathbb{C}P^2$, which is homotopy equivalent but not homeomorphic to $\mathbb{R}P^4\#\mathbb{C}P^2$, is in fact smoothable.

1. INTRODUCTION

In Kirby’s problem list [Kir97, Problem 4.82] and in a recent lecture at MSRI, P. Teichner raised the question of the smoothability of a certain non-orientable 4-manifold. In this note we show that the manifold in question, denoted $\ast\mathbb{R}P^4\#\ast\mathbb{C}P^2$, which is homotopy equivalent but not homeomorphic to $\mathbb{R}P^4\#\mathbb{C}P^2$, is in fact smoothable. The smooth model we construct will have the additional property that its universal cover is diffeomorphic to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. To describe the manifold in question, we remind the reader that one of the first consequences of Freedman’s simply-connected surgery theory was a construction of a manifold $\ast\mathbb{C}P^2$, sometimes called CH in honor of Chern, which is homotopy equivalent but not homeomorphic to $\mathbb{C}P^2$. The manifold $\ast\mathbb{C}P^2$ is not smoothable for classical reasons: it has non-trivial Kirby-Siebenmann invariant $KS \in \mathbb{Z}_2$. Given any simply-connected non-spin manifold $M$, a similar construction produces a homotopy equivalent ‘$\ast$-partner’ $\ast M$ with opposite Kirby-Siebenmann invariant [Teich96]. In 1983, the first author [Rub84] constructed what is in effect the $\ast$-partner of $\mathbb{R}P^3$. The connected sum $\ast\mathbb{C}P^2\#\ast\mathbb{R}P^4$ has trivial KS-invariant and so might expected to be smoothable; on the other hand [HKT94] it is not homeomorphic to $\mathbb{C}P^2\#\mathbb{R}P^4$.

Theorem 1. The manifold $\ast\mathbb{C}P^2\#\ast\mathbb{R}P^4$ has a smooth structure. Moreover, it has a smooth structure such that its universal cover is diffeomorphic to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$.

The classification [HKT94] of non-orientable manifolds with $\pi_1 = \mathbb{Z}_2$ implies that such manifolds which have $b_2 > 1$ are smoothable if and only if their Kirby-Siebenmann invariant vanishes. Together with theorem 1 this yields:

Corollary 2. Let $X$ be a closed non-orientable 4-manifold with $\pi_1(X) = \mathbb{Z}_2$. Then $X$ has a smooth structure if and only if $KS(X) = 0$.

2. CONSTRUCTION OF THE MANIFOLD

The proof of Theorem 1 is constructive; we will find a smooth manifold homeomorphic to $\ast\mathbb{C}P^2\#\ast\mathbb{R}P^4$. The construction uses a homology sphere satisfying the conclusion of the following lemma, whose proof will be given in the next section.

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Lemma 2.1. There is a homology 3–sphere $\Sigma^3$ with the following properties.

(i) $\Sigma$ is obtained by $\pm 1$ surgery on a knot $K$ in $S^3$.
(ii) The Rohlin invariant $\mu(\Sigma) = 1 \pmod 2$.
(iii) $\Sigma$ admits a free, orientation preserving involution $\tau$, which is isotopic to the identity.

Different $\Sigma$’s could in principle give rise to different smooth structures on $\ast \mathbb{CP}^2 \# \ast \mathbb{RP}^4$, but we know of no way to tell them apart. The situation is quite analogous to that for the fake $\mathbb{RP}^4$’s constructed in [FS81].

Proof of Theorem 1. Let $\Sigma$ be a homology 3–sphere as described in the lemma; choose an orientation on $\Sigma$ so that it becomes surgery on a knot with coefficient $= +1$. Items (i) and (ii) are the ingredients in Freedman’s construction [Fre82] of $\ast \mathbb{CP}^2$. That is, let $Y$ be the result of adding a 2–handle to $B^4$ along $K$, with framing 1, then $\partial Y = \Sigma$ and $\ast \mathbb{CP}^2 = Y \cup_{\Sigma} \Delta^4$ where $\Delta^4$ is a contractible 4-manifold with boundary $-\Sigma$. (The sign of the framing is not really important, for the difference between $\ast \mathbb{CP}^2$ and $\ast \mathbb{RP}^4$ will disappear when we connect sum with $\ast \mathbb{RP}^4$.) The non-trivial $\mu$-invariant is readily identified with the Kirby-Siebenmann invariant of $\ast \mathbb{CP}^2$.

Now items (ii) and (iii) are exactly the ingredients for the construction of $\ast \mathbb{RP}^4$ given in [Rub84], i.e.

$$\ast \mathbb{RP}^4 = \Delta^4/(x \in \Sigma \sim \tau(x)) = (\Sigma/\tau \times I) \cup_{\Sigma} \Delta^4$$

(The authors of [HKT94] seem to have been unaware of this earlier construction of $\ast \mathbb{RP}^4$; compare the discussion in [Kir97, Problem 4.74].)

Let $X$ be the smooth manifold obtained as the union of $Y$ and the mapping cylinder of the orbit map of the free involution $\tau$ on $\Sigma$, i.e.

$$X = Y \cup_{\Sigma} (\Sigma/\tau \times I) = Y/(x \in \Sigma \sim \tau(x)).$$

Then $X$ is manifestly smooth, and we claim that it is homeomorphic to $\ast \mathbb{CP}^2 \# \ast \mathbb{RP}^4$. This seems quite plausible, for the construction amounts to performing a sort of connected sum, where instead of removing disks and gluing, we remove the ‘pseudo-disc’ $\Delta^4$ and glue up. Unfortunately, we do not know an elementary proof, and must appeal to the homeomorphism classification theorem of [HKT94].

According to that work, the manifold $\ast \mathbb{CP}^2 \# \ast \mathbb{RP}^4$ is distinguished among non-orientable manifolds with $\pi_1 = \mathbb{Z}_2$ by having $b_2 = 1$, trivial Kirby-Siebenmann invariant, and by a codimension-2 Pin$^c$ Arf-invariant. (The other possible manifolds, up to homeomorphism, with the same homology are $\mathbb{CP}^2 \# \mathbb{RP}^4$, $\ast \mathbb{CP}^2 \# \ast \mathbb{RP}^4$, and $\mathbb{CP}^2 \# \ast \mathbb{RP}^4$.) The Arf-invariant, whose value for $\ast \mathbb{CP}^2 \# \ast \mathbb{RP}^4$ is $\pm 3 \pmod 8$, is that of a surface pulled back from $\mathbb{CP}^N$ via a map $\varphi : X \to \mathbb{CP}^{N+1}$ which classifies $c_\Phi$ of the (primitive) Pin$^c$ structure $\Phi$.

A (topological) Spin$^c$ structure on $\ast \mathbb{CP}^2$ also determines such a map, say $\varphi'$; it is easy to see that (in terms of the decomposition of $\ast \mathbb{CP}^2$ given above) that $\varphi'$ can be taken to be smooth on $Y$, and constant on $\Delta^4$. To be more concrete, the dual surface $F$ could be taken as a Seifert surface of $K$, capped off in the 2-handle. The Arf invariant of $F$ (in $\ast \mathbb{CP}^2$)
is 4 (mod 8), as can be seen from this description of $F$, or by using Rohlin’s theorem as in [IKT94].

The Pin-c structure on $\ast \mathbb{RP}^4$ has for its characteristic class the non-trivial class in $H^2(\ast \mathbb{RP}^4; \mathbb{Z})$. This class is ‘dual’ to a surface in $\ast \mathbb{RP}^4$ which again may be assumed to lie in $\ast \mathbb{RP}^4 - \Delta^4$. By the homotopy invariance of the Arf-invariant for Pin- structures, $\text{Arf}(\ast \mathbb{RP}^4) \equiv \text{Arf}(\mathbb{RP}^4) \equiv \pm 1 \pmod{8}$. There is a unique Pin-c structure on $\Sigma$, so the Pin-c structures on $\ast \mathbb{RP}^4 - \Delta^4$ and $\ast \mathbb{CP}^2 - \Delta^4$ glue up to give a Pin-c structure $\Phi_X$ on $X$. The characteristic class $c_{\Phi_X}$ is clearly dual to the disjoint union of surfaces lying in the two pieces of $X$, so the Arf invariant is $\pm 4 \equiv \pm 3 \pmod{8}$, just as for $\ast \mathbb{CP}^2 \# \ast \mathbb{RP}^4$. Since $X$ is smooth, its Kirby-Siebenmann invariant is trivial, and so $X$ is homeomorphic to $\ast \mathbb{CP}^2 \# \ast \mathbb{RP}^4$.

The additional remark about the universal cover of $X$ being standard may be seen as follows (cf. [FS81]). By the construction of $X$, its cover $\tilde{X} \cong Y \cup \tau \tilde{Y} \cong Y \cup Y$ since $\tau$ is isotopic to the identity. On the other hand, $Y \cup \tilde{Y}$ is obtained by adding two 2-handles to $B^4$, together with a 4-handle. The first is added along $K$, with framing 1, and the second is added along a meridian of $K$, with framing 0. (This is a standard argument in handle theory, see for example [Kir89].) It is then easy to unknot $K$, by repeatedly sliding over the 0-framed handle, resulting in a standard picture of $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

$\square$

3. Proof of Lemma 2.1

In this section, we give two examples of homology spheres satisfying the conclusions of Lemma 2.1. Both examples are Brieskorn spheres, i.e. Seifert-fibered homology spheres of the form $\Sigma(p, q, r)$, where $p, q$, and $r$ are relatively prime odd numbers. The involution $\tau$ is nothing more than multiplication by $-1 \in S^1$ in the natural circle action on $\Sigma(p, q, r)$. The condition that the numbers $p, q$, and $r$ be odd guarantees that $\tau$ is free; since $-1$ is contained in a circle, the involution is isotopic to the identity.

There are many Brieskorn spheres which are integral surgery on a knot—for some examples see [KT90, MM97] or adapt the technique of [CH81]. For most of these constructions one of the indices turns out to be even. One construction is given below, where it is shown that adding a handle (along the curve denoted $\gamma$) to the Brieskorn sphere $\Sigma(5, 9, 13)$ yields $S^3$. Turning the picture upside down shows that $\Sigma(5, 9, 13)$ is integral surgery on a knot in $S^3$. As remarked in the proof of Theorem 1, it doesn’t matter whether the coefficient is positive or negative. Again, the $\mu$-invariant is 1 (from the picture just after blowing down the first $-1$ curve), so this example proves the lemma.

Another construction from the literature which provides Seifert fibered spaces is $rs(p + q)^2 + pq$ surgery on the knot denoted $K_{p,q}(r, s)$ in the recent paper [MM97, §9]. Choosing $p = -13$, $q = 23$, $r = 3$, and $s = 1$ gives the homology sphere $\Sigma(3, 13, 23)$ as $+1$ surgery on a hyperbolic knot. Since $\mu(\Sigma(3, 13, 23)) = 1$, this manifold gives an example which yields the proof of Lemma 2.1. This is the only example of a $\mu$-invariant 1 homology sphere constructible by this method found by a moderately long computer search. It is possible to give a Kirby-calculus proof that $\Sigma(3, 13, 23)$ is surgery on a knot similar to the one for $\Sigma(5, 9, 13)$: aficionados of the subject may wish to check if the knot is the same as the one in the knot from the paper [MM97].
\[ \Sigma(5,9,13) = -\frac{5}{2} -1 -\frac{13}{2} = -2 -3 -1 -7 -2 \]
\[ -\frac{9}{4} \]

\[ \partial \approx \]
blow down -1

\[ \partial \approx \]
blow down \( \gamma \)

Add (-1) framed 2-handle to \( \gamma \)

\[ \partial \approx \]
blow down \( \gamma \)

\[ \approx -2 -0 \approx S^3 \]

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