THE NEWFORM $K$-TYPE AND $p$-ADIC SPHERICAL HARMONICS

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ABSTRACT. Let $K := \text{GL}_n(O)$ denote the maximal compact subgroup of $\text{GL}_n(F)$ with $F$ a nonarchimedean local field. We study the decomposition of the space of square-integrable functions on the unit sphere in $F^n$ into irreducible $K$-modules; for $F = \mathbb{Q}_p$, these are the $p$-adic analogues of spherical harmonics. As an application, we characterise the newform and conductor exponent of a generic irreducible admissible smooth representation of $\text{GL}_n(F)$ in terms of distinguished $K$-types. Finally, we compare our results to analogous results in the archimedean setting.

1. Introduction

1.1. The Space $L^2(S^{n-1})$ as a $K$-Module. Let $F$ be a nonarchimedean local field with ring of integers $O$, maximal prime ideal $p$, and uniformiser $\varpi$, so that $\varpi O = p$ and $O/p \cong \mathbb{F}_q$ for some prime power $q$. Thus either $F$ is a finite extension of the $p$-adic numbers $\mathbb{Q}_p$, or $F$ is the field of formal Laurent series $\mathbb{F}_q[[t]]$, where $\mathbb{F}_q$ is the finite field of prime power order $q$. We let $K_n := \text{GL}_n(O)$ denote the maximal compact subgroup of $\text{GL}_n(F)$, which is unique up to conjugacy; when it is clear from context, we write $K$ in place of $K_n$. Throughout we assume that $n \geq 2$.

The group $K_n$ acts transitively on unit sphere $S^{n-1}$ in $F^n$ via the group action $k \cdot x := xk$ for $k \in K_n$ and $x \in S^{n-1}$, where

$$S^{n-1} := \{x = (x_1, \ldots, x_n) \in F^n : \max \{|x_1|, \ldots, |x_n|\} = 1\};$$

here $| \cdot |$ denotes the absolute value on $F$ normalised such that $|\varpi| = q^{-1}$. The stabiliser of the point $e_n := (0, \ldots, 0, 1) \in S^{n-1}$ is the subgroup

$$K_{n-1, 1} := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in K_n : a \in K_{n-1}, \ b \in \text{Mat}_{(n-1) \times 1}(O) \right\}$$

of $K_n$, which has the structure of the outer semidirect product $K_{n-1} \rtimes O^{n-1}$. It follows that $S^{n-1} \cong K_{n-1, 1}\backslash K_n$. Note that $K_{n-1, 1}$ is the maximal compact subgroup of the mirabolic subgroup

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{GL}_n(F) : a \in \text{GL}_{n-1}(F), \ b \in \text{Mat}_{(n-1) \times 1}(F) \right\}$$

of $\text{GL}_n(F)$.

Let $R$ denote the right regular representation of $K_n \ni k$ on $L^2(S^{n-1}) \ni f$; thus $(R(k) \cdot f)(x) := f(xk)$ for $x \in S^{n-1}$. A natural question to ponder is the following.

**Question.** What is the decomposition of the right regular representation $R$ of $K_n$ on $L^2(S^{n-1})$ into irreducible smooth representations of $K_n$? Equivalently, which irreducible smooth representations of $K_n$ have a nontrivial $K_{n-1, 1}$-fixed vector?

We study this problem in Section 2. While the general classification of irreducible smooth representations of $K_n$ is unknown for $n \geq 3$, we show that the representations having a nontrivial $K_{n-1, 1}$-fixed vector can be explicitly described; in particular, a precise resolution of this question is given in Theorem 2.14. The irreducible smooth representations of interest are indexed by pairs $(\chi, m)$ of characters $\chi$ of $O^\times$ and nonnegative integers $m \geq c(\chi)$, where $c(\chi)$ denotes the conductor exponent of $\chi$. The character $\chi$ is the central character of this representation, while $m$ is its level, namely the minimal nonnegative integer for which this representation factors through

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GL_n(O/p^m). We denote by (τ_{χ,m}, H_{χ,m}(S^{n-1})) the irreducible smooth representation associated to such a pair, where H_{χ,m}(S^{n-1}) is a finite-dimensional vector subspace of L^2(S^{n-1}); this may be thought of as the nonarchimedean analogue of the vector space of spherical harmonics of degree m on the unit sphere in R^n.

1.2. Zonal Spherical Functions. We next study the subspace of locally constant functions in L^2(S^{n-1}) that are K_{n-1,1}-invariant in Section 3. Each irreducible subspace H_{χ,m}(S^{n-1}) of L^2(S^{n-1}) has a one-dimensional subspace of K_{n-1,1}-invariant functions, so that there exists a unique function P_{χ,m}^0 ∈ H_{χ,m}(S^{n-1}) satisfying P_{χ,m}^0(xk') = P_{χ,m}^0(x) for all x ∈ S^{n-1} and k' ∈ K_{n-1,1} and P_{χ,m}^0(ε_n) = 1. We call P_{χ,m}^0 the zonal spherical function on S^{n-1} of character χ and level m. These are the nonarchimedean analogues of zonal spherical harmonics (or ultraspherical polynomials).

We give an explicit formula for P_{χ,m}^0 in Proposition 3.4. Using this, we prove a nonarchimedean analogue of the addition formula for spherical harmonics in Lemma 3.14. A consequence of this is Corollary 3.20, which states that (dim H_{χ,m}(S^{n-1}))P_{χ,m}^0 is the reproducing kernel for H_{χ,m}(S^{n-1}), which implies that each element of H_{χ,m}(S^{n-1}) is equal to a matrix coefficient of τ_{χ,m}.

1.3. The Newform and the Conductor Exponent. We apply this theory of p-adic spherical harmonics in Section 4 to previous work of Jacquet, Piatetski-Shapiro, and Shalika [JP-SS81]. They prove the existence of a distinguished vector, the newform, associated to a given generic irreducible admissible representation (π, V_π) of GL_n(F) (or more generally to an induced representation of Langlands type). This vector is invariant under a certain congruence subgroup K_1(p^m) of K_n and is the minimal such nontrivial vector in the sense that there are no nontrivial vectors invariant under K_1(p^\ell) with \ell < m. This minimal value of m is called the conductor exponent of π and is denoted by c(π).

We give alternative characterisations of the newform and conductor exponent in Theorem 4.16. We show that the conductor exponent c(π) of π is the minimal nonnegative integer m for which there exists a nontrivial K_{n-1,1}-invariant τ_{χ,m}-isotypic vector in V_π for some character χ of O^×; necessarily, χ must then be equal to χ_π, the restriction from F^× to O^× of the central character ω_π of π. We also show that the newform is precisely the nonzero vector, unique up to scalar multiplication, that is K_{n-1,1}-invariant and τ_{χ,\pi,c(π)}-isotypic; for this reason, we name τ_{χ,\pi,c(π)} the newform K-type. Our methods also allow us to describe precisely in Proposition 4.25 the multiplicity with which a representation τ_{χ,m} occurs in the K-type decomposition of π.

These different characterisations open up new avenues of approach to studying properties of the newform. We give a simple example of one such property in Corollary 4.34, where we show that the matrix coefficient associated to the newform may be explicitly described in terms of the zonal spherical function P_{χ,\pi,c(π)}^0 ∈ H_{χ,\pi,c(π)}(S^{n-1}).

1.4. The Archimedean Theory. Finally, in Section 5, we compare our results to analogous results in the archimedean setting. The archimedean analogue of K_n = GL_n(O) is the orthogonal group O(n) if F = R or the unitary group U(n) if F = C. The decomposition of L^2(S^{n-1}) into irreducible K_n-modules in these settings is well-known: it is the classical theory of spherical harmonics. These are the restriction to the unit sphere of homogeneous harmonic polynomials of a given degree if F = R or of a given bidegree if F = C. In both cases, there exist analogues of the zonal spherical functions P_{χ,m}^0.

The archimedean analogue of the theory of newforms and conductor exponents of a generic irreducible admissible smooth representation of GL_n(F) was recently developed by the author [Hum20]. Over archimedean local fields, the approach of Jacquet, Piatetski-Shapiro, and Shalika [JP-SS81] via congruence subgroups is no longer applicable, and instead the development of the theory of newforms is via distinguished K_n-types. Thus our alternative characterisation in the nonarchimedean setting proven in Theorem 4.16 serves to unify these two different settings.
2. \textit{p-adic} Spherical Harmonics

Our goal is to decompose the right regular representation of $K$ on $L^2(S^{n-1})$ into irreducible smooth representations. In place of $L^2(S^{n-1})$, it suffices to study the subspace $C^\infty(S^{n-1}) \subset L^2(S^{n-1})$ of locally constant functions $f : S^{n-1} \to \mathbb{C}$, which is dense in $L^2(S^{n-1})$. Our first observation is that the subspace $C^\infty(S^{n-1})$ may be identified with the induced representation of the trivial representation of $K_{n-1,1}$.

\textbf{Lemma 2.1.} As $K$-modules, the space $C^\infty(S^{n-1})$ is isomorphic to $\text{Ind}_{K_{n-1,1}}^K 1$, the vector space of locally constant functions $\phi : K \to \mathbb{C}$ that satisfy $\phi(k'k) = \phi(k)$ for all $k \in K$ and $k' \in K_{n-1,1}$, upon which $K$ acts via right translations.

\textit{Proof.} Given $f \in C^\infty(S^{n-1})$, define $\phi(k) := f(e_n k)$; then clearly $\phi \in \text{Ind}_{K_{n-1,1}}^K 1$. Conversely, given $\phi \in \text{Ind}_{K_{n-1,1}}^K 1$, $\phi(k)$ is dependent only on $e_n k$. Since for each $x \in S^{n-1}$, there exists some $k \in K$ for which $x = e_n k$, the function $f(x) := \phi(k)$ is well-defined, which gives us an element of $C^\infty(S^{n-1})$. \hfill $\square$

\textbf{2.1. Reduction to $K_1(p^m)$}. We next study subspaces of $C^\infty(S^{n-1})$ consisting of locally constant functions invariant under certain congruence subgroups. For each nonnegative integer $m$, let $K(p^m)$ denote the principal congruence subgroup of level $m$ of $K$, namely

$$K(p^m) := \{ k \in K : k - I_n \in \text{Mat}_{n \times n}(\mathbb{Z}) \},$$

where $I_n$ denotes the $n \times n$ identity matrix. This is a normal subgroup of $K$. These subgroups allow us to construct a filtration of $C^\infty(S^{n-1})$ via the subspaces

$$C^\infty(S^{n-1})_{K(p^m)} := \{ f \in C^\infty(S^{n-1}) : f(\gamma xk) = f(x) \text{ for all } x \in S^{n-1} \text{ and } k \in K(p^m) \}$$

de $K(p^m)$-invariant locally constant functions on $S^{n-1}$. We observe that $C^\infty(S^{n-1})_{K(p^m)}$ is contained in $C^\infty(S^{n-1})_{K(p^\ell)}$ for all nonnegative integers $m \leq \ell$. Furthermore, the union $\bigcup_{m=0}^\infty C^\infty(S^{n-1})_{K(p^m)}$ is dense in $C^\infty(S^{n-1})$.

We also define the congruence subgroup $K_1(p^m)$ of $K$ by

$$K_1(p^m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K : c \in \text{Mat}_{1 \times (n-1)}(\mathbb{Z}), \; d - 1 \in \mathbb{Z} \right\}$$

for each nonnegative integer $m$, so that $K_1(p^0) = K$, while if $m \geq 1$, $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in K_1(p^m)$ implies that $a \in K_{n-1}$ and $d \in O^n$; we make note of the fact that $K_1(p^m)$ contains $K_1(p^\ell)$ and $K(p^\ell)$ as subgroups whenever $\ell \geq m$.

\textbf{Lemma 2.2.} As $K$-modules, $C^\infty(S^{n-1})_{K(p^m)}$ is isomorphic to $\text{Ind}_{K_1(p^m)}^K 1$, the vector space of locally constant functions $\phi : K \to \mathbb{C}$ that satisfy $\phi(k_1 k) = \phi(k)$ for all $k \in K$ and $k_1 \in K_1(p^m)$, upon which $K$ acts via right translations.

\textit{Proof.} From Lemma 2.1, $C^\infty(S^{n-1})_{K(p^m)}$ is isomorphic as a $K$-module to the space of locally constant functions $\phi : K \to \mathbb{C}$ that satisfy $\phi(k'k k'') = \phi(k)$ for all $k' \in K_{n-1,1}$, $k \in K$, and $k'' \in K(p^m)$. Since $K(p^m)$ is a normal subgroup of $K$, this is equal to the space of locally constant functions $\phi : K \to \mathbb{C}$ that satisfy $\phi(k'k' k'') = \phi(k)$ for all $k' \in K_{n-1,1}$, $k'' \in K(p^m)$, and $k \in K$. The result then follows from the fact that $K_{n-1,1} K(p^m) = K_1(p^m)$, which is clearly true if $m = 0$; for $m \geq 1$, it is immediate that $K_{n-1,1} K(p^m) \subset K_1(p^m)$, while the fact that $K_{n-1,1} K(p^m) \supset K_1(p^m)$ can be seen directly, since for $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in K_1(p^m)$, we have that

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a & ab d^{-1} \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -bd^{-1}c \\ 0 & c \end{array} \right). \hfill \square$$

\textbf{Corollary 2.3.} We have that

\begin{equation}
\dim C^\infty(S^{n-1})_{K(p^m)} = \begin{cases} 1 & \text{if } m = 0, \\ q^{(m-1)n}(q^n - 1) & \text{if } m \geq 1. \end{cases}
\end{equation}
Proof. We have that \( \dim C^\infty(S^{n-1})_K^{(p^m)} = \dim \text{Ind}_{K_1(p^m)}^K 1 \) from Lemma 2.2. Since the trivial representation is one-dimensional, the dimension of the monomial representation \( \text{Ind}_{K_1(p^m)}^K 1 \) is simply the index \([K : K_1(p^m)]\) of the subgroup \( K_1(p^m) \) in \( K \), which is precisely the right-hand side of (2.4). \( \square \)

2.2. Reduction to \( K_0(p^m) \). Next, we decompose \( C^\infty(S^{n-1})_K^{(p^m)} \) further into subspaces of locally constant functions with a prescribed central character. Given a character \( \chi \) of \( \mathcal{O}^\times \), we define the subspace of \( K(p^m) \)-invariant locally constant functions on \( S^{n-1} \) with central character \( \chi \) by

\[
C^\infty(S^{n-1})_\chi^{(p^m)} := \left\{ f \in C^\infty(S^{n-1})_K^{(p^m)} : f(ax) = \chi(a)f(x) \text{ for all } a \in \mathcal{O}^\times \text{ and } x \in S^{n-1} \right\}.
\]

Clearly \( C^\infty(S^{n-1})_\chi^{(p^m)} \) is trivial if \( m < c(\chi) \), where \( c(\chi) \) denotes the conductor exponent of \( \chi \), namely the least nonnegative integer \( m \) for which \( \chi \) is trivial on \( 1 + p^m \). Moreover, \( C^\infty(S^{n-1})_\chi^{(p^m)} \) is contained in \( C^\infty(S^{n-1})_K^{(p^m)} \) for all \( \ell \geq m \geq c(\chi) \) and any two subspaces \( C^\infty(S^{n-1})_\chi_1^{(p^m)} \) and \( C^\infty(S^{n-1})_\chi_2^{(p^m)} \) are mutually orthogonal whenever \( \chi_1 \neq \chi_2 \).

For each nonnegative integer \( m \), the congruence subgroup \( K_0(p^m) \) of \( K \) is defined by

\[
K_0(p^m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K : c \in \text{Mat}_{1 \times (n-1)}(p^m) \right\}
\]

for each nonnegative integer \( m \), so that \( K_0(p^0) = K \), while if \( m \geq 1 \), \( (a, b, c, d) \in K_0(p^m) \) implies that \( a \in K_{n-1} \) and \( d \in \mathcal{O}^\times \). We observe that \( K_0(p^m) \) contains \( K_0(p^l) \), \( K_1(p^l) \), and \( K(p^l) \) as subgroups whenever \( \ell \geq m \).

Let \( \mathcal{O}^\times \) denote the set of (continuous) characters \( \chi : \mathcal{O}^\times \to \mathbb{C}^\times \); necessarily the image of such a character is in the unit circle \( \{ z \in \mathbb{C}^\times : |z| = 1 \} \). For \( \chi \in \mathcal{O}^\times \), let \( \psi_\chi \) be the character of \( K_0(p^m) \) given by \( \psi_\chi(k_0) := \chi(d) \), which is a one-dimensional representation of \( K_0(p^m) \); by restriction, this is also a one-dimensional representation of \( K_0(p^m) \) whenever \( m \geq c(\chi) \).

Lemma 2.5. For \( \chi \in \mathcal{O}^\times \) and \( m \geq c(\chi) \), \( C^\infty(S^{n-1})_\chi^{(p^m)} \) is isomorphic as a \( K \)-module to \( \text{Ind}_{K_0(p^m)}^K \psi_\chi \), the vector space of locally constant functions \( f : K \to \mathbb{C} \) that satisfy \( f(k_0k) = \psi_\chi(k_0)f(k) \) for all \( k \in K \) and \( k_0 \in K_0(p^m) \), upon which \( K \) acts via right translations.

Proof. From Lemma 2.2, \( C^\infty(S^{n-1})_\chi^{(p^m)} \) is isomorphic as a \( K \)-module to the space of locally constant functions \( \phi : K \to \mathbb{C} \) that satisfy \( \phi(z(a)k_1k) = \chi(a)\phi(k) \) for all \( z(a) := \text{diag}(a, \ldots, a) \in Z(\mathcal{O}) \), the centre of \( K \), \( k_1 \in K_1(p^m) \), and \( k \in K \). It remains to note that \( Z(\mathcal{O})K_1(p^m) = K_0(p^m) \). \( \square \)

Corollary 2.6. We have that

\[
\dim C^\infty(S^{n-1})_\chi^{(p^m)} = \begin{cases} 1 & \text{if } m = c(\chi) = 0, \\
q^{(m-1)(n-1)}q^m - 1 \quad \text{if } m \geq \max\{c(\chi), 1\}, \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. We have that \( \dim C^\infty(S^{n-1})_\chi^{(p^m)} = \dim \text{Ind}_{K_0(p^m)}^K \psi_\chi \) from Lemma 2.5. As \( \psi_\chi \) is one-dimensional, the dimension of the monomial representation \( \text{Ind}_{K_0(p^m)}^K \psi_\chi \) is simply the index of \( K_0(p^m) \) in \( K \), so that

\[
\dim \text{Ind}_{K_0(p^m)}^K \psi_\chi = \begin{cases} [K : K_0(p^m)] & \text{if } m \geq c(\chi), \\
0 & \text{otherwise},
\end{cases}
\]

which is precisely the right-hand side of (2.7). \( \square \)
Theorem 2.14. For each character $\chi \in \hat{O}$ and for each integer $m \geq c(\chi)$, the $K$-module $\mathcal{H}_{\chi,m}(S^{n-1})$ is irreducible, and we have the orthogonal decompositions

$$C^\infty(S^{n-1})K(p^m) = \bigoplus_{\ell = c(\chi)}^m \mathcal{H}_{\chi,\ell}(S^{n-1}),$$

$$C^\infty(S^{n-1}) = \bigoplus_{c(\chi) = 0}^\infty \bigoplus_{0 \leq c(\chi) \leq m} \mathcal{H}_{\chi,m}(S^{n-1}).$$

Proof. Since $K_1(p^m)$ is a normal subgroup of $K_0(p^m)$ with quotient isomorphic to the finite abelian group $O^\times/(1 + p^n)$, we have that

$$\text{Ind}_{K_0(p^m)}^{K_1(p^m)} 1 = \bigoplus_{\chi \in \hat{O}} \psi_\chi$$

and so by inducing in stages,

$$\text{Ind}_{K_1(p^m)}^{K} 1 = \bigoplus_{\chi \in \hat{O}} \text{Ind}_{K_0(p^m)}^{K} \psi_\chi.$$

Together with Lemma 2.5, this gives the orthogonal decomposition (2.9). \qed

2.3. Irreducible Representations. The spaces $C^\infty(S^{n-1})K(p^m)$ are not irreducible if $m > c(\chi)$ since $C^\infty(S^{n-1})K(p^m)$ contains $C^\infty(S^{n-1})K(p^\ell)$ for all $\ell \in \{c(\chi), \ldots, m - 1\}$. Thus we are led to study the orthogonal complement of $C^\infty(S^{n-1})K(p^{m-1})$ in $C^\infty(S^{n-1})K(p^m)$. For $m \geq c(\chi)$, define

$$\mathcal{H}_{\chi,m}(S^{n-1}) := \begin{cases} C^\infty(S^{n-1})K(p^{\ell(\chi)}) & \text{if } m = c(\chi), \\ C^\infty(S^{n-1})K(p^m) \ominus C^\infty(S^{n-1})K(p^{m-1}) & \text{if } m > c(\chi). \end{cases}$$

As $K$-modules,

$$\mathcal{H}_{\chi,m}(S^{n-1}) \cong \begin{cases} \text{Ind}_{K_0(p^{\ell(\chi)})}^{K} \psi_\chi & \text{if } m = c(\chi), \\ \text{Ind}_{K_0(p^m)}^{K} \psi_\chi \ominus \text{Ind}_{K_0(p^{m-1})}^{K} \psi_\chi & \text{if } m > c(\chi). \end{cases}$$

Lemma 2.13. We have that

$$\dim \mathcal{H}_{\chi,m}(S^{n-1}) = \begin{cases} 1 & \text{if } c(\chi) = m = 0, \\ q^{n-1} - 1 & \frac{q^n - 1}{q - 1} & \text{if } c(\chi) = 0 \text{ and } m = 1, \\ q^{(c(\chi) - 1)(n - 1)}q^n - 1 & \frac{q^n - 1}{q - 1} & \text{if } c(\chi) = m \geq 1, \\ q^{(m-2)(n-1)}(q^n - 1)(q^{n-1} - 1) & \frac{q^n - 1 - q^{n-1}}{q - 1} & \text{if } m > \max\{c(\chi), 1\}. \end{cases}$$

Proof. This follows immediately from (2.7) and (2.11). \qed

The spaces $\mathcal{H}_{\chi,m}(S^{n-1})$ are $K$-invariant; furthermore, any two subspaces $\mathcal{H}_{\chi_1,m_1}(S^{n-1})$ and $\mathcal{H}_{\chi_2,m_2}(S^{n-1})$ are mutually orthogonal whenever either $\chi_1 \neq \chi_2$ or $m_1 \neq m_2$. We claim that these subspaces are irreducible, which thereby completes the decomposition of $C^\infty(S^{n-1})$ into irreducible $K$-modules.

Theorem 2.14. For each character $\chi \in \hat{O}$ and for each integer $m \geq c(\chi)$, the $K$-module $\mathcal{H}_{\chi,m}(S^{n-1})$ is irreducible, and we have the orthogonal decompositions

$$C^\infty(S^{n-1})K(p^m) = \bigoplus_{\ell = c(\chi)}^m \mathcal{H}_{\chi,\ell}(S^{n-1}),$$

$$C^\infty(S^{n-1}) = \bigoplus_{c(\chi) = 0}^\infty \bigoplus_{0 \leq c(\chi) \leq m} \mathcal{H}_{\chi,m}(S^{n-1}).$$
Remark 2.17. Theorem 2.14 is not, in essence, fundamentally new. With \( K_n = \text{GL}_n(O) \) replaced by \( \text{SL}_n(O) \), this decomposition was previously achieved (with scant proofs) by Petrov [Pet82] in a seemingly neglected paper; our method follows that sketched by Petrov and achieves the same decomposition when \( n \geq 3 \). A closely related result, via slightly different methods, is due to Hill [Hil94, Proposition 3.3], who studies instead the decomposition of \( L^2(Z(O)K_{n-1,1}\backslash K_n) \). For \( n = 2 \), Casselman [Cas73b, Proposition 1] studies the decomposition into irreducible representations of \( \text{Ind}_{K_0(p^m)}^K \psi_\chi \); see also [BP17, Section 3.3] for the case \( n = 2 \) and \( \chi = 1 \).

Let \( \hat{K} \) denote the set of equivalence classes of irreducible smooth representations of \( K \), and write \( \tau_{\chi,m} \) for the representation in \( \hat{K} \) given by right translations on the finite-dimensional vector space \( \mathcal{H}_{\chi,m}(S^{n-1}) \). We have now classified precisely which representations in \( \hat{K} \) have a \( K_{n-1,1} \)-fixed vector.

**Corollary 2.18.** For every irreducible smooth representation \( \tau \in \hat{K} \),

\[
\dim \text{Hom}_{K_{n-1,1}}(1, \tau|_{K_{n-1,1}}) = \begin{cases} 
1 & \text{if } \tau = \tau_{\chi,m} \text{ for some } \chi \in \widehat{O}^\times \text{ and } m \geq c(\chi), \\
0 & \text{otherwise.}
\end{cases}
\]

In particular, the subspace

\[
\mathcal{H}_{\chi,m}(S^{n-1})_{K_{n-1,1}} := \{ f \in \mathcal{H}_{\chi,m}(S^{n-1}) : f(xk') = f(x) \text{ for all } x \in S^{n-1} \text{ and } k' \in K_{n-1,1} \}
\]

of \( K_{n-1,1} \)-invariant functions in \( \mathcal{H}_{\chi,m}(S^{n-1}) \) is one-dimensional.

**Proof.** We observe that \( \tau_{\chi,m_1} \) is isomorphic to \( \tau_{\chi,m_2} \) if and only if \( \chi_1 = \chi_2 \) and \( m_1 = m_2 \) by examining the dimensions and central characters of these representations. It follows that \( C^\infty(S^{n-1}) \) is multiplicity-free, so that \( (K_n, K_{n-1,1}) \) is a Gelfand pair. The result then follows via (2.16) and Frobenius reciprocity. \( \square \)

The proof of Theorem 2.14 requires the following lemma.

**Lemma 2.19.** For each nonnegative integer \( m \), we have the double coset decomposition

\[
K = \bigsqcup_{\ell=0}^{m} K_0(p^m) \begin{pmatrix} 1_{n-1} & 0 \\ \varpi^{-\ell}e_{n-1} & 1 \end{pmatrix} K_0(p^m).
\]

For \( n = 2 \), this follows from [Cas73b, Lemma 1] and [Sch02, Lemma 2.1.1], while the same result with \( K_n = \text{GL}_n(O) \) replaced by \( \text{SL}_n(O) \) is implicit in the work of Chang [Cha98].

**Proof.** Let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \) with \( a \in \text{Mat}_{(n-1)\times(n-1)}(O), \ b \in \text{Mat}_{(n-1)\times1}(O), \ c \in \text{Mat}_{1\times(n-1)}(O), \) and \( d \in O \). There are three cases to consider.

1. If \( \max\{|c_1|, \ldots, |c_{n-1}|\} \leq q^{-m} \), then \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(p^m) = K_0(p^m) \begin{pmatrix} 1_{n-1} & 0 \\ \varpi^{-m}e_{n-1} & 1 \end{pmatrix} K_0(p^m) \).

2. If \( \max\{|c_1|, \ldots, |c_{n-1}|\} = q^{-\ell} \) for some \( \ell \in \{1, \ldots, m-1\} \), then \( a \in K_{n-1} \); as \( \varpi^{-\ell}ca^{-1} \in S^{n-2} \), there exists some \( \alpha \in K_{n-1} \) such that \( e_{n-1}^{-1}c^{-1} = \varpi^{-\ell}ca^{-1} \), and we have that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ \varpi^{-\ell}e_{n-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1}a & \alpha^{-1}b \\ 0 & -ca^{-1}b + d \end{pmatrix}.
\]

3. Finally, if \( \max\{|c_1|, \ldots, |c_{n-1}|\} = 1 \), then \( c \in S^{n-2} \). By [Cha98, Lemma 3], there exists \( \beta \in \text{Mat}_{(n-1)\times1}(O) \) such that \( \det(a - \beta c) \in \widehat{O}^\times \), so that \( a - \beta c \in K_{n-1} \). As \( c(a - \beta c)^{-1} \in S^{n-2} \), there exists some \( \alpha \in K_{n-1} \) such that \( e_{n-1}^{-1}c^{-1} = c(a - \beta c)^{-1} \), and we have that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ \varpi^{-1}(a - \beta c) & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1}(a - \beta c) & \alpha^{-1}(b - \beta d) \\ 0 & -c(a - \beta c)^{-1}(b - \beta d) + d \end{pmatrix}.
\]

\( \square \)

**Proof of Theorem 2.14.** For \( m \geq c(\chi) \), we identify \( \text{End}_K(\text{Ind}_{K_0(p^m)}^K \psi_\chi) \) with the space of locally constant functions \( f : K \to \mathbb{C} \) that satisfy \( f(k_0kk_0') = \psi_\chi(k_0)\psi_\chi(k_0')f(k) \) for all \( k \in K \) and \( k_0, k_0' \in K_0(p^m) \). From Lemma 2.19, we deduce that for each integer \( m \geq c(\chi) \),

\[
\dim \text{End}_K(\text{Ind}_{K_0(p^m)}^K \psi_\chi) = m - c(\chi) + 1.
\]
Since \( \text{Ind}^K_{K_0}(p^m) \psi_\chi \cong \bigoplus_{\ell = c(\chi)} \mathcal{H}_{\chi, \ell}(S^{n-1}) \) from (2.12), we conclude that \( \text{End}_K(\mathcal{H}_{\chi, m}(S^{n-1})) \) is one-dimensional by induction, and hence \( \mathcal{H}_{\chi, m}(S^{n-1}) \) is irreducible. Finally, the orthogonal decompositions (2.15) and (2.16) are clear via (2.9) and the fact that the union of the spaces \( C^\infty(S^{n-1})^K(p^m) \) is dense in \( C^\infty(S^{n-1}) \).

\[ \square \]

3. Zonal Spherical Functions

Let

\[ C^\infty(S^{n-1})^{K_{n-1, 1}} := \{ f \in C^\infty(S^{n-1}) : f(x k') = f(x) \text{ for all } x \in S^{n-1} \text{ and } k' \in K_{n-1, 1} \} \]

be the subspace of \( K_{n-1, 1} \)-invariant locally constant functions on the unit sphere. We identify precisely which elements of this lie in \( \mathcal{H}_{\chi, m}(S^{n-1}) \) for each character \( \chi \) of \( \mathcal{O}^\infty \) and nonnegative integer \( m \geq c(\chi) \). We first define a \( K \)-invariant inner product on \( C^\infty(S^{n-1}) \) for \( f_1, f_2 \) via

\[ \langle f_1, f_2 \rangle := \int_K f_1(e_n k) \overline{f_2}(e_n k) \, dk, \]

where \( dk \) denotes the Haar probability measure on the compact group \( K \).

**Lemma 3.1.** The subspace \( C^\infty(S^{n-1})^{K_{n-1, 1}} \cap C^\infty(S^{n-1})^\chi(p^m) \) of \( C^\infty(S^{n-1})^\chi(p^m) \) has dimension \( m - c(\chi) + 1 \) and is spanned by the functions

\[ \phi_{\chi, \ell}(x_1, \ldots, x_n) := \begin{cases} \chi(x_n) & \text{if } \max\{|x_1|, \ldots, |x_n-1|\} \leq q^{-\ell}, \\ 0 & \text{if } q^{-\ell} < \max\{|x_1|, \ldots, |x_n-1|\} \leq 1, \end{cases} \]

for \( \ell \in \{c(\chi), \ldots, m\} \). Furthermore, for \( \ell_1, \ell_2 \geq c(\chi) \),

\[ \langle \phi_{\chi, \ell_1}, \phi_{\chi, \ell_2} \rangle = \begin{cases} 1 & \text{if } \ell_1 = \ell_2 = 0, \\ \frac{q - 1}{q^{(\max\{\ell_1, \ell_2\})(n-1)}(q^n - 1)} & \text{if } \max\{\ell_1, \ell_2\} \geq 1. \end{cases} \]

**Proof.** That the dimension of this subspace is \( m - c(\chi) + 1 \) is a direct consequence of (2.15) and Corollary 2.18. It is then straightforward to see that \( \phi_{\chi, \ell} \) is an element of this subspace for each \( \ell \in \{c(\chi), \ldots, m\} \) and that these are linearly independent. Finally, the identity (3.3) is immediate from the definition (3.2) of \( \phi_{\chi, \ell} \), for this implies that

\[ \langle \phi_{\chi, \ell_1}, \phi_{\chi, \ell_2} \rangle = \text{vol}(K_0(p^{\max\{\ell_1, \ell_2\}})) = \frac{1}{[K : K_0(p^{\max\{\ell_1, \ell_2\}})]}, \]

which is precisely the right-hand side of (3.3). \( \square \)

It is clear that \( \phi_{\chi, c(\chi)} \in \mathcal{H}_{\chi, c(\chi)}(S^{n-1})^{K_{n-1, 1}} \) and that \( \phi_{\chi, c(\chi)}(e_n) = 1 \). We show that there exist similar elements \( P_{\chi, m}^\phi \) in \( \mathcal{H}_{\chi, m}(S^{n-1})^{K_{n-1, 1}} \) for each \( m \geq c(\chi) \), which must be a linear combination of the functions \( \phi_{\chi, c(\chi)}, \ldots, \phi_{\chi, m} \). We deduce the precise linear combination by making use of the fact that \( P_{\chi, m}^\phi \) is orthogonal to \( P_{\chi, j}^\phi \) whenever \( j \in \{c(\chi), \ldots, m - 1\} \).

**Proposition 3.4.** For each \( \chi \in \widehat{\mathcal{O}^\infty} \) and \( m \geq c(\chi) \), there exists a unique locally constant function \( P_{\chi, m}^\phi \) in \( \mathcal{H}_{\chi, m}(S^{n-1}) \) satisfying \( P_{\chi, m}^\phi(x k') = P_{\chi, m}^\phi(x) \) for all \( x \in S^{n-1} \) and \( k' \in K_{n-1, 1} \) and \( P_{\chi, m}^\phi(e_n) = 1 \). We have that

\[ P_{\chi, m}^\phi(x_1, \ldots, x_n) = \begin{cases} \chi(x_n) & \text{if } \max\{|x_1|, \ldots, |x_n-1|\} \leq q^{-m}, \\ \alpha_{\chi, m, m-1} \chi(x_n) & \text{if } m > c(\chi) \text{ and } \max\{|x_1|, \ldots, |x_n-1|\} = q^{-m+1}, \\ 0 & \text{otherwise}, \end{cases} \]

where

\[ \alpha_{\chi, m, m-1} = \begin{cases} -\frac{q - 1}{q(q^{n-1} - 1)} & \text{if } m = 1, \\ -\frac{1}{q^n - 1} & \text{if } m \geq 2. \end{cases} \]
In particular, for all \( k \in K \), we have that

\[
P^0_{\chi,m}(e_n k) = \overline{P^0_{\chi,m}(e_n k^{-1})}.
\]

**Definition 3.8.** We call \( P^0_{\chi,m} \) the zonal spherical function on \( S^{n-1} \) of character \( \chi \) and level \( m \).

**Proof of Proposition 3.4.** Since \( \mathcal{H}_{\chi,m}(S^{n-1})^{K_{n-1}} \) is one-dimensional from Corollary 2.18, there is a unique function \( P^0_{\chi,m} \in \mathcal{H}_{\chi,m}(S^{n-1})^{K_{n-1}} \) satisfying \( P^0_{\chi,m}(e_n) = 1 \). From (3.2), there exist constants \( \alpha_{\chi,m;\ell} \in \mathbb{C} \) for \( \ell \in \{ c(\chi), \ldots, m \} \) such that

\[
P^0_{\chi,m}(x_1, \ldots, x_n) = \sum_{\ell = c(\chi)}^{m} \alpha_{\chi,m;\ell} \varphi_{\chi,m;\ell}(x)
\]

where

\[
\varphi_{\chi,m;\ell}(x_1, \ldots, x_n) := \begin{cases} 
\phi_{\chi,\ell}(x_1, \ldots, x_n) - \phi_{\chi,\ell+1}(x_1, \ldots, x_n) & \text{for } \ell \in \{ c(\chi), \ldots, m-1 \}, \\
\phi_{\chi,m}(x_1, \ldots, x_n) & \text{for } \ell = m,
\end{cases}
\]

\[
= \begin{cases} 
\chi(x_n) & \text{if } \ell \in \{ c(\chi), \ldots, m-1 \} \text{ and } \max\{|x_1|, \ldots, |x_{n-1}|\} = q^{-\ell}, \\
\chi(x_n) & \text{if } \ell = m \text{ and } \max\{|x_1|, \ldots, |x_{n-1}|\} = q^{-m}, \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( P^0_{\chi,m}(e_n) = 1 \), we have that \( \alpha_{\chi,m;\ell} = 1 \). To deduce properties of the remaining coefficients, we make use the fact that

\[
(3.9) \quad \beta_{\chi,m;\ell} := \langle \varphi_{\chi,m;\ell}, \varphi_{\chi,m;\ell} \rangle = \begin{cases} 
\langle \phi_{\chi,\ell}, \phi_{\chi,\ell} \rangle - \langle \phi_{\chi,\ell+1}, \phi_{\chi,\ell+1} \rangle & \text{if } \ell \in \{ c(\chi), \ldots, m-1 \}, \\
\langle \phi_{\chi,m}, \phi_{\chi,m} \rangle & \text{if } \ell = m
\end{cases}
\]

from (3.3), while

\[
(3.10) \quad \langle \varphi_{\chi,m;\ell_1}, \varphi_{\chi,m;\ell_2} \rangle = 0 \text{ if } \ell_1 \neq \ell_2
\]

since these have disjoint support. The zonal spherical function \( P^0_{\chi,m} \) is orthogonal to \( P^0_{\chi,j} \) for all \( j \in \{ c(\chi), \ldots, m-1 \} \) as \( \mathcal{H}_{\chi,m}(S^{n-1}) \) and \( \mathcal{H}_{\chi,j}(S^{n-1}) \) are mutually orthogonal. Writing

\[
P^0_{\chi,j}(x_1, \ldots, x_n) = \sum_{\ell = c(\chi)}^{j-1} \alpha_{\chi,j;\ell} \varphi_{\chi,j;\ell}(x_1, \ldots, x_n) + \sum_{\ell = c(\chi)}^{m} \varphi_{\chi,m;\ell}(x_1, \ldots, x_n),
\]

we deduce from (3.9) and (3.10) that for each \( j \in \{ c(\chi), \ldots, m-1 \} \),

\[
(3.11) \quad \langle P^0_{\chi,m}, P^0_{\chi,j} \rangle = \sum_{\ell = c(\chi)}^{j-1} \alpha_{\chi,m;\ell} \overline{\alpha_{\chi,j;\ell}} \beta_{\chi,m;\ell} + \sum_{\ell = c(\chi)}^{m} \alpha_{\chi,m;\ell} \beta_{\chi,m;\ell} = 0.
\]

Using (3.11), we shall prove by induction that

\[
(3.12) \quad \alpha_{\chi,m;\ell} = \begin{cases} 
1 & \text{if } \ell = m, \\
-\frac{\beta_{\chi,m;m}}{\beta_{\chi,m;m-1}} & \text{if } m > c(\chi) \text{ and } \ell = m-1, \\
0 & \text{otherwise},
\end{cases}
\]

which yields (3.5); the identity (3.6) then follows from (3.3) and (3.9). The base case of (3.12) is \( m = c(\chi) \), so that \( \ell = c(\chi) \), in which case we have that \( \alpha_{\chi,c(\chi);c(\chi)} = 1 \) since \( P^0_{\chi,c(\chi)}(e_n) = 1 \). Now we suppose that (3.12) holds with \( m \) replaced by \( j \) for each \( j \in \{ c(\chi), \ldots, m-1 \} \). From (3.11), we have by the induction hypothesis that for \( m \geq c(\chi) + 2 \),

\[
\langle P^0_{\chi,m}, P^0_{\chi,c(\chi)+1} \rangle - \langle P^0_{\chi,m}, P^0_{\chi,c(\chi)+1} \rangle = \alpha_{\chi,m;c(\chi)} \beta_{\chi,m;c(\chi)} \left( 1 - \frac{\beta_{\chi,c(\chi)+1;c(\chi)+1}}{\beta_{\chi,c(\chi)+1;c(\chi)+1}} \right) = 0.
\]
Thus $\alpha_{\chi,m;c(\chi)} = 0$. Proceeding inductively, we conclude that for all $j \in \{c(\chi), \ldots, m - 2\}$,

$$
\langle P_{\chi,m}^o, P_{\chi,j}^o \rangle - \langle P_{\chi,m}^o, P_{\chi,j+1}^o \rangle = \alpha_{\chi,m;j} \beta_{\chi,m;j} \left( 1 - \frac{\beta_{\chi,j+1;j+1}}{\beta_{\chi,j+1;j}} \right) = 0,
$$

so that $\alpha_{\chi,m;j} = 0$. Finally, for $m \geq c(\chi) + 1$, we have that

$$
\langle P_{\chi,m}^o, P_{\chi,m-1}^o \rangle = \alpha_{\chi,m;m-1} \beta_{\chi,m-1} + \beta_{\chi,m;m} = 0,
$$

which yields the remaining case $\ell = m - 1$ of (3.12).

It remains to prove (3.7). It suffices to show that for all $k \in K$,

$$
(3.13) \quad \phi_{\chi,\ell}(e_n k) = \overline{\phi_{\chi,\ell}(e_n k^{-1})}
$$

for each nonnegative integer $\ell \geq c(\chi)$, since $P_{\chi,m}^o$ may be written as a linear combination of such functions. The identity (3.13) is clear if $k \notin K_0(p^d)$, for then both sides are zero. If $k \in K_0(p^d)$, so that $k^{-1} \in K_0(p^d)$ as well, then $e_n k^t e_n k^{-1} = 1 \in p^d$ as $kk^{-1} = 1_n$, and consequently $\chi(e_n k^t e_n) = \overline{\chi(e_n k^{-1} e_n)}$ since $\ell \geq c(\chi)$, which yields the result. \hfill \Box

The zonal spherical function $P_{\chi,m}^o$ is useful for understanding various properties of the space $\mathcal{H}_{\chi,m}(S^{n-1})$. Our first application is the following lemma, which may be thought of as the addition theorem for $\mathcal{H}_{\chi,m}(S^{n-1})$.

Lemma 3.14. Let $\{Q_j\}$ be an orthonormal basis of $\mathcal{H}_{\chi,m}(S^{n-1})$. Then for any $x \in S^{n-1}$ and $k \in K$, we have that

$$
(3.15) \quad \dim \mathcal{H}_{\chi,m}(S^{n-1}) \sum_{j=1}^{\dim \mathcal{H}_{\chi,m}(S^{n-1})} Q_j(x)\overline{Q_j(e_n k)} = \dim \mathcal{H}_{\chi,m}(S^{n-1}) P_{\chi,m}^o(xk^{-1}).
$$

Proof. Since

$$
(3.16) \quad Q(xk) = (\tau_{\chi,m}(k) \cdot Q)(x) = \sum_{j=1}^{\dim \mathcal{H}_{\chi,m}(S^{n-1})} (\tau_{\chi,m}(k) \cdot Q, Q_j) Q_j(x)
$$

for any $Q \in \mathcal{H}_{\chi,m}(S^{n-1})$ and $x \in S^{n-1}$, we have that for any $k \in K$,

$$
(3.17) \quad \delta_{\ell_1,\ell_2} = \langle Q_{\ell_1}, Q_{\ell_2} \rangle = \langle \tau_{\chi,m}(k^{-1}) \cdot Q_{\ell_1}, \tau_{\chi,m}(k^{-1}) \cdot Q_{\ell_2} \rangle = \sum_{j_1,j_2=1}^{\dim \mathcal{H}_{\chi,m}(S^{n-1})} \langle \tau_{\chi,m}(k^{-1}) \cdot Q_{\ell_1}, Q_{j_2} \rangle \langle Q_{j_2}, \tau_{\chi,m}(k^{-1}) \cdot Q_{\ell_2} \rangle \langle Q_{j_1}, Q_{j_2} \rangle
$$

$$
= \sum_{j=1}^{\dim \mathcal{H}_{\chi,m}(S^{n-1})} \langle \tau_{\chi,m}(k^{-1}) \cdot Q_{\ell_1}, Q_{j} \rangle \langle Q_{j}, \tau_{\chi,m}(k^{-1}) \cdot Q_{\ell_2} \rangle
$$

$$
= \sum_{j=1}^{\dim \mathcal{H}_{\chi,m}(S^{n-1})} \langle Q_{\ell_1}, \tau_{\chi,m}(k) \cdot Q_{j} \rangle \langle \tau_{\chi,m}(k) \cdot Q_{j}, Q_{\ell_2} \rangle.
$$

For fixed $k \in K$, we now define $Q_{e_n k} \in \mathcal{H}_{\chi,m}(S^{n-1})$ by

$$
Q_{e_n k}(x) := \sum_{j=1}^{\dim \mathcal{H}_{\chi,m}(S^{n-1})} Q_j(x)\overline{Q_j(e_n k)}.
$$
Then from (3.16) and (3.17), we have that for all $k_1 \in K$,
\[
Q_{e_n k_1}(x_{k_1}) = \sum_{j=1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} Q_j(x_{k_1}) \overline{Q_j(e_n k_1)}
\]
\[
= \sum_{j, \ell_1, \ell_2 = 1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} (\tau_{x,m}(k_1) \cdot Q_j, Q_{\ell_1}) \langle Q_{\ell_2}, \tau_{x,m}(k_1) \cdot Q_j \rangle Q_{\ell_1}(x) \overline{Q_{\ell_2}(e_n k)}
\]
\[
= \sum_{\ell = 1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} Q_\ell(x) \overline{Q_\ell(e_n k)}
\]
\[
= Q_{e_n k}(x).
\]
We use (3.18) with $x$ replaced by $x k$ and $k_1$ replaced by $k^{-1} k'$ for $k' \in K_{n-1,1}$. Since $e_n k' = e_n$, we deduce that
\[
(\tau_{x,m}(k') \cdot (\tau_{x,m}(k) \cdot Q_{e_n k})) (x) = Q_{e_n k}(x k' k) = Q_{e_n k}(x k) = (\tau_{x,m}(k) \cdot Q_{e_n k})(x)
\]
for all $k' \in K_{n-1,1}$. As $\mathcal{H}_{x,m}(S^{n-1})_{K_{n-1,1}}$ is one-dimensional, it follows that $\tau_{x,m}(k) \cdot Q_{e_n k}$ must be a constant multiple of $P_{x,m}^n$, and this constant is readily seen to be $Q_{e_n k}(e_n k)$ upon taking $x = e_n$. So for all $x \in S^{n-1}$ and $k \in K$,
\[
\sum_{j=1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} Q_j(x) \overline{Q_j(e_n k)} = Q_{e_n k}(x)
\]
\[
= (\tau_{x,m}(k) \cdot Q_{e_n k})(x k^{-1})
\]
\[
= Q_{e_n k}(e_n k) P_{x,m}^n(x k^{-1})
\]
\[
= \sum_{j=1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} |Q_j(e_n k)|^2 P_{x,m}^n(x k^{-1}).
\]
It remains to show that for all $k \in K$,
\[
\sum_{j=1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} |Q_j(e_n k)|^2 = \dim \mathcal{H}_{x,m}(S^{n-1}).
\]
To prove (3.19), we take $x = e_n k$ and $k_1 = k^{-1}$ in (3.18) in order to see that
\[
\sum_{j=1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} |Q_j(e_n k)|^2 = \sum_{j=1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} |Q_j(e_n)|^2
\]
for all $k \in K$ and hence the left-hand side is a constant. Integrating over $K \ni k$, we find that
\[
\sum_{j=1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} |Q_j(e_n k)|^2 = \int_K \sum_{j=1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} |Q_j(e_n k)|^2 dk
\]
\[
= \sum_{j=1}^{\dim \mathcal{H}_{x,m}(S^{n-1})} \langle Q_j, Q_j \rangle
\]
\[
= \dim \mathcal{H}_{x,m}(S^{n-1}).
\]
A simple consequence of Lemma 3.14 is the following result showing that certain matrix coefficients of $\tau_{x,m}$ are themselves elements of $\mathcal{H}_{x,m}(S^{n-1})$.

**Corollary 3.20.** The reproducing kernel for $\mathcal{H}_{x,m}(S^{n-1})$ is $(\dim \mathcal{H}_{x,m}(S^{n-1})) P_{x,m}^n$, so that for all $P \in \mathcal{H}_{x,m}(S^{n-1})$ and $k \in K$,
\[
P(e_n k) = \dim \mathcal{H}_{x,m}(S^{n-1}) \langle \tau(k) \cdot P, P_{x,m}^n \rangle.
\]
In particular,
\[(3.22) \quad \langle P_{\chi,m}^0, P_{\chi,m}^0 \rangle = \frac{1}{\dim H_{\chi,m}(S^{n-1})}.\]

One can also show (3.22) in a more direct fashion by combining (3.3), (3.5), and (3.6).

**Proof.** From (3.15) and (3.16), we have that
\[
P(e_n k) = \sum_{j=1}^{\dim H_{\chi,m}(S^{n-1})} \langle P, Q_j \rangle Q_j(e_n k)
\]
\[
= \langle P, \sum_{j=1}^{\dim H_{\chi,m}(S^{n-1})} Q_j Q_j(e_n k) \rangle
\]
\[
= \langle P, \dim H_{\chi,m}(S^{n-1}) \tau(k^{-1}) \cdot P_{\chi,m}^0 \rangle
\]
\[
= \dim H_{\chi,m}(S^{n-1}) \langle \tau(k) \cdot P, P_{\chi,m}^0 \rangle.
\]
The identity (3.22) then follows upon taking \(P = P_{\chi,m}^0\) and \(k = 1_n\) since \(P_{\chi,m}(e_n) = 1\).

\[\square\]

4. THE NEWFORM \(K\)-TYPE

4.1. The Newform and the Conductor Exponent. Let \((\pi, V_\pi)\) be an induced representation of Langlands type of \(GL_n(F)\). Thus there exist positive integers \(n_1, \ldots, n_r\) for which \(n_1 + \cdots + n_r = n\) and essentially square-integrable representations \((\pi_j, V_{\pi_j})\) of \(GL_{n_j}(F)\) of the form \(\sigma_j \otimes |\det|^s\), where \(\sigma_j\) is square-integrable and \(t_j \in \mathbb{C}\) satisfies \(\Re(t_1) \geq \cdots \geq \Re(t_r)\), such that
\[
\pi = \Ind_{P(F)}^{GL_n(F)} (\bigotimes_{j=1}^r \pi_j),
\]
the representation obtained by normalised parabolic induction from the standard upper parabolic subgroup \(P(F) = P_{(n_1, \ldots, n_r)}(F)\) of \(GL_n(F)\). One can take as a model for \(\pi\) the space of smooth functions \(f : GL_n(F) \to V_{\pi_1} \otimes \cdots \otimes V_{\pi_r}\), upon which \(\pi\) acts via right translations, that satisfy
\[
f(umg) = \prod_{j=1}^r |\det m_j|^{\frac{1}{2} (n_2 - 2(n_1 + \cdots + n_{j-1} - n_j) - n_j)} \left(\bigotimes_{j=1}^r \pi_j(m_j) \cdot f\right)(g)
\]
for all \(u \in N_P(F)\), the unipotent radical of \(P(F)\), \(m = \text{blockdiag}(m_1, \ldots, m_r) \in M_{p}(F)\), the Levi subgroup of \(P(F)\), and \(g \in GL_n(F)\). We note that every generic irreducible admissible smooth representation of \(GL_n(F)\) is isomorphic to some induced representation of Langlands type; see, for example, [JS83].

For each nonnegative integer \(m\) and each character \(\chi \in \widehat{O}^\times\) for which \(0 \leq c(\chi) \leq m\), we define the subspaces
\[
V^{K_1(p^m)}_{\pi} := \{v \in V_\pi : \pi(k) \cdot v = v \text{ for all } k \in K_1(p^m)\},
\]
\[
V^{K_0(p^m),\chi}_{\pi} := \{v \in V_\pi : \pi(k) \cdot v = \psi_\chi(k) v \text{ for all } k \in K_0(p^m)\}.
\]
The former is the subspace of \(V_\pi\) of \(K_1(p^m)\)-invariant vectors; the latter is the subspace of \((K_0(p^m), \psi_\chi)\)-equivariant vectors.

**Lemma 4.1.** Let \((\pi, V_\pi)\) be an induced representation of Langlands type of \(GL_n(F)\). For each nonnegative integer \(m\), the subspace \(V^{K_1(p^m)}_{\pi}\) is the image of the projection map \(\Pi^{K_1(p^m)} : V_\pi \to V_\pi\) given by
\[
(4.2) \quad \Pi^{K_1(p^m)}(v) := \frac{1}{\text{vol}(K_1(p^m))} \int_{K_1(p^m)} \pi(k) \cdot v \, dk,
\]
while $V^K_{\pi_0}(p^m)_{\chi_{\pi}}$ is the image of the projection map $\Pi^K_{\pi_0}(p^m)_{\chi_{\pi}} : V_\pi \to V_\pi$ given by

$$
(4.3) \quad \Pi^K_{\pi_0}(p^m)_{\chi_{\pi}}(v) := \begin{cases} 
\int_{J_K} \pi(k) \cdot v \, dk 
& \text{if } m = c(\chi_{\pi}) = 0, \\
\frac{1}{\text{vol}(K_0(p^m))} \int_{K_0(p^m)} \chi_{\pi}(e_{n} k^t e_{n}) \pi(k) \cdot v \, dk 
& \text{if } m > 0.
\end{cases}
$$

Finally, we have that

$$
(4.4) \quad V^K_{\pi_0}(p^m)_{\chi_{\pi}} = \begin{cases} 
V^K_{\pi_1}(p^m) 
& \text{if } \chi = \chi_{\pi}, \\
\{0\} 
& \text{otherwise},
\end{cases}
$$

where $\chi_{\pi} := \omega_{\pi}|_{O_x}$ with $\omega_{\pi} : F^\times \to \mathbb{C}^\times$ the central character of $\pi$.

**Proof.** The two identities $\Pi^K_{\pi}(p^m)(V_\pi) = V^K_{\pi_1}(p^m)$ and $\Pi^K_{\pi_0}(p^m)_{\chi_{\pi}}(V_\pi) = V^K_{\pi_0}(p^m)_{\chi_{\pi}}$ are clear. Since $Z(O)K_1(p^m) = K_0(p^m)$ and $\pi|_{Z(O)} = \omega_{\pi}|_{O_x} = \chi_{\pi}$, the identities $V^K_{\pi_0}(p^m)_{\chi_{\pi}} = V^K_{\pi_1}(p^m)$ and $V^K_{\pi_0}(p^m)_{\chi_{\pi}} = \{0\}$ for $\chi \neq \chi_{\pi}$ then follow. $\square$

A fundamental result concerning induced representations of Langlands type is the existence of a newform. Jacquet, Piatetski-Shapiro, and Shalika [JP-SS81] have shown that each induced representation of Langlands type $(\pi, V_\pi)$ of $GL_n(F)$ contains a distinguished vector $v^0 \in V_\pi$, the newform, whose complexity is measured in a natural way by a nonnegative integer $c(\pi)$, the conductor exponent. This generalises a result of Casselman [Cas73a], who proved this for $n = 2$, and observed that when $F = \mathbb{Q}_p$ and $\pi$ is the local component of an automorphic representation of $GL_2(\mathbb{A}_Q)$, this is the adèlic reformulation of the Atkin–Lehner theory of newforms for classical modular forms [AL70].

**Theorem 4.5 ([JP-SS81, Théorème (5)]).** Let $(\pi, V_\pi)$ be an induced representation of Langlands type of $GL_n(F)$. There exists a minimal nonnegative integer $m = c(\pi)$ for which the space $V^K_{\pi_1}(p^m) = V^K_{\pi_0}(p^m)_{\chi_{\pi}}$ is nontrivial, in which case it is one-dimensional.

**Definition 4.6.** The nonzero vector $v^0 \in V^K_{\pi_1}(p^m) = V^K_{\pi_0}(p^m)_{\chi_{\pi}}$, unique up to scalar multiplication, is called the newform of $\pi$. The nonnegative integer $c(\pi)$ is called the conductor exponent of $\pi$. Elements of $V^K_{\pi_0}(p^m)_{\chi_{\pi}}$ for $m > c(\pi)$ are called oldforms.

**Remark 4.7.** The proof of [JP-SS81, Théorème (5)] contains a gap; correct proofs were later independently given by Jacquet [Jac12, Theorem 1] and Matringe [Mat13, Corollary 3.3].

**Remark 4.8.** As proven in [JP-SS81], there are other ways to characterise the newform and the conductor exponent of $\pi$ instead of in terms of the subspace $V^K_{\pi_1}(p^m_{(\pi)})$ of $K_1(p^m_{(\pi)})$-invariant vectors in $V_\pi$. The conductor exponent $c(\pi)$ is precisely the nonnegative integer for which the epsilon factor $\varepsilon(s, \pi, \psi)_{\pi}$ associated to $\pi$ is of the form

$$
\varepsilon(s, \pi, \psi) = \varepsilon \left( \frac{1}{2}, \pi, \psi \right) q^{-c(\pi)\left(s - \frac{1}{2}\right)},
$$

where $\psi : F \to \mathbb{C}^\times$ is an unramified additive character of $F$. The newform is such that when viewed in the Whittaker model $W(\pi, \psi)$ of $\pi$, it is the unique Whittaker function $W^o$ satisfying $W^o \left( g \begin{pmatrix} k' & 0 \\ 0 & 1 \end{pmatrix} \right) = W^o(g)$ for all $g \in GL_n(F)$ and $k' \in K_{n-1}$ that is a test vector for the local $GL_n \times GL_{n-1}$ Rankin–Selberg integral whenever the second representation is unramified, so that

$$
\int_{N_{n-1}(F) \backslash GL_{n-1}(F)} W^o \left( g \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) W^{o\prime}(g) |\det g|^{s - \frac{1}{2}} \, dg = L(s, \pi \times \pi'),
$$

for all spherical induced representations of Langlands type $\pi'$ of $GL_{n-1}(F)$ with spherical Whittaker function $W^{o\prime} \in \mathcal{W}(\pi', \psi')$ normalised such that $W^{o\prime}(1_{n-1}) = 1$. 

4.2. The Newform $K$-Type. We shall show that the newform lies in a distinguished $K$-type of $\pi$. To begin, for each irreducible smooth representation $\tau \in \hat{K}$, we define the projection map $\Pi^r : V_\pi \to V_\pi$ by

$$\Pi^r(v) := \int_K \xi^r(k)\pi(k) \cdot v \, dk,$$

where

$$\xi^r(k) := (\dim \tau) \text{Tr}(k^{-1})$$

is the elementary idempotent associated to $\tau$. The image of $V_\pi$ under $\Pi^r$ is the $\tau$-isotypic subspace $V^r_\pi$ of $V_\pi$, which is finite-dimensional since $\pi$ is admissible. We say that $\tau$ is a $K$-type of $\pi$ if $\text{Hom}_K(\tau, \pi|_K)$ is nontrivial, in which case $\dim V^r_\pi = \dim \tau \dim \text{Hom}_K(\tau, \pi|_K) > 0$, and we call $\dim \text{Hom}_K(\tau, \pi|_K)$ the multiplicity of $\tau$ in $\pi$.

In general, the $K$-type decomposition of an induced representation of Langlands type $\pi$ is not known except in special cases. For $n = 2$, this follows from work of Casselman [Cas73b], Silberger [Sil70], and Hansen [Han87]; for $n = 3$ and $\pi$ a principal series representation, this problem has been studied by Campbell and Nevins [CN09, CN10] and Onn and Singla [OS14].

To do so, we define the projection map $\Pi^{K_{n-1,1}} : V_\pi \to V_\pi$ given by

$$\Pi^{K_{n-1,1}}(v) := \int_{K_{n-1,1}} \pi(k') \cdot v \, dk',$$

where $dk'$ denotes the Haar probability measure on the compact group $K_{n-1,1}$, so that the image of $\Pi^{K_{n-1,1}}$ is the subspace of $K_{n-1,1}$-invariant vectors. The composition of the two projections $\Pi^r$ and $\Pi^{K_{n-1,1}}$ is the projection

$$(4.9) \quad \left(\Pi^{r|K_{n-1,1}}\right)(v) := \left(\Pi^{K_{n-1,1}} \circ \Pi^r\right)(v) = \left(\Pi^r \circ \Pi^{K_{n-1,1}}\right)(v) = \int_K \xi^{r|K_{n-1,1}}(k)\pi(k) \cdot v \, dk$$

onto the subspace of $K_{n-1,1}$-invariant $\tau$-isotypic vectors

$$V^{r|K_{n-1,1}}_\pi := \left(\Pi^{r|K_{n-1,1}}\right)(V_\pi) = \{v \in V^r_\pi : \pi(k') \cdot v = v \text{ for all } k' \in K_{n-1,1}\}.$$

Here

$$\xi^{r|K_{n-1,1}}(k) := \int_{K_{n-1,1}} \xi^r(kk') \, dk' = \int_{K_{n-1,1}} \xi^r(1 k') \, dk'.$$

This function has a particularly simple description.

**Lemma 4.10.** For $\tau \in \hat{K}$ and $k \in K$, we have that

$$\xi^{r|K_{n-1,1}}(k) = \begin{cases} 
(\dim \tau_\chi m) P^\circ_\chi m(e_n k^{-1}) & \text{if } \tau = \tau_\chi m \text{ for some } \chi \in \hat{O}^\times \text{ and } m \geq c(\chi), \\
0 & \text{otherwise}.
\end{cases}$$

**Proof.** This follows from Corollaries 2.18 and 3.20. \qed

From Lemma 4.10, we have the following simple consequences.

**Corollary 4.11.** We have that

$$(4.12) \quad \sum_{\ell = 0}^m \sum_{0 \leq c(\chi) \leq \ell} \xi^{r_\chi m|K_{n-1,1}}(k) = \begin{cases} 
1 & \text{if } k \in K_1(p^m), \\
\frac{\text{vol}(K_1(p^m))}{\text{vol}(K(p^m))} & \text{otherwise},
\end{cases}$$

$$(4.13) \quad \sum_{\ell = c(\chi)}^m \xi^{r_\chi m|K_{n-1,1}}(k) = \begin{cases} 
\frac{1}{\text{vol}(K_0(p^m))} \sum(e_n k^t e_n) & \text{if } m = c(\chi) = 0 \text{ and } k \in K, \\
0 & \text{otherwise}.
\end{cases}$$
Consequently, for an induced representation of Langlands type \((\pi, V_\pi)\) of \(\text{GL}_n(F)\), we have that

\[
\Pi^K_1(p^m) = \sum_{\ell=0}^{m} \sum_{\chi \in \hat{O}^\times} \Pi^\chi_{x,\ell}|_{K_n-1,1},
\]

\[
\Pi^K_0(p^m),\chi = \sum_{\ell=c(\chi)}^{m} \Pi^\chi_{x,\ell}|_{K_n-1,1}
\]

for any nonnegative integer \(m\) and any character \(\chi \in \hat{O}^\times\) for which \(0 \leq c(\chi) \leq m\).

**Proof.** The identity (4.13) can be seen by combining Proposition 3.4 and Lemmata 2.13 and 4.10, noting that \(1/\text{vol}(K_0(p^m)) = [K : K_0(p^m)]\). The identity (4.12) follows from (4.13) together with character orthogonality, namely the fact that for \(x \in \hat{O}^\times\),

\[
\sum_{\chi \in \hat{O}^\times} \chi(x) = \begin{cases} 
\#\hat{O}^\times/(1 + p^m) & \text{if } x - 1 \in p^m, \\
0 & \text{otherwise},
\end{cases}
\]

and noting that \(\#\hat{O}^\times/(1 + p^m) = [K_0(p^m) : K_1(p^m)]\). The identities (4.14) and (4.15) for the projections \(\Pi^K_1(p^m)\) and \(\Pi^K_0(p^m),\chi\) in (4.2) and (4.3) in terms of the projections \(\Pi^\chi_{x,\ell}|_{K_n-1,1}\) in (4.9) are immediate consequences of (4.12) and (4.13). \(\square\)

Finally, for any nonnegative integer \(m\), we set

\[
V^K_{\pi, n-1,1}(m) := \bigoplus_{\tau \in \hat{K}} V^\tau_{\pi}|_{K_n-1,1},
\]

the subspace of \(K_{n-1,1}\)-invariant vectors that are \(\tau\)-isotypic for some \(\tau \in \hat{K}\) of level \(c(\tau) = m\); here the level \(c(\tau)\) of \(\tau\) is defined to be the minimal nonnegative integer \(m\) for which \(\ker \tau\) contains \(K(p^m)\), which is necessarily finite since \(\tau\) is smooth. We now show that the newform and the conductor exponent may be characterised in terms of \(V^K_{\pi, n-1,1}(m)\).

**Theorem 4.16.** Let \((\pi, V_\pi)\) be an induced representation of Langlands type of \(\text{GL}_n(F)\). For any nonnegative integer \(m\), we have that

\[
\dim V^K_{\pi, n-1,1}(m) = \begin{cases} 
(m - c(\pi) + n - 2) & \text{if } m \geq c(\pi), \\
0 & \text{otherwise},
\end{cases}
\]

In particular, the minimal nonnegative integer \(m\) for which \(V^K_{\pi, n-1,1}(m)\) is nontrivial is \(m = c(\pi)\). Furthermore, we have that

\[
V^K_{\pi, n-1,1}(m) = \begin{cases} 
V^\tau_{\pi}|_{K_n-1,1} & \text{if } m \geq c(\pi), \\
\{0\} & \text{otherwise},
\end{cases}
\]

and that

\[
V^K_{\pi, n-1,1}(m) = \bigoplus_{\ell=0}^{m} V^\chi_{x,\ell}|_{K_n-1,1} = \begin{cases} 
\bigoplus_{\ell=c(\chi)}^{m} V^\chi_{x,\ell}|_{K_n-1,1} & \text{if } m \geq c(\pi), \\
\{0\} & \text{otherwise}.
\end{cases}
\]

In particular,

\[
V^K_{\pi, n-1,1}(c(\pi)) = V^\tau_{\pi}|_{K_0(p^c(\pi)),\chi} = \bigoplus_{\ell=c(\pi)}^{m} V^\chi_{x,\ell}|_{K_n-1,1}.
\]

**Definition 4.21.** We call \(\tau_{\chi, c(\pi)}\) the newform \(K\)-type of \(\pi\).
Remark 4.22. This gives alternative characterisations of the newform and conductor exponent of $\pi$: we may define the conductor exponent $c(\pi)$ of $\pi$ to be the minimal nonnegative integer $m$ for which the space $V_\pi^{K_{n-1,1}(m)} = V_\pi^{T_{\chi,m}|K_{n-1,1}}$ is nontrivial, while we may define the newform to be the nonzero vector, unique up to scalar multiplication, lying in $V_\pi^{K_{n-1,1}(c(\pi))}$. Of course, this definition is somewhat circular in practice, since, as we shall shortly see, we use Theorem 4.5 (or rather a consequence thereof) in order to prove Theorem 4.16.

Theorem 4.16 gives additional information about the newform not immediately apparent from Theorem 4.5: not only is the newform the unique vector, up to scalar multiplication, that is $K_1(p^{c(\pi)})$-invariant (or, equivalently, $(K_0(p^{c(\pi)}), \psi_{\chi,\pi})$-equivariant), it is also the unique vector, up to scalar multiplication, that is $K_{n-1,1}$-invariant and $\tau_{\chi,\pi}$-isotypic; moreover, there are no nontrivial $K_{n-1,1}$-invariant $\tau_{\chi,m}$-isotypic vectors with $m < c(\pi)$ or $\chi \neq \chi_\pi$. We have also shown that the space $V_\pi^{K_1(p^m)}$ of oldforms of level $m \geq c(\pi)$ decomposes into the direct sum of the subspaces of $K_{n-1,1}$-invariant $\tau_{\chi,m}$-isotypic vectors in $V_\pi$ with $\ell \in \{c(\pi), \ldots, m\}$, each of which has dimension $(\ell-c(\pi)+n-2)$ respectively.

Proof of Theorem 4.16. We first note that $V_\pi^{T_{\chi_1,m_1}|K_{n-1,1}}$ and $V_\pi^{T_{\chi_2,m_2}|K_{n-1,1}}$ are mutually orthogonal whenever either $\chi_1 \neq \chi_2$ or $m_1 \neq m_2$. In conjunction with (4.4) and (4.15), this implies that

$$V_\pi^{K_1(p^m)} = V_\pi^{K_0(p^m), \chi_\pi} = \begin{cases} \bigoplus_{\ell = c(\chi_\pi)}^m V_\pi^{T_{\chi_\pi,\ell}|K_{n-1,1}} & \text{if } m \geq c(\chi_\pi), \\ \{0\} & \text{otherwise.} \end{cases}$$

Since $V_\pi^{K_1(p^m)} = V_\pi^{K_0(p^m), \chi_\pi} = \{0\}$ whenever $m < c(\pi)$ from Theorem 4.5, we deduce that

$$V_\pi^{T_{\chi,m}|K_{n-1,1}} = \{0\} \quad \text{whenever } c(\chi_\pi) \leq m < c(\pi),$$

noting that $c(\chi_\pi) = c(\omega_\pi) \leq c(\pi)$, from which (4.19) and (4.20) both follow.

Next, we have from Lemma 4.10 and the fact that $\pi|_{\mathbb{Z}(\mathcal{O})} = \chi_\pi$ that

$$V_\pi^{K_{n-1,1}(m)} = \begin{cases} V_\pi^{T_{\chi,m}|K_{n-1,1}} & \text{if } m \geq c(\chi_\pi), \\ \{0\} & \text{otherwise.} \end{cases}$$

Together with (4.23), we deduce (4.18).

Finally, from [Ree91, Theorem 1] (which in turn relies on Theorem 4.5), we have that

$$\dim V_\pi^{K_1(p^m)} = \dim V_\pi^{K_0(p^m), \chi_\pi} = \begin{cases} \binom{m-c(\pi)+n-1}{n-1} & \text{if } m \geq c(\pi), \\ 0 & \text{otherwise.} \end{cases}$$

The identity (4.17) then follows from (4.18), (4.19), and induction together with the fact that

$$\sum_{\ell = c(\pi)}^m \left( \binom{\ell-c(\pi)+n-2}{n-2} \right) = \left( \binom{m-c(\pi)+n-1}{n-1} \right). \quad \square$$

We may use Theorem 4.16 to tell us the multiplicity with which an irreducible smooth representations $\tau_{\chi,m} \in \hat{K}$ occurs in the restriction of $\pi$ to $K$; when $\chi_\pi$ is trivial, this was observed by Reeder [Ree94] to follow upon combining results from [Hil94] and [Ree91].

Proposition 4.25. Let $(\pi, V_\pi)$ be an induced representation of Langlands type of $\text{GL}_n(F)$. For any nonnegative integer $m$ and any character $\chi \in \overline{O}\times$ for which $0 \leq c(\chi) \leq m$, we have that

$$\dim \text{Hom}_K(\tau_{\chi,m}, \pi|_K) = \begin{cases} \binom{m-c(\pi)+n-2}{n-2} & \text{if } m \geq c(\pi) \text{ and } \chi = \chi_\pi, \\ 0 & \text{otherwise.} \end{cases}$$
While this can be proved directly using Theorem 4.16, we give an alternate proof via Frobenius reciprocity.

**Proof.** It is clear that \( \text{Hom}_K(\tau_{x,m}, \pi|_K) \) is trivial if \( \chi \neq \chi_\pi \). For \( \chi = \chi_\pi \), by (2.12) and Frobenius reciprocity, we have that

\[
\bigoplus_{\ell = c(\chi_\pi)} \text{Hom}_K(\tau_{x,\ell}, \pi|_K) \cong \text{Hom}_K \left( \text{Ind}_{K_0(p^m)}^K(\psi_{x,\pi}) \right) \cong \text{Hom}_{K_0(p^m)} \left( \psi_{x,\pi}|_{K_0(p^m)} \right).
\]

Since \( \text{Hom}_{K_0(p^m)}(\psi_\chi, \pi|_{K_0(p^m)}) \) is trivial whenever \( \chi \neq \chi_\pi \), this in turn is isomorphic to

\[
\bigoplus_{\chi \in \mathbb{O}_S^\times \atop 0 \leq c(\chi) \leq m} \text{Hom}_{K_0(p^m)}(\psi_\chi, \pi|_{K_0(p^m)}) \cong \text{Hom}_{K_1(p^m)}(1, \pi|_{K_1(p^m)})
\]

by (2.10) and Frobenius reciprocity. By [Ree91, Theorem 1], we have that

\[
\dim \text{Hom}_{K_1(p^m)}(1, \pi|_{K_1(p^m)}) = \begin{cases} 
(m - c(\pi) + n - 1) / n - 1 & \text{if } m \geq c(\pi), \\
0 & \text{otherwise},
\end{cases}
\]

It follows that

\[
\sum_{\ell = c(\chi_\pi)}^m \dim \text{Hom}_K(\tau_{x,\ell}, \pi|_K) = \begin{cases} 
(m - c(\pi) + n - 1) / n - 1 & \text{if } m \geq c(\pi), \\
0 & \text{otherwise},
\end{cases}
\]

from which the result via induction in conjunction with the identity (4.24). \( \square \)

### 4.3. Matrix Coefficients

We now study the properties of certain matrix coefficients of \( \pi \) via matrix coefficients of \( \mathcal{H}_{\chi_n,c(\pi)}(S^{n-1}) \). In order to do so, we explicitly identify \( V^\tau_{\chi_n,c(\pi)} \) with \( \mathcal{H}_{\chi_n,c(\pi)}(S^{n-1}) \). We first observe an identity for the newform.

**Lemma 4.26.** The newform \( v^0 \in V^\tau_{\chi_n,c(\pi)} \) of an induced representation of Langlands type \( (\pi, V_\pi) \) satisfies

\[
(4.27) \quad v^0 = \dim \tau_{x_n,c(\pi)} \int K P^\circ_{x_n,c(\pi)}(e_n k^{-1}) \pi(k) \cdot v^0 \, dk.
\]

**Proof.** This is an immediate consequence of Lemma 4.10 and Theorem 4.16. \( \square \)

The finite-dimensional vector space \( V^\tau_{\chi_n,c(\pi)} \) admits a \( K \)-invariant inner product \( \langle \cdot, \cdot \rangle \). We use such an inner product to construct an isomorphism between \( \mathcal{H}_{\chi_n,c(\pi)}(S^{n-1}) \) and \( V^\tau_{\chi_n,c(\pi)} \).

**Proposition 4.28.** Let \( (\pi, V_\pi) \) be an induced representation of Langlands type with newform \( v^0 \in V^\tau_{\chi_n,c(\pi)} \). An explicit isomorphism between \( \mathcal{H}_{\chi_n,c(\pi)}(S^{n-1}) \ni P = P_\pi \) and \( V^\tau_{\chi_n,c(\pi)} \ni v = v_\pi \) is given by

\[
(4.29) \quad v = \dim \tau_{x_n,c(\pi)} \int K P(e_n k^{-1}) \pi(k) \cdot v^0 \, dk.
\]

For nonzero \( v \in V^\tau_{\chi_n,c(\pi)} \), the inverse is given by

\[
(4.30) \quad P(x) = \left( \dim \tau_{x_n,c(\pi)} \right)^2 \left\langle P, P \right\rangle \int K \frac{\pi(k^{-1}) \cdot v, v^0}{\left\langle v, v \right\rangle} \int K \tau_{x_n,c(\pi)}(k) \cdot P^\circ_{x_n,c(\pi)}(k) \, dk.
\]

Moreover, for all \( P \in \mathcal{H}_{\chi_n,c(\pi)}(S^{n-1}) \) and \( v \in V^\tau_{\chi_n,c(\pi)} \) that are associated via (4.29) and (4.30), we have that

\[
(4.31) \quad \left\langle P, P \right\rangle = \frac{\left\langle v, v \right\rangle P(e_n k^{-1})}{\dim \tau_{x_n,c(\pi)}}.
\]
Proof. We first confirm that for $v$ as in (4.29), we have that $\Pi^\chi_{\pi,c}(v) = v$, so that $v \in V^\chi_{\pi,c}(\pi)$. Indeed, from (4.29),
\[
\Pi^\chi_{\pi,c}(v) = \int_K \xi^\chi_{\pi,c}(k) \pi(k) \cdot v \, dk_1
\]
\[
= \dim \tau_{\chi,c(\pi)} \int_K \xi^\chi_{\pi,c(\pi)}(k) \int_K P(e_n k_2^{-1}) \pi(k_1 k_2) \cdot v^\circ \, dk_2 \, dk_1
\]
\[
= \dim \tau_{\chi,c(\pi)} \int_K \pi(k_2) \cdot v^\circ \int_K \xi^\chi_{\pi,c(\pi)}(k_1) \left( \tau_{\chi,c(\pi)}(k_1) \cdot P \right) (e_n k_2^{-1}) \, dk_1 \, dk_2
\]
\[
= \dim \tau_{\chi,c(\pi)} \int_K P(e_n k^{-1}) \pi(k) \cdot v^\circ \, dk
\]
\[
= v.
\]
Next, we show that (4.30) follows from (4.31). By (3.16), (3.21), (3.22), and Schur orthogonality, we have that for any orthonormal basis $\{Q_j\}$ of $\mathcal{H}_{\chi_{\pi,c}(\pi)}(S^{n-1})$,
\[
P(x) = \sum_{j=1}^{\dim \mathcal{H}_{\chi_{\pi,c}(\pi)}(S^{n-1})} \langle P, Q_j \rangle Q_j(x)
\]
\[
= \sum_{j=1}^{\dim \mathcal{H}_{\chi_{\pi,c}(\pi)}(S^{n-1})} Q_j(x) \left( \dim \tau_{\chi_{\pi,c}(\pi)} \right)^2
\]
\[
\times \int_K \left\langle \tau_{\chi,c(\pi)}(k) \cdot P_{\chi_{\pi,c}(\pi)}^o, Q_j \right\rangle \left\langle \tau_{\chi,c(\pi)}(k^{-1}) \cdot P, P_{\chi_{\pi,c}(\pi)}^o \right\rangle \, dk
\]
\[
= \dim \tau_{\chi,c(\pi)} \int_K P_{\chi_{\pi,c}(\pi)}^o(xk) P(e_n k^{-1}) \, dk.
\]
Assuming (4.31), this is precisely (4.30).

It remains to prove (4.31). We first note that
\[
v = \left( \dim \tau_{\chi_{\pi,c}(\pi)} \right)^2 \int_K \left\langle \tau_{\chi,c(\pi)}(k^{-1}) \cdot P, P_{\chi_{\pi,c}(\pi)}^o \right\rangle \pi(k) \cdot v^\circ \, dk
\]
via (3.21) and (4.29). By (3.21), (4.29), (4.32), and Schur orthogonality, we therefore have that
\[
\langle P, P \rangle v = \left( \dim \tau_{\chi_{\pi,c}(\pi)} \right)^2 \int_K \left\langle \tau_{\chi,c(\pi)}(k^{-1}) \cdot P, P_{\chi_{\pi,c}(\pi)}^o \right\rangle \pi(k) \cdot v^\circ \, dk
\]
\[
= \left( \dim \tau_{\chi_{\pi,c}(\pi)} \right)^3 \int_K \pi(k_1) \cdot v^\circ
\]
\[
\times \int_K \left\langle \tau_{\chi,c(\pi)}(k_2) \cdot \tau_{\chi,c(\pi)}(k_1^{-1}) \cdot P, P \right\rangle \left\langle \tau_{\chi,c(\pi)}(k_2^{-1}) \cdot P, P_{\chi_{\pi,c}(\pi)}^o \right\rangle \, dk_2 \, dk_1
\]
\[
= \left( \dim \tau_{\chi_{\pi,c}(\pi)} \right)^3 \int_K \left\langle \tau_{\chi,c(\pi)}(k_1^{-1}) \cdot P, P \right\rangle
\]
\[
\times \int_K \left\langle \tau_{\chi,c(\pi)}(k_2^{-1}) \cdot P, P_{\chi_{\pi,c}(\pi)}^o \right\rangle \pi(k_1 k_2) \cdot v^\circ \, dk_2 \, dk_1
\]
\[
= \dim \tau_{\chi_{\pi,c}(\pi)} \int_K \left\langle \tau_{\chi,c(\pi)}(k^{-1}) \cdot P, P \right\rangle \pi(k) \cdot v \, dk.
\]
We also have from (4.29) that
\[
\langle v, v \rangle = \dim \tau_{\chi_{\pi,c}(\pi)} \int_K \pi(k) \cdot v^\circ \, dk_1 \, dk_2
\]
We therefore have by (3.21), (4.27), (4.32), (4.33), and Schur orthogonality that
\[
\langle P, P \rangle \langle \pi(k) \cdot v^\circ, v \rangle = \left( \dim \tau_{\chi_{\pi,c}(\pi)} \right)^3 \int_K \int_K \left\langle \tau_{\chi,c(\pi)}(k_1^{-1}) \cdot P_{\chi_{\pi,c}(\pi)}^o, P_{\chi_{\pi,c}(\pi)}^o \right\rangle \left\langle \tau_{\chi,c(\pi)}(k_2) \cdot P, P \right\rangle
\]
\[
\times \pi(kk_1) \cdot v^\circ, \pi(k_2) \cdot v \rangle \, dk_1 \, dk_2
\]
An immediate consequence of the matrix coefficient identity (4.31) is the following.

**Corollary 4.34.** Let $(\pi, V_\pi)$ be an induced representation of Langlands type with newform $v^0 \in V_{\chi, c(\pi)}$. We have that

$$\langle \pi(k) \cdot v^0, v^0 \rangle = P_{\chi, c(\pi)}(e_n k)$$

where $\alpha_{\chi, c(\pi)} \cdot \chi^{-1}$ is as in (3.6).

**Proof.** We take $v = v^0$ in (4.31) and apply (3.5), (3.7), and (3.22).

**Remark 4.35.** From [Tem14, Lemma 3.3], we have that $c(\pi) = c(\chi)$ if and only if $\pi$ is a twist-minimal principal series representation, so that $\pi = \text{Ind}^{GL(F)}_{\mathbb{P}(F)} \bigotimes_{j=1}^{n} \tau_j$ with each $\tau_j$ a character of $F^\times$ and $c(\tau_j) = 0$ for all but at most one $j \in \{1, \ldots, n\}$.

**Remark 4.36.** Suppose that $\pi$ is tempered. One may take as a model for $\pi$ the Whittaker model $W(\pi, \psi)$; the $K$-invariant inner product on the $\tau_{\chi, c(\pi)}$-isotypic subspace of $W(\pi, \psi)$ is then given by

$$\langle W_1, W_2 \rangle := \int_{N_{\pi-1}(F) \backslash GL_{n-1}(F)} W_1 \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W_2 \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \, dg.$$ 

(In fact, this is a $GL_n(F)$-invariant inner product.) The canonical normalisation of the newform $W^0$ in the Whittaker model is such that $W^0(1_n) = 1$. Venkatesh [Ven06, Section 7] has explicitly calculated $\langle W^0, W^0 \rangle$ with respect to this inner product; when $\pi$ is spherical, so that $c(\pi) = 0$, this is simply $L(1, \text{ad} \pi)$.

5. **Archimedean Analogues**

5.1. **Archimedean Spherical Harmonics.** The archimedean analogues of the results in Sections 2 and 3 are well-known; we briefly survey them for the sake of comparison. In place of a nonarchimedean field $F$, we instead work with an archimedean field, which is either $\mathbb{R}$ or $\mathbb{C}$.

5.1.1. $F = \mathbb{R}$. The maximal compact subgroup $K_n$ of $GL_n(\mathbb{R})$ is the orthogonal group $O(n)$. This acts transitively on unit sphere

$$S^{n-1} := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1 \}$$

in $\mathbb{R}^n$ via the group action $k \cdot x := k x$ for $k \in K_n$ and $x \in S^{n-1}$. The stabiliser of the point $e_n := (0, \ldots, 0, 1) \in S^{n-1}$ is the subgroup

$$(5.1) \quad \left\{ \begin{pmatrix} k' & 0 \\ 0 & 1 \end{pmatrix} \in K_n : k' \in K_{n-1} \right\},$$

which we freely identify with $K_{n-1}$. It follows that $S^{n-1} \cong K_{n-1} \backslash K_n$. Note that the subgroup (5.1) does not, at first glance, appear to be the direct archimedean analogue of the subgroup...
$K_{n-1,1}$ as in (1.1), since $K_{n-1,1} \cong K_{n-1} \ltimes O^{n-1}$. In the archimedean setting, on the other hand, there is no analogue of $O^{n-1}$; nonetheless, the subgroup (5.1), like the nonarchimedean subgroup (1.1), is the maximal compact subgroup of the mirabolic subgroup (1.2).

The decomposition of the right regular representation of $K_n$ on $L^2(S^{n-1})$ is precisely the theory of spherical harmonics. This is best understood in terms of homogeneous harmonic polynomials, for which we follow [AH12, Chapter 2].

Given a nonnegative integer $m$, let $\mathcal{H}_m(\mathbb{R}^n)$ denote the vector space consisting of homogeneous harmonic polynomials of degree $m$, namely the set of polynomials $P(x_1, \ldots, x_n)$ in $(x_1, \ldots, x_n) \in \mathbb{R}^n$ that are annihilated by the Laplacian

$$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$$

and satisfy $P(\lambda x_1, \ldots, \lambda x_n) = \lambda^m P(x_1, \ldots, x_n)$ for all $\lambda \in \mathbb{R}$. This space has dimension 1 for $n = 1$ and $m \in \{0, 1\}$ and has dimension

$$\binom{m + n - 2}{n - 2} + \binom{m + n - 3}{n - 3} = \frac{(2m + n - 2)(m + n - 3)!}{m!(n - 2)!} = \frac{2m + n - 2}{m + n - 2}\binom{m + n - 2}{n - 2}$$

for $n \geq 2$.

Let $\mathcal{H}_m(S^{n-1})$ denote the vector space of the restriction of elements of $\mathcal{H}_m(\mathbb{R}^n)$ to the unit sphere; via the homogeneity of elements of $\mathcal{H}_m(\mathbb{R}^n)$, these spaces are isomorphic. Elements of $\mathcal{H}_m(S^{n-1})$ are called spherical harmonics of degree $m$. The group $K_n$ acts on $\mathcal{H}_m(S^{n-1})$ via right translation, which descends to an action on $\mathcal{H}_m(S^{n-1})$. As a $\mathbb{K}_m$-module, $\mathcal{H}_m(S^{n-1})$ is irreducible, and for $n \geq 2$, we have the orthogonal decomposition

$$L^2(S^{n-1}) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S^{n-1}),$$

analogous to the decomposition (2.16) of $C^\infty(S^{n-1})$.

We may view $\mathcal{H}_m(S^{n-1})$ as the real analogue of $\mathcal{H}_{\chi,m}(S^{n-1})$. In turn, the real analogue of $C^\infty(S^{n-1})^{K(p^m)}$ is $\bigoplus_{\ell=0}^{m} \mathcal{H}_\ell(S^{n-1})$, the space of spherical harmonics of degree at most $m$. As $K_1 = O(1) \cong \mathbb{Z}/2\mathbb{Z}$, which is the real analogue of $O^\times$, characters of $K_1$ are of the form $\text{sgn}^\kappa$ for $\kappa \in \{0, 1\}$, where $\text{sgn}(x) := x/|x|$. In particular, the central character $\chi$ of the $K_m$-module $\mathcal{H}_m(S^{n-1})$ is simply $\text{sgn}^m \pmod{2}$, which is determined by the parity of the nonnegative integer $m$. So the real analogue of $C^\infty(S^{n-1})^{K(p^m)}$, in terms of its orthogonal decomposition (2.15), is precisely

$$\bigoplus_{\ell=0}^{m} \mathcal{H}_\ell(S^{n-1})$$

for $\chi = \text{sgn}^\kappa$ with $\kappa \in \{0, 1\}$ and $m \geq \kappa$, while the analogue the orthogonal decomposition (2.9) of $C^\infty(S^{n-1})^{K(p^m)}$ is simply

$$\bigoplus_{\ell=0}^{m} \mathcal{H}_\ell(S^{n-1}) = \bigoplus_{\kappa \in \{0, 1\}} \bigoplus_{\ell=0}^{m} \mathcal{H}_\ell(S^{n-1}).$$

There exists a unique spherical harmonic $P^\circ_m \in \mathcal{H}_m(S^{n-1})$ satisfying $P^\circ_m(e_n) = 1$ and $P^\circ_m(xk) = P^\circ_m(x)$ for all $x = (x_1, \ldots, x_n) \in S^{n-1}$ and $k = (k'/0) \in K_n$ with $k' \in K_{n-1}$, namely

$$P^\circ_m(x_1, \ldots, x_n) := \sum_{\nu=0}^{m} \frac{i^\nu \nu! \Gamma \left( \frac{n-1}{2} \right) (m-\nu)! \nu! \Gamma \left( \frac{\nu+n-1}{2} \right)}{2^{\nu} \left( \frac{\nu}{2} \right)! (m-\nu)! \Gamma \left( \frac{\nu+n-1}{2} \right)} (x_1^2 + \cdots + x_{n-1}^2)^{\nu} x_n^{m-\nu}.$$
takes precisely the same form as the nonarchimedean result given in Lemma 3.14; similarly, \((\dim \mathcal{H}_m(S^{n-1}))P_m^c\) is the reproducing kernel for \(\mathcal{H}_m(S^{n-1})\), akin to Corollary 3.20.

5.1.2. \(F = \mathbb{C}\). The maximal compact subgroup \(K_n\) of \(\text{GL}_n(\mathbb{C})\) is the unitary group \(\text{U}(n)\). This acts transitively on unit sphere
\[ S^{n-1} := \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : z_1\overline{z_1} + \cdots + z_n\overline{z_n} = 1 \} \]
in \(\mathbb{C}^n\) via the group action \(k \cdot x := xk\) for \(k \in K_n\) and \(x \in S^{n-1}\). (It behoves us to point out that as a real topological manifold, this should be viewed as the \((2n - 1)\)-dimensional unit sphere, but we use the notation \(S^{n-1}\) for the sake of consistency.) Just as for the real case, the stabiliser of the point \(e_n := (0, \ldots, 0, 1) \in S^{n-1}\) is the subgroup (5.1), which we freely identify with \(K_{n-1}\), so that \(S^{n-1} \cong K_{n-1}\backslash K_n\).

The decomposition of the right regular representation of \(K_n\) on \(L^2(S^{n-1})\) is again the theory of spherical harmonics, with the additional complexification that one must consider the bidegree of such a spherical harmonic. This is best understood in terms of homogeneous harmonic polynomials, for which we follow [Rud08, Chapter 12].

Given a pair of nonnegative integers \(m_1, m_2\), let \(\mathcal{H}_{m_1,m_2}(\mathbb{C}^n)\) denote the vector space consisting of homogeneous harmonic polynomials of bidegree \((m_1, m_2)\), namely the set of polynomials \(P(z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n})\) in \((z_1, \ldots, z_n) \in \mathbb{C}^n\) that are annihilated by the Laplacian
\[ \Delta = 4\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} \]
and satisfy \(P(\lambda z_1, \ldots, \lambda z_n, \overline{\lambda z_1}, \ldots, \overline{\lambda z_n}) = \lambda^{m_1+n}P(z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n})\) for all \(\lambda \in \mathbb{C}\). This space has dimension 1 for \(n = 1\) and
\[ \frac{(m_1 + m_2 + n - 1)(m_1 + n - 2)!(m_2 + n - 2)!}{m_1!m_2!(n-2)!(n-1)!} = \frac{m_1 + m_2 + n - 1}{n-1} \binom{m_1 + n - 2}{n-2} \binom{m_2 + n - 2}{n-2} \]
for \(n \geq 2\).

Let \(\mathcal{H}_{m_1,m_2}(S^{n-1})\) denote the vector space of the restriction of elements of \(\mathcal{H}_{m_1,m_2}(\mathbb{C}^n)\) to the unit sphere; via the homogeneity of elements of \(\mathcal{H}_{m_1,m_2}(\mathbb{C}^n)\), these spaces are isomorphic. Elements of \(\mathcal{H}_{m_1,m_2}(S^{n-1})\) are called spherical harmonics of bidegree \((m_1, m_2)\). The group \(K_n\) acts on \(\mathcal{H}_{m_1,m_2}(\mathbb{C}^n)\) via right translation, which descends to an action on \(\mathcal{H}_{m_1,m_2}(S^{n-1})\). As a \(K_n\)-module, \(\mathcal{H}_{m_1,m_2}(S^{n-1})\) is irreducible, and for \(n \geq 2\), we have the orthogonal decomposition
\[ L^2(S^{n-1}) = \bigoplus_{m_1,m_2=0}^{\infty} \mathcal{H}_{m_1,m_2}(S^{n-1}), \]
alogous to the decomposition (2.16) of \(C^\infty(S^{n-1})\).

We may view \(\mathcal{H}_{m_1,m_2}(S^{n-1})\) as the real analogue of \(\mathcal{H}_{\chi,m}(S^{n-1})\). In turn, the real analogue of \(C^\infty(S^{n-1})^K(p^m)\) is \(\bigoplus_{m_1+m_2\leq m} \mathcal{H}_{m_1,m_2}(S^{n-1})\), the space of spherical harmonics of total degree at most \(m\). As \(K_1 = \text{U}(1) \cong \mathbb{R}/\mathbb{Z}\), which is the complex analogue of \(O^\times\), characters of \(K_1\) are of the form \(e^{i\kappa \arg}\) for \(\kappa \in \mathbb{Z}\), where \(e^{i\arg(z)} := z^{1/2}/z^{-1/2}\). In particular, the central character \(\chi\) of the \(K_n\)-module \(\mathcal{H}_{m_1,m_2}(S^{n-1})\) is \(e^{i(m_1-m_2)\arg}\), which is determined by the difference of \(m_1\) and \(m_2\). So the complex analogue of \(C^\infty(S^{n-1})^K(p^m)\), in terms of its orthogonal decomposition (2.15), is precisely
\[ \bigoplus_{m_1+m_2\leq m \atop m_1-m_2=\ell} \mathcal{H}_{m_1,m_2}(S^{n-1}) \]
for \(\chi = e^{i\ell\arg}\) with \(m \geq |\ell|\), while the orthogonal decomposition (2.9) of \(C^\infty(S^{n-1})^K(p^m)\) has the complex analogue
\[ \bigoplus_{m_1+m_2\leq m \atop |\ell|\leq m} \bigoplus_{m_1,m_2=\ell} \mathcal{H}_{m_1,m_2}(S^{n-1}). \]
There exists a unique spherical harmonic $P^0_{m_1,m_2} \in \mathcal{H}_{m_1,m_2}(S^{n-1})$ satisfying $P^0_{m_1,m_2}(e_n, e_n) = 1$ and $P^0_{m_1,m_2}(zk, \overline{zk}) = P^0_{m_1,m_2}(z, \overline{z})$ for all $z = (z_1, \ldots, z_n) \in S^{n-1}$ and $k = \left( \begin{smallmatrix} k' & 0 \\ 0 & 1 \end{smallmatrix} \right) \in K_n$ with $k' \in K_{n-1}$, namely
\[
P^0_{m_1,m_2}(z, \overline{z}) := \sum_{\nu=0}^{\min(m_1,m_2)} \frac{(-1)^{\nu} \binom{m_1}{\nu} \binom{m_2}{\nu}}{(\nu+n-2)_{n-2}} (z_1 \overline{z_1} + \cdots + z_{n-1} \overline{z_{n-1}})^{\nu} z_n^{m_1-\nu} \overline{z_n}^{m_2-\nu}.
\]
This is the zonal spherical harmonic on $S^{n-1}$ of bidegree $(m_1, m_2)$, which is the complex analogue of the zonal spherical function $P^0_{\chi,m} \in \mathcal{H}_{\chi,m}(S^{n-1})$. Once more, the addition theorem for $\mathcal{H}_{m_1,m_2}(S^{n-1})$ takes the same form as Lemma 3.14 and $(\dim \mathcal{H}_m(S^{n-1})) P^0_m$ is the reproducing kernel for $\mathcal{H}_m(S^{n-1})$.

5.2. Archimedean Newform Theory for $GL_n$. Finally, we consider the archimedean analogues of the results in Section 4; these are due to Popa [Pop08] for $GL_2$ and to the author [Hum20] for $GL_n$ with $n$ arbitrary.

Let $(\pi, V_\pi)$ be an induced representation of Langlands type of $GL_n(F)$, where $F$ is an archimedean local field, so that $F$ is either $\mathbb{R}$ or $\mathbb{C}$. There is no obvious analogue in the archimedean setting of the nonarchimedean congruence subgroups $K_n(p^m)$ and $K_0(p^m)$ of $K_n$ (though cf. [JN19]). This prevents one from defining the newform and conductor exponent of $\pi$ via the subspace $V_\pi^{K_n(p^m)}$ of $K_n(p^m)$-invariant vectors as in Definition 4.6.

As highlighted in Remark 4.8, one can instead characterise the nonarchimedean newform, when viewed in the Whittaker model, as the unique Whittaker function $W^0 \in \mathcal{W}(\pi, \psi)$ that is right $K_{n-1}$-invariant, with $K_{n-1}$ embedded in $K_n$ as the subgroup (5.1), and that is a test vector for the local $GL_n \times GL_{n-1}$ Rankin–Selberg integral whenever the second representation is unramified. It is shown in [Hum20, Theorem 4.17] that this characterisation does indeed also hold in the archimedean setting, in that there is a unique such Whittaker function satisfying these two conditions. One would also like to show that the conductor exponent is characterised via the epsilon factor $\varepsilon(s, \pi, \psi)$; while this is the case when $F$ is nonarchimedean, this is unfortunately insufficient when $F$ is archimedean, for then $\varepsilon(s, \pi, \psi)$ is simply an integral power of $i$, and this integer is only determined modulo 4.

Nonetheless, the archimedean analogue of Theorem 4.16 holds; more precisely, the archimedean analogues of (4.17) and (4.18) are true via [Hum20, Theorems 4.7 and 4.12]. Here the nonarchimedean subgroup $K_{n-1,1}$ of $K_n$ is replaced by the archimedean subgroup (5.1), which both share the property that they are the stabiliser subgroup of $e_n$ in $F^n$; moreover, the notion of ordering $K$-types by their level in the nonarchimedean setting is replaced by ordering $K$-types by their Howe degree in the archimedean setting (cf. [Hum20, Section 4.2]).

More precisely, in [Hum20, Definition 4.8], the author has defined the conductor exponent $c(\pi)$ of an induced representation of Langlands type $(\pi, V_\pi)$ of $GL_n$ over an archimedean field $F$ to be the minimal nonnegative integer $m$ for which $\pi$ contains a $K_n$-type $\tau$ of Howe degree $m$ having a nontrivial $K_{n-1}$-fixed vector, with $K_{n-1}$ embedded in $K_n$ as the subgroup (5.1). This distinguished $K_n$-type $\tau = \tau^0$ is the newform $K$-type and appears with multiplicity one in $\pi$. The author has also defined the newform in the archimedean setting to be the nonzero vector in $V_\pi$, unique up to scalar multiplication, that is invariant under the subgroup (5.1) and is $\tau^0$-isotypic.

We observed in Remark 4.22 that Theorem 4.16 gives alternative characterisations of the newform and conductor exponent in the nonarchimedean setting in terms of $K$-types. In conjunction with [Hum20, Theorem 4.7], this characterisation thereby unifies the nonarchimedean and archimedean treatments of the newform and the conductor exponent.

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