SECURE DOMINATION IN LICT GRAPHS

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Abstract. For any graph \( G = (V, E) \), lict graph \( \eta(G) \) of a graph \( G \) is the graph whose vertex set is the union of the set of edges and the set of cut-vertices of \( G \) in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of \( G \) are incident. A secure lict dominating set of a graph \( \eta(G) \), is a dominating set \( D \subseteq V(\eta(G)) \) with the property that for each \( v_1 \in (V(\eta(G)) - F) \), there exists \( v_2 \in F \) adjacent to \( v_1 \) such that \((F - \{v_2\}) \cup \{v_1\}\) is a dominating set of \( \eta(G) \). The secure lict dominating number \( \gamma_{ssc}(\eta(G)) \) of \( G \) is a minimum cardinality of a secure lict dominating set of \( G \). In this paper, many bounds on \( \gamma_{ssc}(\eta(G)) \) are obtained and its exact values for some standard graphs are found in terms of parameters of \( G \). Also its relationship with other domination parameters is investigated.

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1. Introduction

The graphs considered here are finite, connected, undirected without loops or multiple edges and without isolated vertices. As usual \( n \) and \( q \) denote the number of vertices and edges of a graph \( G \). For any undefined term or notation in this paper can be found in Harary [1].

A set \( D \subseteq V \) is a dominating set of \( G \) if every vertex in \( V - D \) is adjacent to some vertex in \( D \). The dominating number \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set \( D \). A secure dominating set of \( G \) is a dominating set \( D \subseteq V(G) \) with the property that for each \( u \in V(G) - D \), there exists \( v \in D \) adjacent to \( u \) such that \((D - \{v\}) - \{u\}\) is a dominating set.
The total domination of lict graph has been studied by [2]. The lict graph \( \eta(G) \) of a graph \( G \) is the graph whose vertex set is the union of the set of edges and the set of cut-vertices of \( G \) in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of \( G \) are incident. The secure domination has been intensively studied by [3, 4]. The secure lict dominating set of a graph \( \eta(G) \), is a dominating set \( F \subseteq V(\eta(G)) \) with the property that for each \( v_1 \in (V(\eta(G)) - F) \), there exists \( v_2 \in F \) adjacent to \( v_1 \) such that \((F - \{v_2\}) \cup \{v_1\}\) is a dominating set of \( \eta(G) \). The secure lict dominating number \( \gamma_{se}(\eta(G)) \) of \( G \) is a minimum cardinality of secure lict dominating set of graph \( G \). For complete review on the topic of domination[5]. The vertex independence number \( \beta_0(G) \) is the maximum cardinality among the independent set of vertices of \( G \). \( L(G) \) is the line graph of \( G \), \( \gamma_e(G) \) is edge domination number, \( \gamma'_e(G) \) is the secure edge dominating number, \( \gamma_t(G) \) is the total dominating number, \( \gamma_{ns}(G) \) is the non-split dominating number and \( \chi(G) \) is the chromatic number of \( G \). The degree of a edge [6] is the number of lines adjacent to it. The minimum (maximum) degree of an edge in \( G \) is denoted by \( \delta'(\Delta') \). A subdivision of an edge \( e = uv \) of a graph \( G \) is the replacement of an edge \( e \) by a path \( (u, v, w) \) where \( w \notin E(G) \). The graph obtained from \( G \) by subdividing each edge of \( G \) exactly once is called the subdivision graph of \( G \) and is denoted by \( S(G) \). For any real number \( X, \lceil X \rceil \) denotes the smallest integer not less than \( X \) and \( \lfloor X \rfloor \) denotes the greatest integer not greater than \( X \).

In this paper we established the relationship of this concept with the other domination parameters is investigated.

2. Main results

**Theorem 2.1.** First we list out the exact values of \( \gamma_{se}(\eta(G)) \) for some standard graphs:

(i) For any cycle \( C_n \) with \( n \geq 3 \) vertices,
\[
\gamma_{se}(\eta(C_n)) = \begin{cases} 
1 & n = 3, \\
\left\lceil \frac{n+1}{2} \right\rceil & n \not\equiv 0 \pmod{7}, \\
\left\lfloor \frac{n+1}{2} \right\rfloor & n \equiv 0 \pmod{7}.
\end{cases}
\]

(ii) For any path \( P_n \) with \( n \geq 4 \) vertices, \( \gamma_{se}(\eta(P_n)) = n - 2 \).

(iii) For any star graph \( K_{1,n} \) with \( n \geq 2 \) vertices, \( \gamma_{se}(\eta(K_{1,n})) = 1 \).

(iv) For any wheel graph \( W_n \) with \( n \geq 4 \) vertices, \( \gamma_{se}(\eta(W_n)) = \left\lceil \frac{3n}{2} \right\rceil + 1 \).

(v) For any bipartite graph \( K_{m,n} \) with \( m, n \geq 2 \) vertices, \( \gamma_{se}(\eta(K_{m,n})) = \min\{m,n\} \).

(vi) For any friendship graph \( F_n \) with \( k \) blocks, \( \gamma_{se}(\eta(F_n)) = k \).

(vii) For any complete graph \( K_n \) with \( n \geq 4 \), \( \gamma_{se}(\eta(K_n)) = \left\lfloor \frac{n}{2} \right\rfloor \).

**Theorem 2.2.** Let \( G \) be the connected graph with \( n \geq 3 \) vertices, then \( \gamma_{se}(\eta(G)) = 1 \) if and only if \( G = K_{1,n-1} \) or \( C_3 \).

**Proof.** Necessary: Suppose \( \gamma_{se}(\eta(G)) = 1 \). We consider the following cases:
Case 1: If $G$ is a connected graph with $n = 3$, then $G$ is either $K_{1,2}$ or $C_3$, by using Theorem 2.1(i) and Theorem 2.1(iii), $\gamma_{se}(\eta(G)) = 1$.

Case 2: If $G$ is a connected graph with $n \geq 4$. Let $D = \{e\}$ be the secure dominating set of $\gamma_{se}(\eta(G))$. To prove that $G = K_{1,n-1}$, we assume contrary that $G \neq K_{1,n-1}$. We consider the following subcases:

Subcase 1: Let $F = K_{1,n-1}$ and let the endvertices $v_1, v_2 \in V(F)$, such that the graph $G$ is obtained form $F$ by adding the edge $e_1 = (v_1, v_2) \notin E(F)$. It follows that the set $(D - \{e\}) \cup \{e_1\}$ is not a dominating set of $\eta(G)$. This implies that $D$ is not a dominating set of $\eta(G)$, which is a contradiction. Thus $G = K_{1,n-1}$.

Subcase 2: Let $F = K_{1,n-1}$ and an endvertex $v_1 \in V(F)$, such that the graph $G$ is obtained form $F$ by adding the vertex $v \in V(F)$ and the edge $e_1 = (v, v_1)$. It follows that the set $(D - \{e\}) \cup \{e_1\}$ is not an secure dominating set of $\eta(G)$. This implies that $D$ is not a dominating set of $\eta(G)$, which is a contradiction. Thus $G = K_{1,n-1}$.

Sufficiency: If $G = K_{1,n-1}$ or $G = C_3$, then using Theorem 2.1(i) and Theorem 2.1(iii), $\gamma_{se}(\eta(G)) = 1$.

Theorem 2.3. For any graph $G$, $\gamma_{se}(\eta(G)) \geq \gamma_s'(G)$. Equality holds if $G$ is non-separable.

Proof. Let $D$ be a secure edge dominating set of $G$ and let $B$ be the corresponding vertices of $D$ in $\eta(G)$. We consider the following cases:

Case 1: Suppose the cut-vertices of $G$ are incident with atleast one edge of $D$ in $G$.

Then for each cut-vertex say $v_i$ in $G$ if there exists an vertex $v \in N(v_i)$ in $\eta(G)$ such that $(B - \{v\}) \cup \{v_i\}$ is the dominating set of $\eta(G)$. Then $\gamma_{se}(\eta(G)) = \gamma_s'(G)$. Otherwise $B \cup \{v_i\}$ is the secure dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \geq \gamma_s'(G)$.

Case 2: Suppose if there exists atleast one cut-vertex $v_i$ in $G$ which is not incident with any edge of $D$, then $B \cup \{v_i\}$ is the secure dominating set of $\eta(G)$.

Therefore $\gamma_{se}(\eta(G)) \geq \gamma_s'(G)$.

To prove the equality:

If $G$ is non-separable, then $\eta(G) = L(G)$. Hence $\gamma_{se}(\eta(G)) = \gamma_{se}(L(G)) = \gamma_s'(G)$.

Theorem 2.4. For any graph $G$, $\gamma_{se}(\eta(G)) \geq \gamma_e(G)$.

Proof. Let $D$ be a $\gamma_e$ set of graph $G$. If $D$ is a secure dominating set of graph $\eta(G)$, then

(i) For each $e_i \in E(G) - D$, if there exists an edge $e_1 \in D$, $e_1 \in N(e_i)$, such that the corresponding vertices of $\{(D - e_1) \cup e_i\}$ in $\eta(G)$ is a dominating set of $\eta(G)$. 

\[ \square \]
(ii) For each cut-vertex \( c_i \in G \), if there exists an edge \( e_i \in D \), \( c_i \) is incident with \( e_i \) in \( G \) such that the corresponding vertices of \( \{(D - \{e_i\}) \cup c_i\} \) in \( \eta(G) \) is a dominating set of \( \eta(G) \).

Therefore \( \gamma_{se}(\eta(G)) = \gamma_c(G) \). Otherwise the corresponding vertices of \( \{D \cup e_i \cup c_i\} \) in \( \eta(G) \) is the secure dominating set of \( \eta(G) \). Therefore \( \gamma_{se}(\eta(G)) \geq \gamma_c(G) \).

**Theorem 2.5.** For any Tree \( T \), \( \gamma_{se}(\eta(T)) \leq m \), where ‘\( m’ \) is the number of cut-vertices of \( T \). Equality will hold for \( P_n \) and \( K_{1,m} \).

**Proof.** Let \( A \) be the set of cut-vertices of a graph \( T \) with \( |A| = m \). Since \( A \) covers all the edges of \( T \), therefore in \( \eta(T) \), \( A \) covers all the vertices of \( \eta(T) \). Hence the set \( A \) is the lict dominating set of \( T \). Now for each vertex \( e_i \in V(\eta(T)) \) \(- A \), there exists a vertex \( \{e_i\} \in A \) incident with \( \{e_i\} \) in \( T \) such that \( (A - \{e_i\}) \cup \{e_i\} \) is a lict dominating set of \( T \). Therefore \( \gamma_{se}(\eta(T)) \leq m \).

For Equality:
The result follows from Theorem 2.1(ii) and Theorem 2.1(iii).

**Theorem 2.6.** For any Tree \( T \), \( \gamma_{se}(\eta(T)) + 1 \geq \chi(T) \).

**Proof.** We have \( \chi(T) = 2 \) and \( \gamma_{se}(T) + 1 \geq 2 \), the result follows.

**Theorem 2.7.** If every vertex of \( G \) is adjacent to an end vertex then, \( \gamma_{se}(\eta(G)) = m \), where \( m \) is the number of cut-vertices of \( G \).

**Proof.** Let \( A \) be the set of cut-vertices of \( G \) with \( |A| = m \) and let \( B = \{e_i/e_i \) is incident with \( (v_i,v_j), v_i \in A, d(v_j) = 1 \} \) with \( |B| = m \). Suppose \( \gamma_{se}(\eta(G)) < m \), then \( A \) is not the dominating set of \( \eta(G) \). Hence \( \gamma_{se}(\eta(G)) \geq m \). Now if \( \gamma_{se}(\eta(G)) = |A| \), then for each edge \( e_i \in E(G) \) \(- B \), there exists an edge \( e_j \in N(e_i) \cap B \) such that the corresponding vertices of \( \{(B - e_j) \cup e_i\} \) is the dominating set of \( \eta(G) \) and for every cut-vertex \( v_i \in A \), there exists an edge \( e_j \in B \) incident with \( v_i \) such that the corresponding vertices of \( \{(A - e_j) \cup v_i\} \) in \( \eta(G) \) is the dominating set of \( \eta(G) \). Hence \( \gamma_{se}(\eta(G)) = m \).

**Corollary 2.8.** If every vertex of \( G \) is adjacent to an end vertex, then \( \gamma_{se}(\eta(G)) = \gamma(G) = \gamma_t(G) = \gamma_{ns}(G) \).

**Proof.** Since every vertex of \( G \) is adjacent to an end vertex then, \( \gamma(G) = \gamma_t(G) = \gamma_{ns}(G) = m \) and by using Theorem 2.7, the result follows.

**Theorem 2.9.** For any connected graph \( G \), \( \gamma_{se}(\eta(G)) \leq q - \Delta'(G) + 1 \), \( q \geq 2 \) and \( \Delta' \) is the maximum degree of an edge.

**Proof.** Let \( e \) be an edge with degree \( \Delta' \) and let \( S \) be the set of edges adjacent to \( e \) in \( G \). Then \( E(G) - S \) is the lict dominating set of \( G \). We consider the following cases:
Case 1: If $G$ is a non-separable graph, then for every edge $e_1 \in S$, there exist an edge $e_2 \in E(G) - S$, $e_1 \in N(e_2)$ such that the corresponding vertices of $(E(G) - S) - e_2 \cup \{e_1\}$ in $\eta(G)$ is a dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq q - \Delta(G)$.

Case 2: If $G$ is a separable graph. We consider the following subcases:

(i) If $((E(G) - S) \cup e)$ is a null graph in $G$, then $\Delta(G) = q - 1$ and in the graph $G$, $e = v_1v_2$, where $v_1$ or $v_2$ or both, are the cut-vertices of graph $G$. Therefore by using Theorem 2.7, $\gamma_{se}(\eta(G)) = |\{v_1 \cup v_2\}|$ or $|\{v_1\}$ or $|\{v_2\}| = 2$ or 1. Therefore $\gamma_{se}(\eta(G)) \leq 2 - \Delta(G) + 1$.

(ii) If $((E(G) - S) \cup e)$ is not a null graph and $e_1 = (v_1, v_2)$ where $v_1, v_2$ are the cut-vertices of the graph $G$. Now if $v_1 \notin \gamma_{se}$ set of $\eta(G)$, then $v_2$ is not covered by any vertex corresponding to the vertices of $E(G) - S$ in $\eta(G)$. Therefore the corresponding vertices of $((E(G) - S) \cup \{v_1\})$ in $\eta(G)$ is the secure dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |(E(G) - S) \cup v_2| = q - \Delta(G) + 1$. Otherwise if $\{v_1\}$ is the cut-vertex, then there exist an edge $e \in E(G) - S$, $e$ is incident with $v_1$ such that the corresponding vertices of $((E(G) - S) - e) \cup v_1$ in $\eta(G)$ is a dominating set of $\eta(G)$. Otherwise if $\{v_1, v_2\}$ are not the cut-vertices then for each vertex $e_1$ in $\eta(G)$, there exists a vertex $e_2 \in E(G) - S$ in $\eta(G)$, such that the corresponding vertices of $((E(G) - S) - e_2) \cup e_1$ in $\eta(G)$ is a dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq q - \Delta(G)$.

The result follows from Case (1) and Case (2). □

**Theorem 2.10.** For any connected graph $G$, $\gamma_{se}(\eta(G)) \leq q - 2$, $n \geq 3$.

**Proof.** Let $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let $A = \{e_2, e_3, \ldots, e_{m-1}\}$ with $|A| = q - 2$. We consider the following cases.

Case 1: If $e_1 \in E(G) - A$ is not incident with the cut-vertex say $v_i$, then for each $e_i \in E(G) - A$ there exists an edge $e_j \in N(e_i) \cap A$ such that the corresponding vertices of $\{(A - e_j) \cup e_i\}$ in $\eta(G)$ is dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |A| = q - 2$.

Case 2: If $e_j \in E(G) - A$ is incident with the cut-vertex say $v_i$, then for each $v_i$, there exists an edge $e_j$ which is incident with $v_i$ and $e_j \in A$ such that the corresponding vertices of $\{(A - e_j) \cup e_i\}$ in $\eta(G)$ is dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |A| = q - 2$.

□

**Theorem 2.11.** For any connected graph $G$, $\gamma_{se}(\eta(G)) \leq n - 2$, $n \geq 3$.

**Proof.** Let $V(G) = \{v_1, v_2, \ldots, v_m\}, A = \{v_2, v_3, \ldots, v_{m-1}\}$ with $|A| = n - 2$. Let $F$ be the set of edges which is incident with $\{v_{m-1}, v_m\}$ and $B = \{e_i/e_i \in E(G) - F\}$ with $|B| \leq n - 2$. We consider the following cases.
Case 1: If $e_i \in B$ is not incident with the cut-vertex say $v_i$, then for each $e_j \in E(G) - B$ there exists an edge $e_j \in N(e_i) \cap B$ such that the corresponding vertices of $\{(B - e_j) \cup e_i\}$ is dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |B| = n - 2$.

Case 2: If $e_i \in B$ is incident with the cut-vertex say $v_i$, then for each $v_i$, there exists an edge $e_j$ which is incident with $v_i$ and $e_j \in B$ such that the corresponding vertices of $\{(B - e_j) \cup v_i\}$ is dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |B| = n - 2$.

\[ \square \]

**Theorem 2.12.** For any connected graph $G$, $\gamma_e(G) \leq \gamma_{se}(\eta(G)) \leq \gamma_s(G) + \lceil \frac{n}{3} \rceil$.

**Proof.** Let $D$ be the edge dominating set of $G$. Using Theorem 2.4, we have $\gamma_e(G) \leq \gamma_{se}(\eta(G))$. For the upper bound, Let $A = \{v_i/v_i \notin B\}$ with $\alpha_d(A)$ is not incident with any edge of $D$ with $|A| \leq \lceil \frac{n}{3} \rceil$. Let $B = \{e_i/e_i \text{ is incident with each } v_i, v_i \in A\}$ with $|B| = |A|$. We consider the following cases:

Case 1: If $G$ is non-separable and for each edge $e_i \in E(G) - D$, there exists an edge $e_j \in N(e_i) \cap D$ such that $\{(D - e_j) \cup e_i\}$ is the edge dominating set of $\eta(G)$, then $\gamma_{se}(\eta(G)) \leq |D| = \gamma_e(G)$. Otherwise if the corresponding vertices of $B$ in $\eta(G)$ does not belong to $\gamma_{se}(\eta(G))$, then there exists atleast one vertex in $\eta(G)$ which is not covered by any of the corresponding vertex of $D$ in $\eta(G)$. Therefore $B \in \gamma_{se}(\eta(G))$ set. Hence $\gamma_{se}(G) \leq |D \cup B| = \gamma_s(G) + \lceil \frac{n}{3} \rceil$.

Case 2: If $G$ is separable and if for each edge $e_i \in E(G) - D$ there exists an edge $e_j \in N(e_i) \cap D$ such that the corresponding vertices of $\{(D - e_j) \cup e_i\}$ in $\eta(G)$ is the dominating set of $\eta(G)$ and for each cut-vertex $v_i$, there exists an edge $e_m$ incident with $v_i$ in $G$ such that the corresponding vertices of $\{(D - e_m) \cup v_i\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) = \gamma_s(G)$. Otherwise if the corresponding vertices of $B$ in $\eta(G)$ does not belong to $\gamma_s(\eta(G))$, then there exists atleast one vertex in $\eta(G)$ which is not covered by any vertex corresponding to $D$ in $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |D \cup B| = \gamma_s(G) + \lceil \frac{n}{3} \rceil$.

\[ \square \]

**Theorem 2.13.** For any connected graph $G$, $\gamma_{se}(\eta(G)) \leq \gamma_{s}^0(G) + m$, where $m$ is the number of cut-vertices of $G$.

**Proof.** Let $D$ be a secure edge dominating set of $G$ and let $A = \{v_i/G - v_i \text{ is disconnected}\}$ with $|A| = m$. We consider the following cases:

Case 1: If $m = 0$.

In this case $\eta(G) = L(G)$, the result is obvious.

Case 2: If $m \neq 0$.

Let $C$ be the corresponding vertices of $D$ in $\eta(G)$ and let $F = \{e_i = (v_i, v_j)/e_i \in D \text{ incident with } v_i \in A\}$. We consider the following sub-cases:
If $|F| \neq \phi$ and if $v_j$ is a not a cut-vertex or $e_j \in N(v_i) \in D$, then $\gamma_{sc}(\eta(G)) = \gamma'(G)$. Otherwise there exists at least one vertex in $\eta(G)$ which is not covered by $C$. Therefore $A \in \gamma_{sc}$ set of $\eta(G)$.

Hence $\gamma_{sc}(\eta(G)) \leq |C \cup A| = \gamma'_s(G) + m$.

(ii) If $|F| = \phi$, then $v_j \in A$ is not covered by any vertex of $C$. Therefore $v_i \in \gamma_{sc}$ set of $\eta(G)$. Hence $\gamma_{sc}(\eta(G)) \leq |C \cup A| = \gamma'_s(G) + m$

\[ \square \]

**Theorem 2.14.** For any Tree $T$, if every vertex is adjacent to support vertex then, $\gamma_{se}(\eta(T)) \leq \beta_0(T)$.

**Proof.** Let $D$ be the maximum vertex independence set of $T$ and let $A$ be the $\gamma_{se}$ set of $\eta(G)$. Since every vertex is adjacent to an end vertex, then $A$ will contains all the vertices adjacent to endvertices of $T$ with $|A| \leq \beta_0(T)$. Therefore by using Theorem 2.7, we have $\gamma_{se}(\eta(T)) \leq \beta_0(T)$.

\[ \square \]

**Theorem 2.15.** For any connected graph $G$, $\gamma_{sc}(\eta(G)) \leq \alpha_0(G) + m$, where $‘m’$ is the number of cut-vertices of $G$.

**Proof.** Let $D$ be the minimum vertex covering set of $G$ and $B$ be the set of cut-vertices of $G$ with $|B| = m$. For each vertex $v_i \in D$, choose exactly one edge $e_i \in G$, $e_i$ is incident with $v_i$ such that it covers maximum number of vertices of $G$. Let $F$ be the set of all such edges such that $|F| = |D|$. We consider the following cases:

Case 1: If $m = 0$.

For each edge $e_i \in E(G) - F$, there is an edge $e_j \in F$, $e_j \in N(e_i)$. Since the corresponding vertices of $F$ in $\eta(G)$ will covers all the edges which are incident to $v_i \in V(G)$. Therefore the corresponding vertices of $(F - e_j) \cup e_i$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) \leq \alpha_0(G)$.

Case 2: If $m \neq 0$.

Let $S$ the corresponding vertices of $F$ in $\eta(G)$. If $v_i \in B \cap D$, then for each vertex $v_i$, there exists an edge $e_j \in F$ and $e_j$ is incident with $\{v_i, v_j\}$. $v_j$ is not a cut-vertex, then corresponding vertices of $(F - e_j) \cup v_i$ in $\eta(G)$ is the dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq \alpha_0(G)$.

Otherwise there exists at least one vertex $v_i \in B \cap V(\eta(G))$ which is not covered by $S$, therefore $v_i \in \gamma_{se}(\eta(G)$ set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |D \cup B| = \alpha_0(G) + m$.

The result follows from Case (1) and Case (2).

\[ \square \]

**Theorem 2.16.** For any connected graph $G$, $\gamma_{sc}(\eta(G)) \leq \alpha_1(G) + m$, where $m$ is the number of cut-vertices of $G$.

**Proof.** Let $D$ be the minimum edge covering set of $G$ and $B$ be the set of cut-vertices of $G$ with $|B| = m$. We consider the following cases:
If

For any connected graph $G$, let $\alpha_1(G)$ be the maximum edge independence set of $G$. Hence $\gamma_{se}(\eta(G)) \leq \alpha_1(G)$.

Case 2: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $\{v_1, v_2\} \notin B$. For each edge $e_i \in E(G) - D$, there exists an edge $e_i \in D$, $e_i \in N(e_i)$ and since $e_i$ covers all the edges incident with $v_1$ and $(D - e_i)$ will covers all the edges incident with $v_2$, therefore the corresponding vertices of $\{(D - e_i) \cup e_j\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) \leq \alpha_1(G)$.

Case 3: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $\{v_1, v_2\} \notin B$. For each edge $e_i$ in $E(G) - D$, there exists an edge $e_i$ in $E(G) \cap D$, then $\gamma_1(\eta(G)) \leq \alpha_1(G)$. Otherwise $\{v_1\}$ or $\{v_2\}$ is not covered by any vertices corresponding to $D$ in $\eta(G)$. Therefore $\{v_1 \text{ or } v_2\} \in \gamma_1(\eta(G))$ set. Hence $\gamma_{se}(\eta(G)) \leq |D \cup B| = \alpha_1(G) + m$.

Case 3: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $\{v_1\} \in B, \{v_2\} \notin B$. For each edge $e_j \in E(G) - D$, there exists an edge $e_j$ in $E(G) \cap D$, then $\gamma_1(\eta(G)) \leq \alpha_1(G)$. Otherwise $\{v_1\}$ or $\{v_2\}$ is not covered by any vertices corresponding to $D$ in $\eta(G)$. Therefore $\{v_1 \text{ or } v_2\} \in \gamma_1(\eta(G))$ set. Hence $\gamma_{se}(\eta(G)) \leq |D \cup B| = \alpha_1(G) + m$.

The result follows from Case (1), Case (2) and Case (3).

**Theorem 2.17.** For any connected graph $G$, $\gamma_{se}(\eta(G)) \leq \beta_1(G) + m$, where $m$ is the number of cut-vertices of the graph $G$.

**Proof.** Let $D$ be the maximum edge independence set of $G$ and $B$ be the set of cut-vertices of $G$ with $|B| = m$. We consider the following cases:

Case 1: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $\{v_1, v_2\} \notin B$. For each edge $e_i \in E(G) - D$, there exists an edge $e_j \in D$, $e_i \in N(e_i)$ and since $e_j$ covers all the edges incident with $v_1$ and $(D - e_i)$ will covers all the edges incident with $v_2$, therefore the corresponding vertices of $\{(D - e_i) \cup e_j\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) \leq \beta_1(G)$.

Case 2: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $\{v_1, v_2\} \notin B$. For each edge $e_i$, if there exists an edge adjacent to $e_i$ in $E(G) \cap D$, then $\gamma_1(\eta(G)) \leq \beta_1(G)$. Otherwise $\{v_1\}$ or $\{v_2\}$ is not covered by any vertices corresponding to $D$ in $\eta(G)$. Therefore $\{v_1 \text{ or } v_2\} \in \gamma_1(\eta(G))$ set. Hence $\gamma_{se}(\eta(G)) \leq |D \cup B| = \beta_1(G) + m$.

Case 3: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $\{v_1\} \in B, \{v_2\} \notin B$. For each edge $e_i \in E(G) - D$, there exists an edge $e_j \in D$, $e_i \in N(e_i)$ and since $e_j$ covers all the edges incident with $v_1$ and $(D - e_i)$ will covers all the edges incident with $v_2$, therefore the corresponding vertices of $\{(D - e_i) \cup e_j\}$ in $\eta(G)$ is the dominating set of $\eta(G)$ and for $v_1$ there exists an edge $e_r \in D$ incident with $v_1$ such that the corresponding vertices of $\{(D - v_1) \cup e_r\}$ in $\eta(G)$ is the dominating set of $G$. Hence $\gamma_{se}(\eta(G)) \leq \beta_1(G)$. 

$\square$
For any graph $G$, let $\gamma(G)$ be the secure dominating set of $G$ in $\eta(G)$.

Case 1: \[ \gamma(G) \leq \gamma_{se}(\eta(G)) \]

Proof. Let $D$ be the $\gamma$ set of $G$ and let $B = \{e_i, e_i \in E(G)\}$ incident with $D$ such that $\gamma(G)$ is the secure dominating set of $\eta(G)$ in $G$. If suppose $B$ is the secure dominating set of $G$, then

(i) For each $e_i \in E(G)$, if there exists an edge $e_i \in D$, $e_i \in N(e_i)$, such that the corresponding vertices of $\{\{D-e_i\}\cup e_i\}$ in $\eta(G)$ is a dominating set of $\eta(G)$.

(ii) For each cut-vertex $e_i \in G$, if there exists an edge $e_i \in D$, $e_i$ is a path, such that the corresponding vertices of $\{\{D-e_i\}\cup e_i\}$ in $\eta(G)$ is a dominating set of $\eta(G)$.

Therefore $\gamma_{se}(\eta(G)) = \gamma(G)$. Otherwise the corresponding vertices of $\{\{D-e_i\}\cup e_i\}$ in $\eta(G)$ is the secure dominating set of $\eta(G)$. Therefore $\gamma(G) \leq \gamma_{se}(\eta(G))$.

Theorem 2.19. For any connected graph $G$, $\frac{n}{\Delta+1} \leq \gamma_{se}(\eta(G)) \leq 2q - n$. Furthermore, the upper bound is attained if and only if $G$ is a path.

Proof. Since $\frac{n}{\Delta+1} \leq \gamma(G)$ and using Theorem 2.18. $\gamma(G) \leq \gamma_{se}(\eta(G))$, the lower bound holds.

For upper bound.

Since $G$ is connected, $q \geq n - 1$ and by Theorem 2.11, \[ \gamma_{se}(\eta(G)) \leq n - 2 = 2(n - 1) - n. \] Hence $\gamma_{se}(\eta(G)) \leq 2q - n$.

Now we show that $\gamma_{se}(\eta(G)) = 2q - n$ if and only if $G$ is a path. If $G$ is a path, then by Theorem 2.1(iii), $\gamma_{se}(\eta(G)) = n - 2 = 2(n - 1) - n = 2q - n$. Conversely, suppose $\gamma_{se}(\eta(G)) = 2q - n$. Then by Theorem 2.11, we have $2q - n \leq n - 2$ which implies $q \leq n - 1$. Since $G$ is connected, $G$ must be a tree with $q = n - 1$.

Thus Theorem 2.5, $\gamma_{se}(\eta(G)) \leq n - e$, $e$ is the number of pendent vertices. If $e > 2$, then $\gamma_{se}(\eta(G)) \leq n - e < n - 2 = 2q - n$, a contradiction which shows that $e \leq 2$. But $G$ is a tree, $e \geq 2$. Thus $e = 2$ and $G$ is a path.

Theorem 2.20. For any subdivision graph $G$, $\gamma_{se}(\eta(S(G))) \leq 2\alpha_1 + n_0$, where $n_0$ is the number of vertices that subdivides $E(G)$.

Proof. Let $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ and let $B = \{n_j, n_j \in V(S(G)) - V(G)\} \mid |B| = n_0$. Let $\alpha_1$ be the minimum edge covering set of $G$ and $M = \{n_r, n_r \in B, n_r \text{ in not incident with } \alpha_1 \text{ and } n_r \text{ is the cut-vertex in } S(G)\} \mid |M| \leq n_0$. Let $R$ be the set of edges in $S(G)$ corresponding to the edges in $\alpha_1$ with $|R| = 2\alpha_1$. We consider the following cases:

Case 1: If $|M| = \phi$.

Then for each edge $e_j = (v_i, n_j) \in E(S(G)) - R$, there exists an edge $e_i = (v_i, n_j) \in R \cap N(e_j)$ such that the corresponding vertices of $\{\{R-e_i\}\cup e_j\}$ in $\eta(S(G))$ is the dominating set of $\eta(S(G))$ and for each cut-vertex $v_i \in V(S(G))$, there exists an edge $e_j = (v_i, n_j) \in R \cap e_j$ is incident with $v_i$ such that the corresponding vertices of $\{\{R-e_j\}\cup v_i\}$ in $\eta(S(G))$ is the dominating set of $\eta(S(G))$. Hence $\gamma_{se}(\eta(S(G))) \leq 2\alpha_1$.
Case 1: If $|M| \neq \phi$.  

Now for each cut-vertex say $v_m \in M$, since there is no edge in $R$ incident with $v_m$ and therefore $v_m$ is not covered by the corresponding vertices of $R$ in $\gamma(S(G))$. Therefore $v_m \in \gamma_{se}(\eta(S(G)))$ set. Hence $\gamma_{se}(\eta(S(G))) \leq 2\alpha_1 + n_0$.

\[ \square \]

**Proposition 2.21.** For any graph $G = K_{1,n}$, $\gamma_{c}(S(G)) = n - 1$.

**Theorem 2.22.** For any graph $G = K_{1,n}$, $\gamma_{se}(\eta(S(G))) = n - 1$.

**Proof.** Let $V(G) = \{v_1, v_2, v_3, ..., v_n, d(v_n) = n-1\}$ and $E(G) = \{e_i = (v_n, v_i), i = 1, 2, 3, ..., n-1\}$. Let $B = \{w_i/v_i \text{ is the vertex subdividing } e_i\}$ and $D = \{(v_n, w_i), i = 1, 2, 3, ..., n\}$. The corresponding vertices of $D$ in $\eta(S(G))$ covers all the vertices of $\eta(S(G))$ with $|D| = n - 1$. Now for each edge $(w_i, v_j), i = 1$ to $n - 1 \in E(S(G)) - D$, there exists an edge $(w_i, v_n) \in D \cap N(v_i, v_j)$, such that the corresponding vertices of $\{D - (w_i, v_n)\} \cup \{w_i, v_j\}$ in $\eta(S(G))$ is the dominating set of $\eta(S(G))$ and for $v_n \in V(S(G))$, there exists an edge $(v_n, w_i), i \in 1$ to $n - 1$, incident with $v_n$ such that the corresponding vertices of $\{D - (v_n, w_i)\} \cup \{v_n\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma(S(G)) = |D| = n - 1$. $

\square \)

**Proposition 2.23.** For any graph $G = K_n$, $\gamma_{c}(S(G)) = n - 1$.

**Theorem 2.24.** For any graph $G = K_n$, $\gamma_{se}(\eta(S(G))) = n$.

**Proof.** Let $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ and $A = \{n_j/n_j \in V(S(G)) - V(G), j = 1, 2, 3, 4, 5, ..., \}$ with $|A| = \frac{n(n-1)}{2}$. The set $B = \{e_i = (v_i, n_j), i = 1, 2, ..., n - 1, j = 1, 2, 3, 4, 5, ..., \}$ such that $n_j \in N(v_i)$. The corresponding vertices of $B$ in $\eta(S(G))$ will covers all the vertices of $\eta(S(G))$ with $|B| = n - 1$. Suppose if the corresponding vertices of $B \notin \gamma_{se}(\eta(S(G)))$ set, then there exists at least one edge $(v_i, n_j)$ or $(v_n, v_j)$ in $S(G)$ which is not covered by the corresponding vertices of $B$ in $\eta(S(G))$. Now in $S(G)$, consider the set $C = \{B \cup (v_n, n_j)\}$, then for every edge $e_j = (v_i, n_j) \in E(S(G)) - C$ there exists an edge $e_k = (v_i, n_j) \in C \cap N(e_j)$ such that the corresponding vertices of $\{C - e_j \cup e_j\}$ in $\eta(S(G))$ is the dominating set of $\eta(S(G))$. Hence $\gamma_{se}(\eta(S(G))) = n$.$ \square \)

**Proposition 2.25.** For any graph $G = K_{m,n}$, $\gamma_{c}(S(G)) = m + n - 1, m \geq n, m, n \geq 2$.

**Theorem 2.26.** For any graph $G = K_{m,n}$, $\gamma_{se}(\eta(S(G))) = m + n, m \geq n, m \geq 2$.

**Proof.** Let $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ and $A = \{n_j/n_j \in V(S(G)) - V(G), j = 1, 2, 3, 4, ..., \}$ with $|A| = mn$. The set $B = \{(v_i, n_j), d(v_i) = m, n_j \in N(v_i), d(v_i) = n\}$ and $C = \{(v_i, n_j), d(v_i) = m, n_j \in N(v_i), d(v_i) = n\}$. Then the corresponding vertices of $D = B \cup C$ covers all the vertices of $S(\eta(S(G)))$ with $|D| = m + n - 1$. For $e_1 = (v_1, n_j) \in E(S(G)) - D$, there exists an edge $e_2 = (v_1, n_j)$ adjacent to $e_1$ such that $\{B - e_2 \cup e_1\}$ is a not an edge dominating
set of \( S(G) \). Now in \( S(G) \), the set \( E = D \cup (v_i, n_j), d(v_i) = m, v_i \neq v_n \), then for every edge \( e_i = (v_i, n_j \in E(S(G)) - D \) there exists an edge \( e_k = (v_i, n_j) \in D \) such that \( \{E - e_k \cup e_j\} \) is the secure lict dominating set of \( G \). Hence \( \gamma_{se}(S(G)) = m + n \).

**Proposition 2.27.** For any graph \( G = W_n \), \( \gamma_{se}(S(G)) = n - 1 \).

**Theorem 2.28.** For any graph \( G = W_n \), \( \gamma_{se}(\eta(S(G))) = n \).

**Proof.** Let \( V(G) = \{v_1, v_2, v_3, \ldots, v_n, d(v_n) = n-1\} \) and \( A = \{n_j/n_j \in V(S(W_n)) - V(W_n), j = 1, 2, 3, 4, 5, ..., 2(n-1)\} \) with \( |A| = 2(n-1) \). The set \( B = \{e_i = (v_i, n_j), i = 1, 2, ..., n-1, j = 1, 2, 3, 4, 5, ..., 2(n-1)\} \) such that \( n_j \in N(v_n) \). The corresponding vertices of \( B \) will cover all the vertices of \( \eta(S(G)) \) with \( |B| = n-1 \) and therefore \( B \) is the lict dominating set of \( S(G) \). If suppose \( B \) is the secure lict dominating set of \( S(G) \), then there exists at least one vertex \((v_n, n_j) \) or \((v_i, n_j) \), \( i = 1, 2, ..., n-1 \) in \( \eta(G) \) which is not covered by corresponding vertex of \( B \). Now in \( S(G) \), consider the set \( C = \{B \cup (v_n, n_j)\} \), then for every edge \( e_j = (v_i, n_j) \in E(S(G)) - C \) there exists an edge \( e_k = (v_i, n_j) \in C \cap N(e_j) \) such that the corresponding \( \{C - e_k \cup e_j\} \) is the lict dominating set of \( S(G) \). Hence \( \gamma_{se}(S(G)) = n \).

**Theorem 2.29.** [3]: For any graph \( G \) of order \( n \geq 3 \),

(i) \( \beta_1(G) + \beta_1(\bar{G}) \leq 2\lceil \frac{n}{2} \rceil \).

(ii) \( \beta_1(G) \ast \beta_1(\bar{G}) \leq \lceil \frac{n}{2} \rceil^2 \).

**Theorem 2.30.** For any non-separable connected graph \( G \) of order \( n \geq 3 \) vertices,

(i) \( \gamma_{se}(\eta(G)) + \gamma_{se}(\bar{G}) \leq 2\lceil \frac{n}{2} \rceil \).

(ii) \( \gamma_{se}(\eta(G)) \ast \gamma_{se}(\bar{G}) \leq \lceil \frac{n}{2} \rceil^2 \).

**Proof.** Since \( \beta_1(G) \leq \lceil \frac{n}{2} \rceil \), the result follows from 2.17 and using Theorem 2.29.

**Competing interests**

The authors declare that they have no competing interests.

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