Lectures on Heterotic M-Theory

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March 27, 2022

Abstract

We present three lectures on heterotic $M$-theory and a fourth lecture extending this theory to more general orbifolds. In Lecture 1, Hořava-Witten theory is briefly discussed. We then compactify this theory on Calabi-Yau threefolds, choosing the “standard” embedding of the spin connection in the gauge connection. We derive, in detail, both the five-dimensional effective action and the associated actions of the four-dimensional “end-of-the-world” branes. Lecture 2 is devoted to showing that this theory naturally admits static, $N=1$ supersymmetry preserving BPS three-branes, the minimal vacuum having two such branes. One of these, the “visible” brane, is shown to support a three-generation $E_6$ grand unified theory, whereas the other emerges as the “hidden” brane with unbroken $E_8$ gauge group. Thus heterotic $M$-theory emerges as a fundamental paradigm for so-called “brane world” scenarios of particle physics. In Lecture 3, we introduce the concept of “non-standard” embeddings. These are shown to permit a vast generalization of allowed vacua, leading on the visible brane to new grand unified theories, such as $SO(10)$ and $SU(5)$, and to the standard model $SU(3)_C \times SU(2)_L \times U(1)_Y$. It is demonstrated that non-standard embeddings generically imply the existence of five-branes in the bulk space. The physical properties of these bulk branes is discussed in detail. Finally, in Lecture 4 we move beyond Hořava-Witten theory and consider orbifolds larger than $S^1/Z_2$. For explicitness, we consider
$M$-theory orbifolds on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$, discussing their anomaly structure in detail and completely determining both the untwisted and twisted sector spectra.
1 Lecture 1: The Five-Dimensional Effective Theory

In this first lecture, we introduce our notation and briefly discuss the theory of the strongly coupled heterotic string introduced by Hořava and Witten. In this theory, there is an eleven-dimensional bulk space bounded on either end of the $x^{11}$-direction by two “end-of-the-world” ten-dimensional nine-branes, each supporting an $N = 1$, $E_8$ supergauge theory. We then begin our construction of heterotic $M$-theory by compactifying the Hořava-Witten theory on a Calabi-Yau threefold. This leads to a five-dimensional bulk space bounded at the ends of the fifth dimension by two end-of-the-world four-dimensional three-branes. Assuming, in this lecture, the “standard” embedding of the spin connection into one of the $E_8$ gauge connections we derive, in detail, both the five-dimensional bulk space effective action and the associated actions of the four-dimensional boundary branes. We end this lecture by discussing some of the properties of this effective theory and explicitly giving the $N = 2$ supersymmetry transformations of the bulk space quantum fields.

We begin by briefly reviewing the description of strongly coupled heterotic string theory as 11-dimensional supergravity with boundaries, as given by Hořava and Witten [1, 2]. Our conventions are as follows. We will consider eleven-dimensional spacetime compactified on a Calabi-Yau space $X$, with the subsequent reduction down to four dimensions effectively provided by a double-domain-wall background, corresponding to an $S^1/Z_2$ orbifold. We use coordinates $x^I$ with indices $I, J, K, \cdots = 0, \cdots, 9, 11$ to parameterize the full 11–dimensional space $M_{11}$. Throughout these lectures, when we refer to orbifolds, we will work in the “upstairs” picture with the orbifold $S^1/Z_2$ in the $x^{11}$–direction. We choose the range $x^{11} \in [-\pi \rho, \pi \rho]$ with the endpoints being identified. The $Z_2$ orbifold symmetry acts as $x^{11} \rightarrow -x^{11}$.

Then there exist two ten–dimensional hyperplanes fixed under the $Z_2$ symmetry which we denote by $M_{10}^{(i)}$, $i = 1, 2$. Locally, they are specified by the conditions $x^{11} = 0, \pi \rho$. Barred indices $\bar{I}, \bar{J}, \bar{K}, \cdots = 0, \cdots, 9$ are used for the ten–dimensional space orthogonal to the orbifold. We use indices $A, B, C, \cdots = 4, \cdots 9$ for the Calabi–Yau space. All fields will be required to have a definite behaviour under the $Z_2$ orbifold symmetry in $D = 11$. We demand a bosonic field $\Phi$ to be even or odd; that is, $\Phi(x^{11}) = \pm \Phi(-x^{11})$. For a spinor $\Psi$ the condition is $\Gamma_1 \Psi(-x^{11}) = \Psi(x^{11})$ so that the projection to one of the orbifold planes leads to a ten–dimensional Majorana–Weyl spinor with positive chirality. Spinors in eleven dimensions will be Majorana spinors with 32 real components throughout the paper.

The bosonic part of the action is of the form

$$S = S_{SG} + S_{YM}$$

(1.1)
where $S_{SG}$ is the familiar 11–dimensional supergravity

$$S_{SG} = -\frac{1}{2\kappa^2} \int_{M^{11}} \sqrt{-g} \left[ R + \frac{1}{24} G_{IJKL} G^{IJKL} + \sqrt{2} \frac{1}{1728} \epsilon^{I_{11}I_{11}I_{11}} C_{I_{11}I_{11}I_{11}} G_{I_{11}I_{11}} \ldots G_{I_{11}I_{11}} \right]$$  \hspace{1cm} (1.2)$$

and $S_{YM}$ are the two $E_8$ Yang–Mills theories on the orbifold planes explicitly given by

$$S_{YM} = -\frac{1}{8\pi \kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M^{(1)}_{10}} \sqrt{-g} \left\{ \text{tr}(F^{(1)})^2 - \frac{1}{2} \text{tr}R^2 \right\}$$

$$- \frac{1}{8\pi \kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M^{(2)}_{10}} \sqrt{-g} \left\{ \text{tr}(F^{(2)})^2 - \frac{1}{2} \text{tr}R^2 \right\}.$$  \hspace{1cm} (1.3)$$

Here $F^{(i)}_{IJ}$ are the two $E_8$ gauge field strengths and $C_{IJK}$ is the 3–form with field strength $G_{IJKL} = 24 \partial_I C_{JKL}$. In order for the above theory to be supersymmetric and anomaly free, the Bianchi identity for $G$ should be modified such that

$$(dG)_{11IJKL} = -4\sqrt{2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ J^{(1)} \delta(x^{11}) + J^{(2)} \delta(x^{11} - \pi\rho) \right\}_{IJKL}$$  \hspace{1cm} (1.4)$$

where the sources are given by

$$J^{(i)} = \frac{1}{16\pi^2} \left( \text{tr}F^{(i)} \wedge F^{(i)} - \frac{1}{2} \text{tr}R \wedge R \right).$$  \hspace{1cm} (1.5)$$

Under the $Z_2$ orbifold symmetry, the field components $g_{IJ}, g_{11,11}, C_{IJ11}$ are even, while $g_{I11}, C_{IJK}$ are odd.

The modification of the right hand side of equation (1.4) has important consequences. While the standard embedding of the spin connection of the Calabi–Yau threefold into the gauge connection

$$\text{tr}F^{(1)} \wedge F^{(1)} = \text{tr}R \wedge R$$  \hspace{1cm} (1.6)$$

leads to vanishing source terms in the weakly coupled heterotic string Bianchi identity (which, in turn, allows one to set the antisymmetric tensor gauge field to zero), in the present case, one is left with non–zero sources $\pm \text{tr}R \wedge R$ on the two hyperplanes. This follows from the fact that the sources in the Bianchi identity (1.4) are located on the orbifold planes with the gravitational part distributed equally between the two planes. The consequence is that not all components of the antisymmetric tensor field $G$ can vanish. We find, for the standard embedding (1.6), that all components of $G$ vanish with the exception of

$$G_{ABCD} = -\frac{1}{6} \alpha \epsilon_{ABCD}^{EF} \omega_{EF} \epsilon(x^{11})$$  \hspace{1cm} (1.7)$$
where
\[ \alpha = \frac{1}{8\sqrt{2}\pi^{2/3}} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{C_{\omega}} \text{tr} R \wedge R. \] (1.8)

Here \( \epsilon(x^{11}) \) is the step function which is +1 (−1) for \( x^{11} \) positive (negative) and
\[ v = \int_X \sqrt{\Omega} \] (1.9)

where \( \Omega_{AB} \) is a fixed Calabi–Yau metric and \( v \) is the associated volume of the Calabi–Yau threefold. The two–form \( \omega_{AB} \) is the Kähler form associated with \( \Omega_{AB} \) (that is, \( \omega_{\dot{a}b} = i\Omega_{\dot{a}b} \) where \( a \) and \( \dot{b} \) are holomorphic and anti-holomorphic indices) and \( C_{\omega} \) is the Poincare dual four-cycle of \( \omega \). Furthermore, in deriving this result, we have turned off all Calabi–Yau moduli with the exception of the radial breathing mode. This will be sufficient for all applications dealing with the universal moduli.

Phenomenologically, there is a regime where the universe appears five-dimensional. We would, therefore, like to derive an effective theory in the space consisting of the usual four space-time dimensions and the orbifold. We will, for simplicity, consider the universal zero modes only; that is, the five–dimensional graviton supermultiplet and the breathing mode of the Calabi–Yau space, along with its superpartners. These form a hypermultiplet in five dimensions. Furthermore, to keep the discussion as straightforward as possible, we will not consider boundary gauge matter fields. This simple framework suffices to illustrate our main ideas and was presented as such in \[3\]. The general case was presented in \[4\]. Our five-dimensional conventions are the following. Upon reduction on the Calabi-Yau space we have a five-dimensional spacetime \( M_5 \) labeled by indices \( \alpha, \beta, \gamma, \cdots = 0, \cdots, 3, 11 \). The orbifold fixed planes become four-dimensional with indices \( \mu, \nu, \rho, \cdots = 0, \cdots, 3 \). The 11-dimensional Dirac–matrices \( \Gamma^{I} \) with \( \{ \Gamma^{I}, \Gamma^{J} \} = 2g^{IJ} \) are decomposed as \( \Gamma^{I} = \{ \gamma^a \otimes \lambda, 1 \otimes \lambda^A \} \) where \( \gamma^a \) and \( \lambda^A \) are the five– and six–dimensional Dirac matrices, respectively. Here, \( \lambda \) is the chiral projection matrix in six dimensions with \( \lambda^2 = 1 \). In five dimensions we use symplectic-real spinors \( \psi^i \) where \( i = 1, 2 \) is an \( SU(2) \) index, corresponding to the automorphism group of the \( N = 1 \) supersymmetry algebra in five dimensions. We will follow the conventions given in \[3\].

We can perform the Kaluza-Klein reduction on the metric
\[ ds^2_{11} = V^{-2/3} g_{\alpha\beta} dx^\alpha dx^\beta + V^{1/3} \Omega_{AB} dx^A dx^B. \] (1.10)

Since the compactification is on a Calabi–Yau manifold, the background corresponding to metric \([10]\) preserves eight supercharges, the appropriate number for a reduction down to
five dimensions. It might appear that we are simply performing a standard reduction of 11-dimensional supergravity on a Calabi–Yau space to five dimensions; for example, in the way described in ref. [4]. There are, however, two important ingredients that we have not yet included. One is obviously the existence of the boundary theories. We will return to this point shortly. First, however, let us explain a somewhat unconventional addition to the bulk theory that must be included.

Specifically, for the nonvanishing component $G_{ABCD}$ in eq. (1.7) there is no corresponding zero mode field $\Gamma$. Therefore, in the reduction, we should take this part of $G$ explicitly into account. In the terminology of ref. [8], such an antisymmetric tensor field configuration is called a “non–zero mode”. A more recent name for such a field configuration is a nonvanishing “$G$–flux”. More generally, a non–zero mode is a background antisymmetric tensor field that solves the equations of motion but, unlike antisymmetric tensor field moduli, has nonvanishing field strength. Such configurations, for a $p$–form field strength, can be identified with the cohomology group $H^p(M)$ of the manifold $M$ and, in particular, exist if this cohomology group is nontrivial. In the case under consideration, the relevant cohomology group is $H^4(X)$ which is nontrivial for a Calabi–Yau manifold $X$ since $h^{2,2} = h^{1,1} \geq 1$.

Again, the form of $G_{ABCD}$ in eq. (1.7) is somewhat special, reflecting the fact that we are concentrating here on the universal moduli. In the general case, $G_{ABCD}$ would be a linear combination of all harmonic $(2, 2)$–forms.

The complete configuration for the antisymmetric tensor field that we use in the reduction is given by

$$
C_{\alpha\beta\gamma} , \quad G_{\alpha\beta\gamma\delta} = 24 \partial [\alpha C_{\beta\gamma\delta}]
$$

$$
C_{\alpha AB} = \frac{1}{6} A_{\alpha \omega AB} , \quad G_{\alpha\beta AB} = \mathcal{F}_{\alpha\beta \omega AB} , \quad \mathcal{F}_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} \quad (1.11)
$$

and the non–zero mode is

$$
G_{ABCD} = -\frac{\alpha}{6} \epsilon_{ABCD}^{EF} \omega_{EF} \epsilon(x^{11}) , \quad (1.12)
$$

where $\alpha$ was defined in eq. (1.8). Here, $\omega_{ABC}$ is the harmonic $(3, 0)$ form on the Calabi–Yau space and $\xi$ is the corresponding (complex) scalar zero mode. In addition, we have a five-

\footnote{This can be seen from the mixed part of the Bianchi identity $\partial_{\alpha} G_{ABCD} = 0$ which shows that the constant $\alpha$ in eq. (1.7) cannot be promoted as stands to a five–dimensional field. It is possible to dualize in five dimensions so the constant $\alpha$ is promoted to a five-form field, but we will not pursue this formulation here.}
dimensional vector field $\mathcal{A}_\alpha$ and 3–form $C_{\alpha\beta\gamma}$, which can be dualized to a scalar $\sigma$. The total bulk field content of the five–dimensional theory is then given by the gravity multiplet
\begin{equation}
(g_{\alpha\beta}, \mathcal{A}_\alpha, \psi^i_\alpha)
\end{equation}

(1.13)

together with the universal hypermultiplet
\begin{equation}
(V, \sigma, \xi, \bar{\xi}, \zeta^i).
\end{equation}

(1.14)

Here $\psi^i_\alpha$ and $\zeta^i$ are the gravitini and the hypermultiplet fermions respectively and $i = 1, 2$ since they each form a doublet under the $SU(2)$ automorphism group of $N = 2$ supersymmetry in five dimensions. From their relations to the 11–dimensional fields, it is easy to see that $g_{\mu\nu}, g_{11,11}, A_{11}, \sigma$ must be even under the $Z_2$ action whereas $g_{\mu11}, A_{\mu}, \xi$ must be odd.

Examples of compactifications with non–zero modes in pure 11–dimensional supergravity on various manifolds including Calabi–Yau three–folds have been studied in ref. [9]. There is, however, one important way in which our non–zero mode differs from other non–zero modes in pure 11–dimensional supergravity. Whereas the latter may be viewed as an optional feature of generalized Kaluza-Klein reduction, the non–zero mode in Hořava–Witten theory that we have identified cannot be turned off. This can be seen from the fact that the constant $\alpha$ in expression (1.12) cannot be set to zero, unlike the case in pure 11–dimensional supergravity where it would be arbitrary, since it is fixed by eq. (1.8) in terms of Calabi–Yau data. This fact is, of course, intimately related to the existence of the boundary source terms, particularly in the Bianchi identity (1.4).

Let us now turn to a discussion of the boundary theories. In the five–dimensional space $M_5$ of the reduced theory, the orbifold fixed planes constitute four–dimensional hypersurfaces which we denote by $M^{(i)}_4$, $i = 1, 2$. Clearly, since we have used the standard embedding, there will be an $E_6$ gauge field $A^{(1)}_\mu$ accompanied by gauginos and gauge matter fields on the orbifold plane $M^{(1)}_4$. For simplicity, we will set these gauge matter fields to zero in the following. The field content of the orbifold plane $M^{(2)}_4$ consists of an $E_8$ gauge field $A^{(2)}_\mu$ and the corresponding gauginos. In addition, there is another important boundary effect which results from the non–zero internal gauge field and gravity curvatures. More precisely, for the standard embedding defined in (1.6)
\begin{equation}
\int_X \sqrt{6g}\, \text{tr} F^{(1)}_{AB} F^{(1)AB} = \int_X \sqrt{6g}\, \text{tr} R_{AB} R^{AB} = 16\sqrt{2}\pi v \left(\frac{4\pi}{\kappa}\right)^{2/3} \alpha, \quad F^{(2)}_{AB} = 0.
\end{equation}

(1.15)

In view of the boundary actions (1.3), it follows that we will retain cosmological type terms with opposite signs on the two boundaries. Note that the size of those terms is set by the same constant $\alpha$, given by eq. (1.8), which determines the magnitude of the non–zero mode.
We can now compute the five–dimensional effective action of Hoˇ raˇva–Witten theory. Using the field configuration (1.10)–(1.15) we find from the action (1.1)–(1.3) that

$$S_5 = S_{\text{grav}} + S_{\text{hyper}} + S_{\text{bound}}$$  \hspace{1cm} (1.16)

where

$$S_{\text{grav}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \sqrt{-g} \left[ R + \frac{3}{2} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{\sqrt{2}} \epsilon^{\alpha\beta\gamma\delta\epsilon} A_{\alpha} F_{\beta\gamma} F_{\delta\epsilon} \right]$$  \hspace{1cm} (1.17a)

$$S_{\text{hyper}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \sqrt{-g} \left[ \frac{1}{2} V^{-2} \partial_{\alpha} V \partial^{\alpha} V + 2V^{-1} \partial_{\alpha} \xi \partial^{\alpha} \bar{\xi} + \frac{1}{24} V^2 G_{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} ight. \\
\left. + \frac{\sqrt{2}}{24} \epsilon^{\alpha\beta\gamma\delta} G_{\alpha\beta\gamma\delta} \left( i(\xi \partial_{\epsilon} \bar{\xi} - \bar{\xi} \partial_{\epsilon} \xi) + 2\alpha A_{\epsilon} \right) + \frac{1}{3} V^{-2} \alpha^2 \right]$$  \hspace{1cm} (1.17b)

$$S_{\text{bound}} = -\frac{1}{2\kappa_5^2} \left\{ 2\sqrt{2} \int_{M_4^{(1)}} \sqrt{-g} V^{-1} \alpha - 2\sqrt{2} \int_{M_4^{(2)}} \sqrt{-g} V^{-1} \alpha \right\} \\
- \frac{1}{16\pi\alpha_{\text{GUT}}} \sum_{i=1}^{2} \int_{M_4^{(i)}} \sqrt{-g} V \text{tr} F_{\mu\nu}^{(i)} \right]^2.$$  \hspace{1cm} (1.17c)

In this expression, we have now dropped higher-derivative terms. The 4–form field strength $G_{\alpha\beta\gamma\delta}$ is subject to the Bianchi identity

$$\left( dG \right)_{11\mu\nu\rho\sigma} = -\frac{2\sqrt{2}\pi\kappa_5^2}{\alpha_{\text{GUT}}} \left\{ J^{(1)} \delta(x^{11}) + J^{(2)} \delta(x^{11} - \pi\rho) \right\}_{\mu\nu\rho\sigma}$$  \hspace{1cm} (1.18)

which follows directly from the 11–dimensional Bianchi identity (1.4). The currents $J^{(i)}$ have been defined in eq. (1.5). The five–dimensional Newton constant $\kappa_5$ and the Yang–Mills coupling $\alpha_{\text{GUT}}$ are expressed in terms of 11–dimensional quantities as

$$\kappa_5^2 = \frac{\kappa^2}{v}, \quad \alpha_{\text{GUT}} = \frac{\kappa^2}{2v} \left( \frac{4\pi}{\kappa} \right)^{2/3}.$$  \hspace{1cm} (1.19)

We have checked the consistency of the truncation which leads to the above action by an explicit reduction of the 11–dimensional equations of motion to five dimensions. Note that the potential terms in the bulk and on the boundaries arise precisely from the inclusion of the non–zero mode and the gauge and gravity field strengths, respectively. Since we have compactified on a Calabi–Yau space, we expect the bulk part of the above action to have eight preserved supercharges and, therefore, to correspond to minimal $N = 1$ supergravity in five dimensions. Accordingly, let us compare the result (1.17) to the known $N = 1$ supergravity–matter theories in five dimensions \[3, 10, 11, 12\].
In these theories, the scalar fields in the universal hypermultiplet parameterize a quaternionic manifold with coset structure $\mathcal{M}_Q = SU(2, 1)/SU(2) \times U(1)$. Hence, to compare our action to these we should dualize the three–form $C_{\alpha\beta\gamma}$ to a scalar field $\sigma$ by setting (in the bulk)

$$G_{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{2}} V^{-2} \epsilon_{\alpha\beta\gamma\delta\epsilon} \left( \partial^\epsilon \sigma - i (\xi \partial^\epsilon \bar{\xi} - \bar{\xi} \partial^\epsilon \xi) - 2 \alpha \epsilon(x^{11}) A^\epsilon \right). \quad (1.20)$$

Then the hypermultiplet part of the action $(1.17b)$ can be written as

$$S_{\text{hyper}} = -\frac{v}{2K^2} \int_{M_5} \sqrt{-g} \left[ h_{uv} \nabla_\alpha q^u \nabla^\alpha q^v + \frac{1}{3} V^{-2} \alpha^2 \right] \quad (1.21)$$

where $q^u = (V,\sigma,\xi,\bar{\xi})$. The covariant derivative $\nabla_\alpha$ is defined as $\nabla_\alpha q^u = \partial_\alpha q^u - \alpha \epsilon(x^{11}) A_\alpha k^u$ with $k^u = (0,-2,0,0)$. The sigma model metric $h_{uv} = \partial_u \partial_v K_Q$ can be computed from the Kähler potential

$$K_Q = -\ln(S + \bar{S} - 2C\bar{C}) , \quad S = V + \xi \bar{\xi} + i\sigma , \quad C = \xi . \quad (1.22)$$

Consequently, the hypermultiplet scalars $q^u$ parameterize a Kähler manifold with metric $h_{uv}$. It can be demonstrated that $k^u$ is a Killing vector on this manifold. Using the expressions given in ref. [13], one can show that this manifold is quaternionic with coset structure $\mathcal{M}_Q$. Hence, the terms in eq. (1.21) that are independent of $\alpha$ describe the known form of the universal hypermultiplet action. How do we interpret the extra terms in the hypermultiplet action depending on $\alpha$? A hint is provided by the fact that one of these $\alpha$-dependent terms modifies the flat derivative in the kinetic energy to a generalized derivative $\nabla_\alpha$. This is exactly the combination that we would need if one wanted to gauge the $U(1)$ symmetry on $\mathcal{M}_Q$ corresponding to the Killing vector $k^u$, using the gauge field $A_\alpha$ in the gravity supermultiplet. In fact, investigation of the other terms in the action, including the fermions, shows that the resulting five-dimensional theory is precisely a gauged form of supergravity. Not only is a $U(1)$ isometry of $\mathcal{M}_Q$ gauged, but at the same time a $U(1)$ subgroup of the $SU(2)$ automorphism group is also gauged.

What about the remaining $\alpha$-dependent potential term in the hypermultiplet action? From $D = 4, N = 2$ theories, we are used to the idea that gauging a symmetry of the quaternionic manifold describing hypermultiplets generically introduces potential terms into the action when supersymmetry is preserved (see for instance [14]). Such potential terms can be thought of as the generalization of pure Fayet-Iliopoulos terms. This is precisely what happens in our theory as well, with the gauging of the $U(1)$ subgroup inducing the
α-dependent potential term in (1.21). The general gauged action was discussed in detail in [4].

The phenomenon that the inclusion of non-zero modes leads to gauged supergravity theories has already been observed in type II Calabi-Yau compactifications [13, 16]. From the form of the Killing vector, we see that it is only the scalar field \(\sigma\), dual to the 4–form \(G_{\alpha\beta\gamma\delta}\), which is charged under the \(U(1)\) symmetry. Its charge is fixed by \(\alpha\). We note that this charge is quantized since, suitably normalized, \(\text{tr} R \wedge R\) is an element of \(H^{2,2}(X,\mathbb{Z})\).

To analyze the supersymmetry properties of the solutions shortly to be discussed, we need the supersymmetry variations of the fermions associated with the theory (1.16). They can be obtained either by a reduction of the 11–dimensional gravitino variation or by generalizing the known five–dimensional transformations [6, 12] by matching onto gauged four–dimensional \(N = 2\) theories. It is sufficient for our purposes to keep the bosonic terms only. Both approaches lead to

\[
\delta \psi^i_\alpha = D_\alpha \epsilon^i + \frac{\sqrt{2}i}{8} (\gamma^\beta \gamma^\gamma - 4\delta^\beta_\gamma \gamma^\gamma) F_{\beta\gamma} \epsilon^i - \frac{1}{2} V^{-1/2} \left( \partial_\alpha \xi (\tau_i - i\tau_2)^j - \partial_\alpha \bar{\xi} (\tau_1 + i\tau_2)^j \right) \epsilon^j
\]

\[
= -\frac{\sqrt{2}i}{96} V \epsilon_{\alpha}^{\beta\gamma\delta} G_{\beta\gamma\delta} (\tau_3) e^j \epsilon^i + \frac{\sqrt{2}}{12} \alpha V^{-1} \epsilon (x^{11}) \gamma (\tau_3)^j e^j
\]

\[
\delta \bar{\zeta}^i = \frac{\sqrt{2}}{48} V \epsilon^{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} \gamma e^i - \frac{i}{2} \sqrt{V^{-1}} \gamma (\tau^i - i\tau_2)^j + \partial_\alpha \bar{\xi} (\tau_1 + i\tau_2)^j \right) \epsilon^j
\]

\[
= \frac{i}{2} V^{-1} \gamma \partial \bar{\gamma} \epsilon^i + \frac{i}{\sqrt{2}} \alpha V^{-1} \epsilon (x^{11}) (\tau_3)^j e^j
\]

where \(\tau_i\) are the Pauli spin matrices.

In summary, we see that the relevant five-dimensional effective theory for the reduction of Hořava-Witten theory is a gauged \(N = 1\) supergravity theory with bulk and boundary potentials.

2 Lecture 2: The Domain Wall Solution and Generalizations

In the second lecture, we show that the effective five-dimensional bulk space theory does not have flat space for its static vacuum. Instead, the theory naturally admits static, \(N = 1\) supersymmetry preserving BPS three-branes, the minimal vacuum consisting of two end-of-the-world three-branes. One of these branes, the one with the spin connection embedded in the gauge connection, supports a three generation \(E_6\) grand unified theory and, hence, is
called the “visible” or physical brane. The other brane is the “hidden” brane with an unbroken \(E_8\) supergauge theory. Thus, heterotic \(M\)-theory emerges as a fundamental paradigm for so-called “brane world” scenarios of particle physics. In the second part of this lecture, we generalize the results of Lecture 1 to include, not just the universal hypermultiplet, but all \((1,1)\)-moduli in the bulk space, as well as matter scalar multiplets on the boundary three-branes.

In order to re-construct the \(D = 4, N = 1\) effective theory originally discussed in \([17, 18, 21, 22, 23, 24, 26, 27, 28, 29]\), we expect there to be a three–brane domain wall in five dimensions with a worldvolume lying in the four uncompactified directions. These solutions should break half the supersymmetry of the five–dimensional bulk theory and preserve Poincaré invariance in four dimensions. This domain wall can be viewed as the “vacuum” of the five–dimensional theory, in the sense that it provides the appropriate background for a reduction to the \(D = 4, N = 1\) effective theory.

We notice that the theory \([1, 10]\) has all of the prerequisites necessary for such a three–brane solution to exist. Generally, in order to have a \((D - 2)\)-brane in a \(D\)-dimensional theory, one needs to have a \((D - 1)\)-form field or, equivalently, a cosmological constant. This is familiar from the eight–brane \([30]\) in the massive type IIA supergravity in ten dimensions \([31]\), and has been systematically studied for theories in arbitrary dimension obtained by generalized (Scherk-Schwarz) dimensional reduction \([32]\). In our case, this cosmological term is provided by the bulk potential term in the action \([1, 10]\). From the viewpoint of the bulk theory, we could have multi three–brane solutions with an arbitrary number of parallel branes located at various places in the \(x^{11}\) direction. As is well known, however, elementary brane solutions have singularities at the location of the branes, needing to be supported by source terms. The natural candidates for those source terms, in our case, are the boundary actions. Given the anomaly-cancellation requirements, this restricts the possible solutions to those representing a pair of parallel three–branes corresponding to the orbifold planes.

From the above discussion, it is clear that in order to find a three-brane solution, we should start with the Ansatz

\[
\begin{align*}
\boldsymbol{ds^2} & = a(y)^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2 dy^2 \\
V & = V(y)
\end{align*}
\]

where \(a\) and \(b\) are functions of \(y = x^{11}\) and all other fields vanish. The general solution for
this Ansatz, satisfying the equations of motion derived from action (1.16), is given by

\[
\begin{align*}
    a &= a_0 H^{1/2} \\
    b &= b_0 H^2 \\
    V &= b_0 H^3
\end{align*}
\]  

(2.2)

where \( a_0, b_0 \) and \( c_0 \) are constants. We note that the boundary source terms have fixed the form of the harmonic function \( H \) in the above solution. Without specific information about the sources, the function \( H \) would generically be glued together from an arbitrary number of linear pieces with slopes \( \pm \sqrt{2/3} \alpha \). The edges of each piece would then indicate the location of the source terms. The necessity of matching the boundary sources at \( y = 0 \) and \( \pi \rho \), however, has forced us to consider only two such linear pieces, namely \( y \in [0, \pi \rho] \) and \( y \in [-\pi \rho, 0] \). These pieces are glued together at \( y = 0 \) and \( \pi \rho \) (recall here that we have identified \( \pi \rho \) and \( -\pi \rho \)). Therefore, we have

\[
\partial^2_y H = -\frac{2\sqrt{2}}{3} \alpha (\delta(y) - \delta(y - \pi \rho))
\]  

(2.3)

which shows that the solution represents two parallel three–branes located at the orbifold planes. We stress that this solution solves the five–dimensional theory (1.16) exactly, and is valid to all orders in \( \kappa \).

Of course, we still have to check that our solution preserves half of the supersymmetries. When \( g_{\alpha\beta} \) and \( V \) are the only non–zero fields, the supersymmetry transformations (1.23) simplify to

\[
\begin{align*}
    \delta \psi^i_\alpha &= D_\alpha \epsilon^i + \sqrt{\frac{2}{12}} \alpha \epsilon(y) V^{-1} \gamma_\alpha (\tau_3)^i_j \epsilon^j \\
    \delta \zeta^i &= \frac{i}{2} V^{-1} \gamma_\beta \partial^\beta V \epsilon^i + \frac{i}{\sqrt{2}} \alpha \epsilon(y) V^{-1} (\tau_3)^i_j \epsilon^j.
\end{align*}
\]

The Killing spinor equations \( \delta \psi^i_\alpha = 0, \delta \zeta^i = 0 \) are satisfied for the solution (2.2) if we require that the spinor \( \epsilon^i \) is given by

\[
\epsilon^i = H^{1/4} \epsilon^i_0, \quad \gamma_{11} \epsilon^i_0 = (\tau_3)^i_j \epsilon^j_0
\]  

(2.4)

where \( \epsilon^i_0 \) is a constant symplectic Majorana spinor. This shows that we have indeed found a BPS solution preserving four of the eight bulk supercharges.

Let us discuss the meaning of this solution in some detail. First, we notice that it fits into the general scheme of domain wall solutions in various dimensions. It is, however,
a new solution to the gauged supergravity action (1.16) in five dimensions which has not been constructed previously. In addition, its source terms are naturally provided by the boundary actions resulting from Hořava–Witten theory. Most importantly, it constitutes the fundamental vacuum solution of a phenomenologically relevant theory. The two parallel three–branes of the solution, separated by the bulk, are oriented in the four uncompactified space–time dimensions, and carry the physical low–energy gauge and matter fields. Therefore, from the low–energy point of view where the orbifold is not resolved the three–brane worldvolume is identified with four–dimensional space–time. In this sense the Universe lives on the worldvolume of a three–brane.

Thus far, we have limited the discussion to the universal hypermultiplet only, coupled to \( N = 1 \) five–dimensional gauged supergravity. This result can be extended in a straighforward fashion to include all the \((1,1)\) moduli of the Calabi–Yau threefold. We will not, however, explicitly include the \((2,1)\) sector as it is largely unaffected by the specific structure of Hořava–Witten theory. We now explain the generalized structure of the zero mode fields used in the reduction to five dimensions. We begin with the bulk space. Including the zero modes, the metric is given by

\[
d s^2 = V^{-2/3} g_{\alpha\beta} dx^\alpha dx^\beta + g_{AB} dx^A dx^B
\]  

(2.5)

where \( g_{AB} \) is the metric of the Calabi–Yau space \( X \). Its Kähler form is defined by \( \omega_{\bar{a}b} = i g_{\bar{a}b} \) and can be expanded in terms of the harmonic \((1,1)\)–forms \( \omega_{iAB} \), \( i = 1, \cdots, h^{1,1} \) as

\[
\omega_{AB} = a^i \omega_{iAB} .
\]  

(2.6)

The coefficients \( a^i = a^i(x^\alpha) \) are the \((1,1)\) moduli of the Calabi–Yau space. The Calabi–Yau volume modulus \( V = V(x^\alpha) \) is defined by

\[
V = \frac{1}{v} \int_X \sqrt{\det g}
\]  

(2.7)

where \( \det g \) is the determinant of the Calabi–Yau metric \( g_{AB} \) and \( v \) is defined in (1.9). The modulus \( V \) then measures the Calabi–Yau volume in units of \( v \). The factor \( V^{-2/3} \) in eq. (2.3) has been chosen such that the metric \( g_{\alpha\beta} \) is the five–dimensional Einstein frame metric. Clearly \( V \) is not independent of the \((1,1)\) moduli \( a^i \) but it can be expressed as

\[
V = \frac{1}{6} \mathcal{K}(a) , \quad \mathcal{K}(a) = d_{ijk} a^i a^j a^k
\]  

(2.8)

where \( \mathcal{K}(a) \) is the Kähler potential and \( d_{ijk} \) are the Calabi–Yau intersection numbers.
Let us now turn to the zero modes of the antisymmetric tensor field. We have the potentials and field strengths,

\[ C_{\alpha\beta\gamma} , \quad G_{\alpha\beta\gamma\delta} \]
\[ C_{\alpha AB} = \frac{1}{6} \mathcal{A}_\alpha \omega_{iAB} , \quad G_{\alpha\beta AB} = \mathcal{F}_\alpha^i \omega_{iAB} \]
\[ C_{abc} = \frac{1}{6} \xi \omega_{abc} , \quad G_{aabc} = X_\alpha \omega_{abc} . \]

The five-dimensional fields are therefore an antisymmetric tensor field \( C_{\alpha\beta\gamma} \) with field strength \( G_{\alpha\beta\gamma\delta} \), \( h^{1,1} \) vector fields \( \mathcal{A}_\alpha^i \) with field strengths \( \mathcal{F}_\alpha^i \) and a complex scalar \( \xi \) with field strength \( X_\alpha \) that arises from the harmonic \((3,0)\) form denoted by \( \omega_{abc} \). In the bulk the relations between those fields and their field strengths are simply

\[ G_{\alpha\beta\gamma\delta} = 24 \partial_{[\alpha} C_{\beta\gamma\delta]} \]
\[ \mathcal{F}_\alpha^i = \partial_\alpha \mathcal{A}^i_\alpha - \partial_\beta \mathcal{A}^i_\beta \]
\[ X_\alpha = \partial_\alpha \xi . \]

These relations, however, will receive corrections from the boundary controlled by the 11-dimensional Bianchi identity \((1.4)\). We will derive the associated five-dimensional Bianchi identities later.

Next, we should set up the structure of the boundary fields. The starting point is the standard embedding of the spin connection in the first \( E_8 \) gauge group such that

\[ \text{tr} F^{(1)} \wedge F^{(1)} = \text{tr} R \wedge R . \]

As a result, we have an \( E_6 \) gauge field \( \mathcal{A}_\alpha^{(1)} \) with field strength \( F^{(1)}_{\mu\nu} \) on the first hyperplane and an \( E_8 \) gauge field \( \mathcal{A}_\mu^{(2)} \) with field strength \( F^{(2)}_{\mu\nu} \) on the second hyperplane. In addition, there are \( h^{1,1} \) gauge matter fields from the \((1,1)\) sector on the first plane. They are specified by

\[ \mathcal{A}_b^{(1)} = \bar{A}_b + \omega_{ib} T_{cp} C^{ip} \]

where \( \bar{A}_b \) is the (embedded) spin connection. Furthermore, \( p, q, r, \ldots = 1, \ldots, 27 \) are indices in the fundamental \( 27 \) representation of \( E_6 \) and \( T_{ap} \) are the \((3,27)\) generators of \( E_8 \) that arise in the decomposition under the subgroup \( SU(3) \times E_6 \). Their complex conjugate is denoted by \( T^a_p \). The \( C^{ip} \) are \( h^{1,1} \) complex scalars in the \( 27 \) representation of \( E_6 \). Useful traces for these generators are \( \text{tr}(T_{ap} T^{bp}) = \delta^b_a \delta^q_p \) and \( \text{tr}(T_{ap} T_{bq} T_{cr}) = \omega_{abc} f_{pqr} \) where \( f_{pqr} \) is the totally symmetric tensor that projects out the singlet in \( 27^3 \).
So far, what we have considered is similar to a reduction of pure 11–dimensional supergravity on a Calabi–Yau space, as for example performed in ref. [7], with the addition of gauge and gauge matter fields on the boundaries. An important difference arises, however, because the standard embedding (2.11), unlike in the case of the weakly coupled heterotic string, no longer leads to vanishing sources in the Bianchi identity (1.4). Instead, there is a net five-brane charge, with opposite sources on each fixed plane, proportional to $\pm \text{tr} R \wedge R$.

The nontrivial components of the Bianchi identity (1.4) are given by

$$(dG)_{11ABCD} = -\frac{1}{4\sqrt{2\pi}} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ \delta(x^{11}) - \delta(x^{11} - \pi \rho) \right\} (\text{tr} R \wedge R)_{ABCD} .$$

(2.13)

As a result, the components $G_{ABCD}$ of the antisymmetric tensor field are nonvanishing. We find that

$$G_{ABCD} = -\frac{1}{4V} \alpha^i \epsilon_{ABCD}^{EF} \omega_{iEF} \epsilon(x^{11})$$

(2.14)

where

$$\alpha_i = \frac{1}{8\sqrt{2\pi}} \left( \frac{\kappa}{4\pi} \right)^{2/3} \frac{1}{\sqrt{2}/3} \int_{C_i} \text{tr} R \wedge R .$$

(2.15)

Here, the four–cycles $C_i$ are the Poincare duals of the harmonic $(1,1)$–forms $\omega_i$. The index of the coefficient $\alpha^i$ in the second part of the first equation has been raised using the inverse of the metric

$$G_{ij}(a) = \frac{1}{2V} \int_X \omega_i \wedge (*\omega_j)$$

(2.16)

on the $(1,1)$ moduli space. Note that, while the coefficients $\alpha_i$ with lowered index are truly constants, as is apparent from eq. (2.13), the coefficients $\alpha^i$ depend on the $(1,1)$ moduli $a^i$ since the metric (2.16) does. We can derive an expression for the boundary $\text{tr} F^2$ and $\text{tr} R^2$ terms in the action essential for the reduction of the boundary theories. We have

$$\text{tr} R_{AB} R^{AB} = \text{tr} F_{AB}^{(1)} F^{(1)AB} = 4\sqrt{2\pi} \left( \frac{4\pi}{\kappa} \right)^{2/3} V^{-1} \alpha^i \omega^{AB} \omega_{iAB}$$

(2.17)

while, of course

$$\text{tr} F_{AB}^{(2)} F^{(2)AB} = 0 .$$

(2.18)

The expression (2.14) for $G_{ABCD}$ with $\alpha_i$ as defined in (2.15) is, as previously discussed, the new and somewhat unconventional ingredient in our reduction. This configuration for
the antisymmetric tensor field strength is the generalized nonzero mode or $G$–flux. Generally, a nonzero mode is defined as a nonzero internal antisymmetric tensor field strength $G$ that solves the equation of motion. In contrast, conventional zero modes of an antisymmetric tensor field, like those in eq. (2.10), have vanishing field strength once the moduli fields are set to constants. Since the kinetic term $G^2$ is positive for a nonzero mode it corresponds to a nonzero energy configuration. Given that nonzero modes, for a $p$–form field strength, satisfy

$$dG = d^* G = 0 \quad (2.19)$$

they correspond to harmonic forms of degree $p$. Hence, they can be identified with the $p$th cohomology group $H^p(X)$ of the internal manifold $X$. In the present case, we are dealing with a four–form field strength on a Calabi–Yau threefold $X$ so that the relevant cohomology group is $H^4(X)$. The expression (2.14) is just an expansion of the nonzero mode in terms of the basis of $H^4(X)$. The appearance of all harmonic $(2,2)$ forms shows that it is necessary to include the complete $(1,1)$ sector into the low energy effective action in order to fully describe the nonzero mode. On the other hand, harmonic $(2,1)$ forms do not appear here and are, hence, less important in our context. We stress that the nonzero mode (2.14), for a given Calabi–Yau space, specifies a fixed element in $H^4(X)$ since the coefficients $\alpha_i$ are fixed in terms of Calabi–Yau properties. Thus we see that, correctly normalized, $G$ is in the integer cohomology of the Calabi-Yau manifold. We emphasize that in heterotic $M$-theory, we are not free to turn off the non–zero mode. Its presence is simply dictated by the nonvanishing boundary sources.

Let us now summarize the field content which we have obtained above and discuss how it fits into the multiplets of five–dimensional $N = 1$ supergravity. We know that the gravitational multiplet should contain one vector field, the graviphoton. Thus, since the reduction leads to $h^{1,1}$ vectors, we must have $h^{1,1} - 1$ vector multiplets. This leaves us with the $h^{1,1}$ scalars $a^i$, the complex scalar $\xi$ and the three-form $C_{\alpha\beta\gamma}$. Since there is one scalar in each vector multiplet, we are left with three unaccounted for real scalars (one from the set of $a^i$, and $\xi$) and the three-form. Together, these fields form the “universal hypermultiplet,” universal because it is present independently of the particular form of the Calabi-Yau manifold. From this, it is clear that it must be the overall volume breathing mode $V = \frac{1}{6} d_{ijk} a^i a^j a^k$ that is the additional scalar from the set of the $a^i$ which enters the universal multiplet. The three-form may appear a little unusual, but recall that in five dimensions a three-form is dual to a scalar $\sigma$. Thus, the bosonic sector of the universal hypermultiplet consists of the
four scalars \((V, \sigma, \xi, \bar{\xi})\), as presented previously.

The \(h^{1,1} - 1\) vector multiplet scalars are the remaining \(a^i\). More properly, since the breathing mode \(V\) is already part of a hypermultiplet it should be first scaled out when defining the shape moduli

\[
b^i = V^{-1/3} a^i .
\]

Note that the \(h^{1,1}\) moduli \(b^i\) represent only \(h^{1,1} - 1\) independent degrees of freedom as they satisfy the constraint

\[
K(b) \equiv d_{ijk} b^i b^j b^k = 6 .
\]

The graviton and graviphoton of the gravity multiplet are given by

\[
(g_{\alpha\beta}, \frac{2}{3} b_i A_0^i).
\]

Therefore, in total, the five dimensional bulk theory contains a gravity multiplet, the universal hypermultiplet and \(h^{1,1} - 1\) vector multiplets. The inclusion of the \((2,1)\) sector of the Calabi–Yau space would lead to an additional \(h^{2,1}\) set of hypermultiplets in the theory. Since they will not play a prominent rôle in our context they will not be explicitly included in the following.

On the boundary \(M_4^{(1)}\) we have an \(E_6\) gauge multiplet \((A_{\mu}^{(1)}, \chi^{(1)})\) and \(h^{1,1}\) chiral multiplets \((C^{ip}, \eta^{ip})\) in the fundamental \textbf{27} representation of \(E_6\). Here \(C^{ip}\) denote the complex scalars and \(\eta^{ip}\) the chiral fermions. The other boundary, \(M_4^{(2)}\), carries an \(E_8\) gauge multiplet \((A_{\mu}^{(2)}, \chi^{(2)})\) only. Inclusion of the \((2,1)\) sector would add \(h^{2,1}\) chiral multiplets in the \textbf{27} representation of \(E_6\) to the field content of the boundary \(M_4^{(1)}\). Any even bulk field will also survive on the boundary. Thus, in addition to the four–dimensional part of the metric, the scalars \(b^i\) together with \(A_i^i\), and \(V\) and \(\sigma\) survive on the boundaries. These pair into \(h^{1,1}\) chiral multiplets.

We are now ready to derive the bosonic part of the five–dimensional effective action for the \((1,1)\) sector. Inserting the expressions for the various fields into the 11-dimensional supergravity action (1.1) and dropping higher derivative terms we find

\[
S_5 = S_{\text{grav,vec}} + S_{\text{hyper}} + S_{\text{bound}} + S_{\text{matter}}
\]
with

\[
S_{\text{grav, vec}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \sqrt{-g} \left[ R + G_{ij} \partial_\alpha b^i \partial^\alpha b^j + G_{ij} F^{i}_{\alpha\beta} F^{j}_{\alpha\beta} + \sqrt{2} \frac{\epsilon}{12} \epsilon_{\alpha\beta\gamma\delta\epsilon} d_{ijk} A^i_{\alpha} F^j_{\beta\gamma} F^k_{\delta\epsilon} \right] 
\]

\[
S_{\text{hyper}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \sqrt{-g} \left[ \frac{1}{2} V^{-2} \partial_\alpha V \partial^\alpha V + 2V^{-1} X_\alpha X^\alpha + \frac{1}{24} V^2 G_{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} + \frac{\sqrt{2}}{24} \epsilon_{\alpha\beta\gamma\delta} \epsilon^{\alpha\beta\gamma\delta} \left( i(\xi_\alpha - \bar{\xi} X_\alpha) - 2\epsilon(x^{11}) \alpha_i A^i_\alpha \right) \right] \]

\[
S_{\text{bound}} = -\frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(1)}} \sqrt{-g} V^{-1} \alpha_i b^i + \frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(2)}} \sqrt{-g} V^{-1} \alpha_i b^i \]

\[
S_{\text{matter}} = -\frac{1}{16\pi \alpha_{\text{GUT}}} \sum_{n=1}^{2} \int_{M_4^{(n)}} \sqrt{-g} V \text{tr} F^{(n)}_{\mu\nu} \]

\[
S_{\text{matter}} = -\frac{1}{2\pi \alpha_{\text{GUT}}} \int_{M_4^{(1)}} \sqrt{-g} \left[ G_{ij} (D_\mu C)^i(D_\nu \bar{C})^j + V^{-1} G^{ij} \frac{\partial W}{\partial C^p} \frac{\partial \bar{W}}{\partial \bar{C}^p} + D^{(n)} D^{(n)} \right].
\]

All fields in this action that originate from the 11–dimensional antisym tensor field are subject to a nontrivial Bianchi identity. Specifically, from eq. (1.4) we have

\[
(dG)_{11\mu\nu\rho\sigma} = -\frac{2\sqrt{2} \pi \kappa_5^2}{\alpha_{\text{GUT}}} \left\{ J^{(1)}(x^{11}) + J^{(2)}(x^{11} - \pi \rho) \right\}_{\mu\nu\rho\sigma} \]

\[
(dF^i)_{11\mu\nu} = -\frac{\kappa_5^2}{4\sqrt{2} \pi \alpha_{\text{GUT}}} J^i_{\mu\nu} \delta(x^{11}) \]

\[
(dX)_{11\mu} = -\frac{\kappa_5^2}{4\sqrt{2} \pi \alpha_{\text{GUT}}} J^i_{\mu} \delta(x^{11}) \]

with the currents defined by

\[
J^{(n)}_{\mu\nu\rho\sigma} = \frac{1}{16\pi^2} \left( \text{tr} F^{(n)} \wedge F^{(n)} - \frac{1}{2} \text{tr} R \wedge R \right)_{\mu\nu\rho\sigma} \]

\[
J^i_{\mu\nu} = -2i V^{-1} \Gamma^i_{jk} \left( (D_\mu C)^j (D_\nu \bar{C})^k - (D_\mu \bar{C})^j (D_\nu C)^k \right) \]

\[
J^i_{\mu} = -\frac{i}{2} V^{-1} d_{ijk} f_{pqr} (D_\mu C)^i (D_\nu C)^j (D_\rho C)^k. \]

The five–dimensional Newton constant $\kappa_5$ and the Yang–Mills coupling $\alpha_{\text{GUT}}$ are expressed in terms of 11–dimensional quantities as

\[
\kappa_5^2 = \frac{\kappa^2}{v}, \quad \alpha_{\text{GUT}} = \frac{\kappa^2}{2v} \left( \frac{4\pi}{\kappa} \right)^{2/3}.
\]
We still need to define various quantities in the above action. The metric $G_{ij}$ is given in terms of the Kähler potential $K$ as

$$G_{ij} = -\frac{1}{2} \frac{\partial}{\partial b^i} \frac{\partial}{\partial b^j} \ln K .$$  \hspace{1cm} (2.28)

The corresponding connection $\Gamma^i_{jk}$ is defined as

$$\Gamma^i_{jk} = \frac{1}{2} G^{il} \frac{\partial G_{jk}}{\partial b^l} .$$  \hspace{1cm} (2.29)

We recall that

$$K = d_{ijk} b^i b^j b^k ,$$  \hspace{1cm} (2.30)

where $d_{ijk}$ are the Calabi–Yau intersection numbers. All indices $i, j, k, \cdots$ in the five–dimensional theory are raised and lowered with the metric $G_{ij}$. We recall that the fields $b^i$ are subject to the constraint

$$K = 6$$  \hspace{1cm} (2.31)

which should be taken into account when equations of motion are derived from the above action. Most conveniently, it can be implemented by adding a Lagrange multiplier term $\sqrt{-g} \lambda (K(b) - 6)$ to the bulk action. Furthermore, we need to define the superpotential

$$W = \frac{1}{6} d_{ijk} T^{pqr} C^{ij} C^{pq} C^{qr}$$  \hspace{1cm} (2.32)

and the D–term

$$D^{(u)} = G_{ij} C^j T^{(u)} C^i$$  \hspace{1cm} (2.33)

where $T^{(u)}$, $u = 1, \ldots, 78$ are the $E_6$ generators in the fundamental representation. The consistency of the above theory has been explicitly checked by a reduction of the 11–dimensional equations of motion.

The most notable features of this action, at first sight, are the bulk and boundary potentials for the $(1, 1)$ moduli $V$ and $b^i$ that appear in $S_{\text{hyper}}$ and $S_{\text{bound}}$. Those potentials involve the five–brane charges $\alpha_i$, defined by eq. (2.15), that characterize the nonzero mode. The bulk potential in the hypermultiplet part of the action arises directly from the kinetic term $G^2$ of the antisymmetric tensor field with the expression (2.14) for the nonzero mode inserted. It can therefore be interpreted as the energy contribution of the nonzero mode. The origin of the boundary potentials, on the other hand, can be directly seen from eq. (2.17) and the
10-dimensional boundary actions. Essentially, they arise because the standard embedding leads to nonvanishing internal boundary actions due to the crucial factor $1/2$ in front of the $\text{tr}R^2$ terms. This is in complete analogy with the appearance of nonvanishing sources in the internal part of the Bianchi identity which led us to introduce the nonzero mode. The action presented in (2.23) and (2.24) was first derived in [4].

3 Lecture 3: Bulk Five-Branes and Non-Standard Embeddings

In this third lecture, we begin by discussing the simplest BPS three-brane solution of the generalized five-dimensional heterotic $M$-theory presented in Lecture 2. We then commence a major extension of heterotic $M$-theory. Until now, we have employed the standard embedding of the spin connection of the Calabi-Yau threefold into the gauge connection of the visible brane. However, unlike the case of the weakly coupled heterotic string, there is nothing compelling about the standard embedding in heterotic $M$-theory. Quite the contrary, it is more natural to consider “non-standard” embeddings. Here, we will only briefly discuss such embeddings, referring the reader to the TASI 2001 lectures by Daniel Waldram for details. In this lecture, we focus on one of the important phenomena associated with non-standard embeddings, namely, the appearance of one or more bulk space three-branes (actually, $M5$-branes wrapped on holomorphic curves in the Calabi-Yau threefold). In the second part of this lecture, we will discuss the existence and properties of bulk space wrapped five-branes in detail.

We would now like to find the simplest BPS domain wall solutions of the generalized five-dimensional heterotic $M$-theory. From the above results, it is clear that the proper Ansatz for the type of solutions we are looking for is given by

$$
\begin{align*}
    ds_5^2 &= a(y)^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2 dy^2 \\
    V &= V(y) \\
    b^i &= b^i(y) ,
\end{align*}
$$

(3.1)

where we use $y = x^{11}$ from now on. A solution to the generalized equations of motion is somewhat hard to find, essentially due to the complication caused by the inclusion of all $(1,1)$ moduli and the associated Kähler structure. The trick is to express the solution in terms of certain functions $f^i = f^i(y)$ which are only implicitly defined rather than trying to
find fully explicit formulae. It turns out that those functions are fixed by the equations

\[ d_{ijk} f^j f^k = H_i, \quad H_i = -2\sqrt{2}k\alpha_i |y| + k_i \]  (3.2)

where \( k \) and \( k_i \) are arbitrary constants. Then the solution can be written as

\[
\begin{align*}
V &= \left( \frac{1}{6}d_{ijk} f^i f^j f^k \right)^2 \\
A &= \tilde{k}V^{1/6} \\
B &= kV^{2/3} \\
B^i &= V^{-1/6} f^i
\end{align*}
\]  (3.3)

where \( \tilde{k} \) is another arbitrary constant. We have checked that this solution is indeed a BPS state of the theory; that is, that it preserves four of the eight supercharges. Note that we have chosen the above solution to have no singularities other than those at the two boundaries. Specifically, the harmonic functions \( H_i \) in eq. (3.2) satisfy

\[ H_i'' = 4\sqrt{2}k\alpha_i (\delta(y) - \delta(y - \pi\rho)) , \]  (3.4)

indicating sources at the orbifold planes \( y = 0, \pi\rho \). Recall that we have restricted the range of \( y \) to \( y \in [-\pi\rho, \pi\rho] \) with the endpoints identified. This explains the second delta–function at \( y = \pi\rho \) in the above equation. We conclude that the solution (3.4) represents a multi–charged double domain wall (three–brane) solution with the two walls located at the orbifold planes. It preserves four–dimensional Poincaré invariance as well as four of the eight supercharges.

In Lecture 2, we have presented a related three–brane solution which was less general in that it involved the universal Calabi–Yau modulus \( V \) only. Clearly, we should be able to recover this solution from eq. (3.4) if we consider the specific case \( h^{1,1} = 1 \). Then we have \( d_{111} = 6 \) and it follows from eq. (3.2) that

\[ f^1 = \left( \frac{\sqrt{2}}{3}k\alpha_1 |y| + k_1 \right)^{1/2} . \]  (3.5)

Inserting this into eq. (3.4) provides us with the explicit solution in this case which is given by

\[
\begin{align*}
A &= a_0H^{1/2} \\
B &= b_0H^2 \\
H &= -\frac{\sqrt{2}}{3}\alpha|y| + c_0 , \quad \alpha = \alpha^1 \\
V &= b_0H^3 .
\end{align*}
\]  (3.6)
The constant $a_0, b_0$ and $c_0$ are related to the integration constants in eq. (3.4) by

$$a_0 = \tilde{k} k^{1/2}, \quad b_0 = k^3, \quad c_0 = \frac{k_1}{k}. \quad (3.7)$$

Eq. (3.7) is indeed exactly the solution that was found in Lecture 2. It still represents a double domain wall. However, in contrast to the general solution it couples to one charge $\alpha = \alpha^1$ only. Geometrically, it describes a variation of the five-dimensional metric and the Calabi–Yau volume across the orbifold.

At this point, we introduce an important generalization which greatly expands the scope, theoretical interest and phenomenological implications of heterotic $M$-theory. First, note that all of our previous results have assumed that the gauge field vacuum on the Calabi–Yau threefold is identical to the geometrical spin connection. That is, we have assumed the standard embedding defined in (1.6). Since any Calabi–Yau threefold has holonomy group $SU(3)$, it follows that the spin connection and, hence, the gauge connection has structure group $G = SU(3)$. The four-dimensional low energy theory then exhibits a gauge group $H$ which is the commutant of $G$ in $E_8$. Since $G = SU(3)$, it follows that $H = E_6$, as we discussed above. Although the choice of the standard embedding was natural within the context of weakly coupled heterotic superstring theory, there is no reason to single it out from other gauge vacua in $M$-theory. Indeed, the only constraint on the gauge vacua in heterotic $M$-theory is that they be compatible with $N = 1$ supersymmetry on the boundary planes. That is, the gauge connection on the Calabi–Yau threefold must satisfy the Hermitian Yang–Mills equation, but is otherwise arbitrary. Clearly, it would be of significant interest to demonstrate the existence of gauge vacua other than the standard embedding. For example, if one could construct a “non–standard” embedding gauge vacuum with structure group, say, $G = SU(5) \times \mathbb{Z}_2$, then the low energy gauge group in four-dimensions would be the standard model group $SU(3)_C \times SU(2)_L \times U(1)_Y$. Since the Calabi–Yau threefold has a Euclidean signature and is compact, we will refer to any gauge configuration with structure group $G \subset E_8$ that satisfies the Hermitian Yang–Mills equation as a $G$-instanton. We will, therefore, expand the vacua of heterotic $M$-theory by compactifying Hořava-Witten theory on Calabi–Yau manifolds with $G$-instantons.

Initially, this seems to be a very difficult task, since not a single solution to the Hermitian Yang–Mills equations on a Calabi–Yau threefold is known, with the exception of the standard embedding. However, at this point, some important mathematical results become relevant, which allow us to demonstrate the existence and compute the properties of very large classes of $G$-instantons. The fundamental results in this regard are two–fold. First, it was shown
by Donaldson and Uhlenbeck and Yau that there is a one-to-one correspondence between any $G$-instanton solution of the Hermitian Yang–Mills equation and the existence of a stable holomorphic vector bundle with structure group $G$ over the Calabi–Yau threefold. Given one the other is determined, at least in principle. Now, even though it appears to be very difficult to find solutions of the Hermitian Yang–Mills equations, it was demonstrated by Friedman, Morgan and Witten \[33, 35\] and Donagi \[34\] that one can, rather straightforwardly, construct stable holomorphic vector bundles over Calabi–Yau threefolds. Using, and extending, the technology introduced in these papers, large classes of heterotic $M$-theory vacua with non-standard $G$-bundles have been constructed \[36, 37\]. It was shown in these papers that heterotic $M$-theory vacua corresponding to grand unified theories, with gauge groups such as $SU(5)$ and $SO(10)$ \[36\], and the standard model with gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ \[37\] can, indeed, be constructed in this manner. I will not discuss these holomorphic bundle constructions in these lectures, referring the reader to the TASI lectures by Daniel Waldram. Here, instead, I will discuss an important implication of non-standard $G$-bundle vacua, namely, the necessary appearance of $M5$-branes, wrapped on holomorphic curves, in the bulk space.

Recall from above that anomaly cancellation requires that the Bianchi identity for the four-form field strength $G = dC$ be modified as in equation (1.4). It is useful to rewrite this expression as

$$(dG)_{11\bar{I}\bar{J}\bar{K}\bar{L}} = -4\sqrt{2}\pi \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ J^{(1)} \delta(x^{11}) + J^{(2)} \delta(x^{11} - \pi \rho) \right\}_{\bar{I}\bar{J}\bar{K}\bar{L}}$$

where sources are defined by

$$J^{(n)} = c_2(V^n) - \frac{1}{2} c_2(TX) \quad n = 1, 2, \quad (3.9)$$

and

$$c_2(V^n) = -\frac{1}{16\pi^2} tr F^n \wedge F^n, \quad c_2(TX) = -\frac{1}{16\pi^2} tr R \wedge R, \quad (3.10)$$

$V^n$ is the stable holomorphic vector bundle on the $n$-th plane, $F^n$ is the field strength associated with the gauge theory, and $R$ is the Ricci tensor of the Calabi-Yau manifold. Note that $c_2(V^n)$ and $c_2(TX)$ are the second Chern class of the vector bundle on the $n$-th boundary plane and the second Chern class of the Calabi-Yau tangent bundle respectively. Integrating (3.8) over a five-cycle which spans the orbifold interval and is otherwise an arbitrary four-cycle in the Calabi-Yau three-fold, we find the topological condition that

$$c_2(V^1) + c_2(V^2) - c_2(TX) = 0. \quad (3.11)$$
When $N$ bulk five-branes, located at coordinates $x_i$ for $i = 1, \ldots, N$ in the 11-direction, are present in the vacuum, cancellation of their worldvolume anomalies, as well as the gravitational and gauge anomalies on the orbifold fixed planes, requires that Bianchi identity be further modified to

$$\left(\frac{K}{4\pi}\right)^{2/3} \left(J^{(1)}(x^{11}) + J^{(2)}(x^{11} - \pi \rho) + \Sigma_{i=1}^{N} \hat{J}^{(i)}(x^{11} - x_i)\right)_{IJKL}. \tag{3.12}$$

Each five-brane source $\hat{J}^{(i)}$ is defined to be the four-form which is Poincaré dual to the holomorphic curve in the Calabi-Yau threefold around which the $i$-th five-brane is wrapped. If we define the five-brane class

$$W = \Sigma_{i=1}^{N} \hat{J}^{(i)}, \tag{3.13}$$

then the topological condition (3.11) is modified to

$$c_2(V^1) + c_2(V^2) - c_2(TX) + W = 0. \tag{3.14}$$

The simplest example one can present is the standard embedding, where one fixes the Calabi-Yau three-fold and chooses the two holomorphic vector bundles so that $V^1 = TX$ and $V^2 = 0$. It follows that

$$c_2(V^1) = c_2(TX), \quad c_2(V^2) = 0. \tag{3.15}$$

Note that these Chern classes satisfy the topological condition given in (3.14) with

$$W = 0. \tag{3.16}$$

That is, for the standard embedding there are no $M5$–branes in the bulk space, as we already know from the previous lectures. However, as was shown in [36], most non-standard $G$-bundles correspond to Chern classes that require a non-vanishing five-brane class $W$ in order to be anomaly free. In particular, phenomenologically relevant heterotic $M$-theory vacua, such as those leading to the standard model gauge group with three families of quarks and leptons [37], must have bulk five–branes. We will, therefore, spend the remainder of this lecture discussing the structure and physical properties of bulk space $M5$–branes wrapped on holomorphic curves.

The inclusion of five-branes in the bulk space not only generalizes the types of background one can consider, but also introduces new degrees of freedom into the theory, namely, the
dynamical fields on the five-branes themselves. We will now consider what low-energy fields
survive on one of the five-branes when it is wrapped around a two-cycle in the Calabi–Yau
threefold.

In general, the fields on a single five-brane are as follows [38, 39]. The simplest are the
bosonic coordinates \( X^I \) describing the embedding of the brane into 11-dimensional spacetime.
The additional bosonic field is a world-volume two-form potential \( B \) with field strength
\( H = dB \) satisfying a generalized self-duality condition. For small fluctuations, the duality
condition simplifies to the conventional constraint \( H = *H \). These degrees of freedom are
paired with spacetime fermions \( \theta \), leading to a Green–Schwarz type action, with manifest
spacetime supersymmetry and local kappa-symmetry [40, 41]. (As usual, including the self-
dual field in the action is difficult, but is made possible by either including an auxiliary
field or abandoning a covariant formulation.) For a five-brane in flat space, one can choose
a gauge such that the dynamical fields fall into a six-dimensional massless tensor multiplet
with \( (0,2) \) supersymmetry on the brane world-volume [42, 43]. This multiplet has five scalars
describing the motion in directions transverse to the five-brane, together with the self-dual
tensor \( H \).

For a five-brane embedded in \( S^1/Z_2 \times X \times M_4 \), to preserve Lorentz invariance in \( M_4 \),
3 + 1 dimensions of the five-brane must be left uncompactified. The remaining two spatial
dimensions are then wrapped on a two-cycle of the Calabi–Yau three-fold. To preserve
supersymmetry, the two-cycle must be a holomorphic curve [44, 45]. Thus, from the
point of view of a five-dimensional effective theory on \( S^1/Z_2 \times M_4 \), since two of the five-
brane directions are compactified, it appears as a flat three-brane (or equivalently a domain
wall) located at some point \( x^{11} = x \) on the orbifold. Thus, at low energy, the degrees of
freedom on the brane must fall into four-dimensional supersymmetric multiplets.

An important question is how much supersymmetry is preserved in the low-energy theory.
One way to address this problem is directly from the symmetries of the Green–Schwarz
action, following the discussion for similar brane configurations in [46]. Locally, the 11-
dimensional spacetime \( S^1/Z_2 \times X \times M_4 \) admits eight independent Killing spinors \( \eta \), so should
be described by a theory with eight supercharges. (Globally, only half of the spinors survive
the non-local orbifold quotienting condition \( \Gamma_{11} \eta(-x^{11}) = \eta(x^{11}) \), so that, for instance, the
eleven-dimensional bulk fields lead to \( N = 1 \), not \( N = 2 \), supergravity in four dimensions.)
The Green–Schwarz form of the five-brane action is then invariant under supertranslations
generated by \( \eta \), as well as local kappa-transformations. In general the fermion fields \( \theta \)
transform as (see for instance ref. [43])

\[ \delta \theta = \eta + P_+ \kappa \]  \hspace{1cm} (3.17)

where \( P_+ \) is a projection operator. If the brane configuration is purely bosonic then \( \theta = 0 \) and the variation of the bosonic fields is identically zero. Furthermore, if \( H = 0 \) then the projection operator takes the simple form

\[ P_\pm = \frac{1}{2} \left( 1 \pm \frac{1}{6! \sqrt{g}} \epsilon^{m_1...m_6} \partial_{m_1} X^{I_1} \ldots \partial_{m_6} X^{I_6} \Gamma_{I_1...I_6} \right) \]  \hspace{1cm} (3.18)

where \( \sigma^m, m = 0, \ldots, 5 \) label the coordinates on the five-brane and \( g \) is the determinant of the induced metric

\[ g_{mn} = \partial_m X^I \partial_n X^J g_{IJ} \]. \hspace{1cm} (3.19)

If the brane configuration is invariant for some combination of supertranslation \( \eta \) and kappa-transformation, then we say it is supersymmetric. Now \( \kappa \) is a local parameter which can be chosen at will. Since the projection operators satisfy \( P_+ + P_- = 1 \), we see that for a solution of \( \delta \theta = 0 \), one is required to set \( \kappa = -\eta \), together with imposing the condition

\[ P_- \eta = 0 \]  \hspace{1cm} (3.20)

For a brane wrapped on a two-cycle in the Calabi–Yau space, spanning \( M_4 \) and located at \( x^{11} = x \) in the orbifold interval, we can choose the parameterization

\[ X^\mu = \sigma^\mu, \quad X^A = X^A(\sigma, \bar{\sigma}), \quad X^{11} = x \]  \hspace{1cm} (3.21)

where \( \sigma = \sigma^4 + i\sigma^5 \). The condition (3.20) then reads

\[ -\left( i/\sqrt{g} \right) \partial X^A \bar{\partial} X^B \Gamma^{(4)} \Gamma_{AB} \eta = \eta \]  \hspace{1cm} (3.22)

where we have introduced the four-dimensional chirality operator \( \Gamma^{(4)} = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \). Recalling that on the Calabi–Yau three-fold the Killing spinor satisfies \( \Gamma^b \eta = 0 \), it is easy to show that this condition can only be satisfied if the embedding is holomorphic, that is \( X^a = X^a(\sigma) \), independent of \( \bar{\sigma} \). The condition then further reduces to

\[ \Gamma^{(4)} \eta = i\eta \]  \hspace{1cm} (3.23)

which, given that the spinor has definite chirality in eleven dimensions as well as on the Calabi–Yau space, implies that \( \Gamma^{11} \eta = \eta \), compatible with the global orbifold quotient
condition. Thus, finally, we see that only half of the eight Killing spinors, namely those satisfying (3.23), lead to preserved supersymmetries on the five-brane. Consequently the low-energy four-dimensional theory describing the five-brane dynamics will have $N = 1$ supersymmetry.

The simplest excitations on the five-brane surviving in the low-energy four-dimensional effective theory are the moduli describing the position of the five-brane in eleven dimensions. There is a single modulus $X^{11}$ giving the position of the brane in the orbifold interval. In addition, there is the moduli space of holomorphic curves $C_2$ in $X$ describing the position of the brane in the Calabi–Yau space. This moduli space is generally complicated, and we will not address its detailed structure here. (As an example, the moduli space of genus one curves in K3 is K3 itself [15].) However, we note that these moduli are scalars in four dimensions, and we expect them to arrange themselves as a set of chiral multiplets, with a complex structure presumably inherited from that of the Calabi–Yau manifold.

Now let us consider the reduction of the self-dual three-form degrees of freedom. (Here we are essentially repeating a discussion given in [46, 47].) The holomorphic curve is a Riemann surface and, so, is characterized by its genus $g$. One recalls that the number of independent harmonic one-forms on a Riemann surface is given by $2g$. In addition, there is the harmonic volume two-form $\Omega$. Thus, if we decompose the five-brane world-volume as $C_2 \times M_4$, we can expand $H$ in zero modes as

$$H = da \wedge \Omega + F^u \wedge \lambda_u + h$$  \hspace{1cm} (3.24)

where $\lambda_u$ are a basis $u = 1, \ldots, 2g$ of harmonic one-forms on $C_2$, while the four-dimensional fields are a scalar $a$, $2g$ $U(1)$ vector fields $F^u = dA^u$ and a three-form field strength $h = db$. However, not all these fields are independent because of the self-duality condition $H = *H$. Rather, one easily concludes that

$$h = *da$$  \hspace{1cm} (3.25)

and, hence, that the four-dimensional scalar $a$ and two-form $b$ describe the same degree of freedom. To analyze the vector fields, we introduce the matrix $T^u_v$ defined by

$$*\lambda_u = T^u_v \lambda_v$$  \hspace{1cm} (3.26)

If we choose the basis $\lambda_u$ such that the moduli space metric $\int_{C_2} \lambda_u \wedge (*\lambda_v)$ is the unit matrix, $T$ is antisymmetric and, of course, $T^2 = -1$. The self-duality constraint implies for the vector fields that

$$F^u = T^v_u * F^v.$$  \hspace{1cm} (3.27)
If we choose a basis for $F^u$ such that

$$T = \text{diag} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

(3.28)

with $g$ two by two blocks on the diagonal, one easily concludes that only $g$ of the $2g$ vector fields are independent. In conclusion, for a genus $g$ curve $C_2$, we have found one scalar and $g$ $U(1)$ vector fields from the two-form on the five-brane worldvolume. The scalar has to pair with another scalar to form a chiral $N = 1$ multiplet. The only other universal scalar available is the zero mode of the transverse coordinate $X^{11}$ in the orbifold direction.

Thus, in general, the $N = 1$ low-energy theory of a single five-brane wrapped on a genus $g$ holomorphic curve $C_2$ has gauge group $U(1)^g$ with $g$ $U(1)$ vector multiplets and a universal chiral multiplet with bosonic fields $(a, X^{11})$. Furthermore, there is some number of additional chiral multiplets describing the moduli space of the curve $C_2$ in the Calabi–Yau three-fold.

It is well known that when two regions of the five-brane world-volume in M–theory come into close proximity, new massless states appear [48, 13]. These are associated with membranes stretching between the two nearly overlapping five-brane surfaces. In general, this can lead to enhancement of the gauge symmetry. Let us now consider this possibility, heretofore ignored in our discussion. In general, one can consider two types of brane degeneracy where parts of the five-brane world-volumes are in close proximity. The first, and simplest, is to have $N$ distinct but coincident five-branes, all wrapping the same cycle $C_2$ in the Calabi–Yau space and all located at the same point in the orbifold interval. Here, the new massless states come from membranes stretching between the distinct five-brane world-volumes. The second, and more complicated, situation is where there is a degeneracy of the embedding of a single five-brane, such that parts of the curve $C_2$ become close together in the Calabi–Yau space. In this case, the new states come from membranes stretching between different parts of the same five-brane world-volume [49, 50]. Let us consider these two possibilities separately.

The first case of distinct five-branes is analogous to the M–theory description of $N$ overlapping type IIB D3-branes, which arise as $N$ coincident five-branes wrapping the same cycle in a flat torus. In that case, the $U(1)$ gauge theory on each D3-brane is enhanced to a $U(N)$ theory describing the full collection of branes. Thus, by analogy, in our case we would expect a similar enhancement of each of the $g$ $U(1)$ fields on each five-brane. That is, when wrapped on a holomorphic curve of genus $g$, the full gauge group for the low-energy theory describing $N$ coincident five-branes becomes $U(N)^g$.

The second case is inherently more complicated. It can, however, be clearly elucidated
and studied for Calabi-Yau threefolds which are elliptically fibered. These manifolds consist of a base two-fold, over any point of which is fibered an elliptic curve. At almost all points in the base, the elliptic curve is smooth. However, there is a locus of points, called the discriminant locus, over which the fibers degenerate. These degeneracies have specific characteristics and have been classified by Kodaira \[51\]. If the five-brane is wrapped over a smooth fiber, away from the discriminant locus, then there are no new massless states. However, as the fiber approaches the discriminant it degenerates to a specific Kodaira singularity. Accordingly, the five-brane wrapped on such a fiber begins to “approach itself” near the singularity, leading to new, massless states appearing in the theory. The general theory for computing these massless states was presented for fibers over both the smooth and singular parts of discriminant curves in \[49\] and \[50\] respectively. For example, consider an elliptically fibered Calabi-Yau threefold over an \(F_3\) Hirzebruch base and let the five-brane be wrapped on a fiber near a smooth part of the discriminant curve with Kodaira type \(I_2\). Then, it was shown in \[49\] that, in addition to the usual states, the \(I_2\) degeneracy of the elliptic fiber produces an \(SU(2)\) doublet of massless \(N = 2\) hypermultiplets with unit electric charge. In general, one gets a complicated spectrum of new hypermultiplets and, for sufficiently intricate Kodaira singularities, new non-Abelian vector multiplets as well.

Summarizing the two cases, we see that for \(N\) five-branes wrapping the same curve \(C_2\) of genus \(g\), we expect that the symmetry is enhanced from \(N\) copies of \(U(1)^g\) to \(U(N)^g\). Alternatively, in the second case, even for a single brane we can get new massless states if the holomorphic curve degenerates. These states form hypermultiplets and extended non-Abelian gauge vector multiplets depending on the exact form of the curve degeneracy.

\section{Lecture 4: Beyond Hořava-Witten Theory}

It is of interest to ask whether one can construct other orbifolds of \(M\)-theory beyond the \(S^1/\mathbb{Z}_2\) example of \([1, 2]\). A first step in this direction was taken by Dasgupta and Muhki \[52\] and Witten \[53\] who discussed both local and global anomaly cancellation within the context of \(T^4/\mathbb{Z}_2\) orbifolds. A major generalization of these results was presented in \[54, 55, 56, 57\] and \[58, 59\] where all the \(M\)-theory orbifolds associated with the spacetime \(\mathbb{R}^8 \times K3\) were constructed. In this fourth lecture, we will, for specificity, consider \(M\)-theory orbifolds on \(S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2\). It will be demonstrated, in detail, how such orbifolds can be made anomaly free, completely determining both the twisted and untwisted sector spectra in the process, even on odd dimensional orbifold planes where all anomalies vanish.
The spacetime has topology $\mathbb{R}^6 \times S^1 \times T^4$, where each of the five compact coordinates takes values on the interval $[-\pi, \pi]$ with the endpoints identified. Let $x^\mu$ parameterize the six non-compact dimensions, while $x^i$ and $x^{11}$ parameterize the $T^4$ and $S^1$ factors respectively. Then the $\mathbb{Z}_2$ action on $S^1$ is defined by

$$\alpha : (x^\mu, x^i, x^{11}) \longrightarrow (x^\mu, x^i, -x^{11}) \quad (4.1)$$

whereas the $\mathbb{Z}_2$ action on $T^4$ is

$$\beta : (x^\mu, x^i, x^{11}) \longrightarrow (x^\mu, -x^i, x^{11}). \quad (4.2)$$

The element $\alpha$ leaves invariant the two ten-planes defined by $x^{11} = 0$ and $x^{11} = \pi$, while $\beta$ leaves invariant the sixteen seven-planes defined when the four coordinates $x^i$ individually assume the values 0 or $\pi$. Finally, $\alpha \beta$ leaves invariant the thirty-two six-planes defined when all five compact coordinates individually assume the values 0 or $\pi$. The $\alpha \beta$ six-planes coincide with the intersections of the $\alpha$ ten-planes with the $\beta$ seven-planes.

A gravitational anomaly arises on each ten-plane due to the coupling of chiral projections of the bulk gravitino to currents localized on the fixed planes. Since the two ten-planes are indistinguishable aside from their position, this anomaly is identical on each of the two planes and can be computed by conventional means if proper care is used. The reason why extra care is needed is that each ten-plane anomaly arises from the coupling of eleven-dimensional fermions to ten-dimensional currents, whereas standard index theorem results only apply to ten-dimensional fermions coupled to ten-dimensional currents. If one notes that the index theorem can be applied to the small radius limit where the two ten-planes coincide, then the gravitational anomaly on each individual ten-plane can be computed; it is simply one-half of the index theorem anomaly derived using the “untwisted” sector spectrum in ten-dimensions. By untwisted sector, we mean the $\mathbb{Z}_2$ projection of the eleven-dimensional bulk space supergravity multiplet onto each ten-dimension fixed plane. This untwisted spectrum forms the ten-dimensional $N = 1$ supergravity multiplet containing a graviton, a chiral gravitino, a two-form and a scalar dilaton. We denote by $R$ the ten-dimensional Riemann tensor, regarded as an $SO(9,1)$-valued form.

As pointed out in [1, 2], in addition to the untwisted spectrum, one must allow for the possibility of “twisted” sector $N = 1$ supermultiplets that live on each ten-dimensional orbifold plane only. For the case at hand, the twisted sector spectrum must fall into $N = 1$ Yang-Mills supermultiplets consisting of gauge fields and chiral gauginos. These will give rise to an additional contribution to the gravitational anomaly on each ten-plane, as well
as to mixed and pure-gauge anomalies. However, since the twisted sector fields are ten-dimensional, these anomalies can be computed directly from the standard formulas, without multiplying by one-half. The twisted sector gauge group, the dimension of the gauge group and the gauge field strength on the \( i \)-th ten-plane are denoted by \( \mathcal{G}_i \), \( n_i = \text{dim} \mathcal{G}_i \) and \( F_i \) respectively, for \( i = 1, 2 \).

The quantum mechanical one-loop local chiral anomaly on the \( i \)-th ten-plane is characterized by the twelve-form

\[
I_{12}(1\text{-loop})_i = \frac{1}{4} \left( I_{\text{GRAV}}^{(3/2)}(R) - I_{\text{GRAV}}^{(1/2)}(R) \right) + \frac{1}{2} \left( n_i I_{\text{GRAV}}^{(1/2)}(R) + I_{\text{MIXED}}^{(1/2)}(R, F_i) + I_{\text{GAUGE}}^{(1/2)}(F_i) \right) \tag{4.3}
\]

from which the anomaly arises by descent. The constituent polynomials contributing to the pure gravitational anomaly due to the chiral spin \( 3/2 \) and chiral spin \( 1/2 \) fermions are

\[
I_{\text{GRAV}}^{(3/2)}(R) = \frac{1}{(2\pi)^6!} \left( \frac{55}{56} \text{tr} R^6 - \frac{75}{128} \text{tr} R^4 \wedge \text{tr} R^2 + \frac{35}{512} (\text{tr} R^2)^3 \right) \tag{4.4}
\]

and

\[
I_{\text{GRAV}}^{(1/2)}(R) = \frac{1}{(2\pi)^6!} \left( -\frac{1}{504} \text{tr} R^6 - \frac{1}{384} \text{tr} R^4 \wedge \text{tr} R^2 - \frac{5}{4608} (\text{tr} R^2)^3 \right) \tag{4.5}
\]

respectively, where \( \text{tr} \) is the trace of the \( SO(9,1) \) indices. The polynomials contributing to the mixed and pure-gauge anomalies are due to chiral spin \( 1/2 \) fermions only and are given by

\[
I_{\text{MIXED}}^{(1/2)}(R, F_i) = \frac{1}{(2\pi)^6!} \left( \frac{1}{16} \text{tr} R^4 \wedge \text{Tr} F_i^2 + \frac{5}{64} (\text{tr} R^2)^2 \wedge \text{Tr} F_i^2 \right. \\
\left. - \frac{5}{8} \text{tr} R^2 \wedge \text{Tr} F_i^4 \right) \tag{4.6}
\]

and

\[
I_{\text{GAUGE}}^{(1/2)}(F_i) = \frac{1}{(2\pi)^6!} \text{Tr} F_i^6. \tag{4.7}
\]

Here \( \text{Tr} \) is the trace over the adjoint representation of \( \mathcal{G}_i \). All the anomaly polynomials are computed using standard index theorems. Each term in (4.3) has a factor of 1/2 because the relevant fermions are Majorana-Weyl with half the degrees of freedom of Weyl fermions. The first two terms in (4.3) arise from untwisted sector fermions, whereas the last three terms are contributed by the twisted sector. It follows from the above discussion that the
first two terms must have an additional factor of 1/2, accounting for the overall coefficient of 1/4, whereas the remaining three terms are given exactly by the index theorems.

The quantum anomaly [4.3] would spoil the consistency of the theory were it not to cancel against some sort of classical inflow anomaly. Hence, it is imperative to discern the presence of appropriate local classical counterterms to cancel against (4.3). One begins the analysis of anomaly cancellation by considering the pure tr $R^6$ term in (4.3) which is irreducible and must therefore identically vanish. It follows from the above that this term is

$$-\frac{1}{2(2\pi)^5 6!} \frac{(n_i - 248)}{494} \text{tr} R^6.$$  

(4.8)

Therefore, the tr $R^6$ term will vanish if and only if each gauge group $G_i$ satisfies the constraint

$$n_i = 248.$$  

(4.9)

Without yet specifying which 248-dimensional gauge group is permitted, we substitute 248 for $n_i$ in (4.3) obtaining

$$I_{12}(1\text{-loop})_i = \frac{1}{2(2\pi)^5 6!} \left( -\frac{15}{16} \text{tr} R^4 \wedge \text{tr} R^2 - \frac{15}{64} (\text{tr} R^2)^3 + \frac{1}{16} \text{tr} R^4 \wedge \text{Tr} F_i^2 ight. 
\left. + \frac{5}{64} (\text{tr} R^2)^2 \wedge \text{Tr} F_i^2 - \frac{5}{8} \text{tr} R^2 \wedge \text{Tr} F_i^4 + \text{Tr} F_i^6 \right).$$  

(4.10)

Although non-vanishing, this part of the anomaly is reducible. It follows that it can be made to cancel as long as it can be factorized into the product of two terms, a four-form and an eight-form. A necessary requirement for this to be the case is that

$$\text{Tr} F_i^6 = \frac{1}{24} \text{Tr} F_i^4 \wedge \text{Tr} F_i^2 - \frac{1}{3600} (\text{Tr} F_i^2)^3.$$  

(4.11)

There are two Lie groups with dimension 248 that satisfy this condition, the non-Abelian group $E_8$ and the Abelian group $U(1)^{248}$. Both groups represent allowed twisted matter gauge groups on each ten-plane. Hence, from anomaly considerations alone one can determine the twisted sector on each ten-plane, albeit with a small ambiguity in the allowed twisted sector gauge group. In this paper, we consider only the non-Abelian gauge group $E_8$. Using (4.11) and several $E_8$ trace relations, the anomaly polynomial (4.10) can be re-expressed as follows

$$I_{12}(1\text{-loop})_i = \frac{1}{3} \pi I_{4(i)}^3 + X_8 \wedge I_{4(i)}$$  

(4.12)

where $X_8$ is the eight-form

$$X_8 = \frac{1}{(2\pi)^3 4!} \left( \frac{1}{8} \text{tr} R^4 - \frac{1}{32} (\text{tr} R^2)^2 \right).$$  

(4.13)
and $I_{4(i)}$ is the four-form given by

$$I_{4(i)} = \frac{1}{16\pi^2} \left( \frac{1}{30} \text{Tr} F_i^2 - \frac{1}{2} \text{tr} R^2 \right).$$

(4.14)

Once in this factorized form, the anomaly $I_{12}(1-\text{loop})_i$ can be cancelled as follows.

First, the Bianchi identity $dG = 0$, where $G$ is the field strength of the three-form $C$ in the eleven-dimensional supergravity multiplet, is modified to

$$dG = \sum_{i=1}^{2} I_{4(i)} \wedge \delta^{(1)}_{M_{10}}$$

(4.15)

where $I_{4(i)}$ is the four-form given in (4.14) and $\delta^{(1)}_{M_{10}}$ is a one-form brane current with support on the $i$-th ten-plane. Second, we note that the eleven-dimensional supergravity action contains the terms

$$S = \cdots - \frac{\pi}{3} \int C \wedge G \wedge G + \int G \wedge X_7$$

(4.16)

where $X_7$ satisfies $dX_7 = X_8$. The $CGG$ interaction is required by the minimally-coupled supergravity action, while the $GX_7$ term is an additional higher-derivative interaction necessitated by five-brane anomaly cancellation. Using the modified Bianchi identity (4.15), one can compute the variation of these two terms under Lorentz and gauge transformations. The result is that the $CGG$ and $GX_7$ terms have classical anomalies which descend from the polynomials

$$I_{12}(CGG)_i = -\frac{\pi}{3} I_{4(i)}^3$$

(4.17)

and

$$I_{12}(GX_7)_i = -X_8 \wedge I_{4(i)}.$$ 

(4.18)

respectively. It follows that

$$I_{12}(1-\text{loop})_i + I_{12}(CGG)_i + I_{12}(GX_7)_i = 0$$

(4.19)

and, hence, the total anomaly cancels exactly.

We conclude that the requirement of local anomaly cancellation on the each of the two $S^1/Z_2$ orbifold ten-planes specifies the twisted spectrum of the theory. This specification is almost, but not quite, unique, allowing $N = 1$ vector supermultiplets with either gauge group $E_8$ or $U(1)^{248}$. An important ingredient in this analysis was the fact that the contribution to
the anomaly on each ten-plane from the untwisted sector was a factor of $1/2$ smaller than the index theorem result. This followed from the fact that the index theorem had to be spread over two equivalent ten-planes. A direct consequence of this is that the non-Abelian gauge group on each ten-plane is $E_8$, not $E_8 \times E_8$, and that the gauge group $SO(32)$ is disallowed. Since $S^1/Z_2$ is a subspace of $S^1/Z_2 \times T^4/Z_2$, the results of this section continue to hold on the larger orbifold. We now discuss the cancellation of local anomalies in the other factor space, $T^4/Z_2$.

The quantum anomalies on each of the sixteen indistinguishable seven-planes of the $T^4/Z_2$ orbifold are easy to analyze. In analogy with the ten-planes, an untwisted sector is induced on each seven-plane by the $Z_2$ projection of the eleven-dimensional supergravity multiplet. This untwisted spectrum forms the seven-dimensional $N = 1$ supergravity multiplet consisting of a graviton, a gravitino, three vector fields, a two-form, a real scalar dilaton and a spin 1/2 dilitino. However, unlike the case of a ten-plane, gravitational anomalies cannot be supported on a seven-plane. In fact, since there are no chiral fermions in seven-dimensions, no chiral anomaly of any kind, gravitational or gauge, can arise. Hence, with no local chiral anomalies to cancel, it would appear to be impossible to compute the twisted sector spectrum of any seven-plane. As long as we focus on the seven-planes exclusively, this conclusion is correct. However, as we will see below, the cancellation of the local anomalies on the thirty-two six-dimensional $\alpha \beta$ orbifold planes, formed from the intersection of the $\alpha$ ten-planes with the $\beta$ seven-planes, will require a non-vanishing twisted sector spectrum on each seven-plane and dictate its structure. With this in mind, we now turn to the analysis of anomalies localized on the intersection six-planes in the full $S^1/Z_2 \times T^4/Z_2$ orbifold.

As in the case for the ten-planes, a gravitational anomaly will arise on each six-plane due to the coupling of chiral projections of the bulk gravitino to currents localized on the thirty-two fixed planes. Since the thirty-two six-planes are indistinguishable, the anomaly is the same on each plane and can be computed by conventional means if proper care is taken. Noting that the standard index theorems can be applied to the small radius limit where the thirty-two six-planes coincide, it follows that the gravitational anomaly on each six-plane is simply one-thirty-second of the index theorem anomaly derived using the untwisted sector spectrum in six-dimensions. In this case, the untwisted sector spectrum is the $Z_2 \times Z_2$ projection of the eleven-dimensional bulk supergravity multiplet onto each six-dimensional fixed plane. This untwisted spectrum forms several $N = 1$ six-dimensional supermultiplets. Namely, the supergravity multiplet consisting of a graviton, a chiral gravitino and a self-dual two-form, four hypermultiplets each with four scalars and an anti-chiral hyperino, and
one tensor multiplet with one anti-self-dual two-form, one scalar and an anti-chiral spin 1/2 fermion. A one-loop quantum gravitational anomaly then arises from one chiral spin 3/2 fermion, five anti-chiral spin 1/2 fermions and one each of self-dual and anti-self-dual tensors. However, the anomalies due to the tensors cancel each other. Noting that a chiral anomaly in six-dimensions is characterized by an eight-form, from which the anomaly arises by descent, we find, for the $i$-th six-plane, that

$$I_8(SG)_i = \frac{1}{32} \left( I^{(3/2)}_{GRAV}(R) - 5 I^{(1/2)}_{GRAV}(R) \right)$$  \hspace{1cm} (4.20)$$

where

$$I^{(3/2)}_{GRAV}(R) = \frac{1}{(2\pi)^3 4!} \left( - \frac{49}{48} \text{tr} R^4 + \frac{43}{192} (\text{tr} R^2)^2 \right)$$  \hspace{1cm} (4.21)$$

and

$$I^{(1/2)}_{GRAV}(R) = \frac{1}{(2\pi)^3 4!} \left( - \frac{1}{240} \text{tr} R^4 - \frac{1}{192} (\text{tr} R^2)^2 \right),$$  \hspace{1cm} (4.22)$$

where $R$ is the six-dimensional Riemann tensor, regarded as an $SO(5,1)$-valued form. Note that the terms in brackets in (4.20) are the anomaly as computed by the index theorem. $I_8(SG)_i$ is obtained from that result by dividing by 32.

Noting that each six-plane is embedded in one of the two ten-dimensional planes, we see that there are additional “untwisted” sector fields on each six-plane. These arise from the $\beta \mathbf{Z}_2$ projection of the $N = 1$ $E_8$ Yang-Mills supermultiplet on the associated ten-plane. Such fields are untwisted from the point of view of the six-dimensional plane, although they arise from fields that were in the twisted sector of the ten-plane. In this lecture, we will assume that the $\beta$ action on the ten-dimensional vector multiplets does not break the $E_8$ gauge group. A discussion of the case where $E_8$ is broken to a subgroup by the action of $\beta$ can be found in [55, 56]. A ten-dimensional $N = 1$ vector supermultiplet decomposes in six-dimensions into an $N = 1$ vector multiplet and an $N = 1$ hypermultiplet. However, the action of $\beta$ projects out the hypermultiplet. Therefore, the ten-plane contribution to the untwisted sector of each six-plane is an $N = 1$ $E_8$ vector supermultiplet, which consists of gauge fields and chiral gauginos. The gauginos contribute to the gravitational anomaly on each six-plane, as well as adding mixed and $E_8$ gauge anomalies. Noting that the standard index theorems can be applied to the small radius limit, where each ten-plane shrinks to zero size and, hence, the sixteen six-planes it contains coincide, it follows that the anomaly is simply one-sixteenth of the index theorem result. We find that the one-loop quantum
contribution of this $E_8$ supermultiplet to the gravitational, mixed and $E_8$ gauge anomalies on the $i$-th six-plane is

$$I_8(E_8)_i = \frac{1}{16} \left( 248 I_{GRAV}^{(1/2)}(R) + I_{MIXED}^{(1/2)}(R, F_i) + I_{GAUGE}^{(1/2)}(F_i) \right)$$  \hspace{1cm} (4.23)

where

$$I_{MIXED}^{(1/2)}(R, F_i) = \frac{1}{(2\pi)^{3/4!}} \left( \frac{1}{4} \text{Tr} R^2 \wedge \text{Tr} F_i^2 \right)$$  \hspace{1cm} (4.24)

and

$$I_{GAUGE}^{(1/2)}(F_i) = \frac{1}{(2\pi)^{3/4!}} \left( - \text{Tr} F_i^4 \right).$$  \hspace{1cm} (4.25)

Here Tr is over the adjoint 248 representation of $E_8$. Note that the terms in brackets in (4.23) are the index theorem anomaly. $I_8(E_8)_i$ is obtained from that result by dividing by 16.

Are there other sources of untwisted sector anomalies on a six-plane? The answer is, potentially yes. We note that, in addition to being embedded in one of the two ten-planes, each six-plane is also embedded in one of the sixteen seven-dimensional orbifold planes. In analogy with the discussion above, if there were to be a non-vanishing twisted sector spectrum on each seven-plane, then this could descend under the $\alpha Z_2$ projection as an addition to the untwisted spectrum on each six-plane. This additional untwisted spectrum could then contribute to the chiral anomalies on the six-plane. However, as noted above, a priori, there is no reason for one to believe that there is any twisted sector on a seven-dimensional orbifold plane. Therefore, for the time being, let us assume that there is no such contribution to the six-dimensional anomaly. We will see below that this assumption must be carefully revisited.

As for the ten-dimensional planes, one must allow for the possibility of twisted sector $N = 1$ supermultiplets on each of the thirty-two six-planes. The most general allowed spectrum on the $i$-th six-plane would be $n_{V_i}$ vector multiplets transforming in the adjoint representation of some as yet unspecified gauge group $\mathcal{G}_i$, $n_{H_i}$ hypermultiplets transforming under some representation (possibly reducible) $\mathcal{R}$ of $\mathcal{G}_i$, and $n_{T_i}$ gauge-singlet tensor multiplets. We denote by $\mathcal{F}_i$ the gauge field strength. Since these fields are in the twisted sector, their contribution to the chiral anomalies can be determined directly from the index theorems without modification. We find that the one-loop quantum contribution of the twisted spectrum to the gravitational, mixed and $\mathcal{G}_i$ gauge anomalies on the $i$-th six-plane is

$$I_8(\mathcal{G}_i) = (n_{V_i} - n_{H_i} - n_{T_i}) I_{GRAV}^{(1/2)}(R) - n_{T_i} I_{GRAV}^{(3\text{-form})}(R)$$

$$+ I_{MIXED}^{(1/2)}(R, \mathcal{F}_i) + I_{GAUGE}^{(1/2)}(\mathcal{F}_i)$$  \hspace{1cm} (4.26)
where $I^{(1/2)}_{GRAV}(R)$ is given in (4.22) and
\[
I^{(3\text{-form})}_{GRAV}(R) = \frac{1}{(2\pi)^3 4!} \left( -\frac{7}{60} \text{tr} R^4 + \frac{1}{24} (\text{tr} R^2)^2 \right).
\] (4.27)

Furthermore, the mixed and pure-gauge anomaly polynomials are modified to
\[
I^{(1/2)}_{MIXED}(R,F_i) = \frac{1}{(2\pi)^3 4!} \left( \frac{1}{4} \text{tr} R^2 \wedge \text{trace} F_i^2 \right)
\] (4.28)

and
\[
I^{(1/2)}_{GAUGE}(F_i) = \frac{1}{(2\pi)^3 4!} \left( -\text{trace} F_i^4 \right),
\] (4.29)

where
\[
\text{trace} F_i^n = \text{Tr} F_i^n - \sum_\alpha h_\alpha \text{tr}_\alpha F_i^n.
\] (4.30)

Here \( \text{Tr} \) is an adjoint trace, \( h_\alpha \) is the number of hypermultiplets transforming in the \( \mathcal{R}_\alpha \) representation and \( \text{tr}_\alpha \) is a trace over the \( \mathcal{R}_\alpha \) representation. Note that the total number of vector multiplets is \( n_{Vi} = \text{dim} (\mathcal{G}_i) \), while the total number of hypermultiplets is \( n_{Hi} = \sum_\alpha h_\alpha \times \text{dim} (\mathcal{R}_\alpha) \). The relative minus sign in (4.30) reflects the anti-chirality of the hyperinos.

Combining the contributions from the two untwisted sector sources and the twisted sector, the total one-loop quantum anomaly on the \( i \)-th six-plane is the sum
\[
I_8(1-\text{loop})_i = I_8(SG)_i + I_8(E_8)_i + I_8(\mathcal{G}_i)
\] (4.31)

where \( I_8(SG)_i \), \( I_8(E_8)_i \) and \( I_8(\mathcal{G}_i) \) are given in (4.20), (4.23) and (4.26) respectively.

Unlike the case for the ten-dimensional planes, the classical anomaly associated with the \( GX_7 \) term in the eleven-dimensional action (4.16) can contribute to the irreducible curvature term which, in six-dimensions, is \( \text{tr} R^4 \). Therefore, our next step is to further modify the Bianchi identity for \( G = dC \) from expression (4.15) to
\[
dG = \sum_{i=1}^2 I_{4(i)} \wedge \delta^{(1)}_{M_i} + \sum_{i=1}^{32} g_i \delta^{(5)}_{M_i^6}
\] (4.32)

where \( \delta^{(5)}_{M_i^6} \) has support on the six-planes \( M_i^6 \). As discussed in [54], [55], the magnetic charges \( g_i \) are required to take the values
\[
g_i = -3/4, -1/4, +1/4, ...
\] (4.33)
Using the modified Bianchi identity (4.31), one can compute the variation of the $GX_7$ term under Lorentz and gauge transformations. The result is that this term gives rise to a classical anomaly that descends from the polynomial

$$I_8(GX_7)_i = -g_i X_8$$

(4.34)

where $X_8$ is presented in expression (4.13). The relevant anomaly is then

$$I_8(1\text{-}\text{loop})_i + I_8(GX_7)_i$$

(4.35)

where $I_8(1\text{-}\text{loop})_i$ is given in (4.31). This anomaly spoils the consistency of the theory and, hence, must cancel. One begins the analysis of anomaly cancellation by considering the pure $\text{tr} R^4$ term in (1.33) which is irreducible and must identically vanish. It follows from the above that this term is

$$-\frac{1}{(2\pi)^4 240} (n_{V_i} - n_{H_i} - 29n_{T_i} + 30g_i + 23) \text{tr} R^4.$$  

(4.36)

Therefore, the $\text{tr} R^4$ term will vanish if and only if on each orbifold plane the constraint

$$n_{V_i} - n_{H_i} - 29n_{T_i} + 30g_i + 23 = 0$$

(4.37)

is satisfied. Herein lies a problem, and the main point of paper [57]. Noting from (4.33) that $g_i = c_i/4$ where $c_i = -3, -1, 1, 3, 5, ...$, we see that cancelling the $\text{tr} R^4$ term requires that we satisfy

$$n_{V_i} - n_{H_i} - 29n_{T_i} = (-15c_i - 46)/2.$$  

(4.38)

However, this is not possible since the left hand side of this expression is an integer and the right hand side always half integer. There is only one possible resolution of this problem, which is to carefully review the only assumption that was made above, that is, that there is no twisted sector on a seven-plane and, hence, no contribution of the seven-planes by $\alpha \mathbf{Z}_2$ projection to the untwisted anomaly on a six-plane. As we now show, this assumption is false.

Let us now allow for the possibility that there is a twisted sector of $N = 1$ supermultiplets on each of the sixteen seven-planes. The most general allowed spectrum on the $i$-th seven-plane would be $n_{7V_i}$ vector supermultiplets transforming in the adjoint representation of some as yet unspecified gauge group $G_{7i}$. Each seven-dimensional vector multiplet contains a gauge field, three scalars and a gaugino. With respect to six-dimensions, this vector
multiplet decomposes into an $N = 1$ vector supermultiplet and a single hypermultiplet. Under the $\alpha \mathbb{Z}_2$ projection to each of the two embedded six-planes, the gauge group $G_{7i}$ can be preserved or broken to a subgroup. In either case, we denote the six-dimensional gauge group arising in this manner as $\tilde{G}_i$, define $\tilde{n}_{Vi} = \dim \tilde{G}_i$ and write the associated gauge field strength as $\tilde{F}_i$. In this lecture, for simplicity, we will assume that the gauge group is unbroken by the orbifold projection, that is, $\tilde{G}_i = G_{7i}$. The more general case where it is broken to a subgroup is discussed in [55, 56]. Furthermore, the $\alpha$ action projects out either the six-dimensional vector supermultiplet, in which case the hypermultiplet descends to the six-dimensional untwisted sector, or the six-dimensional hypermultiplet, in which case the vector supermultiplet enters the six-dimensional untwisted sector. We denote by $\tilde{n}_{Hi}$ the number of hypermultiplets arising in the six-dimensional untwisted sector by projection from the seven-plane, and specify their (possibly reducible) representation under $\tilde{G}_i$ as $\tilde{R}$.

Since these fields are in the untwisted sector associated with a single seven-plane, and since there are two six-planes embedded in each seven-plane, their contribution to the quantum anomaly on each six-plane can be determined by taking $1/2$ of the index theorem result. We find that the one-loop quantum contribution of this part of the the untwisted spectrum to the gravitational, mixed and $\tilde{G}_i$ gauge anomalies on the $i$-th six-plane is

$$I^8(1/2)_{GAUGE}(\tilde{F}_i) = \frac{1}{2} \left( I^{(1/2)}_{GRAV}(R) + I^{(1/2)}_{MIXED}(R, \tilde{F}_i) + I^{(1/2)}_{GAUGE}(\tilde{F}_i) \right)$$

where $I^{(1/2)}_{GRAV}(R)$, $I^{(1/2)}_{MIXED}(R, \tilde{F}_i)$ and $I^{(1/2)}_{GAUGE}(\tilde{F}_i)$ are given in (4.22), (4.28) and (4.29) respectively with the gauge and hypermultiplet quantities replaced by their “$\sim$” equivalents.

The total quantum anomaly on the $i$-th six-plane is now modified to

$$I^8(1-\text{loop})_i + I^8(\tilde{G}_i)$$

where $I^8(1-\text{loop})_i$ and $I^8(\tilde{G}_i)$ are given in (4.31) and (4.39) respectively. It follows that the relevant anomaly contributing to, among other things, the irreducible $\text{tr} R^4$ term is modified to

$$I^8(1-\text{loop})_i + I^8(\tilde{G}_i) + I^8(GX_7)_i ,$$

This anomaly spoils the quantum consistency of the theory and, hence, must cancel. We again begin by considering the pure $\text{tr} R^4$ term in (4.41). This term is irreducible and must identically vanish. It follows from the above that this term is

$$-\frac{1}{(2\pi)^3 4! 240} \left( n_{Vi} - n_{Hi} + \frac{1}{2} \tilde{n}_{Vi} - \frac{1}{2} \tilde{n}_{Hi} - 29 n_{Ti} + 30 g_i + 23 \right) \text{tr} R^4 .$$
Therefore, the $\text{tr } R^4$ term will vanish if and only if on each orbifold plane the constraint

$$n_{Vi} - n_{Hi} + \frac{1}{2} \tilde{n}_{Vi} - \frac{1}{2} \tilde{n}_{Hi} - 29 n_{Ti} + 30 g_i + 23 = 0 \quad (4.43)$$

is satisfied. Again, noting that $g_i = c_i/4$ where $c_i = -3, -1, 1, 3, 5, ...$, we see that we must satisfy

$$n_{Vi} - n_{Hi} + \frac{1}{2} \tilde{n}_{Vi} - \frac{1}{2} \tilde{n}_{Hi} - 29 n_{Ti} = \frac{1}{2} (-15c_i - 46) \quad . \quad (4.44)$$

As above, the right hand side is always a half integer. Now, however, because of the addition of the untwisted spectrum arising from the seven-plane, the left hand side can also be chosen to be half integer. Hence, the pure $\text{tr } R^4$ term can be cancelled.

Having cancelled the irreducible $\text{tr } R^4$ term, we now compute the remaining terms in the anomaly eight-form. In addition to the contributions from (4.41), we must also take into account the classical anomaly associated with the $CGG$ term in the eleven-dimensional action (4.16). Using the modified Bianchi identity (4.31), one can compute the variation of the $CGG$ term under Lorentz and gauge transformations. The result is that this term gives rise to a classical anomaly that descends from the polynomial

$$I_8(CGG)_i = -\pi g_i I^2_{4(i)} \quad (4.45)$$

where $I_{4(i)}$ is given in expression (4.14). Adding this anomaly to (4.41), and cancelling the $\text{tr } R^4$ term by imposing constraint (4.43), we can now determine the remaining terms in the anomaly eight-form.

Recall that, in this lecture, we are assuming that the $\beta$ action on the ten-dimensional vector supermultiplet does not break the $E_8$ gauge group. In this case, we can readily show that there can be no twisted sector vector multiplets on any six-plane. Rather than complicate the present discussion, we will simply assume here that gauge field strengths $\mathcal{F}_i$ do not appear. Furthermore, cancellation of the complete anomaly, in the case where $E_8$ is unbroken, requires that $\tilde{\mathcal{G}}_i$ be a product of $U(1)$ factors. Here, we will limit the discussion to the simplest case where

$$\tilde{\mathcal{G}}_i = U(1) \quad (4.46)$$

The $\beta$ action on the seven-dimensional plane then either projects a single vector supermultiplet, or a single chargeless hypermultiplet, onto the untwisted sector of the six-plane. In either case, no $U(1)$ anomaly exists. Hence, the gauge field strengths $\tilde{\mathcal{F}}_i$ also do not appear.
With this in mind, we now compute the remaining terms in the anomaly eight-form. They are

\[
\frac{1}{(2\pi)^3 4! 16} \left( \frac{3}{4} (1 - 4 n_{T_i}) (\text{tr} R^2)^2 + \frac{1}{20} (5 + 8 g_i) \text{tr} R^2 \wedge \text{Tr} F_i^2 \right. \\
- \frac{1}{100} (1 + \frac{4}{3} g_i) \left( \text{Tr} F_i^2 \right)^2 \right) \tag{4.47}
\]

where we have used the $E_8$ trace relation $\text{Tr} F^4 = \frac{1}{100} (\text{Tr} F^2)^2$. Note that, since $n_{T_i}$ is a non-negative integer and $g_i$ must satisfy (4.33), the first two terms of this expression term can never vanish. Furthermore, it is straightforward to show that (4.47) will factor into an exact square, and, hence, be potentially cancelled by a six-plane Green-Schwarz mechanism, if and only if

\[
4 (4 n_{T_i} - 1)(3 + 4 g_i) = (5 + 8 g_i)^2 \tag{4.48}
\]

Again, this equation has no solutions for the allowed values of $n_{T_i}$ and $g_i$. It follows that anomaly (4.47), as it presently stands, cannot be be made to identically vanish or cancel. The resolution of this problem was first described in [54], and consists of the realization that the existence of seven-planes in the theory necessitates the introduction of additional Chern-Simons interactions in the action, one for each seven-plane. The required terms are

\[
S = \cdots + \sum_{i=1}^{16} \int \delta Y_3^{(4)} \wedge G \wedge Y_3^{(0)} \tag{4.49}
\]

where $dY_3^{(0)} = Y_4^{(i)}$ is a gauge-invariant four-form polynomial. $Y_4^{(i)}$ arises from the curvature $R$ and also the field strength $\tilde{F}_i$ associated with the additional adjoint super-gauge fields localized on the $i$-th seven-plane. It is given by

\[
Y_4^{(i)} = \frac{1}{4\pi} \left( -\frac{1}{32} \eta \text{tr} R^2 + \rho \text{tr} \tilde{F}_i \right) \tag{4.50}
\]

where $\eta$ and $\rho$ are rational coefficients. Using the modified Bianchi identity (4.32), one can compute the variation of the $\delta^7 G Y_3$ terms under Lorentz and gauge transformations. The result is that these give rise to a classical anomaly that descends from the polynomial

\[
I_8(\delta^7 G Y_3)_i = -I_4^{(i)} \wedge Y_4^{(i)} \tag{4.51}
\]

where $I_4^{(i)}$ is the four-form given in (4.14).

The total anomaly on the $i$-th six-plane is now modified to

\[
I_8(1-\text{loop})_i + I_8(\tilde{G}_i) + I_8(G X_7)_i + I_8(\text{CGG})_i + I_8(\delta^7 G Y_3)_i \tag{4.52}
\]
where \( I_8(1\text{-loop})_i, I_8(\hat{G}_i), I_8(GX)_i, I_8(CG\bar{G})_i \) and \( I_8(\delta^7GY_3)_i \) are given in (4.31), (4.39), (4.34), (4.45) and (4.51) respectively. Note that for the fixed plane intersection presently under discussion, the field strength \( \tilde{\mathcal{F}}_i \) does not enter the anomaly eight-form (4.47). Therefore, within this context, we must take
\[
\rho = 0.
\] (4.53)

After cancelling the irreducible \( \text{tr} R^4 \) term, the remaining anomaly now becomes
\[
\frac{1}{(2\pi)^4} \left( \frac{3}{4} (1 - 4n_{T_i} - \eta) (\text{tr} R^2)^2 + \frac{1}{20} (5 + 8g_i + \eta) \text{tr} R^2 \wedge \text{Tr} F_i^2 - \frac{1}{100} (1 + \frac{4}{3}g_i) (\text{Tr} F_i^2)^2 \right) (4.54)
\]

Depending on the number of untwisted hypermultiplets, \( n_{T_i} \), these terms can be made to cancel or to factor into the sum of exact squares. In this lecture, we consider the \( n_{T_i} = 0, 1 \) cases only. As discussed in [55, 56], the solutions where \( n_{T_i} \geq 2 \) are related to the \( n_{T_i} = 0, 1 \) solutions by the absorption of one or more five-branes from the bulk space onto the \( i \)-th six-plane.

We first consider the case where
\[ n_{T_i} = 0. \] (4.55)

In this case, no further Green-Schwarz type mechanism in six-dimensions is possible and the anomaly must vanish identically. We see from (4.54) that this is possible if and only if
\[ g_i = -3/4, \quad \eta = 1. \] (4.56)

It is important to note that this solution only exists for a non-vanishing value of parameter \( \eta \). Hence, the additional Chern-Simons interactions (4.49) are essential for the anomaly to vanish identically in the \( n_{T_i} = 0 \) case. Inserting these results into expression (4.43) for the vanishing of the irreducible \( \text{tr} R^4 \) term, and recalling that \( n_{V_i} = 0 \), we find that
\[ -2n_{H_i} + \tilde{n}_{V_i} - \tilde{n}_{H_i} = -1. \] (4.57)

Equation (4.57) can be solved in several ways. Remembering that \( \hat{G}_i = U(1) \), the first solution then consists of allowing the \( U(1) \) hypermultiplet to descend to the six-plane while projecting out the \( U(1) \) vector multiplet. Equation (4.57) is then solved by taking the number of twisted hypermultiplets to vanish. That is, take
\[ \tilde{n}_{H_i} = 1, \quad \tilde{n}_{V_i} = 0, \quad n_{H_i} = 0. \] (4.58)
The second solution follows by doing the reverse, that is, projecting out the $U(1)$ hyper-multiplet and allowing the $U(1)$ vector multiplet to descend to the six-plane. In this case, equation (4.57) is solved by taking

$$\tilde{n}_{Hi} = 0, \quad \tilde{n}_{Vi} = 1, \quad n_{Hi} = 1.$$ (4.59)

Let us now consider the case where

$$n_{Ti} = 1.$$ (4.60)

In this case, the anomaly (4.54) can be removed by a six-dimensional Green-Schwarz mechanism as long as it factors into an exact square. It is straightforward to show that this will be the case if and only if

$$4 \left( 3 + \eta \right) \left( 3 + 4 g_i \right) = \left( 5 + 8 g_i + \eta \right)^2.$$ (4.61)

This equation has two solutions

$$g_i = -3/4, \quad \eta = 1$$ (4.62)

and

$$g_i = 1/4, \quad \eta = 1.$$ (4.63)

Again, note that these solutions require a non-vanishing value of the parameter $\eta$. Hence, the additional Chern-Simons interactions (4.49) are also essential for anomaly factorization in the $n_{Ti} = 1$ case. Inserting these into the expression for the vanishing of the irreducible $tr R^4$ term, and recalling that $n_{Vi} = 0$, we find

$$-2n_{Hi} + \tilde{n}_{Vi} - \tilde{n}_{Hi} = 57$$ (4.64)

and

$$-2n_{Hi} + \tilde{n}_{Vi} - \tilde{n}_{Hi} = -3.$$ (4.65)

The first equation (4.64) cannot be solved within the context of $\tilde{G}_i = U(1)$, since $\tilde{n}_{Vi} \leq 1$. The second equation, however, has two solutions

$$\tilde{n}_{Hi} = 1, \quad \tilde{n}_{Vi} = 0, \quad n_{Hi} = 1$$ (4.66)

and

$$\tilde{n}_{Hi} = 0, \quad \tilde{n}_{Vi} = 1, \quad n_{Hi} = 2.$$ (4.67)
In either case, the anomaly (4.54) factors into an exact square given by
\[- \frac{3}{(2\pi)^3 4! 16} \left( \text{tr} R^2 - \frac{1}{15} \text{Tr} F_i^2 \right)^2. \tag{4.68}\]
The anomaly can now be cancelled by a Green-Schwarz mechanism on the six-plane. First, one alters the Bianchi identity for the anti-self-dual tensor in the twisted sector tensor multiplet from \(dH_{T_i} = 0\), where \(H_{T_i}\) is the tensor field strength three-form, to
\[dH_{T_i} = \frac{1}{16\pi^2} \left( \text{tr} R^2 - \frac{1}{15} \text{Tr} F_i^2 \right). \tag{4.69}\]
Second, additional Chern-Simons terms are added to the action, one for each six-plane. The required terms are
\[S = \cdots - \frac{1}{64\pi} \sum_{i=1}^{32} \int \delta_M^{(5)} \wedge B_{T_i} \wedge \left( \text{tr} R^2 - \frac{1}{15} \text{Tr} F_i^2 \right), \tag{4.70}\]
where \(B_{T_i}\) is the anti-self-dual tensor two-form on the \(i\)-th six-plane. Using Bianchi identity (4.69), one can compute the variation of each such term under Lorentz and gauge transformations. The result is a classical anomaly that descends from an eight-form that exactly cancels expression (4.68). The theory is now anomaly free.

Thus, we have demonstrated, within the context of an explicit orbifold fixed plane intersection where the \(\beta\) \(\mathbb{Z}_2\) projection to the six-plane leaves \(E_8\) unbroken, that all local anomalies can be cancelled. However, this cancellation requires that the intersecting seven-plane support a twisted sector consisting of a \(U(1) N = 1\) vector supermultiplet and an associated Chern-Simons term. This term is of the form (4.49) with \(\eta = 1\) and \(\rho = 0\). The fact that \(\rho = 0\) in this context follows directly from the property that \(E_8\) is unbroken by the \(\beta\) projection.

We conclude that, as has been discussed in detail in \([54, 55, 56, 57]\) and \([58, 59]\), anomaly free \(M\)-theory orbifolds associated with the spacetime \(\mathbb{R}^6 \times K3\) can be constructed in detail, including the entire twisted and untwisted spectra. This work has now been extended to orbifolds of spacetime \(\mathbb{R}^4 \times CY_3\), where \(CY_3\) is a Calabi-Yau threefold, in \([60]\). This last work opens the door to finding realistic standard model-like \(M\)-theory vacua within this context.

### 5 Discussion

Hořava-Witten theory and its compactification on Calabi-Yau threefolds to heterotic \(M\)-theory have stimulated a great deal of both formal \(M\)-theory research as well as discussions
of the associated phenomenology. In addition to the papers referenced in the above lectures, further relevant literature can be found in [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93].

Heterotic $M$-theory has also served as a consistent and phenomenologically relevant venue for studying $M$-theory cosmology. This research comes in two categories. The first consists of work discussing subluminal expansion, Kasner-like solutions and inflation within the context of brane world scenarios associated with heterotic $M$-theory. These results can be found in [81, 82, 83, 84, 85, 86, 87]. Very recently, a new theory of the early universe, called the Ekpyrotic Universe, has been constructed for generic brane world scenarios, including heterotic $M$-theory. In the Ekpyrotic scenario, all expansion is subluminal, with no period of inflation. A nearly scale-invariant spectrum of fluctuations in the microwave background is obtained, not as quantum fluctuations in deSitter space but, rather, as the fluctuations on a bulk brane or end-of-the-world brane as it moves through the fifth-dimension. The fundamental papers on this subject can be found in [88, 89, 90, 91, 92, 93].

Acknowledgements

Burt Ovrut is supported in part by the DOE under contract No. DE-AC02-76-ER-03071.

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