DIFFERENTIAL OPERATORS ON PROJECTIVE MODULES

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Abstract. This paper is a continuation of an earlier paper where the problem of giving explicit formulas for connections on modules was studied. The Kodaira-Sencer map was used to construct explicit formulas for connections on a class of maximal Cohen-Macaulay modules on hypersurface singularities. In this paper we use the notion of a projective basis to give explicit formulas for generalized connections on a class of modules on ellipsoids. We get thus many examples of modules on ellipsoids with non-flat algebraic connections. We use the explicit formulas for connections to prove the first Chern class \( c_1(E) \) of a finitely generated module \( E \) with values in Lie-Rinehart cohomology and DeRham cohomology is zero. We moreover use projective bases to give explicit formulas for higher order connections on projective modules. We define the ring of projectivity of a left \( A \)-module \( E \) and prove this ring is the obstruction for \( E \) to be finitely generated and projective.

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1. Introduction

For vector bundles on complex projective manifolds there are no examples of non-flat connections (see the paper by Bloch and Esnault [3] in the introduction). In fact in Atiyah’s paper [1] it is conjectured that if a holomorphic vector bundle on a complex projective manifold has a connection, then it has a flat connection. One of the subjects of this paper is to indicate that in the affine situation few vector bundles have flat connections.

For a smooth finite rank real vector bundle on a real smooth finite dimensional manifold the existence of a flat connection implies that the bundle is a trivial bundle (see Milnor’s book [10] in the appendix). This is not the case in the algebraic category. In the paper [7] we give examples of non-free Cohen-Macaulay modules on isolated hypersurface singularities with flat algebraic connections. Hence flatness of the algebraic connection does not imply triviality of the module. A trivial vector
bundle on an affine algebraic variety always has a flat connection hence if we look for vector bundles on algebraic varieties with nonflat algebraic connections we must consider nontrivial vector bundles. One of the aims of this paper is to give explicit formulas for nonflat algebraic connections on a large class of nontrivial finite rank vector bundles on ellipsoids (see Theorem 4.2). Another aim is to use projective bases to construct lifts of differential operators on the base variety to the vector bundle via higher order connections (see Definition 5.8). In the smooth category it is known that any local operator is a differential operator (see [11]). In algebra any connection is a differential operator of degree one. Higher order differential operators arise in the theory of \(D\)-modules and crystalline cohomology.

In a previous paper on a related subject the Kodaira-Spencer map and the Atiyah-class was used to give explicit formulas for algebraic connections on maximal Cohen-Macaulay modules on hypersurface singularities (see [7]). In the papers [8] and [9] the notion of an \(I\)-connection was introduced where \(I\) is a left and right \(A\)-module. The purpose of this paper is to introduce the notion of a projective basis for a finitely generated and projective \(A\)-module \(E\) and to use such a basis to give explicit formulas for \(I\)-connections on a class of modules on ellipsoids (see Theorem 2.8). The connections we construct are non-flat in general.

A projective basis for a finitely generated \(A\)-module \(E\) is a set \(e_1,\ldots,e_n\) of generators of \(E\) and \(x_1,\ldots,x_n\) of \(E^*\) satisfying the relation

\[
\sum_i x_i(e)e_i = e
\]

for all elements \(e \in E\). An \(A\)-module \(E\) has a projective basis if and only if it is finitely generated and projective. Using a projective basis \(e_i, x_j\) for a finitely generated projective module \(E\) we construct the fundamental matrix \(M\) associated to \(e_i, x_j\). This is an \(n \times n\)-matrix with coefficients in \(A\). Using the projective basis \(e_i, x_j\) we construct an associated connection

\[
\nabla : \text{Der}_k(A) \to \text{End}_k(E).
\]

In Theorem 5 we prove the following formula:

\[
R_{\nabla}(\delta \wedge \eta) = [\delta(M), \eta(M)]
\]

for any derivations \(\delta, \eta \in \text{Der}_k(A)\). Hence the curvature matrix \(R_{\nabla}(\delta \wedge \eta)\) is given as the Lie-product of the two matrices \(\delta(M), \eta(M)\) where we apply the derivations \(\delta, \eta\) to the coefficients of the matrix \(M\).

We construct a higher order connection

\[
\nabla^l : E \to P^l \otimes_A E
\]

associated to a projective basis for \(E\) (see Proposition 5.5). Here \(P^l\) is the \(l\)’th module of principal parts of the ring \(A\). There is a notion of flatness for higher order connections and the connection \(\nabla^l\) is non-flat in general. We use the class of connections to construct a canonical map

\[
\rho : \text{Diff}_{Z}(A) \to \text{Diff}_{Z}(E)
\]

of left \(A\)-modules (see Equation 5.6.2). The map \(\rho\) is not a map of associative rings in general, hence the left \(A\)-module structure on \(E\) does not lift to a left \(\text{Diff}_{Z}(A)\)-module structure on \(E\). The obstruction to such a lifting is given by the generalized curvature of the higher order connection \(\nabla^l\). To give a left \(\text{Diff}_{Z}(A)\)-module structure on a left \(A\)-module \(E\) is equivalent to giving a stratification on \(E\) in the sense...
of crystalline cohomology (see [2]). The map $\rho$ is a ring-homomorphism if and only if $E$ is a left $\text{Diff}_Z(A)$-module. We show that a canonical connection $\rho$ induced by a projective basis seldom is a ring-homomorphism hence a finitely generated projective module $E$ seldom has a stratification induced by a projective basis. The notion of a stratification is used in the construction of crystalline cohomology (see [2]).

We moreover define the ring of projectivity $\text{end}(E)$ of a left $A$-module $E$ over any commutative unital ring $A$ and prove $\text{end}(E)$ is the obstruction for $E$ to be finitely generated and projective (see Theorem 2.6).

2. The ring of projectivity of a module

Let in this section $A$ be a commutative ring with unit and let $E$ be a left $A$-module. We give several criteria for the $A$-module $E$ to be finitely generated and projective. We use the constructions to give methods to calculate explicit $I$-connections and covariant derivations on finitely generated projective modules.

Let in the following $B = \{e_1, ..., e_n\}$ be a set of elements of $E$ and let $B^* = \{x_1, ..., x_n\}$ be a set of elements of $E^*$.

**Definition 2.1.** We say the sets $\{B, B^*\}$ form a projective basis for $E$ if the following holds:

$$\sum_j x_j(e)e_j = e$$

for all elements $e \in E$.

**Lemma 2.2.** Assume $\{B, B^*\}$ is a projective basis for $E$. It follows the set $B$ generates $E$ hence $E$ is finitely generated as left $A$-module.

**Proof.** Assume $e$ is an element in $E$. We get the equation

$$e = \sum_j x_j(e)e_j$$

hence the set $B$ generates $E$ as left $A$-module since $x_j(e) \in A$ for all elements $e$ in $E$. \qed

Let $F = A\{u_1, ..., u_n\}$ be a free left $A$-module on the basis $C = \{u_1, ..., u_n\}$, let $B = \{e_1, ..., e_n\}$ be a set of elements of $E$ and define the following map:

$$p : F \to E$$

by

$$p(u_i) = e_i.$$ 

Let

$$y_i : F \to A$$

be defined by

$$y_i(\sum a_j u_j) = a_i.$$ 

It follows $y_i = u_i^*$. Let $u = \sum_i a_i u_i \in F$. We get

$$p(u) = \sum_i a_i p(u_i) = \sum_i a_i e_i = \sum_i y_i(u)e_i.$$ 

Define the following map

$$\rho : E^* \otimes_A E \to \text{End}_A(E)$$
by
\[ \rho\left(\sum \phi_i \otimes e_i\right)(e) = \sum \phi_i(e) e_i. \]

**Definition 2.3.** Let \( \text{end}(E) = \text{Coker}(\rho) \) be \( E \)'s ring of projectivity.

Define the following product on \( E^* \otimes_A E^* \):
\[ \bullet : E^* \otimes_A E^* \otimes_A E^* \to E^* \otimes_A E^* \]
by
\[ \phi \otimes u \bullet \psi \otimes v = \psi \otimes \phi(v)u. \]

**Lemma 2.4.** The following holds:

1. \( \{E^* \otimes_A E, \bullet\} \) is an associative ring.
2. \( \rho \) is a map of \( A \)-algebras.
3. \( \text{Im}(\rho) \) is a two sided ideal in \( \text{End}_A(E) \).

**Proof.** Define the following map:
\[ f : E^* \times E \times E^* \times E \to E^* \otimes E \]
by
\[ f(\phi, u, \psi, v) = \psi \otimes \phi(v)u. \]
One checks \( f \) is \( A \)-bilinear in all variables hence we get a well defined product
\[ \bullet : E^* \otimes E \times E^* \otimes E \to E^* \otimes E \]
defined by
\[ (\phi \otimes u) \bullet (\psi \otimes v) = \psi \otimes \phi(v)u. \]
One checks this product is left and right distributive over addition in \( E^* \otimes_A E \).
Hence
\[ x \bullet (y + z) = x \bullet y + x \bullet z \]
and
\[ (y + z) \bullet x = y \bullet x + z \bullet x. \]
We check the product \( \bullet \) is associative:
\[ \phi \otimes u \bullet (\psi \otimes v \bullet \chi \otimes w) = \]
\[ \phi \otimes u \bullet \chi \otimes \psi(w)v = \]
\[ \chi \otimes \phi(\psi(w)v)u = \]
\[ \chi \otimes \psi(w)\phi(v)u = \]
\[ (\psi \otimes \phi(v)u) \bullet \chi \otimes w = \]
\[ (\phi \otimes u \bullet \psi \otimes v) \bullet \chi \otimes w. \]
It follows \( \bullet \) is a product on \( E^* \otimes_A E \) and hence \( E^* \otimes_A E \) is an associative (nonunital) ring. One checks the map \( \rho \) is a map of rings and left \( A \)-modules hence Claim 2.4.2 follows. We check \( \text{Im}(\rho) \) is a two sided ideal. It is clear \( \text{Im}(\rho) \) is an abelian subgroup of \( \text{End}_A(E) \). Assume \( \psi \in \text{End}_A(E) \) and \( \phi \otimes e \in \text{Im}(\rho) \). It follows
\[ \psi \circ \phi \otimes e(x) = \psi(\phi(x)e) = \phi(x)\psi(e). \]
It follows
\[ \psi \circ \phi \otimes e = \rho(\phi \otimes \psi(e)) \]
hence
\[ \psi \circ \phi \otimes e \in \text{Im}(\rho) \]
One checks
\[ \rho(\phi \otimes e) \circ \psi = \rho(\phi \circ \psi \otimes e) \]
hence \( \text{Im}(\rho) \) is a two sided ideal in \( \text{End}_A(E) \) and Claim 2.4.3 follows. The Lemma is proved. \( \square \)

It follows from Lemma 2.4 that \( \text{End}(E) \) is an associative \( A \)-algebra.

Assume \( \nabla : L \rightarrow \text{End}_Z(E) \) is a connection where
\[ \alpha : L \rightarrow \text{Der}_Z(A) \]
is a Lie-Rinehart algebra. This means \( \nabla \) is an \( A \)-linear map and for all \( a \in A \), \( e \in E \) and \( \delta \in L \) it follows
\[ \nabla(\delta)(ae) = a\nabla(\delta)(e) + \delta(a)e. \]
The curvature of the connection \( \nabla \) is the following map:
\[ R_{\nabla} : L \wedge A L \rightarrow \text{End}_A(E) \]
with
\[ R_{\nabla}(\delta \wedge \eta) = [\nabla(\delta), \nabla(\eta)] - \nabla([\delta, \eta]). \]
The curvature \( R_{\nabla} \) is the obstruction for \( \nabla \) to be a map of Lie algebras. We say \( \nabla \) is flat if \( R_{\nabla} \) is zero.

Make the following definitions: The connection \( \nabla \) induce a canonical connection \( \nabla^* \) on \( E^* \) as follows: Let \( \phi \in E^* \) and define
\[ \nabla^*(\delta)(\phi) = \delta \circ \phi - \phi \circ \nabla(\delta). \]
One verifies the map
\[ \nabla^* : L \rightarrow \text{End}_Z(E^*) \]
is a connection. We get a canonical connection
\[ \nabla^* : L \rightarrow \text{End}_Z(E^* \otimes_A E) \]
deﬁned by
\[ \nabla^*(\delta)(\phi \otimes e) = \nabla^*(\delta)(\phi) \otimes e + \phi \otimes \nabla(\delta)(e). \]
There is a canonical connection
\[ \text{ad}\nabla : L \rightarrow \text{End}_Z(\text{End}_A(E)) \]
deﬁned by
\[ \text{ad}\nabla(\delta)(\phi) = \nabla(\delta) \circ \phi - \phi \circ \nabla(\delta). \]

**Lemma 2.5.** The canonical map
\[ \rho : E^* \otimes_A E \rightarrow \text{End}_A(E) \]
is a map of \( L \)-connections. The curvature of \( \text{ad}\nabla \) is as follows:
\[ R_{\text{ad}\nabla}(\delta \wedge \eta)(\phi) = [R_{\nabla}(\delta \wedge \eta), \phi]. \]

**Proof.** We need to check the following: For any element \( \phi \otimes e \in E^* \otimes_A E \)
\[ \text{ad}\nabla(\delta)(\rho(\phi \otimes e)) = \rho(\nabla(\delta)(\phi \otimes e)). \]
We get:
\[ \text{ad}\nabla(\rho(\phi \otimes e))(x) = \nabla(\delta)(\phi(x)e) - \phi(\nabla(\delta)(x))e. \]
We see
\[ \rho(\nabla(\delta)(\phi \otimes e))(x) = \\
\rho(\nabla^*(\delta)(\phi) \otimes e + \phi \otimes \nabla(\delta)(e))(x) = \\
\]
\(\nabla^*(\delta)(\phi)(x)e + \phi(x)\nabla(\delta)(e) = \\
\delta(\phi(x))e - \phi(\nabla(\delta)(x))e + \phi(x)\nabla(\delta)(e).\)

Since \(\nabla\) is a connection it follows
\[
\delta(\phi(x))e + \phi(x)\nabla(\delta)(e) = \nabla(\delta)(\phi(x)e).
\]

It follows
\[ad\nabla(\rho(\phi \otimes e)) = \rho(\nabla(\delta)(\phi \otimes e))\]
and the Lemma is proved. Assume \(\delta, \eta \in L\). We get
\[R_{ad\nabla}(\delta \wedge \eta) = [ad\nabla(\delta), ad\nabla(\eta)] - ad\nabla([\delta, \eta]).\]

It follows
\[R_{ad\nabla}(\delta, \eta)(\phi) = ad\nabla(\delta) \circ ad\nabla(\eta)(\phi) - ad\nabla(\eta) \circ ad\nabla(\delta)(\phi) - ad\nabla([\delta, \eta])(\phi) = \\
ad\nabla(\nabla(\eta) \circ \phi - \phi \circ \nabla(\eta)) - ad\nabla(\eta)(\nabla(\delta) \circ \phi - \phi \circ \nabla(\delta)) \\
- (\nabla([\delta, \eta]) \circ \phi - \phi \circ \nabla([\delta, \eta])) = \\
[\nabla(\delta), \nabla(\eta)] \circ \phi - \phi \circ [\nabla(\eta), \nabla(\delta)] - [\nabla([\delta, \eta]), \phi] = \\
[[\nabla(\delta), \nabla(\eta)], \phi] - [\nabla([\delta, \eta]), \phi] = \\
[R_{\nabla}(\delta \wedge \eta), \phi].\]

It follows
\[R_{ad\nabla}(\delta \wedge \eta)(\phi) = [R_{\nabla}(\delta \wedge \eta), \phi]\]
and the Lemma is proved. \(\square\)

**Theorem 2.6.** The following are equivalent:

\begin{enumerate}
\item[(2.6.1)] \(E\) is a finitely generated projective \(A\)-module.
\item[(2.6.2)] \(E\) has a projective basis.
\item[(2.6.3)] \(id_E\) is in the image of \(\rho\).
\item[(2.6.4)] \(\rho\) is an isomorphism.
\item[(2.6.5)] \(\text{end}(E) = 0\).
\end{enumerate}

Moreover if \(E\) has an \(L\)-connection \(\nabla\) it follows \(\text{end}(E)\) has an \(L\)-connection \(\theta_{\nabla}\). If \(\nabla\) is flat it follows \(\theta_{\nabla}\) is flat.

**Proof.** Define \(x_j = y_j \circ s\) where \(s\) is a section of the map \(p : F \to E\). It follows \(s(e_i) = u_i + y\) where \(y\) is in \(\text{ker}(p)\). Let \(e = e_i\). We get
\[
\sum_j x_j(e)e_j = \sum_j y_j(s(e_i))e_j = \sum_j y_j(u_i + y)e_j = \\
\sum_j y_j(u_i)e_j + \sum_j y_j(y)e_j = \sum_j \delta_{ij}e_j + p(y) = e_i = e
\]

since \(y \in \text{ker}(p)\). It follows \(\sum_j x_j(e_i)e_j = e_i\) for all \(i\). Assume \(e = \sum_i a_i e_i\). We get
\[
\sum_j x_j(e)e_j = \sum_j x_j(\sum_i a_i e_i)e_j = \\
\sum_i a_i \sum_j x_j(e_i)e_j = \sum_i a_i e_i = e.
\]
It follows the sets \( \{B, B^*\} \) form a projective basis for \( E \). Conversely if \( \{e_i, x_j\} \) is a projective basis for \( E \) it follows the map
\[
s(e) = \sum_j x_j(e)u_j
\]
is an \( A \)-linear section of \( p \):
\[
p(s(e)) = p(\sum_j x_j(e)u_j) = \sum_j x_j(e)p(u_j) = \sum_j x_j(e)e_j = e.
\]
The equivalence of \( 2.6.1 \) and \( 2.6.2 \) is shown. We prove the equivalence of \( 2.6.1 \) and \( 2.6.3 \): Let \( \omega = \sum_j x_j \otimes e_j \in E^* \otimes E \) be an element. If \( f(\omega) = id_E \) it follows
\[
\sum_j x_j(e_j) = id_E(e) = e
\]
hence \( \{B, B^*\} \) is a projective basis. It follows \( E \) is finitely generated and projective. Assume conversely \( E \) is finitely generated and projective with projective basis \( \{B, B^*\} \). It follows
\[
f(\sum_j x_j \otimes e_j) = id_E.
\]
We have proved the equivalence of \( 2.6.1 \) and \( 2.6.3 \). We prove the equivalence of \( 2.6.1 \) and \( 2.6.4 \): Assume \( E \) is finitely generated and projective. It follows the map \( f \) is an isomorphism. Assume conversely \( f \) is an isomorphism. It follows there is an element
\[
\sum_j x_j \otimes e_j \in E^* \otimes E
\]
mapping to \( id_E \). The elements \( x_j, e_i \) gives rise to a projective basis \( \{B, B^*\} \) for \( E \) hence by the equivalence above it follows \( E \) is finitely generated and projective. The equivalence between \( 2.6.1 \) and \( 2.6.5 \) is clear.

Assume \( \nabla \) is a connection on \( E \). Let \( ad\nabla \) be the induced connection on \( \text{End}_A(E) \). By Lemma 2.5 it follows the two sided ideal \( Im(\rho) \) is stable under the action of \( ad\nabla \) hence \( ad\nabla \) induces a connection \( \theta_{\nabla} \) on \( \text{end}(E) \). By Lemma 2.6 it follows
\[
R_{ad\nabla}(\delta \wedge \eta)(\phi) = [R_{\nabla}(\delta \wedge \eta), \phi].
\]
Hence if \( \nabla \) is flat it follows \( ad\nabla \) is flat hence the induced connection \( \theta_{\nabla} \) is flat. The Theorem is proved. \( \square \)

Hence the associative ring \( \text{end}(E) \) is the obstruction for the module \( E \) to be finitely generated and projective.

Note: In the paper \( [7] \) we used the Kodaira-Spencer map and Atiyah class to calculate explicit expressions of flat connections on a class of maximal Cohen-Macaulay modules on surface singularities. In this calculation we investigated the endomorphism ring \( \text{End}_A(M) \) for a maximal Cohen-Macaulay module \( M \) and a set of special elements of this ring. It might be this calculation is related to the ring of projectivity \( \text{end}(M) \) of \( M \). This topic will be studied in future papers on the subject.

Consider the following example from \( [7] \): Let \( f = x^m + y^n + z^2 \) be an element of \( K[x, y, z] \) where \( K \) is a field of characteristic zero and let \( A = K[x, y, z]/f \).
Let $\phi, \psi$ be the following matrices:

$$
\phi = \begin{pmatrix}
x^{m-k} & y^{n-l} & 0 & z \\
y^l & -x^k & z & 0 \\
z & 0 & -y^{n-l} & -x^k \\
o & z & x^{m-k} & -y^l \\
\end{pmatrix}
$$

and

$$
\psi = \begin{pmatrix}
x^k & y^{n-l} & 0 & z \\
y^l & -x^{m-k} & 0 & z \\
z & 0 & -y^l & x^k \\
o & z & -x^{m-k} & -y^{n-l} \\
\end{pmatrix}
$$

with $0 \leq k \leq m - 1$ and $0 \leq l \leq n - 1$. It follows the matrices $\phi$ and $\psi$ are a matrix factorization of the polynomial $f$ and we get a periodic resolution

$$
\ldots \rightarrow \psi A^4 \rightarrow \phi A^4 \rightarrow \psi A^4 \rightarrow M(\phi, \psi) \rightarrow 0
$$

where $M = M(\phi, \psi)$ is a maximal Cohen-Macaulay module on $A$. In Theorem 2.1 and 3.3 in [7] we calculated a flat connection

$$
\nabla : \mathcal{V}_M \rightarrow \text{End}_K(M)
$$

for all pairs of matrices $\phi, \psi$ and all $m, n \geq 2$ and $k, l$. The module $M$ is not locally free hence the ring of projectivity $\text{end}(M)$ is non-zero.

**Corollary 2.7.** There is for any pair of matrices $\phi, \psi$ a flat connection

$$
\theta : \mathcal{V}_M \rightarrow \text{End}_K(\text{end}(M)).
$$

**Proof.** The Corollary follows immediately from Theorem 2.6, Theorem 2.1 and Theorem 3.3 in [7].

We get from Corollary 2.7 an $U(\mathcal{V}_M)$-module structure on the associative ring $\text{end}(M)$ where $U(\mathcal{V}_M)$ is the universal enveloping algebra of the Lie-Rinehart algebra $\mathcal{V}_M$.

Assume $E$ has a projective basis $\{B, B^*\}$. Let $I$ be a left and right $A$-module and let $d \in \text{Der}_Z(A, I)$ be a derivation. By [9], Proposition 2.13 there is a characteristic class

$$
c_I(E) \in \text{Ext}_A^1(I \otimes_A E)
$$

with the property $c_I(E) = 0$ if and only if $E$ has an $I$-connection

$$
\nabla : E \rightarrow I \otimes_A E
$$

with

$$
\nabla(ae) = a\nabla(e) + d(a) \otimes e.
$$

If $E$ is projective it follows $\text{Ext}_A^1(I \otimes_A E) = 0$ hence trivially $c_I(E) = 0$. We aim to give an explicit calculation of a generalized connections in this case. Define the following maps:

$$
\nabla : E \rightarrow I \otimes_A E
$$

by

$$
\nabla(e) = \sum_i d(x_i(e)) \otimes e_i.
$$

Define also

$$
\nabla' : \text{Der}_Z(A) \rightarrow \text{End}_Z(E)
$$
by
\[ \nabla'(\delta)(e) = \sum_i \delta(x_i(e))e_i. \]

**Theorem 2.8.** The maps \( \nabla, \nabla' \) are connections on \( E \).

**Proof.** Assume \( a \in A \) and \( e \in E \). We get
\[ \nabla(ae) = \sum_i d(x_i(ae)) \otimes e_i = \]
\[ \sum_i d(ax_i(e)) \otimes e_i = \sum_i ad(x_i(e)) \otimes e_i + \sum_i d(a)x_i(e) \otimes e_i = \]
\[ a \sum_i d(x_i(e)) \otimes e_i + \sum_i d(a) \otimes x_i(e)e_i = \]
\[ a \nabla(e) + d(a) \otimes e. \]

We get moreover
\[ \nabla'(\delta)(ae) = \sum_i \delta(x_i( ae ))e_i = \]
\[ \sum_i \delta(ax_i(e))e_i = \]
\[ \sum_i \delta(a)x_i(e)e_i + a\delta(x_i(e))e_i = \]
\[ a\nabla'(e) + \delta(a)e. \]

It follows the maps \( \nabla, \nabla' \) are connections on \( E \).

We see from Theorem 2.8 that the notion of a projective basis gives rise to explicit formulas for \( I \)-connections where \( I \) is any left and right \( A \)-module and \( E \) is any finitely generated projective \( A \)-module.

**Example 2.9.** Complex line bundles on the projective line.

Let \( C \) be the field of complex numbers and consider the projective line \( \mathbb{P}^1 \) over \( C \). Let \( R \) be the field of real numbers. It follows the underlying real variety \( \mathbb{P}^1(R) \) of \( \mathbb{P}^1 \) is isomorphic to the real 2-sphere \( S^2_R = \text{Spec}(A) \) where \( A = R[x, y, z]/x^2 + y^2 + z^2 - 1 \). Any linebundle \( \mathcal{O}(d) \) on \( \mathbb{P}^1 \) gives rise to a locally free \( A \)-module \( P \) of rank 2. In [9] we proved the linebundle \( \mathcal{O}(d) \) on \( \mathbb{P}^1 \) does not have a classical connection
\[ \nabla : \mathcal{O}(d) \to \Omega^1_{\mathbb{P}^1} \otimes \mathcal{O}(d) \]
for any \( d \geq 1 \). The underlying real rank 2 projective \( A \)-module \( P \) corresponding to \( \mathcal{O}(d)(R) \) has a classical connection
\[ \nabla' : P \to \Omega^1_A \otimes_A P \]
and by the above results we may give explicit formulas for \( \nabla' \) using the notion of a projective basis and Theorem 2.8.
3. A FORMULA FOR THE CURVATURE OF A CONNECTION

Let in this section $A$ be a commutative unital ring over a fixed subring $\mathbb{Z}$ and left $E$ be a finitely generated left projective $A$-module. Let $B = \{e_1, \ldots, e_n\}$ and $B^* = \{x_1, \ldots, x_n\}$ be a projective basis for $E$ and let $p : A\{u_1, \ldots, u_n\} \to E$ be defined by $p(u_i) = e_i$. It has a left $A$-linear section $s$ defined by

$$s(e) = \sum_{i=1}^{n} x_i(e)u_i.$$  

We get from Theorem 2.8 a connection $\nabla : \text{Der}_{\mathbb{Z}}(A) \to \text{End}_{\mathbb{Z}}(E)$ defined by

$$\nabla(\delta)(e) = \sum_{i=1}^{n} \delta(x_i(e))e_i.$$  

The aim of this section is to give a general formula for the trace of the curvature of the connection $\nabla$ using the notion of a projective basis. As a consequence we prove the first Chern class $c_1(E)$ with values in Lie-Rinehart cohomology and algebraic DeRham cohomology is zero for any finitely generated projective $A$-module $E$.

**Definition 3.1.** Let

$$M = M(p, s) = \begin{pmatrix} x_1(e_1) & x_1(e_2) & \cdots & x_1(e_n) \\ x_2(e_1) & x_2(e_2) & \cdots & x_2(e_n) \\ \vdots & \vdots & \ddots & \vdots \\ x_n(e_1) & x_n(e_2) & \cdots & x_n(e_n) \end{pmatrix}_{B,B^*}$$

be the \textit{fundamental matrix} of the split surjection $p$ with respect to the projective basis $B, B^*$.

Let $\delta \in \text{Der}_{\mathbb{Z}}(A)$ be a derivation and let $e = a_1e_1 + \cdots + a_ne_n \in E$ be any element and write

$$e_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_B.$$  

We get the following calculation:

$$\nabla(\delta)(e) = a_1\nabla(\delta)(e_1) + \delta(a_1)e_1 + \cdots + a_n\nabla(\delta)(e_n) + \delta(a_n)e_n.$$  

By definition

$$\nabla(\delta)(e_k) = \delta(x_1(e_k))e_1 + \delta(x_2(e_k))e_2 + \cdots + \delta(x_n(e_k))e_n.$$  

We write this in “matrix form” as follows:

$$\nabla(\delta)(e_k) = \begin{pmatrix} \delta(x_1(e_k)) \\ \delta(x_2(e_k)) \\ \vdots \\ \delta(x_n(e_k)) \end{pmatrix}_B.$$
We will write down explicit formulas for the connection $\nabla$ in matrix-notation using the above notation. By definition
\[
\nabla(\delta)(e_k) = \delta(x_1(e_k))e_1 + \cdots + \delta(x_n(e_k))e_n =
\begin{pmatrix}
\delta(x_1(e_k)) \\
\delta(x_2(e_k)) \\
\vdots \\
\delta(x_n(e_k))
\end{pmatrix}_B = [\delta(x(e_k))]_B.
\]
We get
\[
\nabla(\delta)(e) = a_1[\delta(x(e_1))]_B + \delta(a_1)e_1 + \cdots + a_n[\delta(x(e_n))]_B + \delta(a_n)e_n = ([D_\delta]_B + [\delta(M)]_B)e_B
\]
where we use the following notation:
\[
[D_\delta]_B(e_B) = \begin{pmatrix}
\delta(a_1) \\
\delta(a_2) \\
\vdots \\
\delta(a_n)
\end{pmatrix}_B
\]
and
\[
[\delta(M)]_B = \begin{pmatrix}
\delta(x_1(e_1)) & \delta(x_1(e_2)) & \cdots & \delta(x_1(e_n)) \\
\delta(x_2(e_1)) & \delta(x_2(e_2)) & \cdots & \delta(x_2(e_n)) \\
\vdots & \vdots & \ddots & \vdots \\
\delta(x_n(e_1)) & \delta(x_n(e_2)) & \cdots & \delta(x_n(e_n))
\end{pmatrix}_B.
\]
It follows we have described the endomorphism $\nabla(\delta)$ completely in terms of the fundamental matrix $M = M(p, s)$:
\[
\nabla(\delta) = [D_\delta]_B + [\delta(M)]_B
\]
and the projective basis $\{B, B^*\}$. We want to calculate the curvature $R_{\nabla}$ using $B$ and $B^*$. This is a long but straight forward calculation which we now present. We get
\[
\nabla(\eta)[\nabla(\delta)(e)] =
\]
\[
\nabla(\eta)(a_1 \sum_{i=1}^n \delta(x_i(e_1))e_i + a_2 \sum_{i=1}^n \delta(x_i(e_2))e_i + \cdots + a_n \sum_{i=1}^n \delta(x_i(e_n))e_i)
\]
\[
+ \delta(a_1)e_1 + \delta(a_2)e_2 + \cdots + \delta(a_n)e_n =
\]
\[
\sum_{i=1}^n a_1 \delta(x_i(e_1))\nabla(\eta)(e_i) + \eta(a_1\delta(x_i(e_1)))e_i
\]
\[
+ \sum_{i=1}^n a_2 \delta(x_i(e_1))\nabla(\eta)(e_i) + \eta(a_2\delta(x_i(e_1)))e_i + \cdots +
\]
\[
\sum_{i=1}^n a_n \delta(x_i(e_1))\nabla(\eta)(e_i) + \eta(a_n\delta(x_i(e_1)))e_i
\]
\[
+ \sum_{i=1}^n \delta(a_i)\nabla(\eta)(e_i) + \eta(\delta(a_i))e_i =
\]
\[ \nabla(\eta)(e_1)(\sum_{i=1}^{n} a_i \delta(x_1(e_i))) + \]
\[ \nabla(\eta)(e_2)(\sum_{i=1}^{n} a_i \delta(x_2(e_i))) + \cdots + \]
\[ \nabla(\eta)(e_n)(\sum_{i=1}^{n} a_i \delta(x_n(e_i))) + \]
\[ e_1(\sum_{i=1}^{n} \eta(a_i \delta(x_1(e_i)))) + e_2(\sum_{i=1}^{n} \eta(a_i \delta(x_2(e_i)))) + \cdots + \]
\[ e_n(\sum_{i=1}^{n} \eta(a_i \delta(x_n(e_i)))) + \]
\[ \sum_{i=1}^{n} \delta(a_i)\nabla(\eta)e_i + \eta(\delta(a_i))e_i. \]

When we express the above calculation in matrix-notation we get
(3.1.1)  \[ \nabla(\eta)\nabla(\delta)(e) = \]
\[ [\eta(M)]_B[\delta(M)]_B(e_B) + [\delta(M)]_B[D_\eta]_B(e_B) + \]
\[ [(\eta \circ \delta)(M)]_B(e_B) + [\eta(M)]_B[D_{\delta}]_B(e_B) + [D_{\eta \circ \delta}]_B(e_B). \]

**Theorem 3.2.** The following holds:
\[ R_{\nabla}(\delta, \eta) = [[\delta(M)]_B, [\eta(M)]_B]. \]

**Proof.** By definition
\[ R_{\nabla}(\delta, \eta) = [\nabla(\delta), \nabla(\eta)] - \nabla(\delta, \eta) = \]
\[ \nabla(\delta) \circ \nabla(\eta) - \nabla(\eta) \circ \nabla(\delta) - \nabla(\delta, \eta). \]
Since \( \nabla(\delta) = [D_\delta]_B + [\delta(M)]_B \) we get using Formula 3.1.1 the following calculation:
\[ R_{\nabla}(\delta, \eta)(e) = \]
\[ \nabla(\delta)\nabla(\eta)(e) - \nabla(\eta)\nabla(\delta)(e) - \nabla(\delta, \eta)(e) = \]
\[ [\delta(M)]_B[\eta(M)]_B(e_B) + [\eta(M)]_B[D_\delta]_B(e_B) + \]
\[ + [(\delta \circ \eta)(M)]_B(e_B) + [\delta(M)]_B[D_\eta]_B(e_B) + [D_{\delta \circ \eta}]_B(e_B) \]
\[ - [\eta(M)]_B[\delta(M)]_B(e_B) - [\delta(M)]_B[D_\eta]_B(e_B) - [(\eta \circ \delta)(M)]_B(e_B) \]
\[ - [\eta(M)]_B[D_\delta]_B(e_B) - [D_{\eta \circ \delta}]_B(e_B) \]
\[ - [D_{\delta \circ \eta}]_B(e_B) - [[\delta, \eta](M)]_B(e_B) = \]
\[ [[\delta(M)]_B, [\eta(M)]_B](e_B) \]
and the claim of the Theorem follows. \( \square \)
Since the curvature matrix \( R(\delta \wedge \eta) \) is the commutator of two matrices it follows

\[
\det(R(\delta \wedge \eta)) = 0
\]

when \( n \) is an odd number.

Note: The trace of the commutator \( R = [[\delta(M)]_B, [\eta(M)]_B] \) as an element of \( \text{End}_A(E) \) is non zero in general. If \( e_1, \ldots, e_n, x_1, \ldots, x_n \) is a projective basis the trace of \( R \) is given as follows:

\[
\text{tr}(R) = \sum_{i=1}^{n} x_i(R(e_i)).
\]

The commutator \( [[\delta(M)]_B, [\eta(M)]_B] \) has trace zero as an element of \( \text{End}_A(A^n) \). See [\ref{6}] for an explicit example of a connection \( \nabla \) where \( \text{tr}(R) \neq 0 \).

4. Examples: Algebraic connections on ellipsoids

In this section we use the notions introduced in the previous section to give an explicit construction of connections on modules of Kahler-differentials on ellipsoids.

Let in the following \( A \) be any \( \mathbb{Q} \)-algebra and let \( B = A[x_1, \ldots, x_k] \) be the polynomial ring in \( k \) variables over \( A \). Let \( H = p_1 x_1^{p_1} + \cdots + x_k^{p_k} - 1 \in B \) and consider the \( A \)-algebra \( C = B/H \). In this section we apply the construction in the previous section to give explicit formulas for connections on \( \Omega = \Omega_{C/A} \). Let \( d: C \rightarrow \Omega \) be the universal derivation. Let

\[
dH = p_1 x_1^{p_1-1} dx_1 + \cdots + p_k x_k^{p_k-1} dx_k.
\]

It follows there is an isomorphism

\[
\Omega = C\{dx_1, \ldots, dx_k\}/dH
\]

of left \( C \)-modules. Let \( F = C\{e_1, \ldots, e_n\} \) be the free \( C \)-module on the basis \( \{e_1, \ldots, e_n\} \). Let \( G = p_1 x_1^{p_1-1} e_1 + \cdots + p_k x_k^{p_k-1} e_k \in F \) and let \( Q \) be the left \( C \)-module spanned by \( G \). We get an exact sequence of left \( C \)-modules

\[
0 \rightarrow Q \rightarrow F \rightarrow^p \Omega \rightarrow 0
\]

where the map \( p \) is defined as follows:

\[
p(e_i) = dx_i.
\]

Since the module \( \Omega \) is projective there is a \( C \)-linear splitting \( s \) of \( p \). Define the following map \( s: \Omega \rightarrow F \):

\[
s(dx_i) = e_i - \frac{1}{p_i} x_i G.
\]

Lemma 4.1. The map \( s \) is a left \( C \)-linear section of \( p \).

Proof. We prove the map \( s \) is well defined:

\[
s(dH) = s(p_1 x_1^{p_1-1} dx_1 + \cdots + p_k x_k^{p_k-1} dx_k) =
\]

\[
p_1 x_1^{p_1-1}(e_1 - \frac{1}{p_1} x_1 G) + \cdots + p_k x_k^{p_k-1}(e_k - \frac{1}{p_k} x_k G) =
\]

\[
p_1 x_1^{p_1-1} e_1 + \cdots + p_k x_k^{p_k-1} e_k - x_1^{p_1} G - \cdots - x_k^{p_k} G =
\]
\[
G = (x_1^{p_1} + \cdots + x_k^{p_k})G = G - G = 0.
\]
It follows
\[
s(dH) = 0
\]
hence the map \(s\) is well defined. By definition
\[
s(\omega) = s(a_1dx_1 + \cdots + a_kdx_k) = a_1(e_1 - \frac{1}{p_1}x_1G) + \cdots + a_k(e_k - \frac{1}{p_k}x_kG) = a_1e_1 + \cdots + a_kek + yG.
\]
it follows
\[
p(s(\omega)) = p(a_1e_1 + \cdots + a_kek + yG) = p(a_1e_1 + \cdots + a_kek) + p(yG) = \omega.
\]
The Lemma is proved. \(\square\)

Let \(e_i^* : F \to A\) be coordinate functions on \(F\) and put \(x_i = e_i^* \circ s\). Let \(w_i = \frac{dx_i}{s}\) for \(i = 1, \ldots, k\). Let \(B = \{w_1, \ldots, w_k\}\) and \(B^* = \{x_1, \ldots, x_k\}\). It follows \(\{B, B^*\}\) is a projective basis for the projective module \(\Omega\). Define the following map:
\[
\nabla : \Omega \to \Omega \otimes_A \Omega
\]
by
\[
\nabla(w) = \sum_i d(x_i(w)) \otimes w_i.
\]

**Theorem 4.2.** The map \(\nabla\) is a connection on \(\Omega\).

**Proof.** By Lemma 4.1 it follows the sets \(\{B, B^*\}\) form a projective basis for \(\Omega\). It follows from Theorem 2.8 the map \(\nabla\) is a connection on \(\Omega\). \(\square\)

In the following we calculate explicitly connections and their curvature on a class of ellipsoids.

**Example 4.3. Connections on the two-sphere.**

Let in this example \(K\) be a field of characteristic different from two. Let \(f = x_1^2 + x_2^2 + x_3^2 - 1 \in K[x_1, x_2, x_3]\). Let \(A = K[x_1, x_2, x_3]/f\). It follows \(S = \text{Spec}(A)\) is the two-sphere over \(K\). Let \(\Omega = \Omega_{A/K}\) be the module of Kahler differentials of \(A\) relative to \(K\). It follows \(\Omega = A(dx_1, dx_2, dx_3)/H\) where \(H = x_1dx_1 + x_2dx_1 + x_3dx_3\). Let \(G = x_1u_1 + x_2u_2 + x_3u_3\) where \(A\{u_1, u_2, u_3\}\) is the free \(A\)-module of rank 3. Since \(\text{Der}_K(A)\) is a non-trivial rank two locally free \(A\)-module it follows \(\Omega\) is a non-trivial rank two locally free \(A\)-module.

We get an exact sequence
\[
0 \to (G) \to A\{u_1, u_2, u_3\} \to^p \Omega \to 0
\]
where \(p(u_i) = \frac{dx_i}{s}\). The section \(s\) defined in Lemma 4.1 defines a projective basis \(B, B^*\) and a fundamental matrix \(M = M(p, s)\) for \(\Omega\) with respect to the split surjection \(p\).

Let \(T = \text{Der}_K(A)\). It follows \(T\) is generated by the following derivations:
\[
\partial_1 = x_2\partial_{x_1} - x_1\partial_{x_2}
\]
\[
\partial_2 = x_3\partial_{x_1} - x_1\partial_{x_3}
\]
and
\[
\partial_3 = x_3\partial_{x_2} - x_2\partial_{x_3}.
\]
let $\omega = a_1 dx_1 + a_2 dx_2 + a_3 dx_3 \in \Omega$ and let $\partial \in \text{End}_K(A)$. We define

$$D\partial(\omega) = \partial(a_1) dx_1 + \partial(a_2) dx_2 + \partial(a_3) dx_3 \in \Omega.$$ 

Define the following elements:

$$\nabla(\partial_1) = [D\partial_1]_B + \begin{pmatrix} -2x_1 x_2 & x_1^2 - x_2^2 & -x_2 x_3 \\ x_1^2 - x_2^2 & 2x_1 x_2 & x_1 x_3 \\ -x_2 x_3 & x_1 x_3 & 0 \end{pmatrix} = [D\partial_1]_B + [\partial_1(M)]_B$$

$$\nabla(\partial_2) = [D\partial_2]_B + \begin{pmatrix} -2x_1 x_3 & -x_2 x_3 & x_1^2 - x_3^2 \\ -x_2 x_3 & 0 & x_1 x_2 \\ x_1^2 - x_3^2 & x_1 x_2 & 2x_1 x_3 \end{pmatrix} = [D\partial_2]_B + [\partial_2(M)]_B$$

and

$$\nabla(\partial_3) = [D\partial_3]_B + \begin{pmatrix} 0 & -x_1 x_3 & x_1 x_2 \\ -x_1 x_3 & -2x_2 x_3 & x_1 x_2 \\ x_1 x_2 & x_2^2 - x_3^2 & 2x_2 x_3 \end{pmatrix} = [D\partial_3]_B + [\partial_3(M)]_B$$

**Corollary 4.4.** The maps $\nabla(\partial_1), \nabla(\partial_2)$ and $\nabla(\partial_3)$ define a connection $\nabla : \text{Der}_K(A) \to \text{End}_K(\Omega)$.

**Proof.** One checks the given formulas are the formulas one gets when one makes the connection in Theorem 4.2 explicit. □

The curvature of the connection $\nabla$ is the following map:

$$R_{\nabla} : \text{Der}_K(A) \wedge \text{Der}_K(A) \to \text{End}_K(\Omega)$$

$$R_{\nabla}(\delta \wedge \eta) = [\nabla(\delta), \nabla(\eta)] - \nabla([\delta, \eta]).$$

We calculate $R_{\nabla}(\partial_1 \wedge \partial_2)$ using Theorem 5. The fundamental matrix $M = M(p, s)$ is the following matrix:

$$M = \begin{pmatrix} 1 - x_1^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & 1 - x_2^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & 1 - x_3^2 \end{pmatrix}.$$ 

It follows

$$[\partial_1(M)]_B = \begin{pmatrix} -2x_1 x_2 & x_1^2 - x_2^2 & -x_1 x_3 \\ x_1^2 - x_2^2 & 2x_1 x_2 & x_1 x_3 \\ -x_2 x_3 & x_1 x_3 & 0 \end{pmatrix}$$

and

$$[\partial_2(M)]_B = \begin{pmatrix} -2x_1 x_3 & -x_2 x_3 & x_1^2 - x_3^2 \\ -x_2 x_3 & 0 & x_1 x_2 \\ x_1^2 - x_3^2 & x_1 x_2 & 2x_1 x_3 \end{pmatrix}.$$ 

One calculates using Theorem 5

$$R_{\nabla}(\partial_1 \wedge \partial_2) = [[\partial_1(M)]_B, [\partial_2(M)]_B] = \begin{pmatrix} 0 & x_1 x_3 & -x_1 x_2 \\ -x_1 x_3 & 0 & x_1^2 \\ x_1 x_2 & -x_1^2 & 0 \end{pmatrix}.$$ 

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
be any matrix with $a_{ij}$ independent variables. Consider the characteristic polynomial $P_A(\lambda) = \det(\lambda I - A)$ where $I$ is the $3 \times 3$ identity matrix. It follows

$$P(\lambda) = \lambda^3 - tr(A)\lambda^2 + p_A\lambda - det(A)$$

where

$$p_A = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} + a_{21}a_{12} + a_{31}a_{13} + a_{32}a_{23}.$$ 

Let $A = R_\omega(\partial_1 \wedge \partial_2)$. It follows $p_A = -x_1^2 \neq 0$.

By [7] the connection $\nabla$ define a Chern-class

$$c_1(\Omega) \in H^2(\text{Der}_K(A), A)$$

where $H^2(\text{Der}_K(A), A)$ is the second Lie-Rinehart cohomology of $\text{Der}_K(A)$.

**Corollary 4.5.** The following holds:

1. $\nabla$ is non-flat.  \hspace{2cm} (4.5.1)
2. $tr(R_\omega) = 0$.  \hspace{2cm} (4.5.2)
3. $c_1(\Omega) = 0$ in $H^2(\text{Der}_K(A), A)$.  \hspace{2cm} (4.5.3)

**Proof.** Claim (4.5.1) follows since the map $R_\omega(\partial_1 \wedge \partial_2)$ is a non-zero element of $\text{End}_A(\Omega)$ as one easily checks. Claim (4.5.2) and Claim (4.5.3) follows from an explicit calculation. The Corollary is proved. $\square$

Note: We get examples of non-flat algebraic connections $\nabla : \text{Der}_K(A) \rightarrow \text{End}_K(\Omega)$ defined over any field $K$ of characteristic different from two.

**Example 4.6.** Complex projective manifolds and holomorphic connections.

In [1] the following is conjectured: Let $X$ be a complex projective manifold and let $\mathcal{E}$ be a holomorphic bundle with a holomorphic connection $\nabla$. Then there exist a holomorphic flat connection $\nabla'$. In the affine case as indicated above many “naturally occuring” algebraic connections are non-flat.

5. **Higher order differential operators**

In the litterature one finds many papers devoted to the construction of explicit formulas for connections on maximal Cohen-Macaulay modules on isolated hypersurface singularities (See [7] for one approach to this problem using the Kodaira-Spencer map and the Atiyah class). In this section we use the notion of a projective basis on a finitely generated projective $A$-module to give explicit formulas for $l$-connections

$$\nabla^l : E \rightarrow P^l \otimes_A E$$

where $P^l$ is the $l$'th module of principal parts of the ring $A$. The connections $\nabla^l$ are non-flat in general. We also consider the notion of a stratification and give explicit examples of projective finitely generated module where the canonical connection induced by a projective basis does not give rise to a stratification. The obstruction to this is given by the $(l, k)$-curvature $K^{(l,k)}$ of the $l$-connection $\nabla^l$.

Let in the following $\mathbb{Z}$ be a fixed commutative unital base ring and let $A$ be a commutative $\mathbb{Z}$-algebra. Let $E$ be a left $A$-module.

The existence of a flat connection

$$\nabla : \text{Der}_\mathbb{Z}(A) \rightarrow \text{End}_\mathbb{Z}(E)$$
on E induce a ring homomorphism
\[ \rho : \text{diff}_Z(A) \to \text{End}_Z(E) \]
where \( \text{diff}_Z(A) \) is the \textit{small ring of differential operators} of A. The module E is by definition a left A-module and the induced structure \( \rho \) is a lifting of the left A-module structure to a left \( \text{diff}_Z(A) \)-module structure. The obstruction to this lifting is the curvature of the connection \( \nabla \). The ring \( \text{diff}_Z(A) \) is the associative subring of \( \text{End}_Z(A) \) generated by \( \text{Der}_Z(A) \). In general the ring \( \text{diff}_Z(A) \) is a strict subring of the ring \( \text{Diff}_Z(A) \) - the \textit{ring of differential operators} of A. In the case where A is a regular \( k \)-algebra of finite type over \( k \) where \( k \) is a field of characteristic zero one has an equality
\[ \text{diff}_Z(A) = \text{Diff}_Z(A). \]
In the case when A is non-regular there is a strict inclusion of rings \( \text{diff}_Z(A) \subseteq \text{Diff}_Z(A) \). The higher order connections \( \nabla^l \) we construct are related to the following problem: One wants to lift the left A-module structure on E to a left \( \text{Diff}_Z(A) \)-module structure
\[ \rho : \text{Diff}_Z(A) \to \text{End}_Z(E) \]
on E. The obstruction to such a lifting is given by the notion of \textit{generalized curvature} of the connection \( \nabla^l \).

The ring \( \text{Diff}_Z(A) \) has a filtration \( \text{Diff}^l_Z(A) \) by degree of differential operators and there is an isomorphism
\[ \text{Diff}^l_Z(A) \cong \text{Hom}_A(P^l_{A/Z}, A) \]
where \( P^l = P^l_{A/Z} = A \otimes_Z A/I^{l+1} \) is the \( l \)th module of principal parts of A. Here \( I \subseteq A \otimes_Z A \) is the kernel of the canonical multiplication map. Define the following map
\[ \partial^l : A \to P^l \]
by
\[ \partial^l(a) = \underbrace{1 \otimes a}. \]
There is a canonical projection map
\[ p_l : P^l \to P^{l-1} \]
and an equality \( p_l \circ \partial^l = \partial^{l-1} \) of maps. Let \( E, F \) be left A-modules and define the following:
\[ \text{Diff}^{-1}_Z(E, F) = 0 \]
and
\[ \text{Diff}^l_Z(E, F) = \{ \partial \in \text{Hom}_Z(E, F) : [\partial, a] \in \text{Diff}^{l-1}_Z(E, F) \text{ for all } a \in A \}. \]
By definition \( \text{Diff}^l_Z(A) = \text{Diff}^l_Z(A, A) \) and
\[ \text{Diff}_Z(A) = \cup_{l \geq -1} \text{Diff}^l_Z(A). \]
We get a filtration of left and right A-modules
\[ 0 = \text{Diff}^{-1}_Z(E, F) \subseteq \text{Diff}^0_Z(E, F) \subseteq \cdots \subseteq \text{Diff}^l_Z(E, F) \subseteq \cdots \subseteq \text{Diff}_Z(E, F) \]
where
\[ \text{Diff}_Z(E, F) = \cup_{l \geq -1} \text{Diff}^l_Z(E, F). \]
Consider the following map
\[ d : A \to A \otimes_Z A \]
defined by
\[ d(a) = 1 \otimes a - a \otimes 1. \]

Let \( I_l = \{1, 2, \cdots, l\} \). Given \( a \in A \), let \( \phi_a \in \text{End}_Z(A) \) be the endomorphism which is multiplication with \( a \). Let \( \partial \in \text{End}_Z(A) \).

**Lemma 5.1.** The following holds:

(5.1.1) \[ d(a_1) \cdots d(a_l) = \sum_{H \subseteq I_l} (-1)^{\text{card}(H)} (\prod_{i \in H} a_i) \otimes (\prod_{i \notin H} a_i) \in A \otimes Z A. \]

(5.1.2) \[ [\cdots [\partial, \phi_{a_1}] \cdots] \phi_{a_l}](a) = \sum_{H \subseteq I_l} (-1)^{\text{card}(H)} (\prod_{i \in H} a_i) \partial (\prod_{i \notin H} a_i)a ) \]

**Proof.** The proof of the Lemma follows from [4], Proposition 16.8.8 using induction. \( \square \)

Let \( \partial \in \text{Hom}_Z(E, E) \) be a \( Z \)-linear endomorphism and define the following map:

\[ \phi_{\partial} : A \otimes Z A \otimes_A E \to E \]

by

\[ \phi_{\partial}(a \otimes b \otimes e) = a\partial(be). \]

Let \( I \subseteq A \otimes Z A \) be the kernel of the multiplication map.

**Proposition 5.2.** The following holds:

\( \partial \in \text{Diff}_{Z}^l(E) \) if and only if \( \phi_{\partial}(I^{l+1} (A \otimes Z A) \otimes_A E) = 0 \).

**Proof.** Assume \( \partial \in \text{Diff}_{Z}^l(A) \). It follows from Lemma 5.1

\[ \phi_{\partial}(\prod_{i=1}^{l+1} (1 \otimes a_i - a_i \otimes 1) \otimes e) = \]

\[ \phi_{\partial}(\sum_{H \subseteq I^{l+1}} (-1)^{\text{card}(H)} (\prod_{i \in H} a_i) \otimes (\prod_{i \notin H} a_i) \otimes e) = \]

\[ = \sum_{H \subseteq I^{l+1}} (-1)^{\text{card}(H)} (\prod_{i \in H} a_i) \partial (\prod_{i \notin H} a_i)e ) = \]

\[ [\cdots [\partial, \phi_{a_1}] \cdots] \phi_{a_{l+1}}](e) = 0 \]

since \( \text{Diff}_{Z}^{-1}(A) = 0 \). It follows \( \phi_{\partial}(I^{l+1} (A \otimes Z A \otimes A E)) = 0 \). The converse statement is proved in a similar way and the Proposition is proved. \( \square \)

**Lemma 5.3.** The following holds:

\( \partial^l \in \text{Diff}_{Z}^l(A, P^l) \).

**Proof.** Define the following map

\[ \partial : A \to A \otimes Z A \]

by

\[ \partial(a) = 1 \otimes a. \]

One gets the formula

\[ [\cdots [\partial, \phi_{a_1}] \cdots] \phi_{a_{l+1}}](x) = d(a_1) \cdots d(a_{l+1})(1 \otimes x) \]

where \( d(a) = 1 \otimes a - a \otimes 1 \) and where the product is in \( A \otimes Z A \). There is a natural projection map

\[ p_l : A \otimes Z A \to P^l \]
defined by
\[ p_l(a \otimes b) = a \otimes b. \]
We get \( \partial^l = p_l \circ \partial. \) It follows
\[
[\cdots [\partial^l, \phi_{a_1}] \cdots] \phi_{a_{l+1}}(x) =
\]
\[
[\cdots [\partial, \phi_{a_1}] \cdots] \phi_{a_{l+1}}(x) =
\]
\[
d(a_1)d(a_2) \cdots d(a_{l+1})(1 \otimes x) = 0
\]
hence
\[
[\cdots [\partial^l, \phi_{a_1}] \cdots] \phi_{a_{l+1}} \in \text{Diff}^{-1}_{Z}(A, P^l) = 0.
\]
It follows
\[
\partial^l \in \text{Diff}^l_{Z}(A, P^l)
\]
and the Lemma is proved. \( \square \)

Let \( E \) be a left projective \( A \)-module with projective basis \( B = \{e_1, \ldots, e_n\} \) and \( B^* = \{x_1, \ldots, x_n\} \) and consider the connection
\[
\nabla : E \to I \otimes_A E
\]
defined by
\[
\nabla(e) = \sum_{i=1}^{n} d(x_i(e)) \otimes e_i
\]
where
\[
d : A \to I
\]
is a derivation: \( d \in \text{Der}_{Z}(A, I). \) By Theorem 2.8 it follows \( \nabla \) is a connection on \( E. \)

**Lemma 5.4.** The following holds:
\[
\nabla \in \text{Diff}^1_{Z}(E, I \otimes_A E).
\]
**Proof.** Let \( a \in A \) and consider the following map:
\[
[\nabla, a] : E \to \Omega \otimes_A E.
\]
It follows
\[
[\nabla, a](e) = \nabla(ae) - a\nabla(e) = a\nabla(e) + d(a) \otimes e - a\nabla(e) =
\]
\[
d(a) \otimes e.
\]
We get
\[
[\nabla, a](be) = d(a) \otimes be = b(d(a) \otimes e) = b[\nabla, a](e).
\]
It follows
\[
[\nabla, a] \in \text{Hom}_{A}(E, I \otimes E) = \text{Diff}^0_{Z}(E, I \otimes_A E)
\]
and hence
\[
\nabla \in \text{Diff}^1_{Z}(E, I \otimes_A E).
\]
The Lemma is proved. \( \square \)
Define the following map
\[ \nabla^l : E \to P^l \otimes_A E \]
by
\[ \nabla^l(e) = \sum_{i=1}^{n} \partial^l(x_i(e)) \otimes e_i. \]
Since \( \partial^l \in \text{Diff}_Z^l(A, P^l) \) it follows
\[ \nabla^l \in \text{Hom}_Z(E, P^l \otimes_A E). \]

**Theorem 5.5.** The following holds:
\[ (5.5.1) \quad \nabla^l \in \text{Diff}_Z^l(E, P^l \otimes_A E). \]
\[ (5.5.2) \quad (p_l \otimes 1) \circ \nabla^l = \nabla^l - 1. \]

**Proof.** By Lemma 5.3 the following formula holds:
\[ [\cdots [\nabla^l, \phi_{a_1}] \cdots [\phi_{a_{l+1}}]](e) = \sum_{i=1}^{n} [\cdots [\partial^l, \phi_{a_1}] \cdots [\phi_{a_{l+1}}]](x_i(e)) \otimes e_i = \]
\[ \sum_{i=1}^{n} d(a_1) \cdots d(a_{l+1})(x_i(e)) \otimes e_i = 0 \]
It follows
\[ [\cdots [\nabla^l, \phi_{a_1}] \cdots [\phi_{a_{l+1}}]] \in \text{Diff}_Z^1(E, P^l \otimes_A E) = 0 \]
hence
\[ \nabla^l \in \text{Diff}_Z^l(E, P^l \otimes_A E) \]
and the Proposition is proved. \( \square \)

We get for any left projective \( A \)-module \( E \) and all \( l \geq 1 \) commutative diagrams
\[
\begin{array}{ccc}
 E & \xrightarrow{\nabla^l} & P^l \otimes_A E \\
 \downarrow{\nabla^{l-1}} & \quad & \downarrow{(p_l \otimes 1)} \\
 P^{l-1} \otimes_A E & \quad & \\
\end{array}
\]
of differential operators.

**Definition 5.6.** The map
\[ \nabla^l : E \to P^l \otimes_A E \]
is the \( l \)-connection associated to the projective basis \( B, B^* \).

Note: There \( l \)th module of principal parts \( P^l = A \otimes_Z A/I^{l+1} \) is a commutative unital ring in an obvious way and there is a multiplicative unit \( 1 \in P^l \). One may define the following map
\[ \rho^l : E \to P^l \otimes_A E \]
by
\[ \rho^l(e) = 1 \otimes e. \]
The map \( \rho^l \) is called the universal differential operator for \( E \) of order \( l \). One checks
\[ \rho^l \in \text{Diff}_Z^l(E, P^l \otimes_A E). \]
The map \( \rho^l \) induce an isomorphism
\[ (5.6.1) \quad \text{Diff}_Z^l(E, F) \cong \text{Hom}_A(P^l \otimes_A E, F) \]
of left and right $A$-modules.

Given an $l$'th order differential operator $\partial \in \text{Diff}^l_2(A)$ one gets by Formula \[5.6.1\] an $A$-linear map $\phi_\partial : P^l \to A$ We get using $\nabla^l$ a map

$$\rho(\partial) : E \to E$$

defined by

$$\rho(\partial)(e) = \phi_\partial \otimes 1(\nabla^l(e)) = \sum_{i=1}^n \phi_\partial(\partial^l(x_i(e)))e_i = \sum_{i=1}^n \partial(x_i(e))e_i.$$ 

This defines a map of left $A$-modules

$$\rho^l : \text{Diff}^l_2(A) \to \text{Diff}^l_2(E)$$ 

for all $l \geq 1$. By Theorem \[5.5\] we get commutative diagrams of maps

$$\text{Diff}^l_2(A) \xrightarrow{\rho^l} \text{Diff}^l_2(E)$$

$$\downarrow j \quad \downarrow i$$

$$\text{Diff}^{l+1}_2(A) \xrightarrow{\rho^{l+1}} \text{Diff}^{l+1}_2(E)$$

for all $l \geq 1$ with $i, j$ the canonical inclusion maps. This induce a canonical map

(5.6.2) $$\rho : \text{Diff}_2(A) \to \text{Diff}_2(E)$$

**Example 5.7. Connections on the two-sphere.**

Let in this example $K$ be a field of characteristic zero. Let $f = x_1^2 + x_2^2 + x_3^2 - 1$ be in $K[x_1, x_2, x_3]$. Let $A = K[x_1, x_2, x_3]/f$. It follows $S = \text{Spec}(A)$ is the two-sphere over $K$. Let $\Omega = \Omega_{A/K}$ be the module of Kahler differentials of $A$ relative to $K$. It follows $\Omega = A\{dx_1, dx_2, dx_3\}/H$ where $H = x_1dx_1 + x_2dx_1 + x_3dx_3$. Let $G = x_1u_1 + x_2u_2 + x_3u_3$ where $A\{u_1, u_2, u_3\}$ is the free $A$-module of rank 3. We get an exact sequence

$$0 \to (G) \to A\{u_1, u_2, u_3\} \to^p \Omega \to 0$$

where $p(u_1) = \overline{dx_1}$. Let $s(dx_i) = u_i - x_iG$. It follows $s$ is a left $A$-linear section of $p$. It is well known $\Omega$ is a non-free locally free rank two $A$-module. Let $\text{Der}_K(A)$ be the module of $K$-linear derivations of $A$. It follows $\text{Der}_K(A)$ is generated by the following derivations:

$$\partial_1 = x_2\partial_{x_1} - x_1\partial_{x_2}$$
$$\partial_2 = x_3\partial_{x_1} - x_1\partial_{x_3}$$

and

$$\partial_3 = x_3\partial_{x_2} - x_2\partial_{x_3}.$$ 

Since $S$ is smooth over $K$ it follows $\text{Diff}_K(A)$ is generated as an associative ring by $\partial_1, \partial_2$ and $\partial_3$. There is a map $\rho$ induced by the projective basis $B, B^*$ for $\Omega$ defined as follows:

$$\rho : \text{Diff}_K(A) \to \text{Diff}_K(\Omega)$$

defined by

$$\rho(\partial)(e) = \partial(x_1(e))dx_1 + \partial(x_2(e))dx_3 + \partial(x_3(e))dx_3.$$ 

The map $\rho$ is not a morphism of associative rings for the following reason: One calculates the following:

$$\partial_1 \circ \partial_2 = x_2x_3\partial_{x_1}^2 + x_2\partial_{x_3} + x_1x_2\partial_{x_1} \circ \partial_{x_3} - x_1x_3\partial_{x_1} \circ \partial_{x_2} + x_1^2\partial_{x_2} \circ \partial_{x_3}.$$
One calculates the following:

$$
\rho(\partial_1 \circ \partial_2)(dx_1) =
-2x_1x_3dx_1 + x_1x_3dx_2 - 2x_1x_2dx_3.
$$

One calculates the following:

$$
\rho(\partial_1) \circ \rho(\partial_2)(dx_1) = \nabla(\partial_1)(\nabla(\partial_2)(dx_1)) =
(x_1^2x_2x_3 + 3x_2x_3)dx_1 + (3x_1x_2^2x_3 - x_1x_3)dx_2 + (x_1x_2x_3^2 + 2x_1x_2)dx_3.
$$

Hence

$$\rho(\partial_1 \circ \partial_2) \neq \rho(\partial_1) \circ \rho(\partial_2).$$

It follows \(\rho\) is not a morphism of associative rings.

From Corollary 4.5 it also follows the map \(\rho\) is not a ring homomorphism: Since the connection

$$\nabla : \text{Der}_K(A) \to \text{End}_K(\Omega)$$

is non-flat the \(A\)-module structure on \(\Omega\) does not lift to a \(\text{Diff}_B(A)\)-module structure on \(E\) hence it does not lift to a \(\text{Diff}_B(A)\)-module structure on \(E\).

**Definition 5.8.** We say the map \(\rho\) is the canonical connection induced by the projective basis \(B, B^*\). We say \(E\) is a \(\text{Diff}_Z(A)\)-module if the canonical map \(\rho\) from Equation 5.6.2 is a ring homomorphism.

**Example 5.9.** The free \(A\)-module of rank \(n\).

Assume \(E = A\{e_1, \ldots, e_n\}\) is a free \(A\)-module of rank \(n\) on the basis \(B = \{e_1, \ldots, e_n\}\) and let \(B^*\{x_1, \ldots, x_n\}\) with \(x_i = e_i^* \in E^*\). It follows \(B, B^*\) is a projective basis for the finitely generated projective \(A\)-module \(E\). The map \(\rho\) in this case is the canonical map

$$\rho : \text{Diff}_Z(A) \to \text{Diff}_Z(E)$$

defined by

$$\rho(\partial)(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n \partial(a_i) e_i.$$ 

Hence a differential operator \(\partial \in \text{Diff}_Z(A)\) acts in each coordinate for \(E\). It follows the free \(A\)-module \(E\) of rank \(n\) is a \(\text{Diff}_Z(A)\)-module in a canonical way.

The curvature of the \(l\)-connection \(\nabla^l\) is related to the way the map \(\rho\) deviates from being a map of associative rings.

Recall the following definition from [2] Proposition 2.10:

**Definition 5.10.** A stratification on \(E\) is a collection of isomorphisms

$$\eta_l : P^l \otimes_A E \rightarrow E \otimes_A P^l$$

such that

(5.10.1) \(\eta_l\) is \(P^l\)-linear.

(5.10.2) \(\eta_l\) and \(\eta_k\) are compatible via restriction maps.

(5.10.3) \(\eta_0\) is the identity map.

(5.10.4) The cocycle condition holds for \(\eta_l\).
See [2] for a precise description of the cocycle condition. We say the system \( \{ \theta_l \} \) of right \( A \)-linear maps
\[
\theta_l : E \to E \otimes_A P^l
\]
for all \( l \geq 1 \) with \( \theta_0 = id_E \) is an \( \infty \)-connection on \( E \).

Consider the following diagram:
\[
\begin{array}{ccc}
E \otimes_A P^{l+k} & \xrightarrow{id \otimes \delta^{l,k}} & E \otimes_A P^l \otimes_A P^k \\
\theta_{l+k} \downarrow & & \theta_l \otimes id \\
E & \xrightarrow{\phi_{l,k}} & E \otimes_A P^k
\end{array}
\]

Let \( \phi_{0}^{l,k} = id \otimes \delta^{l,k} \circ \theta_{l+k} \) and \( \phi_{1}^{l,k} = \theta_l \otimes id \circ \theta_k \).

**Definition 5.11.** Let \( K^{l,k} = \phi_{1}^{l,k} - \phi_{0}^{l,k} \) be the \((l,k)\)-curvature of the \( \infty \)-connection \( \{ \theta_l \} \). We say the \( \infty \)-connection \( \{ \theta_l \} \) is flat if \( K^{l,k} = 0 \) for all \( l, k \geq 1 \).

**Proposition 5.12.** The following data are equivalent:

1. A stratification on \( E \).
2. A flat \( \infty \)-connection on \( E \).
3. A left \( \text{Diff}_Z(A) \)-module structure on \( E \).

**Proof.** For a proof see [2], Proposition 2.11. \( \square \)

**Example 5.13.** Stratifications on finitely generated projective modules.

In [2], 2.17 is proved that if \( A \) is a finitely generated algebra over a field \( K \) and \( E \) is a finitely generated left \( A \)-module with a stratification then \( E \) is locally free, hence projective. As indicated in Example 5.7, the canonical connection
\[
\rho : \text{Diff}_Z(A) \to \text{Diff}_Z(E)
\]
on a finitely generated projective module \( E \) constructed using a projective basis \( B, B^* \) is seldom a morphism of rings. Hence the \( A \)-module \( E \) seldom has a stratification in the sense of [2] induced by a projective basis. It is not clear if every connection
\[
\nabla : E \to \Omega \otimes_A E
\]
is induced by a projective basis \( B, B^* \) for \( E \). Given two connections \( \nabla, \nabla' \) their difference \( \phi = \nabla - \nabla' \) is an \( A \)-linear map
\[
\phi : E \to \Omega \otimes_A E
\]
and such a map may be constructed using a projective basis. It is thus unlikely there is an action
\[
\rho : \text{Diff}_K(A) \to \text{Diff}_K(E)
\]
where \( \rho \) is a morphism of associative rings. Hence it is unlikely a finitely generated projective \( A \)-module \( E \) has a stratification. The obstruction to the existence of a stratification is given by the \((l,k)\)-curvature \( K^{l,k} \) from Definition 5.11.

**Example 5.14.** Associative subrings of \( \text{End}_Z(A) \).
In general one may do the following: For any associative subring $R$ of $\text{End}_\mathbb{Z}(A)$ containing $A$ we get a left action

$$\eta : R \rightarrow \text{End}_\mathbb{Z}(E)$$

defined by

$$\eta(\phi)(e) = \sum_{i=1}^{n} \phi(x_i(e))e_i.$$ 

If $a \in A \subseteq R$ acts via $\phi_a$ we get

$$\eta(a)(e) = \sum_{i=1}^{n} \phi_a(x_i(e))e_i = \sum_{i=1}^{n} ax_i(e)e_i = \sum_{i=1}^{n} x_i(ae)e_i = ae$$

hence the $R$-structure on $E$ induced by $\eta$ extends the left $A$-module structure on $E$.

The map $\eta$ is not a map of associative rings in general. If the map $\eta$ is a map of associative rings we get a structure of left $R$-module on $E$ extending the left $A$-module structure. We say the $A$-module structure lifts to $R$. This is a special case of a general problem in deformation theory: Given a map of rings $\phi : A \rightarrow B$ and a left $A$-module $E$. One wants to extend this structure and define a left $B$-module structure on $E$ restricting to the $A$-module structure via the map $\phi$. As indicated in Example 6.1 for a finitely generated projective $A$-module $E$ and a subring $R$ of $\text{End}_\mathbb{Z}(A)$ the left $A$-module structure on $E$ seldom lift to a left $R$-module structure.

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