Modified scattering for the one-dimensional Schrödinger equation with a subcritical dissipative nonlinearity

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Abstract
We study the asymptotic behavior in time of solutions to the one dimensional nonlinear Schrödinger equation with a subcritical dissipative nonlinearity \( \lambda |u|^\alpha u \), where \( 0 < \alpha < 2 \), and \( \lambda \) is a complex constant satisfying \( \text{Im} \lambda > \frac{\alpha}{2 \sqrt{\alpha + 1}} \). For arbitrary large initial data, we present the uniform time decay estimates when \( \frac{4}{3} < \alpha < 2 \), and the large time asymptotics of the solution when \( \frac{2 + \sqrt{145}}{12} < \alpha < 2 \). The proof is based on the vector fields method and a semiclassical analysis method.

Keywords: Schrödinger equation, Decay estimates, Modified scattering, Semiclassical Analysis.

1 Introduction

Background and historical notes. We consider the large time behavior of solutions to the one-dimensional Schrödinger equation

\[
\begin{cases}
  i\partial_t u + \frac{1}{2} \partial_x^2 u + \lambda |u|^\alpha u = 0, & t > 0, \ x \in \mathbb{R}, \\
  u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\tag{1.1}
\]

where \( \alpha > 0 \), \( \lambda \in \mathbb{C} \), and \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) is a complex-valued function. From the physical point of view, (1.1) is said to be a governing equation of the light traveling through optical fibers, in which \( |u(t, x)| \) describes the amplitude of the electric field, \( t \) denotes the position along the fiber and \( x \) stands for the temporal parameter expressing a form of pulse. As for the nonlinear coefficient, \( \text{Re} \lambda \) denotes the magnitude of the nonlinear Kerr effect and \( \text{Im} \lambda \) implies the magnitude of dissipation due to nonlinear Ohm’s law (see e.g. [1]). The equation (1.1) is also a particular case of the more general complex Ginzburg-Landau equation on \( \mathbb{R}^N \)

\[
\partial_t u = e^{i\theta} \Delta u + \zeta |u|^\alpha u
\]

where \( |\theta| \leq \frac{\pi}{4} \) and \( \zeta \in \mathbb{C} \), which is a generic modulation equation that describes the nonlinear evolution of patterns at near-critical conditions. See for instance [6, 15, 18].

There are many papers that studied the global well-posedness problem, decay and the asymptotic behavior of the solution (see [3, 5, 7, 11, 13, 14, 16, 23] and references therein). We use the following classification with respect to the value \( \alpha \): the values \( \alpha > 2 \) we call the super-critical in the scattering problem, the value \( \alpha = 2 \) is the critical one and \( 0 < \alpha < 2 \) we refer to the sub-critical.

Let us recall some known large time asymptotics of (1.1) with \( \lambda \in \mathbb{R} \). Concerning the super-critical case \( \alpha > 2 \), it is well-known that the solution \( u(t) \) behaves like a free solution \( \exp \left( \frac{it}{2} \partial_x^2 \right) \phi \) for \( t \) sufficiently large (10, 21, 22). The strategy for this free asymptotic profile largely relies on the rapid decay of the nonlinearity. More precisely, since \( \int_0^\infty |u(t)|^\alpha dt \approx \int_0^\infty t^{-\alpha/2} dt < \infty \) by expecting that \( u(t) \) decays like a free solution, the nonlinearity can be regarded as negligible in the long time dynamics. As for the sub-critical and critical case \( 0 < \alpha < 2 \), the situation changes. The nonexistence of usual scattering states was obtained [2, 20] by making use of the time decay estimate of solutions obtained from pseudo-conformal conservation law. In the case \( \alpha = 2 \), Ozawa [16] constructed modified wave operators to

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the equation (1.1) for small scattering states, and Hayashi-Naumkin [11] proved the time decay and the large time asymptotics of $u(t)$ for small initial data. According to their results, the small solution $u(t)$ asymptotically tends to a modified free solution. More precisely, there are $\mathbb{C}$-valued $W(x) \in L^\infty \cap L^2$ and $\mathbb{R}$-valued $\Phi(x) \in L^\infty$ such that as $t \to \infty$

$$u(t, x) = \frac{1}{\sqrt{it}} W\left(\frac{x}{t}\right) \exp \left( i \frac{x^2}{2t} + i \lambda |W\left(\frac{x}{t}\right)|^2 \log t + i \Phi\left(\frac{x}{t}\right) \right) + O_{L^\infty}(t^{-1/2})$$

While the subcritical case $0 < \alpha < 2$ seems to be completely open. To our knowledge, there are no results about the precise behavior or any kind of modified scattering of $u$ for large time.

Many works have also dealt with the complex coefficient case. We can only expect the large time asymptotics of (1.1) for the case $\text{Im}\lambda > 0$, since it is proved in [11] that, under the assumptions $\text{Im}\lambda < 0$, $0 < \alpha < \infty$, there exists a class of blowup solutions to (1.1). On the other hand, it is easy to see that

$$\|u(t)\|_{L^2}^2 + 2 \text{Im}\lambda \int_0^t \|u(\tau)\|_{L^{\alpha+2}}^{\alpha+2} d\tau = \|u_0\|_{L^2}^2, \quad \text{(1.2)}$$

which suggests a dissipative structure for $\text{Im}\lambda > 0$. In what follows, we concentrate our attention on the sub-critical and critical case: $0 < \alpha \leq 2$ (For the super-critical case, we refer to the aforementioned works [10] [21] [22], where a super-critical real $\lambda$ were studied, and the ideals are still applicable to a complex $\lambda$ with $\text{Im}\lambda > 0$). The critical case $\alpha = 2$ has been studied in [17], in which the positivity of $\text{Im}\lambda$ visibly affects the decay rate of $\|u(t, x)\|_{L^\infty}$ and, actually, it decays like $(t \log t)^{-1/2}$. Since the nontrivial free solution only decays like $O(t^{-1/2})$, this gain of additional logarithmic time decay reflects a dissipative character. This result is then extended in [13] [14] to the subcritical case. For $\alpha < 2$ is sufficiently close to 2, Kita-Shimomura [13] established the time decay estimates

$$\|u(t, x)\|_{L^\infty} \lesssim t^{-1/\alpha}, \quad \text{for } t \geq 1 \quad \text{(1.3)}$$

and the asymptotic formula of the solutions. In addition, it is proved in [14] that, under the large dissipative assumption

$$\lambda_2 \geq \frac{\alpha |\lambda_1|}{2 \sqrt{\alpha + 1}}, \quad \lambda = \lambda_1 + i \lambda_2, \quad \text{(1.4)}$$

all solutions with initial value in $H^1(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^2 dx)$ satisfy the $L^\infty$ decay estimate (1.3) when $\frac{1+\sqrt{17}}{4} < \alpha < 2$, and possess a large time asymptotic state when $\frac{2+\sqrt{17}}{12} < \alpha < 2$. The strategy used in [13] [14] [17] is to apply the operator $\mathcal{F}U(-t)$ to the equation (1.1), where $U(t) = e^{it/2\Delta}$ is the Schrödinger operator. Using the factorization technique of the Schrödinger operator $U(t)$, they obtain an ODE for $\mathcal{F}U(-t)u(t)$

$$i \partial_t \mathcal{F}U(-t)u(t) = \lambda t^{-\alpha/2} |\mathcal{F}U(-t)u(t)|^\alpha \mathcal{F}U(-t)u(t) + O_{L^\infty}(t^{-\alpha/2-\mu}), \quad 0 < \mu < 1/4, \quad \text{(1.5)}$$

from which, they deduce the large time asymptotics of $\mathcal{F}U(-t)u(t, x)$ and then in the solution $u(t, x)$.

The present work aims to complete the previous results on the time asymptotic behavior of the solutions obtained in [13] [14] [17]. More precisely, for arbitrary large initial data, we present the uniform time decay estimates when $4/3 < \alpha < 2$, and the large time asymptotics of the solution when $\frac{2+\sqrt{17}}{12} < \alpha < 2$.

**Notation and function spaces.** To state our result precisely, we now give some notations. Throughout the paper, $F(\xi)$ denotes the second order constant coefficients classical elliptic symbol, which has an expansion

$$F(\xi) = c_2 \xi^2 + c_1 \xi + c_0 \quad \text{(1.6)}$$

with $c_2 > 0$, $c_1, c_0 \in \mathbb{R}$. We introduce the notations $D_t = \frac{\partial}{\partial t}$, $D = \frac{\partial}{\partial x}$ and the vector field

$$\mathcal{L} = x + t F'(D). \quad \text{(1.7)}$$
For \( \psi \in L^1(\mathbb{R}) \), \( \mathcal{F}\psi \) is represented as \( \mathcal{F}\psi(x) = (2\pi)^{-1/2} \int_\mathbb{R} \psi(z)e^{-izx}dz \). \( [A,B] \) denotes the commutator \( AB-BA \). Different positive constants we denote by the same letter \( C \). We introduce some function spaces. \( S(\mathbb{R}^2) \) denotes the usual two-dimensional Schwarz space. \( L^p = L^p(\mathbb{R}) \) denotes the usual Lebesgue space with the norm \( \|\phi\|_{L^p} = (\int_\mathbb{R} |\phi(x)|^pdx)^{1/p} \) if \( 1 \leq p < \infty \) and \( \|\phi\|_{L^\infty} = \text{ess\ sup} \{ |\phi(x)| : x \in \mathbb{R} \} \). The weighted Sobolev space is defined by \( H^{0,m} = L^2(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^{2m}dx) \).

**Main results.** We are now ready to state the main result.

**Theorem 1.1.** Assume that \( u_0 \in H^{0,1}, 0 < \alpha < 2, \lambda \) satisfies the condition [1.3] and \( \mathcal{L} \) is the vector filed defined in (1.7). Then there exists a unique global solution \( u \in C([0, \infty), L^2) \) to the Cauchy problem

\[
\begin{cases}
(D_t - F(D))u = \lambda |u|^\alpha u, & t > 0, x \in \mathbb{R} \\
u(x, 0) = u_0(x), 
\end{cases}
\]

satisfying

\[
\|u(t, x)\|_{L^2_x} + \|\mathcal{L}u(t, x)\|_{L^2_x} \leq C \|u_0\|_{H^{0,1}}, \quad t \geq 0
\]

and

\[
\|u(t, x)\|_{L^\infty_x} \leq C \|u_0\|_{H^{0,1}} t^{-1/2}, \quad t > 1.
\]

Furthermore, if \( u_0 \in H^{0,2} \) and \( \alpha \geq 1 \), then

\[
\|\mathcal{L}^2u(t, x)\|_{L^2_x} \leq C(\|u_0\|_{H^{0,2}} + \|u_0\|_{H^{0,2}}^{2\alpha+1}) t^{2-\alpha}, \quad t \geq 0.
\]

**Remark 1.1.** Applying the operator \( \mathcal{L} \) to the equation (1.8), we obtain easily the energy inequality

\[
\|\mathcal{L}u(t, \cdot)\|_{L^2} \lesssim \|\mathcal{L}u(1, \cdot)\|_{L^2} + \int_1^t \|u(\tau, \cdot)\|_{L^\infty}^\alpha \|\mathcal{L}u(\tau, \cdot)\|_{L^2} d\tau.
\]

Using Gronwall’s inequality and a priori estimate \( \|u(\tau, \cdot)\|_{L^\infty} \lesssim \epsilon \tau^{-1/2} \), Hayashi and Naumkin [11] obtained a moderate growth rate of \( \|\mathcal{L}u\|_{L^2} \) in the critical case \( \alpha = 2 \), which is essential to close the bootstrap assumption on \( \|u(t, x)\|_{L^\infty} \). While the limit in [Theorem 1.3, part (d)] demonstrates that one can not hope to deduce this from (1.12) for \( \alpha < 2 \) as in [11], even for the small initial data. So we assume the large dissipative condition (1.4) as in [14], which is used in (3.9) to derive the uniform bound (1.9).

Next, we derive the time decay rate of the global solution obtained in Theorem 1.1.

**Theorem 1.2.** Assume that \( u_0 \in H^{0,1}, 4/3 < \alpha < 2, \lambda \) satisfies the condition [1.3] and \( u \) is the global solution obtained in Theorem 1.1. There exists a constant \( C > 0 \) such that for all \( t \geq 1 \),

\[
\|u(t, x)\|_{L^\infty_x} \leq Ct^{-1/\alpha}.
\]

**Remark 1.2.** A similar time decay estimate as in (1.13) was obtained in [14] under the assumptions \( 1 + \sqrt{35}/4 < \alpha \leq 2 \), which was then extended to the case \( 7 + \sqrt{15}/12 < \alpha \leq 2 \) in [12]. We note that \( 4/3 < 1 + \sqrt{35}/4 \approx 1.686 \) and \( 4/3 < 7 + \sqrt{15}/12 \approx 1.586 \). Therefore, Theorem 1.2 is an improvement of the corresponding results in [12, 14].

**Remark 1.3.** The additional assumption \( \alpha > 4/3 \) ensures that the remainder term \( v_N \) decays faster than \( v \) (see (1.22) and (1.28)).

**Remark 1.4.** Theorem 1.2 is valid without any smallness conditions on the initial data. Moreover, the solution decays faster than the free solution. Recall that in one space dimension, the free solution decays like \( t^{-1/2} \).

Finally, we give a large time asymptotic formula for the solutions and show the existence of modified scattering states for a certain range of the exponent in the nonlinear term.
Theorem 1.3. Suppose that the assumptions in Theorem 1.2 are satisfied and
\[ u_0 \in H^{0.1}, \quad 1 + \frac{\sqrt{33}}{4} < \alpha < 2, \]
or
\[ u_0 \in H^{0.2}, \quad 7 + \frac{\sqrt{145}}{12} < \alpha < 2, \]
then the followings hold:
(a) Let
\[ \Phi(t, x) = \int_1^t s^{-\alpha/2} |v_\lambda(s, x)|^\alpha \, ds, \quad (1.14) \]
where \( v_\lambda \) is the function defined in (1.27). There exists a unique complex valued function \( z_+ \in L^\infty_x \cap L^2_x \) such that for some \( \kappa > 0, \)
\[ \| v_\lambda(t, x) \exp (-i(w(x)t + \lambda \Phi(t, x))) - z_+(x) \|_{L^\infty_x \cap L^2_x} = O(t^{-\kappa}) \]
holds as \( t \to \infty \), where \( w(x) = : -(x + c_1)^2/(4c_2) + c_0. \)
(b) Let
\[ K(t, x) = 1 + \frac{2\alpha \lambda_2}{2 - \alpha} |z_+(x)|^\alpha \left( t^{(2-\alpha)/2} - 1 \right), \quad (1.15) \]
\[ \psi_+(x) = \alpha \lambda_2 \int_1^\infty s^{-\alpha/2} (|v_\lambda(s, x)|^\alpha \exp (\alpha \lambda_2 \Phi(s, x)) - |z_+(x)|^\alpha) \, ds, \quad (1.16) \]
and
\[ S(t, x) = \frac{1}{\alpha \lambda_2} \log (K(t, x) + \psi_+(x)). \quad (1.17) \]
The asymptotic formula
\[ u(t, x) = \frac{1}{\sqrt{t}} e^{i \sqrt{w(x)^2 + \lambda S(t, x)}} z_+(\frac{x}{\sqrt{t}}) + O_{L^\infty_x} (t^{-1/2-\kappa}) \cap O_{L^2_t} (t^{-\kappa}) \quad (1.18) \]
holds as \( t \to \infty \), where \( \kappa \) is the same constant as in part (a).
(c) Let \( u_+(x) = \frac{1}{\sqrt{4\pi c_2}} e^{-i \frac{x^2}{2c_2}} (F z_+)(\frac{x}{\sqrt{c_2}}) \), we have the modified linear scattering
\[ \lim_{t \to \infty} \left\| u(t, x) - e^{i \lambda S(t, x)} e^{i P(D)t} u_+(x) \right\|_{L^2_x} = 0. \quad (1.19) \]
(d) If \( u_0 \neq 0 \), then the limit
\[ \lim_{t \to \infty} t^{\frac{\alpha}{2}} \left\| u(t, x) \right\|_{L^\infty_x} = \left( \frac{2 - \alpha}{2 \alpha \lambda_2} \right)^{\frac{1}{\alpha}}, \quad \text{when } \alpha_0 < \alpha < 2, \quad (1.20) \]
exists and is independent of the initial value, where \( \alpha_0 = \frac{5 + \sqrt{85}}{8} \approx 1.804. \)

Remark 1.5. A similar large time asymptotic formula of the solutions is obtained in [14] in the case \( \frac{2 + \sqrt{177}}{12} < \alpha < 2 \). Since \( \frac{2 + \sqrt{177}}{12} \approx 1.587 < \frac{2 + \sqrt{133}}{12} \approx 1.686 < \frac{2 + \sqrt{177}}{12} \approx 1.859 \), we see that Theorem 1.3 generates the result of [14] in the range of \( \alpha. \)

Remark 1.6. The assumption on the lower bound of \( \alpha \) ensures the convergence of the integral (5.28). It can be extended to \( \alpha > \frac{2}{5} \) if we assume some nonvanishing conditions on the initial values. See e.g. [9] [5].
Remark 1.7. According to the asymptotic formulas (1.18) and (1.19), we see that the solution \( u \) is not asymptotically free. By the definition (1.17) of \( S(t, x) \), we can write the modification factor \( e^{i\lambda S(t, x)} \) explicitly:

\[
e^{i\lambda S(t, x)} = \exp \left\{ \frac{i\lambda}{\alpha^2} \log \left\{ 1 + \frac{2\alpha \lambda^2}{\delta_{\alpha}} |z_+(x)|^{\alpha} (t^{(2-\alpha)/2} - 1) + \psi_+(x) \right\} \right\}.
\]

Strategy of the proof. We briefly sketch the strategy used to derive the decay estimate (1.13), which is the key to establishing the large time asymptotics of the solution. We adapt the semiclassical analysis method introduced by Delort [8], see also [19, 23] which are more close to the problem we are considering.

We make first a semiclassical change of variables

\[
u(t, x) = \frac{1}{\sqrt{t}} \psi(t, \frac{x}{t}),
\]

for some new unknown function \( \psi \), that allows to rewrite the equation (1.8) as

\[
(D_t - G^w_h(x \xi + F(\xi)))v = \lambda h^{\alpha/2} |v|^{\alpha} v,
\]

where the semiclassical parameter \( h = \frac{1}{t} \), and the Weyl quantization of a symbol \( a \) is given by

\[
G^w_h(a)u(x) = \frac{1}{2\pi h} \iint e^{\pi(x-y)\xi a(x+y/2, \xi)} u(y)dyd\xi.
\]

By (1.22), the decay estimate (1.13) is equivalent to

\[
\|v(t, x)\|_{L^\infty_t} \leq C t^{1/2 - 1/\alpha}.
\]

If we develop the symbol \( x \xi + F(\xi) \) as follows by using (1.6)

\[
x \xi + F(\xi) = w(x) + \frac{(x + F'(\xi))^2}{4c_2}
\]

with \( w(x) = -\frac{(x + c_1)^2}{4c_2} + c_0 \),

we deduce from (1.23) an ODE for \( v \):

\[
D_t v = w(x)v + \lambda h^{\alpha/2} |v|^{\alpha} v + \frac{1}{4c_2} G^w_h((x + F'(\xi))^2)v
\]

By semiclassical Sobolev inequality, \( \|G^w_h((x + F'(\xi))^2)v\|_{L^\infty_t} \) is controlled by some energy norm of \( v \) that contains the spatial derivative of order three. While deducing this norm from the equation (1.23) via the standard energy method requires \( \alpha \geq 2 \). Instead, we use the operators whose symbols are localized in a neighbourhood of \( M =: \{(x, \xi) \in \mathbb{R}^2 : x + F'(\xi) = 0\} \) of size \( O(\sqrt{h}) \). In that way we can apply Proposition 2.2 to pass uniform norms of the remainders to the \( L^2 \) norm losing only a power \( h^{-1/4} \).

More precisely, we set

\[
v_\lambda = G^w_h(\gamma(x + F'(\xi))\sqrt{h})v,
\]

where \( \gamma \in C_0^\infty(\mathbb{R}) \) satisfying \( \gamma = 1 \) in a neighbourhood of zero. In Lemma 4.1, we will show that \( v_{A^\gamma} =: G^w_h(1 - \gamma(x + F'(\xi))\sqrt{h})v \) satisfies the uniform estimate

\[
\|v_{A^\gamma}(t, x)\|_{L^\infty_t} \lesssim t^{-1/4}.
\]

We see that it suffices to prove the estimate

\[
\|v_\lambda(t, x)\|_{L^\infty_t} \leq C t^{1/2 - 1/\alpha},
\]

\[
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\]
since \( v_\lambda \) decays faster than \( v \) by the assumption \( \alpha > 4/3 \). Applying \( G_h^w(\gamma(\frac{x + F'(\xi)}{\sqrt{h}})) \) to (1.26) and using (1.26) we obtain the ODE for \( v_\lambda \)
\[
D_t v_\lambda = w(x) v_\lambda + \lambda h^{\alpha/2} |v_\lambda|^\alpha v_\lambda + R(v)
\] (1.29)
where the remainder
\[
R(v) = [D_t - G_h^w(x \xi + F(\xi)), G_h^w(\gamma(\frac{x + F'(\xi)}{\sqrt{h}}))]v + \frac{1}{4c_2} G_h^w((x + F'(\xi))^2)v_\lambda
- \lambda h^{\alpha/2} G_h^w(1 - \gamma(\frac{x + F'(\xi)}{\sqrt{h}}))(|v|^\alpha v) + \lambda h^{\alpha/2} (|v|^\alpha v - |v_\lambda|^\alpha v_\lambda)
\]
satisfies the estimate (see Lemmas 4.2–4.5)
\[
\|R(v)\|_{L_x^\infty} \lesssim t^{-5/4} + (\|v_\lambda\|_{L_x^\infty} + \|v\|_{L_x^\infty}) t^{-\alpha/2 - 1/4}.
\]
Note that \( R(v) \) decays faster than the remainder in (1.5) when \( \alpha < 2 \). Performing a bootstrap and a contradiction argument, one finally deduce from the ODE (1.29) the desired \( L^\infty \) estimate for \( v_\lambda \), and then in the solution \( u \). The details can be found in Subsection 4.2.

Outline. The framework of this paper is organized as follows.

In Section 2 we present the definitions and some useful properties of Semiclassical pseudo-differential operators. In Section 3 we establish the global existence and uniqueness of the solution to (1.8). In Section 4 we prove the decay estimates as stated in Theorem 1.2, combining the bootstrap and the contradiction argument. Finally, in Section 5 we establish the asymptotic formulas in Theorem 1.3.

## 2 Semiclassical pseudo-differential operators

The proof of the main theorem will rely on the use of the semiclassical pseudo-differential calculus. For simplicity, we give only the definitions and properties of the operators we shall use. For more properties about semiclassical pseudo-differential operators, we refer to Chapter 7 of the book of Dimassi-Sjöstrand [9] and Chapter 4 of the book of Zworski [24].

**Definition 2.1.** Let \( a(x, \xi) \in \mathcal{S}(\mathbb{R}^2) \) and \( h \in (0, 1] \). Define the Weyl quantization to be the operator \( G_h^w(a) \) acting on \( u \in \mathcal{S}(\mathbb{R}) \) by the formula
\[
G_h^w(a)u = \frac{1}{2\pi h} \int_\mathbb{R} \int_\mathbb{R} e^{ih(x-y)\xi} a(\frac{x+y}{2}, \xi) u(y)dyd\xi.
\]

We have the following boundedness for Weyl quantization.

**Proposition 2.2** (Proposition 2.7 in [23]). Let \( a(\xi) \) be a smooth function satisfying \( |\partial_\xi^\alpha a(\xi)| \leq C_\alpha < \xi^{-1-\alpha} \) for any \( \alpha \in \mathbb{N} \). Then for \( h \in (0, 1] \)
\[
\|G_h^w(a(\frac{x + F'(\xi)}{\sqrt{h}}))\|_{L(L^2, L^\infty)} = O(h^{-\frac{1}{4}}), \quad \|G_h^w(a(\frac{x + F'(\xi)}{\sqrt{h}}))\|_{L(L^2, L^2)} = O(1).
\]

Next, we introduce some useful composition properties for Weyl quantization.

**Proposition 2.3** (Theorem 7.3 in [9]). Suppose that \( a, b \in \mathcal{S}(\mathbb{R}^2) \). Then
\[
G_h^w(ab) = G_h^w(a) \circ G_h^w(b),
\]
where
\[
a^\sharp b(x, \xi) := \frac{1}{(\pi h)^2} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} e^{ih(\eta z - \eta \xi)} a(x + z, \xi + \eta)b(x + y, \xi + \eta)dyd\eta dz d\xi.
\]
Proposition 2.4 (Proposition 2.4 in [23]). Suppose that $a, b \in S(\mathbb{R}^2)$. Then

$$a_x b = ab + \frac{ih}{2} (\partial_x a \partial_{\xi} b - \partial_{\xi} a \partial_x b) + R,$$

where

$$R = \frac{1}{(2\pi)^2} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} e^{\frac{ix}{h}(a z - yz)} \left\{ - \int_0^1 \partial_{\xi} a(x + tz, \xi)(1-t)d\xi \partial_{\xi} b(x + y, \xi + \eta) \\
+ \int_0^1 \int_0^1 \partial_{\xi} a(x + sz, \xi + t\zeta)dsdt \partial_{\eta} \partial_{\eta} b(x + y, \xi + \eta) \\
- \int_0^1 \partial_{\xi} a(x, \xi + t\zeta)(1-t)d\xi \partial_{\eta} b(x + y, \xi + \eta) \right\} d\eta d\zeta d\xi.$$

Lemma 2.1. Assume that $\Gamma_0(\xi)$ is a smooth function, satisfying $|\partial^\alpha \Gamma_0(\xi)| \leq C_\alpha < \xi > ^{1-\alpha}$ for any $\alpha \in \mathbb{N}$. Then we have

$$\Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \frac{x + F'(\xi)}{\sqrt{h}} = \frac{x + F'(\xi)}{\sqrt{h}} \Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) = \Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \left(\frac{x + F'(\xi)}{\sqrt{h}}\right). \quad (2.1)$$

In addition, if $|\partial^\alpha (\Gamma_0(\xi))| \leq C_\alpha < \xi > ^{-1-\alpha}$ for any $\alpha \in \mathbb{N}$, then we have

$$\left(\Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \frac{x + F'(\xi)}{\sqrt{h}}\right) \frac{x + F'(\xi)}{\sqrt{h}} = \Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \left(\frac{x + F'(\xi)}{\sqrt{h}}\right)^2. \quad (2.2)$$

Proof. An application of Proposition [2.4] yields

$$\Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \frac{x + F'(\xi)}{\sqrt{h}} = \Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \frac{x + F'(\xi)}{\sqrt{h}} + \frac{ih}{2} \left( \partial_x \Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \partial_{\xi} \left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \\
- \partial_{\xi} \Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \partial_x \left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \right) + R$$

$$= \Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \left(\frac{x + F'(\xi)}{\sqrt{h}}\right),$$

where $R = 0$, since $\partial_{\xi} (\frac{x + F'(\xi)}{\sqrt{h}}) = \partial_{xx} (\frac{x + F'(\xi)}{\sqrt{h}}) = \partial_{x \xi} (\frac{x + F'(\xi)}{\sqrt{h}}) = 0$. Similarly, we can show that

$$\frac{x + F'(\xi)}{\sqrt{h}} \Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) = \Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \left(\frac{x + F'(\xi)}{\sqrt{h}}\right).$$

This completes the proof of (2.1). Finally, (2.2) follows easily from (2.1):

$$\Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \left(\frac{x + F'(\xi)}{\sqrt{h}}\right)^2 = \left(\Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \frac{x + F'(\xi)}{\sqrt{h}}\right) \frac{x + F'(\xi)}{\sqrt{h}}$$

$$= \left(\Gamma_0\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \frac{x + F'(\xi)}{\sqrt{h}}\right) \frac{x + F'(\xi)}{\sqrt{h}}.$$

\[ \square \]

3 Proof of Theorem 1.1

In this section, we prove the global existence and uniqueness results in Theorem 1.1. We will use the following lemmas.
Lemma 3.1. Assume $f : [1, T] \times \mathbb{R} \to \mathbb{C}$, $T > 1$ is a smooth function, there exists a positive constant $C$ independent of $T$, $f$ such that for all $t \in [1, T]$

$$
\|f(t, x)\|_{L^\infty} \leq Ct^{-1/2} \|f(t, x)\|_{L^2}^{1/2} \|\mathcal{L}f(t, x)\|_{L^2}^{1/2},
$$
(3.1)

$$
\|\mathcal{L}f(t, x)\|_{L^2} \leq C \|f(t, x)\|_{L^\infty}^{1/2} \|\mathcal{L}^2 f(t, x)\|_{L^2}^{1/2}.
$$
(3.2)

Proof. To begin with, it is useful to introduce a certain phase function. Let

$$
\phi(t, x) = \frac{x^2 + 2c_1 t x}{4c_2 t}.
$$

Since

$$
\mathcal{L} = x + t F(D) = x + c_1 t - 2c_2 i t \partial_x,
$$
(3.3)

it is straightforward to check that

$$
- 2c_2 i t \partial_x (f(t, x) e^{i\phi(t, x)}) = e^{i\phi(t, x)} \mathcal{L} f(t, x),
$$
(3.4)

$$
- 4c_2^2 t^2 \partial_{xx} (f(t, x) e^{i\phi(t, x)}) = e^{i\phi(t, x)} \mathcal{L}^2 f(t, x).
$$
(3.5)

From (3.4), we have, for $t \in [1, T],$

$$
\left\| \partial_x (f(t, x) e^{i\phi(t, x)}) \right\|_{L^2} \leq \frac{C}{t} \|\mathcal{L}f(t, x)\|_{L^2}.
$$
(3.6)

On the other hand, using Gagliardo-Nirenberg’s inequality, we obtain

$$
\|f(t, x)\|_{L^\infty} = \|f(t, x) e^{i\phi(t, x)}\|_{L^\infty} \leq C \left\| f(t, x) e^{i\phi(t, x)} \right\|_{L^2}^{1/2} \left\| \partial_x (f(t, x) e^{i\phi(t, x)}) \right\|_{L^2}^{1/2}.
$$

This together with (3.6) yields (3.1).

Similarly, it follows from (3.4) and Gagliardo-Nirenberg’s inequality that

$$
\|\mathcal{L}f(t, x)\|_{L^2} \leq Ct \|f(t, x) e^{i\phi(t, x)}\|_{L^\infty}^{1/2} \left\| \partial_x (f(t, x) e^{i\phi(t, x)}) \right\|_{L^2}^{1/2},
$$

which together with (3.5) gives the desired estimate (3.2). \hfill \Box

Lemma 3.2. Assume $\text{Im}\lambda > 0$, and $u \in C([0, T]; L^2)$ is a solution of (1.3), then we have

$$
\|u(t, \cdot)\|_{L^2} \leq \|u_0\|_{L^2}, \quad t \in [0, T].
$$

Proof of Theorem 1.1. For the initial datum $u_0 \in H^{0, 1}$, the existence and uniqueness of a local strong $L^2$ solution to the Cauchy problem (1.3) easily follow from Strichartz’$’s estimate

$$
\|u\|_{L^{\infty} \cap L^4 L^\infty} \leq C \|u_0\|_{L^2} + C \|u\|_{L^4 L^1} L^1 L^2,
$$

and a standard contraction argument. Moreover, by applying Lemma 3.2 we can extend this local solution to $[0, \infty)$ easily and omit the details.

In what follows, we prove the estimate (1.10)–(1.11) and thus completing the proof of Theorem 1.1.

Estimate of $\|\mathcal{L}u\|_{L^2}$. Applying the operator $\mathcal{L}$ to (1.3), and using the fundamental commutation property $[D_t - F(D), \mathcal{L}] = 0$, we get

$$
(D_t - F(D)) \mathcal{L} u = \lambda \mathcal{L} \|u\|^{\alpha} u.
$$

This implies

$$
\frac{1}{2} \frac{d}{dt} \|\mathcal{L} u\|_{L^2}^2 = -\text{Im} \int_{\mathbb{R}} \lambda \mathcal{L}(\|u\|^{\alpha} u) \overline{u} u \, dx.
$$
On the other hand, from the expressions of $\mathcal{L}$ in (3.3), it is straightforward to check that

\[
\mathcal{L}(|u|^\alpha u) = \frac{\alpha + 2}{2} |u|^\alpha \mathcal{L} u - \frac{\alpha}{2} |u|^\alpha - 2 u \mathcal{L} |u|; \tag{3.7}
\]

and that for $\alpha \geq 1$,

\[
\mathcal{L}^2(|u|^\alpha u) = \frac{\alpha + 2}{2} |u|^\alpha \mathcal{L}^2 u + \frac{\alpha}{2} |u|^\alpha - 2 u \mathcal{L} |u|^\alpha - \frac{\alpha + 2}{2} |u|^{\alpha - 2} \text{Im}(\mathcal{L}u\mathcal{L} u) - \frac{\alpha}{2} |u|^\alpha - 2 u \mathcal{L} |u|^\alpha \mathcal{L} u - \frac{\alpha}{2} |u|^\alpha - 2 u \mathcal{L} |u|^\alpha \mathcal{L} u. \tag{3.8}
\]

From (3.7) and the large dissipative condition (1.4), we have for all $t \geq 0$,

\[
-\text{Im} \left( \lambda \mathcal{L}(|u|^\alpha u) \mathcal{L} u \right) &= -\text{Im} \left( \lambda \frac{\alpha + 2}{2} |u|^\alpha \mathcal{L}^2 u + \frac{\alpha}{2} |u|^\alpha - 2 u \mathcal{L} |u|^\alpha \right) - \frac{\alpha}{2} |u|^\alpha - 2 u \mathcal{L} |u|^\alpha \mathcal{L} u \leq \left( -\frac{\alpha + 2}{2} - \frac{\alpha}{2} |\lambda| \right) |u|^\alpha |\mathcal{L} u|^2 \leq 0. \tag{3.9}
\]

Therefore, for all $t \geq 0$, we have

\[
\|\mathcal{L} u(t)\|_{L^2} \leq \|\mathcal{L} u(0)\|_{L^2} = \|x u_0\|_{L^2}, \tag{3.10}
\]

which together with Lemma 3.2 yields (1.9).

**Estimate of $\|\mathcal{L}^2 u\|_{L^2}$.** Using the commutation relation $[D_t - F(D), \mathcal{L}^2] = 0$, we get

\[
\frac{1}{2} \frac{d}{dt} \|\mathcal{L}^2 u\|_{L^2}^2 = -\text{Im} \int \lambda \mathcal{L}^2(|u|^\alpha u) \cdot \mathcal{L}^2 \mathcal{L} u d\mathbf{x}. \tag{3.11}
\]

From (3.8) and the large dissipative condition (1.4), we have that for all $t \geq 0$,

\[
-\text{Im} \int \lambda \mathcal{L}^2(|u|^\alpha u) \cdot \mathcal{L}^2 \mathcal{L} u d\mathbf{x} \leq -\lambda_2 \frac{\alpha + 2}{2} \int |u|^\alpha |\mathcal{L}^2 u|^2 d\mathbf{x} + \frac{\alpha}{2} |\lambda| \int |u|^\alpha |\mathcal{L}^2 u|^2 d\mathbf{x} + C \int |u|^\alpha |\mathcal{L} u|^2 |\mathcal{L}^2 u| d\mathbf{x} \leq C \|u\|_{\infty}^2 \|\mathcal{L} u\|_{L^\infty} \|\mathcal{L} u\|_{L^2} \|\mathcal{L}^2 u\|_{L^2}. \tag{3.12}
\]

An application of Lemma 3.1 then yields

\[
-\text{Im} \int \lambda \mathcal{L}^2(|u|^\alpha u) \cdot \mathcal{L}^2 \mathcal{L} u d\mathbf{x} \leq C t^{-\alpha/2} \|u\|_{L^2}^{\alpha+1} \|\mathcal{L} u\|_{L^2} \|\mathcal{L}^2 u\|^2_{L^2} \leq C t^{-\alpha/2} \|u_0\|_{H^{\alpha/2}}^{\alpha+1/2} \|\mathcal{L}^2 u\|^2_{L^2}, \tag{3.13}
\]

where the last inequality holds by applying (3.10) and Lemma 3.2. Substituting (3.12) into (3.11), we get

\[
\frac{d}{dt} \|\mathcal{L}^2 u\|_{L^2}^{1/2} \leq C t^{-\alpha/2} \|u_0\|_{H^{\alpha/2}}^{\alpha+1/2}. \tag{3.14}
\]

Hence

\[
\|\mathcal{L}^2 u\|_{L^2} \leq \|u_0\|_{H^{\alpha/2}} + C t^{2-\alpha} \|u_0\|_{H^{\alpha/2}}^{2\alpha+1}, \tag{3.15}
\]

which yields (1.11).

Finally, from Lemma 3.1, Lemma 3.2 and (3.10), the desired decay estimate (1.10) follows. \qed
4 The proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. It is organized in two subsections.

In the first one, we make first a semiclassical change of variables (1.1) and then prove in Lemma 4.1 that $v_{\Lambda^c}$ can be considered as a remainder. In addition, we derive the ODE (4.9) for $v_{\Lambda}$ and then estimate the remainders $R_{1}(v)$, $R_{2}(v)$ in the rest of this subsection.

In the second one, we use the ODE (4.9) to derive the uniform for $v_{\Lambda}$ and then in the solution $u$, combining the bootstrap and the contradiction argument.

4.1 Semiclassical reduction of the problem

We rewrite the problem in the semiclassical framework. Set

$$u(t, x) = \frac{1}{\sqrt{t}} v(t, \frac{x}{t}), \quad h = \frac{1}{t}. \tag{4.1}$$

Then the equation (1.8) is rewritten as

$$(D_t - G_{h}^{w}(x \xi + F(\xi)))v = \lambda t^{-\alpha/2} |v|^\alpha v. \tag{4.2}$$

At the same time, set

$$\tilde{L} = \frac{1}{h} G_{h}^{w}(x + F'(\xi)). \tag{4.3}$$

Indeed, since $F'(\xi) = 2c_2 \xi + c_1$, we have that $\tilde{L} = (x + c_1) t + 2c_2 D_x$ so that

$$Lu(t, x) = \frac{1}{\sqrt{t}} (\tilde{L}v)(t, \frac{x}{t}), \quad L^2 u(t, x) = \frac{1}{\sqrt{t}} (\tilde{L}^2 v)(t, \frac{x}{t}). \tag{4.4}$$

Moreover, one has

$$\|u(t, \cdot)\|_{L^2} = \|v(t, \cdot)\|_{L^2}, \quad \|u(t, \cdot)\|_{L^\infty} = t^{-1/2} \|v(t, \cdot)\|_{L^\infty}, \tag{4.5}$$

and

$$\|Lu(t, \cdot)\|_{L^2} = \|\tilde{L}v(t, \cdot)\|_{L^2}, \quad \|L^2 u(t, \cdot)\|_{L^2} = \|\tilde{L}^2 v(t, \cdot)\|_{L^2}. \tag{4.6}$$

Therefore the proof of Theorem 1.2 reduces to estimate $\|v(t, \cdot)\|_{L^\infty}$. To do so, we decompose $v = v_{\Lambda} + v_{\Lambda^c}$ with

$$v_{\Lambda} = G_h^{w}(\Gamma)v, \tag{4.7}$$

where $\Gamma(x, \xi) = \gamma(t^{x + F'(\xi)} \sqrt{h})$ with $\gamma \in C_0^\infty(\mathbb{R})$ satisfying that $\gamma \equiv 1$ in a neighbourhood of zero.

We have the following $L^\infty$-estimates for $v_{\Lambda^c}$, which shows that $v_{\Lambda}$ can be considered as a small perturbation of $v$.

Lemma 4.1. Assume $u_0 \in H^{0,1}$. Then for all $t \geq 1$,

$$\|v_{\Lambda^c}(t, \cdot)\|_{L^\infty} \leq C \|u_0\|_{H^{0,1}t^{-1/4}} \quad \|v_{\Lambda^c}(t, \cdot)\|_{L^2} \leq C \|u_0\|_{H^{0,1}t^{-1/2}}. \tag{4.8}$$

Proof. Let $\Gamma_{-1}(\xi) = \frac{1 - \gamma(\xi)}{\xi}$, satisfying $|\partial^a \Gamma_{-1}(\xi)| \leq C_{a} < \xi >^{-1-a}$ for any $a \in \mathbb{N}$. Then we can write

$$1 - \Gamma(x, \xi) = \sqrt{h} \Gamma_{-1}(\frac{x + F'(\xi)}{\sqrt{h}})(\frac{x + F'(\xi)}{h}).$$

From Lemma 2.1 and the definition of $\tilde{L}$ in (4.3), we get

$$G_h^{w}(1 - \Gamma)v = \sqrt{h} G_h^{w}(\Gamma_{-1}(\frac{x + F'(\xi)}{\sqrt{h}})) \circ (\tilde{L}v).$$

An application of Proposition 2.2 yields, for all $t \geq 1$,

$$\|G_h^{w}(1 - \Gamma)v\|_{L^\infty} \leq C \|\tilde{L}v\|_{L^2}h^{1/2}, \quad \|G_h^{w}(1 - \Gamma)v\|_{L^2} \leq C \|\tilde{L}v\|_{L^2}h^{1/2}. \tag{4.9}$$

Since $\|\tilde{L}v\|_{L^2} \leq C \|u_0\|_{H^{0,1}}$ by (4.6) and (4.9), we obtain the desired estimate in Lemma 4.1. □
To get the $L^\infty-$ estimates for $v_\lambda$ in large time, we deduce an ODE from the PDE system (4.2):

$$D_t v_\lambda = w(x)v_\lambda + \lambda t^{-\alpha/2} |v_\lambda|^\alpha v_\lambda + t^{-\alpha/2} (R_1(v) + R_2(v))$$

(4.9)

where $w(x) = -(x + c_1^2)/(4c_2) + c_0$ and

$$R_1(v) = t^{\alpha/2} [D_t - G_h^w(x\xi + F(\xi)), G_h^w(\Gamma)]v$$

$$+ t^{\alpha/2} (G_h^w(x\xi + F(\xi)) - w(x)) v_\lambda,$$

$$R_2(v) = -\lambda G_h^w(1 - \Gamma)(|v|^\alpha v) + \lambda (|v|^\alpha v - |v_\lambda|^\alpha v_\lambda).$$

The rest of this subsection is devoted to proving the following estimates for the remainders $R_1(v)$ and $R_2(v)$.

**Proposition 4.1.** For any $t \geq 1$, we have

(a) when $u_0 \in H^{0,1}$: \[
\begin{align*}
\|R_1(v)\|_{L^\infty} &\leq C \|u_0\|_{H^{0,1}} t^{-5/4 + \alpha/2}, \\
\|R_3(v)\|_{L^2} &\leq C \|u_0\|_{H^{0,1}} t^{-3/2 + \alpha/2}, \\
\|R_2(v)\|_{L^\infty} &\leq C \|u_0\|_{H^{0,1}} (\|v\|_{L^\infty}^\alpha + \|v_\lambda\|_{L^\infty}^\alpha) t^{-1/4}, \\
\|R_2(v)\|_{L^2} &\leq C \|u_0\|_{H^{0,1}} \|v\|_{L^\infty}^\alpha + \|v_\lambda\|_{L^\infty}^\alpha) t^{-1/2}.
\end{align*}
\]

(b) when $u_0 \in H^{0,2}$: \[
\begin{align*}
\|R_1(v)\|_{L^\infty} &\leq C (\|u_0\|_{H^{0,2}} + \|u_0\|_{H^{0,2}}^{2\alpha + 1}) t^{1/4 - \alpha/2}, \\
\|R_3(v)\|_{L^2} &\leq C (\|u_0\|_{H^{0,2}} + \|u_0\|_{H^{0,2}}^{2\alpha + 1}) t^{-\alpha/2}, \\
\|R_2(v)\|_{L^\infty} &\leq C (\|u_0\|_{H^{0,2}} + \|u_0\|_{H^{0,2}}^{2\alpha + 1})(\|v\|_{L^\infty}^\alpha + \|v_\lambda\|_{L^\infty}^\alpha) t^{5/4 - \alpha}, \\
\|R_2(v)\|_{L^2} &\leq C (\|u_0\|_{H^{0,2}} + \|u_0\|_{H^{0,2}}^{2\alpha + 1})(\|v\|_{L^\infty}^\alpha + \|v_\lambda\|_{L^\infty}^\alpha) t^{1 - \alpha}.
\end{align*}
\]

The proof of Proposition 11 is split into Lemmas 12, 13, 14 below.

**Lemma 4.2.** If $u_0 \in H^{0,1}$, then for all $t \geq 1$,

$$\| [D_t - G_h^w(x\xi + F(\xi)), G_h^w(\Gamma)]v \|_{L^\infty} \leq C \|\tilde{L}v\|_{L^2} t^{-5/4},$$

$$\| [D_t - G_h^w(x\xi + F(\xi)), G_h^w(\Gamma)]v \|_{L^2} \leq C \|\tilde{L}v\|_{L^2} t^{-3/2}. $$

Furthermore, if $u_0 \in H^{0,2}$ and $\alpha \geq 1$, then for all $t \geq 1$,

$$\| [D_t - G_h^w(x\xi + F(\xi)), G_h^w(\Gamma)]v \|_{L^\infty} \leq C \|\tilde{L}^2v\|_{L^2} t^{-7/4},$$

$$\| [D_t - G_h^w(x\xi + F(\xi)), G_h^w(\Gamma)]v \|_{L^2} \leq C \|\tilde{L}^2v\|_{L^2} t^{-2}. $$

**Proof.** Since $h = t^{-1}$, by a direct computation, we have

$$[D_t, G_h^w(\Gamma)] f = -\frac{h^{1/2}}{2\pi} \int \int e^{i(x-y)\xi/\sqrt{h}} (x-y)\xi (\frac{x+y+ F'(\xi)}{2\sqrt{h}}) f(t,y)dyd\xi$$

$$+ \frac{1}{2\pi} \int \int e^{i(x-y)\xi/\sqrt{h}} \gamma (\frac{x+y+ F'(\xi)}{\sqrt{h}})(x+y) \frac{\sqrt{h}}{2} f(t,y)dyd\xi$$

$$= -h G_h^w(\gamma (\frac{x+y+ F'(\xi)}{\sqrt{h}})(x+y+ F'(\xi)) f(t,y)dyd\xi$$

(4.10)

where we used the fact that

$$\frac{1}{2\pi} \int \int e^{i(x-y)\xi/\sqrt{h}} (x-y)\xi (\frac{x+y+ F'(\xi)}{\sqrt{h}}) f(t,y)dyd\xi$$

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where $\Gamma \xi$.

Then using Propositions 2.3 and 2.4 we write

$$u = \frac{1}{2\pi h} \int \int \int e^{i(x-y)\xi} \frac{x\xi}{\sqrt{h}} \xi \xi' \frac{x+y+F'(\xi)}{\sqrt{h}} f(t, y) dy d\xi,$$

and

$$\eta = \frac{1}{2\pi h} \int \int \int e^{i(x-y)\xi} \frac{x\xi}{\sqrt{h}} \xi \xi' \frac{x+y+F'(\xi)}{\sqrt{h}} f(t, y) dy d\xi.$$

Then using Propositions 2.3 and 2.4 we write

$$[G_h^w(x \xi + F(\xi)), G_h^w(\Gamma \xi)] = ihG_h^w(g'(x + F'(\xi))) \xi \xi' \frac{x + F'(\xi)}{\sqrt{h}} (\xi \xi' \frac{x + F'(\xi)}{\sqrt{h}}) + r_1 - r_2,$$

where

$$r_1 = \frac{1}{(2\pi)^2} \int \int \int e^{i(x-y)\xi} \frac{1}{h} \xi \xi' \frac{x+y+F'(\xi)}{\sqrt{h}} \xi \xi' \frac{x+y+F'(\xi)}{\sqrt{h}} \xi \xi' \frac{x+y+F'(\xi)}{\sqrt{h}} (1-t)(1-t) dt,$$

and

$$r_2 = \frac{1}{(2\pi)^2} \int \int \int e^{i(x-y)\xi} \frac{1}{h} \xi \xi' \frac{x+y+F'(\xi)}{\sqrt{h}} \xi \xi' \frac{x+y+F'(\xi)}{\sqrt{h}} \xi \xi' \frac{x+y+F'(\xi)}{\sqrt{h}} (1-t)(1-t) dt.$$

Since $F'' = 2c_2$ and $\int e^{2\pi x} dx = \delta(y)\pi h$, we have $r_1 = c_2 h^2 \gamma' \frac{x+F'(\xi)}{\sqrt{h}}.$ Similarly, we have $r_2 = c_2 h^2 \gamma' \frac{x+F'(\xi)}{\sqrt{h}}.$ So we deduce from (4.10) and (4.11) that

$$[D_t - G_h^w(x \xi + F(\xi)), G_h^w(\Gamma \xi)]v = \frac{ih}{2} G_h^w(g'(x + F'(\xi))) \frac{x + F'(\xi)}{\sqrt{h}} v.$$  \hspace{1cm} (4.12)

On the other hand, using Lemma 2.1 we can rewrite (4.12) as

$$[D_t - G_h^w(x \xi + F(\xi)), G_h^w(\Gamma \xi)]v = \frac{ih^{3/2}}{2} G_h^w(g'(x + F'(\xi))) \circ (\tilde{L}_v),$$

where $\Gamma_{-1}(\xi) = \frac{\xi}{\xi}$ satisfying $|\partial^a \Gamma_{-1}(\xi)| \leq C_n < \xi >^{-a} \alpha$ for any $\alpha \in \mathbb{N}.$ The above identities and an application of Proposition 2.2 yields the desired estimates in Lemma 4.2.

**Lemma 4.3.** If $u_0 \in H^{0,1},$ then for all $t \geq 1,$

$$\| (G_h^w(x \xi + F(\xi)) - w(x) \|_{L^\infty} \leq C\| \tilde{L}_v \|_{L^2} t^{-5/4},$$

$$\| (G_h^w(x \xi + F(\xi)) - w(x) \|_{L^2} \leq C\| \tilde{L}_v \|_{L^2} t^{-3/2}.$$

Furthermore, if $u_0 \in H^{0,2}$ and $\alpha \geq 1,$ then for all $t \geq 1,$

$$\| (G_h^w(x \xi + F(\xi)) - w(x) \|_{L^\infty} \leq C\| \tilde{L}_v \|_{L^2} t^{-7/4},$$

$$\| (G_h^w(x \xi + F(\xi)) - w(x) \|_{L^2} \leq C\| \tilde{L}_v \|_{L^2} t^{-7/2}.$$

**Proof.** Applying Lemma 2.1 to (1.28) we get

$$G_h^w(x \xi + F(\xi)) - w(x) = \frac{1}{4c_2} G_h^w((x + F'(\xi))^2 \gamma \frac{x+F'(\xi)}{\sqrt{h}}) v,$$

$$= \frac{1}{4c_2} \left\{ h^{3/2} G_h^w(\Gamma_{-3}(\frac{x+F'(\xi)}{\sqrt{h}})) (\tilde{L}_v), \quad u_0 \in H^{0,1}, \right\}

$$h^2 G_h^w(\gamma(\frac{x+F'(\xi)}{\sqrt{h}})) (\tilde{L}_v), \quad u_0 \in H^{0,2},$$

where $\Gamma_{-3}(\xi) = \xi \gamma(\xi)$ satisfying $|\partial^a \Gamma_{-3}(\xi)| \leq C_n < \xi >^{-a} \alpha$ for any $\alpha \in \mathbb{N}.$ The above identities and an application of Proposition 2.2 yields the desired estimates in Lemma 4.3.
Lemma 4.4. If \( u_0 \in H^{0.1} \), then for all \( t \geq 1 \),
\[
\| G_n^w (1 - \Gamma)(|v|^\alpha v) \|_{L^\infty} \leq C \| v \|_{L^\infty} \| \vec{L} v \|_{L^2} t^{-1/4},
\]
\[
\| G_n^w (1 - \Gamma)(|v|^\alpha v) \|_{L^2} \leq C \| v \|_{L^\infty} \| \vec{L} v \|_{L^2} t^{-1/2}.
\]

Furthermore, if \( u_0 \in H^{0.2} \) and \( \alpha \geq 1 \), then for all \( t \geq 1 \),
\[
\| G_n^w (1 - \Gamma)(|v|^\alpha v) \|_{L^\infty} \leq C \| v \|_{L^\infty} \| \vec{L}^2 v \|_{L^2} t^{-3/4},
\]
\[
\| G_n^w (1 - \Gamma)(|v|^\alpha v) \|_{L^2} \leq C \| v \|_{L^\infty} \| \vec{L}^2 v \|_{L^2} t^{-1}.
\]

Proof. We first claim that
\[
\| \vec{L}(|v|^\alpha v) \|_{L^2} \leq C \| v \|_{L^\infty} \| \vec{L} v \|_{L^2},
\]
and for \( \alpha \geq 1 \),
\[
\| \vec{L}^2(|v|^\alpha v) \|_{L^2} \leq C \| v \|_{L^\infty} \| \vec{L}^2 v \|_{L^2}.
\]

We only give the proof of (4.14) as (4.13) can be proved in a similar way. Since
\[
\vec{L}^2(|u|^\alpha u) = t^{-\alpha/2} \vec{L}^2(|v|^\alpha v)(t, \frac{x}{t})
\]
by the second formula in (4.4), it follows from Lemma 3.1 and (3.8) that
\[
\| \vec{L}^2(|v|^\alpha v) \|_{L^2} \lesssim \alpha/2 \left( \| u \|_{L^\infty} \| \vec{L}^2 u \|_{L^2} + \| u \|_{L^\infty} \| \vec{L} u \|_{L^2} \right)
\]
\[
\lesssim \alpha/2 \| u \|_{L^\infty} \| \vec{L}^2 u \|_{L^2},
\]
which together with (4.13) and (4.4) yields the desired estimate (4.14).

We now resume the proof of Lemma 4.4. Using the same method as that used to derive (4.8), one obtains
\[
\| G_n^w (1 - \Gamma)(|v|^\alpha v) \|_{L^\infty} \leq Ch^{1/4} \| \vec{L}(|v|^\alpha v) \|_{L^2}
\]
\[
\| G_n^w (1 - \Gamma)(|v|^\alpha v) \|_{L^2} \leq Ch^{1/2} \| \vec{L}(|v|^\alpha v) \|_{L^2},
\]
which together with (4.13) proves the first part of Lemma 4.4.

When \( u_0 \in H^{0.2} \), we write
\[
1 - \Gamma(x, \xi) = h \Gamma^{-1} \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) \left( \frac{x + F'(\xi)}{h} \right)^2,
\]
where \( \Gamma^{-1}(\xi) = \frac{1 - \gamma(\xi)}{\xi^2} \), satisfying \( |\partial_\xi \Gamma^{-1}(\xi)| \leq C_\alpha \xi > \alpha \) for any \( \alpha \in \mathbb{N} \). Then using Lemma 2.1 and the definition of \( \vec{L} \) in (4.3), one gets
\[
G_n^w (1 - \Gamma)(|v|^\alpha v) = h G_n^w \left( \Gamma^{-1} \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) \right) \circ \vec{L}^2(|v|^\alpha v).
\]

An application of Proposition 2.2 yields, for all \( t \geq 1 \),
\[
\| G_n^w (1 - \Gamma)(|v|^\alpha v) \|_{L^\infty} \leq Ch^{3/4} \| \vec{L}^2(|v|^\alpha v) \|_{L^2},
\]
\[
\| G_n^w (1 - \Gamma)(|v|^\alpha v) \|_{L^2} \leq Ch \| \vec{L}^2(|v|^\alpha v) \|_{L^2}.
\]

This together with (4.14) proves the second part of Lemma 4.4.

Using similar argument as in the proof of Lemma 4.4, we obtain the following lemma easily and omit the details.
Lemma 4.5. If $u_0 \in H^{0.1}$, then for all $t \geq 1$,
\[
\|v_t^\alpha v - v_A^\alpha v_A\|_{L^\infty} \leq C(\|v\|_{L^\infty}^\alpha + \|v_A\|_{L^\infty}^\alpha)\|\tilde{L}v\|_{L^2}t^{-1/4},
\]
\[
\|v_t^\alpha v - v_A^\alpha v_A\|_{L^2} \leq C(\|v\|_{L^\infty}^\alpha + \|v_A\|_{L^\infty}^\alpha)\|\tilde{L}v\|_{L^2}t^{-1/2}.
\]

Furthermore, if $u_0 \in H^{0.2}$ and $\alpha \geq 1$, then for all $t \geq 1$,
\[
\|v_t^\alpha v - v_A^\alpha v_A\|_{L^\infty} \leq C(\|v\|_{L^\infty}^{\alpha} + \|v_A\|_{L^\infty}^{\alpha})\|\tilde{L}^2v\|_{L^2}t^{-3/4},
\]
\[
\|v_t^\alpha v - v_A^\alpha v_A\|_{L^2} \leq C(\|v\|_{L^\infty}^{\alpha} + \|v_A\|_{L^\infty}^{\alpha})\|\tilde{L}^2v\|_{L^2}t^{-1}.
\]

Proof of Proposition 4.1. It follows from (4.10), (4.13) and (4.4) that for all $t \geq 1$,
\[
\|\tilde{L}v(t, \cdot)\|_{L^2} \leq \|u_0\|_{H^{0.1}},
\]
\[
\|\tilde{L}^2v(t, \cdot)\|_{L^2} \leq C(\|u_0\|_{H^{0.2}} + \|u_0\|_{H^{0.2}}^{2} t^{-2}).
\]

Proposition 4.1 then follows from (4.15)-(4.16) and Lemmas 4.2-4.5.

4.2 The rough $L^\infty$ estimate for $v_A$

The goal of this subsection is to derive the $L^\infty$ estimate for $v_A$ from the ODE (4.2) and therefore completing the proof of Theorem 1.2. Notice that the estimate of $R_2(v)$ depends on $\|v_A\|_{L^\infty}$ and the initial value is large, we apply the bootstrap argument to derive the decay estimate (4.25) with $K$ large.

Let $K$ be a sufficiently large constant such that
\[
K > \max\left\{ 2^{1/2+1/\alpha}, 1 \right\}
\]
\[
2 - \frac{\alpha}{2} K^{-\alpha} + C_2 \alpha^2 K^{-1} < \alpha \lambda_2,
\]
where $C_1$, $C_2$ are constants depending on $u_0$ that appear in (4.20), (4.24) respectively.

We assume that $v$ satisfies a bootstrap hypotheses on $t \in [1, T_1]$:
\[
\|v_A(t, x)\|_{L^\infty} \leq 2K t^{1/2-1/\alpha}.
\]

From Lemma 4.1-4.5 and the local decay estimate (4.10), we see that, for $t \in [1, 2]$,
\[
\|v_A(t, x)\|_{L^\infty} \leq \|v(t, x)\|_{L^\infty} + \|v_A(t, x)\|_{L^\infty} \leq C_1 (1 + t^{-1/4}) < K t^{1/2-1/\alpha}.
\]

This implies that $T_1 > 2$. Moreover, it follows from (4.19) and Proposition 4.1 that $\lambda t^{-\alpha/2} |v_A|^\alpha v_A + t^{-\alpha/2} (R_1(v) + R_2(v))$ is integrable on $(1, T_1)$; so that $v_A(t, x) \in C((1, T_1), L^\infty)$ by the equation (4.19).

The following lemma is crucial to close the bootstrap hypotheses (4.19).

Lemma 4.6. Under the assumptions (4.19) and $4/3 < \alpha < 2$, we have that, for all $t \in (1, T_1)$,
\[
\|v_A(t, x)\|_{L^\infty} \leq K t^{1/2-1/\alpha}.
\]

Proof. The proof is inspired by Lemma 2.3 of [13]. We prove it by contradiction argument. Assume there exists some $(t_0, \xi_0) \in (1, T_1) \times \mathbb{R}$ such that $|v_A(t_0, \xi_0)| > K t_0^{1/2-1/\alpha}$. From (4.20) and the continuity of $v_A(t, \xi_0)$, we can find $t_* \in (1, t_0)$ such that
\[
|v_A(t, \xi_0)| > K t_*^{1/2-1/\alpha}, \quad \text{holds for all } t_* \leq t \leq t_0
\]
and furthermore \( |v_\Lambda(t_*, \xi_0)| = K t_*^{1/2-1/\alpha} \). Multiplying \( |v_\Lambda(t, \xi_0)|^{-(\alpha+2)} \overline{v_\Lambda(t, \xi_0)} \) on both hand sides of (4.19) and taking the imaginary part, we obtain

\[
- \frac{1}{\alpha} \frac{d}{dt} |v_\Lambda|^{-\alpha} = -\lambda_2 t^{-\alpha/2} - \text{Im} \left( t^{-\alpha/2} (R_1(v) + R_2(v)) \overline{v_\Lambda} \right) |v_\Lambda|^{-(\alpha+2)}. \tag{4.22}
\]

On the other hand, from Proposition 4.1, we have

\[
\|R_1(v) + R_2(v)\|_{L_\infty} \leq C \left( t^{-5/4+\alpha/2} + (\|v_\Lambda\|_{L_\infty}^2 + \|v_\Lambda^c\|_{L_\infty}^2) t^{-1/4} \right) \leq C 2^{\alpha+2} K^{-\alpha/2}, \tag{4.23}
\]

for all \( t \in (t_*, t_0) \), where we used (4.17), Lemma 2.1 and the bootstrap hypotheses (4.19) to bound \( \|v_\Lambda\|_{L_\infty}^2 + \|v_\Lambda^c\|_{L_\infty}^2 \). It then follows from (4.21)–(4.23) that there exists \( C_2 > 0 \) such that for \( t_* < t < t_0 \)

\[
- \frac{1}{\alpha} \frac{d}{dt} |v_\Lambda(t_0)|^{-\alpha} \leq -\lambda_2 t^{-\alpha/2} + C_2 2^{\alpha+2} K^{-1/4} t^{-3/4-\alpha/2+1/\alpha}. \tag{4.24}
\]

Integrating (4.24) from \( t_* \) to \( t \), we get

\[
|v_\Lambda(t_0)|^{-\alpha} - |v_\Lambda(t_0)|^{-\alpha} \geq \frac{2\alpha \lambda_2}{2-\alpha} \left( t^{1-\alpha/2} - t_*^{1-\alpha/2} \right) - C_2 2^{\alpha+2} K^{-1/4} \int_{t_*}^t s^{-3/4-\alpha/2+1/\alpha} ds.
\]

This inequality together with \( |v_\Lambda(t_0)| = K t_*^{1/2-1/\alpha} \) implies that

\[
\left( t^{1/\alpha-1/2} |v_\Lambda(t_0)| \right)^{-\alpha} \geq \left( \frac{t_*}{t} \right)^{1-\alpha/2} K^{-\alpha} + \frac{2\alpha \lambda_2}{2-\alpha} \left( \frac{t_*}{t} \right)^{1-\alpha/2} - C_2 2^{\alpha+2} K^{-1/4} \int_{t_*}^t s^{-3/4-\alpha/2+1/\alpha} ds =: f(t).
\]

Note that \( f(t_*) = K^{-\alpha} \) and \( f(t) \) is monotone increasing around \( t = t_* \). Indeed, by (4.18)

\[
f'(t_*) = \left( \frac{\alpha-2}{2} K^{-\alpha} + \alpha \lambda_2 - C_2 2^{\alpha+2} K^{-1/4} t_*^{-3/4+1/\alpha} \right) t_*^{-1} > 0.
\]

Thus, if \( t \) is slightly larger than \( t_* \), then \( \left( t^{1/\alpha-1/2} |v_\Lambda(t, \xi_0)| \right)^{-\alpha} > K^{-\alpha} \), which contradicts (4.21), from which Lemma 4.6 follows.

\[\text{Proof of Theorem 1.2} \] Lemma 4.6 and a standard continuation argument imply \( T_1 = \infty \) and that for all \( t \geq 1 \),

\[
\|v_\Lambda(t, x)\|_{L_\infty} \leq 2 K t^{1/2-1/\alpha}. \tag{4.25}
\]

This together with (4.15) and Lemma 4.1 yields \( \|u(t, x)\|_{L_\infty} \lesssim t^{-1/\alpha} \), from which Theorem 1.2 follows.

\section{The proof of Theorem 1.3}

In this section, we prove Theorem 1.3. We only consider the nontrivial case \( u_0 \neq 0 \). To establish the asymptotic formula for the solutions for a wider range of \( \alpha \), we first need to refine the \( L_\infty \) estimate of \( v_\Lambda \).

\subsection{The refined \( L_\infty \) estimate for \( v_\Lambda \)}

In this subsection, we show how to obtain the refined \( L_\infty \) estimate for \( v_\Lambda \) (Lemma 5.1). This will be essential in proving the large time asymptotics of the solutions in Subsection 5.2.
Substituting the rough $L^\infty$ bound (4.25) of $v_\Lambda$ into Proposition 4.11 we can rewrite the equation (4.9) as
\[ D_t v_\Lambda = u(x) v_\Lambda + \lambda t^{-\alpha/2} |v_\Lambda|^\alpha v_\Lambda + t^{-\alpha/2} R(v), \] (5.1)
where $R(v)$ satisfies, for all $t \geq 1$,
(a) $u_0 \in H^{0,1}$:
\[ \|R(v)\|_{L^\infty} \leq C t^{-\frac{3}{4}+\alpha/2}, \] (5.2)
\[ \|R(v)\|_{L^2} \leq C t^{-3/2+\alpha/2}. \] (5.3)
(b) $u_0 \in H^{0,2}$:
\[ \|R(v)\|_{L^\infty} \leq C t^{1/4-\alpha/2}, \] (5.4)
\[ \|R(v)\|_{L^2} \leq C t^{-\alpha/2}. \] (5.5)
For $0 < \varepsilon < 2\alpha\lambda_2$, we define
\[
\begin{cases}
  h_1(\varepsilon) := \frac{3}{4} - \frac{3\alpha}{2} + \frac{2-\alpha}{2\alpha\lambda_2 - \varepsilon}, & \text{when } u_0 \in H^{0,1}, \frac{3+\sqrt{33}}{4} < \alpha < 2, \\
  h_2(\varepsilon) := \frac{3}{4} - \frac{3\alpha}{2} + \frac{2-\alpha}{2\alpha\lambda_2 - \varepsilon}, & \text{when } u_0 \in H^{0,2}, \frac{7+\sqrt{145}}{12} < \alpha < 2.
\end{cases}
\] Then by direct computation, we have
\[
\begin{cases}
  h_1(0) = \frac{3}{4} - \frac{3\alpha}{2} + \frac{2-\alpha}{2\alpha\lambda_2 - \varepsilon} < 0, & \text{when } u_0 \in H^{0,1}, \frac{1+\sqrt{33}}{4} < \alpha < 2, \\
  h_2(0) = \frac{3}{4} - \frac{3\alpha}{2} + \frac{2-\alpha}{2\alpha\lambda_2 - \varepsilon} < 0, & \text{when } u_0 \in H^{0,2}, \frac{7+\sqrt{145}}{12} < \alpha < 2.
\end{cases}
\] (5.6)
By the continuity of $h_1$, $h_2$, we can find $0 < \varepsilon_1 < 2\alpha\lambda_2$ such that for any $0 < \varepsilon_0 < \varepsilon_1$
\[
\frac{3}{4} - \frac{\alpha}{2} + \frac{2-\alpha}{2\alpha\lambda_2 - \varepsilon_0} < 0, \quad \text{when } u_0 \in H^{0,1}, \frac{3+\sqrt{33}}{4} < \alpha < 2, \quad (5.7)
\]
\[
\frac{9}{4} - \frac{3\alpha}{2} + \frac{2-\alpha}{2\alpha\lambda_2 - \varepsilon_0} < 0, \quad \text{when } u_0 \in H^{0,2}, \frac{7+\sqrt{145}}{12} < \alpha < 2. \quad (5.8)
\]
For $0 < \varepsilon_0 < \varepsilon_1$, we define
\[ K_0 = \left( \frac{2-\alpha}{2\alpha\lambda_2 - \varepsilon_0} \right)^{1/\alpha}, \quad T_0 = \max \left\{ \left( \frac{2C_3\alpha}{\varepsilon_0 K_0^{1+\tau}} \right)^{\frac{4\alpha}{3\alpha - 2}}, \varepsilon \right\}, \] (5.9)
where $C_3 > 0$ is the constant in (5.11) that depends on $u_0$.

The following refined $L^\infty$ estimate for $v_\Lambda$ is true:

**Lemma 5.1.** Assume $u_0 \in H^{0,1}$, $\frac{1+\sqrt{33}}{4} < \alpha < 2$ or $u_0 \in H^{0,2}$, $\frac{7+\sqrt{145}}{12} < \alpha < 2$. There exists $T^*(\varepsilon_0) > T_0$, such that for all $t > T^*(\varepsilon_0)$,
\[
\|v_\Lambda(t,x)\|_{L^\infty} \leq K_0 t^{1/2-1/\alpha}.
\]

**Proof.** In what follows, we prove Lemma 5.1 in the case $u_0 \in H^{0,1}$, $\frac{1+\sqrt{33}}{4} < \alpha < 2$ by the contradiction argument. The case $u_0 \in H^{0,2}$, $\frac{7+\sqrt{145}}{12} < \alpha < 2$ follows from a similar argument, but using (5.4)–(5.5), (5.8) instead of (5.2)–(5.3), (5.7).

Assume by contradiction that there exist $\{(t_n, \xi_n)\}_{n=1}^\infty \subset (T_0, \infty) \times \mathbb{R}$ with $t_n$ monotone increases to $\infty$ such that
\[
|v_\Lambda(t_n, \xi_n)| > K_0 t_n^{1/2-1/\alpha}, \quad \text{for all } n \geq 1.
\] (5.10)
We first claim that, for every fixed $n \geq 1$, we have
\[
|v_\Lambda(t, \xi_n)| > K_0 t^{1/2-1/\alpha}, \quad \text{for all } t \in (T_0, t_n).
\] (5.11)
In fact, if this claim is not true, there would exist some $t_n^* \in (T_0, t_n)$ such that
\[
|v_\Lambda(t, \xi_n)| > K_0 t_{n^*}^{1/2-1/\alpha}, \quad \text{holds for all } t \in (t_n^*, t_n),
\] (5.12)
and furthermore \(|v_\Lambda(t_n^*, \xi_n)| = K_0(t_n^*)^{1/2-1/\alpha}\). Multiplying \(|v_\Lambda(t, \xi_n)|^{-(\alpha+2)} v_\Lambda(t, \xi_n)\) on both hands of (5.11) and taking the imaginary part, we obtain, for all \(t \in (t_n^*, t_n)\),
\[
-\frac{1}{\alpha} \frac{d}{dt}|v_\Lambda(t, \xi_n)|^{-\alpha} = -\lambda_2 t^{-\alpha/2} - \text{Im} \left(t^{-\alpha/2} R(v_\Lambda) \right)|v_\Lambda|^{-(\alpha+2)}.
\] (5.13)

Using (5.2) and (5.12) to bound the third term in (5.13), we deduce that there exists \(C_3 > 0\) such that for all \(t \in (t_n^*, t_n)\),
\[
-\frac{1}{\alpha} \frac{d}{dt}|v_\Lambda(t, \xi_n)|^{-\alpha} \leq -\lambda_2 t^{-\alpha/2} + C_3 K_0^{-(\alpha+1)} t^{-3/4-\alpha/2+1/\alpha}.
\] (5.14)

Integrating the above inequality from \(t_n^*\) to \(t\) and using \(|v_\Lambda(t_n^*, \xi_0)| = K_0(t_n^*)^{1/2-1/\alpha}\), we obtain
\[
\left(t^{1/2-1/\alpha}|v_\Lambda(t_n^*, \xi_0)|^{-\alpha}\right)^{\alpha - 1} \geq \left(t_n^* \right)^{1-\alpha/2} K_0^{-\alpha} + \frac{2\alpha \lambda_2}{2-\alpha} \left(1 - \left(t_n^* \right)^{1-\alpha/2}\right)
\]
\[
- C_3 \alpha K_0^{-(\alpha+1)} t^{-1+\alpha/2} \int_{t_n^*}^{t} s^{-3/4-\alpha/2+1/\alpha} ds.
\]

We see that \(g(t_n^*) = K_0^{-\alpha}\) and \(g(t)\) is monotone increasing around \(t = t_n^*\). Indeed,
\[
g'(t_n^*) = \left(\frac{\alpha \lambda_2}{2} K_0^{-\alpha} + \alpha \lambda_2 - C_3 \alpha K_0^{-(\alpha+1)} (t_n^*)^{3/4+1/\alpha}\right) (t_n^*)^{-1} > 0,
\]
where we used (5.9). Thus, if \(t\) is slightly larger than \(t_n^*,\) then \(|t^{1/\alpha-1/2} v_\Lambda(t_n^*, \xi_0)|^{-\alpha} > K_0^{-\alpha}\), which contradicts (5.12). Thus we finish the proof of Claim (5.11).

In what follows, we use Claim (5.11) to derive a contradiction to (5.10). Notice that (5.14) holds for all \(t \in (T_0, t_n)\) by a similar argument used before, but using Claim (5.11) instead of (5.12). Integrating (5.14) from \(T_0\) to \(t_n\), using (5.2) and Claim (5.11), we have, for every \(n \geq 1\),
\[
\left(t_n^{1/\alpha-1/2} |v_\Lambda(t, \xi_0)|^{-\alpha}\right)^{\alpha - 1} \geq \frac{|v_\Lambda(T_0, \xi_0)|^{-\alpha}}{t_n^{1-\alpha/2}} + \frac{2\alpha \lambda_2}{2-\alpha} \left(1 - \left(T_0 \right)^{1-\alpha/2}\right)
\]
\[
- C_3 \alpha K_0^{-(\alpha+1)} t^{-1+\alpha/2} \int_{T_0}^{t_n} s^{-3/4-\alpha/2+1/\alpha} ds.
\] (5.15)

Since \(K_0^{-\alpha} \geq (t_n^{1/\alpha-1/2} |v_\Lambda(t_n^*, \xi_0)|)^{-\alpha}\) by (5.10), and \(|v_\Lambda(T_0, \xi_0)|^{-\alpha} \to 0\) as \(n \to \infty\) by Claim (5.11), we can let \(n \to \infty\) in (5.15) and obtain
\[
K_0^{-\alpha} \geq \frac{2\alpha \lambda_2}{2-\alpha}.
\]

This contradicts the definition of \(K_0\) in (5.9), and thus completing the proof of Lemma 5.1.

\[\square\]

5.2 The proof of Theorem 1.3

We are now in a position to prove Theorem 1.3. We give only the proof for the case \(u_0 \in H^{0,1}, 1 < \beta < \alpha < 2\). The case \(u_0 \in H^{0,1}, \beta = 1+\alpha/2 < \alpha < 2\) follows from a similar argument, but using (5.4)–(5.5), (5.8) instead of (5.2)–(5.3), (5.7). The proof is inspired by Section 5 of [17].

Proof of part (a). Assume \(T^*(\xi_0)\) is the constant in Lemma 5.1 \(\Phi(t, x), K(t, x), \psi_\lambda(x), S(t, x)\) are the functions defined in (1.14)–(1.17), respectively. From the definition of \(\Phi(t, x)\) in (1.14) and Lemma 5.1 we have
\[
\|\Phi(t, x)\|_{L^\infty} \leq \int_1^{T^*(\xi_0)} s^{-\alpha/2} \|v_\Lambda(s, x)\|_{L^\infty} \|v_\Lambda\|_{L^\infty} ds + K_0^{\alpha} \int_{T^*(\xi_0)}^{t} s^{-1} ds
\]
This finishes the proof of part (a).

\[ z(t, x) = v_\Lambda(t, x)e^{-i(w(x)t + \lambda \Phi(t, x))}, \quad t > 1. \]  

(5.17)

From the equation (5.1) and (5.17), we have that for \( t > 1 \),

\[ \partial_t z(t, x) = \frac{iR(v)}{t^{\alpha/2}} e^{-i(w(x)t + \lambda \Phi(t, x))}, \]

so that for all \( t_2 > t_1 > 1 \),

\[ z(t_2, x) - z(t_1, x) = i \int_{t_1}^{t_2} s^{-\alpha/2} R(v) e^{-i(w(x)s + \lambda \Phi(s, x))} ds. \]  

(5.18)

Since \(-1/4 + \lambda_2 K_0^\alpha < 0\) by (5.7) and (5.9), it follows from (5.2)–(5.3), (5.21) and (5.22) that

\[ \|z(t_2, x) - z(t_1, x)\|_{L^\infty_x \cap L^2_x} \lesssim \int_{t_1}^{t_2} s^{-5/4 + \lambda_2 K_0^\alpha} ds \lesssim_{\varepsilon_0} t^{-1/4 + \lambda_2 K_0^\alpha}, \]  

(5.19)

holds for any \( t_2 > t_1 > 1 \). Thus there exists \( z_+(x) \in L^\infty_x \cap L^2_x \) such that

\[ \|z(t, x) - z_+(x)\|_{L^\infty_x \cap L^2_x} \lesssim_{\varepsilon_0} t^{-1/4 + \lambda_2 K_0^\alpha}. \]  

(5.20)

This finishes the proof of part (a).

\[ \square \]

**Proof of part (b).** The first step is to derive the asymptotic formula (5.23) for \( \Phi(t, x) \). Note that

\[ \partial_t \Phi(t, x) = t^{-\alpha/2} |v_\Lambda(t, x)|^\alpha = t^{-\alpha/2} |z(t, x)|^\alpha e^{-\alpha \lambda_2 \Phi(t, x)} \]

for all \( t > 1 \) by the definition of \( z(t, x) \) in (5.17), so that

\[ \partial_t e^{\alpha \lambda_2 \Phi(t, x)} = \alpha \lambda_2 t^{-\alpha/2} |z(t, x)|^\alpha. \]

Integrating the above equation from 1 to \( t \), we get

\[ e^{\alpha \lambda_2 \Phi(t, x)} = 1 + \alpha \lambda_2 \int_1^t s^{-\alpha/2} |z(s, x)|^\alpha ds. \]  

(5.21)

From (1.15), (1.16) and (5.21), we have

\[ e^{\alpha \lambda_2 \Phi(t, x)} - K(t, x) - \psi_+(x) = -\alpha \lambda_2 \int_1^t s^{-\alpha/2} (|z(s, x)|^\alpha - |z_+(x)|^\alpha) ds. \]

Since \( |u|^\alpha - |v|^\alpha \lesssim (|u|^{\alpha-1} + |v|^{\alpha-1}) |u - v| \), and \( z(s, x), z_+(x) \in L^\infty_x \), we deduce from (5.20) and (5.22) that, for all \( t \geq 1 \),

\[ \|e^{\alpha \lambda_2 \Phi(t, x)} - K(t, x) - \psi_+(x)\|_{L^\infty_x \cap L^2_x} \lesssim_{\varepsilon_0} \int_1^t s^{-1/4 - \alpha/2 + \lambda_2 K_0^\alpha} ds \lesssim t^{-\beta}, \]  

(5.23)

where \( \beta = -(3/4 - \alpha/2 + \lambda_2 K_0^\alpha) > 0 \) by (5.8) and (5.10).

We now prove the asymptotic formula (5.23). Since \( |e^{iw(x)t + i\lambda S(t, x)}| = e^{-\lambda_2 \Phi(t, x)} \leq 1 \), we have

\[ e^{iw(x)t + i\lambda S(t, x)} z_+(x) - v_\Lambda(t, x) = e^{i\lambda \Phi(t, x)} z_+(x) + e^{i\lambda \Phi(t, x)} z_+ - v_\Lambda(t, x) \]

\[ \leq \|e^{iw(x)t} e^{i\lambda \Phi(t, x)} z_+ - v_\Lambda(t, x)\|_{L^\infty_x \cap L^2_x} \]

\[ \leq \|e^{iw(x)t + i\lambda S(t, x)} z_+ - v_\Lambda(t, x)\|_{L^\infty_x \cap L^2_x} \]
\[ \|e^{i\lambda_1 S(t,x)}(e^{-\lambda_2 S(t,x)} - e^{-\lambda_2 \Phi(t,x)})z_+(x)\|_{L^\infty_x \cap L^2_x} \]
\[ +\|e^{i\lambda_1 S(t,x)} - e^{i\lambda_1 \Phi(t,x)}e^{-\lambda_2 \Phi(t,x)}z_+(x)\|_{L^\infty_x \cap L^2_x} \]
\[ +\|z_+(x) - z(t,x)\|_{L^\infty_x \cap L^2_x}. \]  
\[ (5.24) \]

Note that \( K(t,x) + \psi_+(x) \geq 1/2 \) for \( t \) sufficiently large by \((5.23)\), so that
\[ \|e^{i\lambda_1 S(t,x)}(e^{-\lambda_2 S(t,x)} - e^{-\lambda_2 \Phi(t,x)})z_+(x)\|_{L^\infty_x \cap L^2_x} \]
\[ = \|e^{i\lambda_1 S(t,x)}(K(t,x) + \psi_+(x))\|_{L^\infty_x \cap L^2_x} \]
\[ \lesssim \|e^{i\lambda_2 \Phi(t,x)\big) - (K(t,x) + \psi_+(x))\|_{L^\infty_x \cap L^2_x} \]
\[ \lesssim \|e^{i\lambda_2 \Phi(t,x)} - (K(t,x) + \psi_+(x))\|_{L^\infty_x \cap L^2_x} \lesssim \tau_0 t^{-\beta/\alpha}. \]
\[ (5.25) \]

Similarly, we have
\[ \|e^{i\lambda_1 S(t,x)} - e^{i\lambda_1 \Phi(t,x)}e^{-\lambda_2 \Phi(t,x)}z_+(x)\|_{L^\infty_x \cap L^2_x} \]
\[ \lesssim \|S(t,x) - \Phi(t,x)\|_{L^\infty_x \cap L^2_x} \]
\[ \lesssim \|e^{i\lambda_2 S(t,x)} - e^{i\lambda_2 \Phi(t,x)}\|_{L^\infty_x \cap L^2_x} \]
\[ = \|e^{i\lambda_2 \Phi(t,x)} - (K(t,x) + \psi_+(x))\|_{L^\infty_x \cap L^2_x} \lesssim \tau_0 t^{-\beta/\alpha}. \]
\[ (5.26) \]

Substituting \((5.19), (5.25) - (5.26)\) into \((5.24)\), we get
\[ \|e^{i\lambda S(t,x)}z_+(x) - v(t,x)\|_{L^\infty_x \cap L^2_x} \lesssim \tau_0 t^{-\gamma}, \]
\[ (5.27) \]
for \( t \) sufficiently large, where \( \gamma := \min\{1/4 - \lambda_2 K_0^{-1}, \beta/\alpha\} < 1/4 \). It then follows from Lemma 4.1 and 5.24 that the asymptotic formula \((1.18)\) holds.

**Proof of part (c).** From the asymptotic formula \((1.18)\), we have
\[ e^{-i\lambda S(t,x)}u(t,x) \]
\[ = e^{-i\Phi(t,x)}\frac{1}{\sqrt{t}}z_+(\frac{x}{\sqrt{t}}) + O_{L^2}(t^{-\gamma}) \]
\[ = \frac{1}{2\pi} \int e^{i(x-y)\xi}e^{-i\Phi(t,x)}\frac{1}{\sqrt{t}}z_+(\frac{y}{\sqrt{t}})e^{i\omega(t,y)\xi}dyd\xi + O_{L^2}(t^{-\gamma}) \]
\[ = \frac{1}{2\pi} \int e^{i\xi \cdot x - itF(\xi)}z_+(y)e^{i\omega(t,y)\xi}dyd\xi + O_{L^2}(t^{-\gamma}) \]
\[ = \frac{1}{2\pi} \int e^{i\xi \cdot x - ic_2(\xi + \frac{\omega(t,y)}{2c_2})^2}z_+(y)e^{i\omega(t,y)\xi}dyd\xi + O_{L^2}(t^{-\gamma}) \]
\[ (5.28) \]
as \( t \to \infty \), where we use \( y\xi + F(\xi) = w(y) + c_2(\xi + \frac{\omega(t,y)}{2c_2})^2 \) in the last step. Moreover, making a change of variables and then using the dominated convergence theorem, we obtain
\[ \lim_{t \to \infty} \frac{1}{2\pi} \int e^{i\xi \cdot x - ic_2(\xi + \frac{\omega(t,y)}{2c_2})^2}z_+(y)e^{i\omega(t,y)\xi}dyd\xi \]
\[ = \lim_{t \to \infty} \frac{1}{2\pi} \int e^{i\xi \cdot x - ic_2(\xi + \frac{\omega(t,y)}{2c_2})}e^{-ic_2(\xi^2)z_+(y)}dyd\xi \]
\[ = \frac{1}{2\pi} \int e^{-ic_2(\xi^2)z_+(y)}dyd\xi \]
\[ = \frac{1}{\sqrt{4\pi c_2}}e^{-i\frac{\pi}{4}c_2(\xi z_+(y))(\frac{x}{2c_2})} \text{ in } L^2. \]
\[ (5.29) \]
The modified scattering formula \((1.19)\) is now an immediate consequence of \((5.28)\) and \((5.29)\).

**Proof of part (d).** From \((4.1)\), Lemma 4.1 and Lemma 5.1 we have
\[ t^{1/\alpha}\|u(t,x)\|_{L^\infty_x} \leq t^{1/\alpha - 1/2}(\|v_\Lambda\|_{L^\infty} + \|v_\Lambda'\|_{L^\infty}) \]
Since $\varepsilon_0$ can be chosen to be arbitrarily small, we obtain

$$
\limsup_{t \to \infty} t^{1/\alpha} \|u(t,x)\|_{L^\infty_x} \leq \left( \frac{2 - \alpha}{2\alpha \lambda_2} \right)^{1/\alpha} + C t^{1/\alpha - 3/4}, \text{ for all } t > T^*(\varepsilon_0).
$$

Therefore, the proof of part (d) reduces to show that

$$
\liminf_{t \to \infty} t^{1/\alpha} \|u(t,x)\|_{L^\infty_x} \geq \left( \frac{2 - \alpha}{2\alpha \lambda_2} \right)^{1/\alpha}.
$$

Assume for a while that we have proved

**Claim 5.1.** If the limit function $z_+(x)$ in (5.20) satisfies $z_+(x) = 0$ for a.e. $x \in \mathbb{R}$, then we must have $u_0 = 0$.

Then since $u_0 \neq 0$, there exists $x_0 \in \mathbb{R}$ such that $z_+(x_0) \neq 0$. So by (5.27) and (1.21), we have

$$
t^{1/\alpha - 1/2} \|v_\alpha(t,x)\|_{L^\infty_x} \geq \frac{t^{1/\alpha - 1/2}|z_+(x_0)|}{(1 + \left( \frac{2\alpha}{2\alpha \lambda_2} \right) |z_+(x_0)|^\alpha(t/\alpha - 1/2) + \psi_+(x_0))^{1/\alpha}} + O_{\varepsilon_0}(t^{1/\alpha - 1/2 + \gamma(\varepsilon_0)}),
$$

where $\gamma(\varepsilon_0) = \min \{ 1/4 - \lambda_2 K_0^\alpha, \beta/\alpha \}$ with $\beta = -3(\alpha - 2) - \lambda_2 K_0^\alpha$ (see (5.23) and (5.27)). Since $\alpha > \frac{5 + \sqrt{29}}{8}$ and $\lambda_2 K_0^\alpha = \left( \frac{2\alpha - 2\alpha \lambda_2}{2\alpha \lambda_2 - 2} \right)$, we have by direct calculation

$$
\lim_{\varepsilon_0 \to 0} \left( \frac{1}{\alpha} - \frac{1}{2} - \gamma(\varepsilon_0) \right) = \frac{1}{\alpha} - \frac{1}{2} - \min \left\{ 1, \frac{\beta}{\alpha}, \frac{2\alpha^2 - \alpha - 4}{4\alpha^2} \right\} < 0.
$$

Therefore, taking $\varepsilon_0 > 0$ sufficiently small, we deduce from (5.31) that

$$
\liminf_{t \to \infty} t^{1/\alpha - 1/2} \|v_\alpha(t,x)\|_{L^\infty_x} \geq \left( \frac{2 - \alpha}{2\alpha \lambda_2} \right)^{1/\alpha}.
$$

This together with (4.1) and Lemma 4.1 yields the limit (5.30).

**Proof of Claim 5.1** The key observation is that the solution decays faster when $z_+ = 0$:

$$
\|u(t,x)\|_{L^\infty_x} \lesssim t^{-3/4}, \quad \|u(t,x)\|_{L_2^x} \lesssim t^{-1/2} \quad \text{for } t > T^*(\varepsilon_0).
$$

In fact, since $z_+ = 0$, it follows from (5.17)–(5.18) that

$$
v_\alpha(t,x) = -i \int_t^\infty e^{i\nu(x)(t-s)+i\lambda(\Phi(t,x)-\Phi(s,x))} s^{-\alpha/2} R(v)(s) ds.
$$

On the other hand, using (1.13) and Lemma 5.1, we have, for $s > t > T^*(\varepsilon_0)$

$$
\|\Phi(t,x) - \Phi(s,x)\|_{L^\infty_x} \leq \int_t^s \tau^{-\alpha/2} \|v_\alpha(\tau,x)\|^\alpha_{L^\infty_x} d\tau \leq K_0^\alpha \log \frac{s}{t}.
$$

Substituting (5.34) to (5.33), and using the $L^\infty$ bound of $R(v)$ in (5.2), we get

$$
\|v_\alpha(t,x)\|_{L^\infty_x} \lesssim \int_t^\infty \left( \frac{s}{t} \right)^\lambda_2 K_0^\alpha s^{-\alpha/2} s^{-5/4 + \alpha/2} ds \lesssim t^{-1/4}, \quad t > T^*(\varepsilon_0).
$$

Similarly, we have

$$
\|v_\alpha(t,x)\|_{L_2^x} \lesssim t^{-1/2}, \quad t > T^*(\varepsilon_0).
$$
The above two inequalities together with (4.1) and Lemma 4.1 yield (5.32).

Next, we apply the decay estimates (5.32) to prove that $u_0 = 0$. Since $z_+ = 0$, it follows from the equation (1.18) and the asymptotic formula (1.18) that

$$u(t, x) = \lambda \int_t^\infty e^{iF(D)(t-s)} |u|^\alpha u(s) ds.$$

Then applying Strichartz’s estimate and Hölder’s inequality, we get

$$\|u\|_{L^4([T, \infty), L^\infty_x)} \lesssim \int_T^\infty \|u(s, x)\|_{L^\infty_x} \|u(s, x)\|_{L^2_x} ds$$

$$\lesssim \|u\|_{L^4([T, \infty), L^\infty_x)} \left( \int_T^\infty \left( \|u(s, x)\|_{L^\infty_x}^{\alpha-1} \|u(s, x)\|_{L^2_x} \right)^{4/3} ds \right)^{3/4}$$

$$\leq C \|u\|_{L^4([T, \infty), L^\infty_x)} T^{1-3\alpha/4},$$

where we use (5.32) in the last step. Since $\alpha > 4/3$, we can choose $T > T^*(\varepsilon_0)$ sufficiently large such that $CT^{1-3\alpha/4} \leq \frac{1}{2}$, so that $\|u\|_{L^4([T, \infty), L^\infty_x)} = 0$. This together with the uniqueness of the solutions implies $u \equiv 0$, from which Claim 5.1 follows.

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