Photonic heat transport from weak to strong coupling

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Superconducting circuits provide a favorable platform for quantum thermodynamic experiments. An important component for such experiments is a heat valve, i.e. a device which allows one to control the heat power flowing through the system. Here we theoretically study the heat valve based on a superconducting quantum interference device (SQUID) coupled to two heat baths via two resonators. The heat current in such system can be tuned by magnetic flux. We investigate how does the heat current modulation depend on the coupling strength $g$ between the SQUID and the resonators. In the weak coupling regime the heat current modulation grows as $g^2$, but, surprisingly, at the intermediate coupling it can be strongly suppressed. This effect is linked to the resonant nature of the heat transport at weak coupling, where the heat current dependence on the magnetic flux is a periodic set of narrow peaks. At the intermediate coupling the peaks become broader and overlap, thus reducing the heat modulation. At very strong coupling the heat modulation grows again and finally saturates at a constant value.

I. INTRODUCTION

Quantum thermodynamics attracts a lot of attention both from the fundamental physics viewpoint and due to potential applications in nanoscale devices [1–3]. In this context, understanding of the heat transport in nanoscale systems is very important [4–10]. Precise control and tuning of the heat power is essential for the design of quantum heat engines [11–17], thermal rectifiers [18–21], transistors [22–23], masers [24] and circulators [17]. Such thermal devices can also be used for heat management in quantum circuits [10, 25]. In superconducting circuits one can control the heat current by tuning the transition frequencies of a qubit [9, 26, 27] and very accurately measure it employing, for example, normal metal - insulator - superconductor junctions as thermometers [28]. The heat transport experiments in superconducting circuits can be performed at very low temperatures, where the photonic heat flux dominates over phononic and electronic contributions [29]. In such systems, the heat can be transmitted over macroscopic distances [30], which permits remote management of the heat.

Thermodynamics of the systems weakly coupled to the environment has been studied extensively, and there also have been many extensions of the theory to the strong coupling regime [31–35]. One of the difficulties in this context is the ambiguity in the definition of heat at strong coupling [36]. Here we consider a system consisting of two small normal metal islands coupled to the two coplanar waveguide resonators. In the weak coupling regime the heat current modulation grows as $g^2$, but, surprisingly, at the intermediate coupling it can be strongly suppressed. The coupling strength between the SQUID and the resonators.

The device depicted in Fig. 1 is supposed to operate as a heat valve, which allows one to tune the heat flux between the resistors by changing the critical current of the SQUID with the magnetic flux. The performance of the valve is characterized by the heat current modulation amplitude, i.e. by the difference between the maximum and the minimum values of the heat current. We investigate how does the heat modulation vary with various parameters and obtain two surprising results. First, we find that the modulation of the heat depends on the coupling strength between the SQUID and the resonators in non-monotonous way. Indeed, the modulation grows with the coupling strength in the weak coupling regime, it almost vanishes at the intermediate coupling, then it grows again and eventually saturates at very strong coupling. Second, the strongest heat modulation is achieved in the weak coupling regime. In our modelling we use feasible parameters [9, 26, 27], and we believe that our predictions can be experimentally tested. Finally, we have derived analytical expressions for the heat flux in various limiting cases in terms of the circuit parameters.

We organize the paper as follows: in Sec. II, we introduce the model and analytically analyze the weak, the intermediate and the strong coupling regimes and in Sec. III we summarize the results.

II. THE MODEL

We consider an electric circuit depicted in Fig. 1. In this circuit, the two normal metal islands, having the same resistances $R$ and kept at constant temperatures $T_1, T_2$, act as heat baths. The temperatures $T_j$ ($j = 1, 2$) can be experimentally monitored using biased normal metal - insulator - superconductor junctions [28]. The two identical superconducting coplanar waveguide λ/4-resonators with characteristic impedance $Z_0$ serve as filters. The resonators are coupled to the SQUID via the capacitors $C_{c,j}$. The frequencies of the resonators, $\omega_1$ and $\omega_2$ may slightly differ to compensate the difference between $C_{c1}$ and $C_{c2}$, as described below.

In this setup, the SQUID can act as a quantum heat
Here \( N_j(\omega) = 1/\left(e^{\hbar\omega/k_BT_j} - 1\right) \) are the Bose functions and \( \tau(\omega, \Phi) \) is the transmission probability, which depends on frequency and magnetic flux. The transmission probability \( \tau(\omega, \Phi) \) equals to the square absolute value of the transmission coefficient \(|S_{21}(\omega, \Phi)|^2\) between the two resistors. Eq. (3) has the familiar form of the Landauer formula for the photon current [41]. For the circuit under consideration, see Fig. 1, the transmission probability \( \tau(\omega, \Phi) \) is given by [21, 38]

\[
\tau(\omega, \Phi) = -\frac{4\Re\left[\frac{1}{Z_1(\omega)}\right]\Re\left[\frac{1}{Z_2(\omega)}\right]}{-i\omega C + \frac{1}{Z_1(\omega)} + \frac{1}{Z_2(\omega)} + \frac{1}{Z_J(\omega, \Phi)}}^2, \tag{4}
\]

where the impedances of the resonators are

\[
Z_j(\omega) = \frac{1}{-i\omega C_{c,j} - iZ_0 \tan\left(\frac{\pi}{2} \frac{\omega}{\omega_j} + i\alpha\right)}, \tag{5}
\]

\[
\alpha = \frac{1}{2} \ln\left(\frac{Z_0 + R}{Z_0 - R}\right), \tag{6}
\]

and the impedance of the SQUID is

\[
Z_{J}(\omega, \Phi) = -i\omega L_{J}(\Phi) = -\frac{i\hbar\omega}{2eI_C(\Phi)} \tag{7}
\]

In Eq. (5) the angular frequencies \( \omega_j \) correspond to the fundamental modes of the uncoupled resonators.

In Figs. 2 and 3 we plot, respectively, the transmission probability (4) and the heat current (3) evaluated numerically. For the numerical simulations we have used the parameter values typical for the experiment: \( Z_0 = 50\Omega \), \( R = 2\Omega \), \( \omega_1/2\pi = \omega_2/2\pi = 8.84 \) GHz, \( C = 58.7 \) fF, \( I_C = 291 \) nA, \( T_2 = 300 \) mK, \( T_1 = 150 \) mK. In the subsequent subsections we discuss various approximate approaches, which allow us to find analytical expressions for the heat current and to understand the underlying physics.

### A. Qualitative discussion

The transmission probability (4) has peaks at frequencies corresponding to the eigenmodes of the whole system “two resonators plus SQUID”. The position, the height and the width of these peaks depend on magnetic flux. In Fig. 2 we plot the function \( \tau(\omega, \Phi) \) for three different values of the coupling strength between the SQUID and the resonators. In this figure and in the rest of the paper, we assume that the maximum value of the SQUID frequency satisfies

\[
\hbar\omega_{1,2} < \omega_j(0) < 3\hbar\omega_{1,2}. \tag{8}
\]

In this case, the SQUID frequency \( \Phi_2 \) crosses only the lowest resonator modes.

In the weak coupling limit (Fig. 2a) the modes of the resonators and of the SQUID are almost independent. They become hybridized only in the vicinity of the

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**FIG. 1.** The schematics of the heat valve under consideration. The hot bath is indicated by red color and cold one - by blue color.
FIG. 2. Panels (a),(b) and (c) — transmission probability (4) versus normalized magnetic Φ/Φ_0 and the frequency f = ω/2π. Panels (d),(e) and (f) — heat current (3) as a function of Φ/Φ_0. (a,d) Weak coupling regime with C_c = 10 fF. The corresponding value of the coupling constant (11) is g/2π = 296 MHz, of the shifted resonator frequency (10) Ω_r/2π = 8.73 GHz and of the damping rate (26) is κ/2π = 451 MHz. (b,e) Intermediate coupling regime with C_c = 1 pF, the coupling constant (25) g/2π = 2.592 GHz, the frequency of the uncoupled mode (27) ω_{unc}/2π = 4.15 GHz, the frequency of the coupled mode (24) Ω_r/2π = 8.425 GHz and κ/2π = 451 MHz. (c,f) Strong coupling regime with C_c = 10^{-7} F, g/2π = 2.975 GHz, ω_{unc}/2π = 14.87 MHz, Ω_r/2π = 8.414 GHz and κ/2π = 451 MHz.

flux point where ω_{J}(Φ) crossed the frequency of the resonators ω_{1,2}. The heat current through the system J(Φ) shows sharp peak at this point and almost vanishes away from it (Fig. 2d). That is why the modulation of the heat current in the weak coupling limit approximately equals to its maximum value.

At the intermediate coupling (Fig. 2b), the hybridization between the resonators and the SQUID becomes significant even far away from the crossing flux point. For this reason, the heat current peaks become broad and overlap (Fig. 2e). Therefore, the magnitude of the heat current modulation drops. In fact, it almost vanishes, see Fig. 2. Another effect, visible in Fig. 2b, is the splitting of the resonator modes into pairs. In each pair, only one of the modes is coupled to the SQUID and is sensitive to the magnetic flux. Namely, it is the mode having voltage antinode in the vicinity of the SQUID, and having the higher frequency of the two modes.

In the strong coupling regime (Fig. 2c) the two lowest lines in the spectrum move to very low frequencies. In this limit, the heat current modulation reappears again. In part, this effect is caused by the divergence of the Bose functions at low frequencies, which makes the relative contribution of these frequencies to the integral (3) more significant. In addition to that, the third hybrid mode with the frequency close to ω_{1,2} depends on the flux and also contributes to the modulation shown in Fig. 2.

In the next three subsections we discuss each of the regimes introduced above in detail.

B. Weak coupling regime

In the weak coupling regime, the Hamiltonian of the combined system "resonators plus SQUID" can be approximately reduced to that of three coupled oscillators [9, 39, 40].

\[
H = \sum_{j=1,2} \hbar \omega_r \left( a_j^+ a_j + \frac{1}{2} \right) + \hbar \omega_J(\Phi) \left( b^+ b + \frac{1}{2} \right) - \hbar g_1 (a_1^+ - a_1) (b^+ b) - \hbar g_2 (a_2^+ - a_2) (b^+ b). \tag{9}
\]

Here \(\Omega_r\) are the frequencies of the two lowest modes of the resonators shifted by presence of the capacitors \(C_{c,j}\),

\[
\Omega_r = \frac{\sqrt{\pi} \omega_j}{\sqrt{\pi + 4 \omega_j Z_0 C_{c,j}}}, \tag{10}
\]

\(a_j\) are the ladder operators of the resonators, \(b\) is the ladder operator of the SQUID, and

\[
g_j = \frac{Z_0 C_{c,j}^2 \omega_j^3}{\pi (C_{c,j} + C_{c,2} + C)} \tag{11}
\]
are the coupling constants between the resonators and the SQUID. In the rest of the paper we consider the symmetric case and assume that the shifted frequencies \( \nu \) are the same for both resonators. This implies that the parameters \( \omega_j \) and \( C_{c,j} \) are not independent. Eqs. \((10)\) and \((11)\) have been derived by expanding the tangents in the resonator impedances \((5)\) as

\[
\tan x = \sum_{n=0}^{\infty} \frac{2x}{n^2 (n + \frac{1}{2})^2 - x^2},
\]

and keeping only the pole in the expansion with \( n = 0 \). Eqs. \((9)\)\((11)\) are valid at small coupling \( g_j \ll \Omega_r \). Below we provide more accurate condition for the weak coupling approximation, which also involves the damping rate of the resonator impedances \((5)\) as

\[
\kappa = \frac{4R_0 \Omega_r}{\pi Z_0}.
\]

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Thus, in this limit the heat modulation grows with the coupling strength as \( g^2 \). In the limit \( g \gg \kappa/2 \) one finds

\[
\Delta J = \frac{2g^2}{\kappa} \hbar \Omega_r [N_2(\Omega_r) - N_1(\Omega_r)].
\]

In the symmetric case \( g_1 = g_2 = g \) and at very weak coupling \( g \ll \kappa/2 \) one finds

\[
\Delta J = \frac{2g^2}{\kappa} \hbar \Omega_r [N_2(\Omega_r) - N_1(\Omega_r)].
\]

In Eq. \((16)\) we have ignored the contributions of the high frequency modes of the resonators to the heat transport. Since in our model the SQUID angular frequency \( \omega_j(\Phi) \) does not cross these modes, in the weak coupling regime they give small contribution.

In Fig. \(2\), the SQUID crosses the resonant frequencies at the flux value \( \Phi_r \approx 0.41 \Phi_0 \). In this case the heat modulation amplitude equals to the maximum value of the heat flux. To find the latter, it is sufficient to consider the range of frequencies \( \omega \sim \omega_j \sim \Omega_r \), where one can accurately approximate the transmission probability \((3)\) as follows

\[
\tau(\omega) = \frac{\kappa^2 g_1^2 g_2^2}{(\omega - \omega_j)(\omega - \nu_r)^2 - (g_1^2 + g_2^2)(\omega - \nu_r)^2}.
\]

Here we have introduced the complex frequency \( \nu_r = \Omega_r - i\kappa/2 \), where

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\kappa = \frac{4R_0 \Omega_r}{\pi Z_0}.
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is the damping rate of the resonator modes.

The heat current \((3)\) with the approximate transmission probability \((14)\) can be evaluated analytically. If the temperatures of the resistors are sufficiently high, \( k_B T_j \gg |\omega_j - \Omega| \), we obtain

\[
J(\Phi) = \frac{2g_1^2 g_2^2 (\omega_j^2 + g_2^2 + \frac{\kappa}{2})}{g_1^2 + g_2^2} \hbar \Omega_r [N_2(\Omega_r) - N_1(\Omega_r)]^2.
\]

The maximum of the heat flux is achieved at the resonance condition \( \omega_j(\Phi) = \Omega \).

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J_{\text{max}} = \frac{2g_1^2 g_2^2 \hbar \Omega_r [N_2(\Omega_r) - N_1(\Omega_r)]}{g_1^2 + g_2^2}.
\]

while the minimum occurs far away from the resonance, i.e. either at zero flux, \( \Phi = 0 \) or at \( \Phi = \Phi_0/2 \). As we have discussed, at weak coupling one always has \( J_{\text{min}} \ll J_{\text{max}} \). Therefore, in this regime the modulation of the heat current is close to its maximum value,

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In Fig. \(3\) we plot the maximum and the minimum values of the heat current, obtained numerically, as a function of the coupling constant \( g \) and compare them with the approximate results. We note that the expressions \((17)\) and \((18)\) well agree with the numerics in the weak coupling regime.

Finally, we provide more accurate condition under which the weak coupling expressions \((16\)\((20)\) are valid,

\[
\frac{2g^2}{\kappa} + \frac{\kappa}{2} \lesssim |\omega_j(0) - \Omega| \lesssim \frac{k_B T_j}{\hbar}.
\]

\[C. \text{ Intermediate coupling regime}\]

In this section we consider the intermediate coupling regime \( g_j \sim \Omega_r \). In this case, the expressions for the coupling constants \( g_j \) \((11)\) and for other parameters should be corrected. To obtain the corrected expressions, we consider the classical Lagrangian of the system. For simplicity, we consider fully symmetric setup and put \( \omega_1 = \omega_2 = \omega_r \), \( C_{c,1} = C_{c,2} = C_c \) and \( g_1 = g_2 = g \).
We also define the effective capacitance of the \(\lambda/4\) resonators, \(C_r = \pi/4Z_0\omega_r\), and their effective inductances \(L_r = 4Z_0/\pi\omega_r\). Afterwards, the classical Lagrangian of the lowest modes of the two resonators interacting with the SQUID is expressed as

\[
\mathcal{L} = \frac{\hbar^2}{8e^2} \sum_{j=1,2} \left( C_r\varphi_j^2 - \frac{\varphi_j^2}{L_r} + C_c(\dot{\varphi}_j - \dot{\varphi})^2 \right) + \frac{\hbar^2}{8e^2} \left( C\dot{\varphi} - \frac{2eI_c(\Phi)}{\hbar}\varphi^2 \right). \tag{22}
\]

Here \(\varphi\) is the Josephson phase of the SQUID and \(\varphi_j\) are the phases describing the resonators. They are related to the electric potentials at the ends of the resonators, adjacent to the coupling capacitors, \(V_j\), as \(\varphi_j = 2eV_j/\hbar\). Diagonalizing the Lagrangian \(\mathcal{L}\), we obtain the corrected expressions for the angular Josephson frequency \(\omega_j\), for the angular frequencies of the resonator modes \(\Omega_r\) and for the coupling constant \(g\):

\[
\omega_j(\Phi) = \sqrt{\frac{2eI_c(\Phi)(C_r + C_r)}{\hbar(C_c + (C + 2C_c)C_r)}}, \tag{23}
\]

\[
\Omega_r = \omega_r \sqrt{\frac{(C + 2C_c)C_r}{CC_c + (C + 2C_c)C_r}}, \tag{24}
\]

\[
g = \omega_r \sqrt{\frac{C_c^2 C_r}{4(C_r + C_r)(C_c + (C + 2C_c)C_r)}}. \tag{25}
\]

In the limit \(C_c \ll C_r\) these expressions match the weak coupling formulas given in the previous section. In addition, if the resistance \(R\) approaches \(Z_0\) one should use more accurate expression for the damping rate,

\[
\kappa = \frac{4RZ_0\omega_r}{\pi|Z_0^2 - R^2|}. \tag{26}
\]

With these corrections, Eq. \(\text{[16]}\) approximately describes the heat current in the intermediate coupling regime. The frequencies of the eigenmodes of the coupled system in the limit \(R \to 0\) are

\[
\omega_{\text{unc}} = \omega_r \sqrt{\frac{C_r}{C_r + C_c}}, \tag{27}
\]

\[
\omega_{\pm} = \sqrt{\frac{\Omega_r^2 + \omega_r^2}{2} \pm \sqrt{(\Omega_r^2 - \omega_r^2)^2 + 32g^2\omega_r^2}} / 2 \tag{28}
\]

for the two hybrid modes. Note that the interaction term in this equation slightly differs from that in Eq. \(\text{[13]}\). As expected, in the limit \(C_c \ll C_r\) these expressions meet those of the previous section.

The main difference between the weak and the intermediate coupling regimes is in the growing value of the minimum heat current. Assuming that \(J_{\text{min}} = J(0)\), from Eq. \(\text{[16]}\) one finds the modulation in the form

\[
\Delta J = -\frac{g^2(\omega_j(0) - \Omega_r)^2\hbar\Omega_r [N_2(\Omega_r) - N_1(\Omega_r)]}{(\omega_j(0) - \Omega_r)^2 + \left(\frac{2g^2}{\kappa} + \frac{\kappa}{2}\right)^2} \tag{29}
\]

This modulation amplitude significantly drops if

\[
\frac{2g^2}{\kappa} + \frac{\kappa}{2} \gg |\omega_j(0) - \Omega_r|, \tag{30}
\]

i.e. as soon as the coupling can no longer be considered weak.

To illustrate these points, in Fig. [2], we show the transmission probability \(\tau(\omega, \Phi)\) in the intermediate coupling regime \(g \sim \Omega_r\). The flux-independent line at \(f \approx 4.4\) GHz corresponds to the uncoupled mode \(\text{[27]}\). The lines corresponding to hybrid modes \(\text{[28]}\) are well separated at all values of magnetic flux. This is why the dependence of the heat current on \(\Phi\) becomes weak and does not exhibit a resonance peak (Fig. [3]). This, in turn, suppresses the heat current modulation, as is evident from Fig. [3].

### D. Strong coupling regime

In the strong coupling limit the hybrid mode \(\omega_{\text{hyb}}(\Phi)\) \(\text{[28]}\) and the uncoupled mode \(\omega_{\text{unc}}\) \(\text{[27]}\) move to low frequencies, where they merge and form a broad peak in \(\tau(\omega, \Phi)\), which depends on \(\Phi\). The mode \(\omega_{\text{hyb}}(\Phi)\) becomes isolated, with pronounced dependence on the magnetic flux due the strong coupling to the SQUID. This behavior of the transmission peaks is visible in Fig. [2]. These effects lead to the reappearing of the heat current modulation in the strong coupling limit.

Formal boundaries of the strong coupling limit, in which one can derive approximate analytic expressions, are

\[
C_c \gg \max\{C_r, C_r/L_r/R^2\}. \tag{31}
\]

In the limit \(C_c \gg \max\{C_r, C_r\}\) Eqs. \(\text{[23]-[25]}\) become

\[
\omega_j(\Phi) = \sqrt{\frac{2eI_c(\Phi)}{\hbar C + 2C_r}}, \tag{32}
\]

\[
\Omega_r = \omega_r \sqrt{\frac{2C_r}{C + 2C_r}}, \quad g = \omega_r \sqrt{\frac{C_r}{4(C + 2C_r)}}. \tag{33}
\]

Furthermore, at low frequencies \(\omega \ll \omega_r\) one can approximate the impedances of the resonators \(\text{[9]}\) as \(Z_1(\omega) = Z_2(\omega) = -i\omega L_r + R\). In this limit, at small capacitance of the SQUID, \(C \ll L_r/R^2\) and for \(C_c \gg L_r/R^2\) the transmission probability at low frequencies acquires the form of a non-Lorentzian peak,

\[
\tau(\omega) = \frac{4R^2\omega^2}{\left(\frac{2eI_c(\Phi)}{\hbar} (\omega^2 L_r^2 + R^2) + 2\omega^2 L_r \right)^2 + 4R^2\omega^2} \tag{34}
\]
Here we have defined the flux dependent dimensionless parameter

\[ A(\Phi) = \frac{\pi e Z_0 I_C(\Phi)}{\hbar \omega_r} = \frac{\pi^2 L_r}{8 L_I(\Phi)}. \]  

(37)

One can work out even more accurate approximation for the low frequency contribution,

\[ J_l(\Phi) = \frac{\kappa k_B (T_2 - T_1) A(\Phi) \kappa^2}{4(2 + A(\Phi))} \frac{A(\Phi)(1 + A(\Phi)) \omega_{unc}^2}{A(\Phi)(1 + A(\Phi)) \omega_{unc}^2}. \]  

(38)

This result can be extended to the intermediate coupling regime, i.e. to the values of \( C_c \) smaller than the condition (31) requires.

The contribution of the mode \( \omega_+ \) (28) to the heat current can be estimated as

\[ J_+ (\Phi) = \frac{\kappa}{4} \hbar \omega_+ (\Phi) |N_2(\omega_+ (\Phi)) - N_1(\omega_+ (\Phi))|, \]  

(39)

where the frequency \( \omega_+ (\Phi) \) is given by Eq. (28). In the limit \( C_c \gg \max(8, C_r) \), where \( g^c = \Omega_r^2 / 8 \), this frequency acquires a simple form

\[ \omega_+ (\Phi) = \sqrt{\Omega_r^2 + \omega_{unc}^2 (\Phi)} = \sqrt{\omega_+^2 (\Phi) + 2 C_r \left( \frac{1}{C + 2 C_r} \right)} \]  

(40)

The total heat current takes the form

\[ J(\Phi) = J_l(\Phi) + J_+(\Phi) + J_{bg}(\Phi), \]  

(41)

where \( J_{bg}(\Phi) \) is the background contribution coming from the modes with frequencies higher than \( \omega_+ \). Interestingly, in the strong coupling regime the heat current has a maximum at \( \Phi = 0.5 \Phi_0 \) and minimum — at \( \Phi = 0 \), see Fig. 2f.

In Fig. 3 we observe the reappearance of the heat current modulation at strong coupling. For the chosen parameters the modulation predominantly comes from the term \( J_+(\Phi) \) (39), although the low frequency part \( J_l(\Phi) \) also gives significant contribution. In the limit \( C_c \to \infty \) and for \( k_B T_{1,2} \gg \omega_+ (\Phi) \) the modulation approaches the limiting value

\[ \Delta J = \left[ \frac{A(1 + A)}{(1 + A)(2 + A)} + \frac{\hbar^2 \omega_{unc}^2}{6 k_B^2 T_1 T_2} \right] \frac{\kappa k_B (T_2 - T_1)}{8} \]  

(42)

where both \( A \) and \( \omega_+ \) are taken at \( \Phi = 0 \).

III. CONCLUSION

In conclusion, we have studied photonic heat transport through a SQUID coupled to the two resonators and two resistors. By tuning the critical current of the SQUID with magnetic flux, one can control the heat power transmitted from the hot resistor to the cold one. This device can be used as a heat valve provided its’ parameters are chosen properly. We study the performance of
the heat valve depending on the coupling strength between the resonators and the SQUID. We find that the main parameter characterizing the performance of such device, namely, the amplitude of modulation of the photonic heat power, non-monotonously varies with the coupling strength. The modulation grows with the coupling strength in the weak coupling regime, then significantly drops at the intermediate coupling and, finally, it reappears again in the strong coupling limit. This unusual behavior is explained by the resonant nature of the heat transport in the system. Indeed, at weak coupling the heat flows through the device only at magnetic flux values corresponding to the resonance condition \( \omega_J(\Phi) = \Omega_r \) and drops to zero away from these values. As a result, the dependence of the heat power on the magnetic flux, \( J(\Phi) \), is given by a periodic set of narrow peaks. At the intermediate coupling these peaks become broader and eventually overlap, thus reducing the heat modulation. The optimal performance of the heat valve is achieved at the boundary between the weak and the intermediate coupling regimes. Our results can help to optimize the design of the low temperature heat valves based on superconducting circuit components.

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