Asymptotics of the Norm of Elliptical Random Vectors

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Abstract: In this paper we consider elliptical random vectors $X$ in $\mathbb{R}^d$, $d \geq 2$ with stochastic representation $ARU$, where $R$ is a positive random radius independent of the random vector $U$ which is uniformly distributed on the unit sphere of $\mathbb{R}^d$ and $A \in \mathbb{R}^{d \times d}$ is a given matrix. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^d$, and let $F$ be the distribution function of $R$. The main result of this paper is an asymptotic expansion of the probability $P\{\|X\| > u\}$ for $F$ in the Gumbel or the Weibull max-domain of attraction. In the special case that $X$ is a mean zero Gaussian random vector our result coincides with the one derived in H"usler et al. (2002).

Key words and phrases: Elliptical distribution; Gaussian distribution; Kotz Type distribution; Gumbel max-domain of Attraction; Tail approximation; Density convergence; Weak convergence.

1 Introduction

Let $X$ be a mean zero Gaussian random vector in $\mathbb{R}^d$, $d \geq 2$ with underlying covariance matrix $\Sigma$. If $\lambda_{(m)} := \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ are the ordered eigenvalues of the matrix $\Sigma$, then in the light of Theorem 1 in H"usler et al. (2002) we have the asymptotic expansion (set $C_* := \prod_{j=m+1}^d (1 - \lambda_j/\lambda_{(m)})^{-1/2}$ and $C_* := 1$ if $m = d$)

$$P\{\|X\| > \sqrt{u\lambda_{(m)}}\} = C_* \frac{2^{1-m/2}}{\Gamma(m/2)} u^{m/2-1} \exp(-u/2), \quad u \to \infty,$$

(1.1)

where $\|x\|$ stands for the Euclidean norm of $x \in \mathbb{R}^d$, $m$ is the multiplicity of $\lambda_1$ i.e., $m := \#\{j : \lambda_j = \lambda_1, 1 \leq j \leq d\}$, and $\Gamma(\cdot)$ is the Gamma function.

It is well-known (see e.g., Cambanis et al. (1981), or Fang et al. (1990)) that $X$ possesses the stochastic representation

$$X \overset{d}{=} ARU,$$

(1.2)

with $R > 0$ such that $R^2$ is Chi-squared distributed with $d$ degrees of freedom being further independent of $U$ which is uniformly distributed on the unit sphere of $\mathbb{R}^d$ and $A$ is a $d \times d$ real matrix satisfying $AA^\top = \Sigma$. Here $d$ and $\top$ stand for equality of distribution functions and the transpose sign, respectively.

If we drop the distributional assumption on $R$ assuming simply that $R > 0$ almost surely with some unknown distribution function $F$ with upper endpoint $x_F \in (0, \infty]$, then the random vector $X$ with stochastic representation (1.2) is an elliptical random vector (see Cambanis et al. (1981)).

In this paper we focus our interest in possible generalisation of (1.1) considering some general elliptical random vector $X$. It is clear that the asymptotics in (1.1) is related to the tail asymptotics of $F$. In the special case that $R^2$ is Chi-squared distributed with $d$ degrees of freedom we have

$$1 - F(u) = (1 + o(1)) \frac{u^{d-2} \exp(-u^2/2)}{2^{d/2-1} \Gamma(d/2)}, \quad u \to \infty$$

(1.3)
implying that $F$ is in the max-domain of attraction of the unit Gumbel distribution function $\Lambda$ i.e.,
\[ \lim_{u \uparrow \mathbb{R}} \frac{1 - F(u + x/w(u))}{1 - F(u)} = \exp(-x), \quad \forall x \in \mathbb{R}, \tag{1.4} \]
where $w(u) = u$, $u > 0$.

Under the max-domain of attraction assumption (1.4) on $F$ we generalise (1.1) in our main result below (see Theorem 1) to the following asymptotic expansion
\[ P\{\|X\| > u\sqrt{\lambda(m)}\} = (1 + o(1))C_u \frac{\Gamma(d/2)}{\Gamma(m/2)} \left( \frac{2}{uw(u)} \right)^{(d-m)/2} \left[ 1 - F(u) \right], \quad u \uparrow x_F \tag{1.5} \]
and obtain also an asymptotic expansion of the density function of $\|X\|$. We derive similar asymptotic results when the distribution function $F$ is in the Weibull max-domain of attraction. Further, considering an elliptical random sample we apply our asymptotic expansions in order to derive convergence in distribution, density convergence and almost sure convergence of the maximum norms.

Brief outline of the rest of the paper: We proceed with a short section dedicated to our notation and some preliminary results. The main results are presented in Section 3. The proofs of all the results are relegated to Section 4.

## 2 Preliminaries

In this section we introduce several notation and briefly discuss elliptical random vectors and max-domain of attraction. Given a random variable $R$ with distribution function $F$ (write $R \sim F$) we denote by $\overline{F}$ the survivor function. If $X$ is Beta distributed with positive parameters $a, b$, then we denote this by $X \sim \text{Beta}(a, b)$. The Beta distribution $\text{Beta}(a, b)$ possesses the density function $x^{a-1}(1-x)^{b-1}/\Gamma(a+b)$, $x \in (0, 1)$.

If $I$ is a non-empty subset of $\{1, \ldots, d\}$, $d \geq 2$, then $|I|$ denotes the number of its elements. For any vector $x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ we define the subvector $x_I$ with respect to $I$ by $x_I := (x_i, i \in I)^\top$. The Euclidean norm of $x_I \in \mathbb{R}^m$ is $\|x_I\| := \sqrt{\sum_{i \in I} x_i^2}$.

In the sequel we write $U^{(k)} := (U_1, \ldots, U_k)^\top, k = 1, \ldots, d$ for a random vector uniformly distributed on the unit sphere of $\mathbb{R}^k$. For notational simplicity we write $U$ instead of $U^{(d)}$.

Basic distributional results for spherical random vectors are derived in Cambanis et al. (1981). By Lemma 2 therein for two non-empty disjoint sets $I, J$ such that $I \cup J = \{1, \ldots, d\}$, and a uniformly distributed random vector $U$ we have the stochastic representation
\[ U_I \overset{d}{=} \sqrt{W_{m,d}} U^{(m)}, \quad U_J \overset{d}{=} (1 - W_{m,d})^{1/2} U^{(d-m)}, \tag{2.1} \]
with $W_{m,d} \sim \text{Beta}(m/2, (d - m)/2)$. Furthermore, $U^{(m)}, U^{(d-m)}, W_{m,d}$ are mutually independent. For our investigation it is crucial that the random vector $U$ is distributional invariant with respect to orthogonal transformations, i.e.,
\[ DU \overset{d}{=} U \tag{2.2} \]
for any orthogonal matrix $D \in \mathbb{R}^{d \times d}$. Furthermore, for two square matrices $A, B$
\[ AU \overset{d}{=} BU \tag{2.3} \]
whenever $BB^\top, AA^\top \in \mathbb{R}^{d \times d}$ is valid.

Next we mention some facts from the univariate extreme value theory: A distribution function $F$ is said to belong to the max-domain of attraction of a univariate extreme value distribution function $H$, if for constants $a_n > 0, b_n, n \geq 1$
\[ \lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| F^n(a_n x + b_n) - H(x) \right| = 0. \tag{2.4} \]
Only three choices for $H$ are possible (see e.g., Galambos (1987), Reiss (1989), Embrechts et al. (1997), Falk et al. (2004), Kotz and Nadarajah (2005), De Haan and Ferreira (2006), or Resnick (2008)), namely the Fréchet distribution, the Gumbel distribution, or the Weibull distribution.

It is well-known that if $F$ with upper endpoint $x_F$ is in the max-domain of attraction of the Fréchet distribution $\Phi_\gamma(x) = \exp(-x^{-\gamma}), x > 0, \gamma \in (0, \infty)$, then necessarily $x_F = \infty$, and furthermore

$$\lim_{u \to \infty} \frac{F(xu)}{F(u)} = x^{-\gamma}, \ \forall x > 0. \quad (2.5)$$

If $H = \Lambda$ with $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$, then as mentioned in the Introduction (2.4) is equivalent to (1.4). Alternatively we write $F \in \text{MDA}(\Lambda, w)$ whenever (1.4) is satisfied.

When (2.4) holds with $H$ the Weibull distribution function $\Psi_\gamma(x) = \exp(-|x|^\gamma), x < 0, \gamma \in (0, \infty)$, then necessarily $x_F$ is finite, and furthermore

$$\lim_{u \to \infty} \frac{F(x_F - x/u)}{F(x_F - 1/u)} = x^\gamma, \ \forall x > 0. \quad (2.6)$$

Clearly, (2.4) means that the sample maxima converges in distribution (after normalisation). If $F$ possesses a density function $f$, then the convergence in (2.4) can be strengthened in several instances to local uniform convergence of the corresponding density functions. In the Gumbel case local uniform convergence follows if for some positive scaling function $w$

$$\lim_{u \to x_F} \frac{f(u + x/w(u))}{f(u)} = \exp(-x), \ \forall x \in \mathbb{R}, \quad (2.7)$$

whereas for $F$ in the Weibull max-domain of attraction it suffices that

$$\lim_{u \to 0} \frac{uf(x_F - u)}{F(x_F - u)} = \gamma. \quad (2.8)$$

See e.g., Reiss and Drees (1992) or Resnick (2008) for more details.

### 3 Main Results

Let $X = (X_1, \ldots, X_d)^\top$ be an elliptical random vector in $\mathbb{R}^d$ with stochastic representation $1.2$, where $R \sim F, F(0) = 0$, and $A \in \mathbb{R}^{d \times d}$ is a given square matrix. We set throughout this paper $\Sigma := AA^\top$ and denote its eigenvalues by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$$

which exist since the matrix $\Sigma$ is semi-positive definite. In the following $m$ stands for the multiplicity of the largest eigenvalue $\lambda_1 := \lambda_{(m)}$.

As already shown in Hüsler et al. (2002) for the Gaussian setup the tail asymptotics of interest depends only on the eigenvalues of $\Sigma$, but not on the covariance matrix $\Sigma$ itself. This must be the case also when $X$ is an elliptical random vector with stochastic representation $1.2$. Indeed, first note that by (2.3) the matrix $\Sigma$ specifies the distribution of $X$ (and not the matrix $A$). Furthermore (2.2) implies

$$\|X\|^2 \overset{d}{=} R^2\left(\lambda_1 U_1^2 + \cdots + \lambda_d U_d^2\right),$$

hence by (2.1)

$$\|X\|^2 \overset{d}{=} \lambda_{(m)} R^2 \left(W_m, d + \frac{1}{\lambda_{(m)}^2} V_{m+1}^2 + \cdots + \frac{1}{\lambda_{(m)}^d} V_{d-m}^d\right)$$

is valid with $V_{d-m} \overset{d}{=} U^{(d-m)}$ and $W_m, d \sim \text{Beta}(m/2, (d - m)/2)$. Furthermore, $R, W_m, d, V_{d-m}$ are mutually independent. Clearly, for any $u > 0$

$$\mathcal{F}(\sqrt{u/\lambda_{(m)}}) = P\{\lambda_{(m)} R^2 > u\} \geq P\{\|X\|^2 \geq u\} \geq P\{\lambda_{(m)} R^2 W_m, d > u\}. \quad (3.1)$$
We show next that for $F$ in the Gumbel or the Weibull max-domain of attraction the asymptotic behaviour of $P\{\|X\|^2 > u\}$ is similar to that of $P\{\lambda(m)R^2W_{m,d} > u\}$. If $F$ is in the max-domain of attraction of $\Phi_\gamma$, $\gamma \in (0,\infty)$, which is equivalent with $X_1$ has distribution function in the max-domain of attraction of $\Phi_\gamma$ (see Hashorva (2006,2007a)), it follows easily that the random vector $(X_1^2, \ldots, X_d^2)^\top$ is regularly varying with index $\gamma/2$. Hence the asymptotic behaviour of $P\{\|X\| > u\}$, $u \uparrow \infty$ can be easily determined. In particular $\|X\|$ has distribution function in the max-domain of attraction of $\Phi_\gamma$. In the sequel we discuss therefore only the Gumbel and the Weibull cases.

### 3.1 Asymptotics in the Gumbel Model

In this section we assume that the distribution function $F$ of the associated random radius $R$ is in the Gumbel max-domain of attraction. Simple instances of distribution functions $F$ in the Gumbel max-domain of attractions are univariate distributions with exponential tails. In our investigation the scaling function $w$ (see (1.4)) plays a crucial role. The following asymptotic properties of $w$ are well-known (see e.g., Resnick (2008))

$$\lim_{u \uparrow x_F} uw(u) = \infty, \quad \text{and} \quad \lim_{u \uparrow x_F} w(u)(x_F - u) = \infty \quad \text{if} \quad x_F < \infty.$$  \hfill (3.2)

Furthermore, $w$ can be defined asymptotically via the mean excess function (see e.g., Embrechts et al. (1997)) as

$$w(u) = \frac{1 + o(1)}{E\{R - u\mid R > u\}}, \quad u \uparrow x_F.$$  \hfill (3.3)

Throughout the rest of the paper $x_F \in (0,\infty]$ denotes the upper endpoint of $F$, and $C_*$ is a positive constant (also defined in the Introduction) given by

$$C_* := \prod_{j=m+1}^d (1 - \lambda_j/\lambda(m))^{-1/2},$$  \hfill (3.4)

where $C_* := 1$ if $m = d$.

Since $X/x_F$, with $x_F \in (0,\infty)$ is again an elliptical random vector, and moreover $F(s/x_F), s \in R$ is in the Gumbel or Weibull max-domain of attraction if $F$ is in the Gumbel or Weibull max-domain of attraction, respectively, we assume in the following without loss of generality that $x_F = 1$ or $x_F = \infty$. Next, we state the main result for the Gumbel setup.

**Theorem 3.1.** Let $X = AU$ be an elliptical random vector in $R^d, d \geq 2$, with $A$ a $d \times d$ real matrix, $R \sim F$ a positive random variable independent of $U$. If $F(0) = 0, x_F \in \{1, \infty\}$ and $F \in MDA(\Lambda, w)$ with $w$ some positive scaling function, then we have

$$P\{\|X\| > u\sqrt{\lambda(m)}\} = (1 + o(1))C_* \frac{\Gamma(d/2)}{\Gamma(m/2)} \left(\frac{2}{uw(u)}\right)^{(d-m)/2} F(u), \quad u \uparrow x_F.$$  \hfill (3.5)

Furthermore, $\|X\|$ possesses the positive density function $h$ with asymptotic behaviour

$$h(u) = (1 + o(1))w(u)P\{\|X\| > u\}, \quad u \uparrow x_F.$$  \hfill (3.6)

We have now the following result:

**Corollary 3.2.** Let $X, X^{(1)}, \ldots, X^{(n)}, n \geq 1$ be independent elliptical random vectors in $R^d, d \geq 2$ with common distribution function $G$ such that $X$ satisfies the assumptions of Theorem 3.1. Assume for simplicity that $\lambda(m) = 1$. Then we have the convergence in distribution

$$\frac{\max_{1 \leq j \leq n}\|X^{(j)}\|-a_n}{b_n} \quad \overset{d}{\to} \quad Y \sim \Lambda, \quad n \to \infty,$$  \hfill (3.7)

where

$$a_n := 1/w(b_n), \quad b_n := H^{-1}(1 - 1/n), \quad n > 1,$$

with $H^{-1}$ the generalised inverse of the distribution function of $\|X\|$. Furthermore, (3.7) can be strengthened to the local uniform convergence of the corresponding density functions.
Remark 3.3. 1. The scaling function \( w \) is self-neglecting (see e.g., Reiss (1989) or Resnick (2008)) i.e.,

\[
\lim_{n \to \infty} w(u + x/w(u))/w(u) = 1, \quad u \uparrow x_F
\]

uniformly for \( x \) in compact sets of \( \mathbb{R} \). In view of (3.7) under the assumptions of Theorem 3.1

\[
\lim_{u \uparrow x_F} \frac{P\{\|X\| > u\sqrt{\Lambda(m)}\}}{F(u)} = 0.
\]

Hence the upper bound for \( P\{\|X\| > u\} \) in (3.1) is not accurate for \( u \) close to \( x_F \). However, the lower bound therein turns out to be asymptotically accurate.

2. Clearly, the local uniform convergence of the density functions of the sample maxima implies the convergence in distribution stated in (3.7). See Resnick (2008) for deeper results on the density convergence of the univariate sample extremes.

We provide next three illustrating examples:

Example 1. (Gaussian random vectors) Let \( X \in \mathbb{R}^d, d \geq 2 \) be Gaussian random vector with covariance matrix \( \Sigma = AA^\top \) and mean zero. As mentioned in the Introduction \( X \) possesses the stochastic representation (1.2), and furthermore \( R^2 \) is Chi-squared distributed with \( d \) degrees of freedom. Since \( F \in MDA(\Lambda, w) \), with the scaling function \( w(u) = u, u > 0 \) (recall (3.6)) Theorem 3.1 yields (3.7) which is obtained in Theorem 1 of Hüler et al. (2002).

Example 2. (\( F \) with finite upper endpoint) Let \( X \overset{d}{=} RAU, R \sim F \) be an elliptical random vector in \( \mathbb{R}^d, d \geq 2 \). Assume that \( F \) has upper endpoint 1, and furthermore

\[
\mathcal{F}(u) = (1 + o(1))c_2 \exp(-c_2/(1 - u)), \quad u \uparrow 1,
\]

with \( c_1, c_2 \) two positive constants. Since for \( w(u) = c_2/(1 - u)^2, u \in (0, 1) \) and any \( s \in \mathbb{R} \) we have

\[
\frac{\mathcal{F}(u + s/w(u))}{\mathcal{F}(u)} = (1 + o(1)) \exp(-c_2[1/(1 - u + s/w(u)) - 1/(1 - u)]) \to \exp(-s), \quad u \uparrow 1,
\]

then \( F \in MDA(\Lambda, w) \). For this example

\[
P\{\|X\| > u\sqrt{\Lambda(m)}\} = (1 + o(1))c_1C_r \frac{\Gamma(d/2)}{\Gamma(m/2)} (2c_2/(1 - u))^{(d-m)/2} \exp(-c_2/(1 - u)), \quad u \uparrow 1.
\]

Example 3. (Kotz Type III elliptical random vectors) Let \( X = ARU \) be a \( d \) dimensional elliptical random vector. Assume that the survivor function \( \mathcal{F} \) of \( R \) satisfies

\[
\mathcal{F}(u) = (1 + o(1))cu^N \exp(-\delta u^r), \quad u \to \infty,
\]

where \( c, \delta, \tau \) are given positive constants and \( N \in \mathbb{R} \). We refer to such \( X \) as a Kotz Type III elliptical random vector. It follows easily that \( F \in MDA(\Lambda, w) \) with the scaling function \( w \) given by

\[
w(u) := \delta \gamma u^{\tau - 1}, \quad u > 0.
\]

In view of Theorem 3.1 we may thus write

\[
P\{\|X\| > u\sqrt{\Lambda(m)}\} = (1 + o(1))C_r \frac{\Gamma(d/2)}{\Gamma(m/2)} (2/(\delta \tau))^{(d-m)/2} u^{(m-d)/2+N} \exp(-\delta u^r), \quad u \to \infty
\]

\[
= (1 + o(1)) Ku^N \exp(-\delta u^r), \quad u \to \infty.
\]

Let \( X^{(1)}, \ldots, X^{(n)}, n \geq 1 \) be independent random vectors in \( \mathbb{R}^d \) with the same distribution function as \( X \), and assume for simplicity that \( \lambda(m) = 1 \). Then the convergence in distribution

\[
\left( \max_{1 \leq j \leq n} \|X^{(j)}\| - a_n \right)/b_n \overset{d}{\to} Y \sim \Lambda, \quad n \to \infty
\]
holds with constants $a_n, b_n$ defined by

$$a_n := (\delta^{-1} \ln n)^{1/\tau - 1/\delta}, \quad b_n := (\delta^{-1} \ln n)^{1/\tau} + a_n\left[\alpha \ln(\delta^{-1} \ln n) + \ln K\right], \quad n > 1.$$ 

The above convergence in distribution implies the convergence in probability

$$\max_{1 \leq j \leq n} \left\|X^{(j)}\right\| \xrightarrow{\text{P}} 0 \cdot n \xrightarrow{\text{P}} \delta^{-1/\tau}, \quad n \to \infty.$$ 

By the Barndorff-Nielsen criterion for the almost sure stability of the sample maxima (see Barndorff-Nielsen (1963), Resnick and Tomkins (1973) or Tomkins (1986)) we retrieve further the almost sure convergence

$$\max_{1 \leq j \leq n} \left\|X^{(j)}\right\| \xrightarrow{\text{a.s}} 0 \cdot n \xrightarrow{\text{a.s}} \delta^{-1/\tau}, \quad n \to \infty. \quad (3.12)$$

Borrowing the idea of Hüsler et al. (2002) we provide next a refinement of (3.12).

**Theorem 3.4.** Let $X, X^{(1)}, \ldots, X^{(n)}$ be independent Kotz Type III random vectors in $\mathbb{R}^d, d \geq 2$ with distribution function $G$. Assume that $X = RAU$ is such that the associated random radius $R$ has tail asymptotics given by (3.3) with $c, \delta, \tau \in (0, \infty), N \in \mathbb{R}$. Assume for simplicity that $\lambda(m) = 1$ and define its multiplicity $m$ as in Theorem 3.1. If $\delta \tau = 1$, then we have

$$P\left\{ \max_{1 \leq j \leq n} \left\|X^{(j)}\right\| \geq b_n^{*} \right\} = \begin{cases} 1 & \text{if } s \geq (d - m)/2 + N/\tau + 1, \\ 0 & \text{if } s < (d - m)/2 + N/\tau + 1, \end{cases} \quad (3.13)$$

where $b_n^{*} := \left[\tau(\ln n + s \ln(\ln n))\right]^{1/\tau}, n > 1$.

**Remark 3.5.** 1. A Kotz Type III random vector $X$ (see Example 3) is a mean zero Gaussian random vector with covariance matrix $\Sigma = AA^\top$ if $N = d - 2, \delta = 1/2, \tau = 2$ implying that (3.13) holds with

$$b_n^{*} := \left[2(\ln n + m/2 \ln(\ln n))\right]^{1/2}, \quad n > 1,$$

which is shown in Theorem 3 of Hüsler et al. (2002).

2. Under the Gaussian setup as shown in Hüsler et al. (2002) the result of Theorem 3.4 can be utilised as a diagnostic tool to detect departure from the Gaussian distribution. In view of our more general result this same tool can be employed for detecting departure from the Kotz Type III multivariate distribution. In the Gaussian case the term $N/\tau$ equals $d/2 - 1$. In the more general setup of Kotz Type III multivariate distribution $N/\tau$ is in general unknown and needs to be estimated.

3. Further refinements of (3.13) dealing also with the case $\delta \tau \neq 1$ can be achieved by borrowing the ideas presented in Embrechts et al. (1997); see Example 3.5.6 and Example 3.5.8 therein.

**3.2 Asymptotics in the Weibull Model**

Next we consider elliptical random vectors $X = ARU$ where $R$ has distribution function $F$ in the Weibull max-domain of attraction. Necessarily the upper endpoint $x_F$ of $F$ is finite. Specifically, we suppose that $x_F = 1$ and (2.6) holds for some $\gamma \in (0, \infty)$. A canonical example of $F$ in the Weibull max-domain of attraction is the Beta distribution (see Example 4 below). As we show in the next result, the asymptotic behaviour of $\|X\|$ is determined by the eigenvalues of $\Sigma$ and the tail asymptotics of $F$.

**Theorem 3.6.** Under the assumptions and the notation of Theorem 3.4 if further $x_F = 1$ and $F$ is in the max-domain of attraction of $\Psi_\gamma, \gamma \in (0, \infty)$, then we have ($u \downarrow 0$)

$$P\{\|X\| > (1 - u)\sqrt{\lambda_m}\} = (1 + o(1))C, \quad \frac{\Gamma(c)}{\Gamma(c + (d - m + 2)/2)} \frac{\Gamma(d/2)}{\Gamma(m/2)} (2u)^{(d - m)/2} F(1 - u). \quad (3.14)$$

Furthermore, $\|X\|$ possesses the positive density function $h$ with asymptotic behaviour

$$h(1 - u) = (1 + o(1)) (\gamma + (d - m)/2) P\{\|X\| > 1 - u\}/u, \quad u \downarrow 0. \quad (3.15)$$
As in the Gumbel setup we utilise (3.14) to derive the asymptotics of the sample maxima.

**Corollary 3.7.** Let $X, X^{(1)}, \ldots, X^{(n)}, n \geq 1$ be independent elliptical random vectors in $\mathbb{R}^d, d \geq 2$ with common distribution function $G$ such that $X$ satisfies the assumptions of Theorem 3.6 and $\lambda_m = 1$. Then the convergence in distribution

$$\frac{\max_{1 \leq j \leq n} \|X^{(j)}\| - 1}{H^{-1}(1 - 1/n)} \overset{d}{\to} Y \sim \Psi_{\gamma + (d - m)/2}, \quad n \to \infty$$

holds with $H^{-1}$ the generalised inverse of the distribution function of $\|X\|$. Furthermore, (3.16) can be strengthened to the local uniform convergence of the corresponding density functions.

We give next an illustrating example.

**Example 4.** Let $X = ARU$ be an elliptical random vector in $\mathbb{R}^d, d \geq 2$. We consider the special case that $R \sim Beta(a, b)$ with $a, b$ two positive constants. Since

$$P\{R > 1 - u\} = (1 + o(1)) \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b + 1)} u^b, \quad u \downarrow 0$$

it follows that $R$ has distribution function in the Weibull max-domain of attraction with index $b$. Consequently, under the assumptions of Theorem 3.6 we obtain

$$P\{\|X\| > (1 - u)\sqrt{\lambda_m}\} = (1 + o(1)) 2^{(d-m)/2} C_m \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b + (d - m + 2)/2) \Gamma(m/2)} u^{(d-m)/2+b}, \quad u \downarrow 0.$$  



### 4 Related Results and Proofs

**Theorem 4.1.** Let $R \sim F, X \sim H, Z_{a,b} \sim Beta(a, b), a,b > 0$ be three independent random variables. Suppose that $x_F \in \{1, \infty\}, F(0) = H(0) = 0$ and $H$ has upper endpoint 1. Assume further that $H$ is in the max-domain of attraction of $\Psi_{\lambda}, \lambda \in (0, \infty)$ and set $Y := R(X - \delta)Z_{a,b} + \delta$ with $\delta \in [0, 1]$.

a) If $F \in MDA(\Lambda, w)$ with some positive scaling function $w$, then we have

$$P\{Y > u\} = (1 + o(1)) \frac{\Gamma(\lambda + 1) \Gamma(a + b)}{\Gamma(a) \Gamma(\gamma + 1) (1 - \delta)uw(u))^{-b}F(u)H(1 - 1/(uw(u)))}, \quad u \uparrow x_F. \quad (4.1)$$

Furthermore, the random variable $Y$ possesses a positive density function $q$ such that

$$\lim_{u \uparrow x_F} \frac{q(u)}{w(u)P\{Y > u\}} = 1. \quad (4.2)$$

b) Suppose that $F$ satisfies (2.6) with $\gamma \in (0, \infty)$ and $x_F = 1$. Then we have

$$P\{Y > u\} = (1 + o(1)) \frac{1}{\gamma + b + \lambda + 1} \Gamma\gamma + 1\Gamma(\gamma + 1) (1 - \delta)^{-b}F(u)H(u), \quad u \uparrow 1. \quad (4.3)$$

Moreover, the density function $q$ satisfies

$$\lim_{u \uparrow 1} \frac{1 - u)q(u)}{P\{Y > u\}} = \gamma + \lambda + b. \quad (4.4)$$

**Proof:** a) We show next the proof only for $x_F = \infty$. When $F$ has a finite upper endpoint the proof follows with similar arguments utilising further (3.2), therefore is omitted here. Since $F$ is rapidly varying (see e.g., Resnick (2008)) i.e., $\lim_{u \to \infty} F(u)c/F(u) = 0$ for any $c \in (1, \infty)$, and recalling that $R$ is independent of the random variable $(X - \delta)Z_{a,b} + \delta$, as in the proof of Theorem 12.3.1 in Berman (1992) for any $c > 0$ we have (set $Y := R(X - \delta)Z_{a,b} + \delta$)

$$P\{Y > u\} = (1 + o(1)) \int_{u}^{\infty} P\{(X - \delta)Z_{a,b} + \delta > u/r\} dF(r), \quad u \to \infty.$$
Define for any \(u, s, y \in (0, \infty)\)
\[
\eta(u) := (uw(u))^{-1}, \quad \delta_u := \eta(u)/(1 - \delta), \quad H_u(y) := H(1 - y\eta(u)), \quad F_u(s) := F(u + s/w(u)).
\]

Transforming the variables we have
\[
P\{Y > u\} = \int_0^{\infty/\eta(u)} \int_0^{-\infty} P\{Z_{a,b} > ((1 + s\eta(u))^{-1} - \delta)/(1 - y\eta(u) - \delta)\} dH_u(y) dF_u(s)
\]
\[
= \int_0^{\infty/\eta(u)} \int_0^{-\infty} P\{Z_{a,b} > 1 - \delta_u(s-y)(1 + o(1))\} dH_u(y) dF_u(s).
\]

By the assumptions on \(F\) and \(H\) we may write
\[
\lim_{u \uparrow x_F} \frac{F_u(s) - F_u(t)}{F(u)} = \exp(-t) - \exp(-s), \quad \lim_{u \uparrow x_F} \frac{H_u(s) - H_u(t)}{H(u)} = t^\lambda - s^\lambda, \quad \forall s, t \in \mathbb{R}.
\]

Furthermore, since \(Z_{a,b} \sim \text{Beta}(a,b)\) for any \(s, y \in \mathbb{R}\) such that \(s \geq y\) we have (recall (3.2))
\[
\lim_{u \uparrow x_F} \delta_u^{-b} P\{Z_{a,b} > 1 - \delta_u(s-y)(1 + o(1))\} = c_{a,b}(s-y)^b,
\]
with \(c_{a,b} := \Gamma(a+b)/\Gamma(a)\Gamma(b+1)\). Hence Fatou Lemma implies
\[
\liminf_{u \uparrow x_F} \frac{P\{Y > u\}}{\delta_u F(u) H(1 - \eta(u))} \geq c_{a,b} \lambda \int_0^{\infty} \exp(-s) \int_0^{\infty} (s-y)^b y^\lambda \{1 - y\} dy ds
\]
\[
= c_{a,b} \lambda \int_0^{\infty} \exp(-s) s^{k+\lambda} \int_0^{\infty} (1 - y)^b y^\lambda \{1 - y\} dy ds
\]
\[
= c_{a,b} \lambda \Gamma(\lambda + b + 1) \Gamma(\lambda)/\Gamma(\lambda + b + 1)
\]
\[
= \frac{\Gamma(\lambda + 1)\Gamma(a + b)}{\Gamma(a)}.
\]

The proof for the \(\limsup\) (which coincides with \(\liminf\)) can be established utilising Lemma 7.5 an Lemma 7.7 in Hashorva (2007a).

Next, the independence of \(X\) and \(Z_{a,b}\) and the fact that \(Z_{a,b}\) possesses a positive density function in \((0,1)\) implies that the random variable \(X^* := (X - \delta)Z_{a,b} + \delta\) possesses a positive density function in \((0,1)\) which we denote by \(g\).

Consequently, since \(X^*\) and \(R\) are independent and \(X^*\) possesses the density function \(g\) it follows that \(Y\) possesses the density function \(q\) given by
\[
q(u) = \int_u^{\infty} g(u/r) \frac{1}{r} dF(r), \quad u \in (0, x_F). \tag{4.5}
\]

The asymptotic behaviour of \(q(u), u \uparrow x_F\) can be established with similar arguments as the proof above leading thus to (4.2).

b) Define next for any \(u \in (0,1)\)
\[
H_u(y) := H(1 - yu), \quad F_u(s) := F(1 - su), \quad s, y \in (0, \infty), \quad \delta_u := u/(1 - \delta).
\]

We may further write \((u \downarrow 0)\)
\[
P\{Y > 1 - u\} = \int_0^1 \int_0^{1-s+o(1)} P\{Z_{a,b} > 1 - \delta_u(1 - s - y)(1 + o(1))\} dH_u(y) dF_u(s).
\]

By the assumptions on \(F\) and \(H\) we have
\[
\lim_{u \downarrow 0} \frac{F_u(s) - F_u(t)}{F(u)} = t^\gamma - s^\gamma, \quad \lim_{u \downarrow 0} \frac{H_u(s) - H_u(t)}{H(u)} = t^\lambda - s^\lambda, \quad \forall s, t \in \mathbb{R}.
\]
As above for any \( s, y \in \mathbb{R} \) such that \( s + y \leq 1 \)
\[
\lim_{u \to 0} \delta_u^{-b} P\{Z_{a,b} > 1 - \delta_u(1 - s - y)(1 + o(1))\} = c_{a,b}(1 - s - y)^b.
\]

Hence we obtain applying Lemma 4.2 in Hashorva (2007b)
\[
P\{Y > 1 - u\} = (1 + o(1)) c_{a,b} \lambda \gamma \int_0^1 s^{\gamma - 1} \int_0^{1-s} (1 - s - y)^b y^{\lambda - 1} dy ds
\]
\[
= (1 + o(1)) c_{a,b} \lambda \gamma \frac{\Gamma(b + 1) \Gamma(\lambda)}{\Gamma(b + \lambda + 1)} \int_0^1 s^{\gamma - 1} (1 - s)^{b+\lambda} ds
\]
\[
= (1 + o(1)) c_{a,b} \lambda \gamma \frac{\Gamma(b + 1) \Gamma(\lambda)}{\Gamma(b + \lambda + 1)} \frac{\Gamma(b + \lambda + 1) \Gamma(\gamma + b + \lambda + 1)}{\Gamma(\gamma + \lambda + 1)} \cdot u \downarrow 0.
\]

The proof of (4.4) follows with similar arguments utilizing further (4.5).

In the next theorem we derive the asymptotic tail behaviour of the product \( XZ_{a,b} \). Its proof is similar to that of Theorem 12.3.1 of Berman (1992) (also Berman (1982, 1983), and Hashorva (2007a)), therefore we omit it here. See Tang and Tsitsiashvili (2004) and Tang (2006, 2008) for recent results on the tail asymptotics of products of random variables.

**Theorem 4.2.** Let \( X \sim F, Z_{a,b} \sim \text{Beta}(a,b), a,b > 0 \) be two independent random variables and let \( Y := X[1 - \delta Z_{a,b}]^\tau, \delta \in (0,1], \tau \in (0, \infty) \). Suppose that \( x_F \in \{1, \infty\} \) and \( F(0) = 0 \).

a) If \( F \in \text{MDA}(\lambda, w) \) with some positive scaling function \( w \), then we have
\[
P\{Y > u\} = (1 + o(1)) \frac{\Gamma(a + b)}{\Gamma(b)} \frac{\lambda}{\gamma} \rho(u) w(u) - a \frac{F(u)}{u} , \ u \uparrow x_F.
\]

Furthermore, the random variable \( Y \) possesses a positive density function \( q \) and
\[
\lim_{u \downarrow x_F} \frac{q(u)}{w(u)} P\{Y > u\} = 1.
\]

b) Suppose that \( F \) satisfies (2.6) with \( \gamma \in (0, \infty) \) and \( x_F = 1 \). Then we have
\[
P\{Y > u\} = (1 + o(1)) \frac{\Gamma(\gamma + 1) \Gamma(a + b)}{\Gamma(b) \Gamma(\gamma + a + 1)} \frac{(1 - u)/((\tau \delta))^{a-1}}{F(u)} , \ u \uparrow 1.
\]

Moreover, the density function \( q \) satisfies
\[
\lim_{u \downarrow 1} \frac{(1 - u)q(u)}{P\{Y > u\}} = \gamma + a.
\]

**Proof of Theorem 3.1** If the multiplicity of \( \lambda_m \) is \( d \), i.e., \( \lambda_1 = \lambda_2 = \cdots = \lambda_d \), then
\[
\|X\|^2 = \lambda_{(m)} R^2 \|U\|^2 = \lambda_{(m)} R^2,
\]
hence the claim follows. Assume next that \( m < d \), and define a random vector \( V_K \) such that \( V_K \overset{d}{=} R \sqrt{W_{m,d}} U^{(m)} \), with \( R \sim F \) being independent of the random variable \( W_{m,d} \sim \text{Beta}(m/2, (d - m)/2) \). We have
\[
\|V_K\|^2 \overset{d}{=} R^2 W_{m,d} \|U^{(m)}\|^2 \overset{d}{=} R^2 W_{m,d} = R_{m}^2,
\]
with \( R_{m} := R \sqrt{W_{m,d}} \). Consequently since \( R \) and \( W_{m,d} \) are independent applying Theorem 4.2 we obtain
\[
P\{\|V_K\| > u\} = (1 + o(1)) \frac{\Gamma(d/2)}{\Gamma(m/2)} \left( \frac{2}{uw(u)} \right)^{(m-d)/2} F(u) , \ u \uparrow x_F.
\]
By the self-neglecting property (recall (3.3)) of the scaling function \(w\) we conclude that the random variable \(\|V_K\|\) has distribution function in the Gumbel max-domain of attraction with the same scaling function \(w\).

Next, since \(X \overset{d}{=} ARU\) is an elliptical random vector we may write

\[
\|X\|^2 \overset{d}{=} \lambda(m)\|Y\|^2,
\]

where \(Y \overset{d}{=} DRU\) with \(D\) a diagonal matrix with \(d_{ii} = 1, i = 1, \ldots, m\) and \(d_{ii} = \frac{\sqrt{\lambda_i}}{\lambda(m)}, i = m + 1, \ldots, d\). Note that \(d_{ii} \in (0, 1)\) for all \(i > m\). Define \(K_i := K_{i-1} \cup \{i\}, i := m + 1, \ldots, d\), with \(K_m := K\). In the light of (2.1)

\[
Y_{K_{m+1}} \overset{d}{=} R_m\left(U^{(m)} \sqrt{W_{m,m+1}}, I_1 \sqrt{(1 - W_{m,m+1})\lambda_{m+1}/\lambda(m)}\right),
\]

where \(I_1\) assumes only two values \(-1, 1\) with equal probability, \(W_{m,m+1} \sim \text{Beta}(m/2, 1/2)\) and the random variables \(I_1, R_m, U^{(m)}, W_{m,m+1}\) are mutually independent. Consequently,

\[
\|X_{K_{m+1}}\|^2 = \lambda(m)R_m^2\left[W_{m,m+1} + \frac{\lambda_{m+1}}{\lambda(m)}(1 - W_{m,m+1})\right] \overset{d}{=} \lambda(m)R_m^2\left[1 - (1 - \frac{\lambda_{m+1}}{\lambda(m)})W_{m,m+1}^*\right]
\]

holds with \(W_{m,m+1}^* \sim \text{Beta}(1/2, m/2)\) being independent of \(R_m\). Hence applying again Theorem 4.2 we obtain

\[
P\{\|X_{K_{m+1}}\| > u\sqrt{\lambda(m)}\} = (1 + o(1))(1 - \frac{\lambda_{m+1}}{\lambda(m)})^{-1/2}P\{\|V_K\| > u\}, \quad u \uparrow x_F.
\]

Similarly

\[
\|X_{K_{m+2}}\|^2 \overset{d}{=} \left(\lambda(m)R_m^2[1 - (1 - \frac{\lambda_{m+1}}{\lambda(m)})W_{m,m+1}^*]\right)W_{m+1,m+2}^*\lambda_{m+2}(1 - W_{m+1,m+2}^*),
\]

where \(R_m, W_{m,m+1}^*, W_{m+1,m+2}^*\) are independent and \(W_{m+1,m+2}^* \sim \text{Beta}((m + 1)/2, 1/2)\). Applying Theorem 4.1 we have

\[
P\{\|X_{K_{m+2}}\| > u\sqrt{\lambda(m)}\} = (1 + o(1))(1 - \frac{\lambda_{m+1}}{\lambda(m)})^{-1/2}(1 - \frac{\lambda_{m+2}}{\lambda(m)})^{-1/2}P\{\|V_K\| > u\}, \quad u \uparrow x_F.
\]

The proof follows now by applying Theorem 4.2 iteratively and utilising (2.1).

\begin{minipage}{0.75\textwidth}
\hspace{0.5cm} \textbf{Proof of Corollary 3.2} The proof follows from the result of Theorem 3.1 and the self-neglecting property of \(w\) in (3.3). \hfill \Box
\end{minipage}

\begin{minipage}{0.75\textwidth}
\hspace{0.5cm} \textbf{Proof of Theorem 3.4} The proof follows by the formula (3.11) utilising further Lemma 1 in Hüsler et al. (2002). \hfill \Box
\end{minipage}

\begin{minipage}{0.75\textwidth}
\hspace{0.5cm} \textbf{Proof of Theorem 3.6} The proof follows along the lines of the proof of Theorem 3.1 utilising further Theorem 4.2 and making use of the asymptotic condition (2.8). \hfill \Box
\end{minipage}

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