Charged Rotating Kaluza-Klein Black Holes in Dilaton Gravity

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Abstract

We obtain a class of slowly rotating charged Kaluza-Klein black hole solutions of the five-dimensional Einstein-Maxwell-dilaton theory with arbitrary dilaton coupling constant. At infinity, the spacetime is effectively four-dimensional. In the absence of the squashing function, our solution reduces to the five-dimensional asymptotically flat slowly rotating charged dilaton black hole solution with two equal angular momenta. We calculate the mass, the angular momentum and the gyromagnetic ratio of these rotating Kaluza-Klein dilaton black holes. It is shown that the dilaton field and the non-trivial asymptotic structure of the solutions modify the gyromagnetic ratio of the black holes. We also find that the gyromagnetic ratio crucially depends on the dilaton coupling constant, $\alpha$, and decreases with increasing $\alpha$ for any size of the compact extra dimension.

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I. INTRODUCTION

The study of black holes in more than four spacetime dimensions is motivated by several reasons. Strong motivation comes from developments in string/M-theory, which is believed to be the most consistent approach to quantum theory of gravity in higher dimensions. In fact, the first successful statistical counting of black hole entropy in string theory was performed for a five-dimensional black hole [1]. This example provides the best laboratory for the microscopic string theory of black holes. Besides, the production of higher-dimensional black holes in future colliders becomes a conceivable possibility in scenarios involving large extra dimensions and TeV-scale gravity. Furthermore, as mathematical objects, black hole spacetimes are among the most important Lorentzian Ricci-flat manifolds in any dimension. While the non-rotating black hole solution to the higher-dimensional Einstein-Maxwell gravity was found several decades ago [2], the counterpart of the Kerr-Newman solution in higher dimensions, that is, the charged generalization of the Myers-Perry solution [3] in higher dimensional Einstein-Maxwell theory, still remains to be found analytically. Indeed, the case of charged rotating black holes in higher dimensions has been discussed in the framework of supergravity theories and string theory [4, 5, 6]. Recently, charged rotating black hole solutions in higher dimensions with a single rotation parameter in the limit of slow rotation have been constructed in [7] (see also [8, 9, 10]).

On the other hand, a scalar field called dilaton appears in the low energy limit of string theory. The presence of the dilaton field has important consequences on the causal structure and the thermodynamic properties of black holes. Thus, much interest has been focused on the study of the dilaton black holes in recent years. While exact dilaton black hole solutions of Einstein-Maxwell-dilaton (EMd) gravity have been constructed by many authors (see e.g. [11, 12, 13, 14, 15, 16, 17]), exact rotating dilaton black hole solutions have been obtained only for some limited values of the dilaton coupling constant [18, 19, 20]. For the general dilaton coupling constant, the properties of charged rotating dilaton black holes only with infinitesimally small charge [21] or small angular momentum in four [22, 23, 24] and five dimensions have been investigated [25]. Recently, charged slowly rotating dilaton black hole solutions in the background of AdS spaces have also been constructed in arbitrary dimensions [26, 27, 28, 29].

Most authors have considered mainly asymptotically flat and stationary higher dimen-
sional black hole solutions since they would be idealized models if such black holes are small enough for us to neglect the tension of a brane or effects of compactness of extra dimensions. However, if not so, we should consider the higher dimensional spacetimes which have another asymptotic structure. Therefore, it is also important to study black hole solutions with a wide class of asymptotic structures. Recently, the black object solutions with nontrivial asymptotic structures have been studied by various authors. For example, squashed Kaluza-Klein black hole solutions [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48] asymptote to the locally flat spacetime, i.e., a twisted $S^1$ fiber bundle over the four-dimensional Minkowski spacetime. In other words, the spacetime is effectively four-dimensional at the infinity. The black ring solutions with the same asymptotic structures were also found [49, 50, 51, 52, 53]. As far as we know, charged Kaluza-Klein dilaton black holes with asymptotically locally flat structures have been constructed in the static case only [38]. In the present work we would like to generalize such static dilaton black holes to the rotating ones. We also investigate the properties of these rotating black holes related to the presence of the dilaton field and the difference of the asymptotic structures. Especially, we want to construct a new class of charged rotating squashed Kaluza-Klein black hole solutions in five-dimensional EMd gravity. We shall also study the properties of the solutions in the various limits. Finally, we investigate the effects of the dilaton field and the twisted compact extra dimension on the angular momentum and the gyromagnetic ratio of these rotating black holes.

II. ROTATING DILATON BLACK HOLES WITH SQUASHED HORIZONS

We consider five-dimensional EMd theory with action

$$S = \frac{1}{16\pi} \int_M d^5 x \sqrt{-g} \left( R - 2 \partial_\mu \Phi \partial^\mu \Phi - e^{-2\alpha \Phi} F_{\mu\nu} F^{\mu\nu} \right) - \frac{1}{8\pi} \int_{\partial M} d^4 x \sqrt{-h} \Theta(h),$$

where $R$ is the scalar curvature, $\Phi$ is the dilaton field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and $A_\mu$ is the electromagnetic potential. $\alpha$ is an arbitrary constant governing the strength of the coupling between the dilaton and the Maxwell field. The last term in Eq. (1) is the Gibbons-Hawking boundary term which is chosen such that the variational principle is well-defined. The manifold $M$ has metric $g_{\mu\nu}$ and covariant derivative.
\( \nabla_{\mu}. \Theta \) is the trace of the extrinsic curvature \( \Theta^{ab} \) of any boundary \( \partial M \) of the manifold \( M \), with induced metric \( h_{ab} \). The equations of motion can be obtained by varying the action \( S \) with respect to the gravitational field \( g_{\mu\nu} \), the dilaton field \( \Phi \) and the gauge field \( A_\mu \), which yields the following field equations

\[
R_{\mu\nu} = 2\partial_\mu \Phi \partial_\nu \Phi + 2e^{-2\alpha \Phi} \left( F_{\mu\eta} F_{\nu}^{\eta} - \frac{1}{6} g_{\mu\nu} F_{\lambda\eta} F^{\lambda\eta} \right),
\]

(2)

\[
\nabla^2 \Phi = -\frac{\alpha}{2} e^{-2\alpha \Phi} F_{\lambda\eta} F^{\lambda\eta},
\]

(3)

\[
\partial_\mu (\sqrt{-g} e^{-2\alpha \Phi} F^{\mu\nu}) = 0.
\]

(4)

We would like to find rotating solutions of the above field equations. For small rotation, we can solve Eqs. (2)-(4) to the first order in the angular momentum parameter \( a \). Inspection of the slowly rotating black hole solutions [29] shows that the only terms in the metric that change to the first order of the angular momentum parameter \( a \) are \( g_{t\phi} \) and \( g_{t\psi} \). Similarly, the dilaton field does not change to \( O(a) \) and, \( A_\phi \) and \( A_\psi \) are the only components of the vector potential that change. Therefore, for an infinitesimal angular momentum, we assume the metric, the gauge potential and the dilaton field being of the following forms

\[
ds^2 = -u(r) dt^2 + h(r) \left[ w(r) dr^2 + \frac{r^2}{4} \left\{ k(r) d\Omega_2^2 + \sigma_3^2 + 2af(r) dt \sigma_3 \right\} \right],
\]

(5)

\[
A = \frac{\sqrt{3} \sinh(\vartheta) \cosh(\vartheta)}{\sqrt{4 + 3\alpha^2}} \left( \frac{r_+^2(r_\infty^2 - r^2)}{(r_+^2 - r^2) \sinh^2(\vartheta)} + \left( \frac{r_\infty^2(r_\infty^2 - r_+^2)}{(r_\infty^2 - r_+^2)r^2}(r_+^2 \cosh^2(\vartheta) - r_\infty^2) \right) \right) \times \left( dt - \frac{a}{2} \sigma_3 \right),
\]

(6)

\[
\Phi(r) = -\frac{3\alpha}{4 + 3\alpha^2} \ln \left( 1 + \frac{r_+^2}{r_\infty^2 - r_+^2} \frac{r_\infty^2 - r^2}{r^2} \sinh^2(\vartheta) \right),
\]

(7)

where \( \sigma_3 = d\psi + \cos \theta d\phi \), \( d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) denotes the metric of the unit two-sphere, \( r_+ \) and \( r_\infty \) are constants. The functions \( u(r), h(r), w(r), k(r) \) and \( f(r) \) should be determined. In the particular case, \( a = 0 \), this metric (5) reduces to the static Kaluza-Klein dilaton black hole solutions [38].

Here, we are looking for the asymptotically locally flat solutions in the case \( a \neq 0 \). Our strategy for obtaining the solution is the perturbative method proposed by Horne and Horowitz [22]. For small \( a \), we can expect to have solutions with \( u(r), h(r), w(r) \) and \( k(r) \) still functions of \( r \) alone. Inserting metric (5), the gauge potential (6) and the dilaton field
(7) into the field equations (2)–(4), one can show that the static part of the metric leads to the following solutions [38]

\[ u(r) = \frac{1 - \frac{r^2}{r_+^2}}{\left(1 - \frac{r^2}{r_\infty^2}\right) h^2(r)} , \]

\[ h(r) = \left[1 + \frac{\frac{r^2}{r_\infty^2} - \frac{r^2}{r_+^2}}{\frac{r^2}{r_\infty^2} - \frac{r^2}{r_+^2}} \sinh^2(\vartheta)\right]^{\frac{4}{4 + 3 \alpha^2}} , \]

\[ w(r) = \frac{(\frac{r^2}{r_\infty^2} - \frac{r^2}{r_+^2})^2 r_\infty^4}{(\frac{r^2}{r_\infty^2} - r^2)^4 \left(1 - \frac{r^2}{r_+^2}\right)} , \]

\[ k(r) = \frac{(\frac{r^2}{r_\infty^2} - \frac{r^2}{r_+^2}) r_\infty^2}{(\frac{r^2}{r_\infty^2} - r^2)^2} , \]

while the rotating part of the metric admits a solution

\[ f(r) = \frac{1}{h^3(r) r^4 (2 - 3 \alpha^2)(\frac{r^2}{r_\infty^2} - \frac{r^2}{r_+^2})(\frac{r^2}{r_+^2} - \frac{r^2}{r_\infty^2} + \frac{r^2}{r_+^2} \sinh^2(\vartheta)) r_+^2 \sinh^2(\vartheta)} \times \left[((6 \alpha^2 - 4) r_\infty^2 r_+^2 + 12 r_\infty^2 r^2 - (4 + 3 \alpha^2)(r_\infty^4 + r^4)) r_+^4 \sinh^2(\vartheta)\right]^{-1} \]

\[ -(\frac{r^2}{r_\infty^2} - \frac{r^2}{r_+^2})(6 \alpha^2 - 4) r_\infty^2 r_+^2 + 12 r_\infty^2 r^2 - (8 + 6 \alpha^2) r^4) r_+^2 \sinh^2(\vartheta)\]

\[ -(4 + 3 \alpha^2)(\frac{r^2}{r_\infty^2} - \frac{r^2}{r_+^2})^2 r^4] . \]

The coordinates \((t, r, \theta, \phi, \psi)\) run the ranges of \(-\infty < t < \infty, 0 < r < r_\infty, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi\) and \(0 \leq \psi < 4\pi\), respectively. The spacetime (5) has the timelike Killing vector field, \(\partial/\partial t\), and the spacelike Killing vector fields with closed orbits, \(\partial/\partial \phi\) and \(\partial/\partial \psi\). To avoid the existence of naked singularities and closed timelike curves on and outside the black hole horizon, we choose the parameters such that \(0 < r_+ < r_\infty\).

The black hole horizon is located at \(r = r_+\). The induced metric on the three-dimensional spatial cross section of the black hole horizon with the time slice is obtained as

\[ ds^2|_{r=r_+, \ t=\text{const.}} = [\cosh(\vartheta)]^{4 + 3 \alpha^2} \frac{r_+^2}{4} \left[k(r_+) d\Omega_{S^2}^2 + \sigma_3^2\right], \]

which implies the shape of horizon is the squashed \(S^3\), a twisted \(S^1\) fiber bundle over an \(S^2\) base space with the different sizes. We see that the function \(k(r)\) causes the deformation of the black hole horizon. We also note that the rotation parameter \(a\) has no contribution to the shape of the horizon, in contrast to the rotating squashed Kaluza-Klein black hole solutions in [30, 37, 40].
III. ASYMPTOTIC STRUCTURE OF THE SOLUTIONS

In the coordinate system \((t, r, \theta, \phi, \psi)\), the metric \(\mathcal{M}\) diverges at \(r = r_\infty\), but we see that this is an apparent singularity and corresponds to the spatial infinity. In order to see this, we introduce the new radial coordinate \(\rho\) given by

\[
\rho = \sqrt{\frac{r_\infty^2 - r_+^2}{2}} \frac{r^2}{r_\infty^2 - r^2}.
\]  

We also define the parameters

\[
\rho_+ = \rho(r_+) = \frac{r_+^2}{2\sqrt{r_\infty^2 - r_+^2}}, \quad \rho_0 = \frac{1}{2}\sqrt{r_\infty^2 - r_+^2}.
\]

In terms of the coordinate \(\rho\), the metric, the gauge potential and the dilaton field can be written as

\[
ds^2 = -U(\rho)dt^2 + V(\rho)d\rho^2 + H(\rho) \left[ \rho(\rho + \rho_0)d\Omega_5^2 + \frac{(\rho_0 + \rho_+)}{\rho + \rho_0} \sigma_3^2 + aF(\rho)d\sigma_3 \right],
\]

\[
A = \frac{\sqrt{2} \sinh(\vartheta) \cosh(\vartheta)}{\sqrt{4 + 3\alpha^2}} \frac{\rho_+(\rho + \rho_0)}{(\rho_0 - \rho_+ \sinh^2(\vartheta))(\rho + \rho_+ \sinh^2(\vartheta))} \left( dt - \frac{a}{2} \sigma_3 \right),
\]

\[
\Phi(\rho) = \frac{-3\alpha}{4 + 3\alpha^2} \ln \left[ 1 + \frac{\rho_+}{\rho} \sinh^2(\vartheta) \right],
\]

where the functions \(U(\rho), V(\rho), H(\rho)\) and \(F(\rho)\) are, respectively, given by

\[
U(\rho) = \left( 1 - \frac{\rho_+}{\rho} \right) \frac{1}{H^2(\rho)},
\]

\[
V(\rho) = H(\rho) \left( \frac{\rho + \rho_0}{\rho - \rho_+} \right),
\]

\[
H(\rho) = \left[ 1 + \frac{\rho_+}{\rho} \sinh^2(\vartheta) \right]^{\frac{4}{4+3\alpha^2}},
\]

\[
F(\rho) = \frac{1}{2(3\alpha^2 - 2) \rho \rho_+ \left( \sinh^2(\vartheta) \rho_+ + \rho_0 \right) \rho_+^3} H^3(\rho) \times \left[ \sinh^2(\vartheta) \left( (4 - 6\alpha^2) \rho + (8 - 3\alpha^2) \rho_0 \right) \right]^{\frac{3}{4+3\alpha^2}}
\]

\[
+ \left( 2(3\alpha^2 - 2) \rho^2 \sinh^2(\vartheta) + (6\alpha^2 + (3\alpha^2 + 4) \sinh^2(\vartheta) - 4) \rho_0^2 \right. 
\]

\[
+ \rho \left( 9\alpha^2 + (3\alpha^2 - 2) \cosh(2\vartheta) + 6 \right) \rho_0 \rho_+^2 
\]

\[
+ \rho_0 \left( \rho \left( -6\alpha^2 + (3\alpha^2 + 4) \text{csch}(\vartheta) + 4 \right) + 12\rho_0 \right) \rho_+ 
\]

\[
+ \left( 3\alpha^2 + 4 \right) \rho^2 \text{csch}^2(\vartheta) \rho_0^2 \right].
\]
The new radial coordinate \( \rho \) runs from 0 to \( \infty \). In the limit \( \rho \to \infty \) (i.e. \( r \to r_\infty \)), the metric (17) reduces to

\[
 ds^2 = -dt^2 + d\rho^2 + \rho^2 d\Omega_{S^2}^2 + \rho_0 (\rho_0 + \rho_+) \sigma_3^2 + a F_\infty dt \sigma_3, \tag{24}
\]

where

\[
 F_\infty = F(\infty) = \frac{(3\alpha^2 + 4) \rho_0 (\rho_+ + \rho_0) \cosh^2(\vartheta) + (3\alpha^2 - 2) \rho_+ [\rho_+ \cosh(2\vartheta) - \rho_+ - 2\rho_0]}{2 (3\alpha^2 - 2) \rho_+ (\sinh^2(\vartheta) \rho_+ - \rho_0)}. \tag{25}
\]

Next, in order to transform the asymptotic frame into the rest frame, we define the coordinate \( \psi_* \) given by

\[
 \psi_* = \psi + \frac{a F_\infty}{2 \rho_0 (\rho_0 + \rho_+)} t. \tag{26}
\]

Then, the metric takes the following asymptotic form

\[
 ds^2 = -dt^2 + d\rho^2 + \rho^2 d\Omega_{S^2}^2 + L^2 \sigma_3^2, \tag{27}
\]

where

\[
 \sigma_3 = d\psi_* + \cos \theta d\phi, \tag{28}
\]

and the size of the extra dimension \( L \) is given by \( L^2 = \rho_0 (\rho_0 + \rho_+) = r_\infty^2/4 \). We see that the spacetime is asymptotically locally flat, i.e., the asymptotic form of the metric is a twisted \( S^1 \) bundle over the four-dimensional Minkowski spacetime.

**IV. VARIOUS LIMITS**

A. \( \alpha \to 0 \)

One may note that in the absence of a non-trivial dilaton, \( \alpha = 0 \), solution (5) reduces to the slowly rotating charged squashed Kaluza-Klein black hole solutions of the five-dimensional Einstein-Maxwell theory with two equal angular momenta. To see this, we introduce a coordinate \( \tilde{r} \) such that

\[
 \tilde{r}^2 = r^2 + \frac{r_+^2}{r_\infty^2 - r_+^2} (r_\infty^2 - r^2) \sinh^2(\vartheta), \tag{29}
\]

and the new parameters

\[
 \tilde{r}_+ = r_+ \cosh(\vartheta), \tag{30}
\]
\[
\tilde{r}_- = \frac{r_+ r_{\infty}}{\sqrt{r_{\infty}^2 - r_+^2}} \sinh(\vartheta), \tag{31}
\]
\[
\tilde{r}_{\infty} = r_{\infty}. \tag{32}
\]

Therefore, the metric and the gauge potential reduce to
\[
ds^2 = -\frac{\tilde{w}(\tilde{r})}{\tilde{w}(\tilde{r}_{\infty})} \, dt^2 + \frac{\tilde{\kappa}(\tilde{r})}{\tilde{w}(\tilde{r})} \, d\tilde{r}^2 + \frac{\tilde{r}_+^2}{4} \left[ \tilde{\kappa}(\tilde{r}) d\Omega^2_{S^2} + \sigma_3^2 + 2a \tilde{f}(\tilde{r}) dt \sigma_3 \right], \tag{33}
\]
\[
\tilde{A} = \frac{\sqrt{3} \tilde{r}_+ \tilde{r}_-}{2 \tilde{r}^2 \sqrt{\tilde{w}(\tilde{r})}} \left( dt - \frac{a}{2} \sigma_3 \right), \tag{34}
\]
where the functions $\tilde{w}(\tilde{r})$, $\tilde{\kappa}(\tilde{r})$ and $\tilde{f}(\tilde{r})$ are now given by
\[
\tilde{w}(\tilde{r}) = \frac{(\tilde{r}_+^2 - \tilde{r}_-^2)(\tilde{r}_+^2 - \tilde{r}_-^2)}{\tilde{r}^4}, \tag{35}
\]
\[
\tilde{\kappa}(\tilde{r}) = \frac{(\tilde{r}_+^2 - \tilde{r}_-^2)(\tilde{r}_{\infty}^2 - \tilde{r}_-^2)}{(\tilde{r}_{\infty}^2 - \tilde{r}_+^2)^2}, \tag{36}
\]
\[
\tilde{f}(\tilde{r}) = \frac{2\tilde{r}_+^4 \tilde{r}_-^2 (\tilde{r}_+^2 \tilde{r}_{\infty}^2 - (\tilde{r}_+^2 + \tilde{r}_-^2) \tilde{r}_{\infty}^2)}{(\tilde{r}_+^2 - \tilde{r}_-^2)(\tilde{r}_{\infty}^2 - \tilde{r}_+^2)\tilde{r}_-^6}, \tag{37}
\]

To avoid the existence of the naked singularities and closed timelike curves on and outside the black hole horizon, we choose the parameters such that $0 < \tilde{r}_- < \tilde{r}_+ < \tilde{r}_{\infty}$. For $\tilde{r}_{\infty} \to \infty$, the solution (33) is just the five-dimensional slowly rotating Kerr-Newman black hole with two equal angular momenta [4]. When the rotation parameter vanishes, $a \to 0$, the solution (33) coincides with the five-dimensional charged static Kaluza-Klein black hole with squashed horizons [35]. For $\tilde{r}_- \to 0$, the solution (33) reduces to the vacuum rotating squashed Kaluza-Klein black hole in the limit of slow rotation [30, 37].

\subsection*{B. $r_{\infty} \to \infty$}

For $r_{\infty} \to \infty$, solution (6) reduces to the five-dimensional slowly rotating charged dilaton black hole solution with two equal angular momenta [29]. To show this, we define the new parameters
\[
\tilde{\alpha} = \alpha, \tag{38}
\]
\[
\tilde{a} = \frac{a}{2}, \tag{39}
\]

\[8\]
then, the metric reduces to
\[ ds^2 = -\tilde{U}(\tilde{r})dt^2 + \frac{d\tilde{r}^2}{W(\tilde{r})} + \tilde{r}^2 \tilde{R}^2(\tilde{r})d\Omega^2_{S^3} + \tilde{a}\tilde{F}(\tilde{r})dt\sigma, \quad (40) \]
where \( d\Omega^2_{S^3} = \frac{1}{4} \left( d\Omega^2_{S^2} + \sigma^2_3 \right) \) denotes the metric of the unit three-sphere, and the functions \( \tilde{U}(\tilde{r}), \tilde{W}(\tilde{r}), \tilde{R}(\tilde{r}) \) and \( \tilde{F}(\tilde{r}) \) are now given by
\[ \tilde{U}(\tilde{r}) = \left( 1 - \frac{\tilde{r}^2}{\tilde{r}^2} \right) \left( 1 - \frac{\tilde{r}^2}{\tilde{r}^2} \right)^{\frac{2-\tilde{\alpha}^2}{2+\tilde{\alpha}^2}}, \quad (41) \]
\[ \tilde{W}(\tilde{r}) = \left( 1 - \frac{\tilde{r}^2}{\tilde{r}^2} \right) \left( 1 - \frac{\tilde{r}^2}{\tilde{r}^2} \right)^{\frac{2+\tilde{\alpha}^2}{2-\tilde{\alpha}^2}}, \quad (42) \]
\[ \tilde{R}(\tilde{r}) = \left( 1 - \frac{\tilde{r}^2}{\tilde{r}^2} \right)^{\frac{\tilde{\alpha}^2}{2(\tilde{\alpha}+\tilde{\alpha}^2)}}, \quad (43) \]
\[ \tilde{F}(\tilde{r}) = \left( 1 - \frac{\tilde{r}^2}{\tilde{r}^2} \right)^{\frac{2-\tilde{\alpha}^2}{2+\tilde{\alpha}^2}} \left[ \frac{2 + \tilde{\alpha}^2}{1 - \tilde{\alpha}^2} \left( 1 - \frac{\tilde{r}^2}{\tilde{r}^2} \right)^{\frac{2(\tilde{\alpha}^2-1)}{2+\tilde{\alpha}^2}} \right] + 2 \left( 1 - \frac{\tilde{r}^2}{\tilde{r}^2} \right). \quad (44) \]
The spacetime (40) asymptotes to the five-dimensional Minkowski spacetime at infinity. In the absence of a dilaton field, \( \tilde{\alpha} = 0 \), the metric (40) reduces to Eq. (33) with \( \tilde{r}_\infty \to \infty \) [7]. When \( \tilde{\alpha} \to 0 \), the solution (40) coincides with the five-dimensional charged static dilaton black hole [12].

C. \( \rho_0 \to 0 \)

We consider the limit \( \rho_0 \to 0 \) with \( \rho_+ \) finite. We introduce the coordinates \( (t', \psi') \) and the parameter \( \alpha' \) defined as
\[ t' = t - \frac{\alpha}{2} \psi, \quad \psi' = \sqrt{\rho_0(\rho_0 + \rho_+)} \psi, \quad \alpha' = \frac{\alpha}{\sqrt{\rho_0(\rho_0 + \rho_+)}}. \quad (45) \]
Then, the metric (17) reduces to
\[ ds^2 = -H^{-2}(\rho) \left( 1 - \frac{\rho_+}{\rho} \right) dt'^2 + H(\rho) \left[ \left( 1 - \frac{\rho_+}{\rho} \right)^{-1} d\rho^2 + \rho^2 d\Omega^2_{S^2} + d\psi'^2 \right], \quad (46) \]
where the function \( H(\rho) \) is given by Eq.(22). This metric (46) coincides with that of charged static dilaton black strings [54].
V. PHYSICAL QUANTITIES

In this section we would like to calculate the mass, the angular momentum and the
gyromagnetic ratio of these rotating Kaluza-Klein dilaton black holes. Starting with (17),
after a few calculations, the Komar mass \( M \) associated with the timelike Killing vector field \( \partial/\partial t \) at infinity and the Komar angular momentumss \( J_\phi \) and \( J_\psi \) associated with the spacelike Killing vector fields \( \partial/\partial \phi \) and \( \partial/\partial \psi \) at infinity can be obtained as

\[
M = \frac{3\pi \rho_+ \sqrt{(\rho_0 + \rho_+)(3\alpha^2 + 4\cosh(2\vartheta))}}{8 + 6\alpha^2},
\]

\[J_\phi = 0,\]

\[
J_\psi = a\pi \rho_+ \sqrt{\rho_0(\rho_0 + \rho_+)} \times \left[ \frac{(5 - 3\alpha^2)\rho_+ - 9\alpha^2\rho_+ +((-8 + 3\alpha^2)\rho_+ - (16 + 3\alpha^2)\rho_0)\cosh(2\vartheta) + 3\rho_+ \cosh(4\vartheta)}{2(4 + 3\alpha^2)(-\rho_- - 2\rho_0 + \rho_+ \cosh(2\vartheta))} \right].
\]

We see that the spacetime (17) has only one angular momentum in the direction of the extra
dimension. For \( r_\infty \rightarrow \infty \), in terms of the parameters \( \tilde{a}, \tilde{r}_\pm \) and \( \tilde{\alpha} \), the mass (47) and the
angular momentum (49) reduce to

\[
M = \frac{3\pi [(2 + \tilde{\alpha}^2)\tilde{r}_+^2 + (2 - \tilde{\alpha}^2)\tilde{r}_-^2]}{8(2 + \tilde{\alpha}^2)},
\]

\[
J = \frac{\pi \tilde{a} [(2 + \tilde{\alpha}^2)\tilde{r}_+^2 + (4 - \tilde{\alpha}^2)\tilde{r}_-^2]}{8(2 + \tilde{\alpha}^2)},
\]

which are the mass and the angular momentum of the five-dimensional charged slowly ro-
tating dilaton black hole with equal rotation parameters (29).

Next, we calculate the gyromagnetic ratio \( g \) of slowly rotating charged Kaluza-Klein dilal-tion black holes. The gyromagnetic ratio is an important characteristic of charged rotating black holes. Indeed, one of the remarkable facts about a Kerr-Newman black hole in four-
dimensional asymptotically flat spacetime is that it can be assigned a gyromagnetic ratio \( g = 2 \), just as an electron in the Dirac theory. It should be noted that, unlike four dimen-sions, the value of the gyromagnetic ratio is not universal in higher dimensions (8). Besides,
scalar fields, such as the dilaton, modify the value of the gyromagnetic ratio of the black
hole and consequently it does not possess the gyromagnetic ratio \( g = 2 \) of the Kerr-Newman
black hole [22]. In our solution, we also expect the modification of the gyromagnetic ratio by the non-trivial Kaluza-Klein asymptotic structure, which is related to the parameter $r_\infty$. The magnetic dipole moment can be defined as

$$\mu = Qa,$$  \hspace{1cm} (52)

where $Q$ denotes the electric charge of the black hole. The gyromagnetic ratio is defined as a constant of proportionality in the equation for the magnetic dipole moment

$$\mu = g\frac{QJ}{2M}.$$  \hspace{1cm} (53)

Substituting $M$ and $J$ from Eqs. (47) and (49), the gyromagnetic ratio can be obtained as

$$g = \frac{6(3\alpha^2 + 4\cosh(2\vartheta))(-\rho_+ - 2\rho_0 + \rho_+ \cosh(2\vartheta))}{\rho_+(5 - 3\alpha^2) - 9\alpha^2\rho_0 + ((-8 + 3\alpha^2)\rho_+ - (16 + 3\alpha^2)\rho_0)\cosh(2\vartheta) + 3\rho_+ \cosh(4\vartheta)}.$$  \hspace{1cm} (54)

In terms of $\tilde{\alpha}$, $\tilde{r}_+$ and $\tilde{r}_\infty$, the gyromagnetic ratio is given by

$$g = \frac{6\left[\left(2 + \tilde{\alpha}^2\right)\tilde{r}_\infty^2 - 4\tilde{r}_+^2\right]\tilde{r}_+^2 + \left(2 - \tilde{\alpha}^2\right)\tilde{r}_+^2\tilde{r}_\infty^2}{2\left(2 + \tilde{\alpha}^2\right)\tilde{r}_\infty^2 - 3\tilde{r}_+^2\right]\tilde{r}_+^2 + \left(4 - \tilde{\alpha}^2\right)\tilde{r}_+^2\tilde{r}_\infty^2}.$$  \hspace{1cm} (55)

We see that the dilaton field and the Kaluza-Klein asymptotic structure modify the value of the gyromagnetic ratio of the five-dimensional Kerr-Newman black hole in the slow rotation limit [7] (see also [55]) through the coupling parameter $\tilde{\alpha}$, which measures the strength of the dilaton-electromagnetic coupling, and the squashing parameter $\tilde{r}_\infty$, which is proportional to the size of compact extra dimension.

In the figure 1 we show the behavior of the gyromagnetic ratio $g$ versus $\tilde{\alpha}$ in the range of parameters $0 < \tilde{r}_- < \tilde{r}_+ < \tilde{r}_\infty$. From this figure 1 we find out that the gyromagnetic ratio decreases with increasing $\tilde{\alpha}$ for any size of the compact extra dimension. In particular, when $r_\infty \to \infty$, the gyromagnetic ratio (55) reduces to

$$g = \frac{6\left[\left(2 + \tilde{\alpha}^2\right)\tilde{r}_\infty^2 - 4\tilde{r}_+^2\right]\tilde{r}_+^2 + \left(2 - \tilde{\alpha}^2\right)\tilde{r}_+^2\tilde{r}_\infty^2}{2\left(2 + \tilde{\alpha}^2\right)\tilde{r}_\infty^2 - 3\tilde{r}_+^2\right]\tilde{r}_+^2 + \left(4 - \tilde{\alpha}^2\right)\tilde{r}_+^2\tilde{r}_\infty^2},$$  \hspace{1cm} (56)

which is the gyromagnetic ratio of the five-dimensional slowly rotating charged dilaton black hole with two equal angular momenta [29]. Moreover, in the absence of a non-trivial dilaton, $\tilde{\alpha} \to 0$, the gyromagnetic ratio (56) reduces to

$$g = 3,$$  \hspace{1cm} (57)

which is the gyromagnetic ratio of the asymptotically flat slowly rotating charged black hole with two equal angular momenta [7].
FIG. 1: The behavior of the gyromagnetic ratio $g$ versus $\tilde{\alpha}$ in various $\tilde{r}_\infty$ for $\tilde{r}_+ = 2\tilde{r}_-$. $\tilde{r}_\infty = 2\tilde{r}_-$ (blue line), $\tilde{r}_\infty = 2.5\tilde{r}_-$ (green line), $\tilde{r}_\infty = 3.5\tilde{r}_-$ (red line), and $\tilde{r}_\infty \to \infty$ (black line).

VI. SUMMARY AND DISCUSSION

To sum up, we have obtained a class of slowly rotating charged Kaluza-Klein black hole solutions of Einstein-Maxwell-dilaton theory with arbitrary dilaton coupling constant in five dimensions. Our investigations are restricted to black holes with two equal angular momenta. At infinity, the metric asymptotically approaches a twisted $S^1$ bundle over the four-dimensional Minkowski spacetime. Our strategy for obtaining this solution is the perturbative method proposed by Horne and Horowitz \cite{22} and solving the equations of motion up to the linear order of the angular momentum parameter. We have started from the non-rotating charged Kaluza-Klein dilaton black hole solutions in five dimensions \cite{38}. Then, we have considered the effect of adding a small amount of rotation parameter $a$ to the solution. We have discarded any terms involving $a^2$ or higher powers in $a$. Inspection of the known rotating black hole solutions shows that the only terms in the metric which change to $O(a)$ are $g_{t\phi}$ and $g_{t\psi}$. Similarly, the dilaton field does not change to $O(a)$. In the absence of the dilaton field ($\alpha = 0$), our solution reduces to the slowly rotating charged Kaluza-Klein black hole solution. We have calculated the angular momentum $J$ and the gyromagnetic ratio $g$ which appear up to the linear order of the angular momentum parameter $a$. Interestingly enough, we found that the dilaton field and the Kaluza-Klein asymptotic structure modify
the value of the gyromagnetic ratio $g$ through the coupling parameter $\alpha$, which measures the strength of the dilaton-electromagnetic coupling, and the squashing parameter $r_\infty$, which is proportional to the size of compact extra dimension. We have also seen that the gyromagnetic ratio crucially depends on the dilaton coupling constant and decreases with increasing $\alpha$ for any size of the compact extra dimension.

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