Space-time orientations, electrodynamics, antiparticles

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Abstract. Two definitions of orientation in space-time are introduced. One is a standard definition found for example in [2] and [3]. The other is a new definition based on the Minkowski geometry of space-time. Parities of differential forms appearing in electrodynamics are analysed. Parities of differential forms based on the standard concept of orientation are those introduced by de Rham [4]. Parities based on the relativistic concept of orientation are the intrinsic space-time version of parities normally assigned to electromagnetic objects in texts on electrodynamics. Such assignments are made by Jackson [5] and also by Landau and Lifshitz [6]. We present two formulations of the dynamics of charged particles corresponding to the two assignments of parities to electromagnetic objects. One is due to Stückelberg [7] and Feynman [8]. The other is an attempt to formulate a classical theory corresponding to Dirac’s quantum interpretation of antiparticles [9] following the publications [1].

1. Orientations of vector spaces
1.1. Standard orientations of vector spaces
Let $V$ be a vector space of dimension $m \neq 0$. We denote by $F(V)$ the space of linear isomorphisms from $V$ to $\mathbb{R}^m$ called frames. Let $G(V)$ be the group of linear automorphisms of $V$. There is a natural group action

$$G(V) \times F(V) \to F(V); (\rho, \xi) \mapsto \xi \circ \rho^{-1}$$

and $F(V)$ is a homogeneous space with respect to this action.

The sets

$$C^E(V) = \{ \rho \in G(V); \det(\rho) > 0 \}$$

and

$$C^P(V) = \{ \rho \in G(V); \det(\rho) < 0 \}$$

are the two connected components of the group $G(V)$. The set $G^E(V) = C^E(V)$ is the component of the unit element. It is a normal subgroup.

The set of orientations $O(V) = F(V)/G^E(V)$ has two elements. This set is a homogeneous space for the quotient group $H(V) = G(V)/G^E(V)$. The sets $C^E(V)$ and $C^P(V)$ are the elements of the quotient group. Symbols $E$ and $P$ will be used to denote these elements. The structure of the group $H(V)$ is simple. The element $E = C^E(V)$ is the unit and the element $P = C^P(V)$ is an involution.
There is an ordered base \((e_1, e_2, \ldots, e_m)\) of \(V\) associated with each frame \(\xi\). If 

\[
\xi(v) = \begin{pmatrix} v^1 \\ \vdots \\ v^m \end{pmatrix},
\]

then \(v = e_\kappa v^\kappa\). For each \(\rho \in G(V)\) the base \((\rho(e_1), \rho(e_2), \ldots, \rho(e_m))\) is associated with the frame \(\xi \circ \rho^{-1}\) if \((e_1, e_2, \ldots, e_m)\) is the base associated with \(\xi\).

### 1.2. Orientations of subspaces

**Inner orientations** of a subspace \(W \subset V\) are elements of the set \(\mathcal{O}_W\). Elements of the set \(\mathcal{O}(V/W)\) are the **outer orientations** of \(W\). An outer orientation \(o''\) of \(W\) can be determined by specifying an inner orientation \(o\) of \(W\) together with an orientation \(o'\) of \(V\). Let \((e_1, \ldots, e_n)\) be the base of \(W\) associated with a frame \(\xi \in o\). This base can be completed to a base \((e'_1, \ldots, e'_m)\) of \(V\) with \((e'_1, \ldots, e'_n) = (e_1, \ldots, e_n)\). The extended base can be chosen to be associated with a frame \(\xi' \in o'\). Let 

\[
\pi: V \rightarrow V/W
\]

be the canonical projection. The sequence

\[
(e''_1, \ldots, e''_{m-n}) = (\pi(e'_{n+1}), \ldots, \pi(e'_m))
\]

is a base of \(V/W\). It determines an orientation \(o''\) of \(V/W\). Hence an outer orientation of \(W\). The outer orientation \(o''\) of \(W\) constructed from \(o \in \mathcal{O}(W)\) and \(o' \in \mathcal{O}(V)\) is the same as the orientation constructed from \(P o\) and \(P o'\).

The subspace \(W = \{0\}\) has no inner orientations. Its outer orientations are the orientations of \(V\). The specification of an outer orientation as a pair of orientations can not be applied.

In the case of the subspace \(W = V\) the quotient space \(V/W\) is of dimension 0. The subspace \(W\) has no outer orientation defined as an orientation of \(V/W\).

### 1.3. Orientation in the Minkowski space-time

Let \(V\) be a vector space of dimension 4 with a Minkowski metric \(g: V \rightarrow V^*\) of signature \((1, 3)\). The **Lorentz group** for this space is the group of linear automorphisms

\[
G(V, g) = \{\rho \in G(V); \ \rho^* \circ g \circ \rho = g\}.
\]

A linear automorphism

\[
\eta: V \rightarrow \mathbb{R}^4
\]

is called a **Lorentz frame** if

\[
(\eta^* \circ g \circ \eta)^{-1} \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} = (v^0, -v^1, -v^2, -v^3)
\]

for each vector

\[
\begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \in \mathbb{R}^4.
\]
We denote by $F(V,g)$ the space of Lorentz frames. This space is a homogeneous space for the group $G(V,g)$ with the natural group action

$$G(V,g) \times F(V,g) \to F(V,g) : (\rho, \xi) \mapsto \xi \circ \rho^{-1}. \quad (11)$$

The light cone

$$LC = \{ v \in V; \langle g(v), v \rangle = 0 \} \quad (12)$$
divides the space $V$ in three disjoint connected regions. There is the region

$$SP = \{ v \in V; \langle g(v), v \rangle < 0 \} \quad (13)$$
of space-like vectors and the region

$$TI = \{ v \in V; \langle g(v), v \rangle > 0 \} \quad (14)$$
of time-like vectors. This region is the union of two disjoint regions $TI_1$ and $TI_2$.

The group $G(V,g)$ has four connected components:

$$C^E(V,g) = \{ \rho \in G(V,g); \det(\rho) = 1, \rho(TI_1) = TI_1 \} \quad (15)$$
$$C^T(V,g) = \{ \rho \in G(V,g); \det(\rho) = -1, \rho(TI_1) = TI_2 \} \quad (16)$$
$$C^S(V,g) = \{ \rho \in G(V,g); \det(\rho) = -1, \rho(TI_1) = TI_1 \} \quad (17)$$
and

$$C^{TS}(V,g) = \{ \rho \in G(V,g); \det(\rho) = 1, \rho(TI_1) = TI_2 \} \quad (18)$$
The component of the unit element $C^E(V,g)$ is a normal subgroup denoted by $G^E(V,g)$.

The set of orientations

$$O(V,g) = F(V,g)/G^E(V,g) \quad (19)$$

has four elements. This set is a homogeneous space for the quotient group $H(V,g) = G(V,g)/G^E(V,g)$. The quotient group is commutative. Its elements are the four components $C^E(V,g), C^T(V,g), C^S(V,g)$, and $C^{TS}(V,g)$ denoted simply by $E, T, S$, and $TS$ respectively. The element $E = C^E(V,g)$ is the unit and all elements are involutions. The composition rule of $T$ with $S$ is incorporated in the notation used.

There is an ordered base $(u_0, u_1, u_2, u_3)$ of $V$ associated with each frame $\eta$. If

$$\eta(v) = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}, \quad (20)$$

then $v = u_\nu v^\nu$. For each $\rho \in G(V,g)$ the base $(\rho(u_0), \rho(u_1), \rho(u_2), \rho(u_3))$ is associated with the frame $\eta \circ \rho^{-1}$ if $(u_0, u_1, u_2, u_3)$ is the base associated with $\eta$. Orthonormality relations

$$\langle g(u_\kappa), u_\lambda \rangle = \begin{cases} 1, & \text{if } \lambda = \kappa = 0 \\ -1, & \text{if } \lambda = \kappa \neq 0 \\ 0, & \text{if } \lambda \neq \kappa \end{cases} \quad (21)$$
follow from (9).
2. Differential forms and Maxwell’s equations

2.1. Differential forms in affine spaces with standard orientations

Let $M$ be an affine space modelled on a vector space $V$. A differential $q$-form on $M$ is a differentiable function

$$A : M \times V^q \times \mathcal{O}(V) \to \mathbb{R}$$

 depending on a point, $q$ vectors and an orientation. It is $q$-linear and totally antisymmetric in its vector arguments. A differential form $A$ is said to be even, if

$$A(x, v_1, v_2, \ldots, v_q, Po) = A(x, v_1, v_2, \ldots, v_q, o).$$

It is said to be odd, if

$$A(x, v_1, v_2, \ldots, v_q, Po) = -A(x, v_1, v_2, \ldots, v_q, o).$$

The vector space of even differential $q$-forms will be denoted by $\Phi^0_q(M)$ and space of odd differential $q$-forms will be denoted by $\Phi^1_q(M)$. We will use the symbol $\Phi^q(M)$ to denote either $\Phi^0_q(M)$ or $\Phi^1_q(M)$.

The exterior product of a $q$-form $A$ with a $q'$-form $A'$ is the $(q+q')$-form

$$A \wedge A' : M \times V^{q+q'} \times \mathcal{O}(V) \to \mathbb{R} : (x, v_1, \ldots, v_{q+q'}, o) \mapsto \sum_{\sigma \in S(q+q')} \frac{\text{sgn}(\sigma)}{q!q'!} A(x, v_{\sigma(1)}, \ldots, v_{\sigma(q)}, o) A'(x, v_{\sigma(q+1)}, \ldots, v_{\sigma(q+q')}, o),$$

where $S(q+q')$ denotes the group of permutations of the set $\{1, \ldots, q+q'\}$ of integers. If both forms $A$ and $A'$ are even or both are odd, the product $A \wedge A'$ is even. In other cases the product is odd.

The exterior product is commutative in the graded sense. If $A$ is a $q$-form and $A'$ is a $q'$-form, then

$$A' \wedge A = (-1)^{qq'} A \wedge A'.$$

The exterior product is associative. The relation

$$A \wedge (A' \wedge A'') = (A \wedge A') \wedge A''$$

holds for any three forms $A$, $A'$ and $A''$.

The exterior differential of a $q$-form $A$ is the $(q+1)$-form

$$d^q A : M \times V^{q+1} \times \mathcal{O}(V) \to \mathbb{R} : (x, v_1, v_2, \ldots, v_{q+1}, o) \mapsto -\sum_{i=1}^{q+1} (-1)^i \left. \frac{d}{ds} A(x + sv_i, v_1, v_2, \ldots, \hat{v}_i, \ldots, v_{q+1}, o) \right|_{s=0}. $$

The parity of the differential $d^q A$ is the same as the parity of the original form $A$. The operator $d$ is a differential in the sense that $ddA = 0$ for each form $A$. 

A form $A$ is said to be \textit{closed} if $dA = 0$. It is said to be \textit{exact} if there is a form $B$ such that $a = dB$. The \textit{Poincaré lemma} states that in an affine space each closed form is exact.

Given a vector field $X : M \to V$ and a form $A \in \Phi^q_p(M)$ we construct forms

$$i_X A : M \times V^{q-1} \times O(V) \to \mathbb{R} : (x, v_1, \ldots, v_{q-1}, o) \mapsto A(x, X(x)v_1, \ldots, v_{q-1}, o)$$

and

$$d_X A = i_X dA + di_X A.$$  

The operator $d_X$ is the \textit{Lie derivative} and can be defined in terms of the one parameter group of diffeomorphisms generated by $X$.

A $q$-form $A$ can be interpreted as a mapping

$$\tilde{A} : M \to \wedge_q^p V^*.$$  

The relation between the form $A$ and the mapping $\tilde{A}$ is expressed by

$$\tilde{A}(x)(v_1, \ldots, v_q, o) = A(x, v_1, \ldots, v_q, o)$$

The exterior product and the exterior differential are extended to this alternative interpretation of forms. Notation

$$\tilde{A} \wedge \tilde{A}' = A \wedge A'$$

and

$$d\tilde{A} = d\tilde{A}$$

will be used.

Electrodynamic phenomena are described in terms of the following geometric quantities:

1) an even 1-form $A$ called the \textit{potential},
2) an even 2-form $F$ called the \textit{electromagnetic field},
3) an odd 2-form $G$ called the \textit{electromagnetic induction},
4) an odd 3-form $J$ called the \textit{current}.

The field $F$ is the exterior differential $dA$ of the potential.

\section*{2.2. Differential forms in the Minkowski space-time}

A relativistic differential $q$-form on the Minkowski space-time $M$ is a differentiable function

$$A : M \times V^q \times O(V) \to \mathbb{R}.$$  

It is $q$-linear and totally antisymmetric in its vector arguments. A differential form $A$ is said to have \textit{even temporal parity} if

$$A(x, v_0, v_1, v_2, v_3, To) = A(x, v_0, v_1, v_2, v_3, o).$$

It is said to have \textit{odd temporal parity}, if

$$A(x, v_0, v_1, v_2, v_3, To) = -A(x, v_0, v_1, v_2, v_3, o).$$

It is said to have \textit{even spatial parity}, if

$$A(x, v_0, v_1, v_2, v_3, So) = A(x, v_0, v_1, v_2, v_3, o).$$
It is said to have odd spatial parity, if
\[ A(x, v_0, v_1, v_2, v_3, s_0) = -A(x, v_0, v_1, v_2, v_3, o). \]  
(40)

For each degree \( q \) there are four spaces \( \Phi_{e,e}^q(M), \Phi_{o,e}^q(M), \Phi_{e,o}^q(M), \) and \( \Phi_{o,o}^q(M) \) of forms of different parities. The first of the two subscripts identifies the temporal parity and the second identifies the spatial parity of the forms. The parities of the exterior product of forms are listed in the following table.

|          | \( \Phi_{e,e}^{q_1} \) | \( \Phi_{o,e}^{q_2} \) | \( \Phi_{e,o}^{q_1} \) | \( \Phi_{o,o}^{q_2} \) |
|----------|-------------------------|-------------------------|-------------------------|-------------------------|
| \( \Phi_{e,e}^{q_1} \) | \( \Phi_{o,e}^{q_1+q_2} \) | \( \Phi_{e,o}^{q_1+q_2} \) | \( \Phi_{e,e}^{q_1} \) | \( \Phi_{o,o}^{q_2} \) |
| \( \Phi_{o,e}^{q_1} \) | \( \Phi_{o,e}^{q_1+q_2} \) | \( \Phi_{o,o}^{q_1+q_2} \) | \( \Phi_{e,e}^{q_1} \) | \( \Phi_{o,o}^{q_2} \) |
| \( \Phi_{e,o}^{q_1} \) | \( \Phi_{o,e}^{q_1+q_2} \) | \( \Phi_{e,o}^{q_1+q_2} \) | \( \Phi_{o,e}^{q_1} \) | \( \Phi_{o,o}^{q_2} \) |
| \( \Phi_{o,o}^{q_1} \) | \( \Phi_{o,e}^{q_1+q_2} \) | \( \Phi_{o,o}^{q_1+q_2} \) | \( \Phi_{o,e}^{q_1} \) | \( \Phi_{o,o}^{q_2} \) |

The exterior differential of a form preserves the parity of the original form.

In terms of relativistic orientations the electrodynamic phenomena are described by the following geometric quantities:
1) the potential \( A \in \Phi_{o,e}^2(M) \),
2) the electromagnetic field \( F = dA \in \Phi_{o,e}^2(M) \),
3) the electromagnetic induction \( G \in \Phi_{e,o}^2(M) \),
4) the current \( J \in \Phi_{e,e}^1(M) \).

2.3. The metric volume in a Minkowski space
Let \( g: V \rightarrow V^* \) be metric tensor of signature \((1,3)\) in a vector space \( V \) of dimension 4. We define an odd 4-covector
\[ \sqrt{|g|}: V^4 \times O(V) \rightarrow \mathbb{R} \]  
(41)

by the formula
\[ \sqrt{|g|}(v_1, v_2, v_3, v_4, o) = \pm \sqrt{\det((g(v_\lambda), v_\lambda))}. \]  
(42)

If vectors \((v_1, v_2, v_3, v_4)\) are dependent, then \(\det((g(v_\lambda), v_\lambda)) = 0\). If the vectors are independent, then they determine an orientation \(o' \in O(V)\). The sign + in the formula is chosen if the orientations \(o\) and \(o'\) agree. Otherwise the sign − is chosen. It follows from elementary properties of determinants that the formula defines an odd 4-covector.

2.4. Maxwell’s equations
The equations:
\[ dF = 0, \]  
\[ dG = -\frac{4\pi}{c} J, \]  
(43)
(44)
and
\[ G = \left( \wedge^2 g^{-1} \circ \tilde{F} \right) \setminus \sqrt{|g|} \]  
(45)
are the Maxwell’s equations and the constitutive relation. These equations are the same with different assignments of parities of the forms involved.
2.5. Maxwell’s equations in the traditional form

The equations

\[ \frac{1}{c} \partial_t \vec{B} - \nabla \times \vec{E} = 0, \quad (46) \]
\[ \nabla \cdot \vec{B} = 0, \quad (47) \]
\[ \partial_t \vec{D} - \nabla \times \vec{H} = -\frac{4\pi}{c} \vec{J}, \quad (48) \]
\[ \nabla \cdot \vec{D} = 4\pi Q. \quad (49) \]

found in standard texts on electrodynamics are the Maxwell’s equations with the spatial components and the temporal components separated. The parity assignments listed in the following table correspond to the parity assignments listed in Section 6.

| The object     | is identified as | temporal parity |
|----------------|------------------|-----------------|
| The even 1-form \( E \) | a vector field    | \( E \)         | odd             |
| The odd 2-form \( B \)    | a pseudovector field | \( B \)         | even            |
| The odd 1-form \( H \)    | a pseudovector field | \( H \)         | even            |
| The even 3-form \( Q \)    | a scalar field    | \( Q \)         | odd             |
| The even 2-form \( J \)    | a vector field    | \( J \)         | even            |
| The even 0-form \( U \)    | a scalar field    | \( U \)         | odd             |
| The even 1-form \( A \)    | a vector field    | \( A \)         | even            |

These parities are not in agreement with parity assignments found in standard texts such as Classical Electrodynamics by John David Jackson [5] and Field Theory by L. D. Landau and E. M. Lifshitz [6]. The assignments made in these texts correspond to the assignments of Section 7. The assignments are listed below.

**Parities according to Landau and Jackson:**

| The object     | is identified as | temporal parity |
|----------------|------------------|-----------------|
| The even 1-form \( E \) | a vector field    | \( E \)         | even            |
| The odd 2-form \( B \)    | a pseudovector field | \( B \)         | odd             |
| The odd 1-form \( H \)    | a pseudovector field | \( H \)         | odd             |
| The even 3-form \( Q \)    | a scalar field    | \( Q \)         | even            |
| The even 2-form \( J \)    | a vector field    | \( J \)         | odd             |
| The even 0-form \( U \)    | a scalar field    | \( U \)         | even            |
| The even 1-form \( A \)    | a vector field    | \( A \)         | odd             |

3. Dynamics of charged particles

3.1. Equations of motion

Lagrange equation:

\[ \pi(s)(\delta \xi(s), o) = \frac{m}{\sqrt{g(\xi'(s)), \xi'(s))}}(g(\xi'(s)), \delta \xi'(s)) \quad (50) \]

and

\[ \pi'(s)(\delta \xi(s), o) = e(o) F(\xi(s), \xi'(s), \delta \xi(s), o). \quad (51) \]
Euler-Lagrange equation:
\[
\frac{m}{\sqrt{\langle g(\xi'(s)), \xi'(s) \rangle}} \left( \langle g(\xi''(s)), \delta \xi(s) \rangle - \langle g(\xi'(s)), \xi''(s) \rangle \langle g(\xi'(s)), \delta \xi(s) \rangle \right) = e(o) F(\xi(s), \xi'(s), \delta \xi(s), o)
\]
(52)

Reparameterizations:
\[
\xi = \xi \circ \gamma, \quad \pi = \pi \circ \gamma, \quad \gamma : \mathbb{R} \to \mathbb{R}.
\]
(53)

3.2. Antiparticles according to St"uckelberg and Feynman
Following St"uckelberg and Feynman we recognize particles an antiparticles as different states of motion of the same physical object. If a world line with one direction is recognized as describing the motion of a particle, then the world line with the opposite direction describes the motion of an antiparticle. It is impossible to tell which of the two possible directions should qualify a directed world line as that of a particle rather than an antiparticle. According to this interpretation both the electron and the positron are objects of the same negative charge. Reparameterizations with negative derivatives are called direction inverting reparameterizations. Equations of motion are preserved by such reparameterizations if the sign of momentum is inverted. Particle and antiparticle states are interchanged.

This theory of antiparticles fits well the natural choice of parities of electromagnetic objects. The potential \( A \) and the field \( F \) are even forms, the charge \( e \) and the momentum \( \pi \) are even objects. The equations of motion (50) and (51) as well as the equation (52) for world lines are Poincaré invariant. Time reflecting transformations interchange particle and antiparticle states.

3.3. A classical version of Dirac’s theory of antiparticles
We present an attempt to formulate a classical mechanics of antiparticles reflecting the features of Dirac’s theory of positrons. This theory is not fully relativistic since it distinguishes between the future and the past.

Vectors in one of the two disjoint regions \( TI_1 \) or \( TI_2 \) inside the light cone are declared as pointing towards the future. Let \( TI_1 \) be this region. Particles have charge \( e \) and antiparticles have the opposite charge \(-e\) both charges are of parity \( o, e \) equal to the parity of the potential \( A \). World lines of particles and antiparticles are directed towards the future. World lines of observers are also directed into the future and energies of particles and antiparticles observed by such observers are positive.

In order to achieve Poincaré invariance of this scheme the parity of the momentum is set to \( o, e \) and time reflecting transformations are accompanied by direction inverting reparameterizations.

3.4. Closing remarks
When pair creations were observed in the Wilson chamber placed in a magnetic field positive charge had to be assigned to positrons in order to obtain the observed trajectory curvatures from the Lorentz force formula with observed velocities. The concept of an antiparticle as a particle of opposite charge fits well the observed situations. The antiparticle is described as it appears to an observer.

St"uckelberg [7] and Feynman [6] considered trajectories of particles and particles as world lines with opposite orientations. Using in the Lorentz force formula vectors tangent to the world lines compatible with their orientations correct curvatures for particle and antiparticle trajectories with the same charge is obtained. Minkowski [1] considered physical reality as existing in space-time rather than being a succession of spatial configurations evolving in time. We find the St"uckelberg and Feynman interpretation of antiparticle states very much in line with the Minkowski concept of physical reality.
Feynman proposed a quantum theory of antiparticles based on the classical mechanics of antiparticles formulated by Stückelberg. A complete implementation of this theory was never reached. Only the Feynman diagrams and the Feynman propagator survived. Dirac formulated his theory of antiparticles as a quantum theory. We have not found in the literature a complete classical description of antiparticles compatible with Dirac theory. The description we propose is probably not the only possible.

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