Stability Properties of Systems of Linear Stochastic Differential Equations with Random Coefficients

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Abstract

This work is concerned with the stability properties of linear stochastic differential equations with random (drift and diffusion) coefficient matrices, and the stability of a corresponding random exponential semigroup. We consider a class of random matrix drift coefficients that involves random perturbations of an exponentially stable flow of deterministic (time-varying) drift matrices. In contrast with more conventional studies, our analysis is not based on the existence of Lyapunov functions and it does not draw on any ergodic properties. These approaches are often difficult to apply in practice when the drift/diffusion coefficients are random. We present rather weak and easily checked perturbation-type conditions for the asymptotic stability properties of time-varying and random linear stochastic differential equations. We provide new log-Lyapunov estimates and exponential contraction inequalities on any time horizon as soon as the fluctuation parameter is sufficiently small. This study yields what seems to be the first result of this type for this class of linear stochastic differential equations with random coefficient matrices.

1 Introduction

This work is concerned with the stability of a system of linear stochastic differential equations (i.e. Ito-type diffusion equations driven by a Wiener process), with the added complexity of time-varying random coefficients. That is, we consider the stability of systems of the form

\[ dX_t^\epsilon = A_t^\epsilon X_t^\epsilon \, dt + (B_t^\epsilon)^{1/2} \, dW_t \]

where \( W_t \) an \( r \)-dimensional Wiener process and \( (A_t^\epsilon, B_t^\epsilon) \) are suitably defined (non-anticipating) random processes indexed by some parameter \( \epsilon \in [0,1] \) and independent of \( (X_0^\epsilon, W_t) \). Detailed technical models are given later. We term these processes: random linear stochastic differential equations, random linear diffusions, or random Ornstein-Ulhenbeck processes\textsuperscript{2}. The regularity and stability properties of this process are far from obvious.

The stability properties of Ito-type diffusion equations with deterministic coefficients (i.e. Wiener processes with non-zero deterministic drift) has been well studied, and we point to the texts

\textsuperscript{1}The “random” predicate is used to denote the randomness of the coefficients. In the case of deterministic coefficients, we term this equation: a stochastic differential equation, or an Ito-type (linear) diffusion process (driven by a Wiener process). Note that Ito-type integrals do not preclude (suitably adapted) random integrands; however, our terminology is used to distinguish those equations with random coefficients and those without.

\textsuperscript{2}“Random” is again used to describe the randomness of the coefficients and we refer to this process as an Ornstein-Ulhenbeck process since we are concerned ultimately with stable processes in an Ornstein-Ulhenbeck-type form.
for a detailed survey of results. Rather independently, systems of ordinary differential equations with random coefficients (i.e. systems of the form above with $B_t' = 0$) have also been considered, and we point to [2, 19, 18, 21, 22, 40, 8, 15, 35] for a collection of results and techniques. Stability results for these latter-type of equations are often difficult to apply in practice. Stability of the special case concerning ordinary differential equations with piece-wise constant (random) coefficients was studied in, e.g., [3, 29] and is applicable to stochastic jump systems [13]. These reference lists are not exhaustive (see also the references therein). We return to some of this literature later, with a more specific technical relationship to our current work.

Combined diffusion-type equations with random coefficients are more general, and stability results are difficult to apply in practice (owing a lot again to the randomness in the coefficients). Nevertheless, these types of model appear (in some fashion) in economics and finance [41, 24], biology [10, 39], mechanics and physics [36], etc. And in mathematical control and systems theory.

Early work by Bismut [6] considered the linear-quadratic optimal control of very general (linear) systems of this form, and in which the control may act on both the drift and the diffusion. Optimal control in this framework is complicated by the need to address the well-posedness of certain backward stochastic differential equations with (the added complexity of) random coefficients. Much work has been considered in this direction and it is mostly beyond the scope of this article; we point to [44, 37, 38, 34] and the references therein.

Nonlinear analogues of this model (i.e. nonlinear stochastic differential equations driven by a Wiener process with random drift/diffusion functions or random inputs to the drift/diffusion) are also of interest, e.g. [36, 40, 33, 44], albeit they are beyond the scope of this work.

Our motivation comes from signal processing, and more particularly the analysis of the long time behaviour of ensemble Kalman-Bucy filters [12]. These particle-type Kalman filters can be viewed as a collection of interacting particles driven by Kalman-Bucy-type update diffusions with an interaction function (i.e. the Kalman gain function) that depends on the sample covariance matrix. That is, the drift coefficient matrices of these time-varying systems depends on the sample covariance matrices of the model (which is itself a random process). Under some natural observability and controllability conditions, the sample covariance matrices can be made as close as needed to the solution of a stable Riccati equation [5, 11]. Then, the difference between the particle sample means and the true signal state can be expressed in terms of random Ornstein-Ulhenbeck processes. In this work, we study the stability of the resulting random Ornstein-Ulhenbeck processes. A related study in one-dimension is given in [4] where stronger results are available.

1.1 Models and Notation

We denote by $M_r = \mathbb{R}^{r \times r}$ the set of $(r \times r)$-square matrices with real entries and $r \geq 1$. We let $S_r \subset M_r$ denote the subset of symmetric matrices, $S_0^r \subset S_r$ the subset of positive semi-definite matrices, and $S_0^+ \subset S_0^r$ the subset of positive definite matrices. Given $B \in S_0^r - S_0^+$ we denote by $B^{1/2}$ a (non-unique) but symmetric square root of $B$ (given by a Cholesky decomposition). When $B \in S_0^+$ we always choose the principal (unique) symmetric square root. We write $A'$ to denote the transposition of a matrix $A$, and $A_{sym} = (A + A')/2$ to denote the symmetric part of $A \in M_r$.

We equip the set $M_r$ with the spectral norm $\|A\| := \|A\|_2 = \sqrt{\lambda_1(AA')}$ where $\lambda_1(\cdot)$ denotes the maximal eigenvalue. Let $tr(A) = \sum_{1 \leq i \leq r} A(i, i)$ denote the trace operator. We also denote by $\mu(A) = \lambda_1(A_{sym})$ its logarithmic norm.

Throughout the remainder, $A : t \in \mathbb{R}_+ := [0, \infty[ \rightarrow A_t \in M_r$ denotes some deterministic flow of matrices satisfying the following condition:
Hypothesis 0 \((H_0)\).

\[
\|A_t - A_\infty\| \leq a e^{-bt} \text{ for some } A_\infty \in \mathcal{M}_r \quad \text{s.t. } \mu(A_\infty) < 0
\]

for any time horizon \(t \geq 0\) and some parameters \(a < \infty\) and \(b > 0\). We set \(c_0 := |\mu(A_\infty)|\) throughout.

Let \(\mathcal{E}_{s,t}(A)\) be the exponential semigroup (a.k.a. propagator, or the state transition matrix) associated with a smooth flow of matrices \(A : t \in \mathbb{R}_+ \mapsto A_t \in \mathcal{M}_r\) defined for any \(s \leq t\) by the forward and backward differential equations,

\[
\partial_t \mathcal{E}_{s,t}(A) = A_t \mathcal{E}_{s,t}(A) \quad \text{and} \quad \partial_s \mathcal{E}_{s,t}(A) = -\mathcal{E}_{s,t}(A) A_s
\]

with \(\mathcal{E}_{s,s}(A) = \text{Id}\), the identity matrix (of appropriate dimension). Equivalently in terms of the matrices \(\mathcal{E}_t(A) := \mathcal{E}_{0,t}(A)\) we have \(\mathcal{E}_{s,t}(A) = \mathcal{E}_t(A) \mathcal{E}_{s,0}(A)^{-1}\).

For any \(s, t > 0\) we recall the logarithmic norm estimate

\[
\log \|\mathcal{E}_{s,s+t}(A)\| \leq \int_s^{s+t} \mu(A_s) \, ds \implies \frac{1}{t} \log \|\mathcal{E}_{s,s+t}(A)\| \leq \mu(A_\infty) + \frac{e^{-bs}}{t} \frac{a}{b} \tag{1}
\]

Hypothesis-Remark 0 \((H'_0)\). With the l.h.s. of this implication \(\Box\) in mind, a straightforward extension would be to consider a relaxing of \((H_0)\) to the case in which \(A_t\) has no fixed point, but is itself just a time-varying stabilising matrix.

We come to the random processes of interest. Let \(A^\epsilon : t \in \mathbb{R}_+ \mapsto A^\epsilon_t \in \mathcal{M}_r\) and \(B^\epsilon : t \in \mathbb{R}_+ \mapsto B^\epsilon_t \in \mathbb{S}^r_{++}\) be a collection of càdlàg random processes defined on a common filtered probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and indexed by some parameter \(\epsilon \in [0, 1]\). Let \(\mathcal{F}^\epsilon_t := \sigma(A^\epsilon_t, B^\epsilon_t), s \leq t\). We require,

\[
\sup_{0 \leq t \leq \epsilon, n} \sup_{t \geq 0} \mathbb{E} (\|A^\epsilon_t\|^n) < \infty \quad \text{and} \quad \rho_n := \sup_{0 \leq t \leq \epsilon, n} \sup_{t \geq 0} \mathbb{E} (\text{tr} (B^\epsilon_t)^n)^{1/n} < \infty \quad \text{for any } n \geq 1 \tag{2}
\]

and for some fluctuation parameters \(\epsilon_{0,n}\) and \(\epsilon_{1,n} \in [0, 1]\).

We let \(X^\epsilon_t\) be the collection of random Ornstein-Uhlenbeck process defined by

\[
\frac{dX^\epsilon_t}{dt} = A^\epsilon_{\infty} X^\epsilon_t \, dt + (B^\epsilon_t)^{1/2} \, dW_t \tag{3}
\]

where \(W_t\) an \(r\)-dimensional Wiener process and we assume that \((X^\epsilon_0, W_t)\) are independent of the stochastic processes \((A^\epsilon_t, B^\epsilon_t)\). We also denote by \(X^\epsilon_{\infty} = x \in \mathbb{R}^r\).

Of course, the independence between \(W_t\) and the stochastic processes \(A^\epsilon_t\) can be relaxed when the process \(B^\epsilon\) is null. Otherwise, this independence is critical for the well-posedness of our results.

The objective of this work is to study the stability properties of the semigroup \(\mathcal{E}_{s,t}(A^\epsilon)\) associated with the stochastic process \(A^\epsilon\), and the stability of the random Ornstein-Uhlenbeck process \((\Box)\).

The analysis of the long time behaviour of the stochastic model \((\Box)\) differs strongly from the analysis of conventional time-invariant and deterministic linear dynamical systems. However, as with general time-varying deterministic linear dynamical systems, the asymptotic behaviour of \((\Box)\) cannot be characterised by the statistical properties of the spectral abscissa of the random matrices \(A^\epsilon_t\). Indeed, unstable semigroups \(\mathcal{E}_{s,t}(A)\) associated with time-varying (deterministic) matrices \(A_t\) with negative eigenvalues are exemplified in \([10, 43]\). Conversely, stable semigroups \(\mathcal{E}_{s,t}(A)\) with \(A_t\) having positive eigenvalues are given by Wu in \([43]\). The same general conclusion holds in the statistical case without (quite strong) additional restrictions on the class of model considered. We seek quite weak and more readily verifiable and practical conditions in this work.
Observe that the solution of (3) is provided by the formula

$$X^\epsilon_t = \mathcal{E}_{s,t}(A^\epsilon)X_0^\epsilon + X_t^{\epsilon,0} \quad \text{with} \quad X_t^{\epsilon,0} = \int_0^t \mathcal{E}_{s,t}(A^\epsilon) (B^\epsilon_s)^{1/2} \, dW_s \quad (4)$$

Note that given $\mathcal{F}_t^\epsilon$, the r.h.s. integral in (4) is an Ito integral since the Brownian motion $(W_s)_{s \leq t}$ is independent of $(A^\epsilon_s, B^\epsilon_s)_{s \leq t}$; i.e. it is non-anticipative due to the independence of the relevant randomness. The process $X_t^{\epsilon,0}$ in (4) is not a martingale and the analysis of its regularity properties is far from obvious.

Note that

$$\mathbb{E}(X^\epsilon_t \mid \mathcal{F}_t^\epsilon) = \mathcal{E}_{s,t}(A^\epsilon)\mathbb{E}(X_0^\epsilon)$$

This implies that the conditional covariance process is given by,

$$\mathbb{E}\left( [X^\epsilon_t - \mathbb{E}(X^\epsilon_t \mid \mathcal{F}_t^\epsilon)][X^\epsilon_t - \mathbb{E}(X^\epsilon_t \mid \mathcal{F}_t^\epsilon)]^\prime \mid \mathcal{F}_t^\epsilon \right)$$

$$= \mathcal{E}_t(A^\epsilon) C_0 \mathcal{E}_t(A^\epsilon)^\prime + \int_0^t \int_0^t \mathcal{E}_{s,t}(A^\epsilon) (B^\epsilon_s)^{1/2} \mathbb{E}(dW_s dW_r) (B^\epsilon_r)^{1/2} \mathcal{E}_{r,t}(A^\epsilon)^\prime \, 1_{(r-s)} \, ds$$

$$= \mathcal{E}_t(A^\epsilon) C_0 \mathcal{E}_t(A^\epsilon)^\prime + \int_0^t \mathcal{E}_{s,t}(A^\epsilon) B^\epsilon_s \mathcal{E}_{s,t}(A^\epsilon)^\prime \, ds$$

Notice that when $B^\epsilon_t = 0$ the Ornstein-Ulhenbeck process (3) resumes to the linear random dynamical equation,

$$\partial_t X^\epsilon_t = A^\epsilon_t X^\epsilon_t \iff X^\epsilon_t = \mathcal{E}(A^\epsilon) X_0^\epsilon \quad (5)$$

whose stability properties have been considered in, e.g., [2] [19] [29] [18] [21] [22] [40] [8] [15]. As previously noted, stability results for systems of the form (5) are often difficult to apply in practice. It is worth noting that the discrete-time version of (5) is given by the recursive equation

$$X^\epsilon_n = A^\epsilon_n X^\epsilon_{n-1} = A^\epsilon_n A^\epsilon_{n-1} \cdots A^\epsilon_1 X_0^\epsilon$$

If $A^\epsilon_n$ is matrix-valued Markov chain, then for ergodic chains, the stability properties of this equation are related to Oseledecc’s multiplicative ergodic theorem [25] [31]. More generally, the stability properties in discrete-time may be related to various theorems concerning the infinite product of stochastic matrices; e.g. see [42] [28] [30]. See also [16] for related results in discrete-time.

Coming to the stability properties of (3). If we assume that for any $s \leq t$ and some $n \geq 4$ we have the exponential estimate

$$\mathbb{E}(\|\mathcal{E}_{s,t}(A^\epsilon)\|^n)^{1/n} \leq \alpha e^{-\beta (t-s)} \quad (6)$$

for some parameters $\alpha < \infty$ and $\beta > 0$. Then, for any $\epsilon \leq \epsilon_{1,2n}$ we find

$$\mathbb{E}(\|X^{\epsilon,x}_t - X^{\epsilon,y}_t\|^n)^{1/n} \leq \alpha e^{-\beta (t-s)} \|x - y\| \quad \text{and} \quad \sup_{t \geq 0} \mathbb{E}\left(\|X^{\epsilon,0}_t\|^{2/n}\right)^{2/n} < \infty \quad (7)$$

In practice, exponential estimates of the form (6) are difficult to obtain mainly because the semigroup $\mathcal{E}_{s,t}(A^\epsilon)$ cannot be represented as elementary matrix exponentials but rather only in terms of Peano-Baker series [32] or sophisticated Magnus exponential series [7] [26]; see also the studies [9] [14] [17]. This elementary result indicates that the stability properties of the random Ornstein-Ulhenbeck process $X^\epsilon_t$ are directly related to the contraction properties of the stochastic semigroup $\mathcal{E}_{s,t}(A^\epsilon)$ with $s \leq t$. This is of course not surprising.
We note that the l.h.s. uniform moment condition in \([2]\) ensures that the random drift process \(A_t^\epsilon\) is uniformly tight, in the sense that for any \(\nu \in [0, 1]\), \(\exists k\) such that \(\sup_{t \geq 0} \mathbb{P}(\|A^\epsilon_t\| \geq k) \leq \nu\). By Prohorov’s theorem this implies that the distributions of the random states \((A^\epsilon_t)_{t \geq 0}\) is relatively compact so there exists at least one limiting invariant distribution \(\pi_\epsilon\) on \(\mathcal{M}_r\). In addition there exists a sequence of random times \(t_n\) such that \(\text{Law}(A^\epsilon_{t_n}) \rightarrow_{n \to \infty} \pi_\epsilon\). The uniqueness property of the invariant measure and the ergodicity properties of the process \((A^\epsilon_t)_{t \geq 0}\) require more sophisticated stochastic analysis.

Assume that the process \(A^\epsilon\) is mean ergodic, in the sense that

\[
\frac{1}{t} \int_0^t \|A^\epsilon_s - \hat{A}_\infty^\epsilon\| \, ds \rightarrow_{t \to \infty} 0 \quad \text{a.s. with} \quad \hat{A}_\infty^\epsilon := \int_{\mathcal{M}_r} \Lambda \pi_\epsilon(d\Lambda) \tag{8}
\]

By the convexity of the maximal eigenvalue functional on symmetric matrices we have the almost sure convergence result

\[
\frac{1}{t} \log \|\mathcal{E}_t(A^\epsilon)\| \leq \frac{1}{t} \int_0^t \mu(A^\epsilon_s) \, ds \rightarrow_{t \to \infty} \mu(\hat{A}_\infty^\epsilon) \leq \int_{\mathcal{M}_r} \mu(\Lambda) \pi_\epsilon(d\Lambda)
\]

This provides a natural condition under which the stochastic semigroup \(\mathcal{E}_t(A^\epsilon)\) is almost surely exponentially stable in terms of the long time behaviour of the process \(A^\epsilon\). Unfortunately in practical situations, the ergodic property \([5]\) of the stochastic matrix valued process \(A^\epsilon\), as well as the condition \(\mu(\hat{A}_\infty^\epsilon) < 0\) can be difficult to check. This approach to stability is related to so-called averaging methods for random (linear) dynamical systems of the form \([5]\); e.g. see \([8, 15, 35]\).

Assume that \(q_1 \text{Id} \leq Q_t \leq q_2 \text{Id}\) is a possibly random solution of the Lyapunov equation

\[
\partial_t Q_t + A_t^\epsilon Q_t + Q_t (A_t^\epsilon)' \leq -q_3 \text{Id}
\]

for some \(q_1, q_2, q_3 > 0\). Then, we have the almost sure contraction estimate

\[
\partial_t [\mathcal{E}_t(A^\epsilon)' Q_t \mathcal{E}_t(A^\epsilon)] \leq -(q_3/q_2) \mathcal{E}_t(A^\epsilon)' Q_t \mathcal{E}_t(A^\epsilon)
\]

\[
\Rightarrow \quad \mathcal{E}_t(A^\epsilon)' \mathcal{E}_t(A^\epsilon) \leq (q_2/q_1) e^{-(q_3/q_2)t} \text{Id}
\]

\[
\Rightarrow \quad \|\mathcal{E}_t(A^\epsilon)\| \leq \alpha e^{-\beta t} \quad \text{with parameters} \quad (\alpha, \beta) = (\sqrt{q_2/q_1}, 2^{-1} q_3/q_2)
\]

Conversely, suppose that \(\|A^\epsilon_t\| \leq \gamma\) is almost surely uniformly bounded and the almost sure contraction estimate just given is satisfied. In this case, for any matrix valued process \(r_1 \text{Id} \leq R_t \leq r_2 \text{Id}\) we have

\[
Q_t := \int_t^\infty \mathcal{E}_{t,u}(A^\epsilon)' R_u \mathcal{E}_{t,u}(A^\epsilon) \, du \leq r_2 \alpha/\beta \quad \Rightarrow \quad \partial_t Q_t + (A_t^\epsilon)' Q_t + Q_t A_t^\epsilon = -R_t \leq -r_1
\]

as well as

\[
\mathcal{E}_{t,s}(A^\epsilon)' \mathcal{E}_{t,s}(A^\epsilon) \geq e^{-\int_{t-s}^{t} \lambda_{\text{max}}(-A^\epsilon)' \text{sym}} \geq e^{-\gamma(t-s)} \quad \Rightarrow \quad Q_t \geq (r_1/\gamma) \text{Id}
\]

This provides sufficient and necessary conditions under which \(\mathcal{E}_t(A^\epsilon)\) is almost surely exponentially stable in terms of the existence of Lyapunov functions. Unfortunately, the design of Lyapunov functions for nonlinear stochastic diffusions \(A^\epsilon\) in matrix spaces is a difficult task. Stability (in the mean) of linear random dynamical equations of the form \([5]\) via Lyapunov methods was considered in early work by Bertram and Sarachik \([2]\) and in \([19]\). However, application of this method in the mean is also typically not practical \([2, 21]\).

The aim of this work is to provide some quantitative stability properties under weaker conditions. Namely, we shall consider the following regularity conditions:
Hypothesis 1 \((H_1)\). Suppose \(H_0\) holds and,
\[
\forall n \geq 1 \quad \exists \epsilon_n \in [0, 1] \quad \text{such that} \quad \forall \epsilon \in [0, \epsilon_n] \quad \text{it follows that} \quad \sup_{t \geq 0} \mathbb{E} (\|A_t - A^\epsilon_t\|^n) \leq c_n \epsilon
\]
We also define throughout \(\epsilon_n(\nu) := \epsilon_n \wedge [\nu^{1/n} c_0/(4 c_n)]\) indexed by \(n \geq 1\) and \(\nu \in [0, 1]\).

Hypothesis 2 \((H_2)\). Suppose \(H_0\) holds and,
\[
\forall n \geq 1 \quad \text{and} \quad \forall \epsilon \in [0, 1] \quad \text{we have the uniform estimates}
\sup_{t \geq 0} \mathbb{E} (\|A_t - A^\epsilon_t\|^n) \leq c_n \epsilon \quad \text{and} \quad \sup_{t \geq 0} \mathbb{E} (\|A^\epsilon_t\|^n) \leq d_1 + \epsilon d_2 n^{1/2}
\]

In the above hypotheses, \(c_n\) and \(d_n\) correspond to a non-decreasing collection of finite non-negative constants. Observe that \((H_2) \implies (H_1)\). For a deterministic flow of perturbed matrices \(A^\epsilon_t\), these two conditions coincide as soon as the l.h.s. condition in \((2)\) is met with \(\epsilon = 0\). For a deterministic flow of perturbed matrices \(A^\epsilon_t\), these two conditions coincide as soon as the l.h.s. condition in \((2)\) is met with \(\epsilon = 0\). Unfortunately, apart from very particular cases (e.g. time-invariant flows, or time-varying commuting matrices) we are aware of no general relaxation to the log-norm condition in the time-varying setting.

Hypothesis-Remark 0 \((H'_0)\). Conditions \((H_1)\) and \((H_2)\) are weak conditions on the moment continuity (not the exponential moments) between \(A_t\) and \(A^\epsilon_t\) in terms of \(\epsilon\). It is possible to relax the strong log-norm condition in \((H_0)\) whenever we have a stability estimate (e.g. like (1)) that respects these continuity properties. Unfortunately, apart from very particular cases (e.g. time-invariant flows, or time-varying commuting matrices) we are aware of no general relaxation to the log-norm condition in the time-varying setting.

We end this section with some comments on the above regularity conditions. Firstly, we highlight that these conditions don’t require any ergodic property on the process \(A^\epsilon_t\), nor any conditions on the limiting log-norm \(\mu(\cdot)\).

When \((H_1)\) is satisfied we have the estimate
\[
\sup_{t \geq 0} \mathbb{E} (\|A^\epsilon_t\|^n) \leq (a + \|A_{\infty}\|) + c_n \epsilon
\]
This ensures that the l.h.s. condition in \((2)\) is met with \(\epsilon_{0,n} = \epsilon_n\) when \((H_1)\) is met. When \((H_2)\) is satisfied then the l.h.s. condition in \((2)\) is trivially met with \(\epsilon_{0,n} = 1\).

This also shows that the r.h.s. condition in \((H_2)\) is met as soon as the l.h.s. condition in \((H_2)\) is satisfied with \(c_n = n^{1/2}\). The latter is often difficult to check for nonlinear diffusion approximation models since the fluctuation analysis of the \(n\)-th error moments often combine Burkholder-Davis-Gundy-type inequalities involving a square root parameter \(n^{1/2}\), with the estimation of \(n\)-th order type moments of \(A^\epsilon_t\).

When \(B^\epsilon\) depends on \(A^\epsilon\) w.r.t. some polynomial type function, the r.h.s. condition in \((2)\) can be readily checked using the moments estimates on \(A^\epsilon\) just discussed.

We may illustrate the satisfaction of \((H_2)\) with a some examples. For instance, \((H_2)\) is satisfied for the spectral norm and sub-Gaussian fluctuations. To be more precise, we set
\[
(\Delta A)_t^\epsilon := \epsilon^{-1} [A_t^\epsilon - A_t] \iff A_t^\epsilon = A_t + \epsilon (\Delta A)_t^\epsilon
\]
In this notation, we have
\[
\sup_{t \geq 0} \mathbb{E} (\|(\Delta A)_t^\epsilon\|^n) \leq d_2 n^{1/2} \implies (H_2) \quad \text{with} \quad c_n = d_2 n^{1/2} \quad \text{and} \quad d_1 = a + \|A_{\infty}\|
\]
Also observe that \((H_2)\) is met for fluctuation matrices with sub-Gaussian entries; that is, when the following condition is met
\[
\forall 1 \leq i, j \leq r, \quad \sup_{t \geq 0} \mathbb{E} (\|(\Delta A)_t^\epsilon(i,j)\|^n) \leq d_2 n^{1/2} \implies (H_2)
\]
2 Statement of the Main Result

In this section we state the main result of this work along with a number of ancillary corollaries of interest on their own.

Our main result takes the following form.

**Theorem 2.1.**

• Suppose the fluctuation estimates in \((H_1)\) are satisfied for \(n = 2\). Then, for any time horizon \(s \geq 0\), and any parameter \(\nu \in [0, 1]\), we have

\[
t \geq \frac{2}{\nu b c_0} \quad \text{and} \quad \epsilon \leq \epsilon_2 \wedge \left( \frac{\nu c_0}{2 c_2} \right)
\]

\[
\Rightarrow \quad \log \|\mathcal{E}_{s,s+t}(A^\epsilon)\| \vee \mathbb{E} \left[\log \|\mathcal{E}_{s,s+t}(A^\epsilon)\|\right] \leq (1 - \nu) \mu(A_\infty) t
\]

(9)

where \(\overline{A}^\epsilon : t \in \mathbb{R}_+ \mapsto \overline{A}^\epsilon_t := \mathbb{E}(A^\epsilon_t)\) is the averaged process.

• Assume \((H_1)\) is satisfied. Then, for any increasing sequence of times \(0 \leq s \leq t_k \uparrow k \to \infty \), the probability of the following event

\[
\limsup_{k \to \infty} \frac{1}{t_k} \log \|\mathcal{E}_{s,t_k}(A^\epsilon)\| < \frac{1}{2} \mu(A_\infty) \quad \text{is greater than} \quad 1 - \nu
\]

(10)

for any \(\nu \in [0, 1]\), as soon as \(\epsilon\) is chosen so that \(\epsilon \leq \epsilon_n(\nu)\) for some \(n \geq 1\).

• Now suppose hypothesis \((H_2)\) is satisfied. Then, for any \(n \geq 1\), any fluctuation parameter \(\epsilon \leq \epsilon_{2,n}\), and any time horizon \(s \geq 0\) we have

\[
T_n \leq t \leq T_n^* \quad \Rightarrow \quad \frac{1}{t} \log \mathbb{E} \left(\|\mathcal{E}_{s,s+t}(A^\epsilon)\|^n\right) \leq \frac{n}{4} \mu(A_\infty)
\]

(11)

where

\[
T_n := \frac{4}{c_0} \log \left(1 + \frac{c_2 n}{c_0} 2^{2 + \frac{1}{2} n}\right) \quad \text{and} \quad T_n^* := \frac{\log \left(1/\epsilon^2\right)}{2\left((4e + (2e)^{1/2}) (d_1 \vee d_2)\right)} + c_0 \wedge \frac{1/\epsilon^2}{2d_2 n}
\]

(12)

and \(\epsilon_{2,n}\) is the smallest parameter such that \(T_n < T_n^*\). Note that \(T_n^* \to \epsilon \to 0 \infty\).

The proof of the above theorem is provided in Section 3. We illustrate the impact of the above theorem with a series of corollaries outlined in the subsequent subsection.

2.1 Corollary Results

Firstly, we consider a collection of corollaries under the hypothesis \((H_1)\). The first corollary is a consequence of the Borel-Cantelli’s lemma applied to (10) in Theorem 2.1.

**Corollary 2.2.** Assume \((H_1)\) is satisfied. Then, for any \(s \geq 0\), any increasing sequence of time horizons \(t_k \uparrow k \to \infty\) and any sequence \(\epsilon_{k_2} \downarrow k_2 \to \infty\) 0 such that \(\sum_{k_2 \geq 1} \epsilon_{k_1}^{n} < \infty\) for some \(n \geq 1\), we have the almost sure Lyapunov estimate

\[
\limsup_{k_2 \to \infty} \limsup_{k_1 \to \infty} \frac{1}{t_{k_1}} \log \|\mathcal{E}_{s,s+t_{k_1}}(A^{\epsilon_{k_2}})\| < \frac{1}{2} \mu(A_\infty)
\]

(13)

The next two results provide some reformulation of the supremum limit estimates stated in (10) and (13) in terms of random relaxation time horizons and random relaxation-type fluctuation parameters.

The first of these two results shows that with a high probability, the semigroup \(\mathcal{E}_{s,t}(A^\epsilon)\) is stable after some possibly random relaxation time horizon, as soon as \(\epsilon\) is chosen sufficiently small.
Corollary 2.3. Assume \((H_1)\) holds. Then, for any increasing sequence of times \(0 \leq s \leq t_k \uparrow_{k \to \infty} \infty\), the probability of the following event,
\[
\left\{ \forall 0 < \nu_2 \leq 1 \quad \exists l \geq 1 \quad \text{such that} \quad \forall k \geq l \quad \text{it holds that} \quad \frac{1}{t_k} \log \|E_{s,s+t_k}(A^\epsilon)\| \leq \frac{1}{2} (1 - \nu_2) \mu(A_{\infty}) \right\}
\]
is greater than \(1 - \nu_1\), for any \(\nu_1 \in ]0, 1[\), as soon as \(\epsilon\) is chosen so that \(\epsilon \leq \epsilon_n(\nu_1)\) for some \(n \geq 1\).

The next result takes this one step further for an almost sure result that comes into effect after some random time and with some sufficiently small fluctuation parameter (where sufficiency in this case is also random).

Corollary 2.4. Assume \((H_1)\) is satisfied. Consider any \(s \geq 0\), any increasing sequence of time horizons \(t_k \uparrow_{k \to \infty} \infty\), and any sequence \(\epsilon_k \downarrow_{k \to \infty} 0\) such that \(\sum_{k \geq 1} \epsilon_k^n < \infty\) for some \(n \geq 1\). Then, we have the almost sure Lyapunov estimate,
\[
\left\{ \forall 0 < \nu_2 \leq 1 \quad \exists l_1, l_2 \geq 1 \quad \text{such that} \quad \forall k_1 \geq l_1, \forall k_2 \geq l_2 \quad \text{it holds that} \quad \frac{1}{t_{k_1}} \log \|E_{s,s+t_{k_1}}(A^{\epsilon_{k_2}})\| \leq \frac{1}{2} (1 - \nu_2) \mu(A_{\infty}) \right\}
\]
for any \(\nu \in ]0, 1[\), as soon as \(\epsilon\) is chosen so that \(\epsilon \leq \epsilon_n(\nu)\) for some \(n \geq 1\).

The next result concerns the stability of the process \((\cdot)\) itself.

Corollary 2.5. Assume \((H_1)\) holds. Then, for any increasing sequence of time horizons \(t_k \uparrow_{k \to \infty} \infty\) and any \(x_1 \neq x_2\), the probability of the following event
\[
\limsup_{k \to \infty} \frac{1}{t_k} \log \|X_{t_k}^{\epsilon,x_1} - X_{t_k}^{\epsilon,x_2}\| < \frac{1}{2} \mu(A_{\infty}) \quad \text{is greater than} \quad 1 - \nu
\]
for any \(\nu \in ]0, 1[\), as soon as \(\epsilon\) is chosen so that \(\epsilon \leq \epsilon_n(\nu)\) for some \(n \geq 1\).

The preceding corollary is a direct consequence of the decomposition
\[
X_{t}^{\epsilon,x_1} - X_{t}^{\epsilon,x_2} = \mathcal{E}_t(A^\epsilon)(x_1 - x_2) \quad \implies \quad \|X_{t}^{\epsilon,x_1} - X_{t}^{\epsilon,x_2}\| \leq \|\mathcal{E}_t(A^\epsilon)\| \|x_1 - x_2\|
\]
Note that \((16)\) in Corollary 2.5 is analogous to \((10)\) in Theorem 2.1 but at the level of the process \((\cdot)\) itself. Analogous results to Corollaries 2.2, 2.3 and 2.4 at the level of the process \((\cdot)\) follow immediately.

Next, we consider a collection of corollaries under the stronger hypothesis \((H_2)\). Firstly, given \((H_2)\), we highlight a fact immediate from \((11)\) and \((12)\), that for any \(n \geq 1\), any \(s \geq 0\), we have
\[
\limsup_{\epsilon \to 0} \frac{1}{T_n} \log \mathbb{E} (\|E_{s,s+T_n}(A^\epsilon)\|^n) \leq \frac{n}{4} \mu(A_{\infty})
\]
The next result provides a fluctuation-type analysis.

Corollary 2.6. Suppose \((H_2)\) is satisfied. For any \(\epsilon \leq \epsilon_{2n}\) we have the fluctuation estimate
\[
T_{2n} \leq s \leq t \leq T_{2n}^\epsilon \quad \implies \quad \epsilon^{-1} \mathbb{E} (\|E_{s,t}(A^\epsilon) - E_{s,t}(A)\|^n)^{1/n} \leq c_n + 4\epsilon^{a/b} \epsilon_{2n}/c_0
\]
By the Burkhölder-Davis-Gundy inequality (e.g. Proposition 4.2 in [1]), for any the other hand, we have

\[ \partial_t [\mathcal{E}_{s,t}(A^t) - \mathcal{E}_{s,t}(A)] = A_t [\mathcal{E}_{s,t}(A^t) - \mathcal{E}_{s,t}(A)] + (A^t_t - A_t) \mathcal{E}_{s,t}(A^t) \]

\[ \implies \mathcal{E}_{s,t}(A^t) - \mathcal{E}_{s,t}(A) = (A^t_s - A_s) + \int_s^t \mathcal{E}_{u,t}(A) (A^t_u - A_u) \mathcal{E}_{s,u}(A^t) \, du \]

\[ \implies \epsilon^{-1} \mathbb{E} (\|\mathcal{E}_{s,t}(A^t) - \mathcal{E}_{s,t}(A)\|^n)^{1/n} \leq c_n + c_2n \epsilon^{a/b} \int_s^t e^{(t-u)} \mu(A_\infty) \mathbb{E} (\|\mathcal{E}_{s,u}(A^t)\|^2)^{1/(2n)} \, du \]

This implies that

\[ T_{2n} \leq s \leq t \leq T_{2n}' \]

\[ \implies \epsilon^{-1} \mathbb{E} (\|\mathcal{E}_{s,t}(A^t) - \mathcal{E}_{s,t}(A)\|^n)^{1/n} \leq c_n + \frac{4c_2n \epsilon^{a/b}}{c_0} e^{(t-s)} \mu(A_\infty)^4 \left( 1 - e^{3(t-s)} \mu(A_\infty)/4 \right) \]

The proof of the corollary is complete.

The next corollary concerns stability in the mean, at the level of the process \( s \) itself, and guaranteed over a relevant (computable) deterministic time interval.

The next corollary establishes and makes precise the relationship alluded to in prior discussion, i.e. \( 6 \Rightarrow 7 \). It is based on the fact that under \( (H_2) \) alone, the result \( 11 \) in Theorem 2.1 establishes an estimate of the form \( 6 \), at least over an interval (which can be chosen as large as needed by reducing the fluctuation parameter).

**Corollary 2.7.** Suppose \( (H_2) \) holds. Then, for any \( n \geq 1 \), \( \epsilon \leq \epsilon_{2,n} \) and any time horizon \( t \) such that \( T_n \leq t \leq T_n' \), we have the contraction inequality,

\[ \mathbb{E} (\|X^{\epsilon,x}_t - X^{\epsilon,y}_t\|^n)^{1/n} \leq \exp \left[ \mu(A_\infty) t/4 \right] \|x_1 - x_2\| \quad (18) \]

In addition, for any \( n \geq 2 \) and any \( \epsilon \leq \epsilon_{2,n} \wedge \epsilon_{1,2n} \), we have the moment estimates

\[ T_n \vee T_{2n} \leq t \leq T_{2n}' \implies \mathbb{E} (\|X^{\epsilon,x}_t\|^n)^{1/n} \leq \exp \left[ \mu(A_\infty) t/4 \right] \|x\| + \kappa_n \quad (19) \]

for some finite constant \( \kappa_n \) whose value only depends on the parameter \( n \) (and possibly on \( r \)).

**Proof.** Observe that

\[ \|X^{\epsilon,x}_t - X^{\epsilon,y}_t\| \leq \|\mathcal{E}_t(A^\epsilon)\| \|x - y\| \quad \text{and} \quad \|X^{\epsilon,x}_t\| \leq \|X^{\epsilon,0}_t\| + \|\mathcal{E}_t(A^\epsilon)\| \|x\| \]

The estimate (18) is a direct consequence of (11) and the l.h.s. estimate in the above display. On the other hand, we have

\[ \|X^{\epsilon,0}_t\|^2 = \left\| \int_0^t \mathcal{E}_{s,t}(A^\epsilon) (B_s^\epsilon)^{1/2} \, dW_s \right\|^2 = \left\| \sum_{1 \leq i \leq r} \left( \sum_{1 \leq j \leq r} \int_0^t \left[ \mathcal{E}_{s,t}(A^\epsilon) (B_s^\epsilon)^{1/2} \right]_{i,j} \, dW_s^j \right)^2 \right\|^{2n} \]

\[ \leq r^{3n-2} \sum_{1 \leq i, j \leq r} \left( \int_0^t \left[ \mathcal{E}_{s,t}(A^\epsilon) (B_s^\epsilon)^{1/2} \right]_{i,j} \, dW_s^j \right)^{2n} \]

By the Burkhölder-Davis-Gundy inequality (e.g. Proposition 4.2 in [1]), for any \( n \geq 1 \) we have

\[ \mathbb{E} \left[ \|X^{\epsilon,0}_t\|^{2n} \mid \mathcal{F}_t \right] \leq c r^{3n-2} (2n)^n \sum_{1 \leq i, j \leq r} \left( \int_0^t \left[ \mathcal{E}_{s,t}(A^\epsilon) (B_s^\epsilon)^{1/2} \right]_{i,j}^2 \, ds \right)^n \]
for some universal constant \( c \). This yields
\[
E \left[ \|X_t^0\|^{2n} | \mathcal{F}_t \right] \leq c (2n)^n \left[ \int_0^t \text{tr} \left[ E_{s,t}(A') B_s^e E_{s,t}(A')' \right] ds \right]^n
\]
\[
\leq c (2n)^n \left[ \int_0^t \|E_{s,t}(A')\|^2 \text{tr} \left[ B_s^e \right] ds \right]^n
\]
where \( c \) may vary from line to line (and depend on \( r \) but not on \( n \)). Combining (2) with the generalized Minkowski inequality and Cauchy-Schwartz inequality we check the estimate
\[
E \left[ \|X_t^0\|^{2n} \right]^{1/n} \leq c^{1/n} n \rho_{2n} \int_0^t E \left( \|E_{s,t}(A')\|^{4n} \right)^{1/(2n)} ds
\]
Assume that condition \((H_2)\) is satisfied. Observe that for any \( T_{4n} \leq t \leq T_{4n}^c \) we have
\[
\int_0^t E \left[ \|E_{s,t}(A')\|^{4n} \right]^{1/(2n)} ds = \int_0^{T_{4n}} E \left[ \|E_{(t-s),(t-s)+s}(A')\|^{4n} \right]^{1/(2n)} ds + \int_{T_{4n}}^t E \left[ \|E_{(t-s),(t-s)+s}(A')\|^{4n} \right]^{1/(2n)} ds
\]
By Theorem 2.1 we have
\[
\int_{T_{4n}}^t E \left[ \|E_{(t-s),(t-s)+s}(A')\|^{4n} \right]^{1/(2n)} ds \leq \int_{T_{4n}}^{T_{4n}} \exp \left[ \frac{8}{2} \mu(A_{\infty}) \right] ds
\]
\[
= \frac{2}{\mu(A_{\infty})} \left[ \exp \left[ \frac{T_{4n}}{2} \mu(A_{\infty}) \right] - \exp \left[ \frac{T_{4n}^c}{2} \mu(A_{\infty}) \right] \right]
\]
In this situation, using (20) we also have
\[
\int_0^{T_{4n}} E \left[ \|E_{(t-s),(t-s)+s}(A')\|^{4n} \right]^{1/(2n)} ds \leq \int_0^{T_{4n}} \left[ \frac{1}{2} e^{8d_1 n s} + e^{\frac{e}{2} \left( 4(1 + (2e)^{1/2} d_2 n s e) e^{(8e^{1/2} d_2 n s e)^2} - 1 \right)} \right]^{1/(2n)} ds
\]
\[
\leq \frac{1}{4d_1} \left[ e^{4d_1 T_{4n}} - 1 \right] + \left[ 2 e^{\frac{e}{2} \left( 1 + (2e)^{1/2} d_2 n T_{4n} \right)} \right]^{1/(2n)} \frac{1}{e^{4d_2}} \left[ e^{4d_2 T_{4n}} - 1 \right]
\]
In the last assertion we have used the fact that \( 4n d_2 T_{4n} \epsilon^2 \leq 1/2 \). These estimates imply that
\[
\sup_{0 \leq t \leq T_{4n}^c} E \left[ \|X_t^0\|^{2n} \right]^{1/(2n)} \leq c n^{1/2} (\rho_{2n} \delta_{2n}^t)^{1/2}
\]
with
\[
\delta_{2n}^t := \frac{1}{4d_1} \left[ e^{4d_1 T_{4n}} - 1 \right] + \left[ 1 + d_2 n T_{4n} e \right]^{1/(2n)} \frac{1}{d_2} \left[ e^{4d_2 T_{4n}} - 1 \right]
\]
Using (11) we conclude that
\[
T_{n} \vee T_{2n} \leq t \leq T_{2n}^c \implies E \left[ \|X_t^0\|^{2n} \right]^{1/(n)} \leq c n^{1/2} (\rho_{2n} \delta_{2n})^{1/2} + e^{\mu(A_{\infty}) t/4} \|x\|
\]
for any \( n \geq 2 \). This ends the proof of the corollary. \( \square \)
3 Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the following technical lemma.

Lemma 3.1. Assume that the r.h.s. estimates in \((H_2)\) are satisfied. Then, for any \(n \geq 1, \epsilon \in [0,1]\) and \(s,t \geq 0\) we have the estimate

\[
E (\|E_{s,t}(A^\epsilon)\|^n) \leq \frac{1}{2} e^{2d_1 nt} + \frac{e}{2} \sqrt{\frac{\epsilon}{\pi}} \left( (1 + (2\epsilon)^{1/2} d_2 n t e) e^{(2\epsilon^{1/2} d_2 n t e)^2} - 1 \right) \quad (20)
\]

Proof. For any \(n \geq 1\) we have

\[
E (\|E_{s,t}(A^\epsilon)\|^n) \leq E \left[ \exp \left( n \int_s^t \|A_u^\epsilon\| \, du \right) \right]
\]

\[
= 1 + \sum_{k \geq 1} \frac{n^k}{k!} \mathbb{E} \left[ \left( \int_s^t \|A_u^\epsilon\| \, du \right)^k \right]
\]

\[
\leq 1 + \sum_{k \geq 1} \frac{n^k}{k!} \left[ \int_s^t \mathbb{E} \left( \|A_u^\epsilon\|^{1/k} \right)^{1/k} \, du \right]^k
\]

\[
\leq 1 + \frac{1}{2} \sum_{k \geq 1} \frac{(2n(t-s))^k}{k!} (d_1^k + d_2^k \epsilon^k k^{k/2})
\]

This implies that

\[
E (\|E_{s,s+t}(A^\epsilon)\|^n) \leq \frac{1}{2} e^{2d_1 nt} + \frac{1}{2} \sum_{k \geq 1} \frac{(3^{3/2} d_2 n t e)^k}{k!} (k/2)^{k/2}
\]

On the other hand, by Stirling approximation we have

\[
\frac{k^k}{(2k)!} \leq \frac{e^k}{\sqrt{4\pi k}} \frac{1}{(2k)^{2k} e^{-2k}} = \frac{e^k}{\sqrt{4\pi}} \frac{1}{\sqrt{k} k^{k} e^{-k}} \leq \frac{e}{\sqrt{4\pi}} \frac{(e/4)^k}{k!}
\]

and

\[
\frac{(2k+1)^{k+1/2}}{(2k+1)!} \leq \frac{1}{\sqrt{2\pi}(2k+1)} \frac{1}{(2k+1)^{(2k+1)/2} e^{-(2k+1)}} \leq \frac{e^2}{\sqrt{4\pi}} \left( \frac{e}{2} \right)^k \frac{1}{k!}
\]

This implies that

\[
\sum_{k \geq 1} \frac{(3^{3/2} d_2 n t e)^k}{(2k)!} k \leq \frac{e}{\sqrt{4\pi}} \sum_{k \geq 1} \frac{(2e)^{1/2} d_2 n t e)^k}{k!} k \leq \frac{e}{\sqrt{4\pi}} \left[ e^{((2e)^{1/2} d_2 n t e)^2} - 1 \right]
\]
and
\[
\sum_{k \geq 0} \frac{(2^{3/2} d nt \epsilon)^{2k+1}}{(2k+1)!} (2k+1)^{k+1/2}
\]
\[
\leq (2^{3/2} d nt \epsilon) \left[ 1 + \sum_{k \geq 1} \frac{(2^{3/2} d nt \epsilon)^{2k}}{(2k+1)!} (2k+1)^{k+1/2} \right]
\]
\[
\leq (2^{3/2} d nt \epsilon) \left[ 1 + \frac{e^2}{\sqrt{4\pi}} \sum_{k \geq 1} \frac{(2e^{1/2} d nt \epsilon)^{2k}}{k!} \right] \leq 2^{1/2} \frac{e^2}{\sqrt{\pi}} d nt \epsilon (2e^{1/2} d nt \epsilon)^2
\]
This implies that
\[
\mathbb{E}(\|E_{s,s+t}(A^\epsilon)\|^n) \leq \frac{1}{2} e^{2d nt} + \frac{1}{4} \sqrt{\pi} e (\mathbb{E}(2e^{1/2} d nt \epsilon)^2 - 1 + 2^{3/2} e d nt \epsilon (2e^{1/2} d nt \epsilon)^2)
\]
\[
\leq \frac{1}{2} e^{2d nt} + \frac{e}{2} \sqrt{\pi} e (1 + (2e)^{1/2} d nt \epsilon (2e^{1/2} d nt \epsilon)^2 - 1)
\]
We conclude that
\[
\mathbb{E}(\|E_{s,s+t}(A^\epsilon)\|^n) \leq \frac{1}{2} e^{2d nt} + \frac{e}{2} \sqrt{\pi} (1 + (2e)^{1/2} d nt \epsilon (2e^{1/2} d nt \epsilon)^2 - 1)
\]
This ends the proof of (20).

Now we come to the proof of Theorem 2.1. Proof of Theorem 2.1: Under (H_1) we have the log-norm estimate
\[
\frac{1}{t} \log \|E_{s,s+t}(A^\epsilon)\| \leq \frac{1}{t} \int_0^t \mu(A_{s+u}^\epsilon) du
\]
\[
\leq \mu(A_\infty) + \frac{1}{t} \int_0^t \|A_\infty - A_{s+u}\| du + \frac{1}{t} \int_0^t \|A_{s+u} - A_{s+u}^\epsilon\| du
\]
\[
\leq \mu(A_\infty) + \frac{a}{b} \frac{e^{-as}}{t} + c_2 \epsilon
\]
In the last assertion we have used the fact that
\[
(H_1) \quad \implies \quad \|A_{s+u} - A_{s+u}^\epsilon\| \leq \mathbb{E}(\|A_{s+u} - A_{s+u}^\epsilon\|^2)^{1/2} \leq c_2 \epsilon
\]
Observe that
\[
e^{-as} \wedge \frac{1}{t} \leq \frac{b}{4a} |\mu(A_\infty)| \quad \text{and} \quad \epsilon \leq \epsilon \wedge \frac{|\mu(P_\infty)|}{4c_2} \quad \implies \quad \frac{1}{t} \log \|E_{s,s+t}(A^\epsilon)\|^2 \leq \mu(A_\infty)
\]
This completes the proof of the l.h.s. of (9). Arguing as above we have the log-norm estimate
\[
\frac{1}{t} \log \|E_{s,s+t}(A^\epsilon)\| - \mu(A_\infty) \leq \frac{1}{t} \int_0^t \|A_{s+u} - A_{s+u}^\epsilon\| du + \frac{a}{b} \frac{e^{-as}}{t}
\]
Observe that
\[
s \vee t \geq h \quad \implies \quad e^{-as} \wedge \frac{1}{t} \leq \frac{b}{4a} |\mu(A_\infty)|
\]
Taking the expectation we check the r.h.s. of (9).

We consider the collection of events
\[ \Omega_{s,t}^\epsilon := \left\{ \|E_{s,s+t}(A^\epsilon)\| \leq \exp \left[ \frac{t}{2} \mu(A_\infty) \right] \right\} \]

Assume \((H_1)\) holds. In this case, applying the Markov inequality, for any \(\epsilon \leq \epsilon_n\) and any \(s \lor t \geq h\) we have the uniform estimates
\[
\mathbb{E} \left( \frac{1}{t} \int_0^t \|A_{s+u} - A_{s+u}^\epsilon\| du \right)^{1/n} \leq \epsilon c_n
\]

\[\Rightarrow \sup_{s \lor t \geq h} \mathbb{P}(\Omega - \Omega_{s,t}^\epsilon)^{1/n} \leq \epsilon \overline{c}_n \quad \text{with} \quad \overline{c}_n := 4c_n/|\mu(A_\infty)| \quad (21)\]

Applying Fatou’s Lemma for any \(h \leq s \leq t_k \uparrow k \to \infty\), we have
\[
\mathbb{P} \left( \forall t \geq 1 \exists k \geq l : \|E_{s,s+t_k}(A^\epsilon)\| > \exp \left[ \frac{t_k}{2} \mu(A_\infty) \right] \right)
\leq \liminf_{k \to \infty} \mathbb{P} \left( \|E_{s,s+t_k}(A^\epsilon)\| > \exp \left[ \frac{t_k}{2} \mu(A_\infty) \right] \right) \leq \epsilon^n \overline{c}_n^n
\]

This implies that for any \(\epsilon \leq \epsilon_n\) we have
\[
\mathbb{P} \left( \limsup_{k \to \infty} \frac{1}{t_k} \log \|E_{s,s+t_k}(A^\epsilon)\|^2 < \mu(A_\infty) \right) \geq 1 - \epsilon^n \overline{c}_n^n
\]

This ends the proof of (10).

Using (20) we have the rather crude estimate
\[
d_{2n} \epsilon^2 \leq 1 \quad \Rightarrow \quad \mathbb{E} \left( \|E_{s,s+t}(A^\epsilon)\|^n \right) \leq \frac{1}{2} e^{2d_n \epsilon t} + e \sqrt{\frac{e}{\pi} e^{(4\epsilon + (2\epsilon)^{1/2}) d_n \epsilon t}}
\leq 2 e^{d_n \epsilon t} \quad \text{with} \quad d = (4\epsilon + (2\epsilon)^{1/2})(d_1 \lor d_2)
\]

Using the Cauchy-Schwartz inequality for any \(n \geq 2\) and any \(\epsilon\) such that \(d_{2n} \epsilon^2 \leq 1\) we have
\[
\mathbb{E} \left( \|E_{s,s+t}(A^\epsilon)\|^{n/2} \right)^{2/n} = \mathbb{E} \left( \|E_{s,s+t}(A^\epsilon)\|^{n/2} 1_{\Omega_{s,t}^\epsilon} \right)^{2/n} + \mathbb{E} \left( \|E_{s,s+t}(A^\epsilon)\|^{n/2} 1_{\Omega - \Omega_{s,t}^\epsilon} \right)^{2/n}
\leq \exp \left[ \frac{t}{2} \mu(A_\infty) \right] + \epsilon \overline{c}_n^{1/n} \exp [td]
\leq (1 + \overline{c}_n^{1/n}) \exp \left[ \frac{t}{4} \mu(A_\infty) \right] \leq \exp \left[ \frac{t}{4} \mu(A_\infty) \right]
\]

with the constant \(\overline{c}_n\) defined in (21) as soon as
\[
\frac{4}{|\mu(A_\infty)|} \log \left( 1 + \overline{c}_n^{1/n} \right) \leq t \leq \frac{1}{d + |\mu(A_\infty)|/2} \log (1/\epsilon)
\]

For instance, we can choose \(\epsilon\) sufficiently small s.t.
\[
c_n^- \leq t \leq c_n^+ \log (1/\epsilon)
\]

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with
\[ c^-_n := \frac{4}{|\mu(A_\infty)|} \log \left(1 + \frac{\tau_n}{2^{1/n}}\right) \quad \text{and} \quad c^+_n := \left(\frac{1}{d + |\mu(A_\infty)|/2} \wedge \frac{1}{2d_2n}\right). \]

In summary, for any \( n \geq 1 \) there exists some finite constants \( c^-_n \leq c^+_n \) such that for any \( \epsilon \leq \exp\left[-(c^-_n/c^+_n)\right] \) and any \( s \geq 0 \) we have
\[ c^-_n \leq t \leq c^+_n \log \left(1/\epsilon\right) \quad \Rightarrow \quad \frac{1}{nt} \log \mathbb{E} \left(\|E_{s,s+t}(A^t)\|^n\right) \leq \frac{1}{4} \mu(A_\infty). \]

This ends the proof of (11). The proof of the theorem is now complete.

References

[1] M.T. Barlow and M. Yor. Semi-martingale inequalities via the Garsia-Rodemich-Rumsey lemma, and applications to local times. Journal of Functional Analysis. vol. 49, no. 2. pp. 198–229 (1982).

[2] J. Bertram and P. Sarachik. Stability of Circuits with Randomly Time-Varying Parameters. IRE Transactions on Circuit Theory. vol. 6, no. 5. pp. 260–270 (1959).

[3] B.H. Bharucha. On the Stability of Randomly Varying Systems. Ph.D. Dissertation. Dept. Elec. Eng., University of California at Berkeley, CA, USA, July (1961).

[4] A.N. Bishop, P. Del Moral, K. Kamatani, and B. Remillard. On one-dimensional Riccati diffusions. arXiv e-print, [arXiv:1711.10065] (2017).

[5] A.N. Bishop, P. Del Moral, and A. Niclas. A perturbation analysis of stochastic matrix Riccati diffusions. arXiv e-print, [arXiv:1709.05071] (2017).

[6] J.M. Bismut. Linear Quadratic Optimal Stochastic Control with Random Coefficients. SIAM Journal on Control and Optimization. vol. 14, no. 3. pp. 419–444 (1976).

[7] S. Blanes, F. Casas, J.A. Oteo, J. Ros. The Magnus expansion and some of its applications. Physics Reports. vol. 470, no. 5-6. pp. 151–238 (2009).

[8] G. Blankenship. Stability of linear differential equations with random coefficients. IEEE Transactions on Automatic Control. vol. 22, no. 5. pp. 834–838 (1977).

[9] R.W. Brockett. Finite Dimensional Linear Systems. Wiley, New York (1970).

[10] W.A. Coppel. Dichotomies in Stability Theory. Springer (1978).

[11] P. Del Moral and J. Tugaut. On the stability and the uniform propagation of chaos properties of ensemble Kalman-Bucy filters. Annals of Applied Probability. vol. 28, no. 2. pp 790–850 (2018). arXiv e-print, [arXiv:1605.09329].

[12] G. Evensen. The Ensemble Kalman Filter: theoretical formulation and practical implementation. Ocean Dynamics. vol. 53, no. 4. pp. 343–367 (2003).

[13] Y. Fang and K.A. Loparo. Stabilization of Continuous-Time Jump Linear Systems. IEEE Transactions on Automatic Control. vol. 47, no. 10. pp 1590–1603 (2002).
[14] R.A. Frazer, W.J. Duncan and A.R. Collar. Elementary Matrices and Some Applications to Dynamics and Differential Equations. Cambridge University Press (1938).

[15] S. Geman. Some averaging and stability results for random differential equations. SIAM Journal on Applied Mathematics. vol. 36, no. 1. pp 86–105 (1979).

[16] L. Guo. Stability of Recursive Stochastic Tracking Algorithms. SIAM Journal on Control and Optimization. vol. 32, no. 5. pp 1195–1225 (1994).

[17] E.L. Ince. Ordinary Differential Equations. Dover, New York (1956).

[18] E.F. Infante. On the stability of some linear nonautonomous random systems. Journal of Applied Mechanics. vol. 35, no. 1. pp. 7–12 (1968).

[19] I.I. Kats and N.N. Krasovskii. On the stability of systems with random parameters. Journal of Applied Mathematics and Mechanics. vol. 24, no. 5. pp. 1225–1246 (1960).

[20] R. Khasminskii. Stochastic Stability of Differential Equations. 2nd edition. Springer-Verlag, Berlin (2011).

[21] F. Kozin. A Survey of Stability of Stochastic Systems. Automatica. vol. 5, no. 1. pp. 95–112 (1969).

[22] F. Kozin and C.M. Wu. On the stability of linear stochastic differential equations. Journal of Applied Mechanics. vol. 40, no. 1. pp. 87–92 (1973).

[23] H.J. Kushner. Stochastic Stability and Control. Academic Press, New York (1967).

[24] A.E.B. Lim and X.Y. Zhou. Mean-Variance Portfolio Selection with Random Parameters in a Complete Market. Mathematics of Operations Research. vol. 27, no. 1. pp. 101–120 (2002).

[25] A. Ludwig. Random Dynamical Systems. Springer-Verlag, Berlin (1998).

[26] W. Magnus. On the exponential solution of differential equations for a linear operator. Communications on Pure and Applied Mathematics. vol. 7, no. 4. pp. 649–673 (1954).

[27] X. Mao. Stochastic Differential Equations and Applications. 2nd edition. Woodhead Publishing Limited, Oxford (2010).

[28] L. Moreau. Stability of multiagent systems with time-dependent communication links. IEEE Transactions on Automatic Control. vol. 50, no. 2. pp. 169–182 (2005).

[29] T. Morozan. Stability of Some Linear Stochastic Systems. Journal of Differential Equations. vol. 3, no. 2. pp. 153–169 (1967).

[30] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multiagent systems. Proceedings of the IEEE. vol. 95, no. 1. pp. 215–223 (2007).

[31] V.I. Oseledec. A multiplicative ergodic theorem. Liapunov characteristic number for dynamical systems. Trans. Moscow Math. Soc. vol. 19. pp. 197–231 (1968).

[32] G. Peano. Intégration par séries des équations différentielles linéaires. Mathematische Annalen. vol. 32, no. 3. pp. 450–456 (1888).
[33] S. Peng. Stochastic Hamilton-Jacobi-Bellman Equations. SIAM Journal on Control and Optimization. vol. 30, no. 2. pp. 284–304 (1992).

[34] H. Pham. Linear quadratic optimal control of conditional McKean-Vlasov equation with random coefficients and applications. Probability, Uncertainty and Quantitative Risk. vol. 1, no.1. December (2016).

[35] V. Solo. On the stability of slowly time-varying linear systems. Mathematics of Control, Signals and Systems. vol. 7 no. 4. pp. 331–350 (1994).

[36] T.T. Soong. Random Differential Equations in Science and Engineering. Academic Press, New York (1973).

[37] S. Tang. General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic Hamilton systems and backward stochastic Riccati equations. SIAM Journal on Control and Optimization. vol. 42, no. 1. pp. 53–75 (2003).

[38] S. Tang. Dynamic programming for general linear quadratic optimal stochastic control with random coefficients. SIAM Journal on Control and Optimization. vol. 53, no. 2. pp. 1082–1106 (2015).

[39] J.L. Tiwari and J.E. Hobbie. Random differential equations as models of ecosystems: Monte Carlo simulation approach. Mathematical Biosciences. vol. 28, no. 1-2. pp. 25–44 (1976).

[40] C.P. Tsokos and W.J. Padgett. Random Integral Equations with Applications to Life Sciences and Engineering. Academic Press, New York (1974).

[41] S.J. Turnovsky. Optimal stabilization policies for deterministic and stochastic linear economic systems. The Review of Economic Studies. vol. 40, no. 1. pp. 79–95 (1973).

[42] J. Wolfowitz. Products of indecomposable, aperiodic, stochastic matrices. Proceedings of the American Mathematical Society. vol. 14, no. 5. pp. 733–737 (1963).

[43] M Wu. A note on stability of linear time-varying systems. IEEE Transactions on Automatic Control. vol. 19, no. 2. pp. 162–162 (1974).

[44] J. Yong and X.Y. Zhou. Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer, New York, (1999).