COX RINGS OF RATIONAL COMPLEXITY ONE \(T\)-VARIETIES

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Abstract. Let \(X\) be a Mori dream space together with an effective torus action of complexity one. In this note, we construct a polyhedral divisor \(D_{\text{Cox}}\) on a suitable finite covering of \(\mathbb{P}^1\) which corresponds to the Cox ring of \(X\). This description allows for a detailed study of torus orbits and deformations of \(\text{Cox}(X)\). Moreover, we present coverings of \(\mathbb{P}^1\) together with an action of a finite abelian group \(A\) in terms of so-called \(A\)-divisors of degree zero on \(\mathbb{P}^1\).

1. Introduction

For simplicity, all varieties appearing in this article are supposed to be complex.

1.1. Mori dream spaces. Let \(X\) be a normal, \(\mathbb{Q}\)-factorial, complete variety with a finitely generated divisor class group \(\text{Cl}(X)\). Thus, \(\text{Cl}_{\mathbb{Q}}(X) := \text{Cl}(X) \otimes \mathbb{Q}\) coincides with \(\text{Pic}_{\mathbb{Q}}(X)\) as well as with the Néron-Severi group \(N_1^Q(X)\). This setting gives rise to the definition of the \(\text{Cl}(X)\)-graded abelian group

\[
\text{Cox}(X) := \bigoplus_{D \in \text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).
\]

Moreover, \(\text{Cox}(X)\) carries a canonical ring structure (the “Cox ring” of \(X\)): If \(\text{Cl}(X)\) is torsion free, then this can be achieved by taking the divisors \(D\) from a fixed section \(\text{Cl}(X) \hookrightarrow \text{Div}(X)\) of the natural surjection \(\text{Div}(X) \twoheadrightarrow \text{Cl}(X)\). If \(\text{Cl}(X)\) has torsion, then one has to work with a presentation of \(\text{Cl}(X)\) by a finitely generated subgroup of \(\text{Div}(X)\) with relations instead. See [HS, Sect.2] for a detailed treatment of this case.

By [HK00], we call \(X\) a Mori dream space (MDS) if \(\text{Cox}(X)\) is a finitely generated \(\mathbb{C}\)-algebra. This property has two important consequences. First, every nef divisor is semi-ample, i.e. the latter property can be checked numerically. Second, since the Cox ring carries the information about all rational maps of \(X\), the data of the minimal model program (MMP) are finite: Not only are the ample, movable, and effective cones \(\text{Nef}(X) \subseteq \text{Mov}(X) \subseteq \text{Eff}(X) \subseteq N_1^Q(X)\) polyhedral, but \(\text{Eff}(X)\) carries a finite polyhedral subdivision such that the birational transformations \(X_i\) of \(X\) correspond to the cells of this subdivision. Actually, the \(X_i\) appear as GIT quotients of the total coordinate space \(\text{Spec Cox}(X)\), and the polyhedral subdivision of \(\text{Eff}(X)\) corresponds to the GIT equivalence classes.

1.2. The toric case. Probably the most well known example of an MDS is the class of toric varieties. Let \(M, N\) be two mutually dual, finitely generated, free abelian groups, and denote by \(M_\mathbb{Q}, N_\mathbb{Q}\) the associated rational vector spaces. For a fan \(\Sigma\) in \(N\) we denote by \(\text{TV}(\Sigma)\) the associated toric variety. It was shown in [Cox95] that \(\text{Cox}(\text{TV}(\Sigma))\) is a polynomial ring whose variables correspond to the rays of \(\Sigma\). This fact, by the way, characterizes toric varieties.

While this describes \(\text{Cox}(\text{TV}(\Sigma))\) completely in algebraic terms, we would like to emphasize a slightly alternative, more polyhedral point of view: By abuse of

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In [HS], the generators and relations of the Cox ring of $X_Y^N$ was generalized to describe normal, used in (1.3). The points $p$ will be recalled in (2.2), but it is very similar to the notion of a $p$-divisor previously.

1.3. Action of the Picard torus. In [AH06] the concept of affine toric varieties was generalized to describe normal, $n$-dimensional, affine varieties $Z$ with an effective action of a torus $T \cong (\mathbb{C}^*)^k$. The combinatorial part of the language is based on the character lattice $M := \text{Hom}(T, \mathbb{C}^*) = \mathbb{Z}^k$ and its dual $N$. While toric varieties as in (1.2) (the case $k = n$) are determined by cones and fans inside $N$, the general situation ($k \leq n$) must also involve an $(n-k)$-dimensional geometric part, i.e. some projective geometric variety $Y$. If $Z$ is given, then $Y = Z/\alpha T$ will be the so-called Chow quotient. Both parts are combined in the notion of a $p$-divisor $D = \sum \Delta_i \otimes D_i$ on $Y$, i.e. a divisor on $Y$ where the coefficients $\Delta_i$ are polyhedra in $N$. Roughly speaking, [AH06] establishes a one-to-one correspondence $Z \leftrightarrow (N, Y, D)$. See [2,4] for the details.

If, as in (1.3), $X$ is an MDS, and if we suppose that $\text{Cl}(X)$ is torsion free, then $Z := \text{Spec} \text{Cox}(X)$ is a normal affine variety, and the $\text{Cl}(X)$-grading encodes an effective action of the so-called Picard torus $T := \text{Hom}(\text{Cl}(X), \mathbb{C}^*)$. Thus, one could ask for a description of $Z$ in terms of some $p$-divisor $D$ on some $Y$. This has been done in [AW], and in the case of smooth MDS surfaces, the result is as follows: $Y = X$ and, up to shifts of the polyhedral coefficients, $D = \sum_{E \subseteq X} \Delta_E \otimes E$ with

$$\Delta_E = \{D \in \text{Eff}(X) \subseteq \text{Cl}_Q(X) \mid (D \cdot E) \geq -1 \text{ and } (D \cdot E') \geq 0 \text{ for } E' \neq E\}$$

where $E, E'$ run through all negative curves in $X$. One easily sees that the common tailcone, cf. (2.3), of the $\Delta_E$ equals $\text{Nef}(X)$ which is dual to $\text{Eff}(X)$. Note that the latter cone carries the degrees of $\text{Cox}(X)$. In the case that $X$ is a del Pezzo surface, the formula for $\Delta_E$ simplifies to $\Delta_E = \bar{\partial}E + \text{Nef}(X) \subseteq \text{Cl}_Q(X)$.

1.4. Complexity one. We now return to the case where our originally given variety $X$ comes with an effective action of a torus $T$. In contrast to (1.2), we assume that this action is of complexity one, i.e. the Chow quotient $Y := X/\alpha^1 T$ is a curve. For $X$ to be an MDS, we must have $Y = \mathbb{P}^1$, and such an $X$ is always representable as $X(S)$ for a so-called divisorial fan $S = \sum_{p \in \mathbb{P}^1} S_p \otimes [p]$. This notion will be recalled in [2,2], but it is very similar to the notion of a $p$-divisor previously used in (1.3). The points $p \in \mathbb{P}^1$ yield divisors $[p]$, and since $X$ is no longer affine, the polyhedral coefficients $\Delta_p$ have been replaced by polyhedral subdivisions $S_p$, so-called slices, of the vector space $N$ with $N := \text{Hom}(\mathbb{C}^*, T)$.

In [HS], the generators and relations of the Cox ring of $X(S)$ were calculated in terms of $S$. More precisely, if $R$ denotes the set of rays of tail $S$ being disjoint to deg $S$, cf. (2.5), and if $p \in \mathbb{P}^1$ runs through the points with non-trivial slices $S_p$, then [HS Corollary 4.9] says that

$$\text{Cox}(X) = \mathbb{C}[S_p \mid p \in R] \otimes_{\mathbb{C}[T_{v,p} \mid v \in S_p(0)]]/\sum_p \beta_p \prod_{v \in S_p(0)} T_{v,p}^{\mu(v)}$$

where $\beta$, runs through all (or a basis of the) linear relations $\sum_p \beta_p \tilde{p} = 0$ among some fixed lifts $\tilde{p} \in \mathbb{C}^\times \setminus \{0\}$ of the points $p \in \mathbb{P}^1$ and $\mu(v)$ denotes the denominator of the vertices $v \in S_p(0) \subseteq N$.
Example 1.1. ([HS] Example 4.4). The three slices $S_0, S_1, S_\infty$ (see Figure 1) for $p = 0, 1, \infty \in \mathbb{P}^1$ with tailfan $\Sigma$ and degree $\deg S$ (see Figure 2) encode the projectivized cotangent bundle $X = \mathbb{P}(\Omega_{\mathbb{P}^2})$ over $\mathbb{P}^2$. The latter is a toric variety, and its torus $T \subseteq \mathbb{P}^2$ still acts on $X$.

\begin{figure}[h]
\begin{center}
\begin{tabular}{ccc}
(a) $S_0$ & (b) $S_1$ & (c) $S_\infty$
\end{tabular}
\end{center}
\caption{Fansy divisor of $\mathbb{P}(\Omega_{\mathbb{P}^2})$.}
\end{figure}

In this example, we have $R = 0$, and there is exactly one relation among $\tilde{p} = (1,0),(1,1),(0,1) \in \mathbb{C}^2$, namely $\beta = (1,-1,1)$. Since all vertices $v_1, v_2 \in S_0(0)$, $v_3, v_4 \in S_1(0)$, and $v_5, v_6 \in S_\infty(0)$ are lattice points, we end up with
\[
\text{Cox}(X) = \mathbb{C}[T_1, \ldots, T_6]/(T_1T_2 - T_3T_4 + T_5T_6).
\]

\begin{figure}[h]
\begin{center}
\begin{tabular}{cc}
(a) $\Sigma = \text{tail } S$ & (b) $\deg S$
\end{tabular}
\end{center}
\caption{Tailfan and degree of $S$.}
\end{figure}

1.5. Cox rings as p-divisors. Our approach to the Cox ring is somewhat different. Keeping the setting from ([AE]), we will describe $\text{Cox}(X)$ (or rather its affine spectrum) as a p-divisor. So far, this is similar to the viewpoint of [AW] described in ([AE]). However, the special feature of the present paper is that the given $T$-action on $X$ is bequeathed to $\text{Cox}(X)$, and by combining it with the action of the Picard torus, $\text{Spec Cox}(X)$ turns into a complexity one variety, too. Our goal is to stay within the language of [AH06] and to present $\text{Cox}(X)$ as a p-divisor $D_{\text{Cox}}$ on a curve $C$.

The combination of the two torus actions might in general involve torsion. Thus, one could try to understand $\text{Spec Cox}(X)$ as a variety with the complexity one action of a diagonalizable group. Then, the Chow quotient $Y$ would be the same $\mathbb{P}^1$ carrying the divisorial fan $S$. However, since there is no general theory of p-divisors for those groups yet, we have to divide out the torsion. Our main result is the description of the now well-defined p-divisor $D_{\text{Cox}}$ in Theorem 4.2. Removing the torsion gives rise to a finite covering $C \to \mathbb{P}^1$, so that our polyhedral divisor $D_{\text{Cox}}$ will live on a curve $C$ of probably higher genus rather than on $\mathbb{P}^1$.

Yet, if the class group $\text{Cl}(X)$ is torsion free, then this will not happen, i.e. $C \to \mathbb{P}^1$ is the identity map. In this case, the p-divisor $D_{\text{Cox}}$ describing $\text{Spec Cox}(X)$ utilizes the very same points $p \in \mathbb{P}^1$ as $S$: Denoting by $p \in \mathcal{P} \subseteq \mathbb{P}^1$ the set of points with
non-trivial slices $S_p$ and by $V := \bigcup_{p \in P} S_p(0)$ the corresponding vertices, we define the compact polytopes
\[ \Delta^c_p := \text{conv}\{e(v)/\mu(v) | v \in S_p(0)\} \subseteq Q^{\mathbb{V} \cup \mathbb{R}} \]
where the $e(v) \in \mathbb{Z}^V$ denote the canonical basis vectors and $\mu(v) \in \mathbb{Z}_{\geq 1}$ is again the smallest positive integer turning $\mu(v)v$ into a lattice point, i.e. transfers it from $N_Q$ into a genuine element of $N$, cf. (2.3). This gives rise to the polyhedral cone
\[ \sigma := Q_{\geq 0} \cdot \bigoplus_{p \in P} \Delta^c_p + Q^R_{\geq 0} \subseteq Q^{\mathbb{V} \cup \mathbb{R}}. \]

**Theorem 1.2.** If $\text{Cl}(X)$ is torsion free, then the $p$-divisor $D_{\text{Cox}}$ of Spec $\text{Cox}(X)$ is, up to shifts of the polyhedral coefficients, given by $\sum_{p \in P}(\Delta^c_p + \sigma) \otimes [p]$ on $\mathbb{P}^1$.

For the details concerning the shifts as well as for the general result that also covers the case of a class group with torsion, see Theorem 1.2. Comparing this with the toric case of (1.2), we observe a similar construction of resolving linear dependencies among elements of $N_Q$ by preserving separate dimensions for each of them. The Cox ring of $T$-varieties of complexity one is therefore closely related to a fiberwise toric Cox construction with respect to the rational map $X \to \mathbb{P}^1$.

**Example 1.3.** Let $\mathcal{S}$ be the divisorial fan of Example 1.1. The three polytopes $\Delta^c_p$ are compact edges, and $\sigma$ becomes a four-dimensional cone over a cube. Hence, according to Example 1.1, the resulting $p$-divisor is that of the affine cone over Grass$(2, 4)$ from [AH08, p. 849].

1.6. The paper is organized as follows: In Section 2 we recall the notions of $p$-divisors and divisorial fans. Section 3 is of independent interest and contains the necessary tools for the description of finite smooth coverings of $\mathbb{P}^1$. The language was chosen to be as close as possible to that of $p$-divisors. This means that we present a covering of $\mathbb{P}^1$ together with an action of a finite abelian group $A$ in terms of an $A$-divisor of degree zero on $\mathbb{P}^1$, cf. Theorem 3.2. Section 4 presents our main result together with its proof, the latter beginning with a diagram comprising all necessary data (1.3). We conclude the paper with a couple of examples in Section 5.

2. Polyhedral Divisors

2.1. Definition of $p$-divisors. First, we recall the basic notions of [AH06]. Let $T$ be a $k$-dimensional affine torus ($\cong (\mathbb{C}^*)^k$), and let $M$, $N$ denote the mutually dual, free abelian groups ($\cong \mathbb{Z}^k$) of characters and one-parameter subgroups, respectively. In particular, $T$ can be recovered as $T = \text{Spec} \mathbb{C}[M] = N \otimes \mathbb{Z} \mathbb{C}^*$. Denote by $M_Q$, $N_Q$ the corresponding vector spaces over $\mathbb{Q}$. Then, for a polyhedral cone $\sigma \subseteq N_Q$, we may consider the semigroup (with respect to Minkowski addition)
\[ \text{Pol}_Q^+(N, \sigma) := \{ \Delta \subseteq N_Q | \Delta = \text{polyhedron with tail } \Delta = \sigma \} \subseteq \text{Pol}_Q(N, \sigma) \]
where tail $\Delta := \{ a \in N_Q | \Delta + a \subseteq \Delta \}$ denotes the *tailcone* of $\Delta$ and $\text{Pol} \supseteq \text{Pol}^+$ is the associated Grothendieck group.

On the other hand, let $Y$ be a normal and semiprojective (i.e. $Y \to Y_0$ is projective over an affine $Y_0$) variety. By $\text{CaDiv}(Y)$ we denote the group of Cartier divisors on $Y$. A $\mathbb{Q}$-Cartier divisor on $Y$ is called *semiample* if it has a positive, base point free multiple. For an element
\[ D = \sum_i \Delta_i \otimes D_i \in \text{Pol}_Q(N, \sigma) \otimes \mathbb{Z} \text{CaDiv}(Y) \]
with $\Delta_i \in \operatorname{Pol}^+\left(N_\mathbb{Q}, \sigma\right)$ and effective divisors $D_i$, we may consider its evaluations

$$
\mathcal{D}(u) := \sum_i \min(\Delta_i, u) D_i \in \operatorname{CaDiv}_\mathbb{Q}(Y)
$$
on elements $u \in \sigma^\vee$. We call $\mathcal{D}$ a polyhedral or $p$-divisor if the $\mathcal{D}(u)$ are semiample and, moreover, big for $u \in \operatorname{int} \sigma^\vee$. The common tailcone $\sigma$ of the coefficients $\Delta_i$ will be denoted by $\operatorname{tail}(\mathcal{D})$. The positivity assumptions imply that $\mathcal{D}(u) + \mathcal{D}(u') \leq \mathcal{D}(u + u')$, so that $\mathcal{O}_Y(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_Y(\mathcal{D}(u))$ becomes a sheaf of rings. We define $\tilde{X}(\mathcal{D}) := \operatorname{Spec}_Y \mathcal{O}_Y(\mathcal{D})$ over $Y$, and we furthermore set $X(\mathcal{D}) := \operatorname{Spec}(Y, \mathcal{O}(\mathcal{D}))$.

The latter space does not change if $\mathcal{D}$ is pulled back via a birational modification $Y' \rightarrow Y$ or if $\mathcal{D}$ is modified by a polyhedral principal divisor on $Y$ where the latter denotes an element in the image of the natural map $N \otimes \mathbb{Z} \mathcal{C}(Y)^* \rightarrow \operatorname{Pol}_\mathbb{Q}(N, \sigma) \otimes \mathbb{Z} \operatorname{CaDiv}(Y)$. Two $p$-divisors that differ by chains of the upper operations are called equivalent. Note that this implies that one can always ask for $Y$ to be smooth.

**Theorem 2.1** ([AH06], Theorems (3.1), (3.4); Corollary (8.12)). The map $\mathcal{D} \mapsto X(\mathcal{D})$ yields a bijection between equivalence classes of $p$-divisors and normal, affine varieties with an effective $T$-action.

Finally, since we always want to assume that $Y$ is projective, we will explicitly allow $\emptyset \in \operatorname{Pol}_\mathbb{Q}(N, \sigma)$ as polyhedral coefficients of $\mathcal{D}$. Then, $\mathcal{D} = \sum_i \Delta_i \otimes D_i$ should be interpreted as $\sum_{\Delta_i \neq \emptyset} \Delta_i \otimes D_i|_{\operatorname{loc} \mathcal{D}}$ with $\operatorname{loc} \mathcal{D} := Y \setminus \cup_{\Delta_i = \emptyset} D_i$.

### 2.2. Complexity one

The situation in fact simplifies a lot in complexity one ($k = n - 1$), because $Y$ becomes a smooth projective curve. For example, introducing the addition rule $\Delta + \emptyset := \emptyset$, we may define the degree of $\mathcal{D} = \sum_i \Delta_i \otimes D_i$ as

$$
\deg \mathcal{D} := \sum_i (\deg D_i) \cdot \Delta_i \in \operatorname{Pol}^+\left(N_\mathbb{Q}, \sigma\right).
$$

In particular, $\deg \emptyset = 0 \iff \operatorname{loc} \mathcal{D} \neq Y \iff \operatorname{loc} \mathcal{D}$ is affine. This easily implies the following criterion: $\mathcal{D}$ is a $p$-divisor if and only if $\deg \mathcal{D} \subseteq \operatorname{tail} \mathcal{D}$ and, additionally, $\mathcal{D}((\emptyset \neq 0) \cdot w)$ is principal for $w \in (\operatorname{tail} \mathcal{D})^\vee$ with $w^\perp \cap (\deg \mathcal{D}) \neq 0$. Note that the latter condition is automatically fulfilled if $Y = \mathbb{P}^1$.

The assignment $\mathcal{D} \mapsto X(\mathcal{D})$ of [21] is functorial. In particular, as was shown in [IS], if $\mathcal{D}$ is a $p$-divisor containing some $\mathcal{D}' = \sum_i \Delta'_i \otimes D_i$ (meaning that $\Delta'_i \subseteq \Delta_i$ for all $i$), then $\mathcal{D}'$ is again a $p$-divisor which induces a $T$-equivariant open embedding $X(\mathcal{D}') \hookrightarrow X(\mathcal{D})$ if and only if $\mathcal{D}' \leq \mathcal{D}$, i.e. if all the coefficients $\Delta'_i \leq \Delta_i$ are faces, and

$$
\deg \mathcal{D}' = \deg \mathcal{D} \cap \operatorname{tail} \mathcal{D}'.
$$

In particular, if $p$-divisors $\mathcal{D}''$ are arranged in a so-called divisorial fan $\mathcal{S} = \{\mathcal{D}''\}$, then we can glue the associated affine $T$-varieties $X(\mathcal{D}'')$ to obtain a separated $X(\mathcal{S}) = \bigcup_j X(\mathcal{D}'')$, cf. [AH06]. The so-called slices $\mathcal{S}_j = \{\Delta''_j\}$ form a polyhedral subdivision in $N_\mathbb{Q}$, and $\operatorname{tail} \mathcal{S} := \{\operatorname{tail} \mathcal{D}_i\}$ is called the tail fan of $\mathcal{S}$. Moreover, the subsets $\deg \mathcal{D} \subseteq \operatorname{tail} \mathcal{D}$ glue to a subset $\deg \mathcal{S} \subseteq \operatorname{tail} \mathcal{S} \subseteq N_\mathbb{Q}$. Roughly speaking, we understand that $\mathcal{S} = \sum_i \mathcal{S}_i \otimes D_i$. Yet, to keep the full information of the divisorial fan $\mathcal{S}$ one needs a labeling of the $\mathcal{S}_i$-cells indicating the $p$-divisor they come from.

However, following [IS], the technical description can be reduced considerably since one may eventually forget about the labeling. Instead, one only needs to mark those cones $\operatorname{tail} D_i$ inside $\operatorname{tail} \mathcal{S}$ which have non-empty $\deg D_i$. The marked cones together with the formal sum $\mathcal{S} = \sum_i \mathcal{S}_i \otimes D_i$ then give the so-called marked fansy divisor associated with the divisorial fan $\mathcal{S}$. Conversely, given a so-called fansy divisor $\mathcal{S}$, i.e. a formal sum $\mathcal{S} = \sum_i \mathcal{S}_i \otimes D_i$ of polyhedral subdivisions $\mathcal{S}_i$ with
common tailfan, one can find a small system of axioms for a marking of the cones of tail \( S \) to yield a divisorial set \( S = \{ \mathcal{D}^\nu \} \), cf. [IS].

### 2.3. Equivariant Weil divisors

Let \( \mathcal{D} \) be a p-divisor of arbitrary complexity. From [AH06] we know that, similar to the toric case, there is a decomposition of \( \tilde{X}(\mathcal{D}) \) into \( T \)-orbits \( \text{orb}(y,F) \) where \( y \in Y \) and \( F \) is a non-empty face of the polyhedron \( \Delta_y := \sum_{D_i \ni y} \Delta_i \) (the neutral element tail \( \mathcal{D} \) serves as the sum of the empty set of summands). The dimension of their closures is \( \dim \text{orb}(y,F) = \dim \mathcal{P} + \text{codim}_X F \). Note that different \( T \)-orbits might be identified via the contraction \( \tilde{X}(\mathcal{D}) \to X(\mathcal{D}) \).

In particular, the set of \( T \)-equivariant prime divisors in \( \tilde{X}(\mathcal{D}) \) is twofold: On the one hand, we have the so-called \emph{vertical} divisors

\[
D_{Z,v} := \text{orb}(\eta(Z), v).
\]

They are associated with prime divisors \( Z \subseteq Y \) (with \( \eta(Z) \) being the generic point of \( Z \)) together with a vertex \( v \in \Delta_Z \). On the other hand, there are the so-called \emph{horizontal} divisors

\[
D_{\varrho} := \text{orb}(\eta(Y), \varrho).
\]

These correspond to rays \( \varrho \) of the cone tail \( D = \Delta_{\eta(Y)} \), where \( \eta(Y) \) denotes the generic point of \( Y \). The \( T \)-equivariant prime divisors in \( X(\mathcal{D}) \) correspond exactly to those on \( \tilde{X}(\mathcal{D}) \) that are not contracted via \( \tilde{X}(\mathcal{D}) \to X(\mathcal{D}) \).

Let us now assume that \( \mathcal{D} \) is of complexity one. Then, the vertical divisors \( D_{(p,v)} \) (with \( p \in Y(\mathbb{C}) \) and \( v \in \Delta_p \)) survive completely in \( X(\mathcal{D}) \). In contrast, \( D_{\varrho} \) becomes contracted if and only if the ray \( \varrho \) is not disjoint from \( \deg \mathcal{D} \). Finally, we denote by \( T \text{-Div} X(\mathcal{D}) \) the free abelian group of the \( T \)-equivariant Weil divisors in \( X(\mathcal{D}) \).

### 2.4. Equivariant principal divisors

Again, let \( \mathcal{D} \) be a p-divisor of arbitrary complexity, and let \( K(Y) \) be the function field of \( Y \). Then we have that \( \mathcal{O}_Y(\mathcal{D}) \subseteq K(Y)[M] \subseteq K(\tilde{X}) = K(X) \). In particular, \( K(Y)[M] \) consists exactly of the semi-invariant, i.e. \( M \)-homogeneous rational functions on \( X \) – and this remains true for the non-affine \( T \)-varieties \( X(S) \) obtained by gluing.

Identifying a ray \( \varrho \) with its primitive generating lattice vector and denoting by \( \mu(v) \) the smallest integer \( k \geq 1 \) for \( v \in N_{\mathbb{Q}} \) such that \( k \cdot v \) is a lattice point, one has the following characterization of \( T \)-equivariant principal divisors on \( \tilde{X} \) or \( X \):

\[\textbf{Theorem 2.2 (PS), Section 3.} \quad \text{Let} \ f(y)\chi^u \in K(Y)[M]. \quad \text{The associated principal divisor on} \ \tilde{X} \text{ or} \ X \text{ is then given by} \]

\[
\text{div}(f(y)\chi^u) = \sum_{\varrho} \langle \varrho, u \rangle D_{\varrho} + \sum_{\langle Z, v \rangle} \mu(v)(\langle v, u \rangle + \text{ord}_Z f) \cdot D_{(Z,v)}
\]

where, if focused on \( X \), one is supposed to omit all prime divisors being contracted via \( \tilde{X} \to X \).

Let us present a short proof that differs from the original one:

**Proof.** It suffices to check the formula on \( \tilde{X}(\mathcal{D}) \). Then everything becomes local, and we can use a formal neighborhood of \( \eta(Z) \) in \( Y \). Thus, \( (Y, \eta(Z)) = (\mathbb{A}^1_k, 0) \) with \( K = K(\eta(Z)) \) being the residue field.

In particular, \( \mathcal{D} = \Delta \otimes [0 \in \mathbb{A}^1_k] \), which is the downrange of a toric situation. Using [AH06, §11], one sees that \( \tilde{X} = TV \left( \text{cone}(\Delta, 1) \subseteq N_{\mathbb{Q}} \oplus \mathbb{Q} \right) \). Moreover, the ray \( \varrho \) corresponds to \( (\varrho, 0) \in N \oplus \mathbb{Z} \), and the vertex \( v \in \Delta \) turns into the ray generated by \( (v, 1) \) or, equivalently, by the primitive lattice generator \( (\mu(v)v, \mu(v)) \in N \oplus \mathbb{Z} \).

On the other hand, \( f(y)\chi^u \) translates into \( f_{\text{ord}_Z f} \cdot \chi^u \), i.e. into \( [u, \text{ord}_Z f] \in M \oplus \)
Z. Pairing the latter element with the upper pairs \((\rho, 0), (\mu(v) v, \mu(v)) \in N \oplus \mathbb{Z}\) completes the proof. 

2.5. Divisor classes and global sections. Let \(X = X(S)\) for a complete divisorial fan \(S = \sum_{P \in \mathcal{P}} S_P \oplus [p]\) on \(Y = \mathbb{P}^1\). In particular, \(\deg S \subseteq |\text{tail} S| = N_Q\). Choose a finite set of points \(\mathcal{P} \subseteq \mathbb{P}^1\) such that \(S_p = \text{tail} S\) for \(p \in \mathbb{P}^1 \setminus \mathcal{P}\). For a vertex \(v\) of some slice of \(S\) we denote by \(p(v) \in \mathbb{P}^1\) the point whose slice \(S_{p(v)}\) we have taken \(v\) from. Let us define the following sets:

\[
\mathcal{V} := \{v \in S_P(0) \mid p \in \mathcal{P}\} \quad \text{and} \quad \mathcal{R} := \{\rho \in (\text{tail} S)(1) \mid \rho \cap \deg S = \emptyset\}.
\]

Using the notation of (2.2), this leads to the definition of the following two natural maps

\[
Q : \mathbb{Z}^{\mathcal{V} \cup \mathcal{R}} \rightarrow \mathbb{Z}^\mathcal{P}/\mathbb{Z} \quad \text{with} \quad e(v) \mapsto \mu(v) \overline{e(p(v))} \quad \text{and} \quad e(\rho) \mapsto 0,
\]

and

\[
\varphi : \mathbb{Z}^{\mathcal{V} \cup \mathcal{R}} \rightarrow N \quad \text{with} \quad e(v) \mapsto \mu(v)v \quad \text{and} \quad e(\rho) \mapsto \rho
\]

with \(e(v)\) and \(e(\rho)\) denoting the natural basis vectors as in (1.5). Since we know from (2.3) that \(\mathbb{Z}^{\mathcal{V} \cup \mathcal{R}*} \subseteq \text{T-Div} X\), the above Theorem 2.2 now yields a direct description of the class group \(\text{Cl}(X)\).

**Corollary 2.3.** For \(X = X(S)\), one has the exact sequence

\[
0 \rightarrow (\mathbb{Z}^\mathcal{P}/\mathbb{Z})^* \oplus M \rightarrow \mathbb{Z}^{\mathcal{V} \cup \mathcal{R}*} \rightarrow \text{Cl}(X) \rightarrow 0
\]

where the first map is induced from \((Q, \varphi)\).

**Proof.** If we dealt with the whole projective line \(\mathbb{P}^1\) instead of the finite subset \(\mathcal{P}\), then \((\mathbb{Z}^\mathcal{P}/\mathbb{Z})^*\) would represent the principal divisors on \(\mathbb{P}^1\), and the formula \(\text{div}(f(y)\chi^v) = (Q, \varphi)^*(\text{div}(f), u)\) of Theorem 2.2 would provide the exactness of the sequence. However, for \(p \in \mathbb{P}^1 \setminus \mathcal{P}\), we have \(S_p = \text{tail} S\), i.e. the corresponding \(\mathbb{Z}\)-summands of the first and second place cancel each other. \(\square\)

Let us eventually come to the description of global sections for \(D \in \mathbb{Z}^{\mathcal{V} \cup \mathcal{R}*} \subseteq \text{T-Div}\). From [PS] we recall the associated polyhedron

\[
\square_D := \text{conv}\{u \in M_Q \mid \langle \rho, u \rangle \geq -\text{coeff}_\rho D \text{ for all } \rho \in \mathcal{R}\}
\]

where \(\text{coeff}_\rho D\) denotes the coefficient of \(D_\rho\) inside \(D\). Moreover, there is a map \(D^* : \square_D \rightarrow \text{Div}_Q \mathbb{P}^1\) with \(\text{coeff}_p D^* (u) := \min\{\langle v, u \rangle + \text{coeff}_{(p,v)} D/\mu(v) \mid v \in S_p(0)\}\). As another direct consequence of Theorem 2.2 one obtains \(\Gamma(X, \mathcal{O}_X(D))(u) = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D^*(u)))\), cf. [PS].

3. Finite Coverings of \(\mathbb{P}^1\)

3.1. Finite covers with abelian group actions. It turns out that the \(p\)-divisors of the Cox rings under investigation do not all live on \(\mathbb{P}^1\), but rather on special branched coverings of it. The latter come with an action of a finite abelian group \(A\) whose quotient is \(\mathbb{P}^1\) again. We will present these coverings in the spirit of \(p\)-divisors, yet with different and unusual coefficients. For basic facts about group cohomology used in the subsequent text see [Ser79] Ch VII, p.109 ff.

Let us point out that algebraic actions of finite abelian groups as well as tori on normal varieties should be considered as special instances of a yet to be developed general theory of actions of diagonalizable groups.
3.2. $A$-divisors on $\mathbb{P}^1$. Let $A, A^*$ be finite and mutually dual abelian groups, i.e. $A^* := \text{Ext}^1_G(A, \mathbb{Z}) = \text{Hom}_G(A, \mathbb{Q}/\mathbb{Z})$. Consider an element $E \in \text{Div}_A^0(\mathbb{P}^1) := A \oplus \text{Div}^0(\mathbb{P}^1)$ which is is equivalent to a group homomorphism $\phi : A^* \to \text{Div}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{P}^1)$. Here, we denote by $\text{Div}^0(\mathbb{P}^1)$ the group of divisors on $\mathbb{P}^1$ which have degree zero. In the following we will construct an $(\neq A)$-fold covering $g : C \to \mathbb{P}^1$ out of these data:

For $a \in A^*$ denote by $\tilde{E}_a$ an arbitrary lifting of the $\mathbb{Q}/\mathbb{Z}$-divisor $E_a := (\mathcal{E}, a)$ to a $\mathbb{Q}$-divisor of degree zero. Then, for $a, b \in A^*$ there exist $f_{a,b} \in K(\mathbb{P}^1)^*$ with

$$\tilde{E}_a + \tilde{E}_b + \text{div}(f_{a,b}) = \tilde{E}_{a+b}.$$ 

These functions $f_{a,b}$ are unique up to constants, and because of $H^2(A^*, \mathbb{C}^*) = 0$ ($\mathbb{C}^*$ is a divisible and therefore injective abelian group), they can be chosen in such a way that

$$f_{a,b} \cdot f_{a+b,c} = f_{a,b+c} \cdot f_{a,c}.$$ 

Indeed, this follows from the fact that $s_{a,b,c} := f_{b,c} f_{a+b,c} f_{a,b+c} f_{a,b}^{-1} \in \mathbb{C}^*$ is a 3-cocycle, and hence arises from elements $s_{a,b} \in \mathbb{C}^*$. Thus, we are allowed to replace the functions $f_{a,b}$ by $f_{a,b}/s_{a,b}$, and the latter finally fulfill the above multiplicative rule. Introducing the map

$$\mathcal{O}_{\mathbb{P}^1}(\tilde{E}_a) \otimes \mathcal{O}_{\mathbb{P}^1}(\tilde{E}_b) \to \mathcal{O}_{\mathbb{P}^1}(\tilde{E}_{a+b})$$

$$f \otimes g \mapsto fg f_{a,b}^{-1}$$

induces an associative multiplication on $\mathcal{A} := \oplus_{a \in A^*} \mathcal{O}_{\mathbb{P}^1}(\tilde{E}_a)$ which eventually gives rise to the definition of the curve $C := \text{Spec}_{\mathcal{A}} A$.

Remark 3.1. 1) The structure of the algebra $\mathcal{A}$ does not depend on the choice of the rational functions $f_{a,b}$. Indeed, let $g_{a,b}$ be another choice. Then, the quotient $s_{a,b} = g_{a,b}/f_{a,b} \in \mathbb{C}^*$ represents a 2-cocycle, and $H^2(A^*, \mathbb{C}^*) = 0$ implies the existence of $s_a \in \mathbb{C}^*$ with $s_{a,b} = s_a s_b s_{a+b}^{-1}$. Thus, the equation $g_{a,b}^{-1} s_a s_b = s_{a+b} f_{a,b}^{-1}$ shows that the multiplication with $s_a$ yields an automorphism of $\Gamma(\mathbb{P}^1, \tilde{E}_a)$ identifying the ring structures induced from the elements $f_{a,b}$ and $g_{a,b}$, respectively.

2) Similarly, $\mathcal{A}$ does not depend on the choice of the rational liftings $\tilde{E}_a$ of the $\mathbb{Q}/\mathbb{Z}$-divisors $E_a$. Indeed, replacing them by $\tilde{E}_a' = \tilde{E}_a + \text{div}(s_a)$ for rational functions $s_a \in K(\mathbb{P}^1)^*$ leads to adjusted elements $f_{a,b}' = f_{a,b} s_{a+b} s_a^{-1} s_b^{-1}$, and thus to mutually compatible isomorphisms $\Gamma(\mathbb{P}^1, \tilde{E}_a') \cong \Gamma(\mathbb{P}^1, \tilde{E}_a)$.

Theorem 3.2. The curve $C$ constructed above is smooth.

The proof of this theorem will be given in paragraph 3.3. In particular it will also allow for a detailed study of the monodromy of the covering.

On the other hand, suppose we are given a smooth projective curve $C$ together with an action of a finite abelian group $A$ such that its quotient is the projective line. Then with a little effort one can construct $A$-divisor on $\mathbb{P}^1$ that recovers the whole picture.

3.3. Cyclic coverings. The usual description of a cyclic $n$-fold covering of $\mathbb{P}^1$ makes use of an invertible sheaf $\mathcal{L}$ on $\mathbb{P}^1$ together with an effective and reduced divisor $D \subseteq \mathbb{P}^1$, and an isomorphism $s : \mathcal{L}^n \cong \mathcal{O}_{\mathbb{P}^1}(-D)$, cf. e.g. [La92] Chapter 4.1. Hence, the sheaf of $\mathcal{O}_{\mathbb{P}^1}$-algebras $\mathcal{A} := \oplus_{i=0}^{n-1} \mathcal{L}^i$ obtains a multiplication via $s : \mathcal{L}^n \to \mathcal{O}_{\mathbb{P}^1}$. If one chooses a divisor $E$ with $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(E)$, then $s$ corresponds to a rational function such that $nE + \text{div}(s) = -D$.

This construction fits into the pattern of 3.2 as follows: Let $A = \mathbb{Z}/n\mathbb{Z}$. If $D$ is a divisor on $\mathbb{P}^1$ with $n|\deg D$, then this gives rise to a map $\mathbb{Z}/n\mathbb{Z} \to \text{Div}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{P}^1)$,
1 \mapsto \frac{1}{n} D. Here, the ordinary divisor \( E \) is used to obtain the liftings to the \( \mathbb{Q} \)-divisors \( \tilde{E}_i := iE + \frac{j}{n} D \) for \( i = 0, \ldots, n-1 \). Finally, the functions

\[
f_{i,j} := \begin{cases} 1 & \text{if } i + j \leq n - 1, \\ s & \text{if } i + j \geq n \end{cases}
\]

fulfill \( \tilde{E}_a + \tilde{E}_b + \text{div}(f_{a,b}) = \tilde{E}_{a+b} \) as well as \( f_{a,b} \cdot f_{a+b,c} = f_{a,b+c} \cdot f_{b,c} \).

Note that the dual language is even simpler. Namely, the covering corresponds exactly to the divisor \( D \) which is to be considered as an element of \( \text{Div}^0_{/\mathbb{Z}} \mathbb{P}^1 \).

3.4. Proof of Theorem 3.2

3.4.1. Let \( p \in \mathbb{P}^1 \) be a point with local parameter \( t \), and identify \( K(\mathbb{P}^1) \) with \( \mathbb{C}(t) \). Furthermore, let \( f_{a,b} = t^{k(a,b)} \varepsilon_{a,b} \) with \( \varepsilon_{a,b} \in \mathbb{C}[t]_{(a)} \) (i.e. \( \varepsilon_{a,b}(0) \neq 0 \)) where \( k(a,b) := \text{ord}_p f_{a,b} \in \mathbb{Z} \). Note that the units \( \varepsilon_{a,b} \) fulfill the same cocycle conditions as the functions \( f_{a,b} \).

We may consider \( E_a \) as a \( \mathbb{Q} \)-divisor with coefficients in the interval \([0,1]\), i.e. it becomes the fractional part \( \{\tilde{E}_a\} \). We now define \( \ell(a) := \text{coeff}_p E_a \in [0,1] \cap \mathbb{Q} \) inducing a linear function \( \ell : A^* \to [0,1] \to \mathbb{Q}/\mathbb{Z} \), i.e. \( \ell \in A \). Note that this is exactly the \( p \)-coefficient of the \( A \)-divisor \( E \). Moreover, we define

\[
\lambda(a,b) := \begin{cases} 0 & \text{if } \ell(a) + \ell(b) < 1 \\ 1 & \text{if } \ell(a) + \ell(b) \geq 1, \end{cases}
\]

which gives us \( \ell(a) + \ell(b) = \ell(a + b) + \lambda(a,b) \). Finally we set \( m(a) := \text{coeff}_p [\tilde{E}_a] \) which yields

\[
m(a) + m(b) + k(a,b) + \lambda(a,b) = m(a + b).
\]

Let us denote the local monomial generators of \( \mathcal{O}_{\mathbb{P}^1}(\tilde{E}_a) \subseteq \mathbb{C}(t) \) by \( s_a := t^{-m(a)} \) \( (a \in A^*) \). Then the set \( \{s_a \mid a \in A^*\} \) with \( s_0 = 1 \) becomes a basis of the stalk \( \mathcal{A}_p \) which is a free \( \mathbb{C}[t]_{(a)} \)-module. Yet the multiplication of the \( s_a \) in \( \mathcal{A}_p \) differs from the one of the monomials in \( \mathbb{C}(t) \):

\[
s_a \circ s_b = t^{-m(a)} \circ t^{-m(b)} = t^{-m(a) - m(b)} \ v_a \ v_b = t^{k(a,b) + \lambda(a,b) - m(a + b)} \ f_{a,b}^{-1} = \varepsilon_{a,b}(t)^{\lambda(a,b)} s_{a+b}.
\]

Thus, we obtain \( \mathcal{A}_p = \mathbb{C}[t, s_a \mid a \in A^*]_{(t)}/(t^{\lambda(a,b)} s_{a+b} - \varepsilon_{a,b}(t) s_a s_b) \) with \( s_0 = 1 \).

Next we set \( N := \text{lcm} \{\text{denominators of } \ell(a) \mid a \in A^*\} \), and \( d(a) := N \cdot \ell(a) \in \mathbb{Z} \).

It follows that \( \gcd \{d(a) \mid a \in A^*\} = 1 \). Hence, there exist integers \( c(a) \) such that \( \sum_{a \in A^*} c(a) d(a) = 1 \), and we define

\[
\gamma := \sum_{a \in A^*} c(a) \cdot a \in A^*.
\]

Then we have

\[
\ell(\gamma) = \ell\left( \sum_{a \in A^*} c(a) \cdot a \right) = \sum_{a \in A^*} c(a) \ell(a) = 1/N,
\]

in other words \( d(\gamma) = 1 \). Moreover, note that \( d(a) = 0 \) implies \( \lambda(a,b) = 0 \) for arbitrary \( b \in A^* \).

Now, consider a closed point \( e = (c_a \in \mathbb{C} \mid a \in A^*) \in C \) with \( q(\mathbb{C}) = p \). This is equivalent to the following two conditions on the coordinates of \( \mathbb{C} \):

\[
c_a = 0 \text{ for } d(a) \geq 1,
\]

and

\[
c_a c_b \varepsilon_{a,b}(0) = c_{a+b} \neq 0 \text{ for } d(a) = 0.
\]
It is not hard to check that both conditions imply the equations for the ideal given above. For the other direction, it suffices to prove the claim for \( c_a = 0 \), \( c_a \neq 0 \) for \( d(a) \geq 1 \) and \( d(a) = 0 \), respectively. Denoting the order of \( a \in A^* \) by \( n_a \), we obtain the following equation for both cases:

\[
s_a^{n_a} = \left( \text{product of units in } \mathbb{C}[t]\right) \cdot t^{\lambda} \cdot s_{n_a} \quad \text{with } \lambda = \begin{cases} \geq 1 & \text{if } d(a) \geq 1 \\ 0 & \text{if } d(a) = 0 \end{cases}
\]

Thus, the claim follows from \( s_{n_a} = s_0 = 1 \) which also yields \( c_{n_a} = 1 \).

3.4.2. Having provided the necessary tools in the previous paragraph, we are now ready to define the following map:

\[
\mathcal{A}_p = \mathbb{C}[t, s_a \mid a \in A^*]_{(t)} / \langle t^{\lambda(a, b)} s_{a+b} - \varepsilon_{a, b}(t) s_a s_b \rangle \xrightarrow{\Phi_a} \mathbb{C}\{u\} \supseteq \mathbb{C}\{u\}_{(u)}
\]

with \( \eta_a(u) \in \mathbb{C}\{u\}^* \), such that \( \eta_a(0) = c_a \) for \( d(a) = 0 \), and for all \( a, b \in A^* \)

\[
\eta_a(u) \eta_b(u) \varepsilon_{a, b}(u) = \eta_{a+b}(u).
\]

The existence of the functions \( \eta_a(u) \) follows from the divisibility of the abelian group \( \mathbb{C}\{u\}^* \). The latter property is also the reason for why we have to enlarge the ring \( \mathbb{C}\{u\}_{(u)} \) to \( \mathbb{C}\{u\}^* \) (corresponding to the non-rationality of the curve \( C \)). Note that \( \Phi_a(s_r) = \eta_r(u^N) u \). Observe that by choosing \( E_0 = 0 \), we can arrange that \( f_{a,0} = \varepsilon_{a,0} = 1 \) and therefore \( \eta_0(u) = 1 \).

\( \Phi_a \) maps \( t \) together with all differences \( (s_a - c_a) \) into the ideal \( (u) \subseteq \mathbb{C}\{u\} \), i.e. \( \Phi_a \) induces a ring homomorphism between the completions

\[
\hat{\Phi}_a : \hat{\mathcal{O}}_{\mathcal{C}}^\infty = (\hat{\mathcal{A}}_p)^\infty \longrightarrow \mathbb{C}\{u\}.
\]

Since we have \( \Phi_a(s_r) = \eta_r(u^N) u \), we conclude that \( \hat{\Phi}_a \) is surjective.

We claim that \( \hat{\Phi}_a \) is an isomorphism. To show that \( \hat{\Phi}_a \) is injective we construct a map \( \hat{\Psi}_a : \mathbb{C}\{u\} \rightarrow (\hat{\mathcal{A}}_p)^\infty \) such that \( \hat{\Psi}_a \circ \hat{\Phi}_a = 1 \). Using the identity \( \varepsilon_{a, b} = \eta_{a+b} / \eta_a \eta_b \), we transform our initial relations \( t^{\lambda(a, b)} s_{a+b} - \varepsilon_{a, b}(t) s_a s_b = 0 \) in \( \mathcal{A}_p \) into

\[
\eta_{a+b}^{-1} s_{a+b} t^{\lambda(a, b)} - \eta_a(t)^{-1} s_a \cdot \eta_b(t)^{-1} s_b = 0.
\]

To simplify our notation we substitute the \( s_a \) by the new variables \( t_a := \eta_a(t)^{-1} s_a \) which transforms the upper relation into

\[
t_{a+b} t^{\lambda(a, b)} - t_a t_b = 0.
\]

Then we define the map \( \hat{\Psi}_a : \mathbb{C}\{u\} \rightarrow (\hat{\mathcal{A}}_p)^\infty \) \( u \mapsto t_u \), and by composition we obtain

\[
(\hat{\Psi}_a \circ \hat{\Phi}_a)(t) = t^N, \quad \text{and} \quad (\hat{\Psi}_a \circ \hat{\Phi}_a)(t_a) = t^d_a.
\]

Thus, it remains to show that \( t = t^N \) and \( t_a = t^d_a \) in the ring \( (\hat{\mathcal{A}}_p)^\infty \). This will be done in (3.4.3).

3.4.3. In an intermediate step, we prove that \( d(a) = d(b) \) implies \( t_a = t_b \) in \( (\hat{\mathcal{A}}_p)^\infty \). First let us restrict ourselves to those elements for which \( d(\cdot) = 0 \). Since \( d(0) = 0 \) and \( t_0 = 1 \) by definition, we have to show that \( t_a = 1 \) in \( (\hat{\mathcal{A}}_p)^\infty \). Yet, this does not involve variables \( t_{a'} \) with \( d(a') \geq 1 \). Hence, it suffices to prove the identity inside the local ring \( (\mathbb{C}[t, t_a \mid d(a) = 0] / (t_{a+b} - t_a t_b))_\mathfrak{m} \), where \( \mathfrak{m} \) is the point that corresponds to the maximal ideal \( (t, t_a - 1) \). Observe that

\[
\mathbb{C}[t, t_a \mid d(a) = 0] / (t_{a+b} - t_a t_b) = \mathbb{C}[t][\ker t \subseteq A^*]
\]
is a commutative group algebra. Furthermore, denoting the order of \( \ker t \) by \( n \), we have the following equalities in \( \mathbb{C}[t][\ker t] \):

\[
(t_a - 1) \cdot (\text{unit}) = \prod_{\xi_k \in \mu_n} (t_a - \xi_k) = t_a^n - 1 = t_0 - 1 = 0.
\]

Consider now elements \( a \in A \) with \( d(a) \geq 1 \). Assume that \( d(a) = d(b) \), and set \( c := a - b \). From \( d(c) = 0 \) we deduce that \( \lambda(b,c) = 0 \) and \( t_c = 1 \). Thus, the relation \( t_a - t_b t_c = 0 \) from the ideal proves our claim.

3.4.4. Let us now come back to the two equations given at the end of (3.4.2). By the relations of the \( t \) the first equality in (3.4.2) now follows from (3.4.3). This completes the proof of Theorem 3.2.

4. The P-Divisor of the Cox Ring

4.1. The map \((Q, \varphi)\). Let \( X = X(S)\) be given by a complete divisorial fan \( S = \sum_{p \in \mathbb{P}^1} S_p \otimes [p] \) on \( \mathbb{P}^1 \). Recall that the map \( Q \) from \( 2.5 \) becomes surjective after tensoring with \( \mathbb{Q} \). Over \( \mathbb{Z} \), however, we define the following two lattices \( \widetilde{N} := \ker Q \subseteq \mathbb{Z}^{\mathbb{P}_1} \), \( L := \text{im} \, Q \subseteq \mathbb{Z}^P/\mathbb{Z}_n \), and we denote by \( A \) the finite abelian group \( A := \text{coker} \, Q \). Note that in all these groups can be assembled in the following exact sequences

\[
0 \to \widetilde{N} \to \mathbb{Z}^{\mathbb{P}_1} \xrightarrow{Q} \mathbb{Z}^P/\mathbb{Z} \to A \to 0 \quad \text{and} \quad 0 \to L \to \mathbb{Z}^P/\mathbb{Z} \to A \to 0.
\]

Furthermore, we choose a section \( s : L \to \mathbb{Z}^{\mathbb{P}_1} \), and by abuse of notation, we denote its rational extension \( s : \mathbb{Z}^P/\mathbb{Z} \to \mathbb{Q}^{\mathbb{P}_1} \) by the very same letter.

Although the next lemma is already a consequence of Corollary 2.3, we believe that the following direct and combinatorial proof sheds some additional light on the whole situation.

**Lemma 4.1.** The map \((Q, \varphi)_Q : \mathbb{Q}^{\mathbb{P}_1} \to (\mathbb{Q}^P/\mathbb{Q}) \oplus N_\mathbb{Q} \) is surjective.

**Proof.** Since \( \deg(D) = \sum_{p \in \mathbb{P}} \text{conv} \, D_p(0) + \text{tail}(D) \subseteq \text{tail}(D) \) for a single \( D \in S, \) every ray \( g \in \text{tail}(D)(1) \) either belongs to \( \mathbb{R} \) (meaning that \( g \cap \deg D = \emptyset \)), or \( g \) intersects \( \sum_{p \in \mathbb{P}} \text{conv} \, D_p(0) \). Thus, every ray \( g \in \text{tail}(S)(1) \) either belongs to \( \mathbb{R} \) or it intersects \( \sum_{p \in \mathbb{P}} \text{conv} \, S_p(0) \) away from the origin. This fact means that non-zero elements of each ray of the tailfan occur in the image of the map \( \varphi_Q : Q^{-1}(0) \to N_\mathbb{Q}, \) i.e. it is surjective.

On the other hand, every slice contains at least one vertex which implies that \( Q_\mathbb{Q} : \mathbb{Q}^P \to \mathbb{Q}^P/\mathbb{Q} \) is also surjective. \( \square \)

4.2. The main polyhedral objects. We briefly recall the polyhedral objects we defined in \( 1.5 \), namely the polytopes

\[
\Delta^c_p = \text{conv} \{ e(v)/\mu(v) \mid v \in S_p(0) \} \subseteq Q^{-1}(e(p)) \subseteq \mathbb{Q}^{\mathbb{P}_1},
\]

and the polyhedral cone

\[
\sigma = \bigoplus_{p \in \mathbb{P}} \Delta^c_p + Q^{\mathbb{R}} = Q^{-1}(0) \cap \mathbb{Q}^{\mathbb{P}_1} \subseteq \widetilde{N}_\mathbb{Q}.
\]

From these data we build

\[
\Delta^c_p := \Delta^c_p + \sigma = Q^{-1}(e(p)) \cap \mathbb{Q}^{\mathbb{P}_1} \quad \text{and} \quad \widetilde{\Delta}_p := \Delta_p - s(e(p)) \subseteq \widetilde{N}_\mathbb{Q}.
\]
Moreover, following Section 3, the cokernel $\pi : \mathbb{Z}^P/\mathbb{Z} \to A$ of $Q$ from (4.1) yields an $A$-divisor of degree zero, i.e. a finite covering $q : C \to \mathbb{P}^1$. Note that for each $c \in C$ the ramification index $e_c \in \mathbb{Z}_{\geq 1}$ only depends on the image $p = q(c) \in \mathbb{P}^1$. While $p \in \mathbb{P}^1 \setminus \mathbb{P}$ leads to $e_c = 1$, the ramification indices of points in the fibers over elements $p \in \mathbb{P}$ equal the order of $\pi(e(p)) \in A$.

**Theorem 4.2.** $\text{Cox}(X)$ corresponds to the $p$-divisor $\mathcal{D}_{\text{Cox}} = \sum_{c \in C} e_c \cdot \tilde{\Delta}_{q(c)} \otimes [c]$ on $C$.

The rest of this section is devoted to the proof of this theorem. Clearly, in the case that $\text{Cl}(X)$ is torsion free, the group $A$ is trivial. Hence, the map $q$ is an isomorphism, and Theorem 4.2 implies Theorem 1.2 from the introduction. The shift mentioned there is the same shift as in the above definition of $\Delta_p$.

**Remark 4.3.** The polyhedral divisor defined in Theorem 4.2 has a complete locus. It is proper since $\deg \mathcal{D}_{\text{Cox}}$ is properly contained in the tailcone, as can easily be seen from the definition (cf. also [AH06, Example 2.12]). Moreover, the handy description of $\mathcal{D}_{\text{Cox}}$ makes it possible to sometimes quickly check some of the singularity criteria for $X(\mathcal{D}_{\text{Cox}})$ presented in [LS, Section 5].

4.3. **The big diagram.** Dualizing the exact sequences from (4.1), we obtain maps like $s^* : \mathbb{Z}^{(\nu \cup \rho)}^* \to L^* \subseteq (\mathbb{Q}^P/\mathbb{Q})^*$, a torsion free lattice $\tilde{M} := \text{Hom}(\tilde{N}, \mathbb{Z})$, and the dual finite abelian group $A^* := \text{Hom}_\mathbb{Z}(\mathbb{A}, \mathbb{Q}/\mathbb{Z}) = \text{Ext}^1_{\mathbb{Z}}(\mathbb{A}, \mathbb{Z})$ as outlined in (3.2). The whole picture may be visualized in the following commutative diagram:

![Diagram](image)

The row containing $L^*$ and $\tilde{M}$ is an exact sequence of free abelian groups. Since $L^*$ is a saturated sublattice, we have that $L^* \cap ((\mathbb{Z}^P/\mathbb{Z})^* \oplus M) = (\mathbb{Z}^P/\mathbb{Z})^*$ within $\mathbb{Z}^{(\nu \cup \rho)}^*$. In particular, $A^* \to \text{Cl}(X)$ is injective. Observe that if $\tilde{M}$ denotes the cokernel of $Q^*$, then we even know that

$$A^* = \text{Tors}(\tilde{M}) \subseteq \text{Tors}(\text{Cl}(X)).$$

Hence, if $\text{Cl}(X)$ is torsion free the finite groups $A^*$ and $A$ become trivial. While the maps $G$, $F$, and $\pi$ are explained by the diagram and the remarks made before, the map $t$ is supposed to be the section induced from $s^*$, i.e. $Q^*s^* + tG = \text{id}$. We will add some details to the remaining maps $\Phi_1, \Phi_2$ appearing in the lower right corner in (4.3).
4.4. The $\hat{M}$- and $\hat{M}$-grading of $\text{Cox}(X)$. Let $f : (\mathbb{Z}^P/\mathbb{Z})^* \to K(P^1)^*$, $E \mapsto f_E$ be a linear map such that $\text{div}(f_E) = E$ for $E \in (\mathbb{Z}^P/\mathbb{Z})^* \subseteq \text{Div}^0 P^1$. We can then deduce from Theorem 2.3 that the map $(f, \chi) : (E, u) \mapsto f_E(y)\chi^u \in K(X)$ is a lifting of
\[ (Q^v, \varphi^v) : (\mathbb{Z}^P/\mathbb{Z})^* \oplus M \to \mathbb{Z}^{(\mathbb{U}\cup\mathbb{R})^*} \subseteq T\text{-Div} X. \]

Now, Corollary 2.3 states that the latter map gives a presentation of $\text{Cl}(X)$. So we are exactly in the setting of [HS, Section 2] used for the definition of the Cox ring under the presence of torsion in the class group. In particular,
\[ \text{Cox}(X) = \bigoplus_{D \in \mathbb{Z}(\mathbb{U}\cup\mathbb{R})^*} \Gamma(X, \mathcal{O}_X(D)) / (1 - f_E\chi^u \mid (E, u) \in (\mathbb{Z}^P/\mathbb{Z})^* \oplus M). \]

The $T$-action on $X$ induces an $M$-grading of all vector spaces $\Gamma(X, \mathcal{O}_X(D))$. On the whole, the sum $\bigoplus_{D \in \mathbb{Z}(\mathbb{U}\cup\mathbb{R})^*} \Gamma(X, \mathcal{O}_X(D))$ admits a $(\mathbb{Z}^{(\mathbb{U}\cup\mathbb{R})^*} \oplus M)$-grading.

In addition, to make the ideal $(1 - f_E\chi^u \mid (E, u) \in (\mathbb{Z}^P/\mathbb{Z})^* \oplus M)$ homogeneous, we have to set the degrees coming from $(\mathbb{Z}^P/\mathbb{Z})^* \oplus M \to \mathbb{Z}^{(\mathbb{U}\cup\mathbb{R})^*} \oplus M$ equal to zero. Since this embedding is represented by the matrix
\[ \begin{pmatrix} Q^v & 0 \\ \varphi^v & \text{id}_M \end{pmatrix}, \]
and since its two columns correspond to the two paths $\Phi_1, \Phi_2$ in the big diagram of [E3], the corresponding exact sequence
\[ 0 \to (\mathbb{Z}^P/\mathbb{Z})^* \oplus M \to \mathbb{Z}^{(\mathbb{U}\cup\mathbb{R})^*} \oplus M \to \hat{M} \to 0 \]
shows that $\text{Cox}(X)$ carries a natural $\hat{M}$-grading. To avoid torsion in the grading group, we downgrade $\text{Cox}(X)$ via $\pi : \hat{M} \to \tilde{M}$.

4.5. Relating the polyhedra $\Delta_p$ and $\tilde{\Delta}_p$ to the Cox sheaf. The multiplication with $\chi^{-u}$ induces isomorphisms between the graded pieces of the above spaces of sections
\[ \chi^{-u} : \Gamma(X, \mathcal{O}_X(D))(u) \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X(D + \varphi^v(u)))(0). \]

In particular, to describe $\text{Cox}(X)$ we may forget all non-zero degrees $u \in M$:
\[ \text{Cox}(X) = \bigoplus_{D \in \mathbb{Z}(\mathbb{U}\cup\mathbb{R})^*} \bigoplus_{u \in M} \Gamma(X, \mathcal{O}_X(D))(u) / (1 - f_E\chi^u \mid E \in (\mathbb{Z}^P/\mathbb{Z})^*). \]

Using the notation and the remarks at the end of [E3] in the first place, and then our knowledge about the vertices of $\Delta_p^v$ (and hence $\tilde{\Delta}_p$) from [L2], we can continue with
\begin{align*}
\text{Cox}(X) & = \bigoplus_{D \in \mathbb{Z}(\mathbb{U}\cup\mathbb{R})^*} \Gamma(P^1, D^v(0)) / (1 - f_E \mid E \in (\mathbb{Z}^P/\mathbb{Z})^*) \\
& = \bigoplus_{D \in \mathbb{Z}(\mathbb{U}\cup\mathbb{R})^*} \Gamma(P^1, \sum_{p \in \text{L}} \min(\Delta_p, D)[p]) / (1 - f_E \mid E \in (\mathbb{Z}^P/\mathbb{Z})^*). 
\end{align*}

Note that multiplication with the elements $f_E$ yields isomorphisms
\[ \Gamma(X, \mathcal{O}_X(D))(0) \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X(D - Q^v(E)))(0). \]

Thus, exactly one summand per fine degree from $\hat{M}$ is needed in the previous direct sum. To make such a choice, we use an arbitrary set theoretical section $\psi : A^* \to L^* \subseteq \mathbb{Z}^{(\mathbb{U}\cup\mathbb{R})^*}$ of $L^* \to A^*$ in the following way:

Let $w \in \hat{M}$. Then $\iota(w) \in \mathbb{Z}^{(\mathbb{U}\cup\mathbb{R})^*}$ is a divisor which represents the single preimage $\text{Ft}(w) \in \hat{M}$ of $w$ via $\pi$. We deduce that $\iota^{-1}(w) = \text{Ft}(w) + A^* \subseteq \hat{M}$, and these elements are represented by the divisors $D := \iota(w) + \psi(a) \subseteq \mathbb{Z}^{(\mathbb{U}\cup\mathbb{R})^*}$ for $a \in A^*$. 

Now we use the orthogonality relations \( \langle \tilde{N}, L^* \rangle = \langle L, \tilde{M} \rangle = 0 \) to obtain the following equalities:

\[
\min(\Delta_p, t(w) + \psi(a)) = \min(\Delta_p, t(w) + \psi(a)) + \langle s(e(p)), t(w) + \psi(a) \rangle = \min(\Delta_p, w) + \langle e(p), s^\ast t(w) + s^\ast(\psi(a)) \rangle = \min(\Delta_p, w) + \langle e(p), \psi(a) \rangle.
\]

Since we have \( e(p) \in \mathbb{Z}^P / \mathbb{Z} \subseteq Q^P / Q = L_Q \), we know that \( (e(p), \psi(a)) \in \mathbf{Q} \). Thus, defining

\[ D := \sum_p \tilde{\Delta}_p \otimes [p] \]

and identifying \( \psi(a) \in L^* \subseteq \tilde{L}_Q = (Q^P / Q)^* \) as the \( \mathbf{Q} \)-divisor \( \sum_p (e(p), \psi(a)) [p] \) on \( \mathbb{P}^1 \), we obtain \( \text{Cox}(X) \) as the global sections of the sheaf

\[ \text{Cox}(X) := \oplus_{w \in \tilde{M}, a \in A^*} \mathcal{O}_{\mathbb{P}^1}(D(w) + \psi(a)). \]

Observe that the ring structure of this sheaf not only makes use of the relations \( D(w) + D(w') \leq D(w + w') \) but also of the rational functions \( f_{\psi(a) + \psi(b) - \psi(a+b)} \) obtained from \( \psi(a) + \psi(b) - \psi(a+b) \in (\mathbb{Z}^P / \mathbb{Z})^* \subseteq L^* \) for \( a, b \in A^* \).

#### 4.6. The covering.

The covering \( q : C \to \mathbb{P}^1 \) is induced from the map \( \pi : \mathbb{P}^1 \to A \) which is the cokernel of \( Q \). This means that the elements \( a \in A^* = \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \) provide \( \mathbb{Q}/\mathbb{Z} \)-divisors \( E_a = \sum_{p \in \mathbb{P}^1} (a \pi)(e(p)) [p] \). According to (3.3) the sheaf \( \mathcal{A} = q_* \mathcal{O}_C \) is then built from their liftings to some \( \tilde{E}_a \in \text{Div}^0_\mathbf{Q} \mathbb{P}^1 \).

Moreover, the fact that \( \psi(a) \in L^* \) maps to \( a \in A^* \) leads to the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & L & \longrightarrow & \mathbb{Z}^P / \mathbb{Z} & \stackrel{\pi}{\longrightarrow} & A & \longrightarrow & 0 \\
& & \psi(a) & \downarrow & \psi(a) & \downarrow & a & \downarrow & 0 \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0
\end{array}
\]

It follows that \( (\overline{e(p)}, \psi(a)) = (a \pi)(\overline{e(p)}) \) in \( \mathbb{Q}/\mathbb{Z} \), i.e. we can choose \( \tilde{E}_a = \psi(a) \).

Furthermore, the elements \( f_{a,b} \) which provide the multiplicative structure of \( \mathcal{A} \) can be defined as \( f_{a,b} = f_{\psi(a) + \psi(b) - \psi(a+b)} \). Thus, using the relation \( \mathcal{O}_{\mathbb{P}^1}(D + E) \supseteq \mathcal{O}_{\mathbb{P}^1}(D) \otimes \mathcal{O}_{\mathbb{P}^1}(E) \) for the \( \mathbb{Q} \)-divisors \( D \) and \( E \), we obtain

\[ \text{Cox}(X) \supseteq \left( \oplus_{w \in \tilde{M}} \mathcal{O}_{\mathbb{P}^1}(D(w)) \right) \otimes \left( \oplus_{a \in A^*} \mathcal{O}_{\mathbb{P}^1}(\tilde{E}_a) \right) = \mathcal{O}_{\mathbb{P}^1}(D) \otimes q_* \mathcal{O}_C. \]

Both sides are \( \tilde{M} \)-graded, and for saturated \( w \) (meaning that \( \min(\tilde{\Delta}_w, w) \in \mathbb{Z} \)), the left hand summand \( D = D(w) \) becomes an ordinary \( \mathbb{Z} \)-divisor. In this case, the inclusion actually becomes an equality.

For any \( \mathbb{Q} \)-divisor \( D = D(w) \) we know that \( |q^* D| \geq |q^* [D]| \), hence

\[ q^* \mathcal{O}_{\mathbb{P}^1}(D) = q^* \mathcal{O}_{\mathbb{P}^1}(|D|) = \mathcal{O}_{\mathbb{P}^1}(q^* [D]) \subseteq \mathcal{O}_{\mathbb{P}^1}(|q^* D|) = \mathcal{O}_{\mathbb{P}^1}(q^* D), \]

and therefore

\[ \mathcal{O}_{\mathbb{P}^1}(D) \otimes q_* \mathcal{O}_C = q_* q^* \mathcal{O}_{\mathbb{P}^1}(D) \subsetneq q_* \mathcal{O}_{\mathbb{P}^1}(q^* D). \]

Combining these relations with the previous inclusion regarding the Cox ring, we finally arrive at

\[ \text{Cox}(X) \supseteq \mathcal{O}_{\mathbb{P}^1}(D) \otimes q_* \mathcal{O}_C \subsetneq q_* \mathcal{O}_{\mathbb{P}^1}(q^* D) \]

with equality for both sides for saturated degrees \( w \in \tilde{M} \). Since the two outer sheaves are normal we have that \( \text{Cox}(X) = q_* \mathcal{O}_{\mathbb{P}^1}(q^* D), \) which then gives

\[ \text{Cox}(X) = \Gamma(\mathbb{P}^1, \text{Cox}(X)) = \Gamma(\mathbb{P}^1, q_* \mathcal{O}_{\mathbb{P}^1}(q^* D)) = \Gamma(C, \mathcal{O}_C(q^* D)). \]

This establishes \( D_{\text{Cox}} = q^* D \) and completes the proof. \( \Box \)
5. Examples

5.1. Cox rings and degenerations. Using the degeneration techniques developed in [Ill, IV], we construct a toric degeneration of \( X(S) = \mathbb{P}(\Omega_{\mathbb{P}^2}) \) from Example 1.1 to the projective cone over the del Pezzo surface of degree six (see also [Süß, Example 5.1]) denoted by \( X' = X(S') \). We have \( \text{tail} S' = \Sigma \), \( Y' = \mathbb{P}^1 \) with the relevant slices pictured in Figure 3. Observe that the marking is the same as for \( S \), namely \( \Sigma(1) \cup \Sigma(2) \).

![Figure 3](image-url)

**Figure 3.** Fansy divisor of a toric degeneration of \( \mathbb{P}(\Omega_{\mathbb{P}^2}) \).

Since \( X(S') \) is toric we know that its Cox ring is a polynomial ring. Applying our recipe, we can see from Figure 3 that the compact part \( \tilde{\Delta}_0 \) is a five-dimensional simplex.

Yet, performing the analogous degeneration on the level of Cox rings, i.e. adding up all polyhedral coefficients of the polyhedral divisor described in Example 1.3, gives a (toric) \( \mathbb{C} \)-algebra which is not a polynomial ring. Observe that the compact part of the only non-trivial polyhedral coefficient is the Minkowski sum of three edges, i.e. a three-dimensional cube.

5.2. Cox rings of log del Pezzo \( \mathbb{C}^* \)-surfaces. We conclude the paper with two examples taken from the classification list of Gorenstein log del Pezzo \( \mathbb{C}^* \)-surfaces in [Süß].

**Example 5.1.** Let \( X(S) \) be the Gorenstein del Pezzo \( \mathbb{C}^* \)-surface of degree three with singularity type \( E_6 \). It has two elliptic fixed points, i.e. \( \mathcal{R} = \emptyset \). The marked fansy divisor \( S \) is illustrated in Figure 4.

![Figure 4](image-url)

**Figure 4.** Fansy divisor of a Gorenstein log del Pezzo \( \mathbb{C}^* \)-surface of singularity type \( E_6 \).

The divisor class group \( \text{Cl} \left( X(S) \right) \) is torsion free of rank one, so \( A = A^* = 0 \) and the covering map \( q : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) is the identity. Using the matrix

\[
Q = \begin{pmatrix}
3 & 0 & -3 & -1 \\
0 & 2 & -3 & -1
\end{pmatrix}
\]
and choosing a suitable section $s$, we obtain a tailcone $\sigma$ which is generated by the rays $(0,1)$ and $(-2,1)$. Furthermore,
\[
\tilde{\Delta}_0 = (0, -1/3) + \sigma,
\tilde{\Delta}_1 = (0, 1/2) + \sigma,
\tilde{\Delta}_\infty = ((0,0), (-1/3,0)) + \sigma,
\]
which eventually gives
\[
D_{\text{Cox}} = \tilde{\Delta}_0 \otimes [0] + \tilde{\Delta}_1 \otimes [1] + \tilde{\Delta}_\infty \otimes [\infty].
\]

**Example 5.2.** Finally, consider the Gorenstein log del Pezzo $\mathbb{C}^*$-surface of degree one with singularity type $E_6$ and $A_2$. It has two elliptic fixed points, so $\mathcal{R} = \emptyset$ as before. The marked fansy divisor $\mathcal{S}$ is pictured in Figure 5.

The divisor class group $\text{Cl}(X(\mathcal{S}))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. As above, using the matrix
\[
Q = \begin{pmatrix}
3 & 0 & -2 & -1 \\
0 & 3 & -2 & -1
\end{pmatrix}
\]
and a suitable section $s$, we derive our combinatorial data of the polyhedral divisor. Namely, the tailcone $\sigma$ is generated by the rays $(3,2)$ and $(0,1)$, whereas the polyhedral coefficients are given by
\[
\tilde{\Delta}_0 = \sigma,
\tilde{\Delta}_1 = (0, 1/3) + \sigma,
\tilde{\Delta}_\infty = ((0,0), (1/2,0)) + \sigma.
\]
Moreover, we also see that $A \cong \mathbb{Z}/3\mathbb{Z}$ gives rise to a 3:1-covering $q : \mathbb{P}^1 \to \mathbb{P}^1$ which is branched over 0 and 1. The polyhedral divisor describing the total coordinate space $\text{Spec} \, \text{Cox} (X(\mathcal{S}))$ is thus given by
\[
D_{\text{Cox}} = 3 \cdot \tilde{\Delta}_0 \otimes [c_0] + 3 \cdot \tilde{\Delta}_1 \otimes [c_1] + \sum_{i=1}^{3} \tilde{\Delta}_\infty \otimes [c^i_\infty],
\]
where $c^i_p \in q^{-1}(p)$ denotes a preimage of $p \in \mathbb{P}^1$.

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