How Many Vertices Does a Random Walk Miss in a Network with Moderately Increasing the Number of Vertices?

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Abstract

Real networks are often dynamic. In response to it, analyses of algorithms on dynamic networks attract more and more attentions in network science and engineering. Random walks on dynamic graphs also have been investigated actively in more than a decade, where in most cases the edge set changes but the vertex set is static. The vertex sets are also dynamic in many real networks. Motivated by a new technology of the analysis of random walks on dynamic graphs, this paper introduces a simple model of graphs with increasing the number of vertices, and presents an analysis of random walks associated with the cover time on such graphs. In particular, we reveal that a random walk asymptotically covers the vertices all but a constant number if the vertex set grows moderately.

Keywords: Cover time, dynamic graph, evolving graph, temporal graph.

1 Introduction

Networks appearing in the real world, such as the Internet, transportation networks, sensor/wireless networks, social networks and chemical dynamics, change their shapes time by time. Nevertheless, what is known about the analyses of algorithms on dynamic networks is quite limited, comparing with a wealth of knowledge on computations in static networks. In response to it, theoretical analyses of models and algorithms on dynamic networks recently attract high attentions, particularly in the context of network science and engineering, concerning such as connectivity, exploration, information spreading, gathering, agreement, sampling, population protocol, random walks and other stochastic processes, see e.g., [28, 27, 21, 8, 33, 23, 9].

Random walk on a graph is a fundamental stochastic process: a walker on a vertex moves to a randomly picked neighbor at each discrete time step. Random walk is a simple and powerful tool in the wide range of computer science, such as randomized search, page rank and MCMC, and so is it in networking science and engineering [9, 33, 5, 34]. The cover time of a random

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walk is the time it takes for a walker to visit all vertices of the graph. The cover time is one of the fundamental quantities of a random walk, see e.g., [3, 1] [26] [18] [17] [15] [2] [25], and it is important with applications such as randomized search. Analyses of random walks on dynamic graphs have been actively developed in the context, where the cover time is a central issue [9] [10] [4] [5] [13] [35] [24] [34] (see Section 1.3 for more detail).

Those existing works, except for Cooper and Frieze [10], about random walks on dynamic networks are concerned only with networks over a static vertex set. However, the real networks change their vertex sets time by time. Motivated by a new analysis technique, this paper investigates random walks on graphs with increasing the number of vertices. A dynamic vertex set causes some technical troubles: it is questionable if the “cover time,” that is a natural quantity for a static vertex set, is also appropriate for a dynamic vertex set, and also it is hopeless, as Cooper and Frieze [10] revealed, to cover vertices beyond a constant ratio when the number of vertices constantly increases. In view of this, we introduce a simple model of growing graphs, and presents an analysis of the number of vertices remaining unvisited by a random walk as a counterpart to the cover time of a random walk on a static vertex set.

1.1 Model and quantities

Example: collection of coupons with increasing the number of types. To introduce our model, let us start with a simple and intuitive example. Suppose you draw a coupon randomly from a finite number of types of coupons every day. A single type of coupon exists on the first day, and a new type of coupon is released at intervals of $n$ days for the number $n$ of existing types of coupons, i.e., you draw from two types of coupons for the second and the third days, draw from three for the fourth to the sixth days, and draw from $n$ for the $\binom{n}{2} + 1$st to the $\binom{n+1}{2}$-th days. It might be difficult to complete all types of coupons because new types are sequentially released. Then, how many types of coupons do you expect to collect? We will prove that you can expect to miss at most two types of coupons. On the other hand, interestingly, the number of uncollected types of coupons diverges to infinity as the days go by if the release intervals are $o(n)$, e.g., $\lceil \sqrt{n} \rceil$ days (see Theorem 1.1).

Coupon collector’s problem is often connected to the cover time of a random walk on a complete graph. Generalizing the above example, we investigate a random walk on a network with moderately increasing the number of vertices. In the network model, we introduce a parameter corresponding to the growth rate of the vertex set, which will be represented by duration, in fact. Then, we will be concerned with the number of unvisited vertices, instead of the cover time.

Random walk on a growing graph. A growing graph is a sequence of graphs $G = G_0, G_1, G_2, \ldots$, where each $G_t = (V_t, E_t)$ is a connected simple undirected graph such that $V_t \subseteq V_{t+1}$. A random walk on a growing graph is a stochastic process $Z = Z_0, Z_1, Z_2, \ldots (Z_t \in V_t)$, where the transition probability from $Z_t$ to $Z_{t+1}$ is provided as a random walk on $G_t$. We remark that $Z_t \in V_{t-1}$ holds for $t = 1, 2, \ldots$, in fact.
This paper is particularly concerned with a simple model of growing graphs with moderate changes. Roughly speaking, a growing graph \( G \) in this paper keeps being a graph \( G(n) \) unchanged for some duration of steps, then changes its shape to \( G(n+1) \) by adding a single vertex and connecting it to \( G(n) \). Let \( \mathcal{G} : \mathbb{N} \to \mathbb{N} \) be a function, denoting the duration of keeping the graph unchanged. Then, \( \mathcal{G} \) is given as \( \mathcal{G}_t = G(n) \) for \( t \) satisfying \( \sum_{i=1}^{n-1} \mathcal{G}(i) \leq t < \sum_{i=1}^{n} \mathcal{G}(i) \) for \( n = 1, 2, \ldots, \), where \( G(n) = (V(n), E(n)) \) is a connected graph such that \( V(n) = \{v_1, \ldots, v_n\} \) and \( E(n) = E(n-1) \cup \{\{v_n, u\} : \text{for some } u \in V(n-1)\} \) for \( v_n \in V(n) \setminus V(n-1) \). Notice that \( \mathcal{G}_0 \) is a graph of a single vertex. In other words, \( \mathcal{G}(n) \) denotes the duration of \( |\mathcal{V}_t| = n \), and hence \( \mathcal{G}(n) = \min\{t : |\mathcal{V}_t| = n+1\} - \min\{t : |\mathcal{V}_t| = n\} \) holds. For convenience, let \( T_n := \sum_{i=1}^{n-1} \mathcal{G}(i) = \min\{t : |\mathcal{V}_t| = n\} \). Figure 1 shows the correspondence between \( \mathcal{G}_t \) and \( G(n) \) in case of \( \mathcal{G}(n) = n \).

This paper is also concerned with a particular model of random walks on growing graphs. For simplicity, we assume that a random walk on a growing graph \( \mathcal{G} \) is temporarily time-homogeneous, meaning that a random walk is formally represented by a common \( n \times n \) transition matrix \( P(n) \) such that \( \Pr[Z_{t+1} = v | Z_t = u] = (P(n))_{u,v} \) when \( \mathcal{G}_t = G(n) \). We simply represent a random walk on a growing graph \( (RWoGG, \text{for short}) \) by a triple \( R = (\mathcal{G}, (G(n))_{n=1}^{\infty}, (P(n))_{n=1}^{\infty}) \).

Then, we are concerned with the number of vertices unvisited by a RWoGG, formally given by

\[
\mathcal{U}_t := |\{v \in \mathcal{V}_{t-1} : v \neq Z_s \text{ for any } s \in \{0, 1, \ldots, t\}\}|
\]

where recall the fact that \( Z_t \in \mathcal{V}_{t-1} \). Particularly, let \( U(n) \) (or simply \( U \) without confusion) denote \( U_{T_{n+1}} \), i.e., \( U(n) = n - \left| \bigcup_{t=0}^{T_{n+1}} \{Z_t\} \right| \), and we will be concerned with it. Remark that \( \mathcal{U}_t \) is monotone nonincreasing for \( t \in (T_n, T_{n+1}] \), and \( U(n-1) + 1 \geq \mathcal{U}_t \geq U(n) \) hold for the same time period.

**Terminology on time-homogeneous Markov chains.** We here briefly introduce other terminology for random walks on static graphs, or time-homogeneous Markov chains, cf. 

Suppose that \( X_0, X_1, X_2, \ldots \) is a random walk on a static graph \( G = (V, E) \) characterized by a time-homogeneous transition matrix \( P = (P(u,v)) \in [0,1]^{V \times V} \) where \( P(u,v) = \Pr[X_{t+1} = v | X_t = u] \). A transition matrix \( P \) is **irreducible** if \( \forall u, v \in V, \exists t > 0, (P^t)_{u,v} > 0 \), and is **aperiodic** if \( \forall v \in V, \gcd\{t > 0 : (P^t)_{u,v} > 0\} = 1 \). An irreducible and aperiodic \( P \) is said to be **ergodic**. A probabilistic distribution \( \pi \) over \( V \) is a **stationary distribution** if it satisfies \( \pi P = \pi \). It is well known that an

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1. For instance, \( G(n) \) is a complete graph, a path graph, an expander graph, etc. of order \( n \) respectively.
2. E.g., \( \mathcal{G}(n) = n \).
3. This is just for convenience of descriptions, but not essential in our later analyses. See also Appendix A.
ergodic $P$ has a unique stationary distribution \[25\]. A random walk is lazy if $P_{u,v} \geq 1/2$ for all $v \in V$, is reversible if $\pi(u)P_{u,v} = \pi(v)P_{v,u}$ hold for all $u, v \in V$, and where $\pi \in [0,1]^V$ is the stationary distribution, and is symmetric if $P_{u,v} = P_{v,u}$ holds for all $u, v \in V$. A simple random walk (resp. simple lazy random walk) on an undirected graph is given by $P_{u,v} = 1/d_u$ for $\{u,v\} \in E$ (resp. $P_{u,v} = 1/(2d_u)$ for $\{u,v\} \in E$ and $P_{u,u} = 1/2$) where $d_u$ is the degree of $u$. The hitting time $t_{hit}$ (also denoted by $t_{hit}(P)$) is given by $t_{hit} := \max_{u,v \in V} E[\min\{t \geq 0 : X_0 = u \text{ and } X_t = v\}]$. The cover time $t_{cov}$ (or $t_{cov}(P)$) is given by $t_{cov} := \max_{u \in V} E[\min\{t \geq 0 : \exists v \in V, \exists s \leq t, X_s = v\}]$. The mixing time $t_{mix}$ is given by $t_{mix} := \min\{t > 0 : (1/2) \max_{u \in V} \sum_{v \in V} |P^t(u,v) - \pi(v)| \leq 1/4\}.$

1.2 Our results

This paper investigates the behavior of $E[U]$ regarding $d$ for a RWoGG $R = (d, (G^{(i)})_{i=1}^\infty, (P^{(i)})_{i=1}^\infty)$, where recall that $U$ is an abbreviation of $U(n) = UT_{n+1}$ denoting the number of vertices unvisited by the random walk at the moment just before a new vertex $v_{n+1}$ is attached (see Section 1.1 for precise). Our results are summarized as follows.

**Complete graph** (Section 2). As an introductory example of our analyses, we firstly concerned with a random walk on a growing complete graph, which corresponds to the example of collecting coupons with new releases in Section 1.1. Let $R_c = (d, (G^{(i)})_{i=1}^\infty, (P^{(i)})_{i=1}^\infty)$ be a random walk on a growing complete graph, where $G^{(i)}$ is a complete graph of order $i$, and $(P^{(i)})_{u,v} = 1/i$ for any $u \in V^{(i)}$ and $v \in V^{(i)}$ (including $u = v$).

**Theorem 1.1.** For $R_c = (d, (G^{(i)})_{i=1}^\infty, (P^{(i)})_{i=1}^\infty)$, the following holds:

1. If there is a constant $C > 0$ such that $d(i) \geq Ci$ for all $i \in [n]$, then $E[U] = O(1)$.
2. If $d(i)/i \to \infty$ as $i \to \infty$, then $E[U] \to 0$ as $n \to \infty$.
3. If $d$ is unbounded (i.e., $d(i) \to \infty$ as $i \to \infty$) and satisfies for all $i \in \mathbb{N}$ that $d(i) \geq d(i+1)/\pi$, then $E[U] = (1-o(1)) \frac{n}{d(n) + 1}$.
4. If $d$ is constant (i.e., $\exists c \in \mathbb{N}, \forall i \in \mathbb{N}$, $d(i) = c$), then $E[U] = (1 - O(n^{-1})) \frac{n}{c+1}$.

Notice that \([1]\) implies that the number of missing types of coupons is at most a constant in expectation, i.e., $E[U_t] = O(1)$ at any time $t$, if $d(i) = \Omega(i)$, while \([2]\) claims a stronger upper bound with a stronger assumption of $d(i) = \omega(i)$ that the expected number of missing types is asymptotic to 0 every time just before a new release (recall the relation between $U$ and $U_t$). \([3]\) claims in case of $d(i) = o(i)$ and $\omega(1)$ that $E[U] \approx \frac{n}{d(n)}$ up to the leading coefficient; for instance, $E[U] \leq n^{\gamma}/C$ holds if $d(i) \geq C i^{1-\gamma}$ as well as $E[U] \geq n^{\gamma}/C$ holds if $d(i) \leq C i^{1-\gamma}$, where $C > 0$ and $\gamma \in [0,1]$ are arbitrary constants common in both equations (See also Proposition 2.2). \([4]\) is the counterpart of \([3]\) for constant $d$. For example, if a new vertex appears every step ($d(i) = 1$), a random walk on a growing complete graph misses a half of the number vertices.

\footnote{Mixing time is usually parametrized by $\epsilon$, but we call $t_{mix} = t_{mix}(P)$ mixing time in this paper \([25]\).}
Upper bound analysis (Section 3). Next, we focus on upper bounds of $E[U]$ with respect to $d$ for RWoGG $(d, (G(i))_i \infty = 1, (P(i))_i \infty = 1)$, in general. For convenience, let $t_{hit}(i)$, $t_{cov}(i)$ and $t_{mix}(i)$ respectively denote the hitting, cover and mixing times of $P(i)$, in the rest of the paper.

To begin with, we remark that it is easy to prove that $E[U] = O(1)$ if $d(i) = \Omega(t_{hit}(i) \log i)$ for any RWoGG using the known fact that the number of unvisited vertices exponentially decays every unit time of $e^{t_{hit}}$ (see e.g. Sections 2.4.3 and 2.6 of [2]; see also Lemma 3.3 in Appendix B). Thus, our interest is in the case that $d(i) = o(t_{hit}(i) \log i)$. We establish the following upper bound of $E[U]$, claiming that $E[U] = O(1)$ if $d(i) = Ct_{hit}(i)$ for $C > 1$, in fact. We remark that the following theorem is an extension of Theorem 1.1(1) and (2) for “a specific random walk on growing complete graphs” to general random walks and graphs.

**Theorem 1.2.** Let $(d, (G(i))_i \infty = 1, (P(i))_i \infty = 1)$ be an arbitrary RWoGG.

1. If there is a constant $C > 1$ such that $d(i) \geq Ct_{hit}(i)$ for all $i \in [n]$, then $E[U] = O(1)$.

2. If $d(i)/t_{hit}(i) \rightarrow \infty$ as $i \rightarrow \infty$, then $E[U] \rightarrow 0$ as $n \rightarrow \infty$.

In Theorem 1.2, we obtain a general upper bound of $E[U]$ in the case of $d(i) \geq (1 + \epsilon)t_{hit}(i)$, where $\epsilon > 0$ is a constant. In contrast, the case of $d(i) \leq (1 + o(1))t_{hit}(i)$ seems not easy: it contains an issue of “short random walks,” that is a challenging topic in the literature of the cover time of multiple random walks, and so on, see e.g., [22]. Henceforth, we focus on lazy and reversible random walks, of which the transition matrices $P(i)$ are known to be (essentially) positive semidefinite. For “rapidly” mixing random walks such that $t_{mix} \ll t_{hit}$, we obtain the following upper bound.

**Theorem 1.3.** Let $(d, (G(i))_i \infty = 1, (P(i))_i \infty = 1)$ be a RWoGG such that $P(i)$ is lazy and reversible. Let $C > 0$ and $\gamma \in [0, 1]$ be arbitrary constants. If $t_{hit}(i)/t_{mix}(i) \geq i^{1-\gamma}/C$ and $d(i) \geq \frac{3Ct_{hit}(i)}{i^{1-\gamma}}$ for all $1 < i \leq n$, then $E[U] \leq \frac{8n^{3\gamma}}{C} + 32$.

Notice that Theorem 1.3 for $\gamma = 0$ claims that $E[U] = O(1)$ if $d(i) = \Theta(t_{hit}(i))$ on the appropriate condition. A natural question remains unsettled whether $E[U] = O(1)$ requires $d(i) = \Omega(t_{hit}(i))$ for any RWoGG $(d, (G(i))_i \infty = 1, (P(i))_i \infty = 1)$. As a consequence of Theorem 1.3, for example, we obtain a bound for degree restricted expander graphs, for which $t_{hit}(i) = O(i)$ and $t_{mix}(i) = O(\log i)$ hold, that $E[U] = O(n^{\gamma})$ if $d(i) = \Omega(i^{1-\gamma})$ for $\gamma \in [0, 1)$; see Corollary 3.5 for detail. We also remark that the upper bound by Theorem 1.3 is tight for growing complete graphs, for which $t_{hit}(i) = \Theta(i)$ and $t_{mix}(i) = \Theta(1)$ hold; see Theorem 1.1(3) (Proposition 2.2(2)) for the lower bound.

Though the condition of $t_{mix} \ll t_{hit}$ covers many interesting examples of rapidly mixing random walks, it misses many examples, such as random walks on paths and lollipop graphs, interested in the context of hitting and cover times. Then, we provide for those examples the following Theorems 1.4 and 1.5.

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5 The transition matrix $P$ of a lazy and reversible random walk is not symmetric in general, but there always exists a diagonal matrix $D$ such that $D^{-1}PD$ is symmetric see e.g., [25].
Theorem 1.4. Let \( (\vartheta, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty}) \) be a RWoGG such that \( P^{(i)} \) is lazy and simple, and that for all \( i (2 < i \leq n) \), \( \frac{|E^{(i)}|}{|E^{(i-1)}|} \leq 1 + \frac{L}{4} \) hold for some positive constant \( L \). Let \( C > 0 \) and \( \gamma \in [0, 1] \) be arbitrary constants. If \( \vartheta(i) \geq \left( \frac{C}{\gamma} + \frac{L+1}{2} \right) t_{hit}(i) \) holds for any \( 1 < i \leq n \), then \( E[U] \leq \sqrt{L + \frac{n^3}{C^2}} \).

We will later give a tight example for Theorem 1.4. Theorem 1.6 gives a lower bound of \( E[U] \) for a growing path (see also Corollary 1.7). We will also demonstrate another example of application of Theorem 1.4 to a growing lollipop graph (see Corollary 3.11), where the static lollipop graph is well-known as a tight example for the bounds \( t_{hit} \) and \( t_{cov} \). We will also demonstrate another example of application of Theorem 1.4 to a growing path (Theorem 1.6 and Corollary 1.7).

Theorem 1.5. Let \( (\vartheta, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty}) \) be a RWoGG such that \( P^{(i)} \) is lazy and symmetric. Let \( C > 0 \) and \( \gamma \in [0, 1] \) be arbitrary constants. If \( \vartheta(i) \geq \left( \frac{C}{\gamma} + \frac{3}{2} \right) t_{hit}(i) \) for all \( 1 < i \leq n \), then \( E[U] \leq \frac{\sqrt{30} \gamma}{C^2} \).

A typical application of Theorem 1.5 is a lazy Metropolis walk with the uniform stationary distribution (see Corollary 3.12 for details), which often appears in the context of Markov chain Monte Carlo. Nonaka et al. [29] proved that the Metropolis achieves \( t_{hit}(i) = O(i^2) \) for any connected graph. The upper bound by Theorem 1.5 is also tight for a Metropolis walk on a growing path (Theorem 1.6 and Corollary 1.7).

A lower bound for a growing path (Section 4). In contrast to upper bounds, an analysis of a lower bound requires more technically complicated arguments. We establish a lower bound of \( E[U] \) for a random walk on a growing path graph, which implies that the upper bound by Theorem 1.5 is tight in the case. Let \( R_p = (\vartheta, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty}) \) be a random walk on a growing path graph, where \( G^{(i)} = (V^{(i)}, E^{(i)}) \) is given by \( V^{(i)} = \{v_1, \ldots, v_i\} \), and \( E^{(i)} = \{\{v_1, v_2\}, \ldots, \{v_{i-1}, v_i\}\} \), and \( P^{(i)} \) is given by

\[
(P^{(i)})_{u,v} = \begin{cases} 
 p & \text{if } u = v = v_1 \text{ or } u = v = v_i, \\
 1-p & \text{if } (u,v) \in \{(v_1,v_2),(v_i,v_{i-1})\}, \\
 q & \text{if } \{u,v\} = \{v_j, v_{j+1}\} \text{ for } j = 2, 3, \ldots, i-1, \\
 1-2q & \text{if } u = v = v_j \text{ for } j = 2, 3, \ldots, i-1, \\
 0 & \text{otherwise} \end{cases}
\]

for two parameters \( p, q \in [0, 1] \) satisfying \( p \geq q \) and \( q \leq 1/2 \) (see Figure 2). For example, if \( (p,q) = (\frac{3}{4}, \frac{1}{4}) \), the corresponding walk is the lazy simple random walk. If \( (p,q) = (\frac{4}{5}, \frac{1}{5}) \) the corresponding one is the lazy Metropolis random walk (see (28) for the definition of Metropolis random walk).

Suppose, for instance, that \( \vartheta(i) = Ci \) for a sufficiently large constant \( C > 0 \). Then, the walker walks \( \sum_{i=1}^{n} \vartheta(i) \approx C^2 n^2 / 2 \) steps in total, which is larger than the cover time of a lazy simple random walk on the path of length \( n \). Thus, one may expect that \( E[U] = O(1) \). However, this is not the case.
### Theorem 1.6
If \( d(i) \leq C i^{2-\gamma} \) in \( R_p \) for some constants \( C > 0 \) and \( \gamma \in [0, 1] \) then \( E[U] = \Omega(n^\gamma/C) \).

Theorems 1.4 to 1.6 imply the following tight bounds of \( E[U] \) on a growing path.

### Corollary 1.7
For \( R_p = (\varnothing, (G(i))_{i=1}^\infty, (P(i))_{i=1}^\infty) \), where \( P(i) \) is the transition matrix of either the lazy simple random walk or the lazy Metropolis random walk. Then

1. If \( d(i) \geq C i^{2-\gamma} \) for some constants \( C > 0 \) and \( \gamma \in [0, 1] \) then \( E[U] = O(n^\gamma/C) \).
2. If \( d(i) \leq C i^{2-\gamma} \) for some constants \( C > 0 \) and \( \gamma \in [0, 1] \) then \( E[U] = \Omega(n^\gamma/C) \).

### 1.3 Related works

The cover time is a fundamental topic of analyses of random walks. Here, we review some representative results about the cover times of random walks on static graphs, and on dynamic graphs.

#### Cover times of random walks on static graphs.

It is known that the cover time of a simple random walk satisfies \( t_{cov} \leq 2m(n-1) \) for any undirected graph, see Aleliunas et al. [3] and Aldous [1]. Mathews [26] devised a technique of upper and lower bounding \( t_{cov} \) by \( t_{hit} \), of which a celebrated implication is \( t_{cov} \leq t_{hit} \log n \). The lolipop graph is famous for \( t_{hit} = \Omega(n^3) \), and hence \( t_{cov} = \Omega(n^3) \). Fiege gave a tight upper bound of the cover times of simple random walks on any graphs such that \( t_{cov} \leq \frac{4}{27}n^3 + O(n^{5/2}) \) in [18], while he in [17] gave a tight lower bound of the cover time of simple random walks on any graphs such that \( t_{cov} \geq n \ln n + o(n \ln n) \), using a Mathews' argument [26]. The connection between the hitting time and electric circuits is well known (see e.g., [15], [2], [25]).

Motivated by a faster covering by a random walk, Ikeda et al. [19] (see also [20]) proposed \( \beta \)-random walk, which makes transitions only using local information, and proved that the cover time of a \( \beta \)-random walk is upper bounded by \( O(n^2 \log n) \) for any graph. Nonaka et al. [29] proved the same bound holds for a Metropolis walk, which is simpler and more popular than \( \beta \)-random walk. Recently, David and Feige [11] (see also [12]) proved that a biased random walk achieves \( O(n^2) \) cover time for any graph, and affirmatively settled the question posed by Ikeda et al. [19].
Cover time of random walks on dynamic graphs. An early work [10] by Cooper and Frieze investigated random walks on “web-graphs,” where the number of vertices increases every constant steps, i.e., corresponding to constant δ in our model, and where $G^{(n)}$ is a preferential attachment graph. Then, they were concerned with the expected proportion of vertices visited by a random walk, and they revealed that it converges to some constant accordingly $\mathbb{E}[U]/n$ converges to some constant in our context, asymptotic to $n$.

There are several results about the cover times of random walks on dynamic graphs, sometimes called “evolving graphs,” with static vertex sets. Avin et al. [4] (see also [5]) investigated the hitting times, mixing times and cover times of random walks on evolving graphs with static vertex sets. They gave a prescribed sequence of graphs on which the hitting time of a simple random walk gets $2^{\Omega(n)}$, and hence the cover time is as well. On the other hand, they proved that the cover time of a max-degree random walk is $O(d_{\text{max}} n^3 (\log n)^2)$ where $d_{\text{max}}$ is the maximum degree of the evolving graph. Denysyuk and Rodrigues [13] were concerned with $\rho$-recurrent family of evolving graphs, where preferable graphs are assumed to appear frequently in the graph sequence. Then, for max-degree random walks on $\rho$-recurrent families, they gave upper and lower bounds of the cover time in terms of the hitting time, as well as gave an upper bound of the mixing time. Lamprou et al. [24] were concerned with two random walks of “random walk with a delay” (RWD), where at each step, the walker chooses an edge of underlying graph and moves when it appears, and “random walk on what is available” (RWA), where the walker chooses an edge of current graph and moves immediately. Then, they investigated the cover times of RWD and RWA for edge-uniform stochastically evolving graphs. Sauerwald and Zanetti [34] extended the argument by Avin et al. [5] in the case that a sequence of graphs have the same stationary distribution, and presented an upper bound $O(n^2)$ of the cover time on $d$-regular dynamic graphs.

Other related works. Saloff-Coste and Zúñiga investigated time-inhomogeneous Markov chains, and provided some Nash and log-Sobolev inequalities [31, 32]. Recently, Cai et al. [7] investigated the relation between the density of edge-Markovian dynamic graphs and mixing times. They showed for fast-changing dynamic graphs that $t_{\text{mix}} = \infty$ in sparse case while $t_{\text{mix}} = O(\log n)$ in dense case. They also showed for slowly-changing dynamic graphs that $t_{\text{mix}} = \Omega(n)$ in sparse case while $t_{\text{mix}} = O(\log n)$ in dense case. Random walk on dynamic graph is also interested in data mining. Yu and McCann [35] presented an analysis on “random walk with restart,” which is used as a measure of proximity between vertices of a graph in the context, over dynamic graphs.

There are many works on other stochastic processes on dynamic graphs, such as exploration, information spreading, rumor spreading, gossiping and voter model, see e.g., [21, 8, 6]. Theoretical analyses of algorithms on dynamic graphs attract high attentions in the context of distributed computing, and there are many works concerning the topics, such as connectivity, exploration, gathering, agreement, flooding and population protocol, on dynamic networks, see e.g., [28, 27, 23].
Figure 3: Correspondence between $Z_t$ and $X_s^{(i)}$ when $\delta(i) = i$. For each $i \in \mathbb{N}$, $(X_s^{(i)})_{s=0,1,...}$ is a random walk on $G^{(i)}$. Note that $X_0^{(i)} = X_{\delta(i)-1}^{(i-1)} = Z_{T_i}$ holds for $i = 2, 3, ...$. In this example, $U(3) = 3 - \left| \bigcup_{t=0}^{T_{i+1}} \{Z_t\} \right| = 3 - \left| \bigcup_{s=0}^{3} \bigcup_{s=0}^{i} \{X_s^{(i)}\} \right|$.

2 Complete Graph

This section proves Theorem 1.1. Throughout this paper, we consider a random walk of length $T_{n+1}$. For convenience, we divide the $T_{n+1}$ step random walk into $n$ random walks each of length $\delta(i)$ (for $i = 1, \ldots, n$). We call each period round. For a round $i \in [n]$, let $(X_s^{(i)})_{s=0}^\infty$ denote a random walk in the $i$-th round (specifically, it is a random walk according to $P^{(i)}$) with the initial state $X_0^{(i)} = Z_{T_i} = X_{\delta(i)-1}^{(i-1)}$. Note that $(X_s^{(i)})_{s=0}^\infty$ is a random walk on $G^{(i)}$. Figure 3 illustrates the correspondence between $Z_t$ and $X_s^{(i)}$ in the case of $\delta(i) = i$.

For $v \in V^{(n)}$ let $\mathcal{E}(v)$ denote the event that $v \notin \bigcup_{t=1}^n \bigcup_{s=0}^\infty \{X_s^{(i)}\} = \bigcup_{t=0}^{T_{n+1}} \{Z_t\}$. In other words, $\mathcal{E}(v)$ means that the random walk $Z_0, Z_1, \ldots, Z_{T_{n+1}}$ does not visit the vertex $v$. For the vertex $v_k$ attached to $G$ at time $T_k$, we see that $\Pr[\mathcal{E}(v_k)] = \prod_{i=k}^n \left(1 - \frac{1}{i} \right)^{\delta(i)}$ holds, and thus

$$E[U] = \sum_{k=1}^n \Pr[\mathcal{E}(v_k)] = \sum_{k=1}^n \prod_{i=k}^n \left(1 - \frac{1}{i} \right)^{\delta(i)}$$

holds. Theorem 1.1 follows the next lemma.

Lemma 2.1. For a function $f : \mathbb{N} \to \mathbb{N}$, let $S(n) := \sum_{k=1}^n \prod_{i=k}^n \left(1 - \frac{1}{i} \right)^{f(i)}$.

(i) If $f(i) \geq Ci$ for some constant $C$, then $S(n) = O(1)$.

(ii) If $f$ satisfies $f(i) \leq f(i+1)$ for all $i \in \mathbb{N}$, then $S(n) \geq \frac{n}{f(n)+1} \left(1 - \frac{1}{n} \right)^{f(n)}$.

(iii) If $f$ satisfies $\frac{f(i)}{i} \geq \frac{f(i+1)}{i+1}$, then for all $n \in \mathbb{N}$, $S(n) \leq \frac{n}{f(n)}$.

(iv) If there is a constant $c \in \mathbb{N}$ such that $f(i) = c$ for all $i \in \mathbb{N}$, then for all $n \in \mathbb{N}$, $S(n) \leq \frac{n}{c+1}$.

Proof of (i). Since $1 + x \leq e^x$, we have

$$S(n) \leq \sum_{k=1}^n \exp \left( - \sum_{i=k}^n \frac{f(i)}{i} \right) \leq \sum_{k=1}^n \exp \left( -(n-k+1)C \right) = O(1).$$

\( \Box \)
Proof of (ii) Observe that $S(1) = 0$ and for all $n \geq 1,$

$$S(n + 1) = \sum_{k=1}^{n+1} \prod_{i=k}^{n+1} \left(1 - \frac{1}{i}\right) f(i) = \left(1 - \frac{1}{n+1}\right) f(n+1) (S(n) + 1).$$  \hspace{1cm} (2)

We prove (ii) by induction on $n$. In the base case, $S(1) = 0$ and we are done. If $S(n) \geq \frac{n}{f(n)} (1 - \frac{1}{n}) f(n),$ then

$$S(n + 1) \geq \frac{n}{f(n)} + 1 \geq \frac{n + 1}{f(n)} \geq \frac{n + 1}{f(n) + 1}.$$  \hspace{1cm} (3)

Here, we used $(1 + x)^r \geq 1 + rx$ in the second inequality and $f(n) \leq f(n + 1)$ in the last inequality.

Combining (2) and (3), $S(n + 1) \geq \left(1 - \frac{1}{n+1}\right) f(n+1) \frac{n+1}{f(n+1) + 1}$ and we are done. \hfill \Box

Proof of (iii). The proof is obtained by induction on $n \geq 1$. When $n = 1$, $S(1) = 0 \leq 1/f(1)$. Assume $S(n) \leq n/f(n)$.

Then,

$$S(n + 1) = \left(1 - \frac{1}{n+1}\right) f(n+1) (S(n) + 1) \leq \frac{n}{f(n)} + 1 \leq \frac{n + 1}{f(n) + 1} = \frac{n + 1}{f(n) + 1}.$$  \hspace{1cm} (4)

Note that $(1 - x)^y \leq 1/(1 + xy)$ for all $x \in [0,1]$ and $y \geq 0$. The second inequality follows from $\frac{1}{f(n) + 1} \leq \frac{1}{n}$. \hfill \Box

Proof of (iv). The proof is obtained by induction on $n$. First $S(1) = 0 \leq 1/(f(1) + 1)$. Assume $S(n) \leq n/(f(n) + 1)$. Then, from (2) and the induction assumption, we have

$$S(n + 1) \leq \frac{n}{f(n) + 1} + 1 = \frac{n+1}{f(n) + 1} \leq \frac{n + 1}{f(n+1) + 1} = \frac{n + 1}{f(n) + 1}.$$  \hspace{1cm} (5)

Note that we use $f(n) = f(n + 1)$ in the first and the last equality. \hfill \Box

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Recall that $E[U] = S(n)$. Statement (1) follows from Lemma 2.1(i).

Statement (3) follows from (ii) and (iii) of Lemma 2.1. Statement (4) follows from (ii) and (iv) of Lemma 2.1.

Now, we prove Statement (2). More precisely, we prove that, for any $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(n) \leq \epsilon$ holds. From the assumption that $\omega(i) = \omega(i)$, for any large constant $C > 0$, we can take $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, $f(i) > Ci$ holds. Fix a constant $C > 0$ and take $i_0$ in this way.

Since $1 + x \leq e^x$ and $f(k)/k > C$ for all $k \geq i_0$, we have

$$S(n) \leq \sum_{i=1}^{i_0} \exp \left(- \sum_{k=i}^{n} \frac{f(k)}{k}\right) + \sum_{i=i_0+1}^{n} \exp \left(- \sum_{k=i}^{n} \frac{f(k)}{k}\right)$$

$$\leq i_0 \exp(-(n-i_0 + 1)C) + \sum_{i=i_0+1}^{n} \exp(-(n-i+1)C)$$

$$\leq i_0 \exp(-(n-i_0 + 1)C) + \frac{e^{-C}}{1 - e^{-C}}.$$
Let \( \epsilon > 0 \) be an arbitrary small constant. Then, take \( C > 0 \) such that \( \frac{\epsilon - C}{1 - e^\epsilon} < \frac{\epsilon}{2} \) holds. According to this constant \( C \), we can take \( i_0 \) such that \( f(i) > Ci \) for all \( i \geq i_0 \) holds. Now \( C \) and \( i_0 \) are fixed. Hence, for sufficiently large \( n \), we have \( i_0 \exp(-(n - i_0 + 1)C) \leq \frac{\epsilon}{2} \). This implies \( S(n) \leq \epsilon \) and we are done.

\[ \square \]

**Remark.** We remark on some monotonicity of \( E[U] \) with respect to \( \mathcal{A} \). Suppose functions \( \mathcal{A}^* \) and \( \mathcal{A} \) satisfy \( \mathcal{A}^*(i) \geq \mathcal{A}(i) \) for all \( i \). Let \( U^*(n) \) and \( U(n) \) respectively denote the numbers of unvisited vertices at the end of \( n \)-th round for \( R^*_c = (\mathcal{A}^*, (G(i))^\infty_{i=1}, (P(i))^\infty_{i=1}) \) and \( R_c = (\mathcal{A}, (G(i))^\infty_{i=1}, (P(i))^\infty_{i=1}) \). Then, \( E[U^*(n)] \leq E[U(n)] \) is clear. From this observation, Lemma 2.1 implies the following proposition, which is a variant of Theorem 1.1 (1), (3) and (4).

**Proposition 2.2.** Let \( C > 0, \gamma \in [0, 1] \) be arbitrary constants. For \( R_c = (\mathcal{A}, (G(i))^\infty_{i=1}, (P(i))^\infty_{i=1}) \), the following holds:

1. If \( \mathcal{A}(i) \geq Ci^{1-\gamma} \) for all \( i \), then \( E[U] \leq n^\gamma \).

2. If \( \mathcal{A}(i) \leq Ci^{1-\gamma} \) for all \( i \), then \( E[U] \geq \frac{n^\gamma}{e + n^{-1}}(1 - \frac{1}{n})^{\gamma n^{1-\gamma}} \geq \frac{n^\gamma}{e} - 1 \).

## 3 Upper Bound Analysis

In this section we show Theorems 1.2 to 1.5. Consider a random walk on a growing graph \( R = (\mathcal{A}, (G(n))^\infty_{n=1}, (P(n))^\infty_{n=1}) \). Recall that, at each round \( i \), \( (X^i_t)_{t=0}^\infty \) denotes the random walk according to \( P(i) \) where \( X^i_0 = X^{(i-1)}_0 \) holds (See Figure 3 for an example). Let \( \pi(i) \) denote the stationary distribution of \( P(i) \). Let \( \tau^i_v := \min\{t \geq 0 : X^i_t = v\} \), i.e., \( \tau^i_v \) denotes the time taken for a random walk \( (X^i_t)_{t=0}^\infty \) to reach \( v \in V(i) \). Note that \( t_{hit}(i) = \max_{u,v \in V} E[\tau^i_v | X^i_0 = u] \). Suppose that the initial position is fixed, i.e., \( X^0_0 = v_1 \). For any round \( k \leq n \), the probability that the walker does not visit the vertex \( v_k \) until the end of the round \( n \) is equal to \( \Pr \left[ \bigwedge_{i=k}^n \left\{ \tau^i_{v_k} > \mathcal{A}(i) \right\} \right] \). Hence we have

\[
\begin{align*}
E[U] &= \sum_{k=1}^{n} \Pr \left[ \bigwedge_{i=k}^{n} \left\{ \tau^i_{v_k} > \mathcal{A}(i) \right\} \right] \\
&= \sum_{k=2}^{n} \sum_{v \in V^{(k-1)}} \Pr \left[ X^0_0 = v \right] \Pr \left[ \bigwedge_{i=k}^{n} \left\{ \tau^i_{v_k} > \mathcal{A}(i) \right\} \right] \\
&\leq \sum_{k=2}^{n} \max_{v \in V^{(k-1)}} \Pr \left[ \bigwedge_{i=k}^{n} \left\{ \tau^i_{v_k} > \mathcal{A}(i) \right\} \right] .
\end{align*}
\]

The second equality follows from \( \Pr[X^{(1)}_1 \neq v_1] = 0 \). The rest of this section is devoted to give upper bounds of (3) (or (4)).

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3.1 Upper bound for large $d$

We show Theorem 1.2 in this section. To begin with, we show the following useful lemma.

**Lemma 3.1.** For any $R = (d, (G^{(i)})_{i=1}^\infty, (P^{(i)})_{i=1}^\infty)$, we have

$$E[U] \leq \sum_{k=2}^n \prod_{v \in V^{(i)}} \Pr[\tau_{v_k}^{(i)} > \vartheta(i)|X_0^{(i)} = v].$$

**Proof.** Consider a fixed vertex $v_k$ with $k > 1$. For a round $i \geq k$ and a vertex $u \in V^{(i)}$, let $E_u^{(i)}$ denote the event that the walker is in vertex $u$ at the end of the $i$-th round without visiting vertex $v_k$ during the round. Formally $E_u^{(i)}(v_k)$ is defined as the event of $\{\tau_{v_k}^{(i)} > \vartheta(i)\} \land \{X_0^{(i)} = u\}$.

Then for any $u_{k-1} \in V^{(k-1)}$,

$$\Pr[\bigwedge_{i=k}^n \{\tau_{v_k}^{(i)} > \vartheta(i)\}|X_0^{(k)} = u_{k-1}] = \sum_{u_k \in V^{(k)}} \cdots \sum_{u_n \in V^{(n)}} \Pr[\bigwedge_{i=k}^n E_{u_i}^{(i)}|X_0^{(k)} = u_{k-1}].$$

To bound (6), we first observe that, for any vertices, $u_{k-1} \in V^{(k-1)}, u_k \in V^{(k)}, \ldots, u_n \in V^{(n)}$,

$$\Pr[\bigwedge_{i=k}^n E_{u_i}^{(i)}|X_0^{(k)} = u_{k-1}] = \frac{\Pr[X_0^{(k)} = u_{k-1}, E_{u_i}^{(i)}]}{\Pr[X_0^{(k)} = u_{k-1}]} \prod_{\ell=k+1}^n \frac{\Pr[X_0^{(k)} = u_{k-1}, \bigwedge_{i=k}^{\ell-1} E_{u_i}^{(i)}]}{\Pr[X_0^{(k)} = u_{k-1}, \bigwedge_{i=k}^{\ell-1} E_{u_i}^{(i)}]}$$

holds. Then, from the definition of the conditional probability, we have

$$\frac{\Pr[X_0^{(k)} = u_{k-1}, E_{u_k}^{(k)}]}{\Pr[X_0^{(k)} = u_{k-1}]} = \Pr[E_{u_k}^{(k)}|X_0^{(k)} = u_{k-1}]$$

and

$$\frac{\Pr[X_0^{(k)} = u_{k-1}, \bigwedge_{i=k}^{\ell-1} E_{u_i}^{(i)}]}{\Pr[X_0^{(k)} = u_{k-1}, \bigwedge_{i=k}^{\ell-1} E_{u_i}^{(i)}]} = \Pr[E_{u_{\ell}}^{(\ell)}|X_0^{(k)} = u_{\ell-1}]$$

We use the Markov property in the second equality. The last equality follows from our assumption of $X_0^{(k)} = X_0^{(\ell)}$. Hence combining (6) to (8), we have

$$\Pr[\bigwedge_{i=k}^n \{\tau_{v_k}^{(i)} > \vartheta(i)\}|X_0^{(k)} = u_{k-1}] = \sum_{u_k \in V^{(k)}} \cdots \sum_{u_n \in V^{(n)}} \Pr[\bigwedge_{i=k}^n E_{u_i}^{(i)}|X_0^{(k)} = u_{k-1}]$$

$$= \prod_{\ell=k}^n \sum_{u_{\ell-1} \in V^{(\ell-1)}} \Pr[\bigwedge_{i=k}^{\ell-1} E_{u_i}^{(i)}|X_0^{(k)} = u_{\ell-1}]$$

$$= \prod_{\ell=k}^n \max_{u_{\ell-1} \in V^{(\ell)}} \sum_{u_{\ell-1} \in V^{(\ell)}} \Pr[\bigwedge_{i=k}^{\ell-1} E_{u_i}^{(i)}|X_0^{(k)} = u_{\ell-1}]$$

$$\leq \prod_{\ell=k}^n \sum_{u_{\ell-1} \in V^{(\ell)}} \Pr[\bigwedge_{i=k}^{\ell-1} E_{u_i}^{(i)}|X_0^{(k)} = u_{\ell-1}] = \prod_{\ell=k}^n \max_{u_{\ell-1} \in V^{(\ell)}} \Pr[\tau_{v_k}^{(i)} > \vartheta(i)|X_0^{(k)} = u_{\ell-1}].$$

We obtain the claim from (5) and (10).
Proof of Theorem 1.2(1) From the Markov inequality, for any \( k \leq i \) and \( v \in V^{(i)} \), we have
\[
\Pr \left[ r_{vk}^{(i)} > \mathfrak{d}(i) \bigg| X_{0}^{(i)} = v \right] \leq \frac{E \left[ r_{vk}^{(i)} \bigg| X_{0}^{(i)} = v \right]}{\mathfrak{d}(i)} \leq \frac{t_{\text{hit}}(i)}{\mathfrak{d}(i)}.
\]
Hence from Lemma 3.1, we obtain
\[
E[U] \leq \sum_{k=1}^{n} \prod_{i=k}^{n} t_{\text{hit}}(k) \frac{t_{\text{hit}}(k)}{\mathfrak{d}(k)} \leq \sum_{k=1}^{n} C^{-k} = \sum_{k=1}^{n} C^{-k} \leq \frac{1}{C-1}.
\]
\[\square\]

Proof of Theorem 1.2(2) For an arbitrary (small) \( \epsilon > 0 \), let \( C = C(\epsilon) = \frac{2}{\epsilon} + 1 \). From assumption on (2), we can take some \( i_0 = i_0(\epsilon) \) such that \( \mathfrak{d}(i) \geq Ct_{\text{hit}}(i) \) for all \( i \geq i_0 \). Let \( K = \max_{i \in [i_0]} \frac{t_{\text{hit}}(i)}{\mathfrak{d}(i)} \).

From Lemma 3.1
\[
E[U] \leq \sum_{i=1}^{i_0} \left( \prod_{k=i}^{i_0} \frac{t_{\text{hit}}(k)}{\mathfrak{d}(k)} \right) \left( \prod_{k=i_0+1}^{n} \frac{t_{\text{hit}}(k)}{\mathfrak{d}(k)} \right) + \sum_{i=i_0+1}^{n} \prod_{k=i_0+1}^{n} \frac{t_{\text{hit}}(k)}{\mathfrak{d}(k)}
\]
\[
\leq C^{-(n-i_0)} \sum_{i=1}^{i_0} K^{i-i_0+1} + \sum_{i=i_0+1}^{n} C^{-(n-i+1)}
\]
\[
= C^{-(n-i_0)} \sum_{i=1}^{i_0} K^{i} + \sum_{i=1}^{n-i_0} C^{-i}
\]
\[
\leq C^{-(n-i_0)} K \frac{1-K^{i_0}}{1-K} + \frac{1}{C-1}.
\]

Then we can take some \( n_0 = n_0(\epsilon) \) satisfying \( C^{-(n-i_0)} K \frac{1-K^{i_0}}{1-K} \leq \epsilon/2 \). Hence for any \( n \geq n_0 \), \( E[U] \leq \epsilon \) and we obtain the claim. \(\square\)

3.2 Upper bound for random walks with small mixing times

In this section we show the following generalized version of Theorem 1.3.

Theorem 3.2. Suppose that \( P^{(i)} \) is reversible and lazy in \( R = (\mathfrak{d}, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty}) \). Let \( N > 0 \) be an arbitrary positive number. If \( \mathfrak{d}(i) \geq \frac{t_{\text{hit}}(i)}{\mathfrak{d}} + 2t_{\text{mix}}(i) \) for all \( i \in [n] \), then \( E[U] \leq 8N + 32 \).

Proof of Theorem 3.2 For all \( i \), it is straightforward to see that
\[
\mathfrak{d}(i) \geq \frac{Ct_{\text{hit}}(i)}{\mathfrak{d}} + \frac{2Ct_{\text{hit}}(i)}{\mathfrak{d}} \geq \frac{t_{\text{hit}}(i)}{n^{\gamma}/C} + 2t_{\text{mix}}(i)
\]
from assumptions. Taking \( N = n^{\gamma}/C \) in Theorem 3.2 we obtain the claim. \(\square\)

To show Theorem 3.2, we introduce following two lemmas. The first one generalizes Lemma 2.4(1).

The second one is a useful variant of Lemma 3.1.
Lemma 3.3. For $f, h : \mathbb{N} \rightarrow \mathbb{N}$ and $n \in \mathbb{N}$, let

$$S(n) := \sum_{k=1}^{n} \prod_{i=k}^{n} \left(1 - \frac{1}{h(i)}\right)^{f(i)}.$$ 

Let $N > 0$ be an arbitrary number. If $f(i) \geq \frac{h(i)}{N}$ for all $i \in [n]$, then $S(n) \leq N$.

Proof. It is easy to check that

$$S(n) \leq \sum_{k=1}^{n} \prod_{i=k}^{n} \exp\left(-\frac{f(i)}{h(i)}\right) = \sum_{k=1}^{n} \exp\left(-\sum_{i=k}^{n} \frac{f(i)}{h(i)}\right) \leq \sum_{k=1}^{n} \exp\left(-\frac{n+k-1}{N}\right)$$

$$= \sum_{k=1}^{n} \exp\left(-\frac{k}{N}\right) \leq \frac{e^{-1/N}}{1 - e^{-1/N}} = \frac{1}{e^{1/N} - 1} \leq N.$$

Note that we use $1 + x \leq e^x$ in the first and the last inequalities. \qed

Lemma 3.4. For any $R = (\vartheta, (G(i))_{i=1}^{\infty}, (P(i))_{i=1}^{\infty})$ and any function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that $s(i) < \vartheta(i)$ holds for all $i$, we have

$$E[U] \leq \sum_{k=2}^{n} \prod_{i=k}^{n} \max_{u \in V(i)} \left(\sum_{v \in V(i)} \left((P(i))^s(i)\right)_{u,v} \Pr \left[\tau(v_k) > \tau(i) - s(i) \bigg| X_v(i) = v\right]\right).$$

Proof. Fix $k \geq 2$ and $i$ satisfying $k \leq i \leq n$. First, for any $u, v \in V(i)$, from the definition of the conditional probability, we observe that

$$\Pr \left[\tau(v_k) > \tau(i), X_{s(i)} = v \bigg| X_0(i) = u\right] = \Pr \left[\tau(v_k) > \tau(i), X_{s(i)} = v, X_0(i) = u, \tau(v_k) > \tau(i)\bigg| X_0(i) = u\right] \Pr \left[X_{s(i)} = v, \tau(v_k) > \tau(i) \bigg| X_0(i) = u\right]$$

holds. We use the Markov property in the third equality. Since

$$\Pr \left[X_{s(i)} = v, \tau(v_k) > \tau(i) \bigg| X_0(i) = u\right] \leq \Pr \left[X_{s(i)} = v, X_0(i) = u\right] = ((P(i))^s(i))_{u,v},$$

we have

$$\Pr \left[\tau(v_k) > \tau(i) \bigg| X_0(i) = u\right] \leq \sum_{v \in V(i)} \left((P(i))^s(i)\right)_{u,v} \Pr \left[\tau(v_k) > \tau(i) - s(i) \bigg| X_0(i) = v\right]$$

for any $u \in V(i)$. Combining Lemma 3.3 and (11), we obtain the claim. \qed
Proof of Theorem 3.2. If $P^{(i)}$ is reversible, for any $i \in [n]$ and $u, v \in V^{(i)}$, some transition matrix $\hat{P}^{(i)} \in [0, 1]^{V^{(i)} \times V^{(i)}}$ exists such that

$$
\left( (P^{(i)})^{2t_{\text{mix}}(i)} \right)_{u,v} = \frac{1}{4} \pi^{(i)}(v) + \frac{3}{4} (\hat{P}^{(i)})_{u,v}
$$

holds (See e.g., p.338 of [25]). Hence it holds for any $u \in V^{(i)}$ that

$$
\sum_{v \in [i]} \left( (P^{(i)})^{2t_{\text{mix}}(i)} \right)_{u,v} \Pr \left[ \tau_{t^{(i)}_{ik}} > \delta(i) - 2t_{\text{mix}}(i) \big| X^{(i)}_0 = v \right] = \frac{1}{4} \sum_{v \in [i]} \pi^{(i)}(v) \Pr \left[ \tau_{t^{(i)}_{ik}} > \delta(i) - 2t_{\text{mix}}(i) \big| X^{(i)}_0 = v \right] + \frac{3}{4} \sum_{v \in [i]} (\hat{P}^{(i)})_{u,v} \Pr \left[ \tau_{t^{(i)}_{ik}} > \delta(i) - 2t_{\text{mix}}(i) \big| X^{(i)}_0 = v \right]
$$

$$
\leq \frac{1}{4} \exp \left( - \frac{\delta(i) - 2t_{\text{mix}}(i)}{t_{\text{hit}(i)}} \right) + \frac{3}{4} \leq \frac{1}{4} \exp \left( - \frac{1}{N} \right) + \frac{3}{4}.
$$

(13)

We use Corollary B.5 in the first inequality. Now, for a positive integer $L$, consider a random variable $X \sim \text{Bin}(L, 1/4)$. Here, $\text{Bin}(L, 1/4)$ is the binomial distribution with parameters $L$ and $1/4$. Then, it is straightforward to see that

$$
\left( \frac{1}{4} \exp \left( - \frac{1}{N} \right) + \frac{3}{4} \right)^L = \sum_{i=0}^{L/8} \binom{L}{i} \left( \frac{1}{4} \exp \left( - \frac{1}{N} \right) \right)^i \left( \frac{3}{4} \right)^{L-i} = \sum_{i=0}^{L/8} \exp \left( - \frac{i}{N} \right) \Pr[X = i] \leq \frac{L}{8} \sum_{i=0}^{L/8} \exp \left( - \frac{i}{N} \right) \Pr[X = i]
$$

$$
\leq \Pr \left[ X \leq \frac{L}{8} \right] + \exp \left( - \frac{L}{8} \right) \leq \exp \left( - \frac{L}{32} \right) + \exp \left( - \frac{L}{8} \right).
$$

(14)

The last inequality follows since

$$
\Pr \left[ X \leq \frac{L}{8} \right] = \Pr \left[ X \leq \frac{E[X]}{2} \right] \leq \exp \left( - \frac{E[X]}{8} \right) = \exp \left( - \frac{L}{32} \right)
$$

holds from the Chernoff inequality Lemma B.2. Thus combining Lemma 3.4 and (13) and (14), we obtain

$$
E[U] \leq \sum_{k=1}^{n} \left( \frac{1}{4} \exp \left( - \frac{1}{N} \right) + \frac{3}{4} \right)^{n-k+1} \leq \frac{n}{k=1} \exp \left( - \frac{n - k + 1}{32} \right) + \exp \left( - \frac{n - k + 1}{8N} \right)
$$

$$
= \sum_{k=1}^{n} \exp \left( - \frac{k}{32} \right) + \sum_{k=1}^{n} \exp \left( - \frac{k}{8N} \right) \leq 32 + 8N.
$$

Example: Degree restricted expander graph. For a graph $G = (V, E)$, let $d_{\text{ave}}(G)$ and $d_{\text{min}}(G)$ denote the average and the minimum degree of $G$, respectively. Suppose that $P$ is the transition matrix of the lazy simple random walk on $G$ and let $\lambda_2(P)$ denote the second largest eigenvalue of $P$. We call a graph $G$ degree restricted expander graph if both $d_{\text{ave}}(G)$ and $d_{\text{min}}(G)$ are upper bounded by some positive constant. For any degree restricted expander graph, we have $t_{\text{hit}}(P) = O(|V|)$ and $t_{\text{mix}}(P) = O(\log |V|)$ (See Lemma B.9 in Appendix B and Theorem 12.4 in [25]). Thus Theorem 1.3 implies the following.
Corollary 3.5. Suppose that $G^{(i)}$ is a degree restricted expander graph and $P^{(i)}$ is the transition matrix of the lazy simple random walk on $G^{(i)}$ in $R = (\Omega, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty})$. Let $\gamma \in [0, 1]$ and $C > 0$ be arbitrary constants. Then two positive constants $K_1, K_2$ satisfying the following exist: If $\theta(i) \geq CK_1i^{1-\gamma} + K_2\log i$ for all $i \in [n]$, then $E[U] \leq 8n^2 \gamma + 32$.

Proof. Since there exist some positive constants $K_1, K_2$ satisfying $t_{hit}(i) \leq K_1i$ and $t_{mix} \leq K_2 \log i$, we obtain the claim from Theorem 3.2.

3.3 Upper bounds for simple or symmetric random walks

This section is devoted to prove Theorem 3.6, which is a generalized version of Theorems 1.4 and 1.5.

Theorem 3.6. Suppose that $P^{(i)}$ is reversible and lazy in $R = (\Omega, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty})$. Let $r_i = \max_{v \in V(i-1)} \frac{\pi^{(i-1)}(v)}{\pi^{(i)}(v)}$ for $1 < i \leq n$. Let $N$ be an arbitrary number. If $\theta(i) \geq \left(\frac{1}{N} + \frac{i(r_i-1)+1}{2} \right) t_{hit}$ for all $i$, then $E[U] \leq N \sqrt{\max_{1 \leq i \leq n} i(r_i - 1) + 1}$.

Proof of Theorem 1.4. Let $d^{(i)}_v$ denote the degree of a vertex $v \in V^{(i)}$ at round $i$. Then, for all $v \in V^{(i)}$,

$$\frac{\pi^{(i-1)}(v)}{\pi^{(i)}(v)} = \frac{d^{(i-1)}_v}{2|E^{(i-1)}|} \leq \frac{|E^{(i)}|}{|E^{(i-1)}|}$$

Note that $d^{(i-1)}_v \leq d^{(i)}_v$ holds from our assumption. Combining the assumptions on $\theta(i)$ and $E^{(i)}$, we have $\theta(i) \geq \frac{t_{hit}(i)}{r_i/C} + \frac{L+1}{2i} t_{hit}(i) \geq \frac{t_{hit}(i)}{n^2/C} + \frac{L+1}{2i} t_{hit}(i)$. Thus we obtain the claim by taking $N = n^2/C$ in Theorem 3.6.

Proof of Theorem 1.5. Since $P^{(i)}$ is symmetric, $r_i = \frac{i}{i-1} \leq \frac{1}{2}$ for all $i > 1$. From the assumption of Theorem 1.5, $\theta(i) \geq \frac{t_{hit}(i)}{r_i/C} + \frac{2t_{hit}(i)}{i} \geq \frac{t_{hit}(i)[2+1]}{n^2/C} + \frac{2t_{hit}(i)[2+1]}{2i}$ for all $1 < i \leq n$. Thus we obtain the claim by taking $N = n^2/C$ in Theorem 3.6.

To show Theorem 3.6, we set the following notations. For two vectors $f, g \in \mathbb{R}$ and a probability vector $\pi \in [0, 1]^V$, let $\langle f, g \rangle_\pi := \sum_{v \in V} \pi(v)f(v)g(v)$. Then, the $\ell_2(\pi)$-norm of $f$ is defined by $\|f\|_2,\pi := \sqrt{\langle f, f \rangle_\pi} = \sqrt{\sum_{v \in V} \pi(v)f(v)^2}$. For two vectors $f, g \in \mathbb{R}^V$ where $g(v) \neq 0$ holds for all $v \in V$, define $\xi_g \in \mathbb{R}^V$ by $\xi_g(v) = \frac{f(v)}{g(v)}$. Note that from these definitions, for any probability vector $\xi \in [0, 1]^V$, $\|\xi - 1(|V|)\|_2,\pi = \|\xi\|_2,\pi - 1$ holds. Here, $1(|n|)$ denotes the $n$-dimensional vector where all elements are equal to one. For a matrix $M \in \mathbb{R}^{V \times V}$ let $\lambda_j(M)$ denote the $j$-th largest (in absolute value) eigenvalue of $M$.

For any round $1 < \ell \leq n$ and $0 \leq t \leq \theta(\ell)$, define a probability vector $\nu^{(\ell)}_t \in [0, 1]^{V^{(\ell)}}$ where

$$\nu^{(\ell)}_t(v) = \Pr[X^{(\ell)}_t = v] \quad (\forall v \in V^{(\ell)}).$$

Furthermore, for any rounds $k, \ell$ satisfying $k - 1 < \ell \leq n - 1$, define $\mu_{\ell}^{(k)} \in [0, 1]^{V^{(\ell)}}$ by

$$\mu_{\ell}^{(k)}(v) = \Pr \left[ \bigwedge_{i=\ell+1}^{n} \left\{ \pi^{(i)} > \theta(i) \right\} \bigg| X^{(\ell)}_{\theta(\ell)} = v \right] \quad (\forall v \in V^{(\ell)}).$$
and \( \mu^{(n)}_{v_k} := 1^{(n)} \). Then, combining the Cauchy-Schwarz inequality, (14), (15) and (16), we have

\[
E[U] = \sum_{k=2}^{n} \sum_{v \in V^{(k-1)}} \nu_{\varnothing^{(k-1)}}^{(k-1)}(v) \mu^{(k-1)}_{v_k} \leq \sum_{k=2}^{n} \sum_{v \in V^{(k-1)}} \frac{\nu_{\varnothing^{(k-1)}}^{(k-1)}(v)}{\pi^{(k-1)}} \sum_{v \in V^{(k-1)}} \pi^{(k-1)}(v) \mu^{(k-1)}_{v_k}(v)^2
\]

\[
= \sum_{k=2}^{n} \left\| \nu_{\varnothing^{(k-1)}}^{(k-1)} \pi^{(k-1)} \right\|_{2, \pi^{(k-1)}} \left\| \mu^{(k-1)}_{v_k} \right\|_{2, \pi^{(k-1)}} = \sum_{k=2}^{n} 1 + \left\| \nu_{\varnothing^{(k-1)}}^{(k-1)} - 1 \right\|_{2, \pi^{(k-1)}} \left\| \mu^{(k-1)}_{v_k} \right\|_{2, \pi^{(k-1)}}^2.
\]

(17)

In the rest of this section, we show the following bounds of \( \left\| \frac{\nu_{\varnothing^{(k)}}^{(k)} - 1}{\pi^{(k)}} \right\|_{2, \pi^{(k)}} \) and \( \left\| \mu^{(k-1)}_{v_k} \right\|_{2, \pi^{(k-1)}} \), from which we immediately derive Theorem 3.6.

**Lemma 3.7.** Suppose that \( P^{(i)} \) is reversible and lazy in \( R = (\varnothing, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty}) \). Let \( r_i = \max_{v \in V^{(i-1)}} \frac{\pi^{(i-1)}(v)}{\pi^{(i)}(v)} \) for \( 1 < i \leq n \). If \( \varnothing(i) \geq \frac{i(r_i-1)+1}{2i(1-\lambda_2(P^{(i)}))} \), then \( \left\| \frac{\nu_{\varnothing^{(k)}}^{(k)} - 1}{\pi^{(k)}} \right\|_{2, \pi^{(k)}} < \max_{1 \leq i \leq n} i(r_i - 1) \)

**Lemma 3.8.** Suppose that \( P^{(i)} \) is reversible and lazy in \( R = (\varnothing, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty}) \). For \( 1 < i \leq n \), let \( r_i := \max_{v \in V^{(i-1)}} \frac{\pi^{(i-1)}(v)}{\pi^{(i)}(v)} \). Let \( N \) be an arbitrary positive number. If \( \varnothing(i) \geq \frac{(\frac{1}{N} + \frac{i-1}{2})}{2i(1-\lambda_2(P^{(i)}))} \), then \( \sum_{k=2}^{n} \left\| \mu^{(k-1)}_{v_k} \right\|_{2, \pi^{(k-1)}} \leq N \)

**Proof of Theorem 3.6.** Suppose \( \varnothing(i) \geq \frac{t\text{hit}(i)}{N} + \frac{i(r_i-1)+1}{2i(1-\lambda_2(P^{(i)}))} \) for all \( 1 < i \leq n \). Then, \( \varnothing(i) \geq \frac{i(r_i-1)+1}{2i(1-\lambda_2(P^{(i)}))} \) from Lemma 3.9. Furthermore, \( \varnothing(i) \geq \frac{t\text{hit}(i)}{N} + \frac{i-1}{2}t\text{hit}(i) \). Thus applying Lemmas 3.7 and 3.8 to (17),

\[
E[U] \leq \sum_{k=2}^{n} \sqrt{\max_{1 \leq i \leq n} i(r_i - 1) + 1} \left\| \mu^{(k-1)}_{v_k} \right\|_{2, \pi^{(k-1)}} \leq \sqrt{N} \max_{1 \leq i \leq n} i(r_i - 1) + 1
\]

and we obtain the claim.

**Proof of Lemma 3.7.** First we show the following lemma. This lemma gives a general upper bound of \( \left\| \frac{\nu_{\varnothing^{(k)}}^{(k)} - 1}{\pi^{(k)}} \right\|_{2, \pi^{(k)}}^2 \) using \( r_i \).

**Lemma 3.9.** Suppose that \( P^{(i)} \) is reversible and lazy in \( R = (\varnothing, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty}) \). Let \( r_i = \max_{v \in V^{(i-1)}} \frac{\pi^{(i-1)}(v)}{\pi^{(i)}(v)} \) for \( 1 < i \leq n \). Then for any round \( 1 \leq k \leq n \),

\[
\left\| \frac{\nu_{\varnothing^{(k)}}^{(k)} - 1}{\pi^{(k)}} \right\|_{2, \pi^{(k)}}^2 \leq \sum_{i=2}^{k} \left( \prod_{j=i}^{k} r_j \lambda_2(P^{(j)})^{2\lambda(j)} \right) \left( 1 - \frac{1}{r_i} \right).
\]
Proof of Lemma 3.7. First we observe that \( \log \left( r_j \left( \frac{j+1}{j} \right) \right) = \log(1+(r_j-1)) + \log(1+\frac{1}{j}) \leq (r_j-1) + \frac{1}{j} \). Hence it holds that
\[
\lambda_2(P^{(j)})^{2\ell(j)} \leq \left( 1 - \left( 1 - \lambda_2(P^{(j)}) \right)^{\frac{\log \left( r_j \left( \frac{j+1}{j} \right) \right)}{1 - \lambda_2(P^{(j)})}} \right) \leq \frac{1}{r_j} \cdot \frac{j}{j+1}.
\]
Applying Lemma 3.9, we obtain
\[
\sum_{i=2}^{k} \left( \prod_{j=1}^{k} r_j \lambda_2(P^{(j)})^{2\ell(j)} \right) \left( 1 - \frac{1}{r_i} \right) \leq \sum_{i=2}^{k} \left( \prod_{j=i}^{j+1} \frac{r_i - 1}{r_i} \right) \leq \sum_{i=2}^{k} \frac{i}{k} (r_i - 1) \leq \max_{1<i \leq n} i(r_i - 1) .
\]

Proof of Lemma 3.4. To obtain the claim, we show the following recurrence inequality:
\[
\left\| \frac{\nu_0^{(\ell)}}{\pi^{(\ell)}} - 1 \right\|_{2,\pi^{(\ell)}}^2 \leq r_{\ell} \lambda_2(P^{(\ell)})^{2\ell} \left( \frac{\nu_0^{(\ell-1)}}{\pi^{(\ell-1)}} - 1 \right)_{2,\pi^{(\ell-1)}}^2 + (r_{\ell} - 1) \lambda_2(P^{(\ell)})^{2\ell} .
\]
Write \( x_\ell = \left\| \frac{\nu_\ell}{\pi^{(\ell)}} - 1 \right\|_{2,\pi^{(\ell)}}^2 \), \( c_\ell = r_{\ell} \lambda_2(P^{(\ell)})^{2\ell} \) and \( d_\ell = (r_{\ell} - 1) \lambda_2(P^{(\ell)})^{2\ell} \) for notational convenience. If (18) holds for any \( \ell > 1 \), applying (18) repeatedly yields
\[
x_k \leq c_k x_{k-1} + d_k \leq c_k c_{k-1} x_{k-2} + c_k d_{k-1} + d_k \leq \cdots \leq \left( \prod_{i=2}^{k} c_i \right) x_1 + \sum_{i=2}^{k} \left( \prod_{j=i+1}^{k} c_j \right) d_i .
\]
Since \( x_1 = \left\| \frac{\nu_1}{\pi^{(1)}} - 1 \right\|_{2,\pi^{(1)}}^2 = 0 \) from definition, we obtain the claim.

Now we proceeds to show (18). From the reversibility of \( P^{(\ell)} \), it is easy to see that, for all \( v \in V^{(\ell)} \),
\[
\left( \frac{\nu_\ell}{\pi^{(\ell)}} \right)(v) = \sum_{u \in V^{(\ell)}} \nu_0^{(\ell)}(u) \left( (P^{(\ell)})^t \right)_{u,v} \frac{\pi^{(\ell)}(v)}{\pi^{(\ell)}(u)} = \sum_{u \in V^{(\ell)}} \nu_0^{(\ell)}(u) \left( (P^{(\ell)})^t \right)_{v,u} = \left( (P^{(\ell)})^t \frac{\nu_0^{(\ell)}}{\pi^{(\ell)}} \right)(v) .
\]
From (19) and Lemma 3.8 it holds that
\[
\left\| \frac{\nu_\ell}{\pi^{(\ell)}} - 1 \right\|_{2,\pi^{(\ell)}}^2 \leq \lambda_2(P^{(\ell)})^2 \left\| \frac{\nu_0^{(\ell)}}{\pi^{(\ell)}} - 1 \right\|_{2,\pi^{(\ell-1)}}^2 = \lambda_2(P^{(\ell)})^2 \left( \left\| \frac{\nu_0^{(\ell)}}{\pi^{(\ell)}} \right\|_{2,\pi^{(\ell-1)}}^2 - 1 \right) .
\]
Furthermore, for a vertex \( v_\ell \) which appears at the round \( \ell \), since \( \nu_0^{(\ell)}(v_\ell) = \Pr[X_0^{(\ell)} = v_\ell] = 0 \) holds, we have
\[
\left\| \frac{\nu_0^{(\ell)}}{\pi^{(\ell)}} \right\|_{2,\pi^{(\ell-1)}}^2 = \sum_{v \in V^{(\ell-1)}} \pi^{(\ell-1)}(v) \nu_0^{(\ell)}(v)^2 \leq \sum_{v \in V^{(\ell-1)}} \pi^{(\ell-1)}(v) \pi^{(\ell-1)}(v) \nu_0^{(\ell-1)}(v)^2 = r_{\ell} \sum_{v \in V^{(\ell-1)}} \pi^{(\ell-1)}(v) \frac{\nu_0^{(\ell-1)}(v)}{\pi^{(\ell-1)}(v)}^2 = r_{\ell} \left\| \frac{\nu_0^{(\ell-1)}}{\pi^{(\ell-1)}} \right\|_{2,\pi^{(\ell-1)}}^2 .
\]
Combining (20) and (21), we obtain (19). □

Proof of Lemma 3.8 The following lemma plays a key role in the proof of Lemma 3.8.

Lemma 3.10. Suppose that $P^{(i)}$ is reversible and lazy in $R = (\emptyset, (G^{(i)})^{\infty}_{i=1}, (P^{(i)})^{\infty}_{i=1})$. For $1 < i \leq n$, let $r_i = \max_{v \in \mathcal{V}(i-1)} \pi^{(i-1)}(v)$. Then, for any round $k$ satisfying $1 < k \leq n$,

$$
\left\| \mu^{(k-1)}_{\pi k} \right\|_{2,\pi(k-1)} \leq \prod_{i=k}^{n} \sqrt{r_i} \left( 1 - \frac{1}{t_{\text{hit}}(i)} \right)^{\frac{\beta(i)}{2}}.
$$

Proof of Lemma 3.8 Since $\log \sqrt{r_i} = \frac{1}{2} \log r_i = \frac{1}{2} \log (1 + (r_i - 1)) \leq \frac{r_i - 1}{2}$, we have

$$
\sqrt{r_i} \left( 1 - \frac{1}{t_{\text{hit}}(i)} \right)^{\frac{\beta(i)}{2}} \leq \left( 1 - \frac{1}{t_{\text{hit}}(i)} \right)^{\beta(i) - \frac{r_i - 1}{2} t_{\text{hit}}(i)} \leq \left( 1 - \frac{1}{t_{\text{hit}}(i)} \right)^{\beta(i) - \frac{1}{t_{\text{hit}}(i)}}.
$$

Thus combining Lemma 3.10 and (22),

$$
\sum_{k=2}^{n} \left\| \mu^{(k-1)}_{\pi k} \right\|_{2,\pi(k-1)} \leq \sum_{k=2}^{n} \prod_{i=k}^{n} \left( 1 - \frac{1}{t_{\text{hit}}(i)} \right)^{\beta(i) - \frac{1}{t_{\text{hit}}(i)}} \leq N.
$$

We invoke Lemma 3.3 in the last inequality. □

Proof of Lemma 3.10 For a transition matrix $P \in [0,1]^{\mathcal{V} \times \mathcal{V}}$ and a vertex $w \in \mathcal{V}$, define $P_{\pi} \in [0,1]^{\mathcal{V} \times \mathcal{V}}$ by

$$(P_{\pi})_{u,v} = \begin{cases} P_{u,v} & \text{(if } u \neq w \text{ and } v \neq w) \\ 0 & \text{(otherwise)} \end{cases}.$$

In other words, $(P_{\pi})_{u,v} = P_{u,v} 1_{u \neq w} 1_{v \neq w}$ for $u, v \in \mathcal{V}$. Note that $P_{\pi}$ is a substochastic matrix (see e.g., Section 3.6.5 of [2]), i.e., $\sum_{v \in \mathcal{V}} (P_{\pi})_{u,v} \leq 1$ holds for any $u \in \mathcal{V}$. Observe for any $u, v \in \mathcal{V}$ and $T > 0$ that

$$(P_{\pi}^{T})_{u,v} = \sum_{v_1 \in \mathcal{V} \setminus \{w\}} \cdots \sum_{v_{T-1} \in \mathcal{V} \setminus \{w\}} 1_{u \neq w} P_{u,v_1} P_{v_1,v_2} \cdots P_{v_{T-1},v} 1_{v \neq w}$$

$$= \Pr \left[ \tau_{\pi} > T, X_{T} = v | X_0 = u \right].$$

(23)

Here, $(X_t)_{t=0}^{\infty}$ denotes a sequence of a random walk according to $P$ and $\tau_{\pi}$ denotes the hitting time to $w$. Note that $(P_{\pi}^{T})_{u,w} = 0$ if $u = w$ or $v = w$.

Consider a fixed $k > 1$. Write $\mu^{(\ell)} = \mu^{(\ell)}_{\pi k}$ and $Q^{(\ell)} = (P^{(\ell)})_{\pi k}$ for notational convenience. The key property for the proof is the following recurrence equation: for all $k - 1 \leq \ell \leq n - 1$ and $v \in \mathcal{V}^{(\ell)}$, it holds that

$$
\mu^{(\ell)}(v) = \left( Q^{(\ell+1)} \mu^{(\ell+1)} \right)(v).
$$

(24)
This equation holds since for any \( u_\ell \in V(\ell) \), combining \((9), (16)\) and \((23)\) yields

\[
\mu(\ell)(u_\ell) = \sum_{u_{\ell+1} \in V(\ell+1)} \cdots \sum_{u_n \in V(n)} \prod_{i=\ell+1}^{n} \left( (P_{\ell_k}(i))^{\delta(i)} \right)_{u_{i-1}, u_i} \\
= \sum_{u_{\ell+1} \in V(\ell+1)} Q^{(\ell+1)}_{u_\ell, u_{\ell+1}} \mu^{(\ell+1)}(u_{\ell+1}).
\]

Using \((24)\) and Corollary \[B.7\], we obtain

\[
\left\lVert \mu^{(\ell)} \right\rVert^2_{2, \pi(\ell)} = \sum_{v \in V(\ell)} \pi^{(\ell)}(v) \mu^{(\ell)}(v)^2 = \sum_{v \in V(\ell)} \frac{\pi^{(\ell)}(v)}{\pi^{(\ell+1)}(v)} \pi^{(\ell+1)}(v) \left( Q^{(\ell+1)} \mu^{(\ell+1)} \right)(v)^2 \\
\leq r_{\ell+1} \sum_{v \in V(\ell+1)} \pi^{(\ell+1)}(v) \left( Q^{(\ell+1)} \mu^{(\ell+1)} \right)(v)^2 = r_{\ell+1} \left\lVert Q^{(\ell+1)} \mu^{(\ell+1)} \right\rVert^2_{2, \pi(\ell+1)} \\
\leq r_{\ell+1} \lambda_1 (Q^{(\ell+1)})^2 \left\lVert \mu^{(\ell+1)} \right\rVert^2_{2, \pi(\ell+1)}.
\]

Hence applying \((25)\) repeatedly, it holds that

\[
\left\lVert \mu^{(\ell)} \right\rVert^2_{2, \pi(\ell)} \leq \prod_{i=\ell+1}^{n} r_i \lambda_1 (Q(i))^2.
\]

From the definition of \(Q(i)\) and \(P_{\ell_k}(i)\), Lemma \[B.6\] implies

\[
\lambda_1(Q(i)) = \lambda_1(P_{\ell_k}(i))^{\delta(i)} \leq \left( 1 - \frac{1}{t_{\text{hit}}(i)} \right)^{\delta(i)}.
\]

Thus we obtain the claim from \((26)\) and \((27)\). \(\square\)

**Example: Lollipop graph.** Consider a growing lollipop graph: We consider \(G^{(i)}\) consisting of the complete graph \(K_{[i/2]}\) and the path graph \(P_{[i/2]}\). Formally, at each round \(i \in [n]\), the set of odd vertices \(V_{o}^{(i)} := \{v_{2i+1} : 1 \leq i \leq \lfloor i/2 \rfloor \}\) forms the complete graph \(K_{[i/2]}\), the set of even vertices \(V_{e}^{(i)} := \{v_{2i} : 1 \leq i \leq \lfloor i/2 \rfloor \}\) forms a path graph, and these two components are connected by \(\{v_1, v_2\}\). Let \(P^{(i)}\) be the transition matrix of the simple lazy random walk on \(G^{(i)}\). For such \(P^{(i)}\), it is well known that \(t_{\text{hit}}(i) = O(i^3)\) (see e.g. \[18\]).

**Corollary 3.11.** Consider \(R = (\mathfrak{d}, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty})\) where \(G^{(i)}\) is the lollipop graph defined above and \((P^{(i)})_{i=1}^{\infty}\) is the transition matrix of the lazy simple random walk on \(G^{(i)}\). Let \(\gamma \in [0,1]\) be an arbitrary constants. If \(\mathfrak{d}(i) \geq C_1 i^{3-\gamma}\) for all \(i\), then \(E[U] \leq C_2 n^\gamma\). Here, \(C_1, C_2\) are some positive constants.

**Proof.** From definition, \(|E(2i)| = 1 + \frac{i(i+1)}{2} + i - 1 = \frac{i(i+1)}{2}\) and \(|E(2i+1)| = 1 + \frac{(i+1)i}{2} + i - 1 = \frac{(i+3)}{2}\). Thus for any \(i\), \(\frac{|E(i)|}{E(2i+1)} \leq 1 + K_1\) for some constant \(K_1\). Furthermore, \(t_{\text{hit}}(i) \leq K_2 i^3\) holds for some constant \(K_2\). Applying Theorem \[1.3\] we obtain the claim. \(\square\)
Example: Metropolis walk. For a given graph $G = (V, E)$, the transition matrix of the lazy Metropolis walk on $G$ is defined by

$$
(P)_{uv} = \begin{cases} 
\frac{1}{2\max\{d_u, d_v\}} & \text{(if } \{u, v\} \in E) \\
1 - \sum_{w: \{u, w\} \in E} (P)_{uw} & \text{(if } u = v) \\
0 & \text{(otherwise)}.
\end{cases}
$$

(28)

Due to the work of Nonaka, Ono, Sadakane and Yamashita [29], we have $t_{\text{hit}}(P) = O(|V|^2)$ for any connected graphs. Since $P$ is symmetric matrix, we can apply Theorem 1.5.

Corollary 3.12. Suppose that $P^{(i)}$ is the lazy Metropolis walk on $G^{(i)}$ in $R = (\mathcal{G}, (G^{(i)})_{i=1}^\infty, (P^{(i)})_{i=1}^\infty)$. Let $\gamma \in [0, 1]$ and $C > 0$ be arbitrary constants. If $\mathcal{H}(i) \geq \left(\frac{C}{\gamma} + \frac{2}{3}\right) t_{\text{hit}}(i)$ for all $1 \leq i \leq n$, then $E[U] \leq \sqrt{n^{\gamma} + C}.$

4 A Lower Bound for a Growing Path

This section is devoted to proving Theorem 1.6. Let $L, R \in [n]$ be parameters satisfying $L < R$. For a vertex $v \in V^{(n)}$, let $\mathcal{E}(v)$ be the event that $v \notin \bigcup_{i=1}^n \bigcup_{t=0}^{\gamma(i)} \{X_t^{(i)}\}$. In other words, $\mathcal{E}(v)$ means that the walker does not visit the vertex $v$ during the walk. For two vertices $v_i, v_j \in V^{(n)}$, we write $v_i \preceq v_j$ if $i \leq j$. Note that, for any two vertices $u \preceq v$ and any round $k \in [n]$, it holds that $\Pr[\mathcal{E}(v) \mid X_0^{(k)} \leq u] \geq \Pr[\mathcal{E}(v) \mid X_0^{(k)} = u]$. Then, we have

$$
E[U] = \sum_{k=1}^n \Pr[\mathcal{E}(v_k)] \geq \sum_{k=R}^n \Pr[\mathcal{E}(v_k)] \geq \sum_{k=R}^n \Pr[\mathcal{E}(v_k) \land X_0^{(k)} \leq v_L] \\
= \sum_{k=R}^n \Pr[\mathcal{E}(v_k) \mid X_0^{(k)} \leq v_L] \Pr[X_0^{(k)} \leq v_L] \\
\geq (n - R) \Pr[\mathcal{E}(v_R) \mid X_0^{(R)} = v_L] \min_{R \leq k \leq n} \left\{ \Pr[X_0^{(k)} \leq v_L] \right\}.
$$

(29)

We will determine the parameters $R$ and $L$ such that $n - R = \Omega(n^\gamma)$, $\Pr[\mathcal{E}(v_R) \mid X_0^{(R)} = v_L] = \Omega(1/C)$ and $\Pr[X_0^{(k)} \leq L] = \Omega(1)$ for all $R \leq k \leq n$. This yields the lower bound $E[U] = \Omega(n^\gamma / C)$.

For fixed parameter $R$, let $T := \sum_{i=R}^n \mathcal{H}(i)$ denote the number of steps of the walk during the last $n - R + 1$ rounds.

Lemma 4.1. Let $L, R \in \mathbb{N}$ be parameters satisfying $L < R$ and let $T := \sum_{i=R}^n \mathcal{H}(i)$. Then, the following holds.

(i) $\Pr[\mathcal{E}(v_R) \mid X_0^{(R)} = v_L] \geq 1 - \frac{T}{4(R - L)^2}$, and

(ii) $\Pr[X_0^{(k)} \leq v_L] \geq 1 - \frac{k}{n}$ for all $k \in [n]$.

Proof of (i). Let $(Z_t)_{t=1}^\infty$ be i.i.d. random variables sampled from the uniform distribution over $\{-1, +1\}$ and $S_c := \sum_{j=0}^c Z_j$ denote the sum. For a vertex $v_i \in V^{(n)}$, let $\text{pos}(v_i) = i$ denote the
Then the complementary event \( \overline{E(v_R)} \) conditioned on \( X_0^{(R)} = v_L \) is equivalent to the event that \( \max_{R \leq i \leq n, 0 \leq j \leq \delta(i)} \{ \text{pos}(X_j^{(i)}) - \text{pos}(X_0^{(R)}) \} \geq R - L \). Moreover, the random variable \( \max_{R \leq i \leq n, 0 \leq j \leq \delta(i)} \{ \text{pos}(X_j^{(i)}) - \text{pos}(X_0^{(R)}) \} \) is dominated \(^6\) by \( \max_{1 \leq c \leq T} |S_c| \) (recall \( T = \sum_{i=R}^{n} \delta(i) \)). This is because the distribution of \( \text{pos}(X_j^{(i)}) - \text{pos}(X_0^{(R)}) \) conditioned on \( \text{pos}(X_j^{(i)}) - \text{pos}(X_{j-1}^{(i)}) \neq 0 \) is uniform on \( \{-1, +1\} \). Thus we obtain
\[
\Pr \left[ \overline{E(v_R)} \middle| X_0^{(R)} = v_L \right] \leq \Pr \left[ \max_{R \leq i \leq n, 0 \leq j \leq \delta(i)} \{ \text{pos}(X_j^{(i)}) - \text{pos}(X_0^{(R)}) \} \geq R - L \middle| X_0^{(R)} = L \right]
\leq \Pr \left[ \max_{1 \leq c \leq T} |S_c| \geq R - L \right]
\leq \frac{\text{Var} [S_T]}{(R - L)^2} = \frac{T}{4(R - L)^2}.
\]

In the last inequality, we used the Kolmogorov inequality (Lemma [B.1]).

**Proof of (ii)** It suffices to show that
\[
\Pr[X_0^{(k)} = v_i] \geq \Pr[X_0^{(k)} = v_{i+1}]
\] (30)
holds for any \( 1 \leq i \leq k - 1 \). To see this, assuming (30), we obtain
\[
\frac{\Pr[X_0^{(k)} \leq v_L]}{L} \geq \Pr[X_0^{(k)} = v_L] \geq \frac{1 - \Pr[X_0^{(k)} \leq L]}{n - L},
\]
which implies the claim (ii). Here, in the second inequality, note that \( \Pr[X_0^{(k)} = v_L] \geq \Pr[X_0^{(k)} = v_j] \) for all \( j > L \) and thus, the average \( \frac{1}{n-L} \sum_{j > L} \Pr[X_0^{(k)} = v_j] \) is at most \( \Pr[X_0^{(k)} = v_L] \).

Now we prove the inequality (30). Let \( x_j^{(i)} \in [0, 1]^V \) denote the distribution of \( X_j^{(i)} \). To simplify notations, for a vector \( y \in [0, 1]^V \), we write \( y[u] \) for the \( u \)-th element of \( y \). We call the distribution \( y \in [0, 1]^V \) monotone if \( y[v_k] \geq y[v_{k+1}] \) holds for any \( 1 \leq k \leq i - 1 \). Our aim here is to prove that \( x_0^{(k)} \) is monotone, which is equivalent to (30).

Indeed, we will prove a stronger statement: \( x_j^{(i)} \) is monotone for any \( i \) and \( j \). We prove this statement inductively. First, the vector \( x_j^{(1)} = (1) \) is obviously monotone. Secondly, if \( x_j^{(i)} \) is monotone, so does \( x_0^{(i+1)} \). To see this, note that \( x_0^{(i+1)} \) is obtained by concatenating \( x_0^{(i)} \) with \( 0 \). More precisely, \( x_0^{(i+1)} \in [0, 1]^{i+1} \) satisfies
\[
x_0^{(i+1)}[j] = \begin{cases} 
    x_0^{(i)}[j] & \text{if } 1 \leq j \leq i, \\
    0 & \text{if } j = i + 1.
\end{cases}
\]

Finally, we check that \( x_j^{(i)} \) is monotone if \( x_j^{(i)} \) is monotone. From (1), we have
\[
x_j^{(i)}[v_k] = \begin{cases} 
    px_j^{(i)}[v_1] + (1 - p)x_j^{(i)}[v_2] & \text{if } k = 1, \\
    qx_j^{(i)}[v_{k-1}] + (1 - 2q)x_j^{(i)}[v_k] + qx_j^{(i)}[v_{k+1}] & \text{if } 1 < k < i, \\
    (1 - p)x_j^{(i)}[v_{i-1}] + px_j^{(i)}[v_i] & \text{if } k = i.
\end{cases}
\]

\(^6\)For two random variables \( X \) and \( Y \), we say \( X \) dominates \( Y \) if, for any \( r \in \mathbb{R}, \Pr[X \geq r] \geq \Pr[Y \geq r] \) holds.
By the induction assumption, \( x_j^{(i)} \) is monotone. Now we check that \( x_j^{(i)} \) is monotone. For \( k = 1 \), since \( p \geq q \), we have
\[
x_j^{(i)}[v_1] - x_j^{(i)}[v_2] = (p - q)(x_j^{(i)}[v_1] - x_j^{(i)}[v_2]) + q(x_j^{(i)}[v_2] - x_j^{(i)}[v_3]) \geq 0.
\]
For \( 1 < k < i - 1 \), since \( q \leq \frac{1}{2} \), we have
\[
x_j^{(i)}[v_i] - x_j^{(i)}[v_{i+1}] = qx_j^{(i)}[v_{k-1}] + (1 - 3q)x_j^{(i)}[v_k] - (1 - 3q)x_j^{(i)}[v_{k+1}] - qx_j^{(i)}[v_{k+2}] \\
\geq (1 - 2q)(x_j^{(i)}[v_k] - x_j^{(i)}[v_{k+1}]) \geq 0.
\]
Finally, for \( k = i \), since \( p \geq q \), we have
\[
x_j^{(i)}[v_{i-1}] - x_j^{(i)}[v_i] = q(x_j^{(i)}[v_{i-2}] - x_j^{(i)}[v_{i-1}]) + (p - q)(x_j^{(i)}[v_{i-1}] - x_j^{(i)}[v_i]) \geq 0.
\]
Therefore \( x_j^{(i)} \) is monotone.

Now we are ready to prove Corollary \ref{cor:hit} Recall \( \delta(i) \leq Ct^{2-\gamma} \). Fix a small positive constant \( \epsilon \) such that \( \epsilon < \min\{1/C, 0.1\} \). Set \( R := n - \epsilon n^\gamma \) and \( L := R - 0.6n \in [0.3n, 0.4n] \). Then we have \( T \leq (n - R)\delta(n) \leq C\epsilon n^2 \leq n^2 \) and thus \( 1 - \frac{T}{4(R-L)^2} \geq 1 - \frac{1}{4 \times 0.36} > 0.3 \) and \( 1 - \frac{1}{n} \geq 0.6 \). Then, from \eqref{eq:hit} and Lemma \ref{lem:hit} we have
\[
\mathbb{E}[U] \geq \epsilon n^\gamma \cdot 0.3 \cdot 0.6 = \Omega\left(\frac{n^\gamma}{C}\right),
\]
which completes the proof of Theorem \ref{thm:hit} (here, we take \( \epsilon > 0 \) such that \( \epsilon = \Omega(1/C) \)).

5 Concluding Remarks

This paper has investigated the expected numbers of vertices remaining unvisited by random walks on growing graphs parametrized by \( \delta \). We have presented some upper bounds of \( \mathbb{E}[U] \) with respect to \( \delta \), where we revealed that \( \mathbb{E}[U] = O(1) \) if \( \delta(i) \geq Ct_{hit}(i) \) for \( C > 1 \) in general (Theorem \ref{thm:hit}), and that \( \mathbb{E}[U] = O(1) \) if \( \delta(i) = \Omega(t_{hit}(i)) \) on some natural assumptions (Theorems \ref{thm:hit1} to \ref{thm:hit3}). We have also presented lower bounds of \( \mathbb{E}[U] \) for random walks on growing complete graphs and on growing path graphs, which imply the upper bounds by Theorems \ref{thm:hit1} to \ref{thm:hit3} are tight in those cases. A general lower bound of \( \mathbb{E}[U] \) is a challenge: a natural question remains unsettled whether \( \mathbb{E}[U] = O(1) \) requires \( \delta(i) = \Omega(t_{hit}(i)) \). A concentration result should be another future work \cite{10}.

In this paper, we have been concerned with a simple model of graphs with the increasing number of vertices, to develop a new technique for analyses of random walks on dynamic graphs. Clearly, it is an interesting and important future work to analyze algorithms on dynamic graphs whose vertex set and edge set are both dynamic.

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A Note on the initial round

For a \( n_0 > 0 \), we consider the case where \( n_0 \) vertices exist at the first round.

**Theorem A.1.** Let \( G^{(i)} = K_{n_0+i} \), i.e., the complete graph with \( n_0 + i \) vertices, and let \( (P^{(i)})_{u,v} = 1/(n_0 + i) \) for all \( u, v \in V^{(i)} \) in \( R = (\mathfrak{d}, (G^{(i)})_{i=1}^{\infty}, (P^{(i)})_{i=1}^{\infty}) \). Let \( N \) be an arbitrary positive number. If \( \mathfrak{d}(i) \geq 2i/N \) for all \( i \), then \( \mathbf{E}[U(n)] \leq 2n_0 + N \).

**Proof.** If \( n \leq n_0 \), \( |V^{(n)}| = n_0 + n \leq 2n_0 \) and we are done. Suppose that \( n > n_0 \). Then it is straightforward to see that

\[
\mathbf{E}[U(n)] = n_0 \prod_{i=1}^{n} \left( 1 - \frac{1}{n_0 + i} \right)^{\mathfrak{d}(i)} + \sum_{k=1}^{n} \prod_{i=k}^{n} \left( 1 - \frac{1}{n_0 + i} \right)^{\mathfrak{d}(i)}
\]

\[
\leq n_0 + n_0 + \sum_{k=n_0+1}^{n} \prod_{i=k}^{n} \left( 1 - \frac{1}{n_0 + i} \right)^{\mathfrak{d}(i)}
\]

\[
\leq 2n_0 + \sum_{k=n_0+1}^{n} \prod_{i=k}^{n} \left( 1 - \frac{1}{2i} \right)^{\mathfrak{d}(i)}
\]

\[
\leq 2n_0 + N.
\]

Note that we use Lemma 3.3 in the last inequality.
B Tools

Lemma B.1 (The Kolmogorov inequality; Theorem 2.5.5 of [16]). Let $Z_1, \ldots, Z_n$ be i.i.d. random variables such that $\mathbb{E}[Z_i] = 0$ and $\mathbb{V}[Z_i] < \infty$. Let $S_i = \sum_{j=1}^i Z_i$. Then,

$$\Pr[\max_{1 \leq j \leq n} |S_j| \geq M] \leq \frac{\mathbb{V}[S_n]}{M^2}.$$ 

Lemma B.2 (The Chernoff inequality (see e.g. Theorem 1.10.5 of [14])). Let $X_1, X_2, \ldots, X_n$ be independent random variables taking values in $[0, 1]$. Let $X = \sum_{i=1}^n X_i$. Let $\delta \in [0, 1]$. Then

$$\Pr[X \leq (1 - \delta) \mathbb{E}[X]] \leq \exp \left( -\frac{\delta^2 \mathbb{E}[X]}{2} \right).$$

Lemma B.3 (See e.g. Sections 2.4.3 of [2]). Consider a random walk on a (static) graph $G = (V, E)$. Then for any $c > 0$ and any $v, u \in V$, $\Pr[\tau_v > c \delta_{hit}|X_0 = u] \leq e^{-c}$.

To see this, divide $c \delta_{hit}$-steps random walk into $c$ independent random walks each of length $\delta_{hit}$. Then, in each walk, the walker does not visit a specific vertex with probability at most $1/e$ from the Markov inequality.

Using Lemma B.3 for any $t \geq \delta_{hit} \log n$, it is easy to see that $\mathbb{E}[t_{hit}] = \sum_{v \in V} \Pr[\tau_v > t|X_0 = u] \leq ne^{-\log n} = 1$. This implies that, for any RWoGG with $d(i) \geq \delta_{hit}(i)$, the number of unvisited vertices is at most 1 in expectation at the end of every round.

Lemma B.4 (Theorem 4.1 of [30]). Let $P \in [0, 1]^{V \times V}$ be an irreducible, reversible and lazy transition matrix over $V$, and let $\pi \in (0, 1)^V$ denote its stationary distribution. Let $(X_t)_{t=0}^\infty$ denote the Markov chain according to $P$. Let $\tau_v(P) = \min\{t \geq 0 : X_t = v\}$ and let $t_{hit}(P) = \max_{u,v \in V} \mathbb{E}_u[\tau_v(P)]$. Then for any $t \geq 0$ and any choice of $h_0, h_1, \ldots, h_t$,

$$\Pr[\forall 0 \leq s \leq t : X_s \neq h_s] \leq \left( 1 - \frac{1}{t_{hit}(P)} \right)^t.$$

Taking $h_i = v \in V$ for all $0 \leq i \leq t$ in Lemma B.4, we immediately obtain the following.

Corollary B.5. Let $P \in [0, 1]^{V \times V}$ be an irreducible, reversible and lazy transition matrix over $V$, and let $\pi \in (0, 1)^V$ denote its stationary distribution. Let $(X_t)_{t=0}^\infty$ denote the Markov chain according to $P$. Let $\tau_v(P) = \min\{t \geq 0 : X_t = v\}$ and let $t_{hit}(P) = \max_{u,v \in V} \mathbb{E}_u[\tau_v(P)]$. Then for any $v \in V$ and $t > 0$,

$$\Pr_\pi[\tau_v(P) > t] \leq \left( 1 - \frac{1}{t_{hit}(P)} \right)^t \leq \exp \left( -\frac{t}{t_{hit}(P)} \right).$$

Lemma B.6 (See Section 3.6.5 of [2] or Theorem 4.1 of [30]). Let $P \in [0, 1]^{V \times V}$ be an irreducible and reversible transition matrix over $V$, and let $\pi \in (0, 1)^V$ denote its stationary distribution. For a
subset $S \subseteq V$, define $P_S \in [0,1]^{V \times V}$ by $(P_S)_{u,v} = P_{u,v}$ for any $u,v \in V \setminus S$ and $(P_S)_{u,v} = 0$ for any $u \in S$ or $v \in S$. Let $\lambda(M)$ denote the largest eigenvalue of a matrix $M$. Then for any $S \notin \{\emptyset, V\}$,

$$\lambda(P_S) \leq 1 - \frac{1}{t_{\text{hit}}(P)}.$$  

Furthermore, for any $S \notin \{\emptyset, V\}$ and any $f \in \mathbb{R}^V$,

$$\langle f, P_S f \rangle_\pi \leq \lambda(P_S) \langle f, f \rangle_\pi.$$  

Since $\|P_S f\|^2_{2,\pi} = \langle P_S f, P_S f \rangle_\pi = \langle f, P_S^2 f \rangle_\pi$, we have the following corollary.

**Corollary B.7.** Let $P \in [0,1]^{V \times V}$ be an irreducible, reversible and lazy transition matrix over $V$, and let $\pi \in (0,1]^V$ denote its stationary distribution. Suppose that $P_S$ is a matrix defined in Lemma B.6. Then for any $S \notin \{\emptyset, V\}$ and any $f \in \mathbb{R}^V$,

$$\|P_S f\|^2_{2,\pi} \leq \lambda_1(P_S) \|f\|^2_{2,\pi} \leq \left(1 - \frac{1}{t_{\text{hit}}(P)}\right)^2 \|f\|^2_{2,\pi}$$

Here, $\lambda_1(M)$ denotes the largest eigenvalue in absolute value of a matrix $M$.

**Lemma B.8** (See e.g. (12.8) of [25]). Let $P \in [0,1]^{V \times V}$ be a reversible transition matrix with respect to $\pi \in (0,1]^V$. Then for any probability vector $f \in [0,1]^V$, $\left\| \frac{f}{\pi} - 1 \right\|^2_{2,\pi} = \left\| \frac{f}{\pi} \right\|^2_{2,\pi} - 1$ and

$$\left\| P \frac{f}{\pi} - 1 \right\|^2_{2,\pi} \leq \lambda_2(P)^2 \left\| \frac{f}{\pi} - 1 \right\|^2_{2,\pi}$$

holds where $\lambda_2(P)$ is the second largest eigenvalue (in absolute value) of $P$.

**Lemma B.9** (Lemmas 4.24 and 4.25 of [2]). Let $P$ be reversible transition matrix and let $\pi$ be its stationary distribution. Then

$$\frac{1}{1 - \lambda_2(P)} \leq t_{\text{hit}}(P) \leq \frac{2}{\pi_{\text{min}}(1 - \lambda_2(P))}.$$