From Cycle Rooted Spanning Forests to the Critical Ising Model: an Explicit Construction

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Abstract: Fisher established an explicit correspondence between the 2-dimensional Ising model defined on a graph $G$ and the dimer model defined on a decorated version $\mathcal{G}$ of this graph (Fisher in J Math Phys 7:1776–1781, 1966). In this paper we explicitly relate the dimer model associated to the critical Ising model and critical cycle rooted spanning forests (CRSFs). This relation is established through characteristic polynomials, whose definition only depends on the respective fundamental domains, and which encode the combinatorics of the model. We first show a matrix-tree type theorem establishing that the dimer characteristic polynomial counts CRSFs of the decorated fundamental domain $\mathcal{G}_1$. Our main result consists in explicitly constructing CRSFs of $\mathcal{G}_1$ counted by the dimer characteristic polynomial, from CRSFs of $G_1$, where edges are assigned Kenyon’s critical weight function (Kenyon in Invent Math 150(2):409–439, 2002); thus proving a relation on the level of configurations between two well known 2-dimensional critical models.

1. Introduction

In [Fis66], Fisher established an explicit correspondence between the 2-dimensional Ising model defined on a graph $G$ and the dimer model defined on a decorated version $\mathcal{G}$ of this graph, known as the Fisher graph of $G$. Since then, dimer techniques have been a powerful tool for tackling the Ising model, see for example the book of [MW73]. More recently, in [BdT10b,BdT10a], we prove fundamental results for the dimer model corresponding to a large class of critical Ising models, by proving explicit formulae for the free energy and for the Gibbs measure.

Critical Ising models we consider are defined on graphs satisfying a geometric property called isoradiality. When the underlying isoradial graph $G$ is infinite and $\mathbb{Z}^2$-periodic, then so is the Fisher graph $\mathcal{G}$, and we let $\mathcal{G}_1 = \mathcal{G}/\mathbb{Z}^2$ be the fundamental domain. The key object involved in the explicit expressions of [BdT10b] for the free

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energy and the Gibbs measure is the critical dimer characteristic polynomial, whose
definition only depends on the fundamental domain $G_1$. This polynomial is a generating
function for configurations related to super-imposed dimer configurations of $G_1$, referred
to as ‘double-dimer’ configurations. By Fisher’s correspondence, this implies that the
dimer characteristic polynomial is a generating function for ‘double-Ising’ configura-
tions.

In [BdT10b], we prove that the dimer characteristic polynomial is equal, up to a con-
stant, to the critical Laplacian characteristic polynomial of $G_1 = G/\mathbb{Z}^2$, where edges
of $G_1$ are assigned Kenyon’s critical weight function [Ken02]. Using a generalization
of Kirchhoff’s matrix tree theorem due to Forman [For93], the latter is shown to be a
generating function for cycle rooted spanning forests (CRSFs) of $G_1$. This suggests
the existence of an explicit relation between ‘double-Ising’ configurations and CRSFs,
which we were not able to find in [BdT10b]. The first result of this paper is a matrix-tree
type theorem, proving that the critical dimer characteristic polynomial is a generating
function for CRSFs of the Fisher graph $G_1$, see Theorem 7 of Sect. 4.2. Then, the main
result of this paper can loosely be stated as follows, refer to Theorem 17 of Sect. 4.4 and
Theorem 29 of Sect. 6.1 for more precise statements.

**Theorem 1.** Consider a critical Ising model defined on an infinite, $\mathbb{Z}^2$-periodic isoradi-
al graph $G$. Then, there exists an explicit way of constructing CRSFs of $G_1$ counted by
the critical dimer characteristic polynomial, from CRSFs of $G_1$ counted by the critical
Laplacian characteristic polynomial.

This exhibits an explicit relation, on the level of configurations, between two well
known models of statistical mechanics: the Ising model and CRSFs at criticality. Note
that such a relation was already suspected by Messikh [Mes06]. Before giving an outline
of the paper, let us make a few comments.

- The main contribution of this paper is the proof of Theorem 1, where we actually
  provide the explicit construction.
- The partition functions (weighted sum of configurations) of both models can be
  expressed from their respective characteristic polynomials, so that it is actually stron-
  ger to work with characteristic polynomials.
- Working with graphs embedded on the torus has the advantage of avoiding bound-
  ary issues, but has the additional difficulty of involving the geometry of the torus,
  with non-trivial cycles occurring in configurations. Working on finite pieces of infi-
  nite graphs, and precisely specifying boundary conditions, would certainly explicitly
  relate double-dimer configurations and spanning trees.
- Spanning trees are a well suited object for defining a height function and prove
  Gaussian fluctuations. Thus, it might be that Theorem 1 could be used to prove
  results which are numerically described in the paper [Wil11].

**Outline of the paper.**

Section 2: Definition of the critical Ising model and of the dimer model. Description of
Fisher’s correspondence relating the two.

Section 3: Definition of the critical dimer and Laplacian characteristic polynomials.
Relation between the Laplacian characteristic polynomial and CRSFs.

Section 4: Statement and proof of Theorem 7 establishing that the critical dimer character-
istic polynomial is a generating function for CRSFs of the Fisher graph $G_1$. Precise statement of Theorem 1.

Section 5: Definition and properties of licit primal/dual edge moves, which are one of
the key ingredients of the correspondence.