A QUANTUM ALGEBRA APPROACH TO MULTIVARIATE ASKEY-WILSON POLYNOMIALS

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Abstract. We study matrix elements of a change of base between two different bases of representations of the quantum algebra $U_q(su(1,1))$. The two bases, which are multivariate versions of Al-Salam–Chihara polynomials, are eigenfunctions of iterated coproducts of twisted primitive elements. The matrix elements are identified with Gasper and Rahman’s multivariate Askey-Wilson polynomials, and from this interpretation we derive their orthogonality relations. Furthermore, the matrix elements are shown to be eigenfunctions of the twisted primitive elements after a change of representation, which gives a quantum algebraic derivation of the fact that the multivariate Askey-Wilson polynomials are solutions of a multivariate bispectral $q$-difference problem.

1. Introduction

In this paper we give an interpretation of Gasper and Rahman’s multivariate Askey-Wilson polynomials [8] in representation theory of the quantum algebra $U_q(su(1,1))$, and we obtain from this interpretation their main properties: orthogonality relations and difference equations.

The univariate Askey-Wilson polynomials [2] are orthogonal polynomials depending on four parameters $a, b, c, d$ and on a parameter $q$. They are given explicitly by

$$p_n(x; a, b, c, d|q) = \frac{(ab, ac, ad; q)_n}{a^n} _4\varphi_3 \left( -q^{-n}, abcdq^{n-1}, ax, a/x; ab, ac, ad; q, q \right),$$

where we use standard notation for $q$-shifted factorials and $q$-hypergeometric functions as in [7]. From the explicit expression (1.1) one sees that $p_n(x)$ is a polynomial in $x + x^{-1}$ of degree $n$. The Askey-Wilson polynomials and their discrete counterparts, the $q$-Racah polynomials (which are essentially also Askey-Wilson polynomials), are on top of the Askey-scheme, see [17], a large scheme consisting of families of orthogonal polynomials of ($q$-)hypergeometric type which are related by limit transitions.

The Askey-Wilson polynomials turned out to be fundamental objects in the representation theory of quantum groups and algebras. Koornwinder [21] gave an interpretation of a two-parameter family of the Askey-Wilson polynomials as zonal spherical functions on the quantum group $SU_q(2)$. Fundamental in this approach is the introduction of twisted primitive elements, which are elements in the quantum algebra $U_q(\mathfrak{su}(2, C))$ that are much like Lie algebra elements. Similar interpretations for the full four-parameter family of Askey-Wilson polynomials were obtained in e.g. [18], [23]. A different interpretation is obtained by Rosengren [24], who introduces a generalized group element (a rediscovery of Babelon’s [3] ‘shifted boundary’) that transforms Koornwinder’s twisted primitive elements into group-like elements. The Askey-Wilson polynomials appear as ‘matrix elements’ of the generalized group element with respect to continuous and discrete bases in a discrete series representations of the quantum algebra $U_q(su(1,1))$. Other interpretations of the Askey-Wilson polynomials, as $3j$ and $6j$-symbols, can be found in e.g. [19], [5], [13].

Gasper and Rahman introduced in [8] multivariate extensions of the Askey-Wilson polynomials. These polynomials can be considered as $q$-analogues of Tratnik’s multivariate Wilson polynomials [27]. It should be remarked that the Gasper and Rahman multivariate Askey-Wilson polynomials are different from the Macdonald-Koornwinder polynomials [20], which are multivariate extensions of Askey-Wilson polynomials as well as extensions of Macdonald polynomials [22] associated to classical root systems. The Gasper and Rahman multivariate Askey-Wilson polynomials in $d$
variables $x_1 + x_1^{-1}, \ldots, x_d + x_d^{-1}$ depend, besides $q$, on $d + 3$ parameters $\alpha_0, \ldots, \alpha_{d+2}$. They can be defined as a nested product of univariate Askey-Wilson polynomials by

$$P_d(m; x; \alpha |q) = \prod_{j=1}^{d} p_{m_j} \left( x_j; \alpha_j q^{M_j-1}, \frac{\alpha_j}{\alpha_j^0}, \frac{\alpha_j q^{M_j-1}}{\alpha_j}, \frac{\alpha_j + 1}{\alpha_j}, \frac{\alpha_j + 1}{\alpha_j} q^{x_j+1} \right),$$

where $m = (m_1, \ldots, m_d)$, $M_j = \sum_{k=1}^d m_k$, $M_0 = 0$, $\alpha = (\alpha_0, \ldots, \alpha_{d+2}) \in \mathbb{C}^{d+3}$, $x_{d+1} = \alpha_{d+2}$. Under appropriate conditions on the parameters these polynomials are orthogonal on the torus $\mathbb{T}^d$, where $\mathbb{T}$ is the unit circle in the complex plane, with respect to the weight function

$$W(x) = \prod_{j=1}^{d} \frac{1}{(x_j^{\pm 1}; q)_{\infty}} \frac{1}{(x_j^{\pm 1}; q)_{\infty}}.$$

Here the $\pm$ symbols in the argument of the $q$-shifted factorials means that we take a product over all possible combinations of $+$ and $-$ signs, e.g.

$$(ab^{+1}c^{+1}; q)_{\infty} = (abc, ab/c, ac/b, a/bc; q)_{\infty}.$$ 

We will recover the orthogonality relations with respect to (1.3) below.

Iliev [15] (see also [12]) showed that the multivariate Askey-Wilson polynomials are eigenfunctions of $d$ commuting difference operators. Furthermore, the multivariate Askey-Wilson polynomials are also eigenfunctions of commuting difference equations in $m$, i.e. they satisfy $d$ independent recurrence relations. In other words, they solve a multivariate bispectral problem in the sense of Duistermaat and Grünbaum [8]. Below we construct the commuting difference operators in a quantum algebra setting.

Just like the univariate Askey-Wilson polynomials, the multivariate Askey-Wilson polynomials have many families of multivariate orthogonal polynomials and functions as limit cases [8], [2], some of which have found natural interpretations and applications in representation theory of quantum algebras and related physical models: 2-variable $q$-Krawtchouk were obtained by Genest, Post and Vinet as matrix elements of $q$-rotations, and they obtained fundamental properties such as orthogonality and difference equations from this interpretation. Rosengren [22] obtained orthogonality for multivariate $q$-Hahn polynomials from their interpretation as nested Clebsch-Gordan coefficients. Genest, Iliev and Vinet [10] obtained from a similar interpretation a difference equation for such $q$-Hahn polynomials, showing that they are wavefunctions for a $q$-deformed quantum Calogero-Gaudin superintegrable systems. Related to this they showed that the multivariate $q$-Racah polynomials appear as $3nj$-coefficients, leading to their orthogonality relation and the duality property. A similar interpretation was also obtained for multivariate $q$-Bessel functions in [13]. The multivariate Askey-Wilson polynomials themselves also appear in representation theory: in [14] Koelink and Van der Jeugt obtained an interpretation of 2-variable Askey-Wilson polynomials as nested Clebsch-Gordan coefficients, and Baseilhac and Martin [11] constructed infinite dimensional representations of the $q$-Onsager algebra using multivariate Askey-Wilson polynomials and Iliev’s corresponding difference operators.

In this paper we extend Rosengren’s interpretation of the Askey-Wilson polynomials to a multivariate setting, and in this way we derive the orthogonality relations and the $q$-difference equations for the multivariate Askey-Wilson polynomials. The main ingredients we use are discrete series representations of $U_q(su(1,1))$, twisted-primitive elements and properties of univariate Al-Salam–Chihara polynomials. The paper is organized as follows. In Section 2 we recall the aspects of representation theory of $U_q(su(1,1))$ that we need in this paper; in particular, we give a representation $\pi$ in terms of $q$-difference operators. In Section 3 we study eigenfunctions of two twisted primitive elements. The eigenfunctions are given in terms of Al-Salam–Chihara polynomials in base $q^2$ and $q^{-2}$. Using properties of these polynomials we introduce two new representations $\rho$ and $\tilde{\rho}$, which are equivalent to the representation $\pi$. In Section 4 we study the matrix elements for a change of base between two different eigenbases of twisted primitive elements. These matrix elements are essentially (univariate) Askey-Wilson polynomials. We show how the fundamental properties of these polynomials are obtained from this interpretation; the orthogonality relations follow essentially directly from their definition as matrix elements, and the difference equations
are shown to correspond to actions of twisted primitive elements in the representations \( \rho \) and \( \tilde{\rho} \). In Sections 2.1 and 3.1 we extend the results in the univariate case to the multivariate setting using \( N \)-fold tensor product representations. In this way we obtain multivariate Askey-Wilson polynomials and their properties from representation theory of the quantum algebra \( U_q(\mathfrak{su}(1,1)) \). The appendix contains some results on asymptotic behavior of functions we use in this paper, as well as an overview of the various Hilbert spaces appearing in this paper.

1.1. Notations and conventions. We assume \( 0 < q < 1 \), unless explicitly stated otherwise. We denote by \( \mathbb{N} \) the set of nonnegative integers, and \( \mathbb{T} \) is the unit circle in the complex plane. For a set \( S \), we write \( F(S) \) for the vector space consisting of complex valued functions on \( S \). If \( S \) is countable, we denote by \( F_0(S) \) the functions with finite support. By \( \mathcal{P} \) we denote the set of Laurent polynomials in \( x_1, \ldots, x_N \) that are invariant under \( x_j \leftrightarrow x_j^{-1} \), or equivalently the set of polynomials in \( x_j + x_j^{-1}, \, j = 1, \ldots, N \) (the number of variables should be clear from the context).

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2. The quantum algebra \( U_q \)

The quantum algebra \( U_q = U_q(\mathfrak{su}(1,1)) \) is the unital, associative, complex algebra generated by \( K, K^{-1}, E \), and \( F \), subject to the relations

\[
KK^{-1} = 1 = K^{-1}K, \\
KE = qEK, \quad KF = q^{-1}FK, \\
EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}}.
\]

\( U_q \) has a \( * \)-structure \( * : U_q \rightarrow U_q \) and a comultiplication \( \Delta : U_q \rightarrow U_q \otimes U_q \) defined on the generators by

\[
K^* = K, \quad E^* = -F, \quad F^* = -E, \quad (K^{-1})^* = K^{-1},
\]

\[
\Delta(K) = K \otimes K, \quad \Delta(E) = K \otimes E + E \otimes K^{-1}, \\
\Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \quad \Delta(F) = K \otimes F + F \otimes K^{-1}.
\]  

2.1. Twisted primitive elements. The following two elements of \( U_q \) play an important role in this paper. For \( s, u \in \mathbb{C}^* \) the twisted primitive elements \( Y_{s,u} \) and \( \tilde{Y}_{s,u} \) are given by

\[
Y_{s,u} = uq^{s \frac{1}{2}}EK - u^{-1}q^{-\frac{s}{2}}FK + \mu_s(K^2 - 1), \\
\tilde{Y}_{s,u} = uq^{-s \frac{1}{2}}EK^{-1} - u^{-1}q^{\frac{s}{2}}FK^{-1} - \mu_s(K^{-2} - 1),
\]

where

\[
\mu_s = \frac{s + s^{-1}}{q^{1-q}}.
\]

In particular, we define \( Y_s = Y_{s,1} \) and \( \tilde{Y}_s = \tilde{Y}_{s,1} \). If we formally write \( K = q^H \) and \( K^{-1} = q^{-H} \), \( \tilde{Y}_{s,u} \) is obtained from \( Y_{s,u} \) by replacing \( q \) by \( q^{-1} \). For \( s \in \mathbb{R}^* \cup \mathbb{T} \) and \( u \in \mathbb{T} \) both \( Y_{s,u} \) and \( \tilde{Y}_{s,u} \) are self-adjoint in \( U_q \). From (2.1) we find

\[
\Delta(Y_{s,u}) = K^2 \otimes Y_{s,u} + Y_{s,u} \otimes 1, \\
\Delta(\tilde{Y}_{s,u}) = \tilde{Y}_{s,u} \otimes K^{-2} + 1 \otimes \tilde{Y}_{s,u}.
\]
2.2. A representation of $U_q^+$. Let $k > 0$, and let $H = H_k$ be the Hilbert space consisting of complex-valued functions on $\mathbb{N}$ with inner product

$$
(f, g)_H = \sum_{n \in \mathbb{N}} f(n)\overline{g(n)} \omega(n),
$$

$$
\omega(n) = \omega_k(n) = q^{n(k-1)} \frac{(q^2; q^2)_n}{(q^{2k}; q^2)_n}.
$$

From the identity $(A; q^{-2})_n = (-A)^n q^{-n(n-1)}(A^{-1}; q^2)_n$ it follows that $\omega$, hence also the inner product, is invariant under $q \leftrightarrow q^{-1}$. We consider the following representation $\pi = \pi_k$ on $F(\mathbb{N})$,

$$
[\pi(K)f](n) = q^{k/2+n} f(n)
$$

$$
[\pi(K^{-1})f](n) = q^{-k/2-n} f(n)
$$

$$
[\pi(E)f](n) = \frac{q^{k+n-1} - q^{-k-n+1}}{q^{1}-q} f(n-1)
$$

$$
[\pi(F)f](n) = \frac{q^{n+1} - q^{-n-1}}{q-1} f(n+1),
$$

with the convention $f(-1) = 0$. This defines an unbounded representation on $H$, where we take $F_0(\mathbb{N})$ as a dense domain. Furthermore, $\pi$ is a $*$-representation on $H$, i.e. $(\pi(X)f, g)_H = (f, \pi(X^* g))_H$ for $f, g \in F_0(\mathbb{N})$. Let us remark that if $X^* = X$, then $\pi(X)$ is a symmetric operator, but not necessarily self-adjoint.

3. Eigenfunctions of twisted primitive elements: Al-Salam–Chihara polynomials

We determine eigenfunction of the difference operators $\pi(Y_s)$ and $\pi(Y_t)$, see also Koelink and Van der Jeugt [19] and Rosengren [24]. In order to assure self-adjointness of the difference operators corresponding to twisted primitive elements we assume from here on that $s, t \in \mathbb{T}$ and $t \in \mathbb{R}$ such that $|t| \geq q^{-1}$. We need the (univariate) Al-Salam–Chihara polynomials, see [16, Section 15.1] and [17, Section 14.8] for details. In this section three different Hilbert spaces $H$, $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are used; for the readers convenience we included a short overview of these Hilbert spaces in the appendix.

3.1. Al-Salam–Chihara polynomials. For $q > 0$, $q \neq 1$, the Al-Salam–Chihara polynomials are Askey-Wilson polynomials (normalized differently from [4.1]) with two parameters equal to zero given by

$$
Q_n(x; a, b | q) = 3\varphi_2 \left( \begin{array}{c}
q^{-n}, ax, a/x \\
ab, 0
\end{array} ; q, q \right)
$$

$$
= (ax)^n (b/x; q)_n \varphi_2 \left( \begin{array}{c}
q^{-n}, ax \\
q^{1-n} x/b
\end{array} ; q, q \right).
$$

They have the symmetry property

$$
Q_n(x; a, b | q) = \left( \frac{a}{b} \right)^n Q_n(x; a, b | q).
$$

The three-term recurrence relation is given by

$$(x + x^{-1})Q_n(x) = \frac{1}{2} (1 - abq^n)Q_{n+1}(x) + (a + b)q^n Q_n(x) + a(1 - q^n)Q_{n-1}(x).$$

If $0 < q < 1$, $|a|, |b| < 1$ and $\overline{a} = b$ the Al-Salam–Chihara polynomials in base $q$ satisfy the orthogonality relations

$$
\frac{1}{4\pi i} \int_{\mathbb{T}} Q_m(x)Q_n(x)w(x; a, b | q) \frac{dx}{x} = \delta_{m,n} \frac{a^{2n}(q; q)_n}{(ab; q)_n},
$$

$$
w(x; a, b | q) = \frac{(q, ab, x^{1+2}; q)_\infty}{(ax^{1+1}, bx^{1+1}; q)_\infty},
$$

and the polynomials form a basis for the corresponding weighted $L^2$-space of functions in $x + x^{-1}$. 

In particular, the dual orthogonality relations

\[ \sum_{y \in \mathbb{Z}} Q_n(y; a, b; q^{-1})Q_n(y; a, b; q^{-1})W(y; a, b; q) = \delta_{m, n} \left( \frac{a}{b} \right)^m \frac{(q; q)_n}{(1/ab; q)_n}, \]

(3.4)

\[ W(y; a, b; q) = \frac{1 - q^{2n/a^2}}{1 - 1/a^2} \frac{(1 - 1/a^2, 1/ab; q)_{m}(bq/a; q)_{\infty}}{(bq/a; q)_{m}(q/a^2; q)_{\infty}} \left( \frac{b}{a} \right)^m q^{m^2}, \quad y = aq^{-m}. \]

Here \(1/ab = \frac{1}{ab}\). Under the conditions above, the \(q^{-1}\)-Al-Salam–Chihara moment problem is determinate, so the polynomials form a basis for the corresponding weighted \(L^2\)-space consisting of functions in \(y+y^{-1}\). From (3.1) it follows that \(Q_n(aq^{-m}; a, b|q^{-1})\) is a polynomial in \(q^n\) of degree \(m\), which can be shown to be a multiple of a little \(q\)-Jacobi polynomial \(p_m(q^n; a^{-1}b, q^{-1}a^{-1}b^{-1}; q)\) using \(q\)-hypergeometric transformations: first transform the \(3\varphi_2\)-function to a \(3\varphi_2\)-function in base \(q^{-1}\) and \(q^{-1}\), and then transform this into a \(2\varphi_1\)-function using (3.1) (III.8);

\[ Q_n(aq^{-m}; a, b; q^{-1}) = \left( -\frac{a}{b} \right)^m q^{-\frac{m(m+1)}{2}} \frac{(bq/a; q)_m}{(q; q)_n} \frac{(1/ab; q)_n}{(q; q)_n} \left( q^{-m-n}; q^m/a^2; q^m/bq/a; q^{1+n} \right). \]

In particular, the dual orthogonality relations

\[ \sum_{n \in \mathbb{N}} Q_n(aq^{-m}; a, b; q^{-1})Q_n(aq^{-m}; a, b; q^{-1}) \left( \frac{bq/a; q)_m}{(1/ab; q)_n} \left( q^{-m}; q^m/a^2; q^m/bq/a; q^{1+n} \right) = \delta_{m, n} W(aq^{-m}; a, b; q). \]

(3.5)

correspond to the orthogonality relations for the little \(q\)-Jacobi polynomials.

The following \(q\)-difference equations will be useful later on.

**Lemma 3.1.** For \(q > 0\), \(q \neq 1\), the Al-Salam–Chihara polynomials satisfy

\[ Q_n(x; a, b|q) = \frac{1-ax}{1-x}Q_n(x; q^{-\frac{1}{2}}, aq^{-\frac{1}{2}}/q^{\frac{1}{2}}|q) + \frac{1-a/x}{1-1/x}Q_n(x; q^{-\frac{1}{2}}, aq^{\frac{1}{2}}/q^{\frac{1}{2}}|q). \]

(3.6)

As a consequence, the following \(q\)-difference equations hold:

\[ q^{-n}Q_n(x; a, b|q) = \frac{(1-ax)(1-bx)}{(1-x)(1-qx^2)}Q_n(x; a, b|q) + \frac{(1-a/x)(1-b/x)}{(1-1/x)(1-q/x^2)}Q_n(x; a, b|q) \]

\[ + \frac{(1+q)(q+ab)-(x+1/x)(aq+bq)}{(1-qx^2)(1-q^2/x^2)}Q_n(x; a, b|q) \]

(3.7)

\[ Q_n(x; a, b|q) = \frac{(1-ax)(1-aqx^2)}{(1-x^2)(1-qx^2)}Q_n(x; aq, b|q) + \frac{(1-a/x)(1-aqx)}{(1-1/x^2)(1-q/x^2)}Q_n(x; aq, b/x|q) \]

\[ + \frac{q(q+1)(1-ax)(1-a/x)}{(1-q^2)(1-q^2/x^2)}Q_n(x; a, b/q|q). \]

(3.8)

Identity (3.7) is the well-known \(q\)-difference equation for the Al-Salam–Chihara polynomials.

**Proof.** Using the explicit expression for the Al-Salam–Chihara polynomials as \(3\varphi_2\)-functions, the right hand side of (3.6) equals

\[ \frac{1-ax}{1-x^2} 3\varphi_2 \left( \begin{array}{c} q^n, ax, a/x \\ ab, 0 \end{array} ; q, q \right) + \frac{1-a/x}{1-1/x^2} 3\varphi_2 \left( \begin{array}{c} q^n, ax, aq/x \\ ab, 0 \end{array} ; q, q \right) \]

\[ = \sum_{j=0}^n \frac{(q^n, q^j, q^j)}{(q, ab; q)_j} \left( \frac{(ax; q)_j, (a/x; q)_j}{1-x^2} + \frac{(ax; q)_j, (a/x; q)_{j+1}}{1-1/x^2} \right). \]

The expression in large brackets is equal to \((ax, ax^{-1}; q)_j\), so that (3.6) follows.

The difference equation (3.7) follows from applying the symmetry (3.3) and then applying (3.6) again on the right hand side of (3.5). The coefficients of \(Q_n(x; a, b)\) can be rewritten to the expression in the lemma. Similarly (3.8) follows from applying (3.6) to itself. \(\Box\)
3.2. Eigenfunctions of \( Y_s \). From (2.22) and (2.24) it follows that \( \pi(Y_s) \) acts as a three-term difference operator on \( F(\mathbb{N}) \) by

\[
(q^{-1} - q)[\pi(Y_s)f](n) = q^{-(k-1)/2}(1 - q^{2k+2n-2})f(n-1)
\]
\[+(s + s^{-1})(q^{k+2n} - 1)f(n) + q^{(k-1)/2}(1 - q^{2n+2})f(n+1).
\]

Using the three-term recurrence relation for Al-Salam--Chihara polynomials we can find eigenfunctions (in the algebraic sense) of \( \pi(Y_s) \). We define

\[
v_{x,s}(n) = v_{x,s,k}(n) = \left(q^{-3k/2}/s\right)\left(q^k; q^2\right)_n Q_n(x; s, q^k, s^{-1} q^k | q^2).
\]

Then \( v_{x,s} \) is an eigenfunction of \( \pi(Y_s) \), i.e.,

\[
[\pi(Y_s)v_{x,s}](n) = \lambda_{x,s} v_{x,s}(n), \quad \lambda_{x,s} = \frac{x + x^{-1} - s - s^{-1}}{q^{-1} - q} = \mu_x - \mu_s.
\]

Note that \( v_{x,s}(n) \) is real-valued for \( x, s \in \mathbb{R} \cap \mathbb{T} \).

We are also interested in eigenfunctions of \( Y_{s,u} \). These can easily be obtained from eigenfunctions of \( Y_s \). Let \( M_u \) be the multiplication operator on \( F(\mathbb{N}) \) defined by \( [M_u f](n) = u^n f(n) \). Then \( \pi(Y_{s,u})M_u = M_u \pi(Y_s) \), and we obtain the following result.

**Lemma 3.2.** For \( u \in \mathbb{T} \),

\[
[\pi(Y_{s,u})M_u v_{x,s}](n) = \lambda_{x,s} M_u v_{x,s}(n).
\]

Let \( \mathcal{H} = \mathcal{H}_{k,s} \) be the Hilbert space consisting of functions on \( \mathbb{T} \) that are \( x \leftrightarrow x^{-1} \) invariant almost everywhere, with inner product

\[
\langle f, g \rangle_{\mathcal{H}} = \frac{1}{4\pi i} \int_{\mathbb{T}} f(x) g(x) w(x) \frac{dx}{x},
\]

where

\[
w(x) = w_{k,s}(x) = w(x; q^k s, q^k / s | q^2) = \frac{(q^2, q^{2k}, x^{k^2}; q^2)_{\infty}}{(q^k s^{1/2}, x^{-k^2}; q^2)_{\infty}}.
\]

The set \( \{ v_{x,s}(n) | n \in \mathbb{N} \} \) is an orthogonal basis for \( \mathcal{H} \) with orthogonality relations

\[
\langle v_{x,s}(n), v_{x,s}(n') \rangle_{\mathcal{H}} = \frac{\delta_{n,n'}}{\omega(n)},
\]

which follows from (9.3). Note that the squared norm \( \omega(n)^{-1} \) is independent of \( s \).

**Proposition 3.3.** Let \( \Lambda = \Lambda_{k,s} : F_0(\mathbb{N}) \rightarrow \mathcal{P} \) be defined by

\[\Lambda f(x) = \langle f, v_{x,s} \rangle_{\mathcal{H}},\]

then \( \Lambda \) intertwines \( \pi(Y_s) \) with multiplication by \( \lambda_{x,s} \). Furthermore, \( \Lambda \) extends to unitary operator \( H \rightarrow \mathcal{H} \).

**Proof.** The intertwining property follows from (3.11) and \( Y_{s}^* = Y_s \); for \( f \in F_0(\mathbb{N}) \),

\[\Lambda(\pi(Y_{s})f)(x) = \pi(Y_{s} f)(x) = \langle f, \pi(Y_{s})v_{x,s} \rangle_{\mathcal{H}} = \langle f, \pi(Y_{s})v_{x,s} \rangle_{\mathcal{H}} = \lambda_{x,s}(\Lambda f)(x).
\]

Note that \( \Lambda f \) is a finite linear combination of Al-Salam--Chihara polynomials, so \( \Lambda f \in \mathcal{P} \). For the unitarity, define for \( m \in \mathbb{N} \) the function \( \delta_m \in H \) by \( \delta_m(n) = \frac{\delta_m}{\omega(n)} \), then \( \{ \delta_m \mid m \in \mathbb{N} \} \) is an orthogonal basis for \( H \) with squared norm \( \| \delta_m \|^2_H = (\omega(m))^{-1} \). Note that \( \langle \Lambda \delta_m \rangle(x) = v_{x,s}(m) \), so that \( \Lambda \) maps an orthogonal basis of \( H \) to an orthogonal basis of \( \mathcal{H} \) with the same norm, from which it follows that \( \Lambda \) extends to a unitary operator.

**Remark 3.4.**

Note that it follows from Proposition 3.3 that \( \pi(Y_s) \) has completely continuous spectrum, which is given by

\[
\{ \lambda_{x,s} \mid x \in \mathbb{T} \} = \left[ -\frac{2}{q^{-1} - q} - \mu_s, \frac{2}{q^{-1} - q} - \mu_s \right].
\]
(i) The function \( v_{x,s}(n) \) is a polynomial of degree \( n \) in \( x + x^{-1} \). Furthermore, from the explicit expression \( (3.1) \) for the Al-Salam–Chihara polynomial and from the symmetry property \( (3.2) \) it follows that \( v_{x,s}(n) \) is also a polynomial in \( s + s^{-1} \) of degree \( n \).

(ii) If we assume \( s \in \mathbb{R} \) such that \(|s| > 1\), instead of \( s \in \mathbb{T} \), the operator \( \pi(Y_s) \) is still self-adjoint, but now finite discrete spectrum will appear if \(|sq^k| > 1\). For simplicity we assume \( s \in \mathbb{T} \) throughout the paper.

The set of eigenfunctions \( \{v_{x,s} \mid x \in \mathbb{T}\} \) is a generalized basis for \( H \) and the set \( \{v_{x,s}(n) \mid n \in \mathbb{N}\} \) is a basis for \( \mathcal{H} \). So using the eigenfunctions \( v_{x,s}(n) \), or actually the corresponding operator \( \Lambda \), we can transfer the action of \( \mathcal{U}_q \) on \( H \) to an action on \( \mathcal{H} \). We define a representation \( \rho = \rho_{k,s} \) of \( \mathcal{U}_q \) on \( \mathcal{P} \) by

\[
\rho(X) = \Lambda \circ \pi(X) \circ \Lambda^{-1}, \quad X \in \mathcal{U}_q.
\]

This extends to a \( * \)-representation on \( \mathcal{H} \). By Proposition \( (3.3) \) we have an explicit expression for \( \rho_k(Y_s) \) as a multiplication operator. In general it seems very difficult to find explicitly the action of \( \rho(X) \) for a given \( X \in \mathcal{U}_q \), but for \( X = K^{-2} \) we can find such an explicit expression using the difference equation \( (3.7) \). We use the following notation for an elementary \( q \)-difference operator: \( [T_f](x) = f(q^2x) \).

**Lemma 3.5.** \( \rho(K^{-2}) \) is the second order \( q \)-difference operator given by

\[
\rho(K^{-2}) = A(x)T + B(x)Id + A(x^{-1})T^{-1},
\]

where

\[
A(x) = A_k(x; s) = \frac{q^{-k}(1 - q^k sx)(1 - q^k x/s)}{(1 - x^2)(1 - q^2x^2)},
\]

\[
B(x) = B_k(x; s) = \frac{q^2(q^{-1} + q)(q^{-k} + q^{-k-1}) - q^2(x + x^{-1})(s + s^{-1})}{(1 - q^2x^2)(1 - q^2/x^2)}.
\]

**Proof.** Let \( f \in \mathcal{P} \). From \( (K^{-2})^* = K^{-2} \) we obtain

\[
[T\rho(K^{-2})f](x) = \langle \pi(K^{-2})(x^{-1}f), v_{x,s} \rangle_H = \langle \Lambda^{-1}f, q^{-k-2}v_{x,s} \rangle_H.
\]

Now we use the difference equation \( (3.7) \) with \( a = sq^k \), \( b = q^k/s \) to rewrite \( q^{-k-2n}v_{x,s}(n) \), and the result follows. \( \square \)

### 3.3. Eigenfunctions of \( \tilde{Y}_t \)

The difference operator \( \pi(Y_t) \) acts on \( f \in F(N) \) by

\[
(q - q^{-1})[\pi(Y_t)f](n) = q^{(k-1)/2}(1 - q^{-2k-2n+2})f(n - 1)
+ (t + t^{-1})(q^{2n-k} - 1)f(n) + q^{(1-k)/2}(1 - q^{-2n-2})f(n + 1).
\]

Note that this is precisely the action of the difference operator \( \pi(Y_t) \) with \( q \) replaced by \( q^{-1} \). So we have the same eigenfunctions, but with \( q \) replaced by \( q^{-1} \); let

\[
\tilde{\psi}_{y,t,k}(n) = \tilde{\psi}_{y,t,k,q^{-1}}(n) = \left( \frac{q^{3k(k-1)/2}}{t} \right)^n \left( \frac{q^{-2k};q^{-2}}{q^{-2};q^{-2}} \right)_n Q_n(y; q^{-k}t, q^{-k}/t | q^{-2}),
\]

then

\[
[\pi(\tilde{Y}_t)\tilde{\psi}_{y,t,k}(n)] = \lambda_{t,y} \tilde{\psi}_{y,t,k}(n).
\]

Note that \( \tilde{\psi}_{y,t,k,q^{-1}}(n) = v_{y,t,k,q^{-1}}(n) \). Eigenfunctions of \( \tilde{Y}_t \), \( u \in \mathbb{T} \), are again directly obtained from the eigenfunctions of \( Y_t \).

**Lemma 3.6.** For \( u \in \mathbb{T} \),

\[
[\pi(\tilde{Y}_t,u)M_a\tilde{\psi}_{y,t,k}(n)] = \lambda_{t,y} M_a\tilde{\psi}_{y,t,k}(n).
\]

We define \( \tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{k,t} \) to be the Hilbert space consisting of functions on the set

\[
S = S_{k,t,q} = \{tq^{-k-2m} \mid m \in \mathbb{N}\}
\]

with inner product

\[
(f, g)_{\tilde{\mathcal{H}}} = \sum_{y \in S} f(y)\overline{g(y)}\tilde{w}(y),
\]
where \( \tilde{w}(y) = \tilde{w}_{k,t}(y) \) is the weight function given by
\[
\tilde{w}(y) = W(y; q^{-k}, q^{-k}/t | q^2) = \frac{1 - q^{4m+2k}/t^2}{1 - q^{2k}/t^2} \frac{(q^{2k}; q^2)_m(q^{2m+2}/t^2; q^2)_\infty q^{-2m^2}}{(q^2; q^2)_m(q^{2k+2}/t^2; q^2)_\infty t^{-2m} q^{2m^2}},
\]
for \( y = tq^{-k-2m}, m \in \mathbb{N} \). From the orthogonality relations (3.3) for the \( q^{-1} \)-Al-Salam–Chiara polynomials we find
\[
\langle \tilde{v}_{t,n}(n), \tilde{v}_{t,n'}(n') \rangle_{\tilde{H}} = \frac{\delta_{n,n'}}{\omega(n)},
\]
and the set \( \{ \tilde{v}_{t,n}(n) \mid n \in \mathbb{N} \} \) is an orthogonal basis for \( \tilde{H} \). Observe that the squared norm \( \omega(n)^{-1} \) is independent of \( t \). From the dual orthogonality relations (3.5) it follows that \( \{ \tilde{v}_{y,t} \mid y \in S \} \) is an orthogonal basis for \( H \) with orthogonality relations
\[
\langle \tilde{v}_{y,t}, \tilde{v}_{y',t} \rangle_H = \frac{\delta_{y,y'}}{w(y)}.
\]

The proof of the following result is similar to the proof of Proposition 3.3.

**Proposition 3.7.** Let \( \tilde{\Lambda} = \tilde{\Lambda}_{k,t} : F_0(\mathbb{N}) \to \mathcal{P} \) be defined by
\[
(\tilde{\Lambda}f)(y) = \langle f, \tilde{v}_{y,t} \rangle_H,
\]
then \( \tilde{\Lambda} \) intertwines \( \pi(\tilde{Y}_t) \) with multiplication by \( \lambda_{t,y} \). Furthermore, \( \tilde{\Lambda} \) extends to a unitary operator \( H \to \tilde{H} \).

Note that \( \pi(\tilde{Y}_t) \) has completely discrete spectrum, which is given by \( \{ \lambda_{t,y} \mid y \in S \} \).

Similar as in the previous subsection we define a representation of \( U_q \) on \( \mathcal{P} \) by
\[
\tilde{\rho}(X) = \tilde{\Lambda} \circ \pi(X) \circ \tilde{\Lambda}^{-1}, \quad X \in U_q.
\]
This defines a *-representation on \( \tilde{H} \). In this case \( \tilde{\rho}(K^2) \) can be given explicitly as a \( q \)-difference operator using the difference equation (3.7) for the Al-Salam–Chiara polynomials. If we denote by \( L_{k,s,q} \) the difference operator \( \rho(K^2) \) given in Lemma 3.5 then by construction \( \tilde{\rho}(K^2) \) is the difference operator \( L_{k,t,q}^{-1} \).

**Lemma 3.8.** \( \tilde{\rho}(K^2) \) is the second order \( q \)-difference operator given by
\[
\tilde{\rho}(K^2) = \tilde{A}(y) T^{-1} + \tilde{B}(y) \text{Id} + \tilde{A}(y)^{-1} T,
\]
where \( \tilde{A}(y) = A_{k,q^{-1}}(y;t) \) and \( \tilde{B}(y) = B_{k,q^{-1}}(y;t) \).

Restricted to the set \( S \) the coefficients \( \tilde{A} \) are given by
\[
\tilde{A}(tq^{-k-2m}) = \frac{q^{k+2} (1 - q^{2k+2m}/t^2) (1 - q^{2k+2m})}{t^2 (1 - q^{2k+4m}/t^2) (1 - q^{2k+4m+2}/t^2)},
\]
\[
\tilde{A}(t^{-1} q^{k+2m}) = \frac{q^k (1 - q^{2m}/t^2) (1 - q^{2m})}{(1 - q^{2k+4m}/t^2) (1 - q^{2k+4m+2}/t^2)}.
\]

**Remark 3.9.** We can now make the connection with Rosengren’s generalized group element, see [21] §4.2. Let \( e_m(n) = \delta_{mn} \), then the generalized group element is essentially the element \( U_{t,u} \) in an appropriate completion of \( U_q \) such that \( \pi(U_{t,u}) : F(\mathbb{N}) \to F(\mathbb{N}) \) is given by \( \pi(U_{t,u}) e_m = M_u \tilde{v}_{tq^{-k-2m},t} \). Then \( \pi(U_{t,u}) e_m \) is an eigenfunction of \( \pi(\tilde{Y}_t) \) with eigenvalue \( \lambda_{t,tq^{-k-2m}} \), so
\[
\pi(U_{t,u}^{-1} \tilde{Y}_t U_{t,u}) e_m = \lambda_{t,tq^{-k-2m}} e_m,
\]
and more general
\[
\pi(U_{t,u}^{-1} X U_{t,u}) = \tilde{\rho}(X) |_{F(S)}, \quad X \in U_q,
\]
where we should identity \( F(\mathbb{N}) \cong F(S) \). Rosengren’s observation that primitive elements are transformed into group-like elements is obtained from \( \pi(K^{\pm 2})e_m = q^{\pm (k+2m)}e_m \), which gives
\[
\pi \left( U_{t,u}^{-1} \tilde{Y}_{t,u} U_{t,u} \right) e_m = \frac{t(1 - q^{-k-2m}) + t^{-1}(1 - q^{k+2m})}{q^{-1} - q} e_m,
\]
\[
\pi \left( \frac{t(1 - K^{-2}) + t^{-1}(1 - K^2)}{q^{-1} - q} \right) e_m.
\]
Furthermore, Stokman \(20\) showed that the assignment \( X \mapsto U_{t,u}^{-1} \tilde{X} U_{t,u} \) transfers the quantum group structure to a dynamical quantum group structure, so the representations \( \rho \) and \( \tilde{\rho} \) can be considered in the context of dynamical quantum groups. We do not use this connection in this paper, but we can recognize the ‘dynamical’ part in the difference operators in Sections 5 and 6.

4. Eigenfunctions of two twisted primitive elements: Askey-Wilson polynomials

In this section we define functions which are eigenfunctions of \( \rho(\tilde{Y}_{t,u}) \) and of \( \tilde{\rho}(Y_{s,u}) \). These functions are multiples of Askey-Wilson polynomials, and we derive properties of the Askey-Wilson polynomials in this way. Moreover, the results in this section serve as a motivation and illustration of the methods we use in Section 6 to study multivariate Askey-Wilson polynomials.

The functions we study in this section are the matrix elements for a change of basis between the discrete basis \( \{ \tilde{v}_{y,t} \mid y \in S \} \) of eigenfunctions of \( \tilde{Y}_t \) and the continuous basis \( \{ v_{x,s} \mid x \in \mathbb{T} \} \) of eigenfunctions of \( Y_s \).

**Definition 4.1.** For \( x \in \mathbb{T} \) and \( y \in S_{k,t,q} \), we define
\[
P_\beta(x,y) = \langle M_u \tilde{v}_{y,t}, v_{x,s} \rangle_H,
\]
where \( \beta \) is the ordered 5-tuple \( \beta = (s,t,u,k,q) \).

Note that \( P_\beta(x,y) = [\Lambda M_u \tilde{v}_{y,t}](x) = [\Lambda M_u v_{x,s}](y) \). It is not a priori clear that the sum \( \sum \) converges, since \( v_{x,s} \notin H \). In the appendix it is shown that the sum converges absolutely under the given conditions on \( x \) and \( y \). We show later on in Lemma 4.6 that \( P_\beta(x,y) \) is essentially an Askey-Wilson polynomial, and for this reason we will sometimes refer to the functions \( P_\beta(x,y) \) as Askey-Wilson polynomials (even though they are not polynomials).

We can derive several fundamental properties of the Askey-Wilson polynomials from our definition (4.1). We start with the orthogonality relations.

**Proposition 4.2.** The set \( \{ P_\beta( \cdot , y) \mid y \in S \} \) is an orthogonal basis for \( H \), with orthogonality relations
\[
\langle P_\beta( \cdot , y), P_\beta( \cdot , y') \rangle_H = \frac{\delta_{y,y'}}{w(y)}.
\]

**Proof.** The orthogonality relations and completeness follows from unitarity of \( \Lambda \) and \( M_u \), and from the orthogonality relations (3.10) for \( v_{y,t} \),
\[
\langle P_\beta( \cdot , y), P_\beta( \cdot , y') \rangle_H = \langle \Lambda M_u \tilde{v}_{y,t}, \Lambda M_u \tilde{v}_{y',t} \rangle_H = \langle \tilde{v}_{y,t}, \tilde{v}_{y',t} \rangle_H = \frac{\delta_{y,y'}}{w(y)}. \]

Our next goal is to obtain difference equations for the Askey-Wilson polynomials \( P_\beta(x,y) \). Using Lemmas 3.2 and 3.6 we see that
\[
[\rho(\tilde{Y}_{t,u}) P_\beta( \cdot , y)](x) = [\Lambda(\pi(\tilde{Y}_{t,u}) M_u \tilde{v}_{y,t})](x) = \lambda_{t,y} P_\beta(x,y),
\]
\[
[\tilde{\rho}(Y_{s,u}) P_\beta(x, \cdot )](y) = [\tilde{\Lambda}(\pi(Y_{s,u}) M_u v_{x,s})](y) = \lambda_{x,s} P_\beta(x,y),
\]
so \( P_\beta(x,y) \) is an eigenfunction of \( \rho(\tilde{Y}_{t,u}) \) and also of \( \tilde{\rho}(Y_{s,u}) \). We will show that the eigenfunction equations (4.2) are essentially the difference equation and three-term recurrence relation for the Askey-Wilson polynomials by realizing \( \rho(\tilde{Y}_{t,u}) \) and \( \tilde{\rho}(Y_{s,u}) \) explicitly as difference operators. To do this we express first \( \tilde{Y}_{t,u} \) in terms of \( Y_s \) and \( K^{-2} \). Similarly, \( Y_{s,u} \) can be expresses in terms of \( \tilde{Y}_t \) and \( K^2 \).
Lemma 4.3.

(i) Let $S, T \in \mathcal{U}_q$ be given by

$$S = K^{-2}(Y_s + \mu_s 1) - \mu_s 1 \quad \text{and} \quad T = \frac{K^{-2}Y_s - Y_s K^{-2}}{q^{-1} - q},$$

then $S$ and $T$ are independent of $s$, and

$$\tilde{Y}_{t,u} = \frac{(u + u^{-1})S + (q\mu - q^{-1}u^{-1})T}{q + q^{-1}} + \mu_t (1 - K^{-2}).$$

Proof. From the definition of $Y_s$, the result follows from the definition of $\tilde{Y}_{t,u}$. The proof for part (ii) is similar. \qed

(ii) Let $\tilde{S}, \tilde{T} \in \mathcal{U}_q$ be given by

$$\tilde{S} = K^2(\tilde{Y}_t - \mu_t 1) + \mu_t 1 \quad \text{and} \quad \tilde{T} = \frac{\tilde{Y}_t K^2 - K^2 \tilde{Y}_t}{q^{-1} - q},$$

then $\tilde{S}$ and $\tilde{T}$ are independent of $t$, and

$$Y_{s,u} = \frac{(u + u^{-1})\tilde{S} + (q^{-1}u^{-1} - qu)\tilde{T}}{q + q^{-1}} - \mu_s (1 - K^{-2}).$$

Proof. From the definition of $Y_s$ and the defining relations for $\mathcal{U}_q$ we find

$$S = q^{-\frac{1}{2}}EK^{-1} - q^{\frac{1}{2}}FK^{-1}, \quad T = q^{-\frac{1}{2}}EK^{-1} + q^{\frac{1}{2}}FK^{-1},$$

which is clearly independent of $s$. Then

$$q^{-\frac{1}{2}}EK^{-1} = \frac{S + qT}{q + q^{-1}}, \quad q^{\frac{1}{2}}K^{-1}F = \frac{q^{-1}T - S}{q + q^{-1}}.$$

Then the result for part (i) follows from the definition of $\tilde{Y}_{t,u}$. The proof for part (ii) is similar. \qed

Using the explicit realizations of $\rho(K^{-2})$ and $\rho(Y_s)$ as difference and multiplication operators from Proposition 3.3 and Lemma 3.5, we can now realize $\rho(\tilde{Y}_{t,u})$ explicitly as a difference operator. Initially this is a difference operator on $\mathcal{P}$, but it can be extended to a difference operator acting on meromorphic functions. Identity (3.2) can then be written as a $q$-difference equation for the Askey-Wilson polynomials $P_\beta(x, y)$.

Proposition 4.4. $\rho(\tilde{Y}_{t,u})$ is the $q$-difference operator given by

$$\rho(\tilde{Y}_{t,u}) = A_\beta(x) T + B_\beta(x) 1d + A_\beta(x^{-1}) T^{-1},$$

where

$$A_\beta(x) = -A(x) \frac{t(1 - qx/ut)(1 - u/qtx)}{q^{-1} - q},$$

$$B_\beta(x) = B(x) \left( \frac{(u + u^{-1})\mu_x}{q^{-1} + q} - \mu_t \right) + \frac{(u + u^{-1})\mu_s}{q^{-1} + q} + \mu_t,$$

where $A$ and $B$ are given in Lemma 3.3. In particular, $P_\beta(x, y)$ satisfies

$$\lambda_{t,u} P_\beta(x, y) = A_\beta(x) P_\beta(xq^2, y) + B_\beta(x) P_\beta(x, y) + A_\beta(x^{-1}) P_\beta(x/q^2, y).$$

A calculation shows that

$$B_\beta(x) = \frac{tq^{-k} + q^k/t + t + 1/t}{q^{-1} - q} + F(x) + F(x^{-1}),$$

with

$$F(x) = \frac{tq^{-k}(1 - q^kx)(1 - q^k x/s)(1 - qux/t)(1 - q/x/ut)}{(q^{-1} - q)(1 - x^2)(1 - q^2 x^2)}$$

$$= -A_\beta(x) \frac{1 - qux/t}{1 - u/qtx}.$$
Proof. \( \rho(K^{-2}) \) is the \( q \)-difference operator from Lemma 4.3 and note that \( \rho(Y_s + \mu_x, 1) \) is multiplication by \( \lambda_{x,s} + \mu_x = \mu_x \). Then \( S \) from Lemma 4.3 is the \( q \)-difference operator given by

\[
\rho(S) = \mu_{q^2 x} A(x) T + \left( \mu_x B(x) - \mu_x \right) \text{Id} + \mu_{q^{-2} x} A(x^{-1}) T^{-1},
\]

and \( T \) is given by

\[
\rho(T) = \frac{\mu_{q^2 x} - \mu_x}{q^2 - q} A(x) T + \frac{\mu_{q^{-2} x} - \mu_x}{q^{-2} - q} A(x^{-1}) T^{-1}.
\]

Expressing \( \tilde{Y}_{t,u} \) in terms of \( S \) and \( T \) using Lemma 4.3 it follows that \( \rho(\tilde{Y}_{t,u}) \) indeed has the form

\[
A_{\beta}(x) T + B_{\beta}(x) \text{Id} + A_{\beta}(x^{-1}) T^{-1}
\]

with \( B_{\beta}(x) \) as stated above, and \( A_{\beta}(x) \) is given by

\[
A_{\beta}(x) = A(x) \left( \frac{(u + u^{-1}) \mu_{q^2 x}}{q^2 - q} + \frac{(qu - q^{-1}u^{-1})(\mu_{q^2 x} - \mu_x)}{(q^2 + q)(q^{-1} - q)} \right).
\]

A small calculation show that the expression between large brackets is equal to

\[
-t \frac{(1 - qx/tu)(1 - u/qtx)}{q^{-1} - q}.
\]

\[\Box\]

In a similar way as in Proposition 4.4 we can realize \( \tilde{\rho}(Y_{s,u}) \) as a \( q \)-difference operator. By construction this operator is obtained from the difference operator \( \rho(\tilde{Y}_{t,u}) \) by replacing \( \beta = (s, t, u, k, q) \) by \( \tilde{\beta} = (t, s, u, k, q^{-1}) \). This immediately leads to a \( q \)-difference equation in \( y \) for \( P_{\beta}(x, y, t, u, k, q) \).

Proposition 4.5. \( \tilde{\rho}(Y_{s,u}) \) is the \( q \)-difference operator given by

\[
\tilde{\rho}(Y_{s,u}) = A_{\tilde{\beta}}(y) T^{-1} + B_{\tilde{\beta}}(y) \text{Id} + A_{\tilde{\beta}}(y^{-1}) T.
\]

In particular, for \( y \in S \) the Askey-Wilson polynomials \( P_{\beta}(x, y) \) satisfy

\[
\lambda_{x,s} P_{\beta}(x, y) = A_{\tilde{\beta}}(y) P_{\beta}(x, y/q^2) + B_{\tilde{\beta}}(y) P_{\beta}(x, y) + A_{\tilde{\beta}}(y^{-1}) P_{\beta}(x, yq^2),
\]

with \( P_{\beta}(x, tq^{-k+2}) = 0 \).

To end this section, let us match the properties of the functions \( P_{\beta}(x, y) \) to properties of standard Askey-Wilson polynomials defined by (1.1), see [2, Chapter 15, 17, Section 14.1]. First we show that \( P_{\beta}(x, y) \) is a multiple of an Askey-Wilson polynomial, see also [24, Proposition 4.2].

Lemma 4.6. Let

\[
(a, b, c, d) = (q^k s, q^k/s, qu/t, q/ut),
\]

then

\[
P_{\beta}(x, y) = (-1)^m d^m q^{-m(m-1)/2} \frac{(acq^{2m}; bcq^{2m}; q^2)_\infty}{(ab; q^2)_m (cx^{-1}; q^2)_\infty} \rho_m(x; a, c, b, d|q^2),
\]

where \( y = tq^{-k-2m} \).

Proof. The proof essentially boils down to comparing the recurrence relation from Proposition 4.5 with the standard Askey-Wilson recurrence relation, which is

\[
(x + x^{-1} - a^{-1} - a) R_n(x) = \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n})(1 - abdq^{2n-2})} (R_{n+1}(x) - R_n(x)) + \frac{a(1 - q^{-n})(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abdq^{2n-1})(1 - abdq^{2n-2})} (R_{n-1}(x) - R_n(x)),
\]

\[(4.4)\]
with $R_{-1}(x) = 0$ and $R_0(x) = 1$, where $R_m(x) = a^n p_n(x; a, b, c, d|q)/(ab, ac, ad; q)_n$. In terms of the Askey-Wilson parameters $(a, b, c, d)$ given by $(a, b, c, d) = \frac{abq}{ac, ad}$ the coefficients of the difference operator in Proposition 4.5 are

\[
A_m^+ = A_{\bar{\beta}}^+(tq^{-k} - 2m) = \frac{(1 - abq^2m)(1 - acq^2m)(1 - beq^2m)(1 - abcdq^{2m-2})}{d^2 - t^2q^{-2m}(q^{-1} - q)(1 - abcdq^{4m})(1 - abcdq^{4m-2})},
\]

\[
A_m^\beta = A_{\bar{\beta}}(t^{-1}q^{k+2m}) = \frac{(1 - q^2m)(1 - adq^{2m-2})(1 - bdq^{2m-2})(1 - cdq^{2m-2})}{dq^{2m-2}(q^{-1} - q)(1 - abcdq^{4m})(1 - abcdq^{4m-2})},
\]

\[
B_m = B_{\bar{\beta}}(tq^{-k} - 2m) = \frac{b + b^{-1} - \sqrt{a/b} - \sqrt{b/a}}{q^{-1} - q} - \frac{A_m^+ 1 - adq^{2m}}{adq^{2m} 1 - beq^{2m-2}} - A_m^- adq^{2m-2}(1 - beq^{2m})
\]

and the functions $P_m(x) = P_{\bar{\beta}}(x, tq^{-k} - 2m)$ satisfy the recurrence relation

\[
\lambda_{x,s} P_m(x) = A_m^w P_{m+1}(x) + B_m P_m(x) + A_m^- P_{m-1}(x).
\]

Note here that

\[
\lambda_{x,s} = \frac{x + x^{-1} - \sqrt{a/b} - \sqrt{b/a}}{q^{-1} - q}.
\]

From 4.3 it follows that the polynomials

\[
\tilde{R}_m(x) = (-1)^m d^{-m} q^{-m(m-1)} p_m(x; a, b, c, d|q^2)
\]

are the unique solution for the same recurrence relation with initial value $\tilde{R}_0(x) = 1$, so that $P_m(x) = \tilde{R}_m(x)\tilde{R}_m(x)$. It remains to evaluate $P_0(x) = P_{\bar{\beta}}(x, tq^{-k})$, which is done in the appendix:

\[
P_{\bar{\beta}}(x, tq^{-k}) = \frac{(q^{k+1} + uq^{\pm 1}/t; q^2)_\infty}{(uq^{\pm 1}/t; q^2)_\infty} = \frac{(ac, bc; q^2)_\infty}{(ex^{\pm 1}; q^2)_\infty}.
\]

Now we can compare properties of $P_{\bar{\beta}}(x, y)$ with properties of the Askey-Wilson polynomials $p_m(x; a, b, c, d|q^2)$. In the proof of Lemma 4.4 we saw that the eigenvalue equation from Proposition 4.5 is the three-term recurrence relation. For the difference equation in Proposition 4.3 observe that the coefficients $A_\beta$ are given in terms of the Askey-Wilson parameters $(a, b, c, d)$ by

\[
A_\beta(x) = -\frac{(1 - eq^{-2}x^{-1})(1 - ax)(1 - bx)(1 - dx)}{\sqrt{abcd}q^{-2}(q^{-1} - q)(1 - x^2)(1 - q^2x^2)}
\]

With this expression it can easily be verified that the difference operator $M^{-1} \circ \rho(\tilde{Y}_{t,u}) \circ M$, where $M$ is multiplication by $P_{\bar{\beta}}(x, tq^{-k}) = C/(ex^{\pm 1}; q^2)$, is the standard Askey-Wilson difference operator.

For the orthogonality relations observe that $d = c$, and then the orthogonality relations in Proposition 4.2 are equivalent to orthogonality relations for $p_m(x; a, b, c, d|q^2)$ with respect to the weight function

\[
w(x) = \frac{(x\pm 1; q^2)_\infty}{(cx^{\pm 1}, dx^{\pm 1}; q^2)_\infty} = \frac{(ax\pm 1, bx\pm 1, cx\pm 1, dx\pm 1; q^2)_\infty}{(x\pm 2; q^2)_\infty},
\]

which is the standard Askey-Wilson weight function.

**Remark 4.7.** From the recurrence relation it follows that $P_{\bar{\beta}}(x, y) = P_{\bar{\beta}}(x, tq^{-k})p(x), x \in T$, for some $p \in \mathcal{P}$. This implies that we can extend $x \mapsto P_{\bar{\beta}}(x, y)$ to a meromorphic function on $\mathbb{C}$ with poles coming from $P_{\bar{\beta}}(x, tq^{-k})$. In particular, $P_{\bar{\beta}}(x, y)$ is also defined for $x = sq^{k+2m}, m \in \mathbb{N}$. Then from the two difference equations we obtain the duality property

\[
P_{\bar{\beta}}(sq^{k+2m}, tq^{-k} - 2m) = P_{\bar{\beta}}(tq^{-k} - 2m, sq^{k+2m}), m, m' \in \mathbb{N},
\]

where $\bar{\beta}$ is the ordered 5-tuple $\bar{\beta} = (t, s, u, k, q^{-1})$. In case $x = sq^{k+2m}$, $m' \in \mathbb{N}$, and $|t| > \sqrt{q^{1+k+2m}}$ it is shown in the appendix that the sum (4.1) converges. In this case the duality property follows directly from Definition 4.1 using $q \leftrightarrow q^{-1}$ invariance of the inner product and $v_{x,s,q^{-1}}(n) = \tilde{v}_{x,s,q}(n)$. 


Identity (4.5) corresponds to the following identity for Askey-Wilson polynomials,

\[
p_{m}(aq^{2m}; a, b, c, d|q^{2}) = \frac{p_{m}(q^{-2m}/a; 1/\bar{a}, 1/\bar{b}, 1/\bar{c}, 1/\bar{d}|q^{-2})}{\bar{a}^{-m}(1/ab, 1/ac, 1/ad; q^{2})_{m}},
\]

where \( \bar{a} = q^{k}/t = \sqrt{abcd}/q^{2}, \quad \bar{b} = q^{k}t = ab/\bar{a}, \quad \bar{c} = qus = ac/\bar{a}, \quad \bar{d} = qsu = ad/\bar{a}. \)

In terms of \( 4\varphi_{3} \)-series this is the identity

\[
4\varphi_{3}\left( q^{-m}, q^{-m'}; a_{1}, a_{2} \atop b_{1}, b_{2}, b_{3} \right) = 4\varphi_{3}\left( q^{m}, q^{m'}, 1/a_{1}, 1/a_{2} \atop 1/b_{1}, 1/b_{2}, 1/b_{3} ; 1/q, 1/q \right), \quad b_{1}b_{2}b_{3}q^{n+m'+1} = a_{1}a_{2}.
\]

5. Multivariate Al-Salam–Chihara polynomials

We extend the results from the previous two sections to a multivariate setting by considering tensor product representations. In this section we consider the multivariate analogs of the Al-Salam–Chihara polynomials \( v_{x, s}(n) \) and \( v_{y, t}(n) \). Before we define the representation we are interested in, let us first introduce some convenient notation. We fix a number \( N \in \mathbb{N}_{\geq 2} \). For \( \mathbf{a} = (a_{1}, a_{2}, \ldots, a_{N}) \) we denote

\[
\mathbf{a} = (a_{N}, a_{N-1}, \ldots, a_{1}), \quad a_{j} = (a_{1}, a_{2}, \ldots, a_{j}), \quad j = 1, \ldots, N; \\
q^{\mathbf{a}} = (q^{a_{1}}, q^{a_{2}}, \ldots, q^{a_{N}}), \\
\Sigma(\mathbf{a}) = (a_{1}, \sum_{j=1}^{2} a_{j}, \ldots, \sum_{j=1}^{N} a_{j}).
\]

Let \( \mathbf{k} = (k_{1}, \ldots, k_{N}) \in (\mathbb{R}_{>0})^{N} \). We denote by \( \pi \) the representation of \( \mathcal{U}_{q}^{\otimes N} \) on \( F(N^{N}) \cong F(\mathbb{N})^{\otimes N} \) given by \( \pi = \pi_{k_{1}} \otimes \cdots \otimes \pi_{k_{N}} \). This is a \( \pi \)-representation on the Hilbert space \( H = H_{k} \), which is the Hilbert space completion of the algebraic tensor product \( H_{k_{1}} \otimes \cdots \otimes H_{k_{N}} \). The inner product for \( H \) is

\[
\langle f, g \rangle_{H} = \sum_{\mathbf{n} \in \mathbb{N}^{N}} f(\mathbf{n})g(\overline{\mathbf{n}}) \omega(\mathbf{n}),
\]

with weight function

\[
\omega(\mathbf{n}) = \omega_{\mathbf{k}}(\mathbf{n}) = \prod_{j=1}^{N} \omega_{k_{j}}(n_{j}) = \prod_{j=1}^{N} \left( \frac{q^{2}; q^{2}}{q^{k_{j}}; q^{2}} \right)_{n_{j}}^{n_{j}} q^{n_{j}(k_{j}-1)}.
\]

Since the univariate weight function \( \omega_{k}(n) = q \leftrightarrow q^{-1} \) invariant, so is the multivariate weight function \( \omega(\mathbf{n}) \).

5.1. Coproducts of twisted primitive elements. We use the following notation for compositions of coproducts. We define \( \Delta^{0} \) to be the identity on \( \mathcal{U}_{q} \), and for \( n \geq 1 \) we define \( \Delta^{n} : \mathcal{U}_{q} \to \mathcal{U}_{q}^{\otimes (n+1)} \) recursively by

\[
\Delta^{n} = (\Delta \otimes 1^{\otimes (n-1)}) \Delta^{n-1}.
\]

Here, and elsewhere, we use the notation \( A \otimes B^{\otimes 0} = A \). Note that we also have

\[
\Delta^{n} = (1^{\otimes (n-1)} \otimes \Delta) \Delta^{n-1},
\]

which follows from coassociativity of \( \Delta \). A useful property of \( \Delta^{n} \) is the following one: if \( \Delta(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)} \), then

\[
\Delta^{n}(X) = \sum_{(X)} \Delta^{n-m-1}(X_{(1)}) \otimes \Delta^{m}(X_{(2)}), \quad m = 0, 1, \ldots, n - 1.
\]

This is easily obtained using induction.
Proposition 5.3. The difference operator \( \pi(Y_s^{(j)}) \) is given by

\[
\pi(Y_s^{(j)}) = \frac{1}{q - q^{-1}} \sum_{i=N-j+1}^{N} U_i^{(j)+}(n) T_i^+ + U_i^{(j)-}(n) T_i^-,
\]

where

\[
U_i^{(j)+}(n) = q^{\sum_{k=1}^{i} (k_i + 2k_i - 1)/2} (1 - q^{2k_i + 2n_i - 2}),
\]

\[
U_i^{(j)-}(n) = q^{\sum_{k=1}^{i} (k_i + 2k_i + 1)/2} (1 - q^{2n_i + 2}),
\]

\[
U_i^{(j)}(n) = q^{\sum_{k=1}^{i} (k_i + 2n_i - s - 1)} (q^{2n_i - k_i} - 1),
\]

The difference operator \( \tilde{\pi}(Y_t^{(j)}) \) is given by

\[
\tilde{\pi}(Y_t^{(j)}) = \frac{1}{q - q^{-1}} \sum_{i=1}^{j} \tilde{U}_i^{(j)+}(n) T_i^+ + \tilde{U}_i^{(j)-}(n) T_i^-,
\]

Lemma 5.1. For \( j, j' = 1, \ldots, N \),

\[
Y_{s,u}^{(j)} Y_{s,u}^{(j')} = Y_{s,u}^{(j')} Y_{s,u}^{(j)} \quad \text{and} \quad \tilde{Y}_{t,u}^{(j)} \tilde{Y}_{t,u}^{(j')} = \tilde{Y}_{t,u}^{(j')} \tilde{Y}_{t,u}^{(j)}.
\]

Proof. We show that \( Y_{s,u}^{(j)} \) commutes with \( Y_{s,u}^{(j')} \), where we assume \( j' < j \). Note that is sufficient to show that \( \Delta^{j'-1}(Y_{s,u}) \) commutes in \( U_q^{(j)} \) with \( 1^{(j-j')} \otimes \Delta^{j'-1}(Y_{s,u}) \). Recall from \( (5.1) \) that \( \Delta(Y_{s,u}) = K^2 \otimes Y_{s,u} + Y_{s,u} \otimes 1 \). Then by \( (5.1) \)

\[
\Delta^{j'-1}(Y_{s,u}) = \Delta^{j'-1}(K^2) \otimes \Delta^{j'-1}(Y_{s,u}) + \Delta^{j'-1}(Y_{s,u}) \otimes \Delta^{j'-1}(1),
\]

and this clearly commutes with \( 1^{(j-j')} \otimes \Delta^{j'-1}(Y_{s,u}) \). The proof for \( \tilde{Y}_{t,u}^{(j)} \) is similar. \( \square \)

In the representation \( \pi \) the elements \( Y_{s,u}^{(j)} \) and \( \tilde{Y}_{t,u}^{(j)} \) become pairwise commuting difference operators acting on \( F(N^N) \), and we are interested in the common eigenfunctions. First we derive an explicit expression for the difference operators. The following expressions will be useful.

Lemma 5.2. For \( j = 1, \ldots, N \),

\[
\Delta^j(Y_{s,u}) = \sum_{n=0}^{j} (K^2)^{j-n} \otimes Y_{s,u} \otimes 1^{(j-n)},
\]

\[
\Delta^j(\tilde{Y}_{s,u}) = \sum_{n=0}^{j} 1^{(j-n)} \otimes \tilde{Y}_{s,u} \otimes (K^2)^{j-n}.
\]

Proof. This follows from repeated application of \( (5.1) \), using the coproducts \( (2.3) \) of \( Y_{s,u} \) and \( \tilde{Y}_{s,u} \),

\[ \Delta(K^{\pm 2}) = K^{\pm 2} \otimes K^{\pm 2}. \] \( \square \)

In order to write down explicit expressions for the difference operators \( \pi(Y_s^{(j)}) \) and \( \pi(\tilde{Y}_t^{(j)}) \) we use the following notation for elementary difference operators on \( F(N^N) \):

\[ [T_i^\pm f](n) = f(n_1, \ldots, n_{i-1}, n_i \pm 1, n_{i+1}, \ldots, n_N). \]

Proposition 5.3. The difference operator \( \pi(Y_s^{(j)}) \) is given by

\[
\pi(Y_s^{(j)}) = \frac{1}{q - q^{-1}} \sum_{i=N-j+1}^{N} U_i^{(j)+}(n) T_i^+ + U_i^{(j)-}(n) T_i^-,
\]

where

\[
U_i^{(j)+}(n) = q^{\sum_{k=1}^{i} (k_i + 2k_i - 1)/2} (1 - q^{2k_i + 2n_i - 2}),
\]

\[
U_i^{(j)-}(n) = q^{\sum_{k=1}^{i} (k_i + 2k_i + 1)/2} (1 - q^{2n_i + 2}),
\]

\[
U_i^{(j)}(n) = q^{\sum_{k=1}^{i} (k_i + 2n_i - s - 1)} (q^{2n_i - k_i} - 1),
\]
where

\[ \hat{U}^{(j)}_i(n) = q^{-\sum_{i=1}^j (k_i+2n_i) - (k_i-1)/2} (1 - q^{-2n_i-2}), \]

\[ \hat{U}_i^{(j)-}(n) = q^{-\sum_{i=1}^j (k_i+2n_i) + (k_i-1)/2} (1 - q^{-2k_i-2n_i+2}), \]

\[ \hat{U}_i^{(j)}(n) = q^{-\sum_{i=1}^j (k_i+2n_i)} (t + t^{-1})(q^{-2n_i+k_i} - 1). \]

**Proof.** This follows from Lemma 5.2, using the actions of \( Y_s, \tilde{Y}_s \) and \( K^{\pm 2} \), see (5.3), (5.13) and (2.4). \( \square \)

### 5.2. Eigenfunctions

We write \( x = (x_1, \ldots, x_N) \), \( y = (y_1, \ldots, y_N) \) and define \( x_{N+1} = s \) and \( y_0 = t \). We define multivariate analogs of the Al-Salam–Chihara polynomials \( \nu_s(n) \) in base \( q^2 \), see (3.10), and multivariate analogs of the Al-Salam–Chihara polynomials \( \tilde{v}_{y,t}(n) \) in base \( q^{-2} \), see (3.14), by

\[ v_x(n) = v_{x,s,k}(n) = \prod_{j=1}^N v_{x_j, x_{j+1}, k_j(n_j)} \]

\[ = \prod_{j=1}^N \left( \frac{q^{-(3k_j-1)/2}}{x_{j+1}} \right)^{n_j} \left( \frac{q^{2k_j}; q^2}{(q^2; q^2)^{n_j}} \right)^{Q_{n_j}(x_j; q^{k_j}x_{j+1}, q^{k_j}x_{j+1} | q^2)}, \]

(5.2)

\[ \tilde{v}_y(n) = \tilde{v}_{y,t,k}(n) = \prod_{j=1}^N \tilde{v}_{y_j, y_{j-1}, k_j(n_j)} \]

\[ = \prod_{j=1}^N \left( \frac{q^{(3k_j-1)/2}}{y_{j-1}} \right)^{n_j} \left( \frac{q^{-2k_j}; q^{-2}}{(-q^2; q^{-2})^{n_j}} \right)^{Q_{n_j}(y_j; q^{-k_j}y_{j-1}, q^{-k_j}y_{j-1} | q^{-2})}. \]

Recall from Remark 5.4 that \( v_{x_j, x_{j+1}, k_j(n_j)} \) is a polynomial in \( x_j + x_j^{-1} \) and in \( x_{j+1} + x_{j+1}^{-1} \), and a similar observation can be made for \( \tilde{v}_{y_j, y_{j-1}, k_j(n_j)} \). So \( v_x(n) \) and \( \tilde{v}_y(n) \) are polynomials in \( N \) variables. Recall that the univariate Al-Salam–Chihara polynomials \( v_{x,s}(n) \) and \( \tilde{v}_{y,s}(n) \) can be obtained from each other by replacing \( q \) by \( q^{-1} \). There is a similar relation for their multivariate analogs, which follows directly from (5.2).

**Lemma 5.4.** The multivariate Al-Salam–Chihara polynomials \( v_{x,s,k,q}(n) \) and \( \tilde{v}_{x,s,k,q}(n) \) are related by

\[ v_{x,s,k,q^{-1}}(n) = \tilde{v}_{x,s,k,q}(n). \]

We show that the multivariate Al-Salam–Chihara polynomials are eigenfunctions of the difference operators \( \pi(Y_s^{(j)}) \) and \( \pi(\tilde{Y}_s^{(j)}) \). For \( N = 2, 3 \) this is proved in [19] Section 4. Slightly more general, we will determine eigenfunction of \( \pi(Y_{t,u}^{(j)}) \) and \( \pi(\tilde{Y}_{t,u}^{(j)}) \). To formulate the result we need the multiplication operator \( M_u \) on \( F(\mathbb{N}^N) \) defined by \( [M_u f(n)] = u^{n_1 + \cdots + n_N} f(n) \).

**Proposition 5.5.** For \( j = 1, \ldots, N \) and \( u \in \mathbb{T} \),

\[ [\pi(Y_{t,u}^{(j)}) M_u v_x(n)] = \lambda_{x_N-j+1,s} M_u v_x(n), \]

\[ [\pi(\tilde{Y}_{t,u}^{(j)}) M_u \tilde{v}_y(n)] = \lambda_{t,y_j} M_u \tilde{v}_y(n). \]

**Proof.** We first set \( u = 1 \) and prove the result for \( \tilde{Y}_t^{(j)} = \tilde{Y}_t^{(j)} \) using induction on \( j \). Let us define for \( j = 1, \ldots, N \), \( \pi_{k_j} = \pi_{s_1} \otimes \cdots \otimes \pi_{k_j} \) and

\[ \tilde{v}_{y_j}(n_j) = \prod_{i=1}^j \tilde{v}_{y_{i}, y_{i-1}, k_i(n_i)} \]

Note that \( \tilde{v}_{y_{j+1}}(n_{j+1}) = \tilde{v}_{y_j}(n_j) v_{y_{j+1}, y_{j+1}, k_{j+1}(n_{j+1})} \). To prove the result for \( u = 1 \) it suffices to prove that

\[ [\pi_{k_j}(\Delta^{-1}(\tilde{Y}_t)) \tilde{v}_{y_j}(n_j)] = \lambda_{t,y_j} \tilde{v}_{y_j}(n_j), \]

(5.3)
for $j = 1, \ldots, N$. Then the result follows from the fact that $\tilde{Y}^{(j)}_t$ acts as $\Delta^j(\tilde{Y}_t)$ on the first $j$ factors of $F_0(N)^{\otimes N}$ and as the identity on the other factors.

For $j = 1$ identity (5.3) follows directly from Proposition 5.7. Assuming (5.3) holds for some $j$, and using

\[ \Delta^j(\tilde{Y}_t) = \Delta^{j-1}(1) \otimes \tilde{Y}_t + \Delta^{j-1}(\tilde{Y}_t) \otimes K^{-2}, \]

we find

\[ \begin{aligned}
\pi_{k_j+1}(\Delta^j(\tilde{Y}_t))\tilde{v}_{y_{j+1}}(n_{j+1}) \\
= \tilde{v}_{y_j}(n_j) \left( \pi_{k_j+1}(\tilde{Y}_t) \tilde{v}_{y_{j+1},y_j,k_j}(n_{j+1}) \right) + \left[ \pi_k(\Delta^{j-1}(\tilde{Y}_t)) \tilde{v}_{y_j}(n_j) \right] \left[ \pi_{k_{j+1}}(K^{-2})\tilde{v}_{y_{j+1},y_j,k_j}(n_{j+1}) \right] \\
= \tilde{v}_{y_j}(n_j) \left( \pi_{k_j+1}(\lambda_t,y_j)K^{-2} + \tilde{Y}_t \tilde{v}_{y_{j+1},y_j,k_j}(n_{j+1}) \right).
\end{aligned} \]

From $\lambda_t,y_j = \mu_t - \mu_y$, we find

\[ \lambda_t,y_j K^{-2} + \tilde{Y}_t = q^{-\frac{t}{2}} E K - q^\frac{t}{2} F K - \mu_y K^{-2} + \mu_1 = \tilde{Y}_y + (\mu_1 - \mu_y). \]

Then using $\pi_k(\tilde{Y}_t)\tilde{v}_{y_{j+1},y_j} = \lambda_{y_j,y_{j+1}}\tilde{v}_{y_j,y_{j+1}}$, we obtain (5.3) for $j + 1$. By induction it follows that $\pi(\tilde{Y}_t)\tilde{v}_y = \lambda_{t,y} \tilde{v}_y$. Finally, from the identities $\pi_k(\tilde{Y}_{t,u})M_u = M_u \pi_k(\tilde{Y}_t)$ and $\pi_k(\tilde{Y}_t)M_u = M_u \pi(\tilde{Y}_t)$ on $F(N)$, and the second identity in Lemma 5.2 it follows that $\pi(\tilde{Y}_{t,u})M_u = M_u \pi(\tilde{Y}_t)$ on $F(N^N)$, which proves the result for $\tilde{Y}_t$. For $\tilde{Y}_t$ the proof runs along the same lines. \qed

Next we define the corresponding Hilbert spaces. We define the weight function $w = w_{k,s}$ on $T^N$ by

\[ w(x) = \prod_{j=1}^N w_{k_j,x_{j+1}}(x_j) = C_k \prod_{j=1}^N \left( \frac{x_{j+1}^{\frac{1}{2}} q^2}{x_j^{\frac{1}{2}} q^2} \right)^\infty, \]

where $C_k$ is the $x$-independent constant given by $C_k = \prod_{j=1}^N (q^2, q^{2k_j}; q^2)^\infty$. The Hilbert space $\mathcal{H} = \mathcal{H}_{k,s}$ consists of functions on $T^N$ which are invariant under $x_j \leftrightarrow x_j^{\pm 1}$ for $j = 1, \ldots, N$, and which have finite norm with respect to the inner product

\[ \langle f, g \rangle_{\mathcal{H}} = \int_{T^N} f(x) \overline{g(x)} w(x) \frac{dx}{x}, \]

where $\frac{dx}{x} = \frac{1}{(2\pi)^N} \prod_{j=1}^N \frac{dx_j}{2\pi}$. The other Hilbert space consist functions on the set

\[ S = S_{k,t,q} = \left\{ t q^{-\sum(k)-\Sigma(m)} \left| m \in N^N \right. \right\}. \]

Let $\tilde{w} = \tilde{w}_{k,t}$ be the weight function on $S$ given by

\[ \tilde{w}(y) = \prod_{j=1}^N \tilde{w}_{k_j,y_j-1}(y_j) = \prod_{j=1}^N \left( \frac{q^{2K_{j-1} + 4M_{j-1} + 2m_j}}{t^2} \right)^{m_j} \left( 1 - \frac{q^{2K_{j-1} + 4M_{j-1}}}{t^2} \right)^{1 - q^{2K_{j-1} + 4M_{j-1}}}, \]

\[ \times \left( q^{2K_{j-1} + 4M_{j-1}} / q^{2K_j}; q^2 \right)^{m_j} \left( q^{2K_{j-1} + 4M_{j-1},2m_j} / t^2, q^2 \right)^\infty, \]

where $y = t q^{-\sum(k)-\Sigma(m)}$, $M_j = \Sigma(m_j) = \sum_{i=1}^j m_i$, $K_j = \Sigma(k_j) = \sum_{i=1}^j k_i$ and $M_0 = 0 = K_0$. The Hilbert space $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{k,t}$ consists of functions on $S$ which have finite norm with respect to the inner product

\[ \langle f, g \rangle_{\tilde{\mathcal{H}}} = \sum_{y \in S} f(y) \overline{g(y)} \tilde{w}(y). \]

From the orthogonality relations of the univariate polynomials we obtain the following ortho-gonality relations for the multivariate ones.

\textbf{Lemma 5.6.}

(i) The set $\{v_n(n) \mid n \in N^N\}$ is an orthogonal basis for $\mathcal{H}$ with orthogonality relations

\[ \langle v_n(n), v_n'(n') \rangle_{\mathcal{H}} = \frac{\delta_{n,n'}}{\omega(n)}. \]
(ii) The set \( \{ \tilde{v}_x(n) \mid n \in \mathbb{N}^N \} \) is an orthogonal basis for \( \tilde{H} \) with orthogonality relations

\[
\langle \tilde{v}_x(n), \tilde{v}_x(n') \rangle_{\tilde{H}} = \frac{\delta_{n,n'}}{\omega(n)}
\]

(iii) The set \( \{ \tilde{v}_y \mid y \in S \} \) is an orthogonal basis for \( H \) with orthogonality relations

\[
\langle \tilde{v}_y, \tilde{v}_{y'} \rangle_H = \frac{\delta_{y,y'}}{\tilde{\omega}(y)}
\]

Proof. Statement (i) follows from (3.12) by writing the inner product as an iterated integral
\[
\int_{x_1} \int_{x_2} \cdots \int_{x_N}
\]
and using that the squared norms of the univariate Al-Salam–Chihara polynomials \( v_{x_j,x_{j+1}}(n_j) \) is independent of \( x_{j+1} \). Statement (ii) is proved in a similar way. Statement (iii) is obtained directly from (3.16) by taking products. \( \square \)

Remark 5.7. The orthogonality in statement (iii) of the functions \( \tilde{v}_y \) with respect to the inner product on \( H \) can be considered as orthogonality relations for multivariate little \( q \)-Jacobi polynomials.

From Proposition 5.5 and Lemma 5.6 we obtain the multivariate analogue of Propositions 5.3 and 5.7.

Proposition 5.8.

(i) Define \( \Lambda = \Lambda_{k,s} : F_0(\mathbb{N}^N) \to \mathcal{P} \) by

\[
(\Lambda f)(X) = \langle f, v_X \rangle_H,
\]

then \( \Lambda \) intertwines \( \pi(Y^{(j)}_s) \), \( j = 1, \ldots, N \), with multiplication by \( \lambda_{x_{N+1-j},s} \), and extends to a unitary operator on \( H \to \tilde{H} \).

(ii) Define \( \tilde{\Lambda} = \tilde{\Lambda}_{k,t} : F_0(\mathbb{N}^N) \to \mathcal{P} \) by

\[
(\tilde{\Lambda} f)(y) = \langle f, \tilde{v}_y \rangle_H,
\]

then \( \tilde{\Lambda} \) intertwines \( \tilde{\pi}(Y^{(j)}_t) \), \( j = 1, \ldots, N \), with multiplication by \( \lambda_{t,y} \), and extends to a unitary operator on \( H \to \tilde{H} \).

Next we define representations on \( \mathcal{P} \) using the above defined intertwining operators \( \Lambda \) and \( \tilde{\Lambda} \).

5.3. Representations on \( \mathcal{P} \). We define a representation \( \rho = \rho_k \) of \( \mathcal{U}_q^{\otimes N} \) on \( \mathcal{P} \) by

\[
\rho(X) = \Lambda \circ \pi(X) \circ \Lambda^{-1}, \quad X \in \mathcal{U}_q^{\otimes N}.
\]

In this representation \( Y^{(j)}_s \), \( j = 1, \ldots, N \), act as multiplication by \( \lambda_{x_{N-j+1},s} \). We define for \( j = 1, \ldots, N \)

\[
K^{-2,(j)} = \Delta^{-1}(K^{-2}) \otimes 1^{\otimes (N-j)} \in \mathcal{U}_q^{\otimes N}.
\]

Our next goal is to realize \( \rho(K^{-2,(j)}) \) as explicit difference operators. To do this we first show that \( \pi_k(K^{-2}) \) acts on a univariate polynomial \( v_{x,s} \) as a ‘dynamical’ difference operator, i.e. it acts as a difference operator in the variable \( x \) and in the parameter \( s \).

Lemma 5.9. The univariate Al-Salam–Chihara polynomials \( v_{x,s}(n) \) satisfy

\[
[\pi_k(K^{-2})v_{x,s}](n) = C(x; s)v_{x,q^k-s,q^k}(n) + D(x; s)v_{x,q^k-s/q^k}(n) + C(x^{-1}; s)v_{x/q^k-s,q^k}(n) + D(x^{-1}; s^{-1})v_{x/q^k-s/q^k}(n) + C(x^{-1}; s^{-1})v_{x/q^k-s/q^k}(n),
\]

where

\[
C(x; s) = C_k(x; s) = \frac{q^{-k}(1-q^kxs)(1-q^{k+2}xs)}{(1-x^2)(1-q^2x^2)},
\]
\[
D(x; s) = D_k(x; s) = \frac{q^{3-k}(q^{-1}+q)(1-q^ksx)(1-q^ks/x)}{(1-q^2x^2)(1-q^2/x^2)}.
\]
where we use the notations
\(5.6\)

Here we use notations similar as in the proof of Proposition 5.5; in particular

\[ \text{Proof.} \]

Let us fix a number \(A\), Al-Salam–Chihara polynomials, which boils down to proving the identities multivariate Al-Salam–Chihara polynomials and for \(i\) with \(A\), \(2\), \(\pi\) ∈ \(\{−1, 0, 1\}^j\) we define
\[
T_\nu = T_1^{v_1} \cdots T_j^{v_j}.
\]

Proposition 5.10. For \(j = 1, \ldots, N\) \(\rho(K^{-2}; j)\) is the q-difference operator on \(P\) given by
\[
\rho(K^{-2}; j) = \sum_{\nu \in \{−1, 0, 1\}^j} V_{\nu}^{(j)}(x) T_\nu
\]
where
\[
V_{\nu}^{(j)}(x) = \prod_{i=1}^j V_{\nu_i}^{(j)}(x),
\]
with
\[
V_{\nu_i}^{(j)}(x) = V_{\nu_i, k_i}^{(j)}(x_j; x_{j+1}) = \begin{cases} A_{k_i}(x_j^{v_i}; x_{j+1}), & \nu_j \neq 0, \\ B_{k_i}(x_j; x_{j+1}), & \nu_j = 0, \end{cases}
\]

and for \(i = 1, \ldots, j−1\)
\[
V_{\nu_i}^{(j)}(x) = \prod_{i=1}^j V_{\nu_i}^{(j)}(x).
\]

Here \(A\) and \(B\) are given in Lemma 5.5 and \(C\) and \(D\) are given in Lemma 5.9. In particular, the multivariate Al-Salam–Chihara polynomials \(v_x(n)\) satisfy
\[
\sum_{\nu \in \{−1, 0, 1\}^j} V_{\nu}^{(j)}(x)[T_\nu v_x(n)](x) = q^{-\sum_{i=1}^j (k_i + 2n_i)} v_x(n), \quad j = 1, \ldots, N.
\]

\[ \text{Proof.} \]

Let us fix a number \(j\). It suffices to prove the q-difference equations for the multivariate Al-Salam–Chihara polynomials, which boils down to proving the identities

\[ \pi_k(\Delta^{−1}(K^{-2}))v_x(n) = \sum_{\nu \in \{−1, 0, 1\}^j} V_{\nu}^{(j)}(x)[T_\nu v_x](n). \]

Here we use notations similar as in the proof of Proposition 5.5 in particular
\[
v_x(n) = \prod_{i=1}^j v_{x_i, x_{i+1}, k_i}(n_i).
\]

We will show that, for \(l = 1, \ldots, j\),
\[
\pi_k(\Delta^{−1}(K^{-2}))v_x = \prod_{i=1}^{l−1} \pi_k(K^{-2})u_{x_i, x_{i+1}, k_i}
\]
\[5.6\]
\[
\times \sum_{(\nu_1, \ldots, \nu_l) \in \{−1, 0, 1\}^{j−l+1}} V_{(\nu_1, \ldots, \nu_l)}^{(j)}(x) T_{(\nu_1, \ldots, \nu_l)} = T_{(\nu_1, \ldots, \nu_l)}^{(\nu_1, \ldots, \nu_l)}.
\]

where we use the notations
\[ V_{(\nu_1, \ldots, \nu_l)}^{(j)}(x) = \prod_{i=1}^j V_{\nu_i}^{(j)}(x) \quad \text{and} \quad T_{(\nu_1, \ldots, \nu_l)} = T_{\nu_1}^{(\nu_1)} \cdots T_{\nu_l}^{(\nu_l)}. \]
For $l = 1$ this is the desired result. We prove (3.9) by backwards induction on $l$. For $l = j$ the identity to prove is

$$
\pi_{k_j}(\Delta^{l-1}(K^{-2})): v_{x_j} = \sum_{i=1}^{j-1} \pi_{k_i}(K^{-2}) v_{x_i,x_{i+1},k_i}
$$

$$
\times \left( A_{k_j}(x_j; x_{j+1})v_{x_j,q^2,x_{j+1},k_j} + B_{k_j}(x_j; x_{j+1})v_{x_j,x_{j+1},k_j} + A_{k_j}(x_j^{-1}; x_{j+1})v_{x_j/q^2,x_{j+1},k_j} \right),
$$

which is valid by Lemma 3.5. Next assume (5.6) holds for some $l$, then

$$
\pi_{k_l}(\Delta^{l-1}(K^{-2})): v_{x_l} = F(x) \pi_{k_{l-1}}(K^{-2}) v_{x_{l-1},x_l,k_{l-1}}
$$

$$
\times \sum_{\nu \in \{-1,0,1\}} V^{(j)}_{\nu} v_{x_l,x_{l+1},k_l},
$$

where $F(x)$ is a function (which can be made explicit) independent of $x_l$. We rewrite the factor $\pi_{k_{l-1}}(K^{-2})v_{x_{l-1},x_l,k_{l-1}}$ depending on the value of $\nu_l$: for $\nu_l = 0$ we use Lemma 3.5, for $\nu_l = 1$ we use the first identity in Lemma 5.9 and for $\nu_l = -1$ we use the second identity in Lemma 5.9. This leads to (5.6) for $l - 1$. □

We define another representation $\tilde{\rho} = \tilde{\rho}_k$ of $U_q^{\otimes N}$ on $\mathcal{P}$ by

$$
\tilde{\rho}(X) = \tilde{\Lambda} \circ \pi(X) \circ \tilde{\Lambda}^{-1}, \quad X \in U_q^{\otimes N}.
$$

By Proposition 5.8, the operator $\tilde{\rho}(\tilde{V}_{\nu,y,t}^{(j)}$) is multiplication by $\lambda_{y,t}$. We also define for $j = 1, \ldots, N$,

$$
K^{2,(j)} = 1^{\otimes (N-j)} \otimes \Delta^{j-1}(K^2) \in U_q^{\otimes N}.
$$

$\tilde{\rho}(K^{2,(j)})$ can be realized as an explicit $q$-difference operator on $\mathcal{P}$. This is done in the same way as we did above for $\rho(K^{-2),(j))$, and formally it is just replacing $q$ by $q^{-1}$. We omit most of the details. First we need the analog of Lemma 5.9 for $\tilde{V}_{y,t}$.

**Lemma 5.11.** The univariate Al-Salam–Chihara polynomials $\tilde{V}_{y,t}(n)$ satisfy

$$
[\pi_k(K^2)]\tilde{V}_{y,t}(n) = \tilde{C}(y,t) \tilde{V}_{y,q^2,t/q^2}(n) + \tilde{D}(y,t) \tilde{V}_{y,t/q^2}(n) + \tilde{C}(y^{-1},t) \tilde{V}_{y,q^2,t/q^2}(n) + \tilde{D}(y^{-1},t^{-1}) \tilde{V}_{y,t/q^2}(n),
$$

where

$$
\tilde{C}(y,t) = \tilde{C}_k(y,t) = \frac{q^k(1-q^{-k}ty)(1-q^{-k^2}ty)}{(1-y^2)(1-y^2/q^2)};
$$

$$
\tilde{D}(y,t) = \tilde{D}_k(y,t) = \frac{q^{k^2}(1-q^{-k}ty)(1-q^{-k^2}ty)}{(1-y^2/q^2)(1-1/x^2q^2)}.
$$

With this lemma we can prove the analog of Proposition 5.11. The following notation will be useful. For $j \in \{1, \ldots, N\}$ and $\nu = (\nu_1, \ldots, \nu_j) \in \{-1,0,1\}^j$ we define

$$
\tilde{T}_\nu = T_{\nu_1} \cdots T_{\nu_j}.
$$

**Proposition 5.12.** For $j = 1, \ldots, N$ $\tilde{\rho}(K^{2,(j)})$ is the $q$-difference operator on $\mathcal{P}$ given by

$$
\tilde{\rho}(K^{2,(j)}) = \sum_{\nu \in \{-1,0,1\}^j} \tilde{V}_{\nu}^{(j)}(y) \tilde{T}_\nu
$$

where

$$
\tilde{V}_{\nu}^{(j)}(y) = \prod_{i=1}^{j} \tilde{V}_{\nu_i}^{(j)}(y),
$$

with

$$
\tilde{V}_{\nu}^{(j)}(y) = \tilde{V}_{\nu_1,\nu_2,\ldots,\nu_{j+1}}^{(j)}(y_{N-j+1}; y_N) = \begin{cases} 
\tilde{A}_{k_{N-j+1}}(y_{N-j+1}; y_N), & \nu_j \neq 0, \\
\tilde{B}_{k_{N-j+1}}(y_{N-j+1}; y_N), & \nu_j = 0,
\end{cases}
$$
where Theorem 6.2.

\[ \text{Definition 6.1.}\]

For \( x \in D^N \) and \( y \in S = S_{k,t,q} \) (see (5.5) for the set \( S_{k,t,q} \)), we define

\[ P_{\beta}(x, y) = \langle M_u \tilde{v}_y, v_x \rangle_H, \]

where \( \beta \) is the ordered \((N + 4)\)-tuple given by \( \beta = (s, t, u, k_1, \ldots, k_N, q) \).

Observe that \( P_{\beta}(x, y) = \Lambda(M_u \tilde{v}_y)(x) = \Lambda(M_u v_x)(y) \).

We first show that these are multiples of the multivariate Askey-Wilson polynomials defined by (1.2).

**Theorem 6.2.** For \( y = t q^{-\Sigma(k) - 2 \Sigma(m)} \in S \),

\[ P_{\beta}(x, y) = C_{\beta}(x, y) P_N(m; x; |q|), \]

where

\[ C_{\beta}(x, y) = \frac{(\alpha_{N+1} \alpha_N + q^{2M_N}; \alpha_{N+1} q^{2M_N}; \alpha_{N+1} q^2; q^2)_{\infty}}{(x; x_1^2; q^2)_{\infty}} \prod_{j=1}^{N} \left( \frac{-\alpha_{N+1}^{-2M_N}; \alpha_{N+1} q^2; q^2}_{\alpha_j^{2M_N}; \alpha_j^2 q^2} \right)_{m_j} q^{-m_j(m_j-1)}, \]

with

\[ \alpha_0 = u, \quad \alpha_{N+2} = s, \quad \alpha_j = uq^{K_j-1+1} / t \quad \text{for} \quad j = 1, \ldots, N + 1, \]

\( M_j = \Sigma(m) \), \( K_j = \Sigma(k) \) and \( M_0 = 0 = K_0 \).

**Proof.** From the definition (5.2) of the multivariate Al-Salam–Chihara polynomials and our definition (1.1) of the univariate Askey-Wilson polynomials we obtain

\[ P_{\beta}(x, y) = \prod_{j=1}^{N} (M_u \tilde{v}_{y_j, y_{j-1}, k_j, v_{x_j, x_{j+1}, k_j}})_{H_k} = \prod_{j=1}^{N} P_{\beta_j}(x_j, y_j), \]

where \( \beta_j = (x_{j+1}, y_{j-1}, u, k_j, q) \). Recall here that \( x_{N+1} = s \) and \( y_0 = t \), and note that \( y_j = y_j-1 q^{-K_j-2M_j} \). By Lemma 4.6 and the symmetry of the Askey-Wilson polynomials \( p_n(x; a, b, c, d |q) \) in its parameters \( a, b, c, d \), a factor \( P_{\beta_j}(x_j, y_j) \) is a multiple of the Askey-Wilson polynomial

\[ p_{m_j}(x; u, t q^{1+K_j-1+2M_j-1}, \frac{1}{u t} q^{1+K_j-1+2M_j-1}, x_{j+1} q^K, x_{j+1} q^{K_j} | q^2), \]
which is the $j$-th factor of the multivariate Askey-Wilson polynomial $P_N(x;\alpha | q^2)$ as defined in (1.2). The expression for $C_\beta(x,y)$ follows from the factor in front of the Askey-Wilson polynomial in Lemma 4.6 i.e.

$$C_\beta(x,y) = \prod_{j=1}^{N} (-ut)^{m_j} q^{-m_j(1+K_{j,1}+2M_{j,1})} q^{-m_j(m_j-1)} \frac{(ux_j^{\pm 1} q^{1+K_j+2M_j}/t; q^2)_\infty}{(q^{2K_j}; q^2)^{m_j} (ux_j^{\pm 1} q^{1+K_{j,1}+2M_{j,1}}/t; q^2)_\infty}.$$  

This simplifies to the expression given in the theorem by cancelling common factors.

Next we derive properties of the functions $P_\beta(x,y)$. We start with orthogonality.

**Theorem 6.3.** The set $\{P_\beta(\cdot, y) \mid y \in S\}$ is an orthogonal basis for $\mathcal{H}$, with orthogonality relations

$$\langle P_\beta(\cdot, y), P_\beta(\cdot, y') \rangle_{\mathcal{H}} = \frac{\delta_{y,y'}}{w(y)}.$$  

**Proof.** The proof is essentially the same as the proof of Proposition 4.2. The orthogonality relations follow from $P_\beta(x,y) = [\Lambda(M_u \tilde{v}_y)](x)$, the orthogonality relations of $\tilde{v}_y$ in Lemma 6.3 and unitarity of $\Lambda$ and $M_u$. $\square$

**Remark 6.4.** From Theorem 6.2 it follows that the orthogonality relations from Theorem 6.3 are equivalent to orthogonality relations of the multivariate Askey-Wilson polynomials $P_N(x;\alpha | q^2)$ with respect to the weight function

$$w(x) = \frac{(\alpha_1 x_1^{\pm 1}, \alpha_1 x_1^{\pm 1}; q^2)_\infty}{(\alpha_{j+1} x_j^{\pm 1} x_j^{\pm 1}/\alpha_j; q^2)_\infty},$$

where $w$ is defined by (5.4). Up to a multiplicative constant $w$ is equal to

$$\prod_{j=1}^{N} \frac{(x_j^{\pm 2}; q^2)_\infty}{(\alpha_{j+1} x_j^{\pm 1} x_j^{\pm 1}/\alpha_j; q^2)_\infty},$$

and $\alpha_1 = \alpha_1/\alpha_0^2$, so we recover the orthogonality relations of $P_N(x;\alpha | q^2)$ with respect to the weight function (5.6) in base $q^2$.

The multivariate Askey-Wilson polynomials $P_\beta(x,y)$ are simultaneous eigenfunctions of the coproducts of the twisted primitive elements, i.e., for $j = 1, \ldots, N$,

$$[\rho(Y_{1,\beta}^{(j)}), P_\beta(\bullet, y)](x) = [\Lambda(\pi(Y_{1,\beta}^{(j)})) M_u \tilde{v}_y](x) = \lambda_{x,y} P_\beta(x,y),$$

$$[\tilde{\rho}(Y_{1,\beta}^{(j)}) P_\beta(x,\bullet)](y) = [\tilde{\Lambda}(\pi(Y_{1,\beta}^{(j)})) M_u v_x](y) = \lambda_{x,y} P_\beta(x,y).$$

Our goal is now to write these eigenvalue equations as explicit $q$-difference equations.

**Theorem 6.5.** For $j = 1, \ldots, N$ $\rho(Y_{1,\beta}^{(j)})$ is the $q$-difference operator given by

$$\rho(Y_{1,\beta}^{(j)}) = \sum_{\nu \in \{-1,0,1\}^j} V_{\nu,\beta}^{(j)}(x) T_{\nu} - \left( \frac{(u+u^{-1}) \mu_{x,j+1}}{q^{-1}+q} - \mu_t \right) \text{Id}$$

where,

$$V_{\nu,\beta}^{(j)}(x) = V_{\nu}^{(j)}(x) \left( \frac{(u+u^{-1}) \mu_{x_j+1,x_j}}{q^{-1}+q} + \frac{(qu-q^{-1}u^{-1})(\mu_{x_{j+1},x_j} - \mu_{x_j})}{(q^{-1}-q)(q^{-1}+q)} - \mu_t \right);$$

with $V_{\nu}^{(j)}$ given in Proposition 5.10. In particular, the multivariate Askey-Wilson polynomials $P_\beta(x,y)$ satisfy

$$\sum_{\nu \in \{-1,0,1\}^j} V_{\nu,\beta}^{(j)}(x) [T_{\nu} P_\beta(\bullet, y)](x) - \left( \frac{(u+u^{-1}) \mu_{x,j+1}}{q^{-1}+q} - \mu_t \right) P_\beta(x,y) = \lambda_{x,y} P_\beta(x,y).$$
Proof. First note that it follows from $\tilde{Y}_{t,u}^{(j)} = \Delta^{j-1}(\tilde{Y}_{t,u}) \otimes 1^{\otimes(N-j)}$ that $\rho_k(\tilde{Y}_{t,u}^{(j)})$ acts only on the variables $x_1, \ldots, x_j$. So we may fix $x_{j+1}, \ldots, x_N$, and consider only the action of $\rho_k(\Delta^{j-1}(\tilde{Y}_{t,u}))$ on appropriate functions in $x_1, \ldots, x_j$.

Using Lemma 13 we can write $\Delta^{j-1}(\tilde{Y}_{t,u})$ in terms of $\Delta^{j-1}(S)$ and $\Delta^{j-1}(T)$, where
\[
\Delta^{j-1}(S) = \Delta^{j-1}(K^{-2})\Delta^{j-1}(Y_{x_{j+1} + \mu x_{j+1}}) - \mu x_{j+1} \Delta^{j-1}(1),
\]
\[
\Delta^{j-1}(T) = \frac{1}{(q - q^{-1})} \left( \Delta^{j-1}(K^{-2}) \Delta^{j-1}(Y_{x_{j+1} + \mu x_{j+1}}) - \Delta^{j-1}(Y_{x_{j+1} + \mu x_{j+1}}) \Delta^{j-1}(K^{-2}) \right).
\]

Note that we use here that $S$ and $T$, which are defined in terms of $Y_s$, are actually independent of $s$. So we can conveniently replace $s$ by $x_{j+1}$. Now we use that $\rho_k(\Delta^{j-1}(Y_{x_{j+1} + \mu x_{j+1}}))$ is multiplication by $\mu x_{j+1}$, and $\rho_k(\Delta^{j-1}(K^{-2}))$ is given as an explicit difference operator in Proposition 5.10. Recall here that $K^{-2}(\cdot) = \Delta^{j-1}(K^{-2}) \otimes 1^{\otimes(N-j)}$. Then
\[
\rho_k(\Delta^{j-1}(S)) = \sum_{\nu \in \{-1,0,1\}} \mu q^{2\nu x_{j+1}} V^{(j)}(\nu) T^\nu - \mu x_{j+1} \text{Id},
\]
and
\[
\rho_k(\Delta^{j-1}(T)) = \frac{1}{q^{-1} - q} \sum_{\nu \in \{-1,0,1\}} (\mu q^{2\nu x_{j+1}} - \mu x_{j+1}) V^{(j)}(\nu) T^\nu.
\]

This gives the following expression for $\rho_k(\Delta^{j-1}(\tilde{Y}_{t,u}))$,
\[
\rho_k(\Delta^{j-1}(\tilde{Y}_{t,u})) = \sum_{\nu \in \{-1,0,1\}} V^{(j)}(\nu \beta) \text{Id} - \frac{(u + u^{-1}) \mu x_{j+1}}{q^{-1} + q} \text{Id},
\]
with
\[
V^{(j)}(\nu \beta) = V^{(j)}(\nu) \left( \frac{(u + u^{-1}) \mu q^{2\nu x_{j+1}}}{q^{-1} + q} + \frac{q u - q^{-1} u^{-1}}{(q^{-1} - q)(q^{-1} + q)} - \mu \right).
\]

To compare the difference equations for the multivariate Askey-Wilson polynomials in Theorem 5.5 with Iliev’s difference equation [13] Proposition 4.2], let us write the coefficients $V^{(j)}(\nu \beta)$ in terms of the parameters $\alpha_0, \ldots, \alpha_{N+1}$ defined by (0.1). We have
\[
V^{(j)}(\nu \beta)(x) = \frac{1}{q^{-1} - q} \prod_{i=0}^{j} V^{(j)}(\nu \beta_i)(x),
\]
where for $i = 0$,
\[
V^{(j)}(\nu \beta_0)(x) = \frac{-q a_0}{\alpha_1} \left( \frac{1 - \alpha_1 x_0^{\nu_1}}{\alpha_1} \right) \left( 1 - \alpha_1 q^{-2} x_1^{-\nu_1} \right), \quad \nu_1 \neq 0,
\]
and for $i = 1, \ldots, j$,
\[
V^{(j)}(\nu \beta_i)(x) = \frac{1 - \alpha_1 x_i^{\nu_1}}{\alpha_1} \left( 1 - \alpha_1 q^{-2} x_i^{-\nu_1} \right), \quad \nu_i \neq 0, \nu_{i+1} = 0,
\]
\[
q^2(q^{-1} + q) \left( \frac{a_0}{\alpha_{i+1}} + \frac{\alpha_{i+1}}{q \alpha_{i+1}} \right) q^2 (x_j + x_j^{-1})(x_{j+1} + x_{j+1}^{-1}) \left( 1 - q^2 x_j^2 \right) \left( 1 - q^2 x_j^{-2} \right), \quad \nu_i = 0, \nu_{i+1} = 0,
\]
\[
\frac{1 - \alpha_1 x_i^{\nu_1}}{\alpha_1} \left( 1 - \alpha_1 q^{-2} x_i^{-\nu_1} \right), \quad \nu_i \neq 0, \nu_{i+1} \neq 0,
\]
\[
q^2 \alpha_i \left( 1 - \alpha_1 x_i^{\nu_1} x_{i+1}^{\nu_1} \right) \left( 1 - \alpha_1 x_i^{\nu_1} x_{i+1}^{\nu_1} \right), \quad \nu_i = 0, \nu_{i+1} \neq 0,
\]
\[
q^2 \alpha_i \left( 1 - \alpha_1 x_i^{\nu_1} x_{i+1}^{\nu_1} \right) \left( 1 - \alpha_1 x_i^{\nu_1} x_{i+1}^{\nu_1} \right), \quad \nu_i = 0, \nu_{i+1} \neq 0.
with the assumption \( v_{j+1} = 0 \). With these expressions and Theorem 6.2 it is a straightforward calculation to show that the difference equations we obtained are equivalent to Iliev’s difference equations.

An explicit expression for the difference operators \( \tilde{\rho}(Y_{t,u}^{(j)}) \) is obtained in the same way as in Theorem 6.5. This gives explicit recurrence relations for the Askey-Wilson polynomials \( P_\beta(x,y) \). We just state the result here.

**Theorem 6.6.** For \( j = 1, \ldots, N \) \( \tilde{\rho}(Y_{t,u}^{(j)}) \) is the \( q \)-difference operator given by

\[
\tilde{\rho}(Y_{t,u}^{(j)}) = \sum_{\nu \in \{-1,0,1\}} \tilde{V}_{\nu,\beta}^{(j)}(y) \tilde{T}_\nu + \left( \frac{(u+u^{-1})\mu y_{N-1}}{q^{-1}+q} - \mu_s \right) \text{Id}
\]

where

\[
\tilde{V}_{\nu,\beta}^{(j)}(y) = -\tilde{V}_{\nu}^{(j)}(y) \left( \frac{(u+u^{-1})\mu_q^{-\nu N}y_N}{q^{-1}+q} + \frac{(qu^{-1}-q^{-1}u)(\mu_q^{-\nu N} y_N - \mu y_N)}{(q^{-1}-q)(q^{-1}+q)} - \mu_s \right),
\]

with \( \tilde{V}_{\nu}^{(j)}(y) \) given in Proposition 5.7.2. In particular, the multivariate Askey-Wilson polynomials \( P_\beta(x,y) \) satisfy

\[
\sum_{\nu \in \{-1,0,1\}} \tilde{V}_{\nu,\beta}^{(j)}(y) \tilde{T}_\nu P_\beta(x,\bullet)(y) + \left( \frac{(u+u^{-1})\mu y_{N-1}}{q^{-1}+q} - \mu_s \right) P_\beta(x,y) = \lambda_{x,y} P_\beta(x,y),
\]

for \( y \in S \).

Note that \( \tilde{V}_{\nu,\beta}^{(j)}(y) = V_{\nu,\beta}^{(j)}(y) \), where \( \tilde{\beta} = (t, s, u, k_1, \ldots, k_1, q^{-1}) \).

### 7. Appendix

#### 7.1. Convergence of the sum for \( P_\beta(x,y) \)

The function \( P_\beta(x,y) \) is defined in (III.1) by

\[
P_\beta(x,y) = \langle M_u \tilde{v}_y, v_{x,s} \rangle_H = \sum_{n=0}^{\infty} \omega(n)v_{x,s}(n)\tilde{v}_y(n)u^n,
\]

where \( y = tq^{-k-2m} \in S \) and \( x \in \mathbb{T} \). The eigenfunctions \( v_{x,s}(n) \) and \( \tilde{v}_y(n) \), see (III.10) and (III.13), are Al-Salam–Chihara polynomials in base \( q^2 \) and \( q^{-2} \), respectively. Using the expressions for these polynomials as \( q \)-hypergeometric functions we show here that the sum for \( P_\beta(x,y) \) converges.

We start with \( v_{x,s}(n) \). Applying transformation formulas [7] (III.2) and then [7] (III.31),(III.1) gives

\[
v_{x,s}(n) = \gamma_n x^n (q^k sx; q^2)^n 2^\varphi_1 \left( \frac{q^{-2n}, q^k sx}{q^2-2n-k sx}; q^2, \frac{sq^{-k}}{x} \right)_{\infty}
\]

\[
= \gamma_n x^n (q^k sx; q^2)^n (sq^k/x; q^2)^{\infty} 2^\varphi_1 \left( \frac{q^{2-k} sx, q^{-2k-2n}}{q^{-2n-k} sx}; q^2, \frac{sq^k}{x} \right)_{\infty}
\]

\[
= \gamma_n c(x)x^n 2^\varphi_1 \left( \frac{q^k sx, q^k x/s}{q^2x^2}; q^2, q^2+2n \right) + \text{idem}(x \leftrightarrow x^{-1})
\]

where

\[
\gamma_n = \frac{q^{-n(k-1)/2}}{(q^2; q^2)^n},
\]

\[
c(x) = \frac{(q^k s/x, q^k/sx; q^2)^\infty}{(1/x^2; q^2)^\infty}.
\]

So for \( n \to \infty \)

\[
v_{x,s}(n) \sim C q^{-n(k-1)/2} \left( c(x)x^n + c(x^{-1})x^{-n} \right), \quad x \in \mathbb{T},
\]

where \( C \) is a constant independent of \( n \).
Next we consider \( \tilde{v}_{q,t}(n) \). This function can be written as

\[
\tilde{v}_{q-t-2m,t}(n) = \gamma_n \tilde{c}_m 2\varphi_1 \left( \frac{q^{-2m}, q^{2m+2k}/t^2}{q^2/t^2}; q^2, q^{2n+2} \right),
\]

with

\[
\gamma_n = t^{-n} q^{(3-k)/2} \frac{(q^{2k}; q^n)_n}{(q^2; q^n)_n},
\]

\[
\tilde{c}_m = (-1)^m t^{2m} q^{-m(m+1)} \frac{(q^2/t^2; q^3)_m}{(q^{2k}; q^2)_m}.
\]

Then for \( n \to \infty \),

\[
(7.3) \quad \tilde{v}_{q-t-2m,t}(n) \sim C t^{-n} q^{(3-k)/2},
\]

where \( C \) is independent of \( n \).

The weight function \( \omega \) satisfies

\[
\omega(n) = q^{n(k-1)} \frac{(q^2; q^n)_n}{(q^{2k}; q^n)_n} = O(q^{n(k-1)}), \quad n \to \infty,
\]

Then from (7.2) and (7.3) it follows that the summand in (7.1) is of order \( O(q/t) \), so that the sum (7.1) converges absolutely (recall that \( |t| \geq q^{-1} \) and \( u \in T \)).

Furthermore, for \( x = sq^{k+2m}, m \in \mathbb{N} \), we have \( c(x) = 0 \), so that

\[
\upsilon_{x,s}(n) \sim C c(x^{-1}) x^{-n}, \quad n \to \infty.
\]

We see that in this case the sum (7.1) converges absolutely if \( |t| > q^{-k+2m} \).

### 7.2. Evaluation of \( P_\beta(x, t q^{-k}) \)

By inserting into (7.1) explicit expressions for \( \upsilon_{x,s}(n) \) and \( \tilde{v}_{q-t-2m,t}(n) \) in terms of \( q \)-hypergeometric functions and using \( x \leftrightarrow x^{-1} \) invariance, we obtain

\[
(7.4) \quad P_\beta(x, tq^{-k}) = \sum_{n=0}^{\infty} \left( \frac{q}{xt} \right)^n \frac{(q^k x/s; q^2)_n}{(q^2; q^2)_n} 2\varphi_1 \left( \frac{q^{2n}, q^k s/x}{q^{2-2n-kx/s}/x}; q^2, sq^{2-k} \right).
\]

We write the \( 2\varphi_1 \) function as a sum and interchange the order of summation

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} A_{n,m}.
\]

Then replace \( n-m = l \) and use the identity

\[
\frac{(q^{-2l-2m}, q^2)_m}{(sq^{k-2l-2m}/x; q^2)_m} = \frac{(q^k x/s; q^2)_l (q^2; q^2)_l}{(q^2; q^2)_l} \frac{(q^{-2n-kx/s}/x; q^2, sq^{2-k})}{(q^{2-2n-kx/s}/x; q^2, sq^{2-k})},
\]

to find

\[
P_\beta(x, tq^{-k}) = \sum_{m=0}^{\infty} \frac{(q^k s/x; q^2)_m}{(q^2; q^2)_m} \left( \frac{q^k x/s}{q^2; q^2} \right)_l \sum_{l=0}^{\infty} \frac{(q^k x/s; q^2)_l}{(q^2; q^2)_l} \left( \frac{q}{xt} \right)^l.
\]

Recall that \( s, u, x \in T \) and \( |t| \geq q^{-1} \), then we see that both sums converge. They can be evaluated using the \( q \)-binomial formula [4 (II.3)], leading to

\[
P_\beta(x, tq^{-k}) = \frac{(q^{k+1} u s^2 \pm 1/t; q^2)_\infty}{(q u x^2 \pm 1/t; q^2)_\infty}.
\]

### 7.3. Overview of various Hilbert spaces

In Sections 3 and 4 we use the following Hilbert spaces of univariate complex-valued functions:

- \( H = H_k \) is the representation space of \( \pi_k \). It is the Hilbert space of functions on \( \mathbb{N} \) with inner product

\[
(f, g)_H = \sum_{n \in \mathbb{N}} f(n) g(n) \omega(n),
\]

\[
\omega(n) = \omega_k(n) = q^{n(k-1)} \frac{(q^2; q^n)_n}{(q^{2k}; q^n)_n}.
\]
In Sections 5 and 6 we use the following Hilbert spaces of multivariate complex-valued functions:

- $\mathcal{H} = \mathcal{H}_{k,s}$ is the representation space of $\rho_{k,s}$. It is the Hilbert space of function on $\mathbb{T}$ which are invariant under $x \leftrightarrow x^{-1}$, with inner product

  $$(f, g)_{\mathcal{H}} = \frac{1}{4\pi i} \int_{\mathbb{T}} f(x) \overline{g(x)} w(x) \frac{dx}{x},$$

  $$w(x) = w_{k,s}(x) = \frac{(q^2, q^{2k}, x^{\pm 2}; q^2)_\infty}{(q^2, q^{2k+2}/t^2; q^2)_\infty} x^{-2m} q^{2m^2},$$

- $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{k,t}$ is the representation space of $\tilde{\rho}_{k,t}$. It is the Hilbert space of functions on $S = S_{k,t,q} = \{ t q^{-k-2m} | m \in \mathbb{N} \}$, with inner product

  $$(f, g)_{\tilde{\mathcal{H}}} = \sum_{y \in S} f(y) \overline{g(y)} \tilde{w}(y),$$

  $$\tilde{w}(y) = \tilde{w}_{k,t}(y) = \frac{1 - q^{4m+2k}/t^2 \sum_{j=1}^{n} (q^{2k}; q^2)_\infty}{1 - q^{2k}/t^2} \sum_{j=1}^{n} (q^{2k}; q^2)_\infty x^{-2m} q^{2m^2},$$

for $y = t q^{-k-2m}, m \in \mathbb{N}$.

In Sections 5 and 6 we use the following Hilbert spaces of multivariate complex-valued functions:

- $H = H_{\pi_k}$ is the representation space of $\pi_k$. It is the Hilbert space of functions on $\mathbb{N}^N$ with inner product

  $$(f, g)_H = \sum_{n \in \mathbb{N}^N} f(n) \overline{g(n)} \omega(n),$$

  $$\omega(n) = \omega_k(n) = \prod_{j=1}^{N} \omega_{k,j}(n_j) = \prod_{j=1}^{N} \frac{(q^2, q^{2k}; q^2)_\infty}{(q^{2k}; q^2)_\infty} q^{n_j(k_j-1)}.$$

- $\mathcal{H} = \mathcal{H}_{k,s}$ is the representation space of $\rho_{k,s}$. It is the Hilbert space of functions on $\mathbb{T}^N$ which are invariant under $x_j \leftrightarrow x_j^{-1}, j = 1, \ldots, N$, with inner product

  $$(f, g)_{\mathcal{H}} = \int_{\mathbb{T}^N} f(x) \overline{g(x)} w(x) \frac{dx}{x},$$

  $$w(x) = w_{k,s}(x) = \prod_{j=1}^{N} w_{k,j,x_{j+1}}(x_j) = \prod_{j=1}^{N} \frac{(q^2, q^{2k}; q^2)_\infty}{(q^{2k}; q^2)_\infty} x^{-2m} q^{2m^2},$$

- $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{k,t}$ is the representation space of $\tilde{\rho}_{k,t}$. It is the Hilbert space of functions on $S = S_{k,t,q} = \{ t q^{-\Sigma(k)-2\Sigma(m)} | m \in \mathbb{N}^N \}$, with inner product

  $$(f, g)_{\tilde{\mathcal{H}}} = \sum_{y \in S} f(y) \overline{g(y)} \tilde{w}(y),$$

  $$\tilde{w}(y) = \prod_{j=1}^{N} \tilde{w}_{k,j,y_{j-1}}(y_j) = \prod_{j=1}^{N} \frac{(q^{2K_j+4M_j-1+2m_j}/t^2)}{(q^{2K_j+4M_j-1}/t^2)} \frac{1 - q^{2M_j+2m_j}/t^2}{1 - q^{2K_j+4M_j-1}/t^2} \times \frac{(q^{2K_j+4M_j-1}/t^2, q^{2k_j}; q^2)_\infty}{(q^{2k_j}; q^2)_\infty} \sum_{j=1}^{N} m_j, K_j = \Sigma(k)_j = \sum_{i=1}^{j} k_i$$

  and $M_0 = 0 = K_0$.

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