ON THE MULTIPlicITIES OF ZEROS OF $\zeta(s)$ AND ITS VALUES OVER SHORT INTERVALS

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Abstract. We investigate bounds for the multiplicities $m(\beta + i\gamma)$, where $\beta + i\gamma$ $(\beta \geq \frac{1}{2}, \gamma > 0)$ denotes complex zeros of $\zeta(s)$. It is seen that the problem can be reduced to the estimation of the integrals of the zeta-function over “very short” intervals. A new, explicit bound for $m(\beta + i\gamma)$ is also derived, which is relevant when $\beta$ is close to unity. The related Karatsuba conjectures are also discussed.

1. Introduction

Let $r = m(\rho)$ denote the multiplicity of the complex zero $\rho = \beta + i\gamma$ of the Riemann zeta-function $\zeta(s)$. It is defined for $\Re s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

and otherwise by analytic continuation. This means that for some $r \in \mathbb{N}$

$$\zeta(\rho) = \zeta'(\rho) = \ldots = \zeta^{(r-1)}(\rho) = 0, \text{ but } \zeta^{(r)}(\rho) \neq 0.$$

All known zeros $\rho$ are simple (i.e., $m(\rho) = 1$), and it may well be that they are all simple, although the proof of this is certainly beyond reach at present. Besides this strongest possible conjecture, A.A. Karatsuba [17] mentions two somewhat weaker conjectures: $m(\rho) \ll 1$ (\forall $\rho$) and $m(\rho)$ is unbounded as $\gamma \to \infty$. He also says that the universality of $\zeta(s)$ (see S.M. Voronin [26]) should include the last conjecture, but that all these “are merely surmises”.

In estimating $m(\rho)$ one may suppose that $\frac{1}{2} \leq \beta < 1$ and that $\gamma > 0$, since $\zeta(s)$ does not vanish for $\Re s \geq 1$, and $1 - \rho$ and $\overline{\rho}$ are zeros of $\zeta(s)$ if $\rho$ is a zero. This follows from $\overline{\zeta(s)} = \zeta(\overline{s})$ and the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) := \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)} \pi^{s-1/2},$$

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where \( \Gamma(s) \) is the familiar gamma-function. For a comprehensive account on \( \zeta(s) \), the reader is referred to the monographs of E.C. Titchmarsh [23] and the author [13].

Several results on the multiplicities of the zeros of the zeta-function were obtained in the author’s paper [14]. In particular, at the end of the paper it was stated that “there is a possibility to bound \( m(\beta + i\gamma) \), provided one has a good lower bound of the form

\[
\int_{\delta}^{2\delta} |\zeta(\beta + i\gamma + i\alpha)|^k \, d\alpha \geq \ell = \ell(\gamma, \delta, k) \quad (0 < \delta < \frac{1}{4}, \beta \geq \frac{1}{2}, \gamma \geq \gamma_0 > 0)
\]

for \( k = 1, 2 \).” Thus the problem is reduced to the evaluation of the moments of \( \zeta(s) \) over “very short” intervals, namely integrals of the form

\[
\int_{\delta}^{2\delta} |\zeta(\beta + i\gamma + i\alpha)|^k \, d\alpha \quad (0 < \delta < \frac{1}{4}),
\]

where \( k \in \mathbb{N} \) is fixed. The interval of integration can be justly called “very short”, since one assumes that \( 0 < \delta < \frac{1}{4} \). One of the aims of this paper is to pursue further this approach and analyze its potential.

We note that zeta zeros with large multiplicities, statistically speaking, are rare. Namely A. Fujii [8] proved in 1975 that

\[
N_j(T) \leq C_1 N(T) e^{-C_2 \sqrt{T}} \quad (T \geq T_0 > 0),
\]

where \( N(T) \), as usual, denotes the number of complex zeros \( \rho \) of \( \zeta(s) \) for which \( 0 < 3\rho \leq T \) (multiplicities counted), while \( N_j(T) \) denotes those zeros counted by \( N(T) \) whose multiplicities are \( j \). Here \( j \geq 1 \) is not necessarily fixed, and \( C_1, C_2 \) are positive constants. A. Fujii [9] in 1981 improved the exponential in (1.3) to \( \exp(-C_2j) \), while M.A. Korolev [19] obtained much later in 2006 explicit numerical values for the constants \( C_1, C_2 \) for the latter bound. Note that we have the identity

\[
\sum_{j=1}^{\infty} N_j(T) = N(T).
\]

If \( j = m(\beta + i\gamma) \) with \( 0 < \gamma \leq T \), then by (2.11) one has \( j \ll \log \gamma \), hence it follows that the sum in (1.5) is finite.

It seems plausible that uniformly, for any given \( j \geq 2 \),

\[
N_j(T) = o(N(T)) \quad (T \to \infty),
\]
which implies that $N(T) \sim N_1(T)$, namely that all the zeros are simple. However, in general, (1.5) is not known yet. It follows from (1.3) if $j = j(T) \to \infty$ as $T \to \infty$. The bound in (1.3) suggests that $N_j(T)$ is a non-increasing function of $j$ for a fixed $T$, but this is not easy to prove. Note that the relation (1.5) certainly cannot hold for $j = 1$, since D.R. Heath-Brown [12] showed that $N_1(T) \gg N(T)$.

In Section 2 and Section 4 we shall deal with lower bounds of the form (1.1) and obtain in Theorem 2 a new lower bound. In Section 3 we shall consider the Karatsuba conjectures involving the quantity

$$F(T, \Delta) := \max_{t \in \left[T, T+\Delta]\right} |\zeta(\frac{1}{2} + it)| \quad (0 < \Delta \leq 1),$$

which is closely related to the integral in (1.1). Finally, in Section 5 we shall employ a complex integration technique to obtain an explicit upper bound for $m(\beta + i\gamma)$, which is relevant when $\beta$ is close to unity.

2. INTEGRALS OVER SHORT INTERVALS

The argument for the estimation of $m(\rho) = r$ that leads to (1.1) is as follows. For fixed $\beta$ such that $\beta \geq \frac{1}{2}$, let $\mathcal{D}$ be the rectangle with vertices

$$\frac{1}{4} - \beta \pm i\log^2 \gamma, \ 2 \pm i\log^2 \gamma, \ \zeta(\rho) = 0, \ \rho = \beta + i\gamma \ (\gamma \geq \gamma_0 > 0),$$

and let $\alpha$ be a parameter for which $0 < \alpha \leq 1$. Since $\rho$ is a zero of $\zeta(s)$ of multiplicity $r$, the function $\zeta(s + \rho) s^{-r}$ is regular at $s = 0$. By the residue theorem we obtain

$$\zeta(\beta + i\gamma + i\alpha) = \frac{1}{2\pi i} \int_{\mathcal{D}} \Gamma(s - i\alpha) \frac{\zeta(s + \rho)}{s^r} \, ds.$$  \hspace{1cm} (2.1)

Namely of the poles of the gamma-factor only $s = i\alpha$ is in $\mathcal{D}$, and it is a simple pole. The unique pole of $\zeta(s + \rho)$, namely $s = 1 - \rho$, lies outside $\mathcal{D}$. This gives, in view of the fast decay of the gamma-function (see e.g., (A.34) of [13]),

$$\zeta(\beta + i\gamma + i\alpha) \ll \alpha^r \left(\gamma(\beta - \frac{1}{4})^{-r} + 2^{-r}\right) \ll \alpha^r \gamma(\beta - \frac{1}{4})^{-r},$$  \hspace{1cm} (2.2)

and the case when $\gamma(\beta - \frac{1}{4})^{-r} \ll 2^{-r}$ is easy, since it implies that

$$\gamma \ll (\beta - \frac{1}{4})^r 2^{-r} \leq \left(\frac{3}{8}\right)^r,$$

and this is impossible if $r \geq r_0$. Hence either $2^{-r} \ll \gamma(\beta - \frac{1}{4})^{-r}$ or

$$r = m(\beta + i\gamma) \ll 1,$$
and this case is covered by the term $O(1)$ in (2.3). It is, of course, possible to insert in the integrand in (2.1) the factor $X^{s-1}$ ($X > 1$), and try to use convexity. This does not appear to give any substantial improvement. Consequently, if $\delta$ is a constant satisfying $0 < \delta < \frac{1}{8}$, then raising (2.2) to the power $k$ and integrating over $\alpha$ we have

$$\int_\delta^{2\delta} |\zeta(\beta + i\gamma + i\alpha)|^k d\alpha \ll \gamma^k (\beta - \frac{1}{\delta})^{-rk} \int_\delta^{2\delta} \alpha^r d\alpha \ll (8\delta)^rk \gamma^k.$$

Thus, recalling (1.1) and taking logarithms, we have

**Theorem 1.** If $\beta \geq \frac{1}{2}$, $\gamma > \gamma_0 > 0$, $k > 0$ and $0 < \delta < \frac{1}{8}$, then with the notation introduced above we have

$$m(\beta + i\gamma) = r \leq \frac{1}{\log \left(\frac{1}{8\delta}\right)} \left(\log \gamma - \frac{1}{k} \log \ell + O(1)\right) + O(1).$$

Therefore (2.3) shows that the upper bound for $m(\beta + i\gamma)$ can be made to depend on $\ell$ in (1.1), that is, on lower bounds for moments of $\zeta(s)$ over very short intervals. We would like to let $\delta \to 0+$ in (2.3) and obtain

$$m(\beta + i\gamma) = o(\log \gamma) \quad (\beta \geq \frac{1}{2}, \gamma \to \infty).$$

This relation is equivalent to

$$\lim_{\delta \to 0^+} \frac{\ell}{\log \left(\frac{1}{8\delta}\right)} = \lim_{\delta \to 0^+} \frac{\ell(\gamma, \delta, k)}{\log \left(\frac{1}{8\delta}\right)} = 0.$$

However, by using the argument on top of p. 219 of E.C. Titchmarsh [23] and the first inequality on p. 230, it follows that (1.1) holds with $\ell = \delta\gamma^{-A/\delta}$. By suitably elaborating the method it follows that even

$$\ell = \delta\gamma^{A\log \delta}$$

is permissible, for some absolute $A > 0$. These bounds, unfortunately, are too weak to yield (2.4). The bound in (2.5) can be compared to the case $\sigma = \frac{1}{2}$ of Theorem 2 in Section 4.

We remark that, on the Lindelöf Hypothesis (LH) that $\zeta(\frac{1}{2} + it) \ll_{\epsilon} |t|^\epsilon$, one has indeed (2.4). Note that $f(x) \ll_{\alpha, \beta, \ldots} g(x)$ (same as $f(x) = O_{\alpha, \beta, \ldots} \{g(x)\}$) means that the implied $\ll$-constant (resp. $O$-constant) depends on $\alpha, \beta, \ldots$. Also on the Riemann Hypothesis (RH, well-known that it implies the LH; see [23]) that $\rho = \frac{1}{2} + i\gamma$ ($\forall \rho$) one has

$$m(\beta + i\gamma) \ll \frac{\log \gamma}{\log \log \gamma}.$$
Furthermore, on the RH, H.L. Montgomery [21] proved that at least 2/3 of the zeta zeros are simple, namely $N_1(T) \geq \frac{2}{3} N(T)$ ($T \geq T_0$). His result was recently improved by H.M. Bui and D.R. Heath-Brown [2] (also on the RH), who obtained the constant $19/27 = 0.703$ in place of 2/3.

It transpires that the estimation of $m(\beta + i\gamma)$ is a very difficult problem, and one which is not satisfactorily solved even under the assumption of the LH or the RH. To see how one obtains (2.4) and (2.6) recall that for $N(T)$, the number of zeros $\beta + i\gamma$ for which $0 < \gamma \leq T$, one has the classical Riemann-von Mangoldt formula (see [13] or [23] for a proof)

$$N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O \left( \frac{1}{T} \right), \tag{2.7}$$

where $S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT)$, and the term $O(1/T)$ is a smooth function. Here $\arg \zeta(\frac{1}{2} + iT)$ is obtained by continuous variation along the segments joining the points $2, 2 + iT, \frac{1}{2} + iT$, starting with the value 0. If $T$ is the ordinate of a zero lying on the critical line, then $S(T) = S(T + 0)$. One has (see [23]) the bounds

$$S(T) \ll \log T, \quad S(T) = o(\log T) \quad (\text{LH}), \quad S(T) \ll \frac{\log T}{\log \log T} \quad (\text{RH}).$$

These bounds combined with (2.7) and the trivial inequality

$$m(\beta + i\gamma) \leq N(\gamma + H) - N(\gamma - H) \quad (0 < H \leq 1) \tag{2.8}$$

easily yield

$$m(\beta + i\gamma) \ll \log \gamma, \quad m(\beta + i\gamma) = o(\log \gamma) \quad (\text{LH}), \quad m(\beta + i\gamma) \ll \frac{\log \gamma}{\log \log \gamma} \quad (\text{RH}),$$

respectively. It seems, however, that these estimates are much too large, and that perhaps one even has

$$m(\beta + i\gamma) \ll \varepsilon (\log \log \gamma)^{1+\varepsilon}, \tag{2.9}$$

which is weaker than the previously stated conjectures, in particular that all zeros are simple. The direct use of pointwise estimates for $S(T)$ certainly cannot give anything close to (2.9), since one has

$$S(T) = \Omega_{\pm} \left( \left( \frac{\log T}{\log \log T} \right)^{1/3} \right), \quad S(T) = \Omega_{\pm} \left( \left( \frac{\log T}{\log \log T} \right)^{1/2} \right) \quad (\text{RH}). \tag{2.10}$$
This was proved proved by K.-M. Tsang [25] (his result is unconditional) and H.L. Montgomery [22], respectively. As usual, \( f(x) = \Omega_\pm(g(x)) \) means that the inequalities
\[
\limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0 \quad \text{and} \quad \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0
\]
both hold. One could use (2.8) with \( H = o(1) \) \((\gamma \to \infty)\) to try to improve the existing bound
\[
(2.11) \quad m(\beta + i\gamma) \ll \log \gamma \quad \left( \frac{1}{2} \leq \beta < 1 \right).
\]
In view of (2.7) this is equivalent to obtaining bounds for \( S(\gamma + H) - S(\gamma - H) \), but no satisfactory results seem to be known for this problem. Note that (2.11) easily follows from (2.7), (2.8) and \( S(T) \ll \log T \). In spite of all the efforts, this is still the best unconditional bound for the whole range \( \frac{1}{2} \leq \beta < 1 \). For an additional discussion concerning \( S(T) \), see Section 6.

3. The Karatsuba conjectures

A function closely related to the integral in (1.1) (when \( \beta = \frac{1}{2}, k = 1 \)) is
\[
(3.1) \quad F(T, \Delta) := \max_{t \in [T, T + \Delta]} |\zeta(\frac{1}{2} + it)| \quad (0 < \Delta \leq 1),
\]
where \( \Delta \) may depend on \( T \). Namely, for a fixed \( k > 0 \), one clearly has
\[
(3.2) \quad \int_0^{2\delta} |\zeta(\frac{1}{2} + i\gamma + i\alpha)|^k d\alpha = \int_0^{\delta} |\zeta(\frac{1}{2} + i\gamma + i\delta + ix)|^k dx \leq \delta F^k(\gamma + \delta, \delta).
\]
The quantity \( F(T, \Delta) \) was introduced and studied by A.A. Karatsuba [15], [16], [17]. He made the following conjectures.

**Conjecture 1.** There exists a positive function \( \Delta = \Delta(T) \to 0 \) as \( T \to \infty \) such that, for some constant \( A > 0 \),
\[
(3.3) \quad F(T, \Delta) \geq T^{-A}.
\]

**Conjecture 2.** Conjecture 1 is valid for \( \Delta = (\log \log T)^{-1} \).

**Conjecture 3.** Conjecture 1 is valid for \( \Delta = (\log T)^{-1} \).

These conjectures have not been proved unconditionally yet. Clearly Conjecture 3 implies Conjecture 2, which in turn implies Conjecture 1. M. Garaev [10] proved that the RH implies Conjecture 3, while Karatsuba himself showed unconditionally that
\[
(3.4) \quad F(T, \Delta) \geq e^{A \log \Delta \log T} \quad (0 < \Delta \leq 1/(\log T)).
\]
Shao-Ji Feng [6] proved that the LH implies Conjecture 1 with an arbitrary constant $A > 0$. Other relevant works on this subject include the papers of M.E. Changa [5], B. Kerr [18] and M.A. Korolev [20].

In view of (3.1) and (3.2) it is seen that the Karatsuba conjectures have their counterparts involving the integral in (3.2). For example, the conjecture

$$\int_{\delta}^{2\delta} |\zeta(\frac{1}{2} + iT + i\alpha)|^k d\alpha \gg T^{-A} \quad (\delta = \delta(T) \to 0)$$

is less stringent than Karatsuba’s Conjecture 1, and similarly for the other two conjectures.

We have

**Theorem 2.** If Conjecture 1 holds, then

$$m\left(\frac{1}{2} + i\gamma\right) = o(\log \gamma) \quad (\gamma \to \infty).$$

**Proof.** The assertion follows from (2.2) with $\beta = \frac{1}{2}, \alpha = \Delta = \Delta(\gamma)$. Namely Conjecture 1 gives

$$\gamma^{-A} \ll \Delta^r \gamma(\beta - \frac{1}{4})^{-r} = (4\Delta)^r \gamma,$$

which implies

$$\left(\frac{1}{4\Delta}\right)^r \ll \gamma^{A+1}.$$

Taking logarithms, we obtain

$$m\left(\frac{1}{2} + i\gamma\right) \log \frac{1}{4\Delta} = r \log \frac{1}{4\Delta} \leq C + (A + 1) \log \gamma \quad (\gamma \geq \gamma_0 > 0),$$

and the assertion readily follows, since

$$\lim_{\gamma \to \infty} \log \frac{1}{4\Delta} \to +\infty$$

by the assumption on $\Delta = \Delta(\gamma)$ in Conjecture 1. This shows again that the LH implies $m\left(\frac{1}{2} + i\gamma\right) = o(\log \gamma)$. A conditional result, similar to Theorem 2, is given by A.A. Karatsuba [17]. Naturally, Conjecture 2 and Conjecture 3, with explicit values of $\Delta = \Delta(T)$ would lead to sharper results on $m\left(\frac{1}{2} + i\gamma\right)$. Open questions are: does (3.5) imply the LH or Conjecture 1?
4. Integrals of $|\zeta(\sigma + it)|$ over very short intervals

We have the following result, which is more general than Karatsuba’s bound (3.4), but of the same strength. The method of proof is different from Karatsuba’s.

**Theorem 3.** For $k > 0, \frac{1}{2} \leq \sigma \leq 1, 0 < \delta \leq \frac{1}{2}, T \geq T_0 > 0$ and a suitable constant $C > 0$ we have

$$(4.1) \quad \int_{T-\delta}^{T+\delta} |\zeta(\sigma + it)|^k \, dt \geq 2\delta T^{-Ck \log(e/\delta)}.$$ 

**Proof.** We start from Th. 9.6 (B) of Titchmarsh’s book [23], namely from the classical formula

$$\log \zeta(s) = \sum_{|t-\gamma| \leq 1} \log(s - \rho) + O(\log t),$$

which is valid unconditionally for $-1 \leq \sigma < 2, s \neq \rho, -\pi < \Im \log(s - \rho) \leq \pi$, where $\rho$ denotes complex zeros of $\zeta(s)$. Since $\Re \log z = \log |z|$, then by taking real parts in this formula it follows that

$$\log |\zeta(s)| = \sum_{|t-\gamma| \leq 1} \log |s - \rho| + O(\log t) \quad (4.2)$$

To get rid of the logarithms one uses (this is a consequence of the arithmetic-geometric means inequality)

$$\log \left\{ \frac{1}{b-a} \int_a^b f(t) \, dt \right\} \geq \frac{1}{b-a} \int_a^b \log f(t) \, dt \quad (4.3)$$

for $a < b, f(t) \in L[a,b]$ and $f(t) > 0$ in $[a,b]$. Hence with

$$a = T - \delta, \quad b = T + \delta, \quad f(t) = |\zeta(\sigma + it)|^k,$$

(4.3) yields

$$\log \left\{ \frac{1}{2\delta} \int_{T-\delta}^{T+\delta} |\zeta(\sigma + i\alpha)|^k \, d\alpha \right\} \geq \frac{1}{2\delta} \int_{T-\delta}^{T+\delta} k \log |\zeta(\sigma + it)| \, dt. \quad (4.4)$$

Note that we have

$$\int_{T-\delta}^{T+\delta} \sum_{|t-\gamma| \leq 1} \log |t - \gamma| \, dt$$

$$= \int_{T-\delta}^{T+\delta} \sum_{|t-\gamma| \leq \delta} \log |t - \gamma| \, dt + \int_{T-\delta}^{T+\delta} \sum_{\delta < |t-\gamma| \leq 1} \log |t - \gamma| \, dt$$

$$= I_1 + I_2,$$
say. But, since \( \log |t - \gamma| \leq 0 \) for \( |t - \gamma| \leq 1 \) and

\[
[\max(T - \delta, \gamma - \delta), \min(T + \delta, \gamma + \delta)] \subseteq [\gamma - \delta, \gamma + \delta],
\]

we obtain

\[
I_1 = \sum_{T - 2\delta \leq \gamma \leq T + 2\delta} \int_{\max(T - \delta, \gamma - \delta)}^{\min(T + \delta, \gamma + \delta)} \log |t - \gamma| \, dt
\]

\[
\geq \sum_{T - 2\delta \leq \gamma \leq T + 2\delta} \int_{\gamma - \delta}^{\gamma + \delta} \log |t - \gamma| \, dt
\]

\[
= \sum_{T - 2\delta \leq \gamma \leq T + 2\delta} \int_{-\delta}^{\delta} \log |u| \, du
\]

\[
= 2(\delta \log \delta - \delta) \sum_{T - 2\delta \leq \gamma \leq T + 2\delta} 1
\]

\[
\geq -C\delta \log(e/\delta) \log T,
\]

since \( \delta \log \delta - \delta < 0 \) for \( 0 < \delta \leq 1 \). We also have, since \( S(T) \ll \log T \) and \( \log \delta < 0 \),

\[
I_2 = \int_{T - \delta}^{T + \delta} \sum_{\delta < |t - \gamma| \leq 1} \log |t - \gamma| \, dt \geq \log \delta \int_{T - \delta}^{T + \delta} \sum_{\delta < |t - \gamma| \leq 1} 1 \, dt \geq -C\delta \log(e/\delta) \log T,
\]

where \( C \) is a positive constant. Therefore from (4.2), (4.4) and the above bounds we obtain

\[
\log \left\{ \frac{1}{2\delta} \int_{T - \delta}^{T + \delta} |\zeta(\sigma + i\alpha)|^k \, d\alpha \right\} \geq -kC \log(e/\delta) \log T,
\]

which implies the lower bound in Theorem 3. This completes the proof. We remark that (4.1) in conjunction with (2.3) produces only the classical bound (2.11).

**Remark 1.** Note that Karatsuba’s function \( F(T, \Delta) \) (see (3.1)) can be connected to the integral of \( \log |\zeta(\frac{1}{2} + it)| \) over a very short interval. Namely, for \( 0 < \Delta \leq 1 \), using (4.3) we have

\[
F(T, \Delta) = \max_{0 \leq \alpha \leq \Delta} |\zeta(\frac{1}{2} + iT + i\alpha)|
\]

\[
\geq \frac{1}{\Delta} \int_{0}^{\Delta} |\zeta(\frac{1}{2} + iT + i\alpha)| \, d\alpha \geq \exp \left\{ \frac{1}{\Delta} \int_{0}^{\Delta} \log |\zeta(\frac{1}{2} + iT + i\alpha)| \, d\alpha \right\}.
\]

Putting \( T_0 = T + \frac{1}{2}\Delta, \delta = \frac{1}{2}\Delta \), it follows that

\[
F(T, \Delta) \geq \exp \left\{ \frac{1}{2\delta} \int_{T_0 - \delta}^{T_0 + \delta} \log |\zeta(\frac{1}{2} + it)| \, dt \right\} \quad (0 < \delta \leq \frac{1}{2}).
\]
The integral in (4.5) is precisely of the type that was dealt with in the proof of Theorem 3.

**Remark 2.** Note that if (4.1) is known to hold for \( k = 1 \), then one can easily deduce that it holds for \( k > 1 \) as well. Namely, by Hölder’s inequality for integrals we have, for \( k > 1 \),

\[
(4.6) \quad \int_{T-\delta}^{T+\delta} |\zeta(\sigma + it)| \, dt \leq \left( \int_{T-\delta}^{T+\delta} |\zeta(\sigma + it)|^k \, dt \right)^{1/k} (2\delta)^{1-1/k}.
\]

Therefore if

\[
\int_{T-\delta}^{T+\delta} |\zeta(\sigma + it)| \, dt \geq 2\delta T^{-C \log\left(\frac{\varepsilon}{\delta}\right)},
\]

one easily obtains (4.1) from (4.6).

5. A bound for multiplicities when \( \beta \) is close to unity

We finally present an explicit bound for \( m(\beta + i\gamma) \), which is relevant when \( \beta \) is close to unity. If such \( \beta \) exists, then the RH cannot hold. The result is

**Theorem 4.** Let \( 5/6 \leq \beta < 1 \). Then we have, for \( \gamma \geq \gamma_0(\varepsilon) \), a suitable constant \( C > 0 \) and any \( \varepsilon > 0 \),

\[
(5.1) \quad m(\beta + i\gamma) \leq C + \frac{13.35\beta}{3(1 - \beta) \log 6 + \beta \log 2} (1 - \beta)^{3/2} \log \gamma + \frac{13.35}{7(3 - 2\beta) + \varepsilon} \log \log \gamma.
\]

**Corollary 1.** For \( 5/6 \leq \beta < 1 \) and \( \gamma \geq \gamma_1 > 0 \), we have

\[
(5.2) \quad m(\beta + i\gamma) \leq 4 \log \log \gamma + 20(1 - \beta)^{3/2} \log \gamma.
\]

**Corollary 2.** If \( m(\beta + i\gamma) \geq 8 \log \log \gamma \) for \( 5/6 \leq \beta < 1 \) and \( \gamma \geq \gamma_2 > 0 \), then

\[
(5.3) \quad \beta \leq 1 - \left( \frac{m(\beta + i\gamma)}{40 \log \gamma} \right)^{2/3}.
\]

One obtains (5.2) and (5.3) by noting that

\[
\frac{13.35\beta}{3(1 - \beta) \log 6 + \beta \log 2} \leq \frac{13.35}{\log 2} = 19.25997\ldots
\]
and that \( m(\beta + \imath \gamma) \geq 8 \log \log \gamma \) implies \( 4 \log \log \gamma \leq \frac{1}{2} m(\beta + \imath \gamma) \). The bound \((5.3)\) says that, if the zero \( \beta + \imath \gamma \) has a large multiplicity, then \( \beta \) cannot be large.

**Proof of Theorem 4.** This result is a sharpening of Theorem 4 of [14], where one had the Vinogradov symbol \( \ll \) instead of explicit inequalities. Let \( \beta \geq 5/6 \), \( r = m(\beta + \imath \gamma) \) and \( E \) be the rectangle with vertices \(-2(1-\beta) \pm 2i \log^2 \gamma, 1 \pm 2i \log^2 \gamma\). If \( X (0 < X \ll \gamma^C) \) is a parameter which will be suitably chosen, then by the residue theorem we obtain

\[
(5.4) \quad \frac{\zeta(1-\beta+\imath \nu)}{(1-\beta)^r} = \frac{1}{2\pi i} \int_{E} X^{s-1+\beta} \Gamma(s - 1 + \beta) \frac{\zeta(s + \nu)}{s^r} \, ds \quad (\nu = \beta + \imath \gamma),
\]

which is similar to \((2.1)\). Namely \(-\beta < -2(1-\beta) < 1 - \beta\), while \( \Gamma(s - 1 + \beta) \) has simple poles at \( s = 1 - \beta, -\beta, -1 - \beta, \ldots \). For the gamma-function we shall use the estimate

\[
\Gamma(w) \ll \frac{e^{-|\text{Im} w|}}{|w|}.
\]

To bound the zeta-factor on the left side of \((5.4)\) we shall use the inequality

\[
(5.5) \quad |\zeta(\sigma + \imath t)| \leq A t^{B(1-\sigma)^{3/2}} \log^{2/3} t \quad (t \geq 3, \quad \frac{1}{2} \leq \sigma \leq 1),
\]

with the currently best known values \( A = 76.2, B = 4.45 \), due to K. Ford [7]. For our purposes it is the value of the constant \( B \) that is relevant. On the left side of \( E \) we have \( \Re(s + \nu) = 3\beta - 2 \geq 1/2 \), since \( \beta \geq 5/6 \) is assumed to hold.

We shall also use the bound

\[
\zeta(1 + \imath t) \gg (\log |t|)^{-2/3} (\log \log |t|)^{-1/3},
\]

which is a consequence of Lemma 12.3 of [13]. Like \((5.5)\), this bound is obtained by an elaboration of the classical method of Vinogradov–Korobov (see e.g., Chapter 6 of [13]) for the estimation of certain exponential sums. It follows then from \((5.4)\) that

\[
(5.6) \quad \frac{(1-\beta)^{-r}}{\log^{2/3} \gamma (\log \log \gamma)^{1/3}} \ll e^{-\log^2 \gamma} + X^\beta + 2^{-r}(1-\beta)^{-r} X^{-3(1-\beta)} \log \gamma \max_{|\nu| \leq \log^2 \gamma} |\zeta(3\beta - 2 + \imath \gamma + \imath \nu)|.
\]

Using \((5.5)\) in \((5.6)\) it follows that

\[
(5.7) \quad (1-\beta)^{-r} \ll L(\gamma) \left( X^\beta + 2^{-r}(1-\beta)^{-r} X^{-3(1-\beta)} \gamma^{3B(1-\beta)^{3/2}} \right),
\]

where for brevity we put

\[
L(\gamma) := (\log \gamma)^{7/3} (\log \log \gamma)^{1/3}.
\]
We multiply (5.7) by $2^r (1 - \beta)^r$ and use $1 - \beta \leq 1/6$ to deduce that
\begin{equation}
2^r \ll 3^{-r} X^\beta + X^{-3(1-\beta)} \gamma^{3B(1-\beta)^{3/2}} L(\gamma).
\end{equation}

Now we choose $X$ in (5.8) so that the two terms on the right-hand side are equal. Thus
\begin{equation}
X = 3^{r/(3-2\beta)} \gamma^{3B(1-\beta)^{3/2}/(3-2\beta)} (\ll \gamma^{C}).
\end{equation}

This gives
\begin{equation}
2^r \ll 3^{-r} 3^{\beta r/(3-2\beta)} \gamma^{3B(1-\beta)^{3/2}/(3-2\beta)} L(\gamma).
\end{equation}

We raise this to the power $3 - 2\beta$ and take logarithms to obtain
\begin{equation}
\begin{split}
& r(3-2\beta) \log 2 + r(3-2\beta) \log 3 - \beta r \log 3 \\
& \leq C_1 + 3B \beta (1 - \beta)^{3/2} \log \gamma + (3 - 2\beta) \log L(\gamma).
\end{split}
\end{equation}

Since the coefficient of $r$ on the left-hand side equals
\begin{equation}
3(1 - \beta) \log 6 + \beta \log 2,
\end{equation}

and
\begin{equation}
(3 - 2\beta) \log L(\gamma) \leq \frac{7(3 - 2\beta) + \varepsilon}{3} \log \log \gamma,
\end{equation}

we obtain the assertion (5.1) of Theorem 4 from (5.9).

6. SOME REMARKS CONCERNING $S(T)$

We conclude with some remarks concerning the function $S(T)$ and its effects on the estimation of $m(\beta + i\gamma)$. In the paper of Goldston–Gonek [11] it is proved, under the RH, that
\begin{equation}
|S(T + H) - S(T)| \leq \left( \frac{1}{2} + o(1) \right) \frac{\log T}{\log \log T} \quad (T \to \infty, 0 < H \leq \sqrt{T}).
\end{equation}

This implies, under the RH, in view of (2.7) and (2.8), the explicit upper bound
\begin{equation}
m(\beta + i\gamma) \leq \left( \frac{1}{2} + o(1) \right) \frac{\log \gamma}{\log \log \gamma} \quad (\frac{1}{2} \leq \beta < 1, \gamma \to \infty),
\end{equation}

on taking $H = 1/\log^2 \gamma$, say. It is known that, unconditionally (see E.C. Titchmarsh [23]) one has,
\begin{equation}
\int_0^T S(t) \, dt \ll \log T.
\end{equation}
From (6.3) it follows that every interval \([T, T + \log^2 T]\) contains a point \(t_0\) for which \(S(t_0) \leq 1\), and a point \(t_1\) for which \(S(t_1) \geq -1\). From this and (6.1) one obtains

\[
S(T) \leq \left( \frac{1}{2} + o(1) \right) \frac{\log T}{\log \log T} \quad \text{(RH, } T \to \infty). \tag{6.4}
\]

The constant one half in (6.4) (and thus also in (6.2)) was improved by Carneiro, Chandee and Milinovich [3] to \(1/4\), and the “\(o(1)\)” term is actually

\[
O\left( \frac{\log \log \log T}{\log \log T} \right).
\]

Generalizations of (6.4) to suitable \(L\)-functions were recently established in a paper by E. Carneiro and R. Finder [4].

A recent unconditional, explicit bound for \(S(T)\) is

\[
|S(T)| \leq 0.111 \log T + 0.275 \log \log T + 2.450,
\]

which is valid for \(T \geq e\). This is a recent result of T. Trudgian [24]. By (2.7) and (2.8) it immediately implies the unconditional bound

\[
m(\beta + i\gamma) \leq 2(0.111 \log \gamma + 0.275 \log \log \gamma + 2.450) \quad \left( \frac{1}{2} \leq \beta < 1, \gamma \geq 14 \right),
\]

which is an explicit version of (2.11).

The largest known values of \(S(T)\) (in absolute value) at present are, for \(T\) less than 29 trillion (\(\approx\) means approximately):

\[
S(T) \approx 3.0214, \quad T \approx 53.365784979; \quad S(T) \approx -3.2281, \quad T \approx 69.976605145.
\]

This was found by S. Wedeniwski [27] and his team in the larger context of searching for the zeros of \(\zeta(s)\) on the critical line. The first 100 billion zeros are simple and lie on the critical line. More extensive calculations are to be found in the forthcoming paper of J.W. Bober and G.A. Hioni [1]. This shows that the values of \(T\) needed for the \(\Omega\)-results in (2.10) to take effect must be extremely large.
REFERENCES

[1] J.W. Bober and G.A. Hiary, New computations of the Riemann zeta function on the critical line, to appear, preprint available at arXiv:1607.00709.
[2] H.M. Bui and D.R. Heath-Brown, On simple zeros of the Riemann zeta-function, Bull. Lond. Math. Soc. 45(2013), no. 5, 953-961.
[3] E. Carneiro, V. Chandee and M.B. Milinovich, Bounding $S(t)$ and $S_1(t)$ on the Riemann hypothesis, Math. Ann. 356(2013), 939-968.
[4] E. Carneiro and R. Finder, On the argument of $L$-functions, Bull. Braz. Math. Soc. (N.S.) 46(2015), no. 4, 601-620.
[5] M.E. Changa, Lower bounds for the Riemann zeta function on the critical line, Math. Notes 76(2004), 859-864.
[6] Shao-Ji Feng, On Karatsuba conjecture and the Lindelöf hypothesis, Acta Arithmetica 114(2004), 295-300.
[7] K. Ford, Vinogradov’s integral and bounds for the Riemann zeta function, Proc. Lond. Math. Soc. (3)85(2002), 565-633.
[8] A. Fujii, On the distribution of the zeros of the Riemann zeta-function in short intervals, Bull. Amer. Math. Soc. 81(1975), 139-142.
[9] A. Fujii, On the zeros of Dirichlet $L$-functions. II. (With corrections to “On the zeros of Dirichlet $L$-functions. I” and the subsequent papers), Trans. Amer. Math. Soc. 267(1981), 33-40.
[10] M.Z. Garaev, Concerning the Karatsuba conjectures, Taiwanese J. Math. 6(2002), 573-580.
[11] D.A. Goldston and S.M. Gonek, A note on $S(t)$ and the zeros of the Riemann zeta-function, Bull. Lond. Math. Soc. 39(3)(2007), 482-486.
[12] D.R. Heath-Brown, Simple zeros of the Riemann zeta-function, Bull. London Math. Soc. 11(1979), 17-18.
[13] A. Ivić, The Riemann zeta-function, John Wiley & Sons, New York 1985 (reissue, Dover, Mineola, New York, 2003).
[14] A. Ivić, On the multiplicity of zeros of the zeta-function, Bulletin CXVIII de l’Académie Serbe des Sciences et des Arts - 1999, Classe des Sciences mathématiques et naturelles, Sciences mathématiques No. 24, pp. 119-131.
[15] A.A. Karatsuba, On lower estimates of the Riemann zeta-function, Dokl. Akad. Nauk 376 (2001), 15-16.
[16] A.A. Karatsuba, Lower bounds for the maximum modulus of $\zeta(s)$ in small domains of the critical strip, Math. Notes 70(2001), 724-726.
[17] A.A. Karatsuba, Zero multiplicity and lower bound estimates of $|\zeta(s)|$, Funct. Approx. Comment. Math. 35(2006), 195-207.
[18] B. Kerr, Lower bounds for the Riemann zeta function on short intervals of the critical line, Archiv Math. 105(2015), 45-53.
[19] M.A. Korolev, On multiple zeros of the Riemann zeta-function, Izv. Math. 70(2006), 427-446; translation from Izv. Ross. Akad. Nauk, Ser. Mat. 70(2006), 3-22.
M.A. Korolev, On large values of the Riemann zeta-function on short segments of the critical line, Acta Arith. 166(2014), 349-390.

H.L. Montgomery, The pair correlation of zeros of the zeta-function, Proc. Symp. Pure Math. 24, AMS, Providence R.I., 1973, 181-193.

H.L. Montgomery, Extreme values of the Riemann zeta-function, Comment. Math. Helv. 52(1977), 511-518.

E.C. Titchmarsh, The theory of the Riemann zeta-function (2nd edition), Oxford University Press, Oxford, 1986.

T. Trudgian, An improved upper bound for the argument of the Riemann zeta-function on the critical line II, J. Number Theory 134(2014), 280-292.

K.-M. Tsang, Some Ω–theorems for the Riemann zeta-function, Acta Arithmetica 46(1986), 369-395.

S.M. Voronin, A theorem on the “universality” of the Riemann zeta-function, Izv. Akad. Nauk SSSR Ser. Mat. 39(1975), no. 3, 475-486.

S. Wedeniwski, Results connected with the first 100 billion zeros of the Riemann zeta function, 2002, at http://piologie.net/math/zeta.result.100billion.zeros.html

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