SUPERORIZATION OF HOMOGENEOUS SPIN MANIFOLDS
AND GEOMETRY OF HOMOGENEOUS SUPERMANIFOLDS

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Abstract: Let $M_0 = G_0/H$ be a (pseudo)-Riemannian homogeneous spin manifold, with reductive decomposition $g_0 = \mathfrak{h} + \mathfrak{m}$ and let $S(M_0)$ be the spin bundle defined by the spin representation $\tilde{\text{Ad}} : H \to \text{GL}_\mathbb{R}(S)$ of the stabilizer $H$. This article studies the superizations of $M_0$, i.e. its extensions to a homogeneous supermanifold $M = G/H$ whose sheaf of superfunctions is isomorphic to $\Lambda(S^*(M_0))$. Here $G$ is the Lie supergroup associated with a certain extension of the Lie algebra of symmetry $g_0$ to an algebra of supersymmetry $\mathfrak{g} = \mathfrak{g}_\mathfrak{r} + \mathfrak{g}_\mathfrak{t} = g_0 + S$ via the Kostant-Koszul construction. Each algebra of supersymmetry naturally determines a flat connection $\nabla^S$ in the spin bundle $S(M_0)$. Killing vectors together with generalized Killing spinors (i.e. $\nabla^S$-parallel spinors) are interpreted as the values of appropriate geometric symmetries of $M$, namely even and odd Killing fields. An explicit formula for the Killing representation of the algebra of supersymmetry is obtained, generalizing some results of Koszul. The generalized spin connection $\nabla^S$ defines a superconnection on $M$, via the super-version of a theorem of Wang.

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Introduction

Lie superalgebras have played an important role in modern physics since the idea of supersymmetry arose. Complex and real simple Lie superalgebras were classified by Kac ([28, 37]). This classification was used to describe algebras of supersymmetry, i.e. extensions of a Lie algebra of symmetry $\mathfrak{g}_0$ to a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$. In this spirit, the classification of extensions of Poincare’ Lie algebras, in all signatures and dimensions, was achieved in [1, 2]. Lorentzian symmetric spin manifolds $M_0 = G_0/H$ appear in constructions of maximally supersymmetric solutions of 11-dimensional supergravity ([9, 14, 15, 16, 23]). In this context, some special, physically relevant algebras of supersymmetry $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ have been considered: the action of the even part $\mathfrak{g}_0 = \mathfrak{h} + \mathfrak{m}$ on the odd part $\mathfrak{g}_1 = S$ is a spin representation, i.e. an extension $\text{ad} |_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{gl}_\mathbb{R}(S)$ of the spin representation $\text{ad} |_{S} : \mathfrak{g}_0 \to \mathfrak{gl}_\mathbb{R}(S)$ of the stability subalgebra $\mathfrak{h}$. In this paper, every such algebra of supersymmetry is called adapted to the spin manifold $(M_0, g, S(M_0))$, where $S(M_0)$ is the spin bundle defined by the spin representation $\text{Ad} : H \to \text{Gl}_{\mathbb{R}}(S)$ of the stabilizer $H$. We remark that notions of generalized Killing spinors appear naturally in the context of supergravity theories ([13]).

"Even" adapted algebras of supersymmetry are used in [10] and [17]. [10] obtains a unified description of homogeneous quaternionic Kähler manifolds of solvable group by means of extended Poincare’ Lie algebras ([1, 2]). This construction has a natural mirror in the setting of supergeometry and leads to the construction of homogeneous quaternionic Kähler supermanifolds, which is also described in [10]. [17] constructs compact real forms of the exceptional Lie algebras $F_4$ and $E_8$ in terms of even adapted algebras of supersymmetry of the spheres $S^8$ and $S^{15}$.

This paper deals with a geometric representation of the algebra of supersymmetry as an algebra of Killing fields on a supermanifold. Recall that the spin bundle $S(M_0)$ of a (pseudo)-Riemannian spin manifold $(M_0, g, S(M_0))$ canonically defines a supermanifold $M = (M_0, \mathcal{A}_M)$ whose sheaf of superfunctions $\mathcal{A}_M$ is isomorphic to the sheaf of sections of the exterior algebra $\Lambda(S^*(M_0))$ of $S^*(M_0)$. Supermanifolds of this type have been studied in [3, 24, 25]. If $M_0 = G_0/H$ is a homogeneous reductive (pseudo)-Riemannian spin manifold then every adapted algebra of supersymmetry $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 = \mathfrak{g}_0 + S$ defines a structure of homogeneous supermanifold $G/H$ on $M$. Here $G$ is the Lie supergroup associated with the super Harish-Chandra pair $(G_0, \mathfrak{g})$ via the Kostant-Koszul construction ([30, 31]). The associated Killing representation of the algebra of supersymmetry

$$\hat{\phi} : \mathfrak{g} \to \text{Der}_\mathbb{R}(\Lambda(S^*(M_0)))$$

(0.1)
recognizes Killing vectors and generalized Killing spinors as "values" of even and odd Killing fields. An explicit description of the representation (0.1) is obtained. Moreover, we give the super-version of the classical theory of invariant connections on a homogeneous manifold and we show that the generalized spin connection $\nabla^S$ defines a $G$-invariant superconnection on $M = G/H$. This could be of interest for future research along the lines of [3, 24, 25].

The paper is structured as follows.

The first section recalls the basic notions of supergeometry; in particular even and odd evaluations
\[ \text{ev}_0 : T_M(U) \to \Gamma(U, TM_0) \quad \text{and} \quad \text{ev}_1 : T_M(U) \to \Gamma(U, (TM)_1) \]
are defined, where $T_M$ and $TM = (TM)_0 + (TM)_1$ are respectively the tangent sheaf and the tangent bundle of a supermanifold $M = (M_0, \mathcal{A}_M)$. The even part $(TM)_0$ of the tangent bundle is naturally isomorphic to the tangent bundle $TM_0$ of the underlying smooth manifold $M_0$ while the odd part $(TM)_1$, in the case $\mathcal{A}_M \cong \Lambda(S^*(M_0))$, is the spin bundle $S(M_0)$.

The second section describes the Lie-Kostant-Koszul theory of Lie supergroups. A supermanifold $G = (G_0, \mathcal{A}_G)$ is a Lie supergroup if it is a group object in the category of supermanifolds. In the seminal paper [30], the notion of super Harish-Chandra pair (shortly sHC pair) is introduced and proved to be equivalent to the notion of Lie supergroup. A sHC pair is a pair $(G_0, g)$, where $G_0$ is a Lie group and $g = g_0 + g_1$ a Lie superalgebra satisfying some consistency conditions. While this notion is probably the most efficient to prove theorems in the category of Lie supergroups and homogeneous supermanifolds, it has the disadvantage of obscuring the geometric meaning of the sheaf of superfunctions $\mathcal{A}_G$ of the associated Lie supergroup $G = (G_0, \mathcal{A}_G)$. [6, 31] describe how to reconstruct the structure sheaf $\mathcal{A}_G$ together with its Hopf superalgebra structure; in particular [31] shows the existence of canonical isomorphisms
\[ \mathcal{A}(G) \cong \text{Hom}_{U(g_0)}(U(g), C^\infty(G_0)) \cong C^\infty(G_0) \otimes \Lambda(g_1^*) \]
and explicitly describes the representation of the Lie superalgebra $g = g_0 + g_1$ by left-invariant vector fields (see Proposition 2.15 and Theorem 2.29). Theorem 2.29 has been proved in [12] with the aid of coalgebra theory. Using this approach, we obtain an analogous description for the representation
\[ \phi : g \to \text{Der}_R(C^\infty(G_0) \otimes \Lambda(g_1^*)) \]
by right-invariant vector fields (see Proposition 2.14 and Theorem 2.30).

Section 3 recalls the basic definitions of action of a Lie supergroup and homogeneous supermanifold and it is an introduction to section 4 where the natural generalization to the category of supermanifolds of a classical theorem of Wang ([26, 27]) is obtained. Wang’s theorem establishes a natural bijective correspondence between the set $\text{Conn}(M_0)_G$ of $G_0$-invariant linear connections on a homogeneous manifold $M_0 = G_0/H_0$ with reductive decomposition $g_0 = h_0 + m_0$ and the set
Hom$_R(\mathfrak{m}, \mathfrak{gl}_R(\mathfrak{m}))^{H_0}$ of Nomizu maps. The analogous result in the super-setting is the following.

**Theorem 0.1.** Let $M = G/H = (G_0/H_0, A_{G/H})$ be a homogeneous supermanifold with reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. There is a bijective correspondence $\text{Conn}(M)^G \cong \text{Hom}_R(\mathfrak{m}, \mathfrak{gl}_R(\mathfrak{m}))^H$ between the set $\text{Conn}(M)$ of $G$-invariant linear connections on $M$ and the set $\text{Hom}_R(\mathfrak{m}, \mathfrak{gl}_R(\mathfrak{m}))^H$ of Nomizu maps.

This result has already been stated in the literature in the case of even stability subgroup ([10]), but, to our knowledge, a proof is missing. The dimension of the space of invariant linear connections on a Poincare’ superspacetime in signature $(r, s)$ is determined.

**Theorem 0.2.** The dimension $D$ of the vector space of invariant connections on a Poincare’ superspacetime of signature $(r, s)$ depends on $r - s \mod 8$ as follows

$$
D \begin{array}{cccccccc}
12 & 24 & 12 & 24 & 12 & 6 & 3 & 6 \\
\end{array}
$$

Section 5 deals with a general method to climb from the level of classical geometry to the level of supergeometry. This procedure is called superization, which is the inverse of evaluation. To every reductive homogeneous pseudo-Riemannian spin manifold $M_0 = G_0/H$, together with an adapted algebra of supersymmetry $\mathfrak{g} = \mathfrak{g}_\mathfrak{g} + \mathfrak{g}_\mathfrak{T} = \mathfrak{g}_0 + S$ (0.2) superization associates a homogeneous supermanifold $M = G/H$ whose sheaf of superfunctions $\mathcal{A}_M$ is isomorphic to $\Lambda(S^*(M_0))$. Each algebra (0.2) determines a flat connection $\nabla^S$ in the spin bundle and generalized Killing spinors are defined as $\nabla^S$-parallel spinors. The Lie superalgebra (0.2) admits a geometric representation (0.1) whose underlying classical geometry is described by Killing vector/spinor maps

$$
\text{ev} \circ \hat{\varphi} : \mathfrak{g}_\mathfrak{T} \to T(M_0), \quad x \mapsto \hat{\varphi}_0(x) \quad (0.3)
$$

$$
\text{ev} \circ \hat{\varphi} : \mathfrak{g}_\mathfrak{T} \to S(M_0), \quad s \mapsto \psi^s \quad (0.4)
$$

We remark that (0.4) does not depend on the odd-odd bracket $[S, S]$ but its lift to the anti-homomorphism (0.1) does. For example, for every $s \in S$, the supervector field $\hat{\varphi}(s)$ coincides with the algebraic interior product $\iota_{\hat{\varphi}(s)} \in \text{Der}(\Lambda(S^*(M_0)))$ only in the odd-commutative case, i.e. when $[S, S] = 0$. On the other hand, the action of $\hat{\varphi}(s)$ is given by the canonical connection $\nabla^\text{can}_{\hat{\varphi}(s)} \in \text{End}(S^*(M_0))$ of the reductive homogeneous manifold $M_0 = G_0/H$ considered as a derivation of $\Lambda(S^*(M_0))$. Each adapted algebra of supersymmetry (0.2) also determines an embedding

$$
T(M_0) \oplus S(M_0) \hookrightarrow \text{Der}_R(\Lambda(S^*(M_0))) \quad (0.5)
$$

$$
X + \psi \mapsto X + \Psi
$$

such that $\text{ev}_T + \text{ev}_\mathfrak{T}$ composed with (0.5) is the identity. The action of $T(M_0)$ is given by the flat spin connection $\nabla^S$ considered as derivation of $\Lambda(S^*(M_0))$ and $\Psi = \iota_{\psi^s}$ if and only if odd-commutativity holds. The flat spin connection $\nabla^S$ can be interpreted, via the super-version of Wang’s theorem, as a $G$-invariant superconnection on the homogeneous supermanifold $G/H$ whose associated Nomizu map is given by (4.12). We check that our setting is compatible with the construction
of the Killing superalgebra in 11-dimensional bosonic supergravity, whose recipe is explained in [13, 14, 16, 39]. Our construction gives an interpretation of the generalized Killing spinors as values of geometric "infinitesimal odd symmetries" as hinted in [39]. The study of adapted algebras of supersymmetry (with possibly as odd part a b-submodule of S) could become the primary object of interest in the search of supergravity solutions with many Killing spinors. Indeed each algebra of supersymmetry encodes the relevant object of the theory: the generalized spin connection $\nabla^S$.

**Notation.** We mainly deal with $\mathbb{Z}_2$-graded real vector spaces. We refer to them by "graded vector spaces" or "supervector spaces". On the other hand, a graded algebra is called a "superalgebra" only if supercommutativity holds. The symbol of exterior algebra is denoted by $\Lambda^x$. The dimension of a supervector space $V$ is defined as the pair $\dim V := \dim V^e \big| \dim V^o$. Let $V = V^e + V^o$, $W = W^e + W^o$ be two supervector spaces.

**Definition 1.1.** A non-zero element $x \in V$ of a supervector space $V = V^e + V^o$ is **homogeneous** of parity $|x| \in \mathbb{Z}_2 = \{0, 1\}$ if $x \in V^{|x|}$. The dimension of $V$ is defined as the pair $\dim V := \dim V^e \big| \dim V^o$. The following decomposition holds

$$\text{Hom} \mathbb{R}(V, W) = \text{Hom} \mathbb{R}(V^e, W^e) \oplus \text{Hom} \mathbb{R}(V^o, W^o) .$$

**Definition 1.3.** The **tensor product** $V \otimes W$ is a supervector space with gradation $(V \otimes W)^e = (V^e \otimes W^e) \oplus (V^o \otimes W^o)$, $(V \otimes W)^o = (V^e \otimes W^o) \oplus (V^o \otimes W^e)$.

**Definition 1.4.** A supervector space $A = A^e + A^o$ is a **(graded) algebra** if there exist **multiplication** $m_A \in \text{Hom} \mathbb{R}(A \otimes A, A)$ and **unity** $1_A \in \text{Hom} \mathbb{R}([\mathbb{R}, A])$ such that $(A, m_A, 1_A)$ is an algebra. **Left and right multiplication** by $a \in A$ are given by

$$L_a : A \to A \quad , \quad R_a : A \to A$$

$$b \mapsto ab \quad , \quad b \mapsto (-1)^{|a||b|} ba .$$

The **supercommutator** is the even morphism $[\cdot, \cdot] \in \text{Hom} \mathbb{R}(A \otimes A, A)$ defined by

$$[a, b] := L_a b - R_a b = ab - (-1)^{|a||b|} ba .$$

$(A, m_A, 1_A)$ is a supercommutative superalgebra (shortly a **superalgebra**) if $[\cdot, \cdot] \equiv 0$.

**Example 1.5.** The exterior algebra $\Lambda^x \mathbb{R}V = \Lambda^x_{\mathbb{R}}^{even} V + \Lambda^x_{\mathbb{R}}^{odd} V$ over an even vector space $V = V^e$ is a superalgebra.
Definition 1.6. A Lie superbracket on a supervector space $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ is an even morphism $\{,\} \in \text{Hom}_R(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ satisfying
\begin{align}
[x, y] &= -(-1)^{|x||y|}y[x, x] \\
[x, [y, z]] &= [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]
\end{align}
for all $x, y, z \in \mathfrak{g}$. The pair $(\mathfrak{g}, [\cdot, \cdot])$ is called a Lie superalgebra.

The Lie superalgebra of endomorphisms $\text{End}_R(V)$ of $V$ is denoted by $\mathfrak{gl}_R(V)$. The definitions of morphism of zero parity (morphism in short) and derivation of Lie superalgebras are the natural ones.

Definition 1.7. Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ be a Lie superalgebra. A representation $\varphi$ of $\mathfrak{g}$ on a supervector space $V = V_0 + V_1$ is a morphism of Lie superalgebras $\varphi : \mathfrak{g} \to \mathfrak{gl}_R(V)$.

An anti-homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}_R(V)$ is also called a "representation".

Definition 1.8 ([38]). A supervector space $C = C_0 + C_1$ is a (graded) coalgebra if there exist comultiplication $\Delta_C \in \text{Hom}_R(C, C \otimes C)$ and counity $\epsilon_C \in \text{Hom}_R(C, R)$ satisfying
\begin{align}
\text{Coassociativity} & \quad (\Delta_C \otimes \text{Id}_C) \circ \Delta_C = (\text{Id}_C \otimes \Delta_C) \circ \Delta_C , \\
\text{Counity} & \quad (\epsilon_C \otimes \text{Id}_C) \circ \Delta_C = (\text{Id}_C \otimes \epsilon_C) \circ \Delta_C = \text{Id}_C .
\end{align}
A coalgebra is denoted by the triple $(C, \Delta_C, \epsilon_C)$.

Sometimes the so-called sigma-notation of [38] is used:
\[ \Delta_C(c) := \sum_i c_{(1)}^i \otimes c_{(2)}^i \]
for every $c \in C$. Symbolically we write
\[ \Delta_C(c) = c_{(1)} \otimes c_{(2)} , \quad (\Delta_C \otimes \text{Id}_C) \circ \Delta_C(c) := c_{(1)} \otimes c_{(2)} \otimes c_{(3)} , \cdots \]
A coalgebra $(C, \Delta_C, \epsilon_C)$ is cocommutative if $c_{(1)} \otimes c_{(2)} = (-1)^{|c_{(1)}||c_{(2)}|}c_{(2)} \otimes c_{(1)}$.

Definition 1.9 ([36]). A coderivation of a coalgebra $(C, \Delta_C, \epsilon_C)$ is an endomorphism $\Phi \in \text{End}_R(C, C)$ such that $\Delta_C \circ \Phi = (\Phi \otimes \text{Id} + \text{Id} \otimes \Phi) \circ \Delta_C$. The space of coderivations is denoted by $\text{CoDer}(C) \ni \Phi$.

Let $(A, m_A, 1_A)$ be a graded algebra and $(C, \Delta_C, \epsilon_C)$ a graded coalgebra.

Definition 1.10 ([38]). The space $\text{Hom}_R(C, A)$ is a graded algebra, over the graded algebra $(C^*, \Delta_C^*, \epsilon_C)$ dual to $C$, with respect to the convolution product
\[ F \ast G := m_A \circ (F \otimes G) \circ \Delta_C \quad F, G \in \text{Hom}_R(C, A) .\]
The triple $(\text{Hom}_R(C, A), \ast, \epsilon_C)$ is the convolution algebra produced from $A$ and $C$.

The importance of coderivations lies in the following fact. The transpose action of a coderivation $\Phi \in \text{CoDer}(C)$ is a derivation of the algebra $\text{Hom}_R(C, A, \ast, \epsilon_C)$. Alternative descriptions of $\text{CoDer}(C)$ are important; with some extra assumptions this space can be identified with another one, called space of formal vector fields (this is the content of Theorem 1.15).

Definition 1.11 ([38]). A (graded) bialgebra $(B, m_B, 1_B, \Delta_B, \epsilon_B)$ is a supervector space $B = B_0 + B_1$ which has the structure of a graded algebra $(B, m_B, 1_B)$ and of a graded coalgebra $(B, \Delta_B, \epsilon_B)$ such that $\Delta_B$ and $\epsilon_B$ are morphisms of algebras.
Definition 1.12 ([35]). The space \( \mathcal{P}(B) \) of primitive elements of a graded bialgebra \((B, m_B, 1_B, \Delta_B, \epsilon_B)\) is defined by \( \mathcal{P}(B) := \{ b \in B | \Delta_B(b) = 1_B \otimes b + b \otimes 1_B \} \). The space \( \text{Hom}_R(B, \mathcal{P}(B)) \) is called the space of formal vector fields of \( \mathcal{P}(B) \).

Definition 1.13 ([38]). An antipode of a graded bialgebra \((H, m_H, 1_H, \Delta_H, \epsilon_H)\) is a convolution inverse \( \delta_H \in \text{Hom}_R(H, H) \) of the identity \( \text{Id}_H \). A graded bialgebra with antipode \((H, m_H, 1_H, \Delta_H, \epsilon_H, \delta_H)\) is called a Hopf (graded) algebra.

Example 1.14 ([37, 38]). The universal enveloping algebra of a Lie superalgebra

\[
\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g}) / \langle x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle
\]

has a natural structure of a cocommutative Hopf graded algebra. In particular, the symmetric algebra \( S(\mathfrak{g}) \) is a cocommutative Hopf superalgebra. The unique extension to \( \mathcal{U}(\mathfrak{g}) \) of a derivation \( D \in \text{Der}_R(\mathfrak{g}, \mathfrak{g}) \) of \( \mathfrak{g} = \mathfrak{g}_- + \mathfrak{g}_+ \) is a coderivation.

Notation: the action of the antipode \( \delta_{\mathcal{U}(\mathfrak{g})} \) on \( u \in \mathcal{U}(\mathfrak{g}) \) is denoted by \( u \mapsto \pi \).

The space \( \text{CoDer}(H) \) of coderivations of a cocommutative Hopf graded algebra \((H, m_H, 1_H, \Delta_H, \epsilon_H, \delta_H)\) is identified with the space of formal vector fields. Indeed

Theorem 1.15 ([36]). The maps

\[
\text{Hom}_R(H, \mathcal{P}(H)) \rightarrow \text{CoDer}(H)
\]

\[
\phi \mapsto c_\phi := \text{Id}_H \ast \phi \quad (1.3)
\]

\[
\phi \mapsto \phi c := \phi \ast \text{Id}_H \quad (1.4)
\]

are bijections satisfying \( \delta_H \ast c_\phi = \phi c \ast \delta_H = \phi \). Moreover \((1.3) = (1.4)\) whenever \((H, m_H, 1_H)\) is a superalgebra.

Let \((\mathfrak{g}, [\cdot, \cdot])\) be a Lie superalgebra and \((C, \Delta_C, \epsilon_C)\) a graded cocommutative coalgebra.

Definition 1.16 ([35]). The space \( \text{Hom}_R(C, \mathfrak{g}) \) is a Lie superalgebra, over the graded algebra \((C^*, \Delta_C^*, \epsilon_C)\) dual to \( C \), with respect to the convolution bracket

\[
[F, G] := [\cdot, \cdot] \circ (F \otimes G) \circ \Delta_C \quad F, G \in \text{Hom}_R(C, \mathfrak{g})
\]

The pair \((\text{Hom}_R(C, \mathfrak{g}), [\cdot, \cdot])\) is the convolution Lie superalgebra produced from \( \mathfrak{g} \) and \( C \).

1.2. Supermanifold.

Definition 1.17 ([30, 33, 40]). A supermanifold \( M \) of dimension \( \dim M = m | n \) is a pair \((M_0, \mathcal{A}_M)\) where \( M_0 \) is a manifold of dimension \( m \) (called the body) and \( \mathcal{A}_M = (\mathcal{A}_M)^\sigma \oplus (\mathcal{A}_M)^\tau \) is a sheaf of superfunctions, i.e., a sheaf of algebras such that for all \( p \in M_0 \) there exists an open neighbourhood \( U \ni p \) such that

\[
\mathcal{A}_M |_U \cong C_M^\infty |_U \otimes \Lambda [\xi_1, \ldots, \xi_n]
\]

where \( C_M^\infty \) denotes the sheaf of smooth functions of \( M_0 \). Sections of \( \mathcal{A}_M \) are called superfunctions of \( M \). If coordinates \( \{ x^i \} \) on \( U \subseteq M_0 \) are given, the set

\[
\{ \eta^k \} = \{ x^1, \ldots, x^m, \xi_1, \ldots, \xi_n \}
\]

is called a set of local coordinates on \( M \).
Locally any superfunction $f \in \mathcal{A}_M(U)$ is of the form
\[ f = \sum_{\alpha} f_\alpha(x^1, \ldots, x^m)\xi^\alpha \] (1.6)
where $f_\alpha(x^1, \ldots, x^m) \in C^\infty_{M_0}(U)$ and $\xi^\alpha := \xi^\alpha_1 \wedge \cdots \wedge \xi^\alpha_n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_2$.

The decomposition $f = f_\phi + f_\phi^* \in \mathcal{A}_M(U)_\phi \oplus \mathcal{A}_M(U)_{\phi^*}$ is locally given by
\[ f_\phi = \sum_{|\alpha|=0} f_\alpha(x^1, \ldots, x^m)\xi^\alpha, \quad f_{\phi^*} = \sum_{|\alpha|=1} f_\alpha(x^1, \ldots, x^m)\xi^\alpha \] ($|\alpha| := \sum_{i=1}^n \alpha_i \in \mathbb{Z}_2$)

Definition 1.17 implies the existence of a canonical epimorphism, called the evaluation map, denoted by
\[ \text{ev} : \mathcal{A}_M \rightarrow C^\infty_{M_0} \]
whose action on a superfunction (1.6) is given by $\hat{f} := \text{ev}(f) = f_0, \ldots, 0(x^1, \ldots, x^n)$.

**Example 1.18.** The supermanifold $\mathbb{R}^{m,n}$ is the pair $(\mathbb{R}^m, \mathcal{A}_{\mathbb{R}^m,n})$ where
\[ \mathcal{A}_{\mathbb{R}^m,n} := C^\infty_{\mathbb{R}^m} \otimes \Lambda[\xi_1, \ldots, \xi_n]. \]

**Example 1.19 ([5]).** Let $M_0$ be a real smooth manifold of dimension $m$, $E$ a rank $n$ vector bundle over $M_0$ and $\Lambda(E)$ the associated exterior bundle. The sheaf
\[ U \rightarrow \mathcal{A}_M(U) := \Gamma(U, \Lambda(E)) \]
defines a supermanifold $M = (M_0, \mathcal{A}_M)$ of dimension $m|n$. Supermanifolds of this form are called split and are $\mathbb{Z}$-graded. Every real smooth supermanifold is non canonically isomorphic to a split supermanifold.

### 1.3. Morphism of supermanifolds and tensor sheaf.

Let $M = (M_0, \mathcal{A}_M)$ and $N = (N_0, \mathcal{A}_N)$ be two supermanifolds.

**Definition 1.20 ([30]).** A morphism $\phi \in \text{Mor}(N, M)$ is a pair $\phi = (\phi_0, \phi^*)$ where $\phi_0 : N_0 \rightarrow M_0$ is a smooth map and $\phi^* : \mathcal{A}_M \rightarrow (\phi_0)_* \mathcal{A}_N$ is a morphism of sheaves of superalgebras over $M_0$ called the pull-back. A diffeomorphism is a morphism $\phi = (\phi_0, \phi^*)$ such that $\phi_0$ is a diffeomorphism and $\phi^*$ is an isomorphism.

Every morphism $\phi = (\phi_0, \phi^*) \in \text{Mor}(N, M)$ is uniquely defined by the pull-back $\phi^* : \mathcal{A}(M) \rightarrow \mathcal{A}(N)$ on global sections of the structure sheaves ([30]).

**Example 1.21.** The evaluation map
\[ \text{ev} : \mathcal{A}(M) \rightarrow C^\infty(M_0) \]
\[ f \mapsto \hat{f} \]
defines the canonical embedding $\phi := (id_{M_0}, \text{ev})$ of the body $(M_0, C^\infty_{M_0})$ inside the supermanifold $(M_0, \mathcal{A}_M)$ while, for every $p \in M_0$, the evaluation at $p$
\[ \text{ev}_p : \mathcal{A}(M) \rightarrow \mathbb{R} \]
\[ f \mapsto \hat{f}(p) := f(p) \]
defines the canonical embedding $p = (p_0, \text{ev}_p) \in \text{Mor}(\mathbb{R}^{0,0}, M)$. Recall that a superfunction $f \in \mathcal{A}(M)$ is not determined by its values $f(p)$, $p \in M_0$. For every fixed supermanifold $N$, denote by $\hat{p} \in \text{Mor}(N, M)$ the constant map $p \in M_0$, i.e. the composition of $p \in \text{Mor}(\mathbb{R}^{0,0}, M)$ with the unique element of $\text{Mor}(N, \mathbb{R}^{0,0})$. 

Example 1.22 ([33]). A morphism \( \phi = (\phi_0, \phi^*) \in \text{Mor}(\mathbb{R}^{m,n}, \mathbb{R}^{m,n}) \) is uniquely defined by the formulae
\[
\begin{align*}
\phi^*(x^i) &= x^i \circ \phi_0 + \sum_{|\alpha|=\bar{\alpha}} \phi^\alpha \xi^\alpha \\
\phi^*(\xi^j) &= \sum_{|\alpha|=|\bar{\alpha}|} \phi^\alpha \phi^* \xi^\alpha \\
\{ \phi^\alpha \}_{|\alpha|=\bar{\alpha}} &\subseteq C^\infty(\mathbb{R}^m)
\end{align*}
\]
and is a change of coordinates whenever it is a diffeomorphism.

Example 1.23 ([22, 33]). The structure sheaf of \( M \times N \) is the completion \( \mathcal{A}_M \otimes \mathcal{A}_N \) of \( \mathcal{A}_M \otimes \mathcal{A}_N \) with respect to Grothendieck’s \( \pi \)-topology. The embedding \( \mathcal{A}(M) \subseteq \mathcal{A}(M) \times \mathcal{A}(N) \) defines the projection of \( M \times N \) onto \( M \).

The sheaf \( \text{Der} \mathcal{A}_M \) of derivations of \( \mathcal{A}_M \) over \( \mathbb{R} \) is a sheaf of left \( \mathcal{A}_M \)-supermodules: \( \text{Der} \mathcal{A}_M = (\text{Der} \mathcal{A}_M)_{\pi} \oplus (\text{Der} \mathcal{A}_M)_{\bar{\pi}} \), where for \( \alpha \in \mathbb{Z}_2 \)
\[
(\text{Der} \mathcal{A}_M)_\alpha := \left\{ X \in \text{End}_\mathcal{A}(\mathcal{A}_M, \mathcal{A}_M)_\alpha \mid X(fg) = X(f)g + (-1)^{|\alpha|f} fX(g) \right\}.
\]

Definition 1.24 ([40]). The sheaf \( T_M := \text{Der} \mathcal{A}_M \) is called the tangent sheaf of the supermanifold \( M = (M_0, \mathcal{A}_M) \). The sections of \( T_M \) are called vector fields. The cotangent sheaf \( T_M^* := \text{Hom} \mathcal{A}_M(T_M, \mathcal{A}_M) \) is the sheaf of graded algebras generated by tensor products (graded over \( \mathcal{A}_M \)) of \( T_M \) and \( T_M^* \). Its sections are called tensor fields.

Example 1.25. A metric on \( M \) is a non-degenerate symmetric even field \( g : T_M \otimes T_M \to \mathcal{A}_M \), i.e. a tensor field of type \((0,2)\) satisfying

(Non degeneracy) \( g(X, -) \) is an isomorphism \( T_M \to T_M^* \).

(Symmetry) \( g(X, Y) = (-1)^{|X||Y|} g(Y, X) \),

(Parity) \( |g(X, Y)| = |X| + |Y| \).

The tangent sheaf \( T_M \) is a sheaf of Lie superalgebras with respect to the bracket
\[
[X, Y] := X \circ Y - (-1)^{|X||Y|} Y \circ X \quad X, Y \in T_M(U)
\]
(1.7)
Any coordinate system (1.5) gives rise to basic even \( \frac{\partial}{\partial x^i} \) and odd \( \frac{\partial}{\partial \xi^j} \) vector fields whose action on (1.6) is
\[
\frac{\partial f}{\partial x^i} = \sum \alpha \frac{\partial f}{\partial x^i} (x^1, \ldots, x^m) \xi^\alpha, \quad \frac{\partial f}{\partial \xi^j} = \sum \alpha_j (-1)^{\alpha_1 + \cdots + \alpha_j - 1} f_{\alpha}(x^1, \ldots, x^m) \xi^{\alpha_1} \wedge \cdots \wedge \xi^{\alpha_j-1} \wedge \xi^{\alpha_{j+1}} \wedge \cdots \wedge \xi^{\alpha_m}.
\]

1.4. Even-value and odd-value.

Let \( \mathcal{A}_{M,p} \) denote the stalk of the sheaf \( \mathcal{A}_M \) at \( p \in M_0 \).

Definition 1.26 ([33]). The tangent space \( T_p M = (T_p M)_\pi \oplus (T_p M)_{\bar{\pi}} \) at \( p \in M_0 \) is the superspace of dimension \( \dim M = m|n \) defined by
\[
T_p M := \{ X \in \text{Hom}_R (\mathcal{A}_{M,p}, \mathbb{R}) \mid X(fg) = X(f)g + f(p)X(g) \}.
\]
Elements of \( T_p M \) are called tangent vectors and
\[
TM := \cup_{p \in M_0} T_p M = \cup_{p \in M_0} (T_p M)_\pi + \cup_{p \in M_0} (T_p M)_{\bar{\pi}} =: (TM)_\pi + (TM)_{\bar{\pi}},
\]
is a graded vector bundle on \( M_0 \) called the tangent bundle of \( M \). The canonical isomorphism \( (T_p M)_\pi \cong T_p M_0 \) gives the canonical identification \( (TM)_\pi \cong TM_0 \).
Definition 1.27. Let $X \in T_M(U)$ be a vector field on $U \ni p$. The tangent vector $X|_p := ev_p \circ X : A_{M,p} \to \mathbb{R}$ is called the value of $X$ at the point $p$. The assignment $U \ni p \mapsto (X|_p)_T \in (T_{p,M})_T \cong T_{p,M_0}$ is a well-defined vector field $\tilde{X}_T \in T_{M_0}(U)$ of $M_0$ called the even-value of $X$. The assignment $U \ni p \mapsto (X|_p)_T \in (T_{p,M})_T$ is a well-defined section $\tilde{X}_T \in \Gamma(U,(TM)_T)$ of $(TM)_T$ called the odd-value of $X$.

Denote by

$$ev_T : T_M(U) \to \Gamma(U,TM_0)$$

$$X \mapsto ev_T(X) := \tilde{X}_T$$ (1.8)

$$ev_T : T_M(U) \to \Gamma(U,(TM)_T)$$

$$X \mapsto ev_T(X) := \tilde{X}_T$$ (1.9)

the operators which send a vector field $X \in T_M(U)$ to its even and odd values. Note that (1.8) and (1.9) do not uniquely determine $X$ and that (1.8) is not a Lie superalgebra morphism, unless dim $M = m \mid 0$. The above definitions are naturally extended to arbitrary tensor fields $T \in T^*_r(M)$. The tensor space at a point $p \in M_0$ is denoted by $\bigoplus_{r,s} T_p M^r \ni T|_p$ and $T(X_1,\ldots,X_s)|_p = T|_p (X_1|_p,\ldots,X_s|_p)$ for all $X_1,\ldots,X_s \in T(M)$.

1.5. $\phi$-vector field.

Let $\phi = (\phi_0,\phi^*) \in \text{Mor} (N,M)$ be a morphism of supermanifolds.

Definition 1.28 ([8]). A $\phi$-vector field on $U \subseteq M_0$ is a linear map

$$X : A_M(U) \to A_N(\phi_0^{-1}U)$$ (1.10)

such that its homogeneous components satisfy

$$X(fg) = (Xf)\phi^*(g) + (-1)^{|X||f|}\phi^*(f)(Xg)$$

for all $f,g \in A_M(U)$. The associated sheaf $T_\phi$ over $M_0$ is a locally free sheaf of left $\phi_0^{-1}A_N$-supermodules of rank $\text{dim} M$. Any $X \in T_\phi(U)$ can be uniquely written in coordinates (1.5) as

$$X = \sum_k f^k \cdot (\phi^* \circ \frac{\partial}{\partial t^k}) \quad f^k \in A_N(\phi_0^{-1}U)$$ (1.11)

Definition 1.29. The differential $\phi_* : T(N) \to T_\phi(M)$ is defined by

$$T(N) \ni Y \mapsto \phi_* Y := Y \circ \phi^* \in T_\phi(M) .$$

Two vector fields $X \in T(M)$, $Y \in T(N)$ are $\phi$-related if $\phi_* Y = \phi^* \circ X \in T_\phi(M)$. The differential $\phi_*|_p : T_p N \to T_{\phi(p)}(M)$ at the point $p \in N_0$ is defined by

$$T_{\phi}(N) \ni v \mapsto v \circ \phi^* \in T_{\phi(p)}M .$$

The morphism $\phi$ is an immersion (resp. submersion) at $p \in N_0$ if $\phi_*|_p$ is injective (resp. surjective). For the local description of these see [33, 40].

If $\phi$ is a diffeomorphism the notation $\phi_* Y := (\phi^{-1})^* \circ Y \circ \phi^* \in T(M)$ is used. In particular, there exists a natural action of the group of diffeomorphisms $\text{Aut} (M)$ on $T^*_r(M)$. The pull-back $\phi^* T \in T_\phi(N)$ of a covariant tensor field $T \in T_\phi(M)$ under $\phi = (\phi_0,\phi^*) \in \text{Mor} (N,M)$ is easily defined using Definition 1.29 and (1.11). The usual functorial property holds.
Lemma 1.30. Let $\psi = (\psi_0, \psi^*) \in \text{Mor}(L, N)$ and $\phi = (\phi_0, \phi^*) \in \text{Mor}(N, M)$ be morphisms of supermanifolds. For all families of vector fields $\{Y_i\} \subseteq T(N)$ and $\{Z_i\} \subseteq T(L)$ satisfying $(\phi \circ \psi)_*(Z_i) = \psi^* \circ Y_i \circ \phi^*$, we have that

$$\psi^*((\phi^* T)(Y_1, \ldots, Y_s)) = ((\phi \circ \psi)^* T)(Z_1, \ldots, Z_s)$$

1.6. Lie derivative.

For every vector field $Y \in T(N)$ there exists a unique derivation $L_Y$ of the tensor sheaf $\oplus_{r,s}(T_N)^r_s$ commuting with contractions and such that

$$L_Y(f) = Yf, \quad L_Y X = [Y, X]$$

for every $f \in A(N)$ and $X \in T(N)$.

Definition 1.31. The representation $\mathcal{L} : T_N \to \mathfrak{gl}_R((T_N)^r_s)$ is called Lie derivative.

Let $\{Y_i\}_{i=1}^N$ be a set of vector fields on a supermanifold $N = (N_0, A_N)$ of dimension $\dim N = m|n$ such that $\text{Span}_R[Y_1|_p, \ldots, Y_s|_p] = (T_p N)_T$ for every $p \in N_0$.

Lemma 1.32. A tensor field $T \in T(N)^r_s$ on $N$ satisfies

i) $T|_p = 0$ for every $p \in N_0$,

ii) $(L_{Y_{i_k}} \cdots L_{Y_{i_1}}(T))|_p = 0$ for every $p \in N_0$ and $1 \leq k \leq n$

is zero.

Proof. See the Appendix.

If $Y \circ \phi^* = 0$, the Lie derivative of a $\phi$-vector field $X \in T_\phi(M)$ is defined by

$$\mathcal{L}_Y(X) := Y \circ X \in T_\phi(M).$$

2. Lie-Kostant-Koszul theory of supergroups

There are various approaches to the theory of Lie supergroups. Subsection 2.1 deals with the categorical approach, i.e. a Lie supergroup is a group object $G = (G_0, A_G)$ in the category of supermanifolds. Subsection 2.2 describes a more algebraic approach. The notion of super Harish-Chandra pair $(G_0, g)$ is introduced and, following the ideas of [6, 31], it is recalled how to explicitly reconstruct the structure sheaf of a Lie supergroup, together with its Hopf superalgebra structure. Lie supergroups are thus equivalent to a global even part together with an infinitesimal odd part and, for this reason, most of the results in Lie supergroup theory find their natural setting here. This approach is refined in subsection 2.3 where an important, more geometric picture is described. Here the behaviour of odd infinitesimal symmetries is better understood and some new formulas related to it are obtained.

2.1. Lie supergroup.

Definition 2.1 ([40]). A supermanifold $G = (G_0, A_G)$ is called a Lie supergroup if there exist morphisms of supermanifolds

$$m = (m_0, m^*) : G \times G \to G, \quad i = (i_0, i^*) : G \to G, \quad e = (e_0, ev_e) : \mathbb{R}^{0,0} \to G$$

satisfying the usual axioms of associativity, identity and taking inverses:

i) $m \circ (id_G \times m) = m \circ (m \times id_G) : G \times G \times G \to G$,

ii) $m \circ (id_G, \hat{e}) = m \circ (\hat{e}, id_G) = id_G : G \to G$,

iii) $m \circ (id_G, i) = m \circ (i, id_G) = \hat{e} : G \to G$. 

An equivalent definition is that the infinite-dimensional superalgebra $\mathcal{A}(G)$ has a structure of (completed) Hopf superalgebra.

**Definition 2.2** ([30]). A supermanifold $G = (G_0, \mathcal{A}_G)$ is called a *Lie supergroup* if there exist morphisms of superalgebras

- **Comultiplication** $m^* : \mathcal{A}(G) \to \mathcal{A}(G) \otimes \mathcal{A}(G)$,
- **Counity** $\text{ev}_e : \mathcal{A}(G) \to \mathbb{R}$,
- **Antipode** $i^* : \mathcal{A}(G) \to \mathcal{A}(G)$,

satisfying the following axioms

$$(\text{Id} \otimes m^*) \circ m^* = (m^* \otimes \text{Id}) \circ m^* : \mathcal{A}(G) \to \mathcal{A}(G) \otimes \mathcal{A}(G) \otimes \mathcal{A}(G),$$

$$(\text{Id} \otimes \text{ev}_e) \circ m^* = (\text{ev}_e \otimes \text{Id}) \circ m^* = \text{Id} : \mathcal{A}(G) \to \mathcal{A}(G),$$

$$m_{\mathcal{A}_G} \circ (\text{Id} \otimes i^*) \circ m^* = m_{\mathcal{A}_G} \circ (i^* \otimes \text{Id}) \circ m^* = \text{ev}_e : \mathcal{A}(G) \to \mathbb{R} \subseteq \mathcal{A}(G).$$

Note that Definition 2.1 and Definition 2.2 reduce to the definition of Lie group if $\dim M = m|0$. In general the body $(G_0, m_0, i_0, e_0)$ is a Lie group.

For every $g \in G_0$ left/right translations by $g$ are the diffeomorphisms

$$L_g : G \doteq \{g\} \times G \leftrightarrow G \times G \overset{m}{\rightarrow} G, \quad R_g : G \doteq G \times \{g\} \leftrightarrow G \times G \overset{m}{\rightarrow} G$$

whose pull-backs on global superfunctions are

$$L_g^* : \mathcal{A}(G) \to \mathcal{A}(G), \quad R_g^* : \mathcal{A}(G) \to \mathcal{A}(G)$$

$$f \mapsto (\text{ev}_g \otimes \text{Id})(m^* f) \quad \quad f \mapsto (\text{Id} \otimes \text{ev}_g)(m^* f).$$

The map $R_{G_0} : G_0 \to \text{Aut} (\mathcal{A}(G))$ (resp. $L_{G_0} : G_0 \to \text{Aut} (\mathcal{A}(G))$) defined by $R_{G_0}(g) := R_g^*$ (resp. $L_{G_0}(g) := L_g^*$) is a group (resp. anti-) homomorphism.

**Definition 2.3** ([40]). A vector field $A \in \mathcal{T}(G)$ on $G$ is said left (resp right)-invariant if

$$(\text{Id} \otimes A) \circ m^* = m^* \circ A \quad \quad (\text{resp. } (A \otimes \text{Id}) \circ m^* = m^* \circ A) \quad (2.1)$$

The Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\mathcal{T}} + \mathfrak{g}_{\mathcal{T}}$ (resp. $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{\mathcal{T}} + \hat{\mathfrak{g}}_{\mathcal{T}}$) is the supervector space of all left (resp. right)-invariant vector fields on $G$ with bracket (1.7).

**Lemma 2.4** ([40]). Let $G = (G_0, \mathcal{A}_G)$ be a Lie supergroup with Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\mathcal{T}} + \mathfrak{g}_{\mathcal{T}}$. The evaluation of a left-invariant vector field at the point $e \in G_0$

$$\mathfrak{g} \rightarrow T_eG, \quad A \mapsto A|_e$$

is an isomorphism of supervector spaces. The inverse map is given by

$$T_eG \rightarrow \mathfrak{g}, \quad v \mapsto A_v := (\text{Id} \otimes v) \circ m^*$$

A similar result holds for the right-invariant vector field $vA \in \hat{\mathfrak{g}}$.

If $A \in \mathfrak{g}_{\mathcal{T}}$, the even value $\text{ev}_{\mathcal{T}} \circ A$ is a left-invariant vector field on $G_0$. Note that invariance by left translations with respect to $G_0$ is not equivalent to (2.1):

**Lemma 2.5.** A vector field $A \in \mathcal{T}(G)$ on $G$ is left-invariant if and only if

$$L_g^* \circ A = A \circ L_g^* \quad \quad \mathcal{L}_B A = 0 \quad (2.2)$$

for all $g \in G_0$ and (odd) $B \in \hat{\mathfrak{g}}$. 
Proof. Let \( A \) be a left-invariant vector field. It is easy to show that it satisfies equations (2.2). Conversely suppose \( A \in \mathcal{T}(G) \) satisfies (2.2) and consider the left-invariant vector field \( A_v \) associated with \( v := A|_e \in T_e G \). The vector field \( A - A_v \) satisfies the hypothesis of Lemma 1.32 and then \( A = A_v \).

The sheaf of derivations of a Lie supergroup is trivial.

**Lemma 2.6.** The sheaf of derivations \( \mathcal{T}_G \) of a Lie supergroup \( G \) is globally trivial; more precisely \( \mathcal{A}_G \otimes \mathfrak{g} \cong \mathcal{T}_G \).

Proof. It follows directly from Nakayama’s Lemma ([40]).

### 2.2. Super Harish-Chandra pair.

An important equivalent way of defining a Lie supergroup is the following.

**Definition 2.7** ([30]). A pair \((G_0, \mathfrak{g})\) consisting of a Lie group \(G_0\) and a Lie superalgebra \(\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1\) is a super Harish-Chandra (shortly sHC) pair if \(\mathfrak{g}_0 = \mathfrak{g}_0\) and if there exists an adjoint action, i.e. a morphism of Lie groups

\[
\text{Ad} : G_0 \longrightarrow \text{Aut}(\mathfrak{g})
\]

such that \(\text{Ad} : G_0 \longrightarrow \text{Aut}(\mathfrak{g}_0)\) is the usual adjoint action and

\[
ad_B A = \frac{d}{dt}|_{t=0} \text{Ad}_{\exp(tB)} A
\]

for every \(A \in \mathfrak{g}\) and \(B \in \mathfrak{g}_0\).

**Theorem 2.8** ([30]). Any Lie supergroup \(G = (G_0, \mathcal{A}_G)\) defines a sHC pair \((G_0, \mathfrak{g})\) where

\[
\text{Ad} : G_0 \longrightarrow \text{Aut}(\mathfrak{g}) \quad , \quad g \mapsto \text{Ad}_g := (A \mapsto R^*_g \circ A \circ R^*_g)
\]

The associated correspondence \((G_0, \mathcal{A}_G) \longrightarrow (G_0, \mathfrak{g})\) is a bijection between the sets of Lie supergroups and of sHC pairs.

Due to Theorem 2.8, Lie supergroups and sHC pairs are synonymous for us.

**Example 2.9.** Denote by \(\mathbb{R}^{r,s}\) the vector space \(\mathbb{R}^{r+s}\) endowed with the standard inner product \(\langle \cdot, \cdot \rangle\) of signature \((r,s)\) and by

\[
\text{Spin}^0_{r,s} \subseteq \text{Spin}_{r,s} \subseteq \text{Cl}^0_{r,s} \subseteq (\text{Cl}_{r,s}, \cdot) \quad , \quad \mathbb{R}^{r,s} \subseteq \text{Cl}_{r,s}
\]

the inclusions of the (connected) Spin group inside (the even part of) the Clifford algebra \((\text{Cl}_{r,s}, \cdot)\) and of \(\mathbb{R}^{r,s}\) inside \(\text{Cl}_{r,s}\). The 2-fold cover \(\xi : \text{Spin}_{r,s} \rightarrow \text{SO}_{r,s}\)

\[
\xi(g) : \mathbb{R}^{r,s} \rightarrow \mathbb{R}^{r,s}
\]

\[
v \mapsto g \cdot v \cdot g^{-1} \quad g \in \text{Spin}_{r,s}, \quad v \in \mathbb{R}^{r,s}
\]

is the vector representation of \(\text{Spin}_{r,s}\). It induces an isomorphism of Lie algebras

\[
\xi_* : \mathfrak{spin}_{r,s} \rightarrow \mathfrak{so}_{r,s}
\]

where \(v_1, v_2 \in \mathbb{R}^{r,s}\). The real spin representation is the restriction to the spin group \(\text{Spin}_{r,s} \subseteq \text{Cl}_{r,s}\) of an irreducible real representation of the Clifford algebra \(\text{Cl}_{r,s}\)

\[
\Delta_{r,s} : \text{Cl}_{r,s} \rightarrow \text{End} \mathbb{R}(S)
\]

The spin representation \(\Delta|_{\text{Spin}_{r,s}} : \text{Spin}_{r,s} \rightarrow \text{End} \mathbb{R}(S)\) is either irreducible or it decomposes into a sum of two irreducible representations \(S^\pm\) depending on \(r-s\) mod.
8 (see [32] for details). The Poincare’ Lie superalgebra \( \mathfrak{g}(\Gamma) = \mathfrak{g}_\mathcal{T} + \mathfrak{g}_\mathcal{T} \) associated with a \( \text{spin}_{r,s} \)-invariant symmetric bilinear map
\[
\Gamma : S \vee S \to \mathbb{R}^{r,s}
\]
is given by the Poincare’ algebra
\[
\mathfrak{g}_\mathcal{T} = \text{spin}_{r,s} \subset \mathbb{R}^{r,s}
\]
as the even part, the odd part \( \mathfrak{g}_\mathcal{T} = S \) considered as a \( \mathfrak{g}_\mathcal{T} \)-module via
\[
[A, s] := \Delta_{r,s}(A)s \quad , \quad [\mathbb{R}^{r,s}, S] := 0 \quad A \in \text{spin}_{r,s}, \quad s \in S
\]
and a bracket \([\cdot, \cdot] : S \vee S \to \mathbb{R}^{r,s}\) given by (2.4). The associated Poincare’ Lie supergroup is a sHC pair \((G_0, \mathfrak{g})\), where
\[
G_0 = \text{Spin}_{r,s} \times \mathbb{R}^{r,s} \quad , \quad \mathfrak{g} = \mathfrak{g}(\Gamma) = (\text{spin}_{r,s} \subset \mathbb{R}^{r,s}) + S
\]
and the morphism \( \text{Ad} : G_0 \to \text{Aut}(\mathfrak{g}) \) is defined, for \( g \in \text{Spin}_{r,s} \) and \( v \in \mathbb{R}^{r,s} \) by
\[
\text{Ad}_g(A + w + s) := (g \cdot A \cdot g^{-1} + g \cdot w \cdot g^{-1} + \Delta_{r,s}(g)s)
\]
\[
\text{Ad}_g(A + w + s) := (A - \xi_s(A)v + w + s)
\]
where \( A \in \text{spin}_{r,s}, \quad w \in \mathbb{R}^{r,s}, \quad s \in S \).

[6, 31] show how to reconstruct, from a given sHC pair \((G_0, \mathfrak{g})\), the structure sheaf of the associated Lie supergroup, together with its Hopf algebra structure. Note that the representation of \( \varphi_0 : \mathfrak{g}_\mathcal{T} \to T(G_0) \) (2.5)
defines a structure of sheaf of left \( \mathcal{U}(\mathfrak{g}_\mathcal{T}) \)-modules on \( C^\infty := (\mathcal{C}_G^\infty, m_{C^\infty}, 1_{C^\infty}) \) and that the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}), m_{\mathcal{U}(\mathfrak{g})}, 1_{\mathcal{U}(\mathfrak{g})}, \Delta_{\mathcal{U}(\mathfrak{g})}, \delta_{\mathcal{U}(\mathfrak{g})} \) of \( \mathfrak{g} \) has the natural structure of left \( \mathcal{U}(\mathfrak{g}_\mathcal{T}) \)-supermodule given by the embedding \( \mathcal{U}(\mathfrak{g}_\mathcal{T}) \subseteq \mathcal{U}(\mathfrak{g}) \).

**Theorem 2.10** ([6, 31]). The structure sheaf \( \mathcal{A}_G \) of the Lie supergroup \((G_0, \mathcal{A}_G)\) associated with a sHC pair \((G_0, \mathfrak{g})\) is the sheaf of convolution algebras given by
\[
U \to \mathcal{A}_G(U) := \text{Hom}_{\mathcal{U}(\mathfrak{g}_\mathcal{T})}(\mathcal{U}(\mathfrak{g}), C^\infty(U)) \quad , \quad F \mapsto F \circ \delta_{\mathcal{U}(\mathfrak{g})}
\]
where \( F_1, F_2 \in \mathcal{A}_G(U) \). The embedding \( \mathcal{U}(\mathfrak{g}_\mathcal{T}) \subseteq \mathcal{U}(\mathfrak{g}) \) induces the evaluation map
\[
\mathcal{A}_G(U) = \text{Hom}_{\mathcal{U}(\mathfrak{g}_\mathcal{T})}(\mathcal{U}(\mathfrak{g}), C^\infty(U)) \to \text{Hom}_{\mathcal{U}(\mathfrak{g}_\mathcal{T})}(\mathcal{U}(\mathfrak{g}_\mathcal{T}), C^\infty(U)) \cong C^\infty(U)
\]
\[
F \mapsto F(1_{\mathcal{U}(\mathfrak{g})})
\]
The structure of Hopf superalgebra of \( \mathcal{A}(G) \) is given by the algebra morphisms:
1) **Comultiplication** \( m^* : \mathcal{A}(G) \to \mathcal{A}(G \times G) \) , \( F \mapsto m^*F \)
\[
(m^*F)(u \otimes v)((g, h)) := F((\text{Ad}_{h^{-1}} u) \cdot v)(gh)
\]
2) **Counity** \( ev_e : \mathcal{A}(G) \to \mathbb{R} \) , \( F \mapsto ev_eF \)
\[
ev_eF := F(1_{\mathcal{U}(\mathfrak{g})})(e)
\]
3) **Antipode** \( i^* : \mathcal{A}(G) \to \mathcal{A}(G) \) , \( F \mapsto i^*F := F^* \)
\[
F^*(u)(g) := F(\text{Ad}_g(u))(g^{-1})
\]
where \( \text{Ad} : G_0 \to \text{Aut} \left( \mathcal{U}(\mathfrak{g}) \right) \) is the unique extension of \((2.3)\), \(e \in G_0\) is the unity of the Lie group \(G_0\) and \(u,v \in \mathcal{U}(\mathfrak{g}),\ g,h \in G_0\). The Lie superalgebra morphism

\[
\varphi : \mathfrak{g} \to T(G) \quad , \quad \varphi(a)F := (u \mapsto (-1)^{|a|} F(ua))
\]

is the representation of \(\mathfrak{g}\) as convolution superalgebras. The superalgebra identification \([\text{Theorem } 2.12]\) is a homomorphism of coalgebras. This implies that \(F\) is the representation of \(G\) as left-invariant vector fields of the Lie supergroup \(G\). The right-invariant vector field associated with \(a \in \mathfrak{g}\) is given by

\[
\mathcal{A}(G) \ni F \mapsto (\varphi(\pi)F)^* \in \mathcal{A}(G)
\]

so that the Lie superalgebra anti-homomorphism

\[
\hat{\varphi} : \mathfrak{g} \to T(G) \quad , \quad \hat{\varphi}(a)(F)(u)(g) = (-1)^{|F||a|} F((\text{Ad}_{g^{-1}} a) \cdot u)(g)
\]

is the representation of \(\mathfrak{g}\) as right-invariant vector fields of the Lie supergroup \(G\).

**Remark 2.11** ([31]). The previous construction can be generalized to associate with any \(\mathfrak{g}\)-manifold (i.e. a manifold \(M_0\) with a representation \(\varphi_0 : \mathfrak{g} \to T(M_0)\)) a supermanifold \(M = (M_0, \mathcal{A}_M)\) with a representation \(\varphi : \mathfrak{g} \to T(M)\) such that \(\text{ev}_\pi \circ \varphi|_{\mathfrak{g}} = \varphi_0\) and \(\text{ev}_\pi \circ \varphi|_{\mathfrak{g}} : \mathfrak{g} \to T_p M|_T\) is an isomorphism for every \(p \in M_0\).

For the sake of completeness, here is how to reconstruct a morphism of Lie supergroups from the associated morphism of sHC pairs. Let \((G_0, \mathfrak{g})\) and \((H_0, \mathfrak{h})\) be two sHC pairs and, for simplicity, assume that \(G_0\) is connected. A morphism \(\phi : \mathfrak{g} \to \mathfrak{h}\) of Lie superalgebras whose even part integrates to a morphism of Lie groups \(\phi_0 : G_0 \to H_0\) defines a morphism of Lie supergroups

\[
\phi^* : \mathcal{A}(H) \to \mathcal{A}(G)
\]

given by

\[
(\phi^* F)(u)(g) := F(\phi u)(\phi_0 g)
\]

where \(F \in \mathcal{A}(H),\ u \in \mathcal{U}(\mathfrak{g}),\ g \in G_0\).

### 2.3. Koszul theory.

**Theorem 2.10** gives a very elegant way to reconstruct the Hopf superalgebra structure of a Lie supergroup \(G\) from its sHC pair \((G_0, \mathfrak{g})\). On the other hand the structure sheaf of a Lie supergroup is trivial and thus, in particular, \(\mathbb{Z}\)-graded.

**Theorem 2.12** ([31]). Let \(G\) be a Lie supergroup and \(\mathfrak{g} = \mathfrak{g}_T^e + \mathfrak{g}_T^o\) its Lie superalgebra. There exists a canonical isomorphism \(\mathcal{A}_G \cong \mathcal{C}^\infty_{\mathfrak{g}_0} \otimes \Lambda(\mathfrak{g}_T^*)\).

**Proof.** It is important to note that for any \(a_1, \ldots, a_p \in \mathfrak{g}_T^o\)

\[
\gamma : \Lambda(\mathfrak{g}_T^o) \to \mathcal{U}(\mathfrak{g}) \quad , \quad a_1 \wedge \cdots \wedge a_p \mapsto \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \text{sgn}(\sigma) a_{\sigma(1)} \cdots a_{\sigma(p)}
\]

is a homomorphism of coalgebras. This implies that

\[
\gamma : \mathcal{U}(\mathfrak{g}_T^o) \otimes \Lambda(\mathfrak{g}_T^o) \to \mathcal{U}(\mathfrak{g}) \quad , \quad u \otimes v \mapsto u \cdot \gamma(v)
\]

is an isomorphism of coalgebras and of left \(\mathcal{U}(\mathfrak{g}_T^o)\)-supermodules (see Theorem 2.22). Thus

\[
\mathcal{A}_G(U) = \text{Hom}_{\mathcal{U}(\mathfrak{g}_T^o)}(\mathcal{U}(\mathfrak{g}), \mathcal{C}^\infty(U)) \to \text{Hom}_{\mathcal{R}(\mathfrak{g}_T^o)}(\Lambda(\mathfrak{g}_T^o), \mathcal{C}^\infty(U))
\]

is an isomorphism of convolution superalgebras. The superalgebra identification

\[
\text{Hom}_{\mathcal{R}(\mathfrak{g}_T^o)}(\Lambda(\mathfrak{g}_T^o), \mathcal{C}^\infty(U)) \to \mathcal{C}^\infty(U) \otimes \Lambda(\mathfrak{g}_T^o)^* \to \mathcal{C}^\infty(U) \otimes \Lambda(\mathfrak{g}_T^o)^\ast
\]
shows that $G = (G_0, A_G)$ is a split supermanifold with respect to the (trivial) vector bundle $G_0 \times \mathfrak{g}_\mathcal{T}$ over $G_0$. Recall that, due to the rule of signs of supergeometry, the last isomorphism $\Lambda(\mathfrak{g}_\mathcal{T})^* \cong \Lambda(\mathfrak{g}_\mathcal{T})$ is given by
\[(a_1 \wedge \cdots \wedge a_p)^* \mapsto (-1)^{(p-1)p/2} a_1^* \wedge \cdots \wedge a_p^*)\]
for $a_1, \ldots, a_p \in \mathfrak{g}_\mathcal{T}$.

The algebra isomorphism
\[A_G(U) \cong C^\infty(U) \otimes \Lambda(\mathfrak{g}_\mathcal{T}) \quad (2.9)\]
can be used to construct a canonical embedding of superalgebras
\[C^\infty(U) \hookrightarrow C^\infty(U) \otimes \Lambda(\mathfrak{g}_\mathcal{T}) \cong A_G(U) \quad (2.10)\]
Given a smooth function $f \in C^\infty(U)$, the corresponding superfunction $f \in A_G(U)$ (denoted by the same symbol, by abuse of notation) is uniquely determined by
\[f(1_{\mathfrak{U}(\mathfrak{g})}) = f \in C^\infty(U) \quad , \quad f(\text{Im} \gamma - \mathbb{R}1_{\mathfrak{U}(\mathfrak{g})}) = 0 \quad (2.11)\]
In particular, for every $a_1, a_2 \in \mathfrak{g}_\mathcal{T}$
\[f(a_1) = 0 \quad , \quad f(2a_1 \cdot a_2) = f([a_1, a_2]) = \varphi_0([a_1, a_2])f \in C^\infty(U)\]
and for every $G \in A_G(U)$ and $a_1, \ldots, a_p \in \Lambda(\mathfrak{g}_\mathcal{T})$
\[(f * G)(\gamma(a_1 \wedge \cdots \wedge a_p)) = f \cdot G(\gamma(a_1 \wedge \cdots \wedge a_p)) = f \cdot g(a_1 \wedge \cdots \wedge a_p) \quad .\]

**Corollary 2.13.** For every $g, h \in G_0$, $u \in \mathcal{U}(\mathfrak{g})$ and $F \in A(G)$
\[L_h^* F(u)(g) = F(u)(hg) \quad , \quad (R_h^* F)(u)(g) = F(\text{Ad}_{h^{-1}} u)(gh) \quad .\]

The antipode, left and right translations with respect to $G_0$ preserve the $\mathbb{Z}$-grading of $A_G$ and, in particular, the embedding (2.10).

**Proof.** The first part of the Corollary is a straightforward consequence of Theorem 2.10. The last assertions follow from the equations
\[\text{Ad}_h \circ \gamma = \gamma \circ \text{Ad}_h \quad , \quad \tilde{\gamma}(a_1 \wedge \cdots \wedge a_p) = (-1)^p \gamma(a_1 \wedge \cdots \wedge a_p)\]
where $h \in G_0$, $a_1, \ldots, a_p \in \mathfrak{g}_\mathcal{T}$.

We want to understand how the representations (2.6) and (2.7) are translated with respect to the canonical isomorphism $A_G \cong C^\infty_{G_0} \otimes \Lambda(\mathfrak{g}_\mathcal{T})$; the results have a geometric role in some applications discussed in section 5. An explicit formula for left-invariant vector fields is recalled ([12, 31]). An analogous formula for the right invariant ones, which to the best of our knowledge has not been described in the literature, is obtained using the results of [12, 35]. The restrictions of the representations (2.6) and (2.7) to $\mathfrak{g}_\mathcal{T}$ are compared to "trivial extensions" in subsection 2.4 while the restrictions to $\mathfrak{g}_\mathcal{T}$ are more complicated to handle and they are studied in subsection 2.5.

**Notation:** The representation of $\mathfrak{g}_\mathcal{T}$ as right-invariant vector fields of the Lie group $G_0$ is denoted by
\[\hat{\varphi}_0 : \mathfrak{g}_\mathcal{T} \to T(G_0) \quad (2.12)\]
2.4. Even symmetries.

The algebra isomorphism (2.9) can be used to construct a canonical embedding of Lie superalgebras

$$T_{G_0}(U) \hookrightarrow \text{Der}_{\Lambda(h^{R}_G)}(\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G)) \subseteq \text{Der}_R(\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G)) \cong T_G(U)$$

(2.13)

Given a vector field $X \in T_{G_0}(U)$, the corresponding vector field $X \in T_G(U)$ (denoted by the same symbol by abuse of notation) acts trivially on $g^{R}_T \subseteq \mathcal{C}^\infty(M_0) \otimes \Lambda(g^{R}_T)$. Composing (2.13) with (2.5) and (2.12) construct two representations

$$\varphi_0 : g^{R}_T \hookrightarrow \text{Der}_{\Lambda(h^{R}_G)}(\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G)) \ , \quad \hat{\varphi}_0 : g^{R}_T \hookrightarrow \text{Der}_{\Lambda(h^{R}_G)}(\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G)) \ .$$

It is not difficult to see that

$$\hat{\varphi}_{0}|_{g_T} = \hat{\varphi}_0 : g_T \rightarrow T(G) \ ,$$

while, for every $a \in g_T$

$$\varphi(a)|_{\mathcal{C}^\infty} = \varphi_0(a) \ , \quad \varphi(a)(g^+_T) = 0 \iff \varphi(a) = \varphi_0(a) \iff [a, g_T] = 0 \ .$$

This behaviour is described in more detail in the following two Propositions.

**Proposition 2.14.** The representation $\hat{\varphi}|_{g_T} : g_T \rightarrow \text{Der}_R(\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G))$ satisfies

$$\hat{\varphi}(a) = \hat{\varphi}_0(a)$$

for every $a \in g_T$. In particular it preserves the $\mathbb{Z}$-gradation of $\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G)$.

**Proof.** Straightforward consequence of (2.7) and the definition of the sheaf $A_G$.

**Proposition 2.15** ([31]). The difference between $\varphi|_{g_T} : g_T \rightarrow \text{Der}_R(\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G))$ and $\varphi_0 : g_T \hookrightarrow \text{Der}_{\Lambda(h^{R}_G)}(\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G))$ is given by the formula

$$[\varphi(a)(f) - \varphi_0(a)(f)](a_1 \wedge \cdots \wedge a_p) = - \sum_{i=1}^{p} f(a_1 \wedge \cdots \wedge [a, a_i] \wedge \cdots \wedge a_p)$$

where $a \in g_T$, $a_1, \ldots, a_p \in g_T$ and $f \in \text{Hom}_R(\Lambda(g^{R}_G), \mathcal{C}^\infty(U))$. The representation $\varphi|_{g_T} : g_T \rightarrow \text{Der}_R(\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G))$ preserves the $\mathbb{Z}$-gradation of $\mathcal{C}^\infty(U) \otimes \Lambda(g^{R}_G)$.

The following example clarifies the situation.

**Example 2.16.** We reconstruct the Poincaré Lie supergroup $G$ associated with the sHC pair $(G_0, g)$ of Example 2.9 in signature $(1, 2)$. Recall that

$$\text{Spin}_{1,2}^0 \cong \text{SL}(2, \mathbb{R})$$

and that the spin module $S$ is a 2-dimensional real vector space $S = [[[s_0, s_1]]]$. The vector representation of $\text{SL}(2, \mathbb{R})$ on $\mathbb{R}^{1,2} = [[[e_0, e_1, e_2]]]$ is the adjoint representation via the identification of vector spaces endowed with a quadratic form

$$(\mathbb{R}^{1,2}, \langle \cdot, \cdot \rangle) \cong (\mathfrak{sl}(2, \mathbb{R}), \det)$$

given by

$$e_0 \cong \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_1 \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad e_2 \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$
An element of $G_0 = \text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^{1,2}$ is denoted by $(g, v) \in \text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^{1,2}$. Fix an $\mathfrak{sl}(2, \mathbb{R})$-invariant symmetric bilinear map $\Gamma : S \vee S \to \mathbb{R}^{1,2}$ and denote by $\mathfrak{g} = \mathfrak{g}(\Gamma) = \mathfrak{g}_\eta + \mathfrak{g}_\tau = (\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^{1,2}) + S$ the Poincaré Lie superalgebra with

$$\{s_\alpha, s_\beta\} = \Gamma^k_{\alpha\beta} e_k$$

for $0 \leq \alpha, \beta \leq 1, 0 \leq k \leq 2$, where we have used the Einstein convention, which is used throughout this example. An element of $\mathfrak{g}$ is denoted by $(A, w, s) \in (\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^{1,2}) + S$ where

$$A = \begin{pmatrix} a_0^0 & a_1^0 \\ a_0^1 & a_1^1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})$$

For every $0 \leq i, j \leq 1$, denote by

$$y_j^i : \text{SL}(2, \mathbb{R}) \to \mathbb{R} \quad x^k : \mathbb{R}^{1,2} \to \mathbb{R} \quad g = \begin{pmatrix} g_0^0 & g_1^0 \\ g_0^1 & g_1^1 \end{pmatrix} \quad g^i_j$$

the coordinates of $\text{SL}(2, \mathbb{R})$ and $\mathbb{R}^{1,2}$. The following linear maps (recall (2.11))

$$y_j^i : \Lambda(S) \to \mathcal{C}^\infty(\text{SL}(2, \mathbb{R})) \quad x^k : \Lambda(S) \to \mathcal{C}^\infty(\mathbb{R}^{1,2}) \quad s_\alpha : \Lambda(S) \to \mathbb{R}$$

are coordinates of $G$. The $\mathcal{U}((\mathfrak{g}_\eta))$-linear maps from $\mathcal{U}((\mathfrak{g}_\eta) \otimes \Lambda(S))$ to $\mathcal{C}^\infty(G_0)$ are denoted by the same letter $y_j^i$, $x^k$, $s_\alpha$, while the associated superfunctions by

$$\mathcal{U}((\mathfrak{g}_\eta)) = \Lambda(S) \to \mathcal{U}(\mathfrak{g}) = \text{Hom}_G(\mathcal{U}(\mathfrak{g}), \mathcal{C}^\infty(G_0))$$

Recall that $y_j^i = Y_j^i \circ \gamma$, $x^k = X^k \circ \gamma$, $s_\alpha = S_\alpha \circ \gamma$. The representation $\varphi : \mathfrak{g} \to \mathcal{T}(G)$ of the Lie superalgebra $\mathfrak{g} = (\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^{1,2} + S$ as left-invariant vector fields of $G$ is

$$\varphi(A) = \varphi_0(A) - a_0^\alpha s_\beta \frac{\partial}{\partial s_\alpha} \quad \varphi_0 = \varphi_0(w)$$

$$\varphi(s_\alpha) = - \frac{\partial}{\partial s_\alpha} - \frac{1}{2} s_\eta \varphi_0(\Gamma^k_{\alpha\eta} e_k)$$

while that as right-invariant vector fields $\hat{\varphi} : \mathfrak{g} \to \mathcal{T}(G)$ is

$$\hat{\varphi}(A) = \hat{\varphi}_0(A) \quad \hat{\varphi}_0 = \hat{\varphi}_0(w) \quad \hat{\varphi}(s_\beta) = y_j^\beta (g^{-1} \frac{\partial}{\partial s_\alpha} [\frac{1}{2} s_\eta \varphi_0(\Gamma^k_{\alpha\eta} e_k)])$$

where $0 \leq \eta \leq 1$. We calculate the comultiplication and the coinverse for the Lie supergroup of $G$ given by the sHC pair $(\mathbb{R}^{1,2}, \mathbb{R}^{1,2} + S)$. By definition

$$m^*(x^k) := m^*(X^k) \circ (\gamma \otimes \gamma)$$

i.e.

$$m^*(x^k)(r \otimes t)(v_0, v_1) = X^k((\text{Ad}_\gamma(v_0) \cdot \gamma(t))(v_0 + v_1) = X^k(\gamma^{-1}(r \cdot \gamma(t)))(v_0 + v_1)$$

where $r, t \in \Lambda(S)$ and $v_0, v_1 \in \mathbb{R}^{1,2}$. The computation of $\gamma^{-1}(r \cdot \gamma(t))$ implies that

$$m^*(x^k) = m^*_{0}(x^k) \otimes 1 + \frac{1}{2} \sum_{0 \leq \alpha, \beta \leq 1} \Gamma^k_{\alpha\beta} \otimes (s_\alpha \otimes s_\beta)^* \in \mathcal{C}^\infty(\mathbb{R}^{1,2} + \mathbb{R}^{1,2}) \otimes (\Lambda(S) \otimes \Lambda(S))^*$$
Denoting by \( \partial^{a} \in C^{\infty}(\mathbb{R}^{1,2}) \otimes \Lambda(S^{*}) \) and \( \theta^{\beta} \in C^{\infty}(\mathbb{R}^{1,2}) \otimes \Lambda(S^{*}) \) global odd coordinates on two copies of the Lie supergroup \((\mathbb{R}^{1,2}, \mathbb{R}^{1,2} + S)\) we get the common forms of comultiplication and coinverse in the mathematical physics literature

\[
m^{*}(x^{k}) = m_{0}^{*}(x^{k}) - \frac{1}{2} \Gamma_{ab}^{k} \partial^{a} \theta^{b} , \quad m^{*}(s^{a}) = \partial^{a} + \theta^{a}
\]

\[
i^{*}(x^{k}) = -x^{k} , \quad i^{*}(s^{a}) = -s^{a} .
\]

2.5. Odd symmetries.

2.5.1. Introduction.

[12, 35] give a proof of Koszul’s Theorem 2.29. The arguments involved in the proof are briefly recalled and then Theorem 2.30 proves an analogous formula. Recall that the symmetric algebra \( S(g) \) of \( g \) has a natural structure of a \( \mathbb{Z} \)-graded cocommutative Hopf superalgebra \((S(g), m_{S(g)}, 1_{S(g)}, \Delta_{S(g)}, \epsilon_{S(g)}, \delta_{S(g)})\).

**Definition 2.17** ([12, 35]). Let \((A, m_{A}, 1_{A})\) be a graded algebra. The convolution algebra \( \text{Hom}_{\mathbb{R}}(S(g), A) \) is the space of *formal functions* on \( g \) with values in \( A \). The algebra \( A \) is embedded as the subalgebra of \( \text{Hom}_{\mathbb{R}}(S(g), A) \) which consists of *constant functions*

\[
a : 1 \mapsto a
\]

\[
S^{n}(g) \mapsto 0 \quad \text{if} \quad n \geq 0
\]

where \( a \in A \).

An analogous definition of constant formal vector fields can be given for the convolution Lie superalgebras \( g_{x} := \text{Hom}_{\mathbb{R}}(S(g), g) \) and \( g_{y} := \text{Hom}_{\mathbb{R}}(\Lambda(\mathfrak{g}), g) \).

**Definition 2.18** ([12, 35]). The map \( x \in g_{x} \) defined by

\[
S^{1}(g) = g \xrightarrow{1_{g}} g , \quad S^{n}(g) \to 0 \quad (n \neq 1)
\]

is the *generic point* of \( g \). Its restriction \( y \in g_{y} \) to \( \Lambda(\mathfrak{g}) \)

\[
\Lambda^{1}(\mathfrak{g}) = \mathfrak{g} \hookrightarrow \mathfrak{g} , \quad \Lambda^{n}(\mathfrak{g}) \to 0 \quad (n \neq 1)
\]

is the *generic point* of \( \mathfrak{g} \).

For every \( n \in \mathbb{N} \) the map \((\text{ad } x)^{n} : g_{x} \to g_{x} \) is a \( S(g)^{*} \)-linear morphism of \( g_{x} \). In particular, for every \( a \in g \subseteq g_{x} \) the map \((\text{ad } x)^{n}(a) : S(g) \to g \) is given by

\[
S^{p}(g) \ni a_{1} \cdots a_{p} \mapsto \begin{cases} 0 & p \neq n \\ \pm \sum_{s \in \Sigma_{p}} \text{ad } a_{s(1)} \circ \cdots \circ \text{ad } a_{s(p)}(a) & p = n \end{cases} \quad (2.14)
\]

where ± is given by the rule of signs in supergeometry. An analogous formula holds for the generic point of \( \mathfrak{g} \). A formal power series in the even variable \( t \) is denoted by

\[
f = \sum_{i=0}^{+\infty} f_{i} t^{i} \in \mathbb{R}[t] .
\]

The following definition makes sense due to equation (2.14).
The derivations \( \text{ad}(a) \) of \( \mathcal{U}(g) \) and \( \Phi c(1) \) of \( \mathcal{S}(g) \) are extensions of the derivation \( \text{ad}(a) \) of \( g \) of \( g \).

2. For every \( b \in S(g) \), \( \gamma^{-1}(a \cdot \gamma(b)) = \Phi c(a) \gamma(b)(1) = \Phi c(\gamma(b)(1)) = \Phi c(b) \).

3. The previous cases give \( \gamma^{-1} \circ R_a \circ \gamma = \Phi c - \Phi c_0 = -\Phi c_{-1} \). \( \square \)
It follows that (2.8) is a coalgebra isomorphism, where \( \mathcal{U}(\mathfrak{g}_T) \otimes \Lambda(\mathfrak{g}_T) \) has the tensor product Hopf-algebra structure.

For every \( c \in \mathbb{R} - \{0\} \), denote the formal power series of \( t \cdot \coth\left(\frac{t}{c}\right) \) and \( \th(t) \) by

\[
p_c(t) := t \cdot \coth\left(\frac{t}{c}\right) = c + \sum_{n=1}^{\infty} b_{2n} \frac{2^n t^{2n}}{c^{2n-1} (2n)!} \in \mathbb{R}[[t]]
\]

\[
q_c(t) := -\th(t) = - \sum_{n=1}^{\infty} b_{2n} \frac{(2^{2n-1} - 2) t^{2n-1}}{c^{2n-1} (2n-1)!} \in \mathbb{R}[[t]]
\]

where \( \{b_p\}_{p \in \mathbb{N}} \) is the sequence of Bernoulli numbers. For \( a \in \mathfrak{g}_T \), denote by

\[
\alpha^a := p_c(\text{ad}(y))(a) \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T)
\]

and

\[
\theta^a := q_c(\text{ad}(y))(a) \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T)
\]

the formal vector fields given by

\[
\alpha^a_1 : a_1 \wedge \cdots \wedge a_{2p} \mapsto \sum_{s \in \Sigma_{2p}} b_{2p} \frac{2^{2p}}{(2p)!} \text{sgn}(s) \text{ad} a_{s(1)} \circ \cdots \circ \text{ad} a_{s(2p)}(a)
\]

and

\[
\theta^a_1 : a_1 \wedge \cdots \wedge a_{2p-1} \mapsto \sum_{s \in \Sigma_{2p-1}} b_{2p} \frac{(2^{2p} - 2) (2p)!} {(2p)!} \text{sgn}(s) \text{ad} a_{s(1)} \circ \cdots \circ \text{ad} a_{s(2p-1)}(a)
\]

2.5.2. Koszul’s theorems.

[12] proves the following

**Theorem 2.24 ([12, 31]).** For every \( a \in \mathfrak{g}_T \), the coderivation of \( \mathcal{U}(\mathfrak{g}_T) \otimes \Lambda(\mathfrak{g}_T) \) induced by \( R_a \in \text{CoDer}_R(\mathcal{U}(\mathfrak{g})) \) via the isomorphism (2.8) is given by

\[
\gamma^{-1} \circ R_a \circ \gamma = [- (m_{\mathcal{U}(\mathfrak{g}_T)} \otimes \text{Id}) \circ (\text{Id} \otimes \alpha^a_1 \otimes \text{Id}) + (\text{Id} \otimes m_{\Lambda(\mathfrak{g}_T)}) \circ (\text{Id} \otimes \theta^a \otimes \text{Id})] \circ (\text{Id}_{\mathcal{U}(\mathfrak{g}_T)} \otimes \Delta_{\Lambda(\mathfrak{g}_T)})
\]

The aim is to prove an analogous result for the coderivation \( L_a \in \text{CoDer}_R(\mathcal{U}(\mathfrak{g})) \). First, we prove the following structural result.

**Proposition 2.25.** For every \( a \in \mathfrak{g}_T \), the coderivation of \( \mathcal{U}(\mathfrak{g}_T) \otimes \Lambda(\mathfrak{g}_T) \) induced by \( L_a \in \text{CoDer}_R(\mathcal{U}(\mathfrak{g})) \) via the isomorphism (2.8) satisfies

\[
\gamma^{-1} \circ L_a \circ \gamma = (\theta^a \otimes \text{Id}) \circ \Delta_{\Lambda(\mathfrak{g}_T)} + m_{\Lambda(\mathfrak{g}_T)} \circ (\theta^a \otimes \text{Id}) \circ \Delta_{\Lambda(\mathfrak{g}_T)}
\]

for some \( \theta^a \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T) \) and \( \epsilon^a \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T) \).

**Proof.** For every \( a \in \mathfrak{g}_T \), Theorem 1.15 applied to the coderivation \( \gamma^{-1} \circ L_a \circ \gamma \in \text{CoDer}(\mathcal{U}(\mathfrak{g}_T) \otimes \Lambda(\mathfrak{g}_T)) \) implies that

\[
\gamma^{-1} \circ L_a \circ \gamma = m_{\mathcal{U}(\mathfrak{g}_T) \otimes \Lambda(\mathfrak{g}_T)} \circ (\theta^a \otimes \text{Id} + \epsilon^a \otimes \text{Id}) \circ \Delta_{\mathcal{U}(\mathfrak{g}_T) \otimes \Lambda(\mathfrak{g}_T)}
\]

for some \( \epsilon^a \in \text{Hom}_R(\mathcal{U}(\mathfrak{g}_T) \otimes \Lambda(\mathfrak{g}_T), \mathfrak{g}_T) \) and \( \theta^a \in \text{Hom}_R(\mathcal{U}(\mathfrak{g}_T) \otimes \Lambda(\mathfrak{g}_T), \mathfrak{g}_T) \). It is then sufficient to restrict the equality to \( \Lambda(\mathfrak{g}_T) \). \( \square \)

Now we determine \( \theta^a \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T) \) and \( \epsilon^a \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T) \). A naive idea could be to obtain results analogous to part 1) of Corollary 2.23. Unfortunately, the map (2.8) does not commute with the odd derivations of \( \mathfrak{g} \), and in particular
with the adjoint action of an element \( a \in \mathfrak{g}_T \). It turns out that handling the antipodes of \( \mathcal{U}(\mathfrak{g}) \), \( \mathcal{U}(\mathfrak{g}_T) \) and \( \Lambda(\mathfrak{g}_T) \) is useful to obtain what we are looking for.

**Lemma 2.26.** For every \( a \in \mathfrak{g}_T \)

\[
-\delta_{\mathcal{U}(\mathfrak{g})} \circ R_a \circ \delta_{\mathcal{U}(\mathfrak{g})} = L_a.
\]

The map induced by the antipode \( \delta_{\mathcal{U}(\mathfrak{g})} \) of \( \mathcal{U}(\mathfrak{g}) \) via the isomorphism (2.8) is given by

\[
\delta_{\mathcal{U}(\mathfrak{g})} \circ \gamma = \gamma \circ (\text{Id} \otimes \Phi_0) \circ (\delta_{\mathcal{U}(\mathfrak{g}_T)} \otimes \text{Id} \otimes \delta_{\Lambda(\mathfrak{g}_T)}) \circ (\Delta_{\mathcal{U}(\mathfrak{g}_T)} \otimes \text{Id})
\]

**Proof.** The proof of the first assertion is straightforward. The proof of the second assertion consists of repeated applications of

\[
\gamma(a_1 \cdots a_p)u_i = u_i \gamma(a_1 \cdots a_p) - \gamma(\Phi_0^\circ_1(a_1 \cdots a_p))
\]
to the equation

\[
\delta_{\mathcal{U}(\mathfrak{g})} \circ \gamma(u_1 \otimes \cdots \otimes u_n \otimes a_1 \cdots a_p) = (-1)^{p+n} \gamma(a_1 \cdots a_p)u_n \cdots u_1
\]
where \( u_i \in \mathfrak{g}_T \) for \( i = 1, \ldots, n \) and \( a_j \in \mathfrak{g}_T \) for \( j = 1, \ldots, p \). \( \square \)

Denote the formal power series of \( p(t) + tq(t) = t \cdot (\coth(t) - th(t)) \) by

\[
e(t) := t \cdot (\coth(t) - th(t)) = \sum_{n=0}^{+\infty} b_{2n} \frac{(-2^{2n+1} + 2^{2n} + 2)t^{2n}}{(2n)!} \in \mathbb{R}[[t]].
\]

For \( a \in \mathfrak{g}_T \), denote by

\[
e^a := \epsilon(\text{ad } y)(a) \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T)\]

the formal vector field given by

\[
e^a : a_1 \wedge \cdots \wedge a_{2p} \mapsto \sum_{\sigma \in \Sigma_{2p}} b_{2p} \frac{(-2^{2p+1} + 2^{2p} + 2)}{(2p)!} \text{sgn}(\sigma) \text{ad } a_{\sigma(1)} \circ \cdots \circ \text{ad } a_{\sigma(2p)}(a)
\]

(2.17)

**Theorem 2.27.** For every \( a \in \mathfrak{g}_T \), the restriction to \( \Lambda(\mathfrak{g}_T) \) of the coderivation of \( \mathcal{U}(\mathfrak{g}_T) \otimes \Lambda(\mathfrak{g}_T) \) induced by \( L_a \in \text{CoDer}(\mathcal{U}(\mathfrak{g})) \) via the isomorphism (2.8) is given by

\[
\gamma^{-1} \circ L_a \circ \gamma = (\theta_1^a \otimes \text{Id}) \circ \Delta_{\Lambda(\mathfrak{g}_T)} + m_{\Lambda(\mathfrak{g}_T)} \circ (e^a \otimes \text{Id}) \circ \Delta_{\Lambda(\mathfrak{g}_T)}
\]

where \( e^a \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T) \) is given by (2.17).

**Example 2.28.** For every \( a, a_1, a_2 \in \mathfrak{g}_T \)

\[
a_\gamma(a_1 \cdot a_2) = -\frac{1}{6} [a_1, [a_2, a]] + \frac{1}{6} [a_2, [a_1, a]] + \gamma(a \cdot a_1 \cdot a_2) + \frac{1}{2} [a_1, a]a_2 - \frac{1}{2} [a_2, a]a_1.
\]

**Proof.** From Theorem 2.24 and Lemma 2.26, the l.h.s. of (2.16) equals

\[
- (\text{Id} \otimes \Phi_0) \circ (\delta_{\mathcal{U}(\mathfrak{g}_T)} \otimes \text{Id} \otimes \delta_{\Lambda(\mathfrak{g}_T)}) \circ (\Delta_{\mathcal{U}(\mathfrak{g}_T)} \otimes \text{Id}) \circ \big[m_{\Lambda(\mathfrak{g}_T)} \circ (\alpha_1^a \otimes \text{Id}) - (\theta_1^a \otimes \text{Id}) \big] \circ \delta_{\Lambda(\mathfrak{g}_T)}.
\]

Using the sigma notation of [38], the previous equation can be re-written as

\[
v \mapsto - (-1)^{|v_1|} \theta_1^a v_{(1)} \otimes v_{(2)} + (-1)^{|v_1|} 1 \otimes \Phi_0^{|v_1|} \theta_1^a v_{(2)} + (-1)^{|v_1|} 1 \otimes \alpha_1^a v_{(1)} \cdot v_{(2)}
\]

\[
= \theta_1^a v_{(1)} \otimes v_{(2)} - 1 \otimes \Phi_0^{|v_1|} \theta_1^a v_{(2)} + 1 \otimes \alpha_1^a v_{(1)} \cdot v_{(2)}
\]
where \( v \in \Lambda(\mathfrak{g}_T) \). The aim is to re-write the second term. Proposition 2.25 states that there exists \( \lambda^a \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T) \) such that

\[
-\theta_0^a \gamma(v_1) v_2 = m_{\Lambda(\mathfrak{g}_T)} \circ (\lambda^a \otimes \text{Id}) \circ \Delta_{\Lambda(\mathfrak{g}_T)}.
\]

The fact that the antipode \( \delta_{\Lambda(\mathfrak{g}_T)} = \delta_{\text{S}(\mathfrak{g})}|_{\Lambda(\mathfrak{g}_T)} \) of \( \Lambda(\mathfrak{g}_T) \) is the convolution inverse of \( \text{Id}_{\Lambda(\mathfrak{g}_T)} \) implies that \( \lambda^a \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), \mathfrak{g}_T) \) equals

\[
| v \mapsto t(\text{ad} y)(\theta_1^a v_1)^2(v_2) = -t \cdot th \left( \frac{\gamma}{2} \right)(\text{ad} y)(a)(v)
\]

which is the assertion of the Theorem. The last equality is a consequence of the following equations. Since \( t(\text{ad} y)(\theta_1^a v_1)^2(v_2) \in \mathfrak{g}_y \) acts non-trivially only on \( \mathfrak{g}_T \subseteq \Lambda(\mathfrak{g}_T) \)

\[
t(\text{ad} y)(\theta_1^a v_1)^2(v_2) = \sum_{j=1}^{2n} (-1)^j [v_j, \theta_1^a (v_1 \cdot v_j \cdot v_2n)]
\]

(2.18)

where \( v = v_1 \cdots v_{2n} \in \Lambda(\mathfrak{g}_T) \). \( v_j \in \mathfrak{g}_T \) for \( j = 1, \ldots, 2n \). Moreover

\[
\theta_1^a (v_1 \cdots v_j \cdots v_{2n}) = \sum_{\sigma \in S_{2n}} \frac{b_{2n}(2^{2n+1} - 2)}{(2n)!} \text{sgn}(\sigma) \text{ad} v_{\sigma(1)} \circ \cdots \circ \text{ad} v_{\sigma(j)} \circ \cdots \circ \text{ad} v_{\sigma(2n)}(a)
\]

which implies that the last term of (2.18) equals

\[
\sum_{j=1}^{2n} \sum_{\sigma \in S_{2n}} (-1)^j \frac{b_{2n}(2^{2n+1} - 2)}{(2n)!} \text{sgn}(\sigma) \text{ad} v_j \circ \text{ad} v_{\sigma(1)} \circ \cdots \circ \text{ad} v_{\sigma(j)} \circ \cdots \circ \text{ad} v_{\sigma(2n)}(a)
\]

\[
= - \sum_{\sigma \in S_{2n}} \frac{b_{2n}(2^{2n+1} - 2)}{(2n)!} \text{sgn}(\sigma) \text{ad} v_{\sigma(1)} \circ \cdots \circ \text{ad} v_{\sigma(2n)}(a) \quad \Box
\]

2.5.3. Left odd symmetries.

The following Theorem of Koszul holds.

**Theorem 2.29** ([12, 31, 35]). For every \( a \in \mathfrak{g}_T \) and \( f \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), C^\infty(U)) \)

\[
\varphi(a)(f) = -\theta_1^a \ast f + (-1)^{|f|} f \circ (\alpha_1^a \ast \text{Id}_{\Lambda(\mathfrak{g}_T)})
\]

(2.19)

where the convolution product \( \theta_1^a \ast f \) is defined via the map \( \varphi_0 : \mathfrak{g}_T \rightarrow T(G_0) \).

**Proof.** Theorem 2.24 implies the relation

\[
(-1)^{|v|} \gamma(v) a = \gamma(\alpha_1^a (v_1)) \cdot v_2) - \theta_1^a (v_1) \cdot \gamma(v_2) = \gamma \circ (\alpha_1^a \ast \text{Id}_{\Lambda(\mathfrak{g}_T)}) - \theta_1^a \ast \gamma
\]

for \( a \in \mathfrak{g}_T, v \in \Lambda(\mathfrak{g}_T) \). It follows that

\[
(-1)^{|f|} \varphi(a)(f)(v) = (-1)^{|f|} \varphi(a)(F)(\gamma(v)) = (-1)^{|f|+1} F(\gamma(v) a)
\]

\[
= (-1)^{|f|} F(\gamma(v) a) = [f \circ (\alpha_1^a \ast \text{Id}_{\Lambda(\mathfrak{g}_T)}) - (-1)^{|f|} \theta_1^a \ast f](v) \quad .
\]

\( \Box \)

For every \( a \in \mathfrak{g}_0 \)

\[
\frac{\partial}{\partial a^*_g} g \simeq a \in \mathfrak{g}_T
\]

(2.20)

i.e. \( \text{ev}_T(\varphi(a)) \) equals the algebraic derivation given by contraction of \( \mathfrak{g}_T \) and \( \mathfrak{g}_T^* \).

On the other hand, if \( [\mathfrak{g}_1, \mathfrak{g}_1] = 0 \), formula (2.19) implies that

\[
\varphi(a)(f)(a_1 \wedge \cdots \wedge a_p) = (-1)^{|f|} f(a \wedge a_1 \wedge \cdots \wedge a_p) \quad ,
\]
implies the relation

\[ M = (g \text{ depends on the bracket } H = \{g, g_T\}) \]

Proposition 2.30, Right odd symmetries.

2.5.4. Right odd symmetries.

**Theorem 2.30.** For every \( a \in g_T \) and \( f \in \text{Hom}_R(\Lambda(\mathfrak{g}_T), C^\infty(U)) \)

\[ \hat{\varphi}(a)(f) = \theta_{1}^{\text{Ad}_g^{-1} a} \ast f + (-1)^{|f|} f \circ (\epsilon^{\text{Ad}_g^{-1} a} \ast \text{Id}_{\Lambda(\mathfrak{g}_T)}) \]  

(2.21)

where the convolution product \( \theta_{1}^{\text{Ad}_g^{-1} a} \ast f \) is defined via the map \( \varphi_0 : g_T \to T(G_0) \).

**Proof.** Theorem 2.27 implies the relation

\[ a \gamma(v) = \gamma(e^a v_1) \cdot v_2 + \theta_{1}^{\epsilon} v_2 \cdot \gamma(v_2) = \gamma \circ \epsilon \ast \text{Id}_{\Lambda(\mathfrak{g}_T)} + \theta_{1}^{\epsilon} \ast \gamma \]

for \( a \in g_T, v \in \Lambda(\mathfrak{g}_T) \). It follows that

\[ (\hat{\varphi}(a)f)(v)(g) = (\hat{\varphi}(a)F)(v)(g) = (-1)^{|f|} F((\text{Ad}_g^{-1} a) \gamma(v))(g) \]

\[ = [(-1)^{|f|} f \circ (\epsilon^{\text{Ad}_g^{-1} a} \ast \text{Id}_{\Lambda(\mathfrak{g}_T)}) + \theta_{1}^{\text{Ad}_g^{-1} a} \ast f](v)(g) \]  

\[ \square \]

Denote a basis of \( g_T \) by \( \{s_{\alpha}\}_{\alpha=1}^{\dim g_T} \) and its dual basis by \( \{s^{\alpha}\}_{\alpha=1}^{\dim g_T} \). For every \( g \in G_0 \) and every natural number \( 1 \leq \beta \leq \dim g_T \), the value of the right-invariant odd vector field \( \hat{\varphi}(s_\beta) \) is given by

\[ \hat{\varphi}(s_\beta) |_g = - \text{Ad}_g^{\beta} (g^{-1}) \otimes \frac{\partial}{\partial s^{\alpha}} |_g \cong \text{Ad}_g^{\beta} (g^{-1}) \otimes s_\alpha \cong \text{Ad}_g^{-1} s_\beta \]  

(2.22)

where

\[ \text{Ad}_g | g_T = \begin{pmatrix} 1 & \text{Ad}_g & \cdots & \text{Ad}_g^{\beta} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \text{Ad}_g & \cdots & \text{Ad}_g^{\beta} \\ 1 & \text{Ad}_g & \cdots & \text{Ad}_g^{\beta} \end{pmatrix} \in \text{End}_R(\mathfrak{g}_T) \]

is the matrix representation of the adjoint action of \( G_0 \) on \( g_T \). Note that \( \text{ev}_T(\hat{\varphi}(a)) \) depends on the bracket \([g_T, g_T]\). Moreover, if \([g_1, g_1] = 0\), formula (2.21) implies that

\[ \hat{\varphi}(a)(f)(a_1 \wedge \cdots \wedge a_p) = (-1)^{|f|} f(\text{Ad}_g^{-1} a \wedge a_1 \wedge \cdots \wedge a_p) \]

, i.e. that

\[ \hat{\varphi}(a) = - \frac{\partial}{\partial (\text{Ad}_g^{-1} a)^\epsilon} \cong \text{Ad}_g^{-1} a \in g_T \]

is the algebraic derivation given by contraction of \( g_T \) and \( g_T \) globally.

### 3. Homogeneous supermanifolds

As in the classical theory, a homogeneous supermanifold can be understood either in terms of a transitive action of a Lie supergroup \( G = (G_0, A_G) \) on a supermanifold \( M = (M_0, A_M) \) (see Definition 3.7) or by means of a quotient of \( G = (G_0, A_G) \) with respect to a closed Lie subsupergroup \( H = (H_0, A_H) \) (see Definition 3.9).
3.1. Action of a Lie supergroup and of a super Harish-Chandra pair.

**Definition 3.1** ([11]). An action of a Lie supergroup $G = (G_0, A_G)$ on a supermanifold $M = (M_0, A_M)$ is a morphism $\rho = (\rho_0, \rho^*) : G \times M \to M$ satisfying

1) $\rho \circ (i\rho_G \times \rho) = \rho \circ (m \times i\rho_M) : G \times G \times M \to M$

2) $\rho \circ (\bar{c}, i\rho_M) = i\rho_M : M \to M.$

The triple $(M_0, A_M, \rho)$ is called a $G$-supermanifold.

**Definition 3.2** ([11]). An action of a sHC pair $(G_0, g)$ on a supermanifold $M = (M_0, A_M)$ is a pair $(\rho_{\varpi}, \hat{\rho})$ consisting of a global action, i.e. a group homomorphism

$\rho_{\varpi} : G_0 \to \text{Aut} (M)$

and a fundamental action, i.e. a Lie superalgebra anti-homomorphism

$\hat{\rho} : g \to T(M)$

satisfying

$\hat{\rho}(B)(f) = (\rho_{\varpi})_*(B)(f) \ := \frac{d}{dt}|_{t=0}(\rho_{\varpi} \circ \exp(tB))^*(f) \quad f \in A(M)$

for every $B \in g_{\varpi}$. The triple $(A_M, \rho_{\varpi}, \hat{\rho})$ is called a $(G_0, g)$-supermanifold.

If there is no danger of confusion, the global diffeomorphisms and the fundamental vector fields are denoted by

$\text{Aut} (M) \ni \rho_{\varpi}(g) := \varpi(g)$, \quad $T(M) \ni \hat{\rho}(A) := \hat{A}$

where $g \in G_0$ and $A \in g$.

**Theorem 3.3** ([11]). Any action $(M_0, A_M, \rho)$ of a Lie supergroup $G = (G_0, A_G)$ defines an action $(\hat{A}_M, \rho_{\varpi}, \hat{\rho})$ of the corresponding sCH pair $(G_0, g)$ where

$g^* := (ev_g \otimes \text{Id}) \circ \rho^* : A(M) \to A(M)$, \quad $\hat{A} := (A|_{e} \otimes \text{Id}) \circ \rho^*$

for all $g \in G_0$ and $A \in g$. The correspondence $\rho \mapsto (\rho_{\varpi}, \hat{\rho})$ is a bijection between the sets of actions of a Lie supergroup and those of the associated sHC pair respectively.

**Example 3.4.** Let $G = (G_0, g)$ be a sHC pair and let $V = V_{\varpi} + V_{\varpi}$ be a finite dimensional supervector space. A representation $\varphi : G \to \text{GL}_R(V)$ of $G$ on $V$ consists of

1) a representation $\varphi_{\varpi} : G_0 \to \text{GL}_R(V)$ of $G_0$ on $V$,

2) a representation $\varphi : g \to \text{gl}_R(V)$ of $g$ on $V$,

such that $(\varphi_{\varpi}^*)_* = \varphi|_{\varpi} : \varpi \to \text{gl}_R(V)_{\varpi}$. The representation (2.3) of Definition 2.7 is called adjoint representation; for any closed Lie subsupergroup $(H_0, h) \subseteq (G_0, g)$ (i.e. $H_0 \subseteq G_0$ closed) it restricts to a well-defined map $Ad_h : H \to \text{GL}_R(g/h)$.

**Example 3.5.** The adjoint action $a : G \times G \to G$ of a Lie supergroup $G = (G_0, A_G)$ on itself is defined as an action of the underlying sHC pair $(G_0, g)$:

$\rho_{\varpi}(g)^* := a^*_g := L^*_g \circ R^*_g$ , \quad $\hat{A} := (A|_{e} \otimes \text{Id}) \circ m^* - A$

for every $g \in G_0$ and $A \in g$.

**Lemma 3.6.** Let $(M_0, A_M, \rho)$ be a $G$-supermanifold, $A \in g$ and $g \in G_0$. Then

1) $(A \otimes \text{Id}) \circ \rho^* = (\text{Id} \otimes \hat{A}) \circ \rho^*$ ,

2) $(e_A \otimes \text{Id}) \circ \rho^* = \rho^* \circ \hat{A}$ ,

3) $g_* \hat{A} = Ad_g \hat{A}$ ,

where $v := A|_{e} \in T_e(G)$. 


3.2. Orbit map and stability subgroup at a point.

Every point \( p \in M_0 \) of a \( G \)-supermanifold \((M_0, \mathcal{A}_M, \rho)\) defines a morphism of supermanifolds
\[
\rho_p = \left( (\rho_p)_0, \rho^*_p \right) : G \rightarrow M,
\]
called the orbit map of the point \( p \in M_0 \), which satisfies
\[
\rho_p \circ R_g = \rho_g \cdot \rho_p, \quad g \circ \rho_p = \rho_p \circ L_g
\]
for every \( g \in G_0 \).

**Definition 3.7 ([21])**. The action \( \rho \) is transitive if \( \rho_p \) is a surjective submersion for every \( p \in M_0 \), i.e. the underlying action of \( G_0 \) on \( M_0 \) is transitive and the map
\[
g \mapsto T_p M, \quad A \mapsto A|_{e \circ \rho^*_p} = \hat{A}|_p
\]
is onto. The triple \((M_0, \mathcal{A}_M, \rho)\) is called a homogeneous \( G \)-supermanifold.

**Definition 3.8 ([21])**. The stability subgroup at \( p \in M_0 \) is the closed Lie subsupergroup \( G_p \) of \( G \) defined by the sHC pair \( G_p := ((G_0)_p, \mathfrak{g}_p) \) where
\[
(G_0)_p := \{ h \in G_0 \mid h(p) = p \}
\]
is the stability subgroup of \( G_0 \) and
\[
\mathfrak{g}_p := \{ B \in \mathfrak{g} \mid \hat{B}|_p = 0 \}
\]
consists of the fundamental vector fields with zero value at the point \( p \in M_0 \). The action \( \text{Ad} : (G_0)_p \rightarrow \text{Aut} (\mathfrak{g}_p) \) is the restriction of \( \text{Ad} : G_0 \rightarrow \text{Aut} (\mathfrak{g}) \).

The sheaf theoretic description of the stability subgroup is more complicated and can be found in [7]. Note that [7] defines the canonical closed embedding \( G_p \hookrightarrow G \) in terms of commutativity of the following diagram
\[
\begin{array}{ccc}
G_p & \longrightarrow & \mathbb{R}^{0,0} \\
\downarrow & & \downarrow p \\
G & \xrightarrow{\rho_p} & M
\end{array}
\]

3.3. Homogeneous supermanifold.

Let \((H, \mathfrak{h}) \subseteq (G, \mathfrak{g})\) be a closed Lie subsupergroup of a Lie supergroup \( G = (G_0, \mathcal{A}_G) \). Denote by
\[
\pi_0 : G_0 \rightarrow G_0/H_0, \quad pr_1 : G \times H \rightarrow G, \quad R_H := m|_{G \times H} : G \times H \rightarrow G
\]
the canonical projections and the right action of \( H \) on \( G \). For every open set \( U \subseteq G_0/H_0 \), define three subalgebras of \( \mathcal{A}_G(\pi_0^{-1}U) \):

i) The superalgebra of \( R_H \)-invariant superfunctions
\[
\mathcal{A}_{G/H}(U) := \{ f \in \mathcal{A}_G(\pi_0^{-1}U) \mid R^*_H f = pr^*_1 f \}.
\]

ii) The superalgebra of \( R_{H_0} \)-invariant superfunctions
\[
\mathcal{A}_{H_0}(U) := \{ f \in \mathcal{A}_G(\pi_0^{-1}U) \mid (R_{h_0})^* f = f, \forall h \in H_0 \}.
\]

iii) The superalgebra of \( \mathfrak{h} \)-invariant superfunctions
\[
A_{\mathfrak{h}}(U) := \{ f \in \mathcal{A}_G(\pi_0^{-1}U) \mid B f = 0, \forall B \in \mathfrak{h} \}.
\]
The associated sheaves over $G_0/H_0$, denoted by $\mathcal{A}_{G/H}$, $\mathcal{A}_{H_0}$, and $\mathcal{A}_h$ satisfy
\[ \mathcal{A}_{G/H} = \mathcal{A}_{H_0} \cap \mathcal{A}_h \] (3.1)

**Theorem 3.9** ([18, 30]). The pair $G/H = (G_0/H_0, \mathcal{A}_{G/H})$ is a supermanifold. There exists a canonical projection $\pi : G \rightarrow G/H$ satisfying

1) $\text{Ker}(\pi_\ast) = \mathcal{A}(G) \otimes h$
2) $\pi_\ast : T_eG \rightarrow T_{o}(G/H)$ is onto with kernel $\mathfrak{h} \subseteq \mathfrak{g} \cong T_eG$ ($o := eH = \pi_0(e)$)

and a canonical left action $\mu : G \times G/H \rightarrow G/H$ such that the following diagram is commutative

\[
\begin{array}{ccc}
G \times G & \xrightarrow{m} & G \\
\downarrow \text{id}_G \times \pi & & \downarrow \pi \\
G \times G/H & \xrightarrow{\mu} & G/H
\end{array}
\]

For every $G$-supermanifold $(M_0, \mathcal{A}_M, \rho)$, the orbit map
\[ \rho_p = ((\rho_p)_0, \rho_p^\ast) : G/G_p \rightarrow M \] (3.2)
is well-defined. Whenever $(M_0, \mathcal{A}_M, \rho)$ is homogeneous, the orbit map (3.2) is a diffeomorphism with $(\text{Id} \otimes \rho_p^\ast) \circ \rho^\ast = \mu^\ast \circ \rho_p^\ast$.

As in the classical case, there exist adapted coordinates for the canonical projection, i.e. $\mathcal{A}_G(\pi_0^{-1}U) \cong \mathcal{A}_{G/H}(U) \otimes \mathcal{A}(H)$ for a suitable open subset $U \subseteq G_0/H_0$. A homogeneous supermanifold $G/H$ is *reductive* if the Lie superalgebra $\mathfrak{g}$ of $G$ can be decomposed into a subsupervector space direct sum
\[ \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \] (3.3)
such that $\text{Ad}_h \mathfrak{m} \subseteq \mathfrak{m}$ for every $h \in H_0$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. If $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, then $G/H$ is *symmetric*.

**Example 3.10.** Let $G$ be a Poincare’ Lie supergroup and $H_0 := \text{Spin}_{r,s}^0 \subseteq G_0$. The Poincare’ superspacetime $M = G/H_0$ is a reductive homogeneous supermanifold.

Lemma 2.6 and Theorem 3.9 imply the following useful description of the space of vector fields of a reductive homogeneous supermanifold.

**Lemma 3.11.** Let $G/H$ be a homogeneous supermanifold with reductive decomposition (3.3). The map
\[ [\mathcal{A}(G) \otimes \mathfrak{m}]_{R_H} \rightarrow T(G/H), \quad X \mapsto X \circ \pi^\ast \] is an isomorphism.

The induced Lie superalgebra structure on $[\mathcal{A}(G) \otimes \mathfrak{m}]_{R_H}$ is given by
\[ [X, Y] = \left[ \sum_i f^i \otimes A_i, \sum_j g^j \otimes B_j \right] := \sum_{i,j} (-1)^{|A_i||g^j|} f^i g^j \otimes [A_i, B_j]_\mathfrak{m} \]
\[ + \sum_j (X g^j) \otimes B_j - \sum_i (-1)^{|X||g^j|} (Y f^i) \otimes A_i \]
where $[A, B] := [A, B]_h + [A, B]_\mathfrak{m} \in \mathfrak{h} + \mathfrak{m} = \mathfrak{g}$ for every $A, B \in \mathfrak{m}$. 
3.4. Isotropy representation.

**Definition 3.12** ([21]). The linear isotropy representation \( \phi = (\varphi, \varphi) : G_p \to \text{GL}_\mathbb{R}(T_pM) \) of the stability subgroup \( G_p := ((G_0)_p, \mathfrak{g}_p) \) on \( T_pM \) is defined by

\[
\begin{cases}
\varphi_{\mathbb{R}} : (G_0)_p \to \text{GL}_\mathbb{R}(T_pM) & h \mapsto h_{\ast, p} \\
\varphi : \mathfrak{g}_p \to \text{gl}_\mathbb{R}(T_pM) & B \mapsto (v \mapsto -[\hat{B}, v])_p = (-1)^{|B||v|} v \circ \hat{B}
\end{cases}
\]

The natural extension of this representation to \( T_p M^r_s \) is also denoted by \( \phi : G_p \to \text{GL}_\mathbb{R}(T_p M^r_s) \) and the set of invariant \((r,s)\)-tensor \( T \in T_p M^r_s \) by \( (T_p M^r_s)^{\phi(G_p)} \).

**Lemma 3.13** ([21]). Let \((M_0, \mathcal{A}_M, \rho)\) be a homogeneous \( G \)-supermanifold. The linear isotropy representation is equivalent to the adjoint representation \( \text{Ad} : G_p \to \text{gl}_\mathbb{R}(\mathfrak{g}/\mathfrak{g}_p) \) via the natural isomorphism \( \mathfrak{g}/\mathfrak{g}_p \cong T_p M \).

**Definition 3.14.** A superfunction \( f \in \mathcal{A}(M) \) (resp. a vector field \( X \in \mathcal{T}(M) \), resp. a 1-form \( \omega \in \mathcal{T}^\ast(M) \)) on a \( G \)-supermanifold \((M_0, \mathcal{A}_M, \rho)\) is \( G \)-invariant if

\[
f \cong 1 \otimes f = \rho^\ast f \in \mathcal{A}(G \times M) \quad \text{(resp. (Id} \otimes X) \circ \rho^\ast = \rho^\ast \circ X, \text{ resp.)}
\]

\[
\omega(Y) \cong 1 \otimes \omega(Y) = (\rho^\ast \omega)(\text{Id} \otimes Y) \in \mathcal{A}(G \times M) \quad \text{for every } Y \in \mathcal{T}(M)
\]

The definition of \( G \)-invariance of a tensor field \( T \in \mathcal{T}^r_s(M) \) follows naturally from Definition 3.14. The set of all \( G \)-invariant tensor fields is denoted by \( \oplus_{r,s} \mathcal{T}^r_s(M)^{G} \).

**Definition 3.15.** A tensor field \( T \in \mathcal{T}^r_s(M) \) on a \((G_0, \mathfrak{g})\)-supermanifold \((\mathcal{A}_M, \rho, \hat{\rho})\) is \((G_0, \mathfrak{g})\)-invariant if it is preserved by the global action of \( G_0 \) and annihilated by the fundamental action of \( \mathfrak{g} \), i.e.

\[
\begin{cases}
g_{\ast}(T) = T & \forall g \in G_0 \\
L_{A \mathfrak{g}} T = 0 & \forall A \in \mathfrak{g}
\end{cases}
\]

(3.4)

The set of all \((G_0, \mathfrak{g})\)-invariant tensor fields is denoted by \( \oplus_{r,s} \mathcal{T}^r_s(M)^{(G_0, \mathfrak{g})} \).

**Theorem 3.16.** Let \((M_0, \mathcal{A}_M, \rho)\) be a homogeneous \( G \)-supermanifold. The evaluation at a point \( p \in M_0 \)

\[
ev_p : \mathcal{T}^r_s(M)^G \to (T_p M^r_s)^{\phi(G_p)}
\]

is an isomorphism between the space \( \mathcal{T}^r_s(M)^G \) of \( G \)-invariant tensor fields of type \((r,s)\) and the space \((T_p M^r_s)^{\phi(G_p)} \) of isotropy invariant tensors of type \((r,s)\). In particular, for vector fields, the inverse map is given by

\[
(T_p M)^{\phi(G_p)} \ni v \to X_v := (\text{Id} \otimes v) \circ \rho^\ast : \mathcal{A}_M \to \mathcal{A}_{G/G_p \rho_p} \cong \mathcal{A}_M
\]

Moreover \( \oplus_{r,s} \mathcal{T}^r_s(M)^G = \oplus_{r,s} \mathcal{T}^r_s(M)^{(G_0, \mathfrak{g})} \).

**Proof.** See the Appendix. \( \square \)

4. Invariant superconnections on a homogeneous supermanifold

This section generalizes a theorem of Wang ([26, 27]), concerning invariant connections on homogeneous manifolds, to the category of supermanifolds. The main Theorem 4.7 is stated in [10] in the case of even stability subgroup.
4.1. Superconnection.

Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{A}_M$-supermodules on $M_0$.

**Definition 4.1** ([11]). A (super)connection on $\mathcal{E}$ is a morphism $\nabla : \mathcal{T}_M \otimes_\mathbb{R} \mathcal{E} \to \mathcal{E}$ of sheaves of supervector spaces such that, for every open subset $U \subseteq M_0$

1) $\nabla f \cdot Z = f \cdot \nabla f \cdot X$
2) $\nabla f = ( X f ) \cdot Z + (-1)^{|X||Y|} f \nabla f \cdot X$

where $f \in \mathcal{A}_M(U)$, $X \in \mathcal{T}_M(U)$, $Z \in \mathcal{E}(U)$. The curvature of $\nabla$ is defined by

$$R(X,Y) := \nabla_X \nabla_Y Z - (-1)^{|X||Y|} \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where $X, Y \in \mathcal{T}_M(U)$ and $Z \in \mathcal{E}(U)$.

A connection on $\mathcal{T}_M$ is called a linear connection on $M$. There is the usual notion of Christoffel symbols, in local coordinates (1.5) on $M$,

$$\nabla_{\frac{\partial}{\partial \eta^k}} \frac{\partial}{\partial \eta^i} = \sum_k \Gamma_{ij} \frac{\partial}{\partial \eta^k}$$

gives elements $\Gamma_{ij} \in \mathcal{A}_M(U)$ of parity $|\Gamma_{ij}| = |\eta^i| + |\eta^j| + |\eta^k|$. The torsion is defined by

$$T(X,Y) := \nabla_X Y - (-1)^{|X||Y|} \nabla^Y X - [X,Y]$$

where $X, Y \in \mathcal{T}_M(U)$. The covariant derivatives of $R$ are denoted by

$$\nabla_{X_1 \ldots X_{r+1}} R := \nabla_{X_1} \circ \ldots \circ \nabla_{X_{r+1}} R$$

where $X_1, \ldots, X_1 \in \mathcal{T}_M(U)$ and $r \in \mathbb{N}$.

**Definition 4.2** ([20]). The infinitesimal holonomy superalgebra $\mathfrak{hol}(\nabla)^{\inf}_p$ at a point $p \in M_0$ of a linear connection $\nabla : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{T}_M$ on $M$ is the linear Lie superalgebra $\mathfrak{hol}(\nabla)^{\inf}_p \subseteq \mathfrak{gl}_\mathbb{R}(T_p M) = \mathfrak{gl}_\mathbb{R}(T_p M)_\pi + \mathfrak{gl}_\mathbb{R}(T_p M)_\pi$ spanned by

$$\nabla_{X_1 \ldots X_1} R_p (X,Y) : T_p M \to T_p M$$

where $X_1, \ldots, X_1, X, Y \in T_p M, r \in \mathbb{N}$.

For the definition of the holonomy supergroup as a sHC pair see [20]. Note that the restriction

$$\tilde{\nabla} := (\nabla|_{\Gamma(TM_0) \otimes \Gamma(TM)}) : \Gamma(TM_0) \otimes \Gamma(TM) \to \Gamma(TM)$$

is a connection on the tangent bundle $TM$, such that $TM_0$ and $(TM)_\pi$ are $\nabla$-stable. The holonomy of this connection is contained in the body of the holonomy supergroup ([20]). This is, in general, a proper inclusion ([20]).

Let $\phi = (\phi_0, \phi^*) \in \text{Mor} (N, M)$ be a morphism of supermanifolds and recall that the sheaf of $\phi$-vector fields (1.10) is denoted by $\mathcal{T}_\phi$.

**Definition 4.3.** A morphism $\nabla^\phi : (\phi_0)_* \mathcal{T}_N \otimes_{\mathcal{T}_\phi} \mathcal{T}_\phi \to \mathcal{T}_\phi$ of sheaves of supervector spaces is a $\phi$-connection if, for every open subset $U \subseteq M_0$

1) $\nabla^\phi f \cdot X = f \cdot \nabla^\phi f \cdot X$
2) $\nabla^\phi f = ( Y f ) \cdot X + (-1)^{|Y||f|} f \nabla^\phi f \cdot X$

where $f \in \mathcal{A}_N(\phi_0^{-1} U)$, $Y \in \mathcal{T}_N(\phi_0^{-1} U)$, $X \in \mathcal{T}_\phi(U)$.
Example 4.4. The pull-back connection \((\phi^*\nabla) : (\phi_0)_*T_N \otimes_{\mathbb{R}} T_{\phi} \to T_{\phi}\) of a linear connection \(\nabla : T_M \otimes_{\mathbb{R}} T_M \to T_M\) is a \(\phi\)-connection defined by the local expression
\[
(\phi^*\nabla)_Y(X) := \sum_k (Yf^k) \cdot (\phi^* \circ \frac{\partial}{\partial \eta^k}) + \sum (\nabla f^i) f^k \cdot (\phi^*\nabla)_Y(\phi^* \circ \frac{\partial}{\partial \eta^k})
\]
where \(Y \in T_N(\phi_0^{-1}U)\) and \(X \in T_{\phi}(U)\) is given by \((\phi \circ \psi)^* \nabla \) \(\psi \in \text{Mor}(L, N)\).

4.2. Nomizu map.

Definition 4.5. A linear connection \(\nabla : T_M \otimes_{\mathbb{R}} T_M \to T_M\) on a \(G\)-supermanifold \((M_0, A_M, \rho)\) is \(G\)-invariant if
\[
(\text{Id} \otimes \nabla_X Y) \circ \rho^* = (\rho^* \nabla)_{\text{Id} \otimes X}((\text{Id} \otimes Y) \circ \rho^*)
\]
for every \(X, Y \in T(M)\). A linear connection \(\nabla : T_M \otimes_{\mathbb{R}} T_M \to T_M\) on a \((G_0, \mathfrak{g})\)-supermanifold \((A_M, \rho_{\mathfrak{g}}, \hat{\rho})\) is \((G_0, \mathfrak{g})\)-invariant if
\[
g_* (\nabla_X Y) = \nabla_{g_* X} g_* Y
\]
and
\[
\mathcal{L}_A(\nabla_X Y) = \nabla_{\mathcal{L}_A X} Y + (\nabla_X \mathcal{L}_A) Y
\]
for every \(X, Y \in T(M)\), \(g \in G_0\) and \(A \in \mathfrak{g}\). The space of all \(G\) (resp. \((G_0, \mathfrak{g})\))-invariant linear connections is denoted by \(\text{Conn}(M)^G\) (resp. by \(\text{Conn}(M)^{(G_0, \mathfrak{g})}\)).

Definition 4.6. An even linear map \(L : \mathfrak{g} \to \mathfrak{gl}_\mathbb{R}(T_p M)\) is a Nomizu map in \(p \in M_0\) if
\[
\begin{align*}
1) \quad L(B) &= -\mathcal{L}_{\hat{g}}(\cdot)|_p \\
2) \quad L(\text{Ad}_h A) &= h_{\mathfrak{g}^p} \circ L(A) \circ h_{\mathfrak{g}^p}^{-1} \\
3) \quad L([B, A]) &= [L(B), L(A)]
\end{align*}
\]
for all \(B \in \mathfrak{g}_p\), \(A \in \mathfrak{g}\), \(h \in (G_0)_p\), \(A \in \mathfrak{g}\), \(B \in \mathfrak{g}_p\).

The space of all Nomizu maps in \(p \in M_0\) is denoted by \(\text{Nom}_p(M)\).

The linear operator
\[
L_X := \nabla_X - \mathcal{L}_X : T(M) \to T(M) \quad X \in T(M)
\]
is \(A(M)\)-linear. Its value \(L_X|_p : T_p M \to T_p M\) at a point \(p \in M_0\) is well-defined.
Let \(\nabla \in \text{Conn}(M)^{(G_0, \mathfrak{g})}\) be a \((G_0, \mathfrak{g})\)-invariant linear connection on \((A_M, \rho_{\mathfrak{g}}, \hat{\rho})\).

The even linear map
\[
L_{\nabla}^p : \mathfrak{g} \to \mathfrak{gl}_\mathbb{R}(T_p M) \quad A \mapsto L_{\nabla}|_p
\]
is called the Nomizu map in \(p\) associated with \(\nabla\).

4.3. Wang’s theorem for supermanifolds.
4.3.1. Main theorem.

**Theorem 4.7.** Let $(M_0, A_M, \rho)$ be a homogeneous G-supermanifold and let $p \in M_0$ be a fixed point. The correspondence

$$\text{Conn}(M)^G \ni \nabla \mapsto L^\nabla_p \in \text{Nom}_p(M)$$

is a bijection. Moreover $\text{Conn}(M)^G = \text{Conn}(M)^{(G_0, g)}$.

**Proof.** The proof is split in five parts.

i) Every $G$-invariant linear connection on a $G$-supermanifold $(M_0, A_M, \rho)$ is $(G_0, g)$-invariant, i.e. $\text{Conn}(M)^G \subseteq \text{Conn}(M)^{(G_0, g)}$.

Let $X, Y \in T(M)$ be vector fields on $M$. Equation (4.3) follows from applying $(ev_y \otimes \text{Id})$ to both sides of (4.2) and using equation (4.1). Equation (4.4) is a (long) calculation in local coordinates (1.5) which uses the following equality

$$-(-1)^{|A||[j]+[i]|}(\nabla p_{\sigma \rho}) \circ \hat{A} = -(-1)^{|A||[j]+[i]|}(\nabla p_{\sigma \rho}) \circ (A_i \otimes \text{Id}) \circ \rho^*$$

$$= -(A_i \otimes \text{Id}) \circ (\text{Id} \circ \nabla p_{\sigma \rho}) \circ \rho^* = -(A_i \otimes \text{Id})((\rho^* \nabla) p_{\sigma \rho}) \circ \rho^*$$.

ii) The Nomizu map (4.5) associated to a $(G_0, g)$-invariant linear connection on a $(G_0, g)$-supermanifold $(A_M, \rho)$ is an element of $\text{Nom}_p(M)$.

Let $v \in T_p M$ be a vector at $p \in M_0$ and $X \in T(M)$ be a vector field on $M$ such that $X|_p = v$. For every \(B \in g_p\)

$$L^\nabla_p(B)(v) = (\nabla_B X - \mathcal{L}_B X)|_p = (\nabla_B X)|_p - \mathcal{L}_B(X)|_p = -\mathcal{L}_B(X)|_p$$.

**Lemma 3.6** together with equation (4.3) imply that

$$L^\nabla_p(\text{Ad}_h A) = (\nabla_{\text{Ad}_h A} - \mathcal{L}_{\text{Ad}_h A})|_p = (\nabla_{h_* A} - \mathcal{L}_{h_* A})|_p = h_* \rho \circ L^\nabla_p(A) \circ h^{-1}_*$$

for every $h \in (G_0)_p$ and $A \in g$. For every $A \in g$ and $B \in g_p$

$$[-\mathcal{L}_B, L^\nabla_A](X)$$

$$= (\mathcal{L}_B \mathcal{L}_A X - (-1)^{|B||A|}(\mathcal{L}_A \mathcal{L}_B X)) + (-\mathcal{L}_B \nabla_A X + (-1)^{|B||A|}\nabla_A \mathcal{L}_B X)$$

$$= \mathcal{L}_{[B, A]} X - \nabla_{[B, A]} X = (\nabla_{[B, A]} - \mathcal{L}_{[B, A]})(X)$$

where the second to last equality follows from (1.2) and (4.4). Evaluating in $p \in M_0$, the assertion follows.

iii) Injectivity of the correspondence (4.6).

Let $\nabla^i : T_M \otimes_{\mathbb{R}} T_M \to T_M$, $i = 1, 2$, be two $(G_0, g)$-invariant linear connections on $M$ such that the associated Nomizu maps $L^1_p, L^2_p : g \to gl(T_p M)$ coincide, i.e.

$$(\nabla^1_A - \mathcal{L}_A)|_p = (\nabla^2_A - \mathcal{L}_A)|_p$$

for every $A \in g$. For every $X \in T(M)$, equation (4.3) and Lemma 3.6 imply that

$$g_*(\nabla^1_A X - \mathcal{L}_A X) = \nabla^i_{g_* A g_* A} X - \mathcal{L}_{g_* A} g_* A X = (\nabla^i_{A d_g A} g_* A X - \mathcal{L}_{A d_g A} g_* A X)$$

Evaluating in $g^{-1}p \in M_0$

$$L^1_{g^{-1}p}(A) = g^{-1}|_p \circ L^1_p(\text{Ad}_g A) \circ g_*|_{g^{-1}p} : T_{g^{-1}p} M \to T_{g^{-1}p} M$$

and then $L^1_q = L^2_q$ for every $q \in M_0$. Equations (4.4) and (1.2) imply that

$$\mathcal{L}_B(L^1_A X) = -L^1_{\mathcal{L}_B A} X + (-1)^{|A||B|} L^1_B \mathcal{L}_B X$$
for every \( A, B \in \mathfrak{g} \) and \( X \in T(M) \). In particular
\[
\mathcal{L}_\mathfrak{g}(L^i_\mathfrak{g}T(M)) \subseteq L^i_\mathfrak{g}T(M)
\]
and iterating (for every \( 0 \leq k \leq \infty \))
\[
\mathcal{L}_{\mathcal{A}_k} \cdots \mathcal{L}_{\mathcal{A}_1}(L^1_\mathcal{A}X) \subseteq L^1_\mathfrak{g}T(M)
\]
where \( A_i \in \mathfrak{g} \) for \( 1 \leq i \leq k \). For every \( 0 \leq k \leq \infty \)
\[
(\mathcal{L}_{\mathcal{A}_k} \cdots \mathcal{L}_{\mathcal{A}_1}(L^1_\mathcal{A}X))_\pi = (\mathcal{L}_{\mathcal{A}_k} \cdots \mathcal{L}_{\mathcal{A}_1}(L^2_\mathcal{A}X))_\pi .
\]
Lemma 1.32 applied to the vector field \( L^1_\mathcal{A}X - L^2_\mathcal{A}X = \nabla^1_\mathcal{A}X - \nabla^2_\mathcal{A}X \) implies that
\( \nabla^1_\mathcal{A}X = \nabla^2_\mathcal{A}X \) and, from Nakayama’s Lemma, \( \nabla^i = \nabla^2 \).

iv) In order to prove surjectivity of the correspondence (4.6), some preliminaries are needed.

Denote \( G_p \) by \( H \) and let \( G/H \) be the associated homogeneous supermanifold (recall Theorem 3.9). Say that a \( \pi \)-connection \( \nabla^\pi : (\pi_0)^* T_G \otimes_R T_\pi \to T_\pi \) is projectable if
\[
R_{H_0} - \text{Invariance} \quad R^*_h \circ (\nabla^\pi_Y X) = \nabla^\pi_{(R^*_h)_Y} (R^*_h \circ X)
\]
\( \mathfrak{h} - \text{Invariance} \quad \nabla^\pi_B (\nabla^\pi_Y X) = \nabla^\pi_{\mathfrak{h}_B Y} X + (-1)^{|Y||B|} \nabla^\pi_B \mathcal{L}_B X
\)
for every \( h \in H_0, Y \in T_G(\pi_0^{-1}U), X \in T_\pi(U), B \in \mathfrak{h} \) and
\[
\text{Horizontalit}\text{y} \quad \nabla^\pi_Y (f \cdot \pi^* \circ X) = (Yf) \cdot (\pi^* \circ X)
\]
for every \( Y \in \mathcal{A}_G(\pi_0^{-1}U) \otimes \mathfrak{h}, f \in \mathcal{A}_G(\pi_0^{-1}U) \) and \( X \in T_{G/H}(U) \). The pull-back
\[
\nabla^\pi_Z := (\pi^* \nabla)
\]
is a bijection between the set of linear connections on \( G/H \) and the set of projectable \( \pi \)-connections. We sketch a proof of this fact. Let \( \nabla : T_{G/H} \otimes_R T_{G/H} \to T_{G/H} \) be a linear connection on \( G/H \). One can check that the pull-back connection
\( \nabla^\pi := (\pi^* \nabla) \) is projectable. Here we only remark that (4.7) is a consequence of (4.1). Consider a projectable \( \pi \)-connection \( \nabla^\pi : (\pi_0)^* T_G \otimes_R T_\pi \to T_\pi \). Using equation (4.9) and adapted coordinates near \( h \in H_0 \subseteq G_0 \) and \( o \in G_0/H_0 \), it is possible to construct (locally near \( o \in U \subseteq G_0/H_0 \)) a map \( \nabla : T_{G/H} \otimes R T_{G/H} \to T_\pi \).

By definition
\[
\nabla^\pi_Z := \nabla^\pi_Y (\pi^* \circ X)
\]
where \( X, Z \in T_{G/H}(U) \) and \( Y \in T_G(\pi_0^{-1}U) \) is \( \pi \)-related to \( Z \). From (4.7) and (4.8), check that locally \( \nabla : T_{G/H} \otimes_R T_{G/H} \to T_{G/H} \). Repeating this construction for every \( gH \in G_0/H_0 \), we get a globally well-defined linear connection on \( G/H \).

Due to Theorem 3.9 and isomorphism (4.10), we need to prove that to any even map \( L : \mathfrak{g} \to \mathfrak{gl}_R(\mathfrak{g}/\mathfrak{h}) \) satisfying
\[
1) \quad L(B) = \text{Ad}_B \text{ for every } B \in \mathfrak{h},
2) \quad L(\text{Ad}_h A) = \text{Ad}_h \circ L(A) \circ \text{Ad}_{h^{-1}} \text{ for every } h \in H_0, A \in \mathfrak{g},
3) \quad L([B, A]) = [L(B), L(A)] \text{ for every } A \in \mathfrak{g}, B \in \mathfrak{h},
\]
there exists a corresponding projectable \( \pi \)-connection. Every \( X \in T_\pi \) can be locally written as
\[
X = \sum_j (g^j \otimes B_j) \circ \pi^*~
\]
where \( g^i \otimes B_j \in A_G \otimes g \) and two such expressions differ by an element of \( A_G \otimes \mathfrak{h} \).

The following \( \pi \)-connection is well-defined

\[
\nabla^\pi : T_G(\pi_0^{-1}U) \otimes_{\mathbb{R}} T_\pi(U) \rightarrow T_\pi(U)
\]

\[
\left( \sum_i (f^i \otimes A_i) \otimes \mathbb{R}X \right) \rightarrow \sum_i f^i \{(A_ig^i)(B_j \circ \pi^*) + (-1)^{|A_i||g^i|}g^i(L(A_i)B_j) \circ \pi^* \}
\]

where \( \sum_i f^i \otimes A_i \in A_G(\pi_0^{-1}U) \otimes g \). It is projectable, for equation (4.7) is a consequence of 2), equation (4.8) of 1) and 3) while equation (4.9) of 1). The condition of \( G \)-invariance follows directly from (4.11).

\( \mathbf{v) } \) The inclusion \( \text{Conn}(M)^{(G_0, \mathfrak{g})} \subseteq \text{Conn}(M)^G \) is now easy to prove. \( \square \)

In the reductive case, the set of \( G \)-invariant linear connections is in bijective correspondence with the set of even linear mappings \( L_m : m \rightarrow \mathfrak{gl}_R(m) \) satisfying

\[
\cdot \ L_m(\text{Ad}_{h})A = \text{Ad}_h \circ L_m(A) \circ \text{Ad}_{h^{-1}} \quad \text{for every } h \in H_0, \ A \in m,
\]

\[
\cdot \ L_m([B, A]) = [\text{ad}_B, L_m(A)] \quad \text{for every } A \in m, \ B \in \mathfrak{h},
\]

where \( L_{|m} = L_m, \ L_{|\mathfrak{h}} = \text{ad}_h \) is the Nomizu map. By abuse of language, we refer to the map \( L_m : m \rightarrow \mathfrak{gl}_R(m) \) as the \( \text{Nomizu map} \). The following corollaries are direct consequences of the main Theorem 4.7.

**Corollary 4.8.** Let \( M = G/H \), be a homogeneous supermanifold with reductive decomposition (3.3). The \( G \)-invariant linear connection associated with the Nomizu map \( L_m : m \rightarrow \mathfrak{gl}_R(m) \) is given by

\[
\nabla : T(M) \otimes_{\mathbb{R}} T(M) \rightarrow T(M)
\]

\[
(f^i \otimes A_i) \otimes (g^j \otimes B_j) \rightarrow f^i \{(A_ig^i)B_j + (-1)^{|A_i||g^i|}g^i(L_m(A_i)B_j)\}
\]

where \( f^i \otimes A_i, \ g^i \otimes B_j \in \left[A(G) \otimes m\right]^{R_H} \cong T(M) \) and we have used the Einstein summation convention.

**Proof.** Lemma 3.11 gives an equivalent description of the space of vector fields of a homogeneous reductive supermanifold \( M = G/H \). \( \square \)

**Definition 4.9.** Let \( G/H \) be a homogeneous supermanifold with reductive decomposition (3.3). The \( \text{canonical connection } \nabla^\text{can} \) (resp. the \( \text{natural torsion-free connection } \nabla^\text{free} \)) is the \( G \)-invariant linear connection associated with

\[
L_m = 0 \quad \text{(resp. } L_m(A) = \frac{1}{2}[A, \cdot]_m) \quad .
\]

If the stability subgroup is even, i.e. \( H = H_0 \), the following corollary holds

**Corollary 4.10.** Let \( H = H_0 \subseteq G \) be a connected closed Lie subgroup of a Lie supergroup \( G \) such that \( M_0 = G_0/H_0 \) is a homogeneous manifold with reductive decomposition \( g_0 = \mathfrak{h} + m_\mathfrak{g}_\mathfrak{g} \). Every pair of even \( \mathfrak{h} \)-invariant linear map \( L_{m_\mathfrak{g}_\mathfrak{g}} \in \text{Hom}_R(m_\mathfrak{g}_\mathfrak{g}, \mathfrak{gl}_R(\mathfrak{g}_\mathfrak{g}))^\mathfrak{h}_\mathfrak{g}_\mathfrak{g} \), \( L_{\mathfrak{g}_\mathfrak{g}} \in \text{Hom}_R(\mathfrak{g}_\mathfrak{g}, \mathfrak{gl}_R(\mathfrak{m}_\mathfrak{g}_\mathfrak{g}))^\mathfrak{h}_\mathfrak{g}_\mathfrak{g} \)

defines an extension of a fixed Nomizu map \( L_0 \in \text{Hom}_R(m_\mathfrak{g}_\mathfrak{g}, \mathfrak{gl}_R(m_\mathfrak{g}_\mathfrak{g}))^\mathfrak{h}_\mathfrak{g}_\mathfrak{g} \) on \( M_0 \) to a Nomizu map \( L_m \in \text{Hom}_R(m, \mathfrak{gl}_R(m))^\mathfrak{h}_\mathfrak{g}_\mathfrak{g} \) on the homogeneous supermanifold \( M = G/H_0 \) with reductive decomposition \( g = \mathfrak{h} + m_\mathfrak{g}_\mathfrak{g} \).

Via Corollary 4.10, the Nomizu map \( L_m \in \text{Hom}_R(m, \mathfrak{gl}_R(m))^\mathfrak{h}_\mathfrak{g}_\mathfrak{g} \) associated with

\[
L_0 = 0 \quad , \quad L_{m_\mathfrak{g}_\mathfrak{g}}|_{\mathfrak{g}_\mathfrak{g}} = \text{ad}_{m_\mathfrak{g}_\mathfrak{g}}|_{\mathfrak{g}_\mathfrak{g}} \quad , \quad L_{\mathfrak{g}_\mathfrak{g}} = 0
\]

defines a \( G \)-invariant linear connection \( \nabla^S \) on \( G/H_0 \) called the \textbf{supersymmetry} connection.
4.3.2. Invariant superconnections on a Poincare' superspacetime.

Invariant linear connections on a Poincare’ superspacetime of signature \((r, s)\) are in bijective correspondence with Spin\(^0_{r,s}\)-equivariant even maps

\[ L : \mathbb{R}^{r,s} + S \rightarrow \mathfrak{gl}(\mathbb{R}^{r,s} + S), \]

i.e. with Spin\(_{r,s}\)-equivariant maps

\[ \mathbb{R}^{r,s} \rightarrow ((\mathbb{R}^{r,s})^* \otimes \mathbb{R}^{r,s}) \oplus (S^* \otimes S) \quad \text{and} \quad S \rightarrow ((\mathbb{R}^{r,s})^* \otimes S) \oplus (S^* \otimes \mathbb{R}^{r,s}). \]

The existence of non-degenerate invariant bilinear forms on \(\mathbb{R}^{r,s}\) and \(S\) (see [1]) implies that \(\mathbb{R}^{r,s}\) and \(S\) are self-dual Spin\(_{r,s}\)-modules, i.e. \((\mathbb{R}^{r,s})^* \cong \mathbb{R}^{r,s}\) and \(S^* \cong \mathbb{R}^{r,s}\).

**Theorem 4.11.** The dimension \(D\) of the vector space of invariant connections on a Poincare’ superspacetime of signature \((r, s)\) depends on \(r - s\) mod 8 as follows

\[
\begin{array}{cccccccc}
 r - s \mod 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 D & 12 & 24 & 12 & 24 & 12 & 6 & 3 & 6 & 12 \\
\end{array}
\]

**Proof.** We want to determine the dimension of the following three spaces

\[
\text{dim}_\mathbb{R} \text{Hom} (\mathbb{R}^{r,s}, \mathbb{R}^{r,s} \otimes \mathbb{R}^{r,s})^\text{spin}_{r,s} = 0 \quad \text{dim}_\mathbb{R} \text{Hom} (\mathbb{R}^{r,s}, S \otimes S)^\text{spin}_{r,s} = \text{dim}_\mathbb{R} \text{C}_{r,s} = \dim \text{Hom} (S, T(M)^{\text{spin}_{r,s}}) = \dim \text{Hom} (S, \mathbb{R}^{r,s} \otimes S)^\text{spin}_{r,s} = \dim \mathbb{R} \text{C}_{r,s},
\]

where \(\text{dim}_\mathbb{R} \text{Hom} (\mathbb{R}^{r,s}, S \otimes S)^\text{spin}_{r,s} = \dim \mathbb{R} \text{Hom} (S, \mathbb{R}^{r,s} \otimes S)^\text{spin}_{r,s} = \dim \mathbb{R} \text{C}_{r,s}\). The assertion of the theorem is then a direct consequence of Corollary 1.3 of [1].

4.3.3. Reconstruction of an invariant connection in local coordinates of \(M\).

Let \((M_0, \mathcal{A}_M, \rho)\) be a homogenous \(G\)-supermanifold of dimension \(\text{dim} M = m|n\). When \(n\) is small the \(G\)-invariant linear connection \(\nabla : \mathcal{T}_M \otimes \mathcal{T}_M \rightarrow \mathcal{T}_M\) associated with a Nomizu map \(L_\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(T_pM)\) can be reconstructed in local coordinates \(\{x^i, \xi_j\}\) of \(M\). For every \(g \in M_0\), consider

\[ L_q(\cdot) := g_*^{-1}|_p \circ L_\rho(\text{Ad}_g \cdot) \circ g_*|_{g^{-1}p} : \mathfrak{g} \rightarrow \mathfrak{gl}(T_qM) \]

where \(q = g^{-1}p \in M_0\) for some \(g \in G_0\). By definition of transitive action, every \(Y \in T(M)\) can be locally written (non-uniquely) on an open set \(U \subseteq M_0\) as

\[ Y = \sum_i f^i A_i \]

where \(\{A_i\}\) is a basis of \(\mathfrak{g}\) and \(f^i \in \mathcal{A}_M(U)\). Write

\[ \nabla_Y X|_q = f^i(q) (L_q(A_i)(X|_q) + (L_{A_i}X)|_q) \]

for every \(X \in \mathcal{T}_M(U)\) and \(q \in U\). This is not sufficient to reconstruct the connection unless \(n = 0\). However, a Taylor expansion with respect to the odd coordinates

\[ \nabla_Y X = (\nabla_Y X)|_q + \sum_{k=1}^n \xi_k (L_{\frac{\partial}{\partial x^k}} \nabla_Y X)|_q + \sum_{k \leq j} \xi_k \xi_j \left( L_{\frac{\partial}{\partial x^k}} L_{\frac{\partial}{\partial x^j}} \nabla_Y X|_q \right) + \ldots \]
to construct $\nabla$. Indeed the condition of $(G_0, g)$-invariance implies that

$$(L_{\frac{\partial}{\partial \xi_k}} \nabla_Y X)|_q = \sum_j (g^j_k (\nabla_{[A_j, Y]} X + (-1)^{|A_j||Y|} \nabla_Y [A_j, X]) - (-1)^{|Y|+|X|}(\nabla_Y X)g^j_k) A_j)|_q$$

where $\frac{\partial}{\partial \xi_k} = \sum_j g^j_k A_j$ is a local (non-unique) description of the odd derivation $\frac{\partial}{\partial \xi_k}$.

### 4.3.4. Curvature, Torsion and Holonomy.

Let $(M_0, A_M, \rho)$ be a homogeneous $G$-supermanifold, $p \in M_0$. The isotropy subgroup $G_p$ is denoted by $H$. The identifications

$$(M_0, A_M, \rho) \cong (G_0/H_0, G/H, \mu), \quad \pi_{\ast, e} : g/h \to T_0 G/H$$

are tacitly used. The curvature and the torsion of a $G$-invariant linear connection $\nabla : T_M \otimes_R T_M \to T_M$ are invariant tensors satisfying for every $A, B \in g$

$$R_o(A, B) = [L(A), L(B)] - L([A, B]),$$

$$T_o(A, B) = L(A)B - (-1)^{|A||B|} L(B)A - [A, B]$$

where $L : g \to \mathfrak{gl}_R (g/h)$ is the Nomizu map associated with $\nabla$. The following lemma then holds.

#### Lemma 4.12.

A $G$-invariant linear connection of a homogeneous $G$-supermanifold is flat if and only if the associated Nomizu map is a Lie superalgebra morphism.

In the reductive case, for every $A, B \in m$

$$R_o(A, B) = [L_m(A), L_m(B)] - L_m([A, B]_m) - [[A, B]_h, \cdot],$$

$$T_o(A, B) = L_m(A)B - (-1)^{|A||B|} L_m(B)A - [A, B]_m.$$  

A $G$-invariant linear connection preserves a $G$-invariant tensor field $T \in T^*_M(M)$, i.e. $\nabla T = 0$, if and only if $L_p(g) \subseteq \{ L \in \mathfrak{gl}_R (T_p M) | L \cdot T|_p = 0 \}$.

#### Definition 4.13.

Let $M = G/H$ be a homogeneous manifold, $\mathfrak{k} \subseteq \mathfrak{gl}_R (g/h)$ a linear Lie superalgebra. A $G$-invariant linear connection is $\mathfrak{k}$-compatible if $L(g) \subseteq \mathfrak{k}$.

#### Example 4.14.

Let $G/H, g = h + m$ be a homogeneous supermanifold with reductive decomposition (3.3). The Levi-Civita connection $\nabla^{LC}$ (see [34] for its existence and unicity) of an invariant metric associated with an $H$-invariant scalar product $g : m \times m \to \mathbb{R}$ is given by

$$L_m(A)B = \frac{1}{2} [A, B]_m + U(A, B)$$

where $U : m \times m \to m$ is the supersymmetric bilinear map given by $(A, B, C \in m)$

$$2g(U(A, B), C) := (-1)^{|B||C|} g(A, [C, B]_m) + (-1)^{|C|(|A|+|B|)} g([C, A]_m, B).$$

In our context, Theorems 5.1 and 8.1 of [20] read as follows.

#### Theorem 4.15 ([20]).

Let $\nabla : T_M \otimes_R T_M \to T_M$ be a $G$-invariant linear connection on a simply connected homogeneous supermanifold $M = G/H$. Then there exists a natural bijective correspondence between

1) Parallel tensors $T \in T^*_s(M)$,

2) Tensors $T_o \in T_o M^*_s$ annihilated by the representation of the infinitesimal holonomy algebra $\mathfrak{hol}(\nabla)^{inf}$ on $T_o M^*_s$. 


We calculate the infinitesimal holonomy algebra of a $G$-invariant connection.

**Theorem 4.16.** The infinitesimal holonomy algebra of a $G$-invariant linear connection $\nabla$ on a homogeneous supermanifold $M = G/H$ is given by

$$\mathfrak{hol}(\nabla)^{\text{inf}} = \mathfrak{r} + [\mathfrak{L}(\mathfrak{g}), \mathfrak{r}] + [\mathfrak{L}(\mathfrak{g}), [\mathfrak{L}(\mathfrak{g}), \mathfrak{r}]] + \cdots$$

where $\mathfrak{L} : \mathfrak{g} \to \mathfrak{gl}_\mathbb{R}(\mathfrak{g}/\mathfrak{h})$ is the Nomizu map associated with $\nabla$ and

$$\mathfrak{r} := \text{Span}_\mathbb{R} \{ [\mathfrak{L}(\mathfrak{A}), \mathfrak{L}(\mathfrak{B})] - \mathfrak{L}([\mathfrak{A}, \mathfrak{B}]) \mid \mathfrak{A}, \mathfrak{B} \in \mathfrak{g} \} \subseteq \mathfrak{gl}_\mathbb{R}(\mathfrak{g}/\mathfrak{h}).$$

**Proof.** Consider $\nabla^\mathfrak{r}_{A_r, \ldots, A_1} R_0(A, B) \in \mathfrak{gl}_\mathbb{R}(\mathfrak{g}/\mathfrak{h})$ for $A_r, \ldots, A_1, A, B \in \mathfrak{g}$. The first two cases are illustrated, the other ones follows similarly.

For $r = 0$, $R_0(A, B) = [\mathfrak{L}(\mathfrak{A}), \mathfrak{L}(\mathfrak{B})] - \mathfrak{L}([\mathfrak{A}, \mathfrak{B}]) \in \mathfrak{r}$.

For $r = 1$,

$$\nabla^\mathfrak{r}_{A_1} R_0(A, B) = \mathfrak{L}(\mathfrak{A}_1) \circ (R_0(A, B)) - R_0(\mathfrak{L}(\mathfrak{A}_1)A, B) - (-1)^{|\mathfrak{A}| |\mathfrak{A}_1|} R_0(A, \mathfrak{L}(\mathfrak{A}_1)B)$$

$$- (-1)^{|\mathfrak{A}_1|(|\mathfrak{A}| + |\mathfrak{B}|)} R_0(A, B) \circ L(\mathfrak{A}_1) = [\mathfrak{L}(\mathfrak{A}_1), R_0(A, B)] \mod \mathfrak{r}.$$ 

$\square$

The proof of the following corollary is as in [27] and thus omitted.

**Corollary 4.17.** The infinitesimal holonomy algebra of a $G$-invariant linear connection $\nabla$ on a homogeneous supermanifold $M = G/H$ with reductive decomposition (3.3) is given by

$$\mathfrak{hol}(\nabla)^{\text{inf}} = \mathfrak{r} + [\mathfrak{L}_\mathfrak{m}(\mathfrak{m}), \mathfrak{r}] + [\mathfrak{L}_\mathfrak{m}(\mathfrak{m}), [\mathfrak{L}_\mathfrak{m}(\mathfrak{m}), \mathfrak{r}]] + \cdots$$

where $\mathfrak{L}_\mathfrak{m} : \mathfrak{m} \to \mathfrak{gl}_\mathbb{R}(\mathfrak{m})$ is the Nomizu map associated with $\nabla$ and

$$\mathfrak{r} := \text{Span}_\mathbb{R} \{ [\mathfrak{L}_\mathfrak{m}(\mathfrak{A}), \mathfrak{L}_\mathfrak{m}(\mathfrak{B})] - \mathfrak{L}_\mathfrak{m}([\mathfrak{A}, \mathfrak{B}]) - [[\mathfrak{A}, \mathfrak{B}]_\mathfrak{h}, \cdot] \mid \mathfrak{A}, \mathfrak{B} \in \mathfrak{m} \} \subseteq \mathfrak{gl}_\mathbb{R}(\mathfrak{m}).$$

In particular, if $\nabla^{\text{can}}$ is the canonical connection and $M_0$ is simply connected, every invariant tensor field is parallel (and viceversa if $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$).

5. **Superizations of a homogeneous spin manifold**

### 5.1. Homogeneous spin manifold.

Let $M_0 = G_0/H$ be a (pseudo)-Riemannian reductive homogeneous space. The Lie algebra $\mathfrak{g}_0$ of the Lie group $G_0$ splits into $\mathfrak{g}_0 = \mathfrak{h} + \mathfrak{m}_\mathfrak{h}$, $\mathfrak{h}$ and $\mathfrak{m}_\mathfrak{h}$ being the Lie algebra of the Lie group $H$ and a $\text{Ad}_H$-invariant complement respectively. The (pseudo)-Riemannian metric on $M_0$ is identified with a $H$-invariant inner product $g \in \text{Bil}_\mathbb{R}(\mathfrak{m}_\mathfrak{h}^H) \subseteq \mathfrak{m}_\mathfrak{h}$. Recall that $\varphi_0 : \mathfrak{g}_0 \to T(G_0)$ is the realization of the (abstract) Lie algebra $\mathfrak{g}_0$ as left-invariant vector fields of $G_0$. It is known that the $H$-principal bundle $\pi_0: G_0 \to M_0$ is a (non-canonical) reduction of the bundle of orthonormal frames of $M_0$. The tangent bundle of $M_0$ is then given by

$$G_0 \times_{\text{Ad}(H)} \varphi_0(\mathfrak{m}_\mathfrak{h}) \cong T(M_0)$$

where an element $[g, \varphi_0(w)] \in G_0 \times_{\text{Ad}(H)} \varphi_0(\mathfrak{m}_\mathfrak{h})$ corresponds to a vector via

$$G_0 \times_{\text{Ad}(H)} \varphi_0(\mathfrak{m}_\mathfrak{h}) \ni [g, \varphi_0(w)] \mapsto (\pi_0)_*g(\varphi_0(w)|_g) \in T_{\pi_0}(M_0).$$

Hence a vector field on $M_0$ is an $H$-equivariant map $X : G_0 \to \varphi_0(\mathfrak{m}_\mathfrak{h})$, i.e. a map $X \in \text{Map}(G_0, \varphi_0(\mathfrak{m}_\mathfrak{h}))$ satisfying the equivariance property

$$X(g) = \text{Ad}_h X(gh)$$
for all \( g \in G_0, h \in H \). In other words, the following identifications hold

\[
[C^\infty(G_0) \otimes \varphi_0(m_{\mathfrak{m}_0})]^H \cong \text{Map}(G_0, \varphi_0(m_{\mathfrak{m}_0}))^H \cong T(M_0)
\]  

(5.1)

These identifications are a special case of Lemma 3.11.

**Definition 5.1** ([4, 19]). A **homogeneous spin structure** of a (pseudo)-Riemannian reductive homogeneous space \((M_0 = G_0/H, g)\) is a homomorphism

\[ \tilde{\text{Ad}} : H \to \text{Spin}(m_{\mathfrak{m}_0}) \]

lifting the isotropy representation to \( \text{Spin}(m_{\mathfrak{m}_0}) \), i.e. such that the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\tilde{\text{Ad}}} & \text{Spin}(m_{\mathfrak{m}_0}) \\
\downarrow{\text{id}_H} & & \downarrow{\xi} \\
H & \xrightarrow{\text{Ad}} & \text{SO}(m_{\mathfrak{m}_0})
\end{array}
\]

commutes. The triple \((M_0, g, \tilde{\text{Ad}})\) is called a **homogeneous spin manifold**. The associated **spin bundle** is defined by

\[ S(M_0) := G_0 \times_{\tilde{\text{Ad}}(H)} S \]

where \( S \) is an irreducible real representation

\[ \Delta : \text{Cl}(m_{\mathfrak{m}_0}) \to \text{End}_\mathbb{R}(S) \]

of the Clifford algebra \( \text{Cl}(m_{\mathfrak{m}_0}) \supseteq \text{Spin}(m_{\mathfrak{m}_0}) \).

If \( G_0 \) is simply connected, the homogeneous spin structures of \( M_0 \) are in one-to-one correspondence with the spin structures of \( M_0 \) ([4]). This condition is tacitly assumed together with the connectedness of \( H \).

Henceforth denote the spin module \( S \) by \( m_{\mathfrak{m}_1} \). A spinor field on \( M_0 \) is an \( H \)-equivariant map \( \psi : G_0 \to m_{\mathfrak{m}_1} \), i.e. a map \( \psi \in \text{Map}(G_0, m_{\mathfrak{m}_1}) \) satisfying

\[ \psi(g) = \Delta \circ \tilde{\text{Ad}}_h \psi(gh) \]

(5.2)

for all \( g \in G_0, h \in H \). Write \( \psi \in S(M_0) = \text{Map}(G_0, m_{\mathfrak{m}_1})^H \). Let \( X, Y \in \text{Map}(G_0, \varphi_0(m_{\mathfrak{m}_1}))^H \) (resp. \( \psi, \zeta \in S(M_0) \)) be vector (resp. spinor) fields of a homogeneous spin manifold \((M_0, g, \text{Ad})\). The canonical connection derivatives

\[ \nabla_X^\text{can} Y \in \text{Map}(G_0, \varphi_0(m_{\mathfrak{m}_1}))^H , \quad \nabla_X^\text{can} \psi \in S(M_0) \]

are the usual derivatives along \( X \), i.e.

\[ (\nabla_X^\text{can} Y)|_g = (\partial_X|_g Y) , \quad (\nabla_X^\text{can} \psi)|_g = (\partial_X|_g \psi) \]

for every \( g \in G_0 \). Recall the following important definition.

**Definition 5.2** ([29]). The **Kosmann Lie derivative** \( \mathcal{L}_X \psi \in S(M_0) \) is defined by

\[ \mathcal{L}_X \psi := \nabla_X^\text{LC} \psi - \mathcal{A}(\nabla^\text{LC} X) \cdot \psi \]

where \( \mathcal{A}(\nabla^\text{LC} X) \in \mathfrak{so}(T_pM_0, g) \cong \mathfrak{spin}(T_pM_0, g) \) is the alternation of the Levi-Civita covariant derivative \( \nabla^\text{LC} X \in \mathfrak{gl}(T_pM_0) \) of \( X \).

The following definition gives extra-structures on \((M_0, g, \tilde{\text{Ad}})\).
Definition 5.3. Every \( \mathfrak{h} \)-invariant bilinear form \( C : m_\mathfrak{m}_\mathfrak{T} \otimes m_\mathfrak{T} \to m_\mathfrak{T} \) defines a generalized Clifford multiplication

\[
T(M_0) \otimes S(M_0) \to S(M_0)
\]

\[
X \otimes \psi \mapsto C(X, \psi) := (g \to C(X(g), \psi(g)))
\]

where \( g \in G_0 \). A spinor field \( \psi \in S(M_0) \) is a generalized Killing spinor if

\[
\nabla_X^S \psi := \nabla^\text{can} X \psi + C(X, \psi) = 0
\]

(5.3)

where \( X \in T(M_0) \). Every \( \mathfrak{h} \)-invariant symmetric bilinear form \( \Gamma : m_\mathfrak{m}_\mathfrak{T} \otimes m_\mathfrak{T} \to m_\mathfrak{m}_\mathfrak{T} \) defines a Dirac current bracket

\[
S(M_0) \otimes S(M_0) \to T(M_0)
\]

\[
\psi \otimes \zeta \mapsto \Gamma(\psi, \zeta) := (g \to \Gamma(\psi(g), \zeta(g))
\]

where \( g \in G_0 \).

The generalized Clifford multiplication \( C \) and the Dirac bracket \( \Gamma \) are some of the objects involved in Definition 5.5.

5.2. Algebra of supersymmetry.

Recall that to a pseudo-Riemannian spin manifold \((M_0, g, S)\) is naturally associated a split supermanifold \( M = (M_0, \mathcal{A}_M) \) whose sheaf of superfunctions \( \mathcal{A}_M \) is the exterior algebra of the (dual) of the spin bundle \( S \), i.e.

\[
\mathcal{A}(M) \cong \Lambda(S^*(M_0))
\]

(5.4)

Supermanifolds of this type have been studied in [3, 24, 25]. In our setting, it is natural to ask whether (5.4) can be endowed with a structure of homogeneous supermanifold.

Definition 5.4. A superization of a homogeneous spin manifold

\[
(M_0 = G_0/H, g, \tilde{\text{Ad}})
\]

(5.5)

with reductive decomposition \( \mathfrak{g}_0 = \mathfrak{h} + m_\mathfrak{m}_\mathfrak{T} \) is a homogeneous structure on the supermanifold \( M = (M_0, \mathcal{A}_M) \) whose sheaf of superfunctions is given by (5.4).

Constructing a superization can be reduced to the following algebraic problem. Recall that the Lie morphism induced by a homogeneous spin structure is given by

\[
\tilde{\text{ad}} := \xi^{-1} \circ \text{ad} : \mathfrak{h} \to \text{spin}(m_\mathfrak{m}_\mathfrak{T})
\]

Definition 5.5. A Lie superalgebra \((\mathfrak{g}, [\cdot, \cdot])\), where

\[
\mathfrak{g} = \mathfrak{g}_\mathfrak{m}_\mathfrak{T} + \mathfrak{g}_\mathfrak{T} := (\mathfrak{h} + m_\mathfrak{m}_\mathfrak{T}) + m_\mathfrak{T}
\]

(5.6)

and the adjoint action of \( \mathfrak{h} \) on \( m_\mathfrak{T} \) is given by

\[
\Delta \circ \tilde{\text{ad}} : \mathfrak{h} \to \mathfrak{g}_\mathfrak{m}_\mathfrak{T}(m_\mathfrak{T})
\]

is called an algebra of supersymmetry adapted to the spin manifold \((M_0, g, \tilde{\text{Ad}})\).

Theorem 5.6. Every algebra of supersymmetry (5.6) adapted to a homogeneous spin manifold (5.5) defines a superization \( M = G/H \) of (5.5).
Proof. Consider the unique sHC pair associated with the algebra of supersymmetry (5.6) \((\pi_{1}(G_{0}) = \{0\})\) is used to prove the existence of the adjoint action of the sHC pair and denote by \(G\) the associated Lie supergroup via Theorem 2.8. The homogeneous supermanifold \(G/H\) is a superization of \(G_{0}/H\). Indeed, using the Koszul realization of the sheaf \(A_{G} \cong C_{0}^{\infty} \otimes \Lambda(\mathfrak{g}_{T}^{\ast})\), for every superfunction \(f \in \text{Hom}_{\mathbb{R}}(\Lambda(\mathfrak{g}_{T}), C_{0}^{\infty}(G_{0}))\) the right action of the Lie group \(H\) on \(h\) is given by

\[
\left( R_{h}^{\ast} f \right)(a)(g) = f(\text{Ad}_{h}^{-1} a)(gh)
\]

where \(g \in G_{0}\) and \(a \in \Lambda(\mathfrak{g}_{T})\). Then the structure sheaf \(A_{G}/H\) satisfies (5.4). \(\Box\)

Henceforth every superization is tacitly assumed to come from an adapted algebra of supersymmetry (5.6) and the notation

\[
C := [\cdot, \cdot]|_{m_{\mathbb{R}} \otimes m_{\mathbb{T}}} : m_{\mathbb{R}} \otimes m_{\mathbb{T}} \rightarrow m_{\mathbb{T}} , \quad \Gamma := \pi_{m_{\mathbb{R}}} \circ [\cdot, \cdot]|_{m_{\mathbb{R}} \otimes m_{\mathbb{T}}} : m_{\mathbb{T}} \otimes m_{\mathbb{T}} \rightarrow m_{\mathbb{R}}
\]

is used. Note that the adjoint representation \(\text{Ad} : G_{0} \rightarrow \text{Aut}(\mathfrak{g})\) of the sHC pair \((G_{0}, \mathfrak{g})\) associated with an adapted algebra of supersymmetry satisfies

\[
\text{Ad}_{h}(a) = \Delta \circ \tilde{\text{Ad}}(a)
\]

where \(h \in H\) and \(a \in m_{\mathbb{T}}\). Thus equation (5.2) can be re-written as

\[
\psi(g) = \text{Ad}_{h} \psi(gh)
\]

where \(g \in G_{0}\) and \(h \in H\). In other words, the identifications

\[
[C_{0}^{\infty}(G_{0}) \otimes m_{\mathbb{T}}]^{R_{H}} \cong \text{Map}(G_{0}, m_{\mathbb{T}})^{H} \cong S(M_{0})
\]

are the analogues of (5.1) and show that vector and spinor fields can be treated equally inside the superization \(M = G/H\).

Lemma 5.7. The tangent bundle \(TM\) of a superization \(M = G/H\) is isomorphic to the direct sum of the tangent bundle \(T(M_{0})\) with the spin bundle \(S(M_{0})\) of the body \(M_{0} = G_{0}/H\). Indeed the projections (1.8) and (1.9) are given by

\[
\begin{align*}
\text{ev}_{\mathbb{T}} : [A(G) \otimes \varphi(m)]^{R_{H}} &\rightarrow [C_{0}^{\infty}(G_{0}) \otimes \varphi(\mathfrak{g}_{T})]^{R_{H}} , \\
\text{ev}_{\mathbb{T}} : [A(G) \otimes \varphi(m)]^{R_{H}} &\rightarrow [C_{0}^{\infty}(G_{0}) \otimes m_{\mathbb{T}}]^{R_{H}} .
\end{align*}
\]

Proof. The assertion follows directly from Lemma 3.11 and the following remark. Equation (2.20) implies that the odd-value at a point \(gH \in G_{0}/H\) of a vector field

\[
\sum_{i} f^{i} \otimes \varphi(a_{i}) \in [A(G) \otimes \varphi(m_{\mathbb{T}})]^{R_{H}}
\]

is given by \(\sum_{i} f^{i}(g) \otimes \varphi(a_{i})|_{g} \cong f(g) \otimes a_{i}, i.e \text{ev}_{\mathbb{T}}(\sum_{i} f^{i} \otimes \varphi(a_{i})) = \sum_{i} \tilde{f}^{i} \otimes a_{i}\). \(\Box\)

5.3. Even and odd Killing fields.

In this subsection we assume, for simplicity, that \((M_{0}, g, \tilde{\text{Ad}})\) is a (pseudo)-Riemannian symmetric spin manifold. In this case, the Levi-Civita connection coincides with the canonical connection. Fix an adapted algebra of supersymmetry (5.6) and denote by \(M = G/H\) the associated superization of \(M_{0} = G_{0}/H\). The space of spinor fields which satisfy the generalized Killing equation (5.3) is denoted by

\[
\mathcal{KS} := \{ \psi \in S(M_{0}) \mid \nabla_{X}^{S} \psi = \nabla_{X}^{LC} \psi + C(X, \psi) = 0 \} \quad (5.7)
\]
Similarly the image of the representation of the Lie algebra $\mathfrak{g}_\Sigma$

$$\hat{\varphi}_0 : \mathfrak{g}_\Sigma \to T(M_0)$$

$$x \mapsto \hat{\varphi}_0(x) = (g \mapsto \varphi_0((\text{Ad}_{g^{-1}} x)_0))$$

by Killing vector fields of $M_0$ is denoted by $KV := \{\hat{\varphi}_0(a) | a \in \mathfrak{g}_\Sigma\}$.

**Definition 5.8.** The anti-homomorphism of Lie superalgebras

$$\hat{\varphi} : \mathfrak{g} \to \text{Der}_\mathbb{R}(\Lambda(S(M_0^*))$$

given by (the passage to the quotient of) the map (2.7) is the **Killing representation** of the algebra of supersymmetry $\mathfrak{g} = \mathfrak{g}_\Sigma + \mathfrak{g}_\Sigma$ on $\Lambda(S^*(M_0))$. For every $a \in \mathfrak{g}$, the fundamental vector field $\hat{\varphi}(a)$ is called a **Killing field**.

Recall that the Killing representation of the algebra of supersymmetry is explicitly described by Proposition 2.14 and Theorem 2.30. The following Theorem is the main result of this subsection. In particular it shows that our construction is consistent with the **symmetry superalgebra construction** in the theoretical physics literature (see all the references cited in Example 5.11). The interpretation of generalized Killing spinors (5.7) as odd-values of the odd Killing fields (2.21) is new. Moreover, vector fields and spinor fields are embedded into a bigger space, naturally endowed with a Lie superalgebra structure, namely $T(M) \cong \text{Der}_\mathbb{R}(\Lambda(S^*(M_0)))$.

**Theorem 5.9.** Let $(M_0, g, \text{Ad})$ be a (pseudo)-Riemannian symmetric spin manifold $M_0 = G_0/H$ together with an adapted algebra of supersymmetry $\mathfrak{g} = \mathfrak{g}_\Sigma + \mathfrak{g}_\Sigma = (\mathfrak{h} + \mathfrak{m}_\Sigma) + S$ and let $M = G/H$ be the associated superization such that $\mathcal{A}(M) = \Lambda(S^*(M_0))$. For every $x \in \mathfrak{g}_\Sigma$ and $s \in S$, the value of the associated Killing field on $M$ is given by

$$\text{ev}_\mathfrak{g}(\hat{\varphi}(x)) = \hat{\varphi}_0(x) \in KV \subseteq T(M_0) \tag{5.8}$$

$$\text{ev}_\mathfrak{g}(\hat{\varphi}(s)) = \psi^s \in KS \subseteq S(M_0) \tag{5.9}$$

where

$$\psi^s := (g \mapsto \text{Ad}_{g^{-1}} s) \tag{5.10}$$

is the (unique) generalized Killing spinor with value $s \in S$ at the point $o \in M_0$. The direct sum of (5.8) together with (5.9) is an isomorphism of Lie superalgebras

$$\text{ev} : \hat{\varphi}(\mathfrak{g}) \to KV + KS \tag{5.11}$$

where the structure of Lie superalgebra of $KV + KS$ is defined through the Kosmann Lie derivative and the Dirac current bracket via

$$[\hat{\varphi}_0(x), \psi^s] := L_{\hat{\varphi}_0(x)} \psi^s, \quad [\psi^s, \psi^t] := -\Gamma(\psi^s, \psi^t)$$

where $x \in \mathfrak{g}_\Sigma$ and $s, t \in S$. Moreover, there exists a canonical embedding

$$\mathcal{F} : T(M_0) \oplus S(M_0) \to \text{Der}_\mathbb{R}(\Lambda(S^*(M_0))) \tag{5.12}$$

$$X + \psi \mapsto X + \Psi$$

such that

i) The linear map $T(M_0) \ni X \mapsto X \in \text{Der}_\mathbb{R}(\Lambda(S^*(M_0)))$ is a morphism of Lie superalgebras,

ii) The linear map (5.12) satisfies $(\text{ev}_\mathfrak{g} + \text{ev}_\mathfrak{g}) \circ \mathcal{F} = \text{Id},$

iii) $\text{ev}_\mathfrak{g} \circ [X, Y] = [X, Y] \in T(M_0),$

iv) $\text{ev}_\mathfrak{g} \circ [X, \Psi] = \nabla_X \psi \in S(M_0),$

v) $\text{ev}_\mathfrak{g} \circ [\Psi, \Psi] = \Gamma(\psi, \psi') \in T(M_0).$
for every \(X,Y \in T(M_0)\) and \(\psi,\psi' \in \mathcal{S}(M_0)\).

**Proof.** Equations (5.8) and (5.9) are direct consequences of Proposition 2.14, Theorem 2.30 and related comments (e.g. equation (2.22)). Note that \(\hat{\psi}(s)\) and \(\psi^s\) bijectively correspond to each other. Equation (5.3) is obviously satisfied, we differentiate equation (5.10) along the vector field

\[
X = \sum_i f^i \otimes \varphi_0(a_i) \in [C^\infty(G_0) \otimes \varphi_0(m_H)]^{RH}
\]

so that

\[
\partial_X \psi^s = \sum_i f^i \otimes \frac{d}{dt}|_{t=0} \psi^s(g \cdot \exp(ta_i)) = \sum_i f^i \otimes \frac{d}{dt}|_{t=0} \Ad_{\exp(-ta_i)} \circ \Ad_{g^{-1}}(s)
\]

\[
= - \sum_i f^i \otimes [a_i, \Ad_{g^{-1}}(s)] = - \sum_i f^i \otimes [a_i, \psi^s(g)] = -C(X, \psi^s)
\]

We prove that the bijective map (5.11) is a Lie superalgebra morphism in the Riemannian case, the pseudo-Riemannian case being analogous. Fix an orthonormal frame \(\{a_i\}_{i=1}^{\dim M_0} \subseteq m_H\), then

\[
\mathcal{L}_X \psi = \sum_i f^i \partial_{\varphi_0(a_i)} \psi - \frac{1}{8} \sum_{i,j} (\partial_{\varphi_0(a_i)} f^j - \partial_{\varphi_0(a_j)} f^i) a_i \cdot a_j \cdot \psi .
\]

If \(X\) is Killing, the previous formula reduces to

\[
\mathcal{L}_X \psi = \sum_i f^i \partial_{\varphi_0(a_i)} \psi + \frac{1}{4} \sum_{i,j} (\partial_{\varphi_0(a_i)} f^j) a_j \cdot a_i \cdot \psi
\]

In particular, when

\[
X = \hat{\varphi}_0(x) = \{ g \mapsto \varphi_0(\Ad_{g^{-1}} x) \}_{m_H}
\]

we get that

\[
\mathcal{L}_{\hat{\varphi}_0(x)} \psi^s = - \sum_i C((\Ad_{g^{-1}} x)_{m_H}, \Ad_{g^{-1}} s) + \frac{1}{4} \sum_i (\partial_{\varphi_0(a_i)} \hat{\varphi}_0(x)) \cdot a_i \cdot \psi^s
\]

\[
= - \sum_i C((\Ad_{g^{-1}} x)_{m_H}, \Ad_{g^{-1}} s) - \frac{1}{4} \sum_i [a_i, \Ad_{g^{-1}} x]_{m_H} \cdot \psi^s
\]

\[
= - \sum_i C((\Ad_{g^{-1}} x)_{m_H}, \Ad_{g^{-1}} s) - \frac{1}{4} \sum_i [a_i, (\Ad_{g^{-1}} x)_\mathfrak{h}] \cdot \psi^s
\]

\[
= - \sum_i (\Ad_{g^{-1}} x)_{m_H}, \Ad_{g^{-1}} s) - [\Ad_{g^{-1}} x, \Ad_{g^{-1}} s] - [\Ad_{g^{-1}} x, \Ad_{g^{-1}} s] - [\Ad_{g^{-1}} x, \Ad_{g^{-1}} s]
\]

It follows that

\[
\ev_T \circ [\hat{\varphi}(x), \hat{\varphi}(s)] = -\ev_T \circ \hat{\varphi}[x,s] = -\psi^{[x,s]} = \mathcal{L}_{\hat{\varphi}(x)} \psi^s
\]

for every \(x \in \mathfrak{g}_T\) and \(s \in \mathfrak{m}_T = S\). The equations

\[
\ev_T \circ [\hat{\varphi}(x), \hat{\varphi}(y)] = [\hat{\varphi}_0(x), \hat{\varphi}_0(y)] , \quad \ev_T \circ [\hat{\varphi}(s), \hat{\varphi}(t)] = -\Gamma(\psi^s, \psi^t)
\]

for every \(x, y \in \mathfrak{g}_T\) and \(s, t \in \mathfrak{m}_T = S\) imply that (5.11) is an isomorphism of Lie superalgebras. The embedding (5.12) is defined by

\[
[C^\infty(G_0) \otimes \varphi_0(m_H)]^{RH} \ni X = \sum_i f^i \otimes \varphi_0(a_i) \mapsto \sum_i f^i \otimes \varphi(a_i) =: X \in [\mathcal{A}(G) \otimes \varphi(m)]^{RH}
\]
\[ [C^\infty(G_0) \otimes \mathfrak{m}]^{RH} \ni \psi = \sum_i f^i \otimes a_i \mapsto \sum_i f^i \otimes \varphi(a_i) =: \Psi \in [A(G) \otimes \varphi(m)]^{RH} \]

The proof of the remaining properties is straightforward using the remarks after Lemma 3.11. □

The following example clarifies the situation.

**Example 5.10.** The notation is the one of Example 2.16 and Example 3.10. The even-value of the odd Killing field

\[ \hat{\varphi}(s_{\beta}) = y^\alpha_\beta (g^{-1}) \left[ -\frac{\partial}{\partial s^\alpha} + \frac{1}{2} s^h \varphi_0 (\Gamma^k_{\alpha \eta} e_k) \right] \in T(M) \]

is obviously zero. The equality

\[ \hat{\varphi}(s_{\beta}) = y^\alpha_\beta (g^{-1}) \left[ \varphi(s_{\alpha}) + s^h \varphi_0 (\Gamma^k_{\alpha \eta} e_k) \right] \pmod{A(G) \otimes \mathfrak{h}} \]

implies that the odd-value is the parallel spinor

\[ y^\alpha_\beta (g^{-1}) s_{\alpha} \in S(M_0) \]

whose image under the embedding (5.12) is given by

\[ y^\alpha_\beta (g^{-1}) \varphi(s_{\alpha}) = y^\alpha_\beta (g^{-1}) \left[ -\frac{\partial}{\partial s^\alpha} - \frac{1}{2} s^h \varphi_0 (\Gamma^k_{\alpha \eta} e_k) \right] \in T(M). \]

The following example describes the symmetry superalgebras of the maximally supersymmetric solutions of (bosonic) 11-dimensional supergravity.

**Example 5.11 ([9, 13, 14, 15, 16, 23]).** A Lorentzian spin manifold \((M_0, g, S)\) together with a closed flux \(F \in \Lambda^2(M_0)\) is a bosonic solution of 11-dimensional supergravity if

\[ \text{Ric}(X, Y) - \frac{1}{2} s g(X, Y) = -\frac{1}{2} g(i_X F, i_Y F) + \frac{1}{6} g(X, Y)|F|^2 \quad (\text{Einstein}) \]

\[ d \ast F = \frac{1}{2} F \wedge F \quad (\text{Maxwell}) \]

for all \(X, Y \in T(M_0)\). A spinor field \(\psi \in S(M_0)\) is supergravity-Killing if

\[ \nabla_X^S \psi := \nabla_X^{LC} \psi + \left( -\frac{1}{12} X \wedge F^2 + \frac{1}{6} (i_X F) \right) \cdot \psi = 0 \]

for every \(X \in T(M_0)\). The spin connection \(\nabla^S\) is the main object of the theory. Indeed the Einstein and Maxwell equations are equivalent to

\[ \sum_i e^*_i : R^\nabla^S (X, e_i) = 0 \]

where \(X \in T(M_0), \{e_i\}\) is a local pseudo-orthonormal frame of \(M_0\) and \(\cdot\) denotes Clifford multiplication. The symmetry superalgebras \(\mathfrak{g} = \mathfrak{g}_\mathfrak{m} + \mathfrak{g}_\mathfrak{t} = (\mathfrak{h} + \mathfrak{m}_\mathfrak{t}) + S\) of maximally supersymmetric solutions, i.e. plane-wave, Freund-Rubin and flat backgrounds, are recalled. Note that the spin connection \(\nabla^S\) is encoded in the bracket \([\mathfrak{m}_\mathfrak{t}, S]\). Theorem 5.9 implies that the supergravity Killing spinors are given by the odd-value (5.10) of the odd-Killing fields of the supermanifold \(\Lambda(S^*(M_0))\).

**Plane-wave.** Let \(\mathbb{R}^{1,10}\) be the vector space \(\mathbb{R}^{11}\), together with the standard inner product \(\langle \cdot, \cdot \rangle\) of signature \((1,10) = (+, -)\). Fix a Witt decomposition

\[ \mathbb{R}^{1,10} = (\mathbb{R}^p \oplus \mathbb{R}^q) \bigoplus E = (\mathbb{R}^p \oplus \mathbb{R}^q) \bigoplus (\oplus_{i=1}^9 \mathbb{R} e_i) \]
with \(\langle e_i, e_j \rangle = -\delta_{ij}\), \(\langle p, E \rangle = \langle q, E \rangle = \langle p, p \rangle = \langle q, q \rangle = 0\) and \(\langle p, q \rangle = 1\). The non-trivial Lie brackets of the Cahen-Wallach symmetric space

\[ g_0 = \mathfrak{h} + \mathfrak{m}_T = (E^* + \mathfrak{so}(3) + \mathfrak{so}(8)) + \mathbb{R} \]

are given by

\[
B = \begin{pmatrix}
\frac{1}{9} & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{9} & 0 & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{9} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{36} & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{36}
\end{pmatrix} \in \text{End } g(E)
\]

are given by

i) \([M_{ij}, e_k^*] = -\delta_{jk} e_i^* + \delta_{ik} e_j^*\),

ii) \([M_{ij}, e_k] = -\delta_{ik} e_j + \delta_{jk} e_i\),

iii) \([e_i^*, e_i] = -B(e_i)p_i\),

iv) \([e_i^*, q] = -B(e_i)e_i\),

v) \([q, e_i] = -e_i^*\),

where, for \(1 \leq i, j \leq 3\) or \(4 \leq i, j \leq 11\), the infinitesimal generators of \(\mathfrak{so}(3) \oplus \mathfrak{so}(8)\)

\[ M_{ij} = e_i \wedge e_j = (e_i \cdot ) e_j - (e_j \cdot ) e_i \in \mathfrak{so}(3) \oplus \mathfrak{so}(8) \]

satisfy the usual mutual relations. The flux is the invariant four form associated with \(F = -q^* \wedge e_1^* \wedge e_2^* \wedge e_3^* \in \Lambda^4 \mathfrak{m}_0^\ast\). The non-trivial even-odd brackets are

i) \([M_{ij}, Q_\pm] = \frac{3}{2} e_i e_j \cdot Q_\pm\),

ii) \([e_i^*, Q_+] = \frac{1}{18} e_i p \cdot Q_+\) if \(1 \leq i \leq 3\),

iii) \([e_i^*, Q_+] = \frac{1}{12} e_i p \cdot Q_+\) if \(4 \leq i \leq 11\),

iv) \([q, Q_+] = \frac{1}{12} I \cdot Q_+\), \([q, Q_-] = \frac{1}{12} I \cdot Q_-\)

\[ I := e_1 e_2 e_3\]

\[ M_{ij} = e_i \wedge e_j \]

\[ M_{ij} = e_i \wedge e_j \]

\[ M_{ij} = e_i \wedge e_j \]

\[ M_{ij} = e_i \wedge e_j \]

\[ M_{ij} = e_i \wedge e_j \]

The 32-dimensional vector space \(S\) is decomposed into two 16-dimensional vector spaces \(S_\pm\) whose elements are denoted by \(Q_\pm \in S_\pm\). The odd-odd bracket is given by

\[ [Q_+, Q_+] = (Q_+, p \cdot Q_+)q + \sum_{i,j \leq 3} \frac{1}{6} (Q_+, I e_i e_j p \cdot Q_+) M_{ij} + \sum_{4 \leq i,j} \frac{1}{12} (Q_+, I e_i e_j p \cdot Q_+) M_{ij}, \]

\[ [Q_+, Q_-] = -\sum_{i=1}^{9} (Q_+, e_i \cdot Q_-) e_i - 3 \sum_{i \leq 3} (Q_+, I e_i \cdot Q_-) e_i^* - 6 \sum_{4 \leq i} (Q_+, I e_i \cdot Q_-) e_i^*, \]

\[ [Q_-, Q_-] = (Q_-, q \cdot Q_-)p \]

[16] gives examples of non-maximally supersymmetric plane-wave solutions. The associated Lie superalgebras are algebra of supersymmetries with odd part the \(\mathfrak{h}\)-submodule \(S_\).
constant, the volume form of the 4-dimensional factor. We describe the symmetry superalgebra \( g = g_T + g_T \) in the first case (the second is similar). The action of

\[
g_T = \mathfrak{so}(2, 3) \oplus \mathfrak{so}(0, 8) = (\mathfrak{so}(1, 3) + \mathbb{R}^{1,3}) \oplus (\mathfrak{so}(0, 7) + \mathbb{R}^{0,7})
\]
on \( g_T \) is given by

\[
i \) \[ M_{ij}, Q \] = \frac{1}{2} e_i e_j \cdot Q \\
ii \) \[ v, Q \] = -\frac{1}{2} v \cdot Q \quad , \quad \[ w, Q \] = \frac{1}{2} w \cdot Q \quad \quad I := \text{dvol(AdS}_4)\]

where \( M_{ij} \in \mathfrak{h} = \mathfrak{so}(1, 3) \oplus \mathfrak{so}(0, 7), v \in \mathbb{R}^{1,3}, w \in \mathbb{R}^{0,7}. \) The odd-odd bracket is given by

\[
[Q, Q] = \sum_{i=1}^{11} (Q, e_i \cdot Q) e_i + \sum_{i,j} (Q, I e_i e_j \cdot Q) M_{ij}.
\]

Minkowski. The symmetry superalgebra of the flat solution \((\mathbb{R}^{1,10}, \langle \cdot, \cdot \rangle), F = 0, \) is the Poincare’ Lie superalgebra in signature \((1, 10)\).

The following example deals with the superconformal algebra. This could be an indication that our setting can be developed in more general situations.

**Example 5.12** ([41]). The even part of the superconformal algebra \( g = g_T + g_T \) of Wess and Zumino is a central extension of \( \mathfrak{so}(2, 4) \):

\[
g_T = \mathbb{R} \cdot 1 + \mathfrak{so}(2, 4) = \mathbb{R} \cdot 1 + (V' + \mathbb{R} \cdot d + \mathfrak{so}(1, 3) + V)
\]

where \( V \) and \( V' \) are two copies of Minkowsky spacetime \((\mathbb{R}^{1,3}, \langle \cdot, \cdot \rangle)\) and the second equality is the usual decomposition of \( \mathfrak{so}(2, 4) \) into infinitesimal generators of special conformal transformations, dilations, rotations and translations of \( \mathbb{R}^{1,3}. \) The spin module in signature \( S_{2,4} \) is the direct sum of two copies of the \( \mathfrak{spin}(1,3) \)-module

\[
g_T = S_{2,4} = S_{1,3} + S'_{1,3}
\]
on which \( \mathfrak{so}(1, 3) \) acts diagonally via the \( \mathfrak{spin}(1,3) \)-representation. The Schur algebra of the \( \mathfrak{so}(1, 3) \)-representation \( S_{1,3} \) is generated by a complex structure \( J. \) The adjoint actions of the central charge \( 1 \in g_T \) and of the dilation generator \( d \in g_T \) on the odd part \( g_T \) are diagonal

\[
[1, (s, s')] = (Js, -Js') , \quad [d, (s, s')] = (\frac{1}{2}s, -\frac{1}{2}s')
\]

where \( s \in S_{1,3} \) and \( s' \in S'_{1,3}, \) while those of \( v \in V \) and \( v' \in V' \) are given by

\[
[v, (s, s')] = \frac{1}{\sqrt{2}}(v \cdot s', 0) \quad , \quad [v', (s, s')] = -\frac{1}{\sqrt{2}}(0, v' \cdot s)
\]

where \( \cdot \) denotes Clifford multiplication of a vector with a spinor. The odd-odd brackets are defined as follows. [2] defines, for any \( 0 \leq k \leq 4, \) an isomorphism

\[
\Gamma^k : \mathfrak{Bil}(S_{1,3})^{\mathfrak{so}(1,3)} \rightarrow \mathfrak{Bil}^k(S_{1,3})^{\mathfrak{so}(1,3)}
\]

\[
\beta \mapsto \Gamma^k_\beta
\]
of the vector space of \( \mathfrak{so}(1,3) \)-invariant bilinear forms on \( S_{1,3} \) with the vector space of \( \mathfrak{so}(1,3) \)-invariant \( \Lambda^k \mathbb{R}^{1,3} \)-valued bilinear forms on \( S_{1,3}, \) where

\[
\langle \Gamma^k_\beta (s \otimes t), v_1 \wedge \cdots \wedge v_k \rangle := \sum_{\pi \in S_k} \text{sgn}(\pi)\beta(v_{\pi(1)} \cdots v_{\pi(k)} \cdot s, t)
\]
for every \( s, t \in S_{1,3} \) and \( v_1, ..., v_k \in \mathbb{R}^{1,3} \). The vector space \( \text{Bil}(S_{1,3})^{\otimes (1,3)} \) is two dimensional and an admissible basis (see [1]) is given by two skew-symmetric bilinear forms \( \beta \) and \( \beta_J := \beta(J, \cdot, \cdot) \). Clifford multiplication of a vector with a spinor is a skew-symmetric (resp. symmetric) operation with respect to \( \beta \) (resp. \( \beta_J \)). The odd-odd brackets are given by

\[
\Gamma_S = \Gamma_1 \cdot \beta, \quad \Gamma_S' = -\Gamma_1 \cdot \beta' \quad \Gamma_{S \otimes S'} = \Gamma_1 + \Gamma_d + \Gamma_{A^2} : S \otimes S' \rightarrow \mathbb{R} \cdot 1 + \mathbb{R} \cdot d + \Lambda^2 \mathbb{R}^{1,3}
\]

where

\[
\Gamma_1 \cong -\frac{3}{2\sqrt{2}} \Gamma_{r \beta}^{0}, \quad \Gamma_d \cong -\frac{1}{\sqrt{2}} \Gamma_{r \beta}^{0}, \quad \Gamma_{A^2} \cong \frac{1}{4\sqrt{2}} \Gamma_{r \beta}^{2}
\]

and \( r \in \mathbb{R} \) is an arbitrary non-zero real constant. The body of the Wess-Zumino Lie supergroup \( G \) is given by

\[
G_0 = \text{Spin}^0(2, 4) \times \mathbb{R} \cong SU(2, 2) \times \mathbb{R}
\]

and the Poincare’ group \( \text{Spin}^0(1, 3) \times \mathbb{R}^{1,3} \) is a Lie subgroup. The projection of the odd Killing fields of \( G \) on \( \mathbb{R}^{1,3} \subseteq G_0 \) gives maps

\[
\psi^s : \mathbb{R}^{1,3} \rightarrow S_{1,3} + S_{1,3}, \quad v \mapsto -\frac{1}{\sqrt{2}} (v \cdot s', 0)
\]

where \( s' \in S_{1,3} \). These are all the (non-parallel) twistor spinors, also called conformal Killing spinors in the literature, of Minkowsky spacetime.

6. Appendix

6.0.1. Proof of Lemma 1.32.

We prove the Lemma only in the case of a vector field \( X \in \mathcal{T}(N) \). The general case is similar. It is enough to prove that \( X = 0 \) on a coordinate patch \( \{x^r, \xi_s\} \). Fix a point \( p \in U \subseteq N_0 \). By \( \mathbb{R} \)-linearity of relation ii), assume that

\[
Y_i|_p = \frac{\partial}{\partial \xi_i}|_p
\]

i.e. there exist \( f_i^r, g_i^s \in \mathcal{A}_M(U) \) such that

\[
Y_i = \sum_{r=1}^m f_i^r \frac{\partial}{\partial x^r} + \sum_{s=1}^n g_i^s \frac{\partial}{\partial \xi_s}
\]

where \( f_i^r(p) = g_i^s(p) = 0 \) for every \( 1 \leq r \leq m, 1 \leq s \neq i \leq n \) and \( g_i^i(p) = 1 \). Every superfunction \( f \in \mathcal{A}_M(U) \) can be expressed as

\[
f = \sum_{s=0}^n \sum_{\alpha_1 < \cdots < \alpha_s} f_{\alpha_1 \cdot \cdot \cdot \alpha_s} \xi_{\alpha_1} \cdot \cdot \cdot \xi_{\alpha_s}
\]

where \( f_{\alpha_1 \cdot \cdot \cdot \alpha_s} \in \mathcal{C}^\infty(U) \). Say that \( f \) is zero up to order \( q \in \mathbb{N} \) if \( f_{\alpha_1 \cdot \cdot \cdot \alpha_s} = 0 \) whenever \( s \leq q \). The local expression of a vector field \( X \) is given by

\[
X|_U = \sum_{r=1}^m f_r^r \frac{\partial}{\partial x^r} + \sum_{s=1}^n g_s^s \frac{\partial}{\partial \xi_s}
\]
where \( f^r, g^s \in A_M(U) \). Say that \( X \) is \textit{zero up to order} \( q \in \mathbb{N} \) if so are all \( f^r, g^s \). Given two local vector fields \( X, Z \), the non-associative operation

\[
Z \cdot X := \sum_{r=1}^{m} (Z f^r) \frac{\partial}{\partial x^r} + \sum_{s=1}^{n} (Z g^s) \frac{\partial}{\partial \xi_s}
\]

is well-defined (\( X \) is as in (6.1)). It satisfies the relation \( \mathcal{L}_Z X = Z \cdot X - (-1)^{|Z||X|} X \cdot Z \). If \( Z \) is zero up to order \( q \), then \( Z \cdot X \) is zero up to order \( q \) as well. Define

\[
X_{i_k \cdots i_1} := \mathcal{L}_{Y_{i_k}} \cdots \mathcal{L}_{Y_{i_1}} X.
\]

By induction on \( q \in \mathbb{N} \), we prove that if \( X \) satisfies i) and ii) for \( 1 \leq k \leq q \) then \( X \) is zero up to order \( q \).

(q = 0) From i), \( f^r = g^s = 0 \), \textit{i.e.} \( X \) is zero up to order 0.

(q = 1) \( X_{i_1} = \mathcal{L}_{Y_{i_1}} X = Y_{i_1} \cdot X \pm X \cdot Y_{i_1} \) and evaluating in \( p \)

\[
0 = X_{i_1}|_p = (Y_{i_1} \cdot X)|_p
\]

because \( X \) is zero up to order 0. Then

\[
0 = (Y_{i_1} \cdot X)|_p = \sum_{r=1}^{m} (Y_{i_1} f^r(p)) \frac{\partial}{\partial x^r} + \sum_{s=1}^{n} (Y_{i_1} g^s(p)) \frac{\partial}{\partial \xi_s}
\]

\[
= \sum_{r=1}^{m} \left( \frac{\partial f^r}{\partial \xi_{i_1}}(p) \right) \frac{\partial}{\partial x^r} + \sum_{s=1}^{n} \left( \frac{\partial g^s}{\partial \xi_{i_1}}(p) \right) \frac{\partial}{\partial \xi_s},
\]

\textit{i.e.} \( X \) is zero up to order 1.

(q = 2) \( X_{i_2 i_1} = \mathcal{L}_{Y_{i_2}} X_{i_1} = Y_{i_2} \cdot X_{i_1} \pm X_{i_1} \cdot Y_{i_2} \) and evaluating in \( p \)

\[
0 = X_{i_2 i_1}|_p = (Y_{i_2} \cdot X_{i_1})|_p
\]

because \( X_{i_1} \) is zero up to order 1 (and so 0). Moreover

\[
(Y_{i_2} \cdot X_{i_1})|_p = (\frac{\partial}{\partial \xi_{i_2}} \cdot (Y_{i_1} \cdot X))|_p = (\frac{\partial}{\partial \xi_{i_2}} (Y_{i_1} \cdot X)|_p = (\frac{\partial}{\partial \xi_{i_2}} (Y_{i_1} \cdot X)|_p
\]

because \( X \) is zero up to order 1. Then

\[
0 = (\frac{\partial}{\partial \xi_{i_2}} \cdot (Y_{i_1} \cdot X))|_p = (\frac{\partial}{\partial \xi_{i_2}} \cdot (\frac{\partial}{\partial \xi_{i_1}} \cdot X))|_p
\]

\[
= \sum_{r=1}^{m} \left( \frac{\partial^2}{\partial \xi_{i_2} \partial \xi_{i_1}} f^r(p) \right) \frac{\partial}{\partial x^r} + \sum_{s=1}^{n} \left( \frac{\partial^2}{\partial \xi_{i_2} \partial \xi_{i_1}} g^s(p) \right) \frac{\partial}{\partial \xi_s} = 0
\]

where the second equality follows from the fact that \( X \) is zero up to order 1.

(q ≥ 3) \( X_{i_q \cdots i_1} = \mathcal{L}_{Y_{i_q}} X_{i_{q-1} \cdots i_1} = Y_{i_q} \cdot X_{i_{q-1} \cdots i_1} \pm X_{i_{q-1} \cdots i_1} \cdot Y_{i_q} \) and evaluating in \( p \)

\[
0 = X_{i_q \cdots i_1}|_p = (Y_{i_q} \cdot X_{i_{q-1} \cdots i_1})|_p
\]

because \( X_{i_{q-1} \cdots i_1} \) is zero up to order 1 (and so 0). Moreover

\[
(Y_{i_q} \cdot X_{i_{q-1} \cdots i_1})|_p = (\frac{\partial}{\partial \xi_{i_q}} \cdot X_{i_{q-1} \cdots i_1})|_p = (\frac{\partial}{\partial \xi_{i_q}} (Y_{i_{q-1}} \cdot X_{i_{q-2} \cdots i_1}))|_p
\]

\[
\pm (\frac{\partial}{\partial \xi_{i_q}} (X_{i_{q-2} \cdots i_1} \cdot Y_{i_q}))|_p = (\frac{\partial}{\partial \xi_{i_q}} (Y_{i_{q-1}} \cdot X_{i_{q-2} \cdots i_1}))|_p
\]
because $X_{i_k-\cdots-i_1}$ is zero up to order 2 (and so 1) by the induction hypothesis. Using the fact that $X_{i_k-i_1}$ is zero up to order $q-k$ (and so $q-k-1$) by the induction hypothesis we get that

$$0 = (Y_q, X_{i_{q-1}^{\cdots}i_1})|_p = \left(\frac{\partial}{\partial \xi_{i_1}} \cdots \frac{\partial}{\partial \xi_{i_{q-1}}} \cdot (Y_{i_{q-2}} \cdots (Y_i \cdot \cdots))\right)|_p$$

$$= \left(\frac{\partial}{\partial \xi_{i_1}} \cdots \frac{\partial}{\partial \xi_{i_{q-2}}} \cdots \frac{\partial}{\partial \xi_{i_1}} \cdot X \cdot \cdots\right)|_p$$

$$= \sum_{r=1}^n \left(\frac{\partial q}{\partial \xi_{i_1}} \cdots \frac{\partial q}{\partial \xi_{i_{r-1}}} f^r(p) \frac{\partial}{\partial x^r} + \sum_{s=1}^n \left(\frac{\partial q}{\partial \xi_{i_1}} \cdots \frac{\partial q}{\partial \xi_{i_s}} g^s(p) \frac{\partial}{\partial x^s}\right)\right)$$

The second to last equality follows from the fact that $X$ is zero up to order $q-1$ by the induction hypothesis. □

6.0.2. Proof of Theorem 3.16.

Every $G$-invariant superfunction $f$ is constant:

$$\rho_p^* f = (\text{Id} \otimes \text{ev}_p) \circ \rho^* f = (\text{Id} \otimes \text{ev}_p)(1 \otimes f) = \tilde{f}(p)1_G = \rho_p^*(\tilde{f}(p)1_M)$$

implies the assertion.

Let $X \in T(M)^G$ be a $G$-invariant vector field. For every $g \in G_0$, applying $(\text{ev}_g \otimes \text{Id})$ to both sides of $(\text{Id} \otimes X) \circ \rho^* = \rho^* \circ X$, we get that $X \circ g^* = g^* \circ X$ while

$$X \circ (A^e \otimes \text{Id}) \circ \rho^* = (-1)^{|X||A|}(A^e \otimes \text{Id}) \circ (\text{Id} \otimes X) \circ \rho^*$$

and

$$(A^e \otimes \text{Id}) \circ \rho^* \circ X = (A^e \otimes \text{Id}) \circ (\text{Id} \otimes X) \circ \rho^*$$

imply that $L_A X = 0$ for every $A \in g$. We have proved that $T(M)^G \subseteq T(M)^{(G_0, \emptyset)}$.

The inclusion

$$\text{ev}_p(T(M)^G) \subseteq \text{ev}_p(T(M)^{(G_0, \emptyset)}) \subseteq (T_p M)\phi(G_p)$$

follows from evaluating (3.4) at $p$ when $g \in (G_0)_p$ and $A \in g_p$. Let $X_1, X_2 \in T(M)^G$ be two $G$-invariant vector fields on $M$ with the same value $X_1|_p = X_2|_p = v$. Applying $(\text{Id} \otimes \text{ev}_p)$ to both sides of $(\text{Id} \otimes X) \circ \rho^* = \rho^* \circ X$, we get that

$$(\text{Id} \otimes v) \circ \rho^* \circ X_1 = \rho^*_p \circ X_2$$

i.e. $X_1 = X_2$. This proves injectivity of the correspondence at the level of vector fields. We prove surjectivity. For every $v \in (T_p M)^\phi(G_p)$ equation (3.5) defines a $G$-invariant vector field through the identification $\rho_p : G/G_p \rightarrow M$. First, equation (3.1) together with

$$R^*_h \circ (\text{Id} \otimes v) \circ \rho^* = (\text{Id} \otimes v) \circ (R^*_h \otimes \text{Id}) \circ \rho^* = (\text{Id} \otimes v) \circ (\text{Id} \otimes h^* \circ \rho^* = (\text{Id} \otimes v) \circ \rho^*$$

$$(1 - 1)|B|v^{|v|} B \circ (\text{Id} \otimes v) \circ \rho^* = (\text{Id} \otimes v) \circ (B \otimes \text{Id}) \circ \rho^* = (\text{Id} \otimes v) \circ (\text{Id} \otimes \tilde{B}) \circ \rho^* = 0$$

where $h \in (G_0)_p$ and $B \in g_p$, implies that $\text{Im}((\text{Id} \otimes v) \circ \rho^*) \subseteq A_G/G_{p}$. Equation

$$(\text{Id} \otimes v) \circ \rho^*(fg) = (\text{Id} \otimes v) \rho^*(f) \cdot \rho_p^*(g) + (-1)^{|f||v|} \rho_p^*(f) \cdot (\text{Id} \otimes v) \rho^*(g)$$

implies that (3.5) is a derivation. It is $G$-invariant, for

$$\rho^* \circ (\rho_p^*)^{-1} \circ ((\text{Id} \otimes v) \circ \rho^*)$$

equals

$$(\text{Id} \otimes (\rho_p^*)^{-1}) \circ (\text{Id} \otimes \text{Id} \otimes v) \circ \rho^* \circ \rho^* = (\text{Id} \otimes (\rho_p^*)^{-1}) \circ (\text{Id} \otimes \text{Id} \otimes v) \circ (\text{Id} \otimes \rho^*) \circ \rho^*$$
These proofs can be generalized naturally to tensor fields of type $(\rho^*_p)^{-1}$.

The inclusion $T(M)^{(G_0, \mathfrak{g})} \subseteq T(M)^G$ is proved as in Lemma 2.5.

Let $\omega \in T^*(M)^G$ be a $G$-invariant 1-form. The inclusion

$$T^*(M)^G \subseteq T^*(M)^{(G_0, \mathfrak{g})}$$

follows from applying $(ev_g \otimes \text{Id})$ and $(A|_e \otimes \text{Id})$ to both sides of $\omega(Y) = (\rho^*\omega)(\text{Id} \otimes Y)$. Thus

$$ev_p(T^*(M)^G) \subseteq ev_p(T^*(M)^{(G_0, \mathfrak{g})}) \subseteq (T^*_p M)^{\phi(G_p)} .$$

Let $\omega_1, \omega_2 \in T^*(M)^G$ be two $G$-invariant 1-forms with the same value $\omega_1|_p = \omega_2|_p = \omega_p$. Applying $(\text{Id} \otimes ev_p)$ to both sides of

$$1 \otimes \omega_i(\hat{A}) = (\rho^*\omega_i)(\text{Id} \otimes \hat{A})$$

we get that

$$\omega_p(\hat{A}|_p) \cdot 1_G = (\text{Id} \otimes ev_p)(\rho^*\omega_i)(\text{Id} \otimes \hat{A}) = (\rho^*_p \omega_i)(A),$$

i.e. $\omega_1 = \omega_2$. Denote $G_p$ by $H$. A 1-form $\omega \in T^*(G)$ is projectable if

$$R_{H_0}\text{-Invariance} \quad \omega = R_{H_0}^*\omega$$

$h$-Invariance

$$B(\omega(Y)) = (-1)^{|\omega|\|B|}\omega(L_B Y) \quad (Y \in T(G))$$

Horizontality

$$\omega|_{A_G \otimes h} = 0$$

for every $h \in H_0, B \in h$. The set of all projectable 1-forms is denoted by $T^*(G)^{R_H}_{\text{hor}}$ and its intersection with the set of $G$-invariant 1-forms by

$$T^*(G)^{L_{G \times R_H}}_{\text{hor}} := T^*(G)^{R_H}_{\text{hor}} \cap T^*(G)^G .$$

The pull-back

$$T^*(G/H) \ni \nu \xrightarrow{\pi^*} \pi^*\nu \in T^*(G)^{R_H}_{\text{hor}}$$

is a bijection such that $\pi^*(T^*(G/H)^G) = T^*(G)^{L_{G \times R_H}}_{\text{hor}}$. A 1-form $\omega \in T^*(G)^{R_H}_{\text{hor}}$ uniquely defines a 1-form $\nu \in T^*(G/H)$ by

$$\nu(X) := \omega(Y)$$

where $X \in T_{G/H}(U), Y \in T_G(S_0^{-1}U)$ are $\pi$-related vector fields. These assertions depend on the existence of adapted coordinates. For every $H$-invariant covector $\omega_o \in (g/h)^*$, the formula

$$\omega(f \otimes A) := (-1)^{|f||\omega_o|} f \omega_o(A) \in A(G) \quad f \otimes A \in A(G) \otimes g$$

defines a 1-form $\omega \in T^*(G)^{L_{G \times R_H}}_{\text{hor}}$ and so a $G$-invariant 1-form on $G/H \cong M$. These proofs can be generalized naturally to tensor fields of type $(r, s)$.

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