Plane one-dimensional MHD flows: symmetries and conservation laws

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Abstract

The paper considers the plane one-dimensional flows for magneto-hydrodynamics in the mass Lagrangian coordinates. The inviscid, thermally non-conducting medium is modeled by a polytropic gas. The equations are examined for symmetries and conservation laws. For the case of the finite electric conductivity we establish Lie group classification, i.e. we describe all cases of the conductivity $\sigma(\rho,p)$ for which there are symmetry extensions. The conservation laws are derived by the direct computation. For the case of the infinite electrical conductivity the equations can be brought into a variational form in the Lagrangian coordinates. Lie group classification is performed for the entropy function as an arbitrary element. Using the variational structure, we employ the Noether theorem for obtaining conservation laws. The conservation laws are also given in the physical variables.

Key words:
Lie point symmetries, conservation laws, Noether theorem, Euler-Lagrange equations
1 Introduction

The equations of magnetohydrodynamics (MHD) describe motion of electrically conducting fluids under the action of the internal forces, which consist of the pressure and electromagnetic forces. These equations describe phenomena related to plasma flows, for example in plasma confinement, as well as physical problems in astrophysics and fluid metals flows.

In the present paper we consider plane one-dimensional MHD flows. The equations which describe such flows will be examined for Lie point symmetries and conservation laws. We assume that medium is inviscid and thermally non-conducting. It is modeled by a polytropic gas. Both cases of finite and infinite electric conductivity are analyzed. The particular case of the infinite electric conductivity corresponds to "freezing" of the magnetic force lines in the trajectories of motion.

Lie point symmetries represent an efficient tool to analyze nonlinear differential equations [1–4]. They are related to fundamental physical principles of the considered models and correspond to important properties of the differential equations:

- Transformations generated by symmetries transfer solutions into another solutions. It allows to find new solutions from the known ones.
- Symmetries of PDEs allow to find particular solutions of a special form, the so called invariant solutions.
- Invariance of variational PDEs is a necessary condition for application of Noether’s theorem, which can be used to find conservation laws.

Lie group symmetries of various versions of MHD equations were considered in many publications. For example, the case of the finite conductivity was investigated in [5]–[6]. The case of the infinite conductivity was examined in [7]–[10]. Invariant solutions were considered in [11]–[15]. It should be noted that most of the papers devoted to applications of Lie group symmetries to MHD consider the case of the infinite conductivity.

Variational methods have many applications in mathematics and physics. If differential equations have a form of Euler-Lagrange equations, there is a possibility to employ the Noether theorem [16]. The theorem allows one to use symmetries of the differential equations which are either variational or divergence symmetries of the Lagrangian function to derive conservation laws. Several other approaches to find conservation laws and other conserved quantities for MHD were recently reviewed in [17].

This paper is organized as follows. The next section provides a short description of the Noether theorem, specified for this paper. Section 3 describes the equations of MHD and restricts them for plane one-dimensional flows. In this section we also introduce mass Lagrangian coordinates. Symmetries and conservation laws of the MHD equations with the finite conductivity are obtained in Sections 4 and 5 respectively. Symmetries and conservation laws for the infinite conductivity are
treated in Sections 6 and 7. Finally, Section 8 provides concluding remarks. Some technical details are extracted into the Appendices.

2 Background theory

In the next section we will describe the magnetohydrodynamics equations and specify them for plane one-dimensional flows, which will be analyzed for admitted Lie point symmetries and conservation laws. The conservation laws will be obtained by direct computations and using the Noether theorem, which can be employed to find conservation laws if the equations have a variational formulation.

2.1 Lie group classification problem

The Lie group classification problem consists of finding all Lie groups admitted by a system of partial differential equations [1–3]. Admitted groups can depend on arbitrary elements (constants and functions of the independent and dependent variables) included in the equations. Practically, the groups are presented by their generators. The generators admitted for all arbitrary elements are called the kernel of the admitted Lie algebras. Lie group classification presents all non-equivalent extensions of the kernel and the corresponding specific form of the arbitrary elements. It is performed with respect to equivalence transformations, which preserve the structure of the equations but may change the arbitrary elements.

2.2 Noether theorem

The Noether theorem [16] (see also [1–4]) can be used to find conservation laws of variational equations with symmetries. Here we present a simplified version of this theorem restricted to second-order PDEs with two independent variables \((t, s)\), which represent time and one spatial coordinate. In this case we need to consider first-order Lagrangian functions

\[ L = L(t, s, \varphi, \varphi_t, \varphi_s), \quad \varphi = (\varphi^1, \ldots, \varphi^m). \]  

(2.1)

The Lagrangian provides the second-order Euler-Lagrange equations

\[ \frac{\delta L}{\delta \varphi^i} = \frac{\partial L}{\partial \varphi^i} - D_t \left( \frac{\partial L}{\partial \varphi_t^i} \right) - D_s \left( \frac{\partial L}{\partial \varphi_s^i} \right) = 0, \quad i = 1, \ldots, m, \]  

(2.2)

where \(D_t\) and \(D_s\) are total differentiation operators with respect to \(t\) and \(s\). Operators \(\frac{\delta}{\delta \varphi^i}\) are called the variational operators.

Lie point symmetries of these differential equations are given by the operators of the form

\[ X = \xi^t(t, s, \varphi) \frac{\partial}{\partial t} + \xi^s(t, s, \varphi) \frac{\partial}{\partial s} + \eta^i(t, s, \varphi) \frac{\partial}{\partial \varphi^i}. \]  

(2.3)
It is assumed that the operator is prolonged to the second-order derivatives, present in the Euler-Lagrange equations, according to the standard prolongation formulas [1][4].

The Noether theorem is based on the following identities. The first identity [2] relates the invariance of the elementary action, which is also called invariance of the Lagrangian, to the conservation laws:

\[ XL + L(D_t \xi^t + D_s \xi^s) = (\eta^j - \xi^t \varphi^j_t - \xi^s \varphi^j_s) \frac{\delta L}{\delta \varphi^j_t} + D_t(N^t L) + D_s(N^s L), \]  

(2.4)

where

\[ N^t = \xi^t + (\eta^j - \xi^s \varphi^j_s) \frac{\partial}{\partial \varphi^j_t}, \quad N^s = \xi^s + (\eta^j - \xi^t \varphi^j_t - \xi^s \varphi^j_s) \frac{\partial}{\partial \varphi^j_s}, \]  

(2.5)

are the Noether operators.

If

\[ XL + L(D_t \xi^t + D_s \xi^s) = 0, \]

the symmetry \( X \) is called a variational symmetry of the Lagrangian. For

\[ XL + L(D_t \xi^t + D_s \xi^s) = D_t B_1 + D_s B_2 \]

with nontrivial \( B_1(t,s,\varphi) \) and \( B_2(t,s,\varphi) \) we say that \( X \) is a divergence symmetry.

The other set of identities [18] (see also [3]) relates the invariance of the Lagrangian to the invariance of the Euler-Lagrange equations:

\[ \frac{\delta}{\delta \varphi^j} (XL + L(D_t \xi^t + D_s \xi^s)) = X \left( \frac{\delta L}{\delta \varphi^j} \right) \]

\[ + \left( \frac{\partial \eta^k}{\partial \varphi^j} - \frac{\partial \xi^t}{\partial \varphi^j} \varphi^k_t - \frac{\partial \xi^s}{\partial \varphi^j} \varphi^k_s + D_t \xi^t + D_s \xi^s \right) \]  

\[ \frac{\delta L}{\delta \varphi^k}, \quad j = 1, 2, \ldots, m. \]  

(2.6)

The Noether theorem is formulated as follows:

**Theorem 2.1** [16] Let the Lagrangian function (2.1) satisfy the equation

\[ XL + L(D_t \xi^t + D_s \xi^s) = D_t B_1 + D_s B_2, \]  

(2.7)

where \( X \) is a generator (2.3) and \( B_i(t,s,\varphi) \), \( i = 1, 2 \). Then the operator \( X \) is a symmetry of the Euler-Lagrange equations (2.2), and the Euler-Lagrange equations possess a conservation law

\[ D_t(N^t L - B_1) + D_s(N^s L - B_2) = 0. \]  

(2.8)
3 Magnetohydrodynamics equations and plane one-dimensional flows

3.1 Three-dimensional MHD

The magnetohydrodynamics equations in Eulerian coordinates can be written in different ways [19–21] (see also [17, 22–24]). For simplicity we take the dimensionless (scaled) form of MHD equations with the finite conductivity

\[ \rho_t + \text{div}(\rho \mathbf{u}) = 0, \]
\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{[\mathbf{i} \times \mathbf{H}]}{\rho}, \]
\[ \varepsilon_t + (\mathbf{u} \cdot \nabla)\varepsilon = -\frac{p}{\rho} \text{div} \mathbf{u} + \frac{1}{\rho} (\mathbf{i} \mathbf{E}), \]
\[ \mathbf{H}_t = \text{rot} [\mathbf{u} \times \mathbf{H}] - \text{rot} \mathbf{E}, \quad \text{div} \mathbf{H} = 0, \]
\[ \mathbf{i} = \sigma \mathbf{E} = \text{rot} \mathbf{H}. \]

Here \( \rho \) is the density, \( p \) is the pressure and \( \varepsilon \) is the internal energy per unit volume. In the three-dimensional space we denote the coordinates and the velocity components as \( \mathbf{x} = (x, y, z) \) and \( \mathbf{u} = (u, v, w) \);

\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \]

is the gradient operator. The equation (3.1c) gives relations for the electric current density \( \mathbf{i} = (i^x, i^y, i^z) \), the electric field \( \mathbf{E} = (E^x, E^y, E^z) \) and the magnetic field \( \mathbf{H} = (H^x, H^y, H^z) \). It contains the electric conductivity \( \sigma = \sigma(\rho, p) \neq 0 \). Note that these relations allow to write down different forms of the MHD system. In particular, \( \mathbf{i} \) and \( \mathbf{E} \) can be eliminated from the system.

The MHD system (3.1) should be supplemented by the equation of state which has the form

\[ \varepsilon = \varepsilon(\rho, p). \]

We consider a medium described by an ideal gas [25–29]

\[ p = \rho RT, \]  

where \( T \) is the temperature and \( R \) is the specific gas constant. The ideal gas is called polytropic if the internal energy function \( \varepsilon \) is linear in the temperature

\[ \varepsilon(T) = c_v T = \frac{RT}{\gamma - 1}, \]

where \( c_v \) is the specific heat capacity measured at constant volume and

\[ \gamma = 1 + \frac{R}{c_v} > 1 \]
is the polytropic constant. Eliminating the temperature from (3.2) and (3.3), we obtain the equation of state

\[ \varepsilon = \frac{1}{\gamma - 1} \frac{p}{\rho}. \] (3.4)

The pressure, the density and the entropy \( \tilde{S} \) are related by the equation \[ p = S \rho^\gamma, \quad S = e^{(\tilde{S} - \tilde{S}_0)/c_v}, \] (3.5)

where \( \tilde{S}_0 \) is constant.

Remark 3.1 The equation of energy conservation can be written in different forms. Instead of the equation (3.1c) it is possible to consider the equation for the pressure

\[ p_t + (\mathbf{u} \cdot \nabla) p + \gamma p \ \text{div} \ \mathbf{u} = (\gamma - 1) (i \mathbf{E}) \] (3.6)

or the equation for the function \( S \), which corresponds to the entropy as given in (3.5),

\[ S_t + (\mathbf{u} \cdot \nabla) S = \frac{\gamma - 1}{\rho^\gamma} (i \mathbf{E}). \] (3.7)

Remark 3.2 For material derivative

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla), \]

i.e. the time derivation along the trajectories, we can rewrite the system (3.1) as

\[ \frac{d}{dt} \rho + \rho \ \text{div} \ \mathbf{u} = 0, \] (3.8a)

\[ \frac{d}{dt} \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{|i \times \mathbf{H}|}{\rho}, \] (3.8b)

\[ \frac{d}{dt} \varepsilon = -\frac{p}{\rho} \ \text{div} \ \mathbf{u} + \frac{1}{\rho} (i \mathbf{E}), \] (3.8c)

\[ \frac{d}{dt} \mathbf{H} = (\mathbf{H} \cdot \nabla) \mathbf{u} - \mathbf{H} \ \text{div} \ \mathbf{u} - \text{rot} \ \mathbf{E}, \quad \text{div} \ \mathbf{H} = 0, \] (3.8d)

\[ i = \sigma \mathbf{E} = \text{rot} \ \mathbf{H}. \] (3.8e)

In this case equations (3.6) and (3.7) get rewritten as

\[ \frac{d}{dt} p + \gamma p \ \text{div} \ \mathbf{u} = (\gamma - 1) (i \mathbf{E}) \] (3.9)

and

\[ \frac{d}{dt} S = \frac{\gamma - 1}{\rho^\gamma} (i \mathbf{E}). \] (3.10)
3.2 Plane one-dimensional flows

The plane one-dimensional MHD flows represent one-dimensional flows of the system (3.1) with all dependent variables being functions of only two independent variables: \( t \) and \( x \). In this case the equations

\[
H_t^x = 0, \tag{3.11a}
\]

\[
\text{div} \ H = \frac{\partial H_x}{\partial x} = 0 \tag{3.11b}
\]

give

\[
H^x = H^0 = \text{const}. \tag{3.12}
\]

The system of equations (3.1) gets reduced to

\[
\rho_t + u \rho_x + \rho u_x = 0, \tag{3.13a}
\]

\[
\rho(u_t + uu_x) + p_x + H^y H^y_x + H^z H^z_x = 0, \tag{3.13b}
\]

\[
\rho(v_t + uv_x) = H^0 H^y_x, \tag{3.13c}
\]

\[
\rho(w_t + uw_x) = H^0 H^z_x, \tag{3.13d}
\]

\[
p_t + up_x + \gamma pu_x = (\gamma - 1)\sigma((E^y)^2 + (E^z)^2), \tag{3.13e}
\]

\[
H^y_t + u H^y_x + H^y u_x = H^0 v_x + E^z_x, \tag{3.13f}
\]

\[
H^z_t + u H^z_x + H^z u_x = H^0 w_x - E^y_x, \tag{3.13g}
\]

\[
\sigma E^y = -H^z_x, \quad \sigma E^z = H^y_x. \tag{3.13h}
\]

Here we use the equation for the pressure (3.6) instead of the equation for the internal energy (3.1c) and eliminate the electric current density \( i \). As we noted earlier, the components \( E^y \) and \( E^z \) can be eliminated. From now the equations (3.11) will be discarded because of (3.12).

3.3 Plane one-dimensional flows in Lagrangian coordinates

The introduction of the mass Lagrangian coordinates \( (s, \eta, \zeta) \) is described in Appendix A. In the Lagrangian coordinates the Eulerian spatial coordinates are given by

\[
x = \varphi(t, s), \quad y = \eta + \psi(t, s), \quad z = \zeta + \chi(t, s), \tag{3.14}
\]

where the functions \( \varphi, \psi \) and \( \chi \) satisfy the equations

\[
\varphi_t(t, s) = u(t, \varphi(t, s)), \quad \varphi_s(t, s) = \frac{1}{\rho(t, \varphi(t, s))}, \tag{3.15a}
\]

\[
\psi_t(t, s) = v(t, \varphi(t, s)), \tag{3.15b}
\]

\[
\chi_t(t, s) = w(t, \varphi(t, s)). \tag{3.15c}
\]
In the mass Lagrangian coordinates \((t, s)\) the equations (3.13), describing the plane one-dimensional MHD flows, take the form

\[
\rho_t = -\rho^2 u_s, \quad \text{(3.16a)}
\]
\[
u_t = -p_s - H^y H^y_s - H^z H^z_s, \quad x_t = u, \quad \text{(3.16b)}
\]
\[
w_t = H^0 H^y_s, \quad y_t = v, \quad \text{(3.16c)}
\]
\[
H^y_t = \rho (H^0 v_s - H^y u_s + E^z_s), \quad \text{(3.16f)}
\]
\[
H^y_s = \rho (H^0 w_s - H^z u_s - E^y_s), \quad \text{(3.16g)}
\]
\[
p_t = -\gamma \rho p u_s + (\gamma - 1) \sigma (E^y)^2 + (E^z)^2, \quad \text{(3.16e)}
\]
\[
s E^y = -\rho H^y_s, \quad s E^z = \rho H^y_s. \quad \text{(3.16h)}
\]

Note that the time differentiation in the system (3.16) is the Lagrangian one, i.e. it is taken along the trajectories. For this reason we add the components of the equation \(x_t = u\).

We remark that in the mass Lagrangian coordinates \((t, s)\) the Eulerian spatial coordinate \(x\) is nonlocal. It is given by the system

\[
x_t = u, \quad x_s = \frac{1}{\rho}, \quad \text{(3.17)}
\]

We also have

\[
y_t = v, \quad \text{(3.18a)}
\]
\[
z_t = w. \quad \text{(3.18b)}
\]

**Remark 3.3** Using equation (3.16a), it is possible to rewrite equations (3.16f) and (3.16g) as the conservation laws

\[
\left( \frac{H^y}{\rho} \right)_t = (H^0 v + E^z)_s, \quad \text{(3.19)}
\]
\[
\left( \frac{H^z}{\rho} \right)_t = (H^0 w - E^y)_s. \quad \text{(3.20)}
\]

4 Lie group classification of equations (3.16) with finite conductivity

In this section we perform a group classification of the system (3.16) with an arbitrary function \(\sigma(\rho, p)\) and an arbitrary constant \(H^0\). Infinitesimal generators of the Lie symmetry group are considered in the form

\[
X = \xi^t \frac{\partial}{\partial t} + \xi^s \frac{\partial}{\partial s} + \eta^x \frac{\partial}{\partial x} + \eta^y \frac{\partial}{\partial y} + \eta^z \frac{\partial}{\partial z} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v} + \eta^w \frac{\partial}{\partial w} + \eta^\rho \frac{\partial}{\partial \rho} + \eta^p \frac{\partial}{\partial p} + \eta^{E^y} \frac{\partial}{\partial E^y} + \eta^{E^z} \frac{\partial}{\partial E^z} + \eta^{H^y} \frac{\partial}{\partial H^y} + \eta^{H^z} \frac{\partial}{\partial H^z}. \quad \text{(4.1)}
\]
The coefficients $\xi_t, \xi^s, \eta^x, \ldots, \eta^{H^z}$ of the generator are functions of the independent and dependent variables $t, s, x, u, \rho, p, E^y, E^z, H^y$ and $H^z$.

The infinitesimal criterion of invariance \cite{1-3} requires

$$X(\mathcal{F})|_{\mathcal{F}=0} = 0,$$

where $\mathcal{F} = 0$ denotes system \eqref{3.16}. Here the generator $X$ is prolonged to all derivatives involved in the system $\mathcal{F} = 0$ according to the standard prolongation formulas \cite{1-3}.

Splitting equation \eqref{4.2} with respect to the first-order derivatives and performing standard simplifications, we derive the classifying equations

$$2(a_6 - a_7 + a_8)\rho\sigma_\rho + 2a_8p\sigma_\rho = (a_6 - 2a_7)\sigma,$$

$$H^0a_8 = 0,$$

$$H^0\eta^y = 0, \quad H^0\eta^y = 0, \quad H^0\eta^z = 0, \quad H^0\eta^z = 0,$$

$$H^0(\eta^y + a_6 - a_7) = 0, \quad H^0(\eta^y + a_5) = 0,$$

$$H^0\eta^w = 0, \quad H^0\eta^w = 0, \quad H^0\eta^z = 0,$$

$$H^0(\eta^w - a_5) = 0, \quad H^0(\eta^w + a_6 - a_7) = 0,$$

where the coefficients of the generator are

$$\xi_t = a_6t + a_1, \quad \xi^s = (2a_6 - a_7 + 2a_8)s + a_2,$$

$$\eta^x = a_4t + a_7x + a_3, \quad \eta^y = f_3t + a_7y - a_5z + f_1, \quad \eta^z = f_4t + a_5y + a_7z + f_2,$$

$$\eta^u = (-a_6 + a_7)u + a_4, \quad \eta^v = (-a_6 + a_7)v - a_5w + f_3, \quad \eta^w = a_5v + (-a_6 + a_7)w + f_4,$$

$$\eta^\rho = 2(a_6 - a_7 + a_8)\rho, \quad \eta^\rho = 2a_8p,$$

$$\eta^{E^y} = (-a_6 + a_7 + a_8)E^y - a_5E^z, \quad \eta^{E^z} = a_5E^y + (-a_6 + a_7 + a_8)E^z,$$

$$\eta^{H^y} = a_8H^y - a_5H^z, \quad \eta^{H^z} = a_5H^y + a_8H^z.$$

\hspace{1cm} (4.4)

Here $a_i, i = 1, \ldots, 8$ are constants and

$$f_i = f_i(s, v, w, y - tv, z - tw), \quad i = 1, \ldots, 4,$$

\hspace{1cm} (4.5)

are arbitrary functions of their arguments.

According to \cite{4-4}, a Lie algebra admitted by system \eqref{3.16} belongs to the ex-
tended Lie algebra with the basis determined by the generators

\[
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial s}, \quad Y_3 = \frac{\partial}{\partial x}, \quad Y_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad Y_5 = \frac{\partial}{\partial y} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} + E^z \frac{\partial}{\partial E^y} - E^y \frac{\partial}{\partial E^z} + H^z \frac{\partial}{\partial H^y} - H^y \frac{\partial}{\partial H^z},
\]

\[
Y_6 = t \frac{\partial}{\partial t} + 2s \frac{\partial}{\partial s} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} + 2 \rho \frac{\partial}{\partial \rho} - E^y \frac{\partial}{\partial E^y} - E^z \frac{\partial}{\partial E^z},
\]

\[
Y_7 = -s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w} - 2 \rho \frac{\partial}{\partial \rho} + E^y \frac{\partial}{\partial E^y} + E^z \frac{\partial}{\partial E^z},
\]

\[
Y_8 = 2s \frac{\partial}{\partial s} + 2 \rho \frac{\partial}{\partial \rho} + 2p \frac{\partial}{\partial p} + E^y \frac{\partial}{\partial E^y} + E^z \frac{\partial}{\partial E^z} + H^y \frac{\partial}{\partial H^y} + H^z \frac{\partial}{\partial H^z},
\]

\[
Y_9 = f_1 \frac{\partial}{\partial y}, \quad Y_{10} = f_2 \frac{\partial}{\partial z}, \quad Y_{11} = f_3 \left( t \frac{\partial}{\partial y} + \frac{\partial}{\partial v} \right), \quad Y_{12} = f_4 \left( t \frac{\partial}{\partial z} + \frac{\partial}{\partial w} \right),
\]

(4.6)

where arbitrary functions \( f_1, \ldots, f_4 \) have the form \( (4.5) \).

For the further analysis of the symmetries it is necessary to consider two cases of \( (3.12) \), namely \( H^0 \neq 0 \) and \( H^0 = 0 \), separately.

### 4.1 Case \( H^0 \neq 0 \)

In this case the group classification of equations is obtained with respect to arbitrary elements \( \sigma(\rho, p) \) and \( H^0 \).

Equivalence transformations allow changing the arbitrary elements while preserving the structure of the equations \([1]\). The generators of the equivalence transformations for system \( (3.16) \) are given in \( (B.2) \), Appendix B. These transformations can be used to scale the function \( \sigma \) and the constant \( H^0 \).

#### 4.1.1 Arbitrary \( \sigma(\rho, p) \)

In the most general case of \( \sigma(\rho, p) \) and \( H^0 \) we find the kernel of the Lie algebras admitted by system \( (3.16) \). It consists of the generators admitted by the system for an arbitrary function \( \sigma(\rho, p) \) and an arbitrary constant \( H^0 \). In order to obtain the kernel, we split \( (4.3) \) with respect to \( \sigma, \sigma_\rho, \sigma_p \) and \( H^0 \). From the resulting equations it immediately follows that

\[
a_6 = a_7 = a_8 = 0.
\]

(4.7)

We also obtain conditions on the functions \( \eta^y, \eta^z, \eta^v \) and \( \eta^w \). From these conditions it follows that the functions \( f_i \) must be less general than given in \( (4.5) \), namely they are functions of \( s \). Finally, we obtain the following kernel of the admitted Lie
algebras

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial s}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_5 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w} + E\frac{\partial}{\partial E} - H\frac{\partial}{\partial H}.
\]

\[
X_6 = h_1(s)\frac{\partial}{\partial y}, \quad X_7 = h_2(s)\frac{\partial}{\partial z}, \quad X_8 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v}, \quad X_9 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial w}.
\] (4.8)

where \(h_1(s)\) and \(h_2(s)\) are arbitrary functions.

4.1.2 Special cases of \(\sigma(\rho, p)\)

Consider system \((4.3)\) with \(H^0 \neq 0\). Taking into account that functions \(f_i\) have the form \((4.5)\), we get

\[
a_8 = 0, \\
\eta^y = b_1 t + a_7 y - a_5 z + h_1(s), \quad \eta^z = b_2 t + a_5 y + a_7 z + h_2(s), \\
\eta^v = (-a_6 + a_7)v - a_5 w + b_1, \quad \eta^w = a_5 v + (-a_6 + a_7)w + b_2,
\] (4.9)

where \(h_1(s)\) and \(h_2(s)\) are arbitrary functions and \(b_1\) and \(b_2\) are constants.

In the case \(a_6 = a_7\) there are no extensions of the kernel \((4.8)\). Hence, the classifying equation can be written as

\[
\rho \sigma_\rho = \alpha \sigma, \quad \alpha = \frac{a_6 - 2a_7}{2(a_6 - a_7)}.
\] (4.10)

The latter equation can be rewritten in the form

\[
(2\alpha - 1)a_6 = 2(\alpha - 1)a_7.
\] (4.11)

The resulting classification, based on this equation, is given in Table 1. The first column of the table gives the dimension \(\text{dim } L\) of the admitted Lie algebra. The extension of the kernel of the admitted Lie algebras \((4.8)\) is given in the second column of the table. The third column gives the corresponding forms of the function \(\sigma\). Here and in the next table \(F\) denotes an arbitrary differentiable function of its argument.

| \(\text{dim } L\) | Extension of kernel \((4.8)\) | \(\sigma(p, \rho)\) |
|-----------------|------------------------|------------------|
| 6              | \(X_{10} = 2(\alpha - 1)Y_6 + (2\alpha - 1)Y_7\) | \(\rho^\alpha F(p)\) |

Table 1: Lie group extensions for \(H^0 \neq 0\).
Remark 4.1 Using equation (3.2), it is possible to express some particular cases \( \sigma(p, \rho) \) in the form \( \tilde{\sigma}(T) \). The presentation of the electric conductivity as a function of the temperature is of interest for physical applications. For the particular case \( F(p) = C p^{-\alpha} \) we obtain \( \tilde{\sigma}(T) = CT^{-\alpha} \).

4.2 Case \( H^0 = 0 \)

For \( H^0 = 0 \) the system of equations (3.16) becomes

\begin{align*}
p_t &= -\rho^2 u_s, \quad & (4.12a) \\
u_t &= -p_s - H^y H^y_s - H^z H^z_s, \quad & x_t = u, \quad & (4.12b) \\
p_t &= -\gamma \rho p u_s + (\gamma - 1) \sigma ((E^y_s)^2 + (E^z_s)^2), \quad & (4.12c) \\
H^y_t &= \rho (-H^y u_s + E^z_s), \quad & (4.12d) \\
H^z_t &= \rho (-H^z u_s - E^y_s), \quad & (4.12e) \\
\sigma E^y &= -\rho H^z_s, \quad \sigma E^z = \rho H^y_s. \quad & (4.12f)
\end{align*}

The remaining four equations

\begin{align*}
v_t &= 0, \quad & y_t = v, \quad & (4.13a) \\
w_t &= 0, \quad & z_t = w. \quad & (4.13b)
\end{align*}

can be analyzed independently. In the rest of this section we discuss the reduced system (4.12). It implies that variables \( y, z, v \) and \( w \) are excluded from the consideration.

Remark 4.2 The equations (4.13) can be easily solved as

\begin{align*}
v &= v_0(s), \quad & y = v_0(s) t + y_0(s, \eta, \zeta), \\
w &= w_0(s), \quad & z = w_0(s) t + z_0(s, \eta, \zeta),
\end{align*}

with functions \( v_0(s), y_0(s, \eta, \zeta), w_0(s) \) and \( z_0(s, \eta, \zeta) \) defined by the initial conditions.

Remark 4.3 Similarly to Remark 3.3 equations (4.12a) and (4.12e) can be rewritten as the conservation laws

\begin{align*}
\left( \frac{H^y}{\rho} \right)_t &= E^z_s, \quad & (4.14) \\
\left( \frac{H^z}{\rho} \right)_t &= -E^y_s. \quad & (4.15)
\end{align*}

The equivalence transformations of the reduced system (4.12) are given in (B.4), Appendix B. They can be used to scale \( \sigma \).
The Lie algebra admitted by system (4.12) belongs to the extended algebra whose basis is defined by the generators $Y_1, Y_2, ..., Y_8$ from (4.6), namely

$$
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial s}, \quad Y_3 = \frac{\partial}{\partial x}, \quad Y_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u},
$$

$$
Y_5 = E^z \frac{\partial}{\partial E^y} - E^y \frac{\partial}{\partial E^z} + H^z \frac{\partial}{\partial H^y} - H^y \frac{\partial}{\partial H^z},
$$

$$
Y_6 = t \frac{\partial}{\partial t} + 2s \frac{\partial}{\partial s} - u \frac{\partial}{\partial u} + 2\rho \frac{\partial}{\partial \rho} - E^y \frac{\partial}{\partial E^y} - E^z \frac{\partial}{\partial E^z},
$$

$$
Y_7 = -s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} - 2\rho \frac{\partial}{\partial \rho} + E^y \frac{\partial}{\partial E^y} + E^z \frac{\partial}{\partial E^z},
$$

$$
Y_8 = 2s \frac{\partial}{\partial s} + 2\rho \frac{\partial}{\partial \rho} + 2p \frac{\partial}{\partial p} + E^y \frac{\partial}{\partial E^y} + E^z \frac{\partial}{\partial E^z} + H^y \frac{\partial}{\partial H^y} + H^z \frac{\partial}{\partial H^z}. \quad (4.16)
$$

Note that these generators are truncated generators (4.6): the variables $y, z, v$ and $w$ are omitted. In the subsequent discussion the corresponding group classification is denoted by $\Theta_1$.

**4.2.1 Arbitrary $\sigma(\rho,p)$**

The kernel of the admitted Lie algebras consists of the following five generators

$$
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial s}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u},
$$

$$
X_5 = E^z \frac{\partial}{\partial E^y} - E^y \frac{\partial}{\partial E^z} + H^z \frac{\partial}{\partial H^y} - H^y \frac{\partial}{\partial H^z}, \quad (4.17)
$$

which are admitted for all $\sigma(\rho,p)$.

**4.2.2 Special cases of $\sigma(\rho,p)$**

For $H^0 = 0$ the classifying system (4.3) reduces to one equation

$$
2(a_6 - a_7 + a_8)\rho \sigma_p + 2a_8 \rho \sigma_p = (a_6 - 2a_7)\sigma. \quad (4.18)
$$

In this case the classification problem is more cumbersome than that for $H^0 \neq 0$. To overcome these difficulties we use an approach, based on the group properties of system (4.12). This approach is applicable when transformations of the symmetry generators under action of the equivalence transformations coincide with transformations of the symmetry generators under action of the inner automorphisms. This method was applied in [30] (see also [31]). It is based on the following idea. The group classification of system (4.12) supposes separation of the generators into dissimilar classes with respect to the equivalence transformation group. This separation leads to the group classification $\Theta_1$ for the extended Lie algebra (4.16). Another classification of the Lie algebra (4.10) can be obtained for the inner automorphisms. We denote this classification by $\Theta_2$. 

13
If the action of the the equivalence transformation group and the action of the inner automorphisms coincide, then any subalgebra of $\Theta_1$ is a subalgebra of $\Theta_2$. Hence, for the group classification one can use subalgebras of $\Theta_2$. This simplifies the group classification problem. Instead of the equivalence transformations, which are generally nonlinear, one can consider the inner automorphisms, which are presented by linear transformations of the generators.

Notice that any subalgebra of $\Theta_1$ includes the kernel of admitted Lie algebras (4.17). Therefore, for obtaining $\Theta_1$ from $\Theta_2$ one can consider only subalgebras of $\Theta_2$ containing the kernel (4.17). This allows avoiding analysis of all subalgebras of $\Theta_2$ that further facilitates the group classification.

Thus, for the group classification we can use the following algorithm.

1. An optimal system of subalgebras $\Theta_2$ is constructed (only subalgebras which contain the kernel of the admitted Lie algebras are needed). This optimal system of subalgebras defines classes of the non-equivalent subalgebras with respect to generator transformations corresponding to the inner automorphisms. As noted before, the inner automorphisms act similarly to the equivalence transformations. Thus, it is possible to use this optimal system for the group classification. From the optimal system of subalgebras $\Theta_2$ one chooses the subalgebras which include the kernel of the admitted Lie algebras. It significantly reduces the number of subalgebras to be considered.

2. For each subalgebra of the optimal system $\Theta_2$, which contains the kernel of the admitted Lie algebras, the coefficients of the basis elements are substituted into the classifying equation. Here it is sufficient to consider the extension of the kernel. Solving the system of the equations obtained for the function $\sigma(\rho, p)$, one obtains non-equivalent cases $\sigma(\rho, p)$ for the group classification.

The algorithm is applied to the generators (4.16) with the kernel of the admitted Lie algebras (4.17) in Appendix C. The equivalence transformations are defined by generators given in (5.4), Appendix B. Here we present only the results. The kernel (4.17) can be extended by operators from the set $\{Y_6, Y_7, Y_8\}$. The possible extensions are one-dimensional subalgebras

\[
\{Y_7\}, \quad \{Y_6 + \alpha Y_7\}, \quad \{Y_8 + \alpha Y_6 + \beta Y_7\},
\]

(4.19)

two-dimensional subalgebras

\[
\{Y_6, Y_7\}, \quad \{Y_8 + \alpha Y_6, Y_7\}, \quad \{Y_8 + \alpha Y_7, Y_6 + \beta Y_7\}
\]

(4.20)

and the three-dimensional subalgebra

\[
\{Y_6, Y_7, Y_8\}.
\]

(4.21)

It remains to find out the corresponding functions $\sigma(\rho, p) \neq 0$. Extensions which lead to $\sigma(\rho, p) = 0$ are to be discarded.
To solve the classifying equations (4.18) for the possible kernel extensions, i.e. for subalgebras given in (4.19), (4.20) and (4.21), we consider the generators from the extensions. For each basis generator of subalgebras, the corresponding coefficients are substituted into equation (4.18). It provides the equations which the function \( \sigma(\rho, p) \) must satisfy in order to admit the considered generator.

As an illustrating example we consider the subalgebra \( \{Y_8 + \alpha Y_7, Y_6 + \beta Y_7\} \). The procedure described above leads to the system

\[
\begin{align*}
    a_6 &= 0, ~ a_7 = \alpha, ~ a_8 = 1 : \quad (1 - \alpha) \rho \sigma_\rho + p \sigma_p &= -\alpha \sigma, \\
    a_6 &= 1, ~ a_7 = \beta, ~ a_8 = 0 : \quad 2(1 - \beta) \rho \sigma_\rho &= (1 - 2\beta) \sigma.
\end{align*}
\]

(4.22)

One can verify that \( \beta = 1 \) leads to the solution \( \sigma \equiv 0 \) that is excluded from the consideration. Thus, the constraint \( \beta \neq 1 \) is imposed on the subalgebra generators.

The solution of the latter system is

\[
\sigma(\rho, p) = C \rho^{2(\beta - 1)} p^{\alpha - 2(\beta - 1)},
\]

(4.23)

where \( C \) is constant. By means of equivalence transformations (B.4), one can set \( C = 1 \).

Similarly we consider the other possible extensions. The results of the calculations for all subalgebras (4.19), (4.20) and (4.21) are presented in Table 2. In the table \( F \) is an arbitrary function and \( C \) is an arbitrary constant, which can be removed by scaling. The cases corresponding to the solution \( \sigma \equiv 0 \) are excluded.

**Remark 4.4** A particular case of the classification presented in Table 2 was carried out for \( H^z = 0, E^y = 0 \) in [6]. The results obtained there provide particular cases of the present classification. For example, consider the exponential case from [6]

\[
\sigma(\rho, p) = e^{a \rho + b p}, \quad a, b = \text{const}, \quad a \geq 0,
\]

(4.24)

which splits into three subcases.

1. If the coefficients \( a \) and \( b \) are arbitrary, the system admits only generators from the kernel (4.17). This case corresponds to arbitrary \( \sigma = \sigma(p, \rho) \).

2. The case \( a = 0 \) and \( b \neq 0 \) corresponds to the extension \( \{Y_8 + \alpha Y_6 + \beta Y_7\} \) with \( \alpha = -2 \) and \( \beta = -1 \) (see Table 2). Thus, for \( F(\rho) = e^\rho \) there exists the additional symmetry

\[
-(Y_8 - 2Y_6 - Y_7) = 2t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} - 2p \frac{\partial}{\partial p} - 2E^z \frac{\partial}{\partial E^z} - H^y \frac{\partial}{\partial H^y}.
\]

(4.25)

3. The case \( a \neq 0 \) and \( b = 0 \) corresponds to the extension \( \{Y_6 + \alpha Y_7\} \) with \( \alpha = 1/2 \). It provides \( F(p) = e^p \) and the additional symmetry

\[
2Y_6 + Y_7 = 2t \frac{\partial}{\partial t} + 3s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} + 2p \frac{\partial}{\partial p} - E^z \frac{\partial}{\partial E^z}.
\]

(4.26)
Table 2: Lie group extensions for $H^0 = 0$.

Remark 4.5 For particular cases of $F(p)$ in the cases $\dim L = 6$ and particular cases of $\alpha$ in the cases $\dim L = 7$, the conductivity can be presented as a function of the temperature. We get the following options

- Case $X_6 = Y_7$
  If $F(p) = Cp^{-1}$, then $\tilde{\sigma}(T) = \tilde{CT}^{-1}$.

- Case $X_6 = Y_6 + \alpha Y_7$
  For $F(p) = Cp^{2(1-\alpha)}$, we get $\tilde{\sigma}(T) = \tilde{CT}^{2(1-\alpha)}$.

- Case $X_6 = Y_8 + \alpha Y_6 + \beta Y_7$
  Function $F(q) = Cq^{\frac{\alpha-2\beta}{2}}$ provides $\tilde{\sigma}(T) = \tilde{CT}^{\frac{\alpha-2\beta}{2}}$.

- Case $X_6 = Y_8 + \alpha Y_6$, $X_7 = Y_7$
  For $\alpha = 0$ we obtain $\tilde{\sigma}(T) = \tilde{CT}^{-1}$.

- Case $X_6 = Y_8 + \alpha Y_7$, $X_7 = Y_6 + \beta Y_7$
  For $\alpha = 0$ we get $\tilde{\sigma}(T) = \tilde{CT}^{\frac{\alpha}{\alpha-1}}$.

5 Conservation laws for the case of finite conductivity

Conservation laws possessed by a system of PDEs with two independent variables $(t, s)$ have the form

$$D_t^L(T^t) + D_s(T^s) = 0.$$  \hspace{1cm} (5.1)

They hold on the solutions of the system. The conservation law densities $T^t$ and $T^s$ for the system (3.16) are functions of the independent and dependent variables $(t, s, x, u, \rho, \rho, E^y, E^z, H^y, H^z)$.

There are several approaches to find conservation laws. If equations have a variational structure, i.e. have the form of Euler-Lagrange equations for some Lagrangian function, one can apply the Noether theorem [16]. It allows to use variational and
divergence symmetries of the Lagrangian function to obtain conservation laws. For equations without variational structure it is possible to introduce additional variables and consider an extended system, which is variational. This approach was called the adjoint equation method [32, 33].

Conservation laws can also be found by direct computation. First, the densities $T^t$ and $T^s$ are differentiated and some derivatives are eliminated with the help of the considered equations and (if necessary) their differential consequences. In the case of evolutionary equations it is standard to eliminate time derivatives. The resulting equation is split for the remaining derivatives and later for the dependent variables in order to find the densities of the conservation law. We will employ the direct method to find conservation laws in this section.

**Remark 5.1** Conservation laws in the Lagrangian coordinates (5.1) can be rewritten in the Eulerian coordinates as

$$D_t^E(eT^t) + D_x(eT^x) = 0.$$  

(5.2)

The total differentiation operators $D^L_t$ and $D^s$ in Lagrangian coordinates $(t,s)$ and the total differentiation operators $D^E_t$ and $D_x$ in Eulerian coordinates $(t,x)$ are related by

$$D^L_t = D^E_t + uD_x, \quad D^s = \frac{1}{\rho}D_x.$$  

(5.3)

The densities of the conservation laws in the Eulerian coordinates are related to the densities of the conservation laws in the Lagrangian coordinates as

$$eT^t = \rho T^t, \quad eT^x = \rho u T^t + T^s.$$  

(5.4)

This relation follows from the identity

$$D^L_t(T^t) + D^s(T^s) = \frac{1}{\rho} \left( D^E_t(\rho T^t) + D_x(\rho u T^t + T^s) \right).$$

Here it is necessary to take into account that the mass Lagrangian coordinate $s$ is a nonlocal dependent variable in the Eulerian coordinates, i.e. it is defined by the equations

$$s_x = \rho, \quad s_t = -\rho u,$$

which follow from the equations (3.15a).

### 5.1 Case $H^0 \neq 0$

#### 5.1.1 Arbitrary conductivity $\sigma(\rho,p)$

Direct computation provides the following 10 conservation laws for the system (3.16) in the general case of $H^0$ and $\sigma(\rho,p)$:

- **mass**
  $$D^L_t \left( \frac{1}{\rho} \right) - D^s(u) = 0;$$

(5.5)
• three momenta

\[ D^L_t (u) + D_s \left( p + \frac{(H^y)^2 + (H^z)^2}{2} \right) = 0, \quad (5.6) \]
\[ D^L_t (v) - D_s (H^0 H^y) = 0, \quad (5.7) \]
\[ D^L_t (w) - D_s (H^0 H^z) = 0; \quad (5.8) \]

• the center of mass conservation laws

\[ D^L_t (tu - x) + D_s \left\{ t \left( p + \frac{(H^y)^2 + (H^z)^2}{2} \right) \right\} = 0, \quad (5.9) \]
\[ D^L_t (tv - y) - D_s (tH^0 H^y) = 0, \quad (5.10) \]
\[ D^L_t (tw - z) - D_s (tH^0 H^z) = 0; \quad (5.11) \]

• magnetic fluxes

\[ D^L_t \left( \frac{H^y}{\rho} \right) - D_s (E^z + H^0 v) = 0, \quad (5.12) \]
\[ D^L_t \left( \frac{H^z}{\rho} \right) + D_s (E^y - H^0 w) = 0; \quad (5.13) \]

• energy

\[ D^L_t \left\{ \frac{1}{2} (u^2 + v^2 + w^2) + \frac{1}{\gamma - 1} \frac{p}{\rho} + \frac{(H^y)^2 + (H^z)^2}{2\rho} \right\} \]
\[ + D_s \left\{ u \left( p + \frac{1}{2} \frac{(H^y)^2 + (H^z)^2}{\rho} \right) + E^y H^z - E^z H^y - H^0 (v H^y + w H^z) \right\} = 0. \quad (5.14) \]

**Remark 5.2** The latter conservation law can be rewritten as

\[ D^L_t \left\{ \frac{1}{2} |u|^2 + \frac{1}{\gamma - 1} \frac{p}{\rho} + \frac{1}{2\rho} |H|^2 \right\} \]
\[ + D_s \left\{ u \left( p + \frac{1}{2} |H|^2 \right) + [E \times H]_1 - H^0 (u \cdot H) \right\} = 0, \quad (5.15) \]

where \([E \times H]_1\) stands for the first component of the vector.

It is worth mentioning that some of these conservation laws are already present in the system (3.16). For example, the conservation of momenta. At the same time other conservation laws, e.g. conservation of energy, hold due to several equations of the system (3.16).
5.1.2 Special cases of conductivity $\sigma(\rho, p)$

There are no particular cases $\sigma(\rho, p)$ which leads to additional conservation laws.

5.2 Case $H^0 = 0$

For $H^0 = 0$ the system of MHD equations (3.16) gets split into the reduced system (4.12) and the four equations (4.13). These two subsystems will be considered separately.

5.2.1 Arbitrary conductivity $\sigma(\rho, p)$

The conservation laws of the reduced system (4.12) are obtained by direct computation. They represent

- conservation of mass

\[ D^L_t \left( \frac{1}{\rho} \right) - D_s (u) = 0; \quad (5.16) \]

- conservation of momentum

\[ D^L_t (u) + D_s \left( p + \frac{(H^y)^2 + (H^z)^2}{2} \right) = 0; \quad (5.17) \]

- motion of the center of mass

\[ D^L_t (tu - x) + D_s \left\{ t \left( p + \frac{(H^y)^2 + (H^z)^2}{2} \right) \right\} = 0; \quad (5.18) \]

- conservation of magnetic fluxes

\[ D^L_t \left( \frac{H^y}{\rho} \right) - D_s (E^z) = 0, \quad (5.19) \]

\[ D^L_t \left( \frac{H^z}{\rho} \right) + D_s (E^y) = 0; \quad (5.20) \]

- conservation of energy

\[
D^L_t \left\{ \frac{1}{2} u^2 + \frac{1}{\gamma - 1} \frac{p}{\rho} + \frac{(H^y)^2 + (H^z)^2}{2\rho} \right\} \\
+ D_s \left\{ u \left( p + \frac{(H^y)^2 + (H^z)^2}{2} \right) + E^y H^z - E^z H^y \right\} = 0, \quad (5.21)
\]

which can be rewritten as (5.15) with $H^0 = 0$. 

19
The system \((4.13)\) has conservation laws of the form

\[
D^L_t \left( T^t(v, w, y - tv, z - tw) \right) = 0, \quad (5.22)
\]

where \(T^t\) is an arbitrary function. These conservation laws include conservation of momenta \((5.7)\) and \((5.8)\)

\[
D^L_t(v) = 0, \quad (5.23)
\]

\[
D^L_t(w) = 0 \quad (5.24)
\]

and conservation laws for the motion of the center of mass \((5.10)\) and \((5.11)\)

\[
D^L_t(y - tv) = 0, \quad (5.25)
\]

\[
D^L_t(z - tv) = 0 \quad (5.26)
\]

as particular cases. In contrast to the case \(H^0 \neq 0\) there is conservation of the angular momentum

\[
D^L_t(zv - yw) = 0. \quad (5.27)
\]

All these conservation laws have trivial coordinate density \(T^s \equiv 0\) in the case \(H^0 = 0\).

The conservation laws \((5.16)-(5.21)\) correspond to the conservation laws \((5.5), (5.6), (5.9), (5.12), (5.13)\) and \((5.14)\) of the case \(H^0 \neq 0\). Combining these conservation laws with the conservation laws \((5.22)\), we conclude that there are more conservation laws for the case \(H^0 = 0\). For example, conservation of the angular momentum \((5.26)\) has no analog for \(H^0 \neq 0\). In addition to this, for \(H^0 = 0\) some of the conservation laws get simplified.

### 5.2.2 Special case of conductivity \(\sigma(\rho, p) = \rho\)

There are several cases with symmetry extensions, which were described in point \(4.2\). For the conservation laws there is only one extension.

If \(\sigma(\rho, p) = \rho\), there are two additional conservation laws

\[
D^L_t \left( \frac{sH^z}{\rho} \right) - D_s(sE^y + H^z) = 0, \quad (5.28)
\]

\[
D^L_t \left( \frac{sH^y}{\rho} \right) + D_s(sE^z - H^y) = 0. \quad (5.29)
\]

Note that the condition \(\sigma = \rho\) has no analog for the infinite conductivity \(\sigma = \infty\). Thus, conservation laws \((5.28)\) and \((5.29)\) hold only for the finite conductivity.
6 Symmetries for the case of infinite conductivity

MHD equations for the case of infinite conductivity ($\sigma = \infty$) can be obtained from the system (3.1) in the limiting case $\sigma \to \infty$. For the plane one-dimensional flows in the mass Lagrangian coordinates we derive from (3.16):

\[
\begin{align*}
\rho_t &= -\rho^2 u_s, \quad \text{(6.1a)} \\
u_t &= -p_s - H^0 H^y_s - H^z H^z_s, \quad x_t = u, \quad \text{(6.1b)} \\
v_t &= H^0 H^y_s, \quad y_t = v, \quad \text{(6.1c)} \\
w_t &= H^0 H^z_s, \quad z_t = w, \quad \text{(6.1d)} \\
p_t &= -\gamma \rho p u_s, \quad \text{(6.1e)} \\
H^y_t &= \rho (H^0 v_s - H^y u_s), \quad \text{(6.1f)} \\
H^z_t &= \rho (H^0 w_s - H^z u_s). \quad \text{(6.1g)}
\end{align*}
\]

**Remark 6.1** Similarly to Remark 3.3 it is possible to rewrite equations (6.1f) and (6.1g) as the conservation laws

\[
\begin{align*}
\left( \frac{H^y}{\rho} \right)_t &= (H^0 v)_s, \quad \text{(6.2)} \\
\left( \frac{H^z}{\rho} \right)_t &= (H^0 w)_s \quad \text{(6.3)}
\end{align*}
\]

with the help of equation (6.1a).

In the mass Lagrangian coordinates we can rewrite the equations (6.2) and (6.3) using functions $\varphi$, $\psi$ and $\chi$, which describe the Eulerian coordinates (3.14), as

\[
(\varphi_s H^y)_t = (H^0 \psi_t)_s, \quad (\varphi_s H^z)_t = (H^0 \chi_t)_s.
\]

Integrating these equations with respect to $t$, we get

\[
\varphi_s H^y = H^0 \psi_s + g_1'(s), \quad \varphi_s H^z = H^0 \chi_s + g_2'(s),
\]

where $g_1(s)$ and $g_2(s)$ are arbitrary functions of integration.

If $H^0 \neq 0$, then one can choose the functions $\psi(t,s)$ and $\chi(t,s)$ such that $g_1(s) = 0$ and $g_2(s) = 0$. In this case

\[
H^y = H^0 \frac{\psi_s}{\varphi_s}, \quad H^z = H^0 \frac{\chi_s}{\varphi_s} \quad \text{(6.4)}
\]

that leads to

\[
y_s = \frac{H^y}{H^0 \rho}, \quad z_s = \frac{H^z}{H^0 \rho}. \quad \text{(6.5)}
\]

It is easy to see that equations (6.1a) and (6.1e) provide

\[
\left( \frac{p}{\rho^2} \right)_t = 0.
\]
Therefore, the entropy function $S$, defined in (3.5), satisfies the equation

$$S_t = 0$$

(6.6)

that represents the conservation of the entropy along trajectories.

### 6.1 Case $H^0 \neq 0$

The equivalence transformations for system (6.1) are given in (B.6), Appendix B. These transformations can be used to scale the constant $H^0$.

The system (6.1) admits the symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial s}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

$$X_5 = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} + w \frac{\partial}{\partial v} + H^y \frac{\partial}{\partial H^y} - H^y \frac{\partial}{\partial H^z},$$

$$X_6 = t \frac{\partial}{\partial t} + 2s \frac{\partial}{\partial s} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} - w \frac{\partial}{\partial w} + 2 \rho \frac{\partial}{\partial \rho},$$

$$X_7 = -s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w} - 2 \rho \frac{\partial}{\partial \rho},$$

$$X_8 = q_1 \left( s, \frac{p}{\rho^\gamma} \right) \frac{\partial}{\partial y}, \quad X_9 = q_2 \left( s, \frac{p}{\rho^\gamma} \right) \frac{\partial}{\partial z}, \quad X_{10} = t \frac{\partial}{\partial y} + \frac{\partial}{\partial v}, \quad X_{11} = t \frac{\partial}{\partial z} + \frac{\partial}{\partial w}. \quad (6.7)$$

Note that $p/\rho^\gamma$ is a function of the entropy (see (3.5)).

### 6.2 Case $H^0 = 0$

For the infinite conductivity and $H^0 = 0$ the system (6.1) is split into the reduced system

$$\rho_t = -\rho^2 u_s,$$

(6.8a)

$$u_t = -p_s - H^y H^\rho_s - H^z H^\rho_z,$$  \quad $x_t = u,$

(6.8b)

$$p_t = -\gamma p u_s,$$

(6.8c)

$$H^y_t = -\rho H^y u_s,$$

(6.8d)

$$H^z_t = -\rho H^z u_s.$$  \quad (6.8e)

and the remaining four equations (4.13), namely

$$v_t = 0, \quad y_t = v,$$

(6.9a)

$$w_t = 0, \quad z_t = w.$$  \quad (6.9b)

**Remark 6.2** Similarly to Remark 3.3 we can rewrite equations (6.8a) and (6.8e) as

$$\left( \frac{H^y}{\rho} \right)_t = 0,$$

(6.10)
\[
\left( \frac{H^z}{\rho} \right)_t = 0. \tag{6.11}
\]

For symmetry properties we discuss only the reduced system (6.8). In the general case of the polytropic constant \( \gamma > 1 \) the system admits the following Lie algebra

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial s}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u},
\]

\[
X_5 = h_1 \left( H^z \frac{\partial}{\partial H^y} - H^y \frac{\partial}{\partial H^z} \right), \quad X_6 = -s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} - 2\rho \frac{\partial}{\partial \rho},
\]

\[
X_7 = t \frac{\partial}{\partial t} + 2s \frac{\partial}{\partial s} - u \frac{\partial}{\partial u} + 2\rho \frac{\partial}{\partial \rho}, \quad X_8 = 2s \frac{\partial}{\partial s} + 2\rho \frac{\partial}{\partial \rho} + 2p \frac{\partial}{\partial p} + H^y \frac{\partial}{\partial H^z} + H^z \frac{\partial}{\partial H^y},
\]  
\tag{6.12}

where

\[
h_1 = h_1 \left( s, \frac{p}{\rho\gamma}, \frac{H^y}{\rho}, \frac{H^z}{\rho} \right)
\]

is an arbitrary function.

For \( \gamma = 2 \) there is the additional generator

\[
X_9 = \rho h_2 \left( \frac{\partial}{\partial H^y} - H^y \frac{\partial}{\partial p} \right), \quad h_2 = h_2 \left( s, \frac{p}{\rho\gamma}, \frac{H^y}{\rho}, \frac{H^z}{\rho} \right). \tag{6.13}
\]

7 Variational approach to conservation laws in the case of infinite conductivity

To the best of our knowledge there is no Lagrangian formulation of the MHD equations in the case of the finite conductivity. However, for the infinite conductivity it is possible to bring the plane one-dimensional MHD flows equations to a variational form.

In Section 6 there was noticed a crucial difference between the cases of the finite and infinite conductivities: the conservation of the entropy along the trajectories (6.6) holds only for the infinite conductivity. This can be easily seen from equation (3.7) rewritten in the Lagrangian coordinates:

\[
S_t = \gamma - 1 \rho \gamma \left( p E \right), \quad i = \sigma E = \text{rot } H.
\]
7.1 Case $H^0 \neq 0$

7.1.1 Variational formulation

We start with the case $H^0 \neq 0$. The system (6.1) with modified equations (6.2), (6.3) and (6.6) takes the form

\begin{align}
\frac{1}{\rho}_t &= u_s, \quad (7.1a) \\
u_t &= -\left(p + \frac{(H^y)^2 + (H^z)^2}{2}\right)_s, \quad x_t = u, \quad (7.1b) \\
v_t &= (H^0 y)_s, \quad y_t = v, \quad (7.1c) \\
w_t &= (H^0 z)_s, \quad z_t = w, \quad (7.1d) \\
S_t &= 0, \quad (7.1e) \\
\frac{(H^y)}{\rho}_t &= (H^0 v)_s, \quad (7.1f) \\
\frac{(H^z)}{\rho}_t &= (H^0 w)_s. \quad (7.1g)
\end{align}

Using functions $\varphi(t, s)$, $\psi(t, s)$ and $\chi(t, s)$, which relate the Eulerian and Lagrangian coordinates (3.14), we get

\begin{align}
u &= \varphi_t, \quad \rho = \frac{1}{\varphi_s}, \quad (7.2) \\
v &= \psi_t, \quad H^y = H^0 \frac{\psi_s}{\varphi_s}, \quad (7.3) \\
w &= \chi_t, \quad H^z = H^0 \frac{\chi_s}{\varphi_s}. \quad (7.4)
\end{align}

This presentation of the physical variables makes the equations (7.1a), (7.1f) and (7.1g) satisfied. We also solve the equation (7.1e) as

\begin{equation}
S(s), \quad (7.5)
\end{equation}

where $S(s)$ is an arbitrary function.

The remaining three equations (7.1b), (7.1c) and (7.1d) can be presented as second-order PDEs

\begin{align}
\varphi_{tt} + \frac{S}{\varphi_s} + (H^0)^2 \frac{\psi_s^2}{2\varphi_s^2} + \frac{\chi_s^2}{2\varphi_s^2} &= 0, \quad (7.6a) \\
\psi_{tt} - (H^0)^2 \left(\frac{\psi_s}{\varphi_s}\right)_s &= 0, \quad (7.6b) \\
\chi_{tt} - (H^0)^2 \left(\frac{\chi_s}{\varphi_s}\right)_s &= 0, \quad (7.6c)
\end{align}
which have variational structure. They are Euler-Lagrange equations (2.2) for the Lagrangian
\[ L = \frac{1}{2} (\varphi_t^2 + \psi_t^2 + \chi_t^2) - \frac{S}{\gamma - 1} \varphi_s^{\gamma-1} - (H_0^2 + (H_z^2))^2 \gamma. \] (7.7)

**Remark 7.1** The Lagrangian function (7.7) has a clear physical interpretation. It equals to the kinetic energy minus the potential energy
\[ L = \frac{1}{2} (u^2 + v^2 + w^2) - \frac{S}{\gamma - 1} \rho^{-1} - \frac{(H_y^2 + (H_z^2))^2}{2\rho}. \]
The potential energy consists of two terms: the internal energy of the gas and the magnetic field energy.

The equivalence transformations for the system (7.6) are given in (B.8), Appendix B. They can be used to scale the function \( S(s) \) and the constant \( H_0 \).

The symmetry generators of the system (7.6) have the form
\[ X = \sum_{k=1}^{12} k_i Y_i, \] (7.8)

where
\[
\begin{aligned}
Y_1 &= \frac{\partial}{\partial t}, \\
Y_2 &= \frac{\partial}{\partial s}, \\
Y_3 &= \frac{\partial}{\partial \varphi}, \\
Y_4 &= \frac{\partial}{\partial \psi}, \\
Y_5 &= \frac{\partial}{\partial \chi}, \\
Y_6 &= t \frac{\partial}{\partial \varphi}, \\
Y_7 &= t \frac{\partial}{\partial \psi}, \\
Y_8 &= t \frac{\partial}{\partial \chi}, \\
Y_9 &= \chi \frac{\partial}{\partial \varphi} - \phi \frac{\partial}{\partial \psi}, \\
Y_{10} &= t \frac{\partial}{\partial t}, \\
Y_{11} &= s \frac{\partial}{\partial s}, \\
Y_{12} &= \varphi \frac{\partial}{\partial \varphi} + \psi \frac{\partial}{\partial \psi} + \chi \frac{\partial}{\partial \chi}.
\end{aligned}
\] (7.9)

Applying the prolonged generator (7.8) to the equations (7.6), we obtain the conditions on the coefficients \( k_i \)
\[
\begin{aligned}
(k_{11} s + k_2) S_s &= \gamma (k_{12} - k_{11}) S, \quad \text{(7.10a)} \\
k_{12} + k_{11} - 2k_{10} &= 0. \quad \text{(7.10b)}
\end{aligned}
\]

The condition (7.10a) is the classifying equation for \( S(s) \). It can be rewritten as
\[ (\alpha_1 s + \alpha_0) S_s = \beta S, \] (7.11)
where \( \alpha_0, \alpha_1 \) and \( \beta \) are constant. The same classifying equation was obtained for gas dynamics equations \([34, 35]\). It was shown that the equation specifies four cases of the entropy function: the general case and three special cases. They are

- arbitrary \( S(s) \);
- \( S(s) = S_0, \ S_0 = \text{const} \);
- \( S(s) = S_0 s^q, \ q \neq 0, \ S_0 = \text{const} \);
- \( S = S_0 e^{q s}, \ q \neq 0, \ S_0 = \text{const} \).
7.1.2 Symmetries

Using the equations (7.10), one obtains the following symmetries for arbitrary \( S(s) \):

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \varphi}, \quad X_3 = \frac{\partial}{\partial \psi}, \quad X_4 = \frac{\partial}{\partial \chi},
\]

\[
X_5 = t \frac{\partial}{\partial \varphi}, \quad X_6 = t \frac{\partial}{\partial \psi}, \quad X_7 = t \frac{\partial}{\partial \chi}, \quad X_8 = \chi \frac{\partial}{\partial \psi} - \psi \frac{\partial}{\partial \chi}.
\] (7.12)

They form the kernel of the admitted Lie algebras. Note that these generators correspond to a subset of generators (4.8).

Symmetry extensions for particular cases of \( S(s) \) are given in Table 3.

| Case | \( S(s) \) | Symmetry Extension | Conditions |
|------|-------------|-------------------|------------|
| 1    | \( S_0 \)   | \( \frac{\partial}{\partial s}, \quad t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s} + \varphi \frac{\partial}{\partial \varphi} + \psi \frac{\partial}{\partial \psi} + \chi \frac{\partial}{\partial \chi} \) | \( q \neq 0 \) |
| 2    | \( S_0 s^q \) | \( (2\gamma + q) t \frac{\partial}{\partial t} + 2\gamma s \frac{\partial}{\partial s} + 2(\gamma + q) \left( \varphi \frac{\partial}{\partial \varphi} + \psi \frac{\partial}{\partial \psi} + \chi \frac{\partial}{\partial \chi} \right) \) | \( q \neq 0 \) |
| 3    | \( S_0 e^{qx} \) | \( q t \frac{\partial}{\partial t} + 2\gamma \frac{\partial}{\partial s} + 2q \left( \varphi \frac{\partial}{\partial \varphi} + \psi \frac{\partial}{\partial \psi} + \chi \frac{\partial}{\partial \chi} \right) \) | \( q \neq 0 \) |

Table 3: Additional symmetries for \( H^0 \neq 0 \). \( S_0 \) is a nonzero constant.

7.1.3 Conservation laws

a) Arbitrary \( S(s) \)

In the case of arbitrary \( S(s) \) the Lagrangian (7.7) admits all eight symmetries (7.12). The symmetries \( X_5, X_6 \) and \( X_7 \) are divergence symmetries with \((B_1, B_2) = (\varphi, 0), (B_1, B_2) = (\phi, 0)\) and \((B_1, B_2) = (\chi, 0)\), respectively. The other symmetries are variational.

Symmetries \( X_1 - X_7 \) provide conservation laws which also exist in the case of the finite conductivity: energy (5.14), momenta (5.6), (5.7), (5.8), motion of center of mass (5.9), (5.10), (5.11).

The symmetry \( X_8 \) gives conservation of the angular momentum

\[
D^L_t (\chi \psi_t - \psi \chi_t) - D_s \left( \frac{(H^0)^2}{\varphi_s} (\chi \psi_s - \psi \chi_s) \right) = 0.
\] (7.13)

In physical variables this conservation law takes the form

\[
D^L_t (zv - yw) + D_s (H^0 (yH^z - zH^y)) = 0.
\] (7.14)
Note that the conservation of the angular momentum also needs the equations (6.5) to hold.

There are also three other conservation laws, namely conservation of mass and magnetic fluxes (in the case of finite conductivity they are (5.5), (5.12) and (5.13), respectively), which were used to introduce potentials \( \varphi, \psi \) and \( \chi \). These three conservation laws as well as the conservation of entropy were used to bring the equations to a variational form. They can not be obtained from the Lagrangian. We conclude that for the arbitrary entropy \( S(s) \) we obtain all conservation laws as of the case of the finite conductivity as well as the conservation of the entropy (6.6) and the angular momentum (7.14).

b) Special case of \( S(s) \)

Table 4 presents additional symmetries of the Lagrangian which are admitted for particular cases of \( S(s) \). These symmetries are variational.

| Case | \( S(s) \) | Symmetry Extension | Conditions |
|------|-------------|-------------------|------------|
| 1    | \( S_0 \)   | \( \frac{\partial}{\partial s} \) |             |
| 2    | \( S_0 s^q \) | \( t \frac{\partial}{\partial t} + 3s \frac{\partial}{\partial s} - \varphi \frac{\partial}{\partial \varphi} - \psi \frac{\partial}{\partial \psi} - \chi \frac{\partial}{\partial \chi} \) | \( q = \frac{4}{3} \gamma \) |

Table 4: Additional variational symmetries for \( H^0 \neq 0 \). \( S_0 \) is a nonzero constant.

The additional symmetries provide the following conservation laws:

- Case \( S(s) = S_0 \)

  The additional symmetry
  \[
  \frac{\partial}{\partial s}
  \]
  leads to the conservation law
  \[
  D_t^L (\varphi_s \varphi_t + \psi_s \psi_t + \chi_s \chi_t) + D_s \left( -\frac{1}{2}(\varphi_t^2 + \psi_t^2 + \chi_t^2) + \frac{\gamma S}{\gamma - 1} \varphi_s^{1-\gamma} \right) = 0. \tag{7.15}
  \]

  In the physical variables it is given as
  \[
  D_t^L \left( \frac{u}{\rho} + \frac{v H_y + w H_z}{H_0 \rho} \right) + D_s \left( -\frac{1}{2}(u^2 + v^2 + w^2) + \frac{\gamma S}{\gamma - 1} \rho^{\gamma-1} \right) = 0. \tag{7.16}
  \]
- Case \( S(s) = S_0 s^q \).
For $q = -\frac{4}{3}\gamma$ there is an additional scaling symmetry
\[
t\frac{\partial}{\partial t} + 3s\frac{\partial}{\partial s} - \varphi\frac{\partial}{\partial \varphi} - \psi\frac{\partial}{\partial \psi} - \chi\frac{\partial}{\partial \chi},
\]
which provides the conservation law
\[
\begin{align*}
D_t \left\{ t \left( \frac{1}{2}(\varphi^2_t + \psi^2_t + \chi^2_t) + \frac{S}{\gamma - 1}\varphi_{s}^{1-\gamma} + \frac{(H^0)^2(\psi^2_s + \chi^2_s)}{2\varphi_s} \right) + 3s(\varphi_s\varphi_t + \psi_s\psi_t + \chi_s\chi_t) + \varphi\varphi_t + \psi\psi_t + \chi\chi_t \right\} \\
+ D_s \left\{ (t\varphi_t + \varphi) \left( S\varphi_s^{-\gamma} + \frac{(H^0)^2(\psi^2_s + \chi^2_s)}{2\varphi_s^2} \right) - (t\psi_t + \psi)\frac{(H^0)^2\psi_s}{\varphi_s} \right. \\
- (t\chi_t + \chi)\frac{(H^0)^2\chi_s}{\varphi_s} + 3s \left( \frac{1}{2}(\varphi^2_t + \psi^2_t + \chi^2_t) + \frac{\gamma S}{\gamma - 1}\varphi_{s}^{1-\gamma} \right) \right\} = 0 \quad (7.17)
\end{align*}
\]
It takes the form
\[
\begin{align*}
D_t \left\{ t \left( \frac{1}{2}(u^2 + v^2 + w^2) + \frac{S}{\gamma - 1}\rho^{\gamma - 1} + \frac{(H^y)^2 + (H^z)^2}{2\rho} \right) \\
+ 3s \left( \frac{u}{\rho} + \frac{vH^y + wH^z}{H^0\rho} \right) + xu + yv + zw \right\} \\
+ D_s \left\{ (tu + x) \left( S\rho^{-\gamma} + \frac{(H^y)^2 + (H^z)^2}{2} \right) - (tv + y)H^0H^y \\
- (tw + z)H^0H^z + 3s \left( -\frac{1}{2}(u^2 + v^2 + w^2) + \frac{\gamma S}{\gamma - 1}\rho^{\gamma - 1} \right) \right\} = 0 \quad (7.18)
\end{align*}
\]
in the physical variables. It should be noted that for conservation law (7.18) we also need the equations (3.17), (3.18) and (6.5), which define $x, y$ and $z$ as nonlocal variables.

### 7.2 Case $H^0 = 0$

#### 7.2.1 Variational formulation

For $H^0 = 0$ we consider the reduced system (6.8). It can be presented as
\[
\begin{align*}
\left( \frac{1}{\rho} \right)_t &= u_s, \\
u_t &= -\left( p + \frac{(H^y)^2 + (H^z)^2}{2} \right)_s, \quad x_t = u, \\
S_t &= 0, \\
\left( \frac{H^y}{\rho} \right)_t &= 0, \\
\left( \frac{H^z}{\rho} \right)_t &= 0,
\end{align*}
\]
where modified equations (6.6), (6.10) and (6.11) are taken into account.

**Remark 7.2** The system (6.8) contains the closed subsystem

\begin{align}
\rho_t &= -\rho^2 u_s, \quad (7.20a) \\
\rho u_t &= -p_s - H_s, \quad x_t = u, \quad (7.20b) \\
p_t &= -\gamma \rho pu_s, \quad (7.20c) \\
H_t &= -2 \rho Hu_s. \quad (7.20d)
\end{align}

where

\[ H = \sqrt{\frac{(H^y)^2 + (H^z)^2}{2}}. \]

It is also possible to develop a variational formulation for this subsystem.

The subsystem is not equivalent to the original system (6.8), but will be equivalent if we add one more equation, namely (6.8d) or (6.8e).

**Remark 7.3** It is easy to see that the equations (7.19) have conservation laws

\[ D_t^L \left\{ T^t \left( S, \frac{H^y}{\rho}, \frac{H^z}{\rho} \right) \right\} = 0. \]

If we consider the complete system, i.e. equations (7.19) and (4.13), then there are more conservation laws with \( T^s \equiv 0 \), namely

\[ D_t^L \left\{ T^t \left( S, \frac{H^y}{\rho}, \frac{H^z}{\rho}, v, w, y - tv, z - tw \right) \right\} = 0. \]

The system (7.19) can be reduced to one variational PDE. For the equation (7.19a) we introduce the potential \( \varphi(t, s) \equiv x \):

\[ \varphi_t = u, \quad \varphi_s = \frac{1}{\rho}. \]

It is used to present the velocity and the density as

\[ u = \varphi, \quad \rho = \frac{1}{\varphi_s}. \quad (7.21) \]

Equations (7.19d) and (7.19e) are solved as

\[ H^y = \rho F(s) = \frac{F(s)}{\varphi_s}, \quad H^z = \rho G(s) = \frac{G(s)}{\varphi_s}, \quad (7.22) \]

where \( F(s) \) and \( G(s) \) are arbitrary functions. Finally, equation (7.19c) gives

\[ S = S(s), \quad (7.23) \]

where \( S(s) \) is arbitrary.
Using (7.21), (7.22) and (7.23), it is possible to rewrite the remaining equations (7.19b) as the following second-order PDE

\[ \varphi_{tt} + \left( S \frac{\varphi_t}{\varphi_s} + A \right)_s = 0, \quad A(s) = \frac{F^2(s) + G^2(s)}{2} \neq 0. \]  

(7.24)

This PDE has variational structure (2.2). It is provided by the Lagrangian function

\[ L = \frac{1}{2} \varphi_t^2 - S \frac{1}{\gamma - 1} \varphi_s^{1-\gamma} - A \frac{\varphi_t}{\varphi_s}. \]

(7.25)

Note that for \( \gamma = 2 \) the last two terms in the PDE (7.24) and in the Lagrangian (7.25) merge: we obtain the PDE

\[ \varphi_{tt} + \left( B \frac{\varphi_t}{\varphi_s} \right)_s = 0, \quad B(s) = \frac{S(s)}{\gamma - 1} + A(s), \]

(7.26)

which is given by the Lagrangian

\[ L = \frac{1}{2} \varphi_t^2 - \frac{B}{\varphi_s}. \]

(7.27)

This particular case coincides with the gas dynamics (MHD equations (7.19) in the absence of the magnetic field). The complete analysis of this case is given in [34,35]. This case needs to be presented separately because of different symmetry and conservation properties.

**Remark 7.4** Searching for a Lagrangian of the form \( L(t,s,\varphi,\varphi_t,\varphi_s) \) for PDE (7.24), we obtain the general form of the Lagrangian

\[ \bar{L} = \alpha L + h, \quad \alpha \neq 0, \quad \alpha = \text{const}, \]

\[ h = D_t^L(C^t(t,s,\varphi)) + D_s(C^s(t,s,\varphi)), \]

where \( L \) is given by (7.25) and the functions \( C^t(t,s,\varphi) \) and \( C^s(t,s,\varphi) \) are arbitrary. Notice that

\[ \frac{\delta h}{\delta \varphi} = 0, \]

(7.28)

it follows that \( h \) does not contribute to the Euler-Lagrange equations.

The Noether identity (2.4) gives

\[ Xh + h(D_t^L(\xi^t) + D_s(\xi^s)) = D_t^L(N^t h) + D_s(N^s h), \]

where \( N^t \) and \( N^s \) are the Noether operators (2.3). It follows that the general Lagrangian \( \bar{L} \) provides the same conservation laws as \( L \). Therefore the term \( h \) (as well as the constant \( \alpha \)) can be discarded.

**Remark 7.5** In the physical variables the Lagrangian function (7.25) takes the form

\[ L = \frac{u^2}{2} - \frac{S}{\gamma - 1} \rho^{\gamma - 1} - \frac{(H^y)^2 + (H^z)^2}{2\rho}. \]

In the following text we will consider the cases \( \gamma \neq 2 \) and \( \gamma = 2 \) separately.
7.2.2 Equivalence transformations

The equivalence transformations for PDEs (7.24) and (7.26) are given in Appendix B. The transformations (B.11) for the PDE (7.24) can be used to scale the functions $S(s)$ and $A(s)$. The equivalence transformation of the PDE (7.26) are the same as for the gas dynamics equation, considered in [35]. They are given in (B.12) and can be used to scale the function $B(s)$.

7.2.3 Symmetries in the general case $\gamma \neq 2$ ($\gamma > 1$)

Symmetries of PDE (7.24) have the form

$$X = \sum_{k=1}^{7} k_i Y_i,$$  (7.29)

where

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial s}, \quad Y_3 = \frac{\partial}{\partial \varphi}, \quad Y_4 = t \frac{\partial}{\partial \varphi}, \quad Y_5 = t \frac{\partial}{\partial t}, \quad Y_6 = s \frac{\partial}{\partial s}, \quad Y_7 = \varphi \frac{\partial}{\partial \varphi}.$$  (7.30)

Applying generator (7.29) to the PDE, we get the following conditions for coefficients $k_i$:

$$(k_6s + k_2)S = ((\gamma + 1)k_7 + (1 - \gamma)k_6 - 2k_5)S;$$  (7.31a)

$$(k_6s + k_2)A = (3k_7 - k_6 - 2k_5)A.$$  (7.31b)

If considered independently, both conditions have the form (7.11) and lead to the same cases of $S(s)$ and $A(s)$ as discussed earlier. Both $S(s)$ and $A(s)$ can be arbitrary, constant, power or exponential functions. However, not all pairs lead to additional symmetries. We obtain the following pairs for the consideration of symmetry extensions:

- arbitrary $(S(s), A(s))$;
- constant $(S(s), A(s)) = (S_0, A_0)$, $S_0 = \text{const}, A_0 = \text{const}$;
- power $(S(s), A(s)) = (S_0s^\alpha, A_0s^\beta)$, $\alpha^2 + \beta^2 \neq 0$, $S_0 = \text{const}, A_0 = \text{const}$;
- exponential $(S(s), A(s)) = (S_0e^{ps}, A_0e^{qs})$, $p^2 + q^2 \neq 0$, $S_0 = \text{const}, A_0 = \text{const}$.

From equations (7.31) we obtain that the kernel of the admitted Lie algebras is defined by the generators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \varphi}, \quad X_3 = t \frac{\partial}{\partial \varphi}.$$  (7.32)
Particular cases of \( S(s) \) and \( A(s) \) which lead to extensions of the admitted symmetry algebra are presented in Table 5.

| Case | \( S(s) \) | \( A(s) \) | Symmetry Extension | Conditions |
|------|----------|----------|-------------------|------------|
| 1    | \( S_0 \) | \( A_0 \) | \( \frac{\partial}{\partial s}, \quad t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s} + \varphi \frac{\partial}{\partial \varphi} \) |            |
| 2    | \( S_0 s^\alpha \) | \( A_0 s^\beta \) | \( (2(\gamma - 2) + 3\alpha - \beta(\gamma + 1))t \frac{\partial}{\partial t} \) \( + 2(\gamma - 2)s \frac{\partial}{\partial s} + (\gamma - 2 + \alpha - \beta)\varphi \frac{\partial}{\partial \varphi} \) | \( \alpha^2 + \beta^2 \neq 0 \) |
| 3    | \( S_0 e^{ps} \) | \( A_0 e^{qs} \) | \( (-q(\gamma + 1) + 3p)t \frac{\partial}{\partial t} + 2(\gamma - 2)\frac{\partial}{\partial s} + 2(p - q)\varphi \frac{\partial}{\partial \varphi} \) | \( p^2 + q^2 \neq 0 \) |

Table 5: Additional symmetries for \( H^0 = 0, \gamma \neq 2 \). \( S_0 \) and \( A_0 \) are nonzero constants.

### 7.2.4 Conservation laws in the general case \( \gamma \neq 2 \) (\( \gamma > 1 \))

**a) Arbitrary** \( S(s) \) and \( A(s) \)

The symmetries (7.32) provide conservation laws of energy, momentum and motion of the center of mass. These conservation laws exist for the finite conductivity. They are (5.21), (5.17) and (5.18), respectively.

The conservation of mass, magnetic fluxes and entropy were used to rewrite the PDE in the variational form and therefore they can not be obtained from the Lagrangian. For the finite conductivity the conservation of mass and magnetic fluxes were given by (5.16), (5.19) and (5.20). The conservation law of the entropy does not hold for the finite conductivity.

We conclude that we obtain the same conservation laws for the reduced system (7.19) as for the corresponding system with the finite conductivity (4.12), and conservation law for the entropy.

**b) Special cases of** \( S(s) \) and \( A(s) \)

There are particular cases of

\[
S(s) \quad \text{and} \quad A(s) = \frac{(H^y)^2 + (H^z)^2}{2\rho^2}
\]

with additional variational symmetries. These cases are specified in Table 6.

32
Table 6: Additional variational symmetries for \( H_0 = 0, \gamma \neq 2 \).
\( S_0 \) and \( A_0 \) are nonzero constants.

The additional symmetries provide the following conservation laws:

- **Case \((S(s), A(s)) = (S_0, A_0)\)**

  There exist the additional conservation law
  \[
  D_L \left( \frac{\partial}{\partial t} \phi s \right) + D_s \left( -\frac{\phi^2}{2} + \frac{\alpha S}{\gamma - 1} \phi^{1-\gamma} + \frac{2A}{\phi^s} \right) = 0. \tag{7.33}
  \]
  In the physical variables it is rewritten as
  \[
  D_L \left( \frac{u}{\rho} \right) + D_s \left( -\frac{u^2}{2} + \frac{\alpha S}{\gamma - 1} \rho^{\gamma-1} + \frac{(H^y)^2 + (H^z)^2}{\rho} \right) = 0. \tag{7.34}
  \]

- **Case \((S(s), A(s)) = (S_0 s^\alpha, A_0 s^\beta)\)**

  For \( \alpha + \beta(\gamma - 3) = -4(\gamma - 2) \) there is the additional conservation law
  \[
  D_L \left\{ \left( 2\beta + 5 \right) t \left( \frac{1}{2} \phi_t^2 + \frac{S}{\gamma - 1} \phi_s^{1-\gamma} + \frac{A}{\phi_s^s} \right) - s \phi_s \phi_t - (\beta + 3) \phi \phi_t \right\} \\
  + D_s \left\{ \left( (2\beta + 5) t \phi_t - (\beta + 3) \phi \right) \left( S \phi_s^{-\gamma} + \frac{A}{\phi_s^s} \right) + s \left( \frac{\phi_t^2}{2} - \frac{\alpha S}{\gamma - 1} \phi_s^{1-\gamma} - \frac{2A}{\phi_s^s} \right) \right\} = 0. \tag{7.35}
  \]
  It is presented in the physical variables as
  \[
  D_L \left\{ \left( 2\beta + 5 \right) t \left( \frac{u^2}{2} + \frac{S}{\gamma - 1} \rho^{\gamma-1} + \frac{(H^y)^2 + (H^z)^2}{2\rho} \right) - \frac{s u}{\rho} - (\beta + 3)xu \right\} \\
  + D_s \left\{ \left( (2\beta + 5) tu - (\beta + 3)x \right) \left( S \rho^{\gamma} + \frac{(H^y)^2 + (H^z)^2}{2} \right) \\
  + s \left( \frac{u^2}{2} - \frac{\gamma S}{\gamma - 1} \rho^{\gamma-1} - \frac{(H^y)^2 + (H^z)^2}{\rho} \right) \right\} = 0. \tag{7.36}
  \]
• Case \((S(s), A(s)) = (S_0 e^{p_s}, A_0 e^{q_s})\)

If \(p + q(\gamma - 3) = 0\), there is the conservation law

\[
D_t^L \left\{ 2qt \left( \frac{1}{2} \varphi_s^2 + \frac{S(s)}{\gamma - 1} \varphi_s^{1-\gamma} + \frac{A}{\varphi_s} \right) - \varphi_s \varphi_t - q \varphi \varphi_t \right\} \\
+ D_s \left\{ q(2t \varphi_t - \varphi) \left( S \varphi_s^{-\gamma} + \frac{A}{\varphi_s^2} \right) \frac{\varphi_s^2}{2} - \frac{\gamma S}{\gamma - 1} \varphi_s^{1-\gamma} - \frac{2A}{\varphi_s} \right\} = 0. \tag{7.37}
\]

In the physical variables it takes the form

\[
D_t^L \left\{ 2qt \left( \frac{u^2}{2} + \frac{S}{\gamma - 1} \rho^{\gamma-1} + \frac{(H^y)^2 + (H^z)^2}{2} \frac{\rho}{\rho} \right) - \frac{u}{\rho} - q \dot{x} \right\} \\
+ D_s \left\{ q(2tu - x) \left( S \rho^\gamma + \frac{(H^y)^2 + (H^z)^2}{2} \frac{\rho}{\rho} \right) - \frac{\gamma S}{\gamma - 1} \rho^{\gamma-1} - \frac{(H^y)^2 + (H^z)^2}{\rho} \right\} = 0. \tag{7.38}
\]

We remark that conservation laws (7.36) and (7.38) in addition to the reduced MHD system (6.8) (equivalently (7.19)) need equations (3.17), which define \(x\) as a nonlocal variable.

### 7.2.5 Symmetries in the special case \(\gamma = 2\)

For \(\gamma = 2\) we consider the variational PDE (7.26) and the corresponding Lagrangian (7.27). This case is equivalent to that of gas dynamic, i.e. MHD equation without the magnetic field. The gas dynamic equations were analyzed in [35]. Therefore we can rely on the results obtained there.

The kernel of the Lie algebras admitted by equation (7.26) consists of the generators

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \varphi}, \quad X_3 = t \frac{\partial}{\partial t} + \frac{\varphi}{\partial \varphi}, \quad X_4 = 3t \frac{\partial}{\partial t} + 2 \varphi \frac{\partial}{\partial \varphi}. \tag{7.39}
\]

The first three symmetries are the same as for \(\gamma \neq 2\). For particular cases \(B(s)\) there are additional symmetries, given in Table 7.

| Case | \(B(s)\) | Symmetry Extension | Conditions |
|------|------------|-------------------|------------|
| 1    | \(B_0\)   | \(\frac{\partial}{\partial s}, \quad t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s} + \varphi \frac{\partial}{\partial \varphi}\) |          |
| 2    | \(B_0 s^\beta\) | \((\beta + 1)t \frac{\partial}{\partial t} - 2s \frac{\partial}{\partial s}\) | \(\beta \neq 0\) |
| 3    | \(B_0 e^{q s}\) | \(q t \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial s}\) | \(q \neq 0\) |
Table 7: Additional symmetries for $H^0 = 0, \gamma = 2$. $B_0$ is a nonzero constant.

7.2.6 Conservation laws in the special case $\gamma = 2$

a) Arbitrary $B(s)$

In the general case of $B(s)$ we obtain the same variational symmetries as in the case $\gamma \neq 2$ with arbitrary $S(s)$ and $A(s)$, namely (7.32). The corresponding conservation laws were discussed in point 7.2.4.

b) Special cases of $B(s)$

For particular cases of $B(s) = S(s)$

\[ B(s) = \frac{S(s)}{\gamma - 1} + \frac{(H_y)^2 + (H_z)^2}{2\rho^2} \]

the Lagrangian (7.27) has additional variational symmetries given in Table 8.

| Case | $B(s)$ | Symmetry Extension | Conditions |
|------|--------|--------------------|------------|
| 1    | $B_0$  | $\partial_s$, $5t \frac{\partial}{\partial t} - s \frac{\partial}{\partial s} + 3\varphi \frac{\partial}{\partial \varphi}$ |            |
| 2    | $B_0 s^\beta$ | $(2\beta + 5)t \frac{\partial}{\partial t} - s \frac{\partial}{\partial s} + (\beta + 3)\varphi \frac{\partial}{\partial \varphi}$ | $\beta \neq 0$ |
| 3    | $B_0 e^q s$ | $2qt \frac{\partial}{\partial t} - \frac{\partial}{\partial s} + q\varphi \frac{\partial}{\partial \varphi}$ | $q \neq 0$ |

Table 8: Additional variational symmetries for $H^0 = 0, \gamma = 2$. $B_0$ is a nonzero constant.

The comparison of these symmetries with those for $\gamma \neq 2$ shows that for $\gamma = 2$ we obtain the symmetries of the generic case $\gamma \neq 2$ and one extension. This extension is the scaling symmetry admitted for $B(s) = B_0$. It provides the conservation law

\[ D_t \left\{ 5t \left( 2\varphi_t^2 + \frac{B}{\varphi_s} \right) - s \varphi_s \varphi_t - 3\varphi \varphi_t \right\} + D_s \left\{ (5t\varphi_t - 3\varphi) \frac{B}{\varphi_s^2} + s \left( \frac{\varphi_t^2}{\varphi_s} - \frac{2B}{\varphi_s} \right) \right\} = 0. \]

(7.40)

In the physical variables it takes the form

\[ D_t \left\{ 5t \left( \frac{u^2}{2} + \frac{S}{\gamma - 1} \rho^{\gamma - 1} + \frac{(H_y)^2 + (H_z)^2}{2\rho} \right) - s \frac{u}{\rho} - 3xu \right\} \\
+ D_s \left\{ (5tu - 3x) \left( S\rho^\gamma + \frac{(H_y)^2 + (H_z)^2}{2} \right) + s \left( \frac{u^2}{2} - \frac{\gamma S}{\gamma - 1} \rho^{\gamma - 1} - \frac{(H_y)^2 + (H_z)^2}{\rho} \right) \right\} = 0. \]

(7.41)
Note that for verification of this conservation law, presented in the physical variables, we also need equations (3.17).

8 Concluding remarks

The paper is devoted to Lie point symmetries and conservation laws of the plane onedimensional MHD flows, described in the mass Lagrangian coordinates by equations (3.16).

The analysis leads to four cases for the electric conductivity $\sigma(\rho,p)$ and $H^x = H^0 = \text{const}$:

1. Finite electric conductivity and $H^0 \neq 0$;
2. Finite electric conductivity and $H^0 = 0$;
3. Infinite electric conductivity and $H^0 \neq 0$;
4. Infinite electric conductivity and $H^0 = 0$.

The latter case splits for the value of the polytropic constant. We get the generic subcase $\gamma \neq 2$ and the special subcase $\gamma = 2$. For $\gamma = 2$ the equations are equivalent to the equations describing the plane one-dimensional flows of the gas dynamics, i.e. the equations without the magnetic field.

For all cases given above we found the admitted Lie point symmetries. For the cases with the finite conductivity it results in the Lie group classifications for $\sigma(\rho,p)$. For the infinite electric conductivity the classifications have the entropy $S$ as an arbitrary element.

The conservation laws for the finite electric conductivity are found by direct computation. For the cases with the infinite conductivity the equations can be brought into the variational forms. Further, the conservation laws were found using the Noether theorem, which was applied to the variational equations. Finally, the conservation laws were converted into the original physical variables.

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**Appendices**

**A. Lagrangian variables**

Let \((\xi, \eta, \zeta)\) be Lagrangian spatial variables. The Eulerian coordinates \((x, y, z)\) and Lagrangian coordinates are related by the equations

\[
x = \tilde{\varphi}(t, \xi, \eta, \zeta), \tag{A.1a}
\]

\[
y = \tilde{\psi}(t, \xi, \eta, \zeta), \tag{A.1b}
\]

\[
z = \tilde{\chi}(t, \xi, \eta, \zeta). \tag{A.1c}
\]
where the functions $\tilde{\phi}, \tilde{\psi}$ and $\tilde{\chi}$ are smooth functions, satisfying the Cauchy problem

$$\tilde{\phi}_t = u_t(t, \tilde{\phi}, \tilde{\psi}, \tilde{\chi}), \quad \tilde{\phi}(0, \xi, \eta, \zeta) = \xi, \quad (A.2a)$$

$$\tilde{\psi}_t = v_t(t, \tilde{\phi}, \tilde{\psi}, \tilde{\chi}), \quad \tilde{\psi}(0, \xi, \eta, \zeta) = \eta, \quad (A.2b)$$

$$\tilde{\chi}_t = w_t(t, \tilde{\phi}, \tilde{\psi}, \tilde{\chi}), \quad \tilde{\chi}(0, \xi, \eta, \zeta) = \zeta. \quad (A.2c)$$

Notice also that due to Euler’s theorem

$$\frac{\partial J}{\partial t} = J \text{div}_e u,$$

where $\text{div}_e u$ is the divergence of the velocity $u$ in the Eulerian coordinates with substituted (A.1), the general solution of the conservation of mass equation has the form

$$\rho(t, \tilde{\phi}(t, \xi, \eta, \zeta), \tilde{\psi}(t, \xi, \eta, \zeta), \tilde{\chi}(t, \xi, \eta, \zeta)) = \rho_0(\xi, \eta, \zeta) \frac{\rho_0(\xi, \eta, \zeta)}{J(t, \xi, \eta, \zeta)}, \quad (A.3)$$

where $\rho_0$ is an arbitrary function and

$$J(t, \xi, \eta, \zeta) = \frac{\partial(\tilde{\phi}, \tilde{\psi}, \tilde{\chi})}{\partial(\xi, \eta, \zeta)}$$

is the Jacobian.

One can show that for the plane one-dimensional flow

$$u = u(t, x), \quad v = v(t, x), \quad w = w(t, x), \quad (A.4)$$

where the functions $u, v$ and $w$ are continuously differentiable functions, it is necessary and sufficient that the functions $\tilde{\phi}, \tilde{\psi}$ and $\tilde{\chi}$ have the form

$$\tilde{\phi}(t, \xi, \eta, \zeta) = \hat{\phi}(t, \xi), \quad (A.5a)$$

$$\tilde{\psi}(t, \xi, \eta, \zeta) = \eta + \hat{\psi}(t, \xi), \quad (A.5b)$$

$$\tilde{\chi}(t, \xi, \eta, \zeta) = \zeta + \hat{\chi}(t, \xi). \quad (A.5c)$$

Indeed, assume that $u, v$ and $w$ satisfy (A.2). Consider the function $\tilde{\phi}$. Differentiating (A.2a) with respect to $\eta$, we obtain that the function $g(t, \xi, \eta, \zeta) = \tilde{\phi}_\eta(t, \xi, \eta, \zeta)$ satisfies the Cauchy problem

$$g_t = u_\eta g, \quad g(0, \xi, \eta, \zeta) = 0.$$

As $g(t, \xi, \eta, \zeta) \equiv 0$ is a solution of the latter Cauchy problem, and due to uniqueness of the solution of this problem, we conclude that $\tilde{\phi}_\eta = 0$. Similarly, we obtain $\tilde{\psi}_\eta = 0$. Therefore, $\tilde{\phi}(t, \xi, \eta, \zeta) = \hat{\phi}(t, \xi)$.

Consider the function $\tilde{\psi}(t, \xi, \eta, \zeta)$. Noting that the function $h = \tilde{\psi}_\eta$ satisfies the Cauchy problem

$$h_t = 0, \quad h(0, \xi, \eta, \zeta) = 1,$$
one obtains that $\tilde{\psi}_\eta = 1$. Similarly, we derive that $\tilde{\psi}_\zeta = 0$, $\tilde{\chi}_\eta = 0$ and $\tilde{\chi}_\zeta = 1$. This gives that

$$\tilde{\psi}(t,\xi,\eta,\zeta) = \eta + \hat{\psi}(t,\xi), \quad \tilde{\chi}(t,\xi,\eta,\zeta) = \zeta + \hat{\chi}(t,\xi).$$

Converse, assume that the relations between the Lagrangian and Eulerian coordinates have the form (A.5). Differentiating (A.5) with respect to $t$, and noting that due to the inverse function theorem the equation $x - \hat{\phi}(t,\xi) = 0$ can be solved with respect to $\xi$, one obtains (A.2).

Considering (A.3) at $t = 0$, we obtain that $\rho_0 = \rho_0(\xi)$. Hence, (A.3) becomes

$$\rho(t,\hat{\phi}(t,\xi)) = \rho_0(\xi).$$

For the mass Lagrangian coordinates one applies the change $s = \alpha(\xi)$, where $\alpha'(\xi) = \rho_0(\xi)$. Hence, the function $\varphi(t,s)$ such that $\varphi(t,\alpha(\xi)) = \hat{\phi}(t,\xi)$ satisfies the conditions

$$\varphi_t(t,s) = u(t,\varphi(t,s)), \quad \varphi_s(t,s) = \frac{1}{\rho(t,\varphi(t,s))}.$$ 

Using similar relations

$$\psi(t,\alpha(\xi)) = \hat{\psi}(t,\xi), \quad \chi(t,\alpha(\xi)) = \hat{\chi}(t,\xi),$$

one obtains

$$\psi_t(t,s) = v(t,\varphi(s,t)), \quad \chi_t(t,s) = w(t,\varphi(s,t)).$$

### B Equivalence transformations

Here we provide the equivalence transformations for the different MHD systems considered in the paper. Equivalence transformations allow to change arbitrary elements while preserving the structure of the equations. The algorithm for finding equivalence transformations is given in [1].

#### B.1 The case of finite conductivity $\sigma(\rho,p)$ and $H^0 \neq 0$

The generators of the equivalence transformations for system (3.16) have the form

$$X^e = \zeta^t \frac{\partial}{\partial t} + \zeta^s \frac{\partial}{\partial s} + \zeta^x \frac{\partial}{\partial x} + \zeta^y \frac{\partial}{\partial y} + \zeta^z \frac{\partial}{\partial z} + \zeta^u \frac{\partial}{\partial u} + \zeta^v \frac{\partial}{\partial v} + \zeta^w \frac{\partial}{\partial w} + \zeta^\rho \frac{\partial}{\partial \rho} + \zeta^p \frac{\partial}{\partial p} + \zeta^{E^y} \frac{\partial}{\partial E^y} + \zeta^{E^z} \frac{\partial}{\partial E^z} + \zeta^{H^y} \frac{\partial}{\partial H^y} + \zeta^{H^z} \frac{\partial}{\partial H^z} + \zeta^\sigma \frac{\partial}{\partial \sigma} + \zeta^{H^0} \frac{\partial}{\partial H^0},$$

(B.1)
where $\zeta^t, \zeta^s, \ldots, \zeta^\sigma, \zeta^{H^0}$ are functions of $t, s, x, u, \rho, p, E^y, E^z, H^y, H^z, \sigma$ and $H^0$. Computation provides the generators

$$X_1^e = \frac{\partial}{\partial t}, \quad X_2^e = \frac{\partial}{\partial s}, \quad X_3^e = \frac{\partial}{\partial x}, \quad X_4^e = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

$$X_5^e = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} + w \frac{\partial}{\partial v} - v \frac{\partial}{\partial w} + E^z \frac{\partial}{\partial E^y} - E^y \frac{\partial}{\partial E^z} + H^z \frac{\partial}{\partial H^y} - H^y \frac{\partial}{\partial H^z},$$

$$X_6^e = t \frac{\partial}{\partial t} + 2s \frac{\partial}{\partial s} - u \frac{\partial}{\partial u} - w \frac{\partial}{\partial w} + 2\rho \frac{\partial}{\partial \rho} - E^y \frac{\partial}{\partial E^y} - E^z \frac{\partial}{\partial E^z} + \sigma \frac{\partial}{\partial \sigma},$$

$$X_7^e = -s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w} - 2\rho \frac{\partial}{\partial \rho} + E^y \frac{\partial}{\partial E^y} + E^z \frac{\partial}{\partial E^z} - 2\sigma \frac{\partial}{\partial \sigma},$$

$$X_8^e = 2s \frac{\partial}{\partial s} + 2\rho \frac{\partial}{\partial \rho} + 2\rho \frac{\partial}{\partial \rho} + E^y \frac{\partial}{\partial E^y} + E^z \frac{\partial}{\partial E^z} + H^y \frac{\partial}{\partial H^y} + H^z \frac{\partial}{\partial H^z} + H^0 \frac{\partial}{\partial H^0},$$

$$X_9^e = \phi^1(s) \frac{\partial}{\partial y}, \quad X_{10}^e = \phi^2(s) \frac{\partial}{\partial z}, \quad X_{11}^e = t \frac{\partial}{\partial y} + \frac{\partial}{\partial v}, \quad X_{12}^e = t \frac{\partial}{\partial z} + \frac{\partial}{\partial w},$$

(B.2)

where $\phi^1(s)$ and $\phi^2(s)$ are arbitrary functions.

### B.2 The case of finite conductivity $\sigma(\rho, p)$ and $H^0 = 0$

The equivalence transformations of the reduced system (4.12) are provided by the generators of the form

$$X^e = \zeta^t \frac{\partial}{\partial t} + \zeta^s \frac{\partial}{\partial s} + \zeta^x \frac{\partial}{\partial x} + \zeta^u \frac{\partial}{\partial u} + \zeta^\rho \frac{\partial}{\partial \rho} + \zeta^\sigma \frac{\partial}{\partial \sigma}$$

$$+ \zeta^{E^y} \frac{\partial}{\partial E^y} + \zeta^{E^z} \frac{\partial}{\partial E^z} + \zeta^{H^y} \frac{\partial}{\partial H^y} + \zeta^{H^z} \frac{\partial}{\partial H^z} + \zeta^{\sigma} \frac{\partial}{\partial \sigma},$$

(B.3)

where $\zeta^t, \zeta^s, \ldots, \zeta^\sigma$ are functions of $t, s, x, u, \rho, p, E^y, E^z, H^y, H^z$ and $\sigma$. We obtain

$$X_1^e = \frac{\partial}{\partial t}, \quad X_2^e = \frac{\partial}{\partial s}, \quad X_3^e = \frac{\partial}{\partial x}, \quad X_4^e = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

$$X_5^e = E^z \frac{\partial}{\partial E^y} - E^y \frac{\partial}{\partial E^z} + H^z \frac{\partial}{\partial H^y} - H^y \frac{\partial}{\partial H^z},$$

$$X_6^e = t \frac{\partial}{\partial t} + 2s \frac{\partial}{\partial s} - u \frac{\partial}{\partial u} + 2\rho \frac{\partial}{\partial \rho} - E^y \frac{\partial}{\partial E^y} - E^z \frac{\partial}{\partial E^z} + \sigma \frac{\partial}{\partial \sigma},$$

$$X_7^e = -s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w} - 2\rho \frac{\partial}{\partial \rho} + E^y \frac{\partial}{\partial E^y} + E^z \frac{\partial}{\partial E^z} - 2\sigma \frac{\partial}{\partial \sigma},$$

$$X_8^e = 2s \frac{\partial}{\partial s} + 2\rho \frac{\partial}{\partial \rho} + 2\rho \frac{\partial}{\partial \rho} + E^y \frac{\partial}{\partial E^y} + E^z \frac{\partial}{\partial E^z} + H^y \frac{\partial}{\partial H^y} + H^z \frac{\partial}{\partial H^z} + H^0 \frac{\partial}{\partial H^0},$$

(B.4)
B.3 The case of infinite conductivity and $H^0 \neq 0$

The equivalence transformations for system (6.1) have the generators

$$X^e = \zeta_t \frac{\partial}{\partial t} + \zeta_s \frac{\partial}{\partial s} + \zeta_x \frac{\partial}{\partial x} + \zeta_y \frac{\partial}{\partial y} + \zeta_z \frac{\partial}{\partial z} + \zeta_u \frac{\partial}{\partial u} + \zeta_v \frac{\partial}{\partial v} + \zeta_w \frac{\partial}{\partial w} + \zeta_{\rho} \frac{\partial}{\partial \rho} + \zeta_p \frac{\partial}{\partial p} + \zeta_{E_y} \frac{\partial}{\partial E_y} + \zeta_{E_z} \frac{\partial}{\partial E_z} + \zeta_{H_y} \frac{\partial}{\partial H_y} + \zeta_{H_z} \frac{\partial}{\partial H_z} + \zeta_{H^0} \frac{\partial}{\partial H^0},$$

(B.5)

where the coefficients $\zeta_t, \zeta_s, \ldots, \zeta_{H^0}$ are functions of $t, s, x, u, \rho, p, E_y, E_z, H_y, H_z$ and $H^0$. Computations lead to the following generators

$$X^e_1 = \frac{\partial}{\partial t}, \quad X^e_2 = \frac{\partial}{\partial s}, \quad X^e_3 = \frac{\partial}{\partial x}, \quad X^e_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X^e_5 = \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} + w \frac{\partial}{\partial v} - v \frac{\partial}{\partial w} + H^z \frac{\partial}{\partial H^y} - H^y \frac{\partial}{\partial H^z},$$

$$X^e_6 = t \frac{\partial}{\partial \rho} + 2s \frac{\partial}{\partial \rho} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} - 2\rho \frac{\partial}{\partial \rho},$$

$$X^e_7 = -s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w} - 2\rho \frac{\partial}{\partial \rho},$$

$$X^e_8 = 2s \frac{\partial}{\partial s} + 2p \frac{\partial}{\partial p} + 2\rho \frac{\partial}{\partial \rho} + H^y \frac{\partial}{\partial H^y} + H^z \frac{\partial}{\partial H^z} + H^0 \frac{\partial}{\partial H^0},$$

$$X^e_9 = f_1 \left( \frac{s}{\rho^\gamma} \right) \frac{\partial}{\partial y}, \quad X^e_{10} = f_2 \left( \frac{s}{\rho^\gamma} \right) \frac{\partial}{\partial z}, \quad X^e_{11} = t \frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}, \quad X^e_{12} = t \frac{\partial}{\partial z} + \frac{\partial}{\partial \psi},$$

(B.6)

where $f_1$ and $f_2$ are arbitrary functions of their arguments.

B.4 Variational approach for infinite conductivity and $H^0 \neq 0$

The general form of the equivalence transformation generators for system (7.6) is

$$X^e = \zeta_t \frac{\partial}{\partial t} + \zeta_s \frac{\partial}{\partial s} + \zeta_{\varphi} \frac{\partial}{\partial \varphi} + \zeta_{\psi} \frac{\partial}{\partial \psi} + \zeta_{\chi} \frac{\partial}{\partial \chi} + \zeta_{H^0} \frac{\partial}{\partial H^0} + \zeta_{S} \frac{\partial}{\partial S},$$

(B.7)

where the coefficients depend on $(t, s, \varphi, \psi, \chi, H^0, S)$. We obtain the following generators

$$X^e_1 = \frac{\partial}{\partial t}, \quad X^e_2 = \frac{\partial}{\partial s}, \quad X^e_3 = \frac{\partial}{\partial \varphi}, \quad X^e_4 = \frac{\partial}{\partial \psi}, \quad X^e_5 = \frac{\partial}{\partial \chi}, \quad X^e_6 = t \frac{\partial}{\partial \varphi}, \quad X^e_7 = t \frac{\partial}{\partial \psi}, \quad X^e_8 = t \frac{\partial}{\partial \chi}, \quad X^e_9 = \chi \frac{\partial}{\partial \psi} - \phi \frac{\partial}{\partial \chi},$$

$$X^e_{10} = \frac{\partial}{\partial t} + s \frac{\partial}{\partial s} + \varphi \frac{\partial}{\partial \varphi} + \psi \frac{\partial}{\partial \psi} + \chi \frac{\partial}{\partial \chi}, \quad X^e_{11} = t \frac{\partial}{\partial \varphi} + 2s \frac{\partial}{\partial s} - 2\gamma S \frac{\partial}{\partial S}, \quad X^e_{12} = (1 - \gamma) t \frac{\partial}{\partial \psi} + 2s \frac{\partial}{\partial s} + \gamma H^0 \frac{\partial}{\partial H^0}.$$

(B.8)
B.5 Variational approach for infinite conductivity and $H^0 = 0$

PDEs (7.24) and (7.26), which correspond to cases $\gamma \neq 2$ and $\gamma = 2$, have arbitrary functions. The equivalence transformations for these PDEs have generators of the forms

$$X^e = \xi^t \frac{\partial}{\partial t} + \xi^s \frac{\partial}{\partial s} + \eta^\varphi \frac{\partial}{\partial \varphi} + \eta^S \frac{\partial}{\partial S} + \eta^A \frac{\partial}{\partial A}$$

and

$$X^e = \xi^t \frac{\partial}{\partial t} + \xi^s \frac{\partial}{\partial s} + \eta^\varphi \frac{\partial}{\partial \varphi} + \eta^B \frac{\partial}{\partial B}.$$ (B.10)

The coefficients of these generators depend on $(t,s,\varphi,S,A)$ and $(t,s,\varphi,B)$, respectively.

Computations provide the generators

$$X_1^e = \frac{\partial}{\partial t}, \quad X_2^e = \frac{\partial}{\partial s}, \quad X_3^e = \frac{\partial}{\partial \varphi}, \quad X_4^e = t \frac{\partial}{\partial \varphi}, \quad X_5^e = \frac{t}{\partial t} + s \frac{\partial}{\partial s} + \varphi \frac{\partial}{\partial \varphi},$$

$$X_6^e = t \frac{\partial}{\partial t} - 2s \frac{\partial}{\partial s} + 2(\gamma - 2)S \frac{\partial}{\partial S}, \quad X_7^e = (1 - \gamma)t \frac{\partial}{\partial t} + 2s \frac{\partial}{\partial s} + 2(\gamma - 2)A \frac{\partial}{\partial A}$$

for equation (7.24). For PDE (7.26), we obtain the generators

$$X_1^e = \frac{\partial}{\partial t}, \quad X_2^e = \frac{\partial}{\partial s}, \quad X_3^e = \frac{\partial}{\partial \varphi}, \quad X_4^e = t \frac{\partial}{\partial \varphi},$$

$$X_5^e = (1 - \gamma)t \frac{\partial}{\partial t} + 2s \frac{\partial}{\partial s}, \quad X_6^e = t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s} + \varphi \frac{\partial}{\partial \varphi}, \quad X_7^e = \frac{t}{\partial t} - 2B \frac{\partial}{\partial B}. \quad (B.11)$$

C Lie algebra extensions for finite conductivity $\sigma(\rho, p)$ and $H^0 = 0$

Here we find extensions of the kernel of the admitted Lie algebras (4.17), which belong to the extended Lie algebra (4.16), by the other generators of the extended Lie algebra, namely by the generators from the set $\{Y_6, Y_7, Y_8\}$.

For this purpose we show that the action of the equivalence transformations defined by the generators (4.16) and the action of the inner automorphisms of the extended Lie algebra (4.16) coincide. Since the kernel is an ideal of the extended algebra it can be extended by subalgebras formed by the remaining generators. Therefore we take the optimal system of subalgebras for the subalgebra of the remaining operators $\{Y_6, Y_7, Y_8\}$. It provides possible extensions of the kernel.

C.1 Action of the equivalence transformations

Consider the change of the coefficients of the generators (4.16) under the variables changes given by the equivalence transformations with generators (B.4). The equivalence transformation groups corresponding to the generators $X_1^e, ..., X_8^e$ act as
follows (the unchanged variables are omitted)

\[ X_1^c : \quad \bar{t} = t + a; \]
\[ X_2^c : \quad \bar{s} = s + a; \]
\[ X_3^c : \quad \bar{x} = x + a; \]
\[ X_4^c : \quad \bar{x} = x + at, \quad \bar{u} = u + a; \]
\[ X_5^c : \quad \bar{E}_y = E_y \cos a + E_z \sin a, \quad \bar{E}_z = E_z \cos a - E_y \sin a, \]
\[ \bar{H}_y = H_y \cos a + H_z \sin a, \quad \bar{H}_z = H_z \cos a - H_y \sin a; \]
\[ X_6^c : \quad \bar{t} = e^a t, \quad \bar{s} = e^{2a} s, \quad \bar{u} = e^{-a} u, \quad \bar{\rho} = e^{2a} \rho, \]
\[ \bar{E}_y = e^{-a} E_y, \quad \bar{E}_z = e^{a} E_z, \quad \bar{\sigma} = e^a \sigma; \]
\[ X_7^c : \quad \bar{s} = e^{-a} s, \quad \bar{x} = e^a x, \quad \bar{u} = e^a u, \quad \bar{\rho} = e^{-2a} \rho, \]
\[ \bar{E}_y = e^a E_y, \quad \bar{E}_z = e^a E_z, \quad \bar{\sigma} = e^{-2a} \sigma; \]
\[ X_8^c : \quad \bar{s} = e^{2a} s, \quad \bar{\rho} = e^{2a} \rho, \quad \bar{\bar{\rho}} = e^{2a} \rho, \]
\[ \bar{E}_y = e^a E_y, \quad \bar{E}_z = e^a E_z, \quad \bar{H}_y = e^a H_y, \quad \bar{H}_z = e^a H_z. \]

Here \( a \) is a group parameter.

Consider transformations defined by a generator of the form

\[ X = \sum_{i=1}^{8} \kappa^i Y_i \]  \hspace{1cm} (C.2)

under the action of the equivalence transformations. An equivalence transformation changes this generator into the generator

\[ X = \sum_{i=1}^{8} \tilde{\kappa}^i \tilde{Y}_i, \]  \hspace{1cm} (C.3)

where the basis generators in the new variables are

\[
\begin{align*}
\dot{Y}_1 &= \frac{\partial}{\partial t}, \quad \dot{Y}_2 = \frac{\partial}{\partial s}, \quad \dot{Y}_3 = \frac{\partial}{\partial x}, \quad \dot{Y}_4 = \bar{t} \frac{\partial}{\partial \bar{t}} + \bar{s} \frac{\partial}{\partial \bar{s}} + \bar{x} \frac{\partial}{\partial \bar{x}}, \\
\dot{Y}_5 &= \bar{E}_z \frac{\partial}{\partial \bar{E}_y} - \bar{E}_y \frac{\partial}{\partial \bar{E}_z} + \bar{H}_z \frac{\partial}{\partial \bar{H}_y} - \bar{H}_y \frac{\partial}{\partial \bar{H}_z}, \\
\dot{Y}_6 &= \bar{t} \frac{\partial}{\partial \bar{t}} + 2s \frac{\partial}{\partial \bar{s}} - \bar{u} \frac{\partial}{\partial \bar{u}} + 2\bar{\rho} \frac{\partial}{\partial \bar{\rho}} - \bar{E}_y \frac{\partial}{\partial \bar{E}_y} - \bar{E}_z \frac{\partial}{\partial \bar{E}_z}, \\
\dot{Y}_7 &= \bar{z} \frac{\partial}{\partial \bar{x}} + \bar{x} \frac{\partial}{\partial \bar{u}} - 2\bar{\rho} \frac{\partial}{\partial \bar{\rho}} + \bar{E}_z \frac{\partial}{\partial \bar{E}_y} + \bar{E}_z \frac{\partial}{\partial \bar{E}_z}, \\
\dot{Y}_8 &= 2s \frac{\partial}{\partial \bar{s}} + 2\bar{\rho} \frac{\partial}{\partial \bar{\rho}} + 2\bar{\rho} \frac{\partial}{\partial \bar{\rho}} + \bar{E}_y \frac{\partial}{\partial \bar{E}_y} + \bar{E}_z \frac{\partial}{\partial \bar{E}_z} + \bar{H}_y \frac{\partial}{\partial \bar{H}_y} + \bar{H}_z \frac{\partial}{\partial \bar{H}_z}. \quad \text{(C.4)}
\end{align*}
\]
For example, consider the change of the generators $Y_1, Y_2, \ldots, Y_8$ under the transformation corresponding to $X_1^e$:

\[ \tilde{t} = t + a. \]

The other variables stay unchanged. According to the variables change in the differential operator formula [1], the generators $Y_4$ and $Y_6$ become

\[
Y_4 = Y_4(\bar{x}) \frac{\partial}{\partial \bar{x}} + Y_4(\bar{u}) \frac{\partial}{\partial \bar{u}} + \cdots = t \frac{\partial}{\partial \bar{x}} + \bar{u} \frac{\partial}{\partial \bar{u}} = \bar{Y}_4 - a \bar{Y}_3, \\
Y_6 = Y_6(\bar{t}) \frac{\partial}{\partial \bar{t}} + Y_6(\bar{s}) \frac{\partial}{\partial \bar{s}} + \cdots = \bar{t} \frac{\partial}{\partial \bar{t}} + \bar{s} \frac{\partial}{\partial \bar{s}} = \bar{Y}_6 - a \bar{Y}_1.
\]

The remaining generators stay unchanged

\[ Y_i = \bar{Y}_i, \quad i \neq 4, 6. \]

From (C.2) and (C.3) we find

\[
\kappa^4 (\bar{Y}_4 - a \bar{Y}_3) + \kappa^6 (\bar{Y}_6 - a \bar{Y}_1) + \sum_{i \neq \{4, 6\}} \kappa^i \bar{Y}_i = \sum_{i=1}^{8} \hat{\kappa}^i \bar{Y}_i.
\]

Hence,

\[
\hat{\kappa}^1 = \kappa^1 - a \kappa^6, \quad \hat{\kappa}^3 = \kappa^3 - a \kappa^4, \quad \hat{\kappa}^i = \kappa^i \quad \text{for} \quad i \neq 1, 3. \quad (C.5)
\]

Similarly we derive the transformations of the the coefficients related to the generators $X_2^e, \ldots, X_8^e$. They are

\[
X_2^e: \quad \hat{\kappa}^2 = \kappa^2 - a(2\kappa^6 - \kappa^7 + 2\kappa^8); \\
X_3^e: \quad \hat{\kappa}^3 = \kappa^3 - a \kappa^7; \\
X_4^e: \quad \hat{\kappa}^3 = \kappa^3 + a \kappa^1, \quad \hat{\kappa}^4 = \kappa^4 + a(\kappa^6 - \kappa^7); \\
X_5^e: \quad \hat{\kappa}^1 = e^{a \kappa^1}, \quad \hat{\kappa}^2 = e^{2a \kappa^2}, \quad \hat{\kappa}^4 = e^{-a \kappa^4}; \\
X_6^e: \quad \hat{\kappa}^2 = e^{-a \kappa^2}, \quad \hat{\kappa}^3 = e^{a \kappa^3}, \quad \hat{\kappa}^4 = e^{a \kappa^4}; \\
X_8^e: \quad \hat{\kappa}^2 = e^{2a \kappa^2};
\]

where the unchanged coefficients are omitted.
C.2 Action of the inner automorphisms

The inner automorphisms are constructed with the help of the commutator table \[1,3\]. We obtain the following commutator table for the generators (4.16)

|      | \(Y_1\) | \(Y_2\) | \(Y_3\) | \(Y_4\) | \(Y_5\) | \(Y_6\) | \(Y_7\) | \(Y_8\) |
|------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(Y_1\) | 0      | 0      | 0      | \(Y_3\) | 0      | 0      |
| \(Y_2\) | 0      | 0      | 0      | 0      | 0      | 0      | 2\(Y_2\) | \(-Y_2\) |
| \(Y_3\) | 0      | 0      | 0      | 0      | 0      | 0      | 0      | \(Y_3\) |
| \(Y_4\) | \(-Y_3\) | 0      | 0      | 0      | \(-Y_4\) | \(Y_4\) | 0      |
| \(Y_5\) | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| \(Y_6\) | \(-Y_1\) | \(-2Y_2\) | 0      | \(Y_4\) | 0      | 0      | 0      | 0      |
| \(Y_7\) | 0      | \(Y_2\) | \(-Y_3\) | \(-Y_4\) | 0      | 0      | 0      | 0      |
| \(Y_8\) | 0      | \(-2Y_2\) | 0      | 0      | 0      | 0      | 0      | 0      |

(C.7)

The inner automorphism \(A_j\) corresponds to the Lie group of transformations with the generator \([1]\) (the minus sign is chosen for convenience)

\[-\kappa^\alpha C^\gamma_{\alpha j} \frac{\partial}{\partial \kappa^\gamma},\]

where the structure constants \(C^\gamma_{\alpha j}\) are found from the commutator table.

As a particular example, consider the inner automorphisms corresponding to the generator \(Y_1\):

\[E_1 = -\kappa^\alpha C^\gamma_{\alpha 1} \frac{\partial}{\partial \kappa^\gamma} = -\kappa^4 \frac{\partial}{\partial \kappa^3} - \kappa^6 \frac{\partial}{\partial \kappa^1}.\]

We obtain the one-parameter group of the inner automorphisms for \(E_1\) integrating the Lie equations

\[\frac{d\tilde{\kappa}^1}{da} = -\tilde{\kappa}^6, \quad \frac{d\tilde{\kappa}^3}{da} = -\tilde{\kappa}^4, \quad \frac{d\tilde{\kappa}^i}{da} = 0, \quad i \neq 1, 3\]

with the initial conditions

\[\tilde{\kappa}^j|_{a=0} = \kappa^j, \quad j = 1, ..., 8.\]

The solution of this Cauchy problem is

\[\tilde{\kappa}^1 = \kappa^1 - a\kappa^6, \quad \tilde{\kappa}^3 = \kappa^3 - a\kappa^4, \quad \tilde{\kappa}^i = \kappa^i \quad \text{for} \quad i \neq 1, 3.\]  

(C.8)

Similarly we obtain the inner automorphisms for the transformations corresponding to the other generators

\[\begin{align*}
Y_2 : & \quad \tilde{\kappa}^2 = \kappa^2 - a(2\kappa^6 - \kappa^7 + 2\kappa^8); \\
Y_3 : & \quad \tilde{\kappa}^3 = \kappa^3 - a\kappa^7; \\
Y_4 : & \quad \tilde{\kappa}^3 = \kappa^3 + a\kappa^1, \quad \tilde{\kappa}^4 = \kappa^4 + a(\kappa^6 - \kappa^7); \\
Y_6 : & \quad \tilde{\kappa}^1 = e^a\kappa^1, \quad \tilde{\kappa}^2 = e^{2a}\kappa^2, \quad \tilde{\kappa}^3 = e^{-a}\kappa^4; \\
Y_7 : & \quad \tilde{\kappa}^2 = e^{-a}\kappa^2, \quad \tilde{\kappa}^3 = e^a\kappa^3, \quad \tilde{\kappa}^4 = e^a\kappa^4; \\
Y_8 : & \quad \tilde{\kappa}^2 = e^{2a}\kappa^2;
\end{align*}\]

(C.9)

where the unchanged coefficients are skipped.
C.3 Extensions of the kernel of the admitted Lie algebras

We observe that the coefficients changes (C.5), (C.6) corresponding to the equivalence transformations coincide with the coefficients changes (C.8), (C.9) for the inner automorphisms. It means that the equivalence transformations act on the generators of the extended Lie algebra the same way as the inner automorphisms. The partition of the admitted Lie algebras into classes with respect to the inner automorphisms coincides with the dissimilar subalgebras with respect to the equivalence transformations. This allows us to use the optimal system of subalgebras for the group classification. Moreover, for the group classification it is necessary to study only subalgebras which include the kernel (4.17). This realizes a significant advantage of the chosen approach: one needs to consider the minimal number of subalgebras.

For low-dimensional Lie algebras calculation of the optimal system of subalgebras (also called the representative list of subalgebras) is relatively easy. For high-dimensional Lie algebras the problem becomes complicated because it requires extensive computations. The difficulties can be facilitated by a two-step algorithm proposed in [36]. This algorithm replaces the problem of constructing the optimal system of high-dimensional subalgebras by a similar problem for lower dimensional subalgebras. Shortly, it can be described as follows.

Let \(L\) be a Lie algebra with the basis \(\{X_1, X_2, \ldots, X_r\}\). Assume that the Lie algebra \(L\) is decomposed as \(L = I \oplus F\), where \(I\) is a proper ideal of the algebra \(L\) and \(F\) is a subalgebra. Then the set of the inner automorphisms \(A = \text{Int } L\) of the Lie algebra \(L\) is decomposed \(A = A_I \cup A_F\), where \(A_I \subseteq I, \quad A_F \subseteq F\).

This means the following [36]. Let \(x \in L\) be decomposed as \(x = x_I + x_F\), where \(x_I \in I\), and \(x_F \in F\). Any automorphism \(B \in A\) can be written as \(B = B_I B_F\), where \(B_I \in A_I\), \(B_F \in A_F\). The automorphisms \(B_I\) and \(B_F\) have the properties:

\[B_I x_F = x_F, \quad \forall x_F \in F, \quad \forall B_I \in A_I,\]
\[B_F x_I \in I, \quad B_F x_F \in F, \quad \forall x_I \in I, \quad \forall x_F \in F, \quad \forall B_F \in A_F.\]

At the first step, an optimal system of subalgebras \(\Theta_{A_F}(F) = \{F_0, F_1, F_2, \ldots, F_p, F_{p+1}\}\) of the algebra \(F\) is formed. Here \(F_0 = \{0\}\), \(F_{p+1} = \{F\}\) and the optimal system of the algebra \(F\) is constructed with respect to the automorphisms \(A_F\). For each subalgebra \(F_j, j = 0, 1, 2, \ldots, p + 1\) one has to find its stabilizer \(\text{St}(F_j) \subset A\):

\[\text{St}(F_j) = \{B \in A \mid B(F_j) = F_j\}.\]

Note that \(\text{St}(F_{p+1}) = A\).

The second step consists of forming the optimal system of subalgebras \(\Theta_A(L)\) of the algebra \(L\) as a collection of \(\Theta_{\text{St}(F_j)}(I \oplus F_j), j = 0, 1, 2, \ldots, p + 1\). If the subalgebra \(F\) can be decomposed, then the two-step algorithm can be used for construction of \(\Theta_{A_F}(F)\).
Following the two-step algorithm, we split the extended algebra (4.16) into the ideal \( \{Y_1, Y_2, Y_3, Y_4, Y_5\} \) and the subalgebra \( \{Y_6, Y_7, Y_8\} \). In the considered case, the ideal coincides with the kernel and the subalgebra is Abelian.

The first step consists of classifying the subalgebra. The optimal system of subalgebras for the Abelian 3-dimensional algebra \( \{X_6, X_7, X_8\} \) was obtained in [37]. It consists of three one-dimensional subalgebras
\[
\{Y_7\}, \quad \{Y_6 + \alpha Y_7\}, \quad \{Y_8 + \alpha Y_6 + \beta Y_7\},
\]
three two-dimensional subalgebras
\[
\{Y_6, Y_7\}, \quad \{Y_8 + \alpha Y_6, Y_7\}, \quad \{Y_8 + \alpha Y_7, Y_6 + \beta Y_7\}
\]
and the whole three-dimensional subalgebra
\[
\{Y_6, Y_7, Y_8\}.
\]

Finally, the cases of the optimal subalgebras are added to the ideal. It turns out to be trivial because the ideal coincides with the kernel \( \{Y_1, Y_2, Y_3, Y_4, Y_5\} \) of the admitted algebras.