ON THE SCHWINGER MODEL ON RIEMANN SURFACES

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ABSTRACT

In this paper the Schwinger model or two dimensional quantum electrodynamics is exactly solved on a Riemann surface providing the explicit expression of the partition function and of the generating functional of the amplitudes between the fermionic currents. This offers one of the few examples, if not the only one, in which it is possible to integrate in an explicit way a gauge field theory interacting with matter on a Riemann surface.

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1. INTRODUCTION

In this paper we solve the Schwinger model \[1\] on a closed and orientable Riemann surface in the absence of topologically nontrivial gauge fields. To this purpose, we compute the partition function of the model and the generating functional of the amplitudes containing fermionic currents. The method used was developed in the recent papers \[2\] and \[3\] and it is able to quantize any abelian gauge field theory on a Riemann surface. Here, we restrict ourselves to the Schwinger model, which describes the quantum electrodynamics in two dimensions or QED\(_2\) because, due to its particular relevance in theoretical physics \[4\], \[5\], \[6\], \[7\], \[8\], \[9\], it provides a significant example of quantum field theory on a manifold. The fact that quantization on a manifold is not a trivial procedure and gives rise to observable phenomena, was discussed in the case of the Schwinger model in ref. \[3\]. For instance, the high energy behavior of QED\(_2\) strongly depends on the topology of the space-time. On a Riemann surface, in fact, the strength of the electromagnetic forces vanishes at short distances, apart from zero modes which are not so easy to determine, while on a disk this asymptotic freedom is not present at all \[3\].

Moreover, it was shown in ref. \[3\] that in the path integral of the Schwinger model defined on a compact manifold without boundary, it is possible to perform the integration over the gauge fields, obtaining as a result an effective theory of fermions. This theory consists in a nonlocal generalization, in the sense of \[10\], of the massless Thirring model \[11\], which will be called here generalized Thirring model or GTM. The nonlocality of the GTM together with the property of asymptotic freedom, make it very interesting by itself. However, the approach of refs. \[2\] and \[3\], intended for the quantization of any abelian gauge field theory, was essentially perturbative and therefore not suitable in order to treat the GTM, which is integrable. For this reason, we concentrate here on the Schwinger model, performing in the path integral also the integration over the fermionic fields which was missing in refs. \[2\] and \[3\]. The result is, as we anticipated at the beginning, a nonperturbative expression of the partition function and of the generating functional of QED\(_2\). In this way, we obtain one of the few examples, if not the only one, in which the explicit integration of a gauge field theory interacting with matter fields is possible on a Riemann surface. Until now, in fact, the only theories which have been quantized on Riemann surfaces are either pure gauge field theories or topological field theories, while
the interaction with the matter fields was restricted to external gauge field without kinetic term [12], [13], [14], [15], [16], [17], [18], [19]. The integrability of the Schwinger model, together with its equivalence with the GTM, make it very interesting not only for string theory, but also in the study of the effects of a curved space-time on the dynamics of the fields. Finally, the quantization of field theory on manifolds can shed some more light in the quantization of general relativity, see for example ref. [20] on this point.

Apart from the calculation of the propagator of the gauge fields, which has been carefully discussed in ref. [2], this paper is selfcontained. In the first part of the next section we briefly review the quantization of the abelian gauge field theories in the Feynman gauge discussed in refs. [2] and [3]. We consider small quantum perturbations $A^{qu}$ around a topologically nontrivial solution of the classical equations of motion $A^I$. In order to evaluate the generating functional of the amplitudes between the fermionic currents of the Schwinger model, we integrate the path integral over the gauge degrees of freedom. In this way we obtain the nonlocal GTM mentioned above as an effective field for the electrons on a Riemann surface. It is important to stress at this point that the inverse procedure, in which one integrates first over the fermionic fields, seems to be not very convenient on curved space-times as it is in the flat case. First of all, in fact, there are complications with the zero modes and the topologically nontrivial solutions of the equations of motion $A^I$. Secondly, also if we succeed in the integration, as a result we should obtain, in analogy with the flat case, a massive theory of vector fields [21], [22]. Massive field theories are not easy to handle on a Riemann surface because, for example, the propagators of massive field theories are still unknown.

In the second part of section 2, we perform the remaining integration over the fermionic degrees of freedom. The advantage of our method is that now we can use previous experience on the Thirring model [13], [23], [24] and apply it to the GTM. Nevertheless, a crucial point in order to solve the GTM is the calculation of the chiral determinant of free fermions in the presence of an external gauge fields $B$. Unfortunately, this calculation becomes involved when $B$ is topologically nontrivial, so we assume here that $B$ belongs to a trivial line bundle on the Riemann surface. Only in this case, in fact, we can compute the determinant using the explicit formula of ref. [13]. For this reason, we have to set $A^I = 0$ in the generating functional of the Schwinger model. The final computation of the generating functional of the correlation functions between the fermionic current is achieved using slight generalizations of the formulas obtained in ref. [23]. The details of the calculation are given in appendix A (generating functional) and appendix B (partition function).
2. INTEGRABILITY OF THE SCHWINGER MODEL ON A RIEMANN SURFACE

We consider here the Schwinger model \([1]\) defined by the following action:

\[
S = \int_M d^2 x \sqrt{g} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i \bar{\psi} \gamma^\mu(x) (D_\mu - ie A_\mu) \psi + e B_\mu \bar{\psi} \gamma^\mu(x) \psi \right] \tag{2.1}
\]

In eq. (2.1) \(e\) is the coupling constant and \(M\) is a closed and orientable Riemann surface of genus \(g\) provided with an Euclidean conformally flat metric \(g_{\mu\nu}, \mu, \nu = 0, 1\). \(g_{\mu\nu}\) is induced by the distance:

\[
ds^2 = e^{-h(x_0, x_1)} (dx_0^2 + dx_1^2) \tag{2.2}
\]

Together with the metric we need also the zweibein \(e^\alpha_\mu, \alpha, \beta = 0, 1\) being the frame indices. The zweibein is defined so that:

\[
g^{\mu\nu}(x) = e^\mu_\alpha e^\nu_\beta \eta^{\alpha\beta} \tag{2.3}
\]

where \(\eta^{\alpha\beta} = \text{diag}(1, 1)\) is the flat metric in the local frame. From this definition it descends that \(g \equiv \det g_{\mu\nu} = (\det e^\alpha_\mu)^2\). Moreover, \(\gamma^\mu(x) \equiv e^\mu_\alpha(x) \gamma^\alpha\), the \(\gamma^\alpha\) denoting the usual \(\gamma\)-matrices expressed in terms of the Pauli matrices \(\sigma_1\) and \(\sigma_2\) as follows:

\[
\gamma^0 = -\sigma_2 \quad \quad \quad \quad \gamma^1 = \sigma_1
\]

Finally, we have defined \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) and \(\bar{\psi} = \psi^\dagger \gamma^0\).

Due to the fact that on a Riemann surface the fermionic fields admit spin structures, we suppose that \(\bar{\psi}\) and \(\psi\) transform according to the following rules when transported along the nontrivial homology cycles \(A_i, i = 1, \ldots, g\), of the Riemann surface \(M\):

\[
\bar{\psi} \rightarrow e^{-2\pi i u_i} \bar{\psi} \quad \quad \quad \quad \psi \rightarrow e^{2\pi i u_i} \psi
\]

Along the homology cycles \(B_i\), we have instead:

\[
\bar{\psi} \rightarrow e^{-2\pi i v_i} \bar{\psi} \quad \quad \quad \quad \psi \rightarrow e^{2\pi i v_i} \psi
\]

\(A_i\) and \(B_i\) are chosen here in such a way that they for a canonical basis for the nontrivial homology cycles \([25]\). We notice also that the covariant derivative \(D_\mu\) for the fermions appearing in eq. (2.1) is given by:

\[
D_\mu = \partial_\mu + \omega_\mu
\]
where $\omega_\mu = \frac{1}{8} e_{\alpha \nu} \nabla_\mu e_\beta^\nu [\gamma^\alpha, \gamma^\beta]$ and $\nabla_\mu$ is the covariant derivative acting on the zweibein $e_\beta^\nu$. However, in two dimensions the spin field $\omega_\mu$ does not contribute to the action (2.1), so that we will set $D_\mu \equiv \partial_\mu$ in the rest of this work.

Here we are interested in computing the partition function of the Schwinger model and the amplitudes of the currents:

$$j^\mu(x) = [\bar{\psi} \gamma^\mu \psi](x) \quad (2.4)$$

In eq. (2.1), the external source of these currents is $B_\mu$.

In order to fix the abelian gauge symmetry of eq. (2.1) and quantize the theory, we impose the covariant gauge:

$$\partial_\mu A^\mu = 0 \quad (2.5)$$

As a consequence, we have to solve the path integral:

$$Z[B_\mu] = \int DA_\mu D\bar{\psi} D\psi e^{-(S+S_{gf})} \quad (2.6)$$

where the gauge fixing term $S_{gf}$ is given by:

$$S_{gf} = \frac{1}{2\lambda} \int_M d^2x \sqrt{g} (\partial_\mu A^\mu)^2 \quad (2.7)$$

We ignore the Faddeev-Popov term, since the ghost are decoupled from all the other fields in the abelian case we are treating. Moreover, we will choose the Feynman gauge, putting $\lambda = 1$ in eq. (2.7).

We start to evaluate eq. (2.6) integrating over the gauge fields. Therefore, we need to compute the following path integral:

$$Z[j^\mu] = \int DA_\mu e^{\int_M d^2x \sqrt{g} [-\frac{1}{4} F^\mu_\nu F^\nu_\mu + ej^\mu A^\mu + \frac{1}{2} (\partial_\mu A^\mu)^2]} \quad (2.8)$$

To this purpose, it is better to use complex coordinates $z = x + ix_0$ and $\bar{z} = x_0 - ix_1$. In the new system of coordinates, the metric is given by:

$$g_{z\bar{z}} = g_{\bar{z}z} = e^{-h(z, \bar{z})} \quad g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad g^{zz} g^{\bar{z}\bar{z}} = 1 \quad (2.9)$$

It is also convenient to decompose the gauge fields in the following way:

$$A_z = A_z^I + A_z^{\text{qu}} + A_z^{\text{har}} \quad A_{\bar{z}} = A_{\bar{z}}^I + A_{\bar{z}}^{\text{qu}} + A_{\bar{z}}^{\text{har}} \quad (2.10)$$
where $A^1_z$ and $A^1_{\bar{z}}$ are instantonic solutions of the equations of motion in the Feynman gauge corresponding to $c_1 = k \in \mathbb{Z}$. Here $c_1$ denotes the first Chern class:

$$c_1 = \frac{1}{2\pi} \int_M d^2z F_{\bar{z}z}(A^1)$$

The explicit form of the fields $A^1$ has been given in ref. [3] and we do not report it.

Let us now return to the equations (2.10). The fields $A^q u$ and $A^q \bar{u}$ denote a small quantum perturbation around the instantonic solutions discussed above. They contain the transverse and longitudinal gauge degrees of freedom, which can be represented as a coexact and exact form respectively using the Hodge decomposition [2], [13]. Finally, $A^\text{har}_z$ and $A^\text{har}_{\bar{z}}$ describe the harmonic components of the gauge fields. They can be written in the following way:

$$A^\text{har}_z dz = 2\pi i (\phi + \bar{\Omega} \theta) \cdot (\Omega - \bar{\Omega})^{-1} \cdot \bar{\omega}(\bar{z}) d\bar{z}$$

(2.11)

$A^\text{har}_{\bar{z}} d\bar{z}$ can be computed from eq. (2.11) by complex conjugation. $\phi_i$ and $\theta_i$ are real numbers and $\Omega_{ij}$, $i, j = 1, \ldots, g$, is the period matrix which can be computed in terms of the harmonic differentials $\omega_i(z) dz$:

$$\Omega_{ij} = \oint_{B_i} \omega_j(z) dz$$

As a consequence, it is easy to see that the path integral (2.8) becomes in complex coordinates:

$$Z[j^\mu] = \int DA^q u_z DA^q u_{\bar{z}} \prod_{i=1}^g d\theta_i d\phi_i e^{-(S^q u + S^1)}$$

(2.12)

with

$$S^q u = \int_M d^2z \left[ g^{z\bar{z}} (\partial_z A^q u_{\bar{z}} \partial_{\bar{z}} A^q u_z + \partial_{\bar{z}} A^q u_z \partial_z A^q u_{\bar{z}}) + e j_z A^q u_z + e j_{\bar{z}} A^q u_{\bar{z}} \right]$$

(2.13)

$$S^1 = e \int_M d^2z \left[ j_z (A^1_z + A^\text{har}_z) + j_{\bar{z}} (A^1_{\bar{z}} + A^\text{har}_{\bar{z}}) \right]$$

(2.14)

Here we have used the following notation: $d^2z \equiv \frac{dz \wedge d\bar{z}}{2i}$. Again, since at the end we will put $A^1 = 0$, we have not shown that the instantonic gauge fields decouple from the action of the quantum fields $S^q u$. A proof is given in ref. [3]. Finally, in complex coordinates the current $j_\mu$ assumes the form:

$$j_z = \bar{\psi}_\theta \psi_\theta \quad j_{\bar{z}} = \bar{\psi}_\bar{\theta} \psi_{\bar{\theta}}$$

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where \( \theta \) and \( \bar{\theta} \) are spinor indices. In order to compute the path integral (2.12), we still need the propagator \( G_{\mu\nu}(z, w) = \langle A^\mu_{\mu}(z, \bar{z}) A^\nu_{\nu}(w, \bar{w}) \rangle \) of the quantum fields. The components of the propagator were already derived in refs. [2] and [3], so that we will report here only the result:

\[
G_{zw}(z, w) = -\int_M d^2 t g_{t\bar{t}} \partial_z K(z, t) \partial_w K(w, t) \tag{2.15}
\]

\[
G_{\bar{z}w}(z, w) = -\int_M d^2 t g_{t\bar{t}} \partial_{\bar{z}} K(z, t) \partial_w K(w, t) \tag{2.16}
\]

where \( K(z, w) \) is the usual scalar Green function defined by the relations:

\[
\partial_z \partial_{\bar{z}} K(z, t) = \delta^{(2)}(z, t) - \frac{g_{z\bar{z}}}{A} \quad A = \int_M d^2 z g_{z\bar{z}} \tag{2.17}
\]

\[
\partial_{\bar{z}} \partial_t K(z, t) = -\delta^{(2)}_{\bar{z}t}(z, t) + \sum_{i,j=1}^g \bar{\omega}_i(\bar{z}) [\text{Im } \Omega]_{ij}^{-1} \omega_j(t) \tag{2.18}
\]

\[
\int_M d^2 t g_{t\bar{t}} K(z, t) = 0 \tag{2.19}
\]

One of the characteristics of the propagator given in eqs. (2.15) and (2.16) that will be important in the following is that it is orthogonal with respect to the zero modes. This can be seen for example from the fact that inserting \( G_{zw}(z, w) \) in the equations of motion coming from the action (2.13), we get:

\[
\partial_z g^{z\bar{z}} \partial_{\bar{z}} G_{zw}(z, w) = \delta^{(2)}_{z\bar{z}}(z, w) - \sum_{i,j=1}^g \bar{\omega}_i(\bar{z}) [\text{Im } \Omega]_{ij}^{-1} \omega_j(w) \tag{2.20}
\]

An analogous equation can be derived for the component \( G_{\bar{z}w}(z, w) \) of the propagator. It is important to notice that in the right hand side of eq. (2.21), the second term is the projector on the space of the zero modes. Moreover, if we consider an external current \( J_t(t, \bar{t}) \) and decompose it as follows:

\[
J_t(t, \bar{t}) = \partial_t \chi(t, \bar{t}) + i \bar{a}_i [\text{Im } \Omega]_{ij}^{-1} \bar{\omega}_j(\bar{t}) \tag{2.21}
\]

with

\[
\bar{a}_i = i \int d^2 t \omega_i(t) J_t(t, \bar{t})
\]

we have:

\[
\partial_{\bar{z}} g^{z\bar{z}} \partial_{\bar{z}} \int d^2 t G_{zt}(z, t) J_t(t, \bar{t}) = \partial_{\bar{z}} \bar{\chi}(z, \bar{z}) \tag{2.22}
\]
A similar formula is valid for the component \( G_{\bar{z}w}(z, w) \):

\[
\partial_{\bar{z}}g^{\bar{z}\bar{z}}\partial_{\bar{z}}\int d^2t G_{\bar{z}\bar{t}}(z, t)J_t(t, \bar{t}) = \partial_{\bar{z}}\chi(z, \bar{z}) \tag{2.23}
\]

showing the orthogonality of the propagator with respect to the harmonic components of the gauge fields.

Now we notice that in eq. (2.12) the integral over the fields \( A^{\mu} \) is gaussian. Therefore, it is possible to eliminate the terms containing the interaction between the current \( j^\mu \) and the fields \( A^{\mu} \) performing a shift of the fields. To this purpose, we define new fields:

\[
A_{\bar{z}}^{\mu} = A_{\bar{z}}^{\mu} + \frac{e}{2} \int d^2w G_{zw}(z, w)J_{\bar{w}}(w, \bar{w})
\]

\[
A_{\bar{z}}^{\mu} = A_{\bar{z}}^{\mu} + \frac{e}{2} \int d^2w G_{\bar{z}w}(z, w)J_w(w, \bar{w})
\]

Inserting the new fields \( A_{\bar{z}}^{\mu} \) in the action \( S^{\mu} \) given in eq. (2.13) we yield:

\[
S^{\mu} = \int_M d^2z \left[ g^{\bar{z}\bar{z}}\partial_{\bar{z}}A_{\bar{z}}^{\mu} \partial_{\bar{z}}A_{\bar{z}}^{\mu} - \frac{e^2}{2} j_{\bar{z}}(z, \bar{z}) \int_M d^2w G_{zw}(z, w)j_{\bar{w}}(w, \bar{w}) + \right.

\left. - \int_M d^2z g^{\bar{z}\bar{z}}\partial_{\bar{z}}A_{\bar{z}}^{\mu} \partial_{\bar{z}}A_{\bar{z}}^{\mu} - \frac{e^2}{2} j_{\bar{z}}(z, \bar{z}) \int_M d^2w G_{\bar{z}w}(z, w)j_w(w, \bar{w}) \right] \tag{2.24}
\]

We notice that we have obtained eq. (2.24) integrating by part in eq. (2.13). This is allowed because the fields \( A^{\mu} \) and \( A_{\bar{z}}^{\mu} \) are orthogonal with respect to the harmonic part. The \( A^{\mu} \) are orthogonal by definition, while the fields \( A_{\bar{z}}^{\mu} \) and \( A_{\bar{z}}^{\mu} \) do not contain any harmonic component due to the properties of orthogonality (2.22) and (2.23) of the propagator. As a consequence, the operators \( \partial_{\bar{z}} \) and \( \partial_{\bar{z}} \) in (2.13) act on a space of (0,1) and (1,0) forms respectively which is orthogonal to the harmonic sector and we are free to integrate by parts in eq. (2.13). Using the action (2.24) in eq. (2.12) and integrating over the fields \( A^{\mu} \) we get, apart from a constant factor:

\[
Z[j^\mu] = \prod_{i=1}^{g} d\theta_i d\phi_i e^{-Si} e^{W[j]} \tag{2.25}
\]

where

\[
W[j] = \frac{e^2}{4} \int_M d^2z d^2w \left[ j_{\bar{z}}(z, \bar{z}) G_{zw}(z, w)j_{\bar{w}}(w, \bar{w}) + j_{\bar{z}}(z, \bar{z}) G_{\bar{z}w}(z, w)j_w(w, \bar{w}) \right] \tag{2.26}
\]
Now we substitute the above result in the path integral of the Schwinger model (2.6) after performing the necessary change of coordinates \((z, \bar{z}) \rightarrow (x_0, x_1)\). The upshot is:

\[
Z[B_\mu] = \int D\bar{\psi} D\psi \prod_{i=1}^q d\theta_i d\phi_i e^{S_{eff}}
\]

(2.27)

where \(S_{eff}\) is the action of the GTM:

\[
S_{eff} = \int_M d^2x \sqrt{g(x)} \left[ i\bar{\psi}\gamma^\mu(x) \left( \partial_\mu - ie(B_\mu + A_\mu^{\text{bar}} + A_\mu^1) \right) \psi + \frac{e^2}{4} \int_M d^2y \sqrt{g(y)}j^\mu(x)G_{\mu\nu}(x,y)j^\nu(y) \right] \]

(2.28)

\(G_{\mu\nu}(x,y)\) is the propagator of the gauge fields written in real coordinates. In terms of the complex components (2.15) and (2.16) we have:

\[
G_{11}(x_0, x_1; y_0, y_1) = -G_{22}(x_0, x_1; y_0, y_1) = \text{Re}[G_{zw}(z, w)]
\]

\[
G_{12}(x_0, x_1; y_0, y_1) = G_{21}(x_0, x_1; y_0, y_1) = -\text{Im}[G_{zw}(z, w)]
\]

In the action (2.28), \(G_{\mu\nu}(x,y)\) plays the role of a relativistic potential through which the fermionic currents interact. Thus \(S_{eff}\) describes the effective theory of fermions after removing the unphysical (in two dimensions) gauge degrees of freedom. The model associated to the action (2.28) has been already discussed in ref. \[3\] and we only notice that this effective theory of the electrons can be viewed as a nonlocal generalization \[2\] of the massless Thirring model \[11\]. As we will show, also this generalization is an integrable model. To verify this, it is convenient to exploit the following formula, which is an extension to our nonlocal case of the analogous formulas given in refs. \[23\], \[24\]:

\[
\exp \left[ -\frac{1}{4} \int_M d^2x d^2y \sqrt{g(x)} \sqrt{g(y)} G_{\mu\nu}(x,y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} \right] \exp \left[ -e \int_M d^2x \sqrt{g(x)} j^\mu B_\mu \right] =
\]

\[
\exp \int d^2x \sqrt{g(x)} \left[ -e j^\mu B_\mu - \frac{e^2}{4} \int_M d^2y \sqrt{g(y)} G_{\mu\nu}(x,y) j^\nu(y) j^\mu(x) \right]
\]

(2.29)

\[Local\;generalizations\;of\;this\;kind\;have\;been\;studied\;for\;example\;in\;ref.\;as\;a\;way\;for\;regularizing\;the\;singularities\;of\;the\;gauge\;field\;theories.\;Here\;the\;regularization\;comes\;from\;the\;fact\;that\;the\;space\;time\;is\;nonflat.\]
The proof of this equation is straightforward (see appendix A). As a consequence of (2.29), the path integral of the Schwinger model can be rewritten in this way:

\[ Z[B_\mu] = \exp \left[ -\frac{1}{4} \int_M d^2x d^2y \sqrt{g(x)} \sqrt{g(y)} G_{\mu\nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} \right] \]

\[
\int D\bar{\psi}D\psi \prod_{i=1}^d d\theta_i \phi_i \exp \left[ -\int_M d^2x \sqrt{g(x)} \left( i\bar{\psi}\gamma^\mu(x)\partial_\mu \psi + e j^\mu(B_\mu + A^\text{har}_\mu + A^1_\mu) \right) \right]
\]

(2.30)

The advantage of having used the formula (2.29) is that now the action in the fermionic path integral becomes gaussian and it amounts to a chiral determinant:

\[
\det Q = \int D\bar{\psi}D\psi \exp \left[ -\int_M d^2x \sqrt{g(x)} \left( i\bar{\psi}\gamma^\mu(x)\partial_\mu \psi + e j^\mu(B_\mu + A^\text{har}_\mu + A^1_\mu) \right) \right]
\]

(2.31)

where \( Q = \gamma^\mu(x) \left( \partial_\mu - ie(B_\mu + A^\text{har}_\mu + A^1_\mu) \right) \). Unfortunately, at least to our knowledge, the above determinant has not yet been computed in the presence of gauge fields \( A^1_\mu \) belonging to a nontrivial topological sector. For this reason, we will set in the following \( A^1_\mu = 0 \) in eq. (2.31). Moreover, we need also some more informations about the external gauge field \( B_\mu \). Here it is convenient to choose \( B_\mu \) purely transverse:

\[ B_\mu(x) = \epsilon_{\mu\nu} \partial^\nu \varphi \]

where \( \varphi \) is a real scalar field. The motivation of this choice is that in any case the longitudinal components of \( B_\mu \) are physically irrelevant in the path integral (2.31) while the harmonic part is already present through the fields \( A^\text{har}_\mu \). Looking at eq. (2.31), we can also conclude that the piece containing the harmonic components \( A^\text{har}_\mu \) factorizes from the rest of the determinant. As a matter of fact, due to the orthogonality properties of the Hodge decomposition [13], we have:

\[
\int_M d^2x \sqrt{g(x)} j^\mu A^\text{har}_\mu = \int_M d^2x \sqrt{g(x)} j^\mu_{\text{har}} A^\text{har}_\mu
\]

where \( j^\mu_{\text{har}} \) is the harmonic component of the current \( j^\mu \) of eq. (2.21). This factorization is evident also from the following formula, derived in [13], which gives the explicit form of the determinant (2.31):

\[
\det Q = \exp \left[ -\frac{e^2}{2\pi} \int_M d^2x \sqrt{g(x)} B_\mu B^\mu \left( \frac{\det(\text{Im}\Omega)A}{\det \triangle_0} \right)^{\frac{1}{2}} \theta \left( \frac{u + \theta}{v + \phi} \right)(0, \Omega) \right]^2
\]

(2.32)
where \( A = \int_M d^2 x \sqrt{g(x)} \) represents the area of \( M \) and \( \det' \triangle_0 \) is the regularized determinant (without the constant zero mode) of the laplacian \( \triangle_0 \). Moreover, \( \theta \left[ \begin{array}{c} u + \theta \\ v + \phi \end{array} \right](0, \Omega) \) is a theta function of periods \( u_i + \theta_i \) along the cycles \( A_i \) and periods \( v_i + \phi_i \) along the cycles \( B_i \).

Inserting the determinant (2.32) in eq. (2.30) we get:

\[
Z[B_\mu] = C \exp \left[ -\frac{1}{4} \int_M d^2 x d^2 y \sqrt{g(x)} \sqrt{g(y)} G_{\mu\nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} \right] \times \exp \left[ -\frac{e^2}{2\pi} \int_M d^2 x \sqrt{g(x)} B_\mu(x) B^\mu(x) \right]
\]

with
\[
C = \left( \frac{\det(\text{Im}\Omega) A}{\det' \triangle_0} \right)^{\frac{1}{2}} \left| \theta \left[ \begin{array}{c} u + \theta \\ v + \phi \end{array} \right](0, \Omega) \right|^2
\]

In order to compute \( Z[B_\mu] \), we use a slight generalization of the formulas contained in [23] showing, see appendix A, that the generating functional satisfies the following equation:

\[
\int_M d^2 x \sqrt{g(x)} \left( \delta^k_\mu \delta^{(2)}(z, x) - \frac{e^2}{2\pi} G^k_\mu(z, x) \right) \frac{\delta}{\delta B_\mu(x)} Z[B_\mu] = -\frac{e^2}{\pi} A^k(z) Z[B_\mu]
\]

(2.34)

To extract the explicit form of the generating functional \( Z[B_\mu] \) from the above equation, we need a kernel \( M^\rho_k(y, z) \) with the property:

\[
\int_M d^2 z \sqrt{g(z)} M^\rho_k(y, z) \left( \delta^k_\mu \delta^{(2)}(z, x) - \frac{e^2}{2\pi} G^k_\mu(z, x) \right) = \delta^\rho_\mu \delta^{(2)}(y, x)
\]

(2.35)

Eq. (2.35) can be solved by iteration supposing that \( \left| \frac{e^2}{2\pi} \right| << 1 \):

\[
M^\rho_k(y, z) = \delta^\rho_\mu \delta^{(2)}(y, z) + \frac{e^2}{2\pi} G^\rho_k(y, z) + \sum_{n=2}^{\infty} \left( \frac{e^2}{2\pi} \right)^n \int_M d^2 x_1 \sqrt{g(x_1)} \ldots d^2 x_{n-1} \sqrt{g(x_{n-1})} G^\rho_{\mu_1}(y, x_1) G^{\mu_1}_{\mu_2}(x_1, x_2) \ldots G^{\mu_{n-1}}_{k}(x_{n-1}, z)
\]

(2.36)

Moreover, exploiting the fact that \( G_{\mu\nu}(x, y) = G_{\nu\mu}(y, x) \), we can easily see that the following equation holds (see appendix A):

\[
M^\rho_k(y, z) = M^{k\rho}(z, y)
\]

(2.37)

\footnote{The symbol \( z \) used in the formula below and in the following is not a complex variable, but represents the two real variables \( z_0, z_1 \in M \).}
Applying to both sides of eq. (2.34) the operator $\sqrt{g(z)} M^\rho_k(y,z)$ and integrating over $z$, we arrive at the following expression for $Z[B_\mu]$:

$$\frac{\delta}{\delta B_\rho(y)} Z[B_\mu] = -\frac{e^2}{\pi} \int_M d^2z \sqrt{g(z)} M^\rho_k(y,z) B^k(z) Z[B_\mu]$$  \hspace{1cm} (2.38)

It is now easy to verify using the property (2.37) of the kernel $M^\rho_k(y,z)$ that the solution of the above equation is given by:

$$Z[B_\mu] = Z_0 \exp \left[ -\frac{e^2}{2\pi} \int_M d^2z d^2y \sqrt{g(z)} \sqrt{g(y)} B_\rho(y) M^\rho_k(y,z) B^k(z) \right]$$  \hspace{1cm} (2.39)

This formula is the explicit expression of the generating functional $Z[B_\mu]$. Deriving eq. (2.39) in $B_\mu$, we are able to compute the amplitudes between the currents (2.4). Still we have to derive the partition function $Z_0$. The explicit calculation is performed in appendix B and here we just state the result:

$$Z_0 = \exp \left[ -\frac{e^2}{\pi} \text{Tr} \log \left( M^\rho_k(y,z) \right) \right]$$  \hspace{1cm} (2.40)

where

$$-\text{Tr} \log \left( M^\rho_k(y,z) \right) = \frac{1}{4} \int_M d^2x \sqrt{g(x)} G^k_k(x,x) + \frac{1}{4} \int_M d^2x d^2y d^2z \sqrt{g(z)} \sqrt{g(y)} \sqrt{g(x)} G^k_k(z,x) G^\rho_k(x,z) + \frac{1}{4} \sum_{n=2}^{\infty} \frac{1}{4(n+1)} \left( \frac{e^2}{2\pi} \right)^n \times$$

$$\int_M d^2x \sqrt{g(x)} d^2z \sqrt{g(z)} d^2x_1 \sqrt{g(x_1)} \cdots d^2x_{n-1} \sqrt{g(x_{n-1})} G^\rho_{\mu_1}(x,x_1) \cdots G^\mu_{n-1} \rho_k(x_{n-1},z) G^k_{\rho}(z,x)$$  \hspace{1cm} (2.41)

3. CONCLUSIONS

In this paper we have shown that the Schwinger model on a Riemann surface is integrable in absence of topologically nontrivial gauge fields. The generating functional and the partition function are given in eqs. (2.39) and (2.40) respectively. Many questions and possible developments remain open. For example, it remains the problem of integrability in the case of nontrivial line bundles. In fact, due to the presence of zero modes, it is
difficult to compute the chiral determinant (2.31) explicitly when $A^I_\mu \neq 0$. In this respect, one of the advantages of our procedure is that, integrating over the gauge fields in the path integral of the Schwinger model, we obtain the GTM, from which one can read many of the properties of QED$_2$, like the high energy behavior discussed above and investigated in ref. [3], which exist also in the presence of topologically nontrivial gauge fields. It is possible to extract these properties from the relativistic potential $G_{\mu\nu}(x, y)$ governing the electromagnetic interactions and whose components are explicitly given in eqs. (2.15), (2.16).

It would also be interesting to see if the GMT defined in eqs. (2.27) and (2.28) can be solved using the inverse scattering method [27]. The nonlocality of the model should have nontrivial consequences on the form of the classical $r-$matrix.

Finally, we stress the fact that two dimensional models quantized on Riemann surfaces represent an important tool in order to study the effects of an external gravitational background on the dynamics. For example, a Riemann surface with punctures which is imbedded in three dimensions has a topology which is similar to that of a wormhole [28]. Moreover, the Riemann surfaces, having a very rich structure, are also very suitable to investigate the possibility of creating quantum mechanical states in curved space-times. As it is well known, in fact, in the canonical quantization of field theories new quantum mechanical states can be generated due to the presence of diffeomorphisms which are not deformable to the identity. In four dimensions these states have been called geons [29]. In two dimensions, the Riemann surfaces with crystallographic group of symmetry provide an excellent way of reproducing this effect. Moreover, the advantage of working in two dimensions is that one can also explicitly construct the quantum mechanical states induced by the topology. This has been done in refs. [28] and [30] for some simple conformal field theories and considering the crystallographic groups $Z_n$ and $D_n$. We hope that the explicit form of the partition function and of the generating functional computed here will allow the extensions of these results to the more physical Schwinger model.

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Appendix A.

First of all, we prove the formula (2.29):

$$\exp\left[\alpha \int_M d^2x d^2y \sqrt{g(x)g(y)} G_{\mu\nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} \right] \exp\left[\beta \int_M d^2u \sqrt{g(u)} j^\sigma(u) B_\sigma(u) \right] =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left[\alpha \int_M d^2x d^2y \sqrt{g(x)g(y)} G_{\mu\nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} \right]^n \exp\left[\beta \int_M d^2u \sqrt{g(u)} j^\sigma B_\sigma \right]$$

(A.1)

Using now the fact that:

$$\exp\left[\alpha \int_M d^2x d^2y \sqrt{g(x)g(y)} G_{\mu\nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} \right] \exp\left[\beta \int_M d^2u \sqrt{g(u)} j^\sigma B_\sigma \right] =$$

$$\beta^2 \int_M d^2x d^2y \sqrt{g(x)g(y)} G_{\mu\nu}(x, y) j^\mu(x) j^\nu(y)$$

and applying it to eq. (A.1), we obtain:

$$\exp\left[\alpha \int_M d^2x d^2y \sqrt{g(x)g(y)} G_{\mu\nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} \right] \exp\left[\beta \int_M d^2u \sqrt{g(u)} j^\sigma(u) B_\sigma(u) \right] =$$

$$\exp\left[\alpha \beta^2 \int_M d^2x d^2y \sqrt{g(x)g(y)} G_{\mu\nu}(x, y) j^\mu(x) j^\nu(y) \right] \exp\left[\beta \int_M d^2u \sqrt{g(u)} j^\sigma B_\sigma \right]$$

Setting $\alpha = -\frac{1}{4}$ and $\beta = -\epsilon$, one recovers exactly eq. (2.29).

Now, we derive eq. (2.34) using a slight generalization of the functional formulas of [23]. Setting $C = 1$ in eq. (2.33), we consider the functional $\frac{\delta Z[B_\mu]}{\delta B_k(z)}$:

$$\frac{\delta Z[B_\mu]}{\delta B_k(z)} = -\frac{e^2}{\pi} \exp\left[-\frac{1}{4} \int_M d^2x d^2y g(x)g(y) G_{\mu\nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} \right] \times$$

$$B^k(z) \exp\left[-\frac{e^2}{2\pi} \int_M d^2x g(x) g^{\mu\nu} B_\mu B_\nu \right]$$

(A.2)

To simplify this equation, we have first to introduce the operators

$$S(\beta) = \exp\left[-\beta \int_M d^2x d^2y g(x)g(y) G_{\mu\nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} \right]$$
and $S^{-1}(\beta) = S(-\beta)$, where $\beta$ is an arbitrary parameter. Now we compute:

$$\frac{d}{d\beta} [S(\beta)B^k(z)S^{-1}(\beta)] =$$

$$-S(\beta) \left[ \int_M d^2 x d^2 y \sqrt{g(x)} \sqrt{g(y)} G_{\mu \nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} B^k(z) \right] S^{-1}(\beta)$$

(A.3)

where $[,]$ denotes the usual commutator. The derivation of the commutator contained in eq. (A.3) is simple and yields:

$$\left[ \int_M d^2 x d^2 y \sqrt{g(x)} \sqrt{g(y)} G_{\mu \nu}(x, y) \frac{\delta^2}{\delta B_\mu(x) \delta B_\nu(y)} B^k(z) \right] = 2 \int_M d^2 x \sqrt{g(x)} G^k_{\mu}(z, x) \frac{\delta}{\delta B_\mu(x)}$$

Therefore, using the fact that $S(\beta)S^{-1}(\beta) = 1$, eq. (A.3) becomes:

$$\frac{d}{d\beta} [S(\beta)B^k(z)S^{-1}(\beta)] = -2 \int_M d^2 x \sqrt{g(x)} G^k_{\mu}(z, x) \frac{\delta}{\delta B_\mu(x)}$$

Integrating this equation in $\beta$ between 0 and $\frac{1}{4}$, we get:

$$[S(\frac{1}{4}), B^k(z)] = -\frac{1}{2} \int_M d^2 x \sqrt{g(x)} G^k_{\mu}(z, x) \frac{\delta}{\delta B_\mu(x)} S(\frac{1}{4})$$

Applying this equation to eq. (A.2), we have:

$$\frac{\delta Z[B_\mu]}{\delta B_k(z)} = -\frac{e^2}{\pi} \left[ B^k(z) - \frac{1}{2} \int_M d^2 x \sqrt{g(x)} G^k_{\mu}(z, x) \frac{\delta}{\delta B_\mu(x)} \right] Z[B_\mu]$$

which is clearly equivalent to eq. (2.34) after setting:

$$\frac{\delta Z[B_\mu]}{\delta B_k(z)} = \int_M d^2 x \sqrt{g(x)} \delta^k_{\mu} \delta^{(2)}(z, x) \frac{\delta Z[B_\mu]}{\delta B_\mu(x)}$$

Finally, we verify eq. (2.37). To this purpose, we exploit the explicit form of $M^\rho_k(y, z)$ given in eq. (2.36). At the zeroth and first order in $\frac{e^2}{2\pi}$, the proof of eq. (2.37) is trivial. At the $n$–th order, $n \geq 2$, we use the property $G^{\mu}_{\nu}(x, y) = G^{\mu}(y, x)$ in order to rewrite the $n$–th element of $M^\rho_k(y, z)$ as follows:

$$\int_M d^2 x_1 \ldots d^2 x_{n-1} \sqrt{g(x_1)} \ldots \sqrt{g(x_{n-1})} G^\rho_{\mu_1}(y, x_1) G^{\mu_1}_{\mu_2}(x_1, x_2) \ldots G^{\mu_{n-1}}_k(x_{n-1}, z) =$$

$$\int_M d^2 x_1 \ldots d^2 x_{n-1} \sqrt{g(x_1)} \ldots \sqrt{g(x_{n-1})} G^{\rho}_{\mu_1}(x_1, y) G^{\mu_1}_{\mu_2}(x_1, x_2) \ldots$$

$$G^{\mu_{n-2}}_{\mu_{n-1}}(x_{n-2}, x_{n-1}) G^k_{\mu_{n-1}}(z, x_{n-1})$$

(A.4)
Now, we lower and rise the summed indices $\mu_1 \ldots \mu_{n-1}$ in eq. (A.4) as follows:

$$G_{\mu_1}^\rho(x_1, y) \rightarrow G_{\mu_1}^{\mu_1 \rho}(x_1, y) \quad G_{\mu_1}^{\mu_1 \rho}(x_i, x_{i+1}) \rightarrow G_{\mu_i}^{\mu_1 \rho}(x_i, x_{i+1})$$

$$G_{\mu_{n-1}}^{k \mu_{n-1}}(z, x_{n-1}) \rightarrow G_{\mu_{n-1}}^{k \mu_{n-1}}(z, x_{n-1})$$

Finally, we change the name of the variables setting $x_{n-i} = x_i$ and $x_i = x_{n-i}$. Combining these two actions together, we have:

$$M_{\rho k}^\rho(y, z) \bigg|_{n-th \ order} = \int_M d^2 x_1 \ldots d^2 x_{n-1} \sqrt{g(x_1)} \ldots \sqrt{g(x_{n-1})} G_{\mu_1}^k(z, x_1) G_{\mu_2}^{\mu_1 \mu_2}(x_1, x_2) \ldots$$

$$G_{\mu_{n-2}}^{\mu_{n-2}}(x_{n-2}, x_{n-1}) G_{\mu_{n-1}}^{\mu_{n-1} \rho}(x_{n-1}, z) \equiv M_{\rho k}^\rho(y, z) \bigg|_{n-th \ order}$$

This concludes our proof of eq. (2.37).

**Appendix B.**

In this appendix we derive the partition function $Z_0$ given in eq. (2.40). To this purpose (see [23]), we consider the following functional:

$$Z_0(\beta) = \exp \left[ -\beta \int_M d^2 x d^2 y \sqrt{g(x)} \sqrt{g(y)} G_{\mu \nu}(x, y) \frac{\delta^2}{\delta B_{\mu}(x) \delta B_{\nu}(y)} \right] \times$$

$$\exp \left[ -\frac{\epsilon^2}{2\pi} \int_M d^2 x \sqrt{g(x)} B_{\mu} B^\mu \right] \bigg|_{B_{\mu}=0}$$

After a simple calculation we find:

$$\frac{d}{d\beta} Z_0(\beta) = - \left[ \int_M d^2 x d^2 y \sqrt{g(x)} \sqrt{g(y)} G_{\mu \nu}(x, y) \frac{\delta}{\delta B_{\mu}(x) \delta B_{\nu}(y)} \right] \bigg|_{B_{\mu}=0}$$

Now we exploit eq. (2.38) in order to evaluate the functional $\frac{\delta Z_0(\beta)}{\delta B_{\nu}(y)}$. Substituting the result in the above equation and recalling the fact that at the end we have to put $B_{\mu} = 0$ everywhere, we obtain:

$$\frac{d}{d\beta} \log Z_0(\beta) = -\frac{\epsilon^2}{\pi} \int_M d^2 x d^2 y \sqrt{g(x)} \sqrt{g(y)} G_{\mu \nu}(x, y) M_{\rho_k}^{\mu \nu}(y, x)$$

(B.1)

In the above equation $M_{\rho_k}^{\mu \nu}(y, x)$ is defined as in eq. (2.36) after the substitution $\frac{\epsilon^2}{2\pi} \rightarrow \frac{\epsilon^2 \beta}{2\pi}$. Integrating both sides of eq. (B.1) in $\beta$ between 0 and 1 we have:

$$Z_0(\beta = 1) = Z_0(\beta = 0) \exp \left[ -\frac{\epsilon^2}{\pi} \int_M d^2 x d^2 y \sqrt{g(x)} \sqrt{g(y)} G_{\mu \nu}(x, y) \int_0^1 d\beta M_{\rho_k}^{\mu \nu}(y, x) \right]$$

Since the primitive in $\beta$ of $M_{\rho_k}^{\mu \nu}(x, y)$ is zero when $\beta = 0$, it is clear that $Z_0(\beta = 0) = 1$. Remembering that $Z_0(\beta = 1) \equiv Z_0$ we obtain eq. (2.41).
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