Exotic twisted equivariant K-theory

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\textbf{A B S T R A C T}

In this paper we introduce exotic twisted $\mathbb{T}$-equivariant K-theory of loop space $LZ$ depending on the (typically non-flat) holonomy line bundle $L^B$ on $LZ$ of a gerbe with connection on $Z$. We define an exotic twisted $\mathbb{T}$-equivariant Chern character on the exotic twisted $\mathbb{T}$-equivariant K-theory of $LZ$ that maps to the exotic twisted $\mathbb{T}$-equivariant cohomology of $LZ$ as previously defined in Han and Mathai (2015).

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0. Introduction

In [13], we introduced exotic twisted $\mathbb{T}$-equivariant cohomology for the loop space $LZ$ of a smooth manifold $Z$ via the invariant differential forms on $LZ$ with coefficients in the (typically non-flat) holonomy line bundle $L^B$ of a gerbe with connection, with differential an equivariantly flat superconnection $\nabla L^B = \iota_K + \bar{H}$ in the sense of [18,20] (c.f. Section 7.1 in [1]), where $K$ is the rotation vector field on $LZ$ and $H$ is a degree 3 circle-invariant form on $LZ$ that is completely determined by $H$, the curvature of the gerbe, cf. [13].

This exotic twisted $\mathbb{T}$-equivariant cohomology theory has two applications.

Firstly, we introduced in [13] the twisted Bismut-Chern character form, generalizing [2], which is a loop space refinement of the twisted Chern character form in [4] and represents classes in the completed periodic exotic twisted $\mathbb{T}$-equivariant cohomology $h^\bullet_{\mathbb{T}}(LZ, \nabla L^B : \bar{H})$ of $LZ$. See also [11,12,16,21,22] for other interesting interpretations and extensions of the Bismut-Chern character. More precisely, we define these in such a way that the following diagram commutes,

\begin{equation}
\begin{array}{ccc}
K^\bullet(Z, H) & \xrightarrow{BCH_H} & h^\bullet_{\mathbb{T}}(LZ, \nabla L^B : \bar{H}) \\
\downarrow Ch_H & & \downarrow \text{res} \\
H^\bullet(\Omega(Z)[[u, u^{-1}]], d + u^{-1}H) & & \end{array}
\end{equation}

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where res is the localization map, degree(\(u\)) = 2.

Secondly, in [13] we established a localization theorem (about the map res) for the completed periodic exotic twisted \(\mathbb{T}\)-equivariant cohomology for loop spaces and apply it to establish \(\mathbb{T}\)-duality in a background flux in type II String Theory from a loop space perspective. Continuing along these lines, we recently used in [14] the exotic twisted \(\mathbb{T}\)-equivariant cohomology to enhance \(\mathbb{T}\)-duality on twisted differential forms on circle bundles, where we also showed that \(\mathbb{T}\)-duality exchanges winding and momentum in a background flux for the first time in the model of [5,6]. For an alternate approach to \(\mathbb{T}\)-duality on loop space using the twisted chiral de Rham cohomology instead, see [17]. See [7] for a review of \(\mathbb{T}\)-duality.

There are several approaches in the literature to the K-theory of loop spaces, and we mention two of them here. The first is [9], who considers Virasoro equivariant (infinite dimensional) vector bundles \(E\) over loop space such that the restriction to the constant loops \(E \big|_M\) decomposes as a direct sum \(\bigoplus_n E_n\) under the action of the infinitesimal generator of the rotation group, where each \(E_n\) is a finite rank vector bundle and \(E_n = 0\) for \(n < n_0\) for some \(n_0\). The second is related to Chas–Sullivan string topology, cf. [15].

In this paper, we introduce exotic twisted \(\mathbb{T}\)-equivariant K-theory, \(K^0_n(L\mathbb{Z}, \nabla^{c_b} : \mathcal{G})\), for the loop space \(L\mathbb{Z}\), where \(\mathcal{G}\) is the weak \(\mathbb{T}\)-invariant gerbe on \(L\mathbb{Z}\) whose Dixmier–Douady class is \(\bar{H}\). We also define the exotic twisted \(\mathbb{T}\)-equivariant Chern character,

\[
Ch_{\nabla^{c_b} : \mathcal{G}} : K^0_n(L\mathbb{Z}, \nabla^{c_b} : \mathcal{G}) \longrightarrow h^\text{even}_n(L\mathbb{Z}, \nabla^{c_b} : \bar{H})
\]

that makes the following diagram commute along the solid arrows (see Remark 3.5):

\[
\begin{array}{ccc}
K^\bullet(Z, H) & \xrightarrow{res} & h^\text{even}_\bullet(L\mathbb{Z}, \nabla^{c_b} : \bar{H}) \\
\downarrow Ch_{\nabla^{c_b} : \mathcal{G}} & & \downarrow h_{\text{res}}^\bullet(L\mathbb{Z}, \nabla^{c_b} : \bar{H}) \\
H^\bullet(\Omega(Z)[u, u^{-1}], d + u^{-1}H) & \xrightarrow{BCh_H} & K^\bullet(Z, H)
\end{array}
\]  

It follows that the exotic twisted \(\mathbb{T}\)-equivariant K-theory is the correct version of K-theory that corresponds via a Chern character map to the exotic twisted \(\mathbb{T}\)-equivariant cohomology as defined in [13]. However we would like to point out that the map \(BCh_H\) does not make the upper triangle of Diagram (0.2) commutative (see Remark 3.6).

The plan of this paper is as follows.

In Section 1, we introduce the concept of weak \(\mathbb{T}\)-invariant gerbes and study the coupling of them to \(\mathbb{T}\)-equivariant line bundles on possibly infinite dimensional good \(\mathbb{T}\)-manifolds. A pair consisting of coupled weak \(\mathbb{T}\)-invariant gerbe and \(\mathbb{T}\)-equivariant line bundle will be the initial input data for an exotic twisted \(\mathbb{T}\)-equivariant K-theory (see Section 3).

In Section 2, we establish the correspondence between the exotic twisted \(\mathbb{T}\)-equivariant cohomology, concerning differential forms on \(M\) with coefficients in a complex line bundle \(\xi\), and certain cohomology theory concerning differential forms on \(S\xi\), the circle bundle of \(\xi\) over \(M\) (see Theorem 2.3). Such a transition from \(M\) to \(S\xi\) is crucial: when we attempted to develop the exotic twisted \(\mathbb{T}\)-equivariant K-theory, we realized that it is difficult to define it on \(M\) itself, instead one needs to work on the circle bundle \(S\xi\). The circle bundle is much larger than \(M\) and allows us more room to construct the correct K-theory, which possesses a Chern character landing into the exotic twisted \(\mathbb{T}\)-equivariant cohomology.

In Section 3, we introduce exotic twisted \(\mathbb{T}\)-equivariant K-theory for possibly infinite dimensional \(\mathbb{T}\)-manifolds, and the exotic twisted \(\mathbb{T}\)-equivariant Chern character that lands into exotic twisted \(\mathbb{T}\)-equivariant cohomology. We also establish the transgression formulae in this context, using a new version of Chern–Simons forms. The odd degree analogue of the theory is also established in this section.

1. Coupling of \(\mathbb{T}\)-equivariant line bundles and weak \(\mathbb{T}\)-invariant gerbes

Let \(M\) be a (possibly infinite dimensional) \(\mathbb{T}\)-manifold. We call \(M\) a good \(\mathbb{T}\)-manifold if \(M\) has an open cover \(\{U_a\}\) such that all finite intersections \(U_{a_0a_1 \cdots a_p} = U_{a_0} \cap U_{a_1} \cdots U_{a_p}\) have trivial \(\mathbb{T}\)-equivariant homotopy groups, for \(j = 0\) and \(j \geq 2\).

Let \(K\) be the Killing vector field of the \(\mathbb{T}\)-action. Denote by \(L_K, \iota_K\) the Lie derivative and contraction along the direction \(K\) respectively.

**Definition 1.1.** The system \((\{U_a\}, H, B_a, A_{\alpha\beta})\) is called a gerbe on \(M\), if

\[
H \in \Omega^3(M), \quad B_a \in \Omega^2(U_a), \quad A_{\alpha\beta} \in \Omega^1(U_{\alpha\beta}),
\]

such that \(\frac{1}{2\pi i} H\) has integral period,

\[
H = dB_a \quad \text{on} \quad U_a,
\]

\[
B_a - B_{\beta} = dA_{\alpha\beta} \quad \text{on} \quad U_{\alpha\beta},
\]

(1.1)
and there exist \( C_{\beta \gamma} \in C^\infty(U_{\beta \gamma}, U(1)) \) such that
\[
A_{\alpha \beta} + A_{\beta \gamma} - A_{\alpha \gamma} = d \ln C_{\beta \gamma}.
\]

It is easy to see that different choices of \( C_{\beta \gamma} \) differ by a \( U(1) \)-valued constant scalar on each connected component of \( U_{\alpha \beta} \).

**Remark 1.2.** Our definition of a gerbe here is slightly more general than the gerbe in the usual sense. We do not require \( C_{\beta \gamma}^{-1} C_{\alpha \beta} C_{\alpha \gamma}^{-1} = 1 \) on each nonempty intersection \( U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \).

**Definition 1.3.** A gerbe \( ([U], H, B_\alpha, A_\alpha) \) is called a **weak** \( \mathbb{T} \)-**invariant gerbe** on \( M \) if
(i) \( H, B_\alpha, A_\alpha \) are all \( \mathbb{T} \)-invariant;
(ii) \( i_K A_{\alpha \beta} + i_K A_{\beta \gamma} - i_K A_{\alpha \gamma} \) takes values in \( 2\pi i \cdot \mathbb{Z} \) on each connected component of \( U_{\alpha \beta} \).

**Remark 1.4.** The second condition is equivalent to
\[
L_K C_{\alpha \beta} = 2\pi i n C_{\alpha \beta}
\]
for some \( n \in \mathbb{Z} \) on each connected component of \( U_{\alpha \beta} \). Actually we have
\[
i_K A_{\alpha \beta} + i_K A_{\beta \gamma} - i_K A_{\alpha \gamma} = i_K \left( C_{\alpha \beta}^{-1} d C_{\alpha \beta} \right) = C_{\alpha \beta}^{-1} i_K d C_{\alpha \beta} = C_{\alpha \beta}^{-1} i_K C_{\alpha \beta}.
\]
If all the \( n \) is equal to 0, i.e. \( C_{\alpha \beta} \)'s are \( \mathbb{T} \)-invariant, we call it a **\( \mathbb{T} \)-invariant gerbe**.

Let \( \xi \) be a \( \mathbb{T} \)-equivariant complex line bundle over \( M \) equipped with a \( \mathbb{T} \)-invariant connection \( \nabla^\xi \).

**Definition 1.5.** The \( \mathbb{T} \)-equivariant line bundle \( (\xi, \nabla^\xi) \) and the weak \( \mathbb{T} \)-invariant gerbe \( ([U], H, B_\alpha, A_\alpha) \) are said to be **coupled** on \( M \) if under some local basis \( \{ s_\alpha \} \) of \( \xi|_{U_\alpha} \),
(i) \( -i_K B_\alpha \) is the connection 1-form of \( \nabla^\xi \) on \( U_\alpha \) for each \( \alpha \);
(ii) \( e^{-i_K A_\alpha} \) is the transition function of \( \xi \) on \( U_\alpha \) for each \( \alpha, \beta \).

**Lemma 1.6.** If the \( \mathbb{T} \)-equivariant line bundle \( (\xi, \nabla^\xi) \) and the weak \( \mathbb{T} \)-invariant gerbe \( ([U], H, B_\alpha, A_\alpha) \) are coupled on \( M \), then the equivariant super connection \( \nabla^\xi = u_K + u^{-1} H \) on \( \xi \) is equivalently flat, i.e.
\[
(\nabla^\xi - u_K + u^{-1} H)^2 + u_K^2 = 0.
\]

**Proof.** The proof is similar to the proof of Lemma 1 in [13]. \( \square \)

We provide some examples of coupled \( \mathbb{T} \)-equivariant line bundles and weak \( \mathbb{T} \)-invariant gerbes.

**Example 1.** Let \( Z \) be a smooth manifold. Let \( \{ U_\alpha \} \) be a *Brylinski open cover* of \( Z \), i.e. \( \{ U_\alpha \} \) is a maximal open cover of \( Z \) with the property that \( H^i(U_{\alpha i}) = 0 \) for \( i = 2, 3 \) where \( U_{\alpha i} = \bigcap_{\alpha \in \tau} U_\alpha, |\tau| < \infty \). Then the free loop space \( LZ \) is good \( \mathbb{T} \)-manifold with the open cover \( \{ LU_\alpha \} \), where the \( \mathbb{T} \)-action is the loop rotating action.

Let
\[
\tau : \Omega^\bullet(U_{\alpha}) \longrightarrow \Omega^{\bullet-1}(LU_{\alpha})
\]
be the transgression map
\[
\tau(\xi) = \int_Z ev^\ast(\xi), \quad \xi \in \Omega^\bullet(U_{\alpha}).
\]

Here \( ev \) is the evaluation map
\[
ev : \mathbb{T} \times LZ \to Z : (t, \gamma) \to \gamma(t).
\]

Let \( \omega \in \Omega^i(Z) \). Define \( \hat{\omega}_s \in \Omega^i(LZ) \) for \( s \in [0, 1] \) by
\[
\hat{\omega}_s(X_1, \ldots, X_i)(\gamma) = \omega(X_1|_{\gamma(s)} \ldots X_i|_{\gamma(s)})
\]
for \( \gamma \in LZ \) and \( X_1, \ldots, X_i \) vector fields on \( LZ \) defined near \( \gamma \). Then one checks that \( d\hat{\omega}_s = \hat{d}\omega_s \). The \( i \)-form, averaging \( \omega \) on the loop space,
\[
\bar{\omega} = \int_0^1 \hat{\omega}_s ds \in \Omega^i(LZ)
\]
is \( \mathbb{T} \)-invariant, that is, \( L_K (\bar{\omega}) = 0 \). Moreover \( \tau(\omega) = i_K \bar{\omega} \). We call \( \bar{\omega} \) the average of \( \omega \).
Let \( (\{U_a\}, H, B_a, A_{\alpha \beta}) \) be a gerbe on \( Z \). Associated to this gerbe, there exists a pair of coupled \( \mathbb{T} \)-equivariant line bundle and weak \( \mathbb{T} \)-invariant gerbe on \( LZ \).

The holonomy of this gerbe is a \( \mathbb{T} \)-equivariant line bundle \( \mathcal{L}^B \rightarrow LZ \) over the loop space \( LZ \), whose construction is detailed in Section 6.2.1 in [10]. \( \mathcal{L}^B \) has \( \mathbb{T} \)-invariant local sections \( \{\sigma_{\alpha}\} \) with respect to \( \{LU_a\} \) such that the transition functions are \( \{e^{-\int_0^1 \chi A_{\alpha \beta}} = e^{-\tau(B_a)}\} \), i.e. \( \sigma_{\alpha} = e^{-\int_0^1 \chi h_{\alpha a}} \alpha \beta \). \( \mathcal{L}^B \) comes with a natural connection, whose definition with respect to the open cover \( \{LU_a\} \) is

\[
\nabla^B \theta = d - i_k \tilde{B}_a = d - \tau(B_a). \tag{1.8}
\]

For more details, cf. 6.2 in [10].

On the other hand, averaging the gerbe \( (\{U_a\}, H, B_a, A_{\alpha \beta}) \) gives rise to a gerbe

\[: (\{LU_a\}, H, \hat{B}_a, \hat{A}_{\alpha \beta}) \]

on \( LZ \). First it is not hard to see that \( \frac{1}{2\pi i} \bar{H} \) still has integral periods. It is evident that

\[
\bar{H} = d\tilde{B}_a \quad \text{on} \quad LU_a,
\]

\[
\bar{B}_a - \tilde{B}_\alpha = d\hat{A}_{\alpha a} \quad \text{on} \quad LU_{\alpha a}.
\]

If on \( U_{\alpha \beta y} \),

\[
A_{\alpha \beta} + A_{\beta y} - A_{\alpha y} = d\ln c_{\alpha \beta y},
\]

then

\[
t_k \hat{A}_{\alpha a} + i_k \hat{A}_{\beta y} - i_k \hat{A}_{\alpha y} = \tau(d\ln c_{\alpha \beta y}) \in 2\pi i\mathbb{Z} \quad \text{on each connected component of} \quad LU_{\alpha \beta y}. \tag{1.11}
\]

By (1.11), if \( x_0 \) is a fixed loop in \( U_{\alpha \beta y} \) and \( x \) is any loop in \( U_{\alpha \beta y} \), then

\[
e^{-\int_0^1 \chi(\hat{A}_{\alpha a} + \hat{A}_{\beta y} - \hat{A}_{\alpha y})} \quad \text{pure imaginary. Then we further have}
\]

\[
\hat{A}_{\alpha a} + \hat{A}_{\beta y} - \hat{A}_{\alpha y} = \tau(d\ln e^{-\int_0^1 \chi(\hat{A}_{\alpha a} + \hat{A}_{\beta y} - \hat{A}_{\alpha y})}), \tag{1.13}
\]

where \( e^{\int_0^1 \chi(\hat{A}_{\alpha a} + \hat{A}_{\beta y} - \hat{A}_{\alpha y})} \) is an \( U(1) \)-valued function on \( LU_{\alpha \beta y} \). Therefore \( \{LU_a\}, \bar{H}, \bar{B}_a, \hat{A}_{\alpha \beta} \) is a gerbe on \( LZ \).

It is obvious that \( H, \bar{B}_a, \hat{A}_{\alpha \beta} \) are all \( \mathbb{T} \)-invariant. Combining (1.11), we see that the gerbe \( (\{LU_a\}, H, \bar{B}_a, \hat{A}_{\alpha \beta}) \) is a weak \( \mathbb{T} \)-invariant gerbe on \( LZ \).

As with the \( \mathbb{T} \)-invariant local sections, the local connection 1-form of \( (\mathcal{L}^B, \nabla^\mathcal{L}^B) \) is

\[
-\tau(B_a) = -i_k \hat{B}_a,
\]

and the transition function of \( \mathcal{L}^B \) is

\[
e^{-\int_0^1 \chi A_{\alpha \beta}} = e^{-\chi \hat{A}_{\alpha \beta}},
\]

we see that \( (\mathcal{L}^B, \nabla^\mathcal{L}^B) \) and \( (\{LU_a\}, \bar{H}, \bar{B}_a, \hat{A}_{\alpha \beta}) \) are coupled on \( LZ \).

**Example 2.** In [5,6], T-duality in a background flux has the following settings. There is a principal circle bundle \( \mathbb{T} \rightarrow Z \rightarrow X \) with a \( \mathbb{T} \)-invariant connection \( \hat{\theta} \) and a background \( \mathbb{T} \)-invariant flux \( H \), which is a \( \mathbb{T} \)-invariant closed 3-form on \( Z \). Let \( \{U_a\} \) be a good cover of \( X \). The cover \( \{\pi^{-1}(U_a)\} \) makes \( Z \) a good \( \mathbb{T} \)-manifold.

The T-dual space \( \hat{\mathbb{T}} \rightarrow \hat{Z} \rightarrow X \) is a principal circle bundle with a \( \hat{\mathbb{T}} \)-invariant connection \( \hat{\theta} \) and a background \( \hat{\mathbb{T}} \)-invariant flux \( \hat{H} \). The cover \( \{\hat{\pi}^{-1}(U_a)\} \) makes \( \hat{Z} \) a good \( \hat{\mathbb{T}} \)-manifold.

Denote \( v, \hat{v} \) the Killing vector field on \( Z, \hat{Z} \) respectively. The gerbe \( (\{\pi^{-1}(U_a)\}, H, B_a, A_{\alpha \beta}) \) on \( Z \) and the gerbe \( (\{\hat{\pi}^{-1}(U_a)\}, \hat{H}, \hat{B}_a, \hat{A}_{\alpha \beta}) \) on \( \hat{Z} \) satisfy the following relations

\[
e^{-i_v A_{\alpha \beta}} = \hat{g}_\alpha \beta, \quad -i_v B_a = \hat{\eta}_a, \quad i_v H = F^{\hat{\theta}} \tag{1.14}
\]

and

\[
e^{-i_{\hat{v}} \hat{A}_{\alpha \beta}} = g_\alpha \beta, \quad -i_{\hat{v}} \hat{B}_a = \eta_a, \quad i_{\hat{v}} \hat{H} = \hat{F}^{\hat{\theta}}, \tag{1.15}
\]

where \( \hat{g}_\alpha \beta \) is the transition functions of the bundle \( \hat{Z}, \hat{\eta}_a \) is the local connection 1-form of \( \hat{\theta} \) on \( U_a \), \( \hat{F}^{\hat{\theta}} \) is the curvature 2-form of \( \hat{\theta} \) on \( X \) and the similar meaning for the notations without hats on the dual side.
In the setting, $B_a, A_{αβ}$ are all chosen to be $T$-invariant. Moreover as $e^{-i_A u_β} = \hat{g}_{α_β}$, we conclude that $i_v A_{αβ} + i_v A_{μβ} - i_v A_{αγ}$ takes values in $2πi \in \mathbb{Z}$ on each $U_{αβγ}$. Therefore $(π^{-1}(U_a)), H, B_a, A_{αβ}$ is a weak $T$-invariant gerbe on $Z$. Similarly $(\hat{π}^{-1}(U_a), \hat{H}, \hat{B}_a, \hat{A}_{αβ})$ is a weak $\hat{T}$-invariant gerbe on $\hat{Z}$.

$(Z, \hat{Θ})$ and the standard representation of the circle on complex plane give rise to a complex line bundle with connection $(\xi, \nabla^ξ)$ on $X$. Dually, there is a similar $(ξ, \nabla^ξ)$ on $X$ coming from $(Z, Θ)$. As

$$e^{-i_A u_β} = \hat{g}_{α_β}, -i_v B_a = \hat{η}_a,$$

the $T$-equivariant line bundle $(π^*ξ, π^*\nabla^ξ)$ and the weak $T$-equivariant gerbe $(\{π^{-1}(U_a)\}, H, B_a, A_{αβ})$ are coupled on $Z$. Dually, the $\hat{T}$-equivariant line bundle $(\hat{π}^*ξ, \hat{π}^*\nabla^ξ)$ and the $\hat{T}$-invariant gerbe $(\{\hat{π}^{-1}(U_a)\}, \hat{H}, \hat{B}_a, \hat{A}_{αβ})$ are coupled on $\hat{Z}$.

2. Exotic twisted equivariant cohomology and $U(1)$-bundles

Let $M$ be a good $T$-manifold. Let $ξ → M$ be a $T$-equivariant Hermitian line bundle over $M$ equipped with a $T$-invariant Hermitian connection $\nabla^ξ$. Let $H ∈ Ω^3(M)$ be a $T$-invariant closed 3-form such that the equivariant superconnection $\nabla^ξ - u_K + u^{-1}H$ is equivariantly flat, i.e.

$$(\nabla^ξ - u_K + u^{-1}H)^2 + u_μ^ξ = 0,$$

where $u$ is a degree 2 indeterminate.

In the previous section, we have seen examples that satisfy these settings.

Let $π : Sξ → M$ be the principal $U(1)$-bundle of $ξ$. Let $v$ be the vertical tangent vector field on $Sξ$, i.e. the Killing vector field of the $U(1)$-action. It is clear that $Sξ$ also admits the induced $T$-action. As the action of $T$ on the fibres of $ξ$ is linear, i.e. $g(λ ⋅ v) = λ ⋅ g(v), ∀g ∈ T, λ ∈ U(1)$, one deduces that the $T$-action and the $U(1)$-action commute. Therefore we have

$$[K, v] = 0.$$ (2.2)

The condition $(\nabla^ξ - u_K + u^{-1}H)^2 + u_μ^ξ = 0$ is equivalent to the following three equalities,

$$\begin{cases} μ^ξ_K = L^ξ_K - [\nabla^ξ, u_K] = L^ξ_K - \nabla^ξ_K = 0 \\
(\nabla^ξ)^2 - u_K H = 0 \\
dH = 0 \end{cases}$$ (2.3)

Let $Θ$ be the connection 1-form on $Sξ$ for $(ξ, \nabla^ξ)$.

**Lemma 2.1.**

$$i_K Θ = 0, \ L_K Θ = 0$$ (2.4)

and

$$dΘ = i_K π^* H.$$ (2.5)

**Proof.** Let $\{U_a\}$ be a $T$-cover of $M$. Choose a $T$-invariant local basis $s_α$ of $ξ$ on $U_a$. Let $η_α$ be the connection 1-form corresponding to $s_α$. By the first relation in (2.3), we have

$$0 = μ^ξ_K(s_α) = (L^ξ_K - [\nabla^ξ, i_K]) s_α = (i_K η_α) ⊗ s_α,$$

and therefore we have

$$i_K η_α = 0.$$ (2.6)

As $s_α$ is $T$-invariant, we get a local $T$-equivariant diffeomorphism $φ_a : U_a × S^1 → \pi^{-1}(U_a)$ such that on the left hand side, $T$ only acts on $U_a$. Then as $φ_a^*(Θ)|_{U_a × S^1} = η_α + dΘ$, we deduce that

$$i_K Θ = 0, \ L_K Θ = 0.$$

By the second relation in (2.3), we get

$$dΘ + \frac{1}{2} Θ^2 - i_K π^* H = 0$$

or

$$dΘ = i_K π^* H.$$

Consider the $C^∞(M)$-module

$$\hat{Ω}^3(Sξ) := \{ω ∈ Ω^3(Sξ)| i_ω = 0, L_ω = -ω\}.$$ (2.7)
Theorem 2.2.

\[ \tilde{\Omega}^*(S\xi)^*[[u, u^{-1}], d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^*H] \]

is a chain complex.

**Proof.** We need to show that:

(i) if \( \omega \in \tilde{\Omega}^*(S\xi)^* \), then

\[ (d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^*H)\omega \in \tilde{\Omega}^*(S\xi)^*; \]

(ii) \( (d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^*H)^2 + uL_K = 0. \)

(i) holds as we have following three equalities,

\[
\begin{align*}
[d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^*H, \iota_v] = & L_v - [\iota_v, \iota_v] - u\iota_K + \iota_v\Theta + u^{-1}\iota_v(\pi^*H) \\
= & L_v + \iota_v\Theta \\
= & 0 \text{ on } \tilde{\Omega}^*(S\xi); \\
[d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^*H, L_v] = & [d, L_v] - \iota_v([\iota_v, L_v] + \iota_vL_v + \Theta + u^{-1}L_v(\pi^*H)) \\
= & 0;
\end{align*}
\]

and

\[
\begin{align*}
[d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^*H, L_K] = & [d, L_K] - \iota_v([\iota_v, L_K] + L_K + \Theta + u^{-1}L_K(\pi^*H)) \\
= & 0.
\end{align*}
\]

To show (ii), we have

\[
\begin{align*}
(d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^*H)^2 = & (d - \iota_v - u\iota_K)^2 + (d - \iota_v - u\iota_K)(\Theta + u^{-1}\pi^*H) + (\Theta + u^{-1}\pi^*H)^2 \\
= & -L_v - uL_K + d\Theta - \iota_v\Theta - \pi^*\iota_vH \\
= & -(L_v - \iota_v\Theta) + (d\Theta - \iota_v\pi^*H) - uL_K \\
= & -uL_K \text{ on } \tilde{\Omega}^*(S\xi). \quad \square
\end{align*}
\]

Let \( \pi^*\xi \) be the pull back bundle of \( \xi \) on \( S\xi \). Clearly this is a trivial bundle which has a canonical global nowhere vanishing section

\[ \gamma : (x, y) \rightarrow y, \quad x \in M, y \in \pi^{-1}(x). \]

Consider the map

\[ f : \Omega^*(M, \xi) \rightarrow \Omega^*(S\xi), \quad \omega \mapsto \gamma^{-1} \cdot \pi^*\omega. \]

Let \( \{U_a\} \) be an \( \mathbb{T} \)-cover of \( M \). Let \( s_a \) be a \( \mathbb{T} \)-invariant local basis of the \( \xi \) on \( U_a \). Suppose \( \omega|_{U_a} = \omega_a \otimes s_a \). Then on \( \pi^{-1}U_a \cong U_a \times S^1 \),

\[
\pi^*\omega = \pi^*(\omega_a) \otimes \pi^*(s_a), \quad \gamma = z \cdot \pi^*(s_a), \quad v = z \cdot \partial_z,
\]

where \( z \) is the complex coordinate on \( S^1 \). Therefore on \( \pi^{-1}U_a \cong U_a \times S^1 \),

\[
\gamma^{-1} \cdot \pi^*\omega = z^{-1} \pi^*(\omega_a)
\]

and

\[
\iota_v(\gamma^{-1} \cdot \pi^*\omega) = 0, \quad L_v(\gamma^{-1} \cdot \pi^*\omega) = -\gamma^{-1} \cdot \pi^*\omega.
\]

Hence we see that

\[ \text{Im}(f) = \tilde{\Omega}^*(S\xi), \quad \ker(f) = \{0\} \]

and therefore get an isomorphism of \( C^\infty(M) \)-modules:

\[ f : \Omega^*(M, \xi) \rightarrow \tilde{\Omega}^*(S\xi). \]
Proof. Let \( f : \Omega^*(M, \xi)^\tau \rightarrow \tilde{\Omega}^*(S\xi)^\tau \). Hence we get an isomorphism of \( C^\infty(M) \)-modules, which we still denote by \( f \):

\[
f : \Omega^*(M, \xi)^\tau \rightarrow \tilde{\Omega}^*(S\xi)^\tau.
\]  

(2.14)

**Theorem 2.3.**

\[
f^* : h^*_\tau(M, \nabla^\xi : H) \rightarrow H^* (\tilde{\Omega}^*(S\xi)^\tau[[u, u^{-1}]], d - \iota_v - u_K + \Theta + u^{-1}\pi^*H) \]

is a chain map and induces an isomorphism on cohomology

\[
f^* : h^*_\tau(M, \nabla^\xi : H) \rightarrow H^* (\tilde{\Omega}^*(S\xi)^\tau[[u, u^{-1}]], d - \iota_v - u_K + \Theta + u^{-1}\pi^*H),
\]

(2.16)

where \( h^*_\tau(M, \nabla^\xi : H) \) is the completed periodic exotic twisted \( \tau \)-equivariant cohomology [13].

3. Exotic twisted equivariant \( K \)-theory and the Chern character

3.1. Gerbe modules and twisted \( K \)-theories

A geometric realization of the gerbe \( \mathcal{G} = (\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta}) \) on \( M \) is \((\{L_{\alpha\beta}, \nabla_{\alpha\beta}^\xi\})\), a collection of trivial line bundles \( L_{\alpha\beta} \rightarrow U_{\alpha\beta} \) with connections \( \nabla_{\alpha\beta}^\xi = d + A_{\alpha\beta} \) such that on \( U_{\alpha\beta\gamma} \) there are connection preserving isomorphisms

\[
L_{\alpha\beta} \otimes L_{\beta\gamma} \cong L_{\alpha\gamma}.
\]  

(3.1)
Note that as here we are using slightly more general version of gerbe (see Definition 1.1 and Remark 1.2), the isomorphisms $L_{αβ} ⊗ L_{ργ} ≅ L_{αγ}$ are not uniquely fixed, but may differ by a multiplication by a locally constant $U(1)$-valued scalar. Then we have

\[(\nabla_{αβ}^L)^2 = F^L_{αβ} = B_β - B_α.\] (3.2)

Let $E = \{E_α\}$ be a collection of (infinite dimensional) Hilbert bundles $E_α \to U_α$ whose structure group is reduced to $U_1$, which are unitary operators on the model Hilbert space $\mathcal{H}$ of the form identity + trace class operator. Here $\mathcal{H}$ denotes the Lie algebra of trace class operators on $\mathcal{H}$. In addition, assume that on the overlaps $U_{αβ}$ that there are isomorphisms

\[\phi_{αβ} : L_{αβ} \otimes E_β \cong E_α,\] (3.3)

which are consistently defined on triple overlaps because of the gerbe property (3.1). More precisely, one has

\[(L_{αβ} \otimes L_{ργ}) \otimes E_γ \cong L_{αγ} \otimes E_γ \cong E_α,\] (3.4)

and

\[L_{αβ} \otimes (L_{ργ} \otimes E_γ) \cong L_{αρ} \otimes E_β \cong E_α.\] (3.5)

Then $\{E_α\}$ is said to be a gerbe module for the gerbe $\{L_{αβ}\}$. A gerbe module connection $\nabla^E$ is a collection of connections $\{\nabla^E_α\}$ of the form $\nabla^E_α = d + A_α^E$ where $A_α^E \in \Omega^1(U_α) \otimes \mathcal{H}$ whose curvature $F^E_α$ on the overlaps $U_{αβ}$ satisfies

\[\phi_{αβ}^{-1}(F^E_α)\phi_{αβ} = F^L_{αβ} + F^E_β.\] (3.6)

Using Eq. (3.2), this becomes

\[\phi_{αβ}^{-1}(B_β + F^E_β)\phi_{αβ} = B_α + F^E_α.\] (3.7)

It follows that $\exp(-B)\ Tr(\exp(-F^E) - I)$ is a globally well defined differential form on $M$ of even degree. Notice that $Tr(I) = ∞$ which is why we need to consider the subtraction.

Let $E = \{E_α\}$ and $E' = \{E'_α\}$ be a gerbe modules for the gerbe $\{L_{αβ}\}$. Then an element of twisted K-theory $K^0(M, G)$ is represented by the pair $(E, E')$. Two such pairs $(E, E')$ and $(G, G')$ are equivalent if $E \oplus G' \oplus K \cong E' \oplus G \oplus K$ as gerbe modules for some gerbe module $K$ for the gerbe $\{L_{αβ}\}$. We can assume without loss of generality that these gerbe modules $E, E'$ are modelled on the same Hilbert space $\mathcal{H}$ after a choice of isomorphism if necessary.

Suppose that $\nabla^E, \nabla^{E'}$ are gerbe module connections on the gerbe modules $E, E'$ respectively. Then we can define the twisted Chern character as

\[Ch_H : K^0(M, G) \to H^{even}(M, H),\]

\[Ch_H(E, E') = \exp(-B)Tr\left(\exp(-F^E) - \exp(-F^{E'})\right).\]

That is a well defined homomorphism is explained in [4,19]. To define the twisted Chern character landing in $(\Omega^*(M)[u, u^{-1}])_{(d+u^{-1}H)-cl}$ simply replace the above formula by

\[Ch_H(E, E') = \exp(-u^{-1}B)Tr\left(\exp(-u^{-1}F^E) - \exp(-u^{-1}F^{E'})\right).\]

The above theory can be extended to equivariant setting with a compact group action on all the data [19].

### 3.2. Exotic twisted equivariant K-theory

Let $M$ be a good $\mathbb{T}$-manifold with a $\mathbb{T}$-invariant cover $\{U_α\}$. Let $ξ \to M$ be a $\mathbb{T}$-equivariant Hermitian line bundle over $M$ equipped with a $\mathbb{T}$-invariant Hermitian connection $\nabla^ξ$. Let $π : Sξ \to M$ be the principal $U(1)$-bundle of $ξ$. Let $G = \{U_α, H, B_α, A_α\}$ be a weak $\mathbb{T}$-invariant gerbe on $M$ and $\{L_{αβ}, \nabla^L_{αβ}\}$ a geometrization of $G$. Assume that $ξ$, $\nabla^ξ$ and $\{U_α, H, B_α, A_α\}$ are coupled on $M$. Denote this system by $\{M, G, (ξ, \nabla^ξ)\}$. Associated to the system $\{M, G, (ξ, \nabla^ξ)\}$, we will introduce a version of twisted K-theory and twisted Chern character in this section.

It is clear that the open cover $\{π^{-1}(U_α)\}$ makes $Sξ$ a good $(\mathbb{T} \times U(1))$-manifold. Here to distinguish the two circle actions, we denote by $\mathbb{T}$ the circle acting on the base $M$ and by $U(1)$ the circle acting on the fibres.

Denote $G^\mathbb{F} := \{(π^{-1}(U_α), π^*H, π^*B_α, π^*A_α)\}$, which is a $(\mathbb{T} \times U(1))$-invariant gerbe on $Sξ$. Let $\{L_{αβ}, \nabla^L_{αβ} = d + π^*A_{αβ}\}$ be the system of $(\mathbb{T} \times U(1))$-line bundles with $(\mathbb{T} \times U(1))$-invariant connections on $U_{αβ} \times U(1)$, which is the geometrization of the gerbe $G^\mathbb{F}$.

Let $v$ be the vertical tangent vector field on $Sξ$, i.e. the Killing vector field of the $U(1)$-action. Let $K$ be the Killing vector field of the $\mathbb{T}$-action. Let $u$ be a degree 2 indeterminate.
Definition 3.1. \( E = \{E_{\alpha}, \nabla^{E_{\alpha}}\} \) is called a \((\mathbb{T} \times U(1))\)-equivariant gerbe module with horizontal connection for the gerbe \( \{\hat{L}_{\alpha\beta}\} \) if

(a) the \((\mathbb{T} \times U(1))\)-invariant connections \( \nabla^{E_{\alpha}} \)'s vanish on the vertical direction, i.e. \( \nabla^{E_{\alpha}}_v \equiv 0 \);

(b) there are \((\mathbb{T} \times U(1))\)-equivariant isomorphisms

\[
\phi_{\alpha\beta} : \hat{L}_{\alpha\beta} \otimes E_{\beta} \cong E_{\alpha},
\]

that define a gerbe module and which respect the connections.

Note that the isomorphisms \( \{\phi_{\alpha\beta}\} \) are consistently defined on triple overlaps because of the type \((3.1)\) property of the gerbe \((\hat{L}_{\alpha\beta}, \nabla^{\hat{L}_{\alpha\beta}} = d + \pi^*A_{\alpha\beta})\).

Let \( (E, E') \) and \( (G, G') \) be two pairs of \((\mathbb{T} \times U(1))\)-equivariant gerbe modules with horizontal connections for the gerbe \( \{\hat{L}_{\alpha\beta}\} \). We say they are equivalent, denoted by

\[
(E, E') \sim (G, G'),
\]

if there exists some \( K \), a \((\mathbb{T} \times U(1))\)-equivariant gerbe modules with horizontal connections. Clearly this is an equivalence relation. As usual, we define

\[
\tilde{K}^G_2(M, \nabla^\xi ; \mathcal{G}) := \{(E, \nabla^E, E', \nabla^{E'})]/\sim\}.
\]

If the horizontal gerbe module connections are forgotten, one defines the exotic twisted \( \mathbb{T} \)-equivariant K-theory of \( \{M, G, (\xi, \nabla^\xi)\} \), denoted as \( K^G_2(M, \nabla^\xi ; \mathcal{G}) \), by

\[
K^G_2(M, \nabla^\xi ; \mathcal{G}) := \{(E, E')]/\sim\}.
\]

Let \( E = \{E_{\alpha}, \nabla^{E_{\alpha}}\} \) be a \((\mathbb{T} \times U(1))\)-equivariant gerbe module with horizontal connection for the gerbe \( \{\hat{L}_{\alpha\beta}\} \). For the equivariant curvatures along the direction \( v + uK \), we have

\[
\begin{align*}
\phi_{\alpha\beta}^{-1}(F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}}) & = (F_{\alpha\beta}^E + \mu_{\hat{L}_{\alpha\beta}}) + (F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}}), \\
\text{where } \mu & \text{ stands for the moment. However}
\end{align*}
\]

\[
F_{\alpha\beta} = \pi^*B_{\beta} - \pi^*B_{\alpha},
\]

\[
\hat{L}_{\alpha\beta} = (u_{\alpha\beta} + u_{\alpha}) \pi^*A_{\alpha\beta} = u_{\alpha\beta} \pi^*A_{\alpha\beta} = 2\pi i u_{\alpha\beta} - 2\pi i u_{\alpha\beta},
\]

where \( u_{\alpha\beta} \) (resp. \( \theta_{\alpha\beta} \)) are the vertical coordinates of \( \pi^{-1}(U_{\alpha\beta}) \) (resp. \( \pi^{-1}(U_{\alpha\beta}) \)). So we have

\[
\begin{align*}
\phi_{\alpha\beta}^{-1}(F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}} + \pi^*B_{\alpha} + 2\pi i u_{\alpha\beta}) & = (F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}} + \pi^*B_{\alpha} + 2\pi i u_{\alpha\beta}), \\
\text{Therefore the forms}
\end{align*}
\]

\[
\exp(-u^{-1}\pi^*B_{\alpha} - 2\pi i u_{\alpha\beta}) Tr\left(\exp(-u^{-1}(F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}})) - 1\right)
\]

can be glued together as a global differential form in \( \Omega^*(S\xi)[[u, u^{-1}]] \). Then let \( E' = \{E'_{\alpha}\} \) be another \((\mathbb{T} \times U(1))\)-equivariant gerbe module for the gerbe \( \{\hat{L}_{\alpha\beta}\} \). Similar to \( E = \{E_{\alpha}, \nabla^{E_{\alpha}}\} \), the forms

\[
\exp(-u^{-1}\pi^*B_{\alpha} - 2\pi i u_{\alpha\beta}) Tr\left(\exp(-u^{-1}(F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}})) - 1\right)
\]

can be glued together as a global differential form in \( \Omega^*(S\xi)[[u, u^{-1}]] \). Then we see that

\[
\exp(-u^{-1}\pi^*B_{\alpha} - 2\pi i u_{\alpha\beta}) Tr\left(\exp(-u^{-1}(F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}})) - \exp(-u^{-1}(F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}})) \right)
\]

\[
\text{glues to a global differential form in } \Omega^*(S\xi)[[u, u^{-1}]]. \text{ Simply denote this form by}
\]

\[
ch_{\xi}^{\mathbb{T}; G}(\nabla^E, \nabla^{E'}) = \exp(-u^{-1}\pi^*B - 2\pi i \theta) Tr\left(-\exp(u^{-1}(F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}})) - \exp(-u^{-1}(F_{\alpha\beta}^E + \mu_{E_{\alpha\beta}})) \right).
\]

Theorem 3.2. (i) The following equalities hold,

\[
\begin{align*}
t_vch_{\xi}^{\mathbb{T}; G}(\nabla^E, \nabla^{E'}) & = 0, \quad L_vch_{\xi}^{\mathbb{T}; G}(\nabla^E, \nabla^{E'}) = -ch_{\xi}^{\mathbb{T}; G}(\nabla^E, \nabla^{E'}), \\
(d - t_v - uK \Theta + u^{-1}\pi^*H)ch_{\xi}^{\mathbb{T}; G}(\nabla^E, \nabla^{E'}) & = 0.
\end{align*}
\]
(ii) If $(\nabla^E_0, \nabla^E_1), (\nabla^E, \nabla^E_1)$ are two horizontal gerbe module connections, then there exists $cs(\nabla^E_0, \nabla^E; \nabla^E_1, \nabla^E_1) \in \hat{\omega}^*(S\xi) \cup [u, u^{-1}]$ such that

$$
ch_{\nabla^E}(\nabla^E_0, \nabla^E) - ch_{\nabla^E}(\nabla^E_1, \nabla^E_1) = (d - \tau_v - \omega_K + \Theta + u^{-1}\pi^*H)cs(\nabla^E_0, \nabla^E; \nabla^E_1, \nabla^E_1).
$$

(3.18)

**Proof.** (i) Consider the local expression

$$
ch_{\nabla^E}(\nabla^E, \nabla^E)|_{\pi^{-1}(U_0)} = \exp(-u^{-1}\pi^*B_0 - 2\pi i\theta_0) Tr\left(\exp(-u^{-1}(F_{0\alpha} + \mu_{\alpha uK})) - \exp(-u^{-1}(F_{0\alpha} + \mu_{\alpha uK}))\right).
$$

Obviously, $\tau_v\pi^*B_0 = 0$. On the other hand, as $\nabla^E_0$ is horizontal connection, we have $\nabla_v^E = 0$, but this is equivalent to $[\nabla^E_0, \tau_v] = L_v$.

Therefore

$$
t_v(F^E_0) = [t_v, (\nabla^E_0)^2] = (L_v - \nabla^E_0 t_v)\nabla^E_0 - \nabla^E_0(L_v - t_v\nabla^E_0) = [\nabla^E_0, L_v] = 0,
$$

as $\nabla^E_0$ is $T \times U(1)$-invariant. Similarly, $t_v(F^E_0) = 0$. We therefore have

$$
t_v ch_{\nabla^E}(\nabla^E_0, \nabla^E)|_{\pi^{-1}(U_0)} = 0.
$$

This shows the first equality in (3.16).

As $\nabla^E_0$ is $T \times U(1)$-invariant, clearly $L_v(F^E_0) = 0$. The moment is

$$
\mu_{\alpha uK}^E = L^v_{\alpha uK} - [t_v + \alpha uK, \nabla^E_0].
$$

Since $[v, K] = 0$, it is easy to see that

$$
L_v\mu_{\alpha uK}^E = 0.
$$

Now $L_v\pi^*B_0 = 0$ and $L_v e^{-2\pi i\theta_0} = -e^{2\pi i\theta_0}$, we have

$$
L_v ch_{\nabla^E}(\nabla^E_0, \nabla^E)|_{\pi^{-1}(U_0)} = -ch_{\nabla^E}(\nabla^E_0, \nabla^E)|_{\pi^{-1}(U_0)}.
$$

This shows the second equality in (3.16).

At last, as $(\xi, \nabla^E)$ and $(\{U_0\}, H, B_0, A_{\alpha \beta})$ are coupled on $M$, one has

$$
2\pi i\theta_0 - \pi^*\pi K B_0 = \Theta|_{\pi^{-1}(U_0)},
$$

where $\Theta$ is the connection 1-form on $S\xi$. Hence

$$
(d - \tau_v - \omega_K)ch_{\nabla^E}(\nabla^E_0, \nabla^E)|_{\pi^{-1}(U_0)} =
$$

$$
= ch_{\nabla^E}(\nabla^E_0, \nabla^E)|_{\pi^{-1}(U_0)} =
$$

$$
\left(\exp(-u^{-1}\pi^*_\alpha B_0 - 2\pi i\theta_0) Tr\left(\exp(-u^{-1}(F_{0\alpha} + \mu_{\alpha uK})) - \exp(-u^{-1}(F_{0\alpha} + \mu_{\alpha uK}))\right)\right)
$$

$$
= \left[\exp(-u^{-1}\pi^*_\alpha B_0 - 2\pi i\theta_0)\left(\exp(-u^{-1}(F_{0\alpha} + \mu_{\alpha uK})) - \exp(-u^{-1}(F_{0\alpha} + \mu_{\alpha uK}))\right)\right]
$$

$$
\cdot Tr\left(-\exp(u^{-1}(F_{0\alpha} + \mu_{\alpha uK})) - \exp(-u^{-1}(F_{0\alpha} + \mu_{\alpha uK}))\right)
$$

$$
= \left[\exp(-u^{-1}\pi^*_\alpha H - 2\pi i\theta_0 - \pi^*_\alpha B_0)\right]
$$

$$
\cdot \left[\exp(-u^{-1}\pi^*_\alpha B_0 - 2\pi i\theta_0) Tr\left(-\exp(u^{-1}(F_{0\alpha} + \mu_{\alpha uK})) - \exp(-u^{-1}(F_{0\alpha} + \mu_{\alpha uK}))\right)\right]
$$

$$
= \left(-u^{-1}\pi^*_\alpha H - \Theta\right)ch_{\nabla^E}(\nabla^E_0, \nabla^E)|_{\pi^{-1}(U_0)},
$$

and therefore

$$
(d - \tau_v - \omega_K + \Theta + u^{-1}\pi^*_\alpha H)ch_{\nabla^E}(\nabla^E_0, \nabla^E)|_{\pi^{-1}(U_0)} = 0.
$$

(ii) Let

$$
\nabla^E_t = (1 - t)\nabla^E_0 + t\nabla^E_1, \quad \nabla^E_t = (1 - t)\nabla^E_0 + t\nabla^E_1
$$

and $F^E_t, F^E_\alpha, \mu^E_t, \mu^E_\alpha$ be the corresponding curvatures and momentums. Let

$$
A_{\alpha \beta}^E = \nabla^E_1 - \nabla^E_0, \quad A_{\alpha \beta}^E = \nabla^E_1 - \nabla^E_0,
$$

We have

$$
\phi_{\alpha \beta}^{-1}(u^{-1}(F^E_\alpha + \mu^E_{\alpha uK,t}) - u^{-1}\pi^*_\alpha B_0 - 2\pi i\theta_0)\delta_{\alpha \beta} = -u^{-1}(F^E_t + \mu^E_{t uK,t}) - u^{-1}\pi^*_\alpha B_0 - 2\pi i\theta_0
$$
and
\[ \phi_{a\beta}^{-1}(-u^{-1}A^a)v\phi_{a\beta} = -u^{-1}A^\beta. \]

Similar equalities hold for \( E' \).

Therefore we have
\[
\exp(-u^{-1}\pi^*B_u - 2\pi it) \cdot \int_0^1 \text{Tr} \left( -u^{-1}A^a \exp(-u^{-1}(F^a_1 + \mu^a_{\mu + uK, 1})) + u^{-1}A^a \exp(-u^{-1}(F^a_1 + \mu^a_{\mu + uK, 1})) \right) dt
\]
\[ = \text{Tr} \left( \exp(-u^{-1}(F^a_1 + \mu^a_{\mu + uK, 1})) \right) - \text{Tr} \left( \exp(-u^{-1}(F^a_0 + \mu^a_{\mu + uK, 0})) \right) \]
\[ - \text{Tr} \left( \exp(-u^{-1}(F^a_0 + \mu^a_{\mu + uK, 0})) \right) \cdot \exp(-u^{-1}(F^a_0 + \mu^a_{\mu + uK, 1})) \].
\]

Then similar to (3.19), we see that
\[
(d - \iota_v - uK)\exp(-u^{-1}\pi^*H)cs(\nabla^E_0, \nabla^E_1; \nabla^E_1, \nabla^E_1) = ch_{\psi^+_G}(\nabla^E_0, \nabla^E_1) - ch_{\psi^+_G}(\nabla^E_0, \nabla^E_0). \]

This theorem shows that \( ch_{\psi^+_G}(\nabla^E, \nabla^E) \) is \((d - \iota_v - uK + \Theta + u^{-1}\pi^*H)\)-closed in \( \widetilde{\Omega}^+(\xi)^T[[u, u^{-1}]] \). Theorem 2.3 then tells us that \( f^{-1} \left( ch_{\psi^+_G}(\nabla^E, \nabla^E) \right) \) is \((\nabla^E - uK + u^{-1}H)\)-closed in \( \Omega^+(M, \xi)^T[[u, u^{-1}]] \).

We call
\[ CS(\nabla^E_0, \nabla^E_1; \nabla^E_1, \nabla^E_1) := f^{-1} \left( cs(\nabla^E_0, \nabla^E_1; \nabla^E_1, \nabla^E_1) \right) \in \Omega^+(M, \xi)[[u, u^{-1}]]. \]

the **exotic twisted equivariant Chern–Simons transgression term.** By (3.20) and Theorem 2.3 (formula (2.22)), one has
\[
Ch_{\psi^+_G}(\nabla^E_1, \nabla^E_1) - ch_{\psi^+_G}(\nabla^E_0, \nabla^E_0) = (\nabla^E - uK + u^{-1}H)CS(\nabla^E_0, \nabla^E_1; \nabla^E_1). \]

We define the **exotic twisted equivariant Chern character** to be:
\[
Ch_{\psi^+_G}(M, \xi) : K^0(M, \xi^\vee) \to h^*_G(M, \xi^\vee : H),
\]
\[ Ch_{\psi^+_G}(E, E') = \left[ f^{-1} \left( ch_{\psi^+_G}(\nabla^E, \nabla^E') \right) \right]. \]

**Remark 3.3.** A natural question is whether the exotic twisted equivariant Chern character is a rational isomorphism. However in the (untwisted) equivariant case, the equivariant Chern character
\[
Ch^G : K^j_G(M) \to H^j_G(M)
\]
where \( H^j_G(M) \) is the even equivariant cohomology for \( j = 0 \) and the odd equivariant cohomology for \( j = 1 \), is not a rational isomorphism, as \( H^j_G(M) = K^j_G(M) \otimes_{\mathbb{R}} R^\infty(G) \). These results are due to Block [3] and Brylinski [8]. We do not explore this further in our context as it is not central to our investigations.

**Remark 3.4.** Assume that the system \([M, G, (\xi, \nabla^E)]\) is trivial, i.e., the \( T \)-action is trivial and the line bundle \( \xi \) is trivial with trivial connection \( V^E = 0 \). Then a \((T \times U(1))\)-equivariant gerbe module with horizontal connection for the gerbe \( \{L_E\} \) on \( SE = M \times S^1 \) in Definition 3.1 can be identified with a gerbe module with connection for the gerbe \( G \) on \( M \). Therefore for the trivial system \([M, G, (\xi, \nabla^E)]\), we have
\[
K^0_G(M, \xi) \cong K^0(G, M). \]

(3.23)
Remark 3.5. Let us apply the constructions of exotic twisted equivariant K theory and exotic twisted equivariant Chern character to the concrete system \( [LZ, \tilde{H}, (E^B, \nabla^B)] \) is Example 1. Let \( i : Z \to LZ \) be the embedding. One sees that when restricting to the fixed point submanifold \( Z \) in \( LZ \), the \( \mathbb{T} \)-action as well as the holonomy line bundle \( E^B \) become trivial and \( i^* \tilde{H} = H \). By Remark 3.4,

\[
K^0_{\mathbb{Z}}(Z, \nabla^B : H) \cong K^0(Z, H). \tag{3.24}
\]

On the other hand, the triviality of the \( \mathbb{T} \)-action and the line bundle on \( Z \) imply that the moments \( \mu_{k+uk'} \)'s in (3.14) all disappear and all the \( \theta_s \)'s are the same, denote it by \( \theta \). Clearly \( \exp(-2\pi i \theta) \otimes s = \gamma^{-1} \), where \( s \) is the global identity section of the trivial circle bundle \( Z \times S^1 \). Then in view of (2.12), we see that when restricted to \( Z \), the exotic twisted equivariant Chern character degenerates to the usual twisted Chern character. This shows us that the diagram (0.2) is commutative.

Remark 3.6. In Diagram (0.2), let \( 0 \neq \Phi \in K^0_{\mathbb{Z}}(LZ, \nabla^B : \mathcal{G}) \) be in the kernel of \( \text{res} \), that is \( \text{res}(\Phi) = 0 \in K^0(LZ, H) \). Then \( BCh_H(\text{res}(\Phi)) = 0 \), but \( Ch_{\nabla^B, \mathcal{G}}(\Phi) \neq 0 \).

3.3. The odd case: gerbe modules

Let \( \mathcal{G} = \{(H, B_\alpha, A_{\alpha\beta})\} \) be a gerbe with connection on \( M \) as above. Let \( E = \{E_\alpha\} \) be a \( U_2 \) gerbe module with module connection \( \nabla^E = \{\nabla^{E_\alpha}\} \). Let \( \phi = \{\phi_\alpha : E_\alpha \to E_\alpha\} \) be an automorphism of the gerbe module \( E \) that respects the \( U_2 \) gerbe module structure, that is, \( \phi_\alpha \in U(E_\alpha) = \{I + A_\alpha \in U(E_\alpha), A_\alpha \text{ a trace class operator}\} \). We also need a compatibility condition on overlaps, \( \psi_{\alpha\beta} \circ \phi_\alpha \circ \psi_{\alpha\beta}^{-1} = \phi_\beta \). Here \( \psi_{\alpha\beta} : L_\alpha \otimes E_\beta \cong E_\alpha \), satisfying associativity by the gerbe condition.

Then odd twisted K-theory \( K^1(M, \mathcal{G}) \) is the abelian group generated by such pairs \( (E, \phi) \) with relations,

1. If \( 0 \to E_1 \to E_2 \to E_3 \to 0 \) is an exact sequence of gerbe modules such that the following diagram commutes,

\[
\begin{array}{cccccc}
0 & \to & E_1 & \to & E_2 & \to & E_3 & \to & 0 \\
& \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \\
0 & \to & E_1 & \to & E_2 & \to & E_3 & \to & 0.
\end{array}
\tag{3.25}
\]

then one has

\[
(E_2, \phi_2) = (E_1, \phi_1) + (E_3, \phi_3).
\]

2. \( (E, \phi_1 \circ \phi_2) = (E, \phi_1) + (E, \phi_2). \)

Then \( \phi^{-1}\nabla^E \phi \) is another module connection for \( E \). As explained in [19],

\[
(\phi^{-1}_\alpha (F^{E_\alpha} \phi_\alpha + B_\alpha)^k - (F^{E_\alpha} + B_\alpha)^k)
\]

are differential forms with values in the trace class endomorphisms of \( E_\alpha \) and

\[
\text{Tr}[\phi^{-1}_\alpha (F^{E_\alpha} \phi_\alpha + B_\alpha)^k - (F^{E_\alpha} + B_\alpha)^k]
\]

patch together to be an even degree differential form on \( M \). Denote it by \( \text{Tr}[\phi^{-1}\nabla^E \phi + B)^k - (F^E + B)^k] \).

Let \( \nabla^E(s) = s\phi^{-1}\nabla^E \phi + (1 - s)\nabla^E \) be a path joining \( \phi^{-1}\nabla^E \phi \) and \( \nabla^E \). Let

\[
A(\phi)(s) = \partial_s \nabla^E(s) = \phi^{-1}\nabla^E \phi - \nabla^E,
\]

which satisfies

\[
A(\phi)_\#(s) = \psi_{\alpha\beta}^{-1} A(\phi)_{\#}(s) \psi_{\alpha\beta}.
\tag{3.28}
\]

Following [19], one defines the odd Chern character form

\[
Ch_H(\nabla^E, \phi) = - \exp(-B) \int_0^1 ds \text{Tr}[A(\phi) \exp(-F^E(s))].
\tag{3.29}
\]

3.4. The odd case: exotic twisted equivariant \( K^1 \)-theory

Let \( M \) be a good \( \mathbb{T} \)-manifold with an \( \mathbb{T} \)-invariant cover \( \{U_\alpha\} \). Let \( \xi \to M \) be a \( \mathbb{T} \)-equivariant Hermitian line bundle over \( M \) equipped with a \( \mathbb{T} \)-invariant Hermitian connection \( \nabla^E \). Let \( \mathcal{G} = \{(U_\alpha), H, B_\alpha, A_{\alpha\beta}\} \) be a weak \( \mathbb{T} \)-invariant gerbe on \( M \). Assume that \( (\xi, \nabla^E) \) and \( (\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta}) \) are coupled on \( M \).

Associated to the system \( \{M, \mathcal{G}, (\xi, \nabla^E)\} \), we will introduce a version of twisted \( K^1 \)-theory and odd twisted Chern character in this section. Adopt the same notations as in Section 3.2.
**Definition 3.7.** The pair $(E, \phi)$ with $E = \{E_\alpha, \nabla^{E_\alpha}\}$ a $U_2$ gerbe module and $\phi = \{\phi_\alpha : E_\alpha \to E_\alpha\}$ a automorphism of the gerbe module respecting the $U_2$ structure, is said to be a $(\mathbb{T} \times U(1))$-equivariant **odd gerbe module with horizontal connection** for the gerbe $\{\hat{L}_\alpha\}$ if

- (a) the $(\mathbb{T} \times U(1))$-invariant connections $\nabla^{E_\alpha}$'s vanish on the vertical direction, i.e. $\nabla^{E_\alpha}_v \equiv 0$;
- (b) there are $(\mathbb{T} \times U(1))$-equivariant isomorphisms

$$\psi_{\alpha\beta} : \hat{L}_\alpha \otimes E_\beta \cong E_\alpha,$$

which respect the connections.

Note that the isomorphisms $\{\psi_{\alpha\beta}\}$ are consistently defined on triple overlaps because of the type (3.1) property of the gerbe $\{(\hat{L}_\alpha, \nabla^{\hat{L}_\alpha} = d + \pi^* A_{\alpha\beta})\}$.

One defines the **exotic twisted $\mathbb{T}$-equivariant $K^1$-theory** of $(\mathcal{M}, \mathcal{G}, (\xi, \nabla^\xi))$, $\mathcal{G}$, denoted as $K_1^\mathbb{T}(\mathcal{M}, \nabla^\xi : \mathcal{G})$, by

$$K_1^\mathbb{T}(\mathcal{M}, \nabla^\xi : \mathcal{G}) := \{(E, \phi) \sim \phi\},$$

(3.30)

where the equivalence relation $\sim$ is analogous to the description in Section 3.3.

Similar to (3.20), the forms

$$\exp(-u^{-1} \pi^* B_u - 2\pi i \theta_u) \int_0^1 ds \text{Tr} \left( \left( -u^{-1} A(\phi)_u(s) \exp(-u^{-1}(F^E_u + \mu^E_u + \mu^K_u)) \right) \right)$$

(3.31)

can be glued together as a global differential form in $\Omega^*(\mathcal{S}^2\xi)[[u, u^{-1}]]$. Simply denote this form by

$$\text{ch}_{F^E}(\mathcal{V}^E, \phi) = \exp(-u^{-1} \pi^* B - 2\pi i \theta) \int_0^1 ds \text{Tr} \left( -u^{-1} A(\phi)(s) \exp(-u^{-1}(F^E_u + \mu^E_u + \mu^K_u)) \right).$$

(3.32)

Then similar to the proof of Theorem 3.2, one can prove that

$$L_\phi \text{ch}_{F^E}(\mathcal{V}^E, \phi) = -\text{ch}_{F^E}(\mathcal{V}^E, \phi),$$

and

$$(d - \iota_\phi - uK + \Theta + u^{-1} \pi^* H) \text{ch}_{F^E}(\mathcal{V}^E, \phi) = 0.$$  

(3.34)

Therefore $\text{ch}_{F^E}(\mathcal{V}^E, \phi)$ is $(d - \iota_\phi - uK + \Theta + u^{-1} \pi^* H)$-closed in $\hat{H}^*(\mathcal{S}^2\xi)^\mathbb{T}[[u, u^{-1}]]$. Theorem 2.3 then tells us that $f^{-1}(\text{ch}_{F^E}(\mathcal{V}^E, \phi))$ is $(\nabla^E - uK + u^{-1} H)$-closed in $\Omega^*(\mathcal{M}, \xi)^\mathbb{T}[[u, u^{-1}]]$. Denote

$$\hat{h}_E^\mathbb{T}(\mathcal{M}, \nabla^\xi : \mathcal{G}) \in h_1^E(M, \nabla^\xi : H).$$

Similar to Proposition 5.1 in [19], one can show that $\text{ch}_{F^E}(\mathcal{V}^E, \phi)$ is independent of the choice of module horizontal connection $\nabla^E$ on $E$ and choice of automorphism $\phi$ of $E$. We define the **exotic twisted equivariant odd Chern character** to be

$$\text{ch}_{F^E} : K_1^\mathbb{T}(\mathcal{M}, \nabla^\xi : \mathcal{G}) \to \hat{h}_E^\mathbb{T}(\mathcal{M}, \nabla^\xi : H).$$

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