Towards a classification of Lorentzian holonomy groups

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Abstract

If the holonomy representation of an \((n+2)\)-dimensional simply-connected Lorentzian manifold \((M,h)\) admits a degenerate invariant subspace its holonomy group is contained in the parabolic group \((\mathbb{R} \times SO(n)) \ltimes \mathbb{R}^n\). The main ingredient of such a holonomy group is the \(SO(n)\)-projection \(G := \text{pr}_{SO(n)}(\text{Hol}_p(M,h))\) and one may ask whether it has to be a Riemannian holonomy group. In this paper we show that this is the case if \(G \subset U(n/2)\) or if the irreducible acting components of \(G\) are simple.

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Introduction

The very first step in a classification of the holonomy groups of semi-Riemannian manifolds is the decomposition theorem of de Rham and Wu ([dR52] for Riemannian manifolds and [Wu64] for general semi-Riemannian manifolds). It asserts that every simply-connected, complete semi-Riemannian manifold is isometric to a product of simply-connected, complete semi-Riemannian manifolds, of which one can be flat and all other are indecomposable (often called “weakly-irreducible”, i.e. with no non-degenerate invariant subspace under holonomy representation). For a Riemannian manifold this theorem asserts that the holonomy representation is completely reducible, i.e. decomposes into factors which are trivial or irreducible, and are again Riemannian holonomy representations. For pseudo-Riemannian manifolds indecomposability is not
the same as irreducibility. We can have degenerate invariant subspaces under holonomy representation.

On the other hand all irreducible factors are known by the Berger classification of possible irreducible semi-Riemannian holonomy groups ([Ber55], [Sim62], [Ale68], [BG72] and [Bry87]). This classification uses an algebraic condition which has to be satisfied by every holonomy group of a torsionfree connection. It follows from the first Bianchi identity and the Amrose-Singer holonomy theorem [AS53] and can be formulated very easily: If \( \mathfrak{h} \) is the Lie algebra of the holonomy group of a torsionfree connection, acting on the vector space \( E \cong T_p M \), then it obeys \( \mathfrak{h} = \{ R(u, v) | u, v \in V, R \in \mathcal{K}(\mathfrak{h}) \} \), where

\[
\mathcal{K}(\mathfrak{h}) := \{ R \in \Lambda^2 V^* \otimes \mathfrak{h} | R(u, v)w + R(v, w)u + R(w, u)v = 0 \text{ for all } u, v, w \in V \}
\]

is the space of curvature endomorphisms.

Lie algebras satisfying this conditions are called Berger algebras. All irreducible Berger algebras are classified in [MS99] and [Sch99].

For non-irreducible, indecomposable holonomy representations (resp. Berger algebras) such a classification is missing.

For a Lorentzian manifold \((M, h)\) of dimension \(m > 2\) the de Rham/Wu–decomposition yields the following two cases:

 Completely reducible: Here \((M, h)\) decomposes into irreducible or flat Riemannian manifolds and a manifold which is an irreducible or flat Lorentzian manifold or \((\mathbb{R}, -dt)\). The irreducible Riemannian holonomies are known, as well as the irreducible Lorentzian one, which has to be the whole \(SO(1, m - 1)\). (The latter follows from the Berger list but was directly proved by [SO01].)

 Not completely reducible: This is equivalent to the existence of a degenerate invariant subspace and entails the existence of exactly one holonomy invariant lightlike subspace. The Lorentzian manifold decomposes into irreducible or flat Riemannian manifolds and a Lorentzian manifold with indecomposable, but non-irreducible holonomy representation, i.e. with invariant lightlike (i.e. one-dimensional) subspace.

Thus in order to classify holonomy groups of simply-connected Lorentzian manifolds one has to find the possible holonomy groups of indecomposable, but non-irreducible Lorentzian manifolds.

The holonomy algebra of such a manifold of dimension \(m := n + 2 > 2\) is contained in \((\mathbb{R} \oplus \mathfrak{so}(n)) \times \mathbb{R}^n\). L. Berard-Bergery and A. Ikemakhen studied in [BI93] the projections of such a holonomy algebra and achieved two important results. The first gives a classification into four types based on the possible projections on \(\mathbb{R}\) and \(\mathbb{R}^n\). For two of these types the projections are coupled and for the remaining two uncoupled to the \(\mathfrak{so}(n)\)-component.
The second result is a decomposition property for the \(\mathfrak{so}(n)\)-projection (see theorem \ref{thm:decomposition}), i.e. there is a decomposition of the representation space into irreducible components and of the Lie algebra into ideals which act irreducible on the components.

The relation between the \(\mathfrak{so}(n)\)-part and the \(\mathbb{R}\)- and \(\mathbb{R}^n\)-parts is understood quite well \cite{Bou}, or very recently \cite{Gal}: If one has a simply-connected, indecomposable, non-irreducible Lorentzian manifold with holonomy of uncoupled type, then, under certain conditions, one can construct a Lorentzian manifold with coupled type holonomy.

Now one may ask: Which algebras can occur as \(\mathfrak{so}(n)\)-projection of an indecomposable, but not-irreducible Lorentzian manifold? Of course it has to satisfy the decomposition property. Riemannian holonomy algebras are the first examples, because there is a method to construct from a given Riemannian manifold an indecomposable Lorentzian manifold with holonomy of uncoupled type for which the \(\mathfrak{so}(n)\)-projection equals to the Riemannian holonomy. Furthermore one can show that the Lorentzian manifold is a \(pp\)-wave if and only if the \(\mathbb{R}\)- and the \(SO(n)\)-component vanish \cite{Lei}.

In \cite{Lei2} we derived an algebraic criterion on the \(\mathfrak{so}(n)\)-component of an indecomposable, non-irreducible, simply-connected Lorentzian manifold \((M,h)\), in analogy to the well known Berger criterion for holonomy algebras. If \(g\) is the \(\mathfrak{so}(n)\)-component of an indecomposable, non-irreducible, simply-connected Lorentzian manifold, acting on an \(n\)-dimensional Euclidean vector space \((E,h)\) then it obeys \(g = \{Q(u)|Q \in B_h(g), u \in E\}\) where \(B_h(g)\) is defined as follows

\[
B_h(g) = \{Q \in E^* \otimes g \mid h(Q(u)v, w) + h(Q(v)w, u) + h(Q(w)u, v) = 0, \forall u, v, w \in E\}.
\]

Since orthogonal Berger algebras do satisfy this criterion we called these algebras weak-Berger algebras. Furthermore we showed that every irreducible weak-Berger algebra, which is contained in \(u(n/2)\) is a Berger algebra, in particular a Riemannian holonomy algebra. This, together with the decomposition property implies that \(g := pr_{\mathfrak{so}(n)}\mathfrak{hol}_p(M, h)\) is a Riemannian holonomy algebra if it is contained in \(u(n/2)\).

In the present paper we prove the following: If \(g\) is a simple weak-Berger algebra, not contained in \(u(n/2)\), which acts irreducible on \(\mathbb{R}^n\), then it is a Berger algebra, and in particular a Riemannian holonomy algebra. This of course applies to the irreducible components of the \(\mathfrak{so}(n)\)-projection of \(\mathfrak{hol}_p(M, h)\). In the proof we proceed analogously to \cite{Sch}, where the holonomy groups of torsion free connections are classified. This will be the main part of this paper and is contained in section \ref{sec:main}.

In the first section we recall the results of \cite{B1} and our results from \cite{Lei2} introducing the notion of weak-Berger algebras. The third section presents again for sake of completeness the proof of the fact that weak-Berger algebras in \(u(n/2)\) are Berger algebras. In the appendix we recall facts about representations of real Lie algebras.

These results leave open the question: Are there semisimple, non simple, irreducible acting Lie algebras, not contained in \(u(n/2)\), which are weak-Berger, but not Berger? We guess that this is not the case, and we intent to apply the methods of the present
paper also in the semisimple case. Up to dimension eleven this was proved very recently by [Gal03] also for algebras not contained in $u(n/2)$. In his paper he studied the space of curvature endomorphisms for subalgebras in $(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$ which are of the types found in [BI93]. Reducing everything to one uncoupled type he proved the other direction of our result: a subalgebra of one of these types is a Berger algebra, if its $\mathfrak{so}(n)$–projection is a weak-Berger algebra (in our terms).

We are aware that the proofs we will present here are a cumbersome case-by-case analysis using the methods of representation theory. It is very desirable to get a direct and more geometric proof of the proposition that every $SO(n)$–projection of an indecomposable, non-irreducible Lorentzian holonomy group is a Riemannian holonomy group, which includes the remaining semisimple case of course.

We want to remark that the starting point of this investigation was the question for the existence of parallel spinors on Lorentzian manifolds. Such a spinor defines a parallel vector field which can be light like. Hence the manifold has an indecomposable, non-irreducible factor. But the existence of parallel spinors on indecomposable Lorentzian manifolds with parallel lightlike vector field depends only on the $SO(n)$–projection. Thus a complete list of the latter would answer this question. In the physically important dimensions below twelve the question for the maximal indecomposable Lorentzian holonomy groups admitting parallel spinors is answered [Bry00], [FO99].

1 Indecomposable Lorentzian holonomy and weak-Berger algebras

1.1 Basic properties

Let $(M,h)$ be an indecomposable, non-irreducible Lorentzian manifold with $\dim M = n + 2 \geq 2$. The holonomy group in a point $p \in M$ acting on $T_pM$ — defined as the group of parallel displacements along loops starting at $p$ — then has a lightlike, one-dimensional invariant subspace $\Xi_p$ which is the fibre of a parallel distribution $\Xi$. This is equivalent to the existence of a recurrent lightlike vector field. The subspace $\Xi^\perp_p$ also is holonomy invariant and the fibre of a parallel distribution $\Xi^\perp$. (We call a distribution parallel if it is closed under $\nabla_U$ for every $U \in TM$.)

With respect to a basis

$$(X, E_1, \ldots, E_n, Z) \subset \Xi_p, \quad \text{i.e. } X \in \Xi_p, E_i \in \Xi^\perp_p$$

with $h(E_i, E_j) = \delta_{ij}$, $h(Z, Z) = h(Z, E_i) = h(X, E_i) = 0$ and $h(X, Z) = 1$ \hfill (1)

the holonomy algebra is contained in the following Lie algebra

$$\mathfrak{hol}_p(M,h) \subset \left\{ \begin{pmatrix} a & u^t & 0 \\ 0 & A & -u \\ 0 & 0^t & -a \end{pmatrix} \right\} a \in \mathbb{R}, u \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n. \hfill (2)$$
Choosing a different basis of type I corresponds to conjugation with an element in $O(1, n+1)$ which respects the form (1). Hence the $\mathfrak{so}(n)$–component is uniquely defined with respect to conjugation in $O(n)$.

The projections of $\mathfrak{hol}_p(M, h)$ on the $\mathbb{R}$– and on the $\mathbb{R}^n$–component are well understood. With respect to these projections there exist four different types (see [Ike90], [BI93] and [Ike96]). For the types I and II the holonomy is equal to $(\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$ resp. $\mathfrak{g} \ltimes \mathbb{R}^n$.

In case of types II and IV the projection on $\mathbb{R}$ is zero, which implies the existence not only of a recurrent lightlike vector field but also of a parallel one. In case of types III and IV the $\mathbb{R}^n$–components are coupled to the $\mathfrak{so}(n)$–component, or more precisely to its center.

In the following shall be $\mathfrak{g} := \text{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}_p(M, h))$. About $\mathfrak{g}$ the in [BI93] is proved

1.1 Theorem. [BI93] Let $\mathfrak{g} := \text{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}_p(M, h))$ be the projection of the holonomy algebra of an indecomposable, non-irreducible, $n+2$–dimensional Lorentzian manifold onto $\mathfrak{so}(n)$. Then $\mathfrak{g}$ satisfies the following decomposition property: There exists a decomposition of $\mathbb{R}^n$ into orthogonal subspaces and of $\mathfrak{g}$ into ideals

$$\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_r \text{ and } \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r,$$

such that $\mathfrak{g}$ acts trivial on $E_0$, $\mathfrak{g}_i$ acts irreducible on $E_i$ and trivial on $E_j$ for $i \neq j$.

This theorem has two important consequences making a further algebraic investigation of $\mathfrak{g}$ possible.

Irreducible acting, connected subgroups of $SO(n)$ are are closed and therefore compact.

Now by the theorem the group $G := \text{pr}_{SO(n)}\text{Hol}_p^0(M, h)$ decomposes in such irreducible acting subgroups. Thus we have as first consequence that $G$ is compact, although the whole holonomy group must not be compact (for such examples see also [BI93]).

The second is, that it suffices to study irreducible acting groups or algebras $\mathfrak{g}$, a fact which is necessary for trying a classification. We will see this in detail in the following section.

We will describe the local situation briefly. Locally there are $n$–dimensional Riemannian submanifolds defined via special coordinates respecting the foliation $\Xi \subset \Xi^\perp$, denoted by $(x, y_1, \ldots, y_n, z)$ with $\frac{\partial}{\partial x} \in \Xi, \frac{\partial}{\partial y_i} \in \Xi^\perp$. The restriction of $h$ to these submanifolds defined by $y_1, \ldots, y_n$ gives a family of Riemannian metrics $g_z$ on it, depending only on the coordinate $z$ (since $\frac{\partial}{\partial x} \in \Xi$, see [BT25], also [Ike96]).

Although these coordinates are unique under certain conditions (see [Bon00]) it is not clear how the Lie algebra $\mathfrak{g}$ can be obtained by the holonomies $\mathfrak{hol}_{p(z)}(g_z)$ of the family of metrics $g_z$. The only known point is, that all these $\mathfrak{hol}_{p(z)}(h_z)$ are contained in $\mathfrak{g}$ [Ike96].

If the dependence on $z$ is trivial — i.e. $g_z \equiv g$ or $g_z \equiv f(z)g$ — then $\mathfrak{g}$ is equal to the holonomy of the Riemannian metric $g$. 
In particular this gives a way to construct indecomposable, non-irreducible Lorentzian manifolds with holonomy equal to \((\mathbb{R} \oplus \text{Riemannian holonomy}) \ltimes \mathbb{R}^n\): Let \((N, g)\) be an \(n\)-dimensional Riemannian manifold, \(\theta\) a closed form on \(N\) and \(q\) a function on \(N \times \mathbb{R}^2\), the latter sufficiently general. Then

\[(M = N \times \mathbb{R}^2, h = dx dz + q dz^2 + \theta dz + f(z)g)\]

is a Lorentzian manifold with holonomy

\[\mathfrak{hol}_{(x,z,p)}(M, h) = (\mathbb{R} \oplus \mathfrak{hol}_p(N, g)) \ltimes \mathbb{R}^n.\]

In case of Riemannian Kähler- and hyper-Kähler manifolds \((N, g)\) these conditions can be weakened [Lei02b].

Furthermore there is a method to construct manifolds with coupled holonomy from manifolds with uncoupled holonomy [Bou00]: If \((M, h)\) is a simply-connected, indecomposable, non-irreducible Lorentzian manifold with uncoupled holonomy \(g \ltimes \mathbb{R}^n\) or \((\mathbb{R} \oplus g) \ltimes \mathbb{R}^n\) such that \(g\) has non-trivial center (and further conditions), then there is a metric \(\tilde{h}\) on \(M\) such that \((M, \tilde{h})\) has holonomy of coupled type and with \(\mathfrak{so}(n)\)-projection \(g\).

In the following we will go an algebraic way, in order to classify the possible algebras \(g\). This algebraic way uses the Bianchi–identity, restricted to \(\mathbb{R}^n\) as representation space of \(g\). This is the aim of the next sections.

### 1.2 Berger and weak-Berger algebras

Here we will introduce the notion of weak-Berger algebras in comparison to Berger algebras. We present some basic properties, in particular a decomposition property and the behavior under complexification. For the details to this section see [Lei02a].

Let \(E\) be a vector space over the field \(\mathbb{K}\) and let \(g \subset \mathfrak{gl}(E)\) be a Lie algebra. Then one defines

\[\mathcal{K}(g) := \{ R \in \Lambda^2 E^* \otimes g \mid R(x, y)z + R(y, z)x + R(z, x)y = 0 \}\]

\[\mathfrak{g} := \text{span}\{ R(x, y) \mid x, y \in E, R \in \mathcal{K}(g) \},\]

and for \(g \subset \mathfrak{so}(E, h)\):

\[B_h(g) := \{ Q \in E^* \otimes g \mid h(Q(x)y, z) + h(Q(y)z, x) + h(Q(z)x, y) = 0 \}\]

\[\mathfrak{g}_h := \text{span}\{ Q(x) \mid x \in E, Q \in B_h(g) \} \].

Then we have the following basic properties.

**1.2 Proposition.** \(\mathcal{K}(g) \subset \Lambda^2 E^* \otimes g\) and \(B_h(g) \subset E^* \otimes g\) are \(g\)-modules. \(\mathfrak{g}\) and \(\mathfrak{g}_h\) are ideals in \(g\).
The representation of \( g \) on \( \mathcal{B}_h(g) \) and \( \mathcal{K}(g) \) is given by the standard and the adjoint representation

\[
(A \cdot Q)(x) = -Q(Ax) + [A, Q(x)]\\
(A \cdot R)(x, y) = -R(Ax, y) - R(x, Ay) + [A, R(x, y)].
\] (3)

\[
(A \cdot R)(x, y) = -R(Ax, y) - R(x, Ay) + [A, R(x, y)].
\] (4)

1.3 Definition. Let \( g \subset \mathfrak{gl}(E) \) be a Lie algebra. Then \( g \) is called **Berger algebra** if \( g = g \). If \( g \subset \mathfrak{so}(E, h) \) is an orthogonal Lie algebra with \( g_h = g \), then we call it **weak-Berger algebra**.

Equivalent to the (weak-)Berger property is the fact that there is no ideal \( h \) in \( g \) such that \( \mathcal{K}(h) = \mathcal{K}(g) \) (resp. \( \mathcal{B}_h(h) = \mathcal{B}_h(g) \)).

The notion “weak-Berger” is satisfied by the following

1.4 Proposition. *Every Berger algebra which is orthogonal is a weak-Berger algebra.*

This proposition has a

1.5 Corollary. Let \( g \subset \mathfrak{so}(E, h) \) be an orthogonal Lie algebra. Then

\[
\text{span}\{R(x, y) + Q(z) | R \in \mathcal{K}(g), Q \in \mathcal{B}_h(g), x, y, z \in E\} \subset g_h.
\] (5)

Concerning the decomposition of Berger and weak-Berger algebras the following proposition holds.

1.6 Proposition. If \( g_1 \subset \mathfrak{gl}(V_1) \), \( g_2 \subset \mathfrak{gl}(V_2) \) and \( g := g_1 \oplus g_2 \subset \mathfrak{gl}(V := V_1 \oplus V_2) \), then it holds:

1. If \( g_1 \) and \( g_2 \) are Berger algebras, then \( g \) is a Berger algebra.

2. If in addition \( g_1 \subset \mathfrak{so}(V_1, h_1) \), \( g_2 \subset \mathfrak{so}(V_2, h_2) \) and \( g := g_1 \oplus g_2 \subset \mathfrak{so}(V := V_1 \oplus V_2, h := h_1 \oplus h_2) \), then holds:

   \( g_1 \) and \( g_2 \) are weak-Berger algebras if and only if \( g \) is a weak-Berger algebra.

The Ambrose-Singer holonomy theorem \[AS53\] then implies that holonomy algebras of torsion free connections — in particular of a Levi-Civita-connection — are Berger algebras. The list of all irreducible Berger algebras is known (\[Ber55\] for orthogonal, non-symmetric Berger algebras, \[Ber57\] for orthogonal symmetric ones, and \[MS99\] in the general affine case).

We should mention that in our notation Berger algebras are not only non-symmetric Berger algebras, as it is sometimes defined. For us only the possibility of being the holonomy algebra of a Riemannian manifold is of interest, symmetric or non symmetric. The \( \mathfrak{so}(n) \)–projection of an indecomposable, non-irreducible Lorentzian manifold is no holonomy algebra, and therefore not necessarily a Berger algebra. But the following statement, which we proved in \[Lei02a\], asserts that it is a weak-Berger algebra.
1.7 Theorem. Let \((M, h)\) be an indecomposable, but non-irreducible, simply connected Lorentzian manifold and \(g = pr_{so(n)}(hol_p(M, h))\). Then \(g\) is a weak-Berger algebra.

From point two of proposition 1.6 we get the following

1.8 Corollary. Let \((M, h)\) be an indecomposable, but non-irreducible Lorentzian manifold and \(g = pr_{so(n)}(hol_p(M, h)) \subset E^* \otimes E\) and \(g = g_1 \oplus \ldots \oplus g_r\) with \(g_i \in so(E_1, h_1)\) the decomposition in irreducible acting ideals from theorem 1.7. Then these \(g_i\) are irreducible weak-Berger algebras.

This corollary ensures that we are at a similar point as in the Riemannian situation, but reaching it by a different way. This is shown schematically in the following diagram:

The proof of the theorem gives another

1.9 Corollary. Let \((M, h)\) be an indecomposable, non-irreducible Lorentzian manifold and \(g = pr_{so(n)}(hol_p(M, h))\). If there exists coordinates \((x, y_1, \ldots, y_n, z)\) of the above form (i.e. respecting the foliation \(\Xi \subset \Xi^\perp\)), with the property that everywhere holds \(R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}) = 0\), then \(g\) is a Berger-algebra.

The aim of the following sections will be to classify all weak-Berger algebras. Before we do this we have to say a word about real and complex (weak-) Berger algebras.

1.3 Real and complex weak-Berger algebras

Because of the above result we have to classify the real weak-Berger algebras. Since we will use the representation theory of complex semisimple Lie algebras we have to describe the transition of a real weak-Berger algebra to its complexification.

First we note that the spaces \(\mathcal{K}(g)\) and \(\mathcal{B}_h(g)\) for \(g \subset so(E, h)\) can be described by the following exact sequences:

\[
0 \rightarrow \mathcal{K}(g) \\ \hookrightarrow \Lambda^2 E^* \otimes g \xrightarrow{\lambda} \Lambda^3 E^* \otimes E
\]

where the map \(\lambda\) is the skew-symmetrization and \(\lambda_h\) the dualization by \(h\) and the skew-symmetrization.
If we now consider a real Lie algebra $\mathfrak{g}$ acting orthogonal on a real vector space $E$, i.e. $\mathfrak{g} \subset \mathfrak{so}(E,h)$, then $h$ extends by complexification (linear in both components) to a non-degenerate complex-bilinear form $h^C$ which is invariant under $\mathfrak{g}^C$, i.e. $\mathfrak{g}^C \subset \mathfrak{so}(E^C,h^C)$. Then the complexification of the above exact sequences gives

\begin{align}
\mathcal{K}(\mathfrak{g})^C &= \mathcal{K}(\mathfrak{g}^C) \\
(\mathcal{B}_h(\mathfrak{g}))^C &= \mathcal{B}_{h^C}(\mathfrak{g}^C).
\end{align}

This implies

**1.10 Proposition.** $\mathfrak{g} \subset \mathfrak{so}(E,h)$ is a (weak-) Berger algebra if and only if $\mathfrak{g}^C \subset \mathfrak{so}(E^C,h^C)$ is a (weak-) Berger algebra.

I.e. complexification preserves the weak-Berger as well as the Berger property. Because of proposition 1.6 it suffices to classify the real weak-Berger algebras which are irreducible. Now irreducibility is a property which is not preserved under complexification. We have to deal with this problem. At a first step one recalls the following definition, distinguishing two cases for a module of a real Lie algebras.

**1.11 Definition.** Let $\mathfrak{g}$ be a real Lie algebra. Irreducible real $\mathfrak{g}$-modules $E$ for which $E^C$ is an irreducible $\mathfrak{g}$-module and irreducible complex modules $V$ for which $V^R$ is a reducible $\mathfrak{g}$-module are called of **real type**. Irreducible real $\mathfrak{g}$-modules $E$ for which $E^C$ is a reducible $\mathfrak{g}$-module and irreducible complex modules $V$ for which $V^R$ is a irreducible $\mathfrak{g}$-module are called of **non-real type**.

This notation corresponds to the distinction of complex irreducible $\mathfrak{g}$-modules into real, complex and quaternionic ones. It makes sense because the complexification of a module of real type is of real type — recall that $(E^C)^R$ is a reducible $\mathfrak{g}$-module — and the realification of a module of non-real type is of non-real type. These relations are described in the appendix A.

In the original papers of Cartan [Car14] and Iwahori [Iwa59], see also [Got78], where these distinction is introduced, a representation of real type is called as representation of **first type** and a representation of non-real type is called of **second type**.

If one now complexifies the Lie algebra $\mathfrak{g}$ too, then $E^C$ becomes a $\mathfrak{g}^C$–module. This transition preserves irreducibility.

**1.12 Lemma.** Let $\mathfrak{g}^C \subset \mathfrak{gl}(V)$ be the complexification of $\mathfrak{g} \subset \mathfrak{gl}(V)$ with a complex $\mathfrak{g}$-module $V$. Then it holds:

1. $\mathfrak{g}$ is irreducible if and only if $\mathfrak{g}^C$ is irreducible.

2. $\mathfrak{g} \subset \mathfrak{so}(V,H)$ if and only if $\mathfrak{g}^C \subset \mathfrak{so}(V,H)$, where $H$ is a symmetric bilinear form.

In the following sections we will describe the weak-Berger property for real and non-real modules of a real Lie algebra $\mathfrak{g}$. 
2 Weak-Berger algebras of real type

In this section we will make efforts to classify weak-Berger algebras of real type, at least the simple ones. The argumentation in this section is analogously to the reasoning in [MS99].

Let \( g_0 \) shall be a real Lie algebra and \( E \) a real irreducible module of real type. Furthermore we suppose \( g_0 \in \mathfrak{so}(E, h) \) with \( h \) positive definite. Then \( E^\mathbb{C} \) is an irreducible \( g_0 \)-module (also of real type). If \( h^\mathbb{C} \) denotes the complexification of \( h \), bilinear in both components we have that \( g_0 \subset \mathfrak{so}(E^\mathbb{C}, h^\mathbb{C}) \).

Now we can extend \( h \) also sesqui-linear on \( E^\mathbb{C} \) and get a hermitian form \( \theta^h \) on \( V \) which is invariant under \( g_0 \). Thus we have \( g_0 \subset \mathfrak{u}(V, \theta^h) \). \( \theta^h \) has the same index as \( h \) (see appendix A).

Since the bilinear form \( h \) we start with is positive definite we can make another simplification. Subalgebras of \( \mathfrak{so}(E, h) \) with positive definite \( h \) are compact and therefore reductive. I.e. its Levi-decomposition is \( g_0 = \mathfrak{z}_0 \oplus \mathfrak{d}_0 \), with center \( \mathfrak{z}_0 \) and semisimple derived algebra \( \mathfrak{d}_0 \). Thus \( g_0^\mathbb{C} = \mathfrak{z} \oplus \mathfrak{d} \) is also reductive. But since it is irreducible by assumption, the Schur lemma implies that the center \( \mathfrak{z} \) is \( \mathbb{C} \text{Id} \) or zero. But \( \mathbb{C} \text{Id} \) is not contained in \( \mathfrak{so}(V, H) \). Thus the center has to be zero and \( g \) is semisimple. Proposition 1.10 gives the following.

2.1 Proposition. If \( g_0 \subset \mathfrak{so}(E, h) \) is a weak-Berger algebra of real type then, \( g_0^\mathbb{C} \subset \mathfrak{so}(E^\mathbb{C}, h^\mathbb{C}) \) is an irreducible weak-Berger algebra. \( E^\mathbb{C} \) is a \( g_0 \)-module of real type and if \( h \) is positive definite then \( g_0^\mathbb{C} \) is semisimple.

If \( g \subset \mathfrak{so}(V, H) \) is an irreducible complex weak-Berger which is semisimple. Then \( g \) has a compact real form \( g_0 \) and if \( V \) is a \( g_0 \)-module of real type, then \( V = E^\mathbb{C} \), \( g_0 \) is unitary with respect to a hermitian form \( \theta \) and \( g_0 \subset \mathfrak{so}(E, h) \) is a weak-Berger algebra of real type. The indices of \( h \) and \( \theta \) are equal.

Proof. The first direction follows obviously from proposition 1.10. That \( E^\mathbb{C} \) is a module of real type holds because of \( (E^\mathbb{C})_R \) is reducible (see appendix proposition A.9).

Since \( g \) is semisimple it has a compact real form \( g_0 \). If \( V \) is a \( g_0 \)-module of real type, then \( g_0 \) is unitary since it is orthogonal (see proposition A.7) and it is \( V = E^\mathbb{C} \) (proposition A.9). By proposition A.15 follows that \( g_0 \) is orthogonal w.r.t. \( h \) which has the same index as \( \theta \). Then the proposition follows by proposition 1.10.

The main point of this proposition is the implication that if \( g_0 \subset \mathfrak{so}(E, h) \) is weak-Berger of real type, then \( g_0^\mathbb{C} \subset \mathfrak{so}(E^\mathbb{C}, h^\mathbb{C}) \) is an irreducible acting, complex semisimple weak-Berger algebra. These we have to classify.

2.2 Remark. Before we start we have to make a remark about definition of holonomy up to conjugation. The \( SO(n) \)-component of an indecomposable, non-irreducible
Lorentzian manifold was defined modulo conjugation in $O(n)$. Hence we shall not distinguish between subalgebras of $\mathfrak{gl}(n, \mathbb{C})$ which are isomorphic under $Ad_\varphi$ where $\varphi$ is an element from $O(n, \mathbb{C})$ and $Ad$ the adjoint action in of $Gl(n, \mathbb{C})$ on $\mathfrak{gl}(n, \mathbb{C})$. We say that an orthogonal representation $\kappa_1$ of a complex semisimple Lie algebra $\mathfrak{g}$ is congruent to an orthogonal representation $\kappa_2$ if there is an element $\varphi \in O(n, \mathbb{C})$ such that the following equivalence of $\mathfrak{g}$-representations is valid: $\kappa_1 \sim Ad_\varphi \circ \kappa_2$. Hence we have to classify semisimple, orthogonal, irreducible acting, complex weak-Berger algebras of real type up to this congruence of representations.

If the automorphism $Ad_\varphi$ is inner, then the representations are equivalent, if it is outer then only congruent.

For semisimple Lie algebras it holds that $Out(\mathfrak{g}) := Aut(\mathfrak{g})/Inn(\mathfrak{g})$ counts the connection components of $Aut(\mathfrak{g})$ and (see for example [OV94]) $Out(\mathfrak{g})$ is isomorphic to the automorphism of the fundamental system, i.e. symmetries of the Dynkin diagram.

For us this becomes relevant in case of $\mathfrak{so}(8, \mathbb{C})$. In the picture one sees that the symmetries of the Dynkin diagram generate the symmetric group $S_3$, i.e. $Out(\mathfrak{so}(8, \mathbb{C})) = S_3$ and it contains the so-called “triality automorphism” which interchanges vector and spin representations of $\mathfrak{so}(8, \mathbb{C})$ without fixing one.

We will use that the automorphism which interchanges the vector representation with one spinor representation and fixes the second spinor representation resp. interchanges the spinor representations and fixes the vector representation comes from $Ad_\varphi$ with $\varphi \in O(n, \mathbb{C})$. Hence the vector and the spinor representations of $\mathfrak{so}(8, \mathbb{C})$ are congruent to each other.

Finally we should remark that compact real forms equivalent to a given one correspond to inner automorphism of $\mathfrak{g}$. Hence the corresponding representations are equivalent.

### 2.1 Irreducible, complex, orthogonal, semisimple Lie algebras

In the following $V$ will be a complex vector space equipped with a non-degenerate symmetric bilinear 2–form $H$. $\mathfrak{g}$ shall be an irreducible acting, complex, semisimple subalgebra of $\mathfrak{so}(V, H)$.

Thus all the tools of root space decomposition and representation theory will apply. Let $t$ be the Cartan subalgebra of $\mathfrak{g}$. We denote by $\Delta \subset t^*$ the roots of $\mathfrak{g}$ and we set $\Delta_0 := \Delta \cup \{0\}$. Then $\mathfrak{g}$ decomposes into its root spaces $\mathfrak{g}_\alpha := \{A \in \mathfrak{g} | [T, A] = \alpha(T) \cdot A \text{ for all } T \in t\} \neq \{0\}$. It is

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha \quad \text{where } \mathfrak{g}_0 = t.$$  

By $\Omega \subset t^*$ we denote the weights of $\mathfrak{g} \subset \mathfrak{so}(V, H)$. Then $V$ decomposes into the weight
spaces \( V_\mu := \{ v \in V | T(v) = \mu(T) \cdot v \text{ for all } T \in \mathfrak{t} \} \neq \{0\} \), i.e.
\[
V = \bigoplus_{\mu \in \Omega} V_\mu.
\]
Now the following holds.

2.3 Proposition. Let \( \mathfrak{g} \subset \mathfrak{so}(V,H) \) be a complex, semisimple Lie algebra with weight space decomposition. Then
\[
V(\mu) \perp V(\lambda) \text{ if and only if } \lambda \neq -\mu.
\]
In particular if \( \mu \) is a weight, then \( -\mu \) too.

Proof. For any \( T \in \mathfrak{t}, u \in V_\mu \) and \( v \in V_\lambda \) we have
\[
0 = H(Tu,v) + H(u,Tv) = (\mu(T) + \lambda(T)) H(u,v).
\]
Now if \( \lambda \neq -\mu \) there is a \( T \) such that \( \mu(T) + \lambda(T) \neq 0 \). But this implies \( V_\lambda \perp V_\mu \).
On the other hand \( V_\mu \perp V_{-\mu} \) would imply \( V_\mu \perp V \) which contradicts the non-degeneracy of \( H \).
Its non-degeneracy also implies that \( \mu \in \mathfrak{t}^* \) is a weight if and only if \( -\mu \) is a weight.

2.2 Irreducible complex weak-Berger algebras

We will now draw consequences from the weak-Berger property. Therefore we consider the space \( B_H(\mathfrak{g}) \) defined by the Bianchi identity. If \( \mathfrak{g} \) is weak-Berger it has to be non-zero, i.e. by proposition 1.2 it is a non-zero \( \mathfrak{g} \)-module. If we denote by \( \Pi \) all its weights then it decomposes into weight spaces
\[
B_H(\mathfrak{g}) = \bigoplus_{\phi \in \Pi} B_\phi.
\]
If \( \Omega \) are the weights of \( V \) then we define the following set
\[
\Gamma := \left\{ \mu + \phi \mid \mu \in \Omega, \phi \in \Pi \text{ and there is an } u \in V_\mu \text{ and a } Q \in B_\phi \text{ such that } Q(u) \neq 0 \right\} \subset \mathfrak{t}^*.
\]
Then one proves a

2.4 Lemma. \( \Gamma \subset \Delta_0 \).

Proof. We have to show that every \( \mu + \phi \in \Gamma \) is a root of \( \mathfrak{g} \). Therefore we consider weight elements \( Q_\phi \in B_\phi \) and \( u_\mu \in V_\mu \) with \( 0 \neq Q_\phi(u_\mu) \). Then for every \( T \in \mathfrak{t} \) holds (because of the definition of the \( \mathfrak{g} \)-module \( B_H(\mathfrak{g}) \)):
\[
[T, Q_\phi(u_\mu)] = (TQ_\phi)(u_\mu) + Q_\phi(T(u_\mu)) = (\phi(T) + \mu(T)) Q_\phi(u_\mu)
\]
I.e. \( \phi + \mu \) is a root or zero.

\[ \square \]
For weak-Berger algebras now the other inclusion is true.

2.5 Proposition. If \( g \subset \mathfrak{so}(V, h) \) is irreducible, semisimple Lie algebra. If it is weak-Berger then \( \Gamma = \Delta_0 \). If \( \Gamma = \Delta_0 \) and \( \text{span}\{Q_\mu(u_\mu) \mid \mu \in \Omega\} = t \) then it is weak-Berger.

Proof. The decomposition of \( \mathcal{B}_H(g) \) and \( V \) into weight spaces and the fact that \( Q_\phi(u_\mu) \in g_{\phi + \mu} \) imply the following inclusion

\[ g_H = \text{span}\{Q_\phi(u_\mu) \mid \phi + \mu \in \Gamma\} \subset \bigoplus_{\beta \in \Gamma} g_\beta. \]

But if \( g = \bigoplus_{\alpha \in \Delta_0} g_\alpha \) is weak-Berger it holds that \( g \subset g_H \) and thus

\[ \bigoplus_{\alpha \in \Delta_0} g_\alpha \subset \bigoplus_{\beta \in \Gamma} g_\beta \subset \bigoplus_{\alpha \in \Delta_0} g_\alpha. \]

But this implies \( \Gamma = \Delta_0 \).

If now \( \Gamma = \Delta_0 \) and \( \text{span}\{Q_\mu(u_\mu) \mid \mu \in \Omega\} = t \) we have that

\[ g_H = \text{span}\{Q_\phi(u_\mu) \mid \phi + \mu \in \Gamma\} = \bigoplus_{\beta \in \Gamma} g_\beta = t \oplus \bigoplus_{\beta \in \Delta} g_\beta = g. \]

This completes the proof. \( \square \)

To derive necessary conditions for the weak Berger property we have to fix a notation.

Let \( \alpha \in \Delta \) be a root. Then we denote by \( \Omega_\alpha \) the following subset of \( \Omega \):

\[ \Omega_\alpha := \{\lambda \in \Omega \mid \lambda + \alpha \in \Omega\}. \]

Then of course \( \alpha + \Omega_\alpha \) are the weights of \( g_\alpha V \).

2.6 Proposition. Let \( g \) be a semisimple Lie algebra with roots \( \Delta \) and \( \Delta_0 = \Delta \cup \{0\} \).

Let \( g \subset \mathfrak{so}(V, H) \) irreducible, weak-Berger with weights \( \Omega \). Then the following properties are satisfied:

(P1) There is a \( \mu \in \Omega \) and a hyperplane \( U \subset t^* \) such that

\[ \Omega \subset \{\mu + \beta \mid \beta \in \Delta_0\} \cup U \cup \{-\mu + \beta \mid \beta \in \Delta_0\}. \tag{8} \]

(PII) For every \( \alpha \in \Delta \) there is a \( \mu_\alpha \in \Omega \) such that

\[ \Omega_\alpha \subset \{\mu_\alpha - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\mu_\alpha + \beta \mid \beta \in \Delta_0\}. \tag{9} \]

Proof. If \( g \) is weak-Berger we have \( \Gamma = \Delta_0 \). We will use this property for \( 0 \in \Delta_0 \) as well as for every \( \alpha \in \Delta \).

(P1) \( \Gamma = \Delta_0 \) implies that there is \( \phi \in \Pi \) and \( \mu \in \Omega \) such that \( 0 = \phi + \mu \) with \( Q \in \mathcal{B}_\phi \) and \( u \in V_\mu \) such that \( 0 \neq Q(u) \in t \), i.e. \( \phi = -\mu \in \Pi \). We fix such \( u, Q \) and \( \mu \). For arbitrary \( \lambda \in \Omega \) then occur the following cases:
Let $(\Pi I)$. Hence $B$ or at least one of $Q$.

But this is $(\Pi I)$. Let now every weight vector $v$ and have that $\alpha H$.

For every $\lambda$, $\mu$ be a weight in $\Omega$ with $\mu \in Q$, $u \in V_{\mu}$, such that $0 \neq Q(u) \in g_{\alpha}$. We fix $Q$ and $u$ for $\alpha$ and have that $\alpha - \mu = \phi \in \Pi$ a weight of $B_{H}$.

Let now $\lambda$ be a weight in $\Omega_\alpha$, i.e. $\lambda + \alpha$ is also a weight. Thus $-\lambda - \alpha$ is a weight. If $v \in V_\lambda$ then $Q(u)v \in V_{\lambda + \alpha}$. Since $H$ is non-degenerate there is a $w \in V_{-\lambda - \alpha}$ such that $H(Q(u)v, w) \neq 0$. Since $Q \in B_{H}(g)$ the Bianchi identity then gives

$$0 = H(Q(u)v, w) + H(Q(v)w, u) + H(Q(w)u, v),$$

i.e. at least one of $Q(v)$ or $Q(w)$ has to be non-zero. Hence we have two cases for $\lambda \in \Omega_\alpha$:

**Case 1:** $Q(v) \neq 0$. This implies $-\mu + \lambda \in \Delta_0$, i.e. $\lambda \in \{\mu + \beta + \beta, \beta \in \Delta_0\}$.

**Case 2:** $Q(w) \neq 0$. This implies $-\mu + \alpha - \lambda = -\mu - \lambda \in \Delta_0$, i.e. $\lambda \in \{-\mu + \beta | \beta \in \Delta_0\}$.

But this is $(\Pi I)$.

Of course it is desirable to find weights $\mu$ and $\mu_\alpha$ which are extremal in order to handle criteria (PI) and (PII). To show in which sense this is possible we need a

**2.7 Lemma.** Let $g \subset \mathfrak{so}(V, H)$ an irreducible acting, complex semisimple Lie algebra. For an extremal weight vector $u \in V_\Lambda$ there is a weight element $Q \in B_{H}(g)$ such that $Q(u) \neq 0$.

**Proof.** Let $u \in V_\Lambda$ be extremal with $Q(u) = 0$ for every weight element $Q$. Since $B_{H}(g) = \bigoplus_{\phi \in \Pi} B_\phi$ the assumption implies $Q(u) = 0$ for all $Q \in B_{H}(g)$. But this gives for every $A \in g$ and every weight element $Q$ that

$$Q(Au) = [A, Q(u)] - (A \cdot Q)(u) = 0.$$

On the other hand $V$ is irreducible and that's why generated as vector space by elements of the form $A_1 \cdot \ldots \cdot A_k \cdot u$ with $A_i \in g$ and $k \in \mathbb{N}$ (see for example [Ser87]). By successive application of $g$ to $u$ we get that $Q(v) = 0$ for every weight element $Q$ and every weight vector $v$. But this gives $Q(v) = 0$ for all $Q \in B_{H}(g)$ and every $v \in V$, hence $B_{H}(g) = 0$. 

\[\square\]
2.8 Proposition. Let $g$ be a semisimple Lie algebra with roots $\Delta$ and $\Delta_0 = \Delta \cup \{0\}$. Let $g \subset \mathfrak{so}(V,H)$ irreducible, weak-Berger with weights $\Omega$. Then there is an ordering of $\Delta$ such that the following holds: If $\Lambda$ is the highest weight of $g \subset \mathfrak{so}(V,H)$ with respect to that ordering, then the following properties are satisfied:

(QI) There is a $\delta \in \Delta_+ \cup \{0\}$ and a hyperplane $U \subset \mathfrak{t}^*$ such that

$$\Omega \subset \{\Lambda - \delta + \beta \mid \beta \in \Delta_0\} \cup U \cup \{-\Lambda + \delta + \beta \mid \beta \in \Delta_0\}. \quad (10)$$

If $\delta$ can not be chosen to be zero, then holds

(QII) There is an $\alpha \in \Delta$ such that

$$\Omega_\alpha \subset \{\Lambda - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\Lambda + \beta \mid \beta \in \Delta_0\}. \quad (11)$$

Proof. First we consider the extremal weights of the representation, i.e. the images of the highest weight under the Weyl group. These do not lie in one hyperplane (because this would imply that all roots lie in one hyperplane). Thus by proposition 2.6 — fixing $\mu \in \Omega$ — there is an extremal weight $\Lambda$ with $\Lambda + \mu \in \Delta_0$ or $\Lambda - \mu \in \Delta_0$. This one we fix.

Since the Weyl group acts transitively on the extremal weights we can find a fundamental root system, i.e. an ordering on the roots, such that $\Lambda$ is the highest weight. With respect to this fundamental root system the roots splits into positive and negative roots $\Delta = \Delta_+ \cup \Delta_-$. This implies

$$\mu = \varepsilon(\Lambda - \delta) \quad (12)$$

with $\delta \in \Delta_+$ and $\varepsilon = \pm 1$.

For an arbitrary $\lambda \in \Omega$ then holds $\lambda \in U = Q(u)^\perp$ or $\lambda + \mu \in \Delta_0$ or $\lambda = \mu \in \Delta_0$. But with (12) this implies that we find an $\beta \in \Delta_0$ such that $\lambda = \pm(\Lambda - \delta) + \beta$ with $\beta \in \Delta_0$. This is (QI). Note that we are still free to choose $\Lambda$ or $-\Lambda$ as highest weight.

Now we suppose that $\delta$ can not be chosen to be zero. Let $v \in V_\Lambda$ be a highest weight vector or $v \in V_{-\Lambda}$. Looking at the proof of proposition 2.6 one has that for all weight elements $Q \in \mathcal{B}(g)$ holds $Q(v) \in g_\alpha$ for a $\alpha \in \Delta$. Since $g$ is weak-Berger $\mathcal{B}(g)$ is non-zero. Thus we get by lemma 2.7 that there is a weight element $Q$ such that $0 \neq Q(v) \in g_\alpha$ and we are done (possibly by making $-\Lambda$ to the highest weight). \qed

Representations of $\mathfrak{sl}(2,\mathbb{C})$ To illustrate how these criteria shall work we apply them to irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$.

2.9 Proposition. Let $V$ be an irreducible, complex, orthogonal $\mathfrak{sl}(2,\mathbb{C})$–module of highest weight $\Lambda$. If it is weak-Berger then $\Lambda \in \{2,4\}$.
Proof. Let $\mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{so}(N, \mathbb{C})$ be an irreducible representation of highest weight $\Lambda$. I.e. $\Lambda(H) = l \in \mathbb{N}$ for $\mathfrak{sl}(2, \mathbb{C}) = \text{span}(H, X, Y)$ where $X$ has the root $\alpha$. Since the representation is orthogonal, $l$ must be even and 0 is a weight. The hypersurface $U$ is the point 0. Now property (3) ensures that $l \in \{2, 4, 6\}$. If $\mu = \Lambda$ we obtain $l \in \{2, 4\}$. If $\mu \neq \Lambda$ we can apply (QII): We have that $\Omega_\alpha = \Omega \setminus \{\Lambda\}$ and $\Omega_{-\alpha} = \Omega \setminus \{-\Lambda\}$. Then (QII) implies that $l \in \{2, 4\}$. □

So we get the first result.

2.10 Corollary. Let $\mathfrak{su}(2) \subset \mathfrak{so}(E, h)$ be a real irreducible weak-Berger algebra of real type. Then it is a Berger algebra. In particular it is equivalent to the Riemannian holonomy representations of $\mathfrak{so}(3, \mathbb{R})$ on $\mathbb{R}^3$ or of the symmetric space of type AI, i.e. $\mathfrak{su}(3)/\mathfrak{so}(3, \mathbb{R})$ in the compact case or $\mathfrak{sl}(3, \mathbb{R})/\mathfrak{so}(3, \mathbb{R})$ in the non-compact case.

2.3 Berger algebras, weak Berger algebras and spanning triples

In this section we will describe a result of [MS99] and [Sch99], where holonomy groups of torsionfree connections, i.e. Berger algebras, are classified. We will describe our results in their language such that we can use a partial result of [Sch99].

For a Berger algebra holds that for every $\alpha \in \Delta_0$ there is a weight element $R \in K(\mathfrak{g})$ and weight vectors $u_1 \in V_{\mu_1}$ and $u_2 \in V_{\mu_2}$ such that $0 \neq R(u_1, u_2) \in \mathfrak{g}_\alpha$. The Bianchi identity then gives for an arbitrary $v \in V$

$$R(u_1, u_2)v = R(v, u_2)u_1 + R(u_1, v)u_2.$$ 

Choosing now $u_1, u_2$ such that $0 \neq R(u_1, u_2) \in \mathfrak{t}$ one gets for any $\lambda \in \Omega$ and $v \in V_\lambda$ that

$$\lambda(R(u_1, u_2))v = R(v, u_2)u_1 + R(u_1, v)u_2.$$ 

This implies $\lambda \in (R(u_1, u_2))^\perp \subset \mathfrak{t}^*$ or $V_\lambda \subset \mathfrak{g}V_{\mu_1} \oplus \mathfrak{g}V_{\mu_2}$. This gives property

(RI) There are weights $\mu_1, \mu_2 \in \Omega$ such that

$$\Omega \subset \{\mu_1 + \beta \mid \beta \in \Delta_0\} \cup U \cup \{\mu_2 + \beta \mid \beta \in \Delta_0\}.$$ 

If one chooses $u_1, u_2$ such that $0 \neq R(u_1, u_2) = A_\alpha \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta$ then one gets for $\lambda \in \Omega$ that $A_\alpha V_\lambda \subset \mathfrak{g}V_{\mu_1} \oplus \mathfrak{g}V_{\mu_2}$. This means that the weights of $A_\alpha V_\lambda$ are contained in $\{\mu_1 + \beta \mid \beta \in \Delta_0\} \cup \{\mu_2 + \beta \mid \beta \in \Delta_0\}$. But this is property

(RII) For every $\alpha \in \Delta$ there are weights $\mu_1, \mu_2 \in \Omega$ such that

$$\Omega_\alpha \subset \{\mu_1 - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{\mu_2 - \alpha + \beta \mid \beta \in \Delta_0\}.$$ 

Of course our (PI) is a special case of (RI) with $\mu_1 = -\mu_2$. (PII) is not a special case of (RII) since $\mu_\alpha + \alpha$ is not a weight apriori.
To describe this situation further in [Sch99] the following definitions are made. We point out that here $\Omega_\alpha$ does not denote the weights of $g_\alpha V$ but the weights $\lambda$ of $V$ such that $\lambda + \alpha$ is a weight.

2.11 Definition. Let $g \subset \text{End}(V)$ be an irreducible acting complex Lie algebra, $\Delta_0$ be the roots and zero of the semisimple part of $g$, $\Omega$ the weights of $g$ and $\Omega_\alpha$ as above.

1. A triple $(\mu_1, \mu_2, \alpha) \in \Omega \times \Omega \times \Delta$ is called spanning triple if

$$\Omega_\alpha \subset \{\mu_1 - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{\mu_2 - \alpha + \beta \mid \beta \in \Delta_0\}.$$ 

2. A spanning triple $(\mu_1, \mu_2, \alpha)$ is called extremal if $\mu_1$ and $\mu_2$ are extremal.

3. A triple $(\mu_1, \mu_2, U)$ with $\mu_1, \mu_2$ extremal weights and $U$ an affine hyperplane in $t^*$ is called planar spanning triple if every extremal weight different from $\mu_1$ and $\mu_2$ is contained in $U$ and $\Omega \subset \{\mu_1 + \beta \mid \beta \in \Delta_0\} \cup U \cup \{\mu_2 + \beta \mid \beta \in \Delta_0\}$.

From (RI) and (RII) in [Sch99] the following proposition is deduced.

2.12 Proposition. [Sch99] Let $g \subset \text{End}(V)$ be an irreducible complex Berger algebra. Then for every root $\alpha \in \Delta$ there is a spanning triple. Furthermore there is an extremal spanning triple or a planar spanning triple.

If we return to the weak-Berger case we can reformulate proposition 2.8 as follows.

2.13 Proposition. Let $g \subset \text{so}(V,H)$ be an irreducible complex weak-Berger algebra. Then there is an extremal weight $\Lambda$ such that one of the following properties is satisfied:

(SI) There is a planar spanning triple of the form $(\Lambda, -\Lambda, U)$.

(SII) There is an $\alpha \in \Delta$ such that $\Omega_\alpha \subset \{\Lambda - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\Lambda + \beta \mid \beta \in \Delta_0\}$.

There is a fundamental system such that the extremal weight in (SI) and (SII) is the highest weight.

Proof. The proof is analogous the the one of proposition 2.8. If there is an $\alpha \in \Delta$ such that the corresponding $\mu_\alpha$ is extremal we are done. If not, then for every extremal weight vector $u \in V_\Lambda$ and every weight element $Q \in \mathcal{B}_g$ holds that $Q(u) \in t^*$. Then by lemma 2.7 there is a $Q$ such that $0 \neq Q(u) \in t^*$. As before this implies

$$\Omega \subset \{\Lambda + \beta \mid \beta \in \Delta_0\} \cup U \cup \{-\Lambda + \beta \mid \beta \in \Delta_0\}.$$ 

To ensure that $(\Lambda, -\Lambda, U)$ is a planar spanning triple we have to show that every extremal weight $\lambda$ different from $\Lambda$ and $-\Lambda$ is contained in $U = Q(u)^{\perp}$. Let $\lambda$ be
extremal and different from Λ and −Λ, \( v_± \in V_{±Λ} \) and \( u \in V_Λ \). Since \( Q(v_±) \in t \) the Bianchi identity gives

\[
0 = H(Q(u)v_+, v_-) + H(Q(v_+)u, v_-) + H(Q(v_-)u, v_+)
\]

\[
= \lambda(Q(u))(v_+ v_-) - \lambda(Q(v_+)) H(v_-, u) + Λ(Q(v_-)) H(u, v_+).
\]

Hence \( \lambda \in U \).

Obviously we are in a slightly different situation as in the Berger case since \(-Λ + α \) is not necessarily a weight and in case it is a weight, it is not extremal in general.

### 2.4 Properties of root systems

In this section we will recall the properties of abstract root systems. Let \((E, ⟨.,.⟩)\) be a euclidian vector space. A finite set of vectors \( ∆ \) is called root system if it satisfies the following properties

1. \( ∆ \) spans \( E \).
2. For every \( α \in ∆ \) the reflection on the hyperplane perpendicular to \( α \) defined by

\[
\rho_α(φ) := φ - \frac{2⟨φ, α⟩}{∥α∥^2} α.
\]

maps \( ∆ \) onto itself.
3. For \( α, β \in ∆ \) the number \( \frac{2⟨β, α⟩}{∥α∥^2} \) is an integer.

A root system is called indecomposable if it does not split into orthogonal subsets. It is called reduced if \( 2α \) is not a root if \( α \) is a root.

The indecomposable, reduced root systems corresponds to the roots of simple Lie algebras. They are classified in a finite list: \( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 \) and \( G_2 \).

The index designates the dimension of \( E \).

We will cite some basic properties of root systems, which can be found for example in [Kna02].

#### 2.14 Proposition. (See for example [Kna02], pp. 149) Let \( ∆ \) be an abstract reduced root system in \((E, ⟨.,.⟩)\).

1. If \( α \in ∆ \), then the only root which is proportional to \( α \) is \(-α\).
2. If \( α, β \in ∆ \), then \( \frac{2⟨β, α⟩}{∥α∥^2} \in \{0, ±1, ±2, ±3\} \). If \( ∆ \) is one of the indecomposable root systems \( ±3 \) occurs only for the root system \( G_2 \). If both roots are non proportional then \( ±2 \) only occurs for \( B_n, C_n, F_4 \) or \( G_2 \).
3. If \( α \) and \( β \) are nonproportional in \( ∆ \) and \( ∥β∥ ≤ ∥α∥ \), then \( \frac{2⟨β, α⟩}{∥α∥^2} \in \{0, ±1\} \).
4. Let be $\alpha, \beta \in \Delta$. If $\langle \alpha, \beta \rangle > 0$, then $\alpha - \beta \in \Delta$. If $\langle \alpha, \beta \rangle < 0$, then $\alpha + \beta \in \Delta$.
I.e. if neither $\alpha - \beta \in \Delta$ nor $\alpha + \beta \in \Delta$, then $\langle \alpha, \beta \rangle = 0$.

5. The subset of $\Delta$ defined by $\{ \beta + k\alpha \in \Delta \cup \{0\} | k \in \mathbb{Z} \}$ is called $\alpha$–string through $\beta$. It has no gaps, i.e. $\beta + k\alpha \in \Delta$ for $-p \leq k \leq q$ with $p, q \geq 0$ and it holds $p - q = \frac{2\langle \beta, \alpha \rangle}{\|\alpha\|^2}$. The maximal length of such string is given by $\max_{\alpha, \beta \in \Delta} \frac{2\langle \beta, \alpha \rangle}{\|\alpha\|^2} + 1$, i.e. it contains at most four roots.

As a consequence of that proposition we get the following lemmata. In these we will refer to long and short roots. This notion is evident because in the indecomposable reduced root systems of type $B_n$, $C_n$, $F_4$ and $G_2$ the roots have two different lengths.

2.15 Lemma. Let $\Delta$ be an indecomposable, reduced root system. Then it holds:

1. If $a\alpha + \beta \in \Delta$ for $a \in \mathbb{N}$ and $a > 1$, then $\langle \alpha, \beta \rangle < 0$ and $\alpha$ is a short root.

2. If $\Delta$ is a root system, where the roots have equal length or if $\alpha$ is a long root, then $\alpha + \beta \in \Delta$ implies $\langle \alpha, \beta \rangle < 0$.

3. Let $\alpha$ and $\beta$ be two short roots. If $\alpha + \beta$ is a long root then $\langle \alpha, \beta \rangle = 0$, if it is a short one then $\langle \alpha, \beta \rangle < 0$. The sum of a short and a long root is a short one.

4. If $\beta$ is a long root in $\Delta \neq G_2$, then there are orthogonal roots $\alpha$ and $\gamma$ such that $\beta = \alpha + \gamma$.

Proof. The proof follows directly from proposition 2.14.

2.16 Lemma. Let $\alpha$ and $\beta$ be two nonproportional roots and $a \in \mathbb{N}$. If $a(\alpha + \beta) \in \Delta$ then $a = 1$.

Proof. If $a > 1$ then $\alpha + \beta$ is not a root. This implies $\langle \alpha, \beta \rangle \geq 0$ and yields for $a(\alpha + \beta) = \gamma \in \Delta$:

$$0 < a(\|\alpha\|^2 + \langle \alpha, \beta \rangle) = \langle \alpha, \gamma \rangle$$

$$0 < a(\langle \alpha, \beta \rangle + \|\beta\|^2) = \langle \gamma, \beta \rangle.$$ 

On the other hand we have

$$\|\gamma\|^2 = a(\langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle).$$

But this gives

$$1 = a\left( \frac{2\langle \gamma, \alpha \rangle}{\|\gamma\|^2} + \frac{2\langle \gamma, \beta \rangle}{\|\gamma\|^2} \right) \geq 2 \text{ in } \mathbb{N}.$$ 

This is a contradiction. Hence $a = 1$. 

The next lemma is a little more general.
2.17 Lemma. Let \( \alpha \) and \( \beta \) be two non-proportional roots in an indecomposable root system and \( a, b \in \mathbb{N} \) with \( a \leq b \) such that \( a\alpha + b\beta \in \Delta \).

1. If \( \Delta \) is not \( G_2 \) then \( a = 1 \). If \( \Delta = A_n, D_n, E_6, E_7, E_8 \) then \( b = 1 \) too. If \( \Delta = B_n, C_n, F_4 \) then \( b \leq 2 \).

2. If \( \Delta = G_2 \) then \( a \leq 2 \) and \( b \leq 3 \).

Proof. We suppose \( a\alpha + b\beta = \gamma \in \Delta \).

First we consider the case \( \langle \alpha, \beta \rangle \geq 0 \). This gives

\[
0 < a\|\alpha\|^2 + b\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle
\]

On the other hand we have \( \|\gamma\|^2 = a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle \) and thus

\[
1 = \frac{a}{2} \frac{2\langle \gamma, \alpha \rangle}{\|\gamma\|^2} + \frac{b}{2} \frac{2\langle \gamma, \beta \rangle}{\|\gamma\|^2}
\]

Hence \( a = 1 \).

Let now be \( \langle \alpha, \beta \rangle < 0 \). This implies, that \( \alpha + \beta =: \delta \) is a root with the property \( \delta - \beta = \alpha \in \Delta \).

Although the above proposition does not assert that this implies \( \langle \delta, \beta \rangle \geq 0 \) we can show this. Suppose that \( \langle \delta, \beta \rangle < 0 \). Hence \( \delta + \beta = \alpha + 2\beta \) is a root. If we exclude the root system \( G_2 \) point 5 of proposition 2.14 implies \( \frac{2\langle \alpha, \delta \rangle}{\|\delta\|^2} = -2 \), i.e. \( \langle \alpha, \beta \rangle = -\|\beta\|^2 \) and finally \( \langle \delta, \beta \rangle = 0 \), which was excluded.

Thus we have that \( \langle \delta, \beta \rangle \geq 0 \). Analogously to the first case we get

\[
1 = \frac{a}{2} \frac{2\langle \gamma, \delta \rangle}{\|\gamma\|^2} + \frac{(b - a)}{2} \frac{2\langle \gamma, \beta \rangle}{\|\gamma\|^2}.
\]

In case that \( a \leq b - a \) we get again that \( a = 1 \). Otherwise we get \( b - a = 1 \), i.e. \( a\delta + \beta = \gamma \). Again by point 5 of proposition 2.14 we get \( p - q = \frac{2\langle \gamma, \delta \rangle}{\|\delta\|^2} \geq 0 \). But this implies \( a \leq 1 \).

The possible values for \( b \) follow also by proposition 2.14.

For \( G_2 \) the possible values of \( a \) and \( b \) can be calculated analogously.

2.18 Lemma. Let \( \eta \) be a long root of an indecomposable root system.

1. Let \( a, b \in \mathbb{N} \) and \( \alpha \in \Delta \) not proportional to \( \eta \) such that \( a\eta + b\alpha \in \Delta \). Then \( a \leq b \), i.e. \( a = 1 \) if \( \Delta \) not equal to \( G_2 \) and \( a \leq 2 \) otherwise.
2. Let $\alpha, \beta$ in $\Delta$ not proportional to $\eta$ and $a \in \mathbb{N}$ such that $a \eta + \alpha + \beta \in \Delta$. Then $a \leq 2$.

Proof. 1.) First we exclude $G_2$ and suppose that $b = 1$, i.e. $a \eta + \alpha = \gamma \in \Delta$. Hence $-p \leq a \leq q$ and

$$|p - q| = \frac{2 |\langle \eta, \alpha \rangle|}{\|\eta\|^2} < 2 \frac{\|\eta\| \cdot \|\eta\|}{\|\eta\|^2} \leq 2,$$

i.e. $|p - q| \leq 1$. But since we have excluded $G_2$ we have that $a = 1$.

For $G_2$ a long root $\eta$ is given by $2e_3 - e_1 - e_2$ with the notations of appendix C of [Kna02]. For this we get the wanted result.

2.) Let $a \eta + \alpha + \beta = \gamma$.

First we consider the case that $\alpha + \beta$ or $\alpha - \gamma$ or $\beta - \gamma$ is a root. If this root is not proportional to $\eta$ we have by the first point that $a \leq 1$. If it is proportional to $\eta$ we get that $a \leq 2$ and we are done.

Now we suppose that neither $\alpha + \beta$ nor $\alpha - \gamma$ nor $\beta - \gamma$ is a root. This implies $\langle \alpha, \beta \rangle \geq 0$, $\langle \alpha, \gamma \rangle \leq 0$ and $\langle \beta, \gamma \rangle \leq 0$. We consider the equations

$$a \langle \eta, \alpha \rangle + \|\alpha\|^2 + \langle \alpha, \beta \rangle = \langle \gamma, \alpha \rangle \leq 0$$
$$a \langle \eta, \beta \rangle + \|\beta\|^2 + \langle \alpha, \beta \rangle = \langle \gamma, \beta \rangle \leq 0$$
$$a \langle \eta, \gamma \rangle + \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle = \|\gamma\|^2 > 0.$$

Hence we have that $\langle \eta, \alpha \rangle < 0$, $\langle \eta, \beta \rangle < 0$ and $\langle \eta, \gamma \rangle > 0$. But since $\eta$ is long, not proportional neither to $\alpha$ nor to $\beta$ we have that

$$\|\eta\|^2 \geq \langle \gamma, \eta \rangle = a \|\eta\|^2 + \langle \alpha, \eta \rangle + \langle \beta, \eta \rangle$$
$$= a \|\eta\|^2 + \langle \alpha, \eta \rangle - \|\alpha\|^2 - \langle \beta, \eta \rangle$$
$$< \langle \alpha, \eta \rangle - \|\alpha\|^2 + \|\beta\|^2 - \langle \beta, \eta \rangle$$
$$< (a - 2) \|\eta\|^2.$$

This gives $a - 2 < 1$ which is the proposition.

2.19 Lemma. Let $\alpha$ be a long root and $\eta$ be a short one with $\langle \alpha, \eta \rangle > 0$, i.e. $\frac{2 \langle \alpha, \eta \rangle}{\|\eta\|^2} \geq 2$.

Then there is a short root $\beta$ with $\beta \not\sim \eta$, $\langle \beta, \alpha \rangle < 0$ and $\langle \beta, \eta \rangle \leq 0$. If the rank of the root system is greater than 2 or if $\frac{2 \langle \alpha, \eta \rangle}{\|\eta\|^2} = 3$ (which can only occur for $G_2$), $\beta$ can be chosen such that $\langle \beta, \eta \rangle < 0$.

Proof. $\langle \alpha, \eta \rangle > 0$ implies that $\eta - \alpha$ is a root, in particular a short one. For the inner product we get

$$\langle \alpha, \eta - \alpha \rangle = \langle \alpha, \eta \rangle - \|\alpha\|^2 < \|\alpha\| \|\eta\| - \|\alpha\|^2 < \|\alpha\|^2 - \|\alpha\|^2 = 0$$
and
\[ \langle \eta, \eta - \alpha \rangle = \|\eta\|^2 - \langle \alpha, \eta \rangle \leq 0. \]
In case of \( \frac{2 \langle \alpha, \eta \rangle}{\|\eta\|^2} = 3 \) the last \( \leq \) is a \(<\), and we are done with the second point in case of \( G_2 \).
If the rank of the root system is greater than 2 this can be seen with the help of the definitions of the reduced, indecomposable root systems (see appendix C of [Kna02]).

\[ \square \]

3 Simple weak-Berger algebras of real type

In this section we will apply the result of proposition 2.13 to simple complex irreducible acting Lie algebras.

We will do this step by step under the following special conditions:

1. The highest weight of the representation is a root.
2. The representation satisfies (SI), i.e. admits a planar spanning triple \((\Lambda, -\Lambda, U)\).
3. The representation satisfies (SII) and has weight zero.
4. The representation satisfies (SII) and does not have weight zero.

Throughout this section the considered Lie algebra is supposed to be different from \( \mathfrak{sl}(2, \mathbb{C}) \).

3.1 Representations with roots as highest weight

3.1 Proposition. Let \( g \subset \mathfrak{so}(N, \mathbb{C}) \) be an irreducible representation of real type of a complex simple Lie algebra different from \( \mathfrak{sl}(2, \mathbb{C}) \) and satisfying (SI) or (SII). If we suppose in addition that there is an extremal weight \( \Lambda \) with \( \Lambda = a\eta \) for a root \( \eta \in \Delta \) and \( a > 0 \), then holds the following:

1. If \( \eta \) is a long root, then \( a = 1 \) and the representation is the adjoint one.
2. If \( \eta \) is a short root, then holds the following for \( a \):
   
   (a) If \( \Delta = B_n \) or \( G_2 \) then \( a = 1, 2 \).
   
   (b) If \( \Delta = C_n \) or \( F_4 \) then \( a = 1 \).

Proof. Let \( \Lambda = a\eta \) with \( \eta \in \Delta \), \( a \in \mathbb{N} \). W.l.o.g. we may suppose that \( \Lambda \) is the extremal weight in the properties (SI) and (SII). (If not then there is an element of the Weyl group \( \sigma \) mapping \( \Lambda \) to the extremal weight of (SI) and (SII) \( \Lambda' \). Then \( \Lambda' = a\sigma\eta \) and \( \sigma\eta \in \Delta \).)

First we show that \( a \in \mathbb{N} \). If we chose a fundamental system \((\pi_1, \ldots, \pi_n)\) such that \( \Lambda = a\eta \) is the highest weight we get that \( \langle \Lambda, \pi_i \rangle = a\langle \eta, \pi_i \rangle \in \mathbb{N} \) for all \( i \). \( a \notin \mathbb{N} \) would
imply that \( \langle \eta, \pi_i \rangle \geq 2 \) for all \( i \) with \( \langle \eta, \pi_i \rangle \neq 0 \). This holds only for the root system \( C_n \) where \( \Lambda = \omega_1 = \frac{1}{2} \eta \). But this representation is symplectic but not orthogonal. (For an explicit formulation of this criterion see [OV94].) So we get \( a \in \mathbb{N} \).

Now we consider two cases.

**Case 1: \( \eta \) is a long root:** In this case the root system of long roots, denoted by \( \Delta_l \) is the orbit of \( \eta \) under the Weyl group. Hence \( a \cdot \Delta_l \) are the extremal weights and \( \Delta \subset \Omega \). This implies \( 0 \in \Omega_\alpha \) for every \( \alpha \in \Delta \).

Furthermore for all roots holds that \( a \cdot \Delta \subset \Omega \). This is true because we can find a short root such that \( \langle \eta, \beta \rangle > 0 \). This implies \( a\eta - a\beta = a(\eta - \beta) \in \Omega \). Applying the Weyl group to this weight we get the property for all short roots.

(SI) Let \( \Lambda \) satisfy (SI), i.e. \( \Lambda \) and \( -\Lambda \) define a planar spanning triple \( (\Lambda, -\Lambda, U) \).

This would imply that every long root different from \( \eta \) lies in the hyperplane \( U \). This is only possible for the root system \( C_n \), because all other root systems have an indecomposable system of long roots. For \( C_n \) holds that \( \Delta_l = A_1 \times \ldots \times A_1 \). But we have still a root \( \beta \) — possibly a short one — such that \( \beta \notin U \) and \( \beta \) not proportional to \( \eta \). This implies \( \Omega \ni a\beta = \Lambda + \gamma = a\eta + \gamma \) with \( \gamma \in \Delta_0 \). Then Lemma 2.16 implies \( a = 1 \).

(SII) Lets suppose that \( \Lambda \) satisfies (SII), i.e. there is an \( \alpha \in \Delta \) such that \( \Omega_\alpha \subset \{ \Lambda - \alpha + \beta | \beta \in \Delta_0 \} \cup \{-\Lambda + \beta | \beta \in \Delta_0 \} \). \( 0 \in \Omega_\alpha \) implies \( 0 = \Lambda - \alpha + \beta = a\eta - a\beta \) or \( 0 = -\Lambda + \beta = -a\eta + \beta \) with \( \beta \in \Delta_0 \). The second is not possible and the first implies by lemma 2.18 that \( a = 1 \) or \( a = 2 \) and \( \eta = \alpha \). In the second case we find a root \( \gamma \neq \alpha \) such that \( \langle \gamma, \alpha \rangle < 0 \), hence \( 2\gamma \in \Omega_\alpha \). Since \( 2\gamma - 2\alpha \notin \Delta \) it has to be \( 2\gamma = \alpha + \beta \), but this is prevented by \( \langle \gamma, \alpha \rangle < 0 \) and lemma 2.15.

Of course if \( \eta \) is a long root the representation is the adjoint one.

**Case 2: \( \eta \) is a short root:** Lets denote by \( \Delta_s \) the root system of short roots. It equals to the orbit of \( \eta \) under the Weyl group. It is a root system of the same rank as \( \Delta \) and all roots have the same length. Clearly \( \Delta_s \subset \Omega \) and \( a \cdot \Delta_s \) are the extremal weights in \( \Omega \). For the root system \( B_n \) the root system of short roots \( \Delta_s \) equals to \( A_1 \times \ldots \times A_1 \), otherwise it is indecomposable.

Furthermore holds the following: If \( a \geq 2 \) then \( \Delta \subset \Omega \). To verify this, we consider a long root \( \beta \in \Delta_l \) with the property that \( \langle \beta, \eta \rangle > 0 \). Such a \( \beta \) always exists. Then we have \( \frac{2\langle \eta, \beta \rangle}{\|\eta\|^2} \geq \frac{2\langle \eta, \beta \rangle}{\|\beta\|^2} \geq 1 \). This implies \( 2\eta - \beta \in \Delta \) (see proposition 2.14). On the other hand \( a \geq 2 \) ensures that \( \Omega \ni s_\beta(2\eta) = 2 \left( \eta - \frac{2\langle \eta, \beta \rangle}{\|\beta\|^2} \beta \right) \). This implies
that the long root $2\eta - \beta$ is a weight. Now applying the Weyl group to $\beta$ shows
that every long root is a weight.

(Si) We suppose that there is a planar spanning triple $(\Lambda, -\Lambda, U)$. This implies
that $a\beta$ lies in the hyperplane $U$ if $\beta$ is a short root. But this is only possible
for $B_n$ because the short roots of all other root systems are indecomposable.
In case of $B_n$ we can at least find a long root $\alpha$ which is not in $U$. Since the
long roots are weights, we have $\alpha = a\eta + \gamma$ or $\alpha = -a\eta + \gamma$ with $\gamma \in \Delta_0$.

(Sii) Suppose that there is an $\alpha \in \Delta$ such that $\Omega_\alpha \subset \{\Lambda - \alpha + \beta | \beta \in \Delta_0\} \cup
\{-\Lambda + \beta | \beta \in \Delta_0\}$. $\Delta \subset \Omega$ implies $0 \in \Omega_\alpha$ for all $\alpha$. $0 = -a\eta + \gamma$ with
$\gamma \in \Delta_0$ implies $a = 1$. Hence if we suppose $a \geq 2$ we must have

$$0 = a\eta - \alpha + \gamma \quad (13)$$

Thus we have to deal with the following cases:

(a) $\alpha = \eta$ and $a = 2$.
(b) $\alpha \not\sim \eta$ and by (ii) of proposition $2.13$ $a \leq \frac{2\langle \eta, \alpha \rangle}{\|\eta\|^2} \leq 3$. I.e. if $a \geq 2$, $\alpha$ is a
long root.

We exclude the first case for any root system different from $B_n$. Set $a = 2
and \alpha = \eta$. If $\Delta \neq B_n$ the short roots are indecomposable, i.e. there is a
short root $\beta$ such that $\beta \not\sim \eta$ and $\langle \beta, \eta \rangle < 0$. Hence $2\beta \in \Omega_\eta$ and $\beta + \eta \in \Delta$.
The existence of a spanning triple implies then $2\beta = \eta + \gamma$ or $2\beta = -2\eta + \gamma$
with $\gamma \in \Delta_0$. The second case is impossible because of lemma $2.16$. The
first implies $2\beta - \eta \in \Delta$. Again this is not possible by $2.15$ and $\langle \beta, \eta \rangle < 0$.
Hence the case (a) is excluded.

Now we consider the case (b). First we show that $a = 3$ is not possible. Set
$a = 3$. We notice that $\langle \eta, \alpha \rangle > 0$ implies $\frac{2\langle \eta, \alpha \rangle}{\|\eta\|^2} \geq 1$ and hence $3\eta - 3\alpha \in \Omega_\alpha$.
Thus we have the alternative $3\eta - 3\alpha = 3\eta - \alpha + \gamma$ or $3\eta - 3\alpha = -3\eta + \gamma$
with $\gamma \in \Delta_0$. The first implies $2\alpha \in \Delta$ and the second $6\eta - 3\alpha \in \Delta$. Both
are not true, hence $a = 3$ is impossible.

We continue with case (b) and have that $\alpha$ is a long root with

$$\frac{2\langle \eta, \alpha \rangle}{\|\eta\|^2} \geq 2, \quad \text{i.e.} \quad 2\eta - \alpha \in \Delta.$$

From now on we suppose, that the root system is different from $G_2$. Then
we have

$$\frac{2\langle \eta, \alpha \rangle}{\|\eta\|^2} = 2. \quad (14)$$

In a next step we will show that under these conditions there is no short
root $\beta$ with

$$\beta \in \Delta_s \text{ with } \langle \alpha, \beta \rangle < 0, \langle \alpha, \eta \rangle < 0 \text{ and } \beta \not\sim \eta. \quad (15)$$
Suppose that there is such a $\beta$. Then the first condition implies that $2\beta \in \Omega_\alpha$ and hence $2\beta = 2\eta - \alpha + \gamma$ or $2\beta = -2\eta + \gamma$ with $\gamma \in \Delta_0$. The latter is not possible. The second implies the following using (14):

$$-2 \geq 2 \cdot \frac{2\langle \beta, \eta \rangle}{||\eta||^2} = 2 \frac{2\langle \eta - \alpha, \eta \rangle}{||\eta||^2} + 2 \frac{2\langle \gamma, \eta \rangle}{||\eta||^2} = 2 + 2 \frac{2\langle \gamma, \eta \rangle}{||\eta||^2}.$$ 

Hence $-4 \geq \frac{2\langle \gamma, \eta \rangle}{||\eta||^2}$ which is impossible.

Now by the lemma 2.19 there is such a $\beta$. Hence for any remaining root systems different from $G_2$ and different from $B_n$ we have that $a = 1$.

All in all we have shown, that for a long root holds $a = 1$ and for a short root $a = 2$ implies $\Delta = B_n$ or $G_2$.

3.2 Corollary. Let $\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})$ be an irreducible complex simple weak Berger algebra different from $\mathfrak{sl}(2, \mathbb{C})$ and with the additional property that the highest weight is of the form $\Lambda = a\eta$ for a root $\eta \in \Delta$. Then $\mathfrak{g}$ is the complexification of a holonomy algebra of a Riemannian manifold or the representation of $G_2$ with highest weight $2\omega_1$.

Proof. Clearly if $\eta$ is a long root the representation is the adjoint one, i.e. the complexification of a holonomy representation of a Lie group with positive definite bi-invariant metric. For a short root $\eta$ we get the following:

$B_n$, $a = 1$: This is the representation of highest weight $\omega_1$, i.e. the standard representation of $\mathfrak{so}(2n + 1, \mathbb{C})$ on $\mathbb{C}^{2n+1}$. Of course this is the complexification of the generic Riemannian holonomy representation.

$B_n$, $a = 2$: This is the representation of highest weight $2\omega_1$. A further analysis shows that this is the complexified representation of the Riemannian symmetric space of type $AI$, i.e. of the symmetric spaces $SU(2n + 1)/SO(2n + 1, \mathbb{R})$, respectively $SL(2n + 1, \mathbb{R})/SO(2n + 1, \mathbb{R})$.

$C_n$, $a = 1$: (for $n \geq 3$) This is the representation of highest weight $\omega_2$. It is the complexified representation of the Riemannian symmetric space of type $AII$, i.e. of the symmetric spaces $SU(2n)/Sp(n)$, respectively $SL(2n, \mathbb{R})/Sp(n)$.

$F_4$, $a = 1$: This is the representation of highest weight $\omega_1$. It is the complexified representation of the Riemannian symmetric space of type $EIV$, i.e. of the symmetric spaces $E_6/F_4$, respectively $E_{6(-26)}/F_4$.

$G_2$, $a = 1$: This is the representation of highest weight $\omega_1$. It is the representation of $G_2$ on $\mathbb{C}^7$, i.e. the complexification of the holonomy of a Riemannian $G_2$–manifold.

$G_2$, $a = 2$: This is the representation $2\omega_1$ of $G_2$. It is a 27-dimensional representation of $G_2$ isomorphic to $\text{Sym}^2_{\mathbb{C}} \mathbb{C}^7$, where $\mathbb{C}^7$ denotes the standard module of $G_2$ and $\text{Sym}^2_{\mathbb{C}} \mathbb{C}^7$ its symmetric, trace free $(2,0)$–tensors. This is the exception, because there is no Riemannian manifold with this complexified holonomy representation.
3.2 Representations with planar spanning triples

Now we consider representations of a simple Lie algebra under the condition that there is a planar spanning triple. The proof of this proposition is a copy of the proof in [Sch99] adding the additional properties of our planar spanning triple.

3.3 Proposition. Let \( g \subset \mathfrak{so}(N, \mathbb{C}) \) be an irreducible representation of real type of a complex simple Lie algebra different from \( \mathfrak{sl}(2, \mathbb{C}) \) and satisfying (SI), i.e. with a planar spanning triple of the form \((\Lambda, -\Lambda, U)\). If there is no root \( \alpha \) such that \( \Lambda = a\alpha \) then \( g \) is of type \( D_n \) with \( n \geq 3 \) and the representation is congruent to the one with highest weight \( \omega_1 \) or \( 2\omega_1 \).

Proof. The condition \( \Lambda \neq a\alpha \) implies that there is no root such that \( -\Lambda = s\alpha(\Lambda) \). The existence of a planar spanning triple then gives that for any \( \alpha \in \Delta \) with \( \langle \Lambda, \alpha \rangle \neq 0 \) the image of the reflection lies in \( U \). If we set \( U = T^\perp \) this gives

\[
\text{For } \alpha \in \Delta \text{ with } \langle \alpha, \Lambda \rangle \neq 0 \text{ holds } \quad \langle \alpha, T \rangle = \frac{\|\alpha\|^2}{2\langle \Lambda, \alpha \rangle} \langle \Lambda, T \rangle \neq 0. \tag{16}
\]

In the following we prove various claims to get the wanted result. We follow completely the lines of reasoning in [Sch99].

Claim 1: For any non-proportional \( \alpha, \beta \in \Delta \) with \( \langle \Lambda, \alpha \rangle \neq 0 \) and \( \langle \Lambda, \beta \rangle \neq 0 \) holds that \( \langle \alpha, \beta \rangle = 0 \) or both have the same length.

To show this we prove that for two such roots hold that they are orthogonal or that \( \langle \Lambda, s_\alpha \beta \rangle = \langle \Lambda, s_\beta \alpha \rangle = 0 \).

Suppose that \( \langle \Lambda, s_\alpha \beta \rangle \neq 0 \). Then (16) gives the following

\[
\|\beta\|^2 = \|s_\alpha \beta\|^2 = \frac{2}{\langle \Lambda, T \rangle} \cdot \langle \Lambda, s_\alpha \beta \rangle \cdot \langle s_\alpha \beta, T \rangle
\]

\[
= \frac{2}{\langle \Lambda, T \rangle} \cdot \left( \langle \Lambda, \beta \rangle - \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \langle \Lambda, \alpha \rangle \right) \cdot \left( \langle \beta, T \rangle - \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \langle \alpha, T \rangle \right)
\]

\[
= 2 \cdot \left( \langle \Lambda, \beta \rangle - \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \langle \Lambda, \alpha \rangle \right) \cdot \left( \frac{\|\beta\|^2}{2\langle \Lambda, \beta \rangle} - \frac{2\langle \alpha, \beta \rangle}{\langle \Lambda, \alpha \rangle} \right)
\]

\[
= 2 \cdot \left( \frac{\|\beta\|^2}{2} - 2\langle \alpha, \beta \rangle \frac{\langle \Lambda, \beta \rangle}{\langle \Lambda, \alpha \rangle} - 2\langle \alpha, \beta \rangle \frac{\langle \Lambda, \alpha \rangle \|\beta\|^2}{\|\alpha\|^2} + 4 \frac{\langle \alpha, \beta \rangle^2}{\|\alpha\|^2} \right).
\]

Subtracting \( \|\beta\|^2 \) and multiplying by the denominators gives

\[
0 = \langle \alpha, \beta \rangle \left( \|\beta\|^2 \langle \Lambda, \alpha \rangle^2 + \|\alpha\|^2 \langle \Lambda, \beta \rangle^2 - 2\langle \beta, \alpha \rangle \langle \Lambda, \alpha \rangle \langle \Lambda, \beta \rangle \right).
\]
But this gives the following pair of equations
\[
0 = \langle \alpha, \beta \rangle \left( \left( \|\beta\| \langle \Lambda, \alpha \rangle + \|\alpha\| \langle \Lambda, \beta \rangle \right) + 2 \left( \|\alpha\| \|\beta\| + \langle \beta, \alpha \rangle \right) \langle \Lambda, \alpha \rangle \langle \Lambda, \beta \rangle \right) > 0
\]
\[
0 = \langle \alpha, \beta \rangle \left( \left( \|\beta\| \langle \Lambda, \alpha \rangle - \|\alpha\| \langle \Lambda, \beta \rangle \right) + 2 \left( \|\alpha\| \|\beta\| - \langle \beta, \alpha \rangle \right) \langle \Lambda, \alpha \rangle \langle \Lambda, \beta \rangle \right) > 0
\]
This implies \( \langle \alpha, \beta \rangle = 0 \) or \( \langle \Lambda, \alpha \rangle \langle \Lambda, \beta \rangle = 0 \), but this was excluded. This argument is symmetric in \( \alpha \) and \( \beta \) hence we get the same result for \( s_\beta \alpha \). Thus we have proved that \( \langle \Lambda, s_\alpha \beta \rangle = \langle \Lambda, s_\beta \alpha \rangle = 0 \) or \( \langle \alpha, \beta \rangle = 0 \).

Now \( \langle \Lambda, s_\alpha \beta \rangle = \langle \Lambda, s_\beta \alpha \rangle = 0 \) implies \( \langle \Lambda, \alpha \rangle = \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \cdot \frac{2\langle \alpha, \beta \rangle}{\|\beta\|^2} \cdot \langle \Lambda, \alpha \rangle \). Since \( \langle \Lambda, \alpha \rangle \) was supposed to be non-zero we have that \( \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \cdot \frac{2\langle \alpha, \beta \rangle}{\|\beta\|^2} = 1 \) which implies — since both factors are in \( \mathbb{Z} \) — that \( \|\alpha\|^2 = \|\beta\|^2 \). This holds if \( \langle \alpha, \beta \rangle \neq 0 \).

Claim 2: All roots in \( \Delta \) have the same length.

Suppose we have short and long roots. Then we can write a long root \( \alpha \) as the sum of two short ones, lets say \( \alpha = \beta + \gamma \). This implies \( \langle \alpha, \beta \rangle \neq 0 \) and \( \langle \alpha, \gamma \rangle \neq 0 \). Since \( \alpha \) is long and \( \beta \) and \( \gamma \) are short we have by the first claim that \( \langle \Lambda, \alpha \rangle \cdot \langle \Lambda, \beta \rangle = 0 \) and \( \langle \Lambda, \alpha \rangle \cdot \langle \Lambda, \gamma \rangle = 0 \). Now \( \langle \Lambda, \alpha \rangle = \langle \Lambda, \beta \rangle + \langle \Lambda, \gamma \rangle \) gives that \( \langle \Lambda, \alpha \rangle = 0 \) for every long root. But this is impossible. Hence all roots have the same length and in particular holds for non-proportional roots
\[
\frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} = \pm 1. \tag{17}
\]

Claim 3: There is an \( a \in \mathbb{N} \) such that for every root \( \alpha \) holds \( \langle \Lambda, \alpha \rangle \in \{0, \pm a\} \). Furthermore \( a \) is less or equal than the length of the roots.

We consider \( \alpha \in \Delta \) with \( \langle \Lambda, \alpha \rangle \neq 0 \) and set \( a := \langle \Lambda, \alpha \rangle \). Then we define the vector space \( A := \text{span} \{ \beta \in \Delta \mid \langle \Lambda, \beta \rangle = \pm a \} \subset \mathfrak{t}^* \). We show that \( A = \mathfrak{t}^* \) and that every root \( \gamma \) with \( \langle \Lambda, \gamma \rangle \notin \{0, \pm a\} \) is orthogonal to \( A \).

To verify \( A = \mathfrak{t}^* \) we show that every root is either in \( A \) or in \( A^\perp \). First consider \( \gamma \in \Delta \) with \( \langle \Lambda, \gamma \rangle = 0 \). If it is not in \( A^\perp \) then there is a root \( \beta \in A \) and a \( \delta \notin A \) such that \( \gamma = \beta + \delta \). But this implies \( 0 = \langle \Lambda, \gamma \rangle = \langle \Lambda, \beta \rangle + \langle \Lambda, \delta \rangle = \pm a + \langle \Lambda, \delta \rangle \). Hence \( \delta \in A \) and therefore \( \gamma \in A \) which is a contradiction. Thus \( \gamma \in A^\perp \).

Now we consider a root \( \gamma \) with \( \langle \Lambda, \gamma \rangle \notin \{0, \pm a\} \). For any \( \beta \) with \( \langle \Lambda, \beta \rangle = \pm a \) then we have because of \( \tag{17} \) that \( \langle \Lambda, s_\beta \gamma \rangle = \langle \Lambda, \gamma \rangle \pm a \neq 0 \). Because of the proof of claim 1 this gives \( \langle \beta, \gamma \rangle = 0 \). Hence \( \gamma \in A^\perp \). Since the root system is indecomposable we have that \( A = \mathfrak{t}^* \). Furthermore we have shown that any root with \( \langle \Lambda, \gamma \rangle \notin \{0, \pm a\} \) is orthogonal to \( A = \mathfrak{t}^* \). Thus the first part of claim 3 is proved.

Now we suppose that \( a > c \) where \( c \) denotes the length of the roots. We consider an \( \alpha \in \Delta \) with \( \langle \Lambda, \alpha \rangle = a \). \( s_\alpha (\Lambda) = \Lambda - \frac{2a}{c} \alpha \) is an extremal weight in \( U \). Then
implies \( \Lambda - 2\alpha \in \Omega \) but not in \( U \). Then the existence of the planar spanning triple \((\Lambda, -\Lambda, U)\) implies \( \Lambda - 2\alpha = -\Lambda + \beta \) for a \( \beta \in \Delta \). Hence

\[
\frac{2\langle \Lambda, \gamma \rangle}{c} = 1 + \frac{2\langle \alpha, \gamma \rangle}{c} = 2
\]

and therefore \( \langle \Lambda, \gamma \rangle = a \) and \( a = c \) which is a contradiction.

Now we consider for any \( \alpha \in \Delta \) the set \( \Delta_\alpha^\perp := \{ \beta \in \Delta \mid \langle \alpha, \beta \rangle = 0 \} \subset \Delta \). This set is a root system, reduced but not necessarily indecomposable. But we can make the following claim.

Claim 4: Let \( \alpha \in \Delta \) with \( \langle \Lambda, \alpha \rangle \neq 0 \). Then one of the following cases holds:

1. \( \Delta_\alpha^\perp \) is orthogonal to \( \Lambda \) or
2. there is a unique \( \beta \in \Delta_\alpha^\perp \) with \( \langle \Lambda, \beta \rangle \neq 0 \) such that
   
   (a) \( \Lambda = \pm \frac{2a}{c}(\alpha + \beta) \) where \( c \) is the lengths of the roots, and
   (b) \( \Delta_\alpha^\perp \) is decomposable with a direct summand \( A_1 = \{ \pm \beta \} \).

Suppose that there is a \( \beta \in \Delta_\alpha^\perp \) with \( \langle \Lambda, \beta \rangle \neq 0 \). W.l.o.g. we can suppose that \( \langle \Lambda, \beta \rangle = \langle \Lambda, \alpha \rangle = \pm a \). \( \langle \alpha, \beta \rangle \) implies then

\[
s_\alpha s_\beta(\Lambda) = \Lambda \mp \frac{2a}{c}(\alpha + \beta).
\]

Now we show with the help of (16) that \( s_\alpha s_\beta(\Lambda) \) is not in \( U \):

\[
\langle s_\alpha s_\beta(\Lambda), T \rangle = \langle \Lambda, T \rangle - \frac{2\langle \Lambda, \alpha \rangle}{\| \alpha \|^2} \langle \alpha, T \rangle - \frac{2\langle \Lambda, \beta \rangle}{\| \beta \|^2} \langle \beta, T \rangle
\]

\[
= -\langle \Lambda, T \rangle \neq 0.
\]

But this implies \( -\Lambda = s_\alpha s_\beta(\Lambda) = \Lambda \pm \frac{2a}{c}(\alpha + \beta) \). By this equation \( \alpha \) determines \( \beta \) uniquely.

We still have to show that such \( \beta \) is orthogonal to all other roots in \( \Delta_\alpha \). For \( \gamma \not\sim \beta \in \Delta_\alpha \) we have

\[
\langle \Lambda, s_\beta \gamma \rangle = \langle \Lambda, \gamma \rangle - \underbrace{2\langle \beta, \gamma \rangle}_{= 0} \frac{\langle \beta, \beta \rangle}{\| \beta \|^2} \langle \Lambda, \beta \rangle.
\]

The uniqueness of \( \beta \) implies that \( \beta \) is orthogonal to \( \Delta_\alpha \).

Claim 5: The root system of \( \mathfrak{g} \) is of type \( A_n \) or \( D_n \).

The only root system with roots of equal length where the root system \( \Delta_\alpha^\perp \) is decomposable for a root \( \alpha \) is \( D_n \). Hence for every root system different from \( D_n \) we have that \( \Delta_\alpha^\perp \perp \Lambda \) by claim 4. Any root system different from \( A_n \) satisfies that \( \text{span}(\Delta_\alpha^\perp) = \alpha^\perp \). Both together imply that for any root system different from \( D_n \) and \( A_n \) we have that \( \alpha = \Lambda \) but this was excluded.
To find the representations of $A_n$ and $D_n$ which obey the above claims we introduce a fundamental system $\Pi = (\pi_1, \ldots, \pi_n)$ which makes $\Lambda$ to the highest weight of the representation. Then we have that $\Lambda = \sum_{k=1}^n m_k \omega_k$ with $m_k \in \mathbb{N} \cup \{0\}$ and $\omega_k$ the fundamental representations. $\langle \omega_i, \pi_j \rangle = \delta_{ij}$ implies $m_i = \langle \Lambda, \pi_i \rangle \in \{0, a\}$. Then we get

Claim 6. The root system is of type $D_n$ and the representation is the $a$-th power of a fundamental representation, i.e. $\Lambda = a \omega_1$.

Applying $\Lambda$ to the root $\sum_{k=1}^n \pi_k$ gives $\sum_{k=1}^n m_k = a$. Applying $\Lambda$ to any of the $\pi_i$ gives that $\sum_{k=1}^n m_k = m_i$ for one $i$.

Now we consider the root system $A_n$. $n = 1$ was excluded from the beginning. Recalling $A_3 \simeq D_3$ we can also exclude $A_3$. Now we impose the condition that the representation is orthogonal. This forces $n$ to be odd and $\Lambda = a \omega_{n+1}$ where $a$ has to be $2$ when $\frac{n+1}{2}$ is odd. Thus we can suppose that $n > 3$. Using the usual notation we consider now the root $\sum_{k=1}^n \pi_k = e_1 - e_{n+1}$ for which holds that $\langle \Lambda, \eta \rangle = a$. Hence by claim 4 we have that $\Delta_\eta$ is orthogonal to $\Lambda$. On the other hand $\Delta_\eta = \{ \pm (e_i - e_j) \mid 2 \leq i < j \leq n \}$ with $n > 3$ is not orthogonal to $a \omega_{n+1} = a \left( e_1 + \ldots + e_{n+1} \right)$. This yields a contradiction.

Finally we show that only the representations of $D_n$ given in the proposition satisfy the derived properties. The fundamental representations of $D_n$ are given by $\omega_i = e_1 + \ldots + e_i$ for $i = 1 \ldots n - 2$ and $\omega_i = \frac{1}{2}(e_1 + \ldots + e_{n-1} \pm e_n)$ for $i = n - 1, n$. Then $\langle a \omega_i, \pi_i \rangle = a$.

On the other hand for the largest root $\eta = e_1 + e_2$ holds

$$\langle a \omega_i, \eta \rangle = \begin{cases} a & : i = 1, n - 1, n \\ 2a & : 2 \leq i \leq n - 2. \end{cases}$$

Hence the representation of $a \omega_i$ with $2 \leq i \leq n - 2$ does not satisfy claim 3. Now we consider for $n > 4$ the representations $\Lambda = \frac{1}{2}(e_1 + \ldots + e_{n-1} \pm e_n)$. For the root $\alpha = e_{n-1} \pm e_n$ holds that $\langle \Lambda, \alpha \rangle = a \neq 0$. The roots $\beta_1 := e_1 - e_2$ and $\gamma := e_1 + e_3$ both satisfy $\langle \Lambda, \beta \rangle = \langle \Lambda, \gamma \rangle = a$ and $\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = 0$. But this is a violation of the uniqueness property in claim 4. Hence $n = 4$.

For $D_4$ it holds that, $\omega_3$ and $\omega_4$ are congruent to $\omega_1$, i.e. there is an involutive automorphism of the Dynkin diagram which interchanges $\omega_1$ with $\omega_3$ respectively $\omega_1$ with $\omega_4$. For $D_3 \simeq A_3$ only the representations $\omega_2$ and $2 \omega_2$ are orthogonal. \qed

Again we get a

3.4 Corollary. Every representation of a Lie algebra which satisfies the conditions of proposition (3.3) is the complexification of a Riemannian holonomy representation.

Proof. The representation with highest weight $\omega_1$ of $D_n$ is the standard representation of $\mathfrak{so}(2n, \mathbb{C})$ in $\mathbb{C}^{2n}$. Hence it is the holonomy representation of a generic Riemannian manifold.
The representation with highest weight $2\omega_1$ is the complexified holonomy representation of a symmetric space of type AI for even dimensions, i.e. of $SU(2n)/SO(2n,\mathbb{R})$ respectively $SL(2n,\mathbb{R})/SO(2n,\mathbb{R})$.

3.3 Representations with the property (SII) and weight zero

Now we will study the property (SII) for representation for which zero is a weight. For this we need a lemma.

3.5 Lemma. Let $g \subset \mathfrak{so}(N,H)$ the irreducible representation of a simple Lie algebra with weights $\Omega$. If $0 \in \Omega$ then

1. $\Delta \subset \Omega$ or
2. the extremal weights are short roots or
3. $\Delta = C_n$ and the representation is a fundamental one with highest weight $\omega_{2k}$ for $k \geq 2$.

Proof. $0 \in \Omega$ implies that there is a $\lambda \in \Omega$ and an $\eta \in \Delta$ such that $0 = \lambda - \eta$, i.e $\lambda = \eta$. Now we consider two cases.

Case 1: $\eta$ is a long root.

Of course we have that the root system of long roots is contained in $\Omega$. We have to show that the short roots are in $\Omega$. This is the case if one short root is in $\Omega$. For this we write $\eta = \alpha + \beta$ where $\alpha$ and $\beta$ are short roots. If $\Delta \neq G_2$ we have that $\langle \alpha, \beta \rangle = 0$. In this case we have that $\frac{2\langle \eta, \alpha \rangle}{||\alpha||^2} = 2$, i.e. $\eta - \alpha = \beta \in \Omega$. For $\Delta = G_2$ we have that $\frac{2\langle \alpha, \beta \rangle}{||\alpha||^2} = 1$ and therefore $\frac{2\langle \eta, \alpha \rangle}{||\alpha||^2} = 3$, i.e. $\eta - \alpha = \beta \in \Omega$ too. Hence also the short roots are weights and we have $\Delta \subset \Omega$.

Case 2: $\eta$ is a short root.

Again the short roots are weights. We have to show that one long root is a weight if $\eta$ is not extremal or that we are in the case of the $C_n$ with the above representations. If $\eta$ is not extremal then exists an $\alpha \in \Delta$ such that $\eta + \alpha \in \Omega$ and $\eta - \alpha \in \Omega$. This $\alpha$ we fix and consider the following cases.

Case A: $\alpha = \eta$, i.e. $2\eta \in \Omega$. If $\Delta \neq G_2$ we find a long root $\beta$ such that $\frac{2\langle \eta, \beta \rangle}{||\eta||^2} = -2$. This implies that $\beta + 2\eta$ is a long root but also a weight. In case of $G_2$ we find a short root $\beta$ with $\langle \eta, \beta \rangle < 0$ and such that $2\eta + \beta \in \Delta$ a long root. This long root is also in $\Omega$ since $\langle \eta, \beta \rangle < 0$.

Case B: $\alpha \not\sim \eta$ and $\langle \alpha, \eta \rangle \neq 0$. First we consider the case where $\alpha$ is a short root.

W.l.o.g. let be $\langle \alpha, \eta \rangle < 0$ Then $\alpha + \eta$ is a root and a weight. If $\Delta$ is different from $C_n$ it is a long root and we are ready. For $C_n$ we have to analyze the situation in detail (see the appendix of [Kna02]): Let $\eta = e_i + e_j$ and
\( \alpha = e_k - e_j \) with \( i \neq k \) be the two short roots. Since \( \Omega \ni \eta - \alpha = 2e_j + e_i - e_k \) we have that \( \frac{2(\eta - \alpha, e_i - e_k)}{\|e_i - e_k\|^2} = 2 \). Hence \( \eta - \alpha - (e_i - e_k) = 2e_j \in \Omega \). But \( 2e_j \) is a long root of \( C_n \) and we are ready.

If \( \alpha \) is a long root we proceed as follows. For \( G_2 \) one of \( \eta \pm \alpha \) is a short root, let's say \( \eta - \alpha \). Then we have that \( \langle \eta + \alpha, \eta - \alpha \rangle < 0 \) hence \( 2\eta \) is a weight and we may argue as in the first case A. If \( \Delta \) is different from \( \Delta \neq C_n \). Hence we have verified \( \Delta \subset \Omega \) in the cases A, B and C. It remains to show that \( \Delta \subset \Omega \) or the representation of \( C_n \) is the one with highest weight \( \omega_{2k} \) with \( k \geq 2 \).

We suppose that \( \Delta \not\subset \Omega \). Hence no long root can be a weight.

First of all we show that under these conditions \( \alpha \) has to be a short root. This is true because \( \frac{2(\eta + \alpha, \eta)}{\|\eta\|^2} = 2 \) implies \( \Omega \ni \eta \pm \alpha - \eta = \pm \alpha \). Hence \( \alpha \) has to be short.

Secondly we note that neither \( \eta + \alpha \) nor \( \eta - \alpha \) can be a root because it would be a long root and a weight. This implies \( n \geq 4 \).

In a third step we show that there is no long root \( \beta \) such that \( \eta + \alpha + \beta \in \Omega \) and \( \eta + \alpha - \beta \in \Omega \). We consider the number

\[
\frac{2(\eta + \alpha \pm \beta, \alpha)}{\|\alpha\|^2} = 2 \pm \frac{2(\alpha, \beta)}{\|\alpha\|^2}.
\]  

If \( \langle \alpha, \beta \rangle = 0 \) we have that \( \eta + \alpha \pm \beta - \alpha = \eta \pm \beta \in \Omega \). But this was excluded (First step or case B). Hence we suppose that \( \frac{2(\alpha, \beta)}{\|\alpha\|^2} = 2 \). We still have that \( \eta + \beta \in \Omega \). We consider the number \( \frac{2(\eta + \beta, \eta)}{\|\eta\|^2} \geq 0 \). If this is not zero we have that

\[
\alpha = e_k - e_j \text{ with } i \neq k \text{ be the two short roots. Since } \Omega \ni \eta - \alpha = 2e_j + e_i - e_k \text{ we have that } \frac{2(\eta - \alpha, e_i - e_k)}{\|e_i - e_k\|^2} = 2. \text{ Hence } \eta - \alpha - (e_i - e_k) = 2e_j \in \Omega. \text{ But } 2e_j \text{ is a long root of } C_n \text{ and we are ready.}
\]

If \( \alpha \) is a long root we proceed as follows. For \( G_2 \) one of \( \eta \pm \alpha \) is a short root, let's say \( \eta - \alpha \). Then we have that \( \langle \eta + \alpha, \eta - \alpha \rangle < 0 \) hence \( 2\eta \) is a weight and we may argue as in the first case A. If \( \Delta \) is different from \( \Delta \neq C_n \). Hence we have verified \( \Delta \subset \Omega \) in the cases A, B and C. It remains to show that \( \Delta \subset \Omega \) or the representation of \( C_n \) is the one with highest weight \( \omega_{2k} \) with \( k \geq 2 \).

We suppose that \( \Delta \not\subset \Omega \). Hence no long root can be a weight.

First of all we show that under these conditions \( \alpha \) has to be a short root. This is true because \( \frac{2(\eta + \alpha, \eta)}{\|\eta\|^2} = 2 \) implies \( \Omega \ni \eta \pm \alpha - \eta = \pm \alpha \). Hence \( \alpha \) has to be short.

Secondly we note that neither \( \eta + \alpha \) nor \( \eta - \alpha \) can be a root because it would be a long root and a weight. This implies \( n \geq 4 \).

In a third step we show that there is no long root \( \beta \) such that \( \eta + \alpha + \beta \in \Omega \) and \( \eta + \alpha - \beta \in \Omega \). We consider the number

\[
\frac{2(\eta + \alpha \pm \beta, \alpha)}{\|\alpha\|^2} = 2 \pm \frac{2(\alpha, \beta)}{\|\alpha\|^2}.
\]  

If \( \langle \alpha, \beta \rangle = 0 \) we have that \( \eta + \alpha \pm \beta - \alpha = \eta \pm \beta \in \Omega \). But this was excluded (First step or case B). Hence we suppose that \( \frac{2(\alpha, \beta)}{\|\alpha\|^2} = 2 \). We still have that \( \eta + \beta \in \Omega \). We consider the number \( \frac{2(\eta + \beta, \eta)}{\|\eta\|^2} \geq 0 \). If this is not zero we have that
\[ \Omega \ni \eta + \beta - \eta = \beta \] which was excluded. Hence \( \frac{2(\eta, \beta)}{||\eta||^2} = -2 \). But this together with \( \frac{2(\eta, \beta)}{||\eta||^2} = 2 \) is a contradiction since the long roots of \( C_n \) are of the form \( \pm 2e_i \) and the short ones of the form \( \pm (e_i \pm e_j) \). Hence if there is a root such that \( \eta + \alpha + \beta \in \Omega \) and \( \eta + \alpha - \beta \in \Omega \), it has to be a short one.

If there is no such \( \beta \) then \( \eta + \alpha \) is extremal. Considering the fundamental weights of \( C_n \) this gives easily that the highest weight of the representation is \( \omega_4 \).

Finally we suppose that there is such a short root \( \beta \). Since \( \beta \) is short equation (18) implies \( \eta \pm \beta \in \Omega \). Since we have excluded case A and B it must hold \( \langle \eta, \beta \rangle = 0 \) and neither \( \eta + \beta \) nor \( \eta - \beta \) is a root. On the other hand the same holds for \( \alpha \) and \( \beta \) since any other would imply that \( \alpha \pm \beta \) is a long root which was excluded or a short root \( \gamma \) orthogonal to \( \eta \) and with \( \eta \pm \gamma \in \Omega \). This way we go on attaining that any extremal weight is the sum of orthogonal short roots whose pairwise sum is no long root. But this is nothing else than the fact that the highest weight of the representation is \( \omega_{2k} \) for \( k \geq 2 \).

All in all we have shown the proposition.

3.6 Proposition. Let \( g \subset \mathfrak{so}(N, \mathbb{C}) \) be an irreducible representation of real type of a complex simple Lie algebra different from \( \mathfrak{sl}(2, \mathbb{C}) \) and satisfying (SII). If \( 0 \in \Omega \) then there is a root \( \alpha \) such that for the extremal weight from property (SII) holds \( \Lambda = a\alpha \) or the representation is congruent to one of the following:

1. \( \Delta = C_4 \) with highest weight \( \omega_4 \).
2. \( \Delta = D_n \) with highest weight \( 2\omega_1 \).

Proof. Let \( \Lambda \) and \( \alpha \) be the extremal weight and the root from property (SII). We suppose that \( \Lambda \) is not the multiple of a root.

First of all we consider the case where \( 0 \in \Omega \). By the previous lemma this is true in the following cases:

(a) \( \Delta \neq C_n \), because in this case \( \Delta \subset \Omega \).

(b) \( \Delta = C_n \) but the highest weight of the representation is not equal to \( \omega_{2k} \) with \( k \geq 2 \), because this again implies \( \Delta \subset \Omega \).

(c) \( \Delta = C_n \) and \( \alpha \) is a short root, because for representations with \( 0 \in \Omega \) holds that the short roots are weights.

For \( 0 \in \Omega \) property (SII) gives \( 0 = \Lambda - \alpha - \beta \) or \( 0 = -\Lambda + \beta \). The second case was excluded thus we have to consider the first case. Suppose that \( \Lambda = \alpha + \beta \) where \( \alpha + \beta \neq \gamma \in \Delta \). In particular \( \alpha + \beta \) is not a root which implies that \( \langle \alpha, \beta \rangle \geq 0 \). We consider three cases.
Case 1: $\Delta = G_2$. In this case $\alpha + \beta \not\sim \gamma \in \Delta$ implies $\langle \alpha, \beta \rangle > 0$ and $\alpha$ and $\beta$ must have different length. Thus we can chose a long root $\gamma$ not proportional neither to $\alpha$ nor to $\beta$ and such that $\langle \alpha, \gamma \rangle < 0$ and $\langle \beta, \gamma \rangle < 0$ which implies $\gamma \in \Omega_\alpha$ as well as $\gamma \in \Omega_\beta$. (SII) implies then $\gamma - \beta \in \Delta$ or $\gamma - \alpha \in \Delta$ or $\gamma + \alpha + \beta \in \Delta$. The first two cases are not possible because of lemma \ref{lem:1}. For the third case we suppose that $\alpha$ is the long root and consider $\frac{2(\gamma + \beta, \alpha)}{||\alpha||^2} = 0$ because $\alpha$ is long and both terms have opposite sign. Hence $\gamma + \alpha + \beta$ can not be a root.

Case 2: $\Delta \neq G_2$ and $\langle \alpha, \beta \rangle > 0$. This implies $\alpha - \beta \in \Delta$. We consider the number $k := \frac{2(\alpha, \alpha + \beta)}{||\alpha||^2} = 2 + \frac{2(\alpha, \beta)}{||\alpha||^2} \geq 3$. Since $G_2$ was excluded we have that $k \in \{3, 4\}$. Hence $\alpha + \beta - k \alpha = \beta - (k-1)\alpha \in \Omega_\alpha$. Then property (SII) implies $\beta - (k-1)\alpha = -\alpha - \beta + \gamma$ with $\gamma \in \Delta_0$, i.e. $2\beta - (k-2)\alpha \in \Delta$. At first this implies $k = 3$ and thus $\frac{2(\alpha, \beta)}{||\beta||^2} = 1$. Secondly we must have $\frac{2(\alpha, \beta)}{||\beta||^2} = 2$, therefore $||\alpha||^2 = 2||\beta||^2$, i.e. $\alpha$ as well as $2\beta - \alpha$ are long roots and $\beta$ and $\beta - \alpha$ are short ones. This implies $\frac{2(\beta - \alpha, \alpha + \beta)}{||\beta - \alpha||^2} = \frac{2(||\beta||^2 - ||\gamma||^2)}{||\beta||^2} = -2$. Hence $\alpha + \beta + 2(\beta - \alpha) = 3\beta - \alpha \in \Omega$ and since $\frac{2(\alpha, \alpha - 3\beta)}{||\alpha||^2} = 2 - 3 = -1$ holds $\alpha - 3\beta \in \Omega_\alpha$. (SII) then gives $\alpha - 3\beta = \beta - \gamma$ or $\alpha - 3\beta = -\beta - \alpha + \gamma$ with $\gamma \in \Delta_0$. But none of these equations can be true.

Case 3: $\langle \alpha, \beta \rangle = 0$ and $\Delta \neq G_2$. Since $\alpha + \beta \not\sim \gamma \in \Delta$ the rank of $\Delta$ has to be greater than 3 or it is $\Delta = D_n$ and $\Lambda = 2e_i$, i.e. $\Lambda = 2w_1$. In the second case we are ready and we exclude this representation in the following. We can suppose $rk\Delta \geq 4$. In this situation we prove the following lemma.

3.7 Lemma. Let $rk\Delta \geq 4$ and let $\Lambda = \alpha + \beta$ be an extremal weight of a representation satisfying property (SII) for $(\Lambda, -\Lambda + \alpha, \alpha)$ with $\beta \in \Delta$ satisfying $\langle \alpha, \beta \rangle = 0$ and $\alpha + \beta \not\sim \gamma \in \Delta$. Then $\Delta$ is a root system with roots of the same length or $\Delta = C_n$ and $\alpha$ and $\beta$ are two short roots.

Proof. Suppose that $\Delta$ has roots of different length.

First we assume that $\beta$ is a long root. We consider the root system $\Delta^\perp_\alpha$ which contains $\beta$. We notice that $\beta$ lies not in an $A_1$ factor of $\Delta^\perp_\alpha$ because otherwise $\alpha + \beta$ would be the multiple of a root. Since $\beta$ is long we find a short root $\gamma \in \Delta^\perp_\alpha$ such that $\frac{2(\gamma, \gamma)}{||\gamma||^2} = -2$. Hence $\alpha + \beta + 2\gamma \in \Omega$ and — since $\frac{2(\alpha, \alpha + \beta + 2\gamma)}{||\alpha||^2} = 2$ — it is $-\alpha - \beta - 2\gamma \in \Omega_\alpha$. But this contradicts property (SII).

Now we suppose that $\alpha$ is a long root. Here we consider the root system $\Delta^\perp_\beta$ containing $\alpha$. Again $\alpha$ lies not in an $A_1$ factor of $\Delta^\perp_\beta$ because otherwise $\alpha + \beta$ would be the multiple of a root. Since $\alpha$ is long we find a short root $\gamma \in \Delta^\perp_\beta$ such that $\frac{2(\gamma, \gamma)}{||\gamma||^2} = -2$. Hence $\alpha + \beta + 2\gamma \in \Omega$. Now we have that $\frac{2(\alpha, \gamma)}{||\alpha||^2} = -1$ and therefore $\frac{2(\alpha, \alpha + \beta + 2\gamma)}{||\alpha||^2} = 2 - 1 = 1$. Thus $-\alpha - \beta - 2\gamma \in \Omega_\alpha$. Again this contradicts (SII).
If $\alpha$ and $\beta$ are short and orthogonal and the root system is not $C_n$, i.e. it is $B_n$ or $F_4$, then the sum of two orthogonal short roots is the multiple of a root.  

Now we prove a second

3.8 Lemma. The assumptions of the previous lemma imply that there is no $\gamma \in \Delta$ such that
\[
\langle \alpha, \gamma \rangle = 0 \text{ and } \frac{2\langle \beta, \gamma \rangle}{\|\gamma\|^2} = 1.
\]  

Proof. Let's suppose that there is a $\gamma \in \Delta$ such that $\langle \alpha, \gamma \rangle = 0$ and $\frac{2\langle \beta, \gamma \rangle}{\|\gamma\|^2} = 1$. In case of $C_n$, $\gamma$ is a short root. We note that both together imply that neither $\alpha + \gamma$ nor $\alpha - \gamma$ is a root. But $\gamma - \beta$ is a root, in case of $C_n$ a short one. Furthermore $\Lambda - \gamma \in \Omega$, hence
\[
\frac{2\langle \Lambda - \gamma, \gamma - \beta \rangle}{\|\gamma - \beta\|^2} = \frac{2\langle \alpha + \beta - \gamma, \gamma - \beta \rangle}{\|\gamma - \beta\|^2} = -2.
\]
Hence $\Lambda - \gamma + 2(\gamma - \beta) = \alpha - \beta + \gamma \in \Omega$. Now $\frac{2\langle \alpha - \beta + \gamma, \alpha \rangle}{\|\alpha\|^2} = 2$, i.e. $-\alpha + \beta - \gamma \in \Omega$. (SII) implies now that $-\alpha + \beta - \gamma = \beta + \delta$ or $-\alpha + \beta - \gamma = -\alpha + \beta + \delta$ for $\delta \in \Delta_0$. But both options are not possible since $\alpha + \gamma$ is not a root and because $\gamma$ is short.  

We conclude that lemma 3.7 left us with representations of $A_n$, $D_n$, $E_6$, $E_7$, $E_8$ or $C_n$ where $\Lambda$ is the sum of two orthogonal (short) roots but not a root.

Now one easily verifies that lemma 3.8 implies $n \leq 4$ and $\Delta \neq A_4$. Hence the remaining representations are $2\omega_1$, $2\omega_3$ and $2\omega_4$ of $D_4$, which are congruent to each other, and $\omega_4$ of $C_4$.

To finish the proof we have to consider the representation of highest weight $\omega_{2k}$ (with $k \geq 2$) of $C_n$ supposing $\alpha$ is a long root. $0 \in \Omega$ implies that the short roots are weights. Let $\beta$ be a short root with $\langle \alpha, \beta \rangle < 0$, i.e. $\beta \in \Omega_\alpha$. (SII) then gives $\beta = \omega_{2k} - \alpha + \delta$ or $\beta = \omega_{2k} - \delta$ for a $\delta \in \Delta_0$. Analyzing roots and fundamental weights of $C_n$ we get that (SII) implies $k = 2$ and $\alpha = 2e_i$ for $1 \leq i \leq 4$. But for $n > 4$ lemma 3.8 applies analogously. The remaining representation is $\omega_4$ of $C_4$.

3.9 Corollary. Let $\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})$ be an orthogonal algebra of real type different from $\mathfrak{sl}(2, \mathbb{C})$ and satisfying (SII). If $0 \in \Omega$, in particular if $\Delta = G_2$, $F_4$ or $E_8$ then it is the complexification of a Riemannian holonomy representation with the exception of $G_2$ in corollary 3.2.

Proof. If $\Lambda$ is the multiple of a root then we are in the situation of corollary 3.2. For $D_n$ the remaining representations are those which appear in corollary 3.4. The representation of highest weight $\omega_4$ of $C_4$ is the complexification of the holonomy representation of the Riemannian symmetric space of type $EI$, i.e. of $E_6/Sp(4)$ resp. $E_{6(6)}/Sp(4)$.
Furthermore analyzing the roots and fundamental representations of the exceptional algebras we notice that every representation of \(G_2, F_4\) and \(E_8\) contains zero as weight. \(\square\)

### 3.4 Representations with the property (SII) where zero is no weight

First we need a

**3.10 Lemma.** Let \(0 \notin \Omega\). Then there is a weight \(\lambda \in \Omega\), such that for every root holds

\[
|\frac{2(\lambda, \alpha)}{\|\alpha\|^2}| \leq 1.
\]

**Proof.** Let \(\lambda\) be a weight and \(\alpha\) a root such that \(\frac{2(\lambda, \alpha)}{\|\alpha\|^2} =: k \geq 2\). If \(k\) is even we have that \(0 \neq \lambda - \frac{k}{2} \alpha \in \Omega\). But for this weight holds \(\frac{2(\lambda - \frac{k}{2} \alpha, \alpha)}{\|\alpha\|^2} = k - k = 0\). If \(k\) is odd we have that \(0 \neq \lambda - \frac{k-1}{2} \alpha \in \Omega\) and \(\frac{2(\lambda - \frac{k-1}{2} \alpha, \alpha)}{\|\alpha\|^2} = 1\). \(\square\)

**3.11 Proposition.** Let \(g \subset so(N, \mathbb{C})\) be an irreducible representation of real type of a complex simple Lie algebra different from \(sl(2, \mathbb{C})\), with \(0 \notin \Omega\) and satisfying (SII). Then \(\frac{2(\lambda, \beta)}{\|\beta\|^2} \leq 3\) for all roots \(\beta \in \Delta\).

**Proof.** Let \(\alpha \in \Delta\) with the property (SII), i.e. \(\Omega_\alpha \subset \{\lambda - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\lambda + \beta \mid \beta \in \Delta_0\}\). By the previous lemma there is a \(\lambda \in \Omega\) such that \(\frac{2(\lambda, \beta)}{\|\beta\|^2} \leq 1\) for all roots \(\beta \in \Delta\).

Applying the Weyl group one can choose \(\lambda\) such that \(\langle \lambda, \alpha \rangle < 0\). \(\langle \lambda, \alpha \rangle < 0\) implies \(\lambda \in \Omega_\alpha\). Hence (SII) gives \(\lambda = \lambda - \gamma\) or \(\lambda = -\lambda + \gamma\) with \(\gamma \in \Delta_0\).

The second case gives for every \(\beta \in \Delta\)

\[
\left|\frac{2(\lambda, \beta)}{\|\beta\|^2}\right| \leq \frac{2(\lambda, \beta)}{\|\beta\|^2} + \frac{2(\gamma, \beta)}{\|\beta\|^2} \leq 3,
\]

because we have excluded \(G_2\).

Thus we have to consider the first case \(\Lambda = \lambda + \alpha + \gamma\) with \(\gamma \in \Delta_0\) and have to verify that

\[
\left|\frac{2(\lambda, \beta)}{\|\beta\|^2}\right| = \left|\frac{2(\lambda, \beta)}{\|\beta\|^2} + \frac{2(\alpha, \beta)}{\|\beta\|^2} + \frac{2(\gamma, \beta)}{\|\beta\|^2}\right| \leq 3.
\]

(20)

for all roots \(\beta \in \Delta\).

For \(\beta = \pm \alpha\) this is satisfied:

\[
\frac{2(\lambda, \alpha)}{\|\alpha\|^2} = \pm \frac{2(\lambda, \alpha)}{\|\alpha\|^2} \pm \frac{2(\gamma, \alpha)}{\|\alpha\|^2} = \mp 1 \pm \frac{2(\gamma, \alpha)}{\|\alpha\|^2} \leq 3.
\]

Now we have to show (20) for all \(\beta \in \Delta\) with \(\beta \not\sim \alpha\). For this we consider three cases.

**Case 1: All roots have the same length.** This implies \(\left|\frac{2(\gamma, \beta)}{\|\beta\|^2}\right| \leq 1\) for all roots which are not proportional to each other. Thus we have (20) for all \(\beta \neq \gamma\):

\[
\left|\frac{2(\lambda, \beta)}{\|\beta\|^2}\right| \leq \left|\frac{2(\lambda, \beta)}{\|\beta\|^2}\right| + \left|\frac{2(\alpha, \beta)}{\|\beta\|^2}\right| + \left|\frac{2(\gamma, \beta)}{\|\beta\|^2}\right| \leq 3.
\]

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For \( \beta = \pm \gamma \) we have

\[
\frac{2\langle \Lambda, \gamma \rangle}{\|\gamma\|^2} = \frac{2\langle \Lambda, \gamma \rangle}{\|\gamma\|^2} + 2\langle \alpha, \gamma \rangle + 2.
\]

This has absolute value \( \geq 4 \) only if \( \langle \lambda, \gamma \rangle > 0 \) and \( \langle \alpha, \gamma \rangle > 0 \). This implies that \( \alpha - \gamma \) is a root. But for this one holds

\[
\frac{2\langle \lambda, \gamma - \alpha \rangle}{\|\gamma - \alpha\|^2} = \frac{2\langle \lambda, \gamma \rangle}{\|\gamma\|^2} - \frac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} = 2
\]

since all roots have the same length. This is a contradiction to the choice of \( \lambda \).

**Case 2:** There are long and short roots and \( \beta \) is a long root. This implies again

\[
\left| \frac{2\langle \gamma, \beta \rangle}{\|\beta\|^2} \right| \leq 1 \quad \text{for all } \beta \text{ which are not proportional to } \gamma. \quad \text{This implies (20) in this case.}
\]

For \( \beta = \pm \gamma \) we argue as above, recalling that \( \gamma - \alpha \) and \( \alpha \) have to be short roots in this case. Hence

\[
\frac{2\langle \lambda, \gamma - \alpha \rangle}{\|\gamma - \alpha\|^2} = \frac{2\langle \lambda, \gamma \rangle}{\|\gamma\|^2} - \frac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} \geq \frac{2\langle \lambda, \gamma \rangle}{\|\gamma\|^2} - \frac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} = \geq 2 \quad \text{which is a contradiction.}
\]

**Case 3:** There are long and short roots and \( \beta \) is a short root.

First we consider the case where \( \beta = \pm \gamma \). Again (20) is not satisfied only if \( \langle \lambda, \gamma \rangle \) and \( \langle \alpha, \gamma \rangle \) are non zero and have the same sign, lets say +.

If \( \alpha \) is a short root too, then because of \( \langle \alpha, \gamma \rangle \neq 0 \) lemma 2.15 gives that \( \alpha - \gamma \) is also a short root. Hence

\[
\frac{2\langle \lambda, \gamma - \alpha \rangle}{\|\gamma - \alpha\|^2} = \frac{2\langle \lambda, \gamma \rangle}{\|\gamma\|^2} - \frac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} = \frac{2\langle \lambda, \gamma \rangle}{\|\gamma\|^2} - \frac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} \geq 2 \quad \text{yields a contradiction.}
\]

If \( \alpha \) is a long root, then \( \gamma - \alpha \) has to be a short one and we get again a contradiction:

\[
\frac{2\langle \lambda, \gamma - \alpha \rangle}{\|\gamma - \alpha\|^2} = \frac{2\langle \lambda, \gamma \rangle}{\|\gamma\|^2} - \frac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} \geq \frac{2\langle \lambda, \gamma \rangle}{\|\gamma\|^2} - \frac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} \geq 2.
\]

Now suppose that \( \beta \neq \gamma \). Then

\[
\frac{2\langle \lambda, \beta \rangle}{\|\beta\|^2} = \frac{2\langle \lambda, \beta \rangle}{\|\beta\|^2} + \frac{2\langle \alpha, \beta \rangle}{\|\beta\|^2} + \frac{2\langle \gamma, \beta \rangle}{\|\beta\|^2}
\]

has absolute value \( \geq 4 \) only if all three right hand side terms have the same sign — lets say they are positive — and at least one of the last two terms has absolute value greater than one, i.e. \( \gamma \) or \( \alpha \) is a long root. If \( \alpha \) is a long root then \( \alpha - \beta \) is a short one and arguing as above gives the contradiction. If \( \alpha \) is a short root then \( \langle \alpha, \beta \rangle > 0 \) implies by lemma 2.15 that \( \beta - \alpha \) is a short root. Again we have a contradiction:

\[
\frac{2\langle \lambda, \beta - \alpha \rangle}{\|\beta - \alpha\|^2} = \frac{2\langle \lambda, \beta \rangle}{\|\beta\|^2} - \frac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} = \frac{2\langle \lambda, \beta \rangle}{\|\beta\|^2} - \frac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} = 2.
\]

\[\blacksquare\]

**3.12 Proposition.** Under the same assumptions as in the previous proposition holds that

\[
\left| \frac{2\langle \Lambda, \eta \rangle}{\|\eta\|^2} \right| \leq 2 \quad \text{for all long roots } \eta.
\]

**Proof.** Let \( \Lambda \) and \( \alpha \) be the extremal weight and the root from property (SII). We suppose that there is a long root \( \eta \) with

\[
\frac{2\langle \Lambda, \eta \rangle}{\|\eta\|^2} = -3 \quad (21)
\]

and derive a contradiction considering different cases.
Case 1: All roots have the same length. By applying the Weyl group we find an extremal weight $\Lambda'$ such that $a := \frac{2(\Lambda', \alpha)}{\|\alpha\|^2} = -3$.

First we find a root $\beta$ with

$$\frac{2(\alpha, \beta)}{\|\beta\|^2} = 1 \text{ and } \frac{2(\Lambda', \beta)}{\|\beta\|^2} \leq -2.$$  

This is obvious: We find a $\beta$ such that $\frac{2(\alpha, \beta)}{\|\beta\|^2} = 1$. If $\frac{2(\Lambda', \beta)}{\|\beta\|^2} \geq -1$ then we consider the root $\alpha - \beta$. It satisfies $\frac{2(\alpha, \alpha - \beta)}{\|\alpha - \beta\|^2} = 1$ and we have

$$\frac{2(\Lambda', \alpha - \beta)}{\|\alpha - \beta\|^2} = -3 - \frac{2(\Lambda', \beta)}{\|\beta\|^2} \leq -2.$$  

Hence we have $\Lambda' + k\beta \in \Omega$ for $0 \leq k \leq 2$ and $\Lambda' + k\alpha \in \Omega$ for $0 \leq k \leq 3$. Furthermore

$$\frac{2(\Lambda' + l\beta, \alpha)}{\|\alpha\|^2} = -3 - \frac{2(\Lambda', \alpha)}{\|\alpha\|^2} = -3 + l.$$  

But this gives $\Lambda' + k\alpha + l\beta \in \Omega_\alpha$ for $0 \leq k \leq 2, 0 \leq k + l \leq 2$.  

Among others (SII) implies the existence of $\gamma_i$ and $\delta_i$ from $\Delta_0$ for $i = 0, 1, 2$ such that the following alternatives must hold

$$\begin{align*}
\Lambda' + \alpha & = \Lambda + \gamma_0 \quad \text{or} \quad \Lambda' & = -\Lambda + \delta_0 & \quad \text{(22)} \\
\Lambda' + 3\alpha & = \Lambda + \gamma_1 \quad \text{or} \quad \Lambda' + 2\alpha & = -\Lambda + \delta_1 & \quad \text{(23)} \\
\Lambda' + \alpha + 2\beta & = \Lambda + \gamma_2 \quad \text{or} \quad \Lambda' + 2\beta & = -\Lambda + \delta_2. & \quad \text{(24)}
\end{align*}$$

First we suppose that the first alternative of (22) holds, i.e $\Lambda' + \alpha = \Lambda + \gamma_0$. Since $a = -3$ and both $\Lambda$ and $\Lambda'$ are extremal we have that $\alpha \neq -\gamma_0$. Hence the first case of (23) can not be true and we have $\Lambda' + 2\alpha = -\Lambda + \delta_1$. We consider now (24): The left side of (22) gives $\Lambda' + 2\beta + \alpha = \Lambda + \gamma_0 + 2\beta$. If the left side of (24) holds, we would have $\gamma_0 = -\beta$. Hence $\Lambda + \beta \in \Omega$ and on the other hand $\Omega \ni \Lambda' + \alpha = \Lambda - \beta$ which contradicts the extremality of $\Lambda$. Thus the right hand side of (24) must be satisfied. From $\Lambda' + 2\alpha = -\Lambda + \delta_1$ follows $\Lambda' + 2\beta = -\Lambda + \delta_1 + 2(\beta - \alpha)$ and therefore $\delta_1 = - (\beta - \alpha)$. Again we have $-\Lambda + (\beta - \alpha) \in \Omega$ and $-\Lambda - (\beta - \alpha) \in \Omega$ which contradicts the extremality of $\Lambda$.

If one starts with the right hand side of (22) we can proceed analogously and get a contradiction in the case where all roots have the same length.

Case 2. The roots have different length and $\alpha$ is a short root. On one hand we find a short root $\beta$ which is orthogonal to $\alpha$ and $\alpha + \beta$ is a long root, and on the other we can find an extremal weight $\Lambda'$ such that

$$\frac{2(\Lambda', \alpha + \beta)}{\|\alpha + \beta\|^2} = -3.$$
Since $\alpha \perp \beta$ we have

$$-3 = \frac{2(\langle \Lambda', \alpha \rangle + \langle \Lambda', \beta \rangle)}{||\alpha||^2 + ||\beta||^2} = \frac{1}{2} \left( \frac{2}{||\alpha||^2} + \frac{2}{||\beta||^2} \right).$$

Because of the previous proposition we get

$$\frac{2(\langle \Lambda', \alpha \rangle)}{||\alpha||^2} = \frac{2(\langle \Lambda', \beta \rangle)}{||\beta||^2} = -3.$$

Hence $\Lambda' + k\alpha + l\beta \in \Omega$ for $0 \leq k, l \leq 3$ and therefore $\Lambda' + k\alpha + l\beta \in \Omega_\alpha$ for $0 \leq k \leq 2$ and $0 \leq l \leq 3$. (SII) implies the following alternatives

$$\Lambda' + \alpha = \Lambda + \gamma_0 \quad \text{or} \quad \Lambda' = -\Lambda + \delta_0 \quad (25)$$
$$\Lambda' + \alpha + 3\beta = \Lambda + \gamma_1 \quad \text{or} \quad \Lambda' + 3\beta = -\Lambda + \delta_1 \quad (26)$$
$$\Lambda' + 2\alpha + 3\beta = \Lambda + \gamma_2 \quad \text{or} \quad \Lambda' + \alpha + 3\beta = -\Lambda + \delta_2 \quad (27)$$
$$\Lambda' + 3\alpha + 2\beta = \Lambda + \gamma_3 \quad \text{or} \quad \Lambda' + 2(\alpha + \beta) = -\Lambda + \delta_3 \quad (28)$$
$$\Lambda' + 3\alpha + 3\beta = \Lambda + \gamma_4 \quad \text{or} \quad \Lambda' + 2\alpha + 3\beta = -\Lambda + \delta_4. \quad (29)$$

If the left hand side of the first alternative is valid then the left hand sides of the remaining four can not be satisfied: For (26) we would have $3\beta = \gamma_1 - \gamma_0$ which is not possible. (27) would imply $3\beta + \alpha = \gamma_2 - \gamma_0$ which is by lemma 2.13 a contradiction since $\alpha \neq -\beta$ and $\gamma_0 \neq -\alpha$. (28) would imply $2(\alpha + \beta) = \gamma_3 - \gamma_0$. Since $\alpha + \beta$ is a long root this would give $\gamma_0 = -\alpha + \beta$ and $\gamma_3 = \alpha + \beta$ which is a contradiction to the extremality of $\Lambda$. (29) would give $2\alpha + 3\beta = \gamma_4 - \gamma_0$ which also is not possible.

Thus for the last four equations the right hand side must hold. Taking everything together we would get $\alpha = \delta_2 - \delta_1 = \delta_4 - \delta_2$ and $\beta = \delta_4 - \delta_3$. This gives $2\alpha = \delta_4 - \delta_1$ and thus

$$\frac{2\langle \delta_4, \alpha \rangle}{||\alpha||^2} - \frac{2\langle \delta_1, \alpha \rangle}{||\alpha||^2} = \frac{4||\alpha||^2}{||\alpha||^2} = 4.$$

The extremality of $\Lambda$ prevents that $\alpha = \delta_4 = -\delta_1$. Hence $\delta_1$ and $\delta_4$ are long roots, in particular

$$\frac{2\langle \delta_4, \alpha \rangle}{||\alpha||^2} = \frac{2\langle \delta_1, \alpha \rangle}{||\alpha||^2} = 2.$$

For $\beta$ again $\beta = \delta_4 = -\delta_3$ can not hold by the extremality of $\Lambda$ and we have

$$0 = \frac{2\langle \beta, \alpha \rangle}{||\alpha||^2} = \frac{2\langle \delta_4, \alpha \rangle}{||\alpha||^2} - \frac{2\langle \delta_3, \alpha \rangle}{||\alpha||^2} = 2 - \frac{2\langle \delta_3, \alpha \rangle}{||\alpha||^2}$$

which forces $\delta_3$ to be a long root too. Now we have a contradiction because the short root $\beta$ is the sum of two long roots. This is impossible.

If we start with the right hand side of the first alternative one proceeds analogously.
Case 3. The roots have different length and $\alpha$ is a long root. In this case we find an extremal weight $\Lambda'$ such that $\frac{2(\langle \Lambda', \alpha \rangle)}{\|\alpha\|^2} = -3$. Now we can write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \perp \alpha_2$ two short roots. As above we get

$$\frac{2(\langle \Lambda', \alpha_1 \rangle)}{\|\alpha_1\|^2} = \frac{2(\langle \Lambda', \alpha_2 \rangle)}{\|\alpha_2\|^2} = -3.$$  \hspace{1cm} (30)

Again this implies $\Lambda' + k\alpha + \ell\beta \in \Omega$ for $0 \leq k, \ell \leq 3$ and therefore $\Lambda' + k\alpha + \ell\beta \in \Omega_\alpha$ for $0 \leq k, \ell \leq 2$. Now (SII) implies the existence of $\gamma_i$ and $\delta_i$ from $\Delta_0$ for $i = 0, \ldots, 8$ such that that the following alternatives must hold

\[
\begin{align*}
(L) & & (R) \\
\Lambda' + \alpha_1 + \alpha_2 & = \Lambda + \gamma_0 & \Lambda' & = -\Lambda + \delta_0 \hspace{0.5cm} (31) \\
\Lambda' + 2\alpha_1 + \alpha_2 & = \Lambda + \gamma_1 & \Lambda' + \alpha_1 & = -\Lambda + \delta_1 \hspace{0.5cm} (32) \\
\Lambda' + 3\alpha_1 + \alpha_2 & = \Lambda + \gamma_2 & \Lambda' + 2\alpha_1 & = -\Lambda + \delta_2 \hspace{0.5cm} (33) \\
\Lambda' + \alpha_1 + 2\alpha_2 & = \Lambda + \gamma_3 & \Lambda' + \alpha_2 & = -\Lambda + \delta_3 \hspace{0.5cm} (34) \\
\Lambda' + \alpha_1 + 3\alpha_2 & = \Lambda + \gamma_4 & \Lambda' + 2\alpha_2 & = -\Lambda + \delta_4 \hspace{0.5cm} (35) \\
\Lambda' + 2\alpha_1 + 2\alpha_2 & = \Lambda + \gamma_5 & \Lambda' + \alpha_1 + \alpha_2 & = -\Lambda + \delta_5 \hspace{0.5cm} (36) \\
\Lambda' + 2\alpha_1 + 3\alpha_2 & = \Lambda + \gamma_6 & \Lambda' + \alpha_1 + 2\alpha_2 & = -\Lambda + \delta_6 \hspace{0.5cm} (37) \\
\Lambda' + 3\alpha_1 + 2\alpha_2 & = \Lambda + \gamma_7 & \Lambda' + 2\alpha_1 + \alpha_2 & = -\Lambda + \delta_7 \hspace{0.5cm} (38) \\
\Lambda' + 3\alpha_1 + 3\alpha_2 & = \Lambda + \gamma_8 & \Lambda' + 2\alpha_1 + 2\alpha_2 & = -\Lambda + \delta_8. \hspace{0.5cm} (39)
\end{align*}
\]

In the following we denote the left hand side formulas with an .L and the right hand side formulas with an .R. Again we suppose that (30.L) is satisfied, i.e. $\Lambda' + \alpha_1 + \alpha_2 = \Lambda + \gamma_0$. Then (30) and the extremality of $\Lambda$ implies that $\gamma_0$ does not equal to $\alpha_i$.

Now (30.L) would imply that $2(\alpha_1 + \alpha_2) = 2\alpha = \gamma_8 - \gamma_0$. Since $\alpha$ is a long root this is not possible and we have (30.R), i.e. $\Lambda' + 2\alpha_1 + 2\alpha_2 = -\Lambda + \delta_8$.

Thinking for a moment gives that (38.L) implies $\gamma_0 = -\alpha_1$ and (37.L) implies $\gamma_0 = -\alpha_2$. On the other hand (38.L) implies $\gamma_0 \neq -\alpha_1$ and (37.L) implies $\gamma_0 \neq -\alpha_2$. Hence (38.L) entails (37.R) and (35.R), as well as (37.L) entails (38.R) and (33.R).

Now we suppose that (38.L) is satisfied. Then we have (39.R), (37.R) and (35.R), i.e.

$$\alpha_2 = \delta_8 - \delta_7 \text{ and } 2\alpha_1 = \delta_8 - \delta_4.$$  

Again because of the extremality of $\Lambda$ these roots are not proportional. It implies $2\alpha_1 - \alpha_2 = \delta_7 - \delta_4$. Now $\alpha_1 \perp \alpha_2$ and $\delta_7 \neq \alpha_1, \neq \alpha_1 - \alpha_2$ (Extremality of $\Lambda$) gives a contradiction.

In the same way we argue supposing that (37.L) holds.
Hence we have shown that neither \((37.L)\) nor \((38.L)\) can be satisfied. Thus we have \((37.R)\) and \((38.R)\). These together with \((39.L)\) are no contradiction, but if one supposes one of the remaining \((32.R), (34.R)\) or \((36.R)\) we get a contradiction. Hence \((32.L), (34.L)\) and \((36.L)\) must be valid. But from these together with \((31.L)\) we derive as above a contradiction.

If we start with the right hand side of the first alternative one proceeds analogously.

All in all we have shown, that the assumption of a long root with \((21)\) leads to a contradiction.

Now we are in a position that we can use results of \([Sch99]\) explicitly. First we will cite them.

3.13 Proposition. \([Sch99]\) Let \(g \subset so(N, \mathbb{C})\) be an irreducible representation of real type of a complex simple Lie algebra different from \(sl(2, \mathbb{C})\). Then it holds:

1. If there is an extremal spanning \((\Lambda_1, \Lambda_2, \alpha)\) triple then there is no weight \(\lambda\) for which exists a pair of orthogonal long roots \(\eta_1\) and \(\eta_2\) such that \(2\left\langle \lambda, \eta_i \right\rangle \left\| \eta_i \right\|^2 = 2\).

2. If furthermore all roots have the same length, then there is no weight \(\lambda\) for which exists a triple of orthogonal roots \(\eta_1 \perp \eta_2 \perp \eta_3 \perp \eta_1\) such that \(2\left\langle \lambda, \eta_i \right\rangle \left\| \eta_i \right\|^2 = 2\) and \(2\left\langle \lambda, \eta_3 \right\rangle \left\| \eta_3 \right\|^2 = 1\).

We will now show that existence of such a pair or triple of roots implies that (SII) defines an extremal spanning pair.

3.14 Proposition. Let \(g \subset so(N, \mathbb{C})\) be an irreducible representation of real type of a complex simple Lie algebra different from \(sl(2, \mathbb{C})\), with \(0 \not\in \Omega\) and satisfying (SII). Then it holds: If there is a pair of orthogonal long roots \(\eta_1\) and \(\eta_2\) such that \(2\left\langle \lambda, \eta_1 \right\rangle \left\| \eta_1 \right\|^2 = 2\) for the extremal weight \(\Lambda\) from the property (SII), then \(\Lambda - \alpha\) is an extremal weight, i.e. (SII) defines an extremal spanning triple.

Proof. Again we argue indirectly considering three different cases for the root \(\alpha\) from the property (SII)

Case 1: All roots have the same length or \(\alpha\) is a long root. Again by applying the Weyl group the indirect assumption implies that there is an extremal weight \(\Lambda'\) and a root long \(\beta\) orthogonal to \(\alpha\) such that \(2\left\langle \Lambda', \alpha \right\rangle \left\| \alpha \right\|^2 = 2\left\langle \Lambda', \beta \right\rangle \left\| \beta \right\|^2 = -2\).

This gives \(\Lambda' + k\alpha + l\beta \in \Omega\) for \(0 \leq k, l \leq 2\) and hence \(\Lambda' + k\alpha + l\beta \in \Omega_\alpha\) for \(0 \leq k \leq 1, 0 \leq l \leq 2\).
Among others (SII) implies the existence of \( \gamma_i \) and \( \delta_i \) from \( \Delta_0 \) for \( i = 0, \ldots, 3 \) such that that the following alternatives must hold

\[
\begin{align*}
(L) & \\
\Lambda' + \alpha &= \Lambda + \gamma_0 \quad \text{or} \quad \Lambda' &= -\Lambda + \delta_0 \quad (40) \\
\Lambda' + 2\alpha &= \Lambda + \gamma_1 \quad \text{or} \quad \Lambda' + \alpha &= -\Lambda + \delta_1 \quad (41) \\
\Lambda' + \alpha + 2\beta &= \Lambda + \gamma_2 \quad \text{or} \quad \Lambda' + 2\beta &= -\Lambda + \delta_2 \quad (42) \\
\Lambda' + 1 + 2\alpha + 2\beta &= \Lambda + \gamma_3 \quad \text{or} \quad \Lambda' + \alpha + 2\beta &= -\Lambda + \delta_3. \quad (43)
\end{align*}
\]

Supposing again (40.L) we conclude that (42.L) and (43.L) cannot hold because \( \beta \) is long and the extremality of \( \Lambda \). Hence (42.R) and (43.R) must be satisfied. Again the extremality of \( \Lambda \) prevents that (41.R) can be valid. Hence we have (41.L).

Now (40.L) gives

\[
\frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = \frac{2\langle \Lambda', \alpha \rangle}{\|\alpha\|^2} + 2 - \frac{2\langle \gamma_0, \alpha \rangle}{\|\alpha\|^2} = -\frac{2\langle \gamma_0, \alpha \rangle}{\|\alpha\|^2}
\]

by assumption.

On the other hand (40.L) together with (41.L) and (42.R) and (43.R) implies that \( \alpha = \gamma_1 - \gamma_0 = \delta_3 - \delta_2 \). We note that \( \gamma_0 \) cannot be equal to 0 and \( \gamma_1 \) not equal to \( \alpha \).

If \( \gamma_0 = -\alpha \) and \( \gamma_1 = 0 \) then \( \Lambda = \Lambda' + 2\alpha \). Then (42.R) and (43.R) imply

\[
\begin{align*}
\langle \delta_2, \alpha \rangle &= 2\langle \Lambda', \alpha \rangle + 2\|\alpha\|^2 = 0 \quad \text{and} \\
\langle \delta_3, \alpha \rangle &= 2\langle \Lambda', \alpha \rangle + 3\|\alpha\|^2 = \|\alpha\|^2.
\end{align*}
\]

Since \( \alpha \) is long this entails \( \delta_2 = 0 \) and \( \delta_3 = \alpha \). Taking now (40.L) and (42.R) together we get \( \Lambda = \alpha - \beta \). But this forces \( 0 \in \Omega \) which was excluded.

Thus we have \( \alpha = \gamma_1 - \gamma_0 \) with non-proportionality. But this implies, since \( \alpha \) is long, that \( \frac{2\langle \gamma_0, \alpha \rangle}{\|\alpha\|^2} = -1 \) and hence \( \frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = 1 \). But this means that \( \Lambda - \alpha \) is an extremal weight.

**Case 2:** There are roots with different length and \( \alpha \) is a short root. By assumption there is a short root \( \gamma \) such that \( \gamma \perp \alpha \) and \( \eta := \alpha + \gamma \) is a long root and an extremal weight \( \Lambda' \) and a long root \( \beta \) such that \( \frac{2\langle \Lambda', \eta \rangle}{\|\eta\|^2} = \frac{2\langle \Lambda', \beta \rangle}{\|\beta\|^2} = -2 \). Analogously to the previous theorem the orthogonality of \( \alpha \) and \( \gamma \) gives

\[
-2 = \frac{1}{2} \left( \frac{2\langle \Lambda', \alpha \rangle}{\|\alpha\|^2} + \frac{2\langle \Lambda', \gamma \rangle}{\|\gamma\|^2} \right).
\]

Hence we have to consider three cases:

(a) \( \frac{2\langle \Lambda', \alpha \rangle}{\|\alpha\|^2} = \frac{2\langle \Lambda', \gamma \rangle}{\|\gamma\|^2} - 2 \),
Then an easy calculation shows that $\langle \alpha, \beta \rangle = \langle \gamma, \beta \rangle = 0$ in each case.

We shall consider the case (a), (b) and (c) separately.

**Case (a):** Here we can proceed completely analogously to the first case 1. We have that $\Lambda' = \gamma_1 - \gamma_0$ non proportional. At least one has to be a short root and $\langle \gamma_0, \alpha \rangle < 0$ and $\langle \gamma_1, \alpha \rangle > 0$. On the other hand, we have $\frac{2(\Lambda, \alpha)}{||\alpha||^2} - \frac{2(\gamma_0, \alpha)}{||\alpha||^2}$ and $\frac{2(\Lambda', \alpha)}{||\alpha||^2} - \frac{2(\gamma_1, \alpha)}{||\alpha||^2} = 2$ by (44) and (45). But this implies that both are short and $\frac{2(\Lambda', \alpha)}{||\alpha||^2} = 1$ which is the proposition.

**Case (b):** $\frac{2(\Lambda', \alpha)}{||\alpha||^2} = -3$ implies $\Lambda' + k\alpha + l\beta \in \Omega_\alpha$ for $0 \leq k, l \leq 2$. (SII) then implies

$$\begin{align*}
(L) & \\
\Lambda' + \alpha & = \Lambda + \gamma_0 \text{ or } \Lambda' = -\Lambda + \delta_0 \quad (44) \\
\Lambda' + 2\alpha & = \Lambda + \gamma_1 \text{ or } \Lambda' + \alpha = -\Lambda + \delta_1 \quad (45) \\
\Lambda' + 3\alpha & = \Lambda + \gamma_2 \text{ or } \Lambda' + 2\alpha = -\Lambda + \delta_2 \quad (46) \\
\Lambda' + 2\alpha + 2\beta & = \Lambda + \gamma_3 \text{ or } \Lambda' + \alpha + 2\beta = -\Lambda + \delta_3 \quad (47) \\
\Lambda' + 3\alpha + 2\beta & = \Lambda + \gamma_4 \text{ or } \Lambda' + 2\alpha + 2\beta = -\Lambda + \delta_3 \quad (48)
\end{align*}$$

Supposing (44) L excludes (47) L and (48) L because $\beta$ is long. Hence (47) R and (48) R are valid and exclude (45) R and (46) L. Hence (45) L and (46) L are satisfied. This gives $\alpha = \gamma_2 - \gamma_1 = \gamma_1 - \gamma_0$ with $\gamma_0$ different from 0 and $-\alpha, \gamma_1$ different from 0 and $\alpha$ and $\gamma_2$ different from $\pm \alpha$. Hence $\alpha + \pm \delta_1$ is a long root with $\alpha \perp \delta_1$. But this gives $\frac{2(\Lambda', \alpha)}{||\alpha||^2} = \frac{2(\Lambda', \alpha)}{||\alpha||^2} + 4 = 1$, i.e. $\Lambda - \alpha$ is an extremal weight.

**Case (c):** Here we have that $\Lambda' + k\gamma + l\beta \in \Omega_\alpha$ for $0 \leq k \leq 3$ and $0 \leq l \leq 2$ since $\frac{2(\Lambda' + k\gamma + l\beta, \alpha)}{||\alpha||^2} = -1$. The equations implied by (SII) lead easily to a contradiction:

$$\begin{align*}
(L) & \\
\Lambda' + \alpha & = \Lambda + \gamma_0 \text{ or } \Lambda' = -\Lambda + \delta_0 \quad (49) \\
\Lambda' + \alpha + 3\gamma & = \Lambda + \gamma_1 \text{ or } \Lambda' + 3\gamma = -\Lambda + \delta_1 \quad (50) \\
\Lambda' + \alpha + 2\beta + 3\gamma & = \Lambda + \gamma_2 \text{ or } \Lambda' + 2\beta + 3\gamma = -\Lambda + \delta_2. \quad (51)
\end{align*}$$

Supposing (49) L excludes (50) L and (51) L. Hence (50) R and (51) R are valid but contradict to each other because $\beta$ is long.
3.15 Proposition. Let \( g \subset \mathfrak{so}(N, \mathbb{C}) \) be an irreducible representation of real type of a complex simple Lie algebra different from \( \mathfrak{sl}(2, \mathbb{C}) \), with \( 0 \not\in \Omega \) and satisfying (SII). If furthermore all roots have the same length, and if there is a triple of orthogonal roots \( \eta_1 \perp \eta_2 \perp \eta_3 \perp \eta_1 \) such that \( \frac{2\langle \Lambda, \eta_i \rangle}{\|\eta_i\|^2} = 2 \) and \( \frac{2\langle \Lambda, \eta_j \rangle}{\|\eta_j\|^2} = \frac{2\langle \Lambda, \eta_k \rangle}{\|\eta_k\|^2} = 1 \) then holds one of the cases

1. \( \Lambda - \alpha \) is an extremal weight, i.e. (SII) defines an extremal spanning triple, or

2. \( \Lambda = \alpha + \frac{1}{2} (\beta + \gamma) \) with roots \( \alpha \perp \beta \perp \gamma \perp \alpha \).

Proof. Let \( \alpha \) be the root determined by (SII). The assumption implies that there is an extremal weight \( \Lambda' \) and roots \( \beta \) and \( \gamma \) such that

\[
\frac{2\langle \Lambda', \alpha \rangle}{\|\alpha\|^2} = -2 \quad \text{and} \quad \frac{2\langle \Lambda, \beta \rangle}{\|\beta\|^2} = \frac{2\langle \Lambda, \gamma \rangle}{\|\gamma\|^2} = -1.
\]

Then \( \Lambda' + k\alpha + l\beta + m\gamma \in \Omega \) for \( k, l, m = 0, 1 \). Hence (SII) implies again

\[
\begin{align*}
(L) & \\
\Lambda' + \alpha & = \Lambda + \gamma_0 \quad \text{or} \quad \Lambda' \quad = \quad -\Lambda + \delta_0 \quad (52) \\
\Lambda' + 2\alpha & = \Lambda + \gamma_1 \quad \text{or} \quad \Lambda' + \alpha \quad = \quad -\Lambda + \delta_1 \quad (53) \\
\Lambda' + \alpha + \beta & = \Lambda + \gamma_2 \quad \text{or} \quad \Lambda' + \beta \quad = \quad -\Lambda + \delta_2 \quad (54) \\
\Lambda' + 2\alpha + \beta & = \Lambda + \gamma_3 \quad \text{or} \quad \Lambda' + \alpha + \beta \quad = \quad -\Lambda + \delta_3 \quad (55) \\
\Lambda' + \alpha + \gamma & = \Lambda + \gamma_4 \quad \text{or} \quad \Lambda' + \gamma \quad = \quad -\Lambda + \delta_4 \quad (56) \\
\Lambda' + 2\alpha + \gamma & = \Lambda + \gamma_5 \quad \text{or} \quad \Lambda' + \alpha + \gamma \quad = \quad -\Lambda + \delta_5 \quad (57) \\
\Lambda' + \alpha + \beta + \gamma & = \Lambda + \gamma_6 \quad \text{or} \quad \Lambda' + \beta + \gamma \quad = \quad -\Lambda + \delta_6 \quad (58) \\
\Lambda' + 2\alpha + \beta + \gamma & = \Lambda + \gamma_7 \quad \text{or} \quad \Lambda' + \alpha + \beta + \gamma \quad = \quad -\Lambda + \delta_7. \quad (59)
\end{align*}
\]

Supposing (52) (L) excludes (59) (R) because of the orthogonality of the roots. Thus it must hold (59) (R). Now we consider two cases:

Case 1: \( \langle \gamma_0, \beta \rangle = \langle \gamma_0, \gamma \rangle = 0 \). This excludes (53) (L), (56) (L) and (58) (L) and implies therefore (54) (R), (56) (R) and (58) (R). The latter together with (59) (R) gives \( \alpha = \delta_7 - \delta_6 \).

Since \( \delta_7 \neq 0 \) this implies \( \langle \alpha, \delta_7 \rangle > 0 \). On the other hand (59) (R) and the assumption gives \( \frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = \frac{2\langle \delta_7, \alpha \rangle}{\|\alpha\|^2} > 0 \). If \( \alpha \neq \delta_7 \) we are done.

But \( \delta_7 = \alpha \) implies \( \Lambda' + \beta + \gamma = -\Lambda = -\Lambda' - \alpha - \gamma_0 \) and hence \( -2 = 2 - \frac{2\langle \alpha, \gamma_0 \rangle}{\|\alpha\|^2} \), i.e. \( \gamma_0 = -\alpha \). Taking everything together we get \( 2\Lambda = 2\alpha - (\beta + \gamma) \).

Case 2: \( \langle \gamma_0, \beta \rangle \) or \( \langle \gamma_0, \gamma \rangle \) not equal to zero. This implies \( \gamma_0 \neq \pm \alpha \) and thus \( \frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = \frac{2\langle \gamma_0, \alpha \rangle}{\|\alpha\|^2} = \pm 1 \) or zero. Now (53) (L) would imply \( \alpha = \gamma_1 - \gamma_0 \), i.e. \( \langle \alpha, \gamma_0 \rangle < 0 \). This would be the proposition.
Hence we suppose (53.R). This together with the starting point (52.L) gives
\[
\Lambda = \frac{1}{2}(\delta_1 - \gamma_0) \quad \text{and} \\
\Lambda' = -\alpha + \frac{1}{2}(\delta_1 + \gamma_0).
\]
The second equation implies using the assumption that \(\langle \alpha, \delta_1 + \gamma_0 \rangle = 0\). For the length of both extremal weights then holds
\[
\|\Lambda\|^2 = \frac{1}{4}(\|\delta_1\|^2 + \|\gamma_0\|^2 - 2\langle \delta_1, \gamma_0 \rangle) \\
\|\Lambda'\|^2 = \|\alpha\|^2 - \langle \alpha, \delta_1 + \gamma_0 \rangle + \frac{1}{4}(\|\delta_1\|^2 + \|\gamma_0\|^2 + 2\langle \delta_1, \gamma_0 \rangle) = 0.
\]
This gives \(0 = \|\alpha\|^2 + \langle \delta_1, \gamma_0 \rangle\). Since all roots have the same length this implies \(\delta_1 = -\gamma_0\). Hence \(\Lambda\) is a root. But this was excluded.

Now using the proposition 3.13 of Schwachhöfer we get a corollary.

3.16 Corollary. Let \(g \subset \mathfrak{so}(N, \mathbb{C})\) be an irreducible representation of real type of a complex simple Lie algebra different from \(\mathfrak{sl}(2, \mathbb{C})\), with \(0 \not\in \Omega\) and satisfying (SII). Then it holds:

1. There is no a pair of orthogonal long roots \(\eta_1\) and \(\eta_2\) such that \(2\langle \Lambda, \eta_i \rangle / \|\eta_i\|^2 = 2\) for the extremal weight \(\Lambda\) from the property (SII).

2. If furthermore all roots have the same length, and if there is a triple of orthogonal roots \(\eta_1 \perp \eta_2 \perp \eta_3 \perp \eta_1\) such that \(2\langle \Lambda, \eta_i \rangle / \|\eta_i\|^2 = 2\) and \(2\langle \Lambda, \eta_2 \rangle / \|\eta_2\|^2 = 2\langle \Lambda, \eta_3 \rangle / \|\eta_3\|^2 = 1\) then \(\Lambda = \alpha + \frac{1}{2}(\beta + \gamma)\) with roots \(\alpha \perp \beta \perp \gamma \perp \alpha\).

Before we apply this corollary we have to deal with the remaining exception in the second point.

3.17 Lemma. If the representation of a simple Lie algebra with roots of the same length has an extremal weight \(\Lambda\) such that \(\Lambda = \alpha + \frac{1}{2}(\beta + \gamma)\) with roots \(\alpha \perp \beta \perp \gamma \perp \alpha\). Then it holds

1. There is no root \(\delta\) such that \(\langle \delta, \beta \rangle = 0, \langle \delta, \gamma \rangle \neq 0\) and \(\delta \not\sim \gamma\).

2. The root system is \(D_n\) and the representation has one of the following highest weights: \(\omega_3\) for arbitrary \(n\), \(\omega_1 + \omega_3\) or \(\omega_1 + \omega_4\) for \(n = 4\) and \(\omega_2\) for \(n = 3\).

Proof. The first point is easy to see: If there is such a \(\delta\) then we have
\[
2\langle \Lambda, \delta \rangle / \|\delta\|^2 = 2\langle \alpha, \delta \rangle / \|\delta\|^2 + 1 / 2 \cdot 2\langle \gamma, \delta \rangle / \|\delta\|^2 = 2\langle \alpha, \delta \rangle / \|\delta\|^2 + 1 / 2 \not\in \mathbb{Z}.
\]
This is a contradiction.

Now we consider the different root systems with roots of constant length.
\(A_n\): Here the assumption means that \(\Lambda = e_i - e_j + \frac{1}{2}(e_p - e_q + e_r - e_s)\) with all indices different from each other. But then \(\frac{2\langle \Lambda, e_i - e_p \rangle}{\|e_i - e_p\|^2}\) is not an integer.

\(D_n\): If \(\alpha = e_i \pm e_j, \beta = e_p \pm e_q \) and \(\gamma = e_r \pm e_s\) with all indices different we get the same contradiction as in the \(A_n\) case. Thus we are left with two cases.

The first is \(\beta + \gamma = e_p + e_q + e_p - e_q = 2e_p\), and hence \(\Lambda = e_i \pm e_j + e_p\). This leads to \(\Lambda = \omega_3\) or for \(n = 3\) to \(\Lambda = \omega_2\).

The second is \(\alpha = e_i + e_j, \beta = e_i - e_j\) and \(\gamma = e_p \pm e_q\). For \(n > 4\) we found a root \(e_p + e_s\) which leads to a contradiction by applying the first point. For \(n = 4\) we have \(\Lambda = \frac{3}{2}e_i + \frac{1}{2}(e_j + e_p \pm e_q)\). But this yields the remaining representations.

\(E_6\): \(E_6\) has two different types of roots:

\[
e_i \pm e_j \quad \text{and} \quad \frac{1}{2}(e_8 - e_7 - e_6 \pm e_5 \pm e_4 \pm e_3 \pm e_2 \pm e_1)\quad \text{even number of minus signs}
\]

The only possibility for \(\beta\) and \(\gamma\) for which the first point yields no contradiction is \(\beta = e_i + e_j\) and \(\gamma = e_i - e_j\). Hence \(\Lambda = \alpha + e_i\). \(\alpha \perp \beta\) and \(\alpha \perp \gamma\) implies \(\alpha = e_p + e_q\).

But then \(2\langle \Lambda, \frac{1}{2}(\ldots) \rangle \notin \mathbb{Z}\).

Proceeding analogously for \(E_7\) and \(E_8\) we prove the second assertion. \(\Box\)

Now using all these properties we can find the representations without weight zero and satisfying (SII).

### 3.18 Proposition

Let \(\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})\) be an irreducible representation of real type of a complex simple Lie algebra different from \(\mathfrak{sl}(2, \mathbb{C})\), with \(0 \notin \Omega\) and satisfying (SII). Then the roots system and the highest weight of the representation is one of the following (modulo congruence):

- \(A_n\): \(\omega_4\) for \(n = 7\).
- \(B_n\): \(\omega_n\) for \(n = 3, 4, 7\).
- \(D_n\): \(\omega_1, 2\omega_1\) for arbitrary \(n\) and \(\omega_8\) for \(n = 8\).

#### Proof

We apply proposition 3.12 and corollary 3.16 to the remaining representations with \(0 \notin \Omega\), i.e. representations of \(A_n\), \(B_n\), \(C_n\), \(D_n\), \(E_6\) and \(E_7\). Therefore we use a fundamental system such that the extremal weight \(\Lambda\) determined by (SII) is the highest weight. It can be written in the fundamental representations \(\Lambda = \sum_{k=1}^{n} m_k \omega_k\) with \(m_k \in \mathbb{N} \cup \{0\}\).

\(A_n\): Proposition 3.12 gives for the largest root

\[
2 \geq \frac{2\langle \Lambda, e_1 - e_n+1 \rangle}{\|e_1 - e_n+1\|^2} = \sum_{k=1}^{n} m_k \langle \omega_k, e_1 - e_n+1 \rangle = \sum_{k=1}^{n} m_k.
\]
Since the representation has to be self dual we have that $m_i = m_{n+1-i}$.

First we consider the case that $\Lambda = \omega_i + \omega_{n+1-i}$. For $n > 2$ we get in case $i > 1$ that $\langle \Lambda, e_2 - e_n \rangle = 2$. But $(e_2 - e_n) \perp (e_1 - e_{n+1})$ gives a contradiction to 3.16 of corollary of corollary 3.16. For $n \geq 2$ it has to be $\Lambda = 2\omega_1 + \omega_n + 1 - i$.

First we consider the case that $\Lambda = \omega_i + \omega_{n+1-i}$. For $n > 2$ we get in case $i > 1$ that $\langle \Lambda, e_2 - e_n \rangle = 2$. But $(e_2 - e_n) \perp (e_1 - e_{n+1})$ gives a contradiction to 3.16 of corollary 3.16. For $n \geq 2$ it has to be $\Lambda = \omega_1 + \omega_n + 1 - i$.

For $n + 1$ even we have to study the case $\Lambda = \omega_{n+1/2}$. This representation is orthogonal if $n + 1/2$ is even. The weights of this representation are given by $\pm e_{k_{1/2}} \pm \ldots \pm e_{k_{n+1/2}}$ where the $\pm$’s are meant to be independent of each other.

We will show that (SII) implies $n \leq 7$.

Hence suppose that there is a root $\alpha$ such that (SII) with $\Lambda$. We have to consider two cases for $\alpha$. The first is that $\alpha = e_i - e_j$ with $1 \leq i \leq n + 1/2 < j \leq n + 1$. W.l.o.g. we take $\alpha = e_{n+1/2} - e_{n+1/2} + 1$ and consider the weight

$$\lambda := e_1 + \ldots e_{n+1/2} - 3 + e_{n+1/2} + 1 + e_{n+1/2} + 2 + e_{n+1/2} + 3.$$  

$\langle \lambda, \alpha \rangle < 0$ implies $\lambda \in \Omega_\alpha$. Then $\lambda - (\Lambda - \alpha) \in \Delta_0$ or $\lambda + \Lambda \in \Delta_0$. We check the first alternative: $\Lambda - \alpha = e_1 + \ldots e_{n+1/2} - 1 + e_{n+1/2} + 1$ implies

$$\lambda - (\Lambda - \alpha) = e_{n+1/2} - 3 + e_{n+1/2} - 2 + e_{n+1/2} + 2 + e_{n+1/2} + 3.$$  

But this is not a root.

For the second alternative we get, recalling that $-e_1 = e_2 + \ldots + e_{n+1}$,

$$\lambda + \Lambda = e_1 + \ldots + e_{n+1/2} - 3 - e_{n+1/2} + 4 - \ldots - e_{n+1}.$$  

This is not a root if $n + 1/2 > 4$, i.e. $n > 7$.

For the second type of root $\alpha = e_i - e_j$ with $1 \leq i < j \leq n + 1/2$ and $n + 1/2 < i < j \leq n + 1$ one derives analogously that $n \leq 5$.

Hence for $\Lambda = \omega_{n+1/2}$ the property (SII) can only be fulfilled if $n \leq 7$. These representations are orthogonal for $n = 7$ and $n = 3$. $A_3$ is isomorphic to $D_3$ and the representation with highest weight $\omega_2$ of $A_3$ is equivalent to the one with $\omega_1$ of $D_3$. 

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\( B_n \): Again proposition 3.12 gives for the largest root
\[
2 \geq \frac{2\langle \Lambda, e_1 + e_2 \rangle}{\| e_1 + e_2 \|^2} = \sum_{k=1}^{n} m_k \langle \omega_k, e_1 + e_2 \rangle = m_1 + 2m_2 + \ldots + 2m_{n-1} + m_n.
\]

The only representations with \( 0 \not\in \Omega \) are those with \( \Lambda = \omega_1 + \omega_n \) and the spin representation \( \Lambda = \omega_n \). There is no possibility to apply the first point of corollary \( \text{3.16} \) But we verify that for \( \Lambda = \omega_1 + \omega_n \) (SII) implies \( n \leq 2 \) and for the spin representation \( \Lambda = \omega_n \) (SII) implies \( n \leq 7 \).

The spin representations: For these we show that (SII) implies \( n \leq 7 \) The spin representation of highest weight \( \Lambda = \frac{1}{2}(e_1 + \ldots + e_n) \) has weights \( \Omega = \left\{ \frac{1}{2}(\varepsilon_1 e_1 + \ldots + \varepsilon_n e_n) | \varepsilon_i = \pm 1 \right\} \). We have to consider three types for the root \( \alpha \): \( \alpha = e_i, \alpha = e_i + e_j \) and \( \alpha = e_i - e_j \).

For the first we can assume w.l.o.g. that \( \alpha = e_1 \). Then \( \Omega_\alpha = \{ \frac{1}{2}(-e_1 + \varepsilon_2e_2 + \ldots + \varepsilon_ne_n) | \varepsilon = \pm 1 \} \). It is \( \Lambda - \alpha = \frac{1}{2}(-e_1 + e_2 + \ldots + e_n) \). Hence for \( \lambda \in \Omega_\alpha \) we have
\[
\Lambda - \alpha - \lambda = \frac{1}{2}((1 - \varepsilon_2)e_2 + \ldots + (1 - \varepsilon_n)e_n) \quad \text{and}
\Lambda + \lambda = \frac{1}{2}(1 + \varepsilon_2)e_2 + \ldots + (1 + \varepsilon_n)e_n)
\]

If (SII) is satisfied at least one of these expression has to be a root. But if \( n \geq 7 \) we can choose \( (\varepsilon_2, \ldots, \varepsilon_n) \) such that none of them is a root.

The second type of root shall be w.l.o.g. \( \alpha = e_1 - e_2 \). In this case \( \Omega_\alpha = \{ \frac{1}{2}(-e_1 + e_2 + e_3e_3 + \ldots + e_n e_n) | \varepsilon_i = \pm 1 \} \) and \( \Lambda - \alpha = \frac{1}{2}(-e_1 + e_2 + e_3 + \ldots + e_n) \). Hence for \( \lambda \in \Omega_\alpha \)
\[
\Lambda - \alpha - \lambda = \frac{1}{2}(e_2 + (1 - \varepsilon_3)e_3 + \ldots + (1 - \varepsilon_n)e_n) \quad \text{and}
\Lambda + \lambda = \frac{1}{2}(2e_2 + (1 + \varepsilon_3)e_3 + \ldots + (1 + \varepsilon_n)e_n)
\]

We can choose \( \lambda \) such that none of them is a roots if \( n \geq 4 \).

Now we consider the last type of root, \( \alpha = e_1 + e_2 \). \( \Omega_\alpha = \{ \frac{1}{2}(-e_1 - e_2 + e_3e_3 + \varepsilon_n e_n) | \varepsilon_i = \pm 1 \} \) and \( \Lambda - \alpha = \frac{1}{2}(-e_1 - e_2 + e_3 + \ldots + e_n) \). Hence for \( \lambda \in \Omega_\alpha \)
\[
\Lambda - \alpha - \lambda = \frac{1}{2}((1 - \varepsilon_3)e_3 + \ldots + (1 - \varepsilon_n)e_n) \quad \text{and}
\Lambda + \lambda = \frac{1}{2}(1 + \varepsilon_3)e_3 + \ldots + (1 + \varepsilon_n)e_n)
\]

We can choose \( \lambda \) such that none of them is a roots if \( n \geq 8 \). Hence if (SII) is satisfied it has to be \( n \leq 7 \) and for \( n = 7 \) the pair of property (SII) is of the shape \( (\Lambda, e_1 + e_2) \).

Now for \( n = 2, n = 5 \) and \( n = 6 \) the spin representations are symplectic but not orthogonal.
The representations of $\Lambda = \omega_1 + \omega_n = \frac{3}{2}e_1 + \frac{1}{2}(e_2 + \ldots + e_n)$. Then the weights are given by $\frac{1}{2}(ae_{k_1} + \varepsilon_2c_{k_2} + \ldots + \varepsilon_ne_{k_n})$ with $a \in \{\pm 1, \pm 3\}$ and $\varepsilon_i = \pm 1$.

For these one shows analogously that (SII) implies $n \leq 2$. For $n = 2$ this representation is symplectic.

$C_n$: For the largest root we get

$$2 \geq \frac{2\langle \Lambda, 2e_1 \rangle}{\|2e_1\|^2} = \sum_{k=1}^{n} m_k \langle \omega_k, e_1 \rangle = \sum_{k=1}^{n} m_k.$$ 

In case that one $m_i = 2$ and all others zero we have that $0 \in \Omega$. Hence we suppose that $\Lambda = \omega_i + \omega_j$ for $i \neq j$. If $i > 1$ we get for the root $2e_2$ which is orthogonal to $2e_1$ that $\frac{2\langle \Lambda, 2e_2 \rangle}{\|2e_2\|^2} = 2$. Thus by 1 of corollary 3.16 we have $i = 1$. But $\Lambda = \omega_1 + \omega_i$ is only orthogonal if $i$ is odd, but if $i$ is odd we have that $0 \in \Omega$.

Hence we have to deal with the case $\Lambda = \omega_1$. This is orthogonal if $i$ is even, but in this case $0 \in \Omega$.

$D_n$: Here we get for the largest root

$$2 \geq \frac{2\langle \Lambda, e_1 + e_2 \rangle}{\|e_1 + e_2\|^2} = \sum_{k=1}^{n} m_k \langle \omega_k, e_1 + e_2 \rangle = m_1 + 2m_2 + \ldots + 2m_{n-2} + m_{n-1} + m_n.$$ 

First we consider the representation where this number is equal to 2.

For the representations $2\omega_n$ and $2\omega_{n-1}$ we have that $0 \in \Omega$.

For the representations $\Lambda = \omega_1 + \omega_n$ and $\Lambda = \omega_1 + \omega_{n-1}$ we get that $n = 4$ or there is no triple as in the second point of proposition 3.16. Thus suppose in this case $n > 4$. We have that $\langle \Lambda, e_1 + e_2 \rangle = 2$ and for the orthogonal roots $\langle \Lambda, e_1 - e_2 \rangle = \langle \Lambda, e_3 \pm e_4 \rangle = 1$. But this contradicts proposition 3.16.

For $\Lambda = \omega_{n-1} + \omega_n = e_1 + \ldots + e_{n-1}$ we have that $0 \not\in \Omega$ implies $n - 1$ even. The first point of corollary 3.16 then gives for $n > 4$ that $2 = \langle \Lambda, e_3 + e_4 \rangle$ which is impossible. Hence $n \leq 4$. Then $1 = \langle \Lambda, e_3 \pm e_4 \rangle$ and the second point of corollary 3.16 imply $n \leq 3$.

Now suppose that $\Lambda = \omega_i$ for $2 \leq i \leq n - 2$. We apply the first point of corollary 3.16. If $n \geq 4$ we get that $\langle \omega_i, e_3 + e_4 \rangle = 2$ for $i \geq 4$ but this was excluded. Hence $i \leq 3$.

In the case $n = 3$ we have that only $\omega_2$ is an orthogonal representation. But for this holds that $0 \in \Omega$.

Thus, to get the assertion of the proposition we have to show that

1. For the spin representations $\Lambda = \omega_{n-1}$ and $\Lambda = \omega_n$ (SII) implies $n \leq 8$
2. $\Lambda = \omega_3$ does not satisfy (SII),

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3. Λ = ω₁ + ω₃ and ω₁ + ω₄ for n = 4 do not satisfy (SII).

The spin representations: For these we show that (SII) implies first n ≤ 8. Because we are interested in the representations modulo congruence it suffices to consider the spin representation of highest weight Λ = \frac{1}{2}(e₁ + \ldots + eₙ) with weights Ω = \{\frac{1}{2}(e₁e₁ + \varepsilon_neₙ)|\varepsilon_i = \pm 1 and \varepsilon_i = -1 for an even number\}.

Analogously as for Bₙ we get for two types of roots α = eᵢ + eⱼ and α = eᵢ − eⱼ that (SII) implies n ≤ 8 (We have to admit one dimension higher because of the sign restriction of the weights).

Now for n odd the spin representation is not self dual, and for n = 6 not orthogonal. For n = 4 it is congruent to ω₁.

Λ = ω₃ = e₁ + e₂ + e₃: Here it is Ω = \{(ε₁e₁ + ε₂e₂ + ε₃e₃)|εᵢ = ± 1} \cup \{±eᵢ\}.

For n = 3 and n = 4 this is a spin representation. Hence suppose n ≥ 5.

For α = e₁ + e₂ we get Λ – α = e₃. Set λ := –e₁ + e₄ + e₅ ∈ Ωₙ. Hence Λ – α – λ = e₃ + e₁ – e₄ – e₅ and Λ + λ = e₂ + e₃ + e₄ + e₅. None is a root, i.e. ω₃ for n ≥ 5 does not satisfy (SII).

For α = e₁ – e₂ we get the same.

Λ = ω₁ + ω₃ and ω₁ + ω₄ for n = 4. These are congruent to each other and as above it can be shown that they do not satisfy (SII).

E₆ and E₇: For these we refer to [Sch99]. There is shown that under the conclusions of proposition 3.12 and 3.13 — which is our situation because of lemma 3.17 — the only remaining representations are the standard representations of E₆ and E₇. But the first is not self dual and the latter symplectic but not orthogonal.

We get the following

3.19 Corollary. Let \mathfrak{g} ⊂ so(N, C) be an orthogonal algebra of real type different from \mathfrak{sl}(2, C). If 0 \not\in Ω and (SII) is satisfied, then it is the complexification of a Riemannian holonomy representation or the spin representation of so(15, C).

Proof. We give the Riemannian manifolds the complexified holonomy representation of which is one of the representations of proposition 3.18.

The representation with highest weight ω₄ of A₇ is the complexified holonomy representation of the symmetric space of type EV, i.e. of E₇/SU(8) resp. E₇₋₂₀/SU(8).

The spin representations of Bₙ for n = 3, 4 are the holonomy representations of a non-symmetric Spin(7)–manifold and of the symmetric space of type FII, i.e. of F₄/Spin(9) resp. F₄₋₂₀/Spin(9). For n = 7 we have an exception.

For Dₙ first we have the standard representation, i.e. the complexified holonomy representation of a generic manifold. The representation with highest weight 2ω₁
is the complexified holonomy representation of the symmetric space of type \(A\), i.e. of \(SU(2n)/SO(2n,\mathbb{R})\), resp. \(SL(2n,\mathbb{R})/SO(2n,\mathbb{R})\). The remaining representation of \(Spin(16)\) is the complexified holonomy representation of the symmetric space of type \(E\), i.e. of \(E_8/Spin(16)\), resp. \(E_8(8)/Spin(16)\).

### 3.5 Consequences for simple weak-Berger algebras of real type

Before we conclude the result we need a lemma to exclude both exceptions.

**3.20 Lemma.** The spin representation of \(B_7\) and the representation of \(G_2\) with two times a short root as highest weight are not weak-Berger.

**Proof.** 1.) Suppose that the spin representation of \(B_7\) is weak-Berger. We have shown that it does not satisfy the property (SI). Hence it obeys (SII). Let \(\Lambda, \alpha\) be the pair of (SII). We choose a fundamental system such that \(\Lambda = \omega_7\) is the highest weight. In the proof of proposition 3.18 we have shown that in this case \(\alpha = e_i + e_j\).

Let now \(Q_\phi\) be the weight element from \(B_H(g)\) and \(u_\Lambda \in V_\Lambda\) such that \(Q_\phi(u_\Lambda) = A_{e_i+e_j} \in g_{e_i+e_j}\). Since \(Q_\phi(u_\Lambda) \in g_{\phi+\Lambda}\) this implies that \(\phi = e_i + e_j - \Lambda\) is a weight of \(B_H(g)\). Hence \(\phi = -\frac{1}{2}(e_1 + \ldots + e_{i-1} + e_i + e_{i+1} + \ldots + e_{j-1} + e_j + e_{j+1} + \ldots + e_7)\) is also an extremal weight of \(V\) and we can consider a weight vector \(u_{-\phi} \in V_{-\phi}\). For this we get \(Q_\phi(u_{-\phi}) \in t\). In case it does not vanish it would define a planar spanning triple \((\phi, -\phi, (Q_\phi(u_{-\phi}))^\perp)\), i.e. (SI) would be satisfied. But this was not possible, and thus \(Q_\phi(u_{-\phi}) = 0\).

On the other hand we have that \(0 \neq Q_\phi(u_\Lambda)u_{-\phi} \in V_\Lambda\) and thus there is a \(v \in V_{-\Lambda}\) such that \(H(Q_\phi(u_\Lambda)u_{-\phi}, v) \neq 0\). Now the Bianchi identity gives

\[
0 = H(Q_\phi(u_\Lambda)u_{-\phi}, v) + H(Q_\phi(u_{-\phi})v, u_\Lambda) + H(Q_\phi(v)u_\Lambda, u_{-\phi}, v).
\]

Hence \(0 \neq Q_\phi(v) \in g_{\phi - \Lambda}\). But \(\phi - \Lambda = -(e_1 + \ldots + e_{i-1} + e_i + e_{i+1} + \ldots + e_{j-1} + e_j + e_{j+1} + \ldots + e_7)\) is not a root, hence \(g_{\phi - \Lambda} = \{0\}\). This is a contradiction.

2.) Suppose that the representation of \(G_2\) with two times a short root as highest weight is weak-Berger. We will argue analogously as for \(B_n\).

In the picture we see the weight lattice of this representation (the arrows represent the roots). Obviously there is no planar spanning triple, because there is no hypersurface which contains all but two extremal weight (see also proof of proposition 3.11).

The weak-Berger property implies that there is a pair \((\Lambda, \alpha)\) such that (SII) is satisfied. We choose a fundamental system such that \(\Lambda = 2\eta\) is the maximal weight.
Using the realization of $G_2$ from the appendix of [Kna02] we have that $\eta = e_3 - e_2$. Now we have to determine the roots for which (SII) is satisfied.

In the picture one can see that the long roots $\alpha$ and $\beta$ satisfy (SII). (We illustrate the situation in detail only for $\alpha$.) Contemplate the picture for a moment one sees that there are no short roots and no other long root for which (SII) can be valid.

Now $\alpha$ and $\beta$ are the only roots with $\langle\Lambda, \alpha\rangle > 0$ and $\langle\Lambda, \beta\rangle > 0$. Hence $\alpha = 2e_3 - e_1 - e_2$ and $\beta = -2e_2 + e_1 + e_3$.

We consider the case where $(\Lambda, \alpha)$ satisfies (SII). There is a weight element $Q_\phi$ from $B_H(g)$ such that $Q_\phi(u_\Lambda) = A_{2e_2-e_1-e_2}$, i.e. $\phi = 2e_3-e_1-e_2-\Lambda = e_2-e_1$. But this is a short root and therefore a weight. Thus we consider $u_{-\phi} \in V_{-\phi}$. Then $Q_\phi(u_{-\phi}) \in t$. Since there is no planar spanning triple it has to be zero. As above the Bianchi identity gives that $\phi - \Lambda$ has to be a root. But $\phi - \Lambda = e_2 - e_1 - 2e_3 + 2e_2 = 3e_2 - 2e_3 - e_1$ is no root.

For $\beta$ one proceeds analogously. \qed

Now we can draw the conclusions from the previous sections. If a Lie algebra acts irreducible of real type the it is semi-simple and obeys the properties (SI) or (SII). The simple Lie algebras with (SI) or (SII) we have listed above. Thus we get

3.21 Theorem. Let $g \subset \mathfrak{so}(N, \mathbb{R})$ be a irreducible weak-Berger algebra of real type. Then it is the holonomy representation of a Riemannian manifold. The conclusion holds in particular if $g$ is simple, of real type and the irreducible component of the $\mathfrak{so}(n)$-projection of an indecomposable, non-irreducible simply connected Lorentzian manifold.

3.22 Remark. Quaternionic symmetric spaces. With the result of course we have covered all simple irreducible acting Riemannian holonomy groups of real type.

If one considers a quaternionic symmetric space $G/Sp(1) \cdot H$ with $H \subset Sp(n)$ then of course $\mathfrak{sp}(1) \oplus \mathfrak{h} \subset \mathfrak{so}(4n, \mathbb{R})$ is a real Berger algebra of real type and thus its complexification is a complex Berger algebra of real type. Then the restriction of this representation to $\mathfrak{h}$ is of quaternionic, i.e. of non-real type, its complexification decomposes into two irreducible components $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$. For this situation in [Sch99] is proved that $\mathfrak{h}^\mathbb{C}_{\mathbb{C}^{2n}}$ is a complex Berger algebra. This result does not collide with our list because this representation is not of real type and hence not orthogonal. $\mathfrak{h} \subset \mathfrak{so}(4n, \mathbb{R})$ is not a real Berger algebra.
4 Weak-Berger algebras of non-real type

In this section we will classify weak-Berger algebras of non-real type, and we will show that these are Berger algebras. For the classification we will use the classification of first prolongations of irreducible complex Lie algebras. We will show that the complexification of the space $\mathcal{B}_h(g_0)$ is isomorphic to the first prolongation of the complexified Lie algebra.

In this section $g_0$ is a real Lie algebra and $E$ a $g_0$-module of non-real type, i.e. $E^C$ is not irreducible. Thus the situation is a little bit more puzzling then in the real case. Since $g_0 \subset \mathfrak{so}(E, h)$ with $h$ positive definite, $g_0$ is compact. For a compact real Lie algebra with module of non-real type the corresponding complex representation of non-real type is not orthogonal but unitary (See appendix A in particular proposition A.19). But if we switch to the complexified algebra the $(g^C, V)$ irreducible remains, but it can no longer be unitary of course. We have to handle this situation.

With the same notations as in appendix A the complex representations space $W = E^C$ splits into the irreducible modules $W = V \oplus V^*$ under $g_0$. This splitting is of course $g^C_0$ invariant.

Now we define the complex Lie algebra

$$g := \left\{ A_{|V} \middle| A \in g^C_0 \subset \mathfrak{so}(W = V \oplus V^*, H) \right\} \subset \mathfrak{gl}(V). \quad (60)$$

Here $H$ denotes again $h^C$. Since the symmetric bilinear form we start with is positive definite the appendix A gives two important results (see proposition A.19):

1. Since $g_0$ is compact there is a positive definite hermitian form $\theta^h$ on $V$ which is the restriction of the sesqui-linear extension of $h$ on $V$, such that $(g_0)_{|V} \subset \mathfrak{u}(V, \theta^h)$.

2. $g$ is not orthogonal, in particular $H_{|V \times V} = 0$. This is the case since modules of non-real type are symplectic if they are self-dual. Thus they can not be orthogonal.

In $g^C_0$ as well as in $g$ we have a conjugation $\overline{-}$ with respect to $g_0$ and $(g_0)_{|V}$ respectively. Since an $A \in g_0$ acts on $V \oplus V^*$ by $A(v + \overline{w}) = Av + \overline{A}w$ we have for $iA \in g^C_0$ that

$$iA(v + \overline{w}) = i(Av + \overline{A}w) = (iAv + \overline{-iA}w).$$

So we write the action of $A \in g^C_0$ with the help of the conjugation in $g$ as follows

$$A(v + \overline{w}) = Av + \overline{A}w. \quad (61)$$

This gives the following Lie algebra isomorphism

$$\varphi : g^C_0 \simeq g \quad A \mapsto A_{|V}. $$

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This is clearly a Lie algebra homomorphism. It is injective because for \( A|_V = B|_V \) holds that \( A(v + \overline{w}) = Av + \overline{Aw} = Bv + \overline{Bw} = B(v + \overline{w}) \) for all \( v, w \in V \), i.e. \( A = B \). By definition it is surjective and \( \varphi^{-1} \) is given by

\[
\varphi^{-1}(A) : v + \overline{w} \mapsto Av + \overline{Aw} \quad \text{for all } A \in \mathfrak{g}.
\] (62)

These notations are needed to show the relation to the first prolongation.

4.1 The first prolongation of a Lie algebra of non-real type

Now we define the first prolongation of an arbitrary Lie algebra \( g \subset \mathfrak{gl}(V) \).

4.1 Definition. The \( g \)-module

\[
g^{(1)} := \{ Q \in V^* \otimes g \mid Q(u)v = Q(v)u \}.
\] (63)

is called first prolongation of \( g \subset \mathfrak{gl}(V) \). Furthermore we set

\[
\tilde{g} := \text{span}\{ Q(u) \in g \mid Q \in g^{(1)}, u \in V \} \subset g,
\]

and if in \( g \) a conjugation \( \overline{\cdot} \) is given:

\[
g^{[1,1]} := \{ R \in V^* \otimes g^{(1)} \mid R(u,v) = -R(v,u) \},
\]

\[
\tilde{g}^{[1,1]} := \text{span}\{ R(u,v) \mid R \in g^{[1,1]}, u \in V, v \in V \} \subset g.
\]

We will now describe the spaces \( \mathcal{B}_H(g^C_0) \) and \( \mathcal{K}(g^C_0) \) — which are essential for the Berger and the weak-Berger property — with the help of the first prolongation of \( g \).

In the setting of the above notations we can now prove the following.

4.2 Proposition. Let \( E \) be a non-real type module of \( g_0 \), orthogonal with respect to a positive definite scalar product \( h \), and \( E^C = V \oplus V^* \) the corresponding \( g^C_0 \) invariant decomposition, \( g \) defined as in (60). Then there is an isomorphism

\[
\phi : \mathcal{B}_H(g^C_0) \simeq g^{(1)}
\]

\[
Q \mapsto Q|_{V \times V}.
\]

Proof. For the prove we will use the \( g_0 \)-invariant hermitian form \( \theta \) on \( V \) which is given by \( \theta(u,v) = h^C(u,\overline{v}) \), where \( \overline{\cdot} \) is the conjugation in \( E^C = V \oplus V^* \) with respect to \( E \).

The linearity of \( \phi \) mapping is clear. we have to show the following:

1.) The definition of \( \phi \) is correct, i.e. for \( Q \in \mathcal{B}_H(g^C_0) \) it is \( Q|_{V \times V} \in g^{(1)} \). We have for every \( u, v, w \in V \) and \( H = h^C_0 \) that

\[
\theta(Q(u)v, w) = h^C_0(Q(u)v, \overline{w}) = h^C_0(Q(v)\overline{w}, u) = 0
\]

since \( h^C_0_{V \times V} = 0 \) (proposition A.40)

\[
h^C_0 \text{ invariant}
\]

\[
h^C_0(Q(v)u, \overline{w}) = \theta(Q(v)u, w),
\]

\[
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\]
i.e. \( Q(u)v = Q(v)u \) which means that \( Q_{V \times V} \in \mathfrak{gl}^{(1)} \).

2.) The homomorphism \( \phi \) is injective. Let \( Q_1 \) and \( Q_2 \) be in \( \mathcal{B}_H(\mathfrak{g}_0^C) \) with \( (Q_1)_{V \times V} = (Q_2)_{V \times V} \). Then it is

a) \( (Q_1)_{V \times V} = (Q_2)_{V \times V} \), since \( Q_1(\overline{w}) = Q_1(u)v = Q_2(u)v = Q_2(\overline{w}) \),

b) \( (Q_1)_{V \times V} = (Q_2)_{V \times V} \), since

\[
\theta(Q_1(\overline{w})v, w) = h^C(Q_1(\overline{w})v, w) = -h^C(v, Q_1(\overline{w})) = h^C(v, Q_2(\overline{w})) = h^C(Q_2(\overline{w})v, w) = \theta(Q_2(\overline{w})v, w).
\]

c) \( (Q_1)_{V \times V} = (Q_2)_{V \times V} \) because of b) with the same argument as in a).

3.) The homomorphism \( \phi \) is surjective. For \( Q \in \mathfrak{gl}^{(1)} \) we define \( \phi^{-1} \) using \( \varphi \):

\[
(\phi^{-1}Q)(u) := \varphi^{-1}(Q(u)) \quad \text{and} \quad (\phi^{-1}Q)(\overline{w}) := \varphi^{-1}(Q(\overline{u})) \in \mathfrak{gl}(E^C),
\]

i.e. \( (\phi^{-1}Q)(u, v) = Q(u)v \) and \( (\phi^{-1}Q)(\overline{w}) = Q(\overline{u})v \).

It is \( (\phi^{-1}Q)(\overline{w}, \overline{v}) = (\phi^{-1}Q)(u, v) \).

Then obviously \( \phi \circ \phi^{-1} = id \), since \( \phi(\phi^{-1}(Q)) = \phi^{-1}(Q)_{V \times V} = Q \).

Because of the symmetry of \( Q \) we have also that \( (\phi^{-1}Q) \in \mathcal{B}_H(\mathfrak{g}_0^C) \):

- For \( u, v \in V, \overline{w} \in \overline{V} \):

\[
H((\phi^{-1}Q)(u)v, \overline{w}) + H((\phi^{-1}Q)(v)\overline{w}, u) + H((\phi^{-1}Q)(\overline{w})u, v) =
\]

\[
= 0 \quad \text{because} \quad H = 0 \text{ on } V \times V
\]

- For \( u \in V, \overline{v}, \overline{w} \in \overline{V} \):

\[
\overline{H((\phi^{-1}Q)(u)\overline{w}, \overline{v}) + H((\phi^{-1}Q)(\overline{v})\overline{w}, u) + H((\phi^{-1}Q)(\overline{w})u, \overline{v})} =
\]

\[
= 0 \quad \text{because} \quad H = 0 \text{ on } V \times V
\]

Terms with entries only from \( V \) or only from \( \overline{V} \) are zero.

Furthermore we show for the space \( \mathcal{K}(\mathfrak{g}) \) an analogous result.
4.3 Proposition. Let $E$ be an orthogonal non-real type module of $\mathfrak{g}_0$ and $E^C = V \oplus \overline{V}$ the corresponding $\mathfrak{g}_0^C$ invariant decomposition, $\mathfrak{g}$ defined as in (60). Suppose that $\theta := \theta^k$ is non-degenerate. Then there is an isomorphism

$$\psi : \mathcal{K}(\mathfrak{g}_0^C) \simeq \mathfrak{g}^{[1,1]}$$

$$R \mapsto R_{|V \times V \times V}.$$ 

Proof. The proof is completely analogous to the previous one.

1.) The definition is correct. We have for $u, v, w \in V$ and $R \in \mathcal{K}(\mathfrak{g}_0^C)$ that

$$\sum_{\in V} R(u, v) \overline{w} = \sum_{\in V} R(v, u) \overline{w} - \sum_{\in V} R(w, u) \overline{v} = 0.$$

but this means that $R(\overline{u}, .)_{|V \times V} \in \mathfrak{g}^{(1)}$.

Further $R(u, v)w = 0$ implies $R(u, v)w = 0$ because

$$\theta(R(u, v)w, z) = h^C(R(u, v)w, \overline{z}) = -h^C(w, R(u, v)\overline{z}) = 0.$$

This implies $R(\overline{u}, \overline{v})w = R(\overline{v}, \overline{u})w = 0$ too.

For a $R \in \mathcal{K}(\mathfrak{g}_0^C)$ we have due to the skew symmetry

$$\frac{R(\overline{u}, v)}{v} \text{ easy calculation } = R(u, \overline{v}) \text{ skew-symm. } = -R(\overline{u}, u),$$

i.e. the restriction of $R$ on $\overline{V} \times V \times V$ is in $\mathfrak{g}^{[1,1]}$.

2.) The homomorphism $\psi$ is injective.

Let $R_1$ and $R_2$ be in $\mathcal{K}(\mathfrak{g}_0^C)$ with $(R_1)_{|V \times V \times V} = (R_2)_{|V \times V \times V}$. Then again via $\theta$ the remaining non zero terms $R_i(\overline{u}, v)w$ are determined by $R_i(\overline{u}, v)w$ which are equal for $i = 1, 2$ and by the skew symmetry of $R$.

3.) The homomorphism $\psi$ is surjective.

We set

$$(\psi^{-1}R)(\overline{u}, v) := \varphi^{-1}R(\overline{u}, v), \quad (\psi^{-1}R)(u, \overline{v}) := \varphi^{-1}(R(\overline{u}, u)) \quad \text{and}$$

$$(\psi^{-1}R)(u, v) := (\psi^{-1}R)(\overline{u}, \overline{v}) := 0.$$

So we have the skew symmetry, i.e. $\psi^{-1}R \in \wedge^2 E^C \otimes \mathfrak{g}_0^C$, because

$$(\psi^{-1}R)(u, \overline{v}) = \varphi^{-1}(R(\overline{u}, v)) = -\varphi^{-1}(R(\overline{u}, u)) = -(\psi^{-1}R)(\overline{u}, v).$$

The Bianchi identity is also satisfied:

- For $u \in \overline{V}, v, w \in V$:

$$0 = (\psi^{-1}R)(\overline{u}, v)w + (\psi^{-1}R)(v, w)\overline{u} + (\psi^{-1}R)(w, \overline{u})v =$$

$$\varphi^{-1}(R(\overline{u}, v)) w + \varphi^{-1}(R(\overline{u}, u)) v =$$

$$\varphi^{-1}(R(\overline{u}, v)) w - \varphi^{-1}(R(\overline{u}, u)) v =$$

$$R(\overline{u}, v)w - R(\overline{u}, u)v = 0.$$
For \( u, v, w \in V \):
\[
\begin{align*}
(\psi^{-1} R)(\overline{u}, v) w + (\psi^{-1} R)(\overline{v}, w) u + (\psi^{-1} R)(w, \overline{u}) v &= \\
\varphi^{-1} (R(\overline{v}, w)) u + \varphi^{-1} (\overline{R(\overline{w}, u)}) v &= \\
-\varphi^{-1} (\overline{R(\overline{w}, v)}) u + \varphi^{-1} (\overline{R(\overline{w}, u)}) v &= \\
-\overline{R(\overline{w}, v)} u + \overline{R(\overline{w}, u)} v = 0
\end{align*}
\]
Terms with entries only from \( V \) or only from \( \overline{V} \) are zero. \( \square \)

In contrary to the previous proof, in this proof we only supposed the fact that \( \theta^h \) is non-degenerate and not that \( h_{[V, V]}^C = 0 \). If we assume \( h \) to be positive definite, then both facts are satisfied.

### 4.2 Consequences for Berger and weak-Berger algebras

Both propositions give three important corollaries.

#### 4.4 Corollary

Let \( h_0 \subset g_0 \subset so(E^C, H) \) be subalgebras of non-real type, \( h \) and \( g \) defined as above. If
\[
(\varphi^{(1)} g) = (\varphi^{(1)} h),
\]
then \( (h_0^C)_H = (g_0^C)_H \). I.e. if in \( g \) exists a proper subalgebra which has the same first prolongation and a compact real form in \( g_0 \) of non-real type, then \( g_0^C \) and therefore \( g_0 \) can not be weak-Berger algebras.

**Proof.** Because of \( Q \in \mathcal{B}_H(h_0^C) \simeq (h_0^C)_H \) we have \( Q(u) \in (h_0^C)_H \) if and only if \( Q(u) \in (h_0^C)_H \).

#### 4.5 Corollary

Let \( g_0 \subset so(E^C, H) \) be a Lie algebra of non-real type, and \( g \) defined as above. Then

1. \( (g_0^C)_H = g_0^C \) (i.e. \( g_0^C \) is a weak-Berger-algebra) if and only if \( g = \tilde{g} \).
2. \( g_0^C = g_0^C \) (i.e. \( g_0^C \) is a Berger-algebra) if and only if \( g = \tilde{g} \).

**Proof.**

1.) First we show the sufficiency: Let \( A \in g_0^C \) be arbitrary. The assumption \( g = \tilde{g} \) gives w.l.o.g. that \( \varphi(A) = Q(u) \) with \( Q \in \mathcal{B}_H(g_0^C) \) and \( u \in V \). But then we have
\[
(\phi^{-1} Q)(u) \overset{\text{per def.}}{=} \varphi^{-1}(Q(u)) = \varphi^{-1}(\varphi(A)) = A,
\]
with \( (\phi^{-1} Q) \in \mathcal{B}_H(g_0^C) \), i.e. \( A \in (g_0^C)_H \).

Now we show the necessity: If \( A \in g \), then the assumption \( g_0^C = (g_0^C)_H \) gives w.l.o.g. that \( \varphi^{-1}(A) = \hat{Q}(u + \overline{v}) \) with \( \hat{Q} \in \mathcal{B}_H(g_0^C) \), \( u \in V \) and \( \overline{v} \in \overline{V} \). But by the isomorphism of the proposition \ref{prop:isomorphism} there is a \( Q \in \mathcal{B}_H(1) \) such that
\[
\varphi^{-1}(A) = \hat{Q}(u + \overline{v}) = (\phi^{-1} Q)(u + \overline{v}) = \varphi^{-1}(Q(u)) + \varphi^{-1}(\overline{Q(v)}).
\]
But this means that
\[ A = \underbrace{Q(u)}_{\in \mathfrak{g}} + \underbrace{Q(v)}_{\in \mathfrak{g}} \in \mathfrak{g}, \]
i.e. \( \mathfrak{g} \subset \mathfrak{g} \).

2.) Both directions are proved completely analogous to 1.)
Suppose that \( \mathfrak{g} = \tilde{\mathfrak{g}} \). Then for \( A \in \mathfrak{g}^C \) one has that \( \varphi(A) = R(\overline{u}, v) \) and
\[
(\psi^{-1}R)(\overline{u}, v) = \varphi^{-1}(R(\overline{u}, v)) = A.
\]
On the other hand we have for \( A \in \mathfrak{g} \) that \( \varphi^{-1}(A) = \tilde{R}(z + \overline{u}, v + \overline{w}) \). This gives
\[
\varphi^{-1}(A) = \tilde{R}(z, \overline{w}) + \tilde{R}(\overline{u}, v) = (\psi^{-1}R)(z, \overline{w}) + (\psi^{-1}R)(\overline{u}, v)
\]
and therefore \( A \in \tilde{\mathfrak{g}} \).

As a result of the previous and this section we have to investigate complex irreducible representations of complex Lie algebras with non-vanishing first prolongation. Fortunately these are classified by Cartan [Car09], Kobayashi and Nagano [KN65] in a rather short list. In the next section we will present this list and check for the entries with the help of the previous corollaries whether they are Berger or weak-Berger algebras.

### 4.3 Lie algebras with non-trivial first prolongation and the result

There are only a few complex Lie algebras \( \mathfrak{g} \) contained irreducibly in \( \mathfrak{gl}(V) \) which have non vanishing first prolongation. The classification is due to Cartan [Car09] and [KN65]. We will cite them following [MS99] in two tables.

#### Table 1

| \( G \) | \( \mathfrak{g} \) | \( V \) | \( \mathfrak{g}^{(1)} \) |
|---|---|---|---|
| 1. | \( \text{Sl}(n, \mathbb{C}) \) | \( \text{sl}(n, \mathbb{C}) \) | \( \mathbb{C}^n, \ n \geq 2 \) | \( (V \otimes \mathfrak{g}^{2V^*})_0 \) |
| 2. | \( \text{Gl}(n, \mathbb{C}) \) | \( \mathfrak{gl}(n, \mathbb{C}) \) | \( \mathbb{C}^n, \ n \geq 1 \) | \( V \otimes \mathfrak{g}^{2V^*} \) |
| 3. | \( \text{Sp}(n, \mathbb{C}) \) | \( \mathfrak{sp}(n, \mathbb{C}) \) | \( \mathbb{C}^{2n}, \ n \geq 2 \) | \( \mathfrak{g}^{3V^*} \) |
| 4. | \( \mathbb{C}^* \times \text{Sp}(n, \mathbb{C}) \) | \( \mathbb{C} \oplus \mathfrak{sp}(n, \mathbb{C}) \), \( \mathbb{C}^{2n}, \ n \geq 2 \) | \( \mathfrak{g}^{3V^*} \) |
Table 2 Complex Lie-groups and algebras with first prolongation $g^{(1)} = V^*$:

|   | $G$       | $g$                  | $V$                   |
|---|-----------|----------------------|-----------------------|
| 1. | $CO(n, \mathbb{C})$ | $co(n, \mathbb{C})$ | $\mathbb{C}^n$, $n \geq 3$ |
| 2. | $Gl(n, \mathbb{C})$ | $gl(n, \mathbb{C})$ | $\mathbb{C}^2$, $n \geq 2$ |
| 3. | $Gl(n, \mathbb{C})$ | $gl(n, \mathbb{C})$ | $\mathbb{C}$, $n \geq 5$ |
| 4. | $Gl(n, \mathbb{C}) \cdot Gl(m, \mathbb{C})$ | $sl(gl(n, \mathbb{C}) \oplus gl(m, \mathbb{C}))$ | $\mathbb{C}^n \otimes \mathbb{C}^m$, $m, n \geq 2$ |
| 5. | $\mathbb{C}^* \cdot Spin(10, \mathbb{C})$ | $\mathbb{C} \oplus spin(10, \mathbb{C})$ | $\Delta_{10}^+ \simeq \mathbb{C}^{16}$ |
| 6. | $\mathbb{C}^* \cdot E_6$ | $\mathbb{C} \oplus \mathfrak{e}_6$ | $\mathbb{C}^{27}$ |

We have to make two remarks about the second table:

The fourth Lie algebra is defined as

$$sl(gl(n, \mathbb{C}) \oplus gl(m, \mathbb{C})) = \{(X, Y) \in gl(n, \mathbb{C}) \oplus gl(m, \mathbb{C}) | tr X + tr Y = 0\} = (gl(n, \mathbb{C}) \oplus gl(m, \mathbb{C})) \cap sl(n + m, \mathbb{C}).$$

The identification with the Lie algebra of the group is given as follows

$$sl(gl(n, \mathbb{C}) \oplus gl(m, \mathbb{C})) \simeq LA(Gl(n, \mathbb{C}) \cdot GL(m, \mathbb{C})) \subset gl(n \cdot m, \mathbb{C})$$

$$(A, B) \mapsto (x \otimes u \mapsto Ax \otimes u - x \otimes Bu).$$

In entry 5, $\Delta_{10}^+$ denotes the irreducible $Spin(10, \mathbb{C})$ spinor module. The representation in 6. is one of the two 27-dimensional, irreducible $\mathfrak{e}_6$ representations, which are conjugate to each other as representations of the compact real form of $\mathfrak{e}_6$.

The algebras of table 1 The first three entries of table 1 are all complexifications of Riemannian holonomy algebras $su(n), u(n)$ acting on $\mathbb{R}^{2n}$ and $sp(n)$ acting on $\mathbb{R}^{4n}$ and therefore Berger algebras.

The fourth has the compact real form $i\mathbb{R} \oplus sp(n) \simeq so(2) \oplus sp(n)$ acting irreducible on $\mathbb{R}^{4n}$ where $i$ id corresponds to the element $J \in u(2n)$. Since the representation of $sp(n)$ on $\mathbb{R}^{4n}$ is of non-real type we are in the situation of corollary 4.4 because $(CId \oplus sp(n, \mathbb{C}))^{(1)} = sp(n, \mathbb{C})^{(1)}$. Hence $\mathbb{C} \oplus sp(2n, \mathbb{C})$ is not a weak-Berger algebra.

The algebras of table 2 If one looks at the unique (up to inner automorphisms) compact real form and the reellification of the Lie algebras and representations in table 2 one sees that they correspond to the holonomy representation of Riemannian symmetric spaces which are Kählerian. This gives the following proposition.
4.6 Proposition. The compact real forms of the algebras in table 2 and the reellification of the representations are equivalent to the holonomy representations of the following Riemannian, Kählerian symmetric spaces (see \[Hel78\]):

| Type   | non-compact | compact          | dim. |
|--------|-------------|------------------|------|
| 1. BD I | SO(0,2)/SO(2) \times SO(n) | SO(2+n)/SO(2) \times SO(n) | 2n   |
| 2. CI   | Sp(n,\mathbb{R})/U(n)       | Sp(n)/U(n)        | n(n+1) |
| 3. D III| SO*(2n)/U(n)               | SO(2n)/U(n)       | n(n-1) |
| 4. A III| SU(n,m)/U(n) \cdot U(m)    | SU(n+m)/U(n) \cdot U(m) | 2nm  |
| 5. E III| (\mathfrak{e}_6(-14), \mathfrak{so}(2) \oplus \mathfrak{so}(10)) | (\mathfrak{e}_6(-78), \mathfrak{so}(2) \oplus \mathfrak{so}(10)) | 32   |
| 6. E VII| (\mathfrak{e}_7(-25), \mathfrak{so}(2) \oplus \mathfrak{e}_6) | (\mathfrak{e}_7(-133), \mathfrak{so}(2) \oplus \mathfrak{e}_6) | 54   |

*Table 3* Riemannian, Kählerian symmetric spaces corresponding to table 2

So we obtain that all algebras corresponding to table 2 are Berger algebras and therefore also weak-Berger algebras.

4.7 Theorem. Let \(\mathfrak{g}\) be a Lie algebra and \(E\) an irreducible \(\mathfrak{g}\)-module of non-real type. If \(\mathfrak{g} \subset \mathfrak{so}(E,h)\) is a weak-Berger algebra then it is a Berger algebra.

Consequences for Lorentzian holonomy All in all we have shown, that every real Lie algebra \(\mathfrak{g}_0\) of non-real type, i.e. contained in \(u(n)\), that can be weak-Berger is a Berger algebra. Further each of these Lie algebras is the holonomy algebra of a Riemannian manifold, the remaining entries of table 1 of non-symmetric ones, and the entries of table 2 of symmetric ones.

Before we apply this to the irreducible components of the \(\mathfrak{so}(n)\)-projection of the holonomy algebra of an indecomposable Lorentzian manifold with light like invariant subspace, we prove a lemma to get the result in full generality.

4.8 Lemma. Let \(\mathfrak{g} \subset u(n) \subset \mathfrak{so}(2n)\) be a Lie algebra with the decomposition property of theorem \[4.4\] i.e. there exists decompositions of \(\mathbb{R}^{2n}\) into orthogonal subspaces and of \(\mathfrak{g}\) into ideals

\[
\mathbb{R}^{2n} = E_0 \oplus E_1 \oplus \ldots \oplus E_r \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r
\]

where \(\mathfrak{g}\) acts trivial on \(E_0\), \(\mathfrak{g}_i\) acts irreducible on \(E_i\) and \(\mathfrak{g}_i(E_j) = \{0\}\) for \(i \neq j\). Then \(\mathfrak{g} \subset u(n)\) implies \(\text{dim } E_i = 2k_i\) and \(\mathfrak{g}_i \subset u(k_i)\) for \(i = 1, \ldots, r\).

*Proof.* Let \(\mathbb{R}^{2n} = \mathbb{C}^n\) and \(\theta\) be the positive definite hermitian form on \(\mathbb{C}^n\). Let \(E_i\) be an invariant subspace on which \(\mathfrak{g}\) acts irreducible. If \(E_i = V^<_{R}\) for a complex vector space \(V^i\), then we can restrict \(\theta\) to \(V^i\). Because \(\theta\) is positive definite it is non-degenerate on \(V^i\) — since \(\theta(v,v) > 0\) for \(v \neq 0\) — we get that \(\mathfrak{g}_i \subset u(V^i, \theta)\), i.e. \(\mathfrak{g} \subset u(k_i)\).

Hence we have to consider a subspace \(E_i\) which is not the reellification of a complex vector space. Let \(J\) be the complex structure on \(\mathbb{R}^{2n}\). We consider the real vector space \(JE_i\), which is invariant under \(\mathfrak{g}\), since \(J\) commutes with \(\mathfrak{g}\). Then the space \(JE_i \cap E_i\)
is contained in $E_i$ as well as in $JE_i$ and invariant under $g$. Because $g$ acts irreducible
on $E_i$ we get two cases. The first is $E_i \cap JE_i = E_i = JE_i$, but this was excluded since
$E_i$ was not a reellification. The second is $E_i \cap JE_i = \{0\}$. So we have two invariant
irreducible subspaces on which $g$ acts simultaneously, i.e. $A(x,Jy) = (Ax,AJy)$, but
this is not possible because of the Borel-Lichnerowicz decomposition property from
theorem 1.1.

4.9 Theorem. Let $(M,h)$ be an indecomposable $n+2$-dimensional Lorentzian manifold
with light like holonomy-invariant subspace. Set $g := \text{pr}_{\mathfrak{so}(n)\text{hol}}(M,h)$ and suppose
$g \subset \mathfrak{u}(n)$. Then $g$ is the holonomy algebra of a Riemannian manifold.

Proof. $g \subset \mathfrak{u}(n)$ is a weak-Berger algebra. Then all the $g_i$ of the decomposition of
theorem 1.1 are unitary because of the lemma and weak-Berger because of corollary
1.6. Hence they are weak-Berger of non-real type. Then $g_i$ corresponds to a compact
real form of the entries of table 1 or 2. But these are all Riemannian holonomy algebras,
and therefore $g$ is a Riemannian holonomy algebra.

4.10 Remark. Quaternionic symmetric spaces. Again we have to make a remark
about quaternionic symmetric spaces (see remark 3.22). If $G/Sp(1) \cdot H$ with $H \subset
Sp(n)$ is a quaternionic symmetric space then the corresponding complex irreducible
representation of $H$ is of quaternionic, i.e. of non real type, and it is Berger [Sch99].
But the real representation of $H$, i.e. the reellification of the complex one, is not. Thats
why it does not occur in the above list. The place of $Sp(1) \cdot H$ would be in a list of
real semisimple, but non-simple, weak-Berger algebras of real type.

A Representations of real Lie algebras

In this appendix we will collect and illustrate some standard facts about representations
of real Lie algebras.

Because of the theorem 1.1 and proposition 1.6 we are interested in irreducible real
representations of real Lie algebras which are orthogonal.

First we will recall some facts about irreducible complex representations of real Lie
algebras, in particular orthogonal or unitary ones.

Then we will use the results of E. Cartan (Car14, see also Got78, pp.363 and Iwa59),
in order to reduce the study of real representations to that of complex ones.

Throughout the whole section $g$ is a real Lie algebra.

A.1 Preliminaries

First of all we recall the Schur-lemma.

A.1 Proposition (Schur-lemma). Let $\kappa_1, \kappa_2$ be irreducible representations of $\mathfrak{g}$ on $\mathbb{K}$-vector spaces $V_1$ and $V_2$. Let $f \in \text{Hom}_g(V_1, V_2)$ be an invariant homomorphism, i.e.

$$f \circ \kappa_1(A) = \kappa_2(A) \circ f$$

for all $A \in \mathfrak{g}$.

Then holds

1. $f$ is zero or an isomorphism, i.e. $V_1 \not\cong V_2$ implies $\text{Hom}_g(V_1, V_2) = 0$.

2. If $V_1 = V_2 =: V$ and if $f$ has an eigenvalue $\lambda \in \mathbb{K}$, then $f = \lambda \text{id}_V$. I.e. if $\mathbb{K} = \mathbb{C}$ and $V_1 = V_2$ we have always $f = \lambda \text{id}$ with $\lambda \in \mathbb{C}$.

For invariant bilinear forms, i.e. forms $\beta$ which satisfy

$$\beta(\kappa(A)u, v) + \beta(u, \kappa(A)v) = 0$$

for all $A \in \mathfrak{g}$

this gives the following consequence.

A.2 Corollary. Let $\kappa$ be an irreducible representation of $\mathfrak{g}$ on a $\mathbb{K}$-vector space $V$ and $\beta$ be the invariant bilinear form. Then $\beta$ is zero or non-degenerate.

If $\mathbb{K} = \mathbb{C}$, then the space of invariant bilinear forms is zero or one-dimensional. It is one-dimensional if and only if $V \cong_{\kappa} V^*$. Then it is generated by a symmetric or an anti-symmetric bilinear form.

This consequence is obvious by applying the Schur-lemma to the endomorphism of $V$, which is induced by two invariant bilinear forms.

For complex representations and invariant sesqui-linear forms, i.e. forms $\theta$ with

$$\theta(\lambda u, v) = \lambda \theta(u, v) \quad \text{and} \quad \theta(u, \lambda v) = \overline{\lambda} \theta(u, v),$$

one has an analogous result.

A.3 Corollary. Let $\kappa$ be an irreducible representation of $\mathfrak{g}$ on a $\mathbb{C}$-vector space $V$.

Every invariant sesqui-linear form is zero or non-degenerate, and the space of invariant sesqui-linear forms is zero or one-dimensional. It is one dimensional if and only if $V \cong_{\kappa} V^*$. In this case it is generated by a hermitian or an anti-hermitian form, and the spaces of invariant hermitian and invariant anti-hermitian forms are one-dimensional real subspaces, identified by the multiplication with $i$.

In these corollaries we refer to the dual and the conjugate representations, which are defined as follows:

$$(\kappa^*(A)\alpha)v = -\alpha(\kappa(A)v)$$

$$\overline{\pi(A)v} = \overline{\kappa(A)v}.$$

A.4 Definition. Let $\kappa$ be an arbitrary representation of a Lie algebra $\mathfrak{g}$ on a $\mathbb{K}$-vector space $V$. 

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1. Then $\kappa$ is called **self-dual** if there is an invariant isomorphism between $V$ and $V^*$. This is equivalent to the existence of an invariant bilinear form $\beta$.

2. If $K = \mathbb{C}$, then $\kappa$ is called **self-conjugate** if there is an invariant isomorphism from $V$ to $\overline{V}$, i.e. there exists an anti-linear bijective mapping $J : V \to V$ which is invariant, i.e. $J \circ \kappa(A) = \kappa(A) \circ J$ for all $A \in \mathfrak{g}$.

It is evident that the existence of an invariant hermitian form or a self-conjugate representation is only possible for **real Lie algebras**.

### A.2 Irreducible complex representations of real Lie algebras

**A.5 Definition.** Let $\kappa$ be an irreducible complex representation of a real Lie algebra $\mathfrak{g}$ on $V$. $\kappa$ is called

- **of real type** if $\kappa$ is self-conjugate with $J^2 = 1$,
- **of quaternionic type** if $\kappa$ is self-conjugate with $J^2 = -1$ and
- **of complex type** if $\kappa$ is not self-conjugate.

From the Schur-lemma it is clear that every complex irreducible representation is either real, complex or quaternionic: If $\kappa$ is self-conjugate, then $J^2$ is a linear automorphism of $\kappa$ so that $J^2 = \lambda \text{id}$. Furthermore $\lambda$ must be real because of

$$\lambda Jv = J^2 Jv = J J^2 v = J \lambda v = \overline{\lambda} J v.$$ 

Dividing $J$ by $\sqrt{|\lambda|}$ one gets $\lambda = \pm 1$.

Now it holds

**A.6 Proposition.** Let $\kappa$ be a complex irreducible representation of a real Lie algebra $\mathfrak{g}$. Then $\kappa$ is of real type if and only if the realification $\kappa_{\mathbb{R}}$ is reducible.

**Proof.** $(\implies)$ Let $\kappa$ be of real type, i.e. there is an anti-linear, invariant automorphism of the complex representation space $V$ with $J^2 = \text{id}$. Then $J$ is $\mathbb{R}$-linear, and $V_{\mathbb{R}}$ splits into invariant, real vector spaces

$$V_\pm = \{ v \in V \mid Jv = \pm v \}$$

$$V_{\mathbb{R}} = V_+ \oplus V_-.$$ 

So $\kappa_{\mathbb{R}}$ is reducible.

$(\impliedby)$ Let $W$ be a real, $\kappa_{\mathbb{R}}$-invariant subspace of $V_{\mathbb{R}}$. On $V_{\mathbb{R}}$ the multiplication with $i$ gives an $\mathbb{R}$-automorphism, which defines two subspaces of $V_{\mathbb{R}}$: $W \cap iW$ and $W + iW$. Then both are complex vector spaces in an obvious way, such that they are complex subspaces of $V$. Since $W$ is $\kappa_{\mathbb{R}}$ invariant, both are $\kappa$ invariant. Since $\kappa$ is irreducible, it remains the case that $W \cap iW = \{0\}$ and $W \oplus iW = V$. But this means that $V = W^\mathbb{C}$ such that $W$ defines a conjugation $J$ in $V$ with the desired properties. \qed
Orthogonal and unitary representations

A.7 Proposition. Let $\kappa$ be an irreducible representation of $\mathfrak{g}$. If $\kappa$ is of complex type, then it can not be both, unitary and self-dual. If $\kappa$ is not of complex type, then it is unitary if and only if it is self-dual. In particular one has for real and quaternionic representations ($J$ denotes the automorphism):

1. If $\kappa$ is of real type, then it is orthogonal if and only if it is unitary with respect to $\theta$ for which holds $J^*\theta = \overline{\theta}$. It is symplectic if and only if it is unitary with respect to $\theta$ satisfying $J^*\theta = -\overline{\theta}$.

2. If $\kappa$ is of quaternionic type, then it is orthogonal if and only if it is unitary with respect to $\theta$ with $J^*\theta = -\overline{\theta}$. It is symplectic if and only if it is unitary with respect to $\theta$ satisfying $J^*\theta = \theta$.

Proof. Unitary is equivalent to $V^* \simeq \kappa \overline{V}$ and therefore self-dual is the same as $V \simeq \kappa \overline{V}$. This gives the proposition. For the remaining single points we get:

1.) Let $\kappa$ be of real type with respect to a real structure $J$. By this $J$ one gets from an invariant bilinear form $\beta$ an invariant sesqui-linear form $\beta(\cdot, J \cdot)$ which is the complex multiple of an invariant hermitian form $\theta$ and vice versa. Then one gets for $\beta$ symmetric/anti-symmetric:

$$J^*\theta(u,v) = \theta(Ju,Jv) = \lambda \beta(Ju,J^2v) = \lambda \beta(Ju,v)$$

$$= \pm \lambda \beta(v,Ju) = \pm \theta(v,u) = \pm \overline{\theta(u,v)}.$$

2.) analogous with $J^2 = -id$. □

A.8 Corollary. If $\kappa$ is positive definite unitary, then it is

1. of real type if and only if it is orthogonal,

2. of complex type if and only if it is not self-dual,

3. of quaternionic type if and only if it is symplectic.

Proof. If $\theta$ is positive definite it can not be $J^*\theta = -\overline{\theta}$. □

A.3 Irreducible real representations

For a real irreducible representation $\rho$ of a real Lie algebra $\mathfrak{g}$ on a real vector space $E$ two cases are possible: $\rho^C$ is irreducible or reducible. We will describe these cases due to results of E. Cartan ([Car14], see also [Got78], pp.363 and [Iwa59]), in order to reduce the study of real representations to that of complex ones.
A.3.1 Representations of real type

A.9 Proposition. Let $\mathfrak{g}$ be a real Lie algebra and $\rho$ a representation of $\mathfrak{g}$ on a real vector space $E$ such that $\rho^C$ is irreducible on $E^C$. Then the complex representation $\rho^C$ is of real type.

If otherwise $\kappa$ is a complex representation of $\mathfrak{g}$ of real type on $V$, then $\kappa$ is the complexification of a real irreducible representation of $\mathfrak{g}$.

Proof. 1.) We show the existence of a $\rho^C$-invariant anti-linear isomorphism $J$ with $J^2 = id$. If we denote by $J$ the conjugation in $E^C$ with respect to $E$, then it is $J^2 = 1$ and we have

$$ J \left( \rho^C(A)(u + iv) \right) = \rho^C(A)(u) - i\rho^C(A)(v) = \rho^C(A)(J(u + iv)) $$

i.e. $J$ is $\rho^C$-invariant.

2.) In the proof of proposition A.6 we had already shown that for complex representations of real type holds that $V = W^C$.

So the following definition makes sense.

A.10 Definition. Irreducible real representations with irreducible complexification and irreducible complex representations with reducible realification (i.e. of real type) are called representations of real type.

We have the following correspondence:

$$ \{\text{real representation of real type}\} / \sim \leftrightarrow \{\text{complex representations of real type}\} / \sim $$

$$ \rho \mapsto \rho^C $$

$$(\kappa_\mathbb{R})_{\text{maximal invariant subspace}} \leftrightarrow \kappa.$$ 

Here $\sim$ denotes the equivalence of representations.

A.3.2 Representations of non-real type

The situation in this case is described by the following

A.11 Proposition. Let $\mathfrak{g}$ be a real Lie algebra and $\rho$ be an irreducible representation of $\mathfrak{g}$ on a real vector space $E$ such that $\rho^C$ is reducible on $E^C$.

1. If $V \subset E^C$ is any invariant subspace of $\rho^C$. Then holds

$$ E^C = V \oplus \overline{V}, $$

where $\overline{\cdot}$ is the conjugation in $E^C$ with respect to $E$. $V$ and $\overline{V}$ are irreducible and unique as maximal invariant proper subspaces. The representations on $V$ and $\overline{V}$ are conjugate to each other.
2. The irreducible representations of \( g \) on \( V \) and on \( \overline{V} \) are of complex or of quaternionic type, its reellifications are equivalent to \( \rho \).

If otherwise \( \kappa \) is a complex irreducible representation of complex or quaternionic type, then \( \kappa \) is the restriction on the maximal invariant proper subspace of the complexification of \( \kappa_\mathbb{R} \).

If \( \kappa \) is of complex type, then exists and \( \kappa_\mathbb{R} \)-invariant complex structure \( J \) on \( V_\mathbb{R} \). \( \kappa \) is of quaternionic type if and only if there exists and \( \kappa_\mathbb{R} \)-invariant quaternionic structure \( (I,J,K) \) on \( V_\mathbb{R} \).

Proof. 1.) Let \( V \subset E^\mathbb{C} \) be any invariant, proper subspace of \( \rho^\mathbb{C} \). Lets denote by \( ^{-} \) the conjugation in \( E^\mathbb{C} \) with respect to \( E \).

We consider \( W := V + \overline{V} \). Now it is \( \overline{W} = W \) which is equivalent to \( W = F^\mathbb{C} \), where \( F = W \cap E \) is a real subspace of \( E \). Since \( W \) is invariant under \( \rho^\mathbb{C} \), \( F \) is invariant under \( \rho \). Now \( \rho \) is irreducible and therefore \( F = E \), i.e. \( V + \overline{V} = E^\mathbb{C} \). Analogously one shows that \( V \cap \overline{V} = \{0\} \), so that one gets

\[
V \oplus \overline{V} = E^\mathbb{C}.
\]

It remains to show that \( V \) is irreducible: This is clear since every invariant subspace \( U \subset V \) is invariant in \( E^\mathbb{C} \), but then holds that \( U \oplus \overline{U} = E^\mathbb{C} \) which implies \( U = V \). \( \overline{V} \) is irreducible too.

Hence we have two irreducible representations of \( g \), one on \( V \) and one on \( \overline{V} \), which are conjugate to each other:

\[
\rho^\mathbb{C}(A)\overline{v} = \overline{\rho^\mathbb{C}v}.
\]

So we will denote it by \( \kappa \) and \( \overline{\kappa} \).

2.) In order to show that \( \kappa \) and \( \overline{\kappa} \) are of complex or quaternionic type, we verify that \( \kappa_\mathbb{R} \) and \( \overline{\kappa_\mathbb{R}} \) are irreducible.

For this we show that \( \kappa_\mathbb{R} \) and \( \overline{\kappa_\mathbb{R}} \) are isomorphic to \( \rho \). The isomorphism between \( V \) and \( E \) is given by

\[
\psi : V_\mathbb{R} \longrightarrow E \\
v \mapsto \frac{1}{2}(v + \overline{v}).
\]

This is obviously an isomorphism of real vector spaces. (Of course this is also an isomorphism between \( \overline{V}_\mathbb{R} \) and \( E \).) It is also invariant since

\[
\psi \circ \kappa_\mathbb{R}(A)(x + iy) = \psi(\rho(A)x + i\rho(A)y) = \rho(A)x = \rho(A)(\psi(x + iy)
\]

for all \( x + iy \in V_\mathbb{R} \).

The existence of the complex and the quaternionic structure on \( V_\mathbb{R} \) is clear. \( \square \)

Again on defines:
A.12 Definition. Irreducible real representations with reducible complexification and irreducible complex representations with irreducible realification are called representations of non-real type (of complex or quaternionic type respectively).

Again we have the correspondence

\[
\begin{align*}
\{ \text{real representations of non-real type} \} & \sim \{ \text{complex representations of non-real type} \} \\
\rho & \mapsto \rho^c|_{\text{maximal invariant subspace}} \\
\kappa & \mapsto \kappa.
\end{align*}
\]

Here $\sim$ denotes the equivalence of representation and $\approx$ the equivalence $\kappa_1 \approx \kappa_2 \iff \kappa_1 \sim \kappa_2$ or $\kappa_1 \sim \overline{\kappa_2}$.

On the real space $E \simeq V_\mathbb{R}$ we have the complex structure $J$, i.e. an $\mathbb{R}$-automorphism with $J^2 = -1$ given by the multiplication with $i$: $Jv = iv$. $J$ commutes with $\rho$ since

\[
\rho(A)(Jv) = \kappa_\mathbb{R}(A)(Jv) = \kappa(A)iv = i\kappa(A)v = J(\kappa(A)v).
\]

One describes the complex vector space $V$ as a subspace in $E^\mathbb{C}$ as follows. One extends the complex structure to an automorphism of $E^\mathbb{C}$ also denoted by $J$ and with the property $J^2 = -1$. Then one defines $V_\pm := \{ v \in E^\mathbb{C} | Jv = \pm iv \} \subset E^\mathbb{C}$ and gets $E^\mathbb{C} = V_+ \oplus V_-$. Furthermore it is

\[
V_\pm = \{ x \mp iJx | x \in E \} \text{ and therefore } V_\pm = V_{\mp}.
\]

Then one has the following isomorphisms, invariant under the corresponding representations:

\[
\begin{align*}
E & \simeq_\mathbb{R} V \\
\frac{1}{2}(v + \overline{v}) & \mapsto v \mapsto \frac{1}{2}(v - iJv) \mapsto \frac{1}{2}(v + iJv).
\end{align*}
\]

A.4 Orthogonal real representations

Let now $\rho$ be a real representation of $\mathfrak{g}$ on $E$ which should be orthogonal (or symplectic) with respect to a (anti)-symmetric bilinear form $h$.

On $E^\mathbb{C}$ $h$ defines a (anti-)symmetric bilinear form by bilinear extension, denoted by $h^\mathbb{C}$ and a (anti-)hermitian form by conjugate linear extension in the second component, denoted by $h'$. Both are invariant under $\rho^c(\mathfrak{g})$. The hermitian form has the same signature as the symmetric form $h$. The existence of an invariant anti-hermitian form is equivalent to the existence of an invariant hermitian form.

For the conjugation in $E^\mathbb{C}$ we have the following relations

\[
h'(u, v) = h^\mathbb{C}(u, \overline{v}) = \overline{h^\mathbb{C}(\overline{u}, v)} = \overline{h'(v, u)}.
\]
A.4.1 Orthogonal or symplectic representations of real type

From these introductory remarks we obtain the following proposition for real type representations which can be found in [Ber55] (for the orthogonal case).

A.13 Proposition. [Ber55] Let $\rho$ be a real representation of real type of a real Lie algebra $g$ on a real vector space $E$, orthogonal or symplectic with respect to $h$. Let $\beta^h$ denote the complex linear and $\theta^h$ the hermitian extension of $h$ on $V = E^\mathbb{C}$. Then both are non-degenerate and $\rho^C$ is orthogonal/symplectic with respect to $\beta^h$ and unitary with respect to $\theta^h$. $\theta^h$ has the same index as $h$ in case $h$ is orthogonal.

This gives a

A.14 Corollary. If $\rho$ is of real type, then the space of invariant bilinear form is one-dimensional and generated by a symmetric or an anti-symmetric form.

The proof is clear because the irreducibility of $\rho^C$ gives that $h^C_1 = h^C_2$, which implies $h_1 = h_2$. □

We will now prove the other direction of proposition A.13.

A.15 Proposition. Let $g$ be a real Lie algebra and $\kappa$ an irreducible, complex representation of real type on $V$, which decomposes $\kappa^R$-invariant into $V = E \oplus iE$, and set $\rho = (\kappa^R)|_E$ the corresponding irreducible real representation. If $\kappa$ is unitary (and therefore self-dual), then $\rho$ is self-dual, i.e. orthogonal or symplectic and we have two cases:

1. If $\kappa$ is orthogonal, then $\rho$ is orthogonal.

2. If $\kappa$ is symplectic, then $\rho$ is symplectic

Proof. Let $\kappa$ be unitary with respect to $\theta$, which defines two bilinear mappings on $E$

$$
\begin{align*}
  h_1(x, y) &= \text{Re} (\theta(x, y)) \quad \text{symmetric} \\
  h_2(x, y) &= \text{Im} (\theta(x, y)) \quad \text{anti-symmetric}.
\end{align*}
$$

Both are $\rho$ invariant. If both are degenerate, then both are zero by the Schur-lemma and so $\theta$ must be zero, which is a contradiction.

1.) If in addition $\kappa$ is orthogonal, then for $\theta$ holds by proposition A.7 that $J^* \theta = \overline{\theta}$, where $J$ is the conjugation of $E$ in $E^\mathbb{C}$. But in this case $h_2$ is zero, because $E = \{v \in V | Jv = v\}$:

$$
  h_2(x, y) = \text{Im} \theta(x, y) = \text{Im} \theta(Jx, Jy) = \text{Im} \overline{\theta(x, y)} = -\text{Im} \theta(x, y) = -h_2(x, y).
$$

Hence $h_1$ must be non degenerate and therefore $\rho$ orthogonal.

2.) If $\kappa$ is symplectic one shows analogously with proposition A.7 that $h_1 = 0$ and therefore $\rho$ symplectic. □
Both results give the following equivalence:

\[
\{ \rho \text{ real, real type, self-dual} \}/\sim \leftrightarrow \{ \kappa \text{ complex, real type, self-dual } \}/\sim
\]

\[
\rho \text{ real, real type, orthogonal/symplectic } \}/\sim \leftrightarrow \{ \kappa \text{ complex, real type, orthogonal/symplectic } \}/\sim
\]

(69)  

(70)

A.4.2 Orthogonal representations of non-real type

For non-real type representations we have the \( \rho^C \)-invariant decomposition \( E^C = V \oplus \overline{V} \).

In a basis, adapted to this decomposition \( h^C \) and \( h' \) are given as follows

\[
h^C = \begin{pmatrix} A & B \\ B^t & \overline{A} \end{pmatrix}
\]

and

\[
h' = \begin{pmatrix} B & A \\ A & B^t \end{pmatrix}
\]

where \( A = A^t \) and \( B^t = \overline{B} \) are quadratic matrices with the dimension of \( V \).

Now one defines a bilinear and a sesqui-linear form on \( V \) resp. on \( \overline{V} \):

\[
\beta^h(u,v) := h^C(u,v) = h'(u,\overline{v}) \quad \text{symmetric/anti-symmetric}
\]

\[
\theta^h(u,v) := h^C(u,v) = h'(u,v) \quad \text{hermitian/anti-hermitian}
\]

for \( u,v \in V \) resp. \( \overline{V} \). Both are invariant under \( \kappa = h^C|_V(\mathfrak{g}) \).

From the Schur-lemma it is clear that at least one of them is non-degenerate, since \( h^C \)

is non-degenerate.

Using the isomorphisms of \([88]\) we can give \( \theta^h \) and \( \beta^h \) explicitly:

\[
\beta^h(x - iJx,y - iJy) = \frac{1}{4} (h(x,y) - h(Jx,Jy) - i(h(Jx,y) + h(x,Jy)))
\]

(71)

\[
\theta^h(x - iJx,y - iJy) = \frac{1}{4} (h(x,y) + h(Jx,Jy) + i(h(x,Jy) - h(Jx,y)))
\]

(72)

Again we have the proposition of Berger (for the orthogonal case).

A.16 Proposition. \([Ber55]\) Let \( \rho \) be a real orthogonal/symplectic representation of non-real type, i.e. \( (E,\rho) = (V_\mathbb{R},\kappa_\mathbb{R}) \). Then \( \kappa \) is invariant under \( \beta^h \) and \( \theta^h \) and at least one of them is non-degenerate, i.e. \( \kappa \) is orthogonal/symplectic or unitary/anti-unitary with respect to \( \beta^h \) or \( \theta^h \).

Furthermore holds: If \( \mathfrak{g} \) contains a real sub-algebra \( \mathfrak{h} \neq 0 \) such that \( \mathfrak{h} = p_\mathbb{R} \) where \( p \) is a complex Lie algebra, then \( \theta^h = 0 \) i.e. \( \beta^h \) non-degenerate.

Proof. We only have to prove the second assertion.

By assumption we have a complex Lie structure on \( \mathfrak{h} \), i.e. a automorphism \( J \) with \( J^2 = -1 \) and \( J \circ ad_X = ad_X \circ J \). As above for vector spaces we have here a Lie algebra decomposition

\[
\mathfrak{g}^C \supset \mathfrak{h}^C = p_+ \oplus p_- \quad \text{with} \quad p_{\pm} = \{ v \in \mathfrak{h}^C | Jv = \pm iv \}.
\]
Then $p \cong \mathfrak{p}_+$. Let now $\rho^C$ be extended to $\mathfrak{g}^C$. Then because of its linearity $h^C$ is invariant under $\rho^C(\mathfrak{g}^C)$. But if we suppose that $\theta^h$ is invariant under $\mathfrak{g}$ we have for a $H \in \mathfrak{h}$ and $\kappa = \rho^C_{\mid V}$ as above

$$
0 = \theta^h(\kappa(JH)v, w) + \theta^h(v, \kappa(JH)w)
$$

$$
p.d. = h^C(\kappa(JH)v, \overline{w}) + h^C(v, \overline{\kappa(JH)w})
$$

$$
= h^C(\rho^C(JH)v, \overline{w}) + h^C(v, \overline{\rho^C(JH)w})
$$

$$
H \in \mathfrak{p}_+ = \begin{cases} 
i \left( h^C(\rho^C(H)v, \overline{w}) - h^C(v, \overline{\rho^C(H)w}) \right) \\ \theta^h \text{ invariant} \equiv 2i\theta^h(\kappa(H)v, w) \end{cases}
$$

for all $H \in \mathfrak{h}$, $v, w \in V$. This means $\mathfrak{h} \subset \ker \kappa = 0$. \hfill \Box

We also can show the other direction.

**A.17 Proposition.** Let $\mathfrak{g}$ be a real Lie algebra, $\kappa$ be a complex representation of non-real type (of complex or quaternionic type), i.e. $\rho = \kappa_{\mathbb{R}}$ is irreducible. Then holds:

1. If $\kappa$ is unitary with respect to $\theta$ or orthogonal with respect to $\beta$, then $\rho$ is orthogonal with respect to $h$ and $\theta^h = \theta$ or $\beta^h = \beta$.

2. If $\kappa$ is anti-unitary with respect to $\theta$ or symplectic with respect to $\beta$, then $\rho$ is symplectic with respect to $h$ and $\theta^h = \theta$ or $\beta^h = \beta$.

**Proof.** We define a bilinear form on $E = V_{\mathbb{R}}$ by

$$
h(x, y) := \text{Re} \ \theta(x - iJx, y - iJy) \quad \text{or} \quad h(x, y) := \text{Re} \ \beta(x - iJx, y - iJy).
$$

This form is invariant and — since $\text{Re} \ iz = -\text{Im} \ z$ — also non-degenerate. (The difference to real type is that here the arguments in $\theta/\beta$ run over the whole complex vector space $V$.) $h$ is symmetric if $\kappa$ is unitary or orthogonal and anti-symmetric if $\beta$ is anti-symmetric or anti-unitary. The fact that the extensions are equal to $\theta$ resp. $\beta$ follows from the formulas (72) and (71). \hfill \Box

Again we have the following correspondence:

$$
\{ \rho \text{ real, non-real type, orthogonal} \} \sim \leftrightarrow \{ \kappa \text{ complex, non-real type, unitary or orthogonal} \}
$$

$$
(73)
$$

$$
\{ \rho \text{ real, non-real type, symplectic} \} \sim \leftrightarrow \{ \kappa \text{ complex, non-real type, symplectic or anti-unitary} \}
$$

$$
(74)
$$

The fact that a complex representation is unitary if and only if it is anti-unitary (the anti-hermitian form is $i\theta$) implies that a real, orthogonal representation of non-real type
with non-degenerate $\theta^h$ on the corresponding complex representation is also symplectic. This corresponds to the equality of real matrix algebras:

$$u(n) = \mathfrak{so}(2n) \cap \mathfrak{sp}(2n)$$

$$= \{ X \in \mathfrak{gl}(2n) \mid X^t = -X \} \cap \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in \mathfrak{gl}(n), B^t = B, C^t = C \right\}$$

$$= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A, B \in \mathfrak{gl}(n), A^t = -A, B^t = B \right\}.$$

I.e. if a complex representation $\kappa$ of non-real type is unitary, then $\kappa_\mathbb{R}$ is orthogonal and symplectic.

Furthermore one proves the following

**A.18 Lemma.** Let $h$ be symmetric, $\beta^h$, $\theta^h$ as above and $J$ the complex structure on $E$. Then holds

1. $\beta^h = 0$ if and only if $h(x, y) = h(Jx, Jy)$ for all $x, y \in E$.
2. $\theta^h = 0$ if and only if $h(x, y) = -h(Jx, Jy)$ for all $x, y \in E$.

**Proof.** If we write every element of $V = V_+$ in the form (67) we get the proposition due to formulas (71) and (72).

We will now prove the main result for the case that $h$ is positive definite.

**A.19 Proposition.** Let $\rho$ be irreducible of non-real type and orthogonal with respect to $h$ where $h$ is positive definite.

Then the corresponding complex representation $\kappa$ of non-real type is unitary, with respect to a positive definite hermitian form, which is the standard hermitian form for representations of compact Lie groups/Lie algebras.

$\kappa$ is not orthogonal, i.e. the linear extension $\beta^h$ of $h$ vanishes on $V \times V$.

**Proof.** We can prove this in two ways.

If $\theta^h$ is degenerate, then it is zero and we have by lemma A.18 that $h(x, x) = -h(Jx, Jx)$. But this is not possible if $h$ is positive definite. So $\theta^h$ is non degenerate and by formula (72) positive definite, since $h$ is positive definite. But the existence of a positive definite hermitian form entails by corollary A.8 for non-real type representations, i.e. of complex or quaternionic type, that the representation can not be orthogonal. So $\beta^h = 0$.

An easier way to argue is that representations of compact Lie algebras are unitary with respect to a standard positive definite hermitian form. This form is unique and thats why equal to $\theta^h$ and by corollary A.8 the representation can not be orthogonal.
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