The singular set of triholomorphic maps into quartic K3 surface

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Abstract. We prove that any weakly triholomorphic map from a compact hyperkähler surface to an algebraic K3 surface defined by a homogeneous polynomial of degree 4 in \( \mathbb{C}P^3 \) has only isolated singularities.

Keywords and Phrases. triholomorphic map, hyperkähler manifold, K3 surface, regularity.

Mathematics Subject Classification (2010). Primary 53C42, 58E20.

1 Introduction

Triholomorphic maps between hyperkähler manifolds arose from the study of higher dimensional gauge theory ([5],[9]). Weakly triholomorphic maps and their regularity were first studied in [6] and [12].

Recall that a hyperkähler manifold is a Riemannian manifold \((M, g)\) with three parallel complex structures \(\{J^1, J^2, J^3\}\) compatible with the metric \(g\) such that \((J^1)^2 = (J^2)^2 = (J^3)^2 = J^1 J^2 J^3 = -\text{id}\). A hyperkähler manifold is of dimension 4m and has a natural \(S^2\)-family of complex structures, which is called hyperkähler \(S^2\). The simplest hyperkähler manifold is the Euclidean space \(\mathbb{R}^{4m}\). It is well-known that the only compact hyperkähler manifolds of dimension 4 are K3 surfaces and complex tori.

A weakly triholomorphic map between two hyperkähler manifolds \((M, I^1, I^2, I^3)\) and \((N, J^1, J^2, J^3)\) is a \(W^{1,2}\)-map \(u : \Omega \subset M \to N\) whose restriction to any ball \(B \subset \Omega\) is a limit of smooth maps from \(B\) to \(N\) in the \(W^{1,2}\)-topology and which satisfies

\[
du = a_{ij} J^j \circ du \circ I^i,
\]

where \(A = (a_{ij})_{1 \leq i,j \leq 3} \in SO(3)\) is a constant matrix which is a stationary harmonic map (c.f. [6]).

An interesting question is to analyze the structure of the singularity of a weakly triholomorphic map. This question has been explored in [6].

Note \(u\) is smooth outside a closed subset \(\text{Sing}_u\) defined by

\[
\text{Sing}_u = \left\{ x \in \Omega \mid \lim_{r \to 0} r^{2-4m} \int_{B(x,r)} |\nabla u|^2 \geq \epsilon \right\},
\]

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where \( \epsilon \) is the constant given by the \( \epsilon \)-regularity of Bethuel ([4]).

By Bethuel’s theorem, the singular set \( \text{Sing}_u \) has vanishing \((4m-2)\)-dimensional Hausdorff measure. In fact, it is expected that \( \text{Sing}_u \) has Hausdorff codimension at least three or possibly four ([2],[15]) in some cases.

Let \( \phi : S^2 \to N \) be a smooth map. A map \( u : \mathbb{R}^4 \to N \) is defined by
\[
u(x, x^4) = \phi\left(\frac{x}{|x|}\right) \quad \text{for any} \quad x \in \mathbb{R}^3 \setminus \{0\}.
\]
Obviously \( u \) is a map of homogeneous degree zero with 1-dimensional singular set. From (1.1) we know that \( u : \mathbb{R}^4 \to N \) is a triholomorphic map with the \( x^4 \)-axis as its singular set if and only if the corresponding map \( \phi \) satisfies
\[
d\phi \circ J_{S^2} = -a_{ij} x^j J^i \circ d\phi, \quad (1.2)
\]
where \( A = (a_{ij})_{1 \leq i,j \leq 3} \in SO(3) \) is a constant matrix, and \( J_{S^2} \) is the standard complex structure on \( S^2 \). Such a smooth map \( \phi \) is called in [6] a holomorphic \( S^2 \) with respect to a complex structure in the hyperkähler \( S^2 \) of \( N \). Chen and Li ([6]) prove that if \( N \) does not admit a holomorphic \( S^2 \) with respect to a complex structure in the hyperkähler \( S^2 \) of \( N \), then the singular set of a weakly triholomorphic map from \( M \) to \( N \) is of codimension 4.

The only known nontrivial example of \( \phi \) satisfying (1.2) is constructed in the case of a non-compact target [7]. In the monograph [1], Atiyah and Hitchin considered the space \( M^0_2 \) of centered 2-monopoles on \( \mathbb{R}^3 \) with finite action. It is a complete hyperkähler manifold of dimension 4. \( SO(3) \) acts on \( M^0_2 \) isometrically and this action lifts to a double covering \( \tilde{M}^0_2 \), which is also a complete hyperkähler manifold of dimension 4. In [7], Chen and Li showed that there does exist a nontrivial map \( \phi \) from \( S^2 \) to \( \tilde{M}^0_2 \) such that the extended map \( u \) from \( \phi \) is a triholomorphic map from \( \mathbb{R}^4 \) to \( \tilde{M}^0_2 \) with the entire \( x^4 \)-axis as singular set.

In this paper, we will focus on the case of compact target. Assume \( N \) is an algebraic K3 surface defined by a homogeneous polynomial of degree 4 in \( \mathbb{C}P^3 \). It is a nonsingular quartic in \( \mathbb{C}P^3 \). For convenience, we call it quartic K3 surface denoted by \( X \). It is a compact hyperkähler manifold of dimension 4, i.e. a hyperkähler surface. A particularly interesting special case is the Fermat quartic \( F \subset \mathbb{C}P^3 \) defined by the equation
\[
x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0,
\]
where \( [(x_0, x_1, x_2, x_3)] \in \mathbb{C}P^3 \).

Looking at the map
\[
h : \quad \mathbb{C}^2 (= \mathbb{R}^4) \to F
\quad (z_1, z_2) \mapsto [z_1, e^{\frac{\pi}{4} i} z_1, z_2, e^{\frac{\pi}{4} i} z_2], \quad (1.3)
\]
which is holomorphic, naturally triholomorphic and singular at the origin, one has to expect at most any weakly triholomorphic map into K3 surface has isolated singularities.

Through proving that the quartic K3 surface does not admit a holomorphic \( S^2 \) with respect to a complex structure in the hyperkähler \( S^2 \), we obtain our main result as follows
Theorem 1.1 Any weakly triholomorphic map from a compact hyperkähler surface to a quartic K3 surface defined by a homogeneous polynomial of degree 4 in \( \mathbb{C}P^3 \) has only isolated singularities.

Our bound of the size of the singular set, in view of the example (1.3), is clearly optimal. It is reasonable to expect this result to be extendable to any compact target. Generally, our result supports the expectation that \( \text{Sing}_u \) has Hausdorff codimension four in the case of a compact target.

2 Quartic K3 surface

Let \( X \) be a nonsingular quartic surface given by a homogeneous polynomial of degree 4 in \( \mathbb{C}P^3 \). Let \( x_i, \ 0 \leq i \leq 3 \) be homogeneous coordinates of the complex projective space \( \mathbb{C}P^3 \). Let \( f(x_0, x_1, x_2, x_3) \) be a homogeneous polynomial of degree 4 in \( \mathbb{C}P^3 \), i.e.

\[
f(tx_0, tx_1, tx_2, tx_3) = t^4 f(x_0, x_1, x_2, x_3),
\]

for all \( t \in \mathbb{C} \).

Then

\[
X = \left\{ [(x_0, x_1, x_2, x_3)] \in \mathbb{C}P^3 \mid f(x_0, x_1, x_2, x_3) = 0 \right\}.
\]

It is well-known that \( X \) is K3 and every K3 surface is hyperkählerian (recall that a manifold is hyperkählerian if it admits a hyperkähler metric) (c.f. \[3,10,14\]).

Set \( U_i = \left\{ [(x_0, x_1, x_2, x_3)] \in \mathbb{C}P^3 \mid x_i \neq 0 \right\} \) and \( \bar{U}_i = X \cap U_i \ (i = 0, 1, 2, 3) \). Then \( \bigcup_{i=0}^3 \bar{U}_i \) is an open covering of \( X \). Let \( z_i = \frac{x_i}{x_0}, \ i = 1, 2, 3 \). Then \( z = (z_1, z_2, z_3) \) form natural holomorphic coordinates on \( U_0(\cong \mathbb{C}^3) \). On \( \bar{U}_0 \), denote

\[
f(1, z_1, z_2, z_3) = \frac{1}{x_0^4} f(x_0, x_1, x_2, x_3), \ f_{z_i} = \frac{\partial f(1, z_1, z_2, z_3)}{\partial z_i}. \tag{2.1}
\]

Then,

\[
f(1, z_1, z_2, z_3) = 0 \text{ on } \bar{U}_0. \tag{2.2}
\]

Applying the exterior differential \( d \) to \( \text{(2.2)} \), we obtain

\[
f_{z_1} dz_1 + f_{z_2} dz_2 + f_{z_3} dz_3 = 0 \text{ on } \bar{U}_0, \tag{2.3}
\]

which implies

\[
f_{z_1} dz_1 \wedge dz_2 = f_{z_2} dz_2 \wedge dz_3 \text{ and } f_{z_1} dz_1 \wedge dz_3 = -f_{z_2} dz_2 \wedge dz_3 \text{ on } \bar{U}_0. \tag{2.4}
\]

Define a 2-form \( \Omega_0 \) on \( \bar{U}_0 \) as follows:

\[
\Omega_0 \mid V_i = \frac{dz_2 \wedge dz_3}{f_{z_1}}, \ \Omega_0 \mid V_2 = \frac{dz_3 \wedge dz_1}{f_{z_2}}, \ \Omega_0 \mid V_3 = \frac{dz_1 \wedge dz_2}{f_{z_3}}, \tag{2.5}
\]

where \( V_i = \bar{U}_0 \cap \left\{ z \in \bar{U}_0 \mid f_{z_i} \neq 0 \right\}, \ (i = 1, 2, 3) \). Here \( \Omega_0 \mid V_i \) means the restriction of the 2-form \( \Omega_0 \) to the open subset \( V_i \).
Since $X$ is non-singular, we have $\tilde{U}_0 = V_1 \cup V_2 \cup V_3$. Then $\Omega_0$ is well-defined on $\tilde{U}_0$. From (2.4) we know
\begin{equation}
\frac{\partial}{\partial x_i}(\Omega_0|_{V_1}) = \frac{\partial}{\partial x_i}(\Omega_0|_{V_2}) = \frac{\partial}{\partial x_i}(\Omega_0|_{V_3}) = 0.
\end{equation}
Therefore, $\Omega_0$ is a holomorphic symplectic form on $\tilde{U}_0$.
On $\tilde{U}_1$, let $y_i = \frac{x_i}{x_1}$ for $i = 0, 2, 3$. Then
\begin{equation}
f(y_0, y_2, y_3) = \frac{1}{x_1^2} f(x_0, x_1, x_2, x_3).
\end{equation}
Define a 2-form $\Omega_1$ on $\tilde{U}_1$ as follows:
\begin{equation}
\Omega_1|_{A_0} = \frac{dy_3 \wedge dy_2}{y_0}, \quad \Omega_1|_{A_2} = \frac{dy_0 \wedge dy_3}{y_2}, \quad \Omega_1|_{A_3} = \frac{dy_2 \wedge dy_0}{y_3},
\end{equation}
where $A_i = \tilde{U}_1 \cap \{z \in \tilde{U}_1 | f_{y_i} \neq 0\}$, $(i = 0, 2, 3)$.
On $\tilde{U}_2$, let $w_i = \frac{x_i}{x_2}$ for $i = 0, 1, 3$. Then
\begin{equation}
f(w_0, w_1, w_3) = \frac{1}{x_2^2} f(x_0, x_1, x_2, x_3).
\end{equation}
Define a 2-form $\Omega_2$ on $\tilde{U}_2$ as follows:
\begin{equation}
\Omega_2|_{B_0} = \frac{dw_1 \wedge dw_3}{w_0}, \quad \Omega_2|_{B_1} = \frac{dw_3 \wedge dw_0}{w_1}, \quad \Omega_2|_{B_3} = \frac{dw_0 \wedge dw_1}{w_3},
\end{equation}
where $B_i = \tilde{U}_2 \cap \{z \in \tilde{U}_2 | f_{w_i} \neq 0\}$, $(i = 0, 1, 3)$.
On $\tilde{U}_3$, let $v_i = \frac{x_i}{x_3}$ for $i = 0, 1, 2$. Then
\begin{equation}
f(v_0, v_1, v_2, 1) = \frac{1}{x_3^2} f(x_0, x_1, x_2, x_3).
\end{equation}
Define a 2-form $\Omega_3$ on $\tilde{U}_3$ as follows:
\begin{equation}
\Omega_3|_{C_0} = \frac{dv_2 \wedge dv_1}{v_0}, \quad \Omega_3|_{C_1} = \frac{dv_0 \wedge dv_2}{v_1}, \quad \Omega_3|_{C_2} = \frac{dv_1 \wedge dv_0}{v_2},
\end{equation}
where $C_i = \tilde{U}_3 \cap \{z \in \tilde{U}_3 | f_{w_i} \neq 0\}$, $(i = 0, 1, 2)$.
Define a form $\Omega$ on $X$ such that
\begin{equation}
\Omega|_{\tilde{U}_1} = \Omega_i.
\end{equation}
Then we claim that $\Omega$ is a well-defined holomorphic symplectic form on $X$. In fact, on $\tilde{U}_0 \cap \tilde{U}_1$, we know
\begin{equation}
\begin{cases}
y_0 = \frac{x_0}{x_1} = \frac{1}{z_1} \\
y_2 = \frac{x_2}{x_1} = \frac{z_2}{z_1} \\
y_3 = \frac{x_3}{x_1} = \frac{z_3}{z_1}
\end{cases}
\begin{cases}
z_1 = \frac{1}{y_0} \\
z_2 = \frac{y_2}{y_0} \\
z_3 = \frac{y_3}{y_0}
\end{cases}
\begin{cases}
dy_2 = \frac{z_1 dz_2 - z_2 dz_1}{z_1^2} \\
dy_3 = \frac{z_1 dz_3 - z_3 dz_1}{z_1^2},
\end{cases}
\end{equation}
\begin{equation}
4
\end{equation}
Assume locally written down as follows. which implies

\[ V = z_1 f_{z_1} + z_2 f_{z_2} + z_3 f_{z_3} \].

Assume \( V_1 \cap A_0 \left( \subset \bar{U}_0 \cap U_1 \right) \neq \emptyset \). Thus,

\[
\Omega_1 \mid_{V_1 \cap A_0} = \frac{dy_3 \wedge dy_2}{f_{y_0}} = \frac{z_1 dz_3 \wedge dz_2 - z_2 dz_3 \wedge dz_1 - z_3 dz_1 \wedge dz_2}{z_1^3 f_{y_0}} = -\frac{z_1 f_{z_1} + z_2 f_{z_2} + z_3 f_{z_3}}{z_1^3 f_{y_0}} dz_2 \wedge dz_3 = \frac{dz_2 \wedge dz_3}{f_{z_1}} = \frac{dz_2 \wedge dz_3}{f_{z_1}^2} = \Omega_0 \mid_{V_1 \cap A_0}.
\]

As the same way, we can also check the other cases.

For each point \( z = (z_1, z_2, z_3) \in V_1 \subset \bar{U}_0 \), the tangent space of \( X \) is as follows,

\[
T_z X = \left\{ a_1 \frac{\partial}{\partial z_1} + a_2 \frac{\partial}{\partial z_2} + a_3 \frac{\partial}{\partial z_3} \in T_z \mathbb{C}P^3 \mid a_1 f_{z_1} + a_2 f_{z_2} + a_3 f_{z_3} = 0, a_1, a_2, a_3 \in \mathbb{C} \right\}
\]

\[
= \left\{ (a_1, a_2, a_3) \in T_z \mathbb{C}P^3 \mid a_1 = -\frac{f_{z_2}}{f_{z_1}} a_2 - \frac{f_{z_3}}{f_{z_1}} a_3, a_1, a_2, a_3 \in \mathbb{C} \right\}
\]

\[
= \left\{ a_2 \left( -\frac{f_{z_2}}{f_{z_1}}, 1, 0 \right) + a_3 \left( -\frac{f_{z_3}}{f_{z_1}}, 0, 1 \right) \mid a_2, a_3 \in \mathbb{C} \right\}
\]

\[
= \text{span}_\mathbb{C} \{ V_1, V_2 \},
\]

where \( V_1 = \left( -\frac{f_{z_2}}{f_{z_1}}, 1, 0 \right), \quad V_2 = \left( -\frac{f_{z_3}}{f_{z_1}}, 0, 1 \right). \)

Thus the corresponding cotangent space of \( X \) is \( (T_z X)^* = \text{span}_\mathbb{C} \{ \varphi^1, \varphi^2 \} \), where \( \varphi^1 = -\frac{f_{z_2}}{f_{z_1}} \alpha dz_1 + (1 - \frac{f_{z_2}}{f_{z_1}})^2 \alpha dz_2 - \frac{f_{z_3}}{f_{z_1}} \alpha dz_3 \) and \( \varphi^2 = -\frac{f_{z_3}}{f_{z_1}} \alpha dz_1 - \frac{f_{z_2}}{f_{z_1}} \alpha dz_2 + (1 - \frac{f_{z_2}}{f_{z_1}})^2 \alpha dz_3 \) with \( \alpha = 1/ \left( 1 + \frac{|f_{z_2}|^2}{|f_{z_1}|^2} + \frac{|f_{z_3}|^2}{|f_{z_1}|^2} \right) \), satisfying \( \varphi^i(V_j) = \delta_{ij} \).

Then it follows from [13] that

**Theorem 2.1** Let \( X \) be a quartic K3 surface defined by a homogeneous polynomial \( f \) of degree 4 in \( \mathbb{C}P^3 \). Then any hyperkähler metrics \( h \) on the complex manifold \( X \) can be locally written down as follows.

For each \( z = (z_1, z_2, z_3) \in V_1 \subset \bar{U}_0 \subset X \), let \( S = \frac{|f_{z_1}|^2 + |f_{z_2}|^2 + |f_{z_3}|^2}{f_{z_1}^2} \) and \( f_{z_1} = |f_{z_1}| e^{i\theta_1} \). Then we have

\[
h = 2 \text{Re} \sum_{i,j=1}^2 h_{ij} \varphi^i \varphi^j,
\]

\[
\]
with

\[
\begin{cases}
    h_{11} = \frac{\rho S}{2|f_{z_1}|} \lambda \\
    h_{12} = -\frac{\rho S}{2|f_{z_1}|} \tau e^{i\theta_1} \\
    h_{21} = h_{12} \\
    h_{22} = \frac{\rho S}{2|f_{z_1}|} \mu
\end{cases}
\]

for some positive constant \( \rho \) and some positive real-valued functions \( \lambda, \mu \) and complex-valued function \( \tau \) on \( V_1 \) satisfying the following conditions:

\[ \lambda \mu = |\tau|^2 + 1. \]

Moreover, the corresponding hyperkähler structure \( \{ J^1, J^2, J^3 \} \) which is compatible with the metric \( h \) is as follows.

\[
\begin{align*}
    J^1 V_1 & = \sqrt{-1} V_1, \quad J^1 V_2 = \sqrt{-1} V_2, \\
    J^2 V_1 & = \tau V_1 + \lambda e^{-i\theta_1} V_2, \quad J^2 V_2 = -\mu e^{-i\theta_1} V_1 - \tau e^{-2i\theta_1} V_2, \\
    J^3 & = J^1 J^2.
\end{align*}
\]

\section{Proof of main theorem}

At first, we give three important lemmas which will be applied in the proof of main theorem.

\textbf{Lemma 3.1 ([6])} Let \( M \) be a compact hyperkähler surface and let \( N \) be a compact hyperkähler manifold. If \( N \) does not admit holomorphic \( S^2 \) with respect to a complex structure in the hyperkähler \( S^2 \), and \( u : M \rightarrow N \) is a weakly (stationary) triholomorphic map, then it is smooth outside of a finite set of points.

\textbf{Lemma 3.2 (Bézout’s theorem [11])} If \( C \) and \( D \) are two projective curves of degrees \( n \) and \( m \) in \( \mathbb{CP}^2 \) which have no common component then they have precisely \( nm \) points of intersection counting multiplicities.

\textbf{Lemma 3.3} Let \( \phi : \Sigma \rightarrow N \) be a smooth isometric immersion from an oriented surface \( \Sigma \) to a 4-dim hyperkähler manifold \( N \). Assume \( h \) is the hyperkähler metric on \( N \) and \( \{ J^1, J^2, J^3 \} \) are three parallel complex structures compatible with the metric \( h \). Let \( \alpha_p \) \( (p = 1, 2, 3) \) be three corresponding Kähler angles of \( \Sigma \) in \( N \). Then we have

\[ \cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 = 1. \]

\textbf{Proof:} Fix \( x \in \Sigma \). We choose the local frame of \( \Sigma \) around \( x \) \{\( e_1, e_2, e_3, e_4 \)\} such that \{\( e_1, e_2 \)\} is the frame of the tangent bundle \( T\Sigma \) and \{\( e_3, e_4 \)\} is the frame of the normal bundle \( (T\Sigma)^\perp \). The Kähler angle \( \alpha_p \) of \( \Sigma \) in \( N \) is defined by

\[ \cos \alpha_p = \omega_{J^p}(e_1, e_2) = (J^p e_1, e_2). \]
We choose suitable \( \{e_3, e_4\} \) such that the complex structure \( J^1 \) has the following form

\[
J^1 = \begin{pmatrix}
0 & \cos \alpha_1 & \sin \alpha_1 & 0 \\
-\cos \alpha_1 & 0 & -\sin \alpha_1 & 0 \\
-\sin \alpha_1 & 0 & \cos \alpha_1 & 0 \\
0 & \sin \alpha_1 & -\cos \alpha_1 & 0
\end{pmatrix}.
\] (3.2)

Let \( J^2, J^3 \) be the following form respectively

\[
J^2 = \begin{pmatrix}
0 & \cos \alpha_2 & b_{13} & b_{14} \\
-\cos \alpha_2 & 0 & b_{23} & b_{24} \\
-b_{13} & -b_{23} & 0 & b_{34} \\
-b_{14} & -b_{24} & -b_{34} & 0
\end{pmatrix},
J^3 = \begin{pmatrix}
0 & \cos \alpha_3 & c_{13} & c_{14} \\
-\cos \alpha_3 & 0 & c_{23} & c_{24} \\
-c_{13} & -c_{23} & 0 & c_{34} \\
-c_{14} & -c_{24} & -c_{34} & 0
\end{pmatrix}.
\] (3.3)

Assume \( \sin \alpha_1 \neq 0 \). Since \( J^3 = J^1 J^2 \), then from (3.2) and (3.3) we have

\[
b_{13} = -b_{24} = -\frac{\cos \alpha_2 \cos \alpha_1}{\sin \alpha_1}, \quad b_{14} = b_{23} = \frac{\cos \alpha_3}{\sin \alpha_1}, \quad b_{34} = \cos \alpha_2,
\]

and

\[
c_{13} = -c_{24} = -\frac{\cos \alpha_3 \cos \alpha_1}{\sin \alpha_1}, \quad c_{14} = c_{23} = -\frac{\cos \alpha_2}{\sin \alpha_1}, \quad c_{34} = \cos \alpha_3.
\]

Thus we obtain (3.1) by \((J^2)^2 = -\text{id}\).

Moreover,

\[
J^2 = \begin{pmatrix}
0 & \cos \alpha_2 & -\frac{\cos \alpha_2 \cos \alpha_1}{\sin \alpha_1} & \frac{\cos \alpha_3 \cos \alpha_1}{\sin \alpha_1} \\
-\frac{\cos \alpha_2 \cos \alpha_1}{\sin \alpha_1} & 0 & \frac{\cos \alpha_3}{\sin \alpha_1} & \frac{\cos \alpha_2}{\sin \alpha_1} \\
-\frac{\cos \alpha_3 \cos \alpha_1}{\sin \alpha_1} & \frac{\cos \alpha_3}{\sin \alpha_1} & 0 & \cos \alpha_2 \\
-\frac{\cos \alpha_3}{\sin \alpha_1} & -\frac{\cos \alpha_2}{\sin \alpha_1} & -\cos \alpha_2 & 0
\end{pmatrix},
\] (3.4)

\[
J^3 = \begin{pmatrix}
0 & \cos \alpha_3 & -\frac{\cos \alpha_3 \cos \alpha_1}{\sin \alpha_1} & \frac{\cos \alpha_2 \cos \alpha_1}{\sin \alpha_1} \\
-\frac{\cos \alpha_3 \cos \alpha_1}{\sin \alpha_1} & 0 & \frac{\cos \alpha_2}{\sin \alpha_1} & \frac{\cos \alpha_3}{\sin \alpha_1} \\
-\frac{\cos \alpha_3}{\sin \alpha_1} & \frac{\cos \alpha_2}{\sin \alpha_1} & 0 & \cos \alpha_3 \\
-\frac{\cos \alpha_3}{\sin \alpha_1} & -\frac{\cos \alpha_2}{\sin \alpha_1} & -\cos \alpha_3 & 0
\end{pmatrix}.
\] (3.5)

In the following, we give the proof of our main theorem.

**Theorem 3.4** Any weakly triholomorphic map from a compact hyperkähler surface to a quartic K3 surface defined by a homogeneous polynomial of degree 4 in \( \mathbb{CP}^3 \) has only isolated singularities.

**Proof:** Let \( X \) be a quartic K3 surface defined by a homogeneous polynomial \( f \) of degree 4 in \( \mathbb{CP}^3 \), which is characterized in section 2. Let \( \phi : S^2 \rightarrow X \) be a smooth map satisfying (1.2). In the following, we will prove that such smooth map \( \phi \) is a constant map, which implies that \( X \) does not admit holomorphic \( S^2 \) with respect to any complex structure in the hyperkähler \( S^2 \). Then from Lemma 3.1, we know that any weakly (stationary)
triholomorphic map from a compact hyperkähler surface to \( X \) is smooth outside a finite set of points. Thus we will finish the proof of our main theorem.

Let \( z = \frac{x^1 + \sqrt{-1}x^2}{x^3} \) be the local holomorphic coordinate on \( S^2 \setminus \{(0,0,1)\} \). Then the standard metric of constant curvature 1 on \( S^2 \) is as follows.

\[
g = \frac{4}{(1 + \overline{z}z)^2} dz d\overline{z}.
\]

Denote
\[
\partial = \frac{\partial}{\partial z}, \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}}
\]

Let \( \phi = [(1, \phi^1, \phi^2, \phi^3)] \), then
\[
d\phi \left( \frac{\partial}{\partial z} \right) = \partial \phi^1 \frac{\partial}{\partial z_1} + \partial \phi^2 \frac{\partial}{\partial z_2} + \partial \phi^3 \frac{\partial}{\partial z_3} + \partial \phi^1 \frac{\partial}{\partial \overline{z}_1} + \partial \phi^2 \frac{\partial}{\partial \overline{z}_2} + \partial \phi^3 \frac{\partial}{\partial \overline{z}_3}
\]

From (2.19) and (3.6), we know that (1.2) is equivalent to
\[
\begin{cases}
\sqrt{-1}(1 + a_{1j}x^j) \partial \phi^2 = -(a_{2j}x^j + \sqrt{-1}a_{3j}x^j) \left( \tau \overline{\partial} \phi^2 - \mu \overline{\partial} \phi^3 \right) \\
\sqrt{-1}(1 + a_{1j}x^j) \partial \phi^3 = -(a_{2j}x^j + \sqrt{-1}a_{3j}x^j) \left( \lambda \overline{\partial} \phi^2 - \tau e^{i\theta_1} \overline{\partial} \phi^3 \right)
\end{cases}
\]

A direct calculation shows
\[
\phi^* h = \frac{\rho S}{|f_{z_1}|(1 + a_{1j}x^j)} \left( \lambda \left| \overline{\partial} \phi^2 \right|^2 + \mu \left| \overline{\partial} \phi^3 \right|^2 + \tau e^{i\theta_1} \overline{\partial} \phi^2 \overline{\partial} \phi^3 - \tau e^{-i\theta_1} \overline{\partial} \phi^3 \overline{\partial} \phi^2 \right) dz d\overline{z},
\]

which implies the corresponding volume form of the pull-back metric \( \phi^* h \) is as follows.
\[
dV_{\phi^* h} = \frac{\sqrt{-1}}{2} \frac{\rho S}{|f_{z_1}|(1 + a_{1j}x^j)} \left( \lambda \left| \overline{\partial} \phi^2 \right|^2 + \mu \left| \overline{\partial} \phi^3 \right|^2 + \tau e^{i\theta_1} \overline{\partial} \phi^2 \overline{\partial} \phi^3 - \tau e^{-i\theta_1} \overline{\partial} \phi^3 \overline{\partial} \phi^2 \right) dz \wedge d\overline{z}.
\]

Let \( \omega_{J^p} \) be the corresponding kähler form which is compatible with the corresponding complex structure \( J^p \) on \( (X, h) \), where \( p = 1, 2, 3 \). Then we have \( \omega_{J^1} = \frac{\sqrt{-1}}{2} h_{J^1} \phi^1 \wedge \overline{\phi^1} \). Let \( \Omega \) be the unique (up to a nonzero constant) holomorphic symplectic form of the quartic K3 surface \( X \). Then we get \( \Omega = \omega_{J^2} + \sqrt{-1} \omega_{J^3} \). From the above section we know \( \Omega |_{V_1} = \frac{dz_2 \wedge d\overline{z}_2}{f_{z_1}} \). Then we have
\[
\phi^* \omega_{J^1} = \frac{\sqrt{-1}}{2} \frac{\rho S}{|f_{z_1}|(1 + a_{1j}x^j)} \left( \lambda \overline{\partial} \phi^2 \wedge \overline{\partial} \phi^2 + \mu \overline{\partial} \phi^3 \wedge \overline{\partial} \phi^3 - \tau e^{i\theta_1} \overline{\partial} \phi^2 \wedge \overline{\partial} \phi^3 - \tau e^{-i\theta_1} \overline{\partial} \phi^3 \wedge \overline{\partial} \phi^2 \right)
\]

\[
= \frac{\sqrt{-1}}{2} \frac{\rho S}{|f_{z_1}|(1 + a_{1j}x^j)} \left( \lambda \left| \overline{\partial} \phi^2 \right|^2 + \mu \left| \overline{\partial} \phi^3 \right|^2 - \tau e^{i\theta_1} \overline{\partial} \phi^2 \overline{\partial} \phi^3 - \tau e^{-i\theta_1} \overline{\partial} \phi^3 \overline{\partial} \phi^2 \right) dz \wedge d\overline{z}
\]

\[
= (-a_{1j}x^j) dV_{\phi^* h},
\]
and

\[ \phi^* \Omega = \frac{d\phi^2 \wedge d\phi^3}{f_{z_1}} \]
\[ = \frac{1}{f_{z_1}} \left( \partial \phi^2 \overline{\partial} \phi^3 - \partial \phi^3 \overline{\partial} \phi^2 \right) dz \wedge d\overline{z} \]
\[ = \frac{(-\sqrt{-1}) a_{2j} x^j + \sqrt{-1} a_{3j} x^j}{|f_{z_1}|(1 + a_{1j} x^j)} \]
\[ \cdot \left( \lambda |\overline{\phi^2}|^2 + \mu |\overline{\phi^3}|^2 - \tau e^{i\theta} \overline{\phi^2} \overline{\partial} \phi^3 - \tau e^{-i\theta} \overline{\phi^3} \overline{\partial} \phi^2 \right) dz \wedge d\overline{z} \]
\[ = -2 \frac{a_{2j} x^j + \sqrt{-1} a_{3j} x^j}{\rho S} dV_{\phi^* h}. \quad (3.11) \]

The kähler angle \( \alpha_p \) \( (p = 1, 2, 3) \) of \( S^2 \) in \( X \) was defined by Chern-Wolfson[8]

\[ \omega_{J^p} = \cos \alpha_p dV_{\phi^* h}. \quad (3.12) \]

It follows from (3.10) and (3.11) that

\[
\begin{cases}
\cos \alpha_1 = -a_{1j} x^j, \\
\cos \alpha_2 = -2 a_{2j} x^j, \\
\cos \alpha_2 = -2 a_{3j} x^j.
\end{cases} \quad (3.13)
\]

From (3.13) and (3.1), we get

\[ S = \frac{2}{\rho} = \text{constant}. \quad (3.14) \]

Assume \( \rho = 2 \). Then we find

\[ S = \frac{|f_{z_1}|^2 + |f_{z_2}|^2 + |f_{z_3}|^2}{|f_{z_1}|^2} = 1, \text{ on } \phi(S^2) \cap V_1 \subset \bar{U}_0 \subset X, \quad (3.15) \]

which implies

\[ f_{z_2} = f_{z_3} = 0, \text{ on } \phi(S^2) \cap V_1. \quad (3.16) \]

Since \( f_{z_1} \neq 0 \), from (2.3) we get

\[ z_1 = \text{constant}, \text{ on } \phi(S^2) \cap V_1. \quad (3.17) \]

Without loss of generality, assume \( z_1 = \sigma \) (constant). Then \( \phi(S^2) \cap V_1 \) is contained in a projective curve of degree 4 in \( \mathbb{C}P^2 \).

Let

\[ g(1, z_2, z_3) = f(1, \sigma, z_2, z_3). \quad (3.18) \]

Then we have

\[ g(1, z_2, z_3) = 0, g_{z_2} = f_{z_2} = 0, g_{z_3} = f_{z_3} = 0, \text{ on } \phi(S^2) \cap V_1. \quad (3.19) \]
Define
\[ C = \{ [1, z_2, z_3] \in \mathbb{C}P^2 : g(1, z_2, z_3) = 0 \}, \] (3.20)
\[ D = \{ [1, z_2, z_3] \in \mathbb{C}P^2 : g_{z_2} = 0 \}, \] (3.21)
and
\[ E = \{ [1, z_2, z_3] \in \mathbb{C}P^2 : g_{z_3} = 0 \}. \] (3.22)

Then \( C, D, E \) are three projective curves of degree 4, 3, 3 in \( \mathbb{C}P^2 \) respectively. Moreover we get
\[ \phi(S^2) \cap V_1 \subset C \cap D \cap E. \] (3.23)

In the following, we prove that \( C \cap D \cap E \) has only finite points by considering three different cases about their common components.

**Case 1:** Assume \( C, D, E \) have common component. Then the corresponding polynomials \( g(1, z_2, z_3), g_{z_2}, g_{z_3} \) have common nonconstant polynomial factor \( P \). Since any two projective curves in \( \mathbb{C}P^2 \) intersect in at least one point, we know that the projective curve given by \( P \) and the one given by \( f_z(1, \sigma, z_2, z_3) \) intersect in at least one point denoted by \([1, b, c] \in \mathbb{C}P^2 \). Then the point \([1, \sigma, b, c] \in \mathbb{C}P^3 \) is a singularity of the polynomial \( f(1, z_1, z_2, z_3) \), which contradicts to the fact that quartic K3 surface given by \( f(1, z_1, z_2, z_3) \) is nonsingular. So there doesn’t exist this case.

**Case 2:** Assume two and only two curves of \( C, D, E \) have common component. Without loss of generality, assume \( C \) and \( D \) have common component. Then \( C \) and \( E \) have no common component. From Lemma 3.2, we know that \( C \) and \( E \) have only finite points of intersection. So \( C \cap D \cap E \) has only finite points in this case.

**Case 3:** Assume any two curves of \( C, D, E \) have no common component. Obviously \( C \cap D \cap E \) has only finite points or \( C \cap D \cap E = \emptyset \) in this case.

Now, we have proven that \( C \cap D \cap E \) has only finite points. It follows from (3.23) that \( \phi(S^2) \cap V_1 \) has only finite points. Similarly, we know that \( \phi(S^2) \cap V_2 \) and \( \phi(S^2) \cap V_3 \) have only finite points respectively. Then we get that \( \phi(S^2) \subset X \) has only finite points. But \( \phi \) is smooth. Hence \( \phi \) must be a constant map.

\[ \square \]

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