On The Cohomological Dimension of Local Cohomology Modules

Vahap Erdoğdu and Tuğba Yıldırım∗

Dedicated to the memory of Alexander Grothendieck

Department of Mathematics
Istanbul Technical University
Maslak, 34469, Istanbul, Turkey

Abstract

Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $M$ an $R$-module with $cd(I, M) = c$. In this article, we are mainly interested in the cohomological dimension of local cohomology modules and we first show that there exists a descending chain of ideals $I = I_c \supseteq I_{c-1} \supseteq \cdots \supseteq I_0$ of $R$ such that for each $0 \leq i \leq c - 1$, $cd(I_i, M) = i$ and that the top local cohomology module $H^i_{I_i}(M)$ is not Artinian. We then give sufficient conditions for an arbitrary non-negative integer $t$ to be a lower bound for the cohomological dimension of a finitely generated $R$-module $M$, and use this to conclude that in non-catenary Noetherian local integral domains, there exist prime ideals that are not set theoretic complete intersection. Finally, we set conditions which determine whether or not a top local cohomology module is Artinian.

Keywords: Local cohomology, Set theoretic complete intersections, Cohomological dimension

2000 Mathematics Subject Classification. 13D45, 14F17.

*Corresponding author
1 Introduction

Throughout, $R$ denote a commutative Noetherian ring with unity, $I$ an ideal of $R$. For an $R$-module $M$, the $i$-th local cohomology module of $M$ with support in $I$ is defined as

$$H^i_I(M) = \lim_{\rightarrow} \text{Ext}^i_{R}(R/I^n, M).$$

For details about the local cohomology modules, we refer the reader to [2] and [5].

One of the important invariant related to local cohomology modules is the cohomological dimension of $M$ with respect to $I$, denoted by $\text{cd}(I, M)$, and defined as:

$$\text{cd}(I, M) = \sup \{ i \in \mathbb{N} | H^i_I(M) \neq 0 \}.$$ 

If $M = R$, we write $\text{cd}(I)$ instead of $\text{cd}(I, R)$.

Recall that the Krull dimension of a nonzero $R$-module $M$, denoted by $\text{dim}(M)$, is the supremum of lengths of chains of prime ideals in the support of $M$ if this supremum exists, and $\infty$ otherwise. If $M$ is finitely generated, then $\text{dim}(M) = \text{dim}(R/\text{Ann}M)$ but this is not the case if $M$ is not finitely generated and it is a well known fact that for a finitely generated $R$-module $M$,

$$\text{grade}(I, M) \leq \text{ht}_M(I) \leq \text{cd}(I, M) \leq \text{dim}(M)$$

where $\text{ht}_M(I) = \text{ht}I(R/\text{Ann}M)$.

There are two interesting questions related to local cohomology modules, the first one is to determine the lower and upper bounds for $\text{cd}(I, M)$ and the second one is to determine whether or not $H^i_I(M)$ is Artinian (see e.g. [1], [3], [4], [6], [7] and [8]).

Our results in this regard are as follows:

In section 2, we show that for an $R$-module $M$ with $\text{cd}(I, M) = c$, there is a descending chain of ideals

$$I = I_c \supseteq I_{c-1} \supseteq \cdots \supseteq I_0$$

of $R$ such that for each $0 \leq i \leq c$, $\text{cd}(I_i, M) = i$, and that the top local cohomology module $H^i_{I_0}(M)$ is not Artinian.

In section 3, we prove a result that gives a sufficient condition for an integer to be a lower bound for the cohomological dimension, $\text{cd}(I, M)$, of $M$ at $I$. One of the important conclusion of this result is that over a Noetherian local
ring \((R, m)\), for a finitely generated \(R\)-module \(M\) of dimension \(n\) and an ideal \(I\) of \(R\) with \(\text{dim}(M/IM) = d \geq 1\), \(n - d\) is a lower bound for \(\text{cd}(I, M)\) and if moreover \(\text{cd}(I, M) = n - d\), then \(H^d_m(H^{n-d}_I(M)) \cong H^n_m(M)\). From this result we also conclude that for any ideal \(I\) of \(R\) with \(\text{dim}(M/IM) = 1\), \(H^{\text{cd}(I, M)} I(M)\) is Artinian if and only if \(\text{cd}(I, M) = n\). As an application of these results, we show that in non-catenary Noetherian local integral domains, there exist prime ideals that are not set theoretic complete intersection.

In section 4, we set conditions under which when one of the two local cohomology modules avoids a Serre subcategory of the category of all \(R\) modules the other one does as well, and use this to determine (under some mild conditions) whether or not a local cohomology module is Artinian.

## 2 Descending Chains With Successive Cohomological Dimensions

In this section, we prove the existence of descending chains of ideals and locally closed sets with successive cohomological dimensions. The main result of this section is the following:

**Theorem 2.1.** Let \(R\) be a Noetherian ring, \(I\) an ideal of \(R\) and \(M\) an \(R\)-module with \(\text{cd}(I, M) = c > 0\). Then there is a descending chain of ideals

\[
I = I_c \supseteq I_{c-1} \supseteq \cdots \supseteq I_0
\]

such that \(\text{cd}(I_i, M) = i\) for all \(0 \leq i \leq c\).

**Proof.** Consider the set

\[
S = \{ J \subsetneq I \mid \text{cd}(J, M) < c \}.
\]

Clearly, the zero ideal belongs to \(S\) and so \(S\) is a non-empty subset of ideals of \(R\). Since \(R\) is Noetherian, \(S\) has a maximal element, say \(I_{c-1}\). We claim that \(\text{cd}(I_{c-1}, M) = c - 1\). To prove this, let \(x \in I \setminus I_{c-1}\). Then \(I_{c-1} + Rx \subsetneq I\). But then it follows from the maximality of \(I_{c-1}\) in \(S\) and Remark 8.1.3 of [2] that

\[
c \leq \text{cd}(I_{c-1} + Rx, M) \leq \text{cd}(I_{c-1}, M) + 1 < c + 1.
\]

Hence \(\text{cd}(I_{c-1} + Rx, M) = c\). Now consider the exact sequence

\[
\cdots \longrightarrow (H_{I_{c-1}}^{c-1}(M))_x \longrightarrow H_{I_{c-1}+Rx}^c(M) \longrightarrow H_{I_{c-1}}^c(M) = 0.
\]
Since $H^{c-1}_{I_{c-1} + R_x}(M)$ is nonzero, it follows that $(H^{c-1}_{I_{c-1}}(M))_x$ is nonzero, then so is $H^{c-1}_{I_{c-1}}(M)$. Therefore the claim follows.

Iterating this argument, one can obtain a descending chain of ideals, as desired. \hfill \square

Recall that a subspace $Z$ of a topological space $X$ is said to be *locally closed*, if it is the intersection of an open and a closed set. Let $X$ be a topological space, $Z \subseteq X$ be a locally closed subset of $X$ and let $F$ be an abelian sheaf on $X$. Then the $i^{th}$ local cohomology group of $F$ with support in $Z$ is denoted by $H^i_Z(X, F)$. We refer the reader to [5] and [9] for its definition and details.

If, in particular, $X = \text{Spec}(R)$ is an affine scheme, where $R$ is a commutative Noetherian ring, and $F = M^\sim$ is the quasi coherent sheaf on $X$ associated to an $R$-module $M$, we write $H^i_Z(M)$ instead of $H^i_Z(X, M^\sim)$.

**Corollary 2.2.** Let $R$ be a Noetherian ring, $M$ an $R$-module and $I$ an ideal of $R$ such that $\text{cd}(I, M) = c > 1$. Then there is a descending chain of locally closed sets

$$T_{c-1} \supseteq T_{c-2} \supseteq \cdots \supseteq T_1$$

in $\text{Spec}(R)$ such that $\text{cd}(T_i, M) = i$ for all $1 \leq i \leq c - 1$.

**Proof.** Let $I$ be an ideal of $R$ with $\text{cd}(I, M) = c > 1$. Then it follows from Theorem 2.1 that there is a descending chain of ideals

$$I = I_c \supseteq I_{c-1} \supseteq \cdots \supseteq I_1 \supseteq I_0$$

such that $\text{cd}(I_i, M) = i$ for all $0 \leq i \leq c$. Let now $U_i = V(I_i)$ and define the locally closed sets $T_i := U_1 \setminus U_{i+1}$. Then it is easy to see that

$$T_{c-1} \supseteq T_{c-2} \supseteq \cdots \supseteq T_1.$$

On the other hand, it follows from Proposition 1.2 of [9] that there is a long exact sequence,

$$\cdots \longrightarrow H^j_{U_1}(M) \longrightarrow H^j_{T_1}(M) \longrightarrow H^{j+1}_{U_{i+1}}(M) \longrightarrow H^{j+1}_{U_1}(M) \longrightarrow \cdots$$

As $H^j_{U_1}(M) \cong H^j_{T_1}(M)$ for all $1 \leq i \leq c - 1$ and for all $j \geq 0$, it follows from the above long exact sequence that $\text{cd}(T_i, M) = i$. \hfill \square
Theorem 2.3. Let $R$ be a Noetherian ring, $I$ an ideal of $R$, $M$ an $R$-module with $\text{cd}(I, M) = c > 0$ and

$$I = I_c \supseteq I_{c-1} \supseteq \cdots \supseteq I_0$$

be the filtration of $I$ as described in Theorem 2.1. Then for each $0 \leq i \leq c-1$, the local cohomology module $H^i_{I_i}(M)$ is not Artinian.

Proof. Let $0 \leq i \leq c-1$ and $x \in I_{i+1} \setminus I_i$ and consider the ideal $I_i + Rx$. Then it follows from the construction of the ideal $I_i$ that $cd(I_i + Rx, M) = i + 1$. Now from Corollary 3.5 of [11], we have the following short exact sequence

$$0 \rightarrow H^1_{I_i}(M) \rightarrow H^{i+1}_{I_i+Rx}(M) \rightarrow H^0_{Rx}(H^{i+1}_{I_i}(M)) \rightarrow 0.$$

Hence $H^1_{Rx}(H^i_{I_i}(M)) \cong H^{i+1}_{I_i+Rx}(M) \neq 0$. But then it follows from Grothendieck’s vanishing theorem that $\dim(H^i_{I_i}(M)) \geq 1$, and so $H^i_{I_i}(M)$ is not Artinian. $\square$

3 Lower Bound For Cohomological Dimension

Our main result of this section is the following theorem which gives a sufficient condition for an integer $t$ to be a lower bound for the cohomological dimension. We use this to conclude several new statements:

Theorem 3.1. Let $R$ be a Noetherian ring, $M$ a finitely generated $R$-module and $I$ an ideal of $R$ with $\dim(M/IM) = d$. Let $t \geq 0$ be an integer. If there exists an ideal $J$ of $R$ such that $H^{d+t}_{I_i+J}(M) \neq 0$, then $t$ is a lower bound for $\text{cd}(I, M)$. Moreover, if $\text{cd}(I, M) = t$, then

$$H^d_J(H^1_I(M)) \cong H^{d+t}_{I_i+J}(M)$$

and $\dim(H^1_I(M)) = d$.

Proof. Consider the Grothendieck’s spectral sequence

$$E^{p,q}_2 = H^p_J(H^q_I(M)) \Rightarrow H^{p+q}_{I_i+J}(M)$$

and look at the stage $p + q = n$. Since $\text{Supp}(H^q_I(M)) \subseteq \text{Supp}(M/IM)$, it follows that for all $q$, $\dim(H^q_I(M)) \leq d$. Therefore for all $p > d$, $E^{p,d+t-p}_2 = 0$. 5
But then since from the hypothesis $H_{I,J}^{d+t}(M)$ does not vanish, there is at least one $p \leq d$ such that

$$E_2^{p,d+t-p} = H_j^p(H_{I,J}^{d+t-p}(M)) \neq 0.$$  

Hence $H_{I,J}^{d+t-p}(M) \neq 0$ and so $cd(I,M) \geq d + t - p \geq t$.

If, in particular, $cd(I,M) = t$, then $E_2^{p,q} = 0$ for all $q > t$. Now from the subsequent stages of the spectral sequence

$$E_{d-k,t+k-1} \longrightarrow E_{d-t} \longrightarrow E_{d+k,-k+1}$$

and the fact that $E_{d-k,t+k-1} = E_{d+k,-k+1} = 0$ for all $k \geq 2$, we have $E_\infty = E_2$. Hence $H_2^d(H_I^1(M)) \cong H_{d+1}^j(M)$.

Since $H_2^d(H_I^1(M)) \neq 0$, it follows from Grothendieck’s vanishing theorem that $dim(H_I^1(M)) \geq d$. On the other hand, since $Supp(H_I^1(M)) \subseteq Supp(M/IM)$, $dim(H_I^1(M)) \leq dim(M/IM) = d$, we conclude that $dim(H_I^1(M)) = d$. \hfill \square

So far, the best known lower bound for $cd(I,M)$ is $ht_M(I)$, we now use our Theorem 3.1 to sharpen this bound to $dim(M) - dim(M/IM) \geq ht_M(I)$.

**Corollary 3.2.** Let $(R,m)$ be a Noetherian local ring, $M$ a finitely generated $R$-module of dimension $n$ and $I$ an ideal of $R$ such that $dim(M/IM) = d$. Then $n - d$ is a lower bound for $cd(I,M)$. Moreover, if $cd(I,M) = n - d$, then

$$H_m^d(H_I^{n-d}(M)) \cong H_m^n(M)$$

and $dim(H_I^{n-d}(M)) = d$.

**Proof.** This follows from Theorem 3.1 and the fact that $H_m^n(M) \neq 0$. \hfill \square

**Corollary 3.3.** Let $(R,m)$ be a Noetherian local ring, $M$ a finitely generated $R$-module of dimension $n$ and $I$ an ideal of $R$ such that $dim(M/IM) = 1$. Then $H_I^{cd(I,M)}(M)$ is Artinian if and only if $cd(I,M) = n$.

**Proof.** Since $dim(M/IM) = 1$, it follows from Corollary 3.2 that $cd(I,M) = n - 1$ or $cd(I,M) = n$. If $cd(I,M) = n$, then by Theorem 7.1.6 of [2], $H_I^n(M)$ is Artinian. If, on the other hand, $cd(I,M) = n - 1$, then it follows from Corollary 3.2 that $dim(H_I^{n-1}(M)) = 1$ and so $H_I^{n-1}(M)$ is non-Artinian. \hfill \square
For an ideal \( I \) of \( R \), it is a well-known fact that \( \text{ht}(I) \leq \text{cd}(I) \leq \text{ara}(I) \), where \( \text{ara}(I) \) denotes the smallest number of elements of \( R \) required to generate \( I \) up to radical. If, in particular, \( \text{ara}(I) = \text{cd}(I) = \text{ht}(I) \), then \( I \) is called a set-theoretic complete intersection ideal. Determining set-theoretic complete intersection ideals is a classical and long-standing problem in commutative algebra and algebraic geometry. Many questions related to an ideal \( I \) to being a set-theoretic complete intersection are still open, see [10] for more details. Varbaro in [12] has shown that under certain conditions there exists ideals \( I \) satisfying the property that \( \text{cd}(I) = \text{ht}(I) \), knowing the existence of ideals with such properties, we have the following:

**Corollary 3.4.** Let \( (R, \mathfrak{m}) \) be a Noetherian local ring of dimension \( n \) and \( I \) an ideal of \( R \) with \( d = \text{dim}(R/I) \) such that \( \text{cd}(I) = \text{ht}(I) = h \). Then \( \text{dim}(R) = \text{ht}(I) + \text{dim}(R/I) \) and
\[
H^{n-h}_m(H^h_I(R)) \cong H^n_m(R).
\]

*Proof.* It follows from Corollary 3.2 that \( \text{dim}(R) - \text{dim}(R/I) \leq \text{cd}(I) \leq \text{ht}(I) \), while the other side of the inequality always holds. Therefore \( \text{dim}(R) = \text{ht}(I) + \text{dim}(R/I) \). Now the required isomorphism follows from Corollary 3.2. \( \square \)

We end this section with the following conclusion:

**Corollary 3.5.** Let \( (R, \mathfrak{m}) \) be a non-catenary Noetherian local domain of dimension \( n \). Then there is at least one prime ideal of \( R \) that is not a set theoretic complete intersection.

*Proof.* Since \( R \) is non-catenary, there is a prime ideal \( \mathfrak{p} \) of \( R \) such that \( \text{ht}(\mathfrak{p}) < n - \text{dim}(R/\mathfrak{p}) \). Then it follows from Corollary 3.2 that \( \text{cd}(\mathfrak{p}) \geq n - \text{dim}(R/\mathfrak{p}) > \text{ht}(\mathfrak{p}) \) and therefore \( \mathfrak{p} \) can not be a set theoretic complete intersection ideal. \( \square \)

### 4 Artinianness of Top Local Cohomology Modules

Recall that a class \( \mathcal{S} \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules, \( C(R) \), when it is closed under taking submodules, quotients and
The main result of this section is the following:

**Theorem 4.1.** Let $R$ be a Noetherian ring, $M$ an $R$-module (not necessarily finitely generated) and let $\mathcal{S}$ be a Serre subcategory of $\mathcal{C}(R)$. Let $I$ and $J$ be two ideals of $R$ such that $H_{I}^{t+i}(H_{I}^{-i}(M)) \in \mathcal{S}$ for all $0 < i \leq c = cd(I, M)$ and $H_{I+J}^{t+c}(M) \notin \mathcal{S}$ for some positive integer $t$. Then $H_{I}^{t}(H_{I}^{c}(M)) \notin \mathcal{S}$.

**Proof.** Consider the Grothendieck’s spectral sequence

$$E_2^{p,q} = H_{I}^{p}(H_{I}^{q}(M)) \implies H_{I+J}^{p+q}(M)$$

and look at the stage $p + q = c + t$. Let now $0 < i \leq c = cd(I, M)$. Since $E_{t+i}^{t+i,c-i} = E_{r}^{t+i,c-i}$ for sufficiently large $r$ and $E_{r}^{t+i,c-i}$ is a subquotient of $E_{t+i}^{t+i,c-i} \in \mathcal{S}$, $E_{t+i}^{t+i,c-i} \in \mathcal{S}$ for all $0 < i \leq c = cd(I, M)$.

On the other hand, since $E_{t+c}^{t+c} = H_{I}^{t}(H_{I}^{c}(M)) \implies H_{I+J}^{t+c}(M)$, there exists a finite filtration

$$0 = \Phi^{t+c+1}H_{I+J}^{t+c} \subseteq \Phi^{t+c}H_{I+J}^{t+c} \subseteq \cdots \subseteq \Phi^{1}H_{I+J}^{t+c} \subseteq \Phi^{0}H_{I+J}^{t+c} = H_{I+J}^{t+c}$$

of $H_{I+J}^{t+c}(M)$ such that $E_{t+c}^{p,q} = \Phi^{p}H_{I+J}^{t+c}/\Phi^{p+1}H_{I+J}^{t+c}$ for all $p + q = t + c$.

Since for all $p < t$, $E_{t+c}^{p,q} = 0$, we have that $\Phi^{t}H_{I+J}^{t+c} = \cdots = \Phi^{1}H_{I+J}^{t+c} = \Phi^{0}H_{I+J}^{t+c} = H_{I+J}^{t+c}$. But then since $E_{t+c}^{t+i,c-i} = \Phi^{t+i}H_{I+J}^{t+c}/\Phi^{t+i+1}H_{I+J}^{t+c} \in \mathcal{S}$ for all $0 < i \leq c$, $\Phi^{t+i}H_{I+J}^{t+c} \in \mathcal{S}$ and so it follows from the short exact sequence

$$0 \longrightarrow \Phi^{t+i}H_{I+J}^{t+c} \longrightarrow H_{I+J}^{t+c}(M) \longrightarrow E_{t+c}^{t+c} \longrightarrow 0$$

that $E_{t+c}^{t+c} \notin \mathcal{S}$. Since $E_{t+c}^{t+c}$ is a subquotient of $E_{t+c}^{t+c}$ and $E_{t+c}^{t+c} \notin \mathcal{S}$, it follows that $E_{t+c}^{t+c} = H_{I}^{t}(H_{I}^{c}(M)) \notin \mathcal{S}$. \hfill \Box

**Corollary 4.2.** Let $(R, m)$ be a Noetherian local ring, $M$ a finitely generated $R$-module of dimension $n$ and $I$ an ideal of $R$ such that $\dim(M/IM) = d$ and $c = cd(I, M) < n$. If $H_{I}^{t}(M)$ is Artinian, then $\dim(H_{I}^{t}(M)) \geq n - i$ for some $n - d \leq i < c$.

**Proof.** We prove the contrapositive of the statement. Let $\mathcal{S}$ be the category of zero module and suppose that $\dim(H_{I}^{t}(M)) < n - i$ for all $n - d \leq i < c$. Then $H_{m}^{n-i}(H_{I}^{t}(M)) = 0 \in \mathcal{S}$ for all $n - d \leq i < c$. But then since $H_{m}^{n}(M) \notin \mathcal{S}$, it follows from Theorem 4.1 that $H_{m}^{n-c}(H_{I}^{t}(M)) \neq 0$. Hence $\dim(H_{I}^{t}(M)) > 0$ and so $H_{I}^{t}(M)$ is not Artinian. \hfill \Box
The purpose of the following corollary is to enlighten the statement of Corollary 4.2.

**Corollary 4.3.** Let \((R, \mathfrak{m})\) be a Noetherian local ring, \(M\) a finitely generated \(R\)-module of dimension \(n\) and \(I\) an ideal of \(R\) such that \(\dim(M/IM) = 2\). Let \(\text{cd}(I, M) = c < n\). Then \(H^c_I(M)\) can not be Artinian unless \(c = n - 1\) and \(H^2_m(H^{n-2}_I(M)) \neq 0\).

**Proof.** Since \(\dim(M/IM) = 2\), it follows from Corollary 3.2 that either \(c = n - 1\) or \(c = n - 2\). If \(c = n - 2\), then from Corollary 3.2 we have that \(\dim(H^c_I(M)) = 2\) and so \(H^c_I(M)\) is not Artinian. If, on the other hand, \(c = n - 1\) and \(H^2_m(H^{n-2}_I(M)) = 0\), then we use Corollary 4.2 to see that \(H^c_I(M)\) is not Artinian. Hence the possibility that \(H^c_I(M)\) is Artinian can occur only in the case when \(c = n - 1\) and \(H^2_m(H^{n-2}_I(M)) \neq 0\). \(\square\)

**References**

[1] M. Aghapournahr and L. Melkersson, *Artinianness of local cohomology modules*, Ark. Mat., 52(2014), 1-10.

[2] M. P. Brodmann and R. Y. Sharp, *Local Cohomology*, 2nd ed., Cambridge Studies in Advanced Mathematics, Vol. 136, Cambridge University Press, Cambridge, 2013. An algebraic introduction with geometric applications.

[3] M. T. Dibaei and A. Vahidi, *Artinian and Non-Artinian Local Cohomology Modules*, Canad. Math. Bull., 54(2011), 619-629.

[4] K. Divaani-Aazar, R. Naghipour, M. Tousi, *Cohomological dimension of certain algebraic varieties*, Proc. Amer. Math. Soc, 130(12) (2002), 3537-3544.

[5] A. Grothendieck (notes by R. Hartshorne), *‘Local cohomology’*, Lect. Notes in Math. 41 Springer-Verlag, 1967.

[6] R. Hartshorne, *Cohomological dimension of algebraic varieties*, Annals of Math., 88(1968), 403-450.

[7] C. Huneke, *Problems in local cohomology*, Free Resolutions in Commutative Algebra and Algebraic Geometry Sundance 90, Res. Notes in Math., Jones and Barlett, 2(1992), 93-108.
[8] C. Huneke and G. Lyubeznik, *On the vanishing of local cohomology modules*, Inv. Math., **102** (1990), 73-93.

[9] G. Lyubeznik, *A partial survey of local cohomology, local cohomology and its applications*, Lect. Notes Pure Appl. Math., **226** (2002), 121-154.

[10] G. Lyubeznik, *A survey of problems and results on the number of defining equations*. Commutative Algebra, Math. Sci. Res. Inst. Publ., **15** (1989), 375-390.

[11] P. Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand. **92** (2003) 161-180.

[12] M. Varbaro, *Cohomological and Projective Dimensions*, Compositio Mathematica, **149**(2013), 1203-1210.