On the Long-Time Behavior of a Wave–Klein-Gordon Coupled System

Zhimeng Ouyang∗

1Department of Mathematics, Brown University

Abstract

We consider a coupled Wave–Klein-Gordon system in 3D, which is a simplified model for the global nonlinear stability of the Minkowski space-time for self-gravitating massive fields. In this paper we study the large-time asymptotic behavior of solutions to such systems, and prove modified wave operators for small data. The key novelty comes from a crucial observation that the modified asymptotic dynamics are dictated by the resonant interactions of the equations. As a consequence, our main results include the derivation of a resonant system with good error bounds, and a detailed description of the global behavior of solutions to the Wave–Klein-Gordon system.

Keywords: Wave–Klein-Gordon System, Quasilinear Dispersive Equations, Modified Scattering, Wave Operators, Resonances.

Contents

1 Introduction 2
1.1 The Wave–Klein-Gordon System 2
1.2 Background and Motivation 3
1.3 Main Result and Method 3

2 Set-up and Overview of the Problem 5
2.1 Reformulation of the Equations: Duhamel’s Formula 5
2.2 Definition and Notation 7
2.2.1 Vector Fields 7
2.2.2 Littlewood–Paley Projections 8
2.2.3 Norms 10
2.3 Resonant System 11
2.3.1 Analysis of the Quadratic Phase Functions 11
2.3.2 Analysis of the (Space-Time) Resonances 12
2.3.3 Extraction of the “Resonant System” 13
2.4 Equations of Perturbation 15
2.5 Main Theorem and Proposition 17
2.6 Strategy of Proof 18
2.7 Organization of the Paper 19

∗zhimeng.ouyang@brown.edu
1 Introduction

1.1 The Wave–Klein-Gordon System

In this paper we are interested in the global behavior of the Wave–Klein-Gordon (W–KG) system in $3 + 1$ space-time dimensions:

$$
\Box u = A^{\alpha\beta} \partial_\alpha u \partial_\beta v + Dv^2, \\
(-\Box + 1)v = u B^{\alpha\beta} \partial_\alpha \partial_\beta v + Euw,
$$

(1.1)

where $\Box := -\partial^2_t + \Delta$ is the d’Alembert operator. The unknowns $u, v : \mathbb{R}^+_t \times \mathbb{R}^3_x \to \mathbb{R}$ are real-valued functions, and $A^{\alpha\beta}, B^{\alpha\beta}, D$ and $E$ are real constants. Without loss of generality we may assume that $A^{\alpha\beta} = A^{\beta\alpha}$ and $B^{\alpha\beta} = B^{\beta\alpha}, \alpha, \beta \in \{0, 1, 2, 3\}$. For convenience we will also assume that $D^{00} = 0$.\footnote{This can be achieved by adding some higher order terms in the second equation in (1.1), which do not change the analysis.}
Our main focus is the description of the asymptotic behavior for small solutions to a prototype of the above system:

\begin{align}
\left( \partial_t^2 - \Delta \right) u &= |\nabla_{t,x} v|^2 + v^2, \\
\left( \partial_t^2 - \Delta + 1 \right) v &= u \Delta v.
\end{align}

This coupled system consists of a semi-linear wave equation for \( u \) and a quasi-linear Klein-Gordon equation for \( v \).

### 1.2 Background and Motivation

The system was derived by LeFloch-Ma [11] as a simplified model for the Einstein-Klein-Gordon (E-KG) system, which describes the coupled evolution of the Lorentzian metric and a self-gravitating massive scalar field.

Intuitively, the deviation of the Lorentzian metric \( g \) from the Minkowski metric is replaced by a scalar function \( u \), and the massive scalar field \( \phi \) is replaced by \( v \). The W-KG system keeps the same linear structure as the Einstein-Klein-Gordon equations in harmonic gauge, but only keeps, schematically, quadratic interactions that involve the massive scalar field (the semilinear terms in the first equation and the quasilinear terms in the second equation coming from the reduced wave operator).

Naturally, the global stability and asymptotic behavior of the Minkowski space-time is a central topic in general relativity. Global solutions have been constructed for W-KG system [6], and for E-KG system [7].

As [6] reveals, when \( t \to \infty \), the W-KG system is asymptotically closer to the so-called resonant system, which captures the key low-frequency resonant interaction of wave component. This further leads to the modified scattering of the KG equation, with a phase correction.

The modified scattering for the Klein-Gordon component can be explained heuristically as follows: The structure of the W-KG system is characterized by the bilinear interactions

\[ \text{Wave} \leftarrow \text{KG} \times \text{KG} \]
\[ \text{KG} \leftarrow \text{Wave} \times \text{KG} \]

1. The Klein-Gordon component is generated linearly by initial data;

2. It interacts with itself in the wave equation to produce a large low-frequency wave component \(|u(t,x)| \approx t^{-1} \varepsilon_0^2\) (if \( t \gg 1 \) and \(|x| \leq ct\)) supported at low frequency \(|\xi| \lesssim 1/t\);

3. This large low-frequency wave component interacts again with the high-frequencies of the Klein-Gordon component to produce growth.

### 1.3 Main Result and Method

As a counterpart of the result above, we also have the existence and uniqueness of the modified wave operator, namely that every possible asymptotic behavior is achieved (i.e. any asymptotic profile leads to a global solution that scatters to it). We intend to justify that for any resonant system solution, there exists a W-KG solution such that they are asymptotically convergent.
Theorem 1.1 (Main Theorem, Formal Version). Assume \((V_{\infty}^{wa}, V_{\infty}^{kg})\) is sufficiently small under higher-order weighted Sobolev norms. Then there exists a unique solution \((u, v)\) with \(t \in [0, \infty)\) of the system (1.2) such that

\[
\lim_{t \to \infty} \left\| \left( \partial_t - i \frac{|\nabla|}{2} \right) u(t) - e^{-it|\nabla|} \left( V_{\infty}^{wa} + \mathcal{H}_{\infty}(t) \right) \right\|_{L^2} = 0, \tag{1.3}
\]

and

\[
\lim_{t \to \infty} \left\| \left( \partial_t - i \langle \nabla \rangle \right) v(t) - e^{-it\langle \nabla \rangle} \left( e^{it\langle \Delta \rangle} V_{\infty}^{kg} + \mathcal{B}_{\infty}(t) \right) \right\|_{L^2} = 0, \tag{1.4}
\]

where \(\mathcal{H}_{\infty}, \mathcal{D}_{\infty}\) and \(\mathcal{B}_{\infty}\) are quantities that explicitly depend on \((V_{\infty}^{wa}, V_{\infty}^{kg})\).

Remark 1.1. (1) This theorem is only a formal version and the precise statement of the theorem (Theorem 2.1) and definition of the quantities are in Section 2.

(2) This theorem reveals that the wave equation undergoes linear scattering and the Klein-Gordon equation undergoes nonlinear modified scattering.

(3) Roughly speaking, for the resonant system solution \((V_{RS}^{wa}, V_{RS}^{kg})\), \(\mathcal{H}_{\infty}\) depends on the low-frequency truncation \(V_{RS, low}^{wa}\) of the wave component, \(\mathcal{D}_{\infty}\) depends on the resonant interaction between \(V_{RS}^{kg}\) and \(V_{RS, low}^{wa}\), and \(\mathcal{B}_{\infty}\) depends on the non-resonant interaction between \(V_{RS}^{kg}\) and \(V_{RS, low}^{wa}\).

(4) Different from \([6]\), our “resonant” system also contains non-resonant interaction term \(\mathcal{B}_{\infty}\), which is a term that decays to zero at the critical speed, so it will not affect the asymptotic behaviors in any essential way. On one hand, this comes from the proof; on the other hand, this indicates that wave operator analysis might capture more detailed time decay information than the modified scattering analysis.

One of the main difficulties comes from the fact that we have a genuine system in the sense that the linear evolution of wave and Klein-Gordon equations admit different modes of propagation, combining to create new dynamics that cannot simply be predicted by looking at each component of the system in isolation.

On a more technical level, it turns out that the main feature of the system above is the slow decay of the low frequencies of wave component \(v\) in the interior of the light cone and in particular along the characteristics associated to the Klein-Gordon operator. This nonlinear effect ultimately leads to modified scattering for the Klein-Gordon component.

Our proof is based on a combination of the energy method and the dispersive analysis. We utilize both the weak formulation (energy equality) and mild formulation (Duhamel’s principle). Energy estimate provides basic control of quantities under Sobolev-type norms. Dispersive estimate offers sufficiently fast time decay in \(L^\infty\) norm to handle the time integral of nonlinear terms.

In particular, for the wave operators, we focus on controlling the perturbation globally using a combination of tools from Fourier and harmonic analysis (Fourier transform, dyadic decomposition), oscillatory integrals (non-stationary/stationary phase analysis), bilinear estimates (pseudo-products operators with singular multipliers), ODE and geometric methods (normal forms, vector fields), and delicately designed function spaces and decay-rate.

We carefully analyze the nonlinear term in the frequency space and bound the resonant and non-resonant interactions in an intricate manner.
2 Set-up and Overview of the Problem

2.1 Reformulation of the Equations: Duhamel’s Formula

Before formulating the problem and explaining the main ideas of the proof, we introduce the following notations: Throughout this paper the Fourier transform is defined as

\[ \hat{f}(\xi) = \mathcal{F}[f](\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) \, dx. \]  

(2.1)

Define the operators on \( \mathbb{R}^3 \)

\[ \Lambda_{wa} := |\nabla| = \mathcal{F}^{-1} |\xi| \mathcal{F}, \quad \Lambda_{kg} := \langle \nabla \rangle = \sqrt{|\nabla|^2 + 1} = \mathcal{F}^{-1} \langle \xi \rangle \mathcal{F}, \]

(2.2)

which relate to the dispersion relations for the wave and Klein-Gordon equation, respectively. We denote also the corresponding multipliers in frequency space by

\[ \Lambda_{wa,+}(\xi) = \Lambda_{wa}(\xi) := |\xi|, \quad \Lambda_{kg,+}(\xi) = \Lambda_{kg}(\xi) := \langle \xi \rangle = \sqrt{|\xi|^2 + 1}, \]

\[ \Lambda_{wa,-}(\xi) = -\Lambda_{wa,+}(\xi), \quad \Lambda_{kg,-}(\xi) = -\Lambda_{kg,+}(\xi). \]

(2.3)

The normalized solutions \( U^{wa}, U^{kg} \) (complex-valued) of the system (2.2) are defined by

\[ U^{wa}(t) := [\partial_t - i\Lambda_{wa}] u(t), \quad U^{kg}(t) := [\partial_t - i\Lambda_{kg}] v(t), \]

(2.4)

along with

\[ U^{wa,+} := U^{wa}, \quad U^{kg,+} := U^{kg}, \]

\[ U^{wa,-} := U^{wa} = [\partial_t + i\Lambda_{wa}] u(t), \quad U^{kg,-} := U^{kg} = [\partial_t + i\Lambda_{kg}] v(t), \]

(2.5)

so that the solutions \( u, v \) can be recovered from \( U^{wa}, U^{kg} \) by the formulas

\[ \partial_t u = \frac{1}{2} \left( U^{wa} + U^{wa} \right) = \text{Re} \left( U^{wa} \right), \quad \partial_t v = \frac{1}{2} \left( U^{kg} + U^{kg} \right) = \text{Re} \left( U^{kg} \right), \]

\[ \Lambda_{wa} u = \frac{1}{2} \left( U^{wa} - U^{wa} \right) = -\text{Im} \left( U^{wa} \right), \quad \Lambda_{kg} v = \frac{1}{2} \left( U^{kg} - U^{kg} \right) = -\text{Im} \left( U^{kg} \right), \]

(2.6)

\[ u = \frac{1}{2\Lambda_{wa}} \left( U^{wa} - U^{wa} \right) = -\frac{1}{|\nabla|} \text{Im} \left( U^{wa} \right), \quad v = \frac{1}{2\Lambda_{kg}} \left( U^{kg} - U^{kg} \right) = -\frac{1}{\langle \nabla \rangle} \text{Im} \left( U^{kg} \right). \]

Conjugating with the linear propagator operator \( e^{it\Lambda} := \mathcal{F}^{-1} e^{it\Lambda(\xi)} \mathcal{F} \), we define the associated (linear) profiles \( V^{wa}, V^{kg} \) by

\[ V^{wa}(t) := e^{it\Lambda_{wa}} U^{wa}(t), \quad V^{kg}(t) := e^{it\Lambda_{kg}} U^{kg}(t), \]

(2.7)

and denote also

\[ V^{wa,+} := V^{wa} = e^{it|\nabla|} U^{wa}(t), \quad V^{kg,+} := V^{kg} = e^{it\langle \nabla \rangle} U^{kg}(t), \]

\[ V^{wa,-} := V^{wa} = e^{-it|\nabla|} U^{wa}(t), \quad V^{kg,-} := V^{kg} = e^{-it\langle \nabla \rangle} U^{kg}(t), \]

(2.8)

which under the Fourier transform become

\[ \hat{V}^{wa,+}(t, \xi) = e^{it|\xi|} \hat{U}^{wa}(t, \xi), \quad \hat{V}^{kg,+}(t, \xi) = e^{it|\xi|} \hat{U}^{kg}(t, \xi), \]

\[ \hat{V}^{wa,-}(t, \xi) = e^{-it|\xi|} \hat{U}^{wa}(t, \xi), \quad \hat{V}^{kg,-}(t, \xi) = e^{-it|\xi|} \hat{U}^{kg}(t, \xi). \]

(2.9)
Remark 2.1. The above definition yields that
\[
\hat{V}_{wa}^{-}(t, \xi) = \hat{V}_{wa}^{+}(t, \xi) = \hat{V}_{wa}^{+}(t, - \xi), \quad \hat{V}_{kg}^{-}(t, \xi) = \hat{V}_{kg}^{+}(t, \xi) = \hat{V}_{kg}^{+}(t, - \xi). \tag{2.10}
\]

Now we reformulate the system (1.2) to derive the Duhamel’s formulas for \( \hat{V}_{wa} \) and \( \hat{V}_{kg} \):

Observing that both equations in (1.2) are of hyperbolic and dispersive type with quadratic nonlinearities, we first write them in terms of the normalized variables \( U_{wa}, U_{kg} \) as quadratic dispersive equations of the form
\[
(\partial_t + i\Lambda_{wa}) U_{wa} = N_{wa} = Q_{wa}(v, v) := |\nabla_{t,x} v|^2 + v^2, \tag{2.11}
\]
\[
(\partial_t + i\Lambda_{kg}) U_{kg} = N_{kg} = Q_{kg}(u, v) := u\Delta v,
\]
where the (quadratic) nonlinearities can be expressed as pseudo-product operators
\[
N_{wa} = T_{a[\xi, \eta]}(v, v) = \sum_{\pm, \pm} T_{a_{\pm \pm}}(U_{kg, \pm}, U_{kg, \pm}),
\]
\[
N_{kg} = T_{b[\xi, \eta]}(u, v) = \sum_{\pm, \pm} T_{b_{\pm \pm}}(U_{kg, \pm}, U_{wa, \pm})
\]
of the general form
\[
T_{m[\xi, \eta]}(f, g) := \mathcal{F}^{-1}\mathcal{Q}(f, g) = \mathcal{F}^{-1}\int m(\xi, \eta)\hat{f}(\xi - \eta)\hat{g}(\eta) \, d\eta
\]
with the symbol-type multiplier \( m(\xi, \eta) \) carrying information of the quadratic interaction (e.g., dispersion/wave-type, derivatives).

Since the normalized solutions \( U_{wa}, U_{kg} \) display oscillations itself, and we want to isolate all the oscillations in a unique factor \( \exp[i\Phi(\xi, \eta)] \), we then need to introduce the profiles \( V_{wa}, V_{kg} \) as in (2.7) so that (2.11) can be rewritten in terms of the new unknowns as
\[
\partial_t V_{wa} = e^{it\Lambda_{wa}} N_{wa} := e^{it\Lambda_{wa}} \sum_{\pm, \pm} T_{a_{\pm \pm}}(e^{it\Lambda_{kg}} V_{kg, \pm}, e^{it\Lambda_{kg}} V_{kg, \pm}), \tag{2.13}
\]
\[
\partial_t V_{kg} = e^{it\Lambda_{kg}} N_{kg} := e^{it\Lambda_{kg}} \sum_{\pm, \pm} T_{b_{\pm \pm}}(e^{it\Lambda_{kg}} V_{kg, \pm}, e^{it\Lambda_{wa}} V_{wa, \pm}), \tag{2.14}
\]
from which we see that \( V_{wa}, V_{kg} \) evolve purely nonlinearly (which is more stable).

Taking the Fourier transform (combined with its differentiation and product-convolution properties) and using the formulas (2.6) and identities (2.9), we obtain
\[
\partial_t \hat{V}_{wa}(t, \xi) = \sum_{t_1, t_2 \in \{+, -\}} \mathbf{1}_{wa}^{t_1 t_2}[V_{kg, t_1}, V_{kg, t_2}](t, \xi), \tag{2.15}
\]
\[
\partial_t \hat{V}_{kg}(t, \xi) = \sum_{t_1, t_2 \in \{+, -\}} \mathbf{1}_{kg}^{t_1 t_2}[V_{kg, t_1}, V_{wa, t_2}](t, \xi), \tag{2.16}
\]
where
\[
\mathbf{1}_{wa}^{t_1 t_2}[F, G](t, \xi) := \frac{1}{4}(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{it\Phi_{wa}^{t_1 t_2}(\xi, \eta)} a_{t_1 t_2}(\xi, \eta) \hat{F}(t, \xi - \eta)\hat{G}(t, \eta) \, d\eta,
\]
\[
\Phi_{wa}^{t_1 t_2}(\xi, \eta) := \Lambda_{wa}(\xi) - \Lambda_{kg, t_1}(\xi - \eta) - \Lambda_{kg, t_2}(\eta) = |\xi| - t_1(\xi - \eta) - t_2(\eta), \tag{2.17}
\]
\[
a_{t_1 t_2}(\xi, \eta) := 1 + \frac{t_1 t_2}{\Lambda_{kg}(\xi - \eta)\Lambda_{kg}(\eta)}[(\xi - \eta) \cdot \eta - 1] = 1 + t_1 t_2 \frac{(\xi - \eta) \cdot \eta - 1}{(\xi - \eta) \cdot \eta}. 
\]
and
\[ \Gamma_{kg}^{(1)}[F,G](t,\xi) := \frac{1}{4}(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{it\Phi_{kg}^{(1)}(\xi,\eta)} b_{1i12}(\xi,\eta) \hat{F}(t,\xi-\eta)\hat{G}(t,\eta) \, d\eta, \]

\[ \Phi_{kg}^{(1)}(\xi,\eta) := \Lambda_{kg}(\xi) - \Lambda_{kg,1}(\xi-\eta) - \Lambda_{wa,2}(\eta) \]

\[ = \langle \xi \rangle - t_1 \langle \xi - \eta \rangle - t_2 |\eta|, \]

\[ b_{1i12}(\xi,\eta) := t_{112} \frac{|\xi - \eta|^2}{\Lambda_{kg}(\xi - \eta)\Lambda_{wa}(\eta)} = t_{112} \frac{|\xi - \eta|^2}{\langle \xi - \eta \rangle |\eta|}. \]  

(2.18)

Our analysis is largely based on the nonlinear phases \( \Phi_{kg}^{(1)} \), which measure the quadratic interactions between different wave-types.

### 2.2 Definition and Notation

Throughout the paper, \( C \) will generally denote a universal constant that may vary from line to line. The notation \( A \lesssim B \) means that \( A \leq CB \) for some universal constant \( C > 0 \); we will use \( \gtrsim \) and \( \simeq \) in a similar standard way.

#### 2.2.1 Vector Fields

The major component of our analysis relies on the energy estimates for solutions of the system \[ \Box. \] These energy estimates involve vector-fields, corresponding to the natural symmetries of the linearized equations.

We denote by \( \partial_0 := \partial_t \) and \( \partial_i := \partial_x \) for \( i = 1, 2, 3 \). Define the Lorentz vector fields \( \Gamma_j \) and the rotation vector fields \( \Omega_{jk} \),

\[ \Gamma_j := x_j \partial_t + t \partial_j, \quad \Omega_{jk} := x_j \partial_k - x_k \partial_j, \]  

(2.19)

for \( j, k = 1, 2, 3 \). These vector fields commute with both the wave operator \( -\Box \) and Klein-Gordon operator \( -\Box + 1 \). For any multi-index \( a = (a_1, a_2, a_3) \in \mathbb{N}^3 \), \( b = (b_1, b_2, b_3) \in \mathbb{N}^3 \), \( \alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^{1+3} \) with \( |a| = \sum_{j=1}^{3} a_j \), \( |b| = \sum_{j=1}^{3} b_j \), \( |\alpha| = \sum_{i=0}^{3} \alpha_i \), we define

\[ \Gamma^a := \Gamma_1^{a_1} \Gamma_2^{a_2} \Gamma_3^{a_3}, \quad \Omega^b := \Omega_1^{b_1} \Omega_2^{b_2} \Omega_3^{b_3}, \quad \partial^\alpha := \partial_0^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}. \]  

(2.20)

For any \( n \in \mathbb{N} \), define \( \mathcal{V}_n \) as the set of differential operators of the form

\[ \mathcal{V}_n := \left\{ \mathcal{L} = \Gamma^a \Omega^b \partial^\alpha : |a| + |b| + |\alpha| \leq n \right\}. \]  

(2.21)

In particular, define \( U_{\mathcal{L}}^{wa} \) and \( U_{\mathcal{L}}^{kg} \) similar to \( U^{wa} \) and \( U^{kg} \), but applied to \( \mathcal{L}[u] \) instead of \( u \). The same notation applies to all the other variables, like \( V_{\mathcal{L}}^{wa} \) and \( V_{\mathcal{L}}^{kg} \):

\[ U_{\mathcal{L}}^{wa}(t) := (\partial_t - i\Lambda_{wa})(\mathcal{L}u)(t), \quad U_{\mathcal{L}}^{kg}(t) := (\partial_t - i\Lambda_{kg})(\mathcal{L}v)(t), \]

\[ V_{\mathcal{L}}^{wa}(t) := e^{it\Lambda_{wa}} U_{\mathcal{L}}^{wa}(t), \quad V_{\mathcal{L}}^{kg}(t) := e^{it\Lambda_{kg}} U_{\mathcal{L}}^{kg}(t). \]  

(2.22)
Similarly, the functions $Lu, Lv$ can be recovered from the normalized variables $U^{wa}_L, U^{kg}_L$ by the formulas
\[
\partial_t(Lu) = \frac{1}{2}(U^{wa}_L + \overline{U^{wa}_L}) = \text{Re}(U^{wa}_L), \quad \partial_t(Lv) = \frac{1}{2}(U^{kg}_L + \overline{U^{kg}_L}) = \text{Re}(U^{kg}_L),
\]
\[
\Lambda_{wa}(Lu) = \frac{1}{2}(U^{wa}_L - \overline{U^{wa}_L}) = -\text{Im}(U^{wa}_L), \quad \Lambda_{kg}(Lv) = \frac{1}{2}(U^{kg}_L - \overline{U^{kg}_L}) = -\text{Im}(U^{kg}_L),
\]
\[
Lu = \frac{1}{2N_{wa}}(U^{wa}_L - \overline{U^{wa}_L}) = -\frac{1}{|\nabla|} \text{Im}(U^{wa}_L), \quad Lv = \frac{1}{2N_{kg}}(U^{kg}_L - \overline{U^{kg}_L}) = -\frac{1}{|\nabla|} \text{Im}(U^{kg}_L).
\]

With the definitions above, The system (1.2)/(2.11) gives, for any $L \in \mathcal{V}_n$,
\[
(\partial_t + i\Lambda_{wa})U^{wa}_L = \Lambda^{wa}_L := L[|\nabla_txt|^2 + v^2],
\]
\[
(\partial_t + i\Lambda_{kg})U^{kg}_L = \Lambda^{kg}_L := L[u\Delta v].
\]

### 2.2.2 Littlewood–Paley Projections

In order to make the following analysis more precise, we need a good way of localization in Fourier space and physical space, which then allows us to parameterize/quantify the variables in terms of time.

We first use a standard (inhomogeneous) dyadic decomposition of the indicator function $1_{[0, +\infty)}$ to localize in $t$: Fix a smooth cutoff function $\tau : \mathbb{R}_+ \to [0, 1]$ supported in $[0, 2]$ and equal to 1 in $[0, 1]$, and define for any $m \in \mathbb{N}$,
\[
\tau_m(t) := \tau(t/2^m) - \tau(t/2^{m-1}), \quad m \geq 1;
\]
\[
\tau_0(t) := 1 - \sum_{m=1}^{+\infty} \tau_m(t) = \tau(t),
\]
so that the sequence of functions $\{\tau_m(t)\}_{m \in \mathbb{N}}$ has the properties
\[
\text{supp} \tau_0 \subseteq [0, 2], \quad \text{supp} \tau_m \subseteq [2^{-m}, 2^{-m+1}] \quad (m \geq 1), \quad \sum_{m=0}^{+\infty} \tau_m(t) = 1. \tag{2.26}
\]

In other words, in the support of $\tau_m$ we have $t \approx 2^m$. Also, we choose the function $\tau$ in such a way that $|\tau_m'(t)| \lesssim 2^{-m}$.

Now we define the (homogeneous) dyadic decomposition\(^\ddagger\) in three dimensions: Fix a smooth radial cutoff function $\varphi : \mathbb{R}^3 \to [0, 1]$ that equals 1 for $|z| \leq 1$ and vanishes for $|z| \geq 2$. For any $k \in \mathbb{Z}$, denote by $k^+ := \max(k, 0)$ and $k^- := \min(k, 0)$. Let
\[
\varphi_k(z) := \varphi(z/2^k) - \varphi(z/2^{k-1}), \quad k \in \mathbb{Z}
\]
be a sequence of functions with the properties
\[
\text{supp} \varphi_k \subseteq \left\{ z \in \mathbb{R}^3 : |z| \in [2^{k-1}, 2^{k+1}] \right\}, \quad \sum_{k \in \mathbb{Z}} \varphi_k(z) \equiv 1 \quad (z \in \mathbb{R}^3 \setminus \{0\}). \tag{2.28}
\]

\(^\ddagger\)The homogeneous dyadic decomposition differs from the inhomogeneous one in that it decomposes also near the origin.
so in the support of $\varphi_k$ we have $|z| \approx 2^k$. We define also

$$\varphi_I(z) := \sum_{k \in I \cap \mathbb{Z}} \varphi_k(z) \quad \text{for any } I \subset \mathbb{R}$$

$$\varphi_{\leq A} := \varphi(-\infty, A], \quad \varphi_{\geq A} := \varphi[A, +\infty), \quad \varphi_{< A} := \varphi(-\infty, A), \quad \varphi_{> A} := \varphi(A, +\infty) \quad \text{for any } A \in \mathbb{R}$$

Let

$$\mathcal{J} := \{(k, j) \in \mathbb{Z} \times \mathbb{Z}_+: k + j \geq 0 \text{ i.e. } j \geq -k^-\}.$$ 

Given any $k \in \mathbb{Z}$, for any $(k, j) \in \mathcal{J}$, we further define

$$\varphi_j^{(k)} := \begin{cases} \varphi_j & j > -k^-; \\ \varphi_{\leq j} & j = -k^-; \end{cases} \quad \text{(2.29)}$$

and notice that, for any $k \in \mathbb{Z}$ fixed,

$$\sum_{j \geq -k^-} \varphi_j^{(k)}(z) \equiv 1 \quad (z \in \mathbb{R}^3 \backslash \{0\}).$$

**Definition 2.1** (Littlewood–Paley Projections). (1) Let $P_k$ ($k \in \mathbb{Z}$) denote the standard Littlewood–Paley projection on $\mathbb{R}^3$ defined by the Fourier multiplier $\xi \mapsto \varphi_k(\xi)$:

$$P_k := \mathcal{F}^{-1} \varphi_k(\xi) \mathcal{F}, \quad \hat{P_k}f(\xi) := \varphi_k(\xi) \hat{f}(\xi). \quad \text{(2.30)}$$

Similarly, for any $A \in \mathbb{R}$, define the operator $P_{\leq A}$ (resp. $P_{> A}$) on $\mathbb{R}^3$ by the Fourier multiplier $\xi \mapsto \varphi_{\leq A}(\xi)$ (resp. $\xi \mapsto \varphi_{> A}(\xi)$).

(2) For $(k, j) \in \mathcal{J}$, let the operator $Q_{jk}$ be defined as

$$Q_{jk}f(x) := \varphi_j^{(k)} \cdot P_k f(x). \quad \text{(2.31)}$$

(3) For $(k, j) \in \mathcal{J}$, let the operator $Q_{jk}$ be defined as

$$Q_{jk}f(x) := P_{[k-2, k+2]} \left\{ \varphi_j^{(k)} \cdot P_k f \right\}(x). \quad \text{(2.32)}$$

(4) Furthermore, let

$$Q_{\leq j}f = \sum_{-k^- \leq j' \leq j} Q_{j'k}f, \quad \text{(2.33)}$$

$$Q_{\geq j}f = \sum_{-k^- \leq j' \leq j} Q_{j'k}f. \quad \text{(2.34)}$$

**Remark 2.2.** (i) In our case we have $\Lambda(\xi) = \omega(|\xi|)$, so we will focus on quantifying the “energy” $|\xi|$. The direction is also important, but can usually be recovered by commuting with the rotation vector-fields.

(ii) In view of the Heisenberg’s uncertainty principle\footnote{It states that the more precisely the position (in physical space) of some particle is determined, the less precisely its momentum (frequency) can be known, and vice versa.}, the operators $Q_{jk}$ are relevant only when $2^j 2^k \gtrsim 1$ i.e. $j + k \gtrsim 0$, which explains the definitions above.
(iii) The Littlewood–Paley operators give a decomposition of the identity (at least formally):

\[
\sum_{k \in \mathbb{Z}} P_k = I, \quad \sum_{k \in \mathbb{Z}} P_k f = f;
\]

\[
\sum_{j \geq -k^*} Q_{jk} = P_k, \quad \sum_{(k,j) \in \mathcal{J}} Q_{jk} f = \sum_{k \in \mathbb{Z}} P_k f = f; \tag{2.35}
\]

\[
\sum_{j \geq -k^*} \mathcal{Q}_{jk} = P_k, \quad \sum_{(k,j) \in \mathcal{J}} \mathcal{Q}_{jk} f = \sum_{k \in \mathbb{Z}} P_k f = f.
\]

2.2.3 Norms

We postulate that the nonlinear system and resonant system have the common convergent asymptotic behavior. On the other hand, they may have different initial data at \( t = 0 \).

In the following, let

\[
N_0 := 40, \quad N_1 := 3, \quad d := 10, \quad \delta = 10^{-10}
\]

\[
N(0) := N_0, \quad N(n) := N_0 - dn \text{ for } n \geq 1,
\]

\[
H(0) := 800, \quad H(n) := 800 - 200n \text{ for } n \geq 1,
\]

and

\[
H'_{wa}(n) = H(n + 1), \quad H''_{ka}(n) = H(n + 1),
\]

\[
N''_{wa}(n) = N(n) - 5, \quad N''_{ka}(n) = N(n) - 5.
\]

We define the norms characterizing the time-independent data (designed for the initial data of the resonant system). Assume \( f(x) \) is a function of certain regularity. \( f_L = \mathcal{L}[f] \) with vector field \( \mathcal{L} \) applied to \( f \). Then, define the “data spaces” (for the initial data) with the norms

\[
\|f\|_{Y_i} = \sup_{n \leq N_i + 2} \sup_{\mathcal{L} \in \mathcal{V}_n} \left\| \left| \nabla \right|^{-\frac{1}{2}} f_L \mathcal{L} \right\|_{H^{N(n-3)}}
\]

\[
+ \sup_{n \leq N_i + 1} \sup_{\mathcal{L} \in \mathcal{V}_n} \sup_{\ell \in \{1,2,3\}} \left( 2^{N(n-2)k^*} 2^{\frac{k^*}{2}} \left\| \varphi_k(\xi) \partial_{\xi_j} f_L \mathcal{L} \right\|_{L^2_\xi} \right),
\]

\[
\|f\|_{Y_2} = \sup_{n \leq N_1 + 2} \sup_{\mathcal{L} \in \mathcal{V}_n} \|f_L\|_{H^{N(n-3)}}
\]

\[
+ \sup_{n \leq N_1 + 1} \sup_{\mathcal{L} \in \mathcal{V}_n} \sup_{\ell \in \{1,2,3\}} \left( 2^{N(n-2)k^*} 2^{k^*} \left\| \varphi_k(\xi) \partial_{\xi_j} f_L \mathcal{L} \right\|_{L^2_\xi} \right).
\]

For \( i = 1,2 \), let

\[
Y_i := \{ f : \|f\|_{Y_i} < \infty \}. \tag{2.38}
\]

Then we define the time-dependent norms. Assume \( f(t,x) \) is a function of certain regularity. \( f_L = \mathcal{L}[f] \) with vector field \( \mathcal{L} \) applied to \( f \).

Define energy norms:

\[
\|f\|_{S_1} = \sup_{t \in [T,\infty)} \sup_{n \leq N_1} \sup_{\mathcal{L} \in \mathcal{V}_n} \left( \langle t \rangle^{H(n)\delta} \left\| \left| \nabla \right|^{-\frac{1}{2}} e^{-itA_{wa}} f_L \mathcal{L} \right\|_{H^{N(n)}} \right),
\]

\[
\|f\|_{T_1} = \sup_{t \in [T,\infty)} \sup_{n \leq N_1 - 1} \sup_{\mathcal{L} \in \mathcal{V}_n} \sup_{\ell \in \{1,2,3\}} \sup_{k \in \mathbb{Z}} \left( \langle t \rangle^{H(n+1)\delta} 2^{N(n+1)k^*} 2^{k^*} \left\| \varphi_k(\xi) \partial_{\xi_j} f_L \mathcal{L} \right\|_{L^2_\xi} \right), \tag{2.39}
\]

\[
\|f\|_{T_2} = \sup_{t \in [T,\infty)} \sup_{n \leq N_1 - 1} \sup_{\mathcal{L} \in \mathcal{V}_n} \sup_{\ell \in \{1,2,3\}} \sup_{k \in \mathbb{Z}} \left( \langle t \rangle^{H(n+1)\delta} 2^{N(n+1)k^*} 2^{k^*} \left\| \varphi_k(\xi) \partial_{\xi_j} f_L \mathcal{L} \right\|_{L^2_\xi} \right), \tag{2.40}
\]
and
\[
\|f\|_{S_2} = \sup_{t \in [T, \infty)} \sup_{n \leq N_1} \sup_{L \in \mathcal{V}_n} \left( \langle t \rangle H(n)^{\delta} \| e^{-it \Lambda_k} f \|_{H^{N(n)}} \right),
\]
(2.41)
\[
\|f\|_{T_2} = \sup_{t \in [T, \infty)} \sup_{n \leq N_1} \sup_{L \in \mathcal{V}_n} \sup_{k \in \mathbb{Z}} \left( \langle t \rangle H(n) 2^{N(n+1)k+} 2^k \| \varphi_k(\xi) \partial_{\xi} \hat{f}_L \|_{L^2_\xi} \right).
\]
(2.42)

We define a variation of energy norms:
\[
\|f\|_{S'_i} = \sup_{t \in [T, \infty)} \sup_{n \leq N_1} \sup_{L \in \mathcal{V}_n} \left( \langle t \rangle (1 + H''_{wa}(n)) 2^{N''_{wa}(n)}k+ 2^{-\frac{1}{2}}k \| \varphi_k(\xi) \hat{f}_L \|_{L^2_\xi} \right),
\]
(2.43)
\[
\|f\|_{T'_i} = \sup_{t \in [T, \infty)} \sup_{n \leq N_1} \sup_{L \in \mathcal{V}_n} \sup_{k \in \mathbb{Z}} \left( \langle t \rangle H''_{wa}(n) 2^{N''_{wa}(n)}k+ 2^{-\frac{1}{2}}k \| \varphi_k(\xi) \partial_{\xi} \left( e^{-it \Lambda_{wa}(\xi)} \hat{f}_L \right) \|_{L^2_\xi} \right),
\]
(2.44)

and
\[
\|f\|_{S'_2} = \sup_{t \in [T, \infty)} \sup_{n \leq N_1} \sup_{L \in \mathcal{V}_n} \left( \langle t \rangle (1 + H''_{kg}(n)) 2^{N''_{kg}(n)}k+ \| \varphi_k(\xi) \hat{f}_L \|_{L^2_\xi} \right),
\]
(2.45)
\[
\|f\|_{T'_2} = \sup_{t \in [T, \infty)} \sup_{n \leq N_1} \sup_{L \in \mathcal{V}_n} \sup_{k \in \mathbb{Z}} \left( \langle t \rangle H''_{kg}(n) 2^{N''_{kg}(n)}k+ \| \varphi_k(\xi) \partial_{\xi} \left( e^{-it \Lambda_{kg}(\xi)} \hat{f}_L \right) \|_{L^2_\xi} \right).
\]
(2.46)

Here \( n(N_1 + 1) = n(N_1) - d \).

\textbf{Remark 2.3}. The norms \( S_i, T_i \) \((i = 1, 2)\) are used to characterize the difference between the W-KG system and the “resonant system”, while the norms \( S'_i, T'_i \) \((i = 1, 2)\) are used to estimate the time derivative of the difference.

Finally, define the “working spaces” (to run the contraction/fixed-point argument) with the norms
\[
\|f\|_{X_1} = \|f\|_{S_1} + \|f\|_{T_1} + \|\partial_t f\|_{S'_1} + \|\partial_t f\|_{T'_1},
\]
(2.47)
\[
\|f\|_{X_2} = \|f\|_{S_2} + \|f\|_{T_2} + \|\partial_t f\|_{S'_2} + \|\partial_t f\|_{T'_2}.
\]
(2.48)

For \( i = 1, 2 \), let
\[
X_i := \{ f : \|f\|_{X_i} < \infty \}.
\]
(2.49)

\subsection{Resonant System}

We start from the Duhamel’s formulas (2.17) and (2.18). We will mainly present the ideas on the derivation of the resonant system.
2.3.1 Analysis of the Quadratic Phase Functions

Let
\[ \Phi_{\sigma\mu\nu} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (2.50) \]
\[ \Phi_{\sigma\mu\nu}(\xi, \eta) := \Lambda_{\sigma}(\xi) - \Lambda_{\mu}(\xi - \eta) - \Lambda_{\nu}(\eta), \]
where
\[ \sigma, \mu, \nu \in \mathcal{P} := \{(wa, +), (wa, -), (kg, +), (kg, -)\}. \quad (2.51) \]

Thus the resonant/stationary-phase analysis is based on essentially only one type of quadratic phase
\[ \Phi(\xi_1, \xi_2) = \Lambda_{kg}(\xi_1) \pm \Lambda_{kg}(\xi_2) \pm \Lambda_{wa}(\xi_1 + \xi_2) \quad (2.52) \]

which satisfies the bound
\[ |\Phi| \gtrsim \frac{(|\xi_1 + \xi_2|)}{(1 + |\xi_1| + |\xi_2|)^2}. \quad (2.53) \]

Based on (2.17) and (2.18), if \(|\xi|, |\xi - \eta|, |\eta| \in [0, b]\) for \(b \geq 1\), then
\[ |\Phi_{wa}^{112}(\xi, \eta)| \geq \frac{|\xi|}{4b^2}, \quad |\Phi_{kg}^{112}(\xi, \eta)| \geq \frac{|\eta|}{4b^2}. \quad (2.54) \]

Therefore, we expect that the interactions where the wave component has low frequency, in particular when \(t|\xi| \lesssim 1\) or \(t|\eta| \lesssim 1\), will play an important role in the analysis.

2.3.2 Analysis of the (Space-Time) Resonances

Define the resonances sets:
- Time resonance set: \(\mathcal{T} := \{(\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3 : \Phi(\xi, \eta) = 0\}\), i.e. stationary over time \(s\).
- Space resonance set: \(\mathcal{I} := \{(\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3 : \nabla_\eta \Phi(\xi, \eta) = 0\text{ or not well-defined}\}\), i.e. stationary in \(\eta\).
- Space-Time resonance set: \(\mathcal{R} := \mathcal{T} \cap \mathcal{I}\), i.e. stationary in both \(s\) and \(\eta\).

By (2.51), (2.17) and (2.18), we can verify that for \(\Phi_{wa}^{112}\):
\[
\begin{cases}
++,-+ & \mathcal{R} = \mathcal{T} = \emptyset \subset \mathcal{I} = \{\xi = 2\eta\}, \\
+,-++ & \mathcal{R} = \mathcal{T} = \mathcal{I} = \{\xi = 0\},
\end{cases} \quad (2.55)
\]

and for \(\Phi_{kg}^{112}\):
\[
\begin{cases}
-++,-- & \mathcal{R} = \mathcal{T} = \emptyset \subset \mathcal{I} = \{\eta = 0\}, \\
++,++ & \mathcal{R} = \mathcal{T} = \mathcal{I} = \{\eta = 0\}.
\end{cases} \quad (2.56)
\]

We call \(\mathcal{R} = \mathcal{T} = \emptyset\) the non-stationary case, which can be handled using the normal form method (integration by parts in time). Otherwise, we call it the stationary case.
2.3.3 Extraction of the “Resonant System”

**Step 1. Low-Frequency Outputs: “Low-Frequency Bulk Term” for Wave-Component.**

We start from (2.15) with \( \iota \) and focus on the cases +− and −+ (when the two inputs have the opposite signs) and look at the contribution of low-frequency outputs:

\[
|\xi| \lesssim \langle t \rangle^{-1+}.
\]

(2.57)

For convenience, we may take the cut-off function

\[
\varphi_{\leq 0}(\xi \langle t \rangle^{\frac{7}{8}}).
\]

(2.58)

We start from (2.15) with \((\iota_1, \iota_2) = (+, -) \) or \((-, +) \) when \(|\xi|\) is small. By Taylor expansion with respect to \(\xi\) up to the first order (here \(\iota_2 = -\iota_1\)),

\[
\Phi^{\iota_1 \iota_2}_{wa}(\xi, \eta) = |\xi| - \iota_1 (\xi - \eta) - \iota_2 (\eta) = |\xi| - \iota_1 \{ (\xi - \eta) - \langle \eta \rangle \}
\]

(2.59)

\[
\rightarrow |\xi| + \iota_1 \frac{\xi \cdot \eta}{\langle \eta \rangle} =: \Phi^{\iota_1 \iota_2, 0}_{wa}(\xi, \eta).
\]

On the other hand, taking \(\xi = 0\), we get

\[
a_{\iota_1 \iota_2}(\xi, \eta) \to 2,
\]

(2.60)

\[
\hat{F}(t, \xi - \eta) \to \hat{F}(t, -\eta).
\]

(2.61)

Define

\[
I_{wa}^{+, 0}(t, \xi) := \frac{1}{2} (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{i t \Phi^{+, 0}_{wa}(\xi, \eta)} V_{kg, +}(t, -\eta) V_{kg, -}(t, \eta) \, d\eta,
\]

(2.62)

\[
I_{wa}^{-, 0}(t, \xi) := \frac{1}{2} (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{i t \Phi^{-, 0}_{wa}(\xi, \eta)} V_{kg, -}(t, -\eta) V_{kg, +}(t, \eta) \, d\eta.
\]

(2.63)

By change of variable \(\eta \to -\eta\) in \(I_{wa}^{+, 0}(t, \xi)\) and noticing that \(V_{kg, -}(t, -\eta) = V_{kg, +}(t, \eta)\), we have

\[
I_{wa}^{+, 0}(t, \xi) = \frac{1}{2} (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{i t (|\xi| - \frac{2\pi}{(\xi, \eta)})} \left| V_{kg, +}(t, \eta) V_{kg, -}(t, -\eta) \right|^2 \, d\eta,
\]

(2.64)

\[
I_{wa}^{-, 0}(t, \xi) = \frac{1}{2} (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{i t (|\xi| - \frac{2\pi}{(\xi, \eta)})} \left| V_{kg, -}(t, \eta) V_{kg, +}(t, -\eta) \right|^2 \, d\eta.
\]

(2.65)

Define

\[
h(t, \xi) := \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{7}{8}}) \cdot \left\{ I_{wa}^{+, 0}(t, \xi) + I_{wa}^{-, 0}(t, \xi) \right\}
\]

(2.66)

\[
= (2\pi)^{-\frac{3}{2}} \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{7}{8}}) \cdot \int_{\mathbb{R}^3} e^{i t (|\xi| - \frac{2\pi}{(\xi, \eta)})} \left| V_{kg}(t, \eta) \right|^2 \, d\eta,
\]

and

\[
H(t, \xi) := \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{7}{8}}) \cdot \left( V_{wa}(0, \xi) + \int_0^t h(s, \xi) \, ds \right)
\]

(2.67)

\[
= (2\pi)^{-\frac{3}{2}} \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{7}{8}}) \cdot \left( V_{wa}(0, \xi) + \int_0^t \int_{\mathbb{R}^3} e^{i s (|\xi| - \frac{2\pi}{(\xi, \eta)})} \left| V_{kg}(s, \eta) \right|^2 \, d\eta \, ds \right).
\]
**Remark 2.4.** Roughly speaking, \( H(t) \sim \hat{V}^{wa}(t) \) and
\[
\partial_t \hat{V}^{wa}(t, \xi) \approx h(t, \xi) + R^{wa}. \tag{2.68}
\]
Here \( R^{wa} \) should be a sufficiently small quantity in the long run.

**Step 2.** **High-Low Interactions: "Phase-Correction/Shift" for KG-Component.**

The next step is to measure the feedback contribution of the low-frequency bulk term \( H(t, \xi) \) to the nonlinear interactions for KG.

Based on (2.56), we mainly focus on the cases ++ and +− and look at the high-low interactions (of high-frequency outputs):
\[
|\eta| \lesssim (t)^{-1+} (\ll |\xi|). \tag{2.69}
\]

For convenience, we may take the cut-off function
\[
\varphi_{\xi} \left( \eta \left( t \right) \right) \tag{2.70}
\]
We start from (2.16) with \(( \iota_1, \iota_2 ) = ( +, + ) \) or \(( +, - ) \) when \(|\eta|\) is small. By Taylor expansion with respect to \( \eta \) up to the first order,
\[
\Phi_{k^g}^{t+2}(\xi, \eta) = \left\{ \langle \xi \rangle - \langle \xi - \eta \rangle - \iota_2 |\eta| = \{ \langle \xi \rangle - \langle \xi - \eta \rangle \} - \iota_2 |\eta| \right\} \rightarrow \frac{\xi \cdot \eta}{\langle \xi \rangle} - \iota_2 |\eta| =: \Phi_{k^g}^{t+2,0}(\xi, \eta). \tag{2.71}
\]
On the other hand, taking \( \eta = 0 \), we get
\[
b_{t+2}(\xi, \eta) \rightarrow \iota_2 \frac{|\xi|^2}{\langle \xi \rangle} \frac{1}{|\eta|}, \tag{2.72}
\]
We formally replace the low-frequency part of \( \hat{V}^{wa, \pm}(t, \eta) \) by the contribution coming from \( H^\pm(t, \eta) \), where \( H^+(t, \eta) = H(t, \eta) = \hat{H}^-(t, -\eta) \) as in (2.67). Define
\[
I_{k^g,0}^{t+2,1}(t, \xi) := \frac{1}{4} (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{it \Phi_{k^g}^{t+2,0}(\xi, \eta) \iota_2} |\xi|^2 \frac{1}{|\eta|} \hat{V}^{k^g}(t, \xi) H^{t+2}(t, \eta) d\eta, \tag{2.74}
\]
By change of variable \( \eta \rightarrow -\eta \) in \( I_{k^g,0}^{t+2,1}(t, \xi) \), we have
\[
I_{k^g,0}^{-1}(t, \xi) = -\frac{1}{4} (2\pi)^{-\frac{3}{2}} \frac{|\xi|^2}{|\xi|} \hat{V}^{k^g}(t, \xi) \int_{\mathbb{R}^3} e^{-it \left( \frac{\xi \cdot \eta}{|\xi|} - |\eta| \right)} \frac{1}{|\eta|} H^-(t, \eta) d\eta \tag{2.75}
\]
Hence,
\[
I_{k^g,0}^{t+2,1}(t, \xi) + I_{k^g,0}^{-1}(t, \xi) = \frac{1}{2} (2\pi)^{-\frac{3}{2}} \frac{|\xi|^2}{|\xi|} \hat{V}^{k^g}(t, \xi) \cdot \text{Im} \left\{ \int_{\mathbb{R}^3} e^{it \left( \frac{\xi \cdot \eta}{|\xi|} \right)} \frac{1}{|\eta|} H(t, \eta) d\eta \right\}. \tag{2.76}
\]
Define
\[
C(t, \xi) := \frac{1}{2} (2\pi)^{-\frac{3}{2}} \frac{|\xi|^2}{|\xi|} \cdot \text{Im} \left\{ \int_{\mathbb{R}^3} e^{it \left( \frac{\xi \cdot \eta}{|\xi|} \right)} \frac{1}{|\eta|} H(t, \eta) d\eta \right\}. \tag{2.77}
\]
Remark 2.5. Roughly speaking, (2.16) is
\[ \partial_t V^{kg}(t, \xi) \approx iC(t, \xi) \cdot \bar{V}^{kg}(t, \xi) + R^{kg}. \]  
(2.78)

Here \( R^{kg} \) should be a sufficiently small quantity in the long run. This shows that (the high frequency of) KG component is essentially transported by the low frequency part of the wave component. Eventually, we will renormalize \( V^{kg} \) to incorporate this modification via a phase correction using the fact that \( C(t, \xi) \in \mathbb{R} \). (This is a similar phenomenon to the one occurring in the scattering-critical equations, such as 1D cubic Schrödinger equation [5, 10] and 2D water wave [9].)

2.4 Equations of Perturbation

In this subsection, we will define the difference between the W-KG system and resonant system, and derive the Duhamel’s principle for this perturbation (difference) term.

Assume \( (V_{\infty}^{wa}, V_{\infty}^{kg}) \in Y_1 \times Y_2 \) are given functions. They are the initial data of the resonant system. Our goal is to show that for resonant system starting from \( (V_{\infty}^{wa}, V_{\infty}^{kg}) \), there exists a solution \( (V^{wa}, V^{kg}) \) to the W-KG system such that their difference goes to zero as \( t \to \infty \).

First, we represent the perturbation \( G^{wa}(t, x), G^{kg}(t, x) \) by
\[ \tilde{G}^{wa}(t, \xi) := V^{wa}(t, \xi) - \left( \widetilde{V}_{\infty}^{wa}(\xi) + \mathcal{H}_{\infty}(t, \xi) \right), \]  
(2.79)
\[ \tilde{G}^{kg}(t, \xi) := V^{kg}(t, \xi) - \left( e^{iD_{\infty}(t, \xi)} V^{kg}_{\infty}(\xi) + \mathfrak{B}_{\infty}(t, \xi) \right). \]  
(2.80)

Here
\[ h_{\infty}(t, \xi) := (2\pi)^{-\frac{3}{2}} \varphi_{\leq 0}(\xi(t)) \cdot \int_{\mathbb{R}^3} e^{i(\left|\xi| - \frac{\xi \eta}{\sqrt{|\eta|}}\right)} \left| \bar{V}^{kg}_{\infty}(\eta) \right|^2 d\eta, \]  
(2.81)
and \( \mathcal{H}_{\infty}, H_{\infty}, C_{\infty}, (C_{\infty}^{\pm}), D_{\infty} \) are defined accordingly:
\[ \mathcal{H}_{\infty}(t, \xi) := \int_0^t h_{\infty}(s, \xi) ds, \]  
(2.82)
\[ H_{\infty}(t, \xi) := \varphi_{\leq 0}(\xi(t)) \cdot \left\{ \widetilde{V}_{\infty}^{wa}(\xi) + \mathcal{H}_{\infty}(t, \xi) \right\}, \]  
(2.83)
\[ C_{\infty}(t, \xi) := \frac{1}{2} (2\pi)^{-\frac{3}{2}} \left| \xi \right|^2 \cdot \Im \left\{ \int_{\mathbb{R}^3} e^{i(\left|\xi| - |\eta|\right)} \left( \frac{1}{|\eta|} H_{\infty}(t, \eta) \right) d\eta \right\}, \]  
(2.84)
\[ D_{\infty}(t, \xi) := \int_0^t C_{\infty}(s, \xi) ds. \]  
(2.85)

Also,
\[ \mathfrak{B}_{\infty}(t, \xi) = -\int_t^\infty e^{i(D_{\infty}(t, \xi) - D_{\infty}(s, \xi))} b_{\infty}(s, \xi) ds, \]  
(2.86)
\[ b_{\infty}(t, \xi) = -\sum_{\epsilon_3 = \pm} \frac{1}{4} (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{i[(\xi + (\xi - \epsilon_3|\eta|) - \epsilon_3|\eta|)]} \left| \frac{\xi - \eta}{\sqrt{|\eta|}} \right|^2 \left( e^{iD_{\infty}V^{kg}_{\infty}} \right)(t, \xi - \eta) H_{\infty}^{\epsilon_3}(t, \eta) d\eta. \]  
(2.87)

Since we already know \( (V_{\infty}^{wa}, V_{\infty}^{kg}) \), to show the well-posedness of \( (V^{wa}, V^{kg}) \), it suffices to consider the well-posedness of \( (G^{wa}, G^{kg}) \).
Remark 2.6. Note two features of this definition:

1. Here, the infinity value should satisfy (in some norm)

\[
\lim_{t \to \infty} \hat{G}^{wa}(t, \xi) = 0, \quad \lim_{t \to \infty} \hat{G}^{kg}(t, \xi) = 0
\]  

(2.88)

Hence, our goal is to show \( \hat{G}^{wa} \) and \( \hat{G}^{kg} \) decay at certain speed.

2. In (2.79) and (2.80), inside the last large parenthesis is the solution of the resonant system starting from \( (V_{wa}^{\infty}, V_{kg}^{\infty}) \). Note that it is slightly different from the expression derived in last subsection. In particular, it contains a non-resonant contribution \( \mathcal{B}_\infty \) and \( b_\infty \). This is due to technical difficulties in nonlinear estimates (see Section 4). Roughly speaking, though non-resonant, this type of term decays pretty slow and will be captured by the asymptotic analysis.

Since \( \hat{V}^-(t, \xi) = \hat{V}(t, -\xi) \), we define \( \hat{G}^{wa,+}(t, \xi) := \hat{G}^{wa}(t, \xi) \) and \( \hat{G}^{wa,-}(t, \xi) := \hat{G}^{wa}(t, -\xi) \). The same convention also applies to \( \hat{G}^{kg}, \hat{V}_{wa}^{\infty}, \hat{V}_{kg}^{\infty}, h_\infty, \mathcal{H}_\infty, H_\infty, D_\infty, b \) and \( \mathcal{B} \). Therefore, we have

\[
\hat{V}^{wa,\pm}(t, \xi) = \hat{G}^{wa,\pm}(t, \xi) + \hat{V}^{wa,\pm}_{\infty}(t, \xi) + \mathcal{H}^{\pm}_{\infty}(t, \xi),
\]

(2.89)

\[
\hat{V}^{kg,\pm}(t, \xi) = \hat{G}^{kg,\pm}(t, \xi) + e^{\mp iD_\infty}(t, \xi)\hat{V}^{kg,\pm}_{\infty}(t, \xi) + \mathcal{B}^{\pm}_{\infty}(t, \xi).
\]

(2.90)

Inserting (2.89) and (2.90) into (2.15) and (2.16), we obtain the differential form for \( (G^{wa}, G^{kg}) \)

\[
\partial_t \hat{G}^{wa}(t, \xi) = -h_\infty(t, \xi)
\]

(2.91)

\[+ \sum_{\iota_1, \iota_2 \in \{\pm\}} I_{wa}^{\iota_1 \iota_2} [\hat{G}^{kg,\iota_1} + e^{iD_\infty}(t, \xi)\hat{V}^{kg,\iota_1}_{\infty} + \mathcal{B}^{\iota_1}_{\infty}, \hat{G}^{kg,\iota_2} + e^{iD_\infty}(t, \xi)\hat{V}^{kg,\iota_2}_{\infty} + \mathcal{B}^{\iota_2}_{\infty}] (t, \xi),
\]

and

\[
\partial_t \hat{G}^{kg}(t, \xi) = -iC_\infty(t, \xi) e^{iD_\infty}(t, \xi)\hat{V}^{kg}_{\infty}(\xi) - b_\infty(t, \xi) - iC_\infty(t, \xi)\mathcal{B}_\infty(t, \xi)
\]

(2.92)

\[+ \sum_{\iota_1, \iota_2 \in \{\pm\}} I_{kg}^{\iota_1 \iota_2} [\hat{G}^{kg,\iota_1} + e^{iD_\infty}(t, \xi)\hat{V}^{kg,\iota_1}_{\infty} + \mathcal{B}^{\iota_1}_{\infty}, \hat{G}^{wa,\iota_2} + \hat{V}^{wa,\iota_2}_{\infty} + \mathcal{H}^{\iota_2}_{\infty}] (t, \xi),
\]

with final data

\[
\hat{G}^{wa}(\infty, \xi) = 0, \quad \hat{G}^{kg}(\infty, \xi) = 0.
\]

(2.93)

Integrating the above from \( t \) to \( \infty \), we obtain the integral form

\[
\hat{G}^{wa}(t, \xi) = \int_t^\infty h_\infty(s, \xi) \, ds
\]

(2.94)

\[\quad - \sum_{\iota_1, \iota_2 \in \{\pm\}} \int_t^\infty I_{wa}^{\iota_1 \iota_2} [\hat{G}^{kg,\iota_1} + e^{iD_\infty}(s, \xi)\hat{V}^{kg,\iota_1}_{\infty} + \mathcal{B}^{\iota_1}_{\infty}, \hat{G}^{kg,\iota_2} + e^{iD_\infty}(s, \xi)\hat{V}^{kg,\iota_2}_{\infty} + \mathcal{B}^{\iota_2}_{\infty}] (s, \xi) \, ds,
\]
and

\[
\tilde{G}^{kg}(t, \xi) = \left( \int_{t}^{\infty} iC_{\infty}(s, \xi)e^{iD_{\infty}(s, \xi)} ds \right) \cdot \tilde{V}_{\infty}^{kg}(\xi) + \int_{t}^{\infty} \left( b_{\infty}(s, \xi) + iC_{\infty}(s, \xi)B_{\infty}(s, \xi) \right) ds
\]

\[
- \sum_{\iota_{1}, \iota_{2} \in \{+, -\}} \int_{t}^{\infty} \Gamma_{kg}^{\iota_{1}\iota_{2}} \left[ G^{\iota_{1}\iota_{2}} \right] \left( t, \xi \right) d\xi
\]

\[
\left[ \tilde{e}^{iD_{\infty}(s, \xi)} - \tilde{e}^{iD_{\infty}(t, \xi)} \right] \cdot \tilde{V}_{\infty}^{kg}(\xi) + \left( B_{\infty}(\xi) - B_{\infty}(t, \xi) \right)
\]

\[
- \sum_{\iota_{1}, \iota_{2} \in \{+, -\}} \int_{t}^{\infty} \Gamma_{kg}^{\iota_{1}\iota_{2}} \left[ G^{\iota_{1}\iota_{2}} \right] \left( t, \xi \right) d\xi.
\]

2.5 Main Theorem and Proposition

Provided that \( (V_{wa}^{kg}, V_{\infty}^{kg}) \in Y_{1} \times Y_{2} \) is given. Our goal is to justify that \( (2.94) \) and \( (2.95) \) can uniquely determine a solution \((G_{wa}^{kg}, G_{\infty}^{kg})\). Below is our main theorem.

**Theorem 2.1** (Main Theorem, Precise Version). There exists \( \varepsilon_{0} > 0 \), such that if the asymptotic profiles (scattering data) \( (V_{wa}^{kg}, V_{\infty}^{kg}) \in Y_{1} \times Y_{2} \) satisfies

\[
\|V_{wa}^{kg}\|_{Y_{1}} \lesssim \varepsilon_{0}, \quad \|V_{\infty}^{kg}\|_{Y_{2}} \lesssim \varepsilon_{0},
\]

then there exists a unique solution \((u, v)\) with \( t \in [0, \infty) \) of the system \( (2.2) \) with associated profiles in \( (2.7) \) such that \( (V_{wa}^{kg}, V_{\infty}^{kg}) \in X_{1} \times X_{2} \) and

\[
\left\| \tilde{V}_{wa}(t, \xi) - \left( \tilde{V}_{wa}(\xi) + H_{\infty}(t, \xi) \right) \right\|_{X_{1}} \lesssim \varepsilon_{0}^{3},
\]

and

\[
\left\| \tilde{V}_{\infty}^{kg}(t, \xi) - \left( \tilde{e}^{iD_{\infty}(t, \xi)} \tilde{V}_{\infty}^{kg}(\xi) + 2B_{\infty}(t, \xi) \right) \right\|_{X_{2}} \lesssim \varepsilon_{0}^{3},
\]

where \( H_{\infty}, D_{\infty} \) and \( B_{\infty} \) are defined in \( (2.82), (2.85), (2.86) \).

We plan to apply the fixed-point argument. Based on \( (2.94) \) and \( (2.95) \), define a mapping

\[
\mathcal{T}: (G_{wa}^{kg}, G_{\infty}^{kg}) \mapsto (\tilde{G}_{wa}^{kg}, \tilde{G}_{\infty}^{kg}),
\]

with \((\tilde{G}_{wa}^{kg}, \tilde{G}_{\infty}^{kg})\) achieving the final data

\[
\tilde{G}_{wa}^{kg}(\infty, \xi) = 0, \quad \tilde{G}_{\infty}^{kg}(\infty, \xi) = 0,
\]

and satisfying the system

\[
\partial_{t}\tilde{G}_{wa}(t, \xi) = -h_{\infty}(t, \xi)
\]

\[
+ \sum_{\iota_{1}, \iota_{2} \in \{+, -\}} \Gamma_{wa}^{\iota_{1}\iota_{2}} \left[ G^{\iota_{1}\iota_{2}} \right] \left( t, \xi \right) d\xi
\]

\[
\left[ \tilde{e}^{iD_{\infty}(t, \xi)} - \tilde{e}^{iD_{\infty}(s, \xi)} \right] \cdot \tilde{V}_{\infty}^{kg}(\xi) + \left( B_{\infty}(\xi) - B_{\infty}(t, \xi) \right)
\]

\[
- \sum_{\iota_{1}, \iota_{2} \in \{+, -\}} \int_{t}^{\infty} \Gamma_{kg}^{\iota_{1}\iota_{2}} \left[ G^{\iota_{1}\iota_{2}} \right] \left( t, \xi \right) d\xi.
\]
We plan to attack it in two steps:

1. **Nonlinear Estimates.**
   We first rewrite the wave equation and Klein-Gordon equation with the Duhamel's principle and estimate the right-hand side nonlinear terms. A direct consequences of this step are the estimates of time derivatives of the unknowns, i.e., $S_i'$ and $T_i'$ norms bound.

2. **Solution Estimates.**
   Then we turn back to the bound of the original solution itself in $S_i$ and $T_i$ norm. Here actually we treat them in completely different fashion.
– Energy estimates: $S_i$ norm is the classical energy norm, so we resort to the energy structure of the perturbed equation, which heavily relies on $S'_i$ norm bounds.

– Duhamel’s formula: $T_i$ norm is much trickier. We utilize the Lorentz vector fields to generate a relation in the mild formulation (we cannot directly take $\partial_{\xi^l}$ derivative in the mild formulation since it will hit the phase function and generate $\langle t \rangle$ growth), which is closely related to $T'_i$ bounds.

In this procedure, we need to utilize both the normal form and vector fields. Normal form helps to improve estimates of time integrals; vector fields helps improve dispersive estimates. We will delicately analyze all kinds of terms showing up in the resonant system and the perturbed equations, and these (preliminary) estimates actually constitute most part of this paper.

Remark 2.7. Essentially, our proof mainly relies on the energy estimates in Sobolev space. Since we need normal form (integration by parts in time), then we must estimate the time derivative. Since we need $Q_{jk}$ estimate and integration by parts in $\eta$, we need $\xi^l$ derivative estimates (Lemma 3.15). Since we need $Q_{jk}$ estimate, we need vector fields (Lemma 3.10).

2.7 Organization of the Paper

This paper is organized as follows: in Section 3 and Section 4, we present several preliminary lemmas about the solution and data; in Section 5 and Section 6, we discuss the nonlinear estimate for $S'_i$ and $T'_i$ norms using the Duhamel’s formula without time integration; finally, in Section 7 and Section 8, we study the energy estimates for $S_i$ norms and the regularity estimates for $T_i$ norms.

3 Preliminaries: Some Lemmas

In this section, we record several preliminary lemmas and analytical tools. Most of them can be found in [6], so we will omit the proofs.

3.1 Basic Analytic Tools

Lemma 3.1 (Stationary Phase). Let $\phi(x) \in C^2(\mathbb{R}^n)$ and $a(x) \in C(\mathbb{R}^n)$. $x_0$ is the only non-degenerate critical point of $\phi$, that is $\nabla_x \phi(x_0) = 0$ and $\det (\nabla_x^2 \phi(x_0)) \neq 0$. Then for $t \in \mathbb{R}^+$, we have

$$\int_{\mathbb{R}^n} e^{it\phi(x)} a(x) dx = \left( \frac{2\pi}{t} \right)^n e^{i\phi(x_0)} a(x_0) \frac{1}{\det (\nabla_x^2 \phi(x_0))} e^{\frac{i}{2} \text{sgn} (\nabla_x^2 \phi(x_0))} \left( a(x_0) + O(t^{-1}) \right) \text{ as } t \to \infty.$$

(3.1)

Here sgn(A) represents the number of positive eigenvalues minus the number of negative eigenvalues.

Proof. This is basically an integration by parts argument. See [2 Section 4.5.3] and [12 Section 8.2].
Lemma 3.2 (Non-Stationary Phase). Let $\phi(x) \in C^1(\mathbb{R}^n)$ and $a(x) \in C^m(\mathbb{R}^n)$. $\nabla_x \phi(x) \neq 0$ for any $x \in \mathbb{R}^n$. Let the operator $\Xi$ be defined as $\Xi[f] = \nabla_x \cdot \left( i \frac{\nabla_x \phi(x)}{|\nabla_x \phi(x)|} f(x) \right)$. Then for $t \in \mathbb{R}^+$, we have

$$
\int_{\mathbb{R}^n} e^{it\phi(x)} a(x) dx = \frac{1}{t^m} \int_{\mathbb{R}^n} e^{it\phi(x)} \Xi^m a(x) dx.
$$

(3.2)

Proof. See [2, Section 4.5.3] and [12, Section 8.2]. □

Lemma 3.3 (Bony’s Decomposition). For $f, g \in L^2$ satisfying

$$
f(x) = \sum_{k_1=-\infty}^{\infty} P_{k_1} f(x), \quad g(x) = \sum_{k_2=-\infty}^{\infty} P_{k_2} g(x),
$$

we have that for $k \in \mathbb{Z}$

$$
P_k(fg) \simeq \sum_{k_1 \simeq k_2 \geq k} P_k \left( (P_{k_1} f)(P_{k_2} g) \right) + \sum_{k_1 \simeq k_2 \geq k+2} P_k \left( (P_{k_1} f)(P_{k_2} g) \right) + \sum_{k_2 \simeq k_1 \geq k+2} P_k \left( (P_{k_1} f)(P_{k_2} g) \right).
$$

(3.3)

The summation is over all possible combination of $k_1, k_2 \in \mathbb{Z}$.

Remark 3.1. Roughly speaking, this lemma provides the effective region of convolution in frequency space. For convenience, we denote this set for fixed $k \in \mathbb{Z}$ by

$$
\chi_k := \left\{ (k_1, k_2) \in \mathbb{Z}^2 : k_1 \simeq k_2 \geq k \text{ or } k_1 \simeq k \geq k_2 + 2 \text{ or } k_2 \simeq k \geq k_1 + 2 \right\},
$$

and we will frequently take the summation over $(k_1, k_2) \in \chi_k$ (sometimes we simply write for short as $\sum_{k_1, k_2})$.

Proof. Let $h(x) = f(x)g(x)$. Hence,

$$
\hat{h}(\xi) = \left( \hat{f} \ast \hat{g} \right)(\xi) = \int_{\mathbb{R}^3} \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta.
$$

(3.5)

Hence, we actually want to find the region that

$$
|\xi| \in [2^{k-1}, 2^{k+1}], \quad |\xi - \eta| \in [2^{k_1-1}, 2^{k_1+1}], \quad |\eta| \in [2^{k_2-1}, 2^{k_2+1}].
$$

(3.6)

Then if $k_1 \simeq k_2$, we know

$$
|\xi - \eta| + |\eta| \simeq [2^{\max(k_1, k_2)+1}, 0],
$$

(3.7)

which means $k \simeq k_1 \simeq k_2$. Otherwise, if $|k_1 - k_2| \geq 2$, we have

$$
|\xi - \eta| + |\eta| \simeq [2^{\max(k_1, k_2)+1}, 2^{\max(k_1, k_2)-1}],
$$

(3.8)

which means $k \simeq \max(k_1, k_2)$. □
3.2 Preliminary Estimates

Lemma 3.4 (Bilinear Operators/Estimates). (i) Assume that \( \ell \geq 2, f_1, \cdots, f_\ell, f_{\ell+1} \in L^2(\mathbb{R}^3) \), and \( M : (\mathbb{R}^3)\ell \to \mathbb{C} \) is a continuously compactly supported function. Then

\[
\int_{(\mathbb{R}^3)^\ell} M(\xi_1, \cdots, \xi_\ell) \hat{f}_1(\xi_1) \cdots \hat{f}_\ell(\xi_\ell) \overline{\hat{f}}_{\ell+1}(\xi_1, \cdots, \xi_\ell) d\xi_1 \cdots d\xi_\ell \leq \|F^{-1}M\|_{L^1((\mathbb{R}^3)^\ell)} \|f_1\|_{L^p_1} \cdots \|f_\ell\|_{L^p_\ell} \|f_{\ell+1}\|_{L^p_{\ell+1}},
\]

for any exponent \( p_1, \cdots, p_\ell, p_{\ell+1} \in [1, \infty] \) satisfying \( \frac{1}{p_1} + \cdots + \frac{1}{p_\ell} + \frac{1}{p_{\ell+1}} = 1 \).

(ii) If \( q, p_1, p_2 \in [1, \infty] \) satisfying \( \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} \), then

\[
\|F^{-1} \left( \int_{\mathbb{R}^3} M(\xi, \eta) \hat{f}(\eta) \overline{\hat{g}}(\xi - \eta) d\eta \right) \|_{L^q_\ell} \leq \|M\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \|f\|_{L^p_1} \|g\|_{L^p_2}. \tag{3.10}
\]

Proof. This is [6, Lemma 3.2].

Lemma 3.5 (Lower Bound for Nonlinear/Quadratic Phase). Assume \(|\xi|, |\eta|, |\xi - \eta| \in [0, b] \) for \( b \geq 1 \). Then we have

\[
\Phi_{wa}^{1/2} \geq \frac{|\xi|}{4b^2}, \quad \Phi_{kg}^{1/2} \geq \frac{|\eta|}{4b^2}. \tag{3.11}
\]

Proof. This is [6, Lemma 3.3].

Lemma 3.6 (Localization). Assume \( k, k_1, k_2 \in \mathbb{Z} \) and \( m(\xi, \eta) \) is a multiplier satisfying \( \|F^{-1}m\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \lesssim 1 \). Let \( \tilde{k} = \max(k, k_1, k_2) \). Then

\[
\left\|F^{-1} \left( \frac{1}{\Phi_{wa}^{1/2}} m(\xi, \eta) \phi_k(\xi) \phi_{k_1}(\xi - \eta) \phi_{k_2}(\eta) \right) \right\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \lesssim 2^{-k} 2^{\tilde{k}} \tag{3.12}
\]

\[
\left\|F^{-1} \left( \frac{1}{\Phi_{kg}^{1/2}} m(\xi, \eta) \phi_k(\xi) \phi_{k_1}(\xi - \eta) \phi_{k_2}(\eta) \right) \right\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \lesssim 2^{-k} 2^{\tilde{k}} \tag{3.13}
\]

Proof. This is [6, Lemma 3.3].

Definition 3.1 (Multipliers). We call a multiplier \( m(\xi, \eta) \) harmless if it satisfies

\[
\left\|F^{-1} \left( \phi_k D^\alpha_m \right) \right\|_{L^1} \lesssim 2^{-|\alpha| |k|} \tag{3.14}
\]

for any \( k \in \mathbb{Z} \) and \( \alpha \in (\mathbb{Z}_+)^3 \).

Lemma 3.7. Assume \( \mu = \{wa, kg\} \) and

\[
(\partial_t + i\Lambda_\mu) U = \mathcal{N}, \tag{3.15}
\]

on \([T, \infty) \times \mathbb{R}^3\). If \( V(t) = e^{i t \Lambda_\mu} U \) and \( \ell = \{1, 2, 3\} \), then for any \( t \in [T, \infty) \), we have

\[
\hat{U}(t, \xi) = i(\partial_{\xi_\ell} \mathcal{N})(t, \xi) + e^{-i t \Lambda_\mu(\xi)} \partial_{t_\ell} \Lambda_\mu(\xi) \hat{V}(t, \xi). \tag{3.16}
\]

Proof. This is [6, Lemma 6.1].
3.3 Linear Estimates

This subsection is about the linear estimates either in the physical space or in the frequency space. In the following, $\sigma > 0$ denotes a sufficiently small constant. When it is present, then the inequality might depend on $\sigma$.

**Lemma 3.8.** For any $f \in L^2(\mathbb{R}^3)$, $(k, j) \in \mathcal{J}$ and $\alpha \in (\mathbb{Z}_+)^3$, we have

$$
\left\| D^\alpha_{\xi} \hat{\mathcal{D}}_{jk} f \right\|_{L^2} \lesssim 2^{|\alpha|j} \left\| \hat{\mathcal{D}}_{jk} f \right\|_{L^2}, \quad \left\| D^\alpha_{\xi} \hat{\mathcal{D}}_{jk} f \right\|_{L^\infty} \lesssim 2^{|\alpha|j} \left\| \hat{\mathcal{D}}_{jk} f \right\|_{L^\infty}.
$$

(3.17)

**Proof.** This is [3] (3.17). \hfill \Box

**Lemma 3.9.** For any $f \in L^2(\mathbb{R}^3)$ and $(k, j) \in \mathcal{J}$, we have

$$
\left\| \hat{Q}_{jk} f - \hat{\mathcal{D}}_{jk} f \right\|_{L^\infty} \lesssim 2^3 j 2^{-4(j+k)} \left\| P_k f \right\|_{L^2}.
$$

(3.18)

**Proof.** This is [3] (3.21). \hfill \Box

**Lemma 3.10.** For any $f \in L^2(\mathbb{R}^3)$ and $(k, j) \in \mathcal{J}$, we have

$$
\left\| \hat{\mathcal{D}}_{jk} f \right\|_{L^\infty} \lesssim \min \left( 2^{3j} \left\| Q_{jk} f \right\|_{L^2}, 2^{\frac{3}{2}k} 2^{-k} 2^\frac{3}{8} \left\| Q_{jk} f \right\|_{H^0_{\Omega}} \right).
$$

(3.19)

**Proof.** This is [3] (3.18). \hfill \Box

**Lemma 3.11** (Dispersive Estimates). For any $f \in L^2(\mathbb{R}^3)$ and $(k, j) \in \mathcal{J}$, we have

$$
\left\| e^{-it\Lambda_{\omega a}} \mathcal{D}_{jk} f \right\|_{L^\infty} \lesssim \min \left( 2^{\frac{3}{2}j}, 2^{\frac{3}{4}j} \langle t \rangle^{-1} \right) \left\| Q_{jk} f \right\|_{L^2},
$$

(3.20)

$$
\left\| e^{-it\Lambda_{k a}} \mathcal{D}_{jk} f \right\|_{L^\infty} \lesssim \min \left( 2^{\frac{3}{2}k}, 2^{\frac{3}{4}k} \langle t \rangle^{-\frac{3}{4}} \right) \left\| Q_{jk} f \right\|_{L^2}.
$$

(3.21)

**Proof.** This is [3] (3.24),(3.28). \hfill \Box

**Lemma 3.12.** For any $f \in L^2(\mathbb{R}^3)$ and $(k, j) \in \mathcal{J}$, if $|t| \geq 1$, we have

$$
\left\| e^{-it\Lambda_{\omega a}} \mathcal{D}_{jk} f \right\|_{L^\infty} \lesssim \langle t \rangle^{-1} 2^\frac{3}{4} \left( 1 + 2^k \langle t \rangle \right) \left\| Q_{jk} f \right\|_{H^0_{\Omega}} \quad \text{if} \; 2^j \leq 2^{-10} \langle t \rangle,
$$

(3.22)

$$
\left\| e^{-it\Lambda_{k a}} \mathcal{D}_{jk} f \right\|_{L^\infty} \lesssim \langle t \rangle^{-1} 2^{2k} \left\| Q_{\leq jk} f \right\|_{L^\infty} \quad \text{if} \; 2^j \lesssim \langle t \rangle^\frac{3}{2} 2^{-\frac{k}{2}}.
$$

(3.23)

**Proof.** This is [3] (3.26),(3.27). \hfill \Box

**Lemma 3.13.** For any $f \in L^2(\mathbb{R}^3)$ and $(k, j) \in \mathcal{J}$, if $1 \leq 2^{2k-20} \langle t \rangle$, we have

$$
\left\| e^{-it\Lambda_{k a}} \mathcal{D}_{jk} f \right\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{3}{2}} 2^{5k} 2^{-k} 2^\frac{3}{4} \left( 1 + 2^{2k} \langle t \rangle \right) \left\| Q_{jk} f \right\|_{H^0_{\Omega}} \quad \text{if} \; 2^j \leq 2^{k-20} \langle t \rangle,
$$

(3.24)

$$
\left\| e^{-it\Lambda_{k a}} \mathcal{D}_{\leq jk} f \right\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{3}{2}} 2^{5k+} \left\| Q_{\leq jk} f \right\|_{L^\infty} \quad \text{if} \; 2^j \lesssim \langle t \rangle^\frac{3}{2}.
$$

(3.25)

**Proof.** This is [3] (3.29),(3.30). \hfill \Box
Lemma 3.14. We have
\[ \|Q_{jk}f\|_{L^p} \lesssim \|\hat{P}_k f\|_{L^p}, \]  
\[ \|Q_{\leq jk}f\|_{L^p} \lesssim \|\hat{P}_k f\|_{L^p}. \]  

Proof. Since \( Q_{jk} = \hat{\varphi}_j \cdot P_k f \), we know
\[ \hat{Q}_{jk} f(\xi) = \int_{\mathbb{R}^3} \hat{\varphi}_j(\xi - \eta) \hat{P}_k f(\eta) d\eta. \]  
Hence,
\[ \|Q_{jk}f\|_{L^p} \lesssim \|\hat{P}_k f\|_{L^p} \|P_k \hat{\varphi}_j\|_{L^1}. \]

The definition yields that \( \|P_k \hat{\varphi}_j\|_{L^1} \lesssim 1 \) regardless of (uniformly in) \( j \) (since \( \varphi \) is in Schwarz space, its Fourier transform must also be in Schwarz space. In particular, it is scaling invariant, so \( j \) does not play a role. ). Hence, our result naturally follows.

Remark 3.2. Roughly speaking, this lemma states that we can use \( P_k \) to bound \( Q_{jk} \). A natural corollary is that we can use \( Q_{jk} \) to bound \( Q_{jk} \).

Lemma 3.15. For \( f \in L^2(\mathbb{R}^3) \) and \( k \in \mathbb{Z} \), let
\[ A_k = \|P_k f\|_{L^2} + \sum_{\ell=1}^3 \|\varphi_k(\xi)(\partial_{\xi_{2\ell}} \hat{f})(\xi)\|_{L^2_\xi}, \]  
\[ B_k = \left( \sum_{j \geq -k} 2^{2j} \|Q_{jk} f\|_{L^2}^2 \right)^{\frac{1}{2}}. \]

Then for any \( k \in \mathbb{Z} \),
\[ A_k \lesssim \sum_{|k' - k| \leq 4} B_{k'}, \]  
and
\[ B_k \lesssim \begin{cases} \sum_{|k' - k| \leq 4} A_{k'} & \text{if } k \geq 0, \\ \sum_{k' \in \mathbb{Z}} A_{k'} 2^{-\frac{|k' - k|}{2}} & \text{if } k \leq 0. \end{cases} \]

Remark 3.3. Roughly speaking, this lemma means that \( A_k \sim B_k \), i.e. they are comparable.

Proof. This is [6, Lemma 3.5], so we omit the proof here.

4 A Priori Estimates

Let
\[ X(t; n) = \langle t \rangle^{-H(n)\delta}, \]  
\[ Y(k, t; n) = \langle t \rangle^{-H(n+1)\delta} 2^{-N(n+1)k^+}, \]  
\[ Z(k; n) = 2^{-N(n-3)k^+}. \]
Remark 4.1. The definition of $\mathcal{L}$ is at $(u, v)$ level. Hence, the rigorous definition of $V_{\mathcal{L}, \infty}^{wa}$ is as follows:

$$V_{\infty}^{wa} \rightarrow U_{\infty}^{wa} \rightarrow u_{\infty, \mathcal{L}} \rightarrow U_{\mathcal{L}, \infty}^{wa} \rightarrow V_{\mathcal{L}, \infty}^{wa}.$$  \hspace{1cm} (4.4)

The same fashion also applies to $V_{\mathcal{L}, \infty}^{kg}$, $V_{\mathcal{L}, \infty}^{wa}$, $V_{\mathcal{L}, \infty}^{kg}$, $G_{\mathcal{L}, \infty}^{wa}$, $G_{\mathcal{L}, \infty}^{kg}$.

4.1 $L^2$ Estimates of $G_{\mathcal{L}}^{wa}$ and $G_{\mathcal{L}}^{kg}$

Lemma 4.1. Assume (2.105) holds. Then we have for $n \leq N_1$,

$$\sup_{L \in \mathcal{V}_n} \left( \left\| \nabla \right\|^{-\frac{1}{2}} G_{\mathcal{L}}^{wa} \right\|_{H^N(n)} + \left\| G_{\mathcal{L}}^{kg} \right\|_{H^N(n)} \right) \lesssim \varepsilon X(t; n),$$  \hspace{1cm} (4.5)

and for $n \leq N_1 - 1$,

$$\sup_{L \in \mathcal{V}_n, t \in \{1, 2, 3\}} \sup_{k \in \mathbb{Z}} \left( 2^{\frac{1}{2}k} \left\| \varphi_k(\xi) \partial_{\xi} G_{\mathcal{L}}^{wa} \right\|_{L^2} + 2^{k^+} \left\| \varphi_k(\xi) \partial_{\xi} G_{\mathcal{L}}^{kg} \right\|_{L^2} \right) \lesssim \varepsilon Y(k, t; n).$$  \hspace{1cm} (4.6)

Proof. This is essentially part (i) of [6, Lemma 4.1]. It can be directly obtained from (2.104), so we omit the proof here.

Remark 4.2. This lemma actually yields the $L^2$ bound for any $k \in \mathbb{Z}$, for $n \leq N_1$,

$$\left\| P_k G_{\mathcal{L}}^{wa}(t) \right\|_{L^2} \lesssim \varepsilon X(t; n) 2^{\frac{1}{2}k} 2^{-N(n)k^+},$$  \hspace{1cm} (4.7)

$$\left\| P_k G_{\mathcal{L}}^{kg}(t) \right\|_{L^2} \lesssim \varepsilon X(t; n) 2^{-N(n)k^+},$$  \hspace{1cm} (4.8)

and for $n \leq N_1 - 1$,

$$\left\| \partial_{\xi} P_k G_{\mathcal{L}}^{wa}(t) \right\|_{L^2} \lesssim \varepsilon Y(k, t; n) 2^{-\frac{1}{2}k},$$  \hspace{1cm} (4.9)

$$\left\| \partial_{\xi} P_k G_{\mathcal{L}}^{kg}(t) \right\|_{L^2} \lesssim \varepsilon Y(k, t; n) 2^{-k^+}.$$  \hspace{1cm} (4.10)

Remark 4.3. Note that taking $\partial_{\xi}$ derivative in the frequency space is equivalent to multiplying $x_\ell$ in the physical space.

Lemma 4.2. Assume (2.105) holds. Then for $n \leq N_1 - 1$ and $(k, j) \in \mathcal{J}$, we have

$$2^j \left\| Q_{jk} G_{\mathcal{L}}^{wa}(t) \right\|_{L^2} \lesssim \varepsilon Y(k, t; n) 2^{-\frac{1}{2}k},$$  \hspace{1cm} (4.11)

$$2^j \left\| Q_{jk} G_{\mathcal{L}}^{kg}(t) \right\|_{L^2} \lesssim \varepsilon Y(k, t; n) 2^{-k^+},$$  \hspace{1cm} (4.12)

and

$$\left\| P_k G_{\mathcal{L}}^{wa}(t) \right\|_{L^2} \lesssim \varepsilon Y(k, t; n) 2^{-\frac{1}{2}k} 2^{k^+},$$  \hspace{1cm} (4.13)

$$\left\| P_k G_{\mathcal{L}}^{kg}(t) \right\|_{L^2} \lesssim \varepsilon Y(k, t; n) 2^{-k^+} 2^{k^+}.$$  \hspace{1cm} (4.14)

Proof. This is essentially part (i) of [6, Lemma 4.1]. (4.11), (4.12) is a direct consequence of Lemma 4.1 and (3.33). When summing over $j \geq -k^-$, we get (4.13), (4.14).
Lemma 4.3. Assume \((2.105)\) holds. Then for \(n \leq N_1 - 1\) and \((k, j) \in J\), we have
\[
\| P_k G_{L}^{\nu a} (t) \|_{L^2} \lesssim \varepsilon \langle t \rangle^{-H(n) \delta} 2^{-N(n) - \frac{1}{2}} k^+ 2^{\frac{1}{2} k}, \tag{4.15}
\]
\[
\| P_k G_{L}^{kq} (t) \|_{L^2} \lesssim \varepsilon \langle t \rangle^{-H(n) \delta} 2^{-N(n) k^+ \delta} k. \tag{4.16}
\]

**Proof.** The \(P_k\) estimates are a combination of Lemma 4.1 (if \(k \geq 0\)) and Lemma 4.2 (if \(k \leq 0\)). \(\square\)

**Remark 4.4.** Lemma 4.3 is an improved version for Lemma 4.2. In particular, (4.43) is favorable in "\(2^k\)"] when \(k \leq 0\). And we will use these bounds in Lemma 4.3 repeatedly in the following energy estimate.

### 4.2 \(L^\infty\) Estimates of \(G_{L}^{\nu a}\) and \(G_{L}^{kq}\)

**Lemma 4.4.** Assume \((2.105)\) holds. Then for \(n \leq N_1 - 1\) and \((k, j) \in J\), we have
\[
\| e^{-it\Lambda_{wa}} \mathcal{D}_{jk} G_{L}^{\nu a} (t) \|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \min \left(2^{-j}, \langle t \rangle^{-1} \right) 2^k, \tag{4.17}
\]
\[
\| e^{-it\Lambda_{kq}} \mathcal{D}_{jk} G_{L}^{kq} (t) \|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \min \left(2^{-j}, 2^{\frac{k}{2}}, 2^{\frac{q}{2} j} \langle t \rangle^{-\frac{1}{2}} \right) 2^{-j} 2^{-k^+}. \tag{4.18}
\]

**Proof.** Combining Lemma 3.11 and Lemma 4.2 this is obvious. \(\square\)

**Lemma 4.5.** Assume \((2.105)\) holds. Then for \(n \leq N_1 - 1\) and \((k, j) \in J\), we have
\[
\sum_{j \geq -k^+} \| e^{-it\Lambda_{wa}} \mathcal{D}_{jk} G_{L}^{\nu a} (t) \|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \min \left(2^{-j}, \langle t \rangle^{-1} \right) 2^k, \tag{4.19}
\]
\[
\sum_{j \geq -k^+} \| e^{-it\Lambda_{kq}} \mathcal{D}_{jk} G_{L}^{kq} (t) \|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \min \left(2^{\frac{k}{2}}, 2^{\frac{q}{2} j} \langle t \rangle^{-\frac{1}{2}}, 2^{\frac{q}{2} k^+} 2^{\frac{q}{2} k} \langle t \rangle^{-1} \right). \tag{4.20}
\]

**Proof.** This is an adaption of (6) (4.7),(4.8)]. Summing up in Lemma 4.4 (4.17), we have
\[
\sum_{j \geq -k^+} \| e^{-it\Lambda_{wa}} \mathcal{D}_{jk} G_{L}^{\nu a} (t) \|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \sum_{j \geq -k^+} \min \left(2^{-j}, \langle t \rangle^{-1} \right) 2^k. \tag{4.21}
\]

For the first term in the minimum of (4.19), we naturally have
\[
\sum_{j \geq -k^+} \| e^{-it\Lambda_{wa}} \mathcal{D}_{jk} G_{L}^{\nu a} (t) \|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \sum_{j \geq -k^+} 2^{-j} 2^k \lesssim \varepsilon Y(k, t; n) 2^{k^-} 2^k. \tag{4.22}
\]

For the second term in the minimum, we divide it into several cases:

- If \(2^{-j} \geq \langle t \rangle^{-1}\), then since \(-k^+ \geq 0\), there are at most \(\ln \langle t \rangle\) such \(j\). Hence, by (4.21) we have the summation
\[
\sum_{2^{-j} \geq \langle t \rangle^{-1}} \| e^{-it\Lambda_{wa}} \mathcal{D}_{jk} G_{L}^{\nu a} \|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \ln \langle t \rangle \langle t \rangle^{-1} 2^k. \tag{4.23}
\]

- If \(2^{-j} \leq \langle t \rangle^{-1}\), then we naturally have
\[
\sum_{2^{-j} \leq \langle t \rangle^{-1}} \| e^{-it\Lambda_{wa}} \mathcal{D}_{jk} G_{L}^{\nu a} \|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \sum_{2^{-j} \leq \langle t \rangle^{-1}} 2^{-j} 2^k \lesssim \varepsilon Y(k, t; n) \langle t \rangle^{-1} 2^k. \tag{4.24}
\]
Hence, to summarize the above cases, we arrive at the desired result (4.19).

On the other hand, for (4.20), we have from (4.18) that

\[
\sum_{j \geq -k^-} \left\| e^{it\Lambda_{ks}} \mathcal{Q}_{jk} G_{L}^{kq} \right\|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \min_{j \geq -k^-} \left( 2^4 2^{-j-2} \cdot 2^{2k+2} (t)^{-\frac{3}{2}} \right). \tag{4.25}
\]

For the first term in the minimum, we have

\[
\sum_{j \geq -k^-} \left\| e^{it\Lambda_{ks}} \mathcal{Q}_{jk} G_{L}^{kq} \right\|_{L^\infty} \lesssim \varepsilon Y(k, t; n) \sum_{j \geq -k^-} 2^{4j} 2^{-j-2} \lesssim \varepsilon Y(k, t; n) 2^{\frac{3k}{2}} 2^{k^+} \tag{4.26}
\]

For the second term in the minimum, \(j\) power is positive, so we cannot directly sum up over \(j\). The key is to combine both bounds in (4.25). We further divide it into two cases:

1. If \(2^j \leq \langle t \rangle \cdot 2^k 2^{-2k^+}\), then \(2^4 2^{-j-2} \cdot 2^{2k+2} (t)^{-\frac{3}{2}} \leq 2^{2k^+} \langle t \rangle^{-\frac{3}{2}} \left( \langle t \rangle^{1/2} 2^{1/2} 2^{-j-2} \right) \leq 2^{2k^+} 2^{k^+} \langle t \rangle^{-1}. \tag{4.27}

2. If \(2^j \geq \langle t \rangle \cdot 2^k 2^{-2k^+}\), then \(2^4 2^{-j-2} \cdot 2^{2k+2} (t)^{-\frac{3}{2}} \leq 2^{2k^+} \langle t \rangle^{-1/2} \left( 2^{-j} 2^{2k^+} \right) \leq 2^{2k^+} 2^{k^+} \langle t \rangle^{-1}. \tag{4.28}

Summarizing the above two cases, we have that

\[
\sum_{j \geq -k^-} \left\| e^{it\Lambda_{ks}} \mathcal{Q}_{jk} G_{L}^{kq} \right\|_{L^\infty} \lesssim 2^{2k^+} 2^{k^+} \langle t \rangle^{-1}. \tag{4.29}
\]

\[
\square
\]

**Lemma 4.6.** Assume \([2,105]\) holds. Then for \(n \leq N_1 - 2\) and \((k, j) \in \mathcal{J}\) with \(1 \leq 2^{k^-} 2^{20} \langle t \rangle\), we have

\[
\sum_{j \geq -k^-} \left\| e^{it\Lambda_{ks}} \mathcal{Q}_{jk} G_{L}^{kq} (t) \right\|_{L^\infty} \lesssim \varepsilon Y(k, t; n+1) \langle t \rangle^{-\frac{3}{2} + \frac{n}{2}} 2^{4k+2} (2^{2k^-} (t)^{-\frac{5}{2} + \frac{n}{2}} \tag{4.30}
\]

for some \(\sigma > 0\).

**Proof.** This is essentially \([6] (4.9)\). Based on Lemma 3.13 we know

\[
\left\| e^{it\Lambda_{ks}} \mathcal{Q}_{jk} G_{L}^{kq} \right\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{3}{2}} 2^{5k^+} 2^{-k^-} 2^{4} \left( 2^{2k^-} (t) \right)^{\frac{5}{2}} \left\| Q_{jk} G_{L}^{kq} \right\|_{H_{01}^{0,1}}. \tag{4.31}
\]

Based on Lemma 4.2 we have

\[
\left\| Q_{jk} G_{L}^{kq} \right\|_{H_{01}^{0,1}} \lesssim \varepsilon Y(k, t; n+1) 2^{-j} 2^{k^+}. \tag{4.32}
\]
Combining the above two, we have
\[ \left\| e^{-it\Lambda G L} \mathcal{O}_{jk} G_{L}^{kg} \right\|_{L^\infty} \lesssim \varepsilon Y(k,t;n+1) (t)^{-\frac{3}{7}} 2^{4k^+} 2^{-k^-} \left( 2^{2k^-} (t) \right)^{\frac{n}{k}}. \tag{4.33} \]
Summing up over \( j \geq -k^- \), we know
\[ \sum_{j \geq -k^-} \left\| e^{-it\Lambda G L} \mathcal{O}_{jk} G_{L}^{kg} \right\|_{L^\infty} \lesssim \varepsilon Y(k,t;n+1) (t)^{-\frac{3}{7}} 2^{4k^+} 2^{-\frac{k^-}{2}} \left( 2^{2k^-} (t) \right)^{\frac{n}{k}}. \tag{4.34} \]
Hence, our result naturally follows. \( \square \)

### 4.3 Estimates of \( V_{L,\infty}^{wa} \) and \( V_{L,\infty}^{kg} \)

In the following, we will record some properties related to the initial data \( (V_{\infty}^{wa}, V_{\infty}^{kg}) \). Their proofs are similar to those of \( G_{\infty}^{wa} \) and \( G_{\infty}^{kg} \) without time decay term, so we omit them here.

**Lemma 4.7.** Assume \([2.105]\) holds. Then we have for \( n \leq N_1 + 2 \),
\[ \left\| P_{L} V_{L,\infty}^{wa} \right\|_{L^2} \lesssim \varepsilon Z(k;n-1) 2^{\frac{1}{2}k}, \tag{4.35} \]
\[ \left\| P_{L} V_{L,\infty}^{kg} \right\|_{L^2} \lesssim \varepsilon Z(k;n-1), \tag{4.36} \]
and for \( n \leq N_1 + 1 \),
\[ \left\| \partial_{t} P_{L} V_{L,\infty}^{wa} \right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{-\frac{1}{2}k}, \tag{4.37} \]
\[ \left\| \partial_{t} P_{L} V_{L,\infty}^{kg} \right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{-k^+}. \tag{4.38} \]

**Proof.** See the proof of Lemma 4.1. \( \square \)

**Lemma 4.8.** Assume \([2.105]\) holds. Then for \( n \leq N_1 + 1 \) and \( (k,j) \in J \), we have
\[ 2^j \left\| Q_{jk} V_{L,\infty}^{wa} \right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{-\frac{1}{2}k}, \tag{4.39} \]
\[ 2^j \left\| Q_{jk} V_{L,\infty}^{kg} \right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{-k^+}, \tag{4.40} \]
and
\[ \left\| P_{L} V_{L,\infty}^{wa} \right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{-\frac{1}{2}k} 2^{k^-}, \tag{4.41} \]
\[ \left\| P_{L} V_{L,\infty}^{kg} \right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{-k^+} 2^{k^-}. \tag{4.42} \]

**Proof.** See the proof of Lemma 4.2. \( \square \)

**Lemma 4.9.** Assume \([2.105]\) holds. Then for \( n \leq N_1 + 1 \) and \( (k,j) \in J \), we have
\[ \left\| P_{L} V_{L,\infty}^{wa} \right\|_{L^2} \lesssim \varepsilon Z(k;n-1) 2^{\frac{1}{2}k^-}, \tag{4.43} \]
\[ \left\| P_{L} V_{L,\infty}^{kg} \right\|_{L^2} \lesssim \varepsilon Z(k;n-1) 2^{k^-}. \tag{4.44} \]
Proof. See the proof of Lemma 4.3.

Lemma 4.10. Assume \((2.105)\) holds. Then for \(n \leq N_1 + 1\) and \((k, j) \in \mathcal{J}\), we have
\[
\|e^{-it\Lambda_\nu a} \mathcal{D}_{jk} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \varepsilon Z(k; n) \min (2^{-j}, \langle t \rangle^{-1}) 2^k,
\]
(4.45)
\[
\|e^{-it\Lambda_\nu a} \mathcal{D}_{jk} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \varepsilon Z(k; n) \min (2^{-j}, 2^{3k} \langle t \rangle^{-\frac{3}{2}}) 2^{-j} 2^{-k}. 
\]
(4.46)

Proof. See the proof of Lemma 4.3.

Lemma 4.11. Assume \((2.105)\) holds. Then for \(n \leq N_1 + 1\) and \((k, j) \in \mathcal{J}\), we have
\[
\sum_{j \geq -k^-} \|e^{-it\Lambda_\nu a} \mathcal{D}_{jk} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \varepsilon Z(k; n) \min \left(2^{-k}, \left(\ln \langle t \rangle \right) \langle t \rangle^{-1} \right) 2^k,
\]
(4.47)
\[
\sum_{j \geq -k^-} \|e^{-it\Lambda_\nu a} \mathcal{D}_{jk} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \varepsilon Z(k; n) \min \left(2^{\frac{1}{2}k+\frac{\varepsilon Z}{4}}, 2^{\frac{1}{2}k-k^{-}}, 2^{\frac{1}{2}k+2^{-k}} \langle t \rangle^{-1} \right).
\]
(4.48)

Proof. See the proof of Lemma 4.3.

Remark 4.5. Note that we have \(\ln \langle t \rangle\) term in the estimate. This is a major troublemaker for later proof since it does not give exactly one order decay.

Lemma 4.12. Assume \((2.105)\) holds. Then for \(n \leq N_1\) and \((k, j) \in \mathcal{J}\) with \(1 \leq 2^{2k^- - 20} \langle t \rangle\), we have
\[
\sum_{j \geq -k^-} \|e^{-it\Lambda_\nu a} \mathcal{D}_{jk} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \varepsilon Z(k; n+1) \langle t \rangle^{-\frac{3}{4} + \frac{\varepsilon Z}{4}} 2^{\frac{1}{2}k+2^{-k}} (\frac{1}{4} - \frac{1}{4})
\]
(4.49)

for some \(\sigma > 0\).

Proof. See the proof of Lemma 4.6.

Lemma 4.13. Assume \((2.105)\) holds. Then for \(n \leq N_1\) and \((k, j) \in \mathcal{J}\), we have
\[
\sum_{j \geq -k^-} \|e^{-it\Lambda_\nu a} \mathcal{D}_{jk} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \varepsilon Z(k; n+1) \langle t \rangle^{-1} 2^k.
\]
(4.50)

Proof. Let \(2^{-j_0} = \langle t \rangle^{\frac{1}{4}} 2^{-\frac{1}{4}}\). We divide it into several cases:

- If \(2^{-k^-} \leq \langle t \rangle^{-1}\), then by Lemma 4.11, we have
\[
\sum_{j \geq -k^-} \|e^{-it\Lambda_\nu a} \mathcal{D}_{jk} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \varepsilon Z(k; n) 2^{-k^-} 2^k \lesssim \varepsilon Z(k; n) \langle t \rangle^{-1} 2^k.
\]
(4.51)

- If \(2^{-k^-} \geq \langle t \rangle^{-1}\) and \(j \leq j_0\), by \((3.28)\) in Lemma 3.12, we have
\[
\|e^{-it\Lambda_\nu a} \mathcal{D}_{\leq j_0} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \langle t \rangle^{-1} 2^{2k} \|Q_{\leq j_0} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty}.
\]
(4.52)

Based on \(Y\) norm definition (or derived from \(L^2\) bounds in Lemma 4.9), we have
\[
\|Q_{\leq j_0} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \|P_k V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \varepsilon Z(k; n) 2^{-k^-}.
\]
(4.53)

Combining them together, we have
\[
\|e^{-it\Lambda_\nu a} \mathcal{D}_{\leq j_0} V_{\mathcal{E}, \infty}^{wa}\|_{L^\infty} \lesssim \varepsilon Z(k; n) \langle t \rangle^{-1} 2^{2k} 2^{-k^-}.
\]
(4.54)
• If $2 k^{-} \geq \langle t \rangle^{-1}$ and $j \geq J_0$ and $2^k \lesssim \langle t \rangle$ and $2^j \lesssim \langle t \rangle$, by (3.22) in Lemma 3.12, we have

$$\| e^{-i t \Lambda_{\omega a}} \mathcal{D}_{\geq J_0 k} V_{L_\infty}^{wq} \|_{L^\infty} \lesssim \sum_{j \geq J_0} \langle t \rangle^{-1} \frac{2^j}{1 + 2^k \langle t \rangle} \| Q_{Jk} V_{L_\infty}^{wq} \|_{H^0_{\Omega}}. \quad (4.55)$$

Lemma 4.8 implies

$$\| Q_{Jk} V_{L_\infty}^{wq} \|_{H^0_{\Omega}} \lesssim \varepsilon Z(k; n) 2^{-j} 2^{-\frac{k}{\sigma}}. \quad (4.56)$$

Combining them together, we have

$$\| e^{-i t \Lambda_{\omega a}} \mathcal{D}_{\geq J_0 k} V_{L_\infty}^{wq} \|_{L^\infty} \lesssim \sum_{j \geq J_0} \varepsilon Z(k; n) \langle t \rangle^{-1} 2^{-j} \left(1 + 2^k \langle t \rangle\right)^{\frac{\sigma}{2}} \lesssim \sum_{j \geq J_0} \varepsilon Z(k; n) \langle t \rangle^{-1} 2^{-j} \left(1 + 2^k \langle t \rangle\right)^{\frac{\sigma}{2}} 2^{-j} 2^{-\frac{k}{\sigma}} 2^{(1 + \frac{\sigma}{2})k}.$$

Then for $\sigma > 0$ small, our result naturally holds.

• If $2 k^{-} \geq \langle t \rangle^{-1}$ and $j \geq J_0$ and $2^k \lesssim \langle t \rangle$ and $2^j \geq \langle t \rangle$, then $2^{-j} \leq \langle t \rangle^{-1}$. Using Lemma 4.10 we have

$$\| e^{-i t \Lambda_{\omega a}} \mathcal{D}_{\geq J_0 k} V_{L_\infty}^{wq} \|_{L^\infty} \lesssim \varepsilon Z(k; n) \sum_{2^{-j} \leq \langle t \rangle^{-1}} 2^{-j} 2^k \lesssim \varepsilon Z(k; n) \langle t \rangle^{-1} 2^k. \quad (4.58)$$

• If $2 k^{-} \geq \langle t \rangle^{-1}$ and $j \geq J_0$ and $2^k \gtrsim \langle t \rangle$, we know $k \geq 0$ and $k = k^+$. Hence, by Sobolev embedding theorem, we know

$$\| e^{-i t \Lambda_{\omega a}} \mathcal{D}_{\geq J_0 k} V_{L_\infty}^{wq} \|_{L^\infty} \lesssim \| P_k V_{L_\infty}^{wq} \|_{L^\infty} \lesssim \| V_{L_\infty}^{wq} \|_{H^2} \lesssim \varepsilon Z(k; n) \| |\nabla|^{-\frac{1}{2}} V_{L_\infty}^{wq} \|_{H^{N(0)}} \lesssim \varepsilon Z(k; n - 1) \langle t \rangle^{-N_0}. \quad (4.59)$$

This suffices to close the proof.

**Remark 4.6.** Compared with Lemma 4.11 the key improvement here is that we get rid of $\ln \langle t \rangle$. This is crucial for later estimates.

**Lemma 4.14.** Assume (2.105) holds. Then for $\mathcal{L} \in \mathcal{V}_n$ with $n = 0, 1, \cdots N_1$, we have

$$\| \mathcal{P}_{j k} V_{L_\infty}^{wq} \|_{L^\infty} \lesssim \varepsilon 2^{\frac{\sigma(k + 1)}{8}} 2^{-\frac{N(n - 2)}{4}} 2^{-N(n - 2)k^+}, \quad (4.60)$$

$$\| P_k V_{L_\infty}^{wq} \|_{L^\infty} \lesssim \varepsilon 2^{-k} 2^{-(N(n - 2) + 1)k^+}, \quad (4.61)$$

and

$$\| \mathcal{P}_{j k} V_{L_\infty}^{kg} \|_{L^\infty} \lesssim \varepsilon 2^{\frac{\sigma(k + 1)}{8}} 2^{-\frac{N(n - 2)}{4}} 2^{-k} 2^{-N(n - 2)k^+}, \quad (4.62)$$

$$\| P_k V_{L_\infty}^{kg} \|_{L^\infty} \lesssim \varepsilon 2^{-\frac{k}{2}} 2^{-(N(n - 2) + 1)k^+}. \quad (4.63)$$
Proof. Based on (3.19) in Lemma 3.10 we know
\[
\| \mathcal{D}_{jk} V_{L,\infty}^{wa} \|_{L_\xi} \lesssim 2^{\frac{3}{2}} 2^{-j k} 2^\frac{3}{8} \| Q_{jk} V_{L,\infty}^{wa} \|_{H_{\Omega}^{0,1}}.
\] (4.64)

Based on Lemma 4.8 we know
\[
\| Q_{jk} V_{L,\infty}^{wa} \|_{H_{\Omega}^{0,1}} \lesssim \varepsilon 2^{-j} 2^{-\frac{1}{2}} 2^{-\frac{1}{3} k} 2^{-N(n-2)k^+}.
\] (4.65)

Combining the above two, we have
\[
\| \mathcal{D}_{jk} V_{L,\infty}^{wa} \|_{L_\xi} \lesssim \varepsilon 2^{-j 2^{-\frac{1}{2} - \frac{1}{3} k} - \frac{1}{2} k - N(n-2)k^+}.
\] (4.66)

Summing up over \( j \geq -k^- \), we know
\[
\| \mathcal{D}_{k} V_{L,\infty}^{wa} \|_{L_\xi} \lesssim \varepsilon 2^{\frac{1}{2} - \frac{1}{3} k} 2^{-\frac{1}{2} - \frac{1}{3} k} 2^{-N(n-2)k^+}.
\] (4.67)

For \( k \geq 0, k = k^+ \) and for \( k \leq 0, k = k^- \). Hence, we have
\[
\| \mathcal{D}_{k} V_{L,\infty}^{wa} \|_{L_\xi} \lesssim \varepsilon 2^{-k} 2^{-(N(n-2)+1)k^+}.
\] (4.68)

Similarly, based on (3.19) in Lemma 3.10 we know
\[
\| \mathcal{D}_{jk} V_{L,\infty}^{kg} \|_{L_\xi} \lesssim 2^{\frac{3}{2}} 2^{-j k} 2^\frac{3}{8} \| Q_{jk} V_{L,\infty}^{kg} \|_{H_{\Omega}^{0,1}}.
\] (4.69)

Based on Lemma 4.8 we know
\[
\| Q_{jk} V_{L,\infty}^{kg} \|_{H_{\Omega}^{0,1}} \lesssim \varepsilon 2^{-j 2^{-(N(n-2)+1)k^+}}.
\] (4.70)

Combining the above two, we have
\[
\| \mathcal{D}_{jk} V_{L,\infty}^{kg} \|_{L_\xi} \lesssim \varepsilon 2^{-j 2^{-\frac{1}{2} - \frac{1}{3} k} - \frac{1}{2} k - (N(n-2)+1)k^+}.
\] (4.71)

Summing up over \( j \geq -k^- \), we know
\[
\| \mathcal{D}_{k} V_{L,\infty}^{kg} \|_{L_\xi} \lesssim \varepsilon 2^{\frac{1}{2} - \frac{1}{3} k} 2^{-\frac{1}{2} - \frac{1}{3} k} 2^{-(N(n-2)+1)k^+}.
\] (4.72)

For \( k \geq 0, k = k^+ \) and for \( k \leq 0, k = k^- \). Hence, we have
\[
\| \mathcal{D}_{k} V_{L,\infty}^{kg} \|_{L_\xi} \lesssim \varepsilon 2^{-k} 2^{-(N(n-2)+1)k^+}.
\] (4.73)
4.4 Estimates of \( h_{\mathcal{L}, \infty} \) and \( \mathcal{H}_{\mathcal{L}, \infty} \)

Remark 4.7. Note that the low-frequency cutoff function in \( h_{\infty} \) and \( \mathcal{H}_{\infty} \) indicates that \( |\xi| \lesssim (t)^{-\frac{7}{8}} \lesssim 1 \). Hence, all estimates in this subsection do not have \( k^+ \) part.

Lemma 4.15. Assume (2.105) holds. Then we have

\[
\| \varphi_k h_{\infty} \|_{L^2_\xi} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{\frac{3}{2}k^-},
\]

and

\[
\| \varphi_k h_{\infty} \|_{L^\infty_\xi} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{-k}.
\]

Also, we have

\[
\| \varphi_k \partial_\xi h_{\infty} \|_{L^2_\xi} \lesssim \varepsilon^2 2^{\frac{1}{2}k^-},
\]

and

\[
\| \varphi_k \partial_\xi h_{\infty} \|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{-k}.
\]

Proof. We know

\[
h_{\infty}(t, \xi) \sim \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{7}{8}}) \cdot \int_{\mathbb{R}^3} e^{i t (|\xi| - \frac{\xi \cdot \eta}{\langle \eta \rangle})} |\hat{V}_k^\xi(\eta)|^2 \; d\eta.
\]

Hence, we integrate by parts in \( \eta \) to obtain

\[
\| \varphi_k h_{\infty} \|_{L^2_\xi} \lesssim \langle t \rangle^{-1} 2^{-k} \| \hat{V}_k^\xi \|_{L^2_\eta} \| \hat{V}_k^\xi \|_{H^1_\eta} \lesssim \varepsilon^2 2^{-k} \langle t \rangle^{-1}.
\]

Also, we know (by Hölder, taking into account the measure of the support of \( \varphi_k \))

\[
\| \varphi_k h_{\infty} \|_{L^2_\xi} \lesssim 2^{\frac{3}{2}k} \| \varphi_k h_{\infty} \|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{\frac{1}{2}k^-} \langle t \rangle^{-1}.
\]

Note that

\[
\partial_\xi h_{\infty}(t, \xi) \sim \langle t \rangle \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{7}{8}}) \cdot \int_{\mathbb{R}^3} e^{i t (|\xi| - \frac{\xi \cdot \eta}{\langle \eta \rangle})} |\hat{V}_k^\xi(\eta)|^2 \; d\eta.
\]

Hence, the \( \partial_\xi h_{\infty} \) estimates follow.

Lemma 4.16. Assume (2.105) holds. Then we have

\[
\| \varphi_k \mathcal{H}_{\infty} \|_{L^2_\xi} \lesssim \varepsilon^2 2^{\frac{3}{2}k^-},
\]

and

\[
\| \varphi_k \mathcal{H}_{\infty} \|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{-k}.
\]

Also, we have

\[
\| \varphi_k \partial_\xi \mathcal{H}_{\infty} \|_{L^2_\xi} \lesssim \varepsilon^2 2^{\frac{3}{2}k^-},
\]

and

\[
\| \varphi_k \partial_\xi \mathcal{H}_{\infty} \|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{-2k}.
\]
Proof. We first integrate over time to obtain
\[
\mathcal{H}_\infty \sim \int_0^t \varphi_{\leq 0}(\xi \langle s \rangle^{\frac{7}{8}}) \int_{\mathbb{R}^3} e^{i s (|\xi| - \frac{\xi \cdot s}{\langle s \rangle})} \left| \widehat{V_{\infty}^{kg}}(\eta) \right|^2 \, d\eta \, ds. \tag{4.86}
\]
\[
\sim \int_{\mathbb{R}^3} \frac{1}{i \left( |\xi| - \frac{\xi \cdot \eta}{\langle \eta \rangle} \right)} \left( e^{i t (|\xi| - \frac{\xi \cdot \eta}{\langle \eta \rangle})} - 1 \right) \left| \widehat{V_{\infty}^{kg}}(\eta) \right|^2 \, d\eta.
\]
Here the cutoff only sets an upper bound for the time integration limit. It will not intervene the integral estimates. A more rigorous version is to understand it as integration by parts in time. Note that
\[
\left| \frac{1}{i \left( |\xi| - \frac{\xi \cdot \eta}{\langle \eta \rangle} \right)} \left( e^{i t (|\xi| - \frac{\xi \cdot \eta}{\langle \eta \rangle})} - 1 \right) \right| \leq |\xi|^{-1}. \tag{4.87}
\]
Hence, we obtain
\[
\| \varphi_k \mathcal{H}_\infty \|_{L^\infty} \lesssim 2^{-k} \left\| \widehat{V_{\infty}^{kg}} \right\|_{L^2_{\eta}} \lesssim \varepsilon^2 2^{-k}, \tag{4.88}
\]
and
\[
\| \varphi_k \mathcal{H}_\infty \|_{L^2_{\xi}} \lesssim 2^{\frac{7}{2} k} \| \varphi_k \mathcal{H}_\infty \|_{L^\infty_{\xi}} \lesssim \varepsilon^2 2^{\frac{7}{2} k}. \tag{4.89}
\]
On the other hand, note that
\[
\partial_{\xi_t} \mathcal{H}_\infty \sim \int_0^t \varphi_{\leq 0}(\xi \langle s \rangle^{\frac{7}{8}}) \int_{\mathbb{R}^3} \langle s \rangle \, e^{i s (|\xi| - \frac{\xi \cdot s}{\langle s \rangle})} \left| \widehat{V_{\infty}^{kg}}(\eta) \right|^2 \, d\eta \, ds. \tag{4.90}
\]
The key is how to handle the extra $\langle s \rangle$. We first integrate by parts in $\eta$ to create $\langle s \rangle^{-1} |\xi|^{-1}$ (thus the net effect is to lose $|\xi|^{-1}$) and then apply the above argument to obtain
\[
\| \varphi_k \partial_{\xi_t} \mathcal{H}_\infty \|_{L^\infty_{\xi}} \lesssim 2^{-2 k} \left\| \widehat{V_{\infty}^{kg}} \right\|_{H^1_{\eta}} \lesssim \varepsilon^2 2^{-2 k}, \tag{4.91}
\]
and
\[
\| \varphi_k \partial_{\xi_t} \mathcal{H}_\infty \|_{L^2_{\xi}} \lesssim 2^{\frac{7}{2} k} \| \varphi_k \partial_{\xi_t} \mathcal{H}_\infty \|_{L^\infty_{\xi}} \lesssim \varepsilon^2 2^{\frac{7}{2} k}. \tag{4.92}
\]
\[\square\]

Remark 4.8. If we directly integrate over time in Lemma 4.15, then there is $\ln \langle t \rangle$ popping out. Hence, we need first integrate in time as in Lemma 4.16 to avoid this mild growth term.

On the other hand, the $\partial_{\xi_t}$ estimate is more obvious. In $h_\infty$ estimate, we lose $\langle t \rangle \sim |\xi|^{-\frac{8}{7}}$. However, in $\mathcal{H}_\infty$ estimate, we only lose $|\xi|^{-1}$.

Lemma 4.17. Assume (2.105) holds. Then for $n \leq N_1 + 1$, we have
\[
\| \varphi_k h_{L,\infty} \|_{L^2_{\xi}} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{\frac{7}{2} k}. \tag{4.93}
\]
Also, we have
\[ \| \varphi_k h \|_{L^\infty_t} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{-k}. \] (4.94)

Also, we have
\[ \| \varphi_k \partial_{\xi_t} h \|_{L^2_t} \lesssim \varepsilon^2 2^k. \] (4.95)

and
\[ \| \varphi_k \partial_{\xi_{\xi_t}} h \|_{L^\infty} \lesssim \varepsilon^2 2^{-k}. \] (4.96)

**Proof.** Denote
\[ \hat{L}[f(\xi)] := \mathcal{F}[\mathcal{F}^{-1} f](\xi), \] (4.97)
and thus by definition (see Remark 4.1 for more specific definition)
\[ \hat{L}[e^{-it\Lambda_{wa}(\xi)} h_{\infty}(t, \xi)] \sim e^{-it\Lambda_{wa}(\xi)} h_{\infty}(t, \xi). \] (4.98)

Note that
\[ e^{-it\Lambda_{wa}(\xi)} h_{\infty}(t, \xi) \sim \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{3}{7}}) \cdot \int_{\mathbb{R}^3} e^{it\langle \xi - \eta \rangle} \overline{V_{\infty}^g(\eta) V_{\infty}^g(\eta)} \, d\eta \] (4.99)
\[ = h_1 + h_2 + h_3, \]

where
\[ h_1(t, \xi) = \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{3}{7}}) \cdot \int_{\mathbb{R}^3} e^{it\langle \xi - \eta \rangle - \langle \eta \rangle} \overline{V_{\infty}^g(\xi - \eta) V_{\infty}^g(\eta)} \, d\eta, \] (4.100)
\[ h_2(t, \xi) = \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{3}{7}}) \cdot \int_{\mathbb{R}^3} \left( e^{it\langle \xi - \eta \rangle} - e^{it\langle \xi - \eta \rangle - \langle \eta \rangle} \right) \overline{V_{\infty}^g(\xi - \eta) V_{\infty}^g(\eta)} \, d\eta, \] (4.101)
\[ h_3(t, \xi) = \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{3}{7}}) \cdot \int_{\mathbb{R}^3} e^{it\langle \xi - \eta \rangle} \left( \overline{V_{\infty}^g(-\eta) - V_{\infty}^g(\xi - \eta)} \right) \overline{V_{\infty}^g(\eta)} \, d\eta. \] (4.102)

Then \( h_1 \) is a perfect convolution. By transferring back and forth between physical space and frequency space, we know
\[ \hat{L}[h_1] \sim \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{3}{7}}) \cdot \int_{\mathbb{R}^3} e^{it\langle \xi - \eta \rangle - \langle \eta \rangle} \overline{V_{L_{1,\infty}^2}(\xi - \eta) V_{L_{2,\infty}^g}(\eta)} \, d\eta. \] (4.103)

Then following the similar proof of Lemma 4.15 we know
\[ \left\| \varphi_k e^{it\Lambda_{wa}(\xi)} \hat{L}[h_1] \right\|_{L^\infty_{\xi_t}} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{-k}. \] (4.104)

On the other hand, using the cutoff, we know
\[ \left| e^{it\langle \xi - \eta \rangle} - e^{it\langle \xi - \eta \rangle - \langle \eta \rangle} \right| \lesssim |\xi|^2 \lesssim \langle t \rangle^{-\frac{2}{7}} \lesssim 1, \] (4.105)
and for $L \in V_n$, 
\[
\left\| \hat{L} \left( e^{it\left( \frac{ax}{\eta} \right)} - e^{it\left( \xi - \eta \right)} \right) \right\|_{L_t^2} \lesssim |\xi|^2 |\xi(t)|^n \lesssim \langle t \rangle^{-\frac{14-n}{8}} \lesssim 1. \quad (4.106)
\]
Hence, following the similar argument of Lemma 4.15, regardless whether $L$ hits the exponential term or $V_{k\eta}^\infty(\xi - \eta)$, we know
\[
\left\| \varphi_k e^{it\Lambda_{wo}(\xi)} \hat{L}[h_2] \right\|_{L_t^\infty} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{-k}. \quad (4.107)
\]
Finally, Minkowski's integral inequality implies
\[
\left\| V_{k\eta}^\infty(-\eta) - V_{k\eta}^\infty(\xi - \eta) \right\|_{L^2} \lesssim \int_0^{|\xi|} \left\| \partial_\sigma V_{k\eta}^\infty(-\eta + c\sigma) \right\|_{L_t^2} \, dc 
\lesssim \int_0^{|\xi|} \left\| \partial_\sigma V_{k\eta}^\infty(-\eta) \right\|_{L_t^2} \, dc \lesssim \varepsilon |\xi| \lesssim \varepsilon \langle t \rangle^{-\frac{7}{8}} \lesssim \varepsilon. \quad (4.108)
\]
Similarly, we have
\[
\left\| \hat{L} \left( V_{k\eta}^\infty(-\eta) - V_{k\eta}^\infty(\xi - \eta) \right) \right\|_{L_t^\infty} \lesssim |\xi| |\xi(t)|^n \lesssim \langle t \rangle^{-\frac{7-n}{8}} \lesssim 1. \quad (4.109)
\]
Hence, following the similar proof of Lemma 4.15, regardless whether $L$ hits the exponential term or $V_{k\eta}^\infty(\xi - \eta)$, we know
\[
\left\| \varphi_k e^{it\Lambda_{wo}(\xi)} \hat{L}[h_3] \right\|_{L_t^\infty} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{-k}. \quad (4.110)
\]
In summary, we have
\[
\left\| \varphi_k h_{L,\infty} \right\|_{L_t^\infty} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{-k}. \quad (4.111)
\]
Then
\[
\left\| \varphi_k h_{L,\infty} \right\|_{L^2} \lesssim 2^{\frac{k}{2}} \left\| \varphi_k h_{L,\infty} \right\|_{L_t^\infty} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{\frac{k}{2}}. \quad (4.112)
\]
The $\partial_{t\xi}$ estimates follow from a similar argument and Lemma 4.15.

\[\square\]

Remark 4.9. It is not difficult to check that among $h_1, h_2, h_3$, actually $h_1$ will dominate since it has the slowest decay. In other words, this theorem states that
\[
h_{L,\infty} \sim \sum_{L_1, L_2} h_{L_1, L_2, \infty} \quad (4.113)
\]
\[
\sim: \sum_{L_1, L_2} \varphi_{\leq 0}(\xi(t)^\frac{7}{8}) \cdot \int_{\mathbb{R}^3} e^{it|\xi|} e^{it(\xi - \eta)} V_{L_1,\infty}^{k\eta} (\xi - \eta) V_{L_2,\infty}^{k\eta} (\eta) \, d\eta.
\]
Hence, we can repeat the above argument to show that
\[
\sum_{\mathcal{L}_1, \mathcal{L}_2} \varphi_{\mathcal{L}_1, \mathcal{L}_2} \approx \sum_{\mathcal{L}_1, \mathcal{L}_2} \varphi_{\leq 0}(\xi (t) \hat{\xi}) \cdot \int_{\mathbb{R}^3} e^{it|\xi|} e^{it(|\xi-\eta|-(\eta-\eta))} \left( e^{iD_{\infty}(t, \eta)} V^{k\eta}_\infty(\xi - \eta) \right)_{\mathcal{L}_1} \left( e^{iD_{\infty}(t, \eta)} V^{k\eta}_\infty(\eta) \right)_{\mathcal{L}_2} d\eta.
\]

This will be used in nonlinear estimates to bound forcing terms.

**Corollary 4.17.1.** Assume (2.103) holds. Then for $n \leq N_1 + 1$, we have
\[
\left\| \varphi_k \left( h_{\mathcal{L}, \infty} - H_{wa}^{low} [V^{k\eta}_{\infty}, \tilde{V}^{k\eta}_{\infty}] \right) \right\|_{L^2_\xi} \lesssim \epsilon^2 \langle t \rangle^{-2} \sum_{k=1}^{2k^-},
\]
and
\[
\left\| \varphi_k \left( h_{\mathcal{L}, \infty} - H_{wa}^{low} [V^{k\eta}_{\infty}, \tilde{V}^{k\eta}_{\infty}] \right) \right\|_{L^\infty_\xi} \lesssim \epsilon^2 \langle t \rangle^{-1} \sum_{k=1}^{2k^-}.
\]

where
\[
H_{wa}^{low} [V^{k\eta}_{\infty}, \tilde{V}^{k\eta}_{\infty}] := \sum_{\mathcal{L}_1, \mathcal{L}_2} \varphi_{\leq 0}(\xi (t) \hat{\xi}) \cdot \int_{\mathbb{R}^3} e^{it|\xi|+|\xi-\eta|-(\eta-\eta)} \left( e^{iD_{\infty}(t, \eta)} V^{k\eta}_\infty(\xi - \eta) \right)_{\mathcal{L}_1} \left( e^{iD_{\infty}(t, \eta)} V^{k\eta}_\infty(\eta) \right)_{\mathcal{L}_2} d\eta.
\]
Lemma 4.18. Assume (2.105) holds. Then for \( n \leq N_1 + 1 \), we have
\[
\| \varphi_k H_{L, \infty} \|_{L^2} \lesssim \varepsilon^2 2^{\frac{1}{2} k^n},
\]
and
\[
\| \varphi_k H_{L, \infty} \|_{L^\infty} \lesssim \varepsilon^2 2^{-k^n}.
\]
Also, we have
\[
\| \varphi_k \partial_\xi \varphi_{H_{L, \infty}} \|_{L^2} \lesssim \varepsilon^2 2^{-2^k 2^{-1} k^n},
\]
and
\[
\| \varphi_k \partial_\xi \varphi_{H_{L, \infty}} \|_{L^\infty} \lesssim \varepsilon^2 2^{-2^k 2^{-2} k^n}.
\]

Proof. Note that
\[
\hat{\mathcal{L}}[e^{-it\Lambda_{\omega\alpha}(\xi)} H_{\infty}(t, \xi)] \sim e^{-it\Lambda_{\omega\alpha}(\xi)} H_{L, \infty}(t, \xi).
\]
From the proof of Lemma 4.16, we first integrate over time to obtain
\[
H_{\infty}(t, \xi) = \int_0^t h_{\infty}(s, \xi) ds \sim \int \frac{1}{|\xi| - \frac{\xi\eta}{(\eta)}} e^{it(|\xi| - \frac{\xi\eta}{(\eta)})} \hat{V}_k^{k^g}(\eta) \hat{V}_k^{k^g}(\eta) d\eta \quad (4.127)
\]
\[
- \int \frac{1}{|\xi| - \frac{\xi\eta}{(\eta)}} \hat{V}_k^{k^g}(\eta) \hat{V}_k^{k^g}(\eta) d\eta.
\]
The first term can be handled as in the proof of Lemma 4.15 and Lemma 4.16. The second term is independent of time, so it can be estimated directly. Then our result naturally follows.

The \( \partial_\xi \) estimates follow from a similar argument and Lemma 4.16.

Lemma 4.19. Assume (2.105) holds. Then for \( n \leq N_1 + 1 \) and \((k, j) \in \mathcal{J}\), we have
\[
\| e^{-it\Lambda_{\omega\alpha}} Q_{jk}(\mathcal{F}^{-1} H_{L, \infty}) \|_{L^\infty} \lesssim \varepsilon^2 \min \left( 2^{-j}, (\langle t \rangle^{-1}) 2^k \right),
\]
and
\[
\sum_{j \geq -k^n} \| e^{-it\Lambda_{\omega\alpha}} Q_{jk}(\mathcal{F}^{-1} H_{L, \infty}) \|_{L^\infty} \lesssim \varepsilon^2 \min \left( 2^k, (\ln \langle t \rangle^{-1}) \langle t \rangle^{-1} \right) 2^k.
\]

Proof. Similar to Lemma 4.10 and Lemma 4.11 using Lemma 4.18.

Lemma 4.20. Assume (2.105) holds. Then for \( n \leq N_1 + 1 \) and \((k, j) \in \mathcal{J}\), we have
\[
\sum_{j \geq -k^n} \| e^{-it\Lambda_{\omega\alpha}} Q_{jk}(\mathcal{F}^{-1} H_{L, \infty}) \|_{L^\infty} \lesssim \varepsilon^2 (\langle t \rangle^{-1}) 2^k.
\]

Proof. Similar to Lemma 4.13 using Lemma 4.18.
4.5 Estimates of $C_\infty$ and $D_\infty$

See the definitions of $C_\infty$ and $D_\infty$ from (2.81) – (2.85).

**Lemma 4.21.** Assume (2.105) holds. Then we have

$$|C_\infty(t, \xi)| \lesssim \varepsilon |\xi| \langle t \rangle^{-1} \ln \langle t \rangle, \quad (4.131)$$

and

$$|D_\infty(t, \xi)| \lesssim \varepsilon |\xi| \langle \ln \langle t \rangle \rangle^2. \quad (4.132)$$

**Proof.** Note that

$$H_\infty(t, \xi) := \varphi_{\leq 0}(\xi \langle t \rangle^{\frac{7}{8}}) \left\{ \sqrt{V_{\infty}}(\xi) + H_\infty(t, \xi) \right\}. \quad (4.133)$$

We directly have

$$\left| \int_{\mathbb{R}^3} e^{i \left( \frac{\xi \cdot \eta}{(t \langle t \rangle)^{\frac{1}{8}}} \right) - \frac{|\eta|}{|\eta|}} \varphi_{\leq 0}(\eta \langle t \rangle^{\frac{7}{8}}) \sqrt{V_{\infty}}(\eta) \, d\eta \right| \lesssim \int_{|\eta| \leq \langle t \rangle} \frac{1}{|\eta|} \sqrt{V_{\infty}}(\eta) \, d\eta \lesssim \varepsilon \langle t \rangle^{-\frac{7}{8}}. \quad (4.134)$$

From Lemma 4.16 we know

$$|H_\infty(t, \xi)| \lesssim \varepsilon^2 |\xi|^{-1}, \quad (4.135)$$

$$|\partial_\xi H_\infty(t, \xi)| \lesssim \varepsilon^2 |\xi|^{-2}. \quad (4.136)$$

Then we decompose the integral

$$\int_{\mathbb{R}^3} e^{i \left( \frac{\xi \cdot \eta}{(t \langle t \rangle)^{\frac{1}{8}}} \right) - \frac{|\eta|}{|\eta|}} \varphi_{\leq 0}(\eta \langle t \rangle^{\frac{7}{8}}) H_\infty(t, \xi) \, d\eta \simeq \int_{|\eta| \leq \langle t \rangle} \frac{1}{|\eta|} H_\infty(t, \eta) \, d\eta. \quad (4.137)$$

For $|\eta| \lesssim \langle t \rangle^{-1}$, we directly integrate to obtain

$$\left| \int_{|\eta| \leq \langle t \rangle^{-1}} \frac{1}{|\eta|} H_\infty(t, \eta) \, d\eta \right| \lesssim \varepsilon^2 \int_{|\eta| \leq \langle t \rangle^{-1}} |\eta|^{-2} \, d\eta \lesssim \varepsilon^2 \langle t \rangle^{-1}. \quad (4.138)$$

For $\langle t \rangle^{-1} \lesssim |\eta| \lesssim \langle t \rangle^{-\frac{7}{8}}$, we integrate by parts in $\eta$ to obtain

$$\left| \int_{\langle t \rangle^{-1} \leq |\eta| \leq \langle t \rangle^{-\frac{7}{8}}} \frac{1}{|\eta|} H_\infty(t, \eta) \, d\eta \right| \lesssim \varepsilon^2 \langle t \rangle^{-1} \int_{\langle t \rangle^{-1} \leq |\eta| \leq \langle t \rangle^{-\frac{7}{8}}} |\eta|^{-3} \, d\eta \lesssim \varepsilon^2 \langle t \rangle^{-1} \ln \langle t \rangle. \quad (4.139)$$

Therefore, from the definition of $C_\infty$, we have

$$|C_\infty(t, \xi)| \lesssim \varepsilon^2 |\xi| \langle t \rangle^{-1} \ln \langle t \rangle. \quad (4.140)$$

Also, the $D_\infty$ estimate follows from integrating the above over time.

**Lemma 4.22.** Assume (2.105) holds. Then we have

$$|
abla_\xi C_\infty(t, \xi)| \lesssim \varepsilon |\xi| \langle t \rangle^{-1} \ln \langle t \rangle, \quad (4.141)$$

and

$$|
abla_\xi D_\infty(t, \xi)| \lesssim \varepsilon |\xi| \langle \ln \langle t \rangle \rangle^2. \quad (4.142)$$
Proof. We first consider $C_\infty$. If $\nabla_\xi$ hits $\frac{\xi}{(\xi)^2}$, then obviously following the same proof as Lemma 4.21, the result holds. Hence, we focus on the case that $\nabla_\xi$ hits the integral.

\[
\nabla_\xi \int_{\mathbb{R}^3} e^{it\left(\frac{\xi}{(\xi)^2}\right)^{-|\eta|}} \frac{1}{|\eta|} \mathcal{H}_\infty(t, \eta) \, d\eta \sim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{it\left(\frac{\xi}{(\xi)^2}\right)^{-|\eta|}} \frac{1}{|\eta|} \mathcal{H}_\infty(t, \eta) \, d\eta (4.143)
\]

Hence, it is almost the same as $C_\infty$ and the same estimate follows. A similar argument can justify $\mathcal{D}_\infty$ case. \qed

Lemma 4.23. Assume (2.105) holds. Then for $1 \leq n \leq N_1 + 1$, we have

\[
\left\| \varphi_k \left[ e^{i\mathcal{D}_\infty(t, \xi)} \hat{V}_{\infty}^{kg}(\xi) \right] \right\|_{L^2} \lesssim \varepsilon \left( \ln (t) \right)^2 Z(k; n - 1)2^k. \quad (4.145)
\]

Also, we have

\[
\left\| \varphi_k \left[ \partial_{\xi} \left( e^{i\mathcal{D}_\infty(t, \xi)} \hat{V}_{\infty}^{kg}(\xi) \right) \right] \right\|_{L^2} \lesssim \varepsilon \left( \ln (t) \right)^2 Z(k; n - 1)2^k. \quad (4.146)
\]

Proof. Let $\hat{W}^{kg}(t, \xi) := e^{i\mathcal{D}_\infty(t, \xi)} \hat{V}_{\infty}^{kg}(\xi)$. By definition

\[
\hat{\mathcal{L}} \left[ e^{-it\Lambda_k(\xi)} \hat{W}^{kg}(t, \xi) \right] \sim e^{-it\Lambda_k(\xi)} \hat{W}^{kg}(t, \xi). \quad (4.147)
\]

Hence, our key is to apply vector fields on $e^{-it\Lambda_k(\xi)} \hat{W}^{kg}(t, \xi)$.

We consider $n = 1$ case. For ordinary derivatives or rotational vector fields, we know

\[
\xi_j \left( e^{-it\Lambda_k(\xi)} \hat{W}^{kg}(t, \xi) \right) = e^{-it\Lambda_k(\xi)} e^{i\mathcal{D}_\infty(t, \xi)} \left( \xi_j \hat{V}_{\infty}^{kg}(\xi) \right), \quad (4.148)
\]

\[
\partial_t \left( e^{-it\Lambda_k(\xi)} \hat{W}^{kg}(t, \xi) \right) = (1 - i|\xi|) e^{-it\Lambda_k(\xi)} \left( \mathcal{C}_\infty(t, \xi) e^{i\mathcal{D}_\infty(t, \xi)} \hat{V}_{\infty}^{kg}(\xi) \right). \quad (4.149)
\]

Hence,

\[
\left\| \hat{P}_k \hat{W}^{kg}_{\mathcal{L}}(t, \xi) \right\|_{L^2} \lesssim (1 + \left\| \xi \mathcal{C}_\infty(t, \xi) \right\|_{L^\infty}) \left\| \hat{V}_{\infty}^{kg}(\xi) \right\|_{L^2} \lesssim \varepsilon Z(k; n - 1)2^k. \quad (4.150)
\]

For rotational vector fields, we know

\[
\Omega_{jk} \left( e^{-it\Lambda_k(\xi)} \hat{W}^{kg}(t, \xi) \right) = e^{-it\Lambda_k(\xi)} \left( \Omega_{jk} e^{i\mathcal{D}_\infty(t, \xi)} \hat{V}_{\infty}^{kg}(\xi) + e^{i\mathcal{D}_\infty(t, \xi)} \left( \Omega_{jk} \hat{V}_{\infty}^{kg}(\xi) \right) \right). \quad (4.151)
\]
For Lorentz vector fields, we know
\[ j_k = 0 \] we have
\[ \sum_k \left\| \psi_k \right\|_{L^2} \lesssim \| \nabla \psi \|_{L^2} + \| \Delta \psi \|_{L^2} \lesssim \varepsilon \ln \langle t \rangle Z(k; n-1) 2^{k-}. \]

For Lorentz vector fields, we know
\[ \Gamma_j \left( e^{-it \Lambda_k \xi} \psi_k (t, \xi) \right) \]
\[ = \varepsilon \left( \xi \xi \right) e^{i D \infty (t, \xi)} \psi_k (t, \xi) + \left( \partial_{\xi} \right) e^{i D \infty (t, \xi)} \psi_k (t, \xi) \]
\[ + \left( \partial_{\xi} \right) \psi_k (t, \xi) e^{i D \infty (t, \xi)} \psi_k (t, \xi) + \varepsilon \left( \ln \langle t \rangle \right)^2 Z(k; n-1) 2^{k-}. \]

In summary, we have verified that for \( n = 1 \)
\[ \left\| P_k \psi_k (t, \xi) \right\|_{L^2} \lesssim \varepsilon \left( \ln \langle t \rangle \right)^2 Z(k; n-1) 2^{k-}. \]

For \( n > 1 \), we may continue the above process. However, the newly created terms are always combination of the above. With the techniques in Lemma \( 4.21 \) and Lemma \( 4.22 \) in hand, the result will follow. The similar argument also applies to \( \partial_{\xi} \) estimate. \( \square \)

**Lemma 4.24.** Assume \( (2.105) \) holds. Then for \( 1 \leq n \leq N_1 + 1 \) and \( (k, j) \in J \), we have
\[ \sum_{j \geq -k^-} \left\| e^{-it \Lambda_k} \mathcal{F}_{j, k} \left\{ \Delta^{-1} \left[ e^{i D \infty (t, \xi)} \psi_k (t, \xi) \right] \right\} \right\|_{L^\infty} \lesssim \varepsilon \left( \ln \langle t \rangle \right)^2 Z(k; n-1) \min \left( 2^{\frac{k}{2}}, 2^{\frac{k}{2}}, 2^{\frac{k}{2}}, 2^{\frac{k}{2}}, (t)^{-1} \right). \]

**Proof.** Similar to Lemma \( 4.11 \) using Lemma \( 4.23 \). \( \square \)

**Lemma 4.25.** Assume \( (2.105) \) holds. Then for \( 0 < n \leq N_1 \) and \( (k, j) \in J \) with \( 1 \leq 2^{k-} - 20 \langle t \rangle \), we have
\[ \sum_{j \geq -k^-} \left\| e^{-it \Lambda_k} \mathcal{F}_{j, k} \left\{ \Delta^{-1} \left[ e^{i D \infty (t, \xi)} \psi_k (t, \xi) \right] \right\} \right\|_{L^\infty} \lesssim \varepsilon \left( \ln \langle t \rangle \right)^2 Z(k; n) \langle t \rangle^{-\frac{\sigma}{2}} 2^{4k^+} 2^{-k-} \left( \frac{1}{2} - \frac{\sigma}{4} \right) \]
for some \( \sigma > 0 \).
Proof. Similar to Lemma 4.12 using Lemma 4.23.

Remark 4.10. For $n = 0$ case, the result also holds, but it does not have $\ln \langle t \rangle$ since there is no vector fields applied. This is extremely crucial.

Lemma 4.26. Assume (2.105) holds. Then for $n \leq N_1 + 1$, we have

$$\left\| \varphi_k \left( \left( iC_\infty(t, \xi) e^{iD_\infty(t, \xi) \widehat{V}_\infty^{kg}(\xi)} \right) - M_{kg}^{low} [e^{iD_\infty \widehat{V}_\infty^{kg}}, H_\infty] \right) \right\|_{L^2_k} \lesssim \varepsilon^2 \langle t \rangle^{-\frac{\gamma}{2}} 2^{-N(n)k^+}, \quad (4.158)$$

where

$$M_{kg}^{low} [e^{iD_\infty \widehat{V}_\infty^{kg}}, H_\infty] \quad (4.159)$$

Proof. We define

$$\hat{L} \left[ e^{-itA_{kg}(\xi)} iC_\infty(t, \xi) e^{iD_\infty(t, \xi) \widehat{V}_\infty^{kg}(\xi)} \right] \sim e^{-itA_{kg}(\xi)} \left( iC_\infty(t, \xi) e^{iD_\infty(t, \xi) \widehat{V}_\infty^{kg}(\xi)} \right) \hat{L}. \quad (4.160)$$

Note that

$$iC_\infty(t, \xi) e^{iD_\infty(t, \xi) \widehat{V}_\infty^{kg}(\xi)} \simeq \sum_{\xi_2 = \pm} \int_{\mathbb{R}^3} e^{it\langle \xi, \xi_2, \eta \rangle} \left| \xi - \eta \right| \left( e^{iD_\infty(t, \xi, \eta) \widehat{V}_\infty^{kg}(\xi, \eta, \eta)} \right) H_\infty^{12}(t, \eta) \mathrm{d}\eta. \quad (4.161)$$

Then we may decompose

$$e^{-itA_{kg}(\xi)} iC_\infty(t, \xi) e^{iD_\infty(t, \xi) \widehat{V}_\infty^{kg}(\xi)} \sim \sum_{\xi_2 = \pm} \left( \mathcal{B}_{1+2}^{+12} + \mathcal{B}_{2+12}^{+12} + \mathcal{B}_{3+12}^{+12} + \mathcal{B}_{4+12}^{+12} \right), \quad (4.162)$$

where

$$\mathcal{B}_{1+2}^{+12} = \sum_{\xi_2 = \pm} \int_{\mathbb{R}^3} e^{it\langle \xi, -\xi, \eta \rangle} \left| \xi - \eta \right|^2 \left( e^{iD_\infty(t, \xi, \eta) \widehat{V}_\infty^{kg}(\xi, \eta)} \right) H_\infty^{12}(t, \eta) \mathrm{d}\eta, \quad (4.163)$$

$$\mathcal{B}_{2+12}^{+12} = \sum_{\xi_2 = \pm} \int_{\mathbb{R}^3} e^{it\langle \xi, -\xi, \eta \rangle} \left( \left| \xi \right|^2 - \left| \xi - \eta \right|^2 \right) \left( e^{iD_\infty(t, \xi, \eta) \widehat{V}_\infty^{kg}(\xi, \eta)} \right) H_\infty^{12}(t, \eta) \mathrm{d}\eta, \quad (4.164)$$

$$\mathcal{B}_{3+12}^{+12} = \sum_{\xi_2 = \pm} \int_{\mathbb{R}^3} \left( e^{it\langle \xi, \xi_2 \rangle} - e^{it\langle -\xi, \xi_2 \rangle} \right) \left| \xi \right|^2 \left( e^{iD_\infty(t, \xi, \eta) \widehat{V}_\infty^{kg}(\xi, \eta)} \right) H_\infty^{12}(t, \eta) \mathrm{d}\eta, \quad (4.165)$$

$$\mathcal{B}_{4+12}^{+12} = \sum_{\xi_2 = \pm} \int_{\mathbb{R}^3} \left( e^{it\langle \xi, \xi_2 \rangle} - e^{it\langle -\xi, \xi_2 \rangle} \right) \left| \xi \right|^2 \left( e^{iD_\infty(t, \xi, \eta) \widehat{V}_\infty^{kg}(\xi, \eta)} \right) H_\infty^{12}(t, \eta) \mathrm{d}\eta. \quad (4.166)$$
Since $\mathcal{R}_{1}^{\pm t_2}$ is a perfect convolution, we know

$$\hat{\mathcal{L}}[\mathcal{R}_{1}^{\pm t_2}] = \sum_{L_1, L_2} \sum_{\pm} \int_{\mathbb{R}^3} e^{it(-\langle \xi - \eta \rangle - t_2|\eta|)} \frac{|\xi - \eta|^2}{|\eta|} \left( e^{iD_\infty(t, \xi - \eta) \overline{V}_\infty} \right)_{L_1} \left( H_{L_2}^{t_2} \right)_{L_2} \, d\eta.$$  \hfill (4.167)

Hence, we only need to bound $\mathcal{R}_{2}^{\pm t_2}, \mathcal{R}_{3}^{\pm t_2}$ and $\mathcal{R}_{4}^{\pm t_2}$.

Note that $H$ incorporates a cutoff function $\varphi_{\leq 0}(\eta \langle t \rangle^{\frac{7}{8}})$ which yields $|\eta| \lesssim \langle t \rangle^{-\frac{7}{8}}$. Since

$$\left| \frac{|\xi|^2}{\langle \xi \rangle} - \frac{|\xi - \eta|^2}{\langle \xi - \eta \rangle} \right| \lesssim |\eta|,$$  \hfill (4.168)

and for $L \in \mathcal{V}_n$

$$\left| \hat{\mathcal{L}} \left( \frac{|\xi|^2}{\langle \xi \rangle} - \frac{|\xi - \eta|^2}{\langle \xi - \eta \rangle} \right) \right| \lesssim |\eta||\eta \langle t \rangle|^n \lesssim |\eta| \langle t \rangle^{\frac{7}{8}} \lesssim |\eta| \langle t \rangle^{\frac{5}{8}},$$  \hfill (4.169)

we have

$$\left\| e^{it\Lambda_{k_0}(\xi)} \hat{\mathcal{L}}[\mathcal{R}_{2}^{\pm t_2}] \right\|_{L^2} \lesssim \sum_{k_1, k_2} \langle t \rangle^{\frac{1}{2} + 2\frac{k_2}{k_1}} \left\| \varphi_{k_1} \left( e^{iD_\infty \overline{V}_{\infty}} \right)_{L_1} \right\|_{L^2} \left\| \varphi_{k_2} H_{L_2}^{t_2} \right\|_{L^2}$$  \hfill (4.170)

$$\lesssim \sum_{k_1, k_2} \langle t \rangle^{\frac{1}{2} - \frac{21}{16}} \left( \varepsilon (\ln \langle t \rangle)^2 2^{-N(n_1)k_1^+} \right) \left( \varepsilon \langle t \rangle^{-\frac{7}{8}} 2^{-N(n_2)k_2^+} \right)$$

$$\lesssim \sum_{k_1, k_2} \varepsilon^2 \langle t \rangle^{-\frac{7}{8}} \left( \ln \langle t \rangle \right)^2 2^{-N(n_1)k_1^+} - N(n_2)k_2^+.$$  

Since

$$\left| e^{it\left( \langle \xi \rangle + \frac{k_0}{\langle \xi \rangle} - t_2|\eta| \right)} - e^{it(-\langle \xi - \eta \rangle - t_2|\eta|)} \right| \lesssim |\eta|^2,$$  \hfill (4.171)

and for $L \in \mathcal{V}_n$

$$\left| \hat{\mathcal{L}} \left( e^{it\left( \langle \xi \rangle + \frac{k_0}{\langle \xi \rangle} - t_2|\eta| \right)} - e^{it(-\langle \xi - \eta \rangle - t_2|\eta|)} \right) \right| \lesssim |\eta|^2 |\eta \langle t \rangle|^n \lesssim |\eta| \langle t \rangle^{-\frac{7}{8}} \lesssim |\eta|,$$  \hfill (4.172)

we have

$$\left\| e^{it\Lambda_{k_0}(\xi)} \hat{\mathcal{L}}[\mathcal{R}_{3}^{\pm t_2}] \right\|_{L^2} \lesssim \sum_{k_1, k_2} 2^{\frac{k_2}{k_1}} \left\| \varphi_{k_1} \left( e^{iD_\infty \overline{V}_{\infty}} \right)_{L_1} \right\|_{L^2} \left\| \varphi_{k_2} H_{L_2}^{t_2} \right\|_{L^2}$$  \hfill (4.173)

$$\lesssim \sum_{k_1, k_2} \langle t \rangle^{-\frac{21}{16}} \left( \varepsilon (\ln \langle t \rangle)^2 2^{-N(n_1)k_1^+} \right) \left( \varepsilon \langle t \rangle^{-\frac{7}{8}} 2^{-N(n_2)k_2^+} \right)$$

$$\lesssim \sum_{k_1, k_2} \varepsilon^2 \langle t \rangle^{-\frac{7}{8}} \left( \ln \langle t \rangle \right)^2 2^{-N(n_1)k_1^+} - N(n_2)k_2^+.$$
For $\mathcal{R}^{+\ell,2}_4$, Minkowski’s integral inequality implies
\[
\left\| V^k_\infty(\xi - \eta) - V^k_\infty(\xi) \right\|_{L^2} \lesssim \int_0^{t} \left\| \partial_\sigma V^k_\infty(\xi - c\sigma) \right\|_{L^2_\xi} \, dc \lesssim \int_0^{t} \left\| \partial_\sigma V^k_\infty(\xi - c\sigma) \right\|_{L^2_\xi} \, dc
\]
(4.174)
\[
\lesssim \int_0^{t} \left\| \partial_\sigma V^k_\infty(\xi) \right\|_{L^2_\xi} \, dc \lesssim |\eta| \left\| \partial_\sigma V^k_\infty(\xi) \right\|_{L^2_\xi}.
\]
Similarly, we have
\[
\left\| e^{iD_\infty(t,\xi-\eta)}V^k_\infty(\xi - \eta) - e^{iD_\infty(t,\xi)}V^k_\infty(\xi) \right\|_{L^2} \lesssim |\eta| \left( \ln \langle t \rangle \right)^2 \left\| \partial_\sigma V^k_\infty(\xi) \right\|_{L^2_\xi}
\]
(4.175)
and
\[
\left\| \left( e^{iD_\infty(t,\xi-\eta)}V^k_\infty(\xi - \eta) \right) - \left( e^{iD_\infty(t,\xi)}V^k_\infty(\xi) \right) \right\|_{L^2} \lesssim |\eta| \left( \ln \langle t \rangle \right)^2 \left\| \partial_\sigma V^k_\infty(\xi) \right\|_{L^2_\xi}
\]
(4.176)
\[
\lesssim |\eta| \langle t \rangle^{\frac{1}{2}} \left( \ln \langle t \rangle \right)^2 \left\| \partial_\sigma V^k_\infty(\xi) \right\|_{L^2_\xi}.
\]
Hence, we have
\[
\left\| e^{iH_k^k(\xi)} \hat{L}[\mathcal{R}^{+\ell,2}_4] \right\|_{L^2}
\]
(4.177)
\[
\lesssim \sum_{k_1, k_2} 2^{\frac{3}{2}k_2} \left\| \varphi_{k_1} \frac{1}{|\eta|} \left( \left( e^{iD_\infty(t,\xi-\eta)}V^k_\infty(\xi - \eta) \right) - \left( e^{iD_\infty(t,\xi)}V^k_\infty(\xi) \right) \right) \right\|_{L^2} \left\| \varphi_{k_2} H^2_{L_\infty,\infty} \right\|_{L^2}
\]
\[
\lesssim \sum_{k_1, k_2} \langle t \rangle^{-\frac{3}{4}} \left( \langle t \rangle^{\frac{1}{2}} \left( \ln \langle t \rangle \right)^2 2^{-N(n_1)k_1} \right) \left( \langle t \rangle^{-\frac{3}{2}} 2^{-N(n_2)k_2} \right)
\]
\[
\lesssim \sum_{k_1, k_2} \langle t \rangle^{\frac{1}{2}} \left( \ln \langle t \rangle \right)^2 2^{-N(n_1)k_1} 2^{-N(n_2)k_2}.
\]
Hence, our result naturally follows. \hfill \square

Remark 4.11. This lemma actually states that
\[
\left( iC_\infty(t, \xi)e^{iD_\infty(t,\xi)}V^k_\infty(\xi) \right)_{L} \sim \sum_{L_1, L_2 \nu_2 = \pm} f^{\nu_2}_{L_1, L_2, \infty}
\]
(4.178)
\[
:= \sum_{L_1, L_2 \nu_2 = \pm} \int_{\mathbb{R}^3} e^{it((\xi) - \langle \xi - \eta \rangle - \xi|\eta|)} 2^{|\xi - \eta|^2 |\eta|} \left( e^{iD_\infty V^k_\infty(\xi - \eta)} \right)_{L_1} \left( H^2_{L_\infty, \infty}(t, \eta) \right)_{L_2} \, d\eta.
\]

4.6 Estimates of $b_\infty$ and $R_\infty$

Lemma 4.27. Assume (2.105) holds. Then for $n \leq N_1 + 1$, we have
\[
\left\| b_\infty \right\|_{L^2} \lesssim \langle t \rangle^{-(N(n)k)}
\]
(4.179)
and
\[
\left\| b_{L, \infty} \right\|_{L^2} \lesssim \langle t \rangle^{-(N(n)k)}
\]
(4.180)
Proof. Note that $H$ has a cutoff function $\varphi_{\leq 0}(\xi \langle t \rangle^{\frac{7}{8}})$. In particular,

$$b_\infty \sim \sum_{k_1, k_2} \int_{\mathbb{R}^3} \varphi_{\leq 0}(\eta \langle t \rangle^{\frac{7}{8}}) e^{i t (\xi + (\xi - \eta) - i 2 |\eta|)} \frac{\xi - \eta}{|\xi - \eta| |\eta|} \left( P_{k_1} e^{i D_\infty \overline{V_{k_2} g_{-}}} \right) \langle t, \xi - \eta \rangle P_{k_2} H_{\infty}^2(\eta) \, d\eta. \quad (4.181)$$

We have

$$\sum_{k_1, k_2} \left\| \sum_{k_1, k_2} \left\| P_{k_1} e^{i D_\infty \overline{V_{k_2} g_{-}}} \right\|_{L^2} \right\|_{L^2} \left\| e^{i t \Lambda_{w_0} P_{k_2} H_{\infty}^{12}} \right\|_{L^\infty} \quad (4.182)$$

$$\leq \sum_{k_1, k_2} 2^{k_1} 2^{-k_2} \left\| P_{k_1} e^{i D_\infty \overline{V_{k_2} g_{-}}} \right\|_{L^2} \left\| e^{i t \Lambda_{w_0} P_{k_2} H_{\infty}^{12}} \right\|_{L^\infty} \quad (4.183)$$

Since $b_\infty$ is a perfect convolution, taking $\mathcal{L}$ just hits every term there. Hence, we have

$$b_{\mathcal{L}_1, \mathcal{L}_2} \sim \sum_{k_1, k_2} \int_{\mathbb{R}^3} \varphi_{\leq 0}(\eta \langle t \rangle^{\frac{7}{8}}) e^{i t (\xi + (\xi - \eta) - i 2 |\eta|)} \frac{\xi - \eta}{|\xi - \eta| |\eta|} \left( e^{i D_\infty \overline{V_{k_2} g_{-}}} \right) \langle t, \xi - \eta \rangle H_{\mathcal{L}_1, \mathcal{L}_2}^1(\eta) \, d\eta. \quad (4.184)$$

Then the rest naturally follows. \qed

**Lemma 4.28.** Assume \textbf{[2.105]} holds. Then for $n \leq N_1 + 1$, we have

$$\| \mathfrak{B}_\infty \|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-\frac{7}{8}} \left( \ln \langle t \rangle \right)^2 2^{-N(n)k}, \quad (4.185)$$

and

$$\| \mathfrak{B}_{\mathcal{L}_1, \mathcal{L}_2} \|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-\frac{7}{8}} \left( \ln \langle t \rangle \right)^2 2^{-N(n)k}. \quad (4.186)$$

**Proof.** Note that

$$\mathfrak{B}_\infty \sim \int_t^\infty e^{i (D_\infty (t, \xi) - D_\infty (s, \xi))} b_\infty (s, \xi) \, ds \quad (4.187)$$

We first integrate by parts in time to gain $\langle t \rangle^{-1}$ decay ($b = 0$ at infinity).

Note that $b_\infty$ and $\mathfrak{B}_\infty$ are only defined for non-resonance case, so we will not create $|\eta|^{-1}$ singularity from phase. Then we have

$$\sum_{k_1, k_2} \left\| \sum_{k_1, k_2} \left\| \varphi_{\leq 0}(\eta \langle t \rangle^{\frac{7}{8}}) e^{i t (\xi + (\xi - \eta) - i 2 |\eta|)} \frac{\xi - \eta}{|\xi - \eta| |\eta|} \right\|_{L^2} \right\|_{L^2} \left\| \right\|_{L^2} \right\|_{L^L} \quad (4.188)$$

$$\lesssim \sum_{k_1, k_2} 2^{\frac{k_1}{2}} 2^{k_1} 2^{-k_2} \left\| P_{k_1} f \right\|_{L^2} \left\| P_{k_2} \mathfrak{B} \right\|_{L^2} \quad (4.189)$$

$$\lesssim \sum_{k_1, k_2} 2^{k_1} 2^{\frac{k_1}{2}} \left\| P_{k_1} f \right\|_{L^2} \left\| P_{k_2} \mathfrak{B} \right\|_{L^2}. \quad (4.190)$$
where \( f \) and \( g \) are \( e^{iD_\infty \overline{V}_{\infty}^{kg}} \) and \( H^\infty_{\delta} \) or their time derivatives. Here, the cutoff takes effect, and we have \( 2^k \lesssim (t)^{-\frac{7}{16}} \). Hence, the rest is obvious. Also, note that

\[
\hat{L} \left[ e^{-it\Lambda_\delta(\xi)} \mathcal{B}_\infty(t, \xi) \right] \sim e^{-it\Lambda_\delta(\xi)} \mathcal{B}_{L,\infty}(t, \xi). \tag{4.188}
\]

The \( \mathcal{L} \) estimate naturally follows. \( \square \)

**Lemma 4.29.** Assume \( (2.105) \) holds. Then for \( 1 \leq n \leq N_1 + 1 \) and \((k, j) \in \mathcal{J} \), we have

\[
\sum_{j \geq -k^-} \left\| e^{-it\Lambda_\delta} \mathcal{Q}_{jk} \mathcal{B}_{L,\infty} \right\|_{L^\infty} \lesssim \varepsilon (t)^{-\frac{7}{16}} \left( \ln (t) \right)^2 Z(k; n - 1) \min \left( 2^k \frac{1}{2} + 2^k \cdot 2^k - 2^k \frac{1}{2} \cdot 2^k \right)^{-1}. \tag{4.189}
\]

**Proof.** Similar to Lemma 4.11 using Lemma 4.28. \( \square \)

**Lemma 4.30.** Assume \( (2.105) \) holds. Then for \( n \leq N_1 \) and \((k, j) \in \mathcal{J} \) with \( 1 \leq 2^{k^-} \cdot 2^k \cdot \langle t \rangle \), we have

\[
\sum_{j \geq -k^-} \left\| e^{-it\Lambda_\delta} \mathcal{Q}_{jk} \mathcal{B}_{L,\infty} \right\|_{L^\infty} \lesssim \varepsilon (t)^{-\frac{7}{16}} \left( \ln (t) \right)^2 Z(k; n) \langle t \rangle^{-\frac{3}{2} + \frac{7}{8} + \frac{3}{8} \cdot 2^{4k} - 2^{-k} \left( \frac{7}{8} - \frac{3}{8} \right)}
\]

for some \( \sigma > 0 \).

**Proof.** Similar to Lemma 4.12 using Lemma 4.28. \( \square \)

### 5 \( S'_1 \) and \( T'_1 \) Estimates for Wave Equation

For \( \mathcal{L} \in \mathcal{V}_n \) with \( n \leq N_1 \), we have (see \( 2.24 \))

\[
(\partial_t + i\Lambda_\omega)U^\omega_{\mathcal{L}} = \mathcal{N}^\omega_{\mathcal{L}} := \mathcal{L} \left[ |\nabla_{t,x}v|^2 + v^2 \right]. \tag{5.1}
\]

Since \( V^\omega_{\mathcal{L}} = e^{it\Lambda_\omega}U^\omega_{\mathcal{L}} \) and \( V^k_{\mathcal{L}} = e^{it\Lambda_\delta}U^k_{\mathcal{L}} \), we have

\[
\partial_t V^\omega_{\mathcal{L}} = e^{it\Lambda_\omega} \mathcal{N}^\omega_{\mathcal{L}}. \tag{5.2}
\]

Due to \( \overline{V^\omega_{\mathcal{L}}} = \overline{G^\omega_{\mathcal{L}}} + \overline{V^\omega_{\mathcal{L}}} + \int_{0}^{t} h_{\mathcal{L},\infty} \), we have

\[
\partial_t \overline{G^\omega_{\mathcal{L}}} = e^{it\Lambda_\omega(\xi)} \mathcal{N}^\omega_{\mathcal{L}} - h_{\mathcal{L},\infty}. \tag{5.3}
\]

In order to control \( \partial_t \overline{G^\omega_{\mathcal{L}}} \), after taking Fourier transform, using Corollary 4.17.2, it suffices to bound

\[
h_{\mathcal{L},\infty} \sim \sum_{\mathcal{L}_1, \mathcal{L}_2} h_{\mathcal{L}_1, \mathcal{L}_2, \infty} \]

\[
: \sim \sum_{\mathcal{L}_1, \mathcal{L}_2} \varphi_{\leq 0}(\xi (t)^{\frac{7}{8}}) \cdot \int_{\mathbb{R}^3} e^{it\xi |e^{it}(\xi - \eta) - \langle \eta \rangle|} \left( e^{iD_\infty \overline{V}_{\infty}^{kg}(\xi - \eta)} \right)_{\mathcal{L}_1} \left( e^{iD_\infty \overline{V}_{\infty}^{kg}(\eta)} \right)_{\mathcal{L}_2} d\eta.
\]
Remark 5.1. Note the fact that for any $\alpha \in \{0, 1, 2, 3\}$ and $\mathcal{L}^* \in \mathcal{V}_n^* = \{\Gamma^\alpha \Omega^b : |a| + |b| \leq n\}$, the commutator

$$[\partial_\alpha, \mathcal{L}^*] = \text{sum of operators of the form } \partial_\beta \mathcal{L}' \text{ for some } \beta \in \{0, 1, 2, 3\}, \mathcal{L}' \in \mathcal{V}_{n-1}^*.$$  

This indicates that interchanging the order of $\mathcal{L}^*$ and $\partial_\alpha$ only produces lower-order terms. Consequently, $U_\mathcal{L} \sim \mathcal{L}[U]$, and the Leibniz rule for $\mathcal{L}$ is valid (up to lower-order terms).

5.1 $S'_1$ Estimates for Wave Equation

Lemma 5.1. Assume \[2.105\] holds and $\mathcal{L} \in \mathcal{V}_n$ with $n \in \{0, 1, \ldots, N_1\}$. We have

$$\left\| \varphi_k \mathbf{1}_{\mathcal{V}_n} \left[ G^{k_1} \mathcal{L}_1, G^{k_2} \mathcal{L}_2 \right] \right\|_{L^2} \lesssim \varepsilon^2 \left( t \right)^{-1} 2^{-N''_{wa}(n)k^+} 2^{\frac{1}{2} k^-}. \quad (5.8)$$

Proof. Without loss of generality, we assume $n_1 \leq n_2$. Note that $a_{1,2}$ does not play a role since $|a_{1,2}| \lesssim 1$. Here for simplicity, we temporarily ignore $t_1$ and $t_2$ superscripts. Our proof mainly relies on two types of bounds:

$$\left\| \varphi_k \mathbf{1}_{\mathcal{V}_n} \left[ P_{k_1} f, P_{k_2} g \right] \right\|_{L^2} \lesssim 2^{\frac{3}{2} \min(k_1, k_2, k)} \left\| P_{k_1} f \right\|_{L^2} \left\| P_{k_2} g \right\|_{L^2}, \quad (5.9)$$

and

$$\left\| \varphi_k \mathbf{1}_{\mathcal{V}_n} \left[ P_{k_1} f, P_{k_2} g \right] \right\|_{L^2} \lesssim \left\| e^{-itA_{kg}} P_{k_1} f \right\|_{L^\infty} \left\| P_{k_2} g \right\|_{L^2}. \quad (5.10)$$

Here we use \[5.5\] and \[5.10\] in different scenarios. In principle, based on Lemma \[5.1\] (the worse scenario is that all $\mathcal{L}$ hit $G^{kg}$, then we cannot use Lemma \[5.3\], we always have

$$\left\| P_{k_2} G_{\mathcal{L}_1}^{kg} \right\|_{L^2} \lesssim \varepsilon \left( t \right)^{-H(n_2)k^+} 2^{-N(n_2)k^+_2}, \quad (5.11)$$

Then we focuses on the bounds of $P_{k_1} G_{\mathcal{L}_1}^{kg}$. Note the following fact:

- We always have $n_1 \leq N_1 - 2$ since $n_1 \leq n_2$ and $n_1 + n_2 \leq N_1$.
- Due to Lemma \[5.3\] we always have $k^+ \lesssim \max(k^+_1, k^+_2)$.

We first discuss the case when $n = N_1 = 3$. We divide it into several cases:
Case I: For $2^{k^-} \lesssim \langle t \rangle^{-1}$, \eqref{eq:5.9} and Lemma \ref{lem:4.3} justify that
\begin{equation}
\sum_{k_1, k_2} \left\| \varphi_k I_{wa} \left[ P_{k_1} G_{L_1}^{k_1}, P_{k_2} G_{L_2}^{k_2} \right] \right\|_{L^2} \lesssim \sum_{k_1, k_2} 2^{\frac{3}{2} \min(k_1, k_2, k)} \left\| P_{k_1} G_{L_1}^{k_1} \right\|_{L^2} \left\| P_{k_2} G_{L_2}^{k_2} \right\|_{L^2} \tag{5.12}
\end{equation}
\begin{align}
&\lesssim \sum_{k_1, k_2} 2^{\frac{1}{2} k^-} \langle t \rangle^{-1} \left( \varepsilon \langle t \rangle - H(n_1) \delta 2^{-N(n_1)k_1^+} 2^{k_1^-} \right) \left( \varepsilon \langle t \rangle - H(n_2) \delta 2^{-N(n_2)k_2^+} \right) \\
&\lesssim \sum_{k_1, k_2} \varepsilon^2 2^{\frac{1}{2} k^-} \langle t \rangle^{-1} \langle t \rangle - (H(n_1) + H(n_2)) \delta 2^{-N(n_1)k_1^+} - N(n_2)k_2^+ 2^{k_1^-}.
\end{align}

Here, we have
\begin{equation}
H(n_1) + H(n_2) \geq H''_{wa}(n), \quad \min \left( N(n_1), N(n_2) \right) \geq N''_{wa}(n). \tag{5.13}
\end{equation}

Case II: For $2^{k^-} \gtrsim \langle t \rangle^{-1}$ and $2^{k^-} \lesssim 2^{k^-}$, \eqref{eq:5.10} and Lemma \ref{lem:4.3} imply that
\begin{equation}
\sum_{k_1, k_2} \left\| \varphi_k I_{wa} \left[ P_{k_1} G_{L_1}^{k_1}, P_{k_2} G_{L_2}^{k_2} \right] \right\|_{L^2} \lesssim \sum_{k_1, k_2} \left\| P_{k_1} e^{it\lambda_{kg}} G_{L_1}^{k_1} \right\|_{L^\infty} \left\| P_{k_2} G_{L_2}^{k_2} \right\|_{L^2} \tag{5.14}
\end{equation}
\begin{align}
&\lesssim \sum_{k_1, k_2} \left( \varepsilon \langle t \rangle - H(n_1+1) \delta 2^{-N(n_1+1)-\frac{3}{2} k_1^+} 2^{\frac{1}{2} k_1^-} \langle t \rangle^{-1} \right) \left( \varepsilon \langle t \rangle - H(n_2) \delta 2^{-N(n_2)k_2^+} \right) \\
&\lesssim \sum_{k_1, k_2} \varepsilon^2 \langle t \rangle^{-1} \langle t \rangle - (H(n_1)+H(n_2)) \delta 2^{-N(n_1+1)-\frac{3}{2} k_1^+} - N(n_2)k_2^+ 2^{\frac{1}{2} k_1^-}.
\end{align}

Note that summation over $k_1$ will result in $2^{\frac{3}{2} k^-}$. Here, we have
\begin{equation}
H(n_1 + 1) + H(n_2) \geq H''_{wa}(n), \quad \min \left( N(n_1 + 1), N(n_2) \right) \geq N''_{wa}(n). \tag{5.15}
\end{equation}

Case III: For $2^{k^-} \gtrsim \langle t \rangle^{-1}$ and $2^{k^-} \lesssim 2^{-k^-} \langle t \rangle^{-1}$, \eqref{eq:5.9} and Lemma \ref{lem:4.3} justify that
\begin{equation}
\sum_{k_1, k_2} \left\| \varphi_k I_{wa} \left[ P_{k_1} G_{L_1}^{k_1}, P_{k_2} G_{L_2}^{k_2} \right] \right\|_{L^2} \lesssim \sum_{k_1, k_2} 2^{\frac{3}{2} \min(k_1, k_2, k)} \left\| P_{k_1} G_{L_1}^{k_1} \right\|_{L^2} \left\| P_{k_2} G_{L_2}^{k_2} \right\|_{L^2} \tag{5.16}
\end{equation}
\begin{align}
&\lesssim \sum_{k_1, k_2} 2^{\frac{3}{2} k^-} \left( \varepsilon \langle t \rangle - H(n_1) \delta 2^{-N(n_1)k_1^+} 2^{k_1^-} \right) \left( \varepsilon \langle t \rangle - H(n_2) \delta 2^{-N(n_2)k_2^+} \right) \\
&\lesssim \sum_{k_1, k_2} \varepsilon^2 2^{\frac{3}{2} k^-} \langle t \rangle^{-1} \langle t \rangle - (H(n_1)+H(n_2)) \delta 2^{-N(n_1)k_1^+} - N(n_2)k_2^+ 2^{k_1^-}.
\end{align}

Note that summation over $k_1$ will result in $2^{-k^-} \langle t \rangle^{-1}$. Here, we have
\begin{equation}
H(n_1) + H(n_2) \geq H''_{wa}(n), \quad \min \left( N(n_1), N(n_2) \right) \geq N''_{wa}(n). \tag{5.17}
\end{equation}
- Case IV: For $2^{2k^-} \gtrsim \langle t \rangle^{-1}$ and $2^{k^-} \gtrsim 2^{k^-}$ and $2^{k^-} \gtrsim 2^{-k^-} \langle t \rangle^{-1}$, (5.10) and Lemma 4.16 (in this case, we naturally have $2^{2k^-} \langle t \rangle \gtrsim 2^{k^-} 2^{-k^-} \langle t \rangle \gtrsim 1$ implies

$$\sum_{k_1, k_2} \left\| \varphi_{k_1} I_{\text{wa}} \left[ P_{k_1} G_{L_1}, P_{k_2} G_{L_2} \right] \right\|_{L^2} \leq \sum_{k_1, k_2} \left\| P_{k_1} e^{-itA_{k_1}G_{L_1}} \right\|_{L^\infty} \left\| P_{k_2} G_{L_2} \right\|_{L^2} \lesssim \sum_{k_1, k_2} \left( \varepsilon \langle t \rangle^{-H(n_1+2)\delta} 2^{-H(n_1+2)-4} k_1^2 k_1^{-2k^-} \langle t \rangle^{-\frac{3}{2} + \frac{\sigma}{8}} \right) \left( \varepsilon \langle t \rangle^{-H(n_2)\delta} 2^{-H(n_2)k_2^2} \right) \lesssim \sum_{k_1, k_2} \varepsilon^2 2^\frac{1}{2} k^- \langle t \rangle^{-\frac{3}{2}} \langle t \rangle^{-(H(n_1+2)+H(n_2))\delta+\frac{\sigma}{8}} 2^{-H(n_1+2)-4} k_1^2 k_1^{-2k^-} \lesssim 2^{-\frac{1}{2} k^-} 2^{-\frac{1}{2} k^-} \lesssim \langle t \rangle^{\frac{1}{2}}.$$

Note that $2^{-\frac{1}{2} k^-} 2^{-\frac{1}{2} k^-} \lesssim \langle t \rangle^{\frac{1}{2}}$. We have

$$\sum_{k_1, k_2} \left\| \varphi_{k_1} I_{\text{wa}} \left[ P_{k_1} G_{L_1}, P_{k_2} G_{L_2} \right] \right\|_{L^2} \lesssim \sum_{k_1, k_2} \varepsilon^2 2^\frac{1}{2} k^- \langle t \rangle^{-1} \langle t \rangle^{-(H(n_1+2)+H(n_2))\delta+\frac{\sigma}{8}} 2^{-H(n_1+2)-4} k_1^2 k_1^{-2k^-} .$$

Here, we have for $\sigma \simeq \delta$

$$H(n_1+2) + H(n_2) - \frac{1}{8} \geq H_{\text{wa}}''(n), \quad \min \left( N(n_1+2) - 4, N(n_2) \right) \geq N_{\text{wa}}''(n) . \quad (5.20)$$

The above discussion works perfectly well for $n = N_1 = 3$ (i.e. $(n_1, n_2) = (0, 3)$ or $(n_1, n_2) = (1, 2)$). For the other $n$, there are subtle problems (a naive application of the above recipe does not give sufficient $N_{\text{wa}}''$, so we need a detailed discussion:

- $n = 2$ (i.e. $(n_1, n_2) = (0, 2)$ or $(n_1, n_2) = (1, 1)$): If $(n_1, n_2) = (0, 2)$, then the above recipe works well. However, if $(n_1, n_2) = (1, 1)$, we need to discuss based on $k, k_1, k_2$ values. Based on Lemma 3.3 if $k \simeq k_2 \gtrsim k_1$ or $k \simeq k_2 \gtrsim k$, we can still use the above recipe; if $k \simeq k_1 \gtrsim k_2$, then in Case I through Case IV, we interchange the status of $G_{k_{L_1}}$ and $G_{k_{L_2}}$ which can improve $N_{\text{wa}}''$ (it is no longer the minimum of two $N$’s terms).

- $n = 1$ (i.e. $(n_1, n_2) = (0, 1)$) or $n = 0$ (i.e. $(n_1, n_2) = (0, 0)$): Based on Lemma 3.3 if $k \simeq k_2 \gtrsim k_1$ or $k \simeq k_2 \gtrsim k$, we can still use the above recipe; if $k \simeq k_1 \gtrsim k_2$, then in Case I through Case IV, we interchange the status of $G_{k_{L_1}}$ and $G_{k_{L_2}}$. $\square$

**Lemma 5.2.** Assume (2.105) holds and $L \in V_n$ with $n \in \{0, 1, \cdots, N_1\}$. We have

$$\left\| \varphi_{k_1} I_{\text{wa}} \left[ e^{itD_{L_1}^c} V_{k_{L_1}} + G_{L_2}, L_2 \right] \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-(1+H(n)\delta)} 2^{-N(n)k^2} 2^\frac{1}{2} k^- , \quad (5.21)$$

$$\left\| \varphi_{k_1} I_{\text{wa}} \left[ e^{itD_{L_1}^c} V_{k_{L_1}} + G_{L_2}, L_2 \right] \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-(1+H(n)\delta)} 2^{-N(n)k^2} 2^\frac{1}{2} k^- . \quad (5.22)$$
Proof. Basically, the proof is similar to that of Lemma 5.1. The only difference is that we do not have $H(n_2)$ time decay, so we should always assign $L^2$ to $G_{L_1}^{k_g,t_1}$ or $G_{L_2}^{k_g,t_2}$. Then $(e^{itD_1^{\perp}V_{\infty}^{k_g,t_1}})_{L_1} + \mathfrak{B}_{L_1,\infty}$ or $(e^{itD_2^{\perp}V_{\infty}^{k_g,t_2}})_{L_2} + \mathfrak{B}_{L_2,\infty}$ may take $L^2$ or $L^\infty$ with Lemma 4.24 following the argument in the proof of Lemma 5.1.

- If only partial $\mathcal{L}$ hits $G^{k_g}$, then we have sufficient time decay and $k^+$ decay since less $\mathcal{L}$ yields faster decay rate. This naturally suppress the $\ln (t)$ from $e^{iD_2 V_{\infty}^{k_g}}$ estimates with $\mathcal{L}$.
- If all $\mathcal{L}$ hits $G^{k_g}$, then this term already achieve the critical time decay and $k^+$ decay in the desired result. In particular, now there is no $\mathcal{L}$ hitting $e^{iD_2 V_{\infty}^{k_g}}$ and thus there is no $\ln (t)$ term.

In both cases, we can achieve the desired result. □

**Lemma 5.3.** Assume $\{\mathfrak{B}^{*}\}$ holds and $\mathcal{L} \in \mathcal{V}_n$ with $n \in \{0, 1, \cdots , N_1\}$. We have

$$\left\| \varphi_k \left\{ \mathcal{I}_{wa}^{t_1,t_2} \left[ \left( e^{itD_1^{\perp}V_{\infty}^{k_g,t_1}} \right)_{L_1}, \left( e^{itD_2^{\perp}V_{\infty}^{k_g,t_2}} \right)_{L_2} \right] - \frac{1}{2} h_{L_1,L_2,\infty} \right\} \right\|_{L^2} \lesssim \varepsilon^2 (t)^{-(1+H(n)\delta)} 2^{-N(n)k^+} 2^{\frac{1}{2}k^-}.$$

**Proof.** First note that all terms related to $\mathfrak{B}_{L_1,\infty}$ can be bounded similar to that of Lemma 5.1 since Lemma 4.28 and Lemma 4.29 provides sufficient time decay. Then we only consider the rest. For the resonant case $(t_1, t_2) = (+, -)$ or $(t_1, t_2) = (-, +)$, we further decompose $\mathcal{I}_{wa}^{t_1,t_2}$ into low-frequency and high-frequency parts:

$$\mathcal{I}_{wa}^{t_1,t_2} \left[ \left( e^{itD_1^{\perp}V_{\infty}^{k_g,t_1}} \right)_{L_1}, \left( e^{itD_2^{\perp}V_{\infty}^{k_g,t_2}} \right)_{L_2} \right] = \varphi_{\leq 0} (\xi \langle t \rangle^{\frac{7}{6}}) \cdot \mathcal{I}_{wa}^{t_1,t_2} \left[ \left( e^{itD_1^{\perp}V_{\infty}^{k_g,t_1}} \right)_{L_1}, \left( e^{itD_2^{\perp}V_{\infty}^{k_g,t_2}} \right)_{L_2} \right]$$

$$+ \varphi_{\geq 0} (\xi \langle t \rangle^{\frac{7}{6}}) \cdot \mathcal{I}_{wa}^{t_1,t_2} \left[ \left( e^{itD_1^{\perp}V_{\infty}^{k_g,t_1}} \right)_{L_1}, \left( e^{itD_2^{\perp}V_{\infty}^{k_g,t_2}} \right)_{L_2} \right].$$

$h_{\infty}$ only corresponds to low-frequency part.

For the low-frequency part, since $\|y\| \sim 2^k \lesssim \langle t \rangle^{-\frac{7}{6}}$, (even without cutoff, using integration by parts in $\eta$ and \[5.9\]), we can get $(t)^{-1}$ decay for free, but cutoff can provide slightly faster decay. Let $g_1 = \left( e^{itD_1^{\perp}V_{\infty}^{k_g,t_1}} \right)_{L_1}$ and $g_2 = \left( e^{itD_2^{\perp}V_{\infty}^{k_g,t_2}} \right)_{L_2}$. Noting that $|a(\xi, \eta) - 2| \lesssim |\xi|$, based on Corollary 4.17.1 and Lemma 4.24, we know the low-frequency part

$$\left\| \varphi_k \left\{ \varphi_{\leq 0} (\xi \langle t \rangle^{\frac{7}{6}}) \cdot \mathcal{I}_{wa}^{t_1,t_2} \left[ \left( e^{itD_1^{\perp}V_{\infty}^{k_g,t_1}} \right)_{L_1}, \left( e^{itD_2^{\perp}V_{\infty}^{k_g,t_2}} \right)_{L_2} \right] - \frac{1}{2} h_{L_1,L_2,\infty} \right\} \right\|_{L^2} \lesssim \varepsilon^2 (t)^{-(1+H(n)\delta)} 2^{-N(n)k^+} 2^{\frac{1}{2}k^-}.$$

$$\lesssim 2^{\frac{1}{2}k} \left\| \varphi_k \left\{ \varphi_{\leq 0} (\xi \langle t \rangle^{\frac{7}{6}}) \cdot \int_{\mathbb{R}^3} e^{it(\xi + (\xi - \eta)(\xi - \eta))} \left( a(\xi, \eta) - 2 \right) \left( e^{iD_\infty V_{\infty}^{k_g} \langle \xi - \eta \rangle} \right)_{L_1} \left( e^{iD_\infty V_{\infty}^{k_g} \langle \eta \rangle} \right)_{L_2} \right\} \right\|_{L^2}$$

$$\lesssim 2^{\frac{1}{2}k} \left\| \varphi_k \left\{ \varphi_{\leq 0} (\xi \langle t \rangle^{\frac{7}{6}}) \cdot \int_{\mathbb{R}^3} e^{it(\xi + (\xi - \eta)(\xi - \eta))} \left( a(\xi, \eta) - 2 \right) \left( e^{iD_\infty V_{\infty}^{k_g} \langle \xi - \eta \rangle} \right)_{L_1} \left( e^{iD_\infty V_{\infty}^{k_g} \langle \eta \rangle} \right)_{L_2} \right\} \right\|_{L^\infty}$$

$$\lesssim 2^{\frac{1}{2}k} \left\| \left( e^{iD_\infty V_{\infty}^{k_g} \langle \xi - \eta \rangle} \right)_{L_1} \right\|_{L^2} \left\| \left( e^{iD_\infty V_{\infty}^{k_g} \langle \eta \rangle} \right)_{L_2} \right\|_{L^2} \lesssim \varepsilon^2 (t)^{-\frac{7}{6}} \left( \ln (t) \right)^{1/2} 2^{\frac{1}{2}k^-}.$$
Here the constant in $\frac{1}{2} h_{L_1, L_\infty}$ is 2, so we have the difference $a - 2$. Previously, we usually ignore the constant in the estimates, but they perfectly match.

For the high-frequency part, since $|\xi| > \frac{a}{2}$, we integrate by parts in $\eta$ twice using Lemma 3.2 to obtain

$$
\sum_{k_1, k_2} \left\| \varphi_k \varphi \geq 0 \left( \xi \langle t \rangle^{\frac{7}{2}} \right) \cdot \tilde{I}_{wa}^{112} \left[ P_{k_1 g_1}, P_{k_2 g_2} \right] \right\|_{L^2} 
\lesssim \sum_{k_1, k_2} 2^{\frac{3}{2}k} \left\| \varphi_k \varphi \geq 0 \left( \xi \langle t \rangle^{\frac{7}{2}} \right) \cdot \tilde{I}_{wa}^{112} \left[ P_{k_1 g_1}, P_{k_2 g_2} \right] \right\|_{L^\infty}
\lesssim \sum_{k_1, k_2} 2^{\frac{3}{2}k} \langle t \rangle^{\frac{7}{2}} \left\| \varphi_k \varphi \geq 0 \right\|_{H^2_{\xi-\eta}} \left\| \tilde{g}_1 \right\|_{H^2_{\eta}}
\lesssim \varepsilon^{2} 2^{-\frac{1}{2}k} \langle t \rangle^{\frac{7}{2}} \sum_{k_1, k_2} 2^{-N(n_1)k_1^+ - N(n_2)k_2^+} 2^{k_1^- + k_2^-}
\lesssim \varepsilon^{2} 2^{\frac{1}{2}k} \langle t \rangle^{\frac{7}{2}} 2^{-N(n)k^+}.
$$

For the non-resonant case $(t_1, t_2) = (+, +)$ or $(t_1, t_2) = (-, -)$, we use stationary phase argument in Lemma 5.1 to justify

$$
\sum_{k_1, k_2} \left\| \varphi_k \tilde{I}_{wa}^{112} \left[ P_{k_1 g_1}, P_{k_2 g_2} \right] \right\|_{L^2}
\lesssim \sum_{k_1, k_2} 2^{\frac{1}{2}k} \left\| \varphi_k \tilde{I}_{wa}^{112} \left[ P_{k_1 g_1}, P_{k_2 g_2} \right] \right\|_{L^\infty}
\lesssim \sum_{k_1, k_2} 2^{\frac{1}{2}k} \langle t \rangle^{\frac{7}{2}} \left\| \varphi_k \tilde{I}_{wa}^{112} \left[ P_{k_1 g_1}, P_{k_2 g_2} \right] \right\|_{L^\infty}
\lesssim \varepsilon^{2} 2^{\frac{1}{2}k} \langle t \rangle^{\frac{7}{2}} \sum_{k_1, k_2} 2^{\frac{1}{2}k^+ - N(n_1)k_1^+ - N(n_2)k_2^+} 2^{k_1^- + k_2^-}
\lesssim \varepsilon^{2} 2^{\frac{1}{2}k} \langle t \rangle^{\frac{7}{2}} 2^{-\frac{19}{8}(N(n)-5)k^+}.
$$

Summarizing all above, we get the desired result. □

Combining Lemma 5.1 Lemma 5.2 and Lemma 5.3 we get the boundedness and contraction theorems:

**Theorem 5.4.** Assume (2.105) holds and $L \in V_n$ with $n \in \{0, 1, \cdots, N_1\}$. We have

$$
\| P_{k} \partial_t G^{wa}_{L_1, L_2} \|_{L^2} \lesssim \varepsilon^{2} \langle t \rangle^{-(1 + H^w(n)\delta)} 2^{-N^w(n)k^+} 2^{\frac{1}{2}k^-}.
$$

**Theorem 5.5.** Assume $(G^{wa}_{1, G^{kw}_{1}}, G^{kw}_{2})$ and $(G^{wa}_{1, G^{kw}_{2}}, G^{kw}_{1})$ are two sets of solutions satisfying (2.105) and $L \in V_n$ with $n \in \{0, 1, \cdots, N_1\}$. We have

$$
\| P_{k} \partial_t (G^{wa}_{1,L} - G^{wa}_{2,L}) \|_{L^2} \lesssim \varepsilon \| G^{kw}_{1} - G^{kw}_{2} \|_{X_2} \langle t \rangle^{-(1 + H^w(n)\delta)} 2^{-N^w(n)k^+} 2^{\frac{1}{2}k^-}.
$$

**5.2 $T^1_\ell$ Estimates for Wave Equation**

**Lemma 5.6.** Assume (2.105) holds and $L \in V_n$ with $n \in \{0, 1, \cdots, N_1 - 1\}$ and $\ell = \{1, 2, 3\}$. We have

$$
\left\| \varphi_k \partial_{\xi} \left\{ e^{-it\Lambda_{wa}(\xi)} \tilde{I}_{wa}^{112} \left[ G^{kw_{1, t_1}}, G^{kw_{1, t_2}} \right] \right\} \right\|_{L^2} \lesssim \varepsilon^{2} \langle t \rangle^{H^w(n)\delta} 2^{-N^w(n)k^+} 2^{\frac{1}{2}k^-}.
$$
Proof. This is similar to that of Lemma 5.1 and of [6, Lemma 4.2]. Since
\[ \mathbf{I}_{wa}^{1/2} \left[ \mathcal{C}^{k_{g,1}} \mathcal{C}^{k_{g,2}} \right] (t, \xi) \sim \int_{\mathbb{R}^3} e^{i t \Phi_{wa}^{1/2}} a_{\iota_{1/2}} (\xi, \eta) \mathcal{C}^{k_{g,1}} \mathcal{L}_1 (t, \xi - \eta) \mathcal{C}^{k_{g,2}} \mathcal{L}_2 (t, \eta) \, d\eta, \] (5.31)
\( \xi \) derivative may hit the phase \( e^{-itA_{wa}e^{it\Phi_{wa}^{1/2}}} \), or the multiplier \( a_{\iota_{1/2}} \) or the function \( \mathcal{C}^{k_{g,1}} (\xi - \eta) \).

- If \( \xi \) derivative hits the phase \( e^{-itA_{wa}e^{it\Phi_{wa}^{1/2}}} \), then we have an extra \( t \) term popping out. Following exactly the same argument as in the proof of Lemma 5.1 we get the desired result.

- If \( \xi \) derivative hits the multiplier \( a_{\iota_{1/2}} \), then nothing happens and we follow the same proof of Lemma 5.1.

- If \( \xi \) derivative hits the function \( \mathcal{C}^{k_{g,1}} (\xi - \eta) \), then we simply use (5.9) and Lemma 4.1 to obtain

\[ \left\| \varphi_k \mathbf{I}_{wa}^{1/2} \left[ \varphi_{k_1} \left( \partial \xi \mathcal{C}^{k_{g,1}} \mathcal{L}_1 \right), \varphi_{k_2} \mathcal{C}^{k_{g,2}} \mathcal{L}_2 \right] \right\|_{L^2} \leq 2^s \min(k_1, k_2, k) \left\| \varphi_{k_1} \left( \partial \xi \mathcal{C}^{k_{g,1}} \mathcal{L}_1 \right) \right\|_{L^2} \left\| \varphi_{k_2} \mathcal{C}^{k_{g,2}} \mathcal{L}_2 \right\|_{L^2} \] (5.32)

\[ \leq \varepsilon \frac{s}{2} \min(k_1, k_2, k) \left\{ \left( H(h(n+1)+H(n)) \delta_0^2 - (n(n+1)+1)(n_1^2 - n_2^2) \right. \right. \right. \] (5.33)

Here, through change of variable in convolution, we can always assume \( k_1 \leq k_2 \), so we know both \( H''_{wa}(n) \) and \( N''_{wa}(n) \) requirements can be met.

Lemma 5.7. Assume \( \mathcal{L} \in \mathcal{V}_n \) with \( n \in \{0, 1, \ldots, N_1 - 1\} \) and \( \ell = \{1, 2, 3\} \). We have

\[ \left\| \varphi_k \partial_{\xi_{\ell}} \left\{ e^{-itA_{wa}(\xi)} \left( \mathbf{I}_{wa}^{1/2} \left[ \mathcal{C}^{k_{g,1}} \mathcal{L}_1, \mathbf{W}_{\ell, L^2} \right] + \mathbf{B}_{\ell, L^2} \right) \right\} \right\|_{L^2} \leq \varepsilon \frac{s}{2} \left( t \right) - H''_{wa}(n) \delta_0^2 - 2N''_{wa}(n) k^+ \] (5.34)

Lemma 5.8. Assume \( \mathcal{L} \in \mathcal{V}_n \) with \( n \in \{0, 1, \ldots, N_1 - 1\} \) and \( \ell = \{1, 2, 3\} \). We have

\[ \left\| \varphi_k \partial_{\xi_{\ell}} \left\{ e^{-itA_{wa}(\xi)} \left( \mathbf{I}_{wa}^{1/2} \left[ \left( e^{itD_{\ell=1}^{1/2}} \mathcal{W}_{\ell=1, \infty} \right) \mathcal{L}_1, \mathbf{B}_{\ell=1, \infty} + \mathcal{C}^{k_{g,1}} \mathcal{L}_2, \mathbf{B}_{\ell=2, \infty} \right] \right\} \right\|_{L^2} \leq \varepsilon \frac{s}{2} \left( t \right) - H''_{wa}(n) \delta_0^2 - 2N''_{wa}(n) k^+ \] (5.35)

Proof. We just apply the same argument as in Lemma 5.3 and Lemma 5.6 to get the desired result.
Combining Lemma 5.6, Lemma 5.7, and Lemma 5.8, we get the boundedness and contraction theorems:

**Theorem 5.9.** Assume \((2.105)\) holds and \(\mathcal{L} \in \mathcal{V}_n\) with \(n \in \{0, 1, \ldots, N_1 - 1\}\). We have

\[
\left\| \varphi_k \partial_t \left\{ \hat{N}^{wa}_1 - e^{-it\Lambda_{wa}(\xi)}h_{\mathcal{L},\infty} \right\} \right\|_{L^2} \lesssim \varepsilon^2 \| t \|^{-H''_{wa}(\eta)} \delta 2^{-N''_{wa}(n)k^+} \frac{1}{2^k}. \tag{5.36}
\]

**Theorem 5.10.** Assume \((G_{1,wa}^1, G_{1,wa}^2)\) and \((G_{2,wa}^1, G_{2,wa}^2)\) are two sets of solutions satisfying \((2.105)\) and \(\mathcal{L} \in \mathcal{V}_n\) with \(n \in \{0, 1, \ldots, N_1 - 1\}\). \(N_{1,wa}^1\) and \(N_{1,wa}^2\) are corresponding nonlinear terms. Then we have

\[
\left\| \varphi_k \partial_t \left\{ e^{-it\Lambda_{wa}(\xi)} \left( \hat{N}^{wa}_1, \hat{N}^{wa}_2 \right) \right\} \right\|_{L^2} \lesssim \varepsilon \left( \left\| G_{1,wa}^1 - G_{2,wa}^1 \right\|_{X_1} + \left\| G_{1,wa}^2 - G_{2,wa}^2 \right\|_{X_2} \right) (t) \| 2^{-H''_{wa}(\eta)} \delta 2^{-N''_{wa}(n)k^+} \frac{1}{2^k}. \tag{5.37}
\]

### 6 \( S'_2 \) and \( T'_2 \) Estimates for Klein-Gordon Equation

Since

\[
(\partial_t + i\Lambda_{kg})U^{'kg}_L = N^{'kg}_L := \mathcal{L} \left[ u \Delta v \right], \tag{6.1}
\]

and \(V^{'wa}_L = e^{it\Lambda_{wa}}U^{'wa}_L\) and \(V^{'kg}_L = e^{it\Lambda_{kg}}U^{'kg}_L\), we have

\[
\partial_t V^{'kg}_L = e^{it\Lambda_{kg}}N^{'kg}_L. \tag{6.2}
\]

Due to \(\tilde{V}^{'kg}_L = \tilde{G}^{'kg}_L + \left( e^{iD_\infty}V^{'kg}_\infty \right)_L + \mathcal{B}_{L,\infty}, \) we have

\[
\partial_t \tilde{G}^{'kg}_L = e^{it\Lambda_{kg}(\xi)}N^{'kg}_L - \left( iC_{\infty} e^{iD_\infty}V^{'kg}_\infty \right)_L - b_{L,\infty} - iC_{\infty} \mathcal{B}_{L,\infty}. \tag{6.3}
\]

Due to Lemma 4.21 and Lemma 4.28, we know \(iC_{\infty} \mathcal{B}_{L,\infty}\) decays very fast. In order to control \(\partial_t \tilde{G}^{'kg}_L\), after taking Fourier transform, using Lemma 4.27, it suffices to bound

\[
\left( iC_{\infty} e^{iD_\infty}V^{'kg}_\infty \right)_L \sim \sum_{L_1, L_2, \ell_2 = \pm} \frac{f^{\ell_2}_{L_1, L_2, \infty}}{L_1, L_2, \infty} \tag{6.4}
\]

\[
= \sum_{L_1, L_2, \ell_2 = \pm} \int_{R^3} e^{it(\xi - (\xi - \eta) - \ell_2 y)} \left| \frac{\xi - \eta}{\xi - \eta} \right|^2 \left( e^{iD_\infty}V^{'kg}_\infty(\xi - \eta) \right) \left( H^{'kg}_\infty(t, \eta) \right) \frac{1}{L_1} \frac{1}{L_2} \left( H^{'kg}_{L_2,\infty}(\eta) \right), \tag{6.5}
\]

and

\[
\mathcal{B}_{L,\infty} \sim \sum_{L_1, L_2, \ell_2 = \pm} b^{\ell_2}_{L_1, L_2, \infty} \tag{6.5}
\]

\[
= \sum_{\ell_1, \ell_2 = \pm} \int_{R^3} \varphi_{\leq 0}(\eta \left( \ell_1 \right)^2) e^{it(\xi + (\xi - \eta) - \ell_2 y)} \left| \frac{\xi - \eta}{\xi - \eta} \right|^2 \left( e^{iD_\infty}V^{'kg}_\infty(\xi - \eta) \right) \left( H^{'kg}_\infty(t, \eta) \right) \frac{1}{L_1} \frac{1}{L_2} \left( H^{'kg}_{L_2,\infty}(\eta) \right), \tag{6.6}
\]

\[
\text{and}
\]

\[
\sum_{\ell_1, \ell_2 = \pm} \left( e^{iD_\infty}V^{'kg}_{L_1 \ell_1, L_2 \ell_2} \right) (t, \xi) \sim \int_{R^3} e^{i\Phi_{\ell_1 \ell_2}} b_{\ell_1 \ell_2}(\xi, \eta) \tilde{V}^{'kg}_{L_1 \ell_1}(t, \xi - \eta) \tilde{V}^{'wa}_{L_2 \ell_2}(t, \eta) d\eta, \tag{6.6}
\]

\[
\sum_{\ell_1, \ell_2 = \pm} \left( e^{iD_\infty}V^{'kg}_{L_1 \ell_1, L_2 \ell_2} \right) (t, \xi) \sim \int_{R^3} e^{i\Phi_{\ell_1 \ell_2}} b_{\ell_1 \ell_2}(\xi, \eta) \tilde{V}^{'kg}_{L_1 \ell_1}(t, \xi - \eta) \tilde{V}^{'wa}_{L_2 \ell_2}(t, \eta) d\eta, \tag{6.6}
\]
for any \( \nu_1, \nu_2 \in \{+, -, \} \), \( \mathcal{L}_1 \in \mathcal{V}_{n_1} \) and \( \mathcal{L}_2 \in \mathcal{V}_{n_2} \) with \( n_1 + n_2 \leq n \). In particular, we may further decompose

\[
\mathbf{I}_{k_1}^{\nu_1} \left[ \mathcal{L}_1, \mathcal{L}_2 \right] \sim \mathbf{I}_{k_1}^{\nu_1} \left[ \mathcal{L}_1, \mathcal{L}_2 \right] \quad \text{(6.7)}
\]

\[
+ \mathbf{I}_{k_2}^{\nu_2} \left[ \mathcal{L}_1, \mathcal{L}_2 \right] + \mathbf{I}_{k_2}^{\nu_2} \left[ \mathcal{L}_1, \mathcal{L}_2 \right] \quad \text{(6.7)}
\]

\[
+ \mathbf{I}_{k_2}^{\nu_2} \left[ \mathcal{L}_1, \mathcal{L}_2 \right] + \mathbf{I}_{k_2}^{\nu_2} \left[ \mathcal{L}_1, \mathcal{L}_2 \right].
\]

### 6.1 \( S'_2 \) Estimates for Klein-Gordon Equation

**Lemma 6.1.** Assume \((2.10)\) holds and \( \mathcal{L} \in \mathcal{V}_n \) with \( n \in \{0, 1, \ldots, N_1\} \). We have

\[
\left\| \varphi_k \mathbf{I}_{k_1}^{\nu_1} \left[ \mathcal{L}_1, \mathcal{L}_2 \right] \left[ G_{k_1}^{\nu_1}, G_{k_2}^{\nu_2} \right] \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-(1 + H''_{k_1}(\eta))} 2^{-N_{k_2}(n)k_1^+}. \tag{6.8}
\]

**Proof.** Note that \( |b_{\nu_1 \nu_2}(\xi, \eta)| \lesssim 2^{k_1^+ + 2k_2^-} \). Here for simplicity, we temporarily ignore \( \nu_1 \) and \( \nu_2 \) superscripts. Our proof mainly relies on two types of bounds

\[
\left\| \varphi_k \mathbf{I}_{k_1}^{\nu_1} \left[ P_{k_1} f, P_{k_2} g \right] \right\|_{L^2} \lesssim 2^{k_1^+ + 2k_2^-} \| P_{k_1} f \|_{L^2} \| P_{k_2} g \|_{L^2}, \tag{6.9}
\]

and

\[
\left\| \varphi_k \mathbf{I}_{k_1}^{\nu_1} \left[ P_{k_1} f, P_{k_2} g \right] \right\|_{L^2} \lesssim 2^{k_1^+ + 2k_2^-} \| P_{k_1} f \|_{L^2} \| P_{k_2} g \|_{L^2}, \tag{6.10}
\]

\[
\left\| \varphi_k \mathbf{I}_{k_1}^{\nu_1} \left[ P_{k_1} f, P_{k_2} g \right] \right\|_{L^2} \lesssim 2^{k_1^+ + 2k_2^-} \| P_{k_1} f \|_{L^2} \| P_{k_2} g \|_{L^2}. \tag{6.11}
\]

Considering Lemma \(3.3\) it suffices to discuss the cases \( k_1 \leq k_2 \) and \( k_2 \leq k_1 \):

- Case 1: \( k_1 \leq k_2 \) and \( n_1 \leq N_1 - 1 \): we use \((6.10)\) combined with Lemma \(4.5\) and Lemma \(4.1\) to get

\[
\sum_{k_1, k_2} \left\| \varphi_k \mathbf{I}_{k_1}^{k_1^+} \left[ P_{k_1} G_{k_1}^{k_1}, P_{k_2} G_{k_2}^{k_2} \right] \right\|_{L^2} \lesssim \sum_{k_1, k_2} 2^{k_1^+ + 2k_2^-} \| P_{k_1} e^{-itL_{k_1}} G_{k_1}^{k_1} \|_{L^\infty} \| P_{k_2} G_{k_2}^{k_2} \|_{L^2}
\]

\[
\lesssim \sum_{k_1, k_2} \left( \varepsilon \langle t \rangle^{-1} \langle t \rangle^{-H(n_1 + 1)} 2^{-N(n_1 + 1) \frac{5}{2} k_1^+ + \frac{5}{2} k_2^-} \right) \left( \varepsilon \langle t \rangle^{-H(n_2)} 2^{-N(n_2)k_2^+ + \frac{5}{2} k_1^+ + \frac{5}{2} k_2^-} \right)
\]

\[
\lesssim \sum_{k_1, k_2} \varepsilon^2 \langle t \rangle^{-1} \langle t \rangle^{-H(n_1 + 1) + H(n_2)} 2^{-N(n_1 + 1) \frac{5}{2} k_1^+ + \frac{5}{2} k_2^-} - N(n_2 + 1) k_2^+ + \frac{5}{2} k_1^+ + \frac{5}{2} k_2^-}
\]

Here, we have

\[
H(n_1 + 1) + H(n_2) \geq H''_{k_1}(n), \quad \min \left( N(n_1 + 1) - \frac{5}{2}, N(n_2 + 1) \right) \geq N''_{k_1}(n). \tag{6.13}
\]
Case II: \( k_1 \leq k_2 \) and \( n_1 = N_1 \): we use (6.11) combined with Lemma 4.1 and Lemma 4.5 to obtain
\[
\sum_{k_1, k_2} \left\| \varphi_k I_{k_2} [P_{k_1} G_{k_1}^{k_2}, P_{k_2} G_{k_2}^{k_2}] \right\|_{L^2} \leq 2^{k_1} 2^{-k_2} \left\| P_{k_1} G_{k_1}^{k_2} \right\|_{L^2} \left\| P_{k_2} e^{it \Lambda_{k_2}} G_{k_2}^{k_2} \right\|_{L^\infty}
\]
\[
\lesssim \sum_{k_1, k_2} (\varepsilon \langle t \rangle)^{-H(N_1)\delta} 2^{-N(N_1)k_1^+} \left( \varepsilon \langle t \rangle^{-1} \langle t \rangle^{-H(1)\delta} (\ln \langle t \rangle) 2^{-N(1)k_2^+} 2^{k_2^-} \right)
\]
\[
\lesssim \sum_{k_1, k_2} (\varepsilon \langle t \rangle)^{-1} \langle t \rangle^{-(H(N_1)+H(1)\delta)} (\ln \langle t \rangle) 2^{-N(N_1)k_1^+} 2^{-N(1)k_2^+} 2^{k_2^-}.
\]
Here, we have
\[
H(N_1) + H(1) \geq H_{k_2}''(N_1), \quad \min \left( N(N_1), N(1) \right) \geq N_{k_2}''(N_1). \tag{6.15}
\]

Case III: \( k_2 \leq k_1 \) and \( n_2 \leq N_1 - 1 \): we use (6.11) combined with Lemma 4.5 and Lemma 4.1 to get
\[
\sum_{k_1, k_2} \left\| \varphi_k I_{k_2} [P_{k_1} G_{k_1}^{k_2}, P_{k_2} G_{k_2}^{k_2}] \right\|_{L^2} \leq 2^{k_1} 2^{-k_2} \left\| P_{k_1} G_{k_1}^{k_2} \right\|_{L^2} \left\| P_{k_2} e^{it \Lambda_{k_2}} G_{k_2}^{k_2} \right\|_{L^\infty}
\]
\[
\lesssim \sum_{k_1, k_2} (\varepsilon \langle t \rangle)^{-H(n_1)\delta} 2^{-N(n_1)k_1^+} \left( \varepsilon \langle t \rangle^{-1} \langle t \rangle^{-H(n_2+1)\delta} (\ln \langle t \rangle) 2^{-N(n_2+1)k_2^+} 2^{k_2^-} \right)
\]
\[
\lesssim \sum_{k_1, k_2} (\varepsilon \langle t \rangle)^{-1} \langle t \rangle^{-(H(n_1)+H(n_2+1)\delta)} (\ln \langle t \rangle) 2^{-(N(n_1)\delta)k_1^+} 2^{-(N(n_2+1)\delta)k_2^+} 2^{k_2^-}.
\]
Here, we have
\[
H(n_1) + H(n_2 + 1) \geq H_{k_2}''(n), \quad \min \left( N(n_1) - 1, N(n_2 + 1) \right) \geq N_{k_2}''(n). \tag{6.17}
\]

Case IV: \( k_2 \leq k_1 \) and \( n_2 = N_1 \): this is the most complicated case. The key is how to control the singular term \( 2^{-k_2} \). We need to bound
\[
\sum_{k_1, k_2} \left\| \varphi_k I_{k_2} [P_{k_1} G_{k_1}^{k_2}, P_{k_2} G_{k_2}^{k_2}] \right\|_{L^2}.
\]

The only possible control of \( P_{k_2} G_{k_2}^{k_2} \) is through Lemma 4.1 as
\[
\left\| P_{k_2} G_{L}^{k_2} \right\|_{L^2} \lesssim \varepsilon \langle t \rangle^{-H(N_1)\delta} 2^{-N(N_1)k_2^+} 2^{\delta k_2^-}.
\]
This is not enough and only provides \( 2^{\frac{1}{2}k_2^-} \). Then it remains to bound \( P_{k_1} G_{k_1}^{k_2} \) part and provide another \( 2^{\frac{1}{2} - k_2^-} \). Notice that this has been achieved in wave nonlinear estimates of Lemma 5.1. We just need to discuss the four cases (replacing \( k^- \) by \( k_2^- \) there) to achieve the desired estimates.
This suffices to close the proof. 

**Lemma 6.2.** Assume \((2.105)\) holds and \(\mathcal{L} \in \mathcal{V}_n\) with \(n \in \{0, 1, \cdots, N_1\}\). We have

\[
\left\| \varphi_k \mathbf{I}_{k}\left( e^{iD_{\mathcal{L}}} v^{k,\mathcal{L}} \right) \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-(1+H(n)\delta)} 2^{-N(n)k^+}, \tag{6.20}
\]

\[
\left\| \varphi_k \mathbf{I}_{k}\left( e^{iD_{\mathcal{L}}^*} v^{k,\mathcal{L}} \right) \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-(1+H(n)\delta)} 2^{-N(n)k^+}. \tag{6.21}
\]

**Proof.** The proof is similar to that of [6, Lemma 4.3] and Lemma 6.1. The only difference is that we do not have \(H(n)\) time decay, so we should always assign \(L^2\) to \(G_{\mathcal{L}_1}\) or \(G_{\mathcal{L}_2}\). The \(V_{\mathcal{L}_2,\infty}^\dagger + H_{\mathcal{L}_2,\infty}\) or \(e^{iD_{\mathcal{L}}^*} v^{k,\mathcal{L}} + \mathcal{B}_{\mathcal{L}_1,\infty}\) may take \(L^2\) or \(L^\infty\) following the argument in the proof of Lemma 6.1. Then based on the similar analysis as in Lemma 5.2 and using the results from Lemma 4.20, Lemma 4.13 and Lemma 4.24, we get the desired result. 

**Lemma 6.3.** Assume \((2.105)\) holds and \(\mathcal{L} \in \mathcal{V}_n\) with \(n \in \{0, 1, \cdots, N_1\}\). We have

\[
\left\| \varphi_k \left( e^{iD_{\mathcal{L}}^*} v^{k,\mathcal{L}} \right) \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-(1+H(n)\delta)} 2^{-N(n)k^+}. \tag{6.22}
\]

**Proof.** Note that all terms related to \(\mathcal{B}_{\mathcal{L}_1,\infty}\) can be bounded as in Lemma 6.1 since \(\mathcal{B}_{\mathcal{L}_1,\infty}\) has sufficient time decay as in Lemma 4.28 and Lemma 4.29. Then we only need to consider the rest. We know

\[
\mathbf{I}_{k}\left( e^{iD_{\mathcal{L}}^*} v^{k,\mathcal{L}} \right) \sim \int_{\mathbb{R}^3} e^{i\Phi_{k}\xi} b_{\mathcal{L}}(\xi, \eta) \left( e^{iD_{\mathcal{L}}^*} v^{k,\mathcal{L}} \right) \langle t, \xi - \eta \rangle \left( V_{\mathcal{L}_2,\infty}^\dagger + H_{\mathcal{L}_2,\infty} \right) \langle t, \eta \rangle \, d\eta. \tag{6.23}
\]

Let \(g_1 = \left( e^{iD_{\mathcal{L}}^*} v^{k,\mathcal{L}} \right) \) and \(g_2 = V_{\mathcal{L}_2,\infty}^\dagger + H_{\mathcal{L}_2,\infty} \). We first discuss the non-resonant case \((\xi_1, \xi_2) = (-, +)\) or \((\xi_1, \xi_2) = (-, -)\): we may decompose

\[
\mathbf{I}_{k}\left[ g_1, g_2 \right](t, \xi) = \mathbf{I}_{k}\left[ g_1, \varphi_0^2(\eta(t)\xi)g_2 \right](t, \xi) + \mathbf{I}_{k}\left[ g_1, \varphi_0^2(\eta(t)\xi)g_2 \right](t, \xi). \tag{6.24}
\]

Here the first term will be handled similarly as the following resonant case. On the other hand, the second term is exactly cancelled by \(H_{\mathcal{L}_1,\mathcal{L}_2,\infty}\).

Next we consider the resonant case \((\xi_1, \xi_2) = (+, +)\) or \((\xi_1, \xi_2) = (+, -)\).

It suffices to bound

\[
\mathbf{I}_{k}\left[ e^{iD_{\mathcal{L}}^*} v^{k,\mathcal{L}} \right](t, \xi) = \int_{\mathbb{R}^3} e^{i\Phi_{k}\xi} b_{\mathcal{L}}(\xi, \eta) \left( \varphi_0^2(\eta(t)\xi)g_2(t, \eta) \right) \, d\eta - \int_{\mathbb{R}^3} e^{i\Phi_{k}\xi} \frac{\xi - \eta}{|\xi - \eta|} \varphi_0^2(\eta(t)\xi) H_{\mathcal{L}_1,\mathcal{L}_2,\infty}(t, \eta) \, d\eta. \tag{6.25}
\]
The major term is $\tilde{g}_2(t, \eta) - H^{1/2}_{L^2, \infty}(t, \eta)$. We know
\begin{equation}
\tilde{g}_2(t, \xi) - H^{1/2}_{L^2, \infty}(t, \xi) = \left(\frac{V^{u_0, g}}{L^2, \infty}(\xi) + H^{1/2}_{L^2, \infty}(t, \xi)\right) - \varphi_{\leq 0}(\xi \langle t \rangle^{2/3}) \left(\frac{V^{u_0, g}}{L^2, \infty}(\xi) + H^{1/2}_{L^2, \infty}(t, \xi)\right) = \varphi_{\geq 0}(\xi \langle t \rangle^{2/3}) V^{u_0, g}(\xi). \tag{6.26}
\end{equation}

We have
\begin{equation}
\left\| \varphi_k \int_{\mathbb{R}^3} \varphi_{\geq 0}(\eta \langle t \rangle^{2/3}) e^{it(\xi - \langle \xi \eta \rangle - i2|\eta|)} \frac{|\xi - \eta|^2}{|\xi - \eta|^2} \tilde{g}_1(t, \xi - \eta) V^{u_0, g}(\eta) \, d\eta \right\|_{L^2} \lesssim \sum_{k_1, k_2} \left\| \varphi_k \int_{\mathbb{R}^3} \varphi_{\geq 0}(\eta \langle t \rangle^{2/3}) e^{it(\xi - \langle \xi \eta \rangle - i2|\eta|)} \frac{|\xi - \eta|^2}{|\xi - \eta|^2} P_{k_1} g_1(t, \xi - \eta) P_{k_2} V^{u_0, g}(\eta) \, d\eta \right\|_{L^2}. \tag{6.27}
\end{equation}

- Case I: If $2k_i^{1/2} \langle t \rangle^{2/3} \lesssim 1$, then using the type of estimate in (6.11), we know
\begin{equation}
\sum_{k_1, k_2} \left\| \varphi_k \int_{\mathbb{R}^3} \varphi_{\geq 0}(\eta \langle t \rangle^{2/3}) e^{it(\xi - \langle \xi \eta \rangle - i2|\eta|)} \frac{|\xi - \eta|^2}{|\xi - \eta|^2} P_{k_1} g_1(t, \xi - \eta) P_{k_2} V^{u_0, g}(\eta) \, d\eta \right\|_{L^2} \lesssim \sum_{k_1, k_2} 2^{k_1^2 + 2k_2^2} \left\| P_{k_1} g_1 \right\|_{L^2} \left\| e^{-i2t\Lambda_0} P_{k_2} V^{u_0, g} \right\|_{L^\infty} \tag{6.28}
\end{equation}
\begin{align*}
\lesssim & \sum_{k_1, k_2} 2^{k_1^2 + 2k_2^2} \left( \varepsilon \left(\ln \langle t \rangle\right) \right)^2 2^{-N(n-4)k_1^2 + 2k_2^2} \\
\lesssim & \sum_{k_1, k_2} \varepsilon^2 \langle t \rangle^{-1} \left(\ln \langle t \rangle\right)^2 2^{-N(n-4)k_1^2 + 2k_2^2 + 3k_1 - N(n-2)k_2^2} \\
\lesssim & \varepsilon^2 \left(\ln \langle t \rangle\right)^2 \sum_{\max(k_1, k_2) \geq k} 2^{-N(n-4)k_1^2 + k_1 - N(n-2)k_2^2} \\
\lesssim & \varepsilon^2 \langle t \rangle^{-1} \left(1 + H(n)\delta\right) 2^{-N(n)k^2}.
\end{align*}

Here in the second inequality we use (4.115) and Lemma 4.13. In the last inequality we may take the summation over $k_1, k_2$ for $\max(k_1, k_2) \geq k$ in view of Lemma 3.3.

- Case II: If $2k_i^{1/2} \langle t \rangle^{2/3} \gtrsim 1$ and $k_1 \gtrsim k_2$ and $k_2 \leq 0$, then we use Lemma 4.125 and Lemma 4.14.
to obtain
\[
\sum_{k_1, k_2} \left\| \varphi_k \int_{\mathbb{R}^3} \varphi_{\geq 0}(\eta \langle \xi \rangle^\frac{7}{8}) \right. \left. e^{it(\xi(\xi-\eta)-\xi\eta)} \frac{|\xi-\eta|^2}{\langle \xi-\eta \rangle^2} \right. P_{k_1} g_1(t, \xi-\eta) P_{k_2} V_{L_2, \infty}^{\text{out}, t_2}(\eta) \, d\eta \right\|_{L^2}
\]
\[
\lesssim \sum_{k_1, k_2} 2^{k_1^+} \left\| e^{-itA_k} P_{k_1} g_1 \right\|_{L^\infty} \left\| \frac{1}{|\eta|} P_{k_2} V_{L_2, \infty}^{\text{out}, t_2} \right\|_{L^2}
\]
\[
\lesssim \sum_{k_1, k_2} 2^{k_1^+} \left\| e^{-itA_k} P_{k_1} g_1 \right\|_{L^\infty} \left\| P_{k_2} V_{L_2, \infty}^{\text{out}, t_2} \right\|_{L^\infty}
\]
\[
\lesssim \sum_{k_1, k_2} \varepsilon^2 (\ln \langle t \rangle)^2 \langle t \rangle^{-\frac{1}{2} + \frac{5}{8}} 2^{-(N(n_1) - 5)} k_1^+ - N(n_2) k_2^+ 2^{-k_1} \left( \frac{1}{2} - \frac{5}{8} \right) - \frac{1}{2} k_2^-
\]
\[
\lesssim \sum_{k_1, k_2} \varepsilon^2 (\ln \langle t \rangle)^2 \langle t \rangle^{-\frac{1}{2} + \frac{5}{8}} 2^{-(N(n_1) - 5)} k_1^+ - N(n_2) k_2^+.
\]

- Case III: If $2^{2k_1^+} \langle t \rangle^\frac{1}{8} \gtrsim 1$ and $k_1 \gtrsim k_2$ and $k_2 \geq 0$, then

\[
\sum_{k_1, k_2} \left\| \varphi_k \int_{\mathbb{R}^3} \varphi_{\geq 0}(\eta \langle \xi \rangle^\frac{7}{8}) \right. \left. e^{it(\xi(\xi-\eta)-\xi\eta)} \frac{|\xi-\eta|^2}{\langle \xi-\eta \rangle^2} \right. P_{k_1} g_1(t, \xi-\eta) P_{k_1} V_{L_2, \infty}^{\text{out}, t_2}(\eta) \, d\eta \right\|_{L^2}
\]
\[
\lesssim \sum_{k_1, k_2} 2^{k_1^+} \left\| e^{-itA_k} P_{k_1} g_1 \right\|_{L^\infty} \left\| P_{k_2} V_{L_2, \infty}^{\text{out}, t_2} \right\|_{L^2}
\]
\[
\lesssim \sum_{k_1, k_2} \varepsilon^2 (\ln \langle t \rangle)^2 \langle t \rangle^{-\frac{1}{2} + \frac{5}{8}} 2^{-(N(n_1) - 5)} k_1^+ - N(n_2) k_2^+ 2^{-k_1} \left( \frac{1}{2} - \frac{5}{8} \right)
\]
\[
\lesssim \sum_{k_1, k_2} \varepsilon^2 (\ln \langle t \rangle)^2 \langle t \rangle^{-\frac{1}{2} + \frac{5}{8}} 2^{-(N(n_1) - 5)} k_1^+ - N(n_2) k_2^+.
\]

- Case IV: If $2^{2k_1^+} \langle t \rangle^\frac{1}{8} \gtrsim 1$ and $k_1 \lesssim k_2$, then we have

\[
\frac{|\xi-\eta|^2}{\langle \xi-\eta \rangle^2} \lesssim |\xi-\eta|.
\]

We first integrate by parts in $\eta$, and then use (4.145) and (4.47) to obtain that

\[
\sum_{k_1, k_2} \left\| \varphi_k \int_{\mathbb{R}^3} \varphi_{\geq 0}(\eta \langle \xi \rangle^\frac{7}{8}) \right. \left. e^{it(\xi(\xi-\eta)-\xi\eta)} \frac{|\xi-\eta|^2}{\langle \xi-\eta \rangle^2} \right. P_{k_1} g_1(t, \xi-\eta) P_{k_1} V_{L_2, \infty}^{\text{out}, t_2}(\eta) \, d\eta \right\|_{L^2}
\]
\[
\lesssim \sum_{k_1, k_2} \langle t \rangle^{-1} \left\| \varphi_k \int_{\mathbb{R}^3} \varphi_{\geq 0}(\eta \langle \xi \rangle^\frac{7}{8}) \right. \left. e^{it(\xi(\xi-\eta)-\xi\eta)} \frac{|\xi-\eta|^2}{\langle \xi-\eta \rangle^2} \right. P_{k_1} g_1(t, \xi-\eta) P_{k_2} V_{L_2, \infty}^{\text{out}, t_2}(\eta) \, d\eta \right\|_{L^2}
\]
\[
\lesssim \sum_{k_1, k_2} \langle t \rangle^{-1} \left\| P_{k_1} g_1 \right\|_{L^2} 2^{-k_2} \left\| e^{-itA_{\omega a}} P_{k_2} V_{L_2, \infty}^{\text{out}, t_2} \right\|_{L^\infty}
\]
\[
\lesssim \sum_{k_1, k_2} \varepsilon^2 \langle t \rangle^{-2} (\ln \langle t \rangle)^3 2^{-(N(n_1)k_1^+ - N(n_2)k_2^+)}.\]
For the term with derivative after integration by parts, we always assign $L^2$, and the other term with $L^\infty$. Note that now $2^{-k_1}$, even present, is not a big deal since it grows at most $\langle t \rangle^{\frac{1}{2}}$.

Combining Lemma 6.1, Lemma 6.2 and Lemma 6.3 we get the boundedness and contraction theorems:

**Theorem 6.4.** Assume (2.105) holds and $L \in \mathcal{V}_n$ with $n \in \{0, 1, \ldots, N_1\}$. We have

$$
\left\| P_k \partial_t G_{kg} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-(1+H''_{kg}(n)\delta)} 2^{-N''_{kg}(n)k^+}.
$$

(6.33)

**Theorem 6.5.** Assume $(G_1^{wa}, G_1^{kg})$ and $(G_2^{wa}, G_2^{kg})$ are two sets of solutions satisfying (2.105) and $L \in \mathcal{V}_n$ with $n \in \{0, 1, \ldots, N_1\}$. We have

$$
\left\| P_k \partial_t (G_{1L}^{kg} - G_{2L}^{kg}) \right\|_{L^2} \lesssim \varepsilon \left( \left\| G_1^{wa} - G_2^{wa} \right\|_{X_1} + \left\| G_1^{kg} - G_2^{kg} \right\|_{X_2} \right) \langle t \rangle^{-(1+H''_{kg}(n)\delta)} 2^{-N''_{kg}(n)k^+}.
$$

(6.34)

### 6.2 $T'_2$ Estimates for Klein-Gordon Equation

**Lemma 6.6.** Assume (2.105) holds and $L \in \mathcal{V}_n$ with $n \in \{0, 1, \ldots, N_1 - 1\}$ and $\ell = \{1, 2, 3\}$. We have

$$
\left\| \varphi_k \partial_{\xi_\ell} \left( e^{-itL_{kg}(\xi_\ell)} I_{kg}^{11,2} [G_{L_1}^{kg,11}, G_{L_2}^{wa,12}] \right) \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-H''_{kg}(n)\delta} 2^{-N''_{kg}(n)k^+}.
$$

(6.35)

**Proof.** This is similar to that of Lemma 6.1 and of [6] Lemma 4.3. Since

$$
I_{kg}^{11,2} [G_{L_1}^{kg,11}, G_{L_2}^{wa,12}] (t, \xi) \sim \int_{\mathbb{R}^3} e^{it\Phi_{kg}^{11,2}} b_{\xi,12} (\xi, \eta) \hat{G}_{L_1}^{kg,11} (t, \xi - \eta) \hat{G}_{L_2}^{wa,12} (t, \eta) d\eta,
$$

(6.36)

the derivative may hit the phase $e^{-itL_{kg} e^{i\Phi_{kg}^{11,2}}}$, or the multiplier $a_{\xi,12}$ or the function $G_{L_1}^{kg,11} (\xi - \eta)$.

- If $\xi_\ell$ derivative hits the phase $e^{-itL_{kg} e^{i\Phi_{kg}^{11,2}}}$, then we have an extra $\langle t \rangle$ term popping out. Following exactly the same argument as in the proof of Lemma 6.1, we get the desired result.

- If $\xi_\ell$ derivative hits the multiplier $b_{\xi,12}$, then nothing happens and we follow the same proof of Lemma 6.1.

- If $\xi_\ell$ derivative hits the function $\hat{G}_{L_1}^{kg,11} (\xi - \eta)$, then we simply use (5.9) and Lemma 4.1 to obtain

$$
\left\| \varphi_k I_{wa}^{11,2} \left[ \varphi_k \partial_{\xi_\ell} \hat{G}_{L_1}^{kg,11}, \varphi_k \partial_{\xi_\ell} \hat{G}_{L_2}^{wa,12} \right] \right\|_{L^2} \lesssim 2^{\frac{3}{2} \min(k_1, k_2) k_2} \left\| \varphi_k \partial_{\xi_\ell} \hat{G}_{L_1}^{kg,11} \right\|_{L^2} \left\| \varphi_k \partial_{\xi_\ell} \hat{G}_{L_2}^{wa,12} \right\|_{L^2} \lesssim \varepsilon^2 2^{\frac{3}{2} \min(k_1, k_2) k_2} \langle t \rangle^{-H(n_1 + 1) + H(n_2)\delta} 2^{-(N(n_1 + 1) + 1)k_1^+} - N(n_2)k_2^+.
$$

This works for $k_1 \leq k_2$. If $k_1 \geq k_2$, then through change of variable in convolution, we can always transfer the derivative to $G_{L_2}^{wa,12}$, so we know both $H''_{kg}(n)$ and $N''_{kg}(n)$ requirements can be met.
Lemma 6.7. Assume (2.105) holds and \( \mathcal{L} \in \mathcal{V}_n \) with \( n \in \{0, 1, \cdots, N_1 - 1\} \) and \( \ell = \{1, 2, 3\} \). We have

\[
\left\| \varphi_k \partial_{\xi_i} \left\{ e^{-it\Lambda_{k_0}(\xi)} I_{k_0}^{11/2} \left[ \hat{v}_{\mathcal{L}_1}^{k_0,11}, \hat{v}_{\mathcal{L}_2}^{k_0,12} \right] \right\} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{H''_{k_0}(n)\delta} 2^{-N''_{k_0}(n)k^+},
\]

(6.38)

\[
\left\| \varphi_k \partial_{\xi_i} \left\{ e^{-it\Lambda_{k_0}(\xi)} I_{k_0}^{11/2} \left[ e^{iD_{\xi}} \hat{v}_{\mathcal{L}_1}^{k_0,11} + \mathcal{B}_{k_0,11,\infty} + H_{k_0,2,\infty}^{11/2} \right] \right\} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{H''_{k_0}(n)\delta} 2^{-N''_{k_0}(n)k^+}.
\]

(6.39)

Proof. Similar to that of Lemma 6.3 and Lemma 6.6.

Lemma 6.8. Assume (2.105) holds and \( \mathcal{L} \in \mathcal{V}_n \) with \( n \in \{0, 1, \cdots, N_1 - 1\} \) and \( \ell = \{1, 2, 3\} \). We have

\[
\left\| \varphi_k \partial_{\xi_i} \left\{ e^{-it\Lambda_{k_0}(\xi)} I_{k_0}^{11/2} \left[ \left( e^{iD_{\xi}} \hat{v}_{\mathcal{L}_1}^{k_0,11} \right) + \mathcal{B}_{k_0,11,\infty} + H_{k_0,2,\infty}^{11/2} \right] - f_{k_0,1}^{11/2} \right\} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{H''_{k_0}(n)\delta} 2^{-N''_{k_0}(n)k^+}.
\]

(6.40)

Proof. We just apply the same argument as in Lemma 6.3 and Lemma 6.6 to get the desired result.

Combining Lemma 6.6, Lemma 6.7 and Lemma 6.8 we get the boundedness and contraction theorems:

Theorem 6.9. Assume (2.105) holds and \( \mathcal{L} \in \mathcal{V}_n \) with \( n \in \{0, 1, \cdots, N_1 - 1\} \) and \( \ell = \{1, 2, 3\} \). We have

\[
\left\| \varphi_k \partial_{\xi_i} \left\{ \tilde{N}_{\mathcal{L}}^{k_0} - e^{-it\Lambda_{k_0}(\xi)} (iC_{\mathcal{L}} e^{iD_{\xi}} \hat{v}_{\mathcal{L}}^{k_0}) - e^{-it\Lambda_{k_0}(\xi)} b_{\mathcal{L},\infty} - e^{-it\Lambda_{k_0}(\xi)} iC_{\mathcal{L}} \mathcal{B}_{\mathcal{L},\infty} \right\} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{H''_{k_0}(n)\delta} 2^{-N''_{k_0}(n)k^+}.
\]

(6.41)

Theorem 6.10. Assume (2.105) holds and \( (G_1^{w_k}, G_1^{k_0}) \) and \( (G_2^{w_k}, G_2^{k_0}) \) are two sets of solutions satisfying (2.105) and \( \mathcal{L} \in \mathcal{V}_n \) with \( n \in \{0, 1, \cdots, N_1 - 1\} \). \( \tilde{N}_1^{k_0} \) and \( \tilde{N}_2^{k_0} \) are corresponding nonlinear terms. Then we have

\[
\left\| \varphi_k \partial_{\xi_i} \left\{ e^{-it\Lambda_{k_0}(\xi)} (\tilde{N}_{1,\mathcal{L}}^{k_0} - \tilde{N}_{2,\mathcal{L}}^{k_0}) \right\} \right\|_{L^2} \lesssim \varepsilon \left( \|G_1^{w_k} - G_2^{w_k}\|_{X_1} + \|G_1^{k_0} - G_2^{k_0}\|_{X_2} \right) \langle t \rangle^{H''_{k_0}(n)\delta} 2^{-N''_{k_0}(n)k^+}.
\]

(6.42)
7 \ S_1 \ and \ T_1 \ Estimates \ for \ Wave \ Equation

7.1 \ S_1 \ Estimates \ for \ Wave \ Equation

We start from
\[ \partial_t \hat{G}_L^{wa} = e^{it\Lambda_{wa}(\xi)} \hat{N}_L^{wa} - h_{L,\infty}. \]  

(7.1)

Multiplying \(|\xi|^{-1} \langle \xi \rangle^{2N(n)} \hat{G}_L^{wa}\) on both sides and integrating over \(\mathbb{R}^3\), we have
\[ \partial_t \mathcal{E}_{wa}^L = \mathcal{R}_{wa}^L, \]

(7.2)

where
\[ \mathcal{E}_{wa}^L := \left\| \left\| \nabla \right\| \right\|_{H^{N(n)}}^2. \]

(7.3)

and
\[ \mathcal{R}_{wa}^L := \int_{\mathbb{R}^3} |\xi|^{-1} \langle \xi \rangle^{2H(n)} \left( e^{it\Lambda_{wa}(\xi)} \hat{N}_L^{wa} - h_{L,\infty} \right) \hat{G}_L^{wa} \, d\xi. \]

(7.4)

Similar to the nonlinear analysis, it suffices to bound the principal part
\[ e^{it\Lambda_{wa}(\xi)} \hat{N}_L^{wa} - h_{L,\infty} = \sum_{\mathcal{L}_1, \mathcal{L}_2} \left( e^{it\Lambda_{wa}(\xi)} \hat{N}_{\mathcal{L}_1, \mathcal{L}_2}^{wa} - h_{\mathcal{L}_1, \mathcal{L}_2, \infty} \right), \]

(7.5)

where for any \(\iota_1, \iota_2 \in \{+,-\}, \mathcal{L}_1 \in \mathcal{V}_{n_1} \) and \(\mathcal{L}_2 \in \mathcal{V}_{n_2} \) with \(n_1 + n_2 \leq n\),
\[ e^{it\Lambda_{wa}(\xi)} \hat{N}_{\mathcal{L}_1, \mathcal{L}_2}^{wa} - h_{\mathcal{L}_1, \mathcal{L}_2, \infty} \]

(7.6)

\[ = \sum_{\iota_1, \iota_2 = \pm} \left\{ I_{wa}^{112} \left[ V_{\mathcal{L}_1}^{kg, \iota_1}, V_{\mathcal{L}_2}^{kg, \iota_2} \right] - \frac{1}{2} h_{\mathcal{L}_1, \mathcal{L}_2, \infty} \right\} \]

\[ = \sum_{\iota_1, \iota_2 = \pm} I_{wa}^{112} \left[ G_{kg, \iota_1}^{1}, G_{kg, \iota_2}^{1} \right] \]

\[ + \sum_{\iota_1, \iota_2 = \pm} I_{wa}^{112} \left[ G_{kg, \iota_1}^{1}, \left( e^{i\iota_1 \mathcal{D}^2_{\mathcal{L}_1} V_{\infty}^{kg, \iota_1}} \right) \mathcal{L}_2 \right] + \mathcal{B}_{L_2, \infty} \]

\[ + \sum_{\iota_1, \iota_2 = \pm} I_{wa}^{112} \left[ \left( e^{i\iota_1 \mathcal{D}^2_{\mathcal{L}_1} V_{\infty}^{kg, \iota_1}} \right) \mathcal{L}_1, 2 \right] + \mathcal{B}_{L_1, \infty} \]

\[ + \frac{1}{2} h_{\mathcal{L}_1, \mathcal{L}_2, \infty} \]

\[ \right\} \]

(7.7)

Remark 7.1. It seems that we can directly apply nonlinear estimates to obtain the desired result. However, this is not available since nonlinear estimate does not have sufficient time decay and \(k^+\) decay, so we have to redo the estimates term by term. In particular, the time integral will play a key role.

Lemma 7.1. Assume \((2.105)\) holds and \(\mathcal{L} \in \mathcal{V}_n \) with \(n \in \{0, 1, \cdots, N_1\} \). We have
\[ \left| \int_{t}^{\infty} \int_{\mathbb{R}^3} |\xi|^{-1} \langle \xi \rangle^{2H(n)} I_{wa}^{112} \left[ G_{\mathcal{L}_1}^{kg, \iota_1}, G_{\mathcal{L}_2}^{kg, \iota_2} \right] \cdot \hat{G}_L^{wa}(s, \xi) \, d\xi ds \right| \lesssim \varepsilon^3 (t)^{-2H(n)\delta}. \]

(7.8)
Proof. Without loss of generality, we assume \( n_1 \leq n_2 \) and \( n_1 + n_2 = n \). Also, we ignore \( t_1 \) and \( t_2 \) when there is no confusion. We intend to bound

\[
\mathcal{I}_{wa}[G_{L_1}^{kg}, G_{L_2}^{kg}, G_{L}^{wa}]: = \int_t^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{i\phi_{wa}} |\xi|^{-1} (\xi)^{2N(n)} a(\xi, \eta) \overrightarrow{G_{L_1}^{kg}} (s, \xi - \eta) \overrightarrow{G_{L_2}^{kg}} (s, \eta) \overrightarrow{G_{L}^{wa}} (s, \xi) d\eta d\xi ds .
\]

(7.9)

Here \( a(\xi, \eta) \) will not play a role in the estimate since \( |a(\xi, \eta)| \lesssim 1 \).

This proof mainly relies on two types of estimates:

\[
\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi_{k_1} f(\xi - \eta) \varphi_{k_2} g(\eta) \varphi_h(\xi) d\eta d\xi \right| \lesssim \| \varphi_{k_1} f \|_{L^p_1} \| \varphi_{k_2} g \|_{L^p_2} \| \varphi_h \|_{L^p} ,
\]

(7.10)

where \( (p_1, p_2, p) \) can be any combination of \( \{2, 2, \infty\} \) (here \( L^\infty \) is in physical space with \( e^{it\Lambda_{wa}} \) or \( e^{it\Lambda_{kg}} \)), and

\[
\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi_{k_1} f(\xi - \eta) \varphi_{k_2} g(\eta) \varphi_h(\xi) d\eta d\xi \right| \lesssim 2^{\frac{3}{2} \min(k_1, k_2)} \| \varphi_{k_1} f \|_{L^2} \| \varphi_{k_2} g \|_{L^2} \| \varphi_h \|_{L^2} .
\]

(7.11)

Define

\[
\mathcal{I}_{wa}^{m,k,k_1,k_2}[G_{L_1}^{kg}, G_{L_2}^{kg}, G_{L}^{wa}]: = \int_t^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_m e^{i\phi_{wa}} 2^{-k} 2^{2N(n)k^+} \overrightarrow{P_{k_1} G_{L_1}^{kg}} (s, \xi - \eta) \overrightarrow{P_{k_2} G_{L_2}^{kg}} (s, \eta) \overrightarrow{P_k G_{L}^{wa}} (s, \xi) d\eta d\xi ds .
\]

(7.12)

If \( n_1 \geq 1 \), then \( n_2 \leq N_1 - 1 \). We integrate by parts in time (normal form transformation) to get

\[
\mathcal{I}_{wa}^{m,k,k_1,k_2} \sim \int_t^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_m \mathcal{K}_{wa}^{m,k,k_1,k_2} [G_{L_1}^{kg}, G_{L_2}^{kg}, G_{L}^{wa}] (s) ds
\]

(7.13)

where

\[
\mathcal{K}_{wa}^{m,k,k_1,k_2}[G_{L_1}^{kg}, G_{L_2}^{kg}, G_{L}^{wa}]: = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_m \frac{e^{i\phi_{wa}}}{\phi_{wa}} 2^{-k} 2^{2N(n)k^+} \overrightarrow{P_{k_1} G_{L_1}^{kg}} (\xi - \eta) \overrightarrow{P_{k_2} G_{L_2}^{kg}} (\eta) \overrightarrow{P_k G_{L}^{wa}} (\xi) d\eta d\xi .
\]

(7.14)

Note the bound that

\[
\left| \frac{e^{i\phi_{wa}}}{\phi_{wa}} \right| \lesssim |\xi|^{-1} \lesssim 2^{-k},
\]

(7.15)

and

\[
|\tau_m| \lesssim 1, \quad |\tau'_m| \lesssim 2^{-m}.
\]

(7.16)
Here, we mainly rely on the bound (7.11) to get
\[
\left\| k^{k,k_1,k_2}[f, g, h] \right\|_{L^2} \lesssim 2^{-2k} 2^{2N(n)k^+} \left| e^{i\Phi_{wa}} \int_{\mathbb{R}^3} \tau_{m} \tilde{P}_{k_1} f(\xi - \eta) \tilde{P}_{k_2} g(\eta) \tilde{P}_k h(\xi) \, d\eta d\xi \right|
\]
\[
\lesssim 2^{-2k} 2^{2N(n)k^+} 2^2 \min(k, k_1, k_2) \left\| P_{k_1} f \right\|_{L^2} \left\| P_{k_2} g \right\|_{L^2} \left\| P_k h \right\|_{L^2}.
\]
Also, based on Lemma 4.1, we have
\[
\left\| P_{k_1} G^{kg}_{L^2} \right\|_{L^2} \lesssim \varepsilon (t)^{-H(n_1)\delta} 2^{-N(n_1)k^+_1},
\]
\[
\left\| P_{k_2} G^{kg}_{L^2} \right\|_{L^2} \lesssim \varepsilon (t)^{-H(n_2)\delta} 2^{-N(n_2)k^+_2},
\]
\[
\left\| P_k G^{na}_{L^2} \right\|_{L^2} \lesssim \varepsilon (t)^{-H(n)\delta} 2^{-N(n)k^+} 2^{\frac{1}{2}k},
\]
and based on nonlinear estimates, we have
\[
\left\| P_{k_1} \partial_t G^{kg}_{L^2} \right\|_{L^2} \lesssim \varepsilon^2 (t)^{-H''_{kg}(n_1)\delta} (t)^{-1} 2^{-N''_{kg}(n_1)k^+_1},
\]
\[
\left\| P_{k_2} \partial_t G^{kg}_{L^2} \right\|_{L^2} \lesssim \varepsilon^2 (t)^{-H''_{kg}(n_2)\delta} (t)^{-1} 2^{-N''_{kg}(n_2)k^+_2},
\]
\[
\left\| P_k \partial_t G^{na}_{L^2} \right\|_{L^2} \lesssim \varepsilon^2 (t)^{-H''_{wa}(n)\delta} (t)^{-1} 2^{-N''_{wa}(n)k^+} 2^{\frac{1}{2}k}.\]

Note the following suffices to close the proof:

- We have sufficient time decay
  \[
  H(n_1) + H(n_2) + H(n) \geq 2H(n),
  \]
  \[
  H''_{kg}(n_1) + H(n_2) + H(n) \geq 2H(n),
  \]
  \[
  H(n_1) + H''_{kg}(n_2) + H(n) \geq 2H(n),
  \]
  \[
  H(n_1) + H(n_2) + H''_{wa}(n) \geq 2H(n).
  \]
  This heavily relies on the fact that \(n_1, n_2 < n\), so the less derivative term always has better time decay.

- We have sufficient \(k^+\) decay
  \[
  \min \left( N(n_1), N(n_2) \right) + N(n) - 5 \geq 2N(n),
  \]
  \[
  \min \left( N''_{kg}(n_1), N(n_2) \right) + N(n) - 5 \geq 2N(n),
  \]
  \[
  \min \left( N(n_1), N''_{kg}(n_2) \right) + N(n) - 5 \geq 2N(n),
  \]
  \[
  \min \left( N(n_1), N(n_2) \right) + N''_{wa}(n) - 5 \geq 2N(n).
  \]
  This heavily relies on the fact that \(n_1, n_2 < n\), so the less derivative term always has better \(k^+\) decay.

- We have sufficient \(k^-\) decay. \(P_k G^{na}_{L^2}\) and \(P_k \partial_t G^{na}_{L^2}\) always provide an extra \(2^{\frac{1}{2}k^-}\).
The only remaining case is when \( n_1 = 0 \) and \( n_2 = n \). (If we continue using integration by parts in time, time decay is enough, but \( k^+ \) decay is not.) Instead, we use (7.10):

- Case \( k_1 \simeq k \geq k_2 \): For \( n \leq 2 \) (time decay is delicate), we choose \( (p_1, p_2, p) = (2, \infty, 2) \) and Lemma 4.5 to get

\[
\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_{m} e^{i \tau_{n}} 2^{-k} 2^{(2N)(k)} \sum_{k} P_{k_1} \sum_{k_2} P_{k_2} \left| \sum_{k} P_{k} \sum_{k} G_{L_1}^{w, a} \left( \xi, t \right) d\eta d\xi \right| \right.
\]

\[
\lesssim 2^{-k} 2^{(2N)(k)} \left( \varepsilon \left( t \right) - H(n_1) \right) 2^{-N(n_1)k_1^+} \left( \varepsilon \left( t \right) - H(n_2) \right) 2^{-N(n_2)k_2^+} \times \left( \varepsilon \left( t \right) - H(n_3) \right) 2^{-N(n_3)k_3^+}
\]

For \( n \geq 2 \) (\( k^+ \) decay is delicate), we choose \( (p_1, p_2, p) = (\infty, 2, 2) \) and Lemma 4.5 to get

\[
\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_{m} e^{i \tau_{n}} 2^{-k} 2^{(2N)(k)} \sum_{k} P_{k_1} \sum_{k_2} P_{k_2} \left| \sum_{k} P_{k} \sum_{k} G_{L_1}^{w, a} \left( \xi, t \right) d\eta d\xi \right| \right.
\]

\[
\lesssim 2^{-k} 2^{(2N)(k)} \left( \varepsilon \left( t \right) - H(n_1) \right) 2^{-N(n_1)k_1^+} \left( \varepsilon \left( t \right) - H(n_2) \right) 2^{-N(n_2)k_2^+} \times \left( \varepsilon \left( t \right) - H(n_3) \right) 2^{-N(n_3)k_3^+}
\]

- Case \( k_2 \simeq k \geq k_1 \): We choose \( (p_1, p_2, p) = (\infty, 2, 2) \) and Lemma 4.5 as above to close the proof.

- Case \( k_1 \simeq k_2 \geq k_1 \): We may integrate by parts in time as above to close the proof (now \( k^+ \) decay is sufficient).

\[ \Box \]

**Lemma 7.2.** Assume (2.105) holds and \( L \in \mathcal{V}_n \) with \( n \in \{0, 1, \ldots, N_1\} \). We have

\[
\left| \int_{t}^{\infty} \int_{\mathbb{R}^3} \left| \xi \right|^{-1} \left( \xi \right) 2H(n) \mathbf{I}[G_{L_1}^{w, a} \left( \xi, t \right) + \mathbf{I}_{L_1, \infty} \left( \xi, t \right)] \cdot \mathbf{G}_{L_2}^{w, a} \left( \xi, t \right) d\xi ds \right| \lesssim \varepsilon^3 \left( t \right)^{-2H(n) \delta},
\]

(7.34)

\[
\left| \int_{t}^{\infty} \int_{\mathbb{R}^3} \left| \xi \right|^{-1} \left( \xi \right) 2H(n) \mathbf{I}[G_{L_1}^{w, a} \left( \xi, t \right) + \mathbf{I}_{L_1, \infty} \left( \xi, t \right)] \cdot \mathbf{G}_{L_2}^{w, a} \left( \xi, t \right) d\xi ds \right| \lesssim \varepsilon^3 \left( t \right)^{-2H(n) \delta}.
\]

(7.35)
Proof. Here we can directly bound
\begin{align}
&\left|\int_t^\infty \int_{\mathbb{R}^3} |\xi|^{-1} \langle \xi \rangle^{2H(n)} \left\{ G_{L_1}^{kg,t_1} \left( e^{i_1 D_1^2} V_{\infty}^{kg} \right) + \mathcal{B}_{L_2,\infty} \right\} \cdot \hat{G}_{L_1}^{wa} (s, \xi) \, ds \right| \tag{7.36}
\end{align}
and
\begin{align}
&\left|\int_t^\infty \int_{\mathbb{R}^3} |\xi|^{-1} \langle \xi \rangle^{2H(n)} \left\{ \left( e^{i_1 D_1^2} V_{\infty}^{kg} \right) _{L_1} + \mathcal{B}_{L_1,\infty} + \left( e^{i_2 D_2^2} V_{\infty}^{kg} \right) _{L_2} \right\} + \mathcal{B}_{L_2,\infty} \right\} \cdot \hat{G}_{L_1}^{wa} (s, \xi) \, ds \right| \tag{7.37}
\end{align}

Then Lemma 5.2 and Lemma 4.1 will do the job.

Lemma 7.3. Assume (2.105) holds and \( L \in \mathcal{V}_n \) with \( n \in \{0, 1, \cdots , N_1\} \). We have

\begin{align}
&\left|\int_t^\infty \int_{\mathbb{R}^3} |\xi|^{-1} \langle \xi \rangle^{2H(n)} \left\{ \left( e^{i_1 D_1^2} V_{\infty}^{kg} \right) _{L_1} + \mathcal{B}_{L_1,\infty} + \left( e^{i_2 D_2^2} V_{\infty}^{kg} \right) _{L_2} \right\} - \frac{1}{2} h_{L_1, L_2, \infty} \right\} \cdot \hat{G}_{L_1}^{wa} (s, \xi) \, ds \right| \tag{7.38}
\end{align}

Proof. Similar to that of Lemma 7.2, Lemma 5.3 and Lemma 4.1 will do the job.

Remark 7.2. The key to this proof is that the nonlinear estimates of forcing term is much better than the quadratic and linear term, so we can directly apply the naive bounds.

Combining Lemma 7.1, Lemma 7.2 and Lemma 7.3 we get the boundedness and contraction theorems:

Theorem 7.4. Assume (2.105) holds and \( L \in \mathcal{V}_n \) with \( n \in \{0, 1, \cdots , N_1\} \). We have

\begin{align}
&\left| \mathcal{E}_{wa}^{L} \left[ G_{L_1}^{wa}, G_{L_2}^{wa} \right] \right| \lesssim \varepsilon^3 \langle t \rangle^{-2H(n)} \delta . \tag{7.39}
\end{align}

Theorem 7.5. Assume (2.105) holds and \( (G_1^{wa}, G_1^{kg}) \) and \( (G_2^{wa}, G_2^{kg}) \) are two sets of solutions satisfying (2.105) and \( L \in \mathcal{V}_n \) with \( n \in \{0, 1, \cdots , N_1\} \). We have

\begin{align}
&\left| \mathcal{E}_{wa}^{L} \left[ G_{1,L_1}^{wa}, G_{1,L_2}^{wa} \right] - \mathcal{E}_{wa}^{L} \left[ G_{2,L_1}^{wa}, G_{2,L_2}^{wa} \right] \right| \lesssim \varepsilon \left( \left\| G_1^{wa} - G_2^{wa} \right\| _{X_1} + \left\| G_1^{kg} - G_2^{kg} \right\| _{X_2} \right) \langle t \rangle^{-2H(n)} \delta . \tag{7.40}
\end{align}

7.2 \( T_1 \) Estimates for Wave Equation

We now estimate \( \xi \) derivative of \( G^{wa} \). Inserting \( \tilde{V}^{wa} = \hat{G}_{L_1}^{wa} + \hat{V}_{\infty}^{wa} - \hat{\mathcal{H}}_{\infty} \) into \( \partial_t \tilde{V}^{wa} = e^{i\Lambda_{wa}(\xi)} \tilde{N}^{wa} \) yields

\begin{align}
&\left[ \partial_t + i\Lambda_{wa}(\xi) \right] \left( e^{-i\Lambda_{wa}(\xi)} \hat{G}_{L_1}^{wa} \right) = \hat{N}_{L_1}^{wa} - e^{-i\Lambda_{wa}(\xi)} h_{L,\infty}. \tag{7.41}
\end{align}
Taking $\mu = wa$ in Lemma 3.7, we know

$$\Gamma_\ell \left( e^{-it \Lambda_{wa}} G_{\mathcal{L}}^{wa} \right) (t, \xi) = i \partial_{\xi} \left[ \hat{\mathcal{N}}_{\mathcal{L}}^{wa} - e^{-it \Lambda_{wa}(\xi)} \hat{h}_{\mathcal{L}, \infty} \right] (t, \xi) + e^{-it \Lambda_{wa}(\xi)} \partial_{\xi} \left[ \Lambda_{wa}(\xi) \hat{G}_{\mathcal{L}}^{wa}(t, \xi) \right].$$

(7.42)

Note that $\Lambda_{wa}(\xi) = |\xi|$. Since

$$\partial_{\xi} \left[ \Lambda_{wa}(\xi) \hat{G}_{\mathcal{L}}^{wa}(t, \xi) \right] = \Lambda_{wa}(\xi) \partial_{\xi} \hat{G}_{\mathcal{L}}^{wa}(t, \xi) + \frac{\xi_{\ell}}{|\xi|} \hat{G}_{\mathcal{L}}^{wa}(t, \xi),$$

(7.43)

we have

$$e^{-it \Lambda_{wa}(\xi)} \Lambda_{wa}(\xi) \left( \partial_{\xi} \hat{G}_{\mathcal{L}}^{wa}(t, \xi) \right) = \Gamma_\ell \left( e^{-it \Lambda_{wa}} G_{\mathcal{L}}^{wa} \right) (t, \xi) - i \partial_{\xi} \left[ \hat{\mathcal{N}}_{\mathcal{L}}^{wa} - e^{-it \Lambda_{wa}(\xi)} \hat{h}_{\mathcal{L}, \infty} \right] (t, \xi) - e^{-it \Lambda_{wa}(\xi)} \frac{\xi_{\ell}}{|\xi|} \hat{G}_{\mathcal{L}}^{wa}(t, \xi).$$

(7.44)

Estimating in $L^2$ with $2^{-\frac{1}{2}k} P_k$, we have

$$2^{\frac{1}{2}k} \left\| \varphi_k \partial_{\xi} \hat{G}_{\mathcal{L}}^{wa} \right\|_{L^2} \lesssim 2^{-\frac{1}{2}k} \left\| \varphi_k \Gamma_\ell \left( e^{-it \Lambda_{wa}} G_{\mathcal{L}}^{wa} \right) \right\|_{L^2} + 2^{\frac{1}{2}k} \left\| \varphi_k \partial_{\xi} \left[ \hat{\mathcal{N}}_{\mathcal{L}}^{wa} - e^{-it \Lambda_{wa}(\xi)} \hat{h}_{\mathcal{L}, \infty} \right] \right\|_{L^2} + 2^{-\frac{1}{2}k} \left\| P_k \hat{G}_{\mathcal{L}}^{wa} \right\|_{L^2}.$$  

(7.45)

Based on energy estimates in Theorem 7.4, we have

$$2^{-\frac{1}{2}k} \left\| \varphi_k \Gamma_\ell \left( e^{-it \Lambda_{wa}} G_{\mathcal{L}}^{wa} \right) \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-H(n+1)\delta} 2^{-N(n+1)k^+},$$

(7.46)

$$2^{-\frac{1}{2}k} \left\| P_k \hat{G}_{\mathcal{L}}^{wa} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-H(n)\delta} 2^{-N(n)k^+}.$$  

(7.47)

Based on nonlinear estimates in Theorem 5.9, we have

$$2^{-\frac{1}{2}k} \left\| \varphi_k \partial_{\xi} \left[ \hat{\mathcal{N}}_{\mathcal{L}}^{wa} - e^{-it \Lambda_{wa}(\xi)} \hat{h}_{\mathcal{L}, \infty} \right] \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-H_{wa}(n)\delta} 2^{-N_{wa}(n)k^+}.$$  

(7.48)

Based on the above analysis, we get the boundedness and contraction theorems:

**Theorem 7.6.** Assume (2.105) holds and $\mathcal{L} \in \mathcal{V}_n$ with $n \in \{0, 1, \cdots, N_1 - 1\}$. We have

$$2^{\frac{1}{2}k} \left\| \varphi_k \partial_{\xi} \hat{G}_{\mathcal{L}}^{wa} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-H(n+1)\delta} 2^{-N(n+1)k^+}.$$  

(7.49)

**Theorem 7.7.** Assume (2.105) holds and $(G_{1}^{wa}, G_{1}^{kg})$ and $(G_{2}^{wa}, G_{2}^{kg})$ are two sets of solutions satisfying (2.105) and $\mathcal{L} \in \mathcal{V}_n$ with $n \in \{0, 1, \cdots, N_1 - 1\}$. We have

$$2^{\frac{1}{2}k} \left\| \varphi_k \partial_{\xi} \left( \hat{G}_{1, \mathcal{L}}^{wa} - \hat{G}_{2, \mathcal{L}}^{wa} \right) \right\|_{L^2} \lesssim \varepsilon^2 \left( \left\| G_{1}^{wa} - G_{2}^{wa} \right\|_{X_1} + \left\| G_{1}^{kg} - G_{2}^{kg} \right\|_{X_2} \right) \langle t \rangle^{-H(n+1)\delta} 2^{-N(n+1)k^+}.  

(7.50)
8 \( S_2 \) and \( T_2 \) Estimates for Klein-Gordon Equation

8.1 \( S_2 \) Estimates for Klein-Gordon Equation

We start from

\[
\partial_t \hat{G}^{kg}_{L} = e^{it\Lambda_k}(\xi) \hat{N}^{kg}_{L} - \left( i C_\infty e^{iD\infty} V^{kg}_\infty \right)_{L} - b_{L,\infty} - i C_\infty \mathfrak{B}_{L,\infty}.
\]  

(8.1)

Multiplying \( \langle \xi \rangle^{2N(n)} \hat{G}^{kg}_{L} \) on both sides and integrating over \( \xi \in \mathbb{R}^3 \), we have

\[
\partial_t \mathcal{E}^{\xi}_{kg} = \mathcal{R}^{\xi}_{kg},
\]

(8.2)

where

\[
\mathcal{E}^{\xi}_{kg} := \left\| G^{kg} \right\|_{H^N(n)}^2,
\]

(8.3)

and

\[
\mathcal{R}^{\xi}_{kg} := \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} \left\{ e^{it\Lambda_k}(\xi) \hat{N}^{kg}_{L,L_2} - \left( i C_\infty e^{iD\infty} V^{kg}_\infty \right)_L \right\} \hat{G}^{kg}_{L}(t, \xi) \, d\xi.
\]

(8.4)

Similar to the nonlinear analysis, it suffices to consider the principal part

\[
e^{it\Lambda_k}(\xi) \hat{N}^{kg}_{L_1,L_2} - \left( i C_\infty e^{iD\infty} V^{kg}_\infty \right)_L - b_{L,\infty} = \sum_{L_1,L_2} \left\{ e^{it\Lambda_k}(\xi) \hat{N}^{kg}_{L_1,L_2} - \sum_{i_2 = \pm} f^{i_2}_{L_1,L_2,\infty} - \sum_{i_2 = \pm} b^{i_2}_{L_1,L_2,\infty} \right\},
\]

(8.5)

where for any \( i_1, i_2 \in \{+, -\}, L_1 \in \mathcal{V}_{n_1} \) and \( L_2 \in \mathcal{V}_{n_2} \) with \( n_1 + n_2 \leq n \)

\[
e^{it\Lambda_k}(\xi) \hat{N}^{kg}_{L_1,L_2} - \sum_{i_2 = \pm} f^{i_2}_{L_1,L_2,\infty} - \sum_{i_2 = \pm} b^{i_2}_{L_1,L_2,\infty}
\]

(8.6)

\[
e^{it\Lambda_k}(\xi) \hat{N}^{kg}_{L_1,L_2} - \sum_{i_2 = \pm} f^{i_2}_{L_1,L_2,\infty} - \sum_{i_2 = \pm} b^{i_2}_{L_1,L_2,\infty}
\]

(8.7)

Lemma 8.1. Assume \( (2.105) \) holds and \( L \in \mathcal{V}_n \) with \( n \in \{0, 1, \cdots, N_1\} \). We have

\[
\left| \int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} \mathcal{I}^{i12}_{kg} [G^{kg}_{L_1}, G^{wa}_{L_2}] \cdot \hat{G}^{kg}_{L}(s, \xi) \, d\xi \, ds \right| \lesssim \varepsilon^3 \langle t \rangle^{-2H(n)\delta}.
\]

(8.8)
Proof. We ignore \( \ell_1 \) and \( \ell_2 \) when there is no confusion. We focus on the bound

\[
I_{kg} [G_{L_1}^{kg}, G_{L_2}^{wa}, G_{L}^{kg}] := \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{i\Phi_{kg}} (\xi)^{2N(n)} b(\xi, \eta) \hat{G}_{L_1}^{kg}(s, \xi-\eta) \hat{G}_{L_2}^{wa}(s, \eta) \hat{G}_{L}^{kg}(s, \xi) \, d\eta d\xi ds.
\]

Here \( b(\xi, \eta) \) will induce \( 2^{k_1+2k_2}2^{-k} \) singularity.

Define

\[
I_{kg}^{m,k,k_1,k_2} [G_{L_1}^{kg}, G_{L_2}^{wa}, G_{L}^{kg}] := \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_m e^{i\Phi_{kg}} 2^{-k_1}2^{-k_2}2^{N(n)k} P_k \hat{G}_{L_1}^{kg}(s, \xi-\eta) \hat{G}_{L_2}^{wa}(s, \eta) \hat{G}_{L}^{kg}(s, \xi) \, d\eta d\xi ds.
\]

Step 1: For \( n_1, n_2 \geq 1 \), we integrate by parts in time to get

\[
I_{kg}^{m,k,k_1,k_2} = \int_0^\infty \int_0^\infty \tau_m K_{kg}^{m,k,k_1,k_2} [G_{L_1}^{kg}, G_{L_2}^{wa}, G_{L}^{kg}] (s) \, ds
\]

where

\[
K_{kg}^{m,k,k_1,k_2} [G_{L_1}^{kg}, G_{L_2}^{wa}, G_{L}^{kg}] := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_m e^{i\Phi_{kg}} 2^{-k_1}2^{-k_2}2^{N(n)k} P_k \hat{G}_{L_1}^{kg}(\xi-\eta) \hat{G}_{L_2}^{wa}(\eta) \hat{G}_{L}^{kg}(\xi) \, d\eta d\xi.
\]

Here, note the bound that

\[
|\frac{e^{i\Phi_{kg}}}{\Phi_{kg}}| \lesssim |\eta|^{-1} \lesssim 2^{-k},
\]

and

\[
|\tau_m| \lesssim 1, \quad |\tau'_m| \lesssim 2^{-m}.
\]

Here, we mainly rely on the bound (5.10) to get

\[
\|K_{kg}^{m,k,k_1,k_2} \hat{f}, \hat{g}, \hat{h}\|_{L^\infty} \lesssim 2^{-2k_1}2^{N(n)k} \frac{|e^{i\Phi_{kg}}|}{\Phi_{kg}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_m \hat{P_k} \hat{f}(\xi-\eta) \hat{P_k} \hat{g}(\eta) \hat{P_k} \hat{h}(\xi) \, d\eta d\xi
\]

\[
\lesssim 2^{-2k_1}2^{N(n)k} 2^{\frac{1}{2} \min(k,k_1,k_2)} \|P_k \hat{f}\|_{L^2} \|P_k \hat{g}\|_{L^2} \|P_k \hat{h}\|_{L^2}
\]

\[
\lesssim 2^{-\frac{1}{2}k_1}2^{N(n)k} \|P_k \hat{f}\|_{L^2} \|P_k \hat{g}\|_{L^2} \|P_k \hat{h}\|_{L^2}.
\]

Also, based on Lemma 4.1, we have

\[
\|P_k G_{L_1}^{kg}\|_{L^2} \lesssim \langle \xi \rangle^{-H(n_1)\delta} 2^{-N(n_1)k},
\]

\[
\|P_k G_{L_2}^{wa}\|_{L^2} \lesssim \langle \xi \rangle^{-H(n_2)\delta} 2^{-N(n_2)k} 2^{\frac{1}{2}k_2},
\]

\[
\|P_k G_{L}^{kg}\|_{L^2} \lesssim \langle \xi \rangle^{-H(n)\delta} 2^{-N(n)k},
\]

where

\[
\delta := \frac{1}{2} (\frac{1}{2} - \frac{1}{2}) = \frac{1}{2}.
\]
and based on nonlinear estimates, we have

\[ \left\| P_{k_1} \partial_t G_{L_1}^{k g} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-1-H''_{k_1}(n_1)\delta} 2^{-N''_{k_1}(n_1)k_1^+}, \tag{8.19} \]

\[ \left\| P_{k_2} \partial_t G_{L_2}^{w a} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-1-H''_{w a}(n_2)\delta} 2^{-N''_{w a}(n_1)k_1^+} 2^{\frac{1}{2}k_2^-}, \tag{8.20} \]

\[ \left\| P_k \partial_t G_{L}^{k g} \right\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-1-H''_{k_1}(n_1)\delta} 2^{-N''_{k_1}(n_1)k_1^+}. \tag{8.21} \]

Then based on a similar argument as in wave equation, this suffices to close the proof.

**Step 2:** When \( n_1 = 0 \) and \( n_2 = n \), we discuss as follows:

- **Case \( k_1 \approx k \geq k_2 \) or \( k_2 \approx k \geq k_1 \):** We choose \((p_1, p_2, p) = (\infty, 2, 2)\) and Lemma 4.5 to get

\[ \left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_n e^{it\Phi_{k_1}} 2^{-k_2} 2^{2N(n)k^+} P_{k_1} \hat{G}_{L_1}^{k g}(\xi-\eta) \hat{P}_{k_2} \hat{G}_{L_2}^{w a}(\eta) \hat{P}_k G_{L}^{k g}(\xi) \, d\eta d\xi \right| \lesssim 2^{-k_2} 2^{2N(n)k^+} \left( \varepsilon \langle t \rangle^{-1-H(n_1+1)\delta} 2^{-(N(n_1+1)-\frac{3}{2}k_1^+\frac{1}{2}k_2^-)} \right) \left( \varepsilon \langle t \rangle^{-H(n_2)\delta} 2^{-N(n_2)k_2^+} 2^{\frac{1}{2}k_1^-} \right) \tag{8.22} \]

- **Case \( k_1 \approx k \leq k_2 \):** If \( n \leq 2 \) (time decay is delicate), we choose \((p_1, p_2, p) = (2, \infty, 2)\) and Lemma 4.5 to get

\[ \left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tau_n e^{it\Phi_{k_1}} 2^{-k_2} 2^{2N(n)k^+} P_{k_1} \hat{G}_{L_1}^{k g}(\xi-\eta) \hat{P}_{k_2} \hat{G}_{L_2}^{w a}(\eta) \hat{P}_k G_{L}^{k g}(\xi) \, d\eta d\xi \right| \lesssim 2^{-k_2} 2^{2N(n)k^+} \varepsilon^3 \langle t \rangle^{-1-H(n_1+1)+H(n_2)+H(n)\delta} 2^{-N(n_1+1)-\frac{3}{2}k_1^+-N(n_2)k_2^+} 2^{\frac{1}{2}k_1^-+\frac{1}{2}k_2^-}. \tag{8.23} \]

If \( n \geq 2 \) (\( N(n) \) is relatively small, but we must find a term providing \( 2^{k_2} \)). However, \( G^{w a} \) in \( L^2 \) only provide \( 2^{\frac{k}{2}} \), then we can integrate by parts in time to close the proof.

**Step 3:** When \( n_1 = n \) and \( n_2 = 0 \), we discuss as follows:

- **Case \( k_1 \approx k \geq k_2 \) or \( k_1 \approx k \geq k_2 \):** choosing \((p_1, p_2, p) = (2, \infty, 2)\) and Lemma 4.5 will do the job as above.

- **Case \( k_2 \approx k \geq k_1 \):** If \( n \leq 2 \), choosing \((p_1, p_2, p) = (\infty, 2, 2)\) and Lemma 4.5 will do the job as above. If \( n \geq 2 \), we can integrate by parts in time to close the proof.
Lemma 8.2. Assume \((2.105)\) holds and \(L \in \mathcal{V}_n\) with \(n \in \{0, 1, \cdots, N_1\}\). We have
\[
\int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} I_{kg}^{112} \left[ G_{L_1}^{kg,11} V_{L_2,\infty}^{wa,12} + H_{L_2,\infty}^{12} \right] \cdot \hat{G}_L^{kg}(s, \xi) d\xi ds \lesssim \varepsilon^3 (t)^{-2H(n)\delta}, \tag{8.24}
\]
\[
\int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} I_{kg}^{112} \left[ (e^{\xi_1 \mathcal{D}_x \xi_1} V_{L_2,\infty})_L + \mathcal{B}_{L_1,\infty} G_{L_2}^{wa,12} \right] \cdot \hat{G}_L^{kg}(s, \xi) d\xi ds \lesssim \varepsilon^3 (t)^{-2H(n)\delta}. \tag{8.25}
\]
Proof. Here we can directly bound
\[
\int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} I_{kg}^{112} \left[ G_{L_1}^{kg,11} V_{L_2,\infty}^{wa,12} + H_{L_2,\infty}^{12} \right] \cdot \hat{G}_L^{kg}(s, \xi) d\xi ds \lesssim \int_t^\infty \left\| \langle \xi \rangle^{H(n)} I_{kg}^{112} \left[ G_{L_1}^{kg,11} V_{L_2,\infty}^{wa,12} + H_{L_2,\infty}^{12} \right] \right\|_{L^2} \left\| \langle \xi \rangle^{H(n)} \hat{G}_L^{kg} \right\|_{L^2} ds, \tag{8.26}
\]
and
\[
\int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} I_{kg}^{112} \left[ (e^{\xi_1 \mathcal{D}_x \xi_1} V_{L_2,\infty})_L + \mathcal{B}_{L_1,\infty} G_{L_2}^{wa,12} \right] \cdot \hat{G}_L^{kg}(s, \xi) d\xi ds \lesssim \int_t^\infty \left\| \langle \xi \rangle^{H(n)} I_{kg}^{112} \left[ (e^{\xi_1 \mathcal{D}_x \xi_1} V_{L_2,\infty})_L + \mathcal{B}_{L_1,\infty} G_{L_2}^{wa,12} \right] \right\|_{L^2} \left\| \langle \xi \rangle^{H(n)} \hat{G}_L^{kg} \right\|_{L^2} ds. \tag{8.27}
\]
Then Lemma 6.2 and Lemma 4.1 will do the job.

Lemma 8.3. Assume \((2.105)\) holds and \(L \in \mathcal{V}_n\) with \(n \in \{0, 1, \cdots, N_1\}\). We have
\[
\int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} I_{kg}^{112} \left[ (e^{\xi_1 \mathcal{D}_x \xi_1} V_{L_2,\infty})_L + \mathcal{B}_{L_1,\infty} G_{L_2}^{wa,12} \right] \cdot \hat{G}_L^{kg}(s, \xi) d\xi ds \lesssim \varepsilon^3 (t)^{-2H(n)\delta}. \tag{8.28}
\]
Proof. Similar to Lemma 8.2, Lemma 6.3 and Lemma 4.1 will do the job.

Combining Lemma 8.1, Lemma 8.2 and Lemma 8.3 we get the boundedness and contraction theorems:

Theorem 8.4. Assume \((2.105)\) holds and \(L \in \mathcal{V}_n\) with \(n \in \{0, 1, \cdots, N_1\}\). We have
\[
\left| \varepsilon_{kg}[G_{L},G_{L}^{kg}] \right| \lesssim \varepsilon^3 (t)^{-2H(n)\delta}. \tag{8.29}
\]

Theorem 8.5. Assume \((2.105)\) holds and \((G_{1}^{wa},G_{1}^{kg})\) and \((G_{2}^{wa},G_{2}^{kg})\) are two sets of solutions satisfying \((2.105)\) and \(L \in \mathcal{V}_n\) with \(n \in \{0, 1, \cdots, N_1\}\). We have
\[
\left| \varepsilon_{kg}[G_{1,L},G_{1,L}^{kg}] - \varepsilon_{kg}[G_{2,L}^{kg},G_{2,L}^{kg}] \right| \lesssim \varepsilon \left( \|G_{1}^{wa} - G_{2}^{wa}\|_{X_1} + \|G_{1}^{kg} - G_{2}^{kg}\|_{X_2} \right) (t)^{-2H(n)\delta}. \tag{8.30}
\]
8.2 $T_2$ Estimates for Klein-Gordon Equation

We now estimate $\xi$ derivative of $\hat{G}^{kg}$. Inserting $\hat{V}^{kg} = \hat{G}^{kg} + e^{iD_\infty \hat{V}^{kg}} + \mathcal{B}_\infty$ into $\partial_t \hat{V}^{kg} = e^{it\Lambda_{kg}(\xi)}\hat{N}^{kg}$ yields

$$[\partial_t + i\Lambda_{kg}(\xi)] \left( e^{-it\Lambda_{kg}(\xi)} \hat{G}^{kg}_L \right) = \hat{N}^{kg}_L - e^{-it\Lambda_{kg}(\xi)} \left\{ i\mathcal{C}_\infty e^{iD_\infty \hat{V}^{kg}} + b_{L,\infty} + i\mathcal{C}_\infty \mathcal{B}_{L,\infty} \right\}. \quad (8.31)$$

Taking $\mu = kg$ in Lemma 3.7, we know

$$\Gamma_\ell \left( e^{-it\Lambda_{kg}(\xi)} \hat{G}^{kg}_L \right)(t, \xi) = i\partial_{\xi_\ell} \left\{ \hat{N}^{kg}_L - e^{-it\Lambda_{kg}(\xi)} \left[ i\mathcal{C}_\infty e^{iD_\infty \hat{V}^{kg}} + b_{L,\infty} + i\mathcal{C}_\infty \mathcal{B}_{L,\infty} \right] \right\}(t, \xi) + e^{-it\Lambda_{kg}(\xi)} \partial_{\xi_\ell} \left[ \Lambda_{kg}(\xi) \hat{G}^{kg}_L(t, \xi) \right]. \quad (8.32)$$

Note that $\Lambda_{kg}(\xi) = \langle \xi \rangle$. Since

$$\partial_{\xi_\ell} \left[ \Lambda_{kg}(\xi) \hat{G}^{kg}_L(t, \xi) \right] = \Lambda_{kg}(\xi) \partial_{\xi_\ell} \hat{G}^{kg}_L(t, \xi) + \frac{\xi_\ell}{\langle \xi \rangle} \hat{G}^{kg}_L(t, \xi), \quad (8.33)$$

we have

$$e^{-it\Lambda_{kg}(\xi)} \Lambda_{kg}(\xi) \left( \partial_{\xi_\ell} \hat{G}^{kg}_L(t, \xi) \right) = \Gamma_\ell \left( e^{-it\Lambda_{kg}(\xi)} \hat{G}^{kg}_L \right)(t, \xi) - e^{-it\Lambda_{kg}(\xi)} \frac{\xi_\ell}{\langle \xi \rangle} \hat{G}^{kg}_L(t, \xi) \quad (8.34)$$

Estimating in $L^2$ with $P_k$, we have

$$2^{k^+} \left\| \varphi_k \partial_{\xi_\ell} \hat{G}^{kg}_L \right\|_{L^2} \lesssim \left\| \varphi_k \Gamma_\ell \left( e^{-it\Lambda_{kg}(\xi)} \hat{G}^{kg}_L \right) \right\|_{L^2} \quad (8.35)$$

Based on energy estimates in Theorem 8.1, we have

$$\left\| \varphi_k \Gamma_\ell \left( e^{-it\Lambda_{kg}(\xi)} \hat{G}^{kg}_L \right) \right\|_{L^2} \lesssim \varepsilon 2^{\frac{3}{2}} (t)^{-H(n+1)\delta} 2^{-N(n+1)k^+}, \quad (8.36)$$

$$\left\| P_k \hat{G}^{kg}_L \right\|_{L^2} \lesssim \varepsilon 2^{\frac{3}{2}} (t)^{-H(n)\delta} 2^{-N(n)k^+}. \quad (8.37)$$

Based on nonlinear estimates in Theorem 6.9, we have

$$\left\| \varphi_k \partial_{\xi_\ell} \left\{ \hat{N}^{kg}_L - e^{-it\Lambda_{kg}(\xi)} \left[ i\mathcal{C}_\infty e^{iD_\infty \hat{V}^{kg}} + b_{L,\infty} + i\mathcal{C}_\infty \mathcal{B}_{L,\infty} \right] \right\} \right\|_{L^2} \lesssim \varepsilon 2^{\frac{3}{2}} (t)^{-H'_n(n)\delta} 2^{-N'_n(n)k^+}. \quad (8.38)$$

Based on the above analysis, we get the boundedness and contraction theorems:
Theorem 8.6. Assume (2.105) holds and $L \in V_n$ with $n \in \{0, 1, \cdots, N_1 - 1\}$. We have
\[
2^{k^+} \left\| \varphi_k \partial_{\xi} \hat{G}_{L}^{k} \right\|_{L^2} \lesssim \varepsilon \left\langle t \right\rangle^{-H(n+1)\delta} 2^{-N(n+1)k^+}. \tag{8.39}
\]

Theorem 8.7. Assume (2.105) holds and $(G_{1, a}, G_{2, a})$ and $(G_{1, b}, G_{2, b})$ are two sets of solutions satisfying (2.105) and $L \in V_n$ with $n \in \{0, 1, \cdots, N_1 - 1\}$. We have
\[
2^{k^+} \left\| \varphi_k \partial_{\xi} \left( \hat{G}_{1, L}^{k} - \hat{G}_{2, L}^{k} \right) \right\|_{L^2} \lesssim \varepsilon \left\langle t \right\rangle^{-H(n+1)\delta} 2^{-N(n+1)k^+}. \tag{8.40}
\]

Acknowledgement
The author would like to thank Professor Benoit Pausader for motivating discussions.

References
[1] Deng, Y.; Pusateri, F.: On the global behavior of weak null quasilinear wave equations. Comm. Pure Appl. Math. 73 (2020), no. 5, 1035–1099.

[2] Evans, L. C.: Partial differential equations. Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, (2010).

[3] Hani, Z.; Pausader, B.; Tzvetkov, N.; Visciglia, N.: Modified scattering for the cubic Schrödinger equation on product spaces and applications. Forum Math. Pi (2015), Vol. 3, e4, 63 pp.

[4] Hani, Z.; Thomann, L.: Asymptotic behavior of the nonlinear Schrödinger equation with harmonic trapping. Comm. Pure Appl. Math. 69 (2016), no. 9, 1727–1776.

[5] Hayashi, N.; Naumkin, P.: Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations. Amer. J. Math. 120 (1998), 369–389.

[6] Ionescu, A. D.; Pausader, B.: On the global regularity for a wave-Klein-Gordon coupled system. Acta Math. Sin. (Engl. Ser.) 35 (2019), no. 6, 933–986.

[7] Ionescu, A. D.; Pausader, B.: The Einstein-Klein-Gordon coupled system: global stability of the Minkowski solution. Preprint (2020), [arXiv:1911.10652v2].

[8] Ionescu, A. D.; Pausader, B.; Wang, X.; Widmayer, K.: On the asymptotic behavior of solutions to the Vlasov-Poisson system. Preprint (2020), [arXiv:2005.03617].

[9] Ionescu, A. D.; Pusateri, F.: Global solutions for the gravity water waves system in 2D. Invent. Math. 199 (2015), no. 3, 653–804.

[10] Kato, J.; Pusateri, F.: A new proof of long-range scattering for critical nonlinear Schrödinger equations. Differential Integral Equations 24 (2011), no. 9–10, 923–940.
[11] LeFloch, P. G.; Ma, Y.: The global nonlinear stability of Minkowski space for self-gravitating massive fields. *Comm. Math. Phys.* 346 (2016), no. 2, 603–665.

[12] Stein, E. M.; Shakarchi, R.: Functional analysis. Introduction to further topics in analysis. Princeton Lectures in Analysis, 4. *Princeton University Press, Princeton, NJ*, (2011).

[13] Wang, Q.: Global Existence for the Einstein equations with massive scalar fields. Lecture at the workshop, January 19–23 (2015), *Mathematical Problems in General Relativity*.

[14] Wang, Q.: An intrinsic hyperboloid approach for Einstein Klein-Gordon equations. *J. Differential Geom.* 115 (2020), no. 1, 27–109.