Drift estimation of the threshold Ornstein-Uhlenbeck
process from continuous and discrete observations

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Supplementary Material

We provide the technical proofs of Lemma 4, Theorem 3, and some related technical lemmas as well as the generalisation of Theorem 1 and 2 to the multi-threshold case as supplementary material.

S1 Proof of Lemma 4

Proof. Without loss of generality, we reduce to prove the statement for threshold $r = 0$. Indeed the quantities $\Omega_{T_N}^{\pm,m} - \Omega_{T_N,N}^{\pm,m}$ and $\mathcal{M}_{T_N}^{\pm,m} - \mathcal{M}_{T_N,N}^{\pm,m}$ for the process $X$ (with threshold $r$) can be written as linear combination (coefficients depending on $m$ and $r$) of the same quantities for the process $X - r$ (which solves (1.1) with threshold at 0 and new drift coefficients $b_\pm - a_\pm r$ and $a_\pm$). We keep denoting as $b_\pm$ (instead of $b_\pm - a_\pm r$) and $a_\pm$ the drift coefficients. In this proof we use the round ground notation
\[ t \triangleq t_k \text{ for } t \in [t_k, t_{k+1}) \subseteq [t_k, t_k + \Delta_N]. \] Moreover, without loss of generality, we assume \( T_N \leq N \) for all \( N \in \mathbb{N} \).

Let us first note that for \( m = 0, 1, 2 \):

\[
\Omega^{\pm,m}_{T_N} - \Omega^{\pm,m}_{T_N,N} = - \sum_{k=1}^{N} J^{(m)}_{k,N}
\]

\[
= \mp \int_0^{T_N} \text{sgn}(X_{[t] \triangle N})X^m_{[t] \triangle N} 1_{\{X_{[t] \triangle N} < 0\}} \, dt + \int_0^{T_N} (X^m_t - X^m_{[t] \triangle N}) 1_{\{\pm X_t > 0\}} \, dt
\]

therefore

\[
\mathbb{E} \left[ |\Omega^{\pm,m}_{T_N} - \Omega^{\pm,m}_{T_N,N}| \right] \leq \int_0^{T_N} \mathbb{E} \left[ |X_{[t] \triangle N}|^m 1_{\{X_{[t] \triangle N} < 0\}} \right] + \mathbb{E} \left[ |X^m_t - X^m_{[t] \triangle N}| \right] \, dt.
\]

Analogously, observe that for \( m = 0, 1 \) it holds

\[
\mathcal{M}^{\pm,m}_{T_N} - \mathcal{M}^{\pm,m}_{T_N,N} = \int_0^{T_N} (X^m_t 1_{\{\pm X_t > 0\}} - X^m_{[t] \triangle N} 1_{\{\pm X_{[t] \triangle N} > 0\}})(b(X_t) - a(X_t)X_t) \, dt
\]

\[
+ \int_0^{T_N} (X^m_t 1_{\{\pm X_t > 0\}} - X^m_{[t] \triangle N} 1_{\{\pm X_{[t] \triangle N} > 0\}})\sigma(X_t) \, dW_t.
\]

Let us rewrite the integrand as

\[
X^m_t 1_{\{\pm X_t > 0\}} - X^m_{[t] \triangle N} 1_{\{\pm X_{[t] \triangle N} > 0\}}
\]

\[
= (X^m_t - X^m_{[t] \triangle N}) 1_{\{\pm X_t > 0\}} - \text{sgn}(X_{[t] \triangle N})X^m_{[t] \triangle N} 1_{\{X_t X_{[t] \triangle N} < 0\}}.
\]
Triangular inequality, Hölder’s inequality, and Itô-isometry imply that

\[
\mathbb{E} \left[ |M_{\cdot}^\pm - M_{\cdot}^{T,N}M_{\cdot}^T_N| \right] \\
\leq \int_0^{T_N} \mathbb{E} \left[ |X_t^m - X_{[\cdot]_{\Delta_N}}^m| (|b_\pm| + a_\pm |X_{[\cdot]_{\Delta_N}}| + a_\pm |X_{[\cdot]_{\Delta_N}}|) \right] dt \\
+ \int_0^{T_N} \mathbb{E} \left[ |X_{[\cdot]_{\Delta_N}}^m| \mathbb{1}_{\{X_{[\cdot]_{\Delta_N}} < 0\}} (|b_\pm| \vee |b_\pm|) + (a_\pm \vee a_\pm) \times (|X_{[\cdot]_{\Delta_N}}| + |X_{[\cdot]_{\Delta_N}} - X_{[\cdot]_{\Delta_N}}|) \right] dt \\
+ \sqrt{2}(\sigma_- \vee \sigma_+) \left( \int_0^{T_N} \mathbb{E} \left[ (X_t^m - X_{[\cdot]_{\Delta_N}}^m)^2 + X_{[\cdot]_{\Delta_N}}^{2m} \mathbb{1}_{\{X_{[\cdot]_{\Delta_N}} < 0\}} \right] dt \right)^{1/2}.
\]

Hence, the proof of Lemma 4, reduces to prove two inequalities:

\[
\int_0^{T_N} \mathbb{E} \left[ |X_t - X_{[\cdot]_{\Delta_N}}|^j |X_{[\cdot]_{\Delta_N}}|^m \right] dt \text{ is } o(T_N^{1/\lambda}) \tag{S1.1}
\]

for all \( j \in \{1, 2, 4\}, m \in \{0, 1, 2\}, \) and

\[
\int_0^{T_N} \mathbb{E} \left[ |X_{[\cdot]_{\Delta_N}}|^m \mathbb{1}_{\{X_{[\cdot]_{\Delta_N}} < 0\}} \right] dt \text{ is } o(T_N^{1/\lambda}) \text{ for } m \in \{0, 1, 2, 3, 4\}. \tag{S1.2}
\]

\textbf{Step 1.} Given \( s \in [0, \infty) \) and \( t \in [0, \Delta_N] \) we show that for every \( j \in \{1, 2, 4\} \) there exists a constant \( C \in (0, \infty) \) depending only on \( j, a_\pm, b_\pm, \sigma_\pm \) such that

\[
\mathbb{E} \left[ |X_{t+s} - X_s|^j |X_s| \right] \leq C t^{j/2} (1 + |X_s|^j).
\] \tag{S1.3}

Let \( \xi_t := X_{t+s} - X_s \) then

\[
\xi_t = \int_0^t (b(\xi_u + X_s) - a(\xi_u + X_s)X_s) - a(\xi_u + X_s) \xi_u du + \int_0^t \sigma(\xi_u + X_s) dW_u^s
\]
where $W^s$ a Wiener process independent of $\sigma(X_u, u \in [0, s])$. So, given $X_s$, $\xi$ is an OU with threshold $-X_s$ (since $X$ has threshold 0). Now, e.g. [Hudde et al., 2021, Corollary 2.5] applied to $\xi$ implies (S1.3).

**Step 2.** (Proof of (S1.1)). Since $X_0$ is distributed as the stationary distribution $\mu$ then $\sup_{u \in [0, \infty)} \mathbb{E}[|X_u|^m] = \mathbb{E}[|X_0|^m] = \int_{-\infty}^{\infty} |x|^m \mu(dx) < \infty$. This, the tower property, and (S1.3) imply that there exists $C \in (0, \infty)$ depending only on $m, j, a_\pm, b_\pm, \sigma_\pm$ such that

$$
\frac{1}{T_N^{j/\lambda}} \int_0^{T_N} \mathbb{E} \left[ |X_t - X_{\lfloor t \rfloor \Delta N}|^j |X_{\lfloor t \rfloor \Delta N}|^m \right] dt 
\leq C \sqrt{\Delta N T_N^{2(1-\lambda^{-1})}} = C \sqrt{\Delta N T_N^{\lambda^{-1}}} \xrightarrow[N \to \infty]{} 0
$$

**Step 3.** (Proof of (S1.2)). Let $s, t \in [0, \infty)$ be fixed such that $t - s \in [0, \Delta_N]$. Let us first note that we just need to consider $\mathbb{E} \left[ 1_{\{\xi_t < 0\}} 1_{\{\xi_s > 0\}} |X_s| \right]$. This, given $X_s$, is bounded by $\mathbb{P}(\tau_{s, \pm} \leq t - s) \leq \mathbb{P}(\tau_{s, \pm} \leq \Delta_N)$ where $\tau_s$ is the first hitting time of the level 0 of the OU process solution to the following SDE: $\xi_u = X_s + \int_0^u b_\pm - a_\pm \xi_v \, dv + \sigma_\pm W^s_v$ with $W^s$ a Brownian motion independent of $\sigma(X_v, v \in [0, s])$. If $a_\pm \neq 0$, [Lipton and Kaushansky, 2020, Section 6.2.1] with $b = -\frac{b_\pm}{\sqrt{a_\pm} \sigma_\pm}, \quad z = \frac{\sqrt{a_\pm}}{\sigma_\pm} X_s - \frac{b_\pm}{\sqrt{a_\pm} \sigma_\pm}, \quad$ and $t = a_\pm \Delta_N$, prove that

$$
\mathbb{P}(\tau_{s, \pm} \leq \Delta_N) = 2e^{-\frac{b_\pm X_s}{\sigma_\pm} \Phi \left( -\frac{\sqrt{a_\pm}}{\sigma_\pm} |X_s| \gamma_N \right)} \quad \text{with} \quad \gamma_N := \frac{e^{-\frac{a_\pm \Delta_N}{2}}}{\sqrt{\sinh(a_\pm \Delta_N)}}.
$$
If \( a_\pm = 0 \) and \( \pm b_\pm < 0 \) then

\[
\mathbb{P}(\tau_s, \pm \leq \Delta_N) = \int_0^{\Delta_N} \frac{|X_s|}{\sigma_\pm \sqrt{2\pi u^3}} \exp \left( -\frac{(X_s - b_\pm u)^2}{2\sigma_\pm^2 u} \right) du \\
\leq \left( 1 + e^{\frac{2|X_s||b_\pm|}{\sigma_\pm^2}} \right) \Phi \left( \frac{|X_s| - |b_\pm|\Delta_N}{\sqrt{2\sigma_\pm^2 \Delta_N}} \right).
\]

Therefore, using the stationary distribution (4.2), to establish Step 3 it suffices to prove that the following quantity vanishes as \( N \to \infty \):

\[
\frac{1}{T_N^{1/\lambda}} \int_0^{T_N} \mathbb{E}\left[ |X_{[t]}|^{m} \mathbb{P}(\tau_{[t]} \Delta_N, \pm \leq \Delta_N) \right] dt.
\]

Let us first consider the case \( a_\pm = 0 \) and \( \pm b_\pm < 0 \). The desired quantity is bounded by

\[
\frac{T_N}{T_N^{1/\lambda}} \int_{-\infty}^{\infty} m(y) dy \int_{\mathbb{R}_\pm} 2|y|^m \exp \left( \frac{2y b_\pm}{(\sigma_\pm)^2} \right) \left( 1 + e^{\frac{2|y||b_\pm|}{\sigma_\pm^2}} \right) \Phi \left( -\frac{|y| - |b_\pm|\Delta_N}{\sqrt{2\sigma_\pm^2 \Delta_N}} \right) dy \\
\leq C_1 \frac{T_N}{T_N^{1/\lambda}} \int_{-\infty}^{\infty} m(y) dy \int_{\mathbb{R}_\pm} \exp(-C_2|y|^2/\Delta_N) dy \leq C_3 T_N^{1-\frac{1}{\lambda}} \sqrt{\Delta_N} \xrightarrow{N \to \infty} 0
\]

for constants \( C_1, C_2, C_3 \in (0, \infty) \) depending on \( a_\pm, b_\pm, \sigma_\pm \). Let us now consider the case \( a_\pm > 0 \) and \( b_\pm \in \mathbb{R} \). The desired quantity is bounded by

\[
\frac{T_N}{T_N^{1/\lambda}} \int_{-\infty}^{\infty} m(y) dy \int_{\mathbb{R}_\pm} 2|y|^m \exp \left( -\frac{y(a_\pm y - 2b_\pm)}{(\sigma_\pm)^2} \right) 2e^{\frac{-b_\pm y}{\sigma_\pm^2}} \Phi \left( -\frac{\sqrt{\sigma_\pm^2} |y|}{\gamma_N} \right) dy \\
\leq C_1 \frac{T_N}{T_N^{1/\lambda}} \int_{-\infty}^{\infty} m(y) dy \int_{\mathbb{R}_\pm} \exp(-C_2|y|^2/\gamma_N^2) dy \leq C_3 \frac{T_N^{1-\frac{1}{\lambda}}}{\gamma_N}
\]

for constants \( C_1, C_2, C_3 \in (0, \infty) \) depending on \( a_\pm, b_\pm, \sigma_\pm \). The latter term vanishes since \( \lim_{N \to \infty} \frac{T_N^{1-\frac{1}{\lambda}}}{\gamma_N} \leq \lim_{N \to \infty} T_N^{1-\frac{1}{\lambda}} \sqrt{\Delta_N} = 0 \). The proof is thus completed. \( \square \)
S2 Proof of Theorem 3

The proof of Item (i) of Theorem 3 is along the lines of the one of Theorem 2.(i).

The proof of Item (ii) of Theorem 3 is based on Lemma 5 and Lemma 6 below.

Let us be more precise. For all $T \in (0, \infty)$ and $N \in \mathbb{N}$ the difference
\[ \left( \hat{a}_T^\pm - \alpha_T^\pm, \hat{b}_T^\pm - \beta_T^\pm \right) \]
can be rewritten, using Theorem 1.(i) and Theorem 3.(i), as an expression involving only terms of the kind
\[ \left( \Omega_{T,N}^{\pm,i} - \Omega_{T}^{\pm,i} \right) \]
for $j \in \{0, 1\}$, $i \in \{0, 1, 2\}$. The convergence $(\hat{a}_T^\pm, \hat{b}_T^\pm, \alpha_T^\pm, \beta_T^\pm) \overset{P}{\longrightarrow} (\alpha_T^\pm, \beta_T^\pm, \alpha_T^\pm, \beta_T^\pm)$ is obtained combining Lemma 2 and Theorem 2.2 in [Crimaldi and Pratelli, 2005] with the convergences in probability in Lemma 5 and Lemma 6 below. The proof of (2.12) relies on Lemma 2, Theorem 2.2 in [Crimaldi and Pratelli, 2005] Lemma 6 and equation (S2.4) in Lemma 5.

Lemma 5. Let $m = 0, 1$. Then $\mathcal{M}_{T,N}^{\pm,m} \overset{P}{\longrightarrow} \mathcal{M}_T^{\pm,m}$ and
\[ N^{1/4}(\mathcal{M}_{T,N}^{\pm,m} - \mathcal{M}_T^{\pm,m}) \overset{\text{stably}}{\longrightarrow} \pm r_m \sqrt{\frac{4\sqrt{T}}{3\sqrt{2\pi}} \sigma_+^2 + \sigma_-^2} B_{L^T_1}^{\ell,T}(X) \quad (\text{S2.4}) \]
where $B$ is a Brownian motion independent of $X$. 

Lemma 6. Let $m \in \mathbb{N}$. Then $\sqrt{N}(\Omega_{T,N}^{\pm,m} - \Omega_T^{\pm,m}) \overset{P}{\longrightarrow} 0$. 

Moreover, if the threshold $r = 0$, then $\sqrt{N^{1+m\varepsilon}}(\Omega_{T,N}^{+,m} - \Omega_{T}^{+,m}) \xrightarrow{\mathbb{P}} 0$ for every $\varepsilon < 1$.

The latter convergence result extends [Lejay and Pigato, 2018, Theorem 4.14], where only $m = 0$ is considered. The proof strategies are the same. In what follows we prove Lemma 5 and Lemma 6 for $X$ solution to (1.1).

**Proof of Lemma 5**

Without loss of generality we can assume $X_0$ deterministic and also reduce ourselves to prove all results of the section in the case of null drift, i.e. $X$ is an oscillating Brownian motion (OBM). Indeed all statements are about convergence in probability or stable convergence, and, once these convergences have been proved for the null drift case, they can be extended to the drifted case (piecewise linear drift) using the fact that Girsanov weight is an exponential martingale and dominated convergence theorem. In the case of convergence in probability one proves that for every sub-sequence there exists a sub-sub-sequence converging a.s., instead stable convergence follows by property (2.8) in Remark 1 and Skorokhod representation theorem.

Therefore, from now on, let $X$ be an OBM with deterministic starting point $X_0$ and let $T \in (0, \infty)$ be fixed.
Before completing the proof Lemma 5, we need the following Lemma 7.

Lemma 7. It holds that

\[ N^{\frac{1}{4}} \left( \sum_{k=0}^{N-1} (X_{(k+1)T/N} - X_{kT/N})^2 1_{\{\pm (X_{kT/N} - r) > 0\}} \right)^{\frac{1}{2}} \rightarrow \mathbb{P} \rightarrow N \rightarrow \infty 0. \]

Proof of Lemma 7. We write \( X^{r,\pm} := (X - r) 1_{\{\pm (X - r) > 0\}} \), \( X_k := X_k^{2/N} \), and \( \Delta_k X := X_{k+1} - X_k \). Moreover we can assume \( r = 0 \) (just note that given \( X \) with threshold 0, \( \eta = X + r \) has threshold \( r \) and \( \Delta_i X = \Delta_i \eta, \Delta_i X^{0,\pm} = \Delta_i \eta^{r,\pm} \)).

We observe that

\[
\sum_{i=0}^{N-1} (\Delta_i X)^2 1_{\{\pm X_i > 0\}} = \sum_{i=0}^{N-1} \Delta_i X \Delta_i X^{0,\pm} + \sum_{i=0}^{N-1} (\Delta_i X) |X_{i+1}| 1_{\{X_i, X_{i+1} < 0\}}.
\]

Proposition 1 in [Mazzonetto 2019](#) (or [Lejay et al. 2019, Proposition 2](#), [Lejay and Pigato 2020, Proof of Lemma 1](#)), combined with property (2.8) in Remark 1, ensures that

\[
\frac{1}{N^{1/4}} \cdot \sum_{i=0}^{N-1} (\Delta_i X) |X_{i+1}| 1_{\{X_i, X_{i+1} < 0\}} \rightarrow \mathbb{P} \rightarrow N \rightarrow \infty 0 \cdot \frac{2\sqrt{2}(\sigma_+^2 + \sigma_-^2)}{3\sqrt{\pi(\sigma_+ + \sigma_-)}} L_{T}^0(X) = 0.
\]

Now, consider the remaining term. In [Lejay and Pigato 2018](#) (cf. proof of Theorem 3.5, page 3594) it is shown that there exists a constant \( C \in \mathbb{R} \)
such that

\[
\sqrt{N} \left( \sum_{k=0}^{N-1} \Delta_k X \Delta_k X^0, \pm \sigma^2_\pm \int_0^T 1_{\{\pm x > 0\}} ds \right)
\]

\[
\xrightarrow{\text{stably}}_{N \to \infty} \sqrt{2T} \sigma^2_\pm \int_0^T 1_{\{\pm x > 0\}} dB_s
\]

where \( B \) is a Brownian motion independent of \( X \). Using (2.8) in Remark 1 with the stably convergent sequence above and the vanishing sequence \( N^{-1/4} \) completes the proof.

We are now ready to prove Lemma 5.

**Proof of Lemma 5** Let \( \{\cdot\}^\pm \) denote positive and negative part. Note that for all \( t \in [0, T] \) it holds \( \mathbb{P}\text{-a.s.} \) that \( (X_t - r)1_{\{\pm(x_t-r) > 0\}} = \pm \{X_t - r\}^\pm \).

Applying Itô-Tanaka formula establishes that the following equalities hold \( \mathbb{P}\text{-a.s.}: \)

\[
(X_T + r)^i \{X_T - r\}^\pm - (X_0 + r)^i \{X_0 - r\}^\pm
\]

\[
= \pm 2i M_T^i \pm i^2 \sigma_{\Delta}^2 \Omega_T^\pm, 0 + \frac{r^i}{2^{1-i}} L_T^r(X),
\]

for \( i = 0, 1 \). Next, note that it holds \( \mathbb{P}\text{-a.s.} \) for all \( i \in \{0, \ldots, N - 1\} \) that

\[
1_{\{\pm(x_i-r) > 0\}} \Delta_i X = \pm \{X_{i+1} - r\}^\pm \mp \{X_i - r\}^\pm \mp 1_{\{(x_i-r)(x_{i+1}-r) < 0\}}|X_{i+1} - r|
\]

and

\[
2X_i 1_{\{\pm(x_i-r) > 0\}} \Delta_i X
\]

\[
= (X_{i+1}^2 - r^2) 1_{\{\pm(x_{i+1}-r) > 0\}} - (X_i^2 - r^2) 1_{\{\pm(x_i-r) > 0\}} - (X_{i+1} - X_i)^2 1_{\{\pm(x_i-r) > 0\}}
\]

\[
\mp 2r 1_{\{(x_i-r)(x_{i+1}-r) < 0\}} |X_{i+1} - r| \mp 1_{\{(x_i-r)(x_{i+1}-r) < 0\}} |X_{i+1} - r| (X_i - r).
\]
Combining this with (2.13) and (S2.5) implies that it holds \( \mathbb{P} \)-a.s. that
\[
\mp 2 (\mathcal{M}^{\pm,0}_{T,N} - \mathcal{M}^{\pm,0}_T) = L^r_{T,N} - L^r_T(X) \quad \text{and} \quad (S2.6)
\]
\[
2 (\mathcal{M}^{\pm,1}_{T,N} - \mathcal{M}^{\pm,1}_T)
= \mp r(L^r_{T,N} - L^r_T(X)) - \sum_{k=0}^{N-1} (X_{k+1} - X_k)^2 1_{\{\pm(x_k-r)>0\}} + \sigma^2_{\pm} \mathcal{Q}^{\pm,0}_T
\]
\[
+ \sum_{k=0}^{N-1} (X_{k+1} - r)|X_{k+1} - r| 1_{\{(x_k-r)(x_{k+1}-r)<0\}}.
\]

The result for \( m = 0 \) follows from [Mazzonetto, 2019, Proposition 2]. For \( m = 1 \), the three terms of the right hand side of (S2.6). The stable convergence of the first terms is ensured by [Mazzonetto, 2019, Proposition 2]. The proof is completed if the remaining terms converge in probability: in this case we have joint stable convergence to the sum of the limits (e.g. by (2.8) in Remark 1). Lemma 7 ensures that
\[
\sum_{k=0}^{N-1} (X_{k+1} - X_k)^2 1_{\{\pm(x_k-r)>0\}} - \sigma^2_{\pm} \mathcal{Q}^{\pm,0}_T
\]
converges to 0 in probability, and it provides the convergence speed. The last term satisfies
\[
\mp \sum_{k=0}^{N-1} (X_{k+1} - r)|X_{k+1} - r| 1_{\{(x_k-r)(x_{k+1}-r)<0\}}
\]
\[
= \mp \frac{1}{\sqrt{N}} \left( \sqrt{N} \sum_{k=0}^{N-1} (X_{k+1} - r)|X_{k+1} - r| 1_{\{(x_k-r)(x_{k+1}-r)<0\}} \right),
\]
which converges to 0 since, by [Mazzonetto, 2019, Proposition 1] or [Lejay et al., 2019, Proposition 2] or [Lejay and Pigato, 2020, Proof of Lemma 1],
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it holds

\[ \sqrt{N} \sum_{i=0}^{N-1} (X_{i+1} - r) |X_{i+1} - r| \mathbf{1}_{\{X_i - r, (X_{i+1} - r) < 0\}} \xrightarrow{\mathbb{P}} \frac{2\sqrt{2}}{3\sqrt{\pi}} (\sigma_+ - \sigma_-) L_T^r(X). \]

The proof is thus completed. \(\square\)

Proof of Lemma 6

As in the previous section, let \(X\) be an OBM with deterministic starting point \(X_0\) and let \(T \in (0, \infty)\) be fixed. We reduce to consider threshold \(r = 0\) because for \(t \in (0, \infty)\) it holds that \(\mathcal{Q}_t^{\pm,m} := \mathcal{Q}_t^{\pm,m}(X, r) = \sum_{k=0}^m \binom{m}{k} r^{m-k} \mathcal{Q}_t^{\pm,k}(X - r, 0)\) and the same holds for \(\mathcal{Q}_t^{\pm,m}\).

The following result follows from Lemma 4.3 in [Lejay and Pigato, 2018] and the scaling property for OBM.

**Lemma 8.** Let \(f\) be a bounded function such that \(\int |x|^k |f(x)|\,dx < \infty\) for \(k = 0, 1, 2\). Then for all \(T \in (0, \infty)\)

\[ (N/T)^{-1/2} \sum_{i=0}^{N-1} f(\sqrt{N/T} X_{iT/N}) \xrightarrow{\mathbb{P}} \lambda_\sigma(f) L_T^0(X) \tag{S2.7} \]

where \(\lambda_\sigma(f) := \left( \frac{1}{\sigma^2} \int_0^\infty f(x)\,dx + \frac{1}{\sigma^2} \int_{-\infty}^0 f(x)\,dx \right).\)

Let us denote by \(\mathcal{G}\) the natural filtration associated to the process \(X\) (or equivalently to its driving BM). Let \(m \in \mathbb{N}\) be fixed. For \(i = 1, \ldots, N,\)
we consider $X_{i-1,N} := X_{(i-1)T/N}$, $G_{i-1,N} := G_{(i-1)T/N}$, and

$$J_{i,N}^{(m)} = \left( \frac{T}{N} X_{i-1,N}^m 1\{\pm X_{i-1,N} \geq 0\} - \int_{(i-1)T}^{iT/N} X_s^m 1\{\pm X_s \geq 0\} \, ds \right)$$

$$= \pm \text{sgn}(X_{i-1,N}) X_{i-1,N}^m \int_{(i-1)T}^{iT/N} 1\{X_{i-1,N} X_s < 0\} \, ds$$

$$+ \int_{(i-1)T}^{iT/N} (X_{i-1,N}^m - X_s^m) 1\{\pm X_s > 0\} \, ds,$$

$$U_{i,N}^{(m)} = J_{i,N}^{(m)} - \mathbb{E}[J_{i,N}^{(m)}|G_{i-1,N}].$$

Observe that $U_{i,N}^{(m)}$ are martingale increments and

$$\Omega_{T,N}^{\pm,m} - \Omega_T^{\pm,m} = \sum_{i=1}^{N} \mathbb{E}[J_{i,N}^{(m)}|G_{i-1,N}] + \sum_{i=1}^{N} U_{i,N}^{(m)}.$$

The following lemma proves the convergence of the two terms.

**Lemma 9.** Let $\varepsilon \in [0, 1)$, $m \in \mathbb{N}$, and let $G$, $J_{i,N}^{(m)}$ and $U_{i,N}^{(m)}$, $i \in \{1, \ldots, N\}$ defined by (S2.8). Then

i) $N^{\frac{1+m\varepsilon}{2}} \sum_{i=1}^{N} \mathbb{E}[J_{i,N}^{(m)}|G_{i-1,N}] \xrightarrow{P} 0$ and

ii) $N^{\frac{1+m\varepsilon}{2}} \sum_{i=1}^{N} U_{i,N}^{(m)} \xrightarrow{P} 0$.

**Proof of Lemma 9.** In this proof we use the following notation: For every

$q \in [0, \infty)$ let $f_m$, $g_{m,q}$, $h_q$ be the real functions satisfying

$$f_m(x) = \begin{cases} \frac{2\sigma_+}{\sigma_- + \sigma_+} \int_0^1 x^m \Phi((-x/(\sigma_+ \sqrt{t})) \, dt & \text{if } x \geq 0 \\ \frac{-2\sigma_-}{\sigma_- + \sigma_+} \int_0^1 x^m \Phi(x/(\sigma_- \sqrt{t})) \, dt & \text{if } x < 0 \end{cases}$$
with \( \Phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{y^2}{2}} \, dy \), \( h_q(x) = |x|^q |f_0(x)| \), and

\[
g_{m,q}(y) := \begin{cases} 
\int_0^1 \int_0^\infty |y^m - x^m|^q \frac{1}{\sqrt{2\pi t} \sigma_+} e^{-\frac{1}{2} \left( \frac{x - y}{\sigma_+} \right)^2} \left( 1 + \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+} e^{-\frac{4xy}{2\sigma_+}} \right) \, dx \, dt & \text{if } y \geq 0 \\
\int_0^1 \int_0^\infty |y^m - x^m|^q \frac{1}{\sqrt{2\pi t} \sigma_-} e^{-\frac{1}{2} \left( \frac{x - y}{\sigma_-} \right)^2} \left( 1 + \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+} e^{-\frac{4xy}{2\sigma_-}} \right) \, dx \, dt & \text{if } y < 0.
\end{cases}
\]

Let us show that the functions above satisfy the assumptions of Lemma 8. Indeed, since \( \Phi(-x)1_{\{x \geq 0\}} \leq \frac{1}{2} e^{-x^2/2} \), it holds

\[
|f_0(x)| \leq \frac{\sigma(x)}{\sigma_- + \sigma_+} \int_0^1 e^{-\frac{t^2 x^2}{2\sigma(x)^2}} \, dt \leq \frac{\sigma(x)}{\sigma_- + \sigma_+} e^{-\frac{x^2}{2\sigma(x)^2}}.
\]

This ensures that the coefficients (defined in (S2.7)) \( \lambda_\sigma(f_m), \lambda_\sigma(h_q) \) are finite. Moreover it can be shown that \( \lambda_\sigma(g_{m,q}) < \infty \) for \( q \in [0, \infty) \). In particular note that

\[
\lambda_\sigma(f_m) = \frac{2(\sigma_+^m - (-\sigma_-)^m)}{\sigma_- + \sigma_+} \int_0^\infty x^m \int_0^1 \Phi(-x/\sqrt{s}) \, ds \, dx.
\]

Hence Lemma 8 shows for all \( q \in [0, \infty) \), \( t \in (0, \infty) \), \( f \in \{f_m, h_q, g_{m,q}\} \) that

\[
\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} f(\sqrt{N/T} X_{iT/N}) \xrightarrow{\text{law}} \frac{\lambda_\sigma(f)}{\sqrt{T}} L_T(X). \quad (S2.9)
\]

Let us first show an easy useful equality, which uses the explicit expression of the transition density of the OBM

\[
q_\sigma(t, \sigma(x)x, \sigma(y)y) = \frac{1}{\sqrt{2\pi t} \sigma(y)} \left( e^{-\frac{(x-y)^2}{2t}} + \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+} \sgn(y) e^{-\frac{(|x|+|y|)^2}{2t}} \right).
\]

(S2.10)
Let $Y$ be an OBM with starting point $Y_0$ and threshold $r = 0$, then for all $c \in \{-1, +1\}$, $q \in [0, \infty)$, Fubini, (S2.10), a change of variable yield
\[
1_{\{cY_0 > 0\}} \mathbb{E} \left[ \int_0^T |Y_0|^q 1_{\{cY_s < 0\}} \, ds \right] = 1_{\{cY_0 > 0\}} \int_0^T |Y_0|^q \mathbb{E} \left[ 1_{\{cY_s < 0\}|Y_0} \right] \, ds
\]
\[
= 1_{\{cY_0 > 0\}} (T/N)^{q+1} \int_0^1 \frac{2\sigma(Y_0)|\sqrt{N/T} Y_0|^q}{\sigma_- + \sigma_+} \Phi(-|\sqrt{N/T} Y_0|/\sigma(Y_0)\sqrt{t})) \, dt.
\]
(S2.11)

Let us now prove Item (i).

First step. Let $i \in \{1, \ldots, N\}$ be fixed.

We prove in this step that
\[
N^{1+m\varepsilon} \sum_{k=0}^{N-1} \mathbb{E} \left[ \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} \pm \text{sgn}(X_{k,N}) X_{k,N}^m \mathbb{I}_{\{X_{k,N} X_s < 0\}} \, ds \right]_{\mathcal{G}_{k,N}} \xrightarrow{P \to \infty} 0.
\]
(S2.12)

Note that the Markov’s property and (S2.11) ensure that
\[
\sqrt{N^{1+m\varepsilon}} \sum_{i=1}^N \mathbb{E} \left[ \int_{\frac{iT}{N}}^{\frac{(i+1)T}{N}} \pm \text{sgn}(X_{(i-1)T/N}) X_{(i-1)T/N}^m \mathbb{I}_{\{X_{(i-1)T/N} X_s < 0\}} \, ds \right]_{\mathcal{G}_{i-1,N}}
\]
\[
= \pm T^{1+m \frac{\varepsilon}{2}} \sum_{i=0}^{N-1} N^{-\frac{1+m(1-\varepsilon)}{2}} f_m(\sqrt{N/T} X_{iT/N}).
\]
(S2.13)

This vanishes because equation (S2.9) holds and $\lambda_\sigma(f_0) = 0$ and when $m \neq 0$ it holds $\varepsilon < 1$. The proof of (S2.12) is thus completed.

Second step. Let $j \in \{1, 2\}$. We prove now that
\[
N^{j(1+m\varepsilon)} \sum_{i=1}^N \mathbb{E} \left[ \int_{\frac{iT}{N}}^{\frac{(i+1)T}{N}} (X_{i-1,N}^m - X_s^m) \mathbb{I}_{\{X_s > 0\}} \, ds \right]_{\mathcal{G}_{i-1,N}} \xrightarrow{P \to \infty} 0.
\]
(S2.14)
By the Markov property, a simple change of variable, Fubini, and the explicit expression of the transition density of the OBM in \((S2.10)\) we obtain for all \(i \in \{1, \ldots, N\}\)

\[
E \left[ \int_{\frac{t}{N}}^{\frac{N}{T}} X^m_{(i-1)T} - X^m_s \, ds \, | \mathcal{G}_{i-1,N} \right] = \frac{T}{N} \int_0^1 E \left[ |X^m_{(i-1)T} - X^m_t| \, 1_{\{X_t > 0\}} | X_{(i-1)T} \right] dt = \left( \frac{T}{N} \right)^{m_i+1} g_{m,j} (\sqrt{N/T} X^m_{(i-1)T}).
\]

Combining \((S2.9)\) with the fact that \(\lambda_\sigma(g_{0,j}) = 0\) and \(\varepsilon < 1\) it follows that the latter quantity converges in probability to 0 with the speed which proves \((S2.14)\). Taking \(j = 1\) establishes Item (i).

**Third step.** (Proof of Item (ii)). Note that Jensen’s inequality implies that

\[
E[(U_{i,N}^{(m)})^2 | \mathcal{G}_{i-1,N}] = E[(J_{i,N}^{(m)})^2 | \mathcal{G}_{i-1,N}] - \left( E[J_{i,N}^{(m)} | \mathcal{G}_{i-1,N}] \right)^2 \leq E[(J_{i,N}^{(m)})^2 | \mathcal{G}_{i-1,N}] \leq E\left[ \frac{2T}{N} X^m_{i-1,N} \int_{\frac{t}{N}}^{\frac{N}{T}} 1_{\{X_{i-1,N} X_s < 0\}} \, ds + \frac{2T}{N} \int_{\frac{t}{N}}^{\frac{N}{T}} (X^m_{i-1,N} - X^m_s)^2 1_{\{X_s > 0\}} \, ds | \mathcal{G}_{i-1,N} \right].
\]

This and \((S2.14)\) with \(j = 2\) ensure that it suffices to prove

\[
N^{\frac{1}{2} + m\varepsilon} E \left[ \frac{2T}{N} X^m_{i-1,N} \int_{\frac{t}{N}}^{\frac{N}{T}} 1_{\{X_{i-1,N} X_s < 0\}} \, ds | \mathcal{G}_{i-1,N} \right] \xrightarrow{\mathbb{P}} 0.
\]

By the Markov’s property and \((S2.11)\) we reduce to study the convergence of

\[
2N^{-\frac{1}{2} + m(\varepsilon - 1)} \sum_{k=0}^{N-1} N^{-\frac{1}{2}} h_{2m} \left( \sqrt{N/T} X_{kT/N} \right).
\]

It follows from \((S2.9)\) that the latter quantity converges to 0 in probability
as \( N \to \infty \). We have therefore obtained that
\[
N^{1+m\varepsilon} \sum_{i=1}^{N} \mathbb{E}[(U_{i,N}^{(m)})^2 | \mathcal{G}_{t-1,N}] \xrightarrow{F} 0.
\]
Applying Theorem 4.4 in \cite{Lejay and Pigato, 2018} completes the proof.

S3 The multi-threshold Ornstein-Uhlenbeck process

Let us consider in this section the multi-threshold version of the threshold OU process, by which we mean the solution to

\[
X_t = X_0 + \int_0^t \sigma(X_s) \, dW_s + \int_0^t (b(X_s) - a(X_s) X_s) \, ds, \quad t \geq 0, \tag{S3.15}
\]

with piecewise constant coefficients \( \sigma, a, \) and \( b \) possibly discontinuous at levels \(-\infty = r_0 < r_1 < \ldots < r_d < r_{d+1} = +\infty, \ d \in \mathbb{N}\). Let \( I_0 = (-\infty, r_1) \) and \( I_j := [r_j, r_{j+1}) \), for \( j \in \{1, \ldots, d\} \). The volatility coefficient is given by

\[
\sigma(x) = \sum_{j=0}^{d} \sigma_j 1_{I_j}(x) > 0 \tag{S3.16}
\]

and similarly the drift coefficients are given by

\[
b(x) = \sum_{j=0}^{d} b_j 1_{I_j}(x) \quad \text{and} \quad a(x) = \sum_{j=0}^{d} a_j 1_{I_j}(x). \tag{S3.17}
\]

In analogy to the result for \( d = 1 \) we also denote \( a_0, b_0, \sigma_0 \) by \( a_-, b_-, \sigma_- \) and \( a_d, b_d, \sigma_d \) by \( a_+, b_+, \sigma_+ \).
S3. THE MULTI-THRESHOLD ORNSTEIN-UHLENBECK PROCESS

S3.1 The regimes of the process

In this section, we establish for which values of the coefficients the process $X$ is (positively or null) recurrent or transient. Recall that we denote scale function and speed measure respectively by $S$ and $m$. The derivative (up to a multiplicative constant) of the scale function satisfies for all $j = 0, 1$ and $k = 2, \ldots, d$

$$S'(x)\mathbf{1}_{I_j}(x) = \frac{1}{s_j(x, r_1)}$$

and

$$S'(x)\mathbf{1}_{I_k}(x) = \frac{1}{s_k(x, r_k) \prod_{i=1}^{k-1} s_i(r_{i+1}, r_i)}$$

where for all $k \in \{0, 1, \ldots, d\}$, $x, r \in \mathbb{R}$ we define the functions

$$s_k(x, r) := \exp\left(\frac{2b_k(x - r) - a_k(x^2 - r^2)}{\sigma_k^2}\right).$$

The speed measure is $m(x) \, dx = \frac{2}{\sigma(x)^2 S'(x)} \, dx$.

Recall that $X$ is recurrent if and only if $\lim_{x \to +\infty} S(x) = +\infty$ and $\lim_{x \to -\infty} S(x) = -\infty$, which happens if and only if $[(a_+ > 0 \text{ and } b_+ \in \mathbb{R})$ or $(a_+ = 0 \text{ and } b_+ \leq 0)]$ and $[(a_- > 0 \text{ and } b_- \in \mathbb{R}) \text{ or } (a_- = 0 \text{ and } b_- \geq 0)]$.

The complementary leads to transience. If $X$ is recurrent and the speed measure is a finite measure, then $X$ is positive recurrent and ergodic. It admits a stationary distribution, denoted by $\mu$, which is the renormalized speed measure. Hence we have the following ergodicity conditions:

$$[(a_+ > 0 \text{ and } b_+ \in \mathbb{R}) \text{ or } (a_+ = 0 \text{ and } b_+ < 0)]$$

and $$[(a_- > 0 \text{ and } b_- \in \mathbb{R}) \text{ or } (a_- = 0 \text{ and } b_- > 0)].$$
These coincide with the conditions for the single threshold case $d = 1$.

**Lemma 10** (Multi-threshold version of Lemma 1). *The speed measure is finite if and only if condition \[(S3.18)\] holds. More precisely, let $C_0 = C_1 = 1$, $C_j = \prod_{k=1}^{j-1} s_k(r_{k+1}, r_k), j = 2, \ldots, d$, let

$$m_{i,j,k} := \frac{\sqrt{\pi}}{\sigma_i \sqrt{a_i}} \exp\left(\frac{a_i}{\sigma_i^2} \left( \frac{b_i}{a_i} - r_j \right)^2 \right) \text{erfc}\left(\frac{-\sqrt{a_i}}{\sigma_i} \left( \frac{b_i}{a_i} - r_k \right) \right) \quad (S3.19)$$

with $i \in \{0, \ldots, d\}$ and $j, k \in \{1, \ldots, d\}$ and let $m_+ = -m_{0,1,1}$ and $m_- = m_{d,d,d}$. Then if $j \in \{0, d\}$

$$\int_{-\infty}^{\infty} 1_{I_j}(y) m(y) \, dy = \begin{cases} +\infty & \text{if } a_+ = 0 \text{ and } b_+ = 0, \\ \frac{C_j}{|b_\pm|} & \text{if } a_+ = 0 \text{ and } (-1)^\pm b_\pm > 0, \\ C_j m_\pm & \text{if } a_+ > 0 \text{ and } b_\pm \in \mathbb{R} \end{cases}$$

and if $j \in \{1, \ldots, d - 1\}$

$$\int_{-\infty}^{\infty} 1_{I_j}(y) m(y) \, dy = \begin{cases} \frac{r_{j+1} - r_j}{\sigma_j^2} & \text{if } a_j = 0 \text{ and } b_j = 0, \\ \frac{C_j}{b_j} \exp\left(\frac{2b_j(r_{j+1} - r_j)}{\sigma_j^2} - 1\right) & \text{if } a_j = 0 \text{ and } b_j \neq 0, \\ C_j (m_{j,j,j} - m_{j,j,j+1}) & \text{if } a_j > 0 \text{ and } b_j \in \mathbb{R}, \\ C_j \int_{r_j}^{r_{j+1}} s_j(x, r_j) \, dx < \infty & \text{if } a_j < 0 \text{ and } b_j \in \mathbb{R}. \end{cases}$$
S3.2 MLE and QMLE from continuous time observations

We assume in this section to observe the process on the time interval \([0, T]\), \(T \in (0, \infty)\). For \(T \in (0, \infty)\), \(m = 0, 1, 2\), and \(j = 0, 1, \ldots, d\) we define

\[
\begin{align*}
M_{j,m,T} &:= \int_0^T X_s^m 1_{I_j}(X_s) \, dX_s \\
Q_{j,m,T} &:= \int_0^T X_s^m 1_{I_j}(X_s) \, ds
\end{align*}
\]

and take as likelihood function \(G_T(a,b)\) and as quasi-likelihood \(\Lambda_T(a,b)\) defined as in Section 2.

**Lemma 11** (Multi-threshold version of Lemma 2). Assume the process is ergodic: condition (S3.18) holds. Then, for all \(i \in \{0, 1, 2\}\), \(j \in \{0, 1, \ldots, d\}\), the quantities \(\Omega_{i,j}\) defined as follows are finite constants:

\[
\Omega_{i,j} := \lim_{t \to \infty} \frac{\sum_{t=1}^n x_i \mu(dx)}{t} = \int_{I_j} x_i \mu(dx) \in \mathbb{R}
\]

**Theorem 4.** i) For every \(T \in (0, \infty)\) the MLE and QMLE are given by

\[
\begin{align*}
(\alpha_T, \beta_T) &= (\alpha_j^{(T)}, \beta_j^{(T)})_{j=0}^d \\
(\alpha_j^{(T)}, \beta_j^{(T)}) &= \left( \frac{\omega^{j,0}_T \Omega_{j,1}^{(T)} - \Omega_{j,0}^{(T)} \omega^{j,1}_T}{\Omega_{j,0}^{(T)} \Omega_{j,1}^{(T)} - (\Omega_{j,1}^{(T)})^2}, \frac{\omega^{j,0}_T \Omega_{j,2}^{(T)} - \Omega_{j,1}^{(T)} \omega^{j,1}_T}{\Omega_{j,0}^{(T)} \Omega_{j,2}^{(T)} - (\Omega_{j,1}^{(T)})^2} \right)
\end{align*}
\]

Assume now that the process is ergodic (i.e., (S3.18) is satisfied).

ii) The following LLN holds, i.e., the estimator is (strongly) consistent:

\[
(\alpha_T, \beta_T) - (a, b) \xrightarrow{a.s.} 0
\]

iii) The following CLT holds:

\[
\sqrt{T} \left( (\alpha_j^{(T)}, \beta_j^{(T)}) - (a_j, b_j) \right)_{j=0}^d \xrightarrow{T \to \infty} (N_j)_{j=0}^d
\]
where \( N_j = \left( N_j^{i,\alpha}, N_j^{i,\beta} \right) \), \( j = 0, 1, \ldots, d \) are independent, independent of \( X \), two-dimensional Gaussian random variables with covariance matrices respectively \( \sigma_j^2 \Gamma_j^{-1} \) and \( \sigma_j^2 \Gamma_j^{-1} \) where

\[
\Gamma_j := \begin{pmatrix}
\overline{\Sigma}_j^{i,2} & -\overline{\Sigma}_j^{i,1} \\
-\overline{\Sigma}_j^{i,1} & \overline{\Sigma}_j^{i,0}
\end{pmatrix},
\]

and \( \overline{\Sigma}_j^{i,i}, i \in \{0, 1, 2\} \) are real constants such that \( \lim_{t \to \infty} \frac{\overline{\Sigma}_j^{i,i}}{t} \overset{a.s.}{=} \overline{\Sigma}_\infty^{i,i} \).

iv) The LAN property holds for the likelihood evaluated at the true parameters \( (a, b) \) with rate of convergence \( \frac{1}{\sqrt{T}} \): there exists a random vector \( A_T \) such that for small perturbations \( \Delta(a, b) := (\Delta a_j, \Delta b_j)_{j=0}^d \) it holds that

\[
\log \frac{G_T((a, b) + \frac{1}{\sqrt{T}} \Delta(a, b))}{G_T(a, b)} - (A_T \cdot \Delta(a, b) - \Delta(a, b) \cdot \Gamma \Delta(a, b))
\]

converges to 0 in probability as \( T \to \infty \). The matrix \( \Gamma \) is the asymptotic Fisher information

\[
\Gamma = \begin{pmatrix}
\sigma^-2 \Gamma_+ & 0_{2\times 2} & \cdots & 0_{2\times 2} \\
0_{2\times 2} & \sigma^-2 \Gamma_{d-1} & \cdots & \vdots \\
\vdots & \vdots & \ddots & 0_{2\times 2} \\
0_{2\times 2} & \cdots & 0_{2\times 2} & \sigma^-2 \Gamma_-
\end{pmatrix}.
\]

In order to study the asymptotic behavior of the estimator, we introduce a different expression for the estimators in (S3.21) based on the following notation. Given \( T \in (0, \infty) \), \( j \in \{0, 1, \ldots, d\} \), and \( i \in \{0, 1\} \) let

\[
M_T^{j,i} := \int_0^T (X_s)^i 1_{I_j}(X_s) \, dW_s.
\]
Observe that \( S3.15 \) yields for \( i \in \{0, 1\}, \ j \in \{0, 1 \ldots, d\} \):

\[
M_{T}^{j,i} = \sigma_{j}M_{T}^{j,i} + b_{j}Q_{T}^{j,i} - a_{j}Q_{T}^{j,i+1}.
\] \( S3.23 \)

Note that \( Q_{T}^{\pm,0}Q_{T}^{\pm,2} - (Q_{T}^{\pm,1})^2 \) is \( \mathbb{P}\)-a.s. positive by Cauchy-Schwarz. By \( S3.23 \), we have the following reformulation of \( S3.21 \).

**Lemma 12** (Multi-threshold version of Lemma 3). Let \( T \in (0, \infty) \) and \( j \in \{0, 1, \ldots, d\} \). Equation \( S3.21 \) can be expressed as

\[
(\alpha_{T}, \beta_{T})_{j} = (a, b)_{j} + \sigma_{j} \left( \frac{M_{T}^{j,0}Q_{T}^{j,2} - M_{T}^{j,1}}{Q_{T}^{j,0}Q_{T}^{j,2} - (Q_{T}^{j,1})^2}, \frac{M_{T}^{j,0}Q_{T}^{j,1} - M_{T}^{j,2}}{Q_{T}^{j,0}Q_{T}^{j,2} - (Q_{T}^{j,1})^2} \right),
\] \( S3.24 \)

that can be rewritten as

\[
\begin{pmatrix}
\alpha_{T}^{(j)} \\
\beta_{T}^{(j)}
\end{pmatrix} = \begin{pmatrix} a_{j} \\ b_{j} \end{pmatrix} + \sigma_{j} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q_{T}^{j,2} & Q_{T}^{j,1} \\ Q_{T}^{j,1} & Q_{T}^{j,0} \end{pmatrix}^{-1} \begin{pmatrix} M_{T}^{j,1} \\ M_{T}^{j,0} \end{pmatrix}.
\] \( S3.25 \)

The proofs of Lemma 12 and of Theorem 4 are easily adapted from the case of a unique threshold \( (d = 1) \) to the multi-threshold case and therefore omitted.

### S3.3 Drift estimation from discrete observations

We assume now to observe the process on the discrete time grid \( 0 = t_{0} < t_{1} < \ldots < t_{N-1} < t_{N} = T \), for \( N \in \mathbb{N} \), \( T \in (0, \infty) \), and set \( \Delta_{N} = \max_{k=1,\ldots,N} \{t_{k} - t_{k-1}\} \). We define \( X_{i} := X_{t_{i}} \) with \( i = 0, \ldots, N \).
The discrete versions of (S3.20) are defined as follows: for \( m = 0, 1, 2 \), \( j = 0, \ldots, d \) let
\[
\mathfrak{M}_{T,N}^{j,m} := \sum_{k=0}^{N-1} (X_{k+1} - X_k) X_k^m 1_j(X_k), \quad \text{and}
\]
\[
\mathfrak{Q}_{T,N}^{j,m} := \sum_{k=0}^{N-1} (t_{k+1} - t_k) X_k^m 1_j(X_k).
\]
(S3.26)

The discretized likelihood \( G_{T,N}(a,b) \) and the discretized quasi-likelihood \( \Lambda_{T,N}(a,b) \) are defined as in Section 2.

For \( N \in \mathbb{N} \) and \( T \in (0, \infty) \) let \((\hat{a}_{T,N}, \hat{b}_{T,N}) = (\hat{a}_{T,N}^{(j)}, \hat{b}_{T,N}^{(j)})_{j=0}^{d}\) with
\[
(\hat{a}_{T,N}^{(j)}, \hat{b}_{T,N}^{(j)}) = \left( \frac{\mathfrak{M}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,1} - \mathfrak{Q}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,1}} {\mathfrak{Q}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,1} - (\mathfrak{Q}_{T,N}^{j,1})^2}, \quad \frac{\mathfrak{M}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,2} - \mathfrak{Q}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,1}} {\mathfrak{Q}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,2} - (\mathfrak{Q}_{T,N}^{j,1})^2} \right). \quad \text{(S3.27)}
\]

**Theorem 5.** Let \((T_N)_{N \in \mathbb{N}}\) be a sequence in \((0, \infty)\). For all \( N \in \mathbb{N} \) let \((\hat{a}_{T,N}, \hat{b}_{T,N})\) be defined as in (S3.27).

i) For every \( N \in \mathbb{N} \) the vector \((\hat{a}_{T,N}, \hat{b}_{T,N})\) maximizes both the likelihood \( G_{T,N}(a,b) \) and the quasi-likelihood \( \Lambda_{T,N}(a,b) \).

Assume that the process is ergodic and that \( X_0 \) follows the stationary distribution \( \mu \). Moreover, assume
\[
\lim_{N \to \infty} T_N = \infty \quad \text{and} \quad \lim_{N \to \infty} \Delta_N = 0.
\]

ii) The following LLN holds: \((\hat{a}_{T_N}, \hat{b}_{T_N}) \xrightarrow{P} (a,b) \) (the estimator is consistent).
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iii) If \( \lim_{N \to \infty} T_N \Delta_N = 0 \), the following CLT holds:

\[
\sqrt{T_N} \left( \left( \hat{a}^{(j)}_{T,N,N}, \hat{b}^{(j)}_{T,N,N} \right) - (a_j, b_j) \right)^d_{j=0} \overset{\text{stably}}{\overset{N \to \infty}{\to}} (\mathcal{N})^d_{j=0}
\]

where \( \mathcal{N} \) is as in Theorem \([4]\).

iv) If \( \lim_{N \to \infty} T_N \Delta_N = 0 \), the analogous of the LAN property holds for the discretized likelihood evaluated at the true parameters with rate of convergence \( \frac{1}{\sqrt{T_N}} \) and asymptotic Fisher information \( \Gamma \) as in Theorem \([4]\).

S3.4 Proof of Theorem \([5]\)

Analogously to the case \( d = 1 \), the main ingredient of the proof is the following lemma.

**Lemma 13** (Multi-threshold version of Lemma 4). Assume the process is ergodic. Let \( X \) be the solution to \([S3.15]\), with \( X_0 \) distributed as the stationary distribution \( \mu \), let \( \lambda \in \{1, 2\} \) be fixed, and let \( (T_N)_{N \in \mathbb{N}} \subset (0, \infty) \) be a sequence satisfying, as \( N \to \infty \), that \( T_N \to \infty \) and \( \lim_{N \to \infty} T_N^{1-1/\lambda} \sqrt{\Delta_N} = 0 \) where \( \Delta_N := \sup_{k=1, \ldots, N} (t_k - t_{k-1}) \). Then for all \( m \in \{0, 1, 2\}, j \in \{0, 1\}, k \in \{0, \ldots, d\} \) it holds

\[
\limsup_{N \to \infty} T_N^{-1/\lambda} \mathbb{E} \left[ |\Omega_{T_N}^{k,m} - \Omega_{T_N,N}^{k,m}| \right] = 0 \quad \text{and} \quad \limsup_{N \to \infty} T_N^{-1/\lambda} \mathbb{E} \left[ |\mathcal{M}_{T_N}^{k,j} - \mathcal{M}_{T_N,N}^{k,j}| \right] = 0
\]

where \( \Omega_{T_N}^{k,m}, \Omega_{T_N,N}^{k,m}, \mathcal{M}_{T_N}^{k,j}, \mathcal{M}_{T_N,N}^{k,j} \) are defined in \([S3.20]\) and \([S3.26]\).
Before providing the proof of Lemma 13, we show how it intervenes in the proof of Items (ii)-(iii) of Theorem 5. For all $N \in \mathbb{N}$

$$(\hat{a}_{T_N,N}, \hat{b}_{T_N,N}) - (a, b) = (\alpha_{T_N,N}, \beta_{T_N,N}) + (\alpha_{T_N,N}, \beta_{T_N,N}) - (a, b)$$

The second term of the sum is handled with Theorem 4 (more precisely Item (iii)) providing the desired limit distribution. The first instead can be rewritten, using equations (S3.21) and (S3.27), as an expression which involves only terms of the kind

$$\left( \frac{\Omega^{j,1}_{T_N,N}}{\Omega^{j,0}_{T_N,N} \Omega^{j,2}_{T_N,N} - \Omega^{j,1}_{T_N,N}^2} - \frac{\Omega^{j,1}_{T_N,N}}{\Omega^{j,0}_{T_N,N} \Omega^{j,2}_{T_N,N} - \Omega^{j,1}_{T_N,N}^2} \right) m^j_k + \left( \frac{\Omega^{j,1}_{T_N,N} (m^{j,k}_{T_N,N} - m^{j,k})}{\Omega^{j,0}_{T_N,N} \Omega^{j,2}_{T_N,N} - \Omega^{j,1}_{T_N,N}^2} \right)$$

for $i \in \{0, 1, 2\}$, $j \in \{0, \ldots, d\}$, $k \in \{0, 1\}$.

Combining Lemma 13 with Lemma 11 and Theorem 2.2 in [Crimaldi and Pratelli, 2005] ensures the consistency of the estimator if $\Delta_N \to 0$ as $N \to \infty$, and if $T_N \Delta_N \to 0$ as $N \to \infty$ then it implies also that

$$\sqrt{T_N} \left( (\hat{a}_{T_N,N}, \hat{b}_{T_N,N}) - (\alpha_{T_N,N}, \beta_{T_N,N}) \right) \xrightarrow{p} 0 \quad \text{as } N \to \infty.$$  

**Proof of Lemma 13** The proof is similar to the one of Lemma 4. We provide here the main idea of the key step, that is the proof of the analogous of (S1.2): for all $j \in \{0, 1, \ldots, d\}$, $m \in \{0, 1, 2, 3, 4\}$

$$\int_0^{T_N} \mathbb{E} \left[ |X_{[\tau]}|_N^m 1_{X_{[\tau]} \in I_j, X_{[\tau]} \notin I_j} \right] dt \text{ is } o(T_N^{1/2}). \quad \text{(S3.28)}$$

We reduce to compute, given $X_{[\tau]}$ for $\tau_x = t_k$, the probability that the first exit time of a standard Brownian motion from a suitable symmetric interval is smaller than $t - t_k$: $p_t := \mathbb{P}(\tau_{X_{[\tau]}}, t_k \leq t - \tau_x)$. Indeed
starting from $X_{t_k} \in I_j$ for some $k \in \{0, \ldots, N\}$, $j \in \{0, \ldots, d\}$ at a suitable distance $R_{X_{t_k}}$ from the boundary of $I_j$, if the Brownian motion driving the OU process does not exit in small time a suitable interval then the OU of parameters $(a_j, b_j, \sigma_j)$ stays in $(X_{t_k} - R_{X_{t_k}}, X_{t_k} + R_{X_{t_k}}) \subset I_j$ because the drift is small. More precisely if $X_{t_k} \in I_j$ let $R_{X_{t_k}} := \min\{X_{t_k} - r_j, r_j + 1 - X_{t_k}\}$, let $\tilde{r}_j \in \{r_j, r_{j+1}\}$ such that $R_{X_{t_k}} = |X_{t_k} - \tilde{r}_j|$, and let

$$B_{X_{t_k}} := (e^{-|a_j|(t_{k+1} - t_k)} - |a_j|(t_{k+1} - t_k))R_{X_{t_k}} - \frac{|b_j - a_j\tilde{r}_j|}{\sigma_{j}}(t_{k+1} - t_k).$$

Note that $\sup_{j=0, \ldots, d} |\tilde{r}_j| < \infty$. If $B_{X_{t_k}} > 0$, then $\tau_{X_{t_k}, I_j}$ is the first exit time of a Brownian motion from the interval $[-B_{X_{t_k}}, B_{X_{t_k}}]$. Moreover for $t \in (t_k, t_{k+1}]$ it holds that $t \mapsto p_t$ is increasing, and

$$p_t \leq 2 \sum_{n=0}^{\infty} \text{erfc} \left( \frac{B_{X_{t_k}} (1+2n)}{\sqrt{t-t_k}} \right) \leq 2 \sum_{n=0}^{\infty} \exp \left( -\frac{B_{X_{t_k}}^2 (1+2n)^2}{t-t_k} \right)$$

and so $p_t \leq 2 \exp \left( -\frac{B_{X_{t_k}}^2}{t-t_k} \right) \left( 1 - \exp \left( -8 \frac{B_{X_{t_k}}^2}{t-t_k} \right) \right)^{-1}$.

Then using that for every $t > 0$, $N \in \mathbb{N}$ it holds $1 = 1_{\{B_{X_{t}}(\Delta_N) > \sqrt{t-|t|} \Delta_N\}} + 1_{\{B_{X_{t}}(\Delta_N) \leq \sqrt{t-|t|} \Delta_N\}}$, we split the integrals into two parts to deal with in two different ways.
Note that

\[ \int_0^{T_N} \mathbb{E} \left[ |X_{[t] \Delta_N}|^m 1_{\{X_{[t] \Delta_N} \in I_j, X_t \notin I_j\}} 1_{\{B_{X_{[t] \Delta_N}} \leq \sqrt{t-[t] \Delta_N}\}} \right] dt \]

\[ \leq \sum_{k=0}^{N-1} (t_{k+1} - t_k) \mathbb{E} \left[ |X_{t_k}|^m 1_{\{X_{t_k} \in I_j\}} 1_{\{B_{X_{t_k}} \leq \sqrt{t_k + 1 - t_k}\}} \right] \]

\[ \leq 2 \left( \sup_{x \in I_j} \mu(x) \right) \sum_{k=0}^{N-1} (t_{k+1} - t_k)^{3/2} \frac{1 + |b_j - a_j \tilde{r}_j|}{\sigma_j} \sqrt{t_{k+1} - t_k} \cdot \left( |\tilde{r}_j| + \frac{\sqrt{t_{k+1} - t_k} + |b_j - a_j \tilde{r}_j| (t_{k+1} - t_k)}{e^{-|a_j|(t_{k+1} - t_k)} - |a_j|(t_{k+1} - t_k)} \right)^m. \]

Therefore for \( N \) big, since \( \Delta_N \) is small, the latter quantity goes like \( \sum_{k=0}^{N-1} (t_{k+1} - t_k)^{3/2} \leq T_N \sqrt{\Delta N} \) and it is \( o(T_N^{1/\lambda}) \).

The other integral satisfies:

\[ \int_0^{T_N} \mathbb{E} \left[ |X_{[t] \Delta_N}|^m 1_{\{X_{[t] \Delta_N} \in I_j, X_t \notin I_j\}} 1_{\{B_{X_{[t] \Delta_N}} > \sqrt{t-[t] \Delta_N}\}} \right] dt \]

\[ \leq \int_0^{T_N} \mathbb{E} \left[ |X_{[t] \Delta_N}|^m p_{X_{[t] \Delta_N}} 1_{\{X_{[t] \Delta_N} \in I_j\}} 1_{\{B_{X_{[t] \Delta_N}} > \sqrt{t-[t] \Delta_N}\}} \right] dt \]

\[ \leq 2 \frac{1}{1 - e^{-8}} \sum_{k=0}^{N-1} \int_{t_{k+1} - t_k}^{t_k} \mathbb{E} \left[ |X_{t_k}|^m e^{-\frac{\beta_j^2}{t} \cdot 1_{\{X_{t_k} \in I_j\}} 1_{\{B_{X_{t_k}} > \sqrt{t}\}}} \right] dt. \]

Let \( g_{j,k} := e^{-|a_j|(t_{k+1} - t_k)} - |a_j|(t_{k+1} - t_k) \), \( f_{j,k} := \frac{|b_j - a_j \tilde{r}_j| (t_{k+1} - t_k)}{g_{j,k}} \), and let \( \mu \)
denote the invariant measure, then
\[
\mathbb{E} \left[ |X_{t_k}|^m e^{-\frac{B^2 X_{t_k}}{2}} 1_{\{X_{t_k} \in I_j\}} 1_{\{B_{X_{t_k}} > r\}} \right]
\]
\[
= \int_{r_{j}}^{r_{j+1}} |x|^m e^{-\frac{\sigma^2_{j,k}(x-\bar{r}_j-I_{j,k})^2}{2}} 1_{\{|x-\bar{r}_j| > f_{j,k} + \frac{\sigma^2}{\sigma_{j,k}}\}} \mu(dx)
\]
\[
\leq C_j \int_{r_{j}}^{r_{j+1}} |x|^m e^{-\frac{\sigma^2_{j,k}(x-f_{j,k})^2}{2}} - \sigma^2_{j,k} |x-\bar{r}_j|^2 e^{-\sigma^2_{j,k}|x-\bar{r}_j|^2} 1_{\{|x| > f_{j,k} + \frac{\sigma^2}{\sigma_{j,k}}\}} \mu(dx)
\]
where the constant \( C_j \) may change from line to line and \(|I_j|\) is the length of the interval. For \( N \) big enough \( \Delta_N << 1, f_{j,k} + \frac{t_k+1-t_k}{g_{j,k}} < |I_j|, g_{j,k} \in [1/2, 1], f_{j,k} \leq 2|b_j - a_j\bar{r}_j|/\sigma_j \) hence the latter integral is bounded from above by
\[
C_j \int_{f_{j,k} + \frac{\sigma^2}{\sigma_{j,k}}}^{|I_j|} |x|^m e^{-\frac{\sigma^2_{j,k}(x-f_{j,k})^2}{2}} - \sigma^2_{j,k} |x-\bar{r}_j|^2 e^{-\sigma^2_{j,k}|x-\bar{r}_j|^2} 1_{\{|x| > f_{j,k} + \frac{\sigma^2}{\sigma_{j,k}}\}} \mu(dx)
\]
\[
= C_j \int_{0}^{\frac{|I_j|}{\sigma^2_{j,k}}} |x|^m e^{-\frac{\sigma^2_{j,k}(x-f_{j,k})^2}{2}} - \sigma^2_{j,k} |x-\bar{r}_j|^2 e^{-\sigma^2_{j,k}|x-\bar{r}_j|^2} \mu(dx)
\]
\[
\leq C_j \sqrt{t} \int_{0}^{+\infty} |x|^m e^{-\frac{\sigma^2_{j,k}(x-f_{j,k})^2}{2}} - \sigma^2_{j,k} |x-\bar{r}_j|^2 e^{-\sigma^2_{j,k}|x-\bar{r}_j|^2} \mu(dx)
\]
and it is $o(T_N^{1/\lambda})$. The proof is thus completed. □

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