Coupled Classical and Quantum Oscillators

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ABSTRACT

Some of the most enduring questions in physics—including the quantum measurement problem and the quantization of gravity—involve the interaction of a quantum system with a classical environment. Two linearly coupled harmonic oscillators provide a simple, exactly soluble model for exploring such interaction. Even the ground state of a pair of identical oscillators exhibits effects on the quantum nature of one oscillator, e.g., a diminution of position uncertainty, and an increase in momentum uncertainty and uncertainty product, from their unperturbed values. Interaction between quantum and classical oscillators is simulated by constructing a quantum state with one oscillator initially in its ground state, the other in a coherent or Glauber state. The subsequent wave function for this state is calculated exactly, both for identical and distinct oscillators. The reduced probability distribution for the quantum oscillator, and its position and momentum expectation values and uncertainties, are obtained from this wave function. The oscillator acquires an oscillation amplitude corresponding to a beating between the normal modes of the system; the behavior of the position and momentum uncertainties can become quite complicated. For oscillators with equal unperturbed frequencies, i.e., at resonance, the uncertainties exhibit a time-dependent quantum squeezing which can be extreme.

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I. INTRODUCTION

Interaction of the quantum and classical worlds—i.e., between a system for which quantum effects predominate and a system operating in a classical limit—lies at the heart of some of the most profound and vexing problems in physics. The quantum measurement problem, of course, is one of the central mysteries of quantum physics; it has engendered concepts from Schrödinger’s Cat and the many-worlds interpretation to consistent histories. The search for a quantum theory of gravitation also entails linking quantum matter sources to apparently classical spacetime geometry, as the Einstein field equations display. While now largely taken for granted, the notion that there must exist a quantum version of gravitation was at one time the subject of some debate [1], and even of experimental test [2]. One longstanding argument that gravitation must be a quantum phenomenon with a classical limit, and not a strictly classical field, is that an ordinary quantum system coupled to a truly classical field would “radiate away” its quantum nature—commutator values, uncertainties, etc.—leading to observable violations of quantum mechanics. A pair of simple harmonic oscillators with linear coupling provides an ideal test bed for exploring this aspect of quantum/classical interaction.

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Two linearly coupled quantum-mechanical simple harmonic oscillators, e.g., two masses on springs, connected by a third spring, constitute an ideally simple model of quantum/classical coupling. First, the system separates into normal modes behaving as independent oscillators, so the evolution of the system from any initial data can be followed exactly. Second, the classical limit of a quantum oscillator is easily described by a coherent or Glauber state \([3,4]\): a Gaussian wave packet of fixed width, the centroid of which follows a classical trajectory. In the present work we analyze the behavior of coupled oscillators with one initially in its quantum ground state, the other initially in such a coherent state. The problem of a quantum oscillator coupled to an actual classical, i.e., \(c\)-number oscillator calls for different techniques and will be examined in a subsequent paper.

The simplest version of the two-oscillator system consists of two identical oscillators, with equal masses, spring constants, and frequencies, plus a connecting spring with its own spring constant. Even in the simplest of quantum states—its ground state—this system reveals interesting effects on the oscillators’ dynamics: The position uncertainty of an individual oscillator is reduced below its uncoupled value, while its momentum uncertainty is increased. (Hence, the second oscillator does not act as a “heat bath,” which would increase both uncertainties.) The product of the uncertainties is increased; the Uncertainty Principle is not violated. If the system is started with one oscillator in its quantum ground state and the other in a quasi-classical coherent state, e.g., if the coupling between them is “turned on” at some initial time, then subsequently the two normal modes evolve one in a coherent state, the other in a modified state termed a displaced squeezed state, e.g., in quantum optics and the analysis of gravitational-wave detectors [4,5]. The initially quantum oscillator acquires an oscillating position expectation value—a “beat” between the normal modes. Its position uncertainty oscillates through values below its uncoupled, ground-state value, while its momentum uncertainty oscillates above its uncoupled value. The product of these uncertainties oscillates through values above \(\hbar/2\), never violating the Uncertainty Principle.

The general two-oscillator system, with arbitrary masses, spring constants, and frequencies, can likewise be analyzed exactly. We construct the wave function for a state with one oscillator initially in its ground state, the other in a coherent state. From this we obtain position and momentum expectation values and uncertainties for the first oscillator. Expectation values behave very like those in the identical-oscillator (“symmetric”) case, but the expressions we find for uncertainties in the fully general case are rather opaque. However, one simple case may be of particular physical importance: that of oscillators with different masses, but equal (uncoupled) frequencies, i.e., that of quantum and classical oscillators interacting “at resonance.” In that case quantum uncertainties clearly exhibit the behaviors found in the symmetric case: Position uncertainty is reduced from its uncoupled quantum value—toward the classical zero value—while momentum uncertainty is increased. The product of these remains always above the Uncertainty Principle bound; the system of course never actually violates quantum mechanics.

The simplest case of two identical oscillators is detailed in Sec. II below. The fully general case is shown in Sec. III: the special but physically important equal-frequency case is treated in Sec. IV. We summarize our results and conclusions in Sec. V. For ease of comparison between quantum and classical regimes, we retain the constant \(\hbar\) explicitly throughout.

II. SYMMETRIC COUPLED OSCILLATORS

The simplest system of two coupled harmonic oscillators consists of two identical, one-dimensional oscillators with linear coupling. For example, two identical masses \(m\), each on a linear spring with spring constant \(k\) [so each oscillator has uncoupled angular frequency \(\omega = (k/m)^{1/2}\)], connected by a third spring with spring constant \(\kappa\), constitute such a system.
A. Hamiltonian and normal modes.

The Hamiltonian for this system can be written

\[ H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2}k x_1^2 + \frac{1}{2}k x_2^2 + \frac{1}{2}\kappa (x_2 - x_1)^2 \]  

(2.1a)

\[ = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2}k x_1^2 + \frac{1}{2}(k + 2\kappa)x_2^2, \]  

(2.1b)

in terms of individual oscillator displacements (from equilibrium) \( x_1 \) and \( x_2 \) and momenta \( p_1 \) and \( p_2 \), or normal-mode coordinates \( x_\pm \) and momenta \( p_\pm \). The normal-mode amplitudes are related to the individual coordinates via

\[ x_+ = \frac{x_1 + x_2}{\sqrt{2}} \]  

(2.2a)

\[ x_- = \frac{x_2 - x_1}{\sqrt{2}}, \]  

(2.2b)

where the scale factor is chosen to make this a unitary transformation. The normal-mode momenta \( p_\pm \), of course, are conjugate to these. The separated form (2.1b) of the Hamiltonian allows the system to be treated as two independent harmonic oscillators.

B. Ground state.

The quantum ground-state wave function of the system can be written as the product of the normal-mode ground states, viz.,

\[ \Psi_0(x_\pm, t) = \left(\frac{m^2\omega\Omega}{\pi^2\hbar^2}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} x_1^2 \right) \exp\left(-\frac{m\Omega}{2\hbar} x_2^2 \right) \exp\left[-\frac{i}{\hbar}(\omega + \Omega)t\right]. \]  

(2.3)

The normal-mode angular frequencies follow from Hamiltonian (2.1b):

\[ \omega = \left(\frac{k}{m}\right)^{1/2} \]  

(2.4a)

for the + mode, and

\[ \Omega = \left(\frac{k + 2\kappa}{m}\right)^{1/2} = \frac{\omega}{\gamma}, \]  

(2.4b)

for the − mode, where the last expression defines the parameter \( \gamma \).

The probability distribution for the position of one oscillator, with the system in this state, is obtained by integrating the probability density over the other oscillator coordinate. The result is

\[ P_1(x_1, t) = \int_{-\infty}^{\infty} \Psi_0^*(x_\pm, t) \Psi_0(x_\pm, t) dx_2 \]

\[ = \left(\frac{m^2\omega\Omega}{\pi^2\hbar^2}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left(-\frac{m\omega}{2\hbar} (x_1 + x_2)^2 - \frac{m\Omega}{2\hbar} (x_2 - x_1)^2 \right) \exp\left(-\frac{2m\omega}{\hbar(1 + \gamma)} x_1^2 \right) dx_2 \]

(2.5)
This is a Gaussian probability distribution with standard deviation

\[ \sigma_z^{(1)} = \sqrt{\left(\frac{\hbar}{2m\omega}\right) \frac{1 + \gamma}{2}}. \]  

The unperturbed vacuum-state probability distribution for the position \( x \) of a single oscillator is

\[ P_0(x, t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{m\omega}{\hbar} x^2\right), \]  

with standard deviation

\[ \sigma_x^{(0)} = \sqrt{\frac{\hbar}{2m\omega}}. \]  

With \( \Omega > \omega \), i.e., \( \gamma < 1 \), the effect of the coupling is to reduce the position uncertainty of the oscillator.

The probability distribution for the momentum of one oscillator can be calculated similarly, or by utilizing the symmetry of a harmonic oscillator under the interchange \( \sqrt{m\omega}x \leftrightarrow p/\sqrt{m\omega} \). The result is

\[ P_1(p_1, t) = \left(\frac{2\gamma}{\pi\hbar m\omega(1 + \gamma)}\right)^{1/2} \exp\left(-\frac{2\gamma}{\hbar m\omega(1 + \gamma)} p_1^2\right), \]  

with standard deviation

\[ \sigma_p^{(1)} = \sqrt{\left(\frac{\hbar m\omega}{2}\right) \frac{1 + \gamma}{2\gamma}}. \]  

These can be compared with the unperturbed results

\[ P_0(p, t) = \left(\frac{1}{\pi\hbar m\omega}\right)^{1/2} \exp\left(-\frac{1}{\hbar m\omega} p^2\right), \]  

\[ \sigma_p^{(0)} = \sqrt{\frac{\hbar m\omega}{2}}. \]  

The coupling increases the momentum uncertainty of the individual oscillator.

Thus the net effects of the coupling, in the ground state of the coupled system, are:

- To decrease the position uncertainty of the individual oscillator;
- To increase the momentum uncertainty of the oscillator;
- To increase its uncertainty product \( \sigma_x \sigma_p \) above the minimum value \( \hbar/2 \);
- To break the symmetry between \( \sqrt{m\omega}x \) and \( p/\sqrt{m\omega} \) for the individual oscillator.

The last is not so unexpected, since the coupling perturbation is in position and not symmetrically in momentum.

The effects of the coupling cannot be characterized by the introduction of a finite temperature for the individual oscillator. That is, the second oscillator does not act as a “heat bath,” even though coupling to
a large number of oscillators should do so. The probability distributions and uncertainties for an oscillator in thermal equilibrium at temperature $T$ are [6]:

$$P_T(x) = \left(\frac{m\omega}{\pi \hbar} \tanh(\beta/2)\right)^{1/2} \exp\left(-\frac{m\omega}{\hbar} \tanh(\beta/2) x^2\right)$$  \hspace{1cm} (2.10a)

$$\sigma_x^{(T)} = \sqrt{\frac{\hbar}{2m\omega} \coth(\beta/2)}$$  \hspace{1cm} (2.10b)

$$P_T(p) = \left(\frac{\tanh(\beta/2)}{\pi \hbar m\omega}\right)^{1/2} \exp\left(-\frac{\tanh(\beta/2)}{\hbar m\omega} p^2\right)$$  \hspace{1cm} (2.10c)

$$\sigma_p^{(T)} = \sqrt{\frac{\hbar m\omega}{2} \coth(\beta/2)}$$  \hspace{1cm} (2.10d)

with $\beta \equiv \hbar \omega / (k_B T)$. The net effects of a finite temperature are to increase both $\sigma_x$ and $\sigma_p$, as well as the uncertainty product $\sigma_x \sigma_p$, while preserving the symmetry between $\sqrt{m\omega}x$ and $p/\sqrt{m\omega}$.

C. Coupled ground/coherent state.

With the system in its ground state, both oscillators display fully quantum dynamics. We model a quantum oscillator coupled to a classical oscillator by a different choice of state: We envision oscillator #1 in its unperturbed ground state, and oscillator #2 in a coherent or Glauber state with classical amplitude $X_0$, at time $t = 0$. We follow the subsequent evolution of this state, and determine the probability distributions, expectation values, and uncertainties for oscillator #1. In effect, oscillator #1 is in its ground state, oscillator #2 is behaving classically with oscillation amplitude $X_0$, and the coupling “turns on” at time $t = 0$. For simplicity, we assume the classical momentum of the second oscillator is zero at $t = 0$, i.e., it is at a maximum of its classical trajectory at that instant.

1. Initial wave function.

The initial wave function for the system, i.e., the wave function at time $t = 0$, for this coupled ground/coherent state is [3,4]:

$$\Psi(x_1, x_2, 0) = \left(\frac{m\omega}{\pi \hbar}\right)^{1/2} \exp\left(-\frac{m\omega}{2\hbar} x_1^2\right) \exp\left(-\frac{m\omega}{2\hbar} (x_2 - X_0)^2\right).$$  \hspace{1cm} (2.11a)

In terms of the normal-mode amplitudes, this is simply

$$\Psi(x_{\pm}, 0) = \left(\frac{m\omega}{\pi \hbar}\right)^{1/2} \exp\left[-\frac{m\omega}{2\hbar} \left(x_+ - \frac{X_0}{\sqrt{2}}\right)^2\right] \exp\left[-\frac{m\omega}{2\hbar} \left(x_- - \frac{X_0}{\sqrt{2}}\right)^2\right].$$  \hspace{1cm} (2.11b)

This corresponds to a coherent state of the $+$ mode, with initial amplitude $X_0/\sqrt{2}$ and momentum zero. The initial state of the $-$ mode, however, is a displaced squeezed state [4,5]: The initial displacement is $X_0/\sqrt{2}$, the initial momentum is zero, but the initial width of the Gaussian wave packet is proportional to $1/\sqrt{\omega}$ rather than the $1/\sqrt{\Omega}$ required for a coherent state of this mode. (In fact this is a displaced “antisqueezed” state, with $1/\sqrt{\omega} > 1/\sqrt{\Omega}$.)
2. Propagation of the wave function

The wave function for all times \( t \geq 0 \) is obtained by applying harmonic-oscillator propagators \([7]\) for the normal modes to the initial wave function \( \Psi(x,0) \). The result is the double integral

\[
\Psi(y, t) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \cos(\Omega t) - i\gamma \sin(\Omega t) \psi_c(y, t) \psi_{ds}(y, t)
\]

where \( y \), at arbitrary time \( t \), are distinguished from coordinates \( x \) at \( t = 0 \). The integral over \( x_+ \) gives a coherent state in \( y_+ \), with classical amplitude \( (X_0/\sqrt{2})\cos(\omega t) \) and momentum \( -m\omega(X_0/\sqrt{2})\sin(\omega t) \). The integral over \( x_- \) gives a displaced squeezed state in \( y_- \). Ultimately, the wave function takes the form

\[
\Psi(y, t) = \psi_c(y_+, t) \psi_{ds}(y_-, t),
\]

with \([3,4]\)

\[
\psi_c(y_+, t) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left( -\frac{im\omega X_0}{2\hbar} \right) \exp \left( \frac{im\omega X_0^* \sin(\omega t) \cos(\omega t)}{4\hbar} \right)
\]

\[
\times \exp \left( -\frac{im\omega X_0}{\hbar \sqrt{2}} \sin(\omega t) \right) \exp \left[ -\frac{m\omega}{2\hbar} \left( y_+ - \frac{X_0}{\sqrt{2}} \cos(\omega t) \right)^2 \right]
\]

and \([4,5]\)

\[
\psi_{ds}(y_-, t) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left( \frac{c}{\sqrt{2}} \sqrt{\frac{\cos(\Omega t) - i\gamma \sin(\Omega t)}{\cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)}} \right)
\]

\[
\times \exp \left( \frac{im\omega \gamma (X_0/\sqrt{2}) \sin(\Omega t) y_-}{\hbar [\cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)]} \right) \exp \left( \frac{c}{\sqrt{2}} \sqrt{\frac{\cos(\Omega t) - \gamma \sin(\Omega t)}{\cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)}} \right)
\]

\[
\times \exp \left[ -\frac{m\omega}{2\hbar \cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)} \left( y_- - \frac{X_0}{\sqrt{2}} \cos(\Omega t) \right)^2 \right],
\]

in terms of normal-mode coordinates \( y_\pm \) and time \( t \geq 0 \).

3. Reduced probability distribution for \( y_1 \).

As before, the probability distribution for the single oscillator coordinate \( y_1 \) is obtained by integrating the full probability density over \( y_2 \). We obtain

\[
P(y_1, t) = \int_{-\infty}^{\infty} |\psi_c(y_+, t) \psi_{ds}(y_-, t)|^2 dy_2
\]
\[
\begin{align*}
\frac{m\omega}{\pi\hbar} \left( \frac{1}{\cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)} \right)^{1/2} \\
\times \int_{-\infty}^{\infty} \exp \left[ -\frac{m\omega}{\hbar} \left( y_+ - \frac{X_0}{\sqrt{2}} \cos(\omega t) \right)^2 \right] \exp \left( -\frac{m\omega}{\hbar} \left( \frac{y_- - \frac{X_0}{\sqrt{2}} \cos(\Omega t)}{\cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)} \right)^2 \right) \, dy_2
\end{align*}
\]

\[
= \left( \frac{2m\omega}{\pi\hbar[1 + \cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)]} \right)^{1/2} \exp \left( -\frac{2m\omega}{\hbar} \left\{ y_1 - \frac{1}{\sqrt{2}} X_0 [\cos(\omega t) - \cos(\Omega t)] \right\}^2 \right)
\]

\[
= \left( \frac{2m\omega}{\pi\hbar[1 + \cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)]} \right)^{1/2} \exp \left( -\frac{2m\omega}{\hbar} \left[ y_1 - X_0 \sin \left( \frac{\Omega - \omega}{2} t \right) \sin \left( \frac{\Omega + \omega}{2} t \right) \right] \right)
\]

(2.14)

treating \( y_+ \) and \( y_- \) as functions of forms (2.2a) and (2.2b) of coordinates \( y_1 \) and \( y_2 \).

4. Quantum expectation values and uncertainties for oscillator #1.

The expectation value and uncertainty for the position of oscillator #1 can be read directly from the Gaussian distribution \( P(y_1, t) \). We find

\[
\langle y_1 \rangle = \frac{X_0}{2} [\cos(\omega t) - \cos(\Omega t)] = X_0 \sin \left( \frac{\Omega - \omega}{2} t \right) \sin \left( \frac{\Omega + \omega}{2} t \right)
\]

and

\[
\sigma^{(1)}_y = \sqrt{\hbar/4m\omega [1 + \cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)]}.
\]

The behavior of \( \langle y_1 \rangle \) can be quite complicated; it can be as large as \( X_0 \) or as negative as \( -X_0 \). In the case of weak coupling, i.e., \( \kappa \ll \bar{k} \) or \( \Omega - \omega \ll \omega \), it oscillates with a slowly oscillating amplitude, exhibiting “beats” between the normal modes. The position uncertainty \( \sigma^{(1)}_y \) oscillates between the unperturbed value \( \sqrt{\hbar/(2m\omega)} \) and the smaller value \( \sqrt{\hbar(1 + \gamma^2)/(4m\omega)} \).

Momentum expectation values and uncertainties cannot be determined from the reduced probability distribution \( P(y_1, t) \); the full wave function \( \Psi(y_\pm, t) \) is needed. The normal-mode relations (2.2a) and (2.2b), plus the chain rule, imply

\[
p_1 = \frac{p_+ - p_-}{\sqrt{2}},
\]

whence follow

\[
\langle p_1 \rangle = \frac{\langle p_+ \rangle - \langle p_- \rangle}{\sqrt{2}}
\]

and

\[
\sigma^{(1)}_p = \sqrt{\frac{\sigma^{(+)}_p^2 + \sigma^{(-)}_p^2}{2}},
\]

(2.16c)
the last since the normal modes are uncorrelated in this state. The necessary expectation values and uncertainties can be read from the momentum-space wave functions for the normal modes, viz.,

\[
\tilde{\psi}_c(p_+ t) = \int_{-\infty}^{\infty} e^{ip_+ y_+ / \hbar} \psi_c(y_+, t) dy_+ \\
= \left( \frac{1}{m \omega \pi \hbar} \right)^{1/4} \exp\left( -i \omega t \right) \exp\left( -im \omega \frac{X_0^2}{2} \cos(\omega t) \sin(\omega t) \right) \\
\times \exp\left( -i \frac{X_0}{\hbar \sqrt{2}} \cos(\omega t) p_+ \right) \exp\left[ -\frac{1}{2m \omega \hbar} \left( p_+ + m \omega \frac{X_0}{\sqrt{2}} \sin(\omega t) \right)^2 \right]
\]

and

\[
\tilde{\psi}_{ds}(p_- t) = \int_{-\infty}^{\infty} e^{ip_- y_- / \hbar} \psi_{ds}(y_-, t) dy_- \\
= \left( \frac{\gamma}{m \Omega \pi \hbar} \right)^{1/4} \left( \frac{\gamma \cos(\Omega t) - i \sin(\Omega t)}{\gamma^2 \cos^2(\Omega t) + \sin^2(\Omega t)} \right)^{1/2} \exp\left( -i \Omega \gamma X_0^2 \cos(\Omega t) \sin(\Omega t) \right) \\
\times \exp\left( -i \frac{\gamma^2 (X_0 / \sqrt{2}) \cos(\Omega t) p_-}{\hbar [\gamma^2 \cos^2(\Omega t) + \sin^2(\Omega t)]} \right) \exp\left( i (1 - \gamma^2) \cos(\Omega t) \sin(\Omega t) \frac{p_-^2}{2m \Omega \hbar [\gamma^2 \cos^2(\Omega t) + \sin^2(\Omega t)]} \right) \\
\times \exp\left[ \frac{-\gamma}{2m \Omega \hbar [\gamma^2 \cos^2(\Omega t) + \sin^2(\Omega t)]} \left( p_- + m \Omega \frac{X_0}{\sqrt{2}} \sin(\Omega t) \right)^2 \right].
\]

These imply

\[
\langle p_+ \rangle = -m \omega \frac{X_0}{\sqrt{2}} \sin(\omega t), \quad (2.18a)
\]

\[
\sigma_p^{(+)} = \sqrt{m \omega \hbar / 2}, \quad (2.18b)
\]

\[
\langle p_- \rangle = -m \Omega \frac{X_0}{\sqrt{2}} \sin(\Omega t), \quad (2.18c)
\]

and

\[
\sigma_p^{(-)} = \sqrt{\frac{m \Omega \hbar}{2 \gamma} [\gamma^2 \cos^2(\Omega t) + \sin^2(\Omega t)]}. \quad (2.18d)
\]

Hence, relations (2.16b) and (2.16c) yield the results

\[
\langle p_1 \rangle = -\frac{m X_0}{2} [\omega \sin(\omega t) - \Omega \sin(\Omega t)] \quad (2.19a)
\]

and

\[
\sigma_p^{(1)} = \sqrt{\frac{m \omega \hbar}{4} \left( 1 + \cos^2(\Omega t) + \frac{1}{\gamma^2} \sin^2(\Omega t) \right)}. \quad (2.19b)
\]

The momentum and position expectation values are related “classically,” via \( \langle p_1 \rangle = m d\langle y_1 \rangle / dt \). The momentum uncertainty oscillates between the unperturbed value \( \sqrt{m \omega \hbar / 2} \) and the larger value \( \sqrt{m \omega \hbar (1 + \gamma^{-2}) / 4} \).
Thus the behavior of oscillator \#1 is perhaps reminiscent of the suggestion that a quantum oscillator coupled to a classical one would lose its quantum nature. In particular, its position uncertainty is reduced below its unperturbed ground-state value. It cannot, however, be reduced arbitrarily close to the classical zero value. For any value of the coupling, i.e., of $\gamma$—and independent of the classical amplitude $X_0$—the uncertainty $\sigma_y^{(1)}$ is never less than $1/\sqrt{2}$ of the unperturbed quantum value. The effect of the coupling could more aptly be described as a quantum “squeezing” than a loss of quantum nature. (Of course, since this model remains a quantum-mechanical system no matter what its state, it could never exhibit an actual violation of quantum mechanics.) The uncertainty product for oscillator \#1 is

$$\sigma_y^{(1)}\sigma_p^{(1)} = \frac{\hbar}{4} \sqrt{\left[1 + \cos^2(\Omega t) + \gamma^2 \sin^2(\Omega t)\right](1 + \cos^2(\Omega t) + \frac{1}{\gamma^2} \sin^2(\Omega t))}$$

$$= \frac{\hbar}{2} \sqrt{1 + \left(\frac{1}{\gamma} - \gamma\right)^2 \frac{1 - \cos^4(\Omega t)}{4}}$$

$$= \frac{\hbar}{4} \sqrt{\left(\frac{1}{\gamma} + \gamma\right)^2 - \left(\frac{1}{\gamma} - \gamma\right)^2 \cos^4(\Omega t)}.$$  (2.20)

This oscillates between the values

$$\frac{\hbar}{2} \leq \sigma_y^{(1)}\sigma_p^{(1)} \leq \frac{\hbar}{4} \left(\frac{1}{\gamma} + \gamma\right),$$  (2.21)

always in accord with the Uncertainty Principle.

### III. GENERAL COUPLED OSCILLATORS

The preceding case of two identical oscillators is rather special. One might expect coupled quantum and classical oscillators to have very different masses and spring constants—including, for example, a quantum oscillator coupled to a mode of a radiation field. The above analysis is readily generalized to the case of oscillators with arbitrary masses $m_1$ and $m_2$ and individual spring constants $k_1$ and $k_2$, hence unperturbed angular frequencies $\omega_1 = \sqrt{k_1/m_1}$ and $\omega_2 = \sqrt{k_2/m_2}$.

#### A. Normal modes.

The Hamiltonian for this two-oscillator system, coupled as before by a spring with spring constant $\kappa$, is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}\kappa(x_2 - x_1)^2.$$  (3.1)

To separate this into normal modes, we first rescale the coordinates and momenta thus:

$$X_1 = \left(\frac{m_1}{m_2}\right)^{1/4} x_1 \quad X_2 = \left(\frac{m_2}{m_1}\right)^{1/4} x_2$$  (3.2a)

$$P_1 = \left(\frac{m_2}{m_1}\right)^{1/4} p_1 \quad P_2 = \left(\frac{m_1}{m_2}\right)^{1/4} p_2.$$  (3.2b)

The Hamiltonian becomes

$$H = \frac{P_1^2}{2\mu} + \frac{P_2^2}{2\mu} + \frac{1}{4}\mu\omega_1^2X_1^2 + \frac{1}{4}\mu\omega_2^2X_2^2$$

$$+ \frac{1}{2}\kappa \left[\left(\frac{m_1}{m_2}\right)^{1/4} X_2 - \left(\frac{m_2}{m_1}\right)^{1/4} X_1\right]^2.$$  (3.3a)

with geometric-mean mass
\[ \mu \equiv (m_1 m_2)^{1/2}. \]  

Now the rotation of coordinates and momenta

\[
\begin{pmatrix}
    x_+ \\
    x_-
\end{pmatrix} = \begin{pmatrix}
    \cos \alpha & \sin \alpha \\
    -\sin \alpha & \cos \alpha
\end{pmatrix} \begin{pmatrix}
    X_1 \\
    X_2
\end{pmatrix},
\]

hence,

\[
\begin{pmatrix}
    p_+ \\
    p_-
\end{pmatrix} = \begin{pmatrix}
    \cos \alpha & \sin \alpha \\
    -\sin \alpha & \cos \alpha
\end{pmatrix} \begin{pmatrix}
    P_1 \\
    P_2
\end{pmatrix},
\]

transfers the Hamiltonian into

\[
H = \frac{p_+^2}{2\mu} + \frac{1}{2}\mu \omega_+^2 x_+^2 + \frac{p_-^2}{2\mu} + \frac{1}{2}\mu \omega_-^2 x_-^2,
\]

given the conditions

\[
\alpha = \frac{1}{2} \arctan \left( \frac{2\kappa/\mu}{\omega_+^2 - \omega_-^2 + \frac{\kappa m_1 - m_2}{\mu}} \right),
\]

\[
\omega_+ = \left\{ \omega_1^2 \cos^2 \alpha + \omega_2^2 \sin^2 \alpha + \frac{\kappa}{\mu} \left[ \left( \frac{m_1}{m_2} \right)^{1/4} \sin \alpha - \left( \frac{m_2}{m_1} \right)^{1/4} \cos \alpha \right]^2 \right\}^{1/2},
\]

\[
\omega_- = \left\{ \omega_1^2 \sin^2 \alpha + \omega_2^2 \cos^2 \alpha + \frac{\kappa}{\mu} \left[ \left( \frac{m_1}{m_2} \right)^{1/4} \cos \alpha + \left( \frac{m_2}{m_1} \right)^{1/4} \sin \alpha \right]^2 \right\}^{1/2},
\]

fixing the rotation angle \( \alpha \) and the normal-mode frequencies \( \omega_\pm \). With this Hamiltonian, the normal modes \( x_\pm \) evolve as independent harmonic oscillators. The combined transformations (3.2a) and (3.4a), between \( (x_1, x_2) \) and \( (x_+, x_-) \), remain a unitary transformation in the quantum-mechanical sense, though not as a matrix for \( m_1 \neq m_2 \).

**B. Coupled ground/coherent state.**

As above, we model a quantum/classical coupling via a state in which oscillator \#1 is in its ground state, and oscillator \#2 is in a coherent state, with classical amplitude \( x_0 \) and momentum zero, at initial time \( t = 0 \). The initial wave function is

\[
\Psi(x_1, x_2, 0) = \left( \frac{m_1 m_2 \omega_1 \omega_2}{\pi^2 \hbar^2} \right)^{1/4} \exp \left( -\frac{m_1 \omega_1 x_1^2 + m_2 \omega_2 (x_2 - x_0)^2}{2\hbar} \right),
\]

in terms of the original oscillator coordinates, or

\[
\Psi(x_\pm, 0) = \left( \frac{\mu^2 \omega_1 \omega_2}{\pi^2 \hbar^2} \right)^{1/4} \exp \left( -\frac{\mu}{2\hbar} \left[ (\omega_1 \cos^2 \alpha + \omega_2 \sin^2 \alpha) x_+^2 + (\omega_1 \sin^2 \alpha + \omega_2 \cos^2 \alpha) x_-^2 \
+ (\omega_2 - \omega_1) \sin(2\alpha) x_+ x_- - 2\omega_2 X_0 (\sin \alpha x_+ + \cos \alpha x_-) + \omega_2 X_0^2 \right] \right),
\]
in terms of normal-mode coordinates, with $X_0 \equiv (m_2/m_1)^{1/4}x_0$ a rescaled classical amplitude.

The time-dependent wave function for this state is again obtained by applying separate harmonic-oscillator propagators $G_\pm$ for the two normal modes. The integrand of the propagation double integral takes the form

$$
G_+G_- \Psi(x_\pm, 0) = \left(\frac{\mu^2 \omega_1 \omega_2}{\pi^2 \hbar^2}\right)^{1/4} \left(\frac{\mu^2 \omega_+ \omega_-}{4\pi^2 i^2 \hbar^2 \sin(\omega_+ t) \sin(\omega_- t)}\right)^{1/2} \exp\left(-\frac{\mu \omega_2 X_0^2}{2\hbar}\right)
\times \exp\left(\frac{i \mu_+ \cot(\omega_+ t)}{2\hbar} y_+^2\right) \exp\left(\frac{i \mu_- \cot(\omega_- t)}{2\hbar} y_-^2\right)
\times \exp\left(-\frac{\mu}{2\hbar}\left[\omega_1 \cos^2 \alpha + \omega_2 \sin^2 \alpha - i \omega_+ \cot(\omega_+ t)\right] x_+^2
+ \left[\omega_1 \sin^2 \alpha + \omega_2 \cos^2 \alpha - i \omega_- \cot(\omega_- t)\right] x_-^2 + (\omega_2 - \omega_1) \sin(2\alpha) x_+ x_-
- 2 \left[\omega_2 X_0 \sin \alpha - i \omega_+ \csc(\omega_+ t) y_+\right] x_+ - 2 \left[\omega_2 X_0 \cos \alpha - i \omega_- \csc(\omega_- t) y_-\right] x_-\right) .
$$

Hence, the propagation integral over $x_\pm$ is a two-dimensional Gaussian integral of the form

$$
\int \exp\left(-\frac{\mu}{2\hbar}\left(A^T S A - D^T A - A^T D\right)\right) d^2 x = \int \exp\left(-\frac{\mu}{2\hbar}\left(B^T S B - D^T S^{-1} D\right)\right) d^2 \xi
= \left(\frac{4\pi^2 \hbar^2}{\mu^2 \det S}\right)^{1/2} \exp\left(\frac{\mu}{2\hbar} D^T S^{-1} D\right) ,
$$

with

$$
A = \begin{pmatrix} x_+ \\ x_- \end{pmatrix} ,
$$

$$
B = A - S^{-1} D = \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} ,
$$

$$
D = \begin{pmatrix} \omega_2 X_0 \sin \alpha - i \omega_+ \csc(\omega_+ t) y_+ \\ \omega_2 X_0 \cos \alpha - i \omega_- \csc(\omega_- t) y_- \end{pmatrix} ,
$$

and

$$
S = \begin{pmatrix} \omega_1 \cos^2 \alpha + \omega_2 \sin^2 \alpha - i \omega_+ \cot(\omega_+ t) & \frac{1}{2}(\omega_2 - \omega_1) \sin(2\alpha) \\ \frac{1}{2}(\omega_2 - \omega_1) \sin(2\alpha) & \omega_1 \sin^2 \alpha + \omega_2 \cos^2 \alpha - i \omega_- \cot(\omega_- t) \end{pmatrix} .
$$
The wave function is then
\[
\Psi(y_\pm, t) = \left( \frac{\mu^2 \omega_1 \omega_2}{\pi^2 \hbar^2} \right)^{1/4} \left( \frac{\omega_+ \omega_-}{i^2 \sin(\omega_+ t) \sin(\omega_- t) \det S} \right)^{1/2} \exp \left( -\frac{\mu \omega_2 X_0^2}{2 \hbar} \right)
\times \exp \left( \frac{\mu}{2 \hbar} (D^T S^{-1} D + i Y^T \Omega_2 Y) \right),
\] (3.10a)

with
\[
Y \equiv \begin{pmatrix} y_+ \\ y_- \end{pmatrix},
\] (3.10b)

and
\[
\Omega_2 \equiv \begin{pmatrix} \omega_+ \cot(\omega_+ t) & 0 \\ 0 & \omega_- \cot(\omega_- t) \end{pmatrix},
\] (3.10c)

for all \( t \geq 0 \). Rewriting column matrix (3.9d) as
\[
D = -i \Omega_1 (Y + i Z),
\] (3.11a)

with
\[
\Omega_1 \equiv \begin{pmatrix} \omega_+ \csc(\omega_+ t) & 0 \\ 0 & \omega_- \csc(\omega_- t) \end{pmatrix}
\] (3.11b)

and
\[
Z \equiv \begin{pmatrix} \omega_2 X_0 \sin \alpha \sin(\omega_+ t) \\ \omega_+ X_0 \cos \alpha \sin(\omega_- t) \end{pmatrix},
\] (3.11c)

puts the wave function into the form
\[
\Psi(Y, t) = \left( \frac{\mu^2 \omega_1 \omega_2}{\pi^2 \hbar^2} \right)^{1/4} \left( \frac{\omega_+ \omega_-}{i^2 \sin(\omega_+ t) \sin(\omega_- t) \det S} \right)^{1/2} \exp \left( -\frac{\mu \omega_2 X_0^2}{2 \hbar} \right)
\times \exp \left( -\frac{\mu}{2 \hbar} \left[ (Y^T + i Z^T) \Omega_1 S^{-1} \Omega_1 (Y + i Z) - i Y^T \Omega_2 Y \right] \right),
\] (3.12)
in terms of the matrices defined above.

To determine probability distributions, expectation values, and uncertainties, it is useful to separate the argument of the exponential in \( \Psi \) into real and imaginary terms. The first matrix of coefficients can be written
\[
\Omega_1 S^{-1} \Omega_1 = U + i V,
\] (3.13)
where $U$ and $V$ are real, symmetric $2 \times 2$ matrices, with $U$ nonsingular and positive definite. The wave function is then

$$
\Psi(Y,t) = \left( \frac{\mu^2 \omega_1 \omega_2}{\pi^2 \hbar^2} \right)^{1/4} \left( \frac{\omega_+ \omega_-}{i^2 \sin(\omega_+ t) \sin(\omega_- t) \det S} \right)^{1/2} \exp \left( -\frac{j \omega_2 X_0^2}{2\hbar} \right) \\
\times \exp \left( -\frac{\mu}{2\hbar} \left[ Y^T U Y - Z^T V Y - Y^T V Z - Z^T U Z \right. \right.
$$

\[ \left. + i Y^T (V - \Omega_2) Y + i (Z^T U Y + Y^T U Z) - i Z^T V Z \right] \right) \\
= \left( \frac{\mu^2 \omega_1 \omega_2}{\pi^2 \hbar^2} \right)^{1/4} \left( \frac{\omega_+ \omega_-}{i^2 \sin(\omega_+ t) \sin(\omega_- t) \det S} \right)^{1/2} \exp \left( -\frac{j \omega_2 X_0^2}{2\hbar} \right) \\
\times \exp \left( -\frac{\mu}{2\hbar} \left[ (Y^T - Z^T V U^{-1} U (Y - U^{-1} V Z) - Z^T V U^{-1} V Z - Z^T U Z \right. \right.
$$

\[ \left. + i Y^T (V - \Omega_2) Y + i (Z^T U Y + Y^T U Z) - i Z^T V Z \right] \right) . \] (3.14)

This can be simplified: Since the matrix $U$ contains no dependence on the classical amplitude $X_0$, normalization of the wave function implies

$$
\omega_2 X_0^2 = Z^T (V U^{-1} V + U) Z , \] (3.15a)

and also

$$
\det U = \frac{\omega_1 \omega_2 \omega_2^2 \omega_-^2 \sin^2(\omega_+ t) \sin^2(\omega_- t) \det S^2} . \] (3.15b)

These can also be verified using the explicit expressions shown in Sec. D below. The first result reduces the wave function to the form

$$
\Psi(Y,t) = \left( \frac{\mu^2 \omega_1 \omega_2}{\pi^2 \hbar^2} \right)^{1/4} \left( \frac{\omega_+ \omega_-}{i^2 \sin(\omega_+ t) \sin(\omega_- t) \det S} \right)^{1/2} \exp \left( \frac{j \omega_2 X_0^2}{2\hbar} \right) \\
\times \exp \left( \frac{\mu}{2\hbar} \left[ Z^T V Z - (Z^T U Y + Y^T U Z) - Y^T (V - \Omega_2) Y \right] \right) \\
\times \exp \left( -\frac{\mu}{2\hbar} (Y^T - Z^T V U^{-1} U (Y - U^{-1} V Z) \right) . \] (3.16)

This wave function is finite or regular for all $y_\pm$ and $t \geq 0$; the components of $U$, $U^{-1}$, $V - \Omega_2$, and $V Z$ contain no vanishing denominators or divergent circular functions. This too can be verified explicitly using the expressions in Sec. D below.

The probability distribution for the normal-mode coordinates is

$$
\mathcal{P}(Y,t) = |\Psi(Y,t)|^2 \\
= \left( \frac{\mu^2 \det U}{\pi^2 \hbar^2} \right)^{1/2} \exp \left( -\frac{\mu}{\hbar} (Y - U^{-1} V Z)^T U (Y - U^{-1} V Z) \right) , \] (3.17)
a Gaussian distribution in two dimensions. To obtain the reduced probability distribution for the single oscillator coordinate $y_1$, the inverse of the transformation to normal-mode coordinates,

$$Y = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} (m_1/m_2)^{1/4} & 0 \\ 0 & (m_2/m_1)^{1/4} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

\[\equiv NY,\] (3.18)

is needed. Here $\mathcal{Y}$ denotes the column containing the oscillator coordinates, and $N$ the product of the two square matrices. The matrix $N$ has unit determinant, but is not orthogonal if masses $m_1$ and $m_2$ are unequal.

Distribution (3.17) can be expressed thus:

$$P(\mathcal{Y},t) = \left(\frac{\mu^2 \det U}{\pi^2 \hbar^2}\right)^{1/2} \exp \left(-\frac{\mu}{\hbar}(\mathcal{Y} - N^{-1}U^{-1}VZ)^T N^TU N (\mathcal{Y} - N^{-1}U^{-1}VZ)\right)$$

(3.19)

as a distribution for $y_1$ and $y_2$. Reduction to $P(y_1,t)$ requires an integral of the general form

$$\int_{-\infty}^{\infty} \exp \left[-(x_1 \ x_2) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] dx_2 = \int_{-\infty}^{\infty} \exp(-ax_1^2 - 2cx_1x_2 - bx_2^2) dx_2$$

$$= \left(\frac{\pi}{b}\right)^{1/2} \exp \left(-\frac{ab - c^2}{b} x_1^2\right).$$ (3.20)

Hence, integrating $P(\mathcal{Y},t)$ over $y_2$ yields

$$P(y_1,t) = \left(\frac{\mu \det U}{\pi \hbar (N^TU N)_{22}}\right)^{1/2} \exp \left(-\frac{\mu \det U}{\hbar (N^TU N)_{22}} \left[y_1 - (N^{-1}U^{-1}VZ)_1\right]^2\right),$$ (3.21)

a normalized, Gaussian distribution for $y_1$.

C. Expectation values and uncertainties.

The expectation value and uncertainty for the position of oscillator #1 in this state can be read off of $P(y_1,t)$. They are

$$\langle y_1 \rangle = (N^{-1}U^{-1}VZ)_1$$ (3.22a)

and

$$\sigma_y^{(1)} = \sqrt{\frac{\hbar}{2\mu \det U} (N^TU N)_{22}},$$ (3.22b)

in terms of the above-defined matrices.

To calculate the momentum expectation value and uncertainty, the full wave function (3.16) must be used. First, we obtain

$$\langle p_1 \rangle = \int \Psi^* \frac{\hbar}{i} \frac{\partial}{\partial y_1} \Psi \ dy_1 \ dy_2$$
\[
\begin{align*}
= \frac{i\mu}{2} & \int \left\{ (1 \ 0) N^T U N (\mathcal{Y} - N^{-1} U^{-1} V Z) + (\mathcal{Y} - N^{-1} U^{-1} V Z)^T N^T U N \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \\
& + i \left[ Z^T U N \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + (1 \ 0) N^T U Z \right] \\
& + i \left[ (1 \ 0) N^T (V - \Omega_2) \mathcal{Y} + \mathcal{Y}^T N^T (V - \Omega_2) N \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right] \right\} P(\mathcal{Y}, t) \, dy_1 \, dy_2 \\
= -\mu \left\{ N^T \left[ U + (V - \Omega_2) U^{-1} V \right] Z \right\}_1 ,
\end{align*}
\]

this last following from the matrix expectation value \( \langle \mathcal{Y} \rangle = N^{-1} U^{-1} V Z \). Second, we find

\[
\langle p_1^2 \rangle = -\hbar^2 \int \Psi^* \frac{\partial^2}{\partial y_1^2} \Psi \, dy_1 \, dy_2
\]

\[
= -\hbar^2 \left[ -\frac{\mu}{\hbar} \left\langle (1 \ 0) N^T U N \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right\rangle + i(1 \ 0) N^T (V - \Omega_2) N \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right] \\
+ \frac{\mu^2}{\hbar^2} \left\langle \left\{ [N^T U N (\mathcal{Y} - N^{-1} U^{-1} V Z)]_1 + i \left[ N^T (V - \Omega_2) N \mathcal{Y} \right]_1 + i (N^T U Z)_1 \right\} \right\rangle
\]

\[
= \mu \hbar \left\{ N^T \left[ U + i (V - \Omega_2) \right] N \right\}_1 \\
- \mu^2 \left\langle (\mathcal{Y} - N^{-1} U^{-1} V Z)^T N^T \left[ U + i (V - \Omega_2) \right] N \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right] \left[ U + i (V - \Omega_2) \right] N (\mathcal{Y} - N^{-1} U^{-1} V Z) \right\rangle \\
+ \mu^2 \left\{ N^T \left[ U + (V - \Omega_2) U^{-1} V \right] Z \right\}_1^2 ,
\]

using the facts the square of the first component of any column matrix \( C \) can always be written

\[
C_1^2 = C^T \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) C ,
\]

and that terms linear in \( \mathcal{Y} - N^{-1} U^{-1} V Z \) appearing in the original squared expression have zero expectation value. The last term in Eq. (3.24) is precisely \( \langle p_1 \rangle^2 \) —without specifying the matrices—leaving

\[
\sigma^{(1)2}_p = \langle p_1^2 \rangle - \langle p_1 \rangle^2
\]

\[
= \mu \hbar \left\{ N^T \left[ U + i (V - \Omega_2) \right] N \right\}_1 \\
- \mu^2 \left\langle (\mathcal{Y} - \langle \mathcal{Y} \rangle)^T N^T \left[ U + i (V - \Omega_2) \right] N \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right] \left[ U + i (V - \Omega_2) \right] N (\mathcal{Y} - \langle \mathcal{Y} \rangle) \right\rangle .
\]

The last expectation value is the integral of a quadratic form times the normalized Gaussian \( P(\mathcal{Y}, t) \). The general form of such an integral is

\[
\left( \frac{\det B}{\pi^n} \right)^{1/2} \int X^T A X e^{-X^T B X} \, d^n x = \frac{1}{2} \text{Tr}(AB^{-1}) ,
\]

(3.27)
for positive-definite $B$, as can be proved by diagonalizing $B$ or by explicit integration of component expressions. This implies
\[
\sigma_p^{(1)2} = \frac{\mu \hbar}{2} \left\{ \text{Tr} \left[ N^T [U + i(V - \Omega_2)] N \right] \right\}_{11}
\]
\[
- \frac{\mu^2}{2} \text{Tr} \left[ N^T [U + i(V - \Omega_2)] N \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} N^T [U + i(V - \Omega_2)] N \begin{pmatrix} \mu N^T U N \end{pmatrix}^{-1} \right] 
\]
\[
= \frac{\mu \hbar}{2} \left\{ N^T [U + (V - \Omega_2) U^{-1}(V - \Omega_2)] N \right\}_{11} .
\]
As expected, this is real and of order $\hbar$, even without using the specific forms of $U$, $N$, $V$, or $\Omega_2$. It implies the momentum uncertainty
\[
\sigma_p^{(1)} = \sqrt{\frac{\mu \hbar}{2} \left\{ N^T [U + (V - \Omega_2) U^{-1}(V - \Omega_2)] N \right\}_{11}} .
\]

The probability distribution for momentum $p_1$ is a Gaussian, with expectation value $\langle p_1 \rangle$ and standard deviation $\sigma_p^{(1)}$.

**D. Explicit expressions.**

Explicit expressions for the wave function, probability distributions, expectation values, and uncertainties in the coupled ground/coherent state—in terms of the individual oscillator masses, frequencies, and coupling—involves an unwieldy number of terms, even with the aid of a computer. It facilitates matters to define the following real and imaginary parts, in terms of the matrices in Eqs. (3.9e), (3.10c), and (3.11b):
\[
M = \Omega_1^{-1} S \Omega_1^{-1} = \mathcal{R} - i \mathcal{I} ,
\]
with real matrices
\[
\mathcal{R} = \begin{pmatrix}
\omega_c \rho^2 & \delta \sin \alpha \cos \alpha \rho \eta \\
\delta \sin \alpha \cos \alpha \rho \eta & \omega_s \eta^2
\end{pmatrix}
\]
and
\[
\mathcal{I} = \Omega_1^{-1} \Omega_2 \Omega_1^{-1} = \begin{pmatrix}
\rho \cos(\omega_+ t) & 0 \\
0 & \eta \cos(\omega_- t)
\end{pmatrix},
\]
and frequency combinations
\[
\omega_c \equiv \omega_1 \cos^2 \alpha + \omega_2 \sin^2 \alpha ,
\]
\[
\omega_s \equiv \omega_1 \sin^2 \alpha + \omega_2 \cos^2 \alpha ,
\]
\[
\delta \equiv \omega_2 - \omega_1 ,
\]
\[
\rho \equiv \frac{\sin(\omega_+ t)}{\omega_+} ,
\]
and
\[
\eta \equiv \frac{\sin(\omega_- t)}{\omega_-} .
\]
The inverse matrix $M^{-1}$ is the matrix in Eq. (3.13), implying
\[
U + i V = \frac{1}{M} (\mathcal{R} - i \mathcal{I}) ,
\]
with $\mathcal{M} \equiv \det M$ and tildes denoting adjoint matrices (transposed matrices of cofactors, not Hermitian conjugates). The matrices appearing in the wave function, *et cetera*, can then be written:

\begin{align}
U &= \frac{1}{|\mathcal{M}|^2}[(\Re \mathcal{M}) \tilde{R} - (\Im \mathcal{M}) \tilde{I}] , \\
U^{-1} &= \frac{1}{|\mathcal{M}|^2 \det U}[(\Re \mathcal{M}) R - (\Im \mathcal{M}) I] , \\
V &= \frac{1}{|\mathcal{M}|^2}[(\Im \mathcal{M}) \tilde{R} + (\Re \mathcal{M}) \tilde{I}] .
\end{align}

The real and imaginary parts of $\mathcal{M}$ are, explicitly,

\begin{align}
\Re \mathcal{M} &= \rho \eta [\omega_1 \omega_2 \rho \eta - \cos(\omega_+ t) \cos(\omega_- t)] , \\
\Im \mathcal{M} &= -\rho \eta [\omega_1 \eta \cos(\omega_+ t) + \omega_2 \rho \cos(\omega_- t)] .
\end{align}

Identity (3.15b) is equivalent to

\begin{equation}
|M|^2 \det U = \omega_1 \omega_2 \rho^2 \eta^2 = \det \mathcal{R} ,
\end{equation}

which can be confirmed by direct evaluation. Finally,

\begin{equation}
(\mathcal{R} - iI)(\bar{\mathcal{R}} - i\bar{I}) = \mathcal{M} \mathbf{1} ,
\end{equation}

where $\mathbf{1}$ denotes the $2 \times 2$ identity matrix, implies the identities

\begin{equation}
\Re \mathcal{M} = \det \mathcal{R} - \det I ,
\end{equation}

as may also be obtained directly, and

\begin{equation}
\mathcal{I} \bar{\mathcal{R}} + \mathcal{R} \bar{I} = -\Im \mathcal{M} \mathbf{1} .
\end{equation}

These simplify the calculations considerably.

Evaluation of the position expectation value $\langle y_1 \rangle$ is straightforward. Results (3.32b) and (3.32c) and identities (3.35b) and (3.35c) imply

\begin{equation}
U^{-1} V = \frac{1}{|\mathcal{M}|^2 \det U} \mathcal{I} \bar{R} .
\end{equation}

Explicit evaluation yields

\begin{equation}
\bar{R} Z = \omega_1 \omega_2 X_0 \rho \eta \begin{pmatrix} \eta \sin \alpha \\ \rho \cos \alpha \end{pmatrix} .
\end{equation}

With identity (3.34) and form (3.30c), this gives

\begin{equation}
\langle Y \rangle = U^{-1} V Z = \begin{pmatrix} X_0 \sin \alpha \cos(\omega_+ t) \\ X_0 \cos \alpha \cos(\omega_- t) \end{pmatrix} ,
\end{equation}

whence follows

\begin{equation}
\langle Y \rangle = N^{-1} U^{-1} V Z
\end{equation}

\begin{equation}
= \begin{pmatrix} (m_2/m_1)^{1/4} X_0 \sin \alpha \cos \alpha [\cos(\omega_+ t) - \cos(\omega_- t)] \\ (m_1/m_2)^{1/4} X_0 \sin^2 \alpha \cos(\omega_+ t) + \cos^2 \alpha \cos(\omega_- t) \end{pmatrix} .
\end{equation}
This gives the desired result:

\[
\langle y_1 \rangle = \frac{m_2}{m_1}^{1/2}x_0 \sin \alpha \cos \alpha \left[ \cos(\omega_+ t) - \cos(\omega_- t) \right],
\]  

(3.39)

in terms of the original classical amplitude \(x_0\) and angle \(\alpha\) and normal-mode frequencies \(\omega_{\pm}\) from Eqs. (3.6a–c).

Evaluation of the position uncertainty \(\sigma_{y}^{(1)}\) requires the matrix

\[
N^TUN = \frac{1}{|M|^2} [ (\Re M) N^T \tilde{R}N - (\Im M) N^T \tilde{I}N ] .
\]  

(3.40)

Direct evaluation of the matrix elements

\[
(N^T \tilde{R}N)_{22} = \left( \frac{m_2}{m_1} \right)^{1/2} \left( \omega_+ \eta^2 \sin^2 \alpha - 2 \delta \rho \eta \sin^2 \alpha \cos^2 \alpha + \omega_c \rho^2 \cos^2 \alpha \right)
\]  

(3.41a)

and

\[
(N^T \tilde{I}N)_{22} = \left( \frac{m_2}{m_1} \right)^{1/2} [ \eta \sin^2 \alpha \cos(\omega_- t) + \rho \cos^2 \alpha \cos(\omega_+ t) ]
\]  

(3.41b)

enables us to obtain the final result

\[
\sigma_{y}^{(1)} = \left\{ \frac{\hbar}{2 \omega_1 \omega_2} \cos^2 \alpha \left[ \frac{\omega_+}{\omega_2} \cos^2(\omega_+ t) + \frac{\omega_2}{\omega_+} \sin^2(\omega_+ t) \right] \right.
\]

\[
+ \sin^2 \alpha \left[ \frac{\omega_+}{\omega_2} \cos^2(\omega_- t) + \frac{\omega_2}{\omega_+} \sin^2(\omega_- t) \right]
\]

\[
+ 2 \frac{\delta}{\omega_2} \cos^2 \alpha \sin^2 \alpha \left( \cos(\omega_+ t) \cos(\omega_- t) - \frac{\omega_1 \omega_2}{\omega_+ \omega_-} \sin(\omega_+ t) \sin(\omega_- t) \right) \}^{1/2} .
\]  

(3.42)

This rather complicated result reduces to form (2.15b) in the equal-mass, equal-frequency case.

The momentum expectation value \(\langle p_1 \rangle\) is obtained from the matrix

\[
U + (V - \Omega_2)U^{-1}V = \frac{1}{|M|^2} [(\Re M) \tilde{R} - (\Im M) \tilde{I}] - \left( \frac{1}{|M|^2} [(\Im M) \tilde{R} + (\Re M) \tilde{I}] + \Omega_2 \right) \frac{1}{|M|^2 \det \tilde{U}} \tilde{R}
\]

\[
= \frac{1}{\omega_1 \omega_2 \rho^2 \eta^2} \left( \begin{array}{cc}
\sin^2(\omega_+ t) & 0 \\
0 & \sin^2(\omega_- t)
\end{array} \right) \tilde{R}. \]

(3.43)

Result (3.37) then yields the column matrix

\[
[U + (V - \Omega_2)U^{-1}V] Z = X_0 \left( \begin{array}{c}
\omega_+ \sin(\omega_+ t) \sin \alpha \\
\omega_- \sin(\omega_- t) \cos \alpha
\end{array} \right),
\]  

(3.44)
hence,
\[
N^T [U + (V - \Omega_2)U^{-1}V] Z = X_0 \left( \begin{array}{c}
(m_1/m_2)^{1/4} \sin \alpha \cos \alpha [\omega_+ \sin(\omega_+ t) - \omega_- \sin(\omega_- t)] \\
(m_2/m_1)^{1/4} [\sin^2 \alpha \omega_+ \sin(\omega_+ t) + \cos^2 \alpha \omega_- \sin(\omega_- t)]
\end{array} \right).
\]
(3.45)

Result (3.23) takes the form
\[
\langle p_1 \rangle = -m_1 x_0 \left( \frac{m_2}{m_1} \right)^{1/2} \sin \alpha \cos \alpha [\omega_+ \sin(\omega_+ t) - \omega_- \sin(\omega_- t)].
\]
(3.46)

As in the symmetric case, the expectation values \( \langle y_1 \rangle \) and \( \langle p_1 \rangle \) are related by \( \langle p_1 \rangle = m_1 \frac{d \langle y_1 \rangle}{dt} \).

The momentum uncertainty \( \sigma_p^{(1)} \) is given by Eq. (3.29). After some manipulation, the requisite matrix takes the form
\[
U + (V - \Omega_2)U^{-1}(V - \Omega_2) = \frac{1}{\det R} \left[ (1 - \Omega_2 \mathcal{I}) \mathcal{R} - \mathcal{R} \Omega_2 \\
+ (\mathcal{R} \mathcal{M}) \Omega_2 \mathcal{R} \Omega_2 - (\mathcal{M} \mathcal{R}) \Omega_2 \mathcal{R} \Omega_2 \right].
\]
(3.47)

Hence, the uncertainty is obtained from the 11 component of
\[
N^T [U + (V - \Omega_2)U^{-1}(V - \Omega_2)] N = \\
\frac{1}{\det R} \left[ N^T \left( \begin{array}{cc}
\sin^2(\omega+ t) & 0 \\
0 & \sin^2(\omega- t)
\end{array} \right) \mathcal{R} N - N^T \mathcal{R} \left( \begin{array}{cc}
\cos^2(\omega+ t) & 0 \\
0 & \cos^2(\omega- t)
\end{array} \right) N \\
+ (\mathcal{R} \mathcal{M}) N^T \Omega_2 \mathcal{R} \Omega_2 N - (\mathcal{M} \mathcal{R}) N^T \Omega_2 \mathcal{R} \Omega_2 N \right].
\]
(3.48)

The components of the individual matrix products on the right-hand side can be obtained by explicit calculation. The results are quite long and unwieldy, even after some trigonometric simplifications. When they are combined, some cancellations and further trigonometric combinations leave the final result

\[
\sigma_p^{(1)} = \left\{ \frac{\hbar m_1 \omega_1}{2} \left[ \cos^2 \alpha \left( \frac{\omega_+}{\omega_1} \cos^2(\omega+ t) + \frac{\omega_-}{\omega_1^2 \omega_2} \sin^2(\omega+ t) \right) \\
+ \sin^2 \alpha \left( \frac{\omega_+}{\omega_1} \cos^2(\omega- t) + \frac{\omega_-}{\omega_1^2 \omega_2} \sin^2(\omega- t) \right) \\
- 2 \frac{\delta}{\omega_1} \cos^2 \alpha \sin^2 \alpha \left( \cos(\omega+ t) \cos(\omega- t) - \frac{\omega_+ \omega_-}{\omega_1^2 \omega_2} \sin(\omega+ t) \sin(\omega- t) \right) \right\}^{1/2}.
\]
(3.49)

This rather complex result agrees with Eq. (2.19b) in the symmetric case.

Similar matrix manipulations can be used to confirm identity (3.15a) explicitly. They can also be used to prove the finiteness of the matrices \( U, VZ, V - \Omega_2 \), and \( U^{-1} \) appearing in the wave function \( \Psi \), thus: The combinations \( UZ + iVZ \) and \( U + i(V - \Omega_2) \) can be written as finite complex matrices divided by the complex number \( \Xi(t) \equiv \mathcal{M}(t)/[\rho(t) \eta(t)] \). Hence, finiteness of all components of \( UZ, VZ, U \), and \( V - \Omega_2 \)
is assured provided $\Xi$ neither vanishes nor approaches arbitrarily close to zero at any time. This number is obtained from Eqs. (3.33a) and (3.33b):
\begin{align}
\Re \Xi(t) &= -\cos(\omega_+ t) \cos(\omega_- t) + \frac{\omega_1 \omega_2}{\omega_+ \omega_-} \sin(\omega_+ t) \sin(\omega_- t) \\
\Im \Xi(t) &= \frac{\omega_+ \cos(\omega_+ t) \sin(\omega_- t) - \omega_- \cos(\omega_- t) \sin(\omega_+ t)}{\omega_+}.
\end{align}

Consequently, $\Xi$ satisfies the identity
\begin{equation}
\omega_1 \omega_2 (\Re \Xi)^2 + \omega_1 \omega_2 (\Im \Xi)^2 = \omega_1 \omega_2 \cos^2(\omega_+ t) \cos^2(\omega_- t) + \frac{\omega_1 \omega_2 \omega_+^2}{\omega_+^2 \omega_-^2} \sin^2(\omega_+ t) \sin^2(\omega_- t) + \frac{\omega_1 \omega_2 \omega_+^2}{\omega_+^2 \omega_-^2} \cos^2(\omega_+ t) \sin^2(\omega_- t) + \frac{\omega_1 \omega_2 \omega_+^2}{\omega_+^2 \omega_-^2} \cos^2(\omega_- t) \sin^2(\omega_+ t).
\end{equation}

With
\begin{align}
\zeta &\equiv \max\{\omega_1, \omega_2, \omega_1 \omega_2\} \\
\lambda &\equiv \min \left\{ \frac{\omega_1 \omega_2 \omega_+^2}{\omega_+^2 \omega_-^2}, \frac{\omega_1 \omega_2 \omega_+^2}{\omega_+^2 \omega_-^2}, \frac{\omega_1 \omega_2 \omega_+^2}{\omega_+^2 \omega_-^2} \right\},
\end{align}

identity (3.51) implies $\zeta |\Xi|^2 \geq \lambda$, i.e.,
\begin{equation}
|\Xi|^2 \geq \frac{\lambda}{\zeta} > 0.
\end{equation}

As required, $\Xi$ is bounded away from zero by a time-independent bound. Likewise, matrix $U^{-1}$ can be written in terms of finite matrices times $\Xi$. But with
\begin{align}
\zeta' &\equiv \min\{\omega_1, \omega_2, \omega_1 \omega_2\} = \omega_1 \omega_2 \\
\lambda' &\equiv \max \left\{ \frac{\omega_1 \omega_2 \omega_+^2}{\omega_+^2 \omega_-^2}, \frac{\omega_1 \omega_2 \omega_+^2}{\omega_+^2 \omega_-^2}, \frac{\omega_1 \omega_2 \omega_+^2}{\omega_+^2 \omega_-^2} \right\},
\end{align}

identity (3.51) also implies $\zeta' |\Xi|^2 \leq \lambda'$, i.e.,
\begin{equation}
|\Xi|^2 \leq \frac{\lambda'}{\zeta'} < \infty.
\end{equation}

With $\Xi$ bounded away from infinity, the finiteness of $U^{-1}$ is also assured.

The expectation values $\langle y_1 \rangle$ and $\langle p_1 \rangle$ for quantum oscillator #1 show the same beating between the normal modes in the general case as in the symmetric case. But the uncertainties $\sigma_y^{(1)}$ and $\sigma_p^{(1)}$, which incorporate the effects of the coupling on the quantum character of the oscillator, are too complicated in the general case to describe easily. It is more illuminating to focus on another special case—one of particular physical significance. The effects of one oscillator on another will certainly be most pronounced when the two are at or near resonance. It will therefore be most useful to focus on the case of two distinct oscillators, with unequal masses $m_1 \neq m_2$ and correspondingly different spring constants, but with equal (unperturbed) frequencies $\omega_1 = \omega_2$. 

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IV. UNEQUAL-MASS OSCILLATORS AT RESONANCE.

The behavior of coupled oscillators of distinct masses but equal unperturbed frequencies is easily obtained from the general results of Sec. III above. In this case the oscillator frequencies and frequency combinations reduce to \( \omega_1 = \omega_2 = \omega_c = \omega_s = \omega \) and \( \delta = 0 \).

A. Normal modes.

The normal-mode rotation angle \( \alpha \) for this case follows from Eq. (3.6a), i.e.,

\[
\tan(2\alpha) = \frac{2\mu}{m_1 - m_2},
\]

which implies

\[
\cos(2\alpha) = \frac{m_1 - m_2}{m_1 + m_2},
\]
\[
\sin(2\alpha) = \frac{2\mu}{m_1 + m_2},
\]

hence

\[
\cos \alpha = \sqrt{\frac{m_1}{m_1 + m_2}}
\]
\[
\sin \alpha = \sqrt{\frac{m_2}{m_1 + m_2}}.
\]

These reduce to \( \alpha = \pi/4 \) in the symmetric case \( m_1 = m_2 \), as in Sec. II above [Eqs. (2.2a) and (2.2b)]. Even with unequal masses, at resonance, angle \( \alpha \) is independent of the oscillator coupling \( \kappa \).

With these results, Eqs. (3.6b) and (3.6c) give the normal-mode frequencies

\[
\omega_+ = \omega
\]
\[
\omega_- \equiv \Omega = \sqrt{\omega^2 + \frac{\kappa(m_1 + m_2)}{\mu^2}}.
\]

The ratio \( \gamma \equiv \omega/\Omega \) can be written

\[
\gamma = \left(1 + \frac{\kappa(m_1 + m_2)}{\mu^2 \omega^2}\right)^{-1/2}.
\]

Here \( \Omega \) and \( \gamma \) do depend on \( \kappa \). Since the masses must obey \( m_1 + m_2 \geq 2\mu \), with equality only for \( m_1 = m_2 \), for given \( \mu \) and \( \omega \) values frequency \( \Omega \) is greater, and ratio \( \gamma \) is smaller, with unequal masses than in the symmetric case.

B. Ground/coherent state: coefficient matrices.

The behavior of the quantum state with oscillator #1 initially in its unperturbed ground state, and oscillator #2 initially in a coherent state with classical amplitude \( x_0 \), follows immediately from the results of Secs. III.B–D above. With the frequencies appropriate to this case, Eq. (3.9e) gives a coefficient matrix \( S \) identical in form to that for the symmetric case, i.e., corresponding to wave function (2.13a–c). Matrices \( \Omega_1 \)
and $\Omega_2$ are likewise the same as in the symmetric case; hence, so are $U$, $U^{-1}$, and $V$. The only matrices different in this case are
\[
N = \frac{1}{\sqrt{\mu(m_1 + m_2)}} \begin{pmatrix} m_1 & m_2 \\ -\mu & \mu \end{pmatrix},
\]
its transpose $N^T$, its inverse
\[
N^{-1} = \frac{1}{\sqrt{\mu(m_1 + m_2)}} \begin{pmatrix} \mu & -m_2 \\ -\mu & m_1 \end{pmatrix},
\]
and the column matrix
\[
Z = x_0 \sqrt{\frac{\mu}{m_1 + m_2}} \begin{pmatrix} \sqrt{m_2/m_1} \sin(\omega t) \\ \gamma \sin(\Omega t) \end{pmatrix}.
\]
This follows from Eqs. (3.11c) and (4.3), expressed in terms of the original classical amplitude $x_0$.

The matrix products required are readily evaluated. They are
\[
N^{-1}U^{-1}VZ = \frac{m_2x_0}{m_1 + m_2} \begin{pmatrix} \cos(\omega t) - \cos(\Omega t) \\ \cos(\omega t) + (m_1/m_2)\cos(\Omega t) \end{pmatrix},
\]
\[
N^TUN = \frac{\omega}{\mu(m_1 + m_2)[\cos^2(\Omega t) + \gamma^2\sin^2(\Omega t)]} \times \begin{pmatrix} \mu^2 + m_1^2[\cos^2(\Omega t) + \gamma^2\sin^2(\Omega t)] & -\mu^2(1 - \gamma^2)\sin^2(\Omega t) \\ -\mu^2(1 - \gamma^2)\sin^2(\Omega t) & \mu^2 + m_2^2[\cos^2(\Omega t) + \gamma^2\sin^2(\Omega t)] \end{pmatrix},
\]
\[
N^T[U + (V - \Omega_2)U^{-1}V]Z = \frac{\mu x_0}{m_1 + m_2} \begin{pmatrix} \omega \sin(\omega t) - \Omega \sin(\Omega t) \\ (m_2/m_1)\omega \sin(\omega t) + \Omega \sin(\Omega t) \end{pmatrix},
\]
and
\[
N^T[U + (V - \Omega_2)U^{-1}(V - \Omega_2)]N = \\
\frac{\omega}{\mu(m_1 + m_2)} \begin{pmatrix} m_1^2 + \mu^2\left[\cos^2(\Omega t) + \frac{1}{\gamma^2}\sin^2(\Omega t)\right] & -\left(\frac{1}{\gamma^2} - 1\right)\mu^2\sin^2(\Omega t) \\ -\left(\frac{1}{\gamma^2} - 1\right)\mu^2\sin^2(\Omega t) & m_2^2 + \mu^2\left[\cos^2(\Omega t) + \frac{1}{\gamma^2}\sin^2(\Omega t)\right] \end{pmatrix}.
\]
C. Expectation values and uncertainties.

The expectation values and uncertainties for position and momentum of oscillator \#1 in this state are obtained from the above matrices via Eqs. (3.22a), (3.22b), (3.23), and (3.29). They yield

\[
\langle y_1 \rangle = x_0 \frac{m_2}{m_1 + m_2} [\cos(\omega t) - \cos(\Omega t)] ,
\]

\[
\sigma_y^{(1)} = \sqrt{\frac{\hbar}{2m_1\omega}} \frac{m_1}{m_1 + m_2} \left(1 + \frac{m_2}{m_1} \frac{m_2}{m_1} \cos^2(\Omega t) + \frac{1}{\gamma^2} \sin^2(\Omega t) \right),
\]

\[
\langle p_1 \rangle = -m_1 x_0 \frac{m_2}{m_1 + m_2} [\omega \sin(\omega t) - \Omega \sin(\Omega t)] ,
\]

\[
\sigma_p^{(1)} = \sqrt{\frac{\hbar m_1\omega}{2}} \frac{m_1}{m_1 + m_2} \left[1 + \frac{m_2}{m_1} \cos^2(\Omega t) + \frac{1}{\gamma^2} \sin^2(\Omega t) \right].
\]

The Gaussian probability distributions for position and momentum of oscillator \#1 follow immediately from these results.

The quantum behavior of oscillator \#1 in this case resembles that in the symmetric case, but exhibits a broader range of possibilities. The position and momentum expectation values satisfy

\[
\langle p_1 \rangle = m_1 \frac{d\langle y_1 \rangle}{dt}.
\]

The position uncertainty oscillates between the unperturbed value and a smaller value:

\[
\sqrt{\frac{\hbar}{2m_1\omega}} \frac{m_1}{m_1 + m_2} \leq \sigma_y^{(1)} \leq \sqrt{\frac{\hbar}{2m_1\omega}}.
\]

The momentum uncertainty compensates:

\[
\sqrt{\frac{\hbar m_1\omega}{2}} \frac{m_1}{m_1 + m_2} \leq \sigma_p^{(1)} \leq \sqrt{\frac{\hbar m_1\omega}{2}} \frac{m_1 + \gamma^2 m_2}{m_1 + m_2}.
\]

The uncertainty product oscillates between the Heisenberg minimum value and a larger value—

\[
\sigma_y^{(1)} \sigma_p^{(1)} = \frac{\hbar}{2} \sqrt{1 + \frac{m_1 m_2}{(m_1 + m_2)^2} \left(\frac{1}{\gamma} - \gamma\right)^2 \sin^2(\Omega t) \left(1 + \frac{m_2}{m_1} \cos^2(\Omega t)\right)}.
\]

implying

\[
\frac{\hbar}{2} \leq \sigma_y^{(1)} \sigma_p^{(1)} \leq \frac{\hbar}{2} \sqrt{1 + \frac{m_1 m_2}{(m_1 + m_2)^2} \left(\frac{1}{\gamma} - \gamma\right)^2}
\]

for \(m_1 \geq m_2\), or

\[
\frac{\hbar}{2} \leq \sigma_y^{(1)} \sigma_p^{(1)} \leq \frac{\hbar}{4} \left(\frac{1}{\gamma} + \gamma\right)
\]

for \(m_1 \leq m_2\)—always satisfying the Heisenberg Uncertainty Principle. However, unlike the symmetric case, for unequal-mass oscillators at resonance the “apparent loss of quantum nature,” i.e., the quantum squeezing of the uncertainties, can be extreme. For a system with \(m_1 \ll m_2\) and strong coupling, i.e., \(\gamma \ll 1\), the minimum value of \(\sigma_y^{(1)}\) can be arbitrarily close to zero, the corresponding maximum value of \(\sigma_p^{(1)}\) arbitrarily large. Of course the squeezing is oscillatory, with \(\sigma_y^{(1)}\) and \(\sigma_p^{(1)}\) returning to their unperturbed quantum values \(\sqrt{\hbar/(2m_1\omega)}\) and \(\sqrt{\hbar m_1\omega/2}\) in each cycle.
V. CONCLUSIONS

By virtue of its exact solubility, both classically and quantum-mechanically, a pair of linearly coupled harmonic oscillators proves to be a useful toy model for probing the interaction of a quantum system with its environment. Even in its ground state, this system displays effects of the coupling on the quantum nature of an oscillator: reduction of its position uncertainty below the unperturbed quantum value; a compensating increase in its momentum uncertainty, yielding an increase in the uncertainty product; breaking of the symmetry between position and momentum variables, a consequence of the position-dependent coupling. These cannot be characterized as finite-temperature effects.

The interaction of a quantum oscillator with a classical one can be simulated via the coupled-oscillator pair in a quantum state with one oscillator initially in its unperturbed ground state, the other initially in a coherent or Glauber state incorporating classical behavior. The subsequent evolution of the wave function is calculated exactly, using ordinary harmonic-oscillator propagators for the normal modes of the system. The reduced probability distribution for the position of the initially quantum oscillator—a Gaussian distribution with time-dependent expectation value and uncertainty—and its position and momentum expectation values and uncertainties all follow from this wave function. The expectation values can be characterized as a “beat” amplitude between the normal modes. The behavior of the uncertainties, i.e., the quantum character of the oscillator, can be quite complicated. For oscillators with equal unperturbed frequencies, e.g., at resonance, this behavior can be described as a time-dependent quantum squeezing: The position uncertainty oscillates through values below the unperturbed value; depending on the relative masses of the original oscillators and the strength of the coupling, its lower bound can be arbitrarily close to zero. The momentum uncertainly oscillates through values larger than its unperturbed value. The uncertainty product oscillates through values larger than the Heisenberg minimum value. Naturally, the system never actually violates the Uncertainty Principle.

Nonetheless, the behavior of the system is certainly reminiscent of the longstanding suggestion that a quantum system coupled to a classical one would “radiate away” its quantum nature. In detail, though, it more closely resembles the quantum squeezing encountered, e.g., in quantum fields coupled to classical backgrounds [8]. Although not explored in the present calculations, more involved effects connecting the quantum and classical worlds, such as decoherence, can be studied exactly and analytically in the two-oscillator system.

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[1] C. Møller, in Les Theories Relativistes de la Gravitation, edited by A. Lichnerowicz and M. A. Tonnelat (CNRS, Paris, 1962); L. Rosenfeld, Nucl. Phys. 40, 353 (1963); B. S. DeWitt, Phys. Rep. 19C, 295 (1975); K. Eppley and E. Hannah, Found. Phys. 7, 51 (1977).

[2] D. N. Page and C. D. Gelker, Phys. Rev. Lett. 47, 979 (1981); B. Hawkins, Phys. Rev. Lett. 48, 520 (1982); L. E. Ballentine, Phys. Rev. Lett. 48, 522 (1982).

[3] E. Schrödinger, Naturwissenschaften 14, 664 (1926); R. J. Glauber, Phys. Rev. Lett. 10, 84 (1963); Phys. Rev. 131, 2766 (1963); J. R. Klauder and B. S. Skagerstam, Coherent States, Applications in Physics and Mathematical Physics (World Scientific, Singapore, 1985), pp. 3–24; see also e.g., L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Non-Relativistic Theory, Third Edition (Pergamon, Oxford, 1977), pp. 71–72.
[4] B. L. Schumaker, Phys. Rep. 135, 317 (1986); W.-M. Zhang, D. H. Feng, and R. Gilmore, Rev. Mod. Phys. 62, 867 (1990).

[5] D. Stoler, Phys. Rev. D1, 3217 (1970); 4, 1925 (1971); E. Y. C. Lu, Lett. Nuovo Cimento 2, 1241 (1971); 3, 585 (1971); J. N. Hollenhorst, Phys. Rev. D19, 1669 (1979); J. Grochmalicki and M. Lewenstein, Phys. Rep. 208, 189 (1991).

[6] A. Messiah, Quantum Mechanics, Volume I (Wiley, New York, 1958), pp. 448–451; R. P. Feynman, Statistical Mechanics (Benjamin, Reading, Massachusetts, 1972), pp. 49–53.

[7] R. P. Feynman, Rev. Mod. Phys. 20, 367 (1948); E. W. Montroll, Commun. Pure Appl. Math. 5, 415 (1952); R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965), pp. 62–63; L. S. Schulman, Techniques and Applications of Path Integration (Wiley, New York, 1981), pp. 37–38.

[8] L. P. Grishchuk and Yu. V. Sidorov, Class. Quantum Grav. 6, L161 (1989); Phys. Rev. D42, 3413 (1990); L. P. Grishchuk, in Proceedings of the Sixth Marcel Grossmann Meeting, edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1992); Class. Quantum Grav. 10, 2449 (1993); Phys. Rev. D50, 7154 (1994); 53, 6784 (1996); Lect. Notes Phys. 562, 167 (2001); L. Grishchuk, H. A. Haus, and K. Bergman, Phys. Rev. D46, 1440 (1992); M. Gasperini and M. Giovannini, Class. Quantum Grav. 10, L133 (1993); Phys. Lett. 301B, 334 (1993); S. Bose and L. P. Grishchuk, Phys. Rev. D66, 043529 (2002).