On the Asymptotic Behaviour of some Positive Semigroups

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Abstract

Similar to the theory of finite Markov chains it is shown that in a Banach space $X$ ordered by a closed cone $K$ with nonempty interior $\text{int}(K)$ a power bounded positive operator $A$ with compact power such that its trajectories for positive vectors eventually flow into $\text{int}(K)$, defines a "limit distribution", i.e. its adjoint operator has a unique fixed point in the dual cone. Moreover, the sequence $\{A^n\}_{n \in \mathbb{N}}$ converges with respect to the strong operator topology and for each functional $f \in X'$ the sequence $\{(A^*)^n(f)\}_{n \in \mathbb{N}}$ converges with respect to the weak*-topology (Theorem 5). If a positive bounded $C_0$-semigroup of linear continuous operators $\{S_t\}_{t \geq 0}$ on a Banach space contains a compact operator and the trajectories of the non-zero vectors $x \in K$ have the property from above then, in particular, $\{S_t\}_{t \geq 0}$ and $\{S^*_t\}_{t \geq 0}$ converge to their limit operator with respect to the operator norm, respectively (Theorem 4). For weakly compact Markov operators in the space of real continuous functions on a compact topological space a corresponding result can be derived, that characterizes the long-term behaviour of regular Markov chains.

1 Introduction

The main purpose of our paper is to show that the method which is used to prove the existence of a limit distribution in the theory of stationary Markov chains (see for example [6], chapt.VII,§7, [8], chapt.IV) can be transferred to a much more general situation. The operator corresponding to the Markov chain is replaced by a positive semigroup of operators acting in a Banach space ordered by a cone with nonempty interior, and the condition of regularity of the Markov chain (in the sense of [8]) is transformed into the condition of strongly positivity of the operator or into an even more general condition (see condition 1) in the Theorems 2 — 5.

In particular, the results generalize Theorem 1 of [5] and show that the limit distribution exists provided the operator of the random walk on a compact space is weakly compact and satisfies the condition of regularity.

The main result (Theorem 4) refers to the case, where a positive $C_0$-semigroup $\{S_t(x)\}_{t \geq 0}$ of linear continuous operators acts in an ordered Banach space $X$. The semigroup of operators is supposed to be uniformly bounded and to contain some compact operator.
Moreover, for each nonzero vector \( x \in K \) its trajectory \( \{ S_t(x) \}_{t \geq 0} \) eventually flows into \( \text{int}(K) \). Then the following alternative takes place: either the operators \( S_t \) for \( t \to +\infty \) converge to 0 with respect to the strong operator topology or the semigroup possesses a common fixed point \( u \) in \( \text{int}(K) \), the adjoint operators \( \{ S_t^* \}_{t \geq 0} \) have a unique fixed point \( f_0 \) ("limit distribution") in the dual cone and finally, \( \{ S_t \}_{t \geq 0} \) and \( \{ S_t^* \}_{t \geq 0} \) both for \( t \to +\infty \) converge to the operator \( A_0 = f_0 \times u \) with respect to the norm topology. Theorem \( \text{[2]} \) shows that the requirement of compactness can be considerably relaxed. However, in this case the convergence of the operators \( \{ S_t \}_{t \geq 0} \) and \( \{ S_t^* \}_{t \geq 0} \) to \( A_0 \) takes place only with respect to the strong operator topology.

The Theorems \( \text{[3]} - \text{[4]} \) deal with sequences \( \{ A^n \}_{n \in \mathbb{N}} \) of iterates of a positive operator \( A \) and are obtained as particular cases of the general results.

## 2 Preliminaries

We remember that a nonempty subset \( K \) in vector space \( X \) is a wedge, if \( x, y \in K, \lambda, \mu \geq 0 \) implies \( \lambda x + \mu y \in K \). If in addition \( x, -x \in K \) implies \( x = 0 \), then \( K \) is a cone. In what follows we consider real normed spaces \( (X, K, \| \cdot \|) \) in which an order is introduced by means of a closed cone \( K \).

### Definition 1

Any complete normed space which is ordered by a closed cone we will call an ordered Banach space and denote it by \( (X, K, \| \cdot \|) \).

Briefly we will write \( X \) instead of \( (X, K, \| \cdot \|) \) and denote its dual by \( X' \). The closed ball in \( X \) with radius \( r > 0 \) and centered at the point \( x \) is denoted by \( B(x; r) \). We use the notations \( x \in K \) and \( x \geq 0 \) synonymously.

A cone \( K \) is said to be generating (or reproducing), if each vector \( x \in X \) has a representation as \( x = x_1 - x_2 \), where \( x_1, x_2 \in K \).

A cone \( K \) is said to be normal, if there exists a positive number \( \delta \), such that \( \| x + y \| \geq \delta \max\{\|x\|, \|y\|\} \) for any \( x, y \in K \).

A cone \( K \) is said to be nonflat, if there exists a positive constant \( \gamma > 0 \), such that each element \( x \in X \) is representable as \( x = x_1 - x_2 \), where \( x_i \in K \) and \( \|x_i\| \leq \gamma \|x\| \) (\( i = 1, 2 \)).

A linear functional defined on \( X \) is said to be positive, if it takes on nonnegative values on all vectors of the cone \( K \). The set of all positive functionals of \( X' \) is called the dual wedge and will be denoted by \( K' \), i.e. \( K' = \{ f \in X : f(x) \geq 0 \text{ for all } x \in K \} \).

The following result goes back to M.G. Krein and V.L. Šmulian (s. [10]).

### Theorem 1

If the cone \( K \) is closed and normal then the wedge \( K' \) is a closed generating cone, i.e. each functional of \( X' \) has a representation as a difference of two positive functionals. Moreover, \( K' \) is nonflat.

### Proof.

For the first part of the theorem see [10]. We restrict ourselves to the proof of the nonflatness of \( K' \). Let be \( B^*_+ = \{ f \in K' : \|f\| \leq 1 \} \) and \( E = B^*_+ - B^*_+ \). According to the Banach - Alaoglu Theorem (s. [7], chapt.III, §3) the set \( B^*_+ \) is weak*-compact, and
therefore $E$ is closed. From the first part it follows that $X' = \bigcup_{n \in \mathbb{N}} nE$, and so 0 is an interior point of $E$, i.e. for some $r > 0$ the ball $B^*(0; r)$ (in $X'$) belongs to $E$. This means $r\frac{\|f\|}{r} \in E$ for each $f \in X'$, $f \neq 0$, and implies that each functional $r\frac{\|f\|}{r}$ can be represented as $f = f_1 - f_2$, where $f_1, f_2 \in B^*_+$. Now $\gamma^* = \frac{1}{r}$ can be taken as the constant of nonflatness of the cone $K'$.

For a convenient refering we list some more properties (s. [9], [10]) of the space $X$, its dual $X'$, of the cone $K$ and its dual cone $K'$ which are frequently used further on.

In the sequel we assume that the cone $K$ is closed and normal and satisfies int$(K) \neq \emptyset$.

a) The cone $K$ is nonflat.

Indeed. Fix $u \in \text{int}(K)$. Then $u$ belongs to $K$ together with some closed ball centered at $u$, i.e. $\overline{B}(u; r) \subset K$ for some $r > 0$. Then for any $x \in X$

$$\frac{\|x\|}{r} u \pm x \in K. \quad (1)$$

Put now $x_1 = \frac{1}{2} \left( \frac{\|x\|}{r} u + x \right)$ and $x_2 = \frac{1}{2} \left( \frac{\|x\|}{r} u - x \right)$. One has $x_1, x_2 \in K$, $x = x_1 - x_2$ and $\|x_i\| \leq \frac{1}{2} \left( \frac{\|x\|}{r} + 1 \right) \|x\|$. As the constant $\gamma$ (of nonflatness of $K$) can be taken the number $\frac{1}{r} \|u\|$. 

b) Each linear positive functional $f$ on $X$ is continuous and satisfies the condition $f(x) > 0$, if $f \in K'$, $f \neq 0$, $x \in \text{int}(K)$.

The relation (1) implies $\mp f(x) \leq \frac{\|x\|}{r} f(u)$, which shows that $f$ is bounded on the unit ball of $X$ and $\|f\| \leq \frac{1}{r} f(u)$. If $f \neq 0$ then $f(u) > 0$.

c) Each additive and positive homogeneous functional $f$ on $K$ with values in the non-negative reals extends uniquely to a linear positive functional on the whole $X$.

Indeed, if $x \in X$ is an arbitrary vector then $x = x_1 - x_2$, where $x_1 \in K$. Put

$$f(x) = f(x_1) - f(x_2).$$

It is easy to see that the functional $f$ is the required extension. We omit the standard proof (based on the nonflatness of $K$) of both the correctness of the definition and the uniqueness of the extension.

d) For any $x \in K, x \neq 0$ there exists a functional $f \in K'$ such that $f(x) > 0$.

Indeed, according to the theorem on a sufficient number of functionals there is a functional $f \in X'$ such that $f(x) \neq 0$. Since $f = f_1 - f_2$ with $f_1, f_2 \in K'$, at least one of the nonnegative numbers $f_1(x), f_2(x)$ is strongly positive.

Remember that a set $D \subset K$ is called a base of the cone $K$, if $D$ is convex and each vector $x \in K$, $x \neq 0$ has a unique representation as $x = \lambda y$, where $\lambda > 0$ and $y \in D$.

In the sequel we are interested in bases of the dual cone $K'$. The existence of interior points in the cone $K$ guarantees that the cone $K'$ possesses a base.
e) Let now $F$ be an arbitrary base of the cone $K'$. Then the closedness of $K$ implies the following important property: If $x, y \in X$ then

$$x \leq y \text{ is equivalent to } f(x) \leq f(y) \text{ for all } f \in F.$$ 

and consequently $x = y$ is equivalent to $f(x) = f(y)$ for all $f \in F$. Moreover, together with b) one has $x \in \text{int}(K)$ if and only if $f(x) > 0$ for each $f \in F$.

f) For an arbitrary fixed element $u \in \text{int}(K)$ denote

$$F = F_u = \{f \in K': f(u) = 1\}.$$ 

Then the set $F$ is bounded, weak*-compact and is a base of the dual cone $K'$.

The relation (1) implies the estimate

$$|f(x)| \leq \frac{1}{r} \|x\| \text{ for any } f \in F, x \in X. \quad (2)$$

Because of its weak*-closedness the set $F$ is weak*-compact by the Banach - Alaoglu Theorem. The set $F$ is convex, and property b) implies that $f(u) > 0$ for $f \in K, f \neq 0$. Therefore, $F$ is a base of the dual cone.

By means of the interior point $u$ of the cone $K$ one can define the following nonnegative functional on $X$

$$\|x\|_u = \inf \{\lambda \geq 0: -\lambda u \leq x \leq \lambda u\}$$

which is called the $u$-norm. Notice that the $u$-norm of an element $x$ can be calculated also by the formula

$$\|x\|_u = \sup \{|f(x)|: f \in F\}.$$ 

It is clear that the $u$-norm is actual a norm and that it is monotone on $K$, i.e. $x \leq y$ implies $\|x\|_u \leq \|y\|_u$.

The $u$-norm is equivalent to the original norm on $X$. Indeed, (2) implies

$$\|x\|_u = \sup \{|f(x)|: f \in F\} \leq \frac{1}{r} \|x\|.$$ 

On the other hand, let $x \in X, f \in X', \|f\| = 1$ and $f(x) = \|x\|$. Then $f = f_1 - f_2$, where $f_1 \in K'$ and $\|f_1\| \leq \gamma^* \|f\|$ and $\gamma^*$ denotes the constant of nonflatness of the cone $K'$. Then

$$\|x\| = f(x) \leq |f_1(x)| + |f_2(x)| \leq f_1(u)\|x\|_u + f_2(u)\|x\|_u \leq C_0\|x\|_u,$$

where $C_0 = 2\gamma^*\|u\|$. 
Summing up we have that for each vector $u \in \text{int}(K)$ there is a constant $C_u > 0$ such that for each $x \in X$
\[ C_u^{-1} \|x\| \leq \|x\|_u \leq C_u \|x\|. \tag{3} \]

We consider now positive operators on $(X, K, \| \cdot \|)$. By $L(X)$ we denote the vector space of all linear continuous operators on $X$, equipped with the usual norm and the order. For $A \in L(X)$ we write $A \geq 0$ if $A(K) \subset K$. Such operators we will call positive. The simple properties of such operators are gathered in the

**Lemma 1** Let $(X, K, \| \cdot \|)$ be an ordered normed real vector space, and let $A$ be a positive linear continuous operator on $X$. Assume there exists a vector $u \in \text{int}(K)$ such that $A(u) = u$. Let $\mathcal{F}_u = \{ f \in K': f(u) = 1 \}$ be the base of $K'$ corresponding to the vector $u$. Then the following statements hold.

(i) The adjoint operator $A^*$ is positive.

(ii) $A^*(\mathcal{F}_u) \subset \mathcal{F}_u$.

(iii) $\|A(x)\|_u \leq \|x\|_u$ for each $x \in X$.

(iv) $\{A^n\}_{n \in \mathbb{N}}$ is a norm bounded sequence in $L(X)$.

(v) $A(x) \in \text{int}(K)$ for each $x \in \text{int}(K)$.

**Proof.**
(i) For an arbitrary vector $f \in K$ we have $A^*(f)(x) = f(A(x)) \geq 0$ for any $x \in K$. Thus $A^*(f)$ belongs to $K'$.

(ii) If $f \in \mathcal{F}_u$ then by (i) $A^*(f) \in K'$. Since $A(u) = u$ and $A^*(f)(u) = f(A(u)) = f(u) = 1$ we obtain $A^*(f) \in \mathcal{F}_u$.

(iii) From (ii) follows that for each $x \in X$ one has
\[
\|A(x)\|_u = \sup_{f \in \mathcal{F}_u} |f(A(x))| = \sup_{f \in \mathcal{F}_u} |(A^*(f))(x)| \leq \sup_{f \in \mathcal{F}_u} |f(x)| = \|x\|_u.
\]

(iv) It is convenient to use the $u$-norm in $X$ which, as was shown in property g), is equivalent to the norm $\| \cdot \|$. Use now the inequality (3) and (iii) then
\[
\|A^n(x)\|_u \leq C_u \|A^n(x)\|_u \leq C_u \|A^{n-1}(x)\|_u \leq \ldots \leq C_u \|x\|_u \leq C_u^2 \|x\| \quad \text{for all} \quad x \in X.
\]
Therefore, $\|A^n\| \leq C_u^2$ for all $n \in \mathbb{N}$.

(v) If $x \in \text{int}(K)$ then in view of (ii) one has $f(A(x)) = (A^*(f))(x) > 0$ for each $f \in \mathcal{F}_u$. By property e) it follows $A(x) \in \text{int}(K)$.

We need also the following auxiliary result concerning positive operators

**Lemma 2** Let $A$ be a positive operator which satisfies the following conditions
1) \( A(\text{int}(K)) \subset \text{int}(K) \);

2) for each vector \( x \in K \), \( x \neq 0 \) there exists a natural \( n_x \) such that \( A^{n_x}(x) \in \text{int}(K) \).

Then for any compact set \( R \subset K \) such that \( 0 \notin R \) there is a natural number \( p \) with \( A^p(R) \subset \text{int}(K) \).

**Proof.** The condition 1) implies

\[
(A^n)^{-1}(\text{int}(K)) \subset (A^{n+j})^{-1}(\text{int}(K))
\]  

for all \( n, j \in \mathbb{N} \).

In view of condition 2) for each vector \( z \in R \) there exists a power \( n_z \) such that \( A^{n_z}(z) \in \text{int}(K) \). Therefore the sets \( (A^{n_z})^{-1}(\text{int}(K)) \) form an open covering of \( R \). Consider any finite subcovering

\[
(A^{n_{z_1}})^{-1}(\text{int}(K)), (A^{n_{z_2}})^{-1}(\text{int}(K)), \ldots, (A^{n_{z_s}})^{-1}(\text{int}(K))
\]

and let be \( p = \max\{n_{z_1}, n_{z_2}, \ldots, n_{z_s}\} \). Then taking into consideration inclusion (4) the family consisting of \( s \) exemplars of \( (A^p)^{-1}(\text{int}(K)) \) also covers the set \( R \). Actually we have \( R \subset (A^p)^{-1}(\text{int}(K)) \). This shows that \( A^p(z) \in \text{int}(K) \) for each \( z \in R \).

**Remark 1** The condition 1) of Lemma 2 is fulfilled, if the operator \( A \) is positive and possesses a fixed point \( u \) such that \( u \in \text{int}(K) \) (s. Lemma (ii)(v)).

### 3 Main results

We need the following notations (3).

**Definition 2** Let \( X \) be a (real) Banach space. A family \( \{S_t\}_{t \geq 0} \) of operators in \( L(X) \) is called a one-parameter semigroup of bounded linear operators if \( S_0 = I \), \( S_{s+t} = S_s S_t \) (\( s, t \geq 0 \)), where \( I \) denotes the identity operator on \( X \).

If, in addition, the function \( t \mapsto S_t \) is continuous with respect to the strong operator topology, i.e. the function \( t \mapsto S_t(x) \) is norm-continuous on \([0, +\infty)\) for each \( x \in X \), then \( \{S_t\}_{t \geq 0} \) is called a strongly continuous semigroup, or also a \( C_0 \)-semigroup.

A \( C_0 \)-semigroup \( \{S_t\}_{t \geq 0} \) in an ordered Banach space is called positive, if each operator \( S_t \) is positive (\( t \geq 0 \)).

The results we are going to prove are valid in real ordered Banach spaces \((X, K, \| \cdot \|)\), briefly denoted by \( X \), where \( K \subset X \) is a closed normal cone which satisfies the condition \( \text{int}(K) \neq \emptyset \).

**Theorem 2** Let \((X, K, \| \cdot \|)\) be an ordered Banach space and \( \{S_t\}_{t \geq 0} \) a positive \( C_0 \)-semigroup of operators in \( L(X) \) which satisfies the following conditions
1) for each vector \( x \in X, x \neq 0 \) there exists a number \( t_x \in [0, +\infty) \) such that \( S_{t_x}(x) \in \text{int}(K) \);

2) for each vector \( x \in K \) its trajectory \( \{ S_t(x) \}_{t \geq 0} \) is relatively compact;

Then the family \( \{ S_t \}_{t \geq 0} \) converges pointwise for \( t \to +\infty \) to some operator \( A_0 \).

If that operator \( A_0 \) is not the zero one, then there exists a vector \( u \in \text{int}(K) \) and a functional \( f_0 \in K' \), \( f_0 \neq 0 \) such that

(i) \( S_t(u) = u, S_t^*(f_0) = f_0 \) for any \( t \geq 0 \) and moreover \( f_0(x) > 0 \) if \( x \in K, x \neq 0 \);

(ii) \( A_0 = f_0 \otimes u \)

(iii) for each \( f \in X' \) one has \( S_t^*(f) \xrightarrow{t \to \infty} A_0^*(f) \) with respect to the weak*-topology \( \sigma(X', X) \);

(iv) \( \lambda = 1 \) is a simple eigenvalue of the operators \( S_t \) and \( S_t^* \) for all \( t > 0 \).

Proof. I. First of all we show that all operators \( S_t \) for \( t \geq 0 \) have a common fixed point in \( \text{int}(K) \), provided the family \( \{ S_t \}_{t \geq 0} \) does not pointwise converge to the zero operator \( 0 \) for \( t \to +\infty \). According to the principle of uniform boundedness the condition 2) implies

\[ \limsup_{t \to +\infty} \| S_t(x_0) \| > 0. \tag{5} \]

holds for some vector \( x_0 \in X \). Since \( K \) is generating we may assume \( x_0 \in K \). Now we show that \( \inf_{t \geq 0} \| S_t(x_0) \| > 0 \). Indeed, if the contrary is assumed we find an increasing sequence \( \{ s_k \}_{k=1}^{\infty} \subset [0, +\infty) \), \( s_k \to +\infty \) such that \( \| S_{s_k}(x_0) \| \to 0 \). Then an arbitrary sequence \( \{ t_n \}_{n=1}^{\infty} \) with \( t_n \to +\infty \) satisfies \( s_{k_n} \leq t_n \leq s_{k_n+1} \), where \( s_{k_n} \to +\infty \) and therefore,

\[ \| S_{t_n}(x_0) \| = \| S_{t_n-s_{k_n}} S_{s_{k_n}}(x_0) \| \leq C \| S_{s_{k_n}}(x_0) \| \xrightarrow{n \to \infty} 0. \]

This contradicts to the inequality \( (5) \).

Consequently, the points of the trajectory \( \{ S_t(x_0) \}_{t \geq 0} \) belong to \( K \) and their norms are separated from zero. Denote by \( Q_0 \) the closure of that trajectory. Then \( Q_0 \) is compact and \( S_t(Q_0) \subset Q_0 \) for any \( t \geq 0 \). Denote the closure of the convex hull of the set \( Q_0 \) by \( Q_1 \). We show now that the zero-vector also does not belong to \( Q_1 \). From \( Q_0 \subset K \) and \( 0 \notin Q_0 \) we find (property d))

\[ Q_0 = \bigcup_{f \in K'} \{ x \in X : f(x) > 0 \}. \]

By selecting a finite covering we have

\[ Q_0 \subset \bigcup_{k=1}^{N} \{ x \in X : f_k(x) > 0 \}, \]
where $f_k \in K'$. Put $g = \sum_{k=1}^{N} f_k$. Then, obviously, $Q_0 \subset \{ x \in X : g(x) > 0 \}$ and therefore, there is some $\sigma > 0$ such that the set $Q_0$ is contained in the closed convex set $K \cap \{ x \in X : g(x) \geq \sigma \}$, which in turn does not contain zero. Now it is clear that 0 does not belong to the closed convex hull $Q_1$ either. Since $S_t(Q_1) \subset Q_1$ for any $t \geq 0$ and since the operators of the family $\{ S_t \}_{t \geq 0}$ commute in pairs we are able to apply the Markov-Kakutani Theorem to that family of operators on $Q_1$ (s. [4], chapt.III.3.2) and to conclude that they possess a common fixed point, say $u \in Q_1$. It is obvious that $u \neq 0$ and in view of condition 1) $u \in \text{int}(K)$. This completes the first step of the proof and allows us to apply the Lemmata 1 and 2 to each of the operators $S_t$. In particular, from Lemma 1(v) we get $S_t(\text{int}(K)) \subset \text{int}(K)$ for any $t > 0$, and from condition 1) there follows that

$$S_s(x) = S_{t_x+x}(x) = S_t(S_{t_x}(x)) \in \text{int}(K) \quad \text{for any } s \geq t_x,$$

i.e. each trajectory starting at $x \in K$ will stay eventually in $\text{int}(K)$.

II. We prove now that for each vector $x \in K$ there exists the limit $\lim_{t \to \infty} S_t(x)$ with respect to the norm.

We introduce the sets

$$\mathcal{F} = \{ f \in K' : f(u) = 1 \}$$

and $\mathcal{S}_+ = \{ x \in K : \max_{f \in \mathcal{F}} f(x) = 1 \}$. As was mentioned above (property f)) $\mathcal{F}$ is a convex weak*-compact set which is a base of the cone $K'$. Both sets $\mathcal{F}$ and $\mathcal{S}_+$ are closed with respect to the norm, and $u \in \mathcal{S}_+$, $0 \notin \mathcal{S}_+$. Observe that $S_t(u) = u$ for any $t > 0$ implies $S_t^*(\mathcal{F}) \subset \mathcal{F}$ (Lemma 1(ii)).

For any vector $x \in X$ and $t \in [0, +\infty)$ we define the numbers

$$M_x(t) = \sup_{f \in \mathcal{F}} f(S_t(x)) \quad \text{and} \quad m_x(t) = \inf_{f \in \mathcal{F}} f(S_t(x)).$$

The equation $f(S_{s+t}(x)) = (S_s^*(f))(S_t(x))$ and the inclusion $S_s^*(\mathcal{F}) \subset \mathcal{F}$ imply

$$m_x(t) \leq m_x(t + s) \leq M_x(t + s) \leq M_x(t) \quad \text{for all } s, t \geq 0. \quad (6)$$

Therefore the functions $M_x(t)$ and $m_x(t)$ are monotone and possess finite limits at infinity. Moreover, since for any $f \in \mathcal{F}$ the inequalities $m_x(t) \leq f(S_t(x)) \leq M_x(t)$ can be written as

$$f(m_x(t) u) \leq f(S_t(x)) \leq f(M_x(t) u),$$

the inequality

$$m_x(t) u \leq S_t(x) \leq M_x(t) u \quad (7)$$

holds for each $x \in K$ (see property e)).

The main aspect of the proof is to establish the relation

$$\delta_x(t) \equiv M_x(t) - m_x(t) \to 0 \quad t \to +\infty \quad (8)$$

for each $x \in X$. 

In view of (9) it suffices to prove that some sequence \( \{\delta_x(kt_0)\}_{k \in \mathbb{N}} \) for \( t_0 > 0 \) converges to 0. Assume by way of contradiction that there is some \( x_0 \in X \) such that \( \delta_{x_0}(t) \not\to 0 \) for \( t \to +\infty \). Due to the monotony of \( \delta_{x_0}(t) \) this means
\[
\delta_{x_0}(t) = M_{x_0}(t) - m_{x_0}(t) \geq \varepsilon
\]
for some \( \varepsilon > 0 \) and all \( t \in [0, \infty) \).
Consider now the set
\[
R_0 = \left\{ \frac{S_t(x_0) - m_{x_0}(t)u}{\delta_{x_0}(t)}, \frac{M_{x_0}(t)u - S_t(x_0)}{\delta_{x_0}(t)} : t \in [0, \infty) \right\}.
\]
In view of condition 2), the inequality (9) and the boundedness of the functions \( M_x(t) \) and \( m_x(t) \), the set \( R_0 \) turns out to be relatively compact. Moreover, it is easy to see that \( R_0 \subset S_+ \). Therefore the closure \( R \) of \( R_0 \) is also contained in \( S_+ \) and, in particular, 0 does not belong to \( R \).
According to the Lemma 2 and the Remark 1 there is a natural number \( p \) such that the compact set \( Q = A^p(R) \) belongs to \( \text{int}(K) \), where \( A = S_1 \).
The bilinear form \( \langle z, f \rangle = f(z) \) is strongly positive on the compact set \( Q \times \mathcal{F} \), where \( Q \) is considered with the norm topology (induced from \( X \)) and \( \mathcal{F} \) with the weak*-topology (s. property f)). The inequalities
\[
|\langle z, f \rangle - \langle y, g \rangle| \leq |\langle z - y, f \rangle| + |\langle y, f - g \rangle| \\
\leq \|z - y\| \|f\| + |\langle y, f - g \rangle|
\]
show that the bilinear form is continuous on the set \( Q \times \mathcal{F} \). Therefore there is some positive number \( \beta \) such that \( f(z) > \beta \) for all \( z \in Q \) and \( f \in \mathcal{F} \). We shall assume \( \beta < \frac{1}{2} \).
The vectors (remember that \( A = S_1 \))
\[
A^p \left( \frac{A^n(x_0) - m_{x_0}(n)u}{M_{x_0}(n) - m_{x_0}(n)} \right) \quad \text{and} \quad A^p \left( \frac{M_{x_0}(n)u - A^n(x_0)}{M_{x_0}(n) - m_{x_0}(n)} \right),
\]
belong to \( Q \) and, consequently, for each \( f \in \mathcal{F} \) we have
\[
f \left( A^p \left( \frac{A^n(x_0) - m_{x_0}(n)u}{M_{x_0}(n) - m_{x_0}(n)} \right) \right) \geq \beta \quad \text{and} \quad f \left( A^p \left( \frac{M_{x_0}(n)u - A^n(x_0)}{M_{x_0}(n) - m_{x_0}(n)} \right) \right) \geq \beta.
\]
This together with \( \beta = f(\beta u) \) implies by e)
\[
A^{n+p}(x_0) \geq m_{x_0}(n) u + \beta (M_{x_0}(n) - m_{x_0}(n)) u \quad \text{and} \\
A^{n+p}(x_0) \leq M_{x_0}(n) u - \beta (M_{x_0}(n) - m_{x_0}(n)) u \quad \text{for any} \ n \in \mathbb{N}.
\]
Put now \( n = kp \). Then
\[
M_{x_0}((k + 1)p) - m_{x_0}((k + 1)p) \leq (1 - 2\beta)(M_{x_0}(kp) - m_{x_0}(kp)),
\]
and therefore
\[ M_{x_0}(k p) - m_{x_0}(k p) \leq (1 - 2\beta)^k (M_{x_0}(0) - m_{x_0}(0)) \to 0. \]

However this contradicts to (9). So the relation (8), i.e. \( M \), III. In order to complete the final part of the proof we denote for each \( x \in X \)
\[ f_0(x) = \lim_{t \to \infty} m_x(t). \]

From the inequalities
\[ m_x(t) u \leq f_0(x) u \leq M_x(t) u \] (10)
and \( f \) by means of passing to the limit we obtain for each \( f \in \mathcal{F} \)
\[ f_0(x) = \lim_{t \to \infty} f(S_t(x)) \]
and so \( f_0 \) is an additive, homogeneous and nonnegative functional on \( X \) such that \( f_0(u) = 1 \).

The inequalities (7) and (10) further imply for \( x \in K \)
\[ -(M_x(t) - m_x(t))) u \leq S_t(x) - f_0(x) u \leq (M_x(t) - m_x(t))) u. \] (11)

Define now the rank-one operator \( A_0 \) by \( A_0 = f_0 \otimes u \), i.e. \( A_0(x) = f_0(x) u \) for \( x \in X \). From (11) and (3) it follows that
\[ ||S_t(x) - A_0(x)|| \leq C_u||S_t(x) - A_0(x)||_u \leq C_u (M_x(t) - m_x(t)) \to 0. \]

This proves the statement (ii) of the theorem.

We finalize the proof of the statements (i) and (iii). Since \( S_{(n+1)t}(x) = S_{nt}(S_t(x)) \) for each \( x \in X, t > 0 \) and \( n \in \mathbb{N} \), after passing to the limits as \( n \to \infty \) we obtain \( f_0(x) u = f_0(S_t(x)) u \) which shows that \( f_0(x) = (S_t^*(f_0))(x) \) for each \( x \in X \), i.e. \( f_0 = S_t^*(f_0) \) for \( t > 0 \). We show that \( f_0(x) > 0 \) if \( x \in K, x \neq 0 \). For such \( x \) there is some \( t_x \) with \( S_{t_x}(x) \in \text{int}(K) \). In view of property b) and the weak*-compactness of \( \mathcal{F} \) we get \( m_x(t_x) > 0 \). Then \( f_0(x) = \lim_{t \to \infty} m_x(t) \geq m_x(t_x) > 0 \).

From the already proved statement (ii) it follows that for each functional \( f \in \mathcal{F} \) the family \( \{S_t^*(f)\} \) converges to \( f_0 \) with respect to the weak*-topology, i.e.
\[ S_t^*(f)(x) = f(S_t(x)) \to f_0(x) \quad \text{for each} \quad x \in X. \]

It remains to notice that due to the facts that any functional \( f \in X' \) is representable as a difference of two nonnegative functionals (s. Theorem I) and that \( \mathcal{F} \) is a base of the dual cone \( K' \), the last relation implies
\[ S_t^*(f) \to f(u)f_0 \quad \text{for each} \quad f \in X' \]
with respect to the weak*-topology. Now (iii) is proved.

It remains to prove (iv), i.e. that $\lambda = 1$ is a simple eigenvalue of the operators $S_t$ and $S_t^*$ for $t > 0$. Indeed, if $u'$ is another fixed point of $S_t$, then $S_{nt}(u') = u'$ for any $n \in \mathbb{N}$. Since $S_{nt}(u') \to f_0(u')u$ one immediately has $u' = f_0(u')u$. That means the eigenspace of the operator $S_t$, corresponding to the eigenvalue $\lambda = 1$, is one-dimensional. A similar argument shows the statement for the adjoint operator.

**Corollary 1** For the operators $S_t$ and $A_0$ for each $t \in [0, \infty)$ and $n \in \mathbb{N}$ there hold the following relations

1. $A_0^n = A_0$;
2. $S_tA_0 = A_0S_t = A_0$;
3. $(S_t - A_0)^n = S_{nt} - A_0$.

**Proof.** a) - c) are obtained by a simple calculation which we will omit.

If the condition 2) of the theorem is replaced by a slightly stronger one, then the operators $S_t^*$, for $t \to \infty$, converge to the operator $A_0^*$ not only in the weak operator topology but also pointwise. The new condition is well known in the theory of Markov chains (see, for example, [11](Lemma V.3.1). We come now to one of our main results.

**Theorem 3** Let $(X, K, \| \cdot \|)$ be an ordered Banach space and $\{S_t\}_{t \geq 0}$ a positive $C_0$-semigroup of operators in $L(X)$ which satisfies the following conditions

1. for each vector $x \in K$, $x \neq 0$ there exists a number $t_x \in [0, \infty)$ such that $S_{t_x}(x) \in \text{int}(K)$;
2. there exist a number $\tau > 0$ and a compact operator $V$ such that $\|S_{\tau} - V\| < 1$;
3. $\sup_{t \geq 0} \|S_t\| < \infty$.

Then all statements of Theorem 2 are valid. Moreover,

(v) the operators $S_t^*$, for $t \to \infty$, converge pointwise to the operator $A_0^*$.

**Proof.** First of all we prove that the condition 2) of Theorem 2 is satisfied. If $x \in X$ then it suffices to show that for an arbitrary fixed $\varepsilon > 0$ the trajectory $\{S_t(x)\}_{t \geq 0}$ possesses a relatively compact $\varepsilon$-net. Put $C = \sup_{t \geq 0} \|S_t\|$, $W = S_\tau - V$ and $q = \|W\|$. Obviously $C < \infty$ and

$$S_{nt} = S^n_{\tau} = W^n + V_n,$$

where $V_n$ is some compact operator and $\|W^n\| \leq q^n \to 0$. 

Fix a sufficiently large $N$ such that $q^N < \varepsilon$. Notice that the trajectory $\{S_t(x)\}_{t \geq 0}$ is contained in the closed ball $B_x = B(0; C\|x\|)$. If $t = N\tau + t', t' > 0$ then $S_{t'}(x) \in B_x$, and therefore

$$\|S_t(x) - V_N(S_{t'}(x))\| = \|(S_t^N - V_N)(S_{t'}(x))\| \leq q^N C\|x\| < C\varepsilon\|x\|.$$ 

Now it is immediate that the relatively compact set

$$\{S_t(x): 0 \leq t \leq N\tau\} \cup V_N(B_x)$$

is a $C\varepsilon\|x\|$-net for the trajectory $\{S_t(x)\}_{t \geq 0}$, and so the condition 2) of the Theorem 2 holds.

We prove now the statement (v). Assume first $A_0 \neq 0$. In this case it suffices to show $\|S^{*}_t(f) - f_0\| \to 0$ for each $f \in F$. Let $\varepsilon$ be an arbitrary positive number and $N, t', V_N$ be the same as above. Let be $H = V_N(B(0; 1))$. In view of the equalities

$$f(S_{N\tau}(x)) = f(V_N(x)) + f(W^N(x)), \quad f_0(x) = (S^*_N(f_0))(x) = f_0(V_N(x)) + f_0(W^N(x))$$

we obtain for $t = N\tau + t'$ the estimate

$$\|S^{*}_t(f) - f_0\| \leq \sup_{\|x\| \leq 1} |S^{*}_t(f)(V_N(x)) - f_0(V_N(x))|$$

$$+ \sup_{\|x\| \leq 1} |S^{*}_t(f)(W^N(x))| + \sup_{\|x\| \leq 1} |f_0(W^N(x))|$$

$$\leq \sup_{y \in H} |S^{*}_{t'}(f)(y) - f_0(y)| + C\|f\| + \|f_0\|q^N$$

$$< \sup_{y \in H} |S^{*}_{t'}(f)(y) - f_0(y)| + (C\|f\| + \|f_0\|)\varepsilon.$$ 

This estimate holds for any $t' > 0$. Because of $S_{t'}(f) \overset{\tau \to \infty}{\longrightarrow} f_0$ with respect to the weak* topology and the relative compactness of the set $H$ the supremum at the right side of the inequality converges to 0 if $t' \to \infty$ by the theorem on uniform convergence on compact sets. Consequently, for sufficiently large $t'$ we obtain

$$\|S^{*}_t(f) - f_0\| < \varepsilon + (C\|f\| + \|f_0\|)\varepsilon,$$

what has to be shown.

If $A_0 = 0$ then for each $f \in X'$ the given proof is applicable if $f_0 = 0$ is assumed.

\begin{corollary}
Under the conditions of Theorem 3 the operator $T := T(t) = I - S_t + A_0$ is invertible for any $t > 0$ and

$$T^{-1} = I + \sum_{n=1}^{\infty} (S_{nt} - A_0),$$

where the series converges pointwise.
\end{corollary}
Proof. We show first that $\ker(T) = \{0\}$ for all $t > 0$. (s. [1], propositions 9.10.2, 9.10.5).
Assume $A_0 \neq 0$. If $T(x_0) = 0$ then apply the operator $A_0$ to the equation $-A_0(x_0) = x_0 - S_t(x_0)$ and by taking into consideration the statements a) and b) of Corollary [1] we see that $A_0(x_0) = f(x_0)u = 0$. Therefore $f_0(x_0) = 0$, and due to statement (i) of the theorem we get $x_0 = 0$.
If $A_0 = 0$, then the operator $S_t$ can not have any nonzero fixed point $x_0$, since in the opposite case there would be $S_{nt}(x_0) = x_0 \not\rightarrow A_0(x_0) = 0$. So,

$$0 = T(x_0) = x_0 - S_t(x_0) + A_0(x_0) = x_0 - S_t(x_0)$$

means $S_t(x_0) = x_0$ and implies that the kernel of the operator $T$ is trivial for any $t > 0$.

The proof of invertibility of the operator $T$ for all $t > 0$ now follows. By keeping the notation of the theorem we put $W = S_\tau - V, q = \|W\|$. Put also $U = S_t - A_0$. Notice that according to the Corollary [1] and Theorem [2] the sequence $U^n = S_{nt} - A_0$ converges to 0 pointwise.

We fix now some $m \in \mathbb{N}$ such that $2Cq^m < 1$, where $C = \sup_{t \geq 0} \|S_t\|$, and prove that $T$ is invertible for $t > m\tau$. Remember that $S_{m\tau} = W^m + V_m$, where $V_m$ is some compact operator. If $t = m\tau + \sigma, \sigma \geq 0$, then $T$ (at the moment $t$) is equal to

$$T = I - S_{m\tau}(S_\sigma - A_0) = I - W^m(S_\sigma - A_0) - V_m(S_\sigma - A_0).$$

Notice that the operator $R = I - W^m(S_\sigma - A_0)$ is invertible because of $\|W^m(S_\sigma - A_0)\| \leq q^m(C + \|A_0\|) \leq 2Cq^m < 1$. Hence

$$T = R(I - R^{-1}V_m(S_\sigma - A_0)).$$

At the same time the operator $I - R^{-1}V_m(S_\sigma - A_0)$ is invertible since, in view of $\ker T = \{0\}$, its kernel is trivial, and the operator $R^{-1}V_m(S_\sigma - A_0)$ is compact together with $V_m$. The invertibility in this case of the operator $T$ is established.

In the case of $0 < t < m\tau$, we use the identity

$$I - U^n = T(I + U + \cdots + U^{n-1}).$$

(13)

If $nt > m\tau$ then by what has been shown above the operator $I - U^n = I - S_{nt} + A_0 = T(nt)$ is invertible. It follows from (13) that also the operator $T = T(t)$ is invertible (since due to the invertibility of the operator $I - U^n$ it shall be injective and surjective).

From the identity (13) and the invertibility of the operator $T$ it follows that

$$T^{-1} - T^{-1}U^n = I + U + \cdots + U^{n-1}. \quad (14)$$

Since $U^n \to 0$ pointwise this proves that the decomposition (12) takes place.

Our next result is

**Theorem 4** Let $(X, K, \| \cdot \|)$ be an ordered Banach space and $\{S_t\}_{t \geq 0}$ a positive $C_0$-semigroup of operators in $L(X)$ which satisfies the following conditions
1) for each vector $x \in K$, $x \neq 0$ there exists a number $t_x \in [0, \infty)$ such that $S_{t_x}(x) \in \text{int}(K)$;

2) for some $\tau > 0$ the operator $S_\tau$ is compact;

3) $\sup_{t \geq 0} \|S_t\| < \infty$.

Then the statements of Theorem 2 are valid. Moreover,

(v) $\|S_t - A_0\| \rightarrow 0$, i.e. the operators $S_t$ (and, of course, the adjoint operators) converge to $A_0$ (to $A_0^*$) with respect to the norm.

Proof. Since the condition 2) of this theorem is stronger than the corresponding condition 2) of the previous theorem and the other ones coincide, it is left to prove only statement (v).

Due to b) of Corollary 1 for $t > \tau$ one has

$$S_t - A_0 = (S_{t-\tau} - A_0) S_\tau,$$

and if $Q$ denotes the closure of the image under $S_\tau$ of the unit ball, therefore

$$\|S_t - A_0\| = \sup_{\|x\| \leq 1} \|(S_{t-\tau} - A_0)S_\tau(x)\| = \sup_{y \in Q} \|(S_{t-\tau} - A_0)(y)\|,$$

Since $S_{t-\tau} \rightarrow A_0$ pointwise as $t \rightarrow \infty$ and $Q$ is compact one has (s. [2], chapt.III §3, prop.5)

$$\sup_{y \in Q} \|(S_{t-\tau} - A_0)(y)\| \rightarrow_{t \rightarrow +\infty} 0.$$

This completes our proof.

Corollary 3 Under the conditions of Theorem 4 the operator $T := T(t) = I - S_t + A_0$ is invertible for $t > 0$ and

$$T^{-1}(t) = I + \sum_{n=1}^{\infty} (S_{nt} - A_0),$$

where the series converges with respect to the norm.

Proof. The invertibility of the operator $T$ has been proved in Corollary 2. Therefore it remains to pass to the limit in the identity (14) by taking into account that $\|U^n\| = \|S_{nt} - A_0\| \rightarrow 0$, as it was shown in the proof of Theorem 4.

Remark 2 By means of the functions $m_x(t), M_x(t)$, which have been introduced during the proof of Theorem 2 an estimate of the value $\|S_t - A_0\|$ might be obtained.

In order to show this we remember some constants (s. properties a), g)): $\gamma$ - the constant of nonflatness of the cone $K$ and $C_u$ a constant which satisfies $\|x\| \leq C_u \|x\|_u$ for each $x \in X$ (s. inequality (3)). Finally denote by $Q_+$ the closure of the set $S_\tau(B_+)$, where $B_+$ is
the intersection of the unit ball $B(0; 1)$ with the cone $K$. Notice that for each $y \in K$ and $f \in \mathcal{F}$ one has

$$m_y(t) \leq f(S_t(y)) \leq M_y(t) \quad \text{and} \quad m_y(t) \leq f(A_0(y)) \leq M_y(t). \quad (16)$$

Then any vector $x \in B(0; 1)$ can be represented as $x = \gamma(x' - x'')$, where $x', x'' \in B_+$. Therefore $t > \tau$ implies

$$\|S_t - A_0\| \leq 2 \gamma \sup_{x \in B_+} \| (S_{t-\tau} - A_0)(S_\tau(x)) \|$$

$$\leq 2 \gamma \sup_{y \in Q_+} \| S_{t-\tau}(y) - A_0(y) \|$$

$$\leq 2 \gamma C_u \sup_{y \in Q_+, f \in \mathcal{F}} \| f(S_{t-\tau}(y)) - f(A_0(y)) \|.$$

In view of the inequality (16) we get for each $f \in \mathcal{F}$

$$|f(S_{t-\tau}(y)) - f(A_0(y))| \leq M_y(t - \tau) - m_y(t - \tau).$$

Together with the previous inequality this yields the required estimate

$$\|S_t - A_0\| \leq 2 \gamma C_u \sup_{y \in Q_+} (M_y(t - \tau) - m_y(t - \tau)).$$

One easy proves that the functions $y \mapsto M_y(t)$, $y \mapsto m_y(t)$ are continuous, and so Dini’s theorem implies that the difference $M_y(t - \tau) - m_y(t - \tau)$, which for $t \to +\infty$ monotonically converges to zero, uniformly decreases to 0 on the compact set $Q_+$. In this way we get another proof of statement (v) of Theorem 4.

It is easy to see that the statements of our Theorems 2 and 4 remain to be valid also for a "discrete" semigroup of operators, i.e. for the sequence of iterates $\{A^n\}_{n \in \mathbb{N}}$ of some positive operator $A$. The proofs of the Theorems 2 and 4 might be adapted to that case, even with some obvious simplifications. Therefore we restrict ourselves with only the formulations.

**Theorem 5** Let $(X, K, \| \cdot \|)$ be an ordered Banach space and $A \in L(X)$ a positive operator which satisfies the following conditions

1) for each vector $x \in K$, $x \neq 0$ there exists a natural number $n_x$ such that $A^{n_x}(x) \in \text{int}(K)$;

2) for each vector $x \in K$ its trajectory $\{A^n(x)\}_{n \in \mathbb{N}}$ is relatively compact.

Then the sequence $\{A^n\}_{n \geq 0}$ pointwise converges to some operator $A_0$. If this operator is not zero, then there exist a vector $u \in \text{int}(K)$ and a functional $f_0 \in K'$ such that
(i) \( A(u) = u, \quad A^*(f_0) = f_0, \quad f_0(u) = 1 \) and moreover \( f_0(x) > 0 \) if \( x \in K, \ x \neq 0 \);
(ii) \( A_0 = f_0 \otimes u \);
(iii) for each \( f \in X' \) one has \( (A^*)^n(f) \xrightarrow{n \to +\infty} A_0^*(f) \) for each \( f \in X' \) with respect to the weak*-topology \( \sigma(X', X) \);
(iv) \( \lambda = 1 \) is a simple eigenvalue of the operators \( A \) and \( A^* \).

**Theorem 6** Let \((X, K, \| \cdot \|)\) an ordered normed space and \( A \in L(X) \) a positive operator which satisfies the following conditions

1) for each vector \( x \in K, \ x \neq 0 \) there exists a natural number \( n_x \) such that \( A^{n_x}(x) \in \text{int}(K) \);
2) some power of \( A \) is a compact operator;
3) \( \sup_{n \in \mathbb{N}} \| A^n \| < +\infty \).

Then the statements of the Theorem remain true. Moreover,

(v) \( \| A^n - A_0 \| \xrightarrow{n \to \infty} 0 \), i.e. the operators \( A^n \) (and, of course, the adjoint operators \( (A^n)^* \)) converge to \( A_0 \) (to \( A_0^* \)) with respect to the norm.

We remark another assertion, which turns out to be a special case of Theorem 6.

**Theorem 7** Let \((X, K, \| \cdot \|)\) an ordered normed space and \( A \in L(X) \) a positive operator which satisfies the following conditions

1) some power of the operator \( A \) is strongly positive, i.e. for some \( p \in \mathbb{N} \) one has \( A^p(K \setminus \{0\}) \subseteq \text{int}(K) \);
2) some power of \( A \) is a compact operator;
3) \( \sup_{n \in \mathbb{N}} \| A^n \| < +\infty \).

Then all statements of the Theorem remain true.

This theorem indeed is a special case of Theorem 6 because its condition 1) is stronger than condition 1) of Theorem 6 and the other assumptions are identical.

At the end we shall shortly deal with two examples. Let \( X \) be either the vector space \( C(Q) \) of all real continuous functions defined on the compact topological space \( Q \) with the cone \( K \) of all nonnegative functions or the vector space \( c \) of all converging real sequences with the cone \( K \) consisting of all nonnegative sequences. The symbol \( 1 \) denotes correspondingly the function identically equal to 1 on \( Q \) or the sequence whose components are all 1. 

\[ \text{In both cases the cone } K \text{ satisfies the condition } \text{int}(K) \neq \emptyset. \]
An operator \( A \in L(X) \) is called a \textit{Markov operator}, if it is positive and \( A(1) = 1 \). We indicate two examples of a compact Markov operator in the spaces \( C([0, 1]) \) and \( c \), respectively, which satisfies the condition 1) of Theorem 6 but does not satisfy the condition 1) of Theorem 7.

**Example 1** Let be \( X = C([0, 1]) \). For some arbitrary fixed number \( \theta \in (0, 1) \) denote the function \( \varphi(t) = (\theta + \sqrt{s})^{-1} \) and put

\[
(A(x))(s) = \varphi(s) \left( \int_0^\theta x(t) \, dt + \int_0^\sqrt{s} x(t) \, dt \right) \quad (x \in C([0, 1])).
\]

Obviously, \( A \) is compact operator in \( C([0, 1]) \) and has the properties

\[
A \geq 0, \quad A(1) = 1, \quad \|A\| = 1.
\]

Therefore the operator \( A \) satisfies the idenitc conditions 2) and 3) of the Theorems 6 and 7.

We show that \( A \) satisfies the condition 1) of Theorem 6 but not the condition 1) of Theorem 7. For \( x \in K \), \( x \neq 0 \) put

\[
p = \sup \{ \tau \in [0, 1]: x(t) = 0 \text{ for } t \in [0, \tau] \}
\]

and define \( p = 0 \) if \( x(0) \neq 0 \).

It is easy to see that \( p < 1 \) and that \( p < \theta \) implies \( (A(x))(s) > 0 \) on \([0, 1]\) because of \( \int_0^\theta x(t) \, dt = \int_0^\theta x(t) \, dt > 0 \).

If further for some \( m \in \mathbb{N} \) the number \( p \) satisfies the inequality

\[
\theta^{\frac{1}{2m}} \leq p < \theta^{\frac{1}{2m-1}},
\]

then an induction argument shows that

\[
(A^{m-1}(x))(0) = 0, \quad \text{but} \quad (A^m(x))(s) > 0 \text{ for } s \in [0, 1],
\]

i.e. \( A^m(x) \in \text{int}(K) \) (\( A^0 = I \)). Consequently, for each \( x \in K \) there exists its individual power \( m_x \) with \( A^{m_x}(x) \in \text{int}(K) \), however a common power, simultaneously for all \( x \in K \), does not exist.

**Example 2** Let an operator \( A \) be defined by a stochastic matrix as follows

\[
A = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots & 0 & 0 & \ldots \\
\frac{1}{2} & \frac{1}{2\pi} & \frac{1}{\pi} & 0 & \ldots & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\frac{1}{2} & \frac{1}{2\pi} & \frac{1}{\pi} & \ldots & \frac{1}{\pi} & \frac{1}{\pi} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix}.
\]
$A$ defines a positive operator from $c$ to $c$, and obviously is a Markov operator. For each $n \in \mathbb{N}$ the matrix $A^n$ is also stochastic. Moreover, the first $n + k$ entries in the $k$-th row of $A^n$ are positive ($k, n = 1, 2, \ldots$). Denote by $e_n$ the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with 1 at the $n$-th position. Then the first coordinate of the vector $A^n(e_{n+2})$ (still) turns out to be zero, all coordinates of $A^{n+1}(e_{n+2})$ are positive and from the second one on, they are all equal, i.e. all coordinates of $A^{n+1}(e_{n+2})$ are separated from 0. For the vector $A^{n+2}(e_{n+2})$ even all coordinates are equal and positive. So, we have $A^n(e_{n+2}) \notin \text{int}(K)$ but $A^{n+1}(e_{n+2}), A^{n+2}(e_{n+2}) \in \text{int}(K)$. On the other hand it is clear that no iterate of $A$ can satisfy the condition $A^n(K \setminus \{0\}) \subset \text{int}(K)$.

The compactness of $A$ follows from the possibility to approximate the operator $A - g \otimes 1$ (and so the operator $A$) by finite-rank operators, where $g$ is the functional generated by the sequence $\{e_n\}_{n \in \mathbb{N}}$.

The limit distribution of the operator $A$, i.e. a vector $f_0$ such that $A^*(f_0) = f_0$, can be calculated as the sequence with the members $c_n = c_{n+1}$, where $c_n = 2^{-\lceil \frac{\log n}{\log 2} \rceil}$ for $n = 1, 2, \ldots$.

**Remark 3** We point out one particular situation, namely the situation of Markov operators, in which Theorem 7 allows us to obtain as a special case some result which has been proved in [6] (chapt.VIII, §7, Th.1).

Theorem 7 holds for any Markov operator $A$ in the space $C(Q)$, if some power of $A$ is a compact operator (in particular, if $A$ is weakly compact) and if $A$ satisfies the condition of regularity: for some $m$ the inequality $A^m(x) > 0$ holds everywhere on $Q$ for each nonnegative function $f \in C(Q)$ that is not identical zero (with other words, $A^m$ is strongly positive).

In this case, the functional $f_0$ whose existence and properties are ensured by the statements of Theorem 7 is called "stationary distribution" (s. [5]) or "limit distribution" (s. [8], chapt.IV). Notice that under the made assumptions for any probability measure $\mu \in C^*(Q)$ one has not only the convergence $(A^*)^n(\mu) \to f_0$ in variation but, according to the statement (v) of Theorem 6 even $\sup_{\mu} \| (A^*)^n(\mu) - f_0 \| \to 0$, i.e. the sequence $\{(A^*)^n\}_{n \in \mathbb{N}}$ converges to the operator $A^*_0$ with respect to the norm.

At the end we notice another remark to Theorem 6 concerning the random walk on a compact space (compare with [6], chapt.VIII, §7 Th.1 and [1], §§6,7). We keep the terminology, which is familiar in the theory of stationary Markov chains and also the notations there (s. [5], [6]).

**Remark 4** Let be $Q$ a metrizable compact space, $\{P_s\}_{s \in Q} \subset C^*(Q)$ a stochastic kernel which satisfies the conditions

1) for each open set $G \subset Q$ the function $s \mapsto P_s(G)$ is continuous on $Q$;

2) for each nonempty open set $G \subset Q$ and each point $s \in Q$ there exists a number $n = n(G, s)$ such that $P^{(n)}_s(G) > 0$, where $P^{(n)}_s$ denotes the $n$-th iterate of the kernel $P_s$. 


Then there exists a probability measure \( P_0 \in C^*(Q) \) such that

(i) \( P_0(G) > 0 \) for any nonempty open subset \( G \);

(ii) \( \sup_{s \in Q} \| P_s^{(n)} - P_0 \| \xrightarrow{n \to \infty} 0 \).

For a short proof we consider the Markov operator \( A \) corresponding to the kernel \( \{ P_s \}_{s \in Q} \), where \( (Ax)(s) = \int_Q x(t) dP_s(t) \). Condition 1) implies that the operator \( A \) is weakly compact (s. [4], Th.9.4.10) and consequently the operator \( A^2 \) is compact (s. [4], sect.9.4.5). From condition 2) one gets that the operator \( A \) meets the condition 1) of Theorem [6].

Therefore, the operator \( A \) satisfies all conditions of Theorem [6]. It remains to notice that \( P_0 \) is that measure which is generated by the functional \( f_0 \). The statement (ii) holds because of

\[
\sup_{s \in Q} \| P_s^{(n)} - P_0 \| \leq \| (A^*)^n - A_0^* \| \xrightarrow{n \to \infty} 0,
\]

where \( A_0 \) is the limit operator from Theorem [6].

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