Partial regularity for elliptic systems with VMO-coefficients

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Abstract. We establish partial Hölder continuity for vector-valued solutions $u : \Omega \to \mathbb{R}^N$ to inhomogeneous elliptic systems of the type:

$$-\text{div}(A(x, u, Du)) = f(x, u, Du) \quad \text{in } \Omega,$$

where the coefficients $A : \Omega \times \mathbb{R}^N \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ are possibly discontinuous with respect to $x$. More precisely, we assume a VMO-condition with respect to the $x$ and continuity with respect to $u$ and prove Hölder continuity of the solutions outside of singular sets.

Keywords. Nonlinear elliptic systems, Partial regularity, VMO-coefficients, $A$-harmonic approximation.

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1 Introduction

In this paper, we consider the second order nonlinear elliptic systems in divergence form of the following type:

$$-\text{div}(A(x, u, Du)) = f(x, u, Du) \quad \text{in } \Omega. \quad (1.1)$$

Here $\Omega$ is bounded domain in $\mathbb{R}^n$, $u$ takes values in $\mathbb{R}^N$ with coefficients $A : \Omega \times \mathbb{R}^N \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

The aim of this paper is to obtain a partial regularity result of weak solutions to (1.1) with discontinuous coefficients. More precisely, we assume that the partial mapping $x \mapsto A(x, u, \xi)/(1 + |\xi|)^p$ has vanishing mean oscillation (VMO), uniformly in $(u, \xi)$. This means that $A$ satisfies an estimate

$$|A(x, u, \xi) - A(\cdot, u, \xi)|_{x_0, \rho} \leq V_{x_0}(x, \rho)(1 + |\xi|)^{p-1},$$

where $V_{x_0} : \mathbb{R}^n \times [0, \rho_1) \to [0, 2L]$ are bounded functions with

$$\lim_{\rho \nearrow 0} V(\rho) = 0, \quad V(\rho) := \sup_{x_0 \in \Omega} \sup_{0 < r \leq \rho} \int_{B_r(x_0) \cap \Omega} V_{x_0}(x, r)dx.$$

We also assume that $u \mapsto A(x, u, \xi)/(1 + |\xi|)^p$ is continuous, that is, there exists a modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ such that an estimate

$$|A(x, u, \xi) - A(x, u_0, \xi)| \leq L\omega(|u - u_0|^2)(1 + |\xi|)^{p-1}$$

holds.

Regularity results under a VMO-condition have been established by Zheng [15] for quasi-linear elliptic systems or integral functionals. General functionals with VMO-coefficients were consider by Ragusa and Tachikawa [14], who generalized the low-dimensional results from problems with continuous coefficients.
to the case of VMO-coefficients. In particular, these results require that the dimension of domain is small, for example, $n \leq p + 2$ is required to obtain the Hölder continuity of the minimizers in [14]. In contrast, Bögelein, Duzaar, Habermann and Scheven [11] give the regularity result for homogeneous nonlinear elliptic system without dimension conditions.

Stronger assumptions such as the Hölder continuity with respect to $(x,u)$ or a Dini-type condition lead to partial $C^1$-regularity with a quantitative modulus of continuity for $Du$; the modulus of continuity can be determined in dependence on the modulus of continuity of the coefficients (cf. Giaquinta and Modica [10], Duzaar and Grotowski [7], Duzaar and Gastel [6], Chen and Tan [5], Qiu [13] and the references therein).

As we knew, we could not expect continuity (and not even boundedness) of the gradient $Du$ under continuous coefficients (or even more relaxed condition). The regularity result with continuous coefficients was already proved in eighties by Campanato [3, 4]. He proves that we could still expect local Hölder continuity of the solution $u$ in special cases, for instance, in lower dimension $n \leq p + 2$. The result with arbitrary dimension was given by Foss and Mingione [8].

Our aim is to extend the homogeneous system result in [1] to inhomogeneous system. Therefore we assume the same structure conditions to coefficients $A$ as in [1]. Under a suitable assumption to inhomogeneous term, we obtain Hölder continuity of weak solution (see Theorem 2.2).

Our proof is based on so-called $A$-harmonic approximation (cf. [7, Lemma 2.1]; see also Lemma 3.2), introduced by Duzaar and Grotowski. They give a simplified (direct) proof of regularity results to the systems with Hölder continuous coefficients and a natural growth condition, without $L^p$-$L^2$-estimates for $Du$.

We close this section by briefly summarizing the notation used in this paper. As mentioned above, we consider a bounded domain $\Omega \subset \mathbb{R}^n$, and maps from $\Omega$ to $\mathbb{R}^N$, where we take $n \geq 2$, $N \geq 1$. For a given set $X$ we denote by $L^n(X)$ the $n$-dimensional Lebesgue measure. We write $B_{\rho}(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$. For bounded set $X \subset \mathbb{R}^n$ with $L^n(X) > 0$, we denote the average of a given function $g \in L^1(X, \mathbb{R}^N)$ by $f_X gdx$, that is, $f_X gdx = \frac{1}{L^n(X)} \int_X gdx$. In particular, we write $g_{\rho,0} := \frac{1}{\mu_{B_{\rho}(x_0) \cap \Omega}} \int_{B_{\rho}(x_0) \cap \Omega} gdx$. We write $\text{Bil}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ for the space of bilinear forms on the space $\text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^N$. We denote $c$ a positive constant, possibly varying from line by line. Special occurrences will be denoted by capital letters $K$, $C_1$, $C_2$ or the like.

2 Statement of the results

Definition 2.1. We say $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p \geq 2$ is a weak solution of (1.1) if $u$ satisfies

$$\int_{\Omega} \langle A(x,u,Du), D\varphi \rangle dx = \int_{\Omega} \langle f, \varphi \rangle dx \quad (2.1)$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^N$ or $\mathbb{R}^{nN}$.

We assume following structure conditions.

(H1) $A(x,u,\xi)$ is differentiable in $\xi$ with continuous derivatives, that is, there exists $L \geq 1$ such that

$$|A(x,u,\xi)| + (1 + |\xi|) |D_\xi A(x,u,\xi)| \leq L(1 + |\xi|)^{p-1} \quad (2.2)$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$ and $\xi \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Moreover, from this we deduce the modulus of continuity function $\mu : [0, \infty) \to [0, \infty)$ such that $\mu$ is bounded, concave, non-decreasing and we have

$$|D_\xi A(x,u,\xi) - D_\xi A(x,u,\xi_0)| \leq L\mu \left( \frac{|\xi - \xi_0|}{1 + |\xi| + |\xi_0|} \right) (1 + |\xi| + |\xi_0|)^{p-2} \quad (2.3)$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$, $\xi, \xi_0 \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Without loss of generality, we may assume $\mu \leq 1.$
(H2) \( A(x,u,ξ) \) is uniformly strongly elliptic, that is, for some \( λ > 0 \) we have
\[
\left\langle D_ξA(x,u,ξ)ν,ν \right\rangle := \sum_{1 ≤ i,β ≤ N, 1 ≤ j,α ≤ n} D_ξ^i A_α^j(x,u,ξ)ν_iν_j ≥ λ|ν|^2(1 + |ξ|)^{p-2} \tag{2.4}
\]
for all \( x ∈ Ω, u ∈ ℝ^N, ξ, ν ∈ \text{Hom}(ℝ^N, ℝ^N) \).

(H3) \( A(x,u,ξ) \) is continuous with respect to \( u \). There exists a bounded, concave and non-decreasing function \( ω : [0,∞) → [0,∞) \) satisfying
\[
|A(x,u,ξ) − A(x,u_0,ξ)| ≤ Lω \left( |u − u_0|^2 \right) (1 + |ξ|)^{p-1} \tag{2.5}
\]
for all \( x ∈ Ω, u, u_0 ∈ ℝ^N, ξ ∈ \text{Hom}(ℝ^N, ℝ^N) \). Without loss of generality, we may assume \( ω ≤ 1 \).

(H4) \( x ↦ A(x,u,ξ)/(1 + |ξ|)^{p-1} \) fulfills the following VMO-conditions uniformly in \( u \) and \( ξ \):
\[
|A(x,u,ξ) − A(x,u,ξ)_{x_0,ρ}| ≤ V_{x_0}(x,ρ)(1 + |ξ|)^{p-1}, \quad \text{for all } x ∈ B_ρ(x_0)
\]
whenever \( x_0 ∈ Ω, 0 < ρ < ρ_0, u ∈ ℝ^N \) and \( ξ ∈ \text{Hom}(ℝ^N, ℝ^N) \), where \( ρ_0 > 0 \) and \( V_{x_0} : ℝ^N × [0,ρ_0] → [0,2L] \) are bounded functions satisfying
\[
\lim_{ρ ↘ 0} V(ρ) = 0, \quad V(ρ) := \sup_{x_0 ∈ Ω} \sup_{0 < ρ ≤ ρ_0} \int_{B_ρ(x_0) \cap Ω} V_{x_0}(x,r)dx. \tag{2.6}
\]

(H5) \( f(x,u,ξ) \) has \( p \)-growth, that is, there exist constants \( a, b ≥ 0 \), with \( a \) possibly depending on \( M > 0 \), such that
\[
|f(x,u,ξ)| ≤ a(M)|ξ|^p + b \tag{2.7}
\]
for all \( x ∈ Ω, u ∈ ℝ^N \) with \( |u| ≤ M \) and \( ξ ∈ \text{Hom}(ℝ^N, ℝ^N) \).

Now, we are ready to state our main theorem.

**Theorem 2.2.** Let \( u ∈ W^{1,p}(Ω, ℝ^N) \cap L^∞(Ω, ℝ^N) \) be a bounded weak solution of \( \text{(1.1)} \) under the structure conditions (H1), (H2), (H3), (H4) and (H5) satisfying \( ||u||_{∞} ≤ M \) and \( 2^{(10−9p)/2}λ > a(M)M \). Then there exists an open set \( Ω_u \subseteq Ω \) with \( L^p(Ω \setminus Ω_u) = 0 \) such that \( u ∈ C^{0,α}_{loc}(Ω_u, ℝ^N) \) for every \( α ∈ (0,1) \). Moreover, we have \( Ω \setminus Ω_u \subseteq Σ_1 \cup Σ_2 \), where
\[
Σ_1 := \left\{ x_0 ∈ Ω : \lim \inf_{ρ ↘ 0} \int_{B_ρ(x_0)} |Du - (Du)_{x_0,ρ}|^p dx > 0 \right\},
\]
\[
Σ_2 := \left\{ x_0 ∈ Ω : \lim \sup_{ρ ↘ 0} |(Du)_{x_0,ρ}| = ∞ \right\}.
\]

## 3 Preliminaries

In this section we present \( A \)-harmonic approximation lemma and some standard estimates for the proof of the regularity theorem.

First we state the definition of \( A \)-harmonic function and recall \( A \)-harmonic approximation lemma as below.
Lemma 3.1 ([7, Section 1]). For a given \( A \in \text{Bil}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^N)) \), we say that \( h \in W^{1,p}(\Omega, \mathbb{R}^N) \) is an \( A \)-harmonic function, if \( h \) satisfies

\[
\int_{\Omega} A(Dh, D\varphi) \, dx = 0
\]

for all \( \varphi \in C_0^\infty(\Omega, \mathbb{R}^N) \).

Lemma 3.2 ([1, Lemma 2.3]). Let \( \lambda > 0, L > 0, p \geq 2 \) and \( n, N \in \mathbb{N} \) with \( n \geq 2 \) given. For every \( \varepsilon > 0 \), there exists a constant \( \delta = \delta(n, N, L, \lambda, \varepsilon) \in (0, 1] \) such that the following holds: assume that \( \gamma \in [0, 1] \) and \( A \in \text{Bil}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^N)) \) with the property

\[
A(\nu, \nu) \geq \lambda|\nu|^2, \quad \text{for all} \ \nu \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N),
\]

\[
A(\nu, \tilde{\nu}) \leq L|\nu| |\tilde{\nu}|, \quad \text{for all} \ \nu, \tilde{\nu} \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N).
\]

Furthermore, let \( g \in W^{1,2}(B_\rho(x_0), \mathbb{R}^N) \) be an approximately \( A \)-harmonic map in sense that there holds

\[
\int_{B_\rho(x_0)} \{ |Dg|^2 + \gamma^{p-2} |Dg|^p \} \, dx \leq 1,
\]

\[
\int_{B_\rho(x_0)} A(Dg, D\varphi) \, dx \leq \delta \sup_{B_\rho(x_0)} |D\varphi|, \quad \text{for all} \ \varphi \in C^1_c(B_\rho(x_0), \mathbb{R}^N).
\]

Then there exists an \( A \)-harmonic function \( h \) that satisfies

\[
\int_{B_\rho(x_0)} \left\{ \left| \frac{h - g}{\rho} \right|^2 + \gamma^{p-2} \left| \frac{h - g}{\rho} \right|^p \right\} \, dx \leq \varepsilon,
\]

\[
\int_{B_\rho(x_0)} \{ |Dh|^2 + \gamma^{p-2} |Dh|^p \} \, dx \leq c(n, p).
\]

Next is a standard estimates for the solutions to homogeneous second order elliptic systems with constant coefficients, due originally to Campanato [2, Teorema 9.2]. For convenience, we state the estimate in a slightly general form than the original one.

Theorem 3.3 ([1, Theorem 2.3]). Consider \( A, \lambda \) and \( L \) as in Lemma 3.2. Then there exists \( C_0 \geq 1 \) depending only on \( n, N, \lambda \) and \( L \) such that any \( A \)-harmonic function \( h \) on \( B_{\rho/2}(x_0) \) satisfies

\[
\left( \frac{\rho}{2} \right)^2 \sup_{B_{\rho/4}(x_0)} |Dh|^2 + \left( \frac{\rho}{2} \right)^4 \sup_{B_{\rho/4}(x_0)} |D^2h|^2 \leq C_0 \left( \frac{\rho}{2} \right)^2 \int_{B_{\rho/2}(x_0)} |Dh|^2 \, dx.
\]

We state the Poincaré inequality (Lemma 3.4) in a convenient form. The proof can be founded in several literature, for example [1, Proposition 3.10].

Lemma 3.4. There exists \( C_p \geq 1 \) depending only on \( n \) such that every \( u \in W^{1,p}(B_\rho(x_0), \mathbb{R}^N) \) satisfies

\[
\int_{B_\rho(x_0)} |u - u_{x_0, \rho}|^p \, dx \leq C_p \rho^p \int_{B_\rho(x_0)} |Du|^p \, dx.
\]

Given a function \( u \in L^2(B_\rho(x_0), \mathbb{R}^N) \), where \( x_0 \in \mathbb{R}^n \) and \( \rho > 0 \). We write \( \ell_{x_0, \rho} \) for the minimizer of the functional

\[
\ell \mapsto \int_{B_\rho(x_0)} |u - \ell|^2 \, dx
\]
among all affine functions $\ell : \mathbb{R}^n \to \mathbb{R}^N$. Let write $\ell_{x_0,\rho}(x) := \ell_{x_0,\rho}(x_0) + D\ell_{x_0,\rho}(x - x_0)$. It is easy to check that $\ell_{x_0,\rho}(x_0) = u_{x_0,\rho}$ and

$$D\ell_{x_0,\rho} = \frac{n + 2}{\rho^2} \int_{B_\rho(x_0)} u \otimes (x - x_0) dx,$$

where $\xi \otimes \zeta = \xi_i \zeta^j$. Based on this formula, elementary calculations yield the following estimates.

**Lemma 3.5** ([12 Lemma 2]). Assume $u \in L^2(B_{\rho}(x_0), \mathbb{R}^N)$, $x_0 \in \mathbb{R}^n$, $\rho > 0$ and $0 < \theta \leq 1$. With $\ell_{x_0,\rho}$ and $\ell_{x_0,\theta \rho}$, we denote the affine functions from $\mathbb{R}^n$ to $\mathbb{R}^N$ defined as above for the radii $\rho$ and $\theta \rho$ respectively. Then we have

$$|D\ell_{x_0,\rho} - D\ell_{x_0,\theta \rho}|^2 \leq \frac{n(n + 2)}{(\theta \rho)^2} \int_{B_{\theta \rho}(x_0)} |u - \ell_{x_0,\rho}|^2 dx,$$

and more generally,

$$|D\ell_{x_0,\rho} - D\ell|^2 \leq \frac{n(n + 2)}{\rho^2} \int_{B_\rho(x_0)} |u - \ell|^2 dx,$$

for all affine functions $\ell : \mathbb{R}^n \to \mathbb{R}^N$.

The estimate (3.12) implies, in particular, that $\ell_{x_0,\rho}$ has the following quasi-minimizing property for the $L^p$-norm. The proof can be founded in [1, Section 2].

**Lemma 3.6.** Consider the minimizer of (3.9), that is, $\ell_{x_0,\rho}$. For any affine functions $\ell : \mathbb{R}^n \to \mathbb{R}^N$ and $p \geq 2$ we have

$$\int_{B_\rho(x_0)} |u - \ell_{x_0,\rho}|^p dx \leq c(n, p) \int_{B_\rho(x_0)} |u - \ell|^p dx.$$

Using Young’s inequality, we obtain the following lemma.

**Lemma 3.7.** Consider fixed $a, b \geq 0$, $p \geq 1$. Then for any $\varepsilon > 0$, there exists $K = K(p, \varepsilon) \geq 0$ satisfying

$$(a + b)^p \leq (1 + \varepsilon)^p a^p + K b^p.$$  

**Proof.** We first consider the case $p = 2k - 1$ for $k \in \mathbb{N}$. By binomial theorem, we have

$$(a + b)^{2k-1} = \sum_{m=0}^{2k-1} \binom{2k-1}{m} a^{2k-1-m} b^m = a^{2k-1} + b^{2k-1} + \sum_{m=1}^{k-1} \binom{2k-1}{m} (a^{2k-1-m} b^m + a^m b^{2k-1-m}).$$

Using Young’s inequality, we obtain

$$\sum_{m=1}^{k-1} \binom{2k-1}{m} (a^{2k-1-m} b^m + a^m b^{2k-1-m}) \leq \sum_{m=1}^{k-1} \binom{2k-1}{m} (\varepsilon' a^{2k-1} + c(k, m, \varepsilon') b^{2k-1}),$$

where $\varepsilon' > 0$ will be fixed later. Thus, we get

$$(a + b)^{2k-1} \leq a^{2k-1} + b^{2k-1} + \sum_{m=1}^{k-1} \binom{2k-1}{m} (\varepsilon' a^{2k-1} + c(k, m, \varepsilon') b^{2k-1})
= \left(1 + \varepsilon' \sum_{m=1}^{k-1} \binom{2k-1}{m}ight) a^{2k-1} + \left(1 + \sum_{m=1}^{k-1} \binom{2k-1}{m} c(k, m, \varepsilon') \right) b^{2k-1}.$$
For any $\varepsilon > 0$ we conclude (3.7) by taking $\varepsilon'$ as $\varepsilon = \varepsilon' \sum_{m=1}^{k-1} \binom{2k-1}{m}$.

In case of $p = 2k$, we may estimate similarly as above, hence we get

$$(a + b)^{2k} = \sum_{m=0}^{2k} \binom{2k}{m} a^{2k-m} b^m = a^{2k} + b^{2k} + \sum_{m=1}^{k-1} \binom{2k}{m} (a^{2k-m} b^m + a^m b^{2k-m}) + \binom{2k}{k} a^k b^k$$

$$\leq a^{2k} + b^{2k} + \sum_{m=1}^{k-1} \binom{2k}{m} (\varepsilon' a^{2k} + c(k, m, \varepsilon') b^{2k}) + \binom{2k}{k} (\varepsilon' a^{2k} + \frac{1}{\varepsilon} b^{2k})$$

$$= \left\{ 1 + \varepsilon' \sum_{m=1}^{k} \binom{2k}{m} \right\} a^{2k} + \left\{ 1 + \sum_{m=1}^{k-1} \binom{2k}{m} c(k, m, \varepsilon') + \frac{1}{\varepsilon} \right\} b^{2k}.$$

This conclude that we have (3.7) for $p \in \mathbb{N}$.

For general $p \geq 1$, let $[p]$ be the greatest integer not greater then $p$. We write

$$(a + b)^p = (a + b)^{[p]} (a + b)^{p-[p]}.$$

By $0 \leq p - [p] < 1$, we have

$$(a + b)^{p-[p]} \leq a^{p-[p]} + b^{p-[p]}.$$

For $\varepsilon' > 0$ to be fixed later, we get

$$(a + b)^{[p]} \leq (1 + \varepsilon') a^{[p]} + K(p, \varepsilon') b^{[p]},$$

since $[p] \in \mathbb{N}$. Combining two estimates, we obtain

$$(a + b)^p \leq \left\{ (1 + \varepsilon') a^{[p]} + K(p, \varepsilon') b^{[p]} \right\} (a^{p-[p]} + b^{p-[p]})$$

$$= (1 + \varepsilon') a^p + K(p, \varepsilon') b^p + (1 + \varepsilon') a^{p-[p]} b^{p-[p]} + K(p, \varepsilon') a^{p-[p]} b^{p-[p]}$$

$$\leq (1 + \varepsilon') a^p + K(p, \varepsilon') b^p + (1 + \varepsilon' + K(p, \varepsilon'))(a^{p-[p]} b^{p-[p]} + a^{p-[p]} b^{[p]}).$$

Again for $\varepsilon'' > 0$ to be fixed later, by using Young’s inequality, we conclude

$$(a + b)^p \leq (1 + \varepsilon') a^p + K(p, \varepsilon') b^p + (1 + \varepsilon' + K(p, \varepsilon'))(\varepsilon'' a^p + c(p, \varepsilon'') b^p).$$

Take $\varepsilon' = \varepsilon/2$ and $\varepsilon'' = \varepsilon'/(1 + \varepsilon' + K(p, \varepsilon'))$, and this complete the proof. \(\square\)

Lemma 3.8 ([14] Lemma 2.1]). For $\delta \geq 0$, and for all $a, b \in \mathbb{R}^k$ we have

$$4^{-(1+2\delta)} \leq \int_0^1 \frac{(1 + |sa + (1-s)b|^2)^{\delta/2}}{(1 + |a|^2 + |b-a|^2)^{\delta/2}} ds \leq 4^\delta. \quad (3.15)$$

4 Proof of the main theorem

To obtain the regularity result (Theorem 2.2), we first prove Caccioppoli-type inequality. In the followings, we define $q > 0$ as the dual exponent of $p \geq 2$, that is, $q = p/(p-1)$. Here we note that $q \leq 2$. 6
Lemma 4.1. Let $u \in W^{1,p} \cap L^\infty$ be a bounded weak solution of the elliptic system (1.1) under the structure condition (H1),(H2),(H3),(H4) and (H5) with satisfying $\|u\|_\infty \leq M$ and $2^{(10-9p)/2} \lambda > a(M)M$. For any $x_0 \in \Omega$ and $p \leq 1$ with $B_p(x_0) \subseteq \Omega$, and any affine functions $\ell : \mathbb{R}^n \to \mathbb{R}^n$ with $|\ell(x_0)| \leq M$, we have the estimate
\[
\int_{B_\rho(x_0)} \left\{ \frac{|Du-D\ell|^2}{(1+|D\ell|)^2} + \frac{|Du-D\ell|^p}{(1+|D\ell|)^p} \right\} dx 
\leq C_1 \left[ \int_{B_\rho(x_0)} \left\{ \frac{|u-\ell|^2}{p^2(1+|D\ell|)^2} + \frac{|u-\ell|^p}{p^p(1+|D\ell|)^p} \right\} dx 
+ \omega \left( \int_{B_\rho(x_0)} |u-\ell(x_0)|^2 dx \right) + V(p) + (a^\delta|D\ell|^\beta + b^\beta) \rho^\beta \right].
\] (4.1)
with the constant $C_1 = C_1(\lambda, p, L, a(M), M) \geq 1$.

Proof. Assume $x_0 \in \Omega$ and $p \leq 1$ satisfy $B_\rho(x_0) \subseteq \Omega$. We take a standard cut-off function $\eta \in C_0^\infty(B_\rho(x_0))$ satisfying $0 \leq \eta \leq 1$, $|D\eta| \leq 4/\rho$, $\eta \equiv 1$ on $B_{\rho/2}(x_0)$. Then $\varphi := \eta^p(u - \ell)$ is admissible as a test function in (2.1), and we obtain
\[
\int_{B_\rho(x_0)} \eta^p \langle A(x, u, Du), Du - D\ell \rangle dx 
= -\int_{B_\rho(x_0)} \langle A(x, u, Du), p\eta^{p-1}D\eta \otimes (u - \ell) \rangle dx 
+ \int_{B_\rho(x_0)} \langle f, \varphi \rangle dx.
\] (4.2)
Furthermore, we have
\[
-\int_{B_\rho(x_0)} \eta^p \langle A(x, u, Du), Du - D\ell \rangle dx 
= \int_{B_\rho(x_0)} \langle A(x, u, Du), p\eta^{p-1}D\eta \otimes (u - \ell) \rangle dx 
- \int_{B_\rho(x_0)} \langle A(x, u, Du), D\varphi \rangle dx,
\] (4.3)
and
\[
\int_{B_{\rho}(x_0)} \langle A(\cdot, \ell(x_0), D\ell) \rangle_{x_0,\rho}, D\varphi \rangle dx = 0.
\] (4.4)
Adding (4.2), (4.3) and (4.4), we obtain
\[
\int_{B_{\rho}(x_0)} \eta^p \langle A(x, u, Du) - A(x, u, D\ell), Du - D\ell \rangle dx 
= -\int_{B_{\rho}(x_0)} \langle A(x, u, Du) - A(x, u, D\ell), p\eta^{p-1}D\eta \otimes (u - \ell) \rangle dx 
- \int_{B_{\rho}(x_0)} \langle A(x, u, D\ell) - A(x, \ell(x_0), D\ell), D\varphi \rangle dx 
- \int_{B_{\rho}(x_0)} \langle A(x, \ell(x_0), D\ell) - (A(\cdot, \ell(x_0), D\ell) \rangle_{x_0,\rho}, D\varphi \rangle dx 
+ \int_{B_{\rho}(x_0)} \langle f, \varphi \rangle dx 
=: I + II + III + IV.
\] (4.5)
The terms I, II, III, IV are defined above. Using the ellipticity condition (H2) to the left-hand side of (4.5), we get
\[
(A(x, u, Du) - A(x, u, D\ell), Du - D\ell) = \int_0^1 \langle D_\xi A(x, u, sDu + (1 - s)D\ell)(Du - D\ell), Du - D\ell \rangle ds \\
\geq \lambda |Du - D\ell|^2 \int_0^1 (1 + |sDu + (1 - s)D\ell|)^{p-2} ds. \tag{4.6}
\]
Then by using (3.15) in Lemma 3.8, we obtain
\[
(A(x, u, Du) - A(x, u, D\ell), Du - D\ell) \\
\geq \lambda |Du - D\ell|^2 \int_0^1 (1 + |sDu + (1 - s)D\ell|^2)^{(p-2)/2} ds \\
\geq 2^{(12 - 9p)/2} \lambda \{ (1 + |D\ell|)^{p-2}|Du - D\ell|^2 + |Du - D\ell|^p \}. \tag{4.7}
\]
For \( \varepsilon > 0 \) to be fixed later, using (H1) and Young’s inequality, we have
\[
|I| \leq \varepsilon \int_{B_\rho(x_0)} p\eta |Du - D\ell|^p dx + \varepsilon^{-q/p} \int_{B_\rho(x_0)} L^q \omega^q \{ |u - \ell(x_0)|^2 \} (1 + |D\ell|)^p dx \\
\leq \varepsilon \int_{B_\rho(x_0)} \eta |Du - D\ell|^p dx + \varepsilon^{-q/p} \int_{B_\rho(x_0)} L^q \omega^q \{ |u - \ell(x_0)|^2 \} (1 + |D\ell|)^p dx \\
\leq \varepsilon \int_{B_\rho(x_0)} \eta |Du - D\ell|^p dx + \varepsilon \int_{B_\rho(x_0)} \frac{|u - \ell|^p}{\rho} dx \\
+ c(p, L, \varepsilon) (1 + |D\ell|)^p \omega \left( \int_{B_\rho(x_0)} |u - \ell(x_0)|^2 dx \right), \tag{4.8}
\]
where we use Jensen’s inequality in the last inequality. We next estimate III by using the VMO-condition (H4) and Young’s inequality, we have
\[
|III| \leq \frac{\varepsilon}{2^{p-1}} \int_{B_\rho(x_0)} \eta^p |Du - D\ell| + \frac{4p|u - \ell|}{\rho} \right)^p dx + \left( \frac{2^{p-1}}{\varepsilon} \right)^{q/p} \int_{B_\rho(x_0)} V_\rho^q(x, \rho) (1 + |D\ell|)^p dx. \tag{4.9}
\]
Then using the fact that \( V_\rho^q V_{\rho}^q \leq (2L)^{q-1} V_\rho \leq 2LV_\rho \), we infer
\[
|III| \leq \varepsilon \int_{B_\rho(x_0)} \eta^p |Du - D\ell|^p dx + c(p, \varepsilon) \int_{B_\rho(x_0)} \frac{|u - \ell|^p}{\rho} dx + c(p, L, \varepsilon) (1 + |D\ell|)^p V(\rho). \tag{4.10}
\]
Lemma 4.3. Assume the same assumption in Lemma 4.1. Then for any $B | \varepsilon \lambda$

Now choose $\varepsilon$ (because the left-hand side inequality of (3.15)

If we insert Remark 4.2.

For $a D\ell$

To use the $A | \rho$

Combining (4.5), (4.7), (4.9), (4.10) and (4.11), and set $\lambda' = 2^{(12-9p)/2} \lambda C \Lambda := \lambda' - 3 \varepsilon - a(1 + \varepsilon')(2M + |D\ell|)$, this gives

Now choose $\varepsilon = \varepsilon(\lambda, p, a(M), M) > 0$ and $\varepsilon' = \varepsilon'(\lambda, p, a(M), M) > 0$ in a right way (for more precise way of choosing $\varepsilon$ and $\varepsilon'$, we refer to [7 Lemma 4.1]), we obtain (4.11).

Remark 4.2. If we insert $p = 2$ to the “smallness condition” $2^{(10-9p)/2} \lambda > a(M)M$, we obtain $\lambda/16 > a(M)M$. On the other hand, we only need $\lambda/2 > a(M)M$ to prove the Caccioppoli-type inequality (Lemma 4.1) since the term $(1 + |sD\ell + (1 - s)\nu|)^{p-2}$ in (4.9) vanishes when $p = 2$. This gap happens because the left-hand side inequality of (3.15) in Lemma 3.7 which we used to estimate $(1 + |sD\ell + (1 - s)\nu|)^{p-2}$ from below, could not take equal when $\delta = p - 2 = 0$.

To use the $A$-harmonic approximation lemma, we need to estimate $\int_{B_\rho(x_0)} A(D(u - \ell), D\varphi)dx$.

Lemma 4.3. Assume the same assumption in Lemma 4.1. Then for any $x_0 \in \Omega$ and $\rho \leq \rho_0$ satisfy $B_{2\rho}(x_0) \subseteq \Omega$, and any affine functions $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^N$ with $|\ell(x_0)| \leq M$, the inequality

$$\int_{B_\rho(x_0)} A(D\varphi, D\varphi)dx \leq C_2(1 + |D\ell|) \left[ \mu^{1/2} \left( \sqrt{\Psi_*(x_0, 2\rho, \ell)} \right)^2 \sqrt{\Psi_*(x_0, 2\rho, \ell)} + \Psi_*(x_0, 2\rho, \ell) + \rho(a|D\ell|^p + b) \right] \sup_{B_\rho(x_0)} |D\varphi|$$ (4.13)
holds for all $\varphi \in C_0^\infty (B_\rho (x_0), \mathbb{R}^N)$ and a constant $C_2 = C_2(n, \lambda, L, p, a(M)) \geq 1$, where

$$A(Dv, D\varphi) := \frac{1}{(1 + |D\ell|)^{p-1}} \left( (D_2 A(\cdot, \ell(x_0), D\ell))_{x_0, \rho} Dv, D\varphi \right),$$

$$\Phi(x_0, \rho, \ell) := \int_{B_\rho (x_0)} \left\{ \frac{|Du - D\ell|^2}{(1 + |D\ell|)^2} + \frac{|Du - D\ell|^p}{(1 + |D\ell|)^p} \right\} dx,$$

$$\Psi(x_0, \rho, \ell) := \int_{B_\rho (x_0)} \left\{ \frac{|u - \ell|^2}{\rho^2 (1 + |D\ell|)^2} + \frac{|u - \ell|^p}{\rho^p (1 + |D\ell|)^p} \right\} dx,$$

$$\Psi_\ast (x_0, \rho, \ell) := \Psi(x_0, \rho, \ell) + \omega \left( \int_{B_\rho (x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) + (a^2 |D\ell|^q + b^q) \rho^q,$$

$$v := u - \ell = u - \ell(x_0) - D\ell(x - x_0).$$

**Proof.** Assume $x_0 \in \Omega$ and $\rho \leq 1$ satisfy $B_{2\rho}(x_0) \subseteq \Omega$. Without loss of generality, we may assume $\sup_{B_\rho (x_0)} |D\varphi| \leq 1$. Note $\sup_{B_\rho (x_0)} |\varphi| \leq \rho \leq 1$. Using the fact that $\int_{B_\rho (x_0)} A(x_0, \xi, \nu) D\varphi dx = 0$, we deduce

$$(1 + |D\ell|)^{p-1} \int_{B_\rho (x_0)} A(Dv, D\varphi) dx$$

$$= \int_{B_\rho (x_0)} \left( \left[ (D_2 A(\cdot, \ell(x_0), D\ell))_{x_0, \rho} - (D_2 A(\cdot, \ell(x_0), D\ell + sDv))_{x_0, \rho} \right] Dv, D\varphi \right) ds dx$$

$$+ \int_{B_\rho (x_0)} \left( A(\cdot, \ell(x_0), Du) - A(x, \ell(x_0), Du) \right) dx$$

$$+ \int_{B_\rho (x_0)} \langle f, \varphi \rangle dx$$

$$= : I + II + III + IV$$

(4.14)

where terms I, II, III, IV are defined above.

Using the modulus of continuity $\mu$ from (H1), Jensen’s inequality and Hölder’s inequality, we estimate

$$|I| \leq c(p, L) (1 + |D\ell|)^{p-1} \int_{B_\rho (x_0)} \mu \left( \frac{|Du - D\ell|}{1 + |D\ell|} \right) \left( \frac{|Du - D\ell|^p}{1 + |D\ell|} \right) dx$$

$$\leq c(1 + |D\ell|)^{p-1} \left[ \mu^{1/2} \left( \sqrt{\Phi(x_0, \rho, \ell)} \right)^2 \Phi(x_0, \rho, \ell) + \mu^{1/2} \left( \Phi^{1/2}(x_0, \rho, \ell) \right)^2 \Phi^{1/4}(x_0, \rho, \ell) \right].$$

(4.15)

The last inequality follows from the fact that $a^{1/p} b^{1/q} a^{1/p} b^{1/(p-2)}/p \leq a^{1/2} b^{1/2} + b$ holds by Young’s inequality.

By using the VMO-condition, Young’s inequality and the bound $V_{\lambda}(x, \rho) \leq 2L$, the term II can be estimated as

$$|II| \leq c(p) (1 + |D\ell|)^{p-1} \int_{B_\rho (x_0)} \left\{ V_{\lambda}(x, \rho) + V_{\lambda}(x, \rho) \frac{|Du - D\ell|^{p-1}}{(1 + |D\ell|)^{p-1}} \right\} dx$$

$$\leq c(1 + |D\ell|)^{p-1} \left[ (1 + (2L)^{p-1}) V(\rho) + \Phi(x_0, \rho, \ell) \right].$$

(4.16)
Similarly, we estimate the term III by using the continuity condition (H3), Young’s inequality, the bound \( \omega \leq 1 \) and Jensen’s inequality. This leads us to
\[
|\text{III}| \leq \int_{B_r(x_0)} (1 + |D\ell| + |Du - D\ell|^{p-1}) \omega \left( |u - \ell(x_0)|^2 \right) dx \\
\leq c(p, L)(1 + |D\ell|^{p-1}) \left[ \omega \left( \int_{B_r(x_0)} |u - \ell(x_0)|^2 dx \right) + \Phi(x_0, \rho, \ell) \right].
\] (4.17)

By using the growth condition (H5) and \( \sup_{B_r(x_0)} |\varphi| \leq \rho \leq 1 \), we have
\[
|\text{IV}| \leq \int_{B_r(x_0)} \rho(a|Du|^p + b) dx \\
\leq 2^{p-1} a(1 + |D\ell|)\Phi(x_0, \rho, \ell) + 2^{p-1} \rho(1 + |D\ell|^{p-1}) (a|Du|^p + b).
\] (4.18)

Combining (4.14) with the estimates (4.15), (4.16), (4.17) and (4.18), we finally arrive at
\[
\int_{B_r(x_0)} A(Du, D\varphi) dx \\
\leq c(p, L, a(M))(1 + |D\ell|) \\
\times \left[ \mu^{1/2} \left( \sqrt{\Phi(x_0, \rho, \ell)} \right) \sqrt{\Phi(x_0, \rho, \ell)} + \Phi(x_0, \rho, \ell) + \Psi_+(x_0, \rho, \ell) + \rho(a|Du|^p + b) \right] \\
\leq C_2(1 + |D\ell|) \left[ \mu^{1/2} \left( \sqrt{\Psi_+(x_0, 2\rho, \ell)} \right) \sqrt{\Psi_+(x_0, 2\rho, \ell)} + \Psi_+(x_0, 2\rho, \ell) + \rho(a|Du|^p + b) \right],
\]
where we use Caccioppoli-type inequality (Lemma 4.4), \( \Phi(x_0, \rho, \ell) \leq C_1 \Psi_+(x_0, 2\rho, \ell) \) and the concavity of \( \mu \) to have \( \mu(cs) \leq c\mu(s) \) for \( c \geq 1 \) at the last step. \( \square \)

From now on, we write \( \Phi(\rho) = \Phi(x_0, \rho, \ell_{x_0, \rho}), \Psi(\rho) = \Psi(x_0, \rho, \ell_{x_0, \rho}), \Psi_+(\rho) = \Psi_+(x_0, \rho, \ell_{x_0, \rho}) \) for \( x_0 \in \Omega \) and \( 0 < \rho \leq 1 \). Here \( \ell_{x_0, \rho} \) is a minimizer of (4.5).

Now we are in the position to establish the excess improvement.

**Lemma 4.4.** Assume the same assumptions with Lemma 4.3. Let \( \theta \in (0, 1/4) \) be arbitrary and impose the following smallness conditions on the excess:

(i) \( \mu^{1/2} \left( \sqrt{\Psi_+(\rho)} \right) + \sqrt{\Psi_+(\rho)} \leq \frac{\delta}{2} \) with the constant \( \delta = \delta(n, N, p, \lambda, L, \theta^{n+p+2}) \) from Lemma 4.3.

(ii) \( \Psi(\rho) \leq \frac{\theta^{n+2}}{4n(n+2)} \).

(iii) \( \gamma(\rho) := \left[ \Psi_+^{1/2}(\rho) + \delta^{-q} \rho^q (a|Du|_{x_0, \rho} + b)^q \right]^{1/q} \leq 1. \)

Then there holds the excess improvement estimate
\[
\Psi(\theta \rho) \leq C_3 \theta^2 \Psi_+(\rho)
\] (4.19)
with a constant \( C_3 \geq 1 \) that depends only on \( n, N, \lambda, L, p, a(M), M \) and \( \theta \).
Proof. We first rescale $u$ and set
\[ w := \frac{u - \ell_{x_0,\rho}}{C_2(1 + |D\ell_{x_0,\rho}|)\gamma}. \]
We claim that $w$ satisfies the assumptions of Lemma 3.2. By Lemma 4.3 with $\rho/2$ and $\ell_{x_0,\rho}$ instead of $\rho$ and $\ell$, and assumption (i), the map $w$ is approximately $\mathcal{A}$-harmonic in the sense that
\[
\int_{B_{\rho/2}(x_0)} \mathcal{A}(Dw, D\varphi) dx \leq \left[ \mu^{1/2} \left( \sqrt{\Psi_*(\rho)} + \sqrt{\Psi_*(\rho)} + \frac{\delta}{2} \right) \sup_{B_{\rho/2}(x_0)} |D\varphi| \right] \leq \delta \sup_{B_{\rho/2}(x_0)} |D\varphi|,
\]
for all $\varphi \in C^\infty_0(B_{\rho/2}(x_0), \mathbb{R}^N)$, with the constant $\delta$ determined by Lemma 3.2 for the choice $\epsilon = \theta^{n+p+2}$. Moreover, the choice of $C_2$, which implies $C_2 \geq C_1$, and the Caccioppoli-type inequality (Lemma 4.1) infer
\[
\int_{B_{\rho/2}(x_0)} \{ |Dw|^2 + \gamma^{p-2} |Dw|^p \} dx \leq \frac{C_1 \Psi_*(\rho)}{C_2^2 \gamma^2} \leq \frac{C_1}{C_2^2} \leq 1.
\]
Thus, Lemma 3.2 ensures the existence of an $\mathcal{A}$-harmonic map $h$ with the properties
\[
\int_{B_{\rho/2}(x_0)} \left\{ \frac{|w - h|^2}{\rho/2} + \gamma^{p-2} \frac{|w - h|^p}{\rho/2} \right\} dx \leq \theta^{n+p+2}, \tag{4.20}
\]
\[
\int_{B_{\rho/2}(x_0)} \{ |Dh|^2 + \gamma^{p-2} |Dh|^p \} dx \leq c(n, p). \tag{4.21}
\]
Since $h$ is $\mathcal{A}$-harmonic, Theorem 3.3 yields the estimate for $s = 2$ as well as for $s = p$
\[
\sup_{B_{\rho/4}(x_0)} |D^2h|^s \leq c(s, n, N, p, \lambda, L) \left( \frac{\rho}{2} \right)^{-s}.
\]
Therefore, using Taylor’s theorem, we have the decay estimate, where $\theta \in (0, 1/4]$ can be chosen arbitrarily:
\[
\gamma^{s-2}(\theta \rho)^{-s} \int_{B_{\rho/2}(x_0)} |w - h(x_0) - Dh(x_0)(x - x_0)|^s dx
\]
\[
\leq 2^{s-1} \gamma^{s-2}(\theta \rho)^{-s} \left[ \int_{B_{\rho/2}(x_0)} |w - h|^s dx + \int_{B_{\rho/2}(x_0)} |h(x) - h(x_0) - Dh(x_0)(x - x_0)|^s dx \right]
\]
\[
\leq c(s, n, N, p, \lambda, L) \theta^2.
\]
Here we applied the energy bound (4.20) for the last estimate. Scaling back to $u$ and using Lemma 3.6 we conclude
\[
(\theta \rho)^{-s} \int_{B_{\rho/2}(x_0)} |u - \ell_{x_0,\rho}|^s dx
\]
\[
\leq c(n, s)(\theta \rho)^{-s} \int_{B_{\rho/2}(x_0)} |u - \ell_{x_0,\rho} - C_2(1 + |D\ell_{x_0,\rho}|) (h(x_0) + Dh(x_0)(x - x_0))|^s dx
\]
\[
= c(n, N, p, \lambda, L, a(M))(\theta \rho)^{-s} \gamma^s (1 + |D\ell_{x_0,\rho}|)^s \int_{B_{\rho/2}(x_0)} |w - h(x_0) - Dh(x_0)(x - x_0)|^s dx
\]
\[
\leq c \gamma^2 (1 + |D\ell_{x_0,\rho}|)^s \theta^2
\]
\[
\leq c \left( 1 + |D\ell_{x_0,\rho}| \right)^s \theta^2 \left[ \Psi_*(\rho) + 2^{q/2} \delta^{-q} \Psi_*(\rho) \right]^{2/q}
\]
\[
\leq c \left( 1 + |D\ell_{x_0,\rho}| \right)^s \theta^2 \Psi_*(\rho). \tag{4.22}
\]
Here we would like to replace the term $|D\ell_{x_0,\rho}|$ on the right-hand side by $|D\ell_{x_0,\theta\rho}|$. For this, we use (3.11) and the assumption (ii) in order to estimate
\[
|D\ell_{x_0,\rho} - D\ell_{x_0,\theta\rho}|^2 \leq \frac{n(n+2)}{\theta n+2} (1 + |D\ell_{x_0,\rho}|^2) \Psi(\rho) \leq \frac{1}{4} (1 + |D\ell_{x_0,\rho}|)^2.
\]
This yields
\[
1 + |D\ell_{x_0,\rho}| \leq 1 + |D\ell_{x_0,\theta\rho}| + |D\ell_{x_0,\rho} - D\ell_{x_0,\theta\rho}| \leq 1 + |D\ell_{x_0,\theta\rho}| + \frac{1}{2} (1 + |D\ell_{x_0,\rho}|),
\]
and after reabsorbing the last term from the right-hand side on the left, we also obtain
\[
1 + |D\ell_{x_0,\rho}| \leq 2 (1 + |D\ell_{x_0,\theta\rho}|).
\]
Plugging this into (4.22), we deduce
\[
(\theta\rho)^{-s} \int_{B_{\rho}(x_0)} |u - \ell_{x_0,\theta\rho}|^s \, dx \leq c(s,n,N,p,\lambda,L,a(M))(1 + |D\ell_{x_0,\theta\rho}|)^s \theta^2 \Psi(\rho)
\]
for $s = 2$ and $s = p$. Dividing by $(1 + |D\ell_{x_0,\theta\rho}|)^s$, then adding the corresponding terms for $s = 2$ and $s = p$, we deduce the claim.

We fix an arbitrarily Hölder exponent $\alpha \in (0,1)$ and define the Campanato-type excess
\[
C_\alpha(x_0,\rho) := \rho^{-2\alpha} \int_{B_{\rho}(x_0)} |u - u_{x_0,\rho}|^2 \, dx.
\]
In the following lemma, we iterate the excess improvement estimate (4.19) from Lemma 4.4 and obtain the boundedness of the two excess functionals, $C_\alpha$ and $\Psi$.

**Lemma 4.5.** Under the same assumption with Lemma 4.4, for every $\alpha \in (0,1)$, there exists constants $\varepsilon_*, \kappa_*, \rho_* > 0$ and $\theta \in (0,1/8]$, all depending at most on $n, N, \lambda, p, L, \alpha, \rho_0, \mu(\cdot), \omega(\cdot), V(\cdot), a(M), b$ and $\mathcal{M}$, such that the conditions
\[
\Psi(\rho) < \varepsilon_*, \quad \text{and} \quad C_\alpha(x_0,\rho) < \kappa_*(A_0)
\]
for all $\rho \in (0,\rho_*)$ with $B_\rho(x_0) \Subset \Omega$, imply
\[
\Psi(\theta^k\rho) < \varepsilon_*, \quad \text{and} \quad C_\alpha(x_0,\theta^k\rho) < \kappa_*(A_k)
\]
respectively, for every $k \in \mathbb{N}$.

**Proof.** We begin by choosing the constants. First, let
\[
\theta := \min \left\{ \left( \frac{1}{16n(n+2)} \right)^{1/(2-2\alpha)}, \frac{1}{\sqrt{4C_3}} \right\} \leq \frac{1}{8},
\]
and after reabsorbing the last term from the right-hand side on the left, we also obtain
\[
1 + |D\ell_{x_0,\rho}| \leq 2 (1 + |D\ell_{x_0,\theta\rho}|).
\]
with the constant $C_3$ determined in Lemma 3.3. In particular, the choice of $\theta = \theta(n, N, \lambda, L, a, M, \alpha) > 0$ fixes the constant $\delta = \delta(n, N, \lambda, L, a, M, \alpha) > 0$ from Lemma 3.2. Next, we fix an $\varepsilon_* = \varepsilon_*(n, N, \lambda, L, a, M, \alpha, \mu) > 0$ sufficiently small to ensure

$$\varepsilon_* \leq \frac{\theta^{n+2}}{16n(n+2)} \quad \text{and} \quad \mu^{1/2} \left( \sqrt{4\varepsilon_*} + \sqrt{4\varepsilon_*} \right) \leq \frac{\delta}{2}.$$  

Then, we choose $\kappa_* = \kappa_*(n, N, \lambda, L, a, M, \alpha, \mu, \omega(\cdot)) > 0$ so small that $\omega(\kappa_*) < \varepsilon_*$. Finally, we fix $\rho_* = \rho_*(n, N, \lambda, L, a, \rho_0, \mu(\cdot), \omega(\cdot), V(\cdot), a, b, M) > 0$ small enough to guarantee

$$\rho_* \leq \min\{\rho_0, \kappa_*^{1/(2-2\alpha)}, 1\}, \quad V(\rho_*) < \varepsilon_* \quad \text{and} \quad \left\{ \left( a \sqrt{n(n+2)} \right)^q + b^q \right\} \rho_*^{2\alpha} < 4\varepsilon_*.$$  

Now we prove the assertion $(A_k)$ by induction. We assume that we have already established $(A_k)$ up to some $k \in \mathbb{N} \cup \{0\}$. We begin with proving the first part of the assertion $(A_{k+1})$, that is, the one concerning $\Psi(\theta^{k+1})$. First, using (3.12) with $\ell \equiv u_{x_n, \theta^k \rho}$, we obtain

$$|D\ell_{x_n, \theta^k \rho}|^2 \leq \frac{n(n+2)}{\theta^k \rho^2} \int_{B_{\rho^k}(x_0)} |n - u_{x_n, \theta^k \rho}|^2 dx = n(n+2)(\theta^k \rho)^{2n-2} C_n(x_0, \theta^k \rho) \leq n(n+2) \rho_*^{2n-2} \kappa_*.$$  

Thus, the assumption $(A_k)$, the choice of $\kappa_*$ and $\rho_*$, and the above estimate infer

$$\Psi_{\ast}(\theta^k \rho) \leq \Psi(\theta^k \rho) + \omega(C_n(x_0, \theta^k \rho) + V(\theta^k \rho) + (a^q |D\ell_{x_n, \theta^k \rho}|^q + b^q)(\theta^k \rho)^q \leq \varepsilon_* + \omega(\kappa_*) + V(\rho_*) + \left( \left( a \sqrt{n(n+2)} \kappa_* \right)^q + b^q \right) \rho_*^{2\alpha} < 4\varepsilon_*.$$  

Now it is easy to check that our choice of $\varepsilon_*$ implies that the smallness condition assumptions (i) and (ii) in Lemma 3.1 are satisfied on the level $\theta^k \rho$, that is, we have

$$\mu^{1/2} \left( \sqrt{\Psi_{\ast}(\theta^k \rho)} + \sqrt{\Psi(\theta^k \rho)} \right) \leq \mu^{1/2} \left( \sqrt{4\varepsilon_*} + \sqrt{4\varepsilon_*} \right) \leq \frac{\delta}{2},$$  

and

$$\Psi(\theta^k \rho) < \varepsilon_* < \frac{\theta^{n+2}}{4n(n+2)}.$$  

Furthermore, we have the smallness condition assumption (iii), that is,

$$\gamma(\theta^k \rho) = \left[ \Psi_{\ast}^{q/2}(\theta^k \rho) + \delta^{-q}(\theta^k \rho)^q (a |D\ell_{x_n, \theta^k \rho}| + b)^q \right]^{1/q} \leq 1.$$  

To check (4.27), first note that $\Psi_{\ast}(\theta^k \rho) < 1$ holds by the estimate (4.24) and the choice of $\varepsilon_*$. This implies

$$\Psi_{\ast}^{q/2}(\theta^k \rho) \leq \Psi_{\ast}^{q/2}(\theta^k \rho) \leq \sqrt{4\varepsilon_*} \leq \frac{\delta}{4}.$$  

Next, using (4.23) and $\rho_*^{2\alpha} \geq 1$, we obtain

$$\delta^{-q}(\theta^k \rho)^q (a |D\ell_{x_n, \theta^k \rho}| + b)^q \leq \delta^{-q} \rho_*^{2\alpha} (a \sqrt{n(n+2)} \kappa_* \rho_*^{2\alpha} + b)^q \leq \delta^{-q} \rho_*^{2\alpha} 2^{q/p} \left\{ \left( a \sqrt{n(n+2)} \right)^q + b^q \right\}.$$  

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Then the choice of $\rho_*$ and $\varepsilon_*$ imply
\[
\delta^{-q}(\theta^k)^q(a|D\ell|_{\rho_*,\theta^k} + b)^q \leq \delta^{-q/2} \varepsilon_*^q \leq 2^{-4+q/n}\delta^{-q} \leq \frac{\delta}{8},
\]
(4.29)
Therefore combining (4.28) and (4.29), we have (4.27). We may thus apply Lemma 4.4 with the radius $\theta^k$ instead of $\rho$, which yields
\[
\Psi(\theta^{k+1}) \leq C_3\theta^2\Psi(\theta^k) < 4C_3\theta^2\varepsilon_* \leq \varepsilon_*,
\]
by the choice of $\theta$. We have thus established the first part of the assertion ($A_{k+1}$) and it remains to prove the second one, that is, the one concerning $C_\alpha(x_0,\theta^{k+1})$. For this aim, we first compute
\[
\frac{1}{(\theta^k)^2}\int_{B_{\theta^k}(x_0)} |u - \ell_{x_0,\theta^k}|^2 \, dx \leq (1 + |D\ell_{x_0,\theta^k}|^2)^2\Psi(\theta^k) \leq 2\varepsilon_* + 2\varepsilon_*|D\ell_{x_0,\theta^k}|^2
\]
where we used the assumption ($A_k$) in the last step. Since $\ell_{x_0,\theta^k}(x) = u_{x_0,\theta^k} + D\ell_{x_0,\theta^k}(x - x_0)$, we can estimate
\[
C_\alpha(x_0,\theta^{k+1}) \leq (\theta^{k+1})^{-2\alpha}\int_{B_{\theta^{k+1}}(x_0)} |u - u_{x_0,\theta^k}|^2 \, dx 
\leq 2(\theta^{k+1})^{-2\alpha}\left[ \int_{B_{\theta^{k+1}}(x_0)} |u - \ell_{x_0,\theta^k}|^2 \, dx + |D\ell_{x_0,\theta^k}|^2(\theta^{k+1})^2 \right] 
\leq 2(\theta^{k+1})^{-2\alpha}\left[ \theta^{-n}\int_{B_{\theta^k}(x_0)} |u - \ell_{x_0,\theta^k}|^2 \, dx + |D\ell_{x_0,\theta^k}|^2(\theta^{k+1})^2 \right] 
\leq 4(\theta^k)^{-2\alpha}[\varepsilon\theta^{-n+2\alpha} + |D\ell_{x_0,\theta^k}|^2(\varepsilon\theta^{-n+2\alpha} + \theta^2)]
\]
Using (4.23) and recalling the choice of $\rho_*$, $\varepsilon_*$ and $\theta$, we deduce
\[
C_\alpha(x_0,\theta^{k+1}) \leq 4\rho_*^{-2\alpha}[\varepsilon\theta^{-n+2\alpha} + n(n+2)\kappa_\alpha\rho_*^{-2\alpha}(\varepsilon\theta^{-n+2\alpha} + \theta^2)] 
\leq \frac{1}{4}\kappa_* + \frac{1}{2}\kappa_* \leq \kappa_*,
\]
This proves the second part of the assertion ($A_{k+1}$) and finally we conclude the proof of the lemma. \qed

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