THE MONOMIAL CONJECTURE AND ORDER IDEALS

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Abstract. In this article first we prove that a special case of the order ideal conjecture, originating from the work of Evans and Griffith in equicharacteristic, implies the monomial conjecture due to M. Hochster. We derive a necessary and sufficient condition for the validity of this special case in terms certain syzygies of canonical modules of normal domains possessing free summands. We also prove some special cases of this observation.

The main focus of this paper is to establish a relation between the monomial conjecture due to Hochster [Ho1] and the assertion on order ideals of minimal generators of syzygies of modules of finite projection dimension, introduced and proved by Evans and Griffith ([E-G1], [E-G2]) for equicharacteristic local rings. The monomial conjecture asserts that given a local ring $R$ and a system of parameters $x_1, \ldots, x_n$ of $R$, $(x_1 \ldots x_n)^t \not\in (x_1^{t+1}, \ldots, x_n^{t+1})$. In [Ho1] Hochster proved this assertion for the equicharacteristic case and proposed it as a conjecture for local rings of all characteristics. He also showed ([Ho1]) that this conjecture is equivalent to the direct summand conjecture for module-finite extension of regular rings. From the equational point of view both these conjectures claim that the polynomial equation $(X_1 \ldots X_n)^t - \sum_{i=1}^n Y_i X_i^{t+1} = 0$ ($X_i, Y_j$s are variables) cannot have a solution $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ in any noetherian ring, unless the height of $(x_1, \ldots, x_n)$ is less than $n$. In the early eighties in order to prove their syzygy theorem ([E-G1], [E-G2]) Evans and Griffith proved an important aspect of order ideals of syzygies over equicharacteristic local rings. We would like to generalize and state their result as a conjecture on arbitrary local rings in the following way.

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Order ideal conjecture. Let \((R, m)\) be a local ring. Let \(M\) be a finitely generated module of finite projective dimension over \(R\) and let \(\text{Syz}^i(M)\) denote its \(i\)th syzygy for \(i > 0\). If \(\beta\) is a minimal generator of \(\text{Syz}^i(M)\), then the order ideal \(O_{\text{Syz}^i(M)}(\beta)(= \{f(\beta) | f \in \text{Hom}_R(\text{Syz}^i(M), R)\})\) has grade \(\geq i\).

For their proof of the syzygy theorem Evans and Griffith actually needed a particular case of this conjecture: \(M\) is locally free on the punctured spectrum and \(R\) is regular local. They reduced the proof of the above conjecture for this special case to the validity of the improved new intersection conjecture (also introduced by them) and proved its validity over equicharacteristic local rings by using big Cohen-Macaulay modules. In the mid-eighties Hochster ([Ho3]) proposed a new conjecture that is deeply homological in nature—the canonical element conjecture. In this conjecture Hochster assigns a canonical element \(\eta_R\) to every local ring \(R\) and asserts that \(\eta_R \neq 0\). He proved this conjecture for the equicharacteristic case. In the same paper Hochster ([Ho2]) showed that the canonical element conjecture is equivalent to the monomial conjecture and it implies the improved new intersection conjecture due to Evans and Griffith. Later this author proved the reverse implication ([D1]). Thus, the monomial conjecture implies a special case of the order ideal conjecture: the case when \(M\) is locally free on the punctured spectrum of \(R\).

In this article we would prove that the validity of the order ideal conjecture over regular local rings (actually a special case of it) implies the monomial conjecture. First we would like to propose the following definition.

Consider a finitely generated module \(M\) of finite projective dimension over a local ring \(R\) with a minimal free resolution \((F_\bullet, \beta_\bullet)\). Let \(\text{Syz}^i(M)\) denote the \(i\)th syzygy of \(M\) in \(F_\bullet\). We say that for \(i \geq 1\), \(\text{Syz}^i(M)\) satisfies the property \((0)\) or simply \((0)\) if for every minimal generator \(\alpha\) of \(\text{Syz}^i(M)\), the ideal \(I\) generated by the entries of \(\alpha\) in \(F_{i-1}\) has grade \(\geq i\). In our main theorem (Theorem 1.4) we prove the following:

**Theorem.** The monomial conjecture is valid for all local rings if for every almost complete intersection ideal \(J\) of height \(d\) in any regular local ring \((R, m)\), \(d < \) dimension of \(R\), \(\text{Syz}^{d+1}(R/J)\) satisfies \((0)\).

The idea involved in the proof of this theorem evolved gradually through our work in ([D5], [D7], [D-G]). Actually our assertion is more specific than what is stated above. We
Corollary (Notations as above). The monomial conjecture is valid for all local rings if for every almost complete intersection ideal \( J \) of height \( d \) in a regular local ring \( (R, m) \), \( K_{d+1}(J; R) \otimes k \to \text{Tor}_{d+1}^R(R/J, k) \) is the 0-map where \( K_\bullet(J; R) \) is the Koszul complex corresponding to \( J \) in \( R \) and \( k = R/m \).

Next we prove a proposition (Prop. 1.5) that replaces the previous assertion involving \( J \) with the canonical module \( \Omega \) of \( R/J \). Let \( J = (x_1, \ldots, x_d, \lambda) \) and let \( x = (x_1, \ldots, x_d) \) denote the ideal generated by an \( R \)-sequence. As a corollary we derive:

The monomial conjecture is valid if \( \text{Tor}_d^R(k, R/x) \to \text{Tor}_d^R(k, R/(x + \Omega)) \) is the zero map (equivalently \( \text{Tor}_d^R(k, \Omega) \to \text{Tor}_d^R(k, R/x) \) is non-zero).

The following proposition (prop. 1.6) provides a characterization of the above statement in terms of syzygies of canonical modules.

Proposition (Notations as above). If \( K_{d+1}(J; R) \otimes k \to \text{Tor}_{d+1}^R(R/J, k) \) is the 0-map then \( \text{Syz}_d^d(\Omega) \) (minimal) has a free summand. Conversely, if \( I \) is an ideal of \( R \) of height \( d \) such that \( R/I \) satisfies the Serre-condition \( S_2 \) and its canonical module \( \Omega \) is such that \( \text{Syz}_d^d(\Omega) \) has a free summand, then \( \text{Tor}_d^R(\Omega, k) \to \text{Tor}_d^R(R/x, k) \) is non-zero, where \( x \) denotes the ideal generated by a maximal \( R \)-sequence in \( I \).

As a corollary to our next theorem we observe the following:

Let \( (R, m) \) be a equicharacteristic regular local ring. Let \( I \) be an ideal in \( R \) of codimension \( d \) and let \( \Omega \) denote the canonical module of \( R/I \). Then \( \text{Syz}_d^d(\Omega) \) has a free summand.

Remark. The above assertion is also valid in the graded equicharacteristic case via the same mode of proof.

In this connection let us recall that for any finitely generated module \( M \) of grade \( g \), no \( \text{Syz}_i^i(M) \) can have a free summand for \( i < g \) ([D2]).

Our next theorem describes the special cases where we can at present prove that \( \text{Syz}_d^d(\Omega) \) possesses a free summand.

Theorem. Let \( (R, m, k) \) be a regular local ring in mixed characteristic \( p > 0 \) and let \( I \) be an ideal of height \( d \) in \( R \). Let \( x \) denote the ideal generated by a maximal \( R \)-sequence contained in \( I \) and let \( \Omega = \text{Hom}_R(R/I, R/x) \) denote the canonical module of \( R/I \). Then
\( \text{Tor}^R_d(\Omega, k) \to \text{Tor}^R_d(R/\mathfrak{x}, k) \) is non-zero (equivalently \( \text{Syz}^d(\Omega) \) possesses a free summand) in the following cases: 1) \( \Omega \) is \( S_3 \) and 2) the mixed characteristic \( p \) is a non-zero-divisor on \( \text{Ext}^{d+1}(R/I, R) \).

In our last proposition (1.7) we derive a sufficient condition for \( \text{Syz}^d(\Omega) \) to possess a free summand. We prove that if \( \Omega' \) denotes the lift of \( \Omega \) in \( R \) via \( R \to R/\mathfrak{x} \) and if at least one of \( x_1, \ldots, x_d \) is contained in \( m\Omega' \), then \( \text{Syz}^d(\Omega) \) has a free summand.

Throughout this work “local” means noetherian local. Over the years different aspects of the monomial conjecture have been studied, special cases have been proved and new equivalent forms have been introduced (see [Bh], [Br-H], [D1], [D2], [D3], [D4], [D5], [D6], [D-G], [Go], [He], [K], [O], [R3], [V]). Statements of four equivalent forms of this set of conjectures, as proposed by Hochster in [Ho3] are given below for the convenience of the reader.

A. Direct Summand Conjecture (DSC).

Let \( R \) be a regular local ring, and let \( i : R \hookrightarrow A \) be a module-finite extension of \( R \). Then \( i \) splits as an \( R \)-module map.

B. Canonical Element Conjecture (CEC).

Let \( A \) be a local ring of dimension \( n \) with maximal ideal \( m \) and residue field \( k \). Let \( S_i \) denote the \( i \)th syzygy of \( k \) in a minimal resolution of \( k \) over \( A \), and let \( \theta_n : \text{Ext}^n_A(k, S_n) \to H^n_m(S_n) \) denote the direct limit map. Then \( \theta_n \) (class of the identity map on \( S_n \)) \( \neq 0 \).

C. Improved New Intersection Conjecture (INIC). Let \( A \) be as before. Let \( F_\bullet \) be a complex of finitely generated free \( A \)-modules,

\[
F_\bullet : 0 \to F_s \to F_{s-1} \to \cdots \to F_1 \to F_0 \to 0,
\]

such that \( \ell(H_i(F_\bullet)) < \infty \) for \( i > 0 \) and \( H_0(F_\bullet) \) has a minimal generator annihilated by a power of the maximal ideal \( m \). Then \( \dim A \leq s \).

D. Let \( (A, m, k = A/m) \) be a local ring of dimension \( n \) which is a homomorphic image of a Gorenstein ring. Let \( \Omega \) denote the canonical module of \( A \). Then the direct limit map \( \text{Ext}^n_A(k, \Omega) \to H^n_m(\Omega) \) is non-null.

Recall that for any equivalent version of this set of conjectures one can assume that the local ring \( A \) is a complete local (normal) domain. We have \( A = R/\mathring{P} \), where \( R \) is a
complete regular local ring. Let \( S = R/y \), where \( y \) is the ideal generated by a maximal \( R \)-sequence contained in \( \tilde{P} \). Then \( A = S/P \), \( P = \tilde{P}/y \). Let \( \Omega \) denote the canonical module for \( A \); \( \Omega \) can be identified with \( \text{Hom}_{S}(A, S) \)—an ideal of \( S \). Let \( E \) denote the injective hull of the residue field of \( S \) (resp. \( R \)) over \( S \) (resp. \( R \)). For any \( S \)-module (\( R \)-module) \( T \), we write \( T^{\vee} \) to denote \( \text{Hom}_{S}(T, E) \) (\( \text{Hom}_{R}(T, E) \)) and \( \dim T \) to denote the Krull dimension of \( T \). These notations will be utilized throughout this article.

Section 1

Our first proposition is an observation due to Strooker and St"uckrad on a characterization of the monomial conjecture ([Str-St"u]). This author independently proved a similar characterization for the direct summand conjecture ([D5]). Since our main focus is on a proof of the monomial conjecture we provide a proof of this result here.

1.1 Proposition (Th. [Str-St"u]). With notations as above, \( A \) satisfies MC if and only if for any system of parameters \( x_{1}, . . . , x_{n} \) of \( S \), \( \Omega \) is not contained in \( (x_{1}, . . . , x_{n}) \).

Proof. Let \( \xi_{1}, . . . , \xi_{n} \) be a system of parameters of \( A \). We can lift it to a system of parameters \( x_{1}, . . . , x_{n} \) for \( S \) such that \( \text{Im}(x_{i}) = \xi_{i}, 1 \leq i \leq n \). Conversely, any system of parameters for \( S \) is a system of parameters for \( A \). Let us write \( \underline{x} = (x_{1}, . . . , x_{n}) \) and \( \underline{\xi} = (\xi_{1}, . . . , \xi_{n}) \). The monomial conjecture for \( A \) is equivalent to the assertion that, for every system of parameters \( \xi_{1}, . . . , \xi_{n} \) of \( A \), the direct limit map \( \alpha : A/\xi \rightarrow H_{m}(A) \) is non-null [Ho1]. Because \( S \) is a complete intersection, the direct limit map \( \beta : S/\underline{x} \rightarrow H_{m}(S) \) is non-null (\( m_{S} = \text{maximal ideal of } S \)). We have the following commutative diagram

\[
\begin{array}{ccc}
S/\underline{x} & \xrightarrow{\beta} & H_{m}(S) \\
\downarrow{\pi} & & \downarrow{\gamma} \\
A/\underline{\xi} & \xrightarrow{\alpha} & H_{m}(A)
\end{array}
\]

where \( \pi \) is induced by the natural surjection \( \eta : S \rightarrow A \) and \( \gamma = H_{m}(\eta) \). This implies that \( \alpha \) is non-null \( \Leftrightarrow \alpha \circ \eta = \gamma \circ \beta \) is non-null \( \Leftrightarrow H_{m}(A)^{\vee} \rightarrow (S/\underline{x})^{\vee} \) is non-null \( \Leftrightarrow \text{Im}(\Omega \rightarrow S/\underline{x}) \) is non-null \( \Leftrightarrow \Omega \not\subseteq \underline{x} \) (recall that, by local duality, \( H_{m}(A)^{\vee} = \Omega \)).

In our next proposition we deduce the validity of the monomial conjecture for all local rings from the validity of the same for all local almost complete intersections. We prove the following:
1.2 Proposition. The monomial conjecture is valid for all local rings if and only if it holds for all local almost complete intersections.

Proof. Suppose MC holds for all local almost complete intersections. Let \( A \) be a complete local domain. Then we have \( A = R/\bar{P} \), where \( R \) is a complete regular local ring. We can choose \( y_1, \ldots, y_r \)—a maximal \( R \) sequence in \( P \) in such a way that \( \bar{P} R_{\bar{P}} = (y_1, \ldots, y_r) R_{\bar{P}} \).

Write \( S = R/y \), where \( y = (y_1, \ldots, y_r) \) and \( P = \bar{P}/y \). Then \( S \) is a complete intersection, \( A = S/P \), \( \dim S = \dim A \), and \( PS_P = 0 \). Let \( \Omega = \text{Hom}_S(A, S) \), the canonical module of \( A \). Consider the primary decomposition of \((0)\) in \( S \): \( 0 = P \cap q_2 \cap \cdots \cap q_h \), where \( q_i \) is \( P_i \)-primary and \( \text{height} \ P = 0 \) for \( 2 \leq i \leq h \). It can be checked easily that \( \Omega = q_2 \cap \cdots \cap q_h \). Choose \( \lambda \in P - \bigcup_{i \geq 2} P_i \). Then \( \Omega = \text{Hom}(S/\lambda S, S) \), and \( S/\lambda S \) is an almost complete intersection. Since \( S/\lambda S \) satisfies MC by assumption, it follows from the above proposition that \( \Omega \) is not contained in the ideal generated by any system of parameters in \( S \). Hence, by Prop. 1.1, \( A \) satisfies MC.

Now we reduce the assertion in the monomial conjecture to a length inequality between \( \text{Tor}_0 \) and \( \text{Tor}_1 \) on regular local rings.

1.3 Proposition. The monomial conjecture is valid for all local rings if and only if for every regular local ring \( R \) and for every pair of ideals \((I, J)\) of \( R \) such that i) \( I \) is a complete intersection, ii) \( J \) is an almost complete intersection (i.e., \( J \) is minimally generated by \((\text{height} \ J + 1)\) elements), iii) \( \text{height} \ I + \text{height} \ J = \dim R \) and iv) \((I + J)\) is primary to the maximal ideal of \( R \), the following length inequality holds:

\[
\ell(R/(I + J)) > \ell(\text{Tor}_1^R(R/I, R/J)).
\]

Proof. First assume that every pair of \((I, J)\), as in the statement of our theorem, satisfies the length inequality:

\[
\ell(R/(I + J)) > \ell(\text{Tor}_1^R(R/I, R/J)).
\]

By Prop. 1.2, we can assume that \( A \) is an almost complete intersection ring of the form \( S/\lambda S \), where \( S \) is a complete intersection and \( \dim S = \dim A \). Let \( \Omega = \text{Hom}(S/\lambda S, S) \) denote the canonical module for \( A \). We consider the short exact sequence

\[
0 \to S/\Omega \xrightarrow{f} S \to S/\lambda S \to 0,
\] (1)
where \( f(\mathfrak{I}) = \lambda \). Let \( x''_1, \ldots, x''_n \) be a system of parameters for \( A \). We can lift \( x''_1, \ldots, x''_n \) to \( x'_1, \ldots, x'_n \) in \( S \) in such a way that \( \{x'_1, \ldots, x'_n\} \) form a system of parameters in \( S \). Let \( \mathfrak{z'} = (x'_1, \ldots, x'_n) \) and \( \mathfrak{z} = (x_1, \ldots, x_n) \). Tensoring (1) with \( S/\mathfrak{z}' \) yields the following exact sequence

\[
0 \to \text{Tor}_1^S(S/\mathfrak{z}', S/\lambda S) \to S/(\Omega + \mathfrak{z}') \xrightarrow{f} S/\mathfrak{z}' \to S/(x' + \lambda S) \to 0,
\]

where \( f \) is induced by \( f \). Then we have the following

\[
\Omega \not\subset \mathfrak{z}' \iff \ell(S/(\mathfrak{z}' + \lambda S)) > \ell(\text{Tor}_1(S/\mathfrak{z}', S/\lambda S)).
\]

As in the proof of the previous proposition, let \( R \) be a complete regular local ring mapping onto \( S \). Now lift \( x'_1, \ldots, x'_n \) to an \( R \)-sequence \( x_1, \ldots, x_n \) in \( R \). Write \( I = \mathfrak{z} \) and \( J = (y_1, \lambda) \). Then (3) translates to \( \ell(R/(I + J)) > \ell(\text{Tor}_1(R/I, R/J)) \), as required in our statement.

For the converse part of the theorem, write \( I = (x_1, \ldots, x_n) \) and \( J = (y_1, \ldots, y_r, \lambda) \), where \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_r \) form \( R \)-sequences such that \( n + r = \dim R \). Let \( S = R/(y_1, \ldots, y_r) \) and \( A = S/\lambda S \), and let \( x'_i = \text{Im}(x_i) \) in \( S \) for \( 1 \leq i \leq n \). Write \( \mathfrak{z}' = (x'_1, \ldots, x'_n) \) and \( \mathfrak{z} = (x_1, \ldots, x_n) \). Since \( I + J \) is primary to the maximal ideal and since both \( R/\mathfrak{z} \) and \( S \) are complete intersections, it follows from [Se] that \( \text{Tor}_i^R(R/\mathfrak{z}, S) = 0 \) for \( i > 0 \). This implies that \( \text{Tor}_i^R(R/I, R/J) = \text{Tor}_i^S(S/\mathfrak{z}', A) \) for \( i \geq 0 \). Let \( \Omega = \text{Hom}_S(A, S) \), the canonical module for \( A \). Now (1)–(3) and the subsequent arguments complete the proof.

1.4. Our main theorem is the following.

**Theorem.** The monomial conjecture is valid for all local rings if for every almost complete intersection ideal \( J \) of height \( d \) in any regular local ring \( R \), \( d < \dim R \), \( \text{Syz}^{d+1}(R/J) \) satisfies (0).

**Proof.** By Prop.1.3 it is enough to prove the following.

**Theorem.** Let \( (R, m) \) be a regular local ring and let \( I, J \) be two ideals of \( R \) such that i) \( R/I \) is Gorenstein and \( J \) is an almost complete intersection, ii) \( I + J \) is \( m \)-primary
and \( iii \) height \( I + \text{height} J = \dim R \). Let \( n = \text{height} I \) and \( d = \text{height} J \). Suppose that \( d < \dim R \) and \( \text{Syz}^{d+1}(R/J) \) satisfies (0). Then

\[
\ell(R/(I+J)) > \ell(\text{Tor}^R_1(R/I, R/J)).
\]

**Proof.** Let \( J = (y_1, \ldots, y_d, \lambda) \) where \( \{y_1, \ldots, y_d\} \) form an \( R \)-sequence. Let \( (K_\bullet, \gamma_\bullet) = K_\bullet(y_1, \ldots, y_d; R) \) denote the Koszul complex corresponding to \( y_1, \ldots, y_d, \lambda \) and let \( H = H_1(y_1, \ldots, y_d, \lambda; R) \). Let \( L_\bullet \) be a minimal free resolution of \( H \) and let \( \psi_\bullet : L_\bullet \to K_\bullet(1) \) be a lift of \( H \hookrightarrow \text{coker} \gamma_2 = G \). We have the following commutative diagram

\[
L_\bullet : \quad \to R^{r_{d+1}} \to R^{r_d} \to R^{r_{d-1}} \to \cdots \to R^{r_0} \to H \to 0
\]

\[
K_\bullet(1) : \quad 0 \to R \xrightarrow{\gamma_{d+1}} R^{d+1} \to \cdots \to R^{d+1} \to G \to 0
\]

The mapping cone of \( \psi_\bullet \) is a free resolution of \( J \) and thereby provides a free resolution of \( R/J \) from which a minimal resolution \( (F_\bullet, \beta_\bullet) \) of \( R/J \) can be extracted.

**Claim.** \( \text{Im} \psi_d = R \).

**Proof of the Claim.** If not, then \( \text{Im} \psi_d \subset m \). Then the copy of \( R = K_{d+1} \) would be a free summand of \( F_{d+1} \). Let \( \alpha \) be the image of \( (1, 0) \) in \( \text{Syz}^{d+1}(R/J) \subset F_d \). Since \( \beta_{d+1}(1, 0) \)

\[
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_d \\
\lambda
\end{pmatrix}
\]

is a part or whole of

\[
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_d \\
\lambda
\end{pmatrix}
\]

the height of the ideal generated by the entries of \( \alpha \) must be less than \( (d + 1) \). This contradicts our hypotheses that \( \text{Syz}^{d+1}(R/J) \) satisfies (0) and hence \( \text{Im} \psi_d = R \).

This means that if \( \eta_\bullet : K_\bullet \to F_\bullet \) lifts the identity map on \( R/J \), then \( \eta_{d+1}(R) \subset m F_{d+1} \).

Recall that the spectral sequences \( \{\text{Ext}_R^p(R/J, \text{Ext}_R^q(R/I, R))\} \) and \( \{\text{Ext}_R^i(\text{Tor}_j^R(R/J, R/I), R)\} \) converge to the same limit when \( p + q = i + j \). Since \( R/I \) is Gorenstein, we have \( \text{Ext}_R^i(R/I, R) = 0 \) for \( i \neq n \) and \( \text{Ext}_R^n(R/I, R) \simeq R/I \). Since \( \ell(R/(I+J)) < \infty \) and \( R \) is regular local of dimension \( (n+d) \), each of the above spectral sequence degenerates. Hence

\[
\text{Ext}_R^{n+d}(\text{Tor}_1^R(R/J, R/I), R) \simeq \text{Ext}_R^{d+1}(R/J, R/I).
\]
By (1) and local duality we have
\[ \ell(\text{Tor}_1^R(R/I, R/J)) = \ell\left(\text{Ext}_{R}^{d+1}(R/J, R/I)\right) \tag{2} \]

We also note that since \( n + d = \dim R \) and \( \ell(R/(I+J)) < \infty \), \( \text{Ext}_R^i(H, R/I) = 0 \) for \( i < d \).

The mapping cone of \( \psi \), leads to the following exact sequence:
\[ 0 \to K \to (\text{a free resolution of } R/J \text{ from the mapping cone of } \psi) \to L(-2) \to 0. \]

Since \( \text{Ext}_{R}^{d-1}(H, R/I) = 0 \) applying \( \text{Hom}_R(\cdot, R/I) \) to this sequence we obtain the following short exact sequence
\[ 0 \to \text{Ext}_{R}^{d+1}(R/J, R/I) \to H^{d+1}(y_1, \ldots, y_d, \lambda; R/I) = R/(I+J). \]

This inclusion cannot be an isomorphism since \( \eta_{d+1}(R) \subset mF_{d+1} \).

Thus, from (2), \( \ell(R/(I+J)) > \ell(\text{Tor}_1^R(R/I, R/J)) \) and the proof of our theorem is complete.

We have the following corollary from the above proof.

**Corollary.** The monomial conjecture is valid if for every almost complete intersection ideal \( J \) of height \( d \) in a regular local ring \((R, m, k = R/m)\), \( K_{d+1}(J; R) \otimes k \to \text{Tor}_{d+1}(R/J, k) \) is the 0-map where \( K_\bullet(J; R) \) is the Koszul complex corresponding to \( J \) in \( R \).

The proof of this corollary follows from our observation after the diagram and the claim in the proof of the above theorem.

1.5. In our next proposition we study the assertion in the above corollary.

**Proposition.** Let \((R, m)\) be a local ring of \( \dim n \). Let \( \{x_1, \ldots, x_d\} \subset m \) be a regular sequence on \( R \) and let \( \lambda \) be a 0-divisor and not a parameter on \( R/x \), \( x = \) the ideal generated by \( x_1, \ldots, x_d \). Let \( \Omega = \text{Hom}(R/(x, \lambda), R/x) \). Let \( K_\bullet(x, \lambda; R) = K_\bullet(x_1, \ldots, x_d, \lambda; R) \) and \( K_\bullet(x; R) \) denote the Koszul complexes corresponding to \( x_1, \ldots, x_d, \lambda \) and \( x_1, \ldots, x_d \) respectively. Let \((L_\bullet, c_\bullet)\) be a minimal free resolution of \( \Omega \). Let \( \psi_\bullet : L_\bullet \to K_\bullet(x, \lambda; \gamma_\bullet)(+1) \) and \( \phi_\bullet : L_\bullet \to K_\bullet(x; \delta_\bullet) \) be lifts of \( \Omega = H_1(x, \lambda; R) \hookrightarrow G = \text{Coker} \gamma_2 \) and \( \Omega \hookrightarrow R/x \) respectively. Then \( \text{Im} \psi_d = R \) if and only if \( \text{Im} \phi_d = R \).
Proof. We have the following commutative diagrams:

\[
\begin{array}{cccccccc}
L_\bullet : & \rightarrow & R^{d+1} & \rightarrow & R^d & \rightarrow & R^{d-1} & \rightarrow & \cdots & \rightarrow & R^0 & \rightarrow & H_1 = \Omega & \rightarrow & 0 \\
& & \downarrow \psi_d & & \downarrow \psi_{d-1} & & & & \downarrow & & \downarrow & & \downarrow & & \\
K_\bullet(x, \lambda; R)(+1) : & 0 & \rightarrow & R & \rightarrow & R^{d+1} & \rightarrow & \cdots & \rightarrow & R^{d+1} & \rightarrow & G & \rightarrow & 0 & \\
\end{array}
\]

(1)

and

\[
\begin{array}{cccccccc}
L_\bullet : & \rightarrow & R^{d+1} & \rightarrow & R^d & \rightarrow & R^{d-1} & \rightarrow & \cdots & \rightarrow & R^0 & \rightarrow & \Omega & \rightarrow & 0 \\
& & \downarrow \phi_d & & \downarrow \phi_{d-1} & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R & \rightarrow & R^d & \rightarrow & \cdots & \rightarrow & R & \rightarrow & R/x & \rightarrow & 0 & \\
\end{array}
\]

(2)

Suppose that \(\text{Im} \psi_d = R\). Let \(\eta_\bullet : K_\bullet(x, \lambda; R)(+1) \rightarrow K_\bullet(x; R)\) denote the natural surjection. Since \(\eta_d = \text{Id}_R\), \(\eta_d \cdot \psi_d : R^d \rightarrow R\) is surjective. Since \(\eta_\bullet \cdot \psi_\bullet : L_\bullet \rightarrow K_\bullet(x; R)\) lifts \(\Omega \hookrightarrow R/x\), \(\eta_\bullet \psi_\bullet\) is homotopic to \(\phi_\bullet\). Hence \(\phi_d(R^d) = R\). Conversely let \(\text{Im} \phi_d = R\).

Let \(\lambda : K_\bullet(x; R) \rightarrow K_\bullet(x; R)\) lift the multiplication by \(\lambda\) on \(R/x\). Since \(\lambda \Omega = 0\) in \(R/x\), \(\lambda \phi_\bullet\) is homotopic to the 0-map: \(L_\bullet \rightarrow K_\bullet(x; R)\). Let \(h_\bullet = (h_i)_{0 \leq i \leq d-1}\) denote corresponding homotopies; \(h_i : R^{d_i} \rightarrow R^d\). Since \(K_\bullet(x, \lambda; R)\) is the mapping cone of \(\lambda : K_\bullet(x; R) \rightarrow K_\bullet(x; R)\), we can define \(\psi'_\bullet : L_\bullet \rightarrow K_\bullet(x; \lambda; R)\), lifting \(\Omega = H_1 \hookrightarrow G\), in the following way: \(\psi'_d(R^{d_i}) = (\phi_i, h_i)\). Then \(\psi'_d(R^d) = R\). Since \(\psi'_\bullet, \psi_\bullet\) both lift the inclusion \(\Omega \hookrightarrow G\), they are homotopic and hence \(\psi_d(R^d) = R\).

Corollary (with notations as above). Let \(J = (x_1, \ldots, x_d, \lambda)\) be an almost complete intersection ideal in a regular local ring \((R, m, k)\) and let \(\Omega\) denote the canonical module of \(R/J\). Then \(K_{d+1}(J; R) \otimes k \rightarrow \text{Tor}_{d+1}^R(R/J, k)\) is the 0-map if and only if \(\text{Tor}_{d}^R(R/x, k) \rightarrow \text{Tor}_{d}^R(R/(x + \Omega), k)\) is the 0-map or equivalently \(\text{Tor}_{d}^R(\Omega, k) \rightarrow \text{Tor}_{d}^R(R/x, k)\) is non-zero.

1.6. The following proposition is partially a consequence of the above proposition.

Proposition (Notations as in the previous corollary). If \(K_{d+1}(J; R) \otimes k \rightarrow \text{Tor}_{d+1}^R(R/J, k)\) is the 0-map then \(\text{Syz}_{d}^R(\Omega)\) (minimal) has a free summand. Conversely, if \(I\) is an ideal of \(R\) of height \(d\) such that \(R/I\) is \(S_2\) and its canonical module \(\Omega\) is such that \(\text{Syz}_{d}^R(\Omega)\) has a free summand, then \(\text{Tor}_{d}^R(\Omega, k) \rightarrow \text{Tor}_{d}^R(R/x, k)\) is non-zero, where \(x\) denotes the ideal generated by a maximal \(R\)-sequence in \(I\).
Proof. Let us assume that \( K_{d+1}(J; R) \otimes k \to \operatorname{Tor}_{d+1}^R(R/J, k) \) is the 0-map. This implies, by diagram (1) in the proof of the above proposition, that \( \psi_d(L_d) = R \). Then we can write \( L_d = R \bigoplus F \) where \( F = \ker \psi_d \). Hence, from the commutativity of the diagram (1) in proposition (1.5), it follows that \( c_{d+1}(L_{d+1}) \subset F \) and this implies that \( \text{Syz}^d(\Omega) \) has a free summand.

Conversely let \((L_\bullet, c_\bullet)\) denote a minimal free resolution of \( \Omega \). If \( \text{Syz}^d(\Omega) \) has a free summand then \( L_d \) has a free generator \( e \) such that \( c_{d+1}^*(e^*) = 0 \) \((-^* = \operatorname{Hom}_R(-, R))\). Let \( L_\bullet^* \) denote the complex \( 0 \to L_0^* \to L_1^* \to \cdots \to L_d^* \to G \to 0 \). Since grade \( \Omega \geq d \), \( L_\bullet^* \) is exact. Then \( \operatorname{im} e^* \in \operatorname{Ext}^d(\Omega, R) \) is non-zero and is a minimal generator of \( G \). Let \( x_1, \ldots, x_d \) be a maximal \( R \) sequence contained in \( I \) and let \( x \) denote the ideal generated by them. Since \( R/I \) is \( S_2 \), we have \( \operatorname{Ext}^d(\Omega, R) \cong \operatorname{Hom}(\Omega, \Omega) \cong R/I \). Let \( G_{d+i} = \operatorname{Coker} c_{d+i}^* \) for \( i \geq 1 \). We consider the following short exact sequences:

\[
0 \to R/I \to G \to \operatorname{Im} c_{d+1}^* \to 0, \quad 0 \to \operatorname{Im} c_{d+1}^* \to L_{d+1}^* \to G_{d+1} \to 0,
\]

\[
0 \to \operatorname{Ext}^{d+1}(\Omega, R) \to G_{d+1} \to \operatorname{Im} c_{d+2}^* \to 0, \quad \ldots,
\]

\[
0 \to \operatorname{Im} c_{n-2}^* \to L_{n-2}^* \to \operatorname{Ext}^{n-2}(\Omega, R) \to 0.
\]

Since \( \Omega \) is \( S_2 \), it follows from the above sequences that grade \( \operatorname{Ext}^i(\Omega, R) \geq i + 2 \) for \( i > d \) and hence it can be easily checked that \( \Omega = \operatorname{Ext}^d_R(G, R) \equiv \operatorname{Ext}^d_R(R/I, R) \). Let \( \alpha \) denote the composite of \( R/\langle x \rangle \to R/I \cong \operatorname{Ext}^d_R(\Omega, R) \to G \), where \( \text{im} \alpha \in R/\langle x \rangle \) goes to \( \text{im} e^* \in \operatorname{Ext}^d(\Omega, R) \). Let \( \phi_\bullet^* : K_\bullet(\langle x \rangle; R) \to L_\bullet^* \) denote a lift of \( \alpha \). Then \( \phi_\bullet : L_\bullet \to K_\bullet(\langle x \rangle; R) \) lifts the injection \( \Omega = \operatorname{Ext}^d(G, R) \hookrightarrow R/\langle x \rangle \), where \( \phi_d(L_d) = R \). Thus, we obtain our required assertion.

In our next theorem we prove that for any ideal in a regular local ring the first part of the above proposition is implied by the validity of the order ideal conjecture.

**Theorem.** Let us assume that the order ideal conjecture is valid for a regular local ring \((R, m, k)\). Let \( I \) be any ideal of \( R \) of codimension \( d \) and let \( \Omega \) denote the canonical module of \( R/I \). Then \( \text{Syz}^d(\Omega) \) has a free summand.

Proof. Let \((F_\bullet, \phi_\bullet) : \to F_d \to F_{d-1} \to \cdots \to F_1 \to R \to 0\) denote a minimal free resolution of \( R/I \) over \( R \). Let \( F_\bullet^* \) denote the complex: \( 0 \to R \xrightarrow{\phi_{d+1}^*} F_{d+1}^* \to \cdots \to F_1^* \to G_d \to 0 \).
where $G_{d+i} = \text{Coker} \phi_{d+i}^*$, for $0 \leq i \leq 1$. Since height of $I = d$, we have $H_i(F_i^*) = 0$ for $i < d$ and $\Omega = \text{Ext}^d(R/I, R) \hookrightarrow G_d$. Let $L_\bullet$ be a minimal free resolution of $\Omega$ and let $\psi_\bullet : L_\bullet \to F_\bullet$ lift $\Omega \hookrightarrow G_d$. Then $\psi_d(L_d) = R$; for otherwise in the minimal free resolution $P_\bullet$ of $G_{d+1}$ obtained from the mapping cone of $\psi_\bullet$, $P_{d+1}$ would contain a copy of $R = L_0^*$ as a summand. Let $I = (y_1, \ldots, y_s)$ where $s = \ell(\text{Tor}_1^R(k, R/I))$. Since height of $I = d$ and $\phi_1^*(1) = (y_1, \ldots, y_s)$, this would contradict the order ideal conjecture. Hence $\psi_d(L_d) = R$. This implies, by a previous observation, that $\text{Syz}^d(\Omega)$ has a free summand.

**Corollary.** Let $(R, m)$ be an equicharacteristic regular local ring and let $I$ be an ideal of $R$ of codimension $d$. Let $\Omega$ denote the canonical module of $R/I$. Then $\text{Syz}^d(\Omega)$ has a free summand.

**Remark.** The above assertion is also valid in the graded equicharacteristic case via the same mode of proof.

Our next theorem describe the cases where we are at present able to prove that $\text{Syz}^d(\Omega)$ possesses a free summand.

**Theorem.** Let $(R, m, k)$ be a regular local ring in mixed characteristic $p > 0$ and let $I$ be an ideal of height $d$ in $R$. Let $x$ denote the ideal generated by a maximal $R$-sequence contained in $I$ and let $\Omega = \text{Hom}_R(R/I, R/x)$ denote the canonical module of $R/I$. Then $\text{Tor}_d^R(\Omega, k) \to \text{Tor}_d^R(R/x, k)$ is non-zero (equivalently $\text{Syz}^d(\Omega)$ possesses a free summand) in the following cases: 1) $\Omega$ is $S_3$ and 2) the mixed characteristic $p$ is a non-zero-divisor on $\text{Ext}^{d+1}(R/I, R)$.

**Proof.** Let $(F_\bullet, \phi_\bullet) : \to F_d \to F_{d-1} \to \cdots \to F_1 \to R \to 0$ denote a minimal free resolution of $R/I$ over $R$. Let $F^*_\bullet$ denote the complex: $0 \to R \xrightarrow{\phi_1^*} F_1^* \to \cdots \to F_d^* \to F_{d+1}^* \to F_{d+2}^* \to 0$. Let $G_{d+i} = \text{Coker} \phi_{d+i}^*$, for $0 \leq i \leq 2$. We have following short exact sequences for $0 \leq i \leq 2$:

$$0 \to \text{Ext}^{d+i}(R/I, R) \to G_{d+i} \to \text{Im} \phi_{d+i+1}^* \to 0.$$ 

Let $F'_\bullet$ denote the complex $F^*_\bullet$ truncated at the $d$th spot i.e. $H_0(F'_\bullet) = G_d$. Since height of $I = d$, $F'_\bullet$ is a minimal free resolution of $G_d$. Let $L_\bullet$ be a minimal free resolution of $\Omega$ and let $\psi_\bullet : L_\bullet \to F'_\bullet$ denote a lift of the inclusion $\Omega = \text{Ext}^d(R/I, R) \hookrightarrow G_d$. 

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Claim. Suppose that either $\Omega$ is $S_3$ or $p$ is a non-zero-divisor on $\text{Ext}_{R}^{d+1}(R/I, R)$. Then $\psi_d(L_d) = R$.

Proof of the Claim. If $\psi_d(L_d) \neq R$, then in the minimal free resolution $P_\bullet$ of $G_{d+1}$ extracted from the mapping cone of $\psi_\bullet$, $P_{d+1}$ would contain the copy of $R = F_0'$ as a free summand. Let $I = (y_1, \ldots, y_r)$ where $r_1 = \text{Tor}_1^R(R/I, k)$. Since height of $I = d$ and $\phi_1^*(1) = (y_1, \ldots, y_r)$, this would imply that $\text{Syz}^{d+1}(G_{d+1})$ does not satisfy $(0)$. Let $S = R/x$; then $\Omega = \text{Ext}_{R}^d(R/I, R) = \text{Hom}_S(R/I, S)$. If $\Omega$ is $S_3$, then $\text{Ext}_S^{1}(R/I, S) = 0$, i.e. $\text{Ext}_{R}^{d+1}(R/I, R) = 0$ and hence $G_{d+1} \simeq \text{Im} \phi_1^{d+2} \hookrightarrow F_{d+2}$. Thus $p$ is a non-zero-divisor on $G_{d+1}$. Let $\overline{R} = R/pR$ and $\overline{G}_{d+1} = G_{d+1}/pG_{d+1}$. Since $\overline{G}_{d+1}$ has finite projective dimension over $\overline{R}$ and $\overline{R}$ is equicharacteristic local ring, $\text{Syz}_{\overline{R}}^{d+1}(G_{d+1})$ must satisfy $(0)$ by the order ideal theorem. This leads to a contradiction and hence $\psi_d(L_d) = R$. If $p$ is a non-zero-divisor on $\text{Ext}_{R}^{d+1}(R/I, R)$, then $p$ is a non-zero-divisor on $G_{d+1}$. Hence, arguing in a similar way as above, it follows that $\psi_d(L_d) = R$ also in this case.

Let $\theta_\bullet : K_\bullet(x; R) \to F_\bullet$ denote a lift of $R/x \to R/I$. Dualizing $\theta_\bullet$ and combining the diagram corresponding to $\theta_\bullet$ with the same corresponding to $\psi_\bullet$ we get our required result.

1.7. Finally we point out a sufficient condition for the assertion in the above theorem to be valid in a more general set-up.

Proposition. Let $(R, m)$ be a local ring of dimension $n$ and let $x_1, \ldots, x_d \in m$ form an $R$-sequence. Let $\lambda$ be a zero-divisor and not a parameter on $R/x$, $x = \text{the ideal generated by} x_1, \ldots, x_d$ and let $\Omega = \text{Hom}(R/(x, \lambda), R/x)$. Let $\Omega'$ be a lift of $\Omega$ in $R$ via the surjection $R \to R/x$. Let $K_\bullet = K_\bullet(x; R; \beta_\bullet)$ and $(L_\bullet, \gamma_\bullet)$ denote the Koszul complex corresponding to $x_1, \ldots, x_d$ and a minimal free resolution of $\Omega$ respectively. Let $\phi_\bullet : L_\bullet \to K_\bullet$ denote a lift of $\Omega \to R/x$. If at least one of $x_1, \ldots, x_d$ is contained in $m\Omega'$, then $\phi_d(L_d) = R$.

Proof. Let $\overline{R} = R/x_1R$, $\overline{L}_\bullet = L_\bullet \otimes \overline{R}$ and $\overline{K}_\bullet = K_\bullet(x; R) \otimes \overline{R}$. Then $H_1(\overline{L}_\bullet) = \Omega$, $H_i(\overline{L}_\bullet) = 0$ for $i > 1$ and $H_1(\overline{K}_\bullet) = R/x = \overline{R}/x$ and $H_i(\overline{K}_\bullet) = 0$ for $i > 1$. We have a canonical surjection $\eta_\bullet(= (\eta_i, i \geq 1)) : \overline{K}_\bullet(+1) \to K_\bullet(x_2, \ldots, x_d; \overline{R})$ such that $\eta_1$ induces the identity map on $H_1(\overline{K}_\bullet)$ to $\overline{R}/x = H_0(K_\bullet(x_2, \ldots, x_d; \overline{R}))$. Let $F_\bullet$ denote a minimal free resolution of $\Omega$ over $\overline{R}$ and let $\theta_\bullet(= (\theta_i)_{i \geq 0}) : F_\bullet \to \overline{L}_\bullet(+1)$ lift the natural inclusion $\Omega = H_1(\overline{L}_\bullet) \hookrightarrow \text{Coker } \overline{\gamma}_2$. Then $\eta_\bullet \overline{\gamma}_\bullet \theta_\bullet : F_\bullet \to K_\bullet(x_2, \ldots, x_d; R)$ lifts the natural inclusion
\( \Omega \hookrightarrow R/x \) over \( \mathbf{R} \). If \( \eta_d \phi_d \theta_{d-1}(F_{d-1}) = \mathbf{R} \), then it follows that \( \phi_d(L_d) = R \). Hence by induction on \( d \) it is enough to consider \( d = 1 \) and \( x = x_d \) is contained in \( m\Omega' \).

Let \( \mu_1, \ldots, \mu_h \) denote a minimal set of generators of \( \Omega' \) over \( R \). Since \( x \in m\Omega' \), there exists \( a_1, \ldots, a_h \in m \) such that \( x = \sum_{i=1}^{h} a_i \mu_i \).

Consider the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Syz}^1(\Omega) \\
\downarrow \phi_1 & & \downarrow \phi_0 \\
0 & \longrightarrow & R \\
\end{array}
\]

\[
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\phi_1 & \phi_0 & \\
R & R & \rightarrow R/xR \\
\end{array}
\]

where \( \phi_0(e_i) = \mu_i, 1 \leq i \leq h \). Since \( x = \sum a_i \mu_i \), the element \( \alpha = (a_1, \ldots, a_h) \in \text{Syz}^1(\Omega) \) is such that \( \phi_0(\alpha) = x \). Hence \( \phi_1(\text{Syz}^1(\Omega)) = R \) and our proof is complete.
References

[A-Br] M. Auslander and M. Bridger, Stable Module Theory, Mem. Amer. Math. Soc. 94 (1969).
[Bh] B. Bhatt, Almost direct summands, Preprint on ArXiv:1109.0356(2011).
[Br-H] W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge Stud. Adv. Math. 39, Cambridge Univ. Press, Cambridge, 1993.
[D1] S. P. Dutta, On the canonical element conjecture, Trans. Amer. Math. Soc. 299 (1987), 803–811.
[D2] S. P. Dutta, Syzygies and homological conjectures, Commutative Algebra, MSRI Publications. 15 (1989), 139–156.
[D3] _____, Dualizing complex and the canonical element conjecture, J. London Math. Soc. 50 (1994), no. 2, 477–487.
[D4] _____, Dualizing complex and the canonical element conjecture, II, J. London Math. Soc. 2 (1997), no. 56, 46–63.
[D5] _____, A note on the monomial conjecture, Trans. Amer. Math. Soc. 350 (1998), 2871–2878.
[D6] _____, Splitting of local cohomology of syzygies of the residue field, J. Algebra 244 (2001), 168–185.
[D7] _____, A connection between two sets of conjectures, The Mathematics Student, Special Centenary Volume of the Indian Mathematical Society (2007), 79–88.
[D-G] S. P. Dutta & P. Griffith, Intersection multiplicities, the canonical element conjecture and the syzygy problem, Michigan Math. J. 57 (2008), 227–247.
[E-G1] E. G. Evans and P. Griffith, The syzygy problem, Ann. of Math. 114 (1981), no. 2, 323–333.
[E-G2] _____, Order ideals, in M. Hochster, J. D. Sally, and C. Huneke (eds.), Commutative Algebra, Math. Sci. Res. Inst. Publ. 15, Springer (1989), 213–225.
[E-G3] _____, A Graded Syzygy Theorem in mixed characteristic, Math. Research Letters 8 (2001), 605–611.
[Go] S. Goto, On the associated graded rings of parameter ideals in Buchsbaum rings, J. Algebra 85 (1983), 490–534.
[He] R. Heitmann, The direct summand conjecture in dimension three, Ann. of Math. 156 (2002), no. 2, 695–712.
[Ho1] M. Hochster, Contracted ideals from integral extensions of regular rings, Nagoya Math. J. 51 (1973), 25–43.
[Ho2] _____, Topics in the homological theory of modules over commutative rings, CBMS Reg. Conf. Ser. Math. 24, Amer. Math. Soc., Providence, RI, 1975.
[Ho3] _____, Canonical elements in local cohomology modules and the direct summand conjecture, J. Algebra 84 (1983), 503–553.
[K] Jee Koh, Degree p extensions of an unmixed regular local ring of mixed characteristic p, J. Algebra 99 (1986), 310–323.
[O] T. Ohi, Direct summand conjecture and descent of flatness, Proc. Amer. Math. Soc. 124 (1996), no. 7, 1967–1968.
[P-S1] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, Inst. Hautes Études Sci. Publ. Math. 42 (1973), 49–119.
[Ro1] P. Roberts, Two applications of dualizing complexes over local rings, Ann. Sci. Ec. Norm. Sup. 4èSés t.9 (1976), 103–106.
[Ro2] _____, Le Théorème d’intersection, C. R. Acad. Sci. Paris, Sér I Math. 304 (1987), 177–180.
[Str-Stü] J. R. Strooker and J. Stückrad, Monomial conjecture and complete intersections, Manuscripta Math. 79 (1993), 153–159.
[V] J. Valez, Splitting results in module-finite extension rings and Koh’s Conjecture, J. Algebra 172 (1995), 454–469.
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