On automorphism group of a possible short algorithm for multiplication of $3 \times 3$ matrices

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To the memory of Irina Dmitrievna Suprunenko

1. Introduction. Basic definitions and the result. Fast matrix multiplication is one of the main issues in the complexity theory. This question may be easily stated in the language of tensor decompositions. Recall necessary definitions.

Let $\tilde{V} = V_1 \otimes \ldots \otimes V_l$ be the tensor product of several finite dimensional vector spaces. A tensor of the form $v_1 \otimes \ldots \otimes v_l$ is called elementary or decomposable. An (additive) decomposition of length $s$ for $w$ is any unordered set $\{w_i \mid i = 1, \ldots, s\}$ of elementary tensors such that $\sum_{i=1}^s w_i = w$.

The rank $\text{rk}(w)$ is the minimal possible decomposition length for $w$.

Let $M_{mn} = M_{m,n}(\mathbb{C}) = \langle e_{ij} \mid 1 \leq i \leq m, \ 1 \leq j \leq n \rangle_{\mathbb{C}}$ be the space of all complex $m \times n$ matrices, where $e_{ij}$ are the usual matrix unities. Consider the following tensor in the space $M_{mn} \otimes M_{np} \otimes M_{pm}$:

$$\mathcal{T} = \sum_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p} e_{ij} \otimes e_{jk} \otimes e_{ki}.$$ 

It is often denoted also by $\langle m, n, p \rangle$, cf. [1] Section 14.2].

The question of main interest in the algebraic complexity theory is the rank of $\mathcal{T}$ (or at least the estimates for this rank), because of the following fact (see [1] Section 15.3]).

**Proposition 1.** Assume that for some $m, n, p \geq 1$ the tensor $\langle m, n, p \rangle$ has a decomposition of length $l$. Then there exists an algorithm for multiplication of $N \times N$ matrices over an arbitrary field of characteristic $0$ of complexity $O(N^\tau)$ arithmetical operations, where $\tau = (3 \ln l)/\ln mnp$.

The following estimates for $\text{rk}(\langle m, n, p \rangle)$ are known: $\text{rk}(\langle 2, 2, 2 \rangle) = 7$ (here the upper estimate follows from the Strassen algorithm, and the lower one is well-known also); $\text{rk}(\langle 2, 2, 3 \rangle) = 11, [2]$; $\text{rk}(\langle 2, 3, 3 \rangle) \in \{14, 15\}$ [23, 1]; $19 \leq \text{rk}(\langle 3, 3, 3 \rangle) \leq 23$ [20, 3]. Some estimates for other $\text{rk}(\langle m, n, p \rangle)$ are known also, but we shall not need them. It should be mentioned that
rk \( (m, n, p) \) is invariant under permutations of \( m, n, p \), and, finally, that always \( \text{rk} \left( (m, n, 1) \right) = mn \).

The most appealing in this field is the following question: what is the rank of \( (3, 3, 3) \)? It was found, by using numerical methods, a lot of new decompositions of length 23, but no one of length \( \leq 22 \). See [24], [13], [27], [31], [21], [22].

One of prospective ways of search for short decompositions of \( (m, n, p) \) is to study decompositions admitting nontrivial symmetry groups. This approach was proposed by the author in preprints [8], [9] and independently by Landsberg and co-authors in [17], [18], [3].

Recall the definitions related to decompositions automorphisms. Let \( \tilde{V} = V_1 \otimes \ldots \otimes V_l \) be as above. To avoid long formulae, we consider only the case \( l = 3 \). Let \( S(\tilde{V}) \) be the group of all nondegenerate linear transformation of \( \tilde{V} \) that preserve the tensor decomposition of this space, but possibly permuting the factors. For example, the transformations of the form

\[
v_1 \otimes v_2 \otimes v_3 \mapsto \alpha(v_1) \otimes \beta(v_2) \otimes \gamma(v_3),
\]

where \( \alpha : V_2 \to V_1 \) and \( \beta : V_1 \to V_2 \) are isomorphisms, and \( \gamma \) is a nondegenerate transformation of \( V_3 \) (note that in this case it is necessary to suppose that \( \dim V_1 = \dim V_2 \)). Obviously, \( S(\tilde{V}) \) preserves the set of decomposable tensors. (Sometimes the elements of \( S(\tilde{V}) \) are called Segre automorphisms, because when acting on the projectivization \( (\tilde{V} \setminus \{0\})/\mathbb{C}^* \) they preserve the Segre variety (the image in the projective space of the set of all nonzero decomposable tensors)). The subgroup of elements of \( S(\tilde{V}) \), corresponding to the trivial permutation of the factors, that is of the form \( A \otimes B \otimes C \), where \( A \in GL(V_1) \), \( B \in GL(V_2) \), and \( C \in GL(V_3) \), will be denoted by \( S^0(\tilde{V}) \).

For a tensor \( w \in \tilde{V} \) call

\[
\Gamma(w) = \{ g \in S(\tilde{V}) \mid gw = w \}
\]

the isometry group of \( w \), and the intersection \( \Gamma^0(w) = \Gamma(w) \cap S^0(\tilde{V}) \) - the small isometry group.

Let \( \mathcal{P} = \{w_1, \ldots, w_l\} \) be a decomposition for \( w \). Consider \( \mathcal{P} \) as a multiset, that is an unordered set some of whose elements may be equal. The automorphism group for \( \mathcal{P} \) is the subgroup of all elements of \( S(\tilde{V}) \) preserving \( \mathcal{P} \):

\[
\text{Aut}(\mathcal{P}) := \{ g \in S(\tilde{V}) \mid g\mathcal{P} = \mathcal{P} \}.
\]

It is clear that \( \text{Aut}(\mathcal{P}) \leq \Gamma(w) \).

The isometry group of \( (m, n, p) \) can be easily described (but the proof is not trivial !). Such a description was obtained in [19], [9], and more accurately in [10]. Specifically, \( \Gamma^0((m, n, p)) \) consists of all transformations of \( M_{mn} \otimes M_{np} \otimes M_{pm} \) of the form

\[
T(a, b, c) : x \otimes y \otimes z \mapsto axb^{-1} \otimes byc^{-1} \otimes cza^{-1},
\]

where \( a \in GL(m, \mathbb{C}) \), \( b \in GL(n, \mathbb{C}) \), and \( c \in GL(p, \mathbb{C}) \) (and observe that \( T(\lambda a, \mu b, \nu c) = T(a, b, c) \), where \( \lambda, \mu, \nu \in \mathbb{C}^* \)). If \( m, n, p \) are pairwise distinct, then \( \Gamma((m, n, p)) \) coincides with \( \Gamma^0 \), whereas if some of \( m, n, p \) are equal, then \( \Gamma \) is the semidirect product \( \Gamma((m, n, p)) = \Gamma^0((m, n, p)) \rtimes Q \), where \( Q \cong \mathbb{Z}_2 \), if two of \( m, n, \) and \( p \) are equal, and \( Q \cong S_3 \) if \( m = n = p \).

In the latter case we can take \( Q = \langle \sigma, \rho \rangle \), where

\[
\sigma : x \otimes y \otimes z \mapsto z \otimes x \otimes y, \quad \rho : x \otimes y \otimes z \mapsto y^t \otimes x^t \otimes z^t
\]
2. A direct sum of tensors. A direct sum of tensors is defined as follows: consider the tensor $w$ of $\tilde{G}_V$ decomposition and $w$. Theorem 2. Suppose that at least one of the six spaces $V_i$, $V_i''$ is of dimension $\leq 2$. Then $\text{rk}(w' + w'') = \text{rk}(w') + \text{rk}(w'')$.

Proof. See [25]. Another proof is contained in [14].

We also need another two general concepts regarding tensors.

1) We say that $w \in V_1 \otimes V_2 \otimes V_3$ and $w' \in V'_1 \otimes V'_2 \otimes V'_3$ are similar, and denote this by $w \sim w'$, if there exist isomorphisms $\varphi_i : V_i \rightarrow V'_i$ such that $(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)w = w'$. Similarly, $w$ and $w'$ are similar up to a permutation of tensor factors, if there exist a permutation $\pi \in S_3$ and isomorphisms $\varphi_i : V_i \rightarrow V'_{\pi i}$ such that $\varphi w = w'$, where $\varphi = \pi \circ (\varphi_1 \otimes \varphi_2 \otimes \varphi_3)$.

Clearly, if two tensors are similar, or similar up to a permutation of factors, then they are of the same rank.

2) Notice that for any two subspaces $V_1', V_1'' \subseteq V_1$ we have

$$(V_1' \otimes V_2 \otimes V_3) \cap (V_1'' \otimes V_2 \otimes V_3) = (V_1' \cap V_1'') \otimes V_2 \otimes V_3.$$
It follows that if \( \{ w_\alpha \in V_1 \otimes V_2 \otimes V_3 \mid \alpha \in A \} \) is any family of tensors, the there exists the least subspace \( U \subseteq V_1 \) such that \( U \otimes V_2 \otimes V_3 \) contains all \( w_\alpha \). This \( U \) will be called the (tensor) projection of the family \( \{ w_\alpha \} \) to \( V_1 \). The projections to other factors are defined similarly.

It is easy to see that the projection of \( \{ w_\alpha \} \) to \( V_1 \) is nothing else but the span of convolutions of the tensors \( w_\alpha \) with all tensors of the form \( l_2 \otimes l_3 \), where \( l_2 \in V^*_2 \), \( l_3 \in V^*_3 \). So it is easy to find the projection of \( T \) to each of the three factors \( M \): it is the whole \( M \).

3. Some identifications. Let \( V = \mathbb{C}^3 \) be the space of all column vectors of height 3. Its dual space \( V^* \) may be identified with the space of rows of length 3, so that \( \langle l, v \rangle = l(v) = lv \), where \( l \) is a row, \( v \) a column (note that \( lv \) is a \( 1 \times 1 \) matrix, that is, a number).

The group \( GL(V) = GL(3, \mathbb{C}) \) acts (on the left) on \( V \) as usually, i.e., \( g(v) = gv \). This group acts (on the left) also on \( V^* \) by \( gl = lg^{-1} \). This actions are compatible in the sense that always \( \langle g(l), g(v) \rangle = \langle lg^{-1}, gv \rangle = (lg^{-1})(gv) = l(v) = \langle l, v \rangle \).

Next, the space \( V \otimes V^* \) can be identified with \( M = M_3(\mathbb{C}) \) by the rule \( v \otimes l \mapsto vl \) (note that the product of column by a row is a \( 3 \times 3 \) matrix). The matrix corresponding to the tensor \( e_i \otimes e^j \) under this identification is the matrix unit \( e_{ij} \) (here \( e_i \) and \( e^i \) are the column and the row, respectively, that have 1 in \( i \)-th position, and 0 in other places).

The space \( V \otimes V^* \) is acted on by the group \( GL(V) \times GL(V) \) by the rule

\[
(g_1, g_2)(v \otimes l) = g_1(v) \otimes g_2(l) = g_1v \otimes lg_2^{-1}.
\]

When \( V \otimes V^* \) identifies with \( M \), the corresponding action on \( M \) is

\[
(g_1, g_2)(x) = g_1xg_2^{-1}.
\]

Let \( V = V_1 \oplus \ldots \oplus V_k \) be a decomposition into a direct sum. Let \( L_i \) be the subspace in the row space \( V^* \), consisting of all \( l \)'s such that \( \langle l, v \rangle = 0 \) for all \( v \in V_j \), \( j \neq i \). Then it is easy to see that \( V^* = L_1 \oplus \ldots \oplus L_k \) and the pairing of \( V_i \) and \( L_i \) is nondegenerate, so that \( L_i \) is identified with \( V^*_i \), and so \( V^* \) identifies with \( V^*_1 \oplus \ldots \oplus V^*_k \).

We apply the identifications described to prove the following.

Proposition 4. Let \( V = V_1 \oplus \ldots \oplus V_m = U_1 \oplus \ldots \oplus U_s \) be two decompositions for \( V \), let \( k \leq \min(m, s) \) and \( K = \bigoplus_{i=1}^k V_i \otimes U_i^* \), and let \( N \) be the sum of all the remaining summands \( V_i \otimes U_j^* \), so that \( M = V \otimes V^* = K \oplus N \). Let \( \zeta \) and \( \xi \) be the components of \( T \) in \( K \otimes M \otimes M \) and \( N \otimes M \otimes M \), respectively. Then

\[
\zeta \sim \bigoplus_{i=1}^k (p_i, q_i, 3),
\]

where \( p_i = \dim V_i \) and \( q_i = \dim U_i \). Moreover, \( \text{rk} \zeta = \sum_{i=1}^k \text{rk} ((p_i, q_i, 3)). \)

Proof. First we reduce the statement to the particular case where \( V_i \) and \( U_j \) are coordinate subspaces, that is \( V_i = \langle e_\alpha \mid \alpha \in I_i \rangle \), \( U_i = \langle e_\alpha \mid \alpha \in J_i \rangle \), where \( I_1, \ldots , I_m \) are disjoint subsets that form a partition of \( \{ 1, 2, 3 \} \), as well as \( J_i \).

Obviously, there exist \( a, b \in GL(V) \) such that \( aV_i \) and \( bU_j \) are coordinate subspaces. Then \( U_j^*b^{-1} = (bU_j)^* \) are coordinate subspaces also.
Consider the transformation \( x \mapsto ab^{-1} \) of \( M \). The image of \( K \) under this transformation is
\[
K_1 = aKb^{-1} = \bigoplus_{i=1}^{k} aV_i \otimes U_i^* b^{-1},
\]
and \( N_1 = aNb^{-1} \) is the sum of the all remaining \( aV_i \otimes U_i^* b^{-1} \).

Consider \( T = T(a, b, E) \). Then \( T \) takes \( K \otimes M \otimes M \) to \( K_1 \otimes M \otimes M \), and \( N \otimes M \otimes M \) to \( N_1 \otimes M \otimes M \). As \( T \) preserves \( T \), \( T \) takes \( \zeta \) to \( \zeta_1 \), and \( \xi \) to \( \xi_1 \), where \( \zeta_1 \) and \( \xi_1 \) are the components of \( T \) in \( K_1 \otimes M \otimes M \) and \( N_1 \otimes M \otimes M \), respectively. Clearly, \( \zeta \sim \zeta_1 \). As we suppose the proposition is true in the case of coordinate subspaces, we see that \( \zeta_1 \sim \bigoplus_{i=1}^{k} (p_i', q_i', 3) \), where \( p_i' = \dim aV_i \), \( q_i' = \dim U_i^* b^{-1} \). But, clearly, \( p_i' = p_i \), \( q_i' = q_i \).

Thus, we can assume that \( V_i \) and \( U_j \) are coordinate subspaces. It is clear that \( K = \langle e_{\alpha \beta} \mid (\alpha, \beta) \in S \rangle \), where \( S = \sqcup_{i=1}^{k} I_i \times J_i \), and \( N = \langle e_{\alpha \beta} \mid (\alpha, \beta) \notin S \rangle \). Hence it is clear that
\[
\zeta = \sum_{(\alpha, \beta) \in S, \ 1 \leq \gamma \leq 3} e_{\alpha \beta} \otimes e_{\beta \gamma} \otimes e_{\gamma \alpha}.
\]
Since the sets \( I_i \times J_i \) are disjoint, and the sets \( \{1, 2, 3\} \times I_i \) are disjoint also, as well as the sets \( J_i \times \{1, 2, 3\} \), it follows that
\[
\zeta = \eta_1 \oplus \ldots \oplus \eta_k,
\]
where
\[
\eta_i = \sum_{\alpha \in I_i, \ \beta \in J_i, \ 1 \leq \gamma \leq 3} e_{\alpha \beta} \otimes e_{\beta \gamma} \otimes e_{\gamma \alpha}.
\]
Finally, it is clear that \( \eta_i \sim \langle |I_i|, |J_i|, 3 \rangle = \langle p_i, q_i, 3 \rangle \). This proves the first claim.

Prove the equality for ranks, that is
\[
\text{rk} \bigoplus_{i=1}^{k} \langle p_i, q_i, 3 \rangle = \sum_{i=1}^{k} \text{rk} \langle p_i, q_i, 3 \rangle. \tag{1}
\]
We can assume that the decomposition contains no trivial summands, that is all \( p_i, q_i \geq 1 \).

If \( k = 1 \), the equality is trivial. If \( k \geq 3 \), then from the inequalities \( p_1 + \ldots + p_k \leq 3 \) and \( q_1 + \ldots + q_k \leq 3 \) we obtain \( k = 3 \) and \( p_i = q_i = 1, i = 1, 2, 3 \). The left-hand side of (1) is the direct sum of three copies of the tensor \( (1, 1, 3) \). One of the subspaces in the tensor product \( M_{11} \otimes M_{13} \otimes M_{31} \), related to \( (1, 1, 3) \), is one-dimensional, so we can apply Proposition 3, whence \( \text{rk} 3 \cdot (1, 1, 3) = 3 \cdot \text{rk} (1, 1, 3) = 9 \).

It remains to treat the case \( k = 2 \). Again, the inequalities \( p_1 + p_2 \leq 3 \) and \( q_1 + q_2 \leq 3 \) imply that either at least one of the pairs \( (p_1, q_1) \) and \( (p_2, q_2) \) is \( (1, 1) \), or one of these pairs is \( (1, 2) \) and the other \( (2, 1) \). In the former case at least one of the two summands \( \langle p_1, q_1, 3 \rangle \) is \( (1, 1, 3) \), and Proposition 3 applies. In the latter case one of the summands is \( (1, 2, 3) \), and the other \( (2, 1, 3) \), so Proposition 3 applies again, as \( M_{1,2} \) is of dimension 2.

4. The proof of the main theorem. Now we begin to prove the main theorem. Assume on the contrary that the homomorphism \( g \mapsto (\alpha(g), \beta(g)) \) is not injective. This means that there exists \( g \in \text{Aut}(\mathcal{P}) \) which preserves all three of the factors \( M \) and fixes all tensors \( w_i \) of the decomposition \( \mathcal{P} = \{ w_i \mid i = 1, \ldots, l \} \). Since \( g \) preserves the factors, it follows that \( g = T(a, b, c) \) for some \( a, b, c \in GL(3, \mathbb{C}) \), and at least one of \( a, b, \) and \( c \) is not a scalar matrix. So we can restate the theorem in the following equivalent form:
Lemma 6. Let \( a, b, c \in \text{GL}(3, \mathbb{C}) \), and at least one of \( a, b, \) and \( c \) is not a scalar matrix. Suppose that \( \{ w_i = x_i \otimes y_i \otimes z_i \mid i = 1, \ldots , l \} \) is a decomposition for \( T \) such that all \( w_i \) are invariant under \( T(a, b, c) \). Then \( l \geq 23 \).

It is this proposition that we are going to prove.

Notice that we can assume (and will assume below), without loss of generality, that \( a \) is not scalar.

We need a lemma.

Lemma 6. Let \( U \) and \( V \) be two spaces, \( A \in \text{GL}(U) \), \( B \in \text{GL}(V) \). Then \( A \otimes B \in \text{GL}(U \otimes V) \) is diagonalizable iff both \( A \) and \( B \) are diagonalizable.

Proof. It is obvious that if both \( A \) and \( B \) are diagonalizable, then \( A \otimes B \) is diagonalizable also. It remains to prove that if one of \( A \) and \( B \), say \( A \), is not diagonalizable, then \( A \otimes B \) is not. There exist \( u_1, u_2 \in U \) and \( \lambda \in \mathbb{C}^* \) such that \( Au_1 = \lambda u_1 \) and \( Au_2 = \lambda u_2 + u_1 \). Also, there exist \( v \in V \) and \( \mu \in \mathbb{C}^* \) such that \( Bv = \mu v \). Now we have \((A \otimes B)(u_1 \otimes v) = \lambda u_1 \otimes v\), \((A \otimes B)(u_2 \otimes v) = Au_2 \otimes Bv = (\lambda u_2 + u_1) \otimes (\mu v) = \lambda u_2 \otimes v + \mu (u_1 \otimes v)\). Therefore the subspace \( \langle u_1 \otimes v, u_2 \otimes v \rangle \subseteq U \otimes V \) is invariant under \( A \otimes B \) and the restriction of \( A \otimes B \) to this subspace is not diagonalizable. So \( A \otimes B \) itself is not diagonalizable. \( \square \)

Statement 7. Under hypothesis of Proposition 5 both \( a \) and \( b \) are diagonalizable (and \( c \) also is).

Proof. Since \( T(a, b, c) \) takes \( w_i \) to itself, the map \( x \mapsto axb^{-1} \) preserves all the lines \( \langle x_i \rangle \). It is obvious that the projection of \( T \) to the first factor \( M_1 \) is contained in the span of all \( x_i \). But this projection is \( M \), so \( \{ x_i \mid i = 1, \ldots , l \} = M \). So the lines which are invariant under the transformation \( x \mapsto axb^{-1} \) generate \( M \). That is, this transformation of \( M \) is diagonalizable.

When we identify \( M \) with \( V \otimes V^* \) this transformation corresponds to \( A \otimes B \), where \( Ax = ax \), \( Bl = lb^{-1} \). It follows from the lemma that both \( A \) and \( B \) are diagonalizable. So \( a \) is diagonalizable, and the transformation \( l \mapsto lb^{-1} \) of \( V^* \) is diagonalizable, whence \( b \) is diagonalizable also. \( \square \)

Let \( \lambda_1, \ldots , \lambda_s \) be all the distinct eigenvalues of \( a \), let \( \mu_1, \ldots , \mu_t \) be the eigenvalues of \( b \), and \( V = V_1 \oplus \ldots \oplus V_s \) and \( V = U_1 \oplus \ldots \oplus U_t \) be the corresponding decompositions into a sum of eigenspaces. Then \( V^* = U_1^* \oplus \ldots \oplus U_t^* \) is the eigenspaces decomposition for \( l \mapsto lb^{-1} \), and the eigenvalue corresponding to \( U_j^* \) is \( \mu_j^{-1} \).

Next, we have

\[
M = V \otimes V^* = \bigoplus_{1 \leq i \leq s, 1 \leq j \leq t} V_i \otimes U_j^* ,
\]

each of the summands \( V_i \otimes U_j^* \) being invariant under \( \Phi \): \( x \mapsto axb^{-1} \), with the eigenvalue \( \lambda_i \mu_j^{-1} \). Hence the set \( \Sigma \) of eigenvalues of \( \Phi \) on \( M \) is the set of all distinct numbers of the form \( \lambda_i \mu_j^{-1} \), and the eigenspace corresponding to \( \sigma \in \Sigma \) is

\[
M_\sigma = \bigoplus_{(i, j) \in S_\sigma} V_i \otimes U_j^* ,
\]

where

\[
S_\sigma = \{(i, j) \mid 1 \leq i \leq s, 1 \leq j \leq t, \lambda_i \mu_j^{-1} = \sigma\} .
\]

Since \( M = \bigoplus_{\sigma \in \Sigma} M_\sigma \), it follows that

\[
M \otimes M \otimes M = \bigoplus_{\sigma \in \Sigma} M_\sigma \otimes M \otimes M .
\]
Let $T_{\sigma}$ be the component of $T$ in $M_{\sigma} \otimes M \otimes M$.

For any tensor $w_{i} = x_{i} \otimes y_{i} \otimes z_{i}$ which is a member of $P$, the $x_{i}$ is an eigenvector for $\Phi$, that is $x_{i} \in M_{\sigma}$ for some $\sigma$. Therefore $T_{\sigma}$ is the sum of all $w_{i}$ such that $x_{i} \in M_{\sigma}$. There are at least $\text{rk}(T_{\sigma})$ of such $w_{i}$. Hence we obtain the inequality

$$l \geq \sum_{\sigma \in \Sigma} \text{rk}(T_{\sigma}).$$

Observe that if $(i, j), (i', j') \in S_{\sigma}$ and are distinct, then $i \neq i', j \neq j'$. So we can apply Proposition 4 to compute the rank of $T_{\sigma}$ (with appropriate renumbering of the spaces $V_{i}$ and $U_{j}$), and obtain

$$\text{rk}(T_{\sigma}) = \sum_{(i, j) \in S_{\sigma}} \text{rk}((d_{i}, f_{j}, 3)), $$

where $d_{i} = \dim V_{i}$ and $f_{j} = \dim U_{j}$. As any pair $(i, j)$ corresponds to some $\sigma$, we see that

$$\sum_{\sigma \in \Sigma} \text{rk}(T_{\sigma}) = \sum_{1 \leq i \leq s, 1 \leq j \leq t} \text{rk}((d_{i}, f_{j}, 3)).$$

It remains to show that the latter sum is $\geq 23$.

Obviously, $d = \{d_{1}, \ldots, d_{s}\}$ and $f = \{f_{1}, \ldots, f_{t}\}$ are partitions of the number 3, and $s \geq 2$, because $a$ is not a scalar matrix. There are three partitions of 3: 3, 21, and 111, where we denote for brevity 21 = \{2, 1\}, 111 = \{1, 1, 1\}. If $d = 21$ and $f = 3$, then $l \geq \text{rk}((2, 3, 3)) + \text{rk}((1, 3, 3)) \geq 14 + 9 = 23$. If $d = 21$ and $f = 21$, then $l \geq \text{rk}((2, 2, 3)) + \text{rk}((2, 1, 3)) + \text{rk}((1, 2, 3)) + \text{rk}((1, 1, 3)) = 11 + 6 + 6 + 3 = 26$. And if $d = 111$, then always $\text{rk}((d_{i}, f_{j}, 3)) = \text{rk}((1, f_{j}, 3)) = 3f_{j}$, whence $\sum_{i,j} \text{rk}((d_{i}, f_{j}, 3)) = 3 \cdot \sum_{j} f_{j} = 9 \sum_{j} f_{j} = 27$.

In the case $d = 21, f = 111$ we can argue in the similar way.

The proof of Proposition 5, and so of Theorem 2, is complete.

5. Finiteness of the set of candidates. Let $P$ be a hypothetical decomposition of length $\leq 22$ for $T$. It follows from Theorem 2 that there are only finitely many possibilities for the isomorphism class of $\text{Aut}(P)$. However, this does not give a guarantee that there are finitely many possibilities for $\text{Aut}(P)$, because for a given finite subgroup $X \leq \Gamma(T)$ the group $\Gamma(T)$ can contain, in general, infinitely many subgroups isomorphic to $X$ (say, all subgroups conjugate with $X$).

It is easy to see, however, that if $X \leq \Gamma(T)$ is a subgroup, $Y = gXg^{-1}$ is a conjugate to it, and $A$ is an $X$-invariant decomposition for $T$, then $B = gA$ is a $Y$-invariant decomposition for $T$ (and, conversely, every $Y$-invariant decomposition of $T$ is $B = gA$, where $A$ is an $X$-invariant decomposition). So when studying the decompositions of $T$ which are invariant under finite subgroups we can restrict our attention and to consider a unique subgroup from each conjugacy class of subgroups.

We have mentioned already that $\Gamma(T) \cong PSL(3, \mathbb{C})^{\times 3} \llhd Q$, where $Q \cong S_{3}$. The group $PSL(3, \mathbb{C})^{\times 3} = \Gamma^{0}(T)$ is an algebraic group, and it is easy to see that the conjugation by an element of $Q$ acts on $\Gamma(T)$ as a polynomial map. Therefore $\Gamma(T)$ is a (non-connected) algebraic group. But it is well known that if $G$ is an algebraic group over an algebraically closed field of characteristic 0, and $X$ is any finite group, then $G$ contains only finitely many conjugacy classes of subgroups isomorphic to $X$. (See, e.g., [30], Theorem 1, or [28], Ch.2, Theorem 17. Actually, in these sources stronger statements are proved.) Thus, there are only finitely many possibilities for $\text{Aut}(P)$, up to conjugacy in $\Gamma(T)$. 

6. Further restrictions. It is clear that the set of conjugacy classes of subgroups of \( \Gamma(T) \cong PSL(3, \mathbb{C})^3 \rtimes S_3 \) that are isomorphic to a subgroup of \( S_{22} \times S_3 \) is very large. To give an observable description of this set is a technically difficult task by itself. The author thinks that this set well may contain billions of groups! So we should obtain further restrictions on possible \( \text{Aut}(\mathcal{P}) \), say to show that this group can not contain certain elements or subgroups.

The aim of this section is to prove the following statement.

**Theorem 8.** If \( \mathcal{P} \) is a decomposition of length \( \leq 22 \) for \( T = (3, 3, 3) \), then \( \text{Aut}(\mathcal{P}) \) does not contain elements of the form \( T(a, E, E) \), where \( a \neq E \) (and the elements \( T(a, b, c) \) such that exactly one of \( a, b, c \) is different from \( E \)).

(Note that the statement in parentheses is a trivial corollary of the first one.)

To prove Theorem 8 it is sufficient to prove the following proposition.

**Proposition 9.** Let \( g = T(a, E, E) \), where \( a \neq E \), let \( w \) be an arbitrary decomposable tensor, and \( \{w, gw, \ldots, g^{l-1}w\} \) be its orbit under cyclic group \( \langle g \rangle \). Then there exist decomposable tensors \( w_1, \ldots, w_k \) such that \( w_1 + \ldots + w_k = w + gw + \ldots + g^{l-1}w \), \( k \leq l \), and all \( w_i \) are \( g \)-invariant.

Indeed, suppose that \( \mathcal{P} \) is a \( g \)-invariant decomposition of \( T \). For each \( \langle g \rangle \)-orbit \( \mathcal{O} \subseteq \mathcal{P} \) there exists a set of decomposable tensors \( \mathcal{O}' \) such that \( |\mathcal{O}'| \leq |\mathcal{O}| \), all elements of \( \mathcal{O}' \) are \( g \)-invariant, and the sum of elements of \( \mathcal{O}' \) is equal to the sum of elements of \( \mathcal{O} \). Replacing all \( \mathcal{O}' \)'s by \( \mathcal{O}' \), we obtain an set of decomposable tensors \( \mathcal{P}' \) such that the sum of elements of \( \mathcal{P}' \) is \( T \), all elements of \( \mathcal{P}' \) are \( g \)-invariant, and \( |\mathcal{P}'| \leq |\mathcal{P}| \). But then \( |\mathcal{P}'| \geq 23 \) by Proposition 5, whence \( |\mathcal{P}| \geq 23 \), a contradiction.

**Lemma 10.** Suppose \( g = T(a, b, c) \) is of finite order \( m \). Then \( g \) can be represented as \( g = T(a_1, b_1, c_1) \), where the order of each of \( a_1, b_1, c_1 \) divides \( m \) (or equals \( m \)).

**Proof.** We have \( \text{id}_{M \otimes M \otimes M} = g^m = T(a^m, b^m, c^m) \), whence \( a^m = \lambda E, b^m = \mu E, c^m = \nu E \). Put \( a_1 = \lambda^{-1/m}a, b_1 = \mu^{-1/m}b, c_1 = \nu^{-1/m}c \). Then \( T(a, b, c) = T(a_1, b_1, c_1) \), and moreover \( a_1^m = b_1^m = c_1^m = E \), \( \square \).

Now begin to prove Proposition 9. When \( l = 1 \) the statement is trivial, so below we assume \( l > 1 \). Let \( m \) be the order of \( g \). Then \( l \) divides \( m \). Moreover, we can assume, by Lemma 10, that \( g = T(a, E, E) \) and the order of \( a \) is \( m \).

Let \( \lambda_1, \ldots, \lambda_l \) be all the distinct eigenvalues of \( a \). Then the eigenvalues of the map \( A: x \mapsto ax \) of \( M \) are the same \( \lambda_i \), the multiplicity of \( \lambda_i \) on \( M \) equals thrice its multiplicity in the usual sense. Also, the eigenvalues of \( B: z \mapsto za^{-1} \) are \( \lambda_i^{-1} \), and the multiplicity of \( \lambda_i^{-1} \) is again thrice the multiplicity of \( \lambda_i \). Thus, we have decompositions

\[
M = K_1 \oplus \ldots \oplus K_l, \quad M = L_1 \oplus \ldots \oplus L_l ,
\]

where

\[
K_i = \{ x \in M \mid ax = \lambda_i x \}, \quad L_i = \{ z \in M \mid za^{-1} = \lambda_i^{-1}z \}.
\]

Let \( w = x \otimes y \otimes z \) be the decomposable tensor as in the hypothesis of the Proposition. We have \( g^i w = a^i x \otimes y \otimes za^{-i} \). Decompose

\[
x = x_1 + \ldots + x_t, \quad x_i \in K_i, \quad z = z_1 + \ldots + z_t, \quad z_i \in L_i.
\]

Hence

\[
w = \sum_{i,j=1}^t x_i \otimes y \otimes z_j.
\]
Notice that \( g \) acts on the subspace \( K_i \otimes M \otimes L_j \) by the multiplication by \( \lambda_i \lambda_j^{-1} \). The latter equals 1 if \( i = j \), and is a nontrivial \( m \)-root of 1 if \( i \neq j \). Hence

\[
\sum_{k=0}^{m-1} g^k w = m \sum_{i=1}^t x_i \otimes y \otimes z_i,
\]

whereas the summands \( x_i \otimes y \otimes z_j \) with \( i \neq j \) give zero when summed over \( \langle g \rangle \).

Some of \( x_i \) or \( z_i \) may vanish. We assume, up to renumbering, that \( x_i \neq 0 \) and \( z_i \neq 0 \) when \( i \leq s \), and \( x_i = 0 \) or \( z_i = 0 \) when \( i > s \). At last, note that since \( g^l w = w \), we have

\[
\sum_{i=0}^{l-1} g^i w = (l/m) \sum_{i=0}^{m-1} g^i w.
\]

Thus, the orbit sum for \( w \) is equal to \( l \sum_{i=1}^s x_i \otimes y \otimes z_i \).

All the tensors \( x_i \otimes y \otimes z_i \) are \( g \)-invariant. So the orbit sum is the sum of \( s \) \( g \)-invariant decomposable tensors. This proves the proposition when \( s \leq l \).

Obviously, \( s \leq t \leq 3 \). As \( l \geq 2 \), the only possible case where \( s > l \) is the case \( s = 3 \), \( l = 2 \). Take this case to a contradiction. We have

\[
x = x_1 + x_2 + x_3, \quad ax = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3,
\]

\[
a^2 x = \lambda_1^2 x_1 + \lambda_2^2 x_2 + \lambda_3^2 x_3.
\]

Since \( \lambda_i \) are pairwise distinct, and \( x_i \) are linearly independent, it follows from Vandermonde that \( x, ax, \) and \( a^2 x \) are independent. Whence \( g^2 w \neq w \), a contradiction.

The proof of Proposition 9, and so of Theorem 8, is complete.

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