1. Introduction

One of the many nice features of the Selberg integral (see [9]), i.e. of
\[
S_n(\alpha, \beta; \gamma) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\gamma} dx_1 \cdots dx_n
\]
(1)
\[
= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j\gamma)\Gamma(\beta + j\gamma)\Gamma(1 + (j + 1)\gamma)}{\Gamma(\alpha + \beta + (n + j - 1)\gamma)\Gamma(1 + \gamma)},
\]
is that it can be used in elementary methods for the study of the distribution of prime numbers. In the most general setting \(\alpha, \beta, \gamma\) are complex numbers such that
\[
\text{Re}(\alpha), \text{Re}(\beta) > 0, \quad \text{Re}(\gamma) > -\min \left\{ \frac{1}{n}, \frac{\text{Re}(\alpha)}{n-1}, \frac{\text{Re}(\beta)}{n-1} \right\}.
\]
Such elementary methods in prime number theory are developments of a method introduced by Gelfond and Schnirelmann [4] (see also [7, Chapter 10] for a detailed analysis on this and some other related methods). We refer the reader to [3] for a survey on the relevance of the integral (1), and to [2] for a general survey on the prime number theory.

For \(\gamma = 1\) the integral (1) evaluates a determinant of Hankel’s type:
\[
\frac{1}{n!} S_n(\alpha, \beta; 1) = \det_{1 \leq i,j \leq n} \left( \int_0^1 x^{\alpha+i+j-3}(1-x)^{\beta-1} dx \right).
\]
This is easily seen by using a classical argument due to Heine (see e.g. [10]). In the papers [8] and [1], independently, [2] was used in order to generalize the Gelfond–Schnirelmann method. In [8] (and in [1] with similar considerations), by taking \(\alpha = \beta = [sn]\) the greatest integer not exceeding \(sn\), for \(n \to \infty\) and \(s = 0.39191162\ldots\), a new elementary proof of the following theorem is obtained:

**Theorem 1.1** (Nair [8]). For all sufficiently large \(x\), we have
\[
\psi_1(x) \geq (0.49517\ldots)x^2 + O(x\log^2 x).
\]

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Here $\psi_1(x)$ is the sum function

$$\psi_1(x) = \sum_{n \leq x} \psi(n),$$

and $\psi(x)$ is Chebyshev’s $\psi$–function. Combining (1) and (2), and evaluating the Euler beta integrals, we obtain

$$\Gamma(\beta)^n \det_{1 \leq i,j \leq n} \left( \frac{\Gamma(\alpha+i+j-2)}{\Gamma(\alpha+\beta+i+j-2)} \right) = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j)\Gamma(\beta+j)j!}{\Gamma(\alpha+\beta+n+j-1)}.$$ 

In [8], the required upper bound of $S_n([sn],[sn];1)$ is obtained by computing the maximum of the integrand in $S_n([sn],[sn];1)$ over $[0,1]^n$, while no use is made of the evaluation (4).

The aim of this paper is twofold. We give a somehow more direct proof of (a generalisation of) (4), and use this formula to obtain the required upper bound when $\alpha = \beta = [sn]$, thus getting a slightly new proof of (3). Though none of these two remarks has special novelty, it seems hopeful that the general setting alluded to above, and even further generalizations occurring when $\gamma$ is a general positive integer (not necessarily 1), might eventually lead to interesting developments.

2. A NICE DETERMINANT

In this section we review the proof of (3) in [1] and [8]. We recall the following evaluation from [6]:

**Lemma 2.1.** [6] Lemma 3] For all indeterminates $X_1, \ldots, X_n, A_2, \ldots, A_n$, $B_2, \ldots, B_n$ we have

$$\det_{1 \leq i,j \leq n} \left( (X_i + B_2) \cdots (X_i + B_j)(X_i + A_{j+1}) \cdots (X_i + A_n) \right) = \prod_{1 \leq i < j \leq n} (X_i - X_j) \prod_{2 \leq i < j \leq n} (B_i - A_j).$$

By choosing $X_i = i$ ($i = 1, \ldots, n$), $B_i = \alpha + i - 3$ and $A_i = \alpha + \beta + i - 3$ ($i = 2, \ldots, n$) one easily gets (4).

From the Stirling formula it easily follows that

$$\log(1! \cdots n!) = \frac{n^2}{2} \log n - \frac{3}{4} n^2 + O(n \log n) \quad (n \to \infty).$$

Let us denote by $\Delta_n(s)$ the quantity in (4) with $\alpha = \beta = [sn]$, where $s$ is a positive parameter to be chosen later. Then

$$\Delta_n(s) = 1! \cdots (n-1)! \left( \frac{[sn]+n-2)!}{1! \cdots [sn]-2)!} \right)^2 \frac{1! \cdots (2[sn]+n-3)!}{1! \cdots (2[sn]+2n-3)!}.$$ 

Therefore

$$\log \Delta_n(s) = \left( \frac{(2s+1)^2}{2} \log(2s+1) - s^2 \log s - (s+1)^2 \log(s+1) - 2(s+1)^2 \log 2 \right) n^2 + O(n \log n) \quad (n \to \infty).$$
On the other hand, using the Pochhammer symbol \((x)_\beta\) for the shifted factorial,
\[
\frac{\Gamma(\beta)\Gamma(\alpha + i + j - 2)}{\Gamma(\alpha + \beta + i + j - 2)} = \frac{(\beta - 1)!}{(\alpha + i + j - 2)_\beta} = \sum_{k=0}^{\beta-1} \binom{\beta - 1}{k} (\alpha + i + j + k - 2)
\]
It follows that
\[
d_{\alpha+\beta+i+j-1} \frac{(\beta - 1)!}{(\alpha + i + j - 2)_\beta} \in \mathbb{Z},
\]
where \(d_m\) denotes the least common multiple of 1, \ldots, \(m\). Hence
\[
d_{\alpha+\beta+i+j-1} \frac{(\beta - 1)!}{(\alpha + i + j - 2)_\beta} \geq 1.
\]
However, due to the above determinant calculation, we get the following improved inequality:
\[
\prod_{i=1}^{n} d_{\alpha+\beta+n+i-3} \frac{(n - i)!}{(\alpha + i - 1)_{\beta+n-1}} \geq 1.
\]
Therefore
\[
\psi_1(2[sn] + 2n) - \psi_1(2[sn] + n) \geq -\log \Delta_n(s).
\]
Thus, after some calculations one arrives at (3).

One may get in the mood of using this special case of [6, Lemma 9], i.e. [6, Lemma 3] with \(X_i + B_j\) replaced by \((X_i + B_j)(X_i - B_j - C)\), \(X_i + A_j\) by \((X_i + A_j)(X_i - A_j - C)\), \(X_i - X_j\) by \((X_i - X_j)(C - X_i - X_j)\) and \(B_i - A_i\) by \((B_i - A_j)(B_i + A_j + C)\). This can be easily obtained by putting \(X_i = -(Y_i - C/2)^2\), \(A_i = (E_i + C/2)^2\) and \(B_i = (F_i + C/2)^2\).

3. A pretentious generalization (but almost suitable for arithmetic progressions)

If we let \(B_i = i\) and \(A_i = \beta + i\) in Lemma 2.1, we obtain the following generalization of (2):
\[
\det_{1 \leq i, j \leq n} \frac{(\beta - 1)!}{(x_i + j + 1)_\beta} = \frac{(\beta - 1)!}{(x_1 + 2)_\beta+n-1 \cdots (x_n + 2)_\beta+n-1} \prod_{1 \leq i < j \leq n} (x_j - x_i).
\]
This implies that
\[
d_{x_1+\beta+n} \cdots d_{x_n+\beta+n} \frac{(\beta - 1)!}{(x_1 + 2)_\beta+n-1 \cdots (x_n + 2)_\beta+n-1} \prod_{1 \leq i < j \leq n} (x_j - x_i) \geq 1,
\]
for all positive integers \(\beta, x_1, \ldots, x_n\) such that \(x_1, \ldots, x_n\) are distinct. The Chudnovsky–Nair determinant is the special case where \(x_1, \ldots, x_n\) are consecutive numbers.

4. Conclusions

No further interesting application seems to come out of considerations of hyperdeterminants [5], when \(\gamma > 1\), or \(q\)-analsogs. Therefore the main contribution of this short note, if any, is to enlighten the improvement from (5) to (6).
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