COMPACT GRADIENT EINSTEIN-TYPE MANIFOLDS WITH BOUNDARY AND CONSTANT SCALAR CURVATURE

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ABSTRACT. Inspired by the study of $V$-static manifold about classification, in this article, we apply the recent results obtained by Freitas and Gomes (Compact gradient Einstein-type manifolds with boundary, 2022) to prove the rigidity results for compact gradient Einstein-type manifolds with nonempty boundary and constant scalar curvature under some suitable pinching conditions. As a special case of gradient Einstein-type manifold, we also give a rigidity result of $(m, \rho)$-quasi-Einstein manifold with boundary.

1. Introduction

Let $(M^n, g)$, $n \geq 3$, be an $n$-dimensional compact Riemannian manifold. In the setting of manifolds without boundary, we say that $(M^n, g)$ be a gradient Einstein-type manifold or, equivalently, that $(M^n, g)$ supports an Einstein-type structure if there are a smooth $u$ on $M$ and a real constant $\rho \in \mathbb{R} \setminus \{0\}$ satisfying

\[ \alpha R + \beta \nabla^2 u + \mu du \otimes du = (\rho R + \lambda) g, \]

where $\alpha, \beta, \mu$ are constants with $(\alpha, \beta, \mu) \neq 0$, $\lambda$ is a smooth function and $R$ is the scalar curvature of the metric $g$. This structure, introduced by Catino et al. [5], unifies various particular cases, such as Ricci solitons, $\rho$-Einstein solitons and Yamabe solitons. In particular, we remark that if $(\alpha, \beta, \mu, \rho) = (1, 1, -\frac{1}{m}, \rho)$, $\lambda \in \mathbb{R}$, $0 < m \leq \infty$, namely,

\[ R + \nabla^2 u - \frac{1}{m} du \otimes du = (\rho R + \lambda) g, \]

then $(M^n, g)$ is called a $(m, \rho)$-quasi-Einstein manifold (see [9]).

By considering $f = e^{\frac{\mu}{\beta}u}$, Eq.(1.1) is equivalent to

\[ \frac{\alpha}{\beta} R + \beta \frac{\mu}{\beta} \nabla^2 f = \Lambda g, \]

where $\Lambda = \frac{1}{\beta}(\rho R + \lambda)$. Nazareno and Gomes [11] proved that a nontrivial, compact, gradient Einstein-type manifold of constant scalar curvature with both $\beta$ and $\mu$ nonzero is isometric to the standard sphere $S^n(r)$. In the insights to Eq.(1.2), recently Freitas and Gomes [8] studied a family of gradient Einstein-type metrics on manifolds with nonempty boundary, namely, there exists a nonconstant smooth function $f$ on $M^n$ satisfying

\[ \nabla^2 f = \frac{\mu}{\beta} f (\Lambda g - \frac{\mu}{\beta} R + \gamma g), \]

\[ f > 0 \quad \text{in} \quad \text{int}(M^n), \]

\[ f = 0 \quad \text{on} \quad \partial M, \]

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for some smooth function \( \Lambda \) and constants \( \alpha, \beta, \mu, \gamma \) with \( \beta \neq 0 \). They provided a complete classification for this family of Einstein-type manifolds that are Einstein.

**Theorem 1.1.** ([8, Theorem 1,Theorem 2]) Let \((M^n, g)\) be a compact gradient Einstein-type manifold with connected boundary \(\partial M\). If \((M^n, g)\) is an Einstein manifold, then it is isometric to a geodesic ball in a simply connected space form for \( \gamma \neq 0 \) or a hemisphere of a round sphere for \( \gamma = 0 \).

For the case of manifolds with constant scalar curvature, they just gave some boundary conditions to prove that \((M^n, g)\) is an Einstein manifold (see [8, Theorem 3, Theorem 4]).

In this paper, we consider a gradient Einstein-type manifold \((M^n, g)\) with nonempty boundary \(\partial M\) such that the metric \( g \) satisfies Eq.\((1.3)\) with constant function \( \Lambda \). For the convenience of calculation, we rewrite the satisfied equation as follow:

\[
\begin{align*}
\delta f \mathring{Ric} + \nabla^2 f &= h g, \\
\text{if } f &> 0 \text{ in int}(M^n), \\
\text{if } f &= 0 \text{ on } \partial M,
\end{align*}
\]

where \( \delta = \frac{\partial}{\partial r} \) and \( h = \theta f + \gamma \) with \( \theta = \frac{4}{n} \Lambda \) being constant.

Observe that Eq.\((1.4)\) is closely related to a \( V \)-static metric (see [10, 6]), thus we first use the zero radial Weyl curvature, i.e. \( W(\cdot, \cdot, \cdot, \nabla f) = 0 \) to classify gradient Einstein-type manifold. Such a condition has been used in \( V \)-static manifold ([1, Corollary 1.5, Corollary 1.6]), quasi-Einstein manifold ([4]) and gradient Ricci soliton ([13]). More precisely, we prove the following theorem.

**Theorem 1.2.** Let \((M^n, g)\), \( n \geq 4 \), be a compact gradient Einstein-type manifold satisfying \((1.4)\) with constant scalar curvature. For \(-1 < \delta < -\frac{n-4}{n-2}\), suppose that \( M^n \) has zero radial Weyl curvature and

\[
|\mathring{Ric}| \leq \frac{\delta(n-1)-2}{(n-2)\delta - n - 2}\frac{R}{\sqrt{n(n-1)}},
\]

where \( \mathring{Ric} \) is the traceless Ricci tensor. If \( \gamma = 0 \) then \((M^n, g)\) is isometric to a hemisphere of a round sphere.

Next, we adapt two different methods to give an important integral formula.

**Theorem 1.3.** Let \((M^n, g)\) be a compact gradient Einstein-type manifold satisfying \((1.4)\) with constant scalar curvature. Then the following integral formula holds:

\[
0 = \frac{2n \delta + n - 4}{2n} \int_M f |\mathring{Ric}|^2 \Delta f dV_g - \frac{4 \delta}{n-2} \int_M f^2 \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jk} dV_g + 2 \delta \int_M f^2 W_{ijkl} \mathring{R}_{ik} \mathring{R}_{jk} dV_g - \delta \int_M f^2 |\nabla \mathring{Ric}|^2 dV_g + \left( \frac{2 \delta(n-1)}{n(n-1)} R - \theta \right) \delta \int_M f^2 |\mathring{Ric}|^2 dV_g - \left( \frac{\delta(n-1)-1}{n^2} R^2 \right) - \frac{\theta}{n+1} R \int_M |\nabla f|^2 dV_g - (1 - \delta) \int_M |\mathring{Ric}(\nabla f)|^2 dV_g + \frac{1}{2} \int_M |\mathring{Ric}|^2 |\nabla f|^2 dV_g,
\]

where \( W \) is the Weyl tensor.

Now, we introduce the definition of Yamabe constant on a Riemannian manifold with nonempty boundary. Given a compact \( n \)-dimensional Riemannian manifold \((M^n, g)\) with boundary \(\partial M\), the Yamabe invariant \( Y(M, \partial M, [g]) \) associated to \((M^n, g)\) is defined by

\[
Y(M, \partial M, [g]) = \inf_{u \in W^{1,2}(M)} \int_M \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + R u^2 \right) dV_g + 2 \mathcal{I}_{\partial M} H u^2 dS_g,
\]

where \( \mathcal{I}_{\partial M} H u^2 dS_g \) is the integral of mean curvature for \( u \) on \( \partial M \).
where \([g]\) is the conformal class of the metric \(g\) and \(H\) is the mean curvature of \(\partial M\). For more details, we refer the readers to [7]. Catino and Baltazar et al. used the Yamabe constant to classify gradient shrinking Ricci soliton and four-dimensional Miao-Tam critical metric, respectively (see [2, 3]). Here we suppose a similar pinching condition with [2, Theorem 2] to obtain the following conclusion.

**Theorem 1.4.** Let \((\mathbb{M}^n, g)\) be an \(n\)-dimensional \((4 \leq n \leq 6)\) compact gradient Einstein-type manifold satisfying (1.4) with constant scalar curvature. If \(-1 < \delta < 0\) and

\[
\left[ \frac{n-1}{8(n-2)} Y(M, \partial M, [g]) - \left( \int_M \left( |W|^2 + \frac{8}{n(n-2)} |\tilde{Ric}|^2 \right)^{\frac{2}{n}} dV_g \right)^{\frac{2}{n}} \right] \Phi(M) 
\geq \frac{\delta - 1}{\delta} \sqrt{\frac{(n-1)^3}{4n(n-2)}} \int_M |\tilde{Ric}|^2 |\nabla f|^2 dV_g,
\]

where \(\Phi(M) = \left( \int_M \frac{2}{n} |\tilde{Ric}|^{\frac{n}{2}} dV_g \right)^{\frac{2}{n}},\) then \((\mathbb{M}^n, g)\) is isometric to a geodesic ball in a simply connected space form if \(\gamma > 0\), or a hemisphere of a round sphere if \(\gamma = 0\).

Since a \((m, \rho)\)-quasi-Einstein manifold is a special gradient Einstein-type manifold, from Theorem 1.4 we obtain a rigidity result of \((m, \rho)\)-quasi-Einstein manifolds with nonempty boundary.

**Corollary 1.1.** Let \((\mathbb{M}^n, g)\) be an \(n\)-dimensional \((4 \leq n \leq 6)\) compact \((m, \rho)\)-quasi-Einstein manifold with nonempty boundary and constant scalar curvature. For \(1 < m < \infty\), if

\[
\left[ \frac{n-1}{8(n-2)} Y(M, \partial M, [g]) - \left( \int_M \left( |W|^2 + \frac{8}{n(n-2)} |\tilde{Ric}|^2 \right)^{\frac{2}{n}} dV_g \right)^{\frac{2}{n}} \right] \Phi(M) 
\geq (m + 1) \sqrt{\frac{(n-1)^3}{4n(n-2)}} \int_M |\tilde{Ric}|^2 |\nabla f|^2 dV_g,
\]

then \((\mathbb{M}^n, g)\) is isometric to a hemisphere of a round sphere.

In order to prove these conclusions, in Section 2 we need review some classical tensors and give some key lemmas, and in Section 3 we will give the proof of our results.

### 2. Preliminaries

In this section we shall collect some fundamental identities and results that will be used in the proof of our results. Recall that on an \(n\)-dimensional Riemannian manifold \((\mathbb{M}^n, g)\) for \(n \geq 3\), the Weyl tensor and the Cotton tensor are respectively defined by

\[
W_{ijkl} = R_{ijkl} - \frac{1}{n} R g_{ij} g_{kl} + \frac{R}{(n-1)(n-2)} (g_{ij} g_{kl} - g_{il} g_{jk})
\]

and

\[
C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} (g_{jk} \nabla_i R - g_{ik} \nabla_j R).
\]

Notice that \(C_{ijk}\) is skew-symmetric in the first two indexes and trac-free in any index, i.e.

\[
C_{ijk} = -C_{jik} \quad \text{and} \quad C_{iik} = C_{iij} = 0.
\]
When $n \geq 4$, the Cotton tensor and Weyl tensor satisfy the following relation:
\[ C_{ijk} = -\frac{n-2}{n-3} \nabla_l W_{ijkl}. \]

For a tensor $T$, we denote by $\hat{T} = T - \frac{trT}{n} g$ the traceless part of $T$.

**Lemma 2.1.** Let $(M^n, g)$ be a compact gradient Einstein-type manifold satisfying (1.4) with constant scalar curvature. Then we have:
\[ \delta f (\nabla_i R_{jk} - \nabla_j R_{ik}) = -R_{ijkl} \nabla_l f - \delta (\nabla_i f R_{jk} - \nabla_j f R_{ik}) + \nabla_i h_{jk} - \nabla_j h_{ik}, \]
\[ (1 + \delta) R_{ij} \nabla_i f = \delta \nabla_j f R + (1 - n) \nabla_j h. \]

**Proof.** Taking the covariant derivative of (1.4), we obtain
\[ \delta (\nabla_i f R_{jk} + f \nabla_i R_{jk}) + \nabla_i \nabla_j f = \nabla_i h_{jk}. \]
Using the formula for the commutation of derivatives and Ricci identity
\[ \nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = R_{ijkl} \nabla_l f, \]
we get the desired equation (2.4). Moreover, since the scalar curvature $R$ is constant,
\[ \nabla_i R_{ij} = \frac{1}{2} \nabla_i R = 0. \]
Thus letting $i = k$ in (2.4) and contracting it will give (2.5). \qed

**Lemma 2.2.** Let $(M^n, g)$ be a compact gradient Einstein-type manifold satisfying (1.4) with constant scalar curvature. Then for $-1 < \delta < 0$ we have
\[ \theta \geq \frac{(n-1)\delta - 1}{n(n-1)} R. \]

In addition, if $\gamma \geq 0$ then $\theta \leq \frac{\delta R}{n}$ and $R > 0$.

**Proof.** Differentiating covariantly (2.5) gives
\[ (1 + \delta) R_{ij} \nabla_i \nabla_j f = \delta \Delta f R + (1 - n) \Delta h. \]
Using (1.4), we have $\Delta f = (-\delta R + n\theta) f + n\gamma$ and $\Delta h = \theta \Delta f$, thus the above formula becomes
\[ (1 + \delta) (-\delta |Ric|^2 + \theta R) f + (1 + \delta) \gamma R = (\delta R + (1 - n) \theta) \Delta f \]
\[ = (\delta R + (1 - n) \theta) [(-\delta R + n\theta) f + n\gamma], \]
that is, 
\[ (n-1) \left[ (n\theta - \delta R) f + n\gamma \right] \left[ \theta - \frac{(n-1)\delta - 1}{n(n-1)} R \right] = \delta (1 + \delta) |\hat{Ric}|^2 f. \]
Here we have used $|Ric|^2 = |\hat{Ric}|^2 + \frac{R^2}{n}$. As $f$ vanishes on the boundary, we have
\[ 0 = \int_M \text{div}(f \nabla f) dV_g = \int_M f \Delta f dV_g + \int_M |\nabla f|^2 dV_g \]
\[ = \int_M f \left[ (n\theta - \delta R) f + n\gamma \right] dV_g + \int_M |\nabla f|^2 dV_g, \]
that is, 
\[ \int_M \left[ (n\theta - \delta R) f + n\gamma \right] f dV_g = -\int_M |\nabla f|^2 dV_g. \]
For $-1 < \delta < 0$, integrating (2.7) over $M$ and using (2.8), we have

$$-(n - 1)\left[\theta - \frac{(n - 1)\delta - 1}{n(n - 1)}R\right] \int_M |\nabla f|^2dV_g = \delta(1 + \delta) \int_M |\text{Ric}|^2f^2dV_g \leq 0,$$

which yields (2.6). Furthermore, if $\gamma \geq 0$ then from (2.8) we obtain

$$(n\theta - \delta R) \int_M f^2dV_g = -n\gamma \int_M fdV_g - \int_M |\nabla f|^2dV_g \leq 0,$$

that means that $\frac{(n - 1)\delta - 1}{n(n - 1)}R \leq \theta \leq \frac{\delta R}{n}$. This shows $R > 0$. Therefore we complete the proof. \hfill \square

**Lemma 2.3.** Let $(M^n, g)$ be a compact gradient Einstein-type manifold satisfying (1.4) with constant scalar curvature. Then we have:

$$\frac{1}{2}\text{div}(f|\nabla|\text{Ric}|^2) = (1 - \delta)|\nabla_i(fC_{ijk}R_{jk}) - \frac{\delta - 1}{2}(\nabla f, |\nabla|\text{Ric}|^2)$$

$$- \frac{1}{n(n - 1)}|\text{Ric}|^2\Delta f - (\delta + 1)f \frac{(n - 2)\delta - n}{n - 2}R_{ij}R_{jk}$$

$$- \frac{1}{2}f|C|^2 + f|\nabla \text{Ric}|^2 - (\delta + 1)f \left[W_{ijkl}R_{ik}R_{jl}\right]$$

$$- \frac{R^3}{n(n - 1)(n - 2)} - n(n - 1)(n - 2)\frac{R|\text{Ric}|^2}{n(n - 1)(n - 2)}$$

$$+ \Delta hR - \nabla_i\nabla_j h R_{ij}.$$

**Proof.** Since $R$ is constant, by (2.2) and (2.4) we compute

$$\nabla_i(-\delta\nabla_j fR_{ik}R_{jk} + R_{ijkl}\nabla_l fR_{jk})$$

$$= -\delta\nabla_i(\nabla_j fR_{ik}R_{jk}) + \nabla_i \left(-\delta C_{ijk}R_{jk}ight)$$

$$- \delta(\nabla_i f|\text{Ric}|^2 - \nabla_j fR_{ik}R_{jk}) + \nabla_i h R - \nabla_j h R_{ij} \right)$$

$$= -\delta \nabla_i(fC_{ijk}R_{jk}) - \delta(\nabla f, |\nabla|\text{Ric}|^2) - \delta|\text{Ric}|^2\Delta f + \Delta h R - \nabla_i\nabla_j h R_{ij}.$$

Here we have used $\nabla_i h R_{ij} = \frac{\delta}{n} \nabla_j R = 0$.

At the same time, by (1.4) we also have

$$\nabla_i(-\delta\nabla_j fR_{ik}R_{jk} + R_{ijkl}\nabla_l fR_{jk})$$

$$= -\delta(\nabla_i \nabla_j fR_{ik}R_{jk} + \nabla_j fR_{ik}\nabla_i R_{jk}) + \nabla_i R_{ijkl}\nabla_l fR_{jk}$$

$$+ R_{ijkl}\nabla_i \nabla_l fR_{jk} + R_{ijkl}\nabla_i f\nabla_l R_{jk}$$

$$= -\delta(\nabla f R_{ik}R_{jk} + h|\text{Ric}|^2 + \nabla_j fR_{ik}\nabla_i R_{jk}) + \nabla_i R_{ijkl}\nabla_l fR_{jk}$$

$$- \delta f R_{ijkl} R_{ij} - h|\text{Ric}|^2 + R_{ijkl}\nabla_l f\nabla_i R_{jk}.$$

From (2.1), using (2.3) one can verify

$$\nabla_1 R_{ijkl} = \nabla_i W_{ijkl} + \frac{1}{n - 2}(\nabla_i R_{jl}g_{ik} - \nabla_i R_{jk}g_{il})$$

$$= -\frac{n - 3}{n - 2}C_{ijkl} + \frac{1}{n - 2}(\nabla_k R_{jl} - \nabla_l R_{jk})$$

$$= -C_{ijkl}.$$


and from (2.4) we obtain

\[ R_{ijkl} \nabla_i f \nabla_i R_{jk} = \left( -\delta f (\nabla_i R_{jk} - \nabla_j R_{ik}) - \delta (\nabla_i f R_{jk} - \nabla_j f R_{ik}) \right. \]
\[ + \nabla_i h_{jk} - \nabla_j h_{ik} \left\} \nabla_i R_{jk} \right. \]
\[ = -\delta f C_{ijk} \nabla_i R_{jk} - \frac{\delta}{2} (\nabla f, \nabla |Ric|^2) + \delta \nabla_j f R_{ik} \nabla_i R_{jk} \]
\[ = -\frac{\delta f}{2} |C|^2 - \frac{\delta}{2} (\nabla f, \nabla |Ric|^2) + \delta \nabla_j f R_{ik} \nabla_i R_{jk}. \]

By the skew-symmetric of \( C_{ijk} \), substituting the above two equations into (2.10) implies

\[ (2.11) \]
\[ \nabla_i (\nabla_j f R_{ik} R_{jk} + R_{ijkl} \nabla_i f R_{jk}) \]
\[ = \delta f (\delta R_{ijk} R_{jk} + R_{ijkl} R_{jl}) + C_{ijk} \nabla_j f R_{ik} \]
\[ - \frac{\delta f}{2} |C|^2 - \frac{\delta}{2} (\nabla f, \nabla |Ric|^2) - (\delta + 1) h |Ric|^2. \]

On the other hand, making use of Ricci identity and (2.2), a straightforward calculation gives (see [1, Eq.(3-3)])

\[ (2.12) \]
\[ \frac{1}{2} f |C|^2 = f |\nabla Ric|^2 + C_{ijk} \nabla_j f R_{ik} + \frac{1}{2} (\nabla f, \nabla |Ric|^2) \]
\[ + f (R_{ij} R_{jk} R_{ki} - R_{ijkl} R_{jl}) - \nabla_j (f \nabla_i R_{jk} R_{ik}). \]

Thus inserting (2.12) into (2.11), we conclude

\[ (2.13) \]
\[ \nabla_i (\nabla_j f R_{ik} R_{jk} + R_{ijkl} \nabla_i f R_{jk}) \]
\[ = (\delta + 1) f (\nabla R_{ijkl} R_{jl} + R_{ijkl} R_{jl}) + \frac{1}{2} \frac{\delta}{2} f |C|^2 - f |\nabla Ric|^2 \]
\[ - \frac{\delta + 1}{2} (\nabla f, \nabla |Ric|^2) - (\delta + 1) h |Ric|^2 + \nabla_j (f \nabla_i R_{jk} R_{ik}) \]
\[ = (\delta + 1) f (\nabla R_{ijkl} R_{jl} + R_{ijkl} R_{jl}) + \frac{1}{2} \frac{\delta}{2} f |C|^2 - f |\nabla Ric|^2 \]
\[ - \frac{\delta + 1}{2} (\nabla f, \nabla |Ric|^2) - (\delta + 1) h |Ric|^2 + \nabla_j (f C_{ijk} R_{ik}) + \frac{1}{2} \text{div}(f \nabla |Ric|^2). \]

Now we combine (2.13) and (2.9) to get

\[ (2.14) \]
\[ \frac{1}{2} \text{div}(f \nabla |Ric|^2) = (1 - \delta) \nabla_i (f C_{ijk} R_{jk}) - \frac{\delta - 1}{2} (\nabla f, \nabla |Ric|^2) - \delta |Ric|^2 \Delta f \]
\[ - (\delta + 1) f ((\nabla - 1) R_{ijkl} R_{jk} + R_{ijkl} R_{jl}) \]
\[ - \frac{\delta}{2} f |C|^2 + f |\nabla Ric|^2 + (\delta + 1) h |Ric|^2 + \Delta h R - \nabla_j \nabla_j h R_{ij}. \]

Contracting equation (1.4) gives \( h = \frac{1}{n} (\Delta f + \delta f R) \), thus (2.14) becomes

\[ \frac{1}{2} \text{div}(f \nabla |Ric|^2) = (1 - \delta) \nabla_i (f C_{ijk} R_{jk}) - \frac{\delta - 1}{2} (\nabla f, \nabla |Ric|^2) - \frac{(n - 1) \delta - 1}{n} |Ric|^2 \Delta f \]
\[ - (\delta + 1) f ((\nabla - 1) R_{ijkl} R_{jk} + R_{ijkl} R_{jl}) \]
\[ - \frac{\delta}{2} f |C|^2 + f |\nabla Ric|^2 + (\delta + 1) h |Ric|^2 + \Delta h R - \nabla_j \nabla_j h R_{ij}. \]

Finally, taking account (2.1) into the above equation, we get the desired equation. \( \square \)
Next we remember that Baltazar-Ribeiro.JR obtained the following divergent formula for any Riemannian manifold with constant scalar curvature.

**Lemma 2.4.** ([1, Lemma 3.1]) Let \((M^n, g)\) be a connected Riemannian manifold with constant scalar curvature and \(f : M \to \mathbb{R}\) be a smooth function defined on \(M\). Then we have

\[
div(f|\nabla Ric|^2) = - f|C|^2 + 2f|\nabla Ric|^2 + \langle \nabla f, \nabla |Ric|^2 \rangle + \frac{2n}{n-2}fR_{ij}R_{ik}R_{jk}
\]

\[
- \frac{4n-2}{(n-1)(n-2)}fR|Ric|^2 - \frac{2}{n(n-2)}fR^3
\]

\[
+ 2\nabla_i(fC_{ijk}R_{jk}) + 2C_{ijk}\nabla_j fR_{ik} - 2fW_{ijkl}R_{ik}R_{jl}.
\]

By Lemma 2.3 and Lemma 2.4, we have

\[
div(f|\nabla Ric|^2) = - \frac{\delta}{2}(\nabla_i(fC_{ijk}R_{jk}) + 2C_{ijk}\nabla_j fR_{ik} - 2fW_{ijkl}R_{ik}R_{jl} + \frac{2n}{n-2}fR_{ij}R_{ik}R_{jk})
\]

\[
+ \delta f|\nabla Ric|^2 - \frac{2n(n-1)}{n(n-2)}\delta fR|\nabla Ric|^2 + \Delta hR - \nabla_i\nabla_j hR_{ij}
\]

\[- (1 - \delta)C_{ijk}\nabla_j fR_{ik} - 2\delta fW_{ijkl}R_{ik}R_{jl}.
\]

**Proof.** Since \(|Ric|^2 = |\hat{Ric}|^2 + \frac{\hat{R}^2}{n}\), combining Lemma 2.3 and Lemma 2.4 to remove the term \(\nabla_i(fC_{ijk}R_{jk})\), we conclude

\[
\delta \text{div}(f|\nabla Ric|^2) = - \frac{2n(n-1)\delta - 2}{n}Ric^2 \Delta f - \frac{2(n-1)(n-2)}{n}R_{ij}R_{ik}R_{jk}
\]

\[
+ \frac{2(\delta + 1)(n-1)(n-2) - 4n(2n-1)}{n(n-2)}\delta fR|Ric|^2 + 2\delta f|\nabla Ric|^2
\]

\[- 2\Delta hR - 2\nabla_i\nabla_j hR_{ij} - 2(1 - \delta)C_{ijk}\nabla_j fR_{ik}
\]

\[- 4\delta fW_{ijkl}R_{ik}R_{jl} + \frac{2(\delta + 1)(n-2) - 4n}{n^2(n-2)}\delta fR^3.
\]

Using \(Ric = \hat{Ric} - \frac{\hat{R}}{n}g\), a direct computation yields

\[
(2.15) \quad \frac{\delta}{2} \text{div}(f|\nabla Ric|^2) = - \frac{2(n-1)\delta - 2}{n}Ric^2 \Delta f - \frac{2(n-1)(n-2) - 4n}{n}R_{ij}R_{ik}R_{jk}
\]

\[
+ \frac{2(\delta + 1)(n-1)(n-2) - 4n(2n-1)}{n(n-2)}\delta fR|Ric|^2 + 2\delta f|\nabla Ric|^2
\]

\[- 2\Delta hR - 2\nabla_i\nabla_j hR_{ij} - 2(1 - \delta)C_{ijk}\nabla_j fR_{ik}
\]

\[- 4\delta fW_{ijkl}R_{ik}R_{jl} + \frac{2(\delta + 1)(n-2) - 4n}{n^2(n-2)}\delta fR^3.
\]

Using \(\nabla_i\nabla_j hR_{ij} = \hat{R}_{ij}R_{ik}R_{jk} + \frac{3}{n}R|Ric|^2 + \frac{\hat{R}^3}{n^2}\),

Substituting (2.16) into the previous equation and a straightforward calculation, we get the desired equation (2.15). \(\square\)

### 3. Proof of results

#### 3.1. Proof of Theorem 1.2.

Since \(R\) is constant, in view of (2.2), Equation (2.4) may be rewritten as

\[
\delta fC_{ijk} = - R_{ijkl}\nabla_i f - \delta(\nabla_i fR_{jk} - \nabla_j fR_{ik}) + \nabla_i h_{gjk} - \nabla_j h_{gik}.
\]
Making use of (2.1) and (2.5), we thus have

\[
\delta f C_{ijk} = -W_{ijkl} \nabla_l f - \frac{1}{n-2} (R_{ik} \nabla_j f + R_{jl} \nabla_i f g_{jk} - R_{al} \nabla_i f g_{jk} - R_{jk} \nabla_i f) \\
+ \frac{R}{(n-1)(n-2)} (\nabla_j f g_{ik} - \nabla_i f g_{jk}) - \delta (\nabla_i f R_{jk} - \nabla_j f R_{ik}) \\
+ \nabla_i h g_{jk} - \nabla_j h g_{ik} \\
= -W_{ijkl} \nabla_l f - \frac{1}{n-2} (n-2) \delta (R_{ik} \nabla_j f - R_{jk} \nabla_i f) \\
+ \frac{1}{(n-1)(n-2)(1+\delta)} R (\nabla_j f g_{ik} - \nabla_i f g_{jk}) \\
+ \frac{1}{(n-2)(1+\delta)} (\nabla_j h g_{ik} - \nabla_i h g_{jk}).
\]

Hence, by the trace-free in any index of $C_{ijk}$, we have

\[
\delta f |C|^2 = -W_{ijkl} \nabla_s f C_{bjk} - \frac{2(1-(n-2)\delta)}{n-2} R_{ik} \nabla_j f C_{ijk}.
\]

If $(M^n, g)$ has zero radial Weyl curvature, namely, $W_{ijkl} \nabla_s f = 0$, then

(3.1) \[
\delta f |C|^2 = -\frac{2(1-(n-2)\delta)}{n-2} R_{ik} \nabla_j f C_{ijk}
\]

and it follows from (2.3) that

\[
0 = \nabla_i (W_{ijkl} \nabla_k f R_{jl}) \\
= \nabla_i W_{ijkl} \nabla_k f R_{jl} + W_{ijkl} \nabla_i \nabla_k R_{jl} + W_{ijkl} \nabla_k f \nabla_i R_{jl} \\
= \frac{n-3}{n-2} C_{ijkl} \nabla_k f R_{jl} - \delta f W_{ijkl} R_{ik} R_{jl},
\]

that is,

(3.2) \[
\delta f W_{ijkl} R_{ik} R_{jl} = -\frac{n-3}{n-2} C_{ijkl} \nabla_j f R_{ik}.
\]

If $\gamma = 0$, we obtain $\Delta f = (-\delta R + n\theta) f$ from (1.4). Since $h = \theta f$ with $\theta$ being constant, it is easy to see that $\Delta h = \theta \Delta f$ and $\nabla_i \nabla_j h = \theta \nabla_i \nabla_j f = \theta (-\delta f R_{ij} + h g_{ij})$. Therefore, by
integrating (2.15) over \( M \), we apply (3.1) and (3.2) to achieve

\[
0 = \frac{\delta(n-1) - 1}{n} \int_M |\tilde{Ric}|^2 dV_g + \frac{(n-2)\delta - n - 2}{n-2} \delta \int_M f \tilde{R}_{ij} \tilde{R}_{ik} \tilde{R}_{jk} dV_g
- \delta \int_M f |\nabla \tilde{Ric}|^2 dV_g + \frac{2\delta(n-1) - 2}{n(n-1)} \delta \int_M f R |\tilde{Ric}|^2 dV_g
- \left[ \theta \frac{n-1}{n} R - \frac{\delta(n-1) - 1}{n^2} R^2 \right] \int_M \Delta f dV_g - \theta \delta \int_M f |\tilde{Ric}|^2 dV_g
- \frac{(n-2)\delta + n - 4}{n - 2} \int_M C_{ijk} \nabla_j f \tilde{R}_{ik} dV_g
= \frac{(n-2)\delta - n - 2}{n - 2} \delta \int_M f \tilde{R}_{ij} \tilde{R}_{ik} \tilde{R}_{jk} dV_g - \delta \int_M f |\nabla \tilde{Ric}|^2 dV_g
+ \left[ \left( - \frac{\delta(n-1)(n-3) - (n-3)}{n(n-1)} R + (n-2)\theta \right) \delta - \theta \right] \int_M f |\tilde{Ric}|^2 dV_g
+ \delta \frac{(n-2)\delta + n - 4}{2(1 - (n-2)\delta)} \int_M f |C|^2 dV_g.
\]

Moreover, recall that the classical Okumura’s lemma [12, Lemma 2.1] implies

\[
\tilde{R}_{ij} \tilde{R}_{ik} \tilde{R}_{jk} \geq - \frac{n - 2}{\sqrt{n(n-1)}} |\tilde{Ric}|^3,
\]

and \( \theta \leq \frac{\delta R}{n} \) and \( R > 0 \) when \( \gamma = 0 \) (see Lemma 2.2), thus we have

\[
0 \geq - \frac{(n-2)\delta - n - 2}{\sqrt{n(n-1)}} \delta \int_M f |\tilde{Ric}|^3 dV_g - \delta \int_M f |\nabla \tilde{Ric}|^2 dV_g
+ \frac{\delta(n-1) - 2}{n(n-1)} R \delta \int_M f |\tilde{Ric}|^2 dV_g + \frac{(n-2)\delta + n - 4}{2(1 - (n-2)\delta)} \int_M f |C|^2 dV_g
= \int_M \left[ \frac{\delta(n-1) - 2}{n(n-1)} R \delta - \frac{(n-2)\delta - n - 2}{\sqrt{n(n-1)}} |\tilde{Ric}| \delta |\tilde{Ric}| \right] f |\tilde{Ric}|^2 dV_g
- \delta \int_M f |\nabla \tilde{Ric}|^2 + \delta \frac{(n-2)\delta + n - 4}{2(1 - (n-2)\delta)} \int_M f |C|^2 dV_g.
\]

Therefore under the assumption of Theorem 1.2, the above inequality shows \( \tilde{Ric} = 0 \), i.e. \( M \) is Einstein. So it suffices to apply Theorem 1.1 to conclude that \( M \) is isometric to a hemisphere of a round sphere. This complete the proof.

3.2. **Proof of Theorem 1.3.** Here we shall give two methods to prove the theorem. The first method is followed from the idea of [2, Theorem 2].
By Lemma 2.5, we have

\[ (3.3) \quad \frac{\delta}{2} \text{div}(f^2 \nabla |\text{Ric}|^2) = \frac{\delta}{2} f \text{div}(f \nabla |\text{Ric}|^2) + \frac{\delta}{2} (f \nabla |\text{Ric}|^2, \nabla f) \]

\[ = f - \frac{\delta(n-1) - 1}{n} |\text{Ric}|^2 \Delta f - \frac{\delta(n-2)\delta - n - 2}{n-2} \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jk} \]

\[ + \delta f |\nabla \text{Ric}|^2 - \frac{2\delta(n-1) - 2}{n(n-1)} \delta f R|\text{Ric}|^2 + \Delta h R - \nabla \nabla h R_{ij} \]

\[ - (1-\delta) C_{ijk} \nabla_j f R_{ik} - 2\delta f W_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} \]

\[ + \frac{\delta}{2} (f \nabla |\text{Ric}|^2, \nabla f). \]

Noticing that \( f \) vanishes on the boundary \( \partial M \) and integrating over \( M \) by part, one has

\[ \int_M (f \nabla |\text{Ric}|^2, \nabla f) dV_g = - \int_M f \Delta f |\text{Ric}|^2 dV_g - \int_M |\text{Ric}|^2 |\nabla f|^2 dV_g. \]

Now, integrating (3.3) over \( M \) and using the above relation, we get

\[ 0 = \frac{\delta(n-1) - 1}{n} \int_M f |\text{Ric}|^2 \Delta f dV_g + \frac{\delta(n-2)\delta - n - 2}{n-2} \int_M f^2 \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jk} dV_g \]

\[ - \delta \int_M f^2 |\nabla \text{Ric}|^2 dV_g + \frac{2\delta(n-1) - 2}{n(n-1)} \delta \int_M f^2 R|\text{Ric}|^2 dV_g \]

\[ - \int_M f \Delta h R dV_g + \int_M f \nabla \nabla h R_{ij} dV_g \]

\[ + (1-\delta) \int_M f C_{ijk} \nabla_j f R_{ik} dV_g + 2\delta \int_M f^2 W_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} dV_g \]

\[ + \frac{\delta}{2} (\int_M f \Delta f |\text{Ric}|^2 dV_g + \int_M |\text{Ric}|^2 |\nabla f|^2 dV_g). \]

For \( h = \theta f + \gamma \), as before we also have that \( \Delta h = \theta \Delta f \) and \( \nabla \nabla h = \theta (-\delta f R_{ij} + h g_{ij}) \). Substituting this into the previous equation yields

\[ (3.4) \quad 0 = \frac{\delta(3n-2) - 2}{2n} \int_M f |\text{Ric}|^2 \Delta f dV_g + \left( \frac{(\delta(n-1) - 1}{n^2} R^2 - \theta \frac{n-1}{n} R \right) \int_M f \Delta f dV_g \]

\[ + \frac{\delta(n-2)\delta - n - 2}{n-2} \int_M f^2 \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jk} dV_g \]

\[ - \delta \int_M f^2 |\nabla \text{Ric}|^2 dV_g + \frac{2\delta(n-1) - 2}{n(n-1)} \delta \int_M f^2 R|\text{Ric}|^2 dV_g \]

\[ - \delta \theta \int_M f^2 |\text{Ric}|^2 dV_g + (1-\delta) \int_M f C_{ijk} \nabla_j f R_{ik} dV_g \]

\[ + 2\delta \int_M f^2 W_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} dV_g + \frac{\delta}{2} \int_M |\text{Ric}|^2 |\nabla f|^2 dV_g. \]

Recalling (2.2) and the constancy of \( R \), we compute

\[ C_{ijk} \nabla_j f R_{ik} = (\nabla_i R_{jk} - \nabla_j R_{ik}) \nabla_j f R_{ik} \]

\[ = R_{ik} \nabla_j f \nabla_i R_{jk} - \frac{1}{2} (\nabla f, \nabla |\text{Ric}|^2) \]

\[ = \mathring{R}_{ik} \nabla_j f \nabla_i \mathring{R}_{jk} - \frac{1}{2} (\nabla f, \nabla |\text{Ric}|^2). \]
Integrating this over $M$ by part, we thus have
\[
\int_M fC_{ijk} \nabla_j fR_{ik} dV_g = \int_M f\dot{R}_{ik} \nabla_j f\dot{R}_{jk} dV_g - \int_M \frac{1}{2} f(\nabla f, \nabla |\dot{\text{Ric}}|^2) dV_g
\]
\[
= -\int_M |\dot{\text{Ric}}(\nabla f)|^2 dV_g - \int_M f\dot{R}_{ik} \nabla_i \nabla_j f dV_g
\]
\[
- \int_M \frac{1}{2} f(\nabla f, \nabla |\dot{\text{Ric}}|^2) dV_g
\]
\[
= -\int_M |\dot{\text{Ric}}(\nabla f)|^2 dV_g + \int_M \delta f^2 \dot{R}_{ik} \dot{R}_{jk} dV_g
\]
\[
+ \frac{n-2}{2n} \int_M f\Delta f |\dot{\text{Ric}}|^2 dV_g + \int_M \frac{1}{2} |\nabla f|^2 |\dot{\text{Ric}}|^2 dV_g.
\]

Inserting this into (3.4) and taking account (2.8), we thus achieve
\[
0 = \frac{2n\delta + n - 4}{2n} \int_M f|\dot{\text{Ric}}|^2 \Delta f dV_g + \left[ \frac{\delta(n-1) - 1}{n^2} R^2 - \theta \frac{n-1}{n} \right] \int_M f\Delta f
\]
\[
- \frac{4\delta}{n-2} \int_M f^2 \dot{R}_{ik} \dot{R}_{jk} dV_g - \delta \int_M f^2 |\nabla \dot{\text{Ric}}|^2 dV_g
\]
\[
+ \frac{2\delta(n-1)}{n(n-1)} \int_M f^2 |\text{Ric}|^2 dV_g - \theta \delta \int_M f^2 |\dot{\text{Ric}}|^2 dV_g
\]
\[
- (1 - \delta) \int_M |\dot{\text{Ric}}(\nabla f)|^2 dV_g + \frac{1}{2} \int_M |\nabla f|^2 |\dot{\text{Ric}}|^2 dV_g + 2\delta \int_M f^2 W_{ijkl} \dot{R}_{ik} \dot{R}_{jl} dV_g
\]
\[
= \frac{2n\delta + n - 4}{2n} \int_M f|\dot{\text{Ric}}|^2 \Delta f dV_g - \frac{4\delta}{n-2} \int_M f^2 \dot{R}_{ik} \dot{R}_{jk} dV_g
\]
\[
+ \frac{2\delta(n-1)}{n(n-1)} R^2 - \theta \frac{n-1}{n} \int_M |\nabla f|^2 dV_g
\]
\[
- \left[ \frac{\delta(n-1) - 1}{n^2} R^2 - \theta \frac{n-1}{n} \right] \int_M |\nabla f|^2 dV_g
\]
\[
- (1 - \delta) \int_M |\dot{\text{Ric}}(\nabla f)|^2 dV_g + \frac{1}{2} \int_M |\dot{\text{Ric}}|^2 |\nabla f|^2 dV_g.
\]

Thus the proof is complete. \qed

Another proof of Theorem 1.3. First we take the covariant derivative of (2.4) to achieve
\[
\delta f(\nabla_i \nabla_j R_{jk} - \nabla_i \nabla_j R_{ik}) = -\delta \nabla_i f(\nabla_j R_{jk} - \nabla_j R_{ik}) - \nabla_i R_{ijkl} \nabla_l f - R_{ijkl} \nabla_i \nabla_l f - R_{ijlk} \nabla_j \nabla_l f - R_{iklj} \nabla_j \nabla_l f - R_{ijlk} \nabla_j \nabla_l f + \nabla_i \nabla_j h_{gjk} - \nabla_i \nabla_j h_{gik}.
\]

Then letting the index $t = i$ and contracting the equation gives
\[
\delta f(\Delta R_{jk} - \nabla_j \nabla_j R_{ik}) = \delta \nabla_i f(\nabla_j R_{jk} - \nabla_j R_{ik}) - R_{ijkl} \nabla_i f - R_{ijlk} \nabla_i \nabla_l f - R_{ijlk} \nabla_i \nabla_l f - R_{iklj} \nabla_j \nabla_l f + \Delta h_{gjk} - \nabla_i \nabla_j h.
\]

From the commutation relations for the second covariant derivative of $R_{ik}$, we have
\[
\nabla_i \nabla_j R_{ik} = \nabla_j \nabla_i R_{ik} + R_{iks} R_{ij} + R_{iks} R_{ij} = R_{iks} R_{ij} + R_{iks} R_{ij}.
\]
On the other hand, from the second Bianchi identities we have

\[(3.7) \quad \nabla_j R_{ijk} \nabla_l f = \nabla_i R_{ik} \nabla_l f - \nabla_k R_{il} \nabla_l f.\]

Thus substituting (3.6) and (3.7) into (3.5), we conclude

\[
\delta f \Delta R_{jk} = \delta(\nabla_i f \nabla_j R_{ik} + f R_{sk} R_{js} + 2f R_{is} R_{ijk}) \\
+ (\nabla_i R_{jk} - \nabla_k R_{ij}) \nabla_l f + (1 + \delta) h R_{jk} \\
- \delta(\Delta f R_{jk} + 2\nabla_i f \nabla_k R_{jk} + \delta f R_{ij} R_{ik}) + \Delta h_{jk} - \nabla_k \nabla_j h \\
= \delta(\nabla_i f \nabla_j R_{ik} + 2f R_{is} R_{ijk}) + ((1 - 2\delta) \nabla_i R_{jk} - \nabla_k R_{jl}) \nabla_l f \\
+ \Delta h_{jk} - \nabla_k \nabla_j h + (\delta - \delta^2) f R_{ij} R_{ik} + [(1 + \delta) h - \delta \Delta f] R_{jk}.
\]

Since \( R \) is constant, using the above equation, we compute

\[
\frac{1}{2} \delta f |\hat{\Delta} \hat{\text{Ric}}|^2 = \frac{1}{2} \delta f |\text{Ric}|^2 = \delta f \Delta R_{jk} + \delta f |\hat{\text{Ric}}|^2 \\
= \delta(\nabla_i f \nabla_j R_{ik} R_{jk} + 2f R_{sk} R_{js} + 2f R_{is} R_{ijk}) \\
+ ((1 - 2\delta) \nabla_i R_{jk} R_{jl} - \nabla_k R_{ij} R_{jk}) \nabla_l f \\
+ \Delta h_{jk} - \nabla_k \nabla_j h + (\delta^2 + 1) f R_{ij} R_{ik} R_{jk} + [(1 + \delta) h - \delta \Delta f] |\text{Ric}|^2 + \delta f |\hat{\text{Ric}}|^2 \\
= 2\delta f R_{jk} R_{ks} R_{ijks} + \frac{1 - 2\delta}{2} \langle \nabla f, |\text{Ric}|^2 \rangle - (1 - \delta) \nabla_k R_{ij} R_{jk} \nabla_l f \\
+ \Delta h_{jk} - \nabla_k \nabla_j h R_{jk} + (\delta^2 + \frac{n(n + 2)\delta}{n - 2}) f R_{ij} R_{ik} R_{jk} \\
+ [(1 + \delta) h - \delta \Delta f] |\text{Ric}|^2 + \delta f |\hat{\text{Ric}}|^2.
\]

Moreover, recalling (2.1) we obtain

\[(3.8) \quad \frac{1}{2} \delta f |\hat{\Delta} \hat{\text{Ric}}|^2 = 2\delta f \left[ W_{ijkl} R_{jk} R_{il} - \frac{2n - 1}{(n - 1)(n - 2)} R |\text{Ric}|^2 + \frac{R^2}{(n - 1)(n - 2)} \right] \\
+ \frac{1 - 2\delta}{2} \langle \nabla f, |\text{Ric}|^2 \rangle - (1 - \delta) \nabla_k R_{ij} R_{jk} \nabla_l f \\
+ \Delta h_{jk} - \nabla_k \nabla_j h R_{jk} + \left( - \delta^2 + \frac{n(n + 2)\delta}{n - 2} \right) f R_{ij} R_{ik} R_{jk} \\
+ [(1 + \delta) h - \delta \Delta f] |\text{Ric}|^2 + \delta f |\hat{\text{Ric}}|^2.
\]

Now, as we known, from Eq.(1.4) one has

\[(3.9) \quad h = \frac{\delta R f + \Delta f}{n} \quad \text{and} \quad \nabla_i \nabla_j h = \theta(-\delta f R_{ij} + h g_{ij}).
\]

Hence, by (2.16) and (3.9), Eq.(3.8) becomes

\[(3.10) \quad \frac{1}{2} \delta f |\hat{\Delta} \hat{\text{Ric}}|^2 = 2\delta f W_{ijkl} \hat{R}_{jk} \hat{R}_{il} - \frac{2(n - 1)\delta^2 - 2\delta}{n(n - 1)} f R |\hat{\text{Ric}}|^2 \\
+ \frac{(1 - 2\delta)}{2} \langle \nabla f, |\hat{\text{Ric}}|^2 \rangle - (1 - \delta) \nabla_k \hat{R}_{ij} \hat{R}_{jk} \nabla_l f \\
+ \theta \frac{n - 1}{n} \Delta f R + \theta \delta f |\hat{\text{Ric}}|^2 + \left( - \delta^2 + \frac{n(n + 2)\delta}{n - 2} \right) f \hat{R}_{ij} \hat{R}_{ik} \hat{R}_{jk} \\
- \frac{(n - 1)\delta - 1}{n} \Delta f |\text{Ric}|^2 + \delta f |\hat{\text{Ric}}|^2.
\]
As \( f \) vanishes on the boundary \( \partial M \), we have

\[
\begin{align*}
(3.11) \quad & \int_M f \langle \nabla f, \nabla |\tilde{\text{Ric}}|^2 \rangle dV_g = 
\frac{1}{2} \int_M \text{div}(f^2 \nabla |\tilde{\text{Ric}}|^2) dV_g - \frac{1}{2} \int_M f^2 \Delta |\tilde{\text{Ric}}|^2 dV_g \\
& = - \frac{1}{2} \int_M f^2 \Delta |\tilde{\text{Ric}}|^2 dV_g,
\end{align*}
\]

\[
(3.12) \quad \int_M f \nabla_k \tilde{R}_{ijkl} \nabla_l f dV_g = \int_M \nabla_k (f \tilde{R}_{ij} \tilde{R}_{jk} \nabla_l f) dV_g - \int_M |\tilde{\text{Ric}}(\nabla f)|^2 dV_g \\
& \quad - \int_M f \tilde{R}_{ij} \tilde{R}_{jk} \nabla_k \nabla_l f dV_g \\
& = - \int_M |\tilde{\text{Ric}}(\nabla f)|^2 dV_g - \int_M f \tilde{R}_{ij} \tilde{R}_{jk} \nabla_k \nabla_l f dV_g.
\]

Multiplying (3.10) by \( f \) and applying (3.11) and (3.12), we deduce

\[
- \frac{1}{2} \int_M f \langle \nabla f, \nabla |\tilde{\text{Ric}}|^2 \rangle dV_g = 2\delta \int_M f^2 W_{ijkl} \tilde{R}_{ij} \tilde{R}_{kl} dV_g - \frac{2(n-1)\delta^2 - 2\delta}{n(n-1)} \int_M f^2 R |\tilde{\text{Ric}}|^2 dV_g \\
+ (1 - \delta) \left( \int_M |\tilde{\text{Ric}}(\nabla f)|^2 + \int_M f \tilde{R}_{ij} \tilde{R}_{jk} (\delta R_{ij} + h_{ij}) \right) dV_g \\
+ \theta \frac{n-1}{n} \int_M f \Delta f R dV_g + \theta \delta \int_M f^2 |\tilde{\text{Ric}}|^2 dV_g \\
+ (-\delta^2 + \frac{(n+2)\delta}{n-2}) \int_M f^2 \tilde{R}_{ij} \tilde{R}_{ik} \tilde{R}_{jk} dV_g \\
- \frac{(n-1)\delta^2 - 1}{n} \int_M f \Delta f |\tilde{\text{Ric}}|^2 dV_g + \delta \int_M f^2 |\nabla \tilde{\text{Ric}}|^2 dV_g,
\]

that is,

\[
0 = - 2\delta \int_M f^2 W_{ijkl} \tilde{R}_{ij} \tilde{R}_{kl} dV_g + (1 - \delta) \int_M |\tilde{\text{Ric}}(\nabla f)|^2 dV_g \\
+ \left[ \theta \delta - \frac{2(n-1)\delta^2 - 2\delta}{n(n-1)} \int_M f^2 |\tilde{\text{Ric}}|^2 dV_g - \frac{1}{2} \int_M |\tilde{\text{Ric}}(\nabla f)|^2 dV_g \\
+ \frac{4\delta}{n-2} \int_M f^2 \tilde{R}_{ij} \tilde{R}_{ik} \tilde{R}_{jk} dV_g + \frac{4 - n - 2\delta}{2n} \int_M f \Delta f |\tilde{\text{Ric}}|^2 dV_g \\
+ \left[ \theta \frac{n-1}{n} R - \frac{(n-1)\delta^2 - 1}{n^2} R^2 \int_M f \Delta f dV_g + \delta \int_M f^2 |\nabla \tilde{\text{Ric}}|^2 dV_g. \right]
\]

Finally, by integrating by part, we also give the desired integral formula. \( \square \)

### 3.3. Proof of Theorem 1.4.

First from (1.5) we have

\[
(3.13) \quad \frac{n-2}{2(n-1)} Y(M, \partial M, [g]) \left( \int_M |u|^2 dV_g \right)^{\frac{n+2}{n-2}} \leq \int_M |\nabla u|^2 dV_g + \frac{n-2}{4(n-1)} \int_M R u^2 dV_g \\
+ \frac{n-2}{2(n-1)} \int_{\partial M} H u^2 dS_g.
\]
for any \( u \in W^{1,2}(M) \). Using Kato inequality \(|\nabla |\hat{\text{Ric}}|^2| \leq |\nabla \text{Ric}|^2\) and choosing \( u = f|\hat{\text{Ric}}|\) in (3.13), Baltazar et al. proved the following inequality (see [2, Eq.(3.16)]):

\[
(3.14) \quad \int_M f^2|\nabla \hat{\text{Ric}}|^2 dV_g \geq \frac{n-2}{4(n-1)} Y(M, \partial M, [g]) \left( \int_M f^{\frac{2n}{n-2}}|\hat{\text{Ric}}|^\frac{2n}{n-2} dV_g \right)^{\frac{n-2}{n}}
\]

\[- (n-2)R \int_M f^2|\hat{\text{Ric}}|^2 dV_g + \int_M f \Delta f|\hat{\text{Ric}}|^2 dV_g.
\]

Meanwhile, a straightforward computation gives (see [2, Eq.(3.19)])

\[
(3.15) \quad |\hat{\text{Ric}}(\nabla f)|^2 \leq \frac{(n-1)\sqrt{2n}}{2n} |\text{Ric}|^2 |\nabla f|^2.
\]

We remark that on every \( n \)-dimensional Riemannian manifold the following estimate holds (see [3, Proposition 2.1]):

\[
| - W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} + \frac{2}{n} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} | \leq \sqrt{\frac{n-2}{2(n-1)}} \left( |W|^2 + \frac{8}{n(n-2)} |\hat{Ric}|^2 \right)^{\frac{2}{n}} |\hat{Ric}|^2.
\]

As \( \delta < 0 \), we have

\[
(3.16) \quad - \delta W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} + \frac{2\delta}{n} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} \geq \delta \sqrt{\frac{n-2}{2(n-1)}} \left( |W|^2 + \frac{8}{n(n-2)} |\hat{Ric}|^2 \right)^{\frac{2}{n}} |\hat{Ric}|^2.
\]

Since \( \Delta f = (-\delta R + n\theta) f + n\gamma \) from (1.4), taking account (3.14), (3.15) and (3.16) into Theorem 1.3 and using H"older inequality, we follow

\[
(3.17) \quad 0 \geq \delta \sqrt{\frac{2(n-2)}{n-1}} \left( \int_M \left( |W|^2 + \frac{8}{n(n-2)} |\hat{Ric}|^2 \right)^{\frac{2}{n}} dV_g \right)^{\frac{n}{2}}
\]

\[- \delta \frac{n-2}{4(n-1)} Y(M, \partial M, [g]) \left( \int_M f^{\frac{2n}{n-2}}|\hat{\text{Ric}}|^\frac{2n}{n-2} dV_g \right)^{\frac{n-2}{n}}
\]

\[+ \left( \frac{8(n-1) - (n-4)^2}{4n(n-1)} R - \theta \right) \delta + \frac{n-4}{2} \theta \int_M f^2|\hat{\text{Ric}}|^2 dV_g
\]

\[+ \frac{n-4}{2} \gamma \int_M f|\hat{\text{Ric}}|^2 dV_g + \left[ \theta \frac{n-1}{n} R - \frac{\delta(n-1)-n}{n^2} R^2 \right] \int_M |\nabla f|^2 dV_g
\]

\[+ \left[ \frac{1}{2} - (1 - \delta) \frac{(n-1)\sqrt{2n}}{2n} \right] \int_M |\hat{\text{Ric}}|^2 |\nabla f|^2 dV_g.
\]

Now, since \(-1 < \delta < 0\), by (2.6) we have

\[
\left( \frac{8(n-1) - (n-4)^2}{4n(n-1)} R - \theta \right) \delta + \frac{n-4}{2} \theta
\]

\[\geq \left( \frac{8(n-1) - (n-4)^2}{4n(n-1)} - \frac{(n-1)\delta - 1}{n(n-1)} \right) R \delta + \frac{(n-4)((n-1)\delta - 1)}{2n(n-1)} R
\]

\[= \frac{4(n-1)\delta^2 + (n^2 - 2n - 4)\delta + 2(n-4)}{4n(n-1)} R > 0 \quad \text{when} \quad 4 \leq n \leq 6.
\]
and
\[ \theta \frac{n - 1}{n} R - \frac{\delta(n - 1) - 1}{n^2} R^2 \geq 0. \]

Hence, under the assumptions of Theorem 1.4, from the inequality (3.17) we have \( \tilde{Ric} \equiv 0 \), i.e. \((M^n, g)\) is an Einstein manifold. Finally, we obtain the desired conclusion in view of Theorem 1.1.

3.4. Proof of Corollary 1.1. For a \((m, \rho)\)-quasi-Einstein manifold, it corresponds to the case where \( \delta = -\frac{1}{m}, \theta = -\frac{\rho R + \lambda}{m}, \) and \( \gamma = 0 \), thus when \( m > 1 \) and \( R \) is constant, we have \(-1 < \delta < 0\) and \( \theta \) is constant. Hence \((M^n, g)\) is isometric to a hemisphere of a round sphere by Theorem 1.4.

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