Non-geometric rough paths on manifolds

John Armstrong\(^1\) | Damiano Brigo\(^2\) | Thomas Cass\(^2\) | Emilio Rossi Ferrucci\(^2\)

\(^1\)Department of Mathematics, King’s College London, London, UK
\(^2\)Department of Mathematics, Imperial College London, London, UK

Correspondence
Emilio Rossi Ferrucci, Department of Mathematics, Imperial College London, 180 Queen’s Gate, South Kensington, London SW7 2AZ, UK.
Email: emilio.rossi-ferrucci16@imperial.ac.uk

Funding information
EPSRC, Grant/Award Number: EP/S026347/1; Centre for Doctoral Training in Financial Computing & Analytics, Grant/Award Number: EP/L015129/1

Abstract
We provide a theory of manifold-valued rough paths of bounded \(3 > p\)-variation, which we do not assume to be geometric. Rough paths are defined in charts, relying on the vector space-valued theory of Friz and Hairer (A course on rough paths, 2014), and coordinate-free (but connection-dependent) definitions of the rough integral of cotangent bundle-valued controlled paths, and of rough differential equations driven by a rough path valued in another manifold, are given. When the path is the realisation of semimartingale, we recover the theory of Itô integration and stochastic differential equations on manifolds (Émery, Stochastic calculus in manifolds, 1989). We proceed to present the extrinsic counterparts to our local formulae, and show how these extend the work in Cass et al. (Proc. Lond. Math. Soc. (3) 111 (2015) 1471–1518) to the setting of non-geometric rough paths and controlled integrands more general than 1-forms. In the last section, we turn to parallel transport and Cartan development: the lack of geometricity leads us to make the choice of a connection on the tangent bundle of the manifold \(TM\), which figures in an Itô correction term in the parallelism rough differential equation; such connection, which is not needed in the geometric/Stratonovich setting, is required to satisfy properties which guarantee...
well-definedness, linearity, and optionally isometric-
ity of parallel transport. We conclude by providing a
few examples that explore the additional subtleties
introduced by our change in perspective.

MSC 2020
53C05, 60L20 (primary)

Contents

INTRODUCTION ...................................... 757
1. BACKGROUND ON ROUGH PATHS ................................ 759
   1.1. $\mathbb{R}^d$-valued rough paths ........................................ 759
   1.2. Controlled paths and rough integration ................................ 762
   1.3. Rough differential equations ........................................... 769
   1.4. Stochastic rough paths ................................................. 771
2. ROUGH PATHS, ROUGH INTEGRATION, AND RDEs ON MANIFOLDS ............ 772
3. THE EXTRINSIC VIEWPOINT ........................................ 781
4. PARALLEL TRANSPORT AND CARTAN DEVELOPMENT .......................... 792
ACKNOWLEDGEMENTS .................................. 816
REFERENCES ........................................ 816

INTRODUCTION

The theory of rough paths, first introduced in [25], has as its primary goal that of providing a
rigorous mathematical framework for the study of differential equations driven by highly irreg-
ular inputs. The roughness of such signals renders the traditional definition of differentiation
and integration inapplicable, and motivates the definition of rough path, a path $X$ accompanied
by functions, satisfying certain algebraic and analytic constraints, which postulate the values of
its (otherwise undefined) iterated integrals. This concept leads to definitions of rough integra-
tion against the rough path $X$ and of rough differential equation (RDE) driven by $X$, which bear
the important feature of being continuous in the signal $X$, according to appropriately defined $p$-
variation norms. Rough path theory applies to a wide variety of settings, including to the case
in which $X$ is given by the realisation of a stochastic process, for which it constitutes a pathwise
approach to stochastic integration, extending the classical stochastic analysis of semimartingales.

An important feature that a rough path can satisfy is that of being geometric: this can be inter-
preted as the statement that it obeys the integration by parts and change of variable laws of first-
order calculus, its irregularity notwithstanding. The theory of geometric rough paths has been the
most studied [20], and applies to semimartingales through the use of the Stratonovich integral.
Other notions of stochastic integration, however, cannot be modelled by geometric rough paths,
the Itô integral being the prime (but not the only [18]) example.

Since smooth manifolds are meant to provide a general setting for ordinary differential calculus
to be carried out, it is natural to ask how ‘rougher’ calculi can be defined in the curved setting. In
the context of stochastic calculus, this question has led to a rich literature on Brownian motion on manifolds. More recently, it has been raised a number of times with regard to rough paths \([5, 8, 11, 15]\) and [4] in the setting of Banach manifolds and arbitrary \(p\)-variation regularity. In all cases, however, only the case of geometric rough paths has been discussed.

The main goal of this paper is to construct a theory of manifold-valued rough paths of bounded \(p\)-variation, with \(p < 3\), which are not required to be geometric. The regularity assumption ensures that we may draw on the familiar setting of [19] for vector space-valued rough paths; dropping this requirement would require the more complex algebraic tools of [22]. Our theory includes defining rough integration and differential equations, both from the intrinsic and extrinsic points of view, and showing how the classical notions of parallel transport and Cartan development can be extended to the case of non-geometric rough paths.

Although the definition of the Itô integral on manifolds has been known for decades, Stratonovich calculus has been preferred in the vast majority of the literature on stochastic differential geometry. Nevertheless, there are phenomena that are best captured by Itô calculus, particularly those which relate to the martingale property. In this spirit, three of the authors recently showed how a concrete problem involving the approximation of stochastic differential equations (SDEs) with ones defined on submanifolds necessitates the use of Itô notation, and that the result naturally provided by projecting the Stratonovich coefficients is suboptimal in general [1, 2]. The reason that Stratonovich integration and geometric rough paths are preferred in differential geometry is that they admit a simple coordinate-free description, as is also remarked on [25, p. 219]. An important point, however, that we wish to make in this paper is the following: an invariant theory of integration against non-geometric rough paths may also be given, albeit one that depends on the choice of a linear connection on the tangent bundle of the manifold. Although geometric rough path theory still retains the important property of being connection-invariant, all rough paths may be treated in a coordinate-free manner, since, while manifolds may not admit global coordinate systems, they always admit covariant derivatives. Overlooking this principle leads to the common misconception that Itô calculus/non-geometric rough integration cannot be carried out on manifolds, even in cases where a connection is already independently and canonically specified, for example, when the manifold is Riemannian. In much of stochastic differential geometry the focus is on the stochastic integral per se, which is viewed as a tool to investigate laws of processes defined on Riemannian Wiener space: in this context it is certainly justifiable to only work with the Stratonovich integral. Our emphasis here, however, is on pathwise integration itself: for this reason, we believe it to be of value to build up the theory in a way that is faithful to the choice of the calculus, as specified through the rough path \(\mathbf{X}\).

This paper is organised as follows: in Section 1 we review the theory of vector space-valued rough paths of bounded \(3 > p\)-variation, controlled rough integrations and RDEs, relying (with a few modifications and additions) on [19].

In Section 2, we develop the theory at the heart of the paper: this entails defining rough paths on manifolds and their controlled integrands in a coordinate-free manner by using pushforwards and pullbacks through charts, showing how the choice of a linear connection gives rise to a definition of rough integral, and defining RDEs in a similar spirit. We follow the ‘transfer principle’ philosophy [17] of replacing all instances of Euclidean spaces with smooth manifolds, which means that both the driving rough path and the solution are valued in (possibly different) manifolds. When we restrict our theory to semimartingales we recover the known framework for Itô integration and SDEs on manifolds [16].

In Section 3, we switch from the local to the extrinsic framework, and show how our theory extends that of [8] to non-geometric integrators and controlled integrands more general than
1-forms. Our broader assumptions require us to make additional nondegeneracy requirements on the path $X$, which are not needed in the local setting. We also remark that in this section we are confining ourselves to the Riemannian case (with the metric being induced by an embedding), while in the rest of the paper we allow for general connections.

Finally, in Section 4 we return to our local coordinate framework to carry out the constructions of parallel transport along rough paths and the resulting notion of Cartan development, or ‘rolling without slipping’, a cornerstone of stochastic differential geometry which yields a convenient way of moving back and forth between the linear and curved setting. Since we are dealing with parallel transport as a $TM$-valued RDE driven by an $M$-valued rough path $X$, the lack of geometricity leads us to require the choice of a connection not just on the tangent bundle of $M$ but also of one on the tangent bundle of the manifold $TM$. The latter connection may not be chosen arbitrarily, and we identify criteria (formulated in terms of the former connection) that guarantee well-definedness, linearity, and, if $M$ is Riemannian, isometricity of parallel transport. Different choices of such connection give rise to different definitions of parallel transport and Cartan development, which are only detectable at a second-order level, and all collapse to the same RDE when the rough path is geometric. Though we develop the theory in the most general way possible, three examples for how a connection on $TM$ may be lifted to one on $TTM$ are drawn from the literature; a case not analysed until now concerns the Levi-Civita connection of the Sasaki metric, which results in parallel transport coinciding with Stratonovich parallel transport. We end by seizing the opportunity to explore a few additional topics in stochastic analysis on manifolds, such as Cartan development in the presence of torsion, with a pathwise emphasis.

We would like to thank Martin Hairer and Zhongmin Qian for their helpful comments, which resulted in an improvement of one of the results in Section 3.

1 | BACKGROUND ON ROUGH PATHS

In this section, we review the core theory of finite-dimensional vector space-valued (controlled) rough paths, and the corresponding notions of rough integrals and RDEs. We refer mainly to [19], with the caveat that we are working in the setting of arbitrary control functions, as opposed to Hölder regularity. The former has the advantage of being a parametrisation-invariant framework, and of allowing us to consider a larger class of paths (for example, all semimartingales, and not just Brownian motion). Other authors have already been treating controlled rough paths in the setting of bounded $p$-variation [9, Subsection 2.4]. When a result in this first section is stated without proof, it is understood that the proof can be found in [19, Chapters 1–10], possibly with trivial modifications needed to adapt the arguments to the case of arbitrary controls. Many of the more quantitative aspects of rough paths are left out, as they will not be relevant for the transposition of the theory to manifolds. Since our vector spaces are finite-dimensional, and since we will rely on arbitrary charts to make the manifold-valued theory coordinate-free, we will use fixed coordinates to express all of our formulae.

1.1 | $\mathbb{R}^d$-valued rough paths

Throughout this document, $p$ will be a real number $\in [1, 3]$: we will not exclude the case of $p \in [1, 2)$ in which the theory reduces to Young integration, and remains valid with trivial adjustments. A control on $[0, T]$ is a continuous function $\omega$ defined on the subdiagonal $\Delta_T := \{(s, t) \in [0, T]^2 |
such that \( \omega(t, t) = 0 \) for \( 0 \leq t \leq T \) and \( \omega(s, u) + \omega(u, t) \leq \omega(s, t) \) for \( 0 \leq s \leq u \leq t \leq T \). The function \( \omega \) will denote a control throughout this document, and should be thought of as being a fixed property of the (rough) path which relates to its parametrisation; the main example is the Hölder control \( \omega(s, t) = t - s \). Given a path \( X : [0, T] \to \mathbb{R}^d \) we will denote its increment \( X_{st} := X_t - X_s \). Let \( C^p_\omega([0, T], \mathbb{R}^d) \) denote the set of \( \mathbb{R}^d \)-valued continuous paths \( X : [0, T] \to \mathbb{R}^d \) with

\[
\sup_{0 \leq s < t \leq T} \frac{|X_{st}|}{\omega(s, t)^{1/p}} < \infty. \tag{1.1}
\]

For there to exist a control \( \omega \) such that the above holds is equivalent to saying that \( X \) is a path of bounded \( p \)-variation [20, Proposition 5.10]; if \( \omega \) is the Hölder control we recover the definition of Hölder regularity. It is trivial to show that this kind of regularity is invariant under smooth maps.

**Lemma 1.1.** Let \( \omega \) be a control, \( X \in C^0_\omega([0, T], \mathbb{R}^d) \) and \( f \in C^{\infty}(\mathbb{R}^d, \mathbb{R}^e) \). Then \( f(X) \in C^p_\omega([0, T], \mathbb{R}^e) \).

Recall that if \( X \in C^0_\omega([0, T], \mathbb{R}^d) \), \( H \in C^q_\omega([0, T], \mathbb{R}^{d \times d}) \) with \( 1/p + 1/q > 1 \) (which happens, in particular, when \( p = q \in [1, 2) \)) we may define the Young integral

\[
\int_s^t H dX := \lim_{n \to \infty} \sum_{[u, v] \in \pi_n} H_{uv} X_{uv}, \tag{1.2}
\]

where \((\pi_n)_n\) is a sequence of partitions on \([s, t]\) with vanishing step size; the resulting path \( \int_s^t H dX \) belongs to \( C^q_\omega([0, T], \mathbb{R}^d) \). When the regularity requirement is no longer satisfied the Riemann sums no longer converge, and the definition of integral will require \( X \) and \( H \) to carry additional structure.

**Definition 1.2** (Rough path). A \( p \)-rough path controlled by \( \omega \) on \([0, T]\), valued in \( \mathbb{R}^d \) consists of a pair \( \mathcal{X} = (X, \mathcal{X}) \) with \( X \in C^0_\omega([0, T], \mathbb{R}^d) \) (the trace) and a continuous function \( \mathcal{X} : \Delta_T \to (\mathbb{R}^d)^{\otimes 2} = \mathbb{R}^{d \times d} \) (the second-order part) satisfying the regularity condition

\[
\sup_{0 \leq s < t \leq T} \frac{|X_{st}|}{\omega(s, t)^{2/p}} < \infty \tag{1.3}
\]

with the property that the *Chen identity* holds: for all \( 0 \leq s \leq u \leq t \leq T \) and \( \alpha, \beta = 1, \ldots, d \)

\[
\mathcal{X}_{st}^{\alpha \beta} = \mathcal{X}_{su}^{\alpha \beta} + X_{su}^\alpha X_{ut}^\beta + \mathcal{X}_{ut}^{\alpha \beta}. \tag{1.4}
\]

We denote the set of all such \( \mathcal{X} \) as \( \mathcal{C}^p_\omega([0, T], \mathbb{R}^d) \) (note the difference in font with \( C \), used for simple paths). Its *bracket* path is given by

\[
[X]_{st}^{\alpha \beta} := X_{st}^\alpha X_{st}^\beta - (\mathcal{X}_{st}^{\alpha \beta} + \mathcal{X}_{st}^{\beta \alpha}). \tag{1.5}
\]

These are indeed the increments of an element of \( C^{p/2}_\omega([0, T], (\mathbb{R}^d)^{\otimes 2}) \)-valued path, where \( \otimes \) denotes symmetric tensor product. We will say that \( \mathcal{X} \) is *geometric* if \( [\mathcal{X}] = 0 \), and denote the set of these with \( \mathcal{C}^p_\omega([0, T], \mathbb{R}^d) \).
The idea is that \( \mathbb{X}_{st} \) represents the value of the (otherwise undefined) integral

\[
\int_s^t \int_s^u dX_r \otimes dX_u = \int_s^t \mathbb{X}_{stu} \otimes dX_u.
\]  

(1.6)

In this interpretation it is easily checked that the Chen relation is simply the statement that the integral \( \int X_u \otimes dX_u \) is additive on consecutive time intervals, and the property of \( \mathbb{X} \) of being geometric represents an integration by parts formula. Relaxing the Chen identity to

\[
\mathbb{X}^{\alpha \beta}_{st} = \mathbb{X}^{\alpha \beta}_{st} + X^{\alpha}_{st} X^{\beta}_{ut} + \varepsilon^{\alpha \beta}_{st} + \mathbb{X}^{\alpha \beta}_{ut} + \varepsilon^{\alpha \beta}_{st}
\]  

for some function of two parameters \( \varepsilon_{st} \in o(\omega(s, t)) \) as \( t \searrow s \) for all \( s \) gives us the definition of almost rough path and space of these denoted with \( \mathcal{E} \) (and \( \mathcal{G} \) for geometric rough paths — we still require \( \mathbb{X} \in \mathcal{F}([0, T], \mathbb{R}^d) \) to satisfy \( [\mathbb{X}] = 0 \) exactly); this definition is motivated by the fact that the functions \( \varepsilon_{st} \) vanish in the limit of a sum over a sequence of partitions:

\[
\sum_{[s, t] \in \pi} \varepsilon_{st} = \sum_{[s, t] \in \pi} \frac{\varepsilon_{st}}{\omega(s, t)} \omega(s, t) \leq \omega(0, T) \sup_{[s, t] \in \pi} \frac{\varepsilon_{st}}{\omega(s, t)} \xrightarrow{|\pi| \to 0} 0
\]  

(1.8)

since \( p < 3 \) and \( O(\omega(s, t)^{3/p}) \subseteq o(\omega(s, t)) \). The same reasoning is also at the root of the following lemma [7, Proposition 3.5; 25, Theorem 3.3.1]. We write \( \approx \) for equality up to an \( \varepsilon_{st} \in o(\omega(s, t)) \) as \( t \searrow s \).

**Lemma 1.3.**

1. If \( \mathbb{X}, \mathbb{Y} \in \mathcal{C}^p([0, T], \mathbb{R}^d) \), \( \mathbb{X} \approx \mathbb{Y} \Rightarrow \mathbb{X} = \mathbb{Y} \).

2. Given \( \mathbb{X} \in \mathcal{F}^p([0, T], \mathbb{R}^d) \), there exists a unique \( \mathbb{X} \in \mathcal{C}^p([0, T], \mathbb{R}^d) \) with \( \mathbb{X} \approx \mathbb{X} \), which is given by

\[
\mathbb{X}_{st} = \lim_{n \to \infty} \bigotimes_{[u, v] \in \pi_n} \mathbb{X}_{uv},
\]  

(1.9)

where \( \pi_n \) is any sequence of partitions of \( [s, t] \) with vanishing step size. Moreover, if \( \mathbb{X} \in \mathcal{F}^p([0, T], \mathbb{R}^d) \), \( \mathbb{X} \in \mathcal{C}^p([0, T], \mathbb{R}^d) \).

Both statements also hold when restricted to the level of paths \( \in \mathcal{C}^p_{\mathbb{X}}([0, T], \mathbb{R}^d) \).

If \( \mathbb{X} \) and \( \mathbb{X} \) are related as in Lemma 1.3(2), we will say that latter is the rough path associated to the former.

Given \( \mathbb{X} \in \mathcal{C}^p([0, T], \mathbb{R}^d) \) we may associate a canonical element \( g \mathbb{X} \in \mathcal{C}^p([0, T], \mathbb{R}^d) \), which we call its geometrisation, with trace equal to that of \( \mathbb{X} \) and

\[
g \mathbb{X}^{\alpha \beta}_{st} := \frac{1}{2} \left( \mathbb{X}^{\alpha \beta}_{st} - \mathbb{X}^{\beta \alpha}_{st} \right) + \frac{1}{2} X^{\alpha}_{st} X^{\beta}_{st}.
\]  

(1.10)

In other words, \( g \mathbb{X} \) has the same antisymmetric part as \( \mathbb{X} \) and symmetric part fixed by the trace and the geometricity condition, and it is easily checked that the Chen identity continues to hold.
1.2 Controlled paths and rough integration

We proceed to define the objects which are, in some sense, dual to rough paths, and are original to [21].

**Definition 1.4** (Controlled path). Let \( X \in C^0_P([0, T], \mathbb{R}^d) \). An \( \mathbb{R}^e \)-valued, \( X \)-controlled path, or element of \( \mathcal{D}_X^P([0, T], \mathbb{R}^e) \) is a pair \( H = (H, H') \), where \( H \in C^0_P([0, T], \mathbb{R}^e) \) (the trace), \( H' \in C^0_P([0, T], \mathbb{R}^{exd}) \) (the Gubinelli derivative of \( H \) with respect to \( X \)), and

\[
R_{st}^k := H_{st}^k - H'_{st}^k X_{st}^\gamma, \quad \sup_{0 \leq s < t \leq T} \omega(s, t)^{2/p} \frac{|R_{st}|}{\omega(s, t)^{2/p}} < \infty. \tag{1.11}
\]

Here \( \mathbb{R}^{exd} \) should be thought of as \( \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e) \) (where \( \mathcal{L} \) means ‘linear maps’). We will identify \( \mathbb{R}^n \)-valued expressions with their coordinate expression throughout this paper, for example, we will write \( X = (X^\gamma), X = (X^\gamma, X^\alpha) \), \( H = (H^k, H'^k) \). We will use \( \approx \) as a shorthand for equality up to \( O(\omega(s, t)^{2/p}) \), that is, (1.11) may be written as \( H_{st}^k \approx H_{st}^k X_{st}^\gamma \).

The following definition and theorem establishes that rough paths should be thought of as integrators, and their controlled paths as a class of admissible integrands.

**Definition/Theorem 1.5** (Rough integral). Let \( X \in \mathcal{C}_P^0([0, T], \mathbb{R}^d) \) and \( H \in \mathcal{D}_X(\mathbb{R}^{exd}) \). We then define, for \( 0 \leq s \leq t \leq T \)

\[
\int_s^t H dX := \lim_{n \to \infty} \sum_{(u, v) \in \pi_n} H_{uv}^\gamma X_{uv}^\gamma + H'_{uv}^\alpha \chi_{uv}^\beta, \tag{1.12}
\]

where \( (\pi_n)_n \) is a sequence of partitions on \([s, t]\) with vanishing step size. This limit exists, is independent of such sequence and is obtained by applying Lemma 1.3(1), restricted to the path level, to

\[
H_{uv}^\gamma X_{uv}^\gamma + H'_{uv}^\alpha \chi_{uv}^\beta. \tag{1.13}
\]

Here \( H_t \) is an \( \mathbb{R}^{exd} \)-valued path and \( H'_t \) is a \( \mathbb{R}^{exd} \)-valued path, with superscripts denoting \( \mathbb{R}^e \)-coordinates and subscripts denoting \( \mathbb{R}^d \)-coordinates; in \( H'^k_{\alpha;st} \), the coordinate of the Gubinelli derivative is \( \alpha \), that is, the controlled path property now reads \( H'^k_{\alpha;st} - H'^{k}_{\alpha;st} X_{st}^\alpha \in O(\omega(s, t)^{2/p}) \).

We will often refer to controlled paths with trace valued in \( \mathbb{R}^{exd} \) as controlled integrands. Clearly if \( \tilde{X} \in \mathcal{C}_P^0([0, T], \mathbb{R}^d) \) we may have substituted it for \( X \) in (1.12) and (1.13). We will often omit the integration extrema: in this case identities are to be intended to hold when the integral is taken on any interval. Also note that it is obvious from the definition that the integral is linear in the integrand and additive on consecutive time intervals.

The condition of \( H \) admitting a Gubinelli derivative with respect to \( X \) is a strong condition, and one can only expect it to be satisfied when \( H \) bears a special relationship with \( X \). One may also ask whether there are conditions on \( X \) under which any Gubinelli derivative \( H' \) is unique: this is not always true, since if \( X \) is too regular inside \( C^0_P([0, T], \mathbb{R}^d) \) the regularity requirement on \( H' \) becomes less stringent. A condition on \( X \) that rules this out, and guarantees uniqueness of the Gubinelli derivative is given by true roughness of \( X \): this means that for all \( s \) in a dense set of \([0, T]\)
and for all $\phi \in (\mathbb{R}^d)^*$

$$\limsup_{t \searrow s} \frac{|\langle \phi, X_{st} \rangle|}{\omega(s, t)^{2/p}} = \infty. \quad (1.14)$$

It is satisfied, for instance, by a.a. sample paths of fractional Brownian motion with Hurst parameter $1/3 < H \leq 1/2$, when considered as elements of $C^{p}$, $1/H < p < 3$.

**Theorem 1.6** (Uniqueness of the Gubinelli derivative). Let $X \in \mathcal{C}^p_\omega([0, T], \mathbb{R}^d)$ with trace $X$ truly rough, $(H, 1'H), (H, 2'H) \in \mathcal{D}_X(\mathbb{R}^d)$. Then $1'H = 2'H$.

A corollary of this result is the uniqueness of the decomposition of the sum of a Young integral and a rough integral.

**Theorem 1.7** (Doob–Meyer for rough paths). Let $X \in \mathcal{C}^p_\omega([0, T], \mathbb{R}^d)$ with trace $X$ truly rough, $Y \in C^{p/2}([0, T], \mathbb{R}^d)$, $1H, 2H \in \mathcal{D}_X(\mathbb{R}^{d\times d})$, $1K, 2K \in \mathcal{C}^p([0, T], \mathbb{R}^{d\times d})$ then

$$\int 1H dX + \int 1K dY = \int 2H dX + \int 2K dY \quad (1.15)$$

implies $1H = 2H$ and $1K = 2K$.

In most cases, as for Example 1.8, the Gubinelli derivative is defined in a canonical manner, and is intended to be computed accordingly, regardless of whether uniqueness holds or not.

**Example 1.8** (Examples of canonically controlled paths).

1. The simplest example of an $X$-controlled path is a smooth function $f \in C^\infty(\mathbb{R}^d, \mathbb{R}^e)$ applied to $X$: its Gubinelli derivative is given by $Df(X)$ (where $Df \in C^\infty(\mathbb{R}^d, \mathbb{R}^{e\times d})$ is the differential of $f$, with coordinates $\partial_y f^k$) since

$$f^k(X)_{st} - \partial_y f^k(X_s)X^y_{st} \in O(|X_{st}|^2) \subseteq O(\omega(s, t)^{2/p}) \quad (1.16)$$

by Taylor’s theorem. The regularity requirements on $f(X)$ and $Df(X)$ are satisfied by Lemma 1.1. We call this $X$-controlled path $f(X)$.

2. Let $X, H$ be as in Definition/Theorem 1.5, then the rough integral $\int_0^T H dX$ admits Gubinelli derivative $H$. We denote the resulting element of $\mathcal{D}_X(\mathbb{R}^e)$ again by $\int H dX$.

3. Assume $H \in \mathcal{D}_X(\mathbb{R}^e)$ and that $K \in C^p_{\omega}([0, T], \mathbb{R}^e)$, then we may use $H'$ as the Gubinelli derivative of $H + K$ and we have that $(H + K, H') \in \mathcal{D}_X(\mathbb{R}^e)$.

**Example 1.9** (Difference of rough integrals against rough paths with common trace). Let $X = (X, 1X), \hat{X} = (X, 2X) \in \mathcal{C}^p_\omega([0, T], \mathbb{R}^d)$, $H \in \mathcal{D}^p_{\omega}((\mathbb{R}^{d\times d})$. Then it is easy to verify that there must exist a path $D \in C^{p/2}_{\omega}(\mathbb{R}^{d\times d})$ such that $\hat{X}_{st} = X_{st} + D_{st}$, and it is easily deduced from the (1.13)

$$\int H d\tilde{X} = \int H dX + \int H' dD \quad (1.17)$$
where the second integral on the right is intended in the sense of Young. An important special case is when \( \mathbf{X} = g \mathbf{X} \) for a rough path \( \mathbf{X} \), in which case \( D = \frac{1}{2} \langle \mathbf{X} \rangle \). Note that this identity also holds at the level of controlled paths, since the Gubinelli derivatives (taken according to Example 1.8(2) and (3)) both coincide with \( H \). We will often use the notation

\[
\circ d \mathbf{X} := d_g \mathbf{X}
\]

which is motivated by Stratonovich calculus (see Remark 1.25).

A controlled path may be transformed into a rough path in a canonical fashion.

**Definition 1.10** (Lift of a controlled path). Let \( \mathbf{X} \in \mathcal{C}^p([0, T], \mathbb{R}^d) \), \( H \in \mathcal{D}(\mathcal{X}(\mathbb{R}^e)) \). Define \( \mathbf{H} \) to be the rough path associated to \( \mathbf{X} \), defined as

\[
(\mathbf{H})_{st} := (H^k_{st}, H^i_j \alpha^j_{\beta s} \mathbf{X}^\alpha_{\beta s})
\]

which is easily verified to belong to \( \mathcal{C}^p([0, T], \mathbb{R}^e) \).

We would also like to show that the operation of lifting restricts to geometric rough paths: this is accomplished in the following lemma.

**Lemma 1.11.** If \( \mathbf{X} \in \mathcal{E}^p([0, T], \mathbb{R}^d) \) then \( \mathbf{H} \in \mathcal{E}^p([0, T], \mathbb{R}^e) \).

**Proof.** We cannot apply Lemma 1.3(2) directly to \( \mathbf{H} \), since its bracket only vanishes up to a \( O(\omega(s, t)^{3/p}) \). We may, however, define \( \mathbf{H} \) by \( (\mathbf{H})^k_{st} := (\mathbf{H})^k_{st} \) and

\[
(\mathbf{H})_{st} := (\mathbf{H})_{st} + \frac{1}{2} R^i_{st} R^i_{st} + H^i_j \alpha^j_{\beta s} \mathbf{X}^\alpha_{\beta s},
\]

where \( R \) is as in (1.11). Then \( (\mathbf{H})_{st} \approx (\mathbf{H})_{st} \) and it is readily checked that \( \mathbf{H} \in \mathcal{E}^p([0, T], \mathbb{R}^e) \), so we now apply Lemma 1.3(1) and (2) to conclude. \( \square \)

**Example 1.12** (Lifts of controlled paths).

1. Given a \( f \in C^\infty(\mathbb{R}^d, \mathbb{R}^e) \) we define \( f \circ \mathbf{X} := \mathbf{H} \) the pushforward of \( \mathbf{X} \) through \( f \), and by Taylor's formula we have

\[
(f \circ \mathbf{X})_{st} \approx (\partial f_k(X_s) X^\gamma_{st} + \frac{1}{2} \partial \alpha^j_{\beta s} f^k(X_s) X^\alpha_{st} X^\beta_{st}, \partial \alpha^j_{\beta s} f^i(X_s) X^\alpha_{st} X^j_{st}).
\]

2. Rough integrals may be lifted to rough paths: if \( \mathbf{H} \) is as in Definition/Theorem 1.5 we abuse the notation once again by setting \( \mathbf{H} \circ d\mathbf{X} := (\mathbf{H})_{st} \) and we have

\[
\int_s^t \mathbf{H} d\mathbf{X} \approx (H^k_{\gamma s} X^\gamma_{st} + H^i_j \alpha^j_{\beta s} \mathbf{X}^\alpha_{\beta s}, H^i_j \alpha^j_{\beta s} \mathbf{X}^\alpha_{\beta s}).
\]
Whenever there is an ambiguity as to whether a function on $\Delta_T$ is a controlled or rough path we will rely on coordinate notation to distinguish these two possibilities, for example, $(\int H \, dX)^k$ are the coordinates of the trace of the controlled/rough path, $(\int H \, dX)^k_\gamma = H^k_\gamma$ are the coordinates of the Gubinelli derivative of the controlled path and $(\int_s^t H \, dX)^j \approx H^j_{\alpha \beta} \times \alpha \beta$ those of the second-order part of the rough path. We will often use coordinate notation inside the integral too, to track the action of the integrand on the integrator, for example, $(\int H \, dX)^k = : \int H^k_\gamma \, dX_\gamma$, with the understanding that we also need the second-order coordinates of $X$ and $H$ to compute this integral (this will help make more complicated expressions clearer).

**Proposition 1.13** (Operations on controlled paths). Let $X \in C^p_\omega([0, T], \mathbb{R}^d)$.

*Change of controlling path.* Let $H \in \mathcal{D}_X(\mathbb{R}^e)$, $K \in \mathcal{D}_H(\mathbb{R}^f)$, then

$$K \ast H' := (K'^e_c, K'^e_{kH'^h}) \in \mathcal{D}_X(\mathbb{R}^f).$$

(1.23)

In particular, if $K = f(H)$ for $f \in C^\infty(\mathbb{R}^e, \mathbb{R}^f)$ we denote this $f \ast H$ and call it the pushforward of $H$ through $f$.

*Leibniz rule.* Let $H \in \mathcal{D}_X(\mathbb{R}^f \times e)$ and $K \in \mathcal{D}_X(\mathbb{R}^g \times f)$, then

$$K \cdot H := (K'^e_c H^e_c + K'^e_{kH'h} + K'^e_{rH'h}) \in \mathcal{D}_X(\mathbb{R}^g \times e).$$

(1.24)

*Pullback.* Let $g \in C^\infty(\mathbb{R}^d, \mathbb{R}^e)$, $H \in \mathcal{D}_g(X)(\mathbb{R}^f \times e)$, then

$$g^*H := (H \ast Dg(X)) \cdot Dg(X) \in \mathcal{D}_X(\mathbb{R}^f \times d)$$

$$= (H'^e_c \partial g^k(X), H'^e_{ij} \partial g^i(X) \partial g^j(X) + H'^e_{k} \partial g^k(X)).$$

(1.25)

**Proof.** Clearly all three paths belong to $C^p_\omega$. We need to check that (1.11) holds in all three cases. In the case of the change of controlling path we have

$$K'^e_{st} - K'^e_{k,s} H'^h_{s} X'^y_{st} \approx_{\geq 2} K'^e_{st} - K'^e_{k,s} H^h_{st} \approx_{\geq 2} 0.$$ 

(1.26)

As for the Leibniz rule, consider the matrix multiplication function

$$m : \mathbb{R}^{g \times f} \times \mathbb{R}^{f \times e} \to \mathbb{R}^{g \times e}, \quad (z'^e_c, y'^e_k) \mapsto (z'^e_c y'^e_k).$$

(1.27)

It is easily verified that $K \cdot H = m \ast (H, K)$, the pushforward of controlled paths being defined in the step above.

The case of the pullback readily follows from its expression as a combination of the two above constructions.

**Proposition 1.14** (Compatibility). Let $X \in \mathcal{C}^p_\omega([0, T], \mathbb{R}^d)$ and $H \in \mathcal{D}_X(\mathbb{R}^e)$.

(1) *Lifting is compatible with change of controlling path in the sense that, for $K \in \mathcal{D}_H(\mathbb{R}^f)$ we have*

$$\uparrow_H K = \uparrow_X (K \ast H'),$$

(1.28)
where $\mathbb{H}$ denotes the second-order part of the rough path $\Uparrow X H$. In particular, for $f \in C^\infty(\mathbb{R}^e, \mathbb{R}^f)$ pushforward of rough and controlled paths are related through lift by $f_* \Uparrow X H = \Uparrow X f_* H$. Moreover, $f_*(g_* X) = (f \circ g)_* X$ for appropriately valued smooth maps $f, g$.

(2) Lifting is compatible with geometrisation in the sense that

$$g_* \Uparrow X H = \Uparrow g_* X H. \quad (1.29)$$

In particular, pushforward of rough paths and rough integration preserve geometricity.

(3) For appropriately value smooth maps $f, g$ and controlled integrands $K$ we have

$$(f \circ g)^* K = g^*(f^* K). \quad (1.30)$$

Proof. As for the first claim, the two rough paths agree on the trace $K$ and second-order part

$$\langle \Uparrow H K \rangle^{ab}_{st} \approx K^{ij}_{st} \alpha_{ij}^a \beta_{ij}^b \approx K^{ij}_{st} K^{ab}_{st} = (\Uparrow X (K * H'))^{ab}. \quad (1.31)$$

Identity of the two rough paths therefore holds by Lemma 1.3(1). Now, taking $K = f(H)$ and the definitions of pushforward, this yields

$$f_* \Uparrow X H = \Uparrow H (f(H)) = \Uparrow X f_* H. \quad (1.32)$$

Taking, furthermore, $H = g(X)$ we obtain

$$f_*(g_* X) = f_* \Uparrow X g(X) = \Uparrow X (f_* g(X)) = \Uparrow X (f \circ g(X)) = (f \circ g)_* X. \quad (1.33)$$

As for the second claim, the two rough paths have the same trace, and therefore the same symmetric part of the second-order part, and antisymmetric part equal to half of

$$g_* (\Uparrow X H)^{ij} - g_* (\Uparrow X H)^{ji} = (\Uparrow X H)^{ij} - (\Uparrow X H)^{ji} \approx H^{ij}_{\alpha\beta} H^{\beta\alpha}_{\alpha\beta} (\Uparrow X H)^{\alpha\beta}_{\alpha\beta} - \Uparrow X H^{\alpha\beta}_{\alpha\beta} \quad (1.34)$$

Therefore, $g_* (\Uparrow X H) \approx \Uparrow g_* X H$ and we conclude again by Lemma 1.3(1).

The final statement is verified using a similar comparison of the expressions in coordinates. □

The following is a rough path version of the Itô lemma. Note how the formula simplifies to a first-order chain rule in the case of $X$ geometric. It is followed by the rough path-version of the Kunita–Watanabe identity, where the bracket path takes the role of quadratic covariation matrix.

**Theorem 1.15** (Itô lemma for rough paths). Let $X \in \mathcal{C}_p^\omega([0, T], \mathbb{R}^d)$ and $f \in C^\infty(\mathbb{R}^d, \mathbb{R}^e)$. Then

$$f(X) = f(X_0) + \int_0^T Df(X) dX + \frac{1}{2} \int_0^T D^2 f(X) d[X]. \quad (1.35)$$

Moreover, the Gubinelli derivatives of the left-hand side and right-hand side, computed canonically according to Example 1.8 agree, thus giving rise to an identity in $\mathcal{D}'(\mathbb{R}^e)$, and after applying $\Uparrow X$, to one in $\mathcal{C}_p^\omega([0, T], \mathbb{R}^e)$ (with the term $f(X_0)$ only influencing the trace).
**Proof.** The path-level statement is proved in [19, Proposition 5.6]. The Gubinelli derivative of the left-hand side according to Example 1.8(1) is $Df(X)$, which coincides with the Gubinelli derivative of the left-hand side according to Example 1.8(2) and (3) (since the bracket path, and thus the Young integral has higher regularity).

Note how the integral of the exact 1-form $Df(X)$ does not require the whole of $X$: this is because its Gubinelli derivative, $D^2f(X)$, is symmetric. Only the symmetric part of $X$ is needed: the pair $(X, \otimes X)$ (with $\otimes$ denoting the symmetrisation operator) is called a reduced rough path.

**Proposition 1.16** (Kunita–Watanabe identity for rough paths). Let $X \in \mathcal{C}^p_{\omega}([0, T], \mathbb{R}^d)$, $H \in \mathscr{D}_X(\mathbb{R}^d)$. Then

$$[\uparrow_X H]_{ij}^j \approx H_{st}^i H_{st}^j - H_{st}^l H_{st}^j (\chi_{st}^{\alpha \beta} + \chi_{st}^{\beta \alpha}),$$

so in particular (if $e = f \times d$ in the second case below)

$$[f_* X]_{ij}^j \approx \int_{s}^{t} \partial_\alpha f(X) \partial_\beta f(X) d[X]^{\alpha \beta}, \quad \left[ \int H dX \right]_{ij}^j = \int_{s}^{t} H_{st}^i H_{st}^j d[X]^{\alpha \beta}.$$

**Proof.** The first claim is immediate from (1.5) and Example 1.12. The bracket of a pushforward is computed as

$$[f_* X]_{ij}^j \approx f^i(X_{st}) f^j(X_{st}) - \partial_\alpha f^i(X_{st}) \partial_\beta f^j(X_{st}) (\chi_{st}^{\alpha \beta} + \chi_{st}^{\beta \alpha})$$

$$\approx \partial_\alpha f^i(X_{st}) \partial_\beta f^j(X_{st}) (X_{st}^{\alpha} X_{st}^{\beta} - (\chi_{st}^{\alpha \beta} + \chi_{st}^{\beta \alpha}))$$

$$= \partial_\alpha f^i(X_{st}) \partial_\beta f^j(X_{st}) [X]_{st}^{\alpha \beta}$$

$$\approx \int \partial_i f(X) \partial_j f(X) d[X]_{ij}^{ij}$$

and since the integral is additive on consecutive intervals we conclude that we have equality by uniqueness in Lemma 1.3(1). As for the rough integral

$$\left[ \int H dX \right]_{ij}^j \approx (H_{\gamma st}^i X_{st}^\gamma + H_{\alpha \beta st}^i \chi_{st}^{\alpha \beta})(H_{\gamma st}^j X_{st}^\gamma + H_{\alpha \beta st}^j \chi_{st}^{\alpha \beta}) - H_{\alpha \beta st}^i H_{\alpha \beta st}^j \chi_{st}^{\alpha \beta}$$

$$\approx H_{\alpha \beta st}^i H_{\alpha \beta st}^j (X_{st}^{\alpha} X_{st}^{\beta} - (\chi_{st}^{\alpha \beta} + \chi_{st}^{\beta \alpha}))$$

$$= H_{\alpha \beta st}^i H_{\alpha \beta st}^j [X]_{st}^{\alpha \beta}$$

$$\approx \int_{s}^{t} H_{st}^i H_{st}^j d[X]_{ij}^{\alpha \beta}$$

and conclude as before that equality holds.

The fact that the rough integral can be canonically considered a rough path in its own right naturally leads to the question of associativity, which is answered in the affirmative.
**Theorem 1.17** (Associativity of the rough integral). Let \( X \in \mathcal{C}_{\omega}^p([0, T], \mathbb{R}^d) \), \( H \in \mathcal{D}_X(\mathbb{R}^{e \times d}) \), \( I := \int H \, dX \in \mathcal{D}_X(\mathbb{R}^e) \), \( Y := \nabla X I \), \( K \in \mathcal{D}_I(\mathbb{R}^{f \times e}) \). Then

\[
\left( \int K \, dY \right) \ast I' = \left( \int (K \ast I') \cdot H \, dX \right) \in \mathcal{D}_X(\mathbb{R}^f).
\] (1.40)

As a result, the identity \( \int K \, dY = \int (K \ast H) \cdot H \, dX \) also holds in \( \mathcal{C}_{\omega}^p([0, T], \mathbb{R}^f) \).

**Proof.** At the level of the trace we have

\[
\int_s^t K^c_k dY^k \approx K^c_k Y^k + K^c_{ij} Y^i Y^j \approx K^c_k \left( H^k Y^k + H^i Y^i + H^j Y^j \right) + K^c_{ij} H^i Y^i + K^c_{ij} H^j Y^j \] (1.41)

which proves the identity of the traces, since \( I' = H \). Their Gubinelli derivatives with respect to \( X \), as computed according to Example 1.8(2) and Proposition 1.13 both coincide with \( (K^c_k H^k \gamma) \). Passing to the lift on this identity we have

\[
\int_X \int (K \ast H) \cdot H \, dX = \int_X \int K \, dY \ast H = \int_X \int K \, dY,
\] (1.42)

where we have used Proposition 1.14(1) in the second identity. This is the identity required in the second statement. \(\square\)

The next proposition expresses the degree to which pushforward of rough paths and pullback of controlled paths fail to be adjoint operators under the rough integral pairing; in particular the adjunction does hold when the integrator is geometric or when \( g \) is an affine map.

**Corollary 1.18.** Let \( X, H, g \) be as in Proposition 1.13 (Pullback). Then

\[
\left( \int H \, d(g^*_X X) \right) \ast D g(X) = \int g^*_X H \, dX + \frac{1}{2} \int H \cdot D^2 g(X) d[X],
\] (1.43)

where, as usual, the identity holds in \( \mathcal{D}_X(\mathbb{R}^e) \) according to Example 1.8 and thus in \( \mathcal{C}_{\omega}^p([0, T], \mathbb{R}^e) \).

**Proof.** Plugging in the expression for \( g^*_X X \) given by Theorem 1.15 and applying Theorem 1.17 we have

\[
\left( \int H \, d(g^*_X X) \right) \ast D g(X) = \left( \int H \, d(\int D g(X) \, dX + \frac{1}{2} \int D^2 g(X) \, d[X]) \right) \ast D g(X)
\]

\[
= \int (H \ast D g(X)) \cdot D g(X) \, dX + \frac{1}{2} \int H \cdot D^2 g(X) \, d[X]
\] (1.44)

\[
= \int g^*_X H \, dX + \frac{1}{2} \int H \cdot D^2 g(X) \, d[X].
\]
As usual, the more regular Young integral only contributes to the trace of the \( X \)-controlled/rough paths in question.

1.3 Rough differential equations

We proceed to discuss a central theme of rough path theory: that of RDEs.

**Definition 1.19.** Let \( \mathbf{X} \in \mathcal{C}_\omega^p([0,T], \mathbb{R}^d), F \in C^\infty(\mathbb{R}^{e+d}, \mathbb{R}^{e \times d}) \). A controlled solution to the RDE

\[
d\mathbf{Y} = F(\mathbf{Y}, \mathbf{X}) d\mathbf{X}, \quad Y_0 = y_0 \tag{1.45}
\]

(which we will write in coordinates as \( d\mathbf{Y}^k = F^k_\gamma(\mathbf{Y}, \mathbf{X}) d\mathbf{X}^\gamma \) when we wish to emphasise the action of the field of linear maps \( F \) on the driver \( \mathbf{X} \)) is an element \( \mathbf{Y} \in \mathcal{D}_X(\mathbb{R}^e) \) such that

\[
\mathbf{Y} = y_0 + \int F^*_\gamma(\mathbf{Y}, \mathbf{X}) d\mathbf{X} \in \mathcal{D}_X(\mathbb{R}^e), \tag{1.46}
\]

where \( F^*_\gamma(\mathbf{Y}, \mathbf{X}) \) is the pushforward of the \( \mathbb{R}^{e+d} \)-valued \( X \)-controlled path with trace \( (\mathbf{Y}, \mathbf{X}) \) and Gubinelli derivative \( (\mathbf{Y}^\prime, 1) \). We will call \( \uparrow_X \mathbf{Y} \) (which we will denote again \( \mathbf{Y} \)) a rough paths solution to (1.45).

We will sometimes write \( d\mathbf{Y} \) (without the bold font for \( \mathbf{Y} \)) on the left-hand side of (1.45) when only referring to the trace level of the solution. Note that the definition of controlled solution implies the requirement \( Y^\prime = F(\mathbf{Y}, \mathbf{X}) \) and

\[
F^*_\gamma(\mathbf{Y}, \mathbf{X}) = (F^k_\gamma(\mathbf{Y}, \mathbf{X}), \partial_\alpha F^k_\beta(\mathbf{Y}, \mathbf{X}) + F^h_\alpha \partial_\beta F^k_\gamma(\mathbf{Y}, \mathbf{X})). \tag{1.47}
\]

Since a solution of either type is entirely determined by its trace and \( F, \mathbf{X} \) we will often just use the term solution without specifying which type we intend.

**Remark 1.20.** Usually only RDEs of the form \( d\mathbf{Y} = F(\mathbf{Y}) d\mathbf{X} \) are considered. (1.45) can be considered as a special case of this by simply ‘doubling the variables’, that is, considering the joint RDE

\[
d\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} 1 \\ F(\mathbf{Y}, \mathbf{X}) \end{pmatrix} d\mathbf{X}. \tag{1.48}
\]

We have chosen to consider RDEs that also depend on \( \mathbf{X} \) since this will become a compulsory requirement when \( \mathbf{X} \) is manifold-valued; this framework is taken from [16], where it is used in the context of manifolds-valued SDEs. A side benefit of introducing this dependence is that rough integrals may now be seen as a particular case of RDEs, namely when \( F \) is independent of \( Y \). In particular, thanks to Theorem 1.15, controlled paths given by 1-forms may be viewed as particular cases of solutions to RDEs driven by the rough path \( (\mathbf{X}, [\mathbf{X}]) \).

**Example 1.21** (RDEs driven by rough paths with common trace). Let \( 1\mathbf{X}, 2\mathbf{X}, D \) be as in Example 1.9. Then we have the following identity of controlled solutions

\[
dY^k = F^k_\gamma(\mathbf{Y}, \mathbf{X}) d\mathbf{X}^\gamma \iff dY^k = F^k_\gamma(\mathbf{Y}, \mathbf{X}) d\mathbf{X}^k + (\partial_\alpha F^k_\beta + F^h_\alpha \partial_\beta F^k_\gamma)(\mathbf{Y}, \mathbf{X}) dD^\alpha \beta. \tag{1.49}
\]
Note this identity does not hold for rough path solutions, for the reason provided in Example 1.12(2). The second expression is an RDE driven by the rough path with trace \( (X, D) \) and second-order part \( \Xi \) (since \( D_{st}^{\alpha \beta} \in O(\omega(s, t)^{p/2}) \) we do not need other components since they are defined canonically by taking (1.6) literally in the sense of Young). This is particularly important when \( X = \mathbb{g}X, D = \frac{1}{2}[X] \) for a rough path \( X \), as it informs us that every RDE may be rewritten as an RDE driven by the geometric rough path \( (\mathbb{g}X, [X]) \).

The following theorem is proved in [8, Corollary 2.17, Theorem 4.2], and its proof carries over to the case of \( X \) non-geometric (thanks to Example 1.21) and with \( F \) depending on \( X \) (thanks to Remark 1.20). We will say that \( Y \) is a controlled/rough path solution up to time \( S \leq T \) if it is a solution to (1.45) where the driving rough path is substituted with \( X|_{[0, R]} \in C^{p}_{\omega([0, R], \mathbb{R}^d)} \) for all \( R < S \). Note that, according to this terminology, a solution up to time \( T \) is a not necessarily a solution on the whole of \([0, T]\) (the former may explode precisely at time \( T \), while for the latter we have \( Y_T = y_0 + \int_0^T F_*(Y, X)dX \): to distinguish the two we will call the latter a global solution.

**Theorem 1.22** (Local existence and uniqueness). Precisely one of the following two possibility holds with respect to (1.45).

1. A global solution exists;
2. There exists an \( S \leq T \) and a solution up to time \( S \), with \( Y|_{[0, S)} \) not contained in any compact set of \( \mathbb{R}^e \).

Moreover, in either case, the solution is unique on the interval on which it is defined.

The following lemma further specifies that the exit time from an open neighbourhood is bounded from below, uniformly in the initial time and initial condition (ranging in a precompact neighbourhood) of an RDE with fixed driver \( X \). It can be found in [8, Corollary 2.17], and its proof carries over to the setting considered here once again by Example 1.21 and Remark 1.20 (and using the obvious fact that \( X|_{[0, T]} \) is compact).

**Lemma 1.23.** Let \( U, V \subseteq \mathbb{R}^e \) be open with \( V \supseteq U \) compact. Then there exists a \( \delta > 0 \) such that for all \( t_0 \in [0, T] \) and \( y_0 \in U \) the unique solution to

\[
dY = F(Y, X)dX, \quad Y_{t_0} = y_0
\]

is defined and satisfies \( Y \in V \) on \([t_0, (t_0 + \delta) \wedge T]\).

Although this is not a paper on global existence, we will need the following lemma that guarantees it in an important special case.

**Lemma 1.24.** Let \( F \) be as in Definition 1.19 with

\[
F^k_*(y, x) = A^k_*(x)y + b^k_*(x)
\]

for some \( A \in C^\infty(\mathbb{R}^d, \mathbb{R}^e \times \mathbb{R}^d), b \in C^\infty(\mathbb{R}^d, \mathbb{R}^e \times \mathbb{R}^d) \). Then (1.45) admits a global solution.
Proof. First of all, observe that

\[
(\partial_\alpha F_k^\beta + F_h^\alpha A_k^\beta)(y, x) = (\partial_\alpha A_k^\beta + A_l^\alpha A_k^\beta)(x)y^h + (\partial_\alpha b_k^\beta + b_h^\alpha A_k^\beta)(x)
\]

has the same form as \( F \): by Example 1.21 we may therefore assume \( \mathbf{X} \) is geometric. To prove the result it does not suffice to invoke the well-known existence of global solutions for linear RDEs, as \( F \) is not linear in \( x \). However, we may assume \( A \) and \( b \) to be bounded with bounded derivatives of all orders, since we only require the values of \( A \) and \( b \) on the compact set \( X_{[0,T]} \) (and may thus multiply them by a mollifier on \( \mathbb{R}^d \) that vanishes outside an open set containing \( X_{[0,T]} \)). Now [20, Theorem 10.53] may be applied to (1.48), with the only caveat that we have to replace \( \nu \) with \( \max_{\gamma}(\|A_{\gamma}\|_\infty, \|b_{\gamma}\|_\infty) \) in its proof (and in that of [20, Lemma 10.52]).

\[\square\]

1.4 Stochastic rough paths

Finally, we address the topic of stochastic processes lifted to rough paths. We denote with \( S(\Omega, [0,T], \mathbb{R}^d) \) the set of \( \mathbb{R}^d \)-valued continuous adapted semimartingales defined up to time \( T \) on some stochastic setup \( (\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}) \) satisfying the usual conditions. We may define \( \tilde{\mathbf{X}}, \hat{\mathbf{X}} \in \mathcal{C}^p([0,T], \mathbb{R}^d) \) almost surely by Stratonovich and Itô integration, respectively,

\[
\tilde{\mathbf{X}}_{\alpha\beta}^{st} := \int_s^t X_{su}^\alpha \circ dX_{u}^\beta, \quad \hat{\mathbf{X}}_{\alpha\beta}^{st} := \int_s^t X_{su}^\alpha dX_{u}^\beta, \quad \tilde{\mathbf{X}}_{\alpha\beta}^{st} = \hat{\mathbf{X}}_{\alpha\beta}^{st} + \frac{1}{2}[X]_{\alpha\beta}^{st},
\]

where \([X]\) denotes the quadratic covariation tensor of \( X \).

Remark 1.25. We have \([\hat{\mathbf{X}}] = [X]\) and \([\tilde{\mathbf{X}}] = 0\) almost surely so \( \tilde{\mathbf{X}} = \tilde{\mathbf{X}} \). In general, rough path theory applied to semimartingales extends the usual stochastic calculus, that is, Itô/Stratonovich stochastic integrals agree almost surely with the path-by-path computed rough integrals with respect to the Itô/Stratonovich lifts [19, Proposition 5.1, Corollary 5.2], and the strong solution to an Itô/Stratonovich SDE coincides almost surely with the path-by-path computed solution to the RDE driven by the Itô/Stratonovich-enhanced rough path [19, Theorem 9.1] (these results are only shown for Brownian integrators, but may be extended to general continuous semimartingales, for example, by reducing to the Brownian case by splitting the integrator into its bounded variation and local martingale parts and applying the Dubins–Schwarz theorem to the latter).

The following is a statement made in the same spirit which will be important later on.

**Proposition 1.26.** Let \( X \in S(\Omega, [0,T]) \) and \( f \in C^\infty(\mathbb{R}^d, \mathbb{R}^e) \). Then \( f \circ \tilde{\mathbf{X}} \) and \( f \circ \hat{\mathbf{X}} \) coincide almost surely with the lifts of the semimartingale \( f(X) \) computed, respectively, through Stratonovich and Itô integration.

**Proof.** We begin with the Itô case. By the classical Itô formula and Remark 1.25 we have that, almost surely

\[
\int_s^t f^i(X) d f^j(X) = \int_s^t f^i(X) \partial_\gamma f^j(X) dX + \frac{1}{2} \int_s^t f^i(X) \partial_\alpha \partial_\beta f(X) d[X]^\alpha\beta
\]

\[
= \int_s^t f^i \partial_\gamma f^j(X) d\hat{\mathbf{X}}^\gamma + \frac{1}{2} \int_s^t f^i \partial_\alpha \partial_\beta f(X) d[\hat{\mathbf{X}}]^\alpha\beta
\]
\[ \approx f^i \partial_y f^j (X_s) X_s^y + (\partial_{\alpha} f^i \partial_{\beta} f^j + f^i \partial_{\alpha \beta} f^j)(X_s) X_s^{\alpha \beta} + \frac{1}{2} f^i \partial_{\alpha \beta} f^j (X_s) \hat{X}_s^{\alpha \beta} \]
\[ = f^i \partial_y f^j (X_s) X_s^y + \partial_{\alpha} f^i \partial_{\beta} f^j (X_s) \hat{X}_s^{\alpha \beta} + \frac{1}{2} f^i \partial_{\alpha \beta} f^j (X_s) X_s^{\alpha \beta} \]
\[ \approx f^i (X_s) f^j (X) \approx f^i (X_s) f^j (X) st + \partial_{\alpha} f^i \partial_{\beta} f^j \hat{X}_s^{\alpha \beta} . \]

(1.54)

Therefore, almost surely
\[ \int_s^t f^i (X) d f^j (X) \approx \int_s^t f^i (X) d f^j (X) - f^i (X_s) f^j (X) st \approx \partial_{\alpha} f^i \partial_{\beta} f^j \hat{X}_s^{\alpha \beta} \]

(1.55)

and we conclude by Lemma 1.3(1). The Stratonovich case is handled analogously, with the only difference that the first-order change of variable formula holds, and that brackets vanish. □

For other examples of stochastic rough paths, which include lifts of Gaussian and Markov processes, we refer to [20, Chapter III]. Though these rough paths are mostly geometric, examples of non-geometric, non-semimartingale stochastic rough paths also exist in the literature [31].

2 \ ROUGH PATHS, ROUGH INTEGRATION, AND RDEs ON MANIFOLDS

In this section, \( M \) and \( N \) will denote smooth \( m \)- and \( n \)-dimensional manifolds respectively. \( \tau M : TM \to M \) denotes the tangent bundle of \( M \); throughout this paper we will identify fibre bundles with their projection. We will denote the tangent map of a smooth map of manifolds \( f : M \to N \) by \( \tau f : TM \to TN \) (a smooth map), and by \( T_s f \) its restriction to the tangent space \( T_s M \) (a linear map). In this subsection we review equivalent notions of a connection on a manifold. Given a smooth fibre bundle \( \pi : E \to M \) we denote with \( \Gamma \pi \) its \( C^\infty M \)-module of smooth sections and \( E_A := \pi^{-1}(A) \) for \( A \subseteq M \), \( E_x := E_{\{x\}} \) for \( x \in M \). We will denote vectors based at \( x \in M \) (that is, elements of \( T_x M \)) as \( U(\mathbf{x}), V(\mathbf{x}), \ldots \), reserving \( U, V, \ldots \) for vector fields, or vectors based at an unspecified point. Given a chart \( \varphi \) we denote with \( \partial_k \varphi(x) \) the basis elements of the tangent space \( T_x M \) defined by \( \varphi \), and we abbreviate \( \partial_k := \partial_k \varphi \) if there is no risk of ambiguity.

Given a control \( \omega \) on \([0, T] \) we say that a continuous path \( X : [0, T] \to M \) lies in \( C^\omega_p([0, T], M) \) if for all \( f \in C^\infty M \), \( f(X) \in C^\omega([0, T], \mathbb{R}) \); this agrees with the ordinary definition when \( M \) is a vector space by Lemma 1.1. Equivalently \( X \in C^\omega_p([0, T], M) \) if for all charts \( \varphi, \varphi(X) \in C^\omega([a, b], \mathbb{R}^m) \) whenever \( X|_{[a, b]} \) is contained in the domain of \( \varphi \).

Example 2.1 (Path in a fibre bundle). Let \( \pi : E \to M \) be a smooth fibre bundle with typical fibre \( \mathbb{R} \). A path \( H \in C^\omega_p([0, T], E) \) is characterised as follows: for every local trivialisation \( \Phi : E_A \to A \times \mathbb{R} \) and for every \( 0 \leq a \leq b \leq T \) such that \( H[a, b] \subseteq E_A \), we have \( \text{pr}_1 \circ \Phi(H|_{[a, b]}) \in C^\omega_p([a, b], A) \) and \( \text{pr}_2 \circ \Phi(H|_{[a, b]}) \in C^\omega_p([a, b], \mathbb{R}) \). Examples of such paths are given by smooth sections \( \sigma \in \Gamma \pi \) evaluated at \( X \in C^\omega_p([a, b], M) \).
We begin with the main definition.

**Definition 2.2** (Rough path on a manifold). Given an atlas \((\varphi : A_\varphi \rightarrow \mathbb{R}^m)_{\varphi}\) of \(M\), an \(M\)-valued \([2,3)\ni p\)-rough path controlled by \(\omega\) on \([0,T]\), \(\mathbf{X} \in \mathcal{C}^p([0,T],M)\), consists of a collection of rough paths \(\overline{\varphi}\mathbf{X} = (\overline{\varphi}_X, \overline{\varphi}_\Xi) \in \mathcal{C}^p([a_\varphi, b_\varphi], \mathbb{R}^m)\), where the intervals \([a_\varphi, b_\varphi]\) are chosen so that the union of their interiors is \((0,T)\), and with the property that for all charts \(\varphi, \overline{\varphi}\) in the atlas such that \([a_\varphi, b_\varphi] \cap [a_{\overline{\varphi}}, b_{\overline{\varphi}}] \neq \emptyset\)

\[(\overline{\varphi} \circ \varphi^{-1})_\ast \mathbf{X} = \overline{\varphi}\mathbf{X} \in \mathcal{C}^p([a_\varphi, b_\varphi] \cap [a_{\overline{\varphi}}, b_{\overline{\varphi}}], \mathbb{R}^m). \tag{2.1}\]

The trace of \(\mathbf{X}\) is the path \(t \mapsto X_t := \varphi^{-1}(\varphi_X t) \in M\) whenever \(t \in [a_\varphi, b_\varphi]\) (independently of \(\varphi\)), \(\mathbf{X} \in \mathcal{C}^p([0,T],M)\). \(\mathbf{X}\) is geometric, \(\mathbf{X} \in \mathcal{G}^p\omega([0,T],M)\), if \(\varphi\mathbf{X}\) is geometric for all \(\varphi\).

We would like to comment that a very similar definition is given in [5]. In the first paper treating (geometric) rough paths on manifolds [10], on the other hand, rough paths are defined, perhaps more ‘intrinsically’, as objects acting on \(\text{Lip}-\gamma\) forms as integrators. There are two advantages to our definition: first and foremost, it lends itself more to computations such as those performed in Section 4. Second, while integrals against non-geometric rough paths on manifolds require the extra data of a connection (see Definition 2.5), Definition 2.2 relies on the smooth structure of \(M\) alone.

To define a rough path on \(M\) with trace \(X\) we only need as many charts as it takes to cover \(X[0,T]\); once the compatibility condition (2.1) is satisfied for one such cover, for any further chart \(\psi\), \(\overline{\psi}\mathbf{X}\) is automatically defined given this condition; moreover, this definition only depends on the smooth structure on \(M\) and not on the particular atlas covering the trace, thanks to Proposition 1.14(1). The definition of \(\mathcal{E}^p_\omega([0,T],M)\) and of the geometrisation map \(\mathcal{E}^p_\omega([0,T],M) \rightarrow \mathcal{G}^p\omega([0,T],M)\) is well-defined in charts in the obvious way, thanks to the fact that pushforward commutes with geometrisation Proposition 1.14(2) (this also guarantees that \(\mathcal{G}^p\omega([0,T],M)\) is well-defined in the first place). The bracket of a manifold-valued rough path is defined in charts, that is,

\[
[\varphi\mathbf{X}]_{st} := [\overline{\varphi}\mathbf{X}]_{st}, \quad [\overline{\varphi}\mathbf{X}]_{st}^{[\alpha,\beta]} \approx \overline{\varphi}_\alpha \overline{\varphi}_\beta (\varphi_X s) [\varphi\mathbf{X}]_{st}^{[\alpha,\beta]} \tag{2.2}
\]

for charts \(\varphi, \overline{\varphi}\), thanks to Proposition 1.16 (and \(\overline{\varphi}_\alpha \overline{\varphi}_\beta\) is the function \(\partial_{\alpha} \varphi \circ \varphi^{-1} \cdot \partial_{\beta} \varphi \circ \varphi^{-1}\); it should be noted that \([\mathbf{X}]\) is not an \(M\)-valued path, and we only rely on its coordinate expression.

Rough paths on manifolds can be pushed forward by smooth maps: if \(f \in C^\infty(M,N)\) and \(\mathbf{X} \in \mathcal{E}^p_\omega([0,T],M)\), \(f \ast \mathbf{X} \in \mathcal{E}^p_\omega([0,T],N)\) is defined by, for a chart \(\psi\) on \(N\)

\[
(\psi \circ f \circ \varphi^{-1})_\ast \overline{\psi}\mathbf{X} \tag{2.3}
\]

independently of the chart \(\varphi\) on \(M\). Following [16] we define an \(M\)-valued semimartingale to be a stochastic process defined on some setup \((\Omega, \mathcal{F}, P)\) satisfying the usual conditions with the property that \(f(X)\) is a real-valued semimartingale for all \(f \in C^\infty M\), and denote the set of those defined on the interval \([0,T]\) as \(\mathcal{S}(\Omega, [0,T]; M)\). If \(M\) is a finite-dimensional \(\mathbb{R}\)-vector space the two notions of \(\mathcal{S}(\Omega, [0,T]; V)\) coincide thanks to Itô’s formula.
Example 2.3 (Itô and Stratonovich rough paths on \( M \)). Let \( X \in S(\Omega, [0, T]; M) \). We can define its Stratonovich and Itô lifts, respectively, by lifting \( \bar{\varphi}X \) to \( \bar{\varphi}X \) and \( \bar{\varphi}X \) defined in (1.53) on all stochastic intervals \([a, b]\) such that \( X_{[a,b]} \subseteq A_{\varphi} \) (the domain of \( \varphi \)) for \( t \in [a, b] \). Crucially, these almost surely define \( M \)-valued stochastic rough paths thanks to Proposition 1.26, and just as in the linear case we have \( \bar{\mathcal{X}} = X \). These definitions, together with Remark 1.25 allow us to restrict all the rough path theory that follows to the semimartingale context and recover the theory of stochastic calculus on manifolds (Stratonovich and Itô integrals, SDEs, etc.) as presented in [16].

We proceed with the definition of controlled paths, specifically in the case of integrands. While for the definition of rough path we used pushforward to force compatibility, for controlled paths we require it through pullbacks.

**Definition 2.4** (Controlled integrand). Let \( X \in C^B_0([0, T], M) \). We define an \( \mathbb{R}^e \)-valued \( X \)-controlled integrand \( H = (\varphi H, \varphi H') \in \mathcal{D}(\mathcal{L}(\tau M, \mathbb{R}^e)) \) to be a collection \( \varphi H \in \mathcal{D}(\mathcal{L}(\tau M, \mathbb{R}^m)) \) with \( \varphi, a_{\varphi}, b_{\varphi} \) as in Definition 2.2 and

\[
(\varphi \circ \varphi^{-1})^* \varphi H = \bar{\varphi}H, \quad \text{that is, } H^{\gamma} = H_f \varphi^{\gamma}(\bar{\varphi}X), \quad H'^{\gamma}_{\alpha\beta} = H'_f \varphi^{\gamma} \varphi^{\alpha}_{\beta} + H_f \varphi^{\gamma}_{\alpha\beta}. \tag{2.4}
\]

The trace of \( H \) is the path \( H := \varphi H \circ T_X \varphi \), which is valued in the fibre of \( X \) of the bundle \( \mathcal{L}(\tau M, \mathbb{R}^e) = (\tau^* M)^e \).

As for \( \mathbb{R}^d \)-valued controlled paths, the most immediate example is given by the evaluation of a 1-form \( \sigma \in \Gamma(\mathcal{L}(\tau M, \mathbb{R}^e)) \): in coordinates this amounts to \( \sigma(X) = (\sigma^k(X), \partial_\alpha \sigma^k(X)) \).

A smooth map of manifolds \( f \in C^\infty(M, N) \) defines the pullback of controlled paths: if \( X \in C^B_0([0, T], M) \) and \( H \in \mathcal{D}(f(X)) (\mathcal{L}(\tau N, \mathbb{R}^e)) \)

\[
\bar{\varphi}(f^* H) := (\psi \circ f \circ \varphi^{-1})^* \psi H \tag{2.5}
\]

is defined independently of the chart \( \varphi \) on \( M \) by Proposition 1.13(Pullback) and is checked to be an element of \( \mathcal{D}(\mathcal{L}(\tau M, \mathbb{R}^e)) \).

We now give the definition of rough integral on manifolds. Note that this definition already exists for the Stratonovich and Itô integral of semimartingale [16, pp. 93, 109], and thanks to Remark 1.25 the notion below extends these when applied to Example 2.3. It is easily checked that the naïve definition of the rough integral in charts \( \int H \, dX := \int H_f \, dX_f \) fails to be coordinate-invariant due to the bracket correction in the change of variable formula; to come up with an intrinsic notion we will rely on a connection on \( \tau M \). We will rely on [24, Chapter 4] for the definition of connection (in the sense of covariant derivative) on \( \tau M \). We will denote connections with \( \nabla \) and their Christoffel symbols with \( \Gamma \), and recall that

\[
(\nabla_U V)^k = U^h \partial_h V^k + U^i V^j \Gamma^k_{ij} \tag{2.6}
\]

We denote \( \mathcal{T} \) the torsion tensor of the connection \( \nabla \), that is, \( \mathcal{T}^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} \). Given two charts \( \varphi, \bar{\varphi} \) defined on overlapping domains, the (non-tensorial) transformation rule of the Christoffel
symbols is
\[
\Gamma_{ij}^k = \partial_k \partial_i \partial_j \Gamma^k + \partial_h \partial_i \partial_j \Gamma^h, \tag{2.7}
\]
where the functions \( \partial \) refer to the derivatives of the change of chart, for example, \( \partial_{ij} = \partial_{ij}(\varphi^h \circ \varphi^{-1}) = \frac{\partial^2(\varphi^h \circ \varphi^{-1})}{\partial x^i \partial x^j} \).

**Definition 2.5.** Assume \( \tau M \) is endowed with a linear connection \( \nabla \) and let \( X \in \mathcal{C}_p^\omega([0, T], \mathcal{M}), H \in \mathcal{D}_X(\mathcal{L}(\tau M, \mathbb{R}^e)) \). We define the rough integral
\[
\int_0^T H \circ X \circ d\nabla X = \sum_{[s_\varphi, t_\varphi]} \int_{s_\varphi}^{t_\varphi} H \circ X \circ d\nabla X + \frac{1}{2} \int_{s_\varphi}^{t_\varphi} H \circ \Gamma^\gamma_{\alpha\beta}(X) \circ d[X]^{\alpha\beta} \in \mathcal{C}_d^p([0, T], \mathbb{R}^e), \tag{2.8}
\]
where we are summing over a finite partition of \([0, \cdot]\) whose intervals \([s_\varphi, t_\varphi]\) are indexed by charts \( \varphi \) with the property that each \([s_\varphi, t_\varphi]\) is contained in the domain of \( \varphi \), and the coordinates in the integrals are taken with respect to these charts. Moreover, this path can be augmented with the unique second-order part \( \approx H^i_{\alpha; \nu} H^j_{\beta; \nu} X^\alpha_{\beta} \) where the coordinates \( \alpha, \beta \) are taken with respect to any chart that contains \( X_{[s, t]} \), thus defining an element of \( \mathcal{C}_d^p([0, T], \mathbb{R}^e) \).

We will sometimes write \( d \circ M \) for \( d \circ \nabla \), especially when more than one manifold is involved. The definition of \( d \circ \nabla \), in particular the presence of the factor \( \frac{1}{2} \) in the second term, is fixed by certain simple requirements, see Remark 2.12. Note how we have defined the rough integral directly as a rough path, without passing through the notion of controlled path.

**Theorem 2.6.** Definition 2.5 is sound: it depends neither on the partition or on the charts chosen for each interval.

**Proof.** We begin by dealing with the trace. The first assertion immediately follows from the second by comparing two integrals taken with respect to two different partitions with that taken with respect to their common refinement (identity holds by additivity of the rough integral on consecutive time intervals). We then consider two charts \( \varphi, \widetilde{\varphi} \), the latter of whose indices we denote using overlines. Then by Corollary 1.18 we have
\[
\int H \circ X \circ d\nabla X = \int H \circ X \circ d\nabla X + \frac{1}{2} \int H \circ \Gamma^\gamma_{\alpha\beta}(X) \circ d[X]^{\alpha\beta}. \tag{2.9}
\]
Moreover, using Proposition 1.16 and (2.7) we have
\[
\int H \circ \Gamma^\gamma_{\alpha\beta}(X) \circ d[X]^{\alpha\beta} = \int (H \circ \partial^\gamma_{\alpha}(X)) \cdot (\partial^\gamma_{\alpha} \partial^\beta_{\alpha} \Gamma^\gamma + \partial^\delta_{\alpha} \partial^\beta_{\delta} \Gamma^\gamma)(X) \cdot (\partial^\delta_{\alpha} \partial^\gamma_{\beta} \circ d[X]^{\alpha\beta})
\]
\[
= \int (H \circ \Gamma^\gamma_{\alpha\beta}(X) + H \circ \partial^\gamma_{\alpha\beta} \circ \partial^\delta_{\alpha} \partial^\gamma_{\beta} \circ d[X]^{\alpha\beta}. \tag{2.10}
\]
Putting these two identities together, we have

\[
\int H_\gamma dX^\gamma + \int H_\gamma \Gamma_\alpha^\gamma(X) d[X]^\alpha{}^\beta = \int H_\gamma dX^\gamma + \int H_\gamma \Gamma_\alpha^\gamma(X) d[X]^\alpha{}^\beta \\
+ \int H_\gamma (\partial_\gamma^\gamma \partial_\alpha^\gamma + \partial_\gamma^\gamma \partial_\alpha^\gamma \partial_\alpha^\gamma)(X) d[X]^\alpha{}^\beta.
\]

(2.11)

But

\[
\partial_\gamma^\gamma \partial_\alpha^\gamma + \partial_\gamma^\gamma \partial_\alpha^\gamma \partial_\alpha^\gamma = \partial_\alpha^\gamma ((\varphi^{-1} \circ \varphi) \circ (\varphi^{-1} \circ \varphi^{-1})) = \partial_\alpha^\gamma \Gamma_\gamma^\gamma = 0
\]

(2.12)

which yields the desired identity. As for the second-order part, we have

\[
H^{i}_{\alpha, \beta} H^{j}_{\beta, \alpha} \approx (H^{i}_{\gamma, \alpha} \partial_\gamma^\gamma(X_s) H^{j}_{\beta, \alpha} \partial_\beta^\beta(X_s) \cdot (\partial_\alpha^\gamma \partial_\beta^\beta(X_s) \cdot X_s)) = H^{i}_{\alpha, \beta} H^{j}_{\beta, \alpha} X_s.
\]

(2.13)

This concludes the proof. □

Note that (2.2) allows us to define the integral of an element of \( K \in C_p([0, T], \mathcal{L}(\tau M \otimes 2, \mathbb{R}^e)) \) above \( X \), against \([X]\) in coordinates as

\[
\int K d[X] := \int K_{\alpha\beta} d[X]^\alpha{}^\beta.
\]

(2.14)

This is the analogue of [16, Definition 3.9] in the rough path context.

Recall that the Hessian of a connection \( \nabla \) on \( \tau M \) is defined as the map

\[
\nabla \nu : C^\infty M \to \Gamma(\tau \nu M \otimes 2), \quad (\nabla \nu f(x), U(x) \otimes V(x)) := (U(x)V - \nabla_U V)f,
\]

where \( V \) is any extension of \( V(x) \) to a local section and \( U(x)V \) denotes composition of vector fields, that is, the differential operator whose action on \( f \in C^\infty M \) is given by \( U(x)(y \mapsto V(y)f) \).

The Hessian can be written in coordinates as

\[
(\nabla \nu f)_{ij} = (\partial_{ij} - \Gamma_{ij} \partial_k)f.
\]

(2.16)

Following [16], we recall what it means for a map to preserve connections. Let \( M \nabla N \nu \) be a linear connection on the tangent bundle of the smooth manifold \( M \) (\( N \)). We will say that \( f \in C^\infty(M, N) \) is affine if

\[
\forall U, V \in \Gamma\tau M \quad T_x f (M \nabla U(x)V) = N \nabla_{T_x f(U(x))} T f(V).
\]

(2.17)

Note that the right-hand side is well-defined, as \( T f(V) \) need only be defined on a curve tangent to \( U(x) \) at \( x \). The name is justified by the fact that the terminology coincides with the usual notion of affinity for smooth maps of Euclidean spaces. Other examples of affine maps are isometries of Riemannian manifolds (Riemannian isomorphisms that is — local isometries are not affine in
general). In terms of the Hessians affinity of $f$ reads

$$T^∞_x f \circ (N^2 g)(f(x)) = M N (g \circ f)(x), \quad g \in C^\infty N.$$ (2.18)

Symmetrising this identity yields the notion of symmetric affinity: this is equivalent to the requirement that $f$ preserve parametrised geodesics, with full affinity holding if $f$ additionally preserves torsion. The most useful characterisation of affinity, however, is the local one

$$(M, N^2 f)_{\alpha\beta}^k (x) := \partial_{\alpha\beta} f^k (x) + N {\Gamma}_ij^k(f(x)) \partial_i f^j (x) - M {\Gamma}_\gamma^{\alpha\beta} \partial_\gamma f^k (x) = 0$$ (2.19)

which symmetrised yields the condition for symmetric affinity:

$$\partial_{\alpha\beta} f^k (x) = \frac{1}{2}(M {\Gamma}_\gamma^{\alpha\beta} + M {\Gamma}_\gamma^{\beta\alpha}) \partial_\gamma f^k (x) - \frac{1}{2}(N {\Gamma}_ij^k + N {\Gamma}_ji^k)(f(x)) \partial_i f^j (x).$$ (2.20)

Of course, there is no difference between the two if both connections are torsion-free. Symmetrised expressions will be of interest to us because of the symmetry of the bracket of a rough path; to lighten the notation we will add $(\alpha\beta)$ in an expression to mean that we are symmetrising it with respect to to the indices $\alpha, \beta$. For instance, symmetric affinity can be written as $(M, N^2 f)_{(\alpha\beta)}^k (x) = 0$ or more succinctly still as $(M, N^2 f)_{(\alpha\beta)}^k (x) = 0$.

**Proposition 2.7** (Properties of the rough integral on manifolds).

**Exact integrands** (cf. [16, p. 109]). For $f \in C^\infty(M, \mathbb{R}^e)$ holds

$$f(X) - f(X_0) = \int_0^d f(X) d\nu X + \frac{1}{2} \int_0^d \nabla^2 f(X) d[X].$$

**Geometric integrators.** $\int H d\nu X$ does not depend on the torsion of $\nabla$, and if $X$ is geometric it is altogether independent of $\nabla$.

**Pushforward–pullback behaviour.** For $X \in \mathcal{C}^p(M, f \in C^\infty(M, N), H \in \mathcal{D}^p f(X) (\mathcal{L}(\tau N, \mathbb{R}^e))$, $M, N$ connections on $N$ and $M$, respectively,

$$\int H d_N f^* X - \int f^* H d_M X = \frac{1}{2} \int H_k (M, N^2 f)_{\alpha\beta}^k (X) d[X]^{\alpha\beta},$$ (2.21)

where $M, N^2 f$ is defined in (2.19). In particular, the right-hand side above vanishes whenever $X$ is geometric or $f$ is symmetrically affine.

**Proof.** It suffices to show all three statements in a single chart. The first follows immediately from (2.16) and Theorem 1.15. The second is evident from the fact that the bracket of a geometric rough path vanishes, and that even when it does not it is a symmetric tensor. The third is handled by using Corollary 1.18.

**Example 2.8** (Tensorial expansion of the rough integral). The Taylor-type approximation

$$\int_s^t H d\nu X \approx H_{r,s} X^\gamma_{st} + H_{\alpha\beta} X^\gamma_{st} + \frac{1}{2} H_{r,s} {\Gamma}_\gamma^{\alpha\beta} (X_s)[X]^{\alpha\beta}_{st}$$ (2.22)
is coordinate-invariant up to $o(\omega(s, t))$, but the single terms in it are not. We may rewrite it as
\[
\int_s^t H \, d\nu X \approx H_{Y,s}(X^Y_{st} + \frac{1}{2} \Gamma_{\alpha\beta}^Y (X_s) X^\alpha_{st} X^\beta_{st}) + (\nabla H)_{\alpha\beta,s} x^\alpha_{st}, \tag{2.23}
\]
where $\nabla_U V := \nabla_U V - \frac{1}{2} \langle T, U \otimes V \rangle$ denotes the torsion-free connection associated to $\nabla$ and for a connection $\nabla$ we define
\[
(\nabla H)_{\alpha\beta} := H'_{\alpha\beta} - H_k \Gamma_{\alpha\beta}^Y (X).
\tag{2.24}
\]
and therefore $(\nabla H)_{\alpha\beta} = H'_{\alpha\beta} - H_k \Gamma_{\alpha\beta}^Y (X)$. $\nabla H$ is defined by analogy with the coordinate expression for the covariant derivative of a 1-form, that is, if $H = \omega(X)$ for a 1-form $\omega \in \Gamma(\tau M, \mathbb{R}^\ell)$, then $\nabla H = \nabla \omega (X)$. Now all four individual terms $H_{Y,s}, (\nabla H)_{\alpha\beta,s}$ (and even $(\nabla H)_{\alpha\beta,s}$), $X^\alpha_{st} + \frac{1}{2} \Gamma_{\alpha\beta}^Y (X_s) X^\alpha X^\beta$ and $x^\alpha_{st}$ transform as tensors, in the latter two cases up to an $o(\omega(s, t))$. Note that omitting the symmetrisation in $\nabla H$ will result in an incorrect expansion, since the accordingly modified expansion (2.23) will not be almost additive, due to the extra term involving the evaluation of the torsion against $X_{su} \wedge X_{ut}$:
\[
\Gamma_{\alpha\beta}^Y (X_s) (X^\alpha_{su} X^\beta_{ut} - X^\beta_{su} X^\alpha_{ut}). \tag{2.25}
\]
If $X$ is geometric the symmetrisation can be omitted by writing the expansion as $H_{Y,s}(X^Y_{st} + \frac{1}{2} \Gamma_{\alpha\beta}^Y (X_s) X^\alpha_{st}) + (\nabla H)_{\alpha\beta,s} x^\alpha_{st}$ (in this case, of course, the connection is purely auxiliary).

**Example 2.9** (Itô–Stratonovich correction on manifolds). We compare the integral of $H \in \mathcal{D}_X (\mathcal{L}(\tau M, \mathbb{R}^\ell))$ against $X \in \mathcal{C}_\omega ([0, T], M)$ and its geometrisation: by Example 1.9 we have, at the path level
\[
\int H \circ dX - \int H \, d\nu X = \frac{1}{2} \int \nabla H \, d[X]. \tag{2.26}
\]
This identity is the analogue of [13, Theorem 5.17] in the context of rough paths. Note how our ability of writing the above Itô–Stratonovich correction formula in terms of an integral against $d[X]$ is due to the fact that we are integrating controlled paths. In our context of rough integration this is a necessity, but in stochastic calculus on manifolds one can integrate a much larger class of $\mathcal{C}(\tau M, \mathbb{R}^\ell)$-valued processes above $X$, and for these the correction formula will involve the quadratic covariations of the components of $H$ and $X$.

We will now define RDEs driven by manifold-valued rough paths and with solutions valued in a second manifold. The semimartingale-analogue of the definition below can be found in [17, p. 428]. A heuristic derivation of the coordinate expression can be derived by writing the ‘intrinsic differential’ on a manifold $P$ with connection as $d_N Z^k := dZ^c + \frac{1}{2} \Gamma^c_{ab} (Z) d[Z]^{ab}$ and writing the identity $d_N Y^k = F^k_{\gamma}(Y, X) d_M X^\gamma$:
\[
dY^k + \frac{1}{2} N_{ij}^k (Y) d[Y]^{ij} = F^k_{\gamma}(Y, X) (dX^\gamma + \frac{1}{2} M_{\alpha\beta}^\gamma (X) d[X]^{\alpha\beta}). \tag{2.27}
\]
Swapping in $d[Y]^{ij} = F^i_\alpha F^j_\beta (Y, X) d[X]^{\alpha \beta}$ (given that terms of regularity $p/2$ do not contribute to the bracket) then yields (2.29) below.

**Definition 2.10** (RDEs on manifolds). Let $V \in \Gamma \mathcal{L}(\tau M, \tau N)$, $X \in \secat([0,T], M)$ and $y_0 \in N$. We define a *solution* to the RDE

$$d_N Y = \mathcal{F}(Y, X) d_M X, \quad Y_0 = y_0$$

(2.28)

to mean a $N$-valued rough path $Y$ with $Y_0 = y_0$ such that for any two charts on $M$ and $N$, and on any interval restricted to which $X$ and $Y$ are contained in the respective domains, the following RDE (in the sense of Definition 1.19) holds

$$d Y^k = \mathcal{F}^k_\gamma (Y, X) d X^\gamma + \frac{1}{2} (\mathcal{F}^k_\gamma (Y, X) M^\gamma_\alpha (X) - N^k_\gamma (Y) F^i_\alpha F^j_\beta (Y, X)) d[X]^{\alpha \beta},$$

(2.29)

where coordinates are taken (invariantly) with respect to the two charts. Note that this implies $Y^{ij}_{st} \approx F^i_\alpha F^j_\beta (Y_{st}, X_{st}) X^{\alpha \beta}_{st}$.

The coordinate-independence check is analogous to that performed in Theorem 2.6 and is therefore omitted. Analogously to the vector space-valued case, notions of global and local solutions can be defined and distinguished, and the smoothness of $F$ ensures local existence and uniqueness of the solution. These results can, as usual, be proved via ‘patching’ and applying Theorem 1.22. Also note that, just as for the rough integral, only the connection modulo its torsion is relevant, and is not relevant at all when $X$ is geometric, in which case the usual coordinate expression $d Y^k = \mathcal{F}^k_\gamma (Y, X) d X^\gamma$ holds: for this reason we shall omit the $M$ and $N$ subscripts to the differentials in this case.

**Remark 2.11** (Connections with Schwartz–Meyer theory). We mention that there exists a different approach to the topics of this section, one that was used in [16] for stochastic calculus on manifolds, following the ideas of [26, 33]. It involves defining a connection-independent ‘second-order integral’ in which the integrator can be viewed as the combination of the semimartingale $X$ and $\frac{1}{2} [X]$, and in which the integrands are valued in appropriately defined second-order cotangent bundles $T^* M$. The Itô and Stratonovich integrals can then be viewed as particular cases of this integral, with different choices of the integrand, the former of which depends on the connection. While this approach could be adapted to the rough path setting (that is, by defining an appropriate class of controlled paths with trace valued in the second-order cotangent bundle), we chose the more direct approach of defining the rough integral in coordinates and showing invariance under change of charts. In our context coordinates are, in any case, necessary, since they are used in the definition of rough and controlled paths.

In [16, Proposition 7.34] the (scalar) Itô integral is characterised as being the unique additive map

$$\int \cdot \ dX : \mathcal{S}(\Omega, [0,T]; \tau^* M) \to \mathcal{S}(\Omega, [0,T]; \mathbb{R})$$

(2.30)

(where the source denotes the set of continuous $T^* M$-valued semimartingales $H$ such that $H_t \in T^*_X M$ for all $t \in [0,T]$) such that that the chain rule (Proposition 2.7, Exact integrands) and the
associativity axiom

\[ \int \lambda d(\int H dX) = \int \lambda H dX \]  

(2.31)

for all \( \lambda \in S(\Omega, [0, T]; \mathbb{R}) \), hold. While this definition has the advantage of not making explicit reference to coordinate charts, its rough path analogue would be cumbersome to formulate, since \( \lambda \) would have to be chosen to be controlled by \( X \), and thus not necessarily by \( \int H dX \).

**Remark 2.12 (Justifying the definition of \( d_Y X \)).** We can justify that Definition 2.5 is the ‘correct’ definition of rough integral as follows: indeed, as long as we are searching for a local expression of the form

\[ \int H d_M X := \int H_\delta \cdot f_\delta(X) dX^\gamma + \int \nabla^\gamma g_\gamma(X) d[X]^\alpha_\beta \]  

(2.32)

for all \( H \) (in particular for \( H = d\varphi(X) \)), that make Proposition 2.7 (Exact integrands) hold, Theorem 1.7 implies (at least in the non-degenerate case of truly rough \( X \)) that we must pick \( f_\delta^\gamma = \delta_\delta^\gamma \) and \( g_\gamma^\alpha_\beta = \frac{1}{2} \Gamma^\gamma_{\alpha_\beta} \). This leads to a justification of Definition 2.10: adapting [16, Definition 6.35] we may characterise the solution \( Y \) as the unique element of \( C^0(\tau N, \mathbb{R}^c) \) (and it is enough to assume this holds for controlled paths given by scalar exact 1-forms) we have

\[ \int H d_N Y = \int K d_M X, \]  

(2.33)

where \( K \in \mathcal{D}_X(\tau M, \mathbb{R}^c) \) is given in coordinates as \( K = (H \ast Y) \cdot F^i_s(Y, X) \), that is,

\[ K_\gamma^c = K^c_i F^k_\gamma(Y, X), \quad K^i_\alpha_\beta = K^i_{ij} F^j_\alpha F^l_\beta(Y, X) + K^c_\gamma(\partial^i_\alpha F^k_\gamma + F^h_\alpha \partial^i_\beta F^k_\gamma)(Y, X). \]  

(2.34)

We will omit this simple check.

**Remark 2.13 (Local existence and uniqueness).** The local existence and uniqueness theorem Theorem 1.22 extends verbatim to the case of RDEs on manifolds Definition 2.10 (where compacts are determined by the manifold topology). This is proven for Schwartz–Meyer SDEs in [17, Theorem 4] through an affine embedding argument, but in light of Remark 2.11 the proof carries over to our rough path setting. In particular, if \( N \) is compact (2.29) always admits a global solution.

When \( M \) is a Euclidean space we may consider autonomous RDEs in which the field \( F \) only depends on the solution; this is not possible in the general case since the tangent spaces \( T_x M \) are not all canonically identified. The next two examples only deals with manifold-valued semimartingales SDEs, but can be viewed in context of RDEs thanks to Example 2.3.

**Example 2.14 (Local martingales).** Recall that if \( M \) is endowed with a connection \( \nabla \), an \( M \)-valued local martingales \( X \) is an \( M \)-valued semimartingale such that for all \( f \in C^\infty M \),

\[ \int f dX = \int f d_M X, \]  

(2.35)
\( f(X) - \frac{1}{2} \int \nabla^2 f(X) d[X] \) is a local martingale in \( \mathbb{R}^d \), or in local coordinates

\[
d_Y X = dX^\gamma + \frac{1}{2} \Gamma^\gamma_{\alpha\beta}(X) d[X]^{\alpha\beta}
\]

is the differential of a local martingale in \( \mathbb{R}^d \). As observed in [17], it is easy to see that the martingale-preserving property of Itô SDEs carries over to the manifold setting: if \( X \) is an \( M \)-valued local martingale, the solution \( Y \) to (2.29) (where \( X \) is given by the Itô lift of \( X \)) is an \( N \)-valued local martingale.

### 3 | THE EXTRINSIC VIEWPOINT

In this paper, we have mostly chosen to adopt a local perspective on differential geometry. This choice was motivated by the fact that the most natural definition of rough and controlled paths involve charts, and that therefore the theory stemming from these notions would most easily be handled using local coordinates. While we shall continue with this approach in the next section, one of our objectives is to compare our results with those of the main published paper on this topic, [8], in which manifolds are handled using an extrinsic approach. To do this, we will revisit the main definitions of the previous section, assuming that all manifolds are smoothly embedded in Euclidean space, and using ambient Euclidean coordinates to express our formulae. We will show that our results do indeed extend those of [8], in which only geometric rough paths and 1-form integrands are considered. One of the most interesting aspect of this section, however, is that for things to generalise in the correct manner to the case of general controlled integrands, additional non-degeneracy hypotheses will have to be placed on the class of integrands; these are always satisfied if \( X \) is truly rough.

We begin with some background on embedded manifolds, following [8]. Let \( M \) be Riemannian and isometrically embedded in \( \mathbb{R}^d \), \( i : M \hookrightarrow \mathbb{R}^d \) (this is always possible by the Nash embedding theorem, for high enough \( d \)). This means that the connection on \( M \) will always be the Levi-Civita connection of the induced metric: this setting is less general than the one considered in the rest of the paper, where non-metric connections with torsion were considered. To precisely distinguish between extrinsic and intrinsic formulae, we will always distinguish between objects on \( M \) (which will be treated using local coordinates, indexed by Greek letters \( \alpha, \beta, \gamma, \ldots \)) and their counterparts on \( \mathcal{M} := i(M) \) (treated using ambient coordinates, indexed by the letters \( a, b, c, \ldots \)). For instance \( T_y M \) and \( T^\perp_y M \) (the normal space) are subspaces of \( T_y \mathbb{R}^d, T^\perp_y \mathbb{R}^d = T_y M \oplus T^\perp_y M \), and \( T_x i : T_x M \rightarrow T_{i(x)} \mathcal{M} \) is an isomorphism.

The geometry of \( \mathcal{M} \) is characterised by the field of orthogonal projection maps

\[
P \in \Gamma \mathcal{L}(\tau_{\mathcal{M}} \mathbb{R}^d, \tau \mathcal{M}), \quad \text{that is, } P(y) \in \mathcal{L}(T_y \mathbb{R}^d, T_y \mathcal{M}), \ y \in \mathcal{M}.
\]

The use of \( \mathcal{M} \) above means we are availing ourselves of the ambient coordinates to think of \( P \) as a \( d \times d \) matrix defined smoothly in \( y \in \mathcal{M} \). In practice \( \mathcal{M} \) is often defined (locally) through a Cartesian equation \( F(y) = 0, F : \mathbb{R}^d \rightarrow \mathbb{R}^{d-m} \) with surjective differential, in which case the projection map is given by

\[
P(y) = DF^\top \circ (DF \circ DF^\top)^{-1} \circ DF(y).
\]
Note that $DF \circ DF^\top$ is invertible thanks to the surjectivity of $DF^\top$. Also note that this expression provides a smooth extension of $P$ to a tubular neighbourhood of $\mathcal{M}$ in $\mathbb{R}^d$, although this depends on $F$ (whereas $P$ only depends on $\mathcal{M}$). We also define

$$Q(y) := I_d - P(y) \in \mathcal{L}(T_y\mathbb{R}^d, T_y^\perp \mathcal{M}), \quad y \in \mathcal{M},$$

the orthogonal projection of $T_y\mathbb{R}^d$ onto the normal bundle of $\mathcal{M}$. Of course, we have

$$P_cP_c(y) = P(y), \quad Q_cQ_c(y) = Q(y), \quad P_cQ_c(y) = 0 = Q_cP_c(y)$$

for $y \in \mathcal{M}$.

Another important map is the Riemannian projection, uniquely determined and defined smoothly in a tubular neighbourhood $A$ of $\mathcal{M}$ in $\mathbb{R}^d$ [30, p. 132]

$$\pi : A \to M, \quad y \mapsto \arg\min_{x \in M} |y - \iota(x)|, \quad \Pi := \iota \circ \pi : A \to \mathcal{M}.$$ (3.5)

The important features of $\pi$ and $\Pi$ are

$$\pi \circ \iota = 1_M, \quad \Pi \circ \Pi = \Pi, \quad \iota = \Pi \circ \iota.$$ (3.6)

We may express the Levi–Civita covariant derivative $\nabla$ of $M$ in ambient coordinates as follows:

$$T_x\iota\nabla_{U(x)}V = P(\iota(x))d\nabla_{T_x\iota U(x)}(T\iota V), \quad \nabla_{U(x)}\omega = d\nabla_{T_x\iota U(x)}(\omega \circ T\pi)$$

for $U(x) \in T_xM, V \in \Gamma\tau M, \omega \in \Gamma\tau^*M$, where we have smoothly extended $T\iota(V)$ to a vector field on a tubular neighbourhood of $\mathcal{M}$ and $d\nabla$ is the canonical covariant derivative on $\mathbb{R}^d$ given by taking directional derivatives

$$d\nabla_{W(y)}Z := W^c\partial_cZ(y), \quad d\nabla_{U(y)}\rho := U^c\partial_c\rho(y).$$ (3.8)

We reiterate that computations are carried out in ambient coordinates, that is, $\partial_c$ is differentiation in the $c$th variable of $\mathbb{R}^d$, and sums go from 1 to $d$.

Differentiating $F \circ \Pi = 0$ (where $F = 0$ is a Cartesian equation defining $\mathcal{M}$), and the fact that $T^\perp_x \mathcal{M}$ is spanned by the gradients of the components of $F$, shows that $D\Pi|_{T^\perp \mathcal{M}} = 0$. Since $\Pi|_{\mathcal{M}} = 1_{\mathcal{M}}, D\Pi|_{T\mathcal{M}} = 1_{T\mathcal{M}}$ we have

$$P(y) = D\Pi(y), \quad y \in \mathcal{M}.$$ (3.9)

Moreover, if $W(y) \in T_y\mathcal{M}$, differentiating $\partial_b \Pi(Y_t) = P_b(Y_t)$ at time 0, where $Y$ is a smooth curve in $M$ with $Y_0 = 0, \dot{Y}_0 = W(y)$ we have $\partial_a P_b W^a(x) = \partial_{ab} \Pi W^a(x)$, or in other words

$$\partial_a P_b P^a(y) = \partial_{ab} \Pi P^a(y), \quad y \in \mathcal{M}$$

(and in particular the left-hand side is independent of the extension of $P$ to a tubular neighbourhood). Another useful fact about the second derivatives of $\Pi$ is the following identity, obtained by
differentiating [3.6], second identity] twice at $y \in M$ and applying (3.9):

$$\partial_{cc} \Pi^{c} P^{c}_{a} P^{c}_{b}(y) + P^{c}_{e} \partial_{ab} \Pi^{c}(y) = \partial_{ab} \Pi(y). \quad (3.11)$$

This implies that for $W(y), Z(y) \in T_{y} M$

$$\partial_{ab} \Pi W^{a} Z^{b}(y) = Q_{c} \partial_{ab} \Pi^{c} W^{a} Z^{b}(y) \in T_{\perp y} \mathcal{M}. \quad (3.12)$$

This coincides with the second fundamental form of $W(y), Z(y)$, that is, $Q(y) d\nabla W(y) Z$, since

$$Q_{c}(y)(d\nabla W(y) Z)^{c} = Q_{c} \partial_{a} Z^{c} W^{a}(y) = Q_{c} \partial_{a} (\Pi^{c} Z^{b}) W^{a}(y) = Q_{c} (\partial_{ab} \Pi^{c} Z^{b} + P^{c}_{b} Z^{b}) W^{a}(y) = Q_{c} \partial_{ab} \Pi^{c} W^{a} Z^{b}(y) \quad (3.13)$$

and is an extrinsic quantity (it cannot be defined on $M$ without the embedding). Finally, we may express the Christoffel symbols of $\nabla$ (with respect to some chart on $M$) through $i, \pi$ as follows:

$$\Gamma_{\alpha \beta}^{\gamma}(x) = \partial_{c} \pi^{\gamma}(i(x)) \partial_{\alpha \beta}^{c}(x). \quad (3.14)$$

To prove this identity, let $\tilde{\partial}_{\alpha}, \tilde{\partial}_{\beta}$ be extensions to a tubular neighbourhood of $M$ of $Ti \partial_{\alpha} = \partial_{\alpha} i, Ti \partial_{\beta} = \partial_{\beta} i$, respectively. (3.7) implies

$$\Gamma_{\alpha \beta}^{\gamma}(x) = \partial_{c} \pi^{\gamma}(d\nabla_{\tilde{\partial}_{\alpha}} \tilde{\partial}_{\beta})(i(x)) \quad (3.15)$$

and

$$(d\nabla_{\tilde{\partial}_{\alpha}} \tilde{\partial}_{\beta})(i(x)) = \partial_{c}(\tilde{\partial}_{\beta} i)(i(x)) \partial_{\alpha}^{c}(x) = \partial_{c}(\tilde{\partial}_{\beta} i \circ \pi)(i(x)) \partial_{\alpha}^{c}(x) = \partial_{c} \pi(i(x)) \partial_{\alpha}^{c}(x) = \partial_{c} \pi^{\gamma}(i(x)) \partial_{\alpha}^{c}(x) \quad (3.16)$$

which concludes the argument.

We are now ready to state when an $\mathbb{R}^{d}$-valued rough path may be considered to lie on $\mathcal{M}$: this will entail not only the obvious requirement on the trace, but also a condition on the second-order part.

**Definition 3.1** (Constrained rough path). Let $X \in \mathcal{C}_{p}^{\omega}([0, T], \mathbb{R}^{d})$. We will say $X$ is constrained to $\mathcal{M}$ if $\Pi, X = X$, and denote the set of $\mathcal{M}$-constrained $p$-rough paths controlled by $\omega$ with $\mathcal{C}_{p}^{\omega}([0, T], \mathcal{M})$ and its subset of geometric ones with $\mathcal{G}_{p}^{\omega}([0, T], \mathcal{M})$. 
The function $\iota_s$ defines bijections $\mathcal{C}_\omega^\theta([0,T],M) \to \mathcal{C}_\omega^\rho([0,T],M)$ with inverse $\pi_s$, but we still choose to distinguish the two notions, since local coordinates are used in the former case, while $\mathcal{C}_\omega^\theta([0,T],M) \subseteq \mathcal{C}_\omega^\rho([0,T],\mathbb{R}^d)$. An equivalent way of stating Definition 3.1 for an $\mathbb{R}^d$-valued rough path $\mathbf{X}$ is as follows: $X \in \mathcal{C}_\omega^\theta([0,T],M)$, or equivalently by Taylor’s formula

$$X^c_{st} = \Pi^c(X)_{st} \approx P^c_d(X)X^d_{st} + \frac{1}{2} \partial_{ab} \Pi^c(X)X^a_{st}X^b_{st}$$

(3.17)

and

$$\chi^cd_{st} \approx P^c_aP^d_b(X)\chi^ab_{st} \Leftrightarrow Q^c_a(X)\chi^ab_{st} \approx 0 \Leftrightarrow Q^d_b(X)\chi^ab_{st} \approx 0.$$ 

(3.18)

Moreover, these imply

$$[X]^cd_{st} \approx P^c_aP^d_b(X)[X]^{ab}_{st}.$$ (3.19)

We note straight away that this definition extends the characterisation [8, Corollary 3.32(2)] to the non-geometric setting; the characterisation [8, Corollary 3.32(1)] (which states that $Q^a_I^b(X_s)(\chi^ab_{st} - \chi^ba_{st}) \approx 0$) does not hold, however, for non-geometric rough paths, as the symmetric part of their second-order part is not determined by their trace (a counterexample is easily found by taking $\mathbf{X} \in \mathcal{C}_\omega^\rho([0,T],M)$ and then adding to $\chi$ any path $Z \in \mathcal{C}_\omega^{\rho/2}([0,T],(\mathbb{R}^d)^\otimes 2)$ such that $Q^d_b(X)Z^ab_{st} \approx 0$).

Instead of defining a notion of ‘constrained controlled path’ we directly define a notion of rough integral ‘on $\mathcal{M}$’ which is valid for any path in $\mathbb{R}^{e \times d}$ that is controlled by the trace of the integrator. We will then show that, under an additional hypothesis on the integrand, this integral only depends on the restriction of the integrand (and indeed just of its trace) to $T_X M$.

**Definition 3.2 (Constrained rough integral).** Let $\mathbf{X} \in \mathcal{C}_\omega^\rho([0,T],\mathcal{M})$, $\mathbf{H} \in \mathcal{D}_\mathcal{X}(\mathbb{R}^{e \times d})$. We define the $\mathcal{M}$-constrained rough integral of $\mathbf{H}$ against $\mathbf{X}$ (both as an element of $\mathcal{D}_\mathcal{X}(\mathbb{R}^e)$ and as one of $\mathcal{C}_\omega^\rho([0,T],\mathbb{R}^e)$) as

$$\int H d_{\mathcal{M}} \mathbf{X} := \int \Pi^*H d\mathbf{X} = \int (H \cdot P(X))d\mathbf{X}.$$ (3.20)

The identity above is shown by the following simple calculation (we will reuse the letters $e,d$ as indices without the risk of ambiguity)

$$\int_s^t \Pi^*H d\mathbf{X} \approx \int_s^t H_{d;st}^e \mathcal{P}^d_e(X)X^e_{st} + (H_{e;st}^f \mathcal{P}^c_eP^f_a(X) + H_{d;st}\partial_{ab}\Pi^d(X)\chi^ab_{st})$$

$$\approx \int_s^t H_{d;st}^e \mathcal{P}^d_e(X)X^e_{st} + (H_{e;st}^f \mathcal{P}^c_eP^f_a(X) + H_{d;st}\partial_{ab}\Pi^d(X)\chi^ab_{st})$$

$$\approx \int_s^t H_{d;st}^e \mathcal{P}^d_e(X)X^e_{st} + (H_{e;st}^f \mathcal{P}^c_eP^f_a(X) + H_{d;st}\partial_{ab}\Pi^d(X)\chi^ab_{st})$$

(3.21)

$$\approx \int_s^t (H \cdot P(X))d\mathbf{X},$$
where we have used that $X$ is constrained to $M$, (3.4) and (3.11); at the level of Gubinelli derivatives/second-order parts the identity is obvious.

Using Corollary 1.18 we compute the correction formula for the traces of the ordinary and constrained rough integrals

$$
\int H \, dX - \int H \, d\Pi_\ast X = \int H \, d\Pi_\ast X - \int \Pi^\ast H \, dX = \frac{1}{2} \int H_c \theta_{ab} \Pi^\ast (X) \, d[X]
$$

(3.22)

while their second-order parts both agree with

$$
\psi_{ij}^{st} \approx H_{c,s}^i H_{d,s}^j \rho_{a}^c \rho_{d}^b (X_s) \gamma_{ab}^{st} \approx H_{a,s}^i H_{b,s}^j \gamma_{ab}^{st}.
$$

(3.23)

In particular, if $X$ is geometric

$$
\int H \, d\mathcal{M}X = \int H \, dX
$$

(3.24)

and hence agrees with [8, Definition 3.24] when restricted to the case of 1-forms (see Example 3.4).

Also note that if $X$ is the Itô or Stratonovich stochastic rough path associated to a semimartingale, the above definition coincides, thanks to Remark 1.25, with the usual Itô and Stratonovich integrals, given in extrinsic form in [13, Definition 5.13].

Now, it is clear that if $\Pi^\ast H$ (or equivalently $\pi^\ast H$, since $\iota_s$ is injective) vanishes, $\int H \, d\mathcal{M}X$ also vanishes, and we may conclude that the integral depends only on the restriction of $H$ to $\mathcal{M}$ in the sense that $\Pi^\ast H = \Pi^\ast K \Rightarrow \int \mathcal{K} \, d\mathcal{M}X = \int H \, d\mathcal{M}X$. This, however, falls short of our goal of generalising [8, Corollary 3.35] (or rather one implication — we will address the second one in Remark 3.6), which states, in our notation, that if $X \in \mathcal{C}_o^\omega([0,T],\mathcal{M})$ then $\int f(X) \, dX = 0$ for all $f \in \Gamma\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ such that $i^\ast f = 0$. The point is that the requirement is only placed on the trace $f(X)$ of the integrand, not on the whole controlled path. Unfortunately, without further assumptions, the obvious generalisation to the setting of general controlled integrands of this statement fails. The example below exhibits two ways in which this can occur.

**Example 3.3.** Take $\mathcal{M}$ to be the unit circle $S^1$ in $\mathbb{R}^2$, so $\Pi$ is given by

$$
\Pi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2, \quad (x,y) \mapsto \frac{(x,y)}{\sqrt{x^2 + y^2}}.
$$

(3.25)

Let $Z \in \mathcal{C}_o^\omega([0,T], \mathbb{R}^2)$ given by

$$
Z_t := (1, 0), \quad Z_{st} := \begin{pmatrix} t-s & 0 \\ 0 & t-s \end{pmatrix}
$$

(3.26)

which satisfies the Chen identity thanks to the constancy of the trace. Define $X := \Pi_\ast Z \in \mathcal{C}_o^\omega([0,T], \mathcal{M})$: it is checked that

$$
X_t = (1, 0), \quad \chi_{st} \approx \begin{pmatrix} 0 & 0 \\ 0 & t-s \end{pmatrix}.
$$

(3.27)
Now let

\[ H_t = (H_{1; t}, H_{2; t}) := (1, 0), \quad H' := 0_{2 \times 2} \]  

(3.28)

Trivially, \( (H, H') \in \mathcal{D}_X(\mathbb{R}^{1 \times d}) \), and we compute

\[
\int_s^t H \, dX \approx H_{d; s}^\alpha d^d c(X_s) X_{st}^c + (H'_{c; d; s}^e X_{st}^c + H_{c, s}^e \partial_{ab} \Pi^c(X_s)) X_{st}^{ab} \\
= H_{1; s} \partial_{22} \Pi^1(X_s) X_{st}^{22} \\
= s - t
\]  

(3.29)

despite the fact that \( H|_{T X M} = 0 \) (and even \( H'|_{T X M \otimes 2} = 0 \)).

Another example is given as follows: let \( \mathcal{M} = \mathbb{R}^d \) with \( d = 2 \) (or embed in \( \mathbb{R}^3 \) if we want non-zero co-dimension) and let \( X \) be the geometric rough path

\[
X_t := (0, 0), \quad X_{st} := \begin{pmatrix} 0 & t - s \\ t - s & 0 \end{pmatrix}
\]  

(3.30)

and \( H \) be given by

\[
H := 0, \quad H'_{st} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]  

(3.31)

Again membership to \( \mathcal{D}_X(\mathbb{R}^{2 \times 1}) \) is trivially satisfied and proceeding as above we compute

\[
\int_s^t H \, dX = H'_{c, d; s}^e X_{st}^c + H_{c, s}^e \partial_{ab} \Pi^c(X_s) X_{st}^{ab} = H'_{12} X_{st}^{12} = t - s.
\]  

(3.32)

To summarise, in the first example we were able to have \( H|_{T X M} = 0 \), \( H'|_{T X M \otimes 2} = 0 \), but the manifold had to be non-flat (\( D^2 \Pi \neq 0 \)) and the rough path had to be chosen to be non-geometric (if not \( X_{22} = (X_{st}^2)^2 \approx 0 \), assuming \( X \) is chosen to be in \( C^{p/2}_\omega ([0, T], \mathbb{R}^d) \), which is necessary to produce a counterexample). In the second example, we were able to choose a geometric rough path, and even a flat manifold, but it was not the case that \( H'|_{T X M \otimes 2} = 0 \) (although it still held that \( H|_{T X M} = 0 \)).

As will be shown later in Corollary 3.9, this type of behaviour can be ruled out whenever \( X \) is truly rough when viewed as being \( \mathcal{M} \)-valued — it is therefore not accidental that in the examples above \( X \) was chosen to be constant (and in particular an element of \( C^{p/2}_\omega ([0, T], \mathbb{R}^d) \)). A case which is instead always well-behaved is that of 1-form integrands.

**Example 3.4** (1-form integrands). Let \( f \in \Gamma \mathcal{L}(\tau \mathbb{R}^d, \mathbb{R}^c) \) be a 1-form defined on \( \mathbb{R}^d \), and assume for the moment that \( f(X)|_{T X M} = 0 \). Then, by differentiating \( f = f_d Q^d \) we obtain

\[
f(X) = (f_d Q^d_c(X), \partial_a f_d Q^d_b(X) - f_d \partial_{ab} \Pi^d(X))
\]
which implies
\[
\Pi^* f(X) = (f_d Q^d_e c(X), \partial_e f_d Q^d_e P^f c(X) - f_d \partial f_e \Pi^d P^f_e (X) + f_d \partial_{ab} \Pi^d)
\]
\[
= (0, f_d \partial_f \Pi^d (P^d_e Q^f_e a f b(X) + Q^e_a P^f_e b(X) + Q^e_a Q^f_e b(X))),
\]
so that
\[
\int_t^s f(X) d_M X \approx f_d \partial_f \Pi^d (P^d_e Q^f_e a f b(X) + Q^e_a P^f_e b(X) + Q^e_a Q^f_e b(X))(X),
\]
so that
\[
\int_t^s f(X) d_M X \approx f_d \partial_f \Pi^d (P^d_e Q^f_e a f b(X) + Q^e_a P^f_e b(X) + Q^e_a Q^f_e b(X)) X_{ab} \approx 0
\]
by (3.18). By linearity, this implies that for a general 1-form \( f \), \( \int f(X) d_M X \) only depends on \( f(X)|_{T_M} \).

The same conclusion follows if we realise that the formula [8, (3.17)]
\[
\int_t^s f(X) d_X X \approx f_d P^d c(X) X_{ab} + (\nabla f)_{ef} (X) P^f e b(X) X_{ab} \approx 0
\]
(3.33)
extends to the case of non-geometric rough paths and that \( \nabla f(x) \) (defined in (3.7)) only depends on \( f(x)|_{T_M} \) for \( x \in M \). This is the extrinsic version of (2.23) applied to 1-form integrands, and the same expansion would hold for arbitrary controlled paths, by defining
\[
(\nabla H)_{ab} := H'_{ab} + H_{c \partial_{ab}} \Pi^c(X).
\]
(3.34)
We have also shown that Definition 3.2 extends [8, Definition 3.24].

**Example 3.5** (Itô–Stratonovich corrections on embedded manifolds). By Proposition 1.14(2) the geometrisation of an \( M \)-constrained rough path is still constrained, and we may use Example 1.9 to compute the trace-level difference of the constrained integrals against \( X \) and its geometrisation as
\[
\int H \circ dX - \int H d_M X = \frac{1}{2} \int (\nabla H)(X),
\]
(3.35)
This is the extrinsic version of Example 2.9.

**Remark 3.6.** In [8, Corollary 3.20], it is shown that, for \( X \in \mathcal{E}_a^p([0, T], \mathbb{R}^d) \) with \( X \) valued in \( M \), the condition
\[
\int f(X) dX = 0 \quad \forall f \in \Gamma \mathcal{L}(\tau \mathbb{R}^d, \mathbb{R}^e) \text{ such that } \Pi^* f = 0
\]
implies (3.18) and thus \( X \in \mathcal{E}_a^p([0, T], M) \). In order to, attempt to generalise this statement to the non-geometric case we must pick which of the integrals in (3.20) to use; in both cases, however, the statement becomes trivial since, and even replacing the quantifier over 1-forms with one over all controlled integrands \( H \), we are dealing with the integral against \( X \) of a controlled path with trace \( H c P c(X) = 0 \): if \( X \) is truly rough this implies that the whole integrand, and thus the integral, vanishes, regardless of the behaviour of \( X \). Note that using the ordinary \( \mathbb{R}^d \)-integral in place of
the constrained integral (the two coincide for geometric rough paths by (3.24)) is not meaningful either: indeed, if (3.36) implied \( X \in C^P ([0, T], \mathcal{M}) \) for non-geometric \( X \), by Example 1.9

\[
\int f(X) \circ dX = \int f(X) dX + \frac{1}{2} \int D f(X) d[X] = \frac{1}{2} \int \partial_a f_b(X) d[X]^{ab}
\]  

which would have to be zero by Theorem 1.7. But this is not the case if we pick \( \mathcal{M}, X \) as in Example 3.3, first example, \( e = 1 \) and \( f(x^1, x^2) = (x^1, x^2) \) (which restricts to 0 on \( T \mathcal{M} \)) we have \([X]^{ab}_{st} = 2 \delta^{a2} \delta^{b2} (s - t)\) and therefore \( \frac{1}{2} \int \partial_a f_b(X) d[X]^{ab} = s - t \neq 0 \), a contradiction.

The only way (that we can think of) to characterise non-geometric rough integrals in terms of ambient ones would be to endow \( \mathbb{R}^d \) with a connection such that \( \mathcal{U} \) is symmetrically affine (which can always be done [17, Lemma 15]) and replacing the integral in (3.36) with the rough integral in \( \mathbb{R}^d \) taken with respect to this connection, in the intrinsic sense of Definition 2.5. This, however, falls short of the goal of characterising constrained rough paths in terms of notions that do not involve manifolds.

**Example 3.7** (Affine subspaces). If \( M \) is an affine subspace of \( \mathbb{R}^d \) then \( P \) is constant, and

\[
\int_s^t H dM_X \approx H_{d,s}^d X^c_{st} + H'_{e,f,s}^e P_a^f X^{ab}_{st}
\]  

and in particular only depends on \( H |_{T_X M}, H' |_{T_X M \otimes 2} \).

We still have not related the constrained rough integral with its intrinsic counterpart, defined in Definition 2.5. This is done as follows.

**Theorem 3.8.** Let \( X \in C^P ([0, T], \mathcal{M}), H \in D_X (\mathbb{R}^{xd}) \). Then

\[
\int H d_M X = \int i^* H d_M \pi_* X
\]  

**Proof.** Applying Proposition 2.7 to \( i \), we obtain

\[
\int H d_M_X = \int \pi^* H (d (i \circ \pi))_* X
\]  

\[
= \int i^* \pi^* H d \pi_* X + \frac{1}{2} \int H_{d, e}^d (X) (M, \nabla^2)_{\alpha \beta}^c (\pi(X)) d[\pi_* X]^{\alpha \beta}.
\]  

Now,

\[
\int i^* \pi^* H d \pi_* X = \int (\Pi \circ i)^* H d \pi_* X = \int i^* H d \pi_* X
\]  

and applying (2.19) and (3.14), for \( x \in M, y := i(x) \)

\[
(M, \nabla^2)^c_{\alpha \beta} (x) = \partial_{\alpha \beta} f' (x) - \Gamma_\alpha ^{\gamma} \partial_\gamma f' (x)
\]

\[
\partial_\alpha f' (x) - \partial_\gamma f' (y) \partial_{\alpha \beta} \partial_\gamma f' (x)
\]
\[
\begin{align*}
&= \partial_{\alpha\beta} f^e(x) - \partial_e (\pi \circ \alpha)(y) \partial_{\alpha\beta} f^e(x) \\
&= \partial_{\alpha\beta} f^e(x) - P^e(x) \partial_{\alpha\beta} f^e(y) \\
&= Q^e(y) \partial_{\alpha\beta} f^e(x)
\end{align*}
\]

which implies
\[
H\pi d^d_c(X) = H\pi d^d_c Q^e(X) \partial_{\alpha\beta} f^e(\pi(X)) = 0
\]
concluding the proof. \(\square\)

The following corollary makes sense in light of the fact that true roughness (1.14) is invariant under diffeomorphisms and can thus be defined for manifold-valued paths, in charts.

**Corollary 3.9.** If \(X\) is such that \(\pi(X)\) is truly rough, \(\int H dM_X\) only depends on \(H|_{T_X M}\).

**Proof.** Let \(H, K\) be such that \(H|_{T_X M} = K|_{T_X M}\). Then by Theorem 3.8
\[
\int K dM_X - \int H dM_X = \int t^*(K - H) d\pi, X = 0
\]
since \((K - H) \circ Tt = 0\) and true roughness of \(X\) imply \(t^*(K - H) = 0\). \(\square\)

We now turn to the extrinsic treatment of RDEs. Let \(\mathcal{N}_l := N \hookrightarrow \mathbb{R}^e\) be a Nash embedding of another Riemannian manifold \(N, \mathcal{N}_l(N) =: N,\) and \(\mathcal{N}_\pi, \mathcal{N}_\Pi\) its projections \((3.5)\) (we will also left superscripts to denote the inclusion/projections relative to \(\mathcal{M}\) accordingly). Let \(F \in \Gamma\mathcal{L}^\omega([0, T], \mathcal{N}_\pi)\) restrict to an element of \(\Gamma\mathcal{L}(\tau \mathcal{M}, \tau \mathcal{N})\) (this means \(F(y, x)\) maps \(T_x \mathcal{M}\) to \(T_y \mathcal{N}\) for \(x \in M, y \in N\); just as in the intrinsic setting the expression \(dY^k = F^k(Y, X) dX^e\) is ill-defined, in the extrinsic setting \(Y\) will, for \(X\) non-geometric, exit \(\mathcal{M}\) when the equation is started on \(\mathcal{M}\). Proceeding heuristically to derive the extrinsic counterpart to the local RDE formula (2.29), with the idea that \(d\mathcal{N}_X = P^e_c(x) d\mathcal{M} X^c\) for an embedded manifold \(\mathcal{N}\) and \(Z \in C^p([0, T], \mathcal{N})\), we interpret
\[
d\mathcal{N}_X Y^k = F^k_c(Y, X) d\mathcal{M} X^c
\]
as \(\mathcal{N}_h^p(Y) dY^h = F^p_c(Y, X) dX^e\) or (using \(X \in C^p_\omega([0, T], \mathcal{M})\), imposing \(Y \in C^p_\omega([0, T], \mathcal{N})\) and using \((3.10)\)) as
\[
\mathcal{N}_h^p(Y_s) Y^h_{st} + \partial_i \mathcal{N}_\Pi^k(Y_s) Y^k_{st} + \partial_i \mathcal{N}_\Pi^k(Y_s) Y^i_{st} \approx F^k_c(Y, X) dM X^c_{st}
\]
\[
Y^k_{st} \approx F^k_c(Y, X) dM X^c_{st}
\]
Note that we have chosen not to expand the right-hand side of the first line into first and second-order parts. Now, by \((3.17)\) applied to \(Y\), we may rewrite this as
\[
Y^k_{st} = \frac{1}{2} \partial_i \mathcal{N}_\Pi^k(Y_s) Y^i_{st} + \partial_i \mathcal{N}_\Pi^k(Y_s) Y^i_{st} \approx F^k_c(Y, X) dM X^c_{st}
\]
or as \((3.48)\) in the definition below.
**Definition 3.10** (Constrained RDE). Given $X \in \mathcal{C}_p^\mathcal{N}([0,T], \mathcal{M})$, $y_0 \in \mathcal{N}$ and $F \in \Gamma(\tau \mathbb{R}^d, \tau \mathbb{R}^e)$ which restricts to an element of $\Gamma(\tau \mathcal{M}, \tau \mathcal{N})$ we will write

$$d_N Y^k = F^k_c(Y, X)dM_cX^c, \quad Y_0 = y_0$$

(3.47)

to mean

$$dY^k = F^k_d(Y, X)^{\mathcal{M}}p^d_c(X)dX^c + \frac{1}{2}\partial_{ij}^N\Pi^k(Y)F^i_aF^j_b(Y, X)d[X]^{ab}, \quad Y_0 = y_0$$

(3.48)

and say that $Y$ solves the $\mathcal{N}$-constrained RDE driven by the $\mathcal{M}$-constrained rough path $X$.

The next proposition legitimises this formula.

**Theorem 3.11.** Let $X, y_0, F$ be as in Definition 3.10.

1. The solution to (3.48) only depends on $(F(y, x)|_{\tau \mathcal{M}})_{x \in \mathcal{M}, y \in \mathcal{N}}$ and belongs to $\mathcal{C}_p^\mathcal{N}([0,T], \mathcal{N})$.
2. If $X$ is geometric, so is $Y$ and the equation can be rewritten as $dY^k = F^k_c(Y, X)dX^c$.
3. $Y \in \mathcal{C}_o^\mathcal{N}([0,T], \mathbb{R}^e)$ satisfies (3.48) if and only if $\mathcal{N}^*\pi Y$ solves the RDE driven by $W := \mathcal{M}^*\pi X$

$$dZ = (T^N\pi \circ F(\mathcal{N}i(Z), \mathcal{M}i(W)) \circ T^M)i)dW$$

(3.49)

in the sense of Definition 2.10.

**Proof.** In this proof, we will draw on the entirety of the theory of Section 3, and will therefore omit the precise equations which motivate our computations. The first part of (1) will be automatically proved once we show (3); we therefore proceed to show that $Y$ is $\mathcal{N}$-constrained. We have, omitting all evaluations at $Y_s$ and $X_s$ and relying on indices to distinguish maps referring to $\mathcal{M}$ and $\mathcal{N}$ (for example, $P^c_d := \mathcal{M}p^c_d(X_s)$, $\partial_{ij}^N \Pi^k := \partial_{ij}^N \Pi^k(Y_s)$)

$$P^k_{h}Y^i_{st} + \frac{1}{2}\partial_{ij}^N \Pi^k Y^i_{st} Y^j_{st} \approx P^k_{h}[F^h_dP^d_cX^c_{st} + (\partial_a F^h_cP^c_{b} + F^h_d \partial_{i} F^c_{i}P^c_{b} + F^h_c \partial_{a} F^c_{a}P^c_{b})X^c_{st} + \frac{1}{2}\partial_{ij}^N \Pi^k F^i_{a}F^j_{b}[X]^{ab}]$$

(3.50)

We calculate

$$P^k_{h}F^h_cP^d_c = F^k_dP^d_c$$

$$P^k_{h}(\partial_a F^h_cP^c_{b} + F^h_c \partial_{a} P^c_{b}) = \partial_a(F^k_cP^c_{b}) - Q^k_{h} \partial_a(F^h_cP^c_{b})$$

$$= \partial_a(F^k_cP^c_{b}) - \partial_a(Q^k_{h}F^h_cP^c_{b})$$

$$= \partial_a(F^k_cP^c_{b})$$

$$= \partial_a F^k_cP^c_{b} + F^k_c \partial_{a} P^c_{b}$$
\[ P^k_h \delta_i F^h P^c_j = F^i_a \delta_i (P^k_h F^h P^c_j) - F^i_a \delta_i P^k_h F^h P^c_j \]

\[ = F^i_a \delta_i (F^k_c P^b_j) - \delta_j \Pi^k F^i_a F^j_l \]

\[ = F^i_a \delta_i F^k_c P^b_j - \delta_j \Pi^k F^i_a F^j_l \]

\[ P^k_h \delta_{ij} \Pi^i_j F^k_a F^j_l [X]_{st}^{ab} \approx P^k_h \delta_{ij} \Pi^i_j F^k_a F^j_l [X]_{st}^{ab} \]

\[ P^k_h \delta_{ij} \Pi^i_j F^k_a F^j_l [X]_{st}^{ab} \approx 0. \quad (3.51) \]

Substituting these in (3.50)

\[ P^k_h Y^h Y^j \]

\[ \approx F^k_a F^l_c X^c_{st} + (\delta_a F^k_c P^b_j + F^k_c \delta_a P^c_j + F^l_a \delta_l F^h P^c_j) X^a_{st} + \frac{1}{2} \delta_{ij} \Pi^k F^i_a F^j_l (X^a_{st} X^b_{st} - 2 X^a_{st} ) \quad (3.52) \]

\[ \approx Y^k. \]

To prove (2), we proceed in a similar fashion: if \( X \) is geometric, we have

\[ Y^k_{st} \approx F^k_a F^l_c X^c_{st} + (\delta_a F^k_c P^b_j + F^k_c \delta_a P^c_j + F^l_a \delta_l F^h P^c_j) X^a_{st} + \frac{1}{2} \delta_{ij} \Pi^k F^i_a F^j_l (X^a_{st} X^b_{st} - 2 X^a_{st} ) \quad (3.53) \]

and \( Y \) is geometric because it is the solution to an RDE driven by an \( \mathbb{R}^d \)-valued geometric rough path.

The proof of (3) is analogous to that of Theorem 3.8 and therefore omitted. \( \square \)

RDEs can be used to generate elements of \( \mathcal{C}_\omega^P([0, T], \mathcal{M}) \) starting from any unconstrained rough path (cf. [8, Example 4.12, Proposition 4.13] for the geometric case).

**Example 3.12** (Projection construction of constrained rough paths). Let \( Z \in \mathcal{C}_\omega^P([0, T], \mathbb{R}^d) \). Then the solution \( X \) to

\[ d_M X^k = P^k_c(X) d[\delta d] Z^c, \quad X_0 = x_0 \in M, \quad (3.54) \]

that is,

\[ dX^k = P^k_c(X) dZ^c + \frac{1}{2} \delta_{ij} \Pi^k F^i_a F^j_l (X) d[Z]^{ab} \quad (3.55) \]

belongs to \( \mathcal{C}_\omega^P([0, T], \mathcal{M}) \) by Theorem 3.11. Here \( P \) and \( \Pi \) refer to the embedded manifold \( \mathcal{M} \). Moreover, it is checked, using (3.17) and (3.18) that if \( Z \in \mathcal{C}_\omega^P([0, T], \mathbb{R}^d) \) with \( Z_0 = x_0 \), then \( X = Z \), that is, this defines a projection \( \mathcal{C}_\omega^P([0, T], \mathbb{R}^d) \to \mathcal{C}_\omega^P([0, T], \mathcal{M}) \).
In this section, we will discuss parallel transport and Cartan development (or ‘rolling without slipping’) along manifold-valued, possibly non-geometric rough paths. The topic has already been addressed in the geometric case (in the extrinsic setting) in [8]; not assuming geometricity however introduces several complications. The literature on Itô calculus of semimartingales on manifolds also features similar topics, and we shall reference such instances throughout the section; however, because of the adjustments that need to be made for the rough path setting, and because of the greater generality with which the theory is approached (even when restricted to semimartingales), the material presented in this section will only depend upon the first two sections of this paper. We will rely on local coordinates for our computations, and will not explore parallel transport and development in the extrinsic context.

We begin with some geometry of fibre bundles. Given a smooth fibre bundle \( \pi : E \to M \) with typical fibre the smooth \( n \)-dimensional manifold \( R \) (note that in general this is not a vector space), its vertical bundle \( V\pi \) is the sub-bundle of \( \tau E \) with total space \( VE := \ker(T\pi : TE \to TM) \), and we have \( V_{e(x)}E = T_{e(x)}E_x \), that is, elements of the total space of \( V\pi \) are vectors tangent to the fibres of \( \pi \). Recall that for a smooth map of manifolds \( f \in C^\infty(P, Q) \) and a fibre bundle \( \rho : D \to Q \), we define the pullback bundle

\[
    f^*\rho : \{(p, d) \in P \times D \mid f(p) = \rho(d)\} := f^*Q \to P, \quad (p, d) \mapsto p
\]

and there is a bundle map \( f^*\rho \to \rho \)

\[
    \begin{array}{ccc}
    f^*Q & \xrightarrow{p} & D \\
    f^*\rho & \downarrow & \downarrow \rho \\
    P & \xrightarrow{f} & Q
    \end{array}
\]

The vertical lift of \( \pi \) is defined as the fibre bundle isomorphism

\[
    \pi^*\pi \to V\pi, \quad E_x \times E_x \ni (e(x), U(x)) \mapsto \nu(e(x))U(x)
\]

\[
    \nu(e(x))U(x)(f \in C^\infty E) := \left. \frac{d}{dt} \right|_0 f(e(x) + tU(x)).
\]

When \( \pi = \tau M \) a connection on \( \tau M \) defines an Ehresmann connection, a vector bundle \( \eta : H \to E \) which is complementary to \( V\pi \), that is, \( H \oplus VE = TE \). To describe this correspondence we first define the horizontal lift (relative to an Ehresmann connection \( \eta : H \to M \) on the fibre bundle \( \pi \)) as the fibre bundle isomorphism

\[
    \mathcal{h} : \pi^*\tau M \to \eta, \quad E_x \times T_xM \ni (e(x), U(x)) \mapsto \mathcal{h}(e(x))U(x) := T_{e(x)}\pi\vert_{H_{e(x)}}^{-1}(U(x)),
\]

that is, \( \mathcal{h} \) is a splitting of the short exact sequence of vector bundles:

\[
    0 \to V\pi \to \tau E \overset{T\pi}{\to} \pi^*\tau M \to 0.
\]
The Ehresmann connection associated to a covariant derivative (where $\pi$ now is a vector bundle) is given in terms of its horizontal lift as

$$\hat{h}(e(x))U(x) := T_x e(U(x)) - \nu(e(x))\nabla_{U(x)} e$$  \hspace{1cm} (4.6)

for any section $e \in \Gamma\pi$ whose value at $x$ is $e(x)$ (the independence on the section $e$ is checked by using the usual characterisation of tensoriality [24, Lemma 2.4], that is, by showing that $\hat{h}(f e(x))U(x) = f(x)\hat{h}(e(x))U(x)$: this is easily done in local coordinates).

If we have a chart $\varphi : A \to \mathbb{R}^m$ for $A \subseteq M$, a chart $\phi : B \to \mathbb{R}^n$ for $B \subseteq R$ (the typical fibre of $\pi$, an arbitrary $n$-dimensional manifold) and a trivialisation $\Phi : E_A \to A \times R$, the triple $(\varphi, \phi, \Phi)$ defines a chart

$$(\varphi \times \phi) \circ \Phi : \Phi^{-1}(A \times B) \to \mathbb{R}^m \times \mathbb{R}^n.$$  \hspace{1cm} (4.7)

We will call the resulting coordinates product coordinates. If $\pi$ is a vector bundle, $R$ can (and always will) be taken equal to $\mathbb{R}^n$ and $\phi$ to the identity, and if $\pi = \tau M$ or $\tau^* M$, $\Phi$ can be defined canonically in terms of $\varphi$ as $T \varphi$ or $T^* \varphi^{-1}$. In these cases, we will speak of induced coordinates.

Convention 4.1. In what follows we will be working on the manifolds $T M$ and $E$. It will therefore be helpful to establish conventions regarding indexing of the product and induced coordinates. In the absence of other manifolds, ambiguities as to the chart, and so on, we will denote with Greek indices $\alpha, \beta, \gamma, ... = 1, ..., m$ the coordinates on $M$, with Latin indices $i, j, k, ... = m + 1, ..., m + n$ the coordinates on $E$ in excess of the aforementioned coordinates of the base space $M$ and with $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, ... = m+1, ..., 2m$ the induced coordinates on $TM$ in excess of those on $M$. More specifically, $\tilde{\gamma} := m + \gamma$, and we will take this into account when using the Einstein convention, for example, $a_{\alpha\beta} b^{\beta\gamma} = \sum_{\beta=1}^{m} a_{\alpha\beta} b^{(m+\beta)\gamma}$. Moreover, we will use capital letters $I, J, K, ... = 1, ..., m + n$ to denote indices that run through all coordinates on $E$, and capital letters $A, B, C, ... = 1, ..., 2m$ to denote indices that run through all the coordinates on $TM$. The following diagrams should help explain this arrangement:

$$E : \begin{pmatrix} x^1, \ldots, x^m, y^1, \ldots, y^n \end{pmatrix}$$  \hspace{1cm} (4.8)

$$TM : \begin{pmatrix} x^1, \ldots, x^m, \tilde{x}^1, \ldots, \tilde{x}^m \end{pmatrix}$$

It is important to point out the following potential source of confusion. If $V(x) \in T_x M$ it can be either viewed as a vector in the vector space $T_x M$, with coordinates $V^\gamma(x)$, or as a point in the manifold $TM$, with coordinates

$$(V(x)^\gamma, V(x)^{\tilde{\gamma}}) = (x^\gamma, V^\gamma(x)).$$  \hspace{1cm} (4.9)
Note the different meaning of $V(x)^r$ and $V'(x)$; in any case, this ambiguity will be avoided by always considering elements as vectors whenever otherwise mentioned. The use of the twiddled indices is seen when considering vectors in $TTM$.

Finally, we mention we will continue to use of Greek/Latin indices will also be used in the separate case in which we are dealing with two different manifolds $M$ and $N$, to distinguish between coordinates on the two manifolds.

In the case of a vector bundle the change of product coordinates from $\varphi, \Phi$ to $\varphi, \Phi$ can be written as (using overlined indices to refer to the latter product coordinates)

$$\partial^\varphi_k(\lambda) = \begin{pmatrix} \partial^\varphi_\gamma(x) & 0 \\ \partial^\varphi_\lambda \gamma_k(x) y^k & \lambda^\varphi_k(x) \end{pmatrix}, \quad (\Phi \circ \Phi^{-1})(x,y) = (x, \lambda(x)y) \quad (4.10)$$

for $x = \pi(y)$ and $\lambda \in C^\infty(\varphi(A), \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$. It is worthwhile to specify this to the cases of $\pi = \tau M$, where $\Phi = T\varphi$ and $\lambda^\gamma = \gamma^\gamma$, respectively, and

$$\partial^\gamma_k(y) = \begin{pmatrix} \partial^\gamma_\gamma(x) & \partial^\gamma_\lambda \gamma_k(x) y^k \\ \partial^\gamma_\lambda \gamma_k(x) y^k & \lambda^\gamma_k(x) \end{pmatrix}, \quad (\Phi \circ \Phi^{-1})(x,y) = (x, \lambda(x)y) \quad (4.11)$$

The expression of the horizontal lift in induced coordinates in the case of $\pi = \tau M$ reads

$$(\mathcal{H}(V)U)^r = U^r, \quad (\mathcal{H}(V)U)^\gamma = -\Gamma^{\gamma}_{\alpha\beta}(x)y^\beta U^\alpha. \quad (4.12)$$

Note that in this case the coordinates of the horizontal lift are not only linear in the vector being lifted, but in the point in $TM$ at which the lift is based.

It will be helpful to have defined the frame bundle $\phi M : FM \to M$, the sub-bundle of $\tau M^\oplus m$ whose fibre at $x \in M$ is given by all $m$-frames (that is, ordered bases) of $T_xM$. Since $FM$ is an open subspace of $TM^\oplus m$ it makes sense to use the product coordinates of the latter for the former; these are canonically defined in terms of a chart on $M$ by pairs $(\lambda, \gamma) \in \{1, \ldots, m\}^2$ with the first referring to the copy of $TM$, that is, if $y \in F_xM$ then $y_\lambda := pr_\lambda(y) \in T_xM$ has coordinates $y^\gamma = y^{(\lambda, \gamma)}$. If $M$ is Riemannian we may additionally consider the orthonormal frame bundle $oM : OM \to M$, that is, the sub-bundle of $\phi M$ with total space consisting of orthonormal frames. We define the fundamental horizontal vector fields $\mathcal{H}_\lambda \in \Gamma TFM$, $\lambda = 1, \ldots, m$ by the property $T_ypr_\lambda(\mathcal{H}_\lambda(y)) = \mathcal{H}(y_\lambda)y_\lambda$, or in coordinates

$$\mathcal{H}_\lambda^\gamma(y) = y^\gamma, \quad \mathcal{H}_\lambda^\mu(y, \gamma) = -\Gamma^{\gamma}_{\alpha\beta}(x)y^\beta y^\alpha \quad (4.13)$$

with $y \in F_xM$. If $M$ is Riemannian and $\nabla$ is metric these vector fields restrict to elements of $\Gamma \tau OM$.

**Example 4.2** (Affinity and fibre bundles). It will be important to consider whether the projection map of a fibre bundle $\pi : E \to M$ is an affine map with respect to to chosen linear connections $\nabla$ on the total space $E$ and $\nabla$ on the base manifold $M$. By (2.19), the condition of $\pi$ of being affine reads in coordinates

$$\partial_\mu \pi^\gamma = \partial_k \pi^\gamma \Gamma^K_{\mu j} - \Gamma^\gamma_{\alpha\beta} \partial_j \pi^\alpha \partial_\mu \pi^\beta, \quad (4.14)$$
where the functions $\tilde{\Gamma}$ denote the Christoffel symbols of $\tilde{\nabla}$. Keeping in mind that $(\varphi \circ \pi \circ T\varphi^{-1})$ is the map $(x^1, \ldots, x^m, y_1, \ldots, y^n) \mapsto (x^1, \ldots, x^m)$ we compute

$$\partial_{ij} \pi^\gamma = 0, \quad \partial_\beta \pi^\alpha = \delta_\beta^\alpha, \quad \partial_k \pi^\gamma = 0$$

and (4.14) becomes

$$\tilde{\Gamma}^\gamma_{\alpha \beta} = \Gamma^\gamma_{\alpha \beta}, \quad \tilde{\Gamma}^\gamma_{\alpha j} = \tilde{\Gamma}^\gamma_{i \beta} = \tilde{\Gamma}^\gamma_{ij} = 0.$$  

(4.16)

Replacing symmetric affinity with affinity results in the above coordinate expressions being symmetrised in the bottom two indices.

We will tackle parallel transport along non-geometric rough paths by first studying the more general case of RDEs with solutions valued in fibre bundles above the manifold in which the driver is valued; we will progressively restrict our attention to more tractable and interesting cases until we reach the case of the horizontal lift, that is, in which the equation is the natural generalisation of the parallel transport equation to non-geometric rough paths; this will then be used to define Cartan (anti)development. We will see that treating non-geometric rough paths entails adding Itô-type corrections to the classical formulae, and that the terms appearing in the resulting equations will have to satisfy second-order conditions for properties that are usually taken for granted (well-definedness, linearity, metricity) to hold.

More precisely, we will consider an $m$-dimensional smooth manifold $M$ whose tangent bundle is endowed with a linear connection $\nabla$ which we will think of as fixed throughout this section; given a fibre bundle $\pi : E \to M$ and a linear connection $\tilde{\nabla}$ on $\tau E$ (note we do not require a connection on the bundle $\pi$), we are interested in equations of the form

$$d\tilde{\nabla}Y = F(Y)d\nabla X, \quad Y_0 = y_0 \in E_0,$$

(4.17)

where $X_0 = o \in M$ is a basepoint on the manifold which will be fixed throughout this section, and $F$ is a section of the bundle $\mathcal{L}_E(\tau M, \tau E)$: the $E$ subscript here means that we are dealing with a bundle over $E$, not $E \times M$, that is, the fibre at $y \in E$ is given by $\mathcal{L}(T_\pi(y)M, T_yE)$. The first thing to note is that such equations are not of the form Definition 2.10, since $F$ is not defined for all pairs $(y, x) \in E \times M$; we proceed to introduce the tools that are needed to give this type of equation a meaning.

The first thing we require of $F$ is that

$$T_y \pi \circ F(y) = 1_{T_yM} \iff \delta_\beta^\alpha = (T_y \pi \circ F(y))^\alpha_\beta = (T_y \pi)^\alpha_\beta F^K_\beta(y) = \delta^K_\beta F^K_\beta(y) = F^K_\beta(y).$$

(4.18)

We will assume this condition to hold throughout this section unless otherwise stated. For $W \in \Gamma \tau M$ we define

$$FW \in \Gamma \tau E, \quad (FW)(y) := F(y)W(\pi(y)).$$

(4.19)

In this section, we will understand all expressions as being evaluated at $(x, y)$ with $y \in E_x$ unless otherwise specified. The following definition will be of importance in the study of non-geometric RDEs on fibre bundles.
Definition 4.3. We define $\tilde{F} := \tilde{F}(\tilde{V}, F)$ by

$$\langle \tilde{F}, U \otimes V \rangle := F \nabla_U V - \tilde{V}_F F V \in TE,$$

for $U, V \in \Gamma\tau M$.  

(4.20)

Lemma 4.4. For $U, V \in \Gamma\tau M$ we have

$$(\tilde{V}_{FU} F V)^\gamma = U^\alpha \partial_\alpha V^\gamma + U^\alpha V^\beta \tilde{\Gamma}_{\alpha \beta}^\gamma + F^i_\alpha U^\alpha \nabla^i V^\gamma + U^\alpha F^j_\beta V^\beta \tilde{\Gamma}_{\alpha \beta}^j$$

$$+ U^\alpha V^\beta \tilde{\Gamma}_{\alpha \beta}^k + F^i_\alpha U^\alpha \nabla^i \tilde{\Gamma}_{\alpha \beta}^k + U^\alpha F^j_\beta V^\beta \tilde{\Gamma}_{\alpha \beta}^k.$$  

(4.21)

so, we have $\tilde{F} \in \Gamma L_E(\tau M \otimes^2, \tau E)$, and

$$\tilde{F}_{\alpha \beta} = 0$$

$$\tilde{F}_{\alpha \beta} = F^k_\gamma \Gamma_{\alpha \beta}^k - (\partial_{\gamma} F^k_\gamma + F^k_\gamma \partial_\beta V^\gamma)$$

(4.22)

(4.23)

Proof. We compute

$$\partial_{\gamma}(FV)^K = \partial_{\gamma} (F^k_\gamma (
abla_{\gamma} \circ \pi))$$

$$= \partial_{\gamma} F^k_\gamma V^\gamma + F^k_\gamma \partial_\beta V^\gamma \partial_\beta \pi^\gamma$$

$$= \begin{cases} 
\partial_{\gamma} V^\gamma & K = \gamma \leq m, I = \alpha \leq m \\
0 & K = \gamma \leq m, I = i > m \\
\partial_i F^k_\gamma V^\gamma & K = k \leq m, I = \alpha \leq m \\
\partial_i F^k_\gamma \partial_\beta V^\gamma & K = k > m, I = \alpha \leq m \\
\partial_i F^k_\gamma V^\gamma & K = k > m, I = i > m.
\end{cases}$$

(4.24)

Substituting these terms in

$$(\tilde{V}_{FU} F V)^K = (FU)^l \partial_l (FV)^K + (FU)^l (FV)^j \tilde{\Gamma}_{ij}^k$$

yields the desired expressions.

We must now show that $\tilde{F}$ is bilinear, thus legitimising our use of the notation $\langle \tilde{F}, U \otimes V \rangle$: this is easily done by computing the right-hand side of (4.20) thanks to the previously computed expression, and seeing that the derivatives of $V$ cancel out, leaving us with the desired expressions for $\tilde{F}_{\alpha \beta}^\gamma$ and $\tilde{F}_{\alpha \beta}^k$. □

The task is now to extend $F$ to all pairs $(y, x)$ where $y$ does not necessarily lie in $E_x$ (the existence of such extensions is proven in [16, Lemma 8.16, Proof of Proposition 8.15]), and to investigate when the resulting (4.17), which can now be understood as in Definition 2.10, is independent of the extension. To do so we introduce the following condition on $\nabla$ and $F$ (with $\nabla$ thought of as fixed).
Condition 4.5. Assuming (4.18) holds, for all \( U \in \Gamma \tau M \) we have

\[
T \pi \nabla_{F_U}(F_U) = \nabla_U U,
\]

that is, \( T \pi \langle \bar{F}, U \otimes U \rangle = 0 \).

We have purposefully stated the condition with two copies of the same vector field \( U \), instead of \( T \pi \nabla_{F_U}(F_V) = \nabla_U V \): this is motivated by polarisation (that is, the statement that for a finite-dimensional vector space \( V \) the set \( \{ u \otimes u \mid u \in V \} \) generates \( V \otimes V \), since \( v \otimes w = \frac{1}{2}((v + w) \otimes (v + w) - v \otimes v - w \otimes w) \)) and the fact that we only need the symmetrisation of this identity, since these terms will turn out to be the coefficients of the bracket. Requiring the unsymmetrised version of the condition would perhaps have been more natural, but is not as sharp; similar comments hold for Conditions 4.11 and 4.23.

Lemma 4.6. Condition 4.5 is equivalent to \( \bar{F} \) restricting to an element of \( \Gamma \mathcal{L}_E(\tau M^{\otimes 2}, \nu \pi) \), or in product coordinates to \( \bar{F}^{\gamma}_{\alpha\beta} = 0 \), that is,

\[
\tilde{\Gamma}^{\gamma}_{\alpha\beta} + \tilde{\Gamma}^{\gamma}_{ij} F^i_{\alpha} + \tilde{\Gamma}^{\gamma}_{\alpha j} F^j_{\beta} + \tilde{\Gamma}^{\gamma}_{ij} F^i_{\alpha} F^j_{\beta} = 0.
\]

Moreover, the condition being satisfied for all choices of \( F \) is equivalent to \( \tau M \) being symmetrically affine with respect to \( \bar{\nabla}, \nabla \) (with the ‘only if’ statement only valid for \( m \geq 2 \)).

Proof. The first characterisation of Condition 4.5 is obvious, and the expression in local coordinates is a direct consequence of Lemma 4.4 and polarisation.

As for the second statement, we must check that the conditions on the symmetrised Christoffel symbols stated in Example 4.2 hold. The ‘if’ part is immediate. For the converse, first of all reading the identity with \( F = 0 \) yields \( \Gamma^{\gamma}_{\alpha\beta} = 0 \), and the identity may be rewritten as

\[
(\tilde{\Gamma}^{\gamma}_{ij} + \tilde{\Gamma}^{\gamma}_{\alpha j}) F^j_{\beta} + (\tilde{\Gamma}^{\gamma}_{ij} + \tilde{\Gamma}^{\gamma}_{\alpha j}) F^i_{\alpha} + (\tilde{\Gamma}^{\gamma}_{ij} + \tilde{\Gamma}^{\gamma}_{\alpha j}) F^i_{\alpha} F^j_{\beta} = 0.
\]

Now read the identity for arbitrary but fixed \( \alpha \neq \beta \) (which is possible since \( m \geq 2 \)) and \( j \), and with \( F^k_{ij} := \delta^k_j \delta_{ij} \) (this is possible since the coefficients \( F^k_{ij} \) are completely arbitrary: we do not even have to argue coordinate-independence, as everything is local and we may take \( F \) to be supported in the domain of the chart): this yields \( \tilde{\Gamma}^{\gamma}_{ij} (\alpha j) = 0 \), reducing our identity to \( (\tilde{\Gamma}^{\gamma}_{ij} + \tilde{\Gamma}^{\gamma}_{ij}) F^i_{\alpha} F^j_{\beta} = 0 \). We may then fix arbitrary \( i, j \) and pick \( F \) exactly as above to conclude \( \Gamma^{\gamma}_{ij} (ij) = 0 \).

The next result establishes the link between the condition and well-definedness of RDEs in fibre bundles.

Theorem 4.7. If Condition 4.5 holds (4.17) is well-defined, that is, it is independent of the extension of \( F \) to an element of \( \Gamma \mathcal{L}_{E \times M}(\tau M, \tau E) \). In this case \( \pi(Y) = X \) and the coordinate expression of the RDE reduces to its vertical component and is given by

\[
dY^k = F^k_{\gamma}(Y) dX^\gamma
\]

\[
+ \frac{1}{2} (F^k_{\gamma}(Y) \Gamma^{\gamma}_{\alpha\beta}(X) - (\Gamma^k_{\alpha\beta} + \Gamma^k_{\alpha j} F^j_{\beta} + \Gamma^k_{ij} F^i_{\alpha} F^j_{\beta})(Y)) d[X]^\alpha\beta
\]
which may be written at the trace level as

\[ dY^k = F^k(Y) \circ dX^\gamma + \frac{1}{2} \tilde{F}^k_{a\beta}(Y) d[X]^{a\beta}. \]  

(4.30)

Moreover, if \( X \) is geometric the equation is always well-defined and independent of the connections \( \nabla \) and \( \tilde{\nabla} \).

**Proof.** Denoting still with \( F = F(y, x) \) an arbitrary extension of \( F \) as a section of the bundle \( \mathcal{L}_{ExM}(\tau M, \tau E) \) (that is, \( F(y, \pi(y)) = F(y) \)), we have that the first \( m \) coordinates of the local form of (4.17) is given by

\[ dY^\gamma = F^\gamma_{a\beta}(Y, X) dX^a + \frac{1}{2} \tilde{F}^\gamma_{(a\beta)}(X) - \tilde{H}^\gamma_{(U)}(Y) F^I_{a\beta}(Y, X) d[X]^{a\beta}, \]  

(4.31)

where we have symmetrised the second-order part thanks to the symmetry of the tensor \([X]\). Note that by (4.18), by hypothesis and Lemma 4.6 we have that on pairs \((Y, X)\) such that \( Y \in T_X M \) the coefficient of \( dX^a \) equals \( \delta^a_{\gamma} \) and that of \( d[X]^{a\beta} \) vanishes. Now consider the RDE defined only locally in the domain of the chart (that is, without the claim that the following is a coordinate-invariant expression)

\[ d \left( \begin{array}{c} Y^\gamma \\ Y^k \end{array} \right) = \left( \begin{array}{c} F^\gamma_{a\beta}(Y, X) dX^a + \frac{1}{2} \tilde{F}^\gamma_{(a\beta)}(X) - \tilde{H}^\gamma_{(U)}(Y) F^I_{a\beta}(Y, X) d[X]^{a\beta} \\ F^k(Y, X) \circ dX^\gamma + \frac{1}{2} \tilde{F}^k_{a\beta}(Y, X) \tilde{H}^\gamma_{(a\beta)}(X) \end{array} \right). \]  

(4.32)

The solution to this RDE stays in the fibre of the trace of the driver \( X \). But the solution to this RDE must also solve (4.31), since it takes its values in the locus in which the coefficients of the two coincide (here we are using the obvious principle that the solution of an RDE does not change if the coefficients are modified away from the solution).

If \( X \) is geometric \([X]\) vanishes altogether and we may show well-definedness in the same manner, and is independent of the connections on the source and target manifolds since the driver is geometric. \( \square \)

For the remainder of this section we assume Condition 4.5 is satisfied unless otherwise stated. An even stronger requirement (which we will instead not assume to hold) would be as follows.

**Condition 4.8.** \( F \nabla_U U = \tilde{\nabla}_U F_U \) for all \( U \in \Gamma \tau M \), that is, \( \tilde{F} = 0 \), or in local coordinates (assuming Condition 4.5 already holds)

\[ F^k_{\gamma (a\beta)}(Y, X) \partial_a F^k_{\beta} + F^h_{a\alpha} \partial_h F^k_{\beta} + \tilde{F}^k_{a\beta}(Y, X) \tilde{F}^I_{a\beta} + F^j_{a\alpha} \tilde{F}^k_{I\beta} + F^j_{a\alpha} \tilde{F}^k_{I\beta} \]  

(4.33)

The following is immediately inferred through Theorem 4.7.

**Corollary 4.9.** If Condition 4.8 holds then the trace-level of (4.17) is well-defined and equivalent to

\[ dY = F(Y) \circ dX. \]  

(4.34)
We will mostly use the geometrised form of our fibre bundle-valued RDEs, regardless of whether Condition 4.8 holds or not, keeping in mind that the second-order part of the solution is given in terms of the original rough path $X$ as $Y_{ij}^{ij} \approx F^i_{\alpha} F^j_{\beta}(Y_s)X_{\alpha\beta}^{ab}$.

For the remainder of this section we let $\pi$ be a vector bundle unless otherwise stated. We will say that $U \in \Gamma E$ is linear if its flow defines vector bundle isomorphisms $\Phi_t : \pi \to \pi$ covering some self-diffeomorphisms $\phi_t$ of $M$:

\[
\begin{array}{ccc}
E & \xrightarrow{\Phi_t} & E \\
\downarrow{\pi} & & \downarrow{\pi} \\
M & \xrightarrow{\phi_t} & M
\end{array}
\]

We will say that $F$ (which we are assuming satisfies (4.18) and Condition 4.5) is linear if $FU$ is a linear vector field for all $U \in \Gamma_M$.

**Lemma 4.10.** The condition of $U \in \Gamma E$ of being linear is equivalent to its coordinate expression being of the form

\[
\bar{U}^\gamma(y) = U^\gamma(x), \quad \bar{U}^k(y) = \bar{U}^k_h(x)y^h \quad (4.35)
\]

with $x = \pi(y)$ and for locally defined functions $U^\gamma$, $\bar{U}^k_h$ on $M$. Linearity of $F$ is equivalent to its coordinate expression being of the form

\[
F^k_\gamma(y) = F^k_\gamma^h(x)y^h \quad (4.36)
\]

for locally defined functions $F^k_\gamma^h$ on $M$.

**Proof.** We may show the first statement locally; the global result is inferred by taking finitely many compositions of the flow map. Linearity of $\bar{U}$ amounts to requiring that $\Phi_t$ map fibres to fibres, or $\Phi^\gamma_t = \phi^\gamma_t$ for maps $\phi_t : M \to M$, and that this mapping be linear, or $\Phi^k_t(y) = A^k_{\gamma h}(x)y^h$ for some linear map $A_t(x) : E \to E_{\phi_t(x)}$. The definition of the flow implies

\[
\frac{d}{dt} \bigg|_{t=0} \Phi_t(y) = \bar{U}(y)
\]

which in turn implies the statement, with $\bar{U}^\gamma = \delta^\gamma_0$ and $\bar{U}^k_h = \delta^k_{h;0}$.

Picking an arbitrary $U \in \Gamma M$, we have $(FU)^\gamma(y) = U^\gamma(x)$ by (4.18) and for $(FU)^k(y) = F^k_\gamma(y)U^\gamma(x)$ to be of the form $\bar{U}^k_h(x)y^h$ for all $U$ we need $F^k_\gamma(y)$ to be of the form $F^k_\gamma^h(x)y^h$ (for the ‘only if’ implication simply pick $U^\gamma = \delta^\gamma_\beta$ with $\beta = 1, \ldots, m$).

Note that the $k$ index in $F^k_\gamma^h(x)y^h$ represents a coordinate in $T_x E$, whereas $h$ represents a coordinate in $E_x$, following Convention 4.1 we will not place a twiddle on the upper index, as we view $F^k_\gamma^h$ as the coordinates of a linear map between vector spaces. For the remainder of this section we will assume $F$ is linear, and we will be concerned with the question of whether this implies that the resulting (4.17) is also linear, that is, that its coordinate expression is linear in the conventional
sense. To this end, we introduce the following condition on $F$ and $\nabla$ (which does not involve the connection $\nabla$ at all, to the extent that $\nabla$ is not defined in terms of it).

**Condition 4.11.** Assume $F$ is linear. $\nabla_{FU}(FU)$, or equivalently $\langle F, U \otimes U \rangle$, is a linear vector field for all $U \in \Gamma \tau M$.

**Lemma 4.12.** Let $F$ be linear and Condition 4.5 hold. Then Condition 4.11 is equivalent to $\bar{F}_{|_{TM^2}}$ lying in the image of the map

$$\Gamma(\tau^*M^2 \otimes \pi^* \otimes \pi) = \Gamma L_M(\tau M^2 \otimes \pi, \pi) \rightarrow \Gamma L_E(\tau M^2, V \pi),$$

(4.37)

$$G \mapsto (e \mapsto (U \otimes V \mapsto \nu(e) \langle G, U \otimes V \otimes e \rangle)),$$

where $\nu(e) : E \rightarrow V e \pi$ denotes the vertical lift isomorphism based at $e$. In other words, we may write its coordinates (symmetrising in the first two indices) as $\bar{F}^{(\alpha \beta)} = 0$ and

$$\bar{F}^{(k)}_{\alpha \beta h} y^h = F^k_{\alpha h} y^h - (\partial^k_{\alpha} F^k_{\beta h} + F^l_{\alpha h} F^k_{\beta l} y^l + F^l_{\alpha h} F^k_{\beta l} y^l + F^l_{\alpha h} F^k_{\beta l} y^l + F^l_{\alpha h} F^k_{\beta l} y^l + F^l_{\alpha h} F^k_{\beta l} y^l + F^l_{\alpha h} F^k_{\beta l} y^l),$$

(4.38)

and it follows that an equivalent formulation of the condition is that the expression

$$\bar{\Gamma}^{(k)}_{\alpha \beta} + \bar{\Gamma}^{(l)}_{\alpha \beta} + \bar{\Gamma}^{(k)}_{\alpha \beta} y^h + \bar{\Gamma}^{(k)}_{\alpha \beta} y^h + \bar{\Gamma}^{(k)}_{\alpha \beta} y^h + \bar{\Gamma}^{(k)}_{\alpha \beta} y^h + \bar{\Gamma}^{(k)}_{\alpha \beta} y^h,$$

(4.39)

is linear in the $y$ coordinates.

Moreover, the condition being satisfied for all choices of $F$ as above (without assuming Condition 4.5 is) is equivalent to the stronger requirement that $\nabla_{\bar{U}} \bar{U}$ be linear for all linear $\bar{U} \in \Gamma \tau E$ (with the ‘only if’ statement only valid for $m \geq 2$), which in coordinates reads

$$\bar{\Gamma}^{(k)}_{\alpha \beta} \text{ constant in } y, \quad \bar{\Gamma}^{(l)}_{\alpha \beta} \text{ constant in } y, \quad \bar{\Gamma}^{(k)}_{\alpha \beta} \text{ constant in } y, \quad \bar{\Gamma}^{(l)}_{\alpha \beta} \text{ constant in } y, \quad \bar{\Gamma}^{(k)}_{\alpha \beta} \text{ constant in } y,$$

(4.40)

Note that in (4.38) we are not able to provide an expression for $\bar{F}^{(k)}_{\alpha \beta h}$, since some of the terms on the right-hand side are nonlinear (recall that the functions $\bar{\Gamma}^{(l)}_{\alpha \beta}$ and $F^i_{\gamma h}$ are evaluated at $x$, but the $\Gamma_{ij}$ are nonlinearly evaluated at $y$, and moreover there are quadratic terms).

**Proof of Lemma 4.12.** The first characterisation is just a reformulation of the second, which is evident by (4.23), Lemma 4.6 and the definition of linear vector field. The third follows from the second by subtracting terms that are already linear in $y$.

As for the second statement, we first observe that linearity of $\nabla_{FU}FU$ without requiring Condition 4.5 entails the additional requirement that (by (4.22), rewritten to account for the linearity of $F$) the expression

$$\Gamma^{(k)}_{\alpha \beta} - (\bar{\Gamma}^{(k)}_{\alpha \beta} + F^i_{\alpha h} \bar{\Gamma}^{(i)}_{\beta h} y^h + F^i_{\alpha h} \bar{\Gamma}^{(i)}_{\beta h} y^h + F^i_{\alpha h} \bar{\Gamma}^{(i)}_{\beta h} y^h + F^i_{\alpha h} \bar{\Gamma}^{(i)}_{\beta h} y^h),$$

(4.41)
(where $\alpha\beta$ denotes symmetrisation) be constant in $y$. Then by arguing as in the proof of Lemma 4.6 by progressively disregarding constant (respectively, linear) terms in (4.41) (respectively, (4.39)) we may conclude that linearity of $\tilde{V}_{FU}FU$ for all $F$ and $U$ as above is equivalent to (4.40).

Now, writing $(\nabla \tilde{U} \tilde{V})^K = \tilde{U}^l \partial_l \tilde{V}^K + \tilde{U}^j \tilde{V}^l \tilde{\Gamma}^K_{lj}$ for $\tilde{U}$, $\tilde{V}$ linear (with notation as in (4.35)) we obtain

$$(\nabla \tilde{U} \tilde{V})^y = U^\alpha \partial_{\alpha} \tilde{V}^y + U^\alpha \tilde{V}^j \tilde{\Gamma}^y_{\alpha j} y^j + \tilde{U}^i \tilde{V}^j \tilde{\Gamma}^y_{ij} y^i + \tilde{U}^i \tilde{V}^j \tilde{\Gamma}^y_{ij} y^j y^i,$$

$$(\nabla \tilde{U} \tilde{V})^k = U^\alpha \partial_{\alpha} \tilde{V}^k y^k + \tilde{U}^i \tilde{V}^l \tilde{\Gamma}^k_{il} y^i + U^\alpha \tilde{V}^j \tilde{\Gamma}^k_{\alpha j} y^j + \tilde{U}^i \tilde{V}^j \tilde{\Gamma}^k_{ij} y^i y^j y^j (4.42)$$

As usual, we rely on the symbols involved to infer whether a function is evaluated at $y \in E$ or at $x = \pi(y)$. We then see, by arbitrariness of $U^r$, $V^r$, $U^k$, $\tilde{V}^k \in C^\infty M$, polarisation, and the usual elimination procedure, that linearity of $\nabla \tilde{U}$ is equivalent to (4.40).

Assuming Condition 4.5 is satisfied we may consider the flow map associated to $F$ and $X$ at times $0 \leq s \leq t \leq T$

$$\Phi_{ts} = \Phi(F, X)_{ts} : E_{X_s} \rightarrow E_{X_t}, \ y \mapsto Y_t$$

(4.43)

where $dY = F(Y) dX$, $Y_s = y$

which is defined as long as $Y_t$ is defined, and by uniqueness we have

$$\Phi_{tu} \circ \Phi_{us} = \Phi_{ts}$$

(4.44)

for $0 \leq s \leq u \leq t \leq T$ whenever one of the two sides is defined. The following theorem justifies our interest in the linearity condition.

**Theorem 4.13.** Let $F$ be linear and satisfy Conditions 4.5 and 4.11. Then (4.17) can be written in coordinates as

$$dY^k = F^k_{\gamma h}(X) Y^h \circ dX^\gamma + \frac{1}{2} \tilde{F}^k_{(\alpha\beta)h}(X) Y^h d[X]^\alpha\beta$$

(4.45)

and admits a global solution. Moreover, $\Phi_{ts}$ defines linear isomorphisms $E_{X_s} \cong E_{X_t}$ for all $0 \leq s \leq t \leq T$. These statements also hold, independently of $F$, if $X$ is geometric.

**Proof.** The first statement is a restatement of (4.29) to the case in which Condition 4.11 is satisfied. We may argue global existence by Theorem 1.22 and Remark 2.13: indeed, assume that there exists $S \leq T$ such that $Y_{[0,S)}$ is not contained in any compact set of $M$. Since $\pi(Y) = X$ on $[0,S)$, we must have that $\lim_{t \rightarrow S^-} \pi(Y_t) = X_S$, that is, $Y$ must ‘explode vertically’. This, however, is not possible either, since if we may pick a system of product coordinates which contains $X_S$, this would mean that the coordinate solution to (4.29) must only be defined for $t < S$, which is ruled out by Lemma 1.24.
Standard uniqueness arguments apply charts to show that $\Phi_{ts}$ is a linear monomorphism (and thus an isomorphism, by dimensionality) when $X_s, X_t$ are contained in a single chart, and these can be combined to yield the global statement by ‘patching’ $X[0,T]$ with finitely many charts and applying (4.44).

We will denote $\Phi_{st} := \Phi_{ts}^{-1}$ for $0 \leq s \leq t \leq T$. We proceed to study the local dynamics satisfied by $t \mapsto \Phi_{t0}$ and $t \mapsto \Phi_{0t}$. Fix coordinates for the vector space $E_0 = E_{X_0}$, which we denote with the symbols $i^o, j^o, k^o ...$; we continue to denote with $\alpha, \beta, \gamma ...$ and $i, j, k ...$ the local coordinates in and above a neighbourhood containing $X_t$; we do not intend for the former indices to bear any relationship with the latter (for example, $k^o$ and $k$ appearing in a common expression have nothing to do with each other).

**Proposition 4.14.** The coordinate expressions $\Phi_{k:o;0}^k$ and $\Phi_{k:o;0}^{k^o}$, respectively, solve the RDEs (at the trace level) driven by $(\mathcal{X}, \mathcal{X})$

\[
\begin{align*}
\frac{d\Phi_{k:o;0}^k}{dt} &= F_{k:o}^h(X_t)\Phi_h^{k:o;0} \circ d\mathcal{X}_t^h + \frac{1}{2} F_{k:o}^{(\alpha\beta)}h(X_t)\Phi_h^{k:o;0}d[\mathcal{X}]_{t}^{\alpha\beta}, \\
\frac{d\Phi_{k:o;0}^{k^o}}{dt} &= -\Phi_{k:o;0}^{k^o} F_{k:o}^h(X_t) \circ d\mathcal{X}_t^h - \frac{1}{2} \Phi_{k:o;0}^{k^o} F_{k:o}^{(\alpha\beta)}k(X_t)d[\mathcal{X}]_{t}^{\alpha\beta}.
\end{align*}
\]

(4.46)

**Proof.** The statement is local, and we may confine ourselves to the domain of a single set of product coordinates containing $X_t$. By Theorem 4.13 we have, for $y \in E_o$

\[
\begin{align*}
(d\Phi_{k:o;0}^{k^o})y^{k^o} &= d(\Phi_{k:o;0}^{k^o})y^{k^o} \\
&= dY_t^k \\
&= F_{k:o}^h(X_t)Y_t^h \circ d\mathcal{X}_t^h + \frac{1}{2} F_{k:o}^{(\alpha\beta)}h(X_t)Y_t^h d[\mathcal{X}]_{t}^{\alpha\beta},
\end{align*}
\]

(4.47)

which we rewrite as

\[
\begin{align*}
\frac{d\Phi_{k:o;0}^{k^o}}{dt} &= -\Phi_{k:o;0}^{k^o} F_{k:o}^h(X_t) \circ d\mathcal{X}_t^h - \frac{1}{2} \Phi_{k:o;0}^{k^o} F_{k:o}^{(\alpha\beta)}k(X_t)d[\mathcal{X}]_{t}^{\alpha\beta}.
\end{align*}
\]

(4.48)

We may therefore conclude, by arbitrariness of $y \in E_o$, that the first of the two RDEs holds. As for the second, we have

\[
\begin{align*}
0 &= \delta_{k:o}^{k^o} \\
&= d(\Phi_{k:o;0}^{k^o}) \Phi_{k:o;0}^{k^o} \\
&= (d\Phi_{k:o;0}^{k^o}) \Phi_{k:o;0}^{k^o} + F_{k:o}^h(X_t)\Phi_h^{k:o;0} \circ d\mathcal{X}_t^h + \frac{1}{2} F_{k:o}^{(\alpha\beta)}h(X_t)\Phi_h^{k:o;0}d[\mathcal{X}]_{t}^{\alpha\beta}
\end{align*}
\]

(4.49)

thus, concluding the proof.
For the remainder of this section we will let \( \pi = \tau M \) unless otherwise stated. An important feature of the equation in this case is that we can integrate the inverse of the flow map to obtain a \( T_\pi M \)-valued rough path. Note that, although we have not explicitly defined controlled integrands with values in an arbitrary finite-dimensional vector space \( V \), this is done simply by choosing a basis of \( V \) and setting \( \mathcal{D}_X(\mathcal{L}(\tau M, V)) := \mathcal{D}_X(\mathcal{L}(\tau M, \mathbb{R}^m)) \) under the corresponding isomorphism \( V \cong \mathbb{R}^m \) (all the needed constructions are easily seen not to depend on the choice of the basis).

The next lemma states the change of coordinate formula satisfied by \( \Gamma^\gamma_{\alpha\beta} \):

**Lemma 4.15.**

\[
\Gamma^\gamma_{\alpha\beta} = \frac{\partial^\gamma}{\partial^\gamma_X} \frac{\partial^\alpha}{\partial^\alpha_X} \frac{\partial^\beta}{\partial^\beta_X} \Gamma^\gamma_{\alpha\beta} - \frac{\partial^\gamma}{\partial^\gamma_X} \frac{\partial^\alpha}{\partial^\alpha_X} \frac{\partial^\beta}{\partial^\beta_X} \Gamma^\gamma_{\alpha\beta}
\]

**Proof.** By (4.11), we have

\[
\Gamma^\gamma_{\alpha\beta} \frac{\partial^\beta}{\partial^\beta_X} y^\beta = \Gamma^\gamma_{\alpha\beta} y^\beta = \Gamma^\gamma_{\alpha}
\]

\[
= \frac{\partial^\gamma}{\partial^\gamma_X} \mathcal{F}^\gamma_{\alpha}
\]

\[
= \frac{\partial^\gamma}{\partial^\gamma_X} \frac{\partial^\alpha}{\partial^\alpha_X} \frac{\partial^\gamma}{\partial^\gamma_X} \mathcal{F}^\gamma_{\alpha}
\]

\[
= \frac{\partial^\gamma}{\partial^\gamma_X} \frac{\partial^\alpha}{\partial^\alpha_X} \frac{\partial^\gamma}{\partial^\gamma_X} \mathcal{F}^\gamma_{\alpha}
\]

\[
= (\frac{\partial^\gamma}{\partial^\gamma_X} \frac{\partial^\alpha}{\partial^\alpha_X} \frac{\partial^\gamma}{\partial^\gamma_X} \mathcal{F}^\gamma_{\alpha}) y^\beta
\]

from which

\[
\Gamma^\gamma_{\alpha\beta} \frac{\partial^\beta}{\partial^\beta_X} = \frac{\partial^\gamma}{\partial^\gamma_X} \frac{\partial^\alpha}{\partial^\alpha_X} \frac{\partial^\gamma}{\partial^\gamma_X} \mathcal{F}^\gamma_{\alpha}
\]

thanks to the arbitrariness of \( y \), and we may conclude. \( \Box \)

As there will be no risk of ambiguity, we shall reassign \( \Gamma^\gamma_{\alpha\beta} := \Gamma^\gamma_{\alpha\beta}^\gamma \), and since now, in view of Lemma 4.12, \( \mathcal{F} \) may be viewed as restricting to an element of \( \Gamma (T_\pi M \otimes 2 \otimes T_\pi M \otimes T M) \) it also makes sense to set \( \mathcal{F}^\delta_{\alpha\beta\gamma} := \mathcal{F}^\delta_{\alpha\beta\gamma} \). The tensor field \( \mathcal{F} \) may now be given the following interpretation: its evaluation against \( (U \otimes V) \otimes W \) consists of taking the (symmetrisation of the) defect in commutativity between covariant derivatives and horizontal lift, \( \kappa \nabla_U V - \nabla_{\kappa U} \kappa V \) and mapping its vertical part at \( W \in TM \) down isomorphically onto \( TM \).

**Proposition 4.16.** \( \Phi_0 \in \mathcal{D}_X(\mathcal{L}(\tau M, T_\pi M)) \), where \( \Phi^\gamma_{\alpha\beta,0t} := -\Phi^{\gamma\gamma}_{\alpha\beta,0t} \mathcal{F}^\gamma_{\alpha\beta}(X_t) \).

**Proof.** The local condition is satisfied in each coordinate chart thanks to Proposition 4.14. We must check that the compatibility condition of Definition 2.4 is met: again, this is obvious at the trace.
level, and for Gubinelli derivatives we have, by Lemma 4.15

$$
\Phi^{\gamma^*}_{\alpha \beta} = -\Phi^{\gamma^*}_{\gamma} F^{\gamma^*}_{\alpha \beta}
= -\Phi^{\gamma^*}_{\gamma} \partial_{\gamma} (\partial_{\alpha}^\gamma \partial_{\beta}^\gamma F^{\gamma^*}_{\alpha \beta} - \partial_{\alpha}^\gamma \partial_{\beta}^\gamma)
= -\Phi^{\gamma^*}_{\gamma} (\partial_{\alpha}^\gamma \partial_{\beta}^\gamma F^{\gamma^*}_{\alpha \beta} - \partial_{\gamma}^\gamma)
= \Phi^{\gamma^*}_{\alpha \beta} \partial_{\alpha}^\gamma \partial_{\beta}^\gamma + \Phi^{\gamma^*}_{\gamma} \partial_{\gamma}^\gamma.
$$

(4.53)

Thus, concluding the proof. □

We now restrict our attention for the last time: from now on we will consider the case in which $F$ is given by the horizontal lift $\mathbf{h}$ unless otherwise stated. In terms of the connections $\tilde{\nabla}$, this means we are interested in differentiating horizontal vector fields with respect to horizontal directions, with Condition 4.5 fixing the horizontal part of such covariant derivatives, while Condition 4.11 and the optional Condition 4.8 impose limitations on their vertical part. The reader who is versed in sub-Riemannian geometry may spot the link with horizontal connections [6, Definition 7.4.1], although it should be remarked that our setting is more specific (that is, not all sub-Riemannian manifolds arise as the total space of a vector or even fibre bundle), and the requirements on the connection is somewhat different (on the one hand we are only interested in $\tilde{\nabla}_U$ with $U$ horizontal, and on the other also consider the vertical components of such covariant derivatives). In coordinates

$$
F^{\gamma^*}_{\alpha \beta} = -\Gamma^{\gamma^*}_{\alpha \beta}
$$

$$
\tilde{F}^{\gamma^*}_{\alpha \beta \gamma} y^\gamma (x^\alpha) = -\Gamma^{\gamma^*}_{\epsilon \alpha} \Gamma^{\epsilon \beta} y^\gamma - \Gamma^{\epsilon \nu} \epsilon_{\alpha \delta} \beta_{\nu} y^\gamma
$$

(4.54)

Note how Lemma 4.15 agrees with (2.7).

We are now in a position to be able to provide the natural generalisation of parallel transport of vectors and Cartan (anti)development to the setting of non-geometric rough paths, with $\tau T M$ endowed with a linear connection. Since the development of a path is not guaranteed to remain in the manifold for all time, it will be helpful to define the following variations of the rough path spaces (note the use of the double closing parenthesis):

$$
\mathcal{C}_p^\omega([0, T], M) := \mathcal{C}_p^\omega([0, T], M) \cup \{X \in \mathcal{C}_p^\omega([0, S], M) \mid \text{for some } S \leq T \text{ and } K \text{ compact } K \subseteq M \text{ such that } X_{[0, S]} \subseteq K\}
$$

(4.55)

$$
\mathcal{C}_p^\omega([0, T], M) := \mathcal{C}_p^\omega([0, T], M) \cup \bigcup_{0 \leq S < T} \mathcal{C}_p^\omega([0, S], M).
$$

Note that $\mathcal{C}_p^\omega([0, T], M) \subseteq \mathcal{C}_p^\omega([0, T], M)$, and we also will use these notations when $M$ is a vector space. Moreover, we will add a modifier in the rough path sets to denote those rough paths
which are started at a specific point. The following notions are defined whenever Conditions 4.5 and 4.11 are met.

**Definition 4.17.** Let \( X \in \mathcal{C}_p^\omega([0, T]), M, o \). We will denote

\[
\Pi(X)_{ts} := \Phi(\theta, X)_{ts} : T_{X_s} M \to T_{X_t} M
\]

which by (4.45) is well-defined for all \( s, t \) at which \( X \) is defined, and call it **parallel transport** of vectors along the non-geometric rough path \( X \). We will denote \( \Pi_{ts} := \Pi(X)_{ts}, \Pi_t := \Pi(X)_{0t} = \Pi^{-1} \) when there is no ambiguity as to the rough path.

**Remark 4.18** (There is no alternate notion of ‘backward parallel transport’). A rough path \( X \) canonically defines a rough path \( \overline{X} = (\overline{X}, \overline{X}_t) \) above the inverted path \( \overline{X}_t := X_{T-t} \). This is done by imposing the Chen identity to hold for all \( 0 \leq s, u, t \leq T \) (not just \( s \leq u \leq t \)), or equivalently by taking (1.6) literally, and results in \( \overline{X}_{su} = -\overline{X}_{T-t, T-s} + X_{T-t, T-s}^{\otimes 2} \) for \( 0 \leq s \leq t \leq T \). It is shown that if \( H \in \mathcal{D}_X \) then \( \overline{H} \in \mathcal{D}_{\overline{X}}, \) where \( \overline{H}_t := H_{T-t}, \) and that

\[
\int_0^T \overline{H} d\overline{X} = - \int_0^T H dX
\]

at the trace level. It can then be concluded (by a uniqueness argument) that

\[
\begin{align*}
\frac{dY}{dt} &= y_0 + \int F(Y) dX \\
\frac{d\overline{Y}}{dt} &= Y_T + \int F(\overline{Y}) d\overline{X} \quad \Rightarrow \quad \overline{Y}_t = Y_{T-t}
\end{align*}
\]

which implies that, denoting with \( \Phi \) the flow map of the RDE defined by \( F, X \) and with \( \overline{\Phi} \) the one defined by \( F, \overline{X}, \overline{\Phi} = \Phi^{-1} \). Therefore, once a rough path is fixed, the definition of \( \\backslash \) given above and the one obtained by defining the parallel transport RDE with respect to \( \overline{X} \) coincide.

**Definition 4.19.** Let \( X \in \mathcal{C}_p^\omega([0, T]), M, o \). Using Proposition 4.16 we will denote

\[
\bigcirc(X) := \int \Pi(X) \nu_X \in \mathcal{C}_p^\omega([0, \leq T]), T_0 M, o \)

which we call the **antidevelopment** of \( X \). If \( Z = \bigcirc(X) \) (up to the time at which \( X \) is defined) we will denote \( X = \bigcirc(Z) \) and call \( X \) the **development** of \( Z \).

In coordinates (4.59) amounts to

\[
d\bigcirc^y (X) = \\backslash^y \nu_X X_t + \frac{1}{2} \nu_{X_t} \Gamma^y_{\alpha\beta} d[X]^{\alpha\beta}.
\]

For the moment we have only defined development of a rough path which already is the antidevelopment of an \( M \)-valued one. If we start from an arbitrary \( Z \in \mathcal{C}_p^\omega([0, T]), T_0 M, o \) with \( Z_0 = 0 \), we would like to invert Definition 4.19 and define its development as the solution to the path-dependent RDE

\[
d\nu \bigcirc(Z) = \Pi(\bigcirc(Z)) dZ, \quad \bigcirc(Z)_0 = o.
\]
Heuristically, this means that in an infinitesimal time interval \([t_0, t_0 + dt]\) we are translating the differential \(dZ_{t_0} \in T_{z_{t_0}} T_0 M\) so that it is based at the origin \(0_0\), parallel-transporting it along the already-developed portion of the rough path \(X_{[0,t_0]} := \mathcal{O}(Z)_{[0,t_0]}\) so that it is now based at \(X_{t_0}\), and then using it to ‘roll \(T_0 M\) on \(M\) along \(Z\) without slipping’ for time \(dt\). The problem, of course, is that we have not defined such (adaptedly) path-dependent RDEs. Moreover, it should be noted that even once this equation is given a meaning, contrary to the case of parallel transport there is no reason why the solution should not explode (see \[14,\text{Corollary 1.36}\] for general criteria that rule this out for \(X\) geometric) (Figure 1). The trick to give (4.61) a meaning is to consider it jointly with a parallel frame: this transforms the path-dependent RDE into a state-dependent one.

**Theorem 4.20.** Let \(Z \in \mathcal{C}_\omega^p([0,T], T_0 M)\). Then \(X = \mathcal{O}(Z) \in \mathcal{C}_\omega^p([0,T], M)\) (possibly up to its exit time from \(M\)) if and only if \(X\) is the unique solution to

\[
\begin{align*}
\begin{cases}
\frac{d}{\gamma^o} = \frac{\gamma^o}{\delta^o} \circ dZ^\delta^o + \frac{1}{2} \bar{F}(\nabla, \gamma^o) \circ d\bar{Z}^\delta^o \\
= -\Gamma^\alpha_{\gamma^o} \frac{\gamma^o}{\delta^o} \circ dZ^\delta^o \\
+ \frac{1}{2} \left[ -\Gamma^\alpha_{\gamma^o} \frac{\gamma^o}{\delta^o} - (\Gamma^\alpha_{\gamma^o} \frac{\gamma^o}{\delta^o} + \Gamma^\alpha_{\gamma^o} \frac{\gamma^o}{\delta^o}) \right] \circ d\bar{Z}^\delta^o \\
\end{cases}
\end{align*}
\]

\[
dX^\gamma = \frac{1}{\gamma^o} \frac{\gamma^o}{\delta^o} \circ d\bar{Z}^\delta^o - \frac{1}{2} \frac{\gamma^o}{\delta^o} \circ d\bar{Z}^\delta^o
\]

\[
X_0 = o, \quad \{\gamma^o\}_{\gamma^o=1,\ldots,m} \text{ any basis of } T_0 M
\]

with the functions \(\Gamma\) evaluated at \(X\), and the functions \(\bar{\Gamma}\) evaluated at \(\gamma^o = \gamma(X)\).
The map \( \bigcirc \) therefore defines a surjective map \( \mathcal{C}^p_{\omega}([0, T], T_0 M, 0_o) \to \mathcal{C}^p_{\omega}([0, T], M, o) \) with right inverse \( \circ : \mathcal{C}^p_{\omega}([0, T], M, o) \to \mathcal{C}^p_{\omega}([0, < T], T_0 M, 0_o) \) (composed with any map prolonging an element of \( \mathcal{C}^p_{\omega}([0, S], T_0 M, 0_o) \) up to time \( T \), for example, trivially). In particular, if \( M \) is compact \( \bigcirc \) takes values in \( \mathcal{C}^p_{\omega}([0, T], M, o) \), that is, development exists for all time.

If \( Z \) is geometric this equation may be stated more elegantly as taking values in the frame bundle \( \phi : FM \to M \), and defined by the fundamental horizontal vector fields, that is,

\[
\d Y = \mathcal{H}_\lambda^\gamma(Y)\d Z^\gamma \quad Y_0 \in F_o M \implies \bigcirc(Z) = \phi M Y, \; Y = \mathcal{H}(X).
\]

In this context, compare (4.62) with [23, p. 86, (3.3.9)], which is stated in the case of \( X \) a Brownian motion, although the formula generalises to more general processes/geometric rough paths. We have decided not to consider frame bundle-valued RDEs in the non-geometric case, since this would require defining a connection on \( FM \), which is a delicate matter (some comments to this effect are provided in [17, p. 439] in the case of the complete lift, though these do not contain an exhaustive description of the connection on \( \tau FM \)). We have preferred to define development in a coordinate-free manner by simply declaring \( X \) to be the development of \( Z \) if \( Z \) is the antidevelopment of \( X \) (as done in Definition 4.19), and only relying on the local description involving the parallel frame (seen as \( m \) vectors which are parallel-transported individually) as an alternative characterisation, useful for explicit computations; in this approach only parallel transport of vectors is needed.

**Proof of Theorem 4.20.** By (4.60) \( X = \bigcirc(Z) \) means

\[
\d Z^\gamma_t = \left\langle Y^\delta, \d X^\gamma \right\rangle + \frac{1}{2} \left\langle Y^\gamma, \Gamma^\gamma_{\alpha\beta} \d[X]^{\alpha\beta} \right\rangle, \quad Z_0 = 0_o
\]

and we have

\[
\left\langle Y^\delta, \d Z^\gamma \right\rangle - \frac{1}{2} \Gamma^\gamma_{\alpha\beta} \left\langle \alpha^\gamma, \beta^\delta \right\rangle \d[Z]^{\alpha\beta} = \left\langle Y^\delta, \frac{1}{2} \left\langle Y^\gamma, \Gamma^\gamma_{\alpha\beta} \d[X]^{\alpha\beta} \right\rangle - \frac{1}{2} \Gamma^\gamma_{\alpha\beta} \left\langle \alpha^\gamma, \beta^\delta \right\rangle \d[X]^{\alpha\beta} \right\rangle
\]

\[
= \d X^\gamma.
\]

By Proposition 4.14 and (4.54), we have

\[
\d \left\langle Y^\delta, \d Z^\gamma \right\rangle = \left\langle F^\gamma_{\epsilon\delta}, \left\langle \frac{1}{2} \left[ \Gamma^\gamma_{\alpha\beta} + \frac{1}{2} F^\gamma_{\alpha\delta} F^\gamma_{\beta\epsilon} \right] \right\rangle \d[X]^{\alpha\beta}
\]

\[
= -\left\langle F^\gamma_{\epsilon\delta}, \left\langle \frac{1}{2} \left[ \Gamma^\gamma_{\alpha\beta} + \frac{1}{2} F^\gamma_{\alpha\delta} F^\gamma_{\beta\epsilon} \right] \right\rangle \right\rangle \d[X]^{\alpha\beta}
\]

\[
= -\left\langle F^\gamma_{\epsilon\delta}, \left\langle \frac{1}{2} \left[ \Gamma^\gamma_{\alpha\beta} + \frac{1}{2} F^\gamma_{\alpha\delta} F^\gamma_{\beta\epsilon} \right] \right\rangle \right\rangle \d[Z]^{\alpha\beta}
\]

\[
= -\left\langle F^\gamma_{\epsilon\delta}, \left\langle \frac{1}{2} \left[ \Gamma^\gamma_{\alpha\beta} + \frac{1}{2} F^\gamma_{\alpha\delta} F^\gamma_{\beta\epsilon} \right] \right\rangle \right\rangle \d[Z]^{\alpha\beta}.
\]
where in the last step the Gubinelli derivative of $-\Gamma_{\beta\gamma}(X)/\varepsilon^\gamma$ with respect to $Z^\varepsilon$ is computed thanks to the previous step and (4.65). Retracing these steps proves the converse. Note that we do not need to show the coordinate invariance of (4.62), as we have shown it is equivalent to $Z = \mathcal{C}(X)$, which is defined in Definition 4.19 without reference to a coordinate system.

The map $\mathcal{C}$ is then well-defined by uniqueness of RDE solutions applied to the $(m + m^2)$-dimensional system in each coordinate patch, and its right inverse is $\mathcal{C}$ by definition. It only remains to show that $\mathcal{C}(Z)$ is either defined up to time $T$ or that it is defined up to and excluding some $S \leq T$ with the image of its trace not contained in any compact of $M$. Assume $(X, //)$ is defined up to time $S$ with $X_{[0,S]}$ contained in a compact $K$ of $M$. Therefore, there exists $t_n \searrow S$ such that $\lim X_{t_n} = \bar{x} \in K$. We now show that for any neighbourhood $V$ of $\bar{x}$ there exists $s_0$ such that $X_{[s_0,S]} \subseteq V$. Consider the image of (4.62) (defined in $V$) through a change of coordinates $\Phi$ that maps the $//$ components to a compact, and extend the resulting coefficients smoothly. Now picking a second neighbourhood $U$ of $\bar{x}$ such that $\text{Im}(U) \subseteq U, \bar{U} \subseteq V$, an application of Lemma 1.23 proves the claim by picking $s_0$ such that $s_0 < S < s_0 + \delta$. (The change of coordinates was necessary because we need to be able to start the equation for $(X, //)$ at an arbitrary point in $TU^m$.) We may then reason as in the proof of [Theorem 4.13] to conclude that $(X, //)[0,S)$ must also lie in a compact of $TM^m$, and a second application of Lemma 1.23 (arguing as in [8, Theorem 4.2]) then shows that the solution may be prolonged past $S$ (or with its limit if $S = T$). This concludes the proof.

The following result is proven in [16, Theorem 8.22] in the case of Stratonovich parallel transport, and interestingly it carries over to the more general case.

**Corollary 4.21.** At the trace level we have

$$\mathcal{C}(X) := \int_0^\gamma \|\gamma\|_s d\gamma_s$$

and we may replace

$$dX^\gamma = //^\gamma \circ dZ^\gamma$$

for the second equation of (4.62).

**Remark 4.22.** We emphasise that this does not mean that the (anti)development of a rough path coincides with that of its geometrisation (including at the trace level): in (4.67) parallel transport is still carried out with reference to the original non-geometric $X$ (and thus depends on the choice of $\bar{\nabla}$), and in the case of development, the first equation of (4.62) still has the $d[Z]$ terms, which are not present when developing $gZ$. Moreover, at the second-order level $X^\alpha_\beta_{st} \approx //^\alpha_{\alpha\beta} //^\beta_{\gamma\gamma} Z^\gamma_\beta_{st}$ locally in terms of the original rough path $Z$.

**Proof of Corollary 4.21.** By Proposition 4.14 and (4.60), we have at the trace level

$$\|\gamma\| \circ dX = \|\gamma\| dX_\gamma + \frac{1}{2} \|\gamma\| \Gamma_{\gamma \alpha \beta} d[X]^{\alpha \beta} = \|\gamma\| d_X$$

and the second claim is proved analogously by using (4.62). \qed
Recall that if $\mathcal{g}$ is a Riemannian metric on $M$ a connection $\nabla$ on $\tau M$ is metric if $\nabla \mathcal{g} = 0$, or in coordinates

$$\mathcal{g}_{ij,k} - \mathcal{g}_{hj} \Gamma_{ki}^h - \mathcal{g}_{ikh} \Gamma_{kj}^h = 0,$$

(4.70)

where indices after the comma denote partial differentiation in the chosen chart, that is, $\mathcal{g}_{ij,k} := \partial_k \mathcal{g}_{ij}$, and $\mathcal{g}_{ij}$ the components of the metric in the same chart ($\mathcal{g}^{ij}$ will denote the inverse of $\mathcal{g}_{ij}$, that is, $\mathcal{g}^{ik} \mathcal{g}_{kj} = \delta^i_j$). We will also use indices after a semicolon to denote covariant differentiation, for example, $\mathcal{g}_{ij;k} := (\nabla \mathcal{g})_{ij,k}$. Note that this does not exclude the presence of torsion; if additionally, $\mathbf{T} = 0$ the connection is uniquely determined as the Levi–Civita connection $\mathcal{g} \nabla$ and is given in coordinates by

$$\mathcal{g}^k_{ij} = \frac{1}{2} \mathcal{g}^{kh}(\mathcal{g}_{hj,i} + \mathcal{g}_{ih,j} - \mathcal{g}_{ij,h}).$$

(4.71)

If $M$ is Riemannian and $\nabla$ is metric we may further ask under what hypotheses the functions $\mathcal{V}_{ts}$ are linear isometries $T_X^s M \cong T_X^t M$. The following condition does not actually require $\mathcal{F}$ to be given by horizontal lift, although we will only apply it in that case.

**Condition 4.23.** Let $\mathcal{g}$ be a Riemannian metric on $M$ and $\nabla$ be $\mathcal{g}$-metric: $\langle \tilde{\mathcal{F}}, U \otimes U \otimes V \rangle \in \Gamma \tau M$ is $\mathcal{g}$-orthogonal to $V$ for all $U, V \in \Gamma \tau M$.

Note that we are not requiring $\tilde{\nabla}$ to be metric with respect to a Riemannian metric on the manifold $TM$. The statement of this condition in coordinates is given in the following lemma, whose proof is immediate by polarisation.

**Lemma 4.24.** In coordinates Condition 4.23 corresponds to

$$\widetilde{\mathcal{F}}_{(\alpha \beta)(\gamma \delta)} = 0.$$

(4.72)

**Theorem 4.25.** If Condition 4.23 holds, or if $X$ is geometric, $\mathcal{V}_{ts}$ is a linear isometry for all $0 \leq s, t \leq T$.

The following pattern has emerged: for each property (well-definedness, linearity, and metricity, each required at the level of generality considered) we have a first-order condition (respectively, (4.18), $\mathcal{F}$ linear, and $\nabla \mathcal{g}$-metric — as shall be seen in the proof below) and a second-order condition (respectively, Conditions 4.5, 4.11, and 4.23). The first-order conditions are necessary when considering the geometric (or even ordinary differential equation (ODE)) case, whereas the second-order conditions become relevant once the driving rough path is no longer geometric. Note how all three conditions are automatically satisfied when Condition 4.8 holds.

**Proof of Theorem 4.25.** We may assume $s = 0$; then for $y, z \in T_0 M$ by Proposition 4.14 we have

$$d\langle \mathcal{g}(X), \mathcal{V}_s \otimes \mathcal{V}_t \rangle$$

$$= d\langle \mathcal{g}_{\alpha \beta} \mathcal{V}_{\alpha} \mathcal{V}_{\beta} \rangle$$

$$= \mathcal{g}_{\alpha \beta, \gamma} \mathcal{V}_{\alpha} \mathcal{V}_{\beta} \circ dX'$$
\[ + g_{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \left( -\Gamma^{\gamma}_{\alpha\beta} \circ d[x]^{\gamma} + \frac{1}{2} \tilde{F}^{\gamma}_{(\xi\eta)(\alpha\beta)} d[x]^{\xi\eta} \right) \]

\[ + g_{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \left( -\Gamma^{\gamma}_{\alpha\beta} \circ d[x]^{\gamma} + \frac{1}{2} \tilde{F}^{\gamma}_{(\xi\eta)(\alpha\beta)} d[x]^{\xi\eta} \right) \]

\[ = \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \left[ \left( g_{\alpha\beta,\gamma} - g_{\alpha\delta} \Gamma^{\delta}_{\gamma\beta} - g_{\delta\beta} \Gamma^{\delta}_{\gamma\alpha} \right) \circ d[X]^{\gamma} + \tilde{F}(\xi\eta)(\alpha\beta) d[x]^{\xi\eta} \right]. \] (4.73)

which vanishes by metricity of \( \nabla \), (4.70) and Condition 4.23 or by vanishing of the bracket in the case of \( X \) geometric (note how by Theorem 1.7 the hypotheses are sharp in the case of \( X \) truly rough).

We will now provide three examples of connections \( \tilde{\nabla} \) on \( \tau M \) which it makes sense to consider. The first two, for which we refer to [35], can be viewed as ‘lifts’ of the connection \( \nabla \) (which is not assumed to be metric or torsion-free), while the third consists of assuming \( M \) is Riemannian, defining a Riemannian metric on the manifold \( TM \), and taking its Levi–Civita connection. Recall that the curvature tensor associated to a connection \( \nabla \) is

\[ \mathcal{R}(U,V)W := \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W, \] (4.74)

where \( [U,V] \) denotes the Lie bracket of vector fields, which vanishes if the vectors are given by the local basis sections \( \partial_k \) defined by a chart. We denote the coefficients

\[ \mathcal{R}^h_{ijk} := \langle R(\partial_i, \partial_j) \partial_k, d^h \rangle = \Gamma^h_{jk,i} - \Gamma^h_{ik,j} + \Gamma^h_{il} \Gamma^l_{jk} - \Gamma^h_{jl} \Gamma^l_{ik} \] (4.75)

and warn the reader that the ordering of the indices is not standard in the literature (this convention is, for instance, the one followed by [24, 35]).

**Example 4.26** (The complete lift of \( \nabla \)). Assume \( \nabla \) is a linear connection on \( \tau M \), which we do not assume to be torsion-free or metric. The complete lift \( \tilde{\nabla} \) of \( \nabla \) is the linear connection on \( \tau TM \) whose Christoffel symbols in induced coordinates (with respect to to any chart \( \varphi \) on \( M \)) are given as functions of the Christoffel symbols \( \Gamma^k_{ij} \) of \( \nabla \) with respect to to \( \varphi \) as follows:

\[ \Gamma^\gamma_{\alpha\beta}(x,y) = \Gamma^\gamma_{\alpha\beta}(x), \quad \Gamma^\gamma_{\alpha\delta}(x,y) = \Gamma^\gamma_{\alpha\delta}(x,y) = 0 \]
\[ \Gamma^\gamma_{\alpha\beta}(x,y) = \partial_\lambda \Gamma^\gamma_{\alpha\beta}(x)y^\lambda, \quad \Gamma^\gamma_{\alpha\delta}(x,y) = \Gamma^\gamma_{\alpha\delta}(x), \quad \Gamma^\gamma_{\alpha\beta}(x,y) = \Gamma^\gamma_{\alpha\beta}(x), \quad \Gamma^\gamma_{\alpha\delta}(x,y) = 0. \] (4.76)

From these and Example 4.2, it follows that \( \tau M \) is an affine map with respect to \( \tilde{\nabla}, \nabla \). This connection admits the following simple description: having defined the complete lift of \( V \in \Gamma TM \) as \( \tilde{V} \in \Gamma \tau TM \) given in induced coordinates by

\[ \tilde{V}^\gamma(x,y) := V^\gamma(x), \quad \tilde{V}^\gamma(x,y) := y^\lambda \partial_\lambda V^\gamma(x) \] (4.77)

(this is checked to be a sound definition; note that no further connection is needed to perform this lift) \( \tilde{\nabla} \) is characterised by the condition

\[ \tilde{\nabla}_U \tilde{V} = \nabla_U \tilde{V}, \quad U, V \in \Gamma \tau M. \] (4.78)
We will only need the local description of $\tilde{V}$. However, we remark that the complete lift can be extended to tensor fields, and in particular to Riemannian metrics $g$, thus yielding a pseudo-Riemannian metric $\tilde{g}$ on $TM$ (with metric signature $(m, m)$) whose components are given by

\[
\begin{pmatrix}
g_{\alpha\beta} & g_{\alpha\tilde{\beta}} \\
g_{\tilde{\alpha}\beta} & g_{\tilde{\alpha}\tilde{\beta}}
\end{pmatrix}(x, y) = \begin{pmatrix}
\partial_\lambda g_{\alpha\phi}(x) y^\lambda & g_{\alpha\phi}(x) \\
g_{\alpha\phi}(x) & 0
\end{pmatrix}.
\] (4.79)

If $V$ is $g$-metric, then $\tilde{V}$ is $\tilde{g}$-metric, and if $V$ is torsion-free then so is $\tilde{V}$; therefore $\tilde{\nabla} = \tilde{g}$. In general, $\tilde{V}$ has the property that its geodesics are given by the Jacobi fields of $\tilde{V}$.

It is easily checked using the theory in this section that Conditions 4.5 and 4.11 are satisfied for all $F$ in the case of the complete lift, and in the case of parallel transport with $V$ torsion-free we have

\[
\begin{pmatrix}
F_{\alpha\beta}^\gamma \\
F_{\alpha\gamma} \beta \\
F_{\alpha\gamma} \beta
\end{pmatrix} = \begin{pmatrix}
\mathcal{R}_{\alpha\beta\gamma

\] (4.80)

Condition 4.23, however, is not satisfied even when $V$ is Levi–Civita, since

\[
\begin{pmatrix}
F_{(\alpha\beta)\gamma} \\
F_{\alpha\beta} \gamma \\
F_{\alpha\gamma} \beta
\end{pmatrix} = \begin{pmatrix}
\mathcal{R}_{\alpha\beta\gamma} + \mathcal{R}_{\beta\delta\gamma} + \mathcal{R}_{\gamma\delta\beta} + \mathcal{R}_{\gamma\alpha\delta} \\
\mathcal{R}_{\alpha\beta\gamma} + \mathcal{R}_{\gamma\beta\delta}
\end{pmatrix} (4.81)

which does not vanish in general. The resulting parallel transport equation was first studied, for semimartingales, in [12] and subsequently in [27, (27)] (we caution the reader that the convention regarding the indices of the curvature tensor differ from the ones used in (4.75)), and it was realised in [17, p. 437] that this type of parallel transport fits into the framework of SDEs of the type defined in Definition 2.10.

Example 4.27 (The horizontal lift of $\nabla$). The second lift of a connection which we examine is the horizontal lift of $\nabla$, which we also denote $\tilde{V}$ (ambiguity will easily be avoided, since we will always use each connection separately). Its Christoffel symbols in induced coordinates are similar to those of the complete lift, with one important difference:

\[
\begin{pmatrix}
\tilde{\Gamma}_{\alpha\beta}^\gamma(x, y) & \Gamma_{\alpha\beta}^\gamma(x, y) \\
\Gamma_{\alpha\beta}^\gamma(x, y) & \tilde{\Gamma}_{\alpha\beta}^\gamma(x, y)
\end{pmatrix} = \begin{pmatrix}
\tilde{\Gamma}_{\alpha\beta}^\gamma(x, y) + \Gamma_{\alpha\beta}^\gamma(x, y) & \tilde{\Gamma}_{\alpha\beta}^\gamma(x, y) \\
\Gamma_{\alpha\beta}^\gamma(x, y) & \tilde{\Gamma}_{\alpha\beta}^\gamma(x, y)
\end{pmatrix} = 0
\] (4.82)

As for the complete lift, $\tau M$ is an affine map with respect to $\tilde{V}, V$; the extra term appearing in $\tilde{\Gamma}_{\alpha\beta}^\gamma$, however, causes $\tilde{V}$ to have nonvanishing torsion in general even if $V$ is torsion-free. Just as for the complete lift, the horizontal lift of a connection is motivated by a broader construction which involves lifting other objects defined on $M$, such as vector fields. However, unlike the case of the complete lift, these lifts require a connection on $\tau M$ to begin with, and are performed in a way which is related to (4.6); we do not provide more details here. If $V$ is $g$-metric, then $\tilde{V}$ is $\tilde{g}$-metric, where $\tilde{g}$ is the pseudo-Riemannian metric (4.79) (although, unlike the case of the complete lift, $\tilde{\nabla} \neq \tilde{\nabla}$ because the former has torsion in general). The characterisation of geodesics of the horizontal lift of a connection is more complicated than that of its complete lift, but it still holds that
\(\tau M\) maps \(\tilde{\nabla}\)-geodesics to \(\nabla\)-geodesics. Moreover, it holds that horizontal lifts of geodesics (namely curves in \(TM\) above geodesics whose tangent vectors are horizontal, that is, parallel transports above geodesics) define geodesics with respect to the horizontal lift: this is seen from [35, p. 115, (9.4)].

Like the complete lift, the horizontal lift results in Conditions 4.5 and 4.11 being satisfied for all \(F\), but in the case of \(F\) given by horizontal lift it additionally satisfies Condition 4.8. Therefore, the resulting parallel transport is, at the trace level, the same as geometric/Stratonovich parallel transport, a conclusion which is also noted in [17, 27].

**Example 4.28** (The Sasaki metric). Let \(g\) be a Riemannian metric on \(M\). We can lift \(g\) to a Riemannian metric \(\tilde{g}\) on \(TM\) (confusion with the previously defined pseudometric will be avoided), called the Sasaki metric by using the horizontal and vertical lift isomorphisms to define the metric within the horizontal and vertical bundles, and declaring them to be orthogonal. In induced coordinates, this is amounts to

\[
\begin{pmatrix}
\tilde{g}_{\alpha\beta}(x, y) \\
\tilde{g}_{\alpha\beta} \\
\end{pmatrix}
= \begin{pmatrix}
g_{\alpha\beta}(x) + g_{\alpha\gamma} \Gamma^\gamma_{\beta\alpha}(x)y^\gamma y^\nu \\
\Gamma^\gamma_{\alpha\beta}(x)y^\gamma \\
\end{pmatrix}
\]

(4.83)

and

\[
\begin{pmatrix}
\tilde{g}_{\alpha\beta}(x, y) \\
\tilde{g}_{\alpha\beta} \\
\end{pmatrix}
= \begin{pmatrix}
g_{\alpha\beta}(x) \\
-G^\gamma_{\alpha\beta}(x)y^\gamma \\
\end{pmatrix}
\]

(4.84)

where the functions \(\Gamma\) are the Christoffel symbols of \(\nabla\). The horizontal lift of \(\nabla\) is \(\tilde{g}\)-metric, but does not coincide with \(\tilde{\nabla}\) due to torsion. We will call \(\tilde{\nabla}\) the Sasaki lift of \(\nabla\) (even though, strictly speaking, it is the metric that we are lifting). The Christoffel symbols of \(\tilde{\nabla}\) in induced coordinates have more complex expressions than the ones for the other two lifts of connections, and are given as functions of the Christoffel symbols of \(\nabla\) and of the components of its curvature tensor by

\[
\begin{align*}
\tilde{\Gamma}^\gamma_{\alpha\beta}(x, y) &= \Gamma^\gamma_{\alpha\beta}(x) + \frac{1}{2} (\mathcal{R}^\gamma_{\mu\delta\alpha\beta}(x)y^\lambda \Gamma^\mu_{\beta\alpha}(x)) y^\lambda y^\mu \\
\tilde{\Gamma}^\gamma_{\alpha\beta}(x, y) &= \frac{1}{2} (\mathcal{R}^\gamma_{\mu\delta\alpha\beta}(x)y^\lambda , \mathcal{R}^\gamma_{\alpha\beta\lambda\delta}(x)y^\lambda y^\mu , \mathcal{R}^\gamma_{\alpha\beta\lambda\delta}(x)y^\lambda y^\mu , \mathcal{R}^\gamma_{\alpha\beta\lambda\delta}(x)y^\lambda y^\mu ) \\
\tilde{\Gamma}^\gamma_{\alpha\beta}(x, y) &= \frac{1}{2} (\mathcal{R}^\gamma_{\mu\delta\alpha\beta}(x)y^\lambda , \mathcal{R}^\gamma_{\alpha\beta\lambda\delta}(x)y^\lambda y^\mu , \mathcal{R}^\gamma_{\alpha\beta\lambda\delta}(x)y^\lambda y^\mu ) \\
\tilde{\Gamma}^\gamma_{\alpha\beta}(x, y) &= \frac{1}{2} (\mathcal{R}^\gamma_{\mu\delta\alpha\beta}(x)y^\lambda , \mathcal{R}^\gamma_{\alpha\beta\lambda\delta}(x)y^\lambda y^\mu , \mathcal{R}^\gamma_{\alpha\beta\lambda\delta}(x)y^\lambda y^\mu ) \\
\end{align*}
\]

(4.85)

These symbols are taken from [32] with one important caveat: the functions \(\mathcal{R}^\gamma_{\alpha\beta\gamma}\) therein have been transcribed into functions \(\mathcal{R}^\gamma_{\alpha\beta\gamma}\) here. This is because the author follows a different ordering in the coordinate expression of the curvature tensor. This convention is not stated in the paper, but it can be deduced by computing any one of the Christoffel symbols involving a curvature...
term. This check can be performed by using the fact that the horizontal lift of $\nabla$ is metric with respect to the Sasaki metric [35, Proposition 7.6] and the following observation: working now on a general Riemannian manifold, if a connection $\nabla$ is $g$-metric, it is not true in general that its symmetrisation $\nabla^c$ is metric: denoting $\mathcal{T}^k_{ij}$ the components of the torsion tensor, we have that the difference between $\nabla$ and the Levi–Civita connection $\mathcal{G}\nabla$ is quantified by the contorsion tensor [29, p. 254]

$$
\mathcal{H}^k_{ij} := \frac{1}{2}(\mathcal{T}^k_{ij} + \mathcal{T}^k_{ji} + \mathcal{T}^k_{ij}), \quad \Gamma^k_{ij} - \mathcal{G}\nabla^k_{ij} = \mathcal{H}^k_{ij}
$$

(4.86)

which has symmetric part $\frac{1}{2}(\mathcal{T}^k_{ij} + \mathcal{T}^k_{ji})$. Let $\tilde{\mathcal{T}}$ denote the torsion tensor of the horizontal lift of $\nabla$: its only nonzero component is given by

$$
\tilde{\mathcal{T}}_{\alpha\beta}^\gamma(x, y) = \frac{1}{2}(\mathcal{R}^\gamma_\alpha\beta\delta(x) y^\delta - \mathcal{R}^\gamma_\delta\alpha\beta(x) y^\delta).
$$

(4.87)

Thus, $\tilde{\mathcal{T}}^\gamma_{\alpha\beta}(x, y) = 0$ and, performing index gymnastics with respect to the Sasaki metric $\tilde{g}$ and using the symmetries of the curvature tensor [24, Proposition 7.4] we obtain

$$
\tilde{\mathcal{H}}_{\alpha\beta}^\gamma(x, y) = \tilde{\mathcal{T}}_{\alpha\beta}^\gamma(x, y) = \frac{1}{2}(\mathcal{R}^\gamma_\alpha\beta\delta(x) y^\delta - \mathcal{R}^\gamma_\delta\alpha\beta(x) y^\delta).
$$

(4.88)

Then, since $\tilde{\mathcal{H}}_{\alpha\beta}^\gamma = \frac{1}{2} \tilde{\mathcal{T}}_{\alpha\beta}^\gamma$, (4.82) yield the value of $\tilde{\mathcal{H}}_{\alpha\beta}^\gamma$ in (4.85). Similarly to the case of the horizontal lift of a connection, the horizontal lift of a Riemannian geodesic is a geodesic with respect to the Sasaki metric.

The question of what ‘Sasaki parallel transport’ amounts to, to our knowledge, has not been considered in the literature, so we answer it here. Like for the horizontal lift, Condition 4.8 is satisfied with respect to the Sasaki lift of $\nabla$ when $F$ is given by horizontal lift, and the resulting definition of parallel transport is therefore equivalent to the geometrised one. However, unlike the complete and horizontal lifts, Condition 4.5 cannot be expected to hold for general $F$, since the Sasaki lift does not make $\tau M$ symmetrically affine: this can be seen, again with reference to Example 4.2, by noting that, for instance $\tilde{\mathcal{H}}_{\alpha\beta}^\gamma(x, y) = \frac{1}{2} \tilde{\mathcal{H}}_{\alpha\beta}^\gamma(x) y^\gamma$ is not, in general, antisymmetric in $\alpha, \beta$. This means we may not, in general, define arbitrary equations (4.17) when $E = TM$ is given the Sasaki lift of $\nabla$.

It would be interesting to explore additional examples of connections on $\tau TM$ which result in notions of parallel transport different from the geometric/Stratonovich one: the Levi–Civita
connection of the Cheeger-Gromoll metric [28] and the connections defined in [3, Section 5] could be worthwhile to test.

**Example 4.29** (Local martingales and Brownian motion). It is well-known that Stratonovich (anti)development preserves local martingales and if $M$ is Riemannian it preserves Brownian motions. In our setting, the first statement always holds in all cases (assuming Conditions 4.5 and 4.11 hold, so that (anti)development is defined), as can be easily seen from the local characterisation of manifold-valued martingales (2.35), and the local expressions (4.60), (4.62). This is also observed (using a different argument) in [17, p. 440].

As for the preservation of Brownian motion, we first recall that the Levy criterion on manifolds [16, Proposition 5.18] immediately implies the following local characterisation of Brownian motion of a Riemannian manifold $(M, \mathcal{G})$: $X$ is a Brownian motion on $M$ if and only if it is a local martingale and

$$d[X]^\alpha_\beta = \mathcal{G}^{\alpha_\beta}(X)dt. \quad (4.89)$$

If $Z$ is a $T_oM$-valued Brownian motion, then if Condition 4.23 holds, we have for $X = d\mathcal{G}(Z)$

$$d[X]^\alpha_\beta = /\alpha/ /\beta d[Z]^{\alpha_\beta}^o = /\alpha/ /\beta d[\delta^{\alpha_\beta}]^o dt = \mathcal{G}^{\alpha_\beta}(X)dt, \quad (4.90)$$

where the last identity holds thanks to Theorem 4.25, and assuming the basis of $T_oM$ to be orthonormal. That antidevelopment maps Brownian motions to Brownian motions under the same hypotheses is checked analogously.

We may therefore conclude that (anti)development defined with respect to the complete, horizontal and Sasaki lifts to preserve local martingales, but only that defined with respect to the horizontal and Sasaki lifts to preserve Brownian motion.

We also note that we should expect (anti)development taken with respect to two different functions $\nabla$ to be different pathwise, even when both satisfy the linearity and metricity conditions. For Brownian motion this might mean that the law of the (anti)developments coincide (that is, they are both Brownian motions), despite the paths defined by the same state $\omega \in \Omega$ being different. Another way of generating pathwise-distinct Brownian motions through (anti)development of the same Brownian motion is by adding a contorsion term (see (4.86)) to the Levi-Civita connection $\nabla$ and taking Stratonovich development. In general, by the Itô isometry the cross-covariance matrix of the Itô antidevelopments $\nabla^o(X)$ and $\mathfrak{C}(X)$ of the same $M$-valued semimartingale $X$ taken with respect to $\nabla$, $\nabla^o$ on the one hand and $\nabla$, $\mathfrak{C}$ on the other is given by $E[\mathfrak{C}^{\alpha_\beta}(X)\mathfrak{C}^{\gamma_\delta}(X)] = E[\int^{\nabla}_{\nabla^o} \lambda^{\alpha_\beta} d[X]^{\alpha_\gamma}]$, with $k, \lambda = 1, 2$ denoting the respective parallel transports above $X$.

**Example 4.30** ($/\nabla$ along Brownian motion on Einstein manifolds with respect to the complete lift). Recall the definition of Ricci tensor

$$\mathcal{R}_{ij} := -\mathcal{R}^k_{ij} = -\mathcal{R}_{ikj} \mathcal{G}^{hk}. \quad (4.91)$$

We assume $(M, \mathcal{G})$ is an Einstein manifold, that is, a Riemannian manifold whose Ricci tensor is proportional to the metric tensor, $\mathcal{R}_{\alpha_\beta} = \lambda \mathcal{G}_{\alpha_\beta}$ with $\lambda \in \mathbb{R}$ (the best known such example is the sphere, in all dimensions). Let $Z$ be a Brownian motion on $T_oM$ and $X$ its Stratonovich
FIGURE 2 Parallel transport along the connection of Example 4.31

FIGURE 3 This figure relates to Example 4.31. The two plots are analogous to Figure 1, with the manifold in question being $\mathbb{R}^3$, endowed with the connection defined above, and the path being developed is the parametrised smooth curve $X_t := (2t \cos(t), 10 \sin(t), 3t)$. The two copies $M$ and $T_0M$ of $\mathbb{R}^3$ are superimposed, with coinciding axes in the first plot. We observe how the two curves are not identical, which would be the case if the connection on $\mathbb{R}^3$ were the Euclidean one. Also note how $X$ and $\mathcal{O}(X)$ are tangent curves at their point of contact.

development, an $M$-valued Brownian motion, and we compare the behaviour of Stratonovich parallel transport $\overline{\parallel}(X)$ with parallel transport defined with respect to to the complete lift $\overline{V}$ of the Levi–Civita connection $V$, which we denote $\overline{\parallel}(X)$. By proceeding as in the proof of [Theorem 4.25] and Example 4.29 we compute

$$d \mathcal{G}(\overline{\parallel}^{\alpha}, \overline{\parallel}^{\beta}) = \frac{1}{2} \mathcal{G}^{\alpha \beta} \overline{\parallel}^{\alpha, \gamma} \overline{\parallel}^{\gamma \eta} \mathcal{G}^{\xi \eta} dt$$

$$= -\frac{1}{2} \mathcal{R}^{\alpha \beta \gamma} \overline{\parallel}^{\gamma} \overline{\parallel}^{\beta} dt$$

$$= -\frac{\lambda}{2} \mathcal{G}(\overline{\parallel}^{\alpha}, \overline{\parallel}^{\beta}) dt$$

which implies $\mathcal{G}(\overline{\parallel}^{\alpha}, \overline{\parallel}^{\beta}) = \exp(-\lambda t/2) \delta^{\alpha \beta}$, and similarly $\mathcal{G}(\overline{\parallel}^{\alpha}, \overline{\parallel}^{\beta}) = \exp(-\lambda t) \delta^{\alpha \beta}$. In other words $\overline{\parallel}(X)$ preserves orthogonality, but not orthonormality, since it consists of a scaling by the above exponential factor. Note that this behaviour of $\overline{\parallel}$ can only be expected to hold along the Brownian motion $X$, and not along $\overline{X} := \mathcal{O}(Z)$, the development of $Z$ taken according to the complete lift of $\overline{V}$, which is not in general a Brownian motion (even given the Einstein assump-
tion): this can be seen by writing $d\bar{X} = \sum_{\gamma^\alpha} \frac{\langle \gamma^\alpha(\bar{X}) \rangle}{\gamma^\beta(\bar{X})} dt$ and by showing that the SDE satisfied by $\sum_{\gamma^\alpha} \frac{\langle \gamma^\alpha(\bar{X}) \rangle}{\gamma^\beta(\bar{X})} \gamma^\beta(\bar{X})$ has an extra drift term when compared to that satisfied by $\varrho^\alpha\beta(\bar{X})$.

**Example 4.31** (Torsion). In general, RDEs of the form Definition 2.10 are independent of the torsion of the connections on the source and target manifolds. For parallel transport, however, torsion of $\nabla$ directly affects the field $F = \dot{h}$ that defines the RDE, and to that extent it influences the definition of $//\gamma$ and therefore that of $\Box$ and $\bigcirc$ (both for the trace and second-order levels of the rough paths considered). The torsion of $\tilde{\nabla}$, instead, plays no role. To exhibit the relevance of torsion for parallel transport and development we need only focus on smooth paths. Take $M = \mathbb{R}^3$ with its canonical coordinates, and $\nabla$ with constant Christoffel symbols $\Gamma^{1}_{23} = 1 = -\Gamma^{1}_{32}, \Gamma^{2}_{31} = 1 = -\Gamma^{2}_{13}, \Gamma^{3}_{12} = 1 = -\Gamma^{3}_{21}$ and $\Gamma^{k}_{ij} = 0$ otherwise. This connection has the same geodesics as the Euclidean connection (straight lines), but, as described in [34], parallel transport along geodesics looks like a spinning rugby ball, as illustrated in Figure 2 for an orthonormal frame. While the Euclidean connection and $\nabla$ agree on geodesics, they define different notions of developments: identifying $T_0\mathbb{R}^3 = \mathbb{R}^3$ we have $\Box = 1$ according to the former, while this is not the case for the latter, as shown in Figure 3.

**ACKNOWLEDGEMENTS**

The work of the third author was supported by the EPSRC Programme Grant ‘DataSig’ EP/S026347/1.

The fourth author’s PhD was funded by the Centre for Doctoral Training in Financial Computing & Analytics, EP/L015129/1.

**JOURNAL INFORMATION**

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

**REFERENCES**

1. J. Armstrong, D. Brigo, and E. R. Ferrucci, *Projections of sdes onto submanifolds*, arXiv:1810.03923, 2018.
2. J. Armstrong, D. Brigo, and E. R. Ferrucci, *Optimal approximation of SDEs on submanifolds: the Itô-vector and Itô-jet projections*, Proc. Lond. Math. Soc. (3) 119 (2019), no. 1, 176–213.
3. M. Arnaudon and A. Thalmaier, *Horizontal martingales in vector bundles*, Séminaire de Probabilités, XXXVI, Lecture Notes in Mathematics, vol. 1801, Springer, Berlin, 2003, pp. 419–456.
4. I. Bailleul, *Rough integrators on Banach manifolds*, Bull. Sci. Math. 151 (2019), 51–65.
5. T. Boutaib and Y. Lyons, *A new definition of rough paths on manifolds*, arXiv:1510.07833v2, 2015.
6. O. Calin and D.-C. Chang, *Sub-Riemannian geometry: general theory and examples*, Encyclopedia of Mathematics and its Applications, vol. 126, Cambridge University Press, Cambridge, 2009, pp. xiv+370. https://doi.org/10.1017/CBO9781139195966.
7. T. Cass, B. K. Driver, N. Lim, and C. Litterer, *On the integration of weakly geometric rough paths*, J. Math. Soc. Japan 68 (2016), no. 4, 1505–1524.
8. T. Cass, B. K. Driver, and C. Litterer, *Constrained rough paths*, Proc. Lond. Math. Soc. (3) 111 (2015), no. 6, 1471–1518.
9. T. Cass and N. Lim, *A Stratonovich-Skorohod integral formula for Gaussian rough paths*, Ann. Probab. 47 (2019), no. 1, 1–60.
10. T. Cass, C. Litterer, and T. Lyons, *Rough paths on manifolds*, arXiv:1102.0998, 2011.
11. T. Cass, C. Litterer, and T. Lyons, *Rough paths on manifolds*, New trends in stochastic analysis and related topics, Interdiscip. Math. Sci., vol. 12, World Science Publications, Hackensack, NJ, 2012, pp. 33–88.
12. D. Dohrn and F. Guerra, *Nelson’s stochastic mechanics on Riemannian manifolds*, Lett. Nuovo Cimento (2) **22** (1978), no. 4, 121–127.
13. B. Driver, *Curved wiener space analysis*, arXiv:math/0403073, 2004. https://arxiv.org/abs/math/0403073.
14. B. Driver, *Global existence of geometric rough flows*, arXiv:1810.03708, 2018.
15. B. K. Driver and J. S. Semko, *Controlled rough paths on manifolds I*, Rev. Mat. Iberoam. **33** (2017), no. 3, 885–950.
16. M. Émery, *Stochastic calculus in manifolds*, Universitext, pp. x+151, Springer, Berlin, 1989. (With an appendix by P.-A. Meyer.)
17. M. Émery, *On two transfer principles in stochastic differential geometry*, Séminaire de Probabilités, XXIV, 1988/89, Lecture Notes in Mathematics, vol. 1426, Springer, Berlin, 1990, pp. 407–441.
18. M. Errami and F. Russo, *n-Covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes*, Stochastic Process. Appl. **104** (2003), no. 2, 259–299.
19. P. K. Friz and M. Hairer, *A course on rough paths*, Universitext, pp. xiv+251, Springer, Cham, 2014. (With an introduction to regularity structures.)
20. P. K. Friz and N. B. Victoir, *Multidimensional stochastic processes as rough paths: Theory and applications*, Cambridge Studies in Advanced Mathematics, vol. 120, pp. xiv+656, Cambridge University Press, Cambridge, 2010.
21. M. Gubinelli, *Controlling rough paths*, J. Funct. Anal. **216** (2004), no. 1, 86–140.
22. M. Hairer and D. Kelly, *Geometric versus non-geometric rough paths*, Ann. Inst. Henri Poincaré Probab. Stat. **51** (2015), no. 1, 207–251.
23. E. P. Hsu, *Stochastic analysis on manifolds*, Graduate Studies in Mathematics, vol. 38, American Mathematical Society, Providence, RI, 2002, pp. xiv+281.
24. J. M. Lee, *Riemannian manifolds*, Graduate Texts in Mathematics, vol. 176, pp. xvi+224, Springer, New York, 1997. (An introduction to curvature.)
25. T. J. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana **14** (1998), no. 2, 215–310.
26. P.-A. Meyer, *Géométrie stochastique sans larmes, i*, Séminaire de probabilités de Strasbourg **15** (1981), 44–102.
27. P.-A. Meyer, *Géometrie différentielle stochastique. II*, Seminar on Probability, XVI, Supplement, Lecture Notes in Mathematics, vol. 921, Springer, Berlin-New York, 1982, pp. 165–207.
28. E. Musso and F. Tricerri, *Riemannian metrics on tangent bundles*, Ann. Mat. Pura Appl. (4) **150** (1988), 1–19.
29. M. Nakahara, *Geometry, topology and physics*, Graduate Student Series in Physics, 2nd ed., Institute of Physics, Bristol, 2003.
30. P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics, Springer, New York, 2006.
31. Z. Qian and X. Xu, *Itô integrals for fractional Brownian motion and applications to option pricing*, arXiv:1803.00335, 2018.
32. S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J. (2) **10** (1958), 338–354.
33. L. Schwartz, *Géométrie différentielle du 2ème ordre, semi-martingales et équations différentielles stochastiques sur une variété différentielle*, Séminaire de probabilités de Strasbourg **S16** (1982), 1–148.
34. User anonymous, *What is torsion in differential geometry intuitively?* MathOverflow. https://mathoverflow.net/q/20510.
35. K. Yano and S. Ishihara, *Tangent and cotangent bundles: differential geometry*, Pure and Applied Mathematics, vol. 16, Marcel Dekker, Inc., New York, 1973, pp. ix+423.