Onset of superconductivity and hysteresis in magnetic field for a long cylinder obtained from a self-consistent solutions of the Ginzburg–Landau equations

G.F. Zharkov
P.N. Lebedev Physical Institute, Russian Academy of Sciences, Moscow, 117924, Russia
(21 March, 2001)

Based on the self-consistent solution of a nonlinear system of one-dimensional GL-equations, the onset and destruction of superconductivity, the phase transitions and hysteresis phenomena are discussed for a cylinder (radius \( R \)) in an axial magnetic field \( (H) \) for arbitrary \( R, \kappa, H, m \) (\( \kappa \) is the GL-parameter, \( m \) is the total vorticity of the system). The edge-suppressed solutions (which are connected with the jumps of magnetization in the states with fixed vorticity \( m \)), the depressed solutions (responsible for the hysteresis in type-II superconductors), and the precursor solutions (which describe the onset of superconductivity in type-I superconductors) are also studied. The limits of applicability of the so-called linear equation approximation are discussed.

74.25.-q, 74.25.Dw

I. INTRODUCTION

The macroscopic GL-theory [1] is widely used for studying the behavior of superconductors in magnetic field. This theory leads to two coupled nonlinear three-dimensional equations for a complex order parameter \( \Psi \) and magnetic field vector-potential \( \mathbf{A} \). Because, in a general case, it is impossible to find the exact solution of this system analytically, the numerical computations are used, or various limiting particular cases are studied. Thus, the London theory [2] follows from the GL-equations [1] in the limit \( \psi \equiv |\Psi| = 1 \), and the Ginzburg approach [3] corresponds to the case of a constant order parameter, which depends on the weak external field \( H, \psi = \psi(H) \). Another approximation was formulated by Saint-James and de Gennes [4], who assumed, that the nucleation of superconductivity in large fields is described by small values of the order parameter (\( \psi \equiv |\Psi| \ll 1 \)). Instead of full system of nonlinear GL-equations, they considered a single linear one-dimensional equation for \( \psi \), what allowed them to find analytically the nucleation field \( H_{c3} = 1.69H_{c2} \) (\( H_{c2} = \phi_0/(2\pi \xi^2) \), \( \xi \) is the coherence length) for a bulk superconducting slab in parallel magnetic field. Later, Saint-James [5] used the same approach to find the oscillation dependence of the nucleation field \( H_{c3} \) versus applied field \( H \) for a very long circular cylinder of finite radius \( R \). In the papers [4,5] the basic conception of the surface superconductivity was laid down, according to which the onset of the superconductivity begins at the fields \( H \sim H_{c3} \) in a form of thin superconducting sheath, layered near the surface of the specimen (the so-called ”giant-vortex state”).

The structure of the giant-vortex solutions with finite values of the order parameter (\( \psi \sim 1 \)) was studied numerically in a number of scattered publications (for the review of early works see [6]). A more systematic study of such solutions, carried out recently in [7], has revealed, that the axially symmetric solution of fixed vorticity \( m \) has different form, depending on the value of the increasing external magnetic field \( H \) (\( m = 0, 1, 2, \ldots \) is the quantum number, ensuring that \( \Psi \) is single valued). In type-II superconductors the giant-vortex solution has the usual Meissner-type space-profile for fields \( H < H_1 \) (where \( H_1(m, R, \kappa) \) is some critical field). At \( H = H_1 \) this form becomes unstable, and for \( H > H_1 \) the giant-vortex solution acquires (by a first-order jump) a new ”edge-suppressed” form [7]. With the field \( H \) further increasing, the edge-suppressed solution degenerates gradually and vanishes finally by a second-order phase transition (\( \psi \rightarrow 0 \)) at some field \( H_2(m, R, \kappa) \). (The field \( H_2(m, R, \kappa) \) is just the critical field, found in [5] from a linear theory; for \( R \gg \xi \) and \( m \sim 1 \) the field \( H_2(m, R, \kappa) \) coincides with \( H_{c2} \); for \( R \gg \xi \) and \( m \gg 1 \) the maximum value of the field \( H_2 \) coincides with \( H_{c2} = 1.69H_{c2} \).)

In type-I superconductors the destruction of the superconducting state occurs at the field \( H_1(m, R, \kappa) \) (i.e. at the field of maximal superheating) by a first-order jump from s-state (having a finite value of \( \psi \sim 1 \)) to n-state (\( \psi \equiv 0 \)). Because in this case there are no solutions with \( \psi \ll 1 \), the linear theory is not applicable, and the superheated s-boundary, found from the full system of equations, deviates substantially from that, predicted in [5].

As is shown in the present paper, the nucleation of superconductivity from n-state in decreasing fields also proceeds differently in type-I and type-II superconducting cylinders.

In type-II cylinders the nucleation begins at \( H_2(m, R, \kappa) \) as a second-order phase transition from n-state, then a small nucleated s-solution grows, repeating the edge-suppressed form down to the field \( H_1(m, R, \kappa) \). At this field the shape of the order parameter starts to deviate gradually from the edge-suppressed form and passes continuously into a new ”depressed” form, which exists
down to some "restoration" field $H_r(m, R, \kappa)$, where the first-order jump to the Meissner-type solution occurs. In the field interval $\Delta = H_I - H_r$ there exist simultaneously two solutions for the same field $H$: one is the usual Meissner-type solution ($\psi \approx 1$), and the second is the depressed solution ($\psi < 1$), which describes the hysteretic $s$-state of the cylinder. The region of parameters, where $\Delta(m, R, \kappa) \geq 0$ (i.e., the phase boundary, where the hysteretic transitions between superconducting states of the same vorticity may exist), is found self-consistently.

In type-I cylinders (in the field decreasing regime) a supercooled $n$-state persist down to some field $H_p$, where the feeble ($\psi \ll 1$) "precursor" solution forms, what corresponds to the second-order $(n, s)$-phase transition, so the field $H_p$ may be found from linear equation approximation [5]. However, the precursor solution exists only in a very small interval ($\Delta_p$) of fields below $H_p$ (the field interval $\Delta_p = H_p - H_r \sim 10^{-2} - 10^{-4} H_p$), after that the first-order restoration of the full Meissner solution ($\psi \approx 1$) occurs. Thus, the nucleation of superconductivity in type-I cylinders is an "almost first-order" phase transition.] The phase region for type-I cylinders, where the magnetically "supercooled" normal state [3] may exist, is also found self-consistently.

As was mentioned above, there are two different regimes to study the effect of the external field penetration into the superconductor interior: the field increase and the field decrease regimes, which should be considered separately. It is demonstrated below, that if the field increases, there exists a critical line between the superconducting and the normal state (and also the regions, where the magnetic hysteresis is possible), are presented. The difference between the depressed and edge-suppressed solutions is explained.

In Sec. IV the critical fields, found self-consistently from the GL-equations for a cylinder in the increasing field, are compared with those, found from the linear approach, for both type-I and type-II superconductors. In Sec. V the analogous comparison is made for the decreasing field. The "supercooling" of the normal state, the precursor solutions and hysteresis in type-I superconducting cylinders are also discussed. Sec. VI contains a short resume and discussion of the results with possible connection to experiment.

II. EQUATIONS

In what follows below, the case is considered of a long superconducting cylinder of radius $R$, in the external magnetic field $H \geq 0$, which is parallel to the cylinder element. Only the radially symmetric (one-dimensional) element is studied. In the cylindrical co-ordinates the system of GL-equations may be written in dimensionless form [7]

$$\frac{d^2U}{dx^2} - \frac{1}{x} \frac{dU}{dx} - \frac{\psi^2 U}{x^2} = 0, \tag{1}$$

$$\frac{d^2\psi}{dx^2} + \frac{1}{x} \frac{d\psi}{dx} + (\psi - \psi^3) - \frac{U^2}{x^2} \psi = 0. \tag{2}$$

The dimensioned potential $A$, field $B$ and current $j_s$ are related to the corresponding dimensionless quantities by the formulae:

$$A = \frac{\phi_0}{2\pi \xi} \frac{U + m}{x}, \quad B = \frac{\phi_0}{2\pi \xi^2} b, \quad b = \frac{1}{x} \frac{dU}{dx}, \quad j_s = \frac{\phi_0}{8\pi^2 \xi^3} - \psi^2 \frac{U}{x}, \quad x = \frac{r}{\xi}. \tag{3}$$

The total vorticity $m$ in (3) specifies how many flux quanta are associated with the vortex, centered at the cylinder axis (the so-called giant-vortex state [5,9]). The equivalent names for $m$, are: vorticity, fluxoid, orbital momentum, magnetic quantum number, winding quantum number.

The boundary conditions to Eq. (1) are:

$$U|_{x=0} = -m, \quad dU/dx|_{x=R_\xi} = h_\xi. \tag{4}$$

where $R_\xi = R/\xi$, $h_\xi = H/H_\xi$, $H_\xi = \phi_0/(2\pi \xi^2)$.

The boundary conditions to Eq. (2) are:

$$d\psi/dx|_{x=0} = 0, \quad d\psi/dx|_{x=R_\xi} = 0 \quad (m = 0), \tag{5}$$

$$\psi|_{x=0} = 0, \quad d\psi/dx|_{x=R_\xi} = 0 \quad (m > 0).$$
Thus, the solutions of the nonlinear equation (6) as \( \psi(x) = K_m(x) \). The analytical form of the (quasi-Kummer) functions \( K_m(x) \), however, is not known, but they may be easily found numerically from Eq. (6) [Sec. IV].

In the limit \( \nu \to 0 \), one can drop the term \( \psi^3 \) from Eq. (6) and consider the linear equation [5]

\[
\frac{d^2 \psi}{dx^2} + \frac{1}{x} \frac{d \psi}{dx} + \left( \psi - \psi^3 \right) - \frac{U_0^2}{x^2} \psi = 0.
\]

with the solutions denoted as \( \psi(x) = K_m(x) \). The function \( K_m(x) \) may be written in the analytical form [10]

\[
K_m(x) = y^{m/2} e^{-y/2} F(y) \quad \text{where} \quad y = \gamma x^2, \gamma = h_\xi/(2).
\]

The function \( F(y) \) satisfies the confluent hypergeometric equation

\[
y F''' + (\mu - y) F' - \nu F = 0
\]

[where \( \mu = m + 1 \) is a positive integer, \( \nu = (1 - h_\xi^{-1})/2 \)],

with a general solution \( F = C_1 F + C_2 \tilde{F} \). The function \( F \) and can be written as an infinite series expansion

\[
F(\nu, \mu, y) = 1 + \frac{\nu y}{\mu \Pi} + \frac{\nu(\nu + 1) y^2}{\mu(\mu + 1) 2!} + \cdots.
\]

[We use the notation \( F(\nu, \mu, y) \) [11–13] for the Kummer functions, instead of \( M(\nu, \mu, y) \), used in [14,15]. If \( \nu \) is a negative integer, or zero, \( F \) reduces to the polinom of power \( \nu \). The function \( F \) has a logarithmic singularity at \( x = 0 \) and sometimes is dropped \( (C_2 = 0) \). However, the function \( F(x) \) has a pre-factor \( y^{m/2} \sim x^m \) (see above), which cancels this singularity, so in a general case \( C_2 \neq 0 \) (for \( m > 0 \)).]

The second of the boundary conditions (5) is equivalent to

\[
\frac{d}{dy} K_m(\nu, \mu, y) \bigg|_{x=R_\xi} = 0,
\]

which is a nonlinear equation for \( \nu = (1 - h_\xi^{-1})/2 \) (if \( m \) and \( R_\xi \) are fixed). From Eq. (9) the proper value of \( h_\xi \) (and also the space-profile \( \psi(x) \)) may be found. In a general case, Eq. (9) has two roots for \( h_\xi \) (if \( m > 0 \)), what corresponds to two different solutions for \( \psi(x) \) (see Sec. IV). The maximum field \( h_\xi \), above which there exists only the solution \( \psi(x) \equiv 0 \), corresponds to the upper critical field \( h_2 \).

[The solutions, found from Eqs. (1)–(9), describe the radially symmetric giant-vortex states of fixed vorticity \( m \). The more complicated multi-vortex solutions (with the same total vorticity \( m = \Sigma_i m_i \), where \( m_i \) is the vorticity of a vortex, situated at some point \( r_i \) on the cylinder cross-section) should be described by the solutions in partial derivatives and will not be considered in the present paper. They may be studied numerically by the methods, used, for instance, in [17–19] to describe the results of experiments [20] with thin mesoscopic discs.]

### III. THE PHASE VIEWS

The solutions of Eqs. (1)–(5) depend on the space coordinate \( x \) and several parameters [for instance, \( \psi(x) = \psi(m, R_\xi, \varphi, h_\xi; x) \); analogously for the potential \( U(x) \) and the field \( b(x) \)]. In this Section a general view is given of the solutions structure in this many-parameter nonlinear problem. Some details of this general picture will be discussed in the ensuing Sections.

#### A. Field increase regime

Let the vorticity \( m \) be fixed \((m = 0, 1, 2, \ldots)\) and consider at first the case \( m = 0 \) (the vortex-free Meissner state). Consider the plane of parameters \((R_\xi, \varphi)\) (Fig. 1(a)). To every point of this plane corresponds a set of solutions of Eqs. (1)–(5), which depend parametrically on the external field \( h_\xi \) (or \( h_\lambda = \varphi^2 h_\xi \)). This set of solutions is unique for each point \((R_\xi, \varphi)\) and may be characterized, for instance, by the field dependence of the maximum value of the order parameter \( \psi_{\max} (h_\xi) \) in this point. It is helpful to look at the plane \((R_\xi, \varphi)\) from above, and to imagine a peep-hole, pierced in arbitrary point of this plane, which allows to see the corresponding dependence \( \psi_{\max} (h_\xi) \). (Such peep-holes are depicted as small circles in Fig. 1(a).) In doing so, one can find three regions in Fig. 1(a), marked as \( s_1^\Delta \), \( s_2^\Delta \) and \( s_0^\Delta \). [The upper index \( \Delta \) denotes a possibility of the hysteresis in these regions (see below). We shall mention Fig. 1(a), depicted on the plane \((R_\xi, \varphi)\) (and similar ones, like those in Fig. 3), as the phase views, to distinguish them from the phase diagrams, depicted on the plane \((R_\xi, h_\xi)\) (as in Fig. 5).]

Fig. 1(b) shows (schematically) the behavior \( \psi_{\max} (h_\xi) \) in the region \( s_2^\Delta \) (the solid curve, the field \( h_\xi \) increases from \( h_\xi = 0 \)). For small \( h_\xi \), the field is almost completely expelled from the superconductor (the Meissner state, \( m = 0, b = 0 \)). This part of the solid curve does not depend on \( h_\xi \) and may be described by the London theory [2] approximation (\( \psi = 1 \)). In larger fields, the order parameter depends on \( h_\xi \) rather weakly. This part
of the curve may be described by the Ginzburg approximation [3] \( \psi = \psi(h_\xi) = \text{const} \). In still larger fields, a full system of GL-equations is needed to describe the behavior \( \psi(h_\xi; x) \). There exists a critical field \( h_1 \), at which the stable Meissner-like solution (the upper section of the solid curve) becomes absolutely unstable relative small perturbations in its shape, and the solution \( \psi(h_\xi; x) \) acquires new stable space-profile, which may be called as the edge-suppressed state [7] (see the lower part of the solid curve, which is labeled also as "tail"; the difference between the various forms of solutions is illustrated in Fig. 2). The transition from the upper to lower solid branch in Fig. 1(b) is a first-order phase transition (with a jump \( \delta_1 \)). For fields \( h_\xi > h_1 \) the order parameter diminishes gradually and vanishes finally at the field \( h_2 \) by a second-order phase transition. Evidently, for fields \( h_\xi \sim h_2 \) (when \( \psi_{\text{max}} \sim 1 \)) the linear equation (7) is applicable.

The width of the tail \( (w = h_2 - h_1) \), where the superconducting state \( (m = 0) \) is destroyed by the second-order phase transition to normal state, diminishes with the diminishing radius \( R_\xi \). The critical radius, at which the tail vanishes \( (w = 0) \), depends on \( \kappa \), and is depicted in Fig. 1(a) by a solid line. This line \( (S_{1 \text{--} H}) \) divides the regions \( (s^3 \Delta) \) of first and \( (s^1 \Delta) \) second-order phase transitions. In the region \( s^3 \Delta \) (marked as "no tails"), the transition to normal state is always of first order, by a jump from a finite value \( \psi_{\text{max}} \) to \( \psi = 0 \).

Type-I superconductors are naturally defined in our case as those, which belong to the region \( s^3 \Delta \), and type-II - to the regions \( s^1 \Delta \), or \( s^0 \Delta \). We see, that the boundary between type-I and type-II superconductors is a complicated function of \( (R_\xi; \kappa) \), it depends on the specimen geometry, and differs [7] from a simple phase boundary \( \kappa = 1/\sqrt{2} \) [8], which describes the case of an infinite (open) superconducting system. For instance, the superconducting cylinder with \( \kappa = 1 \) changes its type from type-II (above the point \( \alpha \) in Fig. 1(a)) to type-I (between the points \( \alpha \) and \( \beta \)), and again to type-II (below the point \( \beta \)).

B. Field decrease regime, type-II region

If the field \( h_\xi \) decreases, starting from the large values \( h > h_2, \psi \equiv 0 \), a weak superconducting state (with \( \psi_{\text{max}} < 1 \)), which nucleates at \( h_2 \), grows gradually into the edge-suppressed form (with \( \psi_{\text{max}} < 1 \) at \( h_\xi = h_1 \)), but for \( h_\xi < h_1 \) the edge-suppressed solution transforms continuously into new "depressed" form (represented by the dotted curve in Fig. 1(b)). In the field interval \( \Delta = h_1 - h \), there exist simultaneously two stable solutions of Eqs. \((1) - (5)\) (the solid and dotted lines in Fig. 1(b)), the depressed solution being responsible for the hysteresis in type-II cylinder. At the restoration field \( h_r \) the depressed solution becomes absolutely unstable and transforms by a first-order jump (\( \delta_r \)) into the Meissner-type form (a solid line; the space-profiles of various solutions are illustrated in Fig. 2). The width of the field interval, \( \Delta = h_1 - h_r \), where hysteresis is possible, diminishes with the diminishing radius \( R_\xi \). The critical radius, at which \( \Delta = 0 \), is presented by the dashed line in Fig. 1(a). No hysteresis is possible below this line.

The first-order jumps (\( \delta_1 \) and \( \delta_1 \) in Fig. 1(b)) also diminish with radius, they disappear simultaneously at the line \( \Delta = 0 \) (Fig. 1(a)). At this line the two disconnected pieces of the solid curve (Fig. 1(b)) met, and it acquires a continuous form, with an inflection point (see Fig. 1(d)). This curve is reversible, in the sense, that for each \( h_\xi \) there exists only one solution, which is independent of the field regime. For even smaller radius \( R_\xi \) (in the region \( s^0 \Delta \), below the line \( \Delta = 0 \) in Fig. 1(a)) the order parameter \( \psi_{\text{max}}(h_\xi) \) behaves as is shown by the dashed line in Fig. 1(d). There is no hysteresis in the region \( s^0 \Delta \), and the transition to normal state is always of second order, independently of \( \kappa \)-value.

C. Type-I region

The order parameter \( \psi_{\text{max}}(h_\xi) \) behaves very differently in the type-I region \( s^3 \Delta \) of Fig. 1(a). If the field increases, and if the radius \( R_\xi \) is sufficiently large, and the parameter \( \kappa \) is sufficiently small, there are no tails in \( s^3 \Delta \) (see Fig. 1(c)). The first-order jump (\( \delta_1 \)) to the normal state occurs at the point \( h_1 \) (where the superconducting solution in the field increase regime becomes absolutely unstable). For fields \( h_\xi > h_1 \) the normal state solution in Fig. 1(c) is absolutely stable, there are no other solutions here, but \( \psi \equiv 0 \).

If the field \( h_\xi \) decreases now below \( h_1 \), the supercooled normal state solution \( (\psi \equiv 0) \) remains stable relative small perturbations with \( \psi \neq 0 \), down to the point \( h_p \). For fields inside the interval \( h_p < h_\xi < h_1 \) there are two stable solutions for \( \psi \); one is \( \psi \equiv 0 \), the other \( (\psi \sim 1) \) is represented by the upper section of the solid curve in Fig. 1(c). (Both solutions are stable, because small perturbations in there shape die down.) Thus, in the field interval \( \Delta = h_1 - h_p \) the supercooling of the normal state is possible [see the hysteresis loop in Fig. 1(c)]. Below \( h_p \) the normal state becomes absolutely unstable (small perturbations grow), and a feeble superconducting solution (or, the "precursor" state, \( \psi \ll 1 \)) establishes in the bulk of a cylinder. (The position of \( h_p \) coincides exactly with the critical field of \( \kappa \)-independent second-order \( (s, n) \)-transitions and may be found from a linear theory [5].) The precursor solution exists down to some restoration field \( h_r \), where it becomes absolutely unstable and the Meissner form restores [notice the presence of first-order jump \( \delta_r \) in Fig. 1(c)]. The field interval \( \Delta_p = h_p - h_r \), where the precursor solution exists, is usually very small \( (\Delta_p \sim 10^{-2} - 10^{-4} h_p) \), so in type-
I superconductors the restoration of superconductivity from a supercooled n-state is "almost first-order" phase transition. The exact value of the restoration field \( h_r \) and the corresponding amplitude of the precursor solution \( \psi_r \) can not be found from the linear theory, but self-consistent solution of the full system of nonlinear GL-equations is required.

If the radius \( R_\xi \) diminishes, the jumps \( \delta_r \) and \( \delta_1 \) also diminish (as well as the interval \( \Delta = h_1 - h_\nu \), see the dashed curves in Fig. 1(c)), they all disappear simultaneously with \( \Delta \to 0 \). The line \( \Delta = 0 \) (for small \( \varkappa \) and \( R_\xi \)) merges with the line \( S_{\text{N}} \) in Fig. 1(a) at the point \( \beta \) (compare the dashed curves in Figs. 1(c) and 1(d)). No supercooling of normal state is possible below the line \( \Delta = 0 \).

D. Examples of co-ordinate dependencies

The co-ordinate dependence of the solutions [as they are seen through the peep-hole, pierced in the type-II point \( R_\xi = 10, \varkappa = 1.2 \) in Fig. 1(a)] is shown in Fig. 2 for different values of the external field \( H (h_\xi = H/H_\xi) \). In the field increasing regime, the sequence of curves, which result from Eqs. (1)–(5), are numerated as \( (0, 1, 2, 2e, K_m) \). The order parameter \( \psi (r) \) (Fig. 2(a)) changes its form continuously from \( 0 (\psi \equiv 1 \at h_\xi = 0) \) to \( 1 (h_\xi = 0.9037) \), and then to \( 2 (h_\xi = 1.0951) \) – this is the critical field, marked as \( h_1 \) in Fig. 1(b)). At this point the solution 2 becomes absolutely unstable and (by a first-order jump, \( \delta_1 = 0.155 \)) acquires the edge-suppressed form \( 2e (h_\xi = 1.0952) \). If the field \( h_\xi \) increases further, the solution \( 2e \) passes continuously into the form \( K_m \) (the Kummer-type solution), which vanishes finally at the critical field \( h_\xi = h_2 = 1.0016 \), where \( \psi \equiv 0 \). Fig. 2(b) illustrates the behavior of the induction \( B (r) \) versus the external magnetic field \( H \). At the point \( h_1 \) the solution 2 switches (by a jump \( \delta_1 \)) from the Meissner form \( 2 \) (the external field is effectively screened out) to the edge-suppressed form \( 2e \) (the external field penetrates the edge region without screening). The curve \( K_m \) may be described by the linear equation approximation \( B = H \).

The sequence of solutions, appearing in Fig. 2 in the field decreasing regime, is different (see Fig. 1(b)). The normal state solution \( \psi (0) \) is absolutely stable for \( h_\xi > h_2 \), and absolutely unstable for \( h_\xi < h_2 \). If the field decreases, a small superconducting solution (of the Kummer-type, \( K_m \)) appears at \( h_\xi = h_2 \) and transforms continuously into the edge-suppressed form \( 2e \) (with \( \psi \sim 1 \) at \( h_\xi = h_2 \)). If the field \( h_\xi \) decreases below \( h_1 \), the solution \( 2e \) keeps transforming continuously into the form \( 1d \) (at \( h_\xi = h_r \)). The intermediate sequence of curves \( (2e \leftrightarrow 1d) \) presents the depressed solutions, which are characterized by a smaller average values of the order parameter \( \bar{\psi} \), in comparison with the Meissner solutions \( (1 \leftrightarrow 2) \). For \( h_\xi < h_r \) the depressed solution \( 1d \) becomes absolutely unstable and transforms (by a jump \( \delta_\xi \)) at \( h_\xi = h_r \) into the Meissner form 1. It is important, that in the interval of fields \( \Delta = h_1 - h_r \) there exist simultaneously (for the same \( h_\xi \)) two independent solutions of Eqs. (1)–(5) (the Meissner and depressed forms). The presence of two solutions means a possibility of hysteresis in the system.

[The Meissner solution can be obtained from Eqs. (1)–(5) by the iteration procedure [21], started with a large trial function \( \bar{\psi} \sim 1 \). The depressed form is obtained by the iterations, started with a small trial function \( \bar{\psi} \sim 0.01 \). Outside the interval \( \Delta = h_1 - h_r \) both iteration procedures produce the same self-consistent result.]

Additional examples of co-ordinate dependencies of the solutions for various \( R \) and \( \varkappa \) may be found in [7]. (The space-profiles of the precursor solutions are depicted below in Fig. 8.)

E. The phase views for \( m=1,2 \)

The analogous phase views on the plane \( (R_\xi, \varkappa) \) for the vortex states \( m = 1 \) and \( m = 2 \) are depicted in Figs. 3(a,b), they are similar to the phase view \( m = 0 \) in Fig. 1(a). The only essential difference is in the presence of the minimal radius \( R_\xi \) [see the dashed lines \( C_{2n} \) in Figs. 3(a,b)], below which only the normal state solutions are possible. This is natural, because in a case of very small radius specimens the intrinsic magnetic field of a vortex is too strong for the survival of the superconducting state.

If the field decreases the sequence of the solutions is \( (0, 2, 2e, K_m) \). In the Meissner-type states \( (0, 1, 2) \) the external field is screened out. At the field \( h_\xi = h_1 \) the jump transformation to the edge-suppressed form \( 2e \leftrightarrow 1d \) occurs (with a jump \( \delta_1 \)). (The concrete values \( h_r, h_1, h_2, \delta_1, \delta_1 \) are given in the caption to Fig. 4.)

In the field decreasing regime the sequence of the appearing solutions is \( (K_m, 2e, 1d, 1, 0) \), with a jump transformation \( \psi \to \bar{\psi} \) from \( 1d \) to 1. In the field interval \( \Delta = h_1 - h_r \) there are two solutions: one is the Meissner-type form (intermediate between 1 and 2), and the other is the depressed form (intermediate between \( 2e \) and \( 1d \)). Within the field interval \( \Delta \) the hysteresis transitions between the Meissner-type and depressed states of the same vorticity \( m \) are possible. For fields outside the interval \( \Delta \) no hysteresis is possible, because only one solution exist there (see Fig. 1(b)): either the Meissner-type solution (for \( h_\xi < h_r \)), or the edge-suppressed solution (for \( h_\xi > h_1 \)).

The giant vortex states with \( m > 2 \) may be studied analogously.
IV. THE PHASE DIAGRAMS

In this Section the comparison is made of the phase diagrams (or, the critical fields $h_2$) for the second-order ($s,n$)-transitions in the increasing magnetic field, found self-consistently and in the linear approximation. [See also Ref. [7a], where the critical fields $h_1$ for the first-order jumps to the edge-suppressed states were studied.]

From the phase view of Fig. 1(a) it follows, that if the magnetic field increases, the order parameter vanishes either gradually [with a tail in the curves of Figs. 1(b,d)], or by the first-order jump [Fig. 1(c)]. Fig. 5(a) is a phase diagram, which shows the dependence of the critical field $h_{sn}^c$ in Fig. 1 [i.e. the field $h_1$ (type-I), or $h_2$ (type-II), for which the transition from $s$- to $n$-state occurs (in the field increase regime),] as a function of $R_\xi$, for different $\kappa$ and $m = 0$. The thick line $K_m$ corresponds to the $(s,n)$-diagram, found from the linear equation (7) with the boundary conditions (5).

As is seen from Fig. 5(a), for $\kappa > 1$ the self-consistent critical curves $h_{sn}^c$ coincide with the universal curve $K_m(h_\xi)$ (found from the linear equation). Thus, for type-II superconductors (with $\kappa > 1$) the linear approach [4,5] gives correct description of the ($s,n$)-boundary.

However, for type-I superconductors (with $\kappa < 1$), the critical curves $h_{sn}^c$ deviate from the universal curve $K_m(h_\xi)$ considerably. The inspection of the asymptotes of the curves $h_{sn}^c$ (for $R_\xi \gg 1$) shows, that they have different functional dependences versus parameter $\kappa$. For type-II superconductors ($\kappa > 1$) the asymptotes are $h_{sn}^c \approx 1$ (they are $\kappa$-independent, in accordance with the linear theory [5]). For type-I superconductors ($\kappa < 1$) the asymptotes behave as $h_{sn}^c \approx \kappa^{-p} (1 < p < 2)$; this means, that the universal linear theory fails for small $\kappa < 1$ and large $R_\xi$. (For sufficiently small radius $R_\xi$ the $(s,n)$-transition is always of second order, so in this case the linear theory is valid for all $\kappa$.)

The analogous conclusions may be drawn from Fig. 5(b) ($m = 1$) and Fig. 5(c) ($m = 2$).

The fact, that the critical fields $h_{sn}^c$ in Fig. 5 for $\kappa < 1$ deviate from the predictions of the linear theory [5], can be attributed to strong nonlinear competition between two characteristic lengths $\lambda$ and $\xi$ in GL-equations. For $\kappa = \lambda/\xi > 1$ the nonlinear equations can be linearized, but for $\kappa = \lambda/\xi < 1$ they can not. (Evidently, the limitations for the linear theory can not be deduced from the linear theory itself, but self-consistent analysis of full nonlinear system of equations is required.)

A. Two solutions of the linear equation

Notice the peculiar behavior of the phase diagrams in Figs. 5(b,c). For sufficiently small $R_\xi$ the line $R_\xi =$const crosses the bottom part of the curves $K_m$ at two points. This means the existence of two solutions $K_m(x)$ of the linear equation (7), which both satisfy the boundary conditions (5) for the same $R_\xi$, but with different $h_\xi$ (see Eq. (9)). These two solutions are depicted in Fig. 6(a) (the solid lines 1 and 2), as they are seen through the peep-holes in Fig. 5(b), pierced along the line $R_\xi = 1.5$ at the points $h_\xi = 0.67$ and $h_\xi = 2.32$ ($m = 1$).

[The insert to Fig. 6(a) illustrates (schematically) the behavior of the solutions of the hypergeometric equation (7) for $m = 1$. The function $K_m(x)$, in a general case, has two extremal points $A$ and $B$, where $K_m'(x) = 0$ (marked by the arrows), whose positions depend on $h_\xi$. The curves $A$ and $B$ in Fig. 6(a) have different functional behavior: the curve $A$ grows monotonously, reaching the extremum at $x = R_\xi = 1.5$. The curve $B$ has a zigzag in the vicinity of $x = R_\xi$ (the zigzag amplitude $\delta$ is small for small $R_\xi$ and is not seen in this scale). The functions $K_m(x)$ for $m > 1$ behave analogously, see Fig. 6(b) for $m = 2$, $R_\xi = 2.5$.]

The existence of two different solutions of Eq. (7) has clear physical interpretation. In the absence of the external field, the vortex is held inside the cylinder by pinning to the boundary (which is the source of the inhomogeneity in otherwise homogeneous system). However, in small radius cylinders the internal vortex field $B_i$ is too strong to be confined inside the mesoscopic sample by the pinning force, and it breaks outside. To prevent the field $B_i$ from leaking to the outer space, it is necessary to impose a finite external field $H$, which helps to keep $B_i$ inside the specimen. (This can be considered as an example of the so-called re-entrant superconductivity, or, the field-enhancement effect.) However, if the field $H$ increases, the superconductivity will be finally destroyed. Two solutions ($A$ and $B$ in Figs. 6(a,b)) describe two different physical situations. The solution $B$ corresponds to the superconducting vortex state ($m > 0$) being destroyed by a large external field $H$. The solution $A$ (with smaller $H$) corresponds to the vortex state being destroyed by the internal field of it’s own vortex.

The solutions $A$ and $B$ in Figs. 6(a,b) are depicted as they are seen through the peep-holes in Figs. 5(b,c), which lie near the opposite sides of the thick curves $K_m$ [these curves represent the states with $\psi_{\text{max}} \ll 1$; the corresponding solutions $\psi(x)$ are the hypergeometric functions $K_m(x)$ of the linear Eq. (7)]. In the intermediate peep-holes $i$ (Fig. 5) the solutions $\psi(x)$ have finite amplitude $\psi_{\text{max}}$, but (if $\psi_{\text{max}}$ is yet sufficiently small) the solution may be approximated by the function $\tilde{K}_m(x)$ of the nonlinear Eq. (6) [see the dashed curves $C$ in Figs. 6(a,b)]. To find $\psi(x)$ with still larger $\psi_{\text{max}}$, it is necessary to solve the full system of Eqs. (1),(2). Such solution is presented by the curve $D$ in Fig. 6(b).
B. Supercooling of the normal state

As was explained in Sec.III, in any peep-hole in the region $s_1^2$ of Fig.1(a) (where $\varkappa > 1$ and $\Delta > 0$) there exist two solutions of Eqs. (1),(2): one is the Meissner-type solution, and the other is the depressed (or partially suppressed) solution, which is responsible for the hysteresis in type-II superconductors (see Fig. 2 for $m = 0$ and Fig. 4 for $m = 1, 2$).

The analogous hysteresis phenomena exist also in type-I superconductors [in the region $s_1^2$ of Fig. 1(a), where $\varkappa < 1$ and $\Delta > 0$; analogously for Fig. 3]. In this case, there are also two solutions: one is again of the Meissner type, but the partially depressed solution (represented by the dotted curve in Fig. 1(b)) degenerates now into the totally depressed normal state of Fig. 1(c).

Thus, in type-I superconductors the second branch of solutions in Fig.1(b) corresponds to the supercooled $n$-state ($\psi(x) \equiv 0$). (Notice, that in type-II superconductors the supercooled normal state can not exist.)

The curve $s_h$ in Fig. 7(a) corresponds to a maximal field, at which the superheated Meissner state ($m = 0$) may still exist in type-I superconductor with $\varkappa = 0.1$ in the field increase regime (i.e. the field $h_2$ in Fig. 1(c)). The curve $s_c$ corresponds to a minimal field, at which the supercooled normal state may still exist in the field decrease regime (i.e. the field $h_r$ in Fig. 1(c)). These two curves merge at the point LCP (the Landau critical point [22]) into a single curve (the calculated difference between $sc$- and $sh$-curves at LCP is less than $1 \cdot 10^{-4}$). The free energies of the superconducting states, which lie between $sc$- and $sh$-curves, depend on the field $h_\xi$, and at some value of $h_\xi$ the difference of free energies $\Delta G = G_s - G_n$ vanishes. The curve $eq$ in Fig. 7(a) shows (schematically) the position of the equilibrium curve ($\Delta G = 0$). All three curves merge at LCP, which is a three-critical point.

The analogous dependences are depicted in Fig. 7(b) for $\varkappa = 0.5$. It is clear, that the width of the region between $sc$- and $sh$-curves diminishes with $\varkappa$ increasing: for $\varkappa = 1$ both curves merge into a single curve and the (normal-state) hysteresis region disappears.

Notice, that for $R_\xi \gg 1$ the maximal supercooling of the normal state in type-I superconducting cylinder is reached in the field $h_\xi = 1$, i.e. in the field $H_\xi = H_{c2} \equiv \phi_0/(2\pi \xi^2)$. To the same conclusion came Ginzburg [3] and Abrikosov [8], who used the approximations $\psi = \text{const}$ and $\varkappa \ll 1$. [The dependence of the superheating field $H_{sh}(h)$ for type-I cylinders was found earlier (in a different parametrization) by Esfandiary and Fink [23], who used self-consistent solutions of the nonlinear equations (the supercooling field $H_{sc}$ was not studied in [23]). (See [6] for the review of early theoretical and experimental works on the problem of hysteresis in type-I superconductors.)]

The comparison of $sc$-curves in Figs. 7(a,b) with the universal $K_m$-curve in Fig. 5(a) reveals, that they are identical and have the following asymptotic behavior: $h_\xi(R_\xi) = 1$ for $R_\xi \gg 1$; $R_\xi(h_\xi) = 2.8/h_\xi$ for $h_\xi > 2$ (see dotted lines $a$ in Figs. 7(a,b)). Notice, that these curves coincide with each other for all $R_\xi$ and $\varkappa < 1$ (not only for $R_\xi \gg 1$ and $\varkappa \ll 1$, as was found in [3,8]). This coincidence is not accidental, because at the supercooling $n$-boundary the precursor solutions appear ($\psi_{max} \ll 1$) which are described by the $\varkappa$-independent linear equation (7).

The precursor solutions are depicted in Fig. 8 for the fields $h_\xi$, which just precede the jumps to the Meissner state with $\psi \sim 1$. The precursor solution shape is $\varkappa$-independent (in accordance with Eq. (7)), but the exact position of the jump and the amplitude of $\psi_{max}$ are $\varkappa$-dependent and should be found from the full system of GL-equations (1), (2). Notice, that the precursor solutions describe the superconductivity nucleation in supercooled type-I cylinder as a bulk (though very feeble) effect. [The detailed study of the precursor solutions behavior would be presented elsewhere.]

C. The free-energy functional

As was mentioned above, the problem of supercooling and superheating in type-I superconductors was considered first by Ginzburg [3], who analysed the behavior of the free energy functional in the approximation $\psi = \text{const}$ ($m = 0$). It is expedient to compare the results of the self-consistent and approximate approaches.

In Fig. 9(a) the exact dependence $\psi_{max}(h_\xi)$ is shown for $\varkappa = 0.5$ at $R_\xi = 3$ and $R_\xi = 1$ ($m = 0$). It is evident, that the width $\Delta$ of the hysteresis region diminishes with $R_\xi$ diminishing, and vanishes at some $R_{\min}$ (at the point LCP in Fig. 7(b)).

In Figs. 9(c) the exact field dependence is shown of the normalized free energy ($\Delta g$) for $\varkappa = 0.5$ and $R_\xi = 3$. (The expressions for $\Delta g$ and magnetization ($-4\pi M$) may be found in [7]). Evidently, the part of the curve $\Delta g(h_\xi)$, which lies to the right of the point $eq$ (where $\Delta g = 0$), corresponds to the metastable s-state with $\Delta g > 0$. In this region s-state is energetically unstable (because n-state has smaller free energy, $\Delta g = 0$). [However, the energetically unstable s-state is stable relative small spatial perturbations.] To the left of the point $eq$ the s-state is energetically more favorable ($\Delta g < 0$), and the n-state (with $\Delta g = 0$) is metastable. [However, the energetically metastable n-state is stable relative small spatial disturbances.] At the equilibrium point $eq$ the transition from one branch of the solution to the other (with smaller free energy) may happen. In this case the system behavior would be totally reversible (without hysteresis).

In Fig. 9(b) the field dependence of the magnetization ($-4\pi M$) is illustrated. In the case of equilibrium transition the reversible jump of the magnetization should be observed at the point $eq$. If the metastable states
are realized, the jumps of the magnetization may happen anywhere between the points $sc$ and $sh$, with the accompanying hysteresis. [There is a principal difference between the jump transitions in Fig. 1(b) and Fig. 8. The first-order jumps in type-II superconductors (Fig. 1(b)) occur within the same state ($m = 0$), while the jump transitions in type-I superconductors (as in Fig. 9) occur between different states ($s$ and $n$)].

Notice, that in the approximation $\psi = \text{const}$ [3] the free energy $\Delta G(\psi)$ has the form, schematically depicted in Fig. 10(a) (the curves $I \div 5$ correspond to the increasing fields $H$). The extremas of these curves define the possible values of the order parameter $\psi_0(H)$, which are depicted (schematically) in Fig. 10(b). [The numerals $I \div 5$ in this figure mark the positions of the extremas on the corresponding curves in Fig. 10(a).] Thus, in the approximate approach [3] there are two solutions for $\psi_0 \neq 0$ in the field interval $H_{sc} < H < H_{sh}$, one (which corresponds to the minimum of the free energy $\Delta G$) is energetically stable, the other (which corresponds to the maximum of the free energy) is energetically unstable.

However, according to the self-consistent theory, in type-I superconductors there exist only one solution with $\psi \neq 0$ [which might be energetically stable, or metastable, depending on $H$ (see Fig. 9(a))], the other (with $\psi \equiv 0$) corresponds to the supercooled normal state. Thus, according to the exact theory, the unstable branch $\psi_0 \neq 0$ of approximate theory does not exist at all.

This contradiction arises because of the following reason. In the rigorous theory the free-energy is a functional, $G[\psi(x)]$, which produces a number, $G$, from a function, $\psi(x)$. The function $\psi(x)$ depends parametrically on $H$, so the functional $G[\psi(x)]$ is a function of $H$. In the Ginzburg approximation the functional $G[\psi(x)]$ is replaced by a function $G(\psi_0, H)$, where $\psi_0$ is considered as an arbitrary number. This function has minima and maxima, what is illustrated by Fig. 10. However, as follows from the self-consistent solution of nonlinear problem, some of the values $\psi_0$ are forbidden. As can be seen from Fig. 9(c), the exact functional $G(H)$ increases with $H$, but it does not reach an extremal point, where $G(0) = 0$, and drops to zero at some value of $H$, where $\psi(H)$ terminates. Thus, the Ginzburg approximation for $G(\psi_0)$ [3] is qualitatively valid, if $\psi(x) \approx \psi_0 = \text{const}$ [see the stable (solid) branch of the curve $\psi_0(H)$ in Fig. 10(b)], but fails for those (forbidden) values of $\psi_0$, which belong to the unstable branch of $\psi_0$ (the dashed curve).

Nevertheless, the point of maximal supercooling, $H_{sc}$, is described by the Ginzburg approximation correctly, because at this point ($\psi_0 = 0$) the precursor state nucleates. [The precursor solution has very small amplitude, $\psi(x) \approx 0$, what always satisfies the condition $[3] \psi(x) \approx \text{const}$, independently of the real space-profile of $\psi(x)$ (see Fig. 8).]

We shall restrict ourselves by these methodical remarks, while comparing the results of the rigorous and approximate approaches.

V. CONCLUSIONS AND DISCUSSION

We have studied in detail the one-dimensional solutions of GL-equations in a case of cylinder geometry. The main new results of the present investigation are summarized below. Mention among them: the phase views, presented in Figs. 1(a) and 3(a,b); the existence of the edge-suppressed and depressed solutions (Figs. 1, 2, 4); the phase diagrams of Fig. 5; the discussion of the applicability of the linear equation approximation (Sec. V); the discussion of two independent solutions of the linear equation (Fig. 6); somewhat novel insight into the problem of hysteresis in type-I superconductors (Figs. 7–9) and the discussion of the previously unknown precursor solutions; the discussion of the free-energy functional and the comparison of the rigorous and approximate approaches to the hysteresis problem (Fig. 10).

In conclusion, some additional comments to the topics, discussed above, should be made.

The edge-suppressed solutions in a cylinder geometry were described first by Fink and Presson in a footnote to their paper [9], but were disregarded as being unstable in the energy space. We consider these solutions (as well as depressed and precursor solutions) as stable in the coordinate space. Mathematically, they are usual axially symmetric solutions, which have different forms in different regions of parameters (due to the nonlinearity of the problem). These solutions describe symmetrical $m$-states, which the physical system may occupy. At the critical field $h_1$ the Meissner state ($m = 0$) becomes unstable and the external field floods the outer region of a cylinder (see the edge-suppressed solution $2e$ in Fig. 2). Another possible mechanism for the field penetration is a creation of a vortex line ($m = 1$) on the cylinder axis (see solutions in Fig. 4). Comparing the free-energies of the competing states, one can find the points of equilibrium transitions [7], and also the field interval, where the edge-suppressed state may exist as a metastable one. The metastable states may manifest themselves in experiments, where such metastability is realized (for instance, in the hysteresis phenomena [24], in paramagnetic Meissner effect [25], etc.), especially in a case of mesoscopic samples (with few number of vortices), when the stabilizing role of the boundary is important.

It is interesting, that similar effects – such as jumps of the magnetization within the states of fixed $m$, the transition to the edge-suppressed form of a giant-vortex solution, the re-entrance of superconductivity and the field-enhancement effects – exist also in a case of mesoscopic disks of small height, studied self-consistently in [17,18]. This means, that these nonlinear effects are common to various sample geometries and are determined mainly by
the parameters, entering the GL-equations, so, the role of the boundary is, probably, not crucial.

No immediate comparison between the present theory and experiment was attempted, partly because the model case of infinitely long cylinder approximates only remotely the thin-disk geometry, mostly used in recent experiments [20]. Moreover, in discussing such experiments, it is necessary to consider also the asymmetric multi-vortex solutions (see the end of Sec. II), what requires using specific numerical methods and large computers (see, for instance, [17,18]). However, a number of qualitative predictions, obtained from the one-dimensional theory, may be used to interpret some of the peculiarities, observed on mesoscopic samples. For instance, it might be possible to attribute some jumps of the magnetization (seen in [20,24,25]) not to transitions between different $m$-states, but occurring within the same $m$-state (due to the order-parameter reconstruction, while it passes to the edge-suppressed form). However, a special experimental analysis would be needed to confirm the existence of such transitions.

Notice, that the precursor solutions, studied above (see Fig. 8), show, that type-I superconductor may be found in two different (metastable) hysteretic states (in addition to the stable Meissner state). One is a supercooled normal metal ($\psi \equiv 0$); this state may persist down to the maximal supercooling field ($h_0$). The other is the supercooled precursor state ($\psi \neq 0$), which nucleates at $h_P$ and survives down to the restoration field ($h_r$), when the first-order transition to the Meissner state ($\psi \approx 1$) occurs. This precursor state is more prominent for values of $\kappa$, laying in the vicinity of $S_{I-II}$-phase boundary (Fig. 1(a)). Though the field interval $\Delta_p$, where the precursor state may be found, is small, it would also be of interest to seek for experimental evidences of its possible existence.

Evidently, further theoretical and experimental study of the problems, discussed above, is desirable.

VI. ACKNOWLEDGMENTS

I am grateful to V.L.Ginzburg for the interest in this work and illuminating discussions. The valuable discussions with F.Peeters, S.Yampolskii and B.Baelus (University of Antwerpen) are also acknowledged with gratitude. I thank prof. R.Gross for hospitality and helpful discussions during my stay at the Walther-Meissner-Institute (Garching), where the final version of this paper was written. The initial stage of this work was partially supported through the grant N00173-99-P-0186.

[1] V.L.Ginzburg, L.D.Landau, Zh.Exp.Teor.Fyz., 10, 1064 (1950).
[2] F. London, Superfluids (Willey, New York, 1950).
[3] V.L.Ginzburg, Zh.Exp.Teor.Fyz. 34, 113 (1958); Soviet Phys.–JETP 7, 78 (1958).
[4] D.Saint-James, P.deGennes, Phys.Lett. 7, 306 (1963).
[5] D.Saint-James, Phys.Lett. 15, 13 (1965).
[6] H.J.Fink, D.S.Mclachlan and B.Roether-Bibby, in Prog. in Low Temp.Phys., v. VIIb, p. 435–516, ed. D.F.Brewer, North Holland Pub., Amsterdam- New York-Oxford (1978).
[7] G.F.Zharkov, V.G.Zharkov, A.Yu.Zvetkov, Phys.Rev.B 61, 12293 (2000); cond-matt/0008217 (2000)(unpublished); G.F.Zharkov, Phys.Rev.B 63, 214502 (2001); Phys.Rev.B 63, 224513 (2001).
[8] A.A.Abrikosov, Fundamentals of the Theory of Metals (North-Holland, Amsterdam, 1988).
[9] H.J.Fink, A.G.Presson, Phys.Rev.B 168, 399 (1968).
[10] R.B.Dingle, Proc.Roy.Soc. A 211 (1952).
[11] L.D.Landau and E.M.Lifshits, Quantum Mechanics (Pergamon, Oxford, 1975).
[12] G.A.Korn and T.M.Korn, Mathematical Handbook (McGraw-Hill Books, New York, 1968).
[13] Yu.N.Orvchinnikov, Sov.Phys.JETP 52, 755 (1980).
[14] V.V.Moshchalkov et al., Phys.Rev.B 55, 11 793 (1997); ibid 60, 10 468 (1999).
[15] V.A.Schweigert, F.M.Peeters, Phys.Rev.B 57, 13 817 (1998).
[16] M.Abramowitz, I.A.Stegun Handbook of Mathematical Functions (Dover, New York, 1970).
[17] V.A.Schweigert, F.M.Peeters et al., Phys.Rev.Lett. 79, 4653 (1997); ibid, 83,2409 (1999); Supralatt. and Microstruct. 25, 1195 (1999); Phys.Rev.B 59, 6039 (1999); ibid, 62, 9663 (2000); Physica C, 332, 266,426,255 (2000).
[18] J.J.Palacios, Phys.Rev.B 57, 10 873 (1998); Physica B, 256-258, 610 (1998); Phys.Rev.Lett. 83, 2409 (1999); 84,1796 (2000).
[19] L.F.Chibotaru et al., Phys.Rev.Lett. 86, 1323 (2001).
[20] A.K.Geim, V.V.Moshchalkov et al., Nature (London) 373, 319 (1995); 390, 259 (1997); 396, 144 (1998); 407, 55 (2000); 408, 833 (2000); Phys.Rev.Lett., 85, 1528 (2000); 86, 1663 (2001).
[21] G.F.Zharkov, V.G.Zharkov, Physica Scripta 57, 664 (1998).
[22] L.D.Landau, Phys.Zs.Sowjetunion 8, 113 (1935); 11, 26 (1937).
[23] M.R.Esfandiary, H.J.Fink, Phys.Lett.A 54, 383 (1975).
[24] O.Buisson et al., Phys.Lett.A 150, 36 (1990).
[25] F.B.Müller-Allinger, A.C.Motta, Phys.Rev.B 59, 8887 (1999).
[26] P.M.Markus, Rev.Mod.Phys. 36, 294 (1964).
[27] A.A.Abrikosov, Sov.Phys.JETP 20, 480 (1965).
[28] J.Matronic, D.Saint-James, Phys.Lett.A 24, 241 (1967).
[29] J.Feder, D.McLachlan, Phys.Rev. 177, 763 (1969).

Figures captions

Fig. 1. (a) – The phase view for a cylinder in the vortex-free Meissner state ($m = 0$) in the field increasing
regime; $s^3_n$ – the region of the first-order phase transitions from $s$- to $n$-state; $s^3_I$ and $s^0_I$ – the regions of second-order phase transitions. The curve $S_{1-1}$ divides the regions of first- and second-order phase transitions.

In the region above the dashed line $\Delta = 0$ the hysteresis transitions are possible; no hysteresis is possible below this line. (b) – Schematic behavior of the order parameter $\psi_{\text{max}}(\xi_2)$ in the region $s^3_I (m = 0, \Delta > 0, w > 0)$. (c) – Schematic behavior of the order parameter in the region $s^3 (m = 0, \Delta > 0, w = 0)$. (d) – Schematic behavior of the order parameter in the region $s^0_I (m = 0, \Delta = 0, w > 0)$. The used notations are explained in the text.

Fig. 2. The coordinate dependences: (a) – for $\psi(x)$ and (b) – for $b(x)$, as they are seen through the peep-hole in Fig. 1(a) at $R_\xi = 10, \kappa = 1.2$ ($m = 0$). The curves are calculated for the dimensionless fields $h_\xi = H/H_\xi; 0 - h_\xi = 0; 1 - h_\xi = 0.6275; 1d - h_\xi = 0.6276; 2 - h_\xi = 0.7605; 2e - h_\xi = 0.7606; K_m - h_\xi = 0.992; \psi_{\text{max}} = 0$ at $h_\xi = 1.000$. The jump $\delta_1$ of $\psi_{\text{max}}$ (see Fig. 1(b)) at the transition $1d \rightarrow 1f$ is $\delta_1 = 2 \cdot 10^{-4}$. The jump at the transition $2\rightarrow 2e$ is $\delta_r = 0.155$. The arrows show how the solutions transform, if the field $h_\xi$ is increased, or decreased. The depressed solutions, responsible for the hysteresis, have the form, intermediate between $2e$ and $1d$.

Fig. 3. The phase views: (a) – for $m = 1$ and (b) – for $m = 2$. Notations are the same, as in Fig. 1(a). Below the line $C_{sn}$ lies the normal-metal region $n$.

Fig. 4. The coordinate dependences $\psi(x)$ and $b(x)$, as they are seen through the peep-holes in Fig. 3(a) at $R_\xi = 10, \kappa = 1.2$ ($m = 1$), and in Fig. 3(b) ($m = 2$). The curves in (a) and (b) are calculated for the fields $h_\xi; 0 - h_\xi = 0; 1 - h_\xi = 0.6367; 1d - h_\xi = 0.6368; 2 - h_\xi = 0.7610; 2e - h_\xi = 0.7611; K_m - h_\xi = 0.986$. The jump at the transition $1d \rightarrow 1f$ is $\delta_1 = 0.018$; the jump at the restoration transition $2\rightarrow 2e$ is $\Delta_r = 0.238$. The curves in (c) and (d) are calculated for the fields $h_\xi; 0 - h_\xi = 0; 1 - h_\xi = 0.6358; 1d - h_\xi = 0.6359; 2 - h_\xi = 0.7619; 2e - h_\xi = 0.7620; K_m - h_\xi = 0.986$. The jump at the transition $1d \rightarrow 1f$ is $\delta_1 = 0.030$; the jump at the restoration transition $2\rightarrow 2e$ is $\delta_r = 0.244$.

Fig. 5. The phase diagrams represent the critical fields $h_\xi = H/H_\xi$, at which the transition from $s$- to $n$-state occurs for a given $R_\xi = R/\xi$ and various $\kappa$ (see the numerals at the curves) and different vorticities $m$: (a) – $m = 0$; (b) – $m = 1$; (c) – $m = 2$. Thick curves $K_m$ correspond to the solutions of linear equation (7). The curves with $\kappa > 1$ are well represented by $K_m$-curve [thus, the linear theory [5] gives correct description of $(s,n)$-boundary]. However, for $\kappa < 1$ (and large $R$) the curves deviate strongly from the curve $K_m$ [thus, for $\kappa < 1$ the linear theory [5] is not applicable]. For sufficiently small $R_\xi$ (when $(s,n)$-transition is of second order) the linear theory is valid for all $\kappa$.

Fig. 6. Two solutions ($1$ and $2$) of the linear equation (7) in the case: (a) – $m = 1$ [$R_\xi = 1.5$, the fields $h_{\xi} = 0.67$ ($1$) and $h_{\xi} = 2.32$ ($2$)]; (b) – $m = 2$ [$R_\xi = 2.5$, the fields $h_{\xi} = 0.291$ ($1$) and $h_{\xi} = 1.742$ ($2$)]. The insert to Fig. 6(a) shows the schematic behavior of two possible solutions. The zigzag amplitude $\delta$ for the solution 2 is not seen in this scale in (a), but is present in (b). (The arrows at the curves 2 show the maximums of $\psi(x)$.) The dashed curves 3 represent solutions of the nonlinear $K_m$-equation (6) in the intermediate peep-hole $i$ [with $h_{\xi} = 1.5$ (a) and $h_{\xi} = 1.6$ (b)]. The dotted curve 4 represents the self-consistent solution, seen through the peep-hole $i'$ ($h_{\xi} = 1$) in Fig. 5(c). [All the curves in Fig. 6 are normalized to $\psi_{\text{max}} = 1$.]

Fig. 7. The upper critical field (the curve $sh$) for superheated $s$-state, and the lower critical field (the curve $sc$) for supercooled $n$-state of the vortex-free Meissner state ($m = 0$) for $\kappa = 0.1$ (a) and $\kappa = 0.5$ (b). For $\kappa = 1$ the $sc$- and $sh$-curves practically coincide. The dotted curve $\alpha \sim 2.8/h_{\xi}$ gives good approximation to $sc$-curve for $R_\xi < 2$. The broken curve eq (schematic) corresponds to the equilibrium transition between $s$- and $n$-states. LCP is the tri-critical Landau point. The curve $sc$ coincides with the phase boundary $K_m$ (dashed line), found from the $\kappa$-independent linear equation (7).

Fig. 8. Precursor solutions for $R_\xi = 3.6, \kappa = 0.9$: (a) – $\psi(x)$, (b) – $b(x)$. Supercooled precursor state nucleates at $h_p = 1.0231$ (with $\psi \approx 0$), and takes forms: $1 - h_\xi = 1.0222; 2 - h_\xi = 1.0210; 3 - h_\xi = 1.1023; 4 - h_\xi = 0.9991$ (this is maximally supercooled precursor state, which at $h_r = 0.9990$ passes into the Meissner form). The dotted line $I_n$ is the normalized (Kummer) function $I(\psi \ll 1)$, which can not be described as $\psi(x) \approx \text{const}$ [3]. However, the exact solution $I$ is small ($\psi(x) \approx 0 = \text{const}$), so the field of maximal supercooling of $n$-state ($h_n$) may be found from the Ginzburg [3] approximation (and also from the linear theory [5]).

Fig. 9. The field dependence of the order parameter $\psi_{\text{max}}(a)$, the magnetization $(M_z = M/H_\xi)$ (b) and the free energy $\Delta G$ (c) in the Meissner state ($m = 0$) of the cylinder with $x = 0.5$ and $R_\xi = 3$. The curve $R_\xi = 1$ in (a) demonstrates, that the hysteresis region $\Delta$ vanishes for small $R_\xi < 1$.

Fig. 10. (a) – Free energy functional $\Delta G$ versus $\psi = \text{const}$, according to the Ginzburg approach [3] (schematic). The sequence of curves 1–5 correlates to the field $H$ increasing. Minimums of $\Delta G$ correspond to stable states, maximums – to unstable states. (b) – The order parameter $\psi_0$, found from the extremums of $\Delta G$. Stable branch is depicted by solid line, unstable – by dashed line. The numerals 1–5 correspond to those in (a). The unstable branch does not exist in the self-consistent theory (see Fig. 9(a)).
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6
Fig. 7

(a) $m=0, \kappa=0.1$

(b) $m=0, \kappa=0.5$
Fig. 8
Fig. 9