Coherent states on Hilbert modules

S Twareque Ali\textsuperscript{1}, T Bhattacharyya\textsuperscript{2} and S S Roy\textsuperscript{3}

\textsuperscript{1} Department of Mathematics and Statistics, Concordia University, 1455 De Maisonneuve Blvd West, Montréal, Québec H3G 1M8, Canada
\textsuperscript{2} Department of Mathematics, Indian Institute of Science, Bengaluru 560012, Karnataka, India
\textsuperscript{3} Department of Mathematics and Statistics, Indian Institute of Science Education and Research, Kolkata, Mohanpur 741252, West Bengal, India

E-mail: stali@math.concordia.ca and tirtha@member.ams.org

Received 6 February 2011, in final form 13 May 2011
Published 6 June 2011
Online at stacks.iop.org/JPhysA/44/275202

Abstract

We generalize the concept of coherent states, traditionally defined as special families of vectors on Hilbert spaces, to Hilbert modules. We show that Hilbert modules over $C^*$-algebras are the natural settings for a generalization of coherent states defined on Hilbert spaces. We consider those Hilbert $C^*$-modules which have a natural left action from another $C^*$-algebra, say $A$. The coherent states are well defined in this case and they behave well with respect to the left action by $A$. Certain classical objects like the Cuntz algebra are related to specific examples of coherent states. Finally we show that coherent states on modules give rise to a completely positive definite kernel between two $C^*$-algebras, in complete analogy to the Hilbert space situation. Related to this, there is a dilation result for positive operator-valued measures, in the sense of Naimark. A number of examples are worked out to illustrate the theory. Some possible physical applications are also mentioned.

PACS numbers: 02.30.Sa, 02.30.Tb, 03.65.Ca

1. Standard coherent states

Coherent states are generically obtained from a reproducing kernel subspace, say $\mathcal{F}_K$ of the Hilbert space $\mathcal{F} = L^2(X, \mu)$ where $\mu$ is a finite measure on the Borel $\sigma$-field of a locally compact topological space $X$. If $\Phi_0, \Phi_1, \ldots, \Phi_n, \ldots$ is any orthonormal basis of $\mathcal{F}_K$, then the reproducing kernel is given by

$$K(x, y) = \sum_k \Phi_k(x) \overline{\Phi_k(y)}.$$  \hfill (1.1)
Using this fact and taking another Hilbert space $\mathcal{K}$ of the same dimension as that of $\mathcal{H}$, the non-normalized coherent states (CS) are defined as

$$|dx\rangle = \sum_k \psi_k \Phi_k(x),$$  \hspace{1cm} (1.2)

where $\psi_1, \psi_2, \ldots, \psi_n, \ldots$ is an orthonormal basis of $\mathcal{K}$. It is then easy to verify that

$$\langle x | y \rangle = K(x, y) \text{ and } \int_X |x\rangle \langle x| \ d\mu(x) = I_{\mathcal{K}},$$  \hspace{1cm} (1.3)

the integral converging in the weak operator topology.

If, furthermore, $K(x, x) = \sum_k |\Phi_k(x)|^2 := \mathcal{N}(x) > 0$, normalized CS can be defined as

$$|\tilde{x}\rangle = \mathcal{N}(x)^{-\frac{1}{2}} |x\rangle,$$

which then satisfy the conditions

$$\||\tilde{x}\rangle\| = 1 \text{ and } \int_X |\tilde{x}\rangle \langle \tilde{x}| \mathcal{N}(x) \ d\mu(x) = I_{\mathcal{K}}.$$

This rather simple construction of an overcomplete family of vectors in a Hilbert space, satisfying a resolution of the identity, turns out to be a powerful tool in many areas of physics and mathematics. Detailed expositions of the theory of coherent states and their applications to mathematics and physics may be found in [4, 16, 20].

The purpose of this paper is to suggest a construction of similar overcomplete families of vectors in Hilbert $C^*$-modules. We call the resulting vectors module-valued coherent states (MVCS). In simple terms, we replace both the set of functions $\Phi_k(x)$ and the vectors $\psi_k$, in the definition of coherent states in (1.2) by elements of Hilbert modules.

It is clear that since the field of complex numbers $\mathbb{C}$ is trivially a $C^*$-algebra, coherent states on Hilbert spaces are special cases of MVCS. The richness of the present generalization will be displayed with a number of examples.

2. Definition and construction of MVCS

Consider two unital $C^*$-algebras $A$ and $B$ and a Hilbert $C^*$-correspondence $E$ from $A$ to $B$. This means that $E$ is a Hilbert $C^*$-module over $B$, with a left action from $A$, i.e. there is a $*$-homomorphism from $A$ into $L(E)$. The term correspondence is now widely used for this structure, see [11, p 148]. Note that $L(E)$ denotes the bounded adjointable operators on $E$ [19, p 8]. A Hilbert $C^*$-correspondence used to be sometimes called a Hilbert $(A, B)$ bimodule, but we refrain from doing so. While $E$ is a Hilbert $C^*$-module under the right action from $B$, there is no left $A$-module structure in general.

Let $(X, \mu)$ be a finite measure space and consider the set of functions

$$\mathcal{F} = \{ F : X \rightarrow E \mid F \text{ is a strongly measurable function} \}.$$

Then, clearly, for any two $F, G$ in $\mathcal{F}$, $x \mapsto \langle F(x) | G(x) \rangle_E$ is a strongly measurable function. Let

$$\mathcal{S} = \{ F \in \mathcal{F} \mid \text{the function } \langle F(x) | F(x) \rangle \text{ is Bochner integrable } \}.$$  \hspace{1cm} (2.1)

Given a strongly measurable function $F$, a necessary and sufficient condition for $\langle F(x) | F(x) \rangle$ to be Bochner integrable is that $\int_X \| \langle F(x) | F(x) \rangle_E \|_B \ d\mu(x) < \infty$. This immediately shows that $\mathcal{S}$ is a complex vector space.

Also, $\mathcal{S}$ is an inner product module over $B$, where the right multiplication and the inner product, respectively, are

$$(F \cdot b)(x) = F(x)b \text{ for all } b \in B, \quad \langle F | G \rangle_{\mathcal{S}} = \int_X \langle F(x) | G(x) \rangle_E \ d\mu(x)$$
on it.
Its completion in the resulting norm \( \| F \|_{\mathcal{B}} = \| (F \mid F)_{\mathcal{B}} \|_{\mathcal{B}}^{1/2} \) is a Hilbert \( C^* \)-module over \( \mathcal{B} \) and can be identified with \( L^2(X) \otimes \mathbf{E} \). There is a natural left action of \( \mathcal{A} \) on \( \mathcal{H} \) because \( \mathbf{E} \) is an \( \mathcal{A} \)-\( \mathcal{B} \)-correspondence.

For \( e \in \mathbf{E} \), we define the map \( \langle e \rangle : \mathbf{E} \rightarrow \mathcal{B} \) by
\[
\langle e \rangle (f) = (e \mid f)_{\mathbf{E}}, \quad f \in \mathbf{E}.
\]
This is an adjointable map. We denote its adjoint by \( \langle e \rangle \). Then \( \langle e \rangle : \mathcal{B} \rightarrow \mathbf{E} \) has the action
\[
\langle e \rangle (b) = eb, \quad b \in \mathcal{B},
\]
so that for \( e_1, e_2 \in \mathbf{E} \),
\[
\langle e_1 \rangle (e_2)(f) = e_1(e_2 \mid f)_{\mathbf{E}}.
\]

We end this section by choosing a set of vectors \( F_0, F_1, \ldots, F_n, \ldots \) (finite or infinite) in the function space \( \mathcal{H} \) (see (2.1)), which are pointwise defined (for all \( x \in X \)) and which satisfy the orthogonality relations
\[
\int_X | F_k(x) \rangle \langle F_\ell(x) | \, d\mu(x) = I_{\mathcal{E}} \delta_{k\ell}.
\]

We now introduce MVCS for two separate situations, highlighting the fact that a Hilbert \( C^* \)-module is a generalization of both a Hilbert space and a \( C^* \)-algebra. The resulting MVCS depend on an auxiliary object \( \mathbf{G} \), which is a Hilbert space in this section and the Cuntz algebras \( \mathcal{O}_n \) or \( \mathcal{O}_\infty \) in section 4.

To proceed with the first construction of MVCS, let \( \mathbf{G} \) be a Hilbert space. In \( \mathbf{G} \) we choose a set of elements, \( \phi_0, \phi_1, \ldots, \phi_n, \ldots \), of the same cardinality as of the \( F_k \), which satisfy
\[
\sum_k | \phi_k \rangle \langle \phi_k | = I_{\mathbf{G}}.
\]
the sum on the left is assumed to converge in the weak operator topology. Note that it follows from (2.4) that any element \( g \in \mathbf{G} \) can be written as a linear combination of the \( \phi_k \):
\[
g = \sum_k | \phi_k \rangle \langle \phi_k | (g) = \sum_k c_k \phi_k, \quad c_k = \langle \phi_k \mid g \rangle_{\mathbf{G}}.
\]

Let \( \mathbf{H} = \mathbf{E} \otimes \mathbf{G} \) denote the exterior tensor product (see, for example, [12, 19]) of \( \mathbf{E} \) and \( \mathbf{G} \), which is then itself a Hilbert module over \( \mathcal{B} \). For each \( x \in X \) and co-isometry \( a \in \mathcal{A} \) (i.e. \( aa^* = id_\mathcal{A} \)), we define the vectors
\[
|x, a \rangle = \sum_k a F_k(x) \otimes \phi_k \in \mathbf{H}
\]
assuming of course that the sum converges in the norm of \( \mathbf{H} \). We call these vectors (non-normalized) MVCS.

**Lemma 2.1.** The MVCS in (2.5) satisfy the resolution of the identity
\[
\int_X | x, a \rangle \langle x, a | \, d\mu(x) = I_{\mathbf{H}},
\]
the integral converging in the sense that for any two \( h_1, h_2 \in \mathbf{H} \),
\[
\int_X \langle h_1 | x, a \rangle_{\mathbf{H}} \langle x, a | h_2 \rangle_{\mathbf{H}} \, d\mu(x) = \langle h_1 | h_2 \rangle_{\mathbf{H}},
\]

as a Bochner integral.
Proof. It is enough to prove the identity on elements in \( H \) of the type \( h_i = e_i \otimes g_i \), with \( e_i \in E \) and \( g_i \in G \), \( i = 1, 2 \). Since these elements form a total set in \( H \), the lemma will be proved by extending by continuity. Indeed,

\[
\int_X \langle h_1 \mid x, a \rangle_H \langle x, a \mid h_2 \rangle_H \, d\mu(x)
\]

\[
= \int_X \sum_k \langle e_1 \otimes g_1 \mid a F_k(x) \otimes \phi_k \rangle_H \cdot \sum_\ell \langle a F_\ell(x) \otimes \phi_\ell \mid e_2 \otimes g_2 \rangle_H \, d\mu(x)
\]

\[
= \sum_{k, \ell} \int_X \langle e_1 \mid a F_k(x) \rangle_E \langle a F_\ell(x) \mid e_2 \rangle_E \langle g_1 \mid \phi_k \rangle_G \langle \phi_\ell \mid g_2 \rangle_G \, d\mu(x)
\]

the interchange of the sums and the integral being easily justified by starting with finite combinations and extending by continuity,

\[
= \sum_{k, \ell} \langle a^* e_1 \mid \int_X F_k(x) F_\ell(x) \, d\mu(x) \rangle \langle g_1 \mid \phi_k \rangle_G \langle \phi_\ell \mid g_2 \rangle_G \]

by virtue of (2.3)

\[
= \langle e_1 \mid e_2 \rangle_E \langle g_1 \mid g_2 \rangle_G \quad \text{in view of (2.4)}
\]

\[
= \langle h_1 \mid h_2 \rangle_H.
\]

This proves that the resolution of the identity holds for vectors of the postulated type. The lemma is proved, as stated earlier, by continuity. \( \square \)

(Note that in the above expressions, we are using the same notation, \( \otimes \), to denote tensor products between different spaces. However, it is clear from the context which spaces are meant in any given instance. Also nothing would really have changed had we taken \( H = G \otimes E \) instead of \( E \otimes G \) in defining the MVCS in (2.5).)

This construction may easily be modified to obtain normalized MVCS under certain conditions. For that, we fix a notation for a certain positive element of \( B \). Let

\[
N(x, a) \overset{\text{def}}{=} \langle x, a \mid x, a \rangle_H = \sum_k \langle F_k(x) \mid a^* a F_k(x) \rangle_E.
\] (2.7)

Lemma 2.2. If \( \phi_1, \phi_2, \ldots \) is an orthonormal basis for \( G \) and \( a \) is a unitary element of \( A \) and \( N(x, \text{id}_A) \) is invertible, then the MVCS constructed above can be normalized, i.e. we can construct MVCS \( \widehat{|x, a\rangle} \) which along with (2.3) and (2.4) also satisfy

\[
\langle \widehat{x, a \mid x, a} \rangle = \text{id}_E \otimes \text{id}_C.
\] (2.8)

Proof. Since \( a \) is unitary, therefore

\[
N(x, a) = N(x, \text{id}_A)
\]

which we have assumed to be invertible and hence \( N(x, a) \) is invertible in \( B \).

We define the normalized MVCS as

\[
\widehat{|x, a\rangle} = |x, a\rangle N(x, a)^{-\frac{1}{2}}.
\] (2.9)
It is then straightforward to verify that these CS satisfy the normalization condition (2.8) and
the resolution of the identity
\[
\int_X |x,a\rangle \mathcal{N}(x,a) \langle x,a| \, d\mu(x) = I_H, \quad H = E \otimes G, \tag{2.10}
\]
which should be compared to (2.6). \qed

3. Vector coherent states and MVCS

We now show how the vector coherent states (VCS) of [6] are obtainable from the
MVCS constructed above. To recall the construction of VCS, consider the Hilbert space
\( H = L^2_{\mathbb{C}N}(X, \mu) \) of \( \mathbb{C}N \)-valued functions on \( X \), with scalar product
\[
\langle f | g \rangle_H = \int_X f(x)^* g(x) \, d\mu(x),
\]
where \( f(x) \) is the column vector with components \( f_i(x) \) and \( f(x)^* \) is the row vector with
components \( f_i^* \). Suppose there exists a reproducing kernel subspace \( H_K \subset L^2_{\mathbb{C}N}(X, \mu) \),
with a (matrix-valued) kernel \( K : X \times X \to M_N(\mathbb{C}) \) (set of all \( N \times N \) complex matrices). Denote by \( P_K \) the projection operator from \( H \) to \( H_K \) and let \( \Phi_0, \Phi_1, \ldots, \Phi_n, \ldots \) be any
orthonormal basis of \( H_K \). Then,
\[
K(x,y) = \sum_k \Phi_k(x)^* \Phi_k(y), \quad K(x,y)^* = K(y,x), \quad \forall x,y \in X \tag{3.1}
\]
Furthermore, for each \( x \in X, \mathcal{N}(x) := K(x,x) = \sum_k \Phi_k(x)^* \Phi_k(x) \) is a positive, invertible
matrix and
\[
\int_X K(x,z)K(z,y) \, d\mu(z) = K(x,y) \tag{3.3}
\]
Let \( \Phi^i_1(x), \Phi^i_2(x), \ldots, \Phi^i_N(x) \) denote the components of the \( N \)-vector \( \Phi^i_k(x) \) in the
orthonormal basis, \( \{ \chi^i \}_{i=1}^N \) of \( \mathbb{C}N \). VCS are now defined to be the elements in \( H_K \):
\[
|x,i\rangle := K(x,x) \chi^i = \sum_k \Phi_k \Phi^i_k(x), \quad x \in X, \quad i = 1, 2, \ldots, N. \tag{3.4}
\]
These satisfy the conditions
\[
\langle x, i | y, j \rangle = K(x,y) \delta_{ij}, \quad \sum_{i=1}^N \int_X |x,i\rangle \langle x,i| \, d\mu(x) = I_{H_K}.
\]
This time, ‘normalized’ VCS are defined as
\[
\langle x,i | = \left[ \text{Tr} (\mathcal{N}(x)) \right]^{-\frac{1}{2}} | x,i \rangle \quad \text{so that} \quad \sum_{i=1}^N \| |x,i\rangle \|^2 = 1, \tag{3.5}
\]
and
\[
\sum_{i=1}^N \int_X |x,i\rangle \langle x,i| \text{Tr} (\mathcal{N}(x)) \, d\mu(x) = I_{H_K}.
\]
Lemma 3.1. The VCS (3.5) are obtainable from a particular set of MVCS constructed using (2.5) above.

Proof. We start by first constructing a family of MVCS by extending the space $X$ over which the VCS (3.5) are defined. Let $d\Omega$ denote the Haar measure (normalized to 1) on the group $SU(N)$. It is known from the general theory of compact groups that

$$\int_{SU(N)} u v^\dagger u^* \, d\Omega(u) = \frac{1}{N} I_N$$

for any normalized vector $v \in \mathbb{C}^N$.

Consider now the domain $X \times SU(N)$ and the orthonormal basis $\{\Phi_i\}$ of the reproducing kernel Hilbert space $\mathcal{H}$ considered above. Define the matrix-valued functions $F_k : X \times SU(N) \rightarrow M_N(\mathbb{C})$,

$$F_k(x, u) = N^{\frac{1}{2}} \sum_{i=1}^{N} u \, \Phi_i^k(x) \, \Phi_i^*,$$

where the $\Phi_i$ are the one-dimensional projection operators, $\chi_i^\dagger \chi_i$, built out of the vectors $\chi_i^\dagger$ in the chosen orthonormal basis of $\mathbb{C}^N$. It is then not difficult to see, using the orthonormality of the vectors $\{\Phi_i\}$ and the relation (3.6), that

$$\int_{X \times SU(N)} F_k(x, u) F_\ell(x, u)^* \, d\mu(x) \, d\Omega_1(u) = I_N \delta_{k, \ell}, \quad k, \ell = 1, 2, \ldots, N. \quad (3.7)$$

Referring to the general construction of MVCS in section 2, we take $\mathcal{A} = \mathcal{B} = M_N(\mathbb{C})$ and $\mathcal{E} = M_N(\mathbb{C})$, considered as a Hilbert module over itself. We take $\mathcal{G}$ to be the Hilbert space $\mathcal{H}$, considered as a Hilbert module over $\mathbb{C}$. The required MVCS are then defined as

$$|x, u, V\rangle = \sum_k V F_k(x, u) \otimes \Phi_k \in \mathcal{H} = M_N(\mathbb{C}) \otimes \mathcal{H}, \quad \text{for all } (x, u) \in X \times SU(N),$$

where $V$ is a unitary element in $SU(N)$. These MVCS satisfy the resolution of the identity

$$\int_{X \times SU(N)} |x, u, V\rangle \langle x, u, V | \, d\mu(x) \, d\Omega(u) = I_{\mathcal{H}}.$$

In order to recover the VCS (3.4) from here, we use the projection operators $P_i(u) = u \Phi_i \Phi_i^* u^*$, $u \in SU(N)$ and simply take the partial trace in $\mathcal{B} = M_N(\mathbb{C})$:

$$|x, i\rangle = \text{Tr}_{\mathcal{G}}[P_i(u) \, |x, u, I_N\rangle]. \quad (3.10)$$

□

4. Representations of Cuntz algebras and MVCS

Let $S_1, S_2, \ldots$ be isometries on a complex separable Hilbert space $\mathcal{K}$ (necessarily infinite dimensional) such that

$$\sum_{j=1}^{\infty} S_j S_j^* = I_{\mathcal{K}}.$$
where the sum converges in the strong operator topology of $B(K)$. Multiplying both sides by $S_i^*$, we obtain
\[ S_i^* + S_i^* \sum_{j \neq i} S_j S_j^* = S_i^* \]
so that
\[ S_i^* \sum_{j \neq i} S_j S_j^* = 0. \]
But $\sum_{j \neq i} S_j S_j^*$ is the projection onto the closure of the span of the ranges of $S_j$ for $j \neq i$. So the range of $S_i$ is orthogonal to the range of $S_j$ for all $j \neq i$. This is a representation of the Cuntz algebra $O_\infty$ with infinitely many generators [15]. In this section, take $G$ to be the $C^*$-algebra generated by the isometries $S_1, S_2, \ldots$. The coherent states are defined as
\[ |x, a \rangle = \left( \sum_{k=1}^{\infty} a \cdot F_k(x) \otimes S_k \right) (\mathcal{N}(x)^{-1/2} \otimes I). \]

We now construct an explicit example of a Cuntz algebra. Let $\omega : \mathbb{N}^0 \rightarrow \mathbb{N}^0 \times \mathbb{N}^0$ be a bijection ($\mathbb{N}^0$ denoting the set of non-zero, positive integers). Consider a Hilbert space $\mathcal{H}$ and let $\{ \phi_n \}$ be its orthonormal basis. Writing $\omega(n) = (k, \ell)$ we define a re-transcription of this basis in the manner
\[ \psi_{k\ell} : = \phi_n = \psi_{\omega(n)}, \quad k, n, \ell \in \mathbb{N}^0. \tag{4.1} \]
Note that the two sets of vectors are exactly the same and satisfy $\langle \phi_m | \phi_n \rangle = \delta_{mn}$ and $\langle \psi_{mn} | \psi_{k\ell} \rangle = \delta_{mk} \delta_{n\ell}$, respectively. Define the family of isometries $S_k, k \in \mathbb{N}^0$ on $\mathcal{H}$, in the manner
\[ S_k \phi_n = \psi_{k\ell}, \quad n \in \mathbb{N}^0. \tag{4.2} \]
Note that this defines an isometry and not a unitary map. Indeed, one has
\[ S_k S_k^* = \delta_{k\ell} I_{\mathcal{H}} \quad \text{and} \quad \sum_{k \in \mathbb{N}^0} S_k S_k^* = \sum_{k \in \mathbb{N}^0} \mathbb{P}_k = I_{\mathcal{H}}, \tag{4.3} \]
$\mathbb{P}_k$ being the projection operator onto the subspace $\mathcal{H}_k$ of $\mathcal{H}$ spanned by the vectors $\psi_{k\ell}, \ell \in \mathbb{N}^0$. Moreover, $S_k S_k^*$ is a partial isometry from $\mathcal{H}_k$ to $\mathcal{H}_k$.

The $C^*$-algebra $O_\infty$, generated by these isometries, is then a Cuntz algebra. Explicit examples of such bijections are well known.

The above construction has an immediate application to a physical situation. We consider the non-normalized version (with $a$ set to the unit element of $\mathcal{A}$)
\[ |x \rangle = \sum_{k=1}^{\infty} F_k(x) \otimes S_k. \]
Let $X = \mathbb{C}$ and $E = L^2(\mathbb{C}, \frac{e^{-|z|^2}}{2\pi} \ dx \ dy)$, $z = \frac{1}{\sqrt{2}}(x + iy)$. We take $F_k : \mathbb{C} \rightarrow \mathbb{C}$ to be the functions
\[ F_k(z) = \frac{z^{k-1}}{\sqrt{(k-1)!}}, \quad k = 1, 2, 3, \ldots. \]
Let $\psi_{k\ell}$ be the complex Hermite polynomials
\[ \psi_{k\ell}(z, \bar{z}) = \frac{(-1)^{\ell+k-2}}{\sqrt{(\ell-1)!}(k-1)!} e^{\bar{z}z} d^{-\ell-1} d^{k-1} e^{-|z|^2}, \quad k, \ell = 1, 2, 3, \ldots. \tag{4.4} \]
These form an orthonormal basis of $L^2(\mathbb{C}, 2 \pi^{-|z|^2} dx dy)$. The coherent states now become

$$|z\rangle = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} S_k,$$

Let $\phi_n$ be as in (4.1) and consider the vectors

$$\xi_{\tau, n} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} S_k \phi_n. \tag{4.5}$$

Then, the vectors (in $L^2(\mathbb{C}, 2 \pi^{-|z|^2} dx dy)$)

$$|z, \tau, n\rangle = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} S_k \xi_{\tau, n} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} \psi_{kn}$$

$(\ell = 1, 2, 3, \ldots, \infty)$ are just the non-normalized versions of the infinite-component vector CS associated with the Landau levels found in [5].

5. Quaternionic coherent states and MVCS

We start by recalling the analytic VCS, built in [7], using powers of matrices from $\mathcal{M}_N(\mathbb{C})$. These are defined as

$$|\mathfrak{z}, i\rangle = \sum_k \frac{3^k}{c_k} \chi^i \otimes \Phi_k \quad \mathfrak{z} \in \mathcal{M}_N(\mathbb{C}), \tag{5.1}$$

where the $c_k$ are the numbers (see, e.g., [7, 17])

$$c_k = \frac{1}{(k+1)(k+2)} \left[ \prod_{j=1}^{k+1} (N+j) - \prod_{j=1}^{k+1} (N-j) \right], \quad k = 0, 1, 2, \ldots.$$

Let $z_{ij}, i, j = 1, 2, \ldots, N$, be the matrix elements of $\mathfrak{z}$. Then, writing

$$F_k(\mathfrak{z}) = \frac{3^k}{\sqrt{c_k}} \quad \text{and} \quad z_{ij} = x_{ij} + iy_{ij},$$

it can be shown that

$$\int_{\mathcal{M}_N(\mathbb{C})} F_k(\mathfrak{z}) F_\ell(\mathfrak{z}^*) \, d\mu(\mathfrak{z}, \mathfrak{z}^*) = \delta_{k\ell} I_N, \quad d\mu(\mathfrak{z}, \mathfrak{z}^*) = \frac{e^{-\text{Tr}[\mathfrak{z}^* \mathfrak{z}]}}{(2\pi)^N} \prod_{i,j=1}^{N} dx_{ij} \, dy_{ij}.$$

Using this fact, it is easy to prove the resolution of the identity

$$\sum_{i=1}^{N} \int_{\mathcal{M}_N(\mathbb{C})} |\mathfrak{z}, i\rangle \langle \mathfrak{z}, i| \, d\mu(\mathfrak{z}, \mathfrak{z}^*) = I_N \otimes I_{\mathcal{H}_K}. \tag{5.2}$$

To construct the related MVCS, we take $E = B = \mathcal{M}_N(\mathbb{C})$. The module $\mathcal{H}$, containing the functions $F_k$, then consists of functions from $\mathcal{M}_N(\mathbb{C})$ to itself. Considering $\mathcal{H}_K$ as a module over $\mathbb{C}$, we may define MVCS in $H = \mathcal{M}_N(\mathbb{C}) \otimes \mathcal{H}_K$ as

$$|\mathfrak{z}, a\rangle = \sum_k a F_k(\mathfrak{z}) \otimes \Phi_k = \sum_k x^k \otimes \Phi_k, \tag{5.3}$$

where $a$ is a unitary element in $\mathcal{M}_N(\mathbb{C})$. These then satisfy the resolution of the identity

$$\int_{\mathcal{M}_N(\mathbb{C})} |\mathfrak{z}, a\rangle \langle \mathfrak{z}, a| \, d\mu(\mathfrak{z}, \mathfrak{z}^*) = I_H. \tag{5.4}$$
In the particular case when $N = 2$, the set $\mathcal{M}_N(\mathbb{C})$ of all complex $2 \times 2$ matrices can be identified with the space of complex quaternions. The resulting MVCS may then be called complex quaternionic MVCS.

VCS of type (5.1), when $z$ is replaced by a real quaternionic variable $q$, have been constructed in [6] and [24], while coherent states in quaternionic Hilbert spaces have been studied in [2]. These latter coherent states, which are a natural generalization of the canonical coherent states to quaternionic quantum mechanics [1], have interesting physical applications. We now construct an analogous family of quaternionic coherent states on a quaternionic Hilbert space. Recall that a quaternionic Hilbert space is a linear vector space over the field of (real) quaternions, $\mathbb{H}$, with the inner product taking values in $\mathbb{H}$. While $\mathbb{H}$ contains the complexes, it is not a $C^*$-algebra. So, strictly speaking, a quaternionic Hilbert space is not a Hilbert $C^*$-module. However, the quaternionic CS we now construct are very similar to the MVCS (5.2).

The quaternionic vector coherent states introduced in [24] are VCS defined on a standard Hilbert space $H$ (over the complex field). We take for $q \in \mathbb{H}$ its representation by $2 \times 2$ complex matrices:

$$q = u(\theta, \phi) \left( \begin{array}{cc} z & 0 \\ 0 & \overline{z} \end{array} \right) u(\theta, \phi)^*, \quad u(\theta, \phi) = \left( \begin{array}{cc} i e^{i\frac{\theta}{2}} \cos \frac{\phi}{2} & -e^{i\frac{\theta}{2}} \sin \frac{\phi}{2} \\ e^{-i\frac{\theta}{2}} \sin \frac{\phi}{2} & -ie^{-i\frac{\theta}{2}} \cos \frac{\phi}{2} \end{array} \right),$$

(5.4)

where $z \in \mathbb{C}$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. Writing $z = re^{i\beta}$, we also have

$$q = r [I_2 \cos \xi + i \sigma(\hat{n}) \sin \xi] = re^{i\beta} e^{i\xi},$$

(5.5)

with

$$I_2 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad \sigma(\hat{n}) = \left( \begin{array}{cc} \cos \theta & e^{i\phi} \sin \theta \\ e^{-i\phi} \sin \theta & -\cos \theta \end{array} \right), \quad [\sigma(\hat{n})]^2 = I_2.$$

Let $\{\Psi_n\}_{n=0}^{\infty}$ be an orthonormal basis of $H$, $\chi^i$, $i = 1, 2$, an orthonormal basis of $\mathbb{C}^2$. Normalized quaternionic vector coherent states are then defined [6, 24] as

$$|q, j\rangle = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{q^n}{\sqrt{n!}} \chi^i \otimes \Psi_n \in \mathbb{C}^2 \otimes H,$$

(5.6)

These vectors satisfy the resolution of the identity

$$\sum_{j=1}^{2} \int_{\mathbb{H}} |q, j\rangle \langle q, j| \, dq = I_2 \otimes I_\mathbb{H}, \quad \mu(q, q^*) = \frac{1}{8\pi^2 r} \sin \theta \, dr \, d\xi \, \sin \theta \, d\theta \, d\phi.$$

(5.7)

Suppose now that $H_{\text{quat}}$ is a Hilbert space over the quaternions. (Multiplication by elements of $\mathbb{H}$ from the right is assumed, i.e. if $\Phi \in H_{\text{quat}}$ and $q \in \mathbb{H}$, then $\Phi q \in H_{\text{quat}}$.) The obvious generalization of the VCS (5.6) to quaternionic coherent states over $H_{\text{quat}}$ is easily written by taking an orthonormal basis $\{\Psi_n^{\text{quat}}\}_{n=0}^{\infty}$ in $H_{\text{quat}}$ and defining the vectors

$$|q\rangle = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{\Psi_n^{\text{quat}} q^n}{\sqrt{n!}} \in H_{\text{quat}}, \quad q \in \mathbb{H}, \quad (q |q\rangle)_{H_{\text{quat}}} = I_2.$$

(5.8)

They satisfy the resolution of the identity

$$\int_{\mathbb{H}} |q\rangle \langle q| \, dv(q, q^*) = I_{H_{\text{quat}}}, \quad dv(q, q^*) = \frac{1}{4\pi^2} r \sin \theta \, dr \, d\xi \, \sin \theta \, d\theta \, d\phi.$$

(5.9)

These coherent states were obtained in [2], where a group theoretical argument was used to construct them. Recently they have also been obtained in [25]. Here we stress their similarity with our general construction over Hilbert $C^*$-modules.
6. Infinite component VCS

As a similar example to the above, but this time involving VCS with an infinite number of components, we consider the VCS

\[ |z, \overline{z}; \ell\rangle = e^{-\frac{1}{2}(|z|^2+|\overline{z}|^2)} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |\Psi_n\rangle, \quad \ell = 0, 1, 2, \ldots, \infty, \quad (z, \overline{z}) \in \mathbb{C} \times \mathbb{C}, \]

(6.1)

where the \( \Psi_n \) form an orthonormal basis in some Hilbert space \( \mathcal{H} \). These VCS are similar to those appearing in the problem of an electron moving in a constant magnetic field and its associated *Landau levels* [5]. They satisfy the normalization condition

\[ \sum_{\ell=0}^{\infty} |z, \overline{z}; \ell \rangle \langle z, \overline{z}; \ell | = 1, \]

and the resolution of the identity

\[ \sum_{\ell=0}^{\infty} \int_{\mathbb{C}} |z, \overline{z}; \ell \rangle \langle z, \overline{z}; \ell | \frac{dx \, dy}{\pi} = I_{\mathbb{H}}, \quad z = x + iy. \]  

(6.2)

In order to construct a family of MVCS corresponding to this set of VCS, we start with a locally compact, unimodular group \( G \) (such as, e.g., \( SU(1, 1) \)), which has a representation, in the discrete series, in an *infinite-dimensional* Hilbert space \( \mathcal{H} \). Let \( G \supseteq g \mapsto U(g) \) be such a unitary irreducible representation and let \( d\mu_G \) denote the Haar measure of \( G \). It is then well known (see, e.g., [4]) that if \( \phi \) is any unit vector in \( \mathcal{H} \), then

\[ \frac{1}{d} \int_G U(g) |\phi \rangle \langle \phi | U(g)^* \, d\mu_G(g) = I_{\mathbb{H}}, \]  

(6.3)

where \( d > 0 \) is a constant, called the *formal dimension* of the representation \( U \). Let \( \{\phi_i\}_{i=1}^{\infty} \) be an orthonormal basis of \( \mathcal{H} \) and \( \mathbb{P}_i = |\phi_i \rangle \langle \phi_i | \) the corresponding one-dimensional projection operators. We define the functions \( F_k: \mathbb{C} \times \mathbb{C} \times G \rightarrow \mathcal{L}(\mathcal{H}) \):

\[ F_k(z, \overline{z}, g) = \frac{1}{d^2} e^{-\frac{1}{2}(|z|^2+|\overline{z}|^2)} \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n!}} \mathbb{P}_n(g), \quad \mathbb{P}_n(g) = U(g) \mathbb{P}_n U(g)^*. \]  

(6.4)

It is then easy to see that

\[ \int_{\mathbb{C} \times \mathbb{C} \times G} F_k(z, \overline{z}, g) F_l(z, \overline{z}, g)^* \frac{dx \, dy}{\pi} \, d\mu_G(g) = \delta_{kl} I_{\mathbb{H}}, \quad z = x + iy. \]

Thus, considering \( \mathcal{L}(\mathcal{H}) \) as a C*-algebra and as a Hilbert module over itself, we again define the MVCS on \( \mathbb{H} = \mathcal{L}(\mathcal{H}) \otimes \mathcal{H} \):

\[ |z, \overline{z}, g; a\rangle = \sum_{k=1}^{\infty} a F_k(z, \overline{z}, g) \otimes \overline{\Psi}_k = \frac{1}{d^2} e^{-\frac{1}{2}(|z|^2+|\overline{z}|^2)} \sum_{k,n} \frac{z^n}{\sqrt{n!}} k! \mathbb{P}_n(g) \otimes \overline{\Psi}_k; \]

where, once more, \( a \) is a unitary element in \( \mathcal{L}(\mathcal{H}) \) and \( \{\overline{\Psi}_k\}_{k=1}^{\infty} \) an orthonormal basis of \( \mathcal{H} \). These MVCS clearly have all the required properties, e.g. the resolution of the identity

\[ \int_{\mathbb{C} \times \mathbb{C} \times G} |z, \overline{z}, g; a\rangle \langle z, \overline{z}, g; a | \frac{dx \, dy}{\pi} \, d\mu_G(g) = I_{\mathbb{H}}, \]

and the VCS can be obtained from them by taking the partial trace in \( \mathcal{L}(\mathcal{H}) \):

\[ |z, \overline{z}, \ell\rangle = \text{Tr}_{\mathcal{L}(\mathcal{H})} [\mathbb{P}_\ell(g) | z, \overline{z}, g; I_{\mathbb{H}} \rangle \].
7. Reproducing kernel, carrier space and a minimal dilation

We start with a description of the very well-known concept of a positive definite kernel and then briefly show how it naturally leads to a vast generalization called a completely positive definite kernel. Given a set $X$, a map $k : X \times X \to \mathbb{C}$ is called a positive definite kernel if for any $n = 1, 2, \ldots$, any $n$ points $x_1, x_2, \ldots, x_n$ in $X$ and any $n$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j k(x_i, x_j) \geq 0.
$$

This immediately and naturally generalizes to the $C^*$-algebra context. Let $k$ map into a $C^*$-algebra $B$ instead of the complex numbers $\mathbb{C}$. Then, $k$ is called a positive definite kernel if for any $n = 1, 2, \ldots$, any $n$ points $x_1, x_2, \ldots, x_n$ in $X$ and any $n$ elements $b_1, b_2, \ldots, b_n$ of $B$, the element

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} b_i^* k(x_i, x_j) b_j
$$

of $B$ is positive. As we note, the key concepts being used are those of involution and positivity in a $C^*$-algebra. Now, given two $C^*$-algebras $A$ and $B$, a set $X$ and a map $k : X \times X \to \mathcal{L}(A, B)$, where $\mathcal{L}(A, B)$ denotes the set of bounded linear operators from $A$ to $B$, the map $k$ called a completely positive definite kernel if for any $n = 1, 2, \ldots$, any $n$ points $x_1, x_2, \ldots, x_n$ in $X$, any $n$ elements $a_1, a_2, \ldots, a_n$ of $A$, and any $n$ elements $b_1, b_2, \ldots, b_n$ of $B$, the element

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} b_i^* k(x_i, x_j) [a_i^* a_j] b_j
$$

of $B$ is positive. Just as there is a Hilbert space that can be constructed from an ordinary scalar-valued positive definite kernel, there is a Hilbert $C^*$-correspondence from $A$ to $B$ that can be constructed out of a completely positive definite kernel. The following theorem, which was shown in ([10], theorem 3.2.3) and can also be found in [8], gives complete clarity.

**Theorem 7.1.** Given a function $K : X \times X \to \mathcal{L}(A, B)$, the following are equivalent.

1. $K$ is a completely positive definite kernel in the sense that the function from $(X \times A) \times (X \times A) \to \mathcal{L}(A, B)$ given by

   $$
   ((x, a), (x', a')) \mapsto K(x', x)[a^* a']
   $$

   is a positive definite kernel in the sense that

   $$
   \sum_{i,j=1}^{N} b_i^* K(x_i, x_j) [a_i^* a_j] b_j \geq 0 \text{ in } B
   $$

   for all $(x_1, a_1), \ldots, (x_N, a_N) \in X \times A$ and $b_1, \ldots, b_N \in B$.

2. $K$ has a Kolmogorov decomposition in the sense of [10], i.e. there exists an $(A, B)$-correspondence $E$ and a mapping $x \mapsto k_x$ from $X$ into $E$ such that

   $$
   K(x, y)[a] = (k_x, a \cdot k_y)_E \text{ for all } a \in A.
   $$

3. $K$ is the reproducing kernel for an $(A, B)$-reproducing kernel correspondence $E = E(K)$, i.e. there is an $(A, B)$-correspondence $E = E(K)$ whose elements are $B$-valued functions on $X \times A$ such that the function $k_x : (x', a') \mapsto K(x', x)[a']$ is in $E(K)$ for each $x \in X$ and has the reproducing property

   $$
   (k_x, a \cdot f)_{E(K)} = (a^* \cdot k_x, f)_{E(K)} = f(x, a) \text{ for all } x \in X \text{ and } a \in A.
   $$
where \( a^* \cdot k_x \) is given by
\[
(a^* \cdot k_x)(x', a') = K(x', x)[a^*a'] = \langle a^* \cdot k_x, a' \cdot k_x \rangle.
\] (7.1)

Corresponding to the MVCS in (2.5), define the kernel \( K : X \times X \rightarrow \mathcal{L}(A, B \otimes C) \) by
\[
K(x, y)a^*a' = \langle x, a | y, a' \rangle = \int_X \langle a \cdot F_k(x) \otimes \phi_k | a' \cdot F_k(y) \otimes \phi_k \rangle_H
\] for all \( x, y \in X \) and \( a, a' \in A \). This is a completely positive definite kernel. In fact, (7.2) gives the Kolmogorov decomposition of the kernel. This kernel has the following reproducing property:
\[
k(x, z)a^*a' = \langle x, a | z, a' \rangle
\]
\[
= \langle x, a \mid \int_X | y, b \rangle \langle y, b | \mu(y) \mid z, a' \rangle \text{ for a co-isometry } b \in A
\]
\[
= \int_X \langle x, a \mid y, b \rangle \langle y, b | z, a' \rangle d\mu(y)
\]
\[
= \int_X k(x, y)a^*b \ k(y, z)b^*a' \ d\mu(y).
\]
In particular, taking \( a = b = \text{id}_A \), we obtain that
\[
k(x, z)a^*a' = \int_X k(x, y)\text{id}_A k(y, z) \ d\mu(y) \ a',
\]
which means that \( k(x, z) = \int_X k(x, y)\text{id}_A k(y, z) \ d\mu(y) \).

We show the existence of an associated reproducing kernel and a carrier Hilbert module, corresponding to a family of MVCS. Let \((X, \mu)\) be a measure space. Consider the Hilbert modules \( E \) over the \( C^* \)-algebra \( B \) and \( G \) over the \( C^* \)-algebra \( C \), \( H = E \otimes G \) and the Hilbert module \( \tilde{H} \), consisting of measurable functions \( F : X \rightarrow E \), which satisfy the ‘square integrability condition’
\[
\left\| \int_X \langle F(x) | F(x) \rangle_E \ d\mu(x) \right\|_B < \infty.
\]
We also require that the elements \( \phi_i \in G \) used to define the non-normalized MVCS in (2.5) satisfy both (2.4) and orthonormality.

Consider now the Hilbert module \( \tilde{H} \), over \( B \otimes C \), consisting of maps \( \tilde{h} : X \rightarrow B \otimes C \), under the (module) inner product
\[
\langle \tilde{h}_1 | \tilde{h}_2 \rangle_{\tilde{H}} = \int_X \tilde{h}_1(x)^* \tilde{h}_2(x) \ d\mu(x).
\]
Recall that our coherent states \(|x, a\rangle\) in (2.5) are elements of the Hilbert module \( H \). Using these MVCS, we now define the linear map \( W : H \rightarrow \tilde{H} \) by
\[
(Wf)(x) = \langle x, \text{id}_A | f \rangle_H, \quad f \in H.
\] (7.3)
That \( W \) is an isometry is then clear, since
\[
\langle Wf | Wf \rangle_{\tilde{H}} = \int_X \langle f | x, \text{id}_A \rangle \langle x, \text{id}_A | f \rangle \ d\mu(x) = \langle f | f \rangle_H.
\]
using (2.6).

**Theorem 7.2.** The range of \( W \) is a complemented submodule of \( \tilde{H} \).
Proof. We denote by $P_K$ the linear operator on $\tilde{H}$ defined by

$$
(P_K \tilde{h})(x) = \int_X K(x, y) \tilde{h}(y) \, d\mu(y) \quad \text{for all} \quad \tilde{h} \in \tilde{H}.
$$

It is then straightforward to verify that $P_K$ is a projection in the $C^*$-algebra $\mathcal{L}(\tilde{H})$ and the range of the isometry in $\tilde{H}$, which we denote by $\mathcal{H}_K$, is a range of the projection $P_K$. The range of a projection is always a complemented submodule. □

We call it a reproducing kernel submodule because it is the image under an isometry of an $A$-$B$ correspondence. The reproducing kernel $K(x, y)$ is $\langle x, \text{id}_A | y, \text{id}_A \rangle$. It follows that $\tilde{H} = \mathcal{H}_K \oplus \mathcal{H}_K^\perp$.

Writing $h_x = W | x, \text{id}_A \rangle \in \tilde{H}$ so that $K(x, y) = \langle h_x | h_y \rangle_{\tilde{H}} = h_x(x) \in \mathcal{B} \otimes \mathcal{C}$, (7.5)

we see that the vectors $h_x$ span the submodule $\mathcal{H}_K$. From (2.6), (7.4) and (7.5), it also follows that

$$
\int_X | h_x \rangle \langle h_x | \, d\mu(x) = P_K.
$$

Note that the vectors $h_x$, $x \in X$, being unitary images in $\mathcal{H}_K$ of the MVCS $| x, \text{id}_A \rangle$ are also themselves MVCS. Furthermore, the submodule $\mathcal{H}_K$ has a natural left action for $a \in A$ given by

$$
(a \cdot \tilde{h})(x) = (Waf)(x) \quad \text{where} \quad \tilde{h} = Wf.
$$

Finally, using the $h_x$, we may define a POV measure on $\mathcal{H}_K$ and obtain a natural dilation of it to a PV measure on $\tilde{H}$. Indeed, the POV measure is defined on the Borel sets $\Delta$ of $X$ as

$$
\nu(\Delta) = \int_\Delta | h_x \rangle \langle h_x | \, d\mu(x) \in \mathcal{L}(\mathcal{H}_K),
$$

and the PV measure $\tilde{\nu}(\Delta)$ by

$$
(\tilde{\nu}(\Delta) \tilde{h})(x) = \chi_\Delta(x) \tilde{h}(x), \quad \tilde{h} \in \tilde{H},
$$

$\chi_\Delta$ being the characteristic function of the set $\Delta$. It is then straightforward to verify that

$$
\nu(\Delta) = P_K \tilde{\nu}(\Delta) P_K.
$$

If $X$ is a locally compact space and the support of the measure $\mu$ is assumed to be the whole of $X$ (i.e. no open set has measure zero), this dilation can easily be shown to be minimal, in the sense of Naimark. In other words, the set of vectors of the type $\tilde{\nu}(\Delta) \tilde{h}$, as $\Delta$ runs through all Borel sets and $\tilde{h}$ through $\mathcal{H}_K$, spans $\tilde{H}$. The proof is an easy adaptation of the proof of the analogous result for Hilbert spaces (see, e.g., [4, p 36]).

The space $\tilde{H}$ acts as a carrier space for the MVCS, the situation with the dilation here being exactly the same as on a Hilbert space. It would appear that the fundamental ingredients for the existence of a family of MVCS are (i) a Hilbert $C^*$-module of the type $\tilde{H}$, consisting of functions from a finite measure space $(X, \mu)$ to the $C^*$-algebra defining the module and (ii) a reproducing kernel submodule contained in this Hilbert module. We plan to discuss these issues in greater detail in a succeeding publication.
8. Possible physical applications

We have referred to the Landau problem above, in connection with the infinite component VCS. Quantum mechanics is based on Hilbert spaces rather than Hilbert modules, so it is expected that MVCS will be most useful in cases where one needs to go beyond the traditional setting of quantum mechanics. One such possibility could be in the theory of the so-called non-commutative quantum mechanics, i.e. quantum mechanics built on a non-commutative configuration space. A well-known example of such a theory is built on a two-dimensional position space, where the two usual position variables \( x \) and \( y \) are replaced by the non-commutative operators \( \hat{x} \) and \( \hat{y} \) (see, for example, [22, 23] and references therein), which satisfy the commutation relations

\[
[\hat{x}, \hat{y}] = i\vartheta,
\]

where \( \vartheta \) is a positive number which measures the amount of non-commutativity between the two spatial coordinates. In this theory, the coordinate space is taken to be a Hilbert space \( \mathcal{H} \) (separable, infinite dimensional), on which the commutation relations (8.1) are irreducibly realized and the state space is then the Hilbert space \( B_2(\mathcal{H}) \) of all Hilbert–Schmidt operators on \( \mathcal{H} \). Coherent states in this theory, which are now taken to be elements in \( B_2(\mathcal{H}) \), are defined using the creation and annihilation operators, \( a = \frac{1}{\sqrt{2\vartheta}}(\hat{x} + i\hat{y}) \), \( a^\dagger = 1\sqrt{2\vartheta}(\hat{x} - i\hat{y}) \), associated with the two non-commuting position operators. For \( z \in \mathbb{C} \), let \( |z\rangle = e^{i\varphi}e^{\pi\vartheta z}|0\rangle \), where \( |0\rangle \) is the vector in \( \mathcal{H} \) for which \( a|0\rangle = 0 \) is a family of the canonical coherent states on \( \mathcal{H} \). The coherent states of the non-commutative theory are defined to be the vectors

\[
|z\rangle = |z\rangle(z), \quad z \in \mathbb{C}
\]

in the ‘non-commutative state space’ \( B_2(\mathcal{H}) \). These states obey the formal resolution of the identity

\[
\int |z\rangle e^{\frac{2\pi}{i\vartheta} \overline{z} \partial z} (z) \frac{dx \, dy}{2\pi \vartheta} = I_q, \quad \text{where} \quad z = \frac{x + iy}{\sqrt{2\vartheta}},
\]

\( I_q \) being the identity operator on \( B_2(\mathcal{H}) \). These states also lead to a formal positive operator-valued measure on \( B_2(\mathcal{H}) \). It has been shown in [22] that an extended version of these coherent states are useful in analyzing the non-local nature of the non-commutative coordinate space. We suggest here a different definition of coherent states, which, as we will argue below, also encapsulate information on the non-locality of the space. Moreover, these coherent states lead to a mathematically well-defined positive operator-valued measure, whose use in analyzing non-locality in the coordinate space here is very much in line with the well-known theory of the joint measurements of non-commuting observables, in this case \( \hat{x} \) and \( \hat{y} \). (Extensive and mathematically rigorous discussions of such joint measurements can be found in the literature on the foundations of quantum mechanics—see, for example, [3, 13, 14, 18, 21].)

In \( \mathcal{H} \) let us choose the so-called Fock basis \(|k\rangle, \quad k = 0, 1, 2, \ldots \), and in \( B_2(\mathcal{H}) \) we take the basis \(|k, \ell\rangle = |k\rangle \ell\rangle \ | k, \ell = 0, 1, 2, 3, \ldots \rangle \), on which the operators \( a, a^\dagger \), defined above, act as

\[
a|k, \ell\rangle = \sqrt{k}|k - 1, \ell\rangle, \quad a^\dagger|k, \ell\rangle = \sqrt{k + 1}|k + 1, \ell\rangle.
\]

With the bijection \( \omega \) used in (4.1), let us define the basis in \( B_2(\mathcal{H}) \),

\[
|n\rangle = |k, \ell\rangle, \quad \text{where} \quad n = \omega(k, \ell),
\]

and the isometries \( S_n \) on \( B_2(\mathcal{H}) \), \( k = 0, 1, 2, \ldots \), (see (4.2)–(4.3)):

\[
S_n|n\rangle = |k, n\rangle, \quad n = 0, 1, 2, \ldots
\]
As in (2.5) we define the set of normalized MVCS:

\[ \eta_z = e^{-\left| z \right|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} S_n, \quad z \in \mathbb{C}, \]  

(8.4)

which are now elements in the Cuntz algebra \( \mathcal{O}_\infty \), generated by the isometries \( S_k, \quad k = 0, 1, 2, \ldots \). These MVCS satisfy the normalization condition

\[ (\eta_z | \eta_z) = I_q, \]

and the resolution of the identity

\[ \int_{\mathbb{C}} |\eta_z| (\eta_z | d\bar{z}) = I_q. \]

The operators

\[ a(\Delta) = \int_{\Delta} |\eta_z| (\eta_z | d\bar{z}), \]

as \( \Delta \) runs through the Borel sets of \( \mathbb{R}^2 \), define a POV measure of the type discussed in the previous section, with values in the set of positive operators on \( B_2(\mathfrak{S}) \). If \( X \in B_2(\mathfrak{S}) \) is a state (i.e. \( \|X\| = Tr[X^*X] = 1 \)), then the real measure

\[ p_X(\Delta) = Tr[X^*a(\Delta)X] \]

gives the joint probability that the non-commutative quantum system, when in the state \( X \), is localized in the region \( \Delta \subset \mathbb{R}^2 \) of position space, i.e. the joint spectra of the operators \( \hat{x} \) and \( \hat{y} \). However, as is well known from the theory of joint measurements of non-commuting observables—in this case \( \hat{x} \) and \( \hat{y} \)—this localization is unsharp, i.e. there is non-locality in the theory. This can be demonstrated by computing the marginals of the POV measure \( a(\Delta) \) in one or the other of the two position variables \( x, y \). The situation is analogous to that of the joint measurement of position and momentum in ordinary quantum mechanics (see, e.g., [3, 13]). A similar POV measure, using VCS, was obtained for such a non-commutative quantum system in [9]. Both in that paper and in [22], the attempt is to capture the intrinsic non-local nature of position space using coherent states. Here, by employing the MVCS (8.4), we do that using standard probabilistic and measure-theoretic constructs, in line with the theory of measurements of joint observables in quantum mechanics. The operators \( a(\Delta) \), which are well-defined operators on the Hilbert space \( B_2(\mathfrak{S}) \), are the operators of localization in position for the non-commutative quantum system.

Finally, let us note that as is clear from our discussion so far, we expect our MVCS to be a particularly useful tool when dealing with quantum systems whose Hamiltonians have discrete spectra, with each energy level being infinitely degenerate. For such Hamiltonians, there is no generally accepted construction of coherent states in the literature. As mentioned earlier, the problem of an electron moving on a two-dimensional plane, in a constant magnetic field perpendicular to this plane, is one such problem. This is the well-known Landau problem and as shown above, MVCS for this problem can also be constructed in more or less the same manner as in the example above. Details of the physical interpretation of such MVCS will be explored in a future publication.

Acknowledgments

STA is supported in part by an NSERC grant. TB is supported in part by DST (Ramanna Fellowship) and UGC SAP Phase IV. SSR is supported in part by the National Board for Higher Mathematics, India.
References

[1] Adler S L 1995 Quaternionic Quantum Mechanics and Quantum Fields (Oxford: Oxford University Press)
[2] Adler S L and Millard A C 1996 Coherent states in quaternionic quantum mechanics J. Math. Phys. 38 2117–26
[3] Ali S T 1985 Stochastic localization, quantum mechanics on phase space and quantum space-time Riv. Nuovo Cimento 8 1–128
[4] Ali S T, Antoine J-P and Gazeau J-P 1999 Coherent States, Wavelets and Their Generalizations (New York: Springer)
[5] Ali S T and Bagarello F 2005 Some physical appearances of vector coherent states and coherent states related to degenerate Hamiltonians J. Math. Phys. 46 053518
[6] Ali S T, Englis M and Gazeau J-P 2004 Vector coherent states from Plancherel’s theorem, Clifford algebras and matrix domains J. Phys. A: Math. Gen. 37 5067–89
[7] Ali S T and Englisi M 2007 Berezin–Toeplitz quantization over matrix domains Contributions in Mathematical Physics: A Tribute to Gérard G. Emch ed S T Ali and K B Sinha (New Delhi: Hindustan Book Agency)
[8] Ball J A, Biswas A, Fang Q and ter Horst S 2009 Multivariable generalizations of the Schur class: positive kernel characterization and transfer function realization Recent Advances in Operator Theory and Applications, Proc. Int. Workshop on Operator Theory and Applications (IWOTA) (Seoul, Korea, 31 July–3 August 2006) (Operator Theory: Advances and Applications vol 187) ed T Ando et al (Basel: Birkhäuser) pp 17–79
[9] Ben Geloun J and Scholtz F G 2009 Coherent states in noncommutative quantum mechanics J. Math. Phys. 50 043505
[10] Barreto S D, Bhat B V R, Liebscher V and Skeide M 2004 Type I product systems of Hilbert modules J. Funct. Anal. 212 121–81
[11] Blackadar B 2006 Operator Algebras (Berlin: Springer)
[12] Brückler F M 1999 Tensor products of $C^*$-algebras, operator spaces and Hilbert $C^*$-modules Math. Commun. 4 257–68
[13] Busch P, Grabowski M and Lahti P 1995 Operational Quantum Physics (Lecture Notes in Physics vol m31) (Berlin: Springer) (second, corrected printing in 1997)
[14] Busch P, Lahti P and Mittelstaedt P 1991 The Quantum Theory of Measurement (Lecture Notes in Physics vol m2) (Berlin: Springer) (second revised edition in 1996)
[15] Cuntz J 1977 Simple $C^*$-algebras generated by isometries Commun. Math. Phys. 57 173–85
[16] Gazeau J-P 2009 Coherent States in Quantum Physics (Weinheim: Wiley-VCH)
[17] Ginibre J 1965 Statistical ensembles of complex, quaternion and real matrices J. Math. Phys. 6 440–9
[18] Holevo A S 2001 Statistical Structure of Quantum Theory (Lecture Notes in Physics vol m67) (Berlin: Springer)
[19] Lance E C 1995 Hilbert $C^*$-modules A Toolkit for Operator Algebraists (London Mathematical Society Lecture Notes Series 210) (Cambridge: Cambridge University Press)
[20] Perelomov A M 1986 Generalized Coherent States and Their Applications (Berlin: Springer)
[21] Prugovečki E 1984 Stochastic Quantum Mechanics and Quantum Spacetime: A Consistent Unification of Relativity and Quantum Theory Based on Stochastic Spaces (Dordrecht: Reidel) (revised printing in 1986)
[22] Rohwer C M, Zloschastiev K G, Gouba L and Scholtz F G 2010 Noncommutative quantum mechanics—a perspective on structure and spatial extent J. Phys. A: Math. Theor. 43 345302
[23] Scholtz F G, Gouba L, Hafver A and Rohwer C M 2009 Formulation, interpretation and application of non-commutative quantum mechanics J. Phys. A: Math. Theor. 42 175303
[24] Thirulogasanthar K and Ali S T 2003 A class of vector coherent states defined over matrix domains J. Math. Phys. 44 5070–83
[25] Thirulogasanthar K, Honnouvo G and Krzyzak A 2010 Coherent states and Hermite polynomials on quaternionic Hilbert spaces J. Phys. A: Math. Theor. 43 385205