A New Line Search Method to Solve the Nonlinear Systems of Monotone Equations

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Abstract: In this study, we suggest a new line search algorithm for solving nonlinear systems of equations such that we combine a monotone technique into a modified line search rule. The new proposed algorithm can decrease the CPU time, the number of iterations and the function evaluations and can increase the efficiency of the approach. Under some standard conditions, the global convergence of the algorithm is proved. Preliminary numerical results shows that the new algorithm is promised for solving nonlinear systems of equations monotone equations.

Key words: Nonlinear system of equations, line search method, Monotone strategy, global convergence, numerical results, iterative method

INTRODUCTION

The nonlinear systems are one of the problems that arise in different fields of science and computational geometry, especially in the interpretation of nonlinear partial differential equations, the problem of specified value, etc. There are situations in which thousands of nonlinear equations can be solved in some independent variables effectively. Thus, finding the roots of nonlinear systems of equations has many applications in numerical and applied mathematics.

Therefore, the focus of many researchers is to find and provide appropriate ways and means to solve these non-linear systems and thus some common algorithms are suggested to solve these problem.

Nonlinear equations are one of the most important problems of multiple scientific uses such as computer science tremolo systems (Ortega and Rheinboldt, 1970; Zeidler, 2013), the first-order necessary condition for the problem of unconstrained convex optimization and also some sub-problems in generalization (Iusem and Solodov, 1997; Shiker and Sahib, 2018).

Since, the fixed points that can be found from the problem of improvement are equal to find the answer of a non-linear system of equations and the systems of nonlinear equations can be converted into problem of the lower squares this indicates a close relationship between the problems of unconstrained optimization and systems of nonlinear equations, so, it is appropriate to use unconstrained optimization algorithms to solve this problem.

One of the two important iterative methods that is used to solve nonlinear system of equations is the line search strategy, the other method is trust region. Here, we focus on the line search method and its framework. This method is fairly simple, so, its understanding and application is easy. However, they are ineffective and have some disadvantage, for example, if the array being searched for contains 30.000 items, to find the value of the last element, the algorithm will have to look at all those 30.000 elements. Typically, if we have a matrix of M elements, the linear search will identify an element in M/2 attempts. For example, if we have a matrix of 40.000 items, the linear search will compare with 20.000 items in a typical case. This is through the possibility to find the search element constantly in the array, so, the number M is always maximum in comparisons. Another disadvantage, on the large scale, the research and convergence of the line search method are slow. So, most of researchers used the monotone strategy to address that problem. Consider the nonlinear system of equations:

$$F(x) = 0$$

where, \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous and monotone, i.e:

$$\langle F(x)-F(y), x-y \rangle \geq 0, \forall x, y \in \mathbb{R}^n$$

By fixed point map or a natural map, some monotone variational inequality can be converted into nonlinear monotonous equations but before that there are some coercive conditions that the basic function has to achieve.
Quasi-Newton methods are considered to be one of the most important algorithms for solving problem (Eq. 1), the methods of Quasi-Newton have been a major advance in the theoretical aspect as a result of the development of solutions to many problems and this is especially, reflected in the analysis of local convergence (Broyden et al., 1973; Dennis and More, 1977). In addition, researchers have done a lot of work to create a global approximation of Quasi-Newton methods for unconstrained optimization problems see (Byrd et al., 1987; Amini et al., 2016; Nocedal, 1980 and Shiker and Amini, 2018).

By Griewank (1986) who is considered to be the closest approximation of global convergence, suggested a derivative-free line search. By Li and Fukushima (2000) had another view by constructing and deducing an example showing that the line search by Griewank (1986) contains in some special cases certain difficulties. As a result of their research and by using the non-monotonous line search method, they suggested a Gauss-Newton based BFGS method to solve nonlinear symmetric equations and a Broydens method to solve nonlinear equations also they proved these methods converge globally (Li and Fukushima, 1999, 2000). However, some of the merit functions such as the quadratic merit function are used to ensure the global approximation of Quasi-Newton.

**MATERIALS AND METHODS**

In this study, the new algorithm is used to solve the nonlinear monotone equations and we proved that it has a global convergence without using merit function. In comparison with BFGS method by Zhou and Li (2008) and PRP method by Cheng (2009), the new method well be more efficient. Now, we will give our algorithm.

**The new algorithm (K)**

**Step 0.** Choose an initial point \( x_0 \in \mathbb{R}^n \) and constants \( \mu_0 \in (0, 1), \rho \in (0, 1), \beta \in (1/2, 1), \sigma \in (0, 1/2), m>0, r>0 \). Let \( k = 0 \)

**Step 1.** Compute the search direction \( d_k \) by:

\[
\mathbf{d}_k = -F(x_k)
\]  

(2)

Stop if \( d_k = 0 \)

**Step 2.** Determine step length \( \alpha_k = \mu^k \beta \) such that \( h_k \) is the smallest nonnegative integer \( h \) satisfies:

\[
-\{F(x_k + \mu^k \beta \mathbf{d}_k), \mathbf{d}_k\} \geq \rho \sigma \mu^k \beta \|F(x_k + \mu^k \beta \mathbf{d}_k)\| \|d_k\|^2
\]  

(3)

Where \( \sigma_i = \frac{\sigma}{1 + \|F(x_i)\|} \)

Let \( z_k = x_k + \alpha_k \mathbf{d}_k \)

Stop if \( \|F(z_k)\| = 0 \)

**Step 3.** Calculate:

\[
x_{k+1} = x_k - \frac{\{F(x_k), x_k - z_k\}}{\|F(z_k)\|^2} F(x_k)
\]  

(4)

Set \( k = k+1 \) Go to Step 1.

**Remark:** The mapping \( F \) is Lipschitz Continuous (LC), satisfies for a positive constant \( L>0 \) that:

\[
\|F(x) - F(y)\| \geq L \|x - y\|, \forall x, y \in \mathbb{R}^n
\]  

(5)

It is clear that \( L+m>m \), so:

\[
\frac{\|F(x_k)\|}{L+m} \leq \|d_k\| \leq \frac{\|F(x_k)\|}{m}
\]  

(6)

Now, we will show that the line search (3) is well-define in a similar way to Solodov and Svaiter (1998). Suppose that for some iteration index \( k \) and for any nonnegative integer \( h \), the line search (3) is not satisfied, i.e.:

\[
-(\{F(x_k + \mu^h \mathbf{d}_k), \mathbf{d}_k\}) \leq \rho \sigma \mu^h \beta \|F(x_k + \mu^h \mathbf{d}_k)\| \|d_k\|^2
\]  

(*)

Now if, we take \( \lim_{h \to \infty} \) for two side to (*):

\[
-\lim_{h \to \infty} \{F(x_k + \mu^h \mathbf{d}_k), \mathbf{d}_k\} \leq \lim_{h \to \infty} \rho \sigma \mu^h \beta \|F(x_k + \mu^h \mathbf{d}_k)\| \|d_k\|^2
\]

\[
\Rightarrow -\{F(x_k), \mathbf{d}_k\} < 0
\]

\[
\Rightarrow -\{F(z_k + \alpha_k \mathbf{d}_k), \mathbf{d}_k\} < 0
\]

(4)

\[
\Rightarrow \alpha_k \{F(z_k + \alpha_k \mathbf{d}_k), \mathbf{d}_k\} < 0
\]

\[
\Rightarrow \alpha_k \{F(z_k + \alpha_k \mathbf{d}_k), \mathbf{d}_k\} < 0
\]

\[
\Rightarrow \alpha_k \|F(z_k)\| \|d_k\|^2 < 0
\]

Then, we have a contradiction, since, it is not possible to have each of \( \alpha_k, \|F(z_k)\| \) and \( \|d_k\|^2 \) less than zero, so, the line search is well-defined.

**Convergence property:** In this study, to obtain the global convergence of our algorithm then, we need the following lemma.

**Lemma 1:** Solodov and Svaiter (1998) let, \( F \) be monotone and \( x, y \in \mathbb{R}^n \) satisfy \( \{F(y), x-y\} > 0 \). Let:

\[
x' = x - \frac{\{F(y), x-y\}}{\|F(y)\|^2} F(y)
\]
Then for any \( x \in \mathbb{R}^n \) such that \( F(x) = 0 \) it holds that:

\[
\|x^* - x\|^2 \leq \|x - x\|^2 + \|x^* - x\|^2
\]

Now, we can state our convergence result by the following theorem similar to Solodov and Svaiter (1998).

**Theorem 1:** Suppose that \( F \) is LC and monotone and let \( \{x_k\} \) be any sequence generated by algorithm (K). Also, we suppose that the solution set of 1 is nonempty. Then for any \( x \) satisfying, we have:

\[
x_{k+1} - x_k = \frac{\langle F(z_k), x_k - z_k \rangle}{\|F(z_k)\|^2} - \frac{\|x_{k+1} - x_k\|^2}{\|F(z_k)\|^2}
\]

In particular, the sequence \( \{x_k\} \) is bounded. Also, its satisfy that either \( \{x_k\} \) is finite and the last iterate is a solution or the sequence is infinite and:

\[
\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0
\]

Furthermore, the sequence \( \{x_k\} \) converges to some \( \pi \) such that \( F(\pi) = 0 \).

**Proof:** First, if the algorithm finishes at some iteration \( k \) then: either \( d_k = 0 \), so by the positive definiteness of \( B_k \), we get \( F(x_k) = 0 \) or \( \|F(x_k)\| = 0 \) in this case \( x_k \) or \( z_k \) will be a solution of 1. Now suppose that \( d_k \neq 0 \) and \( F(x_k) \neq 0 \) for all \( k \), then:

\[
\langle F(z_k), x_k - z_k \rangle = \langle F(z_k), x_k - x_k - d_k \rangle = \langle F(z_k), -\alpha_k d_k \rangle = -\alpha_k \langle F(x_k + \alpha_k d_k), d_k \rangle \geq \rho \sigma_k \|F(z_k)\|\alpha_k \|d_k\|^2 > 0 \\
\langle F(z_k), z_k - d_k \rangle = \langle F(z_k), -\alpha_k d_k \rangle = -\alpha_k \langle F(z_k), d_k \rangle \geq \rho \sigma_k \|F(z_k)\|\alpha_k \|d_k\|^2 > 0
\]

Then:

\[
\langle F(z_k), + x_k z_k, x_k \rangle = \alpha_k \langle F(z_k), + d_k \rangle \geq \rho \sigma_k \|F(z_k)\|\alpha_k \|d_k\|^2 > 0
\]

Let \( \pi \) be any solution of 1 and \( F(\pi) = 0 \). From lemma 1, (4) and (12), we obtain:

\[
\|x_{k+1} - \pi\|^2 \leq \|x_k - \pi\|^2 - \|x_{k+1} - x_k\|^2
\]

In particular, the sequence \( \|x_k - \pi\| \) is decreasing and hence convergent. Consequently, the sequence \( \{x_k\} \) will be bounded and also we have:

\[
\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0
\]

By Eq. 6, it is clear that \( \{d_k\} \) holds to be bounded and so is \( \{z_k\} \). From Eq. 4:

\[
x_{k+1} - x_k = \frac{-\langle F(z_k), x_k - z_k \rangle}{\|F(z_k)\|^2} - \frac{\|x_{k+1} - x_k\|^2}{\|F(z_k)\|^2}
\]

Since, \( \langle F(z_k), x_k - z_k \rangle = \alpha_k \langle F(z_k), d_k \rangle \) then:

\[
x_{k+1} - x_k = \frac{-\langle F(z_k), x_k - z_k \rangle}{\|F(z_k)\|^2} - \frac{\|x_{k+1} - x_k\|^2}{\|F(z_k)\|^2}
\]

From Eq. 9 and 10, we get:

\[
\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0, \lim_{k \to \infty} \alpha_k \|d_k\| = 0
\]

From Eq. 6, we get \( \lim_{k \to \infty} \|F(x_k)\| = 0 \), if \( \lim_{k \to \infty} \|d_k\| = 0 \) then by Eq. 11, we get:

\[
\lim_{k \to \infty} \alpha_k = 0
\]

Now, since \( \{x_k\} \) is bounded and by continuity of \( F \), it is clear that \( \{x_k\} \) has some accumulation point \( \tilde{x} \) with \( F(\tilde{x}) = 0 \). We also have from Eq. 8 that the sequence \( \{\|x_k - \tilde{x}\|\} \) converges. Therefore \( \{x_k\} \) converges to \( \tilde{x} \). Eq. 3 gives us:

\[
\langle F(x_k), \beta d_k \rangle = \sum_{\mu = \alpha_k}^{\beta} \mu \sigma_k \|F(z_k)\|\alpha_k \|d_k\|^2 \|d_k\|^2 > 0
\]

\[
\|F(x_k + \beta d_k)\|^2 > 0
\]

Since, \( \{x_k\}, \{d_k\} \) are bounded, so, we can choose a subsequence, let \( k_{\rightarrow \infty} \) in Eq. 13, we obtain:

\[
\langle F(\tilde{x}), \hat{d} \rangle \leq 0
\]

Such that \( \hat{x} \) and \( \hat{d} \) are limits of subsequences that chosen. Otherwise by Eq. 6 and already familiar argument:

\[
\langle F(\tilde{x}), \hat{d} \rangle > 0
\]

Equation 14 and 15 are a contradiction. Hence, it is not possible to get that:
This finishes the proof.

RESULTS AND DISCUSSION

Numerical results: In this study, we compare the performance of the new method (K) discussed earlier with the following algorithms.

PRP: It is coming from Cheng (2009).

BFGS: It is coming from using the line search by Zhou and Li (2008) with the direction of this study. We wrote all the codes in MATLAB with version R2014a, also the experiments are running on a computer with 4 GB of RAM and CPU 2.30 GHz. The purpose of running the codes is to compare the results of the new algorithm (K) with the algorithms mentioned above.

When \( \|F_k\| < 10^{-8} \) or \( \|F(z_k)\| < 10^{-8} \) or the total number of iterates exceeds 500000 then all the algorithms will be end. In all of the algorithms, the parameters are specified as follows \( \mu = 0.4, \rho = 0.3, \sigma = 0.25, e = 10^{-8} \).

The comparison of these methods is based on three things: \( N_i \) (Number of iterations), \( N_f \) (Number of functions evaluations) and the CPU time. Also, the special dimensions to compare these algorithms are limited to 5000/50000 for the following initial points:

\[
x_2 = (10, 10, ..., 10)^T, \ x_1 = (-10, -10, ..., -10)^T, \ x_2 = (1, 1, ..., 1)^T, \ x_1 = (-1, -1, ..., -1)^T
\]

\[
x_3 = \left(1, \frac{1}{2}, \frac{2}{3}, ..., 1 \right)^T, \ x_4 = (0.1, 0.1, ..., 0.1)^T, \ x_5 = \left(1-\frac{1}{n}, 1-\frac{2}{n}, ..., 0 \right)^T
\]

Numerical results are displayed in Table 1 and 2 the first table contains both of \( N_i \) and \( N_f \) for all algorithms while the second table contains CPU times of these algorithms.

In order to obtain a comprehensive comparison of the results obtained by our proposed algorithm and the two other algorithms used in the comparison, we use the performance profile provided by Dolan and More (2002) as a tool to evaluate these algorithms and compare them through durability and efficiency (Fig. 1-3).

From the comparisons of the results we can see the superiority of the new approach compared to other methods for solving the nonlinear systems of monotone equations. Figure 1 shows the performance for the total of...
| P/Dim. | New | PRP | BFGS |
|--------|-----|-----|------|
|        | SP  | Ni  | Nf   | Ni  | Nf   | Ni  | Nf   |
| P1     | x₀  | 16  | 145  | 188 | 994  | 1255| 8837 |
|        | x₁  | 16  | 145  | 188 | 994  | 1255| 8837 |
|        | x₂  | 14  | 101  | 40  | 194  | 148 | 798  |
|        | x₃  | 14  | 101  | 40  | 194  | 148 | 798  |
|        | x₄  | 15  | 81   | 25  | 115  | 15  | 79   |
|        | x₅  | 10  | 46   | 20  | 97   | 27  | 160  |
|        | x₆  | 19  | 136  | 45  | 202  | 49  | 254  |
|        | x₇  | 19  | 134  | 48  | 230  | 50  | 267  |
| P2     | x₀  | 16  | 145  | 188 | 994  | 1255| 8837 |
|        | x₁  | 14  | 109  | 196 | 1016 | 1327| 9350 |
|        | x₂  | 14  | 101  | 40  | 194  | 148 | 798  |
|        | x₃  | 15  | 121  | 46  | 235  | 65  | 375  |
|        | x₄  | 15  | 81   | 20  | 97   | 27  | 160  |
|        | x₅  | 10  | 46   | 20  | 97   | 27  | 160  |
|        | x₆  | 18  | 126  | 50  | 221  | 49  | 254  |
|        | x₇  | 30  | 250  | 50  | 256  | 50  | 267  |
| P3     | x₀  | 22534| 135605| 149031| 911135| 409228| 2866839|
|        | x₁  | 13699| 87210 | 36187 | 217508 | 149424 | 1057285|
|        | x₂  | 61248| 385783| 99331 | 521291 | 446334 | 3081533|
|        | x₃  | 29703| 149950| 111908 | 587876 | 241250 | 1466562|
|        | x₄  | 60325| 386954| 94078 | 592519 | 102701 | 586236 |
|        | x₅  | 11159| 38646 | 14865 | 49971 | 28066 | 133867 |
|        | x₆  | 9873  | 57261 | 20338 | 104094 | 51190 | 307639 |
|        | x₇  | 10121| 59001 | 20326 | 104018 | 51133 | 307162 |
| P4     | x₀  | 26  | 263  | 8567 | 106957 | 12677 | 163074 |
|        | x₁  | 26  | 272  | 14175 | 200841 | 19577 | 256721 |
|        | x₂  | 23  | 219  | 421  | 5346  | 5072  | 58466  |
|        | x₃  | 146  | 1471 | 5282 | 60829 | 7269  | 83804  |
|        | x₄  | 21  | 185  | 3599 | 35949 | 4134  | 41272  |
|        | x₅  | 1365 | 14992 | 257  | 1809  | 2328  | 20899  |
|        | x₆  | 536  | 4801 | 6679 | 73228 | 4428  | 49192  |
|        | x₇  | 539  | 4828 | 7036 | 77751 | 4525  | 50314  |
| P₅     | x₀  | 88  | 973  | 62668 | 609170 | 228454 | 2427634|
|        | x₁  | 67  | 675  | 61618 | 597442 | 225811 | 2396330|
|        | x₂  | 88  | 973  | 62578 | 608175 | 228235 | 2428065|
|        | x₃  | 86  | 933  | 62398 | 606160 | 227774 | 2419561|
|        | x₄  | 92  | 1041 | 62491 | 607212 | 228010 | 2422394|
|        | x₅  | 91  | 1024 | 62497 | 607267 | 228023 | 2422517|
|        | x₆  | 91  | 1024 | 62516 | 607479 | 228068 | 2423049|
|        | x₇  | 88  | 973  | 62549 | 607843 | 228167 | 2424284|
| P₆     | x₀  | 16  | 91   | 376  | 1916  | 659  | 4044  |
|        | x₁  | 16  | 115  | 378  | 1944  | 2560 | 17934 |
|        | x₂  | 14  | 87   | 40   | 169   | 181  | 1010  |
|        | x₃  | 16  | 91   | 117  | 510   | 659  | 4044  |
|        | x₄  | 15  | 101  | 117  | 510   | 182  | 1023  |
|        | x₅  | 15  | 101  | 117  | 510   | 182  | 1023  |
|        | x₆  | 14  | 87   | 117  | 510   | 181  | 1010  |
|        | x₇  | 14  | 87   | 117  | 510   | 181  | 1010  |
| P₇     | x₀  | 10  | 41   | 350  | 1755  | 606  | 3643  |
|        | x₁  | 12  | 63   | 401  | 2064  | 684  | 4145  |
|        | x₂  | 31  | 65   | 62   | 126   | 62   | 189   |
|        | x₃  | 22  | 164  | 26   | 142   | 97   | 538   |
|        | x₄  | 49  | 100  | 99   | 200   | 25   | 52    |
|        | x₅  | 2584 | 5170 | 5168 | 10338 | 1292 | 2586  |
|        | x₆  | 55  | 112  | 110  | 222   | 110  | 333   |
|        | x₇  | 55  | 112  | 110  | 222   | 110  | 333   |
Table 2: Numerical results (CPU time)

| P/Dim. | SP  | CPU time | New | PRP | BFGS |
|--------|-----|----------|-----|-----|------|
| P1     | 5000 | x0       | 0.5148 | 4.9296 | 40.3730 |
|        | 5000 | x1       | 0.5148 | 4.9608 | 41.4182 |
|        | 5000 | x2       | 0.2652 | 0.7332 | 2.5584  |
|        | 5000 | x3       | 0.3120 | 0.4368 | 2.6502  |
|        | 5000 | x4       | 0.2652 | 0.3744 | 0.5460  |
|        | 5000 | x5       | 0.4212 | 0.7956 | 0.7176  |
|        | 5000 | x6       | 0.3744 | 0.8424 | 0.8892  |
| P2     | 5000 | x0       | 0.5148 | 4.9608 | 42.7910 |
|        | 5000 | x1       | 0.4056 | 5.1012 | 44.7410 |
|        | 5000 | x2       | 0.2964 | 0.6708 | 2.8584  |
|        | 5000 | x3       | 0.3588 | 0.8736 | 1.2636  |
|        | 5000 | x4       | 0.2652 | 0.8904 | 0.2808  |
|        | 5000 | x5       | 0.1560 | 0.3432 | 0.5304  |
|        | 5000 | x6       | 0.3900 | 0.7956 | 0.7176  |
|        | 5000 | x7       | 0.7488 | 0.9828 | 0.8580  |
| P3     | 10000|x0      | 0.5265 | 3.6298 | 1.1398  |
|        | 10000|x1      | 0.3429 | 0.8904 | 0.4389  |
|        | 10000|x2      | 1.4979 | 2.0869 | 1.3758  |
|        | 10000|x3      | 0.5859 | 2.3832 | 0.5998  |
|        | 10000|x4      | 1.5049 | 2.3747 | 0.2346  |
|        | 10000|x5      | 0.1506 | 0.2027 | 0.0535  |
|        | 10000|x6      | 0.2228 | 0.4189 | 0.1233  |
|        | 10000|x7      | 0.2290 | 0.4198 | 0.1229  |
| P4     | 10000|x0      | 0.1716 | 0.7960 | 1.0721  |
|        | 10000|x1      | 0.1716 | 1.5104 | 1.7052  |
|        | 10000|x2      | 0.1248 | 0.0388 | 0.4040  |
|        | 10000|x3      | 1.0140 | 0.4524 | 0.5508  |
|        | 10000|x4      | 0.1248 | 0.2664 | 0.2676  |
|        | 10000|x5      | 10.3116| 0.0143 | 0.1396  |
|        | 10000|x6      | 3.2292 | 0.5561 | 0.3291  |
|        | 10000|x7      | 3.3384 | 0.5779 | 0.3325  |
| P5     | 5000  | x0       | 0.3900 | 2.6088 | 9.1360  |
|        | 5000  | x1       | 0.2808 | 2.5382 | 8.9972  |
|        | 5000  | x2       | 0.4056 | 2.6076 | 9.0797  |
|        | 5000  | x3       | 0.3744 | 2.5844 | 9.0674  |
|        | 5000  | x4       | 0.3744 | 2.5744 | 9.0594  |
|        | 5000  | x5       | 0.3744 | 2.5434 | 9.1320  |
|        | 5000  | x6       | 0.3744 | 2.5518 | 9.1413  |
|        | 5000  | x7       | 0.3744 | 2.5476 | 9.2811  |
| P6     | 5000  | x0       | 0.1560 | 8.4396 | 16.9261 |
|        | 5000  | x1       | 0.2340 | 9.9060 | 18.7045 |
|        | 5000  | x2       | 0.4836 | 0.4836 | 1.7472  |
|        | 5000  | x3       | 0.2808 | 0.9204 | 0.1248  |
|        | 5000  | x4       | 16.8325| 43.9922 | 8.6424  |
|        | 5000  | x5       | 0.4368 | 0.9984 | 0.9672  |
|        | 5000  | x7       | 0.3744 | 1.0608 | 0.9828  |

N_i for the three algorithms, Fig. 2 shows the performance for the total of N_i and Fig. 3 shows the performance for the CPU time. The algorithm K solved about 95, 91 and 79% of the test functions, respectively and has least of N_i, N_f and CPU time among the three methods and will reach to 1 faster than the other algorithms. It means that the new algorithm K is the best algorithm closing to the performance index.

CONCLUSION

From the numerical results obtained through the comparison technique presented in the tables above of different problems with different initial points and dimensions, it is easy to conclude that the performance of the proposed algorithm K is the most efficient and effective in terms of N_i, N_f and the CPU time compared with the two famous algorithms. This can improve the behavior of the new algorithm to solve the nonlinear monotone equations which does not require Jacobian information of the nonlinear equations. The algorithm K is able to calculate the best solution of problem (1), also its global convergence has been created without using any merit functions.

REFERENCES

Amini, K., M.A. Shiker and M. Kimiaei, 2016. A line search trust-region algorithm with nonmonotone adaptive radius for a system of nonlinear equations. 4OR, 14: 133-152.

Broyden, C.G., J.E. Dennis and J.J. More, 1973. On the local and superlinear convergence of quasi-Newton methods. IMA. J. Appl. Math., 12: 223-245.

Byrd, R.H., J. Nocedal and J. Y.X Yuan, 1987. Global convergence of a class of quasi-newton methods on Convex problems. SIAM J. Numer. Anal., 24: 1171-1190.

Cheng, W., 2009. A PRP type method for systems of monotone equations. Math. Comput. Modell., 50: 15-20.

Dennis, J.E. and J.J. More, 1977. Quasi-Newton methods, motivation and theory. SIAM Rev., 19: 46-89.

Dolan, E.D. and J.J. More, 2002. Benchmarking optimization software with performance profiles. Math. Program., 91: 201-213.

Griewank, A., 1986. The global convergence of Broyden-like methods with suitable line search. ANZIAM. J., 28: 75-92.

Iusem, N.A. and V.M. Solodov, 1997. Newton-type methods with generalized distances for constrained optimization. Optim., 41: 257-278.
Li, D. and M. Fukushima, 1999. A globally and superlinearly convergent gauss-newton-based BFGS method for symmetric nonlinear equations. SIAM. J. Numer. Anal., 37: 152-172.
Li, D.H. and M. Fukushima, 2000. A derivative-free line search and global convergence of Broyden-like method for nonlinear equations. Optim. Methods Software, 13: 181-201.
Nocedal, J., 1980. Updating quasi-Newton matrices with limited storage. Math. Comput., 35: 773-782.
Ortega, J.M. and W.C. Rheinboldt, 1970. Iterative Solution of Nonlinear Equations in Several Variables. Vol. 30, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, USA., ISBN-13:978-0-898714-61-6, Pages: 572.
Shiker, M.A. K. and K. Amini, 2018. A new projection-based algorithm for solving a large-scale nonlinear system of monotone equations. Croatian Oper. Res. Rev., 9: 63-73.
Shiker, M.A.K. and Z. Sahib, 2018. A modified technique for solving unconstrained optimization. J. Eng. Applied Sci., 13: 9667-9671.
Solodov, M.V. and B.F. Svaiter, 1998. A Globally Convergent Inexact Newton Method for Systems of Monotone Equations. In: Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods, Fukushima, M. and L. Qi (Eds.). Springer, Boston, Massachusetts, USA., ISBN: 978-1-4419-4805-2, pp: 355-369.
Zeidler, E., 2013. Nonlinear Functional Analysis and its Applications: III: Variational Methods and Optimization. Springer, Berlin, Germany, ISBN:978-1-4612-9529-7, Pages: 651.
Zhou, W.J. and D.H. Li, 2008. A globally convergent BFGS method for nonlinear monotone equations without any merit functions. Math. Comput., 77: 2231-2240.