Non-linear Gravity on Branes and Effective Action

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Abstract
We develop the general formalism to study the low energy regime of the brane world. We apply our formalism to the single brane model where the AdS/CFT correspondence will take an important role. We also consider the two-brane system and show the system is described by the quasi-scalar tensor gravity. Our result provides a basis for predicting CMB fluctuations in the braneworld models.

1 Introduction
The existence of the initial singularity in the standard cosmology is a notorious problem which is expected to be solved by taking into account the quantum effects of the gravity. It is widely accepted that the superstring theory is the most promising candidate for the quantum theory of gravity. One prominent feature of the superstring theory is the existence of the extra dimensions. To fill the gap between this theoretical prediction and our 4-dimensional universe, we need to hide the extra dimensions somehow. The conventional idea is the Kaluza-Klein compactification scenario where the internal dimensions are assumed to be compactified to the Planck scale. Recently, however, a new picture, the so-called braneworld, has emerged thanks to the developments of the non-perturbative aspects of the superstring theory 1 2. In the braneworld picture, the ordinary matter exists on the brane while the gravity can propagate in the bulk. In particular, we are on the brane! Hence, what we would like to know is how the non-linear gravity appears on the brane. In general, it would be difficult to get such a description due to the strong coupling of the bulk degrees of freedom with those of the brane. However, in the low energy regime \( \rho \ll \ell^2 \sigma \), it is possible to obtain the 4-dimensional effective theory approximately 3 4 5. Indeed, this is sufficient except for the extreme situation for which we need more profound understanding of the string theory. For example, if we take the curvature scale \( \ell \) as \( 10^{-16} \) cm, our approximation is valid at the energy scale below \( 10^{11} \) GeV or the gravitational radius greater than \( 10^{-21} \) km. In fact, this is the interesting regime for most astrophysical and cosmological phenomena. In this paper, we review our recent results on this issue 4 5.

In the next section, we will develop the general formalism. In sec.3, we apply our formalism to the single brane model where the AdS/CFT correspondence will play an important role. In sec.4, we consider the two-brane system and show the system is described by the quasi-scalar tensor gravity. Sec.5 is devoted to the conclusion.

2 General Formalism
If there exists no matter, the ground state is the Minkowski spacetime in the 4-dimensional theory. Correspondingly, if there exists no matter on the brane, we would expect the induced metric on the brane is Minkowski and the bulk geometry is the Anti-deSitter spacetime. Indeed, we have such solution

\[ ds^2 = dy^2 + \Omega^2(y) \eta_{\mu\nu} dx^\mu dx^\nu, \]

where \( \Omega^2 = \exp[-2y/\ell] \) is the warp factor. The brane is located at \( y = 0 \) in this coordinate system. To obtain the above solution we have imposed the relation \( \kappa \sigma = 6/\ell \).

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Let us put the small amount of matter on the brane. Then, the brane will be curved and the bulk geometry will be deformed as
\[ ds^2 = dy^2 + (\Omega^2(y)h_{\mu\nu}(x) + \delta g_{\mu\nu}(y, x^\mu)) \, dx^\mu \, dx^\nu, \]
where the boundary condition \( \delta g_{\mu\nu}(y = 0, x^\mu) = 0 \) is imposed so that \( h_{\mu\nu} \) becomes the induced metric on the brane. How the geometry will be deformed is determined by the 5-dimensional Einstein equations:
\[ G^{(5)}_{AB} = \frac{6}{\ell^2} g_{AB} + \delta(y) 8\pi G_N \ell (-\sigma g_{\mu\nu} + T_{\mu\nu}) \delta^A_B, \quad A = (y, \mu), \]
where \( T_{\mu\nu} \) is the energy-momentum tensor of the matter. As we are considering the deviation from the Anti-deSitter spacetime, it is convenient to define the variables
\[ \delta K^\mu_{\nu} = - \frac{1}{2} \delta \left[ g^{\alpha\gamma} g_{\alpha\gamma y} \right] \equiv \delta \Sigma^\mu_{\nu} + \frac{1}{4} \delta^\mu_{\nu} \delta K, \]
where \( \delta \Sigma^\mu_{\nu} = 0 \). In terms of these variables, the Hamiltonian constraint equation becomes
\[ \delta K = - \frac{\ell}{6} \left[ \frac{3}{4} \delta K^2 - \delta \Sigma^\mu_{\nu} \delta \Sigma^\nu_{\mu} - R \right], \]
and the momentum constraint equation reads
\[ \nabla_\lambda \delta \Sigma^\lambda_{\mu} - \frac{3}{4} \nabla_\mu \delta K = 0. \]
The evolution equation in the direction to \( y \) is
\[ \frac{1}{\Omega^4} \left[ \Omega^4 \delta \Sigma^\mu_{\nu} \right]_{y=0} = \delta K \delta \Sigma^\mu_{\nu} - \left[ \frac{4}{R} \right] \text{traceless}. \]
As we have the singular source at the brane position, we must take into account the junction condition,
\[ \frac{2}{\ell} \left[ \delta \Sigma^\mu_{\nu} - \frac{3}{4} \delta^\mu_{\nu} \delta K \right] \bigg|_{y=0} = 8\pi G_N T^\mu_{\nu}, \]
where we have imposed the \( Z_2 \) symmetry on the spacetime.

After solving the bulk equations of motion, the junction condition gives the effective equations of motion for the induced metric. By integrating the evolution equation, we have
\[ \frac{2}{\ell} \delta \Sigma^\mu_{\nu} - \frac{3}{4} \delta^\mu_{\nu} \delta K = - \frac{\ell^2 \chi^\mu_{\nu}}{\Omega^4} - \frac{2}{\ell \Omega^2} \int^y dy \Omega^4 \left[ \frac{4}{R} \delta^\mu_{\nu} - \frac{1}{4} \delta^\mu_{\nu} \delta K \delta \Sigma^\nu_{\mu} \right] - \frac{1}{4} \delta^\mu_{\nu} \frac{3}{4} \delta K^2 - 3 \delta \Sigma^\mu_{\nu} \delta \Sigma^\nu_{\mu} \]
\[ = - \frac{\ell^2 \chi^\mu_{\nu}(x)}{\Omega^4(y)} + \frac{4}{\Omega^4} \chi^\mu_{\nu} + \delta^\mu_{\nu} t^\mu_{\nu}, \]
where we introduced the constants of integration \( \chi^\mu_{\nu} \). This integration can be performed iteratively with the expansion parameter \( \ell^2 R \). The resulting equations take the form
\[ \delta K^\mu_{\nu} (\Omega^2|_{y=0} h_{\mu\nu}) = 8\pi G_N T^\mu_{\nu} + \frac{\ell^2}{\Omega^4} \chi^\mu_{\nu} + t^\mu_{\nu}, \]
where we have included the trivial factor \( \Omega|_{y=0} = 1 \) for memorizing how the warp factor comes in to the effective theory. Here we have decomposed the corrections to the conventional Einstein theory into the nonlocal part \( \chi^\mu_{\nu} \) and the local part \( t^\mu_{\nu} \).
We can expand these corrections in the order of the $\ell^2 R$:

**Nonlocal**
\[
\chi_{\mu\nu} = \chi^{(1)}_{\mu\nu} \mathcal{O}(\ell^2 R) + \chi^{(2)}_{\mu\nu} + \cdots
\]

**Local**
\[
t_{\mu\nu} = t^{(2)}_{\mu\nu} \mathcal{O}(\ell^4 R^2) + t^{(3)}_{\mu\nu} + \cdots
\]

where the expansion of the local tensor starts from the second order because the first order part is already included as the Einstein tensor. Notice that we have $\chi^{(1)}_{\mu\mu} = 0$ because of $\Sigma^\mu_\mu = 0$.

### 3 Single Brane Model (RS2): AdS/CFT correspondence

Now we shall apply the general formalism to the single brane model [4]. A natural boundary condition for the single brane model is to impose the regularity at the Cauchy horizon, namely asymptotically AdS boundary condition. Taking this boundary condition, we obtain $\chi^{(1)}_{\mu\nu} = 0$. Thus, we have recovered Einstein theory at the leading order.

At the next order $\mathcal{O}(\ell^4 R^2)$, the traceless part of $t^{(2)}_{\mu\nu}$ is proportional to

\[
S^\mu_\nu = R^\alpha_\mu R^\alpha_\nu - \frac{1}{3} R R^\mu_\nu - \frac{1}{4} \delta^\mu_\nu (R^\alpha_\beta R^\beta_\alpha - \frac{1}{3} R^2) - \frac{1}{2} \left( R^\alpha_{\mu|\nu} + R^\alpha_{\nu|\mu} - \frac{2}{3} R^\mu_\nu - \Box R^\mu_\nu + \frac{1}{6} \delta^\mu_\nu \Box R \right)
\]

where $S^\mu_\nu$ is transverse and traceless: $S^\mu_\nu|\mu = 0$, $S^\mu_\mu = 0$. The nonlocal part $\chi^{(2)}_{\mu\nu} = \chi^{(2)}_{\mu\nu} + 1/4 h_{\mu\nu} t^{(2)}_{\mu\nu}$ can not be determined by the Einstein equations, but constrained as

\[
\chi^{(2)}_{\mu\mu} = -\frac{1}{8} \left( R^\alpha_\beta R^\beta_\alpha - \frac{1}{3} R^2 \right).
\]

Here, the AdS/CFT correspondence comes in. As the trace anomaly for some supersymmetric theories proportional to the above result, we make the following identification

\[
\chi^{(2)}_{\mu\nu} = \kappa^2 \ell^2 T^\text{CFT}_{\mu\nu}.
\]

Note that the effective number of the CFT fields $\ell^3/\kappa^2 \sim \ell^2/G \sim 10^{66}$ is so huge. This is the regime where AdS/CFT correspondence holds.

Thus, the effective equations of motion become

\[
G^{(4)}_{\mu\nu} = 8\pi G_N T_{\mu\nu} + 8\pi G_N T^\text{CFT}_{\mu\nu} + \alpha S^\mu_\nu.
\]

Seeing Eq.(11), one can read off the effective action

\[
S^\text{eff} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-h} R + S_{\text{matter}} + S^\text{CFT} + \frac{\alpha \ell^2}{16\pi G_N} \int d^4x \sqrt{-h} \left[ R^\mu_\nu R_{\mu\nu} - \frac{1}{3} R^2 \right].
\]

Now one can consider the cosmology using the above effective theory. One can obtain the renormalized action for the CFT, $S^\text{CFT}$, then we can deduce the one point function from the formula

\[
<T^\text{CFT}_{\mu\nu}> = -\frac{2}{\sqrt{-g}} \frac{\delta S^\text{CFT}}{\delta g^{\mu\nu}}.
\]

The two point correlation function can be also calculated as

\[
<T^\text{CFT}_{\mu\nu}(x) T^\text{CFT}_{\lambda\rho}(y)> = -\frac{2}{\sqrt{-g}} \frac{\delta <T^\text{CFT}_{\mu\nu}(x)>}{\delta g^{\lambda\rho}(y)}.
\]
Thus, we obtain the perturbed effective Einstein Equations

$$\delta G_{\mu \nu} = 8\pi G_N \delta T_{\mu \nu} - \frac{1}{2} \int d^4y \sqrt{-g(y)} < T^{\text{CFT}}_{\mu \nu}(x)T^{\text{CFT}}(y) > \delta g_{\lambda \rho} + \alpha \delta \mu^{(2)}_{\mu \nu} .$$

(15)

This is nothing but the integro-differential equation. It is possible in principle to solve numerically the linearized equations of motion in the cosmological situation. We leave this for the future work.

4 Two Brane Model (RS1): Radion

In this section, we will consider the two-brane system which is more realistic from the M-theory point of view [3, 5] (see also [6]). In the two-brane system, the radion field plays an important role. The radion is defined as the distance between the positive tension brane and the negative tension brane, $d(x)$. Now the warp factor $\Omega^2 = \exp[-2d(x)/\ell]$ becomes dynamical variable.

The general formula gives the equation on the positive tension brane:

$$G^{(4)}_{\mu \nu}(h_{\mu \nu}) = \frac{\kappa^2}{\ell} T^{\oplus}_{\mu \nu} + \ell^2 \chi^{(1)}_{\mu \nu}$$

(16)

where $\chi_{\mu \nu}$ represents the effect of the bulk geometry on the brane. Importantly, this equation holds irrespective of the existence of the other brane. Similarly, the equation of motion on the negative tension brane is given by

$$G^{(4)}_{\mu \nu}(f_{\mu \nu} = \Omega^2 h_{\mu \nu}) = -\frac{\kappa^2}{\ell} T^{\ominus}_{\mu \nu} + \ell^2 \Omega^{(1)}_{\mu \nu}$$

(17)

where $f_{\mu \nu}$ is the induced metric on the negative tension brane. Here, the effect of the bulk geometry enhanced by the factor $1/\Omega^4$ since the bulk geometry shrinks towards to the negative tension brane. Although Eqs. (16) and (17) are non-local individually, with undetermined $\chi_{\mu \nu}$, one can combine both equations to reduce them to local equations for each brane. This happens to be possible since $\chi^{(1)}_{\mu \nu}$ appears only algebraically; one can easily eliminate $\chi^{(1)}_{\mu \nu}$ from Eqs. (16) and (17). Defining a new field $\Psi = 1 - \Omega^2$, we find

$$\frac{\ell^3}{2} \chi^{(1)}_{\mu \nu} = -\frac{\kappa^2}{2\Psi} (T^{\oplus}_{\mu \nu} - (1 - \Psi) T^{\ominus}_{\mu \nu})$$

$$\left[ (\Psi_{\mu \nu} - \delta^\mu_{\alpha} \Psi_{|\alpha} |\gamma) + \frac{3}{2(1 - \Psi)} \left( \Psi |\mu \Psi |\nu - \frac{1}{2} \delta^\mu_{\nu} \Psi |\alpha \Psi |\alpha) \right) \right] .$$

The condition $\chi^{(1)}_{\mu \nu} = 0$ gives the equations of motion for the radion field:

$$\square \Psi = \frac{\kappa^2}{3\ell} \left( (1 - \Psi) \left( T^{\oplus} + (1 - \Psi) T^{\ominus} \right) - \frac{1}{2(1 - \Psi)} \Psi^{\mu \nu} \Psi^{\mu \nu} \right) .$$

(18)

Interestingly, we can rearrange the above equations as

$$G^{\mu \nu}_{\mu \nu}(h) = \frac{\kappa^2}{\ell \Psi} T^{\oplus}_{\mu \nu} + \frac{\kappa^2(1 - \Psi)^2}{\ell \Psi} T^{\oplus}_{\mu \nu} + \frac{1}{\Psi} \left( \Psi^{\mu \nu} - \delta^\mu_{\alpha} \Psi |\alpha \right) + \frac{\omega(\Psi)}{\Psi^2} \left( \Psi^{\mu \nu} \Psi^{\mu \nu} - \frac{1}{2} \delta^\mu_{\nu} \Psi |\alpha \Psi |\alpha \right)$$

(19)

and

$$\square \Psi = \frac{\kappa^2}{\ell \Psi} T^{\ominus} + (1 - \Psi) T^{\ominus} - \frac{1}{2\omega(\Psi) + 3} \frac{d\omega(\Psi)}{d\Psi} \Psi^{\mu \nu} \Psi^{\mu \nu} ,$$

(20)

where the coupling function $\omega(\Psi)$ takes the following form: $\omega(\Psi) = 3\Psi/2(1 - \Psi)$. We named this system as the quasi-scalar-tensor theory. Eqs.(19) and (20) can be derived from

$$S_A = \frac{l}{2\kappa^2} \int d^4x \sqrt{-h} \left[ \Psi R(h) - \frac{3}{2(1 - \Psi)} \Psi^{\mu \nu} \Psi^{\mu \nu} \right] + \int d^4x \sqrt{-h} L^{\oplus} + \int d^4x \sqrt{-h} (1 - \Psi)^2 \mathcal{L}^{\ominus} .$$

(21)
This is the effective action on the positive tension brane.

The effective action on the negative tension brane can be also derived in the similar way as
\[ S_B = \frac{1}{2\kappa^2} \int d^4x \sqrt{-f} \left[ \Phi R(f) + \frac{3}{2(1 + \Phi)} \Phi^{\alpha\beta} \Phi_{\alpha\beta} \right] + \int d^4x \sqrt{-f} L^\bigoplus + \int d^4x \sqrt{-f} (1 + \Phi)^2 L^\bigodot, \quad (22) \]
where \( \Phi = 1/\Omega^2 - 1. \)

Thus, we have derived a closed set of equations (19) and (20). By solving these equations, we can know the anisotropic stress \( \chi^{(1)}_{\mu\nu} \) explicitly. Now, we can make a precise predictions on the CMB fluctuations! This will be reported somewhere else.

5 Conclusion

We have developed the general formalism to obtain the effective action in the low energy regime.

In the case of the single brane model, by imposing asymptotically AdS boundary condition, we have obtained the Einstein theory with corrections represented by CFT and higher curvature polynomial. It is suggested that the cosmological perturbation theory in the brane world can be formulated as the integro-differential equations.

In the case of the two-brane model, we have shown that the system is described by the quasi-scalar-tensor theory. Equivalently, it can be regarded as the Einstein theory with the extra energy source, \( \chi^{(1)}_{\mu\nu} \) corresponding to the dark radiation in the homogeneous cosmological case. This turns out to be determined by the radion field and the energy momentum tensors on positive and negative tension branes. The next order corrections due to Kalza-Klein massive modes can be represented by the higher curvature terms. It is interesting to study the cosmological scenario based on the effective action we have derived [1].

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