QCD at High Energies

and Two-Dimensional Field Theory

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Abstract

Previous studies of high-energy scattering in QCD have shown a remarkable correspondence with two-dimensional field theory. In this paper we formulate a simple effective model in which this two-dimensional nature of the interactions is manifest. Starting from the (3+1)-dimensional Yang-Mills action, we implement the high energy limit $s \gg t$ via a scaling argument and we derive from this a simplified effective theory. This effective theory is still (3+1)-dimensional, but we show that its interactions can to leading order be summarized in terms of a two-dimensional sigma-model defined on the transverse plane. Finally, we verify that our formulation is consistent with known perturbative results.
1. Introduction.

High-energy scattering in quantum chromodynamics (QCD) has been the subject of intensive study since the early seventies \[1, 2\]. In particular within the framework of perturbation theory, systematic procedures have been developed for extracting the large \(s\), fixed \(t\), behaviour of each amplitude and for summing these contributions using a leading-log or eikonal approximation scheme \[1, 2\]. A striking feature of the results obtained by these methods is that the contributions at each order take the form of two-dimensional amplitudes, and it has indeed been suspected for some time that there exists an intimate relationship between QCD at high energies and two-dimensional field theory \[3\].

The two-dimensional nature of the interaction can be understood semiclassically as follows. Introduce two light-cone coordinates and two transverse coordinates

\[
\begin{align*}
    x^\alpha &= (x^+, x^-) \\
    z^i &= (y, z)
\end{align*}
\]

with \(x^\pm = x \pm t\), and let us assume that two fast moving particles have very large momenta in the \(x^\pm\) direction, while they remain at a relatively large distance in the \(z\)-direction. Now if we consider the (color) electric field of one of these fast moving charges, it is clear that due to the Lorentz-contraction it will take the form of a shockwave: the field-strength will vanish everywhere, except on a null-hyperplane through the trajectory of the particle. The only physical effect the shockwave can have is that, when a charged test particle passes through it, its wave function \(\psi\) will undergo some instantaneous gauge rotation \(\psi \rightarrow g(z)\psi\). This gauge rotation only depends on the transverse distance between the particles, and thus it indeed appears that all interactions essentially take place within the transverse plane of the shockwave.

Motivated by this simple physical picture, we will in this paper propose a new formulation of high-energy elastic scattering in gauge theory, in which the two-dimensional nature of the interactions is manifest. The method will have some similarities with the eikonal approximation, but appears to be more general. We will interested in the kinematical regime where \(s\) is much larger than \(t\), while \(t\) is also larger that the QCD scale \(\Lambda_{\text{QCD}}\).

\[\text{For the case the particles have only electro-magnetic interactions, this shockwave interaction has been used in } [4, 5] \text{ to give an elegant derivation of the high energy scattering amplitude, reproducing the result obtained via the eikonal approximation } [6, 7]. \text{ Related work in gravity has been done in } [4, 8, 9].\]
In this case it is a good approximation to represent the scattering process between two hadrons as a collection of separate collisions between the individual quarks, and we will concentrate on these individual quark-quark scattering processes. The main simplifying assumption we will make is that all these individual collisions will take place at a very small time and longitudinal length scale of the order of \(1/\sqrt{s}\). Of course, in actual events there will also be very non-trivial dynamics taking place at larger scales, since a scattered quark will interact via secondary collisions with the other quarks inside the hadron, which as a result will fall apart into one or more jets. However, it seems reasonable to assume that the amplitude for the primary collision between the two energetic quarks is independent of the precise details of these subsequent processes, precisely because these occur at a much larger time and length scale.

In section 2 we will use this assumption to derive a simplified model for QCD at high energies. Via a simple scaling argument, similar to the one used in [9], we will isolate from the Yang-Mills action the part of the theory that appears relevant for the dynamics in this regime. Next, in section 3 we will use the reduced theory to reformulate the calculation of elastic scattering amplitudes in terms of a two-dimensional effective field theory, which has the form of a sigma-model defined on the transverse plane. We make contact with some of the results obtained by more standard methods [1, 2] in a concluding section. In an appendix we describe the form of the non-abelian shock-wave solutions, that provide the semiclassical interpretation of the scattering process.

2. QCD in the high-energy limit.

In this section we will formulate a simple effective description of QCD at high energies. We will start from the hypothesis that the typical longitudinal momentum of the dynamical modes in this process grows proportional to the center of mass energy, whereas the typical size of the transversal momenta is determined by the momentum transfer. The same assumption also underlies other approaches to high energy QCD, where it is in particular used to simplify the evaluation of Feynman diagrams. Here, however, we wish to take this high energy limit directly at the level of the action. This procedure has the important advantage that one does not need to fix the gauge before taking the limit, and thus the final effective theory will still be gauge invariant.
2.1. The scaling argument.

We first consider the pure Yang-Mills model, described by

\[ S = \frac{1}{4} \int d^4x \text{tr}(F_{\mu\nu}F^{\mu\nu}). \]  

(2.1)

Here \( F_{\mu\nu} \) is the non-abelian field strength

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e [A_\mu, A_\nu], \quad A_\mu = A_\mu^a \tau^a \]  

(2.2)

where \( e \) is the coupling constant and \( \tau^a \) are the generators of the lie algebra of the gauge group \( G = SU(N) \). To extract that part of the action that is relevant to the high energy forward scattering of quarks, let us consider the behaviour of (2.1) under a rescaling of the longitudinal coordinates

\[ x^\alpha \rightarrow \lambda x^\alpha, \]  

(2.3)

The idea is that by performing this rescaling inside the Yang-Mills action, we will see which part will become strongly or weakly coupled, when we look at the theory at high longitudinal energies. The components of the gauge potential transform under (2.3) as \( A_i \rightarrow A_i \), while \( A_\alpha \rightarrow \lambda^{-1} A_\alpha \). Hence the rescaled Yang-Mills action can be written in the following form

\[ S'_{YM} = \frac{1}{2} \int \text{tr}(E^{\alpha\beta} F_{\alpha\beta} + F_{\alpha i}F^{\alpha i}) + \frac{\lambda^2}{4} \int \text{tr}(E_{\alpha\beta} E^{\alpha\beta} + F_{ij}F^{ij}), \]  

(2.4)

Here we introduced \( E_{\alpha\beta} = -E_{\alpha\beta} \) as an auxiliary field. So far we have done nothing. Indeed, the description of a scattering process with some \( s \) and \( t \) using the standard action is completely equivalent to that using the rescaled action (2.4), provided we also rescale \( s \) to

\[ s' = \lambda^2 s. \]

The point, however, is that we can now use this correspondence and choose

\[ \lambda \sim \frac{1}{\sqrt{s}} \rightarrow 0, \]

so that \( s' \) is kept fixed, and thus reformulate the high energy limit \( s \rightarrow \infty \) in QCD as the \( \lambda \rightarrow 0 \) limit of the rescaled theory (2.4).
The parameter $\lambda$ is not a coupling constant of the original theory, but introduced via the rescaling, so what does it mean to take the limit $\lambda \to 0$? Here we must go back our basic starting point, namely that, before the rescaling (2.3), the typical longitudinal momentum of the dynamical modes is much larger than the typical transversal momentum. For these modes the second term in (2.4) is indeed subdominant, and so, the $\lambda \to 0$ limit in fact corresponds to a truncation of the full theory to these high energy modes. By our assumption, the contribution of the modes removed by this truncation is subleading for $s \gg t$.

Further it seems reasonable to assume that in leading order in $\lambda$ we can neglect the second part of the action (2.4). However, we need to be a little careful here, since one could imagine that the $\lambda \to 0$ limit of the amplitudes of the full theory are not equal to those of the $\lambda = 0$ theory. The reduced theory could for example be singular, in which case the second term in (2.4) must be kept as a regulator. Keeping this in mind, let us however for the moment assume that the $\lambda = 0$ theory is well-defined as it stands, in which case it must give the leading order contribution.

Thus, we propose that high energy scattering in gauge theories can be described by means of the following truncated Yang-Mills action

$$S[A] = \frac{1}{2} \int \text{tr}(E^{\alpha\beta}F_{\alpha\beta} + F_{\alpha i}F^{\alpha i}).$$  (2.5)

In this theory we wish to calculate the quark-quark scattering amplitude. The coupling to charged quark fields $\psi$ is described by the high energy limit of the usual quark action, which takes the form

$$S[\psi, A] = \int \overline{\psi} \gamma^\alpha (\partial_\alpha + e A_\alpha) \psi.$$  (2.6)

Again, this action is obtained from the standard action by performing the rescaling (2.3) and taking limit $\lambda \to 0$. Hence we see that, as was to be expected of a high energy limit, the quarks propagate only in the longitudinal direction and only couple to the components $A_\alpha$ of the gauge potential. Note further that the mass term is subleading, so we can divide $\psi$ into left- and right moving components. It should again be mentioned, however, that the subleading terms of the action could still be important for regulating possible singularities due to the fact that the leading order model is ultra-local in the transverse $z$-direction.
2.2. Relation with Lipatov’s gluon emission vertex.

The first property of the leading order theory described by (2.5) we notice is that the auxiliary field $E^{\alpha\beta}$ has become a Lagrange multiplier imposing the zero-curvature constraint

$$F^- = 0.$$  \hfill (2.7)

In other words, in the high energy limit, the leading order contribution comes from those gauge field configurations that are flat in the longitudinal direction. This is the central observation of this paper. As the above reasoning shows, it is a simple and direct consequence of the fact that we are interested in processes for which the typical longitudinal scale is much smaller than the typical transversal scale.

At first sight this observation seems to be in contradiction with the effective high-energy gluon emission vertex derived by Lipatov, which produces gluon-radiation with $F^- \neq 0$. To clarify this point we will briefly summarize the derivation of Lipatov’s vertex, first from the conventional Yang-Mills action, and then from the high energy effective theory. Lipatov’s vertex describes the effective gluon production relevant to high energy scattering and is given by [12]

$$C_i(k_1, k_2) = -(k_1 + k_2)_i$$
$$C_+(k_1, k_2) = (\alpha + 2\frac{k_1^2}{\beta s})p_+$$
$$C_-(k_1, k_2) = -(\beta + 2\frac{k_2^2}{\alpha s})p_-$$ \hfill (2.8)

Here $p_+$ and $p_-$ denote the momenta of the left and right-moving quarks, $s = 2p_+p_-$, and $k_1 = k_{1,\perp} + \alpha p_+$ and $k_2 = k_{2,\perp} + \beta p_-$ are the respective momenta of the intermediate gluons (see fig 1.)

To derive this result, let us consider the Yang-Mills equations in the presence of a source $j_\alpha$, that represents the energetic quarks. We will implement the high energy limit by assuming that it is chirally conserved

$$D_+j_- = D_-j_+ = 0$$ \hfill (2.9)

Next we solve the Y-M equations perturbatively by expanding the gauge potential $A_\mu$ and the source $j_\alpha$ in powers of the coupling. For our purpose we will need to go to the
Fig 1. This figure explains the notation used in the expression (2.8) of the effective gluon production vertex. The double horizontal lines represent the fast quarks and the other are gluon lines.

next to leading order. Given the leading order solutions

\[ A_i^{(0)} = 0 \]
\[ \partial_\perp^2 A_\alpha^{(0)} = j_\alpha^{(0)} \]  

(2.10)

the next to leading order equations become (in the Lorentz gauge)

\[ \partial^2 A_i^{(1)} = [A_i^{(0)}, \partial_i A_\alpha^{(0)}] \]  
\[ \partial^2 A_\alpha^{(1)} = j_\alpha^{(1)} + \lambda^{-2}[A_\beta^{(0)}, \partial_\beta A_\alpha^{(0)}] \]  

(2.11)  
(2.12)

with \( \partial^2 = \lambda^{-2}\partial_\parallel^2 + \partial_\perp^2 \) and where in the second equation we used that \( \epsilon^{\alpha\beta} \partial_\alpha A_\beta^{(0)} = 0 \). The next to leading order contribution \( j_\alpha^{(1)} \) to the current in (2.12) can be eliminated in terms of \( A_\alpha^{(0)} \), via the conservation equations

\[ \partial_- j_+^{(1)} = [A_-^{(0)}, j_+^{(0)}] \]
\[ \partial_+ j_-^{(1)} = [A_+^{(0)}, j_-^{(0)}] \]  

(2.13)

Using (2.10) this leads to

\[ j_+^{(1)} = \left[ \frac{1}{\partial_-} A_-^{(0)}, \partial_\perp^2 A_+^{(0)} \right] \]
\[ j_-^{(1)} = \left[ \frac{1}{\partial_+} A_+^{(0)}, \partial_\perp^2 A_-^{(0)} \right] \]  

(2.14)

which, inserted into (2.12), gives

\[ \partial^2 A_+^{(1)} = \left[ \frac{1}{\partial_-} A_-^{(0)}, \partial_\perp^2 A_+^{(0)} \right] + \lambda^{-2}[A_-^{(0)}, \partial_\perp A_+^{(0)}] \]  

7
\[
\partial^2 A^{(1)} = \left[ \frac{1}{\partial_+} A^{(0)}_+, \partial_+^2 A^{(0)}_+ \right] + \lambda^{-2} \left[ A^{(0)}_+, \partial_- A^{(0)}_- \right] \tag{2.15}
\]

Combined with (2.11) this is, when translated to momentum space, identical to the effective high energy gluon emission vertex of Lipatov.

Note that if we compute the longitudinal part of the Yang-Mills curvature \( F_{+-} \), to this order, we find a non-zero result

\[
\partial_+ A^{(1)}_+ - \partial_- A^{(1)}_- - [A^{(0)}_+ , A^{(0)}_-] = 2\lambda^2 \left[ \frac{1}{\partial_+^2} [\partial_+, A^{(0)}_+ , \partial_- A^{(0)}_-] \right]\tag{2.16}
\]

As expected, this non-zero result is proportional to (square of) the scaling parameter \( \lambda \), so that it vanishes in the \( \lambda \to 0 \) limit. However, we are not allowed to drop its contribution, because \( F_{+-} \) enters in the equation of motion with a prefactor of \( \lambda^{-2} \).

Now let us consider the analogous computation in the effective theory with \( \lambda = 0 \). To leading order nothing changes, as the solution (2.10) for \( A^{(0)}_\alpha \) automatically satisfies \( F_{+-} = 0 \). The next to leading terms in the equation of motion now read

\[
\partial_+ A^{(1)}_+ - \partial_- A^{(1)}_- - [A^{(0)}_+ , A^{(0)}_-] = 2\lambda^2 \left[ \frac{1}{\partial_+^2} [\partial_+, A^{(0)}_+ , \partial_- A^{(0)}_-] \right] \tag{2.17}
\]

Here the first equation is the flatness equation \( F_{+-} = 0 \) and the second equation replaces the Yang-Mills equation (2.12). Now it may seem that one has made a mistake in putting \( \lambda = 0 \), because it implies that \( F_{+-} = 0 \) exactly, whereas in the full theory there was a small but non-negligible contribution coming from the gauge-field variations with \( F_{+-} \neq 0 \). However, as is usually the case in these type of situations, the role of the fluctuations that are transverse to the constraint surface is taken over by the lagrange multiplier field, in this case \( E^{(1)}_1 \). Indeed, with the help of (2.13) we can solve for \( E^{(1)}_1 \), and find

\[
\partial_\perp^2 E^{(1)}_1 = 2[\partial_+, A^{(0)}_+, \partial_- A^{(0)}_-] \tag{2.18}
\]

Hence the value of \( E^{(1)}_1 \) precisely equals that of \( \lambda^{-2} F_{+-} \) of the finite \( \lambda \) theory. It is now not hard to convince oneself that this correspondence is sufficient to ensure that the perturbation theory in the \( \lambda = 0 \) model is equivalent to the high-energy limit of the conventional theory proposed by Lipatov in [3], at least to this order. This correspondence further supports the validity of our scaling hypothesis.
2.3. Definition of the scattering amplitude.

Encouraged by this further justification of the high-energy lagrangian (2.5) we will now proceed to study the quark-quark scattering amplitude in this theory. The calculations simplify drastically due to the fact that on-shell the gauge-potential $A_\alpha$ is flat. In particular, it follows that the quark propagator for fixed gauge potential is simply represented by the Wilson line of $A_\alpha$. For example, the two-point function of two right-moving quarks reduces to

$$
\langle T(\bar{\psi}(x_1, z_1)\psi(x_2, z_2))\rangle_A = \delta^{(2)}(z_{12}) \left( \delta(x_{12}^-) \theta(x_{12}^+) + \frac{1}{x_{12}^- + i\epsilon} \right) 
\times P \exp(e \int_1^2 dx^+ A_+) (2.19)
$$

Thus we are led to a description of the scattering amplitude in which the quarks are represented by light-like Wilson lines, a prescription that has been proposed previously by Nachtmann in [10]. In his derivation the Wilson lines arise from the coupling of $A_\alpha$ to the quark current by neglecting the recoil terms in the quark propagators. Our argument that $A_\alpha$ is predominantly flat gives a new justification for using this procedure in the high energy limit.

A priori, a single quark-quark scattering-amplitude is by itself not a gauge invariant quantity, since it depends on gauge-rotations at the end-points of the Wilson-lines. One way to get an invariant amplitude would be to consider the scattering of hadron-like objects such as a quarks connected by transversal Wilson-lines and integrated against
some hadron wave-function. However, when \( t \) is much larger than \( \Lambda_{qcd} \) it appears to be meaningful to consider the quark-quark scattering processes individually. In this case the role of the accompanying quarks and gluons inside the hadron is to provide a “frame of reference” for the scattered quark with respect to which one can define the amplitude in an invariant way. To make this concrete, we will follow the procedure advocated in \[10\], and use this reference frame to ‘identify’ the two end-points of the Wilson-line in the sense that gauge-rotations are restricted to act simultaneously and identically at \(+\infty\) and \(-\infty\). In this way by taking the trace of the amplitude in both representations one obtains a meaningful gauge-invariant quantity, which according to \[10\] corresponds to the diffractive term of the amplitude.

Thus, summarizing, we will define the high energy quark-quark scattering amplitude by the expectation value of two light-like Wilson lines

\[
f(s, t) = \frac{is}{2m^2} \int d^2z \, e^{-iq \cdot z} \langle V_+(0) V_-(z) \rangle \quad t = -q_i^2 \quad (2.20)
\]

\[
V_\pm(z) = \text{tr} [\text{P} \exp(e \int_{-\infty}^{\infty} dx^\pm A_\pm(z))] \quad (2.21)
\]

in the theory described by the high energy limit \( (2.5) \) of the QCD lagrangian. The configuration of the two Wilson line operators is depicted in fig 2. The kinematical factor \( \frac{is}{2m^2} \) comes from the conventional normalization of the quark wave-function.

3. Reduction to a Two Dimensional Field Theory.

In this section we will indicate how this model can be further simplified and eventually reduced to a two-dimensional effective field theory. To this end, we will first integrate out the lagrange multiplier field \( E^{\alpha \beta} \) in \( (2.9) \). This produces a functional constraint that restricts the integral longitudinal gauge-field components \( A_\alpha \) to potentials of the form as

\[
A_\alpha = \frac{1}{e} \partial_\alpha U U^{-1} \quad (3.1)
\]

*In fact, it turns out to be a singular limit to take the Wilson lines in \( (2.20) \) exactly light-like. We will later regularize this ‘infrared’ divergence by allowing each line to have a small light-like component.
Here the group element $U$ may still depend on all four coordinates. It can be seen that the Jacobian of this replacement of the variables $E^{\alpha \beta}$ and $A_{\alpha}$ by $U$ is simply equal to 1. When we substitute (3.1) into the action $S$, we get

$$S[U, A_i] = \frac{1}{2e^2} \int \text{tr}(\partial_\alpha (U^{-1} D_i U))^2$$

$$D_i = \partial_i + eA_i$$

(3.2)

So far we have not chosen any gauge, and the above action is indeed manifestly gauge invariant. We can now make use of this fact to choose a gauge in which the longitudinal components $A_{\alpha}$ are completely eliminated via a redefinition of $A_i$ to $\tilde{A}_i = \frac{i}{e} U^{-1} D_i U$. This gauge has one obvious advantage, namely that all local interactions in fact disappear! A subtlety, however, is that, if we wish to be able to send in charged particles, we should not allow for arbitrary gauge transformations at infinity. Some information about the asymptotic values of $U$ will have to survive, since these are the only remaining variables that couple to the quarks.

Notice that if we had chosen some other more standard gauge, such as the Landau gauge $\partial_\mu A^\mu = 0$, it would have been rather difficult to recognize that there are no local interactions. More close inspection, however, would reveal that in such a gauge the theory in fact describes a free field, $A_i$, interacting with a topological field theory, consisting of the group variable $U$ and the ghost fields. Indeed, it appears that many of the Feynman diagram computations done in [1] and [2] can be reinterpreted in this way. We believe that this absence of local four-dimensional interactions is what underlies the apparent two-dimensional nature of high energy QCD.

3.1. The Two-dimensional Model.

We now wish to make this idea more concrete. Let us denote the values of the field $U$ at the end-points of the Wilson lines in (2.20) by $g_1$ and $g_2$ for the right in- and left out-region, and $h_1$ and $h_2$ for the left in- and right out-region, resp. (See fig 3.) Since $A_\alpha$ is flat, the Wilson lines can be expressed in terms of these variables, for example

$$\text{P exp}(e \int_{-\infty}^{\infty} dx^+ A_+(z)) = g_2 g_1^{-1}(z)$$

(3.3)

The group variables $g_A$ and $h_A$ ($A=1,2$) will not be held fixed, but will be treated as dynamical fields. Their physical role is to impose the Gauss-law constraints at the end-points.
through their field equations. In addition, as discussed in the Appendix, these asymptotic modes of $U$ appear naturally in the classical description of the shockwave configurations associated with the fast-moving quarks. Our aim is to get a two-dimensional model that describes the dynamics of these shockwaves. We further impose that the transversal gauge-field $A_i$ takes the same value at the two end-point of each Wilson line, which we will denote be $a_i^\pm(z)$ resp.

It is now clear that the expectation value (2.20) of the Wilson lines, representing the elastic quark-quark scattering amplitude, can in principle be re-expressed as a correlation function in the two-dimensional effective theory of the $x$-independent variables $(g_A, h_B, a_i^\pm, b_i)$

$$\langle V_-(z_L) V_+(z_R) \rangle = \int [dg_A dh_B da_i^\pm] e^{iS[g_A, h_B, a_i^\pm]} \left( g_2 g_1^{-1}(z_L) \right) \left( h_2 h_1^{-1}(z_R) \right),$$  (3.4)

where the two-dimensional effective action $S[g_A, h_B, a_i^\pm]$ is obtained by integrating out all modes of $A_i$, while keeping its asymptotic boundary values fixed. This calculation can be performed explicitly, since the integral over $A_i$ is just gaussian due to the absence of local interactions. Therefore, in the leading semiclassical approximation the only thing we need to know is the action of the classical field configuration with the given boundary conditions. To calculate it we use that for a solution $A_i^{cl}$ of the classical field equations of (3.2), one has.

$$S[g_A, h_B, a_i^\pm] = \frac{1}{2e^2} \int d^2 z \left[ \int_{-\infty}^{\infty} \partial_+(U^{-1} D_i U) \times \int_{-\infty}^{\infty} \partial_-(U^{-1} D_i U) \right]$$  (3.5)

We can now express the right-hand-side directly in terms of the asymptotic values for $U$ and $A_i$ as follows

$$\int_{-\infty}^{\infty} \partial_-(U^{-1} D_i U) = g_2^{-1} D_i^+ g_2 - g_1^{-1} D_i^+ g_1$$

$$\int_{-\infty}^{\infty} \partial_+(U^{-1} D_i U) = h_2^{-1} D_i^- h_2 - h_1^{-1} D_i^- h_1$$  (3.6)

where the covariant derivatives $D_i^\pm$ on the right-hand-side are with respect to $a_i^\pm(z)$. Inserting (3.6) into (3.5) we find that the effective action of the two-dimensional variables $(g_A, h_B, a_i^\pm)$ is given by

$$S[g_A, h_B, a_i^\pm] = \frac{1}{2e^2} \int d^2 z \left[ M^{AB} \left( g_A D_i^+ g_A h_B^{-1} D_i^- h_B \right) \right]$$  (3.7)
where $M^{AB}$ is the $2 \times 2$ matrix

$$M^{AB} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (3.8)$$

and the indices $A \ (B) = 1, 2$ are summed over.

The result (3.7) can be derived in an alternative fashion by doing the functional integral over gluon field $A_i$ with the insertion of delta functionals that keep the value of $A_i$ fixed at the end-points of the Wilson lines. After representing these delta-functionals as a Fourier integral it becomes straightforward to do the gaussian integration. We leave it to the reader to verify that this gives the same result. This second method makes clear that the matrix $M^{AB}$ can be identified with the inverse propagator of the gluon field between the end-points of the Wilson lines. We will make use of this remark in the next subsection.

Thus, the matrix $M^{AB}$ represents a discrete version of the wave-operator $\partial_+ \partial_-$, and thus, in a sense, the two-dimensional model (3.7) has a 3+1-dimensional interpretation in which the longitudinal plane is replaced by the four end-points of the Wilson lines. This interpretation also explains why in the expression (3.4) there is still a factor of $i$ in front of $S$, even though the transverse $z$-plane is euclidean. Indeed, without this factor of $i$ the indefinite signature of the action would have become a problem.

Semi-classically our two-dimensional model can be interpreted as describing high-energy scattering of quarks via shock-waves. This interpretation becomes clear when we consider the classical equations of motion of (3.7)-(3.8). In appendix A we describe the shock wave solutions of the Yang-Mills equations for a single fast-moving quark. In terms of the above model this corresponds to the case where we couple one single classical point charge to $h_2 h_1^{-1}$. It can be easily verified that in this situation the classical equations of motion of the action (3.7) indeed precisely reduce to (A.5)-(A.7), provided we use the Ansatz that $h_2 = h_1 \ , \ a_i^\pm = 0$ and $g_1 = 1$. In other words, the variables $(g_2 g_1^{-1}) (z_L)$ and $(h_2 h_1^{-1}) (z_R)$, which in our model represent the Wilson lines of the quarks, are via their own equation of motion equal to the instantaneous gauge-rotation generated by the shock-wave of the other quark.

Loop corrections are taken into account by adding to the leading result the one-loop effective action induced by the quark and gluon loops. These are given by given by the determinants

$$\log \det (D^2_\alpha + \lambda^2 \partial_\tau^2) \quad (3.9)$$
in leading order for $\lambda^2 \to 0$. These determinants give rise to the usual 1-loop renormalization of the coupling constant, as well as to corrections to the leading order action (3.7). It may be possible to use techniques of two-dimensional conformal field theory to compute these corrections, in a similar way as was done recently in [11] in the context of QCD at high temperatures.

3.2. An infrared problem: How $\log s$ enters.

The above description of the amplitude appears to be essentially independent of the center of mass energy of the two quarks. This is clearly in contradiction with the well-known fact that amplitudes in QCD have a very non-trivial $s$-dependence. The origin of this dependence lies in the infrared divergences of 3+1-dimensional gauge theory. In most standard perturbative treatments of high-energy QCD, these infrared divergences are taken care of by restricting the rapidities of the intermediate gluons to lie in between those of the two fast quarks (see eg. [2]). The size of this rapidity space grows as $\log s$ and in this way amplitudes acquire an overall factor proportional to some power of $\log s$, depending on the number of intermediate gluon propagators. This infrared problem has a direct counterpart in our two-dimensional theory (3.7)-(3.8), namely, the matrix $M^{AB}$ given in (3.8) is not invertible, and as a consequence some of the fields have a singular propagator. Hence to make the model non-singular we will need to introduce an infrared cut-off, in a similar way as in the 3+1-dimensional theory. We can do this as follows.

The inverse $M^{-1}_{AB}$ of the matrix $M^{AB}$ can in fact be identified with the $A_i$ propagator between the different asymptotic regions of the longitudinal plane, i.e. between the different end-points of the Wilson-line operators $V_\pm$. The infrared singularity can be traced back to the fact that the trajectories of these Wilson-lines were taken to be light-like and therefore have an infinite distance in rapidity space. However, for finite quark mass $m$ the classical trajectories of two quarks with centre of mass energy $s$ are related by a finite Lorentz boost with rapidity parameter $\log s/m^2$. Thus, to regularize this infrared problem, we now give the Wilson-lines a small timelike component, such that they coincide with the classical quark trajectories, and, in addition, we let them end after some finite proper time $T$. To be specific, we will choose the end-points of $V_+$ at $(x_+, x_-) = \pm \frac{1}{2} T(p_+, m^2/p_+)$ and of $V_-$ at $(x_+, x_-) = \pm \frac{1}{2} T(m^2/p_-, p_-)$, with $2p_+p_- = s$. Note that in the centre of mass frame $p_+ = p_- = \sqrt{s/2}$ so that in the limit $s \to \infty$, the trajectories of $V_\pm$ indeed
become infinitely long and light-like.

The next step is to evaluate the gluon propagator between these end-points. From the fact that the propagator $\langle T(A_i(x, z)A_j(0, 0)) \rangle$ is proportional to $\log(x^+x^- + i\epsilon)$, we find that, for large but finite $s$, the longitudinal propagator between the different end-points of $V_+$ and $V_-$ becomes essentially independent of the quark mass $m$ and the proper time cut-off $T$. It takes the following form

$$
M^{-1}_{AB} = \begin{pmatrix}
\log(se^{i\pi}) & \log s \\
\log s & \log(se^{i\pi})
\end{pmatrix}
$$

(3.10)

This propagator is schematically depicted in fig 3. In this way we find the following ‘regularized’ form of $M^{AB}$

$$
M^{AB}_{reg} = \begin{pmatrix}
1 + \epsilon & -1 + \epsilon \\
-1 + \epsilon & 1 + \epsilon
\end{pmatrix}
$$

(3.11)

$$
\epsilon^{-1} = 1 - \frac{2i}{\pi} \log s
$$

(3.12)

We want to stress that this derivation of the log $s$-dependence is the direct translation to our situation of the corresponding calculations done in perturbation theory. Equation
(3.11), together with eqns (2.20), (3.4) and (3.7), completes the description of our two-dimensional model for high-energy QCD.

4. Comparison with standard perturbation theory

It is an important and non-trivial test on our model to find out if it is consistent with the known perturbative results [1, 2]. In Appendix B we have summarized the known expression for the quark-quark scattering amplitude to first order in log $s$. The group factors that receive a contribution to this order are depicted in fig 4. We will now indicate how these results can be derived from our effective two-dimensional action. The results (B.2) have been obtained in the Lorentz gauge, so we will choose a corresponding gauge $\partial_i a_i^\pm = 0$. It is further useful to choose the following parametrization of the fields

$$
\begin{align*}
  g_2 g_1^{-1} &= \exp(e\theta) & g_1 h_1^{-1} &= \exp(\bar{e}\chi) \\
  h_2 h_1^{-1} &= \exp(e\phi) & a_i^\pm &= \bar{e}^2 \epsilon_i^j \partial_j \alpha^\pm
\end{align*}
$$

where

$$
\bar{e}^2 = \frac{2e^2}{\pi} \log s
$$

The advantage of this parametrization is that vertex operators that enter in the expression for the quark-quark amplitude are simple exponentials

$$
 f(s, t) = \frac{is}{2m^2} \int d^2 z e^{-iq\cdot z} \left< e^{ie\theta \alpha^a L(0)} e^{ie\phi \alpha^b R(z)} \right>
$$

When we insert the above parametrization into the action $S$, expand into the first two orders in $\bar{e}$, we get

$$
 S = \frac{1}{2} \int d^2 z \text{tr} (\partial_i \theta \partial_i \phi + i(\partial_i \chi)^2 + i\partial_i \alpha^+ \partial_i \alpha^-)
 + \frac{\bar{e}^2}{2} \int d^2 z \text{tr} (\chi [\partial_i \theta, \partial_i \phi] + \frac{1}{2}(\alpha^+ + \alpha^-) [\partial_i \theta, \partial_j \phi] \epsilon^{ij})
 + \frac{\bar{e}^2}{4} \int d^2 z \text{tr} ([\chi, \partial_i \theta] [\chi, \partial_i \phi] + \frac{1}{2} [\partial_i \alpha^+, \theta] [\partial_i \alpha^-, \phi])
$$
Fig 4. These diagrams are the leading order diagrams that contribute to the group factors $G_i$ and $F_2$ as given in (B.3). The double horizontal lines represent the two quarks, and the thick vertical lines are $\theta$-$\phi$ propagators.

Here we dropped terms proportional to $e$ since these are subdominant for large $s$. From this form of the action it is clear that the scattering amplitude indeed has a $\log s$ expansion of the form (B.1). The first terms in this perturbation expansion are obtained by evaluating the diagrams of fig 4, and the one-loop corrections to the first two graphs.

As an example we outline the calculation of the $H$-diagram of fig. 4, because this diagram illustrates in a rather direct way the relation between our approach and the standard theory of [12, 2].

The standard result for this diagram is a sum of many contributions that can be summarized in terms of Lipatov’s effective gluon emission vertex (see eqn (2.8) as [12]

$$A(s, q) = N \frac{e^2}{(2\pi)^2} s \log s \int \frac{d^2 k_1 d^2 k_2 K(k_1, k_2)}{k_1^2 k_2^2 (q-k_1)^2 (q-k_2)^2}$$

(4.5)

with

$$K(k_1, k_2) = C_\mu(k_1, k_2) C_\mu(q-k_1, q-k_2)$$

$$= -q^2 + \frac{k_1^2 (q-k_2)^2}{(k_1-k_2)^2} + \frac{k_2^2 (q-k_1)^2}{(k_1-k_2)^2}$$

(4.6)

In (the above parametrization of) our two-dimensional effective field theory, we only need to add two Feynman diagrams to obtain the full answer for this amplitude, namely the $H$-diagram with an intermediate $\chi$ and $\alpha = \frac{1}{2}(\alpha^+ + \alpha^-)$ line, resp. From the form (4.8) of the action it is readily seen that the sum of these diagrams gives

$$A(s, q) = N \frac{e^2}{(2\pi)^2} s \log s \int \frac{d^2 k_1 d^2 k_2 L(k_1, k_2)}{k_1^2 k_2^2 (q-k_1)^2 (q-k_2)^2 (k_1-k_2)^2}$$

(4.7)

with

$$L(k_1, k_2) = 2[(k_1 \cdot k_2) (q-k_1) \cdot (q-k_2) + (k_1 \cdot k_2) (q-k_1) \cdot (q-k_2)]$$

(4.8)
with \( \tilde{k}_{1,i} = \epsilon_{ij}k_{1,j} \) etc. This expression is indeed identical to the standard result (4.5), because

\[
L(k_1, k_2) = K(k_1, k_2)(k_1 - k_2)^2 \tag{4.9}
\]

This identity is most easily established by checking that both sides coincide for all points where two momenta are equal or one of them vanishes.

This correspondence with the standard theory gives strong evidence that our approximations and reduction method is correct and in principle can be extended to higher orders. Because we have only done computations up to first non-trivial order in \( \log s \), we have in particular not yet seen any indication of the remarkable exponentiation leading to the well-known reggeization of the gluon propagator. It would be interesting to see if this result can be derived from first principles, possibly by extending the methods described here.

5. Concluding Remarks

We have given a simple physical description of the high energy interactions between two quarks, and shown that it can be described in terms of a two-dimensional sigma-model action, given by (3.7). We have obtained this action by evaluating all correlations in the longitudinal direction and keeping only the leading order terms in \( \lambda = t/s \). To obtain 1-loop corrections to this leading order result one would need to evaluate the functional determinants (3.9). This will give rise to the usual renormalization of the coupling \( e \). The physical regime where one can expect this description to be relevant is at short length and time scales, that is \( s >> t > \Lambda_{qcd} \). At these scales confinement has not yet set in, and the color electric field behaves more like an ordinary electro-magnetic field. We should point out, however, that our model fully incorporates the non-abelian character of the gauge fields.

While most of the complicated dynamics of QCD is eliminated in the limit \( \lambda \to 0 \), our model may still contain useful information about the strong coupling regime. In this respect, it would be interesting to further analyze the transversal dynamics described by (3.7). A lot is known about the more conventional two-dimensional sigma-models, and most of this technology can be taken over to our model, even though it has some unusual
features. The theory is perturbatively renormalizable and presumably asymptotically free at short transverse distances. Hence for relatively large values of $t$ perturbation theory is expected to be reliable.

This simple model will of course not be able to describe the full complexity of QCD, and has to be embedded into a more elaborate and sophisticated framework before it can be used as a realistic theory of high energy scattering of hadrons. One could in particular imagine extending our model to include the scattering of gluons and, instead of taking the trace of the Wilson lines, folding the amplitude with a (phenomenological) hadron wave-function that describes the distribution of the quarks and gluons in the transverse space. As a further comment we like to mention that the rescaled 3+1 dimensional action (2.5) in principle can be used to compute gluon production in high-energy collisions. Instead of integrating out the transverse gauge field $A_i$ to get to the two-dimensional model, we can keep the gluons in our description by representing the amplitude in the gluon Fock space. This presumably leads to a 2+1 dimensional model analogous to the extended eikonal model (see e.g. [2]), where in addition to the transverse coordinates $z$ one keeps the rapidity of the produced gluons as an extra coordinate.

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Appendix A: Non-abelian shock-waves.

In this appendix we give a short discussion of the classical shock wave field due to a fast-moving non-abelian charge. One way to obtain such solutions is to start with the classical field for a colored particle at rest, and then apply an infinite Lorentz boost. The precise form of the shock-wave will then of course depend on the static configuration one starts with. Whereas in the abelian theory there is only one candidate, the spherically symmetric Coulomb field, it is known that the non-abelian Yang-Mills equations also have other static solutions with less symmetries and a smaller energy. The same ambiguity shows up here.
Consider the Yang-Mills equations in the presence of a classical source

\[ D^\mu F_{\mu\nu} = j_\nu. \] (A.1)

We are interested in the case where the source is given by a classical point charge that moves at the speed of light. Let the velocity of the particle be in the \( x^- \)-direction and its transversal coordinate be \( z = 0 \), then the only non-vanishing component of the source \( j_\nu \) is

\[ j_+ = \lambda \delta(x^+) \delta^{(2)}(z) \] (A.2)

Here \( \lambda \) is a constant element of the lie algebra of the gauge group, describing the classical non-abelian charge of the particle.

We can restrict the form of the solutions of the Yang-Mills equations (A.1) by using some of the symmetries of the source \( j_\nu \). In particular, we can require that the solutions we are looking for must be invariant under translations in the \( x^- \)-direction and boosts in the \((x^-, x^+)\) plane that leave the hyperplane \( x^+ = 0 \) fixed. These two invariances are already very restrictive: they imply that all fields are independent of \( x^+ \) and \( x^- \), except for a possible discontinuity at \( x^+ = 0 \).

We will restrict ourselves to the simplest type of solutions, which are obtained by assuming that all components of the Yang-Mills curvature vanish except \( F_{++} \). These are then determined by

\[ D^i F_{++} = \lambda \delta(x^+) \delta^{(2)}(z) \] (A.3)
\[ D^+ F_{++} = 0 \]

To write this shock-wave we now choose the gauge \( A_- = 0 \), and use the residual gauge-invariance under \( x^- \)-independent gauge-transformations to put also \( A_+ = 0 \) everywhere. Further, since we assume that \( F_{ij} = 0 \), the transversal component \( A_i \) must be pure gauge as well. Thus the shock-wave can be represented as

\[ A_i = \begin{cases} 0 & \text{for } x^+ < 0 \\ g^{-1} \partial_i g & \text{for } x^+ > 0 \end{cases} \] (A.4)

where group element \( g \) depends only on the transversal coordinate \( z \). Substituting this into (A.3) gives

\[ \hat{D}_i (g^{-1} \partial_i g) = \lambda \delta^{(2)}(z). \] (A.5)
Here, the covariant derivative involves the value of the gauge-field $A_i$ at $x^+ = 0$, i.e. at the position of the shock-wave

$$\hat{D}_i = \partial_i + e[A_i, \ ]_{|x^+=0}$$

(A.6)

So at this point the internal structure of the shock-wave becomes relevant. In principle, the $A_i$ gauge field at $x^+ = 0$ can be different from the field outside the shock wave. So we can take

$$A_i = h^{-1} \partial_i h \quad \text{at} \quad x^+ = 0$$

(A.7)

and for any $A_i$ of this form we can solve (A.3) and obtain a classical solution of (A.1) for the source (A.2).

When the gauge-field $A_i$ at $x^+ = 0$ commutes with $g^{-1} \partial_i g$ in (A.5) the shock-wave takes the ‘abelian’ form

$$g_{\text{abelian}}(z) = \exp\left(\frac{e^2}{4\pi} \lambda \log |z|^2\right)$$

(A.8)

This solution is uniquely singled out if we require that the classical field respects all symmetries of the source.

The physical interpretation of the above field configuration becomes more apparent after we perform a discontinuous gauge rotation and put $A_i = 0$ everywhere. After this $A_+$ will acquire a delta-function singularity at the null-hyperplane $x^+ = 0$, such that the Wilson lines that cross this plane are given by

$$P \exp(e \int_{-\epsilon}^{\epsilon} dx^+ A_+) = g(z)$$

(A.9)

where $g(z)$ solves (A.3). Physically this means that the only physical effect of the shock wave is that when a charged test particle passes through $x^+ = 0$ its wave function $\psi$ will be instantaneously gauge-transformed to $\psi' = g(z)\psi$. 

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Appendix B: Perturbative Results to Order $e^6$.

In this appendix we give the known perturbative result of the high energy quark-quark scattering amplitude, computed in [1, 2]. It is known that the quark-quark amplitude $f(s, t)$ can be expanded in a power series in $\log s$

$$f(s, t) = s \sum_{n=0}^{\infty} A_n(q)(\log s)^n$$  \hspace{1cm} (B.1)

with $t = -q^2$. Using standard perturbative techniques the following results have been obtained for the first two terms $A_0(q)$ and $A_1(q)$, upto order $e^6$ (we use the notation of chapter 12 of [2])

$$A_0(q) = -\frac{e^2}{q^2} G_1 + \frac{ie^4}{2!} I_2(q) G_2 + \frac{e^6}{3!} I_3(q) G_3$$  \hspace{1cm} (B.2)

$$A_1(q) = \frac{Ne^4}{4\pi} I_2(q) G_1 - i\frac{Ne^6}{4\pi} I_3(q) G_2 + \frac{e^6}{4\pi} [2I_3(q) - q^2 I_2(q)] F_2$$

where $G_k$ and $F_2$ are the group factors corresponding to the graphs in fig 4.

$$G_k = (\tau^a_L \otimes \tau^a_R)^k$$  \hspace{1cm} (B.3)

$$F_2 = (f_{abc} \tau^b_L \otimes \tau^c_R)^2$$

where $\tau^a_L$ and $\tau^a_R$ are the non-abelian charges of the left resp. right-moving quark, and $I_2(q)$ and $I_3(q)$ are the (infrared divergent) integrals

$$I_2(q) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2(q-k)^2}$$  \hspace{1cm} (B.4)

$$I_3(q) = \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{1}{k_1^2 k_2^2 (q-k_1-k_2)^2}$$

These result were obtained in the Lorentz gauge.

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