Gravitational energy in stationary spacetimes

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Abstract
Static observers remain on Killing-vector worldlines and measure the rest-mass + kinetic energies of particles moving past them, and the flux of that mechanical energy through space and time. The total mechanical energy is the total flux through a spacelike cut at one time. The difference between the total mass energy and the total mechanical energy is the total gravitational energy, which we prove to be negative for certain classes of systems. For spherical systems, Misner, Thorne and Wheeler define the total gravitational energy in this way. To obtain the gravitational energy density analogous to that of electromagnetism we first use Einstein’s equations with integrations by parts to remove second-order derivatives. Next we apply a conformal transformation to reexpress the scalar 3-curvature of the 3-space. The resulting density is non-local. We repeat the argument for mechanical energies as measured by stationary observers moving orthogonally to constant time slices like the ‘zero angular momentum’ observers of Bardeen who exist even within ergospheres.

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1. Introduction

In classical physics, energy has different forms which are additive. Energy is conserved and one can assess how much is transferred from one form to another. In Einstein’s theory of gravitation different forms of energy in the matter are mixed in inseparable ways with gravitational binding energy. Gravitational energy is diffuse, partly mixed up with various forms of matter energy and partly stored in the gravitational field itself. Alas, the matter is the source of gravity so that it is impossible to disentangle gravitational energy from other forms of energy with or without solving the field equations. This is a great loss that adds to the loss of ‘gravitational force’ and ‘gravitational energy density’ with which it shares a common origin: the principle of equivalence.
Misner, Thorne and Wheeler\textsuperscript{4} [32] gave an expression for the total gravitational binding energy of a spherical system. They also suggest a possible definition for the gravitational energy density in this particular case. Katz [25] realized that their definition of gravitational energy could be extended to all stationary systems and gave explicit expressions for gravitational energy density of stationary systems whose spaces were conformally flat.

Here we give an expression for the gravitational energy density of any stationary space that is asymptotically flat. Our expression is non-local in that it involves the conformal transformation that removes the scalar 3-curvature of the stationary 3-space and it is invariant under transformations of spatial coordinates. However, in specially chosen coordinates (such as isotropic ones when they exist) it may be expressed locally in terms of derivatives of the metric components.

MTW's starting point is the difference between the Schwarzschild mass found asymptotically at large $r$ and the total mechanical energy which is the sum of the rest mass + internal + kinetic energies of the matter. This difference is the gravitational energy. Kinetic energy has to be measured relative to some standard of rest and even when that standard is agreed it will not be conserved; as fluid climbs out of gravitational wells kinetic energy decreases, etc. However, as fluid moves from one place to another there will be a flux of mechanical energy and in relativity we consider flux vectors describing how the mechanical energy moves from one region of spacetime to another. In any stationary spacetime we may use the timelike Killing vector\textsuperscript{5} $\xi^\mu$ to define ‘static’ observers whose velocities $w^\mu = \xi^\mu/\xi$ ($\xi$ is the magnitude of $\xi^\mu$). Such observers assess the mechanical energy of a mass $m_0$ moving with 4-velocity $u^\mu$ to be $m_0 c^2 / \sqrt{1 - v^2/c^2} = m_0 c^2 u_\mu w^\mu$, $v$ is its velocity with respect to the local observer. Likewise a fluid of dust of proper rest-mass density $\rho$ (in its own rest frame) will have a rest-mass energy density in any frame moving with 4-velocity $n^\mu$ of $\rho c^2 u_\mu n^\mu = \rho c^2 / \sqrt{1 - v^2/c^2}$ because the dust particles seem closer together due to Lorentz contraction of their proper volume. Such factors occur in the following expression for rest mass. For a fluid of dust the rest-mass flux vector $\rho u^\mu$ is conserved. The total rest-mass energy is given by

$$M_0 c^2 = \int_\Sigma \rho c^2 u^\mu \sqrt{-g} \, d\Sigma_\mu,$$

where $d\Sigma_\mu = 1/3! \epsilon_{\mu\nu\rho\sigma} \, dx^\nu \wedge dx^\rho \wedge dx^\sigma$, \hfill (1.1)

and the 3-surface $\Sigma$ is any spacelike hypersurface that spans all space. To get the mechanical energy flux vector associated with this fluid as assessed by static observers we must weight the rest-mass flux vector with the mechanical energy per unit rest mass $1/\sqrt{1 - v^2/c^2} = u^\mu w_\mu$. We obtain

$$\rho u^\mu u_\nu w^\nu \sqrt{-g} \, d\Sigma_\mu = T_{\nu\mu} u^\nu \sqrt{-g} \, d\Sigma_\mu.$$ \hfill (1.2)

The total mechanical energy on any cut through spacetime will be

$$E_M = \int_\Sigma T_{\nu\mu} u^\nu \sqrt{-g} \, d\Sigma_\mu.$$ \hfill (1.3)

We have motivated expression (1.3) above by using a fluid made up of dust because the argument is easiest to follow in that special case; however, the final expression in terms of $T_{\nu\mu}$ is not at all confined to a dust fluid nor even to a fluid at all. The same expression would still hold if part of the total $T_{\nu\mu}$ were the Maxwell stress energy–momentum tensor for the

\textsuperscript{4} Better known as MTW. . .

\textsuperscript{5} Indices $\lambda, \mu, \nu, \rho, . . . = 0, 1, 2, 3$; indices $k, l, m, n, . . . = 1, 2, 3$, the metric $g_{\mu\nu}$ has signature $+-- -$ and $g$ is its determinant. Covariant derivatives are indicated by $\nabla$, partial derivatives by $\partial$. And for once neither $G$ nor $c$ are set equal to 1 and $x^0 = ct$. Finally the permutation symbol in 4 dimensions is $\epsilon_{\mu\nu\rho\sigma}$ with $\epsilon_{0123} = 1$ and in 3 dimensions by $\epsilon_{ijk}$ with $\epsilon_{123} = 1$. 
electromagnetic field. Likewise it would still hold for a plasma with an anisotropic pressure tensor or any other physical field.

Whereas formula (1.1) holds for dust on any cut through spacetime, formula (1.3) will give answers that depend on the cut chosen unless $T^\mu_\nu w^\nu$ is a conserved vector which is not generally so. That said, there are many important special cases for which it is conserved. Since

$$D_\mu (T^\mu_\nu w^\nu) = D_\mu \left( T^\mu_\nu \frac{\xi^\nu}{\xi} \right) = -T^\nu_\nu w^\nu \partial_\mu \log \xi,$$

whenever the flux vector $T^\mu_\nu w^\nu$ lies in the equipotential surfaces of constant $\xi$, it is conserved. In stationary spacetimes with purely toroidal motions in $(t, x^1, x^2, \varphi)$ coordinates, $T^\mu_\nu w^\nu$ is only non zero with $\mu$ in the $t$ or $\varphi$ components whereas $\partial_\mu \log \xi$ is only non zero with $\mu$ in the other components so in all such cases $T^\mu_\nu w^\nu$ is conserved. When that occurs, we may evaluate $E_M$ over any cut through spacetime and all will give the same answer. Generally we may always evaluate $E_M$ but should not expect to get an answer independent of the cut chosen. Thus the mechanical energy seen by static observers can depend on the cut through spacetime over which it is evaluated. In practice, there is normally a good time coordinate such as Boyer–Lindquist in Kerr and cuts are chosen to be at constant time.

Following [25] the total gravitational energy is

$$E_G = M c^2 - E_M,$$

where $M c^2$ is the total energy. In this paper we convert the gravitational energy into ‘field energy’ like Maxwell’s $(E^2 + B^2)/8\pi$ in electromagnetism. To do this we first reexpress the $T^\mu_\nu$ in $E_M$ by using Einstein’s equations. Retaining the scalar 3-curvature we remove other second derivatives via integrations by parts. The scalar 3-curvature is reexpressed employing a conformal transformation. Thus the total gravitational energy $E_G$ is found in terms of gravitational field variables.

So far our whole argument has been developed for kinetic energy as seen by the static observers following the Killing vector field but such observers do not exist inside ergospheres. Furthermore Bardeen [2] has emphasized that the static observers have angular momentum in spaces with spin. He has introduced zero-angular-momentum observers (ZAMOs) who move along the normals to the hypersurfaces of constant time. Although such observers move around in azimuth they may be preferred to our static observers because the latter give zero motion to a static body which has angular momentum (backwards). Furthermore these observers exist even within the ergosphere. Thus, t-hypersurface orthogonal observers move with a different 4-velocity field $\tilde{w}^\mu$ so they assess a different kinetic energy than the static observers. According to them the mechanical energy is not $E_M$ but

$$\tilde{E}_M = \int_{\Sigma} T^\mu_\nu \tilde{w}^\nu \sqrt{-g} d\Sigma_\mu.$$

When the mechanical energy is assessed over a hypersurface of constant time, (1.6) seems far more natural than our expression (1.3): for dust the $u^\mu \tilde{w}_\mu = 1/\sqrt{1 - \tilde{v}^2/c^2}$ factor is the same as $u^\mu n_\mu$ because the normal $n^\mu$ to the hypersurface $\Sigma$ is $\tilde{w}^\mu$. Writing $T^\mu_\nu = \rho u^\mu u_\nu$ in the above expression thus yields $(\rho/\sqrt{1 - \tilde{v}^2/c^2})(1/\sqrt{1 - \tilde{v}^2/c^2})$ integrated over all space. The first ‘$\gamma$’ factor comes from the Lorentz contraction and the second from the mechanical energy per unit rest mass. The factors are now the same—they are not normally the same for static observers because for rotating systems their worldlines do not have orthogonal hypersurfaces. Thus there are strong arguments for preferring the hypersurface orthogonal ZAMOs to the static observers and for considering

$$\tilde{E}_G = M c^2 - \tilde{E}_M$$
as the gravitational energy rather than $E_G$. In axial symmetry the ZAMO mechanical energy splits into two terms $E = \Omega J$ as in classical mechanics. We discuss the angular-momentum density of gravitational fields elsewhere.

In what follows we derive the gravitational energy densities for both sets of observers. These densities are different and integrate to different totals. We shall naturally assume that the dominant energy conditions hold [21] so that, in particular,

$$T_{\mu\nu} w^\mu w^\nu > 0 \quad \text{and} \quad T^{\mu\nu} \bar{w}_\mu \bar{w}_\nu > 0. \quad (1.8)$$

A proper evaluation of gravitational energy may be useful in numerical modelling of relativistic stars [37], relativistic ellipsoidal configurations [18] and thick spherical shells [11] among others. Limits of stability are characterized by the relative binding energy $(M - M_B)/M_B$, $M_B$ is the ‘baryonic mass’, which is taken as a relativistic generalization of the $T/|W|$ ratio used in astronomy [6]. However, the more relativistic the configurations, the less representative that ratio is because $(M - M_B)c^2$ is a mixture of all forms of energy and in some models that ‘binding’ energy even changes sign! The $T/|W|$ ratio is better generalized as $E_K/E_G$ where $E_K$ is the bulk kinetic energy as measured by local observers.

There are other properties of gravitational energy which will not be dealt with here but are interesting. For instance, it is not difficult to show that if gravitational energy for static observers is extremal ($\delta E_G = 0$), spacetime must be flat. It might be possible to show, along the lines used by Brill and Deser [7], that $E_G$, near flatness, is a local maximum. It may be true but is as yet unclear that gravitational energy as defined here is always negative. We show that $E_G$ is indeed negative for static and some stationary systems.

2. Stationary spacetimes

Here is a short summary of some basic formulae for stationary spacetimes in which we also fix notations. A good recent summary of present knowledge about stationary spacetimes is given in Beig and Schmidt’s paper [3].

(i) Coordinates adapted to stationary spacetimes, and static observers

Stationary spacetimes have a timelike Killing field $\xi^\mu$. We consider first static observers: they have velocity components $w^\mu$ in the $\xi^\mu$ direction:

$$w^\mu = \frac{\xi^\mu}{\xi} \quad \text{where} \quad \xi = \sqrt{g_{\mu\nu} \xi^\mu \xi^\nu}, \quad (2.1)$$

and are at rest with respect to fixed observers at infinity.

In the projection formalism, the metric is decomposed as follows:

$$\text{d}s^2 = (w_\lambda \text{d}x^\lambda)^2 + (g_{\mu\nu} - w_\mu w_\nu) \text{d}x^\mu \text{d}x^\nu. \quad (2.2)$$

In practical calculations the metric is most often written in coordinates in which it takes the following form [28, 29, 38]:

$$\text{d}s^2 = g_{\mu\nu} \text{d}x^\mu \text{d}x^\nu = f(\text{d}x^0 - A_k \text{d}x^k)^2 - \gamma_{kl} \text{d}x^k \text{d}x^l; \quad (2.3)$$

in these coordinates,

$$\{\xi^\mu\} = \{1, 0, 0, 0\} \quad \text{and} \quad \{w^\mu\} = \{f^{-1/2}, 0, 0, 0\}. \quad (2.4)$$

6 A typical example is the binding energy of Einstein’s spherical shell of self-bound particles [13] (see also [11]). A recent discussion on the sign of the binding energy in static perfect fluids can be found, for instance, in Karkowski and Malec [26].

7 See also [38]; the original formulation of the projection formalism is due to Geroch [16].
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\( f, A_k, \) and \( \gamma_{kl} \) are functions of \( x^k \) only. \( \gamma_{kl} \) plays a special role: indices of tensors in \( \gamma \)-space\(^8\) will be displaced with that metric and its inverse. Thus, for instance, we shall write

\[
\mathcal{A}^k = \gamma^{kl} A_l \quad \text{and} \quad \mathcal{A}^2 = \mathcal{A}^k A_k = \gamma^{kl} A_k A_l > 0. \tag{2.5}
\]

The metric components \( g_{\mu\nu} \) and their inverse \( g^{\mu\nu} \) are

\[
g^{00} = f, \quad g^{0k} = -f A_k, \quad g^{kl} = -\gamma^{kl} + f A_k A_l, \quad g^{00} = f^{-1} - A^2, \quad g^{0l} = -A_l, \quad g^{kl} = -\gamma^{kl}. \tag{2.6}
\]

The determinants \( g \) of \( g_{\mu\nu} \) and \( \gamma \) of \( \gamma_{kl} \) are so related:

\[
\sqrt{-g} = f^{1/2} \sqrt{\gamma}. \tag{2.8}
\]

The metric (2.3) and the components \( w^\mu \), (2.4), keep the same form under a change of coordinates:

\[
x'^0 = x^0 + \tau^0(x^k) \quad \text{and} \quad x'^k = \tau^k(x^l). \tag{2.9}
\]

\( A_k \) and \( \gamma_{kl} \) transform like tensors for \( x'^k = \tau^k(x^l) \). The interesting transformations are however those of \( x^0 \) that leave \( f \) and \( \gamma_{kl} \) invariant but \( A'_k = A_k + \partial_k \tau^0 \). They are associated with different \( x^0 = \text{const} \) hypersurfaces which we need to fix if we want a definite gravitational energy density. The usual gauge condition is\(^9\)

\[
\partial^k (\sqrt{\gamma} A^k) = 0. \tag{2.11}
\]

The dominant terms of the asymptotic solutions for isolated sources are given in MTW [32] on p 456. In properly chosen asymptotically Minkowski orthogonal coordinates at infinity,

\[
f \to 1 - \frac{2m}{r}, \quad A_t \to -2 \frac{\epsilon_{lmn} j^m \delta^{kn}}{r^2}, \tag{2.12}
\]

\[
\gamma_{kl} \to \delta_{kl} \left( 1 + \frac{2m}{r} \right) \quad \text{where} \quad r = \sqrt{x^2 + y^2 + z^2} \to \infty,
\]

and

\[
n^k = \frac{x^k}{r}, \quad \sum_k (n^k)^2 = 1, \quad m = \frac{GM}{c^2}, \quad j^k = \frac{GJ^k}{c^3}. \tag{2.13}
\]

\( Mc^2 \) is the total energy of spacetime and \( \vec{J} = \{ J_k \} = \{-J^k\} \) its angular momentum vector.

Einstein’s equations,

\[
R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \quad \text{with} \quad \kappa = \frac{8\pi G}{c^4}, \tag{2.14}
\]

are better written in terms of projected components that are gauge invariant:

\[
R_{\mu\nu} w^\mu w^\nu, \quad R_{\mu\nu} (g^{\mu\rho} - w^\mu w^\rho) (g^{\nu\sigma} - w^\nu w^\sigma) \quad \text{and} \quad R_{\mu\nu} (g^{\mu\rho} - w^\mu w^\rho) w^\nu \tag{2.15}
\]

\(^8\) Also called the ‘quotient space’ obtained by quotienting spacetime by the action of the stationary isometry [3]. It represents the collection of the orbits of the Killing vectors \( \xi^\mu \). The \( \gamma_{kl} \) give a measure of proper lengths and \( f \) of proper times. See a good discussion in [28]. See also [39]. Indices of \( R_{\mu\nu} \) and \( T_{\mu\nu} \) are always displaced with \( g_{\mu\nu} \) not \( \gamma_{kl} \) or later with \( \tilde{g}_{\mu\nu} \). Thus in (3.20) \( T^k_k = \tilde{g}_{kl} T^{kl} \).

\(^9\) The gauge condition (2.11) reduces the freedom of translations to such \( \tau^0 \) that satisfy the condition \( \nabla^2 \tau^0 = 0 \) (see (2.16) for the definition of \( \nabla^2 \)). If we ask for \( \tau^0 = \text{const} \) at infinity this fixes \( x^0 \) up to a constant.
In these expressions in our coordinates, $\mathcal{A}_k$ must necessarily appear in a curl. Let $\nabla_k$ represent covariant derivatives in $\gamma$-space; we shall write
\begin{equation}
\gamma^{kl}\nabla_l = \nabla^k \quad \text{and} \quad \nabla^k \nabla_k = \nabla^2. \tag{2.16}
\end{equation}
The gauge invariant components of the Ricci tensor, invariant for time translations (2.9) and covariant for spatial coordinate transformations (2.10), are equivalent to [28]:
\begin{equation}
R_{00} = \frac{1}{2} \nabla^2 f - \frac{1}{4 f} \nabla^k f \nabla_k f + f^2 \nabla^{[k} A^{l]} \nabla_{[k} A_{l]}, \tag{2.17}
\end{equation}
\begin{equation}
R^{k l} = g^{k l} R_{\mu \nu} = - \frac{1}{2 f} \nabla^k \nabla^l f + \frac{1}{4 f^2} \nabla^k f \nabla^l f + 2 f \gamma_{m n} \nabla^{[k} A^{n]} \nabla^{l]} A^{m]} + \mathcal{R}^{k l}, \tag{2.18}
\end{equation}
\begin{equation}
R^k = g^{k l} R_{0 l} = - \frac{3}{2} \nabla_k f \nabla^{[l} A^{m]} - \nabla_l (\nabla^{[k} A^{l]}), \tag{2.19}
\end{equation}
and the 4-scalar curvature
\begin{equation}
R = f^{-1} R_{00} - \gamma_{k l} R^{k l}. \tag{2.20}
\end{equation}
In (2.18), $\mathcal{R}^{k l}$ are the components of the Ricci tensor of the $\gamma$-space, not of the $x^0 = \text{const}$ hypersurface. The following linear combination of (2.17) and (2.18) is of particular interest:
\begin{equation}
- \gamma_{k l} R^{k l} - f^{-1} R_{00} = - \gamma_{k l} R^{k l} - 3 f \nabla^{[k} A^{l]} \nabla_{[k} A_{l]} = - \mathcal{R} = 3 f \nabla^{[k} A^{l]} \nabla_{[k} A_{l]}; \tag{2.21}
\end{equation}
$\mathcal{R}$ is the scalar curvature of the $\gamma$-space. The left-hand side of (2.21) can be rewritten using Einstein’s equations (2.14):
\begin{equation}
- \gamma_{k l} R^{k l} - f^{-1} R_{00} = - 2 \kappa T^0_0 + 2 A_k R^k. \tag{2.22}
\end{equation}
Therefore, (2.21) and (2.22) imply
\begin{equation}
- \mathcal{R} = 3 f \nabla^{[k} A^{l]} \nabla_{[k} A_{l]} - 2 A_k R^k = - 2 \kappa T^0_0. \tag{2.23}
\end{equation}
The last two terms on the left-hand side can be recast in another form using (2.19):
\begin{equation}
- 3 f \nabla^{[k} A^{l]} \nabla_{[k} A_{l]} - 2 A_k R^k = - \nabla_{[k} (f A_{l]} \nabla^{k} A^{l]} - \nabla_k (2 f A_l \nabla^{[k} A^{l]}). \tag{2.24}
\end{equation}
Inserting this new expression into (2.23), then transferring $- \nabla_k (2 f A_l \nabla^{[k} A^{l]})$ from left to right we get
\begin{equation}
- \mathcal{R} = - \nabla_{[k} (f A_{l]} \nabla^{k} A^{l]} = - 2 \kappa T^0_0 + \nabla_k (2 f A_l \nabla^{[k} A^{l]}). \tag{2.25}
\end{equation}
Before going further, it is perhaps interesting to rewrite Einstein’s equations in terms of ‘gravoelectric’ and ‘gravomagnetic’ vector fields $\mathcal{E}_k$ and $\mathcal{B}^m$ living in $\gamma$-space in the spirit of [28, 30, 33] and [40]. The gravoelectric potential
\begin{equation}
\psi = \log \sqrt{f}, \tag{2.26}
\end{equation}
and the gravomagnetic vector potential is $\mathcal{A}_k$. $\mathcal{E}_k$ is defined by
\begin{equation}
\mathcal{E}_k = - \partial_k \psi \quad \text{and} \quad \gamma^{k l} \mathcal{E}_k \mathcal{E}_l = \mathcal{E}^2 > 0. \tag{2.27}
\end{equation}
$\mathcal{B}^m$ is divergenceless:\(^{11}\)
\begin{equation}
(\partial_k \mathcal{A}_l - \partial_l \mathcal{A}_k) = \eta_{k l m} \mathcal{B}^m \quad \text{or} \quad \mathcal{B}^m = \eta^{m k l} \partial_k \mathcal{A}_l \quad \text{with} \quad \eta_{k l m} = \sqrt{\mathcal{F}} \epsilon_{k l m}; \tag{2.28}
\end{equation}
\(^{10}\)In the 1959 edition of Landau and Lifshitz there is apparently a printing mistake in the third equation on p 301. What in their notations is written $+\frac{1}{2} f^{m n} f_{n p}$ should be $-\frac{1}{2} f^{m n} f_{n p}$.
\(^{11}\)In [28] and [33], $\mathcal{B}$’s are replaced by $\mathcal{H} = f^{1/2} \mathcal{B}$. This $\mathcal{H}$ is called $\mathcal{B}$ in [25].
in particular,
\[ \nabla_{[\dot{A}]} A_{l]l} = \frac{1}{2} \eta_{kim} \eta^{lij} B_{im} = \frac{1}{2} B^{0m} B_m = \frac{1}{2} B^2 > 0. \]

(2.29)

In terms of \( E \)'s and \( B \)'s the components of the Ricci tensor take the following forms:
\[ f^{-1} R_{00} = - \nabla \cdot E + E^2 + \frac{1}{2} f B^2, \]
\[ R^{kl} = \mathcal{B}^k \mathcal{E}^l - \mathcal{E}^k \mathcal{E}^l + \frac{1}{2} f \left( \gamma^{kl} B^2 - B^k B^l \right) + \mathcal{R}^{kl}, \]
\[ f^{-1} R^k_0 = \frac{1}{2} \eta^{lij}(\mathcal{B}\mathcal{E}_m + 3\mathcal{E}_m B_m) = \frac{1}{2} [-\nabla \times B + 3\mathcal{E} \times B]^k. \]

(2.30)

(2.31)

(2.32)

Also,
\[ R = -2 \nabla \cdot E + 2E^2 - \frac{1}{2} f B^2 - \mathcal{R}, \]
and equality (2.25) takes this form
\[ - \mathcal{R} - f \left( \frac{1}{2} B^2 + B \cdot A \times E \right) = -2 \kappa T^0_0 + \nabla \cdot (f A \times B). \]

(2.33)

(2.34)

Dots and cross products are defined in \( \gamma \)-space.

\( (ii) \) Metric components adapted to observers with velocities orthogonal to constant time slices

The coordinates are the same as in \( (i) \) but observers are different. They have different 4-velocity components; we denote them by \( \tilde{w}^\mu \) instead of \( u^\mu \). \( \tilde{w}^\mu \) is associated with the \((1 + 3)\) decomposition familiar in the Hamiltonian formulation of Einstein’s equations\(^{12}\). By definition, coordinates exist in which
\[ \{ \tilde{\omega}_\mu \} = \{ f^{1/2}, 0, 0, 0 \}. \]

(2.35)

The metric (2.2) with \( \tilde{w}^\mu \) replaces \( w^\mu \) now takes the following form:
\[ dx^2 = f \left( dx^0 \right)^2 - \tilde{\gamma}_{kl} \left( dx^k - W^k dx^0 \right) \left( dx^l - W^l dx^0 \right), \]

(2.36)

in which \( W^k \) are the components of the local coordinate velocities in units of \( c \):
\[ W^k = \tilde{\omega}^k \tilde{\omega}_0 = \frac{dx^k}{dx^0}; \quad \text{also set} \quad W^2 = \tilde{\gamma}_{kl} W^k W^l. \]

(2.37)

The metric components and their inverses are
\[ g_{00} = f - W^2, \quad g_{0i} = W_i = \tilde{\gamma}_{ki} W^k, \quad g_{ij} = -\tilde{\gamma}_{ij}, \]
\[ g^{00} = f^{-1}, \quad g^{0i} = f^{-1} W^i, \quad g^{ij} = -\tilde{\gamma}^{ij} + f^{-1} W^k W^l. \]

(2.38)

(2.39)

The determinant \( g \) is related to the determinant \( \tilde{\gamma} \) of \( \tilde{\gamma}_{ij} \) like this:
\[ \sqrt{-g} = f^{1/2} \sqrt{\tilde{\gamma}}. \]

(2.40)

The metric (2.36) and the velocity components (2.35) keep the same form under a change of coordinates (2.9) and (2.10). Some of Einstein’s equations are now better written in terms of Einstein’s tensor components \( G^\mu^\nu \):
\[ G^\mu^\nu = R^\mu^\nu - \frac{1}{2} g^\mu^\nu R = \kappa T^\mu^\nu. \]

(2.41)

The simplest forms belong to the projected components of \( G^\mu^\nu \), see (2.15), with \( u^\mu \) replaced by \( \tilde{w}^\mu \). In our coordinates \([1]^13\):

\(^{12}\) The original reference is Dirac [12]. See also [1]. Modern versions are given in various books like [32] and [41].

\(^{13}\) This reference contains the clearest and most detailed calculations. We rewrote the formulae in our notations.
\[2 \tilde{f} G^{00} = \tilde{R} + K^2 - K^{kl} K_{kl}, \quad (2.42)\]

\[R_{kl} = \tilde{R}_{kl} + \tilde{\nabla}_{(k} \tilde{\xi}_{l)} - \tilde{\xi}_{k} \tilde{\xi}_{l} + 2 \tilde{f}^{-1/2} K_{k}^{m} \tilde{\nabla}_{(l]} W_{m]} - \tilde{f}^{-1/2} \tilde{\xi}_{m} W_{m} K_{kl} + \tilde{\nabla}_{m} (\tilde{f}^{-1/2} W_{m} K_{kl}), \quad (2.43)\]

\[\tilde{f}^{1/2} G_{k}^{0} = \tilde{f}^{1/2} R_{k}^{0} = -\tilde{\nabla}_{l} (K_{kl} - \delta_{l}^{k} K), \quad (2.44)\]

and

\[R = 2(\tilde{f} G^{00} - \tilde{f}^{k} R_{kl}). \quad (2.45)\]

In these formulae \(\tilde{R}_{kl}\) is the Ricci tensor of the spatial metric \(\tilde{\gamma}_{kl}\); \(\tilde{R} = \tilde{\gamma}_{kl} \tilde{R}_{kl}\) the corresponding scalar curvature while \(K_{kl}\) (the second fundamental form of \(x_{0} = \text{const}\)) is

\[K_{kl} = \tilde{\nabla}_{k} W_{l} - \tilde{\nabla}_{l} W_{k}, \quad (2.46)\]

\(\tilde{\nabla}_{k}\) is a \(\tilde{\gamma}\)-covariant derivative. The time coordinate must be fixed if we wish to integrate over a definite spacelike hypersurface. It is fixed up to a constant by a gauge condition like (2.11). The most common choice in the Hamiltonian formulation and in numerical relativity is maximal spacelike hypersurfaces \(K = 0\) which imply that observers are expansion free, \(D_{\mu} \tilde{w}^{\mu} = 0\). In stationary spacetimes the condition reduces to

\[\tilde{\nabla}_{k} W^{k} = 0 \quad \text{equivalent to} \quad K = 0, \quad (2.47)\]

as follows form (2.46). Note that the condition is identically satisfied in axial symmetry (see (4.15) below).

(iii) Relations

The metrics in the form (2.3)–(2.7) and (2.36)–(2.39) are the same metrics in the same coordinates, adapted to different decompositions associated with different families of observers. The \(\tilde{f}, W^{k}\) and \(\tilde{\gamma}_{kl}\) are related as follows to the \(f, A_{k}\) and \(\gamma_{kl}\):

\[\tilde{f} = \frac{f}{1 - f A^{2}}, \quad W^{k} = -\frac{f A^{k}}{1 - f A^{2}}, \quad \tilde{\gamma}_{kl} = \gamma_{kl} - f A_{k} A_{l}, \quad (2.48)\]

and reciprocally,

\[f = \tilde{f} - W^{2}, \quad A^{k} = -f^{-1} W^{k}, \quad \gamma_{kl} = \tilde{\gamma}_{kl} + f A_{k} A_{l}. \quad (2.49)\]

Note that \(A^{k} \neq \gamma^{kl} A_{l}\), not \(\tilde{\gamma}^{kl} A_{l}\). We now turn to the calculation of gravitational energy.

3. Gravitational energy

(i) As calculated by static observers

If \(w^{k}\) are the velocity components of static observers, the mechanical energy they evaluate on \(x^{0} = \text{const}\) is, as pointed out in (1.3),

\[E_{M} = \int \sqrt{-g} d\Sigma_{\mu} = \int \sqrt{-g} dV \quad \text{where} \quad dV = \sqrt{-g} d^{3}x = (1 - f A^{2})^{-1/2} d\tilde{V}. \quad (3.1)\]

The right-hand side is written in our special coordinates. \(dV\) is the proper volume element in \(\gamma\)-space and not of the \(x^{0} = \text{const}\) hypersurface which is \(d\tilde{V}\). These two coincide in static spacetimes.
Now consider (2.34). Divide by $2\kappa$, multiply by $\sqrt{\gamma}$ and integrate over the whole space. The boundary conditions imply that the integral\(^{14}\) of $\frac{1}{2\kappa} \delta_k (\sqrt{\gamma} f A \times B^k)$ is zero. Note that if the gauge condition (2.11) was not fixed, $\frac{1}{2\kappa} \delta_k (\sqrt{\gamma} f A \times B^k)$ would not be fixed and the remaining integrand in (3.2) would also be gauge dependent. It follows now that the gravitational energy, see (1.5),

$$E_G = Mc^2 - E_M = Mc^2 - \frac{1}{2\kappa} \int_d \left[ R + f \left( \frac{1}{2} B^2 + B \cdot A \times E \right) \right] dV. \quad (3.2)$$

We shall next eliminate $Mc^2$ from (3.2) and obtain an integral that depends entirely on field components and their derivatives\(^{15}\). To this effect, we make a conformal transformation of the $\gamma$-metric, say,

$$\gamma^*_{ij} = e^{2\chi} \gamma_{ij}, \quad (3.3)$$

and set, by analogy with (2.27),

$$\mathcal{F}_k = -\partial_k \chi; \quad \text{also} \quad \gamma^{ij} \mathcal{F}_i \mathcal{F}_j = \mathcal{F}^2. \quad (3.4)$$

The conformal scalar curvatures $R^*$ and $R$ are so related:

$$e^{2\chi} R^* = R + 4\nabla_k \mathcal{F}^k - 2\mathcal{F}^2. \quad (3.5)$$

We then define $\chi$ by the condition that $R^* = 0$, that is

$$R = -4\nabla_k \mathcal{F}^k + 2\mathcal{F}^2, \quad (3.6)$$

or

$$\nabla^2 e^{\chi/2} - \frac{1}{8} R e^{\chi/2} = 0. \quad (3.7)$$

The $\gamma$-metric becomes spherical at large distance and all spherical metrics are conformally flat. Therefore we shall ask the factor $e^{-2\chi}$ to behave like the conformal factor of $\gamma_{kl}$ near infinity as in (2.12):

$$e^{-2\chi} \to 1 + \frac{2m}{r} \quad \text{or} \quad \chi \to -\frac{m}{r} \quad \text{for} \quad r \to \infty. \quad (3.8)$$

Equation (3.7) has been considered by Cantor and Brill [10] who found a necessary and sufficient condition for an asymptotically flat metric to be conformally equivalent to another asymptotically flat metric with zero scalar curvature. They showed that the condition for the existence and uniqueness of a solution of (3.7) is that

$$\int q \cdot \nabla q \, dV \geq -\frac{1}{8} \int q^2 R \, dV \quad (3.9)$$

for all smooth functions $q \neq 0$ of compact support. Note that (2.23) with $R^k_0 = \kappa T^k_0$ can also be written as

$$R + \frac{1}{2} f B^2 = 2\kappa f^{-1} T_{00}. \quad (3.10)$$

\(^{14}\) We use abundantly Stokes theorem [41], or Gauss theorem [28], or Gauss–Ostrogradsky theorem [19] or Green–Ostrogradsky theorem [17] or Green’s theorem [27] or the divergence theorem [6]. Hadamard [20] gives it no name and regards it as a generalization of $\int_a^b f'(x) \, dx = f(b) - f(a)$. It is useful to remember that the divergence theorem applies to divergences of continuous and single valued functions.

\(^{15}\) Note that on a general hypersurface in a general spacetime one may specify the lapse function $f^{1/2}$ arbitrarily by arbitrary time reparametrizations $x^0 = r^0(x^i)$. However, in stationary spacetimes there are privileged coordinates in which the metric has the form (2.3) and the allowed transformations of coordinates are those given by (2.9) and (2.10). Our gauge condition (2.11) fixes the spacelike hypersurface.
This shows that in non-static spacetimes outside matter \(\mathcal{R} = -\frac{1}{2} f B^2 < 0\) and conditions (3.9) are not automatically satisfied\(^{16}\).

If a solution of (3.7) exists, we may replace \(\mathcal{R}\) in (3.2) by its equivalent (3.6) and note that because of (3.8)

\[
\frac{1}{2\kappa} \int V_k(4F_l)\,dV = \frac{2}{\kappa} \int_{r \to \infty} F_l \, dS_l = \frac{2}{\kappa} \left( -\frac{m}{r^2} \right) 4\pi r^2 = -Me^2. \tag{3.11}
\]

As a consequence of (3.11), \(Mc^2\) disappears from (3.2) and we obtain for \(E_G\):

\[
E_G = \frac{1}{\kappa} \int_v \left\{ F^2 + \frac{1}{4} [B^2 + 2B \cdot (A \times \mathcal{E})] \right\} \, dV = \int_v \epsilon_G \, dV; \tag{3.12}
\]

we used (3.1) to reexpress \(dV\) in terms of the proper volume element \(dV\) on the slice \(x^0 = \text{const}\.\) With the gauge condition (2.11), the gravitational energy density \(\epsilon_G\) on the fixed \(x^0 = \text{const}\) hypersurfaces is well defined. It is a scalar for spatial coordinate transformations. Note that it is not purely local as it involves a quantity, \(F^2\), defined by the solution of an elliptic equation (3.7).

In practical calculations of the total gravitational energy, (3.12) is not so good because one needs to solve equation (3.7). We now give an expression for \(E_G\) in terms of metric components and their derivatives. That expression does not depend on the existence of a solution of equation (3.6). To this effect, we first note that calculations near infinity are often simpler in spherical coordinates rather than \(x, y, z\) or axial coordinates \(R, z, \varphi\) as in section 4 (iii). If the metric of the limiting flat space has a general form

\[
dr^2 = (dx^0)^2 - \nabla_k dx^k dx^l, \tag{3.13}
\]

it is useful to take \(dr^2\) as a flat ‘background metric’\(^{17}\). This is done by replacing in \(\mathcal{R}_{kl} = \gamma_{kn} \gamma_{ml} \mathcal{R}^{mn}\) partial derivatives \(\delta_k\) by \(\nabla_k\)-covariant derivatives in the \(\mathcal{Y}\)-background and the \(\gamma\)-Christoffel symbols \(\Gamma^m_{kl}(\gamma)\) by the tensors

\[
\Delta^m_{kl} = \Gamma^m_{kl}(\gamma) - \Gamma^m_{kl} = \frac{1}{2} \gamma^{mn} (\nabla_k \gamma_{nl} + \nabla_l \gamma_{nk} - \nabla_n \gamma_{kl}). \tag{3.14}
\]

\(\Gamma\)’s are Christoffel symbols in \(\mathcal{Y}\)-space. Thus we write \(\mathcal{R}_{kl}\) as follows:

\[
\mathcal{R}_{kl} = \nabla_m \Delta^m_{kl} - \nabla_k \Delta^m_{lm} + \Delta^m_{kl} \Delta^l_{mn} - \Delta^m_{kn} \Delta^l_{nm}. \tag{3.15}
\]

Now remember a familiar identity usually used to construct Einstein’s Lagrangian\(^{18}\):

\[
\mathcal{R} = \mathcal{L} - \nabla_k k^l \quad \text{where} \quad k^l = \gamma^{-1} \nabla_k (\gamma \gamma^{kl}) \quad \text{and} \quad \mathcal{L} = -\gamma^{kl} (\Delta^m_{kl} \Delta^l_{mn} - \Delta^m_{kn} \Delta^l_{nm}). \tag{3.16}
\]

We then substitute \(\mathcal{R}\) given by (3.16) into (3.2). The asymptotic conditions on the metric have the result that, as in (3.11),

\[
\frac{1}{2\kappa} \int V_k k^l \, dV = -Me^2. \tag{3.17}
\]

Thus (3.2) may now be written as

\[
E_G = -\frac{1}{2\kappa} \int_v \left\{ \mathcal{L} + f \left[ \frac{1}{2} B^2 + B \cdot (A \times \mathcal{E}) \right] \right\} \, dV. \tag{3.18}
\]

---

\(^{16}\) The original paper [10] contains precise statements about this theorem and useful references. It gives an example of an axially symmetric space in which the solution of (3.7) does not exist. The reason it does not exist is that \(\mathcal{R}\) is not positive everywhere.

\(^{17}\) The method that consists in introducing a second metric avoids the non-covariance and allows the use of coordinates that are not Minkowski coordinates at spatial infinity. The method goes back to a paper of Rosen [36]. Details on how suitable background metrics and mappings between the backgrounds and physical spacetimes may be introduced, in particular in asymptotically flat spacetimes, can be found in [5]; see also [23].

\(^{18}\) Note the first \(\nabla_k\) and the second \(\nabla_k\).
calculated by observers at rest given in (3.1)). \( L \) is a scalar density because \( \Delta^\prime \)'s are tensors. However, their value depends on the mapping on the flat background which is equivalent to choosing special coordinates. \( E_G \) can be calculated with this expression in terms of the metric components and their first-order derivatives and is correctly given by the integral. Note that if, for instance, \( E_G \) is found to be negative, then gravitational energy is negative irrespective of the choice of coordinates and irrespective of the nature of the sources of gravity. The formula is valuable in this respect. (3.18) is another form of the general formula obtained in [25] (see the appendix).

It is at first surprising that the integrand does not contain derivatives of \( f \) because in the Newtonian approximation it is proportional to the square of \( \partial_k f \). But linearization is usually done in harmonic coordinates which, in particular, imposes (with Minkowski coordinates in the background)

\[
\partial_t (\sqrt{-g} g^{tt}) = 0, \quad \text{or equivalently} \quad \frac{1}{\sqrt{g}} \partial_t (\sqrt{g} g^{tt}) = -\frac{1}{2} g^{tt} \partial_t f = E^t.
\]

This is how \( \partial_t f \) reappears in \( E_G \) in the Newtonian limit.

It is also possible to write the total gravitational energy in terms of field components and a covariant integrand with the non-locality 'shuffled into' a matter tensor integral. To that effect consider (2.30) and (2.31) in the following combination:

\[
\gamma_{kl} R^{kl} - 3 f^{-1} R_{00} = 2 \kappa T_k^k = R + 4 \nabla_l E^l - 4 E^2 - \frac{1}{2} f B^2.
\]

Replace \( R \) by \( L - \nabla_l k^l \), while moving the \( E \) and \( B \) terms to the left-hand side:

\[
4 E^2 + \frac{1}{2} f B^2 + 2 \kappa T_k^k = L + \nabla_l (4 E^l - k^l).
\]

The volume integral of the divergence on the right-hand side is equal to zero because of the asymptotic conditions as follows from (3.11) and (3.17). Thus, after dividing by \( 2 \kappa \), the integral of \( L \) reduces to

\[
\frac{1}{2 \kappa} \int L \, dV = \frac{1}{2 \kappa} \int \left( 2 E^2 + \frac{1}{2} f B^2 \right) \, dV + \int T_k^k \, dV.
\]

Replacing the \( L \) integral in (3.18), we obtain a formula for \( E_G \) in which the non-local character appears in the matter tensor part \( T_k^k \):

\[
E_G = -\frac{1}{2 \kappa} \int \left[ 2 E^2 + \frac{1}{2} f (B^2 + B \cdot A \times E) \right] \, dV - \int T_k^k \, dV.
\]

In particular, for perfect static fluids (\( A = B = 0 \)) with local pressure \( p \) (\( T_k^k = -3p \)):

\[
E_G = -\frac{2}{2 \kappa} \int E^2 \, dV + 3 \int p \, dV.
\]

In weak fields, the first term equals twice the Newtonian gravitational energy and the second term, because of the virial theorem, is equal to minus that same energy.

The important formulae of this subsection are (1) the gravitational energy density as calculated by observers at rest given in (3.12) in terms of a non-local field (which may not always exist), (2) the total gravitational energy expressed in terms of local fields given in (3.18), (3) the total gravitational energy calculated in terms of local fields with a real density, the non-locality being given in terms of the source of gravity, (3.23).

(ii) Gravitational energy for observers moving orthogonal to constant time slices

The mechanical energy is now denoted by \( \tilde{E}_M \) because it is different from \( E_M \). With \( \tilde{w}_\mu \), see (2.35), and the metric, see (2.36),

\[
\tilde{E}_M = \int \sqrt{-g} \, \tilde{w}^\nu \sqrt{g} \, d\Sigma_\mu = \frac{1}{\sqrt{\gamma}} \int T^{00} \tilde{V} \, d\tilde{V} \quad \text{where} \quad d\tilde{V} = \sqrt{\gamma} \, d^3x.
\]
Therefore with $T^{00} \tilde{f} = \frac{1}{2} G^{00} \tilde{f}$ and (2.42), the gravitational energy $\tilde{E}_G$ is now

$$\tilde{E}_G = Mc^2 - \frac{1}{2\kappa} \int_\mathcal{V} (\tilde{R} + \tilde{K}^2 - \tilde{K}^{0i} K_{0i}) \, d\tilde{V}. \quad (3.26)$$

We may again eliminate $Mc^2$ and obtain the energy density\(^{19}\) in terms of a non local field. Thus, by analogy with (3.3) we set

$$\tilde{\gamma}_{i}^{l} = e^{\chi / 2} \tilde{\gamma}_{i}^{l} \quad \text{and} \quad \tilde{\mathcal{F}}_{k} = -\partial_{k} \tilde{\chi}, \quad (3.27)$$

and demand that the scalar curvature of the conformal metric $\tilde{\mathcal{R}}^* = 0$. This defines $\tilde{\chi}$ by an equation similar to (3.6) or (3.7):

$$\tilde{\mathcal{R}} = -4\nabla_{i} \tilde{\mathcal{F}}^{i} + 2 \tilde{\mathcal{F}}^{2} \quad \text{or} \quad \nabla^{2} e^{\frac{\chi}{2}} - \frac{1}{2} \tilde{\mathcal{R}} e^{\frac{\chi}{2}} = 0. \quad (3.28)$$

The asymptotic condition on $\tilde{\chi}$ is the same as (3.8). If $K = 0$ and the dominant energy condition (1.8) is satisfied, then $T^{00} > 0$ and thus $\tilde{\mathcal{R}}$ which is proportional to $T^{00} + K^{0i} K_{0i}$ is positive. It then follows, see below (3.8), that a solution for $\tilde{\chi}$ always exists, whereas we have not proved this for static observers and, instead of (3.26), we obtain,

$$\tilde{E}_G = \frac{1}{k} \int_{\mathcal{V}} \left( \tilde{\mathcal{F}}^{2} - \frac{1}{2} K^{0i} K_{0i} \right) \, d\tilde{V} = \int_{\mathcal{V}} \tilde{\epsilon}_G \, d\tilde{V}, \quad (K = 0). \quad (3.29)$$

Note that the $KK$ term has the opposite sign to the $\tilde{\mathcal{F}}^{2}$ term. The quantity $\tilde{\epsilon}_G$ is the gravitational energy density in $\tilde{\gamma}$-space, i.e., on $x^0 = \text{const}$. For practical calculations of the total gravitational energy, we may use a procedure similar to that used in (3.13)–(3.18), and obtain $\tilde{E}_G$ in terms of local fields:

$$\tilde{E}_G = -\frac{1}{2k} \int_{\mathcal{V}} (\tilde{\mathcal{L}} - K^{0i} K_{0i}) \, d\tilde{V} \quad \text{in which} \quad \tilde{\mathcal{L}} = \mathcal{L}(\mathcal{V} \Rightarrow \tilde{\gamma}), \quad (K = 0), \quad (3.30)$$

and also a formula similar to (3.23):

$$\tilde{E}_G = -\frac{2}{k} \int_{x^0} (\tilde{\mathcal{F}}^{2} - K^{0i} K_{0i}) \, d\tilde{V} - \int (T^{0}_{k} + W^{k} T^{0}_{k}) \, d\tilde{V}, \quad (K = 0). \quad (3.31)$$

### 4. Is gravitational energy negative?

In Einstein’s theory, gravity may be repulsive\(^{20}\). In Newton’s theory, gravity is always attractive and gravitational energy is negative. In relativistic gravity we know, for instance see [25], that even in the linearized theory of gravity, $E_G$ is negative when the source of gravity is a perfect fluid but we do not know if it is true otherwise. Nevertheless, there are important instances in which $E_G < 0$ and some of these are considered here.

(i) Static spacetimes and static observers

In static spacetimes $\mathcal{A} = 0$ and, see (2.25), $\mathcal{R} = 2k T^{0}_{0}$. The energy condition (1.8) is here $T^{00} = f T^{0}_{0} > 0$ and implies $\mathcal{R} > 0$; in this case, as already shown by Cantor [9], a solution of (3.7) exists and

$$E_G = \tilde{E}_G = -\frac{1}{k} \int \mathcal{L} \, dV = -\frac{1}{k} \int \mathcal{F}^{2} \, dV < 0. \quad (4.1)$$

\(^{19}\) Natário [33] wrote (2.30), using (2.14), in the form $\nabla_{i} e^{i} = e^{\chi / 2} f B^2 - f^{-1} x (T^{00} - \frac{1}{2} \nabla_{i} T) \tilde{e}^{2} + \frac{1}{2} f B^2 \tilde{e}^{2}$ the field energy density because of its analogy with electromagnetism. This formal definition is unrelated to our $\epsilon_G$ except in conformastatic spacetimes with metrics $d\tilde{s}^{2} = f (d\tilde{x}^{i})^{2} - f^{-1} \tilde{d}^{2}$, see below, where the two ‘densities’ are indeed equal.

\(^{20}\) This follows from Raychaudhury’s equation [35]. See Ellis and van Elst [14] for a covariant presentation (on the web). Repulsive gravity is discussed in various places see, for instance, [22].
In spherical symmetry, the 3-space is conformally flat and the metric is of the form
\[ ds^2 = f \left( dx^0 \right)^2 - b \left( d\vec{r} \right)^2. \] (4.2)

\( f \) and \( b \) are functions of \( r \). The case has been studied in some detail in [25] where the connection with MTW’s formula for gravitational energy has been given explicitly. Using \( \mathcal{L} \) to calculate \( E_G \) we readily find that
\[ E_G = -\frac{1}{\kappa} \int \left[ \left( b - b^{1/2} \right)^2 \right] dV. \] (4.3)

Since outside matter,
\[ b = \left( 1 + \frac{m}{2r} \right)^4, \] (4.4)

any spherically symmetric stellar model of isotropic radius \( r_M \) has a total energy
\[ E_{\text{stat}} = M c^2 - E_G(r \geq r_M) = M c^2 + \frac{GM^2}{2r_M} = M c^2 \left( 1 + \frac{m}{2r_M} \right). \] (4.5)

In the limiting case of a Schwarzschild black hole, \( r_M = m/2 \) at the horizon and the energy of the hole itself is \( 2Mc^2 \). This is the ‘quasilocal’ energy attributed to the black hole by Brown and York [8].

(ii) Conformastationary spacetimes for static observers

In conformastationary spacetimes [38], \( \gamma_{kl} \) is itself conformally flat but with no special symmetry. If we replace \( f \) by \( e^{2\psi} \), the metric takes the following form:
\[ ds^2 = e^{2\psi} \left( dx^0 - A_k dx^k \right)^2 - e^{-2\chi} \sum_k \left( dx^k \right)^2. \] (4.6)

\( R \) itself has the form of (3.6) and \( E_G \) is given by (3.12) in which \( \mathcal{F}^2 \) is now purely local.

The special case in which \( \chi = \psi \) and thus \( \mathcal{F}_k = \mathcal{E}_k \) has been the object of much scrutiny in empty spacetimes\(^{21} \) and in spacetimes with sources [24]. The gravitational energy for these metrics has been studied in some detail in [25]; the following additional remarks are of some interest\(^{22} \). If \( \mathcal{F}_k = \mathcal{E}_k \), (3.12) can be written in the following form which is slightly more complicated but, as can be seen, it shows that \( E_G \) is manifestly negative:
\[ E_G = -\frac{1}{\kappa} \int f \left[ \frac{1}{4} \left( B + A \times \mathcal{E} \right)^2 + \frac{1}{4} (A \cdot \mathcal{E})^2 + f^{-1} \left( 1 - \frac{1}{4} A^2 \right) \mathcal{E}^2 \right] dV < 0. \] (4.7)

It is manifestly negative because, see (2.7),
\[ g^{00} = f^{-1} \left( 1 - f A^2 \right) = \frac{\det(g_{kl})}{g} > 0. \] (4.8)

Expression (4.7) gives a purely local gravitational energy density when expressed in spatial conformally flat coordinates.

It is interesting to rewrite the last expression in a form which is similar to the energy density of the electromagnetic field. With
\[ A^* = \frac{1}{2} f^{1/2} A, \quad B^* = \nabla \times A^* \quad \text{and} \quad \mathcal{E}^* = -\frac{1}{2} \nabla f^{1/2}, \] (4.9)

(4.7) has this form
\[ E_G = -\frac{1}{\kappa} \int \left[ B^*^2 + 4f^{-1} \left[ (A^* \cdot \mathcal{E}^*)^2 + (1 - A^*^2) \mathcal{E}^*^2 \right] \right] dV < 0. \] (4.10)

\(^{21} \) See [38] for a review of the subject and references to original works. Solutions belong to a class of Einstein–Maxwell equations. With \( \nabla_i A^i = 0 \), the metric is in harmonic coordinates, see [5].

\(^{22} \) Beware of the definition of \( B \) in [25] which is slightly different.
There is a striking resemblance between $E_G$ and the energy of the electromagnetic field $E_{EM}$. The electromagnetic energy momentum tensor is, in standard notation,

$$T_{EM}^{\mu\nu} = \frac{1}{4\pi} \left( F^{\mu\rho} F_{\nu}^{\rho} + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right)$$

where

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \quad (4.11)$$

and so

$$E_{EM} = \int \sqrt{-g} \left( T_{EM}^{\mu\nu} w_{\nu} \right)^{\mu} dS_{\mu} = \int \left( T_{0}^{0} \right)^{\mu}_{\mu} dV.$$  \hspace{1cm} (4.12)

The electric field components are $E_\kappa = F_{\kappa\nu} = \partial_\kappa A_\nu$ while those of the magnetic field are $B_\kappa = \eta^{\kappa\mu\nu} \partial_\mu A_\nu$. In those terms $E_{EM}$ takes the following form in our coordinates (2.3):

$$E_{EM} = \frac{1}{8\pi} \int \left[ B^2 + 4 f^{-1} \left[ (A^* \cdot E)^2 + \left( \frac{1}{4} - A^* \right)^2 E^2 \right] \right] dV > 0,$$  \hspace{1cm} (4.13)

where $B^2 = \gamma_{ij} B^i B^j$ etc. . . . Note that in (4.13) $A^*$ is the gravitational ‘vector potential’ defined in (4.9), not the electromagnetic vector potential $A$.

(iii) Axially symmetric spacetimes and zero angular momentum observers

In axial symmetry, the metric (2.36) takes this special form

$$ds^2 = \tilde{f}(dx^0)^2 - a \sum_{k=1,2} (dx^k)^2 - b(d\varphi - \omega dx^0)^2; \quad (4.14)$$

$\tilde{f}, a, b$ and $\omega = d\varphi/dx^0$, the angular velocity of the zero angular momentum observers as measured at infinity, are all functions of $x^K(K, L, M, \ldots = 1, 2)$. Under these conditions, the second fundamental form components are:

$$K_{LM} = K_{33} = 0 \quad \text{and} \quad K_{L3} = \frac{1}{2} \tilde{f}^{-1/2} b \partial_L \omega. \quad (4.15)$$

Consequently, the gauge condition $K = 0$ holds and $\tilde{\mathbf{F}}$ exists; using (3.29), we find\(^{23}\) that

$$\tilde{E}_G = -\frac{1}{\kappa} \int \left[ \tilde{F}^2 - \frac{1}{4} \tilde{f}^{-1} b \nabla \omega \cdot \nabla \omega \right] dV.$$  \hspace{1cm} (4.16)

A particular case which gives a simple expression for $\tilde{E}_G$ is when the $\tilde{f}$-metric is conformally flat, i.e. $b/R^2 = a$, and

$$ds^2 = \tilde{f}(dx^0)^2 - a[dR^2 + dz^2 + R^2(d\varphi - \omega dx^0)^2]. \quad (4.17)$$

This metric may be written in the following form:

$$ds^2 = f'(dx^0)^2 - A'R d\varphi^2 - a[dR^2 + dz^2 + (1 + a^{-1} f'A^2) R^2 d\varphi^2], \quad (4.18)$$

$$f' = f - a R^2 \omega^2 \quad \text{and} \quad A' = -f^{-1} a R \omega, \quad (4.19)$$

which makes it clear that it is not a conformastationary metric. To calculate the total gravitational energy we better use (3.30) with $\mathcal{L}$ rather than $\tilde{\mathcal{F}}$ and a ‘background’ metric

$$ds^2 = (dx^0)^2 - dR^2 - dz^2 - R^2 d\varphi^2. \quad (4.20)$$

The gravitational energy is

$$\bar{E}_G = -\frac{1}{2\kappa} \int \left( \nabla \log a \cdot \nabla \log a - \frac{1}{2} \tilde{f}^{-1} b \nabla \omega \cdot \nabla \omega \right) dV.$$  \hspace{1cm} (4.21)

\(^{23}\) Note that $\nabla \omega \cdot \nabla \omega = a^{-1} \sum_{k=1,2} (\partial_k \omega)^2$. 

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The ZAMOs see less kinetic energy since they are dragged in the direction in which the source rotates and, therefore, one expects gravitational energy to be less negative. It is surely the case for slowly rotating sources, and presumably the case for fast rotating ones.

One may use (3.30) to calculate the energy of an axially symmetric distribution of matter and, in the limit, of a Kerr black hole. An expression for quasilocal energy of a Kerr black hole, based on paper [8], has been given by Martinez [31]. There exists a great number of quasilocal expressions for energy which give different expressions for the energy of Kerr black holes [4].

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Appendix. Gravitational energy and conservation laws

A different derivation of (3.18), closely related to classical conservation laws in general relativity, was given in [25] and the result appears at first strikingly different from what is given here. The derivation of the original formula is very straightforward if one deals with stationary spacetimes.

We start from the old Freud24[15] complex $\hat{t}^{\mu \nu \lambda}$ antisymmetric in the two upper indices. In the notations25 of [25] where a background metric is used we have

$$\hat{t}^{\mu \nu \lambda} = \frac{1}{\kappa} \hat{g}^{\lambda \rho} D_\sigma (\hat{g}^{\rho \mu} \hat{g}^{\nu \sigma}).$$

A remarkable property of the tensor is this [34]: multiply by any vector field, say, $h^\lambda$; the divergence of the resulting anti-symmetric tensor density, a conserved or divergenceless vector field density, is of the following form:

$$\partial_\nu (\hat{t}^{\mu \nu \lambda} h^\lambda) = (\hat{T}^{\mu \nu} + \hat{t}^{\mu \nu \lambda} \hat{D}_\nu h^\lambda).$$

$T^{\mu \nu}$ is the matter tensor and $t^{\mu \nu}$ is Einstein’s energy ‘complex’ in Rosen’s [36] covariantized form. If $h^\lambda$ is replaced by the field of, for example, observers at rest (see (2.1)), the flux of $\hat{t}^{\mu \nu \lambda} w^\lambda$ through a sphere at infinity, as shown by Freud, is

$$\int \hat{t}^{\mu \nu \lambda} w^\lambda dS_{\mu \nu} = \int_{r \to \infty} \sqrt{-g} t^{\mu \nu \lambda} w^\lambda dS = Mc^2.$$

Since the volume integral of $\hat{T}^{\mu \nu} w^\nu$ is equal to the mechanical energy $E_M$, it follows that gravitational energy is now given by the following expression:

$$E_G = \int (\hat{t}^{\mu \nu \lambda} w^\nu + \hat{t}^{\mu \nu \lambda} \hat{D}_\nu w^\lambda) d\Sigma_w = \int (t^{\mu \nu \lambda} + t^{\mu \nu \lambda} \hat{D}_\nu h^\lambda) dV.$$

It takes some more calculations to show that, if the metric is of the form (2.3), this integral is indeed (3.18).

24 A contemporary accessible version on the web can be found in [34].

25 The notations are those of the present paper with some additions: multiplication by $\sqrt{-g}$ of a tensor like $g^{\mu \nu}$ is represented by a hat like $\hat{g}^{\mu \nu} = \sqrt{-\hat{g}} g^{\mu \nu}$ and, for a change, $\hat{g}^{\mu \nu} = g^{\mu \nu} / \sqrt{-g}$ while $\hat{D}$ stand for 4-covariant derivatives in the $\hat{g}$-background spacetime.
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