Effect of an arbitrary spin orientation on the quadrupolar structure of an extended body in a Schwarzschild spacetime

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The influence of an arbitrary spin orientation on the quadrupolar structure of an extended body moving in a Schwarzschild spacetime is investigated. The body dynamics is described according to the Mathisson-Papapetrou-Dixon model without any restriction on the motion or simplifying assumption on the associated spin vector and quadrupole tensor, generalizing previous works. The equations of motion are solved analytically in the limit of small values of the characteristic length scales associated with the spin and quadrupole variables with respect to the characteristic length of the background curvature. The solution provides all corrections to the circular geodesic on the equatorial plane taken as the reference trajectory due to both dipolar and quadrupolar structure of the body as well as the conditions which the nonvanishing components of the quadrupole tensor must fulfill in order that the problem be self-consistent.

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I. INTRODUCTION

An extended test body moving in a given gravitational background is commonly described according to the Mathisson-Papapetrou-Dixon (MPD) model [1–8]. The equations of motion involve a reference world line, properly set up to represent the center-of-mass line, and a number of vector and tensor fields defined along it through a multipole moment expansion, similarly to the standard nonrelativistic theory. The model is fully determined and self-consistent at the dipolar order, i.e., for an extended body endowed with spin only, providing a set of evolution equations for both the linear and angular momentum of the body. In contrast, there are no evolution equations for the quadrupole as well as higher multipoles, and their evolution is fixed entirely by the body’s internal dynamics. The contribution of higher multipoles appears in the form of additional force and torque terms entering the MPD equations that modify the evolution of both the linear and angular momentum of the body. Therefore, one has to supply the structure of the body as external information, e.g., specifying the equation of state of its matter-energy content. This fact represents a peculiarity of the model itself, which allows for many different approaches. The most natural and simplifying choice consists in considering the body as “quasi-rigid,” i.e., all unspecified quantities describing its shape are taken constant in the frame associated with the 4-momentum of the body itself [9]. Alternatively, one can assume the quadrupole tensor be directly related to the Riemann tensor, having the same symmetry properties. For instance, in Ref. [10] the electric and magnetic parts of the quadrupole tensor have been taken proportional to the electric and magnetic parts of the Riemann tensor, respectively, to study quadrupole deformation effects induced by the tidal field of a black hole on the motion of a spinning body. One can also require that the structure of the body be completely determined by its spin, with a quadratic-in-spin quadrupole tensor.

In the present paper we will relax any such restriction on the body dynamics and structure, allowing it to move off the equatorial plane with an arbitrary orientation of the spin vector. Following the MPD prescriptions, we only require that the body does not perturb significantly the background field, so that backreaction effects can be neglected and the body structure produces very small deviations from geodesic motion. This condition is indeed implicit in the MPD model, and allows to treat the equations of motion perturbatively, in the sense that the natural length scales associated with the body, i.e., the “bare” mass as well as the spin and quadrupole characteristic lengths are taken to be small enough if compared with the length scale associated with the background curvature. The resulting simplified set of differential equations can be integrated analytically. Initial conditions are chosen so that the world line of the extended body has the same starting point and is initially tangent to a timelike circular geodesic on the equatorial plane, taken as a reference world line. The presence of a spin component in the equatorial plane undergoing a simple rotation (which corresponds to parallel transport along the reference circular geodesic) induces an oscillation of the body path in and out of the equatorial plane to first order in spin (i.e., taking into account only corrections which are...
linear in spin). The orthogonal component is instead responsible for an oscillating behavior of the radial distance about the reference radius, while the azimuthal motion undergoes similar oscillations plus an additional secular drift \([22, 23]\). We will show that second order corrections to circular geodesic motion (i.e., spin squared and mass quadrupole terms) introduce secular effects in both radial and polar motions enhancing deviations.

We will then discuss the consequences of that general situation on the quadrupolar structure of the body. Its shape turns out to be varying with time, e.g., passing from nearly spherical to highly deformed configurations during the evolution. A variable mass quadrupole moment is usually generated in a binary system because of the tides produced by the higher mass. In such a situation the net gravitational radiation associated with the motion of the smaller mass is due to its orbit, the time varying tides and the interference between them \([21, 22]\). Furthermore, changes in the gravitational quadrupole moment of the companion star have been shown to be responsible for most of the observed variations in the orbital parameters of binary pulsar systems \([22, 27]\). For instance, recent radio timing observations of the eclipsing millisecond binary pulsar PSR J2051-0827 have provided evidences for variations of the quadrupole moment in its companion \([28–30]\). Such variations together with the spin precession of the companion star have been shown to be responsible for the changes of the orbital period, inclination angle and projected semimajor axis of the binary system. Although the underlying mechanism causing a varying quadrupole moment is most likely of non-gravitational nature (e.g., driven by the magnetic activity in close binaries \([28, 31]\)), the purely gravitational effect discussed here may play a role.

We will follows notations and conventions of Ref. \([32]\). Units are chosen so that \(G = 1 = c\) and the metric signature is \(- + + +\). Greek indices run from 0 to 3, whereas Latin indices from 1 to 3.

II. MPD EQUATIONS IN THE QUADRUPOLE APPROXIMATION

Consider an extended body endowed with structure up to the quadrupole. The MPD equations are

\[
\begin{align*}
\frac{DP^\mu}{d\tau} &= -\frac{1}{2} R^\mu_{\nu\alpha\beta} U^\nu S^{\alpha\beta} - \frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla^\nu R_{\alpha\beta\gamma\delta} \\
&= F^\mu_{\text{(spin)}} + F^\mu_{\text{(quad)}}, \\
\frac{DS^{\mu\nu}}{d\tau} &= 2 P^\mu U^\nu + \frac{4}{3} J^{\alpha\beta\gamma\delta} R^\mu_{\gamma\alpha\beta} \\
&= D^{\mu\nu}_{\text{(spin)}} + D^{\mu\nu}_{\text{(quad)}},
\end{align*}
\]

(2.1)

(2.2)

where \(P^\mu = m u^\mu\) (with \(u \cdot u = -1\)) is the total 4-momentum of the body with mass \(m\), \(S^{\mu\nu}\) is a (antisymmetric) spin tensor, \(J^{\alpha\beta\gamma\delta}\) is the quadrupole tensor, and \(U^\mu = d x^\mu / d\tau\) is the timelike unit tangent vector of the “center of mass line” (with parametric equations \(x^\mu = z^\mu(\tau)\) used to make the multipole reduction, parametrized by the proper time \(\tau\).

In order the model to be mathematically self-consistent the following additional conditions should be imposed \([3, 4]\)

\[
S^{\mu\nu} u_\nu = 0.
\]

(2.3)

Consequently, the spin tensor can be fully represented by a spatial vector (with respect to \(u\)),

\[
S(u)^\alpha = \frac{1}{2} \eta(u)^{\alpha\beta\gamma} S^{\beta\gamma},
\]

(2.4)

where \(\eta(u)^{\alpha\beta\gamma} = \eta_{\mu\alpha\beta\gamma} u^\mu\) is the spatial (with respect to \(u\)) unit volume 3-form with \(\eta_{\alpha\beta\gamma\delta} = \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta}\) the unit volume 4-form and \(\epsilon_{\alpha\beta\gamma\delta} (\epsilon_{0123} = 1)\) the Levi-Civita alternating symbol. It is also useful to introduce the signed magnitude \(s\) of the spin vector

\[
s^2 = S(u)^\beta S(u)_\beta = \frac{1}{2} S_{\mu\nu} S^{\mu\nu},
\]

(2.5)

which is in general not constant along the trajectory of the extended body.

The quadrupole tensor \(J^{\alpha\beta\gamma\delta}\) has the same algebraic symmetries as the Riemann tensor, but enters the MPD equations only through certain combinations, which reduce the number of effective components from 20 to 10 \([23, 33, 34]\). Therefore, it can be written in the form

\[
J^{\alpha\beta\gamma\delta} = 4 u^\mu [X(u)]^{\text{STF}} |_{\gamma\delta} - 2 u^\mu [W(u)]^{\text{STF}} |_{\gamma\delta} - 2 u^\mu [W(u)]^{\text{STF}} |_{\gamma\delta} + 2 u^\mu [W(u)]^{\text{STF}} |_{\gamma\delta} \eta(u)^{\alpha\beta},
\]

(2.6)

where \(X(u)\) and \(W(u)\) are symmetric and trace-free (STF) spatial tensors as measured by an observer co-moving with the body, representing the mass quadrupole moment and the flow (or current) quadrupole moment, respectively (see, e.g., Ref. \([3]\)).

A. Perturbative approach

Consider a pair of world lines emanating from a common spacetime point, one a geodesic with 4-velocity \(U_{\text{(geo)}}\), the other the world line of an extended body deviating from the reference one because of the combined effects of both the spin-curvature and quadrupole-curvature couplings, with 4-velocity \(U\). Introduce a smallness indicator \(\epsilon \ll 1\) to distinguish between the order of multipolar approximation, so that \(S^{\mu\nu} = O(\epsilon)\) and \(J^{\alpha\beta\gamma\delta} = O(\epsilon^2)\). Solutions to the MPD equations can then be found in the general form

\[
\begin{align*}
x^\alpha &= x^\alpha_{\text{(geo)}} + \epsilon x^\alpha_{(1)} + \epsilon^2 x^\alpha_{(2)}, \\
U &= U_{\text{(geo)}} + \epsilon U_{(1)} + \epsilon^2 U_{(2)}.
\end{align*}
\]

(2.7)
The mass \( m = (\mathcal{P} P_{\mu})^{1/2} \) of the body is a conserved quantity to first order and the 4-momentum vector \( P \) is parallel to the 4-velocity \( U \), so that one can assume

\[
m = m_0 + \epsilon^2 m_{(2)} ,
\]
\[
u = U_{(geo)} + \epsilon U_{(1)} + \epsilon^2 U_{(2)} ,
\]
where \( m_0 \) denotes the “bare” mass. Substituting then into the MPD equations (2.11) and (2.12) leads to two different sets of evolution equations for the first order and second order quantities, respectively, neglecting terms of higher order.

It is worth noting that \( U_{(geo)} \) and \( U \) are unit tangent vectors to different timelike world lines, which are parametrized by different proper times: hence, one should use \( \tau_{(geo)} \) as the proper time parameter along \( U_{(geo)} \) and \( \tau \) as the proper time parameter along \( U \). However, recalling the definitions of the proper time parameter along these world lines

\[
d\tau = -U_\alpha dx^\alpha ,
\]
\[
d\tau_{(geo)} = -U_{(geo)\alpha} dx^\alpha_{(geo)}
\]
and using the normalization condition \( U \cdot U = -1 \), one obtains that \( \tau \) and \( \tau_{(geo)} \) can be identified to the second order of approximation, i.e.,

\[
\tau = \tau_{(geo)} + O(\epsilon^2) .
\]

Therefore, although the two world lines are parametrized by different proper times, the latter are synchronized so that \( \tau \) can be used unambiguously for that single proper time parametrization of both world lines.

We are interested in solutions which describe deviations from geodesic motion due to both the spin-curvature force and the quadrupolar force. Hence, we will choose initial conditions so that the world line of the extended body has the same starting point as the reference geodesic, i.e.,

\[
x^{(2)}_{(1)}(0) = 0 = x^{(2)}_{(2)}(0) .
\]

The two world lines in general have not a common unit tangent vector at \( \tau_{(geo)} = 0 = \tau \): as \( \tau \) increases, then, they deviate from each other. We will require that the 4-velocity \( U \) is initially tangent to the geodesic 4-velocity \( U_{(geo)} \), which implies in addition

\[
\frac{dx^{(2)}_{(1)}}{d\tau}(0) = 0 = \frac{dx^{(2)}_{(2)}}{d\tau}(0) .
\]

III. DYNAMICS OF EXTENDED BODIES IN A SCHWARZSCHILD SPACETIME

Consider the Schwarzschild spacetime, with line element written in standard form as

\[
ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,
\]
where \( N \) denotes the lapse function

\[
N = \sqrt{1 - \frac{2M}{r}} .
\]

An orthonormal frame adapted to the static observers following the coordinate time lines with 4-velocity \( n = N^{-1} \partial_t \) is given by

\[
e_\ell = n , \quad e_\phi = N \partial_r ,
\]
\[
e_\theta = \frac{1}{r} \partial_\theta , \quad e_\phi = \frac{1}{r \sin \theta} \partial_\phi ,
\]
with dual \( \omega^\ell = Nd\tau , \omega^\phi = N^{-1} dr , \omega^\theta = r d\theta \) and \( \omega^\phi = r \sin \theta d\phi \).

Let the reference world line be a circular geodesic in the equatorial plane at radius \( r = r_0 \). The associated 4-velocity \( U_{(geo)} = \hat{U}_K \) is

\[
U_K = \gamma_K (n \pm \nu_K e_\phi) = \Gamma_K (\partial_t \pm \zeta_K \partial_\phi) ,
\]
where the \( \pm \) signs refer to co-rotating (+) and counter-rotating (−) motion with respect to increasing values of the azimuthal coordinate, respectively. Here \( \zeta_K \) and \( \nu_K \) (with associated Lorentz factor \( \gamma_K = (1 - \nu_K^2)^{-1/2} \)) denote the Keplerian angular velocity and linear velocity, respectively, and \( \Gamma_K \) is a normalization factor defined by

\[
\zeta_K = \sqrt{\frac{M}{r_0^3}} , \quad \nu_K = \frac{M}{r_0 - 2M} ,
\]
\[
\gamma_K = \sqrt{\frac{r_0 - 2M}{r_0 - 3M}} , \quad \Gamma_K = \frac{\gamma_K \nu_K}{r_0 \zeta_K} .
\]

The circular geodesic is thus described by the parametric equations

\[
t_{(geo)} = t_0 + \Gamma_K \tau , \quad r_{(geo)} = r_0 ,
\]
\[
\theta_{(geo)} = \frac{\pi}{2} , \quad \phi_{(geo)} = \phi_0 \pm \Gamma_K \zeta_K \tau .
\]

It is useful to introduce the unit vector \( \hat{U}_K \) along the azimuthal direction in the local rest space of the circular geodesic, orthogonal to \( U_K \) in the \( t-\phi \) plane, i.e.,

\[
\hat{U}_K = \gamma_K (\pm e_\phi + \nu_K e_\ell) .
\]

An orthonormal frame adapted to \( U_K \equiv E_0 \) is thus given by

\[
E_1 = e_\phi , \quad E_2 = \hat{U}_K , \quad E_3 = -e_\theta ,
\]
with \( E_3 \) aligned with the (positive) \( z \)-axis of a naturally defined Cartesian frame.

A. First order solution

To first order the set of MPD equations (2.11) and (2.12) reduces to

\[
m \frac{DU_{(1)}}{d\tau} = F^{(spin)} + O(\epsilon^2) ,
\]
\[
\frac{DS_{\mu\nu}}{d\tau} = O(\epsilon^2) .
\]
The spin vector must be orthogonal to $U$ due to the supplementary conditions \( [23] \), so that
\[ S = S^t e_{t} + S^\theta e_{\theta} \pm \gamma_{K}^{-1} S^\phi \hat{U}_{K}, \] (3.10)
and turns out to be parallel transported along the reference circular geodesic due to the spin evolution equations, which leads to a simple rotation of the spin components in the $r$-$\phi$ plane within the local rest space of the circular geodesics. The corresponding solution can then be written as
\[ S = s_{\parallel} \cos \alpha e_{t} \pm \sin \alpha \hat{U}_{K} - s_{\perp} e_{\theta}, \] (3.11)
where a polar representation for the spin vector has been conveniently introduced such that $S^t = s_{\parallel} \cos \alpha$, $S^\theta = \gamma_{K} s_{\parallel} \sin \alpha$, and $S^\phi = -s_{\perp}$, with
\[ \alpha(\tau) = \alpha_{0} \mp \gamma_{K} \tau. \] (3.12)
The quantities $s_{\parallel} = [S^t(0)^2 + S^\theta(0)^2/\gamma_{K}^2]^{1/2}$ and $s_{\perp}$ are constant due to the conservation of the spin magnitude.

The solution for the orbit is then given by \( [33] \) adapted to the static observers, the latter writes as $U(1) = U(1)^{\parallel} e_{\parallel}$, leading to the following relations between frame and coordinate components
\[ U^{\parallel}(1) = \pm \nu_{K} \Sigma_{\perp} \left[ \frac{\Omega_{(ep)}}{\zeta_{K}} \sin \Omega_{(ep)} \tau e_{\phi} \right. \]
\[ +2 \left( \cos \Omega_{(ep)} \tau - 1 \right) \hat{U}_{(geo)} \sin \alpha_{0} \cos \Omega_{(orb)} \tau \]
\[ \left. \mp \Gamma_{K} \cos \alpha_{0} \sin \Omega_{(orb)} \tau - \sin \alpha \right] e_{\theta}. \] (3.18)
When decomposed with respect to the frame \( [33] \) and three compatibility conditions involving spin vector components, first order corrections to the orbital frequency governing the geodesic oscillations out of the equatorial plane. The latter frequency together with the spin-precession frequency due to the spin oscillation driving term governs the polar angle oscillations about the equatorial plane. Their ratio
\[ \Omega_{(orb)} / \Omega_{(ep)} = \left(1 - \frac{6M}{r_{0}}\right)^{-1/2} \] (3.17)
will enter most of the relations below, also implying the allowed range for radial distance $r_{0} \geq 6M$.

The solution for $U$ (which at $\tau = 0$ is aligned with the circular geodesic at $r_{0}$) is then given by $U = U_{K} + U(1)$, with
\[ U^{\parallel}(1) = \pm \nu_{K} \Sigma_{\perp} \left[ \frac{\Omega_{(ep)}}{\zeta_{K}} \sin \Omega_{(ep)} \tau e_{\phi} \right. \]
\[ +2 \left( \cos \Omega_{(ep)} \tau - 1 \right) \hat{U}_{(geo)} \sin \alpha_{0} \cos \Omega_{(orb)} \tau \]
\[ \left. \mp \Gamma_{K} \cos \alpha_{0} \sin \Omega_{(orb)} \tau - \sin \alpha \right] e_{\theta}. \] (3.19)

The evolution of the spin vector is completely determined by the first order equations. Therefore, the spin evolution equations \( [22] \) to second order simply provide three algebraic relations between the components of $u(2)$ and $U(2)$ and three compatibility conditions involving spin vector components, first order corrections to the orbit $x^{\parallel}_{(1)}$ and components of the quadrupole tensor. In fact,
\[ D_{(spin)}^{\mu\nu} = 2e^{2} m_{0} (u(2) - U(2))^{\mu\nu} U_{K}^{\nu} + O(e^{3}), \] (3.20)
and $D_{(quad)}^{\mu\nu} = O(e^{3})$. We find
\[ (u_2 - U_2)^{\hat{t}} = \frac{3}{\nu_K (M \Omega_{(orb)})^2} \left[ (\hat{S}^t)^2 - \sigma_+^2 - \frac{4}{3} \left( 2\hat{X}_{11} + \hat{X}_{22} \pm \frac{2}{N} \frac{1 - r_0^2 \zeta_{K}}{r_0 \Omega_{(orb)}} W_{13} \right) \right] = \pm \nu_K (u_2 - U_2)^{\phi}, \]

\[ (u_2 - U_2)^{\vec{r}} = \mp (M \zeta_K) \left[ 3 (M \zeta_K) (r_0 \Omega_{(orb)}) \hat{S}^r \hat{S}^\phi + U^{\vec{r}}_{(1)} \hat{S}^\phi \pm \frac{r_0 \Omega_{(orb)}}{N} \hat{S}^\phi \theta_{(1)} \right], \]

\[ (u_2 - U_2)^{\phi} = -4 (M \zeta_K)^2 \left[ 2\hat{W}_{12} \mp \nu_K \hat{X}_{23} \pm \frac{3}{4} (r_0 \Omega_{(orb)}) \hat{S}^\phi \sigma_\perp \right. \]

\[ \pm \frac{(M \zeta_K) \nu_K^2}{(r_0 \Omega_{(orb)})^2} \left[ (r_0 \Omega_{(orb)})^2 \hat{S}^\phi U^{\vec{r}}_{(1)} + \sigma_\perp U^{\vec{r}}_{(1)} - \hat{S}^r U^{\phi}_{(1)} \right] \]

\[ \left. + \frac{M}{r_0 N^2 r_0 \Omega_{(orb)}} \left[ (r_0 \Omega_{(orb)})^2 (3r_0^2 \zeta_{K}^2 - 2) \right] \hat{S}^\phi \theta_{(1)} \right], \]

where \( \hat{S}^\phi \equiv S^\phi / (m_0 M) \) and

\[ \hat{X}_{ab} = \frac{X(u)_{ab}}{m_0 M^2}, \quad \hat{W}_{ab} = \frac{W(u)_{ab}}{m_0 M^2} \]

are dimensionless quadrupole quantities obtained by suitably rescaling the frame components of the electric and magnetic parts of the quadrupole tensors with respect to the frame (8.8) adapted to the circular geodesics. They have to satisfy the following compatibility conditions

\[ \hat{X}_{12} \pm 2 \nu_K \hat{W}_{23} = \frac{\pm \nu_K}{(M \zeta_K)^2 (r_0 \Omega_{(orb)})^2} \left[ (u_2 - U_2)^{\hat{t}} \pm 3 (M \zeta_K)^2 (r_0 \Omega_{(orb)}) \hat{S}^r \hat{S}^\phi \right], \]

\[ 2 (2\hat{W}_{11} + \hat{W}_{22}) \pm \nu_K \left( 1 - \frac{r_0}{M} \right) \hat{X}_{13} = \frac{3}{2} \nu_K \hat{S}^r \hat{S}_\perp - \frac{1}{4N^2 (r_0 \Omega_{(orb)})^2} \left[ \frac{1}{N^2 (r_0 \Omega_{(orb)})^2} \hat{S}^r U^{\vec{r}}_{(1)} \right] \]

\[ - \frac{r_0}{M} \hat{S}_\perp U^{\vec{r}}_{(1)} \mp \frac{N^2 \nu_K}{(M \zeta_K)(r_0 \Omega_{(orb)})} \hat{S}^r U^{\phi}_{(1)} \right], \]

\[ 2\hat{W}_{12} \mp \nu_K \hat{X}_{23} = -\frac{3 \nu_K^2}{(M \zeta_K)^2 (r_0 \Omega_{(orb)})^2} \left[ (u_2 - U_2)^{\phi} \pm 3 (M \zeta_K)^2 (r_0 \Omega_{(orb)}) \hat{S}^\phi \sigma_\perp \right]. \]

The equations of motion (22) give instead the evolution equations for the second order corrections to the mass of the body \( m_2 \) as well as to the orbit \( x^a(2) \). We find

\[ \frac{r_0 \frac{dm_2}{dt}}{m_0} = \mp (M \zeta_K) \left[ N \hat{S}^r U^{\vec{r}}_{(1)} \pm (r_0 \Omega_{(orb)}) \hat{S}^\phi \theta_{(1)} \right], \]

\[ \frac{r_0 \frac{d^2 r_{(2)}}{dt^2}}{m_0} = 3N^2 (r_0 \Omega_{(orb)})^2 \frac{r_{(2)}}{r_0} \pm \frac{2M}{r_0 \Omega_{(orb)}} \frac{dr_{(2)}}{dt} \pm 3 (r_0 \zeta_K)^2 \Gamma_K \left[ 3 U^{\vec{r}}_{(1)} - (M \zeta_K) (r_0 \Omega_{(orb)}) \left( 3 \frac{r_0}{M} - 7 \right) \sigma_\perp \right] \frac{r_{(1)}}{r_0} \]

\[ - 2 \left( 2 - 3 \frac{M}{r_0} \right) (r_0 \Omega_{(orb)})^2 \frac{\tau_{(1)}}{r_0^2} - \frac{M}{r_0} \theta_{(1)} + \frac{1}{3 \Gamma_K} \left[ (U^{\vec{r}}_{(1)})^2 + (U^{\phi}_{(1)})^2 \right] \]

\[ + 3 (M \zeta_K)(r_0 \Omega_{(orb)}) \left[ \left( 1 - \frac{M}{r_0} \right) \left( U^{\vec{r}}_{(1)} \sigma_\perp \right) + N^2 U^{\phi}_{(1)} \hat{S}^\phi \right] + K, \]

\[ \frac{\frac{d^2 \theta_{(2)}}{dt^2}}{m_0} = - (r_0 \Omega_{(orb)})^2 \theta_{(2)} + \left[ 2 \theta_{(1)} \pm \nu_K \left( 1 + \frac{6M}{r_0} - \frac{15M^2}{r_0^2} \right) \right] \frac{\tau_{(1)}}{r_0} \]

\[ \mp \theta_{(1)} \left[ (r_0 \Omega_{(orb)}) U^{\vec{r}}_{(1)} + (M \zeta_K) \sigma_\perp \right] + 6N (r_0 \Omega_{(orb)})^2 (M \zeta_K)^2 \hat{S}^r \sigma_\perp \]

\[ - \nu_K (r_0 \Omega_{(orb)}) \left[ 1 - \frac{2M}{r_0} + \frac{3M^2}{r_0^2} \right] U^{\phi}_{(1)} \hat{S}^\phi - 2N \left( U^{\vec{r}}_{(1)} - \frac{3}{2} \nu_K (r_0 \Omega_{(orb)}) (M \zeta_K) \hat{S}^\phi \right) U^{\phi}_{(1)}, \]

\[ \frac{\frac{d^2 \phi_{(2)}}{dt^2}}{m_0} = -2 (r_0 \Omega_{(orb)}) \frac{dr_{(2)}}{dt} - 2NU^{\vec{r}}_{(1)} \frac{r_{(2)}}{r_0} \pm (r_0 \Omega_{(orb)}) \theta_{(1)} \left[ 2 U^{\vec{r}}_{(1)} + N^2 (r_0 \zeta_K)^2 \Gamma_K \hat{S}^r \right] \]

\[ + 3 \left[ 2 \frac{M}{r_0} + \frac{M^2}{r_0^2} \right] \frac{\tau_{(1)}}{r_0} U^{\phi}_{(1)}, \]

\[ (3.24) \]
with

\[
K = 6N^2(r_0\zeta_K)^6\Gamma_K^2 \left[ \pm 4N(r_0\zeta_K)\hat{W}_{13} + \left( 1 - \frac{M}{r_0} \right) \hat{X}_{11} + \frac{M}{r_0} \hat{X}_{22} \right],
\tag{3.25}
\]

and

\[
\frac{dt_{(2)}}{d\tau} = \pm \frac{\nu_K^2}{\zeta_K} \frac{d\phi_{(2)}}{d\tau} + \frac{1}{N^2} \frac{r_{(1)}}{r_0} \left[ \frac{3}{2} \left( M\zeta_K \right) \left( r_0\Omega_{(orb)} \right) \frac{r_{(1)}}{r_0} \pm \frac{N\nu_K}{\Gamma_K} U_{(1)}^\phi \right] - \frac{\Gamma_K}{2N^2} (r_0\zeta_K)^2 \theta_{(1)}^2 \\
+ \frac{1}{2N^2\Gamma_K} \left[ (U^r_{(1)})^2 + (U^\theta_{(1)})^2 + \frac{\nu_K^2}{r_0\Omega_{(orb)}} (U_{(1)}^\phi)^2 \right],
\tag{3.26}
\]

from the normalization condition, where the compatibility conditions \(3.28\) have been taken into account.

The equations for \(m_{(2)}\) and \(\theta_{(2)}\) can be integrated straightforwardly. The equations for \(r_{(2)}\) and \(\phi_{(2)}\) are instead coupled. However, taking the derivative of the equation for \(r_{(2)}\) with respect to \(\tau\) and using the equation for \(\phi_{(2)}\) leads to an equation for \(U^r_{(2)}\), which can be easily integrated with initial conditions

\[
U^r_{(2)}(0) = 0, \quad r_0 \frac{dU^r_{(2)}}{d\tau}(0) = K(0) \equiv K_0,
\tag{3.27}
\]

leading to the general solution

\[
U^r_{(2)} = \Omega_{(ep)} (A_0\Omega_{(ep)} \tau - A_1) \cos \Omega_{(ep)} \tau + \Omega_{(ep)} (A_0 - A_2) \sin \Omega_{(ep)} \tau - 2\Omega_{(ep)} A_3 \sin 2\Omega_{(ep)} \tau \pm 2\zeta_K A_4 \sin 2\alpha \\
+ \Omega_{( orb)} \left[ \frac{1}{\Gamma_K} A_5 - A_6 \right] \cos \alpha \sin \Omega_{( orb)} \tau \mp \frac{1}{\Gamma_K} \sin \alpha \cos \Omega_{( orb)} \tau \sin \alpha \\
+ \Omega_{( orb)} \left[ A_5 \pm \frac{1}{\Gamma_K} A_6 \right] \cos \alpha \cos \Omega_{( orb)} \tau \pm \frac{1}{\Gamma_K} \sin \alpha \sin \Omega_{( orb)} \tau \cos \alpha \\
+ \frac{1}{r_0\Omega_{(ep)}} \sin \Omega_{(ep)} \tau \int_0^{\Omega_{(ep)} \tau} \frac{dK}{d\xi} \cos \xi d\xi - \cos \Omega_{(ep)} \tau \int_0^{\Omega_{(ep)} \tau} \frac{dK}{d\xi} \sin \xi d\xi.
\tag{3.28}
\]

In order to obtain an explicit solution, we will assume that the frame components \(\hat{W}_{13}, \hat{X}_{11}\) and \(\hat{X}_{22}\) of the quadrupole tensor, or at least their combination \(3.28\), are constant, implying that \(K = K_0 = \text{const}\). Otherwise, the integrals in the last line of Eq. \(3.28\) remain unspecified and do not allow for a closed form solution. In any case, the solution with \(K = K_0 = \text{const}\) is general enough to capture the main features of the dynamics. The solution is then given by

\[
t_{(2)} = \pm \frac{\nu_K^2}{\zeta_K} \phi_{(2)} + D_1 \sin \Omega_{(ep)} \tau + D_2 \sin 2 \Omega_{(ep)} \tau + D_3 \sin \Omega_{(orb)} \tau + D_4 \sin 2 \Omega_{(orb)} \tau + D_5 (\cos 2 \Omega_{(orb)} \tau - 1) \\
+ D_6 \left[ \cos \alpha \cos \sin \Omega_{(orb)} \tau \mp \frac{1}{\Gamma_K} \sin \alpha \cos \Omega_{(orb)} \tau - \cos \alpha \right] + D_7 (\sin 2 \alpha - \sin 2 \alpha_0),
\]

\[
r_{(2)} = \sin \Omega_{(ep)} \tau (A_0 \Omega_{(ep)} \tau - A_1) + A_2 (\cos \Omega_{(ep)} \tau - 1) + A_3 (\cos 2 \Omega_{(ep)} \tau - 1) + A_4 (\cos 2 \alpha - \cos 2 \alpha_0) \\
+ A_5 \cos \alpha \sin \sin \Omega_{(orb)} \tau \mp \frac{1}{\Gamma_K} \sin \alpha \cos \Omega_{(orb)} \tau - \sin \alpha \\
+ A_6 \cos \alpha \cos \cos \Omega_{(orb)} \tau - \cos \alpha \alpha_0 \pm \frac{1}{\Gamma_K} \sin \alpha \cos \Omega_{(orb)} \tau,
\]

\[
\theta_{(2)} = B_1 (\cos \alpha \cos \Omega_{(ep)} \tau - \cos \alpha_0) + B_2 (\cos \alpha - \cos \alpha_0) + B_3 \sin \Omega_{(orb)} \tau + B_4 \sin \Omega_{(ep)} \tau \\
+ \left[ \cos \alpha \sin \Omega_{(orb)} \tau \mp \frac{1}{\Gamma_K} \sin \alpha \cos \Omega_{(orb)} \tau \right] (B_5 \sin \Omega_{(ep)} \tau + B_6 \Omega_{(ep)} \tau) + B_7 (\cos \Omega_{(orb)} \tau - 1),
\]

\[
\phi_{(2)} = C_0 (\cos \Omega_{(ep)} \tau - 1) + C_1 \sin \Omega_{(ep)} \tau + C_2 \sin 2 \Omega_{(ep)} \tau + (C_3 \cos \Omega_{(ep)} \tau + C_4 \Omega_{(ep)} \tau + C_5 \sin 2 \Omega_{(ep)} \tau \\
+ C_6 (\cos 2 \Omega_{(orb)} \tau - 1) + C_7 \left[ \cos \alpha \sin \cos \Omega_{(orb)} \tau - \sin \alpha \right] \pm \frac{1}{\Gamma_K} \sin \alpha \sin \sin \Omega_{(orb)} \tau \\
+ C_8 \cos \alpha \cos \sin \Omega_{(orb)} \tau \mp \frac{1}{\Gamma_K} \sin \alpha \cos \Omega_{(orb)} \tau - \cos \alpha \alpha_0 \right] + C_9 (\sin 2 \alpha - \sin 2 \alpha_0),
\tag{3.29}
\]
and
\[ m_{(2)} = F_1 \left[ \cos \alpha \sin \alpha \sin \Omega_{(\text{orb})}\tau \mp \frac{1}{\Gamma K} \sin \alpha \left( \sin \alpha \cos \Omega_{(\text{orb})}\tau - \sin \alpha_0 \right) \right] + F_2 (\cos 2\alpha - \cos 2\alpha_0) . \] (3.30)

The explicit expressions for the coefficients are listed in Appendix A. The terms involving the coefficients \( A_0 \) and \( B_6 \) are responsible for secular effects in the radial and polar motion respectively, which may lead to observable effects.

Noticing, in the special case \( \sigma_\perp = 0 \) corresponding to a spin vector \( S = -s_1 e_\theta \) orthogonal to the equatorial plane, the above solution reduces to
\[ t_{(2)} = \pm \frac{\nu_0}{\xi K} \phi_{(2)} + D_1 \sin \Omega_{(ep)}\tau + D_2 \sin 2\Omega_{(ep)}\tau + D_3 \Omega_{(ep)}\tau , \]
\[ r_{(2)} = \sin \Omega_{(ep)}\tau (A_0 \Omega_{(ep)}\tau - A_1) + A_2 (\cos \Omega_{(ep)}\tau - 1) + A_3 (\cos 2\Omega_{(ep)}\tau - 1) , \]
\[ \phi_{(2)} = C_1 \sin \Omega_{(ep)}\tau + C_2 \sin 2\Omega_{(ep)}\tau + (C_3 \cos \Omega_{(ep)}\tau + C_4) \Omega_{(ep)}\tau , \] (3.31)
and \( \theta_{(2)} = 0, m_{(2)} = 0 \), with the limiting expressions for the nonvanishing coefficients easily follow from Eqs. (A1), (A3) and (A4). The motion is confined to the equatorial plane, since \( \theta_{(1)} = 0 \) too. This particular solution reproduces the results of Refs. [23, 33].

In the case \( \sigma_\perp = 0 \) corresponding to a spin vector oscillating in the equatorial plane, we have, instead, \( A_0 = 0 = A_3, B_1 = 0, C_2 = 0 = C_3, D_1 = 0 = D_2 \), implying no secular increase of \( r_{(2)} \) during the evolution and \( \theta_{(2)} = 0 \). Therefore, the motion is oscillating about the reference circular geodesic along both radial and polar directions, due to second order and first order corrections respectively.

One can compute the variation of the radial distance and polar angle after a full revolution, i.e., at the proper time value \( \tau = \tau_* \) such that \( \phi(\tau_*) = 2\pi \). We find
\[ \Delta_r = \frac{r(\tau_*) - r_0}{r_0} = \mp \epsilon \Sigma_\perp (\cos \xi_0 - 1) + \epsilon^2 \left\{ -2\Sigma_\perp^2 \sin \xi_0 (\sin \xi_0 - \xi_0) + \frac{1}{r_0} \left[ A_6 \cos \alpha_0 (\cos \beta_0 - \cos \alpha_0) \right. \right. \]
\[ + \sin \xi_0 (A_0 \xi_0 - A_1) \mp \frac{A_5}{\Gamma K} \sin \alpha_0 (\sin \beta_0 - \sin \alpha_0) + A_2 (\cos \xi_0 - 1) + A_3 (\cos 2\xi_0 - 1) \]
\[ + A_4 (\cos 2\beta_0 - \cos 2\alpha_0) \left\} , \right. \]
\[ \Delta_\theta = \frac{\theta(\tau_*) - \frac{\pi}{2}}{\frac{\pi}{2}} = \mp \epsilon \Sigma_\parallel (\cos \beta_0 - \cos \alpha_0) + \epsilon^2 \frac{\pi}{2} \left\{ \pm 2\Sigma_\parallel (\sin \xi_0 - \xi_0) \mp \frac{2\pi}{\Gamma K \xi_0} (\sin \alpha_0 - \sin \beta_0) \right. \]
\[ + B_1 (\cos \beta_0 \cos \xi_0 - \cos \alpha_0) \mp \frac{1}{\Gamma K} \sin \alpha_0 (B_2 \sin \xi_0 + B_6 \xi_0) + B_2 (\cos \beta_0 - \cos \alpha_0) \]
\[ + B_0 (\sin \beta_0 - \sin \xi_0) \right\} , \] (3.32)

where
\[ \xi_0 = \pm 2\pi \frac{\Omega_{(ep)}}{\Omega_{(\text{orb})}}, \quad \beta_0 = \alpha_0 - \frac{2\pi}{\Gamma K} . \] (3.33)

When the frequencies \( \Omega_{(ep)} \) and \( \Omega_{(\text{orb})} \) are rationally dependent, the variations \( \Delta_r \) and \( \Delta_\theta \) get simpler expressions, since \( \sin \xi_0 = 0 \) and \( \cos \xi_0 = 1 \). Figure 1 shows a typical behavior of both radial and polar variations for selected values of spin and quadrupole parameters. They are both monotonically decreasing at large distances, but \( \Delta_r \) decreases faster than \( \Delta_\theta \). Deviations from the geodesic radius are instead dominant at close distances.

C. Quasi-Keplerian parametrization of the orbit

It is useful to introduce a Keplerian-like parametrization for the \( r-\phi \) motion [36, 39], i.e.,
\[ \frac{2\pi}{P} (t - t_0) = \ell_\tau - e_\tau \sin \ell_\tau , \]
\[ r = a_r (1 - e_r \cos \ell_\tau) , \]
\[ \frac{2\pi}{\Phi (\phi - \phi_0)} = 2 \arctan \left( \sqrt{\frac{1 + e_\phi}{1 - e_\phi}} \tan \frac{\ell_\phi}{2} \right) . \] (3.34)

The quantities \( e_\tau, e_r, \) and \( e_\phi \) are three eccentricities, while \( P \) and \( \Phi \) denote the periods of \( t \) and \( \phi \) motions, respectively (with an abuse of notation for \( P \), not to be confused with the body’s 4-momentum). The quantities \( \ell_\tau, \ell_\tau, \) and \( \ell_\phi \) are functions of the proper time parameter \( \tau \) on the orbit. We find
\[ \ell_\tau = \ell + e_\tau^2 \ell_{(2)}^{(2)} , \]
where $\ell$ is related to observable effects. Indeed, one can measure either the orbital period and the fractional periastron advance \cite{36} defined by

$$ k \equiv \frac{\Phi}{2\pi} - 1. $$

In the weak field limit $u_0 \equiv M/r_0 \ll 1$ the above expressions become

$$ e_t^{\text{wf}} = \mp 6\sigma_\perp u_0^{5/2}(1 + 6u_0)\epsilon - u_0^3 \left[ \frac{2}{3} (5 \cos 2\alpha_0 - 3)\sigma_\parallel^2 + 12 \bar{X}_{11} + 48 \bar{W}_{13} u_0^{1/2} + 12 \lambda_9^2 u_0 \right] \epsilon^2 + O(u_0^4), $$

$$ e_r^{\text{wf}} = \pm 3\sigma_\perp u_0^{3/2}(1 + 4u_0 + 24u_0^2)\epsilon + u_0^2 \left[ \frac{1}{3} (5 \cos 2\alpha_0 - 3)\sigma_\parallel^2 + 6 \bar{X}_{11} + 24 \bar{W}_{13} u_0^{1/2}(1 + 3u_0) + 18 \bar{X}_{11} + 6 \bar{X}_{22} - \frac{1}{6} (5 \cos 2\alpha_0 + 3)\sigma_\parallel^2 - 9 \sigma_\perp^2 \right] u_0 \epsilon^2 + O(u_0^4), $$

$$ e^{\text{wf}} = 2e_r^{\text{wf}} + 18\epsilon_0^3 u_0^2 \sigma_\perp^2 + O(u_0^4), $$

$$ \frac{e^{\text{wf}}}{r_0} = 1 + e_r^{\text{wf}} + 18\epsilon_0^3 u_0^2 \sigma_\perp^2 + O(u_0^4), $$

and

$$ \frac{P^{\text{wf}}}{2\pi M} = \frac{1}{2\pi M} \left( P^{\text{wf}}(0) + \epsilon P^{\text{wf}}(1) + \epsilon^2 P^{\text{wf}}(2) \right). $$

The semi-major axis and the eccentricities turn out to be

$$ a_r = r_0 \pm \epsilon r_0 \Sigma_\perp - \epsilon^2 (A_2 + 2A_3), $$

and

$$ e_t = \pm 2\epsilon \nu_0^2 \Sigma_\perp - \epsilon^2 \frac{\Omega^{(\text{ep})}}{\Gamma_K} \left[ D_1 \pm \nu_0^2 \epsilon \left( C_1 \pm 4\nu_0^2 \Omega^{(\text{orb})} \Sigma_\perp^2 \right) \right], $$

$$ e_r = \pm \epsilon \Sigma_\perp - \epsilon^2 \frac{A_2}{r_0} + \Sigma_\perp^2, $$

$$ e_\phi = \pm 2\epsilon \Sigma_\perp + \epsilon^2 \frac{\Omega^{(\text{ep})}}{\Omega^{(\text{orb})}} C_1 \pm 4\Sigma_\perp^2, $$

respectively, whereas the periods of $t$ and $\phi$ motions read

$$ \frac{P}{2\pi} = \frac{\Gamma_K}{\Omega^{(\text{ep})}} \left[ 1 \pm 2\epsilon \nu_0^2 \Sigma_\perp + \epsilon^2 \frac{\Omega^{(\text{ep})} \Gamma_K}{\Omega^{(\text{orb})}} \left( D_3 \pm \nu_0^2 \epsilon C_4 \right) \right], $$

$$ \frac{\Phi}{2\pi} = \pm \frac{\Omega^{(\text{orb})}}{\Omega^{(\text{ep})}} \left[ 1 \pm 2\epsilon \Sigma_\perp \pm \epsilon^2 \frac{\Omega^{(\text{ep})} \Omega^{(\text{orb})}}{\Omega^{(\text{orb})}} C_4 \right]. $$

Note that the first order corrections to the orbital parameters only depend on $\sigma_\perp$, because the spin precession in the $r$-$\phi$ plane affects only the $\theta$-motion to that order. The interest of such quantities is that they are directly related to observable effects. Indeed, one can measure either the orbital period and the fractional periastron advance \cite{36} defined by

$$ k \equiv \frac{\Phi}{2\pi} - 1. $$

FIG. 1: The behavior of the variation of the radial distance and polar angle after a full revolution as a function of $r_0$ is shown for the choice of parameters $\sigma_\perp = 0.1 = \sigma_\parallel$, $\alpha_0 = \pi/4$ and $K_0 = 0$. The reference circular geodesic is assumed to be co-revolving with respect to increasing values of the azimuthal coordinate.
Simple inspection of the above formulas shows that the second order corrections to both periods involve the square of the spin parameters $\sigma_\perp$ and $\sigma_\parallel$ as well as the quadrupole parameters $\tilde{X}_{ab}$ and $\tilde{W}_{ab}$: the relative weight of $\sigma_\parallel$ to $\sigma_\perp$ contributions behaves as $u_0^3$, whereas that of $\sigma_\parallel$ to $\tilde{X}_{ab}$ and $\tilde{W}_{ab}$ as $u_0$ and $u_0^{3/2}$, respectively, implying an actual dominant role for $\sigma_\parallel$.

As expected, $\sigma_\perp = 0$ implies vanishing of the first order corrections. In the companion case $\sigma_\parallel = 0$ the quadrupolar corrections to $P_{(2)}^{\text{wf}}$ and $\Phi_{(2)}^{\text{wf}}$ reduce to

\[
\frac{P_{(2)}^{\text{wf}}}{2\pi} = -3u_0^{3/2} \left[ 4\tilde{X}_{11} + 16\tilde{W}_{13}u_0^{1/2}(1 + 8u_0) \right] + O(u_0^4),
\]

\[
\Phi_{(2)}^{\text{wf}} = \mp 3u_0^{3/2} \left[ 4\tilde{X}_{11} + 16\tilde{W}_{13}u_0^{1/2}(1 + 6u_0) \right] + \left( 24\tilde{X}_{11} + 4\tilde{X}_{22} - \frac{21}{2}\sigma_\perp^2 \right) u_0 \right] + O(u_0^4).
\]

### D. Evolution of the quadrupolar structure of the body

Having obtained the complete solution of the orbit we can study now the evolution of the quadrupolar structure of the body as given by Eqs. (3.23). They read

\[
\dot{X}_{12} + 2\nu_K \tilde{W}_{23} = \frac{1}{8} \frac{\sigma_\parallel^2}{(r_0\zeta_K)^2} \left\{ (2 - 3r_0^2\zeta_K^2) \sin 2\alpha \right. \\
+ \frac{1}{\Gamma_K} \sin \alpha_0 \left( \sin \alpha \sin \Omega_{(orb)} \tau \pm \frac{1}{\Gamma_K} \cos \alpha \cos \Omega_{(orb)} \tau \right) \\
- \cos \alpha_0 \left( \sin \alpha \cos \Omega_{(orb)} \tau \mp \frac{1}{\Gamma_K} \cos \alpha \sin \Omega_{(orb)} \tau \right) \right\},
\]

\[
2(2\tilde{W}_{11} + \tilde{W}_{22}) \pm \nu_K \left( 1 - \frac{r_0}{M} \right) \tilde{X}_{13} = \frac{1}{4} \frac{\sigma_\parallel^3}{(r_0\zeta_K)^4} \nu_K \left\{ \frac{1}{\Gamma_K} \left[ \cos \alpha_0 \cos \Omega_{(orb)} \tau \pm \frac{1}{\Gamma_K} \sin \alpha_0 \sin \Omega_{(orb)} \tau \right] \\
- \frac{\Omega_{(orb)}^2}{\Omega_{(ep)}^2} \cos \alpha \left[ 1 - 3r_0^2\zeta_K^2 - 9r_0^4\zeta_K^4 - 9r_0^6\zeta_K^6 \right] \\
- 3 \frac{1}{\Gamma_2^2} \left( r_0\zeta_K ight)^2 (2 - 5r_0^2\zeta_K^2) \cos \Omega_{(ep)} \tau \right\},
\]

\[
2\tilde{W}_{12} \mp \nu_K \tilde{X}_{23} = \frac{1}{4} \frac{\sigma_\parallel^3}{(r_0\zeta_K)^4} \nu_K \left\{ \frac{1}{\Gamma_K} \left[ \cos \alpha_0 \sin \Omega_{(orb)} \tau \mp \frac{1}{\Gamma_K} \sin \alpha_0 \cos \Omega_{(orb)} \tau \right] \\
- 3 \frac{\zeta_K}{\Omega_{(ep)}^2} \cos \alpha \sin \Omega_{(ep)} \tau \pm \frac{\Omega_{(orb)}^2}{\Omega_{(ep)}^2} \sin \alpha \left[ 1 - 18r_0^2\zeta_K^2 + 81r_0^4\zeta_K^4 \right]
\right\}.
\]
For simplicity, if one neglects the magnetic components of the quadrupole (i.e., \( W_{0\alpha} = 0 \)), Eqs. \( \text{(3.44)} \) give the temporal evolution of \( \dot{X}_{12}, \dot{X}_{13} \) and \( \dot{X}_{23} \). Recalling that the remaining components \( \dot{X}_{11} \) and \( \dot{X}_{22} \) are constrained by the relation \( \text{(3.47)} \), one can form the electric part of the quadrupole tensor as depending on two quadrupole parameters only (e.g., \( \dot{X}_{11} \) and \( \dot{X}_{22} \), or equivalently \( K_0 \) and \( \dot{X}_{22} \)). The behavior of the frame components \( \dot{X}_{12}, \dot{X}_{13} \) and \( \dot{X}_{23} \) as a function of the proper time \( \tau \) along the orbit of the extended body is shown in Fig. 2 (a) for selected values of the orbital and spin parameters.

The properties of the spatial tensor \( \dot{X} \) thus provide information about the shape of the body, which varies with time. It has zero trace, but \( \text{Tr}[\dot{X}] \) and \( \text{Tr}[\dot{X}^3] \) are nonzero. Therefore, the associated eigenvalue equation results in

\[ \lambda^3 - \frac{1}{2} \text{Tr}[\dot{X}^2] \lambda - \frac{1}{3} \text{Tr}[\dot{X}^3] = 0, \tag{3.45} \]

and can be numerically studied during the evolution. The behavior of the eigenvalues \( \lambda_1 \) and \( -\lambda_2 \) as a function of \( \tau \) is shown in Fig. 2 (b) for the same choice of parameters. The shape of the body significantly changes during the evolution, becoming approximately spherical for those values of \( \tau \) such that \( |\lambda_1, 2| \ll 1 \).

### E. Circular orbits

Finally, it is interesting to consider the special case in which the orbit of the extended body remains circular, which has to be treated separately, due to the choice of initial conditions \( \text{(2.12)} \). This implies that the motion be confined to the equatorial plane with spin vector orthogonal to it, i.e., \( \sigma_{\parallel} = 0 \). The associated 4-velocity is

\[ U = \Gamma (\partial_t + \zeta \partial_\phi), \tag{3.46} \]

with normalization factor

\[ \Gamma = \Gamma_K \left[ 1 \mp \frac{3}{2} \sigma_{\perp} (r_0 \zeta_K)^3 \Gamma_{\alpha K}^2 + \epsilon^2 \tilde{\Gamma}_{(2)} \right], \tag{3.47} \]

and angular velocity

\[ \zeta = \pm \zeta_K \left[ 1 \mp \frac{3}{2} \sigma_{\perp} (r_0 \zeta_K)^3 \epsilon + \epsilon^2 \tilde{\zeta}_{(2)} \right]. \tag{3.48} \]

Here

\[ \tilde{\Gamma}_{(2)} = -\frac{\Gamma_K}{2N^2} K_0 + \frac{9}{8} (r_0 \zeta_K)^6 \Gamma_{\alpha K}^2 (2 - 3 \sigma_{\perp}^2 \zeta_K) \sigma_{\perp}^2, \]

\[ \tilde{\zeta}_{(2)} = -\frac{K_0}{2N^2 (r_0 \zeta_K)^2 \Gamma_{\alpha K}^2} + \frac{9}{8} (r_0 \zeta_K)^6 \sigma_{\perp}^2. \tag{3.49} \]

where \( K_0 = \text{constant is given by Eq. \( \text{(3.24)} \)}. The remaining frame components of the quadrupole tensor must satisfy the following conditions

\[ \dot{X}_{12} \pm 2\nu_K \dot{W}_{23} = 0, \]

\[ 2(\dot{W}_{11} + \dot{W}_{22}) \pm \nu_K \left( 1 - \frac{r_0}{M} \right) \dot{X}_{13} = 0, \]

\[ 2\dot{W}_{12} \mp \nu_K \dot{X}_{23} = 0. \tag{3.50} \]

The parametric equations of the orbit are then given by

\[ t = t_0 + \Gamma \tau, \quad r = r_0, \quad \theta = \frac{\pi}{2}, \quad \phi = \phi_0 + \Omega \tau, \tag{3.51} \]

with orbital angular velocity \( \Omega = \Gamma \zeta \).

The relation between \( U \) and the unit timelike vector \( u \) aligned with the 4-momentum is given by

\[ u - U = -\epsilon^2 N (r_0 \zeta_K)^5 \Gamma_{\alpha K}^2 \left\{ \pm 8 \nu_K (1 - r_0 \zeta_K^2) \dot{W}_{13} + 3 (r_0 \zeta_K)^2 \left[ \sigma_{\perp}^2 + \frac{4}{3} (2 \dot{X}_{11} + \dot{X}_{22}) \right] \right\} \Gamma_{\alpha K}. \tag{3.52} \]

These results generalize those of Ref. \( \text{22} \), where the quadrupole tensor was assumed to be of purely electric type and with vanishing non-diagonal frame components.

### IV. CONCLUDING REMARKS

We have studied the dynamics of an extended body with arbitrary spin orientation and quadrupolar structure in a Schwarzschild spacetime within the framework of the Mathisson-Papapetrou-Dixon model. The equations of motion have been solved analytically in the limit of small values of the characteristic length scales associated with the spin and quadrupole variables with respect to the characteristic length of the background curvature. The solution provides the corrections to the circular geodesic motion on the equatorial plane (taken as the reference trajectory) due to both the spin-curvature and quadrupole-curvature couplings. The component of the spin vector orthogonal to the unperturbed orbital reference plane remains constant, whereas the component in the equatorial plane undergoes precession, due to the evolution equations. The components of the quadrupole tensor, instead, have no associated evolution equations according to the MPD model. However, they turn out to be constrained by several compatibility conditions as a consequence of the spin evolution equations.

In the literature, the most studied case corresponds to the simplest situation of a spin vector with constant magnitude and orthogonal to the orbital plane, i.e., the equatorial plane. Furthermore, most of the components of the quadrupole tensor are assumed either constant or
even vanishing or fully determined by the spin structure itself (spin-induced quadrupole), a fact that leads to a great simplification to the problem. Here, instead, we have allowed for a completely general variability of both the spin vector and quadrupole tensor. As a consequence, the motion is no more confined to a plane and the deviations from the reference circular geodesic are governed by three fundamental frequencies (orbital, epicyclic, spin precession frequency), leading to a general non-periodic motion. More precisely, at the first order (i.e., for a purely spinning particle without quadrupole), the radial and polar motions are decoupled and exhibit periodic oscillations around the reference geodesic (simply periodic in the radial direction). Including the quadrupole in its full generality then implies multi-frequency coupled spatial motions, exhibiting in addition secular drifts which vary with spin orientation. We have computed here the corresponding solution and studied the associated dynamical observables, including periastron advance, orbital period, eccentricities, etc. We have also shown that the quadrupole tensor components have a generally non-periodic oscillating behavior, causing the shape of the body to change along the path. This effect is expected to play a role in explaining the variation of the orbital elements of astrophysical systems, like binary pulsar systems.

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Appendix A: Second order solution coefficients

We list below the integration constants of the second order solution (3.29):

\[
A_0 = -9MN^4u_0^2\frac{\Omega_{(orb)}^6}{\Omega_{(ep)}^2}(1 - 9u_0)\sigma_\perp^2,
\]

\[
A_1 = \pm \frac{1}{6}MN^2u_0\Gamma_\kappa \frac{8 - 33u_0 + 45u_0^2}{1 - 6u_0 - 3u_0^2}\sigma_\parallel \sin 2\alpha_0,
\]

\[
A_2 = -\frac{MK_0}{\Gamma_\kappa^2} + 36N^4Mu_0^3\frac{\Omega_{(orb)}^6}{\Omega_{(ep)}^2}\sigma_\perp^2 + \frac{MN^4u_0^2}{2(r_0\Omega_{(ep)})^2} \frac{2 - 9u_0}{1 - 6u_0 - 3u_0^2}\sigma_\parallel^2
\]
\[ A_3 = \frac{9}{2} MN^4 u_0^2 \frac{\Omega_{\text{orb}}^2}{\Omega_{\text{ep}}^6} (1 - 7u_0)^2 \sigma_{\perp}^2, \]

\[ A_4 = \frac{MN^2}{3 \Gamma_K^2} \sigma_{\parallel}^2, \]

\[ A_5 = \pm \frac{1}{3} MN^4 \Gamma_K \frac{(1 + 3u_0)(2 - 9u_0)}{1 - 6u_0 - 3u_0^2} \sigma_{\parallel}^2, \]

\[ A_6 = -\frac{2}{3} MN^4 \frac{2 - 9u_0}{1 - 6u_0 - 3u_0^2} \sigma_{\parallel}^2. \]

\[ (A1) \]

\[ B_1 = -2N u_0^2 \frac{\Omega_{\text{orb}}^2}{\Omega_{\text{ep}}^2} \frac{1 - 7u_0 + 15u_0^2 - 36u_0^3}{1 - 6u_0 - 3u_0^2} \sigma_{\parallel} \sigma_{\perp}, \]

\[ B_2 = -\frac{1}{6} N \frac{\Omega_{\text{orb}}^2}{\Omega_{\text{ep}}^2} \frac{12u_0 - 24u_0^2}{1 - 6u_0 - 3u_0^2} \sigma_{\parallel} \sigma_{\perp}, \]

\[ B_3 = \mp \frac{1}{6} N \Gamma_K \frac{2 - 33u_0 + 144u_0^2 - 210u_0^3 + 36u_0^4 - 27u_0^5}{1 - 6u_0 - 3u_0^2} \sigma_{\parallel} \sigma_{\perp} \sin\alpha_0, \]

\[ B_4 = \mp N u_0^3 \frac{\Omega_{\text{orb}}}{\Omega_{\text{ep}}^2} \frac{2 - 8u_0 - 9u_0^2 + 63u_0^3}{1 - 6u_0 - 3u_0^2} \sigma_{\parallel} \sigma_{\perp}, \]

\[ B_5 = -6N^3 u_0^2 \frac{\Omega_{\text{orb}}^3}{\Gamma_{\text{ep}}^2} \sigma_{\parallel} \sigma_{\perp}, \]

\[ B_6 = -\frac{1}{2} N u_0^3 \frac{\Omega_{\text{orb}}^3}{\Omega_{\text{ep}}^3} \frac{1 - 21u_0 + 42u_0^2}{1 - 6u_0 - 3u_0^2} \sigma_{\parallel} \sigma_{\perp}, \]

\[ B_7 = -\frac{1}{3} N \frac{1 - 15u_0 + 51u_0^2 - 39u_0^3 - 72u_0^4}{1 - 6u_0 - 3u_0^2} \sigma_{\parallel} \sigma_{\perp} \cos\alpha_0. \]

\[ (A2) \]
\[ C_7 = \frac{1}{3} N^4 \frac{3 - 7u_0}{\Gamma_K (1 - 6u_0 - 3u_0^2 \sigma_\parallel^2)}, \]
\[ C_8 = \pm \frac{1}{3} N^2 \frac{3 - 20u_0 + 39u_0^2}{1 - 6u_0 - 3u_0^2 \sigma_\parallel^2}, \]
\[ C_9 = \frac{1}{24} N^2 u_0 \Gamma_K^3 (8 - 42u_0 + 63u_0^2) \sigma_\parallel^2, \] (A3)

\[ D_1 = 9M_0^{5/2} \frac{\Omega^{(\text{orb})}_3}{\Omega^{(\text{ep})}_0} \sigma_\perp = -4D_2, \]
\[ D_3 = -\frac{3}{4} M_0^{3/2} \frac{\Omega^{(\text{orb})}_3}{\Omega^{(\text{ep})}_0} \left[ \sigma_\parallel^2 + 6u_0 \frac{\Omega^{(\text{orb})}_0}{\Omega^{(\text{ep})}_0} \sigma_\perp^2 \right], \]
\[ D_4 = -\frac{1}{8} M_0^{1/2} [(2 - 3u_0) \cos 2\alpha_0 - 3u_0] \sigma_\parallel^2, \]
\[ D_5 = \pm \frac{1}{4} M_0^{1/2} \Gamma_K \sigma_\parallel^2 \sin 2\alpha_0, \]
\[ D_6 = M_0^{1/2} \sigma_\parallel^2, \]
\[ D_7 = \pm \frac{1}{8} M_0^{1/2} \Gamma_K (2 - 3u_0) \sigma_\parallel^2, \] (A4)

\[ F_1 = \mp m_0 N^2 u_0^2 \Gamma_K \sigma_\parallel^2, \]
\[ F_2 = \frac{1}{4} m_0 N^2 u_0^2 \Gamma_K^2 (2 - 3u_0) \sigma_\parallel^2, \] (A5)

where

\[ N = \sqrt{1 - 2u_0}, \quad \Gamma_K = \frac{1}{\sqrt{1 - 6u_0}} \quad \text{and} \quad \frac{\Omega^{(\text{orb})}_0}{\Omega^{(\text{ep})}_0} = \frac{1}{\sqrt{1 - 6u_0}}. \] (A6)
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