DERIVATION OF FITTS’ LAW FROM THE TASK DYNAMICS MODEL OF SPEECH PRODUCTION

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ABSTRACT

Fitts’ law is a linear equation relating movement time to an index of movement difficulty. The recent finding that Fitts’ law applies to voluntary movement of the vocal tract raises the question of whether the theory of speech production implies Fitts’ law. The present letter establishes a theoretical connection between Fitts’ law and the Task Dynamics model of speech production. We derive a variant of Fitts’ law where the intercept and slope are functions of the parameters of the Task Dynamics model and the index of difficulty is a product logarithm, or Lambert W function, rather than a logarithm.

1 Background

The work of Fitts (1954) provided support for a linear relation between the average duration \( T \) of a movement and an index of difficulty \( I_d = \log(2A/W) \), where \( A \) is the average movement amplitude and \( W \) is the error tolerance. Thus, Fitts’ law can be stated as the linear equation

\[
T = \alpha + \beta I_d,
\]

(1)

where \( \alpha \) is the intercept and \( \beta \) is the slope.

The studies of Lammert, Shadle, Narayanan, and Quatieri (2018) and Kuberski and Gafos (2019) were the first to provide evidence that speech movements of the vocal tract conform with Fitts’ law. Lammert et al. (2018) discovered that Fitts’ law applied to vocal tract constriction movements in the coda position of the syllable (e.g., post-vocalic consonant [k] in the word “pack” [pʰæk]), with a subset of subjects exhibiting Fitts’ law for movements in other syllabic positions as well. Kuberski and Gafos (2019) found that Fitts’ law applied to tongue tip and tongue dorsum constriction movements in the onset position of the syllable when speech rate exceeded a subject-specific threshold. The findings of Lammert et al. (2018) and Kuberski and Gafos (2019) provided evidence that Fitts’ law applies to vocal tract movements just as it does to movements of the upper limb (Fitts, 1954; Fitts & Peterson, 1964), lower limb (Drury, 1975), and postural control (Duarte & Freitas, 2005). Together, these studies support Fitts’ law in a variety of tasks in several domains of voluntary movement.

In addition to inspiring the speech experiments described above, Fitts’ law informed the theory of speech production, with Guenther (1995, §7, “Speaking Rate Effects”) showing that the Directions into Velocities of Articulators model of neural control of speech movements produced movements consistent with Fitts’ law, and Lammert et al. (2018, §2...
The purpose of the present letter is to establish a theoretical connection between Fitts’ law and the Task Dynamics model of speech production (Saltzman & Munhall, 1989). Section 2 presents a derivation of the result that Fitts’ law has intercept α = −1/√k/m and slope β = 1/√k/m, where gestural stiffness k and mass m are parameters of the exponential movement trajectory in the Task Dynamics model of speech production. The index of difficulty involves the product logarithm, or Lambert W function (Corless et al., 1996), rather than the logarithm of existing indices of difficulty. Section 3 compares this derivation with that of Lammert et al. (2018) and shows that the present derivation is more general than that of Lammert et al. (2018) and conforms with the speech articulator movements typically analyzed in studies of vocal tract kinematics.

2 A derivation of Fitts’ law

The present section derives that Fitts’ law has intercept α = −1/√k/m and slope β = 1/√k/m, where gestural stiffness k and mass m are parameters of the exponential movement trajectory in the Task Dynamics model of speech production (Saltzman & Munhall, 1989). The derivation results in an alternative index of difficulty that involves the product logarithm, or Lambert W function (Corless et al., 1996), rather than the logarithm of Fitts’ indices of difficulty (Fitts, 1954). We demonstrate that the alternative index of difficulty is an index of movement difficulty that is qualitatively similar to that of Fitts (1954), as well as those of Lammert et al. (2018) and Kuberski and Gafos (2019) used in speech studies of Fitts’ law.

The equation of motion for a gesture with stiffness k, damping b, and mass m is determined by the second-order linear system

\[ m\ddot{z} + b\dot{z} + kz = 0, \]  

where the parameters satisfy the condition \( b^2 - 4km = 0 \) of critical damping. The fixed point is located at the origin \( z = 0 \). Under these conditions, the solution of the system is the following (Jordan & Smith, 2007, Chapter 1, Section 4).

\[ z(t) = (c_0 + c_1 t) \exp \left( -t\sqrt{k/m} \right) \]  

We solve for \( c_0 \) and \( c_1 \) in terms of initial conditions \( z(0), \dot{z}(0) \) to obtain \( c_0 = z(0) \) and \( c_1 = \dot{z}(0) + z(0)\sqrt{k/m} \).

If a movement begins at time zero, then the residual displacement \( z(\tau) \) from the fixed point after time \( \tau \) is \( z(\tau) = (c_0 + c_1\tau) \exp \left( -\tau\sqrt{k/m} \right) \). Taking the logarithm of both sides and applying the identity \( \log(c_0 + c_1\tau) = \log(1 + c_1\tau/c_0) + \log(c_0) \), we derive the following.

\[ \log(z(\tau)) = \log \left( 1 + \frac{c_1}{c_0} \tau \right) + \log(c_0) - \tau\sqrt{k/m} \]

\[ \iff \tau = \frac{\log \left( 1 + \frac{c_1}{c_0} \tau \right) + \log \left( \frac{c_0}{z(\tau)} \right)}{\sqrt{k/m}}. \]  

In order to solve Equation 5 for \( \tau \), we make the simplifying assumption that the initial velocity \( \dot{z}(0) \) equals zero. Although this assumption limits the generality of the result we are deriving, the assumption is approximately correct in speech kinematic analysis, where speech movements are taken to begin when velocity surpasses a small threshold.

Setting initial velocity equal to zero implies the equation

\[ \left( -\tau\sqrt{k/m} - 1 \right) \exp \left( -\tau\sqrt{k/m} \right) = -\frac{z(\tau)}{bc_0}, \]

which has a solution for \( \tau \). Substituting \( x = -\tau\sqrt{k/m} \), we derive this solution as follows.

\[ (x - 1) \exp(x - 1) = \frac{z(\tau)}{ec_0} \]

\[ x = \ln \left( \frac{z(\tau)}{ec_0} \right) + 1 \]

\[ \iff \tau = -\frac{1}{\sqrt{k/m}} \ln \left( \frac{z(\tau)}{ec_0} \right) - \frac{1}{\sqrt{k/m}} \]
In the above solution, $e$ is Euler’s number and $\omega$ denotes the product logarithm, also known as the Lambert W function (Corless et al., 1996). The product logarithm is defined as the inverse of the function $f(z) = z \exp(z)$. Although the relation defined by this definition is multivalued and has the set of complex numbers as its domain, below we introduce restrictions on the domain and codomain such that the product logarithm is a function whose domain is the real numbers. Note that the solution to Equation 9 above follows from the identity $\omega((x-1) \exp(x-1)) = x-1$ of the product logarithm.

Equating time $\tau$ with movement time $T$, constant $c_0$ with movement amplitude $A$, and residual displacement $z(\tau)$ with error tolerance $W$, we derive that

$$T = \alpha + \beta I'_d,$$

(10)

where intercept $\alpha = -1/\sqrt{k/m}$, slope $\beta = 1/\sqrt{k/m}$, and index of difficulty $I'_d = -\omega(-W/eA)$. Prime indicates that this index of difficulty is not identical to that of Fitts. As movement amplitude $A$ must be greater than error tolerance $W$, we have that the domain of the product logarithm is a real number in the open interval $(-1/e, 0) \subset \mathbb{R}$. In this interval, the product logarithm has two real branches (cf. Figure 1a); an upper branch $\omega_0$ (dashed) and a lower branch $\omega_{-1}$ (solid). Branch $-\omega_{-1}(-W/eA)$ is monotonically decreasing in $A/W$, while branch $-\omega_{-1}(-W/eA)$ is monotonically increasing in $A/W$. Restricting attention to the lower branch $-\omega_{-1}$, we find that the index of difficulty increases with movement amplitude and decreases with error tolerance. The graph of index of difficulty as a function of $A/W$ is similar in form to that of Fitts (1954), taking the value $I'_d = 1$ at $A/W = 1$ and increasing monotonically with $A/W$ (cf. Figure 1b, solid). Furthermore, movement time is a linear function of the index of difficulty (cf. Equation 10). These properties make the prediction of Equation 10 qualitatively similar to the prediction of Fitts’ law, the difference being the index of difficulty assumed: $I'_d = -\omega_{-1}(-W/eA)$ under Equation 10 or $I_d = -\log_2(W/2A)$ under Fitts’ law.

Figure 1: (a) Graph of the real branches of the product logarithm for $x \in (-1/e, 0)$. The principal real branch $\omega_0$ is defined as the upper branch with $\omega \geq -1$ (dashed) and the real branch $\omega_{-1}$ is defined as the lower branch with $\omega < -1$ (solid). (b) Comparison of the index of difficulty $-\omega_{-1}(-W/eA)$ proposed in the present study with others from the literature. The comparison is performed for different values of the ratio $A/W$, where the value $A/W = 1$ is a lower bound on possible movements.
3 Comparison with the existing kinematic derivations

Equation 10 is not the only version of Fitts’ law that has been derived from the Task Dynamics model of speech production. Lammert et al. (2018) derived a version of Fitts’ law by considering the particular solution
\[ z(t) = c_0 \exp \left( -t \sqrt{\frac{k}{m}} \right) \] (11)
to the critically damped linear second-order system of Equation 2, where initial position \( c_0 = z(0) \) and initial velocity \( \dot{z}(t) = -z(0) \sqrt{\frac{k}{m}} \). By considering this particular solution in place of the general solution of Equation 3, the solution to Equation 5 simplifies from a product logarithm to a logarithm, since the constant \( c_1 = \dot{z}(t) + z(0) \sqrt{\frac{k}{m}} \) equals zero.

\[ \tau = \log \left( \frac{1 + c_1 \tau}{c_0} \right) + \log \left( \frac{c_0}{z(\tau)} \right) \bigg/ \sqrt{\frac{k}{m}} . \] (5 revisited)

Substituting movement time \( T \) for time \( \tau \), movement amplitude \( A \) for constant \( c_0 \), and error tolerance \( W \) for residual displacement \( z(\tau) \), we derive the version of Fitts’ law obtained by Lammert et al. (2018, Section “Background: Theoretical framework”, Equation 4).

\[ T = \frac{1}{\sqrt{\frac{k}{m}}} \log \left( \frac{\frac{A}{W}}{1} \right) \] (13)

Lammert et al. (2018) showed the conditions under which this equation is equivalent to the original Fitts’ law (Fitts, 1954).

The reason to prefer the present index of difficulty over the logarithmic index of difficulty of Equation 13 is that the derivation of the present index of difficulty considers the general solution to the critically damped second-order linear system of Equation 2 with initial velocity equal to zero. In contrast, the derivation of Equation 13 replaces the general solution of Equation 3 with the particular solution \( z(t) = c_0 \exp(-t \sqrt{k/m}) \) or, equivalently, considers the general solution of Equation 3 with the particular nonzero initial velocity \( -z(0) \sqrt{k/m} \). The condition of zero initial velocity in the present derivation is approximately correct in speech kinematic analysis, where speech movements are taken to begin when velocity surpasses a small threshold. Moreover, replacing the general solution of Equation 3 with the particular solution \( z(t) = c_0 \exp(-t \sqrt{k/m}) \) implies that the initial velocity \( -z(0) \sqrt{k/m} \) is the peak velocity of the movement (cf. Figure 2, dashed movement traces). The implication that the initial velocity equals the peak velocity does not agree with the observation that peak velocity occurs not at the very start of a movement, but rather closer to the temporal midpoint of a movement (Munhall, Ostry, & Parush, 1985; Ostry, Cooke, & Munhall, 1987; Byrd & Saltzman, 1998; Sorensen & Gafos, 2015).

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Figure 2: Comparison of solution with initial velocity set to zero (solid) and with initial velocity set to $-z(0)\sqrt{k/m}$ (dashed). With zero initial velocity (solid), peak velocity (horizontal line, bottom panel) occurs at time $2/\sqrt{k/m} - c_0/c_1$ (vertical line), where $k$ is gestural stiffness, $m$ is virtual mass, $c_0$ is the initial position $z(0)$, and $c_1 = \dot{z}(0) - z(0)\sqrt{k/m}$.

With initial velocity $-z(0)\sqrt{k/m}$ (dashed), peak velocity occurs at time zero (i.e., is simultaneous with the very start of the movement).

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