Mass-Gaps and Spin Chains for (Super) Membranes

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Abstract

We present a method for computing the non-perturbative mass-gap in the theory of Bosonic membranes in flat background spacetimes. The analysis is extended to the study of membranes coupled to background fluxes as well. The computation of mass-gaps is carried out using a matrix regularization of the membrane Hamiltonians. The mass gap is shown to be naturally organized as an expansion in a 'hidden' parameter, which turns out to be $\frac{1}{d}$: $d$ being the related to the dimensionality of the background space. We then proceed to develop a large $N$ perturbation theory for the membrane/matrix-model Hamiltonians around the quantum/mass corrected effective potential. The same parameter that controls the perturbation theory for the mass gap is also shown to control the Hamiltonian perturbation theory around the effective potential. The large $N$ perturbation theory is then translated into the language of quantum spin chains and the one loop spectra of various Bosonic matrix models are computed by applying the Bethe ansatz to the one-loop effective Hamiltonians for membranes in flat space times. The spin chains corresponding to the large $N$ effective Hamiltonians for the relevant matrix models are generically not integrable. However, we are able to find large integrable sub-sectors for all the spin chains of interest. Moreover, the continuum limits of the spin chains are mapped to integrable Landau-Lifschitz models even if the underlying spin chains are not integrable. Apart from membranes in flat spacetimes, the recently proposed matrix models [hep-th/0607005] for non-critical membranes in plane wave type spacetimes are also analyzed within the paradigm of quantum spin chains. The Bosonic sectors of all the models proposed in [hep-th/0607005] are diagonalized at the one-loop level and an intriguing connection between the existence of supersymmetric vacua and one-loop integrability is also presented.
1 Introduction and Summary

In the present paper we compute the one-loop large \( N \) spectrum of various models of matrix quantum mechanics describing the motion of membranes with and without supersymmetry. The purely Bosonic cases discussed in the paper correspond to spherical membranes in flat backgrounds which we also generalize to include background fluxes. We also study the one loop spectrum of the Bosonic sectors of various supersymmetric matrix models that were recently proposed by Kim and Park \[21\] as the regularized descriptions of supermembranes in non-critical super gravity theories in the background of plane wave like curved spacetimes. The matrix models corresponding to the supersymmetric membranes have explicit quadratic ‘mass’ terms in their Hamiltonians and their spectra are clearly discrete. This allows us to study their large \( N \) spectrum by Bethe ansatz methods which have been very successful in the study of the the BMN matrix model. However, this is not the situation for the non-supersymmetric cases that we study. Since the background spacetime for membrane motion is chosen to be flat, the membrane Hamiltonians do not have mass terms and are ostensibly plagued with the problem of the existence of classical flat directions. Thus a straightforward application of the spin chain techniques to these models of membrane dynamics is not possible. However, the problem of existence of flat directions for non-supersymmetric membranes is entirely an artifact of choosing a particularly difficult classical theory as a starting point for quantization. The flat directions are indeed lifted upon quantization \[23\]. To make use of this simplification, we develop a method for estimating the dynamically generated mass terms in the quantum theory for purely bosonic membranes. The upshot of this technique is that one can develop a controlled large \( N \) perturbation theory both for estimating the spectral gap of the theories as well as for quantizing the theories around the ‘quantum corrected’ effective potential. This quantization naturally allows one to apply the techniques of quantum spin chains and the Bethe ansatz to systematically study the spectrum of non-supersymmetric membranes.

The close connection between matrix quantum mechanics and theories of D branes and membranes has been known for quite sometime \[1, 2\] and one of the principal motivations leading to the present work is the search for methods that might help us understand the gauge- theory/gravity correspondence from the point of view of D branes and other extended objects. A key insight gained from the recent advances in the AdS/CFT correspondence is that integrable structures naturally manifest themselves on both the string and gauge theory sides of the correspondence. As far as the spectrum of closed string like excitations, both from the gauge theory as well as the string theoretic point of view, are concerned, there is by now mounting evidence that they are described by integrable systems. On the gauge theory end, one concerns oneself with the spectrum of anomalous dimensions of single trace local composite operators. To the extent that it has been possible to check so far, this spectrum appears to be the spectrum of an integrable spin chain \[3, 4\]. On the string theory side, the corresponding integrable system is nothing but the world sheet sigma model, which too appears to be integrable, at least at the classical level \[5, 7, 6\]. If one focusses on the gauge theory, then it is not difficult to see that the fundamental system leading to large \( N \) integrability is a quantum mechanical matrix model. This matrix model is nothing but the dilatation operator of the gauge theory, which in the large \( N \) limit can be interpreted as an integrable quantum spin chain \[4, 10, 11, 12\]. Thus, keeping in mind that D brane and membrane world volume theories are nothing but quantum mechanical matrix models, it is perhaps reasonable to expect that integrable systems such as quantum spin chains should also appear naturally in the world volume theories of branes. Indeed, the BMN matrix model \[15\], which can be interpreted as the theory of zero branes in type IIA theory as well as the light cone supermembrane theory of eleven dimensional supergravity in a plane wave background \[18\] is known to be integrable to rather high orders in perturbation theory \[20\]. It provides us an example of the natural emergence of integrable systems in brane dynamics. However, a similar systematic approach that utilizes integrable systems has remained lacking for membrane motions in other backgrounds. Most notably, membrane motion in trivial/flat backgrounds have not been
studied from the point of view of integrable models.

To appreciate the chief obstacle that prevents a straightforward application of quantum spin-chain techniques to matrix regularized descriptions of membranes in flat spacetimes it is worth recalling that both the matrix models mentioned so far, i.e the BMN model and the dilatation operator of $\mathcal{N} = 4$ SYM share a common feature, which is that they have discrete spectra \[15, 9\]. This key common feature is central to being able to analyze them in the large $N$ limit as quantum spin chains. The discrete nature of the spectrum of the models mentioned above manifests itself very transparently at the level of the Hamiltonian in the form of the presence of quadratic 'mass' terms. One can then go to a basis of creation/annihilation operators and carry out a perturbative analysis around the oscillator vacuum. The normal ordered interaction terms can be understood as Hamiltonians of quantum spin chains and their contribution to the spectrum can be studied by applying the Bethe ansatz to the relevant spin chains order by order in perturbation theory. This connection between quantum spin chains and large $N$ Hamiltonian matrix models was first worked out by Rajeev and Lee in \[8\], and it has been extremely useful in the perturbative analysis of the spectrum and integrability properties of the BMN matrix model as well as the dilatation operator of $\mathcal{N} = 4$ SYM \[13, 20, 9\]. Presence of mass terms is a feature that however is not shared by various matrix models of interest. In the BMN matrix model \[15\] the mass terms are a reflection of the fact that the background metric for membrane is a plane wave \[15\]. Matrix models for membrane motions in flat spacetimes do not have such quadratic terms. Thus it may seem that, at least ostensibly, membrane motions in flat spacetimes cannot be studied by the utilizing the connection of matrix models to quantum spin chains that has been so successful in the analysis of the dilatation operator of $\mathcal{N} = 4$ SYM and the BMN matrix model. This is nothing but the standard problem of the existence of 'flat directions' in the Hamiltonian analyses of membrane motions \[16, 17\]. However, there are various examples of membrane Hamiltonians whose spectra are known to be discrete at the quantum mechanical level even though there are apparent flat directions at the classical level. The simplest example of such a scenario is given by the matrix model for the motion of membranes in a $d + 2$ dimensional flat background.

$$ H = \text{Tr} \left( \Pi_i \Pi_i - \frac{1}{4} [X_i, X_j]^2 \right), i, j = 1 \cdots d. \tag{1} $$

The flat directions correspond to the configuration of commuting matrices

$$ [X_i, X_j] = 0 \tag{2} $$

However, semiclassical analyses suggest that the spectrum of the model is indeed discrete \[23, 25, 26, 27\]. In other words, quantum corrections lift the classical flat directions, and the quantum effective potential does indeed acquire a mass term. Indeed, tell-tale signs of integrable behavior in various examples of membrane motions in flat spacetimes have also been reported in the past literature \[28, 29, 30, 31\]. Although, the estimation of the mass gaps of these models has been carried out using various techniques in the past, it would be gratifying to have a method that can enable one to do perturbative analyses around the quantum corrected effective potentials using the techniques of quantum spin chains. It is the development of such techniques that we devote the first part of the paper to. The models that we study to develop our methods correspond to matrix regularized membrane motions in flat background spacetimes with and without fluxes. The main summary of this part of the paper and the corresponding results are as follows.

In section (3) we develop a method for computing the mass gaps in bosonic matrix models. The method is based on a gauge invariant rearrangement of various Feynman diagrams of the quantum mechanical model that allows one to compute the location of the pole of the propagator order by order in perturbation theory. Similar approaches for the computation of mass gaps have also been applied to various Bosonic matrix models as well as to supersymmetric models at finite temperature in the past, (notably in \[28\]). One of
the key insights that we gain from our analysis is the emergence of a new parameter in the problem which controls the perturbative corrections to the dynamically generated mass term for the theory as well as the Hamiltonian perturbation theory around quantum corrected effective potential for the matrix models. Roughly speaking the parameter turns out to be \( \frac{1}{d} \), where \( d \) is the number of matrices in the problem, which is of course related to the dimensionality of the background spacetime for the membrane theory. To put it differently, the obvious coupling constant of the bosonic matrix models, i.e the strength of the commutator squared interaction term, factors out as an overall multiplicative factor in the Hamiltonian analyses of the theories around the effective potential. However, one can form a dimensionless number out of the coupling constant of the matrix model and the dynamical mass term. It is this number that goes as \( \frac{1}{d} \) and appears to control the perturbative expansion of the matrix models.

As mentioned previously, having a mass term in the effective potential, as well as a dynamically generated perturbation parameter, allows one to use the correspondence between large \( N \) matrix models and quantum spin chains to compute the large \( N \) spectrum of the models in perturbation theory. We carry this out in section(4). In this part of the paper we map the one loop effective potentials for the relevant models to Hamiltonians of corresponding quantum spin chains with nearest neighbor interactions. The spin chains corresponding to membrane motion in \( d + 2 \) dimensions have \( so(d) \) as their invariance group. This is a reflection of the \( so(d) \) invariance of the interaction \( \sum_{i,j=1}^{d} \text{Tr}[X_i X_j]^2 \) term of the corresponding matrix models. Having a spin chain emerge as the one loop effective Hamiltonian of the matrix models is one thing, being able to solve it and compute the one loop spectrum of the theory is quite another. If the spin chain turns out to be integrable, in the sense of being part of a family of mutually commuting Hamiltonians derived from some underlying \( R \) matrix, then one can use Bethe ansatz techniques to diagonalize the spin chains\(^1\). However, as has been explained in section(4.1) most of the \( so(d) \) invariant spin chains that emerge as the one loop large \( N \) Hamiltonian are not integrable in the sense described above. Nonetheless, we can still make progress using the following two observations.

1: For \( so(2d) \) or \( so(2d+1) \) spin chains with nearest neighbor interactions, there is always a rather large integrable sub-sector even if the spin chain is not integrable. This sub-sector corresponds the Hilbert space of the spin chain that contains states charged under \( su(d) \) which is contained in \( so(2d) \) or \( so(2d+1) \). Since the number of independent excitations of the spin chains is equal to the rank of the Lie algebra i.e, \( d-1 \) for \( su(d) \) and \( d \) for \( so(2d) \), most of the spectrum of the spin chains of interest to us will be accessible by Bethe ansatz techniques, even though the spin chains are not strictly integrable.

2: If one were interested in the low lying excitations of the spin chain, then it is well known that they are captured by a classical two dimensional non-linear ‘sigma’ model which can be thought of as the continuum limit of the spin chain. For \( so(d) \) invariant spin chains we can show that the corresponding sigma model is always integrable even if the quantum spin chain is not. This understanding is based on realizing the large length \( (J) \) limit of the spin chain as a classical limit and the contraction of the quantum \( R \) matrices of various \( so(d) \) invariant spin chains to a universal classical \( r \) matrix in the large \( J \) limit. Thus one can indeed understand, at least in principle, all the low lying excitations of the spin chains of interest to us irrespective of whether or not they are integrable.

The final parts of chapter(4) are devoted to the one-loop perturbation theory of bosonic models with dynamically generated mass terms. We identify the integrable sub-sectors of the one loop spin chains and explicitly diagonalize them using Bethe ansatz techniques. We also give a formal derivation and proof of the second statement made above in section(4.4). This is carried out by deriving the non-relativistic sigma model for the low lying spectrum of the one loop \( so(d) \) invariant spin chains. We establish the integrability of the sigma models by deriving the lax pairs and monodromy matrices for them. This analysis, also brings

\(^{1}\)There are several books and expository articles on Bethe ansatz techniques. For a pedagogical introduction, see [32]
out, in a rather explicit form, that the large $J$ limit can be regarded as a classical limit, with $\frac{1}{J}$ playing the role of $\hbar$, thus explaining the emergence of classical integrable systems in the continuum limit.

The final parts of the paper are devoted to the study of supersymmetric matrix models obtained recently by Kim and Park from the dimensional reductions of minimally supersymmetric Yang-Mills theories in diverse dimensions [21]. The matrix models that we study in this section have explicit mass terms in them and the focus of our attention will be the large $N$ integrability of these models. Indeed, the mechanism for mass generation that we utilize to study the spectra of various bosonic matrix models will not work in the case of most supersymmetric matrix models. The mass terms for the bosonic models arise by summing over certain classes of self-energy diagrams which will cancel out in the supersymmetric cases against standard fermionic contributions. However, there does exist a fairly large class of supersymmetric matrix models with explicit mass terms in their Hamiltonians; the BMN matrix model is of course an example of such a scenario. The BMN model is of course nothing but the light cone M(atrix) theory hamiltonian in the eleven dimensional plane wave background. It also has at least two other interpretations. It can also be thought of as the matrix regularized supermembrane Hamiltonian in the eleven dimensional plane wave background. As a matrix model, it can just as well also be thought of as a particular dimensional reduction of minimally supersymmetric Yang-Mills theory in ten dimensions (or maximally supersymmetric $N = 4$ Yang-Mills in D=4) down to one dimension. This suggests a connection between dimensional reductions of supersymmetric Yang-Mill theories and super-membrane models in one dimension higher than that of the original super Yang-Mills theory. It is surely instructive to probe this possible connection further. Minimally supersymmetric Yang-Mills theories can, other than in $D = 10$, be defined in dimensions six, four and three and two, and the matrix models obtained by the dimensional reduction of these models to one (time) dimension has recently been accomplished by Kim and Park [21]. One has to be a little careful while relating the dimensional reductions of minimal super Yang-Mills to supermembrane theories. The naive dimensional reduction of super Yang-Mills to one dimension is bound to give a supermembrane theory in flat background. This is of course the original connection made by BFSS [13]. These matrix models do not have mass terms and they suffer from the problem of flat directions which cannot be cured by the method we apply to purely bosonic models in this paper. However, it is possible to add mass terms to the supersymmetric matrix models in a way that does not break any supersymmetries. The resultant mass deformed models can then be interpreted as supermembrane theories not in flat backgrounds, but rather in plane-wave type spaces. All possible supersymmetric mass deformations of the matrix models obtained from the dimensional reductions of minimal super Yang-Mills theories in dimensions six, four, three and two have been worked out by Kim and Park in their paper. The existence of the mass terms in the BMN matrix model is of course what makes it possible to study its large $N$ spectrum using quantum spin chains. Moreover, the spin chains obtained in the pertubative expansion of the matrix model are also known to be integrable to rather high orders in perturbation theory. It is thus natural to ask what kind of quantum spin chains arise in the large $N$ perturbation theory for the mass deformed models obtained in [21], and whether or not these spin chains are integrable. This is the question that we concern ourselves with in section(5).

Following the analysis carried out by Plefka and collaborators [20] we shall focus on the bosonic sectors of the models obtained by Kim and Park as a starting point for investigating integrable behavior in the models. In their paper [21], they were able to show that there are two generic types of possible mass deformations of the matrix models obtained from the dimensional reductions of minimal Yang-Mills theories. The mass deformation of the first kind corresponds to matrix models that have non-trivial maximally supersymmetric vacua. A second type of mass deformation, which has also been carried out by the same authors corresponds to supersymmetric matrix models that do not have non-trivial BPS configurations. In this paper, we are able to show that all the matrix models that do possess non-trivial supersymmetric vacua also correspond to integrable spin chains at one loop. This hints at a strong connection between integrability and super-
symmetry. For the mass deformed models of the second type, the one loop spin chains are not generically integrable. But, as mentioned previously, it is always possible to find sectors of the spin chain that do retain integrability, and thus give us some information about large $N$ spectrum of the matrix models. This analysis is carried out in the final section of the paper.

2 Membranes, Matrix Models and Spin Chains: A Brief Review

In this section we gather together some known results connecting membranes to matrix models, and matrix models to quantum spin chains. A fuller discussion of these connections can be found in several papers. The relation between membrane motion and Hamiltonian matrix models is discussed in depth in [23, 1] while the use of spin chain techniques in the study of matrix models has been elaborated upon in [8, 9]. We shall refer to these papers for a more complete account of the results summarized in this part of the paper.

The action of the membrane in a $d + 2$ dimensional flat background is given by

$$ S = -T \int d^3 \sigma \sqrt{-\det h_{\alpha \beta}},$$

where $T$ is the brane tension, which can be expressed in terms of the basic dimensional parameter in the problem, the Planck length, $l_p$ as

$$ T = \frac{1}{2 \pi l_p^3}. $$(4)

$h_{\alpha \beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$ is the pull back of the spacetime metric to the membrane world volume. $\mu, \nu$ take on values $1 \cdots d + 2$ while $\alpha, \beta$ are the three world volume indices. The Nambu-Goto type membrane action can be replaced by a Polyakov action at the expense of the introduction of a fiducial world volume metric $\gamma$, and the action can be written as

$$ S = \frac{T}{2} \int d^2 \sigma \sqrt{-\gamma (\gamma^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X_\mu - 1)}. $$

(5)

The three diffeomorphism symmetries corresponding to the three worldvolume coordinates can be used to eliminate three of the six metric components, which are fixed as

$$ \gamma_{0i} = 0, \gamma_{00} = -\frac{4}{\nu^2} \det h. $$(6)

One can introduce light front coordinates

$$ X^\pm = \frac{1}{\sqrt{2}} (X^1 \pm X^{D+2}), $$

(7)

and choose the light front gauge $X^+ = \tau$ which eliminates two of the $d + 2$ fields. The gauge fixed Hamiltonian in the light front gauge takes on the form

$$ H = \frac{\nu T}{4} \int d^2 \sigma \left( \dot{X}^i \dot{X}^i + \frac{2}{\nu^2} \{X^i, X^j\} \{X^i, X^j\} \right). $$

(8)

In the above formula, $X^i$ are the transverse degrees of freedom and there are $d$ of them. The curly brackets are the familiar Poisson brackets on the membrane surface, signifying the invariance of the action under the algebra of area preserving diffeomorphisms. The gauge fixed Hamiltonian can be regularized by replacing the
algebra of area preserving diffeomorphisms on the membrane surface by that of finite $N \times N$ matrices. For specificity, one can take the membrane to have the topology of $S^2 \times R$ in which case one can approximate the algebra of area preserving diffeomorphisms on the two sphere by matrices which transform in the $N$ dimensional irreducible representation of $su(2)$. The free parameter $\nu$ which was introduced in the problem while gauge fixing can be set to $N$. The details of the the approximation method leading to the regularization of membrane motions by the quantum mechanics of matrices can be found at various places in the literature; see for example [23, 1]. The final answer for the regularized form of the membrane Hamiltonian in $D + 2$ dimensions is given by the quantum mechanics of $D$ hermitian matrices as

$$H = g^3 \text{Tr} (\Pi_i \Pi_i) - \frac{1}{4g^2} \text{Tr} ([X^i, X^j][X^i, X^j])$$  \hspace{1cm} (9)

where

$$g^3 = 2\pi l_p^3.$$ \hspace{1cm} (10)

One could also allow the background spacetime to have fluxes to which the membrane degrees of freedom can couple. To take the simplest concrete example, one could look at zero-brane quantum mechanics of type IIA string theory in the presence of a non vanishing vev of the four form flux. The dynamical model is described by the quantum mechanics of three hermitian matrices $X_i$, the Euclidean action for which is

$$S = \int dt T \text{Tr} \left( \frac{1}{2} \dot{X}_i^2 - \frac{g^2}{4} [X_i, X_j]^2 + \frac{i\kappa}{6} \epsilon_{ijk} X_i X_j X_k \right).$$ \hspace{1cm} (11)

The cubic interaction term expresses the interaction between the matrix model and the four form flux. The constant four form flux is taken to be

$$F^{(4)}_{i\ell jk} = -2\kappa \epsilon_{ijk}.$$ \hspace{1cm} (12)

$T = \frac{2\pi}{g_s}$ is the zero brane tension. It is convenient to scale the matrices $X_i \rightarrow \frac{1}{\sqrt{T}} X_i$ and define

$$\frac{1}{\sqrt{T}} = g$$ \hspace{1cm} (13)

so that the Euclidean space action takes on the form

$$S = \int dt T \text{Tr} \left( \frac{1}{2} \dot{X}_i^2 - \frac{g^2}{4} [X_i, X_j]^2 + \frac{i\kappa}{3} \epsilon_{ijk} X_i X_j X_k \right).$$ \hspace{1cm} (14)

In the action the dimension of the matrices $X_i$ is $\sqrt{T}$ while those of $g^2$ and $\kappa$ are $\frac{1}{T}$ and $\frac{1}{T}$ respectively. Matrix models with similar cubic couplings also arise when one considers M(atrix) theories in plane wave backgrounds i.e the BMN matrix model and in mass-deformed matrix models obtained from the dimensional reduction of super Yang-Mills theories. For the purposes of this paper, we could turn on such a four-form flux and couple it to the motion of a membrane in a $d + 5$ dimensional flat background. The flux, will pick out three special directions, and at the level of matrix quantum mechanics, three of the $d + 3$ matrices, will couple to the background flux. The resultant matrix mechanics will be a straightforward generalization of the Myers’ model described above. The resultant action would be

$$S = \int dt T \text{Tr} \left( \frac{1}{2} \dot{X}_i^2 - \sum_{i,j=1}^{d+3} \frac{g^2}{4} [X_i, X_j]^2 + \frac{i\kappa}{3} \sum_{a,b,c=1}^{3} \epsilon_{abc} X_i X_j X_k \right).$$ \hspace{1cm} (15)
Several other examples of matrix mechanics involving cubic Chern-Simons type couplings will also be studied in the final parts of the paper in the context of supersymmetric matrix models.

**From Matrix Models to Spin Chains:**

Let us now briefly review the connection between the perturbative dynamics of Hamiltonian matrix models and quantum spin chains. This connection, which was first established by Rajeev and Lee [8], will be central to the computation of the spectrum of various models of interest to us. Let us consider a matrix model Hamiltonian of the kind

\[
H = \text{Tr} \left( \frac{1}{2} (\Pi_i^2 + \mu^2 \Phi_i^2) + \frac{1}{N} \Psi_{ijkl} \Phi^i \Phi^j \Phi^k \Phi^l + \frac{1}{\sqrt{N}} \mu \Psi_{ijkl} \Phi^i \Phi^j \Phi^k \Phi^l \right) \tag{16}
\]

The tensors \( \Psi^3, \Psi^4 \) encode information about the cubic and quartic interaction terms. Looking ahead at the prospect of taking the 't Hooft large \( N \) limit in various analyses that we shall perform, we have incorporated various factors of \( \frac{1}{\sqrt{N}} \) in the generic Hamiltonian above so that it does possess a well defined 't Hooft large \( N \) limit. The chief qualitative difference in the nature of the Hamiltonian given above and the two examples discussed previously [10,15] is the quadratic 'mass' term. The dynamical emergence of such mass terms is of course what a substantial part of the paper will be devoted to. Keeping later applications in mind we recall how the perturbative large \( N \) analysis of matrix model Hamiltonians becomes a problem of diagonalizing quantum spin chains.

Let us start by introducing the matrix creation and annihilation operators

\[
A_i = \frac{1}{\sqrt{2\mu}} (\mu X_i + i \Pi_i) \tag{17}
\]

and their adjoints, in terms of which the Hamiltonian takes on the form

\[
H = \text{Tr} \left( \mu A_i^\dagger A_i + \frac{1}{4N\mu^2} \Psi_{ijkl} [A_i + A_i^\dagger] [A_j + A_j^\dagger] [A_k + A_k^\dagger] [A_l + A_l^\dagger] \right) + \left( \Psi_{ijkl} \left( \frac{1}{2^{1/2} N^{1/2} \mu^{1/2}} \right)^2 (A_i + A_i^\dagger)(A_j + A_j^\dagger)(A_k + A_k^\dagger) \left( A_i + A_i^\dagger \right) \right) \tag{18}
\]

We shall be interested in computing the first order correction to energies of single trace states such as

\[
|i_1 i_2 \cdots i_J > = \frac{1}{\sqrt{N^{J/2}}} \text{Tr} \left( A_{i_1 i_2}^\dagger A_{i_2 i_3}^\dagger \cdots A_{i_J i_1}^\dagger \right) |0 > . \tag{19}
\]

All such single trace states are eigenstates of the free Hamiltonian with an eigenvalue \( \mathcal{E}_0 = \mu J \). The perturbative corrections to the energies of the free Hamiltonian can be arranged in an expansion in powers of \( \frac{1}{\mu^2} \).

\[
\mathcal{E} = \mu J + \frac{1}{\mu^2} \mathcal{E}_1 + \cdots \tag{20}
\]

For an eigenstate of the free Hamiltonian \( |\mathcal{I} > \),

\[
\mathcal{E}_1 = < \mathcal{I} | V | \mathcal{I} > \text{ where } V = V^4 + V^3. \tag{21}
\]

\( V \) is the vertex that contributes to the first order energy shift, and it can be built out of the cubic and quartic terms in [13] as

\[
V^4 = H^4, V^3 = \frac{H^3 (I - \Pi) H^3}{H_0 - J}. \tag{22}
\]
Π is the projector to the subspace of the Hilbert space orthogonal to the one spanned by states of length \( J \), while \( H^3, H^3 \) are the quartic and cubic terms in (18) respectively.

Since only the diagonal matrix elements of \( V \) matter for first order perturbation theory, it is sufficient to keep those terms in \( H^4 \) and \( H^3 \) that have an equal number of creation and annihilation operators. Not all such terms are important if the large \( N \) limit is taken. The terms in \( V \) that have leading order matrix elements of \( O(1) \) are the ones for which normal ordering is compatible with the ordering implied by the trace. i.e they are terms that involve a string of creation operators followed by a string of annihilation operators sitting inside a trace. We can denote such operators by the symbol \( \Theta \); for example,

\[
\Theta_{i,j}^{k,l} = \frac{1}{N} Tr(A^i A^j A^k A^l).
\] (23)

The action of these operators on single trace states can be written down in closed form.

\[
\Theta_{i,j}^{k,l}|i_1 \cdots i_J>= \sum_{m=1}^{J} \delta_{i_m}^{k} \delta_{i_{m+1}}^{l} |i_1 \cdots i_{m-1}i_ji_{m+2} \cdots i_J> + O(\frac{1}{N}).
\] (24)

In other words the operators \( \Theta \) can be thought of as a 'machine' that runs along the entire length of the state and checks if any neighboring indices match the lower indices of the operators, and if they do, it replaces the neighboring indices by the upper indices. This leads to a map between large \( N \) matrix models and quantum spin chains. The single trace states can be thought of as states of a quantum spin chain with periodic boundary conditions, the periodicity being inherited from the cyclicity of the trace. The indices \( i_1 \cdots i_J \) are to be thought of as the 'spins' of the spin chain, which take on two values. A general map between quantum spin chains and large \( N \) matrix models has been proposed and discussed at length in [8] and it has been also used in some detail to study integrability of the dilatation operator of \( \mathcal{N} = 4 \) super Yang-Mills theory in [9, 10]; however for the purposes of the present discussion it is sufficient to restrict ourselves to spin chains with nearest neighbor interactions. The \( so(d) \) invariant operators obtained from \( V \), that survive the large \( N \) limit are \( \Theta_{i,j}^{i,j}, \Theta_{j,i}^{i,j} \) and \( \Theta_{i,i}^{i,j} \). It is clear that \( \Theta_{i,j}^{i,j} \) acts as the number operator, thus in the large \( N \) limit

\[
\Theta_{i,j}^{i,j} = Tr(A^i A^j) = J
\] (25)

The other two operators can be mapped to specific spin chain operators. The action of these spin chain operators on neighboring spins is

\[
\Theta_{j,i}^{i,j} = P_{i,j} \cdot P(i \otimes j) = j \otimes i
\]

\[
\Theta_{i,j}^{j,i} = K_{i,j} \cdot K(i \otimes j) = \delta_{i,j} \sum_{m} (m \otimes m).
\] (26)

In the above equations the arguments of \( K \) and \( P \) are to be interpreted as spins sitting on neighboring sites. \( P \) permutes neighboring spins and \( K \) traces over spin values. The large \( N \) spectrum at one loop for a general \( so(d) \) invariant matrix model is given by a nearest neighbor spin chain of the following generic form.

\[
H_{1-loop} = \mu I_{i,j}^{i,j} + \frac{1}{\mu^2} \sum_{l} (\alpha I_{i,l} + \beta P_{i,l} + \gamma K_{i,l}).
\] (27)

The coefficients \( \alpha, \beta, \gamma \) carry information about the specific details about the matrix model being studied. To avoid various divergences that might potentially arise due to normal ordering issues, it is useful to compute
not the absolute energies of various states, but the energies of various states relative to a reference state. Since \( \sum I_{l+1} = J \) for all states of a given length, all terms proportional to the identity operator drop out when one measures the energies with respect to a particular state. The one loop spin chains that accomplish the task of measuring the energies with respect to a reference state are of the form

\[
\Delta^1 = \frac{1}{\mu^2} \sum_l (\beta P_{l,l+1} + \gamma K_{l,l+1}) \tag{28}
\]

Of great interest in membrane dynamics are the following type of couplings.

\[
\Psi_{ijkl}^4 = -\frac{1}{2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \tag{29}
\]

so that the quartic term can be written as \(-\frac{1}{4} [\Phi^i, \Phi^j]^2\). The particular cubic coupling that will be of importance later in the paper is of the Chern-Simons type

\[
\Psi_{ijk}^3 = iv\epsilon_{ijk}, \tag{30}
\]

where \( \nu \) is a numerical constant. Keeping in mind the various models that we are going to study, we shall list \( \Delta^1 \) for these two classes of models

**I:** \( so(d) \) invariant matrix models for which \( \Psi^3 = 0 \).

\[
\Delta^1 = \frac{1}{2\mu^2} \sum_l \left( \frac{1}{2} K_{l,l+1} - P_{l,l+1} \right) \tag{31}
\]

**II:** \( so(3) \) invariant models for which \( \Psi^3_{ijk} = -iv\epsilon_{ijk} \).

\[
\Delta^1 = \frac{1}{2\mu^2} \sum_l \left( (3\nu^2 - 1) P_{l,l+1} + \frac{1}{2}(1 - 9\nu^2) K_{l,l+1} \right) \tag{32}
\]

### 3 Dynamical Mass Generation via Resummed Perturbation Theory

We shall now revert back to the original matrix models for flat space membrane dynamics \( [10, 15] \) and bring them to a form such that their spectra can be analyzed by quantum spin chain techniques. The crucial ingredient for this to be possible is the presence of a mass term in the matrix model Hamiltonian, the dynamical generation of which shall be the focus of this section. As outlined previously, we would like to set up a well defined perturbation theory for estimating the mass-gaps as well as the spectra of the matrix models of interest to us. On the face of it this might appear to be a self-contradictory goal. Dynamical mass generation is usually regarded as a non-perturbative phenomenon. Indeed, the methods that have been employed to estimate mass-gaps in matrix models in the past, see for example \( [25] \), do indeed make use of non-perturbative techniques, such as Schwinger-Dyson equations, to estimate the spectral gaps. On the other hand, the spin-chain techniques, that we ultimately want to make use of to do a large \( N \) Hamiltonian analysis appear to be an efficient method of doing the usual perturbative expansion. A way out of this impasse is provided if one can discover a new 'hidden' parameter, which in our case will turn out to be \( \frac{1}{d-1} \), in which the spectrum of excitations as well as the mass gap of the theory can be expanded.
Indeed, one of the key points that we would like to bring out from this analysis is the emergence of this new parameter and show that it controls the perturbation theories employed both to estimate the mass gap as well as the one used for carrying out a Hamiltonian quantization of the theory around the quantum corrected effective potential.

We start with the Euclidean action for membrane motion without fluxes\(^{(10)}\), which is

\[
S = \int dt \frac{1}{g^3} Tr \left( \frac{1}{2} \dot{X}_i \dot{X}_i - \frac{1}{2N} \sum_{j>i} [X_i, X_j]^2 \right) . i, j = 1 \cdots d - 2. \tag{33}
\]

We now add and subtract a mass term to the Euclidean action

\[
S = S_m - \hbar \int dt \frac{1}{2g^3} m^2 Tr(X_i X_i) \tag{34}
\]

where

\[
S_m = S + \frac{1}{g^3} \int dt \frac{1}{2} m^2 Tr(X_i X_i). \tag{35}
\]

The goal is not to change the theory but only to rearrange the Feynman diagrams. Thus \(\hbar\) will have to be set equal to one. We shall use the numerical value of \(\hbar\) only at the end of the computation. Its role is that of a loop counting parameter to help us carry out an efficient reorganization of the Feynman diagrams. The action, after appropriate insertions of factors of \(\hbar\), reads as

\[
S = \int dt \frac{1}{2g^3} Tr \left( \dot{X}_i \dot{X}_i + m^2 X_i X_i - \frac{\hbar}{2N} [X_i, X_j]^2 \right) - \hbar \int dt \frac{m^2}{2g^3} Tr(X_i X_i). \tag{36}
\]

Clearly, this is nothing but the original action if \(\hbar\) is set to one. We shall let \(m^2\) admit an expansion in \(\hbar\).

\[
m^2 = m_1^2 + \hbar m_2^2 + \hbar^2 m_3^2 + \cdots \tag{37}
\]

The propagator can be read off and it is

\[
G_2(p) = \langle (X_i)^\delta_a(p)(X_j)^\epsilon(p) \rangle = \frac{\delta_a^\delta_b \delta_{ij}}{g^3 (p^2 + m^2 - \hbar m^2) + \Sigma(p)^{\hbar}}, \tag{38}
\]

where \(\Sigma(p)\) is the self-energy correction to the propagator. Clearly, \(\Sigma(p)\) has an expansion in powers of \(\hbar\),

\[
\Sigma(p) = \hbar \Sigma_1(p) + \hbar^2 \Sigma_2(p) + \cdots \tag{39}
\]

We can now invoke a self-consistency argument, which is that if \(m\) is indeed the gap in the spectrum i.e the location of the pole of the propagator then looking at the form of the propagator above, we see that \(m_1, m_2\) etc must chosen to cancel the perturbative correction to the propagator that arise from the self energy diagrams \(\Sigma_1, \Sigma_2 \cdots\) order by order in perturbation theory. This sets up a series of gap equations that determine the mass gap of the matrix model\(^2\).

So far, we have set up the problem of determining the mass gap as a perturbative computation in \(\hbar\), but of course, \(\hbar\) is an invented parameter and its numerical value is one. That \(\hbar\) is a reasonable parameter to do

\(^2\)This generic technique is known to work rather well for gauge theories as well. For example it has been used to get a remarkably good estimate of the mass gap of pure Yang-Mill in 2+1 dimensions in [24].
perturbative computations in will have to be validated by an explicit demonstration that the corrections to
the mass, when thought of as an expansion in this artificial parameter, turn out to be small in comparison
to the leading order or one-loop estimate. That is indeed the case, and we explicitly demonstrate that in
the next section. As a matter of fact, a two loop computation which is reported below, suggests that what
appears to be an expansion in $\hbar$ is really an expansion in $\frac{1}{d-1}$. This is indeed gratifying, as this generates for
us a bona-fide parameter in which to carry out perturbative computations of the mass gap in the problem.
We shall now proceed to carry out explicit perturbative computations for the mass gap and arrange the
results in a $\frac{1}{d-1}$ expansion. In what follows, we shall only focus on the planar diagrams, which give the
leading order results for the mass-gap in $\frac{1}{N}$.

3.1 Perturbative Corrections to the Mass:

One Loop Mass-Gap:
The one loop correction to the mass is given by the condition that $m^2_1$ cancel out the one-loop self energy
diagram:

$$\frac{hm^2_1}{g^3} = h\Sigma_1.$$  \hfill (40)

$\Sigma_1$ is given by the standard 'tadpole' integral

$$\Sigma_1 = \Upsilon_1 \int \frac{dp}{2\pi p^2 + m^2_1}$$ \hfill (41)

$\Upsilon_1$ is the standard combinatorial factor that counts the number of diagrams contributing to the tadpole
graph in the large $N$ limit. This number may be computed easily enough. To do that, we first observe
that out of the two terms in the potential energy, $Tr(X_iX_jX_iX_j)$ and $Tr(X_iX_jX_i)$, only the the second
one contributes in the large $N$ limit. That is the so, because, it is only in this diagram that the internal
momentum loop of the tadpole diagram results from the Wick contraction of two neighboring $X$ fields
sitting inside a trace, which leads to a contribution of order $N$. Let us suppose, we were computing the
correction to the $<X_1X_1>$ propagator, then clearly, the number of ways in which a $X_1$ field from the
$Tr(X_iX_jX_i)$vertex can be attached to an external line corresponding to the propagator is two. There is
also an overall factor of $d-1$ from the $X$ fields running in the loop. Thus

$$\Upsilon_1 = 2(d-1).$$ \hfill (42)

Thus the gap equation at the one loop order gives the value of the mass-gap to be

$$m^3_1 = (d-1)g^3.$$ \hfill (43)

Two Loop Correction To The Mass:

At the two loop level, there are two different sources of contributions to terms of $O(h^2)$. Since $\hbar$ appears
explicitly in the formula for the propagator, the tadpole diagram, which goes as

$$\int \frac{dp}{2\pi p^2 + m^2 - hm^2}$$ \hfill (44)

will generate a term of order $h^2$. Apart from this there is the usual two loop contribution due to the
'double-scoop' and 'sunset' diagrams$^3$. Thus

$$\Sigma_2 = \frac{1}{2} \frac{g^3}{m^2} I(p)$$ \hfill (45)

$^3$We are borrowing the terminology of [12] for the Feynman diagrams
The first term in the expression for $\Sigma_2$ above comes from the expansion of the formula for the 'one loop' tadpole diagram in higher powers of $\hbar$, while the second term arises from the usual double scoop and sunset two loop Feynman graphs. $I(p)$ being the total contributions of the Feynman integrals, discussed in the appendix. It suffices for now to know the numerical value of these diagrams, which is,

$$I(p) = \frac{1}{2}((d-1)^2 + (d-1)).$$  \hspace{1cm} (46)

(45) explicitly, demonstrates the intertwining of terms, that would naively be considered 'one-loop' diagrams, such as the 'tadpole' diagram with two loop diagrams in the second order contribution to the correction of the mass. Thus the two loop gap equation is:

$$\frac{h(d-1)}{m} + \frac{h^2(d-1)}{2m} - \frac{h^2g^3}{2m^4}((d-1)^2 + (d-1)) = \frac{hm^2}{g^3}.$$ \hspace{1cm} (47)

To solve this equation perturbatively, we can set

$$m^3 = g^3(d-1)(1 + \hbar \delta).$$ \hspace{1cm} (48)

The solution to $O(\hbar)$ is

$$\delta = -\frac{1}{2(d-1)}.$$ \hspace{1cm} (49)

Thus the two loop correction to the mass can be written as

$$m = ((d-1))^{1/3}g(1 - \frac{\hbar}{6(d-1)} + \cdots) = m_1(1 - \frac{\hbar}{6(d-1)} + \cdots).$$ \hspace{1cm} (50)

Hence, we see that the two loop correction to the mass is explicitly suppressed by a factor of $\frac{1}{d-1}$. Thus, as advertised before, the computation of the mass gap can indeed be organized in a $\frac{1}{d-1}$ expansion, at least to the two loop order.

Even when $\frac{1}{d-1} = 1$, we see that the combinatorics and the numerical values of the Feynman integrals involved add up to suppress the two loop contribution. As argued in [25] it is possibly too much to expect that the perturbation series for $m$ actually converges, however it can possibly be regarded as a good asymptotic expansion, of which, we shall keep only the leading order term in later spin-chain computations.

**Aside on the Two Loop Feynman Integrals:**

The contributions of the 'double scoop' and the 'sunset' diagrams can be written as

$$\frac{g^3}{m^4}(\Upsilon_2 I_2 + \Upsilon_3 I_3)$$ \hspace{1cm} (51)

where $\Upsilon_{2,3}$ are the combinatorial factors that count the number of diagrams contributing to the 'double scoop' and the 'sunset' graphs at large $N$. $I_{2,3}$ are the relevant Feynman integrals

$$I_2 = \int \frac{dpdp'}{(2\pi)^2} \frac{1}{(p^2 + 1)(p^{'2} + 1)^2} = \frac{1}{8}$$ \hspace{1cm} (52)

and

$$I_3 = \int \frac{dpdp'}{(2\pi)^2} \frac{1}{(p^2 + 1)(p^{'2} + 1)((p-p^{'})^2 + 1)} = \frac{1}{4\pi^2} \int_{-\pi/2}^{\pi/2} dx \int_{-\pi/2}^{\pi/2} dx' \frac{1}{(\tan(x) - \tan(x'))^2 + 1} = \frac{1}{12}$$ \hspace{1cm} (53)
\( \Upsilon_2 \) is the number of Feynman graphs contributing to the double scoop diagram. This can be computed as follows. Once again, let us assume that we are computing the correction to the \( <X_1X_1> \) propagator. Once again it is easy to convince oneself that the only vertex contributing to the double scoop diagram, just as in the case of the tadpole diagram is \( Tr(X_1X_1X_jX_j) \). The number of ways in which one of the \( X_1 \) fields can be contracted with a an external leg of the propagator is two. The number of ways the two \( X_j \) fields can be contracted with the corresponding fields of the second vertex is also 2. One also picks up a factor of \( d-1 \) from each loop. Thus

\[
\Upsilon_2 = 4(d-1)^2. \tag{54}
\]

In the case of the sunset diagram, we shall have to consider the cases when both the vertices are of the \( Tr(X_iX_iX_jX_j) \) and that when both are of the \( Tr(X_iX_iX_jX_j) \) type. Counting along similar lines as the case above shows that the total number of planar diagrams contributed by the first instance is \( 2(d-1) \) and the number when both the vertices are of the \( Tr(X_iX_iX_jX_j) \) type is \( 4(d-1) \). The case that involves mutual contractions between the vertices of these two types is of lower order in \( \frac{1}{N} \). Thus

\[
\Upsilon_3 = 6(d-1), \tag{55}
\]

leading to

\[
\Upsilon_2 I_2 + \Upsilon_3 I_3 = \frac{1}{2} (d-1) \tag{56}
\]

which was used in \( \ref{56} \).

### 3.2 Mass-Gaps For Models With Chern-Simons Couplings:

The above analysis can be easily extended to estimate the mass gap in flat space membrane motion in the presence of fluxes i.e to the generalization of the Meyers’ model described in \( \ref{15} \). The action that we now consider is

\[
S = \int dt Tr \left( \frac{1}{2} \dot{X}_i^2 - \frac{g^2}{4} [X_i, X_j]^2 + \frac{i g \kappa}{3} \epsilon_{abc} X_a X_b X_c \right). \tag{57}
\]

In the commutator squared interaction term of the action, \( i, j = 1 \cdots d + 3 \) while for the Chern-Simons coupling \( a, b, c = 1, 2, 3 \). The total number of Bosonic matrices in the problem is \( d+3 \), \( d \) of which transform under \( so(d) \), while three of the matrices which are picked out by the anisotropy induced by the flux transform under \( so(3) \). The symmetry group of the problem is obviously \( so(d) \times so(3) \). The one-loop mass for the \( so(d) \) scalars is the same as before. Adapting the result \( \ref{43} \) to the present problem we can read off the one loop masses of the \( so(d) \) scalars to be

\[
m^3_{so(d)} = Ng^2(d+2). \tag{58}
\]

To do a one-loop estimation of the mass gap for the \( so(3) \) scalars we shall have two Feynman diagrams to consider. The first one corresponds to one insertion of the quartic vertex and the second one involves insersions of the cubic vertex. The leading large \( N \) contribution of the insertion of the quartic vertex has already been discussed at length in the previous section. As far as the cubic vertices are concerned, it can be seen, upon carrying out the contraction of the color indices that the leading order large \( N \) contribution comes from the \( Tr(X_1X_2X_3) Tr(X_1X_3X_2) \) vertex. The momentum integrals are all convergent and can be evaluated in closed form by elementary means. The resulting gap equation reads as

\[
(d + 2) \lambda + \frac{\lambda \kappa^2}{m^2_{so(3)}} = m^3_{so(3)}, \tag{59}
\]

13
where,
\[
\lambda = N g^2
\]  
(60)
is the ’t Hooft coupling of the matrix model. To solve this equation we first observe that \(\kappa^2\) and \(\lambda^{2/3}\) have the same dimensions. Thus, we can define a dimensionless parameter \(\beta\) such that
\[
\kappa^2 = \beta \lambda^{2/3}.
\]  
(61)
In the regime where the strength of the four form flux is weak, \(\beta < 1\), we can solve the gap equation order by the ansatz
\[
m_3^3 = C \lambda f(\beta) \left( f(\beta) = 1 + f_1 \beta + f_2 \beta^2 + \cdots \right)
\]  
(62)
The gap equation now becomes a recursion relation for the coefficients \(f_i\) while \(C = d + 2\). The first few coefficients are
\[
f_1 = \frac{1}{(d + 2)^{5/3}}, f_2 = \frac{2}{3((d + 2)^{5/3})} f_1.
\]  
(63)
The higher coefficients in the series fall off as powers of \(\beta^{5/3}\). Thus the mass gaps for the \(so(3)\) scalars arrange itself in a \(\frac{1}{d}\) expansion, and to leading order in \(\frac{1}{d}\) all the \(d + 3\) scalars of the problem have the same mass.

4 Hamiltonian Analysis of the Matrix Models

Having estimated the mass gaps for various Bosonic matrix models, we are now in a position to carry out a Hamiltonian quantization of these models and study their large \(N\) spectrum using the techniques of quantum spin chains. Let us begin by deriving the one-loop energy operator \(\Delta_1\) for the most general Bosonic model analyzed so far, which is the \(so(d) \times so(3)\) symmetric model considered in the previous section.

After incorporating the dynamically generated mass term (62), the Hamiltonian of the matrix model model becomes
\[
H = Tr \left( \frac{1}{2} \Pi_i^2 + \frac{m^2}{2} X_i X_i - g^2 \sum_{i,j=1}^{d+3} [X_i, X_j]^2 + \frac{ig\kappa}{3} \sum_{a,b,c=1}^3 \epsilon_{abc} X_a X_b X_c \right).
\]  
(64)
In writing down the above Hamiltonian, it is of course implied that we are starting from an effective action in which the self energy corrections to the propagators have been accounted for at the one-loop level. Since, the method of re-summed perturbation theory does not affect the vertices of the matrix model, they remain unchanged at the level of the Hamiltonian as well. The only modification is the inclusion of the dynamically generated mass term, which is the same (62) for all the \(d + 3\) scalars at the one loop level.

The mass terms completely lift the classical flat directions and one can now proceed to compute the energy operator. As outlined in the ’introduction’ the perturbative computation is best done in the basis of matrix creation and annihilation operators
\[
A_i = \frac{1}{\sqrt{2m}} (m X_i + i \Pi_i), A_i^\dagger = \frac{1}{\sqrt{2m}} (m X_i - i \Pi_i).
\]  
(65)
The Hamiltonian becomes
\[
H = \lambda^{1/3} Tr \left( \mu A_i^\dagger A_i - \frac{1}{16 N \mu^2} [A_i^\dagger + A_i, A_j^\dagger + A_j]^2 + i \frac{\sqrt{3}}{3(N^{1/2} \lambda^{3/2} \mu^{3/2})} \epsilon_{ijk} (A_i^\dagger + A_j^\dagger) (A_j + A_k) (A_k + A_i^\dagger) \right)
\]  
(66)
where \( \mu \) is defined to be
\[
m = \lambda^{1/3} \mu = \lambda^{1/3} (Cf(\beta))^{1/3}.
\] (67)

Since the symmetry of the problem splits into \( so(d) \times so(3) \) it makes sense to write down the energy operator for the \( so(3) \) and the \( so(d) \) part of the Hamiltonian separately. That can be done easily enough using the basic formulae [31,32]; and the result is
\[
\Delta_{1,so(d)} = -\frac{1}{2\mu^2} \sum_l \left( P_{l,l+1} - \frac{1}{2} K_{l,l+1} \right).
\] (68)

\[
\Delta_{1,so(3)} = \frac{1}{2\mu^2} \sum_l \left( (3\theta^2 - 1) P_{l,l+1} + \frac{1}{2} (1 - 9\theta^2) K_{l,l+1} \right), \quad \theta^2 = \frac{\beta}{\mu^2}.
\] (69)

It is worth noting that \( \mu \) is essentially the same quantity that controls the perturbative expansion of the mass-gap discussed earlier. From the above formulae, it is also clear that \( \mu \) is the dimensionless parameter that controls the Hamiltonian perturbation theory as well.

### 4.1 Integrability:

Having a spin-chain formalism for the one-loop energy operator does not automatically imply that we shall be able to solve for its spectrum. The obvious obstacle in this direction is that generic \( so(d) \) invariant quantum spin chains are not integrable by the methods of the algebraic Bethe ansatz. For an \( so(d) \) invariant quantum spin chain, with nearest neighbor interactions, \( d > 2 \) and spins in the defining representation, to be integrable, the Hamiltonian has to be of the following form [3,33]
\[
H = \alpha \sum_l \left( g' P_{l,l+1} + K_{l,l+1} \right), \quad g' = (1 - \frac{d}{2}).
\] (70)

\( \alpha \) is an arbitrary coupling constant. One is of course free to add a term proportional to the identity operator to the Hamiltonian and not loose integrability. It is easy to see that neither one of [68,69] is of the form required by integrability. However, one can still make progress, as non-integrable spin chains also contain sectors can indeed be studied by the Bethe ansatz. To proceed further, let us briefly recall the construction of integrable spin chains starting from an underlying \( R \)-matrix, which for \( so(d) \) spin chains is [3,33]
\[
R_{12}(u) = u(h g' - u) I_{12} + h(h g' - u) P_{12} + h u K_{12}.
\] (71)

\( g' = (1 - \frac{d}{2}) \). For later uses, we have put in a parameter \( h \) in the \( R \)-matrix to act as a bookkeeping device. The numerical value of \( h \) is one and it is nothing but the strength of non-commutativity of the spin operators at the same lattice point. The \( R \)-matrix is an operator that acts on a tensor product of two vector spaces both of which are \( C^d \) when the spins are in the defining representation of \( so(d) \). The indices 1, 2 refer to the respective vector spaces, while \( u \) is the spectral parameter\(^4\). For a spin chain with \( J \) sites the transfer matrix is built out of the \( R \)-matrix as
\[
T(u) = R_{01}(u) R_{02}(u) \cdots R_{0J}(u).
\] (72)

\(^4\) For a pedagogical introduction to the use of \( R \) matrices in the construction of integrable spin chains, see [32].
The subscript 0 refers to the auxiliary space, which is not a part of the physical Hilbert space of the spin chain, and it will be traced over. Let $T_{0}$ denote the trace over the auxiliary space. The transfer matrix can be obviously be expanded in powers of the spectral parameter

$$T(u) = \sum_{m} u^{m} T_{m}$$

and taking the trace over the auxiliary space of both sides of the above equations produces a generating function for the local conserved charges of the spin chain.

$$t(u) = Tr_{0} T(u) = \sum_{m} u^{m} t_{m}.$$  \hspace{1cm} (74)

Finding a closed form of for the local charges is usually a tedious but straightforward computation; however, we shall be interested in only the first few charges, especially the second one which corresponds to the Hamiltonian. The first charge is

$$t_{0} = Tr_{0} \prod_{l=1}^{J} (\hbar^{2} g')^{J} P_{0l}$$ \hspace{1cm} (75)

It can be shown that this charge corresponds to the discrete lattice shift operator.

$$t_{0}(\mathcal{V}_{1} \otimes \mathcal{V}_{2} \otimes \cdots \otimes \mathcal{V}_{J}) = (\hbar^{2} g')^{J} (\mathcal{V}_{2} \otimes \mathcal{V}_{3} \otimes \cdots \otimes \mathcal{V}_{J} \otimes \mathcal{V}_{1}) \hspace{1cm} (76)$$

The next term in the expansion is

$$t_{1} = \sum_{l=1}^{J} t_{0} (\hbar g' P_{l,l+1} - \hbar I_{l,l+1} + \hbar K_{l,l+1}).$$ \hspace{1cm} (77)

Since $t_{0}$ is a constant of motion and we are free to add constant terms to the Hamiltonian it is easy to see that the most general form for the nearest neighbor Hamiltonian with a $so(d)$ symmetry is

$$H = \alpha \sum_{l=1}^{J} (g' P_{l,l+1} - \beta I_{l,l+1} + K_{l,l+1})$$ \hspace{1cm} (78)

where $\alpha$ and $\beta$ are arbitrary constants. The crucial requirement for integrability is the relative coefficient between the permutation and trace operators which has to be $(1 - \frac{d}{2})$ for an $so(d)$ spin chain. This clearly indicates the lack of integrability of the energy operators \cite{68,69}.

**Integrable Sub-sectors:**

Even when an $so(d)$ spin chain is not integrable it is possible to find integrable sub-sectors of the theory. As has been reviewed in the appendix, if one constructed and $su(d')$ integrable spin chain out of the relevant $\mathcal{R}$ matrix the integrable nearest neighbor Hamiltonian would be

$$H = \alpha \sum_{l=1}^{J} P_{l,l+1}.$$ \hspace{1cm} (79)

$so(d = 2d')$ or $so(d = 2d' + 1)$ of course contains $su(d')$. If we focussed only on spins transforming under this $su(d')$, then at the level of the Hamiltonian, it would correspond to dropping the trace operator, and we shall be left with an integrable $su(d')$ sector of the theory. To find this embedding of $su(d')$ at the level
of the Hilbert space, one needs to isolate a subspace of the Hilbert space of the spin chain such that

1: It is annihilated by the trace operator

2: It remains closed under the action of the spin chain Hamiltonian.

When \( d = 2d' \), one can always find an integrable \( su(d') \) sub-sector for the theory as follows. One can form \( d' \) complex combinations of the \( 2d' \) spins \( i_1 \cdots i_{2d'} \) as

\[
I_1 = i_1 + \sqrt{-1}i_{d'}+1 \cdots \hat{I}_{d'} = i_{d'} + \sqrt{-1}i_{2d'}.
\]

(80)

The sub-space of the spin-chain Hilbert space spanned by states such as

\[
|f> = f_{l_1} \cdots f_{l_J} |I_{l_1} \cdots I_{l_J}>
\]

(81)

involve only traceless \( so(d) \) tensors and are hence annihilated by the trace operator. In this sub-space, the spin chain Hamiltonian involves only the identity and the permutation operators and is hence integrable. When \( d = 2d' + 1 \), the biggest closed integrable sector is the same as the case when \( d = 2d' \). Of course, we cannot use this simplification to diagonalize any part of the Hamiltonian (69), for which we shall resort to some approximate methods described later in the paper. However, the presence of the integrable \( su(d') \) sector allows us to study the spectrum of Bosonic membranes in flat backgrounds in dimensions greater than four.

There is an important exception to the list of integrable spin chains formed out of \( so(d) \) invariant \( R \) matrices discussed above and that corresponds to \( d = 2 \). In four dimensions, which corresponds to the case \( d = 2 \), the above analysis of integrability does not apply and the corresponding one-loop spin chain does turn out to be integrable! Since the Bethe ansatz for this particular case is rather transparent and the generic nature of the spectrum of the \( d = 2 \) case is indeed the same as that of the spin chains corresponding to integrable \( su(d') \) subsectors of membranes in higher dimensions we shall start by analyzing this special case first. We shall then generalize the result to the integrable \( su(d') \) sectors of various bosonic membranes at one loop.

### 4.2 The Special Case of \( d = 2 \)

To solve for the spectrum of the special case \( d = 2 \), the one loop energy operator we shall employ the techniques of coordinate space Bethe ansatz. Let us first write the trace and permutation operators in terms of the Pauli spin matrices as follows.

\[
P_{l,l+1} = \frac{1}{2} \left( I(l) \otimes I(l+1) + \sum_i \sigma^i(l) \otimes \sigma^i(l+1) \right)
\]

\[
K_{l,l+1} = \frac{1}{2} \left( I(l) \otimes I(l+1) + \sigma^x(l) \otimes \sigma^x(l+1) + \sigma^y(l) \otimes \sigma^y(l+1) + 3 \sigma^y(l) \otimes \sigma^y(l+1) \right).
\]

(82)

The energy operator can then be expressed as

\[
\Delta_{1,so(2)} = \frac{1}{8\mu^2} \left( \sum_{l=1}^{J} [\sigma^x(l)\sigma^x(l+1) + \sigma^z(l)\sigma^z(l+1) + 3\sigma^y(l)\sigma^y(l+1)] \right)
\]

(83)

This is nothing but the anisotropic Heisenberg chain, also known as the \( XXZ \) spin chain\(^5\). The spin chain has an obvious symmetry which allows one to carry out rotations in the \( x-z \) plane by rotating \( \sigma^x \) and \( \sigma^z \) among

\(^5\)The name the model refers to the fact that the direction of the anisotropy is usually chosen to be the \( Z \) direction. We can bring the model to that form simply by re-labeling the sigma matrices.
each other. This is nothing but a reflection of the original \(so(2)\) symmetry of the interaction \(Tr[X_1, X_2]^2\) term of the matrix model Hamiltonian. We can also express the hamiltonian in the form

\[
\Delta_{1, so(2)} = \gamma \left( \sum_{l=1}^{J} [S^1_2(l)S^2_1(l+1) + S^2_2(l)S^1_1(l+1) + \delta(S^1_1(l)S^2_2(l+1) + S^2_1(l)S^1_2(l+1))] \right) \quad (84)
\]

In the above equation, \(\gamma = \frac{1}{4\mu^2}\) and \(\delta = -3\). We have introduced the Weyl operators, \(S^i_j(l)\), that check if the spin at the \(l\)th site is equal to \(j\) and if it is, then \(S^i_j(l)\) replaces the value of the spin by \(i\). If the value of the spin at the \(l\)th site does not match with \(j\) then this operator annihilates the state. These operators satisfy the associative Weyl algebra

\[
S^i_j(l)S^k_m(l) = \delta^k_j S^i_m(l) \quad (85)
\]

at the same site and the Weyl operators at different sites commute with each other. With this form \(\Delta_{1, so(2)}\) of the Hamiltonian, it is clear that the Ferromagnetic state, with all the spins up(1) is a ground state of the Hamiltonian.

\[
|0> = |1, 1, 1 \cdots 1>
\]

(86)

zero energy. The complete set of excited states of the Hamiltonian can be built by considering linear combinations of states with a fixed number of down/impurity spins. The one magnon state or the state with a single impurity can be taken to be of the form

\[
|\Psi_1 > = \sum_{l=1}^{J} f(l)|l>
\]

(87)

where \(|l>\) is the state with an upturned spin at the \(l\)th site. If the state is taken to be a plane wave

\[
f(l) = e^{ipl}
\]

(88)

then it is easy to see that it will be a eigenstate of the Hamiltonian with eigenvalue

\[
E(p) = -4\gamma \left( 1 + 2 \sin^2(p \frac{J}{2}) \right). \quad (89)
\]

The allowed values of the momenta are determined from the condition that the state be cyclically symmetric i.e

\[
e^{ipJ} = 1 \Rightarrow p = \frac{2n\pi}{J}. \quad (90)
\]

The one magnon state gives us the dispersion relation \(\Delta_{1, so(2)}\) and we shall have to consider the two magnon state to get the two magnon scattering matrix. The state with two 'down' spins can be written as

\[
|\Psi_2 > = \sum_{\{x_1 < x_2\}}^{J} f(x_1, x_2)|x_1, x_2>
\]

(91)

where it is understood that \(|x_1, x_2>\) is the state with upturned spins at \(x_1\) and \(x_2\). Once again we can make a Bethe ansatz by taking a linear combination of plane waves that interact with each other by exchanging momenta

\[
f(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S(p_1, p_2)e^{i(p_2 x_1 + p_1 x_2)}. \quad (92)
\]
Such a state will be an eigenstate of the Hamiltonian if the two magnon scattering matrix $S(p_1, p_2)$ is given by

$$S(p_1, p_2) = \frac{1 + e^{i(p_1 + p_2)} + 2\delta e^{i p_2}}{1 + e^{i(p_1 + p_2)} + 2\delta e^{i p_1}}$$

(93)

The eigenvalue is given by

$$E(p_1, p_2) = -4\gamma \sum_{i=1,2} \left( 1 + \sin^2 \left( \frac{p_i}{2} \right) \right)$$

(94)

which shows that the energies of the two magnons simply add up. The momenta are determined by the condition that the total phase picked up by one of the magnons by travelling around the spin chain is nothing but the two body scattering matrix, i.e.

$$e^{ip_1 J} = S(p_1, p_2), e^{ip_2 J} = S(p_2, p_1).$$

(95)

These are the Bethe equations for the two unknowns $p_1, p_2$ and they provide us with the complete solution to the two magnon problem. The $XXZ$ quantum spin chain is an integrable system, and one of the manifestations of integrability is the factorized nature of the multi-magnon scattering matrix into two particle $S$ matrices. In other words, the above equations also provide us with the solution to the the $m < J$ magnon problem. The eigenstate with $m$ down spins has energy given by

$$E = -4\gamma \sum_{i=1}^{m} \left( 1 + \sin^2 \left( \frac{p_i}{2} \right) \right)$$

(96)

with the momenta determined by

$$e^{ip_k J} = \prod_{(j\neq k)=1}^{m} S(p_k, p_j).$$

(97)

These are $m$ equations for the $m$ unknown momenta that provide us with the complete spectrum of the spin chain. To get a feel for the qualitative nature of the spectrum as a function of $J$ we can invoke a $BMN$ limit and work in a the dilute gas approximation, where the number of magnons is very small compared to the length of the chain i.e $m <\ll J$. We can write (97) by taking the logarithm of both sides of the equation as

$$p_k J = 2n\pi + \Theta(p_1, p_2).$$

(98)

$\Theta$ is the two body scattering phase shift. In the dilute gas approximation, the magnons do not scatter at all, and have $\Theta \sim 0$. The dispersion relation for the magnons can then be written as

$$E = -4\gamma \left( \sum_{i=1}^{m} \left( 1 + \left( \frac{n_i \pi}{J} \right)^2 \right) \right).$$

(99)

This is the leading order behavior of the spectrum of the theory. There will in general be corrections to this formula involving higher powers of $\frac{1}{J}$. However if we take the limit, where, $\frac{1}{J}$ is small i.e the BMN limit [15], then the above formula, which resulted from a one loop large $N$ perturbative computation should be expected to capture the qualitative nature of the spectrum of the theory to leading order in $J$. 

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4.3 Bethe Ansatz for the Integrable \( SU(d') \) Sub-Sectors

The large \( N \) one loop energy operator in the \( su(d') \) sector of the spin chains can be written as with \( \alpha = -\frac{1}{2\pi} \), as the restriction to the \( su(d') \) sector simply amounts to dropping the trace operator from the spin chain Hamiltonian. The spins for this chain are in the fundamental representation of \( su(d') \). This spin chain is known in the literature as the generalized Heisenberg model. It is known to be integrable and the corresponding Bethe equations take on the form

\[
\left( \frac{u_{m,i} + i\alpha_m \cdot \vec{w}}{u_{m,i} - i\alpha_m \cdot \vec{w}} \right)^J = \prod_{j \neq 1}^{nm} \frac{u_{m,i} - u_{m,j} + i\alpha_m \cdot \vec{a}_m}{u_{m,i} - u_{m,j} - i\alpha_m \cdot \vec{a}_m} \prod_{m' \neq m}^{nm'} \frac{u_{m,i} - u_{m',j} + i\alpha_{m'} \cdot \vec{a}_{m'}}{u_{m,i} - u_{m',j} - i\alpha_{m'} \cdot \vec{a}_{m'}}
\]

\( \alpha_m \) are the simple roots of the Lie algebra and \( w \) is the highest weight of the representation. The details of the derivation of these Bethe equations have been reproduced in the appendix. To use the Bethe equations and derive the results that put this generic case on the same footing as the previous results on the special case of the \( so(2) \) case let us analyze the case of the above equation corresponding to all the impurities of the 1 type. In general, one can have as many types of impurities as the rank of the Lie algebra. However, it is enough to study the special case \( m = 1 \) to get a feel for the nature of the spectrum. In this case, calling \( u_{1,i} \) \( \mu_i \), the Bethe equations become

\[
\left( \frac{\mu_j + i/2}{\mu_j - i/2} \right)^J = \prod_{k \neq j} \frac{\mu_j - \mu_k + i/2}{\mu_j - \mu_k - i/2}
\]

By writing the equation in terms of the momenta \( p_i \)

\[
e^{-ip} = \frac{\mu - i/2}{\mu + i/2}
\]

rather than the rapidities, we can write the Bethe equations as

\[
e^{ip_k J} = \prod_{(j \neq k) = 1}^{m} S(p_k, p_j),
\]

where

\[
S(p_1, p_2) = -\frac{1 + e^{i(p_1 + p_2)} + 2e^{ip_2}}{1 + e^{i(p_1 + p_2)} + 2e^{ip_1}}.
\]

This is completely analogous to the equations for the \( so(2) \) spin chain, except that \( \delta = 1 \) in the present case. That is to be expected, because, the case of having impurities of the 1 type reduces the problem to the usual spin half Heisenberg chain, which is the same as the \( XXZ \) chain with the value of the anisotropy \( \delta = 1 \). As discussed in the appendix, the dispersion relation takes on the form

\[
E = \sum_{i=1}^{m} \epsilon(p_i), \epsilon(p) = -\frac{1}{\mu^2} \cos(p).
\]

As before, one can also take the limit of large \( J \), in which case the magnons/impurities behave as free particles on a circle. In this limit the momenta are given by

\[
p_i = \frac{2n_i \pi}{J},
\]
where \( n_i \) are integers. They can take on any value, but must be constrained to obey the level matching or zero momentum constraint implied by the cyclicity of the trace i.e \( \sum_i n_i = 0 \). The dispersion relation, simplifies to 

\[
\epsilon(n) = - \left( 1 + \frac{2n^2\pi^2}{J^2} \right).
\]  

(107)

If one allowed for other types of impurities, then the form of the Bethe equations would be more complicated, however, the qualitative facts about the nature of the spectrum given by the simplest case above do not change. The effect of the other types of impurities, in a chain of finite length would be to change the allowed values of the momenta. However, the dispersion relation would remain the same. Moreover, only \( u_1 \) enters the dispersion relations, thus the only effect that the other types of impurities would have are indirect. Although the Bethe equations written out in their full form are rather complicated algebraic equations, the lessons learnt from the simplest case of 1 type of impurities carry over to the more general scenario as well.

4.4 Enhancement of Integrability in the large \( J \) Limit

From the previous discussion of the construction of integrable \( so(n) \) invariant spin chains with nearest neighbor interactions from the the standard \( R \) matrix it follows that the form of the Hamiltonian is highly constrained from the requirements of integrability. Only Hamiltonians with very special relative coefficients between the permutation and the trace operator fulfill this requirement. Although, we were able to find rather large integrable subsectors for the \( so(d) \) spin chains, it is worthwhile to ask the question whether or not one can find integrable behavior in some limit of the spin chains without having to resort to the truncation to sub-sectors. If it is indeed possible, then we should be able to gauge the behavior of the complete spectrum of the theory albeit in some approximation scheme. The simplification of the nature of the spectrum in the large \( J \) limits of the various exactly solvable sectors described in the previous sections suggests that the large \( J \) limit might be such an approximation. This is indeed the case. In this final section on the Bosonic matrix models of the paper, we shall formally demonstrate that the large \( J \) limits of the various \( so(d) \) spin chains studied so far can be mapped to integrable non-linear sigma models irrespective of whether the original spin chain is integrable or not.

Since we are interested in spin chains with nearest neighbor interactions, and nearest neighbor interactions are nothing but the lattice Laplacian, the non-linear sigma models we expect to approximate the low lying spectrum of the spin chains will be free sigma models with an appropriate homogeneous space as its target space. However, most classical sigma models of this kind are known to be integrable. Thus, it is reasonable to expect that the low lying excitations of an \( so(d) \) invariant quantum spin chain with nearest neighbor interactions will be described by an integrable classical sigma model even if the original spin chain is not integrable. We shall now proceed to make this idea more precise. We begin by writing the \( R \) matrix in a slightly different but equivalent form

\[
R(u)_{i,j} = I_{i,j} + \frac{\hbar}{u} P_{i,j} + \frac{\hbar}{h g' - u} K_{i,j}, g' = \left( 1 - \frac{d}{2} \right).
\]  

(108)

\( P_{i,j}, K_{i,j} \) are the operators that permute and trace over the spins belonging to the \( i \) and \( j \)th vector spaces respectively. \( \hbar \) is the natural ‘quantum’ deformation parameter in the problem. An alternative way of writing the \( R \) matrix is as a \( d \times d \) operator valued matrix. For example, \( R_{i,j} \) can be expressed as a matrix, each of whole elements is an operator on the \( i \)th vector space as,

\[
R(u)_{\mu\nu} = \delta_{\mu\nu} I(i) + \frac{1}{u} S_{\nu\mu}(i) + \frac{1}{g' - u} S_{\mu\nu}(i), \mu, \nu = 1 \cdots d.
\]  

(109)
In writing the $R$ matrix in the second form, we have introduced the Weyl operators $S_{\mu,\nu}(i)$, which act as $|\mu\rangle\langle\nu|$ on the $i$ th vector space. The Weyl operators satisfy the standard commutation relations,

$$[S_{\mu\nu}(i), S_{\alpha\beta}(j)] = \hbar \delta_{i,j} (\delta_{\nu\alpha} S_{\mu\beta} - \delta_{\mu\beta} S_{\nu\alpha}).$$

(110)

The relations satisfied by the transfer matrix, the so called RTT relations, are written most succinctly as,

$$R_{1,2}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{1,2}(u - v).$$

(111)

1, 2 refer to two auxilliary spaces. It is more insightful to write this equation out in terms of the matrix elements of $T$.

$$\frac{1}{\hbar}[T_{ab}(u), T_{cd}(v)] + \frac{1}{u - v} (T_{ad}(u)T_{cb}(v) - T_{ad}(v)T_{cb}(u)) + \frac{1}{\hbar g' - (u - v)} (\delta_{a,b} T_{lc}(u)T_{ld}(v) - \delta_{c,d} T_{al}(v)T_{bd}(u)) = 0$$

(112)

We shall now write the corresponding RTT relation in the large $J$ limit. To do that we shall define

$$\hbar = \frac{\hbar'}{J}, u', v' = \frac{u}{J}, \frac{v}{J}, S = \frac{S}{J}$$

(113)

In terms of the rescaled variables, the $R$ matrix becomes

$$R(u)_{\mu\nu} = \delta_{\mu\nu} I(i) + \frac{\hbar'}{Ju'} S_{\mu\nu}(i) + \frac{\hbar'}{Ju'} S_{\mu\nu}(i) \approx \delta_{\mu\nu} I(i) + \frac{\hbar'}{Ju'} S_{\mu\nu}(i) + \frac{\hbar'}{Ju'} S_{\mu\nu}(i) + O(\frac{1}{J^2}).$$

(114)

Needless to say that the $S$ operators satisfy

$$[S_{\mu\nu}(i), S_{\alpha\beta}(j)] = \frac{\hbar'}{J} \delta_{i,j} (\delta_{\nu\alpha} S_{\mu\beta} - \delta_{\mu\beta} S_{\nu\alpha}).$$

(115)

From the two preceding equations it is quite clear that the term in the $R$ matrix that involved $g'$, and which was ultimately responsible for the $g'$ dependence of the Hamiltonian, is lower order in $\frac{1}{J}$ and that $\frac{\hbar'}{J}$ plays the role of the deformation parameter in the large $J$ limit. The RTT relations can be written out as

$$\frac{1}{J}[T_{ab}(u'), T_{cd}(v')] + \frac{1}{u' - v'} (T_{ad}(u')T_{cb}(v') - T_{ad}(v')T_{cb}(u')) + \frac{1}{(u' - v')}(\delta_{a,b} T_{lc}(u')T_{ld}(v') - \delta_{c,d} T_{al}(v')T_{bd}(u')) = 0$$

(116)

We also observe that

$$[T_{ab}(u'), T_{cd}(v')] = \frac{\hbar'}{J}\{t_{ab}(u'), t_{cd}(v')\} + O(\frac{1}{J^2})$$

(117)

where $t$, as a matrix is the same as $T$, except that its entries are to be thought of as ordinary functions and not operators. Thus to leading order in $\frac{1}{J}$, the RTT relations may be replaced by

$$\{t_{ab}(u'), t_{cd}(v')\} + \frac{1}{u' - v'} (t_{ad}(u')t_{cb}(v') - t_{ad}(v')t_{cb}(u')) + \frac{1}{(u' - v')}(\delta_{a,b} t_{lc}(u')t_{ld}(v') - \delta_{c,d} t_{al}(v')t_{bd}(u')) = 0,$$

(118)
This simply implies that the matrix elements of $t$ satisfy the following Poisson brackets:

$$\{t(u') \otimes t(v')\} = [r(u' - v'), t(u') \otimes t(v')]$$

(119)

where,

$$r(x - y) = \frac{1}{x - y}(P - K).$$

(120)

To clarify the notation it is worth mentioning that equation (119) is to be thought of as expressing the equality of the actions of both sides of the equation on two copies of a $d$ dimensional vector space $\mathbb{H}$. The main implication of (119) is that $\text{Tr}t$ is the generator of an infinite family of conserved charges i.e.

$$\{\text{Tr}t(u), \text{Tr}t(v)\} = 0.$$  
(121)

Our analysis of large $J$ integrability of the continuum sigma models obtained from (68,69) would be complete if we can show that the relevant sigma models are contained in the family of commuting charges that follow from the classical transfer matrix $t$. This is indeed the case. From a technical point of view, taking the continuum limit of the spin chain amounts to taking its coherent state expectation value in a long coherent state of length $J$ and deriving an effective Hamiltonian in terms of the group parameters characterizing the coherent state. For $\text{so}(d)$ valued spin chains with nearest neighbor interactions this has been worked out in detail in [9], and we shall refer to that paper for the detailed descriptions of many of the results that will be used here. The coherent state construction of [9] when applied to (68,69) generate the following sigma model Hamiltonians

$$H = \alpha \int dx \text{Tr}(\partial_x M \partial_x M).$$

(122)

$\alpha$ is the effective coupling of the sigma model. It is $-\frac{1}{16\mu} \frac{J}{2} P$ for (68) and $\frac{3\theta^2 - 1}{8\mu} \frac{J}{2} P$ for (69). $x$ is nothing but the continuum coordinate along the length of the spin chain. $M(x,t)$ is a $d \times d$ antisymmetric matrix and it satisfies the Poisson brackets

$$\{M(x)_{ij}, M(x')_{kl}\} = \delta(x-x')(\delta_{jk}M_{il}(x) + \delta_{il}M_{jk}(x) - \delta_{ik}M_{jl}(x) - \delta_{jl}M_{ik}(x))$$

(123)

The matrix $M$ is built out of group parameters that parameterize the $\text{so}(d)$ coherent state and it can be written in terms of complex $d$ vectors $Z_i$ that satisfy

$$Z_i Z_i = Z_i^* Z_i = 0, Z_i^* Z_i = 1$$

(124)

as

$$M_{ij} = Z_i Z_j^* - Z_j Z_i^*. $$

(125)

This parametrization along with the various various algebraic identities for $M$ that it leads to have also been discussed in [9]. The equations of motions for $M$ can be derived using the Poisson structure given above

$$\partial_t M = 4\alpha \partial_x [M, \partial_x M]$$

(126)

The equations of motion are equivalent to the following flatness condition.

$$[\partial_t + A_t, \partial_x + A_x] = 0, $$

(127)
where

\[ A_x = \frac{1}{u} M \]

\[ A_t = \frac{4\alpha}{u} [M, \partial_x M] - \frac{4\alpha}{u^2} M \]  

(128)

are the two components of the Lax connection. The Poisson brackets between the spatial components of the Lax connection can be written down as

\[ \{ A_x(x, u), A_x(x', u') \} = \delta(x - x') [r(u - u'), A_v(x, u) \otimes I + I \otimes A_v(x', u')] \]  

(129)

The above equation expresses the Poisson bracket between the dynamical variables as a Lie bracket in so(d). This is the familiar Lie-Poisson structure characteristic of classical integrable two dimensional field theories. The immediate consequence of the Lie-Poisson structure is that

\[ t(u) = P e^{\int_0^J A_x(x) dx} \]  

(130)

satisfies the Poisson brackets. This of course implies the fact that Trt(u) is a generating function of the integrals of motion for the Hamiltonian. This establishes the integrability of the sigma model describing the large J limit of the spin chain. We thus see that all the nearest neighbor spin chains that arise as the one loop energy operators for matrix models with dynamically generated masses correspond to integrable sigma models in the large J limit.

A cautionary remark is in order while discussing the continuum limits of the spin chains. The so(d) invariant spin chain is Ferromagnetic and the continuum limit described above does approximate the spectrum around the Ferromagnetic ground state. However, is anti-ferromagnetic. Thus it is to be understood that the sigma model following from the above construction, when applied to the so(3) case approximates the spectrum of the spin chain around its highest energy state. The sigma model following from the continuum limit of the so(3) chain around the anti-Ferromagnetic vacuum is a level two su(2) Wess-Zumino model, and we shall refer to for the details of that continuum limit.

5 Mass Deformed Super-Membranes and Integrability:

In the final section we shall analyze models of mass deformed supersymmetric quantum mechanics obtained from the dimensional reduction of minimally supersymmetric Yang-Mills theories in various dimensions. The resultant matrix models can be taken to be matrix regularized light cone descriptions of super-membranes propagating in plane wave type backgrounds in one higher dimension than the one in which the original gauge theory was formulated. The gauge theories in question are minimally supersymmetric Yang-Mills theories in spacetime dimensions six four and three. If one carried out a naive dimensional reduction of these Yang-Mills theories to one dimension, the resultant matrix quantum mechanics system would not have any quadratic mass terms. The matrix quantum mechanics models can nevertheless be interpreted as regularized versions of non-critical M(atrix) theory. This is very much along the lines of the original BFSS proposal for M(atrix) theory in eleven dimensions. The gauge theory in question there is of course \( N = 1 \) SYM in \( D = 10 \). In the presence of enough supersymmetries, one cannot be confident of the success of the mechanism for dynamical mass generation that we applied to the case of Bosonic membranes earlier in the paper. There

\footnote{In a recent paper it has been argued that if one allowed for central extensions of the underlying SUSY algebra for the matrix models then mass gaps might be generated even in the presence of supersymmetry. It would of course be interesting to see if the methods of resummed perturbation theory can capture the mass gaps of systems possessing centrally extended supersymmetry.}
might be 'true' flat directions in the quantum mechanical system due to cancelations between Bosonic and
Fermionic self-energy diagrams.\(^7\) However, there does exist a second mechanism for adding mass terms in
supersymmetric matrix models. One can add explicit mass terms to the matrix model Hamiltonian and
ask for for a preservation of all the supersymmetries of the un-deformed model. This greatly constrains the
permissible types of mass deformations, and all possible mass deformations of the matrix models obtained
from dimensional reductions of minimally supersymmetric Yang-Mills theories in various dimensions were
carried out in\(^21\). Sometimes, it is possible to interpret the massive matrix quantum mechanics as being
related to the dimensional reduction of gauge theories, not on \(R^m\) but on \(R \times M^{m-1}\), where \(M\) is a \(m-1\)
dimensional compact manifold. For example the BMN matrix model can be regarded as the dimensional
reduction of \(\mathcal{N} = 4\) SYM on \(R \times S^3\).\(^19\) Obviously, the question as to whether or not there is always such
an interpretation of massive supersymmetric matrix quantum mechanical systems possibly requires further
study. Apart from the connection to Yang-Mills theories, such mass deformed matrix models can be regarded
as light-cone Hamiltonians for super-membranes in plane wave like backgrounds, once again, in one dimension
higher than the dimensionality of the original Yang-Mills theory\(^21\). Thus, they are quite naturally models
for non-critical M(atrix)theory Hamiltonians. Thought of in another way, this is nothing but the extension
of the relation of the BMN matrix model to a critical supermembrane theory\(^18\) in a plane wave background
to non-critical dimensions. Taking a clue from the recently discovered integrable behavior in the BMN
model\(^19\) it is natural to ask, whether or not the matrix models for non-critical membrane correspond to
integrable spin chains in the large \(N\) limit. That is the problem that we shall concern ourselves with in this
part of the paper. As in the BMN model\(^19\), a good starting point for the analysis of integrability is provided
by the Bosonic sectors of these models at one loop. Without integrability in the Bosonic sector at the one
loop level, there is of course no hope of expecting integrability at higher loops. In what follows, we shall
systematically analyze the Bosonic sectors of all models proposed by Kim and Park\(^21\) at the one loop level
at large \(N\).

The various mass deformed models obtained by Kim and Park\(^21\) can be classified into two broad
categories.

I: Models that admit non-trivial supersymmetric configurations saturating the unitary bounds of the corre-
sponding super algebras.

II: Models that do not possess any non-trivial supersymmetric configurations

In what is to follow, we first catalogue all the bosonic subsectors of the models presented in\(^21\). We then go
on to show that all the models that admit supersymmetric vacua also correspond to integrable spin chains
at the one loop level. On the other hand, models that do not have non-trivial supersymmetric vacua, do not
correspond to spin chains that are integrable, but they do possess integrable sub-sectors.

5.1 \(\mathcal{N} = 8\) Super Matrix Quantum Mechanics

As mentioned before, the massive matrix model obtained from \(\mathcal{N} = 1\) SYM in \(D = 10\) is nothing but the
BMN matrix model and it has been studied in substantial detail in the recent past\(^19\). The theory involves
sixteen supercharges, and it shares many common features with \(\mathcal{N} = 4\) SYM on \(R \times S^3\). Among the list of
common features is the integrability of a rather large sub-sector of the matrix model, the so called \(su(2|3)\)
sub-sector to the third order in large \(N\) perturbation theory\(^34\). At the one loop order, the complete theory,
without any truncation to specific sub-sectors also exhibits integrability\(^8\). The next lower dimension that admits a consistent formulation of \(\mathcal{N} = 1\) SYM is six. The mass deformed matrix quantum models obtained from this Yang-Mills theory are of two types. The matrix model that has non-trivial susy vacuum solutions has as its Bosonic part

\[
H = \text{Tr} \left( \frac{1}{2} \Pi_i \Pi_i + \frac{1}{2} \left( \frac{\mu}{3} \right)^2 X_a X_a + \frac{1}{2} \left( \frac{\mu}{6} \right)^2 X_A X_A - \frac{1}{4} [X_i, X_j]^2 + i \mu [X_3, X_4] X_5 \right)
\]

(131)
a = 3, 4, 5 and \(A = 1, 2\), while \(i, j\) in the quartic interaction term run from \(1 \cdots 5\) in the above equation. The mass deformation breaks the \(SO(5)\) symmetry of the original gauge theory down to \(SO(3) \times SO(2)\). This models admits a supersymmetric configuration corresponding to circular motion in the \((1, 2)\) plane and a fuzzy sphere in the \((4, 5, 6)\) directions.

\[
X_1 = R \cos \left( \frac{1}{6} t \mu \right) I, X_2 = R \sin \left( \frac{1}{6} t \mu \right) I, X_a = \frac{1}{3} \mu J_a
\]

(132)

\(\mathcal{J}\) are the standard \(so(3)\) generators satisfying

\[
[J_a, J_b] = i \epsilon_{abc} J_c.
\]

(133)

Clearly, states that involve only Bosonic excitations are eigenstates of the free part of the Hamiltonian, which is a sum of decoupled harmonic oscillators. It thus makes sense to compute the corrections to the tree level energies of the Bosonic excitations of this model. To do such a computation at the one loop level, one will have to take into account the contributions from the Fermionic part of the Hamiltonian, since the Fermions do run in loops. However, as far as the one-loop energy operators is concerned, the analysis of the BMN matrix models shows that contribution of the Fermionic terms to the one loop effective Hamiltonian, when written down in the spin chain language is proportional to the identity operator\([19, 22]\). But since we are interested in measuring the energies with respect to a reference state, that contribution of the Fermions to the energy operator as we have defined it in the paper vanishes. This general observation is true of the present and all the other models that we study in this section. Since the symmetry algebra of the matrix model is \(so(3) \times so(2)\), we can read off the contributions of these sectors to \(\Delta^1\), using the basic relations\([31, 32]\).

\[
\Delta^1_{so(2)} = \frac{18}{\mu^2} \sum_l \left( \frac{1}{2} K_{l,l+1} - P_{l,l+1} \right)
\]

(134)

\[
\Delta^1_{so(3)} = \frac{9}{2 \mu^2} \sum_l \left( 2 P_{l,l+1} - 4 K_{l,l+1} \right)
\]

(135)

The dimensional reduction of six dimensional super Yang-Mills theory also admits a second type of mass deformation, where the bosonic \(so(5)\) symmetry is broken to \(so(4) \times u(1)\). The Bosonic Hamiltonian for this mass deformation is

\[
H = \text{Tr} \left( \sum_{i=1}^5 \Pi_i \Pi_i + \frac{1}{2} \left( \frac{\mu}{3} \right)^2 X_1 X_1 + \frac{1}{2} \left( \frac{\mu}{6} \right)^2 \sum_{A=1}^4 X_A X_A - \frac{1}{4} \sum_{a<b=1}^5 [X_a, X_b]^2 \right)
\]

(136)

\(^8\)An exhaustive account of the one-loop integrability of the BMN matrix model along with its relation to \(\mathcal{N} = 4\)SYM can be found in\([22]\).
In this mass deformation, there are no BPS solutions analogous to the fuzzy sphere solution of the \( so(3) \times so(2) \) case. When one looks at \( \Delta^1 \) for this model, the only non-trivial contribution to it comes from the \( so(4) \) part. As before, using (31) we can write down

\[
\Delta^1_{so(4)} = \frac{18}{\mu^2} \sum_l \left( \frac{1}{2} K_{l,l+1} - P_{l,l+1} \right). \tag{137}
\]

The spins are now in the defining representation of \( so(4) \).

5.2 \( \mathcal{N} = 4 \) Super Matrix Quantum Mechanics

Next in our catalog, are the matrix models obtained from a mass deformation of the dimensional reduction of minimal super Yang-Mills in \( D = 4 \). As in the previous case, one can do two consistent mass deformations\(^{[21]}\). The Bosonic part of the Hamiltonian that admits a maximally supersymmetric fuzzy sphere configuration is

\[
H = \text{Tr} \left( \frac{1}{2} \Pi_i \Pi_i + \frac{1}{2} \left( \frac{\mu}{3} \right)^2 X_a X_a - \frac{1}{4} [X_i, X_j]^2 + i \mu [X_1, X_2 X_3] \right) \tag{138}
\]

This matrix model corresponds to the \( so(3) \) symmetric part of the previous model and the corresponding fuzzy sphere vacua are also the same.

\[
X_a = \frac{1}{3} \mu J_a, [J_a, J_b] = i \epsilon_{abc} J_c. \tag{139}
\]

The one loop energy operator for the scalars of this model is also precisely the same as the one obtained above.

\[
\Delta^1_{so(3)} = \frac{9}{2\mu^2} \sum_l (2P_{l,l+1} - 4K_{l,l+1}) \tag{140}
\]

A second kind of mass deformation of the dimensional reduction of \( D = 4, \mathcal{N} = 1 \) SYM results in a Bosonic Hamiltonian that has the \( so(3) \) symmetry broken down to \( so(2) \times u(1) \). The Hamiltonian for the Bosonic part is

\[
H = \text{Tr} \left( \frac{1}{2} \Pi_i \Pi_i + \frac{1}{72} \mu^2 (X_1 X_1 + X_2 X_2 + 4 X_3 X_3) - \frac{1}{4} [X_i, X_j]^2 \right) \tag{141}
\]

This model too admits a supersymmetric configuration preserving all of the four supercharges\(^{[21]}\). The supersymmetric directions correspond to

\[
X_1 = R \cos \left( \frac{1}{6} \mu t \right) \mathcal{I}, X_2 = R \sin \left( \frac{1}{6} \mu t \right) \mathcal{I}, X_3 = 0 \tag{142}
\]

\[
\Delta^1_{so(2)} = \frac{18}{\mu^2} \sum_l \left( \frac{1}{2} K_{l,l+1} - P_{l,l+1} \right). \tag{143}
\]

5.3 \( \mathcal{N} = 2 \) Super Matrix Quantum Mechanics

We now move on to the final case, which is that of dimensional reduction of three dimensional minimal super Yang-Mills. Unlike the previous cases, there is a unique mass deformed matrix model that one can obtain
from minimal super Yang-Mills in three spacetime dimensions. The bosonic part of the Hamiltonian of this model is

$$H = \text{Tr} \left( \frac{1}{2} \Pi_i \Pi_i + \frac{1}{72} \mu^2 (X_1 X_1 + X_2 X_2) - \frac{1}{4} [X_i, X_j]^2 \right)$$

(144)

There is a maximally supersymmetric BPS configuration in this case as well. It is given by the static solution

$$X_1 = R \cos(\frac{1}{6} \mu \mathcal{I}), \quad X_2 = R \sin(\frac{1}{6} \mu \mathcal{I}).$$

(145)

Finally, we write down the one loop energy operator for the model, which takes on the form:

$$\Delta^1_{\text{so}(2)} = \frac{18}{\mu^2} \sum_i \left( \frac{1}{2} K_{i, i+1} - P_{i, i+1} \right).$$

(146)

**Integrability:**

Having listed all the bosonic one-loop energy operators for the mass deformed models, we can just easily understand whether or not the spin chains are integrable or not. The basic results are summarized in the following table.

| Supercharges | Symmetry | Integrable Sectors |
|--------------|----------|--------------------|
| 8            | $\text{so}(3) \times \text{so}(2)$ | Both $\text{so}(3)$ and $\text{so}(2)$ sectors |
| 8            | $\text{so}(4) \times \text{u}(1)$ | $\text{su}(2) (\in \text{so}(4))$ sector |
| 4            | $\text{so}(3)$ | $\text{so}(3)$ sector |
| 4            | $\text{so}(2) \times \text{u}(1)$ | $\text{so}(2)$ sector |
| 2            | $\text{so}(2)$ | $\text{so}(2)$ sector |

From the summary given above, we see that all the cases that do support non-trivial fuzzy sphere type of supersymmetric configurations give rise to integrable spin chains. The corresponding spin chains have $\text{so}(3)$, $\text{so}(2)$ or their product (corresponding to $\mathcal{N} = 2, 4$ and 8 respectively) as their symmetry groups. Integrability of the $\text{so}(2)$ sectors of the $\mathcal{N} = 8, 4$ and 2 cases follows directly from the realization of $\text{so}(2)$ invariant nearest neighbor spin chains as $xxz$ models discussed previously. We can simply use these results to derive the Bethe ansatz for the three $\text{so}(2)$ cases of interest in the supersymmetric context. This simply amounts to replacing $\mu$ by $\frac{\mu}{\sqrt{3}}$ in the formulae(96,97) for the $\text{so}(2)$ Bethe ansatz given previously.

As far as the two $\text{so}(3)$ invariant sectors of the $\mathcal{N} = 8$ and $\mathcal{N} = 4$ cases are concerned, they are integrable as well! From form of the most general integrable $\text{so}(d)$ spin chain with nearest neighbor interactions given in[78], we see that the $\text{so}(3)$ chains given above have the correct form commensurate with integrability.

To get the spectrum and the Bethe equations, one can use the known Bethe equations for the defining representation of $\text{so}(3)$ [55-57, 58]. Since $\text{so}(3)$ has only one Cartan generator, the Bethe equations only allow for one type of impurity and they are very similar to the $\text{so}(2)$ equations. When written out in terms of the rapidities they read as

$$\left( \frac{u_k + i}{u_k - i} \right)^J = \prod_{j \neq k} \left( \frac{u_k - u_j + i}{u_k - u_j - i} \right),$$

(147)

while the dispersion relation is

$$\epsilon(u) = -\frac{36}{\mu^2} \frac{1}{u^2 + 1}$$

(148)

for the $\text{so}(3)$ chains related to both the $\mathcal{N} = 8$ and $\mathcal{N} = 4$ cases discussed above. This provides the complete one-loop solution for the two $\text{so}(3)$ sectors of interest.
The one example of a mass deformed model that does not have any non-trivial supersymmetric vacua corresponds to the case of the $\mathcal{N} = 8$ matrix quantum mechanics with symmetry group $so(4) \times u(1)$. The $so(4)$ spin chain (137) is clearly not integrable. However, as in the case of the matrix models for Bosonic membrane theories in flat spacetimes discussed previously in the paper, one can find integrable $su(2)$ subsectors of the theory by looking at states formed by excitations corresponding to $Z = X_1 + iX_3$ and $W = X_2 + iX_4$. The resultant spin chain is nothing but the spin one-half Heisenberg model, whose Bethe equations are

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^J = \prod_{j\neq k} \left(\frac{u_k - u_j + i}{u_k - u_j - i}\right).$$

The total energy is given by

$$E = -\sum_i \frac{36}{\mu^2} \left(\frac{u_i^2 - 1/4}{u_i^2 + 1/4}\right).$$

This completes the discussion of the one-loop spectra of the Bosonic sectors of the models proposed in [21]. Whether or not, this understanding of integrability extends to the more general sectors of the models for supermembranes and whether or not integrability is preserved at higher loops are of course open questions that might be of interest.

Before concluding the segment on the analysis of the matrix models given in [21], it is worth remarking that we have left out one particular example that was also presented in [21]. This last example corresponds to $\mathcal{N} = 1$ matrix quantum mechanics and is related to the dimensional reduction of minimal SYM in $D = 2$. This matrix model, which has a single Bosonic and a single Fermionic degree of freedom cannot quite be studied using the formalism presented here, as one needs at least two Bosonic matrices to be able to make it work. However, a different approach, might be in order here. Supersymmetric matrix quantum mechanics systems, involving a single Bosonic matrix degree of freedom can often be mapped to supersymmetric analogs of the Calogero-Sutherland model, see for example [43]. It might be possible to utilize such a connection in this case leading possibly to some exact statements about this model. This possibility probably merits further investigation.

6 Concluding Remarks

In this final section, we briefly return to the analysis of Bosonic membranes and sketch out some directions for future investigations. In sections three and four we presented a mechanism for estimating the non-perturbative mass-gaps in the spectra of Bosonic membranes and also developed an expansion of the matrix model Hamiltonians around the quantum corrected effective potentials using the techniques of quantum spin chains. It is of course implied that we were expanding the Hamiltonian around the trivial/oscillator vacuum around the effective potential. The low-lying excitations around the effective potential were shown to be well described by closed spin chains. In the D-brane picture, the closed spin chains could be thought of as the low-lying closed string like excitations of the membrane. Indeed, the sigma model for the spin chains derived as the continuum limits of the spin chains can be thought of as an effective string sigma model in a background provided by the membrane. The membrane point of view then provides us with an open-string interpretation of the sigma model. One could take this interpretation seriously and ask if there are analogues of bona-fide open string degrees of freedom in the membrane picture as well. This it indeed possible, if one looked at particular $O(N)$ excitations of the matrix models. The closed spin chains arose in our analysis because we focussed on gauge invariant operators built out of traces. However, taking a cue from the analysis
of the dilatation operator of $\mathcal{N} = 4$ SYM, we could just as well have looked at states built out of determinants and sub-determinants. Such operators can also be studied within the paradigm of quantum spin chains, except, one would have open spin chains i.e the analogs of the open string degrees of freedom to consider. Issues related to integrability of these spin chains tend to be rather subtle (see for example) and it might be of interest to investigate these degrees of freedom for bosonic membranes as well.

The complete spectrum of a theory of membranes is of course much richer than what has been discussed in this paper. The richness of the spectrum has to do with the fact that the trivial/oscillator vacuum of the bosonic matrix models is only one of many other vacua that are also present in the theories that we considered. An example of a non-trivial vacuum in a matrix quantum mechanical system was already alluded to in the section on supersymmetric matrix models. These are the so called fuzzy sphere vacua. The Bosonic membranes, coupled to Chern-Simons fluxes that were considered earlier in the paper also possess such vacua. These vacua continue to be present around the quantum/mass corrected effective potentials as well. It is well known that the expansions of the quantum mechanical matrix models around fuzzy sphere vacua leads to gauge theories in $2 + 1$ dimensions. For instance, $D = 2 + 1, \mathcal{N} = 4$ SYM and its connection to the expansion of the BMN matrix model around a fuzzy sphere vacuum has been discussed at length in and . Put differently, quantum mechanical matrix models can be thought of as providing a regularized description of gauge theories in three dimensions. The regularization is accomplished by introducing non-commutativity in the spacial directions. Gauge theories in three dimensions of course have mass-gaps in their spectra. For example, the mass gap in the spectrum of pure Yang-Mills in three dimensions has been computed in. It is thus plausible that the methods for estimating spectral gaps for matrix quantum mechanics that were presented in this paper can well be utilized to compute mass gaps and glueball masses for $D = 2 + 1$ Yang-Mills theories. Such a possibility was also pointed out in. The mass gap in $D = 2 + 1$ Yang-Mills can be estimated in two different ways. In, the gap was computed by applying re-summed perturbation theory to the gauge theory, while in a strong coupling expansion was utilized within the Hamiltonian framework. The answers obtained from both the approaches are strikingly close to each other. To draw a parallel at a matrix model level, the analysis of is very much along the lines of, while, the re-summed perturbation theory used in this paper is closer in spirit to. As mentioned previously, it would be indeed remarkable if these results can be applied to $D = 2 + 1$ Yang-Mills theories with or without supersymmetry. We hope to return to this problem in a future publication.

Finally, it is worth mentioning, that membrane dynamics can also be used to compute certain glueball masses for gauge theories in spaces of finite volume. Since, we can go beyond the computation of the masses and organize the computation of higher excited states of Bosonic membrane theories as quantum spin chain computations, our results might be applicable to improve upon the glueball spectroscopy of gauge theories in spaces of finite volume.

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9For a detailed study of various fuzzy sphere vacua in matrix models of the type considered in the paper see.

10The connection between fuzzy/non-commutative spaces and gauge theories has a rather large literature devoted to it. For a pedagogical introduction we refer to.
7 Appendix A:

In this appendix we outline the Bethe equations for an \( su(n) \) invariant spin chain with Hamiltonian given by

\[
H = - \sum_l P_{l,l+1}.
\] (151)

This discussion is meant to provide the background material for various formulae related to Bethe ansatz techniques throughout the paper. For a comprehensive discussion of \( su(n) \) Bethe ansatz techniques we shall refer to the original papers\[35, 36, 37, 38\].

The spins or the relevant chain are taken to be in the \( (n) \) representation of the algebra, and hence have \( n \) components. To find the spectrum of the Hamiltonian, one starts with the Lax operator that acts on \( \mathcal{H}_l \otimes \mathcal{V}_a \). \( \mathcal{H}_l \) is the 'one particle Hilbert space' associated with the site \( l \) and it is nothing but \( \mathbb{C}^n \). \( \mathcal{V}_a \) is the auxiliary vector space, and as a vector space, it is also \( \mathbb{C}^n \). The Lax operator is\[35\]

\[
L_{l,a} = a(\mu)I_{l,a} + b(\mu)P_{l,a}
\] (152)

where

\[
a(\mu) = \frac{\mu + i/2}{\mu - i/2}, \quad b(\mu) = \frac{i/2}{\mu - i/2}
\] (153)

One builds the transfer matrix from the Lax operator as

\[
T_{J,a}(\mu) = L_{J,a} \cdots L_{1,a}.
\] (154)

This transfer matrix satisfies the Yang-Baxter algebra

\[
R_{ab}(\mu - \nu)T_{J,a}(\mu)T_{J,b}(\nu) = T_{J,b}(\nu)T_{J,a}(\mu)R_{ab}(\mu - \nu).
\] (155)

The trace of the transfer matrix over the auxiliary space \( t(\mu) = \text{Tr}_a T_{J,a}(\mu) \) is the generator of conserved commuting charges, i.e.

\[
[t(\mu), t(\nu)] = 0.
\] (156)

The conserved charges can be obtained in the standard fashion by expanding the transfer matrix around \( \mu = -\frac{i}{2} \).

\[
O' = i \left( \frac{d}{d\mu} \right)^l \ln[t(\mu)t(0)^{-1}]|_{\mu=-\frac{i}{2}}.
\] (157)

In particular,

\[
H = O' - J.
\] (158)

The momentum is given, in terms of the Bethe roots, by

\[
e^{-ip} = \frac{\mu - i/2}{\mu + i/2}
\] (159)

while the dispersion relation

\[
\epsilon(p) = -2 \cos(p)
\] (160)

becomes

\[
\epsilon(\mu) = -2 \left( \frac{\mu^2 - 1/4}{\mu^2 + 1/4} \right).
\] (161)
The Bethe equations are
\[
\left(\frac{u_{m,i} + i\bar{\alpha}_m \bar{w}}{u_{m,i} - i\bar{\alpha}_m \bar{w}}\right)^J = \prod_{j \neq i}^{n_m} \frac{u_{m,i} - u_{m,j} + i\bar{\alpha}_m \bar{\alpha}_m}{u_{m,i} - u_{m,j} - i\bar{\alpha}_m \bar{\alpha}_m} \prod_{m' \neq m}^{n_{m'}} \frac{u_{m,i} - u_{m',j} + i\bar{\alpha}_m \bar{\alpha}_{m'}}{u_{m,i} - u_{m',j} - i\bar{\alpha}_m \bar{\alpha}_{m'}}
\]
(162)
\[
\alpha_m\text{ are the simple roots of the Lie algebra and } w\text{ is the highest weight of the representation. The eigenvalues of the Hamiltonian in terms of the solution of the Bethe equations requires further details of the Bethe ansatz, and we shall simply quote the result here while referring to [35, 36, 37, 38] for a more complete derivation. The total energy, is given in terms of the dispersion relation by}
\[
E = -\sum_{j=1}^{m} \epsilon(u_{1,j}).
\]
(163)
It is important to note however that only roots of the first type enter the dispersion relation, while the other roots only affect the energies indirectly.

To be able to use the Bethe equations, one also needs information about the roots and weights for the Lie algebra, which we also list below for the sake of completeness. For \(su(n)\), the simple roots are given by
\[
\alpha_m = \nu_m - \nu_{m+1}, m = 1 \cdots n - 1,
\]
(164)
while the \(n - 1\) dimensional weights are given by
\[
[\nu_j]_m = \frac{1}{\sqrt{2m(m + 1)}} \left( \sum_{k=1}^{m} \delta_{j,k} - m\delta_{j,m+1} \right).
\]
(165)
The weights satisfy
\[
\nu_m \nu_{m'} = -\frac{1}{2n} + \frac{1}{2} \delta_{mm'},
\]
(166)
while
\[
\alpha_m \alpha_{m'} = \delta_{mm'} - \frac{1}{2} \delta_{m,m'+1}.
\]
(167)
The highest weight in the defining representation \(\bar{w} = \nu_1\).

References

[1] W. Taylor, “M(atrix) theory: Matrix quantum mechanics as a fundamental theory,” Rev. Mod. Phys. 73, 419 (2001) [arXiv:hep-th/0101126].

[2] W. I. Taylor, “The M(atrix) model of M-theory,” arXiv:hep-th/0002016.

[3] J. A. Minahan and K. Zarembo, JHEP 0303, 013 (2003) [arXiv:hep-th/0212208].

[4] N. Beisert and M. Staudacher, “Long-range PSU(2,2—4) Bethe ansaezte for gauge theory and strings, Nucl. Phys. B 727, 1 (2005) [arXiv:hep-th/0504190].

[5] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D 69, 046002 (2004) [arXiv:hep-th/0305116].
[6] L. F. Alday, G. Arutyunov and A. A. Tseytlin, “On integrability of classical superstrings in AdS(5) x S**5,” JHEP 0507, 002 (2005) [arXiv:hep-th/0502240].

[7] G. Arutyunov, S. Frolov and M. Staudacher, “Bethe ansatz for quantum strings,” JHEP 0410, 016 (2004) [arXiv:hep-th/0406256].

[8] C. W. H. Lee and S. G. Rajeev, “Symmetries of large N(c) matrix models for closed strings,” Phys. Rev. Lett. 80, 2285 (1998) [arXiv:hep-th/9711052].

[9] A. Agarwal and S. G. Rajeev, “Yangian symmetries of matrix models and spin chains: The dilatation operator of N = 4 SYM,” Int. J. Mod. Phys. A 20, 5453 (2005) [arXiv:hep-th/0409180].

[10] A. Agarwal and S. G. Rajeev, “The dilatation operator of N = 4 SYM and classical limits of spin chains and matrix models,” Mod. Phys. Lett. A 19, 2549 (2004) [arXiv:hep-th/0405116].

[11] D. Berenstein, D. H. Correa and S. E. Vazquez, “All loop BMN state energies from matrices,” JHEP 0602, 048 (2006) [arXiv:hep-th/0509015].

[12] S. Bellucci, P. Y. Casteillo, J. F. Morales and C. Sochichiu, “Spin bit models from non-planar N = 4 SYM,” Nucl. Phys. B 699, 135 (1989).

[13] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A conjecture,” Phys. Rev. D 55, 5112 (1997) [arXiv:hep-th/9610043].

[14] R. C. Myers, “Dielectric-branes,” JHEP 9912, 022 (1999) [arXiv:hep-th/9910053].

[15] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from N = 4 super Yang Mills,” JHEP 0204, 013 (2002) [arXiv:hep-th/0202021].

[16] J. Hoppe and H. Nicolai, “RELATIVISTIC MINIMAL SURFACES,” Phys. Lett. B 196, 451 (1987).

[17] B. de Wit, M. Luscher and H. Nicolai, “The Supermembrane Is Unstable,” Nucl. Phys. B 320, 135 (1989).

[18] K. Dasgupta, M. M. Sheikh-Jabbari and M. Van Raamsdonk, “Matrix perturbation theory for M-theory on a PP-wave,” JHEP 0205, 056 (2002) [arXiv:hep-th/0205185].

[19] N. w. Kim, T. Klose and J. Plefka, “Plane-wave matrix theory from N = 4 super Yang-Mills on R x S**3,” Nucl. Phys. B 671, 359 (2003) [arXiv:hep-th/0306054].

[20] T. Fischbacher, T. Klose and J. Plefka, “Planar plane-wave matrix theory at the four loop order: Integrability without BMN scaling,” JHEP 0502, 039 (2005) [arXiv:hep-th/0412331].

[21] N. Kim and J. H. Park, “Massive super Yang-Mills quantum mechanics: Classification and the relation to supermembrane,” [arXiv:hep-th/0607005]

[22] T. Klose, PhD Thesis, Humboldt University, 2005.

[23] J. Hoppe, PhD Thesis, MIT, 1982, [www.aei-potsdam.mpg.de/ hoppe]

[24] G. Alexanian and V. P. Nair, “A Selfconsistent Inclusion Of Magnetic Screening For The Quark - Gluon Plasma,” Phys. Lett. B 352, 435 (1995) [arXiv:hep-ph/9504250].
[25] D. Kabat and G. Lifschytz, “Approximations for strongly-coupled supersymmetric quantum mechanics,” Nucl. Phys. B 571, 419 (2000) [arXiv:hep-th/9910001].
[26] L. Boulton, M. P. G. del Moral and A. Restuccia, “The supermembrane with central charges:(2+1)-D NCSYM, confinement and phase transition,” arXiv:hep-th/0609054.
[27] G. Gabadadze, “Modeling the glueball spectrum by a closed bosonic membrane,” Phys. Rev. D 58, 094015 (1998) arXiv:hep-ph/9710402.
[28] J. Hoppe, “MEMBRANES AND INTEGRABLE SYSTEMS, Phys. Lett. B 250, 44 (1990).
[29] C. K. Zachos, D. Fairlie and T. Curtright, “Matrix membranes and integrability,” arXiv:hep-th/9709042.
[30] T. Curtright, D. Fairlie and C. K. Zachos, “Integrable symplectic trilinear interaction terms for matrix membranes,” Phys. Lett. B 405, 37 (1997) arXiv:hep-th/9704037.
[31] J. Hoppe, “Some classical solutions of membrane matrix model equations,” arXiv:hep-th/9702169.
[32] L. D. Faddeev, “How Algebraic Bethe Ansatz works for integrable model,” arXiv:hep-th/9605187.
[33] N. Y. Reshetikhin, “INTEGRABLE MODELS OF QUANTUM ONE-DIMENSIONAL MAGNETS WITH O(N) AND SP(2K) SYMMETRY,” Theor. Math. Phys. 63, 555 (1985) [Teor. Mat. Fiz. 63, 347 (1985)].
[34] N. Beisert, “The su(2—3) dynamic spin chain,” Nucl. Phys. B 682, 487 (2004) arXiv:hep-th/0310252.
[35] B. Sutherland, “A General Model For Multicomponent Quantum Systems,” Phys. Rev. B 12, 3795 (1975).
[36] A. B. Zamolodchikov and V. A. Fateev, “Model Factorized S Matrix And An Integrable Heisenberg Chain With Spin 1,” Yad. Fiz. 32, 581 (1980).
[37] H. M. Babujian, “Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spin S,” Phys. Lett. A 90, 479 (1982).
[38] P. P. Kulish, N. Y. Reshetikhin and E. K. Sklyanin, “Yang-Baxter Equation And Representation Theory: I,” Lett. Math. Phys. 5, 393 (1981).
[39] I. Affleck, “Exact Critical Exponents For Quantum Spin Chains, Nonlinear Sigma Models At Theta = Pi And The Quantum Hall Effect,” Nucl. Phys. B 265, 409 (1986).
[40] G. Ferretti, R. Heise and K. Zarembo, “New integrable structures in large-N QCD,” Phys. Rev. D 70, 074024 (2004) arXiv:hep-th/0404187.
[41] L. D. Faddeev and L. A. Takhtajan, “HAMILTONIAN METHODS IN THE THEORY OF SOLITONS,” Springer,1987.
[42] P. Ramond, “FIELD THEORY: A MODERN PRIMER,” Front. Phys. 74, 1 (1989).
[43] A. Agarwal and A. P. Polychronakos, “BPS operators in N = 4 SYM: Calogero models and 2D fermions,” JHEP 0608, 034 (2006) arXiv:hep-th/0602049.
[44] V. Balasubramanian, D. Berenstein, B. Feng and M. x. Huang, “D-branes in Yang-Mills theory and emergent gauge symmetry,” JHEP 0503, 006 (2005) [arXiv:hep-th/0411205].

[45] A. Agarwal, “Open spin chains in super Yang-Mills at higher loops: Some potential problems with integrability,” JHEP 0608, 027 (2006) [arXiv:hep-th/0603067].

[46] J. M. Maldacena, M. M. Sheikh-Jabbari and M. Van Raamsdouk, “Transverse fivebranes in matrix theory,” JHEP 0301, 038 (2003) [arXiv:hep-th/0211139].

[47] H. Lin and J. M. Maldacena, “Fivebranes from gauge theory,” arXiv:hep-th/0509235.

[48] R. J. Szabo, “Quantum field theory on noncommutative spaces,” Phys. Rept. 378, 207 (2003) [arXiv:hep-th/0109162].

[49] G. Alexanian and V. P. Nair, “A Selfconsistent Inclusion Of Magnetic Screening For The Quark-Gluon Plasma,” Phys. Lett. B 352, 435 (1995) [arXiv:hep-ph/9504256].

[50] D. Karabali and V. P. Nair, “A gauge-invariant Hamiltonian analysis for non-Abelian gauge theories in (2+1) dimensions,” Nucl. Phys. B 464, 135 (1996) [arXiv:hep-th/9510157].

[51] D. Karabali and V. P. Nair, “On the origin of the mass gap for non-Abelian gauge theories in (2+1) dimensions,” Phys. Lett. B 379, 141 (1996) [arXiv:hep-th/9602155].

[52] D. Karabali and V. P. Nair, “Gauge invariance and mass gap in (2+1)-dimensional Yang-Mills theory,” Int. J. Mod. Phys. A 12, 1161 (1997) [arXiv:hep-th/9610002].

[53] D. Karabali, C. j. Kim and V. P. Nair, “On the vacuum wave function and string tension of Yang-Mills theories in (2+1) dimensions,” Phys. Lett. B 434, 103 (1998) [arXiv:hep-th/9804132].

[54] D. P. Jatkar, G. Mandal, S. R. Wadia and K. P. Yogendran, “Matrix dynamics of fuzzy spheres,” JHEP 0201, 039 (2002) [arXiv:hep-th/0110172].