Isoscalar monopole and dipole transitions in $^{24}$Mg, $^{26}$Mg and $^{28}$Si

P. Adsley, V. O. Nesterenko, M. Kimura, L. M. Donaldson, R. Neveling, J. W. Brümmer, D. G. Jenkins, N. Y. Kheswa, J. Kvasil, K. C. W. Li, D. J. Marín-Lámbardi, Z. Mabika, P. Papka, L. Pellegrin, V. Pesudo, B. Rebeiro, P.-G. Reinhard, F. D. Smit, W. Yahia-Cherif

1 School of Physics, University of the Witwatersrand, Johannesburg 2050, South Africa
2 Department of Physics, Stellenbosch University, Private Bag X1, 7602 Matieland, Stellenbosch, South Africa
3 iThemba Laboratory for Accelerator Based Sciences, Somerset West 7129, South Africa
4 Institut de Physique Nucléaire d’Orsay, UMR8608, IN2P3-CNRS, Université Paris Sud 11, 91406 Orsay, France
5 Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region 141980, Russia
6 State University “Dubna”, Dubna, Moscow Region 141980, Russia
7 Moscow Institute of Physics and Technology, Dolgoprudny, Moscow Region, 141701, Russia
8 Department of Physics, Hokkaido University, 060-0810 Sapporo, Japan
9 Reaction Nuclear Data Centre, Faculty of Science, Hokkaido University, 060-0810 Sapporo, Japan
10 Department of Physics, University of York, Heslington, York, YO10 5DD, United Kingdom
11 Institute of Particle and Nuclear Physics, Charles University, CZ-18000, Praha 8, Czech Republic
12 Department of Physics, University of the Western Cape, P/B X17, Bellville 7535, South Africa
13 Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, 01000 Cd. México, México
14 Institut für Theoretische Physik II, Universität Erlangen, D-91058, Erlangen, Germany
15 Université des Sciences et de la Technologie Houari Boumediene (USTHB), Faculté de Physique, B.P. 32 El-Alia,16111 Bab Ezzouar, Algiers, Algeria

(Dated: October 19, 2020)

Background Nuclei in the sd-shell demonstrate a remarkable interplay of cluster and mean-field phenomena. The $N = Z$ nuclei, such as $^{24}$Mg and $^{28}$Si, have been the focus of the theoretical study of both these phenomena in the past. A variety of different cluster structures in these nuclei, characterized by isoscalar dipole and monopole transitions, are predicted. For example, low-energy isoscalar vortical dipole states were predicted in $^{24}$Mg. The cluster and vortical mean-field phenomena can be probed by excitation of isoscalar monopole and dipole states in scattering of isoscalar particles such as deuterons or $\alpha$ particles.

Purpose To investigate, both experimentally and theoretically, the isoscalar dipole ($IS_1$) and monopole ($IS_0$) strengths in three different light nuclei with different properties: stiff prolate $^{24}$Mg, soft prolate $^{26}$Mg and soft oblate $^{28}$Si, and to analyze possible manifestations of clustering and vorticity in these nuclei.

Methods Inelastically scattered $\alpha$ particles were momentum-analysed in the K600 magnetic spectrometer at iThemba LABS, Cape Town, South Africa. The scattered particles were detected in two multi-wire drift chambers and two plastic scintillators placed at the focal plane of the K600. In the theoretical discussion, the Skyrme Quasiparticle Random-Phase Approximation (QRPA) and Antisymmetrized Molecular Dynamics + Generator Coordinate Method (AMD+GCM) were used.

Results A number of isoscalar monopole and dipole transitions were observed in the nuclei studied. Using this information, suggested structural assignments have been made for the various excited states. $IS_1$ and $IS_0$ strengths obtained within QRPA and AMD+GCM are compared with the experimental data. The QRPA calculations lead us to conclude that: i) the mean-field vorticity appears mainly in dipole states with $K = 1$, ii) the dipole (monopole) states should have strong deformation-induced octupole (quadrupole) admixtures, and iii) that near the $\alpha$-particle threshold, there should exist a collective state (with $K = 0$ for prolate nuclei and $K = 1$ for oblate nuclei) with an impressive octupole strength. The results of the AMD+GCM calculations suggest that some observed states may have a mixed (mean-field + cluster) character or correspond to particular cluster configurations.

Conclusion A tentative correspondence between observed states and theoretical states from QRPA and AMD+GCM was established. The QRPA and AMD+GCM analysis shows that low-energy isoscalar dipole states combine cluster and mean-field properties. The QRPA calculations show that the low-energy vorticity is well localized in $^{24}$Mg, fragmented in $^{26}$Mg, and absent in $^{28}$Si.

I. BACKGROUND

Light nuclei demonstrate a remarkable interplay of cluster and mean-field degrees of freedom, see e.g. reviews [1-4]. The exploration of this interplay is a demanding problem which is additionally complicated by the softness of these nuclei and related shape coexistence.
The low-energy isoscalar monopole \((IS0)\) and dipole \((IS1)\) states in light nuclei can serve as fingerprints of clustering \([3, 6]\). \(IS1\) states can also deliver important information on some mean-field features, such as vorticity \([9, 16]\). Understanding the nature of low-energy \(IS0\) and \(IS1\) transitions in light nuclei is thus of great interest.

Clustering in light \(N = Z\) nuclei can manifest itself in low-lying \(IS0\) transitions to \(J^z = 0^+\) states \([3, 8]\). Recent theoretical work has suggested that \(IS1\) excitations may also be used to explore cluster configurations, i.e. the low-lying \(0^+\) states caused by asymmetric clusters have \(1^-\) partner states, thus forming inversion doublets which indicate the symmetry of the cluster configuration \([9]\). In \(N \neq Z\) nuclei, the asymmetric clustering may result in enhanced electric dipole transitions between isoscalar states.

In addition to this clustering behaviour, mean-field structures may also exist. Individual low-lying vortical \(IS1\) states were predicted within the Quasiparticle Random-Phase-Approximation (QRPA) \([9, 11]\) and the Antisymmetrized Molecular Dynamics + Generator Coordinate Method (AMD+GCM) \([12, 16]\). These states should exist in \(^{16}\)Be \([12, 13]\), \(^{12}\)C \([14]\), \(^{16}\)O \([15]\), \(^{20}\)Ne \([10]\) and \(^{24}\)Mg \([9, 11]\) \([16]\).

Such individual low-lying vortical states can be differentiated from the neighbouring excitations and much more easily resolved in experiment. Note that the intrinsic electric vortical flow of nucleons, though widely discussed in recent decades, is still very poorly understood \([17–22]\). The experimental observation and identification of vortical states remains a challenge for the modern experimentalist \([11]\). In this respect, exploration of individual low-lying \(IS1\) vortical states in light nuclei could be used as a promising guide in the experimental design. The \((e, e')\) reaction has been recently suggested to identify the vortical response of nuclei \([11]\). The complementary \((\alpha, \alpha')\) reaction may be used to locate candidates for the \(IS1\) vortical states for these \((e, e')\) measurements.

The light nuclei \(^{24}\)Mg, \(^{26}\)Mg, and \(^{28}\)Si have essentially different properties and thus represent a useful set for the comparative investigation of the interplay between the mean-field and cluster degrees of freedom. These nuclei differ by \(N/Z\) ratio, softness to deformation (stiff \(^{24}\)Mg and soft \(^{26}\)Si and \(^{28}\)Mg), and sign of deformation (prolate \(^{24}\)Mg and oblate \(^{28}\)Si). Therefore, it is interesting to compare the origin and behavior of low-lying \(IS0\) and \(IS1\) strengths in these nuclei, from the perspectives of clustering and vorticity. Many investigations have been performed for each of these nuclei separately (see e.g. Refs. \([7, 23–24]\) for a general view and Refs. \([25, 31, 33–35, 24]^{(24}\)Mg), \([30, 36, 42]^{(26}\)Mg), \([31, 33, 35, 10, 12, 17]^{(28}\)Si for particular studies). We now provide comparative experimental and theoretical analyses of these nuclei.

In this paper, we report \(IS0\) and \(IS1\) strengths in \(^{24}\)Mg, \(^{26}\)Mg and \(^{28}\)Si, determined from \(\alpha\)-particle inelastic scattering at very forward scattering angles (including zero degrees). The data were obtained with the K600 magnetic spectrometer at iThemba LABS (Cape Town, South Africa). The data are limited to excitation energy \(E_x < 16\) MeV so as to avoid the regions dominated by giant resonances, where identification of individual states is difficult without observation of charged-particle decays.

The theoretical analysis is performed within the QRPA model for axially deformed nuclei \([48–52]\) and the AMD+GCM model \([12, 16]\) which can take into account both axial and triaxial quadrupole deformations and describe the evolution of nuclear shape with excitation energy. Moreover, AMD+GCM includes the ability to describe the interplay between mean-field and cluster degrees of freedom. Despite some overlap of QRPA and AMD+GCM, the models basically describe different information on nuclear properties; QRPA treats excited states with a mean-field approach and is therefore suitable for investigation of the nuclear vorticity. Meanwhile, AMD+GCM highlights cluster properties. Altogether, QRPA and AMD+GCM supplement one another and comparison of their results is vital for light nuclei. Our analysis mainly focuses on possible manifestations of clustering and vorticity in \(IS0\) and \(IS1\) states.

The paper is organized as follows. In Secs. II and III, the experimental method and data analysis are briefly outlined. In Sec. IV, the obtained experimental results are reported. In Sec. V, the experimental \(IS1\) and \(IS0\) strengths are compared with QRPA calculations. The vortical and irrotational characters of \(IS1\) states are scrutinized. In Sec. VI, the experimental data are compared with AMD+GCM results. The cluster features of \(IS1\) and \(IS0\) states are inspected. In Sec. VII, the conclusions are offered.

II. EXPERIMENTAL DETAILS

A detailed description of this experiment has been given in two previous papers \([8, 40]\). A brief summary of the experimental method is given here.

A dispersion-matched beam of 200-MeV \(\alpha\) particles was incident on a target and the reaction products were momentum-analysed by the K600 magnetic spectrometer. The focal-plane detectors consisted of two wire chambers giving horizontal and vertical position information, and two plastic scintillating paddles which measured energy deposited at the focal plane.

The spectrometer was used in two different modes to acquire the data: the zero-degree mode in which scattering angles of less than 2 degrees were measured, and the small-angle mode in which the spectrometer aperture covered scattering angles from 2 to 6 degrees.

For the zero-degree measurement, the background resulting from target-induced Coulomb scattering necessitated running the spectrometer in a focus mode in which the scattered particles were focussed onto a vertically narrow horizontal band on the focal plane. In order to obtain a spectrum free from instrumental background, a standard technique used with the iThemba K600 \([53]\) and
the RCNP Grand Raiden [54] magnetic spectrometers was used, in which background spectra are constructed from the regions of the focal plane above and below the focussed band. These background components are then subtracted from the signal spectrum. The vertical focussing required for this technique resulted in the loss of all vertical scattering information and limited the differential cross section for the zero-degree experiment to one point for scattering angles of less than 2 degrees.

For the small-angle measurement, the target-induced Coulomb scattering background was much lower and the spectrometer could be operated in under-focus mode, in which the vertical position on the focal plane corresponds to the vertical scattering angle into the spectrometer aperture. In this case, the scattering angle could be reconstructed from the angle with which the scattered
α particle traversed the focal plane and its vertical position. The angular resolution was around 0.5 degrees (FWHM) and we extracted four points for the differential cross section between 2 and 6 degrees in the laboratory frame. The procedure to calibrate the scattering angles is described in Refs. [8] [53].

III. DATA ANALYSIS

The techniques used for the analysis of the data have been described in more detail in Ref. [8]. In summary, the horizontal focal-plane position was corrected for kinematic and optical aberrations according to the scattering angle into the spectrometer and the vertical focal-plane position.

The scattering angles into the spectrometer were calculated from the vertical position and the angle with which the scattered particle traverses the focal plane; these quantities were calibrated to known scattering trajectories into the spectrometer using a multi-hole collimator at the spectrometer aperture.

Horizontal focal-plane position spectra were generated for each angular region. The calibration of the focal-plane position to excitation energy used well-known states in $^{24}$Mg, $^{20}$Mg and $^{28}$Si [55, 56]. Corrections were made according to the thickness of the relevant targets using energy losses from SRIM [58].

The spectra were fitted using a number of Gaussians with a first-order polynomial used to represent background and continuum. The resolution was around 75 (65) keV (FWHM) for the zero-degree (finite-angle) data. An additional quadratic term was used at $E_x < 9$ MeV for the background from $p(\alpha, \alpha)p$ elastic-scattering reactions from target contaminants. The fitted spectra for $^{24}$Mg and $^{26}$Mg at some angles are shown in Figures 1 and 2. The $^{28}$Si spectra along with a description of the associated fitting procedures can be found in Ref. [8].

To quantify contamination in the targets, elastic-scattering data were taken in the small-angle mode. Population of low-lying states in nuclei contained in the target was observed. For the natural silicon target, small quantities of hydrogen, $^{12}$C, $^{16}$O and $^{29,30}$Si were observed. For the $^{24}$Mg and $^{26}$Mg targets, hydrogen, $^{12}$C and $^{16}$O were also observed but at much lower levels than for the silicon target. From previous experimental studies with the K600 (see e.g. Ref. [59]), the locations of the $^{12}$C and $^{16}$O states are well known and excluded.

IV. EXPERIMENTAL RESULTS

The focus of this paper is on the location and strength of cluster and vortical states. We report monopole ($J^\pi = 0^+$) and dipole ($J^\pi = 1^-$) states in $^{24,26}$Mg and $^{28}$Si. In addition, we discuss states which have received firm or tentative monopole or dipole assignments in previous experimental studies but have not been observed in the present measurement.

The differential cross sections were extracted from the fitted spectra using:

$$\frac{d\sigma}{d\Omega} = \frac{Y}{N\eta\Delta\Omega},$$

where $N$ is the areal density of target ions, $I$ is the integrated charge as given by the current integrator (including the livetime fraction of the data-acquisition system), $\eta$ is the focal plane efficiency and $\Delta\Omega$ is the solid angle of the spectrometer aperture at that scattering angle. The total efficiency, $\eta$, is the product of the efficiencies for each wire plane per Ref. [8]. The uncertainties in the differential cross sections are a combination of the fitting error and Poissonian statistics.

By comparing the experimental differential cross sections to DWBA calculations performed using the code CHUCK3 [62],

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{exp}} = \beta_{R,\lambda}^2 \left(\frac{d\sigma}{d\Omega}\right)_{\text{DWBA}},$$

the transition factors $\beta_{R,\lambda}^2$ were extracted for each dipole ($\lambda = 1$) and monopole ($\lambda = 0$) state. The contribution of the states to the isoscalar dipole and monopole energy-weighted sum rules (EWSRs) were computed. The calculations were performed in accordance with Refs. [28] [56], more details are given in Appendix A.

There is a systematic $\sim 20\%$ uncertainty due to the choice of the optical-model potentials. In the present analysis, we, e.g. find that the well-known $E_x = 7.555$-MeV $J^\pi = 1^-$ state in $^{24}$Mg exhausts 2.6(5)% of the EWSR, which is within the expected systematic deviation due to e.g. choices of optical-model potentials when compared with previous results of 3.1(6)% [20] and 3(1)% [27].

For some of the states, contamination or background in the differential cross sections is problematic. This can occur when the level density is high, e.g. around the $0^+$ states in $^{24}$Mg in the region of $E_x = 13.8 - 14$ MeV, where a third state lies between the two $0^+$ states, or at the minima of the differential cross section where the background is similar in size to the cross section from the state of interest. In these cases, to avoid biasing the extracted transition strengths, a subset of points from the angular distributions has been used for comparison to the DWBA calculation.

Below, in Tables I-V and Figs. 3-4, the monopole and dipole spectra for $^{24}$Mg, $^{26}$Mg, and $^{28}$Si are reported. Some states are discussed in separate subsections; this is done where assignments have been updated or known states have not been observed.

A. $^{24}$Mg

A typical differential cross section for a $J^\pi = 0^+$ state in $^{24}$Mg is shown in Figure 3 and for a $J^\pi = 1^-$ state
in $^{24}$Mg in Figure 4. Similar shapes were used to identify other monopole and dipole states. The $J^\pi = 0^+$ and $J^\pi = 1^-$ levels are summarized in Tables I and II respectively; states with the corresponding $J^\pi$ listed in the ENSDF database [63] are included even when not observed.

### 1. The 10.161-MeV state

A state with $J^\pi = 0^+$ has been reported at 10.161 MeV in $^{24}$Mg($p,p'$)$^{24}$Mg, $^{23}$Na($^3$He,$d$)$^{24}$Mg, $^{25}$Mg($^3$He,$^4$He)$^{24}$Mg and $^{12}$C($^{16}$O,$\alpha$)$^{24}$Mg reactions (see Ref. [63] and references therein). This state is not observed in the present experiment.

![FIG. 3: Differential cross section for the $J^\pi = 0^+$ state at 10.68 MeV in $^{24}$Mg. The data are represented by points with the horizontal error bar delineating the angular range covered. The calculated angle-averaged differential cross section is shown in red.](image)

![FIG. 4: As Figure 3 but for the $J^\pi = 1^-$ state at 11.86 MeV in $^{24}$Mg.](image)

### Table I: $J^\pi = 0^+$ states in $^{24}$Mg.

| $E_x$ [MeV] | $\beta^{2}_{R,0}[10^{-4}]$ | $S_0$ | Comments |
|-------------|-----------------------------|-------|----------|
| 9.30539(24) | 9(2)                        | 1.4(3)| Not observed. |
| 10.161(3)   |                             |       |          |
| 10.6797(4)  | 1.8(4)                      | 0.29(6)|          |
| 11.39(2)    | 0.6(3)                      | 0.12(2)|          |
| 11.7281(10) | 5(1)                        | 1.0(2)|          |
| 13.044(3)   | 2.5(5)                      | 1.1(2)|          |
| 13.37(1)    | 2.5(5)                      | 0.5(1)|          |
| 13.79(1)    | 11(2)                       | 1.7(3)|          |
| 13.89(1)    | 9(2)                        | 2.6(5)|          |
| 15.33(3)    | 6(1)                        | 1.9(4)|          |
| 15.4364(6)  |                             |       |          |
| 15.79(3)    | 3.7(7)                      | 1.1(2)|          |

*From Ref. [63] unless stated otherwise

### Table II: As Table I but for $J^\pi = 1^-$ states in $^{24}$Mg.

| $E_x$ [MeV] | $\beta^{2}_{R,1}[10^{-4}]$ | $S_1$ | Comments |
|-------------|-----------------------------|-------|----------|
| 7.55504(15) | 0.18(16)                    | 2.6(5)|          |
| 8.43773(15) | 2.7(5)                      | 10(2) |          |
| 9.14599(15) | 0.58(12)                    | 2.4(5)|          |
| 11.3898(11) |                             |       | Not observed |
| 11.8649(13) | 2.1(4)                      | 11(2) |          |
| 13.19(2)    | 0.49(10)                    | 2.9(6)|          |

*From Ref. [63] unless stated otherwise

B. $^{26}$Mg

Table [11] summarizes known $J^\pi = 0^+$ states in $^{26}$Mg either listed in the ENSDF database [63] or observed during the present experiment.

Table [14] summarizes known $J^\pi = 1^-$ states along with electrical $B(E1)$s from Refs. [59] and [61]. For the data of Ref. [61], the partial widths of the ground-state decay are...
TABLE III: As in Table I but for $J^\pi = 0^+$ states in $^{26}\text{Mg}$.

| $E_\gamma$ [MeV]$^a$ | $\beta^2_{R,0}$ [10$^{-4}$] | $S_0$ | Comments |
|----------------------|--------------------------|-------|----------|
| 7.200(20)           | Not observed              |       | $J^\pi = (0,1)^+$ [63] |
| 7.428(3)            | Not observed              |       | $J^\pi = (0,1)^+$ [63] |
| 10.159(3)           | Not observed              |       | $J^\pi = (0,1)^+$ [63] |
| 10.74(2)            | Not observed or $J^\pi \neq 0^+$ |       | Part of a multiplet; see text, Refs. [66, 68, 69]. |
| 10.818(1)$^b$       | 7(1)                     | 1.0(2)| Not observed $J = 0$, parity unknown |
| 12.345(2)           |                          |      |          |
| 12.72(2)$^c$        | 6(1)                     | 0.9(2)|          |
| 13.1                | 1.6(3)                   | 0.28(6)|          |
| 13.5                | 2.0(4)                   | 0.37(7)|          |
| 14.88(2)$^c$        | 7(1)                     | 1.4(3)|          |

$^a$From Ref. [63] unless stated otherwise
$^b$See Ref. [64] for a discussion of the energy of this level.
$^c$Present experiment

given and are converted to the reduced matrix element using the relation:

$$\Gamma(\lambda\ell) = \frac{8\pi(\ell + 1)}{e^2(2\ell + 1)!^2} \left(\frac{E_\gamma}{\hbar c}\right)^{2\ell+1} B(\lambda\ell)$$

(3)

for a radiation of multipolarity $\ell$ and type (electric/magnetic) $\lambda$. $E_\gamma$ is the energy of the $\gamma$-ray transition.

1. The 7.062-MeV state

This state is listed in ENSDF [63] but not observed in a previous $^{26}\text{Mg}(\alpha,\alpha')^{26}\text{Mg}$ reaction of Ref. [66]. In the present experiment, a state is observed at $E_x = 7.10$ MeV with a differential cross section that is consistent with a $J^\pi = 1^-$ assignment.

2. The 10.159-MeV state

The $J^\pi = 0^+$ state at $E_x = 10.159$ MeV listed in Ref. [63] is not observed in the present experiment. The state has been previously observed in $^{24}\text{Mg}(t,p)^{26}\text{Mg}$ with $\ell = 0$ [66] and in $^{26}\text{Mg}(p,p')^{26}\text{Mg}$ (see Ref. [63] and references therein). We assume that the state has $T = 1$ if it is populated in $^{24}\text{Mg}(t,p)^{26}\text{Mg}$ reactions. Therefore, population of this state in $^{26}\text{Mg}(\alpha,\alpha')^{26}\text{Mg}$ is unlikely to be isospin-forbidden. The reason why this state is not populated remains unclear.

3. The 10.74-MeV State

Ref. [50] lists a tentative $J^\pi = 0^+$ state at $E_x = 10.74(2)$ MeV. In the present experiment, a state is observed at around $E_x = 10.72(2)$ MeV but the differential cross section is consistent with $J \geq 2$.

4. States in the region of 10.80 to 10.83 MeV

A state with $J^\pi = 1^-$ has been identified at 10.805 MeV in $^{26}\text{Mg}(\gamma,\gamma')^{26}\text{Mg}$ experiments [68]. In a preceding paper focussing on a narrow subset of astrophysically important states in $^{26}\text{Mg}$, we demonstrated that the strong state observed in the $^{26}\text{Mg}(\alpha,\alpha')^{26}\text{Mg}$ reaction has $J^\pi = 0^+$ and is, therefore, evidently a different state from the $J^\pi = 1^-$ state [40]. The existence of multiple states was confirmed by a high-resolution experiment using the Munich Q3D [66].

In the present case, the extraction of the dipole strength is hindered by the close proximity of the strong $J^\pi = 0^+$ state. A higher-resolution inclusive measurement or a coincidence measurement of $^{26}\text{Mg}(\alpha,\alpha')^{26}\text{Mg}$ is necessary for the extraction of the isoscalar dipole transition strength for this state.

5. The 11.321-MeV State

Notably, one $\alpha$-particle cluster state in $^{26}\text{Mg}$ has been identified through direct reactions. The resonance at $E_\gamma = 0.83$ MeV observed in $^{22}\text{Ne}(\alpha,\gamma)^{26}\text{Mg}$ [62, 68] and $^{22}\text{Ne}(\alpha,n)^{25}\text{Mg}$ [69, 71] reactions clearly has a $^{22}\text{Ne}+\alpha$ cluster structure. However, the spin and parity of this state were not clearly assigned in previous $^{26}\text{Mg}(\alpha,\alpha')^{26}\text{Mg}$ reactions including our prior publication [40, 41]. Based on direct measurements of the resonance strengths and the inferred $\alpha$-particle width, the state almost certainly has $J^\pi = 0^+$ or $J^\pi = 1^-$. We do not observe any strong candidate for this state in our present experimental work and, therefore, cannot provide a monopole or dipole transition for the state.

6. The 11.289- and 11.329-MeV states

Both of these states have been identified as $J^\pi = 1^-$ using the reactions of neutrons with $^{25}\text{Mg}$. While $\gamma$-ray partial widths are available, the branching of these states is not, and, therefore, the $B(E1)$ for the ground-state transition cannot be determined.

7. The 12.345-MeV State

A state is listed in Ref. [63] as having $J = 0$ with unknown parity and $\Gamma = 40(5)$ keV. This state is not observed in the present experiment.
TABLE IV: The same as in Table I but for \( J^\pi = 1^- \) states in \( ^{26}\text{Mg} \). Electric dipole reduced transition probabilities \( B(E1) \) from Refs. \[39\] and \[61\] are also shown.

| \( E_x \) [MeV]\(^a\) | \( \beta_{R,1}^2 \) [10\(^{-4}\)] | \( S_1 \) | \( B(E1, 0^+_n \rightarrow 1^-) \) (10\(^{-4}\)e\(^2\) fm\(^2\)) \[39\] | \( B(E1, 0^+_n \rightarrow 1^-) \) (10\(^{-4}\)e\(^2\) fm\(^2\)) \[61\] | Comments |
|---|---|---|---|---|---|
| 7.06190(20) | 0.34(6) | 1.1(2) | | | |
| 7.6973(8) | 0.76(15) | 2.6(5) | 9.4(31) | | |
| 8.5042(4) | 0.21(4) | 0.8(2) | 33.3(41) | | |
| 8.9590(7) | 0.95(19) | 3.9(8) | 12.5(22) | | |
| 9.1395(13) | | | 0.17(4) | | |
| 9.7708(9) | | | 0.58(14) | | |
| 10.87(2) | 0.23(5) | 1.0(2) | | | |
| 10.1031(7) | 0.46(9) | 2.1(4) | 18.9(33) | | |
| 10.495(1) | 0.48(10) | 2.3(5) | | | |
| 10.5733(8) | 0.26(5) | 1.3(3) | | | |
| 10.8057(7) | | | | 0.75(19) | Not cleanly observed due to 10.826-MeV \( J^\pi = 0^+ \) state |
| 10.9491(8) | 0.29(6) | 1.5(3) | | | |
| 11.28558(5) | | | | 1.2(3) | |
| 11.32827(5) | | | | 2.71(42) | |
| 11.51(2)\(^b\) | 0.67(13) | 3.5(7) | | | |

\(^a\)From Ref. \[63\] unless stated otherwise  
\(^b\)Present experiment

8. A possible \( J^\pi = 0^+ \) state at \( E_x \sim 15.8 \text{ MeV} \)

Visual inspection of the \( ^{26}\text{Mg}(\alpha, \alpha')^{26}\text{Mg} \) spectra at various angles suggests that there is possibly a weak \( J^\pi = 0^+ \) state at \( E_x \sim 15.8 \text{ MeV} \). It was not possible to extract a strength for this state due to the high level density in this region. Data over a wider angular range in which a multipole decomposition analysis could be performed may be able to extract this strength.

C. \( ^{28}\text{Si} \)

Data on the states observed in \( ^{28}\text{Si} \) have been previously reported in Ref. \[8\]. In the present paper, we have extended the analysis up to 16 MeV to cover the same range as for the magnesium isotopes. Additional \( J^\pi = 0^+ \) states are observed at 15.02 and 15.76 MeV.

The natural silicon target contains some carbon and oxygen contamination. Carbon and oxygen states which are strongly populated in \( \alpha \)-particle inelastic scattering at \( E_{\alpha} = 200 \text{ MeV} \) are known from previous studies with the K600 \[59\] and are excluded from the reported states.

1. The 11.142- and 11.148-MeV states

As explained in the previous K600 paper on \( ^{28}\text{Si}(\alpha, \alpha')^{28}\text{Si} \), the literature lists two unresolved \( J^\pi = 0^+ \) and \( J^\pi = 2^+ \) states at 11.141 and 11.148 MeV, respectively \[8\]. Further investigation of the existing data on \( ^{28}\text{Si} \) \[75\] has showed that there is, in fact, only one state at this energy which has \( J^\pi = 0^+ \) and, therefore, it is not necessary to include contributions from two states.

TABLE V: As in Table I but for \( J^\pi = 0^+ \) states in \( ^{28}\text{Si} \).

| \( E_x \) [MeV]\(^a\) | \( \beta_{R,0}^2 \) [10\(^{-4}\)] | \( S_0 \) | Comments |
|---|---|---|---|
| 9.71(2) | 2.6(5) | 0.38(8) | |
| 10.81(3) | 2.2(4) | 0.35(7) | |
| 11.142(1) | 5.5(11) | 0.9(2) | See Refs. \[8\] \[73\]. |
| 12.99(2)\(^b\) | 4.3(9) | 0.8(2) | Unresolved multiplet. \[8\] \[63\]. |
| 15.02(3)\(^b\) | 1.4(3) | 0.8(2) | Newly observed. |
| 15.73(3)\(^b\) | 2.3(5) | 0.32(6) | May correspond to a tentative \( E_x = 15.65(5)\)-MeV \( J^\pi = 0^+ \) state \[56\]. |

\(^a\)From Ref. \[63\] unless stated otherwise  
\(^b\)Present experiment

TABLE VI: As in Table I but for \( J^\pi = 1^- \) states in \( ^{28}\text{Si} \).

| \( E_x \) [MeV]\(^a\) | \( \beta_{R,1}^2 \) [10\(^{-4}\)] | \( S_1 \) | Comments |
|---|---|---|---|
| 8.9048(4) | 1.1(2) | 4.8(9) | Confirms a tentative \( J^\pi = 1^- \) assignment \[56\]. |
| 9.9929(17) | 2.3(5) | 11(2) | Confirms a tentative \( J^\pi = 1^- \) assignment \[56\]. |
| 10.994(2) | 1.2(2) | 6(1) | |
| 11.2956(2) | 0.47(10) | 2.5(5) | Confirms a tentative \( J^\pi = 1^- \) assignment \[56\]. |
| 11.58(2)\(^b\) | 0.17(3) | 0.9(2) | |
| 13.95\(^b\) | 0.59(12) | 3.8(8) | |

\(^a\)From Ref. \[63\] unless stated otherwise  
\(^b\)Present experiment
2. The 11.65-MeV state

Ref. 536 reports a tentative $J^\pi = 1^-$ state at $E_x = 11.65(2)$ MeV. This state is not observed in the present experiment.

V. COMPARISON WITH QRPA CALCULATIONS

A. Calculation Scheme

We use a fully self-consistent QRPA approach 39, 50 with the Skyrme force SLy6 76. This force was found to be optimal in the previous calculations of dipole excitations in medium-heavy nuclei 9, 77. The nuclear energy. The volume pairing is treated with the Bardeen-Cooper-Schrieffer (BCS) method 50. The pairing was found to be weak (with a pairing gap about $0.355$ from the experimental data of Stone 78). The dipole 0$^+\to1^+_0$ and 0$^+\to1^+_0$ and 0$^+\to1^+_0$ are obtained from the theory and observed excitations, see the discussion in Ref. 9.

We now consider the vortical and compression isoscalar strengths, $B(IS1K_0)\nu$ and $B(IS1K_0)\nu$, using current-dependent operators from Refs. 9, 11. We need these strengths to estimate the relative vortical and irrotational compression contributions to the dipole states. The current-dependent compression operator includes divergence of the nuclear current and so, can be reduced to Eq. (3) using the continuity equation. For the sake of simplicity, we will further omit the dependence on $\nu$ in rate notations.

B. IS1 strength distributions

1. $^{24}$Mg

In Figure 5, the $(\alpha,\alpha')$ experimental data (transition factors $\beta_{20}^{\exp}$) for $^{24}$Mg (upper) are compared with $B(IS1K_0)\nu$ (middle) for QRPA states with $K = 0$ (red) and $K = 1$ (black). We see that experiment and QRPA give the lowest dipole states at a similar energy, 7.56 and 7.92 MeV, respectively. In QRPA, the states 7.92-MeV (K=1) and 9.56-MeV (K=0) have large $B(IS1)\nu$ responses and so, should be well populated in the $(\alpha,\alpha')$ reaction. However, it is still difficult to establish one-to-one correspondence between these QRPA states and observed excitations, see the discussion in Ref. 9.

In general, QRPA gives many more dipole states between $B(IS1K_0)\nu$ and observed excitations, see the discussion in Ref. 9.

The obtained axial quadrupole deformations are $\beta_2 = 0.536, 0.355$ and $-0.354$ for $^{24,26}$Mg and $^{28}$Si, respectively, meaning that $^{24,26}$Mg are taken to be prolate nuclei while $^{28}$Si is treated as oblate. In the SLy6 calculations, $^{26}$Mg has comparable oblate and prolate energy minima. Following the experimental data of Stone 78 as well as AMD+GCM 79 and Skyrme 80 calculations, the ground-state deformation of $^{26}$Mg is prolate and so, we use the equilibrium deformation $\beta_2 = 0.355$ from the prolate minimum for $^{26}$Mg.

Note that the absolute values obtained for equilibrium deformations are smaller than the experimental ones ($\beta_2^{\exp} = 0.613, 0.484, -0.412$ for $^{24,26}$Mg, $^{28}$Si) 51. This is a common situation for deformation-soft nuclei. Indeed, $\beta_2^{\exp}$ are obtained from the $B(E2)$ values for the transitions in the ground-state rotational bands. However, in soft nuclei, $B(E2)$ values include large dynamical correlations and so, lead to overestimation of the size of the quadrupole deformation, $|\beta_2|$. Therefore, the present observation that $|\beta_2| < |\beta_2^{\exp}|$ is reasonable.

The isoscalar reduced transition probabilities

$$B(IS\lambda\mu_\nu) = |\langle \nu | M(IS\lambda\mu) | 0 \rangle|^2,$$

for the transitions from the ground state $|0\rangle$ with $I^\pi K = 0^+0_{gs}$ to the excited $\nu$-th QRPA state with $I^\pi K = \lambda^\pi \mu$ are calculated using the monopole (IS0) and dipole ($IS1K$) transition operators:

$$\hat{M}(IS0) = \sum_{i=1}^{A} (r^2 Y_{00})_i,$$

$$\hat{M}(IS1K) = \sum_{i=1}^{A} (r^3 Y_{1K})_i, \ K = 0, 1,$$

where $Y_{00} = 1/\sqrt{4\pi}$. To investigate the deformation-induced monopole/quadrupole and dipole/octetopole mixing, we also compute quadrupole $B(IS20)$ and octupole $B(IS3K)$ transition probabilities for isoscalar transitions $0^+0_{gs} \to 2^+0_{\nu}$ and $0^+0_{gs} \to 3^- K_\nu$ using transition operators

$$\hat{M}(IS20) = \sum_{i=1}^{A} (r^2 Y_{20})_i,$$

$$\hat{M}(IS3K) = \sum_{i=1}^{A} (r^3 Y_{3K})_i, \ K = 0, 1.$$
(K=1) and 9.56-MeV (K=0) exhibit fundamental octupole strengths: \( B(IS31) = 715 \text{ fm}^6 \) (21 W.u.) and \( B(IS30) = 2450 \text{ fm}^6 \) (72 W.u.), respectively. Such large \( B(IS3K) \) values originate from two sources: i) collectivity of the states and ii) that the dominant proton and neutron 1ph components of the states \((pp)[211 \uparrow -330 \uparrow], \text{ and } nn[211 \uparrow -330 \uparrow] \) for 7.92-MeV (K=1) state and \( pp[211 \downarrow -101 \downarrow], \text{ and } nn[211 \downarrow -101 \downarrow] \) for 9.56-MeV (K=0) state) fulfill the selection rules for \( E3K \) transitions [82]:

\[
\Delta K = 0 : \Delta N = \pm 1, \pm 3, \Delta n_z = \pm 1, \pm 3, \Delta \Lambda = 0,
\]
\[
\Delta K = 1 : \Delta N = \pm 1, \pm 3, \Delta n_z = 0, \pm 2, \Delta \Lambda = 1.
\]

Here, the single-particle states are specified by Nilsson asymptotic quantum numbers \( Nn\Lambda \) [83], whilst the arrows indicate spin direction. The large \( B(IS3K) \) values signify that 7.92-MeV (K=1) and 9.56-MeV (K=0) states are of a mixed octupole-dipole character. Their leading 1ph components correspond to \( \Delta N = 1 \) transitions between the valence and upper quantum shells, so these states can belong to the Low-Energy Octupole Resonance (LEOR) [23, 51].

As may be seen in Figure 5, both \( IS1K \) and \( IS3K \) distributions can be roughly separated into two groups, the first located below (7-10 MeV) and the second located above (11-14 MeV) the \( \alpha \)-particle threshold \( S_\alpha = 9.3 \text{ MeV} \). Moreover, close to \( S_\alpha \), there is a \( E_x = 9.56-\text{MeV} (K=0) \) state with a huge \( B(IS30) \) strength, which perhaps signals the octupole-deformation softness of the nucleus at this energy. It is reasonable to treat the states below \( S_\alpha \) as being of mean-field origin, while the states close to and above \( S_\alpha \) (including the 9.56-\text{MeV} (K=0) near-threshold state) as including cluster degrees of freedom. This is confirmed by recent AMD+GCM calculations for \( 24^\text{Mg} \) [10], where similar results were obtained: the lowest mean-field 9.2-\text{MeV} (K=1) state is of mean-field character and the 11.1-\text{MeV} (K=0) state has cluster properties.

In Figure 6, the vortical \( B(IS1v) \) and compression \( B(IS1c) \) strengths for \( K = 0 \) and \( K = 1 \) dipole branches in \( 24^\text{Mg} \) are compared. The states with \( B(IS1v) > B(IS1c) \) should be considered as vortical in nature, see e.g. the \( E_x = 7.92-\text{MeV} (K=1) \) state. Instead, the states with \( B(IS1v) < B(IS1c) \) have compression-rotational character. Compressional states can be directly excited in the \( (\alpha,\alpha') \) reaction [23]. The vortical states usually have a minor rotational admixture and, most probably, are weakly excited in the \( (\alpha,\alpha') \) reaction through this admixture. Figure 6 shows that, in accordance with previous QRPA predictions [9, 11], the lowest \( K = 1 \) state at 7.92 MeV is mainly vortical. Moreover, for \( E_x < 14 \text{ MeV} \), the \( K = 1 \) branch exhibits much more

| Nucleus | \( B(IS1v) \) | \( B(IS1c) \) | \( B(IS1v) \) | \( B(IS1c) \) |
|---------|-------------|-------------|-------------|-------------|
| \( 24^\text{Mg} \) | 0.010       | 0.0038      | 0.019       | 0.0033      |
| \( 26^\text{Mg} \) | 0.012       | 0.0011      | 0.028       | 0.0074      |
| \( 28^\text{Si} \) | 0.015       | 0.0011      | 0.029       | 0.0071      |
vorticity than the $K = 0$ branch. The summed $B(IS1\nu)$ and $B(IS1c)$ are reported in Table VII.

The vortical character of the lowest dipole state may be a peculiarity of $^{24}$Mg, potentially unique to that nucleus. At least, this is not the case in $^{26}$Mg and $^{28}$Si, as discussed below. As mentioned above, the vortical 7.92-MeV ($K = 1$) state in $^{24}$Mg is mainly formed by the proton $pp[211 \uparrow -330 \uparrow]$ and neutron $nn[211 \uparrow -330 \uparrow]$ 1ph configurations. Just these configurations produce the vortical flow [10]. The large prolate deformation in $^{24}$Mg downshifts the energy of these configurations, thus making the vortical dipole state the lowest in energy [8] [10]. It is remarkable that the previous AMD+GCM calculations [10] give a very similar result for $^{24}$Mg: that lowest dipole state at $E_x = 9.2$ MeV has vortical ($K = 1$) character and a higher compressional ($K = 0$) state at $E_x = 11.1$ MeV.

2. $^{26}$Mg

In Figure 7, we compare the calculated $B(IS1K)$ and $B(IS3K)$ responses with the $(\alpha, \alpha')$ data. In both experiment and theory, we see numerous dipole states above $E_x \sim 6$ MeV. The fragmentation of the dipole and octupole strengths is somewhat larger than in $^{24}$Mg, which can be explained by the stronger neutron pairing in $^{26}$Mg. As in $^{24}$Mg, the states can be separated into two groups, below and above the threshold $S_x = 10.6$ MeV, and we observe a near-threshold collective $E_x = 9.96$-MeV ($K = 0$) state with an impressive $B(IS30)$ value.

As can be seen in Figure 7, the theory suggests another pattern for the lowest dipole states in $^{26}$Mg. Unlike $^{24}$Mg, where the lowest dipole $K = 1$ state is well separated and exhibits a vortical character, the QRPA dipole spectrum in $^{26}$Mg starts with two almost degenerate $K = 1$ and $K = 0$ states at $E_x \sim 6.6$ MeV. Moreover, as can be seen in Figure 8, these lowest QRPA states in $^{26}$Mg are not vortical.

To understand these results, we should inspect the structure of the lowest 6.60-MeV ($K = 1$) and 6.64-MeV ($K = 0$) QRPA states in $^{26}$Mg. They are dominated by 1ph neutron configurations $nn[211 \uparrow +330 \uparrow]$ and $nn[211 \downarrow -330 \uparrow]$, respectively. The same content explains the quasi-degeneracy of these states. These 1ph configurations have low $B(E1\nu)$ values and are not vortical. The configurations correspond to $F + 1 \rightarrow F + 5$ transitions, where $F$ marks the Fermi level. Both single-particle levels involved in the transition lie above the Fermi level and the transition is active only because of the developed neutron pairing in $^{26}$Mg (but it is suppressed in $^{24}$Mg, where the calculated pairing is negligible).

Note that 1ph excitations $pp[211 \uparrow -330 \uparrow]$ and $nn[211 \uparrow -330 \uparrow]$, which produce the vorticity in the lowest $K = 1$ vortical dipole state in $^{24}$Mg, also exist in $^{26}$Mg, but they are located at a higher energy of $E_x = 8.5 - 9.5$ MeV. Therefore, the distribution of the vorticity is mainly determined by the energy of vortical 1ph configurations.

3. $^{28}$Si

In Figure 9, we present the experimental data and QRPA results for $IS1K$ and $IS3K$ strengths in oblate $^{28}$Si. We see that the theory significantly overestimates the energy of the lowest $K^-$ state: it appears at 8.8 MeV in experiment and at 10.5 MeV in QRPA. So, unlike the experiment, the theory does not suggest any $K^-$ states below the threshold $S_x = 9.98$ MeV. Perhaps this discrepancy is caused by a suboptimal oblate deformation $\beta_2 = -0.354$ used in our calculations. Further, Figure 9 and Table VIII show that the dipole and octupole strengths for $K = 1$ are much larger than for $K = 0$. So, in this nucleus, $K = 1$ states should be more strongly populated in $(\alpha, \alpha')$ than $K = 0$ states.
1− states. Perhaps this is because the present QRPA scheme does not take into account such important factors as triaxiality, shape coexistence, clustering, and complex configurations are omitted.

Nevertheless, the QRPA calculations lead to some interesting and robust results.

1) The strong deformation-induced mixture of the dipole and octupole modes is predicted for most of $K^\pi = 0^−$ and $1^−$ states in $^{24,26}$Mg and in a few particular states in $^{28}$Si. Some mixed states demonstrate impressive octupole transition probabilities $B(IS3K)$. Perhaps these states belong to the so-called LEOR [23, 84].

2) In all three nuclei, the collective state with a large octupole strength is predicted near the $\alpha$-particle thresholds $S_\alpha = 9.3 - 10.6$ MeV. This state has $K = 0$ in $^{24,26}$Mg and $K = 1$ in $^{28}$Si. Most probably, the difference is caused by different signs of the axial deformation in these nuclei.

3) Above the $\alpha$-particle thresholds, fragmented vorticity is found in $K = 1$ states in all three nuclei. Below $S_\alpha$, the picture is different: the vorticity is concentrated in the lowest dipole state at $\sim 8$ MeV in $^{24}$Mg, fragmented between several states at $\sim 8.5 - 9.5$ MeV in $^{26}$Mg, and fully absent in $^{28}$Si. As was discussed, the vorticity is delivered by particular 1ph configurations, which can have a different energy location depending on the nuclear deformation and other factors, e.g., residual interaction. Moreover, these configurations are active only if they are of particle-hole character or supported by the pairing (like in $^{26}$Mg). A particular interplay of these factors in $^{24,26}$Mg and $^{28}$Si leads to the difference in their vorticity distribution.

D. $IS0$ strength distributions

In Figure 11, the $(\alpha, \alpha')$ data for $K^\pi = 0^+$ states in $^{24,26}$Mg and $^{28}$Si (plots (a)-(c)) are compared with QRPA isoscalar monopole strength $B(IS0)$ in the energy interval 0-16 MeV (plots (d)-(f)).

As mentioned above, because of the limitations of the experimental set up, the present $(\alpha, \alpha')$ data cover $E_x = 9 - 16$ MeV. Low-energy 0+ states listed in Tables I, III and V of Section IV are omitted in Figure 11.

Figure 11 shows that, in $^{24}$Mg, the experimental and QRPA strength distributions look rather similar. In $^{26}$Mg, the situation is quite different since QRPA predicts $IS0$ states from around $E_x = 1 - 2$ MeV. In $^{28}$Si, QRPA suggests the onset of 0+ states around $E_x = 4 - 6$ MeV. In all three nuclei, QRPA predicts some 0+ states at 9-16 MeV, which is generally in accordance with the $(\alpha, \alpha')$ data. The QRPA $IS0$ strengths summed over the $E_x = 0 - 16$ MeV interval are 26.1 fm$^4$, 13.76 fm$^4$, and 12.6 fm$^4$ in $^{24}$Mg, $^{26}$Mg, and $^{28}$Si, respectively.

Note that, in the QRPA calculations, the actual number of 0+ states at $E_x < 16$ MeV is much larger than might be seen in Figure 11. In fact, QRPA gives 48 ($^{24}$Mg), 53 ($^{26}$Mg), and 50 ($^{28}$Si) states. However, most
FIG. 11: Experimental $\beta_{2/3}^2$ factors (a-c), QRPA $B(IS0)$ values for $K^\pi = 0^+$ excitations at 0-16 MeV (d-f) and 0-30 MeV (g-i), QRPA $B(IS20)$ values at 0-30 MeV (j-l).

of these states are not seen in plots (d)-(f) because of their very small $B(IS0)$ values.

Plots (g)-(i) in Figure 11 show QRPA $B(IS0)$ strength in the larger energy interval 0-30 MeV including the IsoScalar Giant Monopole Resonance (ISGMR). In deformed nuclei, there is the coupling of monopole and quadrupole modes, see e.g. early studies [23, 85, 86] and recent systematics [87]. In particular, the ISGMR is coupled with the $\lambda \mu = 20$ branch of the IsoScalar Giant Quadrupole Resonance, ISGQR(20). Due to this coupling, a part of the IS0 strength is transferred from the ISGMR to the energy region where the ISGQR(20) branch is located. Thus, we get the deformation-induced splitting of the ISGMR strength into two parts, the main ISGMR fraction and strength located at the energy of the ISGQR(20) component. Since the ISGQR lies below the ISGMR, this strength also appears below the ISGMR. The larger the deformation, the more IS0 strength is transferred to this lower fragment from the main ISGMR, see Ref. [87] for more detail.

In our calculations, $^{24,26}$Mg and $^{28}$Si have large quadrupole deformations and so, we should expect a significant ISGMR splitting. Indeed, the plots (g)-(i) show that the ISGMR in these nuclei is split into the narrow distribution between 15 and 19 MeV and the main wide ISGMR distribution between 20 and 30 MeV. The picture is consistent in prolate $^{24,26}$Mg and oblate $^{28}$Si.

This treatment is justified by the plots (j)-(l), where the strength $B(IS20)$ of quadrupole isoscalar transitions $0^+0_{ax} \rightarrow 2^+0_{0x}$ from the ground state to the rotational quadrupole state built on the band-head $|\nu\rangle$ is exhibited. We see that the ISGQR(20) branch is located at 15-19 MeV, i.e. at the same energy as the narrow IS0 hump. This confirms that the IS0 hump is just the ISGMR part arising due to the deformation-induced ISGMR/ISGQR coupling realized for $K^\pi = 0^+$ states.

Plots (g)-(i) and (j)-(l) highlight some important points. First, the plots (g)-(i) show that the $J^\pi = 0^+$ states in $(\alpha, \alpha')$ data lie just below the ISGMR hump, i.e. basically beyond the ISGMR. Only in $^{24}$Mg, these states perhaps cover the edge of the ISGMR hump. Second, from comparison of the plots (g)-(i) and (j)-(l), we can learn that $K^\pi = 0^+$ states at 0-16 MeV exhibit both strong IS0 and IS20 transitions. They should, therefore, not be treated as solely monopole states but rather as strong mixtures of monopole and quadrupole excitations.

VI. COMPARISON WITH AMD+GCM CALCULATIONS

In this section, we discuss the comparison between the present experimental results and the AMD+GCM calculations for $^{24}$Mg and $^{28}$Si presented in Refs. [31, 6, 89]. These calculations do not take into account all the degrees of freedom of the collective excitations. Therefore, they are not appropriate for the discussion of the
global features of the observed strength distributions. However, AMD+GCM describes the clustering aspects which involves many-particle-many-hole excitations, and hence, can offer a different insight into the low-lying strengths than that from QPRA. From the mean-field side, AMD+GCM takes into account the interplay between axial and triaxial nuclear shapes, which is important for light nuclei.

In Ref. [5], using the AMD+GCM framework, the relationship between the monopole strengths in $^{24}\text{Mg}$ and clustering has been discussed. The $\alpha+^{28}\text{Ne}$, $^{8}\text{Be}+^{16}\text{O}$, $^{12}\text{C}+^{12}\text{C}$ and $5\sigma$ cluster configurations were investigated in addition to the $1ph$ single-particle excitations. It was concluded that several low-lying monopole transitions at energies below the giant monopole resonance can be attributed to the clustering as summarized in Table IX.

As already discussed in previous works on AMD+GCM and QPRA calculations [59–91], the Gogny D1S interaction overestimates the energy of the non-yrast states of $^{24}\text{Mg}$. Therefore, when we compare the AMD+GCM results listed in Table IX with the experiment, it is better to shift down the calculated excitation energies to match with the well-known states. For this purpose, Table IX also lists the calculated excitation energies shifted down by 2.9 MeV so as to reproduce the observed energy ($E_{\text{exp}}=6.4$ MeV) of the $0^+_5$ state. Note that this shift also changes the calculated excitation energy of the $0^+_5$ state (11.7 MeV → 8.8 MeV) close to the observed value (9.3 MeV), which is experimentally well established. For the higher excited states (the $0^+_5$ and $0^+_6$ states), as the observed level density is rather high, the experimental counterparts in the ENDF database [92] are ambiguous.

Table IX should be compared with the present experimental data from Table I and Figure 11. We see that the $0^+_5$ state is out of the acceptance of the present experiment, but the $0^+_5$ state is clearly observed and has the enhanced monopole strength as predicted by AMD+GCM. In Ref. [5], it was concluded that $0^+_5$ is a mixture of the collective and $^{28}\text{Ne}+\alpha$ cluster excitations. Consequently, it is interesting to note that the $0^+_5$ also appears as a prominent peak in the QRPA result (Figure 11). In addition to the $0^+_5$ state, Table I reports a state at 11.7 MeV and a group of states at 13.0–13.9 MeV with the enhanced monopole strengths. These states are of particular interest because their energies are close to the corresponding cluster decay thresholds (9.3 MeV for $^{20}\text{Ne}+\alpha$, 13.9 MeV for $^{12}\text{C}+^{12}\text{C}$, 14.047 MeV for $^{16}\text{O}+2\alpha$ and 14.138 MeV for $^{16}\text{O}+^{8}\text{Be}$) as listed in the Ikeda diagram [93]. Furthermore, these states are also visible in the excitation function reported in another $^{24}\text{Mg}(\alpha,\alpha')^{24}\text{Mg}$ experiment and seem not be reproduced by RPA calculations [25,91]. Therefore, they can be attributed to the cluster resonances. In the AMD+GCM calculations, the candidates of the $^{20}\text{Ne}+\alpha$ and $^{12}\text{C}+^{12}\text{C}$ cluster configurations were predicted at 13.2 and 15.3 MeV (10.3 and 12.4 MeV with the 2.9-MeV shift), respectively. Of course, to firmly establish the assignments of these states, more detailed analysis is indispensable. For example, the differential cross sections of these states should be compared with theoretical predictions in the future. The present experiment probes only a small range of angles and so, is insufficient for thorough comparison with theory.

For $^{28}\text{Si}$, AMD+GCM calculations suggest pairs of $0^+$ and $1^-$ states pertinent to asymmetric cluster configurations, such as $^{24}\text{Mg}+\alpha$, $^{20}\text{Ne}+^{8}\text{Be}$ and $^{16}\text{O}+^{12}\text{C}$ [6]. The predicted results are summarized in Table X. Similar to the $^{24}\text{Mg}$ case, the Gogny D1S interaction systematically overestimates the energies of the non-yrast states, see Figure 6 in Ref. [6]. Therefore, while comparing the AMD+GCM and experimental results, we again use the downshift of the calculated excitation energies, now by 3.3 MeV, to match the energy of the observed $0^+_5$ state. Note that this well-known prolate-deformed state should have a large contribution from the $^{16}\text{O}+^{12}\text{C}$ cluster configuration [94–97]. The value of the energy downshift looks reasonable as it is similar to that introduced for $^{24}\text{Mg}$. With this shift, the energies of other well-known states show reasonable agreement between the AMD+GCM and experimental results. For example, the $2^+$ member of the SuperDeformed (SD) band, which has been experimentally identified at 9.8 MeV in Ref. [74], agrees well with the shifted AMD+GCM state at 9.7 MeV. Furthermore, a couple of the $^{24}\text{Mg}+\alpha$ cluster resonances have been identified around 13 MeV in resonant scattering experiments [98,99], which are close to the shifted AMD $0^+_5$ state at 14.9 MeV.

We now examine the cluster configurations listed in Table X and compare to the present experimental data. Since the monopole ($IS$0) and dipole ($IS$1) transitions have a strong selectivity for the cluster states, the cluster configurations can be classified into two groups, which are strongly populated/hindered in the ($\alpha,\alpha'$) reaction. For example, from a simple theoretical consideration, we can predict that the $0^+_5$ state that is the band head of the prolate band (the lowest $^{16}\text{O}+^{12}\text{C}$ cluster band) should be hindered. See Ref. [101] for details of the hindrance mechanism. It is interesting that the hindrance of the $0^+_5$ state can also be seen in the QRPA results shown in Figure 11. Unfortunately, this state (which is important for validation of the relationship between the monopole transitions and clustering) is out of the acceptance of the present experiment. For the same reason, the AMD+GCM predicts that the SD band head expected at 9.3 MeV should also be hindered. However, in the present experiment, the observed 9.7-MeV $0^+$ state is very close to the 9.8-MeV $2^+$ state and, following Table V, has the enhanced monopole strength in contradiction to the AMD+GCM prediction. This new result requires a more detailed analysis of the SD state in $^{28}\text{Si}$.

At the same time, AMD+GCM predicts an enhancement of the $^{20}\text{Ne}+^{8}\text{Be}$ and $^{24}\text{Mg}+\alpha$ cluster configurations. The pair of the $0^+_5$ and $1^+_2$ states with the $^{20}\text{Ne}+^{8}\text{Be}$ configuration is predicted at $10–11$ MeV, and some fractions of $IS$0 and $IS$1 strengths are indeed experimentally observed in this energy region. This may
TABLE IX: The cluster configurations with significant $B(IS0)$ strengths and their excitation energies $E_x$ in $^{24}$Mg calculated by AMD+GCM and compared with the observed data [92]. The energies $E_{\text{shift}}$ are obtained by a downshift of 2.9 MeV so as to adjust $E_{\text{exp}}$ for the $0^+_1$ state.

| Cluster       | $J^u$ | $E_x$ [MeV] | $B(IS0)$ [fm$^2$] | $E_{\text{shift}}$ [MeV] | $E_{\text{exp}}$ [MeV] | $B(IS0)_{\text{exp}}$ [fm$^2$] |
|---------------|-------|-------------|-------------------|-------------------|-----------------|-----------------|
| $^{20}$Ne+$^8$Be | $0^+_2$ | 5.8        | 16.0              | 2.5               | 4.98            | 14.7            |
| $^{20}$Ne+$^8$Be | $0^+_4$ | 13.8       | 9.3               | 10.5              |                 |                 |
| $^{24}$Mg+$^\alpha$ | $1^+_1$ | 1.6        | 13.0              | 9.6               |                 |                 |
| $^{24}$Mg+$^\alpha$ | $0^+_4$ | 18.2       | 5.1               | 14.9              | 13.0            |                 |
| $^{24}$Mg+$^\alpha$ | $1^+_5$ | 20.6       | 64.0              | 17.3              |                 |                 |
| $^{12}$C+$^{12}$C | $0^+_2$ | 10.0       | 0.0               | 6.7               | 6.69            |                 |
| $^{24}$M+$^\alpha$ (SD) | $0^+_4$ | 15.8       | 0.0               | 12.5              |                 |                 |
| $^{24}$M+$^\alpha$ (SD) | $2^+_5$ | 12.6       | 0.0               | 9.3               | 9.7             |                 |
| $^{24}$M+$^\alpha$ (SD) | $4^+_1$ | 17.6       | 0.0               | 9.7               |                 |                 |
| $^{24}$M+$^\alpha$ (SD) | $4^+_2$ | 18.8       | 0.0               | 15.5              |                 |                 |

TABLE X: The cluster configurations with their excitation energies $E_x$ and transition strengths ($B(IS0)$ for the $0^+$ states and $B(IS1)$ for the $1^+$ states) in $^{28}$Si, calculated within AMD+GCM. The experimental counterparts are taken from Ref. [108] and the present experiment (denoted by bold). The energies $E_{\text{shift}}$ are obtained by a downshift of 3.3 MeV so as to adjust the energy $E_{\text{exp}}$ to 6.69 MeV for the $0^+_1$ state.

| Cluster       | $J^u$ | $E_x$ [MeV] | $B(IS0)$ [fm$^2$] | $E_{\text{shift}}$ [MeV] | $E_{\text{exp}}$ [MeV] | $B(IS0)_{\text{exp}}$ [fm$^2$] |
|---------------|-------|-------------|-------------------|-------------------|-----------------|-----------------|
| $^{20}$Ne+$^8$Be | $0^+_2$ | 5.8        | 16.0              | 2.5               | 4.98            | 14.7            |
| $^{20}$Ne+$^8$Be | $0^+_4$ | 13.8       | 9.3               | 10.5              |                 |                 |
| $^{24}$Mg+$^\alpha$ | $1^+_1$ | 1.6        | 13.0              | 9.6               |                 |                 |
| $^{24}$Mg+$^\alpha$ | $0^+_4$ | 18.2       | 5.1               | 14.9              | 13.0            |                 |
| $^{24}$Mg+$^\alpha$ | $1^+_5$ | 20.6       | 64.0              | 17.3              |                 |                 |
| $^{12}$C+$^{12}$C | $0^+_2$ | 10.0       | 0.0               | 6.7               | 6.69            |                 |
| $^{24}$M+$^\alpha$ (SD) | $0^+_4$ | 15.8       | 0.0               | 12.5              |                 |                 |
| $^{24}$M+$^\alpha$ (SD) | $2^+_5$ | 12.6       | 0.0               | 9.3               | 9.7             |                 |
| $^{24}$M+$^\alpha$ (SD) | $4^+_1$ | 17.6       | 0.0               | 9.7               |                 |                 |
| $^{24}$M+$^\alpha$ (SD) | $4^+_2$ | 18.8       | 0.0               | 15.5              |                 |                 |

be the first indication of the $^{20}$Ne+$^8$Be clustering in $^{28}$Si, which must be confirmed by a more detailed study, e.g. the transfer of $^8$Be to $^{20}$Ne. Other states which are predicted to be strongly populated in the $(\alpha,\alpha')$ reaction are $^{24}$Mg+$\alpha$ cluster states. AMD+GCM study a $1^+_1$ state at 9.6 MeV and $0^+$ and $1^-$ states at approximately 15 and 17-20 MeV. The $1^-$ states at 17-20 MeV are beyond the present experiment. However, several $0^+$ states can be seen at 9.5 MeV and 15 MeV. It is worthwhile to note that the $\alpha$ transfer and $\alpha+^{24}$Mg resonant scattering experiments [95, 96] also report a group of the $\alpha+^{24}$Mg resonances with $J=0^+$ in the same energy region. Therefore, the data of the previous and present experiments as well as the AMD+GCM results look consistent. A more detailed comparison between AMD+GCM and experimental results may be conducted in the future.

VII. CONCLUSIONS

The isoscalar dipole (IS1) and monopole (IS0) excitations of $^{24}$Mg, $^{26}$Mg and $^{28}$Si at the energy interval $E_x = 9 - 16$ MeV have been measured using the $(\alpha,\alpha')$ inelastic-scattering reaction at forward angles (including zero degrees). The experiment was performed at the K600 magnetic spectrometer at iThemba LABS (Cape Town, South Africa). New monopole and dipole states were reported at $E_x = 13 - 15$ MeV.

The extracted IS1 and IS0 strength distributions were compared to the theoretical calculations performed within the QRPA [48-51] and AMD+GCM approaches. The correspondence, at least tentative, between some calculated and observed states was established. This theoretical analysis allows us to draw some important physical conclusions.

First of all, QRPA and AMD+GCM calculations suggest that low-lying IS1 states in light nuclei can have two origins: irrotational cluster (IC) [6] and mean-field (MF) [9, 10, 16]. The MF-states can be irrotational (IMF) and vortical (VMF) [9, 10, 16].

The IC states produce $T=0$ negative-parity cluster bands, which are the doublets of the positive-parity bands based on the monopole states [6]. Some traces of these doublets were found in the comparison of theoretical calculations and experimental data. IC states are actually irrotational dipole oscillations of two clusters which constitute the nucleus relative to one other. These states originate from the reflection-asymmetric form of the nucleus exhibiting the clustering. The negative-parity bands produced by IC states usually have $K=0$.

Instead, the VMF states in light nuclei are mainly of mean-field origin [9, 10, 16] and can exist without clustering. They do not need the reflection-asymmetric nuclear shape and the associated the monopole doublets. Following previous studies [9, 10, 16] and present QRPA calculations, these states produce negative-parity rotational bands, mainly with $K=1$.

Both IC and IMF/VMF states exhibit enhanced IS1 transitions and are usually located near the alpha-particle threshold. In general, IC and IMF/VMF states can be mixed, especially in soft and triaxial nuclei exhibiting $K$-mixing. Nevertheless, the relation to the $K=0$ or $K=1$ bands is perhaps a reasonable indicator for an initial discrimination of IC and VMF states.

Being strongly deformed, $^{24,26}$Mg and $^{28}$Si should exhibit a strong coupling between dipole and octupole modes and between monopole and quadrupole modes. This coupling was confirmed by QRPA calculations, where strong IS3K $(0^+0_{gs} \rightarrow 3^-K\nu)$ and IS20 $(0^+0_{gs} \rightarrow 2^0K\nu)$ transitions were found. So, the theoret-
ically explored states are actually dipole/octupole and monopole/quadrupole mixtures. Further, QRPA predicts that there should exist a specific collective state \((K = 0\) in prolate and \(K = 1\) in oblate nuclei) with an impressive octupole strength near the \(\alpha\)-particle threshold. This near-threshold state manifests the onset of states with cluster features.

Due to triaxiality and significant shape coexistence in \(^{24,26}\text{Mg}\) and \(^{28}\text{Si}\). QRPA results obtained at the fixed axial deformation should be considered as approximate. In addition, QRPA calculations do not include all the dynamical correlations, \(e.g.,\) the coupling with complex configurations. Nevertheless, the main QRPA prediction - of vortical dipole states with enhanced IS1 strength as an alternative to the cluster dipole states - remains robust. In our opinion, more involved calculations may change some details but not this general prediction.

Another interesting QRPA prediction is a change in dipole vorticity below the \(\alpha\)-particle thresholds \(S_\alpha\) in \(^{24,26}\text{Mg}\) and \(^{28}\text{Si}\). Following our analysis, the vorticity is concentrated in the lowest dipole state in \(^{26}\text{Mg}\) at \(\sim 8\) MeV, is fragmented between several states at \(\sim 8.5\)–\(9.5\) MeV in \(^{26}\text{Mg}\), and is fully absent in \(^{28}\text{Si}\). The difference is explained by different energies of \(1p\) configurations responsible for the vorticity. Our explorations confirm the suggestion made in Ref. [9] that \(^{24}\text{Mg}\) is perhaps the unique nucleus with a well-separated low-energy vortical state.

The present \((\alpha, \alpha')\) data do not yet allow confident assignment of the vortical or cluster character of the excitations. However, these data improve our knowledge of the isoscalar monopole and dipole states at the excitation energies where the clustering and vorticity are predicted. This is a necessary and important step in the right direction. The use of the \((\alpha, \alpha')\) reaction at intermediate energies complements other suggested mechanisms for populating cluster and vortical states such as the \((\gamma, \gamma')\) [102], \((e, e')\) [11] and \((d, ^6\text{Li})\) reactions [88] or \((^6\text{Li}, d/t)\), although detailed information on the interior of the nuclei and the vortical mode is likely only available from the \((e, e')\) reaction. Branching ratios and transition strengths of \(\gamma\)-ray transitions from the observed dipole states would provide information on the \(K\) assignment of the levels and should also be a focus of additional future experimental work.

Acknowledgments

The authors thank the Accelerator Group at iThemba LABS for the high-quality dispersion-matched beam provided for this experiment. PA acknowledges support from the Claude Leon Foundation in the form of a postdoctoral fellowship, M. N. Harakeh for providing the BELGEN and FERMDEN codes and helpful advice regarding the DWBA calculations, and Josef Cseh for useful discussions concerning \(^{28}\text{Si}\). RN acknowledges support from the NRF through Grant No. 85509. VON and JK thank Dr. A. Repko for the QRPA code. The work was partly supported by Votruba - Blokhintsev (Czech Republic - BLTP JINR) grant (VON and JK) and a grant of the Czech Science Agency, Project No. 19-14048S (JK). VON and PGR appreciate the Heisenberg-Landau grant (Germany DLTP JINR).

Appendix A: Details of DWBA calculations

In past studies, \(e.g.,\) [25–27], the real part of the potential has been calculated using a folding model, and the imaginary part of the potential has been determined by fitting to elastic-scattering data. Due to time limitations, especially in moving the detectors from the high-dispersion focal plane to the medium-dispersion focal plane of the K600, it was not possible to take elastic-scattering data for this purpose. Instead, the Nolte, Machner and Bojowald optical-model potential was used. For this potential, the reduced radii are \(r_R = 1.245\) fm and \(r_I = 1.570\) fm for the real and imaginary part of the potential, respectively. Other parameters, such as the diffuseness and the depths of the potentials are energy-dependent quantities, which are calculated separately for each entrance and exit channel.

For \(^{24}\text{Mg}\), we employ the quadrupole deformation \(\beta_2 = 0.355\) from Ref. [28]. Using this deformation and the reduced radius of the real potential, we get (with the codes BELGEN and FERMDEN which have been made available at [github.com/padsley/KVICodes]) for the ground-state band, the value \(B(E2) \uparrow = 0.0423\) e²b², which is in good agreement with the experimental value of \(B(E2) \uparrow = 0.0432(19)\) e²b². Using the measured \(B(E2)\) values for \(^{26}\text{Mg}\) and \(^{28}\text{Si}\), we obtain the quadrupole deformations of \(\beta_2 = 0.295\) and \(\beta_2 = -0.255\), respectively. The signs of these deformations (prolate in \(^{26}\text{Mg}\) and oblate in \(^{28}\text{Si}\)) were chosen following the discussion in Sec. V-A. Note that the above parameters of the quadrupole deformation are much smaller than the absolute values for those from the NNDC database [81] (\(\beta_2^{\text{exp}} = 0.613, 0.484, \text{ and } -0.412\) for \(^{24,26}\text{Mg}\) and \(^{28}\text{Si}\)). This is because the NNDC quadrupole deformation parameters are determined assuming a uniform charge distribution, while we use the Fermi distribution for the mass.

Since the radii of the real and imaginary parts of the potential are different, we assumed that the deformation lengths for the real and imaginary parts of each of the potentials are the same:

\[
\beta_R R_R = \beta_I R_I \tag{A1}
\]

where \(R_R = r_R A^{1/3}\), \(R_I = r_I A^{1/3}\), and \(A\) is the mass number of the target [103]. Additionally, following Refs. [23, 50], we assume that the deformation lengths of the potential and the mass distribution are identical, i.e.

\[
\beta_R R_p = \beta_m R_m \tag{A2}
\]

where the mass radius is \(R_m = r_m A^{1/3}\) and \(r_m\) is determined from the reduced radius for the potential
Then, using Eqs. (A1) and (A2), the values of the total and DWBA differential cross sections, see Eq. (2), were obtained. For dipole transitions, we employed the form factors determined by comparing the corresponding experimental and DWBA differential cross sections, see Eq. (2). Then, using Eqs. (A1) and (A2), the values \( \beta_I \) and \( \beta_m \) were obtained.

The percentage of the monopole (\( \lambda = 0 \)) EWSR exhausted by a given state is given by [56]:

\[
S_0 = \frac{\beta^2_{m,0}}{\beta^2_{M,0}},
\]

(A3)

where \( \beta^2_{m,0} \) is the monopole transition strength determined from Eq. (A2) and

\[
\beta^2_{M,0} = \frac{4\pi\hbar^2}{2mAE_x(r^2)}
\]

(A4)

is the total transition strength for the state located at the excitation energy, \( E_x \), and exhausting 100% of the monopole EWSR [56]. Here \( m \) is the nucleon mass and \( \langle r^2 \rangle \) is calculated from the Fermi mass distribution using the Fermi den code.

For dipole (\( \lambda = 1 \)) transitions, the fraction of the EWSR exhausted by a state is given by [56]:

\[
S_1 = \frac{\beta^2_{m,1}}{\beta^2_{M,1}},
\]

(A5)

where \( \beta^2_{m,1} \) is the dipole transition strength, again from Eq. (A2) and

\[
\beta^2_{M,1} = \frac{6\pi\hbar^2}{mAE_x(11\langle r^4 \rangle - \frac{5}{4}\langle r^2 \rangle^2 - 10\epsilon\langle r^2 \rangle)}
\]

(A6)

is the total transition strength for the state lying at excitation energy, \( E_x \), and exhausting 100% of the dipole EWSR [103]. Here \( \langle r^2 \rangle \) and \( \langle r^4 \rangle \) are calculated from the real part of the optical-model potential using Fermi den, and \( R_m \) is the half-density radius of the Fermi mass distribution. The parameter, \( \epsilon \), is generally small compared to the other quantities but is given by:

\[
\epsilon = \frac{\hbar^2}{3mAR^2} \left( \frac{4}{E_2} + \frac{5}{E_0} \right),
\]

(A7)

where \( E_2 = 65A^{-1/3} \text{ MeV} \) is the centroid energy of the isoscalar giant quadrupole resonance and \( E_0 = 80A^{-1/3} \text{ MeV} \) is the centroid energy of the isoscalar giant monopole resonance.
[69] U. Giesen, C. Browne, J. Görres, S. Graff, C. Iliadis, H.-P. Trautvetter, M. Wiescher, W. Harms, K. Kratz, B. Pfeiffer, R.E. Azuma, M. Buckby, and J.D. King, Nucl. Phys. A 561, 95 (1993).
[70] H. W. Drotleff, A. Denker, H. Knee, M. Soine, G. Wolf, J. W. Hammer, U. Greife, C. Rolfs, and H. P. Trautvetter, Astrophys. J. 414, 735 (1993).
[71] M. Jaeger, R. Kunz, A. Mayer, J. W. Hammer, G. Staudt, K. L. Kratz, and B. Pfeiffer, Phys. Rev. Lett. 87, 202501 (2001).
[72] H. Jayatissa, G. Rogachev, V. Goldberg, E. Koshchiy, G. Christian, J. Hooker, S. Ota, B. Roeder, A. Saastamoinen, O. Trippella, S. Upadhyayula, and E. Uberseder, Phys. Lett. B 802, 135267 (2020).
[73] S. Ota, G. Christian, G. Lotay, W. Catford, P. Chowdury, N. J. Hammond, R. V. F. Janssens, T. L. Khoo, T. Lauritsen, D. Seweryniak, T. Davinson, P. J. Woods, A. Jokinen, H. Penttila, F. Haas, and S. Courtin, Phys. Rev. C 86, 064308 (2012).
[74] D. G. Jenkins, C. J. Lister, M. P. Carpenter, P. Chowdury, N. J. Hammond, R. V. F. Janssens, T. L. Khoo, T. Lauritsen, D. Seweryniak, T. Davinson, P. J. Woods, A. Jokinen, H. Penttila, F. Haas, and S. Courtin, Phys. Rev. C 86, 064308 (2012).
[75] P. Adsley, A. Laird, and Z. Meisel, arXiv:1912.11826 (2019).
[76] E. Chabanat, P. Bonche, P. Haensel, J. Meyer, and R. Schaeffer, Nucl. Phys. A 635, 231 (1998).
[77] W. Kleinhia, V. O. Nesterenko, J. Kvasil, P.-G. Reinhard, and P. Vesely, Phys. Rev. C 78, 044313 (2008).
[78] N. Stone, J. Phys. Chem. Ref. Data 44, 031215 (2015).
[79] S. Watanabe, K. Minomo, M. Shimada, S. Tagami, M. Kimura, M. Takechi, M. Fukuda, D. Nishimura, T. Suzuki, T. Matsumoto, Y.R. Shimizu, and M. Yahiro, Phys. Rev. C 89, 044610 (2014).
[80] W. Horiuchi, T. Inakura, T. Nakatsukasa, and Y. Suzuki, Phys. Rev. C 86, 024614 (2012).
[81] Database https://www.nndc.bnl.gov/nudat2.
[82] S.G. Nilsson, Mat. Fys. Medd. Dan. Vid. Selsk. 29, n.16 (1965).
[83] B. Mottelson and S. G. Nilsson, Mat. Fys. Skr. Dan. Vid. Selsk 1, No. 8 (1959).
[84] L.A. Malov, V.O. Nesterenko, and V.G. Soloviev, Phys. Lett. B 64, 247 (1976).
[85] Y. Abgrall, B. Morand, E. Caurier, and B. Grammaticos, Nucl. Phys. A 346, 411 (1980).
[86] S. Jang, Nucl. Phys. A 401, 303 (1983).
[87] J. Kvasil, V. O. Nesterenko, A. Repko, W. Kleinhia, and P.-G. Reinhard, Phys. Rev. C 94, 064302 (2016).
[88] M. Spieker, S. Pascu, A. Zilges, and F. Iachello, Phys. Rev. Lett. 114, 192504 (2015).
[89] Y. Taniguchi and M. Kimura, Phys. Lett. B 800, 135086 (2020).
[90] S. Pérui and H. Goutte, Phys. Rev. C 95 044328 (2008).
[91] S. Pérui and M. Martini, Eur. Phys. J. A 50, 88 (2014).
[92] R. B. Firestone, Nuclear Data Sheets 108, 2319 (2007).
[93] K. Ikeda, N. Takigawa, and H. Horiuchi, Prog. Theor. Phys. Suppl. E 68, 464 (1968).
[94] D. Baye, Nucl. Phys. A 272, 445 (1976).
[95] D. Baye and P.-H. Heenen, Nucl. Phys. A 273, 176 (1977).
[96] Y. Taniguchi, Y. Kanada-En’yo, and M. Kimura, Phys. Rev. C 80, 044316 (2009).
[97] J. Darai, J. Cseh, and D. G. Jenkins, Phys. Rev. C 86, 064309 (2012).
[98] T. Tanabe, K. Haga, M. Yasue, K. Sato, K. Ogino, Y. Kadota, M Tochi, K. Makino, T. Kitahara, and T. Shiba, Nucl. Phys. A 399, 241 (1983).
[99] K. P. Artemov et al., Yad. Fiz. 51, 1220 (1990).
[100] M. Shamsuzzoha Basunia, Nuclear Data Sheets 114, 1189 (2013).
[101] M. Kimura, Y. Chiba, and Y. Taniguchi, J. Phys.: Conf. Ser. 863, 012024 (2017).
[102] F. Iachello, Physics Letters B 160, 1 (1985).
[103] M. N. Harakeh and A. E. L. Dieperink, Phys. Rev. C 23, 2329 (1981).
[104] M. Nolte, H. Machner, and J. Bojowald, Phys. Rev. C 36, 1312 (1987).
[105] M. N. Harakeh, private communication.
[106] github.com/padsley/AngCorPackage.