Coefficient Inequalities for Classes of Univalent Functions Defined by $q-$ Derivatives

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ABSTRACT: Using the principal of subordination and the $q-$derivative, we obtain sharp bounds for some classes of univalent functions.

Key Words: Univalent functions, $q-$derivative , Subordination.

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1. Introduction

Denote by $A$ the class of analytic functions:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U = \{z : z \in \mathbb{C}, |z| < 1\}).$$

(1.1)

For $0 < q < 1$, the $q$-derivative of $f \in A$, is given by (see [4], [5])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0$$

$$= 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

(1.2)

where, $[n]_q = \frac{q^n - 1}{q - 1}$, as $q \to 1^-,[n]_q \to n$, $D_q f(0) = f'(0)$ and $D_q(D_q f(z)) = D^2_q f(z)$. If $\eta(z) = z^n$, then

$$D_q \eta(z) = D_q(z^n) = \frac{q^n - 1}{q - 1} z^{n-1} = [n]_q z^{n-1},$$

$$\lim_{q \to 1^-} D_q \eta(z) = \lim_{q \to 1^-} [n]_q z^{n-1} = nz^{n-1} = \eta'(z).$$

Denote by $P$ the class of analytic functions $\phi$ of positive real part on $U$ with $\phi(0) = 1$, $\Re\{\phi(z)\} > 0$.

Using the $q$-derivative $D_q f(z), f \in A, \alpha \in F, 0 \leq \lambda \leq 1, b \in \mathbb{C}^* = \mathbb{C}/\{0\}$, let

$$\mathcal{H}^\lambda_{q,b}(\alpha) = \left\{ f : 1 + \frac{1}{b} \left[ (1 - \lambda) \left( \frac{zD_q f(z)}{f(z)} \right) + \lambda \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right] \prec \alpha(z) \right\},$$

(1.3)

where $\prec$ denotes the usual subordination (see [7], [3], [2]).

For different choices of $q, b, \lambda, \alpha$ in (1.3), the class $\mathcal{H}^\lambda_{q,b}(\alpha)$, generalizes many classes studied earlier, for example (see Seoudy and Aouf [10], [11], Ravichandran et al. [9], Ali et al. [1] with $p = 1$.

* The third-named author was supported by the Basic Science Research Program through the National Research Foundation of the Republic of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1A3A01050861).

2010 Mathematics Subject Classification: 30C45.

Submitted June 09, 2018. Published November 07, 2018
and Ramachandran et al. [8], with $\alpha = 0$ and $\beta = 1$. Also, we obtain the new class $\mathcal{H}_{q,\theta}^{\gamma,\alpha}(\varkappa)$ for

$$b = e^{-i\theta}(1 - \alpha) \cos \theta, 0 \leq \alpha < 1, |\theta| < \frac{\pi}{2},$$

where

$$\mathcal{H}_{q,\theta}^{\gamma,\alpha}(\varkappa) = \left\{ f : \frac{e^{i\theta} \left( (1 - \lambda) \left( \frac{zD_{q}f(z)}{f(z)} \right) + \lambda \frac{D_{q}(zD_{q}f(z))}{D_{q}f(z)} \right) - \alpha \cos \theta - i \sin \theta}{(1 - \alpha) \cos \theta} < \varkappa(z) \right\}.$$

The following known lemma is needed to establish our results.

**Lemma 1.1** [6]. If $p(z) = 1 + r_{1}z + r_{2}z^{2} + \ldots \in \mathcal{P}$ and $\delta$ is a complex number, then

$$|r_{2} - \delta r_{1}^{2}| \leq 2 \max\{1; |2\delta - 1|\}. \quad (1.4)$$

The result is sharp for the functions given by

$$p(z) = \frac{1 + z}{1 - z^{2}} \text{ and } p(z) = \frac{1 + z}{1 - z}.$$

Also, we note that

$$|r_{2} - \xi r_{1}^{2}| \leq \begin{cases} -4\xi + 2 & if \xi \leq 0, \\ 2 & if 0 \leq \xi \leq 1, \\ 4\xi - 2 & if \xi \geq 1, \end{cases} \quad (1.5)$$

when $\xi < 0$ or $\xi > 1$, the equality holds if and only if $p(z) = (1 + z)/(1 - z)$ or one of its rotations. If $0 < \xi < 1$, then the equality holds if and only if $p(z) = (1 + z^{2})/(1 - z^{2})$ or one of its rotations. If $\xi = 0$, the equality holds if and only if

$$p(z) = \left( \frac{1 + \gamma}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1 - \gamma}{2} \right) \frac{1 - z}{1 + z} \quad (0 \leq \gamma \leq 1)$$

or one of its rotations. If $\gamma = 1$, the equality holds if and only if $p$ is the reciprocal of one of the functions such that equality holds in the case of $\xi = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < \xi < 1$:

$$|r_{2} - \xi r_{1}^{2}| + \xi |r_{1}|^{2} \leq 2 \left( 0 \leq \xi \leq \frac{1}{2} \right)$$

and

$$|r_{2} - \xi r_{1}^{2}| + (1 - \xi) |r_{1}|^{2} \leq 2 \left( \frac{1}{2} \leq \xi \leq 1 \right).$$

2. Main results

We assume in the reminder of this paper that $f \in A, \varkappa \in \mathcal{P}, 0 < q < 1, 0 \leq \lambda \leq 1$ and $b \in \mathbb{C}^{*}$.

**Theorem 2.1.** Let

$$\varkappa(z) = 1 + d_{1}z + d_{2}z^{2} + \ldots \quad (2.1)$$

with $d_{1} > 0$. If $f(z) \in \mathcal{H}_{q,\theta}^{\gamma,\alpha}(\varkappa)$, then

$$|a_{3} - \mu a_{2}^{2}| \leq \frac{|b| |d_{4}|}{2|\lambda q - 1| \left[ 1 + \lambda (\lambda q - 1) \right]} \max \{1, \frac{|d_{2}|}{|d_{1}|} \left[ \frac{1}{\lambda (\lambda q - 1)} \right] \} \left[ 1 + \lambda (\lambda q - 1) - \mu \left( \frac{(\lambda q - 1)^{2}}{(\lambda q - 1)} \right) \right]. \quad (2.2)$$

The result is sharp.
### Proof:
If \( f \in \mathcal{H}^{\lambda}_{q,b}(\mathcal{U}) \), then there is a function \( \omega \), analytic in \( \mathcal{U} \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) such that

\[
1 + \frac{1}{b} \left[ (1 - \lambda) \frac{\omega D_q f(z)}{f(z)} + \lambda \frac{D_q(z D_q f(z))}{D_q f(z)} - 1 \right] = \mathcal{N}(\omega(z)).
\]  

(2.3)

Define the function \( p(z) \) by

\[
p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + r_1 z + r_2 z^2 + \ldots.
\]

(2.4)

We see that \( \Re \{p(z)\} > 0 \) and \( p(0) = 1 \), since \( \omega(z) \) is a Schwarz function. Therefore,

\[
\mathcal{N}(\omega(z)) = \mathcal{N} \left( \frac{p(z) - 1}{p(z) + 1} \right) = \mathcal{N} \left( \frac{1}{2} \left[ r_1 z + \left( r_2 - \frac{r_2^2}{2} \right) z^2 + \left( r_3 - r_1 r_2 + \frac{r_1^3}{4} \right) z^3 + \ldots \right] \right) = 1 + \frac{1}{2} d_1 r_1 z + \left[ \frac{1}{2} d_1 \left( r_2 - \frac{r_2^2}{2} \right) + \frac{1}{4} d_2 r_1^2 \right] z^2 + \ldots.
\]

(2.5)

Equating the coefficients of (2.5) and (2.3), we have

\[
([2]_q - 1 + \lambda([2]_q - 1)^2)a_2 = \frac{1}{2} bd_1 r_1,
\]

\[
([3]_q - 1)[1 + \lambda([3]_q - 1)]a_3 - ([2]_q - 1)[1 + \lambda([2]_q - 1)]a_2^2
= \left( \frac{1}{2} d_1 r_2 - \frac{1}{4} d_1 r_1^2 + \frac{1}{4} d_2 r_1^2 \right) b,
\]

or

\[
a_2 = \frac{bd_1 r_1}{2([2]_q - 1)[1 + \lambda([2]_q - 1)]},
\]

\[
a_3 = \frac{bd_1}{2([3]_q - 1)[1 + \lambda([3]_q - 1)]} \left\{ d_2 - \frac{d_1^2}{2} \left[ 1 - \frac{d_2}{d_1} - \frac{[1 + \lambda([2]_q - 1)] bd_1}{([2]_q - 1)[1 + \lambda([2]_q - 1)]^2} \right] \right\}.
\]

Therefore,

\[
a_3 - \mu a_2^2 = \frac{bd_1}{2([3]_q - 1)[1 + \lambda([3]_q - 1)]} \left( d_2 - \delta d_1^2 \right),
\]

where

\[
\delta = \frac{1}{2} \left\{ 1 - \frac{d_2}{d_1} - \frac{bd_1}{([2]_q - 1)[1 + \lambda([2]_q - 1)]} \left[ 1 + \lambda([2]_q - 1) - \mu \frac{([3]_q - 1)[1 + \lambda([3]_q - 1)]}{([2]_q - 1)} \right] \right\}.
\]

(2.6)

(2.7)

Our result now follows by an application of (1.4). The result is sharp for the functions

\[
1 + \frac{1}{b} \left[ (1 - \lambda) \frac{z D_q f(z)}{f(z)} + \lambda \frac{D_q(z D_q f(z))}{D_q f(z)} - 1 \right] = \mathcal{N}(z^2),
\]

and

\[
1 + \frac{1}{b} \left[ (1 - \lambda) \frac{z D_q f(z)}{f(z)} + \lambda \frac{D_q(z D_q f(z))}{D_q f(z)} - 1 \right] = \mathcal{N}(z).
\]

The proof of Theorem 1 is completed. \( \square \)

### Remark 2.1.
(i) Putting \( \lambda = 0 \) in Theorem 1, we obtain the result of Seoudy and Aouf [10, Theorem 1];

(ii) Putting \( \lambda = 1 \) in Theorem 1, we obtain the result of Seoudy and Aouf [10, Theorem 2];

(iii) Theorem 1 for \( b = 1 \), corrects the result of Ramachandram et al. [8, Theorem 2, \( \alpha = 0, \beta = 1 \)].
Theorem 2.2. Let \( \kappa(z) \) in the form (2.1), with \( d_1 > 0 \) and \( d_3 \geq 0 \). Let
\[
\begin{align*}
\alpha_1 &= \frac{(d_2-d_1)([2]_q-1)^2[1+\lambda([2]_q-1)]^2+(2[2]_q-1)[1+\lambda([2]_q-1)]bd_3^2}{([3]_q-1)[1+\lambda([3]_q-1)]bd_3^2}, \\
\alpha_2 &= \frac{(d_2+d_1)([2]_q-1)^2[1+\lambda([2]_q-1)]^2+(2[2]_q-1)[1+\lambda([2]_q-1)]bd_3^2}{([3]_q-1)[1+\lambda([3]_q-1)]bd_3^2}, \\
\alpha_3 &= \frac{d_3([2]_q-1)^2[1+\lambda([2]_q-1)]^2+(2[2]_q-1)[1+\lambda([2]_q-1)]bd_3^2}{([3]_q-1)[1+\lambda([3]_q-1)]bd_3^2}.
\end{align*}
\]

If \( f(z) \in \mathcal{H}^\kappa_{q,b}(\kappa) \) with \( b > 0 \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases}
\frac{b^2 d^2}{([3]_q-1)[1+\lambda([3]_q-1)]^2} + \frac{b^2 d^2}{[2]_q-1)[1+\lambda([2]_q-1)]^2} \left( \frac{1}{[2]_q-1)[1+\lambda([2]_q-1)]} - \frac{1}{([3]_q-1)[1+\lambda([3]_q-1)]} \right), & \mu \leq \alpha_1, \\
\frac{b^2 d^2}{([3]_q-1)[1+\lambda([3]_q-1)]^2} - \frac{b^2 d^2}{[2]_q-1)[1+\lambda([2]_q-1)]^2} \left( \frac{1}{[2]_q-1)[1+\lambda([2]_q-1)]} - \frac{1}{([3]_q-1)[1+\lambda([3]_q-1)]} \right), & \alpha_1 \leq \mu \leq \alpha_2,
\end{cases}
\]
and if \( \alpha_1 \leq \mu \leq \alpha_3 \), then
\[
|a_3 - \mu a_2^2| + \frac{([2]_q-1)^2[1+\lambda([2]_q-1)]^2}{([3]_q-1)[1+\lambda([3]_q-1)]d_1b} \left( d_1 - d_2 - \frac{bd_3}{([2]_q-1)[1+\lambda([2]_q-1)]} \right)
\times \left( \frac{1}{[2]_q-1)[1+\lambda([2]_q-1)]} - \mu \frac{([3]_q-1)[1+\lambda([3]_q-1)]}{([3]_q-1)[1+\lambda([3]_q-1)]} \right) |a_2|^2 \leq \frac{bd_3}{([3]_q-1)[1+\lambda([3]_q-1)]},
\]
and if \( \alpha_3 \leq \mu \leq \alpha_2 \), then
\[
|a_3 - \mu a_2^2| + \frac{([2]_q-1)^2[1+\lambda([2]_q-1)]^2}{([3]_q-1)[1+\lambda([3]_q-1)]d_1b} \left( d_1 + d_2 + \frac{bd_3}{([2]_q-1)[1+\lambda([2]_q-1)]} \right)
\times \left( \frac{1}{[2]_q-1)[1+\lambda([2]_q-1)]} - \mu \frac{([3]_q-1)[1+\lambda([3]_q-1)]}{([3]_q-1)[1+\lambda([3]_q-1)]} \right) |a_2|^2 \leq \frac{bd_3}{([3]_q-1)[1+\lambda([3]_q-1)]}.
\]

The result is sharp.

Proof: The proof follows by applying (1.5) to (2.6) and (2.7). To show that the bounds are sharp, we define the functions \( \mathcal{K}_{\kappa,k} (k = 2, 3, 4, \ldots) \) by
\[
1 + \frac{1}{b} \left[ (1 - \lambda) \frac{D_q \mathcal{K}_{\kappa,k}(z)}{\mathcal{K}_{\kappa,k}(z)} + \lambda \frac{D_q(z D_q \mathcal{K}_{\kappa,k}(z))}{D_q \mathcal{K}_{\kappa,k}(z)} - 1 \right] = \kappa \left( z^{-1} \right),
\]
\( \mathcal{K}_{\kappa,k}(0) = 0 = \mathcal{K}_{\kappa,k}(0) - 1 \)
and the functions \( \mathcal{F}_\tau \) and \( \mathcal{S}_\tau \) \((0 \leq \tau \leq 1)\) by
\[
1 + \frac{1}{b} \left[ (1 - \lambda) \frac{D_q \mathcal{F}_\tau(z)}{\mathcal{F}_\tau(z)} + \lambda \frac{D_q(z D_q \mathcal{F}_\tau(z))}{D_q \mathcal{F}_\tau(z)} - 1 \right] = \kappa \left( \frac{z + \tau}{1 + \tau z} \right),
\]
\( \mathcal{F}_\tau(0) = 0 = \mathcal{F}_\tau(0) - 1 \)
and
\[
1 + \frac{1}{b} \left[ (1 - \lambda) \frac{D_q \mathcal{S}_\tau(z)}{\mathcal{S}_\tau(z)} + \lambda \frac{D_q(z D_q \mathcal{S}_\tau(z))}{D_q \mathcal{S}_\tau(z)} - 1 \right] = \kappa \left( \frac{1 + \tau z}{z + \tau} \right),
\]
\( \mathcal{S}_\tau(0) = 0 = \mathcal{S}_\tau(0) - 1. \)
The functions \( \mathcal{K}_{\kappa,k}, \mathcal{F}_\lambda \) and \( \mathcal{S}_\lambda \in \mathcal{H}^\kappa_{q,b}(\kappa) \). If \( \mu < \alpha_1 \) or \( \mu > \alpha_2 \), then the equality holds if and only if \( f \) is \( \mathcal{K}_{\kappa,k} \), or one of its rotations. When \( \alpha_1 < \mu < \alpha_2 \), the equality holds if and only if \( f \) is \( \mathcal{K}_{\kappa,k} \), or one of its rotations. If \( \mu = \alpha_1 \), then the equality holds if and only if \( f \) is \( \mathcal{S}_\tau \), or one of its rotations. If \( \mu = \alpha_2 \), then the equality holds if and only if \( f \) is \( \mathcal{S}_\tau \), or one of its rotations. □
Remark 2.2 (i) Taking $q \to 1^-$ and $\lambda = \alpha$, in the above results, we obtain the results of [12, with $\lambda = 0$]; 
(ii) Theorem 2 for $b = 1$, corrects the result of Ramachandram et al. [8, Theorem 1, $\alpha = 0, \beta = 1$]; 
(iii) Putting $\lambda = 0$ in Theorem 2, we obtain the result of Seoudy and Aouf [10, Theorem 3]; 
(iv) Putting $\lambda = 1$ in Theorem 2, we obtain the result of Seoudy and Aouf [10, Theorem 3]; 
(v) Taking $b = e^{-i\theta}(1 - \alpha) \cos \theta$ in the above results, we obtain results for the class $H^{\lambda,\alpha}_{q,\beta}(\kappa)$.

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