GAUGE CHOICE FOR THE YANG-MILLS EQUATIONS USING THE
YANG-MILLS HEAT FLOW AND LOCAL WELL-POSEDNESS IN $H^1$.

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Abstract. In this work, we introduce a novel approach to the problem of gauge choice for the Yang-Mills equation on the Minkowski space $\mathbb{R}^{1+3}$, which uses the Yang-Mills heat flow in a crucial way. As this approach does not possess the drawbacks of the previous approaches (as in [13], [26]), it is expected to be more robust and easily adaptable to other settings.

As the first demonstration of the ‘structure’ offered by this new approach, we will give an alternative proof of the local well-posedness of the Yang-Mills equations for initial data $(A_i, E_i) \in (\dot{H}_x^1 \times L_x^2)$, which is a classical result of S. Klainerman and M. Machedon [13] that had been proved using a different method (local Coulomb gauges). The new proof does not involve localization in space-time, which had been the key drawback of the previous method. Based on the results proved in this paper, a new proof of finite energy global well-posedness of the Yang-Mills equations, also using the Yang-Mills heat flow, is established in the companion article [19].

1. Introduction

In this paper, we address the problem of gauge choice for the Yang-Mills equations on the Minkowski space $\mathbb{R}^{1+3}$, with a non-abelian structure group $\mathfrak{g}$, in the context of low regularity well-posedness of the associated Cauchy problem. The traditional gauge choices in such setting were the (local) Coulomb gauge $\partial^i A_i = 0$ [13] or the temporal gauge $A_0 = 0$ [26]. Each, however, had a shortcoming of its own (to be discussed below), because of which there had not been many results on low regularity solutions to the Yang-Mills equations with large data. In fact, the best result (in terms of the regularity condition on the initial data) along this direction so far has been the local well-posedness of the Yang-Mills equations for data $(A_i, E_i) \in \dot{H}_x^{1/2} \times \dot{H}_x^{-1/2}$ by S. Klainerman and M. Machedon [13], whereas the scaling property of the Yang-Mills equations dictates that the optimal regularity condition should be $(A_i, E_i) \in \dot{H}_x^{1/2} \times \dot{H}_x^{-1/2}$ (for the notations, we refer the reader to [13]).

1We however remark that better results are available in the case of small initial data. See [26] and also the discussion below.

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In this work, we propose a novel approach to this problem using the celebrated Yang-Mills heat flow, which does not possess the drawbacks of the previous gauge choices. As such, this approach is expected to be more robust and have many applications, including that of establishing large data low regularity well-posedness of the Yang-Mills equations. As the first demonstration of the potential of this approach, we give a new proof of the aforementioned local well-posedness result of Klainerman-Machedon [13]. In the companion paper [19], we demonstrate that the main result of [13], namely the finite energy global well-posedness of the Yang-Mills equations on \( \mathbb{R}^{1+3} \), can be proved using the new approach as well.

### 1.1. The Yang-Mills equation on \( \mathbb{R}^{1+3} \)

We will work on the Minkowski space \( \mathbb{R}^{1+3} \). All tensorial indices will be raised and lowered by using the Minkowski metric, which we assume to be of signature \((-+++)\). Moreover, we will adopt the Einstein summation convention of summing up repeated upper and lower indices. Greek indices, such as \( \mu, \nu, \lambda \), will run over \( x^0, x^1, x^2, x^3 \), whereas latin indices, such as \( i, j, k, \ell \), will run only over the spatial indices \( x^1, x^2, x^3 \). We will often write \( t \) instead of \( x^0 \). A \( k \)-fold application of a derivative will be denoted \( \partial^{(k)} \), in contrast to \( \partial^k \) which will always mean the partial derivative in the direction \( x^k \), with its index raised.

Let \( \mathfrak{g} \) be a Lie group and \( \mathfrak{g} \) be its Lie algebra. We will assume that \( \mathfrak{g} \) admits a bi-invariant inner product \( (\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to [0, \infty) \), i.e. an inner product invariant under the adjoint map. For simplicity, we will furthermore take \( \mathfrak{g} \) to be a matrix group. A model example that one should keep in mind is the Lie group \( \mathfrak{g} = SU(n) \) of complex unitary matrices, in which case \( \mathfrak{g} = su(n) \) is the set of complex traceless anti-hermitian matrices and \((A, B) = \text{tr}(AB^*)\).

Below, we will present some geometric concepts we will need in a pragmatic, condensed fashion; for a more thorough treatment, we refer the reader to the standard references [3], [17], [18].

Let us consider a \( \mathfrak{g} \)-valued 1-form \( A_\mu \) on \( \mathbb{R}^{1+3} \), which we call a connection 1-form. For any \( \mathfrak{g} \)-valued tensor \( B \) on \( \mathbb{R}^{1+3} \), we define the associated covariant derivative \( D = (A)D \) by

\[
D_\mu B := \partial_\mu B + [A_\mu, B]
\]

where \( \partial_\mu \) refers to the ordinary directional derivative on \( \mathbb{R}^{1+3} \). Due to the bi-invariance of \((\cdot, \cdot)\), we have the following Leibniz’s rule for \( \mathfrak{g} \)-valued tensors \( B, C \):

\[
\partial_\mu (B, C) = (D_\mu B, C) + (B, D_\mu C)
\]

The commutator of two covariant derivatives gives rise to a \( \mathfrak{g} \)-valued 2-form \( F_{\mu\nu} = F[A]_{\mu\nu} \), which we call the curvature 2-form associated to \( A \), as follows:

\[
D_\mu D_\nu B - D_\nu D_\mu B = [F_{\mu\nu}, B].
\]

It is easy to compute that

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].
\]

From the way \( F_{\mu\nu} \) arises from \( A_\mu \), the following identity, called the Bianchi identity, always holds.

\[\text{(Bianchi)}\]

\[
D_\mu F_{\nu\lambda} + D_\lambda F_{\mu\nu} + D_\nu F_{\lambda\mu} = 0.
\]

The Yang-Mills equations on \( \mathbb{R}^{1+3} \) are the following additional first order equations for \( F_{\mu\nu} \).

\[\text{(YM)}\]

\[
D^\mu F_{\mu\nu} = 0,
\]

where we utilize the Einstein convention of summing repeated lower and upper indices.

An important feature of the Yang-Mills equations is its gauge structure. Given a smooth \( \mathfrak{g} \)-valued function \( U \) on \( \mathbb{R}^{1+3} \), we let \( U \) act on \( A, D, F \) as a gauge transform according to the following rules.

\[
\tilde{A}_\mu = UA_\mu U^{-1} - \partial_\mu U U^{-1}, \quad \tilde{D}_\mu = UD_\mu U^{-1}, \quad \tilde{F}_{\mu\nu} = UF_{\mu\nu} U^{-1}.
\]

We say that a \( \mathfrak{g} \)-valued tensor \( B \) is gauge covariant, or covariant under gauge transforms, if it transforms in the fashion \( \tilde{B} = UBU^{-1} \). Given a gauge covariant \( \mathfrak{g} \)-valued tensor \( B \), its covariant derivative \( D_\mu B \) is also gauge covariant, as the following formula shows.

\[
D_\mu \tilde{B} = U D_\mu B U^{-1}.
\]

\(^2\) A sufficient condition for a bi-invariant inner product to exist is that \( \mathfrak{g} \) is a product of an abelian and a semi-simple Lie groups.
The Yang-Mills equations (YM) are evidently covariant under gauge transforms. This means that a solution to (YM) makes sense only as a class of gauge connections, related to each other by a gauge transform. Accordingly, we will call such a class of smooth connection 1-forms $A$ which solves (YM) a classical solution to (YM).

A choice of a particular representative is called a gauge choice. This is usually done by imposing a condition, called a gauge condition, on the representative. Classical examples of gauge conditions include the temporal gauge condition $A_0 = 0$ and the Coulomb gauge condition $\partial^\nu A_\nu = 0$.

In this work, and also in the companion article [19], we will study the Cauchy problem associated to (YM). The initial data set, which consists of $(A_i, E_i)$ for $i = 1, 2, 3$ with $A_i = A_i(t = 0)$ (magnetic potential) and $E_i = F_0(t = 0)$ (electric field), has to satisfy the constraint equation:

\[\partial^\nu E_\nu[\ell] + [A^\ell, E_\ell] = 0,\]

where $\ell$ is summed only over $\ell = 1, 2, 3$. We remark that this is the $\nu = 0$ component of (YM).

The Yang-Mills equations possess a positive definite conserved quantity $E[t]$, defined by:

\[E(t) := \frac{1}{2} \int_{\mathbb{R}^3} \sum_{\ell = 1,2,3} (F_{0\nu}(t,x), F_{0\nu}(t,x)) + \sum_{k,\ell = 1,2,3, k < \ell} (F_{k\ell}(t,x), F_{k\ell}(t,x)) \, dx\]

We call $E(t)$ the conserved energy of the field at time $t$.

Note that the Yang-Mills equations remain invariant under the scaling

\[x^\alpha \to \lambda x^\alpha, \quad A \to \lambda^{-1} A, \quad F \to \lambda^{-2} F.\]

The norms $\|\partial_\nu A_{i\nu}\|_{L^2}^2$, $\|E_i\|_{L^2}^2$, as well as the conserved energy $E(t)$, of the rescaled field become $\lambda^{-1}$ of that of the original field, which allows us to assume smallness of these quantities by scaling. This reflects the sub-criticality of these quantities compared to the Yang-Mills equations.

1.2. Statement of the Main Theorem. To state the Main Theorem of this paper, we must first to extend the notion of a solution to (YM) by taking the closure of the set of classical solutions in an appropriate topology.

**Definition 1.1** (Admissible solutions). Let $I \subset \mathbb{R}$. We say that a generalized solution $A_\mu$ to the Yang-Mills equations (YM) defined on $I \times \mathbb{R}^3$ is admissible in the temporal gauge $A_0 = 0$ if

\[A_\mu \in C_t(I, \mathcal{D}(1) \cap L^2), \quad \partial_\nu A_\mu \in C_t(I, L^2)\]

and $A_\mu$ can be approximated by representatives of classical solutions in the temporal gauge $A_0 = 0$ in the above topology.

Let us define the corresponding class of initial data sets.

**Definition 1.2** (Admissible $H^1$ initial data set). We say that a pair $(\overline{A}_i, \overline{E}_i)$ of 1-forms on $\mathbb{R}^3$ is an admissible $H^1$ initial data set for the Yang-Mills equations if the following conditions hold:

1. $\overline{A}_i \in H^1 \cap L^2$ and $\overline{E}_i \in L^2$;
2. The constraint equation $\partial^\nu E_\nu[\ell] + [\overline{A}^\ell, \overline{E}_\ell] = 0$ holds in the distributional sense.

Our main theorem is a local well-posedness result for such initial data, within the class of admissible solutions in the temporal gauge.

**Main Theorem ($H^1$ local well-posedness of the Yang-Mills equations, temporal gauge).** Let $(\overline{A}_i, \overline{E}_i)$ be an admissible $H^1$ initial data set, and define $\overline{T} := \|\overline{A}\|_{H^1} + \|\overline{E}\|_{L^2}$. Consider the initial value problem (IVP) for (YM) with $(\overline{A}_i, \overline{E}_i)$ as the initial data.

1. There exists $T^* = T^*(\overline{T}) > 0$, which is non-increasing in $\overline{T}$, such that a unique admissible solution $A_\mu = A_\mu(t,x)$ to the IVP in the temporal gauge $A_0 = 0$ exists on $(-T^*, T^*)$.

Furthermore, the following estimates hold.

\[\sup_i \|\partial_{t,x} A_i\|_{C_t((-T^*, T^*), L^2)} \leq C \sup_i \|\overline{A}_i\|_{H^1} + \sup_i \|\overline{E}_i\|_{L^2}.\]
(1.4) \[ \sup_i \| A_i \|_{C^i((−T∗, T∗), L^2)} \leq C_\Gamma \sup_i \| \mathcal{A}_i \|_{L^2}. \]

(2) Let \((\mathcal{A}_i, \mathcal{E}_i)\) be another admissible \(H^1\) initial data set such that \(\| \mathcal{A}_i \|_{H^1} + \| \mathcal{E}_i \|_{L^2} \leq \mathcal{T}\), and let \(A'_i\) be the corresponding solution given by (1). Then the following estimates for the difference hold.

\[
(1.5) \quad \sup_i \| \partial_t x A_i - \partial_t x A'_i \|_{C^i((−T∗, T∗), L^2)} \leq C_\Gamma (\sup_i \| \mathcal{A}_i - \mathcal{A}'_i \|_{H^1} + \sup_i \| \mathcal{E}_i - \mathcal{E}'_i \|_{L^2}).
\]

\[
(1.6) \quad \sup_i \| A_i - A'_i \|_{C^i((−T∗, T∗), L^2)} \leq C_\Gamma (\sup_i \| \mathcal{A}_i - \mathcal{A}'_i \|_{H^1 \cap L^2} + \sup_i \| \mathcal{E}_i - \mathcal{E}'_i \|_{L^2}).
\]

(3) Finally, the following version of persistence of regularity holds: if \(\partial_t x A_i, \mathcal{E}_i \in H^m_x\) for an integer \(m \geq 0\), then the corresponding solution given by (1) satisfies

\[ \partial_t x A_i \in C^1((-T∗, T∗), H^k) \]

for every pair \((k_1, k_2)\) of nonnegative integers such that \(k_1 + k_2 \leq m\).

We emphasize that the temporal gauge in the statement of the Main Theorem plays a rather minor role. We have chosen to use it mainly because it is a well-known gauge condition that is easy to impose. Another advantage of the temporal gauge is that it is a classical local well-posedness result for smoother data (essentially due to Segal [21] and Eardley-Moncrief [9]) is available, which is useful in the proof of the Main Theorem (Theorem 4.4). However, most of our analysis in this paper takes place under different gauge conditions, defined with the help of the Yang-Mills heat flow, to be introduced below.

The Main Theorem is, in fact, a classical result of S. Klainerman and M. Machedon [13], which has been the best result so far concerning low regularity local well-posedness of \((YM)\) for large data. In both this paper and [13], it is essential to choose an appropriate gauge to reveal the null structure of the quadratic nonlinearities of the wave equations. It is known that such structure is present in the Coulomb gauge \([12]\). Unfortunately, in the case of a non-abelian structural group \(G\), it may not be possible in general to impose the Coulomb gauge condition on an arbitrary initial data. In [13], this issue is avoided by working in a so-called local Coulomb gauge, which is the Coulomb gauge condition imposed on a small domain of dependence. Due to the presence of the constraint equation \([11]\), delicate boundary conditions had to be imposed along the lateral boundary (a cone in the case of the Minkowski space \(\mathbb{R}^{1+3}\)). Because of this, it had been difficult to use this method along with global Fourier analytic techniques such as \(H^{s,b}\) spaces, and thus it has not been extended to initial data with lower regularity than \(H^1_{loc} \times L^2_{loc}\).

Let us also mention an alternative approach to local well-posedness by Tao [26], who worked entirely in the temporal gauge \(A_0 = 0\) to prove local well-posedness for initial data with even lower regularity than \(H^1\). The main idea is that \((YM)\) in the temporal gauge can be cast as a coupled system of wave and transport equations (using the Hodge decomposition of \(A_i\)), where the wave equations possess a null structure similar to that in the Coulomb gauge. Although the temporal gauge has the advantage of being easy to impose (globally in space), the statements proved by this method are unfortunately restricted to small data due to, among other things, the presence of too many time derivatives in the transport equation for \(\partial_t A_i\).

In this paper, we introduce a new approach to the problem of gauge choice which does not have the drawbacks of the methods outlined above. In particular, it does not involve localization in space-time and works well for large initial data. Moreover, the most dangerous quadratic nonlinearities of the wave equations are seen to possess a null structure, which allows us to give a new proof of the Main Theorem. For these reasons, we expect the present approach to be more robust and applicable to other problems as well, such as large data low regularity well-posedness of the Yang-Mills equations and other non-abelian gauge theories.

In the companion paper [19], we prove global well-posedness of \((YM)\) (in the temporal gauge) for this class of initial data, using the positive definite conserved energy of \((YM)\). The proof involves
In simple, heuristic terms, the main idea of the novel approach is to ‘smooth out’ the problem in a ‘geometric fashion’. For the problem under consideration, the ‘smoothed out’ problem is much easier; indeed, recall the classical works of [21] and [9], in which local well-posedness for (possibly large) initial data with higher degree of smoothness was established by working directly in the temporal gauge. The difficulty of the original problem manifests in our approach as the problem of estimating the difference between the solutions to the original and ‘smooth out’ problems. Here, we need to exploit the ‘special structure’ inherent to the Yang-Mills equations. That this is possible by using a smoothing procedure based on the associated geometric flow, the Yang-Mills heat flow in this case, is the main thesis of this work.

The present work advances a relatively new idea in the field of hyperbolic PDEs: To use a geometric parabolic equation to better understand a hyperbolic equation. To the author’s knowledge, this was first used in the work of Klainerman-Rodnianski [15], in which the linear heat equation on a compact 2-manifold was used to develop an invariant form of Littlewood-Paley theory for arbitrary tensors on the manifold. This was applied in [14] and [16] to study the causal geometry of solutions to the Einstein’s equations under very weak hypotheses.

More recently, this idea was carried much further by Tao, who proposed using a nonlinear geometric heat flow to deal with the problem of gauge choice in the context of energy critical wave maps. This approach, called the caloric gauge, was put into use in [28] to study the long term behavior of large energy wave maps on \( \mathbb{R}^{1+2} \). It has also played an important role in the recent study of the related energy critical Schrödinger map problem, in the works [2], [22] and [23].

The basic idea of the caloric gauge is as follows. The associated heat flow (the harmonic map flow) starting from a wave or a Schrödinger map on a fixed time slice converges (under appropriate conditions) to a single point (same for every time slice) as the heat parameter goes to \( \infty \). For this trivial map at infinity, the canonical choice of gauge is clear; parallel-transporting this gauge choice back along the harmonic map flow, we obtain a (in some sense, canonical) gauge choice for the original map, which has been named the caloric gauge by Tao.

We remark that in comparison to the works involving the caloric gauge, the analytic side of this work is simpler. One reason is that we are working with a sub-critical problem, and hence the function spaces used for treating the hyperbolic equations involved are by far less intricate. Another is that, as indicated earlier, our method depends only on the short time smoothing property of the associated heat flow, and as such, does not require understanding the long time behavior of the heat flow as in the other works.

1.3. The Yang-Mills heat flow. Before we give an overview of our proof of the Main Theorem, let us introduce the Yang-Mills heat flow, which is a crucial ingredient of the new approach. Consider an one parameter family of spatial 1-forms \( A_i(s) \) on \( \mathbb{R}^3 \), parameterized by \( s \in [0, s_0] \). We say that \( A_i(s) \) is a Yang-Mills heat flow if it satisfies

\[
\partial_s A_i = D^j F_{ij}, \quad i = 1, 2, 3.
\]

The Yang-Mills heat flow is the gradient flow for the Yang-Mills energy (or the magnetic energy) on \( \mathbb{R}^3 \), which is defined as

\[
B[A_i] := \frac{1}{2} \sum_{1 \leq i < j \leq 3} \int (F_{ij}, F_{ij}) \, dx.
\]

First introduced by Donaldson [8], the Yang-Mills heat flow has been a subject of active research on its own. For more on this heat flow on a 3-dimensional manifold, we refer the reader to [20], [4] and etc.

As indicated earlier, our intention is to use (YMHF) to geometrically smooth out (YM); we must take care, however, since (YMHF) turns out to be not strictly parabolic for \( A_i \), as the highest order terms of the right-hand side of (YMHF) has non-trivial kernel. This phenomenon ultimately

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3Except for Kato’s inequality used in the proof of Lemma 4.5, whose proof is standard and can be found in other sources, such as [22] Proof of Corollary 3.3, as well.
originates from the covariance of the term $D^i F_{gi}$, and can be compensated if gauge transforms which depends on $s$ are used. The system $\text{(YM)}$, as it stands, is not covariant under such gauge transforms (being covariant only under $s$-independent gauge transforms). Therefore, for the purpose of recovering strict parabolicity of the Yang-Mills heat flow, it is useful to reformulate the flow in a fully covariant form.

Along with $A_s$, let us add a new connection component $A_s$, and consider $A_s$ ($a = x^1, x^2, x^3, s$), which is a connection 1-form on the product manifold $\mathbb{R}^3 \times [0, s_0]$. Corresponding to $A_s$, we also define the covariant derivative along the $\partial_s$ direction

$$D_s := \partial_s + [A_s, \cdot].$$

A connection 1-form on $\mathbb{R}^3 \times [0, s_0]$ is said to be a covariant Yang-Mills heat flow if it satisfies

$$(c\text{YMHF})$$

$$F_{si} = D^i F_{ti}, \quad i = 1, 2, 3,$$

where $F_{si}$ is the commutator between $D_s$ and $D_i$, given by the formula

$$F_{si} = \partial_s A_i - \partial_i A_s + [A_s, A_i].$$

As the system $\text{(cYMHF)}$ is underdetermined for $A_s$, we need an additional gauge condition (typically for $A_s$) in order to solve for $A_s$. Choosing $A_s = 0$, we recover $\text{(YM)}$. On the other hand, if we choose $A_s = \partial^s A_t$, then $\text{(cYMHF)}$ becomes strictly parabolic. This may be viewed as a geometric formulation of the ‘compensation-by-gauge-transform’ procedure hinted earlier.

In what follows, the first gauge condition $A_s = 0$ will be called the caloric gauge, following the usage of the term in [27]. On the other hand, the second gauge condition $A_s = \partial^s A_t$ will be dubbed the DeTurck gauge, as the idea of compensating for a non-trivial kernel by a suitable one parameter family of gauge transforms usually, which lies at the heart of the procedure outlined above, goes under the name of DeTurck’s trick.

1.4. Overview of the arguments. Perhaps due to the fact that we deal simultaneously with two nonlinear PDEs, namely $\text{YM}$ and $\text{YMHF}$, each of which with a considerable body of research on its own, the argument of this paper is rather lengthy. To help the reader grasp the main ideas, we would like to present a thorough overview of the paper, with the ambition to indicate each of the major difficulties, as well as their resolutions, without getting into too much technical details. For a shorter, more leisurely overview, we refer the reader to the introduction of [19].

In this overview, instead of the full local well-posedness statement, we will focus on the simpler problem of deriving a local-in-time a priori bound of a solution to $\text{YM}$ in the temporal gauge. In other words, under the assumption that a (suitably smooth and decaying) solution $A^0_{\mu}$ to $\text{YM}$ in the temporal gauge exists on $I \times \mathbb{R}^3$, where $I := (-T_0, T_0) \subset \mathbb{R}$, we aim to prove

$$\|\partial_{t,x} A^\mu_{\alpha}\|_{L^2(I; L^2)} \leq C_0 \overline{\mathcal{T}},$$

where $\overline{\mathcal{T}} := \sum_{i=1,2,3} \|(|A_{\xi}, F_{\xi})\|_{L^2_I \times L^2_x}$ measures the size of the initial data, for $T_0$ sufficiently small compared to $\overline{\mathcal{T}}$.

1. Scaling and set-up of the bootstrap. Observe that, thanks to the scaling and the subcriticality of $\overline{\mathcal{T}}$, it suffices to prove (1.8) for $T_0 = 1$, assuming $\overline{\mathcal{T}}$ is small. We will use a bootstrap argument to establish (1.8). More precisely, under the bootstrap assumption that

$$\|\partial_{t,x} A^\mu_{\alpha}\|_{C^1([-T, T]; L^2)} \leq 2C_0 \overline{\mathcal{T}}$$

holds for $0 < T \leq 1$, we will retrieve (1.8) for $I = (-T, T)$ provided that $\overline{\mathcal{T}}$ is sufficiently small (independent of $T$). Then, by a standard continuity argument, (1.8) will follow for $I = (-1, 1)$.

2. Geometric smoothing of $A^\mu_{\alpha}$ by the (dynamic) Yang-Mills heat flow. As discussed earlier, the main idea of our approach is to smooth out $A^\mu_{\alpha}$ by (essentially) using the covariant Yang-Mills

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4Indeed, up to the top order terms, it is easy to verify that the system looks like $\partial_t A_i = \Delta A_i + (\text{lower order terms}).$

5It has been first introduced by DeTurck in the context of the Ricci flow in [7], and applied in the Yang-Mills heat flow context by Donaldson [3].
heat flow. Let us append a new variable $s$ and extend $A^i_\mu = A^i_\mu(t, x, s)$ to a connection 1-form $A_a = A_a(t, x, s)$ by solving
\[
(d\text{YMHF}) \quad F_{\mu\nu} = D^\ell F_{\ell\mu}, \quad \mu = 0, 1, 2, 3.
\]
with $A_\mu(s = 0) = A^i_\mu$. Note that this system is nothing but $c\text{YMHF}$ with the extra equation $F_{\alpha\beta} = D^\ell F_{\ell\alpha}$ added; we will refer to this as the dynamic Yang-Mills heat flow.

As we would like to utilize the smoothing property of $d\text{YMHF}$, we will impose the DeTurck gauge condition $A_\mu = \partial^\ell A_\ell$. Then $c\text{YMHF}$ will be a strictly parabolic system, and thus can be solved (via Picard iteration) on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$ provided that $\sup_{t \in (-T, T)} ||\partial_s A_\ell(t, s = 0)||_{L^2}^2$ is small enough; see Sections [5] and [6]. The latter condition can be ensured by taking $T$ sufficiently small, thanks to the bootstrap assumption \([1.9]\).

3. The hyperbolic-parabolic-Yang-Mills system and the caloric-temporal gauge. As a result, we have obtained a connection 1-form $A_a = A_a(t, x, s)$ on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$, which satisfies the following system of equations:
\[
(\text{HPYM}) \quad \begin{cases}
F_{\mu\nu} = D^\ell F_{\ell\mu} & \text{on } I \times \mathbb{R}^3 \times [0, 1], \\
D^\mu F_{\mu\nu} = 0 & \text{along } I \times \mathbb{R}^3 \times \{0\},
\end{cases}
\]
as well as the DeTurck gauge condition $A_\mu = \partial^\ell A_\ell$. The system (without the gauge condition) just introduced will be called the hyperbolic-parabolic-Yang-Mills or, in short, $\text{HPYM}$. It is covariant under gauge transformations of the form $U = U(t, x, s)$, which act on various variables in the following fashion:
\[
\tilde{A}_a = U A_a U^{-1} - \partial_a U U^{-1}, \quad \tilde{D}_a = U D_a U^{-1}, \quad \tilde{F}_{ab} = U F_{ab} U^{-1}.
\]
where $a, b = x^0, x^1, x^2, x^3, s$.

We will work with $\text{HPYM}$ in place of $\text{YM}$. Accordingly, instead of $A^i_\mu$, we will work with new variables $\underline{A}_\mu := A_\mu(s = 1)$ and $\partial_s A_\mu(s) (0 < s < 1)$. The former should be viewed as a smoothed-out version of $A^i_\mu$, whereas the latter measures the difference between $\underline{A}_\mu$ and $A^i_\mu$.

In analyzing $d\text{YMHF}$, we indicated that the DeTurck gauge $A_\mu = \partial^\ell A_\ell$ is employed. This choice was advantageous in the sense that the equations for $A_\mu$ were parabolic in this gauge. However, completely different considerations are needed for estimating the evolution in $t$. Here, the gauge condition we will use is
\[
\begin{cases}
A_\mu = 0 & \text{on } I \times \mathbb{R}^3 \times \{(0, 1)\}, \\
\underline{A}_\mu = 0 & \text{on } I \times \mathbb{R}^3 \times \{1\},
\end{cases}
\]
which we dub the caloric-temporal gauge.

Let us briefly motivate our choice of gauge. For $\partial_s A_\mu$ on $(-T, T) \times \mathbb{R}^3 \times (0, 1)$, let us begin by considering the following identity, which is nothing but a rearrangement of the formula \([1.7]\).
\[
\partial_s A_i = F_{si} + D_i A_s.
\]
A simple computation (see Appendix [A]) shows that $F_{si}$ is covariant-divergence-free, i.e. $D^\ell F^\ell_{st} = 0$. In view of this fact, the identity \([1.10]\) may be viewed (heuristically) as a covariant Hodge decomposition of $\partial_s A_i$, where $F_{si}$ is the covariant-divergence-free part and $D_i A_s$, being a pure covariant-gradient term, may be regarded as the ‘covariant-curl-free part’ \([1]\). Recall that the Coulomb gauge condition, which had a plenty of good properties as discussed earlier, is equivalent to having vanishing curl-free part. Proceeding in analogy, we are motivated to set $A_i = 0$ on $(-T, T) \times \mathbb{R}^3 \times (0, 1)$, which is exactly the caloric gauge condition we introduced earlier.

The second gauge condition, $\underline{A}_\mu = 0$, is motivated from the fact that $\underline{A}_\mu$ is expected to be smooth. More precisely, hinted by the works \([21, 9]\), we expect that the increased degree of smoothness of

\[\text{More precisely, } c\text{YMHF} \text{ becomes strictly parabolic, and can be solved by Picard iteration. On the other hand, we can solve for the extra variable } A_0 \text{ using } F_{\alpha\beta} = D^\ell F_{\ell\alpha} \text{ a posteriori, by a process which involves solving only linear equations. For more details, we refer the reader to Section } 6 \text{ Proof of Theorem } 2 \text{ Step 1.}\]

\[\text{Although its covariant curl does not strictly vanish.}\]
$A_0(t = 0)$ will render a delicate choice of gauge (such as the Coulomb gauge) unnecessary, and that an easy choice (such as the temporal gauge $A_0 = 0$) will suffice.

4. **Gauge transform into the caloric-temporal gauge and the initial data estimates.** With these heuristic motivations in mind, let us come back to the problem of establishing the a priori estimate \[ (1.13) \]

In order to proceed, we must perform a gauge transformation on $A_0$, which currently is in the DeTurck gauge, into the caloric-temporal gauge. An inspection of the formula for gauge transformation shows that the desired gauge transform $U$ can be found by solving the following hierarchy of ODEs:

\[
\begin{align}
\partial_t U &= U A_0 \
\partial_s U &= U A_s
\end{align}
\]

where $U(s = 1) = U$. We will choose the initial value for $U(s = 0)$ to be $U(t = 0, s = 1) = I d$. Then, combined with smoothing estimates for (YMHT) in the DeTurck gauge, we arrive at a gauge transform solution (which we still call $A_n$) to (HPYM) in the caloric-temporal gauge, which satisfies the following *initial data estimate.*

\[
(1.12) \quad \begin{cases}
\sup_{0 < s < s_0} s^{-(m+1)/2} \| \partial_x^{(m-1)} \partial_{t,x} F_{a\mu}(t = 0, s) \|_{L^2_x} \leq C_m \sum_{j=1,2,3} \| (\tilde{A}_j, \tilde{F}_j) \|_{\dot{H}^1_x \times L^2_x} \\
\int_{0}^{s_0} s^{-(m+1)} \| \partial_x^{(m-1)} \partial_{t,x} F_{a\mu}(t = 0, s) \|_{L^2_x}^2 \frac{ds}{s} \right)^{1/2} \leq C_m \sum_{j=1,2,3} \| (\tilde{A}_j, \tilde{F}_j) \|_{\dot{H}^1_x \times L^2_x}
\end{cases}
\]

up to some integers $m_0, k_0 > 1$, i.e. $1 \leq m \leq m_0$, $1 \leq k \leq k_0$. Moreover, we obtain a few estimates for the gauge transform $V := U(t = 0, s = 0)$ as well. The weights of $s$ are dictated by scaling (see (1.6) for a more detailed explanation).

The result described in this step is essentially the content of Theorem A, which is stated in Section 4 and proved in Section 6.

5. **Equations of motion of (HPYM).** The next step is to propagate the bounds (1.12) to all $t \in (-T, T)$, by analyzing a system of coupled hyperbolic and parabolic equations derived from (HPYM). To present this system, let us begin by introducing the notion of the Yang-Mills tension field. For a solution $A_n$ to (HPYM) on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$, we define its Yang-Mills tension field $w_\mu(s)$ at $s \in [0, 1]$ by

\[
w_\mu(s) := D^\mu F_{\nu\mu}(s).
\]

The Yang-Mills tension field $w_\mu(s)$ measures the extent to which $A_n(s)$ fails to satisfy the Yang-Mills equations (YM). With $w_\mu$ in hand, we may now state the equations of motion of (HPYM), which are central to the analysis of the $t$-evolution of $A_n$.

\[
\begin{align}
D^\mu D_\mu F_{a\nu} &= 2[F_{a}^{\mu}, F_{\nu\mu}] - 2[F_{\nu}^{\mu}, D_\mu F_{e\nu} + D_\ell F_{\nu\mu}] - D^\ell D_\ell w_\nu + D_\nu D_\ell w_\ell - 2[F_{\nu}^{\ell}, w_\ell], \\
D^\nu F_{\nu\mu} &= w_\mu, \\
D_\nu w_\nu &= D^\ell D_\ell w_\nu + 2[F_{\nu}^{\ell}, w_\ell] + 2[F_{\nu}^{\mu}, D_\mu F_{e\nu} + D_\ell F_{\nu\mu}], \\
D_\ell F_{s\mu} &= D^\ell D_\ell F_{s\mu} - 2[F_{s}^{\ell}, F_{\mu\ell}].
\end{align}
\]

\[8\text{At first sight, one may think that a more natural choice of the initial value } U(t = 0, s = 0) = I d, \text{ as it keeps the initial data set } \tilde{A}_j, \tilde{F}_j \text{ unchanged. However, it turns out that for every } (t, s) \in (-T, T) \times [0, 1], \text{ the gauge transform arising this way is not bounded on } H^m_n \text{ for } m > 1. \text{ As such, it cannot retain the smoothing estimates for (YMHT) in the DeTurck gauge, and thus inappropriate for our purposes. The choice } U(t = 0, s = 1) = I d \text{ on the other hand, avoids this issue, at the cost of introducing a non-trivial gauge transform which we call } V \text{ at } t = 0, s = 0. \text{ See Lemma 5.3 for the relevant estimates.}
\]

\[9\text{We remind the reader that } F_{\nu\mu} = \partial_\nu A_\mu, \text{ thanks to the caloric-temporal gauge condition.}
\]

\[10\text{The system is, roughly speaking, parabolic in the } s\text{-direction and hyperbolic in the } t\text{-direction. Moreover, all the equations we present are covariant.}\]
The underlines of (1.14) signify that each variable is restricted to \( \{s = 1\} \), and the indices \( a, b \) run over \( x^0, x^1, x^2, x^3, s \). Furthermore, \( w_\nu \equiv 0 \) at \( s = 0 \), for all \( \nu = 0, 1, 2, 3 \). The derivation of these equations is deferred to Appendix A.

The equations (1.13) and (1.14) are the main hyperbolic equations of the system, used to estimate \( F_{ri} \) and \( A_\nu \), respectively. Both equations possess terms involving \( w_\mu \) on the right-hand side. The Yang-Mills tension field \( w_\mu \), in turn, is estimated by studying the parabolic equation (1.15). An important point regarding (1.15) is that its data at \( s = 0 \) is zero, thanks to the fact that \( A_\mu(s = 0) \) satisfies (YM).

Next, the equation (1.16) says that each curvature component satisfies a covariant parabolic equation. In view of proving the Main Theorem, of particular interest are the equations

\[
\begin{align*}
(1.17) & \quad D_t F_{si} - D^\ell D_\ell F_{si} = -2[F_{s \ell}^\ell, F_{i\ell}], \\
(1.18) & \quad D_t F_{s0} - D^\ell D_\ell F_{s0} = -2[F_{s \ell}^\ell, F_{0\ell}].
\end{align*}
\]

Thanks to the smoothing property of (1.17), we may (at least heuristically) always exchange derivatives of \( F_{si} \) for an appropriate power of \( s \). The second equation (1.18) will be used to derive estimates for \( F_{s0} \), which, combined with the caloric-temporal gauge condition, leads to the corresponding estimates for \( A_0 \). As \( F_{s0} = -w_0 \), note that the data for (1.18) at \( s = 0 \) is zero as well. This has the implication that \( A_0 \) is, in general, obeys more favorable estimates than \( A_1 \).

6. Analysis of the time evolution. We are now ready to present the key ideas for analyzing the hyperbolic equations of (HYM), namely (1.13) and (1.14); this will be the content of Theorem B, stated in Section 4 and proved in Sections 7–10.

In order to treat (1.13), we need to uncover the aforementioned null structure of the most dangerous quadratic nonlinearity. It turns out that, for the problem under consideration, all quadratic nonlinearities can be treated just by Strichartz and Sobolev inequalities, except for a single term

\[2[A^\ell - \bar{A}_\ell, \partial_t F_{si}].\]

In [12], it was demonstrated that such a term can be written as a linear combination of null forms, provided that \( A_1 - \bar{A}_1 \) satisfied the Coulomb condition \( \partial^\ell (A_1 - \bar{A}_1) = 0 \). Of course, this assumption is not true in our case; nevertheless, combining \( \partial_s A_i = F_{si} \) (from the caloric condition \( A_1 = 0 \)) and the identity \( D^\ell F_{s\ell} = 0 \), we see that the covariant Coulomb condition \( D^\ell (\partial_s A_\ell) = 0 \) is satisfied for each \( \partial_s A_\ell(s), s \in (0, 1) \). This turns out to be sufficient for carrying out an argument similar to [12]. We refer the reader to [10, 12] for more details.

On the other hand, the key point regarding (1.14), which is nothing but the Yang-Mills equations in the temporal gauge with the source \( w_\mu \), is that its data at \( t = 0 \) is smooth. Therefore, we will basically emulate the classical analysis of (YM) in the temporal gauge for initial data with higher degree of smoothness. See [10, 11] for more details.

Provided that \( \mathcal{Z} \) is sufficiently small, the analysis sketched above leads to estimates for \( F_{si}(s) \) and \( \bar{A}_\nu \), such as

\[
\begin{align*}
(1.19) & \quad \sup_{0 < s < t_0} \frac{1}{s^{(m+1)/2}} \| \partial_x^{(m-1)} \partial_t x F_{si}(s) \|_{C_t(1, L^2_x)} \leq C_m \sum_{j = 1, 2, 3} \|(\overline{A}_j, \mathcal{E}_j)\|_{\dot{H}^1 \times L^2}, \\
& \quad \left( \int_0^{t_0} s^{-(m+1)} \| \partial_x^{(m-1)} \partial_t x F_{si}(s) \|_{C_t(1, L^2_x)}^2 \frac{ds}{s} \right)^{1/2} \leq C_m \sum_{j = 1, 2, 3} \|(\overline{A}_j, \mathcal{E}_j)\|_{\dot{H}^1 \times L^2}, \\
& \quad \| \partial_x^{(k-1)} \partial_t x \bar{A}_\nu \|_{C_t(1, L^2_x)} \leq C_k \sum_{j = 1, 2, 3} \|(\overline{A}_j, \mathcal{E}_j)\|_{\dot{H}^1 \times L^2}
\end{align*}
\]

for \( 1 \leq m \leq m_0, 1 \leq k \leq k_0 \).

7. Returning to \( A_1 \). The last step is to translate estimates for \( \partial_s A_i \) and \( \bar{A}_\nu \), such as (1.19), to those for \( A_1 \), so that (1.38) is retrieved. One immediate issue is that the naive approach of integrating the estimates (1.18) in \( s \) fails to bound \( \| \partial_t x A_\mu(s = 0) \|_{C_t(1, L^2_x)} \) by a logarithm. In order to remedy
this issue, let us take the (weakly-parabolic) equation
\[ \partial_s A_i = \Delta A_i - \partial^\ell \partial_i A_\ell + \text{(lower order terms)}. \]

Differentiate by \( \partial_t, x \), multiply by \( \partial_t A_i \) and then integrate the highest order terms by parts over \( \mathbb{R}^3 \times [0, 1] \). This trick, combined with the \( L^2_{\text{A}_s/s} \)-type estimates of (1.19), overcome the logarithmic divergence.\(^1\)

Another issue is that the estimates derived so far, being in the caloric-temporal gauge, are not in the temporal gauge along \( s = 0 \). Therefore, we are required to control the gauge transform back to the temporal gauge along \( s = 0 \), for which appropriate estimates for \( A_0(s = 0) \) in the caloric-temporal gauge are needed; see Lemma 4.6. These are obtained ultimately as a consequence of the analysis of the hyperbolic equations of (HPYM); see Proposition 7.1.

1.5. Outline of the paper. After establishing notations and conventions in Section 2, we gather some preliminary results concerning the linear wave and the heat (or parabolic) equations in Section 3. In particular, for the parabolic equation, we develop what we call the abstract parabolic theory, which allows us to handle various parabolic equations with a unified approach.

We embark on the proof of the Main Theorem in Section 4, where the Main Theorem is reduced to two smaller statements, namely Theorems A and B both of which are concerned with (HPYM). Theorem A roughly addresses Points 2 and 4 in 1.4; More precisely, it starts with a solution \( A^\dagger_{\mu} \) to (YM) in the temporal gauge, and asserts the existence of its extension \( A_\mu \) as a solution to (HPYM) in the caloric-temporal gauge, along with appropriate initial data estimates at \( t = 0 \). The DeTurck gauge is used in an essential way in the proof. On the other hand, Theorem B presents the result of a local-in-time analysis of (HPYM) in the caloric-temporal gauge, corresponding to Point 6 in 1.4.

The aim of the next two sections is to prove Theorem A. In Section 5, we study the covariant Yang-Mills heat flow (cYMHF). In particular, various smoothing estimates for the connection 1-form \( A_\mu \) will be derived in the DeTurck gauge (in 5.1), and estimates for the gauge transform from the DeTurck gauge to the caloric gauge is presented as well (in 5.2). As a byproduct of the analysis, we obtain a proof of local existence of a solution to (YM) (Theorem C), which is used in 19. We remark that the proof is independent of the original one in 20. Next, based on the results proved in Section 5, a proof of Theorem A is given in Section 6.

The remainder of the paper is devoted to a proof of Theorem B. We begin in Section 7 by reducing Theorem A to several smaller statements, namely Propositions 7.1 - 7.4 and Theorems D and E, where the latter two theorems concern estimates for the hyperbolic equations (1.14) and (1.13), respectively. Section 8 is where we derive estimates for solutions to various parabolic equations, and forms the ‘parabolic’ heart of the paper. It is here that our efforts for developing the abstract parabolic theory amply pays off. Equipped with the results from the previous section, we prove Propositions 7.1 - 7.4 in Section 9. The following section, namely Section 10, is where we finally study the wave equations for \( A_i \) and \( F_{si} \). Combined with the parabolic estimates from Section 8, we establish Theorems D and E.

In Appendix A, we give a derivation of various covariant equations from (HPYM). Then finally, in Appendix B we prove estimates for gauge transforms that are deferred in the main body of the paper.

Acknowledgements. The author is deeply indebted to his Ph.D. advisor Sergiu Klainerman, without whose support and constructive criticisms this work would not have been possible. He would also like to thank l’ENS d’Ulm for hospitality, where a major part of this work was done. The author was supported by the Samsung Scholarship.

2. Notations and Conventions

2.1. Schematic notations and conventions. We will often omit the spatial index \( i \); that is, we will write \( A_i, F_{si}, w \) as the shorthands for \( A_i, F_{si}, w_i \), respectively, and so on. A norm of such an

\(^1\)It turns out that such a trick is already needed at the stage of deriving estimates such as (1.19); see Proposition
expression, such as $\|A\|$, is to be understood as the maximum over $i = 1, 2, 3$. (i.e. $\|A\| = \sup_i \|A_i\|$ and etc.)

We will use the notation $O(\phi_1, \ldots, \phi_k)$ to denote a $k$-linear expression in the values of $\phi_1, \ldots, \phi_k$. For example, when $\phi_i$ and the expression itself are scalar-valued, then $O(\phi_1, \ldots, \phi_k) = C\phi_1\phi_2 \cdots \phi_k$ for some constant $C$. In many cases, however, each $\phi_i$ and the expression $O(\phi_1, \ldots, \phi_k)$ will actually be matrix-valued. In such case, $O(\phi_1, \ldots, \phi_k)$ will be a matrix, whose each entry is a $k$-linear functional of the matrices $\phi_i$.

When stating various estimates, we adopt the standard convention of denoting by the same letter $C$ positive constants which are different, possibly line to line. Dependence of $C$ on other parameters will be made explicit by subscripts. Furthermore, we will adopt the convention that $C$ always depends on each of its parameters in a non-decreasing manner, in its respective range, unless otherwise specified. For example, $C_{E,F}$, where $E, F$ range over positive real numbers, is a positive, non-decreasing function of both $E$ and $F$.

2.2. Notations and conventions for the estimates for differences. In the paper, along with estimating a single solution $A_n$ of the Yang-Mills equation, we will also be estimating the difference of two nearby solutions. We will refer to various variables arising from the other solution by putting a prime, e.g. $A'_n$, $F'_{\mu\nu}$, $w'_i$ and etc, and the corresponding differences will be written with a $\delta$, i.e. $\delta A_n := A_n - A'_n$, $\delta F_{\mu\nu} := F_{\mu\nu} - F'_{\mu\nu}$ and $\delta w_i = w_i - w'_i$ and etc.

We will also use equations for differences, which are derived by taking the difference of the equations for the original variables. In writing such equations schematically using the $O$-notation, we will not distinguish between primed and unprimed variables. For example, the expression $O(A, \partial_x (\delta A))$ refers to a sum of bilinear expressions, of which the first factor could be any of $A_i, A'_i$ ($i = 1, 2, 3$), and the second is one of $\partial_i (\delta A_j)$ ($i, j = 1, 2, 3$).

For this purpose, the following rule, which we call the formal Leibniz’s rule for $\delta$, is quite useful:

$$\delta O(\phi_1, \phi_2, \ldots, \phi_k) = O(\delta \phi_1, \phi_2, \ldots, \phi_k) + O(\phi_1, \delta \phi_2, \ldots, \phi_k) + \cdots + O(\phi_1, \phi_2, \ldots, \delta \phi_k).$$

2.3. Small parameters. The following small parameters will be used in this paper.

$$0 < \delta_H \ll \delta_E \ll \delta_P \ll \delta_A \ll 1.$$

In many parts of the argument, we will need an auxiliary small parameter, which may be fixed within that part; for such parameters, we will reserve the letter $\epsilon$, and its variants thereof.

3. Preliminaries

The aim of this section is to gather basic inequalities and preliminary results concerning the linear wave and parabolic equations, which will be rudimentary for our analysis to follow. We also develop what we call the abstract parabolic theory, which is a book-keeping scheme allowing for a unified approach to the diversity of parabolic equations to arise below. In the end, a short discussion is given on the notion of the associated $s$-weights, which is a useful heuristic for figuring out the appropriate weight of $s$ in various instances.

3.1. Basic inequalities. We collect here some basic inequalities that will be frequently used throughout the paper. Let us begin with some inequalities involving Sobolev norms for $\phi \in S_x$, where $S_x$ refers to the space of Schwartz functions on $\mathbb{R}^3$.

Lemma 3.1 (Inequalities for Sobolev norms). Let $\phi \in S_x$. The following statements hold.

- (Sobolev inequality) For $1 \leq q \leq r$, $k \geq 0$ such that $\frac{1}{q} = \frac{1}{r} - k$, we have

$$\|\phi\|_{L^q_x} \leq C\|\phi\|_{\dot{W}^{k,r}_x},$$

where $\dot{W}^{k,r}_x$ is the $L^r$-based homogeneous Sobolev norm of order $k$.

- (Interpolation inequality) For $1 \leq q < \infty$, $k_1 \leq k_0 \leq k_2$, $0 < \theta_1, \theta_2 < 1$ such that $\theta_1 + \theta_2 = 1$ and $k_0 = \theta_1 k_1 + \theta_2 k_2$, we have

$$\|\phi\|_{\dot{W}^{k_0,q}_x} \leq C\|\phi\|^{\theta_1}_{\dot{W}^{k_1,q}_x}\|\phi\|^{\theta_2}_{\dot{W}^{k_2,q}_x}.$$
• **(Gagliardo-Nirenberg inequality)** For $1 \leq q_1, q_2, r \leq \infty$ and $0 < \theta_1, \theta_2 < 1$ such that $\theta_1 + \theta_2 = 1$ and $\frac{1}{r} = \frac{1}{q_1} + \theta_2 \left( \frac{1}{q_1} - 1 \right)$, we have

$$\|\phi\|_{L^r_x} \leq C \|\phi\|_{L^{q_1}_x}^{\theta_1} \|\partial_x \phi\|_{L^{q_2}_x}^{\theta_2}.$$  

**Proof.** These inequalities are standard; we refer the reader to [1]. □

**Lemma 3.2** (Product estimates for homogeneous Sobolev norms). For a triple $(s_0, s_1, s_2)$ of real numbers satisfying

$$s_0 + s_1 + s_2 = 3/2, \quad s_0 + s_1 + s_2 > \max(s_0, s_2),$$

there exists $C > 0$ such that the following product estimate holds for $\phi_1, \phi_2 \in S_x$:

$$\|\phi_1 \phi_2\|_{\dot{H}^{-s_0}_x} \leq C \|\phi_1\|_{\dot{H}^{s_1}_x} \|\phi_2\|_{\dot{H}^{s_2}_x}.$$  

**Proof.** It is a standard result (see, for example, [6]) that given a triple $(s_0, s_1, s_2)$ of real numbers satisfying (3.4), the following inhomogeneous Sobolev product estimate holds for $\phi_1, \phi_2 \in S_x$:

$$\|\phi_1 \phi_2\|_{\dot{H}^{-s_0}_x} \leq C \|\phi_1\|_{\dot{H}^{s_1}_x} \|\phi_2\|_{\dot{H}^{s_2}_x}.$$  

Thanks to the condition $s_0 + s_1 + s_2 = 3/2$, the above estimate implies the homogeneous estimate (3.5) by scaling. □

Next, we state Gronwall’s inequality, which will be useful in several places below.

**Lemma 3.3** (Gronwall’s inequality). Let $(s_0, s_1) \subset \mathbb{R}$ be an interval, $D \geq 0$, and $f(s), r(s)$ non-negative measurable functions on $J$. Suppose that for all $s \in (s_0, s_1)$, the following inequality holds:

$$\sup_{\tau \in (s_0, s]} f(\tau) \leq \int_{s_0}^{s} r(\tau) f(\tau) d\tau + D.$$  

Then for all $s \in (s_0, s_1)$, we have

$$\sup_{\tau \in (s_0, s]} f(\tau) \leq D \exp \left( \int_{s_0}^{s} r(\tau) d\tau \right).$$  

**Proof.** See [24] Lemma 3.3. □

In the course of the paper, we will often perform integration-by-parts arguments which require the functions involved to be sufficiently smooth and decaying sufficiently fast towards the spatial infinity (the latter assumption is used to show that the boundary term which may arise vanishes at infinity). Usually, this issue is usually dealt with by working with Schwartz functions to justify the arguments and then passing to the appropriate limit in the end. In our case, however, the nature of the Yang-Mills equation does not allow us to do so (in particular due to the elliptic constraint equation (1.1)). Instead, we formulate the notion of regular functions, which is weaker than the Schwartz assumption but nevertheless strong enough for our purposes.

**Definition 3.4** (Regular functions). Let $I \subset \mathbb{R}$, $J \subset [0, \infty)$ be intervals.

1. A function $\phi = \phi(x)$ defined on $\mathbb{R}^3$ is regular if $\phi \in H^\infty_x := \cap_{m=0}^{\infty} H^m_x$.
2. A function $\phi = \phi(t, x)$ defined on $I \times \mathbb{R}^3$ is regular if $\phi \in C^\infty_t(I, H^\infty_x) := \cap_{k,m=0}^{\infty} C^k_t(I, H^m_x)$.
3. A function $\psi = \psi(t, x, s)$ defined on $I \times \mathbb{R}^3 \times J$ is regular if $\phi \in C^\infty_{t,x}(I \times J, H^\infty_x) := \cap_{k,m=0}^{\infty} C^k_{t,x}(I \times J, H^m_x)$.

In particular, a regular function is always smooth on its domain. Moreover, Lemmas 3.1 and 3.2 still hold for regular functions, by approximation.
3.2. Estimates for the linear wave equation and the space $\mathring{S}^k$. We summarize the estimates for solutions to an inhomogeneous wave equation that will be used in the following proposition.

**Proposition 3.5** (Wave estimates). Let $\psi, \varphi$ be smooth solutions with a suitable decay towards the spatial infinity (say $\psi, \varphi \in C_0^\infty S_x$) to the inhomogeneous wave equations

$$\Box \psi = N, \quad \Box \varphi = M,$$

on $(-T, T) \times \mathbb{R}^3$. The following estimates hold.

- **($L_t^\infty L_x^2$ estimate)**
  \[\|\partial_{t,x} \psi\|_{L_t^\infty L_x^2((-T,T)\times \mathbb{R}^3)} \leq C \left( \|\psi(t = 0)\|_{H_x^1 R^3} + \|N\|_{L_t^1 L_x^2((-T,T)\times \mathbb{R}^3)} \right)\]

- **($L_t^1 L_x^2$-Strichartz estimate)**
  \[\|\partial_t \psi\|_{L_t^1 L_x^2((-T,T)\times \mathbb{R}^3)} \leq C \left( \|\psi(t = 0)\|_{\dot{H}^{3/2}_x R^3} + \|N\|_{L_t^1 \dot{H}^{1/2}_x((-T,T)\times \mathbb{R}^3)} \right)\]

- **(Null form estimate)** For $Q_{ij}(\psi, \phi) := \partial_i \psi \partial_j \phi - \partial_j \psi \partial_i \phi$, we have
  \[\|Q_{ij}(\psi, \phi)\|_{L_t^1 L_x^2((-T,T)\times \mathbb{R}^3)} \leq C \left( \|\psi(t = 0)\|_{\dot{H}^2_x R^3} + \|\varphi(t = 0)\|_{\dot{H}^1_x R^3} \right) \times C \left( \|\psi(t = 0)\|_{\dot{H}^2_x R^3} + \|\varphi(t = 0)\|_{\dot{H}^1_x R^3} \right)\]

**Proof.** This is a standard material. For the $L_t^\infty L_x^2$ and the Strichartz estimates, we refer the reader to [24, Chapter III]. For the null form estimate, see the original article [11].

Motivated by Proposition 3.5, let us define the norms $\mathring{S}^k$ which will be used as a convenient device for controlling the wave-like behavior of certain dynamic variables. Let $\psi$ be a smooth function on $I \times \mathbb{R}^3$ ($I \subset \mathbb{R}$) which decays sufficiently towards the spatial infinity. We start with the norm $\mathring{S}^1$, which we define by

$$\|\psi\|_{\mathring{S}^1(I)} := \|\partial_{t,x} \psi\|_{L_t^\infty L_x^2} + |I|^{1/2} \Box \psi\|_{L_t^1 L_x^2}.\]

The norms $\mathring{S}^k$ for $k = 2, 3, 4$ are then defined by taking spatial derivatives, i.e.

$$\|\psi\|_{\mathring{S}^k(I)} := \|\partial_x^{(k-1)} \psi\|_{\mathring{S}^1(I)},\]

and we furthermore define $\mathring{S}^k$ for $k \geq 1$ a real number by using fractional derivatives. Note the interpolation property

$$\|\psi\|_{\mathring{S}^{k+\theta}(I)} \leq C\theta \|\psi\|_{\mathring{S}^k(I)}^{1-\theta} \|\psi\|_{\mathring{S}^{k+1}(I)}^{\theta}, \quad 0 < \theta < 1.\]

The following estimates concerning the $\mathring{S}^k$-norms are an immediate consequence of Proposition 3.5 and the fact that regular functions can be approximated by functions in $C_0^\infty S_x$ with respect to each of the norms involved.

**Proposition 3.6.** Let $k \geq 1$ be an integer and $\psi, \phi$ smooth functions on $(-T, T) \times \mathbb{R}^3$ such that $\partial_{t,x} \psi, \partial_{t,x} \phi$ are regular. Then the following estimates hold.

- **($L_t^\infty L_x^2$ estimate)**
  \[\|\partial^{(k-1)}_x \partial_{t,x} \psi\|_{L_t^\infty L_x^2((-T,T)\times \mathbb{R}^3)} \leq \|\psi\|_{\mathring{S}^k(-T,T)},\]

- **($L_t^1 L_x^2$-Strichartz estimate)**
  \[\|\partial^{(k-1)}_x \psi\|_{L_t^1 L_x^2((-T,T)\times \mathbb{R}^3)} \leq C\|\psi\|_{\mathring{S}^{k+1/2}(-T,T)},\]

- **(Null form estimate)**
  \[\|Q_{ij}(\psi, \phi)\|_{L_t^1 L_x^2((-T,T)\times \mathbb{R}^3)} \leq C\|\psi\|_{\mathring{S}^2(-T,T)} \|\phi\|_{\mathring{S}^1(-T,T)}.\]

\[\text{We remark that } \|\cdot\|_{\mathring{S}^k} \text{ is a norm after restricted to regular functions, by Sobolev.}\]
On the other hand, in order to control the $\dot{S}^k$ norm of $\psi$, all one has to do is to estimate the d’Alembertian of $\psi$ along with the initial data. This is the content of the following proposition, which is sometimes referred to as the energy estimate in the literature.

**Proposition 3.7 (Energy estimate).** Let $k \geq 1$ be an integer and $\psi$ a smooth function on $(-T, T) \times \mathbb{R}^3$ such that $\partial_t \partial_x^s \psi$ is regular. Then the following estimate holds.

$$\|\psi\|_{\dot{S}^k(-T, T)} \leq C \left( \|\psi(0)\|_{H^k(x)} + T^{1/2} \|\psi\|_{L^2_{T,T,x}((-T, T) \times \mathbb{R}^3)} \right).$$

**Proof.** After a standard approximation procedure, this is an immediate consequence of (3.4). □

### 3.3. Parabolic-normalized (or $p$-normalized) norms.

The purpose of the rest of this section is to develop a theory of parabolic equations suited to our needs later on.

Given a function $\phi$ on $\mathbb{R}^3$, we consider the operation of scaling by $\lambda > 0$, defined by

$$\phi \rightarrow \phi_\lambda(x) := \phi(x/\lambda).$$

We say that a norm $\| \cdot \|_X$ is **homogeneous** if it is covariant with respect to scaling, i.e. there exists a real number $\ell$ such that

$$\|\phi_\lambda\|_X = \lambda^\ell \|\phi\|_X.$$

The number $\ell$ is called the **degree of homogeneity** of the norm $\| \cdot \|_X$.

Let $\phi$ be a solution to the heat equation $\partial_t \phi - \triangle \phi = 0$ on $\mathbb{R}^3 \times [0, \infty)$. Note that this equation ‘respects’ the scaling $\phi_\lambda(x, s) := \phi(x/\lambda, s/\lambda^2)$, in the sense that any scaled solution to the linear heat equation remains a solution. Moreover, one has **smoothing estimates** of the form $\|\partial_x^k \phi_\lambda(s)\|_{L^p_x} \leq s^{-\delta/2} \|\partial_x^k \phi(0)\|_{L^p_x}$ (for $q \leq p$, $k \geq 0$) which are invariant under this scaling. The norms $\|\partial_x^k\|_{L^p_x}$ and $\|\cdot\|_{L^p_T}$ are homogeneous, and the above estimate can be rewritten as

$$s^{-\ell_1/2} \|\partial_x^k \phi_\lambda(s)\|_{L^p_x} \leq s^{-\ell_2/2} \|\partial_x^k \phi(0)\|_{L^p_x}$$

where $\ell_1$, $\ell_2$ are the degrees of homogeneity of the norms $\|\partial_x^k\|_{L^p_x}$ and $\|\cdot\|_{L^p_T}$, respectively.

Motivated by this example, we will define the notion of **parabolic-normalized**, or $p$-normalized, norms and derivatives. These are designed to facilitate the analysis of parabolic equations by capturing their scaling property.

Consider a homogeneous norm $\| \cdot \|_X$ of degree $2\ell$, which is well-defined for smooth functions $\phi$ on $\mathbb{R}^3$. (i.e. for every smooth $\phi$, $\|\phi\|_X$ is defined uniquely as either a non-negative real number or $\infty$.) We will define its $p$-normalized analogue $\| \cdot \|_{X(s)}$ for each $s > 0$ by

$$\| \cdot \|_{X(s)} := s^{-\ell} \| \cdot \|_X.$$

We will also define the $p$-normalization of space-time norms. As we will be concerned with functions restricted to a time interval, we will adjust the notion of homogeneity of norms as follows. For $I \subset \mathbb{R}$, consider a family of norms $X(I)$ defined for functions $\phi$ defined on $I \times \mathbb{R}^3$. For $\lambda > 0$, consider the scaling $\phi_\lambda(t, x) := \phi(t/\lambda, x/\lambda)$. We will say that $X(I)$ is **homogeneous of degree $\ell$** if

$$\|\phi_\lambda\|_{X(I)} = \lambda^{\ell} \|\phi\|_{X(I)}.$$

As before, we define its $p$-normalized analogue $\| \cdot \|_{X(I, s)}$ as $s^{-\ell} \| \cdot \|_{X(I)}$. Let us furthermore define the parabolic-normalized derivative $\nabla_s \mu(s)$ by $s^{1/2} \partial_s \mu$. Accordingly, for $k > 0$ we define the **homogeneous** $k$-th derivative norm $\| \cdot \|_{\hat{X}^k(s)}$ by

$$\| \cdot \|_{\hat{X}^k(s)} := \|\nabla_x^k \mu(s)\|_{X(s)}.$$

We will also define the parabolic-normalized covariant derivative $D_\mu(s) := s^{1/2} D_\mu$.

We will adopt the convention $X^0 := X$. For $m > 0$ an integer, we define **inhomogeneous** $m$-th derivative norm $\| \cdot \|_{\hat{X}^m(s)}$ by

$$\| \cdot \|_{\hat{X}^m(s)} := \sum_{k=0}^m \| \cdot \|_{\hat{X}^k(s)}.$$
Example 3.8. A few examples of homogeneous norms and their $p$-normalized versions are in order. We will also take this opportunity to fix the notations for the $p$-normalized norms which will be used in the rest of the paper.

1. $X = L^p_J$, in which case the degree of homogeneity is $2\ell = 3/p$. We will define $X = L^p_J$ and $\mathcal{X}^k := \mathcal{W}^k_{p,J}$ as follows.

   \[ \| \cdot \|_{\mathcal{L}^p_J(s)} := s^{-3/(2p)} \| \cdot \|_{L^p_J}, \quad \| \cdot \|_{\mathcal{W}^k_{p,J}(s)} := s^{k/2-3/(2p)} \| \cdot \|_{\mathcal{W}^k_{p,J}}. \]

   The norm $\mathcal{X}^m := \mathcal{W}^m_{p,J}$ will be defined as the sum of $\mathcal{W}^k_{p,J}$ norms for $k = 0, \ldots, m$. In the case $p = 2$, we will use the notation $\mathcal{X}^k := \mathcal{H}^k_{p,J}$ and $\mathcal{X}^{k} := \mathcal{H}^{k}_{p,J}$.

2. Consider a time interval $I \subset \mathbb{R}$. For $X = L^p_I L^q_I (I \times \mathbb{R}^3)$, in which case $2\ell = 1/q + 3/p$, we will write

   \[ \| \cdot \|_{\mathcal{L}^p_I L^q_I(I,s)} := s^{-1/(2q)-3/(2p)} \| \cdot \|_{L^p_I L^q_I}, \quad \| \cdot \|_{\mathcal{W}^k_{p,I,I}(s)} := s^{k/2-1/(2q)-3/(2p)} \| \cdot \|_{L^p_I L^q_I}. \]

   The norms $\mathcal{L}^p_{I,I} \mathcal{W}^k_{p,I,I}(I,s)$ and $\mathcal{L}^q_{I,I} \mathcal{H}^k_{p,I,I}(I,s)$ are defined accordingly.

3. Finally, the norm $\tilde{S}^1$, defined in Section 3.2 for the purpose of hyperbolic estimates, is homogeneous of degree $2\ell = 1/2$, which is the same as $L^\infty_2 \mathcal{H}^1_2$ (i.e. the energy). For $p$-normalized version of $\tilde{S}^1$, we will use a set of notations slightly deviating from the rest in order to keep consistency with the intuition that $\| \phi \|_{\tilde{S}^1}$ is at the level of $L^\infty_2 \mathcal{H}^1_2$. Indeed, for $m, k \geq 1$ and $m$ an integer, we will write

   \[ \| \phi \|_{\tilde{S}^k_{m}} := s^{(k-1)/2-1/4} \| \partial_x (k-1) \phi \|_{L^p_2}, \quad \| \phi \|_{\tilde{S}^m} := \sum_{k=1}^m \| \phi \|_{\tilde{S}^k_{m}}. \]

   For $f = f(s)$ a measurable function defined on an $s$-interval $J \subset (0, \infty)$, we define its $p$-normalized Lebesgue norm $\| f \|_{\mathcal{L}^p_J(s)}$ by

   \[ \| f \|_{\mathcal{L}^p_J(s)} := \int_J |f(s)|^p \frac{ds}{s} \quad \text{for } 1 \leq p < \infty, \quad \text{and } \| f \|_{\mathcal{L}^\infty_J} := \| f \|_{L^\infty_J}. \]

   Given $\ell \geq 0$, we will define the weighted norm $\| f \|_{\mathcal{L}^p_{\ell,p}(J)}$ by

   \[ \| f \|_{\mathcal{L}^p_{\ell,p}(J)} := \| s^\ell f(s) \|_{\mathcal{L}^p(s)}. \]

   Let us consider the case $J = (0, s_0)$ or $J = (0, \infty)$ for some $s_0 > 0$. For $\ell > 0$ and $1 \leq p \leq \infty$, note the obvious computation $\| s^\ell f \|_{\mathcal{L}^p_0(0,s_0)} = C_{\ell,p,s_0}$. Combining this with the Hölder inequality

   \[ \| fg \|_{\mathcal{L}^p_{\ell,p}} \leq \| f \|_{\mathcal{L}^p_{\ell_1,p_1}} \| g \|_{\mathcal{L}^p_{\ell_2,p_2}} \quad \text{for } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \]

   we obtain the following simple lemma.

**Lemma 3.9** (Hölder for $\mathcal{L}^p_{\ell,p}$). Let $\ell, \ell_1, \ell_2 \geq 0, 1 \leq p, p_1, p_2 \leq \infty$ and $f, g$ functions on $J = (0, s_0)$ (or $J = (0, \infty)$) such that $\| f \|_{\mathcal{L}^p_{\ell_1,p_1}}, \| g \|_{\mathcal{L}^p_{\ell_2,p_2}} < \infty$. Then we have

   \[ \| fg \|_{\mathcal{L}^p_{\ell,p}(J)} \leq C S_{\ell,\ell_1,\ell_2} \| f \|_{\mathcal{L}^p_{\ell_1,p_1}(J)} \| g \|_{\mathcal{L}^p_{\ell_2,p_2}(J)} \]

   provided that either $\ell = \ell_1 + \ell_2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, or $\ell > \ell_1 + \ell_2$ and $\frac{1}{p} > \frac{1}{p_1} + \frac{1}{p_2}$. In the former case, $C = 1$, while in the latter case, $C$ depends on $\ell - \ell_1 - \ell_2$ and $\frac{1}{p} - \frac{1}{p_1} - \frac{1}{p_2}$.

We will often use the mixed norm $\| \psi \|_{\mathcal{L}^p s^\ell_{\ell_1,p_1}(J)} := \| |\psi(s)| \mathcal{X}^{\ell_1}_s \|_{\mathcal{L}^p_{\ell_1,p_1}(J)}$ for $\psi = \psi(x, s)$ such that $s \to \| \psi(s) \|_{\mathcal{X}^{\ell_1}_s}$ is measurable. The norms $\mathcal{L}^p s^\ell_{\ell_1,p_1} \mathcal{X}^{\ell_1}_s(J)$ and $\mathcal{L}^p s^\ell_{\ell_1,p_1} \mathcal{X}^{\ell_1}_s(J)$ are defined analogously.

3.4. **Abstract parabolic theory.** Let $J \subset (0, \infty)$ be an $s$-interval. Given a homogeneous norm $X$ and $k \geq 1$ an integer, let us define the (semi-)norm $\mathcal{P}^\ell X^k(J)$ for a smooth function $\psi$ by

   \[ \| \psi \|_{\mathcal{P}^\ell X^k(J)} := \| \psi \|_{\mathcal{L}^\infty_{\ell-1,k-1}(J)} + \| \psi \|_{\mathcal{L}^2_{\ell,k}(J)}. \]

   For $m_0 < m_1$, we will also define the (semi-)norm $\mathcal{P}^\ell X^{m_1}_{m_0}(J)$ by

   \[ \| \psi \|_{\mathcal{P}^\ell X^{m_1}_{m_0}(J)} := \sum_{k=m_0+1}^{m_1} \| \psi \|_{\mathcal{P}^\ell X^k(J)}. \]
We will omit \(m_0\) when \(m_0 = 0\), i.e. \(X^m := X^m_0\).

We remark that despite the notation \(P_\ell X^k\), this norms controls both the \(X^{k-1}\) as well as the \(X^k\) norm of \(\psi\). Note furthermore that \(\|\psi\|_{P_\ell X^{m_0}}\) controls the derivatives of \(\psi\) of order from \(m_0\) to \(m_1\).

**Definition 3.10.** Let \(X\) be a homogeneous norm of degree \(2\ell_0\). We say that \(X\) satisfies the parabolic energy estimate if there exists \(C_X > 0\) such that for all \(\ell \in \mathbb{R}\), \([s_1, s_2] \subset (0, \infty)\) and \(\psi\) smooth and satisfying \(\|\psi\|_{P_\ell X^1(s_1, s_2)} < \infty\), the following estimate holds.

\[
\|\psi\|_{P_\ell X^1(s_1, s_2)} \leq C_X s_1^\ell \|\psi(s_1)\|_{X(s_1)} + C_X (\ell - \ell_0) \|\psi\|_{L_0^\ell X(s_1, s_2)} + C_X \|\partial_s - \Delta\|_{L_0^{\ell_1+1} X(s_1, s_2)}.
\]

(3.15)

The norm \(X\) satisfies the parabolic smoothing estimate if there exists \(C_X > 0\) such that for all \(\ell \in \mathbb{R}\), \([s_1, s_2] \subset (0, \infty)\) and \(\psi\) smooth and satisfying \(\|\psi\|_{P_\ell X^2(s_1, s_2)} < \infty\), the following estimate holds:

\[
\|\psi\|_{P_\ell X^2(s_1, s_2)} \leq C_X s_1^\ell \|\psi(s_1)\|_{X^1(s_1)} + C_X (\ell + 1/2 - \ell_0) \|\psi\|_{L_0^{\ell_1} X^1(s_1, s_2)} + C_X \|\partial_s - \Delta\|_{L_0^{\ell_1+1} X(s_1, s_2)}.
\]

(3.16)

For the purpose of application, we will consider vector-valued solutions \(\psi\) to an inhomogeneous heat equation. The norms \(X, X^\ell, P_\ell X, \text{etc.}\) of a vector-valued function \(\psi\) are defined in the obvious manner of taking the supremum of the respective norm of all components of \(\psi\).

**Theorem 3.11 (Abstract parabolic theory).** Let \(X\) be a homogeneous norm of degree \(2\ell_0\), and \(\psi\) be a vector-valued smooth solution to \(\partial_s \psi - \Delta \psi = N\) on \([0, s_0]\).

1. Let \(X\) satisfy the parabolic energy and smoothing estimates (3.15), (3.16), and \(\psi\) satisfy \(\|\psi\|_{P_0 X^2(0, s_0)} < \infty\). Let \(1 \leq p < \infty\), \(\epsilon > 0\), \(D > 0\) and \(C(s)\) a function defined on \((0, s_0)\) which satisfies

\[
\int_0^{s_0} C(s) \frac{ds}{s} < \infty
\]

for some \(1 \leq p < \infty\), and

\[
\|N\|_{L_0^{\ell+1} X(0, s_0)} + \|N\|_{L_0^{\ell+1} X^1(0, s_0)} \leq \|C(s)\|_{L_0^{\ell+1} X^1(0, s_0)} + \epsilon \|\psi\|_{P_0 X^2(0, s_0)} + D;
\]

(3.17)

for every \(s \in (0, s_0]\).

Then there exists a constant \(\delta_A = \delta_A(C_X, \int_0^{s_0} C(s) \frac{ds}{s}, \epsilon, p) > 0\) such that if

\[
0 < \epsilon < \delta_A,
\]

then the following a priori estimate holds.

\[
\|\psi\|_{P_0 X^2(0, s_0)} \leq C e^{C \int_0^{s_0} C(s) \frac{ds}{s}} (\|\psi(s = 0)\|_X + D),
\]

(3.18)

where \(C\) depends only on \(C_X\) and \(p\).

2. Suppose that \(X\) satisfies the the parabolic smoothing estimate (3.10), and that for some \(\ell \in \mathbb{R}\) and \(0 \leq m_0 \leq m_1\) (where \(m_0, m_1\) are integers) we have \(\|\psi\|_{P_\ell X^{m_0+2}(0, s_0)} < \infty\). Suppose furthermore that for \(m_0 \leq m \leq m_1\), there exists \(\epsilon > 0\) and a non-negative non-decreasing function \(B_m(\cdot)\) such that

\[
\|N\|_{L_0^{\ell+1} X^m(0, s_0)} \leq \epsilon \|\psi\|_{P_\ell X^{m+2}(0, s_0)} + B_m(\|\psi\|_{P_\ell X^{m_0+1}(0, s_0)}).
\]

(3.19)

Then for \(0 < \epsilon < 1/(2C_X)\), the following smoothing estimate holds:

\[
\|\psi\|_{P_\ell X^{m_0+2}(0, s_0)} \leq C
\]

(3.20)

where \(C\) is determined from \(C_X, B_{m_0}, \ldots, B_{m_1}\) and \(\|\psi\|_{P_\ell X^{m_0+1}(0, s_0)}\).

---

13The assumption of smoothness is here only for convenience; we remark that it is not essential in the sense that, by an approximation argument, both (2.10) and (4.10) may be extended to functions \(\psi\) which are not smooth.

14\(\psi\) is a function on \(\mathbb{R}^3 \times [0, s_0]\) or \(I \times \mathbb{R}^3 \times [0, s_0]\) depending on whether \(X\) is for functions on the space or the space-time, respectively.
More precisely, consider the non-decreasing function \( \widetilde{B}_m(r) := (2C_X(\ell - \ell_0 + 1/2)+1)r + 2C_XB_m(r) \). Then \( C \) in \((3.20)\) is given by the composition

\[
C = \widetilde{B}_{m_1} \circ \widetilde{B}_{m_1-1} \circ \cdots \circ \widetilde{B}_{m_0}(\|\psi\|_{\mathcal{P}_{t,\hat{\chi}^{m_0+1}(0,s_0)}}).
\]

**Proof.** Step 1: Proof of \((1)\). Without loss of generality, assume that \( C_X \geq 2 \). Thanks to the hypothesis on \( \psi \) and \((3.17)\), we can apply the parabolic energy estimate \((3.18)\) to obtain

\[
\|\psi\|_{P^0,\chi^2(0,\underline{a})} \leq C_X \|\psi(0)\|_X + C_X(\|C(s)\psi\|_{L^p_0,\chi^1(0,\underline{a})} + \epsilon \|\psi\|_{P^0,\chi^2(0,\underline{s}_0)} + D),
\]

where we have used the fact that \( \lim \inf_{s_1 \to 0} s_1^p \|\psi(s_1)\|_{\chi^1(s_1)} = \|\psi(0)\|_X \). Using again the hypothesis on \( \psi \) and \((3.17)\), we can apply the parabolic smoothing estimate \((3.10)\) and get

\[
\|\psi\|_{P^0,\chi^2(0,\underline{a})} \leq \frac{C_X}{2} \|\psi\|_{L^p_0,\chi^1(0,\underline{a})} + C_X(\|C(s)\psi\|_{L^p_0,\chi^1(0,\underline{a})} + \epsilon \|\psi\|_{P^0,\chi^2(0,\underline{s}_0)} + D),
\]

where we used \( \lim \inf_{\ell_1 \to 0} \ell_1^p \|\psi(\ell_1)\|_{\chi^1(\ell_1)} = 0 \), which holds as \( \|\psi\|_{L^p_0,\chi^1(0,\underline{a})} < \infty \). Using \((3.22)\) to bound the first term on the right-hand side of \((3.23)\), we arrive at

\[
\|\psi\|_{P^0,\chi^2(0,\underline{a})} \leq C_X^2 \|\psi(0)\|_X + C_X(1 + C_X)(\|\psi\|_{L^p_0,\chi^1(0,\underline{a})} + \epsilon \|\psi\|_{P^0,\chi^2(0,\underline{s}_0)} + D),
\]

for every \( 0 < \underline{a} \leq s_0 \).

We will apply Gronwall’s inequality to deal with the term involving \( C(s)\psi \). For convenience, let us make the definition

\[
D' = C_X^2 \|\psi(0)\|_X + C_X(1 + C_X)(\|\psi\|_{P^0,\chi^2(0,\underline{s}_0)} + D).
\]

Recalling the definition of \( \|\psi\|_{P^0,\chi^2(0,\underline{a})} \) and unravelling the definition of \( L^p_{s_0} X \), we see in particular that

\[
\sup_{0 < \ell_0 \leq \underline{a}} s_1^p \|\psi(s)\|_{\chi^1} \leq C_X(1 + C_X) \left( \int_{s_0}^{\underline{a}} C(s)\psi(s) \|\psi(s)\|_{\chi^1} s \frac{ds}{s} \right)^{1/p} + D',
\]

for every \( 0 < \underline{a} \leq s_0 \). Taking the \( p \)-th power, using Gronwall’s inequality and then taking the \( p \)-th root back, we arrive at

\[
\sup_{0 < \ell_0 \leq \underline{a}} s_1^p \|\psi(s)\|_{\chi^1} \leq 2^{1/p} D' \exp \left( \frac{2C_X^p(1 + C_X)^p}{p} \int_{s_0}^{\underline{a}} C(s)^p \frac{ds}{s} \right).
\]

Iterating this bound into \( \|\psi\|_{L^p_0,\chi^1(0,\underline{s}_0)} \) and expanding \( D' \) out, we obtain

\[
\|\psi\|_{P^0,\chi^2(0,\underline{s}_0)} \leq C_0 \left( C_X^2 \|\psi(0)\|_X + C_X(1 + C_X)(\|\psi\|_{P^0,\chi^2(0,\underline{s}_0)} + D) \right),
\]

where \( C_0 = \exp \left( \frac{2C_X^p(1 + C_X)^p}{p} \int_{s_0}^{\underline{a}} C(s)^p \frac{ds}{s} \right) \).

Let us define \( \delta_A := (2C_X(1 + C_X)^{-1})^{-1} \). Then from the hypothesis \( 0 < \epsilon < \delta_A \), we can absorb the term \( C_0 C_X(1 + C_X)\epsilon \|\psi\|_{P^0,\chi^2(0,\underline{s}_0)} \) into the left-hand side. The desired estimate \((3.18)\) follows.

Step 2: Proof of \((2)\). In this step, we will always work on the whole \( s \)-interval \((0, s_0)\).

We claim that under the assumptions of \((2)\), the following inequality holds for \( m_0 \leq m \leq m_1 \):

\[
\|\psi\|_{P^t,\hat{\chi}^{m+2}} \leq \widetilde{B}_m(\|\psi\|_{P^t,\hat{\chi}^{m+1}}).
\]

Assuming the claim, we can start from \( \|\psi\|_{P^t,\hat{\chi}^{m_0+1}} = \|\psi\|_{P^t,\hat{\chi}^{m_0+1}} \) and iterate \((3.24)\) for \( m = m_0, m_0 + 1, \ldots, m_1 \) (using the fact that each \( \widetilde{B}_m \) is non-decreasing) to conclude the proof.

To prove the claim, we use the hypothesis on \( \psi \) and \((3.19)\) to apply the parabolic smoothing estimate \((3.10)\), which gives

\[
\|\psi\|_{P^t,\hat{\chi}^{m+2}} \leq C_X(\ell - \ell_0 + 1/2)\|\psi\|_{L^p_0,\chi^{m+1}} + C_X(\|\psi\|_{P^t,\hat{\chi}^{m+2}} + B_m(\|\psi\|_{P^t,\hat{\chi}^{m+1}})),
\]

where we have used \( \lim \inf_{\ell_1 \to 0} \ell_1^p \|\psi(\ell_1)\|_{\chi^{m+1}(\ell_1)} = 0 \), which holds as \( \|\psi\|_{L^p_0,\chi^{m+1}} < \infty \).
The following statements hold.

(1) Let $\psi$ be a regular function defined on $\mathbb{R}^3 \times J$ (resp. on $I \times \mathbb{R}^3 \times J$), where $J$ is a finite interval. Then for $X = L_2^0$ (resp. $X = L_{1,2}^0$ or $\mathcal{S}_1$), we have

$$
\|\psi\|_{L^p_t X^{s,p}(J)} < \infty
$$

if either $1 \leq p \leq \infty$ and $\ell - \ell_0 + k/2 > 0$, or $p = \infty$ and $\ell - \ell_0 + k/2 = 0$.

(2) Furthermore, the norms $L_2^0$, $L_{1,2}^0$ and $\mathcal{S}_1$ satisfy the parabolic energy and smoothing estimates \ref{energy_estimate}, \ref{smoothing_estimate}.

Proof. By definition, we have

$$
\|\psi\|_{L^p_t X^{s,p}} = \|s^{\ell - \ell_0 + k/2}\|\partial_x^{(k)}\psi(s)\|_{X^s}\|_{L^{p}}.
$$

Since $\sup_{s \in J} \|\partial_x^{(k)}\psi(s)\|_X < \infty$ for each $X$ under consideration when $\psi$ is regular, the first statement follows.

To prove the second statement, let us begin by proving that the norm $L_2^0$ satisfies the parabolic energy estimate \ref{energy_estimate}. In this case, $\ell_0 = 3/4$. Let $\ell \in \mathbb{R}$, $[s_1, s_2] \subset (0, \infty)$ and $\psi$ a smooth (complex-valued) function such that $\|\psi\|_{p_{tX^{s,p}}^2} < \infty$. We may assume that $\|\partial_s - \Delta\psi\|_{L_t^{2+1} L_x^2} < \infty$, as the other case is trivial. Multiplying the equation $(\partial_s - \Delta)\psi$ by $s^{2(\ell - \ell_0)}\overline{\psi}$ and integrating by parts over $[s_1, s_2]$ (where $s_1 \leq s \leq s_2$), we obtain

$$
= \frac{1}{2} s_1^{2(\ell - \ell_0)} \int |\psi(s_1)|^2 \, ds + (\ell - \ell_0) \int_{s_1}^{s_2} s^{2(\ell - \ell_0) + 1} |\partial_x \psi|^2 \, ds
$$

Taking the supremum over $s_1 \leq s \leq s_2$ and rewriting in terms of p-normalized norms, we obtain

$$
= \frac{1}{2} \|\psi\|^2_{L_x^{2+1} L_t^{x=1/2} X^{s_1,s_2}} + \|\psi\|^2_{L_x^{2+1} H_t^{s_1,s_2}}
$$

By Hölder and Lemma \ref{holder}, we can estimate the last term by $\|\partial_s - \Delta\psi\|_{L_t^{2+1} L_x^2}^2$, where the latter can be absorbed into the left-hand side. Taking the square root of both sides, we obtain \ref{energy_estimate} for $L_2^0$.

Next, let us prove that the norm $L_2^0$ satisfies the parabolic smoothing estimate \ref{smoothing_estimate}. Let $\ell \in \mathbb{R}$, $[s_1, s_2] \subset (0, \infty)$ and $\psi$ a smooth (complex-valued) function such that $\|\psi\|_{p_{tX^{s,p}}^2} < \infty$. As before, we assume that $\|\partial_s - \Delta\psi\|_{L_t^{2+1} L_x^2} < \infty$. Multiplying the equation $(\partial_s - \Delta)\psi$ by $s^{2(\ell - \ell_0) + 1}\overline{\psi}$ and integrating by parts over $[s_1, s_2]$ (where $s_1 \leq s \leq s_2$), we obtain

$$
= \frac{1}{2} s_1^{2(\ell - \ell_0) + 1} \int |\partial_x \psi(s_1)|^2 \, ds + (\ell - \ell_0 + 1/2) \int_{s_1}^{s_2} s^{2(\ell - \ell_0) + 1} |\partial_x \psi|^2 \, ds
$$

The following proposition allows us to use Theorem \ref{main_result} in the situations of interest in our work.

**Proposition 3.12.** The following statements hold.

Using the smallness of $\epsilon > 0$, we can absorb $C_X\epsilon\|\psi\|_{p_{tX^{s,p}}^2}$ into the left-hand side. Then adding $\|\psi\|_{L_t^{2+1} X^{s,p}}$ to both sides, we easily obtain

$$
\|\psi\|_{L_t^{2+1} X^{s,p}} \leq (2C_X(\ell - \ell_0 + 1/2) + 1)\|\psi\|_{p_{tX^{s,p}}^2} + 2C_X\mathcal{B}_m(\|\psi\|_{p_{tX^{s,p}}^2}).
$$

Recalling the definition of $\mathcal{B}_m$, this is exactly \ref{main_result}. Theorem 3.11 in the situations of interest in our work.
By a further integration by parts, the second term on the left-hand side is equal to \( \| \psi \|^2_{L^2_t H^2_x(s_1,s_2)} \).

Taking the supremum over \( s_1 \leq s \leq s_2 \) and rewriting in terms of p-normalized norms, we obtain

\[
\frac{1}{2} \| \psi \|^2_{L^2_t H^2_x(s_1,s_2)} + \| \psi \|^2_{L^2_t L^p_x(s_1,s_2)} \leq \frac{1}{2} \| \psi \|^2_{L^2_t H^2_x(s_1,s_2)} + (\ell - \ell_0 + \frac{1}{2}) \| \psi \|^2_{L^2_t L^p_x(s_1,s_2)} + \| (\partial_s - \Delta) \psi \cdot \nabla k \psi \|_{L^2_t L^p_x(s_1,s_2)}.
\]

By Cauchy-Schwarz and Lemma 3.9, we can estimate the last term by

\[
(1/2) \| (\partial_s - \Delta) \psi \|^2_{L^{2+1/2}_t L^2_x(s_1,s_2)} + (1/2) \| \psi \|^2_{L^2_t H^2_x(s_1,s_2)}.
\]

where the latter can be absorbed into the left-hand side. Taking the square root of both sides, we obtain \([3,10]\) for \( L^2_t \).

For the norm \( L^1_t L^p_x \), in which case \( \ell_0 = 1 \), it simply suffices to repeat the above proof with the new value of \( \ell_0 \), and integrate further in time.

Finally, in the case of the norm \( S^1 \), for which \( \ell_0 = 1/4 \), we begin by observing that

\[
\| \psi \|_{L^p_t L^p_x} := \| \nabla_{t,x} \psi \|_{L^p_t L^p_x} + |I|^{1/2} \| s^{3/4} \Box \psi \|_{L^p_t L^p_x} \lesssim s^{1/2} \| \partial_{t,x} \psi (t = 0) \|_{L^p_t L^p_x} + |I|^{1/2} \| s^{3/4} \Box \psi \|_{L^p_t L^p_x}
\]

for every \( \ell \geq 0 \) and \( 1 \leq p \leq \infty \), where \( A \sim B \) means that \( A, B \) are comparable, i.e. there exist \( C > 0 \) such that \( A \leq CB, B \leq CA \). One direction is trivial, whereas the other follows from the energy estimate. Using furthermore the fact that \( \partial_{t,x} \Box \) commute with \( (\partial_s - \Delta) \), this case follows from the last two cases.

\[\square\]

Remark 3.13. From the proof of (3.20), it is evident that the following variant is also true:

Let \( \ell \in \mathbb{R} \), \( 1 \leq p \leq \infty \) and \( k \geq 1 \). For a smooth function \( \psi = \psi(t,x,s) \) such that \( \partial_{t,x} \psi \) is regular, we have \( \| \psi \|_{L^p_t G^k_s} < \infty \) if either \( 1 \leq p \leq \infty \) and \( \ell - 3/4 + k/2 > 0 \), or \( p = \infty \) and \( \ell - 3/4 + k/2 = 0 \).

Remark 3.14. A point that the reader should keep in mind is that, despite the heavy notations and abstract concepts developed in this subsection, the analytic heart of the ‘abstract parabolic theory’ is simply the standard \( L^2 \)-energy integral estimates for the linear heat equation, as we have seen in Proposition 3.12.

The efforts that we had put in this subsection will pay off in various parts below (in particular Sections 5 and 8), as it will allow us to treat the diverse parabolic equations which arise in a unified, economical way.

3.5. Correspondence principle for p-normalized norms. In this subsection, we develop a systematic method of obtaining linear and multi-linear estimates in terms of p-normalized norms, which will be very useful to us later. The idea is to start with an estimate involving the norms of functions independent of the \( s \)-variable, and arrive at the corresponding estimate for \( s \)-dependent functions in terms of the corresponding p-normalized norms by putting appropriate weights of \( s \).

Throughout this subsection, we will denote by \( J \subset (0,\infty) \) an \( s \)-interval, \( \phi_i = \phi_i(s) \) a smooth function independent of \( s \), and \( \psi_i = \psi_i(s,x) \) a smooth function of both \( s \in J \) and \( x \). All norms below will be assumed \( a \) priori to be finite. In application, \( \phi_i \) may be usually taken to be Schwartz in \( x \), and either \( \psi \) or \( \partial_x \psi \) would be regular. The discussion to follow holds also for functions which depend additionally on \( t \).

It is rather cumbersome to give a precise formulation of the Correspondence Principle. We will instead adopt a more pragmatic approach and be satisfied with the following ‘cookbook-recipe’ type statement.

**Correspondence Principle.** Suppose that we are given an estimate (i.e. an inequality) in terms of the norms \( X_i \) of functions \( \phi_i = \phi_i(x) \), all of which are homogeneous. Suppose furthermore that the estimate is scale-invariant, in the sense that both sides transform the same under scaling.

Starting from the usual estimate, make the following substitutions on both sides: \( \phi_i \to \psi_i(s), \partial_s \to \nabla_x(s), X_i \to X_i(s) \). Then the resulting estimate still holds, with the same constant, for every \( s \in J \).
In other words, given an $s$-independent, scale-invariant estimate which involve only homogeneous norms, we obtain its $p$-normalized analogue by replacing the norms and the derivatives by their respective $p$-normalizations. The ‘proof’ of this principle is very simple: For each fixed $s$, the substitution procedure above amounts to applying the usual estimate to $\psi_i(s)$ and multiplying each side by an appropriate weight of $s$. The point is that the same weight works for both sides, thanks to scale-invariance of the estimate that we started with.

**Example 3.15.** Some examples are in order to clarify the use of the principle. We remark that all the estimates below will be used freely in what follows.

1. **(Sobolev)** We begin with the Sobolev inequality (3.1) from Lemma 3.1. Applying the Correspondence Principle, for every $1 \leq q \leq r$, $k \geq 0$ such that $\frac{\alpha'}{q} = \frac{\alpha}{r} - k$, we obtain
   \[ \| \psi(s) \|_{L^r(s)} \leq C \| \psi(s) \|_{L^r_{\alpha}(s)}, \]
   for every $s \in J$.

2. **(Interpolation)** Recall the interpolation inequality (3.2) from Lemma 3.1. Applying the Correspondence Principle, for $1 \leq q < \infty$, $k_1 \leq k_0 \leq k_2$, $0 < \theta_1, \theta_2 < 1$ such that $\theta_1 + \theta_2 = 1$ and $k_0 = \theta_1 k_1 + \theta_2 k_2$, we obtain
   \[ \| \psi(s) \|_{W^{k_0,q}(s)} \leq C \| \psi(s) \|_{W^{k_1,q}(s)}^{\theta_1} \| \psi(s) \|_{W^{k_2,q}(s)}^{\theta_2}, \]
   for every $s \in J$.

3. **(Gagliardo-Nirenberg)** Let us apply the Correspondence Principle to the Gagliardo-Nirenberg inequality (3.3) from Lemma 3.1. Then for $q \leq q_1, q_2 \leq \infty$, $0 < \theta_1, \theta_2 < 1$ such that $\frac{\alpha'}{r} = \theta_1 (\frac{\alpha}{q_1} \frac{r}{q_1} + \frac{\alpha}{q_2} \frac{r}{q_2} - 1)$, we obtain
   \[ \| \psi(s) \|_{L^r_{\alpha}(s)} \leq C \| \psi(s) \|_{L^r_{\alpha}(s)}^{\theta_1} \| \nabla_x \psi(s) \|_{L^r_{\alpha}(s)}^{\theta_2}, \]
   for every $s \in J$.

4. **(Hölder)** Let us start with $\| \phi_1 \phi_2 \|_{L^r_{\alpha}(s)} \leq \| \phi_1 \|_{L^{r_1}_{\alpha}(s)} \| \phi_2 \|_{L^{r_2}_{\alpha}(s)}$, where $1 \leq q_1, q_2, r \leq \infty$ and $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2}$. Applying the Correspondence Principle, for every $s \in J$, we obtain
   \[ \| \psi_1 \psi_2(s) \|_{L^r_{\alpha}(s)} \leq \| \psi_1(s) \|_{L^{r_1}_{\alpha}(s)} \| \psi_2(s) \|_{L^{r_2}_{\alpha}(s)}. \]

All the estimates above extend to functions on $I \times \mathbb{R}^3$ with $I \subset \mathbb{R}$ in the obvious way. In this case, we have the following analogue of the Hölder inequality:

\[ \| \psi(s) \|_{L^r_{\alpha}(s)} \leq s^{-\frac{1}{r_1} - \frac{1}{r_2}} \| I \|_{\frac{1}{r_1} - \frac{1}{r_2}} \| \psi(s) \|_{L^r_{\alpha}(s)}^{r_1} \| \psi(s) \|_{L^r_{\alpha}(s)}^{r_2}, \quad \text{for } 1 \leq q_1 \leq q_2. \]

The following consequence of the Gagliardo-Nirenberg and Sobolev inequalities is useful enough to be separated as a lemma on its own. It provides a substitute for the incorrect $H^{3/2}_x \subset L^r_x$ Sobolev embedding, and has the benefit of being scale-invariant. We will refer to this simply as Gagliardo-Nirenberg for $p$-normalized norms.

**Lemma 3.16 (Gagliardo-Nirenberg).** For every $s \in J$, the following estimate holds.

\[ \| \nabla_x^{(k)} \psi(s) \|_{H^{3/2}_{r,s}(s)} \leq C_k \| \psi(s) \|_{H^{3/2}_{r,s}(s)}^{1/2} \| \psi(s) \|_{H^k_{r,s}(s)}^{1/2}, \]

(3.28)

\[ \leq C_k \| \psi(s) \|_{H^{3/2}_{r,s}(s)} + \| \psi(s) \|_{H^k_{r,s}(s)}. \]

**Proof.** Without loss of generality, assume $k = 0$. To prove the first inequality, by Gagliardo-Nirenberg, interpolation and the Correspondence Principle, it suffices to prove $\| \phi \|_{L^r_x} \leq C \| \phi \|_{H^1_x}$ and $\| \partial_x \phi \|_{L^r_x} \leq C \| \phi \|_{H^2_x}$; the latter two are simple consequences of Sobolev. Next, the second inequality follows from the first by Cauchy-Schwarz.

We remark that in practice, the Correspondence Principle, after multiplying by an appropriate weight of $s$ and integrating over $J$, will often be used in conjunction with Hölder’s inequality for the spaces $L^{r,p}_x$ (Lemma 3.9).

Finally, recall that the notation $O(\psi_1, \psi_2, \ldots, \psi_k)$ refers to a linear combination of expressions in the values of the arguments $\psi_1, \psi_2, \ldots, \psi_k$, where they could in general be vector-valued. It
therefore follows immediately that any multi-linear estimate for the usual product $\|\psi_1 \cdot \psi_2 \cdots \psi_k\|$ for scalar-valued functions $\psi_1, \psi_2, \ldots, \psi_k$ implies the corresponding estimate for $\|\mathcal{O}^{(s)}(\psi_1, \psi_2, \ldots, \psi_k)\|$, where $\psi_1, \psi_2, \ldots, \psi_k$ may now be vector-valued, at the cost of some absolute constant depending on $\mathcal{O}$. This remark will be used repeatedly in the sequel.

3.6. Associated $s$-weights for variables of (HPYM). Let us consider the system (HPYM), introduced in §1.4. To each variable of (HPYM), there is associated a power of $s$ which represents the expected size of the variable in a dimensionless norm (say inverse of the associated $1$).

The associated $s$-weights for the ‘spatial variables’ $A = A_t, F = F_{ij}, F_s = F_{si}$ are derived directly from scaling considerations, and as such easy to determine. Indeed, as we expect that $\|\partial_x A_t\|_{L^2_x}$ should stay bounded for every $t, s$, using the scaling heuristics $\partial_x \sim s^{-1/2}$ and $L^2_x \sim s^{3/4}$, it follows that $A_t \sim s^{-1/4}$. The worst term in $F_{ij}$ is at the level of $\partial_x A_t$, so $F_{ij} \sim s^{-3/4}$, and similarly $F_{si} \sim s^{-5/4}$.

The associated $s$-weights for $w_\nu$ is $s^{-1}$, which is actually better than that which comes from scaling considerations (which is $s^{-5/4}$). To see why, observe that $w_\nu$ satisfies a parabolic equation $(\partial_t - \Delta)w_\nu = (w_\nu)N$ with zero data at $s = 0$. Duhamel’s principle then tells us that $w_\nu \sim s^{(w_\nu)N}$. Looking at the equation (1.15), we see that $(w_\nu)N \sim s^{-2}$, from which we conclude $w_\nu \sim 1$. Note that as $w_0 = -F_{0\nu}$, this shows that the ‘temporal variables’ $A_0, F_{0\nu}$ behave better than their ‘spatial’ counterparts.

We summarize the associated $s$-weights for important variables as follows.

\[
\begin{align*}
A_t & \sim s^{-1/4} & A_0 & \sim s^0 & F_{\mu\nu} & \sim s^{-3/4} \\
F_{si} & \sim s^{-5/4} & F_{0\nu} & \sim s^{-1} & w_\mu & \sim s^{-1}.
\end{align*}
\]

Accordingly, when we control the sizes of these variables, they will be weighted by the inverse of their respective associated weights.

As we always work on a finite $s$-interval $J$ such that $J \subset [0, 1]$, extra powers of $s$ compared to the inverse of the associated $s$-weight should be considered favorable when estimating. For example, it is easier to estimate $\|A_t\|_{L^{1/4+\ell}x, H^3}$ when $\ell > 0$ than $\ell = 0$. (Compare Lemma 8.2 with Proposition 7.2). Informally, when it suffices to control a variable with more power of $s$, say $s^q$, compared to the associated $s$-weight, we will say that there is an extra $s$-weight of $s^q$. Thanks to the sub-critical nature of the problem, such extra weights will be abundant, and this will simplify the analysis in many places.

It is also useful to keep in mind the following heuristics.

\[
\partial_{t,x} D_t, x \sim s^{-1/2}, \quad \partial_s, D_s \sim s^{-1}, \quad L^q L^r_2 \sim s^{1/(2q) + 3/(2r)}
\]

4. Reduction of the Main Theorem to Theorems A and B

In the first subsection, we state and prove some preliminary results that we will need in this section. These include a $H^2$ local well-posedness statement for the Yang-Mills equations in the temporal gauge, an approximation lemma for the initial data and a gauge transform lemma. Next, we will state Theorems A (Estimates for the initial data in the caloric-temporal gauge) and B (Estimates for $t$-evolution in the caloric-temporal gauge), and show that the proof of the Main Theorem is reduced to that of Theorems A and B by a simple bootstrap argument involving a gauge transformation.

4.1. Preliminary results. We will begin this subsection by making a number of important definitions. Let us define the notion of regular solutions, which are smooth solutions with appropriate decay towards the spatial infinity.

**Definition 4.1** (Regular solutions). We say that a representative $A_\mu$ of a classical solution to (YM) is regular if $A_\mu$ is smooth and $\partial_{t,x} A_\mu$ is regular. Furthermore, we say that a smooth solution $A_s$ to (HPYM) is regular if $A_s$ is smooth and $\partial_{t,x} A_\mu, A_s$ are regular.

---

15We remind the reader, that this is a consequence of the original Yang-Mills equations $D^\mu F_{\mu\nu} = 0$ at $s = 0$. 
In relation to regular solutions, we also define the notion of a regular gauge transform, which is basically that which keeps the ‘regularity’ of the connection 1-form.

**Definition 4.2** (Regular gauge transform). We say that a gauge transform $U$ on $I \times \mathbb{R}^3 \times J$ is a regular gauge transform if $U$, $U^{-1}$ are smooth and furthermore

$$U, U^{-1} \in C^\infty_{t,x}(I \times J, L^\infty_x), \quad \partial_{t,x} U, \partial_{t,x} U^{-1} \in C^\infty_{t,x}(I \times J, L^3_x), \quad \partial_{t,x}^{(2)} U, \partial_{t,x}^{(2)} U^{-1} \in C^\infty_{t,x}(I \times J, H^\infty_x).$$

A gauge transform $U$ defined on $I \times \mathbb{R}^3$ is a regular gauge transform if it is a regular gauge transform viewed as an s-independent gauge transform on $I \times \mathbb{R}^3 \times J$ for some $J \subset [0, \infty)$.

We remark that a regular solution (whether to (YM) or (HPYM)) remains regular under a regular gauge transform.

Let us also give the definition of regular initial data sets for (YM).

**Definition 4.3** (Regular initial data sets). We say that an initial data set $(\overline{A}_i, \overline{E}_i)$ to (YM) is regular if, in addition to satisfying the constraint equation (1.1), $\overline{A}_i, \overline{E}_i$ are smooth and $\partial_x \overline{A}_i, \overline{E}_i$ are regular.

Next, let us present some results needed to prove the Main Theorem. The first result we present is a local well-posedness result for initial data with higher regularity. For this purpose, we have an $H^2$ local well-posedness theorem, which is essentially due to Eardley-Moncrief [9]. However, as we do not assume anything on the $L^2$ norm of the initial data $\overline{A}_i$ (in particular, it does not need to belong to $L^2$), we need a minor variant of the theorem proved in [9].

In order to state the theorem, let us define the space $\tilde{H}^2_3$ to be the closure of $S_\mu(\mathbb{R}^3)$ with respect to the partially homogeneous Sobolev norm $\| \phi \|_{\tilde{H}^2_3} := \| \partial_x \phi \|_{H^1_3}$. The point, of course, is that this norm\(^{16}\) does not contain the $L^2$ norm.

**Theorem 4.4** ($H^2$ local well-posedness of Yang-Mills). Let $(\overline{A}_i, \overline{E}_i)$ be an initial data set satisfying (1.1) such that $\partial_x \overline{A}_i, \overline{E}_i \in H^1_3$.

1. There exists $T = T(||(\overline{A}, \overline{E})||_{\tilde{H}^2_3 \times H^1_3}) > 0$, which is non-increasing in $||(\overline{A}, \overline{E})||_{\tilde{H}^2_3 \times H^1_3}$, such that a unique solution $A_\mu$ to (YM) in the temporal gauge satisfying

\[
A_i \in C_t((-T, T), \tilde{H}^2_3) \cap C^1_t((-T, T), H^1_3)
\]

exists on $(-T, T) \times \mathbb{R}^3$.

2. Furthermore, persistence of higher regularity holds, in the following sense: If $\partial_x A, \overline{E} \in H^m_3$ (for an integer $m \geq 1$), then the solution $A_\mu$ obtained in Part (1) satisfies $A_i \in C^m_t((-T, T), \tilde{H}^2_3)$ for non-negative integers $k_1, k_2$ such that $k_1 + k_2 \leq m$.

In particular, if $(\overline{A}_i, \overline{E}_i)$ is a regular initial data set, then the corresponding solution $A_\mu$ is a regular solution to (YM) in the temporal gauge.

3. Finally, we have the following continuation criterion: If $\sup_{t \in (-T, T')} ||\partial_{t,x} A||_{H^1_3} < \infty$, then the solution given by Part (1) can be extended past $(-T', T')$, while retaining the properties stated in Parts (1) and (2).

**Proof.** It is not difficult to see that the iteration scheme introduced in Klainerman-Machedon [13] Proposition 3.1] goes through with the above norm, from which Parts (1) – (3) follow. A cheaper way of proving Theorem 4.4 is to note that $||\overline{A}_i||_{\tilde{H}^2_3(\mathbb{R}^3)} \leq C||\overline{A}_i||_{\tilde{H}^2_3(\mathbb{R}^3)}$, $||\overline{E}_i||_{H^1_3(\mathbb{R}^3)} \leq ||\overline{E}_i||_{H^1_3(\mathbb{R}^3)}$ uniformly for all unit balls in $\mathbb{R}^3$. This allows us to apply the localized local well-posedness statement Proposition 3.1 of [13] to each ball, and glue these local solutions to form a global solution via a domain of dependence argument.

Next, we prove a technical lemma, which shows that an arbitrary admissible $H^1$ initial data set can be approximated by a sequence of regular initial data sets.

---

\(^{16}\)That $\| \cdot \|_{\tilde{H}^2_3}$ is indeed a norm when restricted to $\tilde{H}^2_3$ follows from Sobolev.
Lemma 4.5 (Approximation lemma). Any admissible $H^1$ initial data set $(\mathcal{A}_i, \mathcal{E}_i) \in (\dot{H}^1_x \cap L^6_x) \times L^2_x$ can be approximated by a sequence of regular initial data sets $(\mathcal{A}_{(n)i}, \mathcal{E}_{(n)i})$ satisfying the constraint equation \((1)\). More precisely, the initial data sets $(\mathcal{A}_{(n)i}, \mathcal{E}_{(n)i})$ may be taken to satisfy the following properties.

1. $\mathcal{A}_{(n)i}$ is smooth, compactly supported, and
2. $\mathcal{E}_{(n)i}$ is regular, i.e. $\mathcal{E}_{(n)i} \in H^k_x$ for every integer $k \geq 0$.

Proof. This proof can essentially be read off from [13, Proposition 1.2]. We reproduce it below for the convenience of the reader.

Choose compactly supported, smooth sequences $\mathcal{A}_{(n)i}, \mathcal{E}_{(n)i}$ such that $\mathcal{A}_{(n)i} \to \mathcal{A}_i$ in $\dot{H}^1_x \cap L^6_x$ and $\mathcal{E}_{(n)i} \to \mathcal{E}_i$ in $L^2_x$. Let us denote the covariant derivative associated to $\mathcal{A}_{(n)}$ by $D_{(n)}$. Using the fact that $(\mathcal{A}_i, \mathcal{E}_i)$ satisfies the constraint equation \((1)\) in the distributional sense and the $H^1_x \subset L^6_x$ Sobolev, we see that for any test function $\varphi$,

$$|\int (D'_{(n)} \mathcal{F}_{(n)i}, \varphi) \, dx| = |\int (D'_{(n)} \mathcal{F}_{(n)i} - D' \mathcal{E}_i, \varphi) \, dx|$$

$$= |\int - (\mathcal{F}_{(n)i} - \mathcal{E}_i, D' \varphi) + (\mathcal{A}_{(n)} - \mathcal{A}, \mathcal{F}_{(n)i}) + (\mathcal{A}', \mathcal{F}_{(n)i} - \mathcal{E}_i, \varphi) \, dx|$$

$$\leq \left( \|\mathcal{F}_{(n)i} - \mathcal{E}_i\|_{L^2_x} + \|\mathcal{A}_{(n)} - \mathcal{A}\|_{L^3_x} \|\mathcal{F}_{(n)i}\|_{L^2_x} + \|\mathcal{A}\|_{L^3_x} \|\mathcal{F}_{(n)i} - \mathcal{E}_i\|_{L^2_x} \right) \|\varphi\|_{\dot{H}^{-1}_x}.$$ 

In view of the $L^2_x, L^6_x$ convergence of $\mathcal{A}_{(n)i}, \mathcal{F}_{(n)i}$ to $\mathcal{A}, \mathcal{E}$, respectively, it follows that

$$D'_{(n)} \mathcal{F}_{(n)i} \in \dot{H}^{-1}_x$$

for each $n$, $\|D'_{(n)} \mathcal{F}_{(n)i}\|_{\dot{H}^{-1}_x} \to 0$ as $n \to \infty$,

where $\dot{H}^{-1}_x$ is the dual space of $\dot{H}^1_x$ (defined to be the closure of Schwartz functions on $\mathbb{R}^3$ under the $\dot{H}^1_x$-norm).

Let us now define $\mathcal{F}_{(n)i} := \mathcal{F}_{(n)i} + D_{(n)i} \phi_{(n)}$, where the $g$-valued function $\phi_{(n)}$ is constructed by solving the elliptic equation \((4.2)\)

$$D'_{(n)} D_{(n)i} \phi_{(n)} = -D'_{(n)} \mathcal{F}_{(n)i},$$

imposing a suitable decay condition at infinity; we want, in particular, to have $\phi_{(n)} \in \dot{H}^1_x \cap L^6_x$. This ensures that $(\mathcal{A}_{(n)i}, \mathcal{E}_{(n)i})$ satisfies the constraint equation. Furthermore, in view of the fact that $\mathcal{A}_{(n)}, \mathcal{F}_{(n)}$ are smooth and compactly supported, it is clear that $D_{(n)} \phi_{(n)}$ belongs to any $H^k_x$ for $k \geq 0$, and hence so does $\mathcal{E}_{(n)}$. Therefore, in order to prove the lemma, it is only left to prove $\mathcal{D}_{(n)} \phi_{(n)} \to 0$ in $L^2_x$.

Multiplying \((4.2)\) by $\phi_{(n)}$ and integrating by parts, we obtain

$$\int |D_{(n)} \phi_{(n)}|^2 \, dx \leq \|D'_{(n)} \mathcal{F}_{(n)i}\|_{\dot{H}^{-1}_x} \|\phi_{(n)}\|_{\dot{H}^1_x}.$$ 

On the other hand, expanding out $D_{(n)}$, we have

$$\|\phi_{(n)}\|_{\dot{H}^1_x} \leq \|D_{(n)} \phi_{(n)}\|_{L^2_x} + \|\mathcal{A}_{(n)}\|_{L^3_x} \|\phi_{(n)}\|_{L^6_x}.$$ 

Recall Kato’s inequality (for a proof, see [19, Lemma 4.2]), which shows that $|\partial t |\phi_{(n)}| \leq |D_{(n)}\phi_{(n)}|$ in the distributional sense. Combining this with the $\dot{H}^1_x \subset L^6_x$ Sobolev inequality for $|\phi_{(n)}|$, we get

$$\|\phi_{(n)}\|_{L^6_x} \leq C \|D_{(n)} \phi_{(n)}\|_{L^2_x}.$$ 

Combining \((4.3)\) - \((4.5)\) and canceling a factor of $\|D_{(n)} \phi_{(n)}\|_{L^2_x}$, we arrive at

$$\|D_{(n)} \phi_{(n)}\|_{L^2_x} \leq \|D'_{(n)} \mathcal{F}_{(n)i}\|_{\dot{H}^{-1}_x} (1 + C \|\mathcal{A}_{(n)}\|_{L^2_x}) \to 0,$$

as desired. $\square$

Given a time interval $I \subset \mathbb{R}$, we claim the existence of norms $A_0(I)$ and $\delta A_0(I)$ for $A_0$ and $\delta A_0$ on $I$, respectively, for which the following lemma holds. The significance of these norms will be that they can be used to estimate the gauge transform back to the original temporal gauge.
Lemma 4.6 (Estimates for gauge transform to temporal gauge). Consider the following ODE on \((-T, T) \times \mathbb{R}^3:\)

\[
\begin{cases}
\partial_t V = VA_0 \\
V(t = 0) = \overline{V},
\end{cases}
\]

where we assume that \(A_0\) is smooth and \(A_0(-T, T) < \infty.\)

(1) Suppose that \(\overline{V} = \overline{V}(x)\) is a smooth \(\mathfrak{G}\)-valued function on \(\{t = 0\} \times \mathbb{R}^3\) such that

\[
\nabla \overline{V}, \overline{V}^{-1} \in L^\infty_T, \quad \partial_x \nabla, \partial_x \overline{V}^{-1} \in L^3_T, \quad \partial_x^2 \nabla, \partial_x^2 \overline{V}^{-1} \in L^2_T.
\]

Then there exists a unique solution \(V\) to the ODE, which obeys the following estimates.

\[
\begin{aligned}
\|V\|_{L^\infty_T L^\infty_T(-T,T)} &\leq C_{A_0(-T,T)} \cdot \|\overline{V}\|_{L^\infty_T}, \\
\|\partial_t V\|_{L^2_T L^3_T(-T,T)} &\leq C_{A_0(-T,T)} \cdot (\|\partial_x \overline{V}\|_{L^2_T} + A_0(-T,T) \|\overline{V}\|_{L^\infty_T}), \\
\|\partial_x^2 V\|_{L^2_T L^3_T(-T,T)} &\leq C_{A_0(-T,T)} \cdot (\|\partial_x^2 \overline{V}\|_{L^2_T} + A_0(-T,T) \|\overline{V}\|_{L^\infty_T}).
\end{aligned}
\]  

(2) Let \(A'_0\) be a smooth connection coefficient with \(A'_0(-T, T) < \infty,\) and \(\overline{V}\) a \(\mathfrak{G}\)-valued smooth function on \(\{t = 0\} \times \mathbb{R}^3\) also satisfying the hypotheses of (1). Let \(V'\) be the solution to the ODE (4.6) with \(A_0\) and \(\overline{V}\) replaced by \(A'_0, \overline{V}'\), respectively. Then the difference \(\delta V := V - V'\) satisfies the following estimates.

\[
\begin{aligned}
\|\delta V\|_{L^3_T L^\infty_T(-T,T)} + \|\partial_t \delta V\|_{L^3_T L^3_T(-T,T)} + \|\partial_x \partial_t \delta V(0)\|_{L^3_T} \\
&\leq C_{A_0(-T,T)} \cdot (\|\delta \overline{V}\|_{L^\infty_T} + \|\partial_x \overline{\delta V}\|_{L^3_T} + \|\partial_x \partial_t \overline{V}\|_{L^3_T}) \\
&\quad + C_{A_0(-T,T)} \cdot (\|\overline{V}\|_{L^\infty_T} + \|\partial_x \overline{\partial_t V}\|_{L^3_T} + \|\partial_x \partial_t \partial_x \overline{V}\|_{L^3_T}) \delta A_0(-T,T).
\end{aligned}
\]

(3) Finally, all of the above statement remain true with \(V, \delta V, \overline{V}, \overline{V}^{-1}\) replaced by \(V^{-1}, \delta V^{-1}, V^{-1}, \delta V^{-1}\), respectively.

The precise definition of \(A_0, \delta A_0\) will be given in [7,1] whereas we defer the proof of Lemma 4.6 until Appendix [3].

Next, we prove a simple lemma which will be used to estimate the \(L^2_T\) norm of our solution.

Lemma 4.7. Let \(\psi = \psi(t, x)\) be a function defined on \((-T, T) \times \mathbb{R}^3\) such that \(\psi(0) \in L^3_T\) and \(\partial_t \psi \in C^1_T L^3_T\). Then \(\psi \in C^1_T L^3_T\) and the following estimate holds.

\[
\sup_{t \in (-T, T)} \|\psi(t)\|_{L^3_T} \leq \|\psi(0)\|_{L^3_T} + CT^{1/2} \|\partial_t \psi\|_{L^\infty_T L^3_T}.
\]

Proof. By a standard approximation procedure, it suffices to consider \(\psi = \psi(t, x)\) defined on \((-T, T) \times \mathbb{R}^3\) which is smooth in time and Schwartz in space. For \(t \in (-T, T),\) we estimate via Hölder, Sobolev and the fundamental theorem of calculus as follows:

\[
\begin{aligned}
\|\psi(t) - \psi(0)\|_{L^3_T} &\leq \|\psi(t) - \psi(0)\|_{L^1_T}^{1/2} \|\psi(t) - \psi(0)\|_{L^3_T}^{1/2} \\
&\leq C \int_0^t \|\partial_t \psi(t')\|_{L^3_T} dt' \|\psi(0)\|_{L^1_T}^{1/2} \|\partial_x \psi(0)\|_{L^3_T}^{1/2} \\
&\leq CT^{1/2} \|\partial_t \psi\|_{L^\infty_T L^3_T}^{1/2} \|\partial_x \psi\|_{L^\infty_T L^3_T} \|\partial_x \psi(0)\|_{L^3_T} \leq CT^{1/2} \|\partial_t \psi\|_{L^\infty_T L^3_T} \|\partial_x \psi\|_{L^\infty_T L^3_T}.
\end{aligned}
\]

By the triangle inequality, (4.9) follows.

\[
\square
\]

4.2. Reduction of the Main Theorem. For \(A_a, A_a'\) regular solutions to (HPYM) (defined in [1,3] on \(I \times \mathbb{R}^3 \times [0, 1]\), we claim the existence of norms \(I\) and \(\delta I\) which measure the sizes of \(A_a\) and \(\delta A_a,\) respectively, at \(t = 0\) (i.e. the size of the initial data), such that the theorems below hold. The precise definitions will be given in Section [6].
Theorem A (Estimates for initial data in the caloric-temporal gauge). Let $0 < T \leq 1$, and $A^0_\mu$ a regular solution to the Yang-Mills equation in the temporal gauge $A^0_\mu$ on $(-T, T) \times \mathbb{R}^3$ with the initial data $(\overline{A}_i, \overline{E}_i)$ at $t = 0$. Define $\overline{A} := ||\overline{A}||_{H^1_x} + ||\overline{E}||_{L^2_x}$. Suppose that

$$\sup_{t \in (-T, T)} \sup_i ||A^1_\mu(t)||_{H^1_x} < \delta_\mu,$$

where $\delta_\mu$ is a small constant to be introduced in Proposition 2.8. Then the following statements hold:

1. There exists a regular gauge transform $V = V(t, x)$ on $(-T, T) \times \mathbb{R}^3$ and a regular solution $A_\mu$ to (HPYM) on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$ such that

$$A_\mu(s = 0) = V(A^0_\mu)V^{-1} - \partial_s V V^{-1}.$$

2. Furthermore, the solution $A_\mu$ satisfies the caloric-temporal gauge condition, i.e. $A_\mu = 0$ everywhere and $\overline{A}_0 = 0$.

3. Let $(A')^1_\mu$ be another regular solution to the Yang-Mills equation in the temporal gauge with the initial data $(\overline{A}_i', \overline{E}_i')$ satisfying $\|\overline{A}_i, \overline{E}_i\|_{H^1_x \times L^2_x} \leq \overline{T}$ and (4.10). Let $A'_\mu$ be the solution to (HPYM) in the caloric-temporal gauge obtained from $(A')^1_\mu$ as in Parts (1) and (2). Then the following initial data estimates hold:

$$\overline{T} \leq C_T : \overline{T}, \quad \delta \overline{T} \leq C_T : \delta \overline{T},$$

where $\delta \overline{T} := ||\delta \overline{A}||_{H^1_x} + ||\delta \overline{E}||_{L^2_x}$.

4. Let $V'$ be the gauge transform obtained from $(A')^1_\mu$ as in Part (1), and let us write $\overline{V} := V(t = 0)$, $\overline{V'} := V'(t = 0)$. For the latter two gauge transforms, the following estimates hold:

$$\|\partial_s V\|_{L^2_x} \leq C_T : \overline{T}, \quad \|\partial_s V\|_{L^2_x} + \|\partial_x (\delta V)\|_{L^2_x} \leq C_T : \delta \overline{T}.$$

The same estimates with $V$ and $\delta V$ replaced by $V^{-1}$ and $\delta V^{-1}$, respectively, also hold.

Theorem B (Estimates for t-evolution in the caloric-temporal gauge). Let $0 < T \leq 1$, and $A_\mu$ a regular solution to the hyperbolic-parabolic Yang-Mills system (HPYM) on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$ in the caloric-temporal gauge. Then there exists $\delta_H > 0$ such that if

$$\overline{T} < \delta_H,$$

then the following estimate holds.

$$\sup_{0 \leq s \leq 1} \|\partial_{t,x} A_\mu(s)\|_{C^0((-T, T), L^2_x)} + A_0(-T, T) \leq C \overline{T}.$$

Also, if $A'_\mu$ is an additional solution to (HPYM) on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$ in the caloric-temporal gauge which also satisfies (4.15), then the following estimate for the difference holds as well:

$$\sup_{0 \leq s \leq 1} \|\partial_{t,x} A_\mu - \partial_{t,x} A'_\mu(s)\|_{C^0((-T, T), L^2_x)} + \delta A_0(-T, T) \leq C_T : \delta \overline{T}.$$

Our goal is to prove the Main Theorem, assuming Theorems A and B.

Proof of the Main Theorem. In view of Lemma 4.5 (approximation lemma) and the fact that we are aiming to prove the difference estimates (1.3) and (1.6), we will first consider initial data sets $(\overline{A}_i, \overline{E}_i)$ which are regular in the sense of Definition 4.3. Also, for the purpose of stating the estimates for differences, we will consider an additional regular initial data set $(\overline{A}_i', \overline{E}_i')$. The corresponding solution will be also marked by a prime. The statements in this proof concerning a solution $\overline{A}$ should be understood as being applicable to both $A$ and $A'$.

Observe that $\overline{T}$ does not contain the $L^2_x$ norm of $\overline{A}$, and has the scaling property.

$$\overline{T} \rightarrow \lambda^{-1/2} \overline{T}$$
under the scaling of the Yang-Mills equation (1.12). This allows us to treat the ‘local-in-time, large-data’ case on an equal footing as the ‘unit-time, small-data’ case. More precisely, we will assume by scaling that $T$ is sufficiently small, and prove that the solution to the Yang-Mills equation exists on the time interval $(-1, 1)$. Unravelling the scaling at the end, the Main Theorem will follow. We remark that the length of the time interval of existence obtained by this method will be of size $\sim \|\mathbf{A}, \mathcal{I}\|_{L^2_t \times L^2_x}^2$.

Using Theorem 4.4 we obtain a unique solution $A^1_\mu$ to the hyperbolic Yang-Mills equation (YM) under the temporal gauge condition $A^0_\mu = 0$. We remark that this solution is regular by persistence of regularity. Let us denote by $E$ the largest number $T > 0$ such that the solution $A^1_\mu$ exists smoothly on $(-T, T) \times \mathbb{R}^3$, and furthermore satisfies the following estimates for some $B > 0$ and $C_T > 0$:

$$\left\{ \begin{array}{l} \|\partial_{t,x} A^1_\mu\|_{C_t((-T, T), L^2_x)} \leq B\mathcal{I}, \\ \|\partial_{t,x} A^1_\mu - \partial_{t,x} (A^1_\mu)\|_{C_t((-T, T), L^2_x)} \leq C_T \cdot \delta \mathcal{I}, \\ \|A^1_\mu - (A^1_\mu)\|_{C_t((-T, T), L^2_x)} \leq C_T \cdot \|\mathcal{I}\|_{L^2_x} \cdot \delta \mathcal{I} + C_T \cdot \|\mathbf{A} - \mathcal{I}\|_{L^2_x}. \end{array} \right. \tag{4.18}$$

The goal is to show that $T_1 \geq 1$, provided that $\mathcal{I} > 0$ is small enough.

We will proceed by a bootstrap argument. In view of the continuity of the norms involved, the inequalities (4.18) are satisfied for $T > 0$ sufficiently small if $B \geq 2$ and $C_T \geq 2$, say. Next, we claim that if we assume

$$\|\partial_{t,x} A^1_\mu\|_{C_t((-T, T), L^2_x)} \leq 2B\mathcal{I}. \tag{4.19}$$

then we can recover (4.18) by assuming $\mathcal{I}$ to be small enough and $T \leq 1$.

Assuming the claim holds, let us first complete the proof of the Main Theorem. Indeed, suppose that (4.18) holds for some $0 \leq T < 1$. Applying the difference estimate in (4.18) to infinitesimal translations of $\mathbf{A}, \mathcal{I}$ and using the translation invariance of the Yang-Mills equation, we obtain

$$\|\partial_{t,x} A^1_\mu\|_{C_t((-T, T), L^2_x)} \leq 2B\mathcal{I}. \tag{4.18}$$

This, in turn, allows us to apply Theorem 4.4 ($H^2$ local well-posedness) to ensure that the solution $A^1_\mu$ extends uniquely as a regular solution to a larger time interval $(-T - \epsilon, T + \epsilon)$ for some $\epsilon > 0$. Taking $\epsilon > 0$ smaller if necessary, we can also ensure that the bootstrap assumption (4.19) holds and $T + \epsilon \leq 1$. This, along with the claim, allows us to set up a continuity argument to show that a regular solution $A^1_\mu$ exists uniquely on the time interval $(-1, 1)$ and furthermore satisfies (4.18) with $T = 1$. From (4.18), the estimates (1.3) - (1.6) follow immediately, for regular initial data sets. Then by Lemma 4.5 and the difference estimates (1.5) and (1.6), these results are extended to admissible initial data sets and solutions, which completes the proof of the Main Theorem [3].

Let us now prove the claim. Assuming $2B\mathcal{I} < \delta_p$, we can apply Theorem A. This provides us with a regular gauge transform $V$ and a regular solution $A_0$ to (HPYM) satisfying the caloric-temporal gauge condition, along with the following estimates at $t = 0$:

$$\|\nabla\|_{L^\infty_T} + \|\partial_x \nabla\|_{L^2_T} \leq C_T, \quad \|\partial_x \nabla^2\|_{L^2_T} \leq C_T \cdot \mathcal{I}, \quad \mathcal{I} \leq C_T \cdot \mathcal{I}, \quad \|A_0(t = 0)\|_{L^2_T} \leq C_T \cdot \mathcal{I} + \|\mathbf{A}\|_{L^2_T}.$$

The same estimates as the first two hold with $\nabla$ replaced by $\nabla^{-1}$. We remark that all the constants stated above are independent of $B > 0$. Applying Theorem B with $\mathcal{I}$ small enough (so that $\mathcal{I} \leq C_T \cdot \mathcal{I}$ is also small), we have

$$\sup_{0 \leq s \leq 1} \|\partial_{t,x} A_0(s)\|_{C_t((-T, T), L^2_x)} + \mathcal{A}_0(-T, T) \leq C\mathcal{I} \leq C_T \cdot \mathcal{I}.$$ 

Note that $V$ is a solution to the ODE (1.10), which is unique by the standard ODE theory. Furthermore, in view of the estimates we have for $\mathcal{A}_0(-T, T)$ and $\nabla$ in terms of $\mathcal{I}$, we may invoke

17 We remark that Part (3) of the Main Theorem follows from the persistence of regularity statement in Theorem 4.3.
Lemma 4.6 to estimate the gauge transform $V$ in terms of $\bar{T}$. The same procedure can be used to obtain estimates for $V^{-1}$ in terms of $\bar{T}$. Then using the previous bound for $\partial_{t,x} A_i(s=0)$ and the gauge transform formula

$$A_i^1 = V^{-1}A_i(s=0)V - \partial_t(V^{-1})V,$$

we obtain

$$\|\partial_{t,x} A_i^1\|_{C_t((-T,T),L_x^2)} \leq C_T \cdot \bar{T}.$$  

Applying Lemma 4.7 and the initial data estimate for the $L_x^3$ norm of $\bar{A}$, we also obtain

$$\|A_i^1\|_{C_t((-T,T),L_x^2)} < C_T \cdot \bar{T} + C_{T,\bar{A}}\|\bar{A}\|_{L_x^3}.$$  

Furthermore, applying a similar procedure to the difference, we obtain

$$\|\partial_{t,x} A_i^1 - \partial_{t,x} (A_i^1)\|_{C_t((-T,T),L_x^2)} \leq C_T \cdot \bar{\delta} \bar{T},$$

$$\|A_i^1 - (A_i^1)\|_{C_t((-T,T),L_x^2)} \leq C_T \|\bar{A}\|_{L_x^3} \cdot \bar{\delta} \bar{T} + C_{T,\bar{A}}\|\bar{A} - A^i\|_{L_x^3}.$$  

Therefore, taking $B > 0$ sufficiently large (while keeping $2B\bar{T} < \delta_P$), we recover (1.18). \[
\square
\]

The rest of this paper will be devoted to proofs of Theorems A and B.

5. Analysis of the covariant Yang-Mills heat flow

In this section, which serves as a preliminary to the proof of Theorem A to be given in Section 6, we will study the covariant Yang-Mills heat flow (cYMHF), i.e.

$$F_{si} = D^iF_{t\ell},$$

which is the spatial part of (YMHF).

The original Yang-Mills heat flow (YMHF) corresponds to the special case $A_x = 0$. Compared to (YMHF), which is covariant only under gauge transforms independent of $s$, the group of gauge transforms for (cYMHF) is enlarged to those which may depend on $s$; at the level of the equation, this amounts to the extra freedom of choosing $A_x$. In §5.1 we will see that this additional gauge freedom may be used in our favor to obtain a genuinely (semi-linear) parabolic system of equations. Being parabolic, this system possesses a smoothing property, which lies at the heart of our proof of Theorem A in Section 6. The system is connected to (YMHF) by a gauge transform $U$ solving a certain ODE, for which we will derive various estimates in §5.2. In §5.3 we will analyze a covariant parabolic equation satisfied by $B_i = F_{si}$ for $\nu = 0, 1, 2, 3$. As a byproduct of the results in §5.1 - 5.3 we will obtain a proof of the following local existence result for (YMHF) in §5.4, which is different from the original one given by [20].

**Theorem C** (Local existence for (YMHF) with $\bar{H}_x^1$ initial data). Consider the initial value problem (IVP) for (YMHF) with initial data $\bar{A}_i \in \bar{H}_x^1$ at $s = 0$. Then the following statements hold.

1. There exists a number $s^* = s^*(\|\bar{A}\|_{\bar{H}_x^1}) > 0$, which is non-increasing in $\|\bar{A}\|_{\bar{H}_x^1}$, such that there exists a solution $A_i \in C_s([0,s^*],\bar{H}_x^1)$ to the IVP satisfying

$$\sup_{s \in [0,s^*]} \|A(s)\|_{\bar{H}_x^1} \leq C\|\bar{A}\|_{\bar{H}_x^1}.$$  

(5.1)

2. Let $\bar{A}_i \in \bar{H}_x^1$ be another initial data set such that $\|\bar{A}\|_{\bar{H}_x^1} \leq \|\bar{A}\|_{\bar{H}_x^1}$, and $A'$ the corresponding solution to the IVP on $[0,s^*]$ given in (1). Then the following estimate for the difference $\delta A := A - A'$ holds.

$$\sup_{s \in [0,s^*]} \|\partial_x(\delta A)(s)\|_{\bar{H}_x^2} \leq C\|\delta \bar{A}\|_{\bar{H}_x^2}.$$  

(5.2)

---

18 In order to complete this theorem to a full local well-posedness result, we need to supplement it with a uniqueness statement. We omit such a statement here, as it will not be needed in the sequel. We refer the interested reader to [19], where a proof of uniqueness in the class of regular solutions will be given.
If \( A_i \) is smooth and \( \partial_x A_i, F_{ij} := \partial_i A_j - \partial_j A_i + [A_i, A_j] \) are regular, then the solution \( A_i = A_i(x, s) \) given in (1) is smooth and \( \partial_t A_i, F_{ij} \) are regular on \([0, s^*]\). Furthermore, if \( A_i(t) (t \in I) \) is a one parameter family of initial data such that \( \partial_{t,x} A_i, F_{ij} \) are regular on \( I \times \mathbb{R}^3 \), then \( A_i = A_i(t, x, s) \) is smooth and \( \partial_{t,x} A_i, F_{ij} \) are regular on \( I \times \mathbb{R}^3 \times [0, s^*] \).

This theorem itself is not needed for the rest of this paper, but will be used in [19].

5.1. Estimates for covariant Yang-Mills heat flow in the DeTurck gauge. Here, we will study \( \{\text{YMHE} \} \) in the DeTurck gauge \( A_s = \partial^t A_i \), which makes \( \{\text{YMHE} \} \) a system of (semi-linear) strictly parabolic equations for \( A_i \). Some parts of the standard theory for semi-linear parabolic equations, such as local-wellposedness and smoothing, will be sketched for later use.

Let us begin by deriving the system of equations that we will study. Writing out \( \{\text{YMHE} \} \) in terms of \( A_i, A_s \), we obtain
\[
\begin{align*}
(5.3) \quad & \partial_s A_i = \Delta A_i + 2[A^j, \partial_t A_i] - [A^j, \partial_t A_i] + [A^j, [A_t, A_i]] + \partial_t (A_s - \partial^t A_i) + [A_i, A_s - \partial^t A_i]. \\
(5.4) \quad & \partial_t A_i - \Delta A_i = 2[A^j, \partial_t A_i] - [A^j, \partial_t A_i] + [A^j, [A_t, A_i]],
\end{align*}
\]

The study of this system, which is now strictly parabolic, will be the main subject of this sub-section. For convenience, let us denote the right-hand side of the above equation by \( (A_i)N \), so that \( \partial_t - \Delta)A_i = (A_i)N \). Note that schematically,
\[
(\partial_t - \Delta)A_i = O(A, \partial A) + O(A, A, A).
\]

We will study the initial value problem for (5.3) with the initial data \( A_i(s = 0) = A_i \). We will also consider the difference of two nearby solutions. Given solutions \( A, A' \) to (5.3) with initial data \( A_i, A' \) respectively, we will denote the difference between the solutions and the initial data by \( \delta A_i := A_i - A'_i \) and \( \delta A_i = A_i - \bar{A}_i \), respectively.

Our first result is the local well-posedness for initial data \( A_i \in \dot{H}^1_x \). We refer the reader back to [3.4] for the definition of \( \mathcal{P}^{3/4} \).

**Proposition 5.1** \((\dot{H}^1_x \) local well-posedness of \( \{\text{YMHE} \} \) in the DeTurck gauge). The following statements hold.

1. There exists a number \( \delta_P > 0 \) such that for any initial data \( A_i \in \dot{H}^1_x \) with
\[
(5.5) \quad \|A_i\|_{\dot{H}^1_x} \leq \delta_P,
\]
there exists a unique solution \( A_i = A_i(x, s) \in C^s([0, 1], \dot{H}^1_x) \cap L^2_s((0, 1], H^2_x) \) to the equation (5.3) on \( s \in [0, 1] \), which satisfies
\[
(5.6) \quad \|
\partial_x A_i\|_{\mathcal{P}^{3/4} \dot{H}^2_x([0, 1])} \leq C\|A_i\|_{\dot{H}^1_x}.
\]

2. For \( A_i, A'_i \) solutions to (5.3) with initial data \( A_i, A'_i \) satisfying (5.5) given by (1), respectively, the following estimate hold for the difference \( \delta A_i \)
\[
(5.7) \quad \|
\partial_x (\delta A_i)\|_{\mathcal{P}^{3/4} \dot{H}^2_x([0, 1])} \leq C\|A_i - A'_i\|_{\mathcal{P}^{3/4} \dot{H}^2_x} \|\delta A_i\|_{\dot{H}^1_x}.
\]

3. We have persistence of regularity and smooth dependence on the initial data. In particular, let \( A_i(t) (t \in I \) for some open interval \( I \) ) be a one parameter family of initial data such that \( \partial_{t,x} A_i \) is regular on \( I \times \mathbb{R}^3 \). Then the solution \( A_i \) given by (1) is smooth and \( \partial_{t,x} A_i \) is regular.

**Proof.** This is a standard result. We will only present the proof of the \( a \) priori estimate (5.6), which means that we will assume the existence of a solution \( A_i \) to (5.3) with initial data \( A_i \), which we may assume furthermore to be smooth and \( \partial_x A_i \) is regular. As usual, a small variant of the arguments for the proof of the \( a \) priori estimate leads to estimates needed to run a Picard iteration argument (in a subset of a suitable Banach space), from which existence and uniqueness follows. Similar arguments applied to the equation for the difference \( \delta A_i \) and the differentiated equation for \( \partial_x (\delta A_i) \) will prove \( \dot{H}^1_x \) and persistence of regularity, respectively. For a parametrized family of data, differentiation...
with respect to the parameter yields smooth dependence on the initial data. We will leave these standard details to the interested reader.

In order to derive the a priori estimate for \( \partial_x A_i \), let us differentiate the equation. We then obtain

\[
\partial_x (\partial_x A_i) - \triangle (\partial_x A_i) = s^{-1} \nabla_x \mathcal{O}(A, \nabla_x A) + s^{-1/2} \mathcal{O}(A, \nabla_x A) =: (\partial_x A_i) \mathcal{N}.
\]

Let us work on a subinterval \((0, s) \subset (0, 1)\), assuming the bootstrap assumption \( \|\partial_x A\|_{(p^3/4)H^2_s(0, s)} \leq 10 \epsilon \), where \( \epsilon = \|\mathcal{N}\|_{\dot{H}^1_s} \) is the size of the initial data. As \( \partial_x A \) is regular, we have

\[
\limsup_{s \to 0} \|\partial_x A\|_{(p^3/4)H^2_s(0, s)} = \limsup_{s \to 0} \|\partial_x A\|_{(p^3/4)L^2_s(0, s)} = \|\partial_x A\|_{L^2_s},
\]

so the assumption holds for \( s > 0 \) small enough.

Note the obvious inequalities

\[
\|\partial_x (\phi_1 \partial_x \phi_2)\|_{L^2_s} \leq C\|\phi_1\|_{\dot{H}^{1/2}_s \cap L^\infty_s} \|\phi_2\|_{\dot{H}^1_s}, \quad \|\phi_1 \partial_x \phi_2 \partial_x \phi_3\|_{L^2_s} \leq C\|\phi_1\|_{\dot{H}^1_s} \|\phi_2\|_{\dot{H}^1_s} \|\phi_3\|_{\dot{H}^1_s},
\]

which follow from Hölder and Sobolev. Using the Correspondence Principle, Hölder for \( L^{p,r}_s \) (Lemma 3.10) and Gagliardo-Nirenberg (Lemma 3.13), we obtain

\[
\|s^{-1} \nabla_x \mathcal{O}(\psi_1, \nabla_x \psi_2)\|_{L^{3/4+1,p}_{3/4}L^2_s} = \|\nabla_x \mathcal{O}(\psi_1, \nabla_x \psi_2)\|_{L^{3/4+1,p}_{3/4}L^2_s} \leq C\|\psi_1\|_{L^{3/4+1,\infty}_{3/4} \cap L^\infty_s} \|\psi_2\|_{L^{3/4+2,\infty}_{3/4}} \\
\leq C\|\partial_x \psi_1\|_{L^{3/4+1,\infty}_{3/4} \cap L^\infty_s} \|\partial_x \psi_2\|_{L^{3/4+2,\infty}_{3/4}},
\]

\[
\|s^{-1/2} \mathcal{O}(\psi_1, \psi_2, \nabla_x \psi_3)\|_{L^{3/4+1,p}_{3/4}L^2_s} = \|\mathcal{O}(\psi_1, \psi_2, \nabla_x \psi_3)\|_{L^{3/4+1,p}_{3/4}L^2_s} \leq C\|\psi_1\|_{L^{3/4+1,\infty}_{3/4} \cap L^\infty_s} \|\psi_2\|_{L^{3/4+2,\infty}_{3/4}} \|\psi_3\|_{L^{3/4+2,\infty}_{3/4}} \\
\leq C\|\partial_x \psi_1\|_{L^{3/4+1,\infty}_{3/4} \cap L^\infty_s} \|\partial_x \psi_2\|_{L^{3/4+2,\infty}_{3/4}} \|\partial_x \psi_3\|_{L^{3/4+2,\infty}_{3/4}},
\]

for both \( p = 1, 2 \). Applying these inequalities with \( \psi_j = A \), note that each factor on the right-hand sides of the above two inequalities is controlled by \( \|\partial_x A\|_{(p^3/4)H^2_s} \). Using the bootstrap assumption, we have for \( p = 1, 2 \)

\[
\sup_{s \in [0, 1]} \|\partial_x A\|_{L^{3/4+1,p}L^2_s(0, s)} \leq C(\epsilon + \epsilon^2) \|\partial_x A\|_{(p^3/4)H^2_s(0, s)}. \tag{5.10}
\]

Taking \( \epsilon > 0 \) small enough and applying Theorem 3.11 we beat the bootstrap assumption, i.e. \( \|\partial_x A\|_{(p^3/4)H^2_s} \leq 5 \epsilon \). By a standard bootstrap argument, we conclude that (5.6) holds on \([0, 1]\).

An important property of the parabolic PDE is that it is infinitely and immediately smoothing. Quantitatively, this means that smoother norms of the solution \( A \) becomes controllable for \( s > 0 \) in terms of a rougher norm of the initial data \( \overline{A} \). We will see many manifestations of this property throughout the paper. Here, we give a version of the infinite, immediate smoothing for the solutions to the equation (5.3).

**Proposition 5.2 (Smoothing estimates).** Let \( \overline{A}_i, \overline{A}_i' \) be \( \dot{H}^1_s \) initial data satisfying (5.5), and \( A_i, A_i' \) the corresponding unique solutions to (5.4) given by Proposition 5.1 respectively.

1. For integers \( m \geq 1 \), the following estimate for \( A_i \) (and of course also for \( A_i' \)) holds.

\[
\|\partial_x A\|_{(p^3/4)\dot{H}^{m+2}_{s}(0, 1)} \leq C_m \|\overline{A}\|_{\dot{H}^1_s} \|A\|_{\dot{H}_s^1}. \tag{5.11}
\]

2. Furthermore, for integers \( m \geq 1 \), the following estimate for the difference also holds.

\[
\|\partial_x (\delta A)\|_{(p^3/4)\dot{H}^{m+2}_{s}(0, 1)} \leq C_m \|\overline{A}\|_{\dot{H}^1_s} \|\overline{A}\|_{\dot{H}_s^1} \|\delta A\|_{\dot{H}_s^1}. \tag{5.12}
\]

**Proof.** We now work on the whole interval \( s \in (0, 1] \). We will prove only the non-difference estimate (5.11), as the difference analogue (5.12) can be proved in a similar manner.
By approximation, it suffices to consider $A_i$ such that $\partial_x A_i$ is regular. Using Leibniz’s rule and the estimates \((5.3), (5.9)\), for $m \geq 1$, we immediately obtain
\[
\sup_i \||\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x} \leq C \|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x}^{1/2} \|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x}^{1/2} \|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x} + C \|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x} + C \|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x}^2 \|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x}.
\]

The second and third term on the right hand side can be controlled by $C \|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x}$ and $C \|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x}$, respectively. On the other hand, the first term is problematic as $\|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x}$ is not controlled by $\|\partial_x A_i\|_{L^{3/4+1}_x H^{3/2}_x}$. In this case, we apply Cauchy-Schwarz and estimate it by \(\epsilon^{-1} C \|\partial_x A_i\|_{H^{3/2}_x} + \epsilon \|\partial_x A_i\|_{H^{3/2+2}_x}\).

Therefore, we obtain \((5.10)\) for all $m \geq 1$ with $\psi = \partial_x A_i$, \(X = L_x^2, \ell = 3/4\) and $B_m(r) = Cr^2 + Cr^3$. Applying the second part of Theorem \(5.1\) the smoothing estimate \((5.11)\) easily follows. \(\square\)

The following statement is crucial for estimating the gauge transform into the caloric gauge $A_s = 0$. It is a corollary of the proof of Proposition \(5.1\).

**Corollary 5.3.** Let $\overline{A}_1, \overline{A}_i$ be $\dot{H}^1_x$ initial data satisfying \((5.3)\), and $A_i, A'_i$ the corresponding unique solutions to \((5.4)\) given by Proposition \(5.1\) respectively. Consider also $A_s, A'_s$ given by the equations
\[
A_s = \partial^4 A_i, \quad A'_s = \partial^4 A'_i.
\]

Then the following estimate holds for $\Delta A_s$.
\[
(5.13) \quad \sup_{0 < s \leq 1} \int_s^1 \Delta A_s(s') ds' \leq C_{\|\overline{A}\|_{L^2}} \|\overline{A}\|_{\dot{H}^1_x}.
\]

Furthermore, the following estimate holds for $\Delta (\delta A_s)$.
\[
(5.14) \quad \sup_{0 < s \leq 1} \int_s^1 \Delta (\delta A_s)(s') ds' \leq C_{\|\overline{A}\|_{L^2}} \|\overline{A}\|_{\dot{H}^1_x} \|\delta \overline{A}\|_{\dot{H}^1_x}.
\]

**Proof.** Here, we will only give a proof of \((5.13)\), the proof of \((5.14)\) being similar. Again it suffices to consider $A_i$ such that $\partial_x A_i$ (and therefore $A_s$ also) is regular.

Taking $\partial^4$ of the equations $\partial_\epsilon A_i - \Delta A_i = (A_i) N$, we get a parabolic equation for $A_s$ of the form $\partial_\epsilon A_s - \Delta A_s = \sum \partial_\epsilon (A_i) N$. Integrating this equation from $s$ to $1$, we obtain the following identity.
\[
\int_s^1 \Delta A_s(s') ds' = A_s(1) - A_s(s) - \sum \int_s^1 \partial_\epsilon (A_i) N(s') ds'.
\]

Let us take the $L^2_x$ norm of both sides and take the supremum over $s \in (0, 1]$. The first two terms on the right-hand side are acceptable, in view of the fact that $A_s = O(\partial_x A_i)$. Using Minkowski, it is not difficult to see that in order to estimate the contribution of the last term, it suffices to establish
\[
(5.15) \quad \sup_i \||\partial_x A_i\|_{L^{3/4+1,1}_x L^2_{[0,1]}} \leq C_{\|\overline{A}\|_{L^2}} \|\overline{A}\|_{\dot{H}^1_x}.
\]

This immediately follows by combining \((5.6)\) with \((5.10)\), recalling that $\epsilon = \|\overline{A}\|_{\dot{H}^1_x}$. \(\square\)

### 5.2. Estimates for gauge transform to the caloric gauge $A_s = 0$

In the previous subsection, we analyzed \((5.5)\) under the DeTurck gauge condition $A_s = \partial^4 A_\ell$, which led to a nice system of semi-linear parabolic equations. In this subsection, we present estimates for gauge transforms for the solution to \((5.5)\) in the DeTurck gauge into the caloric gauge $A_s = 0$.

**Lemma 5.4.** Fix $s_0 \in [0, 1]$. Let $\overline{A}_1, \overline{A}_i$ be $\dot{H}^1_x$ initial data sets satisfying \((5.5)\), and $A_i, A'_i$ the corresponding unique solutions to \((5.4)\) in the DeTurck gauge given by Proposition \(5.1\) respectively. Let us consider the following ODE on $\mathbb{R}^3 \times [0, 1]$:
\[
\begin{aligned}
\partial_s U &= U A_s \\
U(s = s_0) &= \text{Id},
\end{aligned}
\]
where we remind the reader that $A_s = \partial^\nu A_\nu$. Then the following statements hold.

1. There exists a unique solution $U$ such that $U(x,s) \in \mathcal{S}$ for all $s \in [0,1]$, and $U$ obeys the following estimates for $m = 2$.

\begin{align}
\|U\|_{L^\infty L^2[0,1]} &\leq C_{\|\bar{A}\|_{H^1}} \|\|\bar{A}\|_{H^1}\|, \\
\|\partial_x U\|_{L^\infty L^2[0,1]} &\leq C_{\|\bar{A}\|_{H^1}} \|\bar{A}\|_{H^1}.
\end{align}

2. Let $U'$ be the solution to the same ODE with $A_s$ replaced by $A'_s$, which possesses the identical initial data $U'(s = 1) = \text{Id}$. The difference $\delta U := U - U'$ satisfies the following estimates for $m = 2$.

\begin{align}
\|\delta U\|_{L^\infty L^2[0,1]} + \|\partial_x (\delta U)\|_{L^\infty L^2[0,1]} &\leq C_{\|\bar{A}\|_{H^1}} \|\delta \bar{A}\|_{H^1}, \\
\|s^{(m-2)/2}\partial_x^{(m)}(\delta U)\|_{L^\infty L^2[0,1]} &\leq C_{m,\|\bar{A}\|_{H^1}} \|\delta \bar{A}\|_{H^1}.
\end{align}

3. Furthermore, if $s_0 = 1$, then 5.17 and 5.19 hold for all integers $m \geq 3$ as well.

4. Finally, all of the above statements with $U$, $\delta U$ replaced by $U^{-1}$, $\delta U^{-1}$, respectively, also hold.

We defer the proof of Lemma 5.4 to Appendix B.

Remark 5.5. We remark that Lemma 5.4 will be for us an analogue of Uhlenbeck’s lemma\textsuperscript{29}, on which the work\textsuperscript{13} crucially rely, in the following sense: Heuristically, an application of this lemma with $s_0 = 1$, combined with the smoothing estimates of Proposition 5.2 amounts to transforming a given initial data set to another whose curl-free part is ‘smoother’. On the other hand, Uhlenbeck’s lemma sets the curl-free part to be exactly zero.

In the simpler case of an abelian gauge gauge theory (i.e. Maxwell’s equations), this heuristic can be demonstrated in a more concrete manner as follows: In this case, the connection component $A_s$ will exist all the way to $s \to \infty$, and will converge to zero in a suitable sense. Note furthermore that $\partial^\nu F_\nu = \partial_i (\partial^\nu A_\nu) - \Delta A_s = 0$. Therefore, this lemma, if applied with ‘$s_0 = \infty$’, transforms the initial data to one such that the curl-free part is zero, i.e. one satisfying the Coulomb gauge condition.

5.3. Linear covariant parabolic equation for $F_{ij}$. In this subsection, we will prove a technical well-posedness proposition for a certain covariant parabolic equation, which is satisfied by $B_i = F_{i\nu}$.

(See Appendix A)

Proposition 5.6. Fix $m \geq 2$, and let $A_i$, $A_s$ be smooth connection coefficients such that $\partial_x A_i$, $A_s$ are regular on $\mathbb{R}^3 \times [0,1]$.

Consider the following initial value problem for the linear parabolic equation

\begin{align}
\begin{cases}
D_x B_i - D^\nu D_\nu B_i = 2[F_{i\nu}, B^\nu], \\
B_i(s = 0) = \bar{B}_i,
\end{cases}
\end{align}

where the initial data $\bar{B}_i$ is regular. Then the following statements hold.

1. There exists a unique regular solution $B_i = B_i(x,s)$ on $[0,1]$ to the problem 5.20.

2. Assume furthermore that we have the following bounds for $A_i$ and $A_s$:

\begin{align}
\sup_{s} \|\nabla_x A_i\|_{L^{\infty}H^{m-1}_{\nu}(0,1)} + \|A_s\|_{L^{\infty}H^{m-1}_{\nu}(0,1)} &\leq C < \infty.
\end{align}

Then the solution $B_i$ obtained in (1) satisfies the following estimate.

\begin{align}
\sup_{s} \|B_i\|_{L^2 H^{m-1}_{\nu}(0,1)} &\leq C \sup_{s} \|\bar{B}_i\|_{L^2}.
\end{align}

\textsuperscript{29}In fact, an inspection of the proof in Appendix\textsuperscript{13} shows that one only needs the assumption $s_0 > 0$.

\textsuperscript{30}Which states, roughly speaking, that there exists a gauge transform (with good regularity properties) which transforms a given connection 1-form $A_i$ into the Coulomb gauge, provided that the $L^4_A$ norm of $F_{ij}$ is small.
(3) Let \( A_i = A_i(t, x, s) \), \( A_s = A_s(t, x, s) \) be a family of coefficients, parametrized by \( t \in I \), such that \( A_i \) is smooth and \( \partial_x A_i, A_s \) are regular on \( I \times \mathbb{R}^3 \times [0, 1] \). Consider the corresponding one parameter family of IVPs \((5.20)\), where the initial data sets \( \mathcal{B}_i(t) \) are also parametrized by \( t \in I \), in such a way that \( \mathcal{B}_i(t) = \mathcal{B}_i(t, x) \) is regular on \( I \times \mathbb{R}^3 \).

Then the solution \( B_i = B_i(t, x, s) \) on \( I \times \mathbb{R}^3 \times [0, 1] \) (obtained by applying (1) to each \( t \)) is regular on \( I \times \mathbb{R}^3 \times [0, 1] \).

Proof. As in the proof of Proposition \( 5.1 \) we will present only the proof of the estimate \( (5.22) \) of Part (2) under the assumption that a regular solution \( B \) already exists. The actual existence, uniqueness, persistence of regularity and stability required to justify Parts (1), (3) follow from a standard Picard iteration argument, which can be set up by a slightly modifying of the argument below. We leave the details of the procedure to the interested reader.

Let us begin by rewriting the equation \( (5.20) \) so that it is manifestly a semi-linear equation for the vector-valued unknown \( B \):

\[
\partial_t B_i - \Delta B_i = 2[A_i, \partial_x B_i] + [\partial_i A_i - A_s, B_i] + [A_i', [A_i, B_i]] + 2[F_{id}, B']
\]

\[
= s^{-1/2} \mathcal{O}(A, \nabla_x B) + s^{-1/2} \mathcal{O}(\nabla_x A, B) + \mathcal{O}(A_s, B) + \mathcal{O}(A, A, B).
\]

Note the following inequalities, which follow easily from Hölder and Sobolev.

\[
(5.23) \quad \begin{cases}
\| \phi_1 \partial_x \phi_2 \|_{L^2} + \| \partial_x \phi_1 \phi_2 \|_{L^2} \leq C \| \phi_1 \|_{H^{3/2} \cap L^\infty} \| \phi_2 \|_{H^1}, \\
\| \phi_1 \phi_2 \|_{L^2} \leq C \| \phi_1 \|_{H^{1/2}} \| \phi_2 \|_{H^1}, \\
\| \phi_1 \phi_2 \|_{L^2} \leq C \| \phi_1 \|_{H^{1/2}} \| \phi_2 \|_{H^1}.
\end{cases}
\]

Fix \((0, s) \subset (0, 1)\). Applying the Correspondence Principle, Hölder for \( L^q_{t,x} \) (Lemma \( 3.9 \)), Gagliardo-Nirenberg (Lemma \( 3.16 \)) and interpolation, we obtain the following set of inequalities on \((0, s)\).

\[
(5.24) \quad \begin{cases}
\| s^{-1/2} \mathcal{O}(\psi_1, \nabla_x \psi_2) \|_{L^{3/4+1,q} \cap L^2} + \| s^{-1/2} \mathcal{O}(\nabla_x \psi_1, \psi_2) \|_{L^{3/4+1,q} \cap L^2} \\
\leq C \| \nabla_x \psi_1 \|_{L^{1/4+\epsilon} \cap L^\infty} \| s^{1/4-\epsilon} \psi_2 \|_{L^{1/4+\epsilon} \cap L^\infty},
\end{cases}
\]

where \( 1 \leq q \leq 2 \) and \( \epsilon' > 0 \) are small enough. Using \((5.24)\) with \( q = 1, 2, \psi_0 = A_s, \psi_1 = A, \psi_2 = B, \) and \( \psi_3 = A \), we obtain \((5.11)\) with \( s = B, X = L^2_t, \ell = 3/4, \epsilon = D = 0, p = 2 \) and \( C(s) = C(C + C^2)s^{1/4-\epsilon'} \). Since \( \| B \|_{p^{3/m} \cap L^\infty} < \infty \) (as \( B \) is regular) and \( C(s)^2 \) is integrable on \((0, 1)\), we can apply the first part of Theorem \( 3.11 \) to conclude that

\[
\sup_i \| B_i \|_{p^{3/m} \cap L^\infty(0, 1)} \leq C \sup_i \| \mathcal{B}_i \|_{L^2}.
\]

Finally, in the case \( m \geq 3 \), let us prove the smoothing estimate \((5.22)\). We use Leibniz’s rule and \((5.24)\) with \( q = 2 \) (and ignoring all extra weights of \( s \)) to estimate \( \| (\partial_s - \Delta) B_i \|_{L^{3/4+1,2} \cap L^\infty} \) by

\[
C(\| \nabla_x A \|_{L^{1/4+\epsilon} \cap L^\infty}, \| A_s \|_{L^{1/4+\epsilon} \cap L^\infty}, \| \nabla_x A \|_{L^{1/4+\epsilon} \cap L^\infty}^2) \| B \|_{L^{1/4+2} \cap L^\infty} \leq C(C + C^2)\| B \|_{p^{3/m} \cap L^\infty}^{1+1}.
\]

for \( 0 \leq k \leq m - 2 \). Using the second part of Theorem \( 5.11 \) \((5.22)\) follows. \( \square \)

By almost the same proof, the following slight variant of Proposition \( 5.1 \) immediately follows.

**Proposition 5.7.** Fix \( m \geq 2 \), and assume that \((A_i, A_s), (A_i', A_s')\) are pairs of smooth connection coefficients such that \( \partial_x A_i, \partial_x A_s', A_s, A_s' \) are regular and \((5.21)\) are satisfied. Assume furthermore

\[
(5.25) \quad \sup_i \| \nabla_x (\delta A_i) \|_{L^{3/4+\epsilon} \cap L^\infty} + \| \delta A_s \|_{L^{3/4+\epsilon} \cap L^\infty} \leq \delta C < \infty.
\]
Consider regular initial data \( B_i, B'_i \) which belongs to \( L^2_x \) and the corresponding solutions \( B_i, B'_i \) on \([0,1]\) to the initial value problems

\[
\begin{align*}
\{ \text{D}_s B_s - \text{D}' \text{D}_s B_s &= 2[F_{t,\ell}, B'_{t}], \\
B_s(s = 0) &= B_i,
\end{align*}
\]

\[
\begin{align*}
\{ \text{D}'_s B'_s - (\text{D}')' \text{D}_s B'_s &= 2[F_{t,\ell}, B'^{t}] , \\
B'_s(s = 0) &= B'_i,
\end{align*}
\]

given by Proposition 5.6. The the following estimate holds.

\[
\sup_i \| \delta B_i \|_{L^2_t L^2_x(0,1)} \leq C_{\mathcal{L}, \| B \|_{L^2_x}} ( \sup_i \| \delta B_i \|_{L^2_t L^2_x} + \delta C ).
\]

(5.26)

5.4. Proof of Theorem \( \square \) Combining the results in 5.1 - 5.3 we can give a proof of Theorem \( \square \) This theorem is needed in \([19]\), but not for the rest of this paper, so the reader is free to skip this subsection insofar as only the Main Theorem is concerned.

**Proof.** The idea is to first use scaling to make the initial data small, and then solve (\( \text{YMHEF} \)) in the DeTurck gauge by the theory we developed in 5.1. Then we will apply Lemma 5.4 with \( s_0 = 0 \) to obtain a solution to the \( \text{YMHEF} \) (which is \( \text{CYMHEF} \)) in the caloric gauge.

We will give a detailed proof of (1) for smooth initial data such that \( \partial_s \mathcal{A}_s \) is regular; then a similar argument leads to (2) for the same class of initial data, at which point we can recover the full statements of (1) and (2) for general initial data sets in \( H^1_3 \) by approximation.

As alluded to earlier, we will assume that \( \mathcal{A}_i \) is smooth and \( \partial_s \mathcal{A}_s \) is regular. Note that \( \text{YMHEF} \) is invariant under the scaling \( A(x,s) \rightarrow \lambda^{-1} A(x/\lambda, s/\lambda^2) \); using this scaling, we may enforce \( \| \mathcal{A} \|_{H^1_3} \leq \delta_p \). We are then in a position to apply Proposition 5.1 from which we obtain a smooth solution \( \tilde{\mathcal{A}}_i \in C_t([0,1], H^1_3) \) to the IVP for \( \text{CYMHEF} \) in the DeTurck gauge, i.e.

\[
\begin{align*}
\{ &\tilde{F}_s = \text{D}' \tilde{F}_s, \quad \tilde{\mathcal{A}}_s = \partial_s \tilde{\mathcal{A}}_s \quad \text{on} \quad \mathbb{R}^3 \times [0,1], \\
&\tilde{\mathcal{A}}_i(s = 0) = \mathcal{A}_i,
\end{align*}
\]

which, by (5.6), obeys

\[
\sup_{s \in [0,1]} \| \tilde{\mathcal{A}}(s) \|_{H^1_3} \leq C_{\| \mathcal{A} \|_{H^1_3}}.
\]

(5.27)

Next, consider a gauge transform \( U = U(x,s) \) which solves the ODE

\[
\begin{align*}
\{ &\partial_s U = U \tilde{\mathcal{A}}_s, \quad \text{on} \quad \mathbb{R}^3 \times [0,1] \\
&U(s = 0) = \text{Id.}
\end{align*}
\]

(5.29)

Note that \( U \) is smooth, as \( \tilde{\mathcal{A}}_s \) is, and furthermore satisfies the following estimates on \( \mathbb{R}^3 \times [0,1] \) thanks to Lemma 5.5:

\[
\left\| U \right\|_{L^\infty_t L^2_x} \leq C_{\| \mathcal{A} \|_{H^1_3}}, \quad \left\| \partial_s U \right\|_{L^\infty_t L^2_x} + \left\| \partial_s^2 U \right\|_{L^\infty_t L^2_x} \leq C_{\| \mathcal{A} \|_{H^1_3}} \| \mathcal{A} \|_{H^1_3}.
\]

(5.30)

The identical estimates hold with \( U \) replaced by \( U^{-1} \) as well.

Let \( \mathcal{A}_i := U \tilde{\mathcal{A}}_i U^{-1} - \partial_s UU^{-1} \) be the connection 1-form obtained by gauge transforming \( \mathcal{A}_i, \tilde{\mathcal{A}}_s \) by \( U \). We remark that \( \mathcal{A}_s = U \tilde{\mathcal{A}}_s U^{-1} - \partial_s UU^{-1} = 0 \) thanks to the above ODE, and therefore \( \mathcal{A}_i \) solves (\( \text{YMHEF} \)), whereas \( \mathcal{A}_i(s = 0) = \tilde{\mathcal{A}}_i(s = 0) = \mathcal{A}_i \) as \( U(s = 0) = \text{Id} \). Therefore, we conclude that \( \mathcal{A}_i \) is a (smooth) solution to the IVP for (\( \text{YMHEF} \)) with the prescribed initial data. Furthermore, from (5.28), (5.30) and the gauge transform formula for \( \mathcal{A}_i \), we see that \( \mathcal{A}_i \in C_t([0,1], H^1_3) \) and also that (5.28) holds with \( \tilde{\mathcal{A}}_i \) replaced by \( \mathcal{A}_i \). Scaling back, we obtain (1).

Next, we will sketch the proof of (3). It suffices to consider the case of smoothly parametrized initial data sets \( \mathcal{A}_i(t) (t \in I) \) such that \( \mathcal{A}_i \) is smooth and \( \partial_{t,x} \mathcal{A}_i, F_{ij} \) are regular on \( I \times \mathbb{R}^3 \). We may furthermore assume that \( I \) is compact, and that \( \sup_{t \in I} \| \mathcal{A}(t) \|_{H^1_x} \leq \delta_p \) by scaling. The aim is to show that the solution \( \mathcal{A}_i \) obtained by applying (1) for each \( t \) is smooth and \( \partial_{t,x} \mathcal{A}_i \) is regular on \( I \times \mathbb{R}^3 \times [0,1] \). As a consequence, \( \mathcal{A}_i = \partial_s \mathcal{A}_s \) is regular; then it is not difficult
to show that the solution $U$ to (5.30) (solved for each $t$) is a regular gauge transform for the initial data at $I = \mathbb{R}^3 \times [0, 1]$. It follows that $A_t = U \hat{A}_t U^{-1} - \partial_t U U^{-1}$ is smooth and $\partial_x A_t$ is regular on $I \times \mathbb{R}^3 \times [0, 1]$, as desired. Next, applying Part (3) of Proposition 5.6 to the equation

$$\partial_t F_{ij} - D^i D_t F_{ij} = -2[F_{ij}, F_{ij}],$$

(where $A_s = 0$ and $B_t = F_{ij} = -F_{ji}$) it follows that $F_{ij}$ is regular on $I \times \mathbb{R}^3 \times [0, 1]$ as well. □

Remark 5.8. The idea of the proof of Theorem A outlined above is not new, and is in fact nothing but the standard DeTurck trick in disguise, first introduced by D. DeTurck for the Ricci flow and introduced in the context of the Yang-Mills heat flow by S. Donaldson. A similar procedure will be used in the next section in our proof of Theorem A, but with an extra twist of choosing $s_0 = 1$ instead of $s_0 = 0$ in Lemma 5.4. This allows us to keep the smoothing estimates through the gauge transform back to $A_s = 0$ (which is not the case for the original DeTurck trick), at the expense of introducing a non-trivial gauge transform for the initial data at $t = s = 0$.

6. Proof of Theorem A: Estimates for the initial data

The goal of this section is to prove Theorem A using the preliminary results established in the previous section.

We begin by giving the precise definitions of $\mathcal{I}$ and $\delta \mathcal{I}$, which had been alluded in Section 5. Let $A_s, A^i_s$ be regular solutions to (HPYM) (which, we remind the reader, was introduced in (4.14) on $I \times \mathbb{R}^3 \times [0, 1]$. We define the norms $\mathcal{I}$ and $\delta \mathcal{I}$ for $F_{si}, A_s$ at $t = 0$ by

$$\mathcal{I} := \sum_{k=1}^{10} \left[ \|\nabla_{t,x} F_s(t = 0)\|_{L_t^{5/4} L_x^{4/3} H_t^{k-1}} + \|\nabla_{t,x} F_s(t = 0)\|_{L_t^{5/4} L_x^{4/3} H_t^{k-1}} \right] + \sum_{k=1}^{31} \|\partial_t A(t = 0)\|_{L_t^{5/4} L_x^{4/3} H_t^{k-1}}.$$

$$\delta \mathcal{I} := \sum_{k=1}^{31} \left[ \|\nabla_{t,x} (\delta F_t)(t = 0)\|_{L_t^{5/4} L_x^{4/3} H_t^{k-1}} + \|\nabla_{t,x} (\delta F_s)(t = 0)\|_{L_t^{5/4} L_x^{4/3} H_t^{k-1}} \right] + \sum_{k=1}^{31} \|\partial_t (\delta A)(t = 0)\|_{L_t^{5/4} L_x^{4/3} H_t^{k-1}}.$$

where we remind the reader the conventions $\|F_s\| = \sup \|F_s\|$ and $\|A_s\| = \sup \|A_s\|$.

Proof of Theorem A. Throughout the proof, let us use the notation $I = (T, T)$. The proof will proceed in a number of steps.

Step 1: Solve (dYMHP) in the DeTurck gauge. The first step is to exhibit a regular solution to (dYMHP), by solving (dYMHP) under the DeTurck gauge condition $A_t = \partial_t A_t$. For the economy of notation, we will denote the solution by $A_s$ in this proof; however, the reader should keep in mind that it is not the $A_s$ in the statement of the theorem, since we are in a different gauge.

We note the reader that in this step and the next, we will mostly be interested in obtaining qualitative statements, such as smoothness of various quantities, etc. These statements will typically depend on smooth norms of $A^i_s$.

We begin by solving (dYMHP), i.e. $F_{si} = D^i D_t F_{si}$ for every $t$, with the initial data $A_i(t, s = 0) = A^i_0(t)$. Let us impose the DeTurck gauge condition $A_t = \partial_t A_t$. Recall that this makes (dYMHP) a system of genuine semi-linear heat equations, which can be solved on the unit $s$-interval $[0, 1]$ provided that the initial data $A^i_0$ is small in a suitable sense. Indeed, for $t \in I$, by Proposition 5.1 and the hypothesis $\|A_i(t)\|_{H^2} < \delta_F$, there exists a unique smooth solution $A_i(t) = A_i(t, s, x)$ to the above system on $0 \leq s \leq 1$. Furthermore, by Part (3) of Proposition 5.1 $\partial_{t,x} A_t$ and $A_s$ are regular.

Our next task is to show that there exists $A_0$ which satisfies the remaining equation $F_{si} = D^i D_t F_{si}$ of (dYMHP). To begin with, let us solve the linear covariant parabolic equation (with smooth coefficients):

$$\begin{cases} D_i B_i = D^i D_t B_i + 2[F^t_{ij}, B_i], \\ B_i(t)|_{t=0} = F^{t}_{ij}(0). \end{cases}$$

The main idea is to recast the ODE in the integral form, differentiate with respect to $t, x, s$ and apply Gronwall’s inequality to deal with the highest order term. See the proof of Proposition 5.1 for an example of this sort of argument.
It is easy to check that the hypotheses of Proposition 5.6 are satisfied, by using the estimates in Proposition 5.2. Therefore, a unique smooth solution $B_i$ to this equation exists on $0 \leq s \leq 1$.

The idea is that $B_i$ should be $F_{i0}$ in the end, as it solves exactly the equation that $F_{i0}$ is supposed to solve. With this in mind, let us extend $A_0$ by formally setting $F_{i0} = D^i B_i$, which leads to the ODE

$$\begin{cases}
\partial_s A_0 = \partial_0 A_s + [A_0, A_s] + D^i B_i \\
A_0|_{s=0} = 0.
\end{cases}$$

This is a linear ODE with smooth coefficients, as $A_s$ is regular. Therefore, by the standard ODE theory, there exists a unique smooth solution $A_0$ on $0 \leq s \leq 1$. Furthermore, in view of the regularity of $A_s, \partial_x A_i$, and $B_i$, we see that $A_0$ is regular. The connection coefficients $A_i, A_s = \partial^i A_t$ and $A_0$ that we have obtained so far constitutes a candidate for the solution to [HPYM].

It now only remains to check that the connection 1-form $A_i$ is indeed a solution to the equation $F_{s\mu} = D^i F_{i\mu}$, along with the condition $A_s = \partial^i A_t$. For this purpose, it suffices to verify that $B_i = F_{i0}$, where $F_{i0}$ is the curvature 2-form given by

$$F_{i0} := \partial_i A_0 - \partial_0 A_i + [A_i, A_0].$$

From the regularity of $\partial_{t,x} A_i$ and $A_0$, it follows that $F_{i0}$ is regular on $I \times \mathbb{R}^3 \times [0,1]$. Then the following lemma shows that $B_i = F_{i0}$ indeed holds.

**Lemma 6.1.** Let $B_i, A_0$ be defined as above, and define $F_{i0} = \partial_i A_0 - \partial_0 A_i + [A_i, A_0]$. Then we have $B_i = F_{i0}$ on $0 \leq s \leq 1$.

**Proof.** Let us begin by computing $D_s F_{i0}$:

$$D_s F_{i0} = D_s F_{i0} - D_0 F_{s0} = D_s F_{00} - D_0 D^i F_{i0} - D_s D^i (F_{i0} - B_i) = D^i D_s F_{i0} + 2[F_i^\ell, F_{i0}] - D_s D^i (F_{i0} - B_i).$$

Subtracting the equation

$$D_s B_i = D^i D_s B_i + 2[F_i^\ell, B_i],$$

from the previous equation, we obtain

$$D_s (\delta F_{i0}) = D^i D_s (\delta F_{i0}) - D^i D_s (\delta F_{i0}) + 2[F_i^\ell, \delta F_{i0}],$$

where $\delta F_{i0} := F_{i0} - B_i$. By construction, note that $\delta F_{i0} = 0$ at $s = 0$. Furthermore, $\delta F_{i0}$ is regular on $I \times \mathbb{R}^3 \times [0,1]$.

To proceed further, let us fix $t \in I$ and $0 \leq s \leq 1$. Taking the bi-invariant inner product of the last equation with $\delta F_{i0}$, summing over $i$, and integrating by parts over $\mathbb{R}^3$, we see that

$$\frac{1}{2} \partial_s \left( \sum_i \int (\delta F_{i0}, \delta F_{i0})(s) \, dx \right)$$

$$= \sum_i \int (D_s D_t (\delta F_{i0}), \delta F_{i0})(s) - (D_t D_s (\delta F_{i0}), \delta F_{i0})(s) + 2([F_i^\ell, \delta F_{i0}], \delta F_{i0})(s) \, dx$$

$$= \sum_i \int -\frac{1}{2} (D_t (\delta F_{i0}) - D_i (\delta F_{i0}), D_t (\delta F_{i0}) - D_i (\delta F_{i0}))(s) + 2([F_i^\ell, \delta F_{i0}], \delta F_{i0})(s) \, dx.$$

The first term on the last line has a favorable sign, and can be thrown away. The remaining term is easily bounded by $C \left( \sup_{t,s} \|F_{i0}(s)\|_{L^2} \right) \left( \sum_i \int (\delta F_{i0}, \delta F_{i0})(s) \, dx \right)$. Since $F_{i0}$ is uniformly bounded on $I \times \mathbb{R}^3 \times [0,1]$ (as $\partial_x A_t$ is regular), we may apply Gronwall’s inequality and conclude that

$$\sum_i \int (\delta F_{i0}, \delta F_{i0})(s) \, dx = 0 \text{ on } 0 \leq s \leq 1.$$  

This shows that $\delta F_{i0} = 0$ on $0 \leq s \leq 1$, as desired. \(\Box\)

---

\footnote{In order to justify the integration by parts carried out on the third line, it suffices to note that $D_t \delta F_{i0}, \delta F_{i0} \in L^2$ for every fixed $t \in I$ and $0 \leq s \leq 1$.}
In sum, we conclude that $A_a$ is a regular solution to (HPYM), which satisfies the DeTurck gauge condition $A_a = \partial^\nu A_t$.

**Step 2 : Construction of a gauge transform to the caloric-temporal gauge.**

In Step 1, we have constructed a regular solution $A_a$ to (HPYM) in the DeTurck gauge $A_s = \partial^\nu A_t$. The next step is to construct a suitable smooth gauge transform $U = U(t, x, s)$ to impose the caloric-temporal gauge condition, i.e. $A_s = 0$ and $A_0 = 0$.

We begin by briefly going over the gauge structure of (HPYM). As before, the gauge transform is a $\mathcal{G}$-valued function $U(t, x, s)$ on $I \times \mathbb{R}^3 \times [0, 1]$ is given by the formulae

$$A_a \rightarrow UA_aU^{-1} - \partial_a UU^{-1} =: \tilde{A}_a, \quad F_{ab} \rightarrow UF_{ab}U^{-1} =: \tilde{F}_{ab}.$$

where $a = (t, x, s)$. As a consequence, any gauge transformed connection 1-form $\tilde{A}_a$ will still solve (HPYM).

From the above discussion, it follows that in order to impose $\tilde{A}_0 = \tilde{A}_0(s = 1) = 0$ and $\tilde{A}_s = 0$, the gauge transform $U(t, s, x)$ must satisfy

$$\begin{cases} \partial_0 U = UA_t, \text{ along } s = 1, \\ \partial_s U = UA_s, \text{ everywhere.} \end{cases}$$

We will solve this system by starting from the identity gauge transform at $(t = 0, s = 1)$. More precisely, let us fix $x \in \mathbb{R}^3$, and first solve the ODE

$$\begin{cases} \partial_0(U(t)) = U(t)A_0(t), \\ U(t = 0) = \text{Id.} \end{cases}$$

along $I \times \{s = 1\}$, where $A_0(t, s = 1) = 0$. Then using these as the initial data, we solve

$$\begin{cases} \partial_0 U(t, s) = U(t, s)A_s(t, s), \\ U(t, s = 1) = U(t). \end{cases}$$

Since both $A_0$ and $A_s = \partial^\nu A_t$ are regular, it is clear, again by the standard ODE theory, that there exists a unique smooth solution $U(t, x, s)$ satisfying the above ODEs. Furthermore, using arguments as in the proof of Proposition [B.1], we can readily prove that $U$ is a regular gauge transform (in the sense of Definition [1.2]).

By construction, the gauge transformed connection 1-form $\tilde{A}_a$ satisfies the following equations.

$$\begin{cases} \tilde{F}_{ab} = D^\nu \tilde{F}_{\nu b}, \\ \tilde{A}_\mu = UA_\mu U^{-1} - \partial_\mu UU^{-1}. \end{cases}$$

Furthermore, it satisfies the caloric-temporal gauge condition, i.e. $\tilde{A}_0 = 0$ and $\tilde{A}_s = 0$. As $A_a$ is a regular solution to (HPYM) and $U$ is a regular gauge transform, it readily follows that $\tilde{A}_\mu$ is a regular solution of (HPYM) as well.

**Step 3 : Quantitative initial data estimates for non-differences, in the caloric-temporal gauge.**

In this step, we will prove the non-difference estimates among the initial data estimates (1.12) – (1.13). From this point on, we must be quantitative, which means that we must make sure that all estimates here depends only on $\mathcal{I}$.

Before we begin, a word of caution on the notation. We will keep the same notation as the previous steps, which means that the $A_a, F_{si}$ in the statement of the theorem are not the same as $\tilde{A}_a, \tilde{F}_{si}$ below, but are rather $\tilde{A}_a, \tilde{F}_{si}$. Accordingly, the gauge transform $V$ in the statement of the theorem is given by $V(t, x) = U(t, x, s = 0)$ (which is regular).

Recalling the definition of $\mathcal{I}$, in order to prove $\mathcal{I} \leq C_{\mathcal{T}} \cdot \mathcal{I}$, we must estimate $\|\nabla_{t,x} F_{si}\|_{L^{5/4}_p H^{3/2}_p}$ for $p = 2, \infty$ and $\|\partial_{t,x} A_a\|_{H^{3/2}_p}$ in terms of $C_{\mathcal{T}} \cdot \mathcal{I}$. On the other hand, to prove (1.13), we need to show

$$\|U(t = 0, s = 0)\|_{L^\infty_p} + \|\partial_x U(t = 0, s = 0)\|_{L^2_p} \leq C_{\mathcal{T}}, \quad \|\partial_t^{(2)} U(t = 0, s = 0)\|_{L^2_p} \leq C_{\mathcal{T}} \cdot \mathcal{I},$$

(6.1)
and the analogous estimates for $U^{-1}$.

By Proposition 5.2 we have the following estimates at $t = 0$ for $m \geq 1$:
\begin{align}
(6.2) \quad \sup_i \|A_i\|_{L^{5/4,\infty}_t \dot{H}^m_x} + \sup_i \|A_i\|_{L^{5/4,2}_t \dot{H}^m_x} \leq C_{m,T} \cdot \mathcal{T}.
\end{align}

Note that the gauge transform $U$ is equal to the identity transform at $t = 0$, $s = 1$. Consequently, we have $\bar{A}_i(t = 0, s = 1) = A_i(t = 0, s = 1)$. Therefore $\|\partial_s \bar{A}\|_{H^{m}_x} \leq C_{T,T} \cdot \mathcal{T}$ follows immediately from (6.2).

Next, we estimate the gauge transform $U$ at $t = 0$. Using Lemma 5.4 it follows that the gauge transform $U$ at $t = 0$ satisfy the following estimates for $m \geq 2$.
\begin{align}
(6.3) \quad \|U\|_{L^\infty_t L^\infty_x} \leq C_{T,T} \cdot \mathcal{T}, \quad \|\partial_s U\|_{L^\infty_t L^2_x} \leq C_{T,T} \cdot \mathcal{T}, \quad \|s^{(m-2)/2} \partial_s^{(m)} U\|_{L^\infty_t L^2_x} \leq C_{m,T} \cdot \mathcal{T}.
\end{align}

The analogous estimates hold for $U^{-1}$. Then (6.1) (or its analogue for $U^{-1}$) follows immediately from (6.3) (or its analogue for $U^{-1}$).

The next step is to estimate $\bar{F}_{si}$. Let us begin by considering the gauge transformed connection 1-form $\bar{A}_i$ at $t = 0$. We claim that they satisfy the estimates
\begin{align}
(6.4) \quad \sup_i \|\bar{A}_i\|_{L^{5/4,\infty}_t \dot{H}^m_x} \leq C_{m,T} \cdot \mathcal{T},
\end{align}
for $m \geq 1$. For $m = 1$, we can compute via Leibniz’s rule
\begin{align*}
\partial_s \bar{A}_i = \partial_s (UA_i U^{-1} - \partial_s UU^{-1})
&= (\partial_s U) A_i U^{-1} + U (\partial_s A_i) U^{-1} + U A_i (\partial_s U^{-1}) - (\partial_s \partial_s U) U^{-1} - \partial_s U (\partial_s U^{-1}).
\end{align*}

Using Hölder, Sobolev and (6.3), it is not difficult to show
\begin{align*}
\|\partial_s \bar{A}_i(s)\|_{L^2_x} \leq C_{T,T} \cdot \|\partial_s A_i(s)\|_{L^2_x} + \mathcal{T}.
\end{align*}
At this point, from (6.2), we obtain (6.4) in the case $m = 1$. Proceeding similarly, we can also prove (6.4) for $m \geq 2$; we omit the details.

From the previous estimates, it is already possible to estimate $\|\nabla_x F_{si}\|_{L^{5/4,\infty}_x \dot{H}^{m}_x}$ for $p = \infty$ but not $p = 2$. This is essentially due to the unpleasant term $-\partial_s UU^{-1}$ in the gauge transformation formula for $A_i$. To estimates both terms at the same time, we argue differently as follows, utilizing the gauge covariance of $F_{si}$. We start by recalling the equation for $F_{si}$ in terms of $A_i$ and ordinary derivatives.
\begin{align*}
F_{si} = D^t F_{ti} = \partial^t \partial_s A_i - \partial_s \partial^t A_t + 2[A^t, \partial_t A_i] + [\partial^t A_t, A_i] - [A^t, \partial_t A_i] + [A^t, [A_t, A_i]].
\end{align*}
Using this formula and (6.2), it is not difficult to prove the following estimates for $m \geq 0$.
\begin{align}
(6.5) \quad \sup_i \|\nabla_x^{(m)} F_{si}\|_{L^{5/4,\infty}_x \dot{L}^2_x} + \sup_i \|\nabla_x^{(m)} F_{si}\|_{L^{5/4,2}_x \dot{L}^2_x} \leq C_{m,T} \cdot \mathcal{T}.
\end{align}

Using interpolation and (6.2) to control $\|\nabla_x^{(m)} A_i\|_{L^{5/4,\infty}_x \dot{L}^2_x}$, it is a routine procedure to replace the ordinary derivatives in (6.5) by covariant derivatives, i.e.
\begin{align}
(6.6) \quad \sup_i \|\nabla_x^{(m)} F_{si}\|_{L^{5/4,\infty}_x \dot{L}^2_x} + \sup_i \|\nabla_x^{(m)} F_{si}\|_{L^{5/4,2}_x \dot{L}^2_x} \leq C_{m,T} \cdot \mathcal{T}.
\end{align}
for $m \geq 0$, where we remind the reader that $D$ is the $p$-normalized $D$, i.e. $D_{pl} := s^{1/2} D_{pl}$. Then using the gauge transform formula $D_{x}^{(m)} \bar{F}_{si} = U (D_{x}^{(m)} F_{si}) U^{-1}$ and (6.3) to estimate $U$ in $L^{\infty}_x L^\infty_x$, we obtain
\begin{align*}
\sup_i \|\nabla_x^{(m)} \bar{F}_{si}\|_{L^{5/4,\infty}_x \dot{L}^2_x} + \sup_i \|\nabla_x^{(m)} \bar{F}_{si}\|_{L^{5/4,2}_x \dot{L}^2_x} \leq C_{m,T} \cdot \mathcal{T}.
\end{align*}
for $m \geq 0$. Using interpolation and (6.4) to control $\|\nabla_x^{(m)} \bar{A}_i\|_{L^{5/4,\infty}_x \dot{L}^2_x}$, we can replace the covariant derivatives by ordinary derivatives, and finally obtain for $m \geq 0$ the following estimate:
\begin{align*}
\sup_i \|\nabla_x^{(m)} \bar{F}_{si}\|_{L^{5/4,\infty}_x \dot{L}^2_x} + \sup_i \|\nabla_x^{(m)} \bar{F}_{si}\|_{L^{5/4,2}_x \dot{L}^2_x} \leq C_{m,T} \cdot \mathcal{T}.
\end{align*}
This gives an adequate control on $\|\nabla_x F_{si}\|_{L^{5/4,p}_x H^m_x}$ for $p = 2, \infty$. 

In order to estimate the terms involving the time derivative, we must look at \( \tilde{F}_{0i} \). Recall that \( \tilde{F}_{0i} \) satisfies the covariant heat equation

\[
\partial_s \tilde{F}_{0i} = \tilde{D}^j \tilde{D}_j \tilde{F}_{0i} - 2[\tilde{F}_{0i}, \tilde{F}_{0i}]_t.
\]

Applying Proposition 5.6 with (6.4), we obtain the following estimates on \( \tilde{F}_{0i} \) for \( m \geq 0 \).

\[
(6.7) \quad \sup_i \| \tilde{F}_{0i} \|_{L_x^{2/4} H_x^m} + \sup_i \| \tilde{F}_{0i} \|_{L_x^{3/4} H_x^{m+1}} \leq C_{m,T} \cdot \mathcal{I}.
\]

Because of our gauge condition \( \tilde{A}_0(s) = 0 \), we have \( \partial_s \tilde{A}_i(s) = \tilde{F}_{0i}(s = 1) \). This immediately proves \( \| \partial_0 \tilde{A} \|_{H_x^m} \leq C \cdot \mathcal{I} \).

In order to estimate the term \( \partial_0 \tilde{F}_{si} \), we need to adapt Step 3 to differences. In particular, we remark that everything we do here is done in \( \tilde{A}_i \) and their spatial derivatives. We estimate \( \tilde{F}_{si} \) by (LYMIF)

\[
(6.9) \quad \tilde{F}_{si} = \tilde{D}^i \tilde{F}_{si} = \partial^i \tilde{F}_{0i} + \tilde{A}_0 \tilde{F}_{si} + \tilde{A}_i \tilde{F}_{0i},
\]

whereas \( \tilde{A}_0 \) is estimated by the equation

\[
(6.10) \quad \tilde{A}_0(s) = -\int_s^1 \tilde{F}_{si}(s') \, ds',
\]

which holds due to the caloric-temporal gauge condition.

Using (6.8), (6.9), (6.10) and the previous estimates for \( \tilde{F}_{0i}, \tilde{F}_{si} \) and \( \tilde{A}_i \), it is not difficult to show

\[
\sup_i \| \nabla_0 \tilde{F}_{si} \|_{L_x^{5/4} L_2^2} + \sup_i \| \nabla_0 \tilde{F}_{si} \|_{L_x^{5/4} L_2^2} \leq C_{m,T} \cdot \mathcal{I}.
\]

Differentiating (6.8), (6.9) and (6.10) appropriate number of times with respect to \( x \), \( \nabla_x (m) \nabla_0 \tilde{F}_{si} \) can be estimated analogously; we leave the details to the reader. This concludes the proof of \( \mathcal{I} \leq C_{m,T} \cdot \mathcal{I} \).

**Step 4: Quantitative Initial Data Estimates for Differences.** In the fourth and the final step, we will prove the difference estimates among (4.12) – (4.14).

Let \( (A, U) \), \( (A', U') \) be constructed according to the Steps 1, 2 from \( A_i', (A')_i^\dagger \), respectively. The idea is to adapt Step 3 to differences. In particular, we remark that everything we do here is done on \( \{ t = 0 \} \).

In order to prove \( \delta \mathcal{I} \leq C_{m,T} \cdot \delta \mathcal{I} \), we must estimate \( \| \nabla_{L_x} (\delta F_{si}) \|_{L_x^{5/4} L_x^2} \) for \( p = 2, 0 \) and \( \| \partial_{t,x} A_i \|_{H_x^m} \) in terms of \( C_{m,T} \cdot \delta \mathcal{I} \). To prove (4.11), we need to show

\[
(6.11) \quad \begin{cases}
\| \delta U(t = 0, s = 0) \|_{L_x^\infty} + \| \partial_x \delta U(t = 0, s = 0) \|_{L_x^2} \leq C_{m,T} \cdot \delta \mathcal{I}, \\
\| \partial_x^2 \delta U(t = 0, s = 0) \|_{L_x^2} \leq C_{m,T} \cdot \delta \mathcal{I},
\end{cases}
\]

and their analogues for \( \delta U^{-1} \).

From Proposition 5.2 we have the following analogue of (6.2) for \( \delta A_i := A_i - A_i' \) at \( t = 0 \), for \( m \geq 1 \).

\[
(6.12) \quad \sup_i \| \delta A_i \|_{L_x^{1/4} H_x^m} + \sup_i \| \delta A_i \|_{L_x^{3/4} H_x^{m+1}} \leq C_{m,T} \cdot \delta \mathcal{I}.
\]

As before, this immediately shows \( \| \partial_x A_i \|_{H_x^m} \leq C_{m,T} \cdot \delta \mathcal{I} \), since the gauge transforms \( U, U' \) equal to the identity \( \text{Id} \) at \( t = 0, s = 1 \). Next, from Lemma 5.4 and the bounds (6.2) (for \( A, A' \)) and (6.12), we have the following analogue of (6.13) for \( \delta U = U - U', \delta U^{-1} = U^{-1} - (U')^{-1} \) at \( t = 0 \), for \( m \geq 2 \).

\[
(6.13) \quad \| \delta U \|_{L_x^\infty L_x^\infty} + \| \partial_x (\delta U) \|_{L_x^\infty L_x^2} \leq C_{m,T} \cdot \delta \mathcal{I}, \quad \| s^{(m-2)/2} \partial_x (\delta U) \|_{L_x^\infty L_x^2} \leq C_{m,T} \cdot \delta \mathcal{I}.
\]
Analogous estimates hold also for $\delta U^{-1}$. We remark that in contrast to (6.3), the right hand side vanishes as $\delta U \to 0$. Since $\delta V$ in the statement of the theorem is exactly $\delta V = \delta U(t = 0, s = 0)$, the estimate (6.11) follows (for both $\delta U$ and $\delta U^{-1}$).

Next, we estimate $\delta F_{si}$ in the statement of the theorem, which in our case is $\delta \tilde{F}_{si} := \tilde{F}_{si} - \tilde{F}'_{si}$. The idea is the same as before, namely we want to use the gauge transform property of $F_{si}, F'_{si}$. As in Step 3, we begin by establishing some bounds for $\delta A_i$. Our starting point is the following formula for the difference $\delta A_i$.

$$\delta \tilde{A}_i = (\delta U) A_i U^{-1} + U' (\delta A_i) U^{-1} + U' A'_i (\delta U^{-1}) - \left( \partial_x (\delta U) \right) U^{-1} - \partial_x U' (\delta U^{-1}).$$

Note that this is nothing but Leibniz’s rule for $\delta$. Differentiating the above formula $m$-times and using the estimates (6.2), (6.12), along with their difference analogues (6.12), (6.13), it is not difficult to prove the analogue of (6.4) for $\delta \tilde{A}_i := \tilde{A}_i - \tilde{A}'_i$, for $m \geq 1$

$$\sup_i \|\delta \tilde{A}_i\|_{L^{1/4,\infty}_{\delta} H^m} \leq C_{m,T} \cdot \delta I.$$

The next step is to estimate $\delta F_{si}$. Using Leibniz’s rule for $\delta$, we obtain the following formula for $\delta F_{si}$:

$$\delta F_{si} = O(\partial^2_x (\delta A)) + O(\delta A, \partial_x A) + O(A, \partial_x (\delta A)) + O(\delta A, A, A).$$

From this formula and the estimates (6.2), (6.12), we obtain the following estimate for $\delta F_{si}$ for $m \geq 0$ as before:

$$\sup_i \|\nabla_x^{(m)} (\delta F_{si})\|_{L^{1/4,\infty}_{\delta} L^2_x} + \sup_i \|\nabla_x^{(m)} (\delta F_{si})\|_{L^{1/4,2}_{\delta} L^2_x} \leq C_{m,T} \cdot \delta I.$$

Equipped with these estimates, we claim that the following estimates are true, for $m \geq 0$.

$$\sup_i \|\delta D_x^{(m)} (\delta F_{si})\|_{L^{1/4,\infty}_{\delta} L^2_x} + \sup_i \|\delta D_x^{(m)} (\delta F_{si})\|_{L^{1/4,2}_{\delta} L^2_x} \leq C_{m,T} \cdot \delta I,$$

where $\delta D_x^{(m)} (\delta F_{si}) := \nabla_x^{(m)} (\delta F_{si}) - \nabla_x^{(m)} (\delta F'_{si})$, and accordingly $\delta \tilde{D}_x^{(m)} (\delta F_{si}) := \tilde{D}_x^{(m)} (\delta F_{si}) - \tilde{D}_x^{(m)} (\delta F'_{si})$.

For $m = 0$, this is an easy consequence of the difference formula

$$\tilde{F}_{si} - \tilde{F}'_{si} = U F_{si} U^{-1} - U' F'_{si} (U')^{-1} = (\delta U) F_{si} U^{-1} + U' (\delta F_{si}) U^{-1} + U' F'_{si} (\delta U^{-1}),$$

and the estimates (6.2), (6.39), (6.14) and (6.15).

For $m = 1$, we begin to see covariant derivatives, which we just write out in terms of ordinary derivatives and connection 1-forms $A_i$. Then we can easily check, using Leibniz’s rule for $\delta$, that the following difference formula holds.

$$\delta (\tilde{D}_x^{(m)} (\delta F_{si})) = U D_x F_{si} U^{-1} - U' D_x F'_{si} (U')^{-1} = O(\delta U, F_{si}, U^{-1}) + O(U, \delta F_{si}, U^{-1}) + O(U, F_{si}, U^{-1}) + O(U, A, F_{si}, U^{-1}) + O(U, A, F_{si}, U^{-1}) + O(U, A, F_{si}, U^{-1}).$$

Using the estimates (6.2), (6.3), (6.13), (6.15), as well as (6.2), (6.12) with interpolation, we obtain (6.16) for $m = 1$.

The cases $m \geq 2$ can be proved in an analogous fashion, namely first computing the difference formula for $\tilde{D}_x^{(m)} (\delta F_{si}) - (\tilde{D}_x^{(m)} F_{si})$ by writing out all covariant derivatives and applying Leibniz’s rule for $\delta$, and then estimating using (6.3), (6.5), (6.13), (6.15), as well as (6.2), (6.12) with interpolation. We leave the tedious details to the interested reader.

Using (6.1) and (6.14), we can easily substitute the covariant derivatives in (6.16) by ordinary derivatives. This proves $\|\nabla_x (\delta F_{si})\|_{L^{1/4,\infty} H^m_x} \leq C_T \cdot \delta I$ for $p = 2, \infty$.

We are now only left to estimate the norms involving $\partial_0$. From Proposition 5.7, we have the following analogue of (6.7) for $\delta F_{0i}$, for $m \geq 0$.

$$\sup_i \|\delta F_{0i}\|_{L^{1/4,\infty}_{\delta} H^m} + \sup_i \|\delta F_{0i}\|_{L^{1/4,2}_{\delta} H^{m+1}} \leq C_{m,T} \cdot \delta I.$$

(6.17)
As before, since $(\partial_0 \tilde{A}_i - \partial_0 \tilde{A}'_i)(t = 0, s = 1) = (\tilde{F}_0 - \tilde{F}'_0)(t = 0, s = 1)$, this proves $\|\partial_0 (\delta \tilde{A})\|_{H^{10}_0} \leq C_T \delta T$.

Finally, we turn to the estimate $\|\nabla_0 (\delta F_i)\|_{L^{5/4}_0 L^6 T} \leq C_T \delta T$ for $p = 2, \infty$. For this purpose, note that we have the following difference versions of (6.8), (6.9), (6.10), as follows (in order).

\[
\partial_0 (\delta \tilde{F}_0) = \partial_s (\delta \tilde{F}_0) + \partial_i (\delta \tilde{F}_0) - [\delta \tilde{A}_0, \tilde{F}_0] - [\tilde{A}'_0, \delta \tilde{F}_0] + [\delta \tilde{A}_1, \tilde{F}_0] + [\tilde{A}'_1, \delta \tilde{F}_0],
\]

\[
\delta \tilde{F}_0 = \partial' (\delta \tilde{F}_0) + [\delta \tilde{A}', \tilde{F}_0] + [(\tilde{A}')', \delta \tilde{F}_0],
\]

\[
\delta \tilde{A}_0 (s) = - \int_s^1 \delta \tilde{F}_0 (s') \, ds'.
\]

Taking the appropriate number of derivatives of the above equations and using the previous bounds, the desired estimate follows. We leave the easy details to the reader. \qed

7. Definition of norms and reduction of Theorem B

The purpose of the rest of the paper is to prove Theorem B. In this section, which is of preliminary nature, we first introduce the various norms which will be used in the sequel, and reduce Theorem B to six smaller statements: Propositions 7.1, 7.2, 7.3 and 7.4 and Theorems 7.5 and 7.6.

7.1. Definition of norms. In this subsection, we define the norms $A_0$, $\tilde{A}_I$, $F$ and $E$, along with their difference analogous.

Let $I \subset \mathbb{R}$ be a time interval. The norms $A_0(I)$ and $\delta A_0(I)$, which are used to estimate the gauge transform back to the temporal gauge $A_0 = 0$ at $s = 0$, are defined by

\[
A_0(I) := \|A_0\|_{L^\infty_T L^2_x} + \|\partial_x A_0\|_{L^1_T L^2_x} + \|A_0\|_{L^1_T L^\infty_x} + \|\partial_x A_0\|_{L^1_T L^\infty_x} + \|\partial_x (\partial A_0)\|_{L^1_T L^2_x},
\]

\[
\delta A_0(I) := \|\delta A_0\|_{L^\infty_T L^2_x} + \|\partial_x (\delta A_0)\|_{L^1_T L^2_x} + \|\delta A_0\|_{L^1_T L^\infty_x} + \|\partial_x (\delta A_0)\|_{L^1_T L^\infty_x} + \|\partial_x (\delta A_0)\|_{L^1_T L^\infty_x},
\]

where $A_0, \delta A_0$ are evaluated at $s = 0$.

The norms $\tilde{A}_I(I)$ and $\delta \tilde{A}_I(I)$, which control the sizes of $\tilde{A}_I$ and $\delta \tilde{A}_I$, respectively, are defined by

\[
\tilde{A}_I(I) := \sup_i \|A_i\|_{L^\infty_T H^{1,1}} + \|\partial_x (\partial \times A)_i\|_{L^\infty_T H^{2,2}} + \sum_{k=1}^{30} \sup_i \|A_i\|_{S^{k}},
\]

\[
\delta \tilde{A}_I(I) := \sup_i \|\delta A_i\|_{L^\infty_T H^{1,1}} + \|\partial_x (\partial \times \delta A)_i\|_{L^\infty_T H^{2,2}} + \sum_{k=1}^{30} \sup_i \|\delta A_i\|_{S^{k}}.
\]

Here, $(\partial_x \times B)_i := \epsilon_{ijk} \partial^j B^k$, where $\epsilon_{ijk}$ is the Levi-Civita symbol, i.e. the completely antisymmetric 3-tensor on $\mathbb{R}^3$ with $\epsilon_{123} = 1$.

Next, let us define the norms $F(I)$ and $\delta F(I)$, which control the sizes of $F_{si}$ and $\delta F_{si}$, respectively.

\[
F(I) := \sum_{k=1}^{10} \left( \sup_i \|F_{si}\|_{L^{5/4}_0 S^k (0,1]} + \sup_i \|F_{si}\|_{L^{9/2}_0 S^k (0,1]} \right),
\]

\[
\delta F(I) := \sum_{k=1}^{10} \left( \sup_i \|\delta F_{si}\|_{L^{5/4}_0 S^k (0,1]} + \sup_i \|\delta F_{si}\|_{L^{9/2}_0 S^k (0,1]} \right).
\]

We remark that $F(I)$ (also $\delta F(I)$) controls far less derivatives compared to $\tilde{A}_I(I)$. Nevertheless, it is still possible to close a bootstrap argument on $F + \tilde{A}_I$, thanks to the fact that $F_{si}$ satisfies a parabolic equation, which gives smoothing effects. The difference between the numbers of controlled derivatives, in turn, allows us to be lenient about the number of derivatives of $\tilde{A}_I$, we use when studying the wave equation for $F_{si}$. We refer the reader to Remark 8.9 for a more detailed discussion.
For \( t \in I \), we define \( \mathcal{E}(t) \) and \( \delta \mathcal{E}(t) \), which control the sizes of low derivatives of \( F_{s0}(t) \) and \( \delta F_{s0}(t) \), respectively, by

\[
\mathcal{E}(t) := \sum_{m=1}^{3} \left( \| F_{s0}(t) \|_{L^1_t \cdot \mathcal{H}^{\infty}_{1}([0,1])} + \| F_{s0}(t) \|_{L^1_t \cdot \mathcal{H}^{\infty}_{m}(0,1)} \right),
\]

\[
\delta \mathcal{E}(t) := \sum_{m=1}^{3} \left( \| \delta F_{s0}(t) \|_{L^1_t \cdot \mathcal{H}^{\infty}_{1}([0,1])} + \| \delta F_{s0}(t) \|_{L^1_t \cdot \mathcal{H}^{\infty}_{m}(0,1)} \right).
\]

We furthermore define \( \mathcal{I}(I) := \sup_{t \in I} \mathcal{E}(t) \) and \( \delta \mathcal{I}(I) := \sup_{t \in I} \delta \mathcal{E}(t) \).

### 7.2. Statement of Propositions 7.1 - 7.4 and Theorems D, E

For the economy of notation, we will omit the dependence of the quantities and norms on the time interval \((-T, T)\); in other words, all quantities and space-time norms below should be understood as being defined over the time interval \((-T, T)\) with \(0 < T \leq 1\).

**Proposition 7.1** (Improved estimates for \( A_0 \)). Let \( A_n, A'_n \) be regular solutions to \([\text{HPYM}]\) in the caloric-temporal gauge and \(0 < T \leq 1\). Then the following estimates hold.

\[
A_n \leq C_{\mathcal{F}, \mathcal{A}} \cdot \mathcal{E} + C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})^2, \tag{7.1}
\]

\[
\delta A_n \leq C_{\mathcal{F}, \mathcal{A}} \cdot \delta \mathcal{E} + C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})(\delta \mathcal{F} + \delta \mathcal{A}) . \tag{7.2}
\]

**Proposition 7.2** (Improved estimates for \( A_i \)). Let \( A_n, A'_n \) be regular solutions to \([\text{HPYM}]\) in the caloric-temporal gauge and \(0 < T \leq 1\). Then the following estimates hold.

\[
\sup_{i} \sup_{0 \leq t \leq 1} \| A_i(s) \|_{\mathcal{F}} \leq C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A}),
\]

\[
\sup_{i} \sup_{0 \leq t \leq 1} \| \delta A_i(s) \|_{\mathcal{F}} \leq C_{\mathcal{F}, \mathcal{A}} \cdot (\delta \mathcal{F} + \delta \mathcal{A}).
\]

**Proposition 7.3** (Estimates for \( \mathcal{E} \)). Let \( A_n, A'_n \) be regular solutions to \([\text{HPYM}]\) in the caloric-temporal gauge and \(0 < T \leq 1\). Suppose furthermore that the smallness assumption

\[
\mathcal{F} + \mathcal{A} \leq \delta \mathcal{E},
\]

holds for sufficiently small \( \delta \mathcal{E} > 0 \). Then the following estimates hold.

\[
\mathcal{E} \leq C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})^2 , \tag{7.3}
\]

\[
\delta \mathcal{E} \leq C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})(\delta \mathcal{F} + \delta \mathcal{A}). \tag{7.4}
\]

**Proposition 7.4** (Continuity properties of \( \mathcal{F}, \mathcal{A} \)). Let \( A_n, A'_n \) be regular solutions to \([\text{HPYM}]\) in the caloric-temporal gauge on some interval \( I_0 := (-T_0, 0) \). For \( \mathcal{F} = \mathcal{F}(I), \mathcal{A} = \mathcal{A}(I) \) (\( I \subseteq I_0 \)) and their difference analogues, the following continuity properties hold.

- The norms \( \mathcal{F}(-T, T) \) and \( \mathcal{A}(-T, T) \) are continuous as a function of \( T \) (where \( 0 < T < T_0 \)).
- Similarly, the norms \( \delta \mathcal{F}(-T, T) \) and \( \delta \mathcal{A}(-T, T) \) are continuous as a function of \( T \).
- We furthermore have

\[
\limsup_{T \to 0^+} \left( \mathcal{F}(-T, T) + \mathcal{A}(-T, T) \right) \leq C \mathcal{I},
\]

\[
\limsup_{T \to 0^+} \left( \delta \mathcal{F}(-T, T) + \delta \mathcal{A}(-T, T) \right) \leq C \delta \mathcal{I}.
\]

**Theorem D** (Hyperbolic estimates for \( \mathcal{A} \)). Let \( A_n, A'_n \) be regular solutions to \([\text{HPYM}]\) in the caloric-temporal gauge and \(0 < T \leq 1\). Then the following estimates hold.

\[
\mathcal{A} \leq C \mathcal{I} + T \left( C_{\mathcal{F}, \mathcal{A}} \cdot \mathcal{E} + C_{\mathcal{E}, \mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})^2 \right) , \tag{7.5}
\]

\[
\delta \mathcal{A} \leq C \delta \mathcal{I} + T \left( C_{\mathcal{F}, \mathcal{A}} \cdot \delta \mathcal{E} + C_{\mathcal{E}, \mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})(\delta \mathcal{E} + \delta \mathcal{F} + \delta \mathcal{A}) \right) . \tag{7.6}
\]
**Theorem E** (Hyperbolic estimates for $F_{si}$). Let $A_0, A'_0$ be regular solutions to \( \text{(HPYM)} \) in the caloric-temporal gauge and $0 < T \leq 1$. Then the following estimates hold.

\[
(7.7) \quad F \leq CT + T^{1/2}C r_{E,F} \cdot (E + F + A)^2,
\]

\[
(7.8) \quad \delta F \leq C \delta I + T^{1/2}C r_{E,F} \cdot (E + F + A)(\delta E + \delta F + \delta A).
\]

A few remarks are in order concerning the above statements.

The significance of Propositions 7.1 and 7.2 is that they allow us to pass from the quantities $F$ and $A$ to the norms of $A_1$ and $A_0$ on the left-hand side of (4.16). Unfortunately, a naive approach to any of these will fail, leading to a logarithmic divergence. The structure of \( \text{(HPYM)} \), therefore, has to be used in a crucial way in order to overcome this.

Proposition 7.3, which will be proved in §8.1, deserves some special remarks. This is a perturbative result for the parabolic equation for $F_{s0}$, meaning that we need some smallness to estimate the nonlinearity. However, the latter fact has the implication that the required smallness cannot come from the size of the time interval, but rather only from the size of the data \( (F + A) \) or the size of the $s$-interval. It turns out that this feature causes a little complication in the proof of global well-posedness in §19. Therefore, in §19, we will prove a modified version of Proposition 7.3 using more covariant techniques to analyze the (covariant) parabolic equation for $F_{0i}$, which allows one to get around this issue.

In this work, to opt for simplicity, we have chosen to fix the $s$-interval to be $[0, 1]$ and make $I$ (therefore $F + A$) small by scaling, exploiting the fact that $I$ is subcritical with respect to the scaling of the equation (compare this with the smallness required for Uhlenbeck’s lemma, which cannot be obtained by scaling). We remark, however, that it would have been just as fine to keep $I$ large and obtain smallness by shrinking the size of the $s$-interval.

The proof of Theorem B will be via a bootstrap argument for $F + A$, and Proposition 7.3 provides the necessary continuity properties. In fact, Proposition 7.3 is a triviality in view of the simplicity of our function spaces and the fact that $A_0, A'_0$ are regular. On the other hand, Theorems D and E obtained by analyzing the hyperbolic equations for $A$ and $F_{s0}$, respectively, give the main driving force of the bootstrap argument. Observe that these estimates themselves do not require any smallness. This will prove to be quite useful in the proof of global well-posedness in §19.

As we need to use some results derived from the parabolic equations of \( \text{(HPYM)} \), we will defer the proofs of Propositions 7.1 - 7.3 along with further discussion, until Section 9. The proofs of Theorems D and E will be the subject of Section 10.

### 7.3. Proof of Theorem B

Assuming the above statements, we are ready to prove Theorem B.

**Proof of Theorem B.** Let $A_0, A'_0$ be regular solutions to \( \text{(HPYM)} \) in the caloric-temporal gauge, defined on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$. As usual, $I$ will control the sizes of both $A_0$ and $A'_0$ at $t = 0$, in the manner described in Theorem A.

Let us prove (4.16). We claim that

\[
(7.9) \quad F(-T, T) + A(-T, T) \leq BI
\]

for a large constant $B$ to be determined later, and $I < \delta H$ with $\delta H > 0$ sufficiently small. By taking $B$ large enough, we obviously have $F(-T', T') + A(-T', T') \leq BI$ for $T' > 0$ sufficiently small by Proposition 7.3. This provides the starting point of the bootstrap argument.

Next, for $0 < T' \leq T$, let us assume the following bootstrap assumption:

\[
(7.10) \quad F(-T', T') + A(-T', T') \leq 2BI.
\]

The goal is to improve this to $F(-T', T') + A(-T', T') \leq BI$.

Taking $2BI$ to be sufficiently small, we can apply Proposition 7.3 and estimate $E \leq C_T A(F + A)^2$. (We remark that in order to close the bootstrap, it is important that $E$ is at least quadratic in $F + A$.) Combining this with Theorems D and E and removing the powers of $T'$ by using the fact that $T' \leq T \leq 1$, we obtain

\[
F(-T', T') + A(-T', T') \leq C I + C_{F(-T', T')} A(-T', T') (F(-T', T') + A(-T', T'))^2.
\]
Using the bootstrap assumption (7.10) and taking $2BT$ to be sufficiently small, we can absorb the last term into the left-hand side and obtain

$$F(-T', T') + A(-T', T') \leq C\mathcal{I}.$$  

Therefore, taking $B$ sufficiently large, we beat the bootstrap assumption, i.e. $F(-T', T') + A(-T', T') \leq 2B\mathcal{I}$. Using this, a standard continuity argument gives (7.9) as desired.

From (7.8), estimate (4.16) follows immediately by Propositions 7.1, 7.2 and 7.3.

Next, let us turn to (4.17). By essentially repeating the above proof for $\delta F + \delta A$, and using the estimate (7.10) as well, we obtain the following difference analogue of (7.9):

$$\delta F(-T, T) + \delta A(-T, T) \leq C_\varepsilon \cdot \delta \mathcal{I}.$$  

From (7.9) and (7.11), estimate (4.17) follows by Propositions 7.1, 7.2 and 7.3.  

8. Parabolic equations of the hyperbolic-parabolic Yang-Mills system

In this section, we analyze the parabolic equations of (HPYM) for the variables $F_{si}, F_{s0}$ and $w_i$. The results of this analysis provide one of the ‘analytic pillars’ of the proof of Theorem B that had been outlined in Section 7, the other ‘pillar’ being the hyperbolic estimates in Section 10. Moreover, the hyperbolic estimates in Section 10 depend heavily on the results of this section as well.

As this section is a bit long, let us start with a brief outline. Beginning in §8.1 with some preliminaries, we prove in §8.2 smoothing estimates for $F_{si}$ (Proposition 8.8), which allow us to control higher derivatives of $\partial_{t,s} F_{si}$ in terms of $F$, provided that we control high enough derivatives of $\partial_{t,s} A$. In §8.3 we also prove that $F_{si}$ itself (i.e. without any derivative) can be controlled in $L^\infty_t L^2_s$ and $L^1_t L^4_s$ by $F + A$ as well (Proposition 8.11). Next, in §8.4 we study the parabolic equation for $F_{s0}$. Two main results of this subsection are Propositions 8.13 and 8.14. The former states that low derivatives of $F_{s0}$ (i.e. $E$) can be controlled under the assumption that $F + A$ is small, whereas the latter says that once $E$ is under control, higher derivatives of $F_{s0}$ can be controlled (without any smallness assumption) as long as high enough derivatives of $A$ are under control. Finally, in §8.5 we derive parabolic estimates for $w_i$ (Proposition 8.17). Although these are similar to those proved for $F_{s0}$, it is important (especially in view of the proof of finite energy global well-posedness in [19]) to note that no smallness of $F + A$ is required in this part.

Throughout the section, we will always work with regular solutions $A_{\alpha}, A_{\alpha}'$ to (HPYM) on $I \times \mathbb{R}^3 \times [0, 1]$, where $I = (-T, T)$.

8.1. Preliminary estimates. Let us begin with a simple integral inequality.

**Lemma 8.1.** For $\delta > 0$ and $1 \leq q \leq p \leq \infty$, the following estimate holds.

$$\| \int_0^1 (s/s')^\delta f(s') \frac{ds'}{s'} \|_{L^p_s(0, 1)} \leq C_{\delta, p,q} \| f \|_{L^q_s(0, 1)}.$$  

**Proof.** This is rather a standard fact about integral operators. By interpolation, it suffices to consider the three cases $(p, q) = (1, 1), (\infty, 1)$ and $(\infty, \infty)$. The first case follows by Fubini, using the fact that $\sup_{0<s'\leq1} \int_0^1 1_{[0,\infty)}(s' - s) (s/s')^\delta ds/\delta \leq C_\delta$, as $\delta > 0$. On the other hand, the second and the third cases (i.e. $p = \infty$ and $q = 1, \infty$) follow by Hölder, using furthermore the fact that $\sup_{0<s, s'\leq1} 1_{[0,\infty)}(s' - s) (s/s')^\delta \leq 1$ and $\sup_{0<s\leq1} \int_0^1 1_{[0,\infty)}(s' - s) (s/s')^\delta ds/\delta \leq C_\delta$, respectively.

By the caloric-temporal gauge condition, we have $\partial_s A_{\alpha} = F_{s\mu}$. Therefore, we can control $A_{\alpha}$ with estimates for $F_{s\mu}$ and $A$. The following two lemmas make this idea precise.

**Lemma 8.2.** Let $X$ be a homogeneous norm of degree $2s_0$. Suppose furthermore that the caloric gauge condition $A_s = 0$ holds. Then for $k, \ell \geq 0$ and $1 \leq q \leq p \leq \infty$ such that $1/4 + k/2 + \ell - s_0 > 0$, the following estimate holds.

$$\| A_i \|_{L^{q/4+k/2}_{s'} L^{q}_{s'}(0, 1)} \leq C(\| F_s \|_{L^{q/4+k}_{s'} L^{q}_{s'}(0, 1)} + \| A \|_{L^q_{s}})$$  

where $C$ depends on $p, q$ and $r(\ell, k, s_0) := 1/4 + k/2 + \ell - s_0$.  

Proof. By the caloric gauge condition $A_s = 0$, it follows that $\partial_s A_i = F_{si}$. By the fundamental theorem of calculus, we have

$$A_i(s) = -\int_s^1 s' F_{si}(s') \frac{ds'}{s'} + A_i,$$

Let us take the $L_s^{1+\ell,p} X^k(0,1)$-norm of both sides. Defining $r(\ell,k,\ell_0) = 1/4 + k/2 + \ell - \ell_0$, we easily compute

$$\| \int_s^1 s' F_{si}(s') \frac{ds'}{s'} \|_{L_s^{1+\ell,p} X^k(0,1)} = \| \int_s^1 (s'/s')^{r(\ell,k,\ell_0)} (s')^{\ell} F_{si}(s') \frac{ds'}{s'} \|_{L_s^{1+p}(0,1)} \leq \| \int_s^1 (s'/s')^{r(\ell,k,\ell_0)} (s')^{\ell} F_{si}(s') \|_{X^k(0,1)} \cdot$$

where on the second line we used $\ell \geq 0$. Since $r > 0$, we can use Lemma 8.1 to estimate the last line by $C_{p,r} \| F_{si} \|_{L_s^{1+\ell,p} X^k(0,1)}$.

On the other hand, $A_i$ is independent of $s$, and therefore

$$\| A_i \|_{L_s^{1+\ell,p} X^k(0,1)} = \| s^{r(\ell,k,\ell_0)} \|_{L_s^{1+p}(0,1)} \| A_i \|_{X^k} \leq C_{p,r} \| A_i \|_{X^k},$$

where the last inequality holds as $r > 0$. \qed

The following analogous lemma for $A_0$, whose proof we omit, can be proved by a similar argument.

**Lemma 8.3.** Let $X$ be a caloric-temporal gauge condition $A_0 = 0$. Suppose furthermore that the caloric-temporal gauge condition $A_s = 0$. Then for $k, \ell \geq 0$ and $1 \leq q \leq p \leq \infty$ such that $k/2 + \ell - \ell_0 > 0$, the following estimate holds.

$$\| A_0 \|_{L_s^{1+\ell,p} X^k(0,1)} \leq C \| F_{s0} \|_{L_s^{1+\ell,p} X^k(0,1)},$$

where $C$ depends on $p$ and $r'(\ell,k,\ell_0) := k/2 + \ell - \ell_0$.

Some of the most frequently used choices of $X$ are $X = \dot{S}^k$ for Lemma 8.2, $X = L_t^2 \dot{H}_x^k$ for Lemma 8.3, and $X = \dot{H}_x^k, \dot{W}_x^k$ for both. Moreover, these lemmas will frequently applied to norms which can be written as a sum of such norms, e.g. $L_s^{1+\ell,p} H_x^m$, which is the sum of $L_s^{1+\ell,p} H_x^k$ norms for $k = 0, \ldots, m$.

As an application of the previous lemmas, we end this subsection with estimates for some components of the curvature 2-form and its covariant derivative.

**Lemma 8.4 (Bounds for $F_{0i}$).** Suppose that the caloric-temporal gauge condition $A_s = 0$. Then:

1. The following estimate holds for $2 \leq p \leq \infty$:

$$\| F_{0i}(t) \|_{L_s^{1+\ell,p} \dot{H}_x^{1/2}} \leq C_{p,k} \left( \| \nabla_0 F_{si}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{1/2}} + \| \partial_\ell A_i(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{1/2}} + \| F_{s0}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{1/2}} + \| \nabla_0 F_{s0}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{1/2}} + \| \nabla_0 F_{si}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{1/2}} + \| \nabla_0 F_{s0}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{1/2}} + \| \nabla_0 F_{si}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{1/2}} + \| \nabla_0 F_{0i}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{1/2}} \right).$$

2. For any $2 \leq p \leq \infty$ and $k \geq 1$ an integer, we have

$$\| F_{0i}(t) \|_{L_s^{1+\ell,p} \dot{H}_x^k} \leq C_{p,k} \left( \| \nabla_0 F_{si}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^k} + \| \partial_\ell A_i(t) \|_{L_s^{1+\ell,2} \dot{H}_x^k} + \| F_{s0}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^k} + \| \nabla_0 F_{s0}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^k} + \| \nabla_0 F_{si}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^k} + \| \nabla_0 F_{s0}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^k} + \| \nabla_0 F_{si}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^k} + \| \nabla_0 F_{0i}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^k} \right).$$

3. For any $2 \leq p \leq \infty$ and $k \geq 0$ an integer, we have

$$\| F_{0i} \|_{L_s^{1+\ell,p} \dot{H}_x^{k+1}} \leq C_{p,k} \left( \| F_{si} \|_{L_s^{1+\ell,2} \dot{H}_x^{k+1}} + \| \partial_\ell A_i \|_{L_s^{1+\ell,2} \dot{H}_x^{k+1}} + T^{1/4} \sup_{t \in I} \| F_{s0}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{k+1/2}} \right. + T^{1/4} \sup_{t \in I} \| \nabla_0 F_{0i}(t) \|_{L_s^{1+\ell,2} \dot{H}_x^{k+1/2}} + \| \partial_\ell A_i \|_{L_s^{1+\ell,2} \dot{H}_x^{k+1}}).$$
Proof. Let us begin with the identity

$$F_{0i} = \partial_0 A_i - \partial_i A_0 + [A_0, A_i] = s^{-1/2} \nabla_0 A_i + s^{-1/2} \nabla_i A_0 + [A_0, A_i],$$

Applying Lemma 8.2 to $s^{-1/2} \nabla_0 A_i$ and Lemma 8.3 to $s^{-1/2} \nabla_i A_0$, the estimates (8.1) and (8.2) are reduced to the product estimates

$$(8.4) \quad ||[A_0, A_i](t)||_{L^{3/4, p} H^{1/2}_s} \leq C_p ||\nabla_x F_{s0}(t)||_{L^{1, 2}_x L^{2}_t} ||\nabla_x F_{st}(t)||_{L^{5/4, 2}_x L^{2}_s} + ||\partial_x A_i(t)||_{L^{2}}),$$

$$(8.5) \quad ||[A_0, A_i](t)||_{L^{3/4, p} H^{1/2}_s} \leq C_p ||\nabla_x F_{s0}(t)||_{L^{1, 2}_x L^{2}_t} ||\nabla_x F_{st}(t)||_{L^{5/4, 2}_x L^{2}_s} + ||\partial_x A_i(t)||_{H^{1/2}_s},$$

respectively.

Let us start with the product estimate

$$(8.6) \quad ||\phi_1 \phi_2||_{H^{1/2}_s} \leq C ||\phi_1||_{H^{1/2}_s} ||\phi_2||_{H^{1/2}_s},$$

which follows from the product rule for homogeneous Sobolev norms (Lemma 3.2). Applying the Correspondence Principle and Lemma 8.3 we obtain

$$||[A_0, A_i](t)||_{L^{3/4, p} H^{1/2}_s} \leq C_p ||A_0(t)||_{L^{0, 3/8, \infty} H^{1/2}_s} ||A_i(t)||_{L^{1/4 + 1/8, p} H^{1/2}_s}.$$

Note the extra weights of $s^{3/8}$ and $s^{1/8}$ for $A_0$ and $A_i$, respectively. Applying Lemma 8.2 to $A_0$ and Lemma 8.3 to $A_i$, the desired estimate (8.1) follows.

The other product estimate (8.6) can be proved by a similar argument, this time starting with $||\phi_1 \phi_2||_{H^{1/2}_s} \leq C ||\phi_1||_{H^{1/2}_s} ||\phi_2||_{H^{1/2}_s}$, (which follows again from Lemma 3.2) instead of (8.6), and using Leibniz’s rule to deal with the cases $k \geq 2$.

Finally, let us turn to (8.3). We use Lemma 8.2 and Strichartz to control $s^{-1/2} \nabla_0 A_i$, and Lemma 8.3 Hölder in time and Sobolev for $s^{-1/2} \nabla_x A_0$. Then we are left to establish

$$(8.7) \quad ||[A_0, A_i]||_{L^{3/4, p} L^{6, 4}_t W^{k, 4}} \leq C_p T^{1/4} \sup_{t \in I} ||\nabla_x F_{s0}(t)||_{L^{1, 2}_x L^{4}_t} ||F_{st}(t)||_{L^{5/4, 2} L^{6}_s} + ||A_i||_{L^{0, 5/16, \infty} H^{1/2}_s}.$$

To prove (8.7), one starts with $||\phi_1 \phi_2||_{L^{6/11, 4}_t L^{4, 4}_x} \leq C ||I||^{1/4} ||\phi_1||_{L^{6/11, \infty} H^{1/2}_s} ||\phi_2||_{L^{6, \infty} H^{1/2}_s}$, (which follows via Hölder and Sobolev) instead of (8.6). Using Leibniz’s rule, the Correspondence Principle and Lemma 3.9 we obtain for $k \geq 0$

$$(8.8) \quad ||[A_0, A_i]||_{L^{3/4, p} L^{6, 4}_t W^{k, 4}} \leq C T^{1/4} \sum_{j=0}^{k} ||A_0||_{L^{0, 5/16, \infty} H^{1/2}_s} ||A_i||_{L^{1/4 + 1/16, p} L^{6, \infty} H^{1/2 - j}_s}.$$

Now we are in position to apply Lemmas 8.2 and 8.3 to $A_i$ and $A_0$, respectively. Using furthermore $||\nabla_x F_{st}||_{L^{5/4, 2} L^{6, \infty} H^{1/2}_s} \leq ||F_{st}||_{L^{5/4, 2} L^{6, \infty} H^{1/2}_s} + ||A_i||_{L^{1/4 + 1/16, p} L^{6, \infty} H^{1/2 - j}_s}$, (8.7) follows. □

By the same proof applied to $\delta F_{0i}$, we obtain the following difference analogue of Lemma 8.4

**Lemma 8.5 (Bounds for $\delta F_{0i}$).** Suppose that the caloric-temporal gauge condition $A_s = 0$, $A_0 = 0$ holds (for both $A$ and $A'$). Then:

1. The following estimate holds for $2 \leq p \leq \infty$:

$$||\delta F_{0i}(t)||_{L^{3/4, p} H^{1/2}_s} \leq C_p (||\nabla_0 (\delta F_{st})(t)||_{L^{5/4, 2} H^{1/2}_s} + ||\partial_0 (\delta A_i)(t)||_{L^{1/2}} + ||\delta F_{st}(t)||_{L^{5/4, 2} H^{1/2}_s})$$

$$+ C ||\nabla_x F_{st}(t)||_{L^{5/4, 2} L^{2}_s} ||\partial_x (\delta A_i)(t)||_{L^{2}_t} + ||\nabla_x (\delta F_{st})(t)||_{L^{5/4, 2} L^{2}_t}$$

$$+ C ||\nabla_x F_{st0}(t)||_{L^{5/4, 2} L^{2}_s} ||\partial_x (\delta A_i)(t)||_{L^{2}_t} + ||\nabla_x (\delta F_{st})(t)||_{L^{5/4, 2} L^{2}_t}$$

2. For any $2 \leq p \leq \infty$ and $k \geq 1$, we have

$$||\delta F_{0i}(t)||_{L^{3/4, p} H^{k}_s} \leq C_p, k (||\nabla_0 (\delta F_{st})(t)||_{L^{5/4, 2} H^{k}_s} + ||\partial_0 (\delta A_i)(t)||_{H^{k}_s} + ||\delta F_{st}(t)||_{L^{5/4, 2} H^{k+1}_s})$$

$$+ C ||\nabla_x F_{st}(t)||_{L^{5/4, 2} H^{k}_s} ||\partial_x (\delta A_i)(t)||_{H^{k}_s} + ||\nabla_x (\delta F_{st})(t)||_{L^{5/4, 2} H^{k}_s}$$

$$+ C ||\nabla_x F_{st0}(t)||_{L^{5/4, 2} H^{k}_s} ||\partial_x (\delta A_i)(t)||_{H^{k}_s} + ||\nabla_x (\delta F_{st})(t)||_{L^{5/4, 2} H^{k}_s}.$$


(3) For any $2 \leq p \leq \infty$ and $k \geq 0$, we have
\[
\|\delta F_0\|_{L_t^{p/4},p; L_x^{5/4}} \leq C_{p,k}(\|\delta F_{st}\|_{L_t^{5/4},2; L_x^{k+3/2}} + \|\delta A\|_{L_x^{k+3/2}} + T^{1/4} \sup_{t \in I} \|F_{st}(t)\|_{L_t^{1/2},H_x^{k+7/4}})
\]
(8.10)
\[
+ T^{1/4}C\|F_{st}\|_{L_t^{5/4},2; L_x^{k+5/2}} \cdot \sup_{t \in I} \|\nabla_t F_{st}(t)\|_{L_t^{1/2},H_x^{k+1}}
\]
\[
+ T^{1/4}C \sup_{t \in I} \|\nabla_t F_{st}(t)\|_{L_t^{1/2},H_x^{k+1}} \cdot \|\delta F_{st}\|_{L_t^{5/4},2; L_x^{k+1}} + \|\delta A\|_{L_x^{k+1}}).
\]

Next, we derive estimates for $D_0 F_{ij}$ and $D_i F_{0j}$. 

**Lemma 8.6** (Bounds for $D_0 F_{ij}$ and $D_i F_{0j}$). Suppose that the caloric-temporal gauge condition $A_s = 0$, $A_0 = 0$ holds.

(1) For any $2 \leq p \leq \infty$ and $k \geq 0$, we have
\[
\|D_0 F_{ij}(t)\|_{L_t^{5/4},p; H_x^k} + \|D_i F_0(t)\|_{L_t^{5/4},p; H_x^k}
\]
\[
\leq C_{p,k} \left( \|\nabla_t F_{st}(t)\|_{L_t^{5/4},2; H_x^{k+1}} + \|\partial_0 A(t)\|_{H_x^{k+1}} + \|F_{st}(t)\|_{L_t^{1/2},H_x^{k+2}}
\]
\[
+ (\|\nabla_t F_{st}(t)\|_{L_t^{5/4},2; H_x^{k+1}} + \|\partial_0 A(t)\|_{H_x^{k+1}} + \|\nabla_t F_{st}(t)\|_{L_t^{1/2},H_x^{k+1}})^2
\]
\[
+ (\|\nabla_t F_{st}(t)\|_{L_t^{5/4},2; H_x^{k+1}} + \|\partial_0 A(t)\|_{H_x^{k+1}} + \|\nabla_t F_{st}(t)\|_{L_t^{1/2},H_x^{k+1}})^3 \right).
\]
(8.11)

(2) For any $2 \leq p \leq \infty$ and $k \geq 0$, we have
\[
\|D_0 F_{ij}\|_{L_t^{5/4},p; L_x^{5/4}} + \|D_i F_0\|_{L_t^{5/4},p; L_x^{5/4}}
\]
\[
\leq C_{p,k} \left( \|F_{st}\|_{L_t^{5/4},2; L_x^{k+5/2}} + \|A\|_{L_x^{k+5/2}} + T^{1/4} \sup_{t \in I} \|F_{st}(t)\|_{L_t^{1/2},H_x^{k+11/4}}
\]
\[
+ (\|F_{st}\|_{L_t^{5/4},2; L_x^{k+2}} + \|A\|_{L_x^{k+2}} + \sup_{t \in I} \|\nabla_t F_{st}(t)\|_{L_t^{1/2},H_x^{k+1}})^2
\]
\[
+ (\|F_{st}\|_{L_t^{5/4},2; L_x^{k+2}} + \|A\|_{L_x^{k+2}} + \sup_{t \in I} \|\nabla_t F_{st}(t)\|_{L_t^{1/2},H_x^{k+1}})^3 \right).
\]
(8.12)

**Proof.** The proof proceeds in a similar manner as Lemma 8.4. We will give a treatment of the contribution of the term $\|D_0 F_{ij}\|$ and leave the similar case of $\|D_i F_0\|$ to the reader.

Our starting point is the schematic identity
\[
(8.13) \quad D_0 F_{ij} = s^{-1} \mathcal{O}(\nabla_t \nabla_x A) + s^{-1/2} \mathcal{O}(A_0, \nabla_x A) + s^{-1/2} \mathcal{O}(A, \nabla_0 A) + \mathcal{O}(A, A, A_0),
\]
which can be checked easily by expanding $D_0 F_{ij}$ in terms of $A_\mu$.

The first term on the right-hand side of (8.13) is acceptable for both (8.11) and (8.12), thanks to Lemma 8.2. Therefore, it remains to treat only the bilinear and trilinear terms in (8.13).

Let us begin with the proof of (8.11). For the bilinear terms (i.e. the second and the third terms), we start with the inequality $\|\phi_1 \phi_2\|_{L_x^2} \leq C \|\phi_1\|_{H_x^1} \|\phi_2\|_{H_x^1}$, which follows from Lemma 8.2. Applying Leibniz’s rule, the Correspondence Principle and Lemma 8.3, we obtain for $k \geq 0$
\[
\|s^{-1/2} \mathcal{O}(A_0, \nabla_x A)(t)\|_{L_t^{5/4},p; H_x^k} \leq C \|\nabla_x A_0\|_{L_t^{5+3/8,\infty} H_x^k} \|\nabla_x A\|_{L_t^{1/4+1/8,\infty} H_x^{k+1}} + \|\nabla_x A\|_{L_t^{1/4+1/8,\infty} H_x^{k+1}} \|\nabla_0 A\|_{L_t^{1/4+1/8,\infty} H_x^{k+1+1}}.
\]

Applying Lemma 8.2 to $A$ and Lemma 8.3 to $A_0$, we see that the bilinear terms on the right-hand side of (8.13) are also okay.

Finally, for the trilinear term, we start with the inequality $\|\phi_1 \phi_2 \phi_3\|_{L_x^2} \leq C \|\phi_1\|_{H_x^1} \|\phi_2\|_{H_x^1} \|\phi_3\|_{H_x^1}$. By Leibniz’s rule, the Correspondence Principle and Lemma 8.3, we obtain for $k \geq 0$
\[
\|\mathcal{O}(A, A, A_0)\|_{L_t^{5/4},p; H_x^k} \leq \|\nabla_x A\|_{L_t^{1/4+1/6,\infty} H_x^k} \|\nabla_x A\|_{L_t^{1/4+1/6,\infty} H_x^k} \|\nabla_x A_0\|_{L_t^{5+5/12,\infty} H_x^k}.
\]

Applying Lemma 8.2 to $A$ and Lemma 8.3 to $A_0$, we see that the last term on the right-hand side of (8.13) is acceptable. This proves (8.11).

Next, let us prove (8.12), which proceeds in an analogous way. For the bilinear terms, we begin with the obvious inequality $\|\phi_1 \phi_2\|_{L_x^1} \leq C \|\phi_1\|_{L_x^1} \|\phi_2\|_{L_x^1}$. Applying Leibniz’s rule, the
Correspondence Principle, Lemma \[\text{Lemma 3.9}\] and Lemma \[\text{Lemma 3.10}\] we obtain for \(k \geq 0\)
\[
\|s^{-1/2}O(A_0, \nabla_x A)\|_{L_t^{5/4}, L_x^{1/2} W_x^{1,4}} + \|s^{-1/2}O(A, \nabla A)\|_{L_t^{5/4}, L_x^{1/2} W_x^{1,4}}
\leq C \sup_{t \in I} \|\nabla_x A_0(t)\|_{L_t^{0+3/8}, H_x^{k+1}} \|\nabla_x A\|_{L_t^{1/4+1/8}, L_x^{5/4} W_x^{1,4}}
+ C \|\nabla_x A\|_{L_t^{1/4+1/8}, L_x^{5/4} W_x^{1,4}} \|\nabla_0 A\|_{L_t^{1/4+1/8}, L_x^{5/4} W_x^{1,4}}.
\]

Using Strichartz and the Correspondence Principle, we can estimate \(\|\nabla_{t,x} A\|_{L_t^{1/4+1/8}, L_x^{5/4} W_x^{1,4}} \leq C \|A\|_{L_t^{1/4+1/8}, L_x^{5/4} \dot{B}_{k+2}^{1,4}}\). Then applying Lemma \[\text{Lemma 8.2}\] to \(A\) and Lemma \[\text{Lemma 8.3}\] to \(A_0\), it easily follows that the bilinear terms on the right-hand side of \[\text{Lemma 8.13}\] are acceptable.

For the trilinear term, we start with the inequality \(\|\phi_1 \phi_2 \phi_3\|_{L_t^{3/2}} \leq C \|\phi_1\|_{L_t^{3/2} L_x^{2}} \|\phi_2\|_{L_t^{3/2} L_x^{2}} \|\phi_3\|_{L_t^{3} L_x^{2}}\). By Leibniz’s rule, the Correspondence Principle and Lemma \[\text{Lemma 8.9}\] we obtain
\[
\|O(A, A, A_0)\|_{L_t^{3/2}, L_x^{1/2} \dot{B}_{k+1}^{1,2}} \leq C \|A\|_{L_t^{1/4+1/8}, L_x^{5/4} \dot{B}_{k+1}^{1,2}} \|A\|_{L_t^{1/4+1/8}, L_x^{5/4} \dot{B}_{k+1}^{1,2}} \sup_{t \in I} \|A_0(t)\|_{L_t^{0+5/12}, L_x^{5/12} \dot{B}_{k+1}^{1,2}}.
\]

Using Strichartz and the Correspondence Principle, let us estimate the first factor \(\|A\|_{L_t^{1/4+1/8}, L_x^{5/4} \dot{B}_{k+1}^{1,2}}\) by \(C \|A\|_{L_t^{1/4+1/8}, L_x^{5/4} \dot{B}_{k+2}^{1,4}}\). Next, using interpolation and the Correspondence Principle, we estimate the second factor \(\|A\|_{L_t^{1/4+1/8}, L_x^{5/4} \dot{B}_{k+1}^{1,2}}\) by \(\|\nabla_x A\|_{L_t^{1/4+1/8}, L_x^{5/4} \dot{H}_{k+1}^{1/2}}\). Finally, for the last factor, let us estimate \(\|A_0(t)\|_{L_t^{0+5/12}, L_x^{5/12} \dot{B}_{k+1}^{1,2}} \leq C \|\nabla_x A_0(t)\|_{L_t^{0+5/12}, L_x^{5/12} \dot{H}_{k+1}^{1/2}}\). At this point, we can simply apply Lemma \[\text{Lemma 8.2}\] to \(A\) and Lemma \[\text{Lemma 8.3}\] to \(A_0\), and conclude that the trilinear term is acceptable as well. This proves \[\text{Lemma 8.12}\].

Finally, by essentially the same proof, we can prove an analogue of Lemma \[\text{Lemma 8.6}\] for \(\delta D_0 F_{ij} := D_0 F_{ij} - D_0' F_{ij}\) and \(\delta D_1 F_{0j} := D_1 F_{0j} - D_1' F_{0j}\), whose statement we give below.

**Lemma 8.7** (Bounds for \(\delta D_0 F_{ij}\) and \(\delta D_1 F_{0j}\)). Suppose that the caloric-temporal gauge condition \(A_s = 0, A_0 = 0\) holds (for both \(A\) and \(A'\)).

1. For any \(2 \leq p \leq \infty\) and \(k \geq 0\), we have
\[
\|\delta D_0 F_{ij}(t)\|_{L_t^{5/4}, L_x^{3/2} H_x^k} + \|\delta D_1 F_{0j}(t)\|_{L_t^{5/4}, L_x^{3/2} H_x^k}
\leq C \left(\|\nabla_{t,x} (\delta F_{si}(t))\|_{L_t^{5/4}, L_x^{3/2, H_x^{k+1}}} + \|\partial_t x (\delta A_s(t))\|_{L_t^{3/2}, L_x^{3/2, H_x^{k+1}}} + \|\nabla_x (\delta F_0(t))\|_{L_t^{1/2}, L_x^{1/2, H_x^{k+1}}}\right),
\]
where \(C = C_{p,k}(\|\nabla_{t,x} F_{si}(t)\|_{L_t^{5/4}, L_x^{3/2, H_x^{k+1}}}, \|\partial_t x A_s(t)\|_{L_t^{3/2}, L_x^{3/2, H_x^{k+1}}}, \|\nabla_x F_0(t)\|_{L_t^{1/2}, L_x^{1/2, H_x^{k+1}}})\) is positive and non-decreasing in its arguments.

2. For any \(2 \leq p \leq \infty\) and \(k \geq 0\), we have
\[
\|\delta D_0 F_{ij}\|_{L_t^{5/4}, L_x^{1/2} W_x^{1,4}} + \|\delta D_1 F_{0j}\|_{L_t^{5/4}, L_x^{1/2} W_x^{1,4}}
\leq C_{p,k}(\|\delta F_{si}\|_{L_t^{5/4}, L_x^{1/2} \dot{B}_{k+2}^{1,4}}, \|\delta A_s\|_{L_t^{0+5/12}, L_x^{5/12} \dot{B}_{k+2}^{1,4}} + T^{1/4} \sup_{t \in I} \|\delta F_0(t)\|_{L_t^{1/2}, L_x^{1/2, H_x^{k+1/4}}})
+ C \left(\|\delta F_0\|_{L_t^{5/4}, L_x^{1/2} \dot{B}_{k+2}^{1,4}}, \|\delta A_s\|_{L_t^{0+5/12}, L_x^{5/12} \dot{B}_{k+2}^{1,4}} + \sup_{t \in I} \|\nabla_x (\delta F_0(t))\|_{L_t^{1/2}, L_x^{1/2, H_x^{k+1}}})\),
\]
where \(C = C_{p,k}(\|\delta F_{si}\|_{L_t^{5/4}, L_x^{1/2} \dot{B}_{k+2}^{1,4}}), \|\delta A_s\|_{L_t^{0+5/12}, L_x^{5/12} \dot{B}_{k+2}^{1,4}}, \sup_{t \in I} \|\nabla_x F_0(t)\|_{L_t^{1/2}, L_x^{1/2, H_x^{k+1}}})\) on the last line is positive and non-decreasing in its arguments.

### 8.2. Parabolic estimates for \(F_{si}\).
Recall that \(F_{si}\) satisfies a covariant parabolic equation. Under the caloric gauge condition \(A_s = 0\), expanding covariant derivatives and \(F_{id}\), we obtain a semi-linear heat equation for \(F_{si}\), which looks schematically as follows.

\[
(F_{si})' := (\partial_s - \triangle) F_{si} = s^{-1/2} O(A, \nabla_x F_s) + s^{-1/2} O(\nabla_x A, F_s) + O(A, A, F_s).
\]

Note that \(F\) already controls some derivatives of \(F_{si}\). Starting from this, the goal is to prove estimates for higher derivatives of \(F_{si}\).

**Proposition 8.8.** Suppose \(0 < T \leq 1\), and that the caloric-temporal gauge condition holds.
(1) For any \( k \geq 0 \), we have
\[
\| \nabla_{t,x} F_{x=1}(t) \|_{L^{5/4} \to \hat{H}^k_+ \{0,1\}} + \| \nabla_{t,x} F_{x=1}(t) \|_{L^{5/4} \to \hat{H}^{k+1}_+ \{0,1\}} \leq C_{k,F} \| \partial_x \Delta(t) \|_{H^k} \cdot F.
\]

(2) For \( 1 \leq k \leq 25 \), we have
\[
\| F_{x=1} \|_{L^{5/4} \to \hat{S}^k(0,1)} + \| F_{x=1} \|_{L^{5/4} \to \hat{S}^{k+1}(0,1)} \leq C_{F,\Delta} \cdot F.
\]

Part (1) of the proposition states, heuristically, that in order to control \( k + 2 \) derivatives of \( F_{x=1} \) in the \( L^2 \) sense, we need \( F \) and a control of \( k + 1 \) derivatives of \( \Delta \). This numerology is important for closing the bootstrap for the quantity \( A \). On the other hand, in Part (2), we obtain a uniform estimate in terms only of \( F \) and \( \Delta \), thanks to the restriction of the range of \( k \). We refer the reader to Remark \[3.9\] for more discussion.

**Proof.** Step 1: Proof of (1). Fix \( t \in (-T, T) \). Let us start with the obvious inequalities
\[
\left\{ \begin{array}{l}
\| \partial_{t,x} \phi_1 \|_{L^1} \leq C \| \partial_{t,x} \phi_1 \|_{H^{1/2}} \| \phi_2 \|_{H^k} + C \| \phi_1 \|_{L^\infty} \| \partial_{t,x} \phi_2 \|_{H^{1/2}}, \\
\| \partial_{t,x} \phi_3 \|_{L^1} \leq C \sum_{\sigma} \| \phi_{\sigma(1)} \|_{H^{1/3}} \| \phi_{\sigma(2)} \|_{H^{1/3}} \| \partial_{t,x} \phi_{\sigma(3)} \|_{H^{1/3}},
\end{array} \right.
\]
where the sum \( \sum_{\sigma} \) is over all permutations \( \sigma \) of \( \{1, 2, 3\} \). These can be proved by using Leibniz’s rule, Hölder and Sobolev.

Using Leibniz’s rule, the Correspondence Principle, Lemma \[3.9\] Gagliardo-Nirenberg (Lemma \[3.10\]) and interpolation, the previous inequalities lead to the following inequalities for \( k \geq 1 \).
\[
\| s^{-1/2} \nabla_{t,x} \mathcal{O}(\psi_1, \nabla_x \psi_2) \|_{L^{5/4} \to H^{k-1}} + \| s^{-1/2} \nabla_{t,x} \mathcal{O}(\nabla_x \psi_1, \psi_2) \|_{L^{5/4} \to H^{k-1}} \leq C \| \nabla_{t,x} \psi_1 \|_{L^{5/4} \to L^{5/4}} \| \nabla_{t,x} \psi_2 \|_{L^{5/4} \to L^{5/4}} + C \| \nabla_{t,x} \psi_1 \|_{L^{5/4} \to L^{5/4}} \| \nabla_{t,x} \psi_2 \|_{L^{5/4} \to L^{5/4}}.
\]

Note the extra weight of \( s^{1/4} \) for \( \psi_1, \psi_3 \). Put \( \psi_1 = A, \psi_2 = F, \psi_3 = A \), and apply Lemma \[8.2\] (with \( \ell = 0 \), \( p = \infty \), \( q = 2 \) and \( X = L^2 \)) for \( \| A \| \). Then for \( k \geq 1 \), we have
\[
\sup_i \| \nabla_{t,x} (F_{x=1} \mathcal{N}) \|_{L^{5/4} \to H^{k-1}} \leq C \| \nabla_{t,x} F \|_{L^{5/4} \to H^{k-1}} + \| \partial_{t,x} \Delta \|_{H^{k-1}} \| \nabla_{t,x} F \|_{L^{5/4} \to H^{k-1}} + C \| \nabla_{t,x} F \|_{L^{5/4} \to H^{k-1}}^2 \| \nabla_{t,x} F \|_{L^{5/4} \to H^{k-1}}.
\]

Combining this with the obvious bound \( \| \nabla_{t,x} F \|_{L^{5/4} \to H^{k-1}} + \| \nabla_{t,x} F \|_{L^{5/4} \to H^{k-1}} \leq F \), we obtain
\[
\| F_{x=1} \|_{L^{5/4} \to \hat{S}^k} \leq F
\]
from the second part of Theorem \[3.11\].

Step 2: Proof of (2). We proceed in a similar fashion. The multilinear estimates are more complicated. On the other hand, as we are aiming to control derivatives of \( F_{x=1} \) only up to order 25 whereas \( A \) controls derivatives of \( \Delta \) up to order 30, we can be relaxed on the number of derivatives falling on \( A \).

For \( \epsilon > 0 \), we claim that the following estimate for \( (F_{x=1}) \mathcal{N} \) holds for \( 1 \leq k \leq 24 \).
\[
\sup_i \| (F_{x=1}) \mathcal{N} \|_{L^{5/4} \to \hat{S}^{k+2}} \leq \epsilon \| F \|_{L^{5/4} \to \hat{S}^{k+2}} + B_{r,k,A}(\| F \|_{L^{5/4} \to \hat{S}^{k+1}}) \| F \|_{L^{5/4} \to \hat{S}^{k+1}}
\]
where \( B_{r,k,A}(r) > 0 \) is non-decreasing in \( r > 0 \). Then for \( \epsilon > 0 \) sufficiently small, the second part of Theorem \[3.11\] can be applied. Combined with the obvious bound \( \| F \|_{L^{5/4} \to \hat{S}^{k+1}} \leq F \), we obtain a bound for \( \| F_{x=1} \|_{L^{5/4} \to \hat{S}^{k+1}} \) which can be computed by \[8.21\]. This leads to \[8.17\], as desired.

Let us now prove the claim. We will begin by establishing the following multilinear estimates for \( \hat{S} \):
\[
\| \phi_1 \partial_x \phi_2 \|_{\hat{S}^1} \leq T^{1/2} \| \phi_1 \|_{\hat{S}^{1/2}} \| \phi_2 \|_{\hat{S}^{1/2}} + \| \phi_1 \|_{\hat{S}^{1/2} \to L^\infty_{t,x}} \| \phi_2 \|_{\hat{S}^{1/2}} + \| \phi_1 \|_{\hat{S}^{1/2} \to L^\infty_{t,x}} \| \phi_2 \|_{\hat{S}^{1/2}}
\]
\[
\| \phi_1 \phi_2 \phi_3 \|_{\hat{S}^1} \leq T^{1/2} \sum_{\sigma} \| \phi_{\sigma(1)} \|_{\hat{S}^{1/2}} \| \phi_{\sigma(2)} \|_{\hat{S}^{1/2}} \| \phi_{\sigma(3)} \|_{\hat{S}^{1/2}}
\]
\[
+ \sum_{\sigma} \| \phi_{\sigma(1)} \|_{\hat{S}^{1/2}} \| \phi_{\sigma(2)} \|_{\hat{S}^{1/2}} \| \phi_{\sigma(3)} \|_{\hat{S}^{1/2}}.
\]
where the sum $\sum_\sigma$ is over all permutations $\sigma$ of $\{1, 2, 3\}$.

For the first inequality of (8.21), it suffices to prove that $\|\phi_t \partial_x \phi_2\|_{L^4_x H^1_x}$ and $T^{1/2}\|\Box(\phi_t \partial_x \phi_2)\|_{L^2_{t,x}}$ can be controlled by the right-hand side. Using Hölder and Sobolev, we can easily bound the former by $\leq C\|\phi_t\|_{L^\infty_x H^3_x} \|\phi_2\|_{L^2_{t,x}}$, which is acceptable. For the latter, using Leibniz’s rule for $\Box$, let us further decompose

$$T^{1/2}\|\Box(\phi_t \partial_x \phi_2)\|_{L^2_{t,x}} \leq 2T^{1/2}\|\partial_t \phi_t \partial_x \phi_2\|_{L^2_{t,x}} + T^{1/2}\|\Box \phi_t \partial_x \phi_2\|_{L^2_{t,x}} + T^{1/2}\|\phi_t \partial_x \Box \phi_2\|_{L^2_{t,x}}.$$  

Using Hölder and the $L^4_t$, Strichartz, we bound the first term by $\leq CT^{1/2}\|\phi_t\|_{S^{3/2}_t} \|\phi_2\|_{S^1_t}$, which is good. For the second term, we use Hölder to put $\Box \phi_t$ in $L^{2}_t L^{6}_x$ and the other in $L^{2}_t L^{3}_x$. Then by Sobolev and the definition of $S^k$, this is bounded by $\|\phi_t\|_{S^{3/2}_t} \|\phi_2\|_{S^2}$. Finally, for the third term, we use Hölder to estimate $\phi_t$ in $L^{\infty}_t \partial_x \phi_2$ in $L^{6}_t L^{2}_x$, which leads to a bound $\leq \|\phi_t\|_{L^{p}_{t,x}} \|\phi_2\|_{S^2}$. This proves the first inequality of (8.21).

The second inequality of (8.21) follows by a similar consideration, first dividing $\|\cdot\|_{S^1}$ into $\|\cdot\|_{L^\infty_x H^1_x}$ and $\|\Box(\cdot)\|_{L^2_{t,x}}$, and then using Leibniz’s rule for $\Box$ to further split the latter. We leave the details to the reader.

Let us prove (8.21) by splitting $(F_k)N$ into its quadratic part $s^{-1/2}O(A, \nabla_x F_s) + s^{-1/2}O(\nabla_x A, F_s)$ and its cubic part $O(A, A, F_s)$. For the quadratic terms, we use the first inequality of (8.21), the Correspondence Principle, Lemma 3.9 and Lemma 3.10. Then for $k \geq 1$ we obtain

$$\|s^{-1/2}O(\psi_1, \nabla_x \psi_2)\|_{L^{5/4+1, 2}_x S^k} \leq CT^{1/2} \sum_{p=0}^{k-1} \|\psi_1\|_{L^{1/p_1}_t \delta^{3/2+p}} \|\psi_2\|_{L^{5/4+1}_t \delta^{3/2+p}} + C\|\psi_1\|_{L^{1/p_1} \delta^{3/2+p}} \|\psi_2\|_{L^{5/4+1/2}_t \delta^{3/2+p}},$$  

(8.22)

where $\ell_1 + \ell_2 = 3/2$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. Note that we have obtained an extra weight of $s^{-1/8}$ for each factor in the last term.

Let $1 \leq k \leq 24$, and apply (8.22) with $(\psi_1, \ell_1, p_1) = (A, 1, 1/4)$, $(\psi_2, \ell_2, p_2) = (F_s, 2, 5/4)$ for $s^{-1/2}O(\nabla_x F_s)$ and vice versa for $s^{-1/2}O(\nabla_x A, F_s)$. We then apply Lemma 3.2 with $X = S^1$, $p = \infty$ and $q = 2$ to control $\|A\|$ in terms of $\|F_s\|$ and $\|A\|$ (here we use the extra weight of $s^{1/8}$). Next, we estimate $\|A\|$ that arises by $A$, which is possible since we only consider $1 \leq k \leq 24$. As a result, we obtain the following inequality:

$$\|s^{-1}O(A, \nabla_x F_s) + s^{-1}O(\nabla_x A, F_s)\|_{L^{5/4+1, 2}_x S^k} \leq CT^{1/2} \sum_{p=0}^{k} (\|F_s\|_{L^{5/4+2}_x \delta^{3/2+p}} + A)\|F_s\|_{L^{5/4+2}_x \delta^{3/2+p}} + C(\|F_s\|_{L^{5/4+2}_x \delta^{3/2+p}} + A,F_s\|_{L^{5/4+2}_x \delta^{3/2+p}}).$$

The last term is acceptable. All summands of the first term on the right-hand side are also acceptable, except for the cases $p = 0, k$. Let us first treat the case $p = 0$. For $\epsilon > 0$, we apply Cauchy-Schwarz to estimate

$$T^{1/2}(\|F_s\|_{L^{5/4+2}_x \delta^{3/2}} + A)\|F_s\|_{L^{5/4+2}_x \delta^{3/2}} \leq (\epsilon/2)\|F_s\|_{L^{5/4+2}_x \delta^{3/2}} + C\|F_s\|_{L^{5/4+2}_x \delta^{3/2}} + A)\|F_s\|_{L^{5/4+2}_x \delta^{3/2}} + A)^2\|F_s\|_{L^{5/4+2}_x \delta^{3/2}},$$

The case $p = k$ is similar. This proves (8.20) for the quadratic terms $s^{-1/2}O(A, \nabla_x F_s) + s^{-1/2}O(\nabla_x A, F_s)$.

Next, let us estimate the contribution of the cubic terms $O(A, A, F_s)$. Starting from the second inequality of (8.21) and applying Leibniz’s rule, the Correspondence Principle, Lemma 3.9 and Lemma 3.10 we obtain the following inequality:

$$\|O(\psi_1, \psi_2, \psi_3)\|_{L^{5/4+1, 2}_x S^k} \leq CT^{1/2} \prod_{j=1, 2, 3} \|\psi_j\|_{L^{5/4+1/12}_t \delta^{3/2+p_j}} + C \prod_{j=1, 2, 3} \|\psi_j\|_{L^{5/4+1/6}_t \delta^{3/2+p_j}},$$

for $\ell_1 + \ell_2 + \ell_3 = 7/4$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{2}$. Note the extra weight of $s^{1/12}$ and $s^{1/6}$ for each factor in the first and second terms on the right-hand side, respectively.
For $1 \leq k \leq 24$, let us put $(\psi_1, \ell_1, p_1) = (A, 1/4, \infty)$, $(\psi_2, \ell_2, p_2) = (A, 1/4, \infty)$ and $(\psi_3, \ell_3, p_3) = (F_s, 5/4, 2)$ in the last inequality, and furthermore apply Lemma 3.2 with $X = \dot{S}^1$, $p = \infty$ and $q = 2$ (which again uses the extra weights of powers of $s$) to control $\|A\|$ by $\|F_s\|$ and $\|A\|$. Then estimating $\|A\|$ by $A$ (which again is possible since $1 \leq k \leq 24$), we finally arrive at

$$\|O(A, A, F_s)\|_{L^{5/4+1/2\delta}_t L^{4}_{x}} \leq C(1 + T^{1/2})(\|F_s\|_{L^{5/4+2\delta_k}_{t,x}} + A^2 \|F_s\|_{L^{5/4+2\delta_k}_{t,x}})$$

which is acceptable. This proves \(\text{(8.20)}.\)

\[\square\]

Remark 8.9. The fixed time parabolic estimate \(\text{(8.10)}\) will let us estimate $A$ in \(\text{(10.1)}\) in terms of $F_s A$, despite the fact that $F$ controls a smaller number of derivatives (of $F_s$) than does $A$ (of $A_f$). This is essentially due to the smoothing property of the parabolic equation satisfied by $F_s$. It will come in handy in \(\text{(10.2)}\) as it allows us to control only a small number of derivatives of $F_s$ to control $F$.

Accordingly, the space-time estimate \(\text{(8.17)}\) (to be used in \(\text{(10.2)}\)) needs to be proved only for a finite range of $k$, which is smaller than the number of derivatives of $A_f$ controlled by $A_f$. This allows us to estimate whatever $\|A_f\|$ that arises by $A_f$; practically, we do not have to worry about the number of derivatives falling on $A_f$. Moreover, we are also allowed to control (the appropriate space-time norm of) less and less derivatives for $F_s$ and $w_i$ (indeed, see \(\text{(8.36)}\) and \(\text{(8.40)}\), respectively), as long as we control enough derivatives to carry out the analysis in \(\text{(10.2)}\) in the end. Again, this lets us forget about the number of derivatives falling on $A_f$ and $F_s$ (resp. $A_f$, $F_s$ and $F_s$) while estimating the space-time norms of $F_s$ and $w_i$.

By essentially the same proof, the following difference analogue of Proposition 8.8 follows.

**Proposition 8.10.** Suppose $0 < T \leq 1$, and that the caloric-temporal gauge condition holds.

1. Let $t \in (-T, T)$. Then for any $k \geq 0$, we have

$$\|\nabla_{t,x}(\delta F_s)(t)\|_{L^{5/4+\delta_k}_{t,x}[0,1]} + \|\nabla_{t,x}(\delta F_s)(t)\|_{L^{5/4+2\delta_k}_{t,x}[0,1]} \leq C_{k,F_s\|\partial_s A(t)\|_{H^2}} \cdot (\delta F + \|\partial_s (\delta A)(t)\|_{H^2}) \tag{8.23}$$

2. For $1 \leq k \leq 25$, we have

$$\|\delta F_s\|_{L^{5/4+\delta_k}_{t,x}[0,1]} + \|\delta F_s\|_{L^{5/4+2\delta_k}_{t,x}[0,1]} \leq C_{F_s A} \cdot (\delta F + \delta A) \tag{8.24}$$

\[8.3.\text{Estimates for } F_s \text{ via integration.}\] We also need some estimates for $F_s$ without any derivatives, which we state below. The idea of the proof is to simply integrate the parabolic equation $\partial_s F_s = \Delta F_s + (F_s)\mathcal{N}$ backwards from $s = 1$.

**Proposition 8.11.** Suppose $0 < T \leq 1$, and that the caloric-temporal gauge condition holds.

1. Let $t \in (-T, T)$. Then we have

$$\|F_s(t)\|_{L^{5/4+\delta_k}_{t,x}[0,1]} + \|F_s(t)\|_{L^{5/4+2\delta_k}_{t,x}[0,1]} \leq C_{F_s A} \cdot (F + A) \tag{8.25}$$

2. We have

$$\|F_s\|_{L^{5/4+\delta_k}_{t,x}[0,1]} + \|F_s\|_{L^{5/4+2\delta_k}_{t,x}[0,1]} \leq C_{F_s A} \cdot (F + A) \tag{8.26}$$

**Proof.** In the proof, all norms will be taken on the interval $s \in (0, 1]$. Let us start with the equation

$$\partial_s F_s = \Delta F_s + (F_s)\mathcal{N}.$$

Using the fundamental theorem of calculus, we obtain for $0 < s \leq 1$ the identity

$$F_s(s) = F_s - \int_0^s s' \Delta F_s(s') \frac{ds'}{s'} - \int_0^s s'(F_s)\mathcal{N}(s') \frac{ds'}{s'} \tag{8.27}$$

To prove \(\text{(8.20)}\) and \(\text{(8.22)}\), let us either fix $t \in (-T, T)$ and take the $L^{5/4+p}_{s} L^2_{x}$ norm of both sides or just take the $L^{5/4+p}_{s} L^2_{x}$ norm, respectively. We will estimate the contribution of each term on the right-hand side of \(\text{(8.22)}\) separately.
For the first term on the right-hand side of (8.27), note the obvious estimates \( \|E_{s,t}(t)\|_{L^4_t L^2_x} \leq C_p \|E_{s,t}(t)\|_{L^p_t L^q_x} \) and \( \|E_{s,t}\|_{L^{5/4}_t L^{12}_x} \leq C_p \|E_{s,t}\|_{L^4_t L^4_x} \). Writing out \( E_{s,t} = O(\partial_t^2 A) + O(A \partial_t A) + O(A A A) \), we see that
\[
\sup_{t \in (-T, T)} \|E_{s,t}(t)\|_{L^2_x} + \|E_{s,t}\|_{L^4_t L^4_x} \leq C A + C A^2 + C A^3,
\]
which is acceptable.

For the second term on the right-hand side of (8.27), let us apply Lemma 8.4 with \( p = 2, \infty \) and \( q = 2 \) to estimate
\[
\left\| \int_s^1 s' \Delta F_{s,t}(s') \frac{ds'}{s'} \right\|_{L^{5/4}_t L^2_x} \leq \left\| \int_s^1 (s/s')^{5/4} (s')^{5/4} \|\nabla_x^2 F_{s,t}(s', s')\| L^5_x(s') \frac{ds'}{s'} \right\|_{L^5_t} \leq C_p \|F_{s,t}(t)\|_{L^{5/4}_t L^5_x} \leq C_p F.
\]
Similarly, for \( p = 2, \infty \), we can prove \( \left\| \int_s^1 s' \Delta F_{s,t}(s') \frac{ds'}{s'} \right\|_{L^{5/4}_t L^4_x} \leq C_p \|F_{s,t}\|_{L^{5/4,2}_t L^{4,2}_x} \leq C_p F. \)
Therefore, the contribution of the second term is okay.

Finally, for the third term on the right-hand side of (8.27), let us first proceed as in the previous case to reduce
\[
\left\| \int_s^1 s' \left((F_{s,t}) N(t, s')\right) \frac{ds'}{s'} \right\|_{L^{5/4}_t L^2_x} \leq C_p \left\| \left((F_{s,t}) N(t)\right) \right\|_{L^{5/4+1+1/4}_t L^2_x}.
\]
Recall that \( (F_{s,t}) N = s^{-1/2} O(A, \nabla_x F_s) + s^{-1/2} O(\nabla_x A, F_s) + O(A, A, F_s) \). Starting from the obvious inequalities
\[
\|\phi_1 \partial_x \phi_2\|_{L^2_x} \leq C \|\phi_1\|_{H^1_x} \|\phi_2\|_{H^{1/2}_x}, \quad \|\phi_1 \partial_t \phi_3\|_{L^2_x} \leq C \prod_{j=1,2,3} \|\phi_j\|_{H^1_x},
\]
and applying the Correspondence Principle, Lemma 8.3 and interpolation, we obtain
\[
\left\| s^{-1/2} O(A, \nabla_x F_s) + s^{-1/2} O(\nabla_x A, F_s) \right\|_{L^{5/4+1+1/4}_t L^2_x} \leq C \|\nabla_x A\|_{L^{5/4+1+1/4}_t L^2_x} \|F_{s,t}\|_{L^{5/4,2}_t L^5_x}
\]
\[
\left\| O(A, A, F_s) \right\|_{L^{5/4+1+1/4}_t L^2_x} \leq C \|A\|_{L^{5/4+1+1/4}_t L^2_x} \|F_{s,t}\|_{L^{5/4,2}_t L^5_x}.
\]
Note the extra weight of \( s^{1/4} \) on each factor of \( A \). This allows us to apply Lemma 8.2 (with \( q = 2 \)) to estimate \( \|A\| \) in terms of \( \|F_s\| \) and \( \|A\| \). From the definition of \( F \) and \( A \), it then follows that \( \left\| \left((F_{s,t}) N(t)\right) \right\|_{L^{5/4+1+1/4}_t L^2_x} \leq C (F + A)^2 + C (F + A)^3 \) uniformly in \( t \in (-T, T) \), which finishes the proof of (8.26).

Finally, as in the previous case, we have
\[
\left\| \int_s^1 s' \left((F_{s,t}) N(s')\right) \frac{ds'}{s'} \right\|_{L^{5/4}_t L^4_x} \leq C_p \left\| \left((F_{s,t}) N(s')\right) \right\|_{L^{5/4+1+1/2}_t L^4_x}.
\]
Using the inequalities
\[
\|\phi_1 \partial_x \phi_2\|_{L^1_t L^{3/2}_x} \leq C \|\phi_1\|_{L^{3/2}_x} \|\phi_2\|_{L^{3/2}_x}, \quad \|\phi_1 \partial_t \phi_3\|_{L^1_t L^{3/2}_x} \leq C T^{1/4} \prod_{j=1,2,3} \|\phi_j\|_{L^p_t L^p_x},
\]
and proceeding as before using the Correspondence Principle, Lemmas 8.3, 8.10 and 8.2, it follows that \( \left\| \left((F_{s,t}) N\right) \right\|_{L^{5/4+1+1/2}_t L^4_x} \leq C (F + A)^2 + C (F + A)^3 \). This concludes the proof of (8.26).

Again with essentially the same proof, the following difference analogue of Proposition 8.11 follows.

**Proposition 8.12.** Suppose \( 0 < T \leq 1 \), and that the caloric-temporal gauge condition holds.

1. Let \( t \in (-T, T) \). Then we have
\[
\|\delta F_{s,t}(t)\|_{L^{5/4}_x L^2_t[0,1]} + \|\delta F_{s,t}(t)\|_{L^{5/4,2}_x L^4_t[0,1]} \leq C F A \cdot (\delta F + \delta A).
\]
2. We have
\[
\|\delta F_{s,t}\|_{L^{5/4}_x L^4_t[0,1]} + \|\delta F_{s,t}\|_{L^{5/4,2}_x L^4_t[0,1]} \leq C F A \cdot (\delta F + \delta A).
\]
8.4. Parabolic estimates for $F_{s_0}$. In this subsection, we will study the parabolic equation (1.13) satisfied by $F_{s_0} = -w_0$. Let us define
\[
(F_{s_0})\mathcal{N} := (\partial_s - \Delta)F_{s_0} = (F_{s_0})\mathcal{N}_{\text{forcing}} + (F_{s_0})\mathcal{N}_{\text{linear}}
\]
where
\[
(F_{s_0})\mathcal{N}_{\text{linear}} = 2s^{-1/2}[A^\ell, \nabla\ell F_{s_0}] + s^{-1/2}[\nabla\ell A^\ell, F_{s_0}] + [A^\ell, [A^\ell, F_{s_0}]],
\]
\[
(F_{s_0})\mathcal{N}_{\text{forcing}} = -2s^{-1/2}[F_{0}^\ell, F_{s_0}].
\]

Our first proposition for $F_{s_0}$ is an a priori parabolic estimate for $\mathcal{E}(t)$, which requires a smallness assumption of some sort.\(^{23}\)

**Proposition 8.13** (Estimate for $\mathcal{E}$). Suppose that the caloric-temporal gauge condition holds, and furthermore that $F + A < \delta_E$ where $\delta_E > 0$ is a sufficiently small constant. Then
\[
(8.30) \sup_{t \in (-T,T)} \mathcal{E}(t) \leq C_{F, A} \cdot (F + A)^2,
\]
where $C_{F, A} = C(F, A)$ can be chosen to be continuous and non-decreasing with respect to both arguments.

**Proof.** Let us fix $t \in (-T, T)$. Define $E := |\partial_\mu|^{-1/2} F_{s_0}$, where $|\partial_\mu|^a := (-\Delta)^a/2$ is the fractional integration operator. From the parabolic equation for $F_{s_0}$, we can derive the following parabolic equation for $E$:
\[
(\partial_s - \Delta)E = s^{1/4}|\nabla x|^{-1/2}(F_{s_0})\mathcal{N},
\]
where $|\nabla x|^a := s^{a/2}|\partial_\mu|^a$ is the $p$-normalization of $|\partial_\mu|^a$. The idea is to work with the new variable $E$, and then translate to the corresponding estimates for $F_{s_0}$ to obtain (8.30).

We begin by making two claims. First, for every small $\epsilon, \epsilon' > 0$, by taking $\delta_E > 0$ sufficiently small, the following estimate holds for $\rho = 1, 2$ and $0 \leq \underline{\rho} \leq 1$:
\[
(8.31) \| (F_{s_0})\mathcal{N} \|_{\mathcal{L}_{\rho, s_0}^{2, 1/2}(0, \underline{\rho})} \leq \epsilon \| E \|_{\mathcal{P}_{3/4}^{1/4}r_{3/4}^{1/4}(0, 1)} + C_{F, A} \cdot (\| s^{1/4-\epsilon'} E \|_{\mathcal{L}_{\rho, s_0}^{3/4, 1/2}(0, 1)} + (F + A)^2).
\]

Second, for $k = 1, 2$, the following estimate holds.
\[
(8.32) \| (F_{s_0})\mathcal{N} \|_{\mathcal{L}_{\rho, s_0}^{2, k-1/2}(0, 1)} \leq \epsilon \| E \|_{\mathcal{P}_{3/4}^{1/4}r_{3/4}^{1/4}(0, 1)} + C_{F, A} \cdot (\| s^{1/4-\epsilon'} E \|_{\mathcal{L}_{\rho, s_0}^{3/4, 1/2}(0, 1)} + (F + A)^2).
\]

Assuming these claims, we can quickly finish the proof. Note that $E = 0$ at $s = 0$, as $F_{s_0} = 0$ there, and that the left-hand side of (8.31) is equal to $\| s^{1/4}|\nabla x|^{-1/2}(F_{s_0})\mathcal{N} \|_{\mathcal{L}_{\rho, s_0}^{3/4, 1/4}(0, 1)}$. Applying the first part of Theorem 3.11 we derive $\| E \|_{\mathcal{P}_{3/4}^{1/4}r_{3/4}^{1/4}(0, 1)} \leq C_{F, A} \cdot (F + A)^2$. Using the preceding estimate and (8.31), an application of the second part of Theorem 3.11 then shows that $\| E \|_{\mathcal{P}_{3/4}^{1/4}r_{3/4}^{1/4}(0, 1)} \leq C_{F, A} \cdot (F + A)^2$. Finally, as $E = s^{1/4}|\nabla x|^{-1/2}F_{s_0}$, it is easy to see that $\mathcal{E}(t) \leq \| E \|_{\mathcal{P}_{3/4}^{1/4}r_{3/4}^{1/4}(0, 1)}$, from which (8.30) follows.

To establish (8.31) and (8.32), we split $(F_{s_0})\mathcal{N}$ into $(F_{s_0})\mathcal{N}_{\text{forcing}}$ and $(F_{s_0})\mathcal{N}_{\text{linear}}$.

- **Case 1:** The contribution of $(F_{s_0})\mathcal{N}_{\text{forcing}}$. In this case, we will work on the whole interval $(0, 1]$. Let us start with the product inequality
\[
\| \phi_1 \phi_2 \|_{r_{3/4}^{1/2}} \leq C \| \phi_1 \|_{r_{3/4}^{1/2}} \| \phi_2 \|_{r_{3/4}^{1/2}},
\]
which follows from Lemma 8.2. Using Leibniz’s rule, the Correspondence Principle and Lemma 8.3 we obtain for $0 \leq k \leq 2$
\[
\| \mathcal{O}(\psi_1, \psi_2) \|_{\mathcal{L}_{\rho, s_0}^{2, k-1/2}} \leq C \sum_{j=0}^{k} \| \psi_1 \|_{\mathcal{L}_{\rho, s_0}^{3/4, j+1/2}} \| \psi_2 \|_{\mathcal{L}_{\rho, s_0}^{3/4, j+1/2}},
\]
where $\frac{1}{2} = \frac{1}{3} - \frac{1}{2}$. Let us put $\psi_1 = F_{0t}, \psi_2 = F_{s_0}$.

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23In our case, as we normalized the $s$-interval to be $[0, 1]$, we will require directly that $F + A$ is sufficiently small. On the other hand, we remark that this proposition can be proved just as well by taking the length of the $s$-interval to be sufficiently small.
In order to estimate $\|F_{0t}\|_{L^2_t H^{1/2}}$ or $\|F_{0t}\|_{L^2_t H^{1/2}}$ with $j > 0$, we apply (8.1) or (an interpolation of) (8.2) of Lemma 8.3 respectively. We then estimate $\|F_{st}\|, \|A\|$ which arise by $F, \Delta$, respectively. (We remark that this is possible as $0 \leq k \leq 2$.)

Next, to estimate $\|F_{st}\|_{L^2_t H^{1/2}}$, we first note, by interpolation, that it suffices to control $\|F_{st}\|_{L^2_t H^{1/2}}$ and $\|F_{st}\|_{L^2_t H^{3/2}}$, to which we then apply Propositions 8.11 and 8.8 respectively. On the other hand, for $\|F_{st}\|_{L^2_t H^{k-j/2+1/2}}$ with $j < k$, we simply apply (after an interpolation) Proposition 8.3. Observe that all of $\|A\|$ which arise can be estimated by $\Delta$. As a result, for $1 \leq p \leq 2$ and $0 \leq k \leq 2$, we obtain

$$
\left((F_{st})_{N_{\text{forcing}}} \right)_{L^2_t H^{k-j/2+1/2}} \leq C_{F, \Delta} \sum_{j=0}^{k} \left( \|F_{st}\|_{L^2_t H^{1/2}} + \|F_{s0}\|_{L^2_t H^{3/2}} \right) + C_{F, \Delta} \|F + \Delta\|^2.
$$

As $E = |\partial_x|^{-1/2}F_{s0}$, note that

$$
\|F_{s0}\|_{L^2_t H^{1/2}} + \|F_{s0}\|_{L^2_t H^{3/2}} = \|E\|_{L^2_t H^{1/2}} + \|E\|_{L^2_t H^{3/2}}.
$$

Note furthermore that the right-hand side is bounded by $\|E\|_{L^2_t H^{3/2}}$. Given $\epsilon > 0$, by taking $\delta_E > 0$ sufficiently small (so that $F + \Delta$ is sufficiently small), we obtain for $k = 0, p = 1, 2$

$$
\left((F_{s0})_{N_{\text{forcing}}} \right)_{L^2_t H^{k-j/2+1/2}} \leq \epsilon \|E\|_{L^2_t H^{3/2}} + C_{F, \Delta} \|F + \Delta\|^2,
$$

and for $k = 1, 2$ (taking $p = 2$)

$$
\left((F_{s0})_{N_{\text{forcing}}} \right)_{L^2_t H^{k-j/2+1/2}} \leq \epsilon \|E\|_{L^2_t H^{3/2}} + \epsilon \|E\|_{L^2_t H^{1/2}} + C_{F, \Delta} \|F + \Delta\|^2,
$$

both of which are acceptable.

- **Case 2:** The contribution of $(F_{s0})_{N_{\text{linear}}}$. Let $\varphi \in (0,1]$; we will work on $(0, \varphi]$ in this case. We will see that for this term, no smallness assumption is needed.

Let us start with the inequalities

$$
\|\phi_1 \phi_2\|_{\dot{H}^{1/2}} \leq C\|\phi_1\|_{\dot{H}^{1/2} \cap L^\infty} \|\phi_2\|_{\dot{H}^{1/2}},
$$

$$
\|\phi_1 \phi_2\|_{\dot{H}^{1/2}} \leq C\|\phi_1\|_{\dot{H}^{1/2}} \|\phi_2\|_{\dot{H}^{1/2}},
$$

$$
\|\phi_1 \phi_2 \phi_3\|_{\dot{H}^{1/2}} \leq C\|\phi_1\|_{\dot{H}^{1/2}} \|\phi_2\|_{\dot{H}^{1/2}} \|\phi_3\|_{\dot{H}^{1/2}}.
$$

To prove the first inequality, note that it is equivalent to the product estimate $\dot{H}^{3/2} \cap L^\infty \dot{H}^{1/2} \subset \dot{H}^{1/2}$ by duality, which in turn follows from interpolation between $(\dot{H}^{3/2} \cap L^\infty) \cdot L^2 \subset L^2$ and $(\dot{H}^{3/2} \cap L^\infty) \cdot \dot{H}^{1/2} \subset \dot{H}^{1/2}$. On the other hand, the second inequality was already used in the previous step. Finally, the third inequality is an easy consequence of the Hardy-Littlewood-Sobolev fractional integration $L^2 \subset \dot{H}^{1/2},$ Hölder and Sobolev.

Let $\epsilon' > 0$. Using the preceding inequalities, along with the Correspondence Principle and Lemma 8.39, we obtain the following inequalities for $p = 1, 2$ on $(0, \varphi]$: 

$$
\begin{align*}
\left\{ \begin{array}{l}
\|s^{-1/2} O(\psi_1, \nabla_x \psi_2)\|_{L^2_t \dot{H}^{1/2}} + \|s^{-1/2} O(\nabla_x \psi_1, \psi_2)\|_{L^2_t \dot{H}^{1/2}} \\
\leq C\varphi' \|\psi_1\|_{L^2_t \dot{H}^{1/2} \cap L^\infty} \|\psi_2\|_{L^2_t \dot{H}^{1/2}},
\end{array} \right.
\end{align*}
(8.33)
\]

$$
\begin{align*}
\left\{ \begin{array}{l}
\|O(\psi_1, \psi_2, \psi_3)\|_{L^2_t \dot{H}^{1/2}} \\
\leq C\varphi' \|\psi_1\|_{L^2_t \dot{H}^{1/2} \cap L^\infty} \|\psi_2\|_{L^2_t \dot{H}^{1/2} \cap L^\infty} \|s^{1/4-\epsilon'} \psi_3\|_{L^2_t \dot{H}^{1/2} \cap L^\infty},
\end{array} \right.
\end{align*}
\]

We remark that the factors of $\varphi'$, which can be estimated by $\leq 1$, arise due to an application of Hölder for $L^p_t$ (Lemma 8.39) with $p = 1$. Taking $\psi_1 = \dot{A}$, $\psi_2 = F_{s0}$, $\psi_3 = A$ and using Lemmas 8.16 and 8.2 and the fact that $s^{1/4-\epsilon'} F_{s0}\|_{L^{5/2}_t \dot{H}^{1/2}} = s^{1/4-\epsilon'} E\|_{L^{5/2}_t \dot{H}^{1/2}}$, we see that

$$
\left((F_{s0})_{N_{\text{linear}}} \right)_{L^2_t \dot{H}^{1/2}} \leq C_{F, \Delta} \|s^{1/4-\epsilon'} E\|_{L^2_t \dot{H}^{1/2}},
$$

for $p = 1, 2$. Combining this with Case 1, (8.31) follows.
Proceeding similarly, but this time applying Leibniz’s rule to \((8.33)\), choosing \(p = 2\) and \(s = 1\), we obtain for \(k = 1, 2\)
\[
\| (F_{s_0}) N_{\text{linear}} \|_{L^2 \cdot H^{s - 1/2}_\tau (0, 1)} \leq C_{F, A} \cdot \| E \|_{p^{3/4} H^{s + 1}_\tau (0, 1)} ;
\]
(we estimated \(s \leq 1\)) from which, along with the previous case, \((8.32)\) follows.

Our next proposition for \(F_{s_0}\) states that once we have a control of \(E (t)\), we can control higher derivatives of \(F_{s_0}\) without any smallness assumption.

**Proposition 8.14 (Parabolic estimates for \(F_{s_0}\)).** Suppose \(0 < T \leq 1\), and that the caloric-temporal gauge condition holds.

1. Let \(t \in (-T, T)\). Then for \(m \geq 4\), we have
\[
\| F_{s_0} (t) \|_{L^1_x \cdot L^\infty_t H^{m - 1}_\tau (0, 1)} + \| F_{s_0} (t) \|_{L^1_x \cdot H^{m}_\tau (0, 1)}
\leq C_{F, A} \cdot (E (t) + (F (F + A)^2)) .
\]
(8.34)

In particular, for \(1 \leq m \leq 31\), we have
\[
\| F_{s_0} (t) \|_{L^1_x \cdot L^\infty_t H^{m - 1}_\tau (0, 1)} + \| F_{s_0} (t) \|_{L^1_x \cdot H^{m}_\tau (0, 1)} \leq C_{F, A} \cdot (E (t) + (F + A)^2)
\]
(8.35)

2. For \(1 \leq m \leq 21\), we have
\[
\| F_{s_0} \|_{L^1_x \cdot L^\infty_t H^{m - 1}_\tau (0, 1)} + \| F_{s_0} \|_{L^1_x \cdot L^\infty_t H^{m}_\tau (0, 1)} \leq C_{F, A} \cdot (E + (F + A)^2)
\]
(8.36)

Part (1) of the proceeding proposition tells us that in order to control \(m\) derivatives of \(F_{s_0}\) uniformly in \(s\) (rather than in the \(L^2_s\) sense), we need to control \(m\) derivatives of \(A\). This fact will be used in an important way to close the estimates for \(A\) in \((10.34)\). On the other hand, as in Proposition \(8.8\) the range of \(k\) in Part (2) was chosen so that we can estimate whatever derivative of \(A\) which arises by \(\Delta\).

**Proof. Step 1: Proof of (1).** Fix \(t \in (-T, T)\). We will be working on the whole interval \((0, 1)\).

Note that \((8.35)\) follows immediately from \((8.34)\) and the definition of \(E (t)\), as \(\| \partial_{t,x} A \|_{L^\infty_\tau H^m} \leq A\). In order to prove \((8.34)\), we begin by claiming that the following estimate holds for \(k \geq 2\):
\[
\| (F_{s_0}) N \|_{L^1_x \cdot L^\infty_t H^{k+1}_\tau} \leq C_{F, \| \partial_{t,x} A \|_{H^m}} \cdot (\| \nabla_x F_{s_0} \|_{L^1_x \cdot H^{k}_\tau} + C_{F, \| \partial_{t,x} A \|_{H^m}} \cdot (F + \| \partial_{t,x} A \|_{H^m})^2)
\]
(8.37)

Assuming the claim, we may apply the second part of Theorem 3.11 along with the bound \(\| F_{s_0} \|_{p^{3/4} H^{k}_\tau} \leq E (t)\), to conclude \((8.34)\).

To prove \((8.37)\), we estimate the contributions of \((F_{s_0}) N_{\text{forcing}}\) and \((F_{s_0}) N_{\text{linear}}\) separately.

- **Case 1.1: The contribution of \((F_{s_0}) N_{\text{forcing}}\).** We start with the simple inequality \(\| \phi_1 \phi_2 \|_{H^m} \leq C \| \phi_1 \|_{H^m} \| \phi_2 \|_{H^{m+2} \cap L^\infty_x}\). Applying Leibniz’s rule, the Correspondence Principle, Lemma 3.9 and Lemma 3.10 we get
\[
\| O (\psi_1, \psi_2) \|_{L^1 \cdot H^{k+1}_\tau} \leq C \| \nabla_x \psi_1 \|_{L^\infty_\tau H^{k-1}_\tau} \| \nabla_x \psi_2 \|_{L^1 \cdot H^{k+1}_\tau},
\]
for \(k \geq 2\).

Let us put \(\psi_1 = F_{s_0} \), \(\psi_2 = F_{s_0} \), and apply Lemma 3.4 to control \(\| \nabla_x F_{s_0} \|_{L^1 \cdot H^{k+1}_\tau}\), in terms of \(\| F \|, \| \Delta \|\) and \(\| F_{s_0} \|\). Then we apply Proposition 8.8 to estimate \(\| F \|\) in terms of \(\nabla F\) and \(\| \Delta \|\). At this point, one may check that all \(\| \Delta \|, \| F_{s_0} \|\) that have arisen may be estimated by \(\| \partial_{t,x} A \|_{H^m}\). As a result, for \(k \geq 2\), we obtain
\[
\| (F_{s_0}) N_{\text{forcing}} \|_{L^1 \cdot H^{k+1}_\tau} \leq C_{F, \| \partial_{t,x} A \|_{H^m}} \cdot (\| \nabla_x F_{s_0} \|_{L^1 \cdot H^{k}_\tau} + C_{F, \| \partial_{t,x} A \|_{H^m}} \cdot (F + \| \partial_{t,x} A \|_{H^m}) \cdot F)
\]
which is good enough for \((8.37)\).
- **Case 1.2: The contribution of \((F_{oa})N_{\text{linear}}\).** Here, let us start from (5.23) in the proof of Proposition 5.6. Applying Leibniz’s rule, the Correspondence Principle, Lemma 3.39 and Lemma 3.16, we obtain

\[
\left\{ \begin{array}{l}
\| s^{-1/2}O(\psi_1, \nabla_x \psi_2) \|_{L^1_x H^k_x} + \| s^{-1/2}O(\nabla_x \psi_1, \psi_2) \|_{L^1_x H^k_x} \\
\leq C \| \nabla_x \psi_1 \|_{L^{1/4+1/4, \infty}_x H^k_x} \| \nabla_x \psi_2 \|_{L^{1/2}_x H^k_x}, \\
\| O(\psi_1, \psi_2, \psi_3) \|_{L^{1/4+1/4, \infty}_x H^k_x} \leq C \| \nabla_x \psi_1 \|_{L^{1/4+1/4, \infty}_x H^k_x} \| \nabla_x \psi_2 \|_{L^{1/2}_x H^k_x} \| \nabla_x \psi_3 \|_{L^{1/4+1/4, \infty}_x H^k_x},
\end{array} \right.
\]

for \( k \geq 1 \).

Note the extra weight of \( s^{1/4} \) on \( \psi_1, \psi_3 \). Let us put \( \psi_1 = A, \psi_2 = F_{oa}, \psi_3 = A \), and apply Lemma 8.2 to control \( \| A \| \) in terms of \( \| F_{oa} \| \) and \( \| A \| \). Then using Proposition 8.8 we can control \( \| F_{oa} \| \) by \( F \) and \( \| A \| \). Observe that all of \( \| A \| \) which have arisen can be estimated by \( \| \partial_t x A \|_{L^2_x} \). As a result, we obtain the estimate

\[
\| (F_{oa})N_{\text{linear}} \|_{L^{1+1,2}_x H^k_x} \leq C_{F, \| \partial_t x A \|_{L^2_x}} \| \nabla_x F_{oa} \|_{L^{1+1,2}_x},
\]

for \( k \geq 1 \). Combining this with the previous case, (8.37) follows.

**Step 2: Proof of (2).** Let \( 0 \leq k \leq 19 \), where the number \( k \) corresponds to the number of times the equation \( (\partial_t - \Delta)F_{oa} = (F_{oa})N \) is differentiated. We remark that its range has been chosen to be small enough so that any norm of \( F_{oa} \) and \( A \) that arises in the argument below can be controlled by \( C_{F, A} \cdot (F + A) \) (by Propositions 8.8 and 8.11) and \( A \), respectively.

We claim that for \( \epsilon' > 0 \) small enough, \( 0 \leq k \leq 19 \) an integer, \( 1 \leq p \leq 2 \) and \( 0 < \ell \leq 1 \), the following estimate holds:

\[
(8.38) \quad \| (F_{oa})N \|_{L^{1+1, p}_x L^{1+2}_x H^k} \leq C_{F, A} : \| s^{1/2-\epsilon'} \nabla_x F_{oa} \|_{L^{1+2}_x L^{2+1}_x} + C_{F, A} : (E + F + A)(F + A).
\]

Assuming (8.38), and taking \( k = 0, p = 1, 2 \), we can apply the first part of Theorem 3.11 to obtain (8.39) in the cases \( m = 1, 2 \). Then taking \( 1 \leq k \leq 19 \) and \( p = 2 \), we can apply the second part of Theorem 3.11 along with the bound (8.38) in the case \( m = 2 \) that was just established, to conclude the rest of (8.38).

As before, in order to prove (8.38), we treat the contributions of \( (F_{oa})N_{\text{forcing}} \) and \( (F_{oa})N_{\text{linear}} \) separately.

- **Case 2.1: The contribution of \((F_{oa})N_{\text{forcing}}\).** We claim that the following estimate holds for \( 0 \leq k \leq 19 \) and \( 1 \leq p \leq 2 \):

\[
(8.39) \quad \| (F_{oa})N_{\text{forcing}} \|_{L^{1+1, p}_x H^k(0, 1)} \leq C_{F, A} : (E + F + A)(F + A).
\]

Note in particular that the right-hand side does not involve \( \| \nabla_x F_{oa} \|_{L^{1+2}_x L^{1+2}_x} \). This is because we can use (8.35) to estimate whatever factor of \( \| F_{oa} \| \) that arises in this case.

In what follows, we work on the whole \( s \)-interval \((0, 1)\). Starting from Hölder’s inequality \( \| \phi_1 \phi_2 \|_{L^1_x} \leq \| \phi_1 \|_{L^{1, p}_x} \| \phi_2 \|_{L^{1, q}_x} \) and using Leibniz’s rule, the Correspondence Principle and Lemma 3.39, we obtain

\[
\| O(\psi_1, \psi_2) \|_{L^{1+1, p}_x H^k_x} \leq C \| \psi_1 \|_{L^{3+1, p}_x L^{1+2}_x W^k_x} \| \psi_2 \|_{L^{3+1, 2}_x L^{1+2}_x W^k_x},
\]

where \( \frac{1}{p} = \frac{1}{q} + \frac{1}{2} \). Let us put \( \psi_1 = F_{oa}, \psi_2 = F_{oa} \) and use Lemma 8.4 to control \( \| F_{oa} \| \) in terms of \( \| F_{oa} \| \) and \( \| A \| \). Then thanks to the assumption \( 0 \leq k \leq 19 \), we can use (8.35), the second part of Proposition 8.5 and the definition of \( A \) to control \( \| F_{oa} \| \), \( \| F_{oa} \| \) and \( \| A \| \) have arisen by \( C_{F, A} : (E + (F + A)^2), C_{F, A} : F + A \) and \( A \), respectively.

On the other hand, to control \( \| F_{oa} \|_{L^{5/4, 2}_x L^{1, 2}_x W^{k, 2}} \), we first use Strichartz to estimate

\[
\| F_{oa} \|_{L^{5/4, 2}_x L^{1, 2}_x W^{k, 2}} \leq C \| F_{oa} \|_{L^{5/4, 2}_x L^{1, 2}_x} + C \| F_{oa} \|_{L^{5/4, 2}_x \mathbb{R}^{k+1/2}},
\]

and then use Propositions 8.11 and 8.8 to estimate the first and the second terms by \( C_{F, A} \cdot (F + A) \) and \( C_{F, A} \cdot F \), respectively. As a result, we obtain (8.39) for \( 1 \leq p \leq 2 \).
Case 2.2: The contribution of \((F_{s0})_{\text{linear}}\). Let 0 < \(s \leq 1\); we will work on the interval \((0, s]\) in this case. Let us begin with the following estimates, which follow immediately from (5.23) by square integrating in \(t\) and using Hölder:

\[
\begin{align*}
\|\phi_1 \partial_t \phi_2\|_{L^2_t L^2_x} + \|\phi_2 \psi_1\|_{L^2_t L^2_x} &\leq C \|\phi_1\|_{L^\infty_t (H^{2/3}_x)} \|\phi_2\|_{L^2_t H^1_x}, \\
\|\phi_1 \phi_2 \phi_3\|_{L^2_t L^2_x} &\leq C \|\phi_1\|_{L^\infty_t H^1_x} \|\phi_2\|_{L^2_t H^1_x} \|\phi_3\|_{L^2_t H^1_x}.
\end{align*}
\]

Using Leibniz’s rule, the Correspondence Principle and Lemma 8.9, we obtain the following inequalities for \(\epsilon' > 0\) small, 0 ≤ \(k \leq 19\) and 1 ≤ \(p \leq q \leq \infty\):

\[
\begin{align*}
&\|s^{-1/2} \mathcal{O}(\psi_1, \nabla_x \psi_2)\|_{L^{1+1,6}_t L^{2/3}_x H^{1/4}_x} + \|s^{-1/2} \mathcal{O}(\nabla_x \psi_1, \psi_2)\|_{L^{1+1,6}_t L^{2/3}_x H^{1/4}_x} \\
&\leq C \sum_{j=0}^k \|\phi_{j+1}\|_{L^1 L^\infty_x} \|s^{-1/2} \nabla_x \psi_2\|_{L^{1+1,6}_t L^{2/3}_x H^{1/4}_x} \|s^{1/4-\epsilon'} \nabla_x \psi_2\|_{L^{1+1,6}_t L^{2/3}_x H^{1/4}_x},
\end{align*}
\]

The factors \(s^{\epsilon'}\) have arisen from applications of Hölder for \(L^{p}_t\) (Lemma 8.9); we estimate them by \(\leq 1\). Let us put \(\psi_1 = A\), \(\psi_2 = F_{s0}\) and \(\psi_3 = A\), and apply Lemma 5.2 to control \(\|A\|\) in terms of \(\|F_s\|\) and \(\mathcal{A}\) (the latter thanks to the range of \(k\)). Then we apply Proposition 8.5 to control \(\|F_s\|\) in terms of \(\mathcal{F}\) and \(\mathcal{A}\) (again using the restriction of the range of \(k\)). As a result, we arrive at

\[
\|F_{s0}\|_{\text{linear}} \leq C_{\mathcal{F}, \mathcal{A}} \|s^{1/4-\epsilon'} \nabla_x F_{s0}\|_{L^{1+1,6}_t L^{2/3}_x H^{1/4}_x},
\]

for \(\epsilon' > 0\) small, 0 ≤ \(k \leq 19\) and 1 ≤ \(p \leq q \leq \infty\). Taking \(q = 2\) and combining with the previous case, we obtain (8.35).

The difference analogues of Propositions 8.13 and 8.14 can be proved in a similar manner, using the non-difference versions which have been just established. We give their statements below, omitting the proof.

**Proposition 8.15** (Estimate for \(\delta \mathcal{E}\)). Suppose that the caloric-temporal gauge condition holds, and furthermore that \(\mathcal{F} + \mathcal{A} < \delta \mathcal{E}\) where \(\delta \mathcal{E} > 0\) is sufficiently small. Then

\[
\begin{align*}
\sup_{t \in (-T, T)} \delta \mathcal{E}(t) &\leq C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})(\delta \mathcal{F} + \delta \mathcal{A}).
\end{align*}
\]

**Proposition 8.16** (Parabolic estimates for \(\delta F_{s0}\)). Suppose \(0 < T \leq 1\), and that the caloric-temporal gauge condition holds. Then the following statements hold.

1. Let \(t \in (-T, T)\). Then for \(m \geq 4\), we have

\[
\begin{align*}
\|\partial_t \mathcal{E}(t)\|_{L^2_t H^m_x} &\leq C_{\mathcal{F}, \mathcal{A}} \|\partial_t \mathcal{E}(t)\|_{L^2_t H^m_x} \\
&\leq C_{\mathcal{F}, \mathcal{A}} \cdot \delta \mathcal{E}(t) + C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{E}(t) + \mathcal{F} + \mathcal{A})(\delta \mathcal{F} + \delta \mathcal{A}).
\end{align*}
\]

In particular, for \(1 \leq m \leq 31\), we have

\[
\begin{align*}
\|\partial_t \mathcal{E}(t)\|_{L^2_t H^m_x} &\leq C_{\mathcal{F}, \mathcal{A}} \cdot \delta \mathcal{E}(t) + C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{E}(t) + \mathcal{F} + \mathcal{A})(\delta \mathcal{F} + \delta \mathcal{A}).
\end{align*}
\]

2. For \(1 \leq m \leq 21\), we have

\[
\begin{align*}
\|\partial_t \mathcal{E}(t)\|_{L^2_t H^m_x} &\leq C_{\mathcal{F}, \mathcal{A}} \cdot \delta \mathcal{E} + C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{E} + \mathcal{F} + \mathcal{A})(\delta \mathcal{F} + \delta \mathcal{A}).
\end{align*}
\]
8.5. Parabolic estimates for \( w_i \). Here we will study the parabolic equation (1.13) satisfied by \( w_i \). Let us define
\[
(w_i)\mathcal{N} := (\partial_s - \Delta)w_i = (w_i)\mathcal{N}_{\text{forcing}} + (w_i)\mathcal{N}_{\text{linear}}
\]
where
\[
(w_i)\mathcal{N}_{\text{linear}} = 2s^{-1/2}[A^\ell, \nabla_\ell w_i] + s^{-1/2}[\nabla_\ell A^\ell, w_i] + [A^\ell, [A^\ell, w_i]] + 2[F^\ell_\ell, w_i],
\]
\[
(w_i)\mathcal{N}_{\text{forcing}} = -2[F_0, D^\ell F_0 + D_0 F^\ell_\ell].
\]

The following proposition proves parabolic estimates for \( w_i \) that we will need in the sequel.

**Proposition 8.17** (Parabolic estimates for \( w_i \)). Suppose \( 0 < T \leq 1 \), and that the caloric-temporal gauge condition holds.

1. Let \( t \in (-T,T) \). For \( 1 \leq m \leq 30 \), we have
\[
\tag{8.44}
\left\| w_i(t) \right\|_{L^1_t\dot{H}^{-1}_x(0,1)} + \left\| w_i(t) \right\|_{L^1_t\dot{H}^{1/2}_x(0,1)} \leq C_{\xi(t), F} \left\| \mathcal{E}(t) + F + \mathcal{A} \right\|^2.
\]
   In the case \( m = 31 \), on the other hand, we have the following estimate.
\[
\tag{8.45}
\left\| w_i(t) \right\|_{L^1_t\dot{H}^{30}_x(0,1)} + \left\| w_i(t) \right\|_{L^1_t\dot{H}^{31}_x(0,1)} \leq C_{\xi(t), F} \left\| \mathcal{E}(t) + F + \mathcal{A} + \left\| \partial_0 \mathcal{A}(t) \right\|_{\dot{H}^{30}_x} \right\|^2.
\]

2. For \( 1 \leq m \leq 16 \), we have
\[
\tag{8.46}
\left\| w_i \right\|_{L^1_t\dot{H}^{-1}_x(0,1)} + \left\| w_i \right\|_{L^1_t\dot{H}^{1/2}_x(0,1)} \leq C_{\xi, F} \left\| \mathcal{E} + F + \mathcal{A} \right\|^2.
\]
   Furthermore, for \( 0 \leq k \leq 14 \), we have the following estimate for \( (w_i)\mathcal{N} \).
\[
\tag{8.47}
\left\| (w_i)\mathcal{N} \right\|_{L^1_t\dot{H}^{k-1/2}_x(0,1)} + \left\| (w_i)\mathcal{N} \right\|_{L^1_t\dot{H}^{k+1/2}_x(0,1)} \leq C_{\xi, F} \left\| \mathcal{E} + F + \mathcal{A} \right\|^2.
\]

**Remark 8.18.** Note that Part (1) of Proposition 8.17 does not require a smallness assumption, as opposed to Proposition 8.13. Moreover, in comparison with Proposition 8.14, we need \( m \) derivatives of \( \mathcal{A} \) (i.e., one more derivative) to estimate \( m \) derivatives of \( w \) uniformly in \( s \).

**Proof.** Step 1: Proof of (1), for \( 1 \leq m \leq 3 \). Fix \( t \in (-T,T) \). Let us define \( v_i := |\partial_x|^{-1/2}w_i \). From the parabolic equation for \( w_i \), we derive the following parabolic equation for \( v_i \):
\[
(\partial_s - \Delta) v_i = s^{1/4}|\nabla_x|^{-1/2}(w_i)\mathcal{N},
\]
where the right-hand side is evaluated at \( t \). Note that \( \left\| w_i \right\|_{L^1_t\dot{H}^k_x} = \left\| v_i \right\|_{L^1_t\dot{H}^{k+1/2}_x} \). The idea, as in the proof of Proposition 8.13, is to derive estimates for \( v_i \) and then to translate to the corresponding estimates for \( w_i \) using the preceding observation.

We will make two claims: First, for \( 0 < s \leq 1 \) and \( 1 \leq p \leq 2 \), the following estimate holds.
\[
\tag{8.48}
\sup_i \left\| (w_i)\mathcal{N} \right\|_{L^p_t\dot{H}^{-1/2}_x(0,1)} \leq C_{\xi(t), F} \left\| \mathcal{E}(t) + F + \mathcal{A} \right\|^2.
\]

Second, for \( k = 1, 2 \), the following estimate holds.
\[
\tag{8.49}
\sup_i \left\| (w_i)\mathcal{N} \right\|_{L^p_t\dot{H}^{k-1/2}_x(0,1)} \leq C_{\xi(t), F} \left\| \mathcal{E}(t) + F + \mathcal{A} \right\|^2.
\]

Note that \( \left\| (w_i)\mathcal{N} \right\|_{L^p_t\dot{H}^k_x(0,1)} = \left\| \mathcal{E}(t) \right\|_{L^p_t\dot{H}^{k+1/2}_x(0,1)} \). Assuming (8.48) and using the preceding observation, we can apply the first part of Theorem 8.11 to \( v_i \) (note furthermore that \( v_i = 0 \) at \( t = 0 \)), from which we obtain a bound on \( \left\| v(i) \right\|_{L^p_t\dot{H}^k_x(0,1)} \). Next, assuming (8.49) and applying the second part of Theorem 8.11 to \( v_i \), we can also control \( \left\| v(i) \right\|_{L^p_t\dot{H}^{k+1/2}_x(0,1)} \) using the fact that \( v_i = s^{1/4}|\nabla_x|^{-1/2}w_i \), now follows.

We are therefore left with the task of establishing (8.48) and (8.49). For this purpose, we divide \( (w_i)\mathcal{N} = (w_i)\mathcal{N}_{\text{forcing}} + (w_i)\mathcal{N}_{\text{linear}} \), and treat each of them separately.
- Case 1.1: Contribution of \((w_i)\)\(N_{\text{forcing}}\). In this case, we work on the whole interval \((0, 1]\). We start with the inequality
\[
\|\phi_1 \phi_2\|_{L_x^{1/2}} \leq \|\phi_1\|_{H_x^{1/2}} \|\phi_2\|_{L_x^2},
\]
which follows from Lemma 3.2. Using Leibniz’s rule, the Correspondence Principle and Lemma 3.3, we arrive at the following inequality for \(k \geq 0\) and \(\frac{1}{p} + \frac{1}{2} = \frac{1}{p} - \frac{1}{2}:
\[
\|O(\psi_1, \psi_2)\|_{L_x^{1/2}} \leq C \|\nabla_x \psi_1\|_{L_x^{2/3-r} L_y^{1/3}} \|\psi_2\|_{L_x^{2/3+2} L_y^{1/3}}.
\]

Let us restrict to \(0 \leq k \leq 2\) and put \(\psi_1 = F_{0, t}, \psi_2 = D^L F_{0, t} + D_0 F_{0, t}^L\). In order to estimate \(\|\nabla_x F_{0, t}\|_{L_x^{2/3-r} L_y^{1/3}}\) and \(\|D^L F_{0, t} + D_0 F_{0, t}^L\|_{L_x^{2/3+2} L_y^{1/3}}\), we apply Lemmas 3.3 (with \(p = r\)) and 3.6 (with \(p = 2\), respectively, from which we obtain an estimate of \(\|\phi_1 \phi_2\|_{L_x^{1/2}}\) in terms of \(\|F_{0, t}\|\), \(\|F_{0, t}\|\) and \(\|A\|\). The latter two types of terms can be estimated by \(F\) and \(A\), respectively. Moreover, using Propositions 3.14, \(\|F_{0, t}\|\) can be estimated by \(E(t), F\) and \(A\). As a result, for \(0 \leq k \leq 2\) and \(1 \leq p \leq 2\), we obtain
\[
\sup_i \|\phi_1 \phi_2 (t)\|_{L_x^{1/2}} \leq C E(t) F A (E(t) + F + A)^2.
\]
which is good enough for (8.48) and (8.49).

- Case 1.2: Contribution of \((w_i)\)\(N_{\text{linear}}\). Note that \((w_i)\)\(N_{\text{linear}}\) has the same schematic form as \((F_{0, t})\)\(N_{\text{linear}}\). Therefore, the same proof as in Case 2 of the proof of Proposition 3.13 gives us the estimates
\[
\sup_i \|\phi_1 \phi_2 (t)\|_{L_x^{1/2}} \leq C F A (E(t) + F + A)^2.
\]
for \(p = 1, 2\), \(0 < \alpha < 1\) and arbitrarily small \(\delta > 0\), and
\[
\sup_i \|\phi_1 \phi_2 (t)\|_{L_x^{1/2}} \leq C F A (E(t) + F + A)^2.
\]
for \(k = 1, 2\). Combined with the previous case, we obtain (8.48) and (8.49).

Step 2: Proof of (1), for \(m \geq 4\). By working with \(w_i\) instead of \(w_i\), we were able to prove the \textit{a priori} estimate (8.44) for low \(m\) by an application of Theorem 3.11. The drawback of this approach, as in the case of \(F_{0, t}\), is that the estimate that we derive is not good enough in terms of the necessary number of derivatives of \(A\). In order to prove (8.44) for higher \(m\), and (8.45) as well, we revert back to the parabolic equation for \(w_i\).

We claim that the following estimate holds for \(k \geq 2\):
\[
\sup_i \|\phi_1 \phi_2 (t)\|_{L_x^{1/2}} \leq C F A (E(t) + F + A)^2.
\]
(8.50)
Assuming the claim, let us first finish the proof of (1). Note that for \(0 \leq k \leq 29\), we have \(\|\partial_t x A\|_{H_x^{k+1}} \leq A\). Therefore, every norm \(\|\partial_t x A\|\) arising in (8.50) for \(2 \leq k \leq 28\) can be estimated by \(A\). Using this, along with the estimate (8.44) for \(1 \leq m \leq 3\) which has been established in Step 1, we can apply the second part of Theorem 3.11 to conclude (8.44) for all \(4 \leq m \leq 30\).

Note, on the other hand, that for \(k = 30\) we only have \(\|\partial_t x A\|_{H_x^{20}} \leq A + \|\partial_t x A\|_{L_x^{20}}\). From (8.50), we therefore obtain the estimate
\[
\sup_i \|\phi_1 \phi_2 (t)\|_{L_x^{1/2}} \leq C F A (E(t) + F + A)^2.
\]
Combining this with the case \(k = 30\) of (8.44), an application of the second part of Theorem 3.11 gives (8.45).

We are therefore only left to prove (8.50). As usual, we will treat \((w_i)\)\(N_{\text{forcing}}\) and \((w_i)\)\(N_{\text{linear}}\) separately, and work on the whole interval \((0, 1]\) in both cases.
- Case 2.1: Contribution of \((w_i)N_{\text{forcing}}\). As in Case 1.1 in the proof of Proposition 8.14 we begin with the inequality \(|\phi_1 \phi_2|_2 \leq C |\phi_1|_{H^{3/2} \cap L_{t,x}^\infty} |\phi_2|_{H_2}^2\) and apply Leibniz’s rule, the Correspondence Principle, Lemma 8.5, and Lemma 8.10. As a result, for \(k \geq 2\), we obtain

\[
\|O(\psi_1, \psi_2)\|_{L_0^{14,2} H^k_{t,x}} \leq C \|\nabla_x \psi_1\|_{L_0^{3/2, \infty} H^k_{t,x}} \|\nabla_x \psi_2\|_{L_0^{3/2, \infty} H^k_{t,x}}.
\]

As in Case 1.1, we put \(\psi_1 = F_{\partial_0} \psi, \psi_2 = D^f F_{\partial_0} + D_0 F_{\psi}^f\), and apply Lemmas 8.3 (with \(p = \infty\)) and 8.3 (with \(p = 2\)), by which we obtain an estimate of \(\|O(\psi_1, \psi_2)\|_{L_0^{14,2} H^k_{t,x}}\) in terms of \(\|F_{\partial_0}\|, \|F_{\psi}\|\) and \(\|\partial_t x A(t)\|_{H^k_{t,x}}\). Using Proposition 8.14 and Proposition 8.8 in order, we can estimate \(\|F_{\partial_0}\|\) and \(\|F_{\psi}\|\) in terms of \(\|E(t), F\|\) and \(\|\partial_t x A\|\). At this point, one may check that all \(\|\partial_t x A\|\) that have arisen can be estimated by \(\|\partial_t x A(t)\|_{H^k_{t,x}}^2\). As a result, we obtain the following estimate for \(k \geq 2\):

\[
\sup_k \|\psi_i\|_{L_0^{14,2} H^k_{t,x}(0,1)} \leq C_r \|E(t), F, \partial_t x A(t)\|_{H^k_{t,x}}^2 + \|\partial_t x A(t)\|_{H^k_{t,x}}^2,
\]

which is good.

- Case 2.2: Contribution of \((w_i)N_{\text{linear}}\). As \((w_i)N_{\text{linear}}\) looks schematically the same as \((F_{\psi})N_{\text{linear}}\), Step 1.2 of the proof of Proposition 8.14 immediately gives

\[
\sup_i \|\psi_i\|_{L_0^{14,2} H^k_{t,x}(0,1)} \leq C \|E(t), F, \partial_t x A(t)\|_{H^k_{t,x}}^2 + \|\partial_t x A(t)\|_{H^k_{t,x}}^2,
\]

for \(k \geq 1\). Combined with the previous case, this proves \((8.50)\), as desired.

**Step 3: Proof of (2).** Let \(0 \leq k \leq 14\), where \(k\) corresponds to the number of times the equation \((\partial_t - \Delta)w_i = (w_i)N_{\text{diff}}\) is differentiated. The range has been chosen so that Proposition 8.14 can be applied to estimate every norm of \(F_{\partial_0}\) which arises in terms of \(E, \|F\|\) and \(\|A\|\), and furthermore so that all \(\|F\|\) and \(\|A\|\) that arise can be estimated by \(C_{E, F, \partial_t x A}\) (by Proposition 8.8 and \(A\), respectively).

We claim that for \(\epsilon > 0\) small enough, \(0 \leq k \leq 14\) an integer, \(1 \leq p \leq q \leq \infty\) and \(0 < s \leq 1\), the following estimate holds:

\[
\sup_i \|\psi_i\|_{L_0^{14,2} H^k_{t,x}(0,1)} \leq C \|E(t), F, \partial_t x A(t)\|_{H^k_{t,x}}^2 + \|\partial_t x A(t)\|_{H^k_{t,x}}^2.
\]

Assuming the claim, let us prove (2). Taking \(k = 0, p = 1, 2\) and \(q = 2\), we may apply the first part of Theorem 3.11 (along with the fact that \(w = 0\) at \(s = 0\)) to obtain (8.40) in the cases \(m = 1, 2\). Combining this with (8.51) in the cases \(1 \leq k \leq 14, p = q = 2\) and \(s = 1\), we can apply the second part of Theorem 3.11 to obtain the rest of (8.40). Finally, considering (8.51) with \(0 \leq k \leq 14\), \(q = 2\) and \(s = 1\), and estimating \(\|\nabla_x u\|_{L_0^{14,2} H^k_{t,x}(0,1)}\) in the first term on the right-hand side by (8.40), we obtain (8.51), which finishes the proof of Part (2).

It therefore only remain to prove (8.51), for which we split \((w_i)N = (w_i)N_{\text{forcing}} + (w_i)N_{\text{linear}}\) as usual.

- Case 3.1: Contribution of \((w_i)N_{\text{forcing}}\). In this case, we work on the whole interval \((0, 1)\).

Let us begin with the inequality \(\|\phi_1 \phi_2\|_{L_{t,x}^4} \leq \|\phi_1\|_{L_{t,x}^4} \|\phi_2\|_{L_{t,x}^4}\). Applying Leibniz’s rule, the Correspondence Principle, Lemma 8.5, and Lemma 8.10, we obtain, for \(k \geq 0\) and \(1 \leq p \leq \infty\),

\[
\|O(\psi_1, \psi_2)\|_{L_0^{14,2} H^k_{t,x}} \leq C \|\psi_1\|_{L_0^{3/2, \infty} H^k_{t,x}} \|\psi_2\|_{L_0^{3/2, \infty} H^k_{t,x}}.
\]

where \(\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2}\). Since \(p \geq 1\), we may choose \(r_1, r_2\) so that \(r_1, r_2 \geq 2\). As before, let us take \(\psi_1 = F_{\partial_0} \psi, \psi_2 = D^f F_{\partial_0} + D_0 F_{\psi}^f\), and apply Lemma 8.3 (with \(p = r_1\)) and Lemma 8.6 (with \(p = r_2\)), respectively. Then we apply Proposition 8.14 and Proposition 8.8 in sequence, where we remark that both can be applied thanks to the restriction \(0 \leq k \leq 14\). As a result, we obtain an estimate of \(\|O(\psi_1, \psi_2)\|_{L_0^{14,2} H^k_{t,x}}\) in terms of \(E, F\) and \(\|\partial_t x A\|\). One may then check that all terms that arise are at least quadratic in the latter three quantities, and furthermore that each \(\|\partial_t x A\|\) which has arisen can be estimated by \(\partial_t x A\), thanks again to the restriction \(0 \leq k \leq 14\). In the end, we obtain, for \(0 \leq k \leq 14\) and \(1 \leq p \leq \infty\), the following estimate:

\[
\sup_i \|\psi_i\|_{L_0^{14,2} H^k_{t,x}(0,1)} \leq C \|E, F, \partial_t x A\| \cdot (E + F + A)^2.
\]
- Case 3.2: Contribution of \((w_i)N_{\text{linear}}\). As before, we utilize the fact that \((w_i)N_{\text{linear}}\) looks schematically the same as \((F_{0\circ})N_{\text{linear}}\). Consequently, Step 2.2 of the proof of Proposition \(8.14\) implies

\[
\sup_i \| (w_i)N_{\text{linear}} \|_{L^{1+r}_{t}L^{2}_{x}H^{2}_{x}(0,0)} \leq C_{F,\mathcal{A}} \cdot \| s^{1/4-\epsilon'} \nabla \cdot \|_{L^{1+r}_{t}L^{2}_{x}H^{2}_{x}(0,0)}
\]

for \(\epsilon' > 0\) small, \(0 \leq k \leq 14, 1 \leq p \leq r \leq \infty\) and \(0 < s \leq 1\). Combined with the previous case, we obtain \(8.51\).

Again, by essentially the same proof, the following difference analogue of Proposition \(8.17\) follows.

**Proposition 8.19** (Parabolic estimates for \(\delta w_{i}\)). Suppose \(0 < T \leq 1\), and that the caloric-temporal gauge condition holds.

1. Let \(t \in (-T,T)\). For \(1 \leq m \leq 30\) we have

\[
\| \delta w_{i}(t) \|_{L^{1}_{t}H^{m-1}_{x}(0,1)} + \| \delta w_{i}(t) \|_{L^{1}_{t}H^{m}_{x}(0,1)} \leq C_{\mathcal{E},F,\mathcal{A}} \cdot (\mathcal{E}(t) + F + \mathcal{A})(\delta \mathcal{E}(t) + \delta F + \delta \mathcal{A})
\]

In the case \(m = 31\), on the other hand, we have the following estimate.

\[
\| \delta w_{i}(t) \|_{L^{1}_{t}H^{30}_{x}(0,1)} + \| \delta w_{i}(t) \|_{L^{1}_{t}H^{30}_{x}(0,1)} \leq C_{\mathcal{E}(t),F,\mathcal{A}} \cdot \| \delta \mathcal{A}(t) \|_{H^{30}_{x}} \cdot (\mathcal{E}(t) + F + \mathcal{A}) (\delta \mathcal{E}(t) + \delta F + \delta \mathcal{A})
\]

2. For \(1 \leq m \leq 16\), we have

\[
\| \delta w_{i} \|_{L^{1}_{t}L^{1}_{x}H^{m-1}_{x}(0,1)} + \| \delta w_{i} \|_{L^{1}_{t}L^{2}_{x}H^{m}_{x}(0,1)} \leq C_{\mathcal{E},F,\mathcal{A}} \cdot (\mathcal{E}(t) + F + \mathcal{A})(\delta \mathcal{E}(t) + \delta F + \delta \mathcal{A})
\]

Furthermore, for \(0 \leq k \leq 14\), we have the following estimate for \((\delta w_{i})N : = (\partial_{a} - \Delta)(\delta w_{i})\).

\[
\| (\delta w_{i})N \|_{L^{2}_{t}L^{2}_{x}H^{2}_{x}(0,1)} + \| (\delta w_{i})N \|_{L^{2}_{t}L^{2}_{x}H^{1}_{x}(0,1)} \leq C_{\mathcal{E},F,\mathcal{A}} \cdot (\mathcal{E}(t) + F + \mathcal{A})(\delta \mathcal{E}(t) + \delta F + \delta \mathcal{A})
\]

9. Proofs of Propositions \(7.1\) - \(7.4\)

In this section, we will sketch the proofs of Propositions \(7.1\) - \(7.4\).

**Proof of Proposition 7.1.** We will give a proof of the non-difference estimate \(7.1\), leaving the similar case of the difference estimate \(7.2\) to the reader.

In what follows, we work on the time interval \(I = (-T,T)\). Recalling the definition of \(A_{0}\), we need to estimate \(\| A_{0}(s = 0) \|_{L^{r}_{t}L^{2}_{x}}, \| \partial_{x}A_{0}(s = 0) \|_{L^{r}_{t}L^{2}_{x}}, \| A_{0}(s = 0) \|_{L^{1}_{t}L^{r}_{x}}, \| \partial_{x}A_{0}(s = 0) \|_{L^{1}_{t}L^{r}_{x}}\) and \(\| \partial_{s}^{(2)}A_{0}(s = 0) \|_{L^{1}_{t}L^{2}_{x}}\) by the right-hand side of \(7.1\).

Using \(\partial_{x}A_{0} = F_{s0}\), the first two terms can be estimated simply by \(C\mathcal{E}\) as follows.

\[
\| A_{0}(s = 0) \|_{L^{r}_{t}L^{2}_{x}} + \| \partial_{x}A_{0}(s = 0) \|_{L^{r}_{t}L^{2}_{x}} \leq \left( \int_{0}^{1} (s')^{1/2}(s')^{2}\| F_{s0}(s') \|_{L^{r}_{x}L^{2}_{x}(s')} \frac{ds'}{s} \right) \leq C\mathcal{E}
\]

For the next two terms, using Hölder in time, it suffices to estimate \(\| A_{0}(s = 0) \|_{L^{1}_{t}L^{r}_{x}}, \| \partial_{x}A_{0}(s = 0) \|_{L^{1}_{t}L^{r}_{x}}\). Using \(8.38\) of Proposition \(8.14\) along with Gagliardo-Nirenberg, interpolation and Sobolev, these are estimated as follows.

\[
\| A_{0}(s = 0) \|_{L^{1}_{t}L^{r}_{x}} \leq \left( \int_{0}^{1} (s')^{1/2}(s')^{2}\| F_{s0}(s') \|_{L^{r}_{x}L^{r}_{x}(s')} + \| \nabla F_{s0}(s') \|_{L^{r}_{x}L^{r}_{x}(s')} \right) \leq C_{F,\mathcal{A}} \cdot \mathcal{E} + C_{F,\mathcal{A}} \cdot (F + \mathcal{A})^{2}
\]

Unfortunately, the same argument applied to the term \(\| \partial_{s}^{(2)}A_{0}(s = 0) \|_{L^{1}_{t}L^{2}_{x}}\) fails by a logarithm. In this case, we make use of the equations \(\partial_{s}A_{0} = F_{s0}\) and the parabolic equation for \(F_{s0}\). Indeed,
let us begin by writing
\[ \Delta A_0(s = 0) = -\int_0^1 \Delta F_{s_0}(s') \, ds' - \int_0^1 \partial_y F_{s_0}(s') \, ds' + \int_s^1 (F_{s_0})_N(s') \, ds' \]
\[ = F_{s_0} + \int_s^1 s'((F_{s_0})_N)(s') \, ds'/s', \]
where on the last line, we used the fact that \( F_{s_0}(s = 0) = -w_0(s = 0) = 0 \). Taking the \( L^2_{t,x} \) norm of the above identity and applying triangle and Minkowski, we obtain
\[ \|\Delta A_0(s = 0)\|_{L^2_{t,x}} \leq \|F_{s_0}\|_{L^2_{t,x}} + \int_s^1 s'\|((F_{s_0})_N)(s')\|_{L^2_{t,x}} \, ds'/s'. \]

The first term can be estimated using (8.36), whereas the last term can be estimated by putting together (8.38) (in the proof of Proposition 8.14) and (8.36). As a consequence, we obtain
\[ \|\Delta A_0(s = 0)\|_{L^2_{t,x}} \leq C F \cdot \mathcal{E} + C F \cdot (\mathcal{F} + \mathcal{A})^2. \]

By a simple integration by parts, it follows that \( \|\partial_x(\Delta A_0(s = 0))\|_{L^2_{t,x}} \leq C \|\Delta A_0(s = 0)\|_{L^2_{t,x}} \). Then by Hölder in time, the desired \( L^1_t L^2_x \)-estimate follows. This completes the proof of (7.1). \( \square \)

**Proof of Proposition 7.2** Again, we will only treat the non-difference case, as the difference case follows by essentially the same arguments.

The goal is to estimate \( \sup_{0 \leq s \leq 1} \|A_i(s)\|_{\mathcal{S}^1} \) in terms of \( \mathcal{F} + \mathcal{A} \). Note that, proceeding naively, one can easily prove the bound
\[ \|A_i(s)\|_{\mathcal{S}^1} \leq \int_s^1 s'\|F_{si}(s')\|_{\mathcal{S}^1} \, ds'/s' + \|\mathcal{A}_i\|_{\mathcal{S}^1} \leq |\log s|^{1/2} \mathcal{F} + \mathcal{A}. \tag{9.1} \]

The essential reason for having a logarithm is that we have an *absolute integral* of \( \|F_{si}(s')\|_{\mathcal{S}^1} \) in the inequality, whereas \( \mathcal{F} \) only controls its *square integral*. The idea then is to somehow replace this absolute integral with a square integral, using the structure of the Yang-Mills system.

We start with the equation satisfied by \( A_i \) under the condition \( A_s = 0 \).
\[ \partial_y A_i = \Delta A_i - \partial^\ell \partial_t A_x + (A_i)^{N''}, \tag{9.2} \]
where
\[ (A_i)^{N''} = \mathcal{O}(A, \partial_x A) + \mathcal{O}(A, A, A). \]

Fix \( t \in (-T, T) \). Let us take \( \partial_{x,y} \) of (9.2), take the bi-invariant inner product\(^23\) with \( \partial_{y,x} A_i \) and integrate over \( \mathbb{R}^3 \times [s, 1] \), for \( 0 < s \leq 1 \). Summing up in \( i \) and performing integration by parts, we obtain the following identity.
\[ \frac{1}{2} \sum_i \int \partial_{x,y} A_i(s)^2 \, dx = \frac{1}{2} \sum_i \int \partial_{x,y} A_i^2 \, dx - \sum_i \int_s^1 \int s'\partial_{x,y}(A_i^{(N''})_1, \partial_{y,x} A_i) \, dx \, ds'/s' \]
\[ + \sum_i \int_s^1 \int s'|\partial_t \partial_{y,x} A_i(s')^2 \, dx \, ds'/s' - \sum_i \int_s^1 \int s'|\partial_{y,x} \partial_t A_i(s')^2 \, dx \, ds'/s'. \]

Take the supremum over \( 0 \leq s \leq 1 \), and apply Cauchy-Schwarz and Hölder to deal with the second term on the right-hand side. Then taking the supremum over \( t \in (-T, T) \) and applying Minkowski, we easily arrive at the following inequality.
\[ \sup_{0 \leq s \leq 1} \|\partial_{y,x} A(s)\|_{L^\infty_{t,x} L^2_x} \leq C \|\partial_{y,x} A\|_{L^\infty_{t,x} L^2_x} + C \left( \int_s^1 s\|\partial_{y,x} A(s)\|^2_{L^\infty_{t,x} L^2_x} \, ds'/s' \right)^{1/2} \]
\[ + C \sup \int_0^1 s\|\partial_{y,x}(A_i^{(N'')})_1(s)\|_{L^\infty_{t,x} L^2_x} \, ds. \]

\(^{23}\)The fact that \( A_0(t, s = 0) \in H^m \) for any \( m \geq 0 \) can be used to show that the boundary terms vanish at the spatial infinity.
\(^{24}\)In fact, for the purpose of this argument, it is possible to use any inner product on \( g \) for which Leibniz's rule holds, so that integration by parts works.
Similarly, taking □ of (9.2), multiplying by □A_i, integrating over (−T,T) × ℝ^3 × [s,1] and etc, we can also prove

\[ \sup_{0 ≤ s ≤ 1} \| □A(s) \|_{L^2_t} ≤ C □A \|_{L^2_t} + C \left( \int_0^1 s \| □A(s) \|_{L^2_t} \frac{ds}{s} \right)^{1/2} \]

\[ + C sup \int_0^1 s □(A\cdot N')(s) \|_{L^2_t} \frac{ds}{s}. \]

Combining the last two inequalities and recalling the definition of the norm $\hat{S}^k$, we get

\[ \sup_{0 ≤ s ≤ 1} \| A(s) \|_{\hat{S}^1} ≤ C □A + C( ∫_0^1 s \| A(s) \|_{S^1} \frac{ds}{s} + C sup \int_0^1 s \| □(A\cdot N')(s) \|_{\hat{S}^1} \frac{ds}{s}. \]

Applying Lemma 8.2 (with $p = q = 2$) to the second term on the right-hand side, we finally arrive at the following inequality.

(9.3) \[ \sup_{0 ≤ s ≤ 1} \| A(s) \|_{\hat{S}^1} ≤ C □A + C( ∫_0^1 s \| A(s) \|_{S^1} \frac{ds}{s} + C sup \int_0^1 s \| □(A\cdot N')(s) \|_{\hat{S}^1} \frac{ds}{s}. \]

All terms on the right-hand side except the last term can be controlled by $C(\mathcal{F} + A)$. Therefore, all that is left to show is that the last term on the right-hand side of (9.3) is okay. To this end, we claim

\[ \sup_i \int_0^1 s \| □(A\cdot N')(s) \|_{\hat{S}^1} \frac{ds}{s} ≤ C(\mathcal{F} + A)^2. \]

Recalling the definition of the $\hat{S}^1$ norm, we must bound the contribution of $\| □(A\cdot N')(s) \|_{L^\infty L^2_t}$ and $T^{1/2} □(A\cdot N')(s) \|_{L^2_t}$. We will only treat the latter (which is slightly more complicated), leaving the former to the reader.

Using the product rule for □, we compute the schematic form of □(A\cdot N') as follows.

$□(A\cdot N') = O(∂μ A, ∂z, A) + O(A, ∂μ A, A) + O(□A, ∂z A) + O(A, ∂z □A) + O(A, A, □A).$

Let us treat each type in order. Terms of the first type are the most dangerous, in the sense that there is absolutely no extra s-weight to spare. Using Cauchy-Schwarz and Strichartz, we have

\[ \int_0^1 sT^{1/2} O(∂μ A(s), ∂z, A(s)) \|_{L^2_t} \frac{ds}{s} \]

\[ ≤ CT^{1/2} \left( \int_0^1 s^{1/2} \| A(s) \|_{S^{1/2}} \frac{ds}{s} \right)^{1/2} \left( \int_0^1 s^{3/2} \| A(s) \|_{S^{1/2}} \frac{ds}{s} \right)^{1/2}. \]

Using Lemma 8.2, the last line can be estimated by $C(\mathcal{F} + A)^2$, which is acceptable.

Terms of the second type can be treated similarly using Hölder, Strichartz and Lemma 8.2, being easier due to the presence of extra s-weights. We estimate these terms as follows.

\[ \int_0^1 sT^{1/2} O(A, ∂μ A(s), ∂z A(s)) \|_{L^2_t} \frac{ds}{s} \]

\[ ≤ CT^{1/2} \int_0^1 s^{1/4} \| A(s) \|_{L^2_t} \left( s^{1/4} \| □(A\cdot N')(s) \|_{L^2_t} \right) \left( s^{1/4} \| □(A\cdot N')(s) \|_{L^2_t} \right) \frac{ds}{s} \]

\[ ≤ CT^{1/2}(A + A)^3. \]

The remaining terms all involve the d’Alembertian □. For these terms, using Hölder, we always put the factor with □ in $L^2_t$ and estimate by the $\hat{S}^k$ norm, whereas the other terms are put in $L^\infty_t$. We will always have some extra s-weight, and thus it is not difficult to show that

\[ \int_0^1 sT^{1/2} O(□A(s), ∂z A(s)) \|_{L^2_t} \frac{ds}{s} ≤ C(\mathcal{F} + A)^2, \]

\[ \int_0^1 sT^{1/2} O(A(s), □A(s)) \|_{L^2_t} \frac{ds}{s} ≤ C(\mathcal{F} + A)^3. \]
As desired, we have therefore proved
\[
\sup_i \int_0^1 s T^{1/2} \| \Box ((A_i N')') (s) \|_{L^2_{t,x}} \frac{ds}{s} \leq C_{F, A} \cdot (F + A)^2.
\]
\[\square\]

Proof of Proposition 10.3 This is an immediate consequence of Propositions 5.3.4 and §3.5.4 \[\square\]

Proof of Proposition 7.4 In fact, this proposition is a triviality in view of the simple definitions of the quantities \( F, A, \delta F, \delta A \) and the fact that \( A_n, A'_n \) are regular solutions to \( [\text{HPYM}] \). \[\square\]

10. Hyperbolic estimates : Proofs of Theorems [D] and [E]

The purpose of this section is to prove Theorems [D] and [E], which are based on analyzing the wave-type equations (1.11) and (1.12) for \( A_i \) and \( F_{si} \), respectively. Note that the system of equations for \( A_i \) is nothing but the Yang-Mills equations with source in the temporal gauge. The standard way of solving this system (see [9]) is by deriving a wave equation for \( F_{\mu \nu} \); due to a technical point, however, we take a slightly different route, which is explained further in §10.4. The wave equation (1.13) for \( F_{si} \), on the other hand, shares many similarities with that for \( A_i \) in the Coulomb gauge. In particular, one can recover the null structure for the most dangerous bilinear interaction \( [A^i, \partial_0 F_{si}] \), which is perhaps the most essential structural feature of the caloric-temporal gauge which makes the whole proof work.

Throughout this section, we work with regular solutions \( A_n, A'_n \) to \( [\text{HPYM}] \) on \((-T, T) \times \mathbb{R}^3 \times [0, 1] \).

10.1. Hyperbolic estimates for \( A_i \) : Proof of Theorem [D]

10.1.1. Equations of motion for \( A_i \). Recall that at \( s = 1 \), the connection coefficients \( A^\mu = A_\mu (s = 1) \) satisfy the hyperbolic Yang-Mills equation with source, i.e.
\[
(D^\nu F_{\mu \nu}) = w, \quad \text{for } \nu = 0, 1, 2, 3.
\]
Furthermore, we have the temporal gauge condition \( A_0 = 0 \).

Recall that \( (\partial \times B)_i := \sum_{j,k} \epsilon_{ijk} \partial_j B_k \), where \( \epsilon_{ijk} \) was the Levi-Civita symbol. In the proposition below, we record the equation of motion of \( A_i \), which are obtained simply by expanding (10.1) in terms of \( A_i \).

Proposition 10.1 (Equations for \( A_i \)). The Yang-Mills equation with source (10.1) is equivalent to the following system of equations.
\[
\partial_0 (\partial_0 F_{\mu \nu}) = - [A^\ell, \partial_0 A_\ell] + w, (10.2)
\]
\[
\Box A_i - \partial_0 (\partial_0 A_i) = - 2[A^\ell, \partial_0 A_\ell] + [A_i, \partial_0 A_\ell] + [A^\ell, \partial_0 A_\ell] - [A^\ell, [A_\ell, A_i]] - w, (10.3)
\]

Taking the curl (i.e. \( \partial \times \cdot \)) of (10.3), we obtain the following wave equation for \( \partial \times A_i \)
\[
\Box (\partial \times A_i) = - \partial \times (2[A^\ell, \partial_0 A_\ell] + [A_i, \partial_0 A_\ell] + [A^\ell, \partial_0 A_\ell] - [A^\ell, [A_\ell, A_i]]), (10.4)
\]

Remark 10.2. The usual procedure of solving (10.1) in temporal gauge consists of first deriving the hyperbolic equation for \( F_{\mu \nu} \), using the Bianchi identity and (10.1). Then one couples these equations with the transport equation
\[
F_{ni} = \partial_0 A_n,
\]
(which follows just from the definition of \( F_{ni} \) and the temporal gauge condition \( A_0 = 0 \)) and solves the system altogether. This is indeed the approach of Eardley-Moncrief [2] and Klainerman-Machedon [13]. A drawback to this approach, however, is that it requires taking a \( t \)-derivative when deriving hyperbolic equations for \( F_{ni} \). In particular, one has to estimate \( \partial_0 w \), which complicates matters in our setting.

The equations that we stated in Proposition 10.1 is the basis for a slightly different approach, which avoids taking \( \partial_0 \) at the expense of using a little bit of Hodge theory. We remark that such an approach had been taken by Tao [26], but with greater complexity than here as the paper was concerned with lower regularity (but small data) solutions to \( [\text{YM}] \).
10.1.2. Proof of Theorem D. In this part, we give a proof of Theorem D.

Proof of Theorem D. In the proof, we will work on the time interval $(-T, T)$, where $0 < T \leq 1$. We will give a rather detailed proof of (7.5). The difference analogue (7.6) can be proved in an analogous manner, whose details we leave to the reader.

Let us begin with a few product estimates.

\begin{align}
\|O(A, \partial_y A)\|_{L^\infty_t H^m_y} &\leq C A^2, & \text{for } 0 \leq m \leq 29, \\
\|O(A, \partial_y A)\|_{L^\infty_t H^m_y} &\leq C A^2, & \text{for } 0 \leq m \leq 30, \\
\|O(A, A, A)\|_{L^\infty_t H^m_y} &\leq C A^3, & \text{for } 0 \leq m \leq 31.
\end{align}

Each of these can be proved by Leibniz’s rule, Hölder and Sobolev, as well as the fact that $\|\partial_y A\|_{L^\infty_t H^{m_0}} + \|\partial_y A\|_{L^\infty_t H^{m_0}} \leq A$. Using the same techniques, we can also prove the following weaker version of (10.6) in the case $m = 30$:

\begin{align}
\|O(A, \partial_y A)\|_{L^\infty_t H^m_y} &\leq C A + C A^2 + C A^2 + C A^2 + C A^2 + (F + A)^2.
\end{align}

Next, observe that $\|u_t\|_{L^\infty_t H^m_y} \leq \sup_{t \in (-T, T)} \|F_0(t)\|_{L^\infty_t H^m_y}$, where the latter can be controlled by (8.35) for $0 \leq m \leq 30$. Combining this with (10.2), (10.3), we obtain the following estimate for $0 \leq m \leq 29$:

\begin{align}
\|\partial_y A(t)\|_{L^\infty_t H^m_y} &\leq C A + C A \cdot E + C A \cdot (F + A)^2.
\end{align}

In the case $m = 30$, replacing the use of (10.5) by (10.8), we have

\begin{align}
\|\partial_y A(t)\|_{L^\infty_t H^m_y} &\leq C A + C A^2 + C A + E + C A^2 + (F + A)^2.
\end{align}

Recall the simple div-curl identity $\sum_{i,j} \|\partial_i B_j\|^2 = \frac{1}{2} \|\partial \times B\|^2 + \|\partial B\|^2$ with $B = A(t)$. Using furthermore (10.3) with $m = 29$ and the fact that $A$ controls $\|A(t)\|_{L^\infty_t H^m_y}$, we obtain the following useful control on $\|\partial_y A(t)\|_{L^\infty_t H^m_y}$:

\begin{align}
\|\partial_y A(t)\|_{L^\infty_t H^m_y} &\leq C A + C A \cdot E + C A \cdot (F + A)^2.
\end{align}

Therefore, (10.9) holds in the case $m = 30$ as well, i.e.

\begin{align}
\|\partial_y A(t)\|_{L^\infty_t H^m_y} &\leq C A + C A \cdot E + C A \cdot (F + A)^2.
\end{align}

Integrating (10.9) with respect to $t$ from $t = 0$, we obtain for $0 \leq m \leq 30$

\begin{align}
\|\partial_y A\|_{L^\infty_t H^m_y} &\leq T \left(C A + C A \cdot (F + A)^2\right).
\end{align}

Next, observe that $\|u_t\|_{L^\infty_t H^m_y} \leq \sup_{t \in (-T, T)} \|u(t)\|_{L^\infty_t H^m_y}$. Combining this observation with (8.44) and (8.45) from Proposition 8.17 as well as (10.10) to control $\|\partial_y A\|_{H^{m_0}_y}$, we have the following estimates for $0 \leq m \leq 30$:

\begin{align}
\|u_t\|_{L^\infty_t H^m_y} &\leq C E \cdot A \cdot (E + F + A)^2.
\end{align}

We are now ready to finish the proof. Let $i = 1, 2, 3$ and $1 \leq m \leq 30$. By the energy inequality and Hölder, we have

\begin{align}
\|A\|_{S^m} &\leq C \|\partial_i A(t = 0)\|_{H^{m-1}} + CT \|A\|_{L^\infty_t H^{m-1}}.
\end{align}

The first term is controlled by $CT$. To control the second term, apply (10.3), (10.6), (10.7), (10.12). Furthermore, use (10.11) to control the contribution of $\partial_y A$. As a result, we obtain

\begin{align}
\|A\|_{S^m} &\leq C T \left(C E \cdot A + C E \cdot A \cdot (E + F + A)^2\right),
\end{align}

for $i = 1, 2, 3$ and $1 \leq m \leq 30$.

Similarly, by the energy inequality and Hölder, we have

\begin{align}
\|\partial \times A\|_{S^{m_0}} &\leq C \|\partial_i (\partial \times A)(t = 0)\|_{H^{m_0}} + CT \|\partial (\partial \times A)\|_{L^\infty_t H^{m_0}}.
\end{align}
The first term is again controlled by $CI$. To control the second term, we apply (10.4), (10.6), (10.7), (10.12); note that this time we do not need an estimate for $\partial^i A_j$. We conclude

$$\|(\partial \times A)_i\|_{S^{\alpha_0}} \leq CI + T C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{E} + \mathcal{F} + \mathcal{A})^2. \tag{10.14}$$

Finally, using the div-curl identity, (10.11) and (10.14), we have

$$\|A\|_{H^1_t} \leq CI + T \left( C_{\mathcal{F}, \mathcal{A}} \cdot \mathcal{E} + C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})^2 \right).$$

This concludes the proof. \( \square \)

10.2. Hyperbolic estimates for $F_{si}$ : Proof of Theorem \( \square \) Let us recall the hyperbolic equation (1.13) satisfied by $F_{si}$:

$$D^\mu D_\mu F_{si} = 2[F_s^\mu, F_{ip}] - D^\ell D_\ell w_i + D_i D^\ell w_\ell - (w_i) N.'$$

Note that we have rewritten $2[F_\ell^\ell, w_\ell] + 2[F^\mu_\ell, D_\mu F_\ell + D_\ell F_{ip}] = (w_i) N$ for convenience.

10.2.1. Semi-linear wave equation for $F_{si}$. Let us begin by rewriting the wave equation for $F_{si}$ in a form more suitable for our analysis. Writing out the covariant derivatives in (1.13), we obtain the following semi-linear wave equation for $F_{si}$.

$$\Box F_{si} = (F_{si}) \mathcal{M}_{\text{quadratic}} + (F_{si}) \mathcal{M}_{\text{cubic}} + (F_{si}) \mathcal{M}_w,$$

where

$$(F_{si}) \mathcal{M}_{\text{quadratic}} := -2[A_\ell^\ell, \partial_\ell F_{si}] + 2[A_0, \partial_0 F_{si}]
+ \partial_0 A_0, F_{si} - \partial_\ell A_\ell, F_{si} - 2[F_\ell^\ell, F_{si}] + 2[F_0, F_{si}],$$

$$(F_{si}) \mathcal{M}_{\text{cubic}} := [A_0, [A_0, F_{si}]] - [A_\ell, [A_\ell, F_{si}]]$$

$$(F_{si}) \mathcal{M}_w := -D_\ell D_\ell w_i + D_i D_\ell w_\ell - (w_i) N.$$

The semi-linear equation for the difference $\delta F_{si} := F_{si} - F_{si}'$ is then given by

$$\Box \delta F_{si} = (\delta F_{si}) \mathcal{M}_{\text{quadratic}} + (\delta F_{si}) \mathcal{M}_{\text{cubic}} + (\delta F_{si}) \mathcal{M}_w,$$

where $(\delta F_{si}) \mathcal{M}_{\text{quadratic}} := (F_{si}) \mathcal{M}_{\text{quadratic}} - (F_{si}') \mathcal{M}_{\text{quadratic}}$, $(\delta F_{si}) \mathcal{M}_{\text{cubic}} := (F_{si}) \mathcal{M}_{\text{cubic}} - (F_{si}') \mathcal{M}_{\text{cubic}}$ and $(\delta F_{si}) \mathcal{M}_w := (F_{si}) \mathcal{M}_w - (F_{si}') \mathcal{M}_w$.

10.2.2. Estimates for quadratic terms. We begin the proof of Theorem \( \square \) by estimating the contribution of quadratic terms.

Lemma 10.3 (Estimates for quadratic terms). Assume $0 < T \leq 1$. For $1 \leq m \leq 10$ and $p = 2, \infty$, the following estimates hold.

$$\sup_i \| (F_{si}) \mathcal{M}_{\text{quadratic}} \|_{L^2_t L^2_x H^{m-1}_x(0,1)} \leq C_{\mathcal{E}, \mathcal{F}, \mathcal{A}} \cdot (\mathcal{E} + \mathcal{F} + \mathcal{A})^2, \tag{10.15}$$

$$\sup_i \| (\delta F_{si}) \mathcal{M}_{\text{quadratic}} \|_{L^2_t L^2_x H^{m-1}_x(0,1)} \leq C_{\mathcal{E}, \mathcal{F}, \mathcal{A}} \cdot (\mathcal{E} + \mathcal{F} + \mathcal{A})(\delta \mathcal{E} + \delta \mathcal{F} + \delta \mathcal{A}). \tag{10.16}$$

Proof. We will give a rather detailed proof of (10.15). The other estimate (10.16) may be proved by first using Leibniz’s rule for $\delta$ to compute $(\delta F_{si}) \mathcal{M}_{\text{quadratic}}$, and then proceeding in an analogous fashion. We will omit the proof of the latter.

Let $1 \leq m \leq 10$ and $p = 2$ or $\infty$. We will work on the whole $s$-interval $(0,1]$. Let us begin with an observation that in order to prove (10.15), it suffices to prove that each of the following can be
bounded by $\mathcal{C}_{\mathcal{E}, F, \nabla} \cdot (\mathcal{E} + \mathcal{F} + \nabla)^2$:

$$\left\{ \begin{array}{l}
\| s^{-1/2}[(A^{\epsilon f})^j, \nabla \nabla_x (m-1) F_{s}]\|_{L_x^{2, p} L_t^2} + \| s^{-1/2}[(A^{df})^j, \nabla \nabla_x (m-1) F_{s}]\|_{L_x^{2, p} L_t^2}, \\
\sum_{j=1}^{m-1} \| s^{-1/2}[(\nabla_x^j A)^k, \nabla \nabla_x (m-j) F_{s}]\|_{L_x^{2, p} L_t^2}, \\
\| s^{-1/2}[A_0, \nabla F_{s}]\|_{L_x^{2, p} L_t^{2, H_0}}, \\
\| s^{-1/2}[\nabla A_0, F_{s}]\|_{L_x^{2, p} L_t^{2, H_0}} - 1, \\
\| [F_{s0}, F_{s0}]\|_{L_x^{2, p} L_t^{2, H_0}}, \end{array} \right. \leq \mathcal{C}_{\mathcal{E}, F, \nabla} \cdot (\mathcal{E} + \mathcal{F} + \nabla)^2.$$

Here, $A^{\epsilon f}$ and $A^{df}$, called the curl-free and the divergence-free parts of $A$, respectively, constitute the Hodge decomposition of $A$, i.e. $A_i = A_i^{\epsilon f} + A_i^{df}$. They are defined by the formulae

$$A^{\epsilon f} := -(-\Delta)^{-1}\partial_t \partial^\ell A, \quad A^{df} := (-\Delta)^{-1}((\partial_t \times (\partial_t \times A)),$$

Let us treat each of them in order.

- Case 1 : Proof of $\| s^{-1/2}[(A^{\epsilon f})^j, \nabla \nabla_x (m-1) F_{s}]\|_{L_x^{2, p} L_t^2} \leq C_{\mathcal{E}, F, \nabla} \cdot (\mathcal{E} + \mathcal{F} + \nabla)^2$.

We claim that the following estimate for $A^{\epsilon f}$ holds.

$$\| A^{\epsilon f}(s)\|_{L_x^{4/3} L_t^2 L_x^\infty} \leq C_{\mathcal{A}} + C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})^2.$$  \hfill (10.17)

Note, on the other hand, that $\| \nabla \nabla_x (m-1) F_{s}\|_{L_x^{4/3} L_t^2 L_x^\infty} \leq \mathcal{F}$ for $1 \leq m \leq 10$. Assuming the claim, the desired estimate then follows immediately by Hölder.

The key to our proof of \hfill (10.14)
is the covariant Coulomb condition satisfied by $F_{si}$

$$\mathcal{D}^\ell F_{si} = 0,$$

which was proved in Appendix A. Writing out the covariant derivative $\mathcal{D}^\ell = \partial^\ell + A^\ell$ and using the relation $F_{si} = \partial_s A_t$, we arrive at the following improved transport equation for $\partial^\ell A_t$.

$$\partial_s(\partial^\ell A_t(s)) = -[A^\ell(s), F_{st}(s)].$$  \hfill (10.18)

Observe furthermore that $\| A^{\epsilon f}(s)\|_{L_x^{4/3} L_t^2 L_x^\infty} = \sup_{0<s\leq 1} \| A^{\epsilon f}(s)\|_{L_x^2 L_x^\infty}$. Our goal, therefore, is to estimate the latter by using (10.18).

Using the fundamental theorem of calculus and Minkowski, we obtain, for $1 \leq r < \infty$, the inequality

$$\sup_{0<s\leq 1} \| \partial^\ell A_t(s)\|_{L_x^r L_x^r} \leq \| \partial^\ell A_t\|_{L_x^r L_x^r} + \int_0^1 \| [A^\ell(s), F_{st}(s)]\|_{L_x^r L_x^r} \, ds.$$  \hfill (10.19)

Let us recall that $(A^{\epsilon f})_i = (-\Delta)^{-1}\partial_t \partial^\ell A_i$ by Hodge theory. It then follows that $\partial_s (A^{\epsilon f})_j = R_i R_j (\partial^\ell A_t)$, where $R_i, R_j$ are Riesz transforms. By elementary harmonic analysis \hfill [25], for $1 < r < \infty$, we have the inequality

$$\| \partial_s A^{\epsilon f}\|_{L_x^r L_x^r} \leq C_r \| \partial^\ell A_t\|_{L_x^r L_x^r}.$$  \hfill (10.19)

On the other hand, using Sobolev and Gagliardo-Nirenberg, we have

$$\| A^{\epsilon f}\|_{L_x^r L_x^r} \leq C \| \partial_s A^{\epsilon f}\|_{L_x^r L_x^r}^{1/3} \| \partial^\ell A^{\epsilon f}\|_{L_x^r L_x^r}^{2/3}.$$  \hfill (10.19)

As a result of these two inequalities, it suffices to bound the $L_x^2$ and $L_x^2 L_x^2$ norms of $\partial^\ell A_t(s)$ using (10.19). For the first term on the right-hand side of (10.19), we obviously have

$$\| \partial^\ell A_t\|_{L_x^2} + \| \partial^\ell A_t\|_{L_x^2 L_x^2} \leq CT^{1/2}(\| \partial^\ell A_t\|_{L_x^2} + \| \partial^\ell A_t\|_{L_x^2 L_x^2}) \leq C_{\mathcal{A}}$$

by Hölder in time. Next, note that the second term on the right-hand side of (10.19) is equal to $\| [A^{\epsilon f}, F_{st}]\|_{L_x^{r, 1} L_x^2} L_x^2$, where $r = \frac{6}{5} + \frac{2}{5}$. In the case $r = 2$, we estimate this, using Lemma 8.2 and Proposition 8.11, as follows.

$$\| [A^{\epsilon f}, F_{st}]\|_{L_x^{2, 1} L_x^2} \leq CT^{1/2}(\| A^{\epsilon f}\|_{L_x^{2, 1} L_x^2} \| F_{st}\|_{L_x^{r, 1} L_x^2} \leq C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{F} + \mathcal{A})^2.$$  \hfill (10.19)
In the other case \( r = 4 \), we proceed similarly, again using Lemma 8.2 and Proposition 8.11

\[
\|A^\ell, F_s\|_{L^{13/8}_x L^{4}_t} \leq C\|s^{1/2} \| L^{1/4}_x L^{\infty}_x F_s\|_{L^{13/8}_x L^{4}_t} \leq C_{F, A}(F + A)^2.
\]

Combining these estimates, we obtain (10.17).

- Case 2 : Proof of \( s^{-1/2}(A^\ell, F_s) \), \( \nabla_t \nabla_x^{(m - 1)} F_s \), \( \|L^{2}_x L^{2}_t \) : Here, we cannot estimate \( A^\ell \) in \( L^{2}_x L^{\infty}_t \). We remark that this is the only place where we utilize the null form estimate. We mark that this is the only place where we utilize the null form estimate.

For \( B = B_4 \) (\( i = 1, 2, 3 \), \( \phi \) smooth and \( B_4, \phi \in \dot{S}^1 \), we claim that the following estimate holds for \( 0 < s \leq 1 \):

\[
(10.20) \quad \|[(B^\ell)^\ell, \partial_t \phi]\|_{L^{2}_x} \leq C(\sup_k \|B_k\|_{\dot{S}^1})\|\phi\|_{\dot{S}^1}.
\]

Assuming the claim, by the Correspondence Principle, we then obtain the estimate

\[
\|s^{-1/2}(T^\ell, \nabla_t \phi)\|_{L^{2}_x L^{2}_t} \leq C(\sup_k \|T_k\|_{L^{1/4}_x L^{\infty}_x})\|\phi\|_{L^{1/4}_x L^{\infty}_x},
\]

for smooth \( T = T_4(s) \) (\( i = 1, 2, 3 \) \( \psi \) such that the right-hand side is finite. Let us take \( T = A, \psi = \nabla_x^{(m - 1)} F_s \). By Proposition 7.2, we have \( \|A\|_{L^{1/4}_x L^{\infty}_x} \leq C_{F, A}(F + A) \), whereas by definition \( \|\nabla_x^{(m - 1)} F_s\|_{L^{1/4}_x L^{\infty}_x} \leq C_F \) for \( 1 \leq m \leq 10 \). The desired estimate therefore follows.

Now, it is only left to prove (10.20). The procedure that we are about to describe is standard, due to Klainerman-Machedon [12], [13]. We reproduce the argument here for the sake of completeness.

Let us first assume that \( B_i \) is Schwartz in \( x \) for every \( t, s \). Then simple Hodge theory tells us that \( B_i^\ell = (\partial \times V_i) \), where

\[
V_i(x) := (-\Delta)^{-1}(\partial \times B_i)_i(x) = \frac{1}{4\pi} \int \left( B(y) \times \frac{(x - y)}{|x - y|^3} \right) dy,
\]

where we suppressed the variables \( t, s \). Substituting \( (B^\ell)^\ell = (\partial \times V)^\ell \) on the left-hand side of (10.20), we have

\[
\|\sum_{j,k,l} \epsilon_{ijk} [\partial_j V_k(s), \partial_l \psi(s)]\|_{L^{2}_x} \leq \frac{1}{2} \sum_{j,k,l} \|Q_{ij}(V_k(s), \psi(s))\|_{L^{2}_x} \leq C(\sup_k \|V_k(s)\|_{\dot{S}^1})\|\psi(s)\|_{\dot{S}^1},
\]

where we remind the reader that \( Q_{ij}(\phi, \psi) = \partial_j \phi \partial_i \psi - \partial_j \phi \partial_i \psi \), and on the last line we used (3.14) of Proposition 3.3 (null form estimate). Since \( \partial_j V_k = (-\Delta)^{-1} \partial_j (\partial \times B_i) \), and \( \|\| \) is an \( L^2 \)-type norm, we see that

\[
\sup_k \|V_k(s)\|_{\dot{S}^1} = \sup_{j,k} \|\partial_j V_k(s)\|_{\dot{S}^1} \leq C \sup_k \|B_k(s)\|_{\dot{S}^1},
\]

from which (10.20) follows, under the additional assumption that \( B_i \) are Schwartz in \( x \). Then, using the quantitative estimate (10.20), it is not difficult to drop the Schwartz assumption by approximation.

- Case 3 : Proof of \( \|s^{-1/2}(\nabla_x^j A^\ell, \nabla_t \nabla_x^{(m - 1 - j)} F_s)\|_{L^{2}_x L^{2}_t} \leq C_{F, A}(F + A)^2 \).

By the Hölder inequality \( L^{1/4}_x L^{4}_x \leq L^{2}_x L^{2}_t \), the Correspondence Principle and Hölder for \( L^{2}_x \) (Lemma 3.9), we immediately obtain the estimate

\[
\|s^{-1/2}(\nabla_x^j A^\ell, \nabla_t \nabla_x^{(m - 1 - j)} F_s)\|_{L^{2}_x L^{2}_t} \leq C \sum_{j=1}^{m-1} \|A\|_{L^{1/4}_x L^{4}_x} \|F_s\|_{L^{1/4}_x L^{4}_x} \|\nabla_x^{j} W_{m-j,4}\|_{L^{4}_x}.
\]

Let us apply Lemma 8.2 to \( \|A\|_{L^{1/4}_x L^{4}_x} \) as \( 1 \leq j \leq m-1 \leq 9 \), this can be estimated by \( C(F + A) \). On the other hand, as \( 1 \leq m-j \leq m-1 \leq 9 \), \( \|F_s\|_{L^{1/4}_x L^{4}_x} \|\nabla_x^{j} W_{m-j,4}\|_{L^{4}_x} \) can be controlled by \( C_F \) via Strichartz. The desired estimate follows.
- Case 4: Proof of \( \|s^{-1/2}[A_0, \nabla_0 F_s]\|_{L^{2,p}_x L^{p}_t H^{m-1}_x} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2 \).

By Leibniz’s rule, the Hölder inequality \( L^2_t L^\infty_x \cdot L^\infty_t L^2_x \subset L^1_{t,x} \), the Correspondence Principle and Hölder for \( L^{\ell,p}_t \) (Lemma 3.9), we have

\[
\|s^{-1/2}[A_0, \nabla_0 F_s]\|_{L^{2,p}_x L^{p}_t H^{m-1}_x} \leq C \sum_{j=0}^{m-1} \|\nabla^{(j)}_x A_0\|_{L^{2+1/4,\infty}_x L^{\infty}_t} \|\nabla_0 F_s\|_{L^{\ell/4,p}_x L^{\infty}_t H^{m-1-j}_x}.
\]

Thanks to the extra weight of \( s^{1/4} \) and the fact that \( 0 \leq j \leq m - 1 \leq 9 \), we can easily prove \( \|\nabla^{(j)}_x A_0\|_{L^{2+1/4,\infty}_x L^{\infty}_t} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2 \) via Lemma 8.19 Gagliardo-Nirenberg (Lemma 3.16) and Proposition 8.14. On the other hand, as \( 0 \leq m - 1 - j \leq 9 \), we have \( \|\nabla_0 F_s\|_{L^{\ell/4,p}_x L^{\infty}_t H^{m-1-j}_x} \leq CF \).

The desired estimate then follows.

- Case 5: Proof of \( \|s^{-1/2}[\nabla_0 A_0, F_s]\|_{L^{2,p}_x L^{p}_t H^{m-1}_x} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2 \).

We claim that the following estimate for \( \nabla_0 A_0 \) holds for \( 0 \leq j \leq 9 \).

\[
\|\nabla^{(j)}_x \nabla_0 A_0\|_{L^{2,p}_x L^{p}_t H^{m-j}_x} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2.
\]

Assuming the claim, let us prove the desired estimate. As in the previous case, we have

\[
\|s^{-1/2}[\nabla_0 A_0, F_s]\|_{L^{2,p}_x L^{p}_t H^{m-1}_x} \leq C \sum_{j=0}^{m-1} \|\nabla^{(j)}_x \nabla_0 A_0\|_{L^{2+1/4,\infty}_x L^{\infty}_t} \|F_s\|_{L^{\ell/4,p}_x L^{\infty}_t H^{m-1-j}_x}.
\]

The factor \( \|\nabla^{(j)}_x \nabla_0 A_0\|_{L^{2+1/4,\infty}_x L^{\infty}_t} \) can be controlled by (10.21). For the other factor, we divide into two cases: For \( 0 \leq j \leq m - 1 \), we have \( \|F_s\|_{L^{\ell/4,p}_x L^{\infty}_t H^{m-1-j}_x} \leq CF \), whereas for \( j = 0 \) we use Proposition 8.11. The desired estimate then follows.

To prove the claim, we begin with the formula \( \partial_0 A_0 = -\int_s^t \partial_0 F_s(s') ds' \). Proceeding as in the proofs of the Lemmas 8.2 and 8.3, we obtain the estimate

\[
\|\nabla^{(j)}_x \nabla_0 A_0\|_{L^{2,p}_x L^{p}_t H^{m-j}_x} \leq C\|\nabla^{(j)}_x \nabla_0 A_0\|_{L^{2,p}_x L^{p}_t H^{m-j}_x}.
\]

In order to estimate the right-hand side, recall the identity \( \partial_0 F_s = \partial_t w_t + [A^\ell, w_t] + [A_0, F_s] \) from Appendix A. It therefore suffices to prove

\[
\|\nabla^{(j)}_x \nabla_t w_t\|_{L^{1,2}_x L^{\infty}_t} + \|\nabla^{(j)}_x [A^\ell, w_t]\|_{L^{1/2,2}_x L^{\infty}_t} + \|\nabla^{(j)}_x [A_0, F_s]\|_{L^{1/2,2}_x L^{\infty}_t} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2,
\]

for \( 0 \leq j \leq 9 \).

By Gagliardo-Nirenberg (Lemma 3.16) and Proposition 8.17 we have \( \|\nabla^{(j)}_x \nabla_t w_t\|_{L^{1,2}_x L^{\infty}_t} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2 \) for \( 0 \leq j \leq 9 \).

Next, by Leibniz’s rule, Hölder, the Correspondence Principle and Lemma 3.9, we obtain

\[
\|\nabla^{(j)}_x [A^\ell, w_t]\|_{L^{1/2,2}_x L^{\infty}_t} \leq C \sum_{j' = 0}^j \|\nabla^{(j')}_x A\|_{L^{1/4+1/4,\infty}_x L^{\infty}_t} \|\nabla^{(j-j')}_x w_t\|_{L^{1,2}_x L^{\infty}_t}.
\]

Note the extra weight of \( s^{1/4} \) on the first factor. As \( 0 \leq j' \leq 9 \), by Lemma 8.2 Gagliardo-Nirenberg (Lemma 3.16) and Proposition 8.13 we have \( \|\nabla^{(j')}_x A\|_{L^{1/4+1/4,\infty}_x L^{\infty}_t} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2 \). On the other hand, \( \|\nabla^{(j-j')}_x w_t\|_{L^{1,2}_x L^{\infty}_t} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2 \) by Gagliardo-Nirenberg (Lemma 3.16) and Proposition 8.17.

Finally, we can show \( \|\nabla^{(j)}_x [A_0, F_s]\|_{L^{1/2,2}_x L^{\infty}_t} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2 \) by proceeding similarly, with applications of Propositions 8.8 and 8.17 replaced by Proposition 8.14. We leave the details to the reader.

- Case 6: Proof of \( \|F_{10}, F_s\|_{L^{2,p}_x L^{p}_t H^{m-1}_x} \leq C_{\varepsilon, F, \Delta} \cdot (\varepsilon + F + \Delta)^2 \).

By Leibniz’s rule, Hölder, the Correspondence Principle and Lemma 3.9, we have

\[
\|F_{10}, F_s\|_{L^{2,p}_x L^{p}_t H^{m-1}_x} \leq C \sum_{j=0}^{m-1} \|\nabla^{(j)}_x F_{10}\|_{L^{2,2}_x L^{\infty}_t} \|F_s\|_{L^{1/4,\infty}_x L^{\infty}_t H^{m-1-j}_x}.
\]
Using Gagliardo-Nirenberg (Lemma 3.10) and Lemma 8.4 combined with Propositions 8.14 8.8 we can prove the following estimate for the first factor (for $0 \leq j \leq 9$):

$$\| \nabla_x^j F_0 \|_{L_2^{3/2} L^\infty} \leq C_{E,F,A} \cdot (E + F + A).$$

For the second factor, we simply apply Proposition 8.14 to conclude $\| F_{00} \|_{L_2^{3/4} L^\infty L_{0}^{m-1}} \leq C_{E,F,A} \cdot (E + F + A)^2$ for $0 \leq m - 1 - j \leq 9$, which is good.

**- Case 7:** Proof of $\| [F_1^2, F_{00}] \|_{L_2^p L^\infty L_{0}^{m-1}} \leq C_{E,F,A} \cdot (E + F + A)^2$.

In this case, we simply expands out $F_{10} = \partial_t A_t - \partial_t A_t + [A_t, A_t]$. Note that the first two terms give additional terms of the form already handled in Step 3, whereas the last term will give us cubic terms which can simply be estimated by using Hölder and Sobolev. For more details, we refer to the proof of Lemma 10.3 below.

10.2.3. **Estimates for cubic terms.** The contribution of cubic terms are much easier to handle compared to quadratic terms. Indeed, we have the following lemma.

**Lemma 10.4** (Estimates for cubic terms). Assume $0 < T \leq 1$. For $1 \leq m \leq 10$ and $p = 2, \infty$, the following estimates hold.

$$\sup_i \| (F_{1})_{i} \cdot M_{\text{cubic}} \|_{L_2^p L^\infty L_{0}^{m-1}(0,1]} \leq C_{E,F,A} \cdot (E + F + A)^3,$$

(10.22)

$$\sup_i \| (\delta F_{1})_{i} \cdot M_{\text{cubic}} \|_{L_2^p L^\infty L_{0}^{m-1}(0,1]} \leq C_{E,F,A} \cdot (E + F + A)^2 (\delta E + \delta F + \delta A).$$

(10.23)

**Proof.** As before, we give a proof of (10.22), leaving the similar proof of the difference version (10.23) to the reader.

Let $1 \leq m \leq 10$ and $p = 2, \infty$. As before, we work on the whole interval $[0,1]$. We begin with the obvious inequality

$$\| \phi_1 \phi_2 \phi_3 \|_{L_2^p} \leq CT^{1/2} \prod_{i=1,2,3} \| \phi_i \|_{L_2^p H_{0}^1},$$

which follows from Hölder and Sobolev. By Leibniz’s rule, the Correspondence Principle and Hölder for $L_2^p$ (Lemma 8.3), we obtain

$$\sup_i \| (F_{1})_{i} \cdot M_{\text{cubic}} \|_{L_2^p L^\infty L_{0}^{m-1}} \leq CT^{1/2} \| \nabla_x A \|_{L_2^{3/4} L^\infty L_{0}^{m-1}} \| \nabla_x F_0 \|_{L_2^{3/4} L^\infty L_{0}^{m-1}} + CT^{1/2} \| \nabla_x A_0 \|_{L_2^{3/4} L^\infty L_{0}^{m-1}} \| \nabla_x F_0 \|_{L_2^{3/4} L^\infty L_{0}^{m-1}}.$$

Note the obvious bound $\| \nabla_x F_0 \|_{L_2^{3/4} L^\infty L_{0}^{m-1}} \leq CF$. Applying Lemma 8.3 to $\| A_0 \|$ (using the extra weight of $s^{1/4}$) and Proposition 8.14, we also obtain $\| \nabla_x A_0 \|_{L_2^{3/4} L^\infty L_{0}^{m-1}} \leq C_{F,A} \cdot (E + F + A)^2$. Finally, we split $\| \nabla_x A \|_{L_2^{3/4} L^\infty L_{0}^{m-1}}$ into $\| A \|_{L_2^{3/4} L^\infty L_{0}^{1}}$ and $\| \nabla (A) \|_{L_2^{3/4} L^\infty L_{0}^{m-2}}$ (where the latter term does not exist in the case $m = 1$). For the former, we apply Proposition 7.7 whereas for the latter we apply Lemma 8.2. We then conclude $\| \nabla_x A \|_{L_2^{3/4} L^\infty L_{0}^{m-1}} \leq C_{F,A} \cdot (F + A)$. Combining all these estimates, (10.22) follows.

10.2.4. **Estimates for terms involving $w$.** Finally, the contribution of $(F_{1})_{i} \cdot M_{w}$ is estimated by the following lemma.

**Lemma 10.5** (Estimates for terms involving $w$). Assume $0 < T \leq 1$. For $1 \leq m \leq 10$ and $p = 2, \infty$, the following estimates hold.

$$\sup_i \| (F_{1})_{i} \cdot M_{w} \|_{L_2^{p} L^\infty L_{0}^{m-1}(0,1]} \leq C_{E,F,A} \cdot (E + F + A)^2,$$

(10.24)

$$\sup_i \| (\delta F_{1})_{i} \cdot M_{w} \|_{L_2^{p} L^\infty L_{0}^{m-1}(0,1]} \leq C_{E,F,A} \cdot (E + F + A)(\delta E + \delta F + \delta A).$$

(10.25)
Proof. As before, we will only give a proof of (10.24), leaving the similar proof of (10.25) to the reader.

Let \( 1 \leq m \leq 10 \) and \( p = 2 \) or \( \infty \). We work on the whole interval \((0, 1]\). Note that, schematically, 
\[
(F_i) \mathcal{M}_w = \partial^\nu \partial_{\nu} w_i - \partial_i \partial_{\nu} w + (w) N + \mathcal{O}(A, \partial_x A) + \mathcal{O}(A, A, w).
\]

By Leibniz’s rule, the Correspondence Principle (from the Hölder inequality \( L_{t,x}^\infty \cdot L_{t,x}^2 \subset L_{t,x}^2 \)) and Lemma 3.9, we obtain the following estimate.

\[
(10.26)
\]

By Lemma 8.2, combined with Proposition 8.8, the following estimate holds for \( 0 \leq j \leq 10 \).

\[
(10.27)
\]

Now (10.24) follows from (10.26), (10.27) and (8.46), (8.47) of Proposition 8.17 thanks to the restriction \( 1 \leq m \leq 10 \).

10.2.5. Completion of the proof. We are now prepared to give a proof of Theorem E.

Proof of Theorem E. Let us begin with (10.24). Recalling the definition of \( \mathcal{F} \), it suffices to show

\[
\| F_{si} \|_{L_{t,x}^{5/4 + p} H^{m-1}} \leq C T^1 + T^{1/2} C_{\mathcal{F}, \mathcal{A}} \cdot (\mathcal{E} + \mathcal{F} + \mathcal{A})^2,
\]

for \( i = 1, 2, 3 \), \( p = 2, \infty \) and \( 1 \leq m \leq 10 \). Starting with the energy inequality and applying the Correspondence Principle, we obtain

\[
\| F_{si} \|_{L_{t,x}^{5/4 + p} H^{m-1}} \leq C \| \nabla_{t,x} F_{si} (t = 0) \|_{L_{t,x}^{5/4 + p} H^{m-1}} + CT^{1/2} \| \nabla_{t,x} F_{si} \|_{L_{t,x}^{5/4 + p} H^{m-1}}.
\]

The first term on the right-hand side is estimated by \( CT \). For the second term, as \( \Box F_{si} = (F_i) \mathcal{M}_{\text{quadratic}} + (F_i) \mathcal{M}_{\text{cubic}} + (F_i) \mathcal{M}_w \), we may apply Lemmas 10.2, 10.3, 10.5 (estimates (10.22) and (10.24)), in particular, to conclude

\[
\| \Box F_{si} \|_{L_{t,x}^{5/4 + 1, p} H^{m-1}} \leq T^{1/2} C_{\mathcal{F}, \mathcal{A}} (\mathcal{E} + \mathcal{F} + \mathcal{A})^2,
\]

which is good.

The proof of (7.8) is basically identical, this time controlling the initial data term by \( C \delta I \) and using (10.16), (10.23), (10.25) in place of (10.15), (10.22), (10.24).

Remark 10.6. Recall that in 13, one has to recover two types of null forms, namely \( P_{ij} (|\partial_x|^{-1} A, A) \) and \( |\partial_x|^{-1} Q_{ij} (A, A) \), in order to prove \( H^1 \) local well-posedness in the Coulomb gauge. An amusing observation is that we did not need to uncover the second type of null forms in our proof.

Appendix A. Derivation of covariant equations of HPYM

Let \( I \subset \mathbb{R} \) be an open interval, \( s_0 > 0 \), and \( A_\alpha \) a connection 1-form on \( I \times \mathbb{R}^3 \times [0, s_0] \) with coordinates \((t, x^1, x^2, x^3, s)\). Recall that a bold-faced latin index \( \alpha \) runs over all the indices corresponding to \( t, x^1, x^2, x^3, s \). As in the introduction, we may define the covariant derivative \( D_\alpha \) associated to \( A_\alpha \). The commutator of the covariant derivatives in turn defines the curvature 2-form \( F_{ab} \). Note that the Bianchi identity holds automatically:

\[
(A.1)
\]

\[
D_\alpha F_{bc} + D_b F_{ca} + D_c F_{ab} = 0.
\]

In this appendix, we will consider a solution \( A_\alpha \) to the hyperbolic-parabolic Yang-Mills system, which we restate here for the reader’s convenience:

\[
\text{(HPYM)} \quad \begin{cases}
F_{\mu} = D^i F_{\mu i} & \text{on } I \times \mathbb{R}^3 \times [0, s_0], \\
D^i F_{\mu i} = 0 & \text{along } I \times \mathbb{R}^3 \times \{0\}.
\end{cases}
\]
Let us also recall the definition of the Yang-Mills tension field $w_\nu$:

$$w_\nu := D^\mu F_{\nu\mu}.$$ 

We will use the convention of using an underline to signify the variable being evaluated at $s = s_0$. For example, $A_\mu = A_\mu(s = s_0), B_\mu = B_\mu(s = s_0)$, and $w_\mu = w_\mu(s = s_0)$ etc. In the main body of the paper, $s_0$ is always set to be 1 by scaling.

The main result of this appendix is the following theorem.

**Theorem A.1** (Covariant equations of the hyperbolic-parabolic Yang-Mills system). Let $A_a$ be a smooth solution to the hyperbolic-parabolic Yang-Mills system (HPYM). Then the following covariant equations hold.

\begin{align}
(A.2) & \quad D^\ell F_{\ell k} = 0, \\
(A.3) & \quad D_0 F_{00} = -D_0 w_0 = -D^\ell w_\ell, \\
(A.4) & \quad D_s F_{ab} = D^\ell D_\ell F_{ab} - 2[F_a^\ell, F_b^\ell], \\
(A.5) & \quad D_s w_\nu = D^\mu D_\mu w_\nu + 2[F_\nu^\ell, w_\ell] + 2[F^{\mu\ell}, D_\mu F_{\nu\ell} + D_\ell F_{\nu\mu}], \\
(A.6) & \quad D^{\nu\mu} D_\mu F_{\nu\nu} = 2[F_s^\mu, F_{\nu\mu}] - 2[F^{\mu\ell}, D_\mu F_{\nu\ell} + D_\ell F_{\nu\mu}] - D^\ell D_\ell w_\nu + D_\mu D^\ell w_\ell - 2[F_\nu^\ell, w_\ell]. \\
(A.7) & \quad D^{\nu\mu} F_{\nu\mu} = w_\nu.
\end{align}

Moreover, we have $w_\nu(s = 0) = 0$.

**Remark A.2.** An inspection of the proof shows that (A.2)–(A.7) hold under the weaker hypothesis that $A_a$ satisfies only (YM). However, the last statement of the theorem is equivalent to (YM) along $\{s = 0\}$.

**Proof.** Let us begin with (A.2). This is a consequence of the following simple computation:

$$D^\ell F_{\ell k} = D^\ell D^k F_{k\ell} = \frac{1}{2} D^\ell D^k F_{k\ell} + \frac{1}{2} D^k D^\ell F_{k\ell} + \frac{1}{2}[F^{\ell k}, F_{k\ell}] = \frac{1}{2}(D^\ell D^k + D^k D^\ell)F_{k\ell} = 0,$$

where we have used anti-symmetry of $F_{k\ell}$ for the last equality.

Next, in order to derive (A.3), we first compute

$$D^{\nu\mu} w_\mu = D^{\nu\mu} D^\ell F_{\nu\mu} = 0,$$

by a computation similar to the preceding one. This give the second equality of (A.3). In order to prove the first equality, we compute

\begin{align}
(A.8) & \quad D^{\nu\mu} F_{\nu\mu} = D^{\nu\mu} D^\ell F_{\ell \nu} = D^\ell D^{\nu\mu} F_{\ell \nu} + [F^{\nu\mu}, F_{\ell \nu}] = D^\ell w_\ell
\end{align}

and note that $D^{\nu\mu} F_{\nu\mu} = D^0 F_{s0} = -D_0 F_{s0}$ by (A.2).

Next, let us derive (A.4). We begin by noting that the equation $F_{sa} = D^t F_{ta}$ holds for $a = t, x, s$; for the last case, we use (A.2). Using this and the Bianchi identity (A.1), we compute

$$D_s F_{ab} = D_a F_{sb} - D_b F_{sa} = D_a D^\ell F_{sb} - D_b D^\ell F_{sa} = D^\ell (D_a F_{tb} - D_b F_{ta}) + [F_a^\ell, F_{tb}] - [F_b^\ell, F_{ta}].$$

In order to prove (A.5), we will use (A.4). We compute as follows.

$$D_s w_\nu = D_\mu D^\mu F_{\nu\mu} = -D^\mu \left( D^\ell D_\ell F_{\nu\mu} - 2[F_\nu^\ell, F_{\mu\ell}] \right) + [F_s^\mu, F_{\nu\mu}]$$

$$= D^\ell D_\ell \left( D^{\nu\mu} F_{\nu\mu} \right) + [F^{\mu\ell}, D_\mu F_{\nu\ell}] + D^\ell [F^{\mu\ell}, F_{\nu\mu}] - 2 D_\mu [F_\nu^\ell, F_{\mu\ell}] + [F_s^\mu, F_{\nu\mu}]$$

$$= D^\ell D_\ell w_\nu + 2[F_\nu^\ell, w_\ell] + 2[F^{\mu\ell}, D_\mu F_{\nu\ell} + D_\ell F_{\nu\mu}].$$

Note that (A.7) is exactly the definition of $w_\nu$ at $s = 1$. We are therefore only left to prove (A.6).
Here, the idea is to start with the Bianchi identity $0 = D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu}$ and to take $D^\mu$ of both sides. The first term on the right-hand side gives the desired term $D^\mu D_\mu F_{\nu\lambda}$. For the second term, we compute
\[ D^\mu D_\nu F_{\lambda\mu} = D_\nu D^\mu F_{\lambda\mu} + [F^\mu_\lambda, F_{\nu\mu}] = D_\nu D^\mu F_{\lambda\mu} - [F^\mu_\lambda, F_{\nu\mu}], \]
and for the third term, we compute, using (A.8),
\[ D^\mu D_\lambda F_{\mu\nu} = D_\lambda D^\mu F_{\mu\nu} + [F^\mu_\nu, F_{\mu\lambda}] = -D_\lambda D^\mu F_{\mu\nu} - [F^\mu_\nu, F_{\mu\lambda}]. \]
Combining these with (A.5), we obtain (A.6). □

Appendix B. Estimates for gauge transforms

In this section, we give proofs of the two gauge transform lemmas stated in the main body of the paper (namely Lemmas 4.6 and 5.4). Let us start with a general proposition concerning the solution to an ODE, which arises from the equations satisfied by the gauge transforms (and their differences).

**Proposition B.1 (ODE estimates).** Let $\Omega \subset \mathbb{R}$ be an interval, $\omega_0 \in \Omega$, and $A = A(\omega, x)$, $F = F(\omega, x)$ a pair of $n \times n$ matrix valued functions on $\Omega \times \mathbb{R}^3$. Consider the following ODE for an $n \times n$ matrix-valued function $U = U(\omega, x)$:

\[ \begin{cases} \partial_\omega U = AU + F, \\ U(\omega_0) = U_0. \end{cases} \]  

The following statements hold.

1. Suppose that $U_0 \in C^1(\Omega) \cap W^{1,3}(\hat{H}_x^3)$, $\mathcal{F} = L^1(C^0\cap W^{1,3}(\hat{H}_x^3))(\Omega)$, and that there exists $D > 0$ such that
\[ \|A\|_{L^1_\omega L^\infty(\Omega)} + \|\partial_\omega A\|_{L^1_\omega L^3(\Omega)} \leq D. \]
Suppose furthermore that
\[ \text{either } \|\partial_\omega^{(2)} A\|_{L^1_\omega L^3(\Omega)} \leq D \text{ or } \sup_{\omega \in \Omega} \left\| \int_{\omega_0}^{\omega} \Delta A(\omega') d\omega' \right\|_{L^2_\omega} \leq D. \]
Then there exists a unique solution $U$ to (B.1) which is continuous in $x$ and obeys the following estimates.

\[ \|U\|_{L^\infty_\omega L^3_x(\Omega)} \leq e^{CD} \|U_0\|_{L^\infty_\omega L^3_x(\Omega)} + \|\mathcal{F}\|_{L^1_\omega L^\infty(\Omega)}, \]

\[ \|\partial_\omega U\|_{L^\infty_\omega L^3_x(\Omega)} \leq e^{CD} \|\partial_\omega U_0\|_{L^2_x} + CDe^{CD} \|U_0\|_{L^\infty_\omega L^3_x(\Omega)} + \|\partial_\omega F\|_{L^1_\omega L^3(\Omega)}, \]

\[ \|\partial_\omega^{(2)} U\|_{L^\infty_\omega L^3_x(\Omega)} \leq e^{CD} \|\partial_\omega^{(2)} U_0\|_{L^2_x} + CDe^{CD} \|U_0\|_{L^\infty_\omega L^3_x(\Omega)} + \|\partial_\omega^{(2)} F\|_{L^1_\omega L^3(\Omega)}. \]

2. In addition to the hypotheses of part (1), suppose furthermore that $F \in L^\infty_\omega (L^3_x \cap \hat{H}_x^3)(\Omega)$ and
\[ \|A\|_{L^\infty_\omega L^3_x(\Omega)} + \|\partial_\omega A\|_{L^\infty_\omega L^3_x(\Omega)} \leq D. \]
Then the unique solution $U$ in (1) furthermore obeys

\[ \|\partial_\omega U\|_{L^\infty_\omega L^3_x(\Omega)} \leq CDe^{CD} \|U_0\|_{L^\infty_\omega L^3_x(\Omega)} + \|\mathcal{F}\|_{L^1_\omega L^\infty(\Omega)}, \]

\[ \|\partial_\omega \partial_\omega U\|_{L^\infty_\omega L^3_x(\Omega)} \leq CDe^{CD} \|\partial_\omega^{(2)} U_0\|_{L^2_x} + CDe^{CD} \|U_0\|_{L^\infty_\omega L^3_x(\Omega)} + \|\partial_\omega F\|_{L^1_\omega L^3(\Omega)}. \]

Finally, we remark that the same conclusion still holds if we replace the ODE by $\partial_\omega U = UA + F$.

**Proof.** With the understanding that all norms are taken over $\Omega$, we will omit writing $\Omega$.

Mollifying $U_0$ and $A, F$ for every $\omega$, we may assume that $A$ and $F$ depends smoothly on $x$. By the standard ODE theory, there then exists a unique solution $U$ to (B.1) which is also smooth in $x$, and whose derivatives obey the equations obtained by differentiating (B.1). We will establish the proposition for such smooth objects; by a standard approximation procedure, the original proposition will then follow.
Proof of (1). We begin by integrating (B.1) to recast it in the following equivalent form.

\[(B.7) \quad U(\omega) = U_0 + \int_{\omega_0}^\omega F(\omega') \, d\omega' + \int_{\omega_0}^\omega A(\omega')U(\omega') \, d\omega'.\]

Taking the supremum in \(x\) of both sides of (B.7), the first estimate (B.2) follows as a simple consequence of Gronwall’s inequality. The second estimate (B.3) can be proved similarly (i.e. by Gronwall and the first estimate), by differentiating the equation (B.7) and taking the \(L^2_x\) norm. Differentiating once more and taking the \(L^2_x\) norm, the third estimate (B.4) easily follows (again by Gronwall and the previous estimates) if one assumes \(\|\partial_x^2 U\|_{L^2_x L^2_x} \leq D\). The case of the alternative hypothesis \(\sup_{\omega \in \Omega} \|\int_{\omega_0}^\omega \Delta A(\omega') \, d\omega'\|_{L^2_x} \leq D\), on the other hand, is not as apparent, and requires a separate argument.

Here we follow the arguments in [13, Proof of Theorem 7]. First, in order to control \(\|\partial_x^2 U\|_{L^2_x}\), observe that it suffices to control \(\|\Delta U\|_{L^\infty_x L^2_x}\), as a simple integration by parts shows that \(\|\partial_x^2 U(\omega)\|_{L^2_x} \leq C\|\Delta U(\omega)\|_{L^2_x}\). Taking \(\Delta\) of (B.7), we obtain

\[
\Delta U(\omega) = \Delta U_0 + \int_{\omega_0}^\omega \Delta F(\omega') \, d\omega' + \int_{\omega_0}^\omega \Delta A(\omega')U(\omega') \, d\omega' + 2\int_{\omega_0}^\omega \partial_x^2 A(\omega') \partial_x U(\omega') \, d\omega' + \int_{\omega_0}^\omega A(\omega') \Delta U(\omega') \, d\omega'.
\]

Let us take the \(L^2_x\) norm of both sides. The contribution of the first two terms are fine by hypotheses. In order to deal with the third term, we use the trick of using (B.7) and Fubini to rewrite it as

\[
\left\| \int_{\omega_0}^\omega \Delta A(\omega') \left( U_0 + \int_{\omega_0}^{\omega'} A(\omega'')U(\omega'') \, d\omega'' \right) \right\|_{L^2_x} \leq \left\| \int_{\omega_0}^\omega \Delta A(\omega') \, d\omega' \right\|_{L^2_x} \|U_0\|_{L^\infty_x L^2_x} + \int_{\omega_0}^\omega \left( \int_{\omega_0}^{\omega'} \Delta A(\omega') \, d\omega' \right) \|A(\omega'')U(\omega'')\|_{L^2_x} \|A(\omega')\|_{L^\infty_x} \|\Delta U(\omega')\|_{L^2_x} \leq D\|U_0\|_{L^\infty_x} + 2D^2\|\Delta U\|_{L^\infty_x L^2_x} \leq CDE\|U_0\|_{L^\infty_x},
\]

where we used (B.2) in the last inequality. For the last two terms, we first use Hölder and Sobolev to estimate

\[
\|2\int_{\omega_0}^\omega \partial_x^2 A(\omega') \partial_x U(\omega') \, d\omega' + \int_{\omega_0}^\omega A(\omega') \Delta U(\omega') \, d\omega'\|_{L^2_x} \leq C\int_{\omega_0}^\omega (\|\partial_x A(\omega')\|_{L^2_x} + \|A(\omega')\|_{L^\infty_x})\|\partial_x^2 U(\omega')\|_{L^2_x} \, d\omega',
\]

at which point we can use Gronwall to conclude (B.4).

Proof of (2). Taking the \(L^2_x\) norm of both sides of the equation \(\partial_x U = AU + F\), the first estimate (B.5) follows immediately by triangle, Hölder, (B.2) and the hypothesis \(\|A\|_{L^\infty_x L^2_x} \leq D\). To prove the second estimate (B.6), we take the \(L^2_x\) norm of both sides of the differentiated equation \(\partial_x \partial_x U = (\partial_x A)U + A \partial_x U + \partial_x F\). Then we use triangle, Hölder, (B.2), (B.3) and the hypotheses. \(\Box\)

Now we are ready to prove Lemmas 4.6 and 5.3.

Proof of Lemma 4.6. Recalling the definition of \(A_0\), (B.7) follow immediately from Proposition (B.1) with \(\omega = t\), \(\Omega = (-T, T)\), \(\omega_0 = 0\), \(A = A_0\), \(F = 0\) and \(D = A_0(-T, T)\). For the difference analogue (B.4), note that \(\delta V\) satisfies the ODE

\[
\partial_t (\delta V) = (\delta V)A_0 + V\delta A_0,
\]

In order to deal with the boundary term, one may multiply \(U\) by a smooth cut-off \(\chi(x/R)\), and then take \(R \to \infty\). Using the fact that \(\|U(\omega)\|_{L^\infty_x} + \|\partial_x U(\omega)\|_{L^2_x} < \infty\), it can be shown that \(\|\partial_t \partial_x (U(\omega)\chi(x/R))\|_{L^2_x} \to \|\partial_t \partial_x U(\omega)\|_{L^2_x}\).
to which Proposition B.1 can also be applied, with \( F = V \delta A_0 \). The appropriate bounds for \( F \) can be proved easily by using \( \delta A_0(-T, T) \) and the previously established estimates for \( V \). This proves (B.8).

Finally, the corresponding estimates for \( V^{-1} \) and \( \delta V^{-1} \) follows in the same manner once one observes that they satisfy the equations

\[
\partial_t V^{-1} = -A_0 V^{-1}, \quad \partial_t \delta V^{-1} = -A_0 \delta V^{-1} - (\delta A_0) V^{-1},
\]

respectively.

\( \square \)

**Proof of Lemma 5.4** By (5.11) of Proposition 5.2, the coefficient matrix \( A_s = \partial^k A \) satisfies

\[
\|s^{k/2} \partial_x^{(k)} A\|_{L^\infty_T L^2_x} \leq C_{k, \|A\|_{H^1}} \|A\|_{H^1}
\]

for all integers \( k \geq 0 \). By interpolation and Gagliardo-Nirenberg, it then easily follows that \( \|A_s\|_{L^1_T L^\infty_x} + \|\partial_x A_s\|_{L^1_T L^2_x} \) is bounded by the right-hand side of (B.9). On the other hand, by Corollary 5.3, \( \sup_{0 \leq s \leq 1} \| \int_0^s \Delta A_s \|_{L^2_x} \) is also bounded by the right-hand side of (B.9). Applying Proposition B.1 with \( \omega = s \), \( \Omega = [0, 1] \), \( s_0 = s_0 \), \( A = A_s \), \( F = 0 \) and \( U_0 = \text{Id} \), we then obtain (5.10) and (5.11) for \( m = 2 \).

Next, assuming \( s_0 = 1 \), let us prove (5.11) for integers \( m \geq 3 \). We will proceed by induction on \( m \). The estimate for \( \|\partial_x^{(m)} U\|_{L^\infty_T L^2_x} \) that we just proved will serve as the base step.

Fix \( m \geq 3 \). Suppose, for the purpose of induction, that we already have the estimates

\[
\|s^{(m-2)/2} \partial_x^{(m)} U\|_{L^\infty_T L^2_x} \leq C_{k, \|A\|_{H^1}} \|A\|_{H^1}
\]

for \( k = 2, 3, \ldots, (m-1) \). Let us differentiate the ODE \( m \)-times and integrate from \( s \) to \( 1 \). Note that \( \partial_x^{(m)} U(1) = \partial_x^{(m)} \text{Id} = 0 \). Taking the \( L^2_x \) norm and multiplying by \( s^{(m-2)/2} \), we obtain

\[
(s/s')^{(m-2)/2} \partial_x^{(m)} U(s)^{1/2} \leq \left( \int_s^1 (s/s')^{(m-2)/2} \|\partial_x^{(m)} U(s')A_s(s')\|_{L^2_x} \right. \frac{ds'}{s'} + \left. \int_s^1 (s/s')^{1/4} \|\partial_x^{(m-1)} A_s(s')\|_{L^2_x} \right)
\]

We remark that here we have written the line element in the scale-invariant form \((ds'/s')\), as it is more convenient for keeping track of \( s'/s \)-weights.) Using Leibniz’s rule and Hölder, we have

\[
(s/s')^{(m-2)/2} \partial_x^{(m)} U(s)^{1/2} \leq (s/s')^{(m-2)/2} \|U(s')\|_{L^\infty_x} \left( (s/s')^{m/2} \|\partial_x^{(m)} A_s(s')\|_{L^2_x} \right)
\]

\[
+ (s/s')^{(m-2)/2} \|\partial_x U(s')\|_{L^2_x} \left( (s')^{m/2} \|\partial_x^{(m-1)} A_s(s')\|_{L^2_x} \right)
\]

\[
+ (s')^{1/4} \sum_{k=0}^{m-2} \left( (s')^{(m-k-2)/2} \|\partial_x^{(m-k-1)} U(s')\|_{L^2_x} \right) \left( (s')^{k/2+3/4} \|\partial_x^{(k)} A_s(s')\|_{L^\infty_x} \right).
\]

To estimate the first two terms on the right hand side, we use Sobolev and (B.9) for \( A_s \), and (5.10) for \( U \). In order to estimate the last term, we use Gagliardo-Nirenberg and (B.9) to estimate \( A_s \), and the induction hypothesis for \( U \). As a consequence, the right-hand side of (B.9) is estimated by

\[
C_{\|A\|_{H^1}} \left( \int_s^1 (s/s')^{(m-2)/2} \frac{ds'}{s'} \right) + \left( \int_s^1 (s/s')^{1/4} \frac{ds'}{s'} \right) + C \int_s^1 (s/s')^{1/4} \|\partial_x^{(m)} U(s')\|_{L^2_x} \frac{ds'}{s'}.
\]

Note that the first term is uniformly bounded in \( s \), since \( m \geq 3 \). The second term, on the other hand, can be treated via Gronwall’s inequality as before, since \( (s')^{1/4} \frac{ds'}{s'} = (s')^{-3/4} \frac{ds'}{s} \) is integrable on \([0, 1] \). As a result, we obtain \( \|s^{(m-2)/2} \partial_x^{(m)} A\|_{L^2_x} \leq C_{m, \|A\|_{H^1}} \|A\|_{H^1} \) as desired.

Next, as in the proof of Lemma 5.1, the corresponding estimates for \( \delta U \), \( U^{-1} \), \( \delta U^{-1} \) can be proved in the same manner, based on the observation that these variables satisfy

\[
\partial_t (\delta U) = (\delta U) A_s + U \delta A_s, \quad \partial_t U^{-1} = -A_s U^{-1}, \quad \partial_t \delta U^{-1} = -A_s \delta U^{-1} - (\delta A_s) U^{-1},
\]

respectively. We leave the precise details to the reader. \( \square \)
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