ON GROUPS PRESENTED BY INVERSE-CLOSED FINITE
CONVERGENT LENGTH-REDUCING REWRITING SYSTEMS

MURRAY ELDER AND ADAM PIGGOTT

Abstract. We show that groups presented by inverse-closed finite convergent length-reducing rewriting systems are characterised by a striking geometric property: their Cayley graphs are geodetic and side-lengths of non-degenerate triangles are uniformly bounded. This leads to a new algebraic result: the group is plain (isomorphic to the free product of finitely many finite groups and copies of \( \mathbb{Z} \)) if and only if a certain relation on the set of non-trivial finite-order elements of the group is transitive on a bounded set. We use this to prove that deciding if a group presented by an inverse-closed finite convergent length-reducing rewriting system is not plain is in \( \mathsf{NP} \). A “yes” answer would disprove a longstanding conjecture of Madlener and Otto from 1987. We also prove that the isomorphism problem for plain groups presented by inverse-closed finite convergent length-reducing rewriting systems is in \( \mathsf{PSPACE} \).

1. Introduction

A group is plain if it is isomorphic to a free product of finitely many finite groups and finitely many copies of \( \mathbb{Z} \). In the 1980s, the following conjecture was framed in an attempt to understand the algebraic structure of groups presented by finite convergent length-reducing rewriting systems.

Conjecture 1 (Madlener and Otto \[17\]). A group \( G \) admits presentation by a finite convergent length-reducing rewriting system if and only if \( G \) is plain.

Diekert \[9\] showed that the groups presented by finite convergent length-reducing rewriting systems (the fclrrs groups) form a proper subclass of the virtually-free (and hence hyperbolic) groups. Showing that every fclrrs group is plain, or otherwise, has proved difficult. Special cases where the length of the rewriting rules are restricted have been shown, including: if left-hand sides of rules have length at most two, in 1984 by Avenhaus, Madlener and Otto \[1\]; if right-hand sides of rules have length at most one, in 1984 by Avenhaus, Madlener and Otto \[1\]; if right-hand sides of rules have length at most two and the generating set is inverse-closed, in 2020 by the present authors \[11\].

One path to resolving Conjecture 1 requires identifying properties enjoyed by fclrrs groups that distinguish them among the virtually-free groups. In this paper we identify a striking geometric property characterising the inverse-closed fclrrs groups (the ifclrrs groups). A graph is geodetic if between any pair of vertices there is a unique shortest path. A triangle in a graph is said to be non-degenerate if its edges are internally disjoint. We say that a group \( G \) has the \( k \)-bounded non-degenerate triangle property with respect to a generating set \( \Sigma \) if \( k \) is a universal bound on the side-lengths of non-degenerate geodesic triangles in \( \Gamma(G, \Sigma) \), the Cayley graph of \( G \) with respect to \( \Sigma \). For a rewriting system \( (\Sigma, T) \), let \( s_T = \max\{|\ell|_\Sigma \mid (\ell, r) \in T\} \) and \( r_T = \max\{|r|_\Sigma \mid (\ell, r) \in T\} \).

Theorem A (Geometric characterisation). Let \( G \) be a group, let \( \Sigma \) be an inverse-closed finite generating set for \( G \), and let \( k \in \mathbb{N} \). Then \( G \) admits presentation by an inverse-closed finite convergent length-reducing rewriting system \( (\Sigma, T) \) with \( r_T \leq k \) if and only if the Cayley graph \( \Gamma(G, \Sigma) \) is geodetic and has the \( k \)-bounded non-degenerate triangle property.

We define a relation \( \sim \) on the set of non-trivial finite-order elements in \( G \) such that \( a \sim b \) if the product \( ab \) has finite order. It follows easily from the normal form theory of free products that in any plain group the relation \( \sim \) is transitive. In general, this property does not distinguish the plain groups among the virtually-free groups; for example, if \( H \) is any finite group then \( \mathbb{Z} \times H \) is a non-plain virtually-free group in which \( \sim \) is transitive. However, using Bass-Serre Theory, information about centralizers in fclrrs groups, and the geometric constraints imposed by Theorem A we prove that the transitivity of \( \sim \) characterizes...
the plain groups among the fcflrs groups (see Lemma 1). For the icfcflrs groups we can sharpen this to checking the transitivity of ~ on a finite set.

**Theorem B** (Algebraic characterisation). If \( G \) is a group presented by an inverse-closed finite convergent length-reducing rewriting system \((\Sigma, T)\), then the following are equivalent:

1. \( G \) is plain;
2. any nontrivial finite-order element in \( G \) is contained in a unique maximal finite subgroup of \( G \);
3. the relation ~ is transitive on the set of non-trivial finite-order elements in \( G \);
4. the relation ~ is transitive on the set of non-trivial finite-order elements in \( G \) of geodesic length (with respect to \( \Sigma \)) at most \( 11\ell_T \).

The equivalence of conditions (1) and (4) in Theorem B allows us to reduce the problem of checking whether or not the group presented by \((\Sigma, T)\) is plain to checking whether or not a finite number of elements have finite order. This can be done efficiently.

**Theorem C** (Detecting plainness). The following decision problem is in \( \text{NP} \): on input an inverse-closed finite convergent length-reducing rewriting system \((\Sigma, T)\), is the group presented by \((\Sigma, T)\) not plain?

A further application of Theorem B concerns the complexity of the isomorphism problem within the class of virtually-free groups. Krstić [16] proved that the isomorphism problem for virtually-free groups is decidable. This was later generalised to all hyperbolic groups by Dahmani and Guirardel [8]. There can be no complexity bound for the isomorphism problem when the inputs are given as arbitrary presentations, since deciding if an arbitrary presentation presents the trivial group is undecidable. Recent work of Sénizergues and Weiß’s [23] shows that the isomorphism problem in virtually-free groups is decidable in doubly exponential space if the input is a context-free grammar for the word problem, and in \( \text{PSPACE} \) if the input is given in the form of virtually-free presentations. A virtually-free presentation of a group \( G \) specifies a free group \( F \) plus a set of representatives \( S \) for \( F \setminus G \) together with relations describing pairwise multiplications of elements from \( F \) and \( S \). Using the results in this paper we are able to prove the same complexity when the input is an inverse-closed finite convergent length-reducing rewriting system presenting a plain group.

**Theorem D** (Isomorphism of plain icfcflrs groups). The isomorphism problem for plain groups presented by inverse-closed finite convergent length-reducing rewriting systems is decidable in \( \text{PSPACE} \).

A contributing factor in the difficulty of Conjecture 1 is a paucity of examples of interesting finite convergent length-reducing rewriting systems that present groups. In 1997, Shapiro asked whether or not the plain groups may be characterized as exactly the groups that admit locally-finite geodetic Cayley graphs [24, p.286]. Theorems B and C are new tools for considering Shapiro’s question.

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2. Preliminaries

If \( \Sigma \) is an alphabet (a non-empty finite set), we write \( \Sigma^* \) for the set of finite words over the alphabet \( \Sigma \), and \(|u|_\Sigma\) for the length of the word \( u \in \Sigma^* \); the empty word, \( \lambda \), is the unique word of length 0. If \( G \) is a group with generating set \( \Sigma \), we write \(|g|_{\Sigma,G}\) for the length of a shortest word in \( (\Sigma \cup \Sigma^{-1})^* \) which evaluates to \( g \). We write: \( u = v \) if \( u, v \in \Sigma^* \) are identical as words; \( u =_G v \) if \( u, v \in \Sigma^* \) and \( u, v \) evaluate to the same element of \( G \); and \( u =_G g \) if \( u, g \in \Sigma^* \) and \( u, g \) evaluates to \( g \). We write \( e_G \) for the identity element in \( G \), and \( B_{\leq r}(G) \) for the subset of \( G \) comprising group elements that may be spelled by a word in \( (\Sigma \cup \Sigma^{-1})^* \) of length not exceeding \( r \).

A length-reducing rewriting system is a pair \((\Sigma, T)\) where \( \Sigma \) is a non-empty alphabet, and \( T \) is a subset of \( \Sigma^* \times \Sigma^* \), called a set of rewriting rules, such that for all \((\ell, r) \in T\) we have that \(|\ell|_{\Sigma} < |r|_{\Sigma}\). We write \( \ell_T = \max\{|\ell|_{\Sigma} \mid (\ell, r) \in T\} \), and \( r_T = \max\{|r|_{\Sigma} \mid (\ell, r) \in T\} \).

The set of rewriting rules determines a relation \( \rightarrow \) on the set \( \Sigma^* \) as follows: \( a \rightarrow b \) if \( a = u\ell v, b = uv \) and \((\ell, r) \in T \). The reflexive and transitive closure of \( \rightarrow \) is denoted \( \twoheadrightarrow \). A word \( u \in \Sigma^* \) is irreducible if no factor is the left-hand side of any rewriting rule, and hence \( u \twoheadrightarrow v \) implies that \( u = v \).

The reflexive, symmetric and closure of \( \rightarrow \) is an equivalence denoted \( \overset{\sim}{\rightarrow} \). The operation of concatenation of representatives is well defined on the set of \( \overset{\sim}{\rightarrow} \)-classes, and hence makes a monoid.
$M = M(\Sigma, T)$. We say that $M$ is the 
\textit{monoid presented by} $(\Sigma, T)$. When the equivalence class of every letter (and hence also the equivalence class of every word) has an inverse, the monoid $M$ is a group and we say it is \textit{the group presented by} $(\Sigma, T)$. We say that $(\Sigma, T)$ (or just $\Sigma$) is \textit{inverse-closed} if for every $a \in \Sigma$, there exists $b \in \Sigma$ such that $ab \rightarrow \lambda$. Clearly, $M$ is a group when $\Sigma$ is inverse-closed.

A rewriting system $(\Sigma, T)$ is \textit{finite} if $\Sigma$ and $T$ are finite sets, and \textit{terminating} (or \textit{noetherian}) if there are no infinite sequences of allowable factor replacements. It is clear that length-reducing rewriting systems are terminating. A rewriting system is called \textit{confluent} if whenever $w \rightarrow x$ and $w \rightarrow y$, there exists $z \in \Sigma^*$ such that $x$ and $y$ both reduce to $z$. A rewriting system is called \textit{convergent} if it is terminating and confluent. In some literature, finite convergent length-reducing rewriting systems are called finite Church-Rosser Thue systems. Since a finite length-reducing rewriting system is necessarily terminating, the well-known Newman’s Lemma [3, p.69] gives that checking a finite list of words (corresponding to ‘critical-pairs’) is enough to determine whether or not the rewriting system is convergent. This can be completed in time that is polynomial in the size of the rewriting system.

If $(\Sigma, T)$ is a finite convergent length-reducing rewriting system, then any element of $M(\Sigma, T)$ is represented by a unique irreducible word $w \in \Sigma^*$, and the word $w$ is the unique \textit{geodesic} (shortest word) among all representatives of the element.

We say that $(\Sigma, T)$ is \textit{normalized} if for any rule $(\ell, r) \in T$ we have that $r$ is irreducible, every proper subword of $\ell$ is irreducible, and $(\ell, r), (\ell, r') \in T$ implies $r = r'$. We say that $(\Sigma, T)$ has \textit{irreducible letters} if each letter in $\Sigma$ is irreducible. We note that if $(\Sigma, T)$ is an inverse-closed finite convergent length-reducing rewriting system that is not normalized or contains reducible letters, then there exist subsets $\Sigma' \subseteq \Sigma$ and $T' \subseteq T$ such that $(\Sigma', T')$ is a normalized inverse-closed finite convergent length-reducing rewriting system with $\Sigma'$ containing only irreducible letters that presents the same group as $(\Sigma, T)$. Moreover it is easy to compute such $\Sigma'$ and $T'$. Therefore, without loss of generality, we may assume that any inverse-closed finite convergent length-reducing rewriting system we consider is normalized with irreducible letters. In such a rewriting system, every rewriting rule is either: $(ab, \lambda)$ for some $a, b \in \Sigma$; or $(u, v)$ for some $u, v \in \Sigma^*$ with $|u|_\Sigma = 1 + |v|_\Sigma \geq 2$. In particular, either every rule has the form $(ab, \lambda)$, in which case the group presented is a free product of cyclic groups [4], or $\ell_T = rT + 1$.

We define the \textit{size} of a rewriting system $(\Sigma, T)$ to be

$$n_T = |\Sigma| + \sum_{(\ell, r) \in T} |\ell r|.$$

Note that $r_T, \ell_T \in \partial(n_T)$.

\section{Geometry of groups presented by rewriting systems}

For a group $G$ and a finite generating set $\Sigma$ (we shall always assume that $\Sigma$ does not contain the identity element $e_G$), the \textit{undirected Cayley graph of} $G$ \textit{with respect to} $\Sigma$ is the locally-finite simple undirected graph $\Gamma = \Gamma(G, \Sigma)$ with vertex set $G$ and in which distinct vertices $g, h \in G$ are adjacent if and only if $g^{-1}h \in \Sigma \cup \Sigma^{-1}$. Each path $\nu_0, \nu_1, \ldots, \nu_n$ in $\Gamma$ is labeled by a word $a_1 \ldots a_n \in (\Sigma \cup \Sigma^{-1})^*$ where $a_i = g_{\nu_i^{-1} \nu_{i-1}}$. A geodesic path in $\Gamma$ from the identity element $e_G$ to $g$ is labelled by a geodesic word $u \in (\Sigma \cup \Sigma^{-1})^*$ with $|u|_\Sigma = |g|_{G, \Sigma}$. A simple undirected connected graph is \textit{geodesic} if any pair of vertices is joined by a unique geodesic path. If $\Gamma(G, \Sigma)$ is geodesic and $g \in G$, we will denote the unique geodesic word evaluating to $g$ by $g_{\gamma}$.

\begin{definition}[Non-degenerate geodesic triangle] Let $\Delta$ be a simple undirected graph. A geodesic triangle in $\Delta$ is the union of three geodesic paths $\alpha = a_0, a_1, \ldots, a_p$, $\beta = b_0, b_1, \ldots, b_q$, $\gamma = c_0, c_1, \ldots, c_r$ such that $a_p = b_0 = c_0$ and $c_r = a_0$. See Figure 12. We denote the geodesic triangle by $(\alpha, \beta, \gamma)$. The geodesic triangle is \textit{non-degenerate} if the vertices $a_i, b_j, c_k$ are all pairwise distinct for $1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r$. Otherwise we say it is \textit{degenerate}.
\end{definition}

Note that if $\Delta$ is geodesic, then $\Delta$ is degenerate when there exist $i_1, i_2, i_3, j_1, j_2, j_3 \in \mathbb{N}$ with $(i_1, i_2, i_3) \neq (0, 0, 0)$ and such that $a_0 = c_r, \ldots, a_{i_1} = c_j, a_{i_2} = c_i, a_{i_3} = c_j, b_0 = b_j, b_2 = c_i, \ldots, b_q = c_0$ as illustrated in Figure 11.

\begin{definition}[Bounded non-degenerate triangle property] Let $\Gamma$ be an undirected graph and $k \in \mathbb{N}$. We say $\Gamma$ has the $k$-\textit{bounded non-degenerate triangle property} ($k$-\textit{bdtp}) if no non-degenerate geodesic triangle in $\Gamma$ has a side-length exceeding $k$.
\end{definition}

We make use of the following notion from [11] (the terminology takes its inspiration from [3]).
Definition 4 \((s\text{-broomlike})\). Let \(\Delta\) be a geodetic graph and \(s\) a positive integer. We say that \(\Delta\) is \(s\)-broomlike if whenever \(a_0, \ldots, a_n, b\) is a path comprising distinct vertices such that \(a_0, \ldots, a_n\) is a geodesic but \(a_0, \ldots, a_n, b\) is not, then the geodesic from \(a_0\) to \(b\) is
\[
a_0, \ldots, a_{n-p}, b_{n-p+1}, \ldots, b_n = b
\]
for \(p \leq s\) and \(b_{n-p+1} \neq a_{n-p+1}\).

![Figure 1](https://example.com/figure1.png)  
(A) A non-degenerate geodesic triangle.  
(B) A degenerate geodesic triangle in a geodetic graph.

Lemma 5 (\([\text{I1}]\) Lemmas 5 and 9). If \(G\) is presented by an inverse-closed finite convergent length-reducing rewriting system \((\Sigma, T)\), then the undirected Cayley graph of \(G\) with respect to \(\Sigma\) is geodetic and \(r_T\)-broomlike.

We now prove our first main result.

Theorem A (Geometric characterisation). Let \(G\) be a group, let \(\Sigma\) be an inverse-closed finite generating set for \(G\), and let \(k \in \mathbb{N}\). Then \(G\) admits presentation by an inverse-closed finite convergent length-reducing rewriting system \((\Sigma, T)\) with \(r_T \leq k\) if and only if the Cayley graph \(\Gamma(G, \Sigma)\) is geodetic and has the \(k\)-bounded non-degenerate triangle property.

Proof. Let \(G\) be presented by an icfclrrs \((\Sigma, T)\). By Lemma\(\text{[\text{I1}]}\) \(\Gamma = \Gamma(\Sigma, T)\) is geodetic and \(r_T\)-broomlike. For any path \(\epsilon \in \Gamma\), we write \(|\epsilon|\) for the length of \(\epsilon\). Suppose that there exists a non-degenerate geodesic triangle in \(\Gamma\) with a side-length exceeding \(r_T\). Let \(f\) be the minimal positive integer such that there exists a non-degenerate triangle \((a_0, b_0, \gamma_0)\) in \(\Gamma\) such that \(|a_0| > r_T\) and \(|a_0| + |b_0| + |\gamma_0| = f\). Let \(\mathcal{A}\) denote the set of all non-degenerate geodesic triangles \((\alpha, \beta, \gamma)\) in \(\Gamma\) such that \(|\alpha| > r_T\) and \(|\alpha| + |\beta| + |\gamma| = f\). Let \(\mathcal{B}\) denote the set of all triangles in \(\mathcal{A}\) for which \(|\beta|\) is maximal among all triangles in \(\mathcal{A}\). Since \(f\) is a fixed integer, the set \(\mathcal{B}\) is well defined. Let \(\mathcal{C}\) denote the set of all triangles in \(\mathcal{B}\) for which \(|\beta|\) is maximal among all triangles in \(\mathcal{B}\). Again, since \(f\) is fixed, \(\mathcal{C}\) is well defined.

Let \((\alpha, \beta, \gamma) \in \mathcal{C}\). Let \(\alpha\) be labelled by the word \(x_1 x_2 \ldots x_p\), and \(\beta\) be labelled by the word \(y_1 y_2 \ldots y_q\), where \(x_i, y_i \in \Sigma\). Without loss of generality we may suppose that the sides are oriented such that
\[
x_1 x_2 \ldots x_p y_1 y_2 \ldots y_q = \gamma e_G.
\]

First we note that \(|\beta| > 1\). If \(|\beta| = 1\), then by the \(r_T\)-broomlike property \(|\alpha| \leq r_T\), which is a contradiction.
The maximality of $|a|$ in $A$, gives that $x_1 \ldots x_py_1$ is not reduced—otherwise $(\alpha y_1,y_2 \ldots y_p, \gamma)$ would be a triangle in $A$ with a longer side. Since $p > \rho_T$, the $\rho_T$-broomlike property gives that there exists $i \geq 1$ and letters $d_{i+1}, \ldots d_p \in \Sigma$ such that $x_1 \ldots x_id_{i+1} \ldots d_p =_G x_1x_2 \ldots x_py_1$ and $x_1 \ldots x_id_{i+1} \ldots d_p$ is a geodesic. Let $\epsilon$ be the path from $c_{\Sigma}$ labelled by $x_1 \ldots x_id_{i+1} \ldots d_p$. Since $\gamma^{-1}$ does not start with $x_1$ and $\epsilon$ does, and the paths are geodesics with the same initial point in a geodetic graph, we have that $\gamma$ and $\epsilon$ are internally disjoint. Let $\beta'$ be the geodesic spelled by $y_2 \ldots y_q$. See Figure 3.

Figure 3. Applying the $\rho_T$-broomlike property to $a\gamma_1$ in the proof of Theorem A.

It follows that $(\epsilon, \beta', \gamma)$ is a non-degenerate geodesic triangle with at least one side-length (that of $\epsilon$) exceeding $\rho_T$ and with $|\epsilon|+|\beta'|+|\gamma|= f-1$. This contradicts the minimality of $f$. Thus every non-degenerate geodesic triangle in $\Gamma$ has side-lengths at most $\rho_T$.

Conversely, suppose that $\Gamma$ has the $k$-bounded non-degenerate triangle property. Define a set of rewriting rules $T$ as follows:

$$T = \{(ab, \lambda) \in \Sigma^* \times \Sigma^* \mid a, b \in \Sigma \text{ such that } ab =_G c_G\} \cup$$

$$\{(a_1a_2 \ldots a_n, b_1b_2 \ldots b_{n-1}) \in \Sigma^* \times \Sigma^* \mid (a_1a_2 \ldots a_{n-1}, a_n, b_{n-1}^{-1}b_{n-2}^{-1} \ldots b_1^{-1})$$

labels a non-degenerate geodesic triangle in $\Gamma(G, \Sigma)$.

By construction, $(\Sigma, T)$ is a finite, inverse-closed, length-reducing rewriting system with $\rho_T \leq k$. Since $\Gamma(G, \Sigma)$ is geodetic, to show that $(\Sigma, T)$ is convergent, it suffices to show that any word labelling a non-geodesic path in $\Gamma(G, \Sigma)$ contains a factor that spells the left-hand side of a rule in $T$. Let $c = c_1c_2 \ldots c_n \in \Sigma^*$ be a word labelling a non-geodesic path in $\Gamma(G, \Sigma)$. Consider first that case that $c$ is not freely reduced. Then there exists an integer $i$ such that $1 \leq i < m$ and $c_i = c_i^{-1}$; clearly $(c_{i+1}, \lambda) \in T$. Now consider the case that $c$ is freely reduced. Let $j$ be the minimal integer such that $1 \leq j < m$ and $c_j = c_j^{-1}$; clearly $(c_1 \ldots c_{j-1}) \in T$. The maximality of $i$ implies that there exists a word $d_1 \ldots d_{j-1} \in \Sigma^*$ with $d_i \neq c_i$ and such that $c_i \ldots c_j =_G d_i \ldots d_{j-1}$. It follows that $((c_1 \ldots c_{j-1}), c_j, (d_{j-1}^{-1} \ldots d_1^{-1}))$ labels a non-degenerate geodesic triangle in $\Gamma(G, \Sigma)$. By hypothesis, $j - i < k$. Thus $(c_1 \ldots c_j, d_1 \ldots d_{j-1}) \in T$.

Finally, we establish that $(\Sigma, T)$ presents $G$. Since $\Sigma$ is inverse-closed, $(\Sigma, T)$ presents a group $\hat{G}$. By construction, $\ell =_G r$ for every rule $(\ell, r) \in T$; it follows that $G$ is a quotient of $\hat{G}$. Any equation that holds in $G$ also holds in $\hat{G}$, because any two words that spell the same element in $G$ will reduce by application of rewriting rules to the unique geodesic representing the group element; it follows that $G \cong \hat{G}$.

4. CENTRALIZERS OF INFINITE ORDER ELEMENTS

For a group $G$ and an element $g \in G$, we write $C_G(g)$ for the centralizer of $g$ in $G$; that is, $C_G(g) = \{t \in G \mid tgt^{-1} = g\}$. In 1988, Madlener and Otto identified the following fact about the centralizers of infinite order elements in fclrrs groups.

**Theorem 6** (Madlener and Otto [18, Corollary 2.4]). Let $G$ be a group presented by a finite convergent length-reducing rewriting system $(\Sigma, T)$. If $g \in G$ is an element of infinite order, then $C_G(g) \cong \mathbb{Z}$.

Using Theorem A we can gain further information about the centralizers in $G$. First we observe the following. For $g, h \in G$, let $\mathfrak{C}_{g,h} = \{t \in G \mid tgt^{-1} = h\}$.

**Lemma 7.** Let $G$ be a group presented by a finite convergent length-reducing rewriting system and let $g, h$ be non-trivial elements that are conjugate in $G$. The following are equivalent:

1. $g$ has infinite order;
2. $C_G(g) \cong \mathbb{Z}$;
(3) $C_G(g)$ has infinite order;
(4) $C_{g,h}$ has infinite order.

**Proof.** That (1) implies (2) is given by Theorem 5. That (2) implies (3) is immediate.

Next we prove that (3) implies (1). We prove the contrapositive. Suppose that $g$ has finite order. Let $j \in C_G(g) \setminus \{e_G\}$. Then $g \in C_G(j)$. If $j$ has infinite order, by Theorem 3 we have that $g$ has infinite order; this contradiction shows that $j$ has finite order and $C_G(g)$ is a torsion subgroup of $G$. In hyperbolic groups, torsion subgroups have finite order [12, Corollaire 36, Chapitre 8]. Hence $C_G(g)$ has finite order.

The equivalence of (3) and (4) is an elementary exercise in group theory as follows. Since $g$ and $h$ are conjugate, $C_{g,h}$ is non-empty; fix $j \in C_{g,h}$. For each $t \in G$ we have $t \in C_{g,h}$ if and only if $tgt^{-1} = h = g^2 g^{-1}$ if and only if $(g^2 t) t^{-1} = g$ if and only if $j^{-1} j \in C_G(g)$. The map $t \mapsto j^{-1} t$ is a bijection from $C_{g,h}$ to $C_G(g)$, and the equivalence of (3) and (4) follows. 

**Lemma 8.** Let $G$ be a group presented by an inverse-closed finite convergent length-reducing rewriting system $(\Sigma, T)$. If $g, h \in B_{\Sigma G}(k) \setminus \{e_G\}$ are conjugate elements of finite order and $t \in G$ is such that $tgt^{-1} = h$, then $|t|_{G, \Sigma} \leq 3r_T + 2k$.

**Proof.** Suppose that $g, h \in B_{\Sigma G}(k) \setminus \{e_G\}$ are conjugate elements of finite order and there exists $t \in G$ such that $tgt^{-1} = h$ and $|t|_{G, \Sigma} \geq 3r_T + 2k + 1$. Let $\gamma_t = w, \gamma_g = u, \gamma_h = v$ be the geodesic words for $t, g, h$ respectively, as shown in Figure 4.

![Figure 4](image-url)

If $x, y$ are labels of vertices in the geodetic graph $\Gamma(G, \Sigma)$, we let $[x, y]$ denote the unique geodesic path starting at $x$ and ending at $y$.

Let $\gamma = [A, D] = \gamma_{t^{-1} h}$ be the geodesic from the vertex marked $A$ to the vertex marked $D$ in Figure 4. Consider travelling along the two geodesics $\gamma, w^{-1}$ starting at $A$. Let $p$ be the last point visited that lies on both paths. Consider travelling on the two geodesics $w, v$ starting at $C$, and let $p'$ be the last point visited that lies on both paths. Note that $[p', p]$ is a subpath of $w$ since $p$ was closer to $C$ than $p'$, the geodesic $[p, D]$ includes $p'$ which means it would be shorter to travel from $A$ to $D$ via $[p, D]$, contradicting that $p$ lies on the geodesics from $A$ to $D$.

In this paragraph we prove that $d(p, C) \leq r_T + k$. We consider cases. Consider first the case that $p = p'$. Then $d(p, C) \leq k$, since $[p', C]$ is a subpath of $v$ which has length at most $k$. Next consider the case that $p \neq p'$. The path $[p, p']$ is a side of a non-degenerate geodesic triangle, so has length at most $r_T$ by Theorem 4. This means $d(p, C) \leq r_T + k$, since $[p', C]$ is a subpath of $v$ which has length at most $k$.

Now let $\rho = [C, B] = \gamma_u$ be the geodesic from the vertex marked $C$ to the vertex marked $B$ in Figure 4 and let $q$ be the last point visited as you travel along the two geodesics $\rho, w$ starting at $C$. Repeating the above argument we have $d(q, A) \leq r_T + k$.

Since $|w|_{\Sigma} > 2(r_T + k)$ we have that $[p, q]$ is a subpath of $w$. Let $\rho_1 = [q, B], \gamma_1 = [D, p], w_1 = [C, p], w_2 = [p, q], \text{ and } w_3 = [q, A]$. Note that $\gamma_1 w_2 \rho_1$ is a geodesic, for if not, it contains a factor of length at most $r_T + 1$ which is the left-hand side of a rewrite rule. This factor cannot lie completely inside $\rho$ nor $\gamma$, so does not include the points $p, q$, and this is impossible. Thus

$$w = \gamma_1 w_2 \rho_1 = w_1 w_2 w_3$$

See Figure 5.

Observe that $|\rho_1|_{\Sigma}, |\gamma_1|_{\Sigma} \leq r_T + k$ since they comprise (at most) one side of non-degenerate triangle, plus a factor of $u$ or $v$, and $|w|_{\Sigma}, |v|_{\Sigma} \leq k$. 


If $|\rho_1|_{\Sigma} = |w_3|_{\Sigma}$, then $\rho_1$ and $w_3$ are paths with the same label that share an initial point and so $A = B$; since we assume that $g \neq e_G$, this is impossible. Therefore one of $\rho_1, w_3$ is shorter. Without loss of generality, assume $|w_3|_{\Sigma} < |\rho_1|_{\Sigma}$. Then $\rho_1 = xw_3$, where $1 \leq |x|_{\Sigma} \leq k + r_T - 1$.

Let $y \in \Sigma^*$ be the geodesic word such that $w_1 = \gamma_1 y$ (note that $|x|_{\Sigma} = |y|_{\Sigma}$). Then we have $w = \gamma_1 y w_2 w_3 = \gamma_1 w_2 x w_3$; this implies that $y w_2 = w_2 x$. From this, an exercise in the combinatorics of words gives that there exist words $r, s \in \Sigma^*$ such that $y = rs$, $x = sr$ and $w_2 = (rs)^n r$ for some $n \in \mathbb{N}$. Thus we have that $w = \gamma_1 (rs)^n r w_3$.

Let $m$ be an integer such that $m > n$. We now show that $\gamma_1 (rs)^m r w_3$ is a geodesic word. Since $|w|_{\Sigma} \geq 3 r_T + 2k + 1$ and $|\gamma_1|_{\Sigma} |w_3|_{\Sigma} \leq r_T + k$, we have that $|(rs)^{n+1} r|_{\Sigma} \geq r_T + 1$. It follows that every length-$(r_T + 1)$ factor of $\gamma_1 (rs)^n r w_3$ is also factor of $\gamma_1 (rs)^{n+1} r w_3$. Since every length-$(r_T + 1)$ factor of $\gamma_1 (rs)^{n+1} r w_3$ is irreducible, every length every length-$(r_T + 1)$ factor of $\gamma_1 (rs)^m r w_3$ is irreducible. Hence $\gamma_1 (rs)^m r w_3$ is a geodesic word.

Since $w_3 u \rho_1^{-1}$ labels a closed path at $q$, we have that $w_3 u w_3^{-1} r^{-1} s^{-1} = G e_G$. It follows that $(\gamma_1 (rs)^m r w_3) u (\gamma_1 (rs)^m r w_3)^{-1} = v$

for all integers $m > n$. This implies that $G_{g, h}$ is infinite, which by Lemma 7 means $g$ does not have finite order, contradicting our hypothesis. \hfill $\Box$

Putting Lemmas 7 and 8 together we have

**Proposition 9 (Centralizers).** Let $G$ be a group presented by an inverse-closed finite convergent length-reducing rewriting system $(\Sigma, T)$. Let $g, h \in B_{G_{\Sigma}} (k) \setminus \{ e_G \}$ be conjugate elements in $G$. The following are equivalent:

1. there exists $t \in G$ such that $t g t^{-1} = h$ and $|t|_{G, \Sigma} > 3 r_T + 2k$;
2. $g$ has infinite order;
3. $G_{G, g}$ is infinite;
4. $G_{G, g}$ has infinite order.

**Remark 10.** It is known that in a $\delta$-hyperbolic group $G$, if two finite-order elements $g, h \in G$ are conjugate, then there exists an element $x$ conjugating $g$ to $h$ such that the length of $x$ can be bounded in terms of $\delta, |\Sigma|, |g|_{G, \Sigma}$ and $|h|_{G, \Sigma}$ [6, Theorem 3.3] (this result extends to elements of infinite order, and to lists of elements). The result above bounds the length of all elements conjugating $g$ to $h$ in the special case that $G$ is a icfclrrs group.

5. **Proving Theorem 11**

In this section we use the geometric insights of Theorem A and the technical information provided by Lemma 8 to prove Theorem B.

Recall that $\sim$ is a relation on the set of non-trivial finite-order elements in $G$ defined by the rule $a \sim b$ if $a b$ has finite order.

**Lemma 11.** If $G$ is a group presented by a finite convergent length-reducing rewriting system $(\Sigma, T)$, then the following are equivalent:

1. $G$ is plain;
2. any nontrivial finite-order element in $G$ is contained in a unique maximal finite subgroup of $G$;
3. the relation $\sim$ is transitive on the set of non-trivial finite-order elements in $G$.

**Proof.** Suppose that $G$ is a group presented by a finite convergent length-reducing rewriting system $(\Sigma, T)$. Since $G$ is virtually-free [4], it is isomorphic to the fundamental group of a finite graph of groups $\mathcal{F}$ with finite vertex groups (and hence also finite edge groups) [13]. Then $\mathcal{F}$ is a connected graph (multiple edges and loops are allowed) in which each vertex is labelled by a group, and each edge is labelled by a group.
and equipped with two homomorphisms from its label to the vertex group(s) to which it is incident. Let $V_1, \ldots, V_p$ be the finite groups labelling vertices in $\mathcal{G}$, and let $E_1, \ldots, E_q$ be the finite groups labelling edges in $\mathcal{G}$. Since $\mathcal{G}$ is connected, $q \geq p - 1$. Without loss of generality, we may assume that $V_i$ is adjacent to $V_{i+1}$ and that the edge incident to $V_i$ and $V_{i+1}$ is labelled $E_i$ for $1 \leq i < p$ (so the edges $E_1, \ldots, E_{p-1}$ form a spanning tree in $\mathcal{G}$). The fundamental group $G_q$ of $\mathcal{G}$ may then be constructed inductively by a sequence of $p - 1$ free products with amalgamation followed by a sequence of $q - p + 1$ HNN extensions, as follows:

- let $G_1 = V_1$;
- for $i = 1, 2, \ldots, p - 1$, let $G_{i+1} = G_i \ast_{E_i} V_{i+1}$ (two copies of $E_i$, one in $V_i$ and one in $V_{i+1}$, are identified);
- for $i = p, p + 1, \ldots, q$, let $G_{i+1} = G_i \ast_{\phi}$, where $\phi$ is an epimorphism mapping one copy of $E_i$ to another (an infinite-order stable letter $t_k$ conjugates one copy of $E_i$ to another according to the epimorphism $\phi$; the two copies live in the same vertex group if the edge is a loop, or distinct vertex groups otherwise).

We note two consequences of the normal form theory of free products with amalgamation and HNN extensions:

- $G_{i+1}$ contains an isomorphic copy of $G_i$ for each $i = 1, 2, \ldots, q - 1$.
- each maximal finite subgroup in $G$ is isomorphic to one of the vertex groups.

Since any non-loop edge may be chosen to be in the spanning tree, any non-loop edge with label $E_k$ that is incident to vertices with labels $V_i$ and $V_j$ corresponds to a subgroup of $G$ that is isomorphic to $V_i \ast_{E_k} V_j$, a free product of finite groups amalgamated over a finite subgroup. Every loop with label $E_k$ and incident to a vertex with label $V_i$ corresponds to a subgroup of $G$ that is isomorphic to $V_i \ast_{E_k}$, an HNN extension of a finite group in which an infinite-order ‘stable letter’ $t_k$ conjugates one embedded copy of $E_k$ to another. It follows from the construction of the fundamental group of $\mathcal{G}$ that $G$ is plain if and only if all edges are labelled by the trivial group. It is therefore useful to limit the relative sizes of groups labelling edges.

Next we show that, without loss of generality, we may assume that any non-trivial group labelling an edge in $\mathcal{G}$ is strictly smaller than any group labelling an incident vertex. Different arguments are needed for non-loop edges and loops. The arguments given below were first used in [10], but are included here for completeness.

In this paragraph we show that, without loss of generality, we may assume that any non-trivial group labelling a non-loop edge in $\mathcal{G}$ is strictly smaller than any group labelling an incident vertex. Suppose that $\mathcal{G}$ contains a non-loop edge with label $E_k$ that is incident to vertices with labels $V_i$ and $V_j$ and $1 < |E_k| = |V_i| \leq |V_j|$. The edge corresponds to a subgroup of $G$ that is isomorphic to $V_i \ast_{E_k} V_j$, but $V_i \ast_{E_k} V_j = V_i$. Simply contracting the edge gives a new graph of groups $\mathcal{G}'$ with one less vertex and a fundamental graph that is isomorphic to $G$. We may therefore, without loss of generality, assume that any group labelling a non-loop edge is strictly smaller than the groups labelling the vertices to which the edge is incident.

Next we show that, because $G$ is presented by a fcrwgs group, any non-trivial group labelling a loop in $\mathcal{G}$ must be strictly smaller than the group labelling the incident vertex. Suppose that $\mathcal{G}$ contains a loop labelled $E_k$ that is incident to a vertex with label $V_i$ such that $1 < |E_k| = |V_i|$. Then $G$ contains a subgroup isomorphic to the HNN extension $V_i \ast_{\phi}$, where $\phi : V_i \to V_i$ is an automorphism. Since $V_i$ is finite, $\phi$ has finite order; say $\phi$ has order $f$. Let $g$ be a non-identity element in $V_i$. Then $t^f$ is an infinite order element that commutes with $g$, contradicting Theorem [9].

We now know that each edge in the graph of groups $\mathcal{G}$ may be classified as being of one of three types:

(E1) An edge with label $E_k$ such that $|E_k| = 1$

(E2) A non-loop edge with label $E_k$ and incident to vertices with labels $V_i$ and $V_j$ such that $1 < |E_k| < |V_i| \leq |V_j|$.

(E3) A loop with label $E_k$ that is incident to a vertex with label $V_i$ such that $1 < |E_k| < |V_i|$.

We are now ready to prove the equivalence of conditions (1) and (2). If $G$ is plain, it follows from the normal form theory of free products that condition (2) holds. Suppose that $G$ is not plain. Then $\mathcal{G}$ must contain an edge of type (E2) or an edge of type (E3). Consider first the case that $\mathcal{G}$ contains an edge of type (E2). Then $G$ contains a subgroup isomorphic to $A \ast B$, where $A$ and $B$ are maximal finite subgroups of $G$ and $1 < |A| \leq |B|$. Then $A$ and $B$ are distinct maximal finite subgroups in $G$ with a non-trivial intersection, and condition (2) fails. Now consider the case that $\mathcal{G}$ contains an edge of type (E3). Then $G$ contains a subgroup isomorphic to $A \ast \phi$, where $A$ is a maximal finite subgroup of $G$, $C$ is a proper subgroup of $A$, $\phi$ is an epimorphism $\phi : C \to A$, and $t \in G$ is an infinite-order element.
It follows that\(\text{t} \in (1)\) trivial elements contained in representatives of different conjugacy classes of maximal finite subgroups non-trivial intersection, and condition (2) fails.

The equivalence of conditions (2) and (3) is immediate. □

**Lemma 12.** If \(G\) is a plain group presented by a finite convergent length-reducing rewriting system \((\Sigma, T)\), then the number of conjugacy classes of maximal finite subgroups in \(G\) is bounded above by \(n_2^2\).

**Proof.** Suppose that \(G\) is a plain group presented by a finite convergent length-reducing rewriting system \((\Sigma, T)\). If \(G\) is torsion free, then the trivial subgroup is the only maximal finite subgroup in \(G\) and the result is clear. Assume that \(G\) is not torsion free. It follows immediately from Lemma 10(2) that non-trivial elements contained in representatives of different conjugacy classes of maximal finite subgroups cannot be conjugate. Hence the number of conjugacy class of non-trivial maximal finite subgroups in \(G\) is bounded by the number of conjugacy classes of non-trivial finite-order elements. Madlener and Otto 19 proved that every non-trivial finite-order element in \(G\) is conjugate to a proper prefix of some left-hand side of a rewriting rule in \(T\). It follows that the number of conjugacy classes of non-trivial finite-order elements in \(G\) is bounded by \(n_2^2\). □

To prove Theorem 14 it suffices to show that if the relation \(\sim\) is not transitive on the set of non-trivial finite-order elements in \(G\), then a triple of group elements witnessing the failure of transitivity can be found in \(B_\infty(11r_T)\). For this we need to develop our understanding of the geodesic structure of finite subgroups in \(G\).

**Definition 13** (Long and short elements). We say that \(g\) is long when \(|g|_{G, \Sigma} > r_T\), and short otherwise.

**Lemma 14.** Let \(G\) be presented by an inverse-closed finite convergent length-reducing rewriting system \((\Sigma, T)\), and let \(H\) be a finite subgroup of \(G\). If \(H\) contains a long element, then there exists a letter \(a \in \Sigma\) and geodesic words \(h_1, \ldots, h_t \in \Sigma^*\) so that \(\{a^{-1}h_1a, \ldots, a^{-1}h_ta\}\) is exactly the set of geodesics representing long elements in \(H\), and at least half of all elements in \(H\) are represented by geodesics of the form \(av\) with \(v \in \Sigma^*\).

**Proof.** For each group element \(g \in G\), we write \(\gamma_g\) for the unique reduced word in \(\Sigma^*\) such that \(|\gamma_g|_T = |g|_{G, \Sigma}\). Suppose that there exists a long element \(h_0 \in H\). Then \(\gamma_{h_0} = ab^{-1}\) for \(a, b \in H\) and some geodesic word \(u \in \Sigma^*\). Let \(A = \{s \in H \mid \gamma_s\) starts with \(a\}\), \(B = \{s \in H \mid \gamma_s\) starts with \(b\}\), \(m_A = |A|\) and \(m_B = |B|\).

For each \(h \in H \setminus A\), consider the geodesic for \(h_0^{-1}h\). If \(\gamma_{h_0^{-1}h}\) does not start with \(b\), then we have \(\gamma_s = v_1v_3, \gamma_{h_0^{-1}h} = v_2v_3\) for \(|v_1|, |v_2| > 0\) since \(\gamma_h\) does not start with \(a\), and then \((h_0^{-1}, v_1, v_2^{-1})\) is a non-degenerate geodesic triangle with \(|h_0^{-1}| > 2s - 2\) (as shown in Figure 6). This contradiction shows that the first letter of \(\gamma_{h_0^{-1}h}\) is \(b\).

\[
\begin{align*}
\text{Figure 6. Proof of Lemma 14}
\end{align*}
\]

Suppose that \(a \neq b\). Note that distinct elements \(j, k \in H \setminus A\) give distinct geodesics \(\gamma_{h_0^{-1}j}\) and \(\gamma_{h_0^{-1}k}\). It follows that \(m_B \geq |H \setminus A| = |H| - m_A\), so

\[
(1)\quad |H| \leq m_A + m_B.
\]

Since \(e_G \not\in A \cup B\) and \(a \neq b\) means \(A, B\) are disjoint subsets of \(H\), then we have

\[
|H| \leq m_A + m_B < |H|.
\]

This contradiction allows us to conclude that \(a = b\).

Since \(a = b\), we have that for each \(h \in H \setminus A\), the first letter of \(\gamma_{h_0^{-1}h}\) is \(a\). Hence \(m_A \geq |H| - m_A\), so \(m_A \geq \frac{|H|}{2}\). The conclusions of the lemma follow immediately. □
By Theorem A if \( G \) is presented by a icfcrs \((\Sigma, T)\), then triangles in \( \Gamma(G, \Sigma) \) are \( r_T \)-thin. A well-known result concerning hyperbolic groups (see [2] 3) then gives that every finite subgroup \( H \) of \( G \) is conjugate to a subgroup contained within \( B_{c_G}(2r_T + 1) \). We can improve this bound for icfcrs groups.

**Proposition 15.** Let \( G \) be presented by an inverse-closed finite convergent length-reducing rewriting system \((\Sigma, T)\). Then every finite subgroup \( H \) of \( G \) is conjugate to a subgroup in \( B_{c_G}(r_T + 2) \).

**Proof.** Let \( H \) be a finite subgroup of \( G \). We shall prove the result by induction on the length of the longest element in \( H \). The result is clearly true in the case that the length of the longest element does not exceed \( r_T + 2 \). Suppose that the result holds in the case that the length of the longest element in \( H \) is at most \( n \) for some \( n \geq r_T + 2 \). Consider the case that the length of the longest element in \( H \) is \( n + 1 \). Let \( h_0 \in H \) be such that \( |\gamma_{h_0}| = |\gamma|_{\Sigma} = n + 1 \). Since \( n + 1 \geq r_T + 2 > r_T \), \( h_0 \) is a long element. By Lemma 14 \( \gamma_{h_0} = aua^{-1} \) for some \( a \in \Sigma \) and some \( u \in \Sigma^* \). Applying Lemma 14 to \( a^{-1}Ha \), because \( a^{-1}h_0a \) is also a long element, yields that \( u = bvb^{-1} \) for some \( b \in \Sigma \) and some \( v \in \Sigma^* \). It follows that \( \gamma_{h_0} = abvb^{-1}a^{-1} \) and \( a \neq b \). Now we consider the lengths of elements \( a^{-1}Ha \). It follows from Lemma 14 that if \( h \in H \) is a long element, then \( |\gamma_{a^{-1}ha}| = |\gamma|_{\Sigma} - 2 \). It is immediate that if \( h \in H \) is a short element (so \( |\gamma|_{\Sigma} = |h|_{G, \Sigma} \leq r_T \)), then \( a^{-1}ha \in B_{c_G}(2s) \). It follows that the length of the longest element in \( a^{-1}H \) does not exceed \( r_T + 2, n - 1 \). The inductive hypothesis gives the result. \( \square \)

**Lemma 16.** Let \( G \) be presented by an inverse-closed finite convergent length-reducing rewriting system \((\Sigma, T)\). If \( J \) is a maximal finite subgroup of \( G \) such that \( J \cap B_{c_G}(r_T + 2) \neq \{e_G\} \), then \( J \subseteq B_{c_G}(11r_T) \).

**Proof.** If \( r_T = 0 \), then \( G \) is a free group and the statement is vacuously true. We therefore assume that \( r_T > 0 \). It follows that \( \varepsilon_T = r_T + 1 \).

Suppose that \( J \) is a maximal finite subgroup of \( G \) such that \( J \cap B_{c_G}(r_T + 2) \neq \{e_G\} \). Let \( J \subseteq \langle J \cap B_{c_G}(r_T + 2) \rangle \setminus \{e_G\} \) and let \( k \in J \setminus \{e_G\} \). We must show that \( k \in B_{c_G}(11r_T) \). By Proposition 15 \( J \) is conjugate to a subgroup \( J' \) contained entirely within \( B_{c_G}(r_T + 2) \). Let \( t \in G \) be such that \( tJ't^{-1} = J \), let \( j' = t^{-1}jt \) and \( k' = t^{-1}kt \). By Lemma 8 with \( g = j' \) and \( h = j \), we have that \(|h|_{G, \Sigma} \leq 3r_T + 2(r_T + 2) \). Since \(|k'| \leq 2r_T + 2, t^{-1} = t \) and \( k = t^{-1}k't \), we have that \(|k| \leq 2(3r_T + 2(r_T + 2)) + r_T + 2 \leq 11r_T + 10 < 11(r_T + 1) = 11r_T \). \( \square \)

**Theorem B (Algebraic characterisation).** If \( G \) is a group presented by an inverse-closed finite convergent length-reducing rewriting system \((\Sigma, T)\), then the following are equivalent:

1. \( G \) is plain;
2. any nontrivial finite-order element in \( G \) is contained in a unique maximal finite subgroup of \( G \);
3. the relation \( \sim \) is transitive on the set of non-trivial finite-order elements in \( G \);
4. the relation \( \sim \) is transitive on the set of non-trivial finite-order elements in \( G \) of geodesic length (with respect to \( \Sigma \)) at most \( 11r_T \).

**Proof.** Suppose that \( G \) is a group presented by a finite convergent length-reducing rewriting system \((\Sigma, T)\). If \( r_T = 0 \), \( G \) is the free product of finitely many cyclic groups [7], but since \( \Sigma \) is inverse-closed, \( G \) is free and the theorem is immediate. So we may assume \( r_T > 0 \) and hence since we have assumed throughout that \((\Sigma, T)\) is normalised, we have \( \varepsilon_T = r_T + 1 \).

The equivalence of conditions (1), (2) and (3) (in a more general context) was established in Lemma 11. We use the notation introduced in the proof of that lemma. It is clear that condition (3) implies condition (4). Thus is suffices to show that if (3) does not hold, then (4) does not hold.

Suppose that \( \sim \) is not transitive. We must show that if \( \mathcal{E} \) contains an edge of type (E2) or type (E3), then the failure of \( \sim \) to be transitive will be witnessed by a triple of elements contained in \( B_{c_G}(11r_T) \).

First consider the case that \( \mathcal{E} \) contains an edge of type (E2). It follows that \( G \) contains finite subgroups \( A, B, C \) with \( A \) and \( B \) maximal finite subgroups, \( A \) not conjugate to \( B \), \( A \cap B = \emptyset \), \( A \cong B \) and \( 1 < \lvert C \rvert < \lvert A \rvert \leq \lvert B \rvert \). By Proposition 15 we may assume that \( A \subseteq B_{c_G}(r_T + 2) \). By Lemma 16 we have that \( B \subseteq B_{c_G}(11r_T) \). Let \( a \in A \), \( b \in B \setminus A \) and \( c \in C \setminus \{e_G\} \). Since \( (A, B) \cong A * C \), we have that \( a \sim c \) and \( a \sim b \), but \( a \not\sim b \). Thus the elements \( a, b, c \) are in \( B_{c_G}(11r_T) \) and witness the failure of \( \sim \) to be transitive.

Finally, consider the case that \( \mathcal{E} \) contains an edge of type (E3). It follows that \( G \) contains finite subgroups \( A, C \) and an infinite order element \( t \) with \( A \) a maximal finite subgroup, \( A \cap tA^{-1} = C \), \( (A, t) \cong A * \phi_t \) and \( 1 < \lvert C \rvert < \lvert A \rvert \). By Proposition 15 we may assume that \( A \subseteq B_{c_G}(r_T + 2) \). By Lemma 16 we have that \( tA^{-1} \subseteq B_{c_G}(11r_T) \). Let \( a \in A \), \( b \in tA^{-1} \setminus A \) and \( c \in A \cap (tA^{-1}) \setminus \{e_G\} \). Since \( (A, tA^{-1}) \cong A * \phi_t \), we have that \( a \sim c \) and \( c \sim b \), but \( a \not\sim b \). Thus the elements \( a, b, c \) are in \( B_{c_G}(11r_T) \) and witness the failure of \( \sim \) to be transitive. \( \square \)
6. Algorithms

We have reduced the problem of deciding plainness to that of deciding whether or not certain elements in the ball of radius $11\ell_T$ have finite order. Narendran and Otto show how to do this in polynomial time. Recall that we defined the size of a rewriting system $(\Sigma, T)$ to be

$$n_T = |\Sigma| + \sum_{(\ell, r) \in T} |\ell r|_\Sigma.$$

**Lemma 17** (Theorem 4.8, Narendran and Otto [20]). Let $G$ be a group presented by an inverse-closed finite convergent length-reducing rewriting system $(\Sigma, T)$ and let $u \in \Sigma^*$. There is a deterministic algorithm to decide whether or not $u$ spells an element of finite order in $G$, which runs in time which is polynomial in $|T|$, $|u|$ and $\mu = \sum_{(\ell, r) \in T} |\ell|_\Sigma$, so polynomial in $|u| n_T$.

**Theorem C** (Detecting plainness). The following decision problem is in NP: on input an inverse-closed finite convergent length-reducing rewriting system $(\Sigma, T)$, is the group presented by $(\Sigma, T)$ not plain?

**Proof.** If the group $G$ presented by $(\Sigma, T)$ is not plain, we can guess elements $u, v, w \in \Sigma^*$ so that $|u| \Sigma, |v| \Sigma, |w| \Sigma \leq 11\ell_T$, and use Lemma 17 to verify that

- each of the words $u, v, w, uv, vw, wv$ spells an element of finite order;
- $uw$ spells an element of infinite order.

Since $|u| \Sigma, |v| \Sigma, |w| \Sigma, |uv| \Sigma, |vw| \Sigma, |wv| \Sigma \in O(n_T)$, by Lemma 17, the algorithm runs in polynomial time in $n_T$. By Theorem B, if the algorithm guesses such a triple, then the relation $\sim$ is not transitive so the group is not plain. If no such triple exists within $(B_{eG}(11\ell_T))^3$, then the group is plain so the algorithm is correct. $\square$

To prove our final theorem, we need to make use of some simple facts.

**Lemma 18.** Let $(\Sigma, T)$ be an icfclrrs. Then $\log_2(|B_{eG}(rT + 2)|) \leq n_T^2$.

**Proof.** Since $\Sigma$ is inverse-closed, there exists a set $T' \subseteq T$ comprising exactly one rule $(ax, \lambda)$ with $x \in \Sigma$ for each $a \in \Sigma$. Then

$$n_T \geq |\Sigma| + 2|\Sigma| + \sum_{(\ell, r) \in T \setminus T'} |\ell r|_\Sigma \geq 3 + rT$$

since $|\Sigma| \geq 1$. We then have that

$$|B_{eG}(rT + 2)| \leq \sum_{i=0}^{rT+2} |\Sigma|^i \leq |\Sigma|^{rT+3}$$

so $\log_2(|B_{eG}(rT + 2)|) \leq (rT + 3) \log_2(|\Sigma|) \leq n_T^2$. $\square$

**Lemma 19.** If $G$ is a finite group, then a minimal generating set for $G$ has at most $\log_2|G|$ elements.

**Proof.** This is a straightforward exercise: let $\{g_1, \ldots, g_m\}$ be a minimal generating set, so $g_i \notin eG$. Let $G_n = \{g_1, \ldots, g_n\}$ for $1 \leq n \leq m$. Then by minimality $g_{n+1} \notin G_n$, so there are at least two cosets $eG_n, g_{n+1}G_n$, so $|G_n+1| \geq 2|G_n|$. By induction $|G| = |G_m| \geq 2^m$. $\square$

We also use this fact.

**Lemma 20.** Let $H_1, \ldots, H_k$ be finite groups, $G \cong H_1 * H_2 * \cdots * H_k * \mathbb{Z} * \cdots * \mathbb{Z}$, and $g \in H_i$. Then $h \in H_i$ if and only if $h, gh$ have finite order.

**Proof.** This follows immediately from the equivalence of conditions (1) and (2) in Theorem B. $\square$

Recall that by Savitch’s theorem [22], a problem that can be solved by a nondeterministic algorithm which uses space $f(n)$ is also in $\text{DSPACE}(f(n)^2)$, so to prove a problem is in $\text{PSPACE}$ it suffices to give a nondeterministic polynomial space algorithm.

**Proposition 21.** On input an inverse-closed finite convergent length-reducing rewriting system $(\Sigma, T)$ which presents a plain group $G$, we can output in space that is polynomial in $n_T$:

- an integer $k \leq n_T^2$;
- a list $L_{\Sigma, T} = (v_1, \ldots, v_k)$ with each $v_i \in \Sigma^*$ of length at most $rT + 2$;
- for each $1 \leq i \leq k$, an integer $p_i \leq 2^{n_T^2}$ written in binary;
- for each $1 \leq i \leq k$, a set $S_i = \{u_1, \ldots, u_{s_i}\}$ with $s_i \leq n_T^2$, $u_j \in \Sigma^*$, $|u_j|_\Sigma \leq rT + 2$;
- an integer $q \leq n_T$. 

so that for each $1 \leq i \leq k$, $\langle S_i \rangle$ is a maximal finite subgroup $H_i \subseteq B_{cG}(r_T + 2)$ of order $p_i$, and

$$G \cong H_1 \ast H_2 \ast \cdots \ast H_k \ast \mathbb{Z} \ast \cdots \ast \mathbb{Z}.$$

Proof. Assume $\Sigma$ is ordered. This induces a shortlex order on $\Sigma^*$.

In our subroutines below, when we say “for $u \in B_{cG}(a) \setminus B_{cG}(b)$” with $a > b \geq 0$ we mean that the subroutine loops through in shortlex order every reduced word $u \in \Sigma^*$ with $b < |u|_\Sigma \leq a$. Note that $B_{cG}(0) = \{e_G\}$. When we say “check that $u$ has finite/infinite order” we mean the subroutine calls the algorithm in Lemma 14.

First we create an ordered list $L_{\Sigma,T} = \langle v_1, v_2, \ldots, v_k \rangle$ that will contain exactly one non-trivial element $v_i$ from exactly one representative $H_i \subseteq B_{cG}(r_T + 2)$ of each conjugacy class of maximal finite subgroups in $G$. By Lemma 15 we know that $k \leq n_T^2$.

**Subroutine 1:** compute the list $L_{\Sigma,T}$

**Input:** $(\Sigma, T)$

Set $L_{\Sigma,T} = \langle \rangle$.

for $u \in B_{cG}(r_T + 2) \setminus \{e_G\}$ do

if $u$ has finite order then

for $t \in B_{cG}(11r_T) \setminus B_{cG}(r_T + 2)$ do

Check if $t$ and $tu$ are both finite order. If both have finite order, by Lemma 16 $u,t$ lie in the same maximal finite subgroup, so we are finished considering $u$ (because $u$ lies in a maximal finite subgroup that is not wholly contained in $B_{cG}(r_T + 2)$). By Lemma 15 we know that $11r_T$ is enough to check, since the maximal finite subgroup containing $u$ lies completely inside $B_{cG}(11r_T)$.

for $t \in B_{cG}(5r_T + 4) \setminus \{e_G\}$ do

for $h \in L_{\Sigma,T}$ do

Check if $tut^{-1}h$ has finite order. If it does, we are finished considering $u$ (because our list already contains a representative from a maximal finite subgroup in the same conjugacy class as the one containing $u$). Note that by Lemma 8 (setting $k = r_T + 2$) the bound of $5r_T + 4$ is enough to check (since $|u|_\Sigma, |h|_\Sigma \leq r_T + 2$).

If $u$ has not been rejected by any of the above steps, append $u$ to $L_{\Sigma,T}$.

The correctness of this subroutine is guaranteed by Lemmas 15 and 8 and the subroutine runs in PSPACE.

Given a word $v_i$ representing a nontrivial element of a maximal finite subgroup $H_i$ contained wholly in $B_{cG}(r_T + 2)$, we can recover in PSPACE the full list of elements of $H_i \setminus \{e_G\}$ as follows:

**Subroutine 2:** recover subgroup

**Input:** $(\Sigma, T); v_i \in L_{\Sigma,T}$.

for $u \in B_{cG}(r_T + 2) \setminus \{e_G\}$ do

Check that $u, uv_i$ have finite order. If so, return $u$.

The correctness of this subroutine is guaranteed by Lemma 16 and by construction of Subroutine 1.

Run Subroutine 1 and store the list $L_{\Sigma,T}$. From this we can read off $k \leq n_T^2$. Then run Subroutine 2 on each entry $v_i$ in $L_{\Sigma,T}$ with a binary counter to compute the size $p_i$ of each $H_i$ (representative of conjugacy class of maximal finite subgroup). Since $H_i \subseteq B_{cG}(r_T + 2)$, by Lemma 13 the integers $p_i$ written in binary require space at most $n_T^2$.

Now we give another subroutine which on input $v$ in the list $L_{\Sigma,T}$ can verify in nondeterministic polynomial space that a given set $S$ is a generating set for a subgroup $H \subseteq B_{cG}(r_T + 2)$ where $v \in H$.
Theorem D \textbf{(Isomorphism problem for plain icfclrrs groups).} The isomorphism problem for plain groups presented by inverse-closed finite convergent length-reducing rewriting systems is decidable in \textbf{PSPACE}.

\textbf{Proof.} Two plain groups given as
\[ H_1 \ast H_2 \ast \ldots \ast H_k \ast \mathbb{Z}^q \ast \ldots \ast \mathbb{Z}^q \ast \mathbb{Z}^{q'} \ast \ldots \ast \mathbb{Z}^{q'} \]
are isomorphic if and only if \( k = k' \), \( q = q' \) and there is a permutation \( \sigma \in S^k \) so that \( H_i \cong H'_\sigma(i) \) for \( 1 \leq i \leq k \).

Assume \((\Sigma, T), (\Sigma', T')\) are the input, and \( N = \max\{n_T, n'_T\} \). The procedure we describe will use polynomial space in \( N \), and be nondeterministic.

Guess the following data and store:
\begin{itemize}
  \item an integer \( q \leq N \);
  \item an integer \( k \leq N^2 \);
\end{itemize}
• a permutation $\sigma$ of length $k$;
• for each $1 \leq i \leq k$, an integer $p_i \leq 2\mathcal{N}^2$ written in binary;
• for each $1 \leq i \leq k$, a set $S_i = \{u_1, \ldots, u_{s_i}\}$ with $s_i \leq \mathcal{N}^2$, $u_j \in \Sigma^*$, and $|u_j| \leq r_T + 2$;
• a list $L' = (z_1, \ldots, z_k)$ with $z_i \in (\Sigma^*)^*$ of length at most $r_T' + 2$;
• for each $1 \leq i \leq k$, maps $f_i : S_i \to (\Sigma^*)^*$ with $|f_i(a)|_\Sigma \leq r_T' + 2$.

Note that this requires $\mathcal{O}(\mathcal{N}^3)$ space: $q \leq \mathcal{N}$; $k, |\sigma| \leq \mathcal{N}^2$; each $p_i$ in binary requires $\mathcal{N}^2$ space and there are $k$ of them, so total $\mathcal{N}^4$ space; each set $S_i$ has at most $\mathcal{N}^2$ words each of length at most $\mathcal{N}$, and there are $k \leq \mathcal{N}^2$ such sets so a total of $\mathcal{N}^5$ space; $L'$ requires $\mathcal{N}^2$ space; each map $f_i$ can be encoded by listing the $s_i \leq \mathcal{N}^2$ images of the generators as words of length at most $\mathcal{N}$, so $f_i$ requires $\mathcal{N}^3$ space and there are $k \leq \mathcal{N}^2$ of them so in total $\mathcal{N}^5$ space is required.

Then perform the following tasks.

1. Run the procedure in Proposition $21$ on input $(\Sigma, T)$ to verify that the output includes $k$ generating sets $S_1, \ldots, S_k$ with $|(S_i)| = p_i$, and the rank of the abelianisation of the plain group presented is $q$.

2. Run the procedure in Proposition $21$ on input $(\Sigma', T')$ to verify that the output includes $k$ subgroups having orders $p_{\sigma^{-1}(1)}, \ldots, p_{\sigma^{-1}(k)}$ (that is, if the $i$-th maximal finite subgroup found by the algorithm is $H_i'$, then $|H_i'| = p_i$), the list $L_{\Sigma', T'}$ is equal to $L'$, and the rank of the abelianisation of the plain group presented is $q$.

So far we have verified that the group presented by $(\Sigma, T)$ is $H_1 \ast H_2 \ast \cdots \ast H_k \ast \mathbb{Z} \ast \cdots \ast \mathbb{Z}$ where $|H_i| = p_i$ and $\langle S_i \rangle = H_i$ for $1 \leq i \leq k$, and the group presented by $(\Sigma', T')$ is $H'_1 \ast H'_2 \ast \cdots \ast H'_k \ast \mathbb{Z} \ast \cdots \ast \mathbb{Z}$ where $|H'_i| = p_i$ and $\langle H'_i \rangle = H'_i$ for $1 \leq i \leq k$.

To complete the verification that the groups are isomorphic, we need to show $H_i \cong H'_{\sigma(i)}$ for $1 \leq i \leq k$.

We do so by showing that each map $f_i$ induces an isomorphism from $H_i$ to $H'_{\sigma(i)}$. We do this as follows. Let $u_1 \in S_1$ be the first element in the list of generators guessed for $S_1$.

1. For all pairs $u, v \in \Sigma^*$ of length at most $r_T + 2$,
   - if $u, v$ are reduced, and if the order of $u, v, uu_1$ and $vu_1$ are finite (using Lemmas $17$ and $20$),
   - check that $f_i(uv) = f_i(u)f_i(v)$ by writing out both sides as words in $(\Sigma^*)^*$ and reducing.

   This shows that $f_i$ is a homomorphism from $H_i$ to $H'_{\sigma(i)}$.

2. Check that $f(S_i)$ is a generating set for $H'_{\sigma(i)}$ using Subroutine $3$ from the proof of Proposition $21$ with input $z_{\sigma(i)}$ and $f(S_i)$.

Once verified, we have that each $f_i$ is a surjective homomorphism between two finite groups of the same size, so $f_i$ is an isomorphism.

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School of Mathematical and Physical Sciences, University of Technology Sydney, Ultimo NSW 2007, Australia

Email address: murray.elder@uts.edu.au

Mathematical Sciences Institute, Australian National University, Canberra ACT 2601, Australia

Email address: adam.piggott@anu.edu.au