Simple Spin Networks as Feynman Graphs

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Abstract

We show how spin networks can be described and evaluated as Feynman integrals over an internal space. This description can, in particular, be applied to the so-called simple SO($D$) spin networks that are of importance for higher-dimensional generalizations of loop quantum gravity. As an illustration of the power of the new formalism, we use it to obtain the asymptotics of an amplitude for the $D$-simplex and show that its oscillatory part is given by the Regge action.

I. INTRODUCTION

Spin networks were originally introduced by Penrose\textsuperscript{[1]} in an attempt to give a combinatorial description of spacetime. Since then they reappeared in many branches of mathematical physics, see, e.g.,\textsuperscript{[2]} and references therein. In particular, spin networks are of fundamental importance in the loop approach to quantum gravity\textsuperscript{[3,4]}. More recently, with a development of the formalism of spin foam quantization\textsuperscript{[5–8]}, it was realized that spin networks that appear in the path integral version of loop quantum gravity are of a very special type. In the case of four spacetime dimensions these special spin networks were discovered in\textsuperscript{[6,7]}. Their higher-dimensional analogs were then described in\textsuperscript{[9]}. These spin networks can be called \textit{simple}: they satisfy a quantum analog of the simplicity constraint which requires a bivector to be a wedge product and, as was realized in\textsuperscript{[9]}, these are in

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certain precise sense the simplest possible spin networks that can be constructed in a given dimension.

In this paper we give a new description of simple spin networks: we show that they can be viewed and evaluated as Feynman graphs. Our construction gives an interesting perspective on these objects, which are of fundamental importance in the loop approach to quantum gravity. It also gives us new technical tools: with the help of the Feynman graph description we will be able to obtain the asymptotics of an amplitude for a $D$-simplex. We find the expected Regge action asymptotics for the amplitude.

The organization of this paper is as follows. In the rest of this section we remind the reader the usual description of spin networks and give the main idea of our construction. It turns out that the description of spin networks as Feynman graphs is quite general and relevant to gauge groups other than $\text{SO}(D)$ considered in [9]. The main idea of our construction is quite simple even when presented in its full generality. Thus, in this section, we do not restrict ourselves to any particular choice of the gauge group and present a general construction. It is then applied in Sec. II to simple $\text{SO}(D)$ spin networks relevant for quantum gravity. As an illustration of the usefulness of the new description we utilize it to derive the asymptotics of the evaluation of a simple $\text{SO}(D)$ spin network in Sec. III.

Let us now remind the reader the standard description of spin networks, see, e.g., [10].

Definition 1 Given a Lie group $G$ (which we assume to be semisimple and compact), spin network is a triple $(\Gamma, \rho, I)$, where:

(i) $\Gamma$ is an oriented graph;
(ii) $\rho$ is a labelling of each edge $e$ by an irreducible unitary representation $\rho_e$ of $G$;
(iii) $I$ is a labelling of each vertex $v$ of $\Gamma$ by an intertwiner $I_v$ mapping the tensor product of incoming representations at $v$ to the product of outgoing representations at $v$.

These data define a function $\phi_{(\Gamma, \rho, I)}$ on $G^E$ invariant under the action of the group at vertices; here $G^E$ is the product of a number of copies of the group $G$, one for each of $E$ edges of $\Gamma$. Thus, spin network can be described as a function which associates a number to each assignment of group elements $g_e$ to the edges $e$:

$$\phi_{(\Gamma, \rho, I)} : G^E \to \mathbb{C}$$

An explicit construction of this function proceeds as follows. First, for each edge $e$ let us consider the operator representing the group element $g_e$ in the representation $\rho_e$. Introducing a basis in the corresponding representation space one can calculate the matrix elements $U_{m}^{\rho_e}(g_e)^n$. One then takes a tensor product of all these matrix elements to obtain a tensor $U(g^E)$; it has one subscript and one superscript for each edge $e$. Then for each vertex $v$ of $\Gamma$ let $S(v)$ be the set of edges having $v$ as “source” and let $T(v)$ be the set of edges having $v$ as “target”. The intertwiner is a map (commuting with the action of $G$):

$$I_v : \otimes_{e \in T(v)} \rho_e \to \otimes_{e \in S(v)} \rho_e.$$ 

We can think of $I_v$ as of a tensor with one superscript for each edge $e \in T(v)$ and one subscript for each edge $e \in S(v)$. One can then form the tensor product of all intertwiners to obtain a tensor $I$, and then take the tensor product $U(g^E) \otimes I$. Note now that each
superscript in $U$ corresponds to a subscript in $I$ and vice versa, because each edge of $\Gamma$ lies in $S(v)$ for one vertex $v$ and in $T(w)$ for one vertex $w$. Therefore, one can contract indices of $U(g^E) \otimes I$ to get a number. This is the value of the function $\phi(\Gamma, \rho, I)$. One can check that the function constructed is invariant under the action of the group $G$, where the group action is that at the vertices.

Before we present our construction we will need the standard notion of a representation of class 1 (see, e.g., [11]). Spin networks that can be represented as Feynman graphs are the ones constructed using only these special representations of $G$.

**Definition 2** Let $\rho$ be an irreducible representation of $G$, and let $H$ be a subgroup of $G$. If the representation space $V^\rho$ contains vectors invariant under $H$, and if all operators $U^\rho(h), h \in H$ are unitary, then $\rho$ is called a representation of class 1 with respect to $H$.

The significance of these representations comes from the fact that they can be realized in the space of functions on the homogeneous space $H \backslash G$. As we describe below, spin networks that are constructed using only representations of class 1 with respect to $H$ can be viewed as Feynman graphs on $H \backslash G$. Simple $SO(D)$ spin networks of [1] are just such spin networks. In this case $H = SO(D - 1)$ and $H \backslash G = S^{D-1}$.

The realization of a representation of class 1 in the space of functions on a homogeneous space $H \backslash G$ is a particular case of a general description of an irreducible representation by shift operators in the space of functions on the group. Let us remind the reader this description. Consider matrix elements

$$U_{x,a}^\rho(g) := (U^\rho(g)x, a),$$

where $x, a$ are vectors from the representation space $V^\rho$. Let us fix $a$. Then the functions $U_{x,a}^\rho(g), x \in V^\rho$ span a subspace in the space $L^2(G)$ of square integrable functions on the group. One can then show that the right regular action of the group $G$ on this subspace gives an irreducible representation equivalent to $\rho$. The scalar product in the representation space is then given by the integral over the group. In the case $\rho$ is a representation of class 1 with respect to $H$, and $a$ is a vector invariant under $H$, the functions $U_{x,a}^\rho(g)$ are constant on the right cosets $Hg$ and can be regarded as functions on the homogeneous space $X = H \backslash G$. The scalar product is then given by an integral over $X$.

We are now ready to describe spin networks constructed from representations of class 1 with respect to $H$ as Feynman graphs on $X$. Let us denote by $P^{(\rho)\circ n}(x), x \in X$ an orthonormal basis in the representation space $\rho$ realized in the space of functions on $X$. The matrix elements of the group operators are then given by:

$$U^\rho(g)_m^n = \int_X dx P_{m}^{(\rho)}(x)P^{(\rho)\circ n}(xg),$$

where $dx$ is the invariant normalized measure on $X$. This gives realization of the matrix elements as integrals over $X$. The other building block necessary to construct a spin network is an intertwiner. Intertwiners can be characterized by their integral kernels. For a $k$-valent vertex one defines the integral kernel $I_v(x_1, \ldots, x_k)$ so that:

$$I_v^{m_1, \ldots, m_k} = \int_X dx_1 \cdots dx_k I_v(x_1, \ldots, x_k) P_{m_1}^{(\rho_1)}(x_1) \cdots P_{m_k}^{(\rho_k)}(x_k) P^{(\rho_1)\circ m_1}(x_{i+1}) \cdots P^{(\rho_k)\circ m_k}(x_k).$$
The integral kernels $I_v(x_1, \ldots, x_k)$ must satisfy the invariance property $I_v(x_1g, \ldots, x_kg) = I_v(x_1, \ldots, x_k)$. A special important set of intertwiners is given by:

$\tilde{I}_v^m(x_1, \ldots, x_k) = \int_X dx \delta(x, x_1) \cdots \delta(x, x_k)$

or

$\tilde{I}_v^{n_1 \ldots n_k} = \int_X dx P_{m_1}^{(\rho_1)}(x) \cdots P_{m_i}^{(\rho_i)}(x) P^{(\rho_{i+1})n_{i+1}}(x) \cdots P^{(\rho_k)n_k}(x)$.

These special intertwiners are the ones that appear in simple spin networks [9]. Exactly for such intertwiners it is possible to represent the spin network evaluation as a Feynman graph. Let us now introduce what can be called Green’s function:

$G^{(\rho)}(x, y) := \sum_n P_{n}^{(\rho)}(x) P^{(\rho)n}(y)$.

This Green’s function satisfies the “propagator” property:

$\int_X dz G^{(\rho)}(x, z) G^{(\rho)}(z, y) = G^{(\rho)}(x, y)$.

Let us also introduce a propagator “in the presence of a source”:

$G^{(\rho)}(x, y; g) := \int_X dz G^{(\rho)}(x, z) G^{(\rho)}(zg, y)$.

It is clear that $G^{(\rho)}(x, y; e) = G^{(\rho)}(x, y)$, where $e$ is the identity element of the group. One can now check that, in the case all spin network intertwiners are of a special type $\tilde{I}_v$ described above, the spin network function $\tilde{\phi}_{(\Gamma, \rho, \tilde{I})}$ of the group elements $g_1, \ldots, g_E$ is given by the Feynman graph with the following set of Feynman rules:

- With every edge $e$ of the graph $\Gamma$ associate a propagator $G^{(\rho_e)}(x, x', g_e)$.

- Take a product of all these data and integrate over one copy of $X$ for each vertex.

These rules can be summarized by the following formula:

$\tilde{\phi}_{(\Gamma, \rho, \tilde{I})}(g_1, \ldots, g_E) = \prod_v \int_X dx_v \prod_e G^{(\rho_e)}(x, x'; g_e)$.

(1)

Thus, in the case intertwiners are given by $\tilde{I}$ the evaluation of a spin network on a string of group elements is given by a Feynman graph: one associates the Green’s function to every edge and integrates over the positions of vertices.

Before we illustrate this general construction on the example of simple $SO(D)$ spin networks, let us note that this construction can be readily generalized to the case of an arbitrary spin network. Indeed, the restriction of representations labelling the spin network to be those of class 1 with respect to a fixed subgroup $H$ was necessary only to guarantee that the resulting Feynman graph lives in the homogeneous space $X = H \backslash G$. It can be dropped at the expense of Feynman graphs becoming graphs in the group manifold. The restriction of intertwiners to be of a special type $\tilde{I}_v$ can be dropped with the result that the set of
Feynman rules specified above changes: in this case one has to associate with every vertex the integral kernel \( I_v(x_1, \ldots, x_k) \) and then integrate over all the arguments. Thus, in the case of arbitrary intertwiners, the evaluation formula takes the form:

\[
\phi_{(\Gamma, \rho, I)}(g_1, \ldots, g_E) = \prod_v \int_{X} dx_v I_v(x_v) \prod_e G^{(\rho_e)}(x, x'; g_e).
\]  

(2)

Here \( x_v \) stands for a string of arguments \( x_1, \ldots, x_k \) of a \( k \)-valent intertwiner, and \( x, x' \) in the argument of the Green’s function \( G^{(\rho_e)}(x, x'; g_e) \) must be the same as those in two intertwiners: \( x \) must the appropriate argument in \( I_v, e \in S(v) \) and \( x' \) must be the argument of \( I(w), e \in T(w) \).

II. SIMPLE \( \text{SO}(D) \) SPIN NETWORKS

In this section we illustrate the general construction presented above on the example of simple \( \text{SO}(D) \) spin networks. Their relevance to quantum gravity in \( D \) dimensions was explained in [9].

Simple \( \text{SO}(D) \) spin networks are the ones constructed from special representations of \( \text{SO}(D) \). As is well-known, group \( \text{SO}(D) \) has a special class of representations, called spherical harmonics, that appear in the decomposition of the space of functions \( L^2(S^{D-1}) \) on \( S^{D-1} \) into irreducible components. Some properties of these representations are described in Appendix A. Using the terminology introduced in Sec. [4] these representations of \( \text{SO}(D) \) can be described as representations of class 1 with respect to \( \text{SO}(D-1) \). They are characterized by a single parameter that we will denote by \( N \) in what follows; \( N \) is required to be an integer. These are the representations that were called simple in [9]. A simple \( \text{SO}(D) \) spin network was defined in [9] as a spin network which is constructed only from simple representations and whose intertwiners are the special intertwiners \( \tilde{I} \) introduced in Sec. [1].

In the case intertwiners are given by \( \tilde{I} \), the value of a simple spin network on a sequence of group elements can be evaluated using the general formula (1). In what follows we will be concerned only with a special case of spin network evaluated on all group elements being equal to the identity element. This “evaluation” of a spin network gives a number that depends only on the graph and on the labelling of its edges by integers \( N_e \). Evaluation of a spin network is of special importance for quantum gravity because this is the way to obtain an amplitude for a spacetime simplex, see [6–8]. Thus, according to our Feynman graph formula (1), the evaluation of a simple spin network is given by

\[
\phi_{(\Gamma, \rho)} = \prod_v \int_{S^{D-1}} dx_v \prod_e G_{N_e}(x, x').
\]  

(3)

Here

\[
G_N(x, y) = \sum_K \chi^K(x) \chi^K(y),
\]  

(4)

where we have introduced an orthonormal basis \( \chi^K, K = (k_1, \ldots, k_{D-3}) \) \( N \geq k_1 \cdots \geq k_{D-3} \geq |k_{D-2}| \) in the representation space (see Appendix A for a construction of such a
basis). The invariance property \( G_N(xg, yg) = G_N(x, y) \) implies that \( G_N(x, y) \) depends only on the scalar product \( (x \cdot y) \), and it is a standard result [1] that

\[
G_N^{(D)}(x, y) = \frac{D + 2N - 2}{D - 2} C_N^{(D-2)/2}(x \cdot y),
\]

(5)

where \( C_N^p \) is the Gegenbauer polynomial, see Appendix [4] for the definition. The expression (3) for the evaluation of a simple spin network is a generalization of the result [12] for the evaluation in \( D = 4 \).

**Example: Evaluation of the \( \Theta \)-graph**

Let us use the above representation of the simple spin network evaluation to compute the evaluation of the \( \Theta \)-graph. This is of importance because the value of the \( \Theta \)-graph appears in the normalization of tri-valent vertices. According to (3) the evaluation is given by:

\[
\Theta^{(D)}(N_1, N_2, N_3) = \int dx dy G_{N_1}(x, y)G_{N_2}(x, y)G_{N_3}(x, y).
\]

(6)

Using the expression of \( G_N \) in terms of a Gegenbauer polynomial (5), this integral can be computed. It is not equal to zero only if \( g = (N_1 + N_2 + N_3)/2 \) is an integer and \( g - N_i \geq 0, i = 1, 2, 3 \). In this case one gets:

\[
\Theta^{(D)}(N_1, N_2, N_3) = \frac{\Gamma(g + 2p)\Gamma(p + 1)}{\Gamma(g + p + 1)\Gamma(2p)} \prod_{i=1}^{3} \left( \frac{(N_i + p)\Gamma(g - N_i + p)}{\Gamma(p + 1)\Gamma(g - N_i + 1)} \right).
\]

(7)

Here \( p = (D - 2)/2 \). To check this result one can check that (3), (6) both satisfy the recurrence relation (implied by (B3)):

\[
\frac{N_1 + 1}{N_1 + p + 1} \Theta^{(D)}(N_1 + 1, N_2, N_3) + \frac{N_1 + 2p - 1}{N_1 + p - 1} \Theta^{(D)}(N_1 - 1, N_2, N_3) = \]

(8)

\[
\frac{N_3 + 1}{N_3 + p + 1} \Theta^{(D)}(N_1, N_2, N_3 + 1) + \frac{N_3 + 2p - 1}{N_3 + p - 1} \Theta^{(D)}(N_1, N_2, N_3 - 1),
\]

(9)

and that \( \Theta(N_1, N_2, 0) \) reproduces the orthogonality relation (B3). For \( D = 4 \) the above expression simplifies:

\[
\Theta^{(4)}(N_1, N_2, N_3) = (N_1 + 1)(N_2 + 1)(N_3 + 1).
\]

(10)

This result can be used to show that the intertwiner used in [9] to define simple spin networks in the case of \( D = 4 \) coincides with the one proposed in [6]. Indeed, the four-valent intertwiner of [9] reads:

\[
I_{2,2}(P_1, P_2, Q_1, Q_2) = \int dx P_1(x)P_2(x)Q_1(x)Q_2(x).
\]

Using the kernel of the identity operator on \( L^2(S^{D-1}) \) given by \( \sum_{N=0}^{\infty} G_N \) we can expand the 4-valent vertex in terms of the sum of the product of two tri-valent ones:
\[ I_{2,2}(\mathcal{P}_1, \mathcal{P}_2, Q_1, Q_2) = \sum_{N=0}^{\infty} \int dx dy \, P_1(x) P_2(x) G_N(x, y) Q_1(y) Q_2(y). \]

In short this can be written as

\[
I^{(D)}_{2,2}(N_1, N_2, N_3, N_4) = \sum_N I^{(D)}_{2,1}(N_1, N_2, N) \cdot I^{(D)}_{1,2}(N, N_3, N_4) = \\
\sum_N \left[ \Theta^{(D)}(N_1, N_2, N) \Theta^{(D)}(N, N_3, N_4) \right]^{1/2} \tilde{I}^{(D)}_{2,1}(N_1, N_2, N) \cdot \tilde{I}^{(D)}_{1,2}(N, N_3, N_4),
\]

where we have introduced the normalized intertwiner

\[
\tilde{I}^{(D)}_{2,1}(N_1, N_2, N_3) = (\Theta^{(D)}(N_1, N_2, N))^{-1/2} I^{(D)}_{2,1}(N_1, N_2, N_3).
\]

In the case of \( D = 4 \), using the result \( \Theta^{(D)} \) for the \( \Theta \)-graph, we get

\[
I^{(D)}_4(N_1, N_2, N_3, N_4) = \\
[(N_1 + 1)(N_2 + 1)(N_3 + 1)(N_4 + 1)]^{1/2} \sum_{N=0}^{\infty} (N + 1) \tilde{I}^{(D)}_{2,1}(N_1, N_2, N) \cdot \tilde{I}^{(D)}_{1,2}(N, N_3, N_4).
\]

Up to an overall normalization factor, this vertex is exactly the vertex given in \( \Theta \).

### III. LARGE SPIN ASYMPTOTICS

In this section we use the Feynman graph representation of the simple spin networks to study the asymptotics of a \( D \)-simplex amplitude for large \( N \). The results of this section generalize those of \( \Theta \) to the case of arbitrary dimension. Most of the labor necessary to get the asymptotics is done in Appendix \( \Theta \). Here we simple use the asymptotics \( \Theta^{(D)} \) of the Gegenbauer polynomial obtained there.

As is explained in Refs. \( \Theta \), the amplitude for a \( D \)-simplex is given by the evaluation of the spin network that is dual to the boundary of the simplex. The \((D-2)\)-simplices are labelled by simple representations of \( SO(D) \), i.e., by integers \( N \). The edges of the spin network dual to the boundary of the simplex are in one-to-one correspondence with the \((D-2)\)-simplices, and inherit the labels of \((D-2)\)-simplices. As one can easily check, all vertices of the spin network in question are \( D \)-valent. All intertwiners are of the special type described in Sec. \( \Theta \) and, thus, the formula \( \Theta \) can be used for the evaluation. Using the asymptotics \( \Theta^{(D)} \) and the formula \( \Theta \) we present an asymptotic evaluation the amplitude: we will use the stationary phase approximation for the integral. Our discussion follows closely that of \( \Theta \).

To get a feeling about the behavior of the amplitude, we will concentrate only on the oscillatory part of \( C_N^{(p)}(\cos \theta) \). Thus, dropping all multiplicative constants, which are unimportant for us, we get

\[
\phi(\Gamma, \rho) \sim \sum_{\{e_{kl}\}} \left( \prod_{k<l} e_{kl} \right) \int_{S^{D-1}} dx_1 \cdots dx_{D+1} e^{i \sum_{k<l} e_{kl}(N_{kl}+p) \theta_{kl} + (1-p) \pi/2},
\]
where the integral is taken over \((D + 1)\) points – vertices of the spin network – on the unit \((D - 1)\)-sphere, and \(k, l\) are indices labelling the vertices \(k, l = 1, \ldots, D + 1\). Thus, a pair \(kl\) labels a spin network edge, and \(\theta_{kl} : \cos \theta_{kl} = x_k \cdot x_l\). The quantity \(\epsilon_{kl}\) takes values \(\pm 1\) and the sum is taken over both possibilities for every edge. The rest of the analysis is exactly the same as in [13]. Taking into account the fact that the variation of the angles satisfy the following identity (see [13]):

\[
\sum_{k<l} V_{kl} \delta \theta_{kl} = 0,
\]

where \(V_{kl}\) are the volumes of \((D - 2)\)-simplices inside a geometric \(D\)-simplex, one finds that all \(\epsilon_{kl}\) are either positive or negative, and that the stationary phase values of \(\theta_{kl}\) are the ones corresponding to a geometric \(D\)-simplex determined by \(N_{kl} + p\) interpreted as volumes of \((D - 2)\)-simplices. Then, in the case the number \(D(D + 1)/2\) of edges in the simplex is even, we get

\[
\phi(\Gamma, \rho) \sim \cos \left( \sum_{k<l} (N_{kl} + p) \theta_{kl} + \kappa \frac{\pi}{4} \right), \tag{11}
\]

where \(\theta_{kl}\) are the higher-dimensional analogs of the dihedral angles of the geometric \(D\)-simplex determined by \(N_{kl} + p\) and

\[
\kappa = \frac{(D + 1)D}{2} (4 - D)
\]

is the integer determined by \(D\). In the case \(D(D + 1)/2\) is odd one gets ‘sin’ instead of ‘cos’ in the asymptotics (11). Thus, the simplex amplitude has the asymptotics of the exponential of the Regge action, as expected.

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APPENDIX A: SIMPLE REPRESENTATIONS OF \(\text{SO}(D)\)

What is referred to in this paper as simple representations of \(\text{SO}(D)\) are the usual spherical harmonics representations. They are irreducible representations of \(\text{SO}(D)\) of class 1 with respect to the subgroup \(\text{SO}(D - 1)\) and, therefore, can be realized in the space of functions on \(S^{D-1}\). This partially explains their relevance for quantum gravity in \(D\) dimensions, where the \((D - 1)\)-sphere has the geometrical meaning of the boundary of the
D-simplex. In this Appendix we review some basic properties of these representations. For more information see, e.g., [11].

The spherical harmonics representations of SO(D) are the most obvious ones: they can be realized in the space of homogeneous polynomials of degree N. Let us denote the space of such polynomials by \( V_N^{(D)} \). Then

\[
\dim V_N^{(D)} = \frac{(N + D - 1)!}{N!(D - 1)!}.
\]

It turns out, however, that the representation in this space is not irreducible. The invariant subspace in \( V_N^{(D)} \) is given, as usual, by the space of polynomials satisfying the Laplace equation in \( \mathbb{R}^D \). Thus, the irreducible representations of this type are realized in the space of homogeneous harmonic polynomials of degree N. Let us denote this space by \( H_N^{(D)} \). As one can show,

\[
\dim H_N^{(D)} = \frac{(2N + D - 2)(D + N - 3)!}{(D - 2)!N!}.
\] (A1)

As we have mentioned, these representations are of the class 1 with respect to SO(D−1). Choosing the upper-left corner embedding of SO(D−1) into SO(D), the vector in \( H_N^{(D)} \) that is invariant under the action of SO(D−1) is given (up to normalization) by \( C_N^p(x_D) \) for \( x = (x_1, \ldots, x_D) \). Here \( p = (D - 2)/2 \) and \( C_N^p(x) \) is the so-called Gegenbauer polynomial defined in the next Appendix.

An explicit basis in \( H_N^{(D)} \) can be constructed by choosing a string of embeddings

\[
\text{SO}(2) \subset \text{SO}(3) \subset \cdots \subset \text{SO}(D - 1) \subset \text{SO}(D).
\]

Then \( H_N^{(D)} \) decomposes into subspaces irreducible with respect to the action of the subgroup SO(D−1). The later again decompose into the irreducible subspaces with respect to the action of SO(D−2) etc. Finally, one arrives at SO(2) whose irreducible representations are 1-dimensional. Thus, we have:

\[
\mathcal{H}_N^{(D)} = \bigoplus_{k_1=0}^N \bigoplus_{k_2=0}^{k_1} \cdots \bigoplus_{k_{D-2}=-k_{D-3}}^{k_{D-3}} V_{k_{D-2}}.
\]

Here \( V_k \) are 1-dimensional representation spaces of SO(2). Note that \( k_{D-2} \) in the last sum runs over both positive and negative values. Thus, a basis in \( \mathcal{H}_N^{(D)} \) can be labelled by a string of integers:

\[
K := (k_1, k_2, \ldots, k_{D-2}), \quad N \geq k_1 \geq k_2 \geq \cdots \geq |k_{D-2}|.
\]

**APPENDIX B: PROPERTIES OF GEGENBAUER POLYNOMIALS**

Gegenbauer polynomials are orthogonal polynomials satisfying many different properties. In this Appendix we review some of them. For more information of Gegenbauer polynomials see, e.g., [11,14].
Let \( p \) be denote a quantity related to the dimension \( D \) according to \( p = (D - 2)/2 \), or \( D = 2p + 2 \). A generating functional for Gegenbauer polynomial is given by:

\[
(1 - 2xr + r^2)^{-p} = \sum_{N=0}^{+\infty} C_N^p(x)r^N. \tag{B1}
\]

Gegenbauer polynomials satisfy the Rodriguez formula:

\[
C_N^p(x) = \frac{(-1)^N(N + 2p - 1)(N + 2p - 2) \cdots (2p)}{2^N N!(N + p - \frac{1}{2})(N + p - \frac{3}{2}) \cdots (p + \frac{1}{2})} \times (1 - x^2)^{-p+rac{1}{2}} \left( \frac{d}{dx}\right)^N(1 - x^2)^{N+p-rac{1}{2}}, \tag{B2}
\]
where the prefactor can also be written as

\[
\frac{(-1)^N \Gamma(N + 2p)\Gamma(p + \frac{1}{2})}{2^N N! \Gamma(2p)\Gamma(N + \frac{1}{2} + p)}
\]

The recurrence formula is given by:

\[
(N + 1)C_{N+1}^p(x) - 2(N + p)xC_N^p(x) + (N + 2p - 1)C_{N-1}^p(x) = 0, \tag{B3}
\]
with \( C_0^p(x) = 1 \) and \( C_1^p(x) = x \). The polynomials satisfy the following differential equation:

\[
\left\{ (1 - x^2)^2 - (2p + 1)x \frac{d}{dx} + N(N + 2p) \right\} C_N^p(x) = 0.
\]

A change of variable \( x = \cos \theta \) puts this in the following form:

\[
\left\{ \left( \frac{d}{d\theta}\right)^2 + 2p \frac{\cos \theta}{\sin \theta} \frac{d}{d\theta} + N(N + 2p) \right\} C_N^p(\cos \theta) = 0. \tag{B4}
\]

The polynomials are normalized as:

\[
C_N^p(1) = \frac{\Gamma(2p + N)}{\Gamma(2p)N!} = \dim \mathcal{H}_N^{(D)} \frac{D - 2}{2N + D - 2}.
\]

where \( \dim \mathcal{H}_N^{(D)} \) is given by (A1). The polynomials satisfy the following orthogonality condition:

\[
\int_{-1}^{+1} dx(1 - x^2)^p \frac{1}{\Gamma} C_N^p C_M^p = \delta_{N,M} \frac{\pi \Gamma(2p + N)}{2^{2p-1}N!(N + p)\Gamma^2(p)}. \tag{B5}
\]
APPENDIX C: ASYMPTOTICS OF THE GEGENBAUER POLYNOMIAL

To get the asymptotics of the Gegenbauer polynomial for large $N$ we use the differential equation (B4). It can be put into a form similar to that of a wave equation by setting

$$C_N^{(p)}(\cos \theta) = f(\theta) \sin^{-p} \theta.$$

One gets:

$$\frac{d^2 f}{d\theta^2} + f \left[ (N + p)^2 - \frac{p(p - 1)}{\sin^2 \theta} \right] = 0.$$

For large $N$ one can neglect the second term in the square brackets and $p$ as compared to $N$ in the first term. Thus, the large $N$ asymptotics is given by

$$C_N^{(p)}(\cos \theta) \sim \frac{A}{\sin^{p} \theta} \sin[(N + p)\theta + \phi],$$

where $\phi$ is a phase and $A$ is a normalization factor, both arbitrary at this stage. It can be constrained by using symmetry properties of $C_N$. From the expression for the generating functional one sees that

$$C_N(-x) = (-1)^N C_N(x).$$

Thus,

$$C_N^{(p)}(\cos(\pi - \theta)) = (-1)^N C_N^{(p)}(\cos \theta).$$

A simple analysis shows that this restricts $\phi$ to be

$$\phi = \frac{(1 - p)}{2} \pi + \pi k,$$

where $k$ is an arbitrary integer. Thus, the ambiguity in $k$ is just the overall sign ambiguity. The constant $A$ can be determined from the normalization condition (B5). One gets:

$$\frac{\pi}{2} A^2 = \frac{\pi \Gamma(2p + N)}{2^{2p-1}N!(N + p)\Gamma^2(p)},$$

or

$$A = \pm \frac{1}{2^{p-1}\Gamma(p)} \left[ \frac{\Gamma(2p + N)}{N!(N + p)} \right]^{1/2}.$$

For large $N$ this behaves as

$$A \sim \pm \frac{N^{p-1}}{2^{p-1}\Gamma(p)}.$$

Using the fact that
and the expression for the derivative of the Gegenbauer polynomial

$$\frac{d}{d\theta} C_{N}^{p} = -2p \sin \theta C_{N-1}^{p+1},$$

we can fix the overall sign to be plus. Thus, finally, we get:

$$C_{N}^{(p)}(\cos \theta) \sim \frac{N^{p-1}}{2^{p-1} \Gamma(p)} \frac{1}{\sin^p \theta} \sin[(N + p)\theta + (1 - p)\pi/2]. \quad (C1)$$
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