Einstein-Born-Infeld black holes with a scalar hair in three-dimensions

S. Habib Mazharimousavi and M. Halilsoy
Department of Physics, Eastern Mediterranean University, Gaziاغosa, north Cyprus, Mersin 10, Turkey.
(Dated: August 18, 2014)

We present a black hole solution in 2 + 1—dimensional Einstein’s theory of gravity coupled with Born-Infeld nonlinear electrodynamic and a massless self-interacting scalar field. The model has five free parameters: mass (M), cosmological constant (ℓ), electric (q) and scalar (r₀) charges and Born-Infeld parameter (β). To attain exact solution for such a highly non-linear system we adjust, i.e. finely tune, the parameters of the theory with the integration constants. In the limit β → 0 we recover the results of Einstein-Maxwell-Scalar theory, obtained before. The self interacting potential admits finite minima apt for the vacuum contribution. Hawking temperature of the model is investigated versus properly tuned parameters.

PACS numbers: 04.20.Jb; 04.40.Nr; 04.70.-s
Keywords: Lower dimensions; Black hole; Nonlinear electrodynamics; Born-Infeld; Scalar field

I. INTRODUCTION

Since the pioneering work of Banados-Teitelboim and Zanelli (BTZ) [1] the subject of 2 + 1—dimensional black holes has attracted much attention and remained deservedly ever a focus of interest due to many reasons. Further to the pure BTZ black hole powered by a mass and a negative cosmological constant the strategy has been to add new sources such as electric / magnetic fields from Maxwell’s theory [2], rotation [2, 3] and various fields [4]. This remains the only possible extension due to the absence of gravitational degree of freedom in the lower dimension. In this situation scalar field coupling to gravity, minimal or nonminimal with self-interacting potential is one such attempts that may come into mind (See [5] and references cited therein). The Brans-Dicke experience in 3 + 1—dimensions with a vast literature behind suggests that a similarly rich structure can be established in the 2 + 1—dimensions as well.

In this line Henneaux et al [6] introduced Einstein’s gravity minimally and non-minimally coupled to a self interacting scalar field. Einstein’s gravity conformally and non-minimally coupled to a scalar field was studied by Hasanpour et al in [7] where they presented exact solutions and their Gravity / CFT correspondences. Also, rotating hairy black hole in 2 + 1—dimensions was considered in [8] while charged hairy black hole was introduced in [5]. Our purpose in this study is to employ self-interacting scalar fields and establish new hairy black holes in 2 + 1—dimensions in analogy with the dilatonic case [9]. In doing this, however, we replace also the linear Maxwell electrodynamics with the nowadays fashionable non-linear electrodynamics (NED). In particular, our choice of NED is the one considered originally by Born andInfeld (BI) [10] with the hope of eliminating the electromagnetic singularities due to point charges. The elimination of singularities in the electromagnetic field unfortunately doesn’t imply the removal of spacetime singularities in a theory of gravity-coupled NED. Rather, the spacetime singularities may undergo significant revision in the presence of NED to replace the linear Maxwell’s theory. Herein, the singularity at r = 0 remains intact but becomes modified, both in powers of r and also with the addition of ln r. Let us add that there are special metrics hosting gravity-coupled NED which are free of spacetime singularities [11].

Our 2 + 1—dimensional model investigated in this paper consists of a non-minimally coupled scalar field (with a potential) coupled to gravity and NED field. We introduce such a model first, by deriving the field equations and solving them. Recently such a model has been considered similar to ours in which the linear Maxwell theory has been employed [5]. Our task is to extend the linear Maxwell Lagrangian to the NED Lagrangian of Born-Infeld in 2 + 1—dimensions [12][13]. In particular limits our model recovers the results obtained before. The self-interacting potential U(ψ), as a function of the scalar field ψ happens in a particular solution to be highly non-linear whereas the scalar field itself is surprisingly simple:

\[ \psi(r) = \left( \frac{r_0}{r + r_0} \right)^{\frac{1}{4}} \]

in which r₀ is a constant such that 0 ≤ r₀ ≤ ̣∞. The scalar field is bounded accordingly as 0 ≤ ψ(r) ≤ 1, and is regular everywhere. As a matter of fact the constant r₀ is the parameter that measures the scalar charge (i.e. the scalar hair) of the black hole in such a model. Similar to the scalar field the static electric field E(r) also happens to be regular in our gravity-coupled NED model in 2 + 1—dimensions. The potential U(ψ) is plotted for chosen parameters which yields projection of a Mexican hat-type picture where the reflection symmetry U(ψ) = U(−ψ) is manifest. The minima of the potential may be considered to represent the vacuum energy of the underlying model field theory. To choose one of the vacua we need to apply spontaneous symmetry breaking which lies beyond our scope in this work. Our analysis reveals that the model yields both black hole and non-black hole solutions in accordance with the proper tuning of the parameters.
Our model Lagrangian admits overall five independent parameters: $\ell$ ($\Lambda = -\frac{1}{\ell^2}$ = the cosmological constant), $q$ (electric charge), $\beta$ (the BI parameter), $M$ (mass) and $r_0$ (scalar charge). The roles of the scalar charge ($r_0$) and BI parameter ($\beta$) are shown explicitly by numerical plottings to highlight their contribution. The role of all parameters in the thermodynamic properties such as Hawking temperature ($T_H$) is displayed numerically versus the horizon radius.

The paper is organized as follows. In Section II we introduce the field equations, present particular solution (with details in Appendix A), investigate the limits and study some thermodynamical properties. The paper ends with Conclusion in Section III.

II. FIELD EQUATIONS AND THE SOLUTIONS

We start with the action (8$\pi$G = 1 = c)

$$S = \frac{1}{2} \int d^3x \sqrt{-g} \left[ R - 8\partial\mu\psi\partial^\mu\psi - 2V(\psi) + L(F) \right]$$

in which $R$ is the Ricci scalar, $\psi$ is the scalar field which is coupled nonminimally to the gravity, $V(\psi)$ is a self-coupling potential of $\psi$ and $L(F)$ is the NED Lagrangian with the Maxwell invariant $F = F_{\alpha\beta}F^{\alpha\beta}$ and electromagnetic 2-form $F = \frac{1}{2} F_{\mu\nu}dx^\mu \wedge dx^\nu$. Let us add that this action differs from the one considered in [5] by a scale transformation in the scalar field and the important fact that $L(F)$ here corresponds to an NED Lagrangian rather than the Maxwell Lagrangian. Variation of the action with respect to $g_{\mu\nu}$ implies

$$G_{\mu\nu}^{\psi} = \tau_{\mu\nu} - V(\psi)\delta_{\mu\nu}$$

in which

$$\tau_{\mu\nu} = 8\partial\mu\psi\partial\nu\psi - 4\partial^\mu\psi\partial^\nu\psi - \{\delta_{\mu\nu} - \nabla_\mu\nabla_\nu + G_{\mu\nu}(\psi)\} \psi^2$$

and

$$T_{\mu\nu} = \frac{1}{2} \left( L_{\delta_{\mu\nu}} - 4F_{\mu\rho}F^{\rho\nu}L_F \right)$$

where $L_F = \frac{d\phi}{r^2}$. Variation of the action with respect to $\psi$ and vector potential $A_\mu$ yields the scalar field equation

$$\Box\psi = \frac{1}{8} \left( R\psi + \frac{dV}{d\psi} \right)$$

and nonlinear Maxwell’s equation

$$d\left( \tilde{F}L_F \right) = 0$$

respectively, in which the dual of $F$. Our line element is static circularly symmetric given by

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\theta^2$$

The NED which we study is the well known BI theory. The BI-Lagrangian is given by

$$L(F) = -\frac{4}{\beta^2} \left( \sqrt{1 + \beta^2 r^2} - 1 \right)$$

in which $\beta > 0$ is the BI parameter such that in the limit $\beta \to 0$ the Lagrangian reduces to the linear Maxwell Lagrangian

$$\lim_{\beta \to 0} L(F) = -F$$

and in the limit $\beta \to \infty$ it vanishes so that the general relativity (GR) limit is found. We also note that our electrodynamic potential is only electric and due to that in the BI Lagrangian the term $G = F_{\mu\nu}F^{\mu\nu}$ is not present. The nonlinear Maxwell equation admits a regular electric field of the form

$$E(r) = \frac{q}{\sqrt{r^2 + \beta^2 q^2}}$$

in which $q \geq 0$ is an integration constant related to the total charge of the black hole. As we have shown in the Appendix A, the field equations admit solution to the field equations as follows

$$V(\psi) = \frac{1}{\sqrt{1 + \frac{r}{r_0}}}$$

in which $r_0 \geq 0$ is a constant, and

$$f(r) = \left[ -M + \frac{q^2}{1 + \beta^2} - 2q^2 \ln \left( 1 + \sqrt{q^2/\beta^2 + r^2} \right) \right]$$

$$\times \left[ (1 + \frac{2r_0}{3r}) + r^2 \left( \frac{1}{r^2} + \frac{2}{\beta^2} \right) \right]$$

$$\frac{2r^2r_0}{3q\beta^3} \ln \left[ \frac{1}{r} \left( q\beta + \sqrt{q^2/\beta^2 + r^2} \right) \right]$$

$$\frac{2r}{\beta^2} \left(\frac{r_0}{3r} + 1\right) \sqrt{q^2/\beta^2 + r^2}. $$

The self-coupled potential is given by

$$V(\psi) = -\frac{1}{\ell^2} + U(\psi)$$

in which

$$U(\psi) = \left( \frac{1}{\ell^2} - \frac{M + 2q^2 \ln A}{3r_0^2} + \frac{q^2}{3r_0^2 \left( 1 + \beta^2 \right)} \right) \psi^6 +$$

$$\frac{2r_0 (\psi^6 - 1) \ln B}{3q\beta^3} + \frac{2q^2 \left( q^2/\beta^2 \psi^4 + (1 - 3\psi^4) r_0^2 \right)}{3r_0 \left( q\beta^2 + \sqrt{\Delta} \right)}$$

$$+ \frac{2q^2 \left( q^2/\beta^2 + 3\beta q_0 r_0 + 2r_0^2 \right)}{3r_0 \beta^2 \left( q\beta^2 + \sqrt{\Delta} \right)}$$

$$+ \frac{2 \beta q \left( q^2/\beta^2 + 1 \right) q + 3r_0 q^2 \psi^4 \sqrt{\Delta}}{3\beta^2 r_0 \left( q\beta^2 + \sqrt{\Delta} \right)}.$$
with the abbreviations
\[
\Delta = (r_0^2 + q^2 \beta^2) \psi^4 + r_0^2 (1 - 2 \psi^2)
\]
\[
A = \frac{r_0 (1 - \psi^2) + \sqrt{\Delta}}{\psi^2 (2 + q \beta)}
\]
and
\[
B = \frac{q \beta \psi^2 + \sqrt{\Delta}}{r_0 (1 - \psi^2)}. 
\]

We note that in terms of \( r \) we have
\[
A = \frac{r + \sqrt{r^2 + q^2 \beta^2}}{2 + q \beta}, 
\]
and
\[
B = \frac{q \beta + \sqrt{r^2 + q^2 \beta^2}}{r} 
\]
which are independent from the scalar charge \( r_0 \). For \( q \beta \rightarrow 0 \) one obtains \( A \rightarrow r \) and \( B \rightarrow 1 \). Similarly
\[
\sqrt{\Delta} = \frac{r_0}{r + r_0} \sqrt{r^2 + q^2 \beta^2} 
\]
which vanishes in the limit \( r_0 \rightarrow 0 \). The general solution found here is a singular black hole solution whose limits and horizons will be investigated in the rest of the paper.

### A. The Limits

The solution given in (11)-(14) for different limits represents the known solutions in 2 + 1–dimensions. The first limit is given with \( r_0 \rightarrow 0 \) which implies \( \psi \rightarrow 0 \). In this setting one finds
\[
\lim_{r_0 \rightarrow 0} f_{BI} (r) = - M + \frac{r^2}{\ell^2} - 2q^2 \ln \left( \frac{r + \sqrt{q^2 \beta^2 + r^2}}{2 + q \beta} \right)
\]
\[
\quad + r^2 \left( \frac{1}{\ell^2} + \frac{2}{\beta^2} \right) - \frac{2r}{\beta^2} \sqrt{q^2 \beta^2 + r^2}. 
\]
which is the black hole solution in Einstein-Born-Infeld (EBI) theory introduced by Cataldo and Garcia (CG) in [12]. We notice that the integration constants in the general solution are finely tuned such that in the EBI limit the solution admits both BTZ and CG-BTZ limits without need for a redefinition of the electric charge. Otherwise it can be seen in the Eq. (29) of Ref. [12] that the CG-BTZ limit has different charge from the original solution (2) of [12]. This form of the solution easily gives BTZ and charged BTZ black holes in the limits when \( \beta \rightarrow \infty \) and \( \beta \rightarrow 0 \) respectively i.e.,
\[
\lim_{\beta \rightarrow \infty} f_{BI} (r) = - M + \frac{r^2}{\ell^2} 
\]
and
\[
f_{CG-BTZ} (r) = \lim_{\beta \rightarrow 0} f_{BI} (r) = - M + \frac{r^2}{\ell^2} - 2q^2 \ln r. 
\]

We note that the limit of the self-coupling potential when \( \psi \rightarrow 0 \) becomes
\[
\lim_{\psi \rightarrow 0} V (\psi) = - \frac{1}{\ell^2} 
\]
which is nothing but the cosmological constant in the action.

The other limit of the solution is given by \( \beta \rightarrow 0 \) which yields
\[
f_{XZ} (r) = \lim_{\beta \rightarrow 0} f (r) = - M + \frac{r^2}{\ell^2} - 2q^2 \left( 1 + \frac{2r}{3r_0} \right) \ln r + \frac{2r_0}{3r} \left( \frac{q^2}{3} - M \right) 
\]
which is the black hole solution in Einstein-Maxwell coupled scalar field found by Xu and Zhao (XZ) in [5]. This limiting solution in the further limit \( q \rightarrow 0 \) becomes
\[
\lim_{q \rightarrow 0} f_{XZ} (r) = - M + \frac{r^2}{\ell^2} - \frac{2r_0M}{3r} 
\]
and when \( r_0 \rightarrow 0 \) gives
\[
\lim_{r_0 \rightarrow 0} f_{XZ} (r) = - M + \frac{r^2}{\ell^2} - 2q^2 \ln r 
\]
which is the charged BTZ in its original form. To complete our discussion we also give the limit of the potential when \( \beta \rightarrow 0 \). This can be found as
\[
\lim_{\beta \rightarrow 0} V (\psi) = - \frac{1}{\ell^2} + \left( \frac{1}{\ell^2} - \frac{M}{3r_0^2} \right) \psi^6 - \frac{2q^2}{3r_0^2} \ln \left( \frac{r_0 (1 - \psi^2)}{\psi^2} \right) - \frac{q^2 \psi^6 (2 \psi^4 + 2q^2 - 7)}{9r_0^2 (1 - \psi^2)^2}. 
\]
We must add that due to the modification made in the action (1) our results are much simpler than those given in [2] but still with a redefinition of the parameters and by rescaling the scalar field one recovers the forms found in [3]. To complete this section we give the form of Ricci scalar in terms of the new parameters:
\[
R = \Pi r_0 + \Xi 
\]
in which
\[
\Pi = \frac{4 (2q\beta (q\beta + r) + r^2)}{(q\beta + r)^2 \beta^2 \chi} - \frac{4 (r + 2q\beta) \chi \varpi}{\beta^3 q (q\beta + \chi)^2} - \frac{4 q (2\chi + r) \varpi}{\beta (r + \chi) (q\beta + \chi)^2} + \frac{8 \beta q^3}{B (r + \chi) (q\beta + \chi)^2}. 
\]
and

\[ \Xi = \frac{4q^2 \left[ 4q^3 \beta^3 - 2(\chi - 2r) rq \beta + (r^2 + 2q^2 \beta^2) (2\chi - r) \right]}{6 (2q \beta (q\beta + r)^2 + \beta q \beta + \chi^2)} - \frac{12q^2 \beta^3}{(q\beta + \chi)^2 r^2} \]

in which \( \chi = \sqrt{q^2 \beta^2 + r^2} \) and \( \omega = \ln \left( \frac{q\beta + \sqrt{q^2 \beta^2 + r^2}}{r} \right) \).

One easily observes that for \( \lim_{r \to 0} R = \Xi \), but to see the structure of the singularity we expand \( R \) about \( r = 0 \) which gives

\[ R = \frac{8q}{3r} + \frac{4r_0}{q^3} \ln r^+ \]

\[ \frac{4r_0}{q^3} (1 - \ln (2q\beta)) - 6 \left( \frac{2}{\beta^2} + \frac{1}{\ell^2} \right) + \mathcal{O} (r) \].

This shows that the singularity is of the order \( \frac{1}{r} \) which, apart from the logarithmic term is weaker than the Einstein-Maxwell-Scalar solution [5], which was of the order \( \frac{1}{r^2} \).

### B. Horizon(s) and Hawking temperature

![Graph showing metric function f(r) versus r for different masses for the black hole. As it is clear, the mass of the central object must be bigger than a certain mass to have black hole solution. Note that this is valid for non-zero cosmological constant i.e., \( \frac{1}{\ell^2} \neq 0 \).](image)

The general solution given in (12), depends on the free parameters and non-zero cosmological constant. It admits one single horizon if \( M_c \leq M \), two horizons if \( M_d < M < M_c \), one degenerate horizon if \( M = M_d \) and no horizon if \( M < M_d \). We comment that, \( M_c \) is found analytically and is expressed as

\[ M_c = 2q^2 \ln \left( 1 + \frac{2}{q\beta} \right) + \frac{q^2}{1 + \beta^2} \]  \hspace{1cm} (33)

while \( M_d \) should be found numerically for each set of parameters. In Fig. 1 and 2 we plot the metric function \( f(r) \) versus \( r \) and the self-coupling potential \( U(\psi) \) versus \( \psi \) to show the effect of mass in forming different cases. We observe that to have a black hole we must have a minimum mass and to have two absolute minimum points for the potential the solution must be a black hole which means that \( M > M_d \). For the case in which the event horizon is present, the Hawking temperature may be determined in terms of the radius of the event horizon \( r_h \). The explicit form of it is expressed as

\[ T_H = \frac{f^\prime (r_h)}{4\pi} = \]

\[ \frac{r_h \left( q\beta + \frac{r_h^2}{q\beta + \eta} \right) (r_0 + r_h) - q (r_0 + 3r_h) (r_0 + r_h)}{\eta (2r_0 + 3r_h) \pi} \left( \frac{r_0 \ln \left( \frac{q\beta + r_h}{r_0} \right) + \frac{3}{2\ell^2}}{\beta^2 (2r_0 + 3r_h) \pi r_h} + \frac{3r_h^2 (r_h - \eta) + (r_0 - 3q\beta) (r_0 + r_h)}{\beta^2 (2r_0 + 3r_h) \pi r_h (q\beta + \eta)} \right), \]  \hspace{1cm} (34)

in which \( \eta = \sqrt{q^2 \beta^2 + r_h^2} \). Fig. 3 displays the effect of \( r_0 \) in \( T_H \).

### III. CONCLUSION

A field theory model of Einstein-Scalar-Born-Infeld is considered in \( 2 + 1 \)-dimensions. Depending on the parameters this naturally admits black hole and non-black hole solutions. This is depicted in Fig. 1 numerically, in which the mass plays a crucial role. The scalar hair dependence of both the self-interacting potential \( U(\psi) \)
and the Hawking temperature are also displayed. The self-interacting potential $U$ (ψ) is highly non-linear with reflection symmetry $U(ψ) = U(−ψ)$. When $U(ψ)$ admits no minima it asymptotes to an infinite potential well of quantum mechanics. With the proper choice of parameters the minima are produced as displayed in Fig. 2. In the limit of (BI parameter) $β → 0$ our results reduce mainly, up to minor scaling, to the ones obtained in Ref. [3]. Our contribution therefore is to extend the hairy black holes of linear-Maxwell theory to nonlinear BI theory in the presence of a self-interacting scalar field. We must admit that exact solutions were obtained at the price of tuning the integration constants. Without such choices finding solution for such a non-linear model field theory remains out of our reach. Finally we add that with the BI addition it is observed from Eq. (32) that the singularity at $r = 0$ modifies from $1/r$ of linear Maxwell theory [5] to the form $1/r + ln r$.

### Appendix: A

The field equations explicitly become

$$G_t^t − τ_t^t − T_t^t + V = 0, \tag{A.1}$$

$$G_r^r − τ_r^r − T_r^r + V = 0, \tag{A.2}$$

$$G_θ^θ − τ_θ^θ − T_θ^θ + V = 0 \tag{A.3}$$

and

$$\frac{1}{r} [fψ' + rf'ψ' + rfψ''] − \frac{1}{8} Rψ − \frac{1}{8} \frac{dV}{dψ} = 0. \tag{A.4}$$

Herein

$$G_t^t = G_r^r = \frac{f'}{2r}, \quad G_θ^θ = \frac{1}{2} f'' , \tag{A.5}$$

$$τ_t^t = \frac{1}{2r} (−4r fψ'' + 4rfψψ'' + 2ψψ' (2f + rf')) + f'ψ^2 \tag{A.6}$$

$$τ_r^r = \frac{(f'ψ + 4fψ')(2rψ' + ψ)}{2r} \tag{A.7}$$

$$τ_θ^θ = −2fψ'^2 + 2f'ψψ' + 2fψψ'' + \frac{1}{2} f''ψ^2 \tag{A.8}$$

$$T_t^t = T_r^r = 2β \frac{(−r^2β^2 + βr√q^2 + r^2β^2 - q^2)}{r√q^2 + r^2β^2} \tag{A.10}$$

and

$$T_θ^θ = \frac{2β^2 (−βr + √q^2 + r^2β^2)}{√q^2 + r^2β^2}. \tag{A.11}$$

After some simplification the field equation can be written as

$$\frac{1}{2r} (−4r fψ'' + 4rfψ'^2 - 2ψψ' (2f + rf')) + f' − T_t^t + V = 0 \tag{A.12}$$

$$\frac{f'}{2r} − \frac{(2rψ' + ψ)(4fψ' + f'ψ)}{2r} − T_r^r + V = 0 \tag{A.13}$$

$$\frac{1}{2} f'' + 2fψ'^2 − 2f'ψψ' + 2fψψ'' - \frac{1}{2} f''ψ^2 − T_θ^θ + V = 0 \tag{A.14}$$

and Eq. (A.4). Next, we subtract (A.13) from (A.12) which simply gives

$$−ψψ'' + 3ψ'^2 = 0. \tag{A.15}$$

This equation admits a solution of the form

$$ψ^2 = \frac{1}{c_1 r + c_2} \tag{A.16}$$

in which $c_1$ and $c_2$ are two integration constants. Hence by redefinition of constants one may write

$$ψ = \frac{μ}{√1 + \frac{r}{r_0}} \tag{A.17}$$
in which $\mu$ and $r_0 > 0$ are two new constants both nonzero. Upon finding $\psi$, one may subtract (A.12) from (A.14) to find a differential equation for only $f(r)$ i.e.

$$r \left(1 - \psi'^2\right) f'' + \left[\psi^2 - 2r\psi\psi' - 1\right] f' + 4f\psi\psi' + 2r \left(T^f_t - T^f_\phi\right) = 0,$$  \hspace{1cm} (A.18)

or explicitly

$$\frac{1}{2} \left(r + r_0\right) \left(r - r_0 \left[\mu^2 - 1\right]\right) r f'' + \left(\mu^2 - 1\right) \left(r + \frac{1}{2} r_0\right) - \frac{r^2}{2} f' - \mu^2 f + r \left(r + r_0\right) \left(T^f_t - T^f_\phi\right) = 0.$$  \hspace{1cm} (A.19)

In this DE there exist four parameters, $\beta, q, \mu$ and $r_0$. The complete solution to this equation is complicated in general but by setting $\mu = 1$ a special solution interesting enough is given by Eq. (12) in the text which includes two new integration constants that are shown by $\ell^2$ and $M$. Up to here, without specifying the form of the potential $V(\psi)$ we found $\psi$ and $f$. It is clear that for an arbitrary potential the rest of the field equations are not in general satisfied. In other words this potential is not independent of the spacetime geometry as well as of the form of the scalar field. This can be considered as a mutual relation between the potential $V(\psi)$ and the final form of the spacetime solution.

Finally one may use one of the Eqs. (A.12)-(A.14) to find the exact form of the potential $V(\psi)$. We comment again that this potential is caused by the spacetime and vice versa. The consistency of the metric function, scalar field and potential can be seen when they satisfy perfectly the last equation (A.4). This is the final test of the correctness of the theory. Once more we add that $V(\psi)$ is not an arbitrary function in the action and this is well known from the outset while finding its exact form is part of the overall problem.

[1] M. Bañados, C. Teitelboim, J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992);
M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48, 1506 (1993).
[2] C. Martín, C. Teitelboim and J. Zanelli, Phys. Rev. D 61, 104013 (2000).
[3] E. W. Mielke and A. A. R. Maggiolo, Phys. Rev. D 68, 104026 (2003).
[4] C. Martínez, R. Troncoso and J. Zanelli, Phys. Rev. D 67, 024008 (2003);
H-J Schmidt and D. Singleton, Phys. Lett. B 721, 294 (2013);
S. Carlip, Class. Quant. Grav. 12, 2853 (1995).
[5] W. Xu and L. Zhao, Phys. Rev. D 87, 124008 (2013).
[6] M. Henneaux, C. Martínez, R. Troncoso, and J. Zanelli, Phys. Rev. D 65, 104007 (2002).
[7] M. Hasanpour, F. Loran and H. Razaghi, Nucl. Phys. B 687, 483 (2003).
[8] K. C. K. Chan and R. B. Mann, Phys. Lett. B 371, 199 (1996);
P. M. Sá and J. P. S. Lemos, Phys. Lett. B 423, 49 (1998);
M. Natsunome and T. Okamura, Phys. Rev. D 62, 064027 (2000);
F. Correa, A. Fuñández and C. Martínez, Phys. Rev. D 87, 027502 (2013);
M. Hortacsu, H. T. Özçelik, and B. Yapsisman, Gen. Rel. and Grav. 35, 1209 (2003);
J. Naji, Eur. Phys. J. C 74, 2697 (2014);
L. Zhao, W. Xu and B. Zhu, Commun. Theor. Phys. 61, 475 (2014).
[9] O. J. C. Dias and J. P. S. Lemos, Phys. Rev. D 64, 064001 (2001).
[10] M. Born and L. Infeld, Foundations of the New Field Theory. Proc. Roy. Soc, A 144, 425 (1934); E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B 163, 123 (1985);
A. Abouelsaood, C. Callan, C. Nappi, and S. Yost, Nucl. Phys. B 280, 599 (1987);
R. G. Leigh, Mod. Phys. Lett. A 4, 2767 (1989);
R. R. Metsaev, M. A. Rahmanov, and A. A. Tseytlin, Phys. Lett. B 193, 207 (1987);
A. A. Tseytlin, Nucl. Phys. B 501, 41 (1997).
[11] J. Bardeen, Proceedings of GR5, Tiflis, U.S.S.R. (1968);
A. Borde, Phys. Rev. D 50, 3692 (1994);
A. Borde, Phys. Rev. D 55, 7615 (1997);
E. Ayon-Beato and A. Garcia, Phys. Rev. Lett. 80, 5056 (1998);
K. A. Bronnikov, Phys. Rev. Lett. 85, 4641 (2000);
K. A. Bronnikov, Phys. Rev. D 63, 044005 (2001);
K. A. Bronnikov, V. N. Melnikov, G. N. Shikin and K. P. Staniukovich, Ann. Phys. (USA) 118, 84 (1979);
S. A. Hayward, Phys. Rev. Lett. 96, 031103 (2006);
M. Cataldo and A. Garcia, Phys. Rev. D 61, 084003 (2000);
S. H. Mazharimousavi, M. Halilsoy, Eur. Phys. J. C 73, 2527 (2013);
S. H. Mazharimousavi, M. Halilsoy and T. Tahamat, Phys. Lett. A 376, 893 (2012).
[12] M. Cataldo, A. Garcia, Phys. Lett. B 28, 456 (1999).
[13] S. H. Hendi, JHEP 03, 065 (2012).
[14] R. Yamazaki and D. Ida, Phys. Rev. D 64, 024009 (2001).