Generalized Ulam-Hyers Stability of Complex Additive Functional Equation

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Abstract. The object of the present paper is to assess speculation of the Hyers-Ulam stability theorem for the complex additive functional equation on abelian groups and stability results have been gotten by a fixed point technique. This technique demonstrates that the stability is identified with some fixed point of an appropriate operator.

1. Introduction
An intriguing and renowned speak conferred by Stanislaw M.Ulam [27] in 1940 triggering the study of stability issues for varied purposeful equations. In his speak, Ulam mentioned variety of necessary unresolved mathematical issues. Among them, a matter regarding the stability of group homomorphisms appeared too abstract for anyone to achieve any conclusion.

"Let \( G \) be group and \( H \) be a metric group with metric \( d(.,.) \). Given \( \epsilon > 0 \) does there exists a \( \delta > 0 \) such that if a function \( f : G \rightarrow H \) satisfies

\[ d(f(xy), f(x)f(y)) < \delta \]

for all \( x, y \in G \), then there exists a homomorphism \( \alpha : G \rightarrow H \) with

\[ d(f(x), \alpha(x)) < \epsilon \]

for all \( x \in G \)."

In the following year, Donald H.Hyers [14] was ready to provides a partial resolution to Ulam’s question that was the primary important breakthrough and step toward a lot of solutions during this space. Since then, an oversized range of papers are printed in reference to varied generalizations of Ulam’s problem and Hyers’s theorem. In 1978, Themistocles M.Rassias [26] succeeded in extending the results of Hyers’s theorem and his exciting result attracted variety of mathematicians who began to be stirred up to research the steadiness issues of many purposeful equations. because of influence of S.M. Ulam, D.H. Hyers within the work of Th.M. Rassias concerning the study of stability issues of purposeful equations, the steadiness development verified by Th.M. Rassias is termed because the Hyers-Ulam-Rassias stability henceforth. For the last thirty five years several results regarding the Hyers-Ulam-Rassias stability of assorted purposeful equations are extensively investigated by variety of authors and there area unit several fascinating results regarding this problem (see [1, 6, 8, 9, 12, 13, 15, 16, 22, 23]).
The general solution and the permanence of the following additive FE

\[ f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y) \]  

(1)

\[ \sum_{i=1}^{m} f(mx_i + \sum_{j=1, j\neq i}^{m} x_j) + f(\sum_{i=1}^{m} x_i) = 2f(\sum_{i=1}^{m} mx_i) \]  

(2)

\[ f\left(\sum_{k=1}^{n} kx_k\right) + \sum_{k=1}^{n} f\left(-kx_k + \sum_{l=1, l\neq k}^{n} lx_l\right) + f\left(x_1 - \sum_{k=2}^{n} kx_k\right) = nf(x_1) + (n-2)\sum_{k=2}^{n} kf(x_k) \]  

(3)

\[ f\left(\sum_{k=1}^{n} kx_k\right) + \sum_{l=2}^{n} f\left(\sum_{k=1, k\neq l}^{n} kx_k - lx_l\right) + f\left(x_1 - \sum_{k=2}^{n} kx_k\right) = (n+1)f(x_1) + (n-3)\sum_{k=2}^{n} kf(x_k) \]  

(4)

\[ f\left(\sum_{p=1}^{n} px_p\right) + f\left(\sum_{p=1}^{n} p(-1)^{p+1} x_p\right) + f\left(x_1 - \sum_{p=2}^{n} px_p\right) = 3f(x_1) + \sum_{p=2}^{n} p(-1)^{p+1} f(x_p) \]  

(5)

were discussed in (see [7, 19, 3, 4, 5]).

In this paper, the authors are to discover the solution and establish Hyers-Ulam stability concerning the complex additive FE

\[ (\lambda^n + 1) f(\frac{e^{in\vartheta} + e^{-in\varphi}}{2}) + (\lambda^n - 1) f(\frac{e^{-in\vartheta} + e^{in\varphi}}{2}) + e^{in\omega} f(\vartheta - \omega) + e^{-in\omega} f(\omega - \vartheta) = 2\lambda^n \cos\left(\frac{n\pi}{3}\right)[f(\vartheta) + f(\omega)] \]  

(6)

with \( n \in \mathbb{Z} \) on abelian group using two different approaches.

2. General Solution

We derive the general solution of FE (6) as follows:

**Lemma 2.1.** Let \( G \) be a group and \( Y \) be a real vector space. Then the function \( f : G \to Y \) satisfies the additive FE

\[ f(x + y) = f(x) + f(y) \]  

(7)

for all \( x, y \in G \) if \( f : G \to Y \) satisfies the FE (6) for all \( \vartheta, \omega \in G \).

**Proof.** Let \( f \) satisfies FE (7). Insert \( x = y = 0 \) in (7), we arrive \( f(0) = 0 \). Put \( x = -y \) in (7), we attain \( f(-y) = -f(y) \) for all \( y \in G \). Henceforth \( f \) is an odd function. By changing \( y \) by \( x \) and \( y \) by \( 2x \) in (7), we arrive

\[ f(2x) = 2f(x) \quad \text{and} \quad f(3x) = 3f(x) \]  

(8)

for all \( \vartheta \in G \). Moreover, for any \( a \), we have

\[ f(ax) = af(x) \]  

(9)

By changing \((x, y)\) by \((e^{in\vartheta}, e^{-in\varphi})\) in (7) and using (9), we get

\[ f\left(e^{in\vartheta} + e^{-in\varphi}\right) = e^{in\vartheta} f(\vartheta) + e^{-in\varphi} f(\omega) \]  

(10)
for all $\vartheta, \omega \in G$. Both side multiply by $(\lambda^n + 1)$ in (10), we arrive
\[
(\lambda^n + 1) f \left( e^{\frac{in\pi}{3}} \vartheta + e^{\frac{in\pi}{3}} \omega \right) = (\lambda^n + 1) e^{\frac{in\pi}{3}} f(\vartheta) + (\lambda^n + 1) e^{\frac{in\pi}{3}} f(\omega)
\] (11)
Replacing $(x,y)$ by $\left( e^{-\frac{in\pi}{3}} \vartheta, e^{\frac{in\pi}{3}} \omega \right)$ in (7) and using (9), we get
\[
f \left( e^{-\frac{in\pi}{3}} \vartheta + e^{\frac{in\pi}{3}} \omega \right) = e^{-\frac{in\pi}{3}} f(\vartheta) + e^{\frac{in\pi}{3}} f(\omega)
\] (12)
for all $\vartheta, \omega \in G$. Both side multiply by $(\lambda^n - 1)$ in (12), we arrive
\[
(\lambda^n - 1) f \left( e^{-\frac{in\pi}{3}} \vartheta + e^{\frac{in\pi}{3}} \omega \right) = (\lambda^n - 1) e^{-\frac{in\pi}{3}} f(\vartheta) + (\lambda^n - 1) e^{\frac{in\pi}{3}} f(\omega)
\] (13)
for all $\vartheta, \omega \in G$. Setting $(x,y)$ by $(\vartheta, \omega)$ in (7), and utilizing the property of odd $f$ we have
\[
f(\vartheta - \omega) = f(\vartheta) - f(\omega)
\] (14)
for all $\vartheta, \omega \in G$. Both side multiply by $e^{-\frac{in\pi}{3}}$ in (14), we arrive
\[
e^{-\frac{in\pi}{3}} f(\vartheta - \omega) = e^{-\frac{in\pi}{3}} f(\vartheta) - e^{-\frac{in\pi}{3}} f(\omega)
\] (15)
for all $\vartheta, \omega \in G$. Setting $(\vartheta, \omega)$ by $(\omega, -\vartheta)$ in (7), and utilizing the property of odd $f$ we have
\[
f(\omega - \vartheta) = f(\omega) - f(\vartheta)
\] (16)
for all $\vartheta, \omega \in G$. Both side multiply by $e^{\frac{in\pi}{3}}$ in (16), we arrive
\[
e^{\frac{in\pi}{3}} f(\omega - \vartheta) = e^{\frac{in\pi}{3}} f(\omega) - e^{\frac{in\pi}{3}} f(\vartheta)
\] (17)
for all $\vartheta, \omega \in G$. Adding (11), (13), (15) and (17), we arrive (6).

3. Stability Results: Direct Method

The mapping is defined as follows: $Df : G \to Y$ by
\[
Df(\vartheta, \omega) = (\lambda^n + 1) f \left( e^{\frac{in\pi}{3}} \vartheta + e^{\frac{in\pi}{3}} \omega \right) + (\lambda^n - 1) f \left( e^{-\frac{in\pi}{3}} \vartheta + e^{\frac{in\pi}{3}} \omega \right)
\]
\[
+ e^{-\frac{in\pi}{3}} f(\vartheta - \omega) + e^{\frac{in\pi}{3}} f(\omega - \vartheta) - 2\lambda^n \cos \left( \frac{n\pi}{3} \right) \left[ f(\vartheta) + f(\omega) \right]
\]
with $n \in \mathbb{Z}$.

**Theorem 3.2.** For $j = \pm 1$ and $M : G \times G \to [0, \infty)$ be a function such that
\[
\lim_{n \to \infty} M(P^{nj}\vartheta, P^{nj}\vartheta) = 0
\] (18)
for all $\vartheta, \omega \in G$. Let $f : G \to Y$ be an odd mapping satisfying
\[
\|Df(\vartheta, \omega)\| \leq M(\vartheta, \omega)
\] (19)
for all $\vartheta, \omega \in G$. Then exists a unique additive mapping $A : G \to Y$ and satisfying FE (6) for
\[
\|f(\vartheta) - A(\vartheta)\| \leq \frac{1}{2\lambda^n P} \sum_{k = \frac{1-j}{2}}^{\infty} \frac{M(P^{kJ}\vartheta, P^{kJ}\vartheta)}{P^{kJ}}
\] (20)
for all $\vartheta \in G$. The mapping $A(\vartheta)$ is defined by
\[
A(\vartheta) = \lim_{n \to \infty} \frac{f(P^{nj}\vartheta)}{P^{nj}}
\] (21)
for all $\vartheta \in G$.  

Proof. Consider \( j = 1 \), by changing \((\vartheta, \omega)\) by \((\vartheta, \vartheta)\) in (19), we get
\[
\|2\lambda^n f \left[ e^{\frac{in\pi}{3}} + e^{-\frac{in\pi}{3}} \right] \vartheta - 4\lambda^n \cos \left( \frac{n\pi}{3} \right) f(\vartheta) \| \leq M(\vartheta, \vartheta) \tag{22}
\]
for all \( \vartheta \in G \). The above inequality can be written as
\[
\|4\lambda^n \cos \left( \frac{n\pi}{3} \right) f(\vartheta) - 2\lambda^n f \left[ 2\cos \left( \frac{n\pi}{3} \right) \right] \vartheta \| \leq M(\vartheta, \vartheta) \tag{23}
\]
for all \( \vartheta \in G \). Bothside divide by \( 4\lambda^n \cos \left( \frac{n\pi}{3} \right) \) in (23), we have
\[
\left\| f(\vartheta) - \frac{f(2\cos \left( \frac{n\pi}{3} \right) \vartheta)}{2\cos \left( \frac{n\pi}{3} \right)} \right\| \leq \frac{M(\vartheta, \vartheta)}{2.2\lambda^n \cos \left( \frac{n\pi}{3} \right)} \tag{24}
\]
for all \( \vartheta \in G \). Consider \( \mathcal{P} = 2\cos \left( \frac{n\pi}{3} \right) \), Now replacing \( \vartheta \) by \( \mathcal{P} \vartheta \) and dividing by \( \mathcal{P} \) in (24), we get
\[
\left\| \frac{f(\mathcal{P}\vartheta)}{\mathcal{P}} - \frac{f(\mathcal{P}^2\vartheta)}{\mathcal{P}^2} \right\| \leq \frac{M(\mathcal{P}\vartheta, \mathcal{P}\vartheta)}{2\lambda^n \mathcal{P}^2} \tag{25}
\]
for all \( \vartheta \in G \). From (24) and (25), we obtain
\[
\left\| f(\vartheta) - \frac{f(\mathcal{P}^n\vartheta)}{\mathcal{P}^n} \right\| \leq \left\| f(\vartheta) - \frac{f(\mathcal{P}\vartheta)}{\mathcal{P}} \right\| + \left\| \frac{f(\mathcal{P}\vartheta)}{\mathcal{P}} - \frac{f(\mathcal{P}^2\vartheta)}{\mathcal{P}^2} \right\|
\leq \frac{1}{2\lambda^n \mathcal{P}} \left[ M(\vartheta, \vartheta) + \frac{M(\mathcal{P}\vartheta, \mathcal{P}\vartheta)}{\mathcal{P}} \right] \tag{26}
\]
for all \( \vartheta \in G \). In any case for \( n \), we get
\[
\left\| f(\vartheta) - \frac{f(\mathcal{P}^n\vartheta)}{\mathcal{P}^n} \right\| \leq \frac{1}{2\lambda^n \mathcal{P}} \sum_{k=0}^{n-1} \frac{M(\mathcal{P}^k\vartheta, \mathcal{P}^k\vartheta)}{\mathcal{P}^k} \tag{27}
\]
for all \( \vartheta \in G \). To demonstrate the convergence of the sequence
\[
\left\{ \frac{f(\mathcal{P}^n\vartheta)}{\mathcal{P}^n} \right\},
\]
By changing \( x \) by \( \mathcal{P}^m x \) and divide by \( \mathcal{P}^m \) in (27), for any \( m, n > 0 \), we attain
\[
\left\| \frac{f(\mathcal{P}^m\vartheta)}{\mathcal{P}^m} - \frac{f(\mathcal{P}^{n+m}\vartheta)}{\mathcal{P}^{n+m}} \right\| = \frac{1}{\mathcal{P}^m} \left\| f(\mathcal{P}^m\vartheta) - \frac{f(\mathcal{P}^{n+m}\vartheta)}{\mathcal{P}^{n+m}} \right\|
\leq \frac{1}{2\lambda^n \mathcal{P}} \sum_{k=0}^{n-1} \frac{M(\mathcal{P}^{k+m}\vartheta, \mathcal{P}^{k+m}\vartheta)}{\mathcal{P}^{k+m}}
\leq \frac{1}{2\lambda^n \mathcal{P}} \sum_{k=0}^{\infty} \frac{M(\mathcal{P}^{k+m}\vartheta, \mathcal{P}^{k+m}\vartheta)}{\mathcal{P}^{k+m}}
\to 0 \text{ as } m \to \infty
for all $\vartheta \in G$. Therefore the sequence $\left\{ \frac{f(P^n \vartheta)}{P^n} \right\}$ is Cauchy. The completeness of $Y$ allow us to consider that, \( \exists \) a map $A : G \to Y$ such that

$$A(\vartheta) = \lim_{n \to \infty} \frac{f(P^n \vartheta)}{P^n} \quad \forall \ \vartheta \in G.$$  

Assuming $n \to \infty$ in (27) one can see that (20) holds for all $\vartheta \in G$. To prove that $A$ satisfies (6), replace $(\vartheta, \omega)$ by $(P^n \vartheta, P^n \omega)$ and divide by $P^n$ in (19), we obtain

$$\frac{1}{P^n} \left\| Df (P^n \vartheta, P^n \omega) \right\| \leq \frac{1}{P^n} M(P^n \vartheta, P^n \omega)$$

for all $\vartheta, \omega \in G$. Letting $n \to \infty$ in the above inequality and using the definition of $A(\vartheta)$, we see that

$$(\lambda^n + 1) A \left( e^{in \frac{\vartheta}{s}} + e^{-in \frac{\vartheta}{s}} \right) + (\lambda^n - 1) A \left( e^{-in \frac{\vartheta}{s}} + e^{in \frac{\vartheta}{s}} \right)$$

$$+ e^{-in \frac{\vartheta}{s}} A(\vartheta - \omega) + e^{in \frac{\vartheta}{s}} A(\omega - \vartheta) = 2\lambda^n \cos \left( \frac{n\pi}{3} \right) [A(\vartheta) + A(\omega)]$$

Therefore $A$ fulfill FE (6) for all $\vartheta, \omega \in G$. To explore $A$ is unique, we let $B(\vartheta)$ be an additional mapping satisfying (6) and (20), then

$$\| A(\vartheta) - B(\vartheta) \| = \frac{1}{P^n} \| A(P^n \vartheta) - B(P^n \vartheta) \|$$

$$\leq \frac{1}{P^n} \left\{ \| A(P^n \vartheta) - f(P^n \vartheta) \| + \| f(P^n \vartheta) - B(P^n \vartheta) \| \right\}$$

$$\leq \frac{1}{\lambda^n P} \sum_{k=0}^{\infty} \frac{M(P^{k+n} \vartheta, P^{k+n} \vartheta)}{P^{k+n}}$$

$$\to 0 \quad \text{as} \quad n \to \infty$$

for all $\vartheta \in G$. Hence $A$ is unique. \( \Box \)

**Corollary 3.3.** Let $\Pi$ and $s$ be non-negative real numbers. Let a function $f : G \to Y$ satisfies the inequality

$$\| Df (\vartheta, \omega) \| \leq \begin{cases} 
\Pi, & s \neq 1; \\
\Pi \left\| \| \vartheta \|^{s} + \| \omega \|^{s} \right\|, & 2s \neq 1; \\
\Pi \left\{ \| \vartheta \|^{s} \| \omega \|^{s} + \left\{ \| \vartheta \|^{2s} + \| \omega \|^{2s} \right\} \right\}, & 2s \neq 1;
\end{cases} \quad (28)$$

for all $\vartheta, \omega \in G$. Then there exists a unique additive function $A : G \to Y$ such that

$$\| f(\vartheta) - A(\vartheta) \| \leq \begin{cases} 
\Pi, & 2\lambda^n \| \vartheta \|^{s} \\
\Pi \left\| \| \vartheta \|^{s} \right\|, & \lambda^n \| \vartheta \| \neq 1; \\
\Pi \left\{ \| \vartheta \|^{s} \| \omega \|^{s} + \left\{ \| \vartheta \|^{2s} + \| \omega \|^{2s} \right\} \right\}, & 3 \Pi \| \vartheta \|^{2s}; \\
2\lambda^n \| \vartheta \| - \| \vartheta \|^{2s}, & 2\lambda^n \| \vartheta \| - \| \vartheta \|^{2s}; \\
2\lambda^n \| \vartheta \| - \| \vartheta \|^{2s}, & 2\lambda^n \| \vartheta \| - \| \vartheta \|^{2s}; \\
2\lambda^n \| \vartheta \| - \| \vartheta \|^{2s}, & 2\lambda^n \| \vartheta \| - \| \vartheta \|^{2s}; \\
\end{cases} \quad (29)$$

for all $\vartheta \in G$.  

4. Stability Results: Fixed Point Method

Let us consider $G$ be a group and $B$ a Banach space, respectively.

**Theorem 4.4.** ([18](The alternative of fixed point)) Suppose that for a complete generalized metric space $(X,d)$ and a strictly contractive mapping $T : X \rightarrow X$ with Lipschitz constant $L$. Then, for each given element $x \in X$, either

$$(B_1) \quad d(T^n x, T^{n+1} x) = \infty \quad \forall \quad n \geq 0,$$

or

$$(B_2) \quad$$

(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$ ;

(ii) The sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$;

(iii) $y^*$ is the unique fixed point of $T$ in the set $Y = \{ y \in X : d(T^n x, y) < \infty \}$;

(iv) $d(y^*, y) \leq \frac{1}{1 - L} d(y, Ty)$ for all $y \in Y$.

**Theorem 4.5.** Let $f : G \rightarrow B$ be a mapping for which there exists functions $\mathcal{M}, \gamma : G \times G \rightarrow [0, \infty)$ with the condition

$$\lim_{t \rightarrow \infty} \frac{\mathcal{M}(\mu^i t \vartheta, \mu^i \omega)}{\mu^i_t} = 0,$$

where

$$\mu^i = \begin{cases} \mathcal{P}, & i = 0, \\ \frac{1}{\mathcal{P}}, & i = 1 \end{cases}$$

satisfying the functional inequality

$$\|Df(\vartheta, \omega)\| \leq \mathcal{M}(\vartheta, \omega)$$

for all $\vartheta, \omega \in G$. If $\exists$ an $L(i) < 1 \exists$ the function

$$\vartheta \rightarrow \gamma(\vartheta) = \frac{1}{2 \lambda^a} \mathcal{M}\left( \frac{\vartheta}{\mathcal{P}} \right),$$

holds the property

$$\gamma(\vartheta) = L \mu^i \gamma\left( \frac{\vartheta}{\mu^i} \right)$$

$\forall \ \vartheta \in G$. Then $\exists$ a unique additive function $A : G \rightarrow B$ satisfying the FE (6) and

$$\|f(\vartheta) - A(\vartheta)\| \leq \frac{L^{1-i}}{1-L} \gamma(\vartheta)$$

holds for all $\vartheta \in G$.

**Proof.** Let us assume $X = \{ p/p : V \rightarrow B, \ p(0) = 0 \}$ and present the generalized metric on $X$,

$$d(p,q) = \inf\{ K \in (0, \infty) : \| p(\vartheta) - q(\vartheta) \| \leq K \gamma(\vartheta), x \in G \}.$$

It is natural to perceive that $(X,d)$ is complete.

The mapping $T : X \rightarrow X$ by

$$Tp(\vartheta) = \frac{1}{\mu^i} p(\mu_i \vartheta), \forall \quad x \in G.$$
For \( p, q \in X \),
\[
  d(p, q) \leq K \Rightarrow \| p(\vartheta) - q(\vartheta) \| \leq K\gamma(\vartheta), \vartheta \in G.
\]
\[
  \Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i \vartheta) - \frac{1}{\mu_i} q(\mu_i \vartheta) \right\| \leq \frac{1}{\mu_i} K\gamma(\mu_i \vartheta), \vartheta \in G,
\]
\[
  \Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i \vartheta) - \frac{1}{\mu_i} q(\mu_i \vartheta) \right\| \leq LK\gamma(\vartheta), \vartheta \in G,
\]
\[
  \Rightarrow \| Tp(\vartheta) - Tq(\vartheta) \| \leq LK\gamma(\vartheta), \vartheta \in G,
\]
\[
  \Rightarrow d(Tp, Tq) \leq LK.
\]

Thus derive
\[
d(Tp, Tq) \leq Ld(p, q),
\]
\( \forall p, q \in X. \)

From (24), we have
\[
  \left\| \frac{1}{P} f(\vartheta) - f(P \vartheta) \right\| \leq \frac{M(\vartheta, \vartheta)}{2\lambda^n P}
\]
for all \( \vartheta \in G \). Using (32) for the case \( i = 0 \), it reduces to
\[
  \left\| \frac{1}{P} f(\vartheta) - f(\vartheta) \right\| \leq \frac{1}{P} \gamma(\vartheta)
\]
for all \( \vartheta \in G \).

i.e., \( d(Tf, f) \leq \frac{1}{P} = L = L^{1-0} = L^{1-i} < \infty \).

By varying again \( \vartheta = \frac{\vartheta}{P} \) in (34), we get
\[
  \left\| f(\vartheta) - P f\left( \frac{\vartheta}{P} \right) \right\| \leq \frac{1}{2\lambda^n} M\left( \frac{\vartheta}{P} \right)
\]
for all \( \vartheta \in G \). Using (32) when \( i = 1 \), it simplifies as
\[
  \left\| f(\vartheta) - P f\left( \frac{\vartheta}{P} \right) \right\| \leq \gamma(\vartheta)
\]
for all \( \vartheta \in G \).

i.e., \( d(f, Tf) \leq 1 = L^0 = L^{1-1} = L^{1-i} < \infty \).

From the above, it appears
\[
d(f, Tf) \leq L^{1-i}.
\]

\( (B_2(i)) \) holds.

By \( (B_2(ii)) \), it pursues that \( \exists \) a fixed point \( A \) of \( T \) in \( X \) \( \ni \)
\[
  A(\vartheta) = \lim_{t \to \infty} \frac{f(\mu_t^i \vartheta)}{\mu_t^i}, \quad \forall \ \vartheta \in G.
\]

Claim that \( A : G \to B \) is additive. To substitute \((\vartheta, \omega)\) by \((\mu_t^i \vartheta, \mu_t^i \omega)\) in (31) and dividing by \( \mu_t^i \), it follows from (30) and (35), \( A \) satisfies (6) for all \( \vartheta, \omega \in G \).
By \((B_2(iii))\), \(A\) is the unparallel fixed point of \(T\) in the set \(Y = \{f \in X : d(Tf, A) < \infty\}\), adopting the alternative fixed point theorem \(A\) is the unique function \(\ni\)
\[
\|f(\vartheta) - A(\vartheta)\| \leq K\gamma(\vartheta)
\]
for all \(\vartheta \in G\) and \(K > 0\). Lastly by \((B_2(iv))\), we acquire that
\[
d(f, A) \leq \frac{1}{1 - L}d(f, Tf)
\]
implying
\[
d(f, A) \leq \frac{L^{1-i}}{1 - L}.
\]
Therefore we infer that
\[
\|f(\vartheta) - A(\vartheta)\| \leq \frac{L^{1-i}}{1 - L}\gamma(\vartheta).
\]
for all \(\vartheta \in G\).

Corollary 4.6. The function \(f : G \to B\) be a mapping and \(\exists\) real numbers \(\Pi\) and \(s \ni\)
\[
\|Df(\vartheta, \omega)\| \leq \left\{\begin{array}{ll}
\Pi, \\
\Pi \{||\vartheta||^s + ||\omega||^s\}, \\
\Pi||\vartheta||^s||\omega||^s, \\
\Pi \{||\vartheta||^s||\omega||^s + \{||\vartheta||^{2s} + ||\omega||^{2s}\}\}, \\
\end{array}\right. 
\]
\[
\forall \vartheta, \omega \in G, \exists \text{ a unique additive function } A : G \to B \ni\]
\[
\|f(\vartheta) - A(\vartheta)\| \leq \left\{\begin{array}{ll}
\Pi, \\
\Pi \{||\vartheta||^s + ||\omega||^s\}, \\
\Pi||\vartheta||^s||\omega||^s, \\
\Pi \{||\vartheta||^s||\omega||^s + \{||\vartheta||^{2s} + ||\omega||^{2s}\}\}, \\
\end{array}\right. (36)
\]
\[
\forall \vartheta, \omega \in G, \exists \text{ a unique additive function } A : G \to B \ni\]
\[
\|f(\vartheta) - A(\vartheta)\| \leq \left\{\begin{array}{ll}
\Pi, \\
\Pi \{||\vartheta||^s + ||\omega||^s\}, \\
\Pi||\vartheta||^s||\omega||^s, \\
\Pi \{||\vartheta||^s||\omega||^s + \{||\vartheta||^{2s} + ||\omega||^{2s}\}\}, \\
\end{array}\right. (37)
\]
for all \(\vartheta \in G\).

Proof. Let us set
\[
\mathcal{M}(\vartheta, \omega) = \left\{\begin{array}{ll}
\Pi, \\
\Pi \{||\vartheta||^s + ||\omega||^s\}, \\
\Pi||\vartheta||^s||\omega||^s, \\
\Pi \{||\vartheta||^s||\omega||^s + \{||\vartheta||^{2s} + ||\omega||^{2s}\}\}.
\end{array}\right.
\]
for all $\vartheta, \omega \in G$. Now

$$M(\mu^i_{\vartheta}, \mu^i_{\omega}) \mu^i_{\vartheta} = \begin{cases} \frac{\Pi}{\mu^i_{\vartheta}} \Pi \left\{ \frac{1}{\mu^i_{\vartheta}} \left( ||\mu^i_{\vartheta}\vartheta||^s + ||\mu^i_{\omega}\vartheta||^s \right) \right\} \\
\frac{\Pi}{\mu^i_{\vartheta}} \left\{ ||\mu^i_{\vartheta}\vartheta||^s ||\mu^i_{\omega}\vartheta||^s \right\} \\
\frac{\Pi}{\mu^i_{\vartheta}} \left\{ ||\mu^i_{\vartheta}\vartheta||^s ||\mu^i_{\omega}\vartheta||^s + \left\{ ||\mu^i_{\vartheta}\vartheta||^{2s} + ||\mu^i_{\omega}\vartheta||^{2s} \right\} \right\} \\
\rightarrow 0 \text{ as } t \to \infty,
\rightarrow 0 \text{ as } t \to \infty,
\rightarrow 0 \text{ as } t \to \infty,
\rightarrow 0 \text{ as } t \to \infty. \end{cases}$$

i.e., (30) is holds. In any case, we have

$$\gamma(\vartheta) = \frac{1}{2\lambda^n} \left[ M \left( \frac{\vartheta}{\bar{P}}, \frac{\vartheta}{\bar{P}} \right) \right].$$

Hence

$$\gamma(\vartheta) = \frac{1}{2\lambda^n} \left[ M \left( \frac{\vartheta}{\bar{P}}, \frac{\vartheta}{\bar{P}} \right) \right] = \begin{cases} \frac{\Pi}{\mu^i_{\vartheta}} \frac{2\lambda^n|\vartheta|}{2\lambda^n|\vartheta|^s}, \\
\frac{\Pi}{\mu^i_{\vartheta}} \frac{2\lambda^n|\vartheta|}{2\lambda^n|\vartheta|^{2s}}, \\
\frac{\Pi}{\mu^i_{\vartheta}} \frac{2\lambda^n|\vartheta|^{2s}}{3\Pi}, \\
\frac{\Pi}{\mu^i_{\vartheta}} \frac{2\lambda^n|\vartheta|^2}{2\lambda^n|\vartheta|^{2s}}. \end{cases}$$

Also,

$$\frac{1}{\mu^i_{\vartheta}} \gamma(\mu^i_{\vartheta}) = \begin{cases} \frac{\Pi}{\mu^i_{\vartheta}} \frac{2\lambda^n|\vartheta|}{2\lambda^n|\vartheta|^s}, \\
\frac{\Pi}{\mu^i_{\vartheta}} \frac{2\lambda^n|\vartheta|}{2\lambda^n|\vartheta|^{2s}}, \\
\frac{\Pi}{\mu^i_{\vartheta}} \frac{2\lambda^n|\vartheta|^{2s}}{3\Pi}, \\
\frac{\Pi}{\mu^i_{\vartheta}} \frac{2\lambda^n|\vartheta|^2}{2\lambda^n|\vartheta|^{2s}}. \end{cases}$$
we prove the subsequent cases for conditions by utilizing (33)

Case: 1 \( L = P \) if \( i = 0 \)
\[
\| f(\vartheta) - A(\vartheta) \| \leq \frac{L^{1-i}}{1-L} \gamma(\vartheta) = \frac{(P^{-1})^{1-i}_0}{1-(P^{-1})^1} \cdot \frac{\Pi}{2\lambda^n P} \\
= \frac{\Pi}{2\lambda^n (P-1)}.
\]

Case: 2 \( L = P \) if \( i = 1 \)
\[
\| f(\vartheta) - A(\vartheta) \| \leq \frac{L^{1-i}}{1-L} \gamma(\vartheta) = \frac{(P)^{1-1}_0}{1-P} \cdot \frac{\Pi}{2\lambda^n P} \\
= \frac{\Pi}{2\lambda^n (1-P)}.
\]

Case: 1 \( L = P^{s-1} \) if \( i = 0 \)
\[
\| f(\vartheta) - A(\vartheta) \| \leq \frac{L^{1-i}}{1-L} \gamma(\vartheta) = \frac{\lambda^n (P^{s-1})^{1-i}_0}{1-P^{s-1}} \cdot \frac{\Pi}{\lambda^n P^s} ||\vartheta||^s \\
= \frac{\Pi}{\lambda^n P^s} ||\vartheta||^s \\
= \frac{\Pi}{\lambda^n (P-1)}.
\]

Case: 2 \( L = \frac{1}{P^{s-1}} \) if \( i = 1 \)
\[
\| f(\vartheta) - A(\vartheta) \| \leq \frac{L^{1-i}}{1-L} \gamma(\vartheta) = \frac{(\frac{1}{P^{s-1}})^{1-i}_0}{1-\frac{1}{P^{s-1}}} \cdot \frac{\Pi}{\lambda^n P^s} ||\vartheta||^s \\
= \frac{\Pi}{\lambda^n P^s} ||\vartheta||^s \\
= \frac{\Pi}{\lambda^n (P^s - 1)}.
\]

Case: 1 \( L = P^{2s-1} \) if \( i = 0 \)
\[
\| f(\vartheta) - A(\vartheta) \| \leq \frac{L^{1-i}}{1-L} \gamma(\vartheta) = \frac{(P^{2s-1})^{1-i}_0}{1-P^{2s-1}} \cdot \frac{\Pi}{2\lambda^n P^{2s}} ||\vartheta||^{2s} \\
= \frac{\Pi}{2\lambda^n P^{2s}} ||\vartheta||^{2s} \\
= \frac{\Pi}{2\lambda^n (P^{2s} - 1)}.
\]

Case: 2 \( L = \frac{1}{P^{2s-1}} \) if \( i = 1 \)
\[
\| f(\vartheta) - A(\vartheta) \| \leq \frac{L^{1-i}}{1-L} \gamma(\vartheta) = \frac{(\frac{1}{P^{2s-1}})^{1-i}_0}{1-\frac{1}{P^{2s-1}}} \cdot \frac{\Pi}{2\lambda^n P^{2s}} ||\vartheta||^{2s} \\
= \frac{\Pi}{2\lambda^n P^{2s}} ||\vartheta||^{2s} \\
= \frac{\Pi}{2\lambda^n (P^{2s} - 1)}.
\]
Case: 1 \( L = p^{2s-1} \) if \( i = 0 \)

\[
\|f(\vartheta) - A(\vartheta)\| \leq \frac{L^{1-i}}{1-L} \gamma(\vartheta) = \frac{(p^{2s-1})^{1-i}}{1-p^{2s-1}} \frac{3\Pi}{2\lambda^n p^{2s}} \|\vartheta\|^2s
\]

\[
= \frac{p^{2s}}{p - p^{2s}} \frac{3\Pi}{2\lambda^n p^{2s}} \|\vartheta\|^2s
\]

\[
= \frac{2\lambda^n}{p - p^{2s}}.
\]

Case: 2 \( L = \frac{1}{p^{2s-t}} \) if \( i = 1 \)

\[
\|f(\vartheta) - A(\vartheta)\| \leq \frac{L^{1-i}}{1-L} \gamma(\vartheta) = \frac{1}{1-p^{2s-t}} \frac{3\Pi}{2\lambda^n p^{2s}} \|\vartheta\|^2s
\]

\[
= \frac{p^{2s}}{p^{2s} - p} \frac{3\Pi}{2\lambda^n p^{2s}} \|\vartheta\|^2s
\]

\[
= \frac{2\lambda^n}{p^{2s} - p}.
\]

5. Conclusion

We attained stability results in both direct and fixed point method in the sense of Hyers Ulam.

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