Smallest Ellipsoid Containing $p$-Sum of Ellipsoids
with Application to Reachability Analysis

Abhishek Halder

Abstract—We study the problem of ellipsoidal bounding of convex set-valued data, where the convex set is obtained by the $p$-sum of finitely many ellipsoids, for any real $p \geq 1$. The notion of $p$-sum appears in the Brunn-Minkowski-Firey theory in convex analysis, and generalizes several well-known set-valued operations such as the Minkowski sum of the summand convex sets (here, ellipsoids). We derive an outer ellipsoidal parameterization for the $p$-sum of a given set of ellipsoids, and compute the tightest such parameterization for two optimality criteria: minimum trace and minimum volume. For such optimal parameterizations, several known results in the system-control literature are recovered as special cases of our general formula. For the minimum volume criterion, our analysis leads to certain fixed point recursion which is proved to be contractive, and found to converge fast in practice. We apply these results to compute the forward reach sets for a linear control system subject to different convex set-valued uncertainty models for the initial condition and control, generated by varying $p \in [1, \infty]$. Our numerical results show that the proposed fixed point algorithm offers more than two orders of magnitude speed-up in computational time for $p = 1$, compared to the existing semidefinite programming approach without significant effect on the numerical accuracy. For $p > 1$, the reach set computation results reported here are novel. Our results are expected to be useful in real-time safety critical applications such as decision making for collision avoidance of autonomous vehicles, where the computational time-scale for reach set calculation needs to be much smaller than the vehicular dynamics time-scale.

Keywords: Ellipsoidal calculus, Firey $p$-sum, outer approximation, optimal ellipsoid, reach sets.

I. INTRODUCTION

Computing an ellipsoid that contains given set-valued data, is central to many applications such as guaranteeing collision avoidance in robotics [1], [2], robust estimation [4], [7], system identification [8], [9], and control [11], [12]. To reduce conservatism, one requires such an ellipsoid to be “smallest” according to some optimality criterion, among all ellipsoids containing the data. Typical examples of optimality criteria are “minimum volume” and “minimum sum of the squared semi-axes”. A common situation arising in practice is the following: the set-valued data itself is described as set operations (e.g. union, intersection, or Minkowski sum) on other ellipsoids. In this paper, we consider computing the smallest ellipsoid that contains the so-called $p$-sum of finitely many ellipsoids, where $p \in [1, \infty]$.

As a set operation, the $p$-sum of convex sets returns a new convex set, which loosely speaking, is a combination of the input convex sets. The notion of $p$-sum was introduced by Firey [13] to generalize the Minkowski sum of convex bodies, and was studied in detail by Lutwak [14], [15], who termed the resulting development as Brunn-Minkowski-Firey theory (also known as $L_p$ Brunn-Minkowski theory, see e.g., [16] Ch. 9.1). In this paper, we derive an outer ellipsoidal parameterization that is guaranteed to contain the $p$-sum of given ellipsoids, and then compute the smallest such outer ellipsoid.

Since the $p$-sum subsumes well-known set operations like the Minkowski sum as special case, we recover known results in the systems-control literature [17]–[20] about the minimum trace and minimum volume outer ellipsoids of such sets, by specializing our optimal parameterization of the $p$-sum of ellipsoids. Furthermore, based on our analytical results, we propose a fixed point recursion to compute the minimum volume outer ellipsoid of the $p$-sum for any real $p \in [1, \infty]$, that has fast convergence. The proposed algorithm not only enables computation for the novel convex uncertainty models (for $p > 1$), it also entails orders of magnitude faster runtime compared to the existing semidefinite programming approach for the $p = 1$ case. Thus, the contribution of this paper is twofold: (i) generalizing several existing results in the literature on outer ellipsoidal parameterization (Section III) of a convex set obtained as set operations on given ellipsoids, and analyzing its optimality (Section IV); (ii) deriving numerical algorithms (Section V) to compute the minimum volume outer ellipsoid containing the $p$-sum of ellipsoids.

To illustrate the numerical algorithms derived in this paper, we compute (Section VI) the smallest outer ellipsoidal approximations for the (forward) reach sets of a discrete-time linear control system subject to set-valued uncertainties in its initial conditions and control. When the initial condition and control sets are ellipsoidal, they model weighted norm bounded uncertainties, and at each time, we are led to compute the smallest outer ellipsoid for the Minkowski sum ($p = 1$ case). It is found that by specializing the proposed algorithms for $p = 1$, we can lower the computational runtime by more than two orders of magnitude compared to the current state-of-the-art, which is to reformulate and solve the same via semidefinite programming. For $p > 1$, the initial conditions and controls belong to $p$-sums of ellipsoidal sets, which are convex but not ellipsoidal, in general. In this case too, the proposed algorithms enable computing the smallest outer ellipsoids for the reach sets.

†The Minkowski sum of two compact convex sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$ is the set $\mathcal{X} + \mathcal{Y} := \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\} \subset \mathbb{R}^d$. 

Abhishek Halder is with the Department of Applied Mathematics and Statistics, University of California, Santa Cruz, CA 95064, USA, ahalder@ucsc.edu
II. Preliminaries

1) Convex Geometry: The support function \( h_K(\cdot) \) of a compact convex set \( K \subset \mathbb{R}^d \), is

\[
h_K(y) := \sup \{ \langle x, y \rangle \mid x \in K, \ y \in \mathbb{R}^d \}, \tag{1}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the standard Euclidean inner product. The function \( h_K : K \to \mathbb{R} \), and can be viewed geometrically as the (signed) perpendicular distance from the origin to the supporting hyperplane of \( K \), which has the outer normal vector \( y \). Thus, the support function returns negative value if and only if the normal vector \( y \) points into the open halfspace containing the origin. The support function \( h_K(\cdot) \) can also be seen as (see e.g., [21, Theorem 13.2]) the Legendre-Fenchel conjugate of the indicator function of the set \( K \), and thus uniquely determines the set \( K \).

The following properties of the support function are well-known:

(i) convexity: \( h_K(y) \) is a convex function in \( y \in \mathbb{R}^d \),
(ii) positive homogeneity: \( h_K(\alpha y) = \alpha h_K(y) \) for \( \alpha > 0 \),
(iii) sub-additivity: \( h_K(y+z) \leq h_K(y)+h_K(z) \) for \( y, z \in \mathbb{R}^d \),
(iv) inclusion: given compact convex sets \( K_1 \) and \( K_2 \), the inclusion \( K_1 \subseteq K_2 \) holds if and only if \( h_{K_1}(y) \leq h_{K_2}(y) \) for all \( y \in \mathbb{R}^d \),
(v) affine transformation: \( h_{AK+b}(y) = h_K(A^\top y) + \langle b, y \rangle \), for \( A \in \mathbb{R}^{d \times d} \), \( b \in \mathbb{R}^d \), \( y \in \mathbb{R}^d \).

Definition 1. \( p \)-Sum of convex sets: Given compact convex sets \( K_1, K_2 \subset \mathbb{R}^d \), their \( p \)-sum \([13]\) is a new compact convex set \( K \subset \mathbb{R}^d \) defined via its support function

\[
h_K(y) = \left(h_{K_1}(y) + h_{K_2}(y)\right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty, \tag{2}
\]
and we write \( K := K_1 +_p K_2 \).

Special cases of the \( p \)-sum are encountered frequently in practice. For example, when \( p = 1 \), the set \( K = K_1 +_1 K_2 \) is the Minkowski sum of \( K_1 \) and \( K_2 \), and

\[
h_{K_1+_1 K_2}(\cdot) = h_{K_1}(\cdot) + h_{K_2}(\cdot).
\]

When \( p = \infty \), the set \( K = K_1 +_\infty K_2 \) is the convex hull of the union of \( K_1 \) and \( K_2 \), and

\[
h_{K_1+_\infty K_2}(\cdot) = \max \{h_{K_1}(\cdot), h_{K_2}(\cdot)\}.
\]

For \( 1 \leq p < q \leq \infty \), we have the inclusion (See Fig. 1):

\[
K_1 \cup K_2 \subseteq K_1 +_q K_2 \subseteq \ldots \subseteq K_1 +_q K_2 \subseteq K_1 +_p K_2 \subseteq \ldots \subseteq K_1 +_1 K_2, \tag{3}
\]
which follows from the support function inequality

\[
h_{K_1+_p K_2}(\cdot) \leq h_{K_1+_q K_2}(\cdot), \quad 1 \leq p < q \leq \infty.
\]

From Definition 1 it is easy to see that the \( p \)-sum is commutative and associative, that is,

\[
K_1 +_p K_2 = K_2 +_p K_1, \tag{4a}
\]
\[
(K_1 +_p K_2) +_p K_3 = K_1 +_p (K_2 +_p K_3), \tag{4b}
\]
where the compact sets \( K_1, K_2, K_3 \) are convex. Furthermore, linear transformation is distributive over \( p \)-sum, i.e.,

\[
A(K_1 +_p K_2) = AK_1 +_p AK_2, \quad A \in \mathbb{R}^{d \times d}, \tag{5}
\]

which is immediate from the aforesaid property (v) and equation (2). We remark that it is often convenient to express \( h_K(\cdot) \) as function of the unit vector \( y/\|y\|_2 \) in \( \mathbb{R}^d \) (see Fig. 2).

Remark 1. At first glance, it might seem odd that the Minkowski sum is defined pointwise as \( X +_1 Y := \{x+y \mid x \in X, y \in Y\} \), which remains well-defined for \( X, Y \) compact (not necessarily convex), but the \( p \)-sum in Definition 1 is given via support functions and requires the summand sets to be convex. Indeed, a pointwise definition for the \( p \)-sum was proposed in [22] for compact summand sets \( X, Y \), given by \( X+_p Y := \{(1-\mu)^{1/p} x + \mu^{1/p} y \mid x \in X, y \in Y, 0 \leq \mu \leq 1\} \), where \( p' \) denotes the Hölder conjugate of \( p \), i.e., \( \frac{1}{p} + \frac{1}{p'} = 1 \). This pointwise definition was shown to reduce to Definition 1 provided the compact summands \( X, Y \) are also convex.

2) Ellipsoids: Let \( S^d_+ \) be the cone of \( d \times d \) symmetric positive definite matrices. For an ellipsoid with center \( q \in \mathbb{R}^d \) and shape matrix \( Q \in S^d_+ \), denoted by

\[
E(q, Q) := \{x \in \mathbb{R}^d \mid \langle x-q, Q^{-1}(x-q) \rangle \leq 1\}, \tag{6}
\]

which reduces to

\[
h_{E(q, Q)}(y) = \langle q, y \rangle + \sqrt{\langle Qy, y \rangle}. \tag{6}
\]

Furthermore, the volume of the ellipsoid \( E(q, Q) \) is given by

\[
\text{vol}(E(q, Q)) = \frac{\text{vol}(B^d_1)}{\sqrt{\det(Q^{-1})}} = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} \sqrt{\det(Q)}, \tag{7}
\]

where \( B^d_1 \) denotes the \( d \)-dimensional unit ball, and \( \Gamma(\cdot) \) denotes the Gamma function.
of this paper:

for ellipsoid will be useful in the later part quadratic form, i.e.,

where \( \theta \) \( E \) and 2). We want to determine an ellipsoid \( p \geq b \) with \( E \h \) \( \geq E \).

Given two ellipsoids \( E_1 \) and \( E_2 \), shown for different values of \( \theta \) \( \in [0,2\pi) \).

An alternative ellipsoidal parameterization can be obtained via a matrix-vector-scalar triple \( (A,b,c) \) encoding the quadratic form, i.e., \( \mathcal{E}(A,b,c) := \{ x \in \mathbb{R}^d : x^\top Ax + 2x^\top b + c \leq 0 \} \). The following relations among \( (A,b,c) \) and \( (q,Q) \) parameterizations for ellipsoid will be useful in the later part of this paper:

\[
A = Q^{-1}, \quad b = -Q^{-1}q, \quad c = q^\top Q^{-1}q - 1, \quad (8a) \\
Q = A^{-1}, \quad q = -Qb. \quad (8b)
\]

III. PARAMETERIZED OUTER ELLIPSOID

Given two ellipsoids \( \mathcal{E}(q_1, Q_1), \mathcal{E}(q_2, Q_2) \subset \mathbb{R}^d \), consider their \( p \)-sum

\[
\mathcal{E}(q_1, Q_1) +_p \mathcal{E}(q_2, Q_2), \quad 1 \leq p \leq \infty, \quad (9)
\]

which is convex but not an ellipsoid in general (see Fig. 1 and 2). We want to determine an ellipsoid \( \mathcal{E}(q, Q) \subset \mathbb{R}^d \), as function of the input ellipsoids, that is guaranteed to contain the \( p \)-sum (9). For this to happen, we must have (from 2 and property (iv) in Section II.1)

\[
(h^p_{\mathcal{E}(q_1, Q_1)}(y))^2 \geq \sum_{i=1,2} (y^\top q_i + \sqrt{y^\top Q_i y})^p, \quad (10)
\]

for all \( y \in \mathbb{R}^d \). In the rest of this paper, we make the following assumption.

**Assumption 1.** The center vectors for the summand ellipsoids in a \( p \)-sum are assumed to be zero, i.e., \( q_1 = q_2 = 0 \).

Under Assumption (1), and \( p = 2 \), it follows from (6) and (9) that (9) is an ellipsoid \( \mathcal{E}(0, Q_1) +_p \mathcal{E}(0, Q_2) \).

For \( p \neq 2 \), the convex set (9) is again not an ellipsoid in general, but we can parameterize an outer ellipsoid \( \mathcal{E}(0, Q) \supseteq \mathcal{E}(0, Q_1) +_p \mathcal{E}(0, Q_2) \) as follows.

For \( \alpha, \gamma > 1 \), let

\[
Q := \alpha Q_1 + \gamma Q_2, \quad \text{and} \quad g_i^2 := y^\top Q_i y \geq 0, \quad i = 1,2. \quad (11)
\]

For the time being, we think of \( \alpha, \gamma > 1 \) as free parameters. We will see that it is possible to re-parameterize \( Q \) in terms of a single parameter \( \beta > 0 \), by expressing both \( \alpha \) and \( \gamma \) as appropriate functions of \( \beta \), while guaranteeing the inclusion of the \( p \)-sum in \( \mathcal{E}(0, Q) \).

With the standing assumptions \( q_1 = q_2 = 0 \), we can re-write (10) as

\[
(\alpha g_1^2 + \gamma g_2^2) \frac{y^\top y}{2} \geq g_1^2 + g_2^2, \\
\Rightarrow (\alpha g_1^2 + \gamma g_2^2)^p \geq g_1^{2p} + g_2^{2p} + 2g_1^p g_2^p. \quad (12)
\]

To proceed further, we need the following Lemma.

**Lemma 1.** A convex function \( f() \) with \( f(0) = 0 \) is superadditive on \([0, \infty)\), i.e., \( f(x + y) \geq f(x) + f(y) \) for all \( x, y \geq 0 \).

**Proof.** For \( 0 \leq \lambda \leq 1, \) by convexity

\[
f(\lambda x) = f(\lambda x + (1 - \lambda)0) \leq \lambda f(x) + (1 - \lambda)f(0) = \lambda f(x).
\]

Therefore, we get

\[
f(x) = f\left(\frac{x}{x + y}(x + y)\right) \leq \frac{x}{x + y} f(x + y),
\]

and

\[
f(y) = f\left(\frac{y}{x + y}(x + y)\right) \leq \frac{y}{x + y} f(x + y),
\]

and adding the last two inequalities yield \( f(x) + f(y) \leq f(x + y) \). \( \square \)

In Theorem 1 below, we use Lemma 1 to derive an explicit parameterization of the shape matrix \( Q(\beta) \in \mathbb{S}^d_+, \beta > 0, \) that guarantees an outer ellipsoidal parameterization containing the \( p \)-sum

\[
\mathcal{E}(0, Q_1) +_p \mathcal{E}(0, Q_2).
\]

**Theorem 1.** Given a scalar \( \beta > 0 \), and a pair of matrices \( Q_1, Q_2 \in \mathbb{S}_d^+ \), let

\[
Q(\beta) = \left(1 + \frac{1}{\beta}\right) \frac{1}{2} Q_1 + \left(1 + \beta\right) \frac{1}{2} Q_2, \quad p \in [1, \infty) \setminus \{2\}. \quad (13)
\]

Then \( \mathcal{E}(0, Q(\beta)) \supseteq \mathcal{E}(0, Q_1) +_p \mathcal{E}(0, Q_2) \).

**Proof.** Since \( f(x) := x^p \) is convex on \([0, \infty)\) for \( 1 \leq p < \infty \), Lemma 1 yields

\[
(\alpha g_1^2 + \gamma g_2^2)^p \geq \alpha^p g_1^{2p} + \gamma^p g_2^{2p} + \xi, \quad \text{where} \ \xi \geq 0. \quad (14)
\]

...
Combining (12) and (14), we obtain
\[(\alpha^p - 1) (g_1^p)^2 + (\gamma^p - 1) (g_2^p)^2 - 2g_1^p g_2^p + \xi \geq 0. \tag{15}\]
Since \(\alpha > 1\), we have \(\alpha^p > 1\) for \(p \geq 1\). Therefore, multiplying both sides of (15) by \((\alpha^p - 1)\), and then adding and subtracting \((g_2^p)^2\), we get
\[
((\alpha^p - 1) g_1^p - g_2^p)^2 + ((\alpha^p - 1) (\gamma^p - 1) - 1) (g_2^p)^2 + (\alpha^p - 1) \xi \geq 0. \tag{16}\]
A sufficient condition to satisfy the inequality (16) is to choose \(\alpha, \gamma > 1\) such that
\[(\alpha^p - 1) (\gamma^p - 1) \geq 1.\]
Letting \(\alpha^p - 1 := \beta^{-1}\) and \(\gamma^p - 1 = \beta, \beta > 0\), and using (11), we arrive at (13). \(\square\)

Remark 2. The parameterization (13) generalizes the outer ellipsoidal parameterization containing the Minkowski sum \((p = 1\) case), well-known in the systems-control literature (see e.g., [17] p. 104, [18], [19]). For this special case \(p = 1\), a discussion about equivalent parameterizations can be found in [23] Section II.A.

IV. OPTIMAL PARAMETERIZATION

To reduce conservatism, it is desired that the parameterized outer ellipsoid \(E(0, Q(\beta))\) in Theorem 1 containing the \(p\)-sum, be as tight as possible. One way to promote “tightness” is by minimizing the sum of the squared semi-axes lengths of \(E(0, Q(\beta))\), which amounts to minimizing trace \((Q(\beta))\) over \(\beta > 0\). Another possible way to promote “tightness” is by minimizing the volume of \(E(0, Q(\beta))\), which, thanks to [7], amounts to minimizing \(\log \det (Q(\beta))\). We next analyze these optimality criteria.

A. Minimum Trace Outer Ellipsoid

We consider the optimization problem
\[
\begin{align*}
\text{minimize} & \quad \text{trace} (Q(\beta)), \\
& \quad \beta > 0
\end{align*}
\tag{17}\]
where \(Q(\beta)\) is given by (13), and let
\[
\beta_{\text{tr}}^* := \text{arg min} \text{trace} (Q(\beta)).
\]
Setting \(\frac{\partial}{\partial \beta} \text{trace} (Q(\beta)) = 0\), and using the linearity of trace operator, straightforward calculation yields
\[
\beta_{\text{tr}}^* = \left( \frac{\text{trace} (Q_1)}{\text{trace} (Q_2)} \right)^{\frac{1}{p-1}}, \quad p \in [1, \infty) \setminus \{2\}, \tag{18}\]
and
\[
\frac{\partial^2}{\partial \beta^2} \text{trace} (Q(\beta)) \bigg|_{\beta = \beta_{\text{tr}}^*} = \frac{1}{p} \left( \frac{1}{p + 1} \right) \left( \beta_{\text{tr}}^* \right)^{\frac{1}{p} - 2} \left( \beta_{\text{tr}}^* + 1 \right)^{\frac{1}{p} - 1} \text{trace} (Q_1) > 0.
\]
The formula (18) generalizes the previously known formula for \(p = 1\) case (minimum trace ellipsoid containing the Minkowski sum of two given ellipsoids) reported in [19], Appendix A.2] and in [17] Lemma 2.5.2(a).

B. Minimum Volume Outer Ellipsoid

Now we consider the optimization problem
\[
\begin{align*}
\text{minimize} & \quad \log \det (Q(\beta)), \\
& \quad \beta > 0
\end{align*}
\tag{19}\]
where \(Q(\beta)\) is given by (13), and let
\[
\beta_{\text{vol}}^* := \text{arg min} \log \det (Q(\beta)). \tag{20}\]
To simplify the first order condition of optimality \(\frac{\partial}{\partial \beta} \log \det (Q(\beta)) = 0\), we notice that the matrix \(R := Q_1^{-1} Q_2\) is diagonalizable in [23] Section III.A, Lemma 1, and denote its spectral decomposition as \(R := S \Lambda S^{-1}\). Further, let the eigenvalues of \(R\) be \(\{\lambda_i\}_{i=1}^d\), which are all positive (see the discussion following Proposition 1 in [23]). Then direct calculation gives
\[
\begin{align*}
\frac{\partial}{\partial \beta} \log \det (Q(\beta)) &= \text{trace} \left( \left( Q(\beta) \right)^{-1} \frac{\partial}{\partial \beta} Q(\beta) \right) \\
&= -\frac{1}{p \beta (1 + \beta)} \text{trace} \left( \left( I + \beta^{-\frac{1}{p}} R \right)^{-1} \left( I - \beta^{3\frac{1}{p}} R \right) \right) \\
&= -\frac{1}{p \beta (1 + \beta)} \text{trace} \left( \left( I + \beta^{-\frac{1}{p}} \Lambda \right)^{-1} \left( I - \beta^{3\frac{1}{p}} \Lambda \right) \right), \tag{21}\end{align*}
\]
wherein the last step follows from substituting \(I = SS^{-1}, R = SAS^{-1}\), and using the invariance of trace of matrix product under cyclic permutation.

Therefore, from (21), the first order optimality condition \(\frac{\partial}{\partial \beta} \log \det (Q(\beta)) = 0\) is equivalent to the following nonlinear algebraic equation:
\[
\sum_{i=1}^d \frac{1 - \beta^{3\frac{1}{p}} \lambda_i}{1 + \beta^{\frac{3}{p}} \lambda_i} = 0, \quad p \in [1, \infty) \setminus \{2\}, \tag{22}\]

To be solved for \(\beta > 0\), with known parameters \(\lambda_i > 0, \quad i = 1, \ldots, d\). If (22) admits unique positive root (which seems non-obvious, and will be proved next), then it would indeed correspond to the argmin in (20) since
\[
\frac{\partial^2}{\partial \beta^2} \log \det (Q(\beta)) \bigg|_{\beta > 0} = \frac{1}{p \beta (1 + \beta)} \sum_{i=1}^d \frac{(2 - \frac{1}{p}) \beta^{2\frac{1}{p}} \lambda_i^2 + (3 - \frac{1}{p}) \beta^{\frac{2}{p}} + \frac{1}{p} \beta^{3\frac{1}{p} - 1} \lambda_i}{(1 + \beta \lambda_i)^2} > 0, \quad \text{for} \quad p \in [1, \infty) \setminus \{2\}. \tag{23}\]

In the following Theorem, we establish the uniqueness of the positive root.

Theorem 2. Given \(\lambda_i > 0, \quad i = 1, \ldots, d\), equation (22) in variable \(\beta\) admits unique positive root.

Proof. We start by rewriting (22) as
\[
\sum_{i=1}^d \left( \beta^{3\frac{1}{p}} - \frac{1}{\lambda_i} \right) \prod_{j=1\atop j \neq i}^d \left( \beta^{\frac{1}{p}} + \frac{1}{\lambda_j} \right) = 0. \tag{24}\]
V. Algorithms and Applications

A. Computing the Minimum Volume Outer Ellipsoid

Thanks to the parameterization (13), computing the minimum volume outer ellipsoid (MVOE) for the \( p \)-sum \( \mathcal{E}(0, Q_1) +_p \mathcal{E}(0, Q_2) \), is reduced to computing \( \beta^*_{\text{vol}} \) introduced in the previous Section. Motivated by the observation that the first order optimality criterion (22) can be rearranged as

\[
\beta^{3-\frac{1}{p}} \sum_{i=1}^{d} \frac{\lambda_i}{1 + \beta^{\frac{1}{p}} \lambda_i} = \frac{1}{1 + \beta^{\frac{1}{p}} \lambda_i},
\]

we consider the fixed point recursion

\[
\beta_{n+1} = g(\beta_n) := \left( \sum_{i=1}^{d} \frac{1}{1 + \beta^{\frac{1}{p}} \lambda_i} \right) \frac{p}{e^{\frac{p}{p-1}}},
\]

where \( n = 0, 1, 2, \ldots \), and \( p \in [1, \infty) \setminus \{2\} \). Furthermore, \( g : \mathbb{R}_+ \to \mathbb{R}_+ \), i.e., the map \( g \) is cone preserving. In the following Theorem, we show that the fixed point recursion (27) converges to a unique positive root (Fig. 3), and is in fact contractive in the Hilbert metric [25, Ch. 2]. Thus, the recursion (27) is indeed an efficient numerical algorithm to compute \( \beta^*_{\text{vol}} \). The recursion (27) and the contraction proof below subsume our previous result [23, Section IV.B] for the \( p = 1 \) case (computing MVOE for the Minkowski sum).

**Theorem 3.** Starting from any initial guess \( \beta_0 \in \mathbb{R}_+ \), the recursion (27) with fixed \( p \in [1, \infty) \setminus \{2\} \), converges to a unique fixed point \( \beta^*_{\text{vol}} \in \mathbb{R}_+ \), i.e.,

\[
\lim_{n \to \infty} g^n(\beta_0) = \beta^*_{\text{vol}}.
\]

**Proof.** For \( \lambda, x > 0 \), consider the positive functions \( f_i := 1/(1 + \lambda_i x^{1/p}) \), where \( i = 1, \ldots, d, p \in [1, \infty) \setminus \{2\} \), and let

\[
\phi(x) := x^p e^{-x}, \quad \text{and} \quad \psi(x) := \frac{\sum f_i}{\sum f_i},
\]

Clearly, \( \phi(x) \) and \( \psi(x) \) are both concave and increasing in \( \mathbb{R}_+ \), and therefore [26, p. 84] so is \( g(\beta_n) = \phi(\psi(\beta_n)) \) as a function of \( \beta_n, n = 0, 1, 2, \ldots \). Consequently (see e.g., the first step in the proof of Theorem 2.1.11 in [27]) the map \( g \) is contractive in Hilbert metric on the cone \( \mathbb{R}_+ \). By Banach contraction mapping theorem, \( g \) admits unique fixed point \( \beta^*_{\text{vol}} \in \mathbb{R}_+ \), and

\[
\lim_{n \to \infty} g^n(\beta_0) = \beta^*_{\text{vol}}.
\]

The rate-of-convergence for (27) is fast in practice, see Fig. 4. Next, we show how the \( p \)-sum computation may arise in the reachability analysis for linear systems.

B. Application to Reachability Analysis for Linear Systems

Consider a discrete-time linear time-invariant (LTI) system

\[
x(t+1) = Fx(t) + Gu(t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^n,
\]

with set-valued uncertainties in initial condition \( x(0) \in \mathcal{X}_0 \subset \mathbb{R}^n \), and control \( u(t) \in \mathcal{U}(t) \subset \mathbb{R}^n \). For \( t = 0, 1, 2, \ldots \), we would like to compute a tight outer ellipsoidal approximation...
the general idea of using support functions as in (29a). This follows from the support function

\[ \mathcal{X}(t) = F^t \mathcal{E}(0, Q_0) + \sum_{k=0}^{t-1} F^{t-k-1} G \mathcal{E}(0, U(t)) . \]  

which can be seen as the sampled version of the continuous-time system \( \dot{x}_1 = x_2 + u_1, \dot{x}_2 = u_2 \), with sampling period \( h > 0 \). To illustrate the proposed algorithms, we will compute the reach sets of (31) for various convex uncertainty models of the form (29) for \( \mathcal{X}_0 \) and \( \mathcal{U}(t) \). For this numerical example, we set the nominal initial condition \( x_0 = 0 \), and nominal control \( u_c(t) \equiv 0 \).

A. The case \( m = n = 1 \)

For \( m = n = 1 \) in (29), both \( \mathcal{X}_0 \) and \( \mathcal{U}(t) \) are ellipsoidal, and hence the reach set \( \mathcal{X}(t) \) for (31), is the Minkowski sum of \( (t+1) \) ellipsoids. Consequently, we are led to compute the MTOE and MVOE of the Minkowski sum \( \mathcal{X}(t) \).

\[ \mathcal{X}(t) = F^t \mathcal{E}(0, Q_0) + \sum_{k=0}^{t-1} F^{t-k-1} G \mathcal{E}(0, U(t)) . \]

Notice that the MTOE admits analytical solution (18), applied pair-wise to the \( (t+1) \) summand ellipsoids in (32), with \( p = 1 \). For this case, the current state-of-the-art for MVOE computation is to reformulate the same as a semidefinite programming (SDP) problem via the \( \mathcal{S} \)-procedure (see e.g., [35] Ch. 3.7.4). Specifically, given \( (t+1) \) ellipsoids \( \mathcal{E}(q_i, Q_i) \) or equivalently \( \mathcal{E}(A_i, b_i, c_i) \) in \( \mathbb{R}^{n_e} \), \( i = 1, \ldots, t+1 \), to compute the MVOE containing their Minkowski sum \( \mathcal{E}(q_i, Q_i) + \cdots + \mathcal{E}(q_{t+1}, Q_{t+1}) \), one solves the SDP problem:

\[ \text{minimize } \log \det A_0^{-1} \]  

subject to

\[ A_0 > 0, \]  

\[ \tau_i \geq 0, \quad i = 1, \ldots, t+1, \]  

\[ E_0 \tilde{A}_i b_i \leq \sum_{i=1}^{t+1} \tau_i \tilde{A}_i b_i c_i \equiv 0, \]  

where \( E_0 \) is the binary matrix of size \( n_e \times (t+1)n_e \) that selects the \( i \)-th vector, \( i = 1, \ldots, t+1 \), from the vertical stacking of \( (t+1) \) vectors, each of size \( n_e \times 1 \); and

\[ E_0 := \sum_{i=1}^{t+1} E_i, \quad \tilde{A}_i := E_i \tilde{A} E_i, \quad \tilde{b}_i := E_i b_i, \quad i = 1, \ldots, t+1. \]

\[ \text{The symbol } \Sigma \text{ in } (32) \text{ stands for the } 1 \text{-sum.} \]
For dynamics (31), the MTOEs and MVOEs of the reach sets \(X(t)\) given by (32) with parameters (35), are shown for \(t = 1, \ldots, 10\), along with the \((t+1)\) summand ellipses in the Minkowski sum (32) for each \(t\). The MTOEs admit analytical solution (18). We compute the MVOEs via three different methods: SDP computation (33)-(34), a root bracketing technique proposed in [23, Section IV.A], and recursion (27).

The argmin pair \((A_0^*, b_0^*)\) associated with the SDP (33)-(34), results the optimal ellipsoid

\[ E_{SDP}^{MVOE} := E(q_{SDP}, Q_{SDP}), \]

where, using (3b), \(Q_{SDP} := (A_0^*)^{-1} \) and \(q_{SDP} := -Q_{SDP}b_0^* \).

Our intent is to compare \(E_{SDP}^{MVOE}\) with \(E_{proposed}^{MVOE}\), where

\[ E_{proposed}^{MVOE} := E(q_1 + \ldots + q_{t+1}, Q(\beta_{vol}^*)), \]

and the parametric form of \(Q(\beta_{vol}^*)\) is given by (13) with \(p = 1\), applied pairwise to the given set of shape matrices \(\{Q_1, \ldots, Q_{t+1}\}\). The numerical value of \(\beta_{vol}^*\) is computed from the fixed point recursion (27) with \(p = 1\), solved pairwise from the set \(\{Q_1, \ldots, Q_{t+1}\}\).

We will see that the MVOE algorithms proposed herein help in reducing computational time, compared to the SDP approach, without sacrificing accuracy. For comparing numerical performance, we implemented both the SDP (via \texttt{cvx} [36]) and our proposed algorithms in MATLAB 2016b, on 2.6 GHz Intel Core i5 processor with 8 GB memory.

For the dynamics (31), we set \(h = 0.3\), and

\[ Q_0 = I_2, \quad U(t) = \left(1 + \cos^2(t)\right) \text{diag}([10, 0.1]), \quad (35) \]

and for each \(t = 1, 2, \ldots\), compute the MTOE and MVOE of the reach set (32). For MVOE computation, we use three different methods: using the SDP (33)-(34), using a root-bracketing algorithm proposed in [23, Section IV.A] to solve (24), and by using the fixed point recursion (27) proposed herein. In Fig. 5 the corresponding MTOEs and MVOEs, as well as the summand ellipses in the Minkowski sum (32) are shown for \(t = 1, \ldots, 10\). In this paper, we do not emphasize the root bracketing method for MVOE computation given in [23, Section IV.A] since that is a custom method for \(n_x = 2\), while the SDP (33)-(34) and the fixed point recursion (27) are valid in any dimensions.

To assess the quality of the outer ellipsoidal approximations shown in Fig. 5 we compare the volumes (in our two-dimensional example, areas) of the MVOEs computed via the three different methods in Table I. The columns in Table I correspond to different time steps while the rows correspond to the three different methods mentioned above. From Table I notice that the MVOE volumes are not monotone in time for any given method (see e.g., the columns for \(t = 7\) and \(t = 8\)).
since the shape matrices \( U(t) \) in (35) are periodic. We notice that the volumes listed in the second and third row in Table I are in close agreement, while they are slightly conservative compared to the same in the first row, which are computed by solving the SDP (33)-(34). Furthermore, the relative numerical error seems to grow (albeit slowly) with \( t \) (with increasing number of summand ellipsoids).

By looking at the computational accuracy comparisons from Table I, it may seem that the SDP approach is superior to the algorithms proposed herein. However, the corresponding computational runtimes plotted in Fig. 6 reveal that solving the fixed point recursion \((27)\) entails orders of magnitude speed-up compared to solving the SDP. Given that the growing interests in reach set computation among practitioners are stemming from real-time safety critical applications (e.g., decision making for collision avoidance in a traffic of autonomous and semi-autonomous cars or drones), the issue of computational runtime becomes significant. Such applications indeed require computing the reach set over a short physical time horizon (typically a moving horizon of few seconds length); the MVOE computational time-scale, then, needs to be much smaller than the dynamics time-scale. The results in Fig. 6 show that the proposed algorithms can be useful in such context as they offer significant computational speed-up without much conservatism.

**B. The Case \( m, n > 1 \)**

We now consider the case \( m, n > 1 \) in \((29)\) with \( m \neq n \), and compute the MVOEs and MTOEs for the reach set \( \mathcal{X}(t) \) of (31) with \( h = 0.3 \), as before. Specifically, in \((29)\), we fix \( m = 2, n = 3 \), \( p_1 = 2.5 \), and \( p_2 = 1.5 \). In words, the set of uncertain initial conditions \( X_0 \) is modeled as a 2.5-sum of two ellipsoids, i.e., \( X_0 = E(0, Q_{01}) + 2.5 \cdot E(0, Q_{02}); \) the control uncertainty set \( U(t) \) is modeled as a 1.5-sum of three (time-varying) ellipsoids, i.e., \( U(t) = E(0, U_{1}(t)) + 1.5 \cdot E(0, U_{1}(t)) + 1.5 \cdot E(0, U_{3}(t)) \). The shape matrices for the set of uncertain initial conditions are randomly generated positive definite matrices:

\[
Q_{01} = \begin{pmatrix} 2.2259 & 0.1992 \\ 0.1992 & 2.4357 \end{pmatrix}, \quad Q_{02} = \begin{pmatrix} 2.3111 & 0.6768 \\ 0.6768 & 2.1848 \end{pmatrix}. \tag{36}
\]

The shape matrices for the set of uncertain controls are chosen as

\[
U_j(t) = \left(1 + \cos^2(jt)\right) \text{diag}([10, 0.1]), \quad j = 1, 2, 3. \tag{37}
\]

Then, the reach set \( \mathcal{X}(t) \) for (31) at any time \( t > 0 \), equals

\[
F^t \left\{ E(0, Q_{10}) + 2.5 \cdot E(0, Q_{20}) \right\} + 1 \sum_{k=0}^{t-1} F^{t-k-1} G \left\{ E(0, U_{1}(t)) + 1.5 \cdot E(0, U_{2}(t)) + 1.5 \cdot E(0, U_{3}(t)) \right\}. \tag{38}
\]

![Fig. 6: Comparison of the computational times for MVOE calculations in Fig. 5 and Table I](image)

The MVOE computational times for the proposed fixed point recursion method (\(--\), third row in Table I) are orders of magnitude faster than the same for the SDP computation (\(--\), first row in Table I). A root bracketing technique to solve (24) proposed in [23, Section IV.A] requires more computational time (\(--\), second row in Table I) than iterating the recursion \((27)\), but is faster than solving the SDP \((33)-(34)\).
which, due to (5), is the 1-sum of $(t+1)$ convex sets, one of them being the 2.5-sum of two ellipsoids, and the remaining $t$ of them each being the 1.5-sum of three ellipsoids.

Unlike the case in Section VI-A where the SDP approach is known in the literature for computing an MVOE of the 1-sum, and served as a baseline algorithm to compare the performance of our proposed algorithms, to the best of our knowledge, no such algorithm is known for the general $p$-sum case. Our proposed algorithms are generic enough to enable the MVOE/MTOE computation in this case. Specifically, at each time $t$, we first compute the MVOEs (resp. MTOEs) for each of the $(t+1)$ summand convex sets in (38) by solving (27) (resp. (18)) pairwise, and then compute the MVOE (resp. MTOE) of the 1-sum of the resulting MVOEs (resp. resulting MTOEs) using the same. In Fig. 7, we show the MVOEs and MTOEs thus computed, for the reach set (38) at $t = 1, \ldots, 10$.

The volumes (in our two dimensional numerical example, areas) of the MVOEs shown in Fig. 7, are listed in Table II. The corresponding computational times are shown in Fig. 8.

Notice that the MVOE computational times reported in Fig. 8 are about two orders of magnitude slower than the same reported in Fig. 6. This is expected since the results in Fig. 8 correspond to computing MVOEs of the 1-sums while the same in Fig. 8 correspond to computing MVOEs of the mixed $p$-sums (in our example, 1-sums, 1.5 sums and 2.5 sums) at any given time $t > 0$, and is indeed consistent with the observation made in Fig. 4 that the rate-of-convergence of recursion (27) decreases with increasing $p \geq 1$. Nevertheless, the computational times shown in Fig. 8 are still smaller than the computational times for the SDP approach in the 1-sum case shown in Fig. 5.

C. Quality of Approximation

It is natural to investigate the quality of approximations for the MTOEs and MVOEs reported herein with respect to the actual reach sets, in terms of the “shapes” of the true and approximating sets. For example, it would be undesirable if the MTOE/MVOE computation promotes “skinny ellipsoids” which are too elongated along some directions and too compressed along others, when the same may not hold for the actual reach sets. This motivates us to compute the (two-sided)
In terms of the support functions:

$$\delta_H(t) = \sup_{s \in S^{n_x-1}} \left| h_{\hat{E}}(t)(s) - h_{\mathcal{X}(t)}(s) \right|,$$  

where $S^{n_x-1}$ denotes the Euclidean unit sphere embedded in $\mathbb{R}^{n_x}$. Thanks to property (iv) in Section II.1, the absolute value in (40) can be dropped. Furthermore, suppose that both $\mathcal{X}(t)$ and $\hat{E}(t)$ are centered at origin, as in Section VI; in particular, $\hat{E}(t) = \mathcal{E}\left(0, \hat{Q}(t)\right)$. Using (2), (6), (29) and (30), we can then rewrite (40) as

$$\delta_H(t) = \sup_{s \in S^{n_x-1}} \left( \frac{1}{2} \left( \sum_{i=1}^{m} \left( s^\top \hat{Q}(t) s \right)^2 + \frac{1}{2} \left( \sum_{s_i \in S^{n_x-1}} \left( s^\top \hat{Q}(t) s \right)^2 \right) \right) \right)^{\frac{1}{4}}$$

In words, $\delta_H(t)$ is the two-sided Hausdorff distance between the reach set of (28) and its outer ellipsoidal approximation at time $t$, provided $\lambda_0$ is the $p_1$-sum of $m$ centered ellipsoids, and $\lambda(t)$ is the $p_2$-sum of $n$ time-varying centered ellipsoids, where $p_1, p_2 \geq 1$.

To illustrate the use of (41), let us consider $m = n = 1$ as in Section VI.A. In this case, (41) can be expressed succinctly as

$$\delta_H(t) = \sup_{s \in S^{n_x-1}} \left\| \hat{Q}^2(t)s \right\|_2 - \sum_{k=0}^{t-1} \left\| M_k^2(t) s \right\|_2,$$  

where $M_k := F^{t-k-1}GU(t)G^\top F^{t-k-1} \in S^{n_x}_+$ for $k = 0, 1, \ldots, t-1$, and $M_t := F^\top Q(t) F^\top \in S^{n_x}_+$. From (42), a simple upper bound for $\delta_H(t)$ follows.

**Proposition 1. (Upper bound for $\delta_H(t)$ when $m = n = 1$)**

$$\delta_H(t) \leq \left\| \hat{Q}^2(t) - \sum_{k=0}^{t} M_k^2(t) \right\|_2.$$

**Proof.** For $s \in S^{n_x-1}$, by the (repeated use of) reverse triangle inequality, we have

$$\left\| \hat{Q}^2(t)s - \sum_{k=0}^{t} M_k^2(t)s \right\|_2 \leq \left\| \hat{Q}^2(t) - \sum_{k=0}^{t} M_k^2(t) \right\|_2 \leq \left\| \hat{Q}^2(t) - \sum_{k=0}^{t} M_k^2(t) \right\|_2.$$
where the last step follows from the sub-multiplicative property of the matrix 2-norm. Since this holds for any \( s \in S_{n+1} \), the same holds for the optimal \( s \) in (42). Hence the result. \( \square \)

We can use the bound derived in Proposition 1 as a conservative numerical estimate for \( \delta_H(t) \). In Fig. 9 we use the data from Section VI-A to plot the upper bound (43) for the Hausdorff distance between the reach set of (28) and its MVOE approximations. The plot shows that the MVOEs computed using the proposed algorithms result in a lower upper bound than the MVOEs computed using the standard SDP approach. This indicates that the proposed MVOEs approximate the reach set quite well compared to the SDP approach, even though their volumes are slightly larger than the SDP MVOEs, as noted in Table I.

VII. Conclusions

Computing a tight ellipsoidal outer approximation of convex set-valued data is necessary in many systems-control problems such as reachability analysis, especially when the convex set is described as set operations on other ellipsoids. Depending on the set operation, isolated results and algorithms are known in the literature to compute the minimum trace and minimum volume outer ellipsoidal approximations. In this paper, we unify such results by considering the \( p \)-sum of ellipsoids, which for different \( p \in [1, \infty] \), generate different convex sets from the summand ellipsoids. Our analytical results lead to efficient numerical algorithms, which are illustrated by the reach set computation for a discrete-time linear control system.

A specific direction of future study is to extend the reach set computation for hybrid systems, by computing the intersection of the guard sets with \( p \)-sum of ellipsoids. We hope that the models and methodologies presented here, will help to expand the ellipsoidal calculus tools [17], [18], [20], [32], [41] for systems-control applications, and motivate further studies to leverage the reported computational benefits in real-time applications.

ACKNOWLEDGEMENT

The author is grateful to Suvrit Sra for suggesting [40] the fixed point iteration (27) for \( p = 1 \). This research was partially supported by a 2018 Faculty Research Grant awarded by the Committee on Research from the University of California, Santa Cruz, and by a 2018 Seed Fund Award from CITRIS and the Banatao Institute at the University of California.

REFERENCES

[1] Y.-K. Choi, J.-W. Chang, W. Wang, M.-S. Kim, and G. Elber, “Continuous Collision Detection for Ellipsoids”, IEEE Transactions on Visualization and Computer Graphics, Vol. 15, No. 2, pp. 311–325, 2009.
[2] A. Best, S. Narang, D. Manocha, “Real-time Reciprocal Collision Avoidance with Elliptical Agents”, 2016 IEEE International Conference on Robotics and Automation (ICRA), pp. 298–305, 2016.
[3] Y. Yan, Q. Ma, G.S. Chirikjian, “Path Planning Based on Closed-form Characterization of Collision-free Configuration-spaces for Ellipsoidal Bodies, Obstacles, and Environments”, Proceedings of the 1st International Workshop on Robot Learning and Planning in Conjunction with 2016 Robotics: Science and Systems, pp. 13–19, 2016.
[4] F.C. Schweppe, “Recursive State Estimation: Unknown but Bounded Errors and System Inputs”, IEEE Transactions on Automatic Control, Vol. 13, No. 1, pp. 22–28, 1968.
[5] D. Bertsekas, and I. Rhodes, “Recursive State Estimation for A Set-Membership Description of Uncertainty”, IEEE Transactions on Automatic Control, Vol. 16, No. 2, pp. 117–128, 1971.
[6] F.M. Schlaepfer, and F.C. Schweppe, “Continuous-time State Estimation Under Disturbances Bounded by Convex Sets”. IEEE Transactions on Automatic Control, Vol. 17, No. 2, pp. 197–205, 1972.
[7] Y.N. Reshetnyak, “Summation of Ellipsoids in the Guaranteed Estimation Problem”, Journal of Applied Mathematics and Mechanics, Vol. 53, No. 2, pp. 193–197, 1989.
[8] E. Fogel, “System Identification via Membership Set Constraints with Energy Constrained Noise”, IEEE Transactions on Automatic Control, Vol. 24, No. 5, pp. 752–758, 1979.
[9] G. Belforte, B. Bona, and V. Cerone, “Parameter Estimation Algorithms for A Set-membership Description of Uncertainty”, Automatica, Vol. 26, No. 5, pp. 887–898, 1990.
[10] R.L. Kosut, M.K. Lau, S.P. Boyd, “Set-membership Identification of Systems with Parametric and Nonparametric Uncertainty”, IEEE Transactions on Automatic Control, Vol. 37, No. 7, pp. 929–941, 1992.
[11] A.B. Kurzhanski, and P. Varaiya, “Optimization of Output Feedback Control under Set-membership Uncertainty”, Journal of Optimization Theory and Applications, Vol. 151, No. 1, pp. 11–32, 2011.
[12] D. Angeli, A. Casavola, G. Franzè, and E. Mosca, “An Ellipsoidal Off-line MPC Scheme for Uncertain Polytopic Discrete-time Systems”, Automatica, Vol. 44, No. 12, pp. 3113–3119, 2008.
[13] W.M.J. Firey, “p-Means of Convex Bodies”, Mathematica Scandinavica, Vol. 10, pp. 17–24, 1962.
[14] E. Lutwak, “The Brunn-Minkowski-Firey Theory I: Mixed Volumes and the Minkowski Problem”. Journal of Differential Geometry, Vol. 38, no. 1, pp. 131–150, 1993.
[15] E. Lutwak, “The Brunn-Minkowski-Firey Theory II: Affine and Geominimal Surface Areas”. Advances in Mathematics, Vol. 118, no. 2, pp. 244–294, 1996.
[16] R. Schneider, Convex Bodies: the Brunn–Minkowski Theory, Encyclopedia of Mathematics and Its Applications, No. 151, Cambridge University Press, 2014.
[17] A.B. Kurzhanski, and I. Vályi, Ellipsoidal Calculus for Estimation and Control, Systems & Control: Foundations and Applications, Birkhäuser Boston and International Institute for Applied Systems Analysis, 1997.
[18] F.C. Schweppe, Uncertain Dynamic Systems. Prentice Hall, Englewood Cliffs, New Jersey, 1973.
[19] D.G. Maksarov, and J.P. Norton, “State Bounding with Ellipsoidal Set Description of the Uncertainty”, International Journal of Control, Vol. 65, No. 5, pp. 847–866, 1996.
[20] A.B. Kurzhanski, and P. Varaiya, Dynamics and Control of Trajectory Tubes. Theory and Computation. Systems & Control. Foundations & Applications. Springer, 2014.
[21] R.T. Rockafellar, Convex Analysis. Princeton Landmarks in Mathematics, Princeton University Press, 1970.
[22] E. Lutwak, D. Yang, and G. Zhang, “The Brunn-Minkowski-Firey Inequality for Nonconvex Sets”, Advances in Applied Mathematics, Vol. 48, No. 2, pp. 407–413, 2012.
[23] A. Halder, “On the Parameterized Computation of Minimum Volume Outer Ellipsoid of Minkowski Sum of Ellipsoids”, 2018 IEEE Conference on Decision and Control, available online: https://arxiv.org/pdf/1803.08157.pdf 2018.
[24] G.H. Hardy, J.E. Littlewood, and G. Pólya, Inequalities. 2nd ed. Cambridge, United Kingdom: Cambridge University Press, 1988.
[25] B. Lemmens, and R. Nussbaum, Nonlinear Perron-Frobenius Theory, Cambridge Tracts in Mathematics, Vol. 189, Cambridge University Press, 2012.
[26] S. Boyd, and L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[27] U. Krause, Positive Dynamical Systems in Discrete Time: Theory, Models, and Applications. Studies in Mathematics, Vol. 62, Walter de Gruyter GmbH & Co KG, 2015.
[28] F.L. Chernousko, “Optimal Guaranteed Estimates of Indeterminacies with the Aid of Ellipsoids. I”, Engineering Cybernetics, Vol. 18, No. 3, pp. 1–9, 1980.
[29] C. Durieu, E. Walter, and B. Polyak, “Multi-Input Multi-Output Ellipsoidal State Bounding”, Journal of Optimization Theory and Applications, Vol. 111, No. 2, pp. 273–303, 2001.
[30] F.L. Chernousko, and A.I. Ovseevich, “Properties of the Optimal Ellipsoids Approximating the Reachable Sets of Uncertain Systems”, Journal of Optimization Theory and Applications, Vol. 120, No. 2, pp. 223–246, 2004.
[31] A.A. Kurzhanskiy, and P. Varaiya, “Ellipsoidal Toolbox (ET)”, 45th IEEE Conference on Decision and Control, pp. 1498–1503, 2006.
[32] A.A. Kurzhanskiy, and P. Varaiya, “Ellipsoidal Techniques for Reachability Analysis of Discrete-time Linear Systems”, IEEE Transactions on Automatic Control, Vol. 52, No. 1, pp. 26–38, 2007.
[33] H.S. Witsenhausen, “Sets of Possible States of Linear Systems Given Perturbed Observations”, IEEE Transactions on Automatic Control, Vol. 13, No. 5, pp. 556–558, 1968.
[34] A. Girard, and C. Le Guernic, “Efficient Reachability Analysis for Linear Systems using Support Functions”, IFAC Proceedings Volumes, Vol. 41, No. 2, pp. 8966–8971, 2008.
[35] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM Studies in Applied Mathematics, Vol. 15, 1994.
[36] M. Grant, and S. Boyd. CVX: Matlab Software for Disciplined Convex Programming, version 2.0 beta. http://cvxr.com/cvx September 2013.
[37] F. John, “Extremum Problems with Inequalities as Subsidiary Conditions”. In Studies and Essays presented to R. Courant on his 60th Birthday, pp. 187–204, Interscience Publishers, 1948.
[38] H. Busemann, “The Foundations of Minkowskian Geometry”. Commentarii Mathematici Helvetici, Vol. 24, No. 1, pp. 156–187, 1950.
[39] A. Ben-Tal, and A. Nemirovski, Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications. MPS-SIAM Series on Optimization, Society for Industrial and Applied Mathematics, 2001.
[40] S. Sra (https://mathoverflow.net/users/8430/suvrit), Positive root of a polynomial, URL (version: 2018-03-23): https://mathoverflow.net/q/280061
[41] L. Ros, A. Sabater, and F. Thomas, “An Ellipsoidal Calculus Based on Propagation and Fusion”, IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), Vol. 32, No. 4, pp. 430–442, 2002.