Thermal fluctuations to thermodynamics of non-rotating BTZ black hole

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In this paper, we discuss the effect of small statistical thermal fluctuations around the equilibrium on the thermodynamics of small non-rotating BTZ black hole. This is done by evaluating the leading-order corrections to the thermodynamical equation of states, namely, entropy, free energy, internal energy, pressure, enthalpy, Gibbs free energy and specific heat quantitatively. In order to analyse the effect of perturbations on the thermodynamics, we plot various graphs and compare corrected and non-corrected thermodynamic quantities with respect to event horizon radius of non-rotating BTZ black hole. We also derive the first-order corrections to isothermal compressibility.

\textbf{Keywords}: Bekenstein entropy; Hawking temperature; Event horizon

\section{Overview and Motivations}

It was first revealed by classical theory of general relativity (GR) that area of event horizon of black hole never decreases \cite{1}. There was a clear indication that thermodynamic principles can be incorporated to the black holes analysis. In the thermodynamic system, the entropy plays a central role to study the thermodynamic properties and thermodynamic evolution takes place in such a way that entropy never decreases. The same effect was observed for the area of event horizon of black holes. This leads to a reasonable interpretation for the area as an entropy of a black hole and we can see that an increase of area of event horizon is reminiscent of law of increase of entropy of thermodynamic system. Also it was found that surface gravity $\kappa$ (which is measure of strength of gravitational field at the event horizon) remains constant over the event horizon surface. This in turn again hints the remniscence with zeroth law of thermodynamics, which states that in thermal equilibrium, there exists a common temperature parameter for the whole thermodynamic system. It is quite obvious from this resemblance that surface gravity $\kappa$ of black hole plays the role of temperature in thermodynamic system \cite{2}. These analogies compelled Bekenstein \cite{3} to propose a quantitative relation between entropy and area of event horizon of black hole. But this proposal seems to contradict the second law of thermodynamics as any thing that falls in black hole could not escape back. Consequently, it is impossible to achieve thermal equilibrium at any non-zero temperature between the black hole and any thermal radiation. However this puzzle was solved by taking the concept of quantum fluctuations into account, which amounts to the hawking evaporation \cite{4}. Since black hole evaporates and it loses the content (information) in it which reduces its size as long as the corrections to the thermodynamics of black hole due to the thermal fluctuations around the equilibrium (which arise from quantum fluctuations) are small. As a result, the Bekenstien entropy relation between the entropy and area also gets correction with some leading-order logarithmic terms \cite{5}.

In order to study the leading-order corrections to thermodynamic properties of black holes, several attempts have been already made. For example, the logarithmic corrections to the entropy of BTZ, string-theoretic and rest other black holes, whose microscopic degrees of freedom are described by underlying conformal field theory (CFT), have been found by using Cardy formula \cite{6, 7}. The correction terms to the entropy are still logarithmic in nature while considering background matter field into account \cite{8, 10}.

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The logarithmic nature of leading-order corrections to the entropy of BTZ black hole is also justified with the exact partition function approach \cite{11}. In fact, the corrections to entropy of black hole turned out to be logarithmic on Rademacher expansion of partition function as well \cite{12}. In the case of string-black hole correspondence \cite{13–16}, the leading-order corrections to entropy of black hole are logarithmic terms. Das et al. in Ref. \cite{17} have shown that no matter what kind of thermodynamic system is dealt with, it always leads to the logarithmic correction.

In recent past, the effects of small thermal fluctuation on the thermodynamics of small black holes have been studied in great details. For instance, the effect of quantum fluctuation on the thermodynamics of Gödel black hole was studied in Ref. \cite{18}. The leading-order corrections to the thermodynamics of Schwarzschild-Beltrami-de-sitter black hole \cite{19} and massive black hole in Ads space \cite{20} have been discussed. In Ref. \cite{21}, the effect of statistical fluctuations on thermodynamics of dilatonic black hole has been found. Again in \cite{22} the leading order corrections to Horava Lifshitz black hole were discussed and it was found that the logarithmic corrections originate from the thermal fluctuations and hence were interpreted as quantum loop corrections. The corrected thermodynamics of quasitopological black hole was examined in \cite{23}. In particular the stability and bound points for quasi topological black hole were studied and it was observed that the correction terms with positive correction parameter resulted in more stable black hole while negative correction parameter introduces instability in such black holes. In \cite{24} corrections to various thermodynamical quantities of Vander walls black hole were evaluated. Also the effect of small statistical fluctuations on equation of state for Vander walls black hole has been discussed. Furthermore in addition to leading order corrections to entropy of small black holes (which are found to be logarithmic in nature as expected), higher order corrections were also analyzed in ref.\cite{25–27}. It turned out that higher order corrections to entropy are inversely proportional to original entropy. In \cite{28} corrections to Reissner-Nordstrom, kerr, and charged Ads black holes were computed and in all these cases it was revealed that these corrections produce remnants. Moreover the effect of thermal fluctuations on various thermodynamical quantities of a modified Hayward black hole was analyzed in \cite{29}. Also the effect of these correction terms on the first law of thermodynamics was examined. The effects of small statistical fluctuations on the thermodynamics of stationary BTZ black holes are studied yet. This provides us an opportunity to bridge this gap.

Here we study the effect of thermal fluctuations on the thermodynamics of non-rotating BTZ black holes. In order to do so, we first derive the leading-order logarithmic corrections due to thermal fluctuations to the entropy of stationary BTZ black hole. In order to study the effect of this correction on the entropy, we do a comparative analysis by plotting a graph between corrected and non-corrected entropy with respect to event horizon radius. From the plot, we find that the entropy of the system is positive valued. Also, there exists a critical horizon radius inside which the corrected entropy is decreasing function. We notice that the correction parameter does not play a significant role for entropy of large sized black hole. However, for the entropy of small sized non-rotating BTZ black hole, role of the correction parameter turns out to be of paramount importance. Furthermore, we derive the the first-order correction to the free energy. To analyze the modification due to the thermal fluctuation, we plot the corrected and non-corrected free energies with respect to the event horizon radius. We observe here that the Helmholtz free energy is decreasing function of radius. There exists a critical horizon where thermal fluctuations do not affect the free energy. For larger black holes (with horizon radius larger than critical horizon) the corrected free energy becomes less negative than its equilibrium values. However, for smaller black holes, the correction terms make the free energy more negative. We also compute the leading-order corrections to the total mass (energy) of non-rotating BTZ black hole with the help of first law of thermodynamics and plot a graph for comparative analysis. From the plot, we see that total energy is monotonically increasing function of event horizon radius. The correction term due to the thermal fluctuations decreases the total energy of the system. We also derive the corrected pressure of BTZ black hole. Here we notice that for large sized black hole the corrected pressure coincides with the equilibrium pressure and becomes saturated. This means that for large black holes the small thermal fluctuations do not affect the pressure considerably. However, for very small black holes the pressure increases to asymptotic value due to the thermal fluctuations. We further derive the enthalpy of the system which is an increasing function of horizon radius for BTZ black holes. The correction terms due to thermal fluctuations decrease the value of enthalpy. As long as the value of correction parameter increases the value of enthalpy decreases. We
also study the effects of thermal fluctuations on Gibbs free energy which has zero value for the equilibrium case. From the plot, we notice that there exist two critical points between which Gibbs free energy is negative valued. Beyond the critical points, Gibbs free energy is positive valued. The corrected Gibbs free energy with larger correction parameter becomes more negative valued for smaller BTZ black holes. Finally, we discuss the corrected specific heat due to thermal fluctuation. From the plot, we infer that specific heat is an increasing function of horizon radius. The equilibrium value of specific heat is positive always which suggests that the BTZ black holes are in stable phase. However, the thermal fluctuations make specific heat negative for small black holes. This suggests that the black holes are in unstable phase due to thermal fluctuations and the thermal fluctuations do not affect the stability of large sized black holes, we also calculate the isothermal compressibility of the system under thermal fluctuations. We find that in equilibrium state takes compressibility diverges which means that system is highly compressible.

The paper is organized in the following way. In section II, we recapitulate the general expression for entropy under thermal fluctuation and compute it for the case of non-rotating BTZ black holes. In section III, we derive various other equations of state to describe the thermodynamics of black holes under the effect of thermal fluctuations. In section IV, we check stability of BTZ black holes under thermal fluctuations. Finally, we discuss results in the last section.

II. LOGARITHMIC CORRECTIONS TO ENTROPY OF NON-ROTATING BTZ BLACK HOLE

Let us start discussion by writing metric for the non-rotating BTZ black hole with cosmological constant $\Lambda = -\frac{1}{l^2}$ given as

$$ds^2 = \left( \frac{r^2}{l^2} - 8G_3M \right) dt^2 + \frac{dr^2}{\left( \frac{r^2}{l^2} - 8G_3M \right)} + r^2 d\theta^2,$$

where $M$ refers to mass of black hole and $G_3$ refers to the three dimensional Newton’s gravitational constant. This metric is of the form

$$ds^2 = f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2,$$

where metric function is given by

$$f(r) = \frac{r^2}{l^2} - 8G_3M.$$

Now, we compute Bekenstein-Hawking temperature from the above metric easily as following:

$$T_H = \frac{f'(r)}{4\pi} \bigg|_{r=r_+} = \frac{r_+}{2\pi l^2}.$$

The value of event-horizon radius $r_+$ can be obtained by setting $f(r) = 0$, which yields

$$r_+ = \sqrt{8G_3Ml}.$$

The corresponding entropy for BTZ non-rotating black hole is given by

$$S_0 = \frac{\pi r_+}{2G_3}.$$

In order to study the effect of thermal perturbations on the entropy of BTZ black hole, we first derive the exact expression for the entropy of BTZ black hole. In this regard, we write partition function describing BTZ black hole as follows

$$Z(\beta) = \int_0^\infty dE \rho(E) e^{-\beta E},$$
where $\beta = \frac{1}{T_H}$ as Boltzmann constant $k = 1$. Now, with the help of the inverse Laplace transform, one can get the density of states

$$\rho(E) = \frac{1}{2\pi i} \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} d\beta Z(\beta) e^{\beta E} = \frac{1}{2\pi i} \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} d\beta e^{S(\beta)}.$$  \hspace{1cm} (8)

Here, $S(\beta) = \ln Z(\beta) + \beta E$ represents the exact entropy for the black hole and this depends on temperature explicitly. If one reduces the size of black hole and expand $S(\beta)$ around equilibrium, then using the method of steepest descent (where $\frac{dS}{d\beta} = 0$ and $\frac{d^2S}{d\beta^2} > 0$), we get

$$S(\beta) = S_0 + \frac{1}{2} (\beta - \beta_0)^2 \frac{d^2S}{d\beta^2} |_{\beta = \beta_0} + \text{(higher order terms)},$$ \hspace{1cm} (9)

where $S_0$ represents the equilibrium (original) value of entropy. By inserting (9) in (8), we have

$$\rho(E) = \frac{e^{S_0}}{2\pi i} \int d\beta e^{\frac{1}{2}(\beta - \beta_0)^2 \frac{d^2S}{d\beta^2}}.$$ \hspace{1cm} (10)

Upon solving this integral, we get

$$\rho(E) = \frac{e^{S_0}}{\sqrt{2\pi \frac{d^2S}{d\beta^2}}}.$$ \hspace{1cm} (11)

Eventually, this leads to

$$S = S_0 - \ln(S_0 T_H^{-2})^{\frac{3}{2}}.$$ \hspace{1cm} (12)

Here, without loss of generality, we may replace the unit factor of second term by a more general correction parameter $\alpha$ as this takes different values too. Thus, the corrected entropy by incorporating small fluctuations around thermal equilibrium is given by

$$S = S_0 - \alpha \ln(S_0 T_H^{-2})^{\frac{3}{2}}.$$ \hspace{1cm} (13)

Moreover, it should be noted that second term in the above equation (which is obviously logarithmic in nature) appears due to small statistical fluctuations around the thermal equilibrium. In other words, we can say that the second term represents the leading-order corrections to entropy of BTZ black hole.

Now, by inserting the values of hawking temperature (4) and Bekenstein entropy (6) to the expression (13), we get the perturbed expression for entropy of non-rotating BTZ black hole as

$$S = \frac{\pi r_+}{2G_3} - \frac{3}{2} \alpha \ln \frac{r_+^3}{8\pi G_3 l^2}.$$ \hspace{1cm} (14)

Here, we observe that the last two terms of above expression, which are the logarithmic in nature, represent corrections to the entropy of stationary BTZ black hole. These corrections appear due to the thermal fluctuations around equilibrium. By setting $\alpha = 0$, one can retain the equilibrium entropy of the system. Here, we can make a comparative analysis between the corrected and uncorrected entropy by plotting a graph between them with respect to the event horizon radius with different possible values of correction parameter. From the FIG. 1 it is evident that the entropy of the system is positive valued. Also, there exists a critical horizon radius below which the corrected entropy is decreasing function. However, similar to the equilibrium entropy behavior, the corrected entropy is an increasing function for the BTZ black holes with larger horizon radius than the critical value. This hints an important point that thermodynamics of large sized black holes is not much affected by the small thermal fluctuations as expected.
III. FIRST-ORDER CORRECTED THERMODYNAMIC EQUATION OF STATES

In this section, we would like to compute thermodynamical equation of states by incorporating small thermal fluctuation to the system. Once we have expressions of entropy and temperature, it is easy to compute various other equations of states. For example, Helmholtz free energy can be evaluated with the help of following formula:

\[ F = -\int SdT_H. \]  (15)

By exploiting relations (4), (14) and (15), we calculate the exact Helmholtz free energy for the BTZ black hole as follows,

\[ F = -\frac{r_+^2}{8G_3l^2} + \frac{3\alpha}{4\pi l^2} r_+ \ln r_+ - \frac{3\alpha}{4\pi l^2} r_+ - \frac{1}{2\pi l^2} \alpha r_+ \ln l^2 \sqrt{8\pi G_3}. \]  (16)

The behavior of Helmholtz free energy with respect to horizon radius can be seen from FIG. 2. Here we see that the Helmholtz free energy shows a decreasing behavior with increasing radius. There exists a critical
horizon where thermal fluctuations do not affect the free energy. For larger black holes (with horizon radius larger than critical horizon) the corrected free energy becomes less negative than its equilibrium values. However, for smaller black holes, the correction terms make the free energy more negative.

The first law of thermodynamics for uncharged stationary BTZ black holes reads

\[ dE = T_H dS, \]  

which upon integration yields the energy

\[ E = \int T_H dS. \]  

Using the values for \( S \) (14), \( T_H \) (14) and (18), we get total energy by incorporating thermal fluctuations for BTZ black hole

\[ E = \frac{r_+^2}{8G_3l^2} - \frac{3\alpha r_+}{4\pi l^2}. \]  

In order to do comparative analysis of corrected and equilibrium total energy, we plot FIG. 3. Here we observe that total energy is monotonically increasing function of event horizon radius. The correction term due to the thermal fluctuations decreases the total energy of the system.

From the area-entropy theorem, we know that the entropy of black hole is proportional to the area covered by the event horizon. So, we can find the volume of BTZ black hole as follows,

\[ V = \int S_0 dr_+ = \frac{\pi r_+^2}{4G_3}. \]  

Since BTZ black holes are considered as the thermodynamic system, so one must be able to calculate other macroscopic parameters such as pressure, which has the following standard definition in thermodynamics:

\[ P = \frac{-dF}{dV} = \frac{dF}{dr_+} \frac{dr_+}{dV}. \]  

So we need to find \( \frac{dF}{dr_+} \) and \( \frac{dV}{dr_+} \). By plugging the values of (16) and (20) in (21), we obtain

\[ P = \frac{1}{2\pi l^2} - \frac{3\alpha G_3}{2\pi^2 l^2 r_+} \ln r_+ + \frac{\alpha G_3}{\pi^2 l^2 r_+} \ln (l^2 \sqrt{8\pi G_3}). \]
This expression refers to exact pressure of BTZ black hole under the effect of small statistical fluctuations. We discuss the effect of the thermal fluctuations on the equilibrium pressure by comparing the plot of corrected and equilibrium pressure with respect to event horizon radius.

From the FIG. 4 it is obvious that for large sized black hole (i.e. for large values of $r_+$) the corrected pressure coincides with the equilibrium pressure and becomes saturated. This means that for large black holes the small thermal fluctuations do not affect the pressure considerably. However, for very small black holes (i.e. for vanishingly small $r_+$) the pressure increases to the large value due to the thermal fluctuations. So, remarkably, the contributions of thermal fluctuations to the pressure of small BTZ black holes can not be ignored. In fact, corrected pressure takes asymptotic values when horizon radius tends to zero.

The enthalpy is an important thermodynamic quantity, equivalent to the total heat content of a system, defined by

$$H = E + PV.$$  \hspace{1cm} (23)

By using the expressions of total energy (19), pressure (22) and volume (20), the corrected enthalpy is calculated by

$$H = \frac{r_+^2}{4G_3l^2} - \frac{3\alpha r_+}{4\pi l^2} r_+ \ln r_+ + \frac{\alpha r_+}{4\pi l^2} \ln (l^2 \sqrt{8\pi G_3}).$$  \hspace{1cm} (24)

From this expression the behavior of enthalpy can be studied. In order to study the effect of thermal fluctuations on the enthalpy, we plot a graph. From the plot, it is evident that the enthalpy is an increasing function of horizon radius. The correction terms due to thermal fluctuations decrease the value of enthalpy. As long as the value of correction parameter increases the value of enthalpy decreases.

The Gibbs free energy is the maximum amount of work that can be extracted from a thermodynamically closed system. This maximum can be attained only in a completely reversible process. In order to compute the corrected Gibbs free energy for the BTZ black holes under the effect of thermal fluctuations, we use following definition:

$$G = F + PV.$$  \hspace{1cm} (25)

Plugging the values of corrected Helmholtz free energy (16), pressure (22), and volume (20) in the above expression, we get the following expression for first-order corrected Gibbs free energy:

$$G = \frac{3\alpha}{8\pi l^2} r_+ \ln r_+ - \frac{3\alpha}{4\pi l^2} r_+ - \frac{\alpha r_+}{4\pi l^2} \ln (l^2 \sqrt{8\pi G_3}).$$  \hspace{1cm} (26)
FIG. 5: Enthalpy vs. the black hole horizon for $G_3$ and $l = 1$. Here, $\alpha = 0$ denoted by blue line, $\alpha = 1$ denoted by red line, and $\alpha = 3$ denoted by black line.

FIG. 6: Gibbs free energy vs. the black hole horizon for $G_3$ and $l = 1$. Here, $\alpha = 1$ denoted by red line, and $\alpha = 3$ denoted by black line.

In order to study the behavior of Gibbs free energy and the effects of thermal fluctuations, we plot a graph. From the figure, we can see that the Gibbs free energy is zero for BTZ black in thermal equilibrium. However, when the small statistical fluctuations around equilibrium is considered, we get a non-zero Gibbs free energy. We notice that there exist two critical points between which Gibbs free energy is negative valued. After the second critical value, it becomes positive valued. The corrected Gibbs free energy with larger correction parameter $\alpha$ becomes more negative valued for smaller BTZ black holes (i.e. $r_+$ is less than the second critical point). However, the corrected Gibbs free energy with larger correction parameter $\alpha$ becomes more positive valued for larger BTZ black holes.

IV. STABILITY

Now, in order to discuss the stability of BTZ black hole, we just compute its specific heat by incorporating small thermal fluctuations. In this regards, one can adopt two different approaches. In first approach, the specific heat of the BTZ black hole will be calculated and the positivity of the specific heat will ensure the local thermal stability of the black holes. In this case, the unstable black holes may
go under phase transition. In second approach, the thermal stability can be investigated by calculating the determinant of Hessian matrix of mass with respect to extensive variables. Here, we will follow the first approach and will check the role of thermal fluctuations on the stability of black holes. By using the standard relation, the specific heat is calculated as

\[ C = \frac{dE}{dT_H} = \frac{dE}{dr_+ dr_+} \frac{\pi r_+}{2 G_3} \frac{3\alpha}{2}. \]  

(27)

We plot a graph to study the behavior of specific heat with respect to horizon radius. From the graph,

FIG. 7: Specific heat vs. the black hole horizon for \( G_3 \) and \( l = 1 \). Here, \( \alpha = 0 \) denoted by blue line, \( \alpha = 1 \) denoted by red line, and \( \alpha = 3 \) denoted by black line.

one can infer that specific heat is an increasing function of horizon radius. The equilibrium value of specific heat is positive always which suggests that the BTZ black holes are in stable phase without any thermal fluctuations. However, the thermal fluctuations cause negative values to the specific heat for small black holes. This suggests that the black holes are in unstable phase due to thermal fluctuations and the thermal fluctuations do not affect the stability of large sized black holes.

Now, let us check the effects of thermal fluctuations on the isothermal compressibility (\( K \)) of uncharged non-rotating BTZ black hole. The isothermal compressibility measures the respective change in volume due to change in pressure. This is calculated by

\[ K = -\frac{1}{V} \frac{dV}{dP} = -\frac{1}{V} \frac{dV}{dr_+} \frac{dr_+}{dP} = \frac{4\pi^2 r_+}{3\alpha G_3(1 + \frac{1}{3} \ln l^2 \sqrt{8\pi G_3 - \ln r_+})}. \]  

(28)

From the resulting expression, it is obvious that when \( \alpha \) takes zero value the \( K \) diverges which means that system is highly compressible. Consequently, we get the zero value of bulk modulus as the compressibility is inverse to the bulk modulus.

V. CONCLUSION

In this work, we have investigated the effect of statistical fluctuations on the small sized BTZ black holes. We have found that the thermal fluctuations around equilibrium lead to a (logarithmic) correction to the entropy of stationary uncharged BTZ black holes. We have plotted a graph between entropy and horizon radius to discuss the effect of thermal fluctuations on the entropy of non-rotating BTZ black hole. We have found that the entropy takes positive values always and there exists a critical point below which the corrected entropy is decreasing function. However, the corrected entropy is an increasing function for the large sized BTZ black holes (for which horizon radius is larger than this critical value).
Moreover, we have calculated the first-order corrected Helmholtz free energy for the BTZ black hole where corrections appear due to the thermal fluctuations. We have checked the behavior of Helmholtz free energy and done comparative analysis between uncorrected and corrected free energies analytically by plotting a graph. We noticed that the Helmholtz free energy is a decreasing function of horizon radius. There exists a critical point where the free energy is unaffected by the thermal fluctuations. For large black holes (with horizon radius larger than the critical horizon), the corrected free energy becomes less negative with respect to its equilibrium values. However, for smaller black holes, the correction terms make the free energy more negative.

We have evaluated the corrected values of total energy (mass) to BTZ black hole with the help of first law of thermodynamics. We have found that total mass of the system is a monotonically increasing function of event horizon radius. The correction term due to the thermal fluctuations decreases the total mass of the system. We have calculated the volume and pressure of the system also. From the plot, we have observed that for large sized black hole the corrected pressure coincides with the equilibrium pressure and becomes saturated. However, for very small black holes (i.e. for vanishingly small horizon radius) the pressure takes asymptotically large values due to the thermal fluctuations. In fact, for large black holes the small thermal fluctuations do not affect the pressure considerably as expected. Remarkably, the contributions of thermal fluctuations to the pressure of small BTZ black holes is considerable. In fact, the corrected pressure has become asymptotic large when horizon radius tends to zero.

Furthermore, we have calculated the corrected expression for the enthalpy. From the plot, we have found that the enthalpy is an increasing function of horizon radius. The thermal fluctuations decrease the value of enthalpy. The enthalpy decreases with the larger values of correction parameter. The non-zero Gibbs free energy has been found due to the thermal fluctuations. From the plot, we have noticed that there exist two critical points. The Gibbs free energy is negative valued for small black holes. It takes positive values for large sized black holes. The corrected Gibbs free energy with larger correction parameter has become more negative valued for smaller BTZ black holes and become more positive valued for larger BTZ black holes.

Finally, we have derived the corrected expression of specific heat for BTZ black hole and found that specific heat is increasing function of event horizon radius. From the graph, it is observed that the equilibrium value of specific heat is always positive which suggests that the BTZ black holes are in stable phase without any thermal fluctuations. However, due to the thermal fluctuations, the specific heat has become negative for small black holes. This suggests that the small black holes are in unstable phase due to thermal fluctuations. We have calculated the compressibility of the system, which diverges in equilibrium state which means that system is highly compressible in equilibrium state. This leads to the zero bulk modulus as the compressibility is inverse to the bulk modulus.

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