An iterative method for Kirchhoff type equations and its applications

Qiuyi Dai

College of Mathematics and Statistics, Hunan Normal University,
Changsha Hunan 410081, PR China

1. Iterative method

Let $A : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with $A(s, t) \geq m$ for some positive constant $m$. For $a, b \geq 0$, we set

$$M_{a,b} = \max \{ A(s, t) : (s, t) \in [0, a] \times [0, b] \}.$$ 

No loss of generality, we may assume $m \geq 1$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. Denote by $L^p(\Omega)$ the standard Lebesgue’s space with norm $\| \cdot \|_p$. Let $H^1_0(\Omega)$ be the standard Sobolev space with norm $\| \nabla \cdot \|_2$, and $2^*$ be the critical exponent for the Sobolev embedding (that is $2^* = +\infty$ if $N \leq 2$, and $2^* = 2N/(N - 2)$ if $N \geq 3$). Let $\Delta$ be the Laplace operator, and $g(x, s)$ be a Holder continuous function defined on $\Omega \times \mathbb{R}$. We consider the following problem of generalized Kirchhoff-Carrier type equations

$$\begin{cases} 
- A(\|u\|_p, \|\nabla u\|_2) \Delta u = g(x, u) & x \in \Omega, \\
u = 0 & x \in \partial \Omega.
\end{cases} \quad (1.1)$$

Assume that $g(x, s)$ satisfies.

(G1): There are two functions $0 \leq \varphi(x) \leq \psi(x)$ such that

$$\begin{cases} 
- \Delta \varphi \leq g(x, \varphi) & x \in \Omega, \\
- \Delta \psi \geq g(x, \psi) & x \in \Omega, \\
\varphi = \psi = 0 & x \in \partial \Omega.
\end{cases} \quad (1.2)$$

(G2): There exists a real number $\alpha$ such that for any $0 < \beta \leq 1$ and $\omega \in [\beta \varphi, \psi]$, there holds

$$- \beta^\alpha \Delta \varphi \leq g(x, \omega) \leq - \Delta \psi.$$ 

This short note devotes to propose an iterative procedure for finding solution of problem (1.1). Iterative procedure based on comparison principle of Kirchhoff type operator itself

E-mail addresses: qiuyidai@aliyun.com.

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has been used to find solution to Kirchhoff equations by many authors (see for example [1, 2, 7, 5, 6, 8]). However, comparison principle may cease to validate for general Kirchhoff type operator except for that possessing some monotonicity property. Instead, our iterative procedure based only on the comparison principle of Laplace operator. Therefore, we only need to put mini restrictions on Kirchhoff type operator. The main result we will prove can be stated as

**Theorem 1.1** Assume that (G1) and (G2) hold, $1 \leq p \leq 2^*$. Let $a = \|\psi\|_p$ and $b = \|\nabla\psi\|_2$. If there exists a function $\varphi_0(x)$ such that

$$\lim_{n \to +\infty} \left( \frac{1}{M_{a,b}} \sum_{k=0}^{n-1} \alpha^k \varphi(x) \right) = \varphi_0(x),$$

then problem (1.1) has at least one nonnegative solution $u(x)$ with property $\varphi_0(x) \leq u(x) \leq \psi(x)$ for any $x \in \Omega$.

**Lemma 1.2** For arbitrary two positive numbers $e$ and $d$, the algebraic system

$$\begin{cases}
  sA(s,t) = e \\
  tA(s,t) = d
\end{cases}$$

has positive solution.

**Proof:** For any fixed $t$, we consider $f(s) = sA(s,t) - e$. Since $f(0) = -e < 0$, and $f(s) \geq ms - e \to +\infty$ as $s \to +\infty$, we infer from the continuity of $f(s)$ that there exists positive number $s(t)$ such that $f(s(t)) = 0$. Moreover $s(t)$ is continuous with respect to $t$ due to the continuity of $A(s,t)$. Next, we consider the function $h(t) = tA(s(t),t) - d$. Since $h(0) = -d < 0$, and $h(t) \geq mt - d \to +\infty$ as $t \to +\infty$, we know that there exists a positive number $t^*$ such that $h(t^*) = 0$. Set $s^* = s(t^*)$. Then $(s^*, t^*)$ is a positive solution of system (1.3). This concludes Lemma 1.2.

**Lemma 1.3** For any $f(x) \in C(\Omega)$, the following problem

$$\begin{cases}
  -A(\|u\|_p, \|\nabla u\|_2)\Delta u = f(x) & x \in \Omega \\
  u = 0 & x \in \partial\Omega
\end{cases}$$

has at least one solution.

**Proof:** If $u$ is a solution of problem (1.4), then it is easy to see that $v = A(\|u\|_p, \|\nabla u\|_2)u$ is a solution of the problem

$$\begin{cases}
  -\Delta v = f(x) & x \in \Omega \\
  v = 0 & x \in \partial\Omega
\end{cases}$$

Since problem (1.5) has solution $v(x)$ for any $f(x) \in C(\Omega)$, we can find a solution of problem (1.4) with the form $u(x) = \frac{v(x)}{A(s,t)}$ provided that $(s,t)$ is a positive solution of the following algebraic system

$$\begin{cases}
  sA(s,t) = \|v\|_p, \\
  tA(s,t) = \|\nabla v\|_2.
\end{cases}$$

2
Therefore, the conclusion of Lemma 1.3 follows from Lemma 1.2.

**Proof of Theorem 1.1:** Let $\varphi$, $\psi$ and $\alpha$ be given in (G1) and (G2). Set $a = \|\psi\|_p$, $b = \|\nabla\psi\|$. We see that $M_{a,b} \geq m \geq 1$. To prove Theorem 1.1, we construct an iterative sequence $\{u_n\}_{n=1}^{+\infty}$ in the following way.

Initially, we set $u_1 = \varphi$. Then we get $u_{n+1}$ from $u_n$ by solving the following problem

\[
\begin{cases}
-A(\|u_{n+1}\|_p, \|\nabla u_{n+1}\|_2)\Delta u_{n+1} = g(x, u_n) & x \in \Omega, \\
u_{n+1} = 0 & x \in \partial\Omega.
\end{cases}
\]  

(1.7)

Obviously, Lemma 1.3 ensures that the sequence $\{u_n\}_{n=1}^{\infty}$ is well defined. Moreover, by induction method, we can claim

\[
(\frac{1}{M_{a,b}})\sum_{k=0}^{n-1} \alpha^k \varphi(x) \leq u_{n+1} \leq \psi(x) \quad \text{and} \quad \|\nabla u_{n+1}\| \leq \|\nabla \psi\| \quad \text{for} \quad n = 1, 2, \cdots.
\]  

(1.8)

In fact, when $n = 1$, by (G1) and (G2) we have

\[
\begin{cases}
-\Delta \varphi \leq -A(\|u_2\|_p, \|\nabla u_2\|_2)\Delta u_2 \leq -\Delta \psi & x \in \Omega, \\
u_2 = \varphi = \psi = 0 & x \in \partial\Omega.
\end{cases}
\]  

(1.9)

From this and the comparison principle of Laplacian, we can deduce

\[
\frac{1}{A(\|u_2\|_p, \|\nabla u_2\|_2)} \varphi(x) \leq u_2 \leq \psi(x) \quad \text{and} \quad \|\nabla u_2\| \leq \|\nabla \psi\|.
\]  

(1.10)

Noting $A(\|u_2\|_p, \|\nabla u_2\|_2) \leq M_{a,b}$, we get

\[
\frac{1}{M_{a,b}} \varphi(x) \leq u_2 \leq \psi(x) \quad \text{and} \quad \|\nabla u_2\| \leq \|\nabla \psi\|.
\]  

(1.11)

This implies that (1.8) is valid for $n = 1$.

Inductively, we assume that (1.8) is valid for $n = l$. That is

\[
(\frac{1}{M_{a,b}})\sum_{k=0}^{l-1} \alpha^k \varphi(x) \leq u_{l+1} \leq \psi(x) \quad \text{and} \quad \|\nabla u_{l+1}\| \leq \|\nabla \psi\|.
\]  

(1.12)

Then, by (G1) and (G2) we have

\[
\begin{cases}
-(\frac{1}{M_{a,b}})\sum_{k=1}^{l} \alpha^k \Delta \varphi \leq -A(\|u_{l+2}\|_p, \|\nabla u_{l+2}\|_2)\Delta u_{l+2} \leq -\Delta \psi & x \in \Omega, \\
u_{l+2} = \varphi = \psi = 0 & x \in \partial\Omega.
\end{cases}
\]  

(1.13)

Therefore, comparison principle of Laplace operator implies

\[
\frac{1}{A(\|u_{l+2}\|_p, \|\nabla u_{l+2}\|_2)}(\frac{1}{M_{a,b}})\sum_{k=1}^{l} \alpha^k \varphi(x) \leq u_{l+2} \leq \psi(x) \quad \text{and} \quad \|\nabla u_{l+2}\| \leq \|\nabla \psi\|.
\]  

(1.14)
Again, by making use of $A(\|u_{l+2}\|_p, \|\nabla u_{l+2}\|_2) \leq M_{a,b}$, we obtain
\[
\sum_{k=a}^{l} \alpha_k \varphi(x) \leq u_{l+2} \leq \psi(x) \quad \text{and} \quad \|\nabla u_{l+2}\| \leq \|\nabla \psi\|. \tag{1.15}
\]
This implies that (1.8) is valid for $n = l + 1$. Therefore, we conclude the claim (1.8).

With (1.8) established, we can deduce from the regularity theory of elliptic equations that
\[
\|u_n\|_{C^2,\tau}(\Omega) \leq C
\]
for some positive constants $C$ and $\tau \in (0, 1)$ independent of $n$. Therefore, up to a subsequence, $\{u_n\}_{n=1}^{\infty}$ converges in $C^2(\Omega)$ to a function $u$ which is obviously a solution of problem (1.1) with property $\varphi_0 \leq u \leq \psi$. This completes the proof of Theorem 1.1.

2. Applications

As applications of Theorem 1.1, we give some concrete examples in this section. The first example is a Dirichlet problem of inhomogeneous Carrier’s equation.

Example 1: Let $f(x) \in C^1(\Omega) \backslash \{0\}$ be a function such that the following problem has a solution
\[
\begin{aligned}
-\Delta \phi &= f(x) \quad x \in \Omega, \\
\phi &\geq 0 \quad x \in \Omega, \\
\phi &= 0 \quad x \in \partial \Omega.
\end{aligned} \tag{2.1}
\]
Obviously, $f(x)$ may change sign. For $d, p > 0$, we consider the following problem with positive parameter $\lambda$.
\[
\begin{aligned}
-(1 + d\|u\|_2^2)\Delta u &= u^p + \lambda f(x) \quad x \in \Omega, \\
u &> 0 \quad x \in \Omega, \\
u &= 0 \quad x \in \partial \Omega.
\end{aligned} \tag{2.2}
\]
It is worth pointing out that Carrier’s operator $-(1 + d\|\cdot\|_2^2)\Delta$ is not a variational operator, and it also lack of comparison principle. Therefore, powerful tools of variational and sub-supersolution method can not be used directly to study problem (2.2). Here, we use Theorem 1.1 to conclude the following result.

Theorem 2.1 There exists a positive number $\lambda_f$ such that problem (2.2) has at least one solution for any $\lambda \in (0, \lambda_f)$.

Proof: Using the notation given in Section 1, we have $g(x, \omega) = \omega^p + \lambda f(x)$. To complete the proof of Theorem 2.1, we only need to show that $g(x, \omega)$ verifies (G1) and (G2). To verify (G1), we let $\phi(x)$ be the solution of problem (2.1). Taking $\varphi = \lambda \phi$, then we have
\[
\begin{aligned}
-\Delta \varphi &= \lambda f(x) \leq \varphi^p + \lambda f(x) \quad x \in \Omega, \\
\varphi &\geq 0 \quad x \in \Omega, \\
\varphi &= 0 \quad x \in \partial \Omega.
\end{aligned} \tag{2.3}
\]
Let $u(x)$ be the solution of the following problem

\[
\begin{cases}
-\Delta u = 1 & x \in \Omega, \\
u = 0 & x \in \partial \Omega.
\end{cases}
\] (2.4)

Choosing $M_0 > 0$ so small that

\[M_0 > M_0^p \max_{x \in \Omega} u^p(x) + M_0^p \max_{x \in \Omega} |f(x)|,\]

and setting $\psi(x) = M_0 u(x)$, we can easily check that

\[
\begin{cases}
-\Delta \psi = M_0 \geq \psi^p + \lambda f(x) & x \in \Omega, \\
\psi = 0 & x \in \partial \Omega
\end{cases}
\] (2.5)

for any $\lambda \in (0, M_0^p)$.

Taking (2.3) and (2.5) into account, we infer from the strong comparison principle for Laplace operator that

\[\varphi(x) < \psi(x) \text{ for } x \in \Omega \text{ and } \lambda \in (0, M_0^p).\] (2.6)

Let $\lambda_f = M_0^p$. Then for any $\lambda \in (0, \lambda_f)$, we have $0 \leq \varphi < \psi$, and

\[
\begin{cases}
-\Delta \varphi \leq \varphi^p + \lambda f(x) & x \in \Omega, \\
-\Delta \psi \geq \psi^p + \lambda f(x) & x \in \Omega, \\
\varphi = \psi = 0 & x \in \partial \Omega
\end{cases}
\] (2.7)

Therefore, (G1) is satisfied. Noting that for any $0 < \beta \leq 1$ and $\omega \in [\beta \varphi, \psi]$ there hold

\[-\Delta \varphi = \lambda f(x) \leq \omega^p + \lambda f(x) \leq \psi^p + \lambda f(x) \leq -\Delta \psi,
\]

we can see that (G2) is satisfied with $\alpha = 0$. Consequently, the conclusion of Theorem 2.1 follows from Theorem 1.1 and the strong comparison principle of Laplace operator.

The second example is a Dirichlet problem of Kirchhoff-Carrier type equation involving cocave-convex nonlinearity.

**Example 2:** Assume $c, d > 0$ be constants, and $0 < q < 1 < p$. For parameter $\mu > 0$, we consider the problem

\[
\begin{cases}
-(1 + c\|u\|_2^2 + d\|\nabla u\|_2^2)\Delta u = \mu u^q + u^p & x \in \Omega, \\
u > 0 & x \in \Omega, \\
u = 0 & x \in \partial \Omega.
\end{cases}
\] (2.8)

By making use of Theorem 1.1, we can prove the following Theorem.

**Theorem 2.2** There exists a positive number $\mu_0$ such that problem (2.8) has at least one solution for any $\mu \in (0, \mu_0)$.
Proof: In this example, we have \( g(x, \omega) = \mu \omega^q + \omega^p \). To verify (G1), we let \( \lambda_1(\Omega) \) be the first eigenvalue of the following eigenvalue problem

\[
\begin{align*}
-\Delta \phi &= \lambda \phi \quad x \in \Omega, \\
\phi &= 0 \quad x \in \partial \Omega,
\end{align*}
\]

and denote by \( \phi_1(x) \) the positive first eigenfunction which is normalized so that \( \max_{x \in \Omega} \phi_1(x) = 1 \). Since \( 0 < q < 1 \), we can choose \( M_0 \) so small that

\[
\lambda_1(\Omega)M_0 \leq \mu M_0^q.
\]

Set \( \varphi(x) = M_0 \phi_1(x) \). Then, we have

\[
\begin{align*}
-\Delta \varphi &= M_0 \lambda_1(\Omega) \phi_1 \leq \mu M_0^q \phi_1^q \leq \mu \varphi^q \quad x \in \Omega, \\
\varphi &= 0 \quad x \in \partial \Omega.
\end{align*}
\]

Let \( \xi(x) \) be the solution of the following problem

\[
\begin{align*}
-\Delta \xi &= 1 \quad x \in \Omega, \\
\xi &= 0 \quad x \in \partial \Omega.
\end{align*}
\]

Since \( p > 1 \), we can choose \( M_1 \) so small that

\[
M_1 > M_1^p \max_{x \in \Omega} \xi(x).
\]

Setting \( \psi(x) = M_1 \xi(x) \), and \( \mu_0 = \frac{M_1 - M_1^p \max_{x \in \Omega} \xi(x)}{M_1^p \max_{x \in \Omega} \xi(x)} \), then, for any \( \mu \in (0, \mu_0) \), we have

\[-\Delta \psi = M_1 > \mu M_1^p \max_{x \in \Omega} \xi(x) + M_1^p \max_{x \in \Omega} \xi(x) \geq \mu \psi^q + \psi^p.
\]

If necessary, we can choose \( M_0 \) even more small so that \( 0 < \varphi(x) \leq \psi(x) \). Therefore, for any \( \mu \in (0, \mu_0) \), we have

\[
\begin{align*}
-\Delta \varphi &\leq \mu \varphi^q + \varphi^p \quad x \in \Omega, \\
-\Delta \psi &\geq \mu \psi^q + \psi^p \quad x \in \Omega, \\
0 < \varphi &\leq \psi \quad x \in \Omega, \\
\varphi = \psi &= 0 \quad x \in \partial \Omega.
\end{align*}
\]

This implies that (G1) hold. Noting that for any \( 0 < \beta \leq 1 \) and \( \omega \in [\beta \varphi, \psi] \), we can infer from (2.10) and (2.12) that

\[-\beta^q \Delta \varphi \leq \mu \beta^q \varphi^q \leq \mu \omega^q + \omega^p \leq \mu \psi^q + \psi^p \leq -\Delta \psi.
\]

This implies that (G2) is valid for \( \alpha = q \). Moreover, we can easily see that

\[
\lim_{n \to +\infty} \sum_{k=0}^{n-1} q^k = \frac{1}{1 - q}.
\]
Therefore, we can infer from Theorem 1.1 that problem (2.8) has at least one solution $u(x)$ with property 

$$0 < \left( \frac{1}{1 + c\|\psi\|^2 + d\|\nabla\psi\|^2} \right)^{1/q} \varphi(x) \leq u(x) \leq \psi(x) \quad \text{for} \quad x \in \Omega.$$ 

This completes the proof of Theorem 2.2.

In the above examples, the operator or nonlinear term have more or less monotonicity property. Here, we give an example whose operator and nonlinear term are all not monotone.

**Example 3:** Assume that $f(x)$ satisfies the condition given in Example 1. We consider the following problem

$$
\begin{cases}
-(1 + d \sin^2(\|\nabla u\|_2)) \Delta u = \sin^2(u) + f(x) & x \in \Omega, \\
u > 0 & x \in \Omega, \\
u = 0 & x \in \partial \Omega.
\end{cases}
$$

(2.13)

By making use of Theorem 1.1, we prove the following result.

**Theorem 2.3** For any $f(x)$ satisfying the condition given in Example 1, Problem (2.13) has at least one solution.

**Proof:** Let $\varphi(x)$ be the solution of problem (2.1). Then we have 

$$-\Delta \varphi = f(x) \leq \sin^2(\varphi) + f(x) \quad \text{for} \quad x \in \Omega.
$$

Let $\xi(x)$ be the solution of the following problem

$$
\begin{cases}
-\Delta \xi = 1 & x \in \Omega, \\
\xi = 0 & x \in \partial \Omega.
\end{cases}
$$

(2.14)

Choosing $M_0 > 1 + \max_{x \in \Omega} |f(x)|$, then $\psi(x) = M_0 \xi(x)$ satisfies 

$$-\Delta \psi = M_0 > \sin^2(\psi) + f(x).$$

Moreover, by strong comparison principle we still have

$$0 \leq \varphi(x) < \psi(x) \quad \text{for} \quad x \in \Omega.$$

Therefore, (G1) is satisfied with the above determined $\varphi$ and $\psi$. Noting that for any $0 < \beta \leq 1$ and $\omega \in [\beta \varphi, \psi]$ we have

$$-\Delta \varphi = f(x) \leq \sin^2(\omega) + f(x) \leq 1 + f(x) < M_0 = -\Delta \psi,$$

(G2) is satisfied with $\alpha = 0$. Consequently, it follows from Theorem 1.1 that problem (2.13) has at least one solution $u(x)$ with property $0 \leq \varphi(x) \leq u(x) \leq \psi(x)$ for any $x \in \Omega$. Finally, the positivity of $u(x)$ follows from the strong comparison principle of Laplace operator. This completes the proof of Theorem 2.3.
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