Howe Duality for an Induced Model of Lattice $U(N)$ Yang–Mills Theory

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Abstract

We propose an approach that views $U(N_c)$ Yang–Mills theory as the critical point of an induced gauge model on the lattice. Similar recent proposals based on the color–flavor transformation rely on taking the limit of an infinite number of infinitely heavy particles. In contrast, we couple a finite number $N_b$ of auxiliary boson flavors to the gauge field and argue that Yang–Mills theory is induced when $N_b$ exceeds $N_c$ and the boson mass is lowered to a critical point. Using the notion of Howe duality we transform the induced gauge model to a dual formulation in terms of local gauge invariant variables. In the abelian case the Howe duality transform turns out to coincide with the standard one, taking weakly coupled $U(1)_{d=4}$ to strongly coupled $\mathbb{Z}_{d=4}$ lattice gauge theory.

1 Introduction

In what has come to be called “induced QCD”, one starts from a theory of auxiliary fields coupled minimally to a gauge field background with gauge group $G$, the notable feature being that the gauge field by itself has no dynamics. A dynamical theory of gauge fields, typically with nonlocal gauge interactions, is induced by elimination of the auxiliary fields. Such an approach \cite{1,2,3,4} has been suggested on the lattice and in the continuum, using scalar fields as well as fermions in the fundamental or adjoint representations of $G$. Perhaps the best known model in the induced QCD category is the Kazakov–Migdal model \cite{4}, which attracted a flurry of interest in 1992–93, presumably because it admits a solution in the large–$N$ limit. There exist two recent papers \cite{5,6} motivated by the color–flavor transformation that revive the old idea of inducing QCD on the lattice.

The big question looming over all these induced gauge models is whether they do indeed lie in the universality class of Yang–Mills theory with gauge group $G$ as intended. In most of the past work, the concern was with nonlocality of the effective gauge field action. To suppress nonlocal contributions, one used the trick of sending both the number of auxiliary fields and their mass to infinity in a specific manner. In the present paper, we propose a rather different approach. We choose the dynamics of the auxiliary fields to be local to begin with, so that the induced gauge field action automatically has that property. Then, fixing the number $N_b$ of auxiliary fields, we tune the mass to a critical point so as to induce Yang–Mills theory. In order for this to work, it is crucial that bosonic scalars (as opposed to fermions) be used.

In more detail, the effective action of the induced gauge model we propose is a sum over elementary plaquettes on a $d$–dimensional lattice: $S = -2N_b \sum_{\mathbf{p}} \text{Re} \text{Tr} \ln (m - U(\partial \mathbf{p}))$, with
mass parameter \( m > 1 \). At unit mass the model has a critical point with a diverging correlation length, allowing a continuum limit to be taken. We claim that the \( m \to 1 \) continuum model flows under renormalization to Yang–Mills theory, provided that \( N_b \) exceeds a certain minimum value. (What that minimum value is depends on the type of gauge group, its rank, and the space–time dimension.) The mathematical basis for this claim is that the local weight function \( U \mapsto |\det(m - U)|^{-2N_b} \) on a classical compact Lie group \( G \) approaches for \( m \to 1 \) the \( \delta \)-function supported at \( U = 1 \), if \( N_b \) is large enough.

The second major theme of the present paper is a duality transformation for the induced gauge model. While “duality” has been very successful for gauge theories in the continuum and with supersymmetries — important examples are electric–magnetic (or Montonen–Olive) duality, which constitutes an important aspect of the Seiberg–Witten solution for the low–energy dynamics of \( N = 2 \) supersymmetric gauge theory with gauge group \( SU(2) \); or the Maldacena conjecture (or AdS/CFT–duality), by which \( N = 4 \) super Yang–Mills theory with gauge group \( SU(N) \) is believed to be dual to a type II-B superstring theory —, our interest here will be in pure gauge theories on the lattice, and without supersymmetry. In this class of quantum field theories duality is a well developed concept only for the abelian case \([7, 8, 9, 10, 11, 12]\), where passing to the dual theory essentially amounts to taking a Fourier transform.

A direct transcription of the abelian duality transform to nonabelian lattice gauge theories makes use of a character expansion of the plaquette statistical weight function (see \[13]\) for a recent reference). Integration over the gauge field then produces sums over products of Racah coefficients of \( G \) [these are higher–rank generalizations of tensor invariants known as \( 3j \), \( 6j \), \( 9j \), \( 12j \) symbols etc. for the case of \( G = SU(2) \)]. The semiclassical asymptotics of such coefficients is not easy to handle, and hence the continuum limit of the theory in this dual formulation remains unclear except in some special situations.

We consider it to be an interesting feature of the determinant–type models we are going to introduce, that an alternative approach is possible. Our induced gauge models can be transformed to a dual description — for any one of the classical compact gauge groups \( G = U(N), \text{Sp}(2N), \text{O}(N) \) — by viewing the determinant weight functions as traces in an (auxiliary) Fock space. The gauge group \( G \) acts in this Fock space as one member of a dual pair of Lie groups, where “dual” is meant in the sense of R. Howe \([14, 15]\). Integration over the gauge fields simply projects on the \( G \) invariant subspace of Fock space. The other member of the Howe dual pair is a noncompact Lie group acting irreducibly on that subspace. This allows a fairly transparent description of what the dual lattice theory is (although we do not yet understand its continuum limit). For \( G = U(1) \) our Howe duality transform reproduces standard abelian duality upon elimination of some redundant degrees of freedom.

Howe pairs underlie the color–flavor transformation \([16, 17]\), which was originally conceived in the context of random matrix theory and disordered electron systems, and has recently been applied to the strong coupling limit of QCD \([18, 19, 20, 21]\). Howe pairs have also been used recently for the investigation of determinant correlations for quantum maps \([22]\).

This paper is organized as follows. In Section 2 we place (local) bosons and/or fermions on the sites of an arbitrary lattice and couple them to gauge fields in the standard manner to induce a pure lattice gauge theory with gauge group \( G \). For the case of \( G = U(N_c) \), Section 3 establishes the \( \delta \)-function property of the induced plaquette weight function when the number of boson flavors \( N_b \geq N_c \). The resulting boson induced gauge model is subjected to a careful investigation in \( d = 1 + 1 \) dimensions in Section 4. We show that its partition function in the
continuum limit (and on any Riemann surface) agrees with the known partition function of some $U(N_c)$ Yang–Mills theory if $N_b \geq N_c + 1$. For $N_b = N_c$ we identify an exotic continuum gauge theory, which is not Yang–Mills but of Cauchy type. Then, after introducing the notion of Howe pairs in the two–dimensional context, we subject the boson induced gauge model to a duality transformation in any space–time dimension in Section 5. We conclude with a summary and listing of open questions in Section 6, commenting in particular on the issue of convergence of the color–flavor transformation when the number of boson flavors is large.

2 Fermion and Boson Induced Gauge Model

It has become standard practice in lattice gauge theory to place the fields on a simple hyper–cubic lattice. While convenient for the purpose of doing numerical calculations, the restriction to hypercubic lattices is rather narrow and special from the perspective of continuum field theory. Since our interest ultimately is in defining and taking a continuum limit, we will set up our formalism on lattices more general than the hypercubic one. This prevents us of from making short cuts and helps us direct our attention to the proper structures.

We take discrete space–time to be some $d$–dimensional complex $\Lambda$ built from oriented $k$–cells. For the cases $k = 0, 1, 2$ these are referred to as sites, links (with a direction), and plaquettes (with a sense of circulation). It will often be convenient to view the $k$–cells of $\Lambda$ as the generators of abelian groups $C_k(\Lambda)$. Their elements, called $k$–chains, are linear combinations of $k$–cells with coefficients in $\mathbb{Z}$. If $c$ is a $k$–chain, then so is $-c$; we get the latter from the former by reversing the orientation for all its $k$–cells. For any reasonable choice of $\Lambda$ there exists a boundary operator $\partial$, which is a linear operator $\partial : C_k(\Lambda) \rightarrow C_{k-1}(\Lambda)$ with the property $\partial \circ \partial = 0$ (the boundary of a boundary always vanishes). For example, the boundary of an oriented link $l$ that begins on site $n_i$ and ends on site $n_f$ is $\partial l = n_f - n_i$; which is the chain consisting of the 0–cells $n_f$ and $n_i$ with coefficient +1 and −1 respectively. The boundary of an oriented plaquette $p$ is the chain of 1–cells $l_i$ surrounding it: $\partial p = \sum_i \pm l_i$, where the plus/minus sign is chosen when the orientations agree/disagree.

The following elaboration on language might help prevent confusion later on: each $k$–cell of $\Lambda$ comes with exactly one of two possible orientations; i.e. the choice of orientation for the $k$–cells is fixed, albeit arbitrary. This means in particular that if the oriented link $l$ is a 1–cell, then $-l$ is not a 1–cell (although it still is an oriented link).

On a $d$–dimensional lattice or cell complex of this general kind, we are going to consider a gauge theory with partition function

$$Z = \int [dU] \int [d\varphi][d\bar{\psi}] e^{-S_f[\psi,\bar{\psi},U]} - S_b[\varphi,\bar{\psi},U],$$

where the action functionals $S_b$ and $S_f$ will be specified shortly. As usual, matrices $U$ taking values in a compact gauge group $G$ are placed on the links of the lattice. More precisely speaking, a lattice gauge field configuration is a mapping from the 1–cells of $\Lambda$ into $G$. The mapping extends to all oriented links by the convention $U(-l) \equiv U(l)^{-1}$, which is motivated by the interpretation of the $U$’s as discrete approximations to the path–ordered line integrals of an underlying gauge field $A$: $U(l) \approx P \exp(\int_l A)$. The a priori statistical weight of the lattice theory is a product of Haar measures $dU$ over the 1–cells of $\Lambda$: $[dU] = \prod_l dU(l)$. Concrete calculations will be carried out for the unitary groups $G = U(N_c)$. 

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In addition to the gauge fields, the theory has a second ingredient: complex bosonic and/or fermionic fields that are placed on the sites of Λ and transform according to the fundamental vector representation of the gauge group. The fermions are denoted by ψ, the bosons by ϕ. They are vectors not only in $N_c$-dimensional color space, but also in a “flavor” space with dimension $N_f$ (fermions) and $N_b$ (bosons).

These bosons and fermions are auxiliary (i.e. unphysical) degrees of freedom introduced solely for the purpose of inducing an effective action for the gauge fields. Unlike the conventional matter fields of lattice gauge theory, they do not propagate all over the lattice. Rather, each one of them is constrained to hop (in the presence of the lattice gauge field $U$) along the boundary chain of some plaquette. In other words, there is a one–to–one correspondence between the 2–cells $p$ of Λ and sets of complex boson and fermion variables, $\{\varphi_p\}$ and $\{\psi_p\}$. The orientation of $p$ determines the sense of circulation of the hopping of the auxiliary particles $\{\varphi_p\}$ and $\{\psi_p\}$; see Figure 1. The integration measure $[d\psi][d\varphi]$ is taken to be the product of flat measures.

To fix the precise details, we must distinguish between gauge groups of two types: those where the vector $(U)$ and covector $(U^{-1T})$ representations are related by an inner automorphism, and others where they are not. The groups $SO(N_c)$, $O(N_c)$ and $Sp(2N_c)$ belong to the former type, the unitary groups $U(N_c)$ and $SU(N_c)$ to the latter. For gauge groups of the former type, the setup we have described — one set of variables per 2–cell — would already suffice. For the latter type, however, and especially for $U(N_c)$, which the present paper focuses on, we have to double the set of auxiliary variables: it will be seen that, to induce a good effective action for the gauge fields, hopping must take place in both the clockwise and the counterclockwise sense for all plaquettes. Thus for each 2–cell $p$ of Λ, we introduce two sets of auxiliary variables, one associated to the 2–cell with its proper orientation (+$p$), and another one where the orientation is reversed (−$p$).

These words are put in formulas as follows. Let $p = \{p, -p\}$, and let $L_p$ denote the “length” of $\pm \partial p$, i.e. the number of links contained in the boundary chain of $\pm p$. Then, fixing some oriented plaquette $p$, we write its boundary as a formal sum of $L_p$ oriented links with positive coefficients: $\partial p = l_{n_1,n_2} + l_{n_2,n_3} + \ldots$, where $\partial l_{n_i,n_{i+1}} = n_{i+1} - n_i$. The 0–cells $n_1, \ldots, n_{L_p}$ are the sites visited by $\partial p$, arranged in ascending order as prescribed by the sense of circulation of $p$. To take notational advantage of the cyclic structure of the boundary chain $\partial p$, we identify $n_1 \equiv n_{L_p+1}$. With these conventions, we put $U_p(n_{j+1},n_j) \equiv U(l_{n_j,n_{j+1}})$, and define the actions $S_f$ and $S_b$ to be

$$S_f[\psi, \bar{\psi}, U] = \sum_{\pm \mathbf{p}} \sum_{j=1}^{L_p} \left( m_{f,p} \bar{\psi}_p(n_j) \psi_p(n_j) - \bar{\psi}_p(n_{j+1}) U_p(n_{j+1},n_j) \psi_p(n_j) \right),$$

$$S_b[\varphi, \bar{\varphi}, U] = \sum_{\pm \mathbf{p}} \sum_{j=1}^{L_p} \left( m_{b,p} \bar{\varphi}_p(n_j) \varphi_p(n_j) - \bar{\varphi}_p(n_{j+1}) U_p(n_{j+1},n_j) \varphi_p(n_j) \right).$$

To keep the expressions transparent, we have suppressed the color $(i = 1, 2, \ldots, N_c)$ and flavor indices $(a = 1, 2, \ldots, N_{f/b})$; it should be clear how to restore them. For example, $\varphi U \varphi$ is short–hand for $\sum_{i_1,i_2=1}^{N_c} \sum_{a=1}^{N_{f/b}} \varphi^{i_1,a} U^{i_1 i_2} \varphi^{i_2,a}$. The notation $\sum_{\pm \mathbf{p}}$ means that each plaquette occurs twice in the sum, once each for the two possible orientations. The parameters $m_{f,p}$ and $m_{b,p}$ are referred to as the (local) fermion and boson masses. We allow for the possibility that they depend on the plaquette label $p$ in general, but on a lattice with translational invariance
and no external gravitational field, we will take them to be constant. The field integral makes sense for any set of $m_{f,p} \in \mathbb{C}$, while the local boson masses must satisfy $\text{Re} m_{b,p} > 1$ for convergence.

The lattice theory so defined is intended to be a discretization of $d$–dimensional (Euclidean) Yang–Mills theory with gauge group $G = U(N_c)$ or $SU(N_c)$. In the second half of the paper we will eliminate the gauge field so as to pass to a dual description. For the time being we address the question whether (1) has a critical point with a continuum limit equivalent to Yang–Mills theory.

The field integrals for the auxiliary fermions $\psi$, $\bar{\psi}$ and bosons $\varphi$, $\bar{\varphi}$ are Gaussian. Carrying them out we obtain a product of determinants (for fermions) and inverse determinants (for bosons), one each for every oriented plaquette $p$. The determinants from the fermions associated with $p$ are $\text{Det}^N_l (m_{f,p} - U(\partial p))$, with $U(\partial p)$ being the ordered product of the $U$’s along the boundary chain of $p$:

$$U(\partial p) \equiv U_p(n_1, n_{L_p}) U_p(n_{L_p}, n_{L_p-1}) \cdots U_p(n_3, n_2) U_p(n_2, n_1).$$

The corresponding factor from the same plaquette with the opposite orientation, $-p$, is just the complex conjugate of this. The Gaussian boson integrals give a similar answer except that the determinants in that case go in the denominator. Thus, combining factors we have

$$Z = \int [dU] \prod_p \left| \frac{\text{Det}(m_{f,p} - U(\partial p))}{\text{Det}(m_{b,p} - U(\partial p))} \right|^{2N_i}.$$  

Note that this is a product over 2–cells, i.e. each oriented plaquette $p$ occurs only once (and the result does not depend on the orientations chosen).

By sending the result of the integration back to the exponent and dropping an irrelevant constant, we obtain

$$Z = \int [dU] e^{-S_{\text{ind},f}[U]-S_{\text{ind},b}[U]} \quad (4)$$
with fermion and boson induced actions

\[ S_{\text{ind},f}[U] = -2N_f \text{Re} \sum_p \text{Tr} \ln \left( 1 - \alpha_{f,p} U(\partial p) \right), \]

\[ S_{\text{ind},b}[U] = +2N_b \text{Re} \sum_p \text{Tr} \ln \left( 1 - \alpha_{b,p} U(\partial p) \right). \]

In place of the fermion and boson masses, we have introduced the coupling parameters \( \alpha_{f,p} \equiv m_{f,p} - L_p \) and \( \alpha_{b,p} \equiv m_{b,p} - L_p \). We take the range of these parameters to be \( -\infty < \alpha_{f,p} < \infty \) and \( -1 < \alpha_{b,p} < 1 \). Note that the partition function \( (4) \) becomes singular in the limit \( \alpha_{b,p} \to 1 \), since the function \( U \mapsto |\text{Det}(1 - \alpha U)|^{-2N_b} \) does so on the codimension one submanifold of matrices \( U \) with at least one eigenvalue equal to unity.

There is an obvious way [6] in which to approach Wilson’s lattice gauge theory \[23, 24, 25\] with action

\[ S^W[U] = -\frac{\beta}{N_c} \sum_p \text{Re} \text{Tr} U(\partial p) \]

from (5) and (6): specializing to the case of a hypercubic lattice, we take the couplings \( \alpha_{b,p} \) and \( \alpha_{f,p} \) to be independent of \( p \). We then send them to zero, and the number of flavors \( N_{f/b} \) to infinity, while keeping the products \( \alpha_{f/b} N_{f/b} \) fixed. Since \( \lim_{N \to \infty} N \ln(1 + x/N) = x \), this limit exactly reproduces the Wilson action \( (7) \), with \( \beta/N_c = 2(N_b \alpha_{b} - N_f \alpha_{f}) \). If only fermions are used, the coupling \( \alpha_{f} = m_{f} - L_{p} \) must be negative, corresponding to a complex mass, say \( m_{f} = |m_{f}| e^{i\pi/L_{p}} \).

It is very unclear to us, however, whether this double limit of an infinite number of infinitely heavy particles is going to lead to any advance in our knowledge about Yang–Mills theory. Certainly, the case of a single boson flavor \( (N_b = 1) \) treated in [6] bears no relation with Yang–Mills theory, let alone QCD.

In the present paper we are going to propose and investigate a more interesting possibility. The partition function \( (4) \) becomes singular when all \( \alpha_{b,p} \) are sent to unity (say, uniformly in \( p \)), which is in fact a critical point with a diverging correlation length. Our main message will be that, on suitable lattices and in high enough space–time dimension, the critical theory with \( \alpha_{b,p} \to 1 \) (and \( \alpha_{f,p} \neq 1 \)) is expected to be in the universality class of \( U(N_c) \) Yang–Mills theory, provided that \( N_b \geq N_c \). The universality conjecture can readily be checked in dimension \( d = 1 + 1 \). There, it will be demonstrated that universality holds for \( N_b > N_c \), while it fails for \( N_b = N_c \).

### 3 The Critical Point \( \alpha_{b} = 1 \)

#### 3.1 Lattice versus continuum

To communicate our argument, we must first recall the general scenario \( [24, 25] \) by which lattice gauge and continuum gauge theories are related. Imagine some smooth \( d \)-dimensional configuration of the gauge field \( A \), and superimpose on it a fine mesh in the form of some \( d \)-dimensional lattice or cell complex \( \Lambda \) with oriented elementary plaquettes \( p \). The rule of translation from the continuum gauge field \( A \) to the lattice field \( U \) is given by

\[ U(\partial p) = P \exp \oint_{\partial p} A = 1 + F(p) + \ldots, \]
where $P$ means path ordering, and $F(p)$ is the field strength (the curvature of the gauge connection $A$) evaluated on $p$ (in the standard setting of a hypercubic lattice, this means that if $p$ is a plaquette based at the site $n$ and is parallel to the $\mu\nu$–plane, then $F(p) = a^2 F_{\mu\nu}(n)$ with $a$ the lattice constant).

Clearly, the lattice approximation will be reasonable if the mesh formed by the lattice is fine enough in order for field curvature effects to be small on the lattice scale. Under such conditions $P\exp \oint_{\partial p} A$, and hence $U(\partial p)$, will be close to unity for every elementary plaquette $p$. Thus, if the lattice approximates the continuum, the plaquette matrices $U(\partial p)$ fluctuate only weakly around the zero–field strength configuration $U(\partial p) = 1$.

Conversely, if the statistical weight of the lattice theory sharply peaks at unity $U(\partial p) = 1$ for all $p$, then the lattice theory has a very large correlation length and is close to a continuum limit. We may then pass to a continuum field theory formulated in terms of the field strength $F$, by expanding around $U(\partial p) = 1$ and using the correspondence $[5]$.

The main principle then is this. Let the lattice gauge theory be given by a product statistical measure $\prod_p w_t(U(\partial p)) [dU]$, where $t$ is some coupling parameter with critical value $t_c$. We require the weight function $w_t(U)$ for $t$ close to $t_c$ to be very strongly localized at the unit element so as to make excursions away from unity statistically very rare. More precisely, we want $w_{t_c}(U) \equiv \delta(U)$ to be the Dirac $\delta$–function supported at the unit element of the gauge group, and $w_t(U)$ for $t \neq t_c$ to be some smeared version thereof. When $t$ moves to $t_c$, the smearing is undone, the lattice statistical weight approaches the product of $\delta$–functions $\prod_p \delta(U(\partial p))$, the correlation length goes to infinity, and the lattice gauge theory converges to a continuum limit. Under favorable conditions, this continuum limit will be Yang–Mills theory with action functional $-S_{YM} \propto \int Tr F \wedge \star F$.

This scenario is realized for the Wilson weight function

$$w^W_\beta(U) = e^{(\beta/N_c) \text{Re} TrU} \int e^{(\beta/N_c) \text{Re} TrU} dU$$

when the parameter $\beta \equiv t$ is sent to $t_c = \infty$. For one thing, one easily shows

$$\lim_{\beta \to \infty} \int_{U(N_c)} f(U) w^W_\beta(U) dU = f(1)$$

for any smooth function $f$ on $U(N_c)$ (or another compact gauge group, for that matter), which is equivalent to saying $\lim_{\beta \to \infty} w^W_\beta(U) = \delta(U)$. Moreover, it is strongly suggested by numerical simulations that the continuum limit approached by Wilson’s lattice gauge theory for $\beta \to \infty$ has all the properties expected of quantum Yang–Mills theory in the continuum. The ultraviolet stability of Wilson’s theory has been established by rigorous analysis [26].

There is of course nothing unique about the Wilson action and many other one–parameter families of weight functions do the same or an even better job. Let us single out one example. In the representation theory of compact semisimple Lie groups $G$ there exists a statement, called the Peter–Weyl theorem [27], which implies that the Dirac $\delta$–function on $G$ can be built up from the complete set of irreducible representations $D^\lambda$ of $G$ as follows:

$$\delta(U) = \text{vol}(G)^{-1} \sum_{\lambda} d_\lambda \text{Tr} D^\lambda(U) , \quad (9)$$
where \( d_\lambda = \text{Tr} D^\lambda(1) \) is the dimension of the representation. Hence, if \( c_2(\lambda) \geq 0 \) is the quadratic Casimir invariant evaluated in the representation \( D^\lambda \), the function

\[
w^\text{HK}_t(U) = \text{vol}(G)^{-1} \sum_{\alpha} e^{-c_2(\lambda)t} d_\lambda \text{Tr} D^\lambda(U) \quad (t > 0)
\]

approaches the \( \delta \)-function as \( t \rightarrow t_c = 0 \). For this reason one expects it to provide (in the limit of small \( t \)) a valid lattice regularization of Yang–Mills theory with gauge group \( G \).

This weight function is called the heat kernel: it solves the heat equation \( \partial_t w_t(U) = \Delta w_t(U) \) (\( \Delta \) being the Laplace–Beltrami operator) on \( G \) with initial condition \( \lim_{t \rightarrow 0^+} w_t(U) = \delta(U) \).

The “time” parameter \( t \) is the analog of \( 1/\beta \) for the Wilson weight function. In the abelian case \( G = U(1) \), the model with weight function \( w^\text{HK}_t \) is known in condensed matter theory as the Villain model.

### 3.2 \( \delta \)-function limit

After these preparations, we are ready to get to the point: we are going to investigate the determinantal weight function

\[
w_{N_b,\alpha}(U) = \left| \text{Det}(1 - \alpha U) \right|^{-2N_b} / \int_G \left| \text{Det}(1 - \alpha U) \right|^{-2N_b} dU
\]

of the boson induced gauge model \( \Box \) with \( \alpha_t = 0 \) and \( \alpha \equiv \alpha_b < 1 \) close to unity. We will be interested mostly in the case \( G = U(N_c) \), but note that the definition of \( w_{N_b,\alpha} \) still makes sense if \( G = U(N_c) \) is replaced by any of its closed subgroups.

As a warm up, we will look at two special cases. The simplest example is \( G = U(1) \) with \( N_b = 1 \), \( U = e^{i\theta} \) and Haar measure \( dU = d\theta \). In that case, elementary manipulations show

\[
w_{1,\alpha}(e^{i\theta}) = (2\pi)^{-1} \left\{ \frac{1 - \alpha^2}{|1 - \alpha e^{i\theta}|^2} \right\} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \alpha^n e^{in\theta},
\]

from which it is seen that \( w_{1,\alpha} \) approaches the Dirac \( \delta \)-function on \( U(1) \) in the limit \( \alpha \rightarrow 1 \). The difference to the Villain model \( \Box \) is that the Gaussian cutoff \( e^{-c_2(n)t} = e^{-n^2t} \) has been replaced by an exponential cutoff \( e^{-|n|t} \), with \( t = \ln(1/\alpha) > 0 \).

Another nice example is \( G = SU(2) \), still with \( N_b = 1 \). Again, by straightforward manipulations on the determinantal weight function \( \Box \) evaluated on \( U = e^{i\theta_3} \in SU(2) \), one finds with \( \int_{SU(2)} dU = 2\pi^2 \):

\[
w_{1,\alpha}(e^{i\theta_3}) = (2\pi^2)^{-1} \left\{ \frac{1 - \alpha^2}{|1 - \alpha e^{i\theta}|^4} \right\} = \frac{1}{2\pi^2} \sum_{n = 0}^{\infty} \alpha^n \left( n + 1 \right) \frac{\sin \left( (n + 1)\theta \right)}{\sin \theta}.
\]

On the right–hand side we recognize the character \( \chi_S(e^{i\theta_3}) = \sin \left( (2S + 1)\theta \right) / \sin \theta \) of the spin \( S = n/2 \) representation of \( SU(2) \), and its dimension \( 2S + 1 = n + 1 \). Both sides extend uniquely to class functions of \( SU(2) \), i.e. to functions on \( SU(2) \) that are invariant under conjugation \( U \mapsto gUg^{-1} \) by \( g \in SU(2) \). As a result, we deduce \( \lim_{\alpha \rightarrow 1} w_{1,\alpha}(U) = \delta_{SU(2)}(U) \) from \( \Box \).

**Fact.** Let \( f : U(N_c) \rightarrow \mathbb{C} \) be an analytic function. Then, for any \( N_b \geq N_c \),

\[
\lim_{\alpha \rightarrow 1} \int_{U(N_c)} f(U)w_{N_b,\alpha}(U)dU = f(1). \quad (12)
\]
Thus we are claiming the desired $\delta$–function property for $N_b \geq N_c$. The proof will be given in the next subsection. For now we make two comments: (i) There is nothing special about $U(N_c)$ in this context, and we expect a similar statement to be true for each of the compact matrix groups $G = SU(N_c), SO(N_c), O(N_c), Sp(2N_c)$. (ii) In view of the Peter–Weyl theorem, the statement \[(12)\] implies that the complete set of irreducible representations of $U(N_c)$ occur in the character expansion of $w_{N_c,\alpha}(U)$ for $N_b \geq N_c$. Conversely, one can show that some representations are missing for $N_b < N_c$, which means that the inequality $N_b \geq N_c$ in the statement cannot be relaxed but is optimal.

3.3 Proof of fact

We will give an elementary proof, and start with a few considerations that simplify it.

First of all, if the function $U \mapsto |\text{Det}(1 - \alpha U)|^{-2N}$ is concentrated near unity $U = 1$ for some value of the exponent $N$, then it will be even more so for exponents larger than $N$. Thus, if the statement is true for some value $N$ of $N_b$, it will certainly be true for all values of $N_b$ greater than $N$. It therefore suffices to show that the statement holds for the borderline case $N_b = N_c$. We abbreviate the notation by writing $w_{\alpha}(U) \equiv w_{N_c,\alpha}(U)$.

Second, since the weight function $w_{\alpha}$ is invariant under conjugation $U \mapsto gUg^{-1}$, the operation of replacing the analytic function $f$ by its average $f^{\text{av}}$ over conjugacy classes,

$$f^{\text{av}}(U) = \text{vol}(U(N_c))^{-1} \int_{U(N_c)} f(gUg^{-1})dg,$$

does not change the value of the integral \[(12)\]. We may therefore assume $f$ to be invariant.

Third, given invariance under conjugation, we may view $w_{\alpha}$ and $f$ as functions on the maximal torus $T = U(1)^{N_c}$ parameterized by the eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_{N_c}}$ of $U$, and we may reduce the integral over $U(N_c)$ to an integral over $T$. Let $J(e^{i\theta_1}, \ldots, e^{i\theta_{N_c}}) = \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2$ be the Jacobian of the polar coordinate map $(U(N_c)/T) \times T \to U(N_c)$. Then we have

$$\int_{U(N_c)} f(U)w_{\alpha}(U)dU = \frac{\langle f \rangle_{\alpha}}{\langle 1 \rangle_{\alpha}},$$

where the angular brackets mean

$$\langle f \rangle_{\alpha} = \int_{[0,2\pi]^{N_c}} f(e^{i\theta_1}, \ldots, e^{i\theta_{N_c}}) J(e^{i\theta_1}, \ldots, e^{i\theta_{N_c}}) \prod_{j=1}^{N_c} |1 - \alpha e^{i\theta_j}|^{2N_c} d\theta_1 \cdots d\theta_{N_c}.$$

To prove the fact \[(12)\] we must show

$$\lim_{\alpha \to 1} \frac{\langle f \rangle_{\alpha}}{\langle 1 \rangle_{\alpha}} = f(1, \ldots, 1) \quad (13)$$

for all analytic functions $f$ on the maximal torus $T = U(1)^{N_c}$ which extend to functions on $U(N_c)$. (Such functions on $T$ are invariant under the Weyl group of $U(N_c)$, i.e. they do not change under permutations of their arguments.)

We will actually establish the limit \[(13)\] for the larger class of all analytic functions $F : T \to \mathbb{C}$. By definition, such functions are absolutely convergent sums of the basic functions $e^{i\sum n_k \theta_k}$.
with integer exponents \( n_k \). It therefore suffices to prove (13) for the complete set of these basic functions. So let

\[
F = e^{i(n_1 \theta_1 + \ldots + n_N \theta_N)}
\]

with any \((n_1, \ldots, n_N) \in \mathbb{Z}^N\). Without loss we may assume that the ordering of variables has been adjusted so that the first \( p \) integers \( n_1, \ldots, n_p \) are positive or zero, while the last \( N_c - p \) integers \( n_{p+1}, \ldots, n_{N_c} \) are negative.

Now we evaluate \( \langle F \rangle_\alpha \) in the limit \( \alpha \to 1 \). The first step is to switch to the variables \( z_k = e^{i \theta_k} \), which yields the expression

\[
\langle F \rangle_\alpha = \frac{1}{N_c^2} \oint_{U(1)^N_c} \frac{z_1^{n_1} \cdots z_{N_c}^{n_{N_c}}}{\prod_{j=1}^{N_c} (z_j - \alpha)^{N_c} (z_j - \alpha^{-1})^{N_c} \prod_{k<l} (z_k - z_l)^2 \prod_{j=p+1}^{N_c} (z_j - \alpha^{-1})^{N_c}} \prod_{j=1}^{N_c} (z_j - \alpha)^{N_c} dz_1 \cdots dz_{N_c}.
\]

By the signs assumed for the integers \( n_1, \ldots, n_{N_c} \), the integrand is obviously regular at zero in the complex plane for each of the variables \( z_1, \ldots, z_p \). Simple power counting shows that the same is true at infinity for \( z_{p+1}, \ldots, z_{N_c} \). (This is to say that the integrand for \( k = p+1, \ldots, N_c \) decays as \( z_k^{-2} \) or faster at infinity.) Therefore, our strategy now is to contract the contour of integration to zero for the first \( p \) variables, and expand it to infinity for the last \( N_c - p \) variables. In doing so, we pick up contributions from the poles of order \( N_c \) at \( z_1 = \ldots = z_p = \alpha \) inside the unit circle \( U(1) \subset \mathbb{C} \), and at \( z_{p+1} = \ldots = z_{N_c} = \alpha^{-1} \) outside the unit circle. By the residue theorem, we then arrive at the exact formula

\[
\langle F \rangle_\alpha = (i/\alpha)^{N_c^2} (2\pi i)^p (-2\pi i)^{N_c-p} (N_c - 1)! \langle 1 \rangle_{\alpha^{-1}} \times \left( \frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_{N_c}} \right)^{N_c-1} \frac{z_1^{n_1} \cdots z_{N_c}^{n_{N_c}} \prod_{k<l} (z_k - z_l)^2 \prod_{j=p+1}^{N_c} (z_j - \alpha^{-1})^{N_c}}{\prod_{j=1}^{N_c} (z_j - \alpha)^{N_c} \prod_{j=p+1}^{N_c} (z_j - \alpha^{-1})^{N_c}} |_{z_1=\ldots=z_p=z_{p+1}=\ldots=z_{N_c}^1=\alpha}.
\]

This expression is divergent at \( \alpha = 1 \). The highest–order singularity, \((\alpha - \alpha^{-1})^{-N_c^2} \), occurs when all derivatives act on \( \prod_{k<l} (z_k - z_l)^2 \) or on the denominator. The order of the singularity is reduced if one or several of the derivatives act on the monomial \( F = z_1^{n_1} \cdots z_{N_c}^{n_{N_c}} \). To compute the limit \( \alpha \to 1 \), it is enough to retain the leading–order singularity. The leading–order singularity is picked up by evaluating \( F \) at \( z_1 = \ldots = z_p = \alpha \) and \( z_{p+1} = \ldots = z_{N_c} = \alpha^{-1} \) and taking it outside of the expression. What is left behind is just the value of the integral \( \langle 1 \rangle_{\alpha} \) obtained by replacing \( F \) by unity. Hence

\[
\langle F \rangle_\alpha = \alpha^{|n_1| + \ldots + |n_{N_c}|} \langle 1 \rangle_{\alpha} + \text{less singular terms}.
\]

So we conclude \( \lim_{\alpha \to 1} \langle F \rangle_\alpha / \langle 1 \rangle_{\alpha} = F |_{z_1=\ldots=z_{N_c}=1} \), and the proof of (12) is complete.

### 4 Continuum Limit in Two Dimensions

The result (12) ensures that on sending all coupling parameters \( \alpha_{b,p} \to 1 \), the induced gauge model (6) for \( N_b \geq N_c \) becomes critical, which allows a continuum limit to be taken on any reasonable direct system of lattices. Based on universality, we expect this continuum limit to be quantum Yang–Mills theory, at least generically.

A precise investigation of the universality conjecture can be made, and will now be made using harmonic analysis on the gauge group, in the simple case of \( d = 1+1 \) dimensions.
There, the universality mechanism at work is the central limit principle in its basic form: computing the two–dimensional theory essentially amounts to taking convolutions of the plaquette distribution $w_{N_b, \alpha_t}(U)dU$ with itself and, by a central limit theorem for compact Lie groups, multiple convolution sends a large class of distributions to the universal heat kernel family $w_{i}^{HK}(U)dU$ written down in (10).

In short, the central limit principle we will exploit is this. (Although we are going to pursue the case of $U(N_c)$, we here state the principle for a semisimple compact Lie group $G$. The extension to $U(N_c)$ will cause minor complications.) Let $w_t(U)dU$ ($t > 0$) be a one–parameter family of smooth $AdG$–invariant distributions on $G$ such that $\lim_{t \to 0} w_t(U) = \delta(U)$. Using the exponential mapping $X \mapsto U = \exp X$ we can pull back the family to a family of distributions $d\mu_t(X)$ on the Lie algebra of $G$ (or, rather, to a domain of injectivity of $\exp$ in $\text{Lie} G$). With respect to $d\mu_t(X)$ we compute the expectation of the Killing form $(X, X) = -\text{Tr} \text{ad}(X)\text{ad}(X)$.

If there exists a “diffusive scaling” limit, i.e. the expectation of $(X, X)/t$ stays finite when $t$ is sent to zero, then a central limit principle applies: denoting the $N$th convolution of $w_t$ with itself by $w_t^N$, we have $\lim_{N \to \infty} w_{t/N}^N = w_{t}^{HK}$.

We will see that the plaquette distribution of the boson induced model (4) for $N_b \geq N_c + 1$ satisfies the diffusive scaling criterion, with $t \sim (1 - \alpha_{b,p})^2$. This will allow us to take a continuum limit which can be considered as a rigorous definition of 2d quantum Yang–Mills theory by the reasoning of Witten [28].

On the other hand, for $N_b = N_c$ the diffusive scaling criterion turns out to be violated! A continuum limit can still be defined, owing to (12). This, however, is not Yang–Mills theory but an unusual theory which, in the “first–order formalism” with an auxiliary Lie $U(N_c)$–valued scalar field $\phi$, is given by an action functional

$$S = -i \int_\Sigma \text{Tr} \phi F + \mu \int_\Sigma \| \phi \|_1 \, d^2x \ , \quad (14)$$

where $\| \phi \|_1 = \sum_{j=1}^{N_c} |\phi_j|$, the $\phi_j$ being the eigenvalues of $\phi$. This exists as a renormalizable theory because the Cauchy distribution on $u(N_c) \equiv \text{Lie} U(N_c)$ approaches under subdivision a distribution which is stable, and yet different from the heat kernel.

We shall begin to substantiate these assertions in Section 4.2.

4.1 One–Plaquette Model

Before we undertake the study of the two–dimensional models where Yang–Mills universality rules, we dispose of those where it does not. As a simple test, we look at a cell complex consisting of a single plaquette $p$ and consider the expectation value of $\text{Tr} U \equiv \text{Tr} U(\partial p)$:

$$W(\alpha_f, \alpha_b) = \frac{1}{Z(\alpha_f, \alpha_b)} \int_{U(N_c)} \text{Tr} U \frac{|\text{Det}(1 - \alpha_f U)|^{2N_f}}{|\text{Det}(1 - \alpha_b U)|^{2N_b}} dU .$$

In $U(N_c)$ Yang–Mills theory the expectation of $\text{Tr} U(C)$ (the holonomy along any loop $C$) goes to $\text{Tr} 1 = N_c$ when the coupling is sent to zero. The same happens with $W(\alpha_f, \alpha_b)$ in the limit $\alpha_f \to 1$, $N_b \geq N_c$, for in that case the statement (12) applies, and gives with $f(U) = \text{Tr} U |\text{Det}(1 - \alpha_f U)|^{2N_f}$:

$$\lim_{\alpha_b \to 1} W(\alpha_f, \alpha_b) = \text{Tr} U|_{U=1} = N_c \quad (N_b \geq N_c, \ \alpha_f \neq 1) .$$
While this is a statement made for the fundamental representation, (12) asserts that a similar result holds true for the trace $\text{Tr} D^{\lambda}(U)$ in any representation $D^{\lambda}$.

What happens in the other cases? We separately look at the boson induced models ($\alpha_f = 0$) with $N_b < N_c$, and at the fermion induced models ($\alpha_b = 0$), starting with the former. Using complex contour integration and residue calculus, we show in Appendix A.2 that the following holds true:

$$\lim_{\alpha_b \to 1} W(0, \alpha_b) = N_b \quad (N_b < N_c).$$

Thus in that case $W(0, 1)$ falls short of reaching the maximal value $N_c$ allowed by the bound $|\text{Tr} U| \leq N_c$. The general dependence can be computed numerically (see Appendix A.2), and is shown in Figure 2 for the case $N_c = 3$. An interesting observation, proved in Appendix A.2 is that for $N_b \leq N_c$ the function $W(0, \alpha_b)$ is exactly linear: $W(0, \alpha_b) = N_b \alpha_b$.

Turning to the fermion induced case, from the invariance of $dU$ under $U \to -U$ and $U \to U^{-1}$, we deduce $W(\alpha_f, 0) = -W(-\alpha_f, 0) = W(1/\alpha_f, 0)$, so it suffices to restrict attention to the interval $-1 \leq \alpha_f \leq 0$. Figure 2 shows numerical results (Appendix A.1) in that range. We observe that $W(\alpha_f, 0)$ is a monotonically increasing function of $-\alpha_f$ for $|\alpha_f| \leq 1$. The global maximum, attained at $\alpha_f = -1$, can be computed analytically by a fermionic variant of the method of Howe pairs described in Section 4.5:

$$W(-1, 0) = \frac{N_f N_c}{N_f + N_c}.$$
the one–plaquette distribution

\[ d\mu_{N_{\text{b}},\alpha}(U) = |\text{Det} (1 - \alpha U)|^{-2N_{\text{b}}} dU . \]  

(15)

Our motivation is that we wish to compute multiple convolution integrals of this distribution with itself, and transforming to the appropriate Fourier (or harmonic) space turns convolutions into simple multiplications.

Recall that by the Peter–Weyl theorem any \( L^2 \)–function on a compact Lie group can be expanded in the matrix entries of its complete set of irreducible representations \( D^\lambda \). Since the distribution function in (15) is a class function of \( U(N_c) \), the expansion proceeds by \( U(N_c) \) characters \( \chi^\lambda(U) \equiv \text{Tr} D^\lambda(U) \):

\[ |\text{Det} (1 - \alpha U)|^{-2N_{\text{b}}} = \sum_{\text{all } \lambda} c^\lambda(\alpha) \chi^\lambda(U) , \]  

(16)

and, by the orthonormality of characters, the expansion coefficients are

\[ c^\lambda(\alpha) = \int_{U(N_c)} |\text{Det} (1 - \alpha U)|^{-2N_{\text{b}}} \chi^\lambda(U^{-1})dU . \]  

(17)

Here, and throughout this section, we normalize Haar measures by \( \int_{U(N_c)} dU = 1 \). From statement (12) we already know the limit

\[ \lim_{\alpha \to 1} \frac{c^\lambda(\alpha)}{c_0(\alpha)} = \chi^\lambda(1) = d^\lambda , \]

which asserts convergence of the ratio of coefficients to the dimension of the representation \( \lambda \). The goal of the current subsection is to gain a more precise understanding of the expansion coefficients \( c^\lambda(\alpha) \) close to \( \alpha = 1 \).

To that end we will use the Cayley map

\[ X \mapsto \gamma(X) = \frac{1 + X}{1 - X} . \]
to express the integral over $U(N_c)$ as an integral over the anti–Hermitian $N_c \times N_c$ matrices, $X \in u(N_c)$. The Jacobian of the Cayley map is $\text{Det}^{-N_c}(1 - X^2)$, which is to say that there exists a flat positive density $dX$ on $u(N_c)$ such that

$$\gamma^*(dU) = \text{Det}^{-N_c}(1 - X^2) \, dX.$$  

Note $-X^2 \geq 0$ for $X \in u(N_c)$. We then immediately have the statements

$$\int \text{Det}^{-N_c}(1 - X^2) \, dX = \int dU = 1,$$ 

$$\int \text{Det}^{-N_b}(1 - X^2) \, dX < 1 \quad \text{for} \quad N_b > N_c, \quad \text{and} \tag{18}$$

$$\int \text{Tr}(-X^2) \, \text{Det}^{-N_b}(1 - X^2) \, dX \leq 1 \quad \text{for} \quad N_b \geq N_c + 1. \tag{19}$$

The last of these follows from the elementary inequality $\text{Tr}(-X^2) \leq \text{Det}(1 - X^2)$. Making two variable transformations in sequence,

$$U = \frac{1 + x}{1 - x}, \quad x = \frac{1 - \alpha}{1 + \alpha} \in u(N_c),$$

we bring the integral formula for the expansion coefficients into the form

$$c_\lambda(\alpha) = (1 - \alpha)^{-2N_c N_b} \left(\frac{1 - \alpha}{1 + \alpha}\right)^{N_c^2} \int \frac{\chi_\lambda\left(\frac{(1 - x)/(1 + x)}{\text{Det}^{-N_c - N_b}(1 - x^2)}\right)}{x = \frac{1 - \alpha}{1 + \alpha}} \text{Det}^{-N_b}(1 - X^2) \, dX.$$ 

In view of the finiteness statement, it is clear that the integral for $c_\lambda(\alpha)$ will localize at unity $U = 1$ in the limit $\alpha \to 1$, for $N_b \geq N_c$. Hence, to get an accurate approximation to these $c_\lambda(\alpha)$ near $\alpha = 1$, the only further input we need is an understanding of the characters $\chi_\lambda(U)$ close to unity.

Let $\lambda$ denote a set of $N_c$ integers, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N_c)$, let $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_{N_c}}) \in U(1)^{N_c}$, and define the elementary antisymmetric torus function $\xi_\lambda$ by

$$\xi_\lambda(e^{i\theta}) = \sum_{\pi \in S_{N_c}} \text{sgn}(\pi) e^{i\lambda_1 \theta_{\pi(1)}} e^{i\lambda_2 \theta_{\pi(2)}} \ldots e^{i\lambda_{N_c} \theta_{\pi(N_c)}} ,$$

where the sum is over all permutations of $\{1, 2, \ldots, N_c\}$. By a classic result of Weyl, the irreducible representations of $U(N_c)$ are in one–to–one correspondence with ordered sets $\lambda$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N_c$, and the character associated with $\lambda$ is the class function $\chi_\lambda(U)$ determined by

$$\chi_\lambda(e^{i\theta}) = \xi_{\lambda + \rho}(e^{i\theta})/\xi_{\rho}(e^{i\theta}) \tag{20}$$

where $\rho = (N_c - 1, N_c - 2, \ldots, 1, 0)$. The character at unity computes the dimension of the representation, which by the Weyl dimension formula is

$$\chi_\lambda(e^{0}) = d_\lambda = \triangle(\lambda + \rho)/\triangle(\rho) \tag{21}.$$
\[ \Delta(\lambda) = \prod_{k<l}(\lambda_k - \lambda_l) \] being the Vandermonde determinant.

Because characters are joint eigenfunctions of the full ring of invariant differential operators, the characters \( \chi_\lambda(U) \) of \( U(N_c) = U(1) \times SU(N_c) \) with central factor \( U(1) \) separate. If we put \( U = e^X \), a \( U(1) \) factor \( e^{(q/N_c) \mathrm{Tr} X} \) with charge \( q(\lambda) = \sum_j \lambda_j \) splits off. Moreover, since the \( SU(N_c) \) part only depends on the traceless part of \( X \in \mathfrak{u}(N_c) \), the characters expand around unity as

\[
\frac{\chi_\lambda(e^X)}{\chi_\lambda(e^0)} = e^{(q/N_c)\mathrm{Tr} X} \times \int_{SU(N_c)} (e^X \cdot e^{-\mathrm{Tr} X/N_c})
\]

\[
= 1 + \frac{q}{N_c} \mathrm{Tr} X + \frac{1}{2} \left( \frac{q^2}{N_c^2} - A \right) (\mathrm{Tr} X)^2 + \frac{A}{2} \mathrm{Tr}(X^2) + \ldots ,
\]

with some coefficient \( A = A(\lambda) \). In parallel, one can directly expand Weyl’s formula for \( \chi_\lambda \) to second order in the angles \( \theta_j \). By doing so and comparing coefficients (or by more sophisticated techniques not discussed here), one finds

\[
A(\lambda) = (N_c^2 - 1)^{-1} (\text{Cas}_2(\lambda) - q(\lambda)^2/N_c) ,
\]

where \( \text{Cas}_2(\lambda) \) is a quadratic Casimir element of \( U(N_c) \),

\[
\text{Cas}_2(\lambda) = \sum_{j=1}^{N_c} \lambda_j (\lambda_j + N_c + 1 - 2j) ,
\]

which is associated with the invariant quadratic form \(-\mathrm{Tr}(X^2)\) in the standard way.

Having gathered enough information about the characters \( \chi_\lambda \) near unity, we now return to the task of computing the behavior of the expansion coefficients \( c_\lambda(\alpha) \) close to \( \alpha = 1 \). If we insert the small-\( X \) expansion of \( \chi_\lambda(e^X) \) into the integral formula for \( c_\lambda(\alpha) \) and take the ratio \( c_\lambda(\alpha)/c_0(\alpha) \), we encounter the second moments of the distribution \( \det^{-N_b}(1 - X^2) dX \):

\[
\mathbb{E}_{N_b} \mathrm{Tr}(X^2) \equiv Z^{-1} \int_{\mathfrak{u}(N_c)} \mathrm{Tr}(X^2) \det^{-N_b}(1 - X^2) dX ,
\]

\[
\mathbb{E}_{N_b} (\mathrm{Tr} X)^2 \equiv Z^{-1} \int_{\mathfrak{u}(N_c)} (\mathrm{Tr} X)^2 \det^{-N_b}(1 - X^2) dX ,
\]

with \( Z = \int_{\mathfrak{u}(N_c)} \det^{-N_b}(1 - X^2) dX \). These are finite for \( N_b \geq N_c + 1 \), by the inequality and \( 0 \leq -\mathrm{Tr}(X)^2 \leq -N_c \mathrm{Tr}(X^2) \). The first moment \( \mathbb{E}_{N_b} \mathrm{Tr} X \) vanishes by parity. It is therefore easy to prove the following statement:

**Fact.** The coefficients \( c_\lambda(\alpha) \) for \( N_b \geq N_c + 1 \) have a Taylor expansion

\[
c_\lambda(\alpha)/c_0(\alpha) = d_\lambda \left( 1 - \frac{1}{2}(1 - \alpha)^2 (B_1 \alpha + B_2 \text{Cas}_2(\lambda)) + R(\alpha) \right) ,
\]

with a remainder term \( R(\alpha) \) that vanishes faster than \((1 - \alpha)^2\) in the limit \( \alpha \to 1 \). The coefficients \( B_1 \) and \( B_2 \) are determined by the linear system of equations

\[
\begin{pmatrix}
-\mathbb{E}_{N_b} (\mathrm{Tr} X)^2/N_c \\
-\mathbb{E}_{N_b} \mathrm{Tr}(X^2)/N_c
\end{pmatrix} = \begin{pmatrix}
N_c & 1 \\
1 & N_c
\end{pmatrix} \begin{pmatrix}
B_1 \\
B_2
\end{pmatrix} .
\]

The leading singularity in the normalization \( c_0(\alpha) \) is

\[
c_0(\alpha) \sim (1 - \alpha)^{-2N_bN_c+N_c^2} .
\]
4.3 Continuum limit

We now place the boson induced gauge model with $N_b \geq N_c + 1$ on a two-dimensional cell complex $\Lambda$ approximating an orientable compact Riemann surface $\Sigma$. Such a complex consists of plaquettes, links and sites, with every link joining at most two plaquettes, and the plaquettes receiving an orientation from $\Sigma$.

To each plaquette $p$ of $\Lambda$, we assign an area $A_p$ determined by some choice of Riemannian structure of $\Sigma$. Measuring area in units of a fundamental area $a^2$, we put $\alpha_p = 1 - \sqrt{A_p/a}$. This specifies the set $\{\alpha\}$ of coupling parameters of our boson induced gauge model on $\Lambda$.

We now focus on the partition function $Z_\Lambda(\{\alpha\}) = \int [dU] \prod_p \left| \text{Det}(1 - \alpha_p U(\partial p)) \right|^{-2N_b}$.

The goal is to demonstrate that, when the cell complex $\Lambda$ is refined so as to approximate $\Sigma$ ever more closely, $Z_\Lambda(\{\alpha\})$ converges to the known partition function of $U(N_c)$ Yang–Mills theory on $\Sigma$, with a particular choice for the $U(1)$ coupling.

Since any $\Sigma$ can be made by gluing of surfaces with disk topology, the first step we take is to compute the (boundary–value) partition function, $\Gamma_\Lambda(\mathcal{U}, \{\alpha\})$, for a two-dimensional cell complex $\Lambda$ approximating a disk, where

$$ \mathcal{U} = U(\partial \Lambda) = \prod_{l \in \partial \Lambda} U(l) $$

is the product of $U$’s along the boundary $\partial \Lambda$, and these boundary $U$’s are held fixed to prescribed values. To compute $\Gamma_\Lambda(\mathcal{U}, \{\alpha\})$, we need to integrate over all the matrices $U$ associated with the links in the interior of $\Lambda$. These link integrations can be carried out in any order, and because $\Lambda$ has disk topology (i.e. every interior link joins exactly two plaquettes), it suffices to show how to do one of them.

Hence, let $l$ be any interior link of $\Lambda$, and let $p$ and $p'$ be the two oriented plaquettes joined by $l$. We are going to do the integral over the link matrix $U(l)$. For that purpose, let the holonomy along the boundary chain of $p$ be

$$ U(\partial p) = U_p(n_1, n_{L_p}) \cdots U_p(n_3, n_2) U_p(n_2, n_1), $$

and similar for $p'$, where our notational conventions are as specified in Section. Without loss we may assume that $l$ is the link joining the sites $n_1 \equiv n'_1$ and $n_2 \equiv n'_2$:

$$ U \equiv U(l) = U_p(n_2, n_1) = (U_{p'}(n'_2, n'_1))^{-1}. $$

Abbreviating the notation by putting

$$ U(\partial p) = WU, \quad U(\partial p') = VU^{-1}, $$

we are then faced with the convolution integral

$$ I(V, W) = \int_{U(N_c)} \left| \text{Det}(1 - \alpha_p VU^{-1}) \right|^{-2N_b} \left| \text{Det}(1 - \alpha_p UW) \right|^{-2N_b} dU. $$

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To carry it out, we use the character expansion $|\text{Det}(1 - \alpha U)|^{-2N_b} = \sum_\lambda c_\lambda(\alpha) \chi_\lambda(U)$ and the fact that characters reproduce under convolution,

$$\int_{U(N_c)} \chi_\lambda(VU^{-1}) \chi_{\lambda'}(UW) dU = \delta_{\lambda,\lambda'} \frac{\chi_\lambda(VW)}{d_\lambda},$$

(28)

to obtain

$$I(V, W) = \sum_\lambda \frac{c_\lambda(\alpha_p)}{d_\lambda} \chi_\lambda(VW).$$

We now iterate the procedure, and successively do all inner link integrations using the convolution law [28] for the characters. The resulting expression for the boundary–value partition function is

$$\Gamma_D(\mathcal{U}, \{\alpha\}) = \sum_\lambda d_\lambda \left( \prod_p \frac{c_\lambda(\alpha_p)}{d_\lambda} \right) \chi_\lambda(\mathcal{U}),$$

where $\mathcal{U} \equiv U(\partial \Lambda) = U_\Lambda(n_{\Lambda_1}, n_{\Lambda_2}) \cdots U_\Lambda(n^2, n_1)$ is the holonomy along $\partial \Lambda$, as before.

Here is the point where we take the continuum limit. If we refine the lattice discretization, and keep on refining it so that $\Lambda$ becomes a closer and ever closer approximation to a disk $D$ (or some other surface $D$ diffeomorphic to a disk), the elementary plaquette areas $A_p$ go to zero, the parameters $\alpha_p = 1 - \sqrt{A_p/a}$ approach unity, and we may eventually use the asymptotic law [28], giving

$$\prod_p \frac{c_\lambda(\alpha_p)}{d_\lambda} = \prod_p \left( 1 - \frac{A_p}{2a^2} (B_1 q(\lambda)^2 + B_2 \text{Cas}_2(\lambda)) \right) c_0(\alpha_p) + \cdots,$$

with corrections that vanish in the limit. Since $\lim_{N \to \infty} (1 - x/N)^N = e^{-x}$, the plaquette areas $A_p$ exponentiate and piece together to yield the total area $\sum_p A_p$ of the surface $D$. To eliminate one irrelevant constant from our expressions, we adopt the convention of measuring area in suitable units: $\mu \equiv (B_2/a^2) \times \sum_p A_p$. Taking the continuum limit then leads to

$$\Gamma_D(\mathcal{U}, \mu) = \sum_\lambda d_\lambda \exp \left( -\frac{\mu}{2} (\text{Cas}_2(\lambda) + (B_1/B_2) q(\lambda)^2) \right) \chi_\lambda(\mathcal{U}).$$

(29)

We have dropped a (diverging) multiplicative constant $\prod_p c_0(\alpha_p)$ that arose from our using an unnormalized statistical weight function. (Physically speaking we made use of our freedom to set the vacuum energy of flat space to zero.)

An expression for $\Gamma_D(\mathcal{U}, \mu)$ of the given form constituted the starting point of Witten’s combinatorial treatment [28]. From it, he computed (among many other things) the Yang–Mills partition function $Z_g(\mu)$ for any orientable compact Riemann surface of genus $g$ and dimensionless area $\mu$. Instead of repeating that computation here, we just quote the answer it entails in the present case:

$$Z_g(\mu) = \sum_\lambda d_\lambda^{-2g+2} e^{-\mu \left( \text{Cas}_2(\lambda) + (B_1/B_2) q(\lambda)^2 \right) / 2},$$

(30)

and verify it for the two simplest examples, the sphere and the torus.

A sphere ($g = 0$) with area $\mu$ can be made by gluing together two disks $D_+$ and $D_-$, say with areas $\mu_+$ and $\mu_- = \mu - \mu_+$. The boundary–value partition functions $\Gamma_{D_+}$ and $\Gamma_{D_-}$
depend only on the total holonomies along $\partial D_+$ and $\partial D_-$ respectively. If the two disks are to fit together to give an oriented sphere, we must have $\partial D_+ = -\partial D_-$, which means that the holonomies are inverse to each other: $U(\partial D_+) = U(\partial D_-)^{-1}$. Thus, introducing $U \equiv U(\partial D_+)$ as the variable of final integration, we obtain

$$Z_{\text{sphere}}(\mu) = \int_{U(N_c)} \Gamma_{D_+}(U, \mu_+) \Gamma_{D_-}(U^{-1}, \mu_-) dU.$$ 

Inserting (29) and doing the resulting integral over a product of two characters with the help of (28), we indeed get the answer (30) for genus $g = 0$.

To manufacture a torus ($g = 1$) with area $\mu$, we take a rectangle $Q$ with the same area, and identify opposite edges. Writing the holonomy along $\partial Q$ as a product over its four edges, $U(\partial Q) = UVU^{-1}V^{-1}$, we have

$$Z_{\text{torus}}(\mu) = \int_{U(N_c)} \int_{U(N_c)} \Gamma_Q(UVV^{-1}V^{-1}, \mu) dU dV.$$ 

To do the first of these two integrals, we use

$$\int_G \chi_\lambda(UAU^{-1}B) dU = \chi_\lambda(A)\chi_\lambda(B)/d_\lambda,$$

which is a consequence of the orthogonality relations obeyed by the matrix entries of an irreducible representation $D^\lambda$. Doing the second integral with (28), we again reproduce the formula (30), now with genus $g = 1$.

Let us summarize. From the $\delta$–function property (12) we always expect the boson induced gauge model (6) with $N_b \geq N_c$ to admit a continuum limit. In two dimensions — and on a direct system of two–dimensional cell complexes converging to a compact Riemann surface —, making the natural assignment $(1 - \alpha_p)^2 \sim A_p$ and taking the continuum limit $A_p \to 0$ sends the disk boundary–value partition function $\Gamma_D$ of the boson induced gauge model to the expression (29), if $N_b \geq N_c + 1$. By the reasoning of (28), this expression is the proper local data to use for $U(N_c)$ Yang–Mills theory, and gives the partition function (30).

More precisely speaking, the situation is this. The gauge group $U(N_c)$ is not semisimple; the number of its quadratic Casimir invariants is two (as opposed to one for a semisimple Lie group), and therefore $U(N_c)$ Yang–Mills theory is not unique but comes as a one–parameter family. The procedure of canonical quantization associates the Casimir invariants $\text{Cas}_2(\lambda)$ and $q(\lambda)^2$ with the action densities $\sum |F_{\mu\nu}^{ij}|^2 d^2x = -\text{Tr}(F \wedge *F)$ and $-\sum F_{\mu\nu}^{ii} F_{\mu\nu}^{jj} d^2x = -(\text{Tr} F) \wedge * (\text{Tr} F)$, respectively. Thus the combination

$$\text{Cas}_2(\lambda) + (B_1/B_2) \ q(\lambda)^2$$

in (30) is the Hamiltonian arising from the action functional

$$-S = \frac{1}{2} \int_{\Sigma} \text{Tr}(F \wedge *F) + \frac{B_1}{2B_2} \int_{\Sigma} (\text{Tr} F) \wedge * (\text{Tr} F)$$

by canonical quantization. The Lagrangian depends on the parameters $B_1$ and $B_2$, which in turn are given by the second moments of the distribution $\text{Det}^{-N_b}(1 - X^2) dX$; see (20).
Note that this distribution becomes Gaussian for $N_b \to \infty$. In that limit, $-\mathbb{E}_{N_b} (\text{Tr} X^2) \sim N_c/2N_b$ and $-\mathbb{E}_{N_b} \text{Tr}(X^2) \sim N_c^2/2N_b$, and we get $B_1/B_2 = 0$. Thus, the coupling $B_1/B_2$ gives some measure of how much the distribution $\text{Det}^{-N_b} (1 - X^2) dX$ differs from a Gaussian distribution. For $N_c = 2$ an easy computation gives

$$-\mathbb{E}_{N_b} \text{Tr}(X^2) = \frac{3}{2N_b - 5} + \frac{1}{2N_b - 3}; \quad -\mathbb{E}_{N_b} (\text{Tr} X)^2 = \frac{3}{2N_b - 5} - \frac{1}{2N_b - 3},$$

which yields $B_1/B_2 = \frac{1}{2}(N_b - 2)^{-1}$. Thus in this case the ratio $B_1/B_2$ for $N_b \geq N_c + 1 = 3$ is always positive.

### 4.4 The Cauchy case $N_b = N_c$

The continuum limit taken in the previous subsection does not go through for $N_b = N_c$. The main obstacle is that the second moments of the Cauchy distribution $\text{Det}^{-N_c} (1 - X^2) dX$ do not exist. The nonexistence can be verified by power counting, and is compatible with the failure of the inequality (19) in that case. It is also signalled by the expectation values of the positive quantities $-\text{Tr}(X^2)$ and $-(\text{Tr} X)^2$ we have just given for $N_b > N_c = 2$, which are seen to become negative at $N_b = N_c = 2$. Thus, although we still have a good integral formula,

$$c_\lambda(\alpha) = (1 - \alpha^2)^{-N_c^2} \int_{\mu(N_c)} \chi_\lambda \left( \frac{1 - \frac{1}{1+\alpha} X}{1 + \frac{1}{1+\alpha} X} \right) \text{Det}^{-N_c} (1 - X^2) dX,$$

we can no longer extract the behavior of $c_\lambda(\alpha)$ near $\alpha = 1$ by expanding the argument of the character $\chi_\lambda$ under the integral sign, as this would lead to a divergent integral. Hence a different approach is needed.

To get some inspiration, we turn to the simple case $N_c = N_b = 1$ with distribution

$$(2\pi)^{-1} \frac{d^\theta}{|1 - \alpha e^{i\theta}| - 2} \big| e^{i\theta} \frac{1 + \alpha + (1 - \alpha)e^{2i\theta}}{1 + \alpha - (1 - \alpha)e^{2i\theta}} \big| = (i\pi)^{-1} (1 - \alpha^2)^{-1} \frac{dx}{1 - x^2} \quad (x \in i\mathbb{R}).$$

The primitive characters of $U(1)$ are $\chi_n(e^{i\theta}) = e^{in\theta} \ (n \in \mathbb{Z})$. The coefficients $c_n(\alpha)$ can easily be written down in closed form:

$$c_n(\alpha) = (1 - \alpha^2)^{-1} \alpha^{|n|}.$$

Obviously, the effect of having a divergent second moment here is to cause a nonanalyticity in the Fourier variable $n$ at $n = 0$. In the previous case of $N_b \geq N_c + 1 = 2$, the ratio $c_n(\alpha)/c_0(\alpha)$ had zero derivative at $\alpha = 1$ by parity symmetry ($x \mapsto -x$), but now

$$\lim_{\alpha \to 1^-} \frac{d}{d\alpha} c_n(\alpha) = |n|,$$

which forces a different scaling of the parameters $\alpha_p$, and thus a different continuum limit. We are going to show that the same situation occurs for all $N_b = N_c \geq 1$.

Consider the formula

$$r_\lambda(\alpha) \equiv \frac{c_\lambda(\alpha)}{c_0(\alpha)} = \int_{\mu(N_c)} \chi_\lambda \left( \frac{1 - \frac{1}{1+\alpha} X}{1 + \frac{1}{1+\alpha} X} \right) \text{Det}^{-N_c} (1 - X^2) dX,$$
which expresses the ratio \( r_\lambda(\alpha) \) as an expectation of the character \( \chi_\lambda \) with respect to the normalized Cauchy distribution \( \text{Det}^{-N_c}(1 - X^2) \, dX \). Because \( |\chi_\lambda(U)| \) is bounded from above by the dimension \( d_\lambda \), the integral makes sense and defines \( r_\lambda(\alpha) \) as a continuous function of \( \alpha \) in the range \(-1 < \alpha < \infty\), and we have

\[
|r_\lambda(\alpha)| \leq d_\lambda \quad (-1 < \alpha < \infty).
\]

\( r_\lambda(\alpha) \) is maximal at \( \alpha = 1 \), where the upper bound \( r_\lambda(1) = \chi_\lambda(1) = d_\lambda \) is attained. Going away from that maximum, \( r_\lambda(\alpha) \) has to decrease (if \( \lambda \neq 0 \)), since nontrivial characters oscillate. From the \( N_b = N_c = 1 \) example (and for reasons which, if not clear already, will soon become evident), we expect a finite first derivative and hence a cusp singularity in \( r_\lambda(\alpha) \) at \( \alpha = 1 \). The task is to compute the slope of \( r_\lambda(\alpha) \) on the \( \alpha < 1 \) side of that cusp.

To that end, we start from the formula

\[
2 \lim_{\alpha \to 1-} \frac{d}{d\alpha} r_\lambda(\alpha) = - \lim_{\epsilon \to 0+} \frac{d}{d\epsilon} \int_{U(N_c)} \chi_\lambda \left( \frac{1 - \epsilon X}{1 + \epsilon X} \right) \text{Det}^{-N_c}(1 - X^2) \, dX.
\]

We would like to take the derivative inside the integral but, as it stands, are not allowed to do so. Since the obstacle arises from the noncompactness of the integration domain, the trick will be to compactify it.

In the first step, we introduce the eigenvalues of \( X \) and the diagonalizing matrix as the new variables of integration. Thus, let \( x = (x_1, \ldots, x_{N_c}) \in (i\mathbb{R})^{N_c} \) be the set of eigenvalues of the anti–Hermitian matrix \( X \), and let \( t^+ \subset t \equiv (i\mathbb{R})^{N_c} \) be the positive Weyl chamber given by \( ix_1 < ix_2 < \ldots < ix_{N_c} \). If \( T = U(1)^{N_c} \) is a maximal torus, say the diagonal matrices in \( U(N_c) \), some dense open set in \( U(N_c) \) is diffeomorphic to \( t^+ \times U(N_c)/T \) by the polar coordinate map \( \psi : (x, gT) \mapsto gxg^{-1} = X \). The Jacobian of this map is \( \Delta(ix)^2 = i^{N_c(N_c-1)} \prod_{k<l}(x_k - x_l)^2 \), which means there exists some positive flat density \( dx \) on \( t^+ \) such that

\[
\psi^*(dX) = dg_T \cdot \Delta(ix)^2 \, dx,
\]

with \( dg_T \) an invariant volume form on \( U(N_c)/T \).

The next step is to write the character \( \chi_\lambda \) as a sum over the integer weight lattice \( L_\lambda \) of the representation \( \lambda \):

\[
\chi_\lambda(e^{i\theta}) = \sum_{\{n\} \in L_\lambda} e^{i \sum n_k \theta_k}.
\]

\( L_\lambda \) is determined in principle by division of the elementary antisymmetric torus functions in Weyl’s formula \( \chi_\lambda(e^{i\theta}) = \xi_{\lambda+\rho}(e^{i\theta})/\xi_{\rho}(e^{i\theta}) \). Inserting the sum into the integral we obtain

\[
-2 \lim_{\alpha \to 1-} \frac{d}{d\alpha} r_\lambda(\alpha) = \text{vol}(U(N_c)/T) \lim_{\epsilon \to 0+} \frac{d}{d\epsilon} \int_{t^+} \sum_{\{n\} \in L_\lambda} \left( \prod_{k=1}^{N_c} \frac{1 - \epsilon x_k}{1 + \epsilon x_k} \right)^{n_k} (1 - x_k^2)^{-N_c} \Delta(ix)^2 \, dx.
\]

For further treatment, the restriction of the domain of integration to the positive Weyl chamber \( t^+ \subset t \) is inconvenient. Because the integrand is invariant w.r.t. the symmetric group (the Weyl group of \( U(N_c) \)), we may actually lift the restriction and integrate over all of \( t = (i\mathbb{R})^{N_c} \). The new integral is \( N_c! \) times the old one, so we divide by that factor.
the scalar factor $N$ Det and then see that the remaining integral is unity by normalization of the Cauchy distribution from the integral. Having done so, we expand the integration manifold to its original form, $X_{gl}$ manifold in the complex space. The integral does not change if we contract the integration manifold $N$ viewed as a differential form (of degree $1$). To compute the remaining integral, we first make the closed integration contours identical for all the variables $x_i$. Now is the point where we compactify. Observe that $\lim_{\alpha \to 1^+} -\frac{d}{d\alpha} r_{\lambda}(\alpha) = \frac{N_c}{\lambda} \frac{1}{1+\lambda}$ as a function of $z \in \mathbb{C}$ satisfies $|f(z)| < 1$ for Re$z > 0$, and $|f(z)|^{-1} < 1$ for Re$z < 0$. Therefore, given some fixed term $\{n\}$ in the sum over weights, we may modify the integration domain at infinity without changing the value of the integral, as follows: for all $k$ with $n_k > 0$ we close the integration contour for the variable $x_k \in i\mathbb{R}$ around the right half-plane Re$z > 0$, whereas for all $k$ with $n_k < 0$ we close around the left half-plane (for $n_k = 0$ we may close either way). Having closed the contours for all the variables, we pull them in from infinity, by holomorphicity. The integration domain, say $C_{\{n\}}$, is now compact. After compactification, we are permitted to differentiate under the integral sign and set $\epsilon$ to zero, which gives

$$\lim_{\alpha \to 1^+} -\frac{d}{d\alpha} r_{\lambda}(\alpha) = \frac{N_c}{\lambda} \frac{1}{1+\lambda} \right|_{\alpha \to 1^+} = \frac{N_c!^{-1} \text{vol}(U(N_c)/T)}{\sum_{\{n\} \in L_{\lambda}} \sum_{k=1}^{N_c} n_k} \int_{C_{\{n\}}} x_k \prod_{l} (1 - x_l^2)^{-N_c} \triangle(iz)^2 \, dx.$$ 

To compute the remaining integral, we first make the closed integration contours identical for all the variables $x_1, \ldots, x_{N_c}$. This is possible to arrange because after setting $\epsilon$ to zero, the point at infinity is no longer a singularity for the variables $x_l$ with $l \neq k$ (although for $l = k$ it still is, because of the presence of the factor $x_k$ in the integrand). Thus we deform from $C_{\{n\}}$ to some $C_{\sigma(n_k)} \times \cdots \times C_{\sigma(n_k)} = C_{\sigma(n_k)}^{N_c}$. The subscript $\sigma(n_k)$ reminds us that $C_{\sigma(n_k)}$ lies in the right or left half–plane — and encircles the pole of $(1 - x_k^2)^{-N_c}$ at $x_k = \pm 1$ — depending on whether $n_k$ is positive or negative, respectively.

Next we symmetrize the integrand, replacing $x_k$ by $N_c^{-1} \sum_l x_l$, and we revert to integrating over a restricted domain (and drop the factor $1/N_c!$), by requiring the variables $x_1, \ldots, x_{N_c}$ to be arranged in ascending order on $C_{\sigma(n_k)}$ according to the orientation of $C_{\sigma(n_k)}$.

Now let $M_{\sigma(n_k)}$ be the adjoint orbit of $U(N_c)$ on $C_{\sigma(n_k)}^{N_c}$. Then, using the polar coordinate map in reverse we arrive at

$$\lim_{\alpha \to 1^+} -\frac{d}{d\alpha} r_{\lambda}(\alpha) = \sum_{\{n\} \in L_{\lambda}} \sum_{k=1}^{N_c} n_k \int_{M_{\sigma(n_k)}} N_c^{-1} \text{Tr} X \, dX \frac{N_c^{-1} \text{Tr} X \, dX}{\text{Det} N_c^{-1} (1 - X^2)}. $$

This integral is easy to compute. The integration domain $M_{\sigma(n_k)}$ is a closed orientable $N_c^2$–manifold in the complex space $\text{gl}(N_c, \mathbb{C})$, and the integrand, a holomorphic density, can be viewed as a differential form (of degree $N_c^2$) which is closed. Therefore, the value of the integral does not change if we contract the integration manifold $M_{\sigma(n_k)}$ to another one in the same homology class enclosing the singular point $X = \text{sgn}(n_k) \times 1$. At that singularity, the scalar factor $N_c^{-1} \text{Tr} X$ takes the value $\text{sgn}(n_k)$. We extract the factor with this value from the integral. Having done so, we expand the integration manifold to its original form, and then see that the remaining integral is unity by normalization of the Cauchy distribution $\text{Det}^{-N_c} (1 - X^2) \, dX$. Thus we get the result

$$\lim_{\alpha \to 1^-} -\frac{d}{d\alpha} r_{\lambda}(\alpha) = d_{\lambda} \text{Cas}_1(\lambda),$$

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where
\[ \text{Cas}_1(\lambda) = d_\lambda^{-1} \sum_{\{n\} \in L_\lambda} \sum_{k=1}^{N_c} |n_k| \] (31)
is some sort of Casimir invariant, as will be explained shortly.

An equivalent way of stating the result is
\[ \frac{c_\lambda(\alpha)}{c_0(\alpha)} = d_\lambda \left( 1 - (1 - \alpha) \text{Cas}_1(\lambda) + \mathcal{R}(\alpha) \right) \quad (\alpha < 1), \]
with a remainder term \( \mathcal{R}(\alpha) \) that vanishes faster than \( 1 - \alpha \) as \( \alpha \to 1 \), from which we deduce
\[ \lim_{N \to \infty} \left( \frac{1}{d_\lambda} \frac{c_\lambda(1 - \mu/N)}{c_0(1 - \mu/N)} \right)^N = e^{-\mu \text{Cas}_1(\lambda)}. \]

Given this formula, the rest of the calculation goes the same way as in Section 4.3, and we just write down the answer. The continuum limit is defined by putting \( \alpha_p = 1 - A_p/a^2 \) and taking the plaquette areas \( A_p \) to zero. Then, setting \( \mu = \sum_p A_p/a^2 \) we have the local data
\[ \Gamma(U, \mu) = \sum_\lambda d_\lambda \exp(-\mu \text{Cas}_1(\lambda)) \chi_\lambda(U), \] (32)
and hence the genus \( g \) partition function
\[ Z_g(\mu) = \sum_\lambda d_\lambda^{-2g+2} e^{-\mu \text{Cas}_1(\lambda)}, \]
as follows again by the reasoning of [28].

What is the corresponding continuum field theory, i.e. what’s the Lagrangian that gives rise to the Hamiltonian \( \text{Cas}_1 \) upon canonical quantization? To answer that question, we first need to sharpen our understanding of \( \text{Cas}_1 \). Recall from Lie theory that Casimir invariants are elements in the center of the universal enveloping algebra (of the Lie algebra at hand), which consists of polynomials in the generators. The invariant \( \text{Cas}_1 \) certainly does not come from a finite-degree polynomial, so we are not entitled to call it a Casimir invariant in the strict sense. However, it is something quite similar. Fix some basis \( \{\tau_A\} \) of \( u(N_c) \) and write \( X \in u(N_c) \) as \( X = \sum_A X_A \tau_A \). Reinterpret \( \tau_A \) as an abstract operator \( \hat{\tau}_A \) that acts in the linear space \( V_\lambda \) of any representation \( D^\lambda \) by \( D^\lambda_{\lambda}(\tau_A) \). Then, viewing the coefficients \( X_A \) as real-valued functions on \( u(N_c) \), consider the formal expression
\[ K_\mu = \int_{u(N_c)} e^{\sum_A X_A \hat{\tau}_A} \mu N_c^2 dX \Det^{N_c} (\mu^2 - X^2)^{-1}. \]

By the replacement \( \hat{\tau}_A \to D^\lambda_{\lambda}(\tau_A) \), this makes sense as a compact operator in every unitary representation space \( V_\lambda \). Moreover, by the invariance of the Cauchy distribution w.r.t. conjugation \( X \to UXU^{-1} \), this operator commutes with all the \( U(N_c) \) generators, so it is a multiple of unity in every irrep by Schur’s lemma. We claim
\[ \text{Cas}_1 = \lim_{\mu \to 0+} \frac{1 - K_\mu}{\mu}. \]
Indeed, since $K_\mu$ acts as a multiple of the identity on every irreducible $V_\lambda$, we might as well take the trace over $V_\lambda$ and divide by the dimension $d_\lambda$. What we then encounter is an integral of the character $\chi_\lambda$ against the Cauchy distribution, and by a slight variant of the calculation done earlier for $r_\lambda(\alpha)$, we find that $K_\mu|_{V_\lambda}$ has a right–hand derivative at $\mu = 0$ and the value of this derivative is $\text{Cas}_1(\lambda)$.

When acting in the Hilbert space of square–integrable class functions $f(U)$ on $U(N_c)$, the operator $e^{\sum N A^\dagger A}$ causes translations $f(U) \mapsto f(e^{X}U) = f(e^{-X}U)$. It follows that the invariant operator $K_\mu$ acting in the same Hilbert space has the integral kernel

$$\langle U' | K_\mu | U \rangle = \int_{u(N_c)} \delta(U^{-1}U' e^{X}) \text{Det}^{-N_c} \left( \mu^2 - X^2 \right) \mu^{N_c^2} dX.$$ 

By definition, $\Gamma(U^{-1}U', \mu)$ in (32) is the kernel of the one–parameter semigroup generated by Cas$_1$. From what we have just shown, the same holds true of $\langle U' | K_\mu | U \rangle$ asymptotically for small $\mu$. Hence we infer the limit relation

$$\Gamma(U^{-1}U', \mu) = \lim_{N \to \infty} \left( K_\mu/N \right)^N \langle U' | U \rangle,$$

which allows us to construct a functional integral representation for $\Gamma(U, \mu)$ in the standard fashion. The resulting field theory is given by the action functional anticipated in the introductory part of the current section. Indeed, expressing the tangent–space $\delta$–function $\delta(U^{-1}U' e^{X})$ by Fourier integration over a conjugate variable $\phi$ with values in $u(N_c)$, we encounter the Fourier transform of the Cauchy distribution:

$$\int_{u(N_c)} e^{i \text{Tr} \phi X} \text{Det}^{-N_c} \left( \mu^2 - X^2 \right) \mu^{N_c^2} dX.$$ 

On substituting $X$ by $\mu X$ this becomes precisely the type of integral we computed in the current subsection, albeit with the character $\chi_\lambda((1 - \epsilon X)/(1 + \epsilon X))$ replaced by the bounded function $e^{i \text{Tr} \phi X}$. Proceeding in the same manner as before we find that the result in the small–$\mu$ limit is approximated by $e^{-\mu\|\phi\|_1}$ with $\| \phi \|_1$ the linear potential in (14).

4.5 A different perspective: Howe duality

In treating the two–dimensional theory, crucial use was made of the expansion of the plaquette distribution function in terms of $U(N_c)$ characters $\chi_\lambda(U)$:

$$\text{Det} (1 - \alpha U)^{-2N_b} = \sum_\lambda c_\lambda(\alpha) \chi_\lambda(U).$$

We now wish to communicate the intriguing fact that the expansion coefficients $c_\lambda(\alpha)$ themselves are characters, of a noncompact Lie group dual to $U(N_c)$ in the sense of R. Howe [15, 14]. Explaining this duality will go a long way toward preparing and setting up the duality transformation carried out in the next section.

The general framework for the kind of duality we are about to describe is what is called the bosonic Fock space in physics (and the Shale–Weil, or metaplectic, or oscillator representation in mathematics). To get started, let there be a single oscillator or boson mode with operators
between bosons of opposite $U(1)$ charges, $\pm a$. Now we extend the formalism further by adding flavor to the boson operators $b_i, b_i^\dagger$. (Of course the canonical commutation relations still hold, the Hermitian scalar product extends in the obvious manner, and the vacuum $|0\rangle$ is annihilated by each of the $b_i$.)

Let $E_{kl}$ be the $N_c \times N_c$ matrix with entry 1 at the intersection of the $k$th row with the $l$th column, and zeroes everywhere else. To a “Hamiltonian” $X = \sum_{kl} X_{kl} E_{kl} \in u(N_c)$, we associate the partition sum

$$\operatorname{Tr}_\mathcal{V} e^{-\mu \sum_i b_i^\dagger b_i + \sum_{kl} X_{kl} b_k^\dagger b_l} = \operatorname{Det} (1 - e^{-\mu + X})^{-1}.$$  

Now we extend the formalism further by adding flavor $a = 1, 2, \ldots, N_b$, and distinguishing between bosons of opposite $U(1)$ charges, $\pm$. Then, if $\hat{N}_{\text{bos}} = \sum a \left( b_+^{j,a\dagger} b_+^{j,a} + b_-^{j,a\dagger} b_-^{j,a} \right)$ is the total boson number, we get

$$\operatorname{Tr}_\mathcal{V} e^{-\mu \hat{N}_{\text{bos}}} e^{\sum_{kl} X_{kl} \sum_a \left( b_+^{k,a\dagger} b_+^{k,a} - b_-^{l,a\dagger} b_-^{l,a} \right)} = |\operatorname{Det} (1 - e^{-\mu + X})|^{-2N_b},$$

which becomes the distribution function we have been working with on identifying $e^{-\mu} = \alpha$ and $e^X = U \in U(N_c)$.

The linear mapping

$$X \mapsto \sum_{a=1}^{N_b} \sum_{k,l=1}^{N_c} X_{kl} \left( b_+^{k,a\dagger} b_+^{k,a} - b_-^{l,a\dagger} b_-^{l,a} \right)$$

is an isomorphism of Lie algebras (i.e. it preserves the commutator), and it is easily seen to exponentiate to a homomorphism of Lie groups, $U \mapsto T_U$. Thus, $U \in U(N_c)$ acts on the bosonic Fock space $\mathcal{V}$ by $T_U$. Using $T_U$ we can write the previous formula in the concise form

$$|\operatorname{Det} (1 - \alpha U)|^{-2N_b} = \operatorname{Tr}_\mathcal{V} \alpha^{\hat{N}_{\text{bos}}} T_U.$$  

This identifies the plaquette statistical weight function of the boson induced gauge model as a trace, or character, in Fock space.

In addition to $U(N_c)$ there is a second group that naturally acts on the bosonic Fock space $\mathcal{V}$. This is the noncompact Lie group $U(N_b, N_c)$. To describe its action, we decompose Lie algebra elements $Y \in u(N_b, N_c)$ into $N_b \times N_b$ blocks as $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with anti–Hermitian $A = \sum A^{ab} E_{ab}$, $D = \sum D^{ab} E_{ab}$, and with $B = \sum B^{ab} E_{ab}$, $C = \sum C^{ab} E_{ab}$ being adjoints of each other; and we assign to $Y$ an operator $\hat{Y} = -Y^\dagger$ on $\mathcal{V}$ by

$$\hat{Y} = \sum_{a,b=1}^{N_b} \sum_{j=1}^{N_c} \left( A^{ab} b_+^{j,a\dagger} b_+^{j,b} + B^{ab} b_+^{j,a\dagger} b_-^{j,b} - C^{ab} b_-^{j,a\dagger} b_+^{j,b} - D^{ab} b_-^{j,a\dagger} b_-^{j,b} \right).$$
The mapping \( Y \mapsto \hat{Y} \) is an isomorphism of \( \mathfrak{u}(N_b, N_b) \) Lie algebras. By exponentiating it, we get a unitary representation of \( U(N_b, N_b) \) on \( \mathcal{V} \).

Thus we have two groups, \( U(N_c) \) and \( U(N_b, N_b) \), acting on the bosonic Fock space. Because \( U(N_c) \) acts on color and \( U(N_b, N_b) \) on flavor, the two group actions commute. This turns out to be a maximal property: you cannot enlarge either one of the two groups without compromising it. Put differently, \( U(N_b, N_b) \) is the centralizer of \( U(N_c) \) inside the big group of symplectic (actually, metaplectic) transformations of \( \mathcal{V} \). R. Howe calls such a pair of Lie groups a dual pair; see [12] for a pedagogical introduction to the subject.

Because \( U(N_c) \) is compact, general theory guarantees that we can decompose \( \mathcal{V} \) into irreducible representation spaces \( V_{\lambda} \) for this group. All \( U(N_c) \) representations of a given type \( \lambda \) are collected into what is called an isotypic component for \( U(N_c) \) in \( \mathcal{V} \). Here comes the main point of the present discussion: as a consequence of the dual pair property, the product \( U(N_b, N_b) \times U(N_c) \) acts irreducibly \([14]\) on every such isotypic component. Thus the decomposition of \( \mathcal{V} \) takes the form of a multiplicity–free sum,

\[
\mathcal{V} = \sum_{\lambda} \hat{V}_{\lambda} \otimes V_{\lambda},
\]

where \( \hat{V}_{\lambda} \) and \( V_{\lambda} \) are irreducible representation spaces for \( U(N_b, N_b) \) and \( U(N_c) \), respectively, and the correspondence \( V_{\lambda} \leftrightarrow \hat{V}_{\lambda} \) is one–to–one.

We can now use this to decompose the trace of the product \( \alpha^{N_{\text{bos}}} T_U \) over \( \mathcal{V} \). The operator \( \alpha^{N_{\text{bos}}} \) is trivial on the second factor of any isotypic component \( \hat{V}_{\lambda} \otimes V_{\lambda} \) in \( \mathcal{V} \), while \( T_U \) is trivial on the first factor. As a result, the trace over every isotypic component separates into two factors, one depending on \( \alpha \) and the other on \( U \):

\[
\text{Tr}_{\hat{V}_{\lambda} \otimes V_{\lambda}} \alpha^{N_{\text{bos}}} T_U = \text{Tr}_{\hat{V}_{\lambda}} \alpha^{N_{\text{bos}}} \times \text{Tr}_{V_{\lambda}} T_U = c_\lambda(\alpha) \chi_{\lambda}(U).
\]

The factors are of a similar kind: both are primitive characters; \( \chi_{\lambda}(U) \) for \( U(N_c) \), and \( c_\lambda(\alpha) \) for \( U(N_b, N_b) \). What we have learned then is this: the coefficient \( c_\lambda(\alpha) \) in the character expansion of the induced statistical weight function,

\[
|\text{Det} (1 - \alpha U)|^{-2N_b} = \text{Tr}_{\mathcal{V}} \alpha^{N_{\text{bos}}} T_U = \sum_{\lambda} c_\lambda(\alpha) \chi_{\lambda}(U),
\]

is itself a character; it is the value on \( \alpha^{N_{\text{bos}}} \) of the \( U(N_b, N_b) \) character associated to the \( U(N_c) \) representation \( \lambda \) by the dual pair correspondence.

Precisely speaking, \( N_{\text{bos}} \) does not represent a generator of the real form \( \mathfrak{u}(N_b, N_b) \), but a generator of the complexified Lie algebra \( \mathfrak{gl}(2N_b, \mathbb{C}) \). (The generator lying in \( \mathfrak{u}(N_b, N_b) \) is \( i\tilde{N}_{\text{bos}} \).) Thus \( \alpha^{N_{\text{bos}}} \) is to be viewed as representing an element of \( \text{GL}(2N_b, \mathbb{C}) \), and the \( c_\lambda(\alpha) \) are obtained from \( U(N_b, N_b) \) characters by analytically continuing to that element.

Because the representation spaces \( \hat{V}_{\lambda} \) are infinite–dimensional, the characters \( c_\lambda(\alpha) = \text{Tr}_{\hat{V}_{\lambda}} \alpha^{N_{\text{bos}}} \) all diverge at the unit element, \( \alpha = 1 \). One of the results we found earlier is \( c_0(\alpha) \sim (1 - \alpha)^{-2N_bN_c + N_c^2} \) near that point. This has a transparent interpretation from the present perspective: taking the trace of \( \alpha^{N_{\text{bos}}} \) over all states of \( \mathcal{V} \) would give \( (1 - \alpha)^{-2N_bN_c} \), since there are \( 2N_bN_c \) boson species. The projection onto \( U(N_c) \) singlets (the trivial representation, \( \lambda = 0 \)) amounts to imposing \( N_c^2 \) constraints, which reduces the degree of the singularity to \( 2N_bN_c - N_c^2 \).
Every irreducible representation $\lambda$ of $U(N_c)$ need not occur in the sum $\mathcal{V} = \sum_\lambda \tilde{V}_\lambda \otimes V_\lambda$. Howe’s statement is that the sum is multiplicity–free, i.e. the multiplicity is at most one, but it can also be zero. For example, for $N_c = 2$ and $N_b = 1$ all the representations $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \geq \lambda_2 > 0$ or $0 > \lambda_1 \geq \lambda_2$ are missing. For $N_b \geq N_c$, however, all irreducible representations of $U(N_c)$ do occur, as is implied by the $\delta$–function property \cite{12}.

5 Duality Transformation

Having established the continuum limit of the 2d induced gauge model \cite{14} (with $N_b \geq N_c + 1$ boson species) to lie in the universality class of 2d Yang–Mills theory, we certainly expect a similar scenario to be true in higher dimension. The simple reason is that going up in dimension enhances the collectivity of the fields, and thereby works in favor of universality. The 2d Cauchy–type theory we found for $N_b = N_c$ (Section 4.4) is a low–dimensional gimmick, and is highly unlikely to persist above two space–time dimensions. Thus in three dimensions and higher, we expect universal Yang–Mills physics in the continuum limit of the boson induced gauge model for all $N_b \geq N_c$.

Motivated by this expectation, we will show in the current section how to pass to a dual version of the induced gauge model. The main tool we are going to use is a variant of the color–flavor transformation \cite{16,17}, which is based on Howe duality and the notion of dual pairs we have just sketched.

As an aside, we mention that the same goal of constructing a dual theory was pursued in \cite{6}, by similar techniques. However, the transformation carried out there does not extend to the good range $N_b \geq N_c$ (in fact, only the case $N_b = 1$ was addressed in that reference) and therefore fails to be of relevance for Yang–Mills theory.

5.1 Abelian duality (review)

By way of preparation and for future reference, we first review the standard construction of a dual theory in the abelian case. The nonabelian duality transform described later on will be seen to reduce to the standard one in the abelian limit.

Consider the partition function of the induced $U(1)$ gauge model \cite{4} with $N_b = 1$ (and no fermions, $N_f = 0$):

$$ Z(\alpha) = \int [dU] \prod_p |1 - \alpha_p U(\partial p)|^{-2}. $$

As before, the model is placed on a $d$–dimensional cell complex $\Lambda$ with boundary operator $\partial$, and all of the real parameters $\alpha_p$ are chosen close to (but less than) unity. To dualize such a model in a concise manner, we need a few basic facts \cite{30} from discrete exterior calculus, which are rapidly summarized in the next paragraph.

Recall that the boundary operator $\partial$ is a linear mapping from the vector space of $k$–chains into the vector space of $(k - 1)$–chains, and the boundary of a boundary is always zero: $\partial \circ \partial = 0$. A $k$–chain $b$ is called closed if $\partial b = 0$. The Poincaré lemma (when applicable) states that every closed $k$–chain $b$ is a boundary: $b = \partial c$. Objects dual to chains are called cochains: a $k$–cochain $\omega$ (the discrete version of a $k$–form) is a linear function that assigns to every $k$–chain $c$ a real number, say $\langle c, \omega \rangle$. The coboundary operator $d : k$–cochains $\rightarrow (k + 1)$–cochains (the
It is in this dual form — as a sum over 2–chains $n$ — that the physics to be expected. For brevity, we do this only for the special case of

\[ \text{let us quickly review how further analysis of the dual theory is carried out, and what is seen to emerge as a special case from the nonabelian duality transform described in the sequel; cf. Section 5.8.} \]

Choosing some surface $S$ with boundary $\partial S = C$, one shifts $n \to n - S$ in order for the sum to be over $n$ with $\partial n = 0$. Since we are assuming trivial homology, the Poincaré lemma applies and guarantees for every closed 2–chain $n$ the existence of a 3–chain $a$ such that $n = \partial a$. Of course, $a$ is not uniquely determined: if $a$ has boundary $n$, then so does the gauge transform $a + \partial \varphi$ by any 4–chain $\varphi$. Thus, solving the constraint $\partial n = 0$ by $n = \partial a$, one arrives at a sum over gauge equivalence classes $[a] = [a + \partial \varphi]$. For the Wegner–Wilson loop one obtains

\[ W(C) = \frac{\sum_{[a]} e^{-\|\partial a - S\|_\alpha}}{\sum_{[a]} e^{-\|\partial a\|_\alpha}}. \tag{33} \]
At this point one usually transcribes the theory from $\Lambda$ to the dual lattice $\Lambda^*$. The transcription proceeds by a canonical isomorphism (known in the cohomological setting as Poincaré duality) which turns $k$–chains on $\Lambda$ into $(4 - k)$–cochains on $\Lambda^*$. In particular, the 3–chains $a$ on $\Lambda$ become $\mathbb{Z}$–valued 1–cochains $a^*$ on $\Lambda^*$, and the boundary $\partial a$ becomes the coboundary $d a^*$. Passing to the dual lattice has the virtue of revealing the true meaning of the dual theory as a lattice gauge theory with gauge group $\mathbb{Z}$. Other than that, the passage is really quite unnecessary, and one may as well continue to work on $\Lambda$ as we do here.

Finally, using Poisson summation one relaxes the values of $a$ from $\mathbb{Z}$ to $\mathbb{R}$, at the expense of inserting a factor $e^{i(a,m)}$ and summing over all closed $\mathbb{Z}$–valued 3–cochains $m$ on $\Lambda$ (they must be closed in order to be well–defined functions on gauge equivalence classes $[a]$). The idea behind this last step is that the $\mathbb{R}$–valued gauge field $a$ should become the dual of the “photon”, and the closed 3–cochain $m$ (which turns into a closed 1–chain on passing to the dual lattice $\Lambda^*$) is to be interpreted as the world lines of magnetic monopoles.

On a hypercubic lattice, and in the Villain approximation

$$e^{-\|\partial a\|_\alpha} \rightarrow e^{-t \sum_p (\partial a_p)^2},$$

the photon field $a$ is now readily integrated out to produce an effective action for the magnetic monopole current $m$. The physics of the resulting model is readily understood when $t$ is sufficiently small: in that case the magnetic monopoles are bound in neutral clusters, leaving the system in a Coulomb phase with a massless photon. Rigorous mathematical control on this scenario has been achieved in [10, 11].

In the present model, that conclusion is less immediate. At very short scales of a few lattice units the photon cannot be free, as the action $\|\partial a\|_\alpha$ is not quadratic but linear (and in fact nonanalytic, by the use of the absolute value)! However, we expect that a sequence of suitable real space renormalization group transformations will cause flow toward the quadratic action. (The renormalization process of thinning the degrees of freedom by summing over the short–distance fluctuations of $n$, should have a similar effect as taking convolutions of the nonanalytic distribution $n \mapsto e^{-\|n\|_\alpha}$, and a generalized version of the central limit theorem should take effect.) We note that the method of Fröhlich and Spencer [11], which does not rely on integrating out the “photon” to produce an effective action for the magnetic monopoles, looks promising for a check on this picture.

In summary, our induced $U(1)$ gauge model is definitely interacting at short distances, but Coulomb behavior and a free photon are expected to emerge at large scales in four dimensions (provided that the $\alpha_p$’s are close enough to unity).

5.2 Nonabelian transform: one–link integral

Our starting point is the partition function (1) on any $d$–dimensional cell complex $\Lambda$. Although our approach is general and can handle fermions as well as bosons, we will restrict our attention to the case of bosons only. Switching the order of integrations, we will first do the gauge field integral and afterwards the auxiliary boson field integral.

For a fixed configuration of the complex boson fields $\varphi$, the Boltzmann weight of the field theory partition function is a product over links. Hence, integrating over the gauge field amounts to doing a set of independent one–link integrals. To write down and compute the one–link integrals, we need a good notation.
Figure 4: The set of plaquettes adjacent to a link \( l \), \( \Pi(l) \), consists of two subsets defined by comparing orientations, which either match \( [\Pi^+(l)] \) or don’t match \( [\Pi^-(l)] \).

Let \( \Pi(l) \) denote the set of oriented plaquettes \( p \) that contain the link \( l \) in their chain of boundary links. In formulas: \( p \in \Pi(l) \) if \( \partial p = \pm l + \ldots \); see Figure 4. The set \( \Pi(l) \) consists of two subsets: the plaquettes whose orientation agrees with that of \( l \) (\( \partial p = +l + \ldots \)), and those where the orientations disagree (\( \partial p = -l + \ldots \)). We write the decomposition into subsets as \( \Pi(l) = \Pi^+(l) \cup \Pi^-(l) \). Note that the cardinalities of the two sets \( \Pi^+ \) and \( \Pi^- \) are the same, since every plaquette occurs twice, once with each orientation.

Now we fix some link \( l \), write \( U \equiv U(l) \) and \( \Pi^\pm \equiv \Pi^\pm(l) \) for short, and denote the two boundary sites of \( l \) by \( n_1 \) and \( n_2 \) (in other words, \( n_1 \) is the site where \( l \) begins, and \( n_2 \) is where \( l \) ends). Then the gauge field integral pertaining to link \( l \) is

\[
I(\varphi, \bar{\varphi}) \equiv \int \exp \left( \sum_{p \in \Pi^+} \bar{\varphi}_p(n_2) U \varphi_p(n_1) \right) \exp \left( \sum_{q \in \Pi^-} \bar{\varphi}_q(n_1) U^{-1} \varphi_q(n_2) \right) dU. \tag{34}
\]

Here we are using the same short-hand notation as in (3): what the quadratic expressions really mean is

\[
\bar{\varphi}_p(n_1) U \varphi_p(n_2) = \sum_{i,j=1}^{N_c} \sum_{a=1}^{N_b} \bar{\varphi}^{i,a}_p(n_2) U_{ij} \varphi^{j,a}_p(n_1).
\]

In what follows we trade the integral over \( U \) for a sum over dual degrees of freedom.

5.3 Quantization

To proceed, we need some definitions. The main idea is to carry out a kind of “quantization” (not to be confused with quantization in the usual sense of passing to a quantum Hamiltonian formulation of the theory): for every color \( i \in \{1, \ldots, N_c\} \), every flavor \( a \in \{1, \ldots, N_b\} \), and every oriented plaquette \( p \in \Pi = \Pi^+ \cup \Pi^- \), we introduce a boson annihilation operator \( b^i_a \) and the adjoint creation operator \( b^{i\dagger}_a \). These operators obey the usual boson commutation relations:

\[
[b^i_a, b^j_b^\dagger] = \delta_{p,q} \delta^{i,j} \delta^{a,b},
\]

(with the \( b \)'s and \( b^\dagger \)'s commuting amongst themselves), and they act in a Fock space \( \mathcal{V} \) with vacuum \(|0\rangle \) characterized by

\[
b^{i\dagger}_a |0\rangle = 0.
\]
Elements of the gauge group, $U(\mathcal{N})$, act on $\mathcal{V}$ by unitary operators $T_\mathcal{U}$ with the properties $T_\mathcal{U}|0\rangle = |0\rangle$ and

\begin{align*}
T_\mathcal{U}b^a_p T_\mathcal{U}^\dagger &= \sum_j b^a_p U^{ji} \quad (p \in \Pi^+) , \\
T_\mathcal{U}b^a_q T_\mathcal{U}^\dagger &= \sum_j b^a_q U^{ji} \quad (q \in \Pi^-) .
\end{align*}

Thus the $b_q$ for $q \in \Pi^-$ transform according to the vector representation of $U(\mathcal{N})$, and the $b_p$ for $p \in \Pi^+$ according to the covector representation ($\bar{U} = U^{-1\dagger}$). For the creation operators $b^\dagger$ the situation is reversed.

Some mathematical background to this setup was given in Section 4.5. There, we explained how $T_\mathcal{U}$ arises by exponentiating an isomorphism of Lie algebras. For present purposes, just knowing the existence of $T_\mathcal{U}$ and its specified properties will be sufficient.

Consider now any one of the factors in the integrand of the one–link integral, for $p \in \Pi^+$. Using the operator $T_\mathcal{U}$ (or rather the part, $T_\mathcal{U}(p)$, acting on the bosons associated with $p$) we can express it as a matrix element between coherent states (index summations suppressed!):

\[ e^{\bar{\varphi}_p(n_2) U \varphi_p(n_1)} = \left\langle 0 | e^{\bar{\varphi}_p(n_2) b_p} T_\mathcal{U}(p) e^{b_p \varphi_p(n_1)} |0\right\rangle . \]

Verification is immediate by the $U(\mathcal{N})$ invariance of the vacuum ($T_\mathcal{U}|0\rangle = |0\rangle$), relation (35) and $\langle 0| e^{\bar{\varphi}_p b_p} |0\rangle = e^{\bar{\varphi}_p \varphi_p}$. Similarly, for $q \in \Pi^-$, we have

\[ e^{\bar{\varphi}_q(n_1) U^{-1} \varphi_q(n_2)} = e^{\bar{\varphi}_q(n_2) U \varphi_q(n_1)} = \left\langle 0 | e^{\bar{\varphi}_q(n_2) b_q} T_\mathcal{U}(q) e^{b_q \varphi_q(n_1)} |0\right\rangle . \]

Here we used $(U^{-1})^{ij} = \bar{U}^{ji}$ and the fact that switching from $p \in \Pi^+$ to $q \in \Pi^-$ interchanges the vector and covector representations of $U(\mathcal{N})$. Thus, although the matrix $U$ was replaced by its complex conjugate $\bar{U}$, the operator whose matrix element we take is still $T_\mathcal{U}$.

### 5.4 Projection on gauge singlets

We return to the task of integrating over $U$. The benefit of “quantization”, i.e. interpreting the integrand of the one–link integral as the matrix element of an operator in Fock space $\mathcal{V}$, is that all dependence on the gauge field matrix $U$ now resides in

\[ T_\mathcal{U} = \prod_{p \in \Pi^+} T_\mathcal{U}(p) \prod_{q \in \Pi^-} T_\mathcal{U}(q) , \]

which is just the operator satisfying the relations (35, 36). Thus, doing the $U$–integral has become very easy; given the last two formulas in the preceding subsection, we immediately express the one–link integral as a coherent state matrix element of $P_0 \equiv \int_{U(\mathcal{N})} T_\mathcal{U} \, dU$:

\begin{align*}
\int_{U(\mathcal{N})} &\exp \left( \sum_{p \in \Pi^+} \bar{\varphi}_p(n_2) U \varphi_p(n_1) + \sum_{q \in \Pi^-} \bar{\varphi}_q(n_1) U^{-1} \varphi_q(n_2) \right) dU = \\
&\left\langle 0 \right| \exp \left( \sum_{p \in \Pi^+} \bar{\varphi}_p(n_2) b_p + \sum_{q \in \Pi^-} \varphi_q(n_2) b_q \right) P_0 \exp \left( \sum_{p \in \Pi^+} b^\dagger_p \varphi_p(n_1) + \sum_{q \in \Pi^-} b^\dagger_q \bar{\varphi}_q(n_1) \right) |0\rangle .
\end{align*}

(37)
What is the meaning of $P_0$? According to [35, 36], the operator $T_U$ acts on Fock space $\mathcal{V}$ by $U(N_c)$ rotations. If $|\psi\rangle$ is any state in $\mathcal{V}$, integrating the rotated state $T_U|\psi\rangle$ against Haar measure $dU$ kills that part of the state which transforms nontrivially under $U(N_c)$, and leaves only the $U(N_c)$ invariant part. Hence $P_0$ is simply the operator that projects on the $U(N_c)$ invariant sector of $\mathcal{V}$!

The present formalism would be useless if that sector — let it be denoted by $\mathcal{V}_0$ — were some complicated and obscure subspace of $\mathcal{V}$. Fortunately, that’s not the case and $\mathcal{V}_0$ has a very transparent description, as follows.

Consider the set of all boson pair creation operators of the form

$$E_{pq}^{ab} = \sum_{i=1}^{N_c} i_p^{i,a\dagger} i_q^{i,b\dagger}, \quad \text{(38)}$$

where $p \in \Pi^+$, $q \in \Pi^-$, and $a, b \in \{1, \ldots, N_b\}$. Because the $i_p^{i,a\dagger}$ transform as a $U(N_c)$ vector and the $i_q^{i,b\dagger}$ as a $U(N_c)$ covector, the contraction formed by summing over equal colors $i = j = 1, \ldots, N_c$ is a $U(N_c)$ scalar. Thus the $E_{pq}^{ab}$ are $U(N_c)$ invariant, and so is every state created by acting with an arbitrary polynomial in these operators on the vacuum:

$$E_{p_1q_1}^{a_1b_1} E_{p_2q_2}^{a_2b_2} \cdots E_{p_nq_n}^{a_nb_n} |0\rangle.$$ \quad \text{(39)}

It is a beautiful and powerful theorem of classical invariant theory [15] that the states [39] span $\mathcal{V}_0$; i.e. every $U(N_c)$ invariant state in $\mathcal{V}$ can be reached by repeatedly acting on the vacuum $|0\rangle$ with the pair creation operators $E_{pq}^{ab}$ and taking linear combinations.

An outline of the reasoning is as follows. Let $r = N_b \times \text{card} \Pi^+$, with $\text{card} \Pi^+$ being the cardinality of the set $\Pi^+$ (which is the same as the cardinality of $\Pi^-$). For a $d$-dimensional hypercubic lattice the value of $r$ is $2(d - 1)$. The antihermitian operators built from the $E_{pq}^{ab}$ and their adjoints, taken together with all their commutators, span a Lie algebra that generates a unitary representation (on $\mathcal{V}$ with the usual Fock space scalar product) of the noncompact group $U(r, r)$. Because the $E_{pq}^{ab}$ are $U(N_c)$ invariant, it is clear that the Fock space actions of the two Lie groups $U(N_c)$ and $U(r, r)$ commute with each other; they in fact centralize each other and constitute a dual pair in the sense of R. Howe.

Recall from Section 4.5 that the representation of a dual pair $U(N_c) \times U(r, r)$ is irreducible on every $U(N_c)$ isotypic component of $\mathcal{V}$. In particular, this implies that $U(r, r)$ acts irreducibly on the $U(N_c)$ invariant sector $\mathcal{V}_0$ of $\mathcal{V}$. This irreducibility is just what we are claiming: every state in $\mathcal{V}_0$ can be reached by acting with the generators of $U(r, r)$ on some $U(N_c)$ invariant reference state. (And, of course, if we take that reference state to be the vacuum $|0\rangle$, it suffices to act with the pair creation operators $E_{pq}^{ab}$.)

Let us summarize where we were prior to the digression characterizing the subspace $\mathcal{V}_0$: we had identified the integral $P_0 = \int_{U(N_c)} T_U dU$ as the projector onto $\mathcal{V}_0$, and we had expressed the one–link integral [34] as a matrix element between coherent states; schematically, [37] is of the form

$$I(\varphi, \bar{\varphi}) = \langle \varphi(n_2)|P_0|\varphi(n_1)\rangle.$$ 

In the following we will think about this matrix element as a trace:

$$I(\varphi, \bar{\varphi}) = \text{Tr}_\mathcal{V}(P_0 A_\varphi) = \text{Tr}_{\mathcal{V}_0} A_\varphi, \quad \text{with } A_\varphi = |\varphi(n_1)\rangle\langle \varphi(n_2)|.$$ 

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We mention in passing that the color–flavor transformation in its standard form, introduced first in a supersymmetric setting in \[1\], and applied to the present context in \[2\], would result at the present stage of development if it was possible to express the trace over \(V_0\) as an integral over \(U(r,r)\) coherent states. However, this can only be done if \(r \leq N_c/2\), and fails for the range of \(r\) values of physical interest. (The problem is a problem of convergence: the \(U(r,r)\) representation spaces \(V_0\) for \(N_c < 2r\) cannot be realized by square–integrable holomorphic sections; cf. the last part of Section \([6]\)) We must therefore proceed in a different manner.

### 5.5 Integration over \(\varphi, \bar{\varphi}\)

We have written the integral \([32]\) over a single matrix \(U \equiv U(1)\) as the trace of some operator \(A_\varphi(l)\) in the \(U(N_c)\) invariant sector \(V_0(l)\) of an auxiliary Fock space \(V(l)\). In the final step taken now, we multiply all one–link integrals together and arrive at an expression for the partition function \([14]\) of the form

\[
Z = \text{Tr}_{V_0} A,
\]

where \(V_0 \equiv \bigotimes_l V_0(l)\) is the tensor product of the \(U(N_c)\) invariant spaces associated with all the links, and the operator \(A\) is the result of integrating \(\prod A_\varphi(l)\) over the boson fields \(\varphi, \bar{\varphi}\). It remains to describe \(A\), which is what we do next.

Recall from Section 2 that the boson \(\varphi_p\) hops from site to site of the chain of boundary links of the oriented plaquette \(p\), and carries a color index \(i \in \{1, \ldots, N_c\}\) and a flavor index \(a \in \{1, \ldots, N_b\}\). \(\varphi_p\) enters through the mass term \(m_{b,p}\) of \([3]\) and, by the manipulations of the previous subsection, it appears as a parameter in the operators \(A_\varphi = \ket{\varphi(n_1)}\bra{\varphi(n_2)}\) (schematic notation). We now write \(m_{b,p} \equiv m_p\) for short.

While the boson fields were originally coupled by the link matrices \(U(l)\), they are now completely uncoupled (the mass term does not couple them, and the operators \(A_\varphi\) under the trace don’t either.) Thus we can do the boson field integration for each component \(\varphi_p^{ia}(n)\) separately. So let us concentrate on some \(\varphi_p^{ia}(n)\) and calculate the corresponding integral.

We start the calculation by noting that \(\varphi_p^{ia}(n)\) and its complex conjugate \(\bar{\varphi}_p^{ia}(n)\) both occur exactly once as a coherent state parameter, multiplying some boson operator \(b\) or \(b^\dagger\) in the exponents on the right–hand side of \([32]\). (The reason is that, since \(n\) is a site visited by the boundary chain of \(p\), there exist exactly two links in \(\partial p\) that are attached to \(n\).) To decide which they multiply — \(b\) or \(b^\dagger\) —, let \(l_1\) and \(l_2\) be those two links in the boundary chain of \(p\) that have the site \(n\) as a boundary point. Recalling that the lattice links \(l\) come with an a priori orientation determined by \(\Lambda\), we are led to distinguish between three cases.

1. Let the site \(n\) be the end point of \(l_1\) and the starting point of \(l_2\), and let the orientations of \(l_1\) and \(l_2\) agree with that of \(\partial p\); see Figure \([5i]\). Then, from \([32]\) we encounter the following integral:

\[
T_p(n) = \int d\varphi_p(n)d\bar{\varphi}_p(n) e^{-m_p|\varphi_p(n)|^2} e^{\frac{1}{2}(b^\dagger_{p,l_2})^k(b_{p,l_2})^k} \varphi(n) \langle 0 | e^{\varphi}(n) b_p(l_1) \rangle.
\]

(The color and flavor indices play no important role here and have been omitted.) Expanding the last two exponentials and doing the Gaussian integral we obtain

\[
T_p(n) = \sum_{k=0}^\infty \frac{m_p^{-k} 1}{k!} (b^\dagger_{p,l_2})^k \langle 0 | b_{p,l_2} \rangle^k.
\]

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Figure 5: For the purpose of doing the boson field integration, one must distinguish between the three types of corner shown here.

Since \((b^\dagger)^k|0\rangle/\sqrt{k!}\) is a normalized state, we see that what the operator \(T_p(n)\) does is to simply transfer bosons from link \(l_1\) to \(l_2\), while weighting each boson with the inverse of the mass \(m_p\). Note that this transfer happens for each color and flavor independently. A second subcase of the present type occurs when the orientations of \(l_1\) and \(l_2\) are opposite to that of \(\partial_p\). In that case the variables \(\varphi_p(n)\) and \(\bar{\varphi}_p(n)\) exchange their roles, but the operator that results on doing the integral is still the same.

2. Next, let the site \(n\) be the starting point of both links \(l_1\) and \(l_2\), and let the orientation of \(\partial_p\) agree (disagree) with that of \(l_1\) (resp. \(l_2\)); see Figure 5(ii). Then from (37) we encounter the integral

\[
C_p(n) = \int d\varphi_p(n)d\bar{\varphi}_p(n) e^{-m_p|\varphi_p(n)|^2} e^{b^\dagger_p(l_1)\varphi_p(n) + b^\dagger_p(l_2)\bar{\varphi}_p(n)} |0\rangle \langle 0| \nonumber
\]

\[
= \sum_{k=0}^{\infty} \frac{m_p^{-k} \langle 0| (b^\dagger_p(l_1)b^\dagger_p(l_2))^k |0\rangle \langle 0|}{k!}.
\]

Clearly, \(C_p(n)\) creates (an indefinite number of) boson pairs in normalized states, weighted by powers of the inverse mass. Again, this happens for each color and flavor independently. If the orientation of \(p\) is reversed, \(\varphi_p(n)\) and \(\bar{\varphi}_p(n)\) again switch roles but the form of the operator \(C_p(n)\) remains unchanged.

3. Finally, let the site \(n\) be the end point of both links \(l_1\) and \(l_2\) (and, although it makes no difference, let the orientation of \(\partial_p\) agree with that of \(l_2\)); see Figure 5(iii). Then from (37) we have

\[
D_p(n) = \int d\varphi_p(n)d\bar{\varphi}_p(n) e^{-m_p|\varphi_p(n)|^2} |0\rangle \langle 0| e^{\varphi_p(n)b_p(l_1) + \bar{\varphi}_p(n)b_p(l_2)} 
\]

\[
= \sum_{k=0}^{\infty} \frac{m_p^{-k} |0\rangle \langle 0| (b_p(l_1)b_p(l_2))^k}{k!}.
\]

The operator \(D_p(n)\) now annihilates pairs of bosons (again in normalized states, and weighted by powers of the inverse mass; and if the orientation of \(p\) is reversed, the final result does not change).
5.6 Summary: dual theory

In summary, doing the integral over the boson fields $\varphi, \bar{\varphi}$ in the Fock space formalism produces operators that either transfer bosons ($T$) from one link to a neighboring one, or create/destroy boson pairs ($C/D$) on adjacent links. In all these processes, the plaquette label is conserved, and so are the color and flavor quantum numbers. (In the latter two processes, pairs of bosons are always created/annihilated in identical color and flavor states.) We are going to need names for the sets of transfer, creation and destruction sites on $\partial p$; let these be $t_p, c_p,$ and $d_p$, respectively.

A simplification now comes from the fact that taking the trace picks out the boson number conserving processes. Using this to extract a common boson mass weight factor, we conclude that the partition function of the theory in the dual representation is

$$Z = \text{Tr}_{V_0} \left( \prod_{\pm p} m_p^{-\hat{N}_p} \prod_{a=1}^{N_c} \prod_{i=1}^{N_c} A^{i,a}_p \right), \quad (40)$$

where $\hat{N}_p$ is the operator counting the total number of bosons associated with $p$, and the operators $A^{i,a}_p$ have the following structure:

$$A^{i,a}_p = \prod_{n \in c_p} C^{i,a}_p(n) \prod_{n \in t_p} T^{i,a}_p(n) \prod_{n \in d_p} D^{i,a}_p(n).$$

The left and right operators create resp. destroy pairs of bosons in identical states, while the middle ones simply transfer bosons. (The operator $A^{i,a}_p$ is illustrated for the case of a triangular plaquette in Figure 6.) The trace runs over the linear space of states $V_0$ spanned by all polynomials

$$E^{a_1 b_1}_{p_1 q_1}(l_1) E^{a_2 b_2}_{p_2 q_2}(l_2) \cdots E^{a_n b_n}_{p_n q_n}(l_n)|0\rangle$$

where $E^{ab}_{pq}(l) = \sum_{i=1}^{N_c} b_i^a (l) b_i^b (l),$ with $p_j \in \Pi^+(l_j)$ and $q_j \in \Pi^-(l_j)$.

While the evaluation of this partition function remains a challenging task in general, there are two conclusions we can draw immediately. To arrive at the first one, we note that one may employ any complete set of linearly independent states to compute the trace (40). If the projection on the $U(N_c)$ invariant sector $V_0$ were absent, we could use a basis of states labeled by occupation numbers $\{n^{i,a}_p(l)\}$, but since the projection acts on the color degrees of freedom, this is not possible unless $N_c = 1$. Nevertheless, the occupation numbers summed over colors,

$$\sum_{i=1}^{N_c} n^{i,a}_p(l) = n^a_p(l),$$

are still good quantum numbers. Now reconsider the triangular plaquette with corners $n_1, n_2, n_3$ in Figure 6. Because the operator $T^{i,a}_p(n_2)$ simply transfers bosons without changing any of their quantum numbers, the contribution of a state in $V_0$ to the trace (40) vanishes unless

$$n^a_p(l_{n_2,n_3}) = n^a_p(l_{n_1,n_2}).$$
Similarly, because $C_{i,a}^j(n_1)$ and $D_{i,a}^j(n_3)$ create and destroy pairs of bosons in identical single–boson states, the contribution of a state to (40) vanishes unless

$$n_{1,a}^i(l_{n_1,n_2}) = n_{1,a}^i(l_{n_1,n_3}); \quad n_{3,a}^i(l_{n_1,n_3}) = n_{3,a}^i(l_{n_2,n_3}).$$

Altogether, this leads to the conclusion that the occupation numbers (summed over colors) must be the same for each of the links in the boundary chain of $p$:

$$\sum_{i=1}^{N_c} n_{1,a}^i(l) \equiv n_{1,a}^i \quad (\text{independent of } l). \quad (41)$$

By identical reasoning, this conclusion holds true not just for the triangular plaquette $p$ in Figure 6, but for any plaquette on $\Lambda$.

The second conclusion we can draw immediately results from the fact that all states in $V_0$ are created by the $U(N_c)$ invariant pair operators $E_{pq}^{ab}(l)$, where $p \in \Pi^+(l)$ and $q \in \Pi^-(l)$. If we write $\tilde{n}_p \equiv \sum_{a=1}^{N_{bf}} n_{p,a}^a$ for the occupation numbers summed over flavors (as well as colors), this implies

$$\sum_{p \in \Pi^+(l)} \tilde{n}_p - \sum_{q \in \Pi^-(l)} \tilde{n}_q = 0 \quad (42)$$

for every link $l$. This equation is easy to interpret: it is equivalent to the constraint on the field strength of the $\mathbb{Z}$–valued gauge field dual to the $U(1)$ gauge field in $U(N_c) = SU(N_c) \times U(1)$. We will come back to this in Section 5.8.

### 5.7 Wegner–Wilson loop

Up to now we have concentrated on the partition function of the induced gauge model. To access its full physics, gauge–invariant correlation functions such as the Wegner–Wilson loop must be computed.

A gauge–invariant correlation function that contains the Wegner–Wilson loop and neatly fits into our formalism is constructed as follows. Let $C$ be any closed oriented contour of total
Doing the integral over $\bar{\psi}_C$, coherent states work formally the same way for fermions and bosons; we can still express the one–link integral (34) as the ordered product

$$U(C) = U(l_L)U(l_{L-1}) \cdots U(l_2)U(l_1),$$

and consider the expectation value $\langle \ldots \rangle$ of the gauge–invariant function $\text{Det}(1 - \alpha_C U(C))$ with respect to the statistical measure given by the partition function of the lattice gauge theory. This generates the Wegner–Wilson loop $W(C)$ in the fundamental vector representation:

$$\langle \text{Det}(1 - \alpha_C U(C)) \rangle = 1 - \alpha_C W(C) + \mathcal{O}(\alpha_C^2), \quad W(C) = \langle \text{Tr} U(C) \rangle.$$

We now set up the calculation of this correlation function, as follows. Let $l_k = n_{k+1} - n_k$, i.e. $n_{k+1}$ is the end point and $n_k$ the starting point of the link $l_k$ in the one–chain of $C$. Introducing complex fermion fields $\psi_C, \bar{\psi}_C$ associated with the sites visited by the contour $C$, we modify the primordial action $S^f_C$ by adding

$$S_{C;\alpha_C} = \sum_{k=1}^{L} \left( \bar{\psi}_C(n_k)\psi_C(n_k) - \alpha_C \frac{1}{L} \bar{\psi}_C(n_{k+1})U(l_k)\psi_C(n_k) \right).$$

By the standard rules of fermionic integration we then have

$$\langle \text{Det}(1 - \alpha_C U(C)) \rangle = \int \frac{e^{-S_b - S_{C;\alpha_C}}}{e^{-S_b}}.$$

To pass to the dual description, we proceed in much the same manner as before. There are a few minor changes in order to take into account the fermions, but these do not affect the general strategy:

The one–link integral is modified for all links $l$ in the contour $C$ by the additional presence of the fermion fields. To carry it out, we again “quantize”, i.e. we introduce colorful fermion operators $f^i_C(l)$ and $f^{i\dagger}_C(l)$ which satisfy the canonical anticommutation relations and act on Fock space with vacuum $|0\rangle$. The $U(N_c)$ transformation behavior is still determined by relative orientation, now between $l$ and $C$:

$$T_{U(l)} f^i_C(l) T_{U(l)}^{\dagger} = \sum_j f^j_C(l) U^{ji}(l), \quad \text{if } C = +1 + \ldots,$$

$$T_{U(l)} f^i_C(l) T_{U(l)}^{\dagger} = \sum_j f^j_C(l) U^{ji}(l), \quad \text{if } C = -1 + \ldots.$$

Coherent states work formally the same way for fermions and bosons; we can still express $e^{\bar{\psi}U\psi}$ as a matrix element of $T_U$ between coherent states $e^{f^{i\dagger} \psi}|0\rangle$.

Doing the integral over $U(l)$ with Haar measure, we still get a Fock space projection operator $P_0(l)$, although the $U(N_c)$ invariant subspace it projects onto, is enlarged for every link $l$ in $C$: there exist additional states, created by boson–fermion pair operators

$$F_{C,q}^a(l) = \sum_{i=1}^{N_c} f^i_C(l)b^a_q(l) \quad (q \in \Pi^{-}(l)) \quad \text{if } C = +1 + \ldots, \quad \text{or} \quad (43)$$

$$F_{p,C}^a(l) = \sum_{i=1}^{N_c} b_{p}^a(l)f^{i\dagger}_C(l) \quad (p \in \Pi^{+}(l)) \quad \text{if } C = -1 + \ldots, \quad (44)$$

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and by invariant pairs involving two fermions. The latter, however, will play no role ultimately, as we only want $W(C)$ in the vector representation (corresponding to a single–quark source), so we refrain from writing them down. The total set of $U(N_c)$ invariant pair operators together with all their (super–)commutators form a certain Lie superalgebra (of $\mathfrak{gl}$ type), $\mathfrak{g}(l)$.

We denote the enlarged $U(N_c)$ invariant subspace associated with a link $l$ on $C$ by $\mathcal{V}_0(l)$. Howe’s theorem [12] on the multiplicity–free action of dual pairs in Fock space [the dual pair now being $U(N_c)$ with $\mathfrak{g}(l)$] still holds in the present generalized setting, so $\mathcal{V}_0'(l)$ is still obtained by acting with all $U(N_c)$ invariant pair operators on the vacuum $|0\rangle$.

Integration over the fermion sources $\psi, \bar{\psi}$ works the same way as for bosons (Section 5.5). The final result of doing the integration (for fixed color quantum number $i$) is an operator which is denoted by $A_C^T$, and is defined as follows. $A_C^T$ is a product over all sites $n$ visited by the contour $C$. In the same way as was shown in detail for the bosons in Section 5.5 the definition of each factor $A_C^T(n)$ depends on the a priori orientation of the links in $C$ that begin or end on $n$. For sites $n$ of type (i) (Figure 5), $A_C^T$ transfers fermions; for sites of type (ii) it creates pairs of fermions; and for sites of type (iii) it annihilates pairs of fermions.

All this discussion is summarized in the following final formula:

$$\langle \text{Det}(1 - \alpha_C U(C)) \rangle = Z^{-1} \text{Tr}_{\mathcal{V}_{0,C}} \left( \alpha_C^{\hat{N}_i/l} \prod_{i=1}^{N_c} A_C^T \prod_{\pm p} \alpha_p^{\hat{N}_p/L_p} \prod_{a=1}^{N_b} A^{c,a}_p \right), \quad (45)$$

where $\hat{N}_i$ counts the total number of fermions, and $\mathcal{V}_{0,C}$ is the tensor product of the spaces $\mathcal{V}_{0}(l)$ (resp. $\mathcal{V}_{0}(l)$) for $l$ contained (resp. not contained) in $C$. The normalization $Z^{-1}$ is given by [40].

### 5.8 Abelian limit: $G = U(1)$

In this subsection we briefly comment on the special case $N_c = 1$, adopting the minimal model $N_b = N_c = 1$. In that model, both the color and flavor degrees of freedom are absent and drastic simplifications occur.

We begin with the expression for the partition function [40] as a trace over $\mathcal{V}_0$. The evaluation of the trace is simplified by the observation that the states of $\mathcal{V}_0$ for $N_c = 1$ are in one–to–one correspondence with sets of boson occupation numbers. Thus for $N_b = 1$ they are labeled by $n_p \in \mathbb{N} \cup \{0\}$. By Eq. (44), these occupation numbers are $l$–independent: $n_p(l) \equiv \tilde{n}_p$. (We put the tilde here to reserve the name $n_p$ for another set of integer variables to be introduced presently.) They also satisfy the constraint (12).

The best way to deal with the constraint is to switch variables. Recall that in the full set of occupation numbers, $\{\tilde{n}_p\}$, every 2–cell $p$ occurs twice: once with its proper orientation ($+p$), and once with its orientation reversed ($-p$). We now switch variables from $\tilde{n}_{\pm p} \in \mathbb{N} \cup \{0\}$ to

$$n_p = \tilde{n}_p - \tilde{n}_{-p} \in \mathbb{Z}, \quad \text{and} \quad l_p = \tilde{n}_p + \tilde{n}_{-p} = |n_p|, |n_p| + 2, |n_p| + 4, \ldots .$$

If we view $p \mapsto n_p$ as a 2–chain $n$ on the $d$–dimensional complex $\Lambda$, the constraint (12) simply says that $n$ is closed: $\partial n = 0$. Indeed, the question whether $p$ belongs to $\Pi^+(l)$ or $\Pi^-(l)$, and hence what sign is attributed to $\tilde{n}_p$ in (12), is decided by comparing orientations. This sign information is concisely encoded in the boundary operator $\partial : 2$–chains $\rightarrow 1$–chains.
The variables \( l_p \), on the other hand, decouple from the problem; the sum over each of them is a geometric series:

\[
\sum_{l_p \in |n_p| + 2(\mathbb{N} \cup \{0\})} \alpha_p^{-2} \alpha_p^{\sum n_p} = (1 - \alpha_p^2)^{-1} \alpha_p^{\sum n_p},
\]

where \( \alpha_p = 1/m_p \) as before. The power \( L_p \) appears because \( \tilde{N}_p \) counts the total number of bosons associated with \( p \) and the \( L_p \) links about \( p \) all have the same boson occupation number [Eq. (41)]. Thus the partition function (40) takes the final form

\[
Z = \prod_p (1 - \alpha_p^2)^{-1} \sum_{n: \partial n = 0} \prod_p \alpha_p^{n_p},
\]

which coincides with the expression that resulted from the standard abelian duality transform; see Section 5.1.

Turning to \( W(C) \), we observe that the same argument that gave constant boson occupation numbers on plaquettes, leads to constant fermion occupation numbers along the contour \( C \). To extract \( W(C) \) from the generating function (45), we must place exactly one fermion on each link \( l \) in \( C \). States in the \( U(1) \) invariant subspace \( \mathcal{V}_0(l) \) are created by the invariant pair operators (43,44), which implies that the creation of a fermion along \( C \) is always accompanied by the creation of exactly one boson. Therefore, along \( C \) the relation (42) is modified to

\[
\epsilon + \sum_{p \in \Pi^+(l)} \tilde{n}_p - \sum_{q \in \Pi^-(l)} \tilde{n}_q = 0,
\]

where \( \epsilon = +1 \) if the orientation of \( C \) agrees with that of \( l \), and \( \epsilon = -1 \) otherwise. If we switch from the variables \( \tilde{n}_p, \tilde{n}_q \) to \( n_p, l_p \), this constraint becomes \( \partial n = -C \). Solving the constraint by setting \( n = \partial a - S \) with \( \partial S = C \), we obtain \( W(C) \) in the form (33).

### 6 Discussion

The distinctive feature of the class of lattice models for \( U(N_c) \) gluodynamics introduced here, is that they are induced from a pre–theory with \( N_b/f \) species of local bosons and/or fermions. The statistical measure of these lattice gauge models is a product over elementary plaquettes, with each factor being a ratio of determinants.

The boson induced models have a critical point at unit mass, which allows a continuum limit to be taken. Our careful study in \( d = 1 + 1 \) dimensions showed that, if \( N_b \geq N_c + 1 \), this continuum limit is \( U(N_c) \) Yang–Mills theory, with a specific ratio of the \( U(1) \) and \( SU(N_c) \) couplings. The ratio goes to unity for \( N_b \to \infty \). In contrast, the continuum limit for \( N_b = N_c \) is not Yang–Mills but an unusual theory, which exists as a (super–)renormalizable quantum theory because the Cauchy distribution on \( u(N_c) \) converges under convolution to a stable family of distributions distinct from the heat kernel family.

Going up in dimension enhances the collectivity of the gauge field (by increasing the number of transverse gluons) and thus works in favor of “universality”. Therefore, whenever our model induces \( U(N_c) \) Yang–Mills theory in two dimensions, we expect it to do so in higher dimensions, at least generically.

Although we concentrated on the special case of the gauge group being \( U(N_c) \), a very similar treatment is possible for all classical compact Lie groups. Proposition (12), which is the
mathematical basis for the existence of a continuum limit, carries over to the normalized distributions \( \text{Det}(1-\alpha U)^{-N_b}dU \) on \( \text{Sp}(2N_c) \) and \( \text{SO}(2N_c) \), with thresholds \( N_b \geq 2N_c + 1 \) and \( N_b \geq 2N_c - 1 \), respectively.

In the second part of the paper we subjected the boson induced gauge model to a duality transformation (a variant of the color–flavor transformation), which demonstrably reduces to standard abelian duality for \( G = U(1) \). The partition function of the theory in the dual formulation is the trace of a color– and flavor–diagonal operator \( \prod A_p^{l,a} \) acting on a tensor product of modules \( V_0(I) \) generated by quadratic \( U(N_c) \) invariants on lattice links \( I \). Again, the dual formulation is not restricted to \( U(N_c) \) but exists for other classical compact gauge groups as well. (This is because all of the compact Lie groups \( U(N_c) \), \( \text{Sp}(2N_c) \) and \( \text{O}(N_c) \) are placed in Howe duality with corresponding families of noncompact Lie groups. For \( SU(N_c) \) and \( \text{SO}(N_c) \) the duality transform is more complicated \[13\] due to the existence of nonquadratic invariants of “baryon” type.)

We do not know at present how far one can push the analysis of the dual theory. We believe it to be possible to develop a combinatorial approach to handle the operator \( \prod A_p^{l,a} \), but the details have not been worked out yet. A more refined understanding of the asymptotics of the modules \( V_0(I) \) at large boson number is also needed. Although they are generated by quadratic invariants, they are not freely generated in the range \( N_b > N_c \), as is evident from the result \( c_0(\alpha) \sim (1-\alpha)^{-2N_bN_c+N_c^2} \) [see \[21\]] for the \( U(N_b,N_b) \) character \( c_0(\alpha) = \text{Tr}_{V_0} \alpha^N \).

We do not understand at present exactly how the Howe duality transform connects with other recent proposals, such as the quantum gravity formulation of \( SU(2)_{d=3} \) Yang–Mills theory in \[13\]. Also, to bring our induced gauge models closer to continuum gauge theories of current interest, one may ask whether they can be extended to models with robust supersymmetry on the lattice \[31\]. Another natural question to ask is whether one can construct a coherent–state resolution of the identity operator on the modules \( V_0 \), and by adding an extra time direction to four–dimensional space–time pass to a five–dimensional continuum theory (of topological gravity?) via the coherent–state path integral method.

Let us finish by commenting on just the issue of existence of a coherent–state resolution. Consider the Howe pair \( U(N_c) \times U(N_b,N_b) \) acting in a Fock space \( \mathcal{V} \) with \( 2N_bN_c \) species of bosons, as described in Section \[156\]. We are interested in the \( U(N_c) \) invariant sector \( V_0 \). In particular, we would like a coherent–state integral representation for the invariant Hermitian scalar product on \( V_0 \). For \( N_c \geq 2N_b \) this is a standard problem with the following standard solution. Let \( M \) be the Hermitian symmetric space formed by dividing \( U(N_b,N_b) \) by its maximal compact subgroup. \( M \) is modeled by complex \( N_b \times N_b \) matrices \( Z \) with noncompact domain \( Z^\dagger Z < 1 \); and \( g \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(N_b,N_b) \) acts on the points \( Z \) of \( M \) by

\[
g \cdot Z = (DZ + C)(BZ + A)^{-1}.
\]

The coherent–state expression for any \( \varphi \in V_0 \) is a holomorphic section \( \varphi(Z) \), transforming under \( g \in U(N_b,N_b) \) as

\[
(D^{N_c}(g)\varphi)(Z) = \text{Det}^{-N_c}(D - ZB) \varphi(g^{-1} \cdot Z).
\]

The invariant Hermitian scalar product of two sections \( \varphi_1 \) and \( \varphi_2 \) is

\[
\langle \varphi_1 | \varphi_2 \rangle = \int_{ZZ^\dagger < 1} \overline{\varphi_1(Z)}\varphi_2(Z) \text{Det}^{N_c-2N_b}(1 - Z^\dagger Z) dZd\bar{Z},
\]

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where \(dZd\bar{Z}\) is a flat density (suitably normalized), and invariance means
\[
\langle \varphi_1 | \varphi_2 \rangle = \langle D^{N_c}(g) \varphi_1 | D^{N_c}(g) \varphi_2 \rangle \quad \text{for} \quad g \in U(N_b, N_b) .
\]
All this makes perfect sense as long as \(N_c\) is big enough (or \(N_b\) small enough). However, when \(N_c\) is decreased below \(2N_b\), the density \(\text{Det}^{-N_c-2N_b}(1 - Z^1\bar{Z}) \) becomes singular at the boundary of the symmetric domain \(Z^1\) \(\leq 1\), and the coherent–state expression for the Hermitian scalar product ceases to exist in the form given. This does not mean that it ceases to exist altogether. Indeed, for the case \(N_b = N_c\) we can easily see how to fix the problem. The singularity at the boundary indicates that the proper measure to use is concentrated in the boundary. Now, the boundary of the \(Z\)–model for \(M\) always contains the unitary group \(U(N_b)\) (the set of solutions of \(Z^1\) \(\leq 1\)) as a \(U(N_b, N_b)\) orbit, i.e. if \(U\) is in \(U(N_b)\), then so is its image \(g^{-1}\cdot U = (D - UB)^{-1}(UA - C)\). A straightforward computation shows that the Haar measure \(dU\) on \(U(N_b)\) transforms under the action of \(g^{-1}\) as \(g^{-1\ast}(dU) = |\text{Det}(D - UB)|^{-2N_b}dU\). For the case \(N_c = N_b \equiv N\) (and only in that case) the factor multiplying \(dU\) is canceled by the multiplier in the transformation law for sections \(\varphi(Z)\), so that the boundary integral
\[
\langle \varphi_1 | \varphi_2 \rangle = \int_{U(N)} \bar{\varphi}_1(U) \varphi_2(U) \ dU
\]
is invariant, and (by uniqueness) is the coherent–state integral representation of the Hermitian scalar product on \(\mathcal{V}_0\). This implies that, while the sections \(\varphi(z)\) are sections on the \(2N^2\)–dimensional symmetric domain \(M\), the complete information about them in the special case \(N_b = N_c \equiv N\) at hand is already encoded in the values they take on approaching the \(N^2\)–dimensional part \(U(N) \subset \partial M\) of the boundary \(\partial M\).

We do not understand the details of the analogous construction of an invariant boundary measure for \(N_b \neq N_c < 2N_b\), although we know on general grounds that such a construction must exist. Having a detailed understanding of that construction would enable us to extend the color–flavor transformation to the whole domain \(N_c < 2N_b\).

\section{Appendix: Calculations for the One–Plaquette Model}

\subsection{Fermion induced model}

We compute the expectation value of the Wilson loop, \(W(\alpha\ell, 0)\), for the fermion induced model on a lattice consisting of a single plaquette (Section 4.1).

Doing the same steps as in the proof of statement (12), Section 3.3, we obtain
\[
W(\alpha\ell, 0) = N_c (e^{i\theta_1})_{\alpha\ell} / \langle 1 | \alpha\ell \rangle ,
\]
where
\[
\langle F \rangle_{\alpha} = \int_{[0, 2\pi]^{N_c}} F(e^{i\vartheta_1}, \ldots) \prod_{j=1}^{N_c} |1 - \alpha e^{i\vartheta_j}|^{2N\ell} \prod_{k<l} |e^{i\vartheta_k} - e^{i\vartheta_l}|^2 \ d\vartheta_1 \cdots d\vartheta_{N_c} .
\]

Putting \(z_k = e^{i\vartheta_k}\), we rewrite this as a multiple complex contour integral over the unit circle \(U(1) \subset \mathbb{C}\):
\[
\langle F \rangle_{\alpha} = (-i)^{N^2} \oint_{U(1)^{N_c}} F(z_1, \ldots) \prod_{j=1}^{N_c} (z_j - \alpha)^{N\ell} (1 - \alpha z_j)^{N\ell} \prod_{k<l} (z_k - z_l)^2 dz_1 \cdots dz_{N_c} . \tag{46}
\]
The integrand is holomorphic on $\mathbb{C}\setminus\{0\}$ for each of the variables $z_k$, which suggests evaluating the integral by contracting all integration contours to zero. There is a pole of order $N_c + N_f$ at zero in each variable of the normalization integral ($F \equiv 1$). In the numerator $\langle z_1 \rangle_\alpha$ of the expectation value, the order of the pole in the distinguished variable $z_1$ is reduced by one. Thus, by contracting the contours and evaluating the residues at zero, we obtain

$$W(\alpha f, 0) = N_c(N_c + N_f - 1) \times \frac{\partial^{N_c + N_f - 2} \partial^{N_c + N_f - 1} \ldots \partial^{N_c + N_f - 1}}{\partial z_{N_c} \partial z_{N_c} \ldots \partial z_{N_c}} f(z_1, \ldots, z_{N_c}; \alpha_f) \bigg|_{z_1 = \ldots = z_{N_c} = 0},$$

where $f$ is the function

$$f(z_1, \ldots, z_{N_c}; \alpha) = \prod_{j=1}^{N_c} (z_j - \alpha)^{N_f}(z_j - \alpha^{-1})^{N_f} \prod_{k<l} (z_k - z_l)^2.$$

For small values of $N_c$ and $N_f$ the derivatives are easy to evaluate using computer algebra. The results obtained in this way for $N_c = 3$ and $N_f = 1, \ldots, 6$ were shown in Figure 3.

**A.2 Boson induced model**

Passing from the fermion induced model to its bosonic analog amounts to doing an analytic continuation, from positive integers $N_f$ to negative integers $-N_b$. Given Eq. (16) of the previous subsection, this continuation yields $W(0, \alpha_b) = N_c \langle z_1 \rangle_{\alpha_b}/\langle 1 \rangle_{\alpha_b}$ with

$$\langle F_\alpha \rangle = (-i)^{N_c^2} \oint_{U(1)^{N_c}} F(z_1, \ldots, z_{N_c}) \prod_{j=1}^{N_c} \prod_{k<l} (z_k - z_l)^2 \prod_{j=1}^{N_c} \prod_{j=1}^{N_c} (z_j - \alpha)^{N_f}(z_j - \alpha^{-1})^{N_f} \prod_{k<l} (z_k - z_l)^2 \prod_{j=1}^{N_c} (1 - \alpha z_j)^{N_b} \prod_{j=1}^{N_c} (1 - \alpha^{-1} z_j)^{N_b} \prod_{k<l} (z_k - z_l)^2 \prod_{j=1}^{N_c} dz_1 \ldots dz_{N_c}. \quad (47)$$

The strategy again is to contract all integration contours to zero. As compared with the fermionic case, we now encounter $N_b$-fold poles at $z_j = \alpha_b$ in addition to the poles at $z_j = 0$ that occur when $N_b$ is less than $N_c$. We separately consider the three cases $N_b > N_c$, $N_b = N_c$, and $N_b < N_c$.

**$N_b > N_c$**: In this case the integrand has an $N_b$-fold pole at $z_j = \alpha_b$ for $j = 1, \ldots, N_c$ and no poles at $z_j = 0$. Application of the residue theorem yields

$$W(0, \alpha_b) = N_c \frac{\partial^{N_b - 1} \ldots \partial^{N_b - 1} z_j f(z_1, \ldots, z_{N_c}; \alpha_b)}{\partial z_{N_c} \partial z_{N_c} \ldots \partial z_{N_c}} \bigg|_{z_1 = \ldots = z_{N_c} = \alpha_b},$$

with $f$ being the function

$$f(z_1, \ldots, z_{N_c}; \alpha) = \prod_{j=1}^{N_c} \frac{z_j^{N_b - N_c}}{(z_j - \alpha^{-1})^{N_f}} \prod_{k<l} (z_k - z_l)^2.$$

Again, for small values of $N_c$ and $N_b$ we have employed computer algebra to evaluate the derivatives. The results for $N_c = 3$ and $N_b = 1, \ldots, 6$ were shown in Figure 4.
where the terms omitted are regular in at least one variable.

The nominal singularities is actually holomorphic in at least one of the variables unless the integrand which is actually holomorphic at $z = \alpha_b$ (and hence holomorphic everywhere in the unit disc) for at least one of the integration variables. A true singularity in every one of the variables occurs only for the terms with equal exponents,

$$\pm (z_1 - \alpha_b)^{N_c-1} \cdots (z_{N_c} - \alpha_b)^{N_c-1},$$

which arise from the permutations with $\pi(j) + \pi'(j) = N_c + 1$ (for all $j$). There exist $N_c!$ of such terms in the double sum, and these are the only ones that give a nonzero contribution to the integral. Looking at $\prod_{k<l}(z_k - z_l)^2$ we see that the resulting poles are simple in each variable, so

$$\langle 1 \rangle_{\alpha_b} = (2\pi)^{N_c} N_c! (1 - \alpha_b^2)^{-N_b^2},$$

and

$$W(0, \alpha_b) = N_c \langle z_1 \rangle_{\alpha_b} / \langle 1 \rangle_{\alpha_b} = N_c \alpha_b,$$

which is the exact result quoted in the main text.

$N_b < N_c$: In this case there exist poles inside the unit circle both at $z_j = \alpha_b$ and at $z_j = 0$. Hence we must investigate the analytic structure of the integrand when $q$ variables are sent to $z = \alpha_b$ and $N_c - q$ variables to $z = 0$. The poles at these locations compete with the zeroes from the numerator $\prod_{k<l}(z_k - z_l)^2$. By power counting one easily sees that the integrand at the nominal singularities is actually holomorphic in at least one of the variables unless $q = N_b$. In other words, nonzero contributions to the integral come only from sending $N_b$ variables to $z = \alpha_b$, and the remaining $N_c - N_b$ variables to $z = 0$. Adapting the Vandermonde determinant expansion of $\prod_{k<l}(z_k - z_l)^2$ to such a location, say $z_1 = \ldots = z_{N_b} = \alpha_b$ and $z_{N_b+1} = \ldots = z_{N_c} = 0$, we see that the corresponding Laurent series starts as follows:

$$\prod_{j=1}^{N_b}(z_j - \alpha_b)^N_b z_j^{N_c-N_b} = N_b! (N_c - N_b)! \prod_{j=1}^{N_b}(z_j - \alpha_b)^{-1} \prod_{j=N_b+1}^{N_c} z_j^{-1} + \ldots,$$

where the terms omitted are regular in at least one variable.

Since there exist \( \binom{N_c}{N_b} \) ways to divide $N_c$ objects into two sets of $N_b$ resp. $N_c - N_b$ objects, and \( \binom{N_c}{N_b} \times N_b! (N_c - N_b)! = N_c! \), the normalization integral is

$$\langle 1 \rangle_{\alpha_b} = (2\pi)^{N_c} N_c! (1 - \alpha_b^2)^{-N_b^2}.$$
In the numerator $\langle z_1 \rangle_{\alpha b}$, a nonzero contribution results only when the distinguished variable $z_1$ is sent to $z = \alpha_b$ (as opposed to $z = 0$). This happens in a fraction $N_b/N_c$ of all cases, and so the Wilson loop expectation value is reduced by this very factor:

$$W(0, \alpha_b) = N_c \frac{\langle z_1 \rangle_{\alpha_b}}{\langle 1 \rangle_{\alpha_b}} = N_c \frac{N_b}{N_c} \alpha_b = N_b \alpha_b .$$

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