The bosonic minimum output entropy conjecture and Lagrangian minimization

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We introduce a new form for the bosonic channel minimal output entropy conjecture, namely that among states with equal input entropy, the thermal states are the ones that have slightest increase in entropy when sent through a infinitesimal thermalizing channel. We then detail a strategy to prove the conjecture through variational techniques. This would lead to the calculation of the classical capacity of a communication channel subject to thermal noise. Our strategy detects input thermal ensembles as possible solutions for the optimal encoding of the channel, lending support to the conjecture. However, it does not seem to be able to exclude the possibility that other input ensembles can attain the channel capacity.

Bosonic communication channels employ bosons such as photons or phonons to carry information\textsuperscript{1–13}. Establishing the fundamental communications capacity of such channels is a primary outstanding scientific question\textsuperscript{14}. It is also a question of considerable social relevance: optical communication channels are pressing down to the level where significant amounts of information are carried by individual photons. It would be useful to know whether we are nearing the limit of communication capacity, or whether quantum effects such as squeezing and entanglement might be used to enhance communication capacity significantly.

Ever since Shannon established the capacity of the continuous classical communication channel with Gaussian noise and loss more than half a century ago\textsuperscript{15}, researchers have sought to establish the capacity of the corresponding quantum channel, the bosonic Gaussian channel that transmits information using bosons such as photons or phonons\textsuperscript{1–13}. The classical capacity\textsuperscript{16} of the bosonic channel with thermal noise and linear loss hinges on the bosonic minimum output entropy conjecture, which states that the vacuum input gives the minimum output von Neumann entropy for a channel with thermal noise\textsuperscript{2–9}. If this conjecture is true, then the channel attains its capacity for coherent state inputs\textsuperscript{1–3, 6–10}. Here we present an infinitesimal version of this conjecture and detail a variational strategy for its proof, using a double Lagrangian constrained minimization.

Since the input that gives the minimum output entropy is a pure state that lies on the boundary of the input space, variational techniques such a Lagrangian methods might at first seem difficult to apply. We circumvent this problem by looking for the minimum output entropy for states with a fixed, non-zero input entropy, and then take the limit in which that input entropy to zero. We find that thermal states are solutions to the Lagrange equations and represent local minima to the output entropy. Unfortunately, unless the gradients of the minimization constraints are linearly independent, one cannot conclude that the minima recovered through the Lagrangian technique are unique\textsuperscript{17}. Thus, even though the results obtained below are consistent with the conjecture (namely input coherent states are identified as minima for the channel output entropy), we cannot conclude that they are the global minima. Namely, we cannot exclude that other input states can give even lower output entropy, thus becoming the constituent of an encoding that achieves the channel capacity. Nonetheless, the approach detailed here constitutes a progress in answering the oldest question of quantum information theory, by casting the problem in a framework that is completely different than what previously analyzed.

\section{I. Conjectures}

Consider a bosonic channel with additive Gaussian thermal noise, e.g. the additive classical noise channel, the thermal-noise channel, or the amplifier channel of Ref.\textsuperscript{2}. Establishing its ultimate classical capacity hinges on the following conjecture\textsuperscript{7–10},

\begin{itemize}
  \item \textit{Conjecture (finite)}: The vacuum input state for the channel gives the minimum output entropy.
\end{itemize}

That is, to get the smallest output entropy, do nothing. If this conjecture is false, then quantum ‘tricks’ could enhance channel capacity. By contrast, if this conjecture is indeed true, then references\textsuperscript{7, 8} show that Gaussian channels with noise and linear loss attain their maximum capacity for coherent state inputs. While highly plausible, the bosonic minimum output entropy conjecture has resisted proof for some time now. Exploiting the fact that the above maps have a semigroup structure and thus admits a Lindblad generator (e.g. see Ref.\textsuperscript{18}), we now extend this conjecture also to the infinitesimal case. Namely we investigate the infinitesimal version of the
additive Gaussian thermal noise channels, i.e.

- **Conjecture (infinitesimal):** Among input states with a fixed entropy, thermal states give the minimum rate of output entropy increase for the infinitesimal additive noise channel.

A proof of the infinitesimal version of the conjecture would establish the truth also of the finite version. In fact, the rate of entropy increase under thermalization is non-negative. If the infinitesimal conjecture were proved, one would conclude that for pure state inputs the vacuum is the zero-temperature thermal state and gives the minimum rate of increase in entropy under thermalization. Then, thermalization would take the vacuum through a sequence of a non-zero temperature thermal states of increasing entropy, each of which would yield the minimum rate of entropy increase for its particular entropy. The path that starts at the vacuum and passes through a sequence of thermal states therefore would give the minimum entropy increase also for non-infinitesimal thermalization, proving also the finite version of the conjecture.

Here we detail a possible strategy for a proof using a Lagrangian technique, enforcing entropy, energy, and dynamical constraints using Lagrange multipliers [17, 19]. Defining a suitable Lagrangian $\Sigma = 0$ and setting its variation $\delta \Sigma = 0$ allows us to identify the potential minima of the output entropy $S$. Symmetries of thermal noise imply that displaced thermal states will also give the minimum thermalization of a bosonic mode according to the Lindblad equation,

$$\frac{d\rho}{dt} = -\frac{\gamma}{2} (2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a + 2a^\dagger pa - aa^\dagger \rho - paa^\dagger)$$

with $\gamma > 0$ governing the rate at which photons are added and subtracted from the mode: over a brief time $\Delta t \ll 1/\gamma$, the average increase in energy for zero-mean input states is $\Delta E = \gamma \Delta t$, independent of $\rho$ as can be easily verified from (2), i.e.

$$\text{tr} [\text{Th}(\rho)a^\dagger a] = \text{tr} [\rho a^\dagger a] + \Delta E + O(\gamma^2 \Delta t^2).$$

Consider now an input state $\rho$ with von Neumann entropy $S(\rho) = -\text{tr} \ln \rho = S_0$. Its entropy increase over time $\Delta t$ can be computed as

$$\Delta S = -\text{tr} \text{Th}(\rho) \ln \text{Th}(\rho) + \text{tr} \rho \ln \rho$$

$$= -\gamma \Delta t \text{tr} N(\rho) \ln \rho + O(\gamma^2 \Delta t^2).$$

To minimize the output entropy for fixed input entropy $S_0$, we introduce a new positive matrix variable $z$ and use the method of Lagrange multipliers to set $z = \text{Th}(\rho)$, as well as enforcing the input entropy constraint. The Lagrange multiplier technique tells us that if our Lagrangian is continuous and has continuous derivatives in the open interior of the set of density matrices, then the absolute minimum of entropy increase either occurs at a stationary point of the Lagrangian or on the boundary of the set. (In infinite-dimensional Hilbert spaces, the gradient of the entropy is everywhere discontinuous. Accordingly, to ensure that our Lagrangian is continuous and has continuous derivatives, we truncate our input Hilbert space at some high photon number $N$, find the minimum output entropy for finite $N$, and take the limit $N \to \infty$. Technical details of this truncation can be found in the Appendix.) We will see that inputs on the boundary maximize entropy increase. Accordingly, the global minimum must be a stationary point of the Lagrangian.

Because the thermalization process of Eq. (3) is covariant under displacements in the phase space, each local minimum is connected to a manifold of local minima with the same output entropy by continuous displacements. Within the set of states with this output entropy,
we pick out the output state with the lowest energy. The only minimum energy stationary points of that the Lagrangian procedure returns are thermal states. So that, if we could exclude other local minima, we could conclude that the thermal state gives the global minimum of entropy production.

### III. INPUTS ON THE BOUNDARY MAXIMIZE ENTROPY INCREASE

First we show that inputs on the boundary give maxima of output entropy. The boundary consists of density matrices with at least one zero eigenvalue, i.e. density matrices with a non-zero kernel, \( K(\rho) \). Let \( P_\perp \) be the projector onto the support of \( \rho \), i.e., onto the subspace orthogonal to \( K \). From the formula for entropy increase, equation (5), we see if infinitesimal thermalization takes any state from subspace perpendicular to the kernel, \( K^{\perp} \), into the kernel, then the rate of entropy increase diverges. That is, for states on the boundary, the rate of entropy increase diverges unless \( \rho a^\dagger a \perp P_\perp, \quad a^\dagger P_\perp a \subseteq P_\perp \). But in the input Hilbert space, there is no nonzero proper subspace that can fulfill both these requirements. Accordingly, states on the boundary give divergent rates of entropy increase.

### IV. STATIONARY POINTS OF THE LAGRANGIAN

Since it does not occur on the boundary, all minima of entropy increase must occur for stationary points of our Lagrangian. Define the Lagrangian

\[
\Sigma = -\text{tr} \ z \ln z + \text{tr} \rho \ln \rho - \alpha (\text{tr} \rho - 1) - \eta (\text{tr} z - 1) - \lambda (\text{tr} \rho \ln \rho - S_0) - \lambda (z - \text{Th}(\rho)). \quad (6)
\]

Here, the \( \alpha \) and \( \eta \) Lagrange multipliers enforce the normalization constraints, \( \lambda \) enforces the input entropy constraint, and the Hermitian matrix of Lagrange multipliers \( \lambda \) enforces the dynamical constraint.

Now find the stationary points of \( \Sigma \). Introduce small variations \( z \rightarrow z + \delta z, \rho \rightarrow \rho + \delta \rho \), together with similar variations in the Lagrange multipliers. We obtain

\[
\delta \Sigma = \text{tr} \delta \rho \left( - \ln z - 1 - \eta - \Lambda \right) + \text{tr} \delta z \left( - \text{tr} \ln z - 1 - \eta - \Lambda \right) + \text{tr} \delta \rho \left( \Lambda + 1 \right) (\ln \rho + 1) - \alpha + \text{Th}(\Lambda) + \delta \lambda (\text{tr} \rho - 1) - \delta \eta (\text{tr} z - 1) - \delta \lambda (\text{tr} \rho \ln \rho - S_0) - \text{tr} \delta \lambda (z - \text{Th}(\rho)). \quad (7)
\]

Here we have used the fact that \( \text{tr} [\text{Th}(\delta \rho) \Lambda] = \text{tr} [\delta \rho \text{Th}(\Lambda)] \); this is easily verified using equation (3). Stationary points of \( \Sigma \) correspond to points such that \( \delta \Sigma = 0 \), which, upon simplification, yields the conditions,

\[
- \ln z - 1 - \eta - \Lambda = 0 \\
(\lambda + 1) (\ln \rho + 1) - \alpha + \text{Th}(\Lambda) = 0 \\
\text{tr} \rho = 1, \quad \text{tr} z = 1 \\
- \text{tr} \rho \ln \rho = S_0, - \text{tr} z \ln z = S \\
z = \text{Th}(\rho). \quad (8)
\]

These equations hold for any cutoff \( N \) for the truncated Hilbert space.

Equation (8) has many solutions. It is not hard to show that, in the limit \( N \rightarrow \infty \), thermal states and displaced thermal states solve equation (8) for suitable choices of the Lagrange multipliers. On the face of it, equation (8) might also have additional solutions, unconnected with thermal states by displacements, that could give lower output entropy than thermal inputs. One strategy to rule out the possibility of such alternative manifolds of minima is to perform a second minimization (on the energy) and verify that only the minimum corresponding to thermal input survives.

Consider a manifold of connected states that give the same local minimum of output entropy. Within this manifold, we find the states that minimize energy as well (because energy is bounded below, such states always exist). We show that thermal inputs and thermal outputs minimize output entropy, and minimize energy within the set of states with that same output entropy. If one could conclude that there is only one manifold of local minima (the one connected to the input thermal state with the specified input entropy), then thermal inputs would give the unique local minimum of output entropy. Unfortunately we have not been able to prove such unicity.

### V. STATIONARY POINTS THAT ALSO MINIMIZE ENERGY

States that give local minima of the output entropy satisfy equation (8). To find states that minimize output energy within each set of connected local minima, we find stationary points of the Lagrangian

\[
\mathcal{L} = \text{tr} \ z a^\dagger a - \mu (\text{tr} z \ln z - S) - \alpha' (\text{tr} \rho - 1) - \eta' (\text{tr} z - 1) - \lambda' (\text{tr} \rho \ln \rho - S_0) - \text{tr} \lambda' (z - \text{Th}(\rho)). \quad (9)
\]

Here, \( S \) is the output entropy for that set of minima. A state within the minimum manifold that minimizes output energy as well as output entropy must extremize both \( \Sigma \) and \( \mathcal{L} \).

Varying \( \mathcal{L} \) and finding its stationary points shows that such a state must satisfy the equations

\[
\mu (\ln z + 1 + a^\dagger a - \eta' - \Lambda') = 0 \\
\lambda' (\ln \rho + 1) - \alpha' + \text{Th}(\Lambda') = 0 \\
\text{tr} \rho = 1, \quad \text{tr} z = 1 \\
- \text{tr} \rho \ln \rho = S_0, - \text{tr} z \ln z = S \\
z = \text{Th}(\rho). \quad (10)
\]

Unfortunately, we have not been able to prove such unicity.
To give a local minimum of output entropy, and within that minimum a local minimum of output energy, and z must simultaneously solve equations (8) and (11) for suitable values of the Lagrange multipliers. (We don’t have to worry about the minimum lying on the boundary of the set of density matrices because points on the boundary have divergent entropy increase: the set of inputs with minimum output entropy S lies in the interior of the set of density matrices.) One solution is obtained by setting \( \mu = 0 \). In this case \( \Lambda' = a' a - \eta' \), and equation (11) immediately implies that \( \rho \) and z are thermal states. All other solutions for \( \mu \neq 0 \) are also thermal, as will now be shown.

The first two lines of equation (8) and the first two lines of equation (11) can now be used to eliminate \( z, \Lambda, \Lambda' \) and to obtain an equation for \( \rho \):

\[
\hat{\lambda}(\ln \rho + 1) - \hat{\alpha} + \text{Th}(a' a / \mu - \eta) = 0. \tag{11}
\]

Here \( \hat{\lambda} = \lambda' / \mu + 1, \hat{\alpha} = \alpha' / \mu + \alpha, \) and \( \hat{\eta} = \eta' / \mu + \eta \).

Since \( \text{Th}(a' a) = a' a + \gamma \Delta t \), equation (11) shows that the only possible form for \( \rho \) is a thermal state:

\[
\rho = (1 - e^{-\beta}) e^{-\beta a' a}. \tag{12}
\]

The output is also thermal:

\[
z = (1 - e^{-\beta'}) e^{-\beta' a' a}. \tag{13}
\]

Here, the inverse temperature \( \beta \) is chosen to give the proper entropy \( S_0 \) for the input state \( \rho \), and \( \beta' \) is chosen to make the energy of the thermal output state equal to the energy for the thermal input state plus \( \Delta E \). The only minimum energy stationary point that the Lagrangian technique finds occurs for a thermal input. We cannot however exclude that other stationary points exist, that the Lagrangian technique fails to identify.

VI. THERMAL STATES AND OUTPUT ENTROPY

For the sake of completeness, we now verify explicitly that this stationary point corresponds to a minimum of the output entropy \( S \). At the extremum, \( \rho \) and \( z = \text{Th}(\rho) \) are thermal states. Look at perturbations \( \Delta \rho, \Delta z \) such that \( \rho + \Delta \rho \) satisfies the input entropy constraint and \( z + \Delta z \) satisfies the output energy and dynamical constraints. The change in \( S \) under these perturbations is

\[
-\text{tr } \Delta z \ln z + O(\Delta z^2) = \beta (\text{tr } (z + \Delta z) a' a - \text{tr } za' a) + O(\Delta z^2) = \beta (E(z + \Delta z) - E(z)) + O(\Delta z^2). \tag{14}
\]

But thermal states not only maximize entropy for fixed energy, they also minimize energy for fixed entropy. That is, the energy of \( \rho + \Delta \rho \) is greater than or equal to the energy of \( \rho \) for \( \Delta \rho \neq 0 \). For the same reason, because \( z + \Delta z \) satisfies the output energy constraint, its energy is greater than or equal to that of \( z \). So the change in \( S \) under a non-zero perturbation of \( \rho \) and \( z \), equation (14), is non-negative. (The change is zero for perturbations that respect the symmetry of the thermal noise. If our perturbation is a translation in \( x-p \) space, we have \( \Delta \rho = -i[\nu a + \bar{v} a' a, \rho], \) and \( \Delta z = -i[\nu a + \bar{v} a' a, z] \). In this case \( E(\Delta z + \Delta z) = E(z) + O(\Delta z^2). \) That is, the stationary point of \( \Sigma \) corresponds to a local minimum of the output entropy \( S \).

We have shown that thermal inputs give one possible minimum entropy increase for all cutoffs \( N \). In the limit \( N \to \infty \) (see the Appendix), thermal states also minimize the rate of entropy increase, and amongst such states possesses a minimum of energy. For fixed input entropy, thermal states minimize the output entropy under infinitesimal thermalization. Building up non-infinitesimal thermalization by repeated infinitesimal thermalization, the path that begins at a thermal state and passes through a sequence of thermal states is a minimum of output entropy. Taking the limit that the input entropy approaches zero (see the Appendix), we find that amongst pure input states, the zero-energy thermal state or vacuum is also a minimum for entropy increase under finite thermalization.

VII. UNIQUENESS OF THE MINIMUM

If the gradients of the constraints of the Lagrangian minimizations are not linearly independent, then not all constrained minima will satisfy the Lagrange equations. Namely, some minima can be undetected by this procedure. Unfortunately, in our second minimization, the gradients of the constraints are linearly dependent. In fact, in the first minimization, on the entropy, we are performing a minimization with the Lagrangian \( \Sigma \) of the form

\[
\nabla f - \sum_j \lambda_j \nabla g_j = 0, \text{ with the constraints } g_j = 0, \tag{15}
\]

where \( f \) is the entropy, \( g_j \) represents the Lagrange constraints introduced in Eq. (11), and \( \lambda_j \) represents the Lagrange multipliers introduced in such equation. Then, in the second minimization, on the energy, we are using the results of the first minimization with the Lagrangian \( \mathcal{L} \), i.e. we are performing a minimization of the form

\[
\nabla g - \mu \nabla f - \sum_j \lambda'_j \nabla g_j = 0
\]

with the constraints \( f = f_{\text{out}}, g_j = 0 \), (16)

where \( g \) is the energy, \( \mu \) and \( \lambda'_j \) are the Lagrange multipliers introduced in Eq. (11), and \( f_{\text{out}} \) is the minimum output entropy that results from the first minimization. Replacing Eq. (15) into (16), it is then clear that the gradients of the constraints in the second Lagrange minimization are not linearly independent, as the gradients of the \( g_j \) appear twice in (16). Generalized treatments
of the Lagrange minimization theorem (e.g. see Sec.3.5 of [17]) show that solutions of the Lagrange equation [16] are all local minima. However, not all the local minima will solve such equation. This means that, although thermal states solve this equation, and more specifically, solve Eq. (10), we are not guaranteed that there might be other (non-thermal) states that may still be local minima although they do not solve the Lagrange equations. These minima might have lower output entropy than the thermal states. There are two ways to conclude the proof of the conjectures. Either one should prove that thermal states are the unique constrained minima, or one should prove that any other constrained minima will have higher output entropy with respect to thermal states.

VIII. DISCUSSION

We have introduced a new form of the minimum output entropy conjecture that refers to infinitesimal channels. We have shown that it is equivalent to the previous conjecture, although it allows one to attack it using very different strategies based on variational techniques. A promising double Lagrangian minimization approach to the proof of the conjecture has been outlined. Unfortunately, it does not permit to prove the conjecture because it cannot exclude the presence of states with output entropy lower than the thermal states that the double Lagrangian technique identifies.

The proof of the bosonic minimum output entropy conjecture would establish that the capacity of the bosonic channel with Gaussian noise and linear loss is attained for coherent state inputs [6,8]. Quantum ‘tricks’ such as squeezing and entanglement would not enhance the channel’s capacity. Instead, sending coherent states would be the optimal strategy. In addition, the additivity of this channel would be immediately derived. However, even in this case the optimal detection strategy may well involve squeezing and entanglement [6,8].

The proof of the minimum output entropy conjecture for bosonic channels in the previous form [6] or in the form detailed here remains an important and extremely challenging open question. Before concluding, it is worth pointing out that some partial progress have been recently presented in Ref. [22].

Appendix

1. Truncation

The Lagrangian method requires that the function to be extremized be continuous on the full input space, and that the gradient of the function be continuous on the open interior of the input space [19]. To insure continuity of entropy and its gradient, we truncate our input Hilbert space at some high photon number \( N \). Since infinitesimal thermalization adds at most one photon, the output Hilbert space is truncated at photon number \( N + 1 \). To include the finite truncation in the exposition above, when a Hamiltonian \( a \dagger a \) appears, this Hamiltonian should be taken to be \( a \dagger a \) restricted to the appropriate truncated Hilbert space. That is, when \( a \dagger a \) occurs with an input density matrix \( \rho \), it is the Hamiltonian restricted to the \( N \)-dimensional input Hilbert space; when \( a \dagger a \) occurs with an output density matrix \( z \), it is the Hamiltonian restricted to the \( N + 1 \)-dimensional output Hilbert space.

Similarly, the input and output spaces for the operators in our Lagrangians are defined on the appropriate truncated Hilbert spaces. The input and output entropies are continuous over the space of density matrices in the truncated Hilbert spaces, and have continuous first derivative in the open interior of these spaces. By making \( N \) sufficiently large we insure that any input state with finite energy is as close as desired to its projection onto the truncated Hilbert space. Note that the equations defining the stationary points of the Lagrangian do not depend on \( N \). That is, for all \( N \), the solution of the Lagrange equations are thermal inputs. Consequently, in the limit \( N \to \infty \), this solution must also be thermal.

2. The limit as input entropy goes to zero

We have shown that for all input entropies strictly greater than zero, \( S_0 > 0 \), the path that starts at a thermal state and ends at a thermal state gives one minimum of the output entropy. We want to show that we can extend this path all the way to \( S_0 = 0 \). As the input entropy goes to zero, the rate of entropy increase diverges. The only way that the path that begins at the vacuum (i.e., the zero-temperature thermal state) could NOT give the minimum increase in entropy would be if the divergent rate of entropy increase at the boundary contributed a finite amount \( \Delta S \) to the overall increase in entropy as \( \Delta t \to 0 \) in equation (5) above, and if \( \Delta S \) were bigger for the vacuum than for some other input state.

We show that this is not the case. Expanding equation (5) to evaluate \( \Delta S \) to next order in \( \Delta t \) for input states on the boundary, we see that the integrated increase in entropy over time \( \Delta t \) goes as \( \Delta S = -\gamma \Delta t \ln \gamma \Delta t \), which goes to zero in the limit \( \Delta t \to 0 \). In other words, although the rate of entropy increase diverges at the boundary, the divergence is a relatively weak, logarithmic one. In the limit that \( \Delta t \to 0 \), the entropy increase at the boundary does not contribute to the overall increase in entropy, even though the rate of entropy increase diverges. Accordingly, the path that starts at the vacuum gives one minimum of the entropy increase.

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