Considerations for Anderson-Bridge Experiment

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Abstract

Various issues concerned with the design and sensitivity of the Anderson bridge are discussed. We analyse the circuit using the tools of Network Analysis to determine the conditions under which the balance may be obtained with greater sensitivity and discuss how to obtain these under practical circumstances in an undergraduate laboratory.
1 Introduction

AC bridges are often used to measure the value of an unknown impedance for example self/mutual inductance of inductors or capacitance of capacitors accurately. A large number of AC bridges are available for the accurate measurement of impedences. An Anderson Bridge is used to measure the self inductance of a coil (Fig. [1]). This is an old experiment and has been a part of the graduation curriculum since ages. In fact the oldest publication on sensitivity of A.C. bridges is Rayleigh’s paper [1]. As it usually happens in such old subjects, modern textbooks have diluted the attention paid to the details and intricacies of the experiment. Most of the textbooks tend to merely state the balance condition without discussing the design of the experiment for greater sensitivity.

Over the years of instructing young graduate students in the lab, the authors have observed unsatisfactory results reported by the students in terms of accuracy. One reason reported by the students was an inability to get a mute on balance condition in the head-phone. That is, the human perception of point of minima rendered results inaccurate. The fact that human ears perceive in decibels makes the situation worse. A question that invariably arose was whether “urbanisation and sound pollution was effectively contributing to the inaccuracies of this experiment (physiological constraints) or rather it was ignorance of the relevant physics contributing.

As a test experiment, a careful experimentalist from among the under-
graduate students was given basic instructions and a copy of Yarwood’s book [2] for review before conducting the Anderson Bridge experiment. Table 1, 2 and 3 list the results reported by this student.

As can be seen on examination of the results listed in the Tables, the results are inaccurate and are scattered and deviated from the known value. Also, notice that the student was not able to resolve the minimum for a range of ‘r’ values (of the order of 50Ω). The results reported by the remaining students of the class by and large suffered from more inaccuracies. Under such circumstances, it warrants a more serious analysis of the experiment.
Table 1: Table lists the experimentally determined value of inductance (L) in bridge with coil L=207.4mH and DC balanced resistances R₁ = 560Ω, R₂ = 552Ω and R₄ = 1490Ω.

| S.No. | C (µF) | r (Ω)     | r₁av (Ω) | L (mH)  |
|-------|--------|-----------|-----------|---------|
| 1.    | 0.05   | 1000-1050 | 1025      | 195.55  |
| 2.    | 0.10   | 400-450   | 425       | 211.00  |
| 3.    | 0.11   | 390-410   | 350       | 217.25  |
| 4.    | 0.12   | 200-250   | 225       | 181.17  |
| 5.    | 0.15   | 170-200   | 185       | 208.43  |
| 6.    | 0.20   | 43-55     | 49        | 196.30  |

Table 2: Table lists the experimentally determined value of inductance (L) in bridge with coil L=240.2mH and DC balanced resistances R₁ = 560Ω, R₂ = 552Ω and R₄ = 1490Ω.

| S.No. | C (µF) | r (Ω)     | r₁av (Ω) | L (mH)  |
|-------|--------|-----------|-----------|---------|
| 1.    | 0.05   | 1250-1310 | 1280      | 233.83  |
| 2.    | 0.10   | 485-515   | 500       | 233.52  |
| 3.    | 0.11   | 390-400   | 395       | 222.20  |
| 4.    | 0.15   | 300-310   | 305       | 209.99  |
| 5.    | 0.20   | 105-115   | 110       | 232.92  |

Table 3: Table lists the experimentally determined value of inductance (L) in bridge with coil L=262.9mH and DC balanced resistances R₁ = 560Ω, R₂ = 552Ω and R₄ = 1490Ω.

| S.No. | C (µF) | r (Ω)     | r₁av (Ω) | L (mH)  |
|-------|--------|-----------|-----------|---------|
| 1.    | 0.05   | 1460-1510 | 1485      | 264.59  |
| 2.    | 0.10   | 650-700   | 675       | 286.05  |
| 3.    | 0.11   | 480-510   | 495       | 255.22  |
| 4.    | 0.12   | 390-410   | 400       | 244.21  |
| 5.    | 0.20   | 100-160   | 130       | 244.92  |

and in process polish our understanding of the same which with time seems to have lost relevance. Our analysis is markedly different from that listed by
Yarwood [2] and therewith cited Rayleigh’s work [1]. However, it utilizes the basic theorems taught in Network Analysis and if not considered a serious contribution, it may be viewed as a different preceptive and an attempt in highlighting the utility of the various Network Theorem in a simple circuit.

Before dwelling on the mathematics and circuit design considerations, for completeness and rendering the article self sufficient for the reader, we explain in short here, the Wheatstone Bridge and standard steps followed for DC and AC balancing of the bridges for the determination of the unknown impedences.

Most of the AC bridges are based on a generalised Wheatstone Bridge circuit. As shown in Fig. 2, the four arms of the D.C. Wheatstone Bridge are replaced by impedences ($Z_A$, $Z_B$, $Z_C$ and $Z_D$), the battery by an A.C. source and the D.C. galvanometer by an A.C. null detector (usually a pair of headphones). Using Kirchoff’s Laws, it can be easily shown that the balance or null condition (i.e. when no current flows through the detector or the potential at the point P becomes equal to that at point R) is given by

$$\frac{Z_A}{Z_B} = \frac{Z_C}{Z_D}. \quad (1)$$

Eqn (1) is a complex equation i.e. it represents two real equations obtained by separately equating the real and imaginary parts of the two sides. It follows from the fact that both amplitude and the phase must be balanced. This implies that to reach the balance condition two different adjustments must
be made. That is, the DC and AC balance conditions have to be obtained one by one. The DC balance is obtained using a D.C. source and moving coil galvanometer by adjusting one of the resistances. After which the battery and the galvanometer are replaced with an AC source and a headphone respectively, without changing the resistances set earlier, the other variable impedance is varied to obtain minimum sound in the headphone.
2 Circuit Designing

Various combinations of the impedences can satisfy the balance conditions of a bridge. For example, in a D.C. Wheatstone Bridge using either \( R_A = R_B = R_C = R_D = 10 \, \Omega \) or \( R_A = R_B = 10 \, \Omega \) and \( R_C = R_D = 1000 \, \Omega \) would balance the bridge as indicated by eqn. (1). However, the bridge is not equally sensitive in both cases. Also interchanging the positions of the source and detecting instrument do not alter the balance conditions (Reciprocity Theorem) but again the bridge may not be equally sensitive in the two cases [2]. The bridge will be more ‘sensitive’ if, in a balanced bridge, changing the impedance in any one of the arms of the bridge from the value \( Z \) required by the balance conditions to value \( Z + \delta Z \) (making the bridge off balance) results in a greater current through the detecting instrument. As explained by Yarwood [2], a D.C. bridge is more sensitive if, \textit{whichever has the greater resistance, galvanometer or battery, is put across the junction of the two lower resistances to the junction of the two higher resistances. Also the unknown resistance should be connected in the Wheatstone bridge between a small ratio arm resistance and a large known variable resistance}. However, it is common practice to make all the four arms of Wheatstone Bridge equal or make them of the same order for the optimum sensitivity. For A.C. bridges also, the same procedure is followed. Below we discuss designing an Anderson Bridge of decent sensitivity.

Most text books derive the DC and AC balance conditions of the Ander-
Figure 3: The ‘π’ shaped circuit between nodes ‘ADE’ is replaced by an equivalent ‘T’ circuit. On rearranging the circuital elements, the complex circuit of fig 1 reduces to a apparently simple circuit shown here. Along with visual simplicity, it renders the computation of potentials trivial.

As stated earlier, we plan to use various theorems taught in a course of Network Analysis to do the same. Hence, to analyse the bridge, we convert the π network consisting of $r$, $R_2$, and $X_c$ and replace it with an equivalent ‘T’ network, whose three impedances would be given as

\begin{align*}
Z_1 &= \frac{rR_2}{r + R_2 + X_c} \\
Z_2 &= \frac{R_2X_c}{r + R_2 + X_c} \\
Z_3 &= \frac{rX_c}{r + R_2 + X_c}
\end{align*}

The circuit shown in Fig. 1 can now be viewed as shown in Fig. 3. The circuit now looks more like the fundamental Wheatstone bridge, which purely due
to its topology looks trivial and overcomes a mental barrier a student might have due to a complex looking circuit.

### 2.1 Balancing Conditions

For the bridge to be balanced, the current in arm ‘EB’ should be zero. This demands the potential $V_{FB}$ to be equal to zero. This potential can be worked out as

$$V_{FB} = V_F - V_B$$

where, using potential divider expression, we can compute $V_F$ and $V_B$. They are expressed as

$$V_F = \left( \frac{Z_2}{R_1 + Z_1 + Z_2} \right) V_{DC}$$
$$V_B = \left( \frac{R_4}{R_3 + r_L + X_L + R_4} \right) V_{DC}$$

giving the potential $V_{FB}$ as

$$V_{FB} = \left( \frac{Z_2}{R_1 + Z_1 + Z_2} - \frac{R_4}{R_3 + r_L + X_L + R_4} \right) V_{DC} \quad (5)$$

On attaining balance of the bridge

$$\frac{Z_2}{R_1 + Z_1 + Z_2} = \frac{R_4}{R_3 + r_L + X_L + R_4} \quad (6)$$
which is same as the condition given by the eqn(1) with the four impedences replaced by the corresponding impedences in the four arms of Fig. 3.

Substituting the expressions of eqn(2), eqn(3) and eqn(4) in eqn(6), we have

\[
\frac{R_2 X_c}{R_1 (r + R_2 + X_c) + r R_2 + R_2 X_c} = \frac{R_4}{R_3 + r_L + X_L + R_4}
\]

Collecting and simplifying the imaginary terms of the above equation, we get

\[
\frac{R_2}{R_4} = \frac{R_1}{R_3 + r_L}
\]

Expression given by eqn(8) is the DC balance condition of the Anderson Bridge. Now collecting the real terms of eqn(7), we have

\[
\frac{L}{C} = \frac{R_4}{R_2} \left[ R_1 R_2 + r (R_1 + R_2) \right]
\]

Eqn (9) is the AC balance condition of the bridge.

The unknowns in the two eqns (8) and (9) are \(r_L\) and \(L\) respectively. To obtain the D.C. balance of the bridge, the circuit is made with ‘r’ shorted, capacitance \(C\) open, headphone replaced by a galvanometer and the A.C. source by a battery or D.C. power supply. Now \(R_3\) is varied until galvanometer shows zero deflection. Eqn. (8) then gives the value of \(r_L\). Now using the complete circuit given in Fig. 1 and leaving the resistances set \(R_1\), \(R_2\) and \(R_3\) undisturbed, AC balance is obtained by varying the resistance ‘r’.
Eqn. (9) can now be used to compute the value of self inductance $L$ applied in the circuit. Thus the unknown impedance $\sqrt{(L\omega)^2 + r_2^2}$ is computed. For the simplicity of circuit designing we suggest taking $R_2 = R_4$, which results in $R_1 = r_L + R_3$ from eqn. (7).

### 2.2 Maximum Power Transfer

For a good audio signal at the head phone, we require that the circuit transfers a substantial amount of power to the head phone. Since the ear senses the intensity of sound, the head phone as a measuring device is being used as a power meter. Good sensitivity hence would be attained as large power is being transfered to the head phone for both balanced and unbalanced (AC) condition. Unbalanced AC condition would be represented by replacing ‘$r$’ (of eqn. 9) with ‘$r + dr$’.

The “Maximum Power Transfer Theorem” [3] demands that the load impedance applied to a circuit should be the complex conjugate of the circuit’s Thevenin impedance. Since the head-phone’s impedance is inductive in nature, i.e. it can be represented as ‘$a + jb$’, the theorem demands the Thevenin impedance should be capacitive in nature, i.e. of the form of ‘$a – jb$’. The Thevenin impedance of the Anderson Bridge circuit ($Z_{TH}$) as seen by the head phone is theoretically determined by shorting the AC source (i.e. we neglect the impedance of the source), giving (see Fig. 4)

$$Z_{TH} = [(R_1 + Z_1)||Z_2] + [(R_3 + r_L + XL)||R_4] + Z_3$$
Figure 4: The Thevenin impedance is calculated by shorting nodes ‘C’ and ‘D’ and evaluating the impedance as seen from nodes ‘EB’.

\[
Z_{TH} = \left[ \frac{(R_1 + Z_1)Z_2}{R_1 + Z_1 + Z_2} + \frac{R_4(R_3 + r_L + X_L)}{R_3 + r_L + X_L + R_4 + Z_3} \right]
\]

Using the D.C. balance condition given by eqn. (8) and imposing \( R_2 = R_4 \), we have

\[
Z_{TH} = \left[ \frac{(R_1 + Z_1)Z_2}{R_1 + Z_1 + Z_2} + \frac{R_2(R_1 + X_L)}{R_1 + X_L + R_2 + Z_3} \right]
\]

The best way to simplify the above expression is to select \( X_c \sim 0 \). This means the AC audio frequency \( (f) \) should be kept large and the capacitor used should have a very large value of ‘C’. From eqn (9), it naturally flows that \( X_L \) has to be considerably large. Selecting a high audio frequency automatically achieves this. Further, \( X_c \sim 0 \) might be difficult to attain.
given the fact that frequency is to be in audible range and very high capacitances are usually electrolytic, thus introducing problems of its own. Hence it would be sufficient to select ‘C’ and ‘f’ such that $|X_c| << |X_L|$ and $(rR_1 + R_1 R_2 + rR_2) >> R_2 |X_c|$. Further, let $R_1 >> R_2$. The justification and advantage of this assumption would be evident later. With these conditions imposed, the Thevenin equivalent impedance reduces to

$$Z_{TH} = \left[ R_2 + \frac{(R_2 + r)X_c}{(r + R_2 + X_c)} \right]$$

(10)

Thus, the basic constraints imposed on the circuit designing has made the Bridge’s Thevenin equivalent impedance capacitive in nature, (eqn. 10 is of the form $a-jb$).

$$Z_{TH} = \left[ R_2 + \frac{(r + R_2)|X_c|^2}{(r + R_2)^2 + |X_c|^2} \right] + \left[ \frac{(r + R_2)^2}{(r + R_2)^2 + |X_c|^2} \right] X_c$$

(11)

As stated earlier, for the maximum power to be transferred, the Bridge’s Thevenin equivalent impedance should be capacitive in nature since the headphone’s impedance is inductive in nature. Also, head phones are usually low impedance devices and hence $X_c$ and $R_2$ should be kept as small as possible. While selection of small $X_c$ was already imposed, an additional requirement is demanded, i.e. $R_2$ should also be of small value.
2.3 Sensitivity

To appreciate the approach of the AC balance point, the sensitivity of the circuit should be good. Here we define the bridge’s sensitivity as \((dP/dr)\), i.e. the power developed across the head-phone should show large change for a small change in ‘\(r\)’. Upon the application of the Maximum Power Transfer theorem, the expression of current can be written as

\[
i = \frac{V_{TH}}{2Re(Z_{TH})}
\]

where \(V_{TH}\) is the Thevenin equivalent voltage. If the Thevenin impedance doesn’t match the impedance of the headphone exactly, the current expression for a circuit designed along the lines discussed above reduces to (the Thevenin voltage would be of the same form as that given by eqn. 5. Notice \(V_{DC}\) has been replaced by \(V_{ac}\), the applied A.C. voltage)

\[
i = \left[ \frac{1}{(R_H + \frac{|X_c|^2}{r^2+|X_c|^2}) + j \left( Z_H - \frac{|X_c|^2}{r^2+|X_c|^2} \right)} \right] V_{ac}'
\]

where \(R_H\) and \(Z_H\) are the real and imaginary part of the impedance associated with the head phone being used. The term \(V_{ac}'\) is given as

\[
V_{ac}' = \left( \frac{Z_2}{R_1 + Z_1 + Z_2} - \frac{R_4}{R_3 + r_L + X_L + R_4} \right) V_{ac}
\]
Using the constraints imposed and eqn (11), this expression reduces to

\[ V'_{ac} = R_2 \left[ \frac{X_c}{R_1(r + R_2 + X_c)} - \frac{1}{R_1 + X_L} \right] V_{ac} \]

Hence, the power dissipated across the head phone is given as

\[ P = ii^* R_H \]

\[ P \approx \left[ \frac{R_H}{\left( R_H + \frac{r|X_c|^2}{r^2 + |X_c|^2} \right)^2 + \left( Z_H - \frac{r^2|X_c|}{r^2 + |X_c|^2} \right)^2} \right] V'_{ac}^2 \]

Differentiating w.r.t. ‘r’ (for simplifying the expression we can assume \( R_H \) and \( Z_H \sim 0 \) which is generally true since head phones are low impedance devices. Further approximation can be made that \( V'_{ac} \) is very small and hence is a shallow function of ‘r’ due to the small values of \( R_2 \) and \( X_c \) in the numerator and large \( R_1 \) and \( X_L \) in the denominator.),

\[ \left. \frac{1}{R_H V'_{ac}^2} \right| \left( \frac{dP}{dr} \right) \approx \frac{2}{r^3} \] (14)

The above expression demands ‘r’ to be of small value for good sensitivity of the bridge. However, till now we have been silent on the nature of ‘r’. For this we revisit eq (9), to understand the effects of our considerations on the nature of ‘r’. For a bridge designed along our suggestions, the AC balance
condition (eq 9) gives

\[ r \approx \frac{L}{R_1 C} \approx X_c \left( \frac{X_L}{R_1} \right) \] (15)

The selection of high frequency, large capacitance with large \( R_1 \) makes sure that ‘\( r \)’, the resistance used to AC balance the bridge, is proportional to the impedance offered by the capacitor. As the case is, in our design AC balance should be achieved with small ‘\( r \)’ which would result in good sensitivity. The interpretations do not vary even if the approximations listed above eqn 14 are not made.

Another design consideration necessary is to make \( R_1 \) large. We summarize here the circuitial conditions one must maintain to get good sensitivity for the Anderson Bridge

1. \( (R_1 = R_3) \gg (R_2 = R_4) \) with \( R_2 \) being very small

2. Select \( X_L \gg X_c \).

3. Audio frequency to be large

We put our analysis to test by designing an Anderson Bridge subject to the conditions listed above and determined the self inductance of a given coil. We report some select observations in Table 4 to highlight our findings. The headphone used as a current detecting instrument had a resistance \( R_H \) of about 100\( \Omega \) and inductance \( L_H \) of about 22\( mH \). The first reading in Table 4 shows our best result, where we ensured that all the conditions
required for proper design along with Maximum Power Transfer Theorem are satisfied. The ability in determining the balance point, in terms of lowest audio intensity, was also remarkable. For increase of $\pm 1\Omega$ in the first reading we could detect increase in sound intensity. For the second reading we took a capacitor ten times smaller than the capacitor used in the previous set of observations. This of course results in a ten fold increase in the value of $X_c$ and ‘r’. The values of other impedences selected here also adhered to the listed conditions. However, the inaccuracy in the value of inductance determined is evident. Even the ability to determine the balance point was compromised in the second design. Similar, inaccuracy is also evident when the audio frequency is decreased to 1 kHz in the third design reducing the value of $X_L$ and increasing that of $X_c$.

Table 4: Table lists the experimentally determined value of inductance (L) in bridge with coil $L=130\text{mH}$ and circuital elements selected as per the analysis described above.

| S.No. | f (KHz) | $R_1$ (Ω) | $R_2$ (Ω) | $R_3$ (Ω) | C (µF) | r (Ω) | L (mH) |
|-------|---------|-----------|-----------|-----------|--------|-------|--------|
| 1.    | 3       | 4700      | 100       | 4770      | 0.10   | 186-188 | 136.7  |
| 2.    | 3       | 4700      | 100       | 4770      | 0.01   | 5570-5720 | 277.5  |
| 3.    | 1       | 4700      | 100       | 4770      | 0.10   | 293-313 | 192.6  |

The design constraints discussed here were not discussed with the experimentalist who reported results of Tables 1, 2 and 3. On examination of the data, we find that only the first reading of Table 2 and Table 3 satisfy stated conditions of circuit design. Not only are these the best results in
terms of returned value of inductance but also in terms of ability to resolve the minimum sound.

**Conclusion**

Based on our derivations using various theorems of Network Analysis, we have analysed how to design an Anderson Bridge with good sensitivity. Various considerations show that for a better sensitivity, one should work at high frequencies and use a low $R_2$ and high capacitance (making $X_L >> X_c$). We hope that the present treatment allows the circuit not just to serve as a method of determining self inductance but also becomes an important pedagogical tool for circuit analysis.

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