Beam-size or MD-effect at colliders and correlations of particles in a beam

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Abstract

For several processes at colliding beams, macroscopically large impact parameters give an essential contribution to the standard cross section. These impact parameters may be much larger than the transverse sizes of the colliding bunches. In that case, the standard calculations have to be essentially modify. The corresponding formulae for such a beam-size effect were given twenty years ago without taking into account correlations of particle coordinates in the beams. In the present paper we derive formulae which necessary to take into account quantitatively the effect of particle correlations in the spectrum of bremsstrahlung as well as in pair production. Besides, we consider critically recent papers [17, 18] in which it was calculated a new additional “subtraction term” related to the coherent contribution into beam-size effect. We show that this result is groundless and point out the origin of the mistake.

1 INTRODUCTION

The so called beam-size or MD-effect is a phenomenon discovered in experiments [1] at the MD-1 detector (the VEPP-4 accelerator with $e^+e^-$ colliding beams, Novosibirsk 1981). It was found out that for ordinary bremsstrahlung, macroscopically large impact parameters should be taken into consideration. These impact parameters may be much larger than the transverse sizes of the interacting particle bunches. In that case, the standard calculations, which do not take into account this fact, will give incorrect results. The detailed description of the MD-effect can be found in review [2].

In 1980–1981 a dedicated study of the process $e^+e^- \rightarrow e^+e^-\gamma$ has been performed at the collider VEPP-4 in Novosibirsk using the detector MD-1 for an energy of the electron and positron beams $E_e = E_p = 1.8$ GeV and in a wide interval of the photon energy $E_\gamma$ from 0.5 MeV to $E_\gamma \approx E_e$. It was observed [1] that the number of measured photons was smaller than that expected. The deviation from the standard calculation reached 30% in the region of small photon energies and vanished for large energies of the photons. Yu.A. Tikhonov [3] pointed out that those impact parameters $\rho$, which give an essential contribution to the standard cross section, reach values of $\rho_m \sim 5$ cm whereas

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the transverse size of the bunch is $\sigma_{\perp} \sim 10^{-3} \text{ cm}$. The limitation of the impact parameters to values $\varrho \lesssim \sigma_{\perp}$ is just the reason for the decreasing number of observed photons.

The first calculations of this effect have been performed in Refs. [4] and [5] using different versions of quasi–classical calculations in the region of large impact parameters. Further experiments, including the measurement of the radiation probability as function of the beam parameters, supported the concept that the effect arises from the limitation of the impact parameters. Later on, the effect of limited impact parameters was taken into account when the single bremsstrahlung was used for measuring the luminosity at the VEPP–4 collider [6] and at the LEP-I collider [7]. In the case of the VEPP–4 experiment [6], it was checked that the luminosities, obtained using either this process or using other reactions (such as the double bremsstrahlung process $e^+e^- \rightarrow e^+e^-\gamma\gamma$, where the MD-effect is absent), agreed with each other.

A general scheme to calculate the finite beam size effect had been developed in paper [8] starting from the quantum description of collisions as an interaction of wave packets forming bunches. Since the effect under discussion is dominated by small momentum transfer, the general formulae can be considerably simplified. The corresponding approximate formulae were given in [8]. In a second step, the transverse motion of the particles in the beams can be neglected. The less exact (but simpler) formulae, which are then found, correspond to the results of Refs. [4] and [5]. It has also been shown that similar effects have to be expected for several other reactions such as bremsstrahlung for colliding $e^-e^-$ beams [9, 10], $e^+e^-\gamma\gamma$ pair production in $e^+e^-$ and $e^-\gamma$ collisions [8]. The corresponding corrections to the standard formulae are now included in programs for simulation of events at linear colliders. The influence of MD-effect on polarization had been considered in Ref. [11]. In 1995 the MD-effect was experimentally observed at the electron-proton collider HERA [12] on the level predicted in [10].

The possibility to create high-energy colliding $\mu^+\mu^-$ beams is now wildly discussed. For several processes at such colliders a new type of beam-size effect will take place — the so called linear beam-size effect [13]. The calculation of this effect had been performed by method developed for MD-effect in [8].

It was realized in last years that MD-effect in bremsstrahlung plays important role for the problem of beam lifetime. At storage rings TRISTAN and LEP-I, the process of a single bremsstrahlung was the dominant mechanism for the particle losses in beams. If electron loses more than 1 % of its energy, it leaves the beam. Since MD-effect reduced considerable the effective cross section of this process, the calculated beam lifetime in these storage rings was larger by about 25 % for TRISTAN [14] and by about 40 % for LEP-I [15] (in accordance with the experimental data) then without taken into account the MD-effect. According to our calculations [16], at B-factories PEP-II and KEKB the MD-effect reduces beam losses due to bremsstrahlung by about 20 %.

It is seen from this brief listing that the MD-effect is a phenomenon interesting from the theoretical point of view and important from the experimental point of view. In the present paper we consider once again the MD-effect having in mind two aims. The main purpose is to take into account correlations of particle coordinates in the beams. Usually these correlations are small, however, more accurate measurement may be sensitive to them. In the present paper we derive formulae which necessary to take into account quantitatively the effect of particle correlations in the spectrum of bremsstrahlung as well as in pair production.

Besides, we would like to consider critically recent papers [17, 18] in which previous
results \cite{4, 5, 8} about bremsstrahlung spectrum had been revised. It was claimed that an additional subtraction related to the coherent contribution has to be done. Analysis, performed in paper \cite{17}, results in the conclusion that this additional “subtraction term” in the spectrum is not important for the MD-1 experiment \cite{11}, but it should be taken into account in processing the HERA experiment \cite{12}; it also may be important for the future experiments at linear $e^+e^-$ colliders. It should be noted that in paper \cite{17} there is no derivation of the starting formulae. On the other hand, in paper \cite{17} there is a general remark that their consideration was motivated by corresponding calculations for bremsstrahlung of ultra-relativistic electrons on oriented crystals. In our critical remark \cite{16} we had shown that the starting formulae of \cite{17} are incorrect. After that new paper \cite{18} was appeared in which there is the “derivation” of the starting formulae used in \cite{17}. Unfortunately, this derivation is incorrect as well.

In the present paper we analyze the coherent and incoherent contributions in the conditions, considered in papers \cite{17, 18}, when the coherent length $l_{\text{coh}}$ is much smaller than the bunch length $l$ but much larger than the mean distance between particles $a$, i.e. $a \ll l_{\text{coh}} \ll l$. We derive expressions for the coherent and incoherent contributions and show that under these conditions the coherent contribution is completely negligible and, therefore, there is no need to revise the previous results. This conclusion is quite natural.

A usual bunch at colliders can be considered as a gaseous media with a smooth particle distribution which has characteristic scales of the order of bunch sizes. In particular, the average particle density in such a bunch has the only scale in the longitudinal direction — the length of the bunch $l$. Therefore, the average field of the bunch has the spectral components in the region of frequencies $\omega = q_z c \sim c/l_{\text{coh}} \ll c/l$ and vanishes in the region of much higher frequencies considered here. On the contrary, in the crystal case there is another scale related to the size of the particle localization in the crystal structure. In this case, the additional subtraction should be taken into account for incoherent contribution. It seems that the electron radiation on oriented crystals played a misleading role for consideration of the MD-effect in \cite{17, 18}. To clarify a question we give our calculations in full details and pointed out the origin of the corresponding mistake in \cite{17, 18}.

In next Section we present the qualitative description of the MD-effect. In Sect. III we discuss our approximations. Basic formulae for coherent and incoherent contributions are given in Sect. IV. Correction to the standard cross section, related to the particle correlations, are derived in Sect. V. Our critical remarks about results of papers \cite{17, 18} are presented in Sect. VI. Some conclusions are given in Sect. VII.
Qualitatively we describe the MD–effect using as an example the $ep \rightarrow ep\gamma$ process.\(^1\) This reaction is defined by the diagrams of Fig. 1 which describe the radiation of the photon by the electron (the contribution of the photon radiation by the proton can be neglected). The large impact parameters $q \gtrsim \sigma_\perp$, where $\sigma_\perp$ is the transverse beam size, correspond to small momentum transfer $\hbar q_\perp \sim (\hbar/q) \lesssim (\hbar/\sigma_\perp)$. In this region, the given reaction can be represented as a Compton scattering (Fig. 2) of the equivalent photon, radiated by the proton, on the electron. The equivalent photons with frequency $\omega$ form a “disk” of radius $\rho_m \sim \gamma_p c/\omega$ where $\gamma_p = E_p/(m_p c^2)$ is the Lorentz-factor of the proton. Indeed, the electromagnetic field of the proton is $\gamma$–times contracted in the direction of motion. Therefore, at distance $q$ from the axis of motion a characteristic longitudinal length of a region occupied by the field can be estimated as $\lambda \sim q/\gamma_p$ which leads to the frequency $\omega \sim c/\lambda \sim \gamma_p c/q$.

In the reference frame connected with the collider, the equivalent photon with energy $\hbar \omega$ and the electron with energy $E_e \gg \hbar \omega$ move toward each other (Fig. 3) and perform a Compton scattering. The characteristics of this process are well known. The main contribution to the Compton scattering is given by the region where the scattered photons fly in a direction opposite to that of the initial photons. For such a backward scattering, the energy of the equivalent photon $\hbar \omega$ and the energy of the final photon $E_\gamma$ and its emission angle $\theta_\gamma$ are related by

$$\hbar \omega = \frac{E_e}{4 \gamma_e^2 (1 - E_\gamma/E_e)} \left[1 + (\gamma_e \theta_\gamma)^2\right]$$

and, therefore, for typical emission angles $\theta_\gamma \lesssim 1/\gamma_e$ one has

$$\hbar \omega \sim \frac{E_\gamma}{4 \gamma_e^2 (1 - E_\gamma/E_e)}.$$ \hspace{1cm} (2)

As a result, we find the radius of the “disk” of equivalent photons with the frequency $\omega$ (corresponding to a final photon with energy $E_\gamma$) as follows:

$$\rho_m = \frac{\gamma_p c}{\omega} \sim \lambda_e 4 \gamma_e \gamma_\gamma \frac{E_e - E_\gamma}{E_\gamma}, \quad \lambda_e = \frac{\hbar}{m_e c} = 3.86 \cdot 10^{-11} \text{ cm}.$$ \hspace{1cm} (3)

\(^1\)Below we use the following notations: $N_e$ and $N_p$ are the numbers of electrons and protons (positrons) in the bunches, $\sigma_z = \ell$ is the longitudinal, $\sigma_x$ and $\sigma_y$ are the horizontal and vertical transverse sizes of the proton (positron) bunch, $\gamma_e = E_e/(m_e c^2)$, $\gamma_p = E_p/(m_p c^2)$ and $r_e = e^2/(m_e c^2)$ is the classical electron radius.
For the HERA collider with $E_p = 820$ GeV and $E_e = 28$ GeV one obtains

$$\varrho_m \gtrsim 1 \text{ cm} \quad \text{for} \quad E_\gamma \lesssim 0.2 \text{ GeV}. \quad (4)$$

Equation (3) is also valid for the $e^- e^+ \to e^- e^+ \gamma$ process with replacement protons by positrons. For the VEPP-4 collider it leads to

$$\varrho_m \gtrsim 1 \text{ cm} \quad \text{for} \quad E_\gamma \lesssim 15 \text{ MeV}, \quad (5)$$

for the PEP-II and KEKB colliders we have

$$\varrho_m \gtrsim 1 \text{ cm} \quad \text{for} \quad E_\gamma \lesssim 0.1 \text{ GeV}. \quad (6)$$

The standard calculation corresponds to the interaction of the photons forming the “disk” with the unbounded flux of electrons. However, the particle beams at the HERA collider have finite transverse beam sizes of the order of $\sigma_\perp \sim 10^{-2}$ cm. Therefore, the equivalent photons from the region $\sigma_\perp \lesssim \varrho \lesssim \varrho_m$ cannot interact with the electrons from the other beam. This leads to the decreasing number of the observed photons and the “observed cross section” $d\sigma_{\text{obs}}$ is smaller than the standard cross section $d\sigma$ calculated for an infinite transverse extension of the electron beam,

$$d\sigma_{\text{obs}} = d\sigma - d\sigma_{\text{cor}}. \quad (7)$$

Here the correction $d\sigma_{\text{cor}}$ can be presented in the form

$$d\sigma_{\text{cor}} = d\sigma_C(\omega, E_e, E_\gamma) \, dn(\omega) \quad (8)$$

where $dn(\omega)$ denotes the number of “missing” equivalent photons and $d\sigma_C$ is the cross section of the Compton scattering. Let us stress that the equivalent photon approximation in this region has a high accuracy (the neglected terms are of the order of $1/\gamma_p$). But for the qualitative description it is sufficient to use the logarithmic approximation in which this number is (see [20], §99)

$$dn = \frac{\alpha \, d\omega \, dq^2_\perp}{\pi \, \omega \, q^2_\perp}. \quad (9)$$

Since $q_\perp \sim 1/\varrho$, we can present the number of “missing” equivalent photons in the form

$$dn = \frac{\alpha \, d\omega \, \varrho^2}{\pi \, \omega \, \varrho^2}. \quad (10)$$

Figure 3: Scattering of equivalent photons, forming the “disk” with radius $\varrho_m = \gamma_p c/\omega$, on the electron beam with radius $\sigma_\perp$. 
with the integration region in \( q \):

\[
\sigma_\perp \lesssim q \lesssim q_m = \frac{\gamma_p c}{\omega}.
\]  

(11)

As a result, this number is equal to

\[
dn(\omega) = 2\frac{\alpha}{\pi} \frac{d\omega}{\omega} \ln \frac{q_m}{\sigma_\perp},
\]

(12)

and the correction to the standard cross section with logarithmic accuracy is\(^2\)

\[
d\sigma_{\text{cor}} = \frac{16}{3} \alpha r_e^2 \frac{dy}{y} (1 - y + \frac{3}{4}y^2) \ln \frac{4\gamma_e \gamma_p (1 - y) \lambda_e}{y \sigma_\perp}, \quad y = \frac{E_\gamma}{E_e}.
\]

(13)

### 3 Approximations

For future linear \( e^+e^- \) colliders the transverse sizes of the beams will change significantly during the time of interaction due to a mutual attraction of very dense beams. However, for most of the ordinary accelerators, including practically all \( e^+e^- \) and \( ep \) storage rings, the change of the transverse beam sizes during the collisions can be neglected. Below we use two main approximations: 1) the particle movement in the bunches has a quasi-classical character; 2) the particle distribution remains practically unchanged during the collision. For definiteness, we use again the \( ep \) collisions as an example.

Therefore, if the proton (electron) bunch moves along (opposite) the direction of \( z \)-axis with the velocity \( v_p \) (\( v_e \)), its density has the form

\[
n_p = n_p(\varrho, z - v_{pt}), \quad n_e = n_e(\varrho, z + v_{et}).
\]

(14)

We also introduce so called “transverse densities”

\[
n_p(\varrho) = \int n_p dz, \quad n_e(\varrho) = \int n_e dz
\]

(15)

which is equal to the total number of protons (electrons) which cross a unit area around the impact parameter \( \varrho \) during the collision. Using the transverse densities, we express the luminosity for collisions of beams whose axes separated by impact parameter \( \varrho \) as

\[
L(\varrho) = \int n_e(\varrho)\, n_p(\varrho + r_\perp) \, d^2r_\perp.
\]

(16)

The usual luminosity for a single collision of \( ep \)-beams \( L_{ep} \) is then

\[
L_{ep} = L(0).
\]

\(^2\)Within this approximation, the standard cross section has the form

\[
d\sigma = d\sigma_{\text{C}} \frac{\alpha}{\pi} \frac{d\omega}{\omega}\frac{dy_\perp^2}{q_\perp^2} = \frac{16}{3} \alpha r_e^2 \frac{dy}{y} (1 - y + \frac{3}{4}y^2) \ln \frac{4\gamma_e \gamma_p (1 - y)}{y} \lambda_e
\]

with the integration region \( \omega/(e\gamma_p) \lesssim q_\perp \lesssim m_e c/\hbar \) corresponding to the impact parameters \( \varrho \) in the interval \( \lambda_e \lesssim \varrho \lesssim \varrho_m \).
Below we consider in detail the case when an electron deflection angle $\theta_e$ is smaller than the typical radiation angle $\sim \gamma^{-1}$. It is easy to estimate the ratio of these angles. The electric $E$ and magnetic $B$ fields of the proton bunch are approximately equal in magnitude, $|E| \approx |B| \sim eN_p/(\sigma_x + \sigma_y)$. These fields are transverse and they deflect the electron into the same direction. In such fields the electron moves a circumference of radius $R \sim \gamma e m_e c^2/(eB)$ and gets the deflection angle $\theta_e \sim l/R$. Therefore, the ratio of these angles is of the order of

$$\frac{\theta_e}{(1/\gamma)} \sim \eta = \frac{r_e N_p}{\sigma_x + \sigma_y}. \quad (18)$$

The parameter $\eta \gg 1$ only for the SLC and future linear $e^+e^-$ colliders, in most of the colliders $\eta \lesssim 1$.

In our consideration we use the equivalent photon approximation. In the region of interest (where impact parameters are large, $\varrho \gtrsim \sigma_\perp$) this simple and transparent method has a high accuracy: the neglected terms are of the order of $1/\gamma$. It should be stressed that the operator quasi-classical method, used in Ref. [17], just coincides in this region with the equivalent photon approximation.

4 COHERENT AND INCOHERENT CONTRIBUTIONS

4.1 General formulae

The corresponding formulae for the number of events in a single collision of the electron and proton bunches can be found in papers [21], [22]. To calculate the MD-effect, we need to know the distribution of equivalent photons (EP) for large values of impact parameters. In this region we can consider the electron–proton scattering as the scattering of electrons on the electromagnetic field of the proton bunch. Replacing this field by the flux of EP with some frequency distribution, we obtain the number of events in the form

$$dN = dL_{\gamma e}(\omega) \, d\sigma_C(\omega, E_e, E_\gamma), \quad dL_{\gamma e}(\omega) = n_\gamma(\varrho, \omega) d\omega n_e(\varrho) \, d^2\varrho. \quad (19)$$

Here $n_e(\varrho)$ is the transverse electron density (15) and $n_\gamma(\varrho, \omega) d\omega$ is the transverse density of EP with the frequencies in the interval from $\omega$ to $\omega + d\omega$. The quantity $dL_{\gamma e}(\omega)$ denotes the differential luminosity for the collisions of EP and electrons and $d\sigma_C(\omega, E_e, E_\gamma)$ is the Compton cross section for the scattering of the equivalent photon with the frequency $\omega$ on the electron.

For comparison with the experimental data the number of events in a single collision of beams $dN$ should be averaged over many collisions of bunches in a given experiment. For example, the typical rate at the HERA collider is less then $1/100$ bremsstrahlung photons in a certain interval of frequencies per a single collision of the beams, therefore, in that experiment the averaging over many collisions of bunches really does exist.

The transverse density of the EP is determined by density of the electromagnetic field for a given frequency, i.e. by $|E_\omega(\varrho)|^2/(4\pi)$, where $E_\omega(\varrho)$ is the spectral component of the collective electric field of the proton bunch. As a result, the transverse density of the EP is

$$n_\gamma(\varrho, \omega) \, d\omega = \frac{c}{4\pi^2} \langle |E_\omega(\varrho)|^2 \rangle \frac{d\omega}{\hbar \omega}, \quad (20)$$
where the sign $\langle \ldots \rangle$ denotes the above mentioned statistical averaging. The field $E_\omega(q)$ itself depends on a distribution of charges in the proton bunch at $t = 0$. We introduce the exact (fluctuating) density of the proton bunch $n(r)$ and the averaging density

$$n_{p}(r) = \langle n(r) \rangle$$  \hspace{1cm} (21)

as well as the corresponding form factor

$$F_{p}(q) = \int n_{p}(r) \, e^{-iqr} \, d^3r$$  \hspace{1cm} (22)

with the normalization

$$F_{p}(0) = \int n_{p}(r) \, d^3r = N_{p}.$$  \hspace{1cm} (23)

In the classical limit

$$n(r) = \sum_{a} \delta(r - r_{a})$$  \hspace{1cm} (24)

where $r_{a}$ is the radius-vector of the $a$-th proton. In these notations, the exact (fluctuating) collective field is

$$E_\omega(q) = -\frac{ie}{\pi c} \int d^2q_{\perp} \frac{q_{\perp} e^{i q_{\perp} \cdot e}}{D(q)} \int d^3r \, n(r) \, e^{-iqr} , \quad D(q) = q_{\perp}^2 + \frac{q_{z}^2}{\gamma_{p}^2}$$  \hspace{1cm} (25)

with $q_{z} = \omega/c$.

As a result, the number of events

$$dN \propto n_{\gamma}(q, \omega) = \frac{\alpha}{4\pi^4 \omega} \int \frac{(q_{\perp} q_{\perp}^{'})}{D(q) D(q')} e^{i(q_{\perp} - q_{\perp}^{'}) \cdot e} S(q, q') \, d^2q_{\perp} \, d^2q_{\perp}'$$  \hspace{1cm} (26)

depends on the beam structure factor

$$S(q, q') = \int S(r, r') \, e^{-i(q_{\perp} - q_{\perp}^{'}) \cdot e} \, d^3r \, d^3r' , \quad S(r, r') = \langle n(r) \, n(r') \rangle$$  \hspace{1cm} (27)

in which

$$q_{z} = q_{z}' = \omega/c.$$  \hspace{1cm} (28)

Below we analyze these formulae in conditions when the coherence length $l_{coh} \sim c/\omega$ is much smaller than the bunch length $l$, but much larger that the mean distance between particles in the beam $a$, i. e. at

$$a \ll \frac{c}{\omega} \sim l_{coh} = \frac{4\gamma_{e}^2 \hbar c}{E_{\gamma}} \, (1 - E_{\gamma}/E_{e}) \ll l.$$  \hspace{1cm} (29)

### 4.2 The beam structure factor

The obtained general formulae include the coherent and incoherent contributions. The coherent contribution is determined by the average field which is given by Eq. (25) with the replacement of the exact density $n(r)$ by the average density $n_{p}(r)$. The averaged density of the proton bunch has a single scale in the longitudinal direction — the length of the bunch $l$. Therefore, the average field of the bunch is essential in the region of
frequencies $\omega = cq_z \lesssim c/l$ and should be small in the region of large frequencies $\omega \gg c/l$.

In particular, if the proton bunch has the Gaussian distribution,

$$n_p(r) = \frac{N_p}{(2\pi)^{3/2}\sigma_x \sigma_y} \exp \left[ -\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2l^2} \right], \quad (30)$$

its form factor is equal to

$$F_p(q) = N_p \exp \left[ -\frac{1}{2} (q_x \sigma_x)^2 - \frac{1}{2} (q_y \sigma_y)^2 - \frac{1}{2} (\omega l/c)^2 \right] \quad (31)$$

and vanishes in the discussed region of frequencies from the interval $(29)$.

If we introduce the density fluctuation

$$\Delta n(r) = n(r) - n_p(r), \quad (32)$$

we can rewrite the average product of densities in the form

$$\langle n(r) n(r') \rangle = n_p(r) n_p(r') + \langle \Delta n(r) \Delta n(r') \rangle. \quad (33)$$

In accordance with this presentation, we split the function $S(r, r')$ in two items called coherent and incoherent contribution:

$$S = S_{\text{coh}} + S_{\text{incoh}}, \quad S_{\text{coh}}(r, r') = n_p(r) n_p(r'), \quad S_{\text{incoh}}(r, r') = \langle \Delta n(r) \Delta n(r') \rangle. \quad (34)$$

The coherent contribution to the structure factor is equal to

$$S_{\text{coh}}(q, q') = F_p(q) F_p^*(q'). \quad (35)$$

This formula was used in Refs. [21], [22] to obtain main characteristics of the coherent bremsstrahlung. It also allows us to obtain the following estimate for the Gaussian beam in the region of interest (at $|q_x|, |q'_x| \lesssim 1/\sigma_x$ and $|q_y|, |q'_y| \lesssim 1/\sigma_y$):

$$S_{\text{coh}}(q, q') \sim N_p^2 \exp \left[ - (\omega l/c)^2 \right]. \quad (36)$$

Let us now consider the incoherent contribution. A bunch at colliders can be treated as a continues media with a smooth average particle distribution. It was shown in Appendix that for such a media the function $S_{\text{incoh}}(r, r')$ is expressed only via the average density $n_p(r)$ and via the correlation function $C(r, r')$ as follows

$$S_{\text{incoh}}(r, r') = \delta(r - r') n_p(r) + C(r, r'). \quad (37)$$

If we neglect the correlations of the particle coordinates in the beam, the correlation function $C(r, r')$ vanishes, and we obtain (taking into account Eq. (28))

$$S_{\text{incoh}}(q, q') = F(q - q'). \quad (38)$$

It is important that this expression is determined only by the transverse average density of the proton bunch and it does not depend on $\omega$. Formula (38) has been used to derive the previous results about MD-effect (for details see review [2] and Sect. V). For the Gaussian beam in the region of interest, we get from (38) a useful estimate

$$S_{\text{incoh}}(q, q') \sim N_p. \quad (39)$$
The correlations of the particle coordinates may arise due to Coulomb interaction of particles in the beam. In this case the characteristic quantity — the correlation length $l_{\text{correl}}$ — is related to the Debye radius. It is evident that the correlations are negligible if the correlation length is much larger than the coherence length, i.e. at $l_{\text{correl}} \gg l_{\text{coh}}$.

According to an estimate [5] it is just the case for the VEPP-4 experiment [1]. In any case, the correlations may give an essential correction to the standard bremsstrahlung cross section only if

$$l_{\text{correl}} \lesssim l_{\text{coh}}.$$  

Therefore, the important quantity is the spectral component of the correlation function:

$$C_{\omega}(r_\perp, r'_\perp) = \int C(r, r') e^{-i\omega(z-z')/c} \, dz \, dz'.$$  

(41)

With this notation the incoherent contribution is now:

$$S_{\text{incoh}}(q, q') = F(q_\perp - q'_\perp) + C_{\omega}(q_\perp, q'_\perp),$$  

(42)

where

$$C_{\omega}(q_\perp, q'_\perp) = \int C_{\omega}(r_\perp, r'_\perp) e^{-i(q_\perp r_\perp - q'_\perp r'_\perp)} \, d^2r_\perp \, d^2r'_\perp.$$  

(43)

## 5 Correction to the Standard Bremsstrahlung Cross Section

Let us compare the coherent and incoherent contributions for the Gaussian beams. In this case, the ratio

$$\frac{dN^{\text{coh}}}{dN^{\text{incoh}}} \sim \frac{S_{\text{coh}}(q, q')}{{S'}_{\text{incoh}}(q, q')} \sim N_p \exp \left[ -\left( \frac{\omega l}{c} \right)^2 \right]$$  

(44)

is determined by the parameters $\omega l/c$. Since $\hbar \omega \sim E_\gamma/[4\gamma e^2(1 - E_\gamma/E_c)]$, it is also useful to introduced the coherence length [29] and the critical energy for the coherent bremsstrahlung

$$E_c = \frac{4\gamma^2 e^2 \hbar c}{l}.$$  

(45)

If the coherence length is large, $l_{\text{coh}} \gtrsim l$, or if the final photon energy is small, $E_\gamma \lesssim E_c$, the parameter $\omega l/c \lesssim 1$ and the coherent contribution is dominant.

On the contrary, in the region of large photon energy, $E_\gamma \gg E_c$, or small coherence length, $l_{\text{coh}} \ll l$, considered here, the incoherent contribution dominates. In particular, for $N_p \sim 10^{11}$ the ratio $dN^{\text{coh}}/dN^{\text{incoh}}$ is small even for $\omega l/c = 6$,

$$\frac{dN^{\text{coh}}}{dN^{\text{incoh}}} \sim N_p e^{-36} \ll 1,$$  

(46)

and the coherent contribution becomes completely negligible. In this case the number of events for bremsstrahlung can be presented in the form (cf. [7])

$$dN^{\text{incoh}} = L_{\text{ep}} \, d\hat{\sigma}_{\text{obs}}, \quad d\sigma_{\text{obs}} = d\sigma - d\sigma_{\text{cor}},$$  

(47)
where $L_{ep}$ is the luminosity of the $ep$-collisions, $d\sigma$ is the standard cross section for the $ep \rightarrow ep\gamma$ process and $d\sigma_{\text{cor}}$ is the correction related to the MD-effect. Then we perform integration over $q_{\perp}$ and $q'_{\perp}$ using the well known equality

$$\int \frac{q_{\perp} e^{iq_{\perp}e}}{q_{\perp}^2 + (1/b)^2} d^2 q_{\perp} = \frac{2\pi i}{b} \frac{q}{\varrho} K_1(q/b)$$

(48)

where $K_n(x)$ denotes the modified Bessel function of third kind with integer index $n$ (McDonald function). As a result, we obtain the correction to the standard cross section in the form (cf. with the approximate formulae (8), (12))

$$d\sigma_{\text{cor}} = d\sigma_C(\omega, E_e, E_\gamma) \frac{\alpha d\omega}{\pi \omega} G(\omega),$$

(49)

where $d\sigma_C$ is the Compton cross section and the function $G(\omega)$ consists of two items

$$G(\omega) = G^{(1)}(\omega) + G^{(2)}(\omega).$$

(50)

The first item represents the previous result for the MD-effect (without taking into account correlations),

$$G^{(1)}(\omega) = \int \frac{d^2 q}{\pi \varrho_m^2} \left[ 1 - \frac{L(q)}{L(0)} \right] K_2^2(\varrho_m/q_m), \quad q_m = \frac{c_\gamma p}{\omega},$$

(51)

where $L(q)$ is defined in (10). Some other useful expressions for $G^{(1)}(\omega)$ as well as its asymptotics can be found in [2]. The second item is directly related to the correlation function (37), (41):

$$G^{(2)}(\omega) = - \int \frac{d^2 q}{\pi \varrho_m^2} \frac{n_e(q)}{L_{ep}} C_\omega (r_{\perp} + \varrho, r'_{\perp} + \varrho) \frac{(r_{\perp} r'_{\perp})}{r_{\perp} r'_{\perp}} K_1(r_{\perp}/q_m)K_1(r'_{\perp}/q_m) d^2 r_{\perp} d^2 r'_{\perp}.$$ 

(52)

Note that the main contribution to $G^{(1)}(\omega)$ (51) is given by the region of large impact parameters (11) while the main contribution to $G^{(2)}(\omega)$ is given by the region $\varrho \sim \sigma_{\perp}$.

The quantity $d\sigma_C d\omega/\omega$ in (49) can be expressed via the energy $E_\gamma$ and the emission angle $\theta_\gamma$ of the final photon as follows (taking into account relation (11))

$$d\sigma_C = \frac{\alpha d\omega}{\pi \omega} = 2\alpha r_e^2 \frac{dy}{y (1 + z)^2} F(y, z), \quad F(y, z) = 2(1 - y) \frac{1 + z^2}{(1 + z)^2} + y^2,$$

(53)

where

$$y = \frac{E_\gamma}{E_e}, \quad z = (\theta_\gamma \gamma_e)^2, \quad r_e = \frac{e^2}{m_e c^2}.$$ 

(54)

6 CRITICAL REMARKS ABOUT RESULTS of Refs. [17, 18]

We derive the final expression for the incoherent contribution from general equations (19), (20) and (25) as a simple consequence of natural assumptions about the particle distribution in a proton bunch. It is useful to rewrite these equations in the form convenient for comparison with the corresponding equations in [17, 18]. To do this, we note that the Compton cross section $d\sigma_C \propto |eM_{\text{Compton}}|^2$ where $eM_{\text{Compton}}$ is the amplitude
of the Compton scattering for the EP with the polarization vector \( \mathbf{e} = \mathbf{E}_\omega (\varrho) / |\mathbf{E}_\omega (\varrho)| \). Therefore, the number of events in a given collisions of beams is proportional to \( |M|^2 \), where

\[
M = \mathbf{E}_\omega (\varrho) M_{\text{Compton}}
\]  

(55)

is related to the probability amplitude of the process. Further, we use Eqs. (24) and (48) and present the collective field of the proton bunch \( \mathbf{E}_\omega (\varrho) \) as a sum of fields of all protons:

\[
\mathbf{E}_\omega (\varrho) = \sum_{a=1}^{N_p} \mathbf{E}_\omega^{(a)} (\varrho), \quad \mathbf{E}_\omega^{(a)} (\varrho) = \frac{2e}{c \gamma_m} \mathbf{g}_a' (\varrho' / \varrho_m) K_1 (\varrho' / \varrho_m) e^{-i \omega z_a / c},
\]  

(56)

where \( \mathbf{g}_a' = \varrho - \varrho_a \) is the impact parameter between the electron and the \( a \)-th proton and the parameter \( \varrho_m = \gamma_p c / \omega \) is the radius of the “disc” of EP (see Fig. 3).

As a consequence, the amplitude \( M \) is the sum

\[
M = \sum_{a=1}^{N_p} m_a e^{-i \omega z_a / c}, \quad m_a = \frac{2e}{c \gamma_m} K_1 (\varrho' / \varrho_m) \frac{\mathbf{g}_a'}{\varrho_a'} M_{\text{Compton}},
\]  

(57)

where the item \( m_a \exp (-i \omega z_a / c) \) is the contribution to \( M \) related to the interaction of the electron with the \( a \)-th proton, while \( |M|^2 \) can be presented as the double sum

\[
|M|^2 = \sum_{a,b} m_a m_b^* e^{-i \omega (z_a - z_b) / c}.
\]  

(58)

We split this sum into sum with \( a = b \) and sum with \( a \neq b \):

\[
|M|^2 = \Sigma_1 + \Sigma_2, \quad \Sigma_1 = \sum_a |m_a|^2, \quad \Sigma_2 = \sum_{a \neq b} m_a m_b^* e^{-i \omega (z_a - z_b) / c}.
\]  

(59)

Equations (57)—(59) can be considered as the same starting formulae in our approach, based on the equivalent photon approximation, and in approach of [17, 18], based on the operator quasi-classical method. However, further calculations are quite different. For simplicity, below we consider the case when we can neglect the correlations between the particle coordinates in the proton bunch.

In our approach, the number of events is proportional to \( |M|^2 \) averaged over collisions of beams, i.e.

\[
dN \propto \langle |M|^2 \rangle = \langle \Sigma_1 \rangle + \langle \Sigma_2 \rangle
\]  

(60)

In the considered region of large frequencies [29], the item \( \langle \Sigma_2 \rangle \), corresponding to the coherent contribution, vanishes,

\[
\langle \Sigma_2 \rangle \ll \langle \Sigma_1 \rangle.
\]  

(61)

Since \( m_a \) does not depend on the longitudinal coordinate \( z_a \), the average value of \( |m_a|^2 \) is determined by the transverse average density of the proton bunch (15),

\[
\langle |m_a|^2 \rangle = \int |m_a|^2 n_p(|r_a|) d^2 r_a = \int |m_a|^2 n_p(|\varrho_a|) d^2 \varrho_a,
\]  

(62)

and it does not depend on index \( a \). Therefore, the item

\[
\langle \Sigma_1 \rangle = N_p \langle |m_a|^2 \rangle
\]  

(63)
leads to the correction, corresponding to $G^{(1)}$ in (59)–(61).

Authors of [17, 18] as the first step had performed averaging over transverse coordinates of the protons. Certainly, after that they get the same expression for $\Sigma_1$ as in (63). For $\Sigma_2$ they had obtained the following expression

$$
\langle \Sigma_2 \rangle_\perp = |\langle m_a \rangle_\perp|^2 Z, \quad Z = \sum_{a \neq b} e^{-i\omega(z_a - z_b)/c},
$$

(64)

where

$$
\langle m_a \rangle_\perp = \int m_a \frac{n_p(\theta_a)}{N_p} d^2 \theta_a.
$$

(65)

The principal mistake in [18] consists in the incorrect calculation of $Z$. It is not difficult to understand its true behavior in the considered region of large frequencies (29). The quantity $Z$ fluctuates near zero for various sets of coordinates $\{z_a\} = z_1, z_2, ..., z_{N_p}$, corresponding to various collisions of beams, and after averaging over many collisions one obtains the estimate (61). This natural behavior of $Z$ is confirmed by numerical calculations given below.

When calculating $Z$, the authors of [18] add and subtract the items with $a = b$, as a consequence,

$$
Z = J - N_p, \quad J = \sum_{a,b} e^{-i\omega(z_a - z_b)/c} = \left| \sum_a e^{-i\omega z_a/c} \right|^2.
$$

(66)

Their next step consists in replacement the sum $J$ by the integral

$$
J \to \left| \int e^{-i\omega z/c} n_p(r) d^3 r \right|^2,
$$

(67)

which is negligible in the considered region. In particular, for the Gaussian beam the replacement (67) means the following:

$$
J \to N_p^2 \exp \left[ -\left( \omega l/c \right)^2 \right] \ll N_p
$$

(68)

(cf. (36), (44) and (46)). This estimate leads to a large negative value of

$$
Z = -N_p
$$

(69)

and to

$$
\langle \Sigma_2 \rangle_\perp = -N_p |\langle m_a \rangle_\perp|^2.
$$

(70)

Just expression (70) is a new “subtraction term” derived in [17, 18].

The mistake of [18] consists in replacement (67). This replacement is true for the region of small frequencies $\omega l/c \ll 1$ when $J = N_p^2$ and $Z = -N_p + J = N_p(N_p - 1) \approx N_p^2$, but such a replacing is completely wrong in the considered region of large frequencies (29). To show this, we perform numerical calculation of the sum $J$. For a given collision of beams we can consider a set of the longitudinal proton coordinates $\{z_a\}$ as a set of random quantities with some distribution $w(z)$. We assume below that

$$
w(z) = \frac{1}{\sqrt{2\pi l}} \exp \left( -\frac{z^2}{2l^2} \right).
$$

(71)
Now the sum
\[
\sum_{a=1}^{N_p} e^{-iqa} = C - iS
\]  
with
\[
C = \sum_{a=1}^{N_p} \cos(qz_a), \quad S = \sum_{a=1}^{N_p} \sin(qz_a), \quad q = \omega/c
\]
is also the random quantity as well as
\[
J = C^2 + S^2.
\]
The quantities \(C\) and \(S\) are the sums of large numbers of random items. Therefore, one can expect that they distribute in accordance with the normal law:
\[
dW/dC = \frac{1}{\sqrt{2\pi N_p \Delta c}} \exp \left[ -\frac{(C - N_p \bar{c})^2}{2N_p(\Delta c)^2} \right],
\]
\[
dW/dS = \frac{1}{\sqrt{2\pi N_p s^2}} \exp \left[ -\frac{S^2}{2N_p s^2} \right],
\]
where \(\Delta c = \sqrt{c^2 - \bar{c}^2}\) and
\[
\bar{c} = \frac{\cos(qz)}{\int_{-\infty}^{+\infty} w(z) \cos(qz) dz} = e^{-(ql)^2/2}, \quad s = \sin(qz) = 0,
\]
\[
\bar{c}^2 = \frac{\cos^2(qz)}{\int_{-\infty}^{+\infty} w(z) \cos^2(qz) dz} = \frac{1}{2} \left(1 + e^{-2(ql)^2}\right), \quad \overline{s^2} = \frac{\sin^2(qz)}{\int_{-\infty}^{+\infty} w(z) \sin^2(qz) dz} = \frac{1}{2} \left(1 - e^{-2(ql)^2}\right).
\]
Numerical calculations had been performed using the generator of random numbers from MATLAB. These calculations confirm the above distributions (75). In particular, it can be seen from Fig. 4 and Fig. 5 where the results of numerical calculations for \(ql = 1, \ N_p = 10^2\) and \(ql = 10, \ N_p = 10^3\) are presented for \(10^4\) various sets of \(\{z_a\}\).

Since
\[
\langle C \rangle = N_p \bar{c}, \quad \langle (C - N_p \bar{c})^2 \rangle = N_p (\bar{c}^2 - \bar{c}^2), \quad \langle S \rangle = 0, \quad \langle S^2 \rangle = N_p \overline{s^2},
\]
we have
\[
\langle J \rangle = N_p + N_p (N_p - 1) \bar{c}^2, \quad \langle Z \rangle = N_p (N_p - 1) \bar{c}^2 \approx N_p^2 \exp \left[-(\omega l/c)^2\right].
\]
Moreover, taking into account that in the considered case \((N_p \bar{c})^2 \ll N_p\), we find that (in contrast to (68), (69))
\[
\langle J \rangle = N_p, \quad |\langle Z \rangle| \ll N_p.
\]
The distribution of the random quantity \(J\) becomes very simple at \(ql \gg 1\):
\[
dW/dJ = \frac{1}{N_p} e^{-J/N_p}.
\]
The results of numerical calculations, presented on Fig. 6, confirm Eq. (80). It should be noted that distribution (80) is rather wide,
\[
\Delta J = \sqrt{\langle J^2 \rangle - \langle J \rangle^2} = N_p,
\]
therefore, the averaging over many various sets of \( \{z_a\} \) is necessary to obtain the stable result for \( \langle J \rangle \).

This consideration shows that the effect, derived in \([17, 18]\), is absent just in the region, discussed in these papers. At the end of this section we reconsider the experiments analyzed in paper \([17]\).

The HERA experiment \([12]\). In this case \( E_c = 27.5 \) GeV and \( l = 8.5 \) cm, therefore, \( E_c = 27 \) keV. For the observed photon energies \( E_\gamma = 2 \div 8 \) GeV, the parameter

\[
\frac{\omega l}{c} \sim \frac{E_\gamma}{E_c} > 10^4,
\]

and the coherent contribution is completely negligible. Therefore, the new correction to the previous results on the level of 10 \%, obtained in \([17]\), is, in fact, absent.

The VEPP-4 experiment \([1]\). In this case \( E_e = 1.84 \) GeV and \( l = 3 \) cm, therefore, \( E_c = 0.34 \) keV. For the observed photon energies \( E_\gamma \gtrsim 1 \) MeV, the parameter

\[
\frac{\omega l}{c} \sim \frac{E_\gamma}{E_c} > 10^3,
\]

and the coherent contribution is completely negligible.

The case of a “typical linear collider” with \( E_e = 500 \) GeV and \( E_\gamma = E_e/1000 \). This example, considered in paper \([17]\), is irrelevant for the discussed problem, since the coherent radiation (called in this case beamstrahlung) at a typical linear collider absolutely dominates in this very region over the ordinary incoherent bremsstrahlung — see, for example, the TESLA project \([23]\) and Sect. 3 of \([18]\).

7 CONCLUDING REMARKS

In the present paper we had performed analysis of the coherent and incoherent contributions to the bremsstrahlung in conditions (29) when the incoherent contribution dominates but large impact parameters give an essential contribution to the standard cross section. In this conditions the known correction (51) to the standard cross section is determined by the transverse distribution of particles in the beams.

We take into account correlations of particles in the beam. The corresponding correction to the standard cross section is given by Eq. (52) and it is determined by correlations of particles in the transverse as well as in longitudinal coordinates.

Through the paper we consider MD-effect in bremsstrahlung. The MD-effect for the \( e^+e^- \) pair production (for example, in the reaction \( \gamma e \rightarrow e^+e^-e^- \)) can be considered in the same manner — for detail see Sect 7.1 from review \([2]\).

We had shown that replacing the sum \( J \) (66) by the integral (67) in conditions (29) is incorrect. As a consequence, papers \([17, 18]\) are incorrect as well.

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APPENDIX. AVERAGE PRODUCT OF PARTICLE DENSITIES IN THE BEAM

In the classical limit [21] the average product of particle densities in the proton beam is given by the double sum over protons in the beam

\[ S(r, r') = \sum_{a,b=1}^{N_p} \langle \delta(r - r_a) \delta(r' - r_b) \rangle. \quad (84) \]

We split this expression into sum with \( a = b \) and sum with \( a \neq b \):

\[ S(r, r') = S_1 + S_2, \quad S_1 = \delta(r - r') \sum_a \langle \delta(r - r_a) \rangle, \quad S_2 = \sum_{a \neq b} \langle \delta(r - r_a) \delta(r' - r_b) \rangle. \quad (85) \]

To perform the averaging, we introduce the average proton distribution function

\[ f(r) = n_p(r)/N_p \]

with the normalization

\[ \int f(r) \, d^3r = 1. \quad (87) \]

It gives

\[ \langle \delta(r - r_a) \rangle = \int \delta(r - r_a) \, f(r_a) \, d^3r_a = f(r). \quad (88) \]

Note, that quantity \( \langle \delta(r - r_a) \rangle \) does not depend on index \( a \) and, therefore,

\[ S_1 = \delta(r - r') N_p f(r) = \delta(r - r') n_p(r). \quad (89) \]

If we can neglect correlations between the particle coordinates, then the average product \( \langle \delta(r - r_a) \delta(r' - r_b) \rangle \) for \( a \neq b \) can be presented as the product of two averaged factors:

\[ \langle \delta(r - r_a) \delta(r' - r_b) \rangle = \langle \delta(r - r_a) \rangle \langle \delta(r' - r_b) \rangle \quad \text{for} \quad a \neq b. \quad (90) \]

As a consequence,

\[ S_2 = \sum_{a \neq b} \langle \delta(r - r_a) \rangle \langle \delta(r' - r_b) \rangle = N_p(N_p - 1) f(r) f(r'). \quad (91) \]

If we do not neglect the correlations between the particle positions, we should introduce the correlation function \( C(r, r') \) as follows

\[ S_2 = \sum_{a \neq b} \langle \delta(r - r_a) \rangle \langle \delta(r' - r_b) \rangle + C(r, r'). \quad (92) \]

In that case we obtain instead of (91) the expression

\[ S_2 = N_p(N_p - 1) f(r) f(r') + C(r, r'). \quad (93) \]

As a result,

\[ S(r, r') = N_p(N_p - 1) f(r) f(r') + \delta(r - r') n_p(r) + C(r, r'). \quad (94) \]

Since in right-hand-side of this equation the first and the second items usually do not compensate each other, we can use approximation

\[ N_p(N_p - 1) \approx N_p^2 \]

and, therefore,

\[ S(r, r') = n_p(r) n_p(r') + \delta(r - r') n_p(r) + C(r, r'). \quad (96) \]
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The curves $s$ and $c$ are given for $dW/dS$ and $dW/dC$, respectively, in accordance with Eqs. (75) and (76) for $q_l = 1$, $N_p = 10^2$. The histograms represent results of numerical calculations for distribution of $S$ and $C$ defined in (73) for $10^4$ various sets of random numbers $\{z_a/l\}$. 

Figure 4: The curves $s$ and $c$ are given for $dW/dS$ and $dW/dC$, respectively, in accordance with Eqs. (75) and (76) for $q_l = 1$, $N_p = 10^2$. The histograms represent results of numerical calculations for distribution of $S$ and $C$ defined in (73) for $10^4$ various sets of random numbers $\{z_a/l\}$. 

[23] The Superconducting electron positron linear collider with an integrated X-ray laser laboratory. Technical design report, DESY 2001-011, ECFA 2001-209, TESLA Report 2001-23, DESY-TESLA-FEL-2001-05, March 2001.
Figure 5: The same as in Fig. 4, but for $q_l = 10$, $N_p = 10^3$ (in this case the curves $s$ and $c$ are practically coincide).

Figure 6: The curve is given for $dW/dJ$ in accordance with Eq. (80) for $q_l = 10$, $N_p = 10^3$. The histogram represents the result of numerical calculations for distribution of $J$ defined in (66), (74) for $10^4$ various sets of random numbers $\{z_{\alpha}/l\}$. 