Non-monotonic hydrodynamic lift force on highly extended polymers near surfaces

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Abstract – The hydrodynamic lift force that polymers experience near boundaries is known to be a crucial element when considering rheological flows of dilute polymer solutions. Here we develop a theory to describe the hydrodynamic lift force on extended polymers flowing near flat surfaces. The lift force is shown to display a non-monotonic character increasing linearly with the distance to the wall \(Z\) in the near-surface regime defined as \(Z < L\), with \(L\) being the contour length of the polymer. At heights \(Z \sim L\) the lift force displays a maximum, and for \(Z > L\) we recover the well-known far-field result in which the force decays as \(Z^{-2}\). Our analytical theory has important implications in understanding adsorption, desorption, and depletion layers of highly extended objects in flow.

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Introduction. – There is a “depletion zone” in flowing solutions of macromolecules which is characterized by a drastic reduction in the concentration of the polymer near a surface [1–6]. Over the past decade it has become clear, after early work by Jendrejack et al., that this behavior is due to a hydrodynamic lift force, which is a direct consequence of polymer stretching under the influence of an external fluid flow [2,3,5–11]. The origin of this lift force arises from a no-slip boundary condition at a solid surface that breaks the fluid flow symmetry around a chain under tension, and the chain feels a resulting upwards force. This force has been widely investigated and applied to a variety of contexts, in particular the description of the aforementioned depletion zone and the case of polymer adsorption [2,3,5–10,12–16]. Ma and Graham were the first to consider this behavior in the context of adsorption, and showed how a dumbbell polymer model could be developed that displayed the qualitative characteristics seen in experiment [2]. This model was further developed by Hoda and Kumar who extended this model to incorporate the effects of electrostatic interactions [9]. Theoretical and simulation work by Sendner and Netz has further investigated this phenomenon in the context of more refined bead-spring models that more explicitly incorporate the correct hydrodynamics [12,14,15]. Despite these advances in the theory, comparison of many of these results with experiment reveal qualitative, but not quantitative agreement. Larson et al. have indicated that the depletion zone near a surface is considerably smaller than predicted by theory, by as much as an order of magnitude [1].

In this letter we provide a general theory that describes the hydrodynamic lift behavior of a polymer at an arbitrary distance from the surface. Our theory provides an explanation of the discrepancies seen between experiment, simulation, and theory for surface depletion, and also an understanding of the lift forces that are relevant for polymer adsorption and desorption. In particular we note that previous models addressing this force, such as the dumbbell model, use an explicit far-field assumption, the validity of which has been questioned by Hernandez-Ortiz et al. and Hoda et al. However, a complete quantitative description of this regime has not yet been developed [9,10]. Here, we provide analytical results that capture the fundamental physical picture in both the far-field and near-surface regimes. Our analytical theory assumes no underlying chain model, and relies only on geometric parameters and flow conditions. It shows for the first time that the lift force on an extended object, like an extended polymer, can become non-monotonic, increasing linearly with the distance from the surface. In the far-field regime

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we recover the well-known results that the force decreases quadratically with the distance. Of further interest is the fact that the crossover from the near-surface to the far-field regime where the lift force is maximal occurs at a height $Z^* \sim L$, where $L$ corresponds to the polymer extension length. Since $L$ can be extremely large, particularly in strong flows, the near-field regime may extend far away from the surface.

**Simulation methods.** – To simulate a single chain in the geometry specified in the paper, we use Brownian Dynamics (BD) simulations. Our polymer is represented by a bead-spring model, which is composed of $2N$ beads $i$ at positions $r_i$ that are held together by a harmonic potential:

$$
\tilde{U}_S = \frac{\kappa}{2} \sum_{i=1}^{2N-1} (\tilde{r}_{i,i+1} - 2)^2.
$$

(1)

The beads interact with other monomers through a Lennard-Jones potential:

$$
\tilde{U}_{LJ} = \tilde{c} \sum_{i,j} \left[ \left( \frac{2}{\tilde{r}_{ij}} \right)^{12} - 2 \left( \frac{2}{\tilde{r}_{ij}} \right)^6 \right].
$$

(2)

There is also a harmonic potential that fixes the polymer at a given height $\tilde{Z}$ from the surface:

$$
\tilde{U}_Z = \left( \sum_{i,j} \tilde{r}_{ij,z}^2 / (2N) - \tilde{Z} \right)^2 / 10,
$$

(3)

where all values designated with a tilde are dimensionless, with distances normalized by the bead radius $a$, energies normalized by $kT$, and times normalized by the characteristic diffusion time $\tau = 6\pi\eta a^3/(kT)$. $\tilde{r}_{ij}$ is the distance between beads $i$ and $j$. $2N$ is the overall number of beads in the chain, $\tilde{k} = 500$ is the bead-bead spring constant, and $\tilde{c}$ is a bead interaction parameter that controls the strength of bead-bead attraction. For this paper we use the value $\tilde{c} = 0.41$, which is typical for a $\theta$-polymer. Beads move through this potential via integration of the Langevin equation:

$$
\frac{\partial}{\partial t} \tilde{r}_i = \tilde{v}_\infty(\tilde{r}_i) - \sum_j \left( \mu_{ij} B \nabla r_j \tilde{U}_{tot}(\tilde{f}) + \nabla r_j \cdot \tilde{D}_{ij} \right) + \xi_i(t),
$$

(4)

where $\tilde{v}_\infty(\tilde{r}_i)$ is the undisturbed solvent flow profile, $\mu_{ij}$ is the Rotne-Prager-Blake mobility matrix [17, 18], $\tilde{U}_{tot} = \tilde{U}_S + \tilde{U}_{LJ} + \tilde{U}_Z$, $\tilde{D}_{ij} = k_B T \mu_{ij}$ is the diffusion tensor, and $\xi_i$ is a random velocity that satisfies $\langle \xi_i \rangle = 0$ and $\langle \xi_i(t) \xi_j(t') \rangle = 2k_B T \mu_{ij} \delta(t-t')$. For $\tilde{v}_\infty(\tilde{r}_i)$, we represent the force loading corresponding to a shear flow with $\tilde{v}_\infty,\tau(\tilde{r}_{i,z}) = \tilde{\gamma}_\tau r_{i,z} - \sum_{j} 2N \tilde{r}_{ij,z}(2N)$, which captures the linear flow increase away from the center of mass of the chain. The Langevin equation is discretized by a time step of $\Delta t = 10^{-4}$, and for a given set of conditions $10^7$ simulation steps are used. For a given value of $\tilde{Z}$, the average lift force $\tilde{F}_L$ on a polymer is given by

$$
\tilde{F}_L = \frac{\sum_{i} 2N \tilde{r}_{ij,z}(2N) - \tilde{Z}}{5}.
$$

(5)

**Theory.** – To develop this model, we only need to consider the geometry of a polymer chain near a surface. Here we provide a description similar to the blob model of a polymer chain under tension by representing the chain as a series of hydrodynamic beads whose length is equal to the ratio of the average length to the average width of the elongated polymer chain $2N = (\Delta X)/(\Delta Z)$ (see fig. 1(a) for a schematic) [19]. $N > 1$ represents a polymer in a strong stretching flow such that the length of the polymer in the $X$-direction is much greater than the $Z$ and $Y$ directions. The polymer can therefore approach the surface without significant entropic penalty up to the point where the height $Z \sim (\Delta Z) \ll N$. Below this height a short-range entropic “excluded-volume” repulsion from the surface will appear. In our model, we do not consider this excluded volume region since it becomes small for large $N$, but it is important to recognize and account for its presence at small $N$. This description of the polymer chain is amenable to the incorporation of molecular theory; for example a straightforward calculation of this geometry in shear flow based on the work of de Gennes yields the relationship $2N = \sqrt{1 + (\tau Z)^2/(4E^2)}$, where $\tau Z = Wi$ is the shear rate rendered dimensionless through the use of the chain relaxation time $\tau Z$ and $E$ is Peterlin’s chain stretching parameter that is 1 at low extension and diverges at high extensions [20]. We introduce a single length scale $2a = (\Delta Z)$ to describe the chain dimension normal to the wall that effectively corresponds to the bead radius. This consequently allows us to represent our system as a bead-spring chain that lies parallel to the surface, in the same spirit as Sendner and Netz (see fig. 1(b) for a schematic) [14, 15]. We can then determine an analytical form for the hydrodynamic lift force of this fully elongated polymer using the Blake-Oseen tensor, which has been shown to provide a good description of hydrodynamics near surfaces in the regime of interest ($Z > a$) [17, 18]. This geometry
conveniently enables us to use a simplified version of the hydrodynamic interaction tensor, and captures the essential physics of the lift force in the limit of high stretching. This approach is also the relevant case for a desorbing or adsorbing polymer, which exhibits extended conformations as it interacts with both the external fluid flow and the surface [15,21]. Understanding a polymer in such a limit is also important when considering the behavior of a polymer in channels that are characterized by small length scales.

The lift force on a polymer, or any other deformable component dispersed in a liquid medium near a surface, is due to the hydrodynamic interactions between individual components and the no-slip boundary condition at the surface. The simplest manifestation of the hydrodynamic interaction tensor is the lift force in the direction normal to the surface, where a no-slip boundary condition must be maintained, Blake introduced an image system that accounted for the aforementioned boundary condition to produce a new mobility tensor $\mu_{ij,B}$ [17].

In our model we consider a finite chain of $2N$ hydrodynamic beads at a distance $Z$ from the surface, each of radius $a$. Using these parameters, we can render all variables dimensionless (rescaled variables denoted by tildes) in terms of distances $a$, energies $kT$, and times $\tau_0 = 6\pi\eta a^3/(kT)$, where $\eta$ is the solvent viscosity. This geometry is shown schematically in fig. 1(b). The tension force $F_T$ along the chain is considered to only be in the $x$-direction, such that the chain is in equilibrium in this configuration. Since we are primarily concerned with the lift forces in the direction normal to the surface, we only need to consider the resulting velocity field in the $z$-direction. The relevant component of the mobility tensor is then just $\mu_{zz}^{\infty,B}(r_x) = (-3rxZ^3)/(2\eta(r_x^2 + 4Z^2)^5/2)$. To evaluate the overall force, one can just define the effect of a given dumbbell centered at the mid-point of the chain that is separated by a distance $4na$ on the velocity field at a dumbbell that is separated by a distance $4ma$, where $n$ and $m$ are just indexing parameters (see fig. 1). The resulting lift force $F_d$ from the $n$ dumbbell on the $m$ dumbbell can be written as

$$F_d(n,m,F_j,\tilde{Z}) = -12\pi\eta a F_T \mu_{zz}^{\infty,B}[\mu_{zz,n,m,B} + \mu_{zz,-n,-m,B}] = 9F_T \tilde{Z}^3 \left[ \frac{(n-m)}{[(n-m)^2 + \tilde{Z}^2]^{5/2}} + \frac{(n+m)}{[(n+m)^2 + \tilde{Z}^2]^{5/2}} \right],$$

where $F_T$ is the magnitude of the force on the $n$-th bead and the rescaled height $\tilde{Z} = Z/a$. To determine the overall lift force due to a given dumbbell on the entire chain $F_{L,d}(n,F_j,\tilde{Z})$ (including the dumbbell itself), we sum over all dumbbells $m = 1$ to $N$:

$$F_{L,d}(n,F_j,\tilde{Z}) = \sum_{m=1}^{N} F_d(n,m,F_j,\tilde{Z})dm = \frac{3F_T \tilde{Z}^3}{16} \left[ (N-n)^3/2 - (N+n)^3/2 \right],$$

where we have used the approximation that $N$ is large enough so that we can replace the sum by a continuous integral from 0 to $N$. To find the total lift force, we perform the summation over all dipoles $n$ and again replace it with an integration:

$$F_L = \sum_{n=1}^{N} F_{L,d} \approx \int_0^{N} F_{L,d}dn = \frac{3\tilde{Z}^3}{16} \int_0^{N} \left[ (N-n)^3/2 - (N+n)^3/2 \right]dn.$$  

Since $F_j$, the overall force on the dumbbell, is the difference of the tension on either side of the bead, we make the replacement of this force with its continuum analogue, $(\partial F_T/\partial n)$. This represents a “kernel” in which the form of the load profile is input into the theory. In this paper we consider a profile that corresponds to shear flows with the extended polymer at a small angle $\theta$; however in principle other profiles may be relevant in other situations. We incorporate the result $(\partial F_T/\partial n) = 6\pi\eta a^3/2\sin(\theta)n$. Performing the integration, we get the result

$$\tilde{F}_{L,shear} = \frac{3}{16}\gamma\tau \sin(\theta) \frac{2N^2\tilde{Z} + \tilde{Z}^3}{\sqrt{4N^2 + \tilde{Z}^2} - \tilde{Z}^2}.$$ 

For the far field, we retain the proper $\tilde{F}_{L,shear} \sim \tilde{Z}^{-2}$ scaling: $\tilde{F}_{L,shear} \approx \frac{3}{16}\gamma\tau \sin(\theta) \frac{N^2}{\tilde{Z}^2}$. It is important to note that this far-field result also matches the $N$-scaling of similar theories, such as the lift force result for rigid rods given by an equivalent approach used by Sendner and Netz, once the $N$-dependence of the $\sin(\theta)$ term is fully accounted for in a given situation [14]. Below a crossover height $\tilde{Z}^* \approx N$, however, there is a near-surface regime where the lift force becomes non-monotonic and the full equation (9) must be used. This theory can now be related to any microscopic theory of choice, with the geometrical parameters $(N, a, \tau)$ corresponding to chain parameters.

**Relation to microscopic theory.** – We can take the de Gennes model of a polymer in flow as a simple example of how to convert a particular chain theory into the geometrical parameters used in the work here presented. A straightforward analysis can show that we obtain the traditional scaling relationship between depletion zone $Z_{dep}$ and Weissenberg number $Wi$ found for the far-field dumbbell model [20]. We use the far-field result for shear
flow, $\tilde{F}_{L, \text{shear}} \approx \frac{3}{2} \gamma \tau \sin (2\theta) N^4 / Z$, and make the following scaling replacements:

$$N \sim Wi, \quad a \sim n^{1/2} b W_i^{1/2}, \quad \sin (2\theta) \sim W_i^{-1}$$

which are appropriate at intermediate values of $W_i$ [20]. $n$ is the degree of polymerization, and $b$ is the monomer size. The translation of $\tau$ to $\tau_Z$ is done by noting that $\tau \sim n a^3 / (kT)$ and $\tau_Z \sim b^3 n^{3/2} / (kT)$. This leads to the result

$$\tilde{F}_L \sim \frac{W_i^2 n^{1/2}}{Z^2},$$

where all distances are now scaled by $b$. The depletion zone length $Z_{\text{dep}}$ is a critical height which can be calculated by counterbalancing the lift force with the diffusive force pushing the polymer back towards the surface, $F_S$. This is a single-value parameter that is typically used in much of the previous literature to succinctly represent the equilibrium concentration distribution near the surface $[2,3,7–10]$. Since there is a no-flux condition at the surface, yielding the condition $\partial c / \partial Z |_{Z=0} = 0$, we can expand the surface concentration profile around $Z = 0$ to get a profile of the form $c(Z) = c(0) + A Z^2 + \cdots$, where $A$ is a constant. If we assume that only the quadratic term in the expansion is the dominant contribution to the equilibrium concentration profile, the resulting diffusive force is thus given by $F_S \sim -\nabla c(Z) \sim -\dot{Z}$. The critical depletion height $Z_{\text{dep}}$ where the diffusive and lift forces balance ($F_S = -\tilde{F}_L$) is thus

$$Z_{\text{dep}} \sim W_i^{2/3} n^{1/2}$$

which is the relationship given in the literature $[2,3,9]$. We add the caveat that these scaling relationships are only appropriate in a finite range of Weissenberg numbers at large distances $Z_{\text{dep}}$ from the surface, and is thus not universal. Separate analysis would have to be performed to obtain this sort of relationship under different conditions; however the underlying geometric theory is completely general.

Order of magnitude calculations can be performed to relate the geometric parameters to the chain parameters. For example, this is done using the relationship (from the above scaling analysis)

$$\frac{2 W_i^2 n^{1/2} b}{Z} \sim \frac{\gamma \tau_{eff} N^4 a}{Z}.$$  \hspace{1cm} (15)

Using the values and relationships $N = 10$, $\gamma \tau_{eff} = 0.01$, and $n^{1/2} b / a \sim W_i^{1/2}$, we obtain the result

$$W_i \sim 5.$$ \hspace{1cm} (16)

This is an order of magnitude result, but it demonstrates that the results presented in this paper are in the relevant shear rate regime to see the effects indicated (lift and depletion).

Results and discussion. – Our results are confirmed by Brownian Dynamics simulations on chains that are extended parallel to a surface. These bead-spring models represent the effective blobs that make up an extended polymer chain. For these simulations, we incorporate the appropriate hydrodynamics by using the Rotne-Prager-Blake tensor and consider the chain with both the absence and presence of thermal fluctuations $[18,22]$. For these simulations we use the force loading profile that corresponds to shear flow, which is given by $\tilde{F}_L \sim -\tilde{F}_S$ for a shear load profile. Solid lines show theoretical results, dashed lines show the effect of bead discretization at low values of $Z$, and dotted lines demonstrate the far-field dumbbell results. Simulation data is also shown, with filled symbols representing data without fluctuations and open symbols representing data including fluctuations. These results collapse onto a single curve (inset) using the scaling $Z \rightarrow Z/N$ and $F \rightarrow F/N^2$.

Fig. 2: (a) Load profile that corresponds to a chain in shear flow, which is placed into eq. (8) to yield eq. (9). It is described by $\bar{\mu} (\partial \tilde{F}_L / \partial n) = \gamma \tau \sin 2\theta (\tilde{F}_{L,x} - \sum_{j} \tilde{F}_{j,x} / (2N))$. (b) Graph of the hydrodynamic contribution to the lift force $\tilde{F}_L$ vs. $Z$ for a shear load profile. Solid lines show theoretical results, dashed lines show the effect of bead discretization at low values of $Z$, and dotted lines demonstrate the far-field dumbbell results. Simulation data is also shown, with filled symbols representing data without fluctuations and open symbols representing data including fluctuations. These results collapse onto a single curve (inset) using the scaling $Z \rightarrow Z/N$ and $F \rightarrow F/N^2$. 

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inset of fig. 2. There are deviations between theory and simulation at small values of \( \tilde{Z} \) due to the discretized representation of the chain in the simulations as opposed to continuous representation of the chain in the theory. Numerical solutions of the discretized version of our theory incorporating only a few terms more accurately reflects the low \( \tilde{Z} \) behavior in our simulations, which is shown as a dashed line in fig. 2(b). We expect that a real chain would more closely resemble the continuous case, as the discretization of the chain is arbitrarily done to develop a convenient simulation model.

The size and shape of the depletion layer near a surface can now be calculated using eq. (9). This is done by describing a potential of mean force \( \tilde{U}_L = -\int_0^{\infty} \tilde{F}_L d\tilde{Z} \). If the only force acting on the polymer in the \( \tilde{Z} \)-direction is the lift force, we can write the concentration \( c(\tilde{Z}) \) of a dilute polymer solution at height \( \tilde{Z} \) from the surface as \( c(\tilde{Z}) = c(\infty) e^{-\tilde{U}_L} \). We plot sample profiles for our model, for the dumbbell model (where we use both the far-field result and the dumbbell result given by Hoda and Kumar) for shear flow in fig. 3(a) [9]. There is a striking difference between these two profiles, with drastic differences appearing due to the significantly lower lift forces close to the surface using our elongated chain model rather than the dumbbell model. We also predict a finite concentration at the wall, at least until sterically or other short-range wall forces become significant, since the effective concentration at the wall, at least until steric or other short-range forces become significant, since the effective lift force potential does not diverge at \( \tilde{Z} = 0 \). These characteristics have been noticed before in experiments, with depletion layer widths that are smaller than the predicted ones by as much as an order of magnitude, and which are often accompanied by finite surface concentrations of polymer [1]. Our model clearly provides an explanation of the origin of the difference between experiment and previous theories, which do not adequately account for near-surface hydrodynamics.

We can also consider flows in a slit, where two walls are present. If the slit height \( H \) is on the same order of magnitude as the polymer contour length \( Na \), we demonstrate extremely large differences between existing theories and our elongated chain model. Since we have introduced an expression for \( \tilde{F}_L(\tilde{Z}, N, \tilde{F}_T) \) that is based off a single image chain, we simply need to consider an infinite series of image chains. For a slit of height \( H \), this becomes \( \tilde{F}_{L, \text{ slit}} = \sum_{j=0}^{\infty} \tilde{F}_L((j+1)H/a + \tilde{Z}, N, \tilde{F}_T) - \tilde{F}_L((j)H/a + \tilde{Z}, N, \tilde{F}_T) \). We plot a sample distribution function in a slit for our chain elongation model and both the far-field and non-far-field dumbbell models in fig. 3(b), where the far-field behavior is held constant. While the two dumbbell cases are essentially identical in this regime, there is a marked difference in the distribution function that results from considering the elongated chain geometry.

These depletion layer calculations emphasize where our elongated-chain theory provides a meaningful improvement over dumbbell-based models, which is under high-shear and near-surface conditions. Higher shear rates result in a longer effective chain, which increases the height \( \tilde{Z}^* \) where there is a crossover from the near-surface to the far-field regimes. This is demonstrated schematically in fig. 3(c), which indicates the extent of the near-surface regime as a function of flow rate (shown here in terms of the Weissenberg number \( Wi = \gamma \tau Z \), for ease of comparison with traditional literature).

Conclusions. – In summary, we have developed an analytical expression for the lift force on a polymer near a surface that significantly improves upon existing
theories by accounting for extended geometries. Computer simulations incorporating the hydrodynamic forces were performed and confirmed these analytical expressions, which give rise to the appearance of non-monotonic lift force behavior. This non-monotonic lift force behavior appears in a near-surface, highly extended regime, and it is not captured by a dumbbell theory. Furthermore, our results have important implications for slit flows in microfluidics and polymer adsorption, where the distance of the polymer from the surface and its contour length are on the same order of magnitude [15]. Figure 3(b) suggests that this result could be verified experimentally using the visualization of the cross-channel distribution fluorescently labelled polymers by applying strong shear flows in narrow channels where this effect is particularly large. Cross-slit flows may also provide a tool to directly examine the lift force on individual polymers.

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REFERENCES

[1] Fang Lin, Hsieh Chih-Chen and Larson Ronald G., Macromolecules, 40 (2007) 8490.
[2] Ma H. B. and Graham M. D., Phys. Fluids, 17 (2005) 083103.
[3] Jendrejack R. M., Schwartz D. C., de Pablo J. J. and Graham M. D., J. Chem. Phys., 120 (2004) 2513.
[4] Criado-Sancho M., Jou D., Del Castillo L. F. and Casas-Vazquez J., Polymer, 41 (2000) 8425.
[5] Chen Y. L., Graham M. D., de Pablo J. J., Jo K. and Schwartz D. C., Macromolecules, 38 (2005) 6680.
[6] Jo Kyubong, Chen Yeng-Long, de Pablo Juan J. and Schwartz David C., Lab Chip, 9 (2009) 2348.
[7] Hoda Nazish and Kumar Satish, Langmuir, 23 (2007) 1741.
[8] Jendrejack R. M., de Pablo J. J. and Graham M. D., J. Chem. Phys., 116 (2002) 7752.
[9] Hoda Nazish and Kumar Satish, J. Rheol., 51 (2007) 799.
[10] Hernandez-Ortiz Juan P., Ma Hongbo, de Pablo Juan J. and Graham Michael D., Phys. Fluids, 18 (2006) 123101.
[11] Jendrejack R. M., Dimalanta E. T., Schwartz D. C., Graham M. D. and de Pablo J. J., Phys. Rev. Lett., 91 (2003) 038102.
[12] Sendner C. and Netz R. R., EPL, 79 (2007) 58004.
[13] Hoda Nazish and Kumar Satish, J. Chem. Phys., 127 (2007) 234902.
[14] Sendner C. and Netz R. R., EPL, 81 (2008) 54006.
[15] Serr A., Sendner C., Müller F., Einert T. R. and Netz R. R., EPL, 92 (2010) 38002.
[16] Watari N., Makino M., Kikuchi N., Larson R. G. and Doi M., J. Chem. Phys., 126 (2007) 094902.
[17] Blake J. R. and Chwang A. T., J. Eng. Math., 8 (1974) 23.
[18] Kim S. and Karrila S. J., Hydrodynamics: Principles and Selected Applications (Butterworth-Heinemann) 1991.
[19] de Gennes Pierre G., Scaling Concepts in Polymer Physics (Cornell Univ. Press) 1979.
[20] de Gennes P. G., J. Chem. Phys., 60 (1974) 5030.
[21] Ladoux B. and Doyle P. S., Europhys. Lett., 52 (2000) 511.
[22] Rotne J. and Prager S., J. Chem. Phys., 50 (1969) 4831.