Differential Geometry on Linear Quantum Groups

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Abstract

An exterior derivative, inner derivation, and Lie derivative are introduced on the quantum group $GL_q(N)$. $SL_q(N)$ is then obtained by constructing matrices with determinant unity, and the induced calculus is found.

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1 Introduction

The general theory of differential calculi on quantum groups is due to Woronowicz [1], and a number of interesting papers have been written since (cf. [2, 3]). In this paper, we present an explicit formulation of the differential geometry on the quantum groups $GL_q(N)$ and $SL_q(N)$. We will show how a calculus incorporating a closed algebra of derivations can be introduced on these quantum groups. We approach the subject from a more physics-oriented perspective, presenting commutation relations between the various matrix elements, differential operators, forms, etc. The Hopf algebraic nature of the subject is deemphasized; there are occasional references to such objects as “antipode”, but in general the focus is on a formulation which is suitable for computations. (A treatment of some of the material contained here using the more mathematical structures of quasitriangular Hopf algebras will be presented in a forthcoming paper [4].)

Many of the conventions and notations we use can be found in [5], as well as other references herein.

2 $GL_q(N)$

2.1 The Quantum Group $GL_q(N)$

The R-matrix for $GL_q(N)$, which of course satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

(1)

is given in [3] as

$$R_{12} = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \lambda \sum_{i > j} e_{ij} \otimes e_{ji},$$

(2)

where $i, j = 1, \ldots, N$, $\lambda = q - q^{-1}$, and $e_{ij}$ is the $N \times N$ unit matrix with lone nonzero element at $(i, j)$. This matrix satisfies the characteristic equation

$$\hat{R}_{12}^2 - \lambda \hat{R}_{12} - 1 = 0,$$

(3)
where $\hat{R}_{12}$ is defined by
\begin{equation}
(\hat{R}_{12})^{ij}_{kl} = (P_{12}R_{12})^{ij}_{kl} = (R_{12})^{ji}_{kl},
\end{equation}
with $P_{12}$ being the permutation matrix which exchanges spaces 1 and 2, as above. The defining representation for the quantum group $GL_q(N)$ is given by matrices $A$ satisfying
\begin{equation}
R_{12}A_1A_2 = A_2A_1R_{12}.
\end{equation}
The determinant of such a matrix can be introduced in the following way: let $\{x^i\}$ be the $N$ coordinates of the quantum hyperplane whose transformation group is $GL_q(N)$, and let $\{dx^i\}$ be the associated differentials. The commutation relations between these quantities which are preserved under such transformations are
\begin{align}
x^j x^i &= q^{-1}(R_{12})^{ij}_{kl}x^k x^l, \\
x^j dx^i &= q(R_{12})^{ij}_{kl}dx^k x^l, \\
dx^j dx^i &= -q(R_{12})^{ij}_{kl}dx^k dx^l.
\end{align}
These commutation relations allow us to define the Levi-Civita tensor as
\begin{equation}
dx^i_1 dx^i_2 \ldots dx^i_N = \epsilon^i_1^{i_2 \ldots i_N} dx^1 dx^2 \ldots dx^N.
\end{equation}
This tensor satisfies the relations
\begin{align}
(R_{0N} \ldots R_{02}R_{01})^{i_0i_1i_2 \ldots i_N}_{j_0j_1j_2 \ldots j_N} \epsilon^i_{q}^{j_1j_2 \ldots j_N} &= \\
(R_{10}R_{20} \ldots R_{N0})^{i_0i_1i_2 \ldots i_N}_{j_0j_1j_2 \ldots j_N} \epsilon^i_{q}^{j_1j_2 \ldots j_N} &= q^{j_{i_0}^{i_1i_2 \ldots i_N}}.
\end{align}
The quantum determinant of $A$, $det_q A$, is defined through the relation
\begin{equation}
A^{i_1}_{j_1} \ldots A^{i_N}_{j_N} \epsilon^j_{q}^{j_1 \ldots j_N} = \epsilon^i_{q}^{i_1 \ldots i_N} det_q A,
\end{equation}
and this definition, together with (3), makes $det_q A$ commute with all elements of $A$. 

2
2.2 The Calculus for $GL_q(N)$

Following the approach of [9], we introduce the exterior derivative $d$ on $GL_q(N)$ as a left action which maps $k$-forms to $(k + 1)$-forms and satisfies the same properties as the undeformed exterior derivative, i.e. it is a linear operator, and for any forms $f$ and $g$,

$$d^2g = 0, \quad d(fg) = (df)g + (-1)^k f(dg),$$  \hspace{1cm} \text{(12)}

where $f$ is a $k$-form. Functions of the elements of $A$ are taken as 0-forms, and we take the elements of $dA$ to be a basis for 1-forms; the commutation relations (first found in [10, 11, 12, 13] but put in R-matrix notation in [14, 15, 16]) are

$$dA_1A_2 = R_{12}^{-1}A_2dA_1R_{21}^{-1}, \quad dA_1dA_2 + R_{12}^{-1}dA_2dA_1R_{21}^{-1} = 0.$$  \hspace{1cm} \text{(13)}

These are consistent with (3) and (5), of course. (Alternatively, we could have taken

$$A_1dA_2 = R_{12}^{-1}dA_2A_1R_{21}^{-1}$$  \hspace{1cm} \text{(14)}

which is also consistent with (3) and (5), and gives the same $dA - dA$ commutation relations as above.)

\textbf{Aside:} It is convenient to introduce the numerical matrix $D$ given by

$$D \equiv q^{2(N-1)}tr_2(P_{12} \tilde{R}_{12}) = diag(1, q^2, \ldots, q^{2(N-1)}),$$  \hspace{1cm} \text{(15)}

where $\tilde{K} = [(K^{tI})^{-1}]^{tI}$ for any $N^2 \times N^2$ matrix $K$. (Here $tr_I$ and $t_I$ denote tracing and transposing with respect to the $I$th pair of indices, respectively.) The definition of the $D$-matrix, together with (4), gives

$$(D^{-1})^tA^tD^tS(A)^t = S(A)^t(D^{-1})^tA^tD^t = 1,$$  \hspace{1cm} \text{(16)}

where $S(A)$, the antipode of $A$, is simply $A^{-1}$. A consequence of this relation is that

$$S^2(A) = DAD^{-1}.$$  \hspace{1cm} \text{(17)}
(5) and (13) together imply the identities
\[ \tilde{R}_{12} = D_1^{-1}R_{12}^{-1}D_1, \quad \tilde{R}_{21} = D_1R_{21}^{-1}D_1^{-1}. \quad (18) \]

All of the above implies two important results: if \( M \) is an \( N \times N \) matrix, then
\[
tr_1(D_1^{-1}R_{12}^{-1}M_1R_{12})^i_j = tr_1(D_1^{-1}R_{21}^{-1}M_1R_{21})^i_j = tr(D^{-1}M)\delta^i_j, \quad (19)
\]
and if the elements of \( M \) commute with the elements of \( A \),
\[
tr(D^{-1}S(A)MA) = tr(D^{-1}M). \quad (20)
\]
For this reason, \( tr(D^{-1}M) \) is called the invariant trace of \( M \).

(5) and (13) go into themselves under the right coaction \( A \mapsto AA' \) and the left coaction \( A \mapsto A'A \), where \( A' \) is a constant (i.e. \( dA'=0 \) \( GL_q(N) \) matrix satisfying (5), and whose elements commute with those of \( A \) and \( dA \). \( d \) is invariant under both these coactions. However, the Cartan-Maurer form
\[
\Omega \equiv S(A)dA \quad (21)
\]
is left-invariant and right-covariant i.e. \( \Omega \mapsto \Omega \) and \( \Omega \mapsto S(A')\Omega A' \) under the respective coactions above. (We could have chosen the left-covariant, right-invariant form \( dA S(A) \) instead.) This allows us to define the left- and right-invariant 1-form \( \xi \) as
\[
\xi \equiv -q^{2N-1}tr(D^{-1}\Omega). \quad (22)
\]
\( \Omega \) satisfies the following equations due to (5) and (13):
\[
\Omega_1 A_2 = A_2 R_{12}^{-1} \Omega_1 R_{21}^{-1}, \quad (23)
\]
\[
\Omega_1 dA_2 + dA_2 R_{12}^{-1} \Omega_1 R_{12} = 0, \quad \Omega_1 R_{21}^{-1} \Omega_2 R_{21} + R_{21}^{-1} \Omega_2 R_{12}^{-1} \Omega_1 = 0.
\]
Using these, (3), and the definition of $D$,

$$dA = \lambda^{-1}[\xi, A], \quad d\Omega = -\Omega^2 = \lambda^{-1}\{\xi, \Omega\}, \quad (24)$$

so $\xi$ is in fact the generator of the exterior derivative. These imply that the exterior derivative of any form $f$ is given by

$$df = \lambda^{-1}[\xi, f]_\pm \quad (25)$$

(where $[,]_\pm$ is a commutator for even-forms, an anticommutator for odd-forms). $\det_q A$ is a 0-form, and the above equations imply that

$$\Omega(\det_q A) = q^{-2}(\det_q A)\Omega,$$

$$d(\det_q A) = -q^{-1}(\det_q A)\xi = -q\xi(\det_q A). \quad (26)$$

(A consequence of these equations is that both $d\xi$ and $\xi^2$ vanish.) The elements of $\Omega$ form a linearly independent basis for 1-forms, and we shall use them instead of the elements of $dA$ from now on.

We now introduce the inner derivation, which we take to be a left action mapping $k$-forms to $(k-1)$-forms. Its action on the $N^2$ elements of $A$ and $\Omega$ is given by introducing $N^2$ vector fields $X^i_j$, and the associated $N^2$ inner derivations are the entries in the matrix $i_X$ whose elements are

$$(i_X)^i_j = i_{X^i_j}. \quad (27)$$

$i_X$ must act on 0- and 1-forms in a way preserving the commutation relations (3) and (23); the appropriate actions are

$$i_{X_1} A_2 = A_2 R_{21} i_{X_1} R_{12},$$

$$R_{21} i_{X_1} R_{12} \Omega_2 + \Omega_2 R_{21} i_{X_1} R_{12} = \frac{1 - R_{21} R_{12}}{\lambda}. \quad (28)$$

This last relation implies

$$i_X \xi + \xi i_X = I. \quad (29)$$
(Notice that by using (3), $\frac{1-R_{21}R_{12}}{\lambda}$ could be replaced by $-\tilde{R}_{12}$ if so desired.)

On $det_q A$, the inner derivation acts as

$$i_X(det_q A) = q^2(det_q A)i_X.$$  \hspace{1cm} (30)

The commutation relations between the inner derivation matrices are similar to the ones for $\Omega$:

$$R_{12}^{-1}i_{X_1}R_{12}i_{X_2} + i_{X_2}R_{21}i_{X_1}R_{12} = 0.$$  \hspace{1cm} (31)

Equations (28) imply that $i_X$ is left-invariant and right-covariant under the respective coactions on $A$.

We may now introduce the Lie derivative matrix $L_X$ in the same way as in the classical theory, i.e. a left action taking $k$-forms to $k$-forms given by

$$L_X \equiv i_Xd + di_X,$$  \hspace{1cm} (32)

where $L_X$ is a matrix with elements $L_{X_{ij}}$ which by definition transforms in the same way as $i_X$ does. The equations already given for $d$ and $i_X$ imply a whole host of relations involving $L_X$:

$$LXd = dL_X,$$

$$R_{21}L_{X_1}R_{12}i_{X_2} - i_{X_2}R_{21}L_{X_1}R_{12} = \lambda^{-1}(R_{21}R_{12}i_{X_2} - i_{X_2}R_{21}R_{12}),$$

$$R_{21}L_{X_1}R_{12}L_{X_2} - L_{X_2}R_{21}L_{X_1}R_{12} = \lambda^{-1}(R_{21}R_{12}L_{X_2} - L_{X_2}R_{21}R_{12}),$$

$$L_{X_1}A_2 = A_2R_{21}L_{X_1}R_{12} + A_2\left(\frac{1 - R_{21}R_{12}}{\lambda}\right),$$

$$R_{21}L_{X_1}R_{12}\Omega_2 - \Omega_2R_{21}L_{X_1}R_{12} = \lambda^{-1}(R_{21}R_{12}\Omega_2 - \Omega_2R_{21}R_{12}),$$

$$L_{X}\xi = \xi L_X,$$  \hspace{1cm} (33)

and for the determinant,

$$L_X(det_q A) = q^2(det_q A)L_X - q(det_q A).$$  \hspace{1cm} (34)

Many of these relations take a much simpler form if we introduce the Lie derivative valued operator $Y$ given by

$$Y = 1 - \lambda L_X,$$  \hspace{1cm} (35)
which, of course, has the same transformation properties as \( L_X \). Using this, we obtain

\[
Y d = d Y,
\]

\[
R_{21} Y_1 R_{12} i_{X_2} = i_{X_2} R_{21} Y_1 R_{12},
\]

\[
R_{21} Y_1 R_{12} Y_2 = Y_2 R_{21} Y_1 R_{12},
\]

\[
Y_1 A_2 = A_2 R_{21} Y_1 R_{12},
\]

\[
R_{21} Y_1 R_{12} \Omega_2 = \Omega_2 R_{21} Y_1 R_{12},
\]

\[
Y_\xi = \xi Y,
\]

(36)

and

\[
Y (\det_q A) = q^2 (\det_q A) Y.
\]

(37)

\( Y \) is useful for more than making pretty equations. Since its leading term is unity, it is invertible. More importantly, we can define a quantity \( \text{Det} Y \), which we identify as the determinant of \( Y \), satisfying

\[
Y (\text{Det} Y) = (\text{Det} Y) Y.
\]

(38)

This quantity is defined through [4]

\[
(Y_{1_N^1}^{(1)} \ldots Y_{1_N^N}^{(N)})^{i_1 \ldots i_N}_{j_1 \ldots j_N} \epsilon^{j_1 \ldots j_N}_q = \epsilon^{i_1 \ldots i_N}_q \text{Det} Y,
\]

(39)

where

\[
Y^{(k)}_{1_N} = \begin{cases} 
(R_{k,N} \ldots R_{k(k+1)})^{-1} Y_k (R_{k,N} \ldots R_{k(k+1)}) & \text{for } k = 1, \ldots, N - 1, \\
Y_N & \text{for } k = N.
\end{cases}
\]

(40)

This determinant is invariant under transformations of \( Y \) (i.e. \( Y \mapsto Y \) for \( A \mapsto AA' \) and \( Y \mapsto S(A')Y A' \) for \( A \mapsto A' A \), with \( Y \) and \( A' \) having commuting elements), and satisfies the following as a consequence of the above equations:

\[
d (\text{Det} Y) = (\text{Det} Y) d,
\]

(36)
\[ (\text{Det} Y)i_X = i_X(\text{Det} Y), \]
\[ (\text{Det} Y)A = q^2 A(\text{Det} Y), \]
\[ (\text{Det} Y)\Omega = \Omega(\text{Det} Y), \]
\[ (\text{Det} Y)\xi = \xi(\text{Det} Y), \]
\[ (\text{Det} Y)(\text{det}_q A) = q^{2N}(\text{det}_q A)(\text{Det} Y). \]  \tag{41}

The above equations for \( \text{Det} Y \) suggest the definition of an operator \( H_0 \) as

\[ \text{Det} Y \equiv q^{2H_0} = 1 + q\lambda[H_0]_q \]  \tag{42}

where

\[ [x]_q = \frac{1 - q^{2x}}{1 - q^2}. \]  \tag{43}

\( H_0 \) commutes with \( Y, d, i_X, \Omega, \) and \( \xi, \) and satisfies

\[ [H_0, A] = A, \quad [H_0, \text{det}_q A] = N(\text{det}_q A). \]  \tag{44}

This operator will be important in the next section.

3 \( SL_q(N) \)

3.1 The Quantum Group \( SL_q(N) \)

There seems to be an obvious way to specify the quantum group \( SL_q(N) \): take the matrix \( A \) and set its determinant to unity. Unfortunately, this doesn’t work. True, \( \text{det}_q A \) as defined in the previous section commutes with the elements of \( A \), but it does not commute with such quantities as \( \Omega \) and \( Y \). Therefore, instead of imposing \( \text{det}_q A = 1 \), we define matrices \( T \) as

\[ T = (\text{det}_q A)^{-\frac{1}{2}} A. \]  \tag{45}

With \( \text{det}_q T \) defined as in \( [11] \), the centrality of \( \text{det}_q A \) automatically gives \( T \) determinant unity. Furthermore, the antipode of \( T \) is also given by \( T^{-1} \).
Therefore, this matrix $T$ is what we identify as an element of the defining representation of $SL_q(N)$, since it also satisfies (5) with $A$ replaced by $T$. However, as we will see in the next section, it becomes convenient to introduce the matrix

$$R_{12} = q^{-\frac{1}{N}} R_{12},$$

(46)

which we identify as the R-matrix for $SL_q(N)$. Thus, we shall write (5) as

$$R_{12} T_1 T_2 = T_2 T_1 R_{12}.$$

(47)

### 3.2 The Calculus for $SL_q(N)$

The exterior derivative on $SL_q(N)$ can be taken to be the same as that introduced on $GL_q(N)$; this is because $T$ is a function of elements of $A$, so its differentials are given by

$$dT = \lambda^{-1} [\xi, T].$$

(48)

Note that this implies that the Cartan-Maurer form $\tilde{\Omega}$ for $SL_q(N)$ is given by

$$\tilde{\Omega} \equiv S(T)dT = q^{\frac{2}{N}} \Omega + q [1/N] q [\xi].$$

(49)

(see (43) for the definition of $[\cdot]_q$.) In the classical limit $q \rightarrow 1$, $\tilde{\Omega}$ is traceless, giving the appropriate reduction from $N^2$ to $N^2 - 1$ independent elements in the Cartan-Maurer matrix 1-form for $SL(N)$.

We have thus found a way to set the determinant of our $SL_q(N)$ matrices to unity; for the calculus of the group, we must do something similar, namely impose a constraint so that the number of independent differential operators is reduced from $N^2$ to $N^2 - 1$. In a way, we have already done this, because (44) and (45) together imply

$$[H_0, T] = 0,$$

(50)

†This relation implies that the matrix of differential forms introduced in (9) is equal to $-q^{2N-1} \Omega$. 

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so that $H_0$ commutes with everything of interest in $SL_q(N)$, i.e. matrices, forms, exterior derivative, etc. Thus, within the context of $SL_q(N)$, $H_0$ is irrelevant, reducing the number of generators from $N^2$ to $N^2 - 1$, as desired. Explicitly, this restriction is accomplished by defining a new Lie derivative valued operator $Z$ by

$$ Z \equiv q^{-\frac{2H_0}{N}}Y, \quad (51) $$

Note that the determinant of $Z$, computed using (39), is unity. This is equivalent to the introduction of a set of $N^2$ “vector fields” $V_{ij}$ through

$$ Z = 1 - \lambda L_X, $$

so that

$$ L_V = L_X + q^{-1}[H_0/N]_{q^{-1}} - q^{-1}\lambda L_X[H_0/N]_{q^{-1}}. \quad (52) $$

The fact that $Det Z = 1$ implies that only $N^2 - 1$ of the elements of $L_V$ are actually independent, which is precisely what we require for $SL_q(N)$. In the classical limit, $H_0 = -tr(L_X)$, so $L_V$ becomes traceless; thus, $V$ contains only $N^2 - 1$ linearly independent vector fields, as we’d expect.

Now that we have obtained all these quantities, we want to find the various relations they satisfy. As a starting point, note that the commutation relations between $\Omega$ and $T$ are given by

$$ \Omega_1 T_2 = q^{\frac{1}{N}}T_2 R_{12}^{-1} \Omega_1 R_{21}^{-1} = T_2 R_{12}^{-1} \Omega_1 R_{21}^{-1}. \quad (53) $$

Here we see the appearance of $R_{12}$, as promised. In fact, there is a general pattern: by using the substitutions $A \to T$, $R_{12} \to R_{12}$, and $L_X \to L_V$, we obtain most of the corresponding relations for $SL_q(N)$. ($\Omega$ remains unchanged, so the last of equations (23) does not have $R_{12}$ in place of $R_{12}$.) $L_V$ satisfies

$$ R_{21} L_{V_1} R_{12} L_{V_2} - L_{V_2} R_{21} L_{V_1} R_{12} = \lambda^{-1}(R_{21} R_{12} L_{V_2} - L_{V_2} R_{21} R_{12}), $$

$$ R_{21} L_{V_1} R_{12} i_X - i_X R_{21} L_{V_1} R_{12} = \lambda^{-1}(R_{21} R_{12} i_X - i_X R_{21} R_{12}). \quad (54) $$

‡When restricted to acting on 0-forms, this operator is identical to the operator $Y$ in [9].
The actions of the various operators on the 0- and 1-forms of $SL_q(N)$ are given by

\begin{align*}
L_{V_1}T_2 &= T_2R_{21}L_{V_1}R_{12} + T_2\left(\frac{1 - R_{21}R_{12}}{\lambda}\right), \\
R_{21}L_{V_1}R_{12}\Omega_2 - \Omega_2R_{21}L_{V_1}R_{12} &= \lambda^{-1}(R_{21}R_{12}\Omega_2 - \Omega_2R_{21}R_{12}), \\
i_{X_1}T_2 &= T_2R_{21}i_{X_1}R_{12}, \\
R_{21}i_{X_1}R_{12}\tilde{\Omega}_2 + \tilde{\Omega}_2R_{21}i_{X_1}R_{12} &= \frac{1 - R_{21}R_{12}}{\lambda}. 
\end{align*}

(55)

As a consequence, $\xi$ satisfies

\begin{equation}
L_V\xi = \xi L_V. 
\end{equation}

(56)

The relations for $Z$ corresponding to (36) are easily obtained by using $L_V = \frac{1 - Z}{\lambda}$ in all of the above equations.

\section{Conclusion}

Many of the relations in this work are not unique to a discussion of $GL_q(N)$ or $SL_q(N)$; for instance, given an R-matrix, the differentials of the quantum hyperplane can be defined and their commutation relations found, and the corresponding Levi-Civita tensor found. This allows the definition of the determinant of a quantum matrix. However, much of this paper is based on the fact that the characteristic equations for $GL_q(N)$ and $SL_q(N)$ are quadratic in $\hat{R}_{12}$. For other quantum groups, the characteristic equation for $\hat{R}_{12}$ may be of higher degree (as in $SO_q(N)$ and $Sp_q(N)$, where the characteristic equations are cubic), in which case relations like (36) will not be consistent with (5). Although others have looked at the calculi of such quantum groups \cite{17}, it remains to be seen whether it is possible to use techniques similar to ours in these cases.

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