A remark on the Laplacian operator which acts on symmetric tensors

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Abstract. More then forty years ago J. H. Samson has defined the Laplacian $\Delta_{\text{sym}}$ acting on the space of symmetric covariant $p$-tensors on an $n$-dimensional Riemannian manifold $(M, g)$. This operator is an analogue of the well known Hodge-de Rham Laplacian $\Delta$ which acts on the space of exterior differential $p$-forms ($1 \leq p \leq n$) on $(M, g)$. In the present paper we will prove that for $n > p = 1$ the operator $\Delta_{\text{sym}}$ is the Yano rough Laplacian and show its spectrum properties on a compact Riemannian manifold.

Key words: Riemannian manifold, second order elliptic differential operator on 1-forms, eigenvalues and eigenforms.

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1. Definitions and notations

Let $(M, g)$ be a compact oriented $C^\infty$-Riemannian manifold of a dimension $n \geq 2$ with the Levi-Civita connection $\nabla$ and let $S^p M$ be a symmetric tensor product of order $p \geq 1$ of a cotangent bundle $T^*M$ of $M$. On the tensor space $S^p M$ on $M$ we have the canonical scalar product $g(\cdot, \cdot)$ and on its $C^\infty$-sections the global scalar product $\langle \varphi, \varphi' \rangle = \int_M \frac{1}{p!} g(\varphi, \varphi') dv$ where $dv$ is the volume element of $(M, g)$.

The covariant derivative $\nabla: C^\infty S^p M \to C^\infty(T^*M \otimes S^p M)$ has the formal adjoint operator $\delta = \nabla^* : C^\infty(T^*M \otimes S^p M) \to C^\infty S^p M$ which is uniquely defined by the formula $\langle \nabla \cdot, \cdot \rangle = \langle \cdot, \delta \cdot \rangle$ (see [1, p. 460]). Furthermore we can define (see also [1, p. 514]) the operator $\delta^* : C^\infty S^p M \to C^\infty S^{p+1} M$ which is the adjoint operator of $\delta : C^\infty S^{p+1} M \to C^\infty S^p M$ with respect to the global product $\langle \cdot, \cdot \rangle$.

More then forty years ago J. H. Samson has defined (see [2]) the Laplacian operator $\Delta_{\text{sym}} = \delta \delta^* - \delta^* \delta : C^\infty S^p M \to C^\infty S^p M$. This operator is an analogue of the well known Hodge-de Rham Laplacian $\Delta : C^\infty \Lambda^p M \to C^\infty \Lambda^p M$ which acts on $C^\infty$-sections of the bundle $\Lambda^p M$ of covariant skew-symmetric tensors of degree $p$ ($1 \leq p \leq n$) on $M$ and is defined by $\Delta = d \delta + \delta d$ for the exterior differential $d : \Lambda^p M \to \Lambda^{p+1} M$ (see [1, p. 34]).

The operator $\Delta_{\text{sym}}$ is studied in the following papers [2]; [3]; [4]; [5] and [6].
This paper is organized as follows. The next section summarizes the basic properties of \( \Delta_{\text{sym}} : C^p S^r M \to C^p S^r M \) for the case \( p = 1 \). Section with the number three expresses our results on infinitesimal conformal and projective transformations. The fourth section of the present paper shows spectrum properties of \( \Delta_{\text{sym}} \) on an \( n \)-dimensional compact Riemannian manifold for the case \( n > p = 1 \). And in the last section we prove a theorem about eigenvalues of the Hodge-de Rham Laplacian \( \Delta \) which acts on closed 1-forms.

2. The Yano rough Laplacian

We proved in [6] that for \( p = 1 \) the Weitzenböck decomposition formula for \( \Delta_{\text{sym}} = \delta \delta^* - \delta^* \delta \) has the form \( \Delta_{\text{sym}} = \delta \nabla - \text{Ric} \) where \( \text{Ric} \) is the Ricci tensor of \((M, g)\) and \( \delta \nabla \) the Bochner rough Laplacian which is also denoted by \( \nabla^* \nabla \) (see [1, p. 54]). Next, thanks to the well-known Weitzenböck decomposition formula \( \Delta = \delta \nabla + \text{Ric} \) for the Hodge-de Rham Laplacian \( \Delta : C^p T^* M \to C^p T^* M \) we concluded that \( \Delta_{\text{sym}} = \Delta - 2 \text{Ric} \). After that, using the equation \( \Delta_{\text{sym}} = \Delta - 2 \text{Ric} \) we can define the differential operator \( \Box : C^p TM \to C^p TM \) such that \( \Box = \Delta - 2 \text{Ric}^* \) for the linear symmetric operator \( \text{Ric}^* \) which is associated with the Ricci tensor \( \text{Ric} \) and defined by the identity \( \text{Ric}(X,Y) = g(\text{Ric}^*X , Y) \) for any \( X,Y \in C^p TM \) (see also [8, p. 40]). In turn, we recall that more then forty years ago the operator \( \Box \) was used by K. Yano (see [8]) for the investigation of local isometric, conformal, affine and projective transformations of compact Riemannian manifolds. Based on the above, we will call \( \Delta_{\text{sym}} \) the Yano rough Laplacian when \( p = 1 \). Hence, the following proposition is true.

**Lemma.** Let \((M, g)\) be an \( n \)-dimensional \((n \geq 2)\) Riemannian manifold. For \( p = 1 \) the Samson Laplacian \( \Delta_{\text{sym}} : C^p S^r M \to C^p S^r M \) is the Yano rough Laplacian.

We recall here that the vector field \( \xi \) on \((M, g)\) is called an infinitesimal harmonic transformation if the one-parameter group of infinitesimal point transformations of \((M, g)\) generated by \( \xi \) consists of harmonic diffeomorphisms (see [6]). In turn, we have proved in [6] that the vector field \( \xi \) is an infinitesimal harmonic transformation on \((M, g)\) if and only if \( \Delta_{\text{sym}} \omega = 0 \) for the 1-form \( \omega \) dual to the vector field \( \xi \) with respect to the metric \( g \), i.e. \( \omega(X) = g(\xi, X) \) for an arbitrary vector field \( X \in C^p TM \). In this case, we adopt the following notation \( \xi := \omega^\# \).

In particular, holomorphic vector fields on nearly Kählerian manifolds (see [9]) and vector fields that transform a Riemannian metrics into Ricci soliton metrics (see [9]) are examples of infinitesimal harmonic transformations. Therefore, all forms which are dual to these vector fields belong to \( \text{Ker} \Delta_{\text{sym}} \).
On the other hand, a vector field $\xi$ is called a *Killing vector field* or, in other words an *infinitesimal isometric transformation* if the one-parameter group of infinitesimal transformations of $(M, g)$ generated by $\xi$ consists of isometric diffeomorphisms. An arbitrary Killing vector field $\xi$ satisfies the condition $\delta^* \omega = 0$ where $\xi := \omega^\mu$. On the other hand, according to the Yano’s theorem (see [8, p. 44]; [10]) a vector field $\xi$ on a compact Riemannian manifold $(M, g)$ is a Killing vector field if and only if $\Delta_{sym} \omega = 0$ and $\delta \omega = 0$. The vector space of 1-forms dual to globally defined Killing vector fields has the finite dimension $k_1(M) \leq \frac{1}{2} n (n + 1)$. The dimension $k_1(M)$ has been named the *first Killing number*. Moreover, we have proved in [7] that the number $k_1(M)$ is a scalar projective invariant of $(M, g)$.

### 3. Conformal Killing and projective Killing 1-forms

A real number $\lambda$, for which there is a form $\omega \in C^\infty T^* M$ (not identically zero) such that $\Delta_{sym} \omega = \lambda \omega$, is called an *eigenvalue* of $\Delta_{sym}$ and the corresponding $\omega \in C^\infty T^* M$ is called an *eigenform* of $\Delta_{sym}$ corresponding to $\lambda$. Next, we consider two examples of eigenforms of $\Delta_{sym}$.

**Conformal Killing vector fields** can be considered as a natural generalization of Killing vector fields. They are also called *infinitesimal conformal transformations* because any conformal Killing vector $\xi$ generates a local one-parameter group of conformal diffeomorphisms of $(M, g)$.

Consider an $n$-dimensional compact orientable Riemannian manifold $(M, g)$. Lichnerowicz has shown (see [8, p. 47]) that a necessary and sufficient condition for $\xi$ to be a *conformal Killing vector field* on $(M, g)$ is

$$\Delta_{sym} \omega + (1 - 2/n) \delta^* \delta \omega = 0 \quad (3.1)$$

for the 1-form $\omega$ dual to the vector field $\xi$ with respect to the metric $g$. This 1-form is called *conformal Killing form* (see, for example, [7]). Let the eigenform $\omega$ of $\Delta_{sym}$ be a conformal Killing form on an $n$-dimensional ($n > 2$) compact and oriented Riemannian manifold $(M, g)$ then

$$\lambda \langle \omega, \omega \rangle = -n^{-1}(n - 2) \langle \omega, \delta^* \delta \omega \rangle = -n^{-1}(n - 2) \langle \delta \omega, \delta \omega \rangle.$$

From these equations, we deduce the following inequality

$$\lambda = -\left(1 - 2/n\right) \frac{\langle \delta \omega, \delta \omega \rangle}{\langle \omega, \omega \rangle} \leq 0.$$

For the second example we consider a *projective Killing vector field* or, in other words an *infinitesimal projective transformation* (see [8, p. 45]) which satisfies the equation $\Delta_{sym} \omega = 2(n + 1)^{-1} \delta^* \delta \omega$ for the form $\omega$ dual to $\xi$. This 1-form will be called *projective Killing form*. Let the eigenform $\omega$ of $\Delta_{sym}$ be a projective Killing form on a compact and oriented Riemannian manifold $(M, g)$. In this case, we have
and consequently the following inequality holds
\[ \lambda = 2(n+1)^{-1} \frac{\langle \delta \omega, \delta \omega \rangle}{\langle \omega, \omega \rangle} \geq 0. \]

4. Spectral properties of the Yano rough Laplacian

We recall that all nonzero eigenforms corresponding to a fixed eigenvalue \( \lambda \) form a vector subspace of \( \mathcal{C}^T \mathcal{V} \) denoted by \( V_\lambda (\mathcal{M}) \) and called the eigenspace corresponding to the eigenvalue \( \lambda \).

The following theorem about eigenvalues of \( \Delta \) and their corresponding forms is valid.

**Theorem 2.** Let \( (\mathcal{M}, g) \) be an \( n \)-dimensional \( (n \geq 2) \) compact and oriented Riemannian manifold and \( \Delta : \mathcal{C}^T \mathcal{V} \rightarrow \mathcal{C}^T \mathcal{V} \) be the Yano rough Laplacian.

1) Suppose the Ricci tensor is negative then an arbitrary eigenvalue \( \lambda \) of \( \Delta \) is positive.

2) The eigenspaces of \( \Delta \) are finite dimensional.

3) The eigenforms corresponding to distinct eigenvalues are orthogonal.

**Proof.**

1) Let \( \varphi \in V_\lambda (\mathcal{M}) \) be a non-zero eigentensor corresponding to the eigenvalue \( \lambda \), that is \( \Delta \varphi = \lambda \varphi \) then we can rewrite the formula \( \Delta \varphi = \delta \nabla - Ric \) in the form
\[ \lambda \langle \varphi, \varphi \rangle = - \int_\mathcal{M} Ric(\xi, \xi) dv + \langle \nabla \varphi, \nabla \varphi \rangle. \] (4.1)

where \( \xi \) is the vector field dual to the 1-form \( \varphi \). If we suppose that the Ricci tensor is negative and we denote by \( -r \) the largest (negative) eigenvalue of matrix \( \|Ric\| \) on \( (\mathcal{M}, g) \) then \( Ric(\xi, \xi) \leq -rg(\xi, \xi) \). In this case from the inequality (4.1), we conclude
\[ \lambda \langle \varphi, \varphi \rangle \geq r \langle \varphi, \varphi \rangle + \langle \nabla \varphi, \nabla \varphi \rangle > 0. \]

2) The eigenspaces of \( \Delta \) are finite dimensional because \( \Delta \) is an elliptic operator.

3) Let \( \lambda_1 \neq \lambda_2 \) and \( \omega_1, \omega_2 \) be the corresponding eigenforms. Then \( \langle \Delta \omega_1, \omega_2 \rangle = \lambda_1 \langle \omega_1, \omega_2 \rangle \) and \( \langle \Delta \omega_1, \omega_1 \rangle = \langle \omega_1, \lambda_1 \omega_1 \rangle = \lambda_2 \langle \omega_1, \omega_1 \rangle \). Therefore \( 0 = (\lambda_1 - \lambda_2) \langle \omega_1, \omega_2 \rangle \) and since \( \lambda_1 \neq \lambda_2 \) it follows that \( \langle \omega_1, \omega_2 \rangle = 0 \), that is, \( \omega_1 \) and \( \omega_2 \) are orthogonal.

In particular, for the case \( n = 2 \) we have the following theorem.

**Theorem 3.** Let \( (\mathcal{M}, g) \) be a 2-dimensional compact and oriented Riemannian manifold. Then the first eigenvalue \( \lambda_1 \) of the Yano rough Laplacian \( \Delta : \mathcal{C}^T \mathcal{V} \rightarrow \mathcal{C}^T \mathcal{V} \) is a non-negative number.
Proof. We compute that 
\[ g(\delta^* \omega, \delta^* \omega) \geq 4 n^{-1} (\delta \omega)^2 \] 
for any \( \omega \in C^* T^* M \). This elementary algebraic fact can be rewritten as
\[ 2^{-1} g(\delta^* \omega, \delta^* \omega) - (\delta \omega)^2 \geq - n^{-1} (n-2) (\delta \omega)^2. \]
Integration by parts yields the following integral inequality
\[ \langle \Delta_{\text{sym}} \omega, \omega \rangle \geq - n^{-1} (n-2) \int_M (\delta \omega)^2 \, dv \]
where the operator \( \Delta_{\text{sym}} \) satisfies the identity
\[ \langle \Delta_{\text{sym}} \omega, \omega \rangle = \langle \delta^* \omega, \delta^* \omega \rangle - \langle \delta \omega, \delta \omega \rangle, \]
which follows immediately from its definition. The inequality proves our theorem.

We consider now the \( n \)-dimensional (\( n \geq 2 \)) Einstein manifold \((M, g)\) where \( \text{Ric} = \frac{s}{n} g \) and \( s \) is a constant (see [1, p. 44]). In this case we can rewrite the formula
\[ \Delta_{\text{sym}} = \Delta - 2 \text{Ric} \]
in the form
\[ \Delta_{\text{sym}} = \Delta - 2 \frac{s}{n} g. \quad (4.2) \]
From (4.2) we conclude that the following theorem is true.

**Theorem 4.** Let \((M, g)\) be an \( n \)-dimensional (\( n \geq 2 \)) compact and oriented Einstein manifold \((M, g)\) then
1) if \( s > 0 \) then any 1-form which is dual to an infinitesimal harmonic transformation is an eigenform of \( \Delta \) corresponding to the eigenvalue \( 2 \frac{s}{n} \) and the converse is also true;
2) if \( s < 0 \) then any harmonic 1-form is an eigenform of \( \Delta_{\text{sym}} \) corresponding to the eigenvalue \( - 2 \frac{s}{n} \) and the converse is also true.

Using the general theory of elliptic operators on a compact \((M, g)\) it can be proved that \( \Delta_{\text{sym}} \) has a discrete spectrum, denoted by \( \text{Spec} \Delta_{\text{sym}} \), consisting of real eigenvalues of finite multiplicity which accumulate only at infinity. In symbols, we have \( \text{Spec} \Delta_{\text{sym}} = \{ 0 \leq \lambda_1 \leq \lambda_2 \leq ... \to +\infty \} \). In addition, if we suppose that the Ricci tensor \( \text{Ric} \) is negative then \( \text{Spec} \Delta_{\text{sym}} = \{ 0 < \lambda_1 \leq \lambda_2 \leq ... \to +\infty \} \). Moreover, here we have the following:

**Theorem 5.** Let \((M, g)\) be an \( n \)-dimensional (\( n \geq 2 \)) compact and oriented Riemannian manifold. Suppose the Ricci tensor \( \text{Ric} \) is negative, then the first eigenvalue \( \lambda_1 \) of the Yano rough Laplacian \( \Delta_{\text{sym}} : C^* T^* M \to C^* T^* M \) satisfies the inequality \( \lambda_1 \geq 2 r \) for the largest (negative) eigenvalue \( - r \) of matrix \( \| \text{Ric} \| \) on \((M, g)\). The equality \( \lambda_1 = 2 r \) is attained for some harmonic eigenform \( \omega \in C^* T^* M \) and in this case the multiplicity of \( \lambda_1 \) is less than or equals to the Betti number \( b_1(M) \).

**Proof.** Let \((M, g)\) be an \( n \)-dimensional compact and oriented Riemannian manifold. Suppose that the Ricci tensor is negative. Denote by \( - r \) the largest (negative) eigenvalue of matrix \( \| \text{Ric} \| \). Then from the formula \( \Delta_{\text{sym}} = \Delta - 2 \text{Ric} \) we obtain the inequality
\[ \langle \Delta_{\text{sym}} \omega, \omega \rangle \geq 2r \langle \omega, \omega \rangle + \langle \Delta \omega, \omega \rangle \]  
\( (4.3) \)

for any \( \omega \in T^*M \). Then for an eigenform \( \omega \) corresponding to an eigenvalue \( \lambda \), \( (4.4) \) becomes the inequalities

\[ \lambda \langle \omega, \omega \rangle \geq 2r \langle \omega, \omega \rangle + \langle \Delta \omega, \omega \rangle \geq 2r \langle \omega, \omega \rangle \]  
\( (4.4) \)

which prove that

\[ \lambda_1 \geq 2r > 0. \]  
\( (4.5) \)

If the equality is valid in \( (4.5) \), then from \( (4.4) \) we obtain \( \Delta \omega = 0 \). In this case \( \omega \) is a harmonic 1-form and, so the multiplicity of \( \lambda_1 \) is less than or equals to the Betti number \( b_1(M) \) because the number of linearly independent (with constant real coefficients) harmonic 1-forms on \((M, g)\) is equal to the Betti number \( b_1(M) \) of \((M, g)\) (see [11]). The proof is complete.

Suppose now that \((\mathbb{H}^n, g_0)\) is a compact \( n \)-dimensional hyperbolic manifold with standard metric \( g_0 \) having constant sectional curvature equal to \(-1\). In this case, from the theorem above we obtain the following corollary.

**Corollary.** Let \((\mathbb{H}^n, g_0)\) be an \( n \)-dimensional compact and oriented hyperbolic manifold then the first eigenvalue \( \lambda_1 \) of the Yano rough Laplacian \( \Delta_{\text{sym}}: C^\infty T^*M \to C^\infty T^*M \) satisfies the inequality \( \lambda_1 \geq 2 \).

The equality \( \lambda_1 = 2 \) is attained if and only if \( n = 2 \). In this case the multiplicity of \( \lambda_1 \) is equal to the Betti number \( b_1(\mathbb{H}^2) \).

**Proof.** Let \((M, g)\) be a compact and oriented model of hyperbolic space \((\mathbb{H}^n, g_0)\) with standard metric \( g_0 \) having constant sectional curvature equal to \(-1\) then \( \lambda_1 \geq 2 \). At the same time it is well known (see [12]) that \( L^2 \)-harmonic \( p \)-forms appear on a simply connected complete hyperbolic manifold \((M, g)\) of constant sectional curvature \(-1\) if and only if \( n = 2p \). Therefore, if \((M, g)\) is a compact and oriented model of hyperbolic space \((\mathbb{H}^n, g_0)\) then the equality \( \lambda_1 = 2 \) is attained if and only if \( n = 2 \). In this case the multiplicity of \( \lambda_1 \) is equal to the Betti number \( b_1(\mathbb{H}^2) \).

5. Appendix

Finally, we prove the following theorem which is dual to the above theorem with the number 5.

**Theorem 6.** Let \((M, g)\) be an \( n \)-dimensional \((n \geq 2)\) compact and oriented Riemannian manifold and \( \mu_1 \) be a first eigenvalue of the Laplacian \( \Delta: C^\infty T^*M \to C^\infty T^*M \) such that the corresponding 1-form \( \omega \in C^\infty T^*M \) is a coclosed form. Moreover, suppose that the Ricci tensor \( \text{Ric} \) is positive, then \( \mu_1 \geq 2 \rho \) for the smallest (positive) eigenvalue \( \rho \) of matrix \( \| \text{Ric} \| \) on \((M, g)\). The equality \( \mu_1 = 2 \rho \) is attained for
some Killing eigenform $\omega \in C^\infty T^*M$ and the multiplicity of $\mu_1$ is less than or equals to the Killing number $k_1(M)$.

**Proof.** Let $\omega$ be a coclosed eigenvalue form of $\Delta$ corresponding to an eigenvalue $\mu$ of $\Delta$ then from the formula $\Delta_{\text{sym}} = \Delta - 2 \text{Ric}$ we obtain the integral equality

$$\mu \langle \omega, \omega \rangle = \langle \delta^* \omega, \delta^* \omega \rangle + 2 \int_M \text{Ric} \langle \xi, \xi \rangle dv$$

where $\xi := \omega^\#$. Now, if we assume that the Ricci tensor is positive and denote by $\rho$ the smallest (positive) eigenvalue of the matrix $\|\text{Ric}\|$, then we have $\text{Ric}(X,X) \geq \rho g(X,X)$ for an arbitrary vector field $X \in C^\infty TM$. In this case, thanks to (5.1), we have $\mu_1 \geq 2 \rho$. On the other hand, if $\mu_1 = 2 \rho$ then from (4.6) we conclude that $\delta^* \omega = 0$. Hence $\xi := \omega^\#$ is a Killing vector field. The theorem is proved.

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