Emergent networks in fractional percolation

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Abstract

Real networks are vulnerable to random failures and malicious attacks. However, when a node is harmed or damaged, it may remain partially functional, which helps to maintain the overall network structure and functionality. In this paper, we study the network structure for a fractional percolation process [Shang, Phys. Rev. E 89, 012813 (2014)], in which the state of a node can be either fully functional (FF), partially functional (PF), or dysfunctional (D). We develop new equations to calculate the relative size of the percolating cluster of FF and PF nodes, that are in agreement with our stochastic simulations. In addition, we find a regime in which the percolating cluster can be described as a coarse-grained bipartite network, namely, as a set of finite groups of FF nodes connected by PF nodes. Moreover, these groups behave as a set of “supernodes” with a power-law degree distribution. Finally, we show how this emergent structure explains the values of several critical exponents around the percolation threshold.
I. INTRODUCTION

Real-world networks, such as social and infrastructure networks, are continuously facing natural and man-made threats that compromise their structure and functionality \[1,2\]. For instance, extreme flooding in urban areas may lead to extensive damage to infrastructures and promote the spread of water-borne and vector-borne diseases \[3,4\]. Similarly, the collapse of an electrical transmission tower, due to poor maintenance, extreme weather, or a malicious attack, could trigger a cascading failure in the power grid \[5-7\]. Because such networks are constantly exposed to failures, many researchers have focused their attention on understanding how damage affects network structure and functionality. In particular, percolation theory \[8-11\] and network science \[12,13\] have been used extensively in these investigations, as they provide tools for assessing the robustness of networks to failures \[11,14-17\].

One of the simplest models to study damaged systems in percolation theory is node percolation \[8,9,11\], in which a fraction \(1 - p\) of nodes are randomly and independently removed (failed or vacant), while the remaining fraction \(p\) of nodes are intact (occupied). Here, \(p\) is also called the control parameter. In this model, several quantities of interest are studied, such as: 1) the distribution of finite cluster sizes \(n_s\) where \(s\) denotes the cluster size, 2) the mean finite cluster size \(\langle s \rangle\), and 3) the probability \(P_\infty\) that a randomly chosen node belongs to a cluster with a macroscopic size, called the percolating cluster or the giant component (GC). It was shown that in node percolation, a second-order phase transition occurs at a critical threshold \(p = p_c\) when the total number of nodes \(N\) tends to infinity. Above this threshold, i.e., in the percolating phase, a GC emerges \((P_\infty > 0)\), whereas below \(p_c\), the system is composed solely of finite clusters \((P_\infty = 0)\), which is called the non-percolating phase. In addition, around \(p \approx p_c\), several magnitudes behave as power-laws, such as \(P_\infty \sim (p - p_c)^\beta\) for \(p > p_c\), \(\langle s \rangle \sim |p - p_c|^{-\gamma}\) (i.e., the mean finite cluster size diverges at the critical point), and \(n_s \sim s^{-\tau}\) at \(p = p_c\), where \(\beta, \gamma,\) and \(\tau\) are called critical exponents \[8,9\]. The value of \(p_c\) for \(N \to \infty\) depends on the network structure or topology. On the contrary, the values of the critical exponents do not depend on the specific structure but rather on the network dimensionality and the node degree-heterogeneity \[8,9,18,19\]. For instance, in square lattices, \(p_c \approx 0.5927\), and in any two-dimensional lattice, \(\beta = 5/36\) and \(\tau = 187/91\) \[8\]. On the other hand,
it was shown that for an uncorrelated random network with a degree distribution \( P(k) \) (i.e., the fraction of nodes with connectivity or degree \( k \)), the threshold \( p_c \) is given by
\[
p_c = 1 / ((k^2)/(k) - 1) \quad [18],
\]
where \( \langle k \rangle \) and \( \langle k^2 \rangle \) are the first- and second-order moments of \( P(k) \), respectively \[20\]. In random homogeneous networks (i.e. with \( \langle k^2 \rangle < \infty \)) the percolation threshold is finite, such as in Erdős-Rényi networks (ER) with a Poisson degree distribution \( P(k) = \langle k \rangle^k \exp(-\langle k \rangle)/k! \). Furthermore, in these types of networks, \( \beta = 1 \) and \( \tau = 5/2 \) \[18\]. On the other hand, for heterogeneous networks (i.e., with \( \langle k^2 \rangle = \infty \)), such as scale-free (SF) networks with a degree distribution \( P(k) \sim k^{-\lambda} \) with \( 2 < \lambda < 3 \), the percolation threshold is zero, implying that the GC is very robust to random node failures. Moreover, using Tauberian theorems, Ref. \[21\] proved that \( \beta = |3 - \lambda|^{-1} \) and \( \tau = 2 + (\lambda - 2)^{-1} \) for \( 2 < \lambda < 4 \), and the percolation transition is higher than second-order. Recently, Radicchi and Castellano \[22\] showed that for random link and node percolation in SF networks with \( \lambda \leq 3 \), the fraction of nodes that belong to the GC behaves above \( p = 0 \) as \( P_\infty \sim p^{\beta_b} \) and \( P_\infty \sim p^{\beta_s} \), respectively, in which the exponents \( \beta_s \) and \( \beta_b \) are related by \( \beta_s = \beta_b + 1 \). Other works have also studied a node percolation (and link percolation) model in random networks with community structure and its critical exponents \[23\,25\].

Besides node percolation, a wide range of percolation models have been proposed to study different types of network failures, such as targeted percolation \[26\], l-hop percolation \[27\], k-core percolation \[28\], and percolation in interdependent networks \[29\]. Most of these models consider that nodes can have only two mutually exclusive states \((n = 2)\): failed and non-failed. However, for some systems, it is more realistic to include additional states in order to study the case where nodes are partially damaged or have different vulnerabilities. To investigate a percolation process in which the number of mutually exclusive states is greater than two \((n > 2)\), Krause et al. \[30, 31\] developed a new type of percolation model called "color-avoiding" percolation that is useful for studying secured-message passing in communication networks. In this model, nodes are separated into different classes or colors, representing a shared vulnerability to failure. On the other hand, Shang \[32\] proposed a percolation process called "fractional percolation" with \( n = 3 \). In this model, nodes can be in one of the following mutually exclusive states: fully functional (FF), partially functional (PF), and dysfunctional (D). FF nodes can be connected to nodes in FF and PF states, while PF nodes only have links to FF nodes, i.e.
they lose their connections with another PF node. This may represent a case in which two partially damaged components in a wireless sensor or an electric network do not have enough energy to communicate with each other, but they can communicate with fully functional components \[32, 33\]. In the simplest version of this model, a fraction \(1 - q\) of nodes are FF, while of the remaining \(q\), a fraction \(1 - r\) is PF and a fraction \(rq\) is D. Here, a giant component is defined as a macroscopic cluster composed of FF and PF nodes. In that work, it was found that the network undergoes a continuous phase transition and the structure is more robust compared to random node percolation. However, the geometrical structure of the network and the critical exponents around the critical point for fractional percolation have not been studied yet.

In this manuscript, we fill these gaps, finding that for a region in the plane \(r \text{ vs } q\), the topology can be described as a coarse-grained bipartite network. In this region, the network is composed of finite clusters of FF nodes that behave like supernodes \[34\] with a SF or power-law degree distribution. Furthermore, we obtain that at \(q = 1 - 1/((k^2)/(k) - 1)\), the fraction of FF and PF nodes belonging to the GC decreases with \(1 - r\) as a power-law function with exponents \(\beta\) and \(\beta^* = \beta + 1\), respectively. For this case, we show that the emergent coarse-grained bipartite network explains the measured value of \(\beta^*\).

Our manuscript is organized as follows: 1) in Sec. II, we present our equations for fractional percolation and compare our theoretical solutions with those of Ref. \[32\], 2) in Sec. III, the critical exponents \(\beta\) and \(\beta^*\) are computed, 3) in Sec. IV, we study a bipartite network in order to explain the values of \(\beta\) and \(\beta^*\), and 4) in Sec. V, we display our conclusions.

II. THEORETICAL EQUATIONS

In this section, we present the equations to compute \(P_\infty\) and \((s)\) for fractional percolation, using that the connections among FF and PF nodes form a semi-bipartite structure.

By definition, a bipartite network is composed of two groups of nodes that we denote \(A\) and \(B\) (for instance, films and actors) in which links only occur between nodes in different groups \[18\]. The degree distribution of each group is denoted as \(P^A(k)\) and \(P^B(k)\). Similarly, in fractional percolation, there are two groups of functional nodes: FF
and PF. In turn, PF nodes cannot be connected to each other but only to FF nodes. However, in contrast to a bipartite structure, an FF node can be connected not only to PF nodes but also to FF nodes. In consequence, this network structure is called "semi-bipartite" [35]. To investigate how the network topology is affected by fractional percolation in the limit of large network size ($N \to \infty$), we will use the generating function formalism which describes the network structure as a branching process [18, 36]. This approach has been applied successfully in previous works to compute different magnitudes, such as $P_\infty$ and $\langle s \rangle$ in several percolation processes [11, 12]. In this approach, for a bipartite network, it is used: 1) the generating function for the degree distribution of group $i = \{A, B\}$,

$$G^i_0[x] = \sum_{k = k_{\text{min}}}^{k_{\text{max}}} P^i(k)x^k$$

where $k_{\text{min}}$ and $k_{\text{max}}$ are the minimum and maximum degrees, and 2) the generating function for the so-called excess degree distribution of group $i = \{A, B\}$,

$$G^i_1[x] = \sum_{k = k_{\text{min}}}^{k_{\text{max}}} kP^i(k)/\langle k \rangle x^{k-1}.$$  

In a previous work on bipartite networks that used the generating function formalism (see Ref. [37]), it was shown that $P_\infty$ can be computed by solving two self-consistent equations, each representing a branching process from one group to the other. Nevertheless, for fractional percolation, we will need an additional equation to consider a branching process between FF nodes, i.e., nodes in the same group. In addition, it is important to note that in fractional percolation, FF and PF nodes have the same degree distribution because they are randomly selected from the same substrate. Thus, FF and PF nodes have the same generating functions for the degree distribution, which we denote as $G_0[x] = \sum_{k = k_{\text{min}}}^{k_{\text{max}}} P(k)x^k$, that is, without any superscript. Similarly, we denote $G_1[x] = \sum_{k = k_{\text{min}}}^{k_{\text{max}}} kP(k)/\langle k \rangle x^{k-1}$ as the generating function for the excess degree distribution for both FF and PF nodes.

The self-consistent equations for fractional percolation are:

$$f_{FF\to FF} = 1 - G_1[qr + (1-q)(1 - f_{FF\to FF}) + (1-r)(1 - f_{PF\to FF})],$$

$$f_{FF\to PF} = 1 - G_1[qr + q(1-r) + (1-q)(1 - f_{PF\to FF})],$$

$$f_{PF\to FF} = 1 - G_1[qr + (1-q)(1 - f_{FF\to FF}) + q(1-r)(1 - f_{PF\to PF})],$$

where:

- $f_{FF\to FF}$ is the probability that a branching process from an FF node to an FF neighbor leads to the GC

- $f_{FF\to PF}$ ($f_{PF\to FF}$) is the probability that a branching process from an FF (PF)
node to a PF (FF) neighbor leads to the GC.

The second r.h.s. term of Eq. (1) accounts for all configurations in which an FF node reached through a link from an FF node does not lead to the GC because its neighbors are either: 1) dysfunctional with probability \( qr \) (see Introduction), 2) fully functional but they do not lead to the GC with probability \( (1-q)(1-f_{FF\rightarrow FF}) \), or 3) partially functional but they do not lead to the GC with probability \( q(1-r)(1-f_{FF\rightarrow PF}) \). On the other hand, the second r.h.s term of Eq. (2) considers all configurations in which the neighbors of a PF node (reached through a link from a FF node) do not lead to the GC because they are either: 1) dysfunctional with probability \( qr \), 2) PF with probability \( q(1-r) \), or 3) FF but they do not lead to the GC with probability \( (1-q)(1-f_{PF\rightarrow FF}) \). Note that the r.h.s of Eqs. (1) and (3) are equal, so \( f_{FF\rightarrow FF} = f_{PF\rightarrow FF} \), and therefore the system of self-consistent equations reduces to

\[
\begin{align*}
    f_{FF\rightarrow FF} &= 1 - G_1[qr + (1-q)(1-f_{FF\rightarrow FF}) + q(1-r)(1-f_{FF\rightarrow PF})], \\
    f_{FF\rightarrow PF} &= 1 - G_1[qr + q(1-r) + (1-q)(1-f_{FF\rightarrow FF})].
\end{align*}
\]

Next, if we change the notation of \( f_{FF\rightarrow FF} \) and \( f_{FF\rightarrow PF} \) to \( f_{FF} \) and \( f_{PF} \), respectively, the above equations can be rewritten as

\[
\begin{align*}
    f_{FF} &= 1 - G_1[qr + (1-q)(1-f_{FF}) + q(1-r)(1-f_{PF})], \\
    f_{PF} &= 1 - G_1[qr + q(1-r) + (1-q)(1-f_{FF})].
\end{align*}
\]

where \( f_{FF} \) (\( f_{PF} \)) can be interpreted simply as the probability that an FF (PF) node reached by traversing a randomly chosen link belongs to the GC.

After solving this system of equations, the fraction of nodes belonging to the GC can be computed as,

\[
P_\infty = P_{\infty}^{FF} + P_{\infty}^{PF},
\]

where the first and second r.h.s terms are the fractions of FF and PF nodes belonging to the GC, respectively, which are described by the following equations

\[
\begin{align*}
    P_{\infty}^{FF} &\equiv (1-q)(1-G_0[qr + (1-q)(1-f_{FF}) + q(1-r)(1-f_{PF})]), \\
    P_{\infty}^{PF} &\equiv q(1-r)(1-G_0[q + (1-q)(1-f_{FF})]).
\end{align*}
\]
FIG. 1: Results for fractional percolation in ER and SF networks. Panel a: $P_\infty$ vs. $q$ for an ER network with $\langle k \rangle = 4$ and several values of $r$: $r = 0$ (black), $r = 0.5$ (red), and $r = 1$ (green). Symbols correspond to our stochastic simulations with $N = 10^5$, the solid lines were obtained from Eq. (8), and the dashed lines were computed from the theory in Ref. [32]. Panel b: $P_\infty$ vs. $q$ for a SF network with $\lambda = 3.5$ and $k_{\text{min}} = 2$ (to ensure the existence of the percolation threshold for $\lambda > 3$ [20, 38]), and using the same parameter values as in panel a. Panel c: $\langle s \rangle$ vs. $q$ for an ER network with $\langle k \rangle = 4$ and $r = 0.5$. The symbols correspond to our stochastic simulations for $N = 10^5$ (red), $N = 4 \times 10^5$ (green), and $N = 16 \times 10^5$ (blue) and the solid line was obtained from Eq. (11). Panel d: $\langle s \rangle$ vs. $q$ for a SF network with $k_{\text{min}} = 2$ and $\lambda = 3.5$, and using the same parameters as in panel c. The stochastic results were obtained over $10^3$ realizations.

We compare the solution of Eq. (8) and the stochastic simulations in Fig. 1a-b for several values of $r$ and for ER and SF networks with $\lambda = 3.5$, obtaining an excellent agreement between theory and simulations. In addition, we also include the curves predicted by Ref. [32], which do not match the stochastic simulation results because the theory in
Ref. [32] uses only one self-consistent equation and does not consider the semi-bipartite structure induced by fractional percolation. The codes of our stochastic simulations (written in Fortran 90) and the main equations are available at GitHub [39].

Besides the relative size of the GC, another quantity of interest in percolation theory is the average finite cluster size \( \langle s \rangle = \sum_{s=1}^{\infty} s n_s \) because it diverges when the system undergoes a continuous phase transition at a critical point (see Introduction). Following a similar reasoning to the derivation of \( P_\infty \), the average finite cluster size is given by

\[
\langle s \rangle = (1 - q) \frac{dH_{0,FF}[x]}{dx} \bigg|_{x=1} + q(1 - r) \frac{dH_{0,PF}[x]}{dx} \bigg|_{x=1},
\]

where \( H_{0,FF}[x] \) and \( H_{0,PF}[x] \) are the generating functions for the cluster size distribution if a randomly chosen node is FF or PF, respectively. The details to calculate \( H_{0,FF}[x] \) and \( H_{0,PF}[x] \) can be found in Appendix [A]. In Fig. 1c-b, we display \( \langle s \rangle \) as a function of \( q \) for \( r = 0.5 \) obtained from Eq. (11) and simulations in ER and SF networks. As expected, we observe that the theoretical solution diverges at a given value \( q_c(r = 0.5) \) for each network topology, and the stochastic simulations converge to this curve as the network size increases. In order to compute \( q_c(r) \) for any value of \( r \), we use that at the critical point in a continuous phase transition, the Jacobian matrix \( J \) of the system of Eqs. (6)-(7) satisfies

\[
\det(J - I) = 0,
\]

where \( \det(\cdot) \) is the determinant function, \( I \) is the identity matrix, and \( J \) is evaluated at \( f_{FF} = f_{PF} = 0 \). After straightforward calculations, we obtain that \( q_c(r) \) is described by the following equation

\[
\frac{(G'_1[1])^2(1-q_c)q_c - (1 - G'_1[1])(1-q_c))}{(G'_1[1])^2(1-q_c)q_c} = r,
\]

where \( G'_1[x] \equiv dG_1[x]/dx \). In particular, for \( r = 1 \), that is, when there are only FF and D nodes, we recover that \( q_c(r = 1) = 1 - 1/G'_1[1] = 1 - 1/((\langle k \rangle^2)/(\langle k \rangle) - 1) \) which is the critical threshold for random node percolation. In Figs. 2a-b, we show the heat-map of \( P_\infty \) in the plane \( q - r \) for ER and SF networks obtained from Eq. (8). In addition, we also include in these figures, the critical line \( q_c(r) \) predicted by Eq. (13) (solid white line), and the vertical line which indicates the value of \( q_c(r = 1) \) (dashed white line). On the right side of the critical line, the network is composed only of finite clusters, whereas on the left
side, a GC of functional nodes (FF and PF) emerges. However, it is important to note that there are two regimes in this percolating phase. For $q_c(r = 1) \leq q \leq q_c(r)$, i.e., in the region between the dashed and solid lines, FF nodes alone cannot form a GC because when $r = 1$ (i.e., when PF nodes are absent), $P_\infty = 0$. Therefore in this regime, the GC structure is composed of finite groups of FF nodes, that we call FF-groups, which are connected by PF nodes, as shown in the schematic illustration in Fig. 2c. In consequence, this structure can be considered as a coarse-grained bipartite network in which one group is composed of PF nodes, and the other group by FF-groups [49]. On the other hand, on the left side of the dashed line ($q < q_c(r = 1)$), the GC emerges regardless of the value of $r$ because even in the extreme case of $r = 1$ when all PF nodes are removed, there are enough FF nodes to form this GC.

In the following sections, we will study how the structure induced by fractional percolation in the percolating phase impacts the critical behavior of several magnitudes at $q_c(r = 1)$. 
FIG. 2: Heat-map of $P_\infty$ in the plane $q$-$r$ for an ER network with $\langle k \rangle = 4$ (panel a) and for a SF network with $k_{\text{min}} = 2$ and $\lambda = 3.5$ (panel b), obtained from Eq. (8). The solid white line corresponds to $q_c(r)$ computed from Eq. (13), and the vertical dashed line indicates the value of $q_c(r = 1)$. Panel c: On the left, we show an illustration of a functional cluster composed of FF nodes (white circles) and PF nodes (gray circles), in the regime $q_c(r = 1) \leq q \leq q_c(r)$. Finite groups of FF nodes (FF-groups) are surrounded by dashed lines. On the right, we show the same cluster but where these FF-groups are replaced by supernodes (squares).

III. CRITICAL BEHAVIOR AT $q = q_c(r = 1)$ IN RANDOM NETWORKS

In this section, we will investigate the asymptotic behavior of the relative size of the GC when $r$ approaches $r = 1$ with $q = q_c(r = 1)$ in random networks. We recall that $r$ only controls the total fraction of PF and D nodes but not the total fraction of FF nodes
which remains fixed for a given value of $q$.

In Figs. 3a and b, we show $P_\infty$ and $P_{\infty}^{FF}$ as functions of $1 - r$ for an ER network with $\langle k \rangle = 4$ at $q = q_c(r = 1) = 1 - 1/\langle k \rangle$, and for a SF network with $\lambda = 3.5$, respectively. We obtain that $P_\infty$ and $P_{\infty}^{FF}$ scale as a power-law with exponent $\beta = 1$ (for the ER network) and $\beta = 2$ (for the SF network) as in random node percolation (see Introduction). On the other hand, Figs. 3a and b show that $P_{\infty}^{PF} \sim (1 - r)^{\beta^*}$ with $\beta^* = 2$ (for the ER network) and $\beta^* = 3$ (for the SF network), which differ from the exponents for $P_\infty$ and $P_{\infty}^{FF}$.

In Fig. 4a and b, from our stochastic simulations at $q = q_c(r = 1)$ and different values of $r$, we observe that the set of FF-groups behaves as a set of “supernodes” (following similar reasoning as in Ref. [34]) with a power-law degree distribution, $\tilde{P}(k) \sim k^{-\tau}$, where:

1. we define the degree of an FF-group as the number $k$ of PF nodes connected to this group,

2. $\tau \in (2, 3)$ is the exponent for the distribution of finite cluster sizes in random node percolation (see Introduction).

Therefore, we obtain that at $q = q_c(r = 1)$, a bipartite network emerges in which “supernodes” follow a heterogeneous degree distribution. We also observe from our simulations that the average mass of an FF-group, $\langle s_{FF} \rangle$, is a linear function of its degree (see insets in Fig. 4).

In the following section, we show that the topology of this emergent bipartite network explains the value of $\beta^*$ observed in Figs. 3a-b.
FIG. 3: $P_\infty$ (black solid line), $P^{EF}_\infty$ (red solid line), and $P^{PF}_\infty$ (green solid line) as functions of $1 - r$ for: an ER network with $\langle k \rangle = 4$ and $q = q_c(r = 1) = 0.75$ (panel a), and a SF network with $\lambda = 3.5$ and $k_{min} = 2$ and $q = q_c(r = 1) \approx 0.7313$ (panel b) in a log-log scale. Solid lines were obtained from Eqs. (8)-(10), and dashed lines represent a power-law fit of $P^{EF}_\infty$ and $P^{PF}_\infty$.

FIG. 4: Degree distribution of FF-groups, $\tilde{P}(k)$, at $q = q_c$ for $r = 0.50$ (black solid line) and $r = 0.90$ (red solid line) for: an ER network with $\langle k \rangle = 4$ and $q = q_c(r = 1) = 0.75$ (panel a), and a SF network with $\lambda = 3.5$ and $k_{min} = 2$ and $q = q_c(r = 1) \approx 0.7313$ (panel b) in a log-log scale. The dashed lines are power-law functions with exponent $\tau$ (see Introduction). Insets: Average mass of an FF-group, $\langle s_{FF} \rangle$, as a function of its degree for $r = 0.50$ (black) and $r = 0.90$ (red) displayed in a double linear scale. Simulations have been averaged over 1000 network realizations with $N = 1.6 \times 10^6$. 
IV. CRITICAL BEHAVIOR IN A BIPARTITE NETWORK

In Sec. II we observed that at $q = q_c(r = 1)$, the GC for fractional percolation is composed of finite FF-groups connected by PF nodes. By merging these finite groups into supernodes (as shown in Fig. 2c), we observed that a bipartite network emerges. Moreover, from our simulations, we obtained that the:

- FF-groups have a degree distribution with a power-law tail with exponent $\tau \in (2, 3)$, where $\tau = 5/2$ for ER networks, and $\tau = 2 + (\lambda - 2)^{-1}$ for SF networks.

- PF nodes have a diluted degree distribution: 1) which is homogeneous if the substrate network is homogeneous, or 2) has a power-law tail with exponent $\lambda$ if the substrate network is a SF network with exponent $\lambda$ (not shown here).

Finally, we obtained from our theoretical equations that the fraction of PF nodes belonging to the GC, $P_{PF}^\infty$, decays faster than $P_\infty$ and $P_{FF}^\infty$ in the limit of $r \to 1$, and it follows a power-law function with exponent $\beta^*$.

In the following, we will introduce and study a simplified model which captures the main topological features of the emergent network in order to explain the value of the exponent $\beta^*$ for an ER network (see Fig. 3a). Specifically, we will consider a bipartite network in which:

- one group, called “sf”, has a pure power-law degree distribution given by $P^{sf}(k) = \epsilon k^{-\tau}$ ($\epsilon$ is a normalization constant) with $\tau = 5/2$,

- the other group, called “h”, has a homogeneous degree distribution $P^h(k)$, i.e., the second-order moment of $P^h(k)$ does not diverge.

The generating functions for the degree distributions if a randomly node is chosen, are:

- for group “sf”: $G_{0,sf}[x] = \sum_{k=1}^{\infty} \epsilon k^{-\tau} x^k = Li_{-\tau}[x]/Li_{-\tau}[1]$ where $Li$ is the Polylogarithm function,

- for group “h”: $G_{0,h}[x] = \sum_{k=0}^{\infty} P^h(k) x^k$,

and the generating functions for the excess degree distributions if a node is reached through a link, are given by:
• for group “sf”: 
\[ G_{1,sf}[x] = \sum_{k=1}^\infty k^\tau \frac{x^k}{\langle k^{sf} \rangle} = x^{-1} Li_{\tau-1}[x]/Li_{\tau-1}[1] \]
where \( \langle k^{sf} \rangle \equiv \sum k^\tau \).

• for group “h”: 
\[ G_{1,h}[x] = \sum_{k=1}^\infty kP^h(k)x^k/\langle k^h \rangle \]
where \( \langle k^h \rangle = \sum kP^h(k) \).

Analogously to fractional percolation in which a fraction \( r \) of PF nodes were removed at \( q = q_c(r = 1) \) whereas FF-groups remained intact (see Sec. III), now we will study a percolation process in our simplified model in which a fraction \( r \) of nodes in group “h” are removed whereas nodes in group “sf” remain undamaged.

Following Ref. [18], the self-consistent equations to compute the GC are given by,

\[ Q_h = r + (1 - r)G_{1,h}[Q_{sf}], \tag{14} \]
\[ Q_{sf} = G_{1,sf}[Q_h], \tag{15} \]

where \( Q_h \) is the probability that a link from group “sf” to group “h” does not lead to the GC because: 1) the node in group “h” is removed with probability \( r \), or 2) the node in group “h” is not removed but is not connected to the GC with probability \( (1 - r)G_{1,h}[Q_{sf}] \).

On the other hand, \( Q_{sf} \) is the probability that a node in group “sf” (reached through a link from group “h”) is not connected to the GC because none of its outgoing links leads to the GC, with probability \( G_{1,sf}[Q_h] \).

Similarly to Eqs. (9) and (10), we now define \( P^h_\infty \) and \( P^{sf}_\infty \) as the fraction of nodes in groups “h” and “sf” that belong to the GC [50], respectively, which are described by the following equations:

\[ P^h_\infty = (1 - r)(1 - G_{0,h}[Q_{sf}]), \tag{16} \]
\[ P^{sf}_\infty = 1 - G_{0,sf}[Q_h]. \tag{17} \]

where the r.h.s. of Eq. (16) accounts for all configurations in which a node in group “h” belongs to the GC because: 1) it is not removed with probability \( 1 - r \), and 2) at least one of its neighbors is connected to the GC with probability \( (1 - G_{0,h}[Q_{sf}]) \). Similarly, the r.h.s. of Eq. (17) accounts for all configurations in which a node in group “sf” belongs to the GC because at least one of its neighbors is connected to this component with probability \( 1 - G_{0,sf}[Q_h] \). It is straightforward from Eqs. (14)-(17) to see that \( r = 1 \) is the critical threshold for this network, i.e., all nodes in group “h” must be removed to destroy the
GC in the limit of large network sizes because the degree distribution of group “sf” is heterogeneous ($\tau < 3$ as mentioned above).

To obtain the critical exponents of $P_h^\infty$ and $P_{sf}^\infty$ in the limit $r \to 1$, we expand the r.h.s of Eq. (14) around $Q_h = 1$ in a Taylor series keeping only terms below first order, and for the r.h.s of Eq. (15) we apply Tauberian theorems \cite{21, 40} around $Q_{sf} = 1$ using that $2 < \tau < 3$, yielding to the following equations,

\begin{align}
Q_h &\sim r + (1 - r) \left(1 + \left(\frac{\left\langle (k_h)^2 \right\rangle}{\left\langle k_h \right\rangle} - 1\right)(Q_{sf} - 1)\right), \\
Q_{sf} &\sim 1 - c_1(1 - Q_h)^{\tau - 2},
\end{align}

where $c_1$ is a constant. The solutions of this system of equations are,

\begin{align}
Q_h &\sim 1 - c_2(1 - r)^{1/\tau}, \\
Q_{sf} &\sim 1 - c_3(1 - r)^{\frac{\tau - 2}{3 - \tau}},
\end{align}

where $c_2$ and $c_3$ are positive constants.

Finally, after expanding the r.h.s of Eqs. (16)-(17) around $Q_{sf} = 1$ and $Q_h = 1$, and combining with Eqs. (20)-(21), we obtain

\begin{align}
P_h^\infty &\sim (1 - r)^{1/\tau}, \\
P_{sf}^\infty &\sim (1 - r)^{1/\tau}.
\end{align}

Therefore, for $\tau = 5/2$ we get that the exponent of $P_h^\infty$ is $(3 - \tau)^{-1} = 2$ which is in agreement with the measured exponent $\beta^*$ for $P_{FF}^\infty$ in Fig. 3a. As a consequence, this result shows that the emergent structure in fractional percolation explains the value of $\beta^*$.

On the other hand, it would be expected that $P_{sf}^\infty$ and $P_{FF}^\infty$ have the same critical exponent. However, by comparing Eqs. (22) and (23), it is clear that the exponent for $P_{sf}^\infty$ is $(3 - \tau)^{-1} = 2$ which is different from the exponent $\beta = 1$ for $P_{FF}^\infty$ (see Fig. 3a). To understand this discrepancy, it should be noted that the size or mass of a node in group “sf” is different from the mass of an FF-group. More specifically, when we compute $P_{sf}^\infty$, we assume that the size of a single node in group “sf” is equal to one. On the contrary, the mass of an FF-group is proportional to its degree $k$ as shown in Fig. 4c. Therefore, based on the aforementioned difference between masses, if we now assign a mass equal to
to each node with degree \( k \) in the group “sf” and recalculate \( P_{sf}^\infty \) accordingly, we obtain (see details in Appendix B)

\[
P_{sf}^\infty \sim (1 - r)^{\frac{1}{\tau - 1}},
\]

where for the homogeneous case (\( \tau = 5/2 \)) the exponent of \( P_{sf}^\infty \) is now \( (3 - \tau)^{-1} - 1 = 1 \) which is in agreement with the measured exponent \( \beta \) in Fig. 3a. Finally, by comparing the exponents of Eqs. (22) and (24), we obtain that \( \beta^* - 1 = \beta \).

Although the exponents in Eqs. (22) and (24) have been derived for the case where the degree distribution of group “h” is homogeneous (with \( \tau = 5/2 \)), if we repeat the same calculations for the following bipartite network:

- the group “h” has a power-law degree distribution with \( 3 < \lambda < 4 \),
- the group “sf” has a pure power-law degree distribution given by \( P_{sf}(k) = \epsilon k^{-\tau} \) (\( \epsilon \) is a normalization constant) with \( \tau = 2 + (\lambda - 2)^{-1} \),

we obtain that Eqs. (22) and (24) still hold, and \( \beta^* - 1 = \beta \). In particular, for \( \lambda = 3.5 \), we get that \( \beta^* = (3 - \tau)^{-1} = (\lambda - 2)/(\lambda - 3) = 3 \), and \( \beta = 2 \) which are in agreement with the measured exponents \( \beta^* \) and \( \beta \) in Fig. 3b.

In Appendix C we show that the relation between \( \beta \) and \( \beta^* \) also holds for fractional percolation in SF networks with \( \lambda = 3.25 \), and \( \lambda = 3.75 \), and in square lattices.

V. SUMMARY AND CONCLUSION

In this manuscript, we study a fractional percolation model in complex networks. We find the exact equations governing the size of the GC and the average size of finite clusters in the limit of large network sizes, using that the connections among FF and PF nodes form a semi-bipartite structure. Moreover, in the \( q - r \) plane, we obtained two functional regimes: one in which the GC does not need PF nodes to emerge and another in which the GC is composed of finite groups of FF nodes (FF-groups) that are connected by PF nodes. In the latter regime, the GC can be described as a coarse-grained bipartite network, and at \( q = q_c(r = 1) \), the FF-groups behave as a set of supernodes with a power-law degree distribution. We also find that at \( q = q_c(r = 1) \) and in the limit of \( r \to 1 \), the fraction of FF and PF nodes behave as power-law functions of \( 1 - r \) with
exponents $\beta$ and $\beta^* = \beta + 1$, respectively, where the value of $\beta$ is the same as in random node percolation. Furthermore, we show that the value of $\beta^*$ can be explained by the emergent bipartite network in which supernodes have a power-law degree distribution. Our present findings could be easily extended to consider more node states ($n > 3$), and also to the case in which $q$ and $r$ depend on the node’s degree. Our work could also be extended to other percolation processes, such as k-core and bootstrap percolation \[41, 43\]. In addition, it would be interesting to characterize the network structure further using recently developed tools and concepts, such as the degree-degree correlation and the spectra of the GC \[44\], as well as articulation points and "bredges" \[45, 46\]. These research directions are left for future work.

VI. ACKNOWLEDGMENTS

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Appendix A: Generating functions for finite cluster sizes, and $\langle s \rangle$

Let $H_0[x] = \sum_{s=0}^{\infty} P(s)x^s$ be the generating function for finite cluster sizes when a node is randomly chosen, where $P(s)$ is the probability that the chosen cluster has size $s$. This function is given by,

$$H_0[x] = qr + (1 - q)H_{0,FF}[x] + q(1 - r)H_{0,PF}[x], \quad (A1)$$

where the first term corresponds to dysfunctional nodes (i.e. clusters with size $s = 0$), the second and third terms correspond to the cases when the chosen node is FF and PF, respectively, and the generating functions $H_{0,FF}[x]$ and $H_{0,PF}[x]$ are given by,

$$H_{0,FF}[x] = xG_0[qr + (1 - q)H_{1,FF}[x] + q(1 - r)H_{1,PF}[x]], \quad (A2)$$

$$H_{0,PF}[x] = xG_0[qr + q(1 - r) + (1 - q)H_{1,FF}[x]]. \quad (A3)$$

In Eq. (A2), the r.h.s accounts for all configurations in which a FF node belongs to a finite cluster because its neighbors are either: 1) D with probability $qr$, 2) FF (PF) and they lead only to a finite cluster of functional nodes with size $s$ whose generating function is given by $H_{1,FF}[x]$ ($H_{1,PF}[x]$). On the other hand, Eq. (A3) has a similar interpretation
as Eq. (A2). Following Refs. [18, 47], the generating functions \( H_{1,FF}[x] \) and \( H_{1,PF}[x] \) are given by the following self-consistent functional equations,

\[
H_{1,FF}[x] = xG_1[qr + (1-q)H_{1,FF}[x] + q(1-r)H_{1,PF}[x]],
\]

\[
H_{1,PF}[x] = xG_1[qr + q(1-r) + (1-q)H_{1,FF}[x]],
\]

which have a similar interpretation as Eqs. (6)-(7). Finally, after solving Eqs. (A1)-(A5), we compute the mean finite cluster size as

\[
\langle s \rangle = \left. \frac{dH_0}{dx} \right|_{x=1} = (1-q)\left. \frac{dH_{0,FF}}{dx} \right|_{x=1} + q(1-r)\left. \frac{dH_{0,PF}}{dx} \right|_{x=1}.
\]

Appendix B: Enlarging a bipartite network

In Sec. IV, we obtained that for a bipartite network, the fraction of nodes in group “sf” that belong to the GC obeys Eq. (17), i.e.,

\[
P_{sf}^\infty = 1 - \sum_{k=1}^{\infty} P_{sf}(k) (Q_h)^k,
\]

where \( P_{sf}(k) = \epsilon k^{-\tau} \) (with \( 2 < \tau < 3 \)), and \( \epsilon \) is a normalization constant. Moreover, we found that in the limit of \( r \to 1 \):

\[
P_{sf}^\infty \sim (1-r)^{\frac{1}{1-\tau}} \sim (1-r)^{\beta^*}.
\]

As mentioned in Sec. IV, in order to compute \( P_{sf}^\infty \), we assumed that the mass of a single node in group “sf” is equal to one. On the contrary, when we compute \( P_{\infty}^{FF} \), we observed that the mass of a single FF-group of connectivity \( k \) is proportional to its degree (see Fig. 4c).

In the following, we recalculate \( P_{sf}^\infty \) for the case in which each node with degree \( k \) has a mass equal to \( k \), i.e., we will “enlarge” the group “sf” \([51]\). More specifically, we have to replace the distribution \( P_{sf}(k) \) in Eq. (B1) by \( kP_{sf}(k)/\langle k^{sf} \rangle \), where \( \langle k^{sf} \rangle = \sum kP_{sf}(k) \) is a normalization constant. By doing this, Eq. (B1) becomes,

\[
P_{sf}^\infty = 1 - \sum_{k=1}^{\infty} \frac{kP_{sf}(k)}{\langle k^{sf} \rangle} (Q_h)^k,
\]

\[
= 1 - \sum_{k=1}^{\infty} \frac{\epsilon k^{-\tau+1}}{\langle k^{sf} \rangle} (Q_h)^k = 1 - Q_h \sum_{k=1}^{\infty} \frac{\epsilon k^{-\tau+1}}{\langle k^{sf} \rangle} (Q_h)^{k-1},
\]

\[
= 1 - Q_h G_{1,svf}[Q_h],
\]
where $G_{1, sf}[x]$ is the generating function for the excess degree distribution of a node in group “sf” (see Sec. IV).

Using Tauberian theorems in the limit of $Q_h \to 1$ (see Sec. 4.3 in [40]), we have that the generating function $G_{1, sf}[x]$ behaves as

$$G_{1, sf}[x] \sim 1 - c(1 - x)^{\tau - 2}, \quad (B6)$$

where $c$ is a constant, and we use the symbol $x \sim y$ to mean that $x/y \to 1$ [48]. Then, if we combine Eqs. (B5) and (B6), we obtain that $P^\infty_{sf}$ can be rewritten as,

$$P^\infty_{sf} \sim 1 - (1 - c(1 - Q_h)^{\tau - 2})^r, \quad (B7)$$

$$\sim c(1 - Q_h)^{\tau - 2}, \quad (B8)$$

$$\sim c(1 - r)^{\beta^* - 1}, \quad (B9)$$

$$\sim c(1 - r)^{\beta^* - 1}, \quad (B10)$$

where in the last two steps we have recalled Eq. (20) and used $\beta^* = 1/(3 - \tau)$ (see Sec. IV). Therefore, after comparing Eqs. (B2) and (B10), we obtain that enlarging the network size has changed the critical exponent of $P^\infty_{sf}$ from $\beta^*$ to $\beta^* - 1$.

Appendix C: Critical exponents for SF networks and square lattices

In Fig. 5, we show for fractional percolation: $P^\infty$, $P^{FF\infty}$, and $P^{PF\infty}$ as functions of $1 - r$ at $q = q_c(r = 1)$ for SF networks with $\lambda = 3.25$ (panel a) and $\lambda = 3.75$ (panel b), obtained from Eqs. (8)-(10). From these figures, we can see that the measured values of the exponents $\beta$ and $\beta^*$ are in agreement with the theoretical ones obtained in Sec. IV, i.e., $\beta^* = 1/(3 - \tau)$ with $\tau = 2 + (\lambda - 2)^{-1}$ (see Introduction), and $\beta = \beta^* - 1 = 1/(\lambda - 3)$ which corresponds to the value of $\beta$ in random node percolation (see Introduction).

Similarly, for square lattices (see Fig. 4), we observe that simulations support the relation $\beta^* = \beta + 1 = 41/36$, where $\beta = 5/36$ (see Introduction). In addition, we also obtain from our simulations that the degree distribution of FF-groups is a power-law function with exponent $\tau$ and the average mass of an FF-group is proportional to its degree (see Fig. 3).
FIG. 5: Log-log plot of $P_{\infty}$ (solid black line), $P_{\infty}^{FF}$ (solid red line), and $P_{\infty}^{PF}$ (solid green line) as functions of $1 - r$ at $q = q_c(r = 1)$ for SF networks with $\lambda = 3.25$ (panel a) and $\lambda = 3.75$ (panel b). Solid lines were obtained from Eqs. (8)-(10) and dashed lines represent a power-law fit.
FIG. 6: Panel a: Log-log plot of $P_\infty$ (empty symbols) and $P_{\infty}^{FF}$ (full symbols) as functions of $1 - r$ at $q = q_c(r = 1) \approx 0.41$ for a square lattice of size $L \times L$ with $L = 1000$ (black circles), and $L = 8000$ (red triangles), and rigid boundary conditions. Symbols are simulations averaged over 500 realizations, and the dashed line corresponds to a power-law function with exponent $5/36$. Panel b: $P_{\infty}^{FF}$ for the same lattice in panel a, for $L = 1000$ (black circles) and $L = 8000$ (red triangles). The dashed line corresponds to a power-law function with exponent $5/36 + 1 = 41/36$. Panel c: Degree distribution of FF-groups $\hat{P}(k)$ at $q = q_c$ for $r = 0.50$, $L = 1000$ (black symbols) and $L = 8000$ (red symbols) in a log-log scale. The dashed line is a power-law function with exponent $\tau$ (see Introduction). In the inset we show the average mass of an FF-group, $\langle s_{FF} \rangle$, as a function of its degree for $r = 0.50$, and $L = 1000$ (black) and $L = 8000$ (red) displayed in a double linear scale. Simulations have been averaged over 1000 network realizations.
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Moreover, the network is composed of only D and PF nodes at $q_c(r = 1) = 1$.
[50] These fractions are relative to the number of nodes in each group.
[51] Qualitatively, this transformation can be pictured as the reverse process illustrated in Fig. 2c, that is, we replace each node in group “sf” with degree $k$ by a finite FF-group whose mass is proportional to $k$. 