Abelian Conjectures of Stark Type
in \( \mathbb{Z}_p \)-Extensions of Totally Real Fields

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1 Introduction

We continue our investigations into complex and \( p \)-adic variants of H. M. Stark’s conjectures \cite{St} for an abelian extension of number fields \( K/k \). We have formulated versions of these conjectures at \( s = 1 \) using so-called ‘twisted zeta-functions’ (attached to additive characters) to replace the more usual \( L \)-functions. The complex version of the conjecture was given in \cite{So3}. In \cite{So4} we formulated an analogous \( p \)-adic conjecture in the case where \( K \) is a (totally) real ray-class field over \( k \), as well as a third ‘combined’ version that related the two previous ones in this case. (For numerical verification of the combined conjecture, see \cite{R-S}). All these conjectures are stated in terms of group-ring-valued regulators similar to those used in \cite{Ru} to formulate fine ‘integral’ variants of Stark’s conjectures at \( s = 0 \).

In Section 2 of the present paper we restate all three versions of the conjectures mentioned above (designated respectively \( C_1 \), \( C_2 \) and \( C_3 \)) in the context of an essentially arbitrary, real abelian extension \( K/k \). However, we give only the most ‘basic’ forms of these conjectures i.e. without any integrality conditions on their solutions, since these conditions are of little direct concern to us in the present paper. (See, however, Remarks 3.3 and 6.1). Instead, our aim – introduced in Section 3 – is to focus on a new aspect of Stark’s Conjectures: under suitable conditions both \( C_1 \) and \( C_2 \) predict the existence of ‘special’ elements in the \( r \)th exterior power of \( \mathbb{Q} \otimes_{\mathbb{Z}} E(K) \) over the group-ring \( \mathbb{Q}\text{Gal}(K/k) \) (where \( r := [k : \mathbb{Q}] \) and \( E(K) \) denotes the unit group of \( K \)). We consider the possible behaviour of these elements as \( K \) varies in a cyclotomic \( \mathbb{Z}_p \)-tower \( K_\infty = \bigcup_{n=0}^\infty K_n \). More specifically, under the additional hypothesis that \( p \) is unramified in \( k \) we formulate a property ‘\( P1 \)’ which requires, roughly speaking, that there exist solutions of the complex conjecture \( C_1 \) for all \( K_n/k, n \geq 1 \), all coming from a single element \( \eta \) of a module \( \bigwedge^r \mathbb{Q}\mathcal{E}_\infty \), which is the exterior power over \( \mathbb{Q}\text{Gal}(K_\infty/k) \) of the norm-coherent sequences of (global) units in the tower (tensored with \( \mathbb{Q} \)). Property

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P1 is shown to hold in a large infinite class of cases with \( K \subset \mathbb{Q}^{ab} \). (Arguments given in Section 3 make it seem plausible that P1 should hold rather more generally. However, this is hard to verify, even numerically, since doing so would seem to require explicitly verifying C1 throughout the tower.) We also enunciate an analogous property ‘P2’ concerning the existence of such an \( \eta \) giving the solutions of the \( p \)-adic conjecture C2 for the \( \mathbb{Z}_p \)-extension (same \( p! \)) that is, for each \( K_n/k \) with \( n \) larger than an explicit constant \( n_2 \). We show that the \( p \)-adic Property P2 follows from the complex one P1 (with the same \( \eta \)) provided we assume also the combined conjecture C3 for all \( K_n/k \) with \( n \geq n_2 \).

Apart from their intrinsic interest and simplicity, the main motivation for introducing properties P1 and, especially, P2 is revealed in Section 4. First, under the strengthened hypothesis that \( p \) splits in \( k \), we construct a doubly infinite sequence of ‘higher’ \( p \)-adic regulators \( \mathcal{R}_{t,n} \) on \( \wedge^r \mathbb{Q}_{E_\infty} \) by replacing the \((p\text{-adic})\) logarithm in the usual group-ring-valued \( p \)-adic regulator with twisted, generalised versions of the Coates-Wiles homomorphisms. We then consider a putative identity between two elements of the group-ring \( \mathbb{C}_p \text{Gal}(K_n/k) \) for some given \( n \geq n_2 \) and \( m \in \mathbb{Z} \). The first element comes from the \( p \)-adic twisted zeta-functions of the extension \( K_n/k \) evaluated at \( s = m \). The second element is (up to an explicit algebraic factor) the value of \( \mathcal{R}_{1-m,n} \) at a hypothetical element \( \eta \) of \( \wedge^r \mathbb{Q}_{E_\infty} \). Thus it is, essentially, a determinant of \((1-m)\)th logarithmic derivatives at level \( n \) of norm-coherent sequences of global units. The existence of an \( \eta \) satisfying this relation for \( m = 1 \) and all \( n \geq n_2 \) amounts to our \( p \)-adic Stark Conjecture C2 for each \( K_n/k \) with \( n \geq n_2 \) as strengthened by saying precisely that \( \eta \) satisfies Property P2. On the other hand, Theorem 4.1 (the main result of this paper) shows that if \( \eta \) satisfies our relation for infinitely many pairs \( (m,n) \) with \( m \in \mathbb{Z} \) and \( n \geq n_2 \), then it satisfies it for all such pairs. The immediate consequence is, of course that if \( \eta \) satisfies P2 then it also satisfies a new, ‘special value conjecture’ of Stark type for \( p \)-adic twisted zeta-functions at any integral \( s \) (and this for each extension \( K_n/k \) with \( n \geq n_2 \)).

In the case \( k = \mathbb{Q} \) of course, a link between the Leopoldt-Kubota \( p \)-adic \( L \)-function and norm-coherent sequences of cyclotomic units was already observed by Iwasawa in [Iw]. Moreover the use of the Coates-Wiles homomorphisms to make this connection is now well-known (see e.g. [Wa]). One way to view the present paper is as an explanation of how a strengthened \( p \)-adic Stark Conjecture at \( s = 1 \) leads to a related and rather precise connection for twisted zeta-functions and more general \( K/k \) as above. On the other hand, a theory due to Perrin-Riou proposes extremely general links between norm-coherent sequences in the first cohomology of a \( p \)-adic representation of \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) and an associated \( p \)-adic \( L \)-function (whose existence is partly conjectural. See [PR2] for details). If this latter theory, or an extension of it, could be made to produce explicit predictions for \( L \)-functions of the extension \( K/k \), then interesting comparisons with our conjectures might follow. So far this seems only to have been done in the above known case of the conjectures, namely \( k = \mathbb{Q} \) (see e.g. [PRI] and references therein).

In section 5 we give the proof of Theorem 4.1 which relies at base on a well known uniqueness principle in \( p \)-adic analysis stating roughly that a non-zero \( p \)-adic power series with bounded coefficients can have only finitely many roots in \( \mathbb{C}_p \). To apply this principle we
use the theory of $p$-adic measures and results of Deligne and Ribet on the analytic behaviour of $p$-adic $L$-functions attached to characters of $\text{Gal}(K_n/k)$. Global Gauss sums intervene in relating the $p$-adic $L$-functions to our $p$-adic twisted zeta-functions, so the proof also requires a rather intricate comparison between these sums and a certain product of local Gauss sums that appears naturally elsewhere. (It should be possible to avoid all Gauss sums by using the twisted zeta-functions directly. However, since their $p$-adic behaviour is less well-understood that of $L$-functions, we have so far done this only for certain classes of real-quadratic $k$, using different methods coming from \[\text{So}\text{4}.\)]

Finally, in Section \[\text{6}\] we show how $P1$ and $P2$ can be weakened by ‘enlarging’ $\bigwedge^r \mathbb{Q}_{\infty}$ to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r \mathcal{U}_{\infty}^{(1)}$. where, $\mathcal{U}_{\infty}^{(1)}$ is the module of norm-coherent sequences of $p$-semilocal units in the tower and the exterior power is taken over the appropriate completed $p$-adic group-ring. Thus, for example, $\widehat{P2}$ weakens $P2$ by assuming only that there exists an element $\hat{\eta} \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r \mathcal{U}_{\infty}^{(1)}$ whose projection to the $n$th level is in the module of exterior powers of global units of $K_n$ for all $n$ and that this projection is also a solution of $C2$ for $K_n/k$ if $n \geq n_2$. The higher regulators $R_{t,n}$ ‘extend’ to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r \mathcal{U}_{\infty}^{(1)}$ and the same methods as in the preceding sections show that the weaker existence hypothesis $\widehat{P2}$ implies the weaker conclusion that there is exists an element $\hat{\eta} \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r \mathcal{U}_{\infty}^{(1)}$ which is global at each finite level and satisfies the special value conjectures at all $s \in \mathbb{Z}$ and $n \geq n_2$ (namely, $\hat{\eta}$ is the element satisfying $\widehat{P2}$).

Some Notation: For the rest of this paper we fix the totally real number field $k$ considered as contained in $\bar{\mathbb{Q}} \subset \mathbb{C}$. In addition to the above notations, we shall write $\mathcal{O}$ for the ring of integers of $k$, $d_k \in \mathbb{N}$ for its absolute discriminant and $S_{\infty}$ for the set of its infinite places. The latter may be identified with the (real) embeddings of $k$ into $\bar{\mathbb{Q}}$ which we denote $\tau_1, \ldots, \tau_r$ and for each $i = 1, \ldots, r$ we fix once and for all an element $\tilde{\tau}_i \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ extending $\tau_i$. For any $m \in \mathbb{Z}_{\geq 1}$ we write $\mu_m$ for the group of all $m$th roots of unity in $\bar{\mathbb{Q}}$ and $\zeta_m$ for its specific generator $\exp(2\pi i/m)$.

## 2 Basic Conjectures

We start by restating Conjectures 3.1, 3.2 and 3.3 of \[\text{So}\text{4},\] referring to \textit{ibid.} for many of the details. First, recall the definitions of the functions $\Phi_{m,T}$ and $\Phi_{m,T,p}$. Let $m$ be a cycle for $k$, namely a formal product $g_3$ where $g$ is a non-zero ideal of $\mathcal{O}$ and $j$ is (the formal product of) a subset of $S_{\infty}$. The ray-class group and the ray-class field of $k$ modulo $m$ will be denoted $\text{Cl}_m(k)$ and $k(m)$ respectively (thus $k \subset k(m) \subset \bar{\mathbb{Q}}$). Let $T$ be any finite subset of prime ideals of $\mathcal{O}$. We defined in \[\text{So}\text{3},\text{So}\text{4}\] a function of one complex variable $s$ taking values in the group-ring $\mathbb{C}\text{Gal}(k(m)/k)$ by setting

$$\Phi_{m,T}(s) = \sum_{\mathfrak{c} \in \text{Cl}_m(k)} Z_T(s; \mathfrak{c} \cdot \mathfrak{w}_m^0) \sigma_{\mathfrak{c}}^{-1}$$

Here $\sigma_{\mathfrak{c}}$ is the element of $\text{Gal}(k(m)/k)$ associated to $\mathfrak{c}$ by the Artin isomorphism and $\{Z_T(s; \mathfrak{c} \cdot \mathfrak{w}_m^0)\}$ is a twisted zeta-function depending on $\mathfrak{c}$. We emphasize that in the present context we
are not assuming — as has been done in some similar ones — that \( g \) is ‘prime to \( T \), i.e. that \( v_p(g) = 0 \) \( \forall p \in T \). We shall say that \( g \) is \( T \)-trivial if and only if it is a product — possibly empty — of distinct primes of \( T \). Recall that \( Z_T(s; \cdot \cdot w_m^0) \) was defined by a Dirichlet series for \( \text{Re}(s) > 1 \) and that it (and hence also \( \Phi_{m,T}(s) \)) is closely related to the \( L \)-functions of the characters of \( \text{Cl}_m(k) \) (see Proposition 2.1). Recall ([So4, Thm. 2.3]) that \( \Phi_{m,T}(s) \) extends to a meromorphic function on \( \mathbb{C} \) with at most a simple pole at \( s = 1 \) and is holomorphic in \( \mathbb{C} \) iff \( g \) is not \( T \)-trivial.

The values of \( \Phi_{m,T} \) at non-positive integers lie in the group ring \( \mathbb{Q}\text{Gal}(k(m)/k) \) (see So4 Lemma 3.2]). Under certain conditions they can be \( p \)-adically interpolated. Let \( p \) be any prime number. We shall use the following notation. Every \( x \in \mathbb{Z}_p^\times \) can be written uniquely as

\[
x = \omega(x)\langle x \rangle
\]

where \( \omega(x) \) is a root of unity in \( \mathbb{Z}_p^\times \) and \( \langle x \rangle \) lies in \( 1 + p\mathbb{Z}_p \) (in \( 1 + 4\mathbb{Z}_2 \) if \( p = 2 \)). Let \( \mathcal{M}(p) \) denote the set of integers \( m \in \mathbb{Z}_{\leq 0} \) such that \( m \equiv 1 \pmod{p - 1} \) \((\pmod{2} \text{ if } p = 2)\). Let \( D(p) \) denote the \( p \)-adic disc \( 1 + 2\mathbb{Z}_p \) and \( D^0(p) \) the punctured disc \( D(p) \setminus \{1\} \). Let \( \mathbb{C}_p \) be a completion of an algebraic closure \( \mathbb{Q}_p \) of \( \mathbb{Q}_p \). Now fix an embedding \( j : \mathbb{Q} \rightarrow \mathbb{C}_p \) and let it act on coefficients to define a homomorphism from \( \mathbb{Q}\text{Gal}(k(m)/k) \) to \( \mathbb{C}_p\text{Gal}(k(m)/k) \). We showed (see So4 Thm./Def. 2.1]) that if \( T \) contains the set \( T_p \) of places of \( k \) dividing \( p \), then there exists a unique, \( p \)-adically continuous function \( \Phi_{m,T,p} = \Phi^{m,T,p}_i \) from \( D^0(p) \) to \( \mathbb{C}_p\text{Gal}(k(m)/k) \) such that \( \Phi_{m,T,p}(m) = j(\Phi_{m,T}(m)) \) for every \( m \in \mathcal{M}(p) \). In fact, \( \Phi_{m,T,p} \) is meromorphic on \( D(p) \) and even holomorphic there if \( g \) is not \( T \)-trivial. It is related to the \( p \)-adic \( L \)-functions which can be attached to the (totally) even characters of \( \text{Cl}_m(k) \) and consequently depends only on \( g \), not on the infinite part \( \mathfrak{o} \) of \( m \). (See Proposition 2.1 for a more precise statement.)

The ‘basic’ conjectures of [So4] concern the values \( \Phi_{m,T}(1) \) and \( \Phi_{m,T,p}(1) \) in the case where \( \mathfrak{o} \) is trivial so that \( k(m) = k(g) \) is a finite real abelian extension of \( k \). Generally, we shall write \( K \) for such an extension (as in the introduction) and \( G \) for \( \text{Gal}(K/k) \). Given also a finite set \( S \) of places \( k \) containing \( S_\infty \), we write \( U_S(K) \) for the group of \( S \)-units of \( K \). We define \( \mathbb{Z}G \)-linear, logarithmic maps \( \lambda_i = \lambda_{K/k,i}: U_S(K) \rightarrow \mathbb{R}G \) and \( \lambda_{i,p} = \lambda^{(j)}_{K/k,i,p}: U_S(K) \rightarrow \mathbb{C}_pG \) by setting \( \lambda_i(u) = \sum_{\sigma \in G} \log |\tilde{\sigma}(u)| \sigma^{-1} \) and \( \lambda_{i,p}(u) = \sum_{\sigma \in G} \log_p(j\tilde{\sigma}(u)) \sigma^{-1} \) for any \( u \in U_S(K) \) where \( \log_p \) denotes Iwasawa’s \( p \)-adic logarithm. Each such map ‘extends’ by \( \mathbb{Q} \)-linearity to \( \mathbb{Q}\text{Us}(K) := \mathbb{Q} \otimes_{\mathbb{Z}} U_S(K) \). Thus we can define unique, \( \mathbb{Q}G \)-linear ‘regulator’ maps \( R_{K/k} = R_{K/k,S} \) and \( R_{K/k,p} = R^{(j)}_{K/k,S,p} \) from the \( r \)th exterior power \( \Lambda^r_{\mathbb{Q}G} \mathbb{Q}\text{Us}(K) \) (noted as an additive \( \mathbb{Q}G \)-module) into \( \mathbb{R}G \) and \( \mathbb{C}_pG \) respectively by setting \( R_{K/k}(u_1 \wedge \ldots \wedge u_r) = \det(\lambda_i(u_1))^{i_1}_{i_{r+1}=1} \) and \( R_{K/k,p}(u_1 \wedge \ldots \wedge u_r) = \det(\lambda_{i,p}(u_1))^{i_1}_{i_{r+1}=1} \) for all \( u_1, \ldots , u_r \in \mathbb{Q}\text{Us}(K) \).

In the case \( K = k(g) \), we write \( G_g \) for \( \text{Gal}(k(g)/k) \) and let \( S(g) \) be the set of places of \( k \) dividing \( g \) together with the infinite ones. Conjecture 3.1 of So4 then reads as follows

**Conjecture C1\( (k,g,T) \)** Suppose that \( g \) is not \( T \)-trivial. Then

\[
\Phi_{g,T}(1) = \frac{2^r}{\sqrt{d_k} \prod_{p \in T} N_p} R_{k(g)/k}(\eta) \quad \text{for some } \eta \in \Lambda^r_{\mathbb{Q}G_g} \mathbb{Q}\text{Us}(g)(k(g))
\]
Note that the conditions of this conjecture already imply that \( \Phi_{g,T}(1) \in \mathbb{R}G_g \) (see Lemma 3.1 of \[So3\]). A \( p \)-adic analogue (Conjecture 3.2 of \[So4\]) can be formulated as

**Conjecture C2\((k, g, T, p)\)** Suppose that \( T \) contains \( T_p \) and that \( g \) is not \( T \)-trivial. Then

\[
\Phi_{g,T,p}(1) = \frac{2^r}{j(\sqrt{d_k}) \prod_{p \in T} Np} R_{k(g)/k,p}(\eta) \quad \text{for some } \eta \in \bigwedge^r_{QG_g} QU_{S(g)}(k(g))
\]

Finally, we also made a combined conjecture (Conjecture 3.3 in \[So4\]):

**Conjecture C3\((k, g, T, p)\)** Under the conditions of C2\((k, g, T, p)\) there exists a simultaneous solution \( \eta \in \bigwedge^r_{QG_g} QU_{S(g)}(k(g)) \) of both (1) and (2).

In (1) and (2) the rational normalising factor \( 2^r / (\prod_{p \in T} Np) \) could of course be absorbed into \( \eta \) but it is more convenient to keep it. The dependence of \( \eta \) on \( T \) and the choices of the \( \tilde{\tau}_i \) — as well as its independence of \( j \) in (2) — are explained in § 3.2 of \[So4\]. In § 3.3 ibid. we examined the relations between these conjectures and Stark’s original conjectures as well as the complex and \( p \)-adic variants given by the author, Rubin, Serre etc. (see \[So3\], \[Ru\], \[Ta\]...). A refined version of Conj. C3\((k, g, T, p)\) was formulated as Conjecture 3.6 of \[So4\]. This was the conjecture tested numerically in \[R-S\] (always with \( k \) quadratic and \( g \) prime to \( T = T_p \)).

Now let \( K \) be a general finite, real, abelian extension of \( k \) with group \( G \). The above conjectures generalise easily from the case \( K = k(g) \) by ‘taking norms from the conductor field’. More precisely, if \( K' \) is any other such extension containing \( K \), with \( G' := \text{Gal}(K'/k) \), we shall write \( \pi_{K'/K} \) for the restriction \( G' \to G \), linearly extended to a homomorphism of groupings. The norm homomorphism \( N_{K'/K} \) defines a \( \pi_{K'/K} \)-semilinear map from \( U_S(K') \) into \( U_S(K) \) for any \( S \), so it induces a homomorphism (also denoted \( N_{K'/K} \)) from \( \bigwedge^r_{QG'} QU_S(K') \) to \( \bigwedge^r_{QG} QU_S(K) \), for which one checks easily that

\[
R_{K/k} \circ N_{K'/K} = \pi_{K'/K} \circ R_{K'/k}
\]

The conductor \( \mathfrak{f}(K) \) of \( K \) is the largest ideal \( \mathfrak{f} \) of \( \mathcal{O} \) such that \( k \subset K \subset k(\mathfrak{f}) \). It can be determined by local analysis of ramification (see \[24\]). In particular, \( S(\mathfrak{f}(K)) \) equals \( S_{\text{ram}}(K/k) \) which we define to be the set of finite places ramified in \( K/k \) (which are also those ramified in \( k(\mathfrak{f}(K))/K \) together with \( S_\infty \)). Let \( \mathcal{D} = \mathcal{D}_{k/Q} \) denote the absolute different. For any \( T \) we denote by \( \mathfrak{f}(K) \) and \( \mathcal{D}' \) the prime-to-\( T \) parts of \( \mathfrak{f}(K) \) and \( \mathcal{D} \) respectively (in the obvious sense) and define a function \( \Phi_{K/k,T} : \mathbb{C} \setminus \{1\} \to \mathbb{C}G \) by setting

\[
\Phi_{K/k,T}(s) := N(\mathfrak{f}(K)\mathcal{D}')^{s-1}\pi_{K(\mathfrak{f}(K))/K}(\Phi_{\mathfrak{f}(K),T}(s))
\]

If \( p \) is a prime number and \( T \) contains \( T_p \) then there exists a \( p \)-adically continuous function \( \Phi_{K/k,T,p} = \Phi_{K/k,T,p}^{(j)} : \mathcal{D}'(p) \to \mathbb{C}_pG \) such that \( \Phi_{K/k,T,p}(m) = j(\Phi_{K/k,T}(m)) \) for every \( m \in \mathcal{M}(p) \). Indeed such a function is clearly unique and given by

\[
\Phi_{K/k,T,p}(s) := \langle N(\mathfrak{f}(K)\mathcal{D}')^{s-1}\pi_{K(\mathfrak{f}(K))/K}(\Phi_{\mathfrak{f}(K),T,p}(s))
\]
If \( f(K) \) is not \( T \)-trivial then \( \Phi_{K/k,T} \) and \( \Phi_{K/k,T,p} \) extend to holomorphic functions on \( \mathbb{C} \) and \( D(p) \) respectively.

**Remark 2.1** In \[So3\] a complex function \( \Phi_{K,m,T} \) rather more general than \( \Phi_{K,k,T} \) was defined by a similar process, but without the factor \( N(f(K))^{s-1} \). Of course, neither this nor the factor \( \langle N(f(K))^{s-1} \rangle \) in \( \langle 3 \rangle \) has any effect on conjectures about values at \( s = 1 \). Both factors are introduced here for purposes of ‘normalisation’ and for conjectures at other integral values of \( s \).

Now take \( K' = k(f(K)) \) and apply \( \pi_{k(f(K))/K} \) to \( \langle 1 \rangle \) and \( \langle 2 \rangle \) with \( g = f(K) \). Using \( \langle 3 \rangle \) (with \( K' = k(f(K)) \)) and replacing \( N_k(f(K))/g \) by \( \eta \), we see that Conjectures \( C1(k, f(K), T) \), \( C2(k, f(K), T, p) \) and \( C3(k, f(K), T, p) \) imply respectively

**Conjecture C1(K/k, T)** Suppose that \( f(K) \) is not \( T \)-trivial. Then

\[
\Phi_{K/k,T}(1) = \frac{2^r}{\sqrt{d_k} \prod_{p \in T} N_p} R_{K/k}(\eta) \quad \text{for some } \eta \in \bigwedge^r \mathbb{Q}U_{S_{ram}}(K) \tag{6}
\]

**Conjecture C2(K/k, T, p)** Suppose that \( T \) contains \( T_p \) and \( f(K) \) is not \( T \)-trivial. Then

\[
\Phi_{K/k,T,p}(1) = \frac{2^r}{j(\sqrt{d_k}) \prod_{p \in T} N_p} R_{K/k,p}(\eta) \quad \text{for some } \eta \in \bigwedge^r \mathbb{Q}U_{S_{ram}}(K) \tag{7}
\]

**Conjecture C3(K/k, T, p)** Under the conditions of \( C2(K/k, T, p) \) there exists a simultaneous solution \( \eta \in \bigwedge^r \mathbb{Q}U_{S_{ram}}(K) \) of both \( \langle 6 \rangle \) and \( \langle 7 \rangle \).

It will sometimes be convenient to use the phrase ‘a solution of \( C1(K/k, T) \)’ (resp. of \( C2(K/k, T, p) \), resp. of \( C3(K/k, T, p) \)) to mean an element \( \eta \) of \( \bigwedge^r \mathbb{Q}U_{S_{ram}}(K) \) satisfying \( \langle 6 \rangle \) (resp. \( \langle 7 \rangle \), resp. \( \langle 6 \rangle \) and \( \langle 7 \rangle \)).

**Remark 2.2** If \( g \) is any non-\( T \)-trivial ideal, then \( C1(k(g)/k, T) \) implies \( C1(k, g, T) \) provided that the ideal \( f(k(g)) \) is also non-\( T \)-trivial. (This is the case, for instance, if \( f(k(g)) = g \), i.e. if \( g \) is a conductor). Indeed if we set \( K = k(g) \) then \( f(K) \) divides \( g \) and \( K \) also equals \( f(K) \). Theorem 2.1 of \[So4\] then implies that \( \Phi_{k,T}(1) \) is a multiple of \( \Phi_{f(K),T}(1) = \Phi_{K/k,T}(1) \) in \( \mathbb{C}Gal(K/k) \) so that \( \langle 6 \rangle \) implies \( \langle 1 \rangle \). Similar arguments work for \( C2 \) and \( C3 \) with similar restrictions.

We record some basic notations and results to be used in investigating these conjectures.

**Lemma 2.1** Suppose that \( K' \supset K \) as above and (for simplicity) that \( S_{ram}(K'/k) \) equals \( S_{ram}(K/k) \) and is disjoint from \( T \). Then

\[
\pi_{K'/K}(\Phi_{K'/k,T}(s)) = \Phi_{K/k,T}(s)
\]

**Proof** \( f(K') \) has the same prime factors as \( f(K) \) and is prime to \( T \), so Theorem 3.2 and Remark 3.1 of \[So3\] give

\[
\pi_{k(f(K'))/k(f(K))}(\Phi_{f(K'),T}(s)) = \left( \frac{Nf(K')}{Nf(K)} \right)^{1-s} \Phi_{f(K),T}(s)
\]

6
Now apply $\pi_{k(K)/K}$ to both sides and use $\pi_{k(K)/K} \circ \pi_{k(K')/k(K)} = \pi_{K'/K} \circ \pi_{k(K')/K'}$ and \([\text{4}]\) (twice).

For any profinite abelian group $A$ we write $A^\dagger$ for the group of (degree-1) complex characters of $A$ which factor through some finite quotient of $A$. Any $\chi \in C^\dagger$ gives rise to a $p$-adic character by composition with $j$ which, by abuse, will also be denoted $\chi$. So also will the associated characters (complex or $p$-adic) of $\text{Gal}(k(f(K))/k)$ and $\text{Cl}_{f(K)}(k)$, obtained from $\chi$ via $\pi_{k(K)/K}$ and the Artin map respectively. The conductor $f(\chi)$ of $\chi$ is that of the subextension $K^{\ker(\chi)}/k$ that it cuts out. Thus $f(K) = \beta f(\chi)$ for some integral ideal $\beta$. We write $f'(\chi)$ and $h'$ for the prime-to-$T$ parts so that $f'(K) = \beta' f'(\chi)$ and $\beta = \beta' \beta_0$ where $\beta_0$ has support in $T$. We write $\hat{\chi}$ for the unique, complex or $p$-adic primitive character on $\text{Cl}_{f(K)}(k)$ associated to $\chi$ and $\hat{\chi}(a)$ for its value on the class of an ideal $a$ which is prime to $f(\chi)$. If $\chi$ is complex then the Gauss sum $g_{f(\chi)}(\hat{\chi}) \in \mathbb{Q}^\times$ defined in [So3 §6.4] will be denoted simply $g(\hat{\chi})$. If $\chi$ is $p$-adic, $\chi = j \circ \phi$ then we define $g(\hat{\chi}) \in \mathbb{C}_p$ to be $j(g(\hat{\phi}))$. (Note that this is not independent of the choice of $j$.) With these conventions the following result relates $\Phi_{K/k,T}(s)$ and $\Phi_{K/k,T,p}(s)$ to the (primitive) complex and $p$-adic $L$-functions, denoted $L(s, \hat{\chi})$ and $L_p(s, \hat{\chi})$ respectively.

**Proposition 2.1** Suppose that $\chi$ is complex and $s \in \mathbb{C} \setminus \{1\}$, respectively $\chi$ is $p$-adic, $T \supset T_p$ and $s \in D^0(p)$. If $\beta_0$ is a product of $t \geq 0$ distinct primes not dividing $f(\chi)$ then

$$
\chi(\Phi_{K/k,T}(s)) = (-1)^t \hat{\chi}^{-1}(\beta_0)g(\hat{\chi})(N(f'(\chi)D'))^{s-1} \prod_{p \text{ prime} \atop p | \beta', \beta f(\chi)} \left(1 - \frac{\hat{\chi}^{-1}(p)}{Np^{1-s}}\right) \times
$$

\[
\prod_{p \text{ prime} \atop p \not| f(\chi)} \left(1 - \frac{\hat{\chi}(p)}{Np^s}\right) L(s, \hat{\chi})
\]

(8)

respectively

$$
\chi(\Phi_{K/k,T,p}(s)) = (-1)^t \hat{\chi}^{-1}(\beta_0)g(\hat{\chi})(N(f'(\chi)D'))^{s-1} \prod_{p \text{ prime} \atop p | \beta', \beta f(\chi)} \left(1 - \frac{\hat{\chi}^{-1}(p)}{Np^{1-s}}\right) \times
$$

\[
\prod_{p \text{ prime} \atop p \not| f(\chi)} \left(1 - \frac{\hat{\chi}(p)}{\omega(Np)(Np)^s}\right) L_p(s, \hat{\chi})
\]

(9)

Otherwise

$$
\chi(\Phi_{K/k,T}(s)) = 0,
\]

(10)

respectively

$$
\chi(\Phi_{K/k,T,p}(s)) = 0
\]

(11)
Lemma 2.2

If one can deduce this lemma from the existence of a solution of \( \text{it also follows unconditionally from Remark 2,} \)

\( R' \) irrelevant here and may be chosen conveniently. Moreover, the

\( \text{(isomorphic to} \) in the subspace (\( \bigwedge \)) as above we set

\( r(S, \chi) := \dim_\mathbb{C}(e_\chi \mathcal{U}_S(K)) = \begin{cases} r + |\{\text{prime ideals } q \in S : \chi|_{G(\mathfrak{q}) = 1}\}| & \text{if } \chi \neq \chi_0 \\ r - 1 + |\{\text{prime ideals } q \in S\}| = |S| - 1 & \text{if } \chi = \chi_0 \end{cases} \) (12)

(Here \( G(\mathfrak{q}) \) denotes the decomposition subgroup of \( G \) associated to \( \mathfrak{q} \). For the second equality, see \([1, \S\S I.3, I.4]\).) Then Proposition 2.1 shows that

\[ T \text{ is prime to } f(K) \implies \text{ord}_{s=1}(\chi(\Phi_{K/k, T}(s))) = r(S_{\text{ram}}(K), \chi) - r \forall \chi \in G^\dagger \] (13)

If \( S \neq S_\infty \) then \( r(S, \chi) \geq r \forall \chi \in G^\dagger \). In this case, we denote by \( e_{S,G,r} \) (respectively \( e_{S,G,>r} \)) the sum of the idempotents \( e_\chi \in \mathbb{C}G \) for those \( \chi \) with \( r(S, \chi) = r \) (respectively \( r(S, \chi) > r \)). Clearly, \( r(S, \chi) \) depends only on the Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \))-orbit \( [\chi] \) of \( \chi \) (so we write also \( r(S, [\chi]) \)). Hence \( e_{S,G,r} \) is a sum of \( e_{[\chi]}'s \) where \( e_{[\chi]} := \sum_{\chi' \in [\chi]} e_{\chi'} \in \mathbb{C}G \) and similarly for \( e_{S,G,>r} \). In particular, \( e_{S,G,r} \) and \( e_{S,G,>r} \) are complementary idempotents of \( \mathbb{C}G \) and \( e_{S,G,>r} := |G| e_{S,G,>r} \) lies in \( \mathbb{Z}G \). For any \( G \)-module we let \( A^{[S,G,r]} := \ker e_{S,G,>r}|A \). If \( A \) is also a \( \mathbb{Q} \)-vector space \( (\text{and } S \neq S_\infty) \) then \( A^{[S,G,r]} = e_{S,G,>r}A = \bigoplus_{r(S, [\chi])=r} e_{[\chi]}A \) on which the \( \mathbb{Q}G \)-action factors through the quotient ring \( e_{S,G,>r} \mathbb{Q}G = \prod_{r(S, [\chi])=r} F_{[\chi]} \). Here \( F_{[\chi]} \) denotes \( e_{[\chi]} \mathbb{Q}G \) which is a field (isomorphic to \( \mathbb{Q}(\chi) \) via \( \chi \)).

Lemma 2.2

If \( S \neq S_\infty \) then \( R_{K/k,S} \) is injective on \( (\bigwedge_{\mathbb{Q}G} \mathbb{Q}U_S(K))^{[S,G,r]} \).

Proof

One can deduce this lemma from the existence of a solution of \( C1(K/k, \emptyset) \) (cf. the proof of Proposition 3.8 (i) in \([So4]\)) and this would suffice for our applications. However, it also follows unconditionally from Remark 2, §1.6 of \([Po1]\). Indeed, the module \( \mathbb{C} \otimes_{\mathbb{Q}} (\bigwedge_{\mathbb{Q}G} \mathbb{Q}U_S(K))^{[S,G,r]} \) in our notation coincides with Popescu’s \( (\mathbb{C} \otimes_{\mathbb{Q}} U_{S,T})_{\sigma, S} \) (the set \( T \) is irrelevant here and may be chosen conveniently). Moreover, the \( \mathbb{C} \)-linear extension of our map \( R_{K/k,S} \) coincides (up to sign) with Popescu’s \( R_W \) on \( \mathbb{C} \otimes_{\mathbb{Q}} U_{S,T} \) provided we take his \( \w_i \) to be the places defined by our \( \pi_i \), for \( i = 1, \ldots, r \). Popescu’s context is function-fields but his proof goes over to number fields without real change. Note also that the Lemma actually holds without the assumption \( S \neq S_\infty \).

As we shall see later, if a solution of \( K \) exists then certain conditions imply that one exists in the subspace \( (\bigwedge_{\mathbb{Q}G} \mathbb{Q}U_{\text{ram}}(K))^{[\text{ram},G,r]} \) and, using the above lemma, that such a solution
Lemma 2.3 Suppose that $S$ contains at least two finite places. Then every element of $(\bigwedge_{\bar{Q}G} \bar{Q}U_S(K))^{[S,G,r]}$ can be written $\frac{1}{c}e_1 \wedge \ldots \wedge e_r$ for some $c \in \mathbb{N}$ and $e_1, \ldots, e_r \in E(K)^{[S,G,r]}$.

**Proof** Let $\chi$ be an element of $G^\dagger$. It is easy to show that $\dim_{\mathbb{C}}(e_\chi \bar{Q}U_S(K)) = r(S,\chi)$ and similarly that $\dim_{\mathbb{C}}(e_\chi \bar{Q}E(K)) = r(S_\infty,\chi)$. But the condition on $S$ shows that if $r(S,\chi) = r$ then $\chi \neq \chi_0$ and also equals $r$. Hence $e_\chi \bar{Q}U_S(K) = e_\chi \bar{Q}E(K)$ for all such $[\chi]$. Summing, we deduce that $(\bar{Q}U_S(K))^{[S,G,r]}$ equals $(\bar{Q}E(K))^{[S,G,r]} = (E(K)^{[S,G,r]})$ and is free of rank $r$ over $\mathbb{Z}_{\infty}[\chi] = \mathbb{Q}G^{[S,G,r]}$. Consequently, every element of $(\bigwedge_{\bar{Q}G} \bar{Q}U_S(K))^{[S,G,r]} = \bigwedge_{\bar{Q}G}^{[S,G,r]}((\bar{Q}U_S(K))^{[S,G,r]})$ can be written as $u_1 \wedge \ldots \wedge u_r$ with $u_1, \ldots, u_r \in E(K)^{[S,G,r]}$. \hfill \Box

3 $\mathbb{Z}_p$-Extensions

For the rest of this paper we fix an extension $K/k$ with Galois group $G$, a prime number $p$ and an embedding $j : \bar{Q} \to \mathbb{C}_p$ all subject to the hypotheses of the previous section as well as the following conditions

$$p \neq 2$$  \hfill (14)

$$S_{ram}(K/k) \text{ contains at least one finite place not dividing } p$$  \hfill (15)

and

$$p \text{ is unramified in } k/\mathbb{Q}$$  \hfill (16)

Condition (14) will be convenient but not necessary in all that follows. Conditions (15) and (16) could possibly be weakened for this section but (16) anyway will need to be strengthened for the next.

For $n = 0, 1, 2, \ldots$ let $\mathbb{B}_n$ be the unique subfield of $\mathbb{Q}(\mu_{p^{n+1}})$ which is cyclic and of degree $p^n$ over $\mathbb{Q}$. For any abelian extension $L$ of $k$ we write $L_n$ for $L\mathbb{B}_n$ (so $L_0 = L$). Thus $K_\infty$ is a totally real abelian extension of $k$ with group $G_n$ say. It is cyclic over $K$ of degree dividing $p^n$. We let $K_\infty = \bigcup_{n \geq 0} K_n$, the cyclotomic $\mathbb{Z}_p$-extension of $K$ and write $G_\infty$ for $\text{Gal}(K_\infty/k)$. Let $n_1 = n_1(K,p)$ be the smallest integer $n$ such that $K_n/\mathbb{Q}$ is ramified at all primes above $p$. Thus $n_1 = 0$ or $1$ and Condition (16) implies that the set $S_{ram}(K_n/k)$ equals $\Sigma = \Sigma(K/k,p) := S_{ram}(K/k) \cup T_p$ for all $n \geq n_1$ hence it contains at least two finite places by Condition (15). In particular, $f(K_n)$ is not $\emptyset$-trivial, so $\Phi_{K_n/k,\emptyset}$ is holomorphic and $C1(K_n/k,\emptyset)$ makes sense.

Lemma 3.1 Suppose $n \geq n_1$ and $\eta$ is a solution of $C1(K_n/k,\emptyset)$ then $e_{\Sigma,G_n,r,\eta}$ is the unique solution lying in $(\bigwedge_{\bar{Q}G_n} \bar{Q}U_S(K_n))^{[\Sigma,G_n,r]}$
Proof The condition that \( \eta \) be a solution of \( C1(K_n/k, \emptyset) \) is clearly equivalent to

\[
\frac{2^r}{\sqrt{d_k}} \chi(R_{K_n/k}(\eta)) = \chi(\Phi_{K_n/k, \emptyset}(1)) \quad \forall \chi \in G^n_k
\]  

(17)

To show that \( e_{\Sigma, G_n, r} \eta \) is also a solution, we must deduce that (17) also holds with \( e_{\Sigma, G_n, r} \eta \) in place of \( \eta \). But \( \chi(R_{K_n/k}(e_{\Sigma, G_n, r} \eta)) \) is equal to \( \chi(e_{\Sigma, G_n, r}) \chi(R_{K_n/k}(\eta)) \) and hence to \( \chi(R_{K_n/k}(\eta)) \) or 0 according as \( r(\Sigma, \chi) = r \) or \( r(\Sigma, \chi) > r \), and in the latter case, (13) shows that one also has \( \chi(\Phi_{K_n/k, \emptyset}(1)) = 0 \), as required. It is clear that \( e_{\Sigma, G_n, r} \eta \) lies in \( \bigwedge^{r}_{\mathbb{Q}G_n} \mathbb{Q}U_{\Sigma}(K_n) \)\(^{[\Sigma, G_n, r]}\) and the uniqueness follows from Lemma 2.2.

We shall be concerned primarily with the complex functions \( \Phi_{K_n/k, \emptyset} \) and \( \Phi_{K_n/k, T_p} \) (for \( n \geq n_1 \)) as well as the \( p \)-adic function \( \Phi^{(j)}_{K_n/k, T_{p^j}} \) which interpolates \( \Phi_{K_n/k, T_p} \). When no confusion is possible we shall abbreviate these three functions respectively as \( \Phi_n \), \( \Phi_{n,(p)} \) and \( \Phi_{n,p} \) and similarly write \( R_n \) and \( R_{n,p} \) for \( R_{K_n/k, \Sigma} \) and \( R^{(j)}_{K_n/k, \Sigma, p^j} \). We write also \( V_n \) for the \( \mathbb{Q} \)-vector space \( \bigwedge^{r}_{\mathbb{Q}G_n} \mathbb{Q}U_{\Sigma}(K_n) \) and \( V_0^n \) for its subspace \( V_n^{[\Sigma, G_n, r]} \) which identifies with \( \bigwedge^{r}_{\mathbb{Q}G_n} \mathbb{Q}E(K_n) \)\(^{[\Sigma, G_n, r]}\) for \( n \geq n_1 \), by Lemma 2.3. Let us assume that \( C1(K_n/k, \emptyset) \) holds for all \( n \geq n_1 \). Lemma 3.1 implies that it has a unique solution in \( V_0^n \), which we denote \( \eta_n \), so that

\[
\frac{2^r}{\sqrt{d_k}} R_n(\eta_n) = \Phi_n(1) \quad \forall n \geq n_1
\]  

(18)

Observe that Lemma 2.3 also implies that for each \( n \geq n_1 \) we have

\[
\eta_n = \frac{1}{c_n} \varepsilon_{1,n} \wedge \ldots \wedge \varepsilon_{r,n} \text{ for some } c_n \in \mathbb{N} \text{ and } \varepsilon_{1,n}, \ldots, \varepsilon_{r,n} \in E(K_n)^{[\Sigma, G_n, r]}
\]  

(19)

Now, the sequence \( (\eta_n)_{n \geq n_1} \) is coherent with respect to norms: Suppose \( n \geq m \geq n_1 \) and write \( N_{n/m} \) and \( \pi_{n/m} \) instead of \( N_{K_n/K_m} \) and \( \pi_{K_n/K_m} \) respectively. Then Equation (3) (with \( S = \Sigma \)) implies that \( R_m(N_{n/m} \eta_n) = \pi_{n/m}(R_n(\eta_n)) \) and Lemma 2.1 implies that \( \pi_{n/m}(\Phi_n(1)) = \Phi_m(1) \). Applying \( \pi_{n/m} \) to (18), it follows that \( N_{n/m} \eta_n \) is a solution of \( C1(K_m/k, \emptyset) \). (Incidentally, this argument also allows that \( C1(K_n/k, \emptyset) \) holds for all \( n \geq n_1 \) if it holds for infinitely many \( n \)). Moreover, if \( \chi \in G^+_m \) then \( \chi \circ \pi_{n/m} \in G^+_n \) and it is easy to see that \( r(\Sigma, \chi) = r(\Sigma, \chi \circ \pi_{n/m}) \). From this it follows without difficulty that \( \pi_{n/m}(e_{\Sigma, G_m, r}) = e_{\Sigma, G_m, r} \). Thus the \( \pi_{n/m} \)-semilinearity of \( N_{n/m} \) gives \( e_{\Sigma, G_m, r} N_{n/m} \eta_n = N_{n/m} e_{\Sigma, G_m, r} \eta_n = 0 \). Therefore \( N_{n/m} \eta_n \) lies in \( V_0^n \) and by uniqueness it follows that \( N_{m/n} \eta_n = \eta_m \). In particular,

\[
(\eta_n)_{n \geq n_1} \in \lim \left\{ \bigwedge^{r}_{\mathbb{Q}G_n} \mathbb{Q}E(K_n) : n \geq n_1 \right\} \subset \lim \left\{ \bigwedge^{r}_{\mathbb{Q}G_n} \mathbb{Q}U_{\Sigma}(K_n) : n \geq n_1 \right\}
\]  

(20)

where the limits are taken with respect to the norm maps \( N_{n/m} \).

The core idea of this paper is to investigate the consequences of a stronger hypothesis than (20) in which, very roughly speaking, one reverses the order of the functors \( \lim \) and \( \bigwedge^{r}_{\mathbb{Q}G_n} \) on the right hand side. Now, on the one hand \( \lim \mathbb{Q}U_{\Sigma}(K_n) \) equals \( \lim U_{T_p}(K_n) \) (since the primes above any finite prime \( q \notin T_p \) are inert and unramified in \( K_{\infty}/K_N \) for some \( N = N(q) \)). On the other, the consequences we have in mind will require norm-coherent
sequences of elements which are units \textit{locally} at primes above \( T_p \). We therefore consider the existence of elements of \( \bigwedge_{\mathbb{Q}G, n} \mathbb{Q}E_\infty(K) \) where \( E_\infty(K) := \lim E(K_n) \) and \( \mathbb{Q}E_\infty(K) = \mathbb{Q} \otimes \mathbb{Z} \ E_\infty(K) \) is treated as a module for the (uncompleted) group-ring \( \mathbb{Q}G_\infty \). We shall write \( \bar{\varepsilon} \) for the image of an element \( \varepsilon \in E_\infty(K) \) in \( \mathbb{Q}E_\infty(K) \) and, for each \( n \geq n_1 \), we denote by \( \beta_n \) the homomorphism from \( \bigwedge_{\mathbb{Q}G, n} \mathbb{Q}E_\infty(K) \) to \( \bigwedge_{\mathbb{Q}G, n} \mathbb{Q}E(K_n) \) (both notated additively) which sends \( \bar{\varepsilon}_1 \wedge \ldots \wedge \bar{\varepsilon}_r \) to \( \bar{\varepsilon}_{1,n} \wedge \ldots \wedge \bar{\varepsilon}_{r,n} \). For any \( K/k \) and \( p \) satisfying (14), (15) and (16) we formulate the

\textbf{Property} \( P_1(K/k, p) \) There exists \( \eta \in \bigwedge_{\mathbb{Q}G, n} \mathbb{Q}E_\infty(K) \) such that, for every \( n \geq n_1 \), \( \beta_n(\eta) \) lies in \( V_n^0 \) and is a solution of \( C_1(K_n/k, \emptyset) \) (the unique solution in \( V_n^0 \)) i.e.

\[ \Phi_n(1) = \frac{2r}{\sqrt{d_k}} R_n(\beta_n(\eta)) \quad \forall n \geq n_1 \]

(We shall say that \( \eta \) demonstrates \( P_1(K/k, p) \)).

\textbf{Remark 3.1} It is unclear at present whether we should expect this for every such pair \( (K/k, p) \), which is why we have called it a ‘property’ (that may or may not be possessed) rather than a conjecture.

\textbf{Remark 3.2} Any \( \eta \) demonstrating \( P_1(K/k, p) \) may be written as a finite sum

\[ \eta = \sum_{w=1}^{W} \frac{1}{c_w} \varepsilon_{i,w}^{(w)} \wedge \ldots \wedge \varepsilon_{r,w}^{(w)} \quad (21) \]

where each \( c_w \) is a positive integer and each \( \varepsilon_{i,w}^{(w)} \) is a norm-coherent sequence \( (\varepsilon_{i,n}^{(w)})_{n \geq n_1} \) with \( \varepsilon_{i,n}^{(w)} \in E(K_n) \) for each \( n \). The condition \( \beta_n(\eta) \in V_n^0 \) implies that \( \eta_n = \beta_n(\eta) \) satisfies (19) for each \( n \geq n_1 \). However, the same assumption at the infinite level (namely that we may take \( W = 1 \) in (21)) appears to be a strict strengthening of \( P_1 \). We shall not consider it further here except to note that it does hold in all the cases with \( K/\mathbb{Q} \) abelian so far considered (see Example 3.2 below).

\textbf{Remark 3.3} Let us define the \textit{denominator} of an element \( \eta \in \bigwedge_{\mathbb{Q}G, n} \mathbb{Q}E(K_n) \) (for any \( n \)) to be the smallest positive integer \( d \) such that \( d\eta \) lies in the lattice which is the image of \( \bigwedge_{\mathbb{Z}G, n} E(K_n) \) in \( \bigwedge_{\mathbb{Q}G, n} \mathbb{Q}E(K_n) \). Property \( P_1(K/k, p) \) implies in particular that the unique solutions of \( C_1(K_n/k, \emptyset) \) lying in \( V_n^0 \) have \textit{bounded denominators} as \( n \) increases. (Indeed, if \( \eta \) is given by (21) and demonstrates \( P_1(K/k, p) \), then for each \( n \geq n_1 \), the denominator of \( \beta_n(\eta) \) clearly divides the l.c.m. of the \( c_w \)’s.)

We consider some special cases of \( P_1(K/k, p) \).

\textbf{Example 3.1} The Case \( k = \mathbb{Q} \). In this case \( r = 1 \), \( K \) is an abelian field and Condition (13) implies that \( \mathfrak{f}(K) = f'p^{n_0} \mathbb{Z} \) for some \( f' \in \mathbb{Z}_{>1} \) prime to \( p \) and \( n_0 \geq 0 \). For every \( n \geq n_1 \) we have \( \mathfrak{f}(K_n) = f_n \mathbb{Z} \) where \( f_n := f'p^{\max(n_0,n+1)} \) so that \( k(\mathfrak{f}(K_n)) \) is the real cyclotomic field \( \mathbb{Q}(\zeta_{f_n})^+ \). For each \( n \geq n_1 \) we let \( \varepsilon_n = \varepsilon_{1,n} \) be the norm from \( \mathbb{Q}(\zeta_{f_n})^+ \) to \( K_n \) of \( (1 - \zeta_{f_n})(1 - \zeta_{f_n}^{-1}) \). Then it can be shown that \( \eta_n := -\frac{1}{2}\varepsilon_n^2 \varepsilon_{1,n}^2 \) lies in \( V_n^0 = (\mathbb{Q}E(K_n))^s(G_n,1) \) and
that if we take \( \bar{\tau} = \bar{\tau}_1 = 1 \in \text{Gal}(\overline{Q}/Q) \) then \( \eta_n \) is a solution of \( C1(K_n/Q, \emptyset) \). (See e.g. §3.5) for the case in which \( K_n \) equals \( Q(\zeta_{n_0})^+ \). The general case follows by ‘taking norms’. Now, is easy to check that the sequence \( (\varepsilon_n)_{n \geq n_0} \) is norm-coherent (in fact, this coherence is already a consequence of (20), at least up to \( \pm 1 \)). Therefore \( \eta := -\frac{1}{2} \otimes (\varepsilon_n)_{n \geq n_0} \) lies in \( \mathcal{Q}_{\infty}(K) \) and \( \eta_n = \beta_n(\eta) \). So \( \eta \) demonstrates \( P1(K/Q, p) \).

Example 3.2 The Case \( K/Q \) Abelian. This generalises the previous case inasmuch as it does not assume that \( k = Q \). In this case, we shall use the methods of [Po3, §3] to sketch a proof that \( P1(K/k, p) \) holds under (14), (15), (16) together with the additional hypothesis

\[ \Sigma(K/k, p) \text{ is precisely the set of places in } k \text{ above } \Sigma(K/Q, p) \quad (22) \]

This allows us to ‘base change’ from the previous example, using induction. Note first that the fixed, extended embeddings \( \bar{\tau}_i \) for \( i = 1, \ldots, r \) (used to define the \( \lambda_{K_n/k,i} \) for any \( n \)) form a set of coset representatives for \( \text{Gal}(\overline{Q}/k) \) in \( \text{Gal}(Q/Q) \). For each \( n \geq n_1(K, p) \) the group-ring \( \mathbb{C} \text{Gal}(K_n/Q) \) may be considered as a free module over \( \mathbb{C}G_n = \mathbb{C} \text{Gal}(K_n/k) \) with basis provided by \( \{ \bar{\tau}_1|K_n, \ldots, \bar{\tau}_r|K_n \} \). For any \( x \in \mathbb{C} \text{Gal}(K_n/Q) \) we write \( \det_{\mathbb{C}G_n}(x) \in \mathbb{C}G_n \) for the determinant of multiplication by \( x \) and, to avoid confusion, we shall denote by \( \eta_n(K/Q) \) and \( \bar{\eta}(K/Q) \) respectively the elements \( \eta_n \in \mathbb{Q}E(K_n) \) and \( \bar{\eta} \in \mathbb{Q}E(K) \) described in the previous example. Using the fact that \( \eta_n(K/Q) \) is a solution of \( C1(K_n/Q, \emptyset) \) and calculating \( \det_{\mathbb{C}G_n} \) by means of the basis described above, we find easily that

\[
\det_{\mathbb{C}G_n}(\frac{1}{2} \Phi_{K_n/Q,0}(1)) = \det_{\mathbb{C}G_n}(\lambda_{K_n/Q,1}(\eta_n(K/Q))) = \det(\lambda_{K_n/k,i}(\bar{\tau}_i^{-1} \eta_n(K/Q)))_{i=1}^r = R_{K_n/k}(\bar{\tau}_1^{-1} \eta_n(K/Q) \wedge \ldots \wedge \bar{\tau}_r^{-1} \eta_n(K/Q)) \quad (23)
\]

for all \( n \geq n_1 \) (provided that \( \lambda_{K_n/Q,1} \) is defined taking ‘\( \bar{\tau}_1 \) for \( Q \)’ to be 1, as in the previous example). On the other hand, it follows from [So3, Thm. 5.1] that for \( n \geq n_1 \), the elements \( \frac{1}{2} \Phi_{K_n/Q,0}(1) \) and \( \sqrt{-d_k} \Phi_{K_n/k,0}(1) \) are equal respectively to the elements which would be denoted \( \Theta_n(K_n/Q, S(0)) \) and \( \Theta_n(r)(K_n/k, S'(0)) \) in [Po2], where \( S = \Sigma(K/Q, p) \) and \( S' = \Sigma(K/k, p) \). Condition (22) therefore puts us in the situation considered by Popescu and his Equation (3) implies that \( \Theta_n(r)(K_n/k, S'(0)) = \det_{\mathbb{C}G_n}(\Theta_n(K_n/Q, S(0))) \). Putting this together with (23) shows that

\[
\sqrt{-d_k} \frac{1}{2r} \Phi_{K_n/k,0}(1) = R_{K_n/k}(\bar{\tau}_1^{-1} \eta_n(K/Q) \wedge \ldots \wedge \bar{\tau}_r^{-1} \eta_n(K/Q))
\]

Also, since \( \eta_n(K/Q) \) lies in \( (\mathbb{Q}E(K_n))^{(\Sigma(K_n/Q,p), \text{Gal}(K_n/Q), 1)} \) and (22) holds, an argument in [Po2] shows that \( \tau_1^{-1} \eta_n(K/Q) \wedge \ldots \wedge \tau_r^{-1} \eta_n(K/Q) \) lies in \( (\bigwedge_{\mathbb{Q}G_n} \mathbb{Q}E(K_n))^{(\Sigma(K_n/k,p), G_n, r)} \). Hence it is a solution of \( C1(K_n/k, \emptyset) \) whenever \( n \geq n_1 \) and it follows immediately that \( \tau_1^{-1} \eta_n(K/Q) \wedge \ldots \wedge \tau_r^{-1} \eta_n(K/Q) \) is an element of \( \bigwedge_{\mathbb{Q}G_{\infty}} \mathbb{Q}E_{\infty}(K) \) demonstrating \( P1(K/k, p) \).

We have also verified \( P1(K/k, p) \) in certain cases with \( K/Q \) abelian for which which (22) does not hold and it is quite possible that the latter hypothesis is in actually unnecessary. On
the other hand, the author knows of no extension of real number fields $K/k$ with $K \not\subset \mathbb{Q}^{ab}$ and such that even the basic Stark Conjecture $C1(K_n/k, \emptyset)$ has been proven for infinitely many $n$, for some $p$. This makes it hard to construct further examples in which $P1$ can be shown either to hold or to fail.

Our main object now is to study a $p$-adic analogue of property $P1(K/k, p)$, in which, roughly speaking, we replace $C1(K_n/k, \emptyset)$ by $C2(K_n/k, T_p, p)$. We must therefore start by examining the passage from $T = \emptyset$ to $T = T_p$ for the complex conjecture $C1$ and this in turn requires a closer analysis of ramifications above $p$ in the extension $K_\infty/k$. Let $L/F$ be a finite abelian field extension. If $L$ and $F$ are local fields and $v \in [-1, \infty)$, we recall that $G(L/F)^v$ denotes the $v$th ramification group (in the ‘upper numbering’, see [Se, Ch. IV]). If $L$ and $F$ are number fields and $v$ is a prime of $\mathcal{O}_F$, then $G(L/F)_p^v$ denotes the $v$th ramification group at any prime $\mathfrak{p}$ of $\mathcal{O}_L$ above $p$, naturally identified with the group $G(L_\mathfrak{p}/F_p)^v$ of the completed extension. The upper numbering is preserved under restriction and (since $L/F$ is abelian) depends only on $[v]$, the ‘integer ceiling’ of $v$. Taking $F$ to be $k$, local and global class field theory give the formula

$$\text{ord}_q(f(L)) = \min\{v \in \mathbb{N} : G(L/k)^v_q = \{1\} \} \quad \text{for all primes } q \text{ of } \mathcal{O} \quad (24)$$

**Lemma 3.2** Suppose that $n, v \geq 1$ are integers and let $p|p$. If $v \leq n$ then $G(k_n/k)^v_p = \text{Gal}(k_n/k_{v-1})$ which is cyclic of order $p^{n-v+1}$. Otherwise, $G(k_n/k)^v_p = \{1\}$.

**Proof** Let $\widehat{k}_n$ and $\widehat{\mathbb{B}}_n$ denote the completions of $k_n$ and $\mathbb{B}_n$ at the unique primes above $p$ and $p$ respectively. Then $\widehat{k}_n$ is the compositum of $\mathbb{B}_n$ with $k_n$. But $k_n/\mathbb{Q}_p$ is unramified by (16). We deduce that $\widehat{k}_n/\mathbb{Q}_p$ is abelian and, easily, that the group $G(\widehat{k}_n/k)^v_p$ equals $G(\widehat{k}_n/\mathbb{Q}_p)^v$. Thus it is a subgroup of $\text{Gal}(\widehat{k}_n/k)$ mapped isomorphically to $G(\mathbb{B}_n/\mathbb{Q}_p)^v$ by restriction to $\widehat{B}_n$. But the latter group is also the image of $G(\mathbb{Q}_p(\zeta_{p^{n+1}})/\mathbb{Q}_p)^v$ which equals $\text{Gal}(\mathbb{Q}_p(\zeta_{p^{n+1}})/\mathbb{Q}_p)^v$ if $v \leq n$, $\{1\}$ if not ([Se, Ch. IV]). Thus if $v \leq n$ then $G(\widehat{B}_n/\mathbb{Q}_p)^v = \text{Gal}(\widehat{B}_n/\mathbb{B}_{v-1})$ and so $G(\widehat{k}_n/k)^v_p$ equals $\text{Gal}(\widehat{k}_n/\mathbb{k}_{v-1})$ and is isomorphic to $\mathbb{Z}/p^{n-v+1}\mathbb{Z}$. Otherwise, $G(\widehat{k}_n/k)^v_p = \{1\}$. Of course, $G(\widehat{k}_n/k)^v_p$ and $\text{Gal}(\widehat{k}_n/\mathbb{k}_{v-1})$ are the images of $G(k_n/k)^v_p$ and $\text{Gal}(k_n/k_{v-1})$ respectively under the natural identification of $\text{Gal}(k_n/k)$ with $\text{Gal}(\widehat{k}_n/k)$, so the result follows. \hfill \Box

Since $k_n/k$ is unramified outside $p$, Lemma 3.2 and (16) give $f(k_n) = p^{n+1}\mathcal{O}$ for all $n \geq 1$. If we define $f_n$ to be $f(k_n)$ for all $n \geq 0$ (so $f_0 = f(K)$) then it follows easily that

$$f_n = \text{l.c.m.}(f_0, f(k_n)) = \text{l.c.m.}(f_0, p^{n+1}\mathcal{O}) \quad \text{for all } n \geq 1 \quad (25)$$

Now define

$$n_2 = n_2(K) = n_2(K/k, p) = \max(\{1\} \cup \{\text{ord}_p(f_0) : p|p\}) = \max(\{n_1\} \cup \{\text{ord}_p(f_0) : p|p\})$$

13
(For the fourth equality, note that \( n_1 \leq 1 \) and that \( \text{ord}_p(f_0) = 0 \) \( \forall \ p||p \) implies that \( K/k \) is unramified above \( p \) and hence that \( n_1 = 1 \).) Equation \((25)\) implies that

\[
\text{the } T_p\text{-part of } f_n \text{ is } p^{n+1}O \text{ for every } n \geq \max(1, n_2 - 1)
\]

\((26)\)

(In particular for every \( n \geq n_2 \).

**Remark 3.4** We are not assuming that the extensions \( K/k \) and \( K_\infty/k \) are linearly disjoint. Thus if we define \( n_3 \geq 0 \) by \( k \cap k_\infty = k_{n_3} \), then it may be that \( n_3 \geq 1 \), in which case \( p^{n_3+1}O = f(k_{n_3}) \) divides \( f_0 \). Thus in all cases \( n_2 \geq n_3 + 1 \), although this inequality may very well be strict.

**Lemma 3.3** Suppose that \( n, v \geq 1 \) are integers with \( v \geq n_2 \) and let \( p||p \). If \( v \leq n \) then \( G(K_n/k)_p^v = \text{Gal}(K_n/K_{v-1}) \) which is cyclic of order \( p^{n-v+1} \). Otherwise \( G(K_n/k)_p^v = \{1\} \).

**Proof** Equation \((24)\) and the definition of \( n_2 \) imply that \( G(K/k)_p^v = 1 \). Hence, restricting \( G(K_n/k)_p^v \) to \( K \) we find that it lies in \( \text{Gal}(K_n/K) \). The restriction to \( k_n \) therefore maps it isomorphically onto \( G(k_n/k)_p^v \). Now apply Lemma \(3.2\) \( \square \).

For any \( \chi \in G_{n_1}^\dagger \) and any prime \( q \) of \( \mathcal{O} \) we clearly have \( \text{ord}_q(f(\chi)) = \min\{v \in \mathbb{N} : G(K_n/k)_q^v \subset \ker \chi\} \). Taking \( v = n \) in the above lemma, we see that if \( n \geq n_2 \) then \( [K_n : K_{n-1}] = p \) and, for every \( \chi \in G_{n_1}^\dagger \),

\[
(f(\chi)|p^{-1}f_n \text{ for some } p||p) \iff \chi(\text{Gal}(K_n/K_{n-1})) = 1 \iff f(\chi)|p^{-1}f_n
\]

\((27)\)

For each \( n \geq n_2 \) we write \( e_n \) for the idempotent of \( \mathbb{Q}G_n \) given by \( 1 - p^{-1} \sum_{\sigma \in \text{Gal}(K_n/K_{n-1})} \sigma \) and note that \( f_n \) is not \( T_p\)-trivial since \( p^2|f_n \) (or by \((15)\)). Thus \( \Phi_{n,(p)} := \Phi_{K_n/k,T_p} \) is holomorphic.

**Proposition 3.1** If \( n \geq n_2 \) then \( \Phi_{n,(p)}(s) = p^{(n+1)(1-s)}e_n\Phi_{n}(s) \) for all \( s \in \mathbb{C} \).

**Proof** It suffices to prove that

\[
\chi(\Phi_{n,(p)}(s)) = p^{(n+1)(1-s)}\chi(e_n\Phi_{n}(s)) \quad \forall \chi \in G_{n_1}^\dagger, \forall s \in \mathbb{C} \setminus \{1\}
\]

\((28)\)

We use Proposition \(2.1\) with \( K = K_n \) (so \( h = f_n|f(\chi)^{-1} \) and suppose first that there exists \( p||p \) such that \( f(\chi)|p^{-1}f_n \). The case \( T = T_p \) of the said Proposition then has \( \text{ord}_p(h) \geq 1 \) but also \( \text{ord}_p(f_n) \geq 2 \) from which it follows that \( h_0 \) is either not square-free or not prime to \( f(\chi) \), hence the L.H.S. of \((28)\) vanishes (for all \( s \in \mathbb{C} \)). But \((27)\) implies that \( \chi(e_n) = 0 \) so the R.H.S. vanishes as well. If no such \( p \) exists then the \( T_p\)-part of \( f(\chi) \) is \( p^{n+1}O \) and in both the cases \( T = T_p \) and \( T = 0 \) of Proposition \(2.1\) every \( p \in T \) divides \( f(\chi) \) but neither \( h \) nor (by \((16)\)) \( D \). Therefore, we can apply \((8)\) in both cases to get (for \( s \neq 1 \)):

\[
\chi(\Phi_{n,(p)}(s)) = g(\hat{\chi})p^{(n+1)(1-s)}N(f(\chi)D)^{s-1} \prod_{p \text{ prime} \neq h, \ p|f(\chi)} \left(1 - \frac{\hat{\chi}^{-1}(p)}{Np^{1-s}}\right)L(s, \hat{\chi})
\]

\[
= p^{(n+1)(1-s)}\chi(\Phi_{n}(s))
\]
But (24) now implies that \( \chi(e_n) = 1 \) in this case. So, again, (23) holds.

If \( \eta \) demonstrates \( P1(K/k, p) \) then for each \( n \geq n_2 \), Proposition 3.1 gives \( \Phi_{n,(p)}(1) = e_n \Phi_n(1) = (2^r/\sqrt{d_k})R_n(e_n\beta_n(\eta)) \) and since (16) implies \( \prod_{p \in T_p} Np = p^r \), it follows that, for each \( n \geq n_2 \), \( p^r e_n\beta_n(\eta) \) is a solution of the complex conjecture \( C1(K_n/k, T_p) \) lying in \( V_n^0 \).

To deduce that it is also a solution of the \( p \)-adic conjecture \( C2(K_n/k, T_p, p) \) it is necessary and sufficient to assume only the existence of some common solution \( \eta'_n \) of \( C1(K_n/k, T_p) \) and \( C2(K_n/k, T_p, p) \) lying in \( V_n \), that is, a solution of \( C3(K_n/k, T_p, p) \). To see this, we use the following claim (proved below)

**Claim 3.1** For any such \( \eta'_n \) (with \( n \geq 1 \)) the element \( e_{\Sigma,G_n,r}\eta'_n \) is also a common solution of \( C1(K_n/k, T_p) \) and \( C2(K_n/k, T_p, p) \).

Indeed, assuming this, \( p^r e_n\beta_n(\eta) \) must be equal to \( e_{\Sigma,G_n,r}\eta'_n \) (since both lie in \( V_n^0 \) on which \( R_n \) is injective), so the former element is also solution of \( C2(K_n/k, T_p, p) \), as required. To sum up, we formulate a \( p \)-adic property potentially satisfied by any \( K/k \) and \( p \) satisfying (14), (15) and (16).

**Property** \( P2(K/k, p) \) There exists \( \eta \in \bigwedge_{\Sigma \in G_n} \mathbb{Q}C_{\infty}(K) \) such that, for every \( n \geq n_2 \), \( \beta_n(\eta) \) lies in \( V_n^0 \) and \( p^r e_n\beta_n(\eta) \) is a solution of \( C2(K_n/k, T_p, p) \), i.e.

\[
\Phi_{n,p}(1) = \frac{2^r}{j(\sqrt{d_k})}R_{n,p}(e_n\beta_n(\eta)) \quad \forall n \geq n_2
\]

(We shall say that \( \eta \) demonstrates \( P2(K/k, p) \)).

The preceding argument gives

**Proposition 3.2** Assume that \( C3(K_n/k, T_p, p) \) holds for all \( n \geq n_2 \). If \( \eta \) demonstrates \( P1(K/k, p) \) then it also demonstrates \( P2(K/k, p) \).

**Proof** It remains only to give the Proof of Claim 3.1. It suffices to show (cf. the proof of Lemma 3.1) that \( \chi(\Phi_{n,(p)}(1)) = \chi(\Phi_{n,(p)}(1)) = 0 \) for any \( \chi \in G_n^\dagger \) with \( r(\Sigma, \chi) > r \). But for any such \( \chi \) the set \( \Sigma(\chi) := \{ p \in \Sigma : \chi(G(p)) = 1 \} \) is non-empty. Moreover, all its elements are clearly prime to \( j(\chi) \).

Now apply Proposition 2.1 and Corollary 2.1 with \( K = K_n, T = T_p \). Suppose first that there exists \( p \in \Sigma(\chi) \cap T_p \) (for example, if \( \chi = \chi_0 \)). In this case, \( \text{ord}_p(\eta_0) = \text{ord}_p(h) = \text{ord}_p(j(K_n)) \geq 2 \) by (23) and the hypothesis that \( n \geq 1 \), so that \( \chi(\Phi_{n,(p)}(s)) \) and \( \chi(\Phi_{n,(p)}(s)) \) vanish identically. If, on the other hand, \( \Sigma(\chi) \cap T_p = \emptyset \) then \( \chi \neq \chi_0 \) and each \( p \in \Sigma(\chi) \) divides \( j' \) thus contributing a factor vanishing at \( s = 1 \) to the first product on the right-hand sides of (28) and (29). The Claim follows.

**Remark 3.5** If \( \eta \) is of the form given in (21) then it is easy to see that for each \( n \geq n_2 \),

\[
p^r e_n\beta_n(\eta) = \sum_{s=1}^{W} \frac{1}{c_s} \varepsilon_{i,n}^{(s)} \wedge \ldots \wedge \varepsilon_{r,n}^{(s)}
\]

where \( \varepsilon_{i,n}^{(s)} \) denotes \( (\varepsilon_{i,n}^{(s)})^{p} / N_{n/(n-1)}\varepsilon_{i,n}^{(s)} = (\varepsilon_{i,n}^{(s)})^{p} / \varepsilon_{i,n}^{(s)} \) (the last equality only makes sense if \( n - 1 \geq n_1 \)).
4 Consequences of Property $P2$

We shall study these by means of the $p$-adic power series attached by Coleman \[Co\] to norm-coherent sequences in certain towers of local fields. First, some basic facts and notations. Let $n \in \mathbb{Z}$, $n \geq -1$. Since $p$ is fixed, we shall usually abbreviate $\zeta_{p^{n+1}}$ to $\zeta_n$ (whenever this abuse cannot cause confusion) and identify this element of $\mathbb{Q}$ with its image in $\mathbb{C}_p$ under $j$. For any subfield $L$ of $\mathbb{C}$ or $\mathbb{C}_p$ we shall write $\tilde{L}_n$ for $L(\zeta_n)$ (so $\tilde{L}_{-1} = L$ and $\tilde{L}_\infty$ for $\bigcup_{n \geq -1} \tilde{L}_n$). For any closed subfield $L$ of $\mathbb{C}_p$ we shall write $\mathcal{A}(L)$ for the subspace of $L[[X]]$ consisting of those power series $h(X) = \sum_{i=0}^{\infty} a_i X^i$ whose norm $||h|| := \sup\{|a_i|_p : i \geq 0\}$ is finite. It is an $L$-Banach space under $|| \cdot ||$. For any closed and open subset $U$ of $\mathbb{Z}_p$, we denote by $\text{Meas}(U,L)$ the $L$-Banach space of of all (bounded) $L$-valued measures on $U$ equipped with the natural norm. ‘Extension by zero’ allows us to regard $\text{Meas}(U,L)$ as the subspace of $\text{Meas}(\mathbb{Z}_p,L)$ consisting of those measures supported on $U$. Moreover, there is a well known correspondence between $\mathcal{A}(L)$ and $\text{Meas}(\mathbb{Z}_p,L)$ which is in fact an isometry of Banach spaces. (More details of the correspondence may be found in [Sc, App. 5,6], [La] or [So2].) The $p$-adic ‘Weierstrass Preparation Theorem’ implies the following well known and useful fact (see e.g. [Wa Cor. 7.4]):

**Uniqueness Principle** If $[L : \mathbb{Q}_p]$ is finite then a non-zero element of $\mathcal{A}(L)$ can have only finitely many roots in $\{a \in \mathbb{C}_p : |a|_p < 1\}$.

Let $H$ be a finite, unramified extension of $\mathbb{Q}_p$ so that $\{\tilde{H}_n\}_{n \geq -1}$ is the division tower over $H$ associated to the multiplicative Lubin-Tate formal group (with group law $X + Y + XY'$) over $\mathbb{Q}_p$. The element $\pi_n := \zeta_n - 1$ is a uniformiser of $\tilde{H}_n$ for all $n \geq 0$ and $[p] \pi_n = (1 + \pi_n)^p - 1 = \pi_{n-1}$ for $n \geq 1$. The Frobenius element $\phi$ of $\text{Gal}(H/\mathbb{Q}_p)$ extends uniquely to a continuous automorphism of $\tilde{H}_\infty$ fixing $\zeta_n$ for all $n$ and acts coefficientwise on the ring $\tilde{H}_\infty[[X]]$ of formal power series. We let $\tilde{\kappa}$ denote the Lubin-Tate isomorphism from $\text{Gal}(\tilde{H}_\infty/H)$ to $\mathbb{Z}_p^\times$ so that $\sigma(\pi_n) = [\tilde{\kappa}(\sigma)] \pi_n = (1 + \pi_n)^{\tilde{\kappa}(\sigma)} - 1$ for all $\sigma \in \text{Gal}(\tilde{H}_\infty/H)$ and all $n$. Thus $\text{Gal}(\tilde{H}_\infty/H)$ also acts on $\mathbb{C}_p[[X]]$ by letting $\sigma \cdot f(X) := f((1 + X)^{\tilde{\kappa}(\sigma)} - 1)$. Let us write $U_\infty(H)$ for $\lim\{U(\tilde{H}_n) : n \geq 0\}$, the inverse limit of the local units with respect to the norm maps $N_{\tilde{H}_m/\tilde{H}_n}$. For each $u = (u_n)_{n \geq 0} \in U_\infty(H)$, it follows from [Co Thm. A] that there exists a formal power series $g \in \mathcal{O}_H[[X]]^\times$ such that $\phi^{-n}g(\pi_n) = u_n$ for all $n \geq 0$ and by the Uniqueness Principle, any infinite subsequence of these equations determines $g$ in $\mathcal{O}_H[[X]]$. Together with norm-coherence this implies that $g$ must satisfy the relation

$$\prod_{\zeta \in \mu_p} g(\zeta(1 + X) - 1) = \phi g((1 + X)^p - 1)$$

(29)

and be independent of the choice of unramified $H$ such that $u \in U_\infty(H)$. We therefore denote it simply $g(u;X)$. Moreover $\text{Gal}(\tilde{H}_\infty/H)$ acts naturally on $U_\infty(H)$ and the Uniqueness Principle also implies that

$$g(\sigma u; X) = \sigma \cdot g(u; X) \quad \forall u \in U_\infty(H), \quad \forall \sigma \in \text{Gal}(\tilde{H}_\infty/H)$$

(30)
Let $g^*(u; X)$ denote the power series $g(u; X)^p/\phi g(u; (1 + X)^p - 1)$. Simple arguments show that it satisfies $\prod_{\zeta \in \mu_p} g^*(u; \zeta(1 + X) - 1) = 1$ and lies in $1 + p\mathcal{O}_H[[X]]$. Therefore,

$$h^*(u; X) := \frac{1}{p} \log(g^*(u; X)) = \sum_{i=1}^{\infty} (-1)^{i-1}(g^*(u; X) - 1)^i/pi$$

defines an element of $\mathcal{O}_H[[X]]$. It follows from the above definitions that $\phi^{-n}h^*(u; \pi_n) = \frac{1}{p}\log_p(u_n^*)$ for all $n \geq 1$ where $u_n^* := u_n^{(p)}/u_{n-1}$, and also that $h(X) = h^*(u; X)$ satisfies the relation

$$\sum_{\zeta \in \mu_p} h(\zeta(1 + X) - 1) = 0 \quad (31)$$

If $L$ is any closed subfield of $\mathbb{C}_p$, we let $\mathcal{A}^*(L)$ denote the subspace of $\mathcal{A}(L)$ consisting of all power series $h \in \mathcal{A}(L)$ satisfying $\mathfrak{B}$-valuation. Under the power series/measure correspondence mentioned above, $\mathcal{A}^*(L)$ corresponds to Meas($\mathbb{Z}_p^\times \backslash L$). Two maps from Meas($\mathbb{Z}_p^\times \backslash L$) to itself are defined by the operation of multiplying a measure respectively by the functions $f_1 : x \mapsto x$ and $f_2 : x \mapsto \omega^{-1}(x)$ (the latter extended by zero from $\mathbb{Z}_p^\times$ to $\mathbb{Z}_p$). It is easy to check that both maps restrict to isometries from Meas($\mathbb{Z}_p^\times \backslash L$) to itself. Multiplication by $f_1$ corresponds to the differential operator $D : h(X) \mapsto (1 + X)h'(X)$ on $\mathcal{A}(L)$, while multiplication by $f_2$ is easily seen to correspond to the map from $\mathcal{A}(L)$ to itself given by

$$V : h(X) \mapsto -\frac{g(\omega)}{p} \sum_{l=1}^{p-1} \omega(l)h(\zeta_l^0(1 + X) - 1)$$

where $g(\omega) := \sum_{l=1}^{p-1} \zeta_l^0\omega^{-1}(l)$.

It follows that $D$ and $V$ restrict to commuting isometries from $\mathcal{A}^*(L)$ to itself (under the $\| \cdot \|$-norm) as does their composite $\tilde{D} = D \circ V = V \circ D : \mathcal{A}(L) \to \mathcal{A}(L)$, corresponding as it does to the multiplication of a measure by the function $f_1f_2 : x \mapsto \langle x \rangle$, extended by zero. For any $h \in \mathcal{A}^*(L)$ and any $t \in \mathbb{Z}$, positive or negative, we shall therefore write simply $D^t h$ (resp. $\tilde{D}^t h$) for $(D|_{\mathcal{A}^*(L)})^t h$ (resp. $(\tilde{D}|_{\mathcal{A}^*(L)})^t h$) which lies in $\mathcal{A}^*(L)$ but is obviously independent of the choice of $L$ given $h$. Clearly, $D^t h = \tilde{D}^t h$ whenever $(p - 1)t$. We can now define some twisted and generalised versions of the Coates-Wiles homomorphisms (compare [Wa, Ch. 13] for example).

If $u$ lies in $U_\infty(H)$ then $\tilde{D}^t h^*(u; X)$ lies in $\mathcal{O}_H[[X]]$. For each $t, n \in \mathbb{Z}$ with $n \geq 1$ we may therefore define a map

$$\delta_{t,n} : U_\infty(H) \to \mathcal{O}_{H_n}$$

$$u \mapsto \phi^{-n}\tilde{D}^t h^*(u; \pi_n)$$

It is easy to see that $\delta_{t,n}$ is a homomorphism (i.e. $\delta_{t,n}(u_1u_2) = \delta_{t,n}(u_1) + \delta_{t,n}(u_2)$) and that

$$\delta_{0,n} (u) = \frac{1}{p} \log_p(u_n^*) \forall n \geq 1 \quad (32)$$

However, for $t \neq 0$, the value of $\delta_{t,n}(u)$ depends on the entire sequence $(u_i^*)_{i \geq 1}$ (or $(u_i)_{i \geq 0}$), not just on $u_n^*$, indeed we have:
Lemma 4.1 Let \( \sigma \in \text{Gal}(\tilde{H}_\infty/H) \), \( u \in U_\infty(H) \) and suppose that \( t,n \in \mathbb{Z} \) with \( n \geq 1 \). Then \( \tilde{D}^t h^*(\sigma u; \pi_n) = \langle \kappa(\sigma) \rangle^t \tilde{D}^t h^*(u; \sigma(\pi_n)) \). In particular, if \( \sigma \) lies in \( \text{Gal}(\tilde{H}_\infty/H_\infty) \) then \( \delta_{t,n}(\sigma u) = \langle \kappa(\sigma) \rangle^t \delta_{t,n}(u) \).

**Proof** For each \( \sigma \in \text{Gal}(\tilde{H}_\infty/H) \) and \( h \in A^*(H) \) one checks that \( D\sigma \cdot h = \hat{\kappa}(\sigma) \sigma \cdot Dh \) and \( V \sigma \cdot h = \omega^{-1}(\hat{\kappa}(\sigma)) \sigma \cdot Vh \) so that \( D\sigma \cdot h = \langle \hat{\kappa}(\sigma) \rangle \sigma \cdot Dh \). It follows that \( \tilde{D}^t \sigma \cdot h = \langle \hat{\kappa}(\sigma) \rangle^t \sigma \cdot \tilde{D}^t h \) for all \( t \in \mathbb{N} \). Since \( \sigma \) and \( \tilde{D} \) map \( A^*(H) \) bijectively to itself, this equation also holds for \( t \in \mathbb{Z} \). Equation (30) implies that \( h^*(\sigma u; X) = \sigma \cdot h^*(u; X) \) and the first statement follows. The second statement clearly follows from the first by applying \( \phi^{-n} \).

We now return to the global situation and the notations of the last section, with \( K/k \) and \( p \) satisfying (14) and (15). For the purposes of the present section however, we replace (16) by the stronger hypothesis

\[
p \text{ splits completely in } k/\mathbb{Q}
\]  

Thus \( p\mathcal{O} = p_1 \cdots p_r \), where the \( p_i \) are distinct prime ideals numbered in such a way that \( p_i \) corresponds to the embedding \( j \circ \tau_i : k \to \mathbb{Q}_p \). Let \( L^{(i)} \) denote the inertia subfield of \( p_i \) in the extension \( \tilde{K}_\infty/k \). We shall need the following consequence of (33)

**Lemma 4.2** For each \( i = 1, \ldots, r \), the restriction map \( \text{Gal}(\tilde{K}_\infty/L^{(i)}) \to \text{Gal}(\tilde{k}_\infty/k) \) is an isomorphism. In particular \( \tilde{L}^{(i)} = \tilde{K}_\infty \forall i \).

**Proof** The extensions \( L^{(i)}/k \) and \( \tilde{k}_\infty/k \) and are respectively unramified and totally ramified at \( p_i \) so the map is surjective. Now, \( \text{Gal}(\tilde{K}_\infty/L^{(i)}) = \text{Gal}(\tilde{K}_\infty/k)^0_{p_i} \). Completing at any prime above \( p_i \), it follows from (33) that \( \text{Gal}(\tilde{K}_\infty/L^{(i)}) \) is isomorphic to a quotient of \( G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)^0 \) hence of \( \mathbb{Z}_p^\times \) by local class field theory. On the other hand, \( \text{Gal}(\tilde{k}_\infty/k) \) is itself isomorphic to \( \mathbb{Z}_p^\times \) so the surjectivity implies injectivity and hence both statements of the lemma. \( \square \)

**Lemma 4.3** Suppose \( n \geq v \geq 0 \) are integers and \( i \in \{1, \ldots, r\} \). Then

(i). \( G(\tilde{L}_n^{(i)}/k)_{p_i} = \text{Gal}(\tilde{L}_n^{(i)}/\tilde{L}_{v-1}) \).

(ii). If \( v \geq n_2 \) then \( G(\tilde{K}_n/k)_{p_i} = \text{Gal}(\tilde{K}_n/\tilde{K}_{v-1}) \).

(iii). \( \tilde{L}^{(i)} = \tilde{K}_m \) for all \( m \in \mathbb{N}, m \geq n_2 - 1 \).

**Proof** For part [1] we complete at a prime of \( \tilde{L}_n^{(i)} \) above \( p_i \). Thus, writing \( H \) for the completion of \( L^{(i)} \) at \( p_i \), it suffices to show that \( G(\tilde{H}_n/\mathbb{Q}_p)^v = \text{Gal}(\tilde{H}_n/\tilde{H}_{v-1}) \). But for every \( t \geq -1 \), \( \tilde{H}_t \) is the compositum of \( H \) (which is unramified over \( \mathbb{Q}_p \)) with \( \mathbb{Q}_p(\mu_{p^{t+1}}) \) so the proof may be concluded along lines to that of Lemma 3.2 mutatis mutandis. Part [iii] follows from Lemma 3.3 on taking \( p = p_i \) and replacing \( K \) by \( K(\mu_p) \) so that \( K(\mu_p)_t = K_t \) for any \( t \in \mathbb{N} \). (Note that the proof of the Lemma 3.2 does not require \( K \) to be totally real. Also, \( n_2(K(\mu_p)) = n_2(K) = n_2 \) since \( \{K(\mu_p)\} = \text{l.c.m.}(f_0, f(k(\mu_p))) = \text{l.c.m.}(f_0, p\mathcal{O}) \).) For
part [iii] we take $v = m + 1$ and let $n \to \infty$ in parts [i] and [ii] to get $\text{Gal}(\tilde{L}_n^i/\tilde{L}_m^i) = G(\tilde{L}_n^i/k)_{p_i}^{m+1}$ and $\text{Gal}(\tilde{K}_n^i/\tilde{K}_m) = G(\tilde{K}_n^i/k)_{p_i}^{m+1}$. Now apply Lemma \ref{lem:12}.

We write $\tilde{G}_\infty$ for $\text{Gal}(\tilde{K}_\infty/k)$, $\Delta$ for $\text{Gal}(\tilde{K}_\infty/K_\infty)$ and $\tilde{\Gamma}(i)$ for $\text{Gal}(\tilde{K}_\infty/L(i))$ (see the diagram \ref{dia:4}). Lemma \ref{lem:4.2} implies the (internal) direct product decomposition for each $i \in \{1, \ldots, r\}$

$$\tilde{G}_\infty = \Delta \times \tilde{\Gamma}(i)$$

(34)

The appropriate restriction maps identify $\Delta$ with the subgroup $\text{Gal}(K/K \cap K_\infty)$ of $G$ and, by \ref{lem:4.1}, with each of the groups $\text{Gal}(L(i)/k)$ (which are therefore isomorphic although the $L(i)$’s may be distinct). For each $i$ and each $n \geq -1$, we shall write $\tilde{\Gamma}(i)$ for the subgroup $\text{Gal}(\tilde{K}_\infty/\tilde{L}_n^i)$ of $\tilde{\Gamma}(i)$ so that if $n \geq n_2 - 1$ then $\tilde{\Gamma}(i) = \text{Gal}(\tilde{K}_\infty/\tilde{K}_n^i)$ independently of $i$. Let $\kappa : \tilde{G}_\infty \to \mathbb{Z}_p^\times$ be the cyclotomic character so that $\gamma(\zeta) = \zeta^{\kappa(\gamma)}$ for all $\zeta \in \mu_p$, and $\gamma \in \tilde{G}_\infty$. Then $\kappa$ factors through $\text{Gal}(\tilde{K}_\infty/k)$ and restricts to an isomorphism $\tilde{\Gamma}(i) \cong \mathbb{Z}_p^\times$ for each $i$ taking $\tilde{\Gamma}(i)$ onto $1 + p^{n+1}\mathbb{Z}_p$ for each $n \geq 0$.

Now fix $i$ and suppose we are given a sequence $(v_n)_{n \geq N}$ for some $N \geq 0$ such that, for all $n \geq N$, $v_n$ lies in $\tilde{L}_n^i$ and satisfies:

$$\text{ord}_p(v_n) = 0 \text{ for all } \mathfrak{Q}|p_i \text{ and } N^i_{L_n^i/L_n^i}v_m = v_n \quad \forall \ m \geq n$$

(35)

By taking norms to $\tilde{L}_n^i$ for each $n$ with $N > n \geq 0$ we can, if necessary, extend $v$ to a sequence $(v_n)_{n \geq 0}$ (also denoted $\tilde{v}$) with $v_n \in \tilde{L}_n^i$ satisfying \ref{eq:35} for all $n \geq 0$. Let $H(i)$ denote the field $j \circ \tilde{\tau}_n(L(i)^0)$ (topological closure in $\mathbb{C}_p$) which is a finite, unramified extension of $\mathbb{Q}_p$. For each $\gamma \in \tilde{G}_\infty$ it is clear that the sequence $j\tilde{\tau}_n\gamma(\tilde{L}_n^i)$ lies in $U_\infty(H(i))$. Suppose $\gamma' \in \tilde{G}_\infty$ fixes $\tilde{L}_n^i$ for some $n \geq 1$. Since $\tilde{K}_\infty/\tilde{L}_n^i$ is totally ramified at $p_i$, we must have $j\tilde{\tau}_n\gamma' = \gamma'j\tilde{\tau}_n\gamma$ for some $\tilde{\gamma} \in \text{Gal}(\tilde{H}_\infty^i/\tilde{L}_n^i)$ and the action on $\mu_{p^\infty}$ shows that $\kappa(\tilde{\gamma}) = \kappa(\gamma) = \langle \kappa(\gamma) \rangle$. Now apply Lemma \ref{lem:4.1} with $H = H(i)$, $\mathfrak{u} = j\tilde{\tau}_n\gamma$ and $\sigma = \tilde{\gamma}$. We deduce easily:

**Lemma 4.4** Fix $t, n \in \mathbb{Z}$, $n \geq 1$ and $i$ and $v$ as above. As $\gamma$ varies in $\tilde{G}_\infty$, the value of $\langle \kappa(\gamma) \rangle^{-1}\delta_{t,n}(j\tilde{\tau}_n\gamma)$ depends only on the image of $\gamma$ in $\text{Gal}(\tilde{L}_n^i/k)$.

We now define the ‘higher $p$-adic regulators’. Suppose $\xi$ is an element of $\mathcal{E}_\infty(K)$ as defined in the last section. Then, for each $n \geq N := \max(n_1, n_2 - 1)$ we have $\varepsilon_n \in K_n \subset \tilde{K}_n^i = \tilde{L}_n^i$ for $i = 1, \ldots, r$ (by Lemma \ref{lem:4.3}(iii)) and for all $n \geq n_2 - 1$ the norm map $N_{L_n^i/K_n}^i = N_{K_n^i/K_n}$ identifies with $N_{K_n^i/K_n}$ by restriction for each $m \geq n$. Therefore $v_n = \varepsilon_n$ satisfies \ref{eq:35} for each $n \geq N$ and every $i$. Clearly, $\text{Gal}(\tilde{K}_\infty/K_\infty)$ fixes $\xi$ and lies in the kernel of $\langle \kappa(\cdot) \rangle$. Therefore, Lemma \ref{lem:4.4} implies that for each $i$ and all $t, n \in \mathbb{Z}$, $n \geq n_2(\geq \max(N, 1))$, the quantity $\langle \kappa(\gamma) \rangle^{-1}\delta_{t,n}(j\tilde{\tau}_n\xi)$ actually depends only on the image of $\gamma \in \tilde{G}_\infty$ in $\text{Gal}(\tilde{K}_n^i \cap K_\infty/k) = G_n$. It follows that, for each such $i, t, n$ the following map (depending on $j$)

$$\tilde{L}_{i,t,n} : \mathcal{E}_\infty(K) \longrightarrow \mathcal{O}_{\tilde{H}_n^i}G_n$$

$$\xi \longmapsto \sum_{\sigma \in G_n} \langle \kappa(\sigma) \rangle^{-1}\delta_{t,n}(j\tilde{\tau}_n(\sigma)(\xi))\sigma^{-1}$$

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By linearity it suffices to prove the equality when
\begin{equation}
\frac{j\tilde{t}\tilde{\sigma}(\varepsilon_n^*)}{j\tilde{t}\tilde{\sigma}(N_{L_{\mathcal{H}}/L_{\mathcal{H}_{n-1}}}(\varepsilon_n))} = \frac{j\tilde{t}\tilde{\sigma}(\varepsilon_n^p)}{j\tilde{t}\tilde{\sigma}(N_{K_n/K_{n-1}}(\varepsilon_n))} = \frac{j\tilde{t}\tilde{\sigma}((pe_n)(\varepsilon_n))}{j\tilde{t}\tilde{\sigma}(N_{\mathcal{H}_{n-1}/\mathcal{H}_{n-2}}(\varepsilon_n))}
\end{equation}

Putting this together with (32) and the definition of \(\lambda_{K_n/k,i,p}\), we get
\begin{equation}
\tilde{L}_{i,0,n}(\varepsilon) = \frac{1}{p}\lambda_{K_n/k,i,p}((pe_n)(\varepsilon_n)) = e_n\lambda_{K_n/k,i,p}(\varepsilon_n) \quad \text{for each } i = 1, \ldots, r \text{ and } n \geq n_2. \quad (36)
\end{equation}

Proceeding by analogy with \(\lambda_{K_n/k,i,p}\), we extend each map \(\tilde{L}_{i,t,n}\) linearly to \(\mathbb{Q}\mathcal{E}_\infty(K)\). Then, for each \(n \geq n_2\) and \(t \in \mathbb{Z}\) we define a unique \(\mathbb{Q}\mathcal{E}_\infty\)-linear, \((\text{higher})\) \(p\)-adic regulator map \(\tilde{R}_{t,n}\) from \(\bigwedge^r_{\mathbb{Q}\mathcal{E}_\infty} \mathbb{Q}\mathcal{E}_\infty(K)\) to \(\mathbb{C}_p\mathcal{G}_n\) by letting \(\tilde{R}_{t,n}(u_1 \wedge \ldots \wedge u_r) = \det(\tilde{L}_{i,t,n}(u_i))_{i=1}^r\) for all \(u_1, \ldots, u_r \in \mathbb{Q}\mathcal{E}_\infty(K)\).

**Lemma 4.5** \(\tilde{R}_{0,n}(\eta) = R_{n,p}(e_n\beta_n(\eta))\) for all \(\eta \in \bigwedge^r_{\mathbb{Q}\mathcal{E}_\infty} \mathbb{Q}\mathcal{E}_\infty(K)\) and every \(n \geq n_2\).

**Proof** By linearity it suffices to prove the equality when \(\eta\) is of the form \(\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r\) with \(\tilde{e}_i \in \mathcal{E}_\infty(K)\) for \(i = 1, \ldots, r\). But then Equation (33) gives \(\tilde{R}_{0,n}(\eta) = \det(e_n\lambda_{K_n/k,i,p}(\varepsilon_n))_{i=1}^r = R_{n,p}(e_n\tilde{e}_1 \wedge \ldots \wedge e_n\tilde{e}_r) = R_{n,p}(e_n\beta_n(\eta))\).

Now let \(\eta\) be any element of \(\bigwedge^r_{\mathbb{Q}\mathcal{E}_\infty} \mathbb{Q}\mathcal{E}_\infty(K)\) and \(m, n\) integers with \(n \geq n_2\) and consider the equation
\begin{equation}
\Phi_{n,p}(m) = \frac{2r}{j(\sqrt{d_k})} \tilde{R}_{1-m,n}(\eta) \quad (37)
\end{equation}

The main result of this paper is

**Theorem 4.1** Suppose that \(K\) and \(p\) satisfy (14), (15) and (33). For a given element \(\eta \in \bigwedge^r_{\mathbb{Q}\mathcal{E}_\infty} \mathbb{Q}\mathcal{E}_\infty(K)\), Equation (37) holds for all pairs \((m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq n_2}\) if and only if it holds for infinitely many such pairs.

The proof of this theorem will occupy the next section. Its relevance here comes largely from the following

**Corollary 4.1** If (14), (15) and (33) hold and \(\eta\) demonstrates \(P2(K/k,p)\) then it satisfies Equation (37) for all \((m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq n_2}\).

**Proof** If \(\eta \in \bigwedge^r_{\mathbb{Q}\mathcal{E}_\infty} \mathbb{Q}\mathcal{E}_\infty(K)\) demonstrates \(P2(K/k,p)\) then it satisfies (37) for \(m = 1\) and all \(n \geq n_2\), by Lemma 4.5 \(\square\)
5 The Proof of Theorem 4.1

To lighten the notation we shall suppress $j$, essentially regarding $\bar{Q}$ simultaneously as a subset of $\mathbb{C}$ and of $\mathbb{C}_p$, whenever this cannot lead to confusion. In addition to the notations $\tilde{\Delta}$ and $\tilde{\Gamma}^{(i)}$ introduced above, we set $\Delta := \text{Gal}(K_\infty/k_\infty)$, $\tilde{\Gamma} := \text{Gal}(\tilde{k}_\infty/k)$ and $\Gamma := \text{Gal}(k_\infty/k)$ which we identify with the subgroup $\text{Gal}(\tilde{k}_\infty/k_0)$ of $\tilde{\Gamma}$ in the obvious way (see the diagram (38)). To further lighten notation we shall regard the isomorphism $\tilde{\Gamma} \cong \mathbb{Z}_p^\times$ induced by $\kappa$ as an identification, under which $\Gamma$ corresponds to the subgroup $1 + p\mathbb{Z}_p$. The restriction to $K_\infty$ gives a surjection from $\tilde{G}_\infty$ to $G_\infty$ which, with the above identifications, fits into the exact commuting diagram (39). The left-hand vertical map here is injective and we regard it henceforth as an inclusion, i.e. we identify $\tilde{\Delta}$ with $\text{Gal}(K_\infty/K_\infty \cap \tilde{k}_\infty) \subset \Delta$. Lemma 4.2
Lemma 5.1
Suppose \( \tilde{\alpha}_i : \mathbb{Z}_p^\times \to \tilde{G}_\infty \) with image \( \tilde{\Gamma}^{(i)} \) which splits the top row of (39). Lemma 4.3 (iii) implies that all the \( \tilde{\alpha}_i \) agree on the subgroup \( 1 + p^{\nu_z} \mathbb{Z}_p \) which they map isomorphically onto \( \text{Gal}(\overline{K}_\infty/\overline{K}_{n_2-1}) \). We fix once and for all an injective homomorphism \( \tilde{\alpha}_i \) with the inclusion \( 1 + p^{\nu_z} \mathbb{Z}_p \hookrightarrow \mathbb{Z}_p^\times \) and the surjection \( \tilde{G}_\infty \to \tilde{G}_\infty \). Thus, whether or not \( \alpha \) actually equals \( \alpha_i \) for some \( i \), we may clearly assume w.l.o.g. that it is chosen to satisfy

\[
\alpha(x) = \tilde{\alpha}_i(x)|_{K_{\infty}} \in \text{Gal}(K_{\infty}/K_{n_2-1}) \quad \text{for all } x \in 1 + p^{\nu_z} \mathbb{Z}_p \text{ and all } i \in \{1, \ldots, r\} \quad (40)
\]

Given such a splitting \( \alpha \), the assignment \( \chi \mapsto (\chi|_{\Delta}, \chi \circ \alpha) \) defines a map \( G_{\infty}^\dagger \to \Delta^\dagger \times \Gamma^\dagger \) which is clearly bijective. We denote by \( \theta \times \psi \in G_{\infty}^\dagger \) the inverse image of \( (\theta, \psi) \in \Delta^\dagger \times \Gamma^\dagger \) under this bijection.

The first step in proving Theorem 4.1 is to break down Equation (37) by characters (linearly extended to the group ring). If \( \eta \in \bigwedge_r^{\mathbb{Q}_{G_{\infty}}} \mathbb{Q} \mathcal{E}_{\infty}(K) \) then for any \( (m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq n_2} \) we have

\[
\Phi_{n,p}(m) = \frac{2^r}{\sqrt{d_k}} \triangleleft \frac{\tilde{R}_{1-m,n}(\eta)}{\gg} \iff \\
\chi(\Phi_{n,p}(m)) = \frac{2^r}{\sqrt{d_k}} \chi(\tilde{R}_{1-m,n}(\eta)) \quad \text{for all } \chi \in G_{\infty}^\dagger \text{ factoring through } G_n \quad (41)
\]

For each character \( \chi \in \tilde{G}_{\infty}^\dagger \) we define an integer \( n(\chi) \in \mathbb{Z}_{\geq -1} \) by

\[
n(\chi) := \max \{ \text{ord}_p(f(\chi)) : i = 1, \ldots, r \} - 1 \\
= \min \{ l \geq -1 : \chi \text{ factors through } \text{Gal}(\overline{L}_i^{(z)}/k) \text{ for } i = 1, \ldots, r \} \\
= \min \{ l \geq -1 : (\chi \circ \tilde{\alpha})|_{1+p^{l+1} \mathbb{Z}_p} = 1 \text{ for } i = 1, \ldots, r \} \quad (42)
\]

Naturally, \( 1+p^0 \mathbb{Z}_p \) is to be interpreted as \( \mathbb{Z}_p^\times \) in the third equality (which then follows from the definition of \( \tilde{\alpha}_i \)). The second equality follows from \( f(\chi) = f(\tilde{K}_{\ker(\chi)}) \), Equation (24) and the fact that \( G(\overline{K}_\infty/k)^{l_1+1} = \text{Gal}(\overline{K}_\infty/\overline{L}_{l_1}^{(z)}) \), by Lemmas 4.2 and 4.3. If \( A \) is any quotient group of \( \tilde{G}_\infty \) we shall regard elements of \( A^\dagger \) as elements of \( \tilde{G}_{\infty}^\dagger \) by inflation. In particular, if \( \chi \in \tilde{G}_{\infty}^\dagger \) and \( \psi \in \Gamma^\dagger \) then \( n(\chi) \) and \( n(\psi) \) make sense.

Lemma 5.1 Suppose \( \chi \) is any element of \( G_{\infty}^\dagger \) with \( \chi = \theta \times \psi \) and \( n \in \mathbb{Z} \) with \( n \geq n_2 - 1 \).

(i). We have the equivalences: \( n(\psi) > n \Leftrightarrow (\psi(1+p^{n_2+1} \mathbb{Z}_p) \neq \{1\}) \Leftrightarrow (n(\chi) > n) \). In particular \( n(\psi) \geq n \Leftrightarrow n(\chi) \geq n_2 \), and in this case \( n(\chi) = n(\psi) = \min \{ l \geq n_2 : \psi(1+p^{l+1} \mathbb{Z}_p) = \{1\} \} \).

(ii). We have the equivalences: \( n(\psi) \leq n \Leftrightarrow (n(\chi) \leq n) \Leftrightarrow (\chi \text{ factors through } G_n) \).
PROOF Part (i) follows easily from (40) and the third equality in (42). The first equivalence in part (ii) follows from (i). For the second, the definition of \( n(\chi) \) and Lemma 5.3 (iii) show that \( (n(\chi) \leq n) \iff (\chi \text{ factors through } \text{Gal}(\tilde{K}_n/k) = \text{Gal}(\tilde{L}_n^{(i)}/k) \forall i) \iff (\chi \text{ factors through } \text{Gal}(\tilde{K}_n \cap K_\infty/k) = G_n). \)

The following result establishes the right-hand equation of (41) in a particularly easy case (the proof is deferred).

**Proposition 5.1** Suppose that \((m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq n_2} \) and \( \chi \in G^\dagger_\infty \) factors through \( G_n \). If \( n(\chi) < n \) then \( \chi(\Phi_{n,p}(m)) = 0 \) and \( \chi(\hat{\mathcal{R}}_{1-m,n}(\eta)) = 0 \) for all \( \eta \in \bigwedge_{Q_{G_\infty}}^r \mathbb{QE}_\infty(K). \)

Let \( G^\dagger_\infty = \{ \chi \in G^\dagger_\infty : n(\chi) \geq n_2 \} \) and \( \Gamma^\dagger = \{ \psi \in \Gamma^\dagger : n(\psi) \geq n_2 \}. \) Then Lemma 5.1 (i) implies that the bijection \( G^\dagger_\infty \to \Delta^\dagger \times \Gamma^\dagger \) restricts to a bijection \( G^\dagger_\infty \to \Delta^\dagger \times \Gamma^\dagger. \) We shall consider functions \( \mathbb{Z} \times \Gamma^\dagger \to \mathbb{C}_p \) and say that such a function \( C \) is of Iwasawa type if and only if there exists a power series \( c(T) \) with bounded coefficients in some finite extension of \( \mathbb{Q}_p \) such that \( C(m, \psi) = c((1 + p)^m \psi(1 + p)^{-1} - 1) \) for every \( (m, \psi) \in \mathbb{Z} \times \Gamma^\dagger. \) (The series converges because \( \psi \) takes values in \( \mu_{p^{\infty}} \).) We write \( \text{Iwa}(\mathbb{Z} \times \Gamma^\dagger) \) for the set of all such functions, considered as a \( \mathbb{Q}_p \)-algebra under pointwise operations. By the Uniqueness Principle, \( c(T) \) is unique given \( C, \) if it exists. Note also that the rôle of \( (1 + p) \) in this definition could be played by any topological generator of \( 1 + p\mathbb{Z}_p; \) the power series \( c \) would change but the set \( \text{Iwa}(\mathbb{Z} \times \Gamma^\dagger) \) would not. For each \( \theta \in \Delta^\dagger \) we define a function

\[
C_\theta : \mathbb{Z} \times \Gamma^\dagger \longrightarrow \mathbb{C}_p
\]

\[
(m, \psi) \longmapsto g(\bar{\chi})^{-1} \langle N\bar{f}(\chi) d_k \rangle^{1-m} \chi \left( \Phi_{n(\psi),p}(m) \right) \text{ where } \chi = \theta \times \psi \in G^\dagger_\infty
\]

Note that here and in what follows, the set \( T \) is implicit and equal to \( T_p. \) Therefore the notations \( \bar{f}(\chi), \bar{f}(K), \bar{f}(K_n) \) etc. represent the prime-to-\( p \) parts of \( f(\chi), f(K), f(K_n) \) etc. Similarly \( \mathcal{D}' \) denotes the prime-to-\( p \) part of \( \mathcal{D}, \) which equals \( \mathcal{D} \) by Condition (33) (or (10)). In particular \( N\bar{f}(\chi) d_k = N(f(\chi)\mathcal{D}'). \) For each \( \theta \in \Delta^\dagger \) and \( \eta \in \bigwedge_{Q_{G_\infty}}^r \mathbb{QE}_\infty(K) \) we define

\[
C_{\theta,\eta} : \mathbb{Z} \times \Gamma^\dagger \longrightarrow \mathbb{C}_p
\]

\[
(m, \psi) \longmapsto \frac{2^r}{\sqrt{d_k}} g(\bar{\chi})^{-1} \langle N\bar{f}(\chi) d_k \rangle^{1-m} \chi (\hat{\mathcal{R}}_{1-m,n}(\psi)(\eta)) \text{ where } \chi = \theta \times \psi \in G^\dagger_\infty
\]

**Proposition 5.2** For each \( \theta \in \Delta^\dagger \) the function \( C_\theta \) lies in \( \text{Iwa}(\mathbb{Z} \times \Gamma^\dagger) \)

**Proposition 5.3** For each \( \theta \in \Delta^\dagger \) and every \( \eta \in \bigwedge_{Q_{G_\infty}}^r \mathbb{QE}_\infty(K) \) the function \( C_{\theta,\eta} \) lies in \( \text{Iwa}(\mathbb{Z} \times \Gamma^\dagger) \)
The proofs of Propositions 5.2 and 5.3 are also deferred while we show how, along with Proposition 5.1, they imply Theorem 4.1. First note that for any choice of \((m, \psi)\) in \(\mathbb{Z} \times \Gamma^t\), \(\theta\) in \(\Delta^t\) and \(\eta \in \bigwedge_{QG_\infty} \mathcal{Q} \mathcal{E}_\infty(K)\), the character \(\chi = \theta \times \psi\) factors through \(G_{n(\chi)} = G_{n(\psi)}\) (by Lemma 5.1) and we have the equivalences:

\[
\begin{align*}
\chi \left( \Phi_{n(\psi),p}(m) \right) &= \frac{2^r}{\sqrt{d_k}} \chi \left( \tilde{R}_{1-m,n(\psi)}(\eta) \right) \quad \text{(with } \chi = \theta \times \psi) \\
\Leftrightarrow \quad C_{\theta}(m, \psi) &= C_{\theta,\mathfrak{n}}(m, \psi) \\
\Leftrightarrow \quad c_\theta((1+p)^m \psi(1+p)^{-1} - 1) &= c_{\theta,\mathfrak{n}}((1+p)^m \psi(1+p)^{-1} - 1)
\end{align*}
\]

where \(c_\theta\) and \(c_{\theta,\mathfrak{n}}\) are the power series whose existence is implied by the statements of Propositions 5.2 and 5.3 respectively. Suppose that \(\eta_0\) is an element of \(\bigwedge_{QG_\infty} \mathcal{Q} \mathcal{E}_\infty(K)\) such that \(\eta_0\) is satisfied for an infinite subset \(Y\), say, of \(\mathbb{Z} \times \mathbb{Z}_{\geq n_2}\) and let \(Y'\) be the inverse image of this set under the map \(\mathbb{Z} \times \Gamma^t \to \mathbb{Z} \times \mathbb{Z}_{\geq n_2}\) sending \((m, \psi)\) to \((m, n(\psi))\). Lemma 5.1 shows that this map is surjective so, in particular, \(Y'\) is also infinite. But the equivalence (41) shows that Equation (43) holds with \(\eta = \eta_0\) for any \(\theta \in \Delta^t\). Hence the same is true of Equation (41). Since the map \((m, \psi) \mapsto ((1+p)^m \psi(1+p)^{-1} - 1)\) is injective on \(Y'\), it follows that for any \(\theta \in \Delta^t\), the power series \(c_\theta\) and \(c_{\theta,\mathfrak{n}}\) agree at infinitely many \(a \in \mathbb{C}_p\) with \(|a|_p < 1\). The Uniqueness Principle therefore implies that \(c_\theta = c_{\theta,\mathfrak{n}}\) for all \(\theta \in \Delta^t\). Hence Equation (44) holds with \(\eta = \eta_0\) for any \(\theta \in \Delta^t\) and \((m, \psi) \in \mathbb{Z} \times \Gamma^t\). Hence the same is true of Equation (43). In other words, for any \(n \geq n_2\), the condition

\[
\chi \left( \Phi_{n,p}(m) \right) = \frac{2^r}{\sqrt{d_k}} \chi \left( \tilde{R}_{1-m,n}(\eta_0) \right) \quad \forall m \in \mathbb{Z}
\]

is satisfied by all \(\chi \in G^t\) with \(n(\chi) = n\). But by Lemma 5.1 again, any other character \(\chi \in G^t\) factoring through \(G_n\) must have \(n(\chi) < n\), in which case Proposition 5.1 implies that it satisfies (45) trivially. Thus the equivalence (41) shows that Equation (43) must hold with \(\eta = \eta_0\) for all \(m \in \mathbb{Z}\) and all \(n \geq n_2\), as required.

In order to prove Proposition 5.2, we first evaluate the term \(\chi \left( \Phi_{n(\psi),p}(m) \right)\) in the definition of \(C_\theta(m, \psi)\). If \(L\) is an infinite abelian extension of \(k\) and \(q\) a prime ideal of \(\mathcal{O}\) then for each \(v \in [-1, \infty)\) we denote by \(G(L/k)^v\) the \(v\)th ramification subgroup (in the upper numbering) of \(\text{Gal}(L/k)\), namely the inverse limit of the \(G(L'/k)^v\) as \(L'/k\) runs through all finite subextensions of \(L/k\). If \(q \nmid p\) and \(v \geq 0\) then the fact that \(k_{\infty}/k\) is unramified at \(q\) implies that \(G(K_{\infty}/k)^v\) is contained in \(\Delta\) and so maps isomorphically onto \(G(K/k)^v\). This implies in particular that for any \(\theta \in \Delta^t\) it makes sense to define an ideal \(\mathfrak{f}'(\theta)\) of \(\mathcal{O}\) which is prime to \(p\) by

\[
\text{ord}_q(\mathfrak{f}'(\theta)) = \min \{ v \in \mathbb{N} : G(K_{\infty}/k)^v \subset \ker(\theta) \} \quad \text{for all primes } q \text{ of } \mathcal{O} \text{ not dividing } p.
\]

Let \(\chi = \theta \times \psi \in G^t\) so in particular \(\theta = \chi|_{\Delta}\). Since \(\mathfrak{f}(\chi) = \mathfrak{f}(K_{\infty}^{\ker(\chi)})\) and \(G(K_{\infty}/k)^v\subset \Delta\) for each \(q \nmid p\) and all \(v \geq 0\), it follows from (24) and (40) that

\[
\mathfrak{f}(\chi) = \mathfrak{f}'(\theta)
\]
(where \( f'(\chi) \) denotes the prime-to-\( p \) part of \( f(\chi) \)). If \( \chi \) lies in \( G^t_\infty \), then \( n(\psi) \geq n_2 \) and \( \chi \) factors through \( G_{n(\psi)} \). The definition of \( n(\chi) \) shows that \( \text{ord}_p(f(\chi)) = n(\chi) + 1 = n(\psi) + 1 \) for some \( i \), hence for all \( i = 1, \ldots, r \), by \([27]\) with \( n = n(\psi) \). Therefore

\[
\chi = \theta \times \psi \in G^t_\infty \implies f(\chi) = p^{n(\psi) + 1} f'(\theta)
\]

(48)

On the other hand, Equations \([25]\) and \([26]\) show that \( f(K_{n(\psi)}) = f_n(\psi) = f'(K)p^{n(\psi) + 1} O \). Thus, applying Proposition \([24]\) (replacing \( K \) by \( K_{n(\psi)} \) and \( T \) by \( T_p \)) we find that \( \mathfrak{h}_0 = O \) and (with a slight change of notation) Equation \([2]\) gives

\[
\theta \in \Delta^t, (m, \psi) \in \mathbb{Z} \times \Gamma^t, \chi = \theta \times \psi \implies \\
\chi(\Phi_{n(\psi),p}(m)) = g(\tilde{\chi}) \langle Nf'(\chi) d_k \rangle^{n-1} \prod_{q \text{ prime } \mid (K), q \notin (a)} B_{\theta,q}(m, \psi) L_p(m, \tilde{\chi})
\]

(49)

where, for each \( \theta \in \Delta^t \) and prime ideal \( q \nmid p\tilde{f}'(\theta) \) of \( O \) we define the function

\[
B_{\theta,q} : \mathbb{Z} \times \Gamma^t \to \mathbb{C}_p
\]

\[
(m, \psi) \mapsto 1 - \frac{\dot{\chi}^{-1}(q)}{\langle Nq \rangle^{1-m}} \text{ where } \chi = \theta \times \psi \in G^t_\infty
\]

For any \( x \in \mathbb{Z}^t_p \) we set \( \ell(x) := \log_p(x)/\log_p(1 + p) \in \mathbb{Z}_p \) so that \( (1 + p)^{\ell(x)} = \langle x \rangle \). We shall need two simple lemmas.

**Lemma 5.2** For each \( \theta \in \Delta^t \) and \( q \nmid p\tilde{f}'(\theta) \) the function \( B_{\theta,q} \) lies in Iwa\((\mathbb{Z} \times \Gamma^t)\)

**Proof** Let \( L_q \) be the inertia subfield of \( q \) in \( k/\kappa \) so that \( L_q \supset k_\kappa \) and the decomposition group \( G(L_q/k)_q^{-1} \) is (infinite) procyclic with canonical topological generator \( \sigma_q \), say, given by the Frobenius at primes above \( q \) in every finite sub-extension. We fix a lift \( \tilde{\sigma}_q \in G_\infty \) of \( \sigma_q \) and let \( \delta_q = \tilde{\sigma}_q \sigma_q k_\kappa^{-1} \in \Delta \). Note that \( \tilde{\sigma}_q k_\kappa \in \text{Gal}(k/k) \) identifies with \( \langle Nq \rangle \in 1 + p\mathbb{Z}_p \) under \( \kappa \). Let \( \theta \in \Delta^t \), \( (m, \psi) \in \mathbb{Z} \times \Gamma^t \) and \( \chi = \theta \times \psi \). The condition \( q \nmid p\tilde{f}'(\theta) = p\tilde{f}'(\chi) \) means that \( \chi \) factors through \( \text{Gal}(L_q/k) \) and \( \dot{\chi}(q) = \chi(\sigma_q) = \dot{\chi}(\tilde{\sigma}_q) = \theta(\delta_q) \psi(\tilde{\sigma}_q k_\kappa) = \theta(\delta_q) \psi(\langle Nq \rangle) \). Thus \( B_{\theta,q}(m, \psi) \) can be written as \( 1 - \theta(\delta_q)^{-1} \langle Nq \rangle^{-1} ((1 + p)^{m \psi(1 + p)^{-1}}) \right)^{\ell(Nq)} \). The lemma now follows from the fact that \( (1 + T)^{\ell(Nq)} \in \mathbb{Z}_p[[T]] \). \( \square \)

Suppose that \( U \) is an open and closed subset of \( \mathbb{Z}^t_p \) and \( \mu \in \text{Meas}(U, L) \) for some finite extension \( L \) of \( \mathbb{Q}_p \). We shall write \( I_{U,\mu} \) for the function \( \mathbb{Z} \times \Gamma^t \to \mathbb{C}_p \) sending \( (m, \psi) \) to \( \int_{x \in U} \langle x \rangle^{-m \psi(\langle x \rangle)} d\mu \).

**Lemma 5.3** For \( U \), \( \mu \) as above, the function \( I_{U,\mu} \) lies in Iwa\((\mathbb{Z} \times \Gamma^t)\).

**Proof** If \( x \in \mathbb{Z}^t_p \) then the Binomial Theorem gives

\[
\langle x \rangle^{-m \psi(\langle x \rangle)} = ((1 + p)^{m \psi(1 + p)^{-1} - \ell(x)}) = \sum_{t=0}^{\infty} \left( \begin{array}{c} -\ell(x) \\ t \end{array} \right) ((1 + p)^{m \psi(1 + p)^{-1}} - 1)^t
\]

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where the series converges uniformly as a function of \( x \in \mathbb{Z}_p^\times \) with values in \( \mathbb{Z}_p[\psi] \). Therefore, \( I_{u,\mu}(m, \psi) \) may be written as \( \sum_{t=0}^{\infty} c_{u,\mu,t} ((1+p)^m \psi (1+p)^{-1})^t \) where \( c_{u,\mu,t} = \int_{x \in U} \left( -\ell(x) \right) \frac{d\mu}{t} \) is bounded as a function of \( t \).

For each \( \theta \in \Delta^1 \), Equation (49) implies that for every \( (m, \psi) \in \mathbb{Z} \times \Gamma^\dagger \)

\[
C_\theta(m, \psi) = \prod_{q \text{ prime} \not\in \{\ell'(K), \ell'-\ell'(\theta)\}} B_{\theta,q}(m, \psi) L_p(m, \theta \times \psi)
\]

(50)

The proof of Proposition 5.2 now splits into two cases according as \( \theta \) is or is not the trivial character.

**Case 1:** \( \theta \neq 1 \) In this case the Proposition will follow from (50) and Lemma 5.2 once we have proven

\[
\theta \neq 1 \implies \text{the map } \mathbb{Z} \times \Gamma^\dagger \ni (m, \psi) \mapsto L_p(m, \theta \times \psi) \text{ is of Iwasawa type.} \tag{51}
\]

We now deduce this implication from the results of [Ri] §4. To do this properly, we first need to connect our notations and set-up with those of *ibid.* For Ribet’s ideal \( 'f' \) we take \( f'(\theta) p \mathcal{O} \) so that Ribet’s group \( 'G' \) identifies with \( G_{\theta,\infty} := \text{Gal}(K_{\theta,\infty}/k) \) where \( K_{\theta,\infty} \) is the union over \( n \) of the strict ray-class fields over \( k \) modulo \( f'(\theta) p^{n+1} \mathcal{O} \). For every \( \psi \in \Gamma^\dagger \) the element of \( G_{\theta,\infty}^\dagger \) which we denote \( \theta \times \psi \) factors through \( \text{Gal}(K_{\theta,\infty}^\ker(\theta)/k) \) and since \( K_{\theta,\infty} \) clearly contains \( K_{\ker(\theta)}^\dagger \), we may regard \( \theta \times \psi \) as an element of \( G_{\theta,\infty}^\dagger \). In particular, we shall take the character of \( G_{\theta,\infty}^\dagger \) denoted \( \varepsilon \) by Ribet to be \( \theta \times 1 \). Our group \( \Gamma \) (identified with \( 1+p\mathbb{Z}_p \)) coincides with Ribet’s. His splitting of \( G_{\theta,\infty}^\dagger \) as \( \Gamma \times A' \) (with \( A = \text{Gal}(K_{\theta,\infty}/k) \)) depends on the choice of a splitting homomorphism \( \tilde{\alpha} \), say, from \( \Gamma \) to \( G_{\theta,\infty}^\dagger \). We may (and shall) choose \( \tilde{\alpha} \) to be a lift of the injective homomorphism \( \Gamma \to \text{Gal}(K_{\theta,\infty}^\ker(\theta)/k) \) obtained by composing our map \( \alpha : \Gamma \to G_{\infty} \) with the natural surjection \( G_{\infty} \to \text{Gal}(K_{\infty}^\ker(\theta)/k) \). Any \( \psi \in \Gamma^\dagger \) inflates to \( \tilde{\psi} \in K_{\theta,\infty}^\dagger \) which is trivial on \( A \) and the above choice means that element of \( G_{\theta,\infty}^\dagger \) denoted \( \tilde{\psi} \varepsilon \) in Ribet’s notation coincides with \( \theta \times \psi \). Moreover, if \( \psi \) lies in \( \Gamma^\dagger \) then [Ri] shows that the prime factors of \( f(\theta \times \psi) \) are those of \( f'\theta p \mathcal{O} \), so that the complex function which would be denoted \( \ell(s, \tilde{\psi} \varepsilon) \) by Ribet in this situation (See pp. 179 and 186 of [Ri]) is exactly equal to our primitive \( L \)-function \( L(s, \theta \times \psi) \) (see e.g. [So4] §2). Interpolating this equality \( p \)-adically for \( s \) in the dense subset \( M(p) \) of \( \mathbb{Z}_p \), it follows that (in an obvious notation)

\[
L_p,\text{Ribet}(m, \tilde{\psi} \varepsilon) = L_p(m, \theta \times \psi) \quad \forall (m, \psi) \in \mathbb{Z} \times \Gamma^\dagger
\]

(52)

Now, \( \theta \neq 1 \) implies that \( \varepsilon \) is non-trivial on \( A' \) and so, taking Ribet’s generator \( '\gamma' \) of \( \Gamma \) to be \( 1+p \), his statements (4.9) and (4.10) imply in this case that there exists a power series \( 'F_c(T)' \) with coefficients in \( R = \mathbb{Z}_p[\theta] \) such that \( L_{p,\text{Ribet}}(s, \tilde{\psi} \varepsilon) = F_c(\psi(1+p)(1+p)^{-s} - 1) \) for all \( s \in \mathbb{Z}_p \) and \( \psi \in \Gamma^\dagger \). Hence, if \( \theta \neq 1 \), Equation (52) gives \( L_p(m, \theta \times \psi) = c_p(1+p)^m \psi(1+p)^{-1} - 1 \) for all \( (m, \psi) \in \mathbb{Z} \times \Gamma^\dagger \), where \( c_p(T) = F_c((1+p)(1+T)^{-1} - 1) \in \mathbb{Z}_p[\theta][[T]] \), thus proving (51).

**Case 2:** \( \theta = 1 \). The comparison of our set-up with Ribet’s, and in particular Equation (52),
goes through exactly as in Case 1. Since Ribet’s ‘ε’ is now the trivial character of $G_{\theta, \infty}$ however, his statements (4.9) and (4.10) do not apply directly. On the other hand, since now $f'(\theta) = \mathcal{O}$, the product in (50) extends over the set of all prime divisors $q$ of $f'(K)$ and Condition (15) says that this set is non-empty. So if we fix $q_0$ dividing $f'(K)$ then Proposition 5.2 will follow from Lemma 5.2 and Equations (50) and (52) provided we can show that

the map $\mathbb{Z} \times \Gamma^\dagger \ni (m, \psi) \mapsto B_{1,\psi_0}(m, \psi)L_{p,\text{Ribet}}(m, \tilde{\psi})$ is of Iwasawa type.  

(53)

The field $K_{\theta, \infty} = K_{1, \infty}$ is now the union over $n$ of the strict ray-class fields over $k$ modulo $p^{n+1}\mathcal{O}$. We (temporarily) write $\pi$ for the natural surjection from $G_{1, \infty} = \text{Gal}(K_{1, \infty}/k)$ to $\Gamma$ (identified with $1 + p\mathbb{Z}_p$) and $\sigma_0$ for the element of $G_{1, \infty}$ which is the limit of the Frobenius at primes above $q_0$ in finite subextensions. For each $(m, \psi) \in \mathbb{Z} \times \Gamma^\dagger$ we substitute $m$ for $s$, $\sigma_0$ for ‘c’ and $\psi$ for ‘ε’ in Ribet’s (4.6) to obtain (in our notation)

$$(1 - \tilde{\psi}(\sigma_0)(Nq_0)^{-m})L_{p,\text{Ribet}}(m, \tilde{\psi}) = \int_{y \in \mathcal{G}_{1, \infty}} \pi(y)^{-1-m} \tilde{\psi}(y)d\lambda_{\sigma_0}$$

where $\lambda_{\sigma_0}$ is a (bounded) measure on the group $\mathcal{G}_{1, \infty}$ with values in $\mathbb{Q}_p$ as constructed by Ribet. We use $\pi$ to push forward the measure $\pi(y)\lambda_{\sigma_0}(y)$ on $\mathcal{G}_{1, \infty}$ to one on $1 + p\mathbb{Z}_p$ giving an element $\mu_{\sigma_0}$, say, of $\text{Meas}(1 + p\mathbb{Z}_p, \mathbb{Q}_p)$. Since also $\tilde{\psi}(\sigma_0) = (1 \times \psi)(q_0)$ by definition, the last equation may be rewritten

$$\frac{B_{1,\psi_0}(m, \psi)}{B_{1,\psi_0}(m, \psi) - 1}L_{p,\text{Ribet}}(m, \tilde{\psi}) = \int_{x \in \Gamma} x^{-m}\psi(x)d\mu_{q_0} = I_{1+p\mathbb{Z}_p,\mu_{q_0}}(m, \psi)$$

Multiplying both sides by $B_{1,\psi_0}(m, \psi) - 1$ and using Lemmas 5.2 and 5.3 establishes (53). This completes the proof of Proposition 5.2.

We now introduce some notation and two lemmas that will be used in the proofs of Propositions 5.1 and 5.3. For any $n \in \mathbb{N}$ and any ($p$-adic) character $\tilde{\psi} \in (\mathbb{Z}_p^\times)^\dagger$ with $1 + p^{n+1}\mathbb{Z}_p \subset \ker \tilde{\psi}$ we choose a set $R_n$ of representatives for $\mathbb{Z}_p^\times$ modulo $1 + p^{n+1}\mathbb{Z}_p$ and define the Gauss sum

$$g_n(\tilde{\psi}) := \sum_{u \in R_n} \zeta_n u \tilde{\psi}(u)^{-1} \in \mathbb{C}_p$$

which is clearly independent of the choice of $R_n$.

**Lemma 5.4** Suppose that $n \geq 1$ and $1 + p^n\mathbb{Z}_p \subset \ker \tilde{\psi}$. Then $g_n(\tilde{\psi}) = 0$.

**Proof** If $R_n$ is a set of representatives as above then, since $n \geq 1$, so is $R'_n := \{u + p^n : u \in R_n\}$. Since $\tilde{\psi}(u) = \tilde{\psi}(u + p^n)$ we get

$$\zeta_0 g_n(\tilde{\psi}) = \zeta_n^p g_n(\tilde{\psi}) = \sum_{u \in R_n} \zeta_n u + p^n \tilde{\psi}(u)^{-1} = \sum_{u' \in R'_n} \zeta_n^u \tilde{\psi}(u')^{-1} = g_n(\tilde{\psi})$$

but $\zeta_0 \neq 1$, so $g_n(\tilde{\psi}) = 0$. □

For $i = 1, \ldots, r$ we let $\tilde{\rho}_i$ be the unique element of $\tilde{\Delta}$ lifting the Frobenius element above $p_i$ in $\text{Gal}(L^{(i)}/k)$ (see (34)).

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Lemma 5.5 Suppose \( i \in \{1, \ldots, r\} \), \( \theta \in \Delta^1 \) and \( \underline{z} \in \mathcal{E}_\infty(K) \). Then there exists a measure \( \mu_{i,\underline{z}} \in \text{Meas}(\mathbb{Z}_p^*, H^{(i)}(\theta)) \) depending only on \( i, \theta \) and \( \underline{z} \) and such that for any \( n \in \mathbb{Z}_{\geq n_2} \) and for all \( (m, \psi) \in \mathbb{Z} \times \Gamma^1 \) satisfying \( n(\psi) \leq n \), we have

\[
\chi(\tilde{L}_{i,1-m,n}(\underline{z})) = g_n(\chi \circ \tilde{\alpha}_i)(\tilde{\rho}_i)^{-n} \int_{x \in \mathbb{Z}_p^*} \langle x \rangle^{1-m}(\chi \circ \tilde{\alpha}_i)(x) d\mu_{i,\underline{z}}
\]

where \( \chi = \theta \times \psi \).

Remark 5.1 Lemma 5.1(iii) implies that \( \chi \) factors through \( G_n \) so the L.H.S. of (54) makes sense. It also implies that \( n(\chi) \leq n \), so \( g_n(\chi \circ \tilde{\alpha}_i) \) makes sense, by (12). The proof will show that \( \mu_{i,\underline{z}} \) actually depends only on \( \theta|_{\Delta} \).

Proof As explained above, the conditions of the Lemma allow us to regard \( \chi \) as a character of \( G_\infty \) (hence of \( \hat{G}_\infty \)) factoring through \( G_n \). The definition of \( \tilde{L}_{i,1-m,n} \) therefore gives \( \chi(\tilde{L}_{i,1-m,n}(\underline{z})) = \sum \langle \kappa(\tilde{\sigma}) \rangle^{m-1} \delta_{1-m,n}(\tilde{\tau}_i \tilde{\alpha}_i(\underline{z})) \chi(\tilde{\sigma})^{-1} \) where \( \tilde{\sigma} \) runs through any set of representatives for \( \hat{G}_\infty \) modulo \( \text{Gal}(\bar{K}_\infty/K_n) \). Choose a set \( R_\Lambda \) of representatives for \( \mathbb{Z}_p^* \) modulo \( 1+p^{n+1}\mathbb{Z}_p \). Then it follows from (32) and the definition of \( \tilde{\alpha}_i \) that \( \{\tilde{\alpha}_i(u)\tilde{d} : u \in R_\Lambda, \tilde{d} \in \Delta\} \) is a set of representatives of \( \hat{G}_\infty \) modulo \( \tilde{\alpha}_i(1+p^{n+1}\mathbb{Z}_p) = \tilde{\Gamma}_{n}^{(i)} \) which equals \( \text{Gal}(\bar{K}_\infty/K_n) \) since \( n \geq n_2 - 1 \). It follows that

\[
\chi(\tilde{L}_{i,1-m,n}(\underline{z})) = |\bar{K}_n : K_n|^{-1} \sum_{u \in R_\Lambda} \sum_{d \in \Delta} \langle u \rangle^{m-1} \delta_{1-m,n}(\tilde{\tau}_i \tilde{\alpha}_i(u)\tilde{d} \underline{z})(\chi \circ \tilde{\alpha}_i)(u)^{-1} \theta(\tilde{d})^{-1}
\]

where \( a := |\bar{K}_0 : K| = |\bar{K}_n : K_n| \) \( \forall n \). The action on \( \mu_{p,\infty} \) shows that \( \tilde{\tau}_i \tilde{\alpha}_i(u) = \tilde{k}^{-1}(u)\tilde{\tau}_i \).

(Recall that \( \tilde{k} \) denotes the Lubin-Tate isomorphism from \( \text{Gal}(\bar{H}_\infty^{(i)}/H^{(i)}) \) to \( \mathbb{Z}_p^* \).) Therefore Lemma 4.1 gives \( \tilde{D}_{i,1-m,n}^{-1}\theta(\tilde{d})^{-1} \phi^{-n} h^{*(\tilde{\tau}_i \tilde{\alpha}_i(u)\tilde{d} \underline{z} ; \pi_n)}(\chi \circ \tilde{\alpha}_i)(u)^{-1} \theta(\tilde{d})^{-1} \)

Now for any \( \underline{z} \in U_\infty(H^{(i)}) \) it is clear that \( \phi^{-n} g(\underline{z} ; X) = g(\phi^{-n} \underline{z} ; X) \) and hence that the same equality holds with \( g^* \) in place of \( g \) on both sides, hence the same with \( h^* \) in place of \( g \). Moreover, it is easy to see that \( \phi^{-n} \tilde{\tau}_i = \tilde{\tau}_i \tilde{\rho}_i^{-n} \). Therefore

\[
a^{-1} \sum_{\tilde{d} \in \Delta} \theta(\tilde{d})^{-1} \phi^{-n} h^{*(\tilde{\tau}_i \tilde{d} \underline{z} ; X)} = a^{-1} \sum_{\tilde{d} \in \Delta} \theta(\tilde{d})^{-1} h^{*(\tilde{\tau}_i \tilde{\rho}_i^{-n} \tilde{d} \underline{z} ; X)} = \theta(\tilde{\rho}_i)^{-n} F(i, \theta, \underline{z} ; X)
\]

where \( F(i, \theta, \underline{z} ; X) \) is defined to be the power series \( a^{-1} \sum_{\tilde{d} \in \Delta} \theta(\tilde{d})^{-1} h^{*(\tilde{\tau}_i \tilde{d} \underline{z} ; X)} \), which lies in \( \mathcal{A}^*(H^{(i)}(\theta)) \). (Since \( p \) is odd, \( a \) divides \( p-1 \), from which it follows easily that \( F(i, \theta, \underline{z} ; X) \)

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are \( r \in E \). Actually has coefficients in the ring of valuation integers of \( H^{(i)}(\theta|_{\Delta}) \). Thus \( F(i, \theta, \varepsilon; X) \) corresponds to an element of \( \text{Meas}(\mathbb{Z}_p^\times, H^{(i)}(\theta)) \) which we denote \( \mu_{i, \theta, \varepsilon} \). As explained above, the formalism of the power series/measure correspondence implies that \( \tilde{D}^{1-m} F(i, \theta, \varepsilon; X) \) corresponds to the measure \( \langle x \rangle^{1-m} \mu_{i, \theta, \varepsilon} \) and also that the value of a power series at \( \zeta - 1 \) (for any \( \zeta \in \mu_{p^\infty} \)) is equal to the integral of the function \( x \mapsto \zeta x \) against the corresponding measure. Putting all this together, we obtain

\[
\chi (\tilde{L}_{i,1-m,n}(\varepsilon)) = \theta(\bar{\rho}_i)^n \sum_{u \in R_n} (\chi \circ \bar{\alpha}_i)(u)^{-1} \int_{x \in \mathbb{Z}_p} \xi_nux \langle x \rangle^{1-m} d\mu_{i, \theta, \varepsilon}
\]

\[
= \theta(\bar{\rho}_i)^n \int_{x \in \mathbb{Z}_p} \left( \sum_{u \in R_n} \xi_nux (\chi \circ \bar{\alpha}_i)(ux)^{-1} \right) \langle x \rangle^{1-m} (\chi \circ \bar{\alpha}_i)(x) d\mu_{i, \theta, \varepsilon}
\]

But, for any \( x \in \mathbb{Z}_p^\times \), the set \( \{ xu : u \in R_n \} \) is clearly another set of representatives of \( \mathbb{Z}_p^\times \) modulo \( 1 + p^{n+1} \mathbb{Z}_p \) and Equation (54) follows.

We can now prove Proposition 5.1. The general form of an element \( \eta \in \bigwedge^r G \mathcal{E}_\infty(K) \) is as in (21). The second equality of Proposition 5.1 will follow by linearity of \( \tilde{\chi} \). We can prove it in the special case

\[
\eta = \varepsilon_1 \wedge \ldots \wedge \varepsilon_r \quad \text{with} \quad \varepsilon_1, \ldots, \varepsilon_r \in \mathcal{E}_\infty(K)
\]

so that, for any \( m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq n_2} \) and \( \chi \in G^\dagger_\infty \) factoring through \( G_n \) we may write

\[
\chi (\tilde{R}_{1-m,n}(\eta)) = \det(\chi (\tilde{L}_{i,1-m,n}(\varepsilon)))_{i,l=1}^{r}
\]

Suppose \( n(\chi) < n \). Then on the one hand \( \chi = \theta \times \psi \) with \( n(\psi) < n \) (by Lemma 5.1 (ii)) so each term \( \chi (\tilde{L}_{i,1-m,n}(\varepsilon)) \) in (56) may be calculated by means of (54). On the other hand Equation (42) implies that for all \( i \) we have \( 1 + p^n \mathbb{Z}_p \subset \ker(\chi \circ \bar{\alpha}_i) \), hence \( g_n(\chi \circ \bar{\alpha}_i) \) vanishes by Lemma 5.4. Thus \( \chi (\tilde{L}_{i,1-m,n}(\varepsilon)) \) is zero for all \( i, l, m \). Hence the equality \( \chi (\tilde{R}_{1-m,n}(\eta)) = 0 \) follows. For the other equality in Proposition 5.1 we note that since \( n(\chi) < n \), Equations (42) and (25) imply that for some \( i \) one has \( \text{ord}_p(f(\chi)) < n + 1 = \text{ord}_p(f_n) \), i.e. \( f(\chi) \mid p^{n-1} f_n \). (In fact, this must hold for all \( i \) by (27), since \( n \geq n_2 \).) The equality \( \chi (\Phi_{n,p}(1)) = 0 \) therefore follows from (11) (cf. the proof of Proposition 3.1). This completes the proof of Proposition 5.1.

For Proposition 5.3 we introduce two new functions as follows. For each \( \theta \in \Delta^\dagger \), \( i = 1, \ldots, r \) and \( \varepsilon \in \mathcal{E}_\infty(K) \) we define \( E_\theta, D_{i, \theta, \varepsilon} : \mathbb{Z} \times \Gamma^\dagger \to \mathbb{C} \) by setting

\[
E_\theta(m, \psi) = \langle N^i(\theta) dk \rangle^{1-m} \psi((N^i(\theta) dk)) \quad \text{and} \quad D_{i, \theta, \varepsilon}(m, \psi) = \int_{x \in \mathbb{Z}_p} \langle x \rangle^{1-m} (\chi \circ \bar{\alpha}_i)(x) d\mu_{i, \theta, \varepsilon}
\]

with \( \chi = \theta \times \psi \in G^\dagger_\infty \) and \( \mu_{i, \theta, \varepsilon} \) as in Lemma 5.5.

**Lemma 5.6** For each \( i = 1, \ldots, r \) and \( \varepsilon \in \mathcal{E}_\infty(K) \) the functions \( E_\theta \) and \( D_{i, \theta, \varepsilon} \) are of Iwasawa type.

**Proof** Clearly, \( E_\theta(m, \psi) = e_\theta((1 + p)m \psi(1 + p)^{-1} - 1) \) where \( e_\theta(T) \) is the power series \( \langle N^i(\theta) dk \rangle (1 + T)^{-t(N^i(\theta) dk)} \in \mathbb{Z}_p[[T]] \). Thus \( E_\theta \) lies in \( \text{Iwa} (\mathbb{Z} \times \Gamma^\dagger) \). As for \( D_{i, \theta, \varepsilon} \) if \( x \in \mathbb{Z}_p^\times \)
then, with our identifications, we find \( \tilde{\alpha}_i(x)|_{k_\infty} = \langle x \rangle = \alpha(\langle x \rangle)|_{k_\infty} \). It follows that the formula 
\[
\nu_i(x) := \tilde{\alpha}_i(x)|_{K_\infty} \alpha(\langle x \rangle)^{-1}
\]
defines a homomorphism \( \nu_i : \mathbb{Z}_p^\times \to \Delta \) (trivial on \( 1 + p^n\mathbb{Z}_p \), by (10)). Thus if \( \chi = \theta \times \psi \in G^\dagger \), then \( \chi(\tilde{\alpha}_i(x)) = \chi(\tilde{\alpha}_i(x)|_{K_\infty}) = (\theta \circ \nu_i)(x) \psi(\langle x \rangle) \). Therefore, if \( (m, \psi) \in \mathbb{Z} \times \Gamma^\dagger \) and \( \chi = \theta \times \psi \), we have \( D_{i,\theta,\xi}(m, \psi) = I_{\mathbb{Z}_p^\times, \mu_i,\theta,\xi}(m, \psi) \) where \( \mu_i,\theta,\xi \in \text{Meas}(\mathbb{Z}_p^\times, H^{(i)}(\theta)) \) denotes the measure \( \langle x \rangle (\theta \circ \nu_i)(x) \mu_i,\theta,\xi(x) \), which depends only on \( i, \theta \) and \( \xi \). The result now follows from Lemma 5.3.

To prove Proposition 5.3, we may again assume (by linearity) that \( \eta \) is as in (55). Combining Equations (54) and (56) (with \( n = n(\psi) \)) and using also (47) and the definitions of \( D_{i,\theta,\xi} \) and \( E_\theta \), we find that for all \( \theta \in \Delta^\dagger \), and all \( (m, \psi) \in \mathbb{Z} \times \Gamma^\dagger \)

\[
C_{\theta,\xi}(m, \psi) = \frac{2^r}{\sqrt{d_k}} \left[ g(\hat{\chi})^{-1} \prod_{i=1}^r g_{n(\psi)}(\chi \circ \tilde{\alpha}_i) \prod_{i=1}^r \theta(\tilde{\rho}_i)^{-n(\psi)} \psi(\langle N\hat{\chi}'(\theta)d_k \rangle)^{-1} \right] \times E_\theta(m, \psi) \det(D_{i,\theta,\xi}(m, \psi))_{t=1}^r
\]

where, as usual, \( \chi \) denotes \( \theta \times \psi \). By Lemma 5.6, the proof of Proposition 5.3 will be complete once we have proven that the quantity in square brackets is of Iwasawa type, when considered as a function of \( (m, \psi) \). However it is evidently independent of \( m \) so this comes down to showing

**Proposition 5.4** For each fixed \( \theta \in \Delta^\dagger \) the quantity

\[
g(\hat{\chi})^{-1} \prod_{i=1}^r g_{n(\psi)}(\chi \circ \tilde{\alpha}_i) \prod_{i=1}^r \theta(\tilde{\rho}_i)^{-n(\psi)} \psi(\langle N\hat{\chi}'(\theta)d_k \rangle)^{-1}
\]

(with \( \chi = \theta \times \psi \)) is independent of the character \( \psi \in \Gamma^\dagger \).

To prove this, we fix \( \theta \in \Delta^\dagger \), and suppose that \( \chi = \theta \times \psi \in G^\dagger_{k_\infty} \) for some \( \psi \in \Gamma^\dagger \). Recall that the character \( \hat{\chi} \) is regarded as a (primitive) character of the ray-class group \( \text{Cl}_{f(\chi)}(k) \) with values either in \( \mathbb{C} \) or (via \( j \)) in \( \mathbb{C}_p \) and that the \( p \)-adic global Gauss sum which we are denoting \( g(\hat{\chi}) \) is simply the embedding by \( j \) of the complex sum attached to \( \hat{\chi} \) as defined in [So3] §6.4. In the notation of the latter we therefore have

\[
g(\hat{\chi}) = \sum_{a \in R} j(e(\text{Tr}_{k/\mathbb{Q}}(a))) \hat{\chi}([af(\chi)\mathcal{D}]_{f(\chi)})^{-1}
\]

where \( e(x) = \exp(2\pi i x) \) and \( R \) is any set of representatives in \( k \) for \( \mathcal{T}(f(\chi), \mathcal{D}^{-1})' \), namely the points of \( k/\mathcal{D}^{-1} \) whose \( \mathcal{O} \)-annihilator is exactly \( f(\chi) \). As explained in ibid., for any \( a \in R \) the ideal \( af(\chi)\mathcal{D} \) is integral and prime to \( f(\chi) \). Furthermore its class \([af(\chi)\mathcal{D}]_{f(\chi)} \) in \( \text{Cl}_{f(\chi)}(k) \) depends only on the class \( a + \mathcal{D}^{-1} \in k/\mathcal{D}^{-1} \) as, of course, does \( e(\text{Tr}_{k/\mathbb{Q}}(a)) \). Let us fix, for each \( \theta \in \Delta^\dagger \), an element \( x_\theta \) of \( k \) satisfying

\[
\begin{align*}
\text{ord}_q(x_\theta) &= -\text{ord}_q(f'(\theta)\mathcal{D}) \text{ for all primes } q | f'(\theta), \\
\text{ord}_p(x_\theta) &\geq 0 \text{ for all } q \mid f'(\theta) \text{ and} \\
\text{ord}_p(x_\theta - 1) &\geq n_2 \text{ for } i = 1, \ldots, r
\end{align*}
\]
These conditions imply that \( \min\{\text{ord}_q(p^{−n(\psi)−1} x_\theta), \text{ord}_q(D^{−1})\} = \text{ord}_q(p^{−n(\psi)−1} f(\theta)^{−1} D^{−1}) \) for every prime \( q \) of \( \mathcal{O} \). (In the case \( q | p \), note that not only \( f(\theta) \) but also \( D \) and \( x_\theta \) are prime to \( p \), by Conditions (33) and (59)). Using (48) we obtain
\[
p^{−n(\psi)−1} x_\theta \mathcal{O} + D^{−1} = p^{−n(\psi)−1} f(\theta)^{−1} D^{−1} = f(\chi)^{−1} D^{−1}
\]
in other words \( p^{−n(\psi)−1} x_\theta \) generates \( f(\chi)^{−1} D^{−1} \) as a (free, rank-1) \( (\mathcal{O}/f(\chi)) \)-module. It follows easily that we may take \( R \) to be \( p^{−n(\psi)−1} x_\theta R_{\tilde{\mathcal{O}}(\chi)} \) where \( R_{\tilde{\mathcal{O}}(\chi)} \) is any set of representatives in \( \mathcal{O} \) of \( (\mathcal{O}/f(\chi))^\times \). Thus, dropping ‘\( j \)’ from the notation once more, we can rewrite \( g(\tilde{x}) \) as
\[
g(\tilde{x}) = \sum_{b \in R_{\tilde{\mathcal{O}}(\chi)}} e(\text{Tr}_{k/\mathbb{Q}}(p^{−n(\psi)−1} x_\theta b)) \tilde{x}([x_\theta b f(\theta) D]\mathcal{O}(\chi))^{-1}
\]
\[
= \tilde{x}([x_\theta f(\theta) D]\mathcal{O}(\chi))^{-1} \sum_{b \in R_{\tilde{\mathcal{O}}(\chi)}} e(p^{−n(\psi)−1} \text{Tr}_{k/\mathbb{Q}}(x_\theta b)) \tilde{x}([b \mathcal{O}]\mathcal{O}(\chi))^{-1} \tag{60}
\]

It will be necessary to compare this explicit expression for the global Gauss sum with the product of the local Gauss sums \( g_{n(\psi)}(\chi \circ \tilde{\alpha}_i) \) etc. Such a comparison is hard to find in the literature. To establish the precise connection we shall therefore recall some basic facts and notations from the idelic framework of class field theory. For any finite abelian extension \( M \) of \( k \) and each place \( v \) of \( k \) we write \( M^v \) for the completion \( M_w \) of \( M \) at some prime \( w \) dividing \( v \) and \( \varphi_v = \varphi_{v,M/k} \) for the local reciprocity map from \( k_v^\times \) to \( \text{Gal}(M^v/k_v) \) identified with the decomposition subgroup \( G(M/k)_v^{-1} \) of \( \text{Gal}(M/k) \) above \( v \). This is independent of the choice of \( w \) given \( v \) and, as \( v \) varies, the maps \( \varphi_v \) give rise to a well-defined global map \( \varphi_{M/k} \) from the idèle group \( \text{Id}(k) \) of \( k \) to \( \text{Gal}(M/k) \), by sending the idèle \( (a_v)_v \) to \( \prod_v \varphi_v M/k(a_v) \). It is well-known that \( \varphi_{M/k} \) is surjective and that \( \ker(\varphi_{M/k}) \) contains \( k^\times \) (embedded diagonally). If \( M/k \) is abelian but infinite, we shall still denote by \( \varphi_{M/k} \) the (not necessarily-surjective) map from \( \text{Id}(k) \) to \( \text{Gal}(M/k) \) which is obtained as the limit of the \( \varphi_{L/k} \) over finite sub-extensions \( L/k \). Similarly, we shall still write \( \varphi_{v,M/k} : k_v^\times \to G(M/k)_v^{-1} \subset \text{Gal}(M/k) \) for the limit of the local reciprocity maps \( \varphi_{v,L/k} \). The content \( c(a) \) of an idèle \( a = (a_v)_v \in \text{Id}(k) \) is defined to be the fractional ideal \( c(a) = \prod_q q^{\text{ord}_q(a_v)} \) where the product runs over all prime ideals \( q \) of \( \mathcal{O} \) (identified with the corresponding finite places). Now let \( \mathfrak{f} \) be an ideal of \( \mathcal{O} \). Every \( a \in \text{Id}(k) \) can be written as \( x d \) where \( x \in k^\times \) and \( d = (d_v)_v \in \text{Id}(k) \) satisfies \( d_v \in 1 + \mathfrak{f} \mathcal{O}_v \) for each \( v | \mathfrak{f} \). We define \( \gamma_{\mathfrak{f}}(a) \) to be the class \( [c(d)]_{\mathfrak{f}} \in \text{Cl}(k) \) of the content of \( d \). This gives a well-defined, surjective homomorphism \( \gamma_{\mathfrak{f}} : \text{Id}(k) \to \text{Cl}(k) \) for which one checks that the composite with the Artin isomorphism \( \text{Cl}(k) \to \text{Gal}(k) / k \) is nothing but \( \varphi_{k(\mathfrak{f})/k} \).

Now the character \( \chi = \theta \times \psi \in G^\times_{\infty} \) as above may be inflated to \( \tilde{G}_{\infty} \) or ‘descended’ to \( \text{Gal}(K_{\infty}/k) \) where \( K_{\infty} \subset k(f(\chi)) \cap \tilde{K}_\infty \) is the subfield that it cuts out. The above remarks (together with the compatibilities of \( \varphi_{K_{\infty}/k}, \varphi_k(\tilde{\mathcal{O}}(\chi))/k, \varphi_{K_{\infty}/k} \) and \( \varphi_{K_{\infty}/k} \) with the various restriction maps) show that, in our notation,
\[
\hat{x} \circ \gamma_{f(\chi)} = \chi \circ \varphi_{K_{\infty}/k} = \chi \circ \varphi_{K_{\infty}/k} \tag{61}
\]
For each \( i = 1, \ldots, r \) the embedding \( \tau_i \) (really \( j \circ \tau_i \)) of \( k \) in \( \mathbb{Q}_p \) extends to an isomorphism \( k_{p_i} \to \mathbb{Q}_p \) which we shall denote \( \tau_i \).
Lemma 5.7 Suppose that $\mathbf{a} = (a_v)_v \in \text{Id}(k)$ is an idèle satisfying $a_v > 0$ for every real place $v$ of $k$. Then $\varphi_{k_\infty/k}(\mathbf{a})$ acts on $\mu_{p_\infty}$ as $Nc(\mathbf{a}) \prod_{i=1}^r \tilde{\tau}_i(a_{p_i})^{-1} \in \mathbb{Z}_p^\times$.

**Proof** We write $\tilde{Q}_\infty$ instead of $Q(\mu_{p_\infty})$. The properties of $\varphi$ imply that $\varphi_{k_\infty/k}(\mathbf{a})|_{\tilde{Q}_\infty} = \varphi_{\tilde{Q}_\infty/Q}(N_k/q\mathbf{a}) = \varphi_{\tilde{Q}_\infty/Q}(\mathbf{s})$ where $\mathbf{s} = (s_v)_v$ is the rational idèle $Nc(\mathbf{a})^{-1}N_k/q\mathbf{a} \in \text{Id}(Q)$. But $s_q \in \mathbb{Z}_q^\times$ for every prime number $q$ and $s_\infty > 0$. Therefore $\varphi_{\tilde{Q}_\infty/Q}(\mathbf{s})$ acts on $\mu_{p_\infty}$ by $s_p^{-1} = Nc(\mathbf{a}) \prod\tilde{\tau}_i(a_{p_i})^{-1}$. (For the last statement, see §5.7, Ch. VII of [C-F].) In fact since every $q \neq p$ is unramified in $\tilde{Q}_\infty/Q$ it follows that $\varphi_{v,\tilde{Q}_\infty/Q}(s_v) = 1$ for each place $v \neq p$. So $\varphi_{\tilde{Q}_\infty/Q}(\mathbf{s}) = \varphi_{p,\tilde{Q}_\infty/Q}(s_p)$ which indeed acts as $s_p^{-1}$ by local class field theory. See the Remark of §7, Ch. XIV of [Se] or §3.7, Ch. VI of [C-F].

For each $b \in k^\times$ and $i = 1, \ldots, r$ we write $\tilde{b}^{(i)} \in \text{Id}(k)$ for the idèle with entry $b^{-1}$ (not $b$) at the place $p_i$ and $1$ everywhere else. If $b$ is a local unit at $p_i$ (resp. $= p$) then the above lemma shows that $\varphi_{k_\infty/k}(\tilde{b}^{(i)})$ acts by $\tau_i(b)$ (resp. acts trivially) on $\mu_{p_\infty}$. Moreover, $\varphi_{L^{(i)}/k}(\tilde{b}^{(i)}) = \varphi_{p_i,L^{(i)}/k}(b^{-1})$ equals 1 (resp. equals the inverse of the Frobenius above $p_i$) by local class field theory, since $L^{(i)}/k$ is unramified at $p_i$. We conclude:

$$\varphi_{K_\infty/k}(\tilde{b}^{(i)}) = \tilde{\alpha}_i(\tau_i(b)) \in \tilde{\Delta}^{(i)}$$

(62)

For any $b \in k^\times$ we also define $\tilde{b}$ (resp. $\tilde{b}'$) to be the idèle with entry $b^{-1}$ at every place dividing $p\theta'$ (resp. dividing $\theta'$) and 1 everywhere else, so that $\tilde{b} = \tilde{b}' \prod\tilde{\tau}_i(b)$. If $b$ is prime to $\tilde{f}(\chi)$ for $\chi \in G_\infty^1$ (i.e. to $p\theta'$) then it is easy to check that $\gamma_{\tilde{f}(\chi)}(\tilde{b}) = [b\mathcal{O}_{\tilde{f}(\chi)}]$ so, by (51),

$$b \in R(\tilde{f}(\chi)) \Rightarrow \tilde{\chi}([b\mathcal{O}_{\tilde{f}(\chi)}]) = \chi(\varphi_{K_\infty/k}(\tilde{b})) = \chi(\varphi_{K_\infty/k}(\tilde{b}')) \prod\tilde{\tau}_i(\varphi_{K_\infty/k}(\tilde{b}^{(i)}))$$

(63)

On the other hand, if $b$ is prime to $\tilde{f}(\theta)$ then for every prime ideal $q|\tilde{f}(\theta)$, local class field theory gives $\varphi_{q,K_\infty/k}(b^{-1}) \in G(K_\infty/k)_q^0 \subset \Delta$ and, moreover, if $\text{ord}_q(b_1) \geq \text{ord}_q(\tilde{f}(\theta))$ then $\varphi_{q,K_\infty/k}(b^{-1})$ lies in $G(K_\infty/k)_q^{\text{ord}_q(\tilde{f}(\theta))}$ hence in $\ker \theta$ by (43). It follows easily that there is a well-defined homomorphism $\theta' : (\mathcal{O}/\tilde{f}(\theta))^\times \to \mathbb{C}_p^\times$, depending only on $\theta$, which sends the class $\tilde{b} \in (\mathcal{O}/\tilde{f}(\theta))^\times$ to $\theta(\varphi_{K_\infty/k}(\tilde{b}')) = \chi(\varphi_{K_\infty/k}(\tilde{b}')) = \chi(\varphi_{K_\infty/k}(\tilde{b}'))$. Combining (62) with (63), we obtain

$$b \in R(\tilde{f}(\chi)) \Rightarrow \tilde{\chi}([b\mathcal{O}_{\tilde{f}(\chi)}]) = \theta'(\tilde{b}) \prod\tilde{\tau}_i(\chi \circ \tilde{\alpha}_i)(\tau_i(b))$$

(64)

Note also that $\varphi_{v,K_\infty/k}(p^{-1})$ lies in $G(K_\infty/k)_v^0 \subset \ker \theta$ for every $v \nmid p\theta'$ hence so also must $\varphi_{K_\infty/k}(\tilde{p})$. Regarding $\tilde{\Delta}$ as a subgroup of $\Delta$ by restriction as usual, it follows that

$$\theta'(\tilde{p}) = \theta(\varphi_{K_\infty/k}(\tilde{p}^{-1})) = \prod\tilde{\tau}_i(\varphi_{K_\infty/k}(\tilde{p}^{(i)}))^{-1} = \prod\tilde{\tau}_i(\theta(\tilde{p}))$$

by (62).

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We turn attention to the first factor of the summand in (60). Define \( f'(\theta) \in \mathbb{Z}_{\geq 1} \) by \( f'(\theta) \cap \mathbb{Z} = f'(\theta)\mathbb{Z} \). Since \( f'(\theta)x_\theta \subset D^{-1} \) (by (57) and (58)) we may also define a function

\[
\xi' = \xi'_{x_\theta} : \mathcal{O}/f'(\theta) \rightarrow \mu_{f'(\theta)}
\]

\[
\bar{b} \mapsto j(e(\text{Tr}_k/Q(x_\theta b)))
\]

and set

\[
g(\theta'; x_\theta) := \sum_{b \in R_{f'(\theta)}} \xi'(\bar{b})\theta'(\bar{b})^{-1}
\]

where \( R_{f'(\theta)} \) is any set of representatives in \( \mathcal{O} \) of \( (\mathcal{O}/f'(\theta))^\times \). (Clearly, \( g(\theta'; x_\theta) \) depends only on \( \theta \) and the choice of \( x_\theta \).

**Lemma 5.8** For any \( b \in \mathcal{O} \) and \( n \in \mathbb{N} \), we have

\[
e(\rho^{-n-1}\text{Tr}(x_\theta b)) = \xi'((\rho^{-n-1}\bar{b}) \prod_{i=1}^r \zeta_{n_i}(x_\theta b))
\]

(66)

Where \( \text{Tr} = \text{Tr}_k/Q, \bar{b} \) denotes the class of \( b \) in \( \mathcal{O}/f'(\theta) \), ‘\( e \)’ stands for \( j \circ e \) and ‘\( \tau_i \)' for \( j \circ \tau_i \) as usual.

**Proof** Note that \( \tau_i(x_\theta b) \) lies in \( \mathbb{Z}_p \) by (58) so \( \zeta_{n_i}(x_\theta b) \) makes sense. We need to prove that the two roots of unity, \( e(\rho^{-n-1}\text{Tr}(x_\theta b)) \) and \( \xi'((\rho^{-n-1}\bar{b}) \zeta_{n_i}(x_\theta b)) \) are equal. But their \( p^{n+1} \)th powers are clearly equal, and so are their \( f'(\theta) \)th powers (note that \( f'(\theta)\text{Tr}(x_\theta b) \) is an integer). Since \( (p^{n+1}, f'(\theta)) = 1 \), the result follows.

Now, if \( \chi = \theta \times \psi \in \mathcal{G}_\infty^1 \), then, as \( b \) runs through \( R_{f'(\theta)} \), so the image of \( b \) in \( (\mathcal{O}/f'(\theta))^\times \) and the images of \( \tau_i(b) \) (for \( r = 1, \ldots, r \)) in \( \mathbb{Z}_p^\times/(1 + p^n(\psi)+1\mathbb{Z}_p) \) all run through these quotients independently of each other. Therefore, fixing any sets \( R_{f'(\theta)} \) and \( R_{n(\psi)} \) of representatives for these quotients and substituting (64) and (66) (with \( n = n(\psi) \)) into the R.H.S. of (60) we obtain

\[
g(\hat{\chi}) = \hat{\chi}(\mathcal{D}(\theta)\hat{f}^\dagger(\theta))^{-1} \sum_{b \in R_{f'(\theta)}} \left[ \xi'(\bar{b}^{-n(\psi)-1}) \theta'(-\bar{b})^{-1} \prod_{i=1}^r \left( \zeta_{n_i}(x_\theta b)(\chi \circ \tilde{\alpha}_i)(\tau_i(b)) \right)^{-1} \right]
\]

\[
= \hat{\chi}(\mathcal{D}(\theta)\hat{f}^\dagger(\theta))^{-1} \left( \sum_{b \in R_{f'(\theta)}} \xi'(\bar{b}^{-n(\psi)-1}) \theta'(-\bar{b})^{-1} \right) \prod_{i=1}^r \left( \sum_{u \in R_{n(\psi)}} \zeta_{n_i}(x_\theta b)(\chi \circ \tilde{\alpha}_i)(u)^{-1} \right)
\]

\[
= \hat{\chi}(\mathcal{D}(\theta)\hat{f}^\dagger(\theta))^{-1} \theta'(\bar{p})^{-n(\psi)-1} g(\theta'; x_\theta) \prod_{i=1}^r [(\chi \circ \tilde{\alpha}_i)(\tau_i(x_\theta))] g_{n(\psi)}(\chi \circ \tilde{\alpha}_i)
\]

(67)

(which shows incidentally that \( g(\theta'; x_\theta) \neq 0 \)). Rearranging (67) and using (55) and the fact (from (10) and (39)) that \( (\chi \circ \tilde{\alpha}_i)(\tau_i(x_\theta)) = \psi(\tau_i(x_\theta)) \) for each \( i \), we find

\[
g(\hat{\chi})^{-1} \prod_{i=1}^r g_{n(\psi)}(\chi \circ \tilde{\alpha}_i) \prod_{i=1}^r \theta'(\bar{p})^{-n(\psi)} \psi(\langle Nf'(\theta)d_k \rangle)^{-1} =
\]
\[\left(\prod_{i=1}^{r} \theta(\rho_i)\right) g(\theta'; x_\theta)^{-1} \left[\hat{\chi}(x_\theta \psi(\theta)_{\mathbb{R}(\chi)}) \psi(N_{\mathbf{k}/\mathbf{Q}} x_\theta)^{-1} \psi((N\psi(\theta)_{d_k})^{-1})\right]\] (68)

Finally, for each $\theta \in \Delta^\dagger$ we fix an idèle $y_\theta = (y_{\theta,v})_v \in \text{Id}(k)$ such that $y_{\theta,v} = 1$ for all places $v$ which are infinite or divide $p\psi(\theta)$ and $c(y_\theta) = x_\theta \psi(\theta)\mathcal{D}$. Then

\[\hat{\chi}(x_\theta \psi(\theta)_{\mathbb{R}(\chi)}) = \hat{\chi}(\gamma_{\mathbb{R}(\chi)}(y_\theta)) = \chi(\varphi_{K_{\infty}/k}(y_\theta))\] (69)

by (61). Also, keeping in mind our identification of $\tilde{\Gamma}$ with $\mathbb{Z}_p^\times$ via $\kappa$ etc., Lemma 5.4 gives

\[
\varphi_{K_{\infty}/k}(y_\theta) |_{k_{\infty}} = \langle \kappa(\varphi_{K_{\infty}/k}(y_\theta)) \rangle = \langle |N_{\kappa/\mathbf{Q}} x_\theta| N\psi(\theta)_{d_k} \rangle = N_{\kappa/\mathbf{Q}} x_\theta (N\psi(\theta)_{d_k}) \in \Gamma
\]

so that the element $z_\theta := \varphi_{K_{\infty}/k}(y_\theta)\alpha(N_{\kappa/\mathbf{Q}} x_\theta (N\psi(\theta)_{d_k}))^{-1}$ lies in $\Delta$. Using Equation (69), it follows that the third factor on the R.H.S. of (68) equals $\chi(z_\theta) = \theta(z_\theta)$ and so it depends only on $\theta$, not on $\psi$ (in fact, it is even independent of the choice of $y_\theta$ given $\theta$). But the other two factors on the R.H.S. of (68) are also independent of $\psi$ so we have proven Proposition 5.4 hence also Proposition 5.3 hence also Theorem 4.1.

\section{Semilocal Weakenings of $P1$ and $P2$}

The aim of this section is to explain the weaker ‘semilocal’ variants of properties $P1$ and $P2$ namely $\widehat{P1}$ and $\widehat{P2}$ which were mentioned in the introduction. We shall also indicate how the constructions and results of Section 4 may be adapted to obtain (for example) an analogue of Corollary 4.1 which assumes only $\widehat{P2}$.

For fixed $p \neq 2$ and any number field $F$, we write $U(F)$ for the $p$-semilocal unit group $(\mathbb{Z}_p \otimes \mathbb{Z} \mathcal{O}_F)^\times$ and $U^{(1)}(F)$ for its Sylow pro-$p$ subgroup. We identify these respectively with the products $\prod_{\mathfrak{p} \mid p} U(F_{\mathfrak{p}})$ and $\prod_{\mathfrak{p} \mid p} U^{(1)}(F_{\mathfrak{p}})$ of the local (resp. principal local) units in the completions of $F$ at all primes $\mathfrak{p}$ dividing $p$. Any embedding $\iota : F \to \mathbb{C}_p$ defines a natural homomorphism $U(F) \to U(\mathbb{C}_p)$ which we shall denote $\tilde{\iota}$. (In the product realisation, $\iota$ extends to an embedding of $F_{\mathfrak{p}} \to \mathbb{C}_p$ for a unique prime $\mathfrak{p} | p$ and $\tilde{\iota}$ is obtained by composing this map with the projection $U(F) \to U(F_{\mathfrak{p}}.)$.) Also, we shall denote by $U^{(1)}_\infty(\mathfrak{p})$ the inverse limit of $U^{(1)}(F_n)$ over $n \geq n_1(\mathfrak{p})$ with respect to the maps $U^{(1)}(F_n) \to U^{(1)}(F_{n'})$ coming from the products of the local norms.

Now suppose $K/k$ and $p$ satisfy (14), (15) and (16) then, for each $n \geq n_1$ we endow $U^{(1)}(K_n)$ with the usual structure of $\mathbb{Z}_p G_{n}$-module and let $a_n : E(K_n) \to U^{(1)}(K_n)$ denote the $\mathbb{Z}G_n$-homomorphism which is the composite of the natural map $E(K_n) \to U(K_n)$ with the usual projection of $U(K_n)$ on $U^{(1)}(K_n)$. (Thus $\ker(a_n) = \{ \pm 1 \}$ and Leopoldt’s conjecture – which we need not assume here – predicts that $\text{Im}(a_n)$ spans a $\mathbb{Z}_p$-submodule of co-rank 1 in $U^{(1)}(K_n)$.) The map $\wedge[p] a_n$ gives rise to an obvious $\mathbb{Q}G_n$-linear map $b_n$, say, from $\wedge[p]_{\mathbb{Q}G_n} E(K_n) = \mathbb{Q} \otimes_{\mathbb{Z} G_n} E(K_n)$ to $\mathbb{Q} \otimes_{\mathbb{Z} G_n} \wedge[p]_{\mathbb{Z} G_n} U^{(1)}(K_n)$. Now $U^{(1)}(K)$ is naturally a module for the completed group-ring $\mathbb{Z}_p[[G_{\infty}]]$ (namely, the inverse limit over $n \geq n_1$ of the rings $\mathbb{Z}_p G_n$ w.r.t. the natural restriction maps $\mathbb{Z}_p G_n \to \mathbb{Z}_p G_{n'}$). We may therefore form the
\[ \Lambda_{QG_n}^r QE_\infty(K) = Q \otimes \Lambda_{ZG_n}^r E(K) \quad b_n \quad Q_p \otimes_{Z_p} \Lambda_{Z_p[G_n]}^r U_\infty^{(1)}(K) \]

\[ \Lambda_{QG_n}^r QE(K_n) = Q \otimes \Lambda_{ZG_n}^r E(K_n) \quad b_n \quad Q_p \otimes_{Z_p} \Lambda_{Z_p[G_n]}^r U_\infty^{(1)}(K_n) \]

(70)

exterior power \( \Lambda_{Z_p[G_n]}^r U_\infty^{(1)}(K) \) which is equipped with a natural map \( \hat{\beta}_n \) to \( \Lambda_{Z_p[G_n]}^r U_\infty^{(1)}(K_n) \) for each \( n \geq n_1 \). Furthermore, the map \( a_\infty = \lim_{\leftarrow} a_n : E_\infty(K) \to U_\infty^{(1)}(K) \) naturally defines a homomorphism \( b_\infty \) from \( \Lambda_{ZG_n}^r E_\infty(K) \) to \( \Lambda_{Z_p[G_n]}^r U_\infty^{(1)}(K) \). We now have the commutative diagram (70) where we have extended \( \hat{\beta}_n \) \( \forall n \geq n_1 \) and \( b_\infty \) linearly to the tensor products. We formulate

Property \( \hat{P}1(K/k, p) \) There exists an element \( \hat{\eta} \in Q_p \otimes_{Z_p} \Lambda_{Z_p[G_n]}^r U_\infty^{(1)}(K) \) such that, for every \( n \geq n_1 \) we have \( \hat{\beta}_n(\hat{\eta}) = b_n(\eta_n) \) where \( \eta_n \) lies in \( V_n^0 \) and is a solution of \( C1(K_n/k, \emptyset) \) (the unique solution in \( V_n^0 \)) i.e.

\[ \Phi_n(1) = \frac{2^r}{\sqrt{d_k}} R_n(\eta_n) \quad \forall n \geq n_1 \]

(We say that \( \hat{\eta} \) demonstrates \( \hat{P}1(K/k, p) \)).

Remark 6.1 This property is implied by \( P1(K/k, p) \). Indeed, if \( \eta \) demonstrates \( P1(K/k, p) \) then clearly \( b_\infty(\eta) \) demonstrates \( \hat{P}1(K/k, p) \). However the converse implication is not obviously true. For example, \( \hat{P}1(K/k, p) \) at most implies that the \( p \)-part of the denominator of \( \eta_n \) is bounded as \( b \to \infty \). (Compare Remark 3.3).

One way to formulate \( \hat{P}2(K/k, p) \) would simply be to replace the above condition that \( \eta_n \in V_n^0 \) be a solution of \( C1(K_n/k, \emptyset) \) \( \forall n \geq n_1 \) by the requirement that \( p^r e_n \eta_n \) be a solution of \( C2(K_n/k, T_p, p) \) \( \forall n \geq n_2 \), or indeed the equivalent equation involving \( R_n(p \eta_n) \). There is, however, a neater, equivalent formulation, using the fact that the map \( \hat{R}_{n,p} \) can be ‘extended’ to \( Q_p \otimes_{Z_p} \Lambda_{Z_p[G_n]}^r U_\infty^{(1)}(K_n) \) (i.e. it factors through \( b_n \)). Indeed, for each \( i = 1, \ldots, r \) we may ‘extend’ \( \lambda_{K_n/k,i,p} \) to a \( Z_p G_n \)-linear map \( \hat{\lambda}_{K_n/k,i,p} : U_\infty^{(1)}(K_n) \to \mathbb{C}_p \) by replacing the embedding \( j_{\tilde{r}_i} \) with \( j_{\tilde{r}_i} \) in the definition of \( \lambda_{K_n/k,i,p} \). Then we obtain a unique \( Q_p G_n \)-linear regulator map \( \hat{R}_{n,p} = \hat{R}_{K_n/k,p} \) from \( Q_p \otimes_{Z_p} \Lambda_{Z_p[G_n]}^r U_\infty^{(1)}(K_n) \) to \( \mathbb{C}_p G_n \) by setting \( \hat{R}_{n,p}(1 \otimes (u_1 \wedge \ldots \wedge u_r)) := \det(\hat{\lambda}_{K_n/k,i,p}(u_s))_{s=1}^r \). It should be clear that \( \hat{R}_{n,p} \circ b_n = R_{n,p} \).

For all \( K/k \) and \( p \) satisfying (14), (15) and (16) we formulate
Property \( \widehat{P}2(K/k, p) \) There exists an element \( \hat{\eta} \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r_{\mathbb{Z}_p[[G_\infty]]} U_\infty^{(1)}(K) \) such that, for every \( n \geq n_2 \), \( \hat{\beta}_n(\hat{\eta}) \) lies in \( b_n(V_n^0) \) and satisfies
\[
\Phi_{n,p}(1) = \frac{2^r}{j(\sqrt{d_k})} \hat{R}_{n,p}(e_n \hat{\beta}_n(\hat{\eta}))
\]
(We say that \( \hat{\eta} \) demonstrates \( \widehat{P}2(K/k, p) \)).

**Remark 6.2** The exact analogue of Proposition 3.2 now holds, namely one may replace \( P1 \) by \( \widehat{P}1 \) and \( P2 \) by \( \widehat{P}2 \) (and \( \eta \) by \( \hat{\eta} \)) in the statement of this proposition, the proof remaining essentially the same.

Next, we strengthen the hypothesis (13) to (33) and sketch the analogues of some of the constructions of Section 4. Just as before, for any \( i = 1, \ldots, r \) and \( u = (u_n)_n \in U_\infty^{(1)}(K) \), we use the norm maps to \( \tilde{L}_n \) for \( N > n \geq 0 \) to extend the subsequence \( (u_n)_{n \geq N} \) (where \( N = \max(n_1, n_2 - 1) \)) to an element of \( U_\infty^{(1)}(\tilde{L}_0^{(1)}) \). Then we apply \( j_{\tilde{r}_i} \) to get an element of \( U_\infty^{(1)}(H^{(1)}) \). The analogue of Lemma 4.4 obviously holds with \( u \) and \( j_{\tilde{r}_i} \) in place of \( \begin{array}{c} \eta \end{array} \) and \( j_{\tilde{r}_i} \). Thus for any \( (t, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq n_2} \) we get a well-defined homomorphism \( \hat{L}_{i,t,n} : U_\infty^{(1)}(K) \rightarrow \hat{O}_{\tilde{H}^{(i)}}G_n \) by simply replacing \( j_{\tilde{r}_i} \) by \( j_{\tilde{r}_i} \) (and \( \tilde{r}_i \in \mathcal{E}_\infty(K) \) by \( u \in U_\infty^{(1)}(K) \)). The analogue of Equation (36) holds with \( \hat{L}_{i,t,n} \), \( \hat{\lambda}_{K_n/k,i,p} \) and \( \tilde{r}_i \) replaced by \( \hat{L}_{i,t,n} \), \( \hat{\lambda}_{K_n/k,i,p} \) and \( u \) respectively. Moreover, we have

**Lemma 6.1** For each \( i, n, t \) as above, the map \( \hat{L}_{i,t,n} \) is \( \mathbb{Z}_p[[G_\infty]] \)-linear with \( \mathbb{Z}_p[[G_\infty]] \) acting on the group-ring \( \hat{O}_{\tilde{H}^{(i)}}G_n \) through the quotient \( \mathbb{Z}_p G_n \).

**Proof (Sketch.)** \( \hat{L}_{i,t,n} \) is clearly \( \mathbb{Z}_p \)-linear and also \( G_\infty \)-linear with \( G_\infty \) acting through the \( G_n \). In particular, it is \( \mathbb{Z}_p G_\infty \)-linear. Let \( I_n = \ker([\mathbb{Z}_p[[G_\infty]]] \rightarrow \mathbb{Z}_p G_n) \). Then \( \mathbb{Z}_p[[G_\infty]] = \mathbb{Z}_p G_\infty + I_n \) so it suffices to prove that \( \hat{L}_{i,t,n}(zu) = 0 \) for all \( u \in U_\infty^{(1)}(K), z \in I_n \). But \( n \geq n_2 \) implies that \( \text{Gal}(K_\infty/K_\infty) \) is isomorphic to \( \mathbb{Z}_p \), with topological generator \( \gamma_n \), say, and a compactness argument (for example) shows that \( I_n = (\gamma_n - 1)\mathbb{Z}_p[[G_\infty]] \). Thus \( \hat{L}_{i,t,n}(zu) = \hat{L}_{i,t,n}((\gamma_n - 1)z'u) = \hat{L}_{i,t,n}(\gamma_n z'u) - \hat{L}_{i,t,n}(z'u) = 0 \) since the image of \( \gamma_n \) in \( G_n \) is 1. \( \square \)

It follows that for each \( (t, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq n_2} \) there is a unique, well-defined \( \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G_\infty]] \)-linear higher regulator map \( \hat{R}_{t,n} \) from \( \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r_{\mathbb{Z}_p[[G_\infty]]} U_\infty^{(1)}(K) \) to \( \mathbb{C}_p G_n \) satisfying \( \hat{R}_{t,n}(1 \otimes (u_1 \wedge \ldots \wedge u_r)) = \det(\hat{L}_{i,t,n}(u_s))_{i,s=1} \). It should be clear that \( \hat{R}_{t,n} \) ‘extends’ \( \hat{R}_{t,n} \) in the sense that \( \hat{R}_{t,n} \circ b_\infty = \hat{R}_{t,n} \). Moreover, the analogue of Lemma 4.3 holds, namely
\[
\hat{R}_{0,n}(\hat{\eta}) = \hat{R}_{n,p}(e_n \hat{\beta}_n(\hat{\eta})) \quad \forall \hat{\eta} \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r_{\mathbb{Z}_p[[G_\infty]]} U_\infty^{(1)}(K), \forall n \geq n_2
\]
(71)

The (hypothetical) relation (37) is of course replaced by the following one, for \( \hat{\eta} \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r_{\mathbb{Z}_p[[G_\infty]]} U_\infty^{(1)}(K) \) and \( (m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq n_2} \):
\[
\Phi_{n,p}(m) = \frac{2^r}{j(\sqrt{d_k})} \hat{R}_{1-m,n}(\hat{\eta})
\]
(72)
and the most important thing to check is that Theorem 4.1 now extends as follows.

**Theorem 6.1** Suppose that $K$ and $p$ satisfy (14), (15) and (33). For a given element $\hat{\eta} \in \mathbb{Q}_p \otimes \mathbb{Z}_p \wedge \mathbb{Z}_p[[G_{\infty}]] \mathcal{U}_\infty^{(1)}(K)$, Equation (72) holds for all pairs $(m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq n_2}$ if and only if it holds for infinitely many such pairs.

**Proof** The key Lemma 5.5 in the proof of Theorem 4.1 works with $u \in \mathcal{U}_\infty^{(1)}(K)$ and $\hat{\mathcal{L}}_{i,1-m,n}$ in place of $\mathcal{E}_\infty(K)$ and $\hat{\mathcal{L}}_{i,1-m,n}$. Indeed, the construction of the required measure (now $\mu_{i,\theta,u}$) works exactly as before because, crucially, it depends only on the power series attached norm-coherent sequences of local units (now those coming from $j\hat{\tau}_i$ applied to $u$ and its various Galois conjugates). Having thus ‘extended’ this lemma, the other aspects of the proof Theorem 4.1 go across identically, *mutatis mutandis*, as the reader may verify. \(\square\)

Finally, from Equation (71) and Theorem 6.1 we deduce the following analogue of Corollary 4.1.

**Corollary 6.1** If (14), (15) and (33) hold and $\hat{\eta} \in \mathbb{Q}_p \otimes \mathbb{Z}_p \wedge \mathbb{Z}_p[[G_{\infty}]] \mathcal{U}_\infty^{(1)}(K))$ demonstrates $\hat{P}2(K/k,p)$ then it satisfies Equation (72) for all $(m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq n_2}$. \(\square\)

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