Non- minimally Coupled Scalar Fields, Holst Action and Black Hole Mechanics

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The paper deals with the extension of the Weak Isolated Horizon (WIH) formulation to the non-minimally coupled scalar fields. In the first part of the paper, we construct the appropriate Holst type action to incorporate non-minimal scalar field and construct the covariant phase space of the theory. Using this covariant phase space, we prove the laws of black hole mechanics and show that with a gauge fixing, the symplectic structure on the horizon reduces to that of a U(1) Chern-Simons theory. The level of the Chern-Simons theory is shown to depend on the non-minimally coupled scalar field.

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I. INTRODUCTION

General theory of relativity (GR) is the foremost theory of gravity and has passed a number of experimental tests. However, some future experiment might reveal deviations from GR. This is expected because GR can be thought of as an effective description of a fundamental theory of underlying quantum structure. Effective classical limits of quantum gravity should contain all possible interactions with dimensionful length parameters representing scales at which new degrees of freedom emerge to play crucial roles. The idea is quite similar to the Fermi theory of weak interactions. It is expected that as one goes higher in the energy scale, newer degrees of freedom emerge. In other words, the degrees of freedom relevant to quantum gravity are suppressed at lower energies (for example, the energy scale at Tevatron) and are revealed only at scales of Planck length. Thus, it is very natural to construct other effective theories which have a limit to GR (at lower energies) while show considerable deviations at higher energies. Examples of such theories include the Brans-Dicke theory, the Einstein-Cartan theory, extra dimensional models inspired by String theories and many more.

The scalar-tensor theories are perhaps the most popular alternative to GR. The first of its kind was the Brans-Dicke theory [1]. In conformity to the equivalence principle, this theory also considers gravity in terms of spacetime curvature. In addition, there exists a massless scalar field in the spacetime which together with the gravitational constant $G$, determines the coupling strength of gravity to matter. In some limit, the standard GR equations are recovered. This theory is interesting because it includes the possibility of variation of effective gravitational constant influenced by the scalar field, which can be constrained by direct astronomical observations like the solar system test. In this paper, we shall not commit ourselves to a particular form of the scalar-tensor theory like the Brans-Dicke. Instead, the most general coupling will be studied. It must be emphasised that these theories are different from GR.

To exemplify, consider the Brans-Dicke theory with a non-minimal scalar coupling in the so called Jordan frame (or String frame) with metric $g$. In the Einstein frame with metric $\tilde{g}$ (conformally related to $g$ by the scalar field), the theory can be equivalently represented by a theory of a minimally coupled scalar field in curved spacetime. It might then seem that the theory is the same as GR (with scalar field sources) in different variables. The crucial difference is that test particles move along the geodesics determined by $g$ and will not (in general) coincide with that of $\tilde{g}$ [1]. One is free to choose the frame for comfortable calculations. In the Jordan frame, the gravitational field equations are different and in the Einstein frame, one has to account for the changes in the matter equations. Either of these leads to departure from GR.

The aim of the present paper is to extend the formalism of Weak Isolated Horizons (WIH) for non-minimally coupled scalar fields (for details of isolated horizon formulation see [2] and for its applications, see [3]). The rationale for such an extension is the following. We have argued that alternate effective descriptions of gravity (other than GR) can exist in which new degrees of freedom can become important at higher energies (say the Planck scale). Black holes are ideal laboratories to look for effects of such non-standard degrees of freedom. If a non-minimally coupled scalar field becomes important at Planck scales, it can leave its imprint on the entropy of these black holes. (String theories for example, predict existence of such scalar fields.) Thus, if one is able to determine the entropy of these black hole horizons, the exact dependence on these scalar fields will be clarified. The loop approach is one of the many ways of calculating entropy [4, 5] (see [6] for other approaches). Here, one uses the formalism of isolated horizons.

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to determine the effective topological theory on the black hole horizon, quantise it and count the states. Through a canonical analysis of the Palatini action, it was determined that this topological theory is a $U(1)$ Chern-Simons theory $[7]$. Entropy is then calculated by quantising the Chern-Simons theory $[4, 5]$. Holst’s modification of the Palatini action is another possible theory of gravity in 4 dimensions $[8]$. This action is in fact the starting point for Loop Quantum Gravity (LQG) (see $[9]$ for a detailed comparison of the Palatini and the Holst action). It is also possible to study the formalism of isolated horizons using the Holst action. Indeed, it is possible to construct the covariant phase space of the Holst action admitting a WIH as an inner boundary. Through this completely covariant formulation, we also proved that the boundary symplectic structure is that of a $U(1)$ Chern-Simons theory $[9]$. Entropy of WIH can be calculated just as before by quantisation of the topological theory. Thus, if the Holst action is further modified to include a non-minimal scalar field coupled to gravity, the WIH formulation will provide an ideal set-up to study the effects of these non-canonical fields on black hole entropy. In other words, the Holst action modified to include non-minimally scalar field and extended to WIH formulation will precisely be able to tell us, in a covariant framework, to what extent the scalar field contribute to the entropy of black holes in these theories.

To proceed for such calculations, we need to go through a series of steps. We shall first modify the boundary conditions for WIH making it amiable to the case of non-minimal coupling. We will follow the modified boundary conditions already stated in $[10]$. The zeroth law for the black holes will simply follow from these conditions. Secondly, we shall have to modify the Holst action to include the non-minimal scalar field coupling $^1$. This is non-trivial because we have to ensure that the phase space of this theory can obtained from non-minimal Palatini theory $[10]$ by a one parameter canonical transformation (just like in the standard case of minimal coupling). Then the covariant phase space for the non-minimally coupled Holst action admitting a WIH as the internal boundary needs to be constructed. The first law for the WIHs in this theory can be proved using the symplectic structure constructed on this covariant phase space. The first law for these horizons are expected to be modified induced by the non-minimal scalar coupling. This result was already obtained previously for the Weakly Isolated Horizons using the non-minimally coupled Palatini action. We shall rederive these results using the non-minimal Holst action and for a more general class of horizons. This derivation of the first law (from the symplectic structure of Holst action) will be non-trivial because the extra contributions form the Holst action precisely cancel so as to lead to the standard results from the Palatini theory $[10]$. We shall then show that on the phase space containing spherical horizons of fixed area and show that the surface symplectic structure acquires the structure of a Chern-Simons theory although the scalar field will appear as a label of the Chern-Simons theory (and hence the entropy). This possibility was already predicted from Killing Horizon framework $[11]$. Such a dependence on scalar field was also derived through canonical formalism in $[12]$. We put this result on a firmer basis by rederiving in a completely covariant way.

The plan of the paper is as follows. First section will contain a quick introduction of the WIH formalism. The zeroth law will also be proved in this section. The second section will be used to define the non-minimally coupled Holst action and then will be followed by construction of the space of solution of the theory admitting a WIH as an internal boundary. The first law for the black holes in this theory will be carried out in the next section. We will also provide an alternative derivation of the first law. The fourth section will be devoted to the derivation of the Chern-Simons boundary symplectic structure on the phase space of fixed area.

II. WEAK ISOLATED HORIZONS

We provide a brief introduction to Weak Isolated Horizons (WIH). (see $[9, 13]$ for details). Let $M$ be a four-manifold equipped with a metric $g_{ab}$ of signature $(-, +, +, +)$ and $V_a$ be the covariant derivative compatible with $g_{ab}$. Consider a null hypersurface $\Delta$ in $M$. The surface $\Delta$ naturally admits an equivalence class of null normals $[\xi\ell^a]$, $\xi$ being any arbitrary positive function. We denote by $q_{ab} \equiv g^ab$ the degenerate intrinsic metric on $\Delta$ induced by $g_{ab}$. The expansion $\theta_{(\ell)}$ of the null normal $\ell^a$ is then defined by $\theta_{(\ell)} = q^{ab}V_a\ell_b$, where $V_a$ is the covariant derivative compatible with $g_{ab}$. We shall work with the null tetrad basis $(\ell, n, m, \bar{m})$ such that $1 = -n \cdot \ell = m \cdot \bar{m}$ and all other scalar products vanish. This is specially suited for the present problem since one of the null normals $\ell^a$ matches with one of the vectors in the equivalence class $[\xi\ell^a]$. The spacetime metric, in terms of this null basis is then given by $g_{ab} = -2\ell_a\ell_b + 2m_a\bar{m}_b$.

We shall now impose a minimal set of boundary conditions on the null surface $\Delta$ so that effectively the surface behaves as a black hole horizon $[8, 10, 13]$. Since the null surface is generated by an equivalence class of null normals $[\xi\ell^a]$, it is natural to ensure that the boundary conditions hold for the entire equivalence class. This would seem to

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$^1$ We shall use the acronym ‘non-minimally coupled Holst(Palatini) action’ to mean the Holst(Palatini) action modified to include that effect of scalar fields non-minimally coupled to gravity

$^2$ indices that are not explicitly intrinsic on $\Delta$ will be pulled back and $\Delta$ means that the equality holds only on $\Delta$
imply that we need infinite number of boundary conditions, one for each \( \ell \) in \([\xi \ell]\). However, we shall see that in a restricted class of the function \( \xi \) (which we will specify in this subsection, see eqn (2.6), the boundary conditions are such that they are satisfied for any \( \ell \) in \([\xi \ell]\) if they hold for one \( \ell \).

The null surface \( \Lambda \) generated by the equivalence class \([\xi \ell]\) will be called a \textit{weak isolated horizon} (WIH) in \((\mathcal{M}, g_{ab})\) if the following conditions are satisfied:

1. \( \Lambda \) is topologically \( S^2 \otimes \mathbb{R} \).
2. The expansion \( \theta_{(\xi \ell)} = 0 \) for any \( \xi \ell \) in the equivalence class.
3. The equations of motion hold on the surface \( \Lambda \) and the non-minimally coupled scalar field \( \phi \) is such that \( \mathcal{L}_\xi \phi = 0 \).
4. There exists a one-form \( \omega_{(\xi \ell)} \) on \( \Lambda \) so that \( \mathcal{L}_\xi \omega_{(\xi \ell)} = 0 \).

All the boundary conditions, except the fourth one, hold for the entire equivalence class if they hold for one representative of the equivalence class \([\xi \ell]\). Each null normal \( \xi \ell \) is geodetic by construction

\[
\xi \ell \nabla_a (\xi \ell)^b = \kappa_{(\xi \ell)} \xi \ell^b
\]

where \( \kappa_{(\xi \ell)} \) is the acceleration of \( \xi \ell \). It is easy to see from (2.1) that the acceleration varies in the equivalence class

\[
\kappa_{(\xi \ell)} = \xi \kappa_{(\ell)} + \mathcal{L}_\xi \ell.
\]

The boundary conditions then imply that each null normal in the equivalence class \([\xi \ell]\) is twist-free, shear-free and

\[
\nabla_a \bar{\ell}^b = \omega_a^{(\ell)} \bar{\ell}^b
\]

where \( \omega_a^{(\ell)} \) is a one-form on \( \Lambda \) associated with the null normal \( \ell \) which varies in the equivalence class as

\[
\omega_{(\xi \ell)} = \omega^{(\ell)} + d \ln \xi,
\]

where \( d \) is the exterior derivative in \( \Lambda \). It follows that each \( \xi \ell \) in the class is a \textit{Killing vector field} on \( \Lambda \), namely \( \mathcal{L}_\xi q_{ab} = 0 \). It also follows that the curvature of \( \omega_{(\xi \ell)} \) is (see appendix of [9] for details)

\[
d\omega^{(\xi \ell)} = 2(\text{Im} \Psi_2)^2 \epsilon,
\]

where \( \text{Im} \Psi_2 = C_{abcd} \ell^a \bar{m}^b m^c \bar{n}^d \) is a complex scalar associated with the Weyl-tensor \( C_{abcd} \) and \( ^2\epsilon = \text{Im} \wedge \bar{m} \) is the area two-form of the cross-sections (these are \( \upsilon = \) constant sections where \( \upsilon \) is the affine parameter of \( \ell \) such that \( \mathcal{L}_\xi \upsilon = 1 \) ) of \( \Lambda \). From (2.4) it is obvious that (2.5) will hold for all \( \omega_{(\xi \ell)} \) in the class. Since each \( \xi \ell \) is Killing, the area two-form \( ^2\epsilon \) is preserved under its Lie-flow \( \mathcal{L}_{(\xi \ell)}^2 \epsilon_{ab} = 0 \).

The fourth boundary condition is \textit{not} valid for any all null normals. This condition can be viewed as a restriction on the function \( \xi \) so that in this restricted class, any null-normal in \([\xi \ell]\) will satisfy all the boundary conditions if it holds one. We restrict the choice of \( \xi \)s to

\[
\xi = c e^{-\kappa_{(\ell)}} + \kappa_{(\xi \ell)}/\kappa_{(\ell)},
\]

where \( c \) is a nonzero function satisfying \( \mathcal{L}_\xi c = 0 \) and \( \upsilon \) is the affine parameter of \( \ell \). In the rest of the paper we choose \( c \equiv \text{constant} \). It is easy to see that the fourth boundary condition gives

\[
\mathcal{L}_{(\xi \ell)} \omega_{(\xi \ell)} = d\kappa_{(\xi \ell)} + \kappa_{(\xi \ell)} = 0
\]

for any \( \xi \) belonging to the restricted class (2.6). This is equivalent to the \textit{zeroth law} which states that the surface gravity associated with each \( \xi \ell \) in the equivalence class is constant on \( \Lambda \). The restricted class (2.6) admits a \( \xi \equiv c e^{-\kappa_{(\ell)}} \) such that \( \kappa_{(\xi \ell)} \equiv 0 \) when \( \kappa_{(\ell)} \neq 0 \). For obvious reasons such a WIH will be called \textit{extremal}. Thus, the restricted class of null normals, as opposed to the constant class of null normals \([\ell \ell]\), contains both extremal and non-extremal horizons. In other words, WIH boundary conditions are sufficiently weak to accommodate both types of horizons.

We shall work with spherical horizons so that the scalar field is spherically symmetric. The boundary condition then imply that the field is constant on the horizon. The case for non-spherical horizons will be dealt with elsewhere.
III. NON-MINIMAL COUPLING OF HOLST ACTION AND WIH

The Holst action non-minimally coupled to a scalar field is given by the following action:

\[
S[e, A, \phi] = \int_M d^4x \ e \left[ \frac{1}{16\pi G} f(\phi) e^a_I e^b_J \left( F(A)_{IJ}^a - \frac{1}{2\gamma} \epsilon_{KL}^{IJ} F(A)^{KL}_{ab} \right) - \frac{1}{2} K(\phi) \partial_a \phi \partial_b \phi \ e^a_I e^b_J n^{IJ} - V(\phi) \right],
\]

(3.1)

where

\[
K(\phi) = [1 + (3/16\pi G)(f'(\phi)/f(\phi))]^2.
\]

(3.2)

Here \(e^a_I\) is the tetrad, \(e\) its determinant, \(F_{ab}^{IJ}\) is the curvature of the connection \(A_{IJ}\), and \(\gamma\) is a fixed but arbitrary number and is called the Barbero-Immirzi parameter.

Consider the case when the manifold has no inner boundary. Variation with respect to \(A\) yields the equation of motion for the connection:

\[
\nabla_a \left( f(\phi) e e^a_I \varepsilon^I_{I\ell} e^b_J \right) = 0,
\]

(3.3)

where \(\nabla_a\) is the covariant derivative operator corresponding to the connection \(A_{IJ}\) and acts both on the spacetime and the Lorentz indices. The boundary contributions at the spatial infinity can be taken care from the asymptotic flatness conditions. The equation (3.3) has a dependence of the scalar field. To solve this equation, assume that the rescalings of the tetrads [10]:

\[
\hat{e}^a_I = (p e^a_I)
\]

(3.4)

where \(p = 1/\sqrt{f(\phi)}\) is well-defined and non-degenerate. The determinant \(e\) is then also rescaled so that \(p^4 \hat{e} = e\). It then follows that the equation (3.3) can be rewritten as:

\[
\nabla_a \left( \hat{e} \hat{e}^I_{I\ell} \hat{e}^b_J \right) = 0.
\]

(3.5)

The form of the equation (3.5) simply suggests that \(A_{IJ}\) is the unique Lorentz spin connection compatible with \(\hat{e}^a_I\). Also, when the above equation of motion is satisfied, i.e. when the connection \(A_{IJ}\) is the spin connection, the second term in the action is precisely the Bianchi identity and hence is zero. Furthermore, when the connection is a spin connection, the the standard non-minimally coupled Einstein- Hilbert action can be recovered up to a surface term [10]. The extra non-minimally coupled Holst modification in fact induces canonical transformation on the phase space labelled by \(\gamma\). While the symplectic structure can be shown to be immune to this canonical transformation, the quantum theory is sensitive to these \(\gamma\) sectors.

![Diagram](image)

FIG. 1: \(M_\pm\) are two partial Cauchy surfaces enclosing a region of space-time and intersecting \(\Delta\) in the 2-spheres \(S_\pm\) respectively and extend to spatial infinity \(i^0\). Another Cauchy slice \(M\) is drawn which intersects \(\Delta\) in \(S_\Delta\).

Tetards and Connection on \(\Delta\)

In this subsection, we make a detailed study of the Holst action in a spacetime region which is bounded by an inner boundary (a WIH) and two Cauchy surfaces \(M_+\) and \(M_-\) extended to spatial infinity (see fig. 1). The variation
of the fields is subject to the WIH boundary conditions on \( \Lambda \) and the asymptotic flatness conditions at spatial infinity. For convenience let us choose a fixed set of internal null vectors \((\bar{\ell}^I, n^I, m^I, \bar{m}^I)\) on \( \Lambda \) such that \( \partial_a(\bar{\ell}^I, n^I, m^I, \bar{m}^I) = 0 \) (this partially fixes the internal Lorentz frame). Given these internal null vectors and the tetrads \( e^I_a \), we can construct the null vectors \((\ell^I, n^I, m^I, \bar{m}^I)\) through \( \ell_a = e^I_a \bar{\ell}_I \) etc.

In case WIH is an inner boundary, we must take appropriate care to verify the variational principle while the equation of motion are determined. This requires expressions of the tetrads and the connection pulled back to the WIH. On WIH \( \Lambda \), the expression of wedge product of tetrads is given by

\[
\mathcal{e}^{I}_{a} \wedge \mathcal{e}^{J}_{b} = -2 n_a \wedge m_b \ell^{I} \bar{m}^{J} - 2 n_a \wedge \bar{m}_b \ell^{I} m^{J} + 2i m^I \bar{m}^J \mathcal{\epsilon}_{ab}
\]

Using this expression for the tetrad products \((3.6)\), and the expansion for the internal epsilon tensor \( \mathcal{\epsilon}_{IJKL} = 4!\ell_I n_J m_K \bar{m}_L \), the expression for \( \mathcal{\Sigma}_{ab}^{IJ} \), restricted to \( \Delta \) is

\[
\sum_{ab}^{IJ} = 2 \ell^{I} \bar{m}^{J} \mathcal{\epsilon}_{ab} + 2 n_a \wedge (im_a \ell^{I} \bar{m}^{J} - i\bar{m}_a \ell^{I} m^{J})
\]

To find an expression for the connection \( \mathcal{A}_{IJ} \) on \( \Delta \), we need the covariant derivatives of null normals pulled back and restricted to \( \Delta \). We shall use the Newman- Penrose formalism (see appendix of [9] for details). Note that the covariant derivative does not annihilate the metric \( g_{ab} \) but the conformal metric \( p^{-2} g_{ab} \). However, the inner product is still taken with respect to the metric \( g_{ab} \), see eqn. \((3.3)\). This leads to a modification of the connection compared to the standard minimally coupled case, as is collected below:

\[
\nabla_\mu \ell^b \equiv \omega_a^{(\ell)} \ell^b
\]

\[

abla_\mu n^b \equiv - (\omega_a^{(\ell)} + 2 \nabla_a p) n^b + \bar{U}^{(m)}_a m^b + U^{(m)}_a \bar{m}^b
\]

\[

abla_\mu m^b \equiv U^{(m)}_a \ell^b + V^{(m)}_a m^b
\]

\[

abla_\mu \bar{m}^b \equiv \bar{U}^{(m)}_a \ell^b - (V^{(m)}_a + 2 \nabla_a p) \bar{m}^b
\]

where, the superscripts for each of the one forms keep track of their dependencies on the rescaling of the corresponding null normals. The expressions of the one forms \( \omega_a^{(\ell)} \), \( U^{(m)}_a \), \( \bar{U}^{(m)}_a \) and \( V^{(m)}_a \) can be written in terms of the null normals and are as follows:

\[
\omega_a^{(\ell)} = - (\epsilon + \bar{\epsilon}) n_a + (\tilde{\alpha} + \bar{\beta}) \bar{m}_a + (\alpha + \bar{\beta}) m_a
\]

\[
U^{(m)}_a = - \bar{\pi} n_a + \bar{\mu} m_a + \bar{\lambda} \bar{m}_a
\]

\[
V^{(m)}_a = - (\epsilon - \bar{\epsilon}) n_a + (\beta - \bar{\alpha}) \bar{m}_a + (\alpha - \bar{\beta}) m_a
\]

The part of the connection \( V^{(m)} \) is purely imaginary.

We can use this information to find the connection. Since the internal null vectors are fixed, we get

\[
\nabla_\mu \ell^I \equiv A^{(\ell)}_\mu \ell^I.
\]

This tetrad is annihilated by the covariant derivative, \( \nabla_\mu \ell^I = 0 \). Then the equation \((3.13)\) gives: \( A^{(\ell)}_\mu \ell^I \equiv \omega_a^{(\ell)} \ell^I \).

Written in a more compact form, this reduces to

\[
A^{(\ell)}_\mu \ell^I \equiv -2 (\omega_a^{(\ell)} + \nabla_a p) \ell_{[nn]I} + Q_{II},
\]

where, the one form \( Q_{IJ} \) is such that \( Q_{IJ} \ell^I = 0 \). This construction can be followed for other null vectors \( n_l, m_l, \bar{m}_l \). The connections obtained for these other null vectors \( n, m, \bar{m} \) will complement each other. Combining all the expression for these connections, we get the complete expression for the connection \( A_{\mu}^{IJ} \)

\[
A_{\mu}^{IJ} \equiv -2 (\omega_a^{(\ell)} + \nabla_a p) \ell_{[nn]I} + 2 U^{(m)}_a \ell_{[nm]I} + 2 \bar{U}^{(m)}_a \ell_{[n\bar{m}]I} + 2 (V^{(m)} + \nabla_a p) m_{[\bar{m}]I}
\]
We define the following connection for ease of computation

$$A_{ij}^{(H)} := \frac{1}{2}(A_{ij} - \frac{\gamma}{2} \epsilon_{ij}^{KL} A_{KL})$$ (3.16)

This leads to the following form of the connection:

$$A_{ij}^{(H)} \equiv \ell_{[im]} \left( -\omega_{a}^{(f)} + i\gamma V^{(m)}_{a} \right) + m_{[il]m} \left( V^{(l)}_{a} - i\gamma \omega_{a}^{(f)} \right) + \ell_{[il]m} \left( U^{(l)}_{a} + i\gamma U^{(m)}_{a} \right) - \ell_{[iml]} (1 - i\gamma) V^{a}_{p} \ell_{[pm]}$$(3.17)

Let us at this stage point out the result of the rescaling of the null normal $\ell^{a}$ on the various quantities of interest. Firstly, for $\ell^{a} \rightarrow \xi^{a} \ell^{a}$, we have:

$$\omega_{a}^{(\ell)} \rightarrow \omega_{a}^{(\xi \ell)} = \omega_{a}^{(\ell)} + \nabla_{a} \ln \xi$$ (3.18)

Since the normalization of $\ell^{a}$ and $n^{a}$ is connected, we must have $n^{a} \rightarrow \frac{n^{a}}{\xi}$ when $\ell^{a} \rightarrow \xi \ell^{a}$. Then the effect of the rescaling can be seen to be:

$$\nabla_{a} \left( \frac{n^{b}}{\xi} \right) \equiv -\omega_{a}^{(\xi \ell)} \left( \frac{n^{b}}{\xi} \right) + U_{a}^{(\ell, m)} m^{b} + U_{a}^{(\xi, m)} \bar{m}^{b}$$ (3.19)

Thus, under this transformation, the components of the connection transform as

$$\omega_{a}^{(\ell)} \rightarrow \omega_{a}^{(\xi \ell)} = \omega_{a}^{(\ell)} + \nabla_{a} \ln \xi,$$

$$U_{a}^{(\ell, m)} \rightarrow U_{a}^{(\xi, m)} \equiv \frac{U_{a}^{(\ell, m)}}{\xi} \quad \text{and} \quad U_{a}^{(\xi, m)} \rightarrow U_{a}^{(\xi, m)} \equiv \frac{U_{a}^{(\xi, m)}}{\xi}$$ (3.20)

We can also independently rescale the other set of null vectors $m, \bar{m}$ of the null tetrad. This rescaling is completely free of any information about the rescaling in the $\ell, n$ sector. Since $\xi$ rescaling controls the extremality of the horizon, this means that whatever be the nature of the horizon (extremal or non-extremal) rescaling of $m, \bar{m}$ is always possible. Let, $m \rightarrow f m$ and $\bar{m} \rightarrow \frac{\bar{m}}{f}$, where $f$ is any function on $\Delta$. Note that this implies that $f$ must be a pure phase of the form $e^{i\theta}$. The transformation are:

$$\nabla_{a} \left( f m^{b} \right) \equiv U_{a}^{(\ell, fm)} m^{b} + V_{a}^{(fm)} \left( f m^{b} \right)$$

$$\nabla_{a} \left( \frac{\bar{m}}{f} \right) \equiv \bar{U}_{a}^{(\ell, fm)} m^{b} - V_{a}^{(fm)} \left( \frac{\bar{m}}{f} \right)$$ (3.21)

The transformation rules are as follows for the one forms $U_{a}^{(\ell, m)}, U_{a}^{(\ell, fm)}$ and $V_{a}^{(m)}$ are as follows:

$$U_{a}^{(\ell, m)} \rightarrow U_{a}^{(\ell, fm)} \equiv \frac{U_{a}^{(\ell, m)}}{f}$$

$$U_{a}^{(\ell, fm)} \rightarrow U_{a}^{(\ell, fm)} \equiv f U_{a}^{(\ell, fm)}$$

$$V_{a}^{(m)} \rightarrow V_{a}^{(fm)} \equiv V_{a}^{(m)} + \nabla_{a} \ln f$$ (3.22)

The part of the connection $\omega^{(f)}$ and $V^{(m)}$ transform as abelian gauge field whereas the other parts of connections only rescale.

**Variation of the Action**

We take the Lagrangian 4-form appropriate to the action of the Holst action to be

$$-16\pi\gamma L = \gamma f(\phi) \Sigma_{ij} \wedge F^{ij} - f(\phi) e_{i} \wedge e_{j} \wedge F^{ij} - \gamma d(f(\phi) \Sigma_{ij} \wedge A^{ij}) + d(f(\phi) e_{i} \wedge e_{j} \wedge A^{ij}) - 8\pi G K(\phi)^* d\phi \wedge d\phi + 16\pi G V(\phi) e,$$ (3.23)
The construction of the symplectic current from here is standard. The current is 
\[ \Theta \] symplectic one-form well defined.

\[ \delta J = \text{orientation of the spacetime foliation into account and get} \]

\[ \int_{\Delta} \left[ (iV^{(m)} + \gamma \omega^{(f)}) + (1 + \gamma) \ dp \right] \wedge \delta(f(\phi)^{2}e) \] (3.24)

We will argue that the term is zero and hence the action principle is well defined. The nature of argument is almost similar to that in [9]. We however repeat the arguments for the sake of completeness. First of all, the field configurations over which the variations are taken are such that they satisfy the standard boundary conditions at infinity and the WIH boundary conditions at \( \Delta \). This immediately implies that the scalar field does not affect the variation of the action. We only have to worry about the other terms. The weak isolation condition implies that \( \mathcal{L}_{\ell}(e^{(f)} \equiv 0 \) though there is no such condition on \( V^{(m)} \). However interestingly, \( d\omega^{(f)} \) and \( dV^{(m)} \) are proportional to \( \text{e}^{2} \) and hence inner product with \( \ell^{\mu} \) of these quantities are zero. This implies that for variations among field configurations with null normals in the equivalence class, we have \( \mathcal{L}_{\ell}(e^{(f)} \equiv d(\xi_{\ell}(e^{(f)})) \) and \( \mathcal{L}_{\ell}V^{(m)} \equiv d(\xi(e - \bar{e})) \). This implies that on the application of \( \mathcal{L}_{\ell} \), the integral goes to the initial and the final cross section of \( \Delta \). However, the variation of the fields for example \( \delta \text{e}^{2} \) is zero at the initial and final hypersurface by the standard rules of variational principle. Thus the integral is lie dragged by any null normal in the equivalence class. In other words, the integral in zero at the initial and the final hypersurface and is lie dragged on \( \Delta \). Thus, the entire integral is zero and the action principle is well defined.

The Symplectic Structure

The construction of the symplectic structure from a given Lagrangian is detailed in [14]. One first extracts the symplectic one-form (spacetime three-form in 4-dimensions) from the variation of the Lagrangian such that \( \delta L = d\Theta(\delta) \) where \( \delta \) is an arbitrary vector field in the phase space. In the present case, we have

\[ 16\pi G \gamma \Theta(\delta) = -2 \delta(f(\phi)e^{(f)} \wedge e^{(f)} + A_{ij}^{(f)} + K(\phi) \cdot d\phi \delta \phi \] (3.25)

The construction of the symplectic current from here is standard. The current is \( J(\delta_{1}, \delta_{2}) := \delta_{1}\Theta(\delta_{2}) - \delta_{2}\Theta(\delta_{1}) \). The current is closed on-shell \( i.e. \ df = 0 \). The resulting Symplectic Current is:

\[ J(\delta_{1}, \delta_{2}) := \frac{1}{8\pi G \gamma} \left( \delta_{1} \left( f(\phi) e^{(f)} \wedge e^{(f)} \right) \wedge \delta_{2} \left( A_{ij} - \frac{\gamma}{2} e^{(f)} K_{ij} A_{Kl} \right) \right) - K(\phi) \left\{ \delta_{2} \left( *d\phi \right) \delta_{2}(\phi) \right\} \] (3.26)

Since \( df = 0 \), upon integrating the symplectic current over \( M \), we get contributions only from the boundaries under consideration

\[ \int_{M_{+} \cup M_{-} \cup \partial M} J(\delta_{1}, \delta_{2}) = 0 \] (3.27)

The boundary conditions at infinity ensure that the integral of the symplectic current at spatial infinity vanishes. To construct the symplectic structure we must be careful that no data flows out of the phase space because of our choice of foliation. In other words, the symplectic structure should be independent of the choice of foliation. To this end, we introduce potentials,

1. \( \mathcal{L}_{\ell}(\psi) \equiv \xi^{(f)} \omega^{(f)} \equiv K_{\ell} \)
2. \( \mathcal{L}_{\ell}(\mu^{(m)}) \equiv i\xi^{(f)} V_{\mu}^{(m)} \equiv i\xi(e - \bar{e}) \)

which satisfy the boundary conditions that they are zero at the initial cross-section of \( \Delta \) so that the additive ambiguities in them are removed. We choose \( \psi(\ell) = 0 \) and \( \mu^{(m)} = 0 \) at \( S_{-} \). This potentials imply that \( J(\delta_{1}, \delta_{2}) \equiv dJ(\delta_{1}, \delta_{2}) \)

\[ J(\delta_{1}, \delta_{2})_{\mid_{\Delta}} \equiv d \left[ \frac{1}{8\pi G \gamma} \left\{ \delta_{2}(f(\phi)^{2} e) \delta_{2}(\mu^{(m)} + \gamma\psi(\ell)) - (1 \leftrightarrow 2) \right\} \right] \] (3.28)

With this simplification, the integrals of \( J(\delta_{1}, \delta_{2}) \) on \( \Delta \) will be taken to the boundaries \( S_{+} \) of \( \Delta \). We take a particular orientation of the spacetime foliation into account and get

\[ \left( \int_{M_{+}} - \int_{M_{-}} \right) J(\delta_{1}, \delta_{2}) \equiv \frac{1}{8\pi G \gamma} \left( \int_{S_{+}} - \int_{S_{-}} \right) \delta_{2}(f(\phi)^{2} e) \delta_{2}(\mu^{(m)} + \gamma\psi(\ell)) - (1 \leftrightarrow 2) \]} (3.29)
The construction of symplectic current is independent of our choice foliation and hence all the phase space information can be obtained from this symplectic current by staying on any arbitrary foliation. We choose a particular Cauchy surface \( M \) which intersects \( \Delta \) in the sphere \( S_\Delta \) so that

\[
\Omega(\delta_1, \delta_2) := \frac{1}{8\pi G} \int_M \left[ \delta_1(f(\phi) \, e^l \wedge e^l) \wedge \delta_2 A_{ij}^{(H)} - \delta_2(f(\phi) \, e^l \wedge e^l) \wedge \delta_1 A_{ij}^{(H)} \right] \\
+ \frac{1}{8\pi G} \int_{S_\Delta} \left[ \delta_1(f(\phi) \, \psi_m - \gamma \psi_{(0)}) - \delta_2(f(\phi) \, \psi_m + \gamma \psi_{(0)}) \right] \\
+ \int_M K(\phi) \left[ \delta_1('d\phi') \delta_2 'd\phi' - \delta_2('d\phi') \delta_1 'd\phi' \right]
\]

(3.30)

The symplectic structure (3.30) obtained from the non-minimally coupled Holst action has some additional terms compared to that obtained form the non-minimally coupled Palatini theory [10]. The Palatini symplectic structure can be read-off by collecting the \( \gamma \) independent terms. We shall see that some miraculous cancellations among the \( \gamma \) dependent terms lead to the usual result of first law as is obtained from the Palatini theory.

IV. THE FIRST LAW

We have already stressed that WIH is a local definition of horizon unlike the event horizon or Killing horizon. The first law for event horizons studies variations of quantities defined (or normalised) with respect to spatial infinity. In the present case, we want the first law to relate variations of local quantities that are defined only at the horizon without any reference to the rest of the spacetime. In other words, we expect that the first law based on the definition of WIH should involve only locally defined quantities. For WIH, surface gravity \( \kappa_{(\xi)} \) has already been defined at the horizon. We now must define energy locally. In spacetime, energy is associated with a timelike Killing vector field. Given any timelike vector field \( W^a \) in spacetime, it naturally induces a vector field \( \delta W \) in the phase space. The phase space vector field \( \delta W \) is the generator of time translation in the phase space. If time translation is a canonical transformation in the phase space then \( \delta W \) defines a Hamiltonian function \( H_W \) for us. The vector field \( \delta W \) is globally Hamiltonian if and only if \( X_W(\delta) = \Omega(\delta, \delta W) = \delta H_W \) for any vector field \( \delta \) in the phase space. On WIH, the vector fields \( W^a \) are restricted by the condition that it should be tangential on \( \Delta \). Just like the usual advanced time coordinate, analog of ‘time’ translation on WIH is along the null direction (two other are spacelike). It is generated by the vector field \( [\xi^a] \). For global solutions this null normal vector field becomes timelike outside the horizon and is expected to match with the asymptotic time-translation for asymptotically flat spacetimes.

We want to find out if the flow generated by the phase space vector field \( \delta \xi^a \) is Hamiltonian. The action of the phase space vector field \( \delta \xi^a \) on tensor fields is the lie flow \( \xi^a \) generated by the vector field \( \xi^a \). For the above symplectic structure, \( X_{\xi^a}(\delta) \) gets contribution from both the bulk and the surface symplectic structure. The bulk term, thanks to the equation of motion satisfied by the fields and their variations, contributes only through the boundaries of the Cauchy surface \( M \), which are the 2- spheres \( S_\Delta \) and \( S_\infty \) respectively:

\[
X_{\xi^a}(\delta)|_M = -\frac{1}{8\pi G} \xi^a(\delta) (f(\phi) \, A_\Delta) - \frac{i}{8\pi G} \int_{S_\Delta} \xi (\bar{e} - \bar{e}) \delta(f(\phi) \, \psi) + \delta E_{(\xi^a)}
\]

(4.1)

where, \( A_\Delta = \int_{S_\Delta} e \) is the area of \( S_\Delta \) and \( E_{(\xi^a)} \) is the ADM energy arising out of the integral at \( S_\infty \), assuming that the asymptotic limit of the integral matches with the vector field \( \xi^a \) at infinity.

The one- form \( X_{\xi^a}(\delta) \) also gets contribution from the surface symplectic structure (the arguments are similar to that in [9,13]). The action of \( \delta_{(\xi^a)} \) cannot be interpreted as \( \xi^a(\delta) \) when acting on potentials. To determine the action, we proceed as follows. For the case of \( \psi_{(0)} \), it is clear that since variation of \( \psi_{(0)} \) is completely determined by \( \kappa_{(\xi)} \), \( \delta_{(\xi^a)} \psi_{(0)} = 0 \). However, \( \psi_{(1)} = \psi_{(0)} + \bar{e} \) implies that \( \delta_{(\xi^a)} \psi_{(0)} = -\bar{e} \xi^a \). For the other potential, observe that \( \delta_{(\xi^a)} \mu_m - \bar{e} \) satisfies the differential equation \( \xi^a \delta_{(\xi^a)} \mu_m = 0 \) with the boundary condition at \( \mu_m = 0 \) at the point \( v = 0 \). This implies that because \( (\bar{e} - \bar{e}) = 0 \) at \( v = 0 \), the action is \( \delta_{(\xi^a)} \mu_m = \bar{e} \xi^a \). The considerations above leads to

\[
X_{\xi^a}(\delta)|_{S_\Delta} = -\frac{1}{8\pi G} \xi^a(\delta) (f(\phi) \, A_\Delta) + \frac{i}{8\pi G} \int_{S_\Delta} \xi (\bar{e} - \bar{e}) \delta(f(\phi) \, \psi) + \delta E_{(\xi^a)}
\]

(4.2)

Combining the two equations (4.1) and (4.2), we get:

\[
X_{\xi^a}(\delta) = -\frac{1}{8\pi G} \kappa_{(\xi^a)} (f(\phi) \, A_\Delta) + \delta E_{(\xi^a)}
\]

(4.3)
For $\delta_\ell$ to be Hamiltonian, the surface gravity $\kappa_{(\ell)}$ must be a function of area $\mathcal{A}_\Lambda$ only. This is reasonable since the phase space is characterized by area (and charges) and so $\kappa_{(\ell)}$ can only be a function of these quantities. The exact functional dependence of $\kappa$ on area is undetermined. This is a fundamental result of the generalization to the generalised class of null normals $[\xi^{\ell}]$. In the constant class of null normals, there is no contribution from the surface symplectic structure. In the present case, the precise contribution (and cancellations) from the bulk (4.1) and the boundary (4.2) leads to the physically meaningful variation.

Previous results also imply that there exists a locally defined function $E_\Lambda$ such that

$$\delta E_\Lambda \equiv \frac{1}{8\pi G} \kappa_{(\ell)} \delta (f(\psi) \mathcal{A}_\Lambda) \tag{4.4}$$

such that $H_\ell = E_{(\ell)} - E_\Lambda$ where $H_\ell$ is the associated Hamiltonian function $X_{(\ell)}(\delta) = \delta H_\ell$. We shall interpret $E_\Lambda$ as the locally defined energy of the WIH and (4.4) as the first law of the WIH. $H_\ell$ receives contributions both from the bulk as well as the boundary symplectic structures has information of energy of the region between the WIH and spatial infinity. The ADM energy $E_{(\ell)}$ is the sum total of these two energies. The explicit form of $E_\Lambda$ can be determined iff the functional dependence of $\kappa_{(\ell)}$ on area is known.

**Inclusion of rotation**

A WIH is spherically symmetric if the geometry of the $\nu = \text{constant}$ ($\nu$ is the affine parameter of $\ell$) sections of $\Delta$ is spherical. So in addition to $\xi^{\ell}$, such horizons admit three other local spacelike Killing vector fields that are tangent to the cross-sections. To include rotations, we consider the horizon to have a symmetry about some axis $q^a$. For spherical symmetry, $q^a$ is one of the three Killing vectors. The metric for this case will be given by

$$ds^2 = r_\Lambda^2 \left( d\delta^2 + F(\delta)^2 d\varphi^2 \right) \tag{4.5}$$

- where, $F(\delta) = \sin \vartheta$ for spherical symmetry. We shall keep the function as $F(\delta)$ for further calculations. The corresponding vector fields can also be found out easily. They are given by:

$$m^a = \frac{1}{r_\Lambda \sqrt{2}} \left[ \frac{\partial}{\partial \delta} \right]^a + i \frac{\partial}{\partial \vartheta} \left( \frac{\partial}{\partial \vartheta} \right)^a \tag{4.6}$$

The volume element of the sphere is $\epsilon = i m \wedge m = r_\Lambda^2 F(\delta) (d\delta \wedge d\vartheta)$. Let us now turn to the symplectic structure. The symplectic structure of the non-minimally coupled Holst action (see 3.30) admits a canonical transformation to the symplectic structure of the non-minimally coupled Palatini action. We shall not show this explicitly, this can be derived fairly easily from the results in appendix of [9].

$$\Omega(\delta_1, \delta_2) = \frac{1}{16\pi G} \int_M \left[ \delta_2 (f(\psi) \Sigma^a) \wedge \delta_1 A_a - \delta_1 (f(\psi) \Sigma^a) \wedge \delta_2 A_a \right]$$

$$-\frac{1}{8\pi G} \int_{S_\Lambda} \left[ \delta_2 (f(\psi) \epsilon) \delta_1 \psi_{(\ell)} - \delta_1 (f(\psi) \epsilon) \delta_2 \psi_{(\ell)} \right] . \tag{4.7}$$

As in the derivation of the first law, the Killing vector $q^a$ on the spacetime induces a vector field $\delta_q$ on the phase space. We now ask whether the flow generated by $\delta_q$ on the phase space is a Hamiltonian. In that case, we can call the Hamiltonian function as angular momentum. In the symplectic structure, the only contribution comes from the bulk symplectic structure. The surface symplectic structure has no contribution. This can be shown as follows: First, the area element is Lie-dragged by the vector field $q$. The other term in the symplectic structure is also zero by the following argument. We know that $\delta_q \kappa_{(\ell)} = 0 = \delta_q(\ell, \nabla \psi)$. This implies that $[\delta_q, \ell] \nabla \psi + \ell(\delta_q \psi) = 0$. But the first term is zero by the conditions of symmetry and to keep the foliation fixed. This then implies that $\ell(\delta_q \psi) = 0$, i.e. the function $(\delta_q \psi)$ is a constant along $\ell$. However, since $(\delta_q \psi) = 0$ on $S_{-\ell}$, this implies that $(\delta_q \psi) = 0$ on each $S_{\Lambda}$ on $\Delta$. This result holds true for all $\ell$ in the class $[\xi^{\ell}]$.

Proceeding as in previous sections, the bulk symplectic structure contributes only through the boundaries of $M$ which are at $\Delta$ and at infinity. The contribution of the bulk symplectic structure to the bulk of $M$ vanishes due to
the equation of motion satisfied by the fields \((e, A)\) and the linearised equations satisfied by them. So, we only get

contribution from the surface term of the bulk symplectic structure. Thus, the symplectic structure reduces to:

\[
\Omega \left( \delta, \delta_0 \right) = \frac{-1}{8\pi G} \oint_{S_\Lambda} \left[ \left[ g | \omega \right] \delta (f(\phi) e) - f(\phi) \left[ g | e \right] \wedge \delta \omega \right]
\]  

(4.8)

We can determine the terms in the symplectic structure explicitly. The first term is found out as follows:

\[
\frac{\partial}{\partial \theta} \omega = \frac{i r_A F(\theta)}{\sqrt{2}} (\pi - \bar{\pi});
\]

\[
\left[ \frac{\partial}{\partial \theta} \omega \right] \delta e = \frac{i}{\sqrt{2}}(\pi - \bar{\pi})(2\delta r_A F + r_A \delta F) e
\]

(4.9)

The second term in the expression of symplectic structure is also determined in a similar fashion. The result is:

\[
\frac{\partial}{\partial \theta} \epsilon = -r_A^2 F(\theta) d\theta
\]

\[
\left[ \frac{\partial}{\partial \theta} \epsilon \right] \wedge \delta \omega = \frac{i}{\sqrt{2}} \delta [r_A F(\pi - \bar{\pi})] e,
\]

(4.10)

where, we have used the decomposition of \(\omega\) in terms of the null normals (see eqn 3.12). Putting these relations in the symplectic structure (eqn 4.8) and taking into account of the fact that \(r_A F \delta e = (2\delta r_A F + r_A \delta F) e\), we get

\[
\Omega \left( \delta, \delta_0 \right) = \frac{-1}{8\pi G} \oint_{S_\Lambda} \delta \left\{ \frac{if(\phi)}{\sqrt{2}}(\pi - \bar{\pi}) r_A F \epsilon \right\}
\]

(4.11)

Note that the symplectic structure is now a total variation. This is precisely the necessary and sufficient condition for which there exists a Hamiltonian vector field \(\delta_0\), i.e., the vector field is a phase space symmetry \((E_0, \Omega = 0, \text{everywhere on } \Gamma)\). Then one can define a function \(f^{(\omega)}_A\) which will generate diffeomorphism along the particular vector field \(\theta\) such that for all vector fields \(\delta\) on \(\Gamma\),

\[
\delta f^{(\omega)}_A = \Omega(\delta, \delta_0)
\]

(4.12)

Using the expressions in the eqn 4.10 and the expression for \(\epsilon\) and \(\omega\), we get

\[
\Omega_{\theta} \left( \delta, \delta_0 \right) = \frac{-1}{8\pi G} \oint_{S_\Lambda} \delta \left\{ f(\phi) \left( \frac{\partial}{\partial \theta} \omega \right) \epsilon \right\}
\]

\[
= \frac{1}{8\pi G} \oint_{S_\Lambda} \delta \left[ \left( f(\phi) \left( \frac{\partial}{\partial \theta} \epsilon \right) \wedge \omega \right) \right],
\]

(4.13)

where the second term in the above expression eqn. 4.13 is obtained by noting that \(\left[ g | \omega \right] \epsilon = -\left[ g | e \right] \wedge \omega\). The angular momentum at the horizon corresponding to the vector field \(\theta\) is thus defined to be

\[
f^{(\omega)}_A = \frac{1}{8\pi G} \oint_{S_\Lambda} \left[ f(\phi) \left( \frac{\partial}{\partial \theta} \epsilon \right) \wedge \omega \right]
\]

\[
= \frac{-1}{8\pi G} \oint_{S_\Lambda} \left[ f(\phi) \left( \frac{\partial}{\partial \theta} \omega \right) \epsilon \right]
\]

(4.14)

Let us now define the function \(F(\theta)\) as follows

\[
F = \frac{\partial \phi}{\partial \theta}
\]

(4.15)

Then the expression for the angular momentum reduces to:

\[
f^{(\omega)}_A = \frac{-r_A^2}{8\pi G} \oint_{S_\Lambda} F d\theta \wedge \omega = \frac{-r_A^2}{8\pi G} \oint_{S_\Lambda} \mathbf{n} f(\phi) d\omega
\]

(4.16)
Now, note that we have previously defined the curvature of the part of the connection $\bar{d}\omega = 2(\text{Im}\Psi_2 e)$ where $\text{Im}\Psi_2$ provides us the information of the rotation. Thus, if the solution is such that the Weyl tensor gives a contribution to $\text{Im}\Psi_2$, then the horizon admits an angular momentum. For spherical symmetric solutions however, $\text{Im}\Psi_2 = 0$. Thus for the case of spherical solutions, the angular momentum is zero. Also note that $\text{Im}\Psi_2$ is gauge invariant and thus so is $f^{(0)}_\lambda$. This is interesting since the result is independent of the rescaling freedom of $[\xi \ell^a]$ the notion of angular momentum also makes sense for any NH. It is important to note that if we have any arbitrary vector field tangent to $S_\lambda$, one can still define a Hamiltonian $J_\lambda$ which will generate diffeomorphisms along that vector field. However, we will not be able to identify that $J_\lambda$ to angular momentum as it is intimately connected to symmetries. We have obtained this result for a metric which has a symmetry along some direction $\vartheta^\lambda$. This gave us the angular momentum for the rotation about the axis. In the case of spherical symmetry, the angular momentum is zero because $\text{Im}\Psi_2 = 0$. We also have two more Killing vectors. It is not very difficult to see that these two Killing vectors give their corresponding angular momenta. However, again the momenta are proportional to $\text{Im}\Psi_2 = 0$ and hence are zero for spherical metric. These results were previously derived in [10]. However, we rederive using a choice of coordinate system for the sphere.

The first law for the rotating WIH can now be written down fairly easily by taking into account the first laws for the two hamiltonian vector fields considered above. Consider a vector field $W^a = \xi \ell^a - \Omega W^\vartheta$ on the spacetime. The first law for this vector field will be:

$$\delta E_A \equiv \frac{1}{8\pi G}\kappa(\xi \ell^a)\delta(f(\phi)\mathcal{A}_A) + \Omega(\psi)\delta J_A$$

(4.17)

V. SPHERICAL HORIZON AND CHERN-SIMONS SYMPLECTIC STRUCTURE

Arguments of Bekenstein and Hawking, based on the laws of black hole mechanics and semiclassical calculations, tells us that the entropy of black holes are equal to quarter of their areas. However, such an interpretation of entropy as area needs to be backed up by microstate counting a la Boltzmann. Knowledge of microstates lies beyond the domain of classical theory because the laws of the microscopic world are quantum mechanical. Thus, one needs a quantum theory of spacetime for a satisfactory calculation of entropy. One of the statistical interpretations of black hole entropy is the loop approach based on Loop Quantum Gravity [4, 5, 7]. Here, instead of knowing the microscopic degrees of freedom of the entire spacetime, it is proposed that we consider their effects on the black hole horizon. The basic idea is that the essential features of the black hole spacetime are captured by some effective degrees of freedom on the horizon which originate because of the interaction of bulk and the boundary of the spacetime. Isolated horizon formulation is relevant because such surfaces capture the essential features of a black hole spacetime. One determines the effective theory induced at an isolated horizon, quantize it and count the appropriate quantum states. This turn out to be consistent with the semiclassical estimates made by Bekenstein and Hawking. The effective theory on the horizon can only be a theory of the topological kind, namely it must be insensitive to the metric on the horizon. This is because the horizon is a null surface and therefore cannot support a physical particle. The above papers show, through a detailed canonical phase space analysis, that the effective theory on the horizon is Chern-Simons type, more precisely a $U(1)$ Chern-Simons theory.

The main objective of the section is to find out the symplectic structure of the effective field theory on a spherically symmetric WIH of a fixed area, starting from the non-minimally coupled Holst action in a completely covariant framework (the derivation is similar in spirit to that in [9]). We shall show that the claims in the abovementioned papers are reinforced independent of any slicing. Since the horizon is spherically symmetric, the boundary conditions imply that the scalar field is constant on the horizon. Once these conditions are fulfilled, it follows from the Einstein’s equations that:

$$\Phi_{11} + \frac{1}{8}R - \frac{1}{2}\Lambda \equiv 4\pi Ge$$

(5.1)

where, $\Phi_{11} = \frac{1}{2}R_{ab}(\ell^a n^b + m^a \bar{m}^b)$ and $e$ is spherically symmetric having contribution from the scalar field the other minimally coupled fields like Maxwell fields etc. The equation (5.1) implies that for spacetimes with cosmological constant zero, the term $\Phi_{11} + \frac{1}{8}R$ is spherically symmetric. Further, it is not difficult to check that (see [9] for details)

$$\Psi_2 + \frac{1}{12}R \equiv \xi \ell^a \mu + \kappa(\ell)\mu$$

(5.2)

where $\mu, \kappa(\ell)$ are as previously defined (see [3, 12]). All the terms on the right hand side of (5.2) are real and $R$ is real, the term $\text{Im}\Psi_2 = 0$, i.e., $d\omega^{(0)} = 0$. Then, the term $(\text{Re}\Psi_2 + \frac{1}{12}R)$ is spherically symmetric.
Using the equations (5.2) and (5.1), we see that the term \( F := (\Psi_2^{(H)} - \Phi_{11} - \frac{\ell^2}{2}) \) is again spherically symmetric. Moreover, \( F \) is constant over \( \Delta \). We want to find a value for this constant. It turns out that (see appendix of [9] for detailed calculations)

\[
dV^{(m)} = -2iF^2e
\]  

(5.3)

The connection \( iV^{(m)} \) is precisely the connection on the sphere \( S^2 \). Using Gauss- Bonnet theorem, we get

\[
(\Psi_2^{(H)} - \Phi_{11} - \frac{R}{24}) = -\frac{2\pi}{\mathcal{A}}
\]  

(5.4)

Now, we define the connection for \( V^{(H)} = -im[I\mu]A^{(H)}I] = (iV^{(m)} + \gamma\omega^{(I)}/2 \). Clearly, \( dV^{(H)} = idV^{(m)}/2 \) since on spherically symmetric horizon \( d\omega^{(I)} = 0 \). It follows from (5.4) that

\[
2e = -\frac{\mathcal{A}}{2\pi} dV^{(H)}
\]  

(5.5)

We shall use this expression (5.5) for the surface contribution to symplectic current for fixed areas phase space. We shall see that the surface symplectic structure is a \( U(1) \) Chern- Simons theory symplectic structure .

In (3.28), the potential \( \psi_{\ell}(\theta) \) is a function of \( \ell \) only while \( \mu_{(m)} \) is still a function of \( (\ell, \theta, q) \). A simple calculation gives

\[
\int A_{\ell}(\theta) = \frac{1}{8\pi G\gamma} (\int S_\ell - \int S_\ell \{\delta_1 \mu_{(m)} \delta_2^2 e - (1 \leftrightarrow 2)\}
\]  

(5.6)

Now recall that the one-form \( m \) has a rescaling freedom given by \( m \to g m(\ell, \theta, q) = e^{-i\mu_{(m)}(\ell, \theta, q)} m(0, \theta, q) \). This gives (see (3.22) for the transformation rules)

\[
V^{(m)} \to V^{(m)} e^{\gamma} = V^{(m)} - id\mu_{(m)}
\]

\[
V^{(H)} \to V^{(H)} e^{\gamma} = V^{(H)} + \frac{1}{2}d\mu_{(m)}(\ell)
\]  

(5.7)

To proceed further, we use the expression (5.5) in the symplectic current and integrate by parts and again use (5.7). This gives the following expression for the current

\[
\int A_{\ell}(\theta) = \frac{2f(\phi_s, F, \mathcal{A}^e)}{8\pi G\gamma} (\int S_\ell - \int S_\ell \{\delta_1 V^{(H)} \wedge \delta_2 V^{(H)} - (1 \leftrightarrow 2)\}
\]

(5.8)

Now note that \( V^{(H)} e^{\gamma} \) is a function of \( (\ell, \theta, q) \) whereas \( V^{(H)} \) has only the dependence on \( (\theta, q) \) and \( \mu_{(m)} \) is \( \ell \) dependent. In other words, the \( \ell \) dependence of \( V^{(H)} e^{\gamma} \) has been transferred to \( \mu_{(m)} \) leaving \( V^{(H)} \) only with the angular dependence. Using this information, we get

\[
\int A_{\ell}(\theta) = \frac{f(\phi_s, F, \mathcal{A}^e)}{8\pi G\gamma} (\int S_\ell - \int S_\ell \{\delta_1 V^{(H)} \wedge \delta_2 V^{(H)} - (1 \leftrightarrow 2)\}
\]  

(5.9)

which is identical to the Chern-Simons symplectic structure. The full symplectic structure for the spherically symmetric and fixed phase space admitting a WIH

\[
\Omega(\delta_1, \delta_2) = \frac{1}{8\pi G\gamma} \int \left[ \delta_1 (\ell^2 \wedge \ell^2) - \delta_2 (\ell^2 \wedge \ell^2) + \delta_1 A^{(H)}_{IJ} - \delta_2 A^{(H)}_{IJ} \right] - \frac{\mathcal{A}}{8\pi G\gamma} \int \left[ \delta_1 V^{(H)} \wedge \delta_2 V^{(H)} \right]
\]  

(5.10)

The boundary symplectic structure can be identified with that of a \( U(1) \) Chern-Simons theory (because the symplectic structure involves only one type of connection \( V^{(H)} e^{\gamma} \) with level \( k = f(\phi_s, F, \mathcal{A}^e)/4\pi G\gamma \). Upon quantization the level becomes an integer. So \( f(\phi_s, F, \mathcal{A}^e)/4\pi G\gamma \) has to be an integer in the quantum theory. This is a highly nontrivial result of the WIH formulation. WIH phase space includes both extremal as well as non-extremal types of horizons and what we find here is that the effective theory on the boundary is insensitive to these two types of horizons. The Chern-Simons gauge field does not see the \( \ell \) scaling of the null normal \( \ell^2 \), which controls the value of surface gravity for the horizon. In the quantum theory one essentially counts the surface states of the quantum Chern-Simons theory and hence, the entropy function is also expected to be insensitive to two types of horizons.
VI. DISCUSSIONS

In this paper, we have studied the classical phase space of non-minimally coupled Host action admitting a WIH. The boundary conditions for the WIH amiable for the non-minimal scalar coupling is stronger than the boundary conditions in [3,13]. The non-minimally coupled scalar field is constrained so that it remains constant along any $\ell^a$ in the class $[\xi^\ell]$. In the standard case of minimal couplings, the constancy of the field along $\ell^a$ arises because of the energy conditions. Non-minimally coupled scalar fields violate this energy conditions. Thus, this condition is put in by hand and suffices for our discussion. The zeroth law for the WIH is a result of these boundary conditions. Next, we have constructed the symplectic structure on this phase space and proved the first law. The essence of this derivation is similar to that in [3,13]. We also included rotations and showed in a detailed derivation how the first law is changed. We have not included any minimally coupled fields like the Maxwell field. The result of such inclusion is standard and can be derived as in [3,13]. Subsequently, we have shown that on a fixed area phase space containing spherical horizons, the symplectic structure is that of a $U(1)$ Chern-Simons theory. This result was only derived for the constant class of null-normals [$c\ell^a$] (Weakly Isolated Horizons) in a covariant formalism (see [12]). We rederive these results for WIH in a covariant phase space formulation. It is easy to map the covariant phase space results to canonical phase space (see appendix of [9]).

Let us point out the importance of these results. It is known that string theories generally predict non-minimal scalar field couplings. Moreover, extremal black holes are ubiquitous in such theories [15]. The standard prescription for calculating the entropy of these extremal black holes is the Wald’s Noether charge approach [11]. It is well known that this formulation is not well suited for the extremal black holes. Indeed, in this case the extremal and non-extremal black holes belong to distinct phase spaces (see [9] for a detailed discussion). Thus, it is difficult to make sense of extremal limits of quantities defined for non-extremal black holes. On the other hand, WIH naturally encompass both extremal and non-extremal black horizons. The space of solutions admitting WIH internal boundary has both the extremal and non-extremal solutions on the same footing. On this phase space, extremal black holes can be defined as a limit of sequence of non-extremal black holes. The laws of black hole mechanics for these two type of horizons are clarified in this formulation. We have also showed that the topological theory induced on the horizon is a $U(1)$ Chern-Simons theory induced on WIH is irrespective of the nature of extremality of the solution. Indeed the Chern-Simons gauge field $V^{(H)}$ does not depend on the $\xi$ function which controls the extremality/non-extremality of the horizon. It will be argued thus that if the space of solutions has both extremal and non-extremal horizons, the entropy is same for both species. This is because, one essentially counts the surface states of the quantum Chern-Simons theory and hence, the entropy function is also expected to be insensitive to two types of horizons. In other words, WIH is ideally suited for study of extremal and non-extremal black holes in the same footing. Thus, the formalism is ideally suited for study of black holes in string theory.

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