Transitive bounded-degree 2-expanders from regular 2-expanders

Eyal Karni$^1$  Tali Kaufman$^2$

$^1$Bar Ilan University, ISRAEL.  eyalk5@gmail.com
$^2$ Bar Ilan University, ISRAEL.  kaufmant@mit.edu

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Abstract

A two-dimensional simplicial complex is called $d$-regular if every edge of it is contained in exactly $d$ distinct triangles. It is called $\epsilon$-expanding if its up-down two-dimensional random walk has a normalized maximal eigenvalue which is at most $1 - \epsilon$. In this work, we present a class of bounded degree 2-dimensional expanders, which is the result of a small 2-complex action on a vertex set. The resulted complexes are fully transitive, meaning the automorphism group acts transitively on their faces.

Such two-dimensional expanders are rare! Known constructions of such bounded degree two-dimensional expander families are obtained from deep algebraic reasonings (e.g. coset geometries).

We show that given a small $d$-regular two-dimensional $\epsilon$-expander, there exists an $\epsilon' = \epsilon'(\epsilon)$ and a family of bounded degree two-dimensional simplicial complexes with a number of vertices goes to infinity, such that each complex in the family satisfies the following properties:

• It is $4d$-regular.
• The link of each vertex in the complex is the same regular graph (up to isomorphism).
• It is $\epsilon'$ expanding.
• It is transitive.

The family of expanders that we get is explicit if the one-skeleton of the small complex is a complete multipartite graph, and it is random in the case of (almost) general $d$-regular complex. For the randomized construction, we use results on expanding generators in a product of simple Lie groups. This construction is inspired by ideas that occur in the zig-zag product for graphs. It can be seen as a loose two-dimensional analog of the replacement product.
Part I
Overview

1 General Introduction

The notion of expansion is a key one in graphs, having many applications to a variety of mathematical problems. It generally means how easy is it to get from one vertex set to another one on a random walk on a graph. Among its applications are practical such as error correcting codes [Sip94] which could be used in communication, and theoretical as it was used to prove the PCP theorem [Din07].

Over the last two decades or so, there have been various attempts to generalize this notion to higher dimensions.

The usual useful notion is that of (pure) simplicial complex:

**Definition 1.** Simplicial complex $C$ is a collection of subsets of set $X$ such that for any $f \in C$, any subset of $f$ is in $C$. Members of $C$ are called faces, where the dimension of a face $f$ is $|f| - 1$.

The set of all faces of dimension $k$ is denoted $C(k)$. Vertices are 0-faces, edges are 1-faces and triangles are 2-faces.

There has been a growing interest in this field, motivated partially by its usefulness to constructing quantum error correcting codes. It was speculated that it holds the key for solving some theoretical questions that are out of reach for expander graphs otherwise (for example [DK17]).

The various attempts led to various definitions for expansion in high dimensions. In graphs, one would find various definitions for expansion that are essentially the same. Expansion in the notion of sets is equivalent to expansion in term of random walks, which is the same as pseudo-randomness. The extension of these properties to the high dimensional case, turned out to yield inherently different notions of expansion, where some of them are even contradicting.

We are interested in a certain expansion property of 2-complexes, that is the (two-dimensional) random walk convergence.

**Definition 2 (Random walk).** On a 2-complex $C$ is defined to be a sequence of edges $E_0, E_1, \cdots \in C(1)$ such that:

1. $E_0$ is chosen in some initial probability distribution $p_0$ on $C(1)$
2. for every $i > 0$, $E_i$ is chosen uniformly from the neighbors of $E_{i-1}$. That is the set of $f \in C(1)$ s.t. $E_{i-1} \cup f$ is in $C(2)$

This is equivalent to a random walk on graph $G_{walk}$:

**Definition 3 (Random Walk Graph).** The random walk graph of a complex $C$, $G_{walk}(C)$ is defined by $V(G_{walk}) = C(1)$, where $E \sim E' \iff \exists \sigma \in C(2)$ s.t. $E, E' \in \sigma$

$(a \sim b$ suggests that $a$ is adjacent to $b$ in the graph)

**Definition 4 (Expander).** We say $C$ is an $\epsilon$-expander if $\lambda(G_{walk}) < 1 - \epsilon$, where $\lambda(G)$ is the maximal non-trivial (normalized) eigenvalue of the graph.

This property is useful in the setting of agreement expanders [DK17], and has multiple relations to other notions of expansion [KM16].

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1 Original definition by [KM16]. Based on the definition in [Con19]
2 We assume the matrices are normalized by default
Good expanders are both sparse, translated to having a low degree, and have good expansion properties. While randomizing edges in a graph would typically yield a good expander family, that is not the case in high dimensions. And specifically there are not many ways that are simple and combinatorial in nature.

Until recently, the prime example of bounded-degree high dimensional expander was Ramanujan Complexes (ie [LSV05]), which yields the best possible expansion properties.

We are also interested in symmetric or transitive complexes:

**Definition 5** (Transitive Complex). An automorphism of complex $C$ is a function

$$\phi : V(C) \rightarrow V(C)$$

that is bijective and such that $\phi(v) \in C \iff v \in C$.

A complex $C$ is called transitive if its automorphism group act transitively on its faces. For every $x, y \in C$ where $|x| = |y|$, exists an automorphism $\phi$ s.t. $\phi(x) = y$. That means that the automorphism group acts transitively on vertices, edges and so on.

Such constructions are rare. Some coset geometries are known to be transitive [KO17]. There is an open conjecture that suggest that SRW (simple random walk) on a transitive expanders exhibits a cut-off phenomenon [LP16]. That was proved only for Ramanujan graphs so far (also in [LP16]).

### 1.1 Combinatorial Constructions of High Dimensional Expanders

The innovation of David Conlon in his construction [Con17] was that he used combinatorial method to allow one to take random set of generators of Cayley graphs and to provide a 2-complex built upon this that satisfies some HDE expansion properties including the geometric-overlapping property and the 2D random walk. The drawback of this construction is that it is built upon abelian groups, as they have a poor expansion properties [AR94].

[CLP18] studied regular graphs whose links are regular. They achieved some bounds concerning the maximal non-trivial eigenvalue for such graphs. And they constructed 2-expanders from expander graphs whose random walk converge rapidly [LPS16].

[LMY19] provided a construction called LocalDensifier that takes a small complex $H$ with expansion properties, and a graph $G$ and generates a large complex where all high order random walks have a constant spectral gap. In their construction, the vertex set of the new complex is $V(G) \times V(H)$. They used technique of decomposing the Markov chain into a restriction chain and a projection chain.

In the very recent [AL20], they obtained a bound on the second eigenvalue in the $k$-dimensional random walk in terms of the maximal second eigenvalue of the links. Using this method, it is straight forward to obtain a bound on the spectral gap of the LocalDensifier construction.

### 2 General Overview

We want to take a small general (abstract simplicial-)complex and generate a complex that is transitive regular expander, and has isomorphic link. We need the following definitions:

**Definition 6.** For complex $C$:

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3 We disregard geometric overlapping here
4 See [BHP14] for explanation of this property.
5 [AR94, Proposition 3]  
6 The graphs should be of high girth
• the link of a vertex $v$ is the complex $C_v$ defined by $\{ \sigma \setminus v \mid \sigma \in \mathcal{C} \text{ where } v \in \sigma \}$

• The $k$-skeleton denoted $C_k$ is $\{ \sigma \mid |\sigma| = k - 1 \text{ where } \sigma \in \mathcal{C} \}$

**Definition 7** (Upper-regular complex). A 2-complex is $d$-regular if every edge is contained in exactly $k$ triangles (see §2.1 for more details)

We want to prove the following theorem:

**Theorem 1** (Non-formal version of theorem). Given a complex that is $d$-regular, with minimal additional requirements on coloring, exists $\epsilon' = \epsilon'(\epsilon)$ and a family of simplicial complexes with a number of vertices goes to infinity, such that each complex in the family has the following properties:

• It is $4d$-regular.

• The link of each vertex is the same regular graph (up to isomorphism).

• It is $\epsilon'$ expanding.

• It is transitive.

We will do this in several steps. First, we introduce a general structure called Schreier complex that represents an action of a complex on a set of vertices. We do so in section 3.

Afterwards, we will focus on a private 2-dimensional case of Schreier complex.

We show a method to analyze expansion of a large complex from its small composing complex assuming they sit together in a structure that we call CTS (commutative triplets structure). This structure is a special case of Schreier complex in which there is a small 2-complex that acts on a group, and additional conditions are satisfied.

We do present it in section 4 where the exact proofs with all the details are in part II.

We will show that it is possible to generate a CTS from a general 2-complex using a construction we call HDZ. That in general would only give us an obscure expression for the expansion. In order to make it meaningful, we based our complex on product of simple Lie groups, and rely on expansion properties of random generators in Cayley graphs. We present it in section 5 while the proofs are in part III.

Finally, we will show that Schreier complexes are transitive when they are based upon a transitive group action, as in the case of HDZ.

The proof is short and will be brought here (section 6).

### 3 Schreier Complex

We remind the reader the definition of Cayley graph:

**Definition 8.** Given a group $G$ and a set $S \subset G$, $\text{Cay}(G, S)$ is defined to be the graph with vertex set $G$, and edge set $\{(sg, g) \mid s \in S, g \in G\}$

**Definition 9** (Schreier Graph). Given a group $G$ that acts on a set $X$ and a set of generator $S \subseteq G$, the Schreier graph $\text{Shr}(G, X, S)$ is a graph, whose vertices are labeled by elements of $X$ and there is an edge $(x_1, x_2)$ iff there exist $s \in S$ such that $x_2 = s \cdot x_1$ (the action of $s$ on $x_1$).

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7 $\chi$-strongly-colorable (Strong coloring or rainbow coloring means that every triangle has vertices of 3 different colors), $\chi$ is even and there is a coloring in which the edges that involve vertices of colors $a, b$ incident to at least 2 distinct vertices of each color

8 with the minimal requirements as in footnote 7
Let us focus on the case where $S$ is symmetric and each generator in $S$ is composed of two commutative steps namely $s = s_1 s_2$ where $[s_1, s_2] = 0$. Then $S \subset S_1 S_2$ where $S_1$ and $S_2$ commute.

Let us look at a specific edge $E = \{s_1 s_2 g, g\}$

We wish to describe this edge as a right action of $G$ upon $P(G)$ or using the following product

$$\Psi : P(G) \times G \rightarrow P(G)$$

$$(A, g) \rightarrow A \cdot g = \{ag \mid a \in A\}$$

In the product form, our edge has two descriptions:

$$E = \{s_2, s_1^{-1}\} \cdot s_1 g = \{s_2^{-1}, s_1\} \cdot s_2 g$$

And we can say

$$E(Cay(G, S)) = E(S) \cdot g$$

where $E(\tilde{G})$ denotes the edge set of graph $\tilde{G}$, and $S$ a graph with the following edges

$$E(S) = \{\{s_1, s_2\} \mid s_1^{-1}s_2 \in S, s_1 \in S_1\}$$

Now, we can generalize this in two different ways -

We can generalize $S$ to be a complex instead of graph. And we can generalize the group product to an action of a group on a set.

**Definition 10** (Complex Action on a vertex set). Let $G$ a group that acts on a set $X$. Let $S$ be a complex such that $S \subset P(G)$. We wish to describe an action of an complex on a vertex set by defining the following product:

$$\Psi : S \times X \rightarrow P(X)$$

$$\sigma \cdot x := \Psi(\sigma, x) = \{\sigma \cdot x \mid \sigma \in S\}$$

**Definition 11** (Schreier complex). Let $G$ a group that acts on a set $X$. Let $S$ be a complex such that $S \subset P(G)$. We define the Schreier complex of $S$ on $X$ by

$$Sc[S, X, G] := S \cdot X = \{\sigma \cdot x \mid x \in X, \sigma \in S\}$$

using the product $\Psi$.

**Remark.** Notice that is not a direct generalization of the Schreier graph, but inspired one. If the action is a left product in a group we will write $Sc[S, G]$. If the action is transitive, then the complex is transitive lemma.

We think this construction has some potential that has yet to be fulfilled. For example, the Toric code could be interpreted as an action of 3-complex over the set $\mathbb{Z}_m \times \mathbb{Z}_m$ [Kit03].

**Remark.** The CTS construction and all the proofs that are related to expansion, only deal with the case that the action is a group product and that the dimension of the complex is 2.

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9. power set of $G$

10. A fact that would prove beneficial in a generalized construction
4 Commutative Triplets Structure

Definition 12 (CTS - Shorted ). Given a group $G$, and a 2-complex $S$, We define

$$\mathcal{C} = S\mathcal{c}[S,G]$$

where the involved action is a left group product.

Provided that

A $S$ is $d$-regular

B $S(1)$ is a collection of commutative generators

C $S(1)$ is symmetric

D The action by the complex $S$ resembles a free action: $\tau \cdot g = \tau' \cdot g'$ only in the trivial case

E $S^1$ is connected

We say that the Schreier complex $\mathcal{C}$ is also a commutative triplets structure, denoted by

$$\mathcal{C} = CTS[S,G]$$

The conditions will be explained shortly.

Our analysis is based upon describing the walk in terms of equivalent graphs and analyzing them. Condition A would make the required graphs regular, which would allow us to rely upon known theorems of the zig-zag product (Theorem 2) to analyze them.

As an edge in $\mathcal{C}$ is naturally $\Psi(\tau, g)$, we describe it by a pair (center, type) as follows: We think of a function $E: G \times S(1) \rightarrow V(G_{\text{walk}})$

$$E(g, \tau) = \tau \cdot g$$

that given a (center, type) translate it into the corresponding edge. This function is similar to $\Psi$, where the arguments are in reverse order.

Definition 13 (Center & Type). A 2-edge $w$ is in center $c$ if

$$\exists \tau \in C(1) \text{ s.t. } E(c, \tau) = w$$

and $\tau$ would be called the type of the edge (that is a 2-edge in $S$).

Our 2-edge $\mathcal{E}$ from before would be described as of type $\{s_1^{-1}, s_2\}$ in center $s_1 g$ (equivalently, as type $\{s_2^{-1}, s_1\}$ in center $s_2 g$).

Each edge is contained in two centers due to the combination of the commutativity requirement (Condition B) and the symmetry requirement (Condition C). The fact that it is in exactly 2 centers is by Condition C.

Sharing centers would be central for the analysis. That would mean that the random walk mix between centers. Between centers the random walk is governed by a graph $G_{\text{dual}}$ that is to be introduced. And in the vicinity of each center the random walk is governed by the applied local structure that is the small complex $S$. We need Condition 2 for the complex to have expansion properties.

11 we included the important conditions - see definition 31 for a full version

12 Using this CTS notation assures additionally that it satisfies these conditions

13 Easier to think about this way, and it encompasses standard definition of replacement product.

14 It simplifies the analysis but seems non-essential. As more shared centers seems to produce just a better expansion.
4.1 Main Theorem

Before we present the main theorem, we will introduce several graphs. The first three are enough to present the theorem, while careful examination of the $G_{zig}$ is crucial for the proof.

1. $L = G_{walk}(S)$

2. $G_{dual}$ graph
   
   We define $G_{dual}$ as the graph on $G$ with edge set $\{\{g, \tau g\} | \tau \in S(1)\}$.
   
   This time we use another kind of product, in which $\{s_1, s_2\}g := s_1s_2g$
   
   This product is well defined because the two generators $s_1, s_2$ commute (Condition B).
   
   $G_{dual}$ is undirected, as $S$ is symmetric (Condition C). We also have an equivalent definition for $G_{dual}$.
   
   First, we look at the down-up random walk graph $G_{walk}$ (which is also part the Poincaré dual complex). That is a random walk on triangles (through 2-edges).

Definition 14. The vertex set of $G_{walk}$ is $S(2) \times G$

   Equivalently:
   
   $V(G_{walk}) = \{\Psi^{-1}(\sigma) | \sigma \in S(2)\}$

   (i.e. the vertex $\{a, b, c\}, g$ correspond to $\{ag, bg, cg\}$).

   An edge exists if the corresponding triangles intersect.

   We now ”forget” the involves triangle in $S$, by a projection of $E(G_{walk})$ into its second component ($G$). The result is called $G_{dual}$\footnote{The proof of this equivalence is left to the reader. It won’t be relied upon.}. Hence, $G_{dual}$ encompasses the edges in $G_{walk}$ that are between two different centers, ignoring their types.

3. $G_{rep} := G_{dual} \otimes L$ (detailed definition is \footnote{The proof of this equivalence is left to the reader. It won’t be relied upon.}).

   This is the replacement product of $G_{rep}$ and $G_{dual}$ \footnote{The proof of this equivalence is left to the reader. It won’t be relied upon.}. In general, a replacement product takes a graph $\tilde{G}$ which is $d$-regular, and a graph $\tilde{H}$ on $d$ edges, and returns a graph on vertex set $V(\tilde{G}) \times V(\tilde{H})$.

   It requires a certain order of the neighbors of $\tilde{g}$. Alternatively, a function $\phi_v : V(\tilde{H}) \to \tilde{G}$ that is bijective.

   \[
   E^{red} = \{(v, \tau) \sim (v, \tau') \text{ if } \tau \sim \tau' \text{ on } \tilde{H}\}
   \]

   \[
   E^{blue} = \{(v, \tau) \sim (u, \tau') \text{ if } u \sim v \text{ and } \phi_u(\tau) = u, \phi_u(\tau') = v\}
   \]

   The edge set of the replacement graph is $E^{red} \cup E^{blue}$.

   In our case, $\phi_\tau(g) = \tau g$.

4. $G_{zigzag} = G_{dual} \otimes \otimes L$ (definition \footnote{The proof of this equivalence is left to the reader. It won’t be relied upon.}).

   In general, the zig-zag product of two graphs $G \otimes H$ is built upon the mentioned replacement product (on the same vertices), by taking all the 3-paths of the form red-blue-red in the corresponding replacement product. This is a well-known and widely used construction aimed at providing expansion while reducing the degree of a large graph \footnote{The proof of this equivalence is left to the reader. It won’t be relied upon.}.

   In our case, $G_{zigzag}$ has the same vertices as $G_{rep}$, where its edges are the collection of the red-blue-red edges in $G_{rep}$. 

5. \( G_{\text{zig}} \) The graph defined by the subgraph of \( G_{\text{rep}} \) that is the collection of all the paths in the form blue-red and red, from any vertex. This is unique to our construction. We also define an operator \( T \) that is the adjacency matrix of this graph.

We can now present the main theorem of the CTS part:

**Theorem 3.** (CTS Theorem)

\[
\lambda(G_{\text{walk}}(C)) \leq \sqrt{\frac{1}{2} + \frac{1}{2} \lambda(G_{\text{dual}}(\mathcal{Z}_L))}
\]

where \( A\mathcal{Z}_L B \) is the zig-zag product between graphs \( A \) and \( B \).

The expansion properties of the zig-zag product are described by the following theorem by Reingold, Vadhan and Wigderson (originally [RVW02, Theorem 4.3]). The theorem reads:

**Theorem 2.** If \( G_1 \) is an \((N_1, D_1, \lambda_1)\)-graph and \( G_2 \) is a \((D_1, D_2, \lambda_2)\)-graph then \( G_1 \mathcal{Z}_L G_2 \) is a \((N_1 \cdot D_1, D_2, f(\lambda_1, \lambda_2))\)-graph, where \( f \) is a function that satisfies:

- \( f(a, b) \leq a + b \)
- \( f < 1 \) where \( a, b < 1 \)

We call it the "zig-zag" function. So, we have

**Corollary 3.** \( \lambda(G_{\text{walk}}) \leq \sqrt{\frac{1}{2} + \frac{1}{2} f(\alpha, \beta)} \) where \( \alpha = \lambda(G_{\text{dual}}) \), \( \beta = \lambda(G_{\text{walk}}) \) and \( f \) is the zig-zag function

Thus, the expansion properties of \( C \) are somehow the middle ground between the expansion properties of the involved complexes (\( S \) and \( G_{\text{dual}} \)), in a way similar to that of the Zig-zag product.

We know that some specific important examples are already a CTS, namely:

- The construction by Conlon [CTZ18].
- The construction by Chapman, Linal and Peled [CLP18].

Therefore, we can use this corollary as a "black box". We do so in part IV for these constructions.

### 4.2 The course of the proof

We want to relate the expansion properties of the different presented graphs to the expansion of the complex \( C \). We do it (non-formally here) in several steps. First, there are some observation we would like to make.

We can say the following about \( G_{\text{rep}} \):

- A blue edge connects every two vertices \( v, w \in G \times S(1) \) that are identifiable under \( E \) (\( E(v) = E(w) \)). By lemma 3 that means that \( v = g, \tau \) and \( w = \tau g, \tau^{-1} \) for a certain \( g \in G \), \( \tau \in S(1) \). (It is always true that \( E(\tau g, \tau^{-1}) = E(g, \tau) \))
- A red edge connects \( v = g, \tau \) and \( w = g, \tau' \), if \( E(v), E(w) \) are both in center \( g \), and are connected by an edge in \( G_{\text{walk}} \). That is equivalent to the condition that \( \tau \) is adjacent to \( \tau' \) in \( L \) (by lemma 3).

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16. Here it refers to a specific case. See section 21
17. More specifically, section 20 for Conlon’s and section 21 for the other one.
Therefore, when looking at edge \((v, w)\) of \(G_{\text{zig}}\). If it is a red-edge then \(E(v) \sim E(w)\), since they are both in the same center. If it is a blue-red-edge then \(E(v) \sim E(w)\) too, since the ends of the blue edge are identifiable. We got a graph homomorphism (lemma 7), but there is even a stronger relation.

From here, we will take several steps.

In step 1, we show that \(G_{\text{zig}}\) is a lift of \(G_{\text{walk}}\).

A lift of graph \(\tilde{G}\) is a graph on \(V(\tilde{G}) \times A\) for some set \(A\), such that the neighborhood of each vertex is kept (More details in [HLW06]). Each neighbor of \(v \in V(\tilde{G})\) of \(\tilde{G}\) (quotient graph) translates into a neighbor of \(v, a\) in the lift (the formal definition is 35).

The mapping between the neighborhoods is done by the function \(E\) (which is called a covering map).

**Lemma 8.** We denote \(\Gamma_{\tilde{G}(v)}\) for the set of neighbors of \(v\) in graph \(\tilde{G}\).

For every vertex \(v \in V(\tilde{G}_{\text{rep}})\), the mapping

\[
E : \Gamma_{G_{\text{zig}}}(v) \longrightarrow \Gamma_{G_{\text{walk}}}(E(v))
\]

is bijective

A main property of a lift is that its spectrum contains all the eigenvalues of the quotient graph:

**Corollary 2.** \(\lambda(G_{\text{walk}}) \leq \lambda(G_{\text{zig}})\)

In step 2, we want to relate \(\lambda(G_{\text{zig}})\) to \(\lambda(G_{\text{zigzag}})\). Because of the similarity between \(G_{\text{zig}}\) and \(G_{\text{zigzag}}\), we can say that two random steps in \(G_{\text{zig}}\) in probability \(1/2\) has the same effect on convergence as a single step in \(G_{\text{zigzag}}\). We conclude (lemma 9)

\[
\lambda(G_{\text{zig}}) \leq \sqrt{\frac{1}{2} + \frac{1}{2} \lambda(G_{\text{zigzag}})}
\]

Finally, we combine them all:

\[
\lambda(G_{\text{walk}}) \leq \lambda(G_{\text{zig}}) \leq \sqrt{\frac{1}{2} + \frac{1}{2} \lambda(G_{\text{dual} \otimes L})}
\]

5 HDZ Overview

So far we have discussed CTS. In CTS one gets an expanding complex \(\mathcal{C}\) given a complex \(\mathcal{S}\), only if \(\mathcal{S}\) satisfies demanding conditions, mentioned in definition 12. We would like to replicate the required properties in a more general settings, with minimal restrictions on the small complex.

We first assume a general complex \(\mathcal{A}\). We do require it to be connected, \(d\)-regular and \(\chi\)-strongly colorable (Strong coloring or rainbow coloring means that every triangle has vertices of 3 different colors). We choose \(\chi\) different groups \(G_i\). We will generate a complex \(\mathcal{C}\) called HDZ that is a special case of CTS. The vertex set of \(\mathcal{C}\) would be:

\[
\mathcal{G} = G_1 \times G_2 \times \ldots \times G_\chi
\]

and that would satisfy all the requirements for CTS.

To do so, we map the vertices of color \(i\) in \(\mathcal{A}\) bijectively to set of generators \(F_i \subset G_i\). We have an isomorphic complex \(\mathcal{S}\) over the vertex set \(\mathcal{F} := \bigcup_{i \in [\chi]} F_i\), generated by the mapping

\[
\phi : \mathcal{A}(0) \rightarrow \mathcal{F}(0)
\]

\[
\phi(V_i^c) = (F_c)_i.
\]
As there are only polychromatic edges in $A$, there are only edges of different generator sets in $S$. That is every edge in $S$ satisfies $E \in \{F_i, F_j\}$ for some $i \neq j$. The fact that it is built upon group tensor product, means that different 2-edges commute. Finally, we define $HDZ^-$, the weak version of $HDZ$ to be $C = Cs[S, G]$ with left group product in $G$ as the action. We denote it by $HDZ^-[A, C, G, F_1, \ldots F_\chi]$. For a formal definition see section 14.1.

We get that Condition D (the action is freelike action) and Condition B (the underlying generators of the edges commute) are satisfied automatically. All the other conditions except Condition C are satisfied too (claim 2). Condition C translates into a condition we call $Inv$ on $A$ (definition 39).

Suppose condition $Inv$ is satisfied for $A$. Now we get that $HDZ$ is a special case of CTS, and we can use the CTS Theorem (theorem 3) to get the expansion rate based on the expansion of $G_{dual}$ and $G_{walk}(A)$. But even if it is not satisfied, using the HPOWER mechanism (next section & section 16.1) solves this issue.

The $HDZ$ is inspired by the construction of [CLP18]. And it is in fact a generalization of a special case of it.

### 5.1 HPOWER and Property INV

(the full details are in section 16) For the theorem to be useful, we wish to have complexes that satisfies property $Inv$.

**Claim 1.** We can claim so immediately in several cases:

- Property $Inv$ is satisfied if $A$ is a commutative triplet structure.
- Property $Inv$ is satisfied if the 1-skeleton of $A$ isomorphic to some $Cay(G', F')$ where $G'$ is an abelian group and $F'$ is symmetric.
- Property $Inv$ is satisfied if the complex 1-skeleton is complex $\chi$-partite graph.

But in case the complex doesn’t satisfy it, we use an auxiliary construction called $HPOWER$. $HPOWER[A]$ is another complex that maintains the expansion properties of $A$ while ensuring that condition $Condition C$ is satisfied.

**Definition 15.** $HPOWER[A]$ is a complex defined by

$$E(HPOWER[A]) = \{0, 1\} \times E(A)$$

We analyze the expansion of $HPOWER[A]$ and we get that it maintains (lemma 13) the expansion of $A$ for any complex. The analysis is done by the trace method. So we define $HDZ^+$ to be the same as $HDZ^-$, where $A$ is replaced by $HPOWER[A]$ (formal definition in section 14.1).

### 5.2 Analyzing the expansion of $G_{dual}$

So far, we have assured that all the conditions for the CTS structure are satisfied, so we can express the expansion rate of $HDZ$ in terms of the expansion of $G_{dual}$ and $G_{walk}(A)$ by using theorem 3. We still don’t know the expansion of $G_{dual}$ in general. In case the complex contains every possible edge, that means for us that every polychromatic edge is in the graph, that would be easy to handle.

We calculate the eigenvalues explicit and get this theorem:

**Theorem 5.** Let $A$ be a $d$-regular complex that is a $d$-regular, strongly $\chi$-colorable (Strong coloring or rainbow coloring means that every triangle has vertices of 3 different colors), such that the 1-skeleton of $A$ is a complete $\chi$-partite graph.
Let $C : V(A) \to [\chi]$ be a coloring of $A$. We denote $V_c$ the vertices of color $c$.

Let $Cay(G_1, F_1) \ldots Cay(G_{\chi}, F_{\chi})$ be a collection of Cayley graphs such that $|F_c| = |V_c|$ for every $c \in [\chi]$ and such that

$$\max(\lambda(Cay(G_i, F_i))) \leq \nu$$

Then, exists a complex

$$C[A, C, G, F_1, \ldots F_{\chi}]$$

which is a $3A(2)$-regular $2d$-regular transitive with $G = \prod G_i$ as its vertex set. And

$$\lambda(G_{\text{walk}}(C)) \leq \sqrt{\frac{1}{2} + \frac{1}{2} f \left( \frac{N - (\chi - 1) + (\chi - 1)\nu}{N}, \lambda(G_{\text{walk}}(A)) \right)}$$

where $f$ is the "zig-zag function" and $N = (\frac{\chi}{2})$

This theorem is proved in section 15.

5.3 Expressing $\lambda(G_{dual})$

We would like to express the expansion of $G_{dual}$ by Cayley graph on each coordinate separately. However, it is not easy in a more generalize settings.

We call edges that that look like $\{F_a, F_b\}$, edges of template $ab$. We denote the set of edges of template $ab$ by $S_{ab}$.

We first want to get an expression for $\lambda(G_{dual})$ based on the graphs $M_{cd} := Cay(G_cG_d, S_{cd})$.

To do so, we use an old theorem [Bar79, Abstract] that proves that we can decompose the pairs into several unordered partitions of distinct pairs such that any pair appears in one partition exactly once, and the pairs in each partition are disjoint. This allows us to rethink about random walk in $G_{dual}$, in which we first choose a partition and then chose a pair in this partition. From this observation, we can calculate the spectral gap of $G_{dual}$ (lemma 14).

5.4 Lie groups

(details in section 18)

We want to reduce the condition on $M_{ij}$ to condition on the corresponding projection of the edges on $G_i$ and $G_j$. That is $Cay(G_i, P_k(S_{ij}))$ where $k \in \{i, j\}$.

$P_k$ is the natural projection ( $P_k : G_iG_j \to G_k$).

We know that this kind of projection maintains expansion properties for product of simple lie groups. [Bre+13, Proposition 8.4] states informally that, given $\{s_1, s_2, \ldots s_k\}$ that expand in $G_1$ and $\{t_1, t_2, \ldots t_k\}$ that expand in $G_2$, assuming they are both simple lie groups with no common factors, then $\{s_1t_1, \ldots, s_kt_k\}$ expands in $G_1G_2$.

So, if for every $ij$, every $k \in \{i, j\}$, every $Cay(G_i, P_k(S_{ij}))$ is a good expander, we are done. Generally, these are different subgraphs of $Cay(G_i, F_i)$. We can’t say much generally here. The question is: Is there a good chance that they will be good expanders?

[Bre+13, Theorem 1.2] says(informally) that choosing uniformly randomly 2 generators leads to expanding Cayley graph in probably that is overwhelming.

Now, we want to randomize $K$ generators uniformly independently, and we want to get that every pair of them expand. So, Lovász local lemma comes to our help. We are able to prove the following lemma:

---

18 Formal relation in proposition 8
19 $Cay(G_1, \{s_1, \ldots s_k\})$ is an expander
20 It is quoted in lemma 16
21 At least $1 - 2^{\Omega(n^c)}$ where $n$ is the number of vertices in the graph and constant $c$
Lemma 16. Suppose that $G$ is a finite simple group of Lie type.

Let $f_1, \ldots, f_K \in G$ where $f_i$ are chosen uniformly independently in random, and $K$ is bounded. Then from a certain $N$ in probability at least $1 - \frac{C}{G^e} \binom{K}{2}$, $\{f_i, f_j\}$ is $e$-expanding where $C, \delta, N, \epsilon > 0$ depend only on $K$ and $\text{rk}(G)$.

We conclude that randomization of elements in product of simple Lie groups could provide a random model for bounded degree complexes. We get our main theorem:

Theorem 7 (HDZ theorem). Let $A$ be a complex that is $d$-regular, $\chi$-strongly-colorable ($\chi$ is even), and the edges that involves vertices of colors $a, b$ incident to at least 2 distinct vertices of each color $23$. Suppose it also has a connected 1-skeleton.

Let $C : V(A) \to [\chi]$ be a coloring of $A$. $K_c$ is the number of vertices of color $c$.

Let $G = G_1 \times G_2 \times \ldots \times G_\chi$ where $G_i$ are product of at most $r$ finite simple (or quasisimple) groups of Lie type of rank at most $r$. Additionally, no simple factor of $G_i$ is isomorphic to a simple factor of $G_j$ for $i \neq j$.

Let $F_1, F_2, \ldots, F_\chi$ symmetric subsets of the corresponding groups of corresponding sizes $2K_1, \ldots, 2K_\chi$ chosen uniformly independently.

Then in probability at least $1 - O(|G|^{\delta})$ where $G_i$ is the smallest component of $G$, and from a certain $N \forall i |G_i| > N$, the complex $C = \text{HDZ}^+[A, C, G, F_1, \ldots, F_\chi]$ has the following properties:

- Its vertex set is $G$
- The degree of each vertex is $24A(2)$
- It is $4d$-regular.
- The link of each vertex is the same regular graph (up to isomorphism).
- It is $\epsilon'$ expanding.
- It is transitive.

where $\epsilon', \delta, N$ depend only on ranks of the groups and the choice of $A$.

The transitivity properties are followed from a standard argument in section 6. And similarly we get the links are isomorphic and regular in 10.1

6 Symmetric properties

Lemma 1. Suppose that $X = X_1 \times X_2 \times \ldots \times X_k$ and that $F_i \subset X_i$ where this action is transitive (i.e. left group product). Let $S$ a $(k-1)$-complex s.t.

$$S(k-1) \subset \bigcup_{m \in \text{k-tuples in } [\chi]} \{F_{m_1}, \ldots, F_{m_k}\}$$

Then, the complex $S^c[S, X]$ is transitive.

\[22\text{where } |G| \geq N\]
\[23\text{Formally: } |P^c(E^{a,b}_i(A))| \geq 2\text{ where}

- $c \in \{a, b\}$
- $E^{a,b}_i$ are the edges between colors $a$ and $b$ 
- $P^c$ is a projection into $V^i$
Proof Sketch. We take $A = \{a_1, a_2, \ldots, a_\chi\}, B = \{b_1, \ldots, b_\chi\}$ where $A, B \in E(C)$. We want to find $f$ automorphism of $C$ such that $f(A) = B$. We take $f$ s.t.

$$f_i \cdot a_i = b_i$$

It is an automorphism because of the way the Schreier complex is constructed $\sigma \cdot X = \sigma \cdot f(X)$ for $\sigma \in S$. $f_i : X \to X$ is onto and therefore bijective in every coordinate.

**Corollary 1.** The HDZ structure is transitive (immediate)

### 7 Organization of the paper

Part I provides an informal overview of the paper, with the key points. The reset of the sections should provide the full details of the involved definitions and theorems. This is true, except the Schreier complex defined in II.

In III, we provide preliminary knowledge and full details for the CTS construction (some of them are also needed for part III).

In IV, we provide the details for the HDZ construction.

In V, we provide some additional applications for known constructions.
Part II
Commutative Triplet Structure

8 Preliminaries

Here we list different definitions we need, and some notations.

These definitions are built upon in the current part, and the next one part [III]. We repeat here some definitions from the beginning to make the part self-contained. We do rely on the previously introduced Schreier graph.

8.1 General Graph

Definition 16. Regarding a $d$-regular expander graph:

- $G$ is said to be $\epsilon$-expander
- $\sigma(G)$ is the spectral gap of graph $G$
- $\lambda(G)$ is the normalized nontrivial eigen-value of $G$

if the following relation persist:

$$\sigma(G) = d(1 - \lambda(G)) = d\epsilon$$

8.2 General Complex

Definition 17 (Basic definitions). Let $\mathcal{C}$ be a complex.

- The set of its triangles will be denoted $\mathcal{C}(2)$.
- The 1-edges of $\mathcal{C}$ are $\mathcal{C}(1) := \{\{u, v\} | \{u, v, w\} \in \mathcal{C}(2) \text{ for some } w \in V(\mathcal{C})\}$
- The 1-skeleton of $\mathcal{C}$ is denoted $\mathcal{C}^1$. The vertex set of it is $V$, and the edge set is $\mathcal{C}(1)$.
- The link of a vertex $v$ is the graph $\mathcal{C}_v$ with the edge set

$$\{E \setminus v | E \in E(\mathcal{C}) \text{ s.t. } v \in E\}$$

- A complex $\mathcal{C}$ is called 1-connected if $\mathcal{C}^1$ is connected graph
- A complex is of dimension $d$ if the maximally size edge are of size $d + 1$

Remark. We will usually assume the complex is 2-dimensional unless otherwise specified

8.2.1 Regular Complexes

Definition 18 (Regular). A $d$-dimensional complex is $k$-(upper)regular if every $(d-1)$-edge is contained in exactly $k$ $d$-edges.

A 2-complex is $d$-regular if every edge is contained in exactly $k$ triangles

There are numerous examples of such complexes.

---

24 We will mostly deal with 2-dimensional ones so the definitions are not completely general.
25 A special case of the definition in the index.
• Some coest geometries \cite{KO17}
• The flag complexes $S(d, q)$ \cite[section 10.3]{LLR18}
• Conlon’s construction \cite{Con19}
• Any pseudo-manifold is an example in which $d = 2$.

In the case that the complex has top edges of size $k + 1$, and the $k$-edges are exactly $\binom{k}{V}$, this is a special case of a design with parameters $(|V(C)|, k, k + 1, d)$. Designs have been vastly studied, see \cite{Kee14}.

\section*{8.3 Random Walks}

\textbf{Definition 19 (Random walk).} on a 2-complex $C$ is defined to be a sequence of edges $E_0, E_1, \cdots \in C(1)$ such that:

1. $E_0$ is chosen in some initial probability distribution $p_0$ on $C(1)$
2. for every $i > 0$, $E_i$ is chosen uniformly from the neighbors of $E_{i-1}$. That is the set of $f \in C(1)$ s.t. $E_{i-1} \cup f$ is in $C(2)$

This is equivalent to a random walk on graph $G_{\text{walk}}$.

\textbf{Definition 20 (Random walk graph).} The random walk graph of a complex $C$, $G_{\text{walk}}(C)$ is defined by $V(G_{\text{walk}}) = C(1)$, where $E \sim E' \iff \exists \sigma \in C(2)$ s.t. $E, E' \in \sigma$

($a \sim b$ suggests that $a$ is adjacent to $b$ in the graph)

\section*{8.4 Cartesian Product}

\textbf{Definition 21 (Cartesian Product).} Given $G, H$ graphs $G \Box H$ is a Cartesian product of graphs where:

- $V(G \Box H) = V(G) \times V(H)$
- $(u, u') \sim (v, v')$ iff either:
  - $u = v$ and $u' \sim^H v'$
  - $u' = v'$ and $u \sim^G v$

Cartesian product $A \Box B$ maintains the lower spectral gap among the graphs:

\textbf{Lemma 2.} If $M, N$ are $d, d'$ regular graphs

$$\sigma(M \Box N) = \min(\sigma(M), \sigma(N))$$

where $\sigma$ is the spectral gap.

\textbf{Proof.}

$$G := M \Box N$$
$$D := \text{deg}(G) = d + d'$$

We have that

$$A_G = A_M \otimes I + I \otimes A_N$$

\footnote{Original definition by \cite{KM14}}

15
So, any eigenvector $v = x \otimes y$ where $x, y$ has eigenvalues $\lambda, \mu$ yields

$$A_Gv = (d\lambda + d'\mu)v$$

If we let $\lambda, \mu = \lambda(M), \lambda(N)$. We assume without loss of generalization

$$\min (\sigma(M), \sigma(N)) = \sigma(M)$$

Equivalently,

$$\max\{d\lambda, d'\mu\} = d\lambda$$

Then

$$D\lambda(G) = d\lambda + d' = d\lambda + D - d$$

$$\sigma(G) = D(1 - \lambda(G)) = d(1 - \lambda(M)) = \sigma(M)$$

8.4.1 Johnson Graph

**Definition 22** (Johnson graph). $J(S, n)$ is the Johnson graph

$$V(J) = \binom{S}{n}$$ and $v \sim v'$ if $|v \cap v'| = n - 1$

For example, in case $n = 2$

$$\{a, b\} \sim \{c, d\}$$ if both sets share one element

The Johnson graph is a well studied object, and appear naturally when we talk about random walk on complexes. It is known that $\lambda(J(S, 2)) = \frac{S - 4}{2(S - 2)}$

9 CTS definitions

9.1 Scherier Complex

**Definition 23** (Complex Action on a vertex set). Let $G$ a group that acts on a set $X$. Let $S$ be a complex s.t. $S \subset P(G)$. We wish to describe an action of an complex on a vertex set by defining the following product:

$$\Psi : S \times X \longrightarrow P(X)$$

$$\sigma \cdot x := \Psi(\sigma, x) = \{\sigma \cdot x \mid \sigma \in S\}$$

**Definition 24** (Schreier complex). Let $G$ a group that acts on a set $X$. Let $S$ be a complex s.t. $S \subset P(G)$. We define the Schreier complex of $S$ on $X$ by

$$Sc[S, X, G] := S \cdot X = \{\sigma \cdot x \mid x \in X, \sigma \in S\}$$

using the product $\Psi$.

**Remark.** Notice that is not a direct generalization of the Schreier graph, but inspired one. If the action is a left product in a group we will write $Sc[S, G]$. If the action is transitive, then the complex is transitive lemma 1.
9.2 CTS definitions

Given $S$ a complex, and $G$ a group, we will describe $\mathcal{C} = CTS[S, G]$.

**Definition 25 (Types).** $\mathcal{T} = S(1)$ is the set of types ($\tau = \{\tau_1, \tau_2\}$ is an element in $\mathcal{T}$).

We have the ordered version of it

$\mathcal{T}_o = \{\text{the 2-ordered-subsets of } S\}$

$(t = (t_1, t_2)$ is an element in $\mathcal{T}_o)$

We have a natural inverse in $\mathcal{T}_o$. That is

$$(a, b)^{-1} := (b^{-1}, a^{-1})$$

**Definition 26 (E function).** $E$ gets the edge by center and type

$E : G \times \mathcal{T} \to \mathcal{C}(1)$

$E(g, \tau) = \{\tau_1g, \tau_2g\}$

**Definition 27 (Center).** An edge $w$ is in center $c$ if

$\exists \tau \text{ s.t } E(c, \tau) = w$

**Definition 28 (Center function).** $c : E^1 \to P(G)$ is a function that returns the set of centers of an edge

**Definition 29 ($G_{\text{walk}}$).** $G_{\text{walk}}$ is defined to be the auxiliary graph of the random walk on edges $G_{\text{walk}}(\mathcal{C})$.

**Definition 30 (Graph $L$).** We define $L := G_{\text{walk}}(S)$

9.3 Formal definition

**Definition 31 (CTS - formal).** Given a group $G$, and a 2-complex $S$, We define

$\mathcal{C} = Sc[S, G]$

where the involved action is a left group product. With $\mathcal{T} = S(1)$, Provided that

0 $\{s, s^{-1}\} \notin S \forall s \in S(0)$

A $S$ is $d$-regular

B $\mathcal{T}$ is a collection of commutative generators

$\{a, b\} \in \mathcal{T} \implies ab = ba$

C $\mathcal{T}$ is symmetric

$\{a, b\} \in \mathcal{T} \iff \{a^{-1}, b^{-1}\} \in \mathcal{T}$

D The action by the complex $S$ resembles a free action. Namely, $\tau \cdot g = \tau' \cdot g'$ only in the trivial case: that is if $\tau' = \tau^{-1}$ and $g' = \tau g$. Equivalently: for $t \neq t' \in \mathcal{T}_o$, s.t.

$$t_1 t_2^{-1} = t'_1(t'_2)^{-1} \implies$$

$$t_2 = t_1^{-1} \text{ and } t'_1 = t_2^{-1}$$
E $S^1$ is connected

We say that the Schreier complex $C$ is also a commutative triplets structure, denoted by $\mathcal{C} = CTS[S, G]$

**Definition 32** ($G_{dual}$). We define $G_{dual}$ to be $Cay(G, T)$ (aka $Cay(G, S(1))$)

$$\{ (g, \tau \cdot g) \mid \tau \in T \}$$

**Remark.** Here we abuse notation:

For $\tau = \{a, b\}$, $\tau \cdot g$ corresponds to $abg = bag$

Notice that since $T$ is symmetric, this graph is well-defined and undirected.

Here we prove some properties of the structure.

### 10 Basic properties

We assume all along a commutative triplet structure $S$ that is $\tilde{d}$-regular (**Condition A**).

We look at $L$ (definition 30).

$L$ is $2\tilde{d}$ regular because there are $\tilde{d}$ triangles to choose from. Each introduces $2$ distinct edges to choose from.

As an illustrative example, in case of Conlon (section 20),

$$L = J(S, 2)$$

where $J$ is the Johnson graph (definition 22) Notice that $L$ is a subgraph of the $J(S(1), 2)$ graph, because two elements must have an intersection of size 1, in order to possibly be adjacent.

We can describe a random walk on the complex as a random walk on types (that is random walk on $L$), and a random walk on centers. We intend to construct a graph on the types and centers that would reflect this.

As in the case of the Conlon’s construction, the commutativity requirement translates into an edge being in two centers. And being in two centers leads to expansion properties of the complex as the random walk progresses.

**Lemma 3.** For $C$ triplet structure if $E(g, \tau) = E(g', \tau')$ where $g \neq g'$ then

$$g' = \tau g$$

$$\tau' = \tau^{-1}$$

(by abusing notation)

**Remark.** We implicitly say that $\tau$ is a symmetric edge.

That is for $\tau = \{t_1, t_2\}$

$$t_1t_2 = t_1t_2$$

and so

$$\tau' = \{t_2^{-1}, t_1^{-1}\}$$

and

$$g' = t_2t_1g = t_1t_2g$$

---

27 Using this CTS function assures that it satisfies all the conditions

28 Because if $\tau \cup c$ and $\tau \cup c'$ both contain $\{a, b\}$, then $\tau = \{a, b\}$.

29 This intuition was largely inherited from Conlon’s talk at a conference by the IIAS (Israel institute for advanced studies) in April 2018. He just didn’t go as far.
Proof. Let \( t, t' \) be the corresponding ordered types.

That without loss of generalization corresponds to \( t_1g = t'_1g' \) and \( t_2g = t'_2g' \). Of course, if \( t = t' \) we get a contradiction.

Then

\[
t_1t_2^{-1} = (t'_1)(t'_2)^{-1}
\]

and by Condition \( D \)

\[
t'_2 = (t_1)^{-1}

t'_1 = (t_2)^{-1}
\]

we get that \( t_2t_1 = t_1t_2 \). By

\[
g' = (t'_2)^{-1}t_2g
\]

we get that \( g' = \tau g \) and \( \tau' = \tau^{-1} \).

\[\square\]

Lemma 4. Every edge is in exactly two centers (see definition \( \exists \))

Proof. First \( |c(\mathcal{E})| \geq 1 \) by the definition of \( \mathcal{C} \).

But for \( t = (a, b) \) by Condition \( D \) we can see that

\[
E(g, t) = E(tg, t^{-1})
\]

Suppose \( |c(\mathcal{E})| > 2 \),

\[
E(g, \tau) = E(g', \tau')
\]

then, \( \tau' = \tau^{-1} \). If exists another \( g'', \tau' \) s.t. \( E(g'', \tau') = E(g, \tau) \) then by lemma \( 3 \)

\[
g'' = \tau g = g'
\]

\[
\tau'' = \tau^{-1} = \tau'
\]

\[\square\]

Lemma 5. In \( G_{walk} \) on \( CTS[\mathcal{S}, G] \) where it is \( \bar{d} \)-regular

1. \( \mathcal{E} \sim \mathcal{E}' \) iff exists \( c \) s.t. \( \mathcal{E} = E(c, \tau) \mathcal{E}' = E(c, \tau') \) and \( \tau \sim^L \tau' \)

2. The walk is \( 4\bar{d} \) regular

Proof.

1. Let’s check when \( \mathcal{E} \) is contained in a certain triangle \( \sigma \).

The triangle \( \sigma \) is \( \tilde{s} \cdot g \) where \( \tilde{s} \in \mathcal{S} \) and \( g \) is distinct for any \( \sigma \). So, any 2-subset of it is guaranteed to be of the form \( \{s_o g, s_g g\} \). That means that having a common center is essential for being adjacent in \( G_{walk} \).

If \( \mathcal{E} \sim \mathcal{E}' \), we must have that \( c = c(\mathcal{E}) \cap c(\mathcal{E}') \) exists, so we define \( \mathcal{E}' = E_o(c, t') \) and \( \mathcal{E} = E_o(c, t) \).

There can’t be 2 such centers in the intersection. Suppose they both belong to centers \( c, c' \), then \( c' = tc \) from lemma \( 3 \) and similarly \( c' = t'c \).

We define \( \tau = \{t_1, t_2\} \) and similarly for \( \tau' \).

\[
\mathcal{E} = \{\tau_1 c, \tau_2 c\} \mathcal{E}' = \{\tau'_1 c, \tau'_2 c\} \sigma = \{sc, s'c, s''c\}
\]
We can see that the condition \( \{ \tau_1, \tau_2 \}, \{ \tau'_1, \tau'_2 \} \subset \{ s, s', s'' \} \) is equivalent to the condition \( \mathcal{E}, \mathcal{E}' \subset \sigma \).

2. For every center \( c \) of \( \mathcal{E} \), we have \( \mathcal{E} = E(c, \tau) \) for a certain \( \tau \). And for every \( \tau' \) s.t. \( \tau' \sim L \tau \), \( \tau' \) induces a distinct edge \( \mathcal{E}' = E(c, \tau') \). \( L \) is 2\( \tilde{d} \)-regular. And we have \( d(\mathcal{E}) = |c(\mathcal{E})|2\tilde{d} \), where \( |c(\mathcal{E})| = 2 \) by lemma \( \text{V} \).

Now we assume \( \mathcal{E} = E(c, \tau) \), and we want to prove that we got a distinct set of edges. We haven’t counted the same edge twice, since that would mean that there exists an edge \( E' = E(c_1, \tau') \) which has two different centers \( c_1, c_2 \) that are also centers of \( \mathcal{E} \). We can assume that \( c_1 = c \), and \( c_2 = \tau'c \). The other center of \( \mathcal{E} \) is \( \tau'c \) and is equal to \( c_2 \). Since the generators commutate, \( \tau = \tau' \).

\[ \square \]

10.1 **Link of a vertex**

(Non-compulsory addition) Our analysis isn’t based upon the link, in contrary to traditional analysis methods. Instead, we will derive the expansion properties from the random walk on the small complex.

We will look at the link of an element in \( \mathcal{C} \). We will call this graph \( \mathcal{C}_g \) or just \( G_{\text{link}}(\mathcal{C}) \) because they are all isomorphic.

**Lemma 6.** The link of vertex \( g \) isomorphic to graph \( G_{\text{link}} \)

\[
V(G_{\text{link}}) = \{ xy^{-1} \mid \{ x, y \} \in T \}
\]

\[
ac^{-1} \sim bc^{-1} \text{ for every triangle } \{ a, b, c \} \text{ in } \mathcal{S}
\]

*Proof.* Let \( h = c^{-1}g \), for every choice of \( c \in V(\mathcal{S}) \) and \( h \in G \).

The triangles in center \( h \) are described by \( \{ ah, bh, ch \} \) for every \( \sigma = \{ a, b, c \} \in \mathcal{S} \). Equivalently:

\[
\{ ac^{-1}g, bc^{-1}g, g \mid g \in G \}
\]

that would be translated to the edge set of \( G_{\text{link}} \):

\[
\{ ac^{-1}, bc^{-1} \mid \text{for every triangle } \{ a, b, c \} \text{ in } \mathcal{S} \}
\]

\( ac^{-1} = bc^{-1} \) only in the trivial case (Condition \( \text{D} \)).

\[ \square \]

*Remark.* Now, the link of a single element \( c \) is \( \mathcal{S}_c \) and it looks like:

\[
\{ \{ a, b \} \mid \{ a, b, c \} \in \mathcal{S}(2) \}
\]

And is isomorphic to

\[
\{ \{ ac^{-1}, bc^{-1} \} \mid \{ a, b, c \} \in \mathcal{S}(2) \}
\]

So, the link of a single vertex seems like a union of all the links of the vertices in \( \mathcal{S} \). It is highly depending upon the structure of \( \mathcal{S} \).

\[ \text{30} \]The other case is easy to see

\[ \text{31} \]Each edge has at least one common center with \( \mathcal{E} \)
11 Replacement graph properties

**Definition 33.** Given a commutative triplet structure \( C = CT_S[G,S] \),

We define \( G_{rep} \) that stands for a replacement product graph \( G_{rep} := G_{dual}(C)L \)

And more specifically, in \( G_{rep} \), the vertex set is

\[ V_{rep} = G \times T \]

we define for \( v \in G \),

\[ \phi_v : T \rightarrow G \]

\[ \phi_v(\tau) = \tau \cdot v = \tau_2 \tau_1 v \]

We have

\[ v = abu \iff (u, \{a, b\}) \sim (v, \{a^{-1}, b^{-1}\}) \]

and

\[ E^{red} = \{(v, \tau) \sim (v, \tau') \text{ if } \tau \sim \tau' \text{ on } L\} \]

\[ E^{blue} = \{(v, \tau) \sim (u, \tau') \text{ if } u \sim v \text{ and } \phi_v(\tau) = u, \phi_u(\tau') = v\} \]

\[ E(G_{rep}) = E^{blue} \cup E^{red} \]

\( E^{blue} \) will be defined by the adjacency matrix \( P_{blue} \) (or simply \( P_B \))

\( E^{red} \) will be defined by the adjacency matrix \( P_{red} \) (or \( P_R \))

\( P_{R|g} \) signifies restriction of \( P_R \) to elements \( g, \tau \). This would be of course exactly an instance of the graph \( L \). So, \( P_R \) is \( \tilde{d} \)-regular.

**Definition 34 (Zig-Zag graph ).** We define the zig-zag graph over \( V(G_{rep}) \) by the operator \( P_R P_B P_R \) (see [HLW06]).

12 Operator \( T \)

We define \( T \) (will act over \( V(G_{rep}) \)) by

\[ T = \frac{1}{2} P_R + \frac{1}{2} P_R P_B \]

(the matrices are normalized)

Thus, \( T \) induces a subgraph \( G_{zig} \) that is a subgraph of \( G_{rep} \).

We define an inverse function to \( E : G \times T \rightarrow E^1 \) that is

\[ \gamma : E^1 \rightarrow P(G \times T) \]

Notice that \( \gamma \) is the labeling function that gives the vertices in \( G_{walk} \) the corresponding name in \( G_{rep} \).

The random walk described by \( T \) over \( G_{rep} \) and the random walk over \( G_{walk} \) are very similar, as demonstrated in the following section:

---

32 The following are standard definitions. Some of them were taken from this excellent lecture about zig-zag product [Dik].

33 Notice that we need that the generators will commute here. The action is defined as in subsection 31.

34 That is \( \{g, \tau \mid \tau \in T\} \)
12.1 Relation to $G_{\text{walk}}$

The definition of lift is the following ([HLW06, Definition 6.1]):

**Definition 35 (Lift).** Let $G$ and $H$ be two graphs. We say that a function $f : V(H) \rightarrow V(G)$ is a covering map if for every $v \in V(C)$, $f$ maps the neighbor set $\Gamma_H(v)$ of $v$ one-to-one and onto $\Gamma_G(f(v))$. If there exists a covering function from $C$ to $G$, we say that $H$ is a lift of $G$ or that $G$ is a quotient of $C$.

We are about to prove that $G_{\text{zig}}$ is a lift of $G_{\text{walk}}$. To do so, we show that there is a graph homomorphism between the graphs, and then we show that the neighborhood are transfered bijectively.

**Definition 36 ($G_{\text{zig}}$).** The graph $G_{\text{zig}}$ is defined as the induced graph of $T$ on the vertices of $G_{\text{rep}}$. This is equivalent to the definition presented

**Lemma 7.** There is graph homomorphism between $G_{\text{zig}}$ and $G_{\text{walk}}$. That is the function

$E : V(G_{\text{zig}}) \rightarrow V(G_{\text{walk}})$

(defined earlier) such that

$E(g, \tau) \sim_{G_{\text{walk}}} E(g', \tau')$

This mapping is 2-1.

**Proof.** We denote $e_v$ for vertex $v$ in $G_{\text{rep}}$.

Either

$e_{g', \tau'} \frac{1}{2} Pr e_{g, \tau} = \frac{1}{4d}$

or

$e_{g', \tau'} \frac{1}{2} PrPb e_{g, \tau} = \frac{1}{4d}$

In the first case, we have that $g = g'$ and $\tau \sim L \tau'$. So we are done by lemma 5 for $c = g$.

In the second case, we have that $\tau g = g'$ and $\tau^{-1} \sim L \tau'$. So, by the same lemma, $E(g, \tau) \sim E(\tau g, \tau^{-1})$. But as mentioned, $E(\tau g, \tau^{-1}) = E(g, \tau)$. 

**Lemma 8.** We denote $\Gamma_{\hat{G}(v)}$ for the set of neighbors of $v$ in graph $\hat{G}$.

For every vertex $v \in V(G_{\text{rep}})$, the mapping

$E : \Gamma_{G_{\text{zig}}}(v) \rightarrow \Gamma_{G_{\text{walk}}}(E(v))$

is bijective

**Proof.** The mapping is well-defined because it is a graph homomorphism. It is enough to prove that the mapping is onto(because the sets are equal in size, every vertex in both graphs has 4$d$ neighbors). Suppose $E \sim E(v)$ . By lemma 5 we can assume that $E(v) = E(g, \tau_1)$ and $E = E(g, \tau_2)$, where $\tau_1 \sim L \tau_2$. Therefore, $v$ is either $g, \tau_1$ or $\tau_1 g, \tau_1^{-1}$. In any case, it is easy to see that $g, \tau_2 \in \Gamma_{G_{\text{rep}}}(v)$

The last lemma assured that the $G_{\text{zig}}$ is a lift of the graph $G_{\text{walk}}$ as defined by 35.

It is well known that a lift has all the eigenvalues of the quotient(for example, see [BL06]). We get as a conclusion that $T$ has all the eigenvalues of $G_{\text{walk}}$. So, we can deduce:

**Corollary 2.** $\lambda(G_{\text{walk}}) \leq \lambda(G_{\text{zig}})$

---

35 There is a directed edge from $g, \tau$ to $g', \tau'$ or the other way around

36 Because $Pb$ impose a condition on the center
13 Global Properties of CS

13.1 Bounding the convergence rate of CTS

Now we want to get a bound on the convergence rate of the walk on $G_{walk}$ in terms of the walk on $T$. And then to get a bound on the random walk on $T$.

We define $\pi$ to be the uniform distribution on $V(G_{walk})$.
We define $\pi'$ to be the uniform distribution on all vertices of $V(G_{rep})$.

Notice that $\pi'$ on any vertex is half the value of $\pi$. Let $V_+$ be the space $x \perp \pi'$ where $x \in \mathbb{R}^{V(G_{rep})}$.
We prove here that $T_+$ is well-defined.$^37$

**Fact 2.** The operator $T$ satisfies the following:

1. $T \pi' = \pi'$
2. $V_+ (2) \subset V_+$

**Proof.**

1. We have

$$T \pi' = \frac{1}{2} P_R \pi' + \frac{1}{2} P_R P_B \pi' = P_R \pi' = \pi'$$

Since $P_B \pi' = \pi'$. That is because $\forall x, \tau$

$$(P_B \pi')(e_{x,\tau}) = \pi'(e_{x,\tau}^{-1}) = \pi'(e_{\tau, x})$$

2. Suppose $y \perp \pi'$. Then

$$\langle \pi', Ty \rangle = \langle \pi', P_R y + P_R P_B y \rangle = \langle \pi', P_R y \rangle + \langle \pi', P_R P_B y \rangle = (P_R \pi', y) + \langle P_R \pi', P_B y \rangle = (P_B \pi', y) = 0$$

**Lemma 9.** $\lambda(G_{zig}) \leq \sqrt{\frac{1}{2} + \frac{1}{2} \lambda(G_{dual}(\mathbb{Z}) L)}$

**Proof.**

$$T = \frac{1}{2} P_R + \frac{1}{2} P_R P_B$$

So, we have:

$$T^2 = \frac{1}{4} [P_R^2 + P_R^2 P_B + P_R P_B P_R + P_R P_B P_R P_B]$$

We will bound $\|T^2\|_+$

$$\|P_R^2 + P_R^2 P_B\|_+ \leq 2$$

$^37$ $T$ restricted to $V_+$
\[ \|P_R P_B P_R + P_R P_B P_R P_B \|_+ \leq \|P_R P_B P_R \|_+ + \|P_B P_R P_B \|_+ \leq 2 \|P_R P_B \|_+ \]

(Notice that \(P_B x \perp \pi'\) if \(x \perp \pi\))

So we have that

\[ \|T^2\|_+ \leq \frac{1}{2} + \frac{1}{2} \lambda(G_{\text{dual}} \otimes L) \]

That assures that we have a rapid convergence as the eigenvalues of \(T^2\) are bounded away from 1.

We have our main theorem:

**Theorem 3.** (CTS Theorem)

\[ \lambda(G_{\text{walk}}(\mathcal{C})) \leq \sqrt{\frac{1}{2} + \frac{1}{2} \lambda(G_{\text{dual}} \otimes L)} \]

where \(A \otimes B\) is the zig-zag product between graphs \(A\) and \(B\).

**Proof.** Immediate from corollary 2 and lemma 9.

**Corollary 3.** \(\lambda(G_{\text{walk}}) \leq \sqrt{\frac{1}{2} + \frac{1}{2} f(\alpha, \beta)}\) where \(\alpha = \lambda(G_{\text{dual}})\), \(\beta = \lambda(G_{\text{walk}})\) and \(f\) is the zig-zag function

**Proof.** We now wish to bound \(\lambda(G_{\text{dual}} \otimes L)\) to get a bound on \(G_{\text{walk}}\). We can rely on known theorems about the zig-zag product. We use here the following theorem by Reingold, Vadhan and Wigderson (originally [RVW02, Theorem 4.3]).

**Theorem 4.** If \(G_1\) is an \((N_1, D_1, \lambda_1)\) -graph and \(G_2\) is a \((D_1, D_2, \lambda_2)\) -graph then

\(G_1 \otimes G_2\) is a \((N_1 \cdot D_1, D_2^2, f(\lambda_1, \lambda_2))\) -graph, where \(f(\lambda_1, \lambda_2) \leq \lambda_1 + \lambda_2\) and \(f(\lambda_1, \lambda_2) < 1\) when \(\lambda_1, \lambda_2 < 1\).

\(f\) is the function:

\[ f(\lambda_1, \lambda_2) = \frac{1}{2} (1 - \lambda_2^2) \lambda_1 + \frac{1}{2} \sqrt{(1 - \lambda_2^2)^2 \lambda_1^2 + 4 \lambda_2^2} \quad (\star) \]

We call it the "zig-zag function".

Some of its properties are studied there. It is better (lower) when \(\lambda_1\) and \(\lambda_2\) are worse. And less than 1 if \(\lambda_1\) and \(\lambda_2\) are less than 1. This assures the resulted graph is an expander when the original graphs are.

**Corollary 4.** Let \(G\) be a group. Let \(S \subset \binom{G}{3}\) be a set of triangles.

Suppose \(\mathcal{C} = CTS[G, S]\) is a \(k\)-edge regular (where \(k\) is bounded by \(D\)) commutative triplet structure (satisfies conditions \(0 \cdot E\)). Suppose further that \(\text{Cay}(G, (S/2))\) is \(\epsilon\)-expander. Then, \(2D\)-random walk on \(\mathcal{C}\) converges rapidly with some rate \(\alpha(D, \epsilon) < 1\).
Proof. Since $C = CTS[G, S]$ is of bounded-degree, the number of vertices of $L$ is bounded. By Condition $\mathcal{D}$, $L$ is connected. So, each graph $L$ has convergence rate less than 1. There are only finitely many possibilities, so there is a number $\beta'(D) < 1$ that is the maximal convergence rate for all the graphs $L$.

We can use corollary 3 and get:

$$\lambda(G_{walk})^2 \leq \frac{1}{2} + \frac{1}{2}f(1 - \epsilon, \beta') < 1$$
Part III
HDZ

14 Main Part

14.1 HDZ definition

We describe here the procedure of making the HDZ complex, formally defining the complex $H$

$$H = HDZ[A, C, F_1, F_2, \ldots, F_\chi]$$

given initial complex $A$ with coloring $C$ and group $G$ as described above.

We have two variants of it $HDZ^-$, $HDZ^+$ which consists of two, or respectively 3 steps.

1. If the original complex doesn’t satisfy property Inv (definition 39), we need to convert it to a complex that does

$$A' = HPOWER[A]$$

while preserving the expansion. See section 16. ( otherwise $A' = A$)

2. We convert $A'$ to complex $S$ over the group $G$ that is isomorphic to $A$. We call it

$$S = CONV[A', C, F_1, F_2, \ldots, F_\chi]$$

We do so in section 14.2

3. We simply plug it into the mechanism of commutative triplet structure.

$$H = Cts[G, S]$$

We prove it satisfies the required conditions in section 14.3.

$HDZ^+$ is with the additional step 1.
$HDZ^-$ is without it.

To conclude\(^\text{38}\):

$$HDZ^-[A, C, G, F_1, F_2, \ldots, F_\chi] := Cts[G, CONV[A, C, F_1, F_2, \ldots, F_\chi]]$$

$$HDZ^+[A, arguments] := HDZ^-[HPOWER[A], arguments]$$

We take $HDZ$ to be $HDZ^-$ if $A$ satisfies property Inv and $HDZ^+$ otherwise. We will specify the needed variant in each case we handle. It has implications on the size and degree of the complex and how similar it is to $A$.

The $HDZ^+$ variant always works and yields the same convergence rate.

14.2 The CONV mechanism

Let $A$ be a complex that is a regular and strongly $\chi$-colorable complex. (Strong coloring or rainbow coloring means that every triangle has vertices of 3 different colors)

Let $C : V(A) \to [\chi]$ be a coloring of $A$. We denote $V^c$ the vertices of color $c$ where $V^c_i$ is $i$-th element of $V^c$. $K_c$ is the number of vertices of color $c$, and we assume it is even.

Let $G = G_1 \times G_2 \ldots \times G_\chi$ Suppose we have $F_1, F_2, \ldots, F_\chi$ symmetric subsets of the corre-
sponding groups, where $|F_c| = |V_c|$ for every color $c \in [\chi]$.

We use a useful notation here:

$(c, n)$ is the element of $G$ that is $F_c^c$. Given $g = (g_1, g_2, \ldots, g_\chi)$,

$$(c, n) \cdot (g_1, g_2, \ldots, g_\chi) = (g_1, g_2, \ldots, F_c^c \cdot g_c, \ldots, g_\chi)$$

We use here the notion introduced earlier. We define:

$$F = \bigcup_{i \in I} F_i$$

**Definition 37** (CONV mechanism).

$$S = \text{CONV}[A, C, F_1, F_2, \ldots, F_\chi]$$

is a complex on the vertex set $F$ generated by the mapping

$$\phi : A(0) \to F$$

$$\phi(V_c^c) = (F_c)_i.$$  

where we assume that $F_c$ is ordered such that it satisfies property $\tilde{\text{Inv}}$:

**Definition 38.** Property $\tilde{\text{Inv}}$ is satisfied if for every $i \in [K_c]$:

$$(F_c^c)^{-1} = \left(F_c^{i+K_c/2}\right)$$

And there is no element in $F_c$ of order 2. (The indices are taken as modulo $K_c$)

We use this map to generate complex $S$ on vertices $F$, where $\{a, b, c\} \rightarrow \{\phi(a), \phi(b), \phi(c)\}$. That is clearly isomorphic to $C$.

We have this definition:

**Definition 39.** We say that $A$ satisfies property $\text{Inv}$ if $K_c$ is symmetric for every $c \in [\chi]$, and for every $c, d \in [\chi]$ and $i, j \in [K_c]$

$$\{V_i^c, V_j^d\} \in A(1) \iff \{V_{i+K_c/2}^c, V_{j+K_c/2}^d\} \in A(1)$$

where the indices are taken modulo $K_c$.

**Definition 40.** An complex $S$ is edge symmetric if $S \subset P(G)$ for some group $G$ and

$$\{a, b\} \in S(1) \iff \{a^{-1}, b^{-1}\} \in S(1)$$

for the inverse in the group.

It is easy to see that if complex $A$ satisfies property $\text{Inv}$, then $S$ has symmetric edge set.

### 14.3 CTS Requirements

**Claim 2.** Let $S$ a 2-complex s.t.

$$S(2) \subset \bigcup_{m \in \{3\text{-tuples in} [\chi]\}} \{F_{m_1}, \ldots, F_{m_3}\}$$

that is edge-symmetric (definition 40) and 1-connected (definition 17).

We let

$$C = \text{St}(G, S)$$

Then, $C$ is a standard commutative triplet structure (CTS for short).
Proof. We verify that the required conditions are satisfied. We can see that any $\tau \in T$ is given by $\{(n,i),(m,j)\}$ for a certain $i \neq j$ where $1 \leq i,j \leq K$ and $n,m$ correspond to $F_m,F_n$. We call $mn$ the template of the edge.

Condition $A$ is satisfied by the fact that $S$ is regular. Condition $B$ is satisfied because $S$ is commutative. Condition $C$ is satisfied because it is symmetric. There is no element $\{f,f^{-1}\}$ because $i,j$ are different (Condition $D$). Condition $E$ is satisfied because it is 1-connected.

14.4 Basic properties

Lemma 10. Assuming $A$ is $d$-regular, and

$$C = HDZ^{-1}[A,C,G,F_1,F_2,\ldots,F_\chi]$$

is a valid CTS. Then $C$ has the following properties:

1. $|G|$ vertices
2. The degree of each vertex is $3|A(2)|$
3. $2d$-regular

Proof. $S$ is isomorphic to $A$.

$V(C) = G$ by definition.

Let’s look at vertex $x \in G$. Every selection of $\bar{s} \in S(2)$ yields 3 different options for a center of edge. Namely,

$$c = \bar{s}_i^{-1}x \text{ for } i \in \{1,2,3\}$$

This induces $\bar{s}c$ as the triangle that contains it (by abuse of notion).

These are all the options. So, there are $3|A(2)|$ options in total.

The regularity of the random walk on $C$ is 4 times the regularity of $S$. As the random walk includes two new edges for every triangle, the regularity of $C$ is $2d$. This is also clear, since an edge is in two centers, and each induces $d$ distinct triplets that contains the edge.

14.5 The convergence rate

Now, $G_{\text{dual}}$ is defined to be the graph (definition $32$)

$$G_{\text{dual}} = Cay(G,T)$$

Where

$$T = S(1)$$

This is an abuse of notation.

Another way to look at it, the mapping $\phi$ induces a function $\mu$ that relate the vertices of $A(1)$ to $F^2$ defined by
\[ \mu({a, b}) = \phi_c(a)\phi_c(b) \]

And we have
\[ G_{\text{dual}} = \text{Cay}(G, \mu(A(1))) \]

We have proved that this is a commutative triplet structure. So we can finally use the main theorem, and get:
\[ \lambda(G_{\text{walk}}(C)) \leq \sqrt{\frac{1}{2} + \frac{1}{2} \lambda(G_{\text{dual}} \circ G_{\text{walk}}(A))} \]

We will combine all that we have concluded so far:

**Proposition 1.** Let \( A \) be a complex that is a regular and strongly \( \chi \)-colorable complex. (Strong coloring or rainbow coloring means that every triangle has vertices of 3 different colors)

Let \( C : V(A) \rightarrow [\chi] \) be a coloring of \( A \). We denote \( V^c \) the vertices of color \( c \) where \( V^c_i \) is \( i \)-th element of \( V^c \). \( K_c \) is the number of vertices of color \( c \) (and it is even).

Let \( G = G_1 \times G_2 \cdots \times G_\chi \) where \( G_i \) are groups. Suppose we have \( F_1, F_2, \ldots, F_\chi \) subsets of the corresponding groups.

We require that:

- \( |F_c| = K_c \)
- \( A \) satisfies property Inv (definition 39).

The complex that is defined by
\[ C = HDZ^{-\chi}[A, C, G, F_1, F_2, \ldots, F_\chi] \]

satisfies:
\[ \lambda(G_{\text{walk}}(C)) \leq \sqrt{\frac{1}{2} + \frac{1}{2} f(\lambda(G_{\text{dual}}), \lambda(G_{\text{walk}}(A)))} \]

where (for \( \mu \) defined earlier)
\[ G_{\text{dual}} = \text{Cay}(G, \mu(A(1))) \]

and \( f \) is the "zig-zag function".

### 14.5.1 Two problems

To successfully use what we have so far, we need two conditions. The first is that Inv property should be satisfied. We will handle this in the section 16.

The second hurdle is that we can’t always assure that \( G_{\text{dual}} \) is a good enough expander. However, in certain cases, we can.

We reduce the expression for the expansion of \( G_{\text{dual}} \) to something that is occasionally more manageable in section 17. And we get a concrete result in section 18.

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39 At least in one case
15 Full 1-skeleton case

We calculate $\lambda(G_{\text{dual}})$ in the case the complex has full 1-skeleton. Since the complex is strongly $\chi$-colorable, full means that all the possible 2-edges are present, namely, that the 1-skeleton of it is a $\chi$-multipartite graph.

In this case, we have no problem with $\text{Inv}$ or with calculating $G_{\text{dual}}$. We know property $\text{Inv}$ is satisfied as it contains the required edges.

Lemma 11. Let $\mathcal{A}$ be a $d$-regular, strongly $\chi$-colorable 2-complex. (Strong coloring or rainbow coloring means that every triangle has vertices of 3 different colors)

Let $C : V(\mathcal{A}) \rightarrow [\chi]$ be a coloring of $\mathcal{A}$. We denote $V^c$ the vertices of color $c$.

Let $\text{Cay}(G_1, F_1) \ldots \text{Cay}(G_\chi, F_\chi)$ be a collection of cayley graphs such that $|F_c| = |V^c|$ for every $c \in [\chi]$. With $N = \binom{\chi}{2}$, we have

$$\lambda(G_{\text{dual}}) = N - (\chi - 1) + (\chi - 1)\nu$$

Proof. We know that

$$\mu(A(1)) = F^2$$

$$F^2 := \bigcup_{i \neq j \in [\chi]} F_i F_j$$

So

$$G_{\text{dual}} = \text{Cay}(\mathcal{G}, F^2)$$

Therefore if we set

$$A_i = \text{Cay}(G_i, S_i)$$

Then we can define:

$$A_{ij} := I \times \ldots \times A_i \times I \times \ldots \times A_j \times \ldots \times I$$

that is the adjacency matrix of $\text{Cay}(\mathcal{G}, S_{ij})$ in our case, where

$$S_{ij} = S_i \times S_j$$

And $G_{\text{dual}}$ is just a union of all possible $\text{Cay}(\mathcal{G}, S_{ij})$.

We let $N = \binom{\chi}{2}$

$$G_{\text{dual}} = \frac{1}{N} \sum_{(i,j) \in \binom{\chi}{2}} A_{ij}$$

We let $y = x_1 \otimes x_2 \ldots \otimes x_\chi$ be a collection of eigenvectors, where $x_i$ is an eigenvector of $A_i$ with $\mu_i$ as its eigenvalue.

$$G_{\text{dual}} y = \frac{1}{N} \sum_{(i,j) \in \binom{\chi}{2}} A_{ij} y = \frac{1}{N} \sum_{(i,j) \in \binom{\chi}{2}} \mu_i \mu_j y$$

That is why the eigenvalue is

$$\frac{1}{N} \left( \sum_{(i,j) \in \binom{\chi}{2}} \mu_i \mu_j \right)$$

Now, to maximize it we choose all the eigenvalue to be 1, except $\mu_1$ which would be max$_i (\lambda_2(A_i))$. The number of times $\mu_1$ appears in the sum is $\chi - 1$.

So, we have

$$\frac{N - (\chi - 1) + (\chi - 1)\nu}{N}$$

as the maximal eigenvalue.
We now have

**Theorem 5.** Let $\mathcal{A}$ be a $d$-regular complex that is a $d$-regular, strongly $\chi$-colorable (Strong coloring or rainbow coloring means that every triangle has vertices of 3 different colors), such that the 1-skeleton of $\mathcal{A}$ is a complete $\chi$-partite graph.

Let $C : \text{V}(\mathcal{A}) \to [\chi]$ be a coloring of $\mathcal{A}$. We denote $V_c$ the vertices of color $c$.

Let $\text{Cay}(G_1, F_1) \ldots \text{Cay}(G_\chi, F_\chi)$ be a collection of Cayley graphs such that $|F_c| = |V_c|$ for every $c \in [\chi]$ and such that

$$\max(\lambda(\text{Cay}(G_i, F_i))) \leq \nu$$

Then, exists a complex

$$C[\mathcal{A}, C, G_1, \ldots G_\chi]$$

which is a $3\mathcal{A}(2)$-regular $2d$-regular transitive with $G = \prod G_i$ as its vertex set. And

$$\lambda(G_{\text{walk}}(C)) \leq \sqrt{\frac{1}{2} + \frac{1}{2} f \left( \frac{N - (\chi - 1)}{N}, \lambda(G_{\text{walk}}(\mathcal{A})) \right)}$$

where $f$ is the "zig-zag function" and $N = \binom{\chi}{2}$

This theorem is proved in section 15.

**Proof.** We use proposition 1 with $\mathcal{A}$.

We know property $\text{Inv}$ is satisfied. And it is already $d$-regular. The properties are from lemma 10.

We have $\lambda(G_{\text{dual}})$ from lemma 11.

An interesting application of this is the case the complex is complete (complete 3-partite graph). This case is called the 3-product case.

### 16 Property Inv and HPOWER

For the theorem to be useful, we wish to have complexes that satisfies property $\text{Inv}$. We can claim so immediately in several cases.

**Claim 3.** Property $\text{Inv}$ is satisfied if $\mathcal{A}$ is a commutative triplet structure.

**Proof.** Obvious, since it is one of the requirements (Condition 4).

What is interesting about this claim is that, we can composite several HDZ constructions together, or start with a known CTS, such as Conlon’s construction, and continue with HDZ, yielding larger and larger constructions.

**Claim 4.** Property $\text{Inv}$ is satisfied if the 1-skeleton of $\mathcal{A}$ isomorphic to some $\text{Cay}(G’, F’)$ where $G’$ is an abelian group and $F’$ is symmetric.

**Proof.** We prove that if $(g, g’) \in A(1)$ then $(g^{-1}, (g’)^{-1}) \in A(1)$. In this case,

$$g’ = fg$$

for $f \in F’$. And surely

$$(fg)^{-1} = f^{-1}g^{-1} = g’^{-1}$$

Since $f$ symmetric, $f^{-1} \in F’$. 

16.1 HPOWER

Even if we can’t prove our original complex $A$ satisfies Inv, we can just generate a new complex from $A$ that does. What we do is we define a power of complex that is similar to normal product by a full complex, and we prove it preserves the expansion. We only need to do product by complete complex of 2 vertices, but it could be easily extended to complete complex of general $k$-vertices. And possibly could be a useful tool in other situations.

Claim 5. For any $d$-regular complex $A$ s.t. $G_{walk}(A)$ is $\epsilon$-expander, we define

$$A' = HPOWER[A]$$

that has the following properties:

- $|V(A')| = 2|V(A)|$.
- $|A'(1)| = 4|A(1)|$.
- $|A'(2)| = 8|A(2)|$.
- $(A')$ is $2d$ edge regular
- $G_{walk}(A')$ is $\epsilon$-expander

**Definition 41.** $A'$ is over $(0, 1) \times V(A)$:

For every $e \in E(A)$

$$\{0, 1\} \times e \in E(A')$$

We define

$$V := V(A), V' := V(A')$$

We think of $G_{Walk}(A')$ as a graph on vertex set

$$U' := V(G_{Walk}) = \binom{V'}{2}$$

The walk is by the matrix $A'_{walk}$. And similarly for $G_{Walk}(A)$.

**Definition 42.** We define a projection $p : E(A) \to E(A')$

$$x \times e \to e$$

where $x \in \{0, 1\}, e \in E(A)$.

Similarly, We define a projection $q : E(A) \to \{0, 1\}$

$$x \times e \to x$$

16.2 Properties of HPOWER

The first 3 claims are obvious.

Let $J$ be the Johnson graph $J(V', 2)$

Claim 6. For $v, w \in U'$

$$v \sim_{A'_{walk}} w \iff v \sim_J w \text{ and } pv \sim_{A'_{walk}} pw$$
Let’s explain the condition.
Assuming
\[ \{(a, x), (b, y)\} \sim \{(a', x'), (b', y')\} \]
If the walk is valid, one of the vertices of \( U' \) is common to both of them. This is equal to the condition that should be adjacent in \( J \).

The second condition is that under \( p \), they are adjacent in \( A \). It is true if
\[ \{x, y\} \sim A \{x', y'\} \]
We don’t demand any conditions on \( a, a', b', b \) as the edges are defined as a tensor product on edges.

**Lemma 12.** With complex \( A \) and \( A' = \text{HPower}[A] \)
\[ \text{tr}(A'^{2m}) = \text{tr}(A^{2m})2^{2m+1} \]
**Proof.** Now, we inspect the circles of length \( 2m \) of \( A'_{\text{walk}} \). We can easily see that given \( u_1, \ldots, u_{2m+1} = u_1 \in U' \) a circle in \( A'_{\text{walk}} \) then
\[ pu_1, \ldots pu_{2m+1} = pu_1 \in U \] a circle in \( A_{\text{walk}} \)
So, this is a necessary condition. We will see that it is also sufficient.
Let \( u_1 \ldots u_{2m+1} = u_1 \in U' \) a circle in \( A'_{\text{walk}} \).
We assume
\[ u_j = \{(a, x), (b, y)\} \]
As long as one element in \( u_j \) is kept, and the condition on \( pu_{j+1} \) is satisfied, the step is legal.
Assuming we keep \( (a, x) \), the next one could be \( \{(a, x), (0/1, z)\} \), provided that \( \{x, y, z\} \) is a valid triangle in \( A_{\text{walk}} \). So, we can decide on \( q(u_{j+1} \setminus u_j) \) where \( j \in 2 \ldots 2m \). That means we have freedom of 2 choices per step. We can also see that if the random walk on \( A \) is \( d \)-regular, the random walk on \( A' \) is \( 2d \)-regular.
All in all for every \( u \)
\[ A'^{2m}_{uu} = 2^{2m-1} A^{2m}_{pu,pu} \]
Now
\[ \sum_{u \in U'} A'^{2m}_{uu} = \sum_{u \in U'} 2^{2m-1} A^{2m}_{pu,pu} \]
We notice that in the sum every \( A^{2m}_{vv} \) for every \( v \in U \) is obtained 4 times.
\[ = \sum_{v \in U} 2^{2m+1} A^{2m}_{vv} \]
Therefore, we have that
\[ \text{tr}(A'^{2m}) = \text{tr}(A^{2m})2^{2m+1} \]

**Lemma 13.** With complex \( A \) and \( A' = \text{HPower}[A] \)
\[ \lambda(G_{\text{walk}}(A)) = \lambda(G_{\text{walk}}(A)) \]
Proof. Let $P' = \frac{A'}{2d}$, the normalized version of $A'$.

$$tr(P^{2m}) = \sqrt{\sum \lambda_i^{2m}} = \sqrt{\lambda^{2m}(1+o(1))} = \lambda^m + o(\lambda^m)$$

This is also true for $P$.

And

$$tr(P'^{2m}) = \frac{tr(A'^{2m})}{(2d)^{2m}} = \frac{tr(P^{2m})^{2m+1}}{(2d)^{2m}} = tr(P^{2m})^2$$

So,

$$\lambda' = \lim_{m \to \infty} m \sqrt{\frac{tr(P'^{2m})}{m}} = \lim_{m \to \infty} m \sqrt{\frac{2^m}{2m} tr(P^{2m})} = \lambda$$

Claim 7. There is a coloring $C$, order on $V$ such that complex $A$ satisfies property Inv (definition [34]).

Proof. We can see that if $a \in V$ then there is no triangle that contains $\{(0, a), (1, a)\}$, as this would suggest that $\{a, a\}$ is an edge in $A$. Therefore, given a vertex $V_c^i$, we define

$$V_c^{ic} = (0, a)$$

$$V_{i+K_c} = (1, a)$$

This is a valid coloring. The reason is that if $\{(x, a), (y, b), (z, c)\}$ is a triangle iff $\{a, b, c\}$ is a triple. It is straightforward to see that property Inv is satisfied.

17 Reducing $G_{dual}$

In this section we analyze the expansion properties of $G_{dual}$.

17.0.1 Definitions

We have several definitions here.

We call $S_{cd}$ the generators obtained by the edges of index (originally colors) $c, d \in [\chi]$.

$$S_{cd} := \{(c, i) \cdot (d, j) \mid \{(c, i), (d, j)\} \in S(1)\}$$

And

$$S_{ijk} := \bigcup_{\{c,d\} \in \{i,j,k\}} S_{cd}$$

$$M_{cd} := Cay(G_cG_d, S_{cd})$$

$$M_{ck}^k := Cay(G_cG_d, P_k(S_{cd}))$$

where $P_k$ is the natural projection $P_k : G_cG_d \to G_k$ for $k \in \{c, d\}$.

Lemma 14. Let $N = \chi - 1$. Exists $U_1 \ldots U_N$ sets of 2-elements in $\left(\binom{[\chi]}{2}\right)$ such that

$$\bigcup U_i = \binom{[\chi]}{2}$$

and each 2-element appears once in one of the $U_i$s s.t.

$$\sigma(G_{dual}) \geq \sum_{i=1}^{N} \min_{ij \in U_i} \sigma(M_{ij})$$

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Proof. We quote from [Bar79, Abstract]

If \( h \mid n \) then the \( h \)-element subsets of an \( n \)-element set can be partitioned into \( \begin{pmatrix} n-1 \\ h-1 \end{pmatrix} \) classes so that every class contains \( n/h \) disjoint \( h \)-element sets and every \( h \)-element set appears in exactly one class. \(^{30}\)

Let’s apply this for \( h = 2 \) and \( n = \chi \). We set \( N = \chi - 1 \). Using the theorem, we get \( U_1 \ldots U_N \) sets such that

\[
\bigcup U_i = \left\lceil \frac{\chi}{2} \right\rceil
\]

And such that for \( U_i = \{u_1, u_2 \ldots u_{\chi/2}\} \)

\[
u_1 \cup u_2 \ldots \cup u_{\chi/2} = \lceil \chi \rceil
\]

For each set \( U_i \), we relate a graph

\[
G^l := \square_{(ij) \in U_i} M_{ij}
\]

Notice that \( G^l \) is on the same vertices as \( G_{\text{dual}} \).

On a r.w. on \( G_{\text{dual}} \), we chose uniformly an edge \( u \) from \( S_{cd} \) in probability that is \( \propto |S_{cd}| \) and move to the incident vertex.

Claim 8. The last method of selection is further equivalent to the following:
Choosing \( U_k \) in probability \( \propto \sum_{\{lm\} \in U_k} |S^{lm}| \), then choosing an edge uniformly from \( G_l \).

All in all we concluded that

\[
A_{\text{cay}} = \sum_{k=1}^{N} a_k A^k
\]

for

\[
a_k = \frac{\sum_{\{lm\} \in U_k} |S^{lm}|}{d(G_{\text{dual}})}
\]

Notice that all \( A^k \) are distinct. Generally (lemma 2):

\[
\sigma(A \square B) = \min(\sigma(A), \sigma(B))
\]

Where \( \sigma \) is the spectral gap of the graph \( \sigma(A) \). And specifically,

\[
\sigma(G^l) = \min_{(ij) \in U_l} \sigma(C^{ij})
\]

We want to plug it in.

\[
\lambda(G_{\text{dual}}) \leq \sum_{l=1}^{N} a_l \lambda(G^l)
\]

\[
1 - \lambda(G_{\text{dual}}) \geq \sum_{l=1}^{N} a_l (1 - \lambda(G^l))
\]

\[= \sum_{l=1}^{N} \frac{d(G^l)}{d(G_{\text{dual}})} (1 - \lambda(G^l))\]

\(^{30}\)I would like to thank Liran Katzir for pointing me to the paper which has precisely the wanted claim.
So

\[ \sigma(G_{\text{dual}}) \geq \sum_{l=1}^{N} \sigma(G^{l}) \]

And finally,

\[ \sigma(G_{\text{dual}}) \geq \sum_{l=1}^{N} \min_{ij \in U_l} (\sigma(M_{ij})) \]

We state the lemma that we got:

**Lemma 15.** Let \( G = G_1 \times G_2 \times \ldots \times G_{\chi} \) where \( G_i \) are groups. Suppose we have \( F_1, F_2, \ldots, F_{\chi} \) symmetric subsets of the corresponding groups

Let \( S \) a 2-complex s.t.

\[ S(2) \subset \bigcup_{m \in (\chi^3)} \{ F_{m_1}, F_{m_2}, F_{m_3} \} \]

For simplicity, we assume \( 2 \mid \chi \) and \( \chi \geq 3 \).

Suppose that \( \exists \nu < 1 \) s.t. \( \forall ij \in (\chi^3) \)

\[ \min (\sigma(\text{Cay}(G_iG_j, S_{ij}))) \geq \frac{(1 - \nu)|S(1)|}{\chi - 1} \]  

\( (\star) \)

Then, the complex that is defined by \( \mathcal{C} = Sc[S, G] \) satisfies:

\[ \lambda(\mathcal{C}) \leq \sqrt{\frac{1}{2} + \frac{1}{2} f(\nu, \lambda(\text{G}_{\text{walk}}(A)))} \]

**Proof.** If condition \( \star \) is true, then all the graphs \( G^k \) has spectral gap of at least

\[ \frac{(1 - \nu)|S(1)|}{(\chi - 1)^2} \]

Then, we have by lemma 15

\[ \sigma(G_{\text{dual}}) \geq \sum_{l=1}^{N} \sigma(G^{l}) \geq (1 - \nu)|S(1)| \]

\[ \lambda(G_{\text{dual}}) \leq \nu \]

\( \square \)

18 Random model on Lie groups

Unfortunately, we have no out of the box way to verify that indeed condition \( \star \) is satisfied. But we can verify it in case of Lie groups when we randomize elements.

In this section, we use the notation as appears in [Tao15]:

**Definition 43** (Expanding set). \( \{a, b\} \) is \( \epsilon \)-expanding for group \( G \) if \( \text{Cay}(G, \{a, b, a^{-1}, b^{-1}\}) \) is \( \epsilon \)-expanding.

First we prove this lemma:
Lemma 16. Suppose that $G$ is a finite simple group of Lie type.

Let $f_1, \ldots, f_K \in G$ where $f_i$ are chosen uniformly independently in random, and $K$ is bounded. Then from a certain $N$ in probability at least $1 - \frac{C}{G^{\delta'}}(K/2)$, $\{f_i, f_j\}$ is $\epsilon$-expanding where $C, \delta', N, \epsilon > 0$ depend only on $K$ and $\text{rk}(G)$.

Proof. We randomize $f_1 \ldots f_K$ and we want that every pair of $\{f_i, f_j\}$ would be $\epsilon$-expanding. For this, we use the following theorem:

**Theorem 6.** ([Bre+13, Theorem 1.2] Random pairs of elements are expanding). Suppose that $G$ is a finite simple group of Lie type and that $a, b \in G$ are selected uniformly at random. Then with probability at least $1 - C|G|^{-\delta}$, $\{a, b\}$ is $\epsilon$-expanding for some $C, \epsilon, \delta > 0$ depending only on the rank of $G$.

And we combine it with the asymmetric case of Lovász local lemma [AS04, lemma 5.1.1 on pg.64].

Let $E_{ij}$ the event in which $\text{Cay}(G, \{f_i, f_j\})$ doesn’t generate $G$ for this $\epsilon(\text{rk}(G))$. Then

$$\Pr(E_{ij}) \leq C|G|^{-\delta}$$

We define a dependency graph of $E_{ij}$ (we identify it with the vertex $\{i, j\}$). This is essentially Johnson graph (definition 22) $J(K, 2)$, where

$$\{i, j\} \sim \{i, k\} \text{ if } k \neq j$$

$$\{i, j\} \sim \{j, k\} \text{ if } k \neq i$$

It has this form because $E_{ij}$ is mutually independent of $E_{kl}$ where $\{k, l\} \cap \{i, j\} = \emptyset$. That means that every vertex $E_{ij}$ has $2(K - 1)$ neighbors.

Now, we assign a number to each event $E_{ij}$

$$x(E_{ij}) = \frac{C}{|G|^\frac{\delta}{2}}$$

For large enough $|G|$, that assures that $\forall i, j$

$$\Pr(E_{ij}) \leq C|G|^{-\delta} \leq x(E_{ij}) \prod_{B \in \Gamma(E_{ij})} (1 - x(B)) \quad \text{(X)}$$

We will see why the second inequality is true. Let’s define

$$Y := \prod_{B \in \Gamma(E_{ij})} (1 - x(B)) = \left[1 - \frac{C}{|G|^\frac{\delta}{2}}\right]^{2(K-1)} \geq 1 - \frac{2(K-1)C}{|G|^\frac{\delta}{2}}$$

We set

$$N = (4(K - 1)C)^{\frac{2}{\delta}}$$

and we assume $|G| \geq N$.

$N$ depends only on the rank and the choice of $\mathcal{A}$.

We get

$$Y \geq \frac{1}{2}$$

\[\text{[source: Bre+13, Theorem 1.2] Random pairs of elements are expanding.}\]

\[\text{[AS04, lemma 5.1.1 on pg.64].}\]

\[\text{[source: Bre+13, Theorem 1.2] Random pairs of elements are expanding.}\]

\[\text{[source: Bre+13, Theorem 1.2] Random pairs of elements are expanding.}\]

\[\text{[source: Bre+13, Theorem 1.2] Random pairs of elements are expanding.}\]
Equation $\Box$ is satisfied if $|G|^2 > 2$, which is indeed the case.

According to Lovász local lemma [AS04, lemma 5.1.1 on pg. 64], if $E_{ij}$ are mutually independent of all the events that are not its neighbors, and there is an assignment $x(E_{ij})$ in $[0,1]$ s.t. equation $\Box$ is satisfied, then

$$\Pr \left( \bigcap_{ij} \overline{E}_{ij} \right) \geq \prod_{ij} (1 - x(E_{ij})) = \left( 1 - \frac{C}{|G|^2} \right)^{K} \geq 1 - \frac{C}{|G|^2} \left( \frac{|G|^2}{2} \right)$$

We can now describe the expansion properties of a randomly generated HDZ:

**Proposition 2.** Let $A$ be a complex that is $d$-regular, $\chi$-strongly-colorable, s.t. for every projection $P^k, c, c'$

$$|P^k(E^1_{cc'}(A))| \geq 2$$

where $E^1_{cc'}$ are the edges between colors $c$ and $c'$, $P^k$ in a projection into $V^k (k \in \{c,c'\})$.

Let $C : V(A) \to [\chi]$ be a coloring of $A$. $K_c$ is the number of vertices of color $c$. Suppose $A$ satisfies property $\text{Inv}$.

We let $G = G_1 \times G_2 \ldots \times G_\chi$ where $G_i$ are product of at most $r$ finite simple (or quasisimple) groups of Lie type of rank at most $r$. Additionally, no simple factor of $G_i$ is isomorphic to a simple factor of $G_j$ for $i \neq j$. We assume $\forall i |G_i| \geq N$ where $N$ depends only on the ranks of $G_i$ and the choice of $A$.

We randomize $F_1, F_2, \ldots, F_\chi$ subsets of the corresponding groups of corresponding sizes $K_1, K_2, \ldots, K_\chi$ uniformly independently. We order them such that $\text{Inv}$ is satisfied.

We let $C$ be

$$\text{HDZ}^+[A, C, G, F_1, F_2, \ldots, F_\chi]$$

Then, in probability at least $1 - O(|G_1|^\delta)$ where $G_1$ is the smallest component of $G$, random walk on $C$ converges with rate $\lambda$ where $\lambda, \delta$ depend only on ranks of $G_i$ and the choice of $A$.

**Proof.** We want to reduce the condition on $M_{cc'}$ to condition on the corresponding projection of the edges on $G_c$ and $G'_c$. That is on

$$M^k_{cc'} := \text{Cay}(G_c, P^k(S_{cc'}))$$

To do so, we use the following proposition:

**Proposition 3.** ([Bre+13, Proposition 8.4. j]) let $r \in N$ and $\epsilon > 0$. suppose $G = G_1G_2$, where $G_1$ and $G_2$ are products of at most $r$ finite simple (or quasisimple) groups of Lie type of rank at most $r$. Suppose that no simple factor of $G_1$ is isomorphic to a simple factor of $G_2$. If $x_1 = x_1^{(1)}, x_2^{(2)}, x_k = x_k^{(1)}, x_k^{(2)}$ are chosen so that $\left\{ x_1^{(1)}, \ldots, x_k^{(1)} \right\}$ and $\left\{ x_1^{(2)}, \ldots, x_k^{(2)} \right\}$ are both $\epsilon$-expanding generating subsets in $G_1$ and $G_2$ respectively, then $\{x_1, \ldots, x_k\}$ is $\delta$-expanding in $G$ for some $\delta = \delta(\epsilon, r) > 0$
Let’s assume that for every $c, c’, \text{ and for every } k \in \{c, c’\}$, $M_{cc'}^k$ is $\epsilon$-expander.

Then exists $\delta(rk(G_c), rk(G_c'), \epsilon)$ s.t. $M_{cc'}$ is a $\delta$-expander for every $c, c’$. We can take the minimum $\delta$, and get that condition $\blacklozenge$ is satisfied.

Next, we rely on randomization properties of Lie groups in order to assure that $M_{cc'}^k$ are all $\epsilon$-expanders.

We use lemma 16 for every color $c$ separately, with the corresponding $G_c, K_c$ and get $\epsilon$-expansion. The probabilities are independent. For every projection $P^k, c, c’$

\[ |P^k(E_{cc'}^{1}(A))| \geq 2 \]

That means that $M_{cc'}^k$ generated by at least two elements. So we have that in probability at least $\prod_{c \in [\chi]}(1 - \frac{C(G_c)}{|G|} \frac{(K_c)}{2})$, every $M_{cc'}^k$ is a $\epsilon$ expander, for $\epsilon = min(\epsilon_c)$.

If we set $\delta''$ to be the minimal $\delta'$, then in probability at least $1 - O(|G|\delta'')$ where $G_i$ is the smallest component, condition $\blacklozenge$ is satisfied, and we can use lemma 15 as $S$ is defined over the correct vertices. Therefore, the complex has a convergence rate of at least $\lambda$, depending only on ranks and the choice of $\mathcal{A}$.

\[ \Box \]

19 Main theorem

We now turn to proof the main theorem, for which need to combine the expansion with the symmetry and the links properties.

**Theorem 7** (HDZ theorem). Let $\mathcal{A}$ be a complex that is $d$-regular, $\chi$-strongly-colorable ($\chi$ is even), and the edges that involves vertices of colors $a, b$ incident to at least 2 distinct vertices of each color $43$. Suppose it also has a connected 1-skeleton.

Let $C : V(\mathcal{A}) \rightarrow [\chi]$ be a coloring of $\mathcal{A}$. $K_c$ is the number of vertices of color $c$.

Let $\mathcal{G} = G_1 \times G_2 \ldots \times G_\chi$ where $G_i$ are product of at most $r$ finite simple (or quasisimple) groups of Lie type of rank at most $r$. Additionally, no simple factor of $G_i$ is isomorphic to a simple factor of $G_j$ for $i \neq j$.

Let $F_1, F_2, \ldots, F_\chi$ symmetric subsets of the corresponding groups of corresponding sizes $2K_1, \ldots, 2K_\chi$ chosen uniformly independently.

Then in probability at least $1 - O(|G_i|\delta)$ where $G_i$ is the smallest component of $\mathcal{G}$, and from a certain $N$ (or $|G_i| > N$ ), the complex $C = HDZ^+[\mathcal{A}, C, \mathcal{G}, F_1, \ldots, F_\chi]$ has the following properties:

- Its vertex set is $\mathcal{G}$
- The degree of each vertex is $24\mathcal{A}(2)$
- It is $4d$-regular.
- The link of each vertex is the same regular graph (up to isomorphism).
- It is $\epsilon'$ expanding.
- It is transitive.

\[ \Box \]

\[ \text{Formally: } |P^c(E_{ab}^{1}(A))| \geq 2 \text{ where} \]

\[ c \in \{a, b\} \]

\[ E_{ab}^{1} \text{ are the edges between colors } a \text{ and } b \]

\[ P^c \text{ is a projection into } V^c \]

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where \( \varepsilon', \delta, N \) depend only on ranks of the groups and the choice of \( A \).

Proof. We have \( A \) that is \( d \)-regular and \( \chi \)-strongly colorable with a coloring \( C \).

Let \( F_1, F_2, \ldots, F_\chi \) symmetric subsets of the corresponding groups of corresponding sizes \( 2K_1, \ldots, 2K_\chi \) chosen uniformly independently.

We have \( A' = HPOWER[A] \) with corresponding coloring function \( C' \).

By claim \( 5 \), \( A' \) is \( 2d \)-regular, \( \chi \)-colorable and satisfies property \( Inv \) (the other regularity properties of \( A' \) are also obtained, i.e. \( A'(2) = 8A(2) \)).

We use proposition \( 2 \) to get the required expansion in the required probability.

The regularity properties are obtained from lemma \( 10 \) as \( C = HDZ[A', C', F_1 \ldots F_\chi] \).

We let

\[
S = CONV[A', C, F_1, F_2, \ldots, F_\chi]
\]

our complex is

\[
C = Sc[S, G]
\]

We can use this fact in lemma \( 4 \) to get that \( C \) is transitive.

Then, finally, we use the lemma about the links to get the link properties (lemma \( 6 \)).
Part IV
Additional Applications

Here we show how the method provide better convergence rate for two known constructions.

20 The Construction By Conlon

We can see that a special case of this construction is the construction by Conlon [Con17]. This would be an illustrative example.

Conlon looked at $\text{Cay}(G, S)$ where $S$ is a set of generators with no non-trivial 4-cycles, and $G = \mathbb{F}_2^n$. He built a complex $C$ which is based upon triangles of this graph. The triples of $C$ are composed of 3 different vertices adjacent to the same vertex, and were divided to cliques naturally.

We have a set $S \subset G$ s.t. $S = S^{-1}$.
We define the Conlon’s complex $C$ by its triangles:

$$C(2) = \{s_3 g, s_2 g, s_1 g \mid s_1, s_2, s_3 \in S \text{ distinct and } g \in G\}$$

In our case it is enough to define $S = \left(\frac{S}{3}\right)$ and $C = CTS[G, S]$. We require that there are no non-trivial 4-cycles in $\text{Cay}(G, S)$ for Condition [D] to be satisfied. Indeed, this is equivalent, because a non-trivial 4-cycle in $\text{Cay}(G, S)$ is

$$abcd = e$$

$a, b, c, d \in S$ s.t. $\{c, d\} \neq \{a, b\}$
Which contradicts Condition [D].

So far we have described a small generalization of Conlon’s construction. To describe it specifically, we require that $G = \mathbb{F}_2^n$ and the product is additive.

20.1 The random walk in Conlon case

We describe the random walk in Conlon’s case, in terms of types and centers. Suppose we start at $\{s_1 g, s_2 g\}$ or $E(g, \{s_1, s_2\})$. In each step, we pick first a center our edge is contained in. That is, we choose $k \in \{1, 2\}$. So we will be either in center $g$ or in center $s_1 s_2 g$ in probability $1/2$.
Now we choose another $s \in S$, where $s \neq s_1, s_2$.

Then we look at the triangle that contains the edge $\{s_3 g, s_2 g, s_1 g\}$ for $s \in S$. The type of the new edge is $\{s, s_j\}$ for $j \in \{1, 2\}$.

So the choices are exactly $2d$, where $d$ is the degree of $L$. The graph $L$ is exactly $J(S, 2)$ defined earlier, so $d$ is $2(S - 2)$. There are $4(S - 2)$ choices all in all.

Looking it as a random walk over $G_{rep}$, selecting a center corresponds to selecting either the first operand or second operand in $T$. And selecting the new edge type corresponds to an action by $P_R$.

20.2 Result

We have the following result:
Corollary 5. Assuming

\[ G = \mathbb{F}_2' \]

\[ S \subset G \] such that

\[ a + b = c + d \] only in trivial case (\( \{a, b, c, d\} \subset S \))

Then for

\[ S = \binom{S}{3} \quad H = CTS[G, S] \]

in terms of the original graph,

\[ \lambda = \lambda(Cay(G, S)) \]

we have that the convergence rate on 2D-random walk is

\[ \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} \lambda^2 + O \left( \frac{1}{d} \right) \]

Proof. Directly from [3] for the defined CTS

Remark. In terms of expansion of the auxiliary graph, we get as \( \varepsilon \to 1 \), asymptotic behavior of

\[ 1 - \frac{\sqrt{3}}{2} - \frac{(\varepsilon - 1)^2}{2\sqrt{3}} \]

compared to

\[ \frac{\varepsilon^4}{215} \]

achieved in [Con17]. This is asymptotically better\(^{45}\).

\[ \text{21 The 3-product case} \]

Chapman, Linal and Peled described a construction called Polygraph in the paper [CLP18]. We will describe it very briefly, and refer the reader to the paper for further explanation.

In this construction, one takes a graph \( G \) with large enough girth and a multiset of numbers \( S \). And one defines a graph \( G_S \) called polygraph.

The vertices of \( G_S \) are \( V(G)^m \) (tensor product)

Two vertices \((x_1 \ldots x_n), (y_1 \ldots y_n)\) are adjacent if the collection \((d(x_i, y_i) \mid 1 \leq i \leq m)\) is equal as a multiset to \( S \), where \( d \) is the distance function on the graph.

Finally, one takes the cliques complex of this complex \( C_{G^3} \).

In the HDZ construction, If we take \( G = G_1 \times G_2 \times G_3 \) with \( S_1, S_2, S_3 \) as generators and \( A = K_3^3 \) (the complete 3-partite graph), we get a very similar construction as the [1,1,0] construction in the paper.

Namely, we get a complex \( C = HDZ(G, A, C, F_1, F_2, F_3) \) with triangles

\[ T = \{ s_1g_1, s_2g_2, s_3g_3 \mid g_i \in G_i, s_i \in S_i \} \]

For \( S = [1, 1, 0] \), \( C_{G^3} \) is the same as the complex \( C \), in the specific case \( G_1 = G_2 = G_3 \). So, we provide a slight generalization of the [1,1,0] case, as we allow taking different base graphs.

On the other hand, we force all the graphs to be Cayley graph.

\(^{45}\)It is possible that link analysis would yield even better results

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Definition 44 (3-Product-Case). Given $G_1, G_2, G_3$ groups, with $S_i \subset G_i$ s.t.

1. $S_i = S_i^{-1}$
2. $d := |S_i| = |S_j|
3. $|G_i| = |G_j|

We define

$$G := G_1 \times G_2 \times G_3$$

We can also describe it as a HDZ. by defining $S$ to be the 2-complex with faces

$$S(2) = \{S_1, S_2, S_3\}$$

as $HDZ[S, C, \mathcal{G}, S_1, S_2, S_3]$ with the obvious coloring (vertex $S_i$ is in color $i$) and

$$H := CTS[G, S]$$

And we have the following corollary:

Corollary 6. Given $\lambda_i = \lambda(Cay(G_i, S_i))$ ordered s.t. $\lambda_1 \leq \lambda_2 \leq \lambda_3$ the random walk on $H$ defined in definition 44 converges with rate $\sqrt{\frac{1}{2} + \frac{1}{2} f\left(\frac{1 + 2\lambda_3}{3}, \frac{1}{2}\right)}$ where $f$ is the zig-zag function\footnote{That means $\{s_1, s_2, s_3\}$ where $s_i \in S_i$}

$\left( f(a, b) \leq a + b, f < 1 \text{ where } a, b < 1 \right)$.\footnote{originally defined in \cite[Theorem 3.2]{RVW02}}

Proof. Directly from theorem 5

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