Topological properties of spaces of Baire functions

S. Gabriyelyan

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, P.O. 653, Israel

Abstract

A fundamental result proved by Bourgain, Fremlin and Talagrand states that the space $B_1(M)$ of Baire one functions over a Polish space $M$ is an angelic space. Stegall extended this result by showing that the class $B_1(M,E)$ of Baire one functions valued in a normed space $E$ is angelic. These results motivate our study of various topological properties in the classes $B_\alpha(X,G)$ of Baire-$\alpha$ functions, where $\alpha$ is a nonzero countable ordinal, $G$ is a metrizable non-precompact abelian group and $X$ is a $G$-Tychonoff first countable space. In particular, we show that (1) $B_\alpha(X,G)$ is a $\kappa$-Fréchet–Urysohn space and hence it is an Ascoli space, and (2) $B_\alpha(X,G)$ is a $k$-space iff $X$ is countable.

Keywords: Baire functions, $\kappa$-Fréchet–Urysohn, Ascoli, $k$-space, normal space

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1. Introduction

For Tychonoff (=completely regular and Hausdorff) spaces $X$ and $Y$, we denote by $C(X,Y)$ the family of all continuous functions from $X$ to $Y$. We say that $X$ is $Y$-Tychonoff if for every closed subset $A$ of $X$, a point $x \in X \setminus A$, and two distinct points $y, z \in Y$ there exists $f \in C(X,Y)$ such that $f(x) = y$ and $f(A) = \{z\}$, i.e., $x$ and $A$ are completely separated by a continuous function from $X$ to $Y$. In what follows we consider only $Y$-Tychonoff spaces to have the space $C(X,Y)$ sufficiently rich in the sense of the definition of $Y$-Tychonoff spaces. If $Y$ is an absolutely convex subset of a locally convex space, then every Tychonoff space $X$ is $Y$-Tychonoff (see Lemma 3.3 below).

The spaces of Baire class $\alpha$ functions were defined and studied by René Baire in his PhD thesis [1]. Let $Y$ be a Tychonoff space and let $X$ be a $Y$-Tychonoff space. The Baire class zero $B_0(X,Y)$ is the class $C(X,Y)$, and the Baire class one $B_1(X,Y)$ is the family of all functions from $X$ to $Y$ which are pointwise limits of sequences of continuous functions. If $\alpha > 1$ is a countable ordinal, the class $B_\alpha(X,Y)$ of Baire-$\alpha$ functions is the family of all functions $f : X \to Y$ such that there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq \bigcup_{i<\alpha} B_i(X,Y)$ which pointwise converges to $f$. The spaces $B_\alpha(X,Y)$ are endowed with the pointwise topology induced from the direct product $Y^X$. If $Y = \mathbb{R}$, set $B_\alpha(X) := B_\alpha(X,\mathbb{R})$.

The most important case is the case when $X = M$ is a Polish space. The compact subsets of $B_1(M)$ (called Rosenthal compact) have been studied intensively by Rosenthal [27], Bourgain, Fremlin and Talagrand [6], Godefroy [14], Todorčević [30] and others. The following fundamental result is proved in [6].

**Theorem 1.1** ([6]). If $X$ is a Polish space, then $B_1(M)$ is angelic.
Recall that a Hausdorff topological space $X$ is called an *angelic space* if (1) every relatively countably compact subset of $X$ is relatively compact, and (2) any compact subspace of $X$ is Fréchet–Urysohn. In [29, Corollary 7], Stegall generalized Theorem 1.1 as follows:

**Theorem 1.2** ([29]). If $X$ is a Polish space and $G$ is a metric space, then $B_1(M, G)$ is angelic.

In particular, if $G$ is a Banach space, then $B_1(M, G)$ is angelic. This last result was generalized by Mercourakis and Stamati in [22, Theorem 1.8].

For the general case when $X$ is a Tychonoff space, it is known that the tightness of $B_1(X)$ is equal to $\sup_{n\in\mathbb{N}} l(X^n)$, where $l(Y)$ is the Lindelöf number of a space $Y$ and $X_{\aleph_0}$ is the $\aleph_0$-modification of $X$, see [25]. Lorch noted in [20] that $X_{\aleph_0}$ coincides with the minimal topology on $X$ generated by all functions of the first Baire class, which shows that usually the space $X_{\aleph_0}$ is sufficiently complicated. In [26], Pytkeev showed that if $X$ is a Čech-complete Lindelöf space, then $B_1(X)$ is a $q$-space if and only if $X$ is perfectly normal.

The aforementioned results motivate a more detailed study of topological properties of the spaces $B_\alpha(X, Y)$. In this paper we concentrate mostly on the case when $X$ is a Tychonoff space and $Y = G$ is an abelian non-precompact metrizable group and $X$ is a $G$-Tychonoff first countable space. We examine the spaces $B_\alpha(X, G)$ in that case and having one of the topological properties described in the diagram below (which also shows relationships between the considered properties):

\[
\begin{aligned}
\text{countable} & \quad \text{cs}^*\text{-character} \\
\text{separable} & \quad \text{metric} \\
\text{metric} & \quad \Rightarrow \text{Fréchet–Urysohn} \\
\Rightarrow & \quad \text{sequential} \\
\Rightarrow & \quad \text{k-space} \\
\Rightarrow & \quad \text{countably tight} \\
\Rightarrow & \quad \text{paracompact} \\
\Rightarrow & \quad \text{normal} \\
\Rightarrow & \quad \text{countably tight} \\
\Rightarrow & \quad \text{Ascoli}
\end{aligned}
\]

Note that the implication “$\kappa$-Fréchet–Urysohn$\Rightarrow$Ascoli” is proved in Theorem 2.5 below, other implications in the diagram are well known (all relevant definitions are given below in Sections 2 and 3).

Our main result is the following theorem.

**Theorem 1.3.** Let $G$ be a non-precompact abelian metrizable group, $X$ a $G$-Tychonoff first countable space and let $H$ be a subgroup of $G^X$ containing $B_1(X, G)$. Then:

(A) $H$ is a $\kappa$-Fréchet–Urysohn space and hence is an Ascoli space.

(B) The following assertions are equivalent:
   (i) $X$ is countable;
   (ii) $H$ is a metrizable space and $H = G^X$;
   (iii) $H$ has countable tightness;
   (iv) $H$ has countable $cs^*$-character;
   (v) $H$ is a $\sigma$-space;
   (vi) $H$ is a $k$-space.

   If in addition $B_2(X, G) \subseteq H$, then (i)-(vi) are equivalent to
   (vii) $H$ is a normal space.

(C) If $B_2(X, G) \subseteq H$, then $H$ is a Lindelöf space if and only if $X$ is countable and $G$ is separable.
(D) $H$ is Čech-complete if and only if $X$ is countable and $G$ is complete.

In Theorem [1.3] the assumption on $G$ of being non-precompact is essential for items (vi), (vii) and (C)-(D) (if $X$ is any discrete space and $G$ is compact, then $C(X, G) = G^X$ is a compact group). In Corollary [3.13] we consider the properties of being a locally compact, cosmic, analytic, or $K$-analytic space.

The paper is organized as follows. In Section 2 we give a new characterization of $\kappa$-Fréchet–Urysohn spaces (Theorem 2.1) which implies (A) of Theorem [1.3]. In Theorem 2.5, we prove that every $\kappa$-Fréchet–Urysohn space $X$ is Ascoli. Applying Theorem 2.5 and some of the main results from [2, 3, 9, 10, 13] we characterize the $\kappa$-Fréchet–Urysohness in various important classes of locally convex spaces including strict (LF)-spaces and free locally convex spaces. In Section 3 we prove Theorem [1.3] using several more general results. From that results it follows that (B)-(D) hold also for subspaces $H$ of $G^X$ containing $B_1(X, S)$ or $B_2(X, S)$, where $S$ is the closed unit ball of an infinite-dimensional normed space.

2. The $\kappa$-Fréchet–Urysohn property for locally convex spaces

Following Arhangel’skii, a topological space $X$ is said to be $\kappa$-Fréchet–Urysohn if for every open subset $U$ of $X$ and every $x \in \overline{U}$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq U$ converging to $x$. Clearly, every Fréchet–Urysohn space is $\kappa$-Fréchet–Urysohn. In [19, Theorem 3.3], Liu and Ludwig showed that a topological space $X$ is $\kappa$-Fréchet–Urysohn if and only if $X$ is a $\kappa$-pseudo open image of a metric space. It is known that there are $\kappa$-Fréchet–Urysohn spaces which are not $k$-spaces, and there are sequential spaces which are not $\kappa$-Fréchet–Urysohn, see [19] or Proposition 2.9 below. In the next theorem we give a new characterization of $\kappa$-Fréchet–Urysohn spaces. The closure of a subset $A$ of a topological space $X$ is denoted by $\overline{A}$ or $\operatorname{cl}_X(A)$.

**Theorem 2.1.** A topological space $X$ is $\kappa$-Fréchet–Urysohn if and only if each point $x \in X$ is contained in a dense $\kappa$-Fréchet–Urysohn subspace of $X$.

**Proof.** The necessity is clear. To prove the sufficiency, fix an open subset $U$ of $X$ and a point $x \in \overline{U}$. Let $Y$ be a dense $\kappa$-Fréchet–Urysohn subspace of $X$ containing $x$. Then $V := U \cap Y$ is an open subset of $Y$. We claim that $x \in \operatorname{cl}_Y(V)$. Indeed, if $W \subseteq Y$ is an open neighborhood of $x$ in $Y$, take an open $W' \subseteq X$ such that $W = W' \cap Y$. As $x \in \overline{U}$, the set $W' \cap U$ is a nonempty open subset of $X$. Since $Y$ is dense in $X$ the set $(W' \cap U) \cap (W' \cap Y) \cap (U \cap Y) = W \cap V$ is not empty. Thus $x \in \operatorname{cl}_Y(V)$ and the claim is proved. Finally, since $Y$ is $\kappa$-Fréchet–Urysohn there is a sequence $\{y_n\}_{n \in \mathbb{N}} \subseteq V \subseteq U$ converging to $x$. \(\square\)

**Corollary 2.2.** Let $Y$ be a dense subset of a homogeneous space (in particular, a topological group) $X$. If $Y$ is $\kappa$-Fréchet–Urysohn, then $X$ is also a $\kappa$-Fréchet–Urysohn space.

**Proof.** Fix an arbitrary $y_0 \in Y$. Let $x \in X$. Take a homeomorphism $h$ of $X$ such that $h(y_0) = x$. Then $x \in h(Y)$ and $h(Y)$ is a $\kappa$-Fréchet–Urysohn space. Therefore, each element of $X$ is contained in a dense $\kappa$-Fréchet–Urysohn subspace of $X$ and Theorem 2.1 applies. \(\square\)

In [19, Theorem 4.1], Liu and Ludwig proved that the product of a family of bi-sequential spaces is $\kappa$-Fréchet–Urysohn. Note that any countable product of bi-sequential spaces is bi-sequential, see [23, Proposition 3.D.3]. On the other hand, countable products of $W$-spaces are $W$-spaces ([15, Theorem 4.1]) and there are $W$-spaces which are not bi-sequential ([15, Example 5.1]). Taking into account that bi-sequential spaces and $W$-spaces are Fréchet–Urysohn spaces, the next corollary essentially generalizes Theorem 4.1 of [19].

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Corollary 2.3. Let \( \{X_i : i \in I\} \) be a family of topological spaces such that \( \prod_{i \in I'} X_i \) is Fréchet–Urysohn for any countable subset \( I' \) of \( I \). Then the space \( X = \prod_{i \in I} X_i \) is \( \kappa \)-Fréchet–Urysohn.

Proof. For every \( z = (z_i) \in X \), set
\[
\sigma(z) := \{x = (x_i) \in X : \{i : x_i \neq z_i\} \text{ is finite}\}.
\]
Clearly, \( \sigma(z) \) is a dense subspace of \( X \). Proposition 2.6 of \([11]\) states that \( \sigma(z) \) is Fréchet–Urysohn. Thus, by Theorem 2.1, \( \kappa \) is \( \kappa \)-Fréchet–Urysohn. \( \square \)

Corollary 2.4. Let \( G \) be a nontrivial metrizable group with the identity \( e \), \( \kappa \) be a cardinal and let \( H \) be a subgroup of the product \( G^\kappa \) containing
\[
\bigoplus_{\kappa} G := \{f \in G^\kappa : \text{supp}(f) := \{i \in \kappa : f(i) \neq e\} \text{ is finite}\}.
\]
Then \( H \) is a \( \kappa \)-Fréchet–Urysohn space.

Proof. Proposition 2.6 of \([11]\) implies that \( \bigoplus_{\kappa} G \) is Fréchet–Urysohn. Clearly, \( \bigoplus_{\kappa} G \) is dense in \( G^\kappa \) and hence in \( H \). Thus, by Corollary 2.2, \( H \) is \( \kappa \)-Fréchet–Urysohn. \( \square \)

Let \( X \) be a Tychonoff space. Denote by \( C_k(X) \) and \( C_p(X) \) the space \( C(X) \) of all real-valued continuous functions on \( X \) endowed with the compact-open topology and the pointwise topology, respectively. Following \([3]\), \( X \) is called an Ascoli space if every compact subset \( K \) of \( C_k(X) \) is evenly continuous (i.e., if the map \( (f, x) \mapsto f(x) \) is continuous as a map from \( K \times X \) to \( \mathbb{R} \)). In \([8]\) we noticed that \( X \) is Ascoli if and only if every compact subset of \( C_k(X) \) is equicontinuous. The classical Ascoli theorem \([7\), Theorem 3.4.20\] states that every \( k \)-space is Ascoli. Now we prove the following somewhat unexpected result.

Theorem 2.5. Each \( \kappa \)-Fréchet–Urysohn Tychonoff space \( X \) is Ascoli.

Proof. Suppose for a contradiction that \( X \) is not an Ascoli space. Then there exists a compact set \( K \) in \( C_k(X) \) which is not equicontinuous at some point \( z \in X \). Therefore there is \( \varepsilon_0 > 0 \) such that for every open neighborhood \( U \) of \( z \) there exists a function \( f_U \in K \) for which the open set \( W_{f_U} := \{x \in U : |f_U(x) - f_U(z)| > \varepsilon_0\} \) is not empty (note that \( z \notin W_{f_U} \subseteq U \)). Set
\[
W := \bigcup \{W_{f_U} : U \text{ is an open neighborhood of } z\}.
\]
Then \( W \) is an open subset of \( X \) such that \( z \notin \overline{W} \). As \( X \) is \( \kappa \)-Fréchet–Urysohn, there is a sequence \( \{x_n : n \in \mathbb{N}\} \subseteq W \) converging to \( z \). For every \( n \in \mathbb{N} \), choose an open neighborhood \( U_n \) of \( z \) such that \( x_n \in W_{f_{U_n}} \) and, therefore,
\[
|f_{U_n}(x_n) - f_{U_n}(z)| > \varepsilon_0 \quad (\text{for all } n \in \mathbb{N}). \tag{2.1}
\]
Set \( S := \{x_n : n \in \mathbb{N}\} \cup \{z\} \). Then \( S \) is a compact subset of \( X \). Denote by \( p \) the restriction map \( p : C_k(X) \to C_k(S), p(f) = f|_S \). Then \( p(K) \) is a compact subset of the Banach space \( C_k(S) \). Applying the Ascoli theorem to the compact space \( S \) we obtain that the sequence \( \{p(f_{U_n})\}_{n \in \mathbb{N}} \subseteq p(K) \) is equicontinuous at \( z \in S \) and, therefore, there is an \( N \in \mathbb{N} \) such that
\[
|f_{U_n}(x_i) - f_{U_n}(z)| < \frac{\varepsilon_0}{2} \quad \text{for all } i \geq N \text{ and } n \in \mathbb{N}.
\]
In particular, for \( i = n = N \) we obtain \( |f_{U_N}(x_N) - f_{U_N}(z)| < \frac{\varepsilon_0}{2} \). But this contradicts (2.1). Thus \( X \) is an Ascoli space. \( \square \)
In the rest of this section, using Theorem 2.5 and some of the main results from \([2, 8, 9, 10, 13]\), we characterize the \(\kappa\)-Fréchet–Urysohnness in various important classes of locally convex spaces.

In \([28, \text{Theorem 2.1}]\), Sakai showed that \(C_p(X)\) is \(\kappa\)-Fréchet–Urysohn if and only if \(X\) has the property \((\kappa)\). In \([11]\) we proved that if \(C_p(X)\) is Ascoli, then it is \(\kappa\)-Fréchet–Urysohn. These results and Theorem 2.5 immediately imply Corollary 2.6.

**Corollary 2.6.** Let \(X\) be a Tychonoff space. Then \(C_p(X)\) is Ascoli if and only if \(X\) has the property \((\kappa)\).

The following corollary strengthens Theorem 1.3 of \([11]\).

**Corollary 2.7.** Let \(X\) be a \v{C}ech-complete space. Then \(C_p(X)\) is Ascoli if and only if \(X\) is scattered.

**Proof.** If \(C_p(X)\) is Ascoli, then \(X\) is scattered by Theorem 1.3 of \([11]\). Conversely, if \(X\) is scattered, then, by Corollary 3.8 of \([28]\), \(X\) has the property \((\kappa)\). Thus, by Corollary 2.6, \(C_p(X)\) is Ascoli. \(\square\)

Let \(E\) be a locally convex space over a field \(F\), where \(F = \mathbb{R}\) or \(\mathbb{C}\), and let \(E'\) be the dual space of \(E\). If \(E\) is a Banach space, denote by \(B\) the closed unit ball of \(E\) and set \(B_w := (B, \sigma(E, E')|_B)\), where \(\sigma(E, E')\) is the weak topology on \(E\).

**Corollary 2.8.**

(i) If \(E\) is a Banach space, then \(B_w\) is \(\kappa\)-Fréchet–Urysohn if and only if \(E\) does not contain an isomorphic copy of \(\ell_1\).

(ii) A Fréchet space \(E\) over \(F\) is \(\kappa\)-Fréchet–Urysohn in the weak topology if and only if \(E = F^N\) for some \(N \leq \omega\).

(iii) If \(X\) is a \(\mu\)-space and a \(kR\)-space, then \(C_k(X)\) is \(\kappa\)-Fréchet–Urysohn in the weak topology if and only if \(X\) is discrete.

(iv) The weak* dual space of a metrizable barrelled space \(E\) is \(\kappa\)-Fréchet–Urysohn if and only if \(E\) is finite-dimensional.

**Proof.** (i) Theorem 1.9 of \([13]\) or Theorem 6.1.1 and Corollary 1.7 of \([2]\) state that \(B_w\) is Ascoli if and only if \(B_w\) is Fréchet–Urysohn if and only if \(E\) does not contain an isomorphic copy of \(\ell_1\). Now Theorem 2.5 applies.

(ii) Corollary 1.7 of \([8]\) states that \(E\) is Ascoli in the weak topology if and only if \(E = F^N\) for some \(N \leq \omega\). This result and Theorem 2.5 imply the desired.

(iii) Corollary 1.9 of \([8]\) states that \(C_k(X)\) is Ascoli in the weak topology if and only if \(X\) is discrete. Now the assertion follows from Theorem 2.5 and Corollary 2.3.

(iv) Corollary 1.14 of \([8]\) states that the weak* dual space of \(E\) is Ascoli if and only if \(E\) is finite-dimensional, and Theorem 2.5 applies. \(\square\)

Now we consider direct locally convex sums of locally convex spaces. The simplest infinite direct sum of lcs is the space \(\varphi\), the direct locally convex sum \(\bigoplus_{n \in \mathbb{N}} E_n\) with \(E_n = F\) for all \(n \in \mathbb{N}\). It is well known that \(\varphi\) is a sequential non-Fréchet–Urysohn space, see Example 1 of \([24]\). Below we strengthen the negative part of this assertion.

**Proposition 2.9.** \(\varphi\) is a sequential non-\(\kappa\)-Fréchet–Urysohn space.

**Proof.** The space \(\varphi\) is sequential by \([24, \text{Example 1}]\). To show that \(\varphi\) is not \(\kappa\)-Fréchet–Urysohn, we consider elements of \(\varphi\) as functions from \(\mathbb{N}\) to \(F\) with finite support. Recall that the sets of the form

\[\{ f \in \varphi : |f(n)| < \varepsilon_n \text{ for every } n \in \mathbb{N} \},\]  

(2.2)
where \( \varepsilon_n > 0 \) for all \( n \in \mathbb{N} \), form a basis at 0 of \( \varphi \) (see for example \cite[Example 1]{24}). For every \( n, k \in \mathbb{N} \), set
\[
U_{n,k} := \left\{ f \in \varphi : |f(1)| > \frac{1}{2n} \text{ and } |f(n)| > \frac{1}{2k} \right\},
\]
and set \( U := \bigcup_{n,k \in \mathbb{N}} U_{n,k} \). It is easy to see that all the sets \( U_{n,k} \) are open in \( \varphi \) and \( 0 \notin U_{n,k} \). Hence \( U \) is an open subset of \( \varphi \) such that \( 0 \notin U \). To show that \( \varphi \) is not \( \kappa \)-Fréchet–Urysohn, it suffices to prove that (A) \( 0 \in \overline{U} \), and (B) there is no sequences in \( U \) converging to \( 0 \).

(A) Let \( W \) be a basic neighborhood of zero in \( \varphi \) of the form \([2,2]\). Choose an \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \varepsilon_1 \), and take \( k \in \mathbb{N} \) such that \( \frac{1}{k} < \varepsilon_n \). It is clear that \( U_{n,k} \cap W \) is not empty. Thus \( 0 \in \overline{U} \).

(B) Suppose for a contradiction that there is a sequence \( S = \{f_j\}_{j \in \mathbb{N}} \) in \( U \) converging to \( 0 \). For every \( j \in \mathbb{N} \), take \( n_j, k_j \in \mathbb{N} \) such that \( f_j \in U_{n_j,k_j} \). Since \( f_j \to 0 \), the definition of \( U_{n,k} \) implies that \( \frac{1}{2n_j} < |f_j(1)| \to 0 \), and hence \( n_j \to \infty \). Without loss of generality we can assume that \( 1 < n_1 < n_2 < \cdots \). For every \( n \in \mathbb{N} \), define \( \varepsilon_n = \frac{1}{n} k_j \) if \( n = n_j \) for some \( j \in \mathbb{N} \), and \( \varepsilon_n = 1 \) otherwise. Set
\[
V := \left\{ f \in \varphi : |f(n)| < \varepsilon_n \text{ for every } n \in \mathbb{N} \right\}.
\]
Then, \( V \) is a neighborhood of \( 0 \), and the construction of \( U_{n,k} \) implies that \( V \cap U_{n_j,k_j} = \emptyset \) for every \( j \in \mathbb{N} \). Thus \( S \cap V = \emptyset \) and hence \( f_j \not\to 0 \), a contradiction. \( \Box \)

**Corollary 2.10.** An infinite direct sum of (non-trivial) locally convex spaces is not \( \kappa \)-Fréchet–Urysohn.

**Proof.** Let \( L = \bigoplus_{i \in I} E_i \) be the direct locally convex sum of an infinite family \( \{E_i\}_{i \in I} \) of locally convex spaces. It is well known that every \( E_i \) can be represented as a direct sum \( F \oplus E_i' \). Therefore \( L \) contains \( \varphi \) as a direct summand. Since the projection of \( L \) onto \( \varphi \) is open and the \( \kappa \)-Fréchet–Urysohn property is preserved under open maps (see Proposition 3.3 of \cite{14}), Proposition \( \ref{2.9} \) implies that \( L \) is not a \( \kappa \)-Fréchet–Urysohn space. \( \Box \)

Recall that a **strict (LF)**-space \( E \) is the direct limit \( E = \overline{s\text{-ind}}_{n \geq 1} E_n \) of an increasing sequence
\[
E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots
\]
of Fréchet (= locally convex complete metric linear) spaces in the category of locally convex spaces and continuous linear maps. The space \( \mathcal{D}(\Omega) \) of test functions over an open subset \( \Omega \) of \( \mathbb{R}^n \) is one of the most famous and important examples of strict (LF)-spaces which are not Fréchet.

**Corollary 2.11.** A strict (LF)-space \( E \) is \( \kappa \)-Fréchet–Urysohn if and only if \( E \) is a Fréchet space.

**Proof.** Theorem 1.2 of \cite{9} states that \( E \) is an Ascoli space if and only if \( E \) is a Fréchet space or \( E = \varphi \). Now the assertion follows from Theorem \( \ref{2.9} \) and Proposition \( \ref{2.9} \). \( \Box \)

Consequently, \( \mathcal{D}(\Omega) \) is not a \( \kappa \)-Fréchet–Urysohn space.

One of the most important classes of locally convex spaces is the class of free locally convex spaces. Following \cite{21}, the **free locally convex space** \( L(X) \) on a Tychonoff space \( X \) is a pair consisting of a locally convex space \( L(X) \) and a continuous map \( i : X \to L(X) \) such that every continuous map \( f \) from \( X \) to a locally convex space \( E \) gives rise to a unique continuous linear operator \( f : L(X) \to E \) with \( f = f \circ i \). The free locally convex space \( L(X) \) always exists and is essentially unique.

**Corollary 2.12.** Let \( X \) be a Tychonoff space. Then \( L(X) \) is a \( \kappa \)-Fréchet–Urysohn space if and only if \( X \) is finite.
Proof. It is well known that $L(D)$ over a countably infinite discrete space $D$ is topologically isomorphic to $\varphi$. By Theorem 1.2 of [10], $L(X)$ is an Ascoli space if and only if $X$ is a countable discrete space. This fact, Theorem 2.5 and Proposition 2.9 immediately imply the assertion. □

3. Proof of Theorem 1.3

We start from several lemmas in which we construct special functions from $B_1(X, G)$. For every $g \in G$, define $g : X \to G$ by $g(x) = g$ for every $x \in X$.

Lemma 3.1. Let $G$ be a nontrivial abelian (Hausdorff) topological group and let $X$ be a $G$-Tychonoff space. If $a_1, \ldots, a_l \in X$ are distinct points, $U_1, \ldots, U_l$ are pairwise disjoint open neighborhoods of $a_1, \ldots, a_l$, respectively, and $g_0, g_1, \ldots, g_l \in G$, then there is a continuous function $f : X \to G$ such that $f(X \setminus \bigcup_{i=1}^l U_i) = \{g_0\}$ and $f(a_i) = g_i$ for every $i = 1, \ldots, l$.

Proof. Since $X$ is $G$-Tychonoff, for every $i = 1, \ldots, l$, there is a continuous function $f_i : X \to G$ such that $f_i(X \setminus U_i) = \{0\}$ and $f_i(a_i) = g_i - g_0$. Set $f := f_1 + \cdots + f_l + g_0$. Then $f$ is as desired. □

Corollary 3.2. Let $G$ be a nontrivial abelian group and let $X = \{x_n\}_{n \in \mathbb{N}}$ be a countable $G$-Tychonoff first countable space. Then $B_1(X, G) = G^X$.

Proof. Fix $f \in G^X$. For every $n \in \mathbb{N}$, by Lemma 3.1 there is a continuous function $f_n : X \to G$ such that $f_n(x_k) = f(x_k)$ for every $k = 1, \ldots, n$. Then $f_n \to f$ in $G^X$. Thus $f \in B_1(X, G)$. □

Let $G$ be a nontrivial abelian topological group with zero 0 and let $X$ be a $G$-Tychonoff space. Recall that the sets of the form $$[F; V] := \{f \in G^X : f(F) \subseteq V\},$$

where $F$ is a finite subset of $X$ and $V$ is an open neighborhood of 0, form a base of the pointwise topology on $G^X$ at zero function 0. For a function $f : X \to G$ and a subset $A \subseteq G$, the set $\text{supp}(f) := f^{-1}(G \setminus \{0\})$ is called the support of $f$ and define

$$\sigma(f) := \{h \in G^X : \{x \in X : h(x) \neq f(x)\} \text{ is finite}\},$$

$$\sigma(f, A) := \{h \in \sigma(f) : h(x) \in A \text{ for every } x \in X \text{ such that } h(x) \neq f(x)\}.$$  

Also we set $\sigma(f, g) := \sigma(f, \{g\})$.

Lemma 3.3. Let $G$ be an absolutely convex subset of a locally convex space $E$, and let $X$ be a Tychonoff first countable space. Then:

(i) $X$ is a $G$-Tychonoff space.
(ii) Let $\{U_n\}_{n < N}$, $1 < N \leq \infty$, be a disjoint family of open subsets of $X$ and let $x_n \in U_n$ for every $n < N$. Then, for every $g_1, g_2, \ldots, g_N \in G$, the function

$$f(x) := \begin{cases} g_k, & \text{if } x = x_k \text{ for some } k < N, \\ g_N, & \text{if } x \in X \setminus \{x_k : k < N\} \end{cases}$$

belongs to $B_1(X, G)$. In particular, $\sigma(g, G) \subseteq B_1(X, G)$ for every $g \in G$.  



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Proof. (i) Let $x \in X$ and $A$ be a closed subset of $X$ such that $x \notin A$. If $g, h$ are two distinct points in $G$, then the closed interval $[h, g] := \{h + \alpha(g - h) : \alpha \in [0, 1]\} \subseteq G$ is topologically isomorphic to $[0, 1]$. As $X$ is Tychonoff, one can find a continuous function $f : X \to [h, g]$ such that $f(x) = g$ and $f(A) = \{h\}$.

(ii) First we prove the following assertion which is similar to Lemma 3.1

Claim 1. If $a_1, \ldots, a_l \in X$ are distinct points, $U_1, \ldots, U_l$ are pairwise disjoint open neighborhoods of $a_1, \ldots, a_l$, respectively, and $g_0, g_1, \ldots, g_l \in G$, then there is a continuous function $f : X \to G$ such that $\text{Im}(f)$ is a bounded subset of $G$ and hence $f(a_i) = g_i$ for every $i = 1, \ldots, l$.

Indeed, for every $k = 1, \ldots, l$, by the proof of (i), there is a continuous function $f_k : X \to [0, g_k - g_0]$ such that $f_k(X \setminus U_k) = \{0\}$ and $f_k(a_k) = g_k - g_0$. Set $f := f_1 + \cdots + f_l + g_0$. We can prove that $\text{Im}(f) \subseteq G$: if $x \in U_k$ for some $k = 1, \ldots, l$, we obtain $f_k(x) = \alpha(g_k - g_0)$ for some $\alpha \in [0, 1]$ and hence $f(x) = f_k(x) + g_0(x) = \alpha(g_k - g_0) + g_0 = \alpha g_k + (1 - \alpha) g_0 \in G$.

Now, for every $k < N$, choose a decreasing open base $\{U_{n,k}\}_{n \in \mathbb{N}}$ at $x_k$ such that $\overline{U_{1,k}} \subseteq U_k$.

Case 1. Assume that $N$ is finite. For every $n \in \mathbb{N}$, by Claim 1, choose a continuous function $f_n : X \to G$ such that $f_n(X \setminus U_{n,k}) = \{g_N\}$ and $f_n(x_k) = g_k$ for every $k = 1, \ldots, N - 1$. It is clear that $f_n \to f$ in the pointwise topology. Thus $f \in B_1(X, G)$.

Case 2. Assume that $N = \infty$. For every $n \in \mathbb{N}$, by Claim 1, choose a continuous function $f_n : X \to G$ such that $f_n(X \setminus U_{n,k}) = \{g_N\}$ and $f_n(x_k) = g_k$ for every $k = 1, \ldots, n$. It is easy to see that $f_n \to f$ in the pointwise topology. Thus $f \in B_1(X, G)$.

Finally, the inclusion $\sigma(g, G) \subseteq B_1(X, G)$ follows from the trivial fact that for every finite subset $\{x_1, \ldots, x_n\}$ of $X$ there is a disjoint family $\{U_1, \ldots, U_n\}$ of open sets of $X$ such that $x_i \in U_i$ for every $i = 1, \ldots, n$. □

Using Lemma 3.1 instead of Claim 1 in the proof of (ii) of Lemma 3.3 one can prove the following result.

Lemma 3.4. Let $G$ be a nontrivial abelian topological group and let $X$ be an infinite $G$-Tychonoff first countable space. Let $\{U_n\}_{n < N}, 1 < N \leq \infty,$ be a disjoint family of open subsets of $X$ and let $x_n \in U_n$ for every $n < N$. Then, for every $g_1, g_2, \ldots, g_N \in G$, the function

\[
f(x) := \begin{cases} 
g_n, & \text{if } x = x_n \text{ for some } n < N, 
g_N, & \text{if } x \in X \setminus \{x_n : n < N\} \end{cases}
\]

belongs to $B_1(X, G)$. In particular, $\sigma(g, G) \subseteq B_1(X, G)$ for every $g \in G$.

Let $E$ be a locally convex space (lcs for short) and let $X$ be a first countable Tychonoff space (so $X$ is $E$-Tychonoff, see Lemma 3.3). A function $f : X \to E$ is called bounded if the image $\text{Im}(f)$ of $f$ is a bounded subset of $E$. If $S$ is a nontrivial absolutely convex subset of $E$, let $B_{\|\cdot\|_S}(X, S)$ be the family of all bounded functions from $B_n(X, S)$.

Theorem 3.5. Let $G$ be a nontrivial abelian metrizable group and let $X$ be a $G$-Tychonoff first countable space. Assume that a subgroup $H$ of $G^X$ satisfies one of the following conditions:

(a) $H$ contains $B_1(X, G)$;
(b) $G$ is a metrizable lcs and $B_1^0(X, G) \subseteq H$.

Then $H$ is a $\kappa$-Fréchet–Urysohn space and hence an Ascoli space.

**Proof.** By Lemma 3.3 and Lemma 3.4 we have $\sigma(0) \subseteq H$. Therefore, by Corollary 2.4 $H$ is a $\kappa$-Fréchet–Urysohn space. Thus, by Theorem 2.5 $H$ is an Ascoli space. \qed

Recall that a Tychonoff space $X$

- is a $\sigma$-space if $X$ has a $\sigma$-locally finite network (see [16] for details);
- is a $k$-space if for each non-closed subset $A \subseteq X$ there is a compact subset $K \subseteq X$ such that $K \cap A$ is not closed in $K$;
- has countable tightness if whenever $A \subseteq X$ and $x \in \overline{A}$, then $x \in \overline{B}$ for some countable $B \subseteq A$;
- has countable $cs^*$-character if $X$ has a countable $cs^*$-network at each point $x \in X$ (i.e., there is a countable family $\mathcal{N}_x$ of subsets of $X$ such that for each sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ converging to $x$ and for each neighborhood $O_x$ of $x$ there is a set $N \in \mathcal{N}_x$ such that $x \in N \subseteq O_x$ and the set $\{n \in \mathbb{N} : x_n \in N\}$ is infinite);

It is well known that every compact subset of a $\sigma$-space is metrizable, see [16]. Topological groups with countable $cs^*$-character are studied in [3]. In [12] we proved that a Baire topological vector space $E$ is metrizable if and only if $E$ has countable $cs^*$-character, and the same metrizability condition holds also for $b$-Baire-like locally convex spaces.

**Theorem 3.6.** Let $G$ be a nontrivial abelian metrizable group, $X$ be a $G$-Tychonoff first countable space and let $H$ be a subspace of $G^X$ containing $\sigma(0,g) \cup \{g\}$ for some nonzero $g \in G$. Then the following assertions are equivalent:

(i) $X$ is countable;
(ii) $H$ is a metrizable space;
(iii) $H$ has countable tightness;
(iv) $H$ is a $\sigma$-space;
(v) $H$ has countable $cs^*$-character.

In particular, (i)-(v) are equivalent if $H$ satisfies one of the following conditions:

(a) $H$ contains $B_1(X, G)$;
(b) $G$ is a metrizable lcs and $B_1^0(X, S) \subseteq H$, where $S$ is a nontrivial absolutely convex subset of $G$.

**Proof.** (i)$\Rightarrow$(ii) follows from the fact that $H$ is a subspace of the metrizable group $G^X$, and the implications (ii)$\Rightarrow$(iii)-(v) are clear.

(iii)$\Rightarrow$(i): Suppose for a contradiction that $X$ is uncountable. Fix an open neighborhood $V$ of $0 \in G$ such that $0 \not\in g + V$. Set $A := \sigma(0,g) \subseteq H$. It is easy to see that $g \in \overline{A} \setminus A$, where the closure is taken in $H$. Let $B$ be a countable subset of $A$. Set $D := \bigcup \{\text{supp}(h) : h \in B\}$. Then $D$ is a countable subset of $X$. As $X$ is uncountable there is $z \in X \setminus D$. Set $W := \{\{z\}; V\}$. Then $h(z) = 0 \not\in g + V$ for every $h \in B$, and hence $(g + W) \cap B = \emptyset$. So $g \not\in \overline{B}$. Thus the tightness of $H$ is uncountable, a contradiction.

(iv)$\Rightarrow$(i): Suppose for a contradiction that $X$ is uncountable. Since every compact subset of a $\sigma$-space is metrizable ([12, Corollary 4.7]), it is sufficient to find a compact subset $K$ of $H$ which is not metrizable. For every $x \in X$, define a function $\delta_{x,g} : X \to G$ by

$$\delta_{x,g}(x) = g, \quad \delta_{x,g}(y) = 0 \text{ if } y \neq x.$$
It is clear that $\delta_{x,g} \in \sigma(0,g)$. Set $K := \{0\} \cup \{\delta_{x,g} : x \in X\} \subseteq H$. Then any neighborhood of 0 contains all but finitely many of $\delta_{x,g}$s. Therefore $K$ is a compact subset of $H$. However, $K$ does not have countable base at 0 because $X$ is uncountable. This contradiction shows that $X$ must be countable. It is easy to see that $K$ is topologically isomorphic to the one point compactification of a discrete space of cardinality $|X|$.

(v)$\Rightarrow$(i): Suppose for a contradiction that $X$ is uncountable. Consider the compact subset $K$ of $H$ defined in the previous paragraph. Then also $K$ has countable $cs^*$-character as a subspace of $H$. However, by Proposition 9 of [3], the $cs^*$-character of $K$ is uncountable. This contradiction shows that $X$ is countable.

In the cases (a)-(b), by Lemmas 3.3 and 3.4, there is a nonzero $g \in G$ or $g \in S$ such that

\begin{align*}
\sigma(0,g) &\cup \{g\} \subseteq H,
\end{align*}

and the last assertion follows.

A sequence $\{g_n\}_{n \in \mathbb{N}}$ in an abelian topological group $G$ is called uniformly discrete if there is an open symmetric neighborhood $W$ of 0 in $G$ such that

\begin{align*}
%(1) &\quad 0 \not\in g_k + (4)W \text{ for every } k \in \mathbb{N}, \text{ and } \\
%(2) &\quad g_k - g_n \not\in (4)W \text{ for all distinct } k,n \in \mathbb{N}.
\end{align*}

**Remark 3.7.** It follows from Theorem 5 of [4] that:

(i) any non-precompact abelian group $G$ contains a uniformly discrete sequence, and

(ii) if $G$ is an infinite-dimensional normed space, then the closed unit ball $S$ of $G$ contains a uniformly discrete sequence (otherwise, $S$ would be precompact and hence $G$ would be finite-dimensional).

It is well known that for every uncountable cardinal $\kappa$, the space $\mathbb{Z}^\kappa$ is neither a $k$-space nor a normal space. In the next two theorems we essentially generalize this result.

**Theorem 3.8.** Let $G$ be an abelian metrizable group with zero 0 containing a uniformly discrete sequence $\{g_n\}_{n \in \mathbb{N}}$, $X$ be a $G$-Tychonoff first countable space and let $H$ be a subspace of $G^X$ containing $\{0\} \cup \bigcup_{n \in \mathbb{N}} \sigma(g_n,0)$. Then $H$ is a $k$-space if and only if $X$ is countable.

**Proof.** If $X$ is countable, then $H$ being a subspace of the metrizable group $G^X$ is metrizable, and hence $H$ is a $k$-space. Assume that $H$ is a $k$-space and suppose for a contradiction that $X$ is uncountable.

Denote by $A$ the set of all functions $f : X \to G$ for which there is an $n \in \mathbb{N}$ such that $|X \setminus \text{supp}(f)| \leq n$ and $f(x) = g_n$ for every $x \in \text{supp}(f)$ (so $f \in \sigma(g_n,0)$). By assumption, $A \subseteq H$. Clearly, $0 \in \overline{A} \setminus A$, where the closure is taken in $H$. Therefore, to prove that $H$ is not a $k$-space it is sufficient to show that the set $A \cap K$ is closed in $K$ for every compact subset $K$ of $H$. Set $B := A \cap K$. We have to show that $\overline{B} = B$. Fix an arbitrary $f \in \overline{B}$, so $f \in K$.

Claim 1. $f(X) \subseteq \{0, g_1, g_2, \ldots \}$. Indeed, fix an arbitrary $x \in X$. Then (1)-(2), the compactness of $K$ and the definition of $A$ imply that there is $k(x) \in \mathbb{N}$ such that $h(x) = \{0, g_1, \ldots, g_{k(x)}\}$ for every $h \in A \cap K$. Therefore, also $f(x) \in \{0, g_1, \ldots, g_{k(x)}\}$.

Claim 2. $\sigma(0) \cap \overline{B} = \emptyset$. Indeed, fix an arbitrary $t \in \sigma(0)$. Since $X$ is uncountable, there are $s \in \mathbb{N}$ and an uncountable subset $Y$ of $X$ such that $k(x) = s$ for every $x \in Y$ (see Claim 1) and $Y \cap \text{supp}(t) = \emptyset$. Let $C$ be a subset of $Y$ such that $|C| > 2s$ (so $C \cap \text{supp}(t) = \emptyset$). Then, by (1)-(2) and the definition of $A$, every $h \in (t + [C;W]) \cap A$ must satisfy $h(x) = 0$ for every $x \in C$. Now, the definition of $A$ implies that $h(x) = g_s$ for some $k \geq |C| > 2s$ and each $x \in \text{supp}(h)$. As $Y \cap \text{supp}(h) \neq \emptyset$, we obtain that $g(y) = g_s \not\in \{0, g_1, \ldots, g_s\}$ for every $y \in Y \cap \text{supp}(h)$. Therefore $h \not\in A \cap K$ and $(t + [C;W]) \cap (A \cap K) = \emptyset$. Thus $t \not\in \overline{B}$ and the claim is proved.
Claim 3. There is an $m \in \mathbb{N}$ such that $f(x) = g_m$ for every $x \in \text{supp}(f)$. Indeed, suppose for a contradiction that there are $y, z \in \text{supp}(f)$ such that $f(y) \neq f(z)$. Since every $h \in A$ takes only one nonzero value, Claim 1 and (‡) imply $(f + \{y, z\}; W) \cap A = \emptyset$. Thus $f \not\in A$, a contradiction.

Claim 4. If $C := \{x : f(x) = 0\}$, then $|C| \leq m$. Indeed, suppose for a contradiction that $|C| > m$. Choose a finite subset $C_0$ of $C$ such that $|C_0| > m$. Claim 2 implies that the support of $f$ is infinite. Therefore we can choose $x \in \text{supp}(f) \subseteq X \setminus C_0$ (so $f(x) = g_m$ by Claim 3) and define $V := \{x\} \cup C_0; W\}$. Then $(f + V) \cap H$ is a neighborhood of $f$ such that, by the definition of $A$ and (‡)-($\dagger$), the intersection $(f + V) \cap A$ is empty. Thus $f \not\in A$, a contradiction.

Finally, Claims 3 and 4 imply that $f \in A$, and hence $f \in B$. Thus $\overline{B} = B$. $\square$

Corollary 3.9. Let $S$ be an absolutely convex subset of a metrizable lcs $E$ containing a uniformly discrete sequence $\{g_n\}_{n \in \mathbb{N}}$. $X$ be a Tychonoff first countable space and let $H$ be a subspace of $E^X$ containing $B_b^0(X, S)$. Then $H$ is a $k$-space if and only if $X$ is countable.

Proof. By Lemma 3.8, $\{0\} \cup \bigcup_{n \in \mathbb{N}} \sigma(g_n, 0) \subseteq B_b^0(X, S) \subseteq H$, and Theorem 3.8 applies. $\square$

For example, Corollary 3.9 holds if $S$ is the closed unit ball of an infinite-dimensional normed space $E$, see Remark 3.7.

Theorem 3.10. Let $G$ be an abelian metrizable group containing a uniformly discrete sequence $\{g_n\}_{n \in \mathbb{N}}$. Set $T := \{0\} \cup \{g_n\}_{n \in \mathbb{N}}$. Assume that $X$ is a $G$-Tychonoff first countable space and let $H$ be a subspace of $G^X$ containing pointwise limits of sequences from $\sigma(g_1, T) \cup \sigma(0, T)$. Then $H$ is a normal space if and only if $X$ is countable.

Proof. If $X$ is countable, then $H$ being a subgroup of the metrizable group $G^X$ is metrizable, and hence $H$ is a normal space. Assume that $H$ is a normal space and suppose for a contradiction that $X$ is uncountable.

Denote by $D$ the family of all functions $f \in G^X$ such that

1. $\text{supp}(f)$ is finite;
2. if $x \in \text{supp}(f)$, there is $n \in \mathbb{N}$ such that $f(x) = g_n$;
3. $|\{x : f(x) = g_n\}| \leq 1$ for every $n \in \mathbb{N}$.

By assumption, $D \subseteq \sigma(0, T) \subseteq H$. Set $A := \text{cl}_H(D)$.

Claim 1. If $f \in A$, then

4. $f(\text{supp}(f)) \subseteq \{g_n : n \in \mathbb{N}\}$;
5. if $f \neq 0$, then $|\{x \in \text{supp}(f) : f(x) = g_n\}| \leq 1$ for every $n \in \mathbb{N}$.

Indeed, let $x \in \text{supp}(f)$. Assuming that $f(x) \notin \{g_n : n \in \mathbb{N}\}$, by (‡)-($\dagger$), we can find an open neighborhood $W' \subseteq W$ of $0 \in G$ such that $(f(x) + W') \cap T = \emptyset$. Then, by (2), $(f + \{x\}; W') \cap D = \emptyset$. Thus $f \not\in A$. This contradiction proves (4). To prove (5), suppose that $f(y) = f(z) = g_n$ for some $n \in \mathbb{N}$ and distinct $y, z \in \text{supp}(f)$. Then, by (‡)-($\dagger$) and (2)-(3), $(f + \{y, z\}; W) \cap D = \emptyset$. Hence $f \not\in A$, a contradiction. The claim is proved.

Set $g_0 := 0$ and denote by $E$ the family of all functions $f \in G^X$ such that

6. $\text{supp}(f - g_1)$ is finite;
7. if $x \in \text{supp}(f - g_1)$, there is $n \in \{0, 2, 3, \ldots\}$ such that $f(x) = g_n$;
8. $|\{x : f(x) = g_n\}| \leq 1$ for every $n \in \{0, 2, 3, \ldots\}$.

By assumption, $E \subseteq \sigma(g_1, T) \subseteq H$. Set $B := \text{cl}_H(E)$.

Claim 2. If $f \in B$, then
Therefore, by assumption, \( f(\text{supp}(f - g_1)) \subseteq \{g_0, g_2, g_3, \ldots\}; \)

(10) if \( f \neq g_1 \), then \( |\{x \in \text{supp}(f) : f(x) = g_n\}| \leq 1 \) for every \( n \in \{0, 2, 3, \ldots\}. \)

We omit the proof of Claim 2 because it is similar to the proof of Claim 1.

**Claim 3.** \( A \cap B = \emptyset \). Indeed, fix \( f \in A \). It follows from (4) and (5) that \( \text{supp}(f) \) is countable. Since \( X \) is uncountable, (9) and (10) imply that \( f \notin B \). Thus \( A \cap B = \emptyset \), and the claim is proved.

As \( H \) is normal, there are disjoint open sets \( U, V \subseteq H \) such that \( A \subseteq U \) and \( B \subseteq V \).

Now we define inductively a sequence \( \{f_n : n \in \mathbb{N}\} \subseteq A \) as follows. Set \( f_1 := 0 \). Choose a finite subset \( F_1 \subseteq X \) and a neighborhood \( W_1 \subseteq W \) of \( 0 \in G \) such that \( (f_1 + [F_1; W]) \cap H \subseteq U \). Assume that we found functions \( f_1, \ldots, f_n \in A \), finite sets

\[
F_1 = \{x_1, \ldots, x_{m(1)}\}, \ldots, F_n = \{x_1, \ldots, x_{m(n)}\}
\]

with \( m(1) < \cdots < m(n) \), and a decreasing sequence \( W_1 \supseteq \cdots \supseteq W_n \) of open neighborhoods of zero in \( G \) such that \( (f_i + [F_i; W]) \cap H \subseteq U \) for every \( i = 1, \ldots, n \). Define \( f_{n+1} : X \rightarrow G \) by

\[
f_{n+1}(x) := \begin{cases} 
g_k, & \text{if } x = x_k \text{ for some } 1 \leq k \leq m(n), \\
g_1, & \text{if } x \in X \setminus \{x_1, \ldots, x_{m(n)}\}. 
\end{cases}
\]

Then \( f_{n+1} \in D \subseteq A \). So there are distinct points \( x_{m(n)+1}, \ldots, x_{m(n+1)} \in X \) and an open neighborhood \( W_{n+1} \subseteq W_n \) of \( 0 \in G \) such that \( (f_{n+1} + [F_{n+1}; W]) \cap H \subseteq U \), where \( F_{n+1} = \{x_1, \ldots, x_{m(n+1)}\} \). The induction is now complete.

For every \( n \in \mathbb{N} \), define \( h_n \in E \subseteq B \) by

\[
h_n(x) := \begin{cases} 
g_k, & \text{if } x = x_k \text{ for some } 1 \leq k \leq m(n), \\
g_1, & \text{if } x \in X \setminus \{x_1, \ldots, x_{m(n)}\}. 
\end{cases}
\]

Clearly, the sequence \( \{h_n\} \) converges in \( G^X \) to the function

\[
h(x) := \begin{cases} 
g_k, & \text{if } x = x_k \text{ for some } k \in \mathbb{N}, \\
g_1, & \text{if } x \in X \setminus \{x_k\}_{k \in \mathbb{N}}. 
\end{cases}
\]

Therefore, by assumption, \( h \in H \) and hence \( h \in B \).

Choose a finite subset \( R \) of \( X \) and an open neighborhood \( W_0 \) of zero in \( G \) such that \( (h + [R; W_0]) \cap H \subseteq V \). Since \( R \) is finite, there is an \( n \in \mathbb{N} \) such that \( \{x_k\}_{k \in \mathbb{N}} \cap R = F_n \cap R \). Now we define a function \( t : X \rightarrow G \) by

\[
t(x) := \begin{cases} 
g_k, & \text{if } x = x_k \text{ for some } 1 \leq k \leq m(n), \\
g_1, & \text{if } x \in X \setminus F_{n+1}. 
\end{cases}
\]

Therefore \( t \in \sigma(g_1, T) \subseteq H \). By construction, \( t(x) = h(x) \) for every \( x \in R \), and hence

\[
t \in (h + [R; W_0]) \cap H \subseteq V.
\]

On the other hand, \( t(x) = f_{n+1}(x) \) for every \( x \in F_{n+1} \), and hence

\[
t \in (f_{n+1} + [F_{n+1}; W]) \cap H \subseteq U.
\]

Therefore \( t \in U \cap V = \emptyset \), a contradiction. Thus \( X \) must be countable. \( \square \)
Corollary 3.11. Let $E$ be an infinite-dimensional normed space, $X$ a first countable Tychonoff space and let $H$ contain $B^b_1(X, S)$, where $S$ is the closed unit ball of $E$. Then $H$ is a normal space if and only if $X$ is countable.

Proof. By Lemma 3.3 $\sigma(g, S) \subseteq B^b_1(X, E)$ for every $g \in S$. By Remark 3.7 $S$ contains a uniformly discrete sequence $T$ which is of course bounded. As $B^b_2(X, S) \subseteq H$, $H$ contains pointwise limits of sequence from $\sigma(g_1, T) \cup \sigma(0, T)$, where $g_1 \in T$. Now Theorem 3.10 applies. □

Corollary 3.12. Let $G$ be a non-precompact abelian metrizable group, $X$ a $G$-Tychonoff first countable space and let $H$ be a subspace of $G^X$ containing $B_2(X, G)$. Then the following assertions are equivalent:

(i) $X$ is countable and $G$ is separable;
(ii) $H$ is a Lindelöf space.

Proof. (i)⇒(ii): If $X$ is countable and $G$ is separable, then $H$ is a subspace of the separable metrizable group $G^X$. Thus $H$ is a Lindelöf space.

(ii)⇒(i): The group $G$ contains a uniformly discrete sequence, see Remark 3.7. Therefore, by Lemma 3.3 and Theorem 3.10 the space $X$ is countable and hence, by Corollary 3.2 $H = G^X$. So also $G$ is Lindelöf. Being metrizable $G$ must be separable. □

Now we are ready to prove Theorem 1.3

Proof of Theorem 1.3 (A) follows from Theorem 3.5.

(B) The equivalence of (i)-(v) follows from Theorem 3.6 and Corollary 3.2. (i) and (vi) are equivalent by Lemma 3.4 and Theorem 3.8 and (i) and (vii) are equivalent by Lemma 3.4 and Theorem 3.10.

(C) follows from Corollary 3.12.

(D) Assume that $H$ is Čech-complete. Then $H$ is a $k$-space ([7, Theorem 3.9.5]), and (B) implies that $X$ is countable and $H = G^X$. Then $G$ being a closed subgroup of $G^X$ is also Čech-complete. Being metrizable $G$ must be complete. Conversely, assume that $X$ is countable and $G$ is complete. Then, by Corollary 3.2 $B_1(X, G) = G^X \subseteq H$. Hence $H = G^X$ is a complete metrizable group. Thus $H$ is Čech-complete. □

Recall that a Tychonoff space $X$ is

- **cosmic** if it is a continuous image of a separable metrizable space;
- **analytic** if it is a continuous image of a Polish space;
- **$K$-analytic** if it is the image under usco compact-valued map defined on $\mathbb{N}^\mathbb{N}$.

It is clear that every cosmic space is separable. The next corollary completes Theorem 1.3.

Corollary 3.13. Let $G$ be a non-precompact abelian metrizable group, $X$ a $G$-Tychonoff first countable space and let $H$ be a subgroup of $G^X$ containing $B_1(X, G)$. Then:

(E) $H$ is locally compact if and only if $X$ is finite and $G$ is locally compact.
(F) $H$ is cosmic if and only if $X$ is countable and $G$ is separable.
(G) $H$ is analytic if and only if $X$ is countable and $G$ is analytic.
(H) If $B_2(X, G) \subseteq H$, then $H$ is $K$-analytic if and only if $H$ is analytic if and only if $X$ is countable and $G$ is analytic.
Proof. (E) Assume that \( H \) is locally compact. Then \( H \) is Čech-complete, and hence, by (B) and (D) of Theorem 1.3, \( X \) is countable, \( G \) is complete and \( H = G^X \). As \( G \) is not compact, Theorem 3.3.13 of [7] implies that \( X \) is finite and \( G \) is locally compact. The converse assertion is trivial.

(F) Assume that \( H \) is cosmic. Then \( H \) is a \( \sigma \)-space, and hence \( X \) is countable and \( H = G^X \), see (B) of Theorem 1.3. Therefore, \( G \) is also cosmic, and hence \( G \) is separable. Conversely, if \( X \) is countable and \( G \) is separable, then, by Corollary 3.2, \( H = G^X \). Therefore, \( H \) is cosmic.

(G) Assume that \( H \) is analytic. Then \( H \) is cosmic and hence, by (F), \( X \) is countable and \( G \) is separable. Corollary 3.2 implies that \( H = G^X \), and hence \( G \) is analytic. Conversely, if \( X \) is countable and \( G \) is analytic, then, by Corollary 3.2, \( H = G^X \). Since the countable product of analytic spaces is analytic, we obtain that \( H \) is an analytic space.

(H) Assume that \( H \) is \( K \)-analytic. Then \( H \) is Lindelöf, see Proposition 3.4 of [17]. Then (C) of Theorem 1.3 implies that \( X \) is countable and \( G \) is separable. By Corollary 3.2 we obtain \( H = G^X \). Thus, by Proposition 3.3 of [17], \( G \) is \( K \)-analytic. As \( G \) is metrizable, it is analytic, see [17, Proposition 6.3]. Conversely, if \( X \) is countable and \( G \) is analytic, then, by (G), \( H \) is analytic and hence is \( K \)-analytic.

We do not know whether the additional inclusion \( B_2(X, G) \subseteq H \) is essential in Theorem 1.3 and Corollary 3.13.

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