ON A FLOW OF TRANSFORMATIONS OF A WIENER SPACE

J. NAJNIUDEL, D. STROOCK, M. YOR

Abstract. In this paper, we define, via Fourier transform, an ergodic flow of transformations of a Wiener space which preserves the law of the Ornstein-Uhlenbeck process and which interpolates the iterations of a transformation previously defined by Jeulin and Yor. Then, we give a more explicit expression for this flow, and we construct from it a continuous gaussian process indexed by $\mathbb{R}^2$, such that all its restriction obtained by fixing the first coordinate are Ornstein-Uhlenbeck processes.

1. Introduction

An abstract Wiener space is a triple $(H,E,\mathcal{W})$ consisting of a separable, real Hilbert space $H$, a separable real Banach space $E$ in which $H$ is continuously embedded as a dense subspace, and a Borel probability measure $\mathcal{W}$ on $E$ with the property that, for each $x^* \in E^*$, the $\mathcal{W}$-distribution of the map $x \in E \mapsto \langle x, x^* \rangle \in \mathbb{R}$, from $E$ to $\mathbb{R}$, is a centered gaussian random distribution with variance $\|h_x^*\|^2_H$, where $h_x^*$ is the element of $H$ determined by $\langle h, h_x^* \rangle_H = \langle h, x^* \rangle$ for all $h \in H$. See Chapter 8 of [5] for more information on this topic.

Because $\{h_{x^*} : x^* \in E^*\}$ is dense in $H$ and $\|h_{x^*}\|_H = \|\langle \cdot, x^* \rangle\|_{L^2(\mathcal{W})}$, there is a unique isometry, known as the Paley–Wiener map, $I : H \mapsto L^2(\mathcal{W})$ such that $I(h) = \langle \cdot, x^* \rangle$ if $h = h_{x^*}$. In fact, for each $h \in H$, $I(h)$ under $\mathcal{W}$ is a centered Gaussian variable with variance $\|h\|^2_H$. Because, when $h = h_{x^*}$, $I(h)$ provides an extension of $(\cdot, h)_H$ to $E$, for intuitive purposes one can think of $x \sim [I(h)](x)$ as a giving meaning to the inner product $x \sim (x, h)_H$, although for general $h$ this will be defined only up to a set of $\mathcal{W}$-measure 0.

An important property of abstract Wiener spaces is that they are invariant under orthogonal transformations on $H$. To be precise, given an orthogonal transformation $O$ on $H$, there is a $\mathcal{W}$-almost surely unique $T_O : E \mapsto E$ with the property that, for each $h \in H$, $I(h) \circ T_O = I(O^\top h)$ $\mathcal{W}$-almost surely. Notice that this is the relation which one would predict if one thinks of $[I(h)](x)$ as the inner product of $x$ with $h$. In general, $T_O$ can be constructed by choosing $\{x_m^* : m \geq 1\} \subseteq E^*$ so the $\{h_{x_m^*} : m \geq 1\}$ is an orthonormal basis in $H$ and then taking

$$T_Ox = \sum_{m=1}^{\infty} \langle x, x_m^* \rangle O h_{x_m^*},$$

where the series converges in $E$ for $\mathcal{W}$-almost every $x$ as well as in $L^p(\mathcal{W}; E)$ for every $p \in [1, \infty)$. See Theorem 8.3.14 in [5] for details. In the case when $O$ admits an extension as a continuous map on $E$ into itself, $T_O$ can be the taken equal to that extension. In any case, it is an easy matter to check that the measure $\mathcal{W}$ is preserved by $T_O$. Less obvious is a theorem, originally formulated by I.M. Segal (cf. [6]), which says that $T_O$ is ergodic if and only $O$ admits no non-trivial, finite dimensional, invariant subspace. Equivalently, $T_O$
is ergodic if and only if the complexification $O_c$ has a continuous spectrum as a unitary operator on the complexification $H_c$ of $H$.

The classical Wiener space provides a rich source of examples to which the preceding applies. Namely, take $H = H_0^1$ to be the space of absolutely continuous $h \in \Theta$ whose derivative $\dot{h}$ is in $L^2([0, \infty))$, and set $\|h\|_{H_0^1} = \|\dot{h}\|_{L^2([0,\infty))}$. Then $H_0^1$ with norm $\| \cdot \|_{H_0^1}$ is a separable Hilbert space. Next, take $E = \Theta$, where $\Theta$ is the space of continuous paths $\theta : [0, \infty) \rightarrow \mathbb{R}$ such that $\theta(0) = 0$ and

$$\frac{|\theta(t)|}{t^\frac{3}{2} \log(e + |\log t|)} \rightarrow 0 \quad \text{as} \ t > 0 \ \text{tends to} \ 0 \ \text{or} \ \infty,$$

and set

$$\|\theta\|_{\Theta} = \sup_{t > 0} \frac{|\theta(t)|}{t^\frac{3}{2} \log(e + |\log t|)}.$$

Then $\Theta$ with norm $\| \cdot \|_{\Theta}$ is a separable Banach space in which $H_0^1$ is continuously embedded as a dense subspace. Finally, the renowned theorem of Wiener combined with the Brownian law of the iterated logarithm says that there is a Borel probability measure $W_{H_0^1}$ on $\Theta$ for which $(H_0^1, \Theta, W_{H_0^1})$ is an abstract Wiener space. Indeed, it is the classical Wiener space on which the abstraction is modeled, and $W_{H_0^1}$ is the distribution of an $\mathbb{R}$-valued Brownian motion.

One of the simplest examples of an orthogonal transformation on $H_0^1$ for which the associated transformation on $\Theta$ is ergodic is the Brownian scaling map $S_\alpha$ given by

$$S_\alpha \theta(t) = \frac{1}{\sqrt{\alpha}} \theta(\alpha t)$$

for $\alpha > 0$. It is an easy matter to check that the restriction $O_\alpha$ of $S_\alpha$ to $H_0^1$ is orthogonal, and so, since $S_\alpha$ is continuous on $\Theta$, we can take $T_{O_\alpha} = S_\alpha$. Furthermore, as long as $\alpha \neq 1$, an elementary computation shows that $\lim_{n \rightarrow \infty} (g, O_\alpha^n h)_{H_0^1} = 0$, first for smooth $g, h \in H_0^1$ with compact support in $(0, \infty)$ and thence for all $g, h \in H_0^1$. Hence, when $\alpha \neq 1$, $O_\alpha$ admits no non-trivial, finite dimensional subspace, and therefore $S_\alpha$ is ergodic; and so, by the Birkoff’s Individual Ergodic Theorem, for $p \in [1, \infty)$ and $f \in L^p(W_{H_0^1})$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} f \circ S_\alpha^m = \int f \, dW_{H_0^1}$$

both $W_{H_0^1}$-almost surely and in $L^p(W_{H_0^1})$. Moreover, since $\{S_\alpha : \alpha \in (0, \infty)\}$ is a multiplicative semigroup in the sense that $S_{\alpha \beta} = S_\alpha \circ S_\beta$, one has the continuous parameter version

$$\lim_{a \rightarrow \infty} \frac{1}{\log a} \int_1^a \frac{f \circ S_\alpha}{\alpha} \, d\alpha = \int f \, dW_{H_0^1}$$

of the preceding result.

A more challenging ergodic transformation of the classical Wiener space was studied by Jeulin and Yor (see [1], [2] and [4]), and, in the framework of this article, it is obtained by considering the transformation $O$ on $H_0^1$, defined by

$$[Oh](t) = h(t) - \int_0^t \frac{h(s)}{s} \, ds.$$  \hfill (1.1)
An elementary calculation shows that $\mathcal{O}$ is orthogonal. Moreover, $\mathcal{O}$ admits a continuous extension to $\Theta$ given by replacing $h \in H_0^1$ in (1.1) by $\theta \in \Theta$. That is

$$[T_\mathcal{O}\theta] = \theta(t) - \int_0^t \frac{\theta(s)}{s} \, ds \quad \text{for } \theta \in \Theta \text{ and } t \geq 0. \tag{1.2}$$

In addition, one can check that $\lim_{n \to \infty} (g, O^n h)_{H_0^1} = 0$ for all $g, h \in H_0^1$, which proves that $T_\mathcal{O}$ is ergodic for $\mathcal{W}_{H_0^1}$.

In order to study the transformation $T_\mathcal{O}$ in greater detail, it will be convenient to reformulate it in terms of the Ornstein–Uhlenbeck process. That is, take $H^U$ to be the space of absolutely continuous functions $h : \mathbb{R} \to \mathbb{R}$ such that

$$\|h\|_{H^U} \equiv \sqrt{\int_{\mathbb{R}} (\frac{1}{2} h(t)^2 + \dot{h}(t)^2) \, dt} < \infty.$$

Then $H^U$ becomes a separable Hilbert space with norm $\| \cdot \|_{H^U}$. Moreover, the map $F : H_0^1 \to H^U$ given by

$$[F(g)](t) = e^{-\frac{t}{2}} g(e^t), \quad \text{for } g \in H_0^1 \text{ and } t \in \mathbb{R}, \tag{1.3}$$

is an isometric surjection which extends as an isometry from $\Theta$ onto Banach space $\mathcal{U}$ of continuous $\omega : \to \mathbb{R}$ satisfying $\lim_{|t| \to \infty} \frac{\omega(t)}{\log |t|} = 0$ with norm $\|\omega\|_{\mathcal{U}} = \sup_{t \in \mathbb{R}} (\log(e + |t|))^{-1} |\omega(t)|$.

Thus, $(H^U, \mathcal{U}, \mathcal{W}_{H^U})$ is an abstract Wiener space, where $\mathcal{W}_{H^U} = F^* \mathcal{W}_{H_0^1}$ is the image of $\mathcal{W}_{H_0^1}$ under the map $F$. In fact, $\mathcal{W}_{H^U}$ is the distribution of a standard, reversible Ornstein–Uhlenbeck process.

Note that the scaling transformations for the classical Wiener space become translations in the Ornstein–Uhlenbeck setting. Namely, for each $\alpha > 0$, $F \circ S_\alpha = \tau_{\log \alpha} \circ F$, where $\tau_s$ denotes the time-translation map given by $[\tau_s \omega](t) = \omega(s + t)$. Thus, for $s \neq 0$, the results proved about the scaling maps say that $\tau_s$ is an ergodic transformation for $\mathcal{W}_{H^U}$. In particular, for $p \in [1, \infty)$ and $f \in L^p(\mathcal{W}_{H^U})$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} f \circ \tau_{ns} = \lim_{T \to \infty} \frac{1}{T} \int_0^T f \circ \tau_s \, ds = \int f \, d\mathcal{W}_{H^U}$$

both $\mathcal{W}_{H^U}$-almost surely and in $L^p(\mathcal{W}_{H^U})$.

The main goal of this article is to show that the reformulation of transformation $T_\mathcal{O}$ coming from the Jeulin–Yor transformation in terms of the Ornstein–Uhlenbeck process allows us to embed $T_\mathcal{O}$ in a continuous-time flow of transformations on the space $\mathcal{U}$, each of which is $\mathcal{W}_{H_0^1}$-measure preserving and all but one of which is ergodic. In Section 2 this flow is described via Fourier transforms. In Section 3 a direct and more explicit expression, involving hypergeometric functions and principal values, is computed. In Section 4 we study the two-parameter gaussian process which is induced by the flow introduced in Section 2. In particular, we compute its covariance and prove that it admits a version which is jointly continuous in its parameters.

2. Preliminary description of the flow

Let $\mathcal{O}$ and $T_\mathcal{O}$ be the transformations on $H_0^1$ and $\Theta$ given by (1.1) and (1.2), and recall the unitary map $F : H_0^1 \to H^U$ in (1.3) and its continuous extension as an isometry from
Clearly, the inverse of $F$ is given by

$$F^{-1}(\omega)(t) = \sqrt{t} \omega(\log t) \quad \text{for } t > 0.$$  

Because $F$ is unitary and $O$ is orthogonal on $H^1_0$, $-F \circ O \circ F^{-1}$ is an orthogonal transformation on $H^U$, and because

$$S := -F \circ T_O \circ F^{-1}$$

is continuous extension of $-F \circ O \circ F^{-1}$ to $U$, we can identify $S$ as $T_{-F \circ O \circ F^{-1}}$.

Another expression for action of $S$ is

$$\left[ S(\omega) \right](t) = -\omega(t) + \int_0^\infty e^{-\frac{s}{2}} \omega(t-s) \, ds \quad \text{for } t \in \mathbb{R}.$$  

Equivalently,

$$S(\omega) = \omega * \mu,$$

where $\mu$ is the finite, signed measure $\mu$ given by

$$\mu := -\delta_0 + e^{-\frac{t}{2}} \mathbf{1}_{t \geq 0} \, dt.$$  

To confirm that $\omega * \mu$ is well-defined as a Lebesgue integral and that it maps $U$ continuously into itself, note that, for any $\omega \in U$ and $t \in \mathbb{R}$,

$$\int_0^\infty e^{-\frac{s}{2}} |\omega(t-s)| \, ds \leq \|\omega\|_U \int_0^\infty e^{-\frac{s}{2}} \log(e + |t| + s) \, ds$$

$$ \leq \|\omega\|_U \log(e + |t|) \int_0^\infty e^{-\frac{s}{2}} (1 + s) \, ds \leq 9 \|\omega\|_U \log(e + |t|)$$

The Fourier transform $\hat{\mu}$ of $\mu$ is given by

$$\hat{\mu}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda x} \, d\mu(x) = -1 + \int_0^\infty e^{-x(1/2+i\lambda)} \, dx = -1 + \frac{1}{1/2+i\lambda} = \frac{1 - 2i\lambda}{1 + 2i\lambda} = e^{-2i \text{Arctg}(2\lambda)},$$

Hence, for all $h \in H^U$ and $\lambda \in \mathbb{R}$,

$$\hat{h} * \mu(\lambda) = e^{-2i \text{Arctg}(2\lambda)} \hat{h}(\lambda),$$

which, since

$$\|h\|_{H^U}^2 = \frac{1}{8\pi} \int_{\mathbb{R}} |\hat{h}(\lambda)|^2 (1 + 4\lambda^2) \, d\lambda,$$

provides another proof that $S \upharpoonright H^U$ is isometric.

The preceding, and especially (2.1), suggests a natural way to embed $S \upharpoonright H^U$ into a continuous group of orthogonal transformations. Namely, for $u \in \mathbb{R}$, let $\mu^u$ to be the unique tempered distribution whose Fourier transform is given by

$$\hat{\mu^u}(\lambda) = e^{-2i u \text{Arctg}(2\lambda)},$$

and define $S^u \varphi = \varphi * \mu^u$ for $\varphi$ in the Schwartz test function class $\mathcal{S}$ of smooth functions which, together with all their derivatives, are rapidly decreasing. Because

$$\hat{S^u \varphi}(\lambda) = e^{-2i u \text{Arctg}(2\lambda)} \hat{\varphi}(\lambda),$$
it is obvious that $S^u$ has a unique extension as an orthogonal transformation on $H^U$, which we will again denote by $S^u$. Furthermore, it is clear that $S^{u+v} = S^u \circ S^v$ for all $u, v \in \mathbb{R}$. Finally, for all $g, h \in H^U, u \in \mathbb{R},$

$$(g, S^u h)_{H^U} = \frac{1}{8\pi} \int_{\mathbb{R}} \frac{\hat{g}(\lambda)}{2} \frac{\lambda}{\pi} e^{-2iu\text{Arctg}(2\lambda)} (1 + 4\lambda^2) d\lambda$$

$$= \frac{1}{16\pi} \int_{-\pi/2}^{\pi/2} \left| \frac{\lambda}{2} \right| \frac{\lambda}{\pi} e^{-2iu\tau} d\tau,$$

where

$$\frac{1}{16\pi} \int_{-\pi/2}^{\pi/2} \left| \frac{\lambda}{2} \right| \left( \frac{\lambda}{\pi} \right) e^{-2iu\tau} d\tau = \frac{1}{8\pi} \int_{\mathbb{R}} |\hat{g}(\lambda)||\hat{h}(\lambda)| (1 + 4\lambda^2) d\lambda$$

$$\leq \frac{1}{8\pi} \left( \int_{\mathbb{R}} |\hat{g}(\lambda)|^2 (1 + 4\lambda^2) d\lambda \right)^{1/2} \left( \int_{\mathbb{R}} |\hat{h}(\lambda)|^2 (1 + 4\lambda^2) d\lambda \right)^{1/2} = ||g||_{H^U} ||h||_{H^U} < \infty.$$ 

Hence, by Riemann–Lebesgue lemma, shows that $(g, S^u h)_{H^U}$ tends to zero when $|u|$ goes to infinity.

Now define the associated transformations $S^u := T_{S^u}$ on $\mathcal{U}$ for each $u \in \mathbb{R}$. By the general theory summarized in the introduction and the preceding discussion, we know that \{S^u : u \in \mathbb{R}\} is a flow of $\mathcal{W}_{H^U}$-measure preserving transformations and that, for each $u \neq 0$, $S^u$ is ergodic.

3. A more explicit expression

So far we know very little about the transformations $S^u$ for general $u \in \mathbb{R}$. By getting a handle on the tempered distributions $\mu^{su}$, in this section we will attempt to find out a little more.

We begin with the case when $u$ is an integer $n \in \mathbb{Z}$. Recalling that $\mu = -\delta_0 + e^{-\frac{i}{2}t}1_{t \geq 0} dt$, one can use induction to check that, for $n \geq 0$,

$$\mu^n = (-1)^n (\delta_0 + e^{-\frac{i}{2}t}1_{t \geq 0} dt),$$

where $L_n$ is the nth Laguerre polynomial. Indeed, the Laguerre polynomials satisfy the following relations: for all $n \geq 0$,

$$L_n(0) = 1$$

and for all $n \geq 0, t \in \mathbb{R},$

$$L'_{n+1}(t) = L'_n(t) - L_n(t).$$

Similarly, starting from $\mu^{-1} = -\delta_0 + e^{\frac{i}{2}t}1_{t \geq 0} dt$, one finds that

$$\mu^n = (-1)^n (\delta_0 + e^{\frac{i}{2}t}1_{t \leq 0} dt)$$

for $n \leq 0$. In particular, $\mu^n$ is a finite, signed measure for $n \in \mathbb{Z}$ and $S^n\omega$ can be identified as $\mu^n * \omega$ for all $\omega \in \mathcal{U}$ and $n \in \mathbb{Z}.$

As the next result shows, when $u \notin \mathbb{Z}$, $\mu^{su}$ is more singular tempered distribution than a finite, signed measure.

**Proposition 3.1.** For each $u \notin \mathbb{Z}$, the distribution $\mu^{su}$ is given by the following formula:

$$\mu^{su} = \cos(\pi u)\delta_0(x) + \frac{\sin(\pi u)}{\pi \nu(1/x)} + \Phi_u(x), \quad (3.1)$$
where $\text{pv}$ denotes the principal value, and $\Phi_u \in L^2(\mathbb{R})$ is the function for which $\Phi_u(x)$ equals
\[
e^{-|x|/2} \left( -\frac{u \sin(\pi u)}{\pi} \sum_{k=0}^{\infty} \frac{(1 - u \text{sgn}(x)) k |x|^k}{k!(k+1)!} \left[ \frac{\Gamma'}{\Gamma}(1 + k - u \text{sgn}(x)) \frac{\Gamma'}{\Gamma}(1 + k) \right] - \frac{\Gamma'}{\Gamma}(2 + k) \right) + \frac{\sin(\pi u)}{\pi x} - \sin \pi u \frac{1}{\pi x},
\]
$\Gamma'/\Gamma$ being the logarithmic derivative of the Euler gamma function and $(\ldots)_k$ being the Pochhammer symbol.

**Proof.** Define the functions $\psi_u$ and $\theta_u$ from $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ to $\mathbb{R}$ so that $\theta_u(x) = e^{-\pi x} \psi_u(x)$ and $\psi_u(x)$ equals
\[
e^{-|x|/2} \left( -\frac{u \sin(\pi u)}{\pi} \sum_{k=0}^{\infty} \frac{(1 - u \text{sgn}(x)) k |x|^k}{k!(k+1)!} \left[ \frac{\Gamma'}{\Gamma}(1 + k - u \text{sgn}(x)) \frac{\Gamma'}{\Gamma}(1 + k) \right] - \frac{\Gamma'}{\Gamma}(2 + k) \right) + \frac{\sin(\pi u)}{\pi x}.
\]
From Lebedev [3], p. 264, equation (9.10.6), with the parameters $\alpha = 1 - u$ or $\alpha = 1 + u$, $n = 1$, $z = x$ or $z = -x$, the function $\psi_u$ satisfies, for all $x \in \mathbb{R}^*$, the differential equation:
\[
x \psi_u''(x) + (2 - |x|) \psi_u'(x) + (u - \text{sgn}(x)) \psi_u(x) = 0,
\]
and grows at most polynomially at infinity. One then deduces that $\theta_u$ decreases as least exponentially at infinity, and satisfies (for $x \neq 0$) the following equation:
\[
x \theta_u''(x) + 2 \theta_u'(x) + \left( u - \frac{x}{4} \right) \theta_u(x) = 0.
\]
(3.2)
At the same time, by writing
\[
e^{-|x|/2} = (e^{-|x|/2} - 1) + 1
\]
and expanding $\theta_u(x)$ accordingly, we obtain:
\[
\theta_u(x) = \frac{\sin(\pi u)}{\pi x} - \frac{u \sin(\pi u)}{\pi} \left[ \frac{\Gamma'}{\Gamma}(1 - u \text{sgn}(x)) \frac{\Gamma'}{\Gamma}(1) \right] - \frac{\Gamma'}{\Gamma}(2) \right) + \frac{\sin(\pi u)}{\pi x} \text{sgn}(x) + \eta_u(x),
\]
where $\eta_u^{(1)}, \eta_u^{(2)}, \eta_u^{(3)}, \eta_u^{(4)}$ are all smooth functions. The derivatives of the functions $x, |x|, x \log |x|, |x| \log |x|$ in the sense of the distributions are obtained by interpreting their ordinary derivatives as distributions. Similarly, the product by $x$ of their second distributional derivatives are obtained by multiplying their ordinary second derivatives by $x$. Hence, both $\eta_u''(x)$ and $x \eta_u''(x)$ as distributions can be obtained by computing $\eta_u'(x)$ and $x \eta_u'(x)$ as functions on $\mathbb{R}^*$.

Now, let $\nu_u$ be the distribution given by the expression:
\[
\nu_u(x) = \cos(\pi u) \delta_0(x) + \frac{\sin(\pi u)}{\pi} \text{pv}(1/x) + \left[ \theta_u(x) - \frac{\sin(\pi u)}{\pi x} \right].
\]
Note that the term in brackets, in the definition of \( \nu_u \), is a locally integrable function, and that \( \nu_u \) coincides with the function \( \theta_u \) in the complement of the neighborhood of zero. Let us now prove that \( \nu_u \) satisfies the analog of the equation (3.2), in the sense of the distributions. One has:

\[
\nu_u(x) = \cos(\pi u)\delta_0(x) + \frac{\sin(\pi u)}{\pi}pv(1/x) - \frac{u\sin(\pi u)}{\pi}\left[\frac{\Gamma'}{\Gamma}(1 - u\text{sgn}(x)) - \frac{\Gamma'}{\Gamma}(1) - \frac{\Gamma'}{\Gamma}(2) + \log(|x|)\right] - \frac{\sin(\pi u)}{2\pi}\text{sgn}(x) + \eta_u(x).
\]

Since

\[
\frac{\Gamma'}{\Gamma}(1 + u) - \frac{\Gamma'}{\Gamma}(1 - u) = \frac{d}{du}\left(\Gamma(1 + u)\Gamma(1 - u)\right) = \frac{d}{du}(\pi u/\sin(\pi u)) = \frac{1}{u} - \pi \cot(\pi u),
\]

one obtains, after straightforward computation,

\[
\nu_u(x) = \cos(\pi)\delta_0(x) + \frac{\sin(\pi u)}{\pi}pv(1/x) - \frac{u\cos(\pi u)}{2}\text{sgn}(x) - \frac{u\sin(\pi u)}{\pi}\log(|x|) + c(u) + \eta_u(x),
\]

where \( c(u) \) does not depend on \( x \). One deduces that

\[
\nu_u(x) = \cos(\pi u)\delta_0(x) + \frac{\sin(\pi u)}{\pi}pv(1/x) + \chi_{u,1}(x),
\]

where \( \chi_{u,1} \) denotes a locally integrable function. Moreover,

\[
\nu'_u(x) = \cos(\pi u)\delta_0'(x) - \frac{\sin(\pi u)}{\pi}fp(1/x^2) - u\cos(\pi u)\delta_0(x) - \frac{u\sin(\pi u)}{\pi}pv(1/x) + \eta'_u(x),
\]

where \( fp(1/x^2) \) denotes the finite part of \( 1/x^2 \), and then

\[
x\nu'_u(x) = -\cos(\pi u)\delta_0(x) - \frac{\sin(\pi u)}{\pi}pv(1/x) - \frac{u\sin(\pi u)}{\pi} + x\eta'_u(x).
\]

By differentiating again, one obtains:

\[
\nu''_u(x) + x\nu'_u(x) = -\cos(\pi u)\delta_0'(x) + \frac{\sin(\pi u)}{\pi}fp(1/x^2) + \eta''_u(x) + x\eta'_u(x).
\]

Therefore,

\[
x\nu''_u(x) + 2\nu'_u(x) + \left(u - \frac{x}{4}\right)\nu_u(x) = \chi_{u,2}(x) + \left(-\cos(\pi u)\delta_0'(x) + \frac{\sin(\pi u)}{\pi}fp(1/x^2)\right)
\]

\[
+ \left(\cos(\pi u)\delta_0'(x) - \frac{\sin(\pi u)}{\pi}fp(1/x^2) - u\cos(\pi u)\delta_0(x) - \frac{u\sin(\pi u)}{\pi}pv(1/x)\right)
\]

\[
+ u \left(\cos(\pi u)\delta_0(x) + \frac{\sin(\pi u)}{\pi}pv(1/x)\right) = \chi_{u,2}(x),
\]

where \( \chi_{u,2} \) is a locally integrable function. Since \( \theta_u \) satisfies (3.2), \( \chi_{u,2} \) is identically zero. Hence, \( \nu_u \) is a tempered distribution solving the differential equation:

\[
x\nu''_u(x) + 2\nu'_u(x) + \left(u - \frac{x}{4}\right)\nu_u(x) = 0,
\]

or equivalently,

\[
\frac{x}{4}\nu_u(x) - \frac{d^2}{dx^2}(x\nu_u(x)) - u\nu_u(x) = 0.
\]
Multiplying by $-4i$ and taking the Fourier transform (in the sense of the distributions), one deduces:

$$\hat{\nu}_u'(\lambda)(1 + 4\lambda^2) = -4i\nu_u(\lambda).$$

This linear equation admits a unique solution, up to a multiplicative factor $c$:

$$\hat{\nu}_u(\lambda) = c \exp\left(\int_0^\lambda \frac{-4i\lambda}{1 + 4\lambda^2} dt\right) = c \exp(-2i\lambda \arctg(2\lambda)).$$

Hence, $\nu_u$ is proportional to $\mu^{*u}$. In order to determine the constant $c$, let us observe that the distribution $\nu_{u,0}$ given by

$$\nu_{u,0}(x) = \nu_u(x) - c \cos(\pi u)\delta_0(x) - \frac{c\sin(\pi u)}{\pi}pv(1/x)$$

admits the Fourier transform:

$$\widehat{\nu}_{u,0}(\lambda) = c e^{-2i\lambda \arctg(2\lambda)} - c e^{-\pi iu \sgn(\lambda)}.$$

One deduces that $\nu_{u,0}$ is a function in $L^2$, which implies that $\nu_{u,0}$ is also a function in $L^2$, and then locally integrable. Since the last term in (3.3) is also a locally integrable function, one deduces that $c = 1$, and then

$$\mu^{*u} = \nu_u,$$

which proves Proposition 3.1.

The reasonably explicit expression for $\mu^{*u}$ found in Proposition 3.1 yields a reasonably explicit expression for the action of $S^u$. Indeed, only the term $pv(1/x)$ is a source of concern. However, convolution with respect of $pv(1/x)$ is, apart from a multiplicative constant, just the Hilbert transform, whose properties are well-known. In particular, it is a translation invariant, bounded map on $L^2(\mathbb{R})$, and as such it is also a bounded map on $H^U$. Thus, we can unambiguously write $S^u(h) = h * \mu^{*u}$ for all $h \in H^U$. On the other hand, the interpretation of $\omega * \mu^{*u}$ for $\omega \in \mathcal{U}$ needs some thought. No doubt, $\omega * \mu^{*u}$ is well-defined as an element of $\mathcal{S}'$, the space tempered distributions, but it is not immediately obvious that it is can be represented by an element of $\mathcal{U}$ or, if it can, that the element of $\mathcal{U}$ which represents it can be identified as $S^u\omega$. In fact, the best that we should expect is that such statements will be true of $\mathcal{W}_{W^U}$-almost every $\omega \in \mathcal{U}$. The following result justifies that expectation.

**Proposition 3.2.** For $\mathcal{W}_{W^U}$-almost every $\omega \in \mathcal{U}$, the tempered distribution $\omega * \mu^{*u}$ is represented by an element of $\mathcal{U}$ which can be can be identified as $S^u\omega$.

**Proof.** Recall that, for $\varphi \in \mathcal{S}$, $\varphi * \mu^{*-u}$ is the element of $\mathcal{S}$ whose Fourier transform is given by

$$\widehat{\varphi * \mu^{*-u}}(\lambda) = \widehat{\varphi}(\lambda)e^{2i\lambda \arctg(2\lambda)}$$

for all $\lambda \in \mathbb{R}$.

Also, if $T \in \mathcal{S}'$, then $T * \mu^{*u}$ is the tempered distribution whose action on $\varphi \in \mathcal{S}$ is given by

$$\langle \varphi, T * \mu^{*u} \rangle_{\mathcal{S}'} = \langle \varphi * \mu^{*-u}, T \rangle_{\mathcal{S}'}.$$

Now choose an orthonormal basis $\{h_n : n \geq 1\}$ for $H^U$ all of whose members are elements of $\mathcal{S}$, and, for each $n \geq 1$, set $g_n = \frac{1}{4}h_n + h_n^\prime$. Next, think of $g_n$ as the element of $\mathcal{U}^*$ whose action on $\omega \in \mathcal{U}$ is given by

$$u(\omega, g_n)_{\mathcal{U}^*} = \langle g_n, \omega \rangle_{\mathcal{S}'}.$$
It is then an easy matter to check that, in the notation of the introduction, \( h_n = h_{g_n} \). Hence, if \( B \) is the subset of \( \omega \in U \) for which

\[
\omega = \lim_{n \to \infty} \sum_{m=1}^{n} \xi(g_n, \omega) \mathcal{H} h_n \quad \text{and} \quad S^u \omega = \lim_{n \to \infty} \sum_{m=1}^{n} \xi(g_n, \omega) \mathcal{H} h_n * \mu^u,
\]

where the convergence is in \( U \), then \( \mathcal{W}_{H^u}(B) = 1 \).

Now let \( \omega \in B \). Then, for each \( \varphi \in \mathcal{S} \),

\[
\mathcal{S} \langle \varphi, \omega * \mu^u \rangle_{\mathcal{H}} = \mathcal{S} \langle \varphi * \mu^u, \omega \rangle_{\mathcal{H}} = \lim_{n \to \infty} \sum_{m=1}^{n} \mathcal{S} \langle g_n, \omega \rangle_{\mathcal{H}} \mathcal{S} \langle \varphi, h_n * \mu^u \rangle_{\mathcal{H}},
\]

\[
= \lim_{n \to \infty} \sum_{m=1}^{n} \mathcal{S} \langle g_n, \omega \rangle_{\mathcal{H}} \mathcal{S} \langle \varphi, S^u h_n \rangle_{\mathcal{H}} = \mathcal{S} \langle \varphi, S^u \omega \rangle_{\mathcal{H}}.
\]

Thus, for \( \omega \in B \), \( \omega * \mu^u \in \mathcal{S} \) is represented by \( S^u \omega \in U \).

\[\square\]

4. A TWO PARAMETER GAUSSIAN PROCESS

By construction, \( \{S^u \omega(t) : (u, t) \in \mathbb{R}^2\} \) is a gaussian family in \( L^2(\mathcal{W}_{H^u}) \). In this concluding section, we will show that this family admits a modification which is jointly continuous in \( (u, t) \).

Let \( \varphi, \psi \in \mathcal{S} \) and \( u, v \in \mathbb{R}^2 \) be given. Then, by Proposition 3.2 for \( \mathcal{W}_{H^u} \)-almost every \( \omega \in U \),

\[
\int_{\mathbb{R}^2} \varphi(s) \psi(t)(S^u(\omega))(s)(S^v(\omega))(t) \, ds dt = \mathcal{S} \langle \varphi, \omega * \mu^u \rangle_{\mathcal{H}} \mathcal{S} \langle \psi, \omega * \mu^v \rangle_{\mathcal{H}},
\]

where the integral in the left-hand side is absolutely convergent. Because \( \mathbb{E}_{\mathcal{W}_{H^u}} [S^u \omega(t)^2] \) is finite and independent of \( (u, t) \in \mathbb{R}^2 \), by taking the expectation with respect to \( \mathcal{W}_{H^u} \) and using (2.2), one can pass from this to

\[
\int_{\mathbb{R}^2} \varphi(s) \psi(t) \mathbb{E}_{\mathcal{W}_{H^u}} \left[ (S^u(\omega))(s)(S^v(\omega))(t) \right] \, ds dt = \mathbb{E}_{\mathcal{W}_{H^u}} \left[ \mathcal{S} \langle \varphi, \omega * \mu^u \rangle_{\mathcal{H}} \mathcal{S} \langle \psi, \omega * \mu^v \rangle_{\mathcal{H}} \right]
\]

\[
= \frac{2}{\pi} \int_{-\infty}^{\infty} e^{i(u-v) \text{Arctg}(2\lambda)} \frac{\varphi(\lambda) \overline{\psi}(\lambda) d\lambda}{1 + 4\lambda^2} = \frac{2}{\pi} \iint_{\mathbb{R}^3} e^{[i((t-s)\lambda + 2(u-v) \text{Arctg}(2\lambda)]} \varphi(s) \psi(t) \, ds dt d\lambda.
\]

Hence,

\[
\mathbb{E}_{\mathcal{W}_{H^u}} [S^u(\omega))(s)(S^v(\omega))(t)] = \frac{2}{\pi} \int_{-\infty}^{\infty} e^{[i((t-s)\lambda + 2(u-v) \text{Arctg}(2\lambda)]} \frac{d\lambda}{1 + 4\lambda^2}, \quad (4.1)
\]

first for almost every and then, by continuity, for all \( (s, t) \in \mathbb{R}^2 \). In particular, we now know that the \( \mathcal{W}_{H^u} \)-distribution of \( \{S^u(\omega))(t) : (u, t) \in \mathbb{R}^2\} \) is stationary.

To show that there is a continuous version of this process, we will use Kolmogorov’s continuity criterion, which, because it is stationary and gaussian, comes down to showing that

\[
|1 - \mathbb{E}_{\mathcal{W}_{H^u}} [(S^u(\omega))(s)(S^v(\omega))(t)]| \leq C \left| \frac{1}{(u, s) - (v, t)} \right|^{\alpha}
\]

\[9\]
for some $C < \infty$ and $\alpha > 0$. But
\[
|1 - \mathbb{E}_{\mathcal{W}_\omega}(S^u(s)(S^v(\omega))(t))| \leq \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{1 + 4\lambda^2} |e^{i(t-s)\lambda + 2(u-v)\arctg(2\lambda)}| - 1 \]
\[
\leq \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{1 + 4\lambda^2} |e^{i(t-s)\lambda} - 1| + \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{1 + 4\lambda^2} |e^{2i(u-v)\arctg(2\lambda)} - 1| \]
\[
\leq \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{1 + 4\lambda^2} (|t - s||\lambda| \wedge 2) + \frac{4}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{1 + 4\lambda^2} |(u - v)\arctg(2\lambda)|,
\]
and, after simple estimation, this shows that
\[
|1 - \mathbb{E}(S^u(s)(S^v(\omega))(t))| \leq C \left[|u - v| + |t - s| \left(1 + \log \left(1 + \frac{1}{(t - s)^2}\right)\right)\right],
\]
where $C < \infty$. Clearly, the desired conclusion follows.

**Remark 4.1.** A question about filtrations comes naturally when one considers the group of transformations $(S^u)_{u \in \mathbb{R}}$ on the space $\mathcal{U}$. Indeed, for all $t, u \in \mathbb{R}$, let $\mathcal{F}^u_t$ be the $\sigma$-algebra generated by the $\mathcal{W}^u$-negligible subsets of $\mathcal{U}$ of and the variables $(S^u(\omega))(s)$, for $s \in (-\infty, t]$ (these variables are well-defined up to a negligible set). From the results of Jeulin and Yor, one quite easily deduces the following properties of the filtrations of the form $(\mathcal{F}^u_t)_{t \in \mathbb{R}}$ for $u \in \mathbb{R}$:

- For all $t, u \in \mathbb{R}$, $\mathcal{F}^u_t$ is generated by $\mathcal{F}^{u+1}_t$ and $(S^u(\omega))(t)$.
- For all $t, u \in \mathbb{R}$, $\mathcal{F}^{u+1}_t$ and $(S^u(\omega))(t)$ are independent under $\mathcal{W}^u$.
- For all $t, u \in \mathbb{R}$, the decreasing intersection of $\mathcal{F}^{u+n}_t$ for $n \in \mathbb{Z}$ is trivial (i.e. it satisfies the zero-one law).
- If $u \in \mathbb{R}$ is fixed, the $\sigma$-algebra generated by $\mathcal{F}^{u+n}_t$ for $t \in \mathbb{R}$ does not depend on $n \in \mathbb{Z}$.

All these statements concern the sequence of filtrations $(\mathcal{F}^{u+n})_{n \in \mathbb{Z}}$ for fixed $u \in \mathbb{R}$. A natural question arises: how can these results be extended to the continuous family of filtrations $(\mathcal{F}^u)_{u \in \mathbb{R}}$? Unfortunately, for the moment, we have no answer to this question (in particular the family does not seem to be decreasing with $u$).

**References**

[1] T. Jeulin, M. Yor, *Filtration des ponts browniens et équations différentielles stochastiques linéaires*, Séminaire de Probabilités, XXIV, Lecture Notes in Math., vol. 1426, Springer-Verlag, 1990, pp. 227–265.

[2] P.-A. Meyer, *Sur une transformation du mouvement brownien due à Jeulin et Yor*, Séminaire de Probabilités, XXVIII, Lecture Notes in Math., vol. 1583, Springer-Verlag, 1994, pp. 98–101.

[3] N.-N. Lebedev, *Special functions and their applications* (Translated from Russian), Dover, New York, 1972.

[4] M. Yor, *Some aspects of Brownian motion. Part I: Some Special Functionals*, Lectures in Mathematics, Ed. ETH Zürich, Birkhäuser, 1992.

[5] D. Stroock, *Probability Theory, an Analytic View, 2nd edition*, Cambridge University Press, 2011.

[6] D. Stroock, *Some thoughts about Segal’s ergodic theorem*, Colloq. Math. vol. 118 #1, pp. 89-105, 2010.