The Trilobite and Crab, a full explanation

Chaim Goodman-Strauss
Univ. Arkansas
strauss@uark.edu

The “trilobite and crab”, shown above, admit tilings of the plane but admit only non-periodic tilings, and so are an aperiodic set of tiles. They are among the very simplest aperiodic set of aperiodic tiles known in \( E^2 \) — there are only a few other pairs known and only one other, Amman’s A2 [10], has as few translation classes, i.e. appears in so few orientations, eight. (It is still a well-known open question whether or not there is a single aperiodic tile.) A complete bibliography appears in [7].

The pair is derived from the “trilobite and cross” tiles, described in [4] (which generalize to an aperiodic pair of tiles in all \( E^n \geq 3 \)). The proof that the trilobite and cross tiles are aperiodic is a fairly simple combinatorial check that the tiles can form larger patches with the same combinatorial structure, which can then be assembled into still larger patches, ad infinitum, and thus can tile the plane. Conversely, in any tiling by these tiles, the trilobites must lie in such a hierarchy of patches, showing no such tiling can have a translational period.

However, like the “Pegasus” pair of tiles [8], the trilobite and cross have unusual “tip-to-tip” matching rules; we can easily recompose these into three tiles with matching rules that are completely encoded geometrically (by “bumps and nicks”) or by colored edges, as shown at right below. (The tiles can be adjusted to have areas 1, \( \epsilon \) and \( \epsilon^2 \), as in [11].)

This raises the natural question: can we conflate two of these tiles, giving us the crab tile, and have an aperiodic pair? We certainly will allow a richer variety of local configurations, giving more complex combinatorial structure.

It is quite remarkable that the proof complexity of the following theorem seems to be quite high:

**Theorem** The trilobite and crab are an aperiodic pair of tiles.

Because of the undecidability of the domino problem, it is certainly the case that as we enumerate all possible sets of tiles, among those that are aperiodic, the length of the shortest proof that
they are so cannot be bounded by any computable function (see [6]). But is amazing, at least
to me, that this kicks in so readily.

As I wrote in [4], a full proof of this theorem is “not worth the readers time”, but it is worth
having as a striking example of this phenomenon. Moreover, there seem to be many interesting
possibilities for exploiting this complexity, such as programming within defects of the tiling.

So, at last, here is a full proof, given in a graphical shorthand, drawn on a square grid.

*Proof of the Theorem:*

We will encode the trilobite and crab thusly:

Here is an encoding of a typical configuration:

Our proof is essentially that the matching rules enforce hierarchical configurations such as this
one:

We adopt several further conventions: We take \[
\] to mean \[
\], \[
\], or \[
\]. We take \[
\] to mean \[
\], or \[
\], and we take \[
\] to mean a trilobite in any orientation.

In each case of the proof, solid black objects are given, and gray ones are implied, with numbers
giving the order of the implication; green indicates reduction to an earlier case, blue means a further subcase, and red indicates a contradiction.

**Axioms**

With [diagram of axioms],

1. Each square and corner of a square grid must be covered, and

2. By parity, any pair of trilobites separated by crabs must be oriented as: [diagram of oriented trilobites]

We now have

**Several Elementary Lemmas** (The helpful paper included at the end of these notes will be useful for checking these and all our arguments.)

[Diagrams of initial cases]

Similarly, each of:

[Diagrams of initial cases]

**Enumerating initial cases** We consider the possible tiles around a trilobite, naming the configurations similarly to [diagram], writing T for trilobite, O for crab, or * for either.

[Diagrams of initial cases]

TTT,OTO and OOO arise within the combinatorial structure we seek:
However (as in [4]) we must take special care with $OTT$ and $TTO$ and ensure that $OOT$, $TOT$ and $TOO$ are forbidden entirely.

$TOT$ is forbidden:

$*OT$ is forbidden:

Taking each of these cases in turn, and noting that that the later cases reduce to the first (but indicating all of the implied tiles), we have:

Chains of $*TO$’s are forbidden  As in [4], the trilobite and crab do admit chains of alternating $TTO$’s and $OTT$’s, but we must show that any $*TO$ can only appear in this way.
This case is more complex still:

Finally we can conclude:

Any *T0 in a tiling must therefore occur in an infinite chain of alternating TTO’s and OTT’s and as in [4], there must always be a corresponding tiling, formed by sliding half the plane one tile along this diagonal. In this new tiling, all trilobites will be TTT, OTO or 000.
**Establishing the induction**  Every trilobite of type $TTT$ in effect is a larger trilobite, but we *still* must check that these large tiles satisfy our axioms. The difficulty is ensuring that only the top left alignment is allowed:

One last case breaks into still further subcases:
Finally, after all of this, we have that, in any tiling by the trilobite and crab, every trilobite is in a larger trilobite, or after a shift along a chain of $\ast T \circ$'s this is so, and that these larger trilobites satisfy the axioms as before.

We may therefore induct, and the proof, as in [4], is finally complete!
References

[1] R. Amman, B. Grunbaum and G.C. Shepherd, *Aperiodic tiles*, Discrete and Computational Geometry 8 (1992) 1-25.

[2] R. Berger, *The undecidability of the domino problem*, Memoirs Am. Math. Soc. 66 (1966).

[3] C. Goodman-Strauss, *Matching rules and substitution tilings*, Annals of Math. 147 (1998), 181-223.

[4] C. Goodman-Strauss, *A small aperiodic set of planar tiles*, Europ. J. Combinatorics 20 (1999), 375-384.

[5] C. Goodman-Strauss, *An aperiodic pair of tiles in $E^n$ for all $n \geq 3$*, Europ. J. Combinatorics 20 (1999), 385-395.

[6] C. Goodman-Strauss, *Can't Decide? Undecide!*, Notices A.M.S. 57 (2010), 343-356.

[7] C. Goodman-Strauss, *Lots of Aperiodic Sets of Tiles*, arXiv

[8] C. Goodman-Strauss, *The Pegasus tiles*, arXiv

[9] C. Goodman-Strauss, *Matching rules for the sphinx substitution tiling*, arXiv

[10] B. Grünbaum and G.C. Shepherd, *Tilings and patterns*, W.H. Freeman and Co. (1987).

[11] R. Penrose *Remarks on Tiling: details of a $(1 + \epsilon + \epsilon^2)$-aperiodic set*, The mathematics long range aperiodic order, NATO Adv. Sci. Inst. Ser. C. Math. Phys. Sci. 489 (1997), 467-497.
Useful paper for checking cases