Vanishing of the conformal anomaly for strings in a gravitational wave

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Abstract.
Using the non-symmetric-connection approach proposed by Osborn, we demonstrate that, for a bosonic string in a specially chosen plane-fronted gravitational wave and an axion background, the conformal anomaly vanishes at the two-loop level. Under some conditions, the anomaly vanishes at all orders.

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1. Introduction.
In a recent paper [1] Osborn carried out a two-loop calculation of the $\beta$-functions of a bosonic string in the presence of a non-trivial background metric $g_{\mu\nu}$ and of an antisymmetric axion field $b_{\mu\nu}$. He used dimensional regularization, extending the two-dimensional antisymmetric tensor $\epsilon_{\sigma\tau}$ into a complex structure in $2n$ dimensions. An immediate consequence of his procedure was to combine the metric and the axion into a single, non-symmetric tensor $t_{\mu\nu}$; the three-form $H_{\mu\nu\rho}$ associated to the axion appears as the torsion of the non-symmetric connection.

In this Letter we apply Osborn’s conceptual framework to an explicit example, namely to string propagation in a generalized pp wave (plane-fronted gravitational wave with parallel rays) of the type discussed by Brinkmann [2], whose metric is

\begin{equation}
    g_{\mu\nu}dx^\mu dx^\nu = -2dudv + 2a_i(u, x)dx^i du + g_{ij}(u, x)dx^i dx^j + k(u, x)du^2.
\end{equation}

Brinkmann’s metrics (1.1) are more general than the well-known [3] plane waves in four dimensions, with $a_i = 0$: a non trivial ‘vector potential’ $a_i$ can arise, if the spacetime is at least five dimensional. All these metrics admit a covariantly constant null vector, $\partial/\partial v$, and, assuming the vacuum Einstein equations $R_{\mu\nu} = 0$ are satisfied, have zero scalar curvature.

String propagation in an ordinary pp wave with flat transverse metric $g_{ij} = \delta_{ij}$ and with zero vector potential $a_i = 0$ was previously considered [4-5], and it was found that the vacuum Einstein equations $R_{\mu\nu} = 0$ imply the vanishing of the conformal anomaly to all orders in $\sigma$-model perturbation theory. (Including axions and dilatons yields more general conditions [5-7]). In the Appendix of his conference talk [5] Horowitz shortly discussed the case of a non-trivial $a_i$, but he concluded that the anomaly does not in general vanish.

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However, in the above papers, the terms coming from the curvature and from the axion are treated separately, and are independently set to vanish. The question arises, therefore, whether some cancellation between the various terms can take place. We show here that this can indeed happen: the vector potential can act as a counterterm, canceling the contribution to the \( \beta \)-function of the axion (at least at the two-loop level). This is similar to what happens in the Wess-Zumino model \([8]\), where the conformal invariance at the quantum level can be restored by adding a topological term with a suitable coefficient.

The calculation is particularly clear in Osborn’s unified framework which appears hence to be the ideal way to treat the higher-order contributions to the \( \beta \)-function equations.

2. Non-symmetric connections.

Let us first outline, following the ideas of Osborn \([1]\), how the metric and axion fields can be unified into a single framework, by allowing for non-symmetric connections. Let \( g_{\mu\nu} \) be a \( D \)-dimensional Lorentz metric, and \( b_{\mu\nu} \) the antisymmetric tensor which corresponds to the axion. Set \( t_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu} \), and define a non-symmetric connection by

\[
\Gamma^+{}_{\mu\rho\nu} = \Gamma^{\rho}{}_{\mu\nu} + H^{\rho}{}_{\mu\nu},
\]

where the \( \Gamma^{\rho}_{\mu\nu} \)'s are the Christoffel symbols of the metric, and \( H = \frac{1}{6} H_{\mu\nu\rho} dx^\rho \wedge dx^\mu \wedge dx^\nu \) is the three-form \( \frac{1}{2} db \), \( H_{\rho\mu\nu} = \frac{1}{2} \left( \partial_\rho b_{\mu\nu} + \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} \right) \). (Since the non-symmetric quantities \( \Gamma^+ \), etc. do not have the usual symmetry properties, the position of the indices is important).

The metric-Christoffels form the symmetric part, \( \Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu} \), and the axion field is the antisymmetric part, \( H^{\rho}_{\mu\nu} = \Gamma^+{}_{\rho\mu\nu} \). Remarkably, the non-symmetric connection is expressed as

\[
\Gamma^+{}_{\rho\mu\nu} = \frac{1}{2} \left( \partial_\rho t_{\mu\nu} + \partial_\mu t_{\nu\rho} - \partial_\nu t_{\rho\mu} \right).
\]

Indices are raised and lowered by the metric, \( g_{\mu\nu} \). The curvature is

\[
R^+{}_{\sigma\mu\nu} = \partial_\mu \Gamma^+{}_{\sigma\nu\rho} - \partial_\nu \Gamma^+{}_{\sigma\mu\rho} + \Gamma^+{}_{\rho\alpha\nu} \Gamma^{\alpha}{}_{\sigma\mu} - \Gamma^+{}_{\nu\alpha\rho} \Gamma^{\alpha}{}_{\sigma\mu}.
\]

\( R^+{}_{\alpha\beta\gamma\delta} \) is antisymmetric in the first two as well as in the last two indices. In terms of the symmetric and antisymmetric parts, the non-symmetric Riemann tensor can be also written as

\[
R^+{}_{\sigma\mu\nu} = R^{\rho}_{\sigma\mu\nu} + \nabla_\mu H^\rho_{\nu\sigma} - \nabla_\nu H^\rho_{\mu\sigma} + H^\rho_{\mu\alpha} H^\alpha_{\nu\sigma} - H^\rho_{\nu\alpha} H^\alpha_{\mu\sigma},
\]

where \( \nabla \) is the metric covariant derivative. The commutator of the covariant derivatives \( \nabla^+ \) is now related to the curvature and the torsion. We shall need its expression for 4-component tensors

\[
[\nabla^+_\mu, \nabla^+_\nu] Y_{\alpha\beta\gamma\delta} = -2 H_{\sigma\mu\nu} \nabla^+ Y_{\alpha\beta\gamma\delta} + R^+{}_{\alpha\mu\nu} Y_{\gamma\beta\delta} - R^+{}_{\beta\mu\nu} Y_{\alpha\gamma\delta} - R^+{}_{\gamma\mu\nu} Y_{\alpha\beta\delta} - R^+{}_{\delta\mu\nu} Y_{\alpha\beta\gamma},
\]

showing that \( H \) plays indeed the role of a torsion tensor.

As for any connection, the curvature satisfies a Bianchi identity. In our case, it can be worked out as

\[
\nabla^+_\rho R^+{}_{\alpha} = -2 H^\sigma_{[\nu} R^+{}_{\alpha]} - 2 H^\sigma_{[\nu} R^+{}_{\alpha]} = 0,
\]

where \( '[\lambda]' \) means ‘no permutation in \( \lambda \).’ Since the three-form \( H \) is closed, \( dH = 0 \), the curvature tensor satisfies the identity

\[
\nabla^+_\rho H_{\nu\rho} = \frac{3}{2} R^+{}_{[\nu\rho\sigma]}.\]
The Ricci tensor \( R^+_{\mu\nu} = R^+_{\mu\nu} \) is found to be

\[
(2.8) \quad R^+_{\mu\nu} = R_{\mu\nu} + \nabla_\alpha H^\alpha_{\nu\mu} - H^\rho_{\rho\mu} H^\rho_{\nu\alpha}.
\]

Its symmetric part, \( R_{\mu\nu} - H^\rho_{\rho\mu} H^\rho_{\nu\alpha} \), is recognized here as the first-order contribution to the metric \( \beta \)-function, and the antisymmetric part, \( \nabla_\alpha H^\alpha_{\nu\sigma} \), is the contribution for the \( b_{\mu\nu} \)-\( \beta \)-function [6, 7]. The first-order anomaly-cancellation condition is, therefore, unified into

\[
(2.9) \quad R^+_{\mu\nu} = 0,
\]

which generalizes the standard vacuum Einstein equations to connections with torsion.

3. Solvable examples.

As an example, let us consider the non-symmetric tensor

\[
(3.1) \quad t_{\mu\nu} = \begin{pmatrix} \delta_{ij} & 0 & a_i + b_i \\ 0 & 0 & -2 \\ a_j - b_j & 0 & k \end{pmatrix} = \begin{pmatrix} \delta_{ij} & 0 & a_i \\ 0 & 0 & -1 \\ a_j - 1 & k & -b_j & 1 & 0 \end{pmatrix},
\]

which clearly describes an axion field \( b_{\mu\nu}(x, u) \), with non-trivial \( iu \) components in a Brinkmann metric \( dx^2 + 2a_i(x, u)dx^i + k(x, u)du^2 - 2du dv \).

The components of the connection are readily calculated to give the Riemann curvature tensor. The non-zero components of this tensor are

\[
R^+_{jk \alpha} = \frac{1}{2} \partial_\alpha \left[ \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} + \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} \right] \delta^{ij},
\]

\[
R^+_{uj \alpha} = -\frac{1}{2} \partial_\alpha \left[ \frac{\partial a_j}{\partial x^u} - \frac{\partial a_u}{\partial x^j} + \frac{\partial b_j}{\partial x^u} - \frac{\partial b_u}{\partial x^j} \right] \delta^{ju},
\]

\[
R^+_{ji \alpha} = \left[ \partial_j \left( \frac{\partial a_i}{\partial u} - \frac{1}{2} \frac{\partial k}{\partial x^i} \right) - \frac{1}{2} \partial_u \left( \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} + \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} \right) \right] \delta^{ij},
\]

as well as those obtained from these by antisymmetry. The first-order conditions for the anomaly cancellation are, therefore,

\[
(3.3) \quad \left\{ \begin{array}{l}
R^+_{ju} = \frac{1}{2} \partial^u \left[ \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} + \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} \right] = 0, \\
R^+_{aj} = -\frac{1}{2} \partial^j \left[ \frac{\partial a_i}{\partial x^a} - \frac{\partial a_a}{\partial x^j} + \frac{\partial b_i}{\partial x^a} - \frac{\partial b_a}{\partial x^j} \right] = 0, \\
R^+_{uu} = \partial^u \left( \frac{\partial a_i}{\partial u} - \frac{1}{2} \frac{\partial k}{\partial x^i} \right) - \frac{1}{4} \left( \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} \right)^2 + \frac{1}{4} \left( \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} \right)^2 = 0.
\end{array} \right.
\]

(Here the square of \( f_{ij} \) means \( f_{ij} f^{ij} \).)
These equations can be solved. Let, for example, both the vector potential and the axion field be linear in the transverse coordinates,

\begin{equation}
\begin{aligned}
a_i = \frac{1}{2} a_{ij}(u)x^j \\
b_i = \frac{1}{2} b_{ij}(u)x^j.
\end{aligned}
\end{equation}

Then the only non-zero component of $H$ is $H_{iju} = \frac{1}{2}(b_{ij} - b_{ji})$. Similarly, $\partial_i a_j - \partial_j a_i = \frac{1}{2}(a_{ij} - a_{ji})$. The two upper equations in (3.3) are identically satisfied, and the last one reduces to the Poisson equation

\begin{equation}
\Delta k = \frac{1}{2}(a_{ij}a^{ij} - b_{ij}b^{ij}),
\end{equation}

which generalizes the constraint given in Ref. [5] to the case of a non-zero vector potential.

So far we have only studied the first-order condition for anomaly cancellation. Adapting an argument of Horowitz and Steif [5] to our situation, we can prove, however, that, for the linear choice (3.4), the higher order terms in the perturbation expansion of the anomaly vanish at all orders.

These terms are in fact tensors with two indices, constructed from the Riemann tensor $R^+$, its covariant derivatives, and from $H$. (The covariant derivatives of $H$ can be expressed in terms of $R^+$, by Eq. (2.7)). Observe that all components of the Riemann tensor (3.2) with three spatial indices are zero because the $a_{ij}$ are now assumed position-independent. Similarly, $H$ is independent of the transverse coordinates and its nonvanishing components have an $u$ index. Therefore, all tensors with two indices, composed only from the powers of $R^+$ and of $H$, will be zero. The reason is that these expressions will involve summation for at least one of the $u$ indices of these tensors, which yields zero, as a consequence of the particular form of the metric.

As an illustration of this mechanism, we show here that the ‘square’ of the $H$ tensor is zero. Let us introduce the notation $H_\mu^\nu = H_\mu^{\rho\kappa}H_{\nu\rho\kappa}$, and let $H^2 = (H^2)_\mu^\nu = H_{\mu\nu}\rho\sigma$. The only non-zero component of $(H^2)_\mu^\nu$ is $(H^2)_{uu}$. Therefore, $(H^2)^u_u = g^{u\alpha}(H^2)_{u\alpha} = 0$ since $g^{uu} = 0$. The vanishing of all higher powers can be shown similarly.

Next, in terms containing $R^+_{\mu\rho\sigma}$ and its derivatives, one has to contract at least one $u$-index, which can either be an index of $R^+_{\mu\rho\sigma}$ or of $\nabla^+_{\mu}$. But all terms with at least one upper $u$ index vanish. Those terms constructed out of $R^+$ and from $H$ vanish in the same way.

Finally, $\nabla^+_{\nu}\nabla^+_{\rho}R^+_{\mu\rho\sigma}$ are related to the covariant derivatives of the Ricci tensor by the (generalized) Bianchi identity. Contracting Eq. (2.6) in the indices $\nu$ and $\alpha$, we get in fact

\begin{equation}
\nabla^+_{\nu}R^+_{\lambda\beta\kappa} - \nabla^+_{\beta}R^+_{\lambda\kappa} + \nabla^+_{\kappa}R^+_{\lambda\beta} + 2H^\beta_{\gamma}R^+_{\nu\lambda\kappa} - 2H^\kappa_{\gamma}R^+_{\nu\lambda\beta} - 2H^\sigma_{\gamma}R^+_{\nu\lambda\sigma} = 0.
\end{equation}

The sum $2H^\beta_{\gamma}R^+_{\nu\lambda\kappa} - 2H^\kappa_{\gamma}R^+_{\nu\lambda\beta}$ is antisymmetric in $\beta$ and $\kappa$. But, after the summation, these last indices only can be $u$ and add, therefore, to zero. Thus, the generalized Einstein equation $R^+_{\mu\nu} = 0$ implies hence the vanishing of the covariant derivatives of any order of $\nabla^+_{\nu}R^+_{\lambda\beta\kappa} = 0$. Then $\nabla^+_{\nu}\nabla^+_{\rho}R^+_{\mu\rho\sigma}$ vanishes also, since interchanging the order of the covariant derivatives introduces the curvature and the torsion, cf. Eq. (2.5), and such terms have already been shown to vanish.

Let us mention that if $k$ is quadratic in the transverse coordinates, $k = k_{ij}(u)x^ix^j$, which corresponds to having an exact plane wave, the anomaly cancellation can be shown even non-perturbatively [9], using the standard technique [4-5].

Observe that the vector-potential and the axion enter the formulae (3.3) with opposite signs, opening the possibility for some cancellations to take place. Another, even more interesting, solution can be found in fact by choosing

\begin{equation}
a_i = \pm b_i,
\end{equation}
so that the $t_{\mu\nu}$ form an upper (lower) triangular matrix. (The vector-potential can now be an arbitrary function of $x$ and $u$). Choose, e.g., the plus sign. Then the second eqn. in (3.3) is automatically satisfied, while the two others reduce to

$$\begin{align*}
R^+_{ju} &= \partial^j \left( \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} \right) = 0, \quad (3.8) \\
R^+_{au} &= \partial^i \left( \frac{\partial a_i}{\partial u} - \frac{1}{2} \frac{\partial}{\partial x^i} \right).
\end{align*}$$

The first of these equations is a vacuum Maxwell equation,

$$\partial_i f_{ij} = 0 \quad \text{where} \quad f_{ij} = \partial_i a_j - \partial_j a_i, \quad (3.9)$$

and the second is

$$\partial_a \text{div} a - \frac{1}{2} \Delta k = 0. \quad (3.10)$$

Inserting any solution of the vacuum Maxwell equation (3.9) into Eq. (3.10) we get a Poisson equation for $k$; in the Coulomb gauge $\text{div} a = 0$ one even gets Laplace’s equation. All such solutions satisfy the anomaly cancellation condition at the one-loop level.

Remarkably, our ‘upper-triangular’ choice works also at the two-loop level. Using the results of Osborn [1], the two-loop contribution to the anomaly can in fact be worked out to get

$$(3.11) -2R^+_{\lambda\mu\nu} R^+_{\alpha\kappa\lambda\mu} + 3R^+_{\lambda[\mu\nu]} R^+_{\alpha\kappa\lambda\mu} + 4(H^2)^{\kappa\lambda} R^+_{\alpha\kappa\beta\lambda} + 2q \nabla^\alpha \partial_\beta H^2,$$

where $q$ is a constant, and $H^2_{\mu\nu}$, and $H^2$ was defined earlier. (The term proportional to $q$ can actually be eliminated by an appropriate redefinition of the fields [1]). From (3.2) one sees that for the triangular choice $a_i = b_i$ one has $R^+_{ijkl} = 0$, so that the two first terms in Eq. (3.11) are zero. Next, $(H^2)^{\mu\nu}$ is non-zero only for $\mu = \nu = v, (H^2)^{vv}$. But $R^+$ with at least one lower v-index is zero, because the metric admits a covariantly constant null vector. Therefore, the third term vanishes. Finally, the last term vanishes also, since $H^2 = 0$ as we have seen above.

One may wonder if the anomaly vanishes at all orders without requiring the vector potential to be linear. This is not completely unlikely, due to the special structure coming from to the complex structure of Osborn.

One can easily see that if all higher-order terms of the perturbation expansion of the $\beta$-function have (after an appropriate redefinition of the fields) the form

$$\beta_{\mu\nu} = Y^{+\lambda\rho}_\mu \nu R^+_{\mu\sigma\lambda\rho} \quad (3.12)$$

for some tensor $Y^{+\lambda\rho}_\mu \nu$, then the triangular configurations are anomaly free at all orders. The only non-zero components of the Riemann tensor $R^+$ are in fact $R^+_{ijk\mu}$, $R^+_{ij\mu\nu}$, and those coming from the antisymmetry in the first and second pairs of indices. Thus, one of the summation indices $\sigma\lambda\rho$ should be an $u$; but then the term in (3.12) is zero as we have seen above.

Another soluble case would arise by replacing Eq. (3.7) by $a_i \mp b_i = c_{ij}(u)x^j$ with $c_{ij}$ antisymmetric.

Let us mention finally, that a dilaton can be included as well.

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Note added. After this work was completed, we discovered a recent paper by Tseltlin [10], containing some similar ideas. We also learned that the relation between the axion and the torsion was considered before by E. Braaten, T. L. Curtright and C. K. Zachos, Nucl. Phys. B260, 630 (1985), and that our formula (3.6) for the two-loop beta-function appears also in R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B293, 385 (1987). We are indebted to Professor A. Tseytlin for these references.

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