ASYMPTOTIC BEHAVIOR OF CLASS GROUPS AND CYCLOTOMIC IWASA THEORY OF ELLIPTIC CURVES

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Abstract. In this article, we study a relation between certain quotients of ideal class groups and the cyclotomic Iwasawa module $X_\infty$ of the Pontrjagin dual of the fine Selmer group of an elliptic curve $E$ defined over $\mathbb{Q}$. We consider the Galois extension field $K^E_n$ of $\mathbb{Q}$ generated by coordinates of all $p^n$-torsion points of $E$, and introduce a quotient $A^E_n$ of the $p$-Sylow subgroup of the ideal class group of $K^E_n$ cut out by the modulo $p^n$ Galois representation $E[p^n]$. We describe the asymptotic behavior of $A^E_n$ by using the Iwasawa module $X_\infty$. In particular, under certain conditions, we obtain an asymptotic formula as Iwasawa’s class number formula on the order of $A^E_n$ by using Iwasawa’s invariants of $X_\infty$.

1. Introduction

Let $E$ be an elliptic curve over $\mathbb{Q}$. For each $N \in \mathbb{Z}_{>0}$, we denote by $E[N]$ the subgroup of $E(\overline{\mathbb{Q}})$ consisting of elements annihilated by $N$. Fix an odd prime number $p$ at which $E$ has good reduction. For each $n \in \mathbb{Z}_{>0}$, we put $K^E_n := \mathbb{Q}(E[p^n])$, and $h_n := \text{ord}_p \#(\text{Cl}(O_{K^E_n}) \otimes \mathbb{Z}_p)$, where $\text{ord}_p$ denotes the additive $p$-adic valuation normalized by $\text{ord}_p(p) = 1$ and $\text{Cl}(O_{K^E_n})$ is the ideal class group of the ring of integers $O_{K^E_n}$. In recent papers [SY1], [SY2], and [Hi], there has been renewal of interest in an asymptotic behavior of the class numbers $\{h_n\}_{n \geq 0}$ along the tower of number fields $K^E_n$. It has been shown that an asymptotic inequality which gives a lower bound of $\{h_n\}_{n \geq 0}$ in terms of the Mordell-Weil rank $\text{rank}_{\mathbb{Z}} E(\mathbb{Q})$ of $E$ (cf. Remark 1.12). For some generalizations of these results including abelian varieties over a number field, see [Ga] and [Oh1]. In these works, the divisible part of the fine Selmer group $\text{Sel}_p(\mathbb{Q}, E[p^\infty])$ (cf. Definition 4.3) plays important roles.

We define a quotient $A^E_n$ of $\text{Cl}(O_{K^E_n}) \otimes \mathbb{Z}_p$, which is cut out by the Galois representation $E[p^n]$ (see (1.2) below). In this paper, we shall describe the asymptotic behavior of $A^E_n$ by using the fine Selmer group $\text{Sel}_p(K_n, E[p^n])$, where we put $K_n := \mathbb{Q}(\mu_{p^n})$. As an application of our result, we shall show an asymptotic formula on the order of $A^E_n$ using Iwasawa’s $\mu$ and $\lambda$-invariants of the cyclotomic Iwasawa module associated with the fine Selmer group of the elliptic curve $E$, as “Iwasawa’s class number formula” ([Iw]).

1.1. The statements of the main results. In order to state our main results, let us introduce some notation. For each $N \in \mathbb{Z}_{>0}$, we denote by $\mu_N := \mu_N(\overline{\mathbb{Q}})$ the group of $N$-th roots of unity. For each $m \in \mathbb{Z}_{\geq 0}$, we define $K_m := \mathbb{Q}(\mu_{p^m})$ (in particular,
we put $K_0 := \mathbb{Q}$, and set $K_\infty := \bigcup_{m \geq 0} K_m$. For each $m_1, m_2 \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with $m_2 > m_1$, we set $G_{m_2,m_1} := \text{Gal}(K_{m_2}/K_{m_1})$, and put $\Delta := G_{1,0} \simeq (\mathbb{Z}/p\mathbb{Z})^\times$. For any $m \geq 1$, we have $G_{m,0} = \Delta \times G_{m,1}$. We can regard $\mathbb{Z}_p[\Delta]$ as a subring of $\mathbb{Z}_p[G_{m,0}]$. We put $\hat{\Delta} := \text{Hom}(\Delta, \mathbb{Z}_p^\times)$. For each $\chi \in \hat{\Delta}$, we define $E_p(\chi) := E_p$ to be the $\mathbb{Z}_p[\Delta]$-algebra where $\Delta$ acts via $\chi$, and for a $\mathbb{Z}_p[\Delta]$-module $M$, we set $M_\chi := M \otimes_{\mathbb{Z}_p[\Delta]} E_p(\chi)$. We have $M = \bigoplus_{\chi \in \hat{\Delta}} M_\chi$ because $p$ is odd. For each $m, n \in \mathbb{Z}_{\geq 0}$, we define

$$R_{m,n} := \mathbb{Z}/p^n\mathbb{Z}[G_{m,0}] = \mathbb{Z}_p/p^n\mathbb{Z}_p[\text{Gal}(K_m/\mathbb{Q})],$$

and put $R_n := R_{n,n}$. For each number field $L$, that is, a finite extension of $\mathbb{Q}$, and each $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let $\text{Sel}(L, E[p^n])$ be the Selmer group in the classical sense, and $\text{Sel}_p(L, E[p^n])$ the kernel of the localization map

$$\text{Sel}(L, E[p^n]) \longrightarrow \prod_{\wp \mid p} H^1(L, E[p^n])$$

which is called the fine Selmer group (for details, see Definition 4.3 and Remark 4.6, later). For each $m, n \in \mathbb{Z}_{\geq 0}$, the fine Selmer group $\text{Sel}_p(K_m, E[p^n])$ becomes an $R_{m,n}$-module. For any $n \in \mathbb{Z}_{\geq 0}$, the field $K^E_n = \mathbb{Q}(E[p^n])$ contains $\mu_{p^n}$ and hence $K^E_n \supseteq K_n = \mathbb{Q}(\mu_{p^n})$ because of the Weil pairing $E[p^n] \times E[p^n] \longrightarrow \mu_{p^n}$ ([Sil, Chapter III, Corollary 8.1.1]). Let

$$(p_n^E)^\vee : \text{Gal}(K^E_n/\mathbb{Q})^{\text{op}} \longrightarrow \text{Aut}_{\mathbb{Z}_p}(E[p^n]^{\vee}) = GL_2(\mathbb{Z}/p^n\mathbb{Z})$$

be the right action of $G_Q$ on the Pontrjagin dual $E[p^n]^{\vee} := \text{Hom}_{\mathbb{Z}_p}(E[p^n], \mathbb{Z}/p^n\mathbb{Z})$ of $E[p^n]$. We define a $\mathbb{Z}_p$-module $A^E_n$ by

$$(1.1) \quad A^E_n := (M_2(\mathbb{Z}/p^n\mathbb{Z}), (p_n^E)^\vee) \otimes_{\mathbb{Z}_p[\text{Gal}(K^E_n/K_n)]} \text{Cl}((\mathcal{O}_{K^E_n}[1/p]),$$

where $M_2(\mathbb{Z}/p^n\mathbb{Z})$ denotes the matrix algebra of degree two over $\mathbb{Z}/p^n\mathbb{Z}$. We can define a $\mathbb{Z}_p$-linear left action of $G_Q := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $A^E_n$ by

$$\sigma(A \otimes [a]) := A((p_n^E)^\vee(\sigma^{-1}) \otimes [a])$$

for each $\sigma \in G_Q$, $A \in M_2(\mathbb{Z}/p^n\mathbb{Z})$ and $[a] \in \text{Cl}((\mathcal{O}_{K^E_n}[1/p])$. Since every $\sigma \in G_K$ acts trivially on $A^E_n$, we may regard $A^E_n$ as an $R_n$-module. We denote by $(A^E_n)^\vee = \text{Hom}_{\mathbb{Z}_p}(A^E_n, \mathbb{Z}/p^n\mathbb{Z})$ the Pontrjagin dual of $A^E_n$. The following theorem is the main result of our paper.

**Theorem 1.1 (Theorem 5.16).** Let $E$ be an elliptic curve over $\mathbb{Q}$, and $p$ an odd prime number where $E$ has good reduction. Suppose that $E$ satisfies the following conditions (C1), (C2) and (C3).

(C1) The Galois representation $\rho_n^E : G_{K_{\infty}} := \text{Gal}(\overline{\mathbb{Q}}/K_{\infty}) \longrightarrow \text{Aut}_{\mathbb{F}_p}(E[p]) \simeq GL_2(\mathbb{F}_p)$

is absolutely irreducible over $\mathbb{F}_p$.

(C2) For any $n \in \mathbb{Z}_{\geq 1}$ and any place $v$ of $K_n$ where the base change $E_{K_n,v}$ of $E$ has potentially multiplicative reduction, we have $E(K_n,v)[p] = 0$.

(C3) If $E$ has complex multiplication, the ring $\text{End}(E)$ of endomorphisms of $E$ defined over $\mathbb{Q}$ is the maximal order of an imaginary quadratic field.

Then, there exists a family of $R_n$-homomorphisms

$$r_n : \text{Sel}_p(K_n, E[p^n])^{\oplus 2} \longrightarrow (A^E_n)^\vee$$
such that the kernel $\text{Ker}(r_n)$ and the cokernel $\text{Coker}(r_n)$ are finite with order bounded independently of $n$.

**Remark 1.2.** As we see Proposition 3.1 below, the condition (C1) is satisfied if the following condition (C1)$_{\text{str}}$ holds:

(C1)$_{\text{str}}$ The Galois representation

$$\rho^E = \rho^{E,p} : G_\mathbb{Q} \rightarrow \text{Aut}_{\mathbb{Z}}(T_p(E)) \simeq GL_2(\mathbb{Z}_p)$$

is surjective.

Note that if $E$ does not have complex multiplication, then the map $\rho^E$ is surjective for all but finitely many prime number $p$ by Serre’s open image theorem [Se3].

**Remark 1.3.** In § 3, we show that for any elliptic curve $E$ over $\mathbb{Q}$, there exists a quadratic twist $E'/\mathbb{Q}$ of $E$ which satisfies the condition (C2) (Proposition 3.2).

**Remark 1.4.** If the condition (C1) for $E$ is satisfied, then the ring homomorphism $\mathbb{Z}_p[G_{K_n}] \rightarrow M_2(\mathbb{F}_p)$ induced by $\rho^E_1 = (\rho^E \mod p)$ is surjective, where $M_2(\mathbb{F}_p)$ is the matrix algebra of degree two over $\mathbb{F}_p$. Hence, with the aid of Nakayama’s lemma for finitely generated $\mathbb{Z}_p$-modules, the condition (C1) for $E$ implies that the map

$$(\rho^E_1)^\vee : \mathbb{Z}_p[G_{K_n}] \rightarrow M_2(\mathbb{Z}/p^n\mathbb{Z})$$

induced by (1.1) is surjective. Under the assumption of (C1), we can regard $A_n^E$ as a quotient of $\text{Cl}(\mathcal{O}_{K_E})$.

**Remark 1.5.** For each $n \in \mathbb{Z}_{\geq 1}$, we define an $R_n$-module

$$S_n := \text{Hom}_{\mathbb{Z}_p[\text{Gal}(K_{E,F}/K_n)]}(\text{Cl}(\mathcal{O}_{K_E}^F[1/p]) \otimes_{\mathbb{Z}} \mathbb{Z}_p, E[p^n]).$$

In § 5, we shall prove Theorem 1.1 by constructing $\text{Gal}(K_n/\mathbb{Q})$-homomorphisms

$$\text{Sel}_p(K_n, E[p^n])^{\oplus 2} \rightarrow S_n^{\oplus 2} \leftarrow \leftarrow (A_n^E)^\vee,$$

where the orders of the kernel and the cokernel of the former map are bounded and the latter is an isomorphism.

**Remark 1.6.** In [PS], under certain assumptions on $(E, p)$, Prasad and Shekhar studied a relation between $\text{Sel}_p(\mathbb{Q}, E[p])$ and $\widetilde{S} := \text{Hom}_{\mathbb{Z}_p}(\text{Cl}(\mathcal{O}_{K_E}^F) \otimes_{\mathbb{Z}} \mathbb{F}_p, E[p])$. Here, we give a remark on a relation between $\widetilde{S}$ and our $A_n^E$. Let $1 \in \Delta$ be the trivial character. Note that $S_{1,1}$ in the sense of Remark 1.5 is an $\mathbb{F}_p$-subspace of $\widetilde{S}$. Moreover, if $E(\mathbb{Q}_p)[p] = \{0\}$, then the natural injection $S_{1,1} \hookrightarrow \widetilde{S}$ becomes an isomorphism. Indeed, in such case, for any $f \in \widetilde{S}$ and any prime ideal $\mathfrak{p}$ of $K_E^F$, it follows from the comparison of the action of the decomposition group at $\mathfrak{p}$ in $\text{Gal}(K_E^F/\mathbb{Q})$ that we have $f([\mathfrak{p}] \otimes 1) = 0$. Hence by Remark 1.5, we deduce that if $E(\mathbb{Q}_p)[p] = \{0\}$, then we have $A_{1,1}^E \simeq \widetilde{S}^{\oplus 2}$.

Here, we shall note that Theorem 1.1 gives a description of the asymptotic behavior of the higher Fitting ideals of the $\mathbb{Z}_p$-modules $A_n^E$. Let $M$ be a finitely generated $\mathbb{Z}_p$-module. For each $i \in \mathbb{Z}_{\geq 0}$, we denote by $\text{Fitt}_{\mathbb{Z}_p,i}(M)$ the $i$-th Fitting ideal of $M$ (cf. Definition 2.1), and put

$$\Phi_i(M) := \text{ord}_p(\text{Fitt}_{\mathbb{Z}_p,i}(M)) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$
The sequence \( \{ \Phi_i(M) \}_{i \geq 0} \) determines the isomorphism class of the \( \mathbb{Z}_p \)-module \( M \) (see Remark 2.4). There is an equality \( \Phi_i(A_{n,\chi}^E) = \Phi_i((A_{n,\chi-1}^E)') \) for any \( \chi \in \hat{\Delta} \) because \( A_{n,\chi}^E \) is non-canonically isomorphic to \( (A_{n,\chi-1}^E)' = \text{Hom}_{\mathbb{Z}_p}(A_{n,\chi-1}^E, \mathbb{Z}/p^n\mathbb{Z}) \) as a \( \mathbb{Z}_p \)-module. Similarly, we have \( \Phi_i(A_{n,n}^E) = \Phi_i((A_{n,n}^E)' ) \). Let \( \{ a_n \}_n \) and \( \{ b_n \}_n \) be sequences of real numbers. We write \( a_n > b_n \) if it holds that \( \lim \inf_{n \to \infty} (a_n - b_n) > -\infty \), namely, if the sequence \( \{ a_n - b_n \}_n \) is bounded below. If \( a_n > b_n \) and \( b_n > a_n \), then we write \( a_n \sim b_n \). For a family of homomorphisms \( f_n : M_n \to M'_n \) of finitely generated torsion \( \mathbb{Z}_p \)-modules if the order of \( \text{Ker}(f_n) \) and that of \( \text{Coker}(f_n) \) are bounded independently of \( n \), then we have \( \Phi_i(M_n) \sim \Phi_i(M'_n) \) for any \( i \in \mathbb{Z}_{\geq 0} \) (Lemma 2.8). Theorem 1.1 implies the following corollary:

**Corollary 1.7.** Let \( E \) be an elliptic curve over \( \mathbb{Q} \), and \( p \) an odd prime number where \( E \) has good reduction. Suppose that \( E \) satisfies the conditions (C1), (C2) and (C3). Then, for any \( i \in \mathbb{Z}_{\geq 0} \) and \( \chi \in \hat{\Delta} \), it holds

\[
\Phi_i(A_{n,\chi}^E) = \Phi_i((A_{n,\chi-1}^E)') \sim \Phi_i \left( \text{Sel}_p(K_n, E[p^n])^{\oplus 2} \right),
\]

and moreover, we have \( \Phi_i(A_{n,n}^E) = \Phi_i((A_{n,n}^E)') \sim \Phi_i \left( \text{Sel}_p(K_n, E[p^n])^{\oplus 2} \right). \)

1.2. **Asymptotic formulas as Iwasawa’s class number formula.** For each \( \chi \in \hat{\Delta} \), we put \( h_{n,\chi}^E := \text{ord}_p(\#A_{n,\chi}^E) \), and \( h_n^E := \text{ord}_p(\#A_n^E) = \sum_{\chi \in \hat{\Delta}} h_{n,\chi}^E \). Since \( A_n^E \) is a quotient of \( \text{Cl}(O_{K_n^E}) \) as noted in Remark 1.4, we have

\[
h_n := \text{ord}_p(\# \text{Cl}(O_{K_n^E}) \otimes \mathbb{Z}_p) \geq h_n^E.
\]

As we shall see below, Corollary 1.7 for \( i = 0 \) gives a description of asymptotic behavior of \( h_n^E \) like “Iwasawa’s class number formula”. Let us introduce Iwasawa theoretic notation. We put \( \Gamma := \mathcal{G}_{\infty,1} = \text{Gal}(K_\infty/K_1) \). There is a non-canonical isomorphism \( \Gamma \simeq \mathbb{Z}_p \) and fix a topological generator \( \gamma \in \Gamma \). We set \( \Lambda := \mathbb{Z}_p[\Gamma] \). There exists an isomorphism \( \Lambda \xrightarrow{\sim} \mathbb{Z}_p[T] \) of \( \mathbb{Z}_p \)-algebras sending \( \gamma \) to \( 1 + T \). For each \( m, n \in \mathbb{Z}_{\geq 0} \), we define

\[
\Lambda_{m,n} := \mathbb{Z}/p^n\mathbb{Z}[\mathcal{G}_{m,1}] \simeq \Lambda / (p^n, \gamma p^{m-1} - 1),
\]

and put \( \Lambda_n := \Lambda_{0,n} \). Since we have \( \mathcal{G}_{m,0} = \Delta \times \mathcal{G}_{m,1} \), the equality \( R_{m,n} = \Lambda_{m,n}[\Delta] \) holds. In the following, we introduce the Iwasawa module of the Pontrjagin dual of the fine Selmer groups. Write

\[
\text{Sel}_p(K_\infty, E[p^{\infty}]) := \lim_{m \to \infty} \text{Sel}_p(K_m, E[p^{\infty}]).
\]

For any \( m, n \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \), define

\[
X_{m,n} := \text{Sel}_p(K_m, E[p^n])' := \text{Hom}_{\mathbb{Z}_p}(\text{Sel}_p(K_m, E[p^n]), \mathbb{Q}_p/\mathbb{Z}_p),
\]

and put \( X_n := X_{0,n} \). It is known that the \( \Lambda \)-module \( X_\infty \) is finitely generated and torsion (\([\text{I}3\text{a}]\)). Take any \( \chi \in \hat{\Delta} \). The control theorem of the fine Selmer groups (Corollary 4.9) implies that

\[
\Phi_0(X_{\infty,\chi} \otimes \Lambda_n) \sim \Phi_0(X_n, \chi) \sim \Phi_0 \left( \text{Sel}_p(K_n, E[p^n])^{\oplus 2} \right). \tag{1.3}
\]

Since \( X_{\infty,\chi} \) is a finitely generated torsion \( \Lambda \)-module, we can define Iwasawa’s \( \mu \) and \( \lambda \)-invariants \( \mu(X_{\infty,\chi}) \) and \( \lambda(X_{\infty,\chi}) \) of the \( \Lambda \)-module \( X_{\infty,\chi} \): for any finitely generated torsion \( \Lambda \)-module \( M \), the characteristic ideal \( \text{char}_{\Lambda}(M) \) of the \( \Lambda \)-module \( M \) is generated
by an element $p^{\mu(M)}f(\gamma - 1) \in \Lambda$ for some distinguished polynomial $f(T) \in \mathbb{Z}_p[T]$ of degree $\lambda(M)$. By (1.3), we have

$$\Phi_0(X_{\infty, \chi} \otimes \Lambda_n) \sim \mu(X_{\infty, \chi})p^n + \lambda(X_{\infty, \chi})n.$$ 

**Corollary 1.7** for $i = 0$ combined with the additivity of $\Phi_0$, $\mu$ and $\lambda$, implies the following.

**Corollary 1.8.** Let $E$ be an elliptic curve over $\mathbb{Q}$, and $p$ an odd prime number where $E$ has good reduction. Suppose that $E$ satisfies the conditions (C1), (C2) and (C3). Then, for any $\chi \in \hat{\Delta}$, we have

$$h_{n, \chi}^E \sim 2(\mu(X_{\infty, \chi})p^n + \lambda(X_{\infty, \chi})n),$$

and moreover, $h_n^E \sim 2(\mu(X_{\infty})p^n + \lambda(X_{\infty})n)$.

As we note below, by assuming the Iwasawa main conjecture for elliptic curves, the constants $\mu(X_{\infty})$ and $\lambda(X_{\infty})$ are described in terms of Kato’s Euler systems. Let us recall the Iwasawa main conjecture (in the formulation using Kato’s Euler systems). By using Euler systems of Beilinson–Kato elements, Kato constructed a $\Lambda$-submodule $Z$ of $H^1$, where we set

$$H^q = H^q(T_p(E)) := \lim_{\rightarrow m} H^1(K_m, T_p(E))$$

for each $q \in \mathbb{Z}_{\geq 0}$ (or the construction of $Z$, see [Ka, Theorem 12.6] for the Galois representation $T = T_p(E) \subseteq V_{Q_p}(f_E)$, where $f_E$ is the cuspidal attached to $E$). The Iwasawa main conjecture for $(f_E, p, \chi)$ with $\chi \in \hat{\Delta}$ in the sense of [Ka, Conjecture 12.10] (combined with [Ka, Theorem 12.6]) predicts the equality

$$(1.4)\quad \operatorname{char}_{\Lambda}(H^2_{\chi}) = \operatorname{char}_{\Lambda}(H^1_{\chi}/Z_\chi).$$

Since $E$ has good reduction at $p$, for the left hand side of (1.4), we have $\operatorname{char}_{\Lambda}(X_{\infty, \chi}) = \operatorname{char}_{\Lambda}(H^2_{\chi})$ because of the following:

- By the limit of the Poitou-Tate exact sequence, our $X_{\infty}$ coincides with

$$H^2(T_p(E))_m := \operatorname{Ker}(H^2 \rightarrow H^2_{\text{loc}} := \lim_{\rightarrow m} H^1(Q_{p^m}(\mu_{p^m}), T_p(E))).$$

(see, for instance, the proof of [Oh2, Proposition 3.17]).

- When $E$ has good reduction at $p$, the local duality of the Galois cohomology and Imai’s result [im] imply that the order of $H^2_{\text{loc}}$ is finite, and hence the index of $H^2(T_p(E))_m$ in $H^2(T_p(E))$ is finite.

By using the Euler systems, Kato proved that the half side of (1.4), that is, the inclusion

$$\operatorname{char}_{\Lambda}(H^2_{\chi}) \supseteq \operatorname{char}_{\Lambda}(H^1_{\chi}/Z_\chi)$$

holds for any $\chi \in \hat{\Delta}$ under the following condition which is satisfied when (C1)$_{\text{str}}$ holds:

The image of the Galois representation

$$\rho^E|_{G_{K_{\infty}}}: G_{K_{\infty}} \rightarrow \operatorname{Aut}_{\mathbb{Z}_p}(T_p(E)) \simeq GL_2(\mathbb{Z}_p)$$

contains $SL_2(\mathbb{Z}_p)$

(See [Ka, Theorem 13.4]. Note that (C1)$_{\text{str}}$ implies the assumption (3) in [Ka, Theorem 13.4]). By summarizing all $\chi$-parts, the following corollary follows from Corollary 1.8.
Corollary 1.9. Let $E$ be an elliptic curve over $\mathbb{Q}$, and $p$ an odd prime number where $E$ has good reduction.

1. Suppose that $E$ satisfies the conditions $(C1)_{str}$ and $(C2)$. Then, we have
   \[ h_n^E < 2 \left( \mu(H^1/Z) p^n + \lambda(H^1/Z) n \right). \]

2. Suppose that $E$ satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. Let $\chi_0 \in \hat{\Delta}$. Then, if the Iwasawa main conjecture for $(f_E, p, \chi_0)$ holds, we have
   \[ h_{n, \chi_0}^E \sim 2 \left( \mu(H^1_{\chi_0}/Z_{\chi_0}) p^n + \lambda(H^1_{\chi_0}/Z_{\chi_0}) n \right). \]

In particular, if the Iwasawa main conjecture for $(f_E, p, \chi)$ holds for every $\chi \in \hat{\Delta}$, then we have
   \[ h_n^E \sim 2 \left( \mu(H^1/Z) p^n + \lambda(H^1/Z) n \right). \]

Let $1 \in \hat{\Delta}$ be the trivial character. In [SU], Skinner and Urban proved the Iwasawa main conjecture for $(f_E, p, 1)$ with the following conditions:

- The pair $(E, p)$ satisfies $(C1)_{str}$.
- The elliptic curve $E$ has good ordinary reduction at $p$.
- There exists a prime number $\ell_0$ where $E$ has multiplicative reduction.

(See [SU, Theorem 3.33].) These conditions are satisfied when $E$ is semistable, and $p$ is a prime number of good ordinary reduction satisfying $p \geq 11$ (see [SU, Theorem 3.34]).

We obtain the following corollary.

Corollary 1.10. Suppose that $E$ is semistable, and let $p$ be a prime number with $p \geq 11$ where $E$ has good ordinary reduction. If $E$ satisfies the condition $(C2)$, then we have
   \[ h_n^E \sim 2 \left( \mu(H^1_1/Z_1) p^n + \lambda(H^1_1/Z_1) n \right). \]

Let us see the relation between our results and previous works on the asymptotic behavior of $h_n$. By the arguments in [Oh1, §4.1], for any number field $L$, we have
   \[ \text{corank}_{L_p} \text{Sel}_p(L, E[p^\infty]) \geq \text{rank}_L E(L) - [L : \mathbb{Q}]. \]

(Indeed, the fine Selmer group $\text{Sel}_p(L, E[p^\infty])$ contains the kernel of
\[ E(L) \otimes_\mathbb{Z} \mathbb{Q}_p/Z_p \rightarrow E(L \otimes_\mathbb{Q} \mathbb{Q}_p) \otimes_\mathbb{Z} \mathbb{Q}_p/Z_p = \prod_{v|p} E(L_v) \otimes_\mathbb{Z} \mathbb{Q}_p/Z_p, \]
and we have
   \[ \text{corank}_{L_p} \left( \prod_{v|p} E(L_v) \otimes_\mathbb{Z} \mathbb{Q}_p/Z_p \right) = \sum_{v|p} [L_v : \mathbb{Q}_p] = [L : \mathbb{Q}]. \]

By the control theorem of fine Selmer groups (Corollary 4.9 and Remark 4.10), we deduce that
   \[ \lambda(X_\infty) \geq \text{rank}_L E(K_m) - \varphi(p^m) \]
for any $m \in \mathbb{Z}_{\geq 0}$, where $\varphi$ denotes Euler’s totient function. Thus, Corollary 1.8 implies the following.

Corollary 1.11. Let $E$ be an elliptic curve over $\mathbb{Q}$ which has good reduction at an odd prime $p$. Suppose that $E$ satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. Then, for any fixed $m \in \mathbb{Z}_{\geq 0}$, we have
   \[ h_n \geq h_n^E > 2(r_m - \varphi(p^m)) n \]
as $n \to \infty$, where we put $r_m := \text{rank}_L E(K_m)$. 

Remark 1.12. The assertion of Corollary 1.11 for \( m = 0 \) implies the “asymptotic parts” of the results by [SY1], [SY2] and [Hi], and that for general \( m \geq 0 \) implies \([Oh1]\) for the \( p\)-adic representation \( T_p(E) = \lim_{\rightarrow n} E[p^n] \) of \( G_{K_m} \). (Here, the “asymptotic parts” means the assertions without description of constant error factors.) Our results, in particular Theorem 1.1 and Corollary 1.8, can be regarded as a refinement of them in the following senses.

- Corollary 1.8 determines the quotient \( A^E_n \) of the ideal class group \( \text{Cl}(O_{K^E}) \), whose growth is described by the fine Selmer groups.
- Theorem 1.1 describes not only the asymptotic behavior of the order of \( A^E_n \) but also asymptotic behavior of the \( R_n \)-module (and in particular, \( \mathbb{Z}_p \)-module) structure.

Example 1.13. Let \( E \) be the elliptic curve over \( \mathbb{Q} \) of the LMFDB label 5077.a1 (the Cremona label 5077a1), which is defined by the equation

\[
y^2 + y = x^3 - 7x + 6,
\]

and set \( p := 7 \). It is known the following ([LMF]):

(i) The elliptic curve \( E \) does not have CM, and \( (E, p) \) satisfies \( (C1)_{str} \).
(ii) The conductor of \( E \) is 5077, which is a prime number, and \( E \) has non-split multiplicative reduction at 5077.
(iii) The rank of \( E(\mathbb{Q}) \) is 3.
(iv) Let \( \tilde{X} := \text{Sel}(\mathbb{Q}_\infty, E[7\infty])^\vee \) be the Iwasawa module of the Pontrjagin dual of the classical Selmer group of \( E \) over the cyclotomic \( \mathbb{Z}_T \)-extension field \( \mathbb{Q}_\infty \) of \( \mathbb{Q} \). We have \( \mu(\tilde{X}) = 0 \), and \( \lambda(\tilde{X}) = 3 \).

The properties (iii) and (iv) imply that we have \( \text{char}_\Lambda(\tilde{X}) = (\gamma - 1)^3\Lambda \). We further obtain \( \text{char}_\Lambda(X_{\infty,1}) = (\gamma - 1)^2\Lambda \) (see, for instance, [Wu, VI.10]). This implies that \( \mu(X_{\infty,1}) = 0 \) and \( \lambda(X_{\infty,1}) = 2 \). Moreover, we can show that the pair \( (E, p) \) satisfies the condition \( (C2) \) (see Example 3.7 in §3.3). Therefore, we obtain

\[
h^E_{n,1} \sim 2n.
\]

Notation. Let \( L/F \) be a Galois extension with \( G = \text{Gal}(L/F) \), and \( M \) a topological abelian group equipped with a \( \mathbb{Z} \)-linear action of \( G \). For each \( i \in \mathbb{Z}_{\geq 0} \), we denote by \( H^i(L/F, M) = H^i_{\text{cont}}(G, M) \) the \( i \)-th continuous Galois cohomology group. If \( L \) is a separable closure of \( F \), then we write \( H^1(F, M) = H^1(L/F, M) \). When \( F \) is a non-archimedean local field, we denote by \( F^\text{ur} \) the maximal unramified extension of \( F \). We define \( H^i_{ur}(F, M) = \text{Ker}(H^1(F, M) \rightarrow H^1(F^\text{ur}, M)) \) (cf. [Ru, Definition 1.3.1]).

For a \( \mathbb{Z}_p \)-module \( A \), let \( A_{\text{div}} \) denote its maximal divisible subgroup. For an abelian group \( M \) and an endomorphism \( f \) of \( M \), we put \( M[f] := \text{Ker}(f) \). In particular, if \( M \) is a module over a ring \( R \), then, for each \( a \in R \), we set \( M[a] := \{ x \in M \mid ax = 0 \} \). For an elliptic curve \( E \) over a field \( K \) and a field extension \( L/K \), we will denote by \( E_L := E \otimes_K L \) the base change to \( L \).

2. The higher Fitting ideals

Definition 2.1 (cf. [Ei, Section 20.2]). Let \( R \) be a commutative ring, and \( M \) a finitely presented \( R \)-module given by a presentation

\[
R^m \xrightarrow{A} R^n \rightarrow M \rightarrow 0
\]
with \( m \geq n \). We define the \( i \)-th Fitting ideal \( \text{Fitt}_{R,i}(M) \) of the \( R \)-module \( M \) to be the ideal of \( R \) generated by \( (n - i) \times (n - i) \) minors (that is, the determinants of the submatrices) of the matrix \( A \). When \( i \geq n \), we define \( \text{Fitt}_{R,i}(M) := R \).

**Remark 2.2.** The ideal \( \text{Fitt}_{R,i}(M) \) in Definition 2.1 does not depend on the choice of the presentation (2.1) ([Ei, Corollary-Definition 20.4]).

**Remark 2.3.** The higher Fitting ideals are compatible with base change in the following sense: Let \( R \) be a commutative ring, and \( M \) a finitely presented \( R \)-module. Then, for any \( R \)-algebra \( S \) and any \( i \in \mathbb{Z}_{\geq 0} \), we have \( \text{Fitt}_{S,i}(S \otimes_R M) = \text{Fitt}_{R,i}(M)S \) ([Ei, Corollary 20.5]).

**Remark 2.4.** Let \( R \) be a PID, and suppose that \( M \) is a finitely generated \( R \)-module. By the structure theorem of finitely generated modules over a PID, the \( R \)-module \( M \) is isomorphic to an elementary \( R \)-module \( R^{d_1} \oplus \bigoplus_{j=1}^s R/d_jR \) with a sequence \( \{ d_j \} \subseteq R \cap R^\times \) satisfying \( d_j | d_{j-1} \) for every \( j \). We have

\[
\text{Fitt}_{R,i}(M) = \begin{cases} 
\{ 0 \} & \text{if } i < r, \\
\left( \prod_{j=i-r+1}^s d_j \right) R & \text{if } r \leq i < s + r, \\
R & \text{if } i \geq s + r.
\end{cases}
\]

(2.2)

In particular, the higher Fitting ideals \( \{ \text{Fitt}_{R,i}(M) \} \) determine the isomorphism class of the \( R \)-module \( M \).

**Remark 2.5.** Let \( R \) be a commutative ring, and \( M \) an \( R \)-module with the presentation (2.1). Let \( N \) be an \( R \)-submodule of \( M \).

1. For any \( i \in \mathbb{Z}_{\geq 0} \), we have \( \text{Fitt}_{R,i}(M) \subseteq \text{Fitt}_{R,i}(M/N) \). Indeed, we have a presentation of \( M/N \) of the form \( R^{m+k} \xrightarrow{(A|B)} R^n \rightarrow M/N \rightarrow 0 \), and every \( (n - i) \times (n - i) \) minor of \( A \) becomes an \( (n - i) \times (n - i) \) minor of the augmented matrix \((A | B)\).

2. Suppose that \( R = \mathbb{Z}_p \), and \( M \) is a torsion \( \mathbb{Z}_p \)-module. For any finitely generated torsion \( \mathbb{Z}_p \)-module \( L \), we denote by \( L^\vee = \text{Hom}_{\mathbb{Z}_p}(L, \mathbb{Q}_p/\mathbb{Z}_p) \) the Pontrjagin dual of \( L \). The dual \( N^\vee \) is a quotient of \( M^\vee \), and there are non-canonical isomorphisms \( M \approx M^\vee \) and \( N \approx N^\vee \). By (1), we have

\[ \text{Fitt}_{R,i}(M) = \text{Fitt}_{R,i}(M^\vee) \subseteq \text{Fitt}_{R,i}(N^\vee) = \text{Fitt}_{R,i}(N) \]

for any \( i \in \mathbb{Z}_{\geq 0} \).

As in § 1, we introduce the following notation:

**Definition 2.6.** Let \( M \) be a finitely generated torsion \( \mathbb{Z}_p \)-module. For each \( i \in \mathbb{Z}_{\geq 0} \), we define

\[ \Phi_i(M) = \text{ord}_p(\text{Fitt}_{\mathbb{Z}_p,i}(M)) := \min \{ m \in \mathbb{Z}_{\geq 0} \mid p^m \in \text{Fitt}_{\mathbb{Z}_p,i}(M) \} \].

If \( M \) is a torsion \( \mathbb{Z}_p \)-module isomorphic to \( \bigoplus_{j=1}^s \mathbb{Z}_p/p^{e_j} \mathbb{Z}_p \) with a decreasing sequence \( \{ e_j \} \subseteq \mathbb{Z}_{\geq 0} \), then

\[
\Phi_i(M) = \begin{cases} 
\sum_{j=i+1}^s e_j & \text{if } 0 \leq i < s, \\
0 & \text{if } i \geq s.
\end{cases}
\]

(2.3)
immediately follows from (2.2). In particular, we have \( \Phi_0(M) = \text{ord}_p(\#M) \). As noted in §1, the isomorphism class of a finitely generated torsion \( \mathbb{Z}_p \)-module \( M \) is determined by \( \{ \Phi_i(M) \}_i \) by (2.3).

**Lemma 2.7.** Let \( M \) be a finitely generated torsion \( \mathbb{Z}_p \)-module. Then, for any \( i \in \mathbb{Z}_{\geq 0} \), we have

\[
\Phi_i(M) = \min_{(a_1, \ldots, a_i) \in M^i} \text{ord}_p \left( \# \left( \frac{M}{\sum_{j=1}^{i} \mathbb{Z}_p a_j} \right) \right).
\]

**Proof.** By the structure theorem, we have \( M = \bigoplus_{j=1}^{s} (\mathbb{Z}/p^{e_j} \mathbb{Z})m_j \), where the sequence \( \{ e_j \} \subseteq \mathbb{Z}_{\geq 0} \) is decreasing. For any \( j \in \mathbb{Z} \) with \( 1 \leq j \leq s \), the annihilator of \( m_j \in M \) is \( p^{e_j} \mathbb{Z}_p \). Fix any \( i \in \mathbb{Z}_{\geq 0} \). If \( i = 0 \) or \( i \geq s \), then the assertion of Lemma 2.7 is clear. Now, we assume \( 1 \leq i \leq s - 1 \). Put \( N_0 := \sum_{j=1}^{i} \mathbb{Z}_p m_j \). We have

\[
\text{ord}_p(\#(M/N_0)) = \text{ord}_p \left( \# \left( \bigoplus_{j=i+1}^{s} (\mathbb{Z}/p^{e_j} \mathbb{Z})m_j \right) \right) = \sum_{j=i+1}^{s} e_j \overset{(2.3)}{=} \Phi_i(M).
\]

Take any \( a_1, \ldots, a_i \in M \), and put \( N := \sum_{j=1}^{i} \mathbb{Z}_p a_j \). In order to prove Lemma 2.7, it suffices to show \( \Phi_i(M) \leq \text{ord}_p(\#(M/N)) \). Let \( \pi_N : \mathbb{Z}_p^i \longrightarrow N \) be the surjection given by the generators \( a_1, \ldots, a_i \in N \), and take a presentation

\[
\mathbb{Z}_p^k \xrightarrow{A} \mathbb{Z}_p^i \xrightarrow{\pi_N} N \longrightarrow 0
\]

for some \( k \geq 1 \). Since \( M/N \) is a torsion \( \mathbb{Z}_p \)-module, there is a square presentation

\[
0 \longrightarrow \mathbb{Z}_p^l \xrightarrow{B} \mathbb{Z}_p^t \longrightarrow M/N \longrightarrow 0
\]

by the structure theorem. This gives a presentation

\[
\mathbb{Z}_p^{k+l} \xrightarrow{C} \mathbb{Z}_p^{l+t} \longrightarrow M \longrightarrow 0
\]

with \( C = \begin{pmatrix} A & * \\ O & B \end{pmatrix} \). We obtain \( \#(M/N) = \det B \in \text{Fitt}_{\mathbb{Z}_p^i}(M/N) \). This implies \( \#(M/N) \geq \Phi_i(M) \). \( \square \)

Let \( \{ a_n \}_n \) and \( \{ b_n \}_n \) be sequences of real numbers. We write \( a_n \succ b_n \) if it holds that \( \liminf_{n \to \infty} (a_n - b_n) > -\infty \), namely, if the sequence \( \{ a_n - b_n \}_n \) is bounded below. If \( a_n \succ b_n \) and \( b_n \succ a_n \), then we write \( a_n \sim b_n \).

**Lemma 2.8.** Let \( \{ M_n \}_{n \geq 0} \) be a sequence of finitely generated torsion \( \mathbb{Z}_p \)-modules, and suppose that for each \( n \in \mathbb{Z}_{\geq 0} \), a \( \mathbb{Z}_p \)-submodule \( N_n \) of \( M_n \) is given. Then, the following hold.

1. If \( \{ (M_n : N_n) \}_{n \geq 0} \) is bounded, then we have \( \Phi_i(M_n) \sim \Phi_i(N_n) \) for any \( i \in \mathbb{Z}_{\geq 0} \).
2. If \( \{ \#N_n \}_{n \geq 0} \) is bounded, then we have \( \Phi_i(M_n) \sim \Phi_i(M_n/N_n) \) for any \( i \in \mathbb{Z}_{\geq 0} \).

**Proof.** Let us show the assertion (1). Suppose that there exists some \( B \in \mathbb{Z}_{\geq 0} \) such that \( (M_n : N_n) \leq p^B \) for any \( n \in \mathbb{Z}_{\geq 0} \). Since \( N_n \) is a submodule of \( M_n \), by Remark 2.5...
(2), we have $\Phi_i(M_n) \geq \Phi_i(N_n)$. In order to prove the assertion (1), it suffices to show that $\Phi_i(M_n) \leq \Phi_i(N_n) + B$. By Lemma 2.7, there exist $a_1, \ldots, a_i \in N_n$ such that

$$\text{ord}_p \left( \# \left( \frac{N_n}{\sum_{j=1}^i \mathbb{Z}_p a_j} \right) \right) = \Phi_i(N_n).$$

Since $(M_n : N_n) \leq p^B$, Lemma 2.7 implies that

$$\Phi_i(M_n) \leq \text{ord}_p \left( \# \left( \frac{M_n}{\sum_{j=1}^i \mathbb{Z}_p a_j} \right) \right) \leq \Phi_i(N_n) + B.$$

Accordingly, we obtain $\Phi_i(M_n) \sim \Phi_i(N_n)$, and the assertion (1) is verified. By taking the Pontrjagin dual, the assertion (2) immediately follows from (1). \hfill $\square$

3. The conditions (C1) and (C2)

Until the end of this note, we use the following notation: Fix an odd prime number $p$. Let $E$ be an elliptic curve over $\mathbb{Q}$. We denote by $D_E$ the discriminant of the minimal Weierstrass model for $E$ over $\mathbb{Z}$. We define the $p$-adic Tate module $T_p(E)$ by $T_p(E) := \lim_{\to} E[p^n]$, and put $V_p(E) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(E)$. As in §1, for each $n \in \mathbb{Z}_{\geq 0}$, we define $K_n^E = \mathbb{Q}(E[p^n])$, and $K_n = \mathbb{Q}(\mu_{p^n})$. Put also $K_\infty = \bigcup_{n \geq 0} K_n^E$ and $K_\infty = \bigcup_{n \geq 0} K_n$.

In this section, we review some results on the conditions (C1) and (C2) referred in Theorem 1.1 under the additional assumption that $E$ has good reduction at $p$. First, we recall the conditions:

(C1) The restriction $\rho^E_1 : G_{K_\infty} \longrightarrow \text{Aut}_{\mathbb{F}_p}(E[p]) \simeq GL_2(\mathbb{F}_p)$

to $G_{K_\infty}$ of the mod $p$ Galois representation $\rho^E_1 : G_\mathbb{Q} \longrightarrow \text{Aut}_{\mathbb{F}_p}(E[p])$ is absolutely irreducible over $\mathbb{F}_p$.

(C2) For any $n \in \mathbb{Z}_{\geq 1}$ and any place $v$ of $K_n$ with the base change $E_{K_n,v}$ of $E$ has potentially multiplicative reduction, we have $E(K_{n,v})[p] = 0$.

3.1. Remarks on (C1) and (C2). In this paragraph, we shall show some properties relates to (C1) and (C2) mentioned in §1. First, let us verify the following property, which is noted in Remark 1.2.

Proposition 3.1. The condition (C1) is satisfied if the Galois representation

$$\rho^E : G_\mathbb{Q} \longrightarrow \text{Aut}_{\mathbb{Z}_p}(T_p(E)) \simeq GL_2(\mathbb{Z}_p)$$

is surjective.

Proof. It is enough to show that the image of $\rho^E : G_{K_\infty} \longrightarrow GL_2(\mathbb{F}_p)$ generates $\text{End}_{\mathbb{F}_p}(E[p]) \simeq M_2(\mathbb{F}_p)$ over $\mathbb{F}_p$. By using the Weil pairing, $G_\mathbb{Q}$ acts on $\bigwedge^2_{\mathbb{Z}_p} T_p(E) \simeq T_p(\mu)$ via the cyclotomic character $\chi$, where $T_p(\mu) := \lim_{\to} \mu_{p^n}$ (cf. [Si1, Chapter V, Section 2]). We obtain the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G_{K_\infty} & \longrightarrow & G_\mathbb{Q} & \longrightarrow & \text{Gal}(K_\infty/\mathbb{Q}) & \longrightarrow & 0 \\
& & \downarrow \rho^E & & \downarrow \simeq \chi & & & & \\
0 & \longrightarrow & \text{SL}_2(\mathbb{Z}_p) & \longrightarrow & GL_2(\mathbb{Z}_p) & \longrightarrow & \mathbb{Z}_p^\times & \longrightarrow & 0.
\end{array}
$$
The assumption implies that the image of the restriction \( \rho^E|_{G_{K_w}} \) coincides with \( SL_2(\mathbb{Z}_p) \). By taking the mod \( p \) reduction, \( SL_2(\mathbb{F}_p) = \rho^E_1(G_{K_w}) \) and this generates \( M_2(\mathbb{F}_p) \) over \( \mathbb{F}_p \). □

Next, let us see the following property referred in Remark 1.3.

**Proposition 3.2.** There exists a quadratic twist \( E'/\mathbb{Q} \) of \( E \) which satisfies the condition (C2).

**Proof.** For each prime number \( \ell \), put \( L_\ell := \mathbb{Q}_\ell(\mu_{p^\infty}) \). Suppose that \( E \) is defined by the Weierstrass equation \( y^2 = x^3 + ax + b \) with \( a, b \in \mathbb{Q} \), and let \( S(E) \) be the set of all the prime numbers at which \( E \) has potentially multiplicative reduction. As \( E \) has good reduction at \( p \), we have \( p \not\in S(E) \). For each \( \ell \in S(E) \), we fix an embedding \( \iota_\ell: \mathbb{Q} \hookrightarrow \mathbb{Q}_\ell \), and regard \( \mu_{p^\infty} \) as a subgroup of \( \mathbb{Q}_\ell^\times \). Note that under these notations, the elliptic curve \( E \) satisfies the condition (C2) if and only if \( E(L_\ell)[p] = 0 \) for any \( \ell \in S(E) \). In order to show the assertion of Proposition 3.2, we may suppose that \( E \) does not satisfy the condition (C2). In particular, the set \( S(E) \) is not empty. We define

\[
S_0(E) := \{ \ell \in S(E) \mid E(L_\ell)[p] \neq 0, 2 \nmid |\mathbb{Q}_\ell(\mu_p):\mathbb{Q}_\ell| \}, \\
S_1(E) := \{ \ell \in S(E) \mid E(L_\ell)[p] \neq 0, 2 \nmid |\mathbb{Q}_\ell(\mu_p):\mathbb{Q}_\ell| \}
\]

and put \( N_1^* := \prod_{\ell \in S_1(E)} (\ell')^* \), where for each odd prime number \( \ell \), we write \( \ell^* := (-1)^{\frac{\ell-1}{2}} \ell^* \), and put \( 2^* := 2 \). For each odd \( \ell \in S(E) \setminus S_1(E) \), we put

\[
\varepsilon_\ell := \prod_{\ell' \in S_1(E)} \left( \left( \frac{\ell'}{\ell} \right) \right),
\]

where \( \left( \frac{\cdot}{\ell} \right) \) denotes the Legendre symbol modulo \( \ell \). By Dirichlet’s theorem on arithmetic progressions, there exists an odd prime number \( q \) prime to \( p \) such that \( q \equiv 1 \mod 4 \), and

\[
\left( \frac{q}{\ell} \right) = \begin{cases} 
-\varepsilon_\ell & \text{if } \ell \in S_0(E), \\
\varepsilon_\ell & \text{if } \ell \notin S_0(E),
\end{cases}
\]

for any odd \( \ell \in S(E) \setminus S_1(E) \). Moreover, we may suppose that

\[
qN_1^* \equiv \begin{cases} 
1 \mod 8 & \text{if } 2 \in S(E) \setminus (S_0(E) \cup S_1(E)), \\
5 \mod 8 & \text{if } 2 \in S_0(E).
\end{cases}
\]

Take such a prime number \( q \), and let \( E' \) be a quadratic twist of \( E \) defined by the Weierstrass equation \( qN_1^* y^2 = x^3 + ax + b \). We have an equality \( S(E') = S(E) \) because \( E \) and \( E' \) are isomorphic over the field \( \mathbb{Q}(\sqrt{qN_1^*}) \).

Let us show that \( E' \) satisfies (C2). In the following, We prove \( E'(L_\ell)[p] = 0 \) for any \( \ell \in S(E') \). Take any \( \ell \in S(E') \).

(\textbf{The case } \ell \not\in S_0(E) \cup S_1(E)) First, we suppose that \( \ell \) does not belong to \( S_0(E) \cup S_1(E) \).

\textbf{Claim 1.} The prime \( \ell \) splits in \( \mathbb{Q}(\sqrt{qN_1^*})/\mathbb{Q} \).
Claim 2. The action of $G_L$ on $E[p]$ is unipotent.

Proof. Take a basis $\{P, Q\}$ of $E[p]$ as an $\mathbb{F}_p$-vector space with $Q \in E[p] \setminus \mathbb{F}_p P$. Recall that the Weil pairing $\varepsilon : E[p] \times E[p] \to \mathbb{F}_p$ is alternating and $G_L$-equivariant ([Si1, Chapter III, Section 8]). As $\mu_p \subseteq L$, we have $\sigma(e(P, Q)) = e(P, Q)$ for any $\sigma \in G_L$. On the other hand, $\sigma(e(P, Q)) = e(\sigma P, \sigma Q) = e(P, \sigma Q - Q) = 1$. Here, the element of the form $\sigma Q - Q$ is in the kernel of $E[p] \to \mu_p; T \mapsto e(P, T)$ which is generated by $P$ so that $\sigma Q - Q = aP$ for some $a \in \mathbb{F}_p$. According to the fixed basis above, the action of $\sigma$ is written as $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ which is unipotent. \hfill \square

Claim 3. The extension $L_\ell(\sqrt{qN_1^*})/L_\ell$ is quadratic.

Proof. Let us show the claim by dividing into three cases.

(i) Suppose that $\ell \in S_0(E)$, and $\ell$ is odd. The equalities
\[
\left(\frac{qN_1^*}{\ell}\right) = -\prod_{\ell' \in S_1(E)} \left(\frac{\ell'}{\ell}\right) = -1
\]

imply that the prime $\ell$ is inert in the quadratic extension $\mathbb{Q}(\sqrt{qN_1^*})/\mathbb{Q}$. For the prime 2 does not divit $[Q_\ell(\mu_p) : \mathbb{Q}]$, we have $Q_\ell(\sqrt{qN_1^*}) \not\subseteq L_\ell$. Hence, the extension $L_\ell(\sqrt{qN_1^*})/L_\ell$ is non-trivial.

(ii) Suppose that $\ell = 2 \in S_0(E)$. The extension $L_2 = \mathbb{Q}_2(\mu_{p^2})/\mathbb{Q}_2$ does not contain quadratic extension fields of $\mathbb{Q}_2$. Since we have $qN_1^* \equiv 5 \mod 8$, the prime 2 is inert in the extension $\mathbb{Q}(\sqrt{qN_1^*})/\mathbb{Q}$. Thus, the extension $L_2(\sqrt{qN_1^*})/L_2$ is non-trivial.

(iii) Suppose that $\ell \not\in S_0(E)$. Then $\ell \in S_1(E)$ and thus $\ell \mid N_1^*$. This implies that the prime $\ell$ is ramified in the extension $\mathbb{Q}(\sqrt{qN_1^*})/\mathbb{Q}$. We also have $L_\ell(\sqrt{qN_1^*}) \not\subseteq L_\ell$ because $L_\ell/\mathbb{Q}_\ell$ is unramified.

In each cases, the extension $L_\ell(\sqrt{qN_1^*})/L_\ell$ is quadratic. \hfill \square

From Claim 2 above, there exists a basis $\{P, Q\}$ of $E[p]$ as $\mathbb{F}_p$-vector space such that $G_L$ acts trivially on $\mathbb{F}_p P$, and also $E[p]/\mathbb{F}_p P$ which is generated by the residue class represented by $Q \in E[p] \setminus \mathbb{F}_p P$. We have an isomorphism $f : E[p] \otimes \mathbb{F}_p(\psi) \to E'[p]$. If
The following Lemma 3.4 gives some conditions equivalent to (C2).

**Lemma 3.3.** Suppose that $E$ has potentially multiplicative reduction at $\ell$ ($\neq p$). Then, the elliptic curve $E_{K^E_{\ell}}$ has split multiplicative reduction at every place of $K^E_{\ell} = \mathbb{Q}(E[p])$ above $\ell$.

**Proof.** We may assume that the $j$-invariant $j(E)$ is not equal to 0 or 1728 because $E$ has potentially good reduction at all primes in such cases ([Si1, Chapter VII, Proposition 5.5]). By [Si2, Chapter V, Lemma 5.2], there exist elements $q, \gamma \in \mathbb{Q}_{\ell}^\times$ with $\text{ord}_{\ell}(q) > 0$ such that $E_{\ell}(\sqrt{\gamma})$ has split multiplicative reduction, and we have a $G_{\mathbb{Q}_{\ell}}$-equivariant isomorphism

$$f : E[p^\infty] \xrightarrow{\sim} (\mathbb{Q}_{\ell}^\times / q^\infty)[p^\infty] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi),$$

where $\chi : G_{\mathbb{Q}_{\ell}} \to \mathbb{Z}_p^\times$ is the trivial character or the quadratic character attached to the extension $\mathbb{Q}_{\ell}(\sqrt{\gamma})/\mathbb{Q}_{\ell}$. In order to prove the assertion, it is sufficient to show that $\sqrt{\gamma} \in \mathbb{Q}_{\ell}(E(\mathbb{Q}_{\ell}))[p]$. Since the Weil paring

$$E(\mathbb{Q}_{\ell})[p] \times E(\mathbb{Q}_{\ell})[p] \to \mu_p(\mathbb{Q}_{\ell}) = \mu_p$$

preserves the action of $G_{\mathbb{Q}_{\ell}}$, we have $\mu_p \subseteq \mathbb{Q}_{\ell}(E(\mathbb{Q}_{\ell})[p])$. If $\sqrt{\gamma} \notin \mu_p$, then $\sqrt{\gamma} \in \mathbb{Q}_{\ell}(E(\mathbb{Q}_{\ell})[p])$. Suppose that $\sqrt{\gamma} \notin \mu_p$. The fields $F_1 := \mathbb{Q}_{\ell}(\sqrt{\gamma})$ and $\mathbb{Q}_{\ell}(\mu_p)$ are linearly disjoint over $\mathbb{Q}_{\ell}$. Moreover, as $p$ is odd, the fields $F_1$ and $F_2 := \mathbb{Q}_{\ell}(\mu_p, \sqrt{\gamma})$ are linearly disjoint over $\mathbb{Q}_{\ell}$. Put $\tilde{F} := \mathbb{Q}_p(\mu_p, \sqrt{\gamma}, \sqrt{\gamma}) = F_2 F_1$. By the isomorphism $f$, we have $\mathbb{Q}_{\ell}(E(\mathbb{Q}_{\ell})[p], \sqrt{\gamma}) = \tilde{F}$. It holds that $F_2 \subseteq \mathbb{Q}_p(\mu_p, \sqrt{\gamma})$. Consequently, the elliptic curve $E_{K^E_{\ell}}$ has split multiplicative reduction at every place of $K^E_{\ell}$. □

The following Lemma 3.4 gives some conditions equivalent to (C2).

**Lemma 3.4.** Let $\ell$ be a prime number. Suppose that $E$ has potentially multiplicative reduction at $\ell$. Then, the following are equivalent:

(a) For any $n \in \mathbb{Z}_{>1}$ and any place $v$ of $K_n$ above $\ell$, we have $E(K_n,v)[p] = 0$.

(b) For any $n \in \mathbb{Z}_{>1}$ and any place $w$ of $K^E_n$ above $\ell$ where the base change $E_{K^E_{n,w}}$ of $E$ has split multiplicative reduction, we have

$$H^0 \left( K_{n,v}, E(\mathbb{K}_{n,w}^E)[p^\infty]_{\text{div}} \right) = 0.$$

Here, we denote by $v$ the place of $K_n$ below $w$. (Note that the absolute Galois group $G_{K_n,v}$ acts on $E(\mathbb{K}_{n,w}^E)[p^\infty]_{\text{div}}$ because the extension $\mathbb{K}_{n,w}^E/K_n,v$ is Galois.)

(c) For any $n \in \mathbb{Z}_{>1}$ and any place $w$ of $K^E_n$ above $\ell$ at which $E_{K_n^E}$ has split multiplicative reduction, we have

$$H^0 \left( K_{n,v}, E[{p^\infty}] / E(\mathbb{K}_{n,w}^E)[p^\infty]_{\text{div}} \right) = 0,$$

where $v$ denotes the place of $K_n$ below $w$. □
Lemma 3.4. For any \( n \in \mathbb{Z}_{>1} \), any place \( v \) of \( K \) above \( \ell \) and any subquotient \( \mathbb{Z}_p[G_{K_{n,v}}] \)-module \( M \) of \( E[p^\infty] \), we have
\[ H^0(K_{n,v}, M) = 0. \]

**Remark 3.5.** Recall that the condition (C2) holds if and only if for any prime number \( \ell \) with \( E \) has potentially multiplicative reduction, the condition (a) in Lemma 3.4 holds. As we are assuming \( E \) has good reduction at \( p \), the prime number \( \ell \neq p \).

**Proof of Lemma 3.4.** (a) \( \Rightarrow \) (b): Suppose that the base change \( E_{K_{n,w}}^E \) has split multiplicative reduction for \( n \geq 1 \) and a place \( w \) of \( K \) above \( \ell \), we have
\[ H^0(K_{n,v}, E(K_{n,w}^E)[p^\infty]_{\text{div}}) = E(K_{n,v})[p^\infty]_{\text{div}} \subseteq E(K_{n,v})[p^\infty]. \]
The latter group is trivial because of \( E(K_{n,v})[p] = 0 \).

(d) \( \Rightarrow \) (a): Take any \( n \geq 1 \), and any place \( v \) of \( K \) above \( \ell \). As \( E[p] \) is a submodule of \( E[p^\infty] \), the condition (d) implies \( E(K_{n,v})[p] = H^0(K_{n,v}, E[p]) = 0 \).

(b) \( \Leftrightarrow \) (c): Suppose that \( w \) is a place of \( K \), where \( E_{K_{n,w}}^E \) has split multiplicative reduction, and let \( v \) be a place of \( K \) below \( w \). The elliptic curve is isomorphic to a Tate curve \( \mathbb{G}_m/q_w^\mathbb{Z} \) ([Si2, Chapter V, Theorem 3.1]). Since \( \ell \neq p \) and \( K_{n,w} \) is an extension of \( q_\ell \), the extension \( K_{n,w}(\mu_\ell) \) is unramified ([Se1, CHAPITRE IV, §4, PROPOSITION 16]) so that we have \( E(K_{n,w}^E)[p^\infty]_{\text{div}} \simeq \mu_p\times q_w^\mathbb{Z} \).

By the Weil pairing, we have a natural \( G_{K_{n,v}} \)-equivariant isomorphism
\[ (E(K_{n,w}^E)[p^\infty]_{\text{div}})[p] \simeq \text{Hom}_{\mathbb{Z}_p} \left( \frac{E[p^\infty]}{E(K_{n,w}^E)[p^\infty]_{\text{div}}[p]}, \mu_p \right) \]
for any \( n \in \mathbb{Z}_{>1} \). As \( G_{K_{n,v}} \) acts trivially on \( \mu_p \), we deduce that (b) and (c) are equivalent.

(b) \& (c) \( \Rightarrow \) (a): Take any \( n \geq 1 \), and any place \( v \) of \( K \) above \( \ell \). By Lemma 3.3, the base change \( E_{K_{n,w}}^E \) of \( E \) has split multiplicative reduction for some place \( w \) of \( K \) above \( v \). The short exact sequence
\[ 0 \longrightarrow E(K_{n,w}^E)[p^\infty]_{\text{div}} \longrightarrow E(K_{n,w}^E)[p^\infty] \longrightarrow \frac{E(K_{n,w}^E)[p^\infty]}{E(K_{n,w}^E)[p^\infty]_{\text{div}}} \longrightarrow 0 \]
induces the exact sequence
\[ H^0(K_{n,v}, E(K_{n,w}^E)[p^\infty]_{\text{div}}) \longrightarrow E(K_{n,v})[p^\infty] \longrightarrow H^0 \left( K_{n,v}, \frac{E(K_{n,w}^E)[p^\infty]}{E(K_{n,w}^E)[p^\infty]_{\text{div}}} \right) \]
by the equality \( H^0(K_{n,v}, E(K_{n,w}^E)[p^\infty]) = E(K_{n,v})[p^\infty] \). From the condition (b), we have \( H^0(K_{n,v}, E(K_{n,w}^E)[p^\infty]_{\text{div}}) = 0 \). As the functor \( H^0(K_{n,v}, -) \) is left exact, the condition (c) implies
\[ H^0 \left( K_{n,v}, \frac{E(K_{n,w}^E)[p^\infty]}{E(K_{n,w}^E)[p^\infty]_{\text{div}}} \right) \subseteq H^0 \left( K_{n,v}, \frac{E[p^\infty]}{E(K_{n,w}^E)[p^\infty]_{\text{div}}} \right) = 0. \]
From the exact sequence (3.2), we obtain \( E(K_{n,v})[p] \subseteq E(K_{n,v})[p^\infty] = 0 \) and this implies the condition (a).
(b) & (c) ⇒ (d): For any $n \geq 1$ and any place $v$ of $K_n$ above $\ell$, take any subquotient $\mathbb{Z}_p[G_{K_{n,v}}]$-module $M$ of $E[p^\infty]$. From Lemma 3.3, the elliptic curve $E_{K_{n,w}}$ has split multiplicative reduction for some place $w$ of $K_n$ above $v$. Every Jordan–Hölder constituent (that is, composition factors of the Jordan–Hölder series) of the $\mathbb{Z}_p[G_{K_{n,v}}]$-module $E[p]$ (and hence every simple subquotient of $E[p^\infty]$) is isomorphic to $(E(K_{n,v})^E)[p^\infty]_{\text{div}}$ or $(E(K_{n,v}^E)[p^\infty]_{\text{div}})[p]$, which are one dimensional representation of $G_{K_{n,v}}$ over $\mathbb{F}_p$. As $M$ is a subquotient $\mathbb{Z}_p[G_{K_{n,v}}]$-module of $E[p^\infty]$, every simple subquotient of $M$ is isomorphic to a Jordan–Hölder constituents of $E[p]$. The conditions (b) and (c) imply (d). This completes the proof of Lemma 3.4 □

3.3. Example of (C2). It is obvious that if $E$ has potentially good reduction everywhere, then $(E,p)$ satisfies the condition (C2). Here, we introduce an example of $(E,p)$ satisfying (C2) such that $E$ has multiplicative reduction at some primes. The following proposition is useful to find such a pair $(E,p)$.

**Proposition 3.6.** Let $\ell$ be a prime number distinct from $p$. Suppose that $E$ has non-split multiplicative reduction at $\ell$. We also assume that $p \equiv 3 \mod 4$, and $-p$ is quadratic residue modulo $\ell$. Then, it holds that $E(K_{n,v})[p] = 0$ for any $n \in \mathbb{Z}_{\geq 0}$, and any place $v$ of $K_n$ above $\ell$.

**Proof.** Fix any embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, and regard $\mu_{p^\infty}$ as a subgroup of $\overline{\mathbb{Q}}_\ell^\times$. Let $q, \gamma \in \mathbb{Q}_\ell$ be as in the proof of Lemma 3.3. We have $\sqrt{-p} \not\in \mathbb{Q}_\ell$ because $E$ has non-split multiplicative reduction at $\ell$. Let $\chi: G_{\mathbb{Q}_\ell} \rightarrow \mathbb{Z}_p^\times$ be the quadratic character attached to $\mathbb{Q}_\ell(\sqrt{-p})/\mathbb{Q}_\ell$. We have a $G_{\mathbb{Q}_\ell}$-equivariant isomorphism

$$f: E[p^\infty] \longrightarrow (\overline{\mathbb{Q}}_\ell^\times / q^{\mathbb{Z}})[p^\infty] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi).$$

In order to prove Proposition 3.6, it suffices to show that

$$(3.3) \quad H^0(\mathbb{Q}_\ell(\mu_{p^\infty}), (\overline{\mathbb{Q}}_\ell^\times / q^{\mathbb{Z}})[p^\infty] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi)) = 0.$$ 

It holds that $\sqrt{-p} \in \mathbb{Q}_\ell$, because $p \equiv 3 \mod 4$, and $-p$ is quadratic residue modulo $\ell$. For this reason, the extension degree $[\mathbb{Q}_\ell(\mu_{p^\infty}) : \mathbb{Q}_\ell]$ is odd. This implies that $\mathbb{Q}_\ell(\mu_{p^\infty})$ never contains any quadratic extension field of $\mathbb{Q}_\ell$ because $p$ is odd. We obtain

$$H^0(\mathbb{Q}_\ell(\mu_{p^\infty}), \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi)) \simeq H^0(\mathbb{Q}_\ell(\mu_{p^\infty}), (\mathbb{Q}_p / \mathbb{Z}_p)(\chi)) = 0.$$ 

Suppose that (3.3) does not hold. There exists an element

$$P \in H^0(\mathbb{Q}_\ell(\mu_{p^\infty}), (\overline{\mathbb{Q}}_\ell^\times / q^{\mathbb{Z}})[p^\infty] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi))$$

of order $p$. Let $\zeta$ be a primitive $p$-th root of unity, and $\sigma \in G_{\mathbb{Q}_\ell(\mu_{p^\infty})}$ an element satisfying $\chi(\sigma) = -1$. Note that $\mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi)$, we have $\sigma(\zeta \otimes 1) = \zeta \otimes (1)$. By taking the Weil pairing $e: E[p] \times E[p] \rightarrow \mu_p$, we obtain

$$e(\zeta \otimes 1, P) = \sigma(e(\zeta \otimes 1, P)) = e(\sigma(\zeta \otimes 1), \sigma P) = e(\zeta \otimes (1), P) = e(\zeta \otimes 1, P)^{-1}.$$ 

As $p$ is odd, this contradicts to the fact that the Weil pairing $e$ is $G_{\mathbb{Q}_\ell}$-equivariant. Consequently, the assertion (3.3) holds. □

**Example 3.7.** Let $(E,p)$ be as in Example 1.13. Then, the elliptic curve $E$ has good reduction outside the prime $5077$, and it has non-split multiplicative reduction at $5077$. Since $p = 7 \equiv 3 \mod 4$, and $-7$ is a quadratic residue modulo $5077$, Proposition 3.6 implies that $(E,p)$ satisfies the condition (C2).
4. Selmer Groups

In this section, we shall recall the definition of the fine Selmer groups of an elliptic curve, and introduce some preliminary results related to Selmer groups. In § 4.2, we shall review preliminary results in the Iwasawa theoretical setting. We keep the notation and the assumptions in § 3.

4.1. Definition of Selmer groups. Let $K$ be a number field, that is, a finite extension field of $\mathbb{Q}$. First, let us recall Bloch–Kato’s finite local conditions.

**Definition 4.1** ([Ru, Definition 1.3.4, Remark 1.3.6]). Let $v$ be any place of $K$. We define $H^1_f(K_v, V_p(E))$ by

$$H^1_f(K_v, V_p(E)) := \begin{cases} H^1(\text{cris})(K_v, V_p(E)) & \text{if } v \nmid p, \\ \operatorname{Ker}(H^1(K_v, V_p(E)) \to H^1(K_v, B_{\text{cris}} \otimes_{\mathbb{Q}_p} V_p(E))) & \text{if } v \mid p, \\ 0 & \text{if } v \mid \infty, \end{cases}$$

where $B_{\text{cris}}$ is Fontaine’s $p$-adic ring and $v \mid \infty$ means that $v$ is an infinite place in $K$. We define $H^1_f(K_v, E[p^n]) \subseteq H^1_f(K_v, E[p^\infty])$ and $H^1_f(K_v, T_p(E)) \subseteq H^1_f(K_v, T_p(E))$ to be the image and the inverse image, respectively, of $H^1_f(K_v, V_p(E))$ under the natural maps $H^1_f(K_v, T_p(E)) \to H^1_f(K_v, V_p(E)) \to H^1_f(K_v, E[p^n])$. For each $n \in \mathbb{Z}_{>0}$, we define $H^1_f(K_v, E[p^n])$ to be the inverse image of $H^1_f(K_v, E[p^n])$ by the natural map

\[ t_{n,v} : H^1(K_v, E[p^n]) \to H^1_f(K_v, E[p^n]). \]

The subgroup $H^1_f(K_v, E[p^n])$ coincides with the image of $H^1_f(K_v, T_p(E))$ under the map $H^1_f(K, T_p(E)) \to H^1_f(K_v, E[p^n])$ induced by $T_p(E) \to T_p(E)/p^nT_p(E) \cong E[p^n]$ ([Ru, Lemma 1.3.8]).

**Remark 4.2.** Let $v$ be any finite place of $K$ not above $p$. Suppose that $E_K$ has good reduction at $v$. The $p$-adic Tate module $T_p(E)$ is unramified at $v$ (from the “easy” direction of the Néron-Ogg-Shafarevich criterion [Si1, Chapter VII, Theorem 7.1]) so that $H^1_f(K_v, E[p^n])$ coincides with $H^1(\text{cris})(K_v, E[p^n])$ (cf. [Ru, Lemma 1.3.8]), for each $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. Furthermore, the inflation-restriction exact sequence (e.g., [Ru, Proposition B.2.5]) gives a natural isomorphism

$$H^1(K_v, E[p^n]) \cong H^1_f(K_v, E[p^n]).$$

**Definition 4.3** (the fine Selmer group). For each $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, we define the fine Selmer group $\text{Sel}_p(K, E[p^n])$ to be the kernel of

$$H^1(K, E[p^n]) \to \prod_{u \mid p} H^1(K_u, E[p^n]) \times \prod_{v \nmid p} \frac{H^1(K_v, E[p^n])}{H^1_f(K_v, E[p^n])},$$

where $u$ runs through all the places of $K$ above $p$, and $v$ runs through all the places of $K$ not above $p$.

**Remark 4.4.** When $v$ is an infinite place of $K$, the cohomology group $H^1(K_v, E[p^n])$ is annihilated by at most 2 for each $n \in \mathbb{Z}_{>1} \cup \{\infty\}$. Since we are considering the odd prime $p$, we have $H^1(K_v, E[p^n]) = 0$. Because of this, we may not care about infinite places in the following.
Remark 4.5. We denote by $\Sigma_K$ the set of places of $K$ above the prime divisors of $pD_E$ and all infinite places and by $K_{\Sigma}$ the maximal algebraic extension field of $K$ unramified outside $\Sigma_K$. Then, for each $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, the kernel of the natural map

$$H^1(K, E[p^n]) \to \prod_{v \in \Sigma_K} H^1(K_v, E[p^n])$$

coincides with $H^1(K_{\Sigma}/K, E[p^n])$ ([Ru, Lemma 1.5.3]). The fine Selmer group $\text{Sel}_p(K, E[p^n])$ can be regarded as a subgroup of $H^1(K_{\Sigma}/K, E[p^n])$.

Remark 4.6. Here, we give a remark on the relation between $\text{Sel}_p(K, E[p^n])$ and the classical Selmer group. Take any $n \in \mathbb{Z}_{>0}$. Recall that the classical Selmer group $\text{Sel}(K, E[p^n])$ is defined by

$$\text{Sel}(K, E[p^n]) := \text{Ker} \left( H^1(K, E[p^n]) \to \prod_v H^1(K_v, E[p^n]) \right),$$

where $v$ runs through all the finite places of $K$, and $H^1_{cl}(K_v, E[p^n])$ denotes the image of the homomorphism

$$E(K_v) = H^0(K_v, E(\overline{K}_v)) \to H^1(K_v, E[p^n])$$

induced by the short exact sequence

$$0 \to E[p^n] \xrightarrow{\cdot p^n} E(\overline{K}_v) \xrightarrow{\times p^n} E(\overline{K}_v) \to 0.$$

For any $n \in \mathbb{Z}_{>0}$, there exists a short exact sequence

$$0 \to E(K) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z} \to \text{Sel}(K, E[p^n]) \to \text{III}(E_K/K)[p^n] \to 0,$$

where $\text{III}(E_K/K)$ denotes the Tate-Shafarevich group of $E_K/K$. For each finite place $v$ of $K$, we have

$$H^1_{cl}(K_v, E[p^n]) = H^1_f(K_v, E[p^n]).$$

It holds that

$$\text{Sel}_p(K, E[p^n]) = \text{Ker} \left( \text{Sel}(K, E[p^n]) \to \prod_{v | p} H^1(K_v, E[p^n]) \right).$$

4.2. Preliminaries of Iwasawa theory. For each place $v$ of $K_1$, we denote by $D_v$ the decomposition subgroup of the Galois group $\Gamma := G_{\infty,1} = \text{Gal}(K_{\infty}/K_1)$ at $v$, and define

$$A_v := \begin{cases} \text{Ann}_{\mathbb{Z}_p[D_v]} \left( \frac{E(K_{\infty,v})[p^\infty]}{E(K_{\infty,w})[p^\infty]} \right) & \text{if } v \mid p, \\ \text{Ann}_{\mathbb{Z}_p[D_v]} \left( H^1(K_{\infty,w}/K_{\infty,v}, E(K_{\infty,w})[p^\infty]) \right) & \text{if } v \notmid p, \end{cases}$$

where $w$ is a place of $K_{\infty}$ above $v$. We set

$$A_N := \prod_{v | pD_E} A_v \mathbb{Z}_p[\Gamma].$$

Proposition 4.7 (Control theorem, [Ru, Proposition 7.4.4]). Suppose that $E$ satisfies the condition (C1). Let $m, n \in \mathbb{Z}_{>0}$ be any integers. Then, the following hold:

1. The restriction map $H^1(K_m, E[p^\infty]) \to H^1(K_{\infty}, E[p^\infty])$ is injective.
2. The natural map $H^1(K_m, E[p^n]) \to H^1(K_{\infty}, E[p^n])[p^n]$ is injective.
The cokernel of the restriction map
\[ \text{Sel}_p(K_m, E[p^\infty]) \rightarrow H^0(K_m, \text{Sel}_p(K_{\infty}, E[p^\infty])) \]
is finite, and annihilated by \( A_N \).

The cokernel of the natural map
\[ \text{Sel}_p(K_m, E[p^n]) \rightarrow \text{Sel}_p(K_m, E[p^\infty])[p^n] \]
is finite, and independent of \( n \).

Remark 4.8. In [Ru, Proposition 7.4.4] two additional assumptions, namely Assumption 7.1.4 and Assumption 7.1.5, are assumed, but the arguments in the proof of [Ru, Proposition 7.4.4] do not need Assumption 7.1.4. In our setting, it follows from Hasse-Weil’s theorem that the \( \mathbb{Z}_p[G_K] \)-module \( T_p(E) \) satisfies Assumption 7.1.5. We also note that (C1) for \( E \) implies

\[ A_{\text{glob}} := \text{Ann}_{\mathbb{Z}_p[\Gamma]}(E(K_{\infty})) = \mathbb{Z}_p[\Gamma]. \]

As in §1, we fix a topological generator \( \gamma \) of \( \Gamma \). For each \( m, n \in \mathbb{Z}_{\geq 1} \), we put
\[ \Lambda_{m,n} := \mathbb{Z}/p^n\mathbb{Z}[\text{Gal}(K_m/K_1)] \cong \Lambda/(\gamma^p - 1, p^n). \]
Recall that for any \( m, n \in \mathbb{Z}_{\geq 0} \), we put \( X_{m,n} := \text{Sel}_p(K_m, E[p^n]) \). By Proposition 4.7, we immediately obtain the following corollary.

Corollary 4.9. There exists an integer \( \nu_X \) such that for any \( m, n \in \mathbb{Z}_{> 0} \), the orders of the kernel and the cokernel of \( X_{\infty,1} \otimes_{\Lambda} \Lambda_{m,n} \rightarrow X_{m,n} \) are at most \( p^{\nu_X} \).

Remark 4.10. Recall that \( \Delta = \text{Gal}(K_1/\mathbb{Q}) \). Take any \( n \in \mathbb{Z}_{> 0} \). Since the order of \( \Delta \) is prime to \( p \), we have
\[ H^0(\Delta, \text{Sel}_p(K_1, E[p^n])) \cong \text{Sel}_p(\mathbb{Q}, E[p^n]), \]
and hence \( (X_{1,n})_1 \cong X_{0,n} \), where \( 1 \in \hat{\Delta} \) denotes the trivial character. By Corollary 4.9, the orders of the kernel and the cokernel of
\[ X_{\infty,1} \otimes_{\Lambda} \Lambda_{1,n} \rightarrow (X_{1,n})_1 \cong X_{0,n} \]
are at most \( p^{\nu_X} \).

5. Proof of Main Results

In this section, we shall prove our main results, in particular, Theorem 1.1. We keep the notation in §3 and we suppose that the elliptic curve \( E \) over \( \mathbb{Q} \) has good reduction at an odd prime number \( p \).

5.1. Boundedness of the order of Galois cohomology. In this paragraph, let us prove the following Proposition 5.1, which is related to the boundedness of the order of the kernel and the cokernel of the restriction map
\[ H^1(K_1, E[p^n]) \rightarrow H^1(K_{E_1}, E[p^n]). \]

Proposition 5.1. Suppose that the elliptic curve \( E \) satisfies the conditions (C1) and (C3). Then, for any \( i \in \{1, 2\} \), the set
\[ \{ \#H^i(K^n_{E_1}/K_1, E[p^n]) \}_{n \geq 0} \]
is bounded.

In order to prove Proposition 5.1, we need the following lemmas.
Lemma 5.2. We assume the condition (C1) and also \( E \) has complex multiplication by an order \( \mathcal{O} \) of an imaginary quadratic field \( \mathbb{Q}(\sqrt{-d}) \). Then, the fields \( \mathbb{Q}(\sqrt{-d}) \) and \( K_\infty \) are linearly disjoint over \( \mathbb{Q} \).

Proof. Assume \( \mathbb{Q}(\sqrt{-d}) \subseteq K_\infty \) for the contradiction. As \( E \) is defined over \( \mathbb{Q} \), every endomorphism of \( E \) is defined over \( \mathbb{Q}(\sqrt{-d}) \) ([Si2, Chapter II, Theorem 2.2 (b)]), hence over \( K_\infty \). Recall that \( E[p] \) is a free \( \mathcal{O}/p\mathcal{O} \)-module of rank 1 ([Si2, Chapter II, Proposition 1.4]). The two dimensional representation \( \rho_1^E : G_{K_\infty} \to \text{Aut}_{\mathbb{F}_p}(E[p]) \) is given by a character \( G_{K_\infty} \to \text{Aut}_{\mathcal{O}\otimes\mathbb{Z}_p}(E[p]) \simeq (\mathcal{O}/p\mathcal{O})^\times \). This contradicts to (C1).

Lemma 5.3. Suppose that \( E \) satisfies (C1) and (C3). Then, for any \( i \in \mathbb{Z}_{\geq 0} \), it holds that \( H^i(K_{\infty}^E / K_\infty, V_p(E)) = 0 \).

Proof. (The case: non CM) First, suppose that \( E \) does not have complex multiplication. Recall that \( G_{\mathbb{Q}} \) acts on \( \bigoplus_{n \geq 2} T_p(E) \) via the cyclotomic character (cf. [Si1, Chapter V, Section 2]). By Serre’s open image theorem [Sc3], the image \( H \) of the Galois representation

\[
\rho^E : \text{Gal}(K_{\infty}^E / K_\infty) \to \text{Aut}_{\mathbb{Z}_p}(T_p(E)) \simeq GL_2(\mathbb{Z}_p)
\]

becomes an open subgroup of \( SL_2(\mathbb{Z}_p) \). There exists an open normal standard pro-\( p \) subgroup \( U \) of \( H \) ([DDMS, 8.29 Theorem]), because \( H \) is a \( p \)-adic Lie group, By [La, Chapter V, (2.4.9) Théorème], we have

\[
H^q(U, V_p(E)) = H^q(Lie(U), V_p(E))
\]

for any \( q \geq 0 \). Since \( Lie(U) \) is an open Lie-subalgebra of \( sl_2(\mathbb{Z}_p) := \{ A \in pM_2(\mathbb{Z}_p) \mid \text{Tr} A = 0 \} \),

a matrix of the form

\[
\begin{pmatrix}
1 + p^n & 0 \\
0 & -(1 + p^n)
\end{pmatrix}
\]

for some \( n \) belongs to \( Lie(U) \). By [Sc2, THÉORÈM 1], we obtain \( H^q(Lie(U), V_p(E)) = 0 \). Hence, the Hochschild–Serre spectral sequence

\[
E_2^{pq} = H^p(H/U, H^q(U, V_p(E))) \Rightarrow H^{p+q}(H, V_p(E))
\]

implies that \( H^i(K_{\infty}^E / K_\infty, V_p(E)) = H^i(H, V_p(E)) = 0 \) for any \( i \geq 0 \).

(The case: CM) Next, let us assume that \( E \) has complex multiplication. By the assumption (C3), the ring \( \text{End}(E) \) of endomorphisms of \( E \) defined over \( \overline{\mathbb{Q}} \) is the maximal order \( \mathcal{O} \) of an imaginary quadratic field \( L := \mathbb{Q}(\sqrt{-d}) \). Put \( L_\infty^E = LK_\infty^E \). Since \( E \) is defined over \( \mathbb{Q} \), every element of \( \text{End}(E) \) is defined over \( L \) ([Si2, Chapter II, Theorem 2.2(b)]). Consider the representation \( \rho : G_L \to \text{Aut}(T_p(E)) \) which is arising from the action of \( G_L \) on \( T_p(E) \). This factors through an injective homomorphism \( \text{Gal}(L_\infty^E / L) \to \text{Aut}(T_p(E)) \) which is also denoted by \( \rho \). The Tate module \( T_p(E) = \lim_{\to} E[p^n] \) is a free \( \mathcal{O} \otimes \mathbb{Z}_p \)-module of rank 1 because \( E[p^n] \) is a free \( \mathcal{O}/p^n\mathcal{O} \)-module of rank 1 ([Si2, Chapter II, Proposition 1.4]). As we noted above, every endomorphism of \( E \) is defined over \( L \), the action of \( \text{Gal}(L_\infty^E / L) \) commutes with the
scalar multiplication by $\sigma$, and we obtain the commutative diagram

$$
\begin{array}{c}
\text{Gal}(L^E_/L) \\
\rho_o \downarrow \quad \downarrow \rho
\end{array}
\begin{array}{c}
\text{Aut}(T_p(E)) \\
\text{Aut}_{\sigma \otimes \mathbb{Z}_p}(T_p(E)) \cong (\mathfrak{o} \otimes \mathbb{Z}_p)^{\times}.
\end{array}
$$

(5.1)

In particular, the extension $L^E_/L$ is an abelian extension. The short exact sequence

$$0 \longrightarrow \text{Gal}(L^E_/L) \longrightarrow \text{Gal}(L^E_/\mathbb{Q}) \longrightarrow \text{Gal}(L_/\mathbb{Q}) \longrightarrow 0$$

induces the action of $\text{Gal}(L_/\mathbb{Q})$ to $\text{Gal}(L^E_/L)$. In fact, let $c$ be the unique generator of $\text{Gal}(L_/\mathbb{Q})$ and take $\tilde{c} \in \text{Gal}(L^E_/\mathbb{Q})$ a lift of $c$. The action of $\text{Gal}(L_/\mathbb{Q})$ on $\text{Gal}(L^E_/L)$ is given by $\sigma \mapsto \tilde{c} \sigma \tilde{c}^{-1}$. The induced map $\rho_o$ preserves the action of $\text{Gal}(L_/\mathbb{Q})$. Let $\pi^{\times}: \text{Aut}_{\mathfrak{o} \otimes \mathbb{Z}_p}(T_p(E)) \longrightarrow (\mathfrak{o} \otimes \mathbb{Z}_p)^{\times} / \mathfrak{o}^{\times}$ be the natural surjection. We denote by $H'$ the image of $\rho^{\times}_{E'}$, and by $\overline{H}'$ that of $\pi^{\times} \circ \rho^{\times}_{E'}$. Let $L_{\overline{H}'}$ be the maximal subfield of $L^E_/L$ fixed by the kernel of $\pi^{\times} \circ \rho^{\times}_{E'}$. We have

$$\text{Gal}(L^E_/L) \cong H' \subset (\mathfrak{o} \otimes \mathbb{Z}_p)^{\times}, \quad \text{and} \quad \text{Gal}(L_/L) \cong \overline{H}' \subseteq (\mathfrak{o} \otimes \mathbb{Z}_p)^{\times} / \mathfrak{o}^{\times}.
$$

Claim 1. The extension $L_{\overline{H}'} / L$ is the maximal abelian extension unramified outside $p$.

Proof. The elliptic curve $E$ is defined over $\mathbb{Q}$ so that the class number of $L$ is 1 ([Si2, Chapter II, Theorem 4.1]). We denote by $L_{\overline{H}'}$ be the fixed field of $L^E_/n := L(E[p^n])$ by the kernel of the composition

$$\text{Gal}(L^E_/L) \longrightarrow \text{Aut}_{\mathfrak{o} \otimes \mathbb{Z}_p}(E[p^n]) \longrightarrow \text{Aut}_{\mathfrak{o} \otimes \mathbb{Z}_p}(E[p^n]) / \text{Aut}(E) \cong (\mathfrak{o} / p^n \mathfrak{o})^{\times} / \mathfrak{o}^{\times}.$$

By the theory of complex multiplication ([Si2, Chapter II, Theorem 5.6]), $L_{\overline{H}'}$ is the ray class field of $L$ modulo $p^n \mathfrak{o}$. The claim follows from $L_{\overline{H}'} = \bigcup_n L_{\overline{H}'}$.

By the global class field theory, the above claim implies that the group $\overline{H}' \cong \text{Gal}(L_{\overline{H}'} / L)$ has a quotient isomorphic to $\mathbb{Z}_p^2$ (see, for instance, [Wa, Chapter 13, Proposition 13.2 and Theorem 13.4]). The subgroup $H'$ is open in $(\mathfrak{o} \otimes \mathbb{Z}_p)^{\times}$, and in particular, the complex conjugate $c$ acts non-trivially on $H'$.

Claim 2. The field $K^E_/\mathbb{Q}$ contains $L = \mathbb{Q}(\sqrt{-d})$.

Proof. If $K^E_/\mathbb{Q}$ and $L$ are linearly disjoint over $\mathbb{Q}$, then the extension $L^E_/\mathbb{Q} = L K^E_/\mathbb{Q}$ becomes abelian. Therefore, the complex conjugate $c$ acts on $\text{Gal}(L^E_/L)$ trivially, and it acts on $H'$ via $\rho_o$. This contradicts to the fact that $c$ acts on $H'$ non-trivially.

From the above claim, we have $L^E_/ = L K^E_/ = K^E_/\mathbb{Q}$.

Claim 3. There exists a lift $\tilde{c} \in \text{Gal}(K^E_/\mathbb{Q})$ of $c$ whose order is two such that

$$\text{Gal}(K^E_/\mathbb{Q}) = (\tilde{c}) \times \text{Gal}(K^E_/L) \cong (\tilde{c}) \times H'.$$
Proof. From Claim 2, we have $K^E_\infty \supseteq L$. Fix an embedding $\iota_C : K^E_\infty \hookrightarrow \mathbb{C}$. Consider the following short exact sequence:

$$
0 \xrightarrow{} \text{Gal}(K^E_\infty /L) \xrightarrow{} \text{Gal}(K^E_\infty /Q) \xrightarrow{} \text{Gal}(L/Q) \xrightarrow{} 0
$$

The embedding $\iota_C$ induces a splitting of this short exact sequence which sends $c$ to the restriction $\tilde{c} \in \text{Gal}(K^E_\infty /Q)$ of the complex conjugation after regarding $K^E_\infty$ as a subfield of $\mathbb{C}$ via $\iota_C$. This splitting gives $\text{Gal}(K^E_\infty /Q) \simeq \langle \tilde{c} \rangle \rtimes H'$.

Claim 4. Putting $L_\infty = LK_\infty$, we have $L_{\mathfrak{P}} \cap Q^{ab} = L_\infty$.

Proof. By Lemma 5.2, the fields $K_\infty$ and $L = Q(\sqrt{-d})$ are linearly disjoint. The composition field $L_\infty = K_\infty L$ is an abelian extension of $Q$ so that $L_\infty \subseteq Q^{ab}$. The extension $K_\infty = Q(\mu_p) = \bigcup_n Q(\mu_{p^n})$ of $Q$ is unramified outside $p$ and hence the extension $L_\infty = K_\infty L/L$ is unramified outside $p$. Let us show that $L_{\mathfrak{P}} \cap Q^{ab} = L_\infty$. Claim 1 implies that $L_{\mathfrak{P}} \cap Q^{ab} \supseteq L_\infty$ because the extension $L_\infty/L$ is unramified outside $p$. Accordingly, it suffices to show that $L_{\mathfrak{P}} \cap Q^{ab} \subseteq L_\infty$. As $E$ is defined over $Q$, the class number of $L$ is one. Put $p^* := (-1)^{(p-1)/2}p$. Lemma 5.2 implies that $Q(\sqrt{p^*})$ and $L$ are linearly disjoint over $Q$ because $Q(\sqrt{p^*})$ is contained in $K_1 = Q(\mu_p)$. We deduce that $p$ is unramified in $L/Q$. In fact, if $p$ were ramified in $L/Q$, the Hilbert class field of $L$ would contain the quadratic extension $L(\sqrt{p^*})/L$. Since $L$ is the imaginary quadratic field of class number one, there exists a unique prime $q_L \in \{2, 3, 7, 8, 11, 19, 43, 67, 163\}$ which is ramified in $L/Q$.

For each prime $\ell$, we denote by $I_\ell$ the inertia subgroup of $\text{Gal}((L_{\mathfrak{P}} \cap Q^{ab})/Q)$ at $\ell$. We define $L_1$ to be the subfield of $L_{\mathfrak{P}} \cap Q^{ab}$ fixed by $I_p$, and $L_2$ to be that fixed by $I_{q_L}$. The extension $L_{\mathfrak{P}}/L$ is unramified outside $p$, and $L$ has class number one. The extension $(L_{\mathfrak{P}} \cap Q^{ab})/L$ does not contain the proper extension field of $L$ where every place above $p$ is unramified. As $p$ is unramified in $L/Q$, we obtain $L_1 = L$. The field $L_2$ coincides with the maximal intermediate field of $(L_{\mathfrak{P}} \cap Q^{ab})/L$ unramified outside $p$ because the extension $L_{\mathfrak{P}}/Q$ is unramified outside $\{p, q_L\}$. The inclusion $L_2 \subseteq K_\infty$ holds, because $K_\infty/Q$ is the maximal abelian extension unramified outside $p$. As a result, we obtain $L_1L_2 \subseteq L_\infty$. Additionally, the extension $L_{\mathfrak{P}} \cap Q^{ab}$ of $L_1 = L$ is unramified outside $p$. In particular, the extension $(L_{\mathfrak{P}} \cap Q^{ab})/L_1$ is unramified at $q_L$. Because of this, we have

$$I_p \cap I_{q_L} = \text{Gal}((L_{\mathfrak{P}} \cap Q^{ab})/L_1) \cap I_{q_L} = \{1\}.$$  

Consequently, we deduce that $L_{\mathfrak{P}} \cap Q^{ab} = L_1L_2 \subseteq L_\infty$.

By this Claim 4, the abelianization of the Galois group $\text{Gal}(L_{\mathfrak{P}}/Q)$ is

$$\text{Gal}(L_{\mathfrak{P}}/Q)^{ab} = \text{Gal}(L_{\mathfrak{P}}/Q^{ab}/Q) = \text{Gal}(L_\infty/Q).$$

The abelianization $\text{Gal}(L_{\mathfrak{P}}/Q)^{ab}$ is the maximal quotient of $\text{Gal}(L_{\mathfrak{P}}/Q)$ where $c$ acts trivially, and we have $\text{Gal}(L_{\mathfrak{P}}/Q) \simeq \langle \tilde{c} \rangle \rtimes \overline{H}$ by Claim 3. Therefore, we obtain

$$\text{Gal}(L_\infty/Q)^{(5.3)} = \text{Gal}(L_{\mathfrak{P}}/Q)^{ab}
\simeq \left( \langle \tilde{c} \rangle \rtimes \overline{H} \right) / \left( \langle \tilde{c} \rangle \rtimes (1 - c)\overline{H} \right)
\simeq \overline{H} / (1 - c)\overline{H}.$$
(Here, the group operation of $\overline{H'}$ is written in additive manner.) Let $H'_\infty$ be the inverse image of $\{1 - c\overline{H'}\}$ by $\pi_{\sigma^*}|_{H'}: H' \subseteq (\mathfrak{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\times} \longrightarrow (\mathfrak{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\times}/\mathfrak{o}^{\times}$. By (5.2), we have

$$\text{Gal}(L_{\overline{\pi}}/L_\infty) \simeq (1 - c)\overline{H'}, \quad \text{and} \quad \text{Gal}(L_\infty^{E}/L_\infty) \overset{\text{Claim 2}}{=} \text{Gal}(K_\infty^{E}/L_\infty) \simeq H'_\infty.$$ 

By Lemma 5.2, the fields $K_\infty$ and $L$ are linearly disjoint over $\mathbb{Q}$. We obtain an isomorphism $\text{Gal}(L_\infty/K_\infty) \simeq \text{Gal}(L/\mathbb{Q})$ and an exact sequence

$$0 \longrightarrow \text{Gal}(K_\infty^{E}/L_\infty) \longrightarrow \text{Gal}(K_\infty^{E}/K_\infty) \longrightarrow \text{Gal}(L_\infty/K_\infty) \longrightarrow 0.$$ 

There exists a lift $\overline{\mathcal{C}} \in \text{Gal}(K_\infty^{E}/K_\infty)$ of $c$. Note that $\overline{\mathcal{C}}$ and $\text{Gal}(K_\infty^{E}/L_\infty) \simeq H'_\infty$ generate $\text{Gal}(K_\infty^{E}/K_\infty)$, and we have $(\text{Gal}(K_\infty^{E}/K_\infty): H'_\infty) = 2$.

\[
\begin{array}{c}
\text{Claim 5. We have } H'_\infty \subseteq H'[1+(1+c)^2]. \text{ Here, the } (1+c)^2\text{-torsion part of a } \mathbb{Z}[\text{Gal}(L/\mathbb{Q})]\text{-module } M \text{ is denoted by } M[(1+c)^2]. \\
\text{Proof. Note that } (1 - c)\overline{H'} \text{ is contained in } \overline{H'} [1 + c] \text{ and } \mathfrak{o}^{\times} \text{ is contained in } H'[1 + c]. \\
\text{For any } x \in H'_\infty = \pi_{\sigma^*}^{-1}((1 - c)\overline{H'}), \text{ we have } \pi_{\sigma^*}(x) \in (1 - c)\overline{H'} \subseteq \overline{H'} [1 + c]. \text{ For } (1 + c)x \in \text{Ker}(\pi_{\sigma^*}) = \mathfrak{o}^{\times} \subseteq H'[1 + c], \text{ we obtain } (1 + c)^2x = (1 + c)(1 + c)x = 0. \quad \Box \\
\text{Put } V := (\mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}_p \simeq \mathbb{Q}_p^2. \text{ Since } c \text{ acts on } V \text{ non-trivially, and } 1 + p \in V \text{ is a non-trivial element fixed by } c, \text{ the eigenvalues of the action of } c \text{ on } V \text{ are } 1 \text{ and } -1. \\
\text{The group } (\mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\times}[(1+c)^2] \text{ has a subgroup of finite index which is isomorphic to } \mathbb{Z}_p^2. \text{ This implies that there exists an element } x \in H'_\infty \text{ of infinite order such that the closure } H_\infty \text{ of } \langle x \rangle \text{ has finite index in } H'_\infty. \text{ Fix an embedding } \iota_p: L \hookrightarrow \mathbb{Q}_p. \text{ The embedding } \\
\iota_p \text{ induces the ring homomorphism } \overline{\iota}_p: \mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \mathbb{Q}_p \text{ sending } a \otimes b \text{ to } \iota_p(a)b. \text{ The eigenvalues of the action of } x \text{ on } V_p(E) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \text{ are } \overline{\iota}_p(x) \text{ and } \overline{\iota}_p(c(x)) = \overline{\iota}_p(x)^{-1}. \text{ We obtain } V_p(E)[x - 1] = 0 \text{ and } V_p(E)/(1 - x) = 0. \text{ Note that } H_\infty \text{ is topologically generated by } x. \text{ By } [\text{NSW}, (1.7.7) \text{ Proposition]} \text{ combined with } [\text{Ta}, (2.2) \text{ Corollary} \text{ and } (2.3) \text{ Proposition}], \text{ it holds that } H^q(H_\infty, V_p(E)) = 0 \text{ for any } q \geq 0. \text{ Let us identify } H'_\infty \text{ with } \text{Gal}(K_\infty^{E}/L_\infty). \text{ We may regard } H_\infty \text{ as a normal subgroup of } \text{Gal}(K_\infty^{E}/K_\infty) \text{ because } c \text{ acts on } H_\infty \text{ by } x \mapsto x^{-1}. \text{ Hence, by the Hochschild–Serre spectral sequence} \\
E_2^{pq} = H^p(\text{Gal}(K_\infty^{E}/K_\infty)/H_\infty, H^q(H_\infty, V_p(E))) \Rightarrow H^{p+q}(K_\infty^{E}/K_\infty, V_p(E)), \text{ we deduce that } H^i(K_\infty^{E}/K_\infty, V_p(E)) = 0 \text{ for any } i \geq 0. \quad \Box}
\end{array}
\]
In the proof of Proposition 5.1, we use a corollary of the following well-known lemma called topological Nakayama’s lemma.

**Lemma 5.4** (Topological Nakayama’s lemma). Let \((R, m)\) be a Noetherian complete local ring whose residue field is finite, and \(M\) a compact Hausdorff \(R\)-module. Suppose that \(\dim_{R/m} M/mM < \infty\). Then, the \(R\)-module \(M\) is finitely generated.

**Proof.** Since \(M\) is compact, for any neighborhood \(U\) of \(0 \in M\), there exists an integer \(n \in \mathbb{Z}_{\geq 0}\) such that \(m^nM \subseteq U\). As \(M\) is Hausdorff, we obtain \(\bigcap_{n \geq 0} m^nM = 0\). By [El, Exercise 7.2], we deduce that \(M\) is finitely generated over \(R\) if \(\dim_{R/m} M/mM < \infty\). (See [Wa, Lemma 13.16] for the proof of Lemma 5.4 in the case when \(R = \mathbb{Z}_p[T]\).) \(\square\)

**Corollary 5.5.** Let \(M\) be a torsion \(\mathbb{Z}_p\)-module satisfying \(\dim_{\mathbb{Z}_p} M[p] < \infty\). Then, it holds that \(M\) is a cofinitely generated \(\mathbb{Z}_p\)-module.

**Proof.** We regard \(M\) as a topological group equipped with the discrete topology. By applying Lemma 5.4 to the Pontryagin dual of \(M\), we obtain Corollary 5.5. \(\square\)

**Proof of Proposition 5.1.** Take any \(i \in \{1, 2\}\). Let \(j \in \mathbb{Z}\) be any integer satisfying \(0 \leq j \leq i\). The group \(\text{Gal}(K^E_{\infty}/K_{\infty})\) is topologically finitely presented because it is isomorphic to a closed subgroup of \(GL_2(\mathbb{Z}_p)\). This implies that \(H^j(K^E_{\infty}/K_{\infty}, E[p])\) is of finite order. The long exact sequence arising from the short exact sequence \(0 \to E[p] \to E[p^\infty] \xrightarrow{p} E[p^\infty] \to 0\) induces the surjective homomorphism \(H^j(K^E_{\infty}/K_{\infty}, E[p]) \to H^j(K^E_{\infty}/K_{\infty}, E[p^\infty])[p]\). In particular, we have

\[
\dim_{\mathbb{Z}_p} H^j(K^E_{\infty}/K_{\infty}, E[p^\infty])[p] \leq \dim_{\mathbb{Z}_p} H^j(K^E_{\infty}/K_{\infty}, E[p]) < \infty.
\]

By Corollary 5.5, it holds that \(H^j(K^E_{\infty}/K_{\infty}, E[p^\infty])\) is cofinitely generated over \(\mathbb{Z}_p\). Moreover, the short exact sequence \(0 \to T_p(E) \to V_p(E) \to E[p^\infty] \to 0\) induces

\[
H^j(K^E_{\infty}/K_{\infty}, V_p(E)) \to H^j(K^E_{\infty}/K_{\infty}, E[p^\infty]) \to H^{j+1}(K^E_{\infty}/K_{\infty}, V_p(E)).
\]

Since \(H^{j+1}(K^E_{\infty}/K_{\infty}, T_p(E))\) does not have a non-trivial divisible \(\mathbb{Z}_p\)-submodule by [Ta, (2.1) Proposition], it follows from Lemma 5.3 that \(\#H^j(K^E_{\infty}/K_{\infty}, E[p^\infty]) < \infty\). Take any \(n \in \mathbb{Z}_{\geq 0}\). As \(K_{\infty}/K_n\) is a pro-cyclic extension, the Hochschild–Serre spectral sequence

\[
E_2^{pq} = H^p(K_{\infty}/K_n, H^q(K^E_{\infty}/K_{\infty}, E[p^\infty])) \Rightarrow H^{p+q}(K^E_{\infty}/K_n, E[p^\infty])
\]

implies that

\[
\#H^i(K^E_{\infty}/K_n, E[p^\infty]) \leq \prod_{q \leq i} \{ \#H^q(K^E_{\infty}/K_{\infty}, E[p^\infty]) \} < \infty.
\]

Therefore, the sequence \(\{ \#H^i(K^E_{\infty}/K_n, E[p^\infty]) \}_{n \geq 0}\) is bounded. The exact sequence

\[
H^{i-1}(K_{\infty}/K_n, E[p^\infty])/p^n \to H^i(K^E_{\infty}/K_n, E[p^n]) \to H^i(K^E_{\infty}/K_n, E[p^\infty])[p^n]
\]

implies that \(\{ \#H^i(K^E_{\infty}/K_n, E[p^n]) \}_{n \geq 0}\) is bounded. The inflation map

\[
H^1(K^E_{\infty}/K_n, E[p^n]) \to H^1(K^E_{\infty}/K_n, E[p^n])
\]

is injective ([Ru, Proposition B.2.5]). The assertion of Proposition 5.1 for \(i = 1\) follows from this. In order to prove Proposition 5.1 for \(i = 2\), by considering the inflation-restriction sequence

\[
H^1(K^E_{\infty}/K_n, E[p^n]) \xrightarrow{\text{Gal}(K^E_{\infty}/K_n)} H^2(K^E_{\infty}/K_n, E[p^n]) \to H^2(K^E_{\infty}/K_n, E[p^n])
\]
Lemma 5.6 \[ \text{[Ru, Proposition B.2.5 (ii)]}, \text{it suffices to show that the order of} \\
\quad H^0(K_n, \text{Hom}(Gal(K^E_{\infty}/K^E_n), E[p^n])) \\
\text{is bounded. Put} \ H_{n,m} := H^0(K_n, \text{Hom}(Gal(K^E_{\infty}/K^E_n), E[p^m])). \text{The short exact sequence} \\
0 \to E[p] \to E[p^n] \to E[p^{n-1}] \to 0 \text{induces an exact sequence} \\
0 \to H_{n,1} \to H_{n,m} \to H_{n,m-1}. \\
\text{The lemma below (Lemma 5.6) says that there exists an integer} N \text{ such that} \\
H_{n,1} = H^0(K_n, \text{Hom}(Gal(K^E_{\infty}/K^E_n), E[p])) = 0 \\
\text{for all} \ n \geq N. \text{Thus, we have a sequence of injective homomorphisms} \\
\begin{array}{c}
H_{n,m} \hookrightarrow H_{n,m-1} \hookrightarrow \cdots \hookrightarrow H_{n,1}.
\end{array} \\
\text{The lemma below again implies} H_{n,1} = 0. \text{In particular, we have} \\
H_{n,n} = H^0(K_n, \text{Hom}(Gal(K^E_{\infty}/K^E_n), E[p^n])) = 0 \\
\text{for all} \ n \geq N. \text{Therefore, the sequence} \ \{ \#H^0(K_n, \text{Hom}(Gal(K^E_{\infty}/K^E_n), E[p^n])) \}_{n \geq 0} \text{is bounded.} \\
\] 

\textbf{Lemma 5.6.} Suppose that \( E \) satisfies (C1) and (C3). There exists an integer \( N \) such that \\
\[ H^0(K_m, \text{Hom}(Gal(K^E_{\infty}/K^E_m), E[p])) = 0 \]
for any \( m \in \mathbb{Z}_{\geq N}. \)

\textit{Proof. (The case: non-CM)} First, suppose that \( E \) does not have complex multiplication. The representation \( \rho^E : G_\mathbb{Q} \to \text{Aut}(T_p(E)) \simeq GL_2(\mathbb{Z}_p) \) factors through \( Gal(K^E_{\infty}/\mathbb{Q}) \to GL_2(\mathbb{Z}_p) \) and is also denoted by \( \rho^E \). In this non-CM case, Serre’s open image theorem \[ \text{Se3} \] implies that the group \( \rho^E(Gal(K^E_{\infty}/\mathbb{Q})) \) is an open subgroup of \( GL_2(\mathbb{Z}_p) \). We can take an integer \( N \in \mathbb{Z}_{\geq 1} \) such that \( \rho^E(Gal(K^E_{\infty}/\mathbb{Q})) \) contains \( 1 + p^N M_2(\mathbb{Z}_p) \). Take any \( m \in \mathbb{Z}_{\geq N} \). As we have \\
\[ H^0(K_m, \text{Hom}(Gal(K^E_{\infty}/K^E_m), E[p])) = \text{Hom}_{Gal(K^E_{\infty}/K_m)}(Gal(K^E_{\infty}/K^E_m), E[p])), \]
\text{it is enough to show that there is no non-trivial} \( Gal(K^E_{\infty}/K_m) \)-equivariant homomorphism \( Gal(K^E_{\infty}/K^E_m) \to E[p] \). The commutative diagram \\
\begin{array}{c}
\text{Gal}(K^E_{\infty}/\mathbb{Q}) \xleftarrow{\rho^E} \text{Aut}(T_p(E)) \simeq GL_2(\mathbb{Z}_p) \\
\downarrow \\
\text{Gal}(K^E_{m}/\mathbb{Q}) \xleftarrow{\rho^E} \text{Aut}(E[p^m]) \simeq GL_2(\mathbb{Z}/p^m \mathbb{Z})
\end{array}
\text{indicates that} \ \rho^E(\text{Gal}(K^E_{\infty}/K^E_m)) \subseteq 1 + p^m M_2(\mathbb{Z}_p). \text{As we have} \\
1 + p^m M_2(\mathbb{Z}_p) \subseteq 1 + p^N M_2(\mathbb{Z}_p) \subseteq \rho^E(\text{Gal}(K^E_{\infty}/\mathbb{Q})), \\
\text{it holds} \\
\rho^E(\text{Gal}(K^E_{\infty}/K^E_m)) = \rho^E(\text{Gal}(K^E_{\infty}/\mathbb{Q})) \cap (1 + p^m M_2(\mathbb{Z}_p)) = 1 + p^m M_2(\mathbb{Z}_p). \\
\text{Hence, every group homomorphism} \ f : Gal(K^E_{m}/K^E_m) \to E[p] \text{factors through} \\
\text{Gal}(K^E_{m+1}/K^E_m) \simeq gl_2(\mathbb{F}_p) = M_2(\mathbb{F}_p). \]
The group $G := \rho^E(\text{Gal}(K^E_{\infty}/K_m)) \subseteq GL_2(\mathbb{Z}_p)$ acts on $M_2(\mathbb{F}_p)$ via the conjugate action, and we have $M_2(\mathbb{F}_p) = \mathbb{F}_p \oplus \mathfrak{sl}_2(\mathbb{F}_p)$ as $\mathbb{F}_p[G]$-modules, where we set

$$\mathfrak{sl}_2(\mathbb{F}_p) := \{ A \in M_2(\mathbb{F}_p) \mid \text{Tr} \, A = 0 \}.$$  

The condition $(C1)$ for $E$ implies that there is no non-trivial $G$-equivariant homomorphism $\mathbb{F}_p \rightarrow E[p]$. Suppose that there is a non-trivial $G$-equivariant homomorphism $f: \mathfrak{sl}_2(\mathbb{F}_p) \rightarrow E[p]$, and show that this assumption leads to a contradiction. Put $V := \text{Ker}(f)$. By $(C1)$, we have $\dim_{\mathbb{F}_p} \text{im}(f) = 2$, and $\dim_{\mathbb{F}_p} V = 1$. Take any non-zero $A \in V$.

- First, let us suppose that $A$ is nilpotent. In this case, there exists a matrix $P \in GL_2(\mathbb{F}_p)$ such that $A = P \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P^{-1}$. Since $G$ acts via the conjugate action on the space $V := \mathbb{F}_p A$, for any $B \in G$ there exists $a \in \mathbb{F}_p^\times$ such that $BAB^{-1} = aA$. This implies that if $v \in \mathbb{F}_p$ is an eigenvector of $A$, then $Bv$ is also an eigenvector of $A$. As a result, the group $G$ is contained in the Borel subgroup $P \begin{pmatrix} \mathbb{F}_p^\times & \mathbb{F}_p \\ 0 & \mathbb{F}_p^\times \end{pmatrix} P^{-1}$ of $GL_2(\mathbb{F}_p)$. This implies that $G$ acts on the subspace $W \subseteq \mathfrak{sl}_2(\mathbb{F}_p) \setminus V$ generated by $P \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P^{-1}$. In fact, for $Q = P \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} P^{-1} \in G$, we have $QP \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P^{-1}Q^{-1} \in W$. The image of $W$ by $f$ spans a proper $G$-stable $\mathbb{F}_p$-subspace of $E[p]$. This contradicts to $(C1)$.

- Next, suppose that $A$ is not nilpotent. If we assume the matrix $A \in \mathfrak{sl}_2(\mathbb{F}_p)$ has one eigenvalue $\alpha$, then $0 = \text{Tr} \, A = 2\alpha$. Since $p$ is odd, we have $\alpha = 0$ and $A$ is nilpotent. The matrix $A$ has two distinct eigenvalues $\alpha, -\alpha$ in $\mathbb{F}_p$. Since $V$ is stable under the conjugate action of $G$, for any $B \in G$, there exists some $a \in \mathbb{F}_p^\times$ such that $BAB^{-1} = aA$. For each eigenvalue $\beta \in \{ \alpha, -\alpha \}$ of $A$, we denote by $V_{\beta} \subseteq \mathbb{F}_p^2$ the eigenspace associated with the eigenvalue $\beta$. Take any non-zero $v \in V_{\alpha}$. Note that $Bv$ is also an eigenvector of $A$, for we have $BA = aAB$. Suppose that $Bv \in V_{-\alpha}$. The group $G$ acts on $\{ V_\alpha, V_{-\alpha} \}$ transitively, and $G$ has a subgroup of index 2. This contradicts to the fact that $G$ is a pro-$p$-group. Because of this, we obtain $Bv \in V_\alpha$. This implies that $V_\alpha$ is $G$-stable. This contradicts to $(C1)$.

Hence, there is no non-trivial $G$-equivariant homomorphism $\mathfrak{sl}_2(\mathbb{F}_p) \rightarrow E[p]$, and the assertion for the non-CM case is finished.

(The case: CM) Suppose that $E$ has complex multiplication. By the assumption (C3), the ring $\text{End}(E)$ is the maximal order $\mathfrak{o}$ of some imaginary quadratic field $L := \mathbb{Q}(\sqrt{-d})$. As we shall see below, in this case, we can take $N := 1$. Take any $m \in \mathbb{Z}_{\geq 1}$, and put $G := \text{Gal}(K^E_{\infty}/K_m)$. Let $H'_m$ be the subgroup of $(\mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ corresponding to $\text{Gal}(K^E_{\infty}/L_m)$ by

$$\rho^E_\mathfrak{o}: \text{Gal}(K^E_{\infty}/L) \rightarrow \text{Aut}_{\mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Z}_p}(T_{p}(E)) = (\mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times,$$

where $L_m = K_m L$ (cf. (5.1)). Recall that $L = \mathbb{Q}(\sqrt{-d})$ and $K_{\infty}$ are linearly disjoint over $\mathbb{Q}$ (Lemma 5.2) and $LK^E_{\infty} = K^E_{\infty}$ by Claim 2 in the proof of Lemma 5.3. There exists a lift $\tilde{c}_m \in G = \text{Gal}(K^E_{\infty}/K_m)$ of the generator $c \in \text{Gal}(L/\mathbb{Q})$. Note that $G$ is generated by $\tilde{c}_m$ and $H'_m$, and $H'_m$ is a normal subgroup of $G$ of index two.
Claim. There exists a non-trivial element of $H'_m$ whose order is prime to $p$.

Proof. Suppose that $H'_m$ has no non-trivial element whose order is prime to $p$ for the contradiction. Then $H'_m$ becomes a pro-$p$ group, and hence there exists a non-zero element $a \in E[p]$ fixed by $H'_m$ (cf. [Sel, CHAPITRE IX, §1, LEMME 2]).

- If $a$ is an eigenvector of $\bar{c}_m$, then $a$ spans a proper $G$-stable $\mathbb{F}_p$-subspace of $E[p]$.
- Let us suppose that $a$ is not an eigenvector of $\bar{c}_m$. Note that $H'_m$ acts trivially on both $a$ and $\bar{c}_m(a)$, for $H'_m$ is a normal subgroup of $G$. Since $E[p]$ is spanned by $\{a, \bar{c}_m(a)\}$ over $\mathbb{F}_p$, the action of $H'_m$ on $E[p]$ is trivial. The action of $G$ on $E[p]$ factors through the cyclic group $G/H'_m$ of order two, especially prime to $p$, generated by the image of $\bar{c}_m$.

In any cases, it contradicts to (C1). As a result, there exists a non-trivial element $H'_m$ whose order is prime to $p$. □

5.2. The kernel and the cokernel of the restriction maps. The goal of this subsection is the following proposition which is a key of the proof of Theorem 1.1.

Proposition 5.7. Let $p$ be a prime number at which the elliptic curve $E$ has good reduction. Suppose that $E$ satisfies the conditions (C1), (C2) and (C3). Let $\text{res}_n := \text{res}_{n}^{\text{Sel}} : \text{Sel}_p(K_n, E[p^n]) \to H^0(K_n, \text{Sel}_p(K_n^E, E[p^n]))$.

be the restriction map. Then, the following hold.

1. There exists a non-negative integer $\nu_{\text{Ker}}^{\text{Ker}}$ such that

$$\# \text{Ker}(\text{res}_n) \leq p^{\nu_{\text{Ker}}^{\text{Ker}}}$$

for any $n \in \mathbb{Z}_{\geq 0}$.

2. There exists a non-negative integer $\nu_{\text{Coker}}^{\text{Coker}}$ such that

$$\# \text{Coker}(\text{res}_n) \leq p^{\nu_{\text{Coker}}^{\text{Coker}}}$$

for any $n \in \mathbb{Z}_{\geq 0}$.

In order to prove Proposition 5.7, we need the following theorem:

Theorem 5.8. Let $\ell$ be a prime number, and $F/\mathbb{Q}_\ell$ a finite extension. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{F}$, and regard $\mu_{p^n}$ as a subset of $\overline{F}$. If $\ell$ is distinct from $p$, suppose that $E_{F(\mu_{p^n})}$ has additive reduction for any $n \geq 1$. Then, the sequence

$$\{ \# E(F(\mu_{p^n}[p^n]) \}_{n \geq 0}$$

is bounded.
Lemma 3.3. The base change 

\[ E_n,0(F_n) := \pi^{-1}(\overline{E}_{n,0}(\kappa_n)) \]

of the reduction map \( \pi \) induces a short exact sequence

\[ 0 \rightarrow E_{n,1}(F_n) \rightarrow E_{n,0}(F_n) \rightarrow \overline{E}_{n,0}(\kappa_n) \rightarrow 0, \]

where the left term \( E_{n,1}(F_n) \) is defined by the exactness ([Si1, Chapter VII, Proposition 2.1]). From the assumption that \( E_n \) has additive reduction, the order of the quotient \( E_n(F_n)/E_n,0(F_n) \) is at most 4 ([Si1, Chapter VII, Theorem 6.1]). Hence, it is enough to show that \( \{ \#E_{n,0}(F_n)|p^\infty \} \) is bounded. The above sequence (5.4) induces

\[ 0 \rightarrow E_{n,1}(F_n)[p^m] \rightarrow E_{n,0}[p^m] \rightarrow \overline{E}_{n,0}(\kappa_n)[p^m] \rightarrow E_{n,1}(F_n)/p^mE_{n,1}(F_n) \]

for any \( m \geq 1 \). Since \( E_{n,1}(F_n) \) is written by the group associated to the formal group law and has no non-trivial points of order \( p^m \) ([Si1, Chapter VII, Proposition 3.1]), we obtain \( E_{n,1}(F_n)[p^m] = E_{n,0}[p^m]E_{n,1}(F_n) = 0 \). From the assumption that \( E_n \) has additive reduction again, it follows that \( \overline{E}_{n,0}(\kappa_n) \) is isomorphic to the additive group \( \kappa_n \) ([Si1, Chapter III, Exercise 3.5]) so that \( \overline{E}_{n,0}(\kappa_n)[p^m] = 0 \). The assertion follows from this. \( \Box \)

Lemma 5.9. Suppose that \( E \) has potentially multiplicative reduction at a prime number \( \ell \) (distinct from \( p \)). Then, there exists an integer \( N_\ell \) such that for any \( n \in \mathbb{Z}_{\geq N_\ell} \) and any place \( w \) of \( K_{n,\infty} \) above \( \ell \), we have \( p^nE(K_{n,w}^E)[p^\infty] = 0 \).

Proof. Suppose that \( E \) has potentially multiplicative reduction at a prime \( \ell \).

Claim. There exists a finite Galois extension field \( L \) of \( \mathbb{Q} \) contained in \( K_{n,\infty}^E \) satisfying the following conditions:

(a) The elliptic curve \( E_L \) has split multiplicative reduction at every place of \( L \) above \( \ell \).

(b) Every place of \( L \) above \( \ell \) is inert in \( L_\infty := L(\mu_{p^\infty})/L \).

(c) There exists \( N \in \mathbb{Z}_{\geq 1} \) such that \( K_N \subseteq L \subseteq K_{n,\infty}^E \).

Proof. By Lemma 3.3, the base change \( E_{K_{n,\infty}^E} \) has split multiplicative reduction at every place of \( K_{n,\infty}^E \). Take any integer \( N \in \mathbb{Z}_{>0} \) satisfying \( \mu_{p^N} \subseteq \mathbb{Q}_\ell(E(\overline{\mathbb{Q}_\ell})[p]) \), and put \( L := K_{n,\infty}^E(\mu_{p^N}) \). As \( \mu_{p^N} \subseteq K_{n,\infty}^E \), the conditions (a) and (c) are satisfied. Note that \( L_\infty := K_{n,\infty}^E(\mu_{p^\infty})/K_{n,\infty}^E \) is a (cyclotomic) \( \mathbb{Z}_p \)-extension, and our choice of \( N \) implies that the group \( \text{Gal}(L_\infty/L) \) becomes a proper subgroup of the decomposition group of \( \text{Gal}(L_\infty/K_{n,\infty}^E) \) at any place \( v \) of \( K_{n,\infty}^E \) above \( \ell \). The condition (b) is satisfied. \( \Box \)

In order to prove Lemma 5.9, it suffices to show that there exists an integer \( N' \in \mathbb{Z}_{>0} \) such that for any \( n \in \mathbb{Z}_{\geq N'} \) and any place \( w \) of \( K_{n,\infty}^E \) above \( \ell \), it holds that \( E(K_{n,w}^E)[p^\infty] = E[p^n] \). For the field \( L \) and \( N \in \mathbb{Z}_{\geq 1} \) given in the above claim, take any \( n \in \mathbb{Z}_{\geq N} \) and any place \( w \) of \( K_{n,\infty}^E \) above \( \ell \). Let \( u \) be the place of \( L \) below \( w \). Since \( E_{Lu} \) has split multiplicative reduction, we have a \( G_{Lu} \)-invariant isomorphism

\[ E(L_u) \xrightarrow{\sim} T_u^\infty/q^\mathbb{Z} \]
for some \( q \in L_a \) with \( \text{ord}_a(q) > 0 \). Recall that every place of \( L \) above \( \ell \) is inert in \( L_{\infty}/L \). By the isomorphism (5.5), if \( n \geq N_0 := N + \text{ord}_a(q) \), then we have \( E(K_{n,w}^E)[p^n] = E[p^n] \), and \( E(K_{n,w}^E)[p^{\infty}] \cong \mu_{p^{\infty}} \times q^{0-n}E / q^zE \).

\[ \square \]

**Lemma 5.10** ([BK, Example 3.11]). For any prime number \( \ell \) distinct from \( p \) and any finite extension \( F'/\mathbb{Q}_\ell \), it holds that \( H^1(F', E[p^{\infty}]) = 0 \).

For each \( n \in \mathbb{Z}_{\geq 0} \), we denote by \( \Sigma_{n,p} \) the set of all the finite places \( v \) of \( K_n \) above \( p \), and \( \Sigma_{n,\text{bad}} \) by the set of all the finite places \( v \) of \( K_n \) where \( E_{K_n,w} \) has bad reduction. We put \( \Sigma_n := \Sigma_{n,p} \cup \Sigma_{n,\text{bad}} \), and define \( \Sigma_n^0 \) to be the subset of \( \Sigma_n \) consisting of all the places \( v \) which lies below some \( w \in \Sigma_m \) for every \( m \in \mathbb{Z}_{\geq n} \). Namely, we have

\[
\Sigma_n^0 = \left\{ v \in \Sigma_n \setminus \Sigma_{n,p} \mid \text{for any } m \geq 0, \text{ the elliptic curve } E_{K_m} \text{ has bad reduction for some } w \mid v \right\}.
\]

**Proof of Proposition 5.7.** In this proof, once we fix \( n \in \mathbb{Z}_{\geq 0} \) and simplify the notation \( H^1(F'/F, E[p^n]) = H^1(F'/F) \) for an extension \( F'/F \). We denote by \( K_n, \Sigma_n \) the maximal unramified extension of \( K_n \) outside \( \Sigma_n \). As noted in Remark 4.5, the fine Selmer group \( \text{Sel}_p(K_n, E[p^n]) \) is a subgroup of \( H^1(K_n, \Sigma_n, K_p) \). The Hochschild–Serre spectral sequence gives the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ker}(\text{res}_n^{\text{loc}}) & \longrightarrow & \text{Coker}(i_n) & \longrightarrow & \text{Coker}(i_n^E) \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & H^1(K_n^{E}/K_n) & \overset{\text{inf}}{\longrightarrow} & H^1(K_n, \Sigma_n/K_n) & \overset{\text{res}_n}{\longrightarrow} & H^1(K_n, \Sigma_n/K_n)^{G_{K_n}} & \longrightarrow & H^2(K_n^{E}/K_n) \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \text{Ker}(\text{res}_n^{\text{Sel}}) & \longrightarrow & \text{Sel}_p(K_n, E[p^n]) & \overset{\text{res}_n^{\text{Sel}}}{\longrightarrow} & \text{Sel}_p(K_n^{E}, E[p^n])^{G_{K_n}} & \longrightarrow & \text{Coker}(\text{res}_n^{\text{Sel}}). \\
\end{array}
\]

The snake lemma induces the exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ker}(\text{res}_n^{\text{Sel}}) & \longrightarrow & H^1(K_n^{E}/K_n) & \longrightarrow & \text{Ker}(\text{res}_n^{\text{loc}}) \\
& & \downarrow & & \delta & & \downarrow \\
& & \text{Coker}(\text{res}_n^{\text{Sel}}) & \longrightarrow & H^2(K_n^{E}/K_n). \\
\end{array}
\]

By Proposition 5.1, the order of \( H^1(K_n^{E}/K_n) = H^1(K_n^{E}/K_n, E[p^n]) \) is bounded independently of \( n \), and so is the kernel \( \text{Ker}(\text{res}_n^{\text{Sel}}) \).

Let us investigate the cokernel of \( \text{res}_n^{\text{Sel}} \). For each finite place \( v \) in \( K_n \), we define restriction maps

\[
\begin{align*}
\text{res}_{n,v}^{\text{loc}} & : H^1(K_{n,v}, E[p^n]) \longrightarrow H^0\left( K_n, \prod_{w|v} H^1(K_{n,w}^{E}, E[p^n]) \right), \\
\text{res}_{n,v}^{f} & : H^1(K_{n,v}, E[p^n]) \longrightarrow H^0\left( K_n, \prod_{w|v} H^1(K_{n,w}^{E}, E[p^n]) \right), \quad \text{and} \\
\text{res}_{n,v}^{\text{loc}} & : H^1(K_{n,v}, E[p^n]) \longrightarrow H^0\left( K_n, \prod_{w|v} H^1(K_{n,w}^{E}, E[p^n]) \right).
\end{align*}
\]
These maps induce the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^1_f(K_{n,v}) & \rightarrow & H^1(K_{n,v}) & \rightarrow & H^1(\ker(\text{res}_n^{\text{loc}})) & \rightarrow & 0 \\
\downarrow \text{res}_{n,v}^{\text{loc}} & & \downarrow \text{res}_{n,v}^{\text{loc}} & & \downarrow \text{res}_{n,v}^{\text{loc}} & & \downarrow \text{res}_{n,v}^{\text{loc}} & & \\
0 & \rightarrow & \left( \prod_{w \mid v} H^1_f(K_{n,w}^E) \right)^{G_{K_n}} & \rightarrow & \left( \prod_{w \mid v} H^1(K_{n,w}^E) \right)^{G_{K_n}} & \rightarrow & \left( \prod_{w \mid v} H^1(\ker(\text{res}_n^{\text{loc}})) \right)^{G_{K_n}} & \rightarrow & 0
\end{array}
\]

(5.8)

By Proposition 5.1, the group \( H^2(K_n^E/K_n) = H^2(K_n^E/K_n, E[p^n]) \) is finite and its order is bounded independently of \( n \). From the exact sequence (5.7), it is enough to give a bound for \( \{ \# \ker(\text{res}_n^{\text{loc}}) \}_{n \geq 0} \). By diagram chase and the definition of the fine Selmer groups (Definition 4.3), we have an injective homomorphism

\[
\ker(\text{res}_n^{\text{loc}}) \subset \prod_{v \mid p} \ker(\text{res}_n^{\text{loc}}) \times \prod_{v \mid p} \ker(\text{res}_n^{\text{loc}}).
\]

(5.9)

When \( E_{K_n} \) has good reduction at a finite place \( v \nmid p \) of \( K_n \), then the Tate module \( T_\ell(E) \) for the prime number \( \ell \) with \( v \mid \ell \) is unramified ([Sil, Chapter VII, Theorem 7.1]). We have \( H^1_f(K_{n,v}^E, E[p^n]) = H^1_{ur}(K_{n,v}^E, E[p^n]) \) and \( H^1_f(K_{n,w}^E, E[p^n]) = H^1_{ur}(K_{n,w}^E, E[p^n]) \) ([Ru, Lemma 1.3.8 (ii)]). Moreover, the extension \( K_{n,w}^E/K_{n,v}^E \) is unramified for any \( w \mid v \) as \( E[p^n] \) is unramified. From the definition of the unramified cohomology (cf. Notation), we have a commutative diagram

\[
\begin{array}{ccc}
H^1(K_{n,v}^E, E[p^n]) & \rightarrow & \prod_{w \mid v} \left( H^1(K_{n,w}^E, E[p^n]) \right)^{G_{K_n}} \\
\downarrow \text{res}_{n,v}^{\text{loc}} & & \downarrow \\
H^1(K_{n,v}^{ur}, E[p^n]) & \rightarrow & \prod_{w \mid v} H^1(K_{n,w}^{ur}, E[p^n])^{G_{K_n}}
\end{array}
\]

From the inflation-restriction sequence ([Ru, Proposition B.2.5 (i)]), the kernel of the bottom horizontal map \( \text{res}_{n,v}^{\text{loc}} \) is \( \bigcap_{w \mid v} H^1(K_{n,w}^{ur}/K_{n,v}^{ur}, E[p^n])^{G_{K_n}} = 0 \) and the map \( \text{res}_{n,v}^{\text{loc}} \) is injective. In particular, we have \( \ker(\text{res}_n^{\text{loc}}) = 0 \) for any finite place \( v \notin \Sigma_n \). This implies that the order of \( \ker(\text{res}_n^{\text{loc}}) \) is bounded independently of \( n \) for the case \( v \nmid p \) and \( v \notin \Sigma_n^0 \). From (5.9), in order to prove Proposition 5.7 (2), it is left to show the following assertions:

- For \( v \mid p \), the sequence \( \{ \# \ker(\text{res}_n^{\text{loc}}) \}_{n \geq 0} \) is bounded.
- For \( v \nmid p \), and \( v \in \Sigma_n^0 \), the sequence \( \{ \# \ker(\text{res}_n^{\text{loc}}) \}_{n \geq 0} \) is bounded.

By applying the snake lemma to the diagram (5.8), there is an exact sequence

\[
0 \rightarrow \ker(\text{res}_n^{f,v}) \rightarrow \ker(\text{res}_n^{\text{loc},v}) \rightarrow \ker(\text{res}_n^{\text{loc},v}) \rightarrow \coker(\text{res}_n^{f,v}).
\]

The assertion (2) in Proposition 5.7 follows from the lemma below (Lemma 5.11). □
By definition (cf. (5.6)), we have

$$\Sigma_0^0 = \left\{ \ell : \text{prime number} \mid \text{for any } m \geq 0, \text{the elliptic curve } E_{K_m} \text{ has bad reduction at a place above } \ell \right\}.$$ 

We are assuming $E$ has good reduction at $p$, so that $p \not\in \Sigma_0^0$.

**Lemma 5.11.** (1) For any prime number $\ell \in \Sigma_0^0$ (distinct from $p$), the set

$$\{ \# \text{Ker}(\text{res}_{n,v}^{loc}) \mid n \geq 0, v \mid \ell \}$$

is bounded.

(2) For the fixed prime $p$, the set

$$\{ \# \text{Ker}(\text{res}_{n,v}^{loc}) \mid n \geq 0, v \mid p \}$$

is bounded.

(3) For any prime number $\ell \in \Sigma_0^0$ (distinct from $p$), the set

$$\{ \# \text{Coker}(\text{res}_{n,v}^f) \mid n \geq 0, v \mid \ell \}$$

is bounded.

**Proof.** First, we prove the following claim.

**Claim 1.** There exists a finite Galois extension field $L$ of $\mathbb{Q}$ contained in $K^E_\infty = \mathbb{Q}(E[p^{\infty}])$ satisfying the following conditions.

(a) The elliptic curve $E_L$ has semistable reduction everywhere.
(b) The elliptic curve $E_L$ has split multiplicative reduction at every place $u$ of $L$ above a prime number $q$ where $E$ has potentially multiplicative reduction.
(c) Every place of $L$ above every $\ell \in \Sigma_0^0$ is inert in $L_\infty := L(\mu_{p^{\infty}})/L$.
(d) There exists an integer $N \in \mathbb{Z}_{\geq 0}$ such that $K_N \subseteq L \subseteq K^E_N$.

**Proof.** Let $q_0$ be a prime number where $E$ has potentially good additive reduction. Since $q_0$ is distinct from $p$, the order of the image of $G_{\mathbb{Q}_{\Sigma_0^0}}$ in $\text{Aut}_{\mathbb{Q}_p}(T_p(E))$ is finite ([S1, Chapter VII, Theorem 7.1]). This implies that there exists an intermediate field $F^{(q_0)}_{pg}$ of $K^E_\infty/\mathbb{Q}$ such that $F^{(q_0)}_{pg}/\mathbb{Q}$ is a finite Galois extension, and $E^{(q_0)}_{F_{pg}}$ has good reduction at every place above $q_0$. Let $F_{pg}$ be the composite of the fields $F^{(q)}_{pg}$ where $q$ runs all the prime numbers where $E$ has potentially good additive reduction. By Lemma 3.3, the composite field $L := F_{pg}K^E_1$ satisfies the conditions (a) and (b). Moreover, take a sufficiently large $N \in \mathbb{Z}_{\geq 0}$, and replace $L$ with $L(\mu_{p^{N}})$, the additional conditions (c) and (d) follow from the similar arguments in the proof of Lemma 5.9. Note that $L = F_{pg}K^E(\mu_{p^{N}})$ is a finite Galois extension field of $\mathbb{Q}$ contained in $K^E_\infty$. 

Put $L_n := L(\mu_{p^{n}})$ for each $n \geq 1$. Take any prime number $\ell \in \Sigma_0^0$. Fix a place $w_\infty$ of $K^E_\infty$ above $\ell$. For any $m \in \mathbb{Z}_{\geq N}$, denote by $w_m$ the place of $K^E_m$ below $w_\infty$ and by $u_m$ the place of $L_m$ below $w_\infty$ respectively.

Let us prove the assertion (1). Take any $n \in \mathbb{Z}_{\geq N}$, and let $v = v_n$ be the place of $K_N$ below $w_\infty$. For the fixed place $w_n$, by identifying $G_{K_{n,v}}$ with the decomposition subgroup of $G_{K_n}$ at $v$, we consider $H^1(K^E_{n,w_n}, E[p^n])$ as an $G_{K_{n,w_n}}$-module and $\prod_{u \mid \ell} H^1(K^E_{n,w_n}, E[p^n])$ is isomorphic to the induced module $\text{Ind}^{G_{K_{n,w_n}}}_{G_{K_n}}(H^1(K^E_{n,w_n}, E[p^n]))$. 


Shapiro’s lemma gives an isomorphism

\[(5.10) \quad H^0 \left( K_n, \prod_{w | v} H^1(K_{n,w,E}^E, E[p^n]) \right) \cong H^0 \left( K_{n,v}, H^1(K_{n,w_n,E}^E, E[p^n]) \right)\]

(cf. [NSW, (1.6.4) Proposition]). By the Hochschild-Serre exact sequence ([Ru, Proposition B.2.5 (ii)]), we obtain the following commutative diagram whose rows are exact:

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Ker}(\text{res}_{n,v}^{\text{loc}}) & \rightarrow & H^1(K_{n,v}, E[p^n]) & \rightarrow & \prod_{w | v} H^1(K_{n,w,E}^E, E[p^n])^{G_{K_n}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^1(K_{n,w_n,E}^E, E[p^n]) & \rightarrow & H^1(K_{n,v}, E[p^n]) & \rightarrow & H^1(K_{n,w_n,E}^E, E[p^n])^{G_{K_n}} \\
\end{array}
\]

\[(5.10) \cong \]

It holds that

\[(5.11) \quad \text{Ker}(\text{res}_{n,v}^{\text{loc}}) \cong H^1(K_{n,w_n,E}^E / K_{n,v,E}^E, E[p^n]).\]

The order of \(\text{Ker}(\text{res}_{n,v}^{\text{loc}})\) depends only on the prime number \(\ell\) and the positive integer \(n\) (in particular, it is independent of the choice of the place \(w_\infty\) of \(K_{\infty}^E\) above the fixed prime number \(\ell\)). For any intermediate field \(M\) of \(K_{n,w_n,E}^E / K_{n,v,E}^E\) which is Galois over \(K_{n,v,E}^E\), we have an exact sequence

\[
0 \rightarrow Y_n(M) \rightarrow \text{Ker}(\text{res}_{n,v}^{\text{loc}}) \rightarrow Z_n(M),
\]

where we put

\[
Y_n(M) := H^1(M / K_{n,v,E}^E, E(M)[p^n]), \quad \text{and} \quad Z_n(M) := H^0(K_{n,v,E}^E, H^1(K_{n,w_n,E}^E / M, E[p^n])).
\]

First, let us study the cases when \(\ell \neq p\). Recall that \(E_{K_{n,w_n,E}}\) has good or split multiplicative reduction and \(E_{K_{n,v,E}}\) has additive reduction from the very definition of \(\Sigma_0\).

**The case: Potentially good reduction with \(\ell \neq p\).** Suppose that \(\ell \neq p\), and \(E\) has potentially good reduction at \(\ell\). Let \(M_n\) be the maximal subfield of \(K_{n,w_n,E}^E\) which is unramified above \(K_{n,v,E}^E\). As the extension \(M_n / K_{n,v,E}^E\) is cyclic, we have

\[
H^1(M_n / K_{n,v,E}^E, E(M_n)[p^n]) \cong \hat{H}^{-1}(M_n / K_{n,v,E}^E, E(M_n)[p^n])
\]

\[
= \frac{\text{Ker} \left( N_{M_n/K_{n,v,E}^E} : E(M_n)[p^n] \rightarrow E(K_{n,v,E}^E)[p^n] \right)}{\langle \text{Frob}_v - 1 \rangle},
\]

where \(\hat{H}^*\) stands for the Tate cohomology group, \(N_{M_n/K_{n,v,E}^E}\) is the norm map and \(\text{Frob}_v\) is the Frobenius automorphism at \(v\) which is a generator of the cyclic group.
Gal(M_n/K_{n,v}) (cf. [Se1, Chapitre VIII, §4]). There are (in)equalities below:

\[
\begin{align*}
\#Y_n(M_n) &= \#H^1(M_n/K_{n,v}, E(M_n)[p^n]) \\
&= \#\hat{H}^{-1}(M_n/K_{n,v}, E(M_n)[p^n]) \\
&\leq \# \left( \frac{E(M_n)[p^n]}{\langle \text{Frob}_v - 1 \rangle} \right) \\
&= \#(E(M_n)[p^n])[\text{Frob}_v - 1] \\
&= \#E(K_{n,v})[p^n] \\
&\leq \#E(K_{n,v})[p^\infty].
\end{align*}
\]

From Theorem 5.8, the sequence \{ \#Y_n(M_n) \}_{n \geq 0} is bounded. Let us study \( Z_n(M_n) \).

Note that \( K_{n,w_n}/L_{n,u_n} \) is unramified because \( E_{L_n,u_n} \) has good reduction. Since \( K_{n,w_n}/M_n \) is totally ramified, we have

\[
[K_{n,w_n} : M_n] = [L_{n,u_n} : L_{n,u_n} \cap M_n] \leq [L_{n,u_n} : K_{n,v}] \leq [L : K_N].
\]

This implies that

\[
\sup_{n \geq N} \#Z_n(M_n) \leq \sup_{n \geq N} \#H^1(K_{n,w_n}/M_n, E[p^n]) < \infty.
\]

Consequently, the set \{ \#\ker(res_{n,v}) | n \geq 0, v | \ell \} is bounded.

**The case: Potentially multiplicative reduction with \( \ell \neq p \).** Suppose that \( \ell \neq p \), and \( E \) has potentially multiplicative reduction at \( \ell \). Put \( Y_n := Y_n(L_{n,u_n}) \) and \( Z_n := Z_n(L_{n,u_n}) \). Let \( u := u_N \) be the place of \( L \) below \( u_n \). The elliptic curve \( E_{L_u} \) is isomorphic to a Tate curve \( \mathbb{G}_m/q_u^\mathbb{Z} \), and in particular, we have a \( G_{L_u} \)-equivariant isomorphism

\[
E[p^\infty] \simeq \mu_{p^\infty} \times q_u^\mathbb{Z}/q_u^\mathbb{Z}.
\]

This implies that \( K_{\infty,w_\infty}/L_{\infty,u_\infty} \) is a totally ramified cyclic extension, where \( u_\infty \) is the place of \( L_\infty \) below \( w_\infty \). Fix a topological generator \( \tau \in \Gal(K_{\infty,w_\infty}/L_{\infty,u_\infty}) \). Since \( L_{\infty,u_\infty}/L_u \) is unramified, the homomorphism

\[
\Gal(K_{\infty,w_\infty}/L_{\infty,u_\infty}) \to \Gal(K_{n,w_n}/L_{n,u_n}); \sigma \mapsto \sigma |_{K_{n,w_n}}
\]

is surjective. Firstly, we show that \{ \#Y_n \}_{n \geq 0} is bounded. We define

\[
E'_n := (E(K_{n,w_n})[p^\infty], [p^n]).
\]

The isomorphism (5.12) implies that \( E'_n \) is isomorphic to \( \mu_{p^n} \) as a \( \mathbb{Z}_p[G_{L,u}] \)-module. Note that \( E'_n \) is \( G_{K_{n,v}} \)-stable as \( K_{n,w_n}/K_{n,v} \) is a Galois extension. We obtain an exact sequence

\[
Y'_n \longrightarrow Y_n \longrightarrow Y''_n,
\]

where we put

\[
Y'_n := H^1(L_{n,u_n}/K_{n,v}, E'_n), \quad \text{and} \quad Y''_n := H^1(L_{n,u_n}/K_{n,v}, E(L_{n,u_n})[p^n]/E'_n).
\]

Let us study \( Y''_n \). Note that \( \tau \) acts on \( T_p(E) \) non-trivially and unipotently. Putting \( \nu_\tau := \ord_p(\#(T_p(E)/\langle \tau - 1 \rangle)_{\text{tor}}) \), we have

\[
\#(E(L_{n,u_n})[p^n]/E'_n) \leq \#(E(L_{\infty,u_{\infty}})[p^\infty]/E'_n) = p^\nu_\tau.
\]
It follows that the sequence \( \{ \#Y_n \}_{n \geq 0} \) is bounded, because of the inequality \( |L_{n,u} : K_{n,v}| \leq [L : \mathbb{Q}] \). Let us consider \( Y_n' \). We define \( H_n \) to be the maximal subgroup of \( \text{Gal}(L_{n,u} / K_{n,v}) \) acting trivially on \( E_n' \) and \( L_n' \) the maximal subfield of \( L_{n,u} \) fixed by \( H_n \). Now, we consider an exact sequence

\[
0 \rightarrow H^1(L_n' / K_{n,v}, E_n') \rightarrow Y_n' \rightarrow H^1(L_{n,u} / L_n', E_n')
\]

By (C2) for \( E \), we know \( H^0(K_{n,v}, E_n') = 0 \) (Lemma 3.4). Since \( L_n' / K_{n,v} \) is cyclic, we have \( H^1(L_n' / K_{n,v}, E_n') \simeq \hat{H}^{-1}(L_n' / K_{n,v}, E_n') \) (cf. [Sel, CHAPITRE VIII, §4]). For \( E_n' \) is finite, its Herbrand quotient is trivial so that \( \# \hat{H}^{-1}(L_n' / K_{n,v}, E_n') = \# \hat{H}^0(L_n' / K_{n,v}, E_n') \) ([Sel, CHAPITRE VIII, §4, PROPOSITION 8]). Therefore, we have

\[
\#H^1(L_n' / K_{n,v}, E_n') = \#\hat{H}^0(L_n' / K_{n,v}, E_n') \leq \#H^0(K_{n,v}, E_n') = 1.
\]

Since \( L_{n,u} / L_n' \) is a cyclic extension whose order is at most \( [L : \mathbb{Q}] \), we have

\[
\#H^1(L_{n,u} / L_n', E_n') = \#H^0(L_{n,u} / L_n', E_n') = \#H^0(L_{n,u} / L_n', E_n') \leq [L : \mathbb{Q}].
\]

The sequence \( \{ \#Y_n \}_{n \geq 0} \) is bounded. This implies that \( \{ \#Z_n \}_{n \geq 0} \) is bounded by (5.13). Secondly, let us show that \( \{ \#Z_n \}_{n \geq 0} \) is bounded. We have an exact sequence

\[
\begin{align*}
H^0(L_{n,u}, E[p^n] / E_n') & \delta_n \rightarrow H^1(K_{n,w}^E / L_{n,u}, E_n') \\
& \rightarrow H^1(K_{n,w}^E / L_{n,u}, E[p^n] / E_n').
\end{align*}
\]

We put

\[
Z'_n := H^0(K_{n,v}, \text{Coker}(\delta_n)), \quad \text{and} \quad Z''_n := H^0(K_{n,v}, H^1(K_{n,w}^E / L_{n,u}, E[p^n] / E_n')).
\]

In order to prove that \( \{ \#Z_n \}_{n \geq 0} \) is bounded, it suffices to show that \( \{ \#Z'_n \}_{n \geq 0} \) and \( \{ \#Z''_n \}_{n \geq 0} \) are bounded. Let us show that \( \{ \#Z'_n \} \) is bounded. The generator \( \tau_n := \tau_{K_{n,w}^E / L_{n,u}} \) of the Galois group \( \text{Gal}(K_{n,w}^E / L_{n,u}) \) acts trivially on \( E[p^n] / E_n' \). It holds that

\[
H^0(L_{n,u}, E[p^n] / E_n') = E[p^n] / E_n' \simeq \mathbb{Z} / p^n.
\]

We also have an isomorphism

\[
H^1(K_{n,w}^E / L_{n,u}, E_n') = \text{Hom}(\text{Gal}(K_{n,w}^E / L_{n,u}), E_n') \xrightarrow{\simeq} E_n'[p^{M_n}]
\]

given by the evaluation at \( \tau_n \), where \( M_n := \text{ord}_p([K_{n,w}^E : L_{n,u}]) \). We denote by \( \overline{E}_n' \) the image of \( E_n' \) in \( E[p^n] / \langle \tau - 1 \rangle \). Its order is bounded as \( \#\overline{E}_n' \leq p^{\nu_n} \). By definition, the coboundary map \( \delta_n \) is given by

\[
\delta_n : E[p^n] / E_n' \rightarrow E_n'[p^{M_n}], \quad (P \mod E_n') \mapsto \langle \tau - 1 \rangle P.
\]

We obtain

\[
\#Z'_n \leq \#\text{Coker}(\delta_n) \leq \#\overline{E}_n' \leq p^{\nu_n}.
\]

Finally, let us show that \( \{ \#Z''_n \}_{n \geq 0} \) is bounded. Note that we have an injective homomorphism

\[
H^1(K_{n,w}^E / L_{n,u}, E[p^n] / E_n') \xrightarrow{E[p^n] / E_n' \langle \tau - 1 \rangle} E[p^n] / E_n'.
\]
By (C2) for $E$ and Lemma 3.4, it holds that $\Xi_n^\ell = H^0(K_{n,v}, E[p^n]/E'_{n,v}) = 0$. This implies that \{ $\#Z_n$ \}$_{n \geq 0}$ is bounded. Hence, we deduce that \{ $\#\ker(\text{res}_{n,v}^{\text{loc}})$ \}$_{n \geq 0, v \mid \ell}$ is bounded.

Now, suppose that $\ell \neq p$, and let us show the assertion (3) of Lemma 5.11. Again, take any $n \in \mathbb{Z}_{\geq N}$, and let $v = v_n$ be the place of $K_n$ below $w_\infty$. The order of $\text{Coker}(\text{res}_{n,v}^{f})$ depends only on $\ell$ and $n$. By the short exact sequence $0 \rightarrow E[p^n] \rightarrow E[p^\infty] \rightarrow E[p^\infty]/E'_{n,v} \rightarrow 0$, there is a short exact sequence

$$0 \rightarrow A_n^0 \xrightarrow{\delta} H^1(K_{n,v}, E[p^n]) \xrightarrow{\iota_{n,v}} H^1(K_{n,v}, E[p^\infty])[p^n],$$

where $A_n^0 := E(K_{n,v})[p^n] \otimes_{\mathbb{Z}_p} (\mathbb{Z}/p^n \mathbb{Z})$. Recall that $H^1_1(K_{n,v}, E[p^n])$ is defined to be the inverse image of $H^1_1(K_{n,v}, E[p^\infty])$ by $\iota_{n,v}$ (cf. (4.1)). Thus, the map $\iota_{n,v}$ induces the short exact sequence $0 \rightarrow A_n^0 \rightarrow B_n^0 \rightarrow C_n^0$, where

$$B_n^0 := H^1_1(K_{n,v}, E[p^n]), \text{ and } C_n^0 := H^1_1(K_{n,v}, E[p^\infty])[p^n].$$

Furthermore, we obtain a commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & A_n^0 \\
\downarrow a_n & & \downarrow b_n \\
0 & \rightarrow & B_n^0 \\
\downarrow c_n & & \downarrow c_n \\
0 & \rightarrow & C_n^0
\end{array}$$

whose rows are exact, where

$$A_n^1 := H^0 \left( K_n, \prod_{w \mid v} E(K_{n,w})[p^\infty] \otimes_{\mathbb{Z}_p} (\mathbb{Z}/p^n \mathbb{Z}) \right),$$

$$B_n^1 := H^0 \left( K_n, \prod_{w \mid v} H^1_1(K_{n,w}, E[p^n]) \right),$$

$$C_n^1 := H^0 \left( K_n, \prod_{w \mid v} H^1_1(K_{n,w}, E[p^\infty])[p^n] \right),$$

and the arrows $a_n$ and $c_n$ are restriction maps, and $b_n = \text{res}_{n,v}^{f}$. By Lemma 5.10, we have $C_n^0 = C_n^1 = 0$. In order to prove Lemma 5.11 (3), it suffices to show that the sequence \{ $\#\text{Coker}(a_n)$ \}$_{n \geq N}$ is bounded. By the exact sequence

$$0 \rightarrow p^n E(K_{n,v}^E)[p^\infty] \rightarrow E(K_{n,v}^E)[p^\infty] \rightarrow E(K_{n,v}^E)[p^\infty] \otimes_{\mathbb{Z}_p} (\mathbb{Z}/p^n \mathbb{Z}) \rightarrow 0,$$

using Shapiro’s lemma as in (5.10), we obtain an exact sequence

$$E(K_{n,v})[p^\infty] \xrightarrow{a_n} A_n^1 \rightarrow H^1 \left( K_n^E/K_n, \prod_{w \mid v} p^n E(K_{n,w}^E)[p^\infty] \right) =: \Xi_n.$$

In order to prove that \{ $\#\text{Coker}(a_n)$ \}$_{n \in \mathbb{Z}_{\geq N}}$ is bounded, it suffices to show that \{ $\#\Xi_n$ \}$_{n}$ is bounded. Fix a place $w_n$ of $K_n^E$. We have

$$\Xi_n = H^1(K_{n,w_n}^E/K_{n,v}, p^n E(K_{n,w_n}^E)[p^\infty]).$$
For any intermediate field $M$ of $K_{n,w_n}^E/K_{n,v}$ which is Galois over $K_{n,v}$, we have the inflation-restriction exact sequence

$$0 \rightarrow \Xi'_n(M) \rightarrow \Xi_n \rightarrow \Xi''_n(M),$$

where we put

$$\Xi'_n(M) := H^1\left(M/K_{n,v}, H^0(M, p^n E(K_{n,w_n}^E)[p^\infty])\right), \quad \text{and}$$

$$\Xi''_n(M) := H^0\left(K_{n,v}, H^1\left(K_{n,w_n}^E/M, p^n E(K_{n,w_n}^E)[p^\infty]\right)\right).$$

Recall that $K_{n}^E$ contains $L$, the elliptic curve $E_{K_{n}^E}$ has semistable reduction everywhere. Let $u_n$ be the place of $L_n := L(\mu_{p^n})$ below $w_n$.

**The case: Good reduction** Suppose that $E_{L_{n,u_n}}$ has good reduction. Let $M_n$ be the maximal subfield of $K_{n,w_n}^E$ which is unramified over $K_{n,v}$. By similar arguments of the boundedness of $\{\#Y_n(M_n)\}_{n \geq 0}$ for **The case: Potentially good reduction with $\ell \neq p$** in the proof of (1), we have

$$\#\Xi'_n(M_n) = \#H^1(M_n/K_{n,v}, p^n E(K_{n,w_n}^E)[p^\infty])$$

$$\leq \#\left(\frac{p^n E(M_n)[p^\infty]}{\langle \text{Frob}_n - 1 \rangle}\right)$$

$$= \#p^n E(K_{n,v})[p^\infty]$$

$$\leq \#E(K_{n,v})[p^\infty].$$

**Theorem 5.8 and 1 implies that the sequence $\{\#\Xi'_n(M_n)\}_{n \geq 0}$ is bounded. Moreover, as noted in the proof of the boundedness of $\{\#Z_n(M_n)\}_{n \geq 0}$ in **The case: Potentially good reduction with $\ell \neq p$**, the sequence $\{\#Z_n(M_n)\}_{n \geq N}$ is bounded, and hence $\{\#\Xi'_n(M_n)\}_{n \geq N}$ is bounded.

**The case: Multiplicative reduction** Suppose that $E_{L_{n,u_n}}$ has multiplicative reduction. Put $\Xi'_n := \Xi'_n(L_{n,u_n})$ and $\Xi''_n := \Xi''_n(L_{n,u_n})$. In this case, Lemma 5.9 implies that $\Xi'_n = 0$ and $\Xi''_n = 0$ for sufficiently large $n$, and in particular, the sequences $\{\#\Xi'_n\}_{n \geq N}$ and $\{\#\Xi''_n\}_{n \geq N}$ is bounded.

By the above arguments, we deduce that in any case, the set $\{\#\Xi'_n\}_{n \geq N}$ is bounded and so is $\{\#\Xi''_n\}_{n \geq N}$. Accordingly, the assertion Lemma 5.11 (3) is proved.

Let us show the assertion (2). Here, we study the case when $\ell = p$. Recall that by our assumption, the elliptic curve $E$ has good reduction at $p$.

**The case: Good ordinary reduction** Suppose that the elliptic curve $E$ has good ordinary reduction at $p$. In this case, there exists a $G_{Q_p}$-stable $\mathbb{Z}/p^n\mathbb{Z}$-submodule $\text{Fil}E[p^n]$ of $E[p^n]$ of rank one such that the inertia group $I_p$ of $G_{Q_p}$ acts via the cyclotomic character on $\text{Fil}E[p^n]$, and trivially on $E[p^n]/\text{Fil}E[p^n]$. Fix a generator $P_n$ of the cyclic $\mathbb{Z}_p$-module $\text{Fil}E[p^n]$ and a lift $Q_n \in E[p^n]$ of a generator of the cyclic $\mathbb{Z}_p$-module $\overline{Q}_n \in E[p^n]/\text{Fil}E[p^n]$. The pair $(P_n, Q_n)$ becomes a basis of the free $\mathbb{Z}/p^n\mathbb{Z}$-module of rank two. Let $M_n$ be the maximal subfield of $K_{n,w_n}^E$ which is unramified over $K_{n,v}$, and put $I_n := \text{Gal}(K_{n,w_n}^E/M_n)$. Since $I_n$ acts trivially on $\text{Fil}E[p^n]$ and $E[p^n]/\text{Fil}E[p^n]$, the group $I_n$ is a cyclic group which is generated by an element acting on $E[p^n]$ via a unipotent matrix

$$U = \left(\begin{array}{cc} 1 & x_n \\ 0 & 1 \end{array}\right) \in M_2(\mathbb{Z}/p^n\mathbb{Z})$$
under the basis \((P_n, Q_n)\). Fix a lift \(\tau \in \text{Gal}(K_{n,w_n}^{E} / K_{n,v})\) of the Frobenius \(\text{Frob}_v \in \text{Gal}(M_n/K_{n,v})\). The filtration \(\text{Fil}_p E[p^n]\) is stable under the action of \(\text{Gal}(K_{n,w_n}^{E} / K_{n,v})\), and the Weil pairing \(e: E[p^n] \times E[p^n] \to \mu_p\) is an alternative pairing preserving the action of \(\text{Gal}(K_{n,w_n}^{E} / K_{n,v})\) ([Si1, Chapter III, Section 8]). Accordingly, the fixed lift \(\tau\) acts on \(E[p^n]\) by a matrix

\[
A = \begin{pmatrix} a_n & b_n \\ 0 & a_{n-1} \end{pmatrix} \in M_2(\mathbb{Z}/p^n\mathbb{Z})
\]

for some \(a_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times\) and \(b_n \in \mathbb{Z}/p^n\mathbb{Z}\). We can define \(a := (a_n)_n \in \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times\). Since \(E\) has good reduction at \(p\), Theorem 5.8 implies that \(a^k \neq 1\) for any \(k \in \mathbb{Z}_{>0}\). In fact, if \(a^k = 1\), then for any \(m \in \mathbb{Z}_{\geq 0}\), the group \(\text{Fil}_p E[p^m]\) of order \(p^m\) is contained in \(E(\mathbb{Q}_p^\mu(\mu_{p^m}))\), and contradicts to Theorem 5.8. Here, we denote by \(\mathbb{Q}_p^\mu\) the unramified extension field of \(\mathbb{Q}_p\) of degree \(k\). It holds that

\[
AUA^{-1} = \begin{pmatrix} 1 & a_n^2 x_n \\ 0 & 1 \end{pmatrix} = U^{a_n^2}.
\]

By the short exact sequence \(0 \to \text{Fil}_p E[p^n] \to E[p^n] \to E[p^n]/\text{Fil}_p E[p^n] \to 0\) and (5.11), we obtain an exact sequence

\[
Y_n \to \ker(\text{res}^{b_0}_{n,v}) \to Z_n,
\]

where

\[
Y_n := H^1(K_{n,w_n}^{E} / K_{n,v}, \text{Fil}_p E[p^n]), \quad Z_n := H^1(K_{n,w_n}^{E} / K_{n,v}, E[p^n]/\text{Fil}_p E[p^n]).
\]

In order to show that \(\{ \# \ker(\text{res}^{b_0}_{n,v}) \}_{n \geq 0}\) is bounded, it is sufficient to prove that both \(\{ Y_n \}_{n \geq 0}\) and \(\{ Z_n \}_{n \geq 0}\) are bounded.

First, let us study the order of \(Z_n\). Since \(I_n\) acts trivially on \(E[p^n]/\text{Fil}_p E[p^n]\), we have an exact sequence

\[
0 \to Z'_n \to Z_n \to Z''_n,
\]

where

\[
Z'_n := H^1(M_n/K_{n,v}, E[p^n]/\text{Fil}_p E[p^n]), \quad Z''_n := H^0(K_{n,v}, H^1(K_{n,w_n}^{E} / M_n, E[p^n]/\text{Fil}_p E[p^n])).
\]

Since \(a \neq 1\), we have \(\#(E[p^\infty]/\text{Fil}_p E[p^\infty])[a^{-1} - 1] < \infty\), and

\[
\# Z'_n \leq \# \left( \frac{E[p^n]/\text{Fil}_p E[p^n]}{(\tau - 1)} \right)
= \# \left( \frac{E[p^n]/\text{Fil}_p E[p^n]}{a^{-1} - 1} \right)
= \#(E[p^n]/\text{Fil}_p E[p^n])[a^{-1} - 1]
\leq \#(E[p^\infty]/\text{Fil}_p E[p^\infty])[a^{-1} - 1].
\]

The sequence \(\{ Z'_n \}_{n \geq 0}\) is bounded. Let us consider the order of \(Z''_n\). The matrix presentation (5.14) implies that the Galois group \(\text{Gal}(M_n/K_{n,v}) = \langle \text{Frob}_v \rangle\) acts on \(E[p^n]/\text{Fil}_p E[p^n]\) via the character \(\text{Frob}_v \mapsto a_n^{-1}\), and (5.15) implies that \(\text{Frob}_v \in \)
Gal($M_n/K_{n,v}$) acts on $I_n$ via the character Frob$_v \mapsto a_n$. Since $a^3 \neq 1$, namely $a^2 \neq a^{-1}$, there exists an integer $m_0 \in \mathbb{Z}_{>0}$ such that $a^{m_0} \neq a^{-m_0}$. We have
\[ Z''_n \subseteq \text{Hom}(I_n, E[p^{m_0-1}]/\text{Fil}_E[p^{m_0-1}]). \]

Since $I_n$ is cyclic, the sequence \( \{ \# \text{Hom}(I_n, E[p^{m_0-1}]/\text{Fil}_E[p^{m_0-1}]) \}_{n \geq 0} \) is bounded and so is \( \{ Z''_n \}_{n \geq 0} \). As a result, the sequence \( \{ \# Z_n \}_{n \geq 0} \) is bounded from (5.17).

The boundedness of \( \{ \# Y_n \} \) follows from the arguments in the previous paragraph just by replacing $E[p^m]/\text{Fil}_E[p^m]$ with $\text{Fil}_E[p^m]$, where the Galois group $\text{Gal}(M_n/K_{n,v}) = (\text{Frob}_v)$ acts via the character $\text{Frob}_v \mapsto a_n$. By the short exact sequence (5.16) we deduce that \( \{ \# \text{Ker}(\text{res}_{n,v}) \}_{n \geq 0} \) is bounded.

**The case: Good supersingular reduction** Suppose that $E$ has good supersingular reduction at $p$. In order to prove that the sequence \( \{ \# \text{Ker}(\text{res}_{n,v}) | n \geq 0, v \mid p \} \) is bounded, by (5.11) it suffices to show that $H^1(K_{n,v}/K_{n,v}, E[p^n]) = 0$ for any $n \geq 0$. The short exact sequence $0 \rightarrow E[p] \rightarrow E[p^{n+1}] \rightarrow E[p^n] \rightarrow 0$ induces the exact sequence
\[ H^1(K_{n,v}/K_{n,v}, E[p]) \rightarrow H^1(K_{n,v}^E/K_{n,v}, E[p^{n+1}]) \rightarrow H^1(K_{n,v}/K_{n,v}, E[p^n]). \]

By induction on $m$, it is enough to show that $H^1(K_{n,v}^E/K_{n,v}, E[p]) = 0$. We denote the inertia subgroup of $G_{Q_p}$ by $I_{Q_p} := \text{Gal}(\overline{Q_p}/Q_p^\text{ur})$, and the wild inertia subgroup by $I_{Q_p}^w := \text{Gal}(\overline{Q_p}/Q_p^\text{tame}) \subseteq I_{Q_p}$, where $Q_p^\text{tame}$ is the maximal tamely ramified extension of $Q_p$. Let $I_{Q_p}^n := I_{Q_p}/I_{Q_p}^n \simeq \lim_{n \rightarrow \infty} F_{p^n}$ be the tame inertia group of $G_{Q_p}$ (cf. [Se3, 1.3, Proposition 2]), and $\psi: I_{Q_p}^n \rightarrow F_{p^n}$ the character induced by the natural projection $\lim_{n \rightarrow \infty} F_{p^n} \rightarrow F_{p^n}$. The characters $\psi$ and $\psi^p$ form the fundamental characters of level 2 (cf. [Se3, 1.7]). By [Se3, 1.11, Proposition 12], the following hold.

- The action of the wild inertia subgroup $I_{Q_p}^n$ on $E[p]$ is trivial, so that the action of the inertia group $I_{Q_p}$ of $G_{Q_p}$ on $E[p]$ factors through $I_{Q_p}^n$.
- The group $E[p]$ has a structure of $F_{p^2}$-vector space of dimension 1.
- The image of $I_{Q_p}$ in $\text{Aut}(E[p])$ is a cyclic group of order $p^2 - 1$.
- The action of $I_{Q_p}^n$ on $E[p]$ is given by the fundamental character $\psi$ of level 2.

Let us regard $E[p]$ as an $F_{p^2}$-vector space, and consider the $F_{p^2}$-vector space $E[p] \otimes_{F_p} F_{p^2}$, which is the extension of scalar of $E[p]$. By the properties of $E[p]$ noted above, the action of $I_{Q_p}^n$ on $E[p] \otimes_{F_p} F_{p^2}$ is given by the matrix

\[ \begin{pmatrix} \psi & 0 \\ 0 & \psi^p \end{pmatrix} \]

after taking a suitable $F_{p^2}$-basis $E[p] \otimes_{F_p} F_{p^2}$ (cf. [Se3, 1.9, Corollaire 3], see also [Ed, 2.6 Theorem] which is a result on modulo $p$ Galois representations attached to modular forms with coefficients in $\overline{F}_p$). Let $F$ be the maximal unramified extension field of $Q_p$ contained in $K_{l,v}^E$. Put $F_n := F(\mu_{p^n})$. We have the following inflation-restriction exact sequences:

\[ H^1(F_n/Q_p(\mu_{p^n}), H^0(F_n, E[p])) \rightarrow H^1(K_{n,v}/K_{n,v}, E[p]) \]

(5.19)
and
\[ H^1(K_{1,w_1}^E(\mu_{p^n})/F_n, E[p]) \to H^1(K_{n,w_n}^E/F_n, E[p]) \]
(5.20)

\[ H^1(K_{n,w_n}^E/K_{1,w_1}^E(\mu_{p^n}), E[p])^{G_{F_n}} = \text{Hom}_{\mathbb{Z}[G_{F_n}]}(\text{Gal}(F'_n/K_{1,w_1}^E(\mu_{p^n})), E[p]), \]

where \( F'_n \) is the maximal abelian extension field of \( K_{1,w_1}^E(\mu_{p^n}) \) contained in \( K_{n,w_n}^E \).

**Claim 2.** We have \( H^0(F_n, E[p]) = 0 \) and \( H^1(K_{1,w_1}^E(\mu_{p^n})/F_n, E[p]) = 0. \)

**Proof.** We may assume \( n \geq 1 \). Since \( F'/\mathbb{Q}_p \) is unramified, the ramification index of \( F_n/\mathbb{Q}_p \) is \((p-1)p^{n-1}\), which is not divisible by \([K_{1,w_1}^E:F] = p^2 - 1\). This implies that the restrictions of \( \psi \) and \( \psi^p \) on \( I_{\mathbb{Q}_p} \cap G_{F_n} \) are non-trivial, and by (5.18), we have
\[ H^0(F_n, E[p]) \leq H^0(F_n, E[p] \otimes_{\mathbb{F}_p} \mathbb{F}_p^2) = 0. \]

Furthermore, the extension \( K_{1,w_1}^E(\mu_{p^n})/F_n \) is finite cyclic. By using the Herbrand quotient of the Tate cohomology groups ([Sel, CHAPITRE VIII, §4, PROPOSITION 8]), we have
\[ \#H^1(K_{1,w_1}^E(\mu_{p^n})/F_n, E[p]) = \#H^1(K_{1,w_1}^E(\mu_{p^n})/F_n, E[p]) \]
\[ = \#H^0(K_{1,w_1}^E(\mu_{p^n})/F_n, E[p]) \]
\[ \leq \#H^0(F_n, E[p]) = 1. \]

Because of this, we obtain the claim. \( \square \)

Applying Claim 2, the exact sequences (5.19) and (5.20) give
\[ \#H^1(K_{n,w_n}^E/K_{1,w_1}, E[p]) \leq \#H^1(K_{n,w_n}^E/F_n, E[p]) \]
\[ \leq \#\text{Hom}_{\mathbb{Z}[G_{F_n}]}(\text{Gal}(F'_n/K_{1,w_1}^E(\mu_{p^n})), E[p]). \]

Now, we shall show that
\[ \text{Hom}_{\mathbb{Z}[G_{F_n}]}(\text{Gal}(F'_n/K_{1,w_1}^E(\mu_{p^n})), E[p]) = 0. \]

(5.21)

For each \( m \in \mathbb{Z} \) with \( 1 \leq m \leq n \), we define the subgroup \( \text{Fil}^m \) of \( \text{Gal}(F'_n/K_{1,w_1}^E(\mu_{p^n})) \) to be the image of \( \text{Gal}(K_{n,w_n}^E/K_{m,w_m}^E(\mu_{p^n})) \) by the natural map
\[ \text{Gal}(K_{n,w_n}^E/K_{1,w_1}^E(\mu_{p^n})) \to \text{Gal}(F'_n/K_{1,w_1}^E(\mu_{p^n})). \]

The family \( \{ \text{Fil}^m \}_{m} \) becomes a \( G_{F_n} \)-stable descending filtration of \( \text{Gal}(F'_n/K_{1,w_1}^E(\mu_{p^n})). \)

In order to show (5.21), it suffices to show that
\[ \text{Hom}_{\mathbb{Z}[G_{F_n}]}(\text{Fil}^m/\text{Fil}^{m+1}, E[p] \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}) = 0. \]

Take an \( \mathbb{F}_{p^2} \)-basis \( B_1 \) of \( E[p] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2} \) which gives the presentation of the action of \( I_{\mathbb{Q}_p} \) by the matrix (5.18), and for each \( m \in \mathbb{Z} \) with \( 2 \leq m \leq n \), fix a basis \( B_m \) of \( E[p^m] \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2} \) which is a lift of \( B_1 \). Since \( \text{Gal}(K_{m,w_m +1}/K_{m,w_m}^E) \) is a normal subgroup of \( \text{Gal}(K_{m+1,w_{m+1}}^E/\mathbb{Q}_p) \), it is stable under the conjugate action of \( G_{F_n} \). Recall that we have a \( G_{F_n} \)-stable injection
\[ \text{Gal}(K_{m+1,w_{m+1}}^E/K_{m,w_m}^E) \to \text{Ker}(\text{Aut}(E[p^m] \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}) \to \text{Aut}(E[p^m] \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}) \]
\[ \to 1 + p^m M_2(\mathbb{Z}_{p^2}/p^{m+1} \mathbb{Z}_{p^2}) \simeq M_2(\mathbb{F}_{p^2}), \]
where the action of $\sigma \in G_{F_n}$ on $M_2(\mathbb{F}_{p^2})$ is defined by the conjugate action of the matrix
\[
\begin{pmatrix}
\psi(\sigma) & 0 \\
0 & \psi^p(\sigma)
\end{pmatrix}.
\]
Since $\text{Fil}^m/\text{Fil}^{m+1}$ is a quotient of $\text{Gal}(K^{E}_{m+1,w_{m+1}}(\mu_{p^n})/K^{E}_{m,w_m}(\mu_{p^n}))$ by definition, and the restriction
\[
\text{Gal}(K^{E}_{m+1,w_{m+1}}(\mu_{p^n})/K^{E}_{m,w_m}(\mu_{p^n})) \rightarrow \text{Gal}(K^{E}_{m+1,w_{m+1}}/K^{E}_{m,w_m})
\]
is an injective homomorphism, we can regard $\text{Fil}^m/\text{Fil}^{m+1}$ as a $G_{F_n}$-stable subquotient of $M_2(\mathbb{F}_{p^2})$. Let us study the $\mathbb{F}_{p^2}[G_F]$-module structure of $M_2(\mathbb{F}_{p^2})$. Take any $\sigma \in G_{F_n}$. It holds that
\[
\begin{pmatrix}
\psi(\sigma) & 0 \\
0 & \psi^p(\sigma)
\end{pmatrix} = \begin{pmatrix} a & 0 \\
0 & b \end{pmatrix} \text{ for any } a, b \in \mathbb{F}_{p^2},
\]
and
\[
\begin{pmatrix}
\psi(\sigma) & 0 \\
0 & \psi^p(\sigma)
\end{pmatrix}^{-1} = \begin{pmatrix} \psi^{-1}(\sigma) & 0 \\
0 & \psi^{-1}(\sigma) \end{pmatrix}.
\]
Note that $\psi \neq \psi^{p-1}$, and $\psi \neq \psi^{1-p}$. It holds that $M_2(\mathbb{F}_{p^2})$ is a semisimple $\mathbb{F}_{p^2}[G_{F_n}]$-module, and there is no simple $\mathbb{F}_{p^2}[G_{F_n}]$-submodule of $M_2(\mathbb{F}_{p^2})$ which is isomorphic to an $\mathbb{F}_{p^2}[G_{F_n}]$-submodule of $E[p] \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$. This implies
\[
\text{Hom}_{\mathbb{Z}[G_{F_n}]}(\text{Fil}^m/\text{Fil}^{m+1}, E[p] \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}) = 0,
\]
and we obtain (5.21). Consequently, we have
\[
\text{Ker}(\text{res}_{n,v}^{\text{loc}}) = H^1(K^{E}_{n,w_n}/K_{n,v}(\mu_{p^n}), E[p^n]) = 0.
\]
By the above arguments, we deduce that $\{ \# \text{Ker}(\text{res}_{n,v}^{\text{loc}}) \}_{n \geq 0, v \mid p}$ is bounded. This completes the proof of Lemma 5.11. \hfill \Box

5.3. **Proof of Theorem 1.1.** Recall that $\Sigma_{0,\text{bad}}$ denotes the set of prime numbers where $E$ has bad reduction. As $E$ has good reduction at $p$ the prime $p$ does not belong to $\Sigma_{0,\text{bad}}$.

**Lemma 5.12.** Suppose that $E$ satisfies (C2). Let $\ell \in \Sigma_{0,\text{bad}}$. For each $n \in \mathbb{Z}_{\geq 0}$ and $i \in \{0, 1, 2\}$, we put
\[
\mathcal{H}^i_j(\ell, n) := H^i \left( K_n, \prod_{w \mid \ell} \frac{H^1_w(K^{E}_{n,w}, E[p^n])}{H^1_w(K^{E}_{n,w}, E[p^n]) \cap H^1_w(K_{n,w}, E[p^n])} \right), \quad \text{and}
\]
\[
\mathcal{H}^i_{\text{un}}(\ell, n) := H^i \left( K_n, \prod_{w \mid \ell} \frac{H^1_w(K^{E}_{n,w}, E[p^n])}{H^1_w(K^{E}_{n,w}, E[p^n]) \cap H^1_w(K_{n,w}, E[p^n])} \right).
\]
Then, there exists an integer $N'_{\ell} \in \mathbb{Z}_{\geq 1}$ such that for any $n \in \mathbb{Z}_{\geq N'_{\ell}}$ and $i \in \{0, 1, 2\}$, it holds that $\mathcal{H}^i_j(\ell, n) = 0$ and $\mathcal{H}^i_{\text{un}}(\ell, n) = 0$. 

Remark 4.2, the base change Ru. By 5.10 Ru). We obtain Lemma 3.4, all the Jordan–Hölder constituents \(\ell\) takes any Frob \(\pi\). It follows from the condition (C2) for \(f\), Lemma 1.3.5 (iii), we have \(\mathcal{H}_f^i(\ell, n) = 0\) and \(\mathcal{H}_u^i(\ell, n) = 0\) for any \(n \in \mathbb{Z}_{\geq n_0}\) and \(i \in \{0, 1, 2\}\).

**The case: Potentially multiplicative reduction at \(\ell\)** Next, suppose that \(E\) has potentially multiplicative reduction at \(\ell\). Let \(N_\ell \in \mathbb{Z}_{\geq 1}\) be as in Lemma 5.9. By Lemma 3.3, the base change \(E_{K_{\ell}^E}\) has split multiplicative reduction at every \(w \mid \ell\). Take any \(n \in \mathbb{Z}_{\geq N_\ell}\), and let \(v\) be any place of \(K_n\) above \(\ell\). For any place \(w\) of \(K_{\ell}^E\) above \(v\), \(E_{K_{\ell}^E,w}\) is isomorphic to a Tate curve \(\mathbb{G}_m/q_w^m\). By Shapiro’s lemma as in (5.10), we have

\[
\mathcal{H}_F^i(\ell, n) \simeq H^1\left(K_{n,v}, \frac{H^1_{\mathcal{F}}(K_{n,w}^E, E[p^n])}{H^1_{\mathcal{F}}(K_{n,w}^E, E[p^n]) \cap H^1_{\mathcal{F}}(K_{n,w}^E, E[p^n])}\right)
\]

for each \(\mathcal{F} \in \{f, ur\}\) and \(i \in \{0, 1, 2\}\).

Let us show that \(\mathcal{H}_u^i(\ell, n) = 0\) for each \(i\). The natural surjective homomorphism \(T_p(E) \longrightarrow T_p(E)/p^nT_p(E) \simeq E[p^n]\) induces a map

\[
\pi_{n,w} : H^1(K_{n,w}^E, T_p(E)) \longrightarrow H^1(K_{n,w}^E, E[p^n]).
\]

We note that \(H^1_{ur}(K_{n,w}^E, T_p(E))\) is contained in the inverse image of \(H^1_{ur}(K_{n,w}^E, E[p^n])\) by the map \(\pi_{n,w}\). By [Ru, Lemma 1.3.8], the image of \(H^1_{ur}(K_{n,w}^E, T_p(E))\) by \(\pi_{n,w}\) coincides with \(H^1_{ur}(K_{n,w}^E, E[p^n])\). Here, we have \(H^1_{ur}(K_{n,w}^E, T_p(E)) \subseteq H^1_{ur}(K_{n,w}^E, T_p(E))\) with finite index ([Ru, Lemma 1.3.5 (ii)]). The map \(\pi_{n,w}\) induces a surjection

\[
\pi_{n,w}^f : H^1_{\mathcal{F}}(K_{n,w}^E, T_p(E)) \longrightarrow H^1_{ur}(K_{n,w}^E, E[p^n]) \cap H^1_{\mathcal{F}}(K_{n,w}^E, E[p^n]).
\]

By [Ru, Lemma 1.3.5 (iii)], we have

\[
\frac{H^1(K_{n,w}^E, T_p(E))}{H^1_{ur}(K_{n,w}^E, T_p(E))} = \left(\frac{E(K_{n,w}^E)[p^\infty]}{E(K_{n,w}^E)[p^\infty]}\right)^{\text{Frob}_u=1},
\]

where \(\text{Frob}_u \in \text{Gal}(K_{n,w}^E/K_{n,w}^E)\) is the Frobenius automorphism. Since \(\pi_{n,w}^f\) is surjective, all the Jordan–Hölder constituents \(J_i/J_{i-1}\) of the composition series

\[
0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_t := \frac{H^1_{ur}(K_{n,w}^E, E[p^n])}{H^1_{ur}(K_{n,w}^E, E[p^n]) \cap H^1_{\mathcal{F}}(K_{n,w}^E, E[p^n])}
\]

as \(\mathbb{Z}_p[G_{K,v}]-\text{modules}\) are isomorphic to

\[
\frac{E(K_{n,w}^E)[p^\infty]}{E(K_{n,w}^E)[p^\infty]} \left[\frac{E[p^\infty]}{\text{div}}\right] = \left(\frac{E_K[p^\infty]}{\text{div}}\right) = (\mu_p \times q_w^{-1}Z)/(\mu_p \times q_w^2).
\]

It follows from the condition (C2) for \(E\) and Lemma 3.4 that

\[
H^0\left(K_{n,v}, \frac{E(K_{n,w}^E)[p^\infty]}{E(K_{n,w}^E)[p^\infty]}\right) \subseteq H^0\left(K_{n,v}, \frac{E(K_{n,w}^E)[p^\infty]}{E(K_{n,w}^E)[p^\infty]}\right) = 0.
\]

By induction on \(i\), we have \(H^0(K_{n,v}, J_i) = 0\). In particular, we obtain

\[
H^0(K_{n,v}, J_1) = \mathcal{H}_f^0(\ell, n) = 0.
\]
The condition (C2) for $E$ and Lemma 3.4 also imply the equality

$$H^0\left( K_{n,u}, \text{Hom}_{Z_p} \left( \frac{H^1_\text{ur}(K_{n,w}^E, E[p^n])}{H^1_\text{ur}(K_{n,w}^E, E[p^n]) \cap H^1_\text{ur}(K_{n,w}^E, E[p^n])}, \mu_p \right) \right) = 0.$$  

By the local duality of the Galois cohomology ([NSW, (7.2.6) Theorem]), we also have $\mathcal{H}_j^2(\ell, n) = 0$. Moreover, as we have $\ell \neq p$, the local Euler–Poincaré characteristic

$$\frac{\# \mathcal{H}_j^0(\ell, n) \# \mathcal{H}_j^2(\ell, n)}{\# \mathcal{H}_j^1(\ell, n)}$$

is equal to 1 ([NSW, (7.3.1) Theorem]). We obtain $\mathcal{H}_j^1(\ell, n) = 0$.

Next, let us show that $\mathcal{H}_j^1(\ell, n) = 0$ for each $i$. The inclusion $E[p^n] \subseteq E[p^\infty]$ induces a homomorphism

$$\iota_{n,w} : H^1(K_{n,w}^E, E[p^n]) \to H^1(K_{n,w}^E, E[p^\infty]).$$

Recall that $H^1(K_{n,w}^E, E[p^n])$ is defined to be the inverse image of $H^1(K_{n,w}^E, E[p^\infty])$ by the natural map $\iota_{n,w}$ (cf. [Ru, Remark 1.3.9]). From Lemma 5.10, we have

(5.22)  

$$H^1(K_{n,w}^E, E[p^n]) = \text{Ker}(\iota_{n,w}).$$

By [Ru, Lemma 1.3.2 (i)], we have

$$H^1_\text{ur}(K_{n,w}^E, E[p^n]) \simeq \frac{E(K_{n,w}^E, E[p^n])}{(\text{Frob}_w - 1)}.$$  

The latter group is isomorphic to $E[p^n] = E(K_{n,w}^E[p^n]$ because of $K_{n,w}^E = \mathbb{Q}(E[p^n])$. The image of $H^1_\text{ur}(K_{n,w}^E, E[p^n])$ by $\iota_{n,w}$ is contained in

$$H^1_\text{ur}(K_{n,w}^E, E[p^\infty]) \simeq \frac{E(K_{n,w}^E[p^\infty])}{(\text{Frob}_w - 1)},$$

and we have

$$\iota_{n,w}(H^1_\text{ur}(K_{n,w}^E, E[p^n])) = \left( \frac{E(K_{n,w}^E[p^\infty])}{(\text{Frob}_w - 1)} \right)[p^n].$$

By (5.22), the map $\iota_{n,w}$ induces

$$\frac{H^1_\text{ur}(K_{n,w}^E, E[p^n])}{H^1_\text{ur}(K_{n,w}^E, E[p^n]) \cap H^1_\text{ur}(K_{n,w}^E, E[p^n])} \simeq \left( \frac{E(K_{n,w}^E[p^\infty])}{(\text{Frob}_w - 1)} \right)[p^n].$$

Therefore, by (C2) and Lemma 3.4, we have $\mathcal{H}_j^0(\ell, n) = 0$. Moreover, similar to the proof of $\mathcal{H}_j^1(\ell, n) = 0$, by using the local duality theorem and the local Euler–Poincaré characteristic formula, we deduce that $\mathcal{H}_j^1(\ell, n) = 0$ and $\mathcal{H}_j^2(\ell, n) = 0$. This completes the proof of Lemma 5.12. 

**Corollary 5.13.** Suppose that $E$ satisfies (C2). Let $\ell$ be a prime number (distinct from $p$) at which $E$ has bad reduction. Then, there exists an integer $N'_{\ell} \in \mathbb{Z}_{\geq 1}$ such that for any $n \in \mathbb{Z}_{\geq N'_{\ell}}$ and any $F \in \{f, \text{ur}\}$, the natural map

$$\left( \prod_{w | \ell} \frac{H^1(K_{n,w}^E, E[p^n])}{H^1_\text{ur}(K_{n,w}^E, E[p^n]) \cap H^1_\text{ur}(K_{n,w}^E, E[p^n])} \right)^{G_{K_{n,w}}} \to \left( \prod_{w | \ell} \frac{H^1(K_{n,w}^E, E[p^n])}{H^1_\text{ur}(K_{n,w}^E, E[p^n])} \right)^{G_{K_{n,w}}}$$

is an isomorphism.
Proof. Take $N'_r \in \mathbb{Z}_{\geq 1}$ as in Lemma 5.12. For $n \geq N'_r$, to simplify the notation, we put $H^j_F(K_n^E) := H^j_F(K_n^E, E[p^n])$ ($F' \in \{ 0, \text{ur}, f \}$). The short exact sequences

$$0 \rightarrow \frac{H^j_F(K_{n,w})}{H^j_f(K_{n,w}) \cap H^1_{ur}(K_{n,w})} \rightarrow \frac{H^1_F(K_{n,w})}{H^1_f(K_{n,w}) \cap H^1_{ur}(K_{n,w})} \rightarrow \frac{H^1_F(K_{n,w})}{H^1_f(K_{n,w})} \rightarrow 0$$

for all place $w$ above $\ell$ induce the cohomological long exact sequence

$$H^0_F(\ell, n) \rightarrow \left( \prod_{w|\ell} \frac{H^1_F(K_{n,w})}{H^1_f(K_{n,w}) \cap H^1_{ur}(K_{n,w})} \right)^{G_{K_n}} \rightarrow \left( \prod_{w|\ell} \frac{H^1_F(K_{n,w})}{H^1_f(K_{n,w})} \right)^{G_{K_n}} \rightarrow H^1_F(\ell, n).$$

Lemma 5.12 implies that the map $h$ is an isomorphism.

Recalling from § 1, let

$$\rho_n^E : \text{Gal}(K_n^E / \mathbb{Q}) \rightarrow \text{Aut}_{\mathbb{Z}_p}(E[p^n]) \cong GL_2(\mathbb{Z}/p^n),$$

and

$$(\rho_n^E)^{\vee} : \text{Gal}(K_n^E / \mathbb{Q})^{\text{opp}} \rightarrow \text{Aut}_{\mathbb{Z}_p}(E[p^n]^{\vee}) = GL_2(\mathbb{Z}/p^n)$$

be the Galois representations arising from the action on $E[p^n]$ and the right action of the Pontrjagin dual $E[p^n]^{\vee} = \text{Hom}_{\mathbb{Z}_p}(E[p^n], \mathbb{Z}/p^n)$ of $E[p^n]$. For each $n \in \mathbb{Z}_{\geq 0} \cup \{ \infty \}$, we define $R_n = \mathbb{Z}/p^n[\text{Gal}(K_n^E / \mathbb{Q})]$-modules

$$S_n := \text{Hom}_{\mathbb{Z}_p}[\text{Gal}(K_n^E / K_n)](\text{Cl}(\mathcal{O}_{K_n^E}[1/p]) \otimes_{\mathbb{Z}} \mathbb{Z}_p, E[p^n])$$

and

$$A_n^{\mathbb{E}} := (M_2(\mathbb{Z}/p^n\mathbb{Z}), (\rho_n^E)^{\vee} \otimes_{\mathbb{Z}[\text{Gal}(K_n^E / K_n)]} \text{Cl}(\mathcal{O}_{K_n^E}[1/p])).$$

Lemma 5.14. For each $n \in \mathbb{Z}_{\geq 1}$, there exists a $\text{Gal}(K_n^E / \mathbb{Q})$-equivariant isomorphism

$$(A_n^{\mathbb{E}})^{\vee} \cong \hat{S}_n^\oplus.$$ Proof. There is a $\text{Gal}(K_n^E / \mathbb{Q})$-equivariant isomorphism

$$(M_2(\mathbb{Z}/p^n\mathbb{Z}), \rho_n^E) \simeq E[p^n]^{\oplus 2}$$

so that we have $\text{Gal}(K_n^E / \mathbb{Q})$-equivariant isomorphisms

$$\text{Hom}_{\mathbb{Z}_p}(A_n^{\mathbb{E}}, \mathbb{Z}/p^n\mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}_p}[\text{Gal}(K_n^E / K_n)] \left( \text{Cl}(\mathcal{O}_{K_n^E}[1/p]) \otimes_{\mathbb{Z}} \mathbb{Z}_p, (M_2(\mathbb{Z}/p^n\mathbb{Z}), \rho_n^E) \right) \cong \hat{S}_n^\oplus.$$ This shows the assertion.

Lemma 5.15. There exists an integer $N \in \mathbb{Z}_{\geq 1}$ such that, for any $n \in \mathbb{Z}_{\geq N}$, we have an isomorphism

$$S_n \simeq H^0(K_n, \text{Sel}_p(K_n^E, E[p^n])).$$

Proof. Let $H^0_n$ be the maximal subextension of the $p$-Hilbert class field of $K_n^E$ which is completely split at primes above $p$. From the global class field theory, the ideal class
Corollary 5.13 that, for each prime 

Remark 4.2

Therefore, we obtain

\[ S_n \simeq \text{Hom}_{\mathbb{Z}_p}[\text{Gal}(K_n^E/K_n)](\text{Gal}(H_n^E/K_n^E), E[p^n]) \]

By the very definition of \( H_1^1 \), there exists an injective homomorphism

\[ H_1^1(K_{n,w}^E, E[p^n]) / H_1^1(K_{n,w}^E, E[p^n]) \hookrightarrow H_1^1(K_{n,w}^E, E[p^n]) \]

and hence \( S_n \) is isomorphic to the kernel of

\[ (5.23) \quad H_1^1(K_n^E, E[p^n])^{G_{K_n}} \rightarrow \left( \prod_{w \mid p} H_1^1(K_{n,w}^E, E[p^n]) \times \prod_{w \mid p} H_1^1(K_{n,w}^E, E[p^n]) \right)^{G_{K_n}}. \]

It follows from Corollary 5.13 that, for each prime \( \ell \in \Sigma_{0,\text{bad}} \), there exists an integer \( N_\ell' \in \mathbb{Z}_{\geq 1} \) such that

\[ (5.24) \quad \left( \prod_{w \mid \ell} H_1^1(K_{n,w}^E, E[p^n])^{G_{K_n}} \right)^{G_{K_n}} \simeq \left( \prod_{w \mid \ell} H_1^1(K_{n,w}^E, E[p^n]) \right)^{G_{K_n}} \]

for any \( n \geq N_\ell' \). Now, we put \( N := \max \{ N_\ell' \mid \ell \in \Sigma_{0,\text{bad}} \} \). For any \( n \geq N \), we have

\[ H^0(K_n, \text{Sel}_p(K_n^E, E[p^n])) \]

\[ = \ker \left( H_1^1(K_n^E, E[p^n])^{G_{K_n}} \rightarrow \left( \prod_{w \mid p} H_1^1(K_{n,w}^E, E[p^n]) \times \prod_{w \mid p} H_1^1(K_{n,w}^E, E[p^n]) \right)^{G_{K_n}} \right) \]

\[ \overset{\langle \rangle}{=} \ker \left( H_1^1(K_n^E, E[p^n])^{G_{K_n}} \rightarrow \left( \prod_{w \mid p} H_1^1(K_{n,w}^E, E[p^n]) \times \prod_{w \mid p} H_1^1(K_{n,w}^E, E[p^n]) \right)^{G_{K_n}} \right) \]

\[ \overset{(5.23)}{=} S_n. \]

Here, the second equality \( \langle \rangle \) follows from (5.24) for a bad prime \( \ell \neq p \) and Remark 4.2 for a good prime \( \ell \neq p \).

Theorem 5.16. Suppose that \( E \) satisfies the conditions (C1), (C2) and (C3). Then, there exists a family of \( R_n \)-homomorphisms

\[ r_n : \text{Sel}(K_n, E[p^n])^{\otimes 2} \rightarrow (A_n^E)'^{\vee} \]

such that the kernel \( \ker(r_n) \) and the cokernel \( \text{Coker}(r_n) \) are finite with order bounded independently of \( n \).
Lemma 5.15.

Proof. By Proposition 5.7 and Lemma 5.15, there exists $N \in \mathbb{Z}_{\geq 1}$, the order of the kernel and that of the cokernel of the map

$$\text{Sel}_p(K_n, E[p^n])^{\mathbb{Z}/2} \xrightarrow{(\text{res}_{\text{Sel}})^{\mathbb{Z}/2}} H^0(K_n, \text{Sel}_p(K_n^E, E[p^n]))^{\mathbb{Z}/2} \simeq S_n^{\mathbb{Z}/2}$$

are at most $p^{2n_{\text{res}}}$ and $p^{2n_{\text{coker}}}$ respectively for all $n \geq N$. By Lemma 5.14, there is an isomorphism $S_n^{\mathbb{Z}/2} \simeq (A_n^E)^\vee$. Since $\text{Sel}_p(K_n, E[p^n])^{\mathbb{Z}/2}$ and $(A_n^E)^\vee$ are finite for any $n < N$, this completes the proof of Theorem 5.16. □

References

[BK] Bloch, S. and Kato, K., *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, 333–400, Progr. Math. 86, Birkhäuser Boston, Boston, MA, 1990.

[DDMS] Dixon, J. D., Du Sautoy, M. P., Mann, A., and Segal, D., *Analytic pro-p groups*, Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, (2003).

[Ed] Edixhoven, B., *The weight in Serre’s conjectures on modular forms*, Invent. Math. 109, (1992), 563–594.

[Ei] Eisenbud, D., *Commutative Algebra: with a View Toward Algebraic Theory*, Grad. Texts in Math., vol. 150, Springer-Verlag (1995).

[Ga] Garnek, J., *On class numbers of division fields of abelian varieties*, J. Théor. Nombres Bordeaux, 31 (2019), no. 1, 227–242.

[Hi] Hiranouchi, T., *Local torsion primes and the class numbers associated to an elliptic curve over Q*, Hiroshima Math. J. 49 (2019), no. 1, 117–128.

[Im] Imai, H., *A remark on the rational points of abelian varieties with values in cyclotomic $\mathbb{Z}_p$-extensions*, Proc. Japan Acad. 51 (1975), 12–16.

[Iw] Iwasawa, K., *On $\mathbb{Z}_p$-extensions of algebraic number fields*, Ann. of Math. 98 (1973), 246–326.

[Ka] Kato, K., *p-adic Hodge theory and values of zeta functions of modular forms*, no. 295, 2004, Cohomologies $p$-adiques et applications arithmétiques. III, pp. ix, 117–290.

[La] Lazard, M., *Groupes analytiques $p$-adiques*, Inst. Hautes Études Sci. Publ. Math., 26 (1965), 389–603.

[LMF] The LMFDB Collaboration, *The L-functions and modular forms database*, Elliptic curve with LMFDB label 5077.a1 (Cremona label 5077a1), https://www.lmfdb.org/EllipticCurve/Q/5077/a/1/.

[Ne] Neukirch, J., *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[NSW] Neukirch, J., Schmidt, A. and Wingberg, K., *Cohomology of number fields*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008.

[Oh1] Ohshita, T., *Asymptotic lower bound of class numbers along a Galois representation*, J. Number Theory 211 (2020), 95–112.

[Oh2] , *On higher Fitting ideals of certain Iwasawa modules associated with Galois representations and Euler systems*, Kyoto J. Math. 61 (2021), no. 1, 1–95.

[PS] Prasad, D. and Shekhar, S., *Relating the Tate–Shafarevich group of an elliptic curve with the class group*, Pacific Journal of Mathematics 312 (1) (2021), 203–218.

[Ru] Rubin, K., *Euler systems*, Hermann Weyl lectures, Ann. of Math. Studies, vol. 147, Princeton Univ. Press (2000).

[SY1] Saito, F. and Yamauchi, T., *On the class numbers of the fields of the $p^n$-torsion points of certain elliptic curves over Q*, J. Number Theory 156 (2015), 277–289.

[SY2] , *On the class numbers of the fields of the $p^n$-torsion points of elliptic curves over Q*, J. Théor. Nombres Bordeaux 30 (2018), no. 3, 893–915.

[Se1] Serre, J.-P., *Corps locaux*, Hermann, Paris, 1968, Deuxième édition, Publications de l’Université de Nancago, No. VIII.
[Sc2] Sur les groupes de congruence des variétés abéliennes. II, Izv. Akad. Nauk SSSR Ser. Mat., 1971, Volume 35, Issue 4, 731–737.

[Sc3] Propriétés galoisiennes des points d’ordre fini des courbes elliptiques, Invent. Math. 15 (4) (1972), 259–331.

[Si1] Silverman, J. H., The arithmetic of elliptic curves, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009.

[Si2] Advanced topic in the arithmetic of elliptic curves, Graduate Texts in Mathematics, vol. 151, Springer, Dordrecht, 2013.

[SU] Skinner, C. and Urban, E., The Iwasawa Main Conjectures for GL2, Invent. Math. 195 (2014), 1–277.

[Ta] Relation between $K_2$ and Galois cohomology, Invent. Math. 36 (1976), 257–274.

[Wa] Washington, L., Introduction to Cyclotomic Fields, 2nd edition, Grad. Texts in Math., vol. 83, Springer-Verlag (1997).

[Wu] Wuthrich, C. The fine Selmer group and height pairings, Ph.D. thesis, University of Cambridge, UK, 2004.

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