Bertotti-Robinson solutions of $D = 5$ Einstein-Maxwell-Chern-Simons-Lambda theory

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We present a series of new solutions in five-dimensional Einstein-Maxwell-Chern-Simons theory with an arbitrary Chern-Simons coupling $\gamma$ and a cosmological constant $\Lambda$. For general $\gamma$ and $\Lambda$ we give various generalizations of the Bertotti-Robinson solutions supported by electric and magnetic fluxes, some of which presumably describe the near-horizon regions of black strings or black rings. Among them there is a solution which could apply to the horizon of a topological AdS black ring in gauged minimal supergravity. Others are horizonless and geodesically complete. We also construct extremal asymptotically flat multi-string solutions for $\Lambda = 0$ and arbitrary $\gamma$.

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I. INTRODUCTION

Five-dimensional supergravity is an interesting proving ground for string theory. Its Lagrangian, obtainable by toroidal dimensional reduction of eleven-dimensional supergravity, contains a Chern-Simons term for the Maxwell field inherited from reduction of the corresponding four-form term. Though it does not influence the Einstein equations, it modifies the Maxwell equations and consequently the gravitational field too. Surprisingly enough, its presence is crucial for the enhancement of hidden symmetries of five-dimensional Einstein-Maxwell theory in the case of field configurations possessing two commuting spacetime Killing vectors. For such configurations the theory reduces to a three-dimensional sigma model realizing a harmonic map from the spacetime manifold to the homogeneous space $G/H = G_{2(+2)}/((SL(2, R) \times SL(2, R))$ [1–4]. Owing to this hidden symmetry, a generating technique had been developed [1, 2], which opened the way to derive new charged rotating black rings and general black strings [5–10].

In various physical contexts one is also interested in the more general Einstein-Maxwell theory containing a Chern-Simons term with an arbitrary coupling constant $\gamma$ and a cosmological constant $\Lambda$:

$$S_5 = \frac{1}{16\pi G_5} \int d^5x \sqrt{|g(5)|} \left[ R(5) - \frac{1}{4} F_{\mu\nu}(5) F^{\mu\nu}(5) - 2\Lambda - \frac{\gamma}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} F_{(5)\mu\nu} F_{(5)\rho\sigma} A_{(5)\lambda} \right],$$

where $F_{(5)} = dA_{(5)}$, $\mu, \nu, \cdots = 1, \cdots, 5$, and $\epsilon^{\mu\nu\rho\sigma\lambda}$ is an antisymmetric symbol whose signs will be detailed later. For $\gamma = 1, \Lambda = 0$ this is the action of minimal five-dimensional supergravity, for $\gamma = 1, \Lambda < 0$ the action of minimal gauged
supergravity, while for $\gamma = 0$ it is the Einstein-Maxwell (EM) action. Restricted to field configurations possessing two commuting Killing vectors, this theory reduces to a three-dimensional gravitating sigma model coupled to a potential originating from the cosmological constant term. For $\gamma \neq 1$ the target space of this sigma model is not a symmetric space (the isometry group is solvable), so there are no non-trivial hidden symmetries which could be used to generate exact solutions. Moreover, the potential term is not invariant under target space isometries apart from some trivial ones. Not surprisingly, no exact charged rotating black hole solutions are known in the pure Einstein-Maxwell ($\gamma = 0$) theory even for $\Lambda = 0$, though their existence was demonstrated perturbatively [13–15] and numerically [16]. A similar situation holds for charged black rings: static solutions in EM theory have been found by Ida and Uchida [17], and generalized to EM-dilaton theory in [11, 18]. But stationary charged black ring solutions of pure EM theory are not known in a closed form, though their existence again was confirmed both perturbatively (for small charges [19]) and numerically [20]. For $\gamma \neq 1$, 0 exact charged black hole/ring solutions are not known either, while numerical black hole solutions have been constructed [21] and shown to have unusual properties for $\gamma > 1$ such as rotation in the sense opposite to the angular momentum and a negative horizon mass with positive asymptotic mass.

It is therefore of interest to explore other tools to generate exact solutions of the general EMCSL action (1.1) which could shed some light on the nature of black objects in this theory. A particular motivation for this lies in the still unsolved question about the existence of asymptotically $AdS$ and $dS$ black rings in presence of the cosmological constant. The issue of charged five-dimensional black objects with cosmological constant was discussed previously in a number of papers. Supersymmetric $AdS_5$ black holes were obtained by several authors [22]. General non-extremal rotating black holes in minimal five-dimensional gauged supergravity were constructed by Chong, Cvetic, Liu and Pope [23]. Charged squashed black holes in EM theory ($\gamma = 0$) with cosmological constant were constructed numerically in [24]. Black strings with cosmological constant were studied in [25]. The issue of black rings turns out to be more subtle. General considerations do not prevent their existence both for positive and negative cosmological constants: the additional centripetal/centrifugal force acting on the $S^1$ can be balanced by tuning the angular momentum along the $S^1$. And indeed, Chu and Dai [26] have found analytically asymptotically $dS$ black rings within the $N = 4$ de Sitter supergravity (see also [27]). As in the asymptotically flat (AF) case the asymptotically de Sitter black holes/rings may have horizon topologies $S^3$ (or a quotient), $S^1 \times S^2$ and $T^3$. For a negative cosmological constant other topologies can be anticipated, namely $S^1 \times H^2$ where $H^2$ stands for a negative curvature hyperbolic two-surface. Meantime, no analytical solutions are known for black rings with negative cosmological constant. Approximate “thin” black rings were obtained by Caldarelli, Emparan and Rodriguez [28] using the “blackfolds” approach applicable in arbitrary dimensions [29]. These approximate solutions exist for both signs of the cosmological constant and smoothly go into a straight black string in the limit of an infinite $S^1$ radius. Moreover, the possibility of black Saturns with non-flat asymptotics was also indicated. Supersymmetric black rings with $AdS_5$ asymptotics and compact horizons were investigated in [30] for a negative cosmological constant.

Lacking exact globally defined solutions, it is tempting to explore local solutions in the vicinity of the event horizons of the presumed black objects. This is a particularly fruitful approach in the extremal case. Usually the near-horizon limits are themselves exact solutions of the same theory, as in the case of the near-horizon limit of extremal four-dimensional EM black holes, which is the Bertotti-Robinson (BR) solution with geometry $AdS_2 \times S^2$ supported by a monopole electric or magnetic field. Typically, the near-horizon solutions possess a larger isometry group than the full black hole solutions, therefore they can be obtained by different solution-generating techniques. All known exactly five-dimensional supergravity solutions exhibit enhancement of isometries in the near-horizon region to $SO(2,1) \times U(1)^2$ (with $U(1)$ factors standing for rotational symmetries). This is valid for stationary and asymptotically AdS solutions, and also persists in presence of scalar fields with a potential [32] and with account for higher-curvature corrections. In $D$ dimensions the enhanced symmetry includes the $U(1)^{D-3}$ rotational symmetry. Note that in this and more general theories certain properties of extremal black holes including topology and thermodynamics can be extracted from the near-horizon solutions using Sen’s entropy function approach [31].

Keeping in mind the importance of the BR solution in the four-dimensional EM theory we explore here similar exact solutions within the five-dimensional EMCSA theory with arbitrary $\gamma$ and $\Lambda$. Presumably these could be near-horizon limits of black holes/rings with various topologies, including the above-mentioned case of the hyperbolic topology $S^1 \times H^2$. As was observed several decades ago, the $n$-dimensional Einstein-Maxwell theories can be compactified to $(n-2)$-dimensional space-time, the two extra space dimensions being curved into a two-sphere through the action of a monopole magnetic field living on that two-sphere [33]. This mechanism was later generalized to the Freund-Rubin compactification of $(d-s)$ or $s$ dimensions by an $s$-form field [34]. The consistency of general sphere compactifications of supergravity actions was extensively discussed in the past (see, e.g. [35]) showing that a consistent sphere truncation of gravity coupled to a form field and (possibly) to a dilaton is possible only in a limited number of cases.

Our approach consists in compactifying the theory (1.1) on a constant curvature two-space $\Sigma_2$ of positive, zero or negative curvature. We consider only compactification through a direct product ansatz, as in the Freund-Rubin case, thereby avoiding the consistency problems associated with non-abelian Kaluza-Klein gauge fields. This compactification, carried out in Sec. 2, results in the three-dimensional EMCSA gravity, with an additional constraint on the
scalar curvature. In Sec. 3 we present a number of new non-trivial solutions of the five-dimensional EMCSA theory of the BR-type which are obtained by uplifting BTZ, self-dual and Gödel solutions of the constrained three-dimensional theory. These solutions are applicable, in particular, to gauged and ungauged $D = 5$ supergravities, but they are also valid in the EMCSA theory with more general values of parameters. In Sec. 4 we perform an alternative toroidal reduction of the five-dimensional EMCSA to three dimensions, deriving a gravity coupled sigma model with a potential. Some of the solutions listed in Sec. 3 correspond to null geodesics of the target space. In the case of a vanishing cosmological constant, the reduced three-space is flat, enabling the generalization of these solutions to new classes of non-asymptotically flat or asymptotically flat multi-center solutions of the five-dimensional theory with arbitrary $\gamma$. The technical proof that the solutions of Sec. 3 are the only solutions with constant curvature two-space sections is outlined in the Appendix.

II. REDUCTION ON CONSTANT CURVATURE TWO-SPACES

The main idea underlying the present paper is that the five-dimensional theory (1.1) may be reduced, by monopole compactification on constant curvature two-surfaces $\Sigma_2$, to three-dimensional EMCSA:

$$S_{(3)} = \frac{1}{2k} \int d^3x \left[ \sqrt{|g|} \left( R - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} - 2\lambda \right) - \frac{\mu}{4} \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} A_\gamma \right]$$  \hspace{1cm} (2.1)

with an additional constraint, which admits several classes of non-trivial exact stationary solutions, leading to non-asymptotically flat stationary solutions of the original five-dimensional theory with the structure $M_3 \times \Sigma_2$.

The five-dimensional Maxwell-Chern-Simons and Einstein equations following from the action (1.1) are

$$\partial_\mu (\sqrt{|g|} F_{\mu\nu}^{(5)}) = \frac{\gamma}{4\sqrt{3}} \epsilon^{\nu\rho\sigma\tau\lambda} F_{(5)\rho\sigma} F_{(5)\tau\lambda},$$  \hspace{1cm} (2.2)

$$R_{(5)\nu}^{\mu} - \frac{1}{2} R_{(5)}^{\mu} \delta_\nu^\mu = \frac{1}{2} F_{(5)\mu} F_{(5)\nu} - \frac{1}{8} F_{(5)}^2 \delta_\nu^\mu - \Lambda \delta_\nu^\mu.$$  \hspace{1cm} (2.3)

In Eq. (2.2) we need to fix a sign convention for the five-dimensional antisymmetric symbol. Throughout this paper we will assume that $\epsilon^{12345} = +1$, with the spacetime coordinates numbered according to their order of appearance in the relevant five-dimensional metric. Let us assume for the five-dimensional metric and the vector potential the direct product ansatz

$$ds^2_{(5)} = g_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta + a^2 d\Sigma_k, \quad A_{(5)} = A_\alpha(x^\gamma) dx^\alpha + e f_k d\varphi,$$  \hspace{1cm} (2.4)

where $\alpha, \beta, \gamma = 1, 2, 3$, and the two-metrics for $k = \pm 1, 0$ are

$$d\Sigma_1 = d\theta^2 + \sin^2 \theta d\varphi^2, \quad d\Sigma_0 = d\theta^2 + \theta^2 d\varphi^2, \quad d\Sigma_{-1} = d\theta^2 + \sinh^2 \theta d\varphi^2$$ \hspace{1cm} (2.5)

$$f_1 = -\cos \theta, \quad f_0 = \frac{1}{2} \theta^2, \quad f_{-1} = \cosh \theta$$ \hspace{1cm} (2.6)

with $\varphi \in [0, 2\pi]$ and $\theta \in [0, \pi]$ for $k = 1$ and $\theta \in [0, \infty]$ for $k = 0, -1$. Here the moduli $a$ and $a^2$ are taken to be constant and real (though for generality we do not assume outright $a^2$ to be positive). The corresponding five dimensional geometric quantities are

$$\sqrt{|g(5)|} = \sqrt{|g||a^2|} |\partial_\theta f_k|, \quad R_{(5)\theta}^{\theta} = R_{(5)\varphi}^{\varphi} = ka^{-2},$$  \hspace{1cm} (2.7)

and the Maxwell tensor decomposes as

$$F_{(5)\mu\varphi} = e \partial_\theta \delta_k, \quad F_{(5)}^{\theta} = \frac{e}{a^2} \partial_\theta \delta_k, \quad F = dA.$$  \hspace{1cm} (2.8)

Inserting the ansatze (2.4) in the field equations[55], we find that the equations (2.2) for $\nu = 4, 5$ are trivially satisfied, while for $\nu = \beta$ they reduce to

$$\partial_\alpha (\sqrt{|g|} F^{\alpha\beta}) = \frac{\gamma e}{\sqrt{3a^2}} \epsilon^{\beta\gamma\delta} F_{\gamma\delta}.$$  \hspace{1cm} (2.9)
The Einstein equations (2.3) lead to the system
\[
R_{\alpha \beta} - \frac{1}{2} R \delta_{\alpha \beta} = \frac{1}{2} F^\alpha \gamma F_{\beta \gamma} - \frac{1}{8} F^2 \delta_{\alpha \beta} + \left( \frac{4k a^2 - e^2}{4a^4} - \Lambda \right) \delta_{\alpha \beta},
\]
\[
F^2 = \frac{4(e^2 - 3ka^2)}{a^4} + 8\Lambda.
\] (2.10)

It is straightforward to check that the reduced equations derive from the action (2.1) with \(\kappa = 2G_5/|a^2|\) and the following identification of parameters:
\[
\lambda = \Lambda + (e^2 - 4ka^2)/4a^4, \quad \mu = g/|a^2|, \quad (g = 2\gamma e/\sqrt{3}),
\] (2.11)
with the additional constraint on the three-dimensional scalar curvature
\[
R = (e^2 - 6ka^2)/2a^4 + 4\Lambda.
\] (2.12)

Inverting the above relations for \(k \neq 0\) and \(\gamma \neq 0\) leads to
\[
a^2 = \frac{k \gamma^2}{3\mu^2/16 + (\Lambda - \lambda)\gamma^2}, \quad e = \frac{\sqrt{3}\gamma \mu}{2[3\mu^2/16 + (\Lambda - \lambda)\gamma^2]},
\] (2.13)
and the constraint
\[
\mathcal{R} = \Lambda + 3\lambda - 3\mu^2/16\gamma^2 = \Lambda + 3\lambda - e^2/4a^4.
\] (2.14)

The equations (2.13) break down for \(k = 0\), in which case the parameters \(\mu, \lambda\) are related by
\[
\lambda = \Lambda + \frac{3\mu^2}{16\gamma^2},
\] (2.15)
and for \(\gamma = 0\), which leads to \(\mu = 0\).

### III. SOLUTIONS WITH THREE COMMUTING KILLING VECTORS

The three-dimensional theory defined by the action (2.1) is Maxwell-Chern-Simons electrodynamics (or Maxwell electrodynamics in the limiting case \(\gamma = 0\)) coupled to cosmological Einstein gravity. Several classes of exact solutions to this theory with two commuting Killing vectors and constant Ricci scalar are known [38–40]. The proof that these are the only solutions with constant scalar curvature is rather involved and is given in the Appendix. After uplift to five dimensions according to (2.4) these solutions will lead to Bertotti-Robinson-like solutions of EMCSΛ5 with three commuting Killing vectors.

#### A. BTZ class

The first class corresponds to neutral (vacuum) three-dimensional solutions with
\[
e^2 = 3ka^2 - 2\Lambda a^4, \quad \mathcal{R} = 6\lambda = 3\Lambda - \frac{3k}{2a^2},
\] (3.1)
These exist irrespective of the value of the Chern-Simons coupling constant \(\gamma\). The constant curvature three-space is \(dS_3\) for \(\lambda > 0\), Minkowski and its coordinate transforms for \(\lambda = 0\), and \(AdS_3\) and its coordinate transforms, the BTZ black holes, for \(\lambda < 0\). We first concentrate on this last case, before discussing briefly the two other cases \(\lambda = 0\) and \(\lambda > 0\).

The BTZ black hole is a vacuum solution of three-dimensional gravity with negative \(\lambda = -l^{-2}\), which restricts the parameters of the five-dimensional EMCSΛ theory by
\[
k - 2\Lambda a^2 > 0.
\] (3.2)
This may be considered as a restriction on the five-dimensional cosmological constant for any given $k$. Combining the BTZ vacuum black hole with a constant curvature two-surface, we obtain the following two-parameter family of solutions generically valid for all $k$:

$$\text{ds}_5^2 = -N^2 \text{d}t^2 + \frac{\text{d}r^2}{N^2} + r^2 (\text{d}\phi + N^2 \text{d}t)^2 + a^2 \text{d}\Sigma_k,$$

(3.3)

where $\text{d}\Sigma_k$ is given by (2.5), and

$$N^2 = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}, \quad N^2 = \frac{J}{2r^2}.$$

(3.4)

These solutions exist independently of the irrelevant Chern-Simons coupling constant $\gamma$, including the cases of five-dimensional minimal supergravity and pure Einstein-Maxwell theory. Generically, they are supported by magnetic fluxes along $\Sigma_k$, which are parameterized by $e$ given by

$$e^2 = \frac{4a^4}{l^2} + 2ka^2,$$

(3.5)

$e = 0$ corresponding to $k = -1$ and $\Lambda < 0$. The local isometry group is $SO(2,2) \times SO(3)$ for $k = +1$, $SO(2,2) \times GL(2,R)$ for $k = 0$ and $SO(2,2) \times SO(2,1)$ for $k = -1$.

The following particular cases are worth mentioning:

1. **Minimal supergravity**

   In the case $\Lambda = 0$, the above relations give, for $a^2 > 0$,

   $$k = +1, \quad a^2 = \frac{l^2}{4}, \quad e^2 = \frac{3l^2}{4}.$$

   (3.6)

   In this case the radius of $AdS$ is twice the radius of the two-sphere. In the case of minimal supergravity, the solution (3.3) with $\phi \in R$ coincides with the decoupling (near-horizon) limit of the general five-dimensional black string [10]. With $\phi$ periodically identified, the solution (3.3) may be interpreted as a NAF black ring rotating along the $S^1$.

   Moreover, it is the near-horizon limit of the asymptotically flat black ring with horizon $S^1 \times S^2$.

2. **Gauged supergravity**

   The intriguing question about the possible existence of asymptotically AdS black rings in gauged supergravity is still open. No supersymmetric black rings are possible in this case, but there is no proof either of the non-existence of non-BPS rings. Our solution (3.3) with $\Lambda < 0$ and $\phi$ periodically identified exists in all the three versions $k = 1, k = 0, k = -1$, i.e. with horizon topologies $S^1 \times S^2$, $S^1 \times R^2$, and $S^1 \times H^2$. It remains an open question whether these are indeed the near-horizon limits of topological asymptotically AdS black rings, but as themselves they can be regarded as NAF and non-asymptotically AdS rings of various topologies rotating along $S^1$.

3. **Rings in De Sitter**

   Another yet unsolved issue is the possibility of black rings with positive cosmological constant, asymptotically dS. As an argument to support this, we may consider our solutions (3.3) for $k = 1$ and $a^2 < 1/2\Lambda$ as the presumed near-horizon limit of such rings. In any case, one can interpret these solutions as NAF and non-asymptotically dS rings of topology $S^1 \times S^2$, rotating along $S^1$.

4. **Vacuum solutions**

   For $\Lambda < 0$, the special case $k = -1, l^2 = 2a^2 = -3/\Lambda$ leads to a two-parameter $(M, J)$ vacuum family of NAF topological black rings, rotating along $S^1$, with horizon topology $S^1 \times H^2$. We are not aware whether this family of locally $AdS_3 \times H^2$ topological solutions of the vacuum five-dimensional Einstein equations has been reported elsewhere.
5. Case $\lambda = 0$

In this case, the constraints (3.1) imply $e^2 = 2k^2a^2$. For $k = +1$ ($e^2 = 2a^2 = \Lambda^{-1}$, the resulting five-dimensional metrics include Minkowski $\mathbb{R}^3 \times S^2$, Rindler $\mathbb{R}^3 \times S^1 \times S^2$, and the metric generated from a special coordinate transform of three-dimensional Minkowski spacetime [41, 42]:

$$ds^2_{(5)} = r^2 dt^2 + 2 dt dz + dr^2 + \frac{1}{2\Lambda} (d\theta^2 + \sin^2 \theta d\varphi^2) .$$ (3.7)

The other possibility $k = 0$ ($e = 0$, $\Lambda = 0$), leads to flat five-dimensional vacuum solutions, including the $k = 0$ partner of (3.7):

$$ds^2_{(5)} = -r dt^2 + 2 dt dz + dr^2 + d\theta^2 + d\varphi^2 .$$ (3.8)

6. Case $\lambda > 0$

In this case, $k = +1$, $\Lambda > 0$, and the five-dimensional metric is the product $dS_3 \times S^2$.

B. Self-dual class

The second case is that of the “self-dual” solutions of [38] and [43] which asymptote to the extreme ($J = Ml$) BTZ solution (3.3) for $\lambda < 0$ or to the three-dimensional flat metric of (3.7) for $\lambda = 0$. For these solutions, $F^2 = 0$ (but $F_{\alpha \beta} \neq 0$), and the constant Ricci scalar has again the BTZ value $\mathcal{R} = 6\lambda$. The corresponding five-dimensional solution is, for $\lambda = -l^{-2}$, $\gamma \neq 0$,

$$ds^2_{(5)} = \frac{2}{l} \left[ -(r - lM_{\mu}(r)/2) dt^2 - lM_{\mu}(r) dt dz + (r + lM_{\mu}(r)/2) dz^2 \right]$$

$$+ \frac{l^2 dr^2}{4 r^2} + a^2 d\Sigma_k ,$$ (3.9)

$$A_{(5)} = q \left( \frac{2r}{l} \right)^{-\mu l/2} (dt - dz) + e_{f_k} d\varphi , \quad M_{\mu}(r) = M - \frac{q^2 \mu l}{4(\mu l + 1)} \left( \frac{2r}{l} \right)^{-\mu l}$$

($\mu l \neq 0, -1$). These solutions depend on the three independent parameters $a$ (entering $l$, $\mu$ and $e$), and the two dimensionless parameters $M$ and $q$. For $q = 0$ this reduces to the extreme ($J^2 = M l^2$) five-dimensional BTZ solution (3.3) after the coordinate transformations $r^2_{BTZ} = l(r + M l/2)$, $t_{BTZ} = \sqrt{2} t$, $\varphi_{BTZ} = \sqrt{2} z/l$. These metrics have generically five Killing vectors (the obvious $\partial_t$, $\partial_z$ and the three isometries of $\Sigma_k$), except in the special case $\mu l = -2$, which is the intersection (3.17) of the self-dual and Gödel classes, with seven Killing vectors.

In the case $\gamma = \mu = 0$ of five-dimensional Einstein-Maxwell theory, the solution degenerates to the solution (derived from the solution to three-dimensional EMA theory given in Eq. (29) of [47]) with five-dimensional metric given by (3.9), and

$$A_{(5)} = q \ln \left( \frac{r}{r_0} \right) (dt - dz) + e_{f_k} d\varphi , \quad M_0(r) = q^2 \ln \left( \frac{r}{r_0} \right) .$$ (3.10)

The definition of the mass function $M_{\mu}(r)$ in (3.9) also breaks down for $\mu l = -1$, in which case it must be replaced by $M_{\mu}(r) = M + (q^2/2l) r \ln(2r/l)$.

The self-dual solution for the case $\lambda = 0$ [38] leads to the five-dimensional solution

$$ds^2_{(5)} = - \left( \alpha + \beta r + \frac{q^2}{4} e^{-2\nu} \right) dt^2 + 2 dt dz + dr^2 + \frac{1}{2\Lambda} (d\theta^2 + \sin^2 \theta d\varphi^2) ,$$

$$A_{(5)} = q e^{-\nu} dt - \frac{1}{\sqrt{\Lambda}} \cos \theta d\varphi \quad (\mu = \frac{2\gamma}{\sqrt{3} \Lambda}) .$$ (3.11)

This is horizonless and geodesically complete for $\gamma > 0$. 


C. Gödel class

The third class, corresponding to so-called three-dimensional Gödel black holes (no relation with the five-dimensional Gödel black holes), was given in [39] and [40] (in the case where the Chern-Simons term for gravity is absent). These solutions are closely related to the warped $\text{AdS}_5$ black hole solutions (3.15) of topologically massive gravity [38, 40, 44–46]. Using the notations of [40], the three-dimensional solutions, characterized by a dimensionless constant $\Omega = 0$; the parameter $\Omega$ can be restored by the local parameter relabellings $t \rightarrow t$, and the coordinate relabellings of Eqs. (3.16) and (3.27) of [40] (with $c = -1$ to ensure reality of $A(5)$, $\kappa = 1/2$, and the coordinate relabellings $t \rightarrow z$, $\varphi \rightarrow t$) is

$$ds^2 = -\frac{\beta^2}{\beta^2 - 1}(r^2 - m^2)dt^2 + \frac{1}{\beta^2 - 1}[(\beta^2 - 1)dz - (r + \Omega(\beta^2 - 1))dt]^2 + a^2 \left[\frac{a^2}{g^2(\beta^2 - 1)}dr^2 + d\Omega^2\right],$$

$$A(5) = \sqrt{2}[(\beta^2 - 1)dz - (r + \Omega(\beta^2 - 1))dt] + e f_k d\varphi,$$

where $m^2$ and $\Omega$ are two real parameters [56] (the other parameters $\beta^2$ and $e$ depend on the compactification scale $a$ through (3.13) and (3.14)). In the following we will take $\Omega = 0$: the parameter $\Omega$ can be restored by the local coordinate transformation $z \rightarrow z - \Omega t$. The isometry group of these metrics is $SO(2, 1) \times SO(2) \times SO(3)$ for $\gamma < \sqrt{3}/2$, $SO(2, 1) \times SO(2) \times GL(2, R)$ for $\gamma = \sqrt{3}/2$, and $SO(2, 1) \times SO(2) \times SO(2, 1)$ for $\gamma > \sqrt{3}/2$ [40]. Similarly to the metrics (3.3) of the BTZ class, they are geodesically complete.

1. Bertotti-Robinson string

The geometry described by the metric (3.15) depends on the range of values of the real parameter $\beta^2$. The solution in the case of pure Einstein-Maxwell theory ($\gamma = 0$, implying $k = +1$), is obtained by rescaling the coordinate $z \rightarrow \beta^{-1}z$, taking the limit $\beta^2 \rightarrow \infty$ with $\beta^2 g^2 = e^2(1 - \Lambda e^2)/2$ fixed, and gauging away the constant $A_5$:

$$ds^2 = -(r^2 - m^2)dt^2 + dz^2 + e^2 \left[\frac{dr^2}{1 - \Lambda e^2 r^2 - m^2} + d\theta^2 + \sin^2 \theta d\varphi^2\right],$$

$$A_5 = -\sqrt{2}r d\theta - e \cos \theta d\varphi,$$

(the three-dimensional reduced solution was previously given in [47], Eq. (25)). This is the five-dimensional embedding of a four-dimensional dyonic Bertotti-Robinson solution (with $F_{(4)}^2 = 4\Lambda$ for the $T^2_z$ energy-momentum tensor component to vanish). The geometry is $\text{AdS}_5 \times S^2 \times S^1$.

2. Rotating Bertotti-Robinson

For $\beta^2 > 1$, the solution (3.15), with a regular horizon, is a five-dimensional analogue of the four-dimensional rotating Bertotti-Robinson solution $RBR_-$ of EMDA [48]. Similarly to the case of $RBR_-$, this is not a black hole:
the parameter $m$ of this solution can be transformed away by a global coordinate transformation, and its three- 
dimensional mass and momentum, computed according to the prescriptions of [40], vanish.

For $\beta^2 = 1$, the solution (3.15) is replaced by [40]

$$
\begin{align*}
\text{d}s^2_{(5)} &= -(r^2 + \alpha) \text{d}t^2 - 2r \text{d}t \text{d}z + a^2 \left( \frac{a^2 \text{d}r^2}{g^2 - r^2} + \text{d}\Sigma_k \right), \\
A_{(5)} &= -\sqrt{2} r \text{d}t + e f_k \text{d}\varphi,
\end{align*}
$$

with $\alpha$ a free parameter.

3. $H^2 \times \Sigma_2 \times R$

For $0 < \beta^2 < 1$, the coordinate transformation $t \rightarrow (\overline{\gamma}/m)(1 - \beta^2)^{1/2}\psi$, $z \rightarrow (1 - \beta^2)^{-1/2}t$, $r \rightarrow m \cosh \chi$, with

$$
\overline{\gamma} = -a^2/\beta^2 g,
$$

transforms (3.15) into

$$
\begin{align*}
\text{d}s^2_{(5)} &= -(\text{d}t - \overline{\gamma} \cosh \chi \text{d}\psi)^2 + \overline{\gamma}^2 \beta^2 (\text{d}\chi^2 + \sinh^2 \chi \text{d}\psi^2) + a^2 \text{d}\Sigma_k, \\
A_{(5)} &= -\sqrt{2}(1 - \beta^2)(\text{d}t - \overline{\gamma} \cosh \chi \text{d}\psi) + e f_k \text{d}\varphi.
\end{align*}
$$

This metric is horizonless and geodesically complete provided $\psi$ is an angle (period $2\pi$).

4. $R^2 \times \Sigma_2 \times R$

For $\beta^2 = 0$, the regular solution, derived from (3.22) of [40] with appropriate coordinate transformations, is

$$
\begin{align*}
\text{d}s^2_{(5)} &= -\left( \text{d}t + \frac{\mu}{2} x^2 \text{d}\psi \right)^2 + \text{d}x^2 + x^2 \text{d}\psi^2 + a^2 \text{d}\Sigma_k, \\
A_{(5)} &= -\sqrt{2} \left( \text{d}t + \frac{\mu}{2} x^2 \text{d}\psi \right) + e f_k \text{d}\varphi.
\end{align*}
$$

5. $S^2 \times \Sigma_2 \times R$ with NUT

Finally, for $\beta^2 < 0$, the five-dimensional metric (3.15) is replaced by

$$
\begin{align*}
\text{d}s^2_{(5)} &= -\frac{\overline{\gamma}^2}{\beta^2 + 1} (r^2 - m^2) \text{d}t^2 - \frac{1}{\beta^2 + 1} [(\beta^2 + 1) \text{d}z + r \text{d}t]^2 \\
&\quad + a^2 \left[ -\frac{a^2}{g^2 \beta^2} \frac{\text{d}r^2}{r^2 - m^2} + \text{d}\Sigma_k \right]
\end{align*}
$$

(where $\overline{\gamma}^2 = -\beta^2$), with signature $(- - - + +)$ in the coordinate range $r^2 > m^2$. However the Minkowskian signature $(+ - - + +)$ is recovered in the range $r^2 < m^2$, which suggests carrying out the coordinate transformation $t \rightarrow (\overline{\gamma}/m)(1 + \overline{\gamma})^{1/2}\psi$, $z \rightarrow (1 + \overline{\gamma})^{-1/2}t$, $r \rightarrow m \cos \chi$, with $\overline{\gamma}$ given by (3.18), leading to

$$
\begin{align*}
\text{d}s^2_{(5)} &= -(\text{d}t - \overline{\gamma} \cos \chi \text{d}\psi)^2 + \overline{\gamma}^2 \beta^2 (\text{d}\chi^2 + \sin^2 \chi \text{d}\psi^2) + a^2 \text{d}\Sigma_k, \\
A_{(5)} &= -\sqrt{2}(1 + \overline{\gamma}^2)(\text{d}t - \overline{\gamma} \cos \chi \text{d}\psi) + e f_k \text{d}\varphi.
\end{align*}
$$

(3.22)
6. Minimal supergravity

The corresponding Gödel solution is (3.22) with

\[ k = -1, \quad \beta^2 = \frac{1}{8}, \quad \beta g_\beta^2 = a^2, \quad \beta = g. \] (3.23)

By transforming the coordinates \( \theta \) and \( \chi \) to \( x = \cosh \theta, \ y = \cos \chi \), this solution may be written in the symmetrical form

\[ ds^2_{(5)} = -(dt - gy d\psi)^2 + \frac{g^2}{8} \left[ \frac{dy^2}{1-y^2} + (1-y^2)d\psi^2 + \frac{dx^2}{x^2-1} + (x^2-1)d\varphi^2 \right] \]
\[ A_{(5)} = -\frac{3}{2} (dt - gy d\psi) + \frac{\sqrt{3}}{2} gx d\varphi, \] (3.24)

with \( x^2 > 1, \ y^2 < 1 \).

Let us note that, for \( \gamma > \sqrt{3}/2 \), Eq. (3.14) can also be solved by \( k = +1, \ a^2 = -\alpha^2 < 0 \) (reduction on a timelike two-sphere). In the range \( r^2 > m^2 \), the resulting metric for minimal supergravity may be written in the form (3.19), with

\[ k = +1, \quad \beta^2 = -\frac{1}{8}, \quad \beta g_\beta^2 = -\pi^2, \quad \beta = -g. \] (3.25)

This metric, with the unphysical signature \((- - - - -\)), is related to (3.22) by the analytical continuation \( \chi \leftrightarrow i\chi, \ \theta \leftrightarrow i\theta, \ \psi \leftrightarrow -\psi, \ \varphi \leftrightarrow -\varphi \). The corresponding symmetrical form of this “anti-Gödel” solution, obtained by putting \( x = \cos \theta, \ y = \cosh \chi \), is again (3.24), but with \( g \rightarrow -g \), and \( x^2 < 1, \ y^2 > 1 \). Remarkably, as we shall show in a forthcoming paper [49], the Gödel and anti-Gödel solutions, with different spacetime signatures, can also be transformed into each other by \( G_{2(5)+2} \) sigma-model transformations.

IV. MULTICENTER SOLUTIONS

A. Toroidal reduction

All the solutions of EMCSA5 discussed above admit three commuting Killing vectors. In this case, beside reduction on a constant curvature two-surface, one can also carry out toroidal reduction relative to any two \( \partial_a \) \((a = 1,2)\) of these three Killing vectors, according to the \( GL(2,R)\)-covariant Kaluza-Klein ansatz

\[ ds^2_{(5)} = \lambda_{ab}(dx^a + \alpha^a dx^i)(dx^b + \alpha^b dx^j) + \tau^{-1} h_{ij} dx^i dx^j, \] (4.1)
\[ A_{(5)} = \sqrt{3}(\psi_a dx^a + A_i dx^i) \] (4.2)

\( (i,j = 3,4,5) \) where \( \tau = -\text{det} \lambda \). The Maxwell and Kaluza-Klein vector fields are then dualized to scalar potentials \( \nu \) (magnetic [57]) and \( \omega \) (twist). In performing this dualization, we must take care that the scalar potential \( \tau \) can be positive (for most of the solutions considered here) or negative (in the special case of the Gödel solutions (3.15) with \( \beta^2 < 0 \) and (5–) signature). In this case \( \sqrt{|g_{(5)}|} = \varepsilon \tau \sqrt{h} \), where \( \varepsilon = \text{sign}(\tau) \), and the dualization equations of [1, 2] are modified to

\[ F^{ij} = a^{ij} \partial^a \psi_a - a^{ai} \partial^j \psi_a + \varepsilon \frac{1}{\tau \sqrt{h}} \epsilon^{ijk} \eta_k, \quad \eta_k = \partial_k \nu + \gamma e^{ab} \psi_a \partial_k \psi_b \] (4.3)

and

\[ \lambda_{ab} G^{abij} = \varepsilon \frac{1}{\tau \sqrt{h}} \epsilon^{ijk} V_{ak}, \quad V_{ak} = \partial_k \omega_a - \psi_a \left( 3 \partial_k \nu + \gamma e^{ce} \psi_b \partial_k \psi_c \right) \] (4.4)

with \( G^{abij}_{ij} \equiv \partial_i a^a_j - \partial_j a^b_i \). After dualization, the reduced field equations derive from the three-dimensional gravity coupled sigma model with a potential

\[ S_3 = \int d^3 x \sqrt{h} \left( -R + \frac{1}{2} G_{AB} \frac{\partial \Phi^A}{\partial x^i} \frac{\partial \Phi^B}{\partial x^j} h^{ij} + U \right), \] (4.5)
where $R$ is the Ricci scalar of the metric $h_{ij}$ (not to be confused with the three-dimensional Ricci scalar $\mathcal{R}$ of Sec. 2), $\Phi^A$ ($A = 1, \ldots, 8$) are the eight moduli $\lambda_{ab}$, $\omega_a$, $\psi_a$, $\nu$. The potential arises owing to the cosmological constant and depends on the three moduli $\lambda_{ab}$ through the only determinant variable $\tau$:

$$ U = 2\Lambda \tau^{-1} . \quad (4.6) $$

The metric of the eight-dimensional target space is:

$$ dS^2 = \mathcal{G}_{AB} d\Phi^A d\Phi^B = \frac{1}{2} \text{Tr}(\lambda^{-1} d\lambda \lambda^{-1} d\lambda) + \frac{1}{2} \tau^{-2} d\tau^2 - \tau^{-1} V^T \lambda^{-1} V $$

$$ + 3 \left( d\psi^T \lambda^{-1} d\psi - \tau^{-1} \eta^2 \right), \quad (4.7) $$

where $\lambda$ is the $2 \times 2$ matrix of elements $\lambda_{ab}$, and $\psi, V$ the column matrices of elements $\psi_a, V_a$. In the case $\gamma = 1$ the target space is a symmetric space with $G_{2(2)}$ isometry group, while for general $\gamma$ (including $\gamma = 0$) it is not a symmetric space and the isometry group is solvable.

In the case of the solutions given in the preceding section, the moduli $\Phi^A$ depend on the three-space coordinates through a single scalar function which we will denote $\sigma(x)$. The equations of motion then reduce to

$$ \frac{D^2 \Phi^A}{d\sigma^2} \sigma^i \sigma^j + \hat{\Phi}^A \nabla^2 \sigma = G^{AB} \partial_B U , \quad (4.8) $$

$$ R_{ij} = \frac{1}{2} \frac{dS^2}{d\sigma^2} \sigma^i \sigma^j + U h_{ij} . \quad (4.9) $$

where $\hat{\Phi}_A$ stands for the derivative over $\sigma$, $D$ denotes the covariant derivative with respect to the target space metric $\mathcal{G}_{AB}$, and $\nabla$ is the covariant derivative with respect to the three-dimensional reduced metric $h_{ij}$. In the case $\Lambda = 0$, the potential $U$ vanishes and, as shown in [50], the function $\sigma(x)$ can be chosen to be harmonic,

$$ \nabla^2 \sigma = 0 , \quad (4.10) $$

so that Eqs. (4.8) reduce to the equations for the target space geodesics. Null geodesics lead to a Ricci-flat, hence flat, reduced three-space of metric $h_{ij}$ [52, 53]. In that case, the Laplacian $\nabla^2$ becomes a linear operator, so that an arbitrary number of harmonic functions may be superposed, leading to a multicenter solution

$$ \sigma(x) = \epsilon + \sum_i \frac{a_i}{|\bar{x} - \bar{x}_i|} . \quad (4.11) $$

(in the case where the flat three-space is $R^3$).

In what follows we restrict to the case $\Lambda = 0$. In the present case, the target space metric (4.7) does not depend on the three cyclic coordinates $\omega_a$ and $\nu$ arising from dualization, so that the corresponding conjugate momenta $\Pi^a$ and $3\mathcal{P}$ are constants of the motion:

$$ \Pi^a = \frac{2}{\tau} \lambda_{ab} V_b , \quad \mathcal{P} = \frac{2}{\tau} \eta - \Pi^a \psi_a , \quad (4.12) $$

where we have defined

$$ \eta = \dot{\nu} + \gamma \psi^T J \dot{\psi} , \quad \mathcal{P} = \dot{\omega} - \left( 3\eta - 2\gamma \psi^T J \dot{\psi} \right) \psi , \quad (4.13) $$

where $\dot{\cdot} \equiv d/d\sigma$, and $J$ is the $2 \times 2$ matrix $\epsilon_{ab}$. The five remaining geodesic equations and the null geodesic condition $dS^2 = 0$ read

$$ 6 \left( \lambda^{-1} \dot{\psi} \right) - \gamma (4\Pi^T \psi + 6\mathcal{P}) (J \dot{\psi}) - 2\gamma (\Pi^T \psi) (J \psi) - \left( \frac{3\tau}{2} (\Pi^T \psi + \mathcal{P}) + 2\gamma \psi^T J \dot{\psi} \right) \Pi = 0 , \quad (4.14) $$

$$ \dot{\chi} + \text{Tr} \dot{\chi} + 3\lambda^{-1} \psi^T \dot{\psi} = \frac{\tau}{4} \left[ \Pi^T \lambda + \Pi^T \lambda \Pi + 3(\Pi^T \psi + \mathcal{P})^2 \right] , \quad (4.15) $$

$$ \frac{1}{2} \text{Tr}(\dot{\chi}^2) + \frac{1}{2} (\text{Tr} \chi)^2 - \frac{\tau}{4} \left[ \Pi^T \lambda \Pi + 3(\Pi^T \psi + \mathcal{P})^2 \right] + 3\dot{\psi}^T \lambda^{-1} \dot{\psi} = 0 , \quad (4.16) $$
where $\chi \equiv \lambda^{-1} \dot{\lambda}$.

While it seems difficult to systematically solve this system of equations, it is easy to promote the special solutions presented in Sect. 2 to multicenter (null geodesic) solutions, provided that, after toroidal reduction relative to $\partial_\tau$ and $\partial_\xi$, the reduced metric $\delta s^2 = h_{ij} dx^i dx^j$ is flat, with the three possibilities $\delta s^2 = dr^2 + r^2 d\Sigma_1, \pm dr^2 + d\Sigma_0, \text{or} -dr^2 + r^2 d\Sigma_{-1}$. For $\Lambda = 0$, this is the case for the self-dual solution (3.9) with $k = +1, l = 2a$, as well as its $q = 0$ limit, the extreme BTZ solution (3.3) with $J^2 = M^2 l^2$, for the special vacuum solution (3.8), for the Gödel solution (3.17) with $\beta^2 = 1 (\gamma = 1/2, k = +1, g^2 = a^2)$, and for the extreme ($\mu^2 = 0$) Gödel solutions (3.16), (3.15) (with $k = +1, g^2 \beta^2 = a^2$) and (3.21) (with $k = -1, \beta^2 = a^2$).

### B. BTZ and self-dual solutions

We first consider, in the case $\Lambda = 0$, the self-dual solution (3.9) with $k = +1, l = 2a$, which contains for $q = 0$ the extreme BTZ solution. In the case of a generic Chern-Simons coupling constant $\gamma$, the solution (3.9) can be generalized by replacing the harmonic function $a/r$ by an arbitrary harmonic function $\sigma(\vec{x})$,

$$\begin{align*}
\delta s^2_{(5)} &= \sigma^{-1} du dv + \left( M - \frac{3c^2 \gamma}{4\gamma \pm 1} \right) du^2 + \sigma^2 dz^2, \\
A_{(5)} &= \sqrt{3} \left[ c \sigma^{\pm 2\gamma} du \pm A_3 \right] (\nabla \wedge A_3 = \nabla \sigma),
\end{align*}$$

with $u = z - t, v = z + t, c = q/\sqrt{3}$ [58]. The linear superposition (4.11) leads to multicenter solutions of EMCSA5, which are asymptotic to the one-center solution (3.9) for $\epsilon = 0$, and asymptotically Minkowskian (up to a gauge transformation) for $\epsilon \neq 0$. These are to our knowledge the first multi-string solutions of EMCSA5 (multi-hole solutions were considered in [54]).

The one-center asymptotically flat solution may be written in the ADM form:

$$\begin{align*}
\delta s^2_{(5)} &= -\frac{r^2}{(r + a)^2 R^2} dt^2 + R^2 \left( \frac{dz}{(r + a) R^2} dt \right)^2 + \left( \frac{r + a}{r} \right)^2 dr^2 + (r + a)^2 d\Omega_3^2, \\
A_{(5)} &= \sqrt{3} \left[ c \left( \frac{r}{r + a} \right)^{\pm 2\gamma} (dz - dt) \pm a \cos \theta d\varphi \right] \left( R^2 = M + \frac{r}{r + a} - \frac{3c^2 \gamma}{4\gamma \pm 1} \left( \frac{r}{r + a} \right)^{\mp 2\gamma} \right).
\end{align*}$$

For the lower sign the metric has a double horizon at $r = 0$. However this horizon is generically not regular. The first integral for geodesic motion in the metric (4.18) reads

$$r^2 + U(r) = \Pi'_v M$$

with the effective potential

$$U(r) = \Pi_u \Pi_v \frac{r}{r + a} + \Pi^2 \frac{3c^2 \gamma}{4\gamma \pm 1} \left( \frac{r}{r + a} \right)^{\mp 4\gamma} + \frac{L^2 r^2}{(r + a)^4} - \frac{\varepsilon r^2}{(r + a)^2},$$

where $\Pi_u, \Pi_v$ are the constant momenta conjugate to the cyclic coordinates $u$ and $v$, $L$ is the orbital angular momentum, and $\varepsilon = -1, 0,$ or $+1$ for timelike, null, or spacelike geodesics. It is clear that for the lower sign geodesics can be analytically continued inside the horizon $r = 0$ only for integer values of $4\gamma$. So (taking into account the fact that the form of the solution (4.18) breaks down for $\gamma = 0$ and $\gamma = 1/4$), the AF solution (4.18) is an extreme black string for the down sign and $\gamma = (n + 2)/4, n$ integer (or all real $\gamma$ for $c = 0$). For the upper sign, geodesics with $\Pi_v \neq 0$ are reflected by an infinite potential barrier, while spacelike geodesics with $\Pi_v = 0$ are either reflected or attain $r = 0$ only asymptotically, so that the spacetime is geodesically complete.

For $\gamma = 1/4$ and the lower sign the solution (4.17) is replaced by

$$\begin{align*}
\delta s^2_{(5)} &= \sigma^{-1} du dv + \left( M + \frac{3c^2}{4} \sigma^{-1} \ln \sigma \right) du^2 + \sigma^2 dz^2, \\
A_{(5)} &= \sqrt{3} \left[ c \sigma^{-1/2} du - A_3 \right],
\end{align*}$$

and for $\gamma = 0$ (Einstein-Maxwell theory) it is replaced by

$$\begin{align*}
\delta s^2_{(5)} &= \sigma^{-1} du dv + \left( M - 3c^2 \ln \sigma \right) du^2 + \sigma^2 dz^2, \\
A_{(5)} &= \sqrt{3} [c \ln \sigma du \pm A_3].
\end{align*}$$
In the case of the special vacuum solution (3.8), \( r \) is one of the cartesian coordinates of the reduced three-space and is a harmonic function on that space, so that the generalisation to a multicenter solution is the vacuum solution [51, 52]

\[ds^2_{(5)} = 2du dv - \sigma du^2 + dz^2\]  

(with \( u = t, v = z \)).

C. Gödel solutions

The first obvious candidate multicenter solution in the Gödel class is the \( \gamma = 1/2, \beta^2 = 1 \) solution (3.17) with \( k = +1, g^2 = a^2 \). Actually, it turns out that, after a trivial coordinate transformation \( z \propto u, t \propto v \), this is just an instance of the self-dual solution (4.17) for \( \gamma = 1/2 \) and the lower sign, with \( M \propto \alpha \).

The Einstein-Maxwell solution (3.16) (\( \gamma = 0 \)) with \( m^2 = 0, \Lambda = 0 \) leads to the multicenter solution

\[ds^2_{(5)} = -\sigma^{-2}dt^2 + dz^2 + \sigma^2d\xi^2,\]
\[A_{(5)} = -\sqrt{2} \sigma^{-1}dt + A_3,\]  

which is the trivial five-dimensional embedding of a dyonic Majumdar-Papapetrou solution.

From the generic \( m^2 = 0 \) Gödel solution (3.15) for \( 0 < \gamma < \sqrt{3}/2 \) with \( k = +1, g^2 \beta^2 = a^2 \), we derive the multicenter solution

\[ds^2_{(5)} = -(\sigma^{-1}dt + dz)^2 + \beta^2d\xi^2 + \sigma^2d\xi^2,\]
\[A_{(5)} = -\sqrt{2} \sigma^{-1}dt - \sqrt{3} 2\gamma \beta A_3 \left( \beta^2 = \frac{3}{8\gamma^2} - \frac{1}{2} \right),\]  

which may be viewed as a deformation of (4.24). For \( \beta^2 > 1 (\gamma < 1/2) \), the one-center asymptotically flat solution, generalized by the local coordinate transformation \( t \rightarrow t - \varpi z \) (with \( \varpi \) a second parameter)

\[ds^2_{(5)} = -\frac{\beta^2r^2}{(r + a)^2} dt^2 + R^2(dz - N^2 dt)^2 + \frac{(r + a)^2}{r^2}dr^2 + (r + a)^2 d\Omega_2^2,\]
\[A_{(5)} = a \left[ \frac{\sqrt{2}}{r + a}(dt - \varpi dz) - \varpi \sqrt{3} 2\gamma \beta \cos \theta d\varphi \right],\]  

with

\[R^2 = \beta^2 - (1 - \varpi^2) - \frac{2a\varpi(1 - \varpi^2)}{r + a} + \frac{a^2\varpi^2}{(r + a)^2},\]
\[N^z = \frac{r(1 - \varpi^2) + a}{(r + a)^2 R^2};\]

is a dyonic extreme black string.

Finally, for \( \gamma > \sqrt{3}/2 \),

\[ds^2_{(5)} = -(\sigma^{-1} dt + dz)^2 - \beta^2 d\xi^2 + \sigma^2 d\xi^2,\]
\[A_{(5)} = -\sqrt{2} \sigma^{-1} dt - \sqrt{3} 2\gamma \beta A_3 \left( \beta^2 = \frac{1}{2} - \frac{3}{8\gamma^2} \right),\]  

where \( \sigma(\vec{x}) \) is harmonic on the Minkowskian reduced metric \( d\vec{x}^2 = \eta_{ij} dx^i dx^j \). The signature of the metric (4.27) is \((--+++)) \). A Minkowskian multicenter solution may be obtained from the \( \Lambda = 0 \) Gödel solution (3.22) with \( k = -1, \overline{g^2 \beta^2} = a^2 \). By transforming the \( H^2 \) coordinates \((\theta, \varphi)\) to coordinates \((r, z)\) such that

\[d\Sigma_{-1} = \frac{dr^2}{r^2} + r^2 dz^2, \quad f_{-1} = r,\]  

we obtain the multicenter solution with NUTs:

\[ds^2_{(5)} = -\left( dt - \beta^{-1} A_3 \right)^2 + \sigma^{-2} dz^2 + \sigma^2 d\xi^2,\]
\[A_{(5)} = -\sqrt{2}(1 + \beta^2) \beta A_3 - \sqrt{3} 2\gamma \beta \sigma^{-1} dz.\]
V. SUMMARY

In this paper we have investigated solutions of the general five-dimensional EMCSΛ theory — containing as particular cases minimal ungauged and gauged supergravities— using known exact solutions of three-dimensional Einstein-Maxwell gravity with a Chern-Simons term. Our main tool was dimensional reduction, assuming the five-dimensional spacetime to be the direct product of a constant curvature surface $\Sigma_2$ of positive, zero or negative curvature ($k = 1$, 0 or $-1$) and a three-dimensional spacetime. The reduced theory is then the three-dimensional EMCSΛ with a constraint on the scalar curvature, which can be satisfied by three classes of solutions found earlier, namely, the BTZ, self-dual and Gõdel classes. Promoting these to five dimensions, we have constructed new EMCSΛ5 solutions of the generalized Bertotti-Robinson type which could be near-horizon limits of black string and black ring solutions.

The three-parameter BTZ class of solutions is geodesically complete and exists for an arbitrary (irrelevant) Chern-Simons coupling $\gamma$. Generically these solutions are non-vacuum, being supported by magnetic fluxes along the constant curvature two-surfaces. Their isometry groups are $SO(2, 2) \times SO(3)$ for $k = +1$, $SO(2, 2) \times GL(2, R)$ for $k = 0$ and $SO(2, 2) \times SO(2, 1)$ for $k = -1$. In the minimal supergravity case $\Lambda = 0$, $\gamma = 1$ only spherical sections are possible, and the corresponding solutions are near-horizon limits of black strings. For a negative cosmological constant this class of solutions exists in all the three versions $k = 1$, 0, or $-1$, with horizon topologies $S^1 \times S^2$, $S^3 \times R^2$ and $S^1 \times H^2$ respectively. The latter two could be near-horizon limits of topological asymptotically AdS black rings, though the existence of the global solutions remains to be checked. There is a special case with zero magnetic flux, when we get a two-parameter family of vacuum solutions which can be seen as non-asymptotically flat topological black rings with the horizon $S^1 \times H^2$. These are (presumably new) locally AdS$_3 \times H^2$ solutions of vacuum five-dimensional Einstein equations. For a positive cosmological constant the solutions exist in the $k = 1$ version only. Their interpretation is similar, but the relevant asymptotics is De Sitter.

The second class of EMCSΛ5 solutions, depending also on three parameters, generalize the extremal Schwarzschild–de Sitter and the BTZ class, which they asymptote. They are supported by a magnetic flux together with an independent dyonic electromagnetic field. The solutions of the third three-parameter class are generated from three-dimensional Gõdel black holes. They exist with $k = +1$ for $\gamma < \sqrt{3}/2$, $k = 0$ for $\gamma = \sqrt{3}/2$ and $k = -1$ for $\gamma > \sqrt{3}/2$. The isometry groups are $SO(2, 1) \times SO(2) \times SO(3)$, $SO(2, 1) \times SO(2) \times GL(2, R)$, and $SO(2, 1) \times SO(2) \times SO(2, 1)$ respectively. Subclasses include solutions analogue to previously found rotating BR solutions in dilaton-axion gravity, horizonless and geodesically complete solutions with spacetime topology $H^2 \times \Sigma_2 \times R$, and NUTty solutions with spacetime topology $S^2 \times \Sigma_2 \times R$. In the case of minimal $D = 5$ supergravity, the NUTty $S^2 \times H^2 \times R$ solution (3.24) has been shown [49] to be a near-extreme, near-bolt limit of an asymptotically flat solitonic string solution of EMCSΛ5.

We have then shown that some of the BR solutions thus constructed can be promoted to NAF or AF multicenter solutions. For this purpose we have performed an alternative toroidal compactification of EMCSΛ5 to three dimensions, leading to a sigma-model representation with a potential. For $\Lambda = 0$ this potential vanishes, in which case null geodesics of the target space give rise to exact solutions of the five-dimensional equations with a flat reduced three-space. The corresponding BR solutions may be promoted to multicenter solutions by redefinition of the associated harmonic functions. We have thus identified three families of multi-string solutions of EMCSΛ5, the two “self-dual” families (4.17), and the Gõdel family (4.25) (for $\gamma < \sqrt{3}/2$) or (4.29) (for $\gamma > \sqrt{3}/2$). An unexpected by-product of our analysis is the construction of new closed-form asymptotically flat solutions (4.18) generated by a dyonic electromagnetic field along with a magnetic flux. For the lower sign these are regular extreme black strings for a discrete set of values of the Chern-Simons coupling constant (including the minimal supergravity case $\gamma = 1$), while for the upper sign they are geodesically complete.

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Appendix: Constant scalar curvature solutions to three-dimensional EMCS\(\Lambda\) theory

The ansatz \[40\]
\[d s^2 = \lambda_{ab}(\rho) \, d x^a \, d x^b + \mu^{-2}(\rho) R^{-2}(\rho) \, d \rho^2, \quad A = \psi_a(\rho) \, d x^a\]  
(A.1)

where \(\lambda\) is the \(2 \times 2\) matrix

\[\lambda = \begin{pmatrix} T + X & Y \\ Y & T - X \end{pmatrix},\]  
(A.2)

and \(R^2 \equiv X^2\) is the Minkowski pseudo-norm of the “vector” \(X(\rho) = (T, X, Y)\),

\[X^2 = \eta_{ij} \, X^i \, X^j = -T^2 + X^2 + Y^2,\]  
(A.3)

reduces the equations of three-dimensional EMCS\(\Lambda\) theory to the Maxwell-Chern-Simons, Einstein and Hamiltonian constraint equations

\[S' = \frac{2}{R^2} X \wedge S,\]  
(A.4)

\[X'' = \frac{2}{R^2} \left[ \frac{2}{R^2} (S \cdot X) X - S \right],\]  
(A.5)

\[X' \cdot X'' + 4 \frac{\lambda}{\mu^2} = 0,\]  
(A.6)

where \(S\) is the null vector

\[S = \frac{1}{4} \left( \psi_0^2 + \psi_1^2, \psi_0^2 - \psi_1^2, 2 \psi_0 \psi_1 \right), \quad \text{\((S^2 = 0)\)}\]  
(A.7)

\[\cdot = \partial/\partial \rho, \text{ and the wedge product is defined by } (X \wedge Y)^i = \eta^{ij} \epsilon_{jkl} X^k Y^l \text{ (with } \epsilon_{012} = +1).\]

To the above equations we must add the constraint (2.14)

\[R \equiv \mu^2 \left[ \frac{1}{2} X^2 - (X^2)'' \right] = \Lambda + 3\lambda - 3\mu^2/16\gamma^2.\]  
(A.8)

Combining equations (A.6) and (A.8), we obtain the relations

\[(R^2)'' = 2b, \quad X \cdot X'' = c, \quad X' \cdot X'' = b - c,\]  
(A.9)

with \(b = -(\Lambda + \lambda)/\mu^2 + 3/16\gamma^2, c = (\Lambda - 3\lambda)/\mu^2 - 3/16\gamma^2\). From (A.5) it follows that

\[S \cdot X = \frac{c}{2} R^2.\]  
(A.10)

Noting that \((S \cdot X)' = S \cdot X'\) from (A.4), we derive from (A.10)

\[(S \cdot X)'' = S \cdot X'' + S' \cdot X' = c^2 - \frac{2}{R^2} L \cdot S,\]  
(A.11)

where we have defined

\[L = X \wedge X'.\]  
(A.12)

Comparing (A.11), (A.10) and (A.9), we obtain

\[L \cdot S = \frac{c(c - b)}{2} R^2.\]  
(A.13)

Next, noting that \(L' \cdot S = 0\) from (A.4), we obtain

\[(L \cdot S)' = \frac{2}{R^2} (X \wedge X') \cdot (X \wedge S) = \frac{2}{R^2} \left[ (S \cdot X)(X \cdot X') - R^2(S \cdot X') \right] = -\frac{c}{2} (R^2)' .\]  
(A.14)
Comparing with (A.13), we derive the constraint
\[ c(c - b + 1)(R^2)' = 0 \]  \tag{A.15}

The first possibility \( c = 0 \) means that \( S \cdot X = 0 \), which is equivalent to the self-duality condition \([43]\) \( F^{\alpha\beta} F_{\alpha\beta} = 0 \), which leads either to the self-dual solutions (3.9) for \( \lambda < 0 \) and (3.11) for \( \lambda = 0 \), or to the vacuum solutions of the BTZ class.

Consider now the second possibility, \( b - c = 1 \). Squaring Eq. (A.5), we find that
\[ X''^2 = 0 \]  \tag{A.16}
while the last equation (A.9), which now reads \( X'^2 = 1 \), implies that \( X' \cdot X'' = 0 \). The fact that the null vector \( X'' \) is orthogonal to \( X' \) implies that the wedge product \( X' \wedge X'' \) is collinear with \( X'' \):
\[ X' \wedge X'' = \pm X'' \]  \tag{A.17}
This relation, together with Eq. (A.5), may be used to transform Eq. (A.4) to
\[ S' = -X \wedge X'' = \mp X \wedge (X' \wedge X'') \]
\[ = \mp (X \cdot X')X'' \pm (X \cdot X'')X' = \mp \frac{1}{2} (R^2)'X'' \pm cX'. \]  \tag{A.18}
Taking the scalar product of (A.18) with the vector \( L \) and using (A.5) and (A.13), we obtain
\[ L \cdot S' = \mp \frac{c}{2} (R^2)' \]  \tag{A.19}
which upon comparison with (A.14) shows that we should take the upper sign in the preceding equations. Finally, Eq. (A.18) with the upper sign may be compared with the direct derivative of Eq. (A.5),
\[ S' = -\frac{1}{2} (R^2)X'' - \frac{1}{2} (R^2)'X'' + cX', \]  \tag{A.20}
leading to the conclusion that
\[ X''' = 0. \]  \tag{A.21}
This equation is integrated by
\[ X = \alpha \rho^2 + \beta \rho + \gamma, \quad (\alpha^2 = 0, \quad \alpha \wedge \beta = -\alpha), \]  \tag{A.22}
the vector relations between the constant vectors \( \alpha, \beta, \gamma \) following from (A.16) and (A.17). We recognize in (A.22) the quadratic ansatz which was used in [40] to derive the Gödel solutions to three-dimensional EMCSA.

For the last possibility, \( (R^2)' = 0 \), we can adapt the above argument to show that
\[ X' \wedge X'' = qX'' \quad (q^2 = b - c). \]  \tag{A.23}
Eq. (A.18) is now replaced by
\[ S' = \frac{c}{q} X'. \]  \tag{A.24}
Taking the scalar product with \( X' \) and using (A.11) and (A.13), we obtain
\[ S' \cdot X' = cq = cq^2. \]  \tag{A.25}
The first solution, \( c = 0 \), leads to a subcase of the self-dual class. The second solution, \( q = 0 \), means that \( X'' \) is collinear with \( X' \) so that \( X \cdot X'' = c = 0 \), leading again to a subcase of the self-dual class. And the last solution, \( q = 1 \), means \( b - c = 1 \), corresponding to a subclass of the Gödel class.

[1] A. Bouchareb, C. M. Chen, G. Clément, D. V. Gal’tsov, N. G. Scherbluk and T. Wolf, Phys. Rev. D 76, 104032 (2007), Erratum, Phys. Rev. D 78, 029901 (2008) [arXiv:0708.2361].
[54] K. Matsuno, H. Ishihara, M. Kimura and T. Tatsuoka, Phys. Rev. D 86, 104054 (2012) [arXiv:1208.5536].

[55] Dimensional reduction in the action (1.1) does not produce a correct three-dimensional action.

[56] These solutions were obtained in [40] under the assumption \( \mu > 0 \), i.e. on account of the second Eq. (2.13) \( \gamma e > 0 \).

[57] The magnetic potential \( \mu \) of [1, 2] is noted here \( \nu \) to avoid confusion with the Chern-Simons coupling constant.

[58] We have changed a sign in \( A_\circ \) because our coordinate transformation implies \( \epsilon_{uv} = -\epsilon_{tz} \).