Topological Superfluid in P-band Optical Lattice

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In this paper by studying p-band fermionic system with nearest neighbor attractive interaction we find translation symmetry protected $\mathbb{Z}_2$ topological superfluid (TSF) that is characterized by a special fermion parity pattern at high symmetry points in momentum space \( k = (0,0), (0,\pi), (\pi,0), (\pi,\pi) \). Such $\mathbb{Z}_2$ TSF supports the robust Majorana edge modes and a new type of low energy excitation - (supersymmetric) $\mathbb{Z}_2$ link-excitations. In the end we address a possible realization of such interacting p-band fermions with $\mathbb{Z}_2$ TSF.

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Superconductivity/superfluidity is a paired state in many-body charged/neutral fermion systems introduced by Bardeen, Cooper and Schrieffer (BCS). In BCS’s theory, the glue for the Cooper pair is phonon - the collective mode of atoms in solids. To describe such a quantum ordered state, a local order parameter \( \Delta_{\mathbf{k}} = \langle \hat{c}_{\mathbf{k}}^\dagger \hat{c}_{-\mathbf{k}} \rangle \) is introduced that differs in different superconducting (SC) states (for example, s-wave, p-wave, d-wave, etc.). Recently, people found that SC states with the same local order parameter may have different topological properties, that leads to the concept of "topological superconductivity (TSC)". According to the characterization of "ten-fold way" of random matrix, there are three types of TSCs in two dimensions: D-type chiral TSC without time reversal symmetry \( (p_x \pm ip_y \text{-wave}) \), s-wave with strong spin-orbital coupling \( \cdots \), C-type chiral TSC without time reversal symmetry \( (d_z \pm id_y \text{-wave}) \) and DIII-type \( \mathbb{Z}_2 \) TSC with time reversal symmetry \( (p_x + ip_y) \times (p_x - ip_y) \text{-wave} \).

Recently, rapid advances in trapping and manipulating ultracold atoms in optical lattice have made it possible to simulate topological superfluid (TSF) - a kind of "SC" in a neutral fermion system. For the paired states in optical lattices, the glues cannot be the phonons. Instead, people may have an additional degree of freedom to obtain the TSF by considering p-band fermions with nearest neighbor attractive interaction on a square optical lattice and found that the ground state may be a \( p_x \times p_y \text{-wave} \) SF - the \( p_x \)-band fermions are paired into \( p_x \)-wave SF state and the \( p_y \)-band fermions are paired into \( p_y \)-wave SF state. Such a \( p_x \times p_y \text{-wave} \) paired state is translation symmetry protected \( \mathbb{Z}_2 \) TSF against arbitrary translation invariant perturbations. In addition, we found that this \( \mathbb{Z}_2 \) TSF supports the robust Majorana edge modes and there exists a new type of low energy excitations - (supersymmetric) \( \mathbb{Z}_2 \) link-excitations. Let us explain why there exist \( \mathbb{Z}_2 \) TSF beyond "ten-fold way" : if one enforces translation invariant on the system, there may exist sixteen symmetry protected TSCs/TSFs on a two dimensional (2D) lattice that are characterized by a special fermion parity pattern at four high symmetry points \( (k = (0,0), (0,\pi), (\pi,0), (\pi,\pi)) \).

Mean field phase diagram and \( \mathbb{Z}_2 \) TSF: In this paper we studied (spinless) p-band fermions with nearest neighbor attractive interactions on square lattice

\[
H_{\text{eff}} = t \sum_i \hat{c}_{i\sigma}^\dagger \hat{c}_{i+\sigma} + t \sum_i \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\uparrow} - U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - U_0 \sum_{\langle i,j \rangle} \hat{n}_{i\downarrow} \hat{n}_{j\uparrow} - \mu \sum_i \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\uparrow} - \Delta_{\sigma} \sum_i \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma} + h.c.
\]

where \( p_x \)-band fermions only hop along x-direction and \( p_y \)-band fermions only hop along y-direction. \( U \) and \( U_0 \) are the on-site interaction strength and the nearest neighbor interaction strength, respectively \( (U > 0, U_0 > 0) \). We denote orbitals index by pseudo-spin \( \sigma = (p_x, p_y) \equiv (\uparrow, \downarrow) \) and \( \langle i,j \rangle \) represents two sites on a nearest-neighbor link. In this paper we only consider the half filling case with \( \mu = 0 \).

The model in Eq.\ref{eq:hamiltonian} is unstable against superfluid (SF) orders that are described by \( \Delta_x \) and \( \Delta_y \) for s-wave and p-wave order parameters, where \( \Delta_x = \langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\uparrow} \rangle, \Delta_y = \langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\uparrow} \rangle \). The effective Hamiltonian becomes

\[
H_{\text{eff}} = t \sum_i \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\uparrow} + U_0 \Delta_p \sum_i \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\uparrow} - U \Delta_x \sum_i \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\uparrow} + U_0 \Delta_y \sum_i \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\uparrow} + h.c.
\]

One can see that \( p_x \)-band fermions only pair along x-direction, \( \langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\uparrow} \rangle \neq 0, \langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\uparrow} \rangle = 0 \) and \( p_y \)-band fermions only pair along y-direction, \( \langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\uparrow} \rangle \neq 0, \langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\uparrow} \rangle = 0 \). For this reason we call the p-wave SF as \( p_x \times p_y \)-wave state. In the SF state, the total fermion
physical indices identify two distinct SF states: SF with trivial topological indices \( \zeta \) where \( \Theta(x) = 1 \) if \( x > 0 \) and \( \Theta(x) = 0 \) if \( x < 0 \). Thus we classify fermionic parities at high symmetry points by defining four \( \mathbb{Z}_2 \) topological invariants\(^1\): for \( k = (0,0) \), \( (\pi,\pi) \), we have a trivial result as \( \zeta_{k=(0,0)} = \zeta_{k=(\pi,\pi)} = 0 \); for \( k = (0,\pi) \), \( (\pi,0) \), we have

\[
\zeta_{k=(0,\pi)} = \zeta_{k=(\pi,0)} = 1 - \Theta (\mu^2 + U^2 \Delta^2 - 4t^2) \tag{3}
\]

where \( \Theta(x) = 1 \) if \( x > 0 \) and \( \Theta(x) = 0 \) if \( x < 0 \). Thus we identify two distinct SF states: SF with trivial topological indices \( \zeta_{k=(0,0)} = \zeta_{k=(\pi,\pi)} = 0 \) in region I \( (\Delta_s \neq 0, \Delta_p = 0) \) and region II \( (\Delta_s \neq 0, \Delta_p 
eq 0) \) and \( \mathbb{Z}_2 \) TSFs with nontrivial topological indices \( \zeta_{k=(0,\pi)} = \zeta_{k=(\pi,0)} = 1 \) in region III \( (\Delta_s \neq 0, \Delta_p \neq 0) \) and IV \( (\Delta_s = 0, \Delta_p \neq 0) \). From topological indices of \( \mathbb{Z}_2 \) TSF \( (0110)\) we found a special fermion parity pattern: even fermion parity at \( k = (0,0) \) and \( k = (\pi,\pi) \) and odd fermion parity at \( k = (0,\pi) \), \( (\pi,0) \). This is the reason why we call it \( \mathbb{Z}_2 \) TSF. The SF states in region V \( (\Delta_s \neq 0, \Delta_p \neq 0) \) is gapless which we are not interest in. In the following parts, we will focus on \( \mathbb{Z}_2 \) TSFs in region III and region IV.

Majorana edge state: In this part, we study the edge states of \( \mathbb{Z}_2 \) TSFs. Fig.2 illustrates the edge states in region III and IV. In the region IV \( (\Delta_s = 0, \Delta_p \neq 0) \), \( \mathbb{Z}_2 \) TSF behaviors like two coupled 1D p-wave SCs, one along x-direction, and the other along y-direction. Because each 1D p-wave SCs with effective Hamiltonian

\[
H_{\text{eff}} = t \sum \hat{c}_{\uparrow,i}^\dagger \hat{c}_{\downarrow,i+1} + e_{xy} - \Delta_p U \hat{c}_{\uparrow,i}^\dagger \hat{c}_{\downarrow,i} e_{xy} + h.c.
\]

is really a Majorana chain, we get zero energy states at the boundary\(^2\). Indeed, the zero energy states can be mapped to a Majorana fermion system with a flat band exactly (See right figure in Fig.2). In the region of III \( (\Delta_s \neq 0, \Delta_p = 0) \), the Majorana fermion system is still stable but has dispersion\(^3\) (See left figure in Fig.2).

On the contrary, in the region I and II, there are no edge states at all.

\( \mathbb{Z}_2 \) link-excitation: Next, we study the low energy excitations in the \( \mathbb{Z}_2 \) TSF states in region III and IV. One excitation is the Bogolubov quasi-particle which has an energy gap \( E_{qp} \) in \( \mathbb{Z}_2 \) TSFs (See blue line in Fig.3). Another excitation is the collective mode - the Goldstone mode due to spontaneous global U(1) symmetry breaking. In this part we emphasis that there may exist another type of low energy excitations for TSF on a square lattice - the \( \mathbb{Z}_2 \) link-excitation.

A \( \mathbb{Z}_2 \) link-excitation is defined as the sign-switching of SC order parameter on a link position \( I = (I,J) = (i_0, i_0 + e_x/e_y) \) as \( \Delta_p \rightarrow \{ -\Delta_p, \ i = i_0 \ \}. \) From this definition, one can see that its particle number is conserved mod 2. This is the reason why we call it \( \mathbb{Z}_2 \) link-excitation. By exact diagonalization calculations, we obtain the excited energy \( E_{\text{link}} \) for a single \( \mathbb{Z}_2 \) link-excitation in Fig.3 (See magenta line). From it one can see that the \( \mathbb{Z}_2 \) link-excitation even have lower energy than quasi-particle excitation and becomes relevant to low energy physics.

We point out that an important property of \( \mathbb{Z}_2 \) link-excitation is the existence of a bound state (zero mode) on it for the \( \mathbb{Z}_2 \) TSFs states both in region III and IV. The inset of Fig.3 shows the bound state on a \( \mathbb{Z}_2 \) link-excitation in the region IV. Then we define that \( E_+ \) is the energy for the empty bound state (or the even parity state) and \( E_- \) is the energy for the occupied bound state (or odd parity state). In the region IV, when \( U_0 \Delta_p \neq t \), there may exist energy splitting between \( E_+ \) and \( E_- \) due

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\( \Delta \) indicates a gapless state that we are not interest in. The blue lines with dots are the super-\( \mathbb{Z}_2 \) TSFs.

![FIG. 1: Phase diagram: region I and region II are trivial SC states with \( \Delta_s \neq 0, \Delta_p = 0 \) and \( \Delta_s \neq 0, \Delta_p = 0 \), respectively; Yellow regions (region III and region IV) are \( \mathbb{Z}_2 \) TSFs with \( \Delta_s \neq 0, \Delta_p \neq 0 \) and \( \Delta_s = 0, \Delta_p = 0 \), respectively; region V is a gapless state that we are not interest in. The blue lines with dots are the super-\( \mathbb{Z}_2 \) TSFs.](image1)

![FIG. 2: Left figure: Edge states of TSF in region III which have a symmetry protected zero energy mode at \( k_x = 0, \pi \); Right figure: Edge states of TSF in region IV which have a flat band.](image2)
to quantum tunneling effect, $\Delta E = E_+ - E_-$. By exact diagonalization calculations, we derive $\Delta E$ in Fig.3 (See dash violet line). The occupied bound state with odd fermion parity has lower energy than the empty bound state with even fermion parity, $\Delta E > 0$ (we have set the chemical potential to be zero). Now the $Z_2$ link-excitation can be regarded as a fermion when the energy scale is smaller than $\Delta E$.

Particularly, in the region IV, when $U_0\Delta_p = t$, the energy splitting of the two bound states becomes zero $\Delta E = 0$. For this case, the energy dispersion of quasi-particles becomes a constant or the Bogoliubov quasi-particles have a flat band, that corresponds to a quasi-particle with divergent mass. Thus the quantum tunneling effect by exchanging quasi-particles is frozen and the energy is degenerate for an occupied zero mode and an empty zero mode. Now the $Z_2$ link-excitation of even parity bound state is a Boson and that of odd parity state is a Fermion, that is really emergent supersymmetry. And we call this TSF with supersymmetric $Z_2$ link-excitation to be super-$Z_2$ TSF.

We also study the $Z_2$ link-exitations in region III and find that their properties are the same as those in region IV (see the dotted line in region III of Fig.1 that corresponds to the super-$Z_2$ TSF).

$Z_2$ vortex: After recognizing the properties of $Z_2$ link-excitation, we turn to study $Z_2$ vortex. For a SF on square lattice, $Z_2$ vortex is really a $\pi$-flux on a plaquette which is confined at zero temperature and cannot be real excitation. On one hand, the $Z_2$ link-exitations can be regarded as bound states of a pair of $Z_2$ vortices on nearest neighbor plaquettes ($A$ and $B$) (See Fig.4). On the other hand, the $Z_2$ vortex can be regarded as the end of non-local string of $Z_2$ link-exitations and is generated by a string of link-exitations.

When $Z_2$ link-excitation is a fermion ($U_0\Delta_p \neq t$), due to fermion parity constraint, $Z_2$ vortex cannot hop from a plaquette to nearest neighbor plaquette by creating an additional fermionic $Z_2$ link-excitation which violates the fermion parity conservation. Thus there must exist two types of $Z_2$ vortices: $e$-type $Z_2$ vortex on $A$ sub-plaquette and the $m$-type $Z_2$ vortex on $B$ sub-plaquette. Each type of $Z_2$ vortex lies on each sub-plaquette. In addition, there is mutual-seminion statistics between $e$-type $Z_2$ vortex and $m$-type $Z_2$ vortex. And an $e$-type $Z_2$ vortex and an $m$-type $Z_2$ vortex annihilate with each other into a fermionic $Z_2$ link-excitation. Now the system has 4 superselection sectors: 1 (the vacuum), $e$ ($e$-type $Z_2$ vortex), $m$ ($m$-type $Z_2$ vortex), $\psi$ (fermionic $Z_2$ link-excitation) and the fusion rule is $e \times e = m \times m = \psi \times \psi = 1$, $e \times m = \psi$, $e \times \psi = m$, $m \times \psi = e$. The fusion rule is the same as that of the $Z_2$ topological order in toric code model and Wen’s plaquette model

For the super-$Z_2$ TSF, we find that $Z_2$ vortex has internal degree of freedom - it can be either $e$-type $Z_2$ vortex or $m$-type $Z_2$ vortex. That is each $Z_2$ vortex has two components as $\sigma = (\begin{array}{c} e \\ m \end{array})$. When same types of $Z_2$-vortices ($ee$, $mm$) annihilate each other, we have a bosonic $Z_2$ link-excitation which belongs to the vacuum sector; When different types of $Z_2$-vortices ($em$) annihilate each other, we have a fermionic $Z_2$ link-excitation. For this reason we call it super-$Z_2$-vortex. See the schemes in Fig.4. The nonAbelian representation of the fusion rule for the supersymmetric case is $\sigma \times \sigma = 1 + \psi$ where $\sigma$ is a ma-

FIG. 3: Three energy scales in the TSF states: $E_{qp}$, $E_{link}$, $\Delta E$ (The scale is $t$). The left inset is the scheme of the energy spectrum of quasi-particles, of which $E_{qp}$ is the energy gap of Bogoliubov quasi-particle and $\Delta E$ is the energy splitting of the bound state of $Z_2$ link-excitation. The right inset figure shows the bound state on a $Z_2$ link-excitation with $U/t = 0.3, U_0/t = 3.25$. $E_{link}$ is the excited energy of a $Z_2$ link-excitation.

FIG. 4: Left figure: Illustration of the relationship between fermionic $Z_2$ link-excitation (the green link) and $e$-type $Z_2$ vortex, $m$-type $Z_2$ vortex (the shadow plaquettes) of $Z_2$ TSF. There are two sub-plaquettes, one for $e$-type $Z_2$ vortex, the other for $m$-type $Z_2$ vortex; Right figure: Illustration of the relationship between supersymmetric $Z_2$ link-excitation (the green link) and two-component super-$Z_2$ vortex (the shadow plaquettes) of super-$Z_2$ TSF.
trix. This equation is much similar to that of the non-Abelian TSC\(^{20,21}\), where the vortex section \(\sigma\) is not a matrix. Just for this reason, super-Z\(_2\) TSF may be another possible candidate for the fault-tolerant topological quantum computation.

**Physics realization:** In general, to realize such exotic \(\mathbb{Z}_2\) TSF, we need to design a spinless p-orbital fermionic model with significant nearest neighbor attractive interaction. Unfortunately, for p-band fermions the direct nearest neighbor attracting interaction is too small to be considered. A candidate is an interacting mixture of ultracold bosons on s-band and spinless fermions on p-band in 2D square optical lattice\(^7\), of which the effective Hamiltonian is \(H = \sum_{\delta=s,p}(H_\delta - \mu_\delta \hat{N}_\delta) - U_{sp} \sum_i \hat{n}_{s,i} \hat{n}_{p,i}\) where \(H_s\) and \(H_p\) describe the hopping terms of s-orbital bosons and p-orbital fermions: \(H_s = -t_s \sum_{\langle i,j \rangle} \hat{b}_{s,i}^\dagger \hat{b}_{s,j} + U/2 \sum_i \hat{n}_{s,i} (\hat{n}_{s,i} - 1)\) and \(H_p = t_p \sum_i \hat{f}_{i+e_x}^\dagger \hat{f}_i^\dagger + t_p \sum_i \hat{f}_{i+e_y}^\dagger \hat{f}_i^\dagger - t_L \sum_i \hat{f}_{i+e_x}^\dagger \hat{f}_{i+e_y}^\dagger - t_L \sum_i \hat{f}_{i+e_y}^\dagger \hat{f}_{i+e_x}^\dagger\), respectively. \(t_s\) is the hopping parameter for s-band bosons and \(t_p/t_L\) is hopping parameter for p-band fermions on neighbor sites parallel/perpendicular to the bond direction. Pseudo-spin \(\{\uparrow, \downarrow\}\) denotes orbital index \((p_x, p_y)\). Due to \(t_L \ll t_p\), we set \(t_L\) to be zero. \(U_{sp} > 0\) denotes the attractive interaction strength between s-band bosons and p-band fermions and \(U_s > 0\) denotes the repulsive interaction strength between s-band bosons. \(\mu_\delta\) is the chemical potential and \(\hat{N}_\delta = \sum_i \hat{n}_{\delta,i}\) is the particle number operator where \(\hat{n}_{s,i} = \hat{b}_{s,i}^\dagger \hat{b}_{s,i}\) and \(\hat{n}_{p,i} = \hat{f}_{i+e_x}^\dagger \hat{f}_i^\dagger + \hat{f}_{i+e_y}^\dagger \hat{f}_i^\dagger\). Here, we consider the Mott insulator phase of s-band bosons with unit filling and a half filling case of p-band fermions. By tuning the interaction between s-orbital bosons and p-orbital fermions for the case of \(U_s > U_{sp} > 0\), an s-orbital boson and a p-orbital fermion may bind into a composite p-orbital fermion with both effective on-site and effective nearest neighbor attractive interaction (See detailed calculations in Ref\(^{22}\)). In this case, the corresponding parameters in Eq.[1] are \(t \to 2t_s t_p/U_{sp}\), \(U \to U_s - 4U_{sp}\), \(U_0 \to -2t_s^2/U_s - t_p^2/U_{sp}\), respectively. Here \(\hat{c}_i\) is the operator of the composite fermion as \(\hat{b}_i + \hat{f}_i\). From the fact \(U_0 < 0\), there always exists effective nearest neighbor attractive interaction that will lead to \(p_x \times p_y\)-wave pairing between composite fermion of the same p-orbital. If \(4U_{sp} > U_s > U_{sp}\), we have an additional on-site attractive interaction that will leads to s-wave pairing between composite fermion of the opposite p-orbital. When the effective nearest neighbor attracting interaction dominates, the ground state is just the \(\mathbb{Z}_2\) TSF.

**Conclusion :** In this paper we studied the interacting p-band fermions on square lattice and found the ground state may be translation symmetry protected \(\mathbb{Z}_2\) topological superfluid beyond the characterization of “ten-fold way”. Such \(\mathbb{Z}_2\) TSF has even fermion parity at \(k = (0, 0)\) and \(k = (\pi, \pi)\), odd fermion parity at \(k = (0, \pi)\) and \(k = (\pi, 0)\). This \(\mathbb{Z}_2\) TSF supports the robust Majorana edge modes and there exists a new type of low energy excitation - (supersymmetric) \(\mathbb{Z}_2\) link-excitation. In the end we address a possible realization of such interacting p-band fermions - a mixture of ultracold bosons on s-band and spinless fermions on p-band in 2D square optical lattice.

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The effective model of this Bose-Fermion mixture for the case of $U_s > U_{sp} > 0$ is lightly different from that in Eq.[1] due to the band-depedent anisotropic nearest neighbor attracting interactions. However, in mean field level, the two models have the similar global phase diagram.