COMMENT

Comment on ‘Exact results for survival probability in the multistate Landau–Zener model’

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Abstract

We correct the proof of the Brundobler–Elser formula (BEF) provided in Volkov and Ostrovsky (2004 \textit{J. Phys. B: At. Mol. Opt. Phys.} \textbf{37} 4069) and continued in the appendix of Volkov and Ostrovsky (2005 \textit{J. Phys. B: At. Mol. Opt. Phys.} \textbf{38} 907). After showing that some changes of variables employed in these articles are used erroneously, we propose an alternative change of variables which solves the problem. In our proof, we reveal the connection between the BEF for a general \(N\)-level Landau–Zener system and the exactly solvable bow-tie model. The special importance of the diabatic levels with maximum/minimum slope is emphasized throughout.

1. Introduction

In [1], Volkov and Ostrovsky (referred to below as V-O) proposed a rigorous mathematical proof of the Brundobler–Elser formula (BEF) \cite{3} for a general \(N\)-level Landau–Zener (LZ) system. The approach of [1] was to consider a formal solution in terms of an infinite series in powers of the coupling constants, with further calculation of every term and summation of the resulting algebraic series. Following this strategy, these authors have demonstrated the exact cancellations of many possible quantum paths and have reduced the problem to the calculation of a sub-series (5.1) \cite{1}.

In the calculation of this remaining sub-series (5.1) in [1] V-O made the error of assuming a wrong domain of integration for their variables \(x_j\) \cite{1}.

In [2], whose main text is dedicated to a different subject, V-O included an appendix in which a new attempt is made at calculating the sub-series (5.1). After acknowledging the error made in [1], V-O proceeded to employ a different change of variables, previously proposed by Kayanuma and Fukuchi in [5].

We would like to point out that this derivation contained in the appendix of [2] is still erroneous. After equation (A.4) \cite{2} the authors claim that the integrand is symmetrical under the permutation of variables \(x_0, \ldots, x_m\). It is, however, straightforward to check, by direct examination of the terms containing the fourth power in the coupling
constants $V_i$, that this is not true; nor is it symmetrical under the exchange of pairs of variables $(x_0, y_0), \ldots, (x_m, y_m)$.

The following example will support these considerations: the term corresponding to $l = m = 2$ in equation (5.1) of [1] reads

$$
\sum_{k_1 \neq 1}^N |V_{k_1}|^2 \sum_{k_2 \neq 1}^N |V_{k_2}|^2 \int_{-\infty}^\infty dt_1 \int_{-\infty}^\infty dt_2 \int_{-\infty}^\infty dt_3 \int_{-\infty}^\infty dt_4 \times \exp \left[ i (\varepsilon_1 - \varepsilon_{k_1}) (t_4 - t_3) + i (\varepsilon_1 - \varepsilon_{k_2}) (t_2 - t_1) \right] \times \frac{1}{2} (\beta_1 - \beta_{k_1}) (t_2^2 - t_1^2) + \frac{1}{2} (\beta_1 - \beta_{k_2}) (t_4^2 - t_3^2) \right].
$$

(1)

Let us consider the simplest model beyond the two-level LZ-system, namely the one with $N = 3$. Then, the sum above has four terms

$$I_{22} = |V_{12}|^4 \int_{-\infty}^\infty dt_1 \int_{-\infty}^\infty dt_2 \int_{-\infty}^\infty dt_3 \int_{-\infty}^\infty dt_4 \times \exp \left[ i (\varepsilon_1 - \varepsilon_2) (t_4 - t_3 + t_2 - t_1) + \frac{1}{2} (\beta_1 - \beta_2) (t_2^2 + t_4^2 - t_1^2 - t_3^2) \right],
$$

(2)

$$I_{33} = |V_{13}|^4 \int_{-\infty}^\infty dt_1 \int_{-\infty}^\infty dt_2 \int_{-\infty}^\infty dt_3 \int_{-\infty}^\infty dt_4 \times \exp \left[ i (\varepsilon_1 - \varepsilon_3) (t_4 - t_1 + t_2 - t_3) + \frac{1}{2} (\beta_1 - \beta_3) (t_2^2 + t_4^2 - t_1^2 - t_3^2) \right],
$$

(3)

$$I_{23} = |V_{12}|^2 |V_{13}|^2 \int_{-\infty}^\infty dt_1 \int_{-\infty}^\infty dt_2 \int_{-\infty}^\infty dt_3 \int_{-\infty}^\infty dt_4 \times \exp [i (\varepsilon_1 - \varepsilon_2)(t_4 - t_3) + i (\varepsilon_1 - \varepsilon_3)(t_2 - t_1)] \times \frac{1}{2} (\beta_1 - \beta_2)(t_2^2 - t_3^2) + \frac{1}{2} (\beta_1 - \beta_3)(t_2^2 - t_3^2) \right],
$$

(4)

$$I_{32} = |V_{13}|^2 |V_{12}|^2 \int_{-\infty}^\infty dt_1 \int_{-\infty}^\infty dt_2 \int_{-\infty}^\infty dt_3 \int_{-\infty}^\infty dt_4 \times \exp [i (\varepsilon_1 - \varepsilon_3)(t_4 - t_3) + i (\varepsilon_1 - \varepsilon_2)(t_2 - t_1)] \times \frac{1}{2} (\beta_1 - \beta_3)(t_2^2 - t_3^2) + \frac{1}{2} (\beta_1 - \beta_2)(t_2^2 - t_3^2) \right].
$$

(5)

The change of variables [5] used by V-O in the appendix of [2] amounts to

- $t_1 = x_1, \quad x_1 \in (-\infty, \infty)$
- $t_2 = x_1 + y_1, \quad y_1 \in [0, \infty)$
- $t_3 = x_2 + y_1, \quad x_2 \in [x_1, \infty)$
- $t_4 = x_2 + y_1 + y_2, \quad y_2 \in [0, \infty)$.

The integrals $I_{22}$ and $I_{33}$ can be easily evaluated (see [5]), e.g.

$$I_{22} = |V_{12}|^4 \int_{-\infty}^\infty dx_2 \int_{-\infty}^\infty dx_1 \int_0^\infty dy_2 \int_0^\infty dy_1 \times \exp \left[ i (\varepsilon_1 - \varepsilon_2)(y_1 + y_2) + \frac{1}{2} (\beta_1 - \beta_2)(y_1^2 + y_2^2) + i (\beta_1 - \beta_2)(y_1x_1 + y_2x_2) \right]
$$

$$= |V_{12}|^4 \frac{1}{2} \int_{-\infty}^\infty dx_2 \int_{-\infty}^\infty dx_1 \int_0^\infty dy_2 \int_0^\infty dy_1 \times \exp \left[ i (\varepsilon_1 - \varepsilon_2)(y_1 + y_2) + \frac{1}{2} (\beta_1 - \beta_2)(y_1^2 + y_2^2) + i (\beta_1 - \beta_2)(y_1x_1 + y_2x_2) \right]
$$
\[
I_{23} + I_{32} = |V_{12}|^4 |V_{13}|^2 \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \exp \left\{ \frac{i}{2} (\beta_1 - \beta_2)(2x_2 y_2 + y_1^2) + \frac{i}{2} (\beta_1 - \beta_3)(2x_1 y_1 + y_1^2) + \frac{i}{2} (\beta_1 - \beta_3)(2x_2 y_2 + y_2^2) \right. \\
+ \left. \frac{i}{2} (\beta_1 - \beta_2)(2x_1 y_1 + y_1^2) + i(\beta_1 - \beta_3) y_1 y_2 \right\}.
\]

(7)

and, similarly, \( I_{13} = \frac{|V_{13}|^4}{2} \left( \frac{\pi}{|\beta_1 - \beta_3|} \right)^2 \).

The sum of the remaining two integrals reads

\[
= |V_{12}|^4 \frac{1}{2} \int_{0}^{\infty} dy_1 \int_{0}^{\infty} dy_2 \frac{2\pi}{|\beta_1 - \beta_2|} \delta(y_1) \frac{2\pi}{|\beta_1 - \beta_2|} \delta(y_2) \\
\times \exp \left\{ \frac{i}{2} (\epsilon_1 - \epsilon_2)(y_1 + y_2) + \frac{1}{2} (\beta_1 - \beta_2)(y_1 + y_2)^2 \right\} = \frac{|V_{12}|^4}{2} \left( \frac{\pi}{|\beta_1 - \beta_2|} \right)^2
\]

(6)

 Contrary to the claims made by V-O after equation (A.4) of [2], the integrand of (7) is clearly \textit{not symmetric} with respect to interchanging the pairs \((x_1, y_1)\) and \((x_2, y_2)\). Indeed, the terms \( (\beta_1 - \beta_2) y_2 \neq (\beta_1 - \beta_3) y_1 \) belong to different terms of the sum in the integrand and are left unchanged by the permutation. Therefore, one cannot extend the domain of integration of the variable \( x_2 \) to \((-\infty, +\infty)\) as proposed by V-O in equation (A.5) of [2].

While the mathematical errors made by V-O are presented above in detail only for completeness, there is also a more general argument as to why the calculation of the series (5.1) in both [1] and [2] is erroneous. One can note that this series represents a formal solution for a special case of the multi-state LZ-model, in which all states interact with only one special level with slope \( \beta_1 \). Series (5.1) is the probability amplitude for the system to remain on this special level after all crossings if only this level has been initially populated. This observation does \textit{not} assume at all that the slope \( \beta_1 \) is an extremum. Therefore, if the proof provided by V-O in [1] and [2] were correct, then their summation of the series (5.1) that \textit{does not} use the assumption of a maximal/minimal slope for level 1 could be applied to such a LZ-model with an \textit{arbitrary} slope \( \beta_1 \).

However, it is well known from numerical simulations [3] of this model that if \( \beta_1 \) is \textit{not} an extremum, then the BEF does \textit{not} hold, and the transition probability for \( 1 \rightarrow 1 \) depends essentially on the parameters \( \epsilon_i \). Also, the exactly solvable case of the above example in which all \( \epsilon_i = 0 \), the bow-tie model [4], proves that the diagonal elements of the \( S \) matrix are \textit{different} from the BEF \textit{except} for the levels with \textit{maximal/minimal} slope.

The total ignorance of this very important property of the slope \( \beta_1 \) being an \textit{extremum} in the calculation of series (5.1) by V-O has actually been the trigger that prompted our search for their mathematical mistakes.

2. Proof of the BEF

We will split the proof of the BEF into three steps. The first step has been correctly performed in [1] and we only briefly mention here the main results in order to introduce the notation and as a starting point for the next steps.
It has been shown in [1] that if the slope $\beta_1$ is maximal/minimal, then the transition amplitude for the transition $1 \rightarrow 1$ has the form

$$S_{11} \equiv \langle 1|\hat{U}(\infty, -\infty)|1 \rangle = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=2}^{N} \sum_{k_2=2}^{N} \cdots \sum_{k_m=2}^{N} \left| V_{1k_1} \right|^2 \left| V_{1k_2} \right|^2 \cdots \left| V_{1k_m} \right|^2$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\tau_1} \int_{-\infty}^{\tau_2} \cdots \int_{-\infty}^{\tau_{2m-1}} \int_{-\infty}^{\tau_{2m}} \exp \left[ \frac{1}{2} \left( \frac{B(k_1)}{2} (\tau_1^2 - \tau_2^2) + \frac{B(k_2)}{2} (\tau_3^2 - \tau_4^2) + \cdots + \frac{B(k_m)}{2} (\tau_{2m-1}^2 - \tau_{2m}^2) \right) \right] \exp[iE(k_1)(\tau_1 - \tau_2) + iE(k_2)(\tau_3 - \tau_4) + \cdots + iE(k_m)(\tau_{2m-1} - \tau_{2m})], \quad (8)$$

where

$$B(k_j) \equiv \beta_1 - \beta_{k_j} \quad \text{and} \quad E(k_j) \equiv \varepsilon_{k_j}, \quad j = 1, 2, \ldots, m. \quad (9)$$

$\hat{U}$ is the time-evolution operator and $N$ is the number of levels in the system.

Series (8) is equivalent to the series (5.1) in [1]. From this point on, Volkov and Ostrovsky no longer use the property of $\beta_1$ being maximal/minimal in [1] and in the appendix of [2], and the changes of variables they employ are used erroneously, as shown in the introduction.

In the second step we prove that if $\beta_1$ is maximal/minimal, then $S_{11} \equiv \langle 1|\hat{U}(\infty, -\infty)|1 \rangle$ does not depend on $E(l_j)$, for any $l_j = 2, 3, \ldots, N$ and any $j = 1, 2, \ldots, m$.

From equation (8) it follows that

$$\frac{\partial S_{11}}{\partial E(l_j)} = \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=2}^{N} \sum_{k_2=2}^{N} \cdots \sum_{k_m=2}^{N} \left| V_{1k_1} \right|^2 \left| V_{1k_2} \right|^2 \cdots \left| V_{1k_m} \right|^2$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\tau_1} \int_{-\infty}^{\tau_2} \cdots \int_{-\infty}^{\tau_{2m-1}} \int_{-\infty}^{\tau_{2m}} \exp \left[ \frac{1}{2} \left( \frac{B(k_1)}{2} (\tau_1^2 - \tau_2^2) + \frac{B(k_2)}{2} (\tau_3^2 - \tau_4^2) + \cdots + \frac{B(k_m)}{2} (\tau_{2m-1}^2 - \tau_{2m}^2) \right) \right] \exp[iE(k_1)(\tau_1 - \tau_2) + iE(k_2)(\tau_3 - \tau_4) + \cdots + iE(k_m)(\tau_{2m-1} - \tau_{2m})], \quad (10)$$

where $\delta(l_j, k_{\rho})$ is the Kronecker delta.

Next, we introduce the well-known change of variables (see, e.g., [7], p 114)

$$\tau_1 = x_1 \in (-\infty, \infty)$$

$$\tau_{j+1} = x_j - x_{j+1}, \quad \text{with} \quad x_{j+1} \in [0, \infty), \quad j = 1, 2, \ldots, 2m - 1. \quad (11)$$

From equations (10) and (11) it follows that

$$\frac{\partial S_{11}}{\partial E(l_j)} = \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=2}^{N} \sum_{k_2=2}^{N} \cdots \sum_{k_m=2}^{N} \left| V_{1k_1} \right|^2 \left| V_{1k_2} \right|^2 \cdots \left| V_{1k_m} \right|^2$$

$$\times \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\tau_1} dx_2 \int_{-\infty}^{\tau_2} dx_3 \cdots \int_{-\infty}^{\tau_{2m-1}} \int_{-\infty}^{\tau_{2m}} \exp \left[ \frac{1}{2} \left( \frac{B(k_1)}{2} (x_1^2 - x_2^2) + \frac{B(k_2)}{2} (x_3^2 - x_4^2) + \cdots + \frac{B(k_m)}{2} (x_{2m-1}^2 - x_{2m}^2) \right) \right] \exp[iE(k_1)(x_1 - x_2) + iE(k_2)(x_3 - x_4) + \cdots + iE(k_m)(x_{2m-1} - x_{2m})], \quad (12)$$
where
\[
F(x_1, x_2, x_3, \ldots, x_{2m}) = \exp \left[ -\frac{i}{2} G(x_2, x_4, \ldots, x_{2m-2}, x_{2m}) \right] \\
\times \exp \left[ i \left( B(k_1) x_2 + B(k_2) x_4 + \cdots + B(k_m) x_{2m} \right) \right] \\
\times \exp \left[ -i \left( B(k_2) x_4 + B(k_3) x_6 + \cdots + B(k_m) x_{2m} \right) \right] \\
\times \exp \left[ -i \left( B(k_3) x_6 + B(k_4) x_8 + \cdots + B(k_m) x_{2m} \right) \right] \\
\vdots \\
\times \exp \left[ -i \left( B(k_m) x_{2m} \right) \right].
\] (13)

and
\[
G(x_2, x_4, \ldots, x_{2m-2}, x_{2m}) = B(k_1)(x_2^2) + B(k_2)(2x_4x_2 + x_4^2) \\
+ B(k_3)(2x_6x_2 + 2x_6x_4 + x_6^2) \\
\vdots \\
+ B(k_m)(2x_{2m}x_2 + 2x_{2m}x_4 + \cdots + 2x_{2m}x_{2m-2} + x_{2m}^2) \\
- 2\left[ E(k_1)x_2 + E(k_2)x_4 + \cdots + E(k_m)x_{2m} \right].
\] (14)

Upon integrating over \( x_1 \) in equation (12) one obtains
\[
\frac{\partial S_{11}}{\partial E(l_j)} = \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=2}^{N} \sum_{k_2=2}^{N} \cdots \sum_{k_m=2}^{N} |V_{1k_1}|^2 |V_{1k_2}|^2 \cdots |V_{1k_m}|^2 \\
\times \int_0^\infty dx_2 \int_0^\infty dx_3 \cdots \int_0^\infty dx_{2m} \\
\times i\left[ \delta(l_j, k_1)x_2 + \delta(l_j, k_2)x_4 + \cdots + \delta(l_j, k_m)x_{2m} \right] \\
\times \exp \left[ -\frac{i}{2} G(x_2, x_4, \ldots, x_{2m-2}, x_{2m}) \right] \\
\times 2\pi \delta(B(k_1)x_2 + B(k_2)x_4 + \cdots + B(k_m)x_{2m}) \\
\times \exp \left[ -i \left( B(k_2)x_4 + B(k_3)x_6 \right) + \cdots + B(k_m)x_{2m} \right] \\
\times \exp \left[ -i \left( B(k_3)x_6 + B(k_4)x_8 \right) + \cdots + B(k_m)x_{2m} \right] \\
\vdots \\
\times \exp \left[ -i \left( B(k_m)x_{2m} \right) \right].
\] (15)

From equation (9) it follows that, for any \( k_j, B(k_j) > 0 \) if \( \beta_1 \) is maximal, and \( B(k_j) < 0 \) if \( \beta_1 \) is minimal. Therefore, \( \delta(B(k_1)x_2 + B(k_2)x_4 + \cdots + B(k_m)x_{2m}) = 0 \) unless \( x_2 = x_4 = \cdots = x_{2m} = 0 \). The presence of the term
\[
[\delta(l_j, k_1)x_2 + \delta(l_j, k_2)x_4 + \cdots + \delta(l_j, k_m)x_{2m}]
\] (16)
in the integrands of equation (15) ensures that each of the integrals is zero.

This argument can be made rigorous by regularizing the behaviour of integrals at \( \pm \infty \), as follows:
\[
I_q \equiv \int_{-\infty}^{\infty} dx_1 \int_0^{\infty} dx_2 \int_0^{\infty} dx_3 \cdots \int_0^{\infty} dx_{2m} F(x_1, \ldots, x_{2m}) x_{2q} \\
= \lim_{n \to 0^+} \left\{ \int_0^{\infty} dx_2 \int_0^{\infty} dx_4 \cdots \int_0^{\infty} dx_{2m} \int_0^{\infty} dx_1 \int_0^{\infty} dx_3 \cdots \int_0^{\infty} dx_{2m-1} \\
\times x_{2q} F(x_1, \ldots, x_{2m}) \exp(-\eta(x_1 + x_3 + \cdots + x_{2m-1})) \right\}
\]
that

\[ \delta(B(k_1), B(k_2), \ldots, B(k_m)) \]

for any \( q = 1, 2, \ldots, m \). Upon integrating over \( x_j \) with odd \( j \) in equation (17), one obtains

\[ I_q = \lim_{\eta \to 0} \int_0^\infty dx_2 \int_0^\infty dx_4 \cdots \int_0^\infty dx_{2m} W(x_2, x_4, \ldots, x_{2m}; \eta), \]

where

\[ W(x_2, x_4, \ldots, x_{2m}; \eta) = \exp \left[ -\frac{i}{2} G(x_2, x_4, \ldots, x_{2m}) \right] \]

\[ \times \left[ \frac{B(k_1)x_2 + B(k_2)x_4 + \cdots + B(k_m)x_{2m}}{\eta^2} + \eta^2 \right] \]

\[ \times \frac{1}{i[B(k_2)x_4 + B(k_3)x_6 + \cdots + B(k_m)x_{2m} + \eta]} \]

\[ \vdots \]

\[ \times \frac{1}{i[B(k_m)x_{2m} + \eta]} \]

(19)

Since either \( B(k_j) > 0 \) or \( B(k_j) < 0 \) for any \( k_j \), from equations (18) and (19) it follows that

\[ |I_q| \leq \lim_{\eta \to 0} \int_0^\infty dx_2 \int_0^\infty dx_4 \cdots \int_0^\infty dx_{2m} |W(x_2, x_4, \ldots, x_{2m}; \eta)| \]

\[ = \lim_{\eta \to 0} \int_0^\infty dx_2 \int_0^\infty dx_4 \cdots \int_0^\infty dx_{2m} \frac{2\eta x_{2q}}{[B(k_1)x_2 + B(k_2)x_4 + \cdots + B(k_m)x_{2m}]^2 + \eta^2} \]

\[ \times \frac{1}{\sqrt{[B(k_2)x_4 + B(k_3)x_6 + \cdots + B(k_m)x_{2m}]^2 + \eta^2}} \cdots \frac{1}{\sqrt{[B(k_m)x_{2m}]^2 + \eta^2}}. \]

(20)

For any continuous function \( f \) defined on a compact interval \([a, b]\) the relation \( \int_a^b f(x) dx = f(\xi)(b - a) \) holds, for some \( \xi \in [a, b] \). Hence, equation (20) reduces to

\[ |I_q| \leq \frac{1}{|B(k_1)B(k_2)\cdots B(k_m)|} \lim_{\eta \to 0} \frac{2\eta}{|B(k_q)|} \frac{1}{(\xi_1 + \xi_2 + \cdots + \xi_m)^2 + 1} \]

\[ \times \frac{1}{\sqrt{(\xi_2 + \xi_3 + \cdots + \xi_m)^2 + 1}} \cdots \frac{1}{\sqrt{\xi_m^2 + 1}} = 0, \]

(21)

where \( \xi_j \in [0, 1] \), for any \( j = 1, 2, \ldots, m \). Equation (21) shows that \( I_q = 0 \) for any \( q = 1, 2, \ldots, m \), and consequently \( \delta_B(x_1, x_3, \ldots, x_{2m}) = 0 \) for any \( E(l_j) \).

The fact that \( \beta_1 \) is maximal/minimal has played a key role in the arguments above. These arguments fail to hold if \( \beta_1 \) is not an extremum, since then the argument of \( \delta(B(k_1)x_1 + B(k_2)x_4 + \cdots + B(k_m)x_{2m}) \) would have other zeros, besides the obvious \( x_2 = x_4 = \cdots = x_{2m} = 0 \).
In the last step, we reveal the connection between the BEF for a general \( N \)-level LZ-system and the exactly solvable bow-tie model \[4\], in which all levels interact with only one special level (SL). Indeed, since \( S_{11} \) does not depend on \( E(l_j) \) if \( \beta_1 \) is maximal/minimal, we can safely set \( E(l_j) = 0 \) for any \( l_j = 2, 3, \ldots, N \). The form for \( S_{11} \) (equation (8)) with all \( E(l_j) = 0 \) is exactly what one obtains for a bow-tie model if the SL has slope \( \beta_1 \), and a perturbation expansion is being developed for it.

Since the Brundobler–Elser conjecture for the bow-tie model was proven analytically by Ostrovsky and Nakamura in \[4\], we only cite their statement from page 6947 of \[4\]: ‘This hypothesis is confirmed within the present model’, and refer to their work for further details.

This ends the proof of the Brundobler–Elser formula for a general \( N \)-level LZ-system. Before drawing the conclusions, it is worthwhile to mention that, preceding \[1\], an alternative proof of the BEF, based on analytic continuation in imaginary time, was proposed by Shytov in \[6\].

3. Conclusions

In summary, we revealed the deficiencies of the BEF proof proposed by Volkov and Ostrovsky in \[1\] and the appendix of \[2\]. We emphasized that using the important property of the slope \( \beta_1 \) being an extremum only to arrive at the sub-series (5.1) in their proof is not enough, regardless of the changes of variables they attempted, and this property still has to play a crucial role in evaluating (5.1).

We have constructed a proof of the BEF that corrects for this shortcoming. Indeed, in our proof, the fact that the slope \( \beta_1 \) is maximal/minimal had to be used for three different purposes: (1) to arrive at equation (5.1) as was shown in \[1\]; (2) to prove that \( S_{11} \) does not depend on \( E(l_j) \); (3) to employ the fact that the Brundobler–Elser conjecture for maximum/minimum slope was proved for the particular case of the bow-tie model \[4\].

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Note added. After this work was submitted to the publication we received a reply by Volkov and Ostrovsky \[8\] in which these authors now verify, by direct calculation, the important point which they missed before in \[1\] and the appendix of \[2\], and which we have advocated throughout our comment, namely that in order for the BEF to hold the important property of \( \beta_1 \) being an extreme slope (ES) must be used manifestly in the evaluation of series (5.1). In their reply V-O abandon the old changes of variables and use this time the same well-known change of variables (11) (see, e.g., \[7\], p 114) as we have employed. We note that this change of variables, while previously used by V-O in a different context (e.g., in the proof of the no-go theorem \[2\]), was never employed in their attempts of calculating series (5.1) in \[1\] and the appendix of \[2\]. The point at issue in our comment is, however, not about what changes of variables should be or not be used in the proof of the BEF. Mathematical inconsistencies aside, it was definitely not the V-O’s choice of integration variables (in \[1\] and the appendix of \[2\]) that prompted our comment, but their total neglect of the very important ES-property of \( \beta_1 \) in evaluating series (5.1).

Though only the evaluation of fourth-order terms is presented in \[8\], our claim that one can set \( \epsilon_j = 0 \) only if \( \beta_1 \) is an ES is reconfirmed, together with the fact that this ES-property must still be used even after setting \( \epsilon_j = 0 \) in order to prove the Brundobler–Elser conjecture.

References

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