N. ALAA
I. MOUNIR

Weak solutions for some reaction-diffusion systems with balance law and critical growth with respect to the gradient

Annales mathématiques Blaise Pascal, tome 8, n° 2 (2001), p. 1-19

<http://www.numdam.org/item?id=AMBP_2001__8_2_1_0>
Weak solutions for some Reaction-Diffusion Systems with Balance law and Critical growth with respect to the Gradient.*

N. Alaa† & I. Mounir‡

Abstract

This paper is concerned with the existence of weak solutions for $2 \times 2$ reaction-diffusion systems for which two main properties hold: the positivity of the solutions and the triangular structure. Moreover, the nonlinear terms have critical growth with respect to the gradient.

AMS subject classification :35J20, 35J25, 35J65, 45H15.

1 Introduction

This paper deals with existence results for the following Reaction-Diffusion system:

$$\begin{align*}
-\Delta u &= f(x, u, v, \nabla u, \nabla v) + F(x) \quad \text{on } \Omega \\
-\Delta v &= g(x, u, v, \nabla u, \nabla v) + G(x) \quad \text{on } \Omega \\
u = v &= 0 \quad \text{on } \partial \Omega
\end{align*}$$

(1)

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $-\Delta$ denotes the Laplacian operator on $\Omega$ with Dirichlet boundary conditions. Since we are essentially concerned with systems frequently encountered in applications, we restrict ourself to the case of positive solutions satisfying the triangular structure. These two main properties are ensured (respectively) by the following hypotheses

$$\begin{align*}
f(x, 0, v, p, q), g(x, u, 0, p, q) &\geq 0, \\
F(x), G(x) &\geq 0,
\end{align*}$$

(2)

for all $(u, v, p, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $x \in \Omega$.

*This work was partially supported by the "Action Concertée SPM02/01" program, between the University Cadi Ayyad FST Gueliz of Marrakech-Morocco and the University of Picardie Amiens-France.

†Faculty des Sciences et Techniques Gueliz, Département de Mathématiques et Informatique. B.P.618 Marrakech-Maroc. E-mail: alaa@fstg-marrakech.ac.ma

‡Faculté des Sciences Semlalia, Département de Mathématiques. B.P. 2390 Marrakech-Maroc. E-mail: ilhamounir@ucam.ac.ma
\[
\begin{align*}
\begin{cases}
  f(x, u, v, p, q) + g(x, u, v, p, q) & \leq 0 \\
  f(x, u, v, p, q) & \leq 0 \\
  \text{for all } (u, v, p, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \text{ and a.e. } x \in \Omega.
\end{cases}
\end{align*}
\]

When \( f \) and \( g \) does not depend on the gradient, an extensive literature has dealt with this kind of problems (especially for the parabolic version), existence results have been given in [9], [10], [13], [14]. Excellent surveys treating reaction-diffusion systems include Rothe [16] and Smoller [17]. If \( f \) and \( g \) does depend on the gradient, an existence theorem has been proved in [12], by means of \( L^1 \) method introduced in [13], when the growth of the nonlinearities with respect to the gradient is subquadratic, namely

\[
|f| + |g| \leq C(|u| + |v|)^\alpha (|\nabla u\alpha + |\nabla v\alpha + K)
\]

\( C \geq 0, C \) is nondecreasing; \( K \in L^1(\Omega) \) and 1 \( \leq \alpha < 2 \).

Our objective is to investigate the case \( \alpha = 2 \). This critical growth with respect to the gradient creates some difficulties in the passage to the limit for the approximating problem and the \( L^1 \) method can not be applied in this case. We adopt a different approach based on techniques introduced in [7] for the case of elliptic equations to deal with exponential test function of the truncations. We refer the reader to [4], [6], [7], [15] for a general survey of this method. Let us point out here that the parabolic version of such systems with \( L^2 \)-data has been recently treated by the same authors, see [2]. Typical model where the results of this paper can be applied is the following

\[
\begin{align*}
\begin{cases}
  -\Delta u & = -\rho_2(u, v) |\nabla u|^2 + F & \text{on } \Omega \\
  -\Delta v & = \rho_2(u, v) |\nabla u|^2 + G & \text{on } \Omega \\
  u = v = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

where the functions \( \rho_i, i = 1, 2 \), are nonnegative continuous bounded with respect to \( v \) such that \( \rho_2 \leq \rho_1 \) and where \( |.| \) denotes the euclidian norm in \( \mathbb{R}^N \).

We have organised this paper in the following manner. In section 2, we give the precise setting of the problem and state our main result. In section 3, we truncate the system and establish suitable a priori estimates. Finally, we prove the convergence of the truncated problem to some solution of our system. The difficulties in this section are similar to those in [7], [15] and the techniques are of the same spirit. But specific new difficulties owed to the nature of the system must be handled.

\section{Statement of the result}

\subsection{Assumptions}

We first introduce the notion of solution to the problem (1) used here.
Definition 1 We say that \((u, v)\) is a weak solution of (1) if

\[
\begin{align*}
    u, v &\in W^{1,1}_0(\Omega), \\
    f(., u, v, \nabla u, \nabla v), g(., u, v, \nabla u, \nabla v) &\in L^1(\Omega), \\
    -\Delta u &= f(., u, v, \nabla u, \nabla v) + F \quad \text{in } \mathcal{D}'(\Omega) \\
    -\Delta v &= g(., u, v, \nabla u, \nabla v) + G \quad \text{in } \mathcal{D}'(\Omega).
\end{align*}
\]

Throughout this note we will assume that:

- \(H1/ f, g : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R} \) are measurable
- \(H2/ f, g : \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R} \) are continuous for almost every \(x \in \Omega\).
- \(H3/ |f(x, u, v, p, q)| \leq C_1(|u|)(L(x) + ||p||^2 + ||q||^\alpha), \) where \(C_1 : [0, \infty) \to [0, \infty)\) is non-decreasing, \(L \in L^1(\Omega),\) \(1 \leq \alpha < 2.\)
- \(H4/ |g(x, u, v, p, q)| \leq C_2(|u|, |v|)(K(x) + ||p||^2 + ||q||^2), \) where \(C_2 : [0, \infty)^2 \to [0, \infty)\) is non-decreasing, and \(K \in L^1(\Omega).\)
- \(H5/ F, G \in L^1(\Omega).\)

Remark 1 One can deduce from assumptions (2) and (3) that

\[ f(x, 0, 0, p, q) = g(x, 0, 0, p, q) \text{ for all } p, q \in \mathbb{R}^N. \]

2.2 The main result

Theorem 2 Assume that (2), (3) and \(H1/ - H5/\) hold. Then there exists a positive weak solution of (1).

Before giving the proof of this theorem, let us denote by \(T_k\) the truncation function

\[ T_k(s) = \max(-k, \min(s, k)) \quad k \in \mathbb{R}^+. \]

3 Proof of Theorem 2

3.1 Approximating scheme

Let us define \(\psi_n\) a truncation function by \(\psi_n \in C_0^\infty(\mathbb{R}),\) \(0 \leq \psi_n \leq 1,\) and

\[
\psi_n(r) = \begin{cases} 
1 & \text{si } |r| \leq n \\
0 & \text{si } |r| \geq n + 1.
\end{cases}
\]

Consider the following two functions

\[ f_n(x, u, v, p, q) = \psi_n(|u| + |v| + ||p|| + ||q||)f(x, u, v, p, q), \]
\[ g_n(x, u, v, p, q) = \psi_n(|u| + |v| + |p| + |q|)g(x, u, v, p, q). \]

It is easily seen that \( f_n, g_n \) satisfy the same properties as \( f \) and \( g \). Moreover \(|f_n| + |g_n| \leq \eta_n(x) \in L^1(\Omega)\). By a direct application of the Leray-Schauder fixed point theorem one can prove that the system

\[
\begin{aligned}
-\Delta u_n &= f_n(x, u_n, v_n, \nabla u_n, \nabla v_n) + F(x) \quad \text{in } D'(\Omega) \\
-\Delta v_n &= g_n(x, u_n, v_n, \nabla u_n, \nabla v_n) + G(x) \quad \text{in } D'(\Omega)
\end{aligned}
\]

(4) has a weak positive solution \((u_n, v_n)\) in \( W^{1,q}_0(\Omega) \) with \( 1 \leq q < \frac{N}{N-1} \) (see [12] for more details).

### 3.2 A priori estimates

Lemma 3 let \( u_n, v_n \in W^{1,q}_0(\Omega) \) be nonnegative sequences such that

\[
\begin{aligned}
-\Delta u_n &= f_n + F \quad \text{in } D'(\Omega) \\
-\Delta v_n &= g_n + G \quad \text{in } D'(\Omega)
\end{aligned}
\]

(5)

\[ f_n \leq 0, \quad f_n + g_n \leq 0; \quad F, G \in L^1(\Omega); \quad F, G \geq 0. \]

Then

i/ There exists a constant \( R_1 \) depending on \( \|F\|_{L^1(\Omega)} \), \( \|G\|_{L^1(\Omega)} \) such that:

\[ \int_{\Omega} |f_n(x, u_n, v_n, \nabla u_n, \nabla v_n)| + |g_n(x, u_n, v_n, \nabla u_n, \nabla v_n)| \leq R_1. \]

ii/ The sequence \((u_n, v_n)\) is relatively compact in \( W^{1,q}_0(\Omega) \times W^{1,q}_0(\Omega) \) for all \( 1 \leq q < \frac{N}{N-1} \).

Proof. i/ Consider the equations satisfied by \( u_n \) and \( v_n \), we can write

\[
\begin{aligned}
u_n, v_n \in W^{1,q}_0(\Omega) \\
-f_n = \Delta u_n + F \quad \text{in } D'(\Omega) \\
-g_n = \Delta v_n + G \quad \text{in } D'(\Omega).
\end{aligned}
\]

(6)

Since \( u_n, v_n \geq 0 \) and \(-\Delta \) is dissipative in \( L^1(\Omega) \) (see [8]), then

\[ \int_{\Omega} \Delta u_n \leq 0, \quad \int_{\Omega} \Delta v_n \leq 0. \]

Integrating (6) on \( \Omega \) yields

\[ -\int_{\Omega} f_n \leq \int_{\Omega} F. \]
The fact that \( f_n \leq 0 \) and \( F \in L^1(\Omega) \) allows us to conclude
\[
\int_{\Omega} |f_n| = - \int_{\Omega} f_n \leq \|F\|_{L^1(\Omega)}.
\]

Similarly, we get
\[
\int_{\Omega} |f_n + g_n| = - \int_{\Omega} f_n + g_n \leq \int_{\Omega} F + G \leq \|F\|_{L^1(\Omega)} + \|G\|_{L^1(\Omega)}.
\]

Therefore
\[
\int_{\Omega} |g_n| \leq \int_{\Omega} |f_n + g_n| + \int_{\Omega} |f_n| \leq 2 \|F\|_{L^1(\Omega)} + \|G\|_{L^1(\Omega)}.
\]

**ii/** This assertion follows by a direct application of a result in [8]. Indeed the applications

\[ f_n \rightarrow u_n, \quad g_n \rightarrow v_n \]

are compact from \( L^1(\Omega) \) into \( W^{1,q}_0(\Omega) \), with \( 1 \leq q < \frac{N}{N-1} \). Therefore, since by **i/** the nonlinearities \( f_n \) and \( g_n \) are uniformly bounded in \( L^1(\Omega) \), we obtain the required result.

**Lemma 4** Let \((u_n, v_n)\) be as in the previous lemma. Then

**i/ a/** \( \lim_{h \to \infty} \int_{[u_n \geq h]} |f_n| = 0 \) uniformly on \( n \).

**b/** \( \lim_{h \to \infty} \int_{[u_n + v_n \geq h]} |f_n| = 0 \) uniformly on \( n \).

**ii/** There exists a constant \( R_2 \) depending on \( k \), \( \|F\|_{L^1(\Omega)} \) and \( \|G\|_{L^1(\Omega)} \) such that
\[
\int_{\Omega} |\nabla T_k(u_n)|^2 + \int_{\Omega} |\nabla T_k(v_n)|^2 \leq R_2.
\]

**iii/** There exists a constant \( R_3 \) depending on \( k \), \( \|F\|_{L^1(\Omega)} \), \( \|G\|_{L^1(\Omega)} \) such that
\[
\int_{\Omega} |\nabla T_k(u_n + v_n)|^2 \leq R_3.
\]

**iv/** Moreover
\[
\lim_{h \to \infty} \frac{1}{h} \int_{\Omega} |\nabla T_h(u_n)|^2 = 0 \quad \text{uniformly on } n.
\]

**Remark 2** The same results can be found in [6] and [15].
Proof. \( i/a \) We first define the following test-function for every \( t, h > 0 \)

\[
P_{t,h}(s) = \begin{cases} 
0 & \text{if } 0 \leq s < h \\
\frac{s - h}{t} & \text{if } h \leq s \leq t + h \\
1 & \text{if } s > t + h.
\end{cases}
\]

Writing

\[
\int_{\Omega} \nabla u_n \nabla P_{t,h}(u_n) - \int_{\Omega} f_n P_{t,h}(u_n) = \int_{\Omega} F P_{t,h}(u_n),
\]

and using the fact that \( f_n \leq 0 \) and \( P_{t,h}(u_n) \geq 0 \) yield

\[
\frac{1}{t} \int_{[h \leq u_n \leq t+h]} |\nabla u_n|^2 - \int_{[u_n \geq t+h]} f_n \leq \int_{[u_n \geq t+h]} F P_{t,h}(u_n).
\]

Since \( \frac{1}{t} \int_{[h \leq u_n \leq t+h]} |\nabla u_n|^2 \geq 0 \), we get

\[
\int_{[u_n \geq t+h]} |f_n| \leq \int_{[u_n \geq t+h]} F P_{t,h}(u_n) \leq \int_{[u_n \geq t+h]} |F|.
\]

Thanks to Lebesgue's theorem, we have by passing to the limit as \( t \) tends to 0

\[
\int_{[u_n \geq t+h]} |f_n| \leq \int_{[u_n \geq t+h]} |F|.
\]

But

\[
||u_n \geq h|| = \int_{[u_n \geq h]} dx \leq h^{-1} ||u_n||_{L^1} \leq C h^{-1},
\]

since \( u_n \) is bounded in \( L^1(\Omega) \) by lemma 3.

On the other hand, since \( F \in L^1(\Omega) \), we have

\[
\int_A |F| \to 0 \text{ as } |A| \to 0.
\]

Therefore

\[
\sup_n \int_{[u_n \geq h]} |f_n| \leq \sup_A \int_{[u_n \geq h]} |F| \leq C h^{-1} \to 0
\]

We conclude that

\[
\lim_{h \to \infty} \int_{[u_n \geq h]} |f_n| = 0 \quad \text{uniformly on } n.
\]
b/ The main idea is to consider the equation satisfied by $2u_n + v_n$, and to take $P_{t,h}(2u_n + v_n)$ as a test function. We obtain

$$\frac{1}{t} \int_{[h \leq 2u_n + v_n \leq t+h]} |\nabla (2u_n + v_n)|^2 - \int_{[2u_n + v_n \geq t+h]} (2f_n + g_n)P_{t,h}(2u_n + v_n)$$

$$\leq \int_{[2u_n + v_n \geq t+h]} (2F + G)P_{t,h}(2u_n + v_n)$$

Since $f_n \leq 0$, $f_n + g_n \leq 0$ and $P_{t,h}(2u_n + v_n) \geq 0$, we obtain

$$\int_{[2u_n + v_n \geq t+h]} |f_n|P_{t,h}(2u_n + v_n) \leq \int_{[2u_n + v_n \geq t+h]} (2F + G)P_{t,h}(2u_n + v_n)$$

and

$$\int_{[2u_n + v_n \geq t+h]} |g_n|P_{t,h}(2u_n + v_n) \leq \int_{[2u_n + v_n \geq t+h]} (2F + G)P_{t,h}(2u_n + v_n)$$

The rest of the proof runs as in the previous step.

ii/ We multiply the first equation in (5) by $T_k(u_n)$ and we integrate on $\Omega$, we obtain

$$\int_{\Omega} |\nabla T_k(u_n)|^2 = \int_{\Omega} f_n T_k(u_n) + \int_{\Omega} FT_k(u_n) \leq \int_{\Omega} FT_k(u_n),$$

since $f_n T_k(u_n) \leq 0$. We then have

$$\int_{\Omega} |\nabla T_k(u_n)|^2 \leq k \|F\|_{L^1(\Omega)}.$$

In the same way, we multiply the second equation in (5) by $T_k(v_n)$ and we integrate on $\Omega$, we obtain

$$\int_{\Omega} |\nabla T_k(v_n)|^2 = \int_{\Omega} g_n T_k(v_n) + \int_{\Omega} GT_k(v_n) \leq \int_{\Omega} (G + |g_n|)T_k(v_n).$$

We then have

$$\int_{\Omega} |\nabla T_k(v_n)|^2 \leq k(R_1 + \|G\|_{L^1}).$$

iii/ follows trivially from ii/.

iv/ We first remark that $u_n$ satisfies

$$-\Delta u_n \leq F, \text{ in } D'(\Omega)$$
if we multiply this inequality by $T_h(u_n)$ and integrate on $\Omega$, we obtain for every $M > 0$

$$\int_{\Omega} |\nabla T_h(u_n)|^2 \leq \int_{\Omega \cap [u_n \leq M]} T_h(u_n)F + \int_{\Omega \cap [u_n > M]} T_h(u_n)F \leq M \int_{\Omega} F + h \int_{\Omega} F I_{[u_n > M]}.$$ 

Hence

$$\frac{1}{h} \int_{\Omega} |\nabla T_h(u_n)|^2 \leq \frac{M}{h} \|F\|_{L^1} + \int_{\Omega} F I_{[u_n > M]}.$$ 

Fix $\varepsilon > 0$. Since $u_n$ is bounded in $L^1(\Omega)$, we have $\|[u_n > k]\| \leq C k^{-1}$. Therefore, there exists $k_\varepsilon$ independent of $n$ such that

$$\int_{\Omega} F I_{[u_n > k_\varepsilon]} \leq \frac{\varepsilon}{2}.$$ 

Taking $M = k_\varepsilon$ an letting $h$ tend to infinity, we obtain the desired conclusion.

The last assertion in lemma 3 allows us to ensure the existence of a subsequence still denoted by $(u_n, v_n)$ such that

\[
\begin{align*}
&\quad u_n \rightharpoonup u & \text{in } W^{1,q}_0(\Omega) \text{ strongly.} \\
& u_n \rightarrow u & \text{a.e in } \Omega. \\
& \nabla u_n \rightarrow \nabla u & \text{a.e in } \Omega. \\
& v_n \rightarrow v & \text{in } W^{1,q}_0(\Omega) \text{ strongly.} \\
& v_n \rightarrow v & \text{a.e in } \Omega. \\
& \nabla v_n \rightarrow \nabla v & \text{a.e in } \Omega.
\end{align*}
\]

In the next step, we will show that this subsequence $(u_n, v_n)$ satisfies some useful properties.

**Lemma 5** Suppose that $u_n$, $v_n$, $u$ and $v$ are as above.

i/ If

$$|f_n| \leq C_1 (|u_n|)(|\nabla u_n|^2 + |\nabla v_n|^\alpha + L)$$

$C_1 \geq 0$, $C_1$ is nondecreasing; $L \in L^1(\Omega)$; $1 \leq \alpha < 2$.

Then for each fixed $k$

$$\lim_{n \to \infty} \int_{\Omega} |\nabla T_k(u_n) - \nabla T_k(u)|^2 I_{[u_n + v_n \leq k]} = 0.$$ 

ii/ If

$$|g_n| \leq C_2 (|u_n|, |v_n|)(|\nabla u_n|^2 + |\nabla v_n|^2 + K)$$

$C_2 \geq 0$, $C_2$ is nondecreasing; $K \in L^1(\Omega)$.
Then for each fixed $k$

$$\lim_{n \to \infty} \int_{\Omega} |\nabla T_k(u_n + v_n) - \nabla T_k(u + v)|^2 \chi_{(u_n + v_n \leq s)} = 0.$$ 

Proof. i) Let

$$\varphi(s) = s \exp(\mu s^2),$$

where: $\mu \geq \max \left( \frac{C_1^2(k)}{4}, C_2^2(k, k) \right)$. An easy calculation allows us to write

$$\varphi'(s) - C_1(k) |\varphi(s)| > \frac{1}{2},$$

$$\varphi'(s) - 2C_2(k, k) |\varphi(s)| > \frac{1}{2}.$$

Let us also define the following function

$$H \in C^1(\mathbb{R}), 0 \leq H(s) \leq 1 \forall s \in \mathbb{R},$$

$$H(s) = \begin{cases} 0 & \text{if } |s| \geq 1, \\ 1 & \text{if } |s| \leq \frac{1}{2}. \end{cases}$$

Now let $h$ and $k$ be positive real numbers such that $k < h$ and take $\varphi(T_k(u_n) - T_k(u))H \left( \frac{u_n + v_n}{h} \right)$ as a test function in the first equation of (5). We have

$$- \int_{\Omega} \Delta u_n \varphi(T_k(u_n) - T_k(u))H \left( \frac{u_n + v_n}{h} \right) = J_1 + J_2$$

where

$$J_1 = \int_{\Omega} f_n \varphi(T_k(u_n) - T_k(u))H \left( \frac{u_n + v_n}{h} \right)$$

$$J_2 = \int_{\Omega} F \varphi(T_k(u_n) - T_k(u))H \left( \frac{u_n + v_n}{h} \right).$$

For sake of brevity, we will denote by

$$\xi_{k,n} = (T_k(u_n) - T_k(u))$$

$$I = - \int_{\Omega} \Delta u_n \varphi(\xi_{k,n})H \left( \frac{u_n + v_n}{h} \right).$$

Integration by part yields

$$I = \int_{\Omega} \nabla u_n (\nabla T_k(u_n) - \nabla T_k(u))\varphi'(\xi_{k,n})H \left( \frac{u_n + v_n}{h} \right) + \frac{1}{h} \int_{\Omega} \nabla u_n \nabla (u_n + v_n) \varphi(\xi_{k,n})H \left( \frac{u_n + v_n}{h} \right)$$

$$= I_1 + I_2.$$
For $I_1$, we have
\[
I_1 = - \int_{[u_n > k]} \nabla u_n \nabla T_k(u) \varphi'(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right) + \\
+ \int_{[u_n \leq k]} |\nabla (T_k(u_n) - T_k(u))|^2 \varphi'(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right) + \\
+ \int_{[u_n \leq k]} \nabla T_k(u) \nabla (T_k(u_n) - T_k(u)) \varphi'(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right)
\]

\[
= I_{1.1} + I_{1.2} + I_{1.3}.
\]

For the term $J_1$, we have
\[
J_1 = \int_{[u_n \leq k]} f_n \varphi(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right) + \int_{[u_n \geq k]} f_n \varphi(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right)
\]

\[
\leq \int_{[u_n \leq k]} f_n \varphi(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right),
\]

since $\varphi(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right) \geq 0$ on $[u_n > k]$ ($H \geq 0, \xi_{k,n} \geq 0$ on $[u_n > k]$) and $f_n \leq 0$ by hypotheses. Therefore
\[
J_1 \leq C_1(k) \int_{[u_n \leq k]} L(x) |\varphi(\xi_{k,n})| H\left(\frac{u_n + v_n}{h}\right)
\]

\[
+ C_1(k) \int_{[u_n \leq k]} |
abla u_n|^2 |\varphi(\xi_{k,n})| H\left(\frac{u_n + v_n}{h}\right)
\]

\[
+ C_1(k) \int_{[u_n \leq k]} |
abla T_k(u_n)|^2 |\varphi(\xi_{k,n})| H\left(\frac{u_n + v_n}{h}\right)
\]

\[
= J_{1.1} + J_{1.2} + J_{1.3}.
\]

Consequently by equality (7), we have
\[
I_{1.2} + I_2 - J_{1.2} \leq J_{1.1} + J_{1.3} + J_2 - I_{1.1} - I_{1.3}.
\]  

(8)

One can see that $I_{1.1}$ can be written as
\[
I_{1.1} = - \int_{\Omega} \nabla T_k(u_n) \nabla T_k(u) \varphi'(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right) \chi_{[u_n \geq k]} \chi_{[u \geq k]}
\]

\[
- \int_{\Omega} \nabla T_k(u_n) \nabla T_k(u) \varphi'(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right) \chi_{[u_n \geq k]} \chi_{[u < k]}.
\]
Since $\nabla T_k(u)\chi_{[u \geq k]} = 0$ a.e in $\Omega$, and $\chi_{[u \geq k]}\chi_{[u < k]} \to 0$ a.e in $\Omega$ as $n \to \infty$,

$$\lim_{n \to \infty} I_{1.1} = 0.$$ 

On the other hand, $\nabla T_k(u_n)$ is bounded in $L^2(\Omega)$ and converges to $\nabla T_k(u)$ a.e., and $\left| \varphi'(\xi_{k,n})H\left(\frac{u_n + v_n}{h}\right) \right| \leq C |\varphi(2k)|$. Therefore, it follows from ([11] Lemme 1.3 p12) that $\nabla(T_k(u_n) - T_k(u))\varphi'(\xi_{k,n})H\left(\frac{u_n + v_n}{h}\right)$ converges weakly to 0 in $L^2(\Omega)$.

Then $\lim_{n \to \infty} I_{1.3} = 0$.

Now we investigate $I_2$. Since $u_n$ and $u_n + v_n$ satisfy the hypotheses of the previous lemma, we get

$$|I_2| \leq \frac{1}{2h} \int_{\Omega} |\nabla T_k(u_n)|^2 \left| \varphi'(\xi_{k,n})H\left(\frac{u_n + v_n}{h}\right) \right|$$

$$+ \frac{1}{2h} \int_{\Omega} |\nabla T_k(u_n + v_n)|^2 \left| \varphi'(\xi_{k,n})H\left(\frac{u_n + v_n}{h}\right) \right|.$$ 

Then $\lim_{n \to \infty} |I_2| = 0$ uniformly on $n$.

For the term $J_{1.1}$, we use Lebesgue’s theorem to conclude that $\lim_{n \to \infty} J_{1.1} = 0$.

For $J_{1.2}$, we write

$$J_{1.2} = C_1(k) \int_{[u_n \leq k]} |\nabla(T_k(u_n) - T_k(u))|^2 \left| \varphi'(\xi_{k,n}) \right| H\left(\frac{u_n + v_n}{h}\right)$$

$$+ 2C_1(k) \int_{[u_n \leq k]} \nabla T_k(u_n) \nabla T_k(u) \left| \varphi'(\xi_{k,n}) \right| H\left(\frac{u_n + v_n}{h}\right)$$

$$- C_1(k) \int_{[u_n \leq k]} |\nabla T_k(u)|^2 \left| \varphi'(\xi_{k,n}) \right| H\left(\frac{u_n + v_n}{h}\right).$$

Since $|\varphi'(\xi_{k,n})| H\left(\frac{u_n + v_n}{h}\right) \chi_{[u_n \leq k]}$ converges to 0 a.e., and

$|\varphi'(\xi_{k,n})| H\left(\frac{u_n + v_n}{h}\right) \chi_{[u_n \leq k]} \leq \varphi(2k)$ and $\nabla(T_k(u_n))$ is bounded in $L^2(\Omega)$, it follows from ([11] lemme 1.3 p12) that $\nabla T_k(u_n) |\varphi'(\xi_{k,n})| H\left(\frac{u_n + v_n}{h}\right) \chi_{[u_n \leq k]}$ converges weakly to 0 in $L^2(\Omega)$. This implies that the second term of this equality goes to zero as $n$ tends to infinity. Concerning the last term, we remark that $|\varphi'(\xi_{k,n})| H\left(\frac{u_n + v_n}{h}\right) \chi_{[u_n \leq k]} \to 0$ a.e. on $\Omega$ and

$|\nabla T_k(u)|^2 |\varphi'(\xi_{k,n})| H\left(\frac{u_n + v_n}{h}\right) \chi_{[u_n \leq k]} \leq |\nabla T_k(u)|^2 \varphi(2k) \in L^1(\Omega)$. Thanks to
Lebesgue's theorem, we then have
\[
\lim_{{n \to \infty}} \int_{{[u_n \leq k]}} \left| \nabla T_h(u) \right|^2 |\varphi(\xi_{k,n})| H\left( \frac{u_n + v_n}{h} \right) = 0.
\]

To investigate the remaining term \( J_{1.3} \), we apply Hölder's inequality as follows: choose \( \beta \) such that \( \alpha - 1 < \beta < 1 \), we have
\[
J_{1.3} \leq C_1(k) \int_{{[u_n \leq k]}} |\nabla T_h(v_n)|^\alpha |\xi_{k,n}| \exp(\mu \xi_{k,n}^2) H\left( \frac{u_n + v_n}{h} \right) \\
\leq C(k) \left( \int_\Omega |\nabla T_h(v_n)|^2 \xi_{k,n}^{\frac{\alpha}{2}} \right)^{\frac{\beta}{2}} \left( \int_\Omega \left| \xi_{k,n} \right|^{\frac{2(1 - \beta)}{2 - \beta}} \right)^{\frac{2 - \beta}{2}},
\]
\[
\leq C(k) \left( \int_\Omega |\nabla T_h(v_n)|^2 \right)^{\frac{\beta}{2}} \left( \int_\Omega \left| \xi_{k,n} \right|^2 \right)^{\frac{1 + \frac{4}{4} - \alpha - \beta}{4}}.
\]

Now we use lemma 4-(ii) to obtain
\[
J_{1.3} \leq C(k) (h^2 \|F\|_{L^1} + h \|G\|_{L^1})^\beta \left( \int_\Omega \left| \xi_{k,n} \right|^2 \right)^{\frac{1 + \frac{4}{4} - \alpha - \beta}{4}}.
\]

Passing to the limit as \( n \) tends to infinity (for fixed \( h, k \)), the strong convergence of \( \xi_{k,n} \) to 0 in \( L^2(\Omega) \) yields: \( \limsup J_{1.3} = 0 \).

For the last term \( J_2 \), we have by a direct application of Lebesgue's theorem
\[
\lim_{{n \to \infty}} J_2 = 0,
\]

since \( F \in L^1(\Omega) \), and \( |\varphi(\xi_{k,n})H\left( \frac{u_n + v_n}{h} \right) | \leq C |\varphi(2k)| \).

In view of inequality (8), we have shown that for \( k, h \) fixed
\[
\limsup \left( I_{1.2} + I_2 - J_{1.3} \right) \leq 0.
\]

Then
\[
\limsup \left( \limsup \left( I_{1.2} + I_2 - J_{1.3} \right) \right) \leq 0.
\]

But \( \lim I_2 = 0 \) uniformly on \( n \), this should be \( \limsup \left( \limsup |I_2| \right) = \lim \left( \limsup |I_2| \right) = 0 \). It follows
\[
\limsup \left( \limsup \left( I_{1.2} - J_{1.3} \right) \right) \leq 0.
\]
Therefore
\[
\limsup_{h \to \infty} \left( \limsup_{n \to \infty} \int_{\Omega} |\nabla (T_k(u_n) - T_k(u))|^2 \left[ \varphi'(\xi_{k,n}) - C_1(k) |\varphi(\xi_{k,n})| \right] H\left(\frac{u_n + v_n}{h}\right) \right) \leq 0.
\]

By the choice of \(\mu\), we deduce that
\[
\limsup_{h \to \infty} \left( \limsup_{n \to \infty} \int_{\Omega} |\nabla T_k(u_n) - \nabla T_k(u)|^2 H\left(\frac{u_n + v_n}{h}\right) \right) = 0.
\]

On the other hand
\[
\int_{[u_n + v_n \leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 H\left(\frac{u_n + v_n}{h}\right) \leq
\int_{\Omega} |\nabla T_k(u_n) - \nabla T_k(u)|^2 H\left(\frac{u_n + v_n}{h}\right).
\]

Then for every fixed \(h, k\) such that \(k < h\)
\[
\limsup_{h \to \infty} \left( \limsup_{n \to \infty} \int_{[u_n + v_n \leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 H\left(\frac{u_n + v_n}{h}\right) \right) = 0.
\]

Choose \(2k < h\). By the definition of \(H\), we get \(H\left(\frac{u_n + v_n}{h}\right) = 1\) on \([u_n + v_n \leq k]\).

Hence, this should be a limsup
\[
\lim_{n \to \infty} \int_{\Omega} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \chi_{[u_n + v_n \leq k]} \leq
\limsup_{n \to \infty} \int_{\Omega} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \chi_{[u_n + v_n \leq k]} = 0.
\]

Therefore
\[
\lim_{n \to \infty} \int_{\Omega} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \chi_{[u_n + v_n \leq k]} = 0.
\]

\(ii/\) Consider the equation satisfied by \(u_n + v_n\)
\[-\Delta(u_n + v_n) = f_n + g_n + F + G,
\]
and use again \(\varphi(T_k(u_n + v_n) - T_k(u + v))H\left(\frac{u_n + v_n}{h}\right)\) as a test function where \(h\) and \(k\) such that \(0 < k < h\). We get
\[
\int_{\Omega} \nabla(u_n + v_n) \nabla(T_k(u_n + v_n) - T_k(u + v)) \varphi'(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right)
+ \frac{1}{h} \int_{\Omega} |\nabla(u_n + v_n)|^2 \varphi(\xi_{k,n}) H\left(\frac{u_n + v_n}{h}\right) = K_2 + K_3,
\]

(9)
where we denote by

\[ \xi_{k,n} = T_k(u_n + v_n) - T_k(u + v) \]

\[ K_1 = \int \nabla(u_n + v_n) \nabla(T_k(u_n + v_n) - T_k(u + v)) \phi'(\xi_{k,n}^*) H(\frac{u_n + v_n}{h}) \]

\[ K_2 = \frac{1}{h} \int \Omega |\nabla(u_n + v_n)|^2 \phi(\xi_{k,n}^*) H(\frac{u_n + v_n}{h}) \]

\[ K_3 = \int \Omega (f_n + g_n) \phi'(\xi_{k,n}^*) H(\frac{u_n + v_n}{h}) \]

\[ K_4 = \int \Omega (F + G) \phi'(\xi_{k,n}^*) H(\frac{u_n + v_n}{h}) \]

The proof of this assertion follows closely the steps used in the proof of the previous one, it suffices to replace \( u_n \) by \( u_n + v_n \) and \( u \) by \( u + v \). Hence we obtain

\[
\lim_{n \to \infty} (\limsup_{n \to \infty} K_2) = 0
\]

\[
\lim_{n \to \infty} K_4 = 0,
\]

for the term \( K_1 \) we have

\[
K_1 = \int_{[u_n + v_n > k]} \nabla(u_n + v_n) \nabla T_k(u + v) \phi'(\xi_{k,n}^*) H(\frac{u_n + v_n}{h})
\]

\[
+ \int_{[u_n + v_n \leq k]} |T_k(u_n + v_n) - T_k(u + v)|^2 \phi'(\xi_{k,n}^*) H(\frac{u_n + v_n}{h})
\]

\[
+ \int_{[u_n + v_n \leq k]} \nabla T_k(u + v)(T_k(u_n + v_n) - T_k(u + v)) \phi'(\xi_{k,n}^*) H(\frac{u_n + v_n}{h})
\]

\[
= K_{1.1} + K_{1.2} + K_{1.3}.
\]

It is easily seen that \( K_1 \) can be treated in the same way as \( I_1 \).

The only difference is the investigation of the term \( K_3 \), indeed

\[
K_3 = \int_{[u_n + v_n \leq k]} (f_n + g_n) \phi(\xi_{k,n}^*) H(\frac{u_n + v_n}{h})
\]

\[
+ \int_{[u_n + v_n > k]} (f_n + g_n) \phi(\xi_{k,n}^*) H(\frac{u_n + v_n}{h})
\]

\[
\leq \int_{[u_n + v_n \leq k]} (f_n + g_n) \phi(\xi_{k,n}^*) H(\frac{u_n + v_n}{h}),
\]
since \( \varphi(\xi_{k,n}) \geq 0 \) on \([u_n + v_n \leq k]\) and \( H \geq 0, f_n + g_n \leq 0 \) by hypotheses. Hence

\[
|K_3| \leq C_1(k) \int_{[u_n + v_n \leq k]} |\nabla u_n|^2 |\varphi(\xi_{k,n}')| H\left(\frac{u_n + v_n}{h}\right) \\
+ C_1(k) \int_{[u_n + v_n \leq k]} (|\nabla u_n|^2 + L) |\varphi(\xi_{k,n}')| H\left(\frac{u_n + v_n}{h}\right) \\
+ C_2(k,k) \int_{[u_n + v_n \leq k]} (|\nabla u_n|^2 + |\nabla v_n|^2 + K) |\varphi(\xi_{k,n}')| H\left(\frac{u_n + v_n}{h}\right) \\
= K_{3.1} + K_{3.2} + K_{3.3}.
\]

Therefore equality (9) implies

\[
K_{1.2} + K_{2} - K_{3.3} \leq K_{3.1} + K_{3.2} + K_{4} - K_{1.1} - K_{1.3}.
\] (10)

We have

\[
K_{3.1} = C_1(k) \int_{[u_n + v_n \leq k]} |\nabla u_n|^2 |\varphi(\xi_{k,n}')| H\left(\frac{u_n + v_n}{h}\right) \\
\leq 2C_1(k) \int_{[u_n + v_n \leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 |\varphi(\xi_{k,n}')| H\left(\frac{u_n + v_n}{h}\right) \\
+ 2C_1(k) \int_{[u_n + v_n \leq k]} |\nabla T_k(u)|^2 |\varphi(\xi_{k,n}')| H\left(\frac{u_n + v_n}{h}\right) \\
\leq 2C_1(k) |\varphi(2k)| \int_{\Omega} |\nabla T_k(u_n) - \nabla T_k(u)|^2 H\left(\frac{u_n + v_n}{h}\right) \\
+ 2C_1(k) \int_{\Omega} |\nabla T_k(u)|^2 |\varphi(\xi_{k,n}')| H\left(\frac{u_n + v_n}{h}\right).
\]

By using the first assertion (i) and Lebesgue’s theorem for the second integral, we obtain

\[
\limsup_{h \to \infty} \left( \lim_{n \to \infty} K_{3.1} \right) = 0.
\]

As for the term \( J_{1.3} \), we write

\[
\lim_{n \to \infty} K_{3.2} = 0.
\]
Let us now investigate $K_{3.3}$. We first remark that this term can be controlled as follows:

$$
K_{3.3} \leq C_2(k, k) \int_{[u_n + v_n \leq k]} (3|\nabla u_n|^2 + 2|\nabla (u_n + v_n)|^2 + K) \left| \varphi'(\xi_{k,n}^n) \right| H\left(\frac{u_n + v_n}{h}\right)
$$

$$
\leq C_2(k, k) \int_{[u_n + v_n \leq k]} (3|\nabla T_k(u_n)|^2 + K) \left| \varphi'(\xi_{k,n}^n) \right| H\left(\frac{u_n + v_n}{h}\right)
$$

$$
+ 2C_2(k, k) \int_{[u_n + v_n \leq k]} |\nabla T_k(u_n + v_n)|^2 \left| \varphi'(\xi_{k,n}^n) \right| H\left(\frac{u_n + v_n}{h}\right).
$$

The first integral can be dropped by using the same arguments as before. In conclusion, we have

$$
\lim \sup \lim \sup_n \int_{[u_n + v_n \leq k]} |\nabla T_k(u_n + v_n) - \nabla T_k(u + v)|^2 \left| \varphi'(\xi_{k,n}^n) - 2C_2(k, k) \left| \varphi'(\xi_{k,n}^n) \right| H\left(\frac{u_n + v_n}{h}\right) \right| = 0.
$$

As in the previous assertion, we obtain

$$
\lim_{n \to \infty} \int_{\Omega} |\nabla T_k(u_n + v_n) - \nabla T_k(u + v)|^2 \chi_{[u_n + v_n \leq k]} = 0.
$$

### 3.3 Convergence

The aim of this paragraph is to show that $(u, v)$ obtained before is in fact solution of the problem (1) in the sense of definition 1.

By the continuity of the functions $f_n$ and $g_n$, we deduce

$$
f_n(x, u_n, v_n, \nabla u_n, \nabla v_n) \to f(x, u, v, \nabla u, \nabla v) \quad \text{a.e in } \Omega.
$$

$$
g_n(x, u_n, v_n, \nabla u_n, \nabla v_n) \to g(x, u, v, \nabla u, \nabla v) \quad \text{a.e in } \Omega.
$$

These almost pointwise convergences are not sufficient to ensure that $(u, v)$ is a solution of (1). In fact, we have to prove that the previous convergences are in $L^1(\Omega)$. In view of Vitali's theorem, we have to show that $f_n$ and $g_n$ are equi-integrable in $L^1(\Omega)$.

**Lemma 6** The sequences $(f_n(x, u_n, v_n, \nabla u_n, \nabla v_n))_n$ and $(g_n(x, u_n, v_n, \nabla u_n, \nabla v_n))_n$ are equi-integrable in $L^1(\Omega)$.

**Proof.** Let $A$ be a measurable subset of $\Omega$, we have

$$
\int_A |f_n(x, u_n, v_n, \nabla u_n, \nabla v_n)| = \int_A |f_n| + \int_{\partial A \cap [u_n + v_n > k]} |f_n| + \int_{\partial A \cap [u_n + v_n \leq k]} |f_n|.
$$

$$
\leq \int_A |f_n| + \int_{\partial A \cap [u_n + v_n > k]} |f_n| + \int_{\partial A \cap [u_n + v_n \leq k]} |f_n|.
$$
Thanks to lemma 4, we obtain $\forall \varepsilon > 0 \exists k_0$ such that if $k \geq k_0$ then

$$\int_{A \cap [|u_n + v_n| \leq k]} |f_n(x, u_n, v_n, \nabla u_n, \nabla v_n)| \leq \frac{\varepsilon}{4}$$

for all $n$.

Hypothesis $(H3)$ implies that for all $k > k_0$

$$\int_{A} |f_n(x, u_n, v_n, \nabla u_n, \nabla v_n)| \leq \frac{\varepsilon}{4} + C_1(k) \left( \int_{A} L(x) + \int_{A \cap [u_n + v_n \leq k]} |\nabla T_k(u_n)|^2 \right) + C_1(k) \int_{A \cap [u_n + v_n \leq k]} |\nabla T_k(v_n)|^2.$$

Using Hölder’s inequality for $\alpha < 2$ and Lemma 4(ii), we obtain for the third integral

$$C_1(k) \int_{A \cap [u_n + v_n \leq k]} |\nabla T_k(v_n)|^\alpha \leq C_1(k) \left[ \int_{A} |\nabla T_k(v_n)|^2 \right]^{\frac{\alpha}{2}} |A|^{\frac{2-\alpha}{2}} \leq C_1(k) R_2^{\frac{\alpha}{2}} |A|^{\frac{2-\alpha}{2}} \leq \frac{\varepsilon}{4},$$

whenever $|A| \leq \delta_1$, with $\delta_1 = (\frac{\varepsilon}{4} C_1(k) R_2^{\frac{\alpha}{2}})^{2-\alpha}$.

To deal with the second integral we write

$$\int_{A \cap [u_n + v_n \leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 + 2 \int_{A \cap [u_n + v_n \leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2.$$

According to lemma 5, $|\nabla T_k(u_n) - \nabla T_k(u)|^2 \chi_{[u_n + v_n \leq k]}$ is equi-integrable in $L^1(\Omega)$ since it converges strongly to 0 in $L^1(\Omega)$. So, there exists $\delta_2$ such that if $|A| \leq \delta_2$, then $2C_1(k) \int_{A \cap [u_n + v_n \leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \leq \frac{\varepsilon}{4}$.

On the other hand $L, |\nabla T_k(u)|^2 \in L^1(\Omega)$, therefore there exists $\delta_3$ such that

$$C_1(k) \left[ 2 \int_{A} |\nabla T_k(u)|^2 + \int_{A} L(x) \right] \leq \frac{\varepsilon}{4},$$

whenever $|A| \leq \delta_3$. Choose $\delta_0 = \inf(\delta_1, \delta_2, \delta_3)$, if $|A| \leq \delta_0$ we obtain

$$\int_{A} |f_n(x, u_n, v_n, \nabla u_n, \nabla v_n)| \leq \varepsilon.$$
Similarly, we get
\[ \int_A |g_n| = \int_{A \cap [u_n + v_n \leq k]} |g_n| + \int_{A \cap [u_n + v_n > k]} |g_n|. \]
\[ \leq \frac{\varepsilon}{4} + \int_{A \cap [u_n + v_n \leq k]} |g_n|. \]

By hypothesis \( H4/ \), we have
\[ \int_A |g_n| \leq \frac{\varepsilon}{4} + C_2(k, k) \int_{A \cap [u_n + v_n \leq k]} \left( K(x) + |\nabla T_k(u_n)|^2 + |\nabla v_n|^2 \right). \]

Therefore
\[ \int_A |g_n| \leq \frac{\varepsilon}{4} + C_2(k, k) \int_{A \cap [u_n + v_n \leq k]} K(x) + (C_2(k, k) + 2) \int_{A \cap [u_n + v_n \leq k]} |\nabla T_k(u_n)|^2 + + 2C_2(k, k) \int_{A \cap [u_n + v_n \leq k]} |\nabla T_k(u_n + v_n)|^2. \]

Arguing in the same way as before, we obtain the required result.

Acknowledgment. The authors would like to take the opportunity here to thank the referee for the interest he gave to this work and for his valuable suggestions and corrections.

References

[1] N. Alaa, solutions faibles d'équations paraboliques quasi-linéaires avec données initiales mesures. Ann. Math. Blaise Pascal, 3 N°2, pp. 1 - 15(1996).

[2] N. Alaa, I. Mounir, Global existence for reaction-diffusion systems with mass control and critical growth with respect to the gradient. Journal of Mathematical Analysis and Applications 253, pp.532 – 557, (2001).

[3] N. Alaa, M. Pierre. Weak solutions for some quasilinear elliptic equations with data measures. SIAM J. Math. Anal, vol 24, n°1, p. 23 – 35, (1993).

[4] A. Bensoussan, L. Boccardo and F.Murat, On a non linear partial differential equation having natural growth terms and unbounded solution, Ann. Inst. Henri Poincaré. vol 5, n°4, p. 347 – 364(1988).

[5] L. Boccardo, F. Murat, J. P. Puel, Existence results for some quasilinear parabolic equations, Nonlinear Analysis. Theory. Methods & Applications, 13, N° 4, pp. 373 – 392(1989).
[6] L. Boccardo, T. Gallouet. Strongly nonlinear elliptic equations having natural growth terms in $L^1$ data. Nonlinear Anal. T.M.A., vol 19, pp. 573 - 579, (1992).

[7] L. Boccardo, F. Murat, J. P. Puel. Existence de solutions faibles des équations elliptiques quasi-lineaires à croissance quadratique. Nonlinear P.D.E. and their applications, Collège de France Seminar, Vol IV, ed. by H. Brezis & J. L. Lions. Research Notes in Mathematics 84, Pitman, London, pp. 19 – 73, (1983).

[8] H. Brezis, W. Strauss, Semi-linear elliptic equation in $L^1$, J. Math. Soc. Japan, Vol. 25, pp: 565 – 590(1973).

[9] W. E. Fitzgibbon, J. Morgan, R. Sanders. Global existence and boundedness for class of inhomogeneous semilinear parabolic systems, Nonlinear Analysis, Theory Methods and Applications, Vol. 19,N°9,pp.885 – 899(1992).

[10] S. L. Hollis, R. H. Martin, M.Pierre, Global Existence and Boundeness in Reaction-Diffusion Systems, SIAM. J. Math. Anal. 18, pp. 744 – 761, (1987).

[11] J. L. Lions, quelques méthodes de résolutions des problèmes aux limites non linéaires, Dunod-Gauthier Villars(1969).

[12] F. Maach. Existence pour des systèmes de réaction-diffusion quasi-lineaires avec loi de balance. Thèse de Doctorat de l’Université Henri Poincaré, Nancy I,(1994).

[13] R. H. Martin, M. Pierre, Nonlinear reaction-diffusion systems, in "Nonlinear Equations in the Applied Sciences", ed W. F. Ames, C. Rogers and Kapell, Academic Press (1991), Notes and reports in Mathematics in Science and Engineering.

[14] M. Pierre, An $L^1$ method to prove global existence in some reaction-diffusion systems, in Contributions to nonlinear partial differential equations, J. I et P. L. Lions, ed Pitman Res. Notes in Math. Series, pp. 220 – 231(1987).

[15] A. Porretta. Existence for elliptic equations in $L^1$ having lower order terms with natural growth.Portugaliae Mathematica Vol. 57 Fasc.2 – 2000.

[16] F. Rothe, Global Solutions of Reaction-Diffusion Systems, Lectures Notes in Math. 1072, Springer-Verlag (1984).

[17] J. Smoller. Shock waves and reaction-diffusion equations, in Comprehensive Studies in Mathematics, Vol. 258. Springer, New York (1984).