FURTHER IDENTITIES FOR \( c_{mk} \) AND \( a_{mk} \)-WEIGHTED SUMS
AND A REMARK ON A REPRESENTATION OF
PYTHAGORAS’ EQUATION

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Abstract. We present some properties of the expansion coefficients \( a_{mk} \) and \( c_{mk} \) of a pair of dual bases,

\[ n^m = \sum_{k=2}^{m} c_{mk} \psi_k(n), \]

and

\[ \psi_m(n) = n + (m - 1)(n - 1)B_{n-1,m-1}, \]

we introduced earlier in arXiv:2207.01935v1. Here, \( B_{a,b} = (a + b)!/(a!b!) \) is a binomial coefficient. We extend the knowledge on the \( c_{mk} \) by giving an explicit expression for them in terms of the Stirling numbers of the second kind. From the interchangeability of the indices of the binomial coefficients, we formulate the central identity we use in this work:

\[ \psi_m(n) - n = \psi_n(m) - m. \]

With this equation, we evaluate sums of the form

\[ T_m^a = \sum_{k=2}^{m} c_{mk} k^a. \]

Explicitly, the case \( T_m^1 \) is handled. Furthermore, we indicate connections of \( T_m^2 \) and \( T_m^3 \) to the Mersenne numbers (general integer exponent) and the OEIS entry A024023. We conclude with a small remark on how we can represent Pythagoras’ equation in terms of the \( a_{mk} \) coefficients.

1. Introduction

Prior[3], we have shown that a power sum of \( a \)-th order (or hyper-sum),

\[ S^a_m(n) = \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_2=1}^{n} \sum_{\nu_3=1}^{n} \nu_1^m, \]

can be replaced by a polynomial expansion

\[ S^a_m(n) = \sum_{k=2}^{m} c_{mk} \psi_k^{(a)}(n), \]

where the polynomials \( \psi_k^{(a)}(n) \) are given by

\[ \psi_k^{(a)}(n) = B_{a+1,n-1} + \frac{m(m - 1)}{m + a} (n - 1)B_{m+a-1,n-1}. \]

The expansion coefficients \( c_{mk} \) and polynomials are introduced and discussed in detail in Ref. [3]. \( B_{ab} = (a + b)!/(a!b!) \) denotes a binomial coefficient.

In the following, if not stated otherwise, we require \( n \in \mathbb{N} \) and \( m \geq 2 \) (although we might readily include \( m = 1 \), this case is considered separately as stated in Ref. [3]).

In the following we furthermore consider only the \( \psi \)-polynomials for \( a = 0 \),

\[ \psi_0^m(n) = n + (m - 1)(n - 1)B_{n-1,m-1} =: \psi_m(n). \]
Trivially, we can expand the polynomials $\psi_m(n)$ in powers of $n$. Through this expansion, we obtain the coefficients $a_{mk}$:

$$\psi_m(n) = \sum_{k=2}^{m} a_{mk} n^k,$$

where we have already argued using Vieta’s formulas, that the coefficients $a_{m0}$ and $a_{m1}$ are both equal to zero. As J. L. Cereceda points out in a nice addendum to the author’s work, they have the closed form

$$a_{mk} = \frac{[m-1]_{k-1} - [m-1]_k}{(m-2)!},$$

where $[n]_m$ is an unsigned Stirling number of the first kind. Using the recursion relation for the Stirling numbers of the first kind (cf. Table 250 in Ref. [2]),

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n - 1 \\ k \end{bmatrix}(n-1) + \begin{bmatrix} n - 1 \\ k - 1 \end{bmatrix},$$

we may write down a similar relation for the coefficients $a_{mk}$,

$$a_{mk} = a_{m-1,k} + \frac{a_{m-1,k-1}}{m-2}.$$

The dual expansion, i.e. the expansion of the monomial $n^m$ in terms of the $\psi_m(n)$,

$$n^m = \sum_{k=2}^{m} c_{mk} \psi_k(n),$$

defines the coefficients $c_{mk}$ which can be used in the power sum in Eq. (1). Cereceda gives furthermore several identities and properties which the author had not touched yet in his published work. In particular, here, we will use the following two identities given by Cereceda (Eqs. (4) and (8) ibid.):

$$\sum_{k=2}^{m} c_{mk} = 1$$

and

$$\sum_{k=2}^{m} c_{mk} B_{k-1,n-1} = \frac{n^{m-1} - n}{n-1}.$$

Those are easily obtained by inserting $n = 1$ into Eq. (9) and by inserting the explicit form of $\psi_m(n)$, Eq. (4), respectively.

The equivalent identity to Eq. (10) is given by inserting $n = 1$ into Eq. (5):

$$\sum_{k=2}^{m} a_{mk} = 1.$$

From Eqs. (10) and (12) it is clear, that any constant term $C$ may be pulled inside a sum of sequence elements $A_k$ which is weighted by $c_{mk}$ or $a_{mk}$, that is

$$\sum_{k=2}^{m} c_{mk} A_k - C = \sum_{k=2}^{m} c_{mk}(A_k - C)$$

and

$$\sum_{k=2}^{m} a_{mk} A_k - C = \sum_{k=2}^{m} a_{mk}(A_k - C).$$

We can read the two expansions, Eqs. (5) and (9), as vector–matrix products of a coefficient vector and a vector of polynomial values. The latter vector can
be interpreted as the left-product of what one might call here a “vector of choice” 
\((0, 0, \ldots, 1, 0, \ldots, 0)^T\) and a polynomial matrix. We denote the coefficient vec-
tors by \(c_m = (c_{m\mu})_{2 \leq \mu \leq m}\) and \(a_m = (a_{m\mu})_{2 \leq \mu \leq m}\). For the matrices we write 
\(\Psi_m(n) = (\psi_{m\nu}(\nu))_{2 \leq \mu \leq m, 1 \leq \nu \leq n}\) and \(V_m(n) = (n^m_{\nu})_{2 \leq \mu \leq m, 1 \leq \nu \leq n}\) (as this is a kind of 
Vandermonde matrix). With this, we write for the right-product of the polynomial 
matrix with the coefficient vector
\[(1, 2^m, \ldots, n^m) = c_m^T \Psi_m(n)\]
and
\[(1, \psi_m(2), \ldots, \psi_m(n)) = a_m^T V_m(n).\]
In the following, we consider the triple products or matrix elements \((\alpha, m \geq 2)\)

\[T_\alpha^m := c_m^T \Psi_m(\alpha) c_\alpha = \sum_{l=2}^{\alpha} c_{\alpha l} l^\alpha.\]

The corresponding matrix elements
\[U_\alpha^m := a_m^T V_m(\alpha) a_\alpha = \sum_{l=2}^{\alpha} a_{\alpha l} \psi_m(l), \quad (\alpha, m \geq 2),\]

will not be discussed further in this manuscript. On the other hand, we will explic-
itly handle the special cases \(T_1^m\) and \(U_1^m\).  

2. Explicit form for the \(c_{mk}\) coefficients

We start from Eq. (11), which we want to write in the following way:

\[\sum_{k=2}^{m} c_{mk} \frac{n^{k-1}}{(k-2)!} = \frac{n^m - n}{n - 1},\]

where \(n^{k-1} = \prod_{j=0}^{k-2} (n + j)\) means Knuth’s notation for the rising factorial.

We can expand the right-hand side using polynomial division:

\[\frac{n^m - n}{n - 1} = n^{m-1} + n^{m-2} + \ldots + n = \sum_{l=2}^{m} n^{l-1}.\]

We know that we can express powers in terms of the falling or rising factorials by
means of the Stirling numbers of the second kind. Using the equation\(^2\) p. 250]

\[x^n = \sum_k \binom{n}{k} (-1)^{n-k} x^k,\]

to substitute the powers in Eq. (20) and replacing the right-hand side in Eq. (19),
we have

\[\sum_{k=2}^{m} c_{mk} \frac{n^{k-1}}{(k-2)!} = \sum_{l=2}^{m} \left( \sum_{j=2}^{l-1} \frac{l-1}{j} (-1)^{l-1-j} \right) n^7.\]

We switch the summation order and resolve a factor of 1 into \((j-1)!/(j-1)!:\)

\[\sum_{k=2}^{m} c_{mk} \frac{n^{k-1}}{(k-2)!} = \sum_{j} \left( \sum_{l=2}^{m} \frac{l-1}{j} (-1)^{l-1-j} (j-1)! \right) \frac{n^7}{(j-1)!}.\]

By comparison of coefficients \((k - 1 = j)\), we find

\[c_{mk} = (k - 2)! \sum_{l=2}^{m} \frac{l-1}{k-1} (-1)^{l-k}.\]
3. EXCHANGING \( m \) AND \( n \) IN \( \psi_m(n) \)

Before we present the identities for the sums shown in the introductory section, we want to consider a property of the polynomials \( \psi_m(n) \), which deems important for deriving these identities, first.

**Corollary 1.** The polynomial \( \psi_m(n) - n \) is symmetric under exchange of \( n \) and \( m \), that is

\[
\psi_m(n) - n = \psi_n(m) - m
\]

or

\[
\psi_m(n) = \psi_n(m) + n - m.
\]

\[25\]

**Nota bene:** we can see that through this, we are able to extend the definition of the function \( \psi_m(n) \) to \( m = 0 \).

\[27\]

\[\psi_0(n) := \lim_{m \to 0} (\psi_n(m) - m + n) = n + \lim_{m \to 0} \psi_n(m),\]

where we can find the remaining limit by considering the value of the polynomial \( \psi_m(n) \), in contrast to its binomial coefficient form (which is undefined for \( m = 0 \)). If the limit of the polynomial \((n \in \mathbb{R})\) exists, so does the limit of the sequence. Clearly, the limit is zero. Thus, we find that \( \psi_0(n) = \psi_1(n) \), that is

\[28\]

\[\psi_0(n) = n.\]

4. THE WEIGHTED SUMS

Next we want to present the values of the sums

\[ T^1_m = \sum_{k=2}^m c_{mk} k \quad \text{and} \quad U^1_m = \sum_{l=2}^m a_{ml} l. \]

First, we want to write Eq. (11) in a form which stresses that we are dealing with a polynomial. We begin with Eq. (9) and insert the explicit form for \( \psi_m(n) \):

\[29\]

\[
\sum_{k=2}^m c_{mk} \left[ n + (n - 1) \frac{1}{(k - 2)!} \prod_{l=0}^{k-2} (n + l) \right] = n^m.
\]

After further manipulation, this equation becomes

\[30\]

\[
\sum_{k=2}^m c_{mk} \frac{1}{(k - 2)!} \prod_{l=0}^{k-2} (n + l) = \frac{n^m - n}{n - 1}.
\]

Next, consider the limit for \( n \to 1 \):

\[31\]

\[
\sum_{k=2}^m c_{mk} \frac{1}{(k - 2)!} \prod_{l=0}^{k-2} (l + 1) = \lim_{n \to 1} \frac{n^m - n}{n - 1}.
\]

On the left-hand side, we can further cancel the product term with the factorial and arrive at

\[32\]

\[
\sum_{k=2}^m c_{mk} (k - 1) = \lim_{n \to 1} \frac{n^m - n}{n - 1}.
\]

By means of Eq. (10), this can be further simplified to

\[33\]

\[
\sum_{k=2}^m c_{mk} k = 1 + \lim_{n \to 1} \frac{n^m - n}{n - 1}.
\]
If the limit exists for the rational function \( \frac{x^m - x}{x - 1} \), \( x \in \mathbb{R} \), its value will be the same for the sequence we are dealing with, originally. We have
\[
\lim_{n \to 1} \frac{n^m - n}{n - 1} = \lim_{x \to 1} \frac{x^m - x}{x - 1} = \lim_{x \to 1} (mx^{m-1} - 1),
\]
where in the last step we used l’Hôpital’s rule. Finally, we have the equation
\[
\sum_{k=2}^{m} c_{mk} k = m. \quad (35)
\]
For finding the corresponding expression in which the \( a_{mk} \) coefficients are used, we can exploit the orthogonality of the coefficient vectors,
\[
\sum_{k=2}^{m} a_{ml} c_{lk} = \delta_{mk}. \quad (36)
\]
Multiplying with \( a_{mk} \) from the left on Eq. (35) and summing over \( k \), we have
\[
\sum_{k=2}^{m} a_{mk} k = \sum_{k=2}^{m} a_{mk} \sum_{l=2}^{k} c_{kl}. \quad (37)
\]
We can decouple the summation boundaries by using \( c_{kl} = 0 \) for \( l < 2 \lor l > m \):
\[
\sum_{k=2}^{m} a_{mk} k = \sum_{k=2}^{m} a_{mk} \sum_{l=2}^{m} c_{kl}. \quad (38)
\]
Then, after interchanging the summation order and using Eq. (36), we finally have
\[
\sum_{k=2}^{m} a_{mk} k = m. \quad (39)
\]
5. The matrix elements \( T_\alpha m \) and \( U_\alpha m \) for \( \alpha > 1 \)

At last we consider \( T_\alpha m \), for \( \alpha > 1 \). Again, we use Eq. (9) to expand \( k^\alpha \):
\[
\sum_{k=2}^{m} c_{mk} k^\alpha = \sum_{k=2}^{m} c_{mk} \sum_{l=2}^{\alpha} c_{al} \psi_l (k). \quad (40)
\]
Subtract and add \( k \) within the sum over \( l \). By means of Corrolary 1 we interchange \( k \) and \( l \) for \( \psi_l (k) - k \):
\[
\sum_{k=2}^{m} c_{mk} \sum_{l=2}^{\alpha} c_{al} [\psi_l (k) - k + k] = \sum_{l=2}^{\alpha} \sum_{k=2}^{m} c_{al} c_{mk} [\psi_k (l) - l + k]. \quad (41)
\]
Using again the expansion in terms of the \( c_{mk} \), we can reformulate this as
\[
T_\alpha m = \sum_{l=2}^{\alpha} c_{al} l^m - \sum_{k,l} c_{mk} c_{al} l + \sum_{k,l} c_{mk} c_{al} k. \quad (42)
\]
Using first Eq. (10) and then Eq. (35) for the two last terms and substituting the sum on the right-hand side by \( T_\alpha m \), we have
\[
T_\alpha m - m = T_\alpha m - \alpha. \quad (43)
\]
The nice property of Eq. (43) is that we always can use it to shorten or lengthen \( T_\alpha m \) by interchanging the indices accordingly. In particular, if \( \alpha = 2 \) or \( \alpha = 3 \) the sums can be shortened to have a single term:
\[
T_2^2 - m = \sum_{k=2}^{m} c_{mk} k (k - 1) = 2^m - 2, \quad (44)
\]
or
\begin{equation}
T^3_m - m = \sum_{k=2}^{m} c_{mk}(k-1)k(k+1) = 3^m - 3.
\end{equation}

We want to point out, that these equations relate the coefficients $c_{mk}$ to the Mersenne numbers \cite{4} and to the OEIS entry A024023 \cite{5}. Also, note that $T^3_m - m$ is coincidentally the double of $\sum_{k=2}^{m} c_{mk} \binom{k}{3}$ as used by Cereceda \cite{1}. A similar relation can be found for the case $m = 3$ (cf. the identities following Eq. (8) in Ref. \cite{1}).

6. Another Identity using the closed form of the $a_{mk}$ Coefficients

We can start from Eq. (9) and use Corollary \cite{1} to find
\begin{equation}
n^m = \sum_{k=2}^{m} c_{mk} \psi_n(k) - m + n.
\end{equation}

On the right-hand side, we expand $\psi_n(k)$ using Eq. (5). Shuffling all other terms to the left-hand side, we have
\begin{equation}
n^m - n + m = \sum_{k=2}^{m} c_{mk} \sum_{l=2}^{n} a_{nl}k^l.
\end{equation}

We switch the summation order and substitute with $T^j_m$:
\begin{equation}
n^m - n + m = \sum_{l=2}^{n} a_{nl}T^j_m.
\end{equation}

In particular, for the simple cases with only one non-zero $c_{mk}$, $m = 2$ and $m = 3$, we have the two equations
\begin{align*}
n^2 - n + 2 &= \sum_{l=2}^{n} 2^l a_{nl} \quad \text{and} \quad n^3 - n + 3 = \sum_{l=2}^{n} 3^l a_{nl},
\end{align*}
the forms of which are strongly reminiscent of the binary and ternary representations of a number (mind however, that the $a_{nl}$ have rational values in general).

7. Remark on Pythagoras’ equation

Given Eq. (48), we can reformulate equations of the form
\begin{equation}
N - \sum_{i=1}^{N} q_i m_i - q_N = 0,
\end{equation}
insofar we have
\begin{equation}
\sum_{i=1}^{N-1} \left( q_i \sum_{l=2}^{q_i} a_{q_i,l}T^j_m \right) - \sum_{j=2}^{q_N} a_{q_N,j}T^j_m - \sum_{i=1}^{N-1} q_i + q_N - (N-2)m = 0.
\end{equation}

We want to consider the example $N = 3$, with $q_1 = A$, $q_2 = B$, $q_3 = C$, and $m = 2$, i.e. $A^2 + B^2 - C^2 = 0$. Pythagoras’ equation has then the form\footnote{Another representation of this equation in terms of the Mersenne numbers comes to mind immediately: \( \sum_{l=2}^{\max(A,B,C)} (2^l - 1)(a_{Al} + a_{Bl} - a_{Cl}) = \frac{C^2 - A^2 - B^2}{2} \).}
\begin{equation}
\sum_{l=2}^{\max(A,B,C)} 2^l(a_{Al} + a_{Bl} - a_{Cl}) = 2 + C - A - B.
\end{equation}

The solutions $(A, B, C)$ of this equation are---of course---the Pythagorean triples. It is intriguing how the simple addition and subtraction of squares is transformed into a sum over weighted hypercubes of common sidelength and “a bit”, $2 + C - A - B$.\footnote{Another representation of this equation in terms of the Mersenne numbers comes to mind immediately: \( \sum_{l=2}^{\max(A,B,C)} (2^l - 1)(a_{Al} + a_{Bl} - a_{Cl}) = \frac{C^2 - A^2 - B^2}{2} \).}
REFERENCES

[1] José L. Cereceda. A note on another approach on power sums. 2022. DOI: 10.48550/ARXIV.2208.06751 URL: https://arxiv.org/abs/2208.06751.

[2] R.L. Graham, D.E. Knuth, and O. Patashnik. Concrete Mathematics: A Foundation for Computer Science. Addison-Wesley, 1994. ISBN: 9780201558029.

[3] Christoph Muschielok. Another Approach on Power Sums. 2022. DOI: 10.48550/ARXIV.2207.01935 URL: https://arxiv.org/abs/2207.01935.

[4] OEIS Foundation Inc. (2022). Entry A000225 in The On-Line Encyclopedia of Integer Sequences. 2022. URL: https://oeis.org/A000225 (visited on 07/08/2022).

[5] OEIS Foundation Inc. (2022). Entry A024023 in The On-Line Encyclopedia of Integer Sequences. 2022. URL: https://oeis.org/A024023 (visited on 07/08/2022).

[6] OEIS Foundation Inc. (2022). Entry A355570 in The On-Line Encyclopedia of Integer Sequences. 2022. URL: https://oeis.org/A355570 (visited on 07/08/2022).

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