Proof of Brouwer’s Conjecture

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Abstract We prove the Brouwer’s Conjecture

Let $A$ be $n \times n$ incidence matrix of simple undirected graph $G$:

\[ a_{i,j} = \begin{cases} 
1, & \text{iff } (i, j) \in G, \\
0, & \text{otherwise}.
\end{cases} \]

Define Laplacian $L(G)$ of $G$ as follows

\[ L(G) = D - A, \]

were diagonal $n \times n$ matrix $D$ has entries

\[ d_i = |\{j : (i, j) \in G\}|. \]

We have $\sum d_i = 2m$, were $m$ is number of edges in $G$. Considering $G$ as directed graph with some choice of ordering of vertices in $G$ define $m \times n$ matrix $B$:

\[ b_{i,j} = \begin{cases} 
1, & \text{if } j \text{ starting vertex in edge } i, \\
-1, & \text{if } j \text{ pendant vertex in edge } i, \\
0, & \text{otherwise}
\end{cases} \]

Then $G = B^T B$ and hence eigenvalues of matrix $L(G)$ are nonnegative:

\[ 0 = \mu_n(L(G)) \leq \mu_1(L(G)) \leq \ldots \leq \mu_1(L(G)). \]

Brouwer’s Conjecture. [2]. For every graph $G \subset 2^n$ and integer $1 \leq t \leq n - 1$, the following inequality is valid:

\[ S_t(G) = \sum_{i=1}^t \mu_i(L(G)) \leq m + \binom{t+1}{2}, \quad t = 1, \ldots, n. \]

For convenience, we denote

\[ \Delta_t(G) = S_t(G) - m(G) - \binom{t+1}{2}. \]

Whenever $\Delta_t(G) \leq 0$, we say that $G$ satisfy $BC_t$.

It is known to be valid for trees [8], for $k = 1, 2, n - 1, n$, for unicyclic and bicyclic graphs [12], for regular graphs [4], for $n \leq 10$ it was checked by A. Brouwer using a computer. In [5] was proved that Brouwer’s conjecture holds asymptotically almost surely.

Before the proof of Brouwer’s conjecture (we call it next Conjecture) we introduce some consequences of the its validity.

Define

\[ \bar{d}(G) = \{\bar{d}_1, \ldots, \bar{d}_n\}, \quad \bar{d}_i = |\{j : d_j \geq i\}|. \]
We say that the set $E$ of edges is compressed if from $e = (i < j) \in E$ it follows that $e = (i_1 < j_1) \in E$, were $i_1 \leq i, j_1 \leq j$. Define threshold graph $\hat{G}$ as the graph which set of edges is equivalent (up to permutations on $[n]$) to compressed set.

Grone - Merris conjecture [6], which was proved by Bai [7] says that the following upper bound is valid

$$\sum_{i=1}^{t} \mu_i(L(G)) \leq \sum_{i=1}^{t} \bar{d}_i(G).$$

It is known [9] that for threshold graphs there is equality in the last relation.

Define Laplacian energy of graph as follows

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i(L(G)) - \frac{2m}{n} \right|.$$

Define graph $G$ on $n$ nodes with $m$ edges is spectrally threshold dominated (III) if for each $k \in [n]$ there is a threshold graph $\hat{G}$ having the same number of nodes and edges satisfying

$$\sum_{i=1}^{t} \mu_i(L(G)) \leq \sum_{i=1}^{t} \sum_{i=1}^{t} \mu_i(L(\hat{G})) = \sum_{i=1}^{t} \bar{d}_i(L(\hat{G})).$$

In [11] was stated

**Conjecture 1.** All graphs are spectrally threshold dominated.

The main result of the paper [11] is the following

**Theorem 1.** For each spectrally threshold dominated graph $G$ there exists a threshold graph with the same number of nodes and edges whose Laplacian energy is at least as large as that of $G$.

In [11] was proved that all (finite) graphs are spectrally threshold dominated iff Conjecture is valid. Hence proving Brouwer’s conjecture we prove conjecture III.

In [11] for each $m$ and $n$ were found the extremal threshold graphs $G$ which deliver the maximum to the $LE(G)$ over all extremal threshold graphs and as it follows now over all graphs with given $m, n$.

# 1 Preliminary Remarks

Here we gather preliminary results that will be useful later.

Let $\overline{G} = (\binom{[n]}{2}) - G$ denote the complement of $G$. Then, (9):

$$\mu_i(L(G)) = n - \mu_{n-i}(L(G)), \; i = 1, \ldots, n - 1,$$

The following duality result will be key in our work. It follows directly from the proof of Theorem 6 in [8], by including the proper $\Delta’s$ in the calculation.
Theorem 2 ([10]). For every graph $G$,

$$\Delta_t(G) = \Delta_{n-t-1}(\bar{G})$$  \hspace{1cm} (1)

In particular, $G$ satisfies $BC_t$ if and only if $\bar{G}$ satisfies $BC_{n-t-1}$.

On the other hand, once $G$ satisfy $BC_t$, the graph obtained by adding an empty vertex, $G \cup \{v\}$, trivially satisfy $BC_t$. Then, from Theorem 2 we conclude that the graph obtained by adding a complete vertex, $G'$ satisfies $BC_{t+1}$:

$$\Delta_{t+1}(G') = \Delta_{t+1}(G \cup \{v\}) = \Delta_{n-t-1}(\bar{G} \cup \{v\}) = \Delta_{n-t-1}(\bar{G}) = \Delta_t(G).$$  \hspace{1cm} (2)

Given $G \subset \binom{[n]}{2}$, we define the threshold family of $G$, $\mathcal{T}(G)$, as the family of all graphs obtained by $G$ by adding complete or empty vertices. Note that the family of threshold graphs defined in the Introduction coincides with $\mathcal{T}(\emptyset)$. We also conclude that $G$ satisfy Brouwer’s Conjecture if and only if an element in $\mathcal{T}(G)$ does so. The identities $\Delta_{t+1}(G \cup \{v\}) = \Delta_t(G)$ and (2) gives us:

Lemma 1. Brouwer's Conjecture is valid for every $n$ and $t$ provided that $BC_t'$ holds for every graph $G$ with $n'$ vertices where $t' = \frac{n'-1}{2}$ if $n'$ is odd or $t'$ equal to either $\frac{n'-2}{2}$ or $\frac{n'}{2}$ if $n'$ is even.

We call the explicit $t'$'s in Lemma 1 as the middle $t$'s. In what follows we will consider an inductive approach on $n$ to prove that $BC_t$ holds for the middle $t$'s, whenever it holds for middle $t$'s for graphs with less vertices. To this end, we remove one vertex of $G$ and derive a special basis of $\mathbb{R}^n$ where explicit bounds can be inferred. To this aim, we recall the following formula for $L(G)$:

$$\langle L(G)v, v \rangle = \frac{1}{2} \sum_{p \sim q} (v_p - v_q)^2, \hspace{1cm} (3)$$

and recall that

$$S_t(G) = \max \left\{ \sum_{i=1}^{t} \langle L(G)x_i, x_i \rangle \left| x_1, \ldots, x_t, \langle x_i, x_j \rangle = \delta_{ij} \right. \right\},$$

$$= \max \left\{ \text{tr}(L(G)|_V) \left| V \text{ is a } t \text{ dimensional subspace of } \mathbb{R}^n \right. \right\} \hspace{1cm} (4)$$

for $\{z_1, \ldots, z_n\}$ an orthonormal set of eigenvectors corresponding to non-increasing eigenvalues, and $z_n = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$. From the last equality we conclude that

$$S_t(G) = \sum_{i=1}^{t} \langle L(G)x_i, x_i \rangle$$

for any orthonormal basis $\{x_1, \ldots, x_t\}$ of $\text{span}\{z_1, \ldots, z_t\}$.
Lemma 2. For every $p \in [n]$, there is a basis $\{x_1(p), ..., x_t(p)\}$ of $\text{span}\{z_1, ..., z_t\}$ such that $x_t(p)$ is the only vector with a possibly non-zero $p$th coordinate. Likewise, there is a basis $\{x_{t+1}(p), ..., x_{n-1}(p)\}$ of $\text{span}\{z_{t+1}, ..., z_{n-1}\}$ such that $x_{t+1}(p)$ is the only vector with non-zero $p$th coordinate.

Proof. For simplicity, we prove only the first part for $p = 1$ and omit the notation $(p)$ in $x_i(p)$. Given a collection of orthonormal vectors $\mathcal{W} = \{w_1, ..., w_t\}$, denote

$$l_p(\mathcal{W}) = (w_{1p}, ..., w_{tp})$$

as the vector defined by the $p$th coordinates. For the proof, we take $w_i = z_i$, the $i$th eigenvector and write $|l_i|^2 = \sum_i z_i^2$. Let $A$ be an orthonormal $t \times t$ matrix whose last column, $A_t$, has coordinates

$$\alpha_{t,1} = \frac{z_{1,1}}{|l_1|}, ..., \alpha_{t,t} = \frac{z_{t,1}}{|l_1|}.$$  

Let $Z$ be the matrix whose $i$th column is $z_i$. Then the columns of $X =ZA$ forms an orthonormal basis for $\text{span}\{z_1, ..., z_t\}$. The first coordinate of the $i$th column in $X$ is given by

$$\sum_{j=1}^t \alpha_{i,j} z_{i,j,1} = |l_1| \langle A_i, A_t \rangle = |l_1| \delta_{i,t}. \quad \square$$

From now on we fix a basis as in Lemma 2 with $p = 1$ and denote it by $\{x_1, ..., x_t, x_{t+1}, ..., x_{n-1}\}$. We further assume $0 < x_{t,1}, x_{t+1,1} < \sqrt{\frac{n-1}{n}}$, since the extremal cases are easily dealt with (see the discussion below and Proposition 1).

Note that $x_t$ is given explicitly by:

$$x_t = \frac{1}{|l_1|} \sum_{i=1}^t \frac{z_{i,1}}{z_i}.$$  

This form is used to conclude the main bound in this paper (Proposition 1). The existence of $x_t$ also allows our induction step. Let $x_1, ..., x_t$ be as in Lemma 2. Given $G \subset \binom{[n]}{2}$, consider the subgraph $G'$ obtained by removing the first vertex of $G$, together with its edges. We have that

$$S_t(G) = \sum_{i=1}^t \langle L(G)x_i, x_i \rangle = \sum_{i=1}^{t-1} \langle L(G')x_i, x_i \rangle + \sum_{i=1}^{t-1} \sum_{j=1}^{t-1} x_{j,i}^2 + \langle L(G)x_t, x_t \rangle \leq S_{t-1}(G') + \omega_1 + \langle L(G)x_t, x_t \rangle,$$

where

$$\omega_p = \sum_{(p,q) \in E} \sum_{i=1}^{t-1} (x_{i,p}(p) - x_{i,q}(p))^2,$$

and the last inequality follows from (4). In particular, if $G'$ satisfies $BC_{t-1}$, then $G$ satisfies $BC_t$ if

$$\omega_p + \langle L(G)x_t(p), x_t(p) \rangle \leq t + d_p,$$  

where $d_p$ is the degree of vertex $p$. This completes the proof of Proposition 1.
for some $p$. Equivalently, we can work with the dual graph, $\bar{G}$, and show that $BC_t$ holds if
\[
\bar{\omega}_p + \langle L(G)x_{t+1}(p), x_{t+1}(p) \rangle \leq \bar{t} + \bar{d}_p.
\] (9)

Where here we take $x_{t+1}(p)$ as the only vector with (possibly) non-zero $p$th-coordinate, and
\[
\bar{t} = n - 1 - t, \quad \bar{d}_p = n - 1 - d_p,
\]
\[
\bar{\omega}_p = \sum_{q:(p,q) \in \bar{E}} \sum_{i=t+2}^{n-1} (x_{i,p}(p) - x_{i,q}(p))^2.
\]

The key elements in the paper are the following bounds on $\langle L(G)x_t, x_t \rangle$ and $\omega_1$.

**Proposition 1.** Let $x_t(1) = x_t$ be as in Lemma 2 and $x_{t,1} > 0$. Then,
\[
\langle x_t, L(G)x_t \rangle \leq \left\{ \begin{array}{ll}
d_1 + \sqrt{d_1 \frac{1-x_t^2}{x_t^2}}, & x_{t,1}^2 \geq \frac{d_1}{d_1+1}; \\
\frac{n d_1}{n-1} + \sqrt{\frac{d_1 d_2}{n-1} - \frac{d_1 d_1}{n-1}} \frac{1-x_t^2}{x_t^2}, & x_{t,1}^2 < \frac{d_1}{d_1+1}.
\end{array} \right.
\] (10)

Likewise,
\[
\langle x_{t+1}, L(\bar{G})x_{t+1} \rangle \leq \left\{ \begin{array}{ll}
\bar{d}_1 + \sqrt{\bar{d}_1 \frac{1-x_{t+1}^2}{x_{t+1}^2}}, & x_{t+1,1}^2 \geq \frac{\bar{d}_1}{\bar{d}_1+1}; \\
\frac{n \bar{d}_1}{n-1} + \sqrt{\bar{d}_1 \left(1 - \frac{\bar{d}_1}{n-1}\right) \frac{1-x_{t+1}^2}{x_{t+1}^2}}, & x_{t+1,1}^2 < \frac{\bar{d}_1}{\bar{d}_1+1}.
\end{array} \right.
\]

**Proof.** By eventually replacing $x_t$ by $-x_t$, we assume that $x_{t,1} > 0$. Denote $(L(G)x_t)_1$ as the first coordinate of $L(G)x_t$. Let us first show that
\[
\langle x_t, L(G)x_t \rangle = \frac{(L(G)x_t)_1}{|l_1|},
\] (11)
then the Proposition will follow by Jensen inequality.

Let $\{z_1, ..., z_t\}$ be as in Lemma 2. Using (9), we conclude that
\[
\langle x_t, L(G)x_t \rangle = \sum_{i=1}^{t} \langle z_i, L(G)z_i \rangle \frac{z_i^2}{|l_1|^2} = \frac{1}{|l_1|^2} \sum_{i=1}^{t} \mu_i z_i^2.
\]
On the other hand,
\[
(L(G)x_t)_1 = \left( \sum_{i=1}^{t} \frac{\mu_i z_i |l_1|}{|l_1|} \right)_1 = \frac{1}{|l_1|} \sum_{i=1}^{t} \mu_i z_i^2,
\]
concluding the claim. Nonetheless:

$$\langle x_t, L(G)x_t \rangle x_{t,1} = d_1x_{1,t} - \sum_{p:(1,p)\in E} x_{t,p} \leq d_1x_{t,1} + \sum_{p:(1,p)\in \bar{E}} x_{t,p}$$

and

$$\left| \sum_{p:(1,p)\in E} x_{t,p} \right| = \left| \sum_{p:(1,p)\in \bar{E}} x_{t,p} + x_{t,1} \right| \leq x_{t,1} + \left| \sum_{p:(1,p)\in \bar{E}} x_{t,p} \right|.$$ 

Thus

$$(x_t, L(G)x_t)x_{t,1} \leq d_1x_{1,t} + \min \left\{ \left| \sum_{p:(1,p)\in E} x_{t,p} \right|, x_{t,1} + \left| \sum_{p:(1,p)\in \bar{E}} x_{t,p} \right| \right\}.$$ 

Using Jensen inequality we obtain:

$$\left| \sum_{p:(1,p)\in E} x_{t,p} \right| \leq \sqrt{d_1x}, \quad \left| \sum_{p:(1,p)\in \bar{E}} x_{t,p} \right| \leq \sqrt{d_1(1-x_{t,1}^2)-x},$$

where $x = \sum_{p:(1,p)\in E} x_{t,p}^2$. Therefore,

$$\min \left\{ \left| \sum_{p:(1,p)\in E} x_{t,p} \right|, x_{t,1} + \left| \sum_{p:(1,p)\in \bar{E}} x_{t,p} \right| \right\} \leq \max_{x\in[0,1-x_{t,1}^2]} \min \left\{ \sqrt{d_1x}, x_{t,1} + \sqrt{(n-d_1-1)(1-x_{t,1}^2)-x} \right\}$$

$$= \begin{cases} \frac{x_{t,1}d_1}{n-1} + \sqrt{\frac{d_1d_{i+1}}{n-1}(1-\frac{n}{n-1}x_{t,1}^2)}, & x_{t,1}^2 \geq \frac{d_1}{d_{i+1}}; \\ \sqrt{d_1(1-x_{t,1}^2)}, & \text{otherwise}. \end{cases}$$

To conclude the last equality, first observe that $\sqrt{d_1x}$ is increasing with respect to $x$, thus $\sqrt{d_1x} \leq \sqrt{d_1(1-x_{t,1}^2)}$. On the other hand, the first bound on $x_{t,1}^2$ is equivalent to

$$\sqrt{d_1(1-x_{t,1}^2)} \leq x_{t,1},$$

making $\sqrt{d_1(1-x_{t,1}^2)}$ the solution to the max min problem. Otherwise, since $x \mapsto \sqrt{d_1(1-x_{t,1}^2)-x}$ is decreasing, the max min is achieved when

$$\sqrt{d_1x} = x_{t,1} + \sqrt{(n-d_1-1)(1-x_{t,1}^2)-x}. \quad (12)$$
We manipulate this equation as follows:

\[
\begin{align*}
\sqrt{d_1 x} - x_{t,1} = \\
(n-1)x - 2\sqrt{d_1 x}x + x_{t,1}^2 - d_1(1-x_{t,1}^2) = 0
\end{align*}
\]

\[
\begin{align*}
x = \frac{\sqrt{d_1 x_{t,1}}}{n-1} + \sqrt{\left(\frac{\sqrt{d_1 x_{t,1}}}{n-1}\right)^2 - \frac{x_{t,1}^2 - d_1(1-x_{t,1})}{n-1}}
\end{align*}
\]

\[
\begin{align*}
= \frac{\sqrt{d_1 x_{t,1}}}{n-1} + \sqrt{d_1 x_{t,1}^2 - (n-1)x_{t,1}^2 + (n-1)d_1(1-x_{t,1})}
\end{align*}
\]

\[
\begin{align*}
= \frac{\sqrt{d_1 x_{t,1}}}{n-1} + \sqrt{d_1(1-x_{t,1}^2) + \frac{1}{n-1}x_{t,1}^2}
\end{align*}
\]

\[
\begin{align*}
= \frac{\sqrt{d_1 x_{t,1}}}{n-1} + \sqrt{d_1(1-x_{t,1})}
\end{align*}
\]

The result is concluded by multiplying the last expression by $\sqrt{d_1}$. \hfill \square

Before proceeding, we remark the following inequality that follows from the last proof.

**Lemma 3.** Let $y = |\sum_{p:(1,p)\in E} x_{t,p}|$, then, $y < \sqrt{\frac{d_1 d_2}{n-1}}$.

**Proof.** Indeed

\[
y = \left| \sum_{p:(1,p)\in E} x_{t,p} \right| = x = \left| \sum_{p:(1,p)\in E} x_{t,p} + x_{t,1} \right| \leq \max_{x\in[0,1]} \left\{ \sqrt{d_1 x}, \sqrt{d_1(1-x)} \right\}
\]

Note that max min in the rhs of last inequality achieved at $x$ which satisfied equality

\[
\sqrt{d_1 x} = \sqrt{d_1(1-x)}
\]

or

\[
x = \frac{\bar{d}_1}{n-1}.
\]

From this equality Lemma follows. \hfill \square

An extra inequality is also needed. Recall that $x_t, x_{t+1}$ are the only vectors in $\{x_1, \ldots, x_{n-1}\}$ with non-zero first coordinates. To motivate the next inequality, we also recall that the first vertex is complete if and only if the vector

\[
v_1 = \left( \sqrt{\frac{n-1}{n}}, \frac{1}{\sqrt{n(n-1)}}, \ldots, \frac{1}{\sqrt{n(n-1)}} \right)
\]

is in the span of $\{x_1, \ldots, x_t\}$. Next, we measure how much this vector does not belong to this $t$-subspace.
Since $v_0$ is orthogonal to $\{x_1, \ldots, x_{t-1}, x_{t+2}, \ldots, x_{n-1}, z_n\}$, there exists $0 < \lambda < 1$ and a vector $y = (0, y_2, \ldots, y_n)$, $\sum_{p=2}^{n} y_p = 0$, $\sum_{p=2}^{n} y_p^2 = 1$ such that

$$x_t = v_1 \sqrt{\lambda} + \sqrt{1 - \lambda} y;$$

$$x_{t+1} = v_1 \sqrt{1 - \lambda} - \sqrt{\lambda} y.$$

Further denote:

$$B = \sum_{p < q, (p, q) \in E} (y_p - y_q)^2, \quad \bar{B} = \sum_{p < q, (p, q) \in \bar{E}} (y_p - y_q)^2.$$

Then, Lemma 3 gives:

$$\langle x_t, L(G)x_t \rangle = \lambda d_1 \frac{n}{n-1} + (1 - \lambda) B - 2 \sqrt{\lambda(1 - \lambda)} \sum_{p : (1, p) \in E} x_{t,p} \quad \text{(13)}$$

$$\leq \lambda d_1 \frac{n}{n-1} + (1 - \lambda) B + 2 \sqrt{\lambda(1 - \lambda)} d_1 \left(1 - \frac{d_1}{n-1}\right);$$

$$\langle x_{t+1}, L(\bar{G})x_{t+1} \rangle = (1 - \lambda) \bar{d}_1 \frac{n}{n-1} + \lambda \bar{B} - 2 \sqrt{\lambda(1 - \lambda)} y$$

$$\leq (1 - \lambda) \bar{d}_1 \frac{n}{n-1} + \lambda \bar{B} + 2 \sqrt{\lambda(1 - \lambda)} \bar{d}_1 \left(1 - \frac{\bar{d}_1}{n-1}\right). \quad \text{(14)}$$

Optimisation over $\lambda$ deliver the following bounds:

**Proposition 2.** Let $x_t$ be as above. Then,

$$\langle x_t, L(G)x_t \rangle \leq \frac{d_1 \frac{n}{n-1} + B}{2} + \frac{1}{2} \sqrt{ \left( d_1 \frac{n}{n-1} - B \right)^2 + 4 d_1 \left(1 - \frac{d_1}{n-1}\right) };$$

$$\langle x_{t+1}, L(\bar{G})x_{t+1} \rangle \leq \frac{\bar{d}_1 \frac{n}{n-1} + \bar{B}}{2} + \frac{1}{2} \sqrt{ \left( \bar{d}_1 \frac{n}{n-1} - \bar{B} \right)^2 + 4 \bar{d}_1 \left(1 - \frac{\bar{d}_1}{n-1}\right) }.$$

**Proof.** We maximize the expression in (13) for $0 < \lambda < 1$. To this aim, we analyze the derivative of the expression with respect to $\lambda$:

$$d_1 \frac{n}{n-1} - B + \sqrt{ \left( d_1 \frac{n}{n-1} - B \right)^2 + 4 d_1 \left(1 - \frac{d_1}{n-1}\right) }; \quad \text{(15)}$$

Observe that the derivative goes to $+\infty$ and $-\infty$ as $\lambda$ goes to 0 and 1, respectively. Therefore, we conclude that the maximum is in the interior. On the other hand, equalling (15) to zero gives:

$$\lambda^2 - \lambda + \frac{1}{4 + A^2} = 0, \quad A = \frac{d_1 \frac{n}{n-1} - B}{\sqrt{d_1 \frac{n}{n-1}}}. \quad \text{9}$$
The maximum is achieved at:
\[ \lambda_{\pm} = \frac{1}{2} \left( 1 \pm \frac{A}{\sqrt{4 + A^2}} \right). \]

The proof is concluded by replacing \( \lambda \) by \( \lambda_{\pm} \) in (13), observing that \( \lambda_{\pm} = 1 - \lambda_{\mp} \).

As an alternative to using Proposition 2, BC will hold if we can prove at least one of the following inequalities:

\[ \frac{d_1(n-1)+B}{2} + \frac{1}{2} \sqrt{\left( \frac{d_1}{n-1} - B \right)^2 + 4d_1 \left( 1 - \frac{d_1}{n-1} \right)} + \omega_1 \leq t + d_1; \quad (16) \]

\[ \frac{\bar{d}_1(n-1)+\bar{B}}{2} + \frac{1}{2} \sqrt{\left( \frac{\bar{d}_1}{n-1} - \bar{B} \right)^2 + 4\bar{d}_1 \left( 1 - \frac{\bar{d}_1}{n-1} \right)} + \bar{\omega}_1 \leq \bar{t} + \bar{d}_1. \quad (17) \]

For the bound on \( \omega_\tau \), we remark that \( x_t(p) \) can be somehow uniquely determined if \( l_p \neq 0 \).

### 2 Proof of Conjecture

We use induction on \( n \) to prove BC and assume that BC is true for \( n \leq 15 \). For \( n \leq 15 \) BC can be checked using modern computer.

Let \( \{x_i, i \in [n-1]\} \) be the set of eigenvectors of \( L(G) \). Considering Grassmannian frame \( F \) with row set \( \{x_i, i \in [t]\} \) and complement frame \( \bar{F} \) with row set \( \{x_i, i \in [t+1,n-1]\} \) we note that orthogonal transformation of columns preserve these frames and we can assume that Grassmannian frames \( H_{1,p}(\{x_i(p), j \in [t]\}), H_{2,p}(\{x_i(p), j \in [t+1,n-1]\}) \), are generated by vectors \( x_j(p) = (x_{j,1}(p), x_{j,2}(p), \ldots, x_{j,\bar{d}_1+1}(p), 0, x_{j,\bar{d}_1+2}(p), \ldots, x_{j,n}(p)) \), \( j \in [n-1] \setminus \{t,t+1\} \), \( x_j(p) = (x_{j,1}(p), x_{j,2}(p), \ldots, x_{j,n}(p)) \), \( j \in \{t,t+1\} \). Note, that

\[ x_{t,1}^2 + x_{t+1,1}^2 = \frac{n-1}{n}. \]

W.l.o.g. we can assume that \( p = 1 \) and \( x_{t,1} \in \left( 0, \sqrt{(n-1)/n} \right) \). Next we skip \( p = 1 \) in the notations.

As a first step, we observe that the case \( x_{t,1}^2 \leq \frac{d_1}{d_1+1} \) (respectively, \( x_{t+1,1}^2 \leq \frac{d_1}{d_1+1} \)) is easily discarded.

**Lemma 4.** Suppose that either \( |\ell_i|^2 < \frac{d_1}{d_1+1} \) or \( |\bar{\ell}_i|^2 < \frac{\bar{d}_1}{\bar{d}_1+1} \) for some \( i \). Then, BC holds for \( G \).

**Proof.** To prove BC for \( n \) and \( x_{t,1}^2 \geq \frac{d_1}{d_1+1} \), assuming that it is true for \( n-1 \) it is sufficient to prove the inequality

\[ d_1 + \sqrt{d_1 \frac{1-x_{t,1}^2}{x_{t,1}^2}} + \omega_1 \leq d_1 + t \]

or

\[ \omega_1 \leq t - \sqrt{d_1 \frac{1-x_{t,1}^2}{x_{t,1}^2}}. \]
Last inequality is trivial, since

\[
\sqrt{d_1 \frac{1-x_{t,1}^2}{x_{t,1}^2}} \leq \sqrt{d_1 \frac{1-d_1}{d_{t+1}}} \leq 1.
\]

The same consideration proves BC when \( x_{t+1,1}^2 \geq \frac{d_{t+1}}{d_1} \).

Putting (8) and Proposition 1 together we conclude that \( BC_t \) holds for \( G \) if one of the following inequalities is true:

\[
\frac{d_1}{n-1} + \sqrt{d_1 \left(1 - \frac{d_1}{n-1}\right) \frac{1-n^{-1}x_{t,1}^2}{x_{t,1}} + \omega_1} \leq t; \tag{18}
\]

\[
\frac{d_1}{n-1} + \sqrt{d_1 \left(1 - \frac{d_1}{n-1}\right) \frac{1-n^{-1}x_{t+1,1}^2}{x_{t+1,1}} + \tilde{\omega}_1} \leq \tilde{t}. \tag{19}
\]

For the remaining of the paper, we consider \( t = \frac{n}{2} \) when \( n = 2t \) and \( t = \frac{n-1}{2} \) when \( n = 2t + 1 \).

Assume at first that \( \omega_1 < t(1 - \delta) \).

Then using (18) we obtain the inequality

\[
d_1 \left(1 - \frac{d_1}{n-1}\right) \frac{1-n^{-1}x_{t,1}^2}{x_{t,1}^2} \leq \left(\frac{n}{2} \delta - 1\right)^2
\]

or

\[
x_{t,1}^2 \geq \frac{z\overline{z}(n-1)}{z\overline{z}n + \left(\frac{n}{2} \delta - 1\right)^2}.
\]

The last inequality is satisfied if

\[
x_{t,1}^2 \geq \frac{2}{n\delta^2}, \ \delta > \frac{2\sqrt{2}}{(\sqrt{2} - 1)n}.
\]

It is left to consider the reverse condition:

\[
x_{t,1}^2 < \frac{2}{n\delta^2}. \tag{20}
\]

In this case, we have:

\[
d_1 < \omega_1 + \sum_{p: (1,p) \in E} (x_{t,p} - x_{t,1})^2 < d_1 x_{t,1}^2 + 2\sqrt{d_1} x_{t,1} + 1 + \omega_1 < \frac{2}{\delta^2} + \frac{2\sqrt{2}}{\delta} + 1 + \omega_1 < \frac{2}{\delta^2} + 2\sqrt{2} + 1 + t(1 - \delta) < \frac{n-1}{n} t, \ \delta > 2n^{-1/3}.
\]

We thus conclude the proof with the following result:
Lemma 5. $G$ satisfies $BC_t$ if \([21]\) holds.

Proof. We have

$$\frac{d_1 n}{2} + B + \frac{1}{2} \sqrt{\left(\frac{d_1 n}{n-1} - B\right)^2 + 4d_1 \left(1 - \frac{d_1}{n-1}\right)} + d_1 \leq d_1 + t;$$

$$\sqrt{\left(\frac{d_1 n}{n-1} - B\right)^2 + 4d_1 \left(1 - \frac{d_1}{n-1}\right)} \leq 2t - \frac{n}{n-1} - B.$$

Assume that $B \leq t - \frac{2}{\delta}$, then

$$4d_1 \left(1 - \frac{d_1}{n-1}\right) \leq \left(2t - \frac{n}{n-1} - B\right)^2 - \left(\frac{n}{n-1} - B\right)^2.$$

Hence we need to prove inequality

$$d_1 \left(1 - \frac{d_1}{n-1}\right) \leq \left(\frac{n}{n-1} - B\right)(t - B).$$

Hence we have inequality

$$d_1 < t \left(1 - \frac{\delta}{\delta}\right)$$

when $\frac{\delta}{\delta} < \delta < 1$, $t - B > \frac{2}{\delta}$.

At last, if $B > t - \frac{2}{\delta}$, then $\bar{B} < n - t + \frac{2}{\delta} = \frac{n}{2} + \frac{2}{\delta}$, $\bar{d}_1 > n - 1 - t \left(1 - \frac{\delta}{\delta}\right) > \frac{n}{2} \left(1 + \frac{\delta}{\delta}\right)$.

From other side

$$\bar{B} \geq \sum_{p,(1,p) \in \bar{E}} \left(x_{t+1,1} - x_{t+1,p}\right)^2 \geq \frac{\bar{d}_1 x_{t+1,1}^2}{1 + \bar{x}_{t+1,1}^2} - 2|x_{t+1,1}|\sqrt{\bar{d}_1(1 - x_{t+1,1}^2)}$$

$$> \frac{n}{2} \left(1 + \frac{\delta}{16}\right) \left(\frac{n-1}{n} - \frac{2}{n\delta^2}\right) - 2\sqrt{\frac{n}{2} \left(1 + \delta\right)} > \frac{n}{2} + \frac{2}{\delta}.$$

Last inequality is true when $\delta > 2n^{-1/3}$. This contradiction complete the proof in the case when there exists $i \in [n]$, s.t. $\omega_i \leq t(1 - \delta)$.

Next we assume that $\omega_i > t(1 - \delta)$, $i \in [n]$. Then $d_i \geq \omega_i > t(1 - \delta)$. Assume that there exists $p \in [n]$ s.t. $t(1 + \delta) \geq d_p \geq t(1 - \delta)$. Taking into account that $\omega_q > t(1 - \delta)$ we have

$$\sum_{q:p,(p,q) \in \bar{E}} \sum_{i=1}^t x_{i,q}^2 < t - t(1 - \delta) = t\delta.$$

Hence

$$\frac{1}{n - d_p} \sum_{q:p,(p,q) \in \bar{E}} \sum_{i=1}^t x_{i,q}^2 < \frac{t\delta}{n - 1 - d_p} < \frac{t\delta}{n - t - t\delta} < \frac{\delta}{1 - \delta}. $$

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We have
\[ P\left( \sum_{i=1}^{t} x_{i,q}^2 > \frac{\sqrt{\delta}}{1 - \delta} \right) < \sqrt{\delta}. \]

Hence
\[ \sum_{i=1}^{t} x_{i,p}^2 < \frac{\sqrt{\delta}}{1 - \delta} \]

for \( q \in J, J \subset \{ q : (p, q) \in \bar{E} \}, |J| > (1 - \sqrt{\delta})(1 - \delta)t. \)

Hence for the arbitrary \( q \in J \)
\[ d_q < \omega_q + \sum_{k \in J}(x_{t,q} - x_{t,k})^2 < d_q x_{i,q}^2 + 2\sqrt{d_q|x_{i,q}|} + t < d_p \frac{\sqrt{\delta}}{1 - \delta} + 2\sqrt{n} \frac{\sqrt{\delta}}{1 - \delta} + t \]

or
\[ d_q < t(1 + 2\sqrt{\delta}), n > 36\delta^{-3/2}, p \in J. \]

The same argument for \( \bar{d}_q, q \in \bar{J} \subset \{ q : (p, q) \in E \}, |\bar{J}| > (1 - \sqrt{\delta})(1 - \delta)t \) reaches the same inequality
\[ \bar{d}_q < t(1 + 2\sqrt{\delta}), n > 36\delta^{-3/2}. \]

Renumbering vertices we can assume that
\[ t(1 - 2\sqrt{\delta}) \leq d_q, \bar{d}_q \leq t(1 + 2\sqrt{\delta}), q \in [2(1 - \sqrt{\delta})t(1 - \delta)] > [n(1 - 2\sqrt{\delta})]. \]

In the remaining part \([n] \setminus [n(1 - 2\sqrt{\delta})]\), we use the trivial bound \( d_p, \bar{d}_p < n. \) \( G \) satisfies BC if
\[ \sum_{i=1}^{t} \mu_i(L(G)) \leq \sum_{i=1}^{t} d_i + \sqrt{2mt} < 2n^2\sqrt{\delta} + t^2(1 + 2\sqrt{\delta}) + n\sqrt{n} < m + \frac{t + 1}{2} = m + \frac{n/2 + 1}{2}. \]

Brouwer conjecture BC\(_{t-1}\) for graph \( \bar{G} \) is valid if
\[ \sum_{i=1}^{t-1} \mu_i(L(\bar{G})) \leq \sum_{i=1}^{t-1} d_i + \sqrt{2mt} < 2n^2\sqrt{\delta} + t^2(1 + 2\sqrt{\delta}) + n\sqrt{n} < \bar{m} + \frac{n/2}{2}. \]

Both inequalities could not be violated for small \( \delta \), because \( m + \bar{m} = \binom{n}{2} \).

Because for BC to be true it is sufficient that one of these inequalities to be valid, what completes the proof whenever there is \( i \in [n] \) such that \( t(1 + \delta) \geq d_i \geq t(1 - \delta). \)

Assume now that \( d_i \geq t(1 + \delta), i \in [n] \). Then \( d_i \leq t(1 - \delta). \)

BC\(_{t-1}\) is true if
\[ \sum_{i=1}^{t-1} \mu_i(L(\bar{G})) \leq \sum_{i=1}^{t-1} \bar{d}_i + \sqrt{2mt} \leq t(t - 1)(1 - \delta) + \sqrt{nm} \leq t(t - 1)(1 - \delta) + n\sqrt{n} \leq \bar{m} + \frac{n(n - 2)}{8}, \]

which is true because BC is obviously true when \( \bar{m} < \frac{n}{2} \). \( \square \)
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