Excellent graphs with respect to domination: subgraphs induced by minimum dominating sets

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Abstract

A graph $G = (V(G), E(G))$ is $\gamma$-excellent if $V(G)$ is the union of all $\gamma$-sets of $G$, where $\gamma$ stands for the domination number of $G$ and a $\gamma$-set is a dominating set of cardinality $\gamma$. Let $I$ be a set of all mutually nonisomorphic graphs and let $\emptyset \neq H \subseteq I$. In this paper, the study of the $H$-$\gamma$-excellent graphs is initiated. A graph $G$ is $H$-$\gamma$-excellent if the following conditions hold: (i) for every $H \in I$ and for each $x \in V(G)$ there exists an induced subgraph $H_x$ of $G$ such that $H$ and $H_x$ are isomorphic, $x \in V(H_x)$ and $V(H_x)$ is a subset of some $\gamma$-set of $G$, and (ii) the vertex set of every induced subgraph $H$ of $G$, which is isomorphic to some element of $I$, is a subset of some $\gamma$-set of $G$. We consider some well-known graphs, including cycles, trees and some cartesian products of two graphs, and for every considered graph we describe its largest set $H \subseteq I$ for which the graph is $H$-$\gamma$-excellent. Results on $\gamma$-excellent regular graphs and on a generalized lexicographic product of graphs are presented. Several open problems and questions are also posed.

Keywords: domination number; excellent graph; graph product.

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1. Introduction

All graphs in this paper will be finite, simple, and undirected. We use [8] as a reference for terminology and notation which are not explicitly defined here. In a graph $G = (V(G), E(G))$, for a subset $S \subseteq V(G)$ the subgraph induced by $S$ is the graph $(S)$ with vertex set $S$ and two vertices in $(S)$ are adjacent if and only if they are adjacent in $G$. The complement $\overline{G}$ of $G$ is the graph whose vertex set is $V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are nonadjacent in $G$. The union of two disjoint graphs $G$ and $H$ is denoted by $G \cup H$. For any vertex $x$ of a graph $G$, $N_G(x)$ denotes the set of all neighbors of $x$ in $G$, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of $x$ is $deg_G(x) = |N_G(x)|$. The minimum and maximum degrees of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $S \subseteq V(G)$, let $N_G[S] = \cup_{v \in S} N_G[v]$. Let $X \subseteq V(G)$ and $x \in X$. The X-private neighborhood of $x$, denoted by $mn_G[x, X]$ or simply by $mn[x, X]$ (if the graph is clear from the context), is the set $\{y \in V(G) \mid N[y] \cap X = \{x\}\}$. A leaf is a vertex of degree one and a support vertex is a vertex adjacent to a leaf. A vertex which separates two other vertices of the same component is a cut-vertex, and an edge separating its ends is a bridge. The distance $dist_G(x, y)$ in $G$ of two vertices $x, y$ is the length of a shortest $x - y$ path in $G$; if no such path exists, we set $dist_G(x, y) := \infty$. An isomorphism of two graphs $G$ and $H$ is a bijection $f : V(G) \to V(H)$ between the vertex sets of $G$ and $H$ such that any two vertices $u$ and $v$ of $G$ are adjacent in $G$ if and only if $f(u)$ and $f(v)$ are adjacent in $H$. If an isomorphism exists between two graphs, then the graphs are called isomorphic and denoted as $G \cong H$. We write $K_n$ for the complete graph of order $n$ and $P_n$ for the path on $n$ vertices. Let $C_m$ denote the cycle of length $m$. A complete $r$-partite graph $K_{n_1, n_2, \ldots, n_r}$ is a graph whose vertex set can be partitioned into $r$ parts, say $V_1, V_2, \ldots, V_r$, such that (a) no two vertices within the same part are adjacent, (b) there is an edge between every two vertices of different parts of the partition, and (c) $|V_i| = n_i$, $i = 1, 2, \ldots, r$. The 1-corona, denoted $cor(U)$, of a graph $U$ is the graph obtained from $U$ by adding a degree-one neighbor to every vertex of $U$. We use the notation $[k]$ for $\{1, 2, \ldots, k\}$.

An independent set is a set of vertices in a graph, no two of which are adjacent. The independence number of $G$, denoted $\beta_0(G)$, is the maximum size of an independent set in $G$. A subset $D \subseteq V(G)$ is called a dominating set (or a total dominating set) in $G$, if for each $x \in V(G) - D$ (or for each $x \in V(G)$, respectively) there exists a vertex $y \in D$ adjacent to $x$. A dominating set $R$ of a graph $G$ is a restrained dominating set (or an outer-connected dominating set) in $G$, if every vertex in $V(G) - R$ is adjacent to a vertex in a vertex in $V(G) - R$ (or $V(G) - R$ induces a connected graph, respectively). The minimum number of vertices of a dominating set in $G$ is the domination number $\gamma(G)$ of $G$. Analogously the total domination number $\gamma_t(G)$, the restrained domination number $\gamma_r(G)$ and the outer-connected domination number $\gamma_{oc}(G)$ are defined. The minimum
cardinality of a set $S$ which is simultaneously total dominating and restrained dominating in $G$ is called the \textit{total restrained domination number} $\gamma_{tr}(G)$ of $G$. The minimum cardinality of a set $S$ which is simultaneously total dominating and outer-connected dominating in $G$ is called the \textit{total outer-connected domination number} $\gamma_{toc}(G)$ of $G$. The \textit{independent domination number} of $G$, denoted by $\iota(G)$, is the minimum size of an independent dominating set.

For a graph $G$, let $\pi$ be a graphical property that can be possessed, or satisfied by the subsets of $V$. For example, being a maximal complete subgraph, a maximal independent set, acyclic, a closed/open neighborhood, a minimal dominating set, etc. Suppose that $f_x$ and $F_x$ are the associated graph invariants: the minimum and maximum cardinalities of a set with property $\pi$. Let $\mu \in \{f_x, F_x\}$. For a graph $G$, denote by $h_{\mu}(G)$ the family of all subsets of $V(G)$ each of which has property $\pi$ and cardinality $\mu(G)$. Each element of $h_{\mu}(G)$ is called a $\mu$-set of $G$. Fricke et al. \cite{6} define a graph $G$ to be $\mu$-excellent if each its vertex belongs to some $\mu$-set. Perhaps historically the first results on $\mu$-excellent graphs were published by Berge \cite{1} who introduced the class of $B$-graphs consisting of all graphs in which every vertex is in a maximum independent set. Of course all $B$-graphs form the class of $\beta_0$-excellent graphs. The study of excellent graphs with respect to the some domination related parameters was initiated by Fricke et al. \cite{6} and continued e.g. in \cite{3,9,10,14,18,20,23}.

In this paper we focus on the following subclass of the class of $\mu$-excellent graphs.

**Definition 1.1.** Let $I$ be a set of all mutually nonisomorphic graphs and $\emptyset \neq \mathcal{H} \subseteq I$. We say that a graph $G$ is $\mathcal{H}$-$\mu$-excellent if the following hold:

(i) For each $H \in \mathcal{H}$ and for each $x \in V(G)$ there exists an induced subgraph $H_x$ of $G$ such that $H$ and $H_x$ are isomorphic, $x \in V(H_x)$ and $V(H_x)$ is a subset of some $\mu$-set of $G$.

(ii) For each induced subgraph $H$ of $G$, which is isomorphic to some element of $\mathcal{H}$, there is a $\mu$-set of $G$ having $V(H)$ as a subset.

By the above definition it immediately follows that each $\mathcal{H}$-$\mu$-excellent graph is $\mu$-excellent. If a graph $G$ is $\mathcal{H}$-$\mu$-excellent and $\mathcal{H}$ contains only one element, e.g. $\mathcal{H} = \{H\}$, we sometimes omit the brackets and say that a graph $G$ is $H$-$\mu$-excellent.

Define the $\mu$-excellent family of induced subgraphs of a $\mu$-excellent graph $G$, denoted by $G(\mu)$, as the family of all graphs $H \in I$ for which $G$ is $H$-$\mu$-excellent. The next two observations are obvious.

**Observation 1.1.** If $G$ is a $\mu$-excellent graph, then $\{K_1\} \subseteq G(\mu)$ and $\mu(G) \geq \max\{|V(H)| \mid H \in G(\mu)\}$.

**Observation 1.2.** Let a graph $G$ be both $\mu$-excellent and $\nu$-excellent. If the set of all $\mu$-sets and the set of all $\nu$-sets of $G$ coincide, then $G(\mu) = G(\nu)$.

As first examples of $\mathcal{H}$-$\mu$-excellent graphs let us consider the case $\mu = \beta_0$. Clearly, any $\beta_0$-excellent graph $G$ is $\{K_1, \overline{K_{\beta_0}(G)}\}$-$\beta_0$-excellent. A graph is $r$-extendable if every independent set of size $r$ is contained in a maximum independent set (Dean and Zito \cite{4}). Clearly, a graph is $\{K_1, \overline{K_2}, ..., \overline{K_s}\}$-$\beta_0$-excellent if and only if it is $s$-extendable for all $s = 1, 2, ..., r$. Plummer \cite{15} define a graph $G$ to be \textit{well covered} whenever $G$ is $k$-extendable for every integer $k$. In other words, a graph $G$ is well covered if and only if $G(\beta_0) = \{K_1, \overline{K_2}, ..., \overline{K_{\beta_0}(G)}\}$.

In this paper we concentrate mainly on excellent graphs with respect to the domination number $\gamma$. We give basic terminologies and notations in the rest of this section. In Section 2 we describe the $\gamma$-excellent family of induced subgraphs for some well known graphs. In Section 3 we show that, under appropriate restrictions, the generalized lexicographic product of graphs has the same excellent family of induced subgraphs with respect to six domination-related parameters. Section 4 contains results on $\gamma$-excellent regular graphs and trees. We conclude in Section 5 with some open problems.

## 2. Examples

Here we find the $\gamma$-excellent family of induced subgraphs of some well known graphs.

**Example 2.1.** Let $G$ be a connected graph with $\gamma(G) = 2$. In \cite{11} it is proved that (in our terminology) $G$ is $K_2$-$\gamma$-excellent if and only if $G$ is a complete $r$-partite graph $K_{n_1,n_2,...,n_r}$, $n_i \geq 2$, $i = 1, 2, ..., r \geq 2$. Clearly $K_2, 2, ..., 2(\gamma) = \{K_1, K_2, \overline{K_2}\}$ and $K_{n_1,n_2,...,n_r}(\gamma) = \{K_1, K_2\}$ when $n_s \geq 3$ for some $s \in [r]$.

**Example 2.2.** Let $\nu \in \{\gamma, i\}$. Then all the following hold:

(i) \textit{(folklore)} $\nu(P_n) = \lceil n/3 \rceil$ and $\nu(C_r) = \lceil r/3 \rceil$. $C_r$ is $\nu$-excellent for all $r \geq 3$. $P_n$ is $\nu$-excellent if and only if $n = 2$ or $n \equiv 1 \pmod{3}$.

(ii) $P_n(\nu) = \{K_1\}$ when $n \in \{1, 2\} \cup \{7, 10, \ldots\}$ and $P_4(\nu) = \{K_1, \overline{K_2}\}$.
(iii) $C_5(\nu) = \{K_1, K_2\}$ and $C_3r(\nu) = C_{5+3r}(\nu) = \{K_1\}$, $r \geq 1$.

(iv) $C_7(\gamma) = \{K_1, K_2, \overline{K_2}, K_3\}$, and $C_{3r+1}(\gamma) = \{K_1, K_2, \overline{K_2}\}$ for $r \neq 2$.

(v) $C_7(\delta) = \{K_1, \overline{K_2}, \overline{K_3}\}$ and $C_{3r+1}(\delta) = \{K_1, \overline{K_2}\}$ for $r \neq 2$.

The proof is straightforward and hence we omit it.

Denote by (CEA) the class of all graphs $G$ such that $\gamma(G + e) \neq \gamma(G)$ for all $e \in E(G)$.

**Example 2.3.** Let a noncomplete graph $G$ be in (CEA). It is well known fact that any two nonatomic vertices of $G$ belong to some $\gamma$-set of $G$ (Sumner and Blitch [21]). In other words, $G$ is $\{K_1, K_2\}$-$\gamma$-excellent graph.

**Proposition 2.1.** Let $G$ be a graph with $\beta_0(G) = \gamma(G) = s$. Then $G$ is $\{K_1, K_2, K_3, \ldots, K_s\}$-$\gamma$-excellent and $G(\nu) = G(\beta_0) = \{K_1, \ldots, K_s\}$ (for the second conclusion, see [15]).

**Proof.** Every independent set of $G$ is a subset of a maximal independent set. Since each maximal independent set is always a dominating set and $\beta_0(G) = \gamma(G) = s$, the result immediately follows.

The Cartesian product of two graphs $G$ and $H$ is the graph $G \square H$ whose vertex set is the Cartesian product of the sets $V(G)$ and $V(H)$. Two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \square H$ precisely when either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. It is clear from this definition that $G \square H$ is $H$-$\gamma$-excellent if and only if $G$ is $H$-$\gamma$-excellent.

**Example 2.4.** Let $G = K_m \square K_n$, $n \geq m \geq 2$. Then $G(\nu) = G(\beta_0) = \{K_1, \ldots, K_m\}$. If $n > m$, then $G(\gamma) = \{K_1, \ldots, K_m\}$. If $n = m$, then $G(\gamma) = \{K_1, \ldots, K_m\}$. Suppose that $G = H$-$\gamma$-excellent. Then there is a $\gamma$-set $D$ of $G$ such that $\langle D \rangle$ has an induced subgraph $H_1 \simeq H$. Assume that $H$ has at least one edge.

**Case 1:** $m < n$. Clearly $|A_1 \cap D| = 1$ for all $i = 1, 2, \ldots, m$. Because of symmetry, we assume without loss of generality that $D \cap B_j$ is empty for all $j > m$. Define now the set $D'' = \{x, r \mid x, r \in D\}$. Since $H$ is not edgeless, $|D \cap B_j| > 1$ for some $j \leq m$. But then $|D'' \cap A_j| > 1$, which means that $D''$ is not a $\gamma$-set of $G$. Since $\langle D'' \rangle \simeq \langle D \rangle$, $G$ is not $H$-$\gamma$-excellent. Thus, $G(\gamma) = \{K_1, \ldots, K_m\}$.

**Case 2:** $m = n$. Obviously in this case exactly one of $|A_1 \cap D| = 1$ for all $i = 1, 2, \ldots, m$ and $|B_j \cap D| = 1$ for all $j = 1, 2, \ldots, m$ holds. Say the first is valid. Let $R_1$ be a $l$-order component of $\langle H \rangle$ for some $l \geq 2$. For the sake of symmetry, we can assume that all elements of $R_1$ are in $B_1$ and $D \cap B_1$ is empty for all $s \in [p]$. Clearly $p \leq m - l + 1$. Suppose that $\langle D \rangle$ has another nontrivial component. Then the difference $m - p$ is not less than $l$. Define the set $D_1 = (D - V(R_1)) \cup \{x_{1, p+1}, x_{1, p+2}, \ldots, x_{1, p+1}\}$. Clearly $D_1$ is not a $\gamma$-set of $G$ and $\langle D_1 \rangle \simeq \langle D \rangle$. Thus $R_1$ is the only nontrivial component of $\langle D \rangle$. Hence $H$ is either a complete graph or a union of complete and edgeless graph. Finally, it is easy to see that for each such a graph $H$, $G$ is $H$-$\gamma$-excellent.

We need the following negative result.

**Theorem 2.1.** There is no $P_3$-$\gamma$-excellent graph $G$ with $\gamma(G) = 3$.

**Proof.** Assume that $G$ is a $P_3$-$\gamma$-excellent, $\gamma(G) = 3$ and $x_1, x_2, x_3$ is an induced path in $G$. Since $X = \{x_1, x_2, x_3\}$ is a $\gamma$-set of $G$, there is $y_i \in pm(x_i, X)$, $i = 1, 2, 3$. Then $\{x_1, x_2, y_2\}$ is a $\gamma$-set of $G$, which implies $y_2 y_3 \in E(G)$. But now no vertex of the induced path $y_2, y_3, x_3$ is adjacent to $x_1$, a contradiction.

**Example 2.5.** $\overline{K_3} \square \overline{K_3}(\gamma) = \{K_1, K_2, \overline{K_2}, K_1 \cup K_2, \overline{K_3}, K_3\}$ when $n \geq 3$, and $\overline{K_3} \square \overline{K_3}(\gamma) = \{K_1, K_2, \overline{K_2}, K_1 \cup K_2, \overline{K_3}, K_3\}$ when $n \geq 3$.

**Proof.** First note that $\overline{K_3} \square \overline{K_3} \simeq K_3 \square K_3$ and by Example 2.1 it immediately follows that $\overline{K_3} \square \overline{K_3}(\gamma) = \{K_1, K_2, \overline{K_2}, K_1 \cup K_2, \overline{K_3}, K_3\}$. So, let $n \geq 4$ and $m$ be an integer such that $n \geq m \geq 3$. It is well known that $\gamma(\overline{K_m} \square \overline{K_n}) = 3 \leq m = i(\overline{K_m} \square \overline{K_n})$. Let us consider the graph $G_{m,n} = \overline{K_m} \square \overline{K_n}$ as a $m \times n$ array of vertices $\{a_{i,j} \mid (1 \leq i \leq m) \land (1 \leq j \leq n)\}$, with an adjacency $\{a_{i,j} \mid (V(G_{m,n}) - (Y_i \cup Z_j), Y_i = \cup^m_{r=1} \{a_{i,k}\}$ and $Z_j = \cup^m_{r=1} \{a_{r,j}\}$. Remark now that:
(a) \( \{a_{i,j}, a_{k,l}, a_{r,s}\} \simeq K_3 \) if and only if both 3-tuples \((i,k,r)\) and \((j,l,s)\) consist of paired distinct integers. The vertices of each triangle of \(G_{m,n}\) form a \(\gamma\)-set. Every two adjacent vertices \(a_{i,j}\) and \(a_{k,l}\) belong to a triangle.

(b) All induced subgraphs isomorphic to \(K_1 \cup K_2\) are \(\{\{a_{i,j}, a_{k,l}, a_{i,j}\}\} \) and \(\{\{a_{i,j}, a_{k,l}, a_{k,j}\}\}\), where \(i \neq k\) and \(j \neq l\). The vertices of each such a subgraph form a \(\gamma\)-set. Every two vertices belong to an induced subgraph isomorphic to \(K_1 \cup K_2\).

(c) Each 3-cardinality subset of \(Z_3\) is independent and it is not dominating.

Theorem 2.1 together with (a)-(c) immediately lead to the required.

To continue we need the following theorem and definitions.

**Theorem 2.2.** \([5]\) \(\gamma(G \square H) \geq \min\{|V(G)|, |V(H)|\}\) for any two arbitrary graphs \(G\) and \(H\).

A \(G\)-layer of the Cartesian product \(G \square H\) is the set \(\{(u,y) \mid u \in V(G)\}\), where \(y \in V(H)\). Analogously an \(H\)-layer is the set \(\{(x,v) \mid v \in V(H)\}\), where \(x \in V(G)\). A subgraph of \(G \square H\) induced by a \(G\)-layer or an \(H\)-layer is isomorphic to \(G\) or \(H\), respectively.

**Theorem 2.3.** Let \(H\) be a connected noncomplete \(n\)-order graph and \(p \geq n \geq 3\). If each induced subgraph of \(K_p \square H\) which is isomorphic to \(H\) has as a vertex set some \(H\)-layer, then \(\gamma(K_p \square H) = n\) and \(K_p \square H\) is a \(H\)-\(\gamma\)-excellent graph.

**Proof.** Each \(H\)-layer of \(K_p \square H\) is a dominating set of \(K_p \square H\). Hence \(\gamma(K_p \square H) \leq |V(H)| = n\). Since \(p \geq n\), by Theorem 2.2 we have that each \(H\)-layer is a \(\gamma\)-set of \(K_p \square H\). It remains to note that clearly each vertex of \(K_p \square H\) belongs to some \(H\)-layer.

The next example serves as an illustration of the above theorem.

**Example 2.6.** If \(p \geq n \geq 5\), then the graph \(K_p \square C_n\) is \(C_n\)-\(\gamma\)-excellent.

**Proof.** Let \(H\) be an induced subgraph of \(K_p \square C_n\) which is isomorphic to \(C_r\). It is easy to see that if the vertex set of \(H\) is not a \(C_n\)-layer, then either \(r \in \{3,4\}\) or \(r \geq n + 2\). The required immediately follows by Theorem 2.3.

3. Generalized lexicographic product

Let \(G\) be a graph with vertex set \(V(G) = \{1,2,...,n\}\) and let \(\Phi = (F_1,F_2,...,F_n)\) be an ordered \(n\)-tuple of paired disjoint graphs. Denote by \(G[\Phi]\) the graph with vertex set \(\cup_{i=1}^{n} V(F_i)\) and edge set defined as follows: (a) \(F_1,F_2,...,F_n\) are induced subgraphs of \(G[\Phi]\), and (b) if \(x \in V(F_i), y \in V(F_j), i,j \in \{n\}\) and \(i \neq j\), then \(xy \in E(G[\Phi])\) if and only if \(ij \in E(G)\). A graph \(G[\Phi]\) is called the generalized lexicographic product of \(G\) and \(\Phi\). If \(F_i \cong F\) for every \(i = 1,2,...,n\), then \(G[\Phi]\) becomes the standard lexicographic product \(G[F^n]\). Each subset \(U = \{u_1,u_2,...,u_n\} \subseteq V(G[\Phi])\) such that \(u_i \in V(F_i)\), for every \(i \in \{n\}\), is called a \(G\)-layer. From the definition of \(G[\Phi]\) it immediately follow:

(A) \((\text{folklore})\) \(G[\Phi] \simeq G\) if and only if \(G[\Phi] = G[K_1]\), \(G[F] \simeq F\) if and only if \(G \simeq K_1\). If \(G\) has at least two vertices, then \(G[\Phi]\) is connected if and only if \(G\) is connected. If \(G\) is edgeless, then \(G[\Phi] = \cup_{i=1}^{n} F_i\). For any \(G\)-layer \(U = \{u_1,u_2,...,u_n\}\) the bijection \(f: V(G) \rightarrow U\) defined by \(f(i) = u_i \in V(F_i)\) is an isomorphism between \(G\) and \(U\). For any \(x \in V(F_i)\) and \(y \in V(F_j), i \neq j\), is fulfilled \(dist_{G[\Phi]}(x,y) = dist_G(i,j)\). The equality \(dist_{G[\Phi]}(x,y) = dist_G(i,j)\) will be used in the sequel without specific references.

**Theorem 3.1.** Given a graph \(G[\Phi]\), where \(G\) is connected of order \(n \geq 2\) and \(|V(F_k)| \geq 3\) for all \(k \in \{n\}\). Then \(G[\Phi] \langle \gamma \rangle = G[\Phi] \langle \gamma_{oc} \rangle = G[\Phi] \langle \gamma_{oc} \rangle\) and \(G[\Phi] \langle \gamma_{oc} \rangle = G[\Phi] \langle \gamma_{oc} \rangle\). If \(\gamma(F_k) \geq 3\) for all \(k \in \{n\}\), then \(G[\Phi] \langle \gamma \rangle = G[\Phi] \langle \gamma_{oc} \rangle = G[\Phi] \langle \gamma_{oc} \rangle\).

**Proof.** Let \(\mu \in \{\gamma,\gamma_{oc}\}\) and \(D\) a \(\mu\)-set of \(G[\Phi]\). Assume there is \(i \in \{n\}\) such that \(V(F_i) \cap D = \{v_1,v_2,...,v_r\}\), where \(r \geq 2\). Then clearly for each \(j \in N(i), V(F_j) \cap D\) is empty and for any \(u_j \in V(F_j)\) the set \((D - \{v_2,...,v_r\}) \cup \{u_j\}\) is a dominating set of \(G[\Phi]\) or a total dominating set of \(G[\Phi]\) depending on whether \(\mu = \gamma\) or \(\mu = \gamma_{oc}\), respectively. Hence \(r = 2\). Since \(G\) is connected of order \(n \geq 2\) and \(|V(F_i)| \geq 3\) for all \(i \in \{n\}\), the graph \((V(G[\Phi]) - D)\) is connected. Therefore the first two equality chains are correct.

Finally, let \(D_1\) be a \(\gamma\)-set of \(G[\Phi]\) and \(\gamma(F_k) \geq 3\) for all \(k \in \{n\}\). Then clearly for every \(i \in \{n\}\) the sets \(D\) and \(V(F_i)\) must have no more than one element in common. But this immediately implies that \(D_1\) is a total dominating set of \(G[\Phi]\). Thus, the last equality chain holds.
Theorem 3.2. Given a graph $G[\Phi]$, where $G$ is connected of order $n \geq 2$ and $F_k$ is complete with $|V(F_k)| \geq 2$ for all $k \in [n]$. Then $G[\Phi]$ is $K_\gamma$-$\gamma$-excellent if and only if $G$ is $K_\gamma$-$\gamma$-excellent.

Proof. Recall that any $G$-layer of $G[\Phi]$ induces a graph isomorphic to $G$. We need the following claim.

Claim 1. (i) Each $\gamma$-set $D$ of $G[\Phi]$ is contained in a $G$-layer of $G[\Phi]$; moreover, $D$ is a $\gamma$-set of each subgraph of $G[\Phi]$ that is induced by a $G$-layer containing $D$. (ii) If $D^*$ is a $\gamma$-set of some subgraph of $G[\Phi]$ that is induced by a $G$-layer, then $D^*$ is a $\gamma$-set of $G[\Phi]$.

Proof of Claim 1. If $D$ is a $\gamma$-set of $G[\Phi]$, then since all $F_i$’s are complete $|D \cap V(F_i)| \leq 1$ for all $i \in [n]$. But then $D$ is a dominating set of any subgraph of $G[\Phi]$ that is induced by a $G$-layer containing $D$. In particular this leads to $\gamma(G[\Phi]) \leq \gamma(G)$.

If $D^*$ is a $\gamma$-set of some subgraph of $G[\Phi]$ that is induced by a $G$-layer, then again by the fact that all $F_i$’s are complete, it follows that $D^*$ is a dominating set of $G[\Phi]$. This clearly leads to $\gamma(G[\Phi]) \geq \gamma(G)$.

Thus $\gamma(G[\Phi]) = \gamma(G)$ implying the required.

$\square$

\[ \Rightarrow \] Choose $u \in V(G[\Phi])$ arbitrarily. Then there is a $G$-layer $U$ containing $u$. Since $G$ is $K_\gamma$-$\gamma$-excellent, there is a $\gamma$-set $D^*$ of $U$ that contains $s$ paired nonadjacent vertices one of which is $u$. By Claim 1, $D^*$ is a $\gamma$-set of $G[\Phi]$.

If $R$ is a $s$-vertex independent set in $G[\Phi]$, then since all $F_i$’s are complete graphs, $R$ is a subset of some $G$-layer. The rest is as above.

Theorem 4.1. Let $G$ be a $n$-order graph with minimum degree $\delta$. Then $\gamma(G) \leq n\delta/(3\delta - 1)$ when $\delta \in \{3, 4, 5\}$ (see [16], [19] and [22], respectively).

For any 5-regular graph $G$ with $\gamma(G) = 3$, the bound stated in Theorem 4.1 can be improved by 3.

Proposition 4.1. Let $G$ be a 5-regular graph with $\gamma(G) = 3$. Then $n \geq 12$.

Proof. By Theorem 4.1 we have $n \geq 9$. Since there is no 5-regular graphs of odd order, $n \geq 10$ is even. Note that there are exactly sixty 5-regular graphs of order 10 [12, 13]. Their adjacency lists can be found in [13]. A simple verification shows that each of these graphs has the domination number equals to 2.

4. Regular graphs and trees

To present the next results on regular graphs, we need the following theorem.

Theorem 4.2. Let $G$ be a $s$-regular $K_r$-$\gamma$-excellent $n$-order connected graph with $\gamma(G) = r$, where $n > s \geq r \geq 3$. Then the following assertions hold.

(i) $n \leq r(s - r + 2)$.

(ii) If $r = 3$, then $s \geq 4$ with equality if and only if $n = 9$ and $G$ is one of the graphs depicted in Fig. 1.

(iii) If $r = 3$ and $s = 5$, then $n = 12$.

Figure 1: The two 4-regular $K_3$-$\gamma$-excellent graphs of order 9. The graph on the right is $K_3 \square K_3$. 
Proof. (i) Let $H \cong K_r$ be a subgraph of $G$. Each vertex of $H$ is adjacent to $s - r + 1$ vertices outside $V(H)$. Hence $n \leq r + r(s - r + 1) = r(s - r + 2)$.

(ii) Since $r = 3$, we have $\gamma(G) = 3$ and $n \leq 3s - 3$. By Theorem 4.1 we obtain $8 \leq n$ when $s = 3$ and $9 \leq n$ when $s \geq 4$. Thus $s \geq 4$ and if the equality holds, then $n = 9$. There are exactly 16 4-regular graphs of order 9 [13]. An immediate verification shows that among them only the graphs depicted in Fig.1 are $K_3$-$\gamma$-excellent.

(iii) By (i), $n \leq 12$ and by Proposition 4.1, $n \geq 12$.

Note that the connected 5-regular $K_3$-$\gamma$-excellent graph depicted in Fig. 2 has order 12.

![Figure 2: A 5-regular $K_3$-$\gamma$-excellent connected graph on 12 vertices.](image)

Proof. Using induction on $k$ we easily obtain from Lemma 4.1 that $\{x\} = V^-(G_1) \cap V^-(G_2) \cap \ldots \cap V^-(G_k)$ and $\gamma(G) = \gamma(G_1) + \gamma(G_2) + \ldots + \gamma(G_k) - k + 1$. Consider any induced subgraph $R$ of $G$, which is isomorphic to $H$. Since $H$ is connected and without cut-vertices, $R$ is an induced subgraph of some $G_i$, say without loss of generality, $i = 1$. Then there is a $\gamma$-set $D_i$ of $G_i$ for which $R$ is an induced subgraph of $(D_i)$. Let $D_i$ be a $\gamma$-set of $G_i - x$, $i = 2, 3, \ldots, k$. Since $x \in V^-(G_i)$, $|D_i| = \gamma(G_i) - 1$. Then $D = \cup_{i=1}^k D_i$ is a $\gamma$-set of $G$ and $R$ is an induced subgraph of $(D)$.

Define a vertex labeling of a tree $T$ as a function $S : V(T) \to \{0, 1\}$. A labeled tree $T$ is denoted by a pair $(T, S)$. Let $0_T$ and $1_T$ be the sets of vertices assigned the values 0 and 1, respectively. In a labeled 1-corona tree $T$ of order at least four all its leaves are in $0_T$ and all its support vertices form $1_T$.

Let $\mathcal{T}$ be the family of labeled trees $(T, S)$ that can be obtained from a sequence of labeled trees $\tau : (T^1, S^1), \ldots, (T^j, S^j)$, $(j \geq 1)$, such that $(T^1, S^1)$ is a labeled 1-corona tree of order at least four and $(T, S) = (T^j, S^j)$, and, if $j \geq 2$, $(T^{j+1}, S^{j+1})$ can be obtained recursively from $(T^j, S^j)$ by the following operation (a visual example of this operation is given in Figure 3):

Operation $O$. The labeled tree $(T^{j+1}, S^{j+1})$ is obtained from vertex disjoint $(T^j, S^j)$ and a labeled 1-corona tree $G_j$ in such a way that $T^{j+1} = (T^j \cdot G_j)(u, v : u)$, where (a) $u \in 0_{T^j}$, $v \in 0_{G_j}$ and $u \in 0_{T^{j+1}}$, and (b) $0_{T^{j+1}} = 0_{T^j} \cup 0_{G_j} - \{v\}$ and $1_{T^{j+1}} = 1_{T^j} \cup 1_{G_j}$.

Now we are in a position to present a (reformulated) constructive characterization of $\gamma$-excellent trees.

Theorem 4.4. [17] For any tree $T$ of order at least four the following are equivalent:

(i) $T$ is $\gamma$-excellent.

(ii) There is labeling $S : V(T) \to \{0, 1\}$ such that $(T, S)$ is in $\mathcal{T}$.

Moreover, if $(T, S)$ is in $\mathcal{T}$, then $0_T = V^-(T)$, $0_T$ is a $\gamma$-set of $T$ and $1_T = V^+(T)$. In particular, all leaves of $T$ are in $V^-(T)$. 

\[ \text{Figure 3: An example of Operation } O. \text{ A labeled } 1\text{-corona tree } (T^1, S^1) \text{ and } G_j \text{ are shown.} \]
Another constructive characterization of the $\gamma$-excellent trees can be found in [3]. To prove our last result we need the following lemma.

**Lemma 4.2.** Let $G$ be a connected graph and $x \in V^-(G)$.

(i) If $xy$ is a bridge in $G$, then no $\gamma$-set of $G$ contains both $x$ and $y$.

(ii) If $x$ and $z$ are bridges in $G$, then no $\gamma$-set of $G$ contains both $y$ and $z$.

**Proof.** (i) Clearly, we can consider $G$ as a coalescence $(F \cdot H)(x)$, where without loss of generality, $y \in V(F)$ and $x$ is a leaf of $F$. Suppose $D$ is a $\gamma$-set of $G$ and $x, y \notin D$. Then $D \cap V(H)$ and $D \cap V(F)$ are dominating sets of $H$ and $F$, respectively. Moreover, since $x$ is a leaf in $F$, $D \cap V(F)$ is not a $\gamma$-set of $F$. Hence $|D| = |D \cap V(H)| + |D \cap V(F)| - 1 \geq \gamma(H) + (\gamma(F) + 1) - 1$, a contradiction with Lemma 4.1.

(ii) Let as in (i), $G = (F \cdot H)(x), y \in V(F)$ and $x$ is a leaf of $F$. Hence $z \in V(H)$. Let $D$ be a $\gamma$-set of $G$ and $y, z \in D$. By (i), $x \notin D$ and then $D \cap V(H)$ and $D \cap V(F)$ are dominating sets of $H$ and $F$, respectively. This implies $|D| = |D \cap V(H)| + |D \cap V(F)| \geq \gamma(H) + \gamma(F)$, a contradiction with Lemma 4.1. 

**Theorem 4.5.** Let $T$ be a $\gamma$-excellent tree of order at least four.

(a) If $T$ has a cut-vertex belonging to $V^-(T)$, then $T(\gamma) = \{K_1\}$.

(b) If no cut-vertex of $T$ is in $V^-(T)$, then $T$ is a 1-corona tree and $T(\gamma) = \{K_1, \ldots, K_r\}$, where $2r = |V(T)|$.

**Proof.** Suppose $H \in T(\gamma)$ and $H$ is not edgeless. Let $D$ be a $\gamma$-set of $T$ and $R \simeq H$ be an induced subgraph of $\langle D \rangle$. Choose arbitrarily an edge $xy$ of $R$. Clearly both $x$ and $y$ are not leaves and by Lemma 4.2, neither $x$ nor $y$ is a cut-vertex belonging to $V^-(T)$. Hence $x, y \in V^+(T)$, because of Theorem 4.4. Now we choose $xy$ so that $x$ is a leaf in $R$. By Theorem 4.4, a vertex $y$ has a neighbor $z \in V^-(T)$. Lemma 4.2 now implies $N[z] \cap D = \{y\}$. But then the graph $R_z = \langle V(R - x) \cup \{z\} \rangle$ is isomorphic to $R$. Since $z \in V^-(T)$ and $yz \in E(T)$, Lemma 4.2 shows that no $\gamma$-set of $T$ contains both $y$ and $z$. Thus, we arrive to a contradiction.

Therefore, $T(\gamma)$ contains only edgeless graphs. By Theorem 4.4 $V^-(T)$ is a $\gamma$-set of $T$. Assume first that there is a cut-vertex $x \in V^-(T)$. Then for any two neighbors $y$ and $z$ of $x$ the set $V_1 = (V^-(T) - \{x\}) \cup \{y, z\}$ is independent of cardinality $\gamma(T) + 1$. Suppose $T$ is $K_2, \gamma$-excellent for some $r \geq 2$. Choose any cardinality $r$ subset $V_1$ of $(V^-(T) - \{x\}) \cup \{y, z\}$ that contains both $y$ and $z$. Now by Lemma 4.2, we conclude that no $\gamma$-set of $T$ has $V_1$ as a subset. Thus, $T(\gamma) = \{K_1\}$.

Finally, let $V^-(T)$ contains only leaves. By Theorem 4.4, $T$ is a 1-corona tree. Clearly $\gamma(T) = i(T) = \beta_0(T) = r$ and then the required now follows by Proposition 2.1. 

**5. Open problems and questions**

We conclude the paper by listing some interesting problems and directions for further research.

- For which ordered pairs $(r, s)$ there are $s$-regular $K_r$-excellent graphs of order $r(s - r + 2)$ (see Theorem 4.2)? Find all 12-order 5-regular $K_5$-$\gamma$-excellent graphs.

- Characterize/describe all graphs $F$ such that there is no $F$-$\mu$-excellent graph $G$ with $\mu(G) = |V(F)|$ (see Observation 1.1). Recall that there is no $P_3$-$\gamma$-excellent graph $G$ with $\gamma(G) = 3$ (Theorem 2.1).

- Let $b$ be a positive integer. Denote by $\mathcal{A}(b)$ the class of all $\mu$-excellent connected graphs $G$ for which $\mu(G) = b$ and $|G| \mu(G)$ is maximum. It might be interesting for the reader to investigate these classes at least when $b$ is small. Note that we already know that $\mathcal{A}(\gamma, 1)$ consists of all complete graphs, and all connected graphs obtained from $K_{2n}$, $n \geq 2$, by removing a perfect matching form $\mathcal{A}(\gamma, 2)$ (Example 2.1). In addition, by Example 2.4 we have $\gamma(K_3 \square K_3) = 3, K_3 \square K_3(\gamma) = \{K_1, K_2, K_3, K_4\}$ and by Theorem 2.1 we know that there is no $P_3$-$\gamma$-excellent graph $G$ with $\gamma(G) = 3$. Thus, $K_3 \square K_3$ belongs to $\mathcal{A}(\gamma, 3)$ and $|K_3 \square K_3(\gamma)| = 6$. Find $\mathcal{A}(\gamma, 3)$.
• Find $T(\mu)$ for each $\mu$-excellent tree $T$, where $\mu \in \{i, \gamma, \gamma_R\}$ and $\gamma_R$ stand for the Roman domination number (see [9], [10] and [18], respectively).

• Find graphs $H$ such that each induced subgraph of $K_p \Box H$ which is isomorphic to $H$ has as a vertex set some $H$-layer (see Theorem 2.3).

• Characterize/describe all connected $K_2^* \gamma$-excellent graphs $G$ with $\gamma(G) = 2$.

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