Phases of a rotating Bose-Einstein condensate with anharmonic confinement

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We examine an effectively repulsive Bose-Einstein condensate of atoms that rotates in a quadratic-plus-quartic potential. With use of a variational method we identify the three possible phases of the system (multiple quantization, single quantization, and a mixed phase) as a function of the rotational frequency of the gas and of the coupling constant. The derived phase diagram is shown to be universal and the continuous transitions to be exact in the limit of weak coupling and small anharmonicity. The variational results are found to be consistent with numerical solutions of the Gross-Pitaevskii equation.

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I. INTRODUCTION

When rotated, superfluids are expected to show a structure of quantized vortex states. This effect has been confirmed experimentally in Bose-Einstein condensates of atoms which typically are confined in harmonic potentials [1–5]. In contrast to traditional experiments on superfluid liquid helium contained in a “bucket”, the presence of a harmonic potential introduces the trap frequency, \(\omega\), which sets the scale for the critical frequency of rotation of the gas \(\Omega_c\), for vortex formation. The same frequency \(\omega\) also sets an upper bound for the rotational frequency of the atoms. When \(\Omega = \omega\), the centrifugal force exactly cancels the confining force, and the atoms fly apart. In harmonically trapped Bose-Einstein condensates, the cloud starts to rotate when \(\Omega = \Omega_c\), and a vortex state forms at its center. As \(\Omega\) increases, more vortices appear and eventually form an array. As indicated, \(\Omega\) cannot exceed the trap frequency, \(\omega\).

Theoretical studies suggest that this picture becomes richer when an anharmonic term is added to the trapping potential [6–14]. The presence of the anharmonic term leads to two significant differences. First, \(\Omega\) is no longer bounded by \(\omega\). Second, the system can exhibit multiply-quantized vortex states, i.e., vortex states with more than one units of circulation (as opposed to singly-quantized vortex states which have one unit of circulation.) Indeed, for fixed interaction strength and sufficiently large \(\Omega\), the gas always exists in a state of multiple quantization. In the opposite extreme of fixed \(\Omega\) and large interaction strength, the gas forms an array of vortices as in the case of harmonic confinement. In the intermediate regime where both parameters are large, there is a third phase which contains both an array of vortices and a strong suppression of the density in the middle of the cloud. This state is a mixed phase with a multiply-quantized vortex state at the center of the cloud and singly-quantized vortices around it. Recently, the experiment of Ref. [15] managed to create vortices in an anharmonic trapping potential and investigated the phases of the system as a function of \(\Omega\). Reference [16] has examined the dynamics of the gas in the presence of vortices.

The purpose of the present study is to offer a description of the transitions between phases of multiple and single quantization. These transitions can be of either first or second order. In the limit of weak coupling and small anharmonicity, this will lead to an exact description of the continuous (i.e., second-order) transitions. In the same limit, the phase diagram as a function of the strength of the interaction versus \(\Omega\) will be seen to be universal. This phase diagram turns out to be very rich. Of particular importance are the triple points which we discover, which indicate the meeting of one phase of multiple quantization with two other phases which are distinct combinations of singly- and multiply-quantized vortex states with discrete rotational symmetries. In fact, we encounter an entire family of triple points.

We present our model in Sec. II. We then examine separately continuous phase transitions in Sec. III and discontinuous phase transitions in Sec. IV. In Sec. V, we demonstrate the universality of the phase diagram and the location of the phase boundaries describing continuous transitions in the limit of small anharmonicity and weak coupling. We offer a physical interpretation of our results in Sec. VI. Section VII contains the phase diagram that results from the solution of the exact eigenvalue problem for the anharmonic confining potential and Sec. VIII discusses the direct numerical solutions of the Gross-Pitaevskii equation, which are in agreement with our variational approach. Finally Sec. IX gives a summary of our results and conclusions.

II. MODEL – APPROACH

We start by assuming a confining potential of the form

\[ V(\rho) = \frac{1}{2} M \omega^2 \rho^2 [1 + \lambda (\frac{\rho}{a_0})^2]. \]  (1)
Here $\rho$ is the cylindrical polar coordinate, $M$ is the atomic mass, $a_0 = (\hbar/M\omega)^{1/2}$ is the oscillator length, and $\lambda$ is a small dimensionless constant. In the experiment of Ref. [15], $\lambda \approx 10^{-3}$. We neglect the trapping potential along the $z$ axis since the cloud rotates about this direction and instead assume a constant density per unit length $\sigma = N/Z$, where $N$ is the number of atoms and $Z$ is the width of the gas along the $z$ axis. The energy levels of the non-interacting (two-dimensional) problem are thus given by

$$\epsilon = (2n_r + |m| + 1)\hbar\omega,$$

where $n_r$ is the radial quantum number, and $m$ is the quantum number corresponding to the angular momentum. This equation emphasizes the large degeneracy which is associated with the many different ways of distributing $L$ units of angular momentum to $N$ atoms. As shown in Refs. [17, 18], this degeneracy is lifted by the interactions. One of the basic conclusions of these studies is that repulsive interactions always favor singly-quantized vortex states. On the other hand, when $\lambda > 0$ and the interaction is sufficiently weak, it is energetically favorable for the system to form multiply-quantized vortex states. As the interaction increases, the effective repulsion between vortices ultimately results in the splitting of multiply-quantized vortex states into singly-quantized states [6–14].

Here we shall account for the interactions with the usual assumption of a contact potential

$$V_{\text{int}} = \frac{1}{2} U_0 \sum_{i \neq j} \delta(r_i - r_j),$$

where $U_0 = 4\pi\hbar^2 a / M$ is the strength of the effective two-body interaction with $a$ equal to the scattering length for atom-atom collisions. The interaction is assumed repulsive, $a > 0$. (Reference [11] has examined an effectively-attractive Bose-Einstein condensate confined in an anharmonic potential.) The dimensionless quantity which plays the role of a coupling constant in our two-dimensional problem is thus $\sigma a$, since the typical atom density $n$ is $\sim N/(\pi a_0^2 Z)$ in the limit of weak interactions that we consider here. Therefore, the typical interaction energy $nU_0$ is $\sim \sigma a \hbar \omega$.

As shown in Ref. [14], the phase transition between multiple and single quantization occurs when $\sigma a \approx \lambda / \alpha$, where $\alpha$ is a dimensionless constant of order $10^{-1}$. Given that $\lambda \approx 10^{-3}$ in Ref. [15], $\sigma a$ is less than unity in the region of the transition. This fact allows us to adopt a variational approach where the unperturbed states are restricted to the nodeless eigenstates of the harmonic potential with angular momentum $\hbar \lambda$

$$\Phi_m(\rho, \phi) = \frac{1}{(m!\pi a_0^2 Z)^{1/2}} \left( \frac{\rho}{a_0} \right)^{|m|} e^{im\phi} e^{-\rho^2/2a_0^2}.$$  

Here $\phi$ is the angle in cylindrical polar coordinates. In this basis we can write the order parameter as

$$\Psi(\rho, \phi) = \sum_m c_m \Phi_m,$$

where the coefficients $c_m$ are variational parameters (assumed to be real without loss of generality). As a first step, we calculate the energy. From Ref. [14] we find that the energy per particle in the rotating frame in the state $\Psi$ given by Eq. (5) is

$$\mathcal{E} = \frac{E}{N} = \frac{1}{S} \sum_m \epsilon_m c_m^2 + \frac{1}{S} \sum_{m,n,l,k} c_m c_n c_l c_k \langle m, n | V_{\text{int}} | l, k \rangle \delta_{m+n,l+k},$$

where $S = \sum_m c_m^2$, and

$$\epsilon_m = (|m| - m \frac{\Omega}{\omega} + \frac{\lambda}{2} (|m| + 1)(|m| + 2)$$

is the single-particle energy in the rotating frame measured with respect to the zero-point energy, $\hbar \omega$. Here the matrix elements of the interaction are given as

$$\langle m, n | V_{\text{int}} | l, k \rangle = \sigma a \frac{(|m| + |n|)!}{2^{|m|+|n|} \sqrt{|m|!|n|!|l|!|k|!}} \delta_{m+n,l+k}.$$  

As shown in Ref. [14], for sufficiently small $\sigma a$ this energy is minimized when only one component $m = m_0$ in the above sum is nonzero, i.e., $c_m = 0$ for $m \neq m_0$. The critical frequencies, which denote the lower limit on the absolute stability of the state $m_0$, are given by [14]

$$\frac{\Omega m_0}{\omega} = \frac{m_0}{|m_0|} [1 + \lambda(|m_0| + 1) - \sigma a \frac{(2|m_0| - 2)!}{2^{2|m_0|-1}(|m_0| - 1)!|m_0|!}].$$

The (straight) solid lines in Fig. 1 show the phase boundaries in the $\Omega / \omega - \sigma a$ plane as given by Eq. (9). The transitions across these phase boundaries are discontinuous and of first order.

As $\sigma a$ increases, however, the states with a single component $m = m_0$ become unstable resulting in both continuous as well as discontinuous phase transitions. In Sec. III we identify the phase boundaries within a given sector $m_0$ as continuous transitions which are denoted as dashed lines in Fig. 1. Clearly, so long as $\sigma a$ is smaller than the value for which a given solid line is first cut by a dashed line, the straight solid lines correctly locate the discontinuous phase transitions between multiply-quantized vortices. On the other hand, for somewhat larger values of $\sigma a$ there is a competition between the mixed state that lies to the right of each solid line and the multiply-quantized state to its left. As a result of this competition, the straight lines of Eq. (9) no longer describe the phase boundary correctly; the correct boundary is pushed to the left as shown in the dotted-dashed lines in Fig. 1. The determination of this boundary is given in Sec. IV.
FIG. 1. The phase diagram in the $\Omega/\omega - \sigma a$ plane for $\lambda = 0.05$. The straight lines show the phase boundary between multiply-quantized vortices given by Eq. (9), which denote discontinuous transitions. The dotted lines give the phase boundary between $m_0 = 2$ and $(m_1, m_0, m_2) = (0, 2, 4)$ (left), and between $m_0 = 3$ and $(m_1, m_0, m_2) = (0, 3, 6)$ (right), which denote continuous transitions. The dotted-dashed lines show the phase boundary between $m_0 = 1$ and $(m_1, m_0, m_2) = (0, 2, 4)$ (left), and between $m_0 = 2$ and $(0, 3, 6)$ (right), which denote discontinuous transitions. The straight lines do not have any physical meaning for $\sigma a$ higher than the point where they are cut by the dashed lines, as the phase boundary is given by the dotted-dashed lines.

FIG. 2. The phase diagram of vortex states in a quadratic-plus-quartic potential in the $\Omega/\omega - \sigma a$ plane for $\lambda = 0.05$ with $m_0 = 0, \ldots, 15$. The solid lines denote discontinuous transitions, and the dashed lines continuous transitions. The first circle on the left indicates the triple point, and the other circles show triple points on the lines of continuous transitions for $m_0 = 8, 10, 13$.

III. CONTINUOUS TRANSITIONS

We begin our consideration of the continuous transitions by showing that along the phase boundary, where only the $m = m_0$ state is present (and thus $c_m = 0$ for $m \neq m_0$), the derivative of the energy with respect to all the $c_m$ vanishes. Having established that this pure state is an extremum, we will then examine the matrix that results by calculating the second derivatives of the energy with respect to any $c_m$ and $c_n$. The criterion for the stability of the pure state $m_0$ is then the positivity of the eigenvalues of this matrix. An instability and, hence, the phase boundary occurs when one eigenvalue of this matrix becomes negative.

It is important to note that close to the phase boundary one needs to keep only those terms in the interaction energy [i.e., in the last term of Eq. (6)] which are at most bilinear in the $c_m$ for $m \neq m_0$. This implies that the only two terms which must be retained are those with $\langle m_0, m_i | V_{\text{int}} | m_0, m_i \rangle$ ($i = 1, 2$) and $\langle m_0, m_0 | V_{\text{int}} | m_1, m_2 \rangle$ with $m_1 + m_2 = 2m_0$. The second term is the only off-diagonal element in the second-derivative matrix. This observation simplifies the problem significantly since the matrix of second derivatives is block diagonal with the dimensionality of each block being equal to two at most. The problem reduces to diagonalizing $2 \times 2$ matrices.

Given the form of the interaction, Eq. (8), it is elementary to verify that all derivatives $\partial E / \partial c_{m_n}$ do indeed vanish at the phase boundary as a consequence of angular momentum conservation [i.e., the delta function in Eq. (11)]. Turning to the second derivatives, $\partial^2 E / \partial c_m \partial c_n$, explicit calculation reveals that $\partial^2 E / \partial c_{m_0}^2 = 0$ and that

$$\frac{\partial^2 E}{\partial c_{m_i}^2} = 2\hbar \omega (\epsilon_{m_i} - \epsilon_{m_0}) + [8 \langle m_0, m_i | V_{\text{int}} | m_0, m_i \rangle - 4 \langle m_0, m_0 | V_{\text{int}} | m_0, m_0 \rangle],$$

(10)

and

$$\frac{\partial^2 E}{\partial c_{m_i} \partial c_{m_j}} = 4 \langle m_0, m_0 | V_{\text{int}} | m_1, m_2 \rangle \delta_{2m_0, m_1 + m_2}.$$  

(11)

Using Eqs. (8), (10), and (11), we can now construct and diagonalize the resulting matrix, which is block diagonal as a consequence of the delta function in Eq. (11). Evidently, the highest dimensionality of each block is $2 \times 2$. The result is shown as the dashed curves in Fig. 2 for $m_0 \leq 15$. For $m_0 \leq 6$, we find that the most unstable mode is that which involves the states with $(m_1, m_0, m_2) = (0, m_0, 2m_0)$. However, as $\Omega$ and $m_0$ increase, it becomes energetically unfavorable to have a significant atom density close to the center of the cloud. This is due to the form of the effective potential.
triplet point. Note that the slope of the phase boundary (i.e., the dashed line) changes at this point.

For higher values of $m_0$ this pattern persists with transitions from some $m_0$ to either one set of $(m_1, m_0, m_2)$ extending over the whole region of stability of the $m_0$ state or with a transition from $m_0$ to one set of states $(m_1, m_0, m_2)$ initially, and then to some other set $(m'_1, m_0, m'_2)$. The second alternative is realized for $m_0 = 10$ [which is initially unstable to $(3, 10, 17)$ and finally to $(2, 10, 18)$] and for $m_0 = 13$ [which is initially unstable to $(5, 13, 21)$ and finally to $(4, 13, 22)$]. These triple points are indicated by the final two circles in Fig. 2. For $m_0 \leq 15$, these are the only two patterns found. Note that we cannot exclude the possibility of more triple points along a given segment of the second-order line for larger values of $m_0$. In addition, $m_1$ is found to be an increasing function of $m_0$. We offer a physical interpretation of these results in Sec. VI.

IV. DISCONTINUOUS TRANSITIONS

Turning to the discontinuous transitions, it is instructive to consider a specific example between the sectors with $m_0 = 1$ and $m_0 = 2$ (see Fig. 1). The value of $\sigma a$ at which the first dashed line cuts the solid line is 0.4226. For $\sigma a < 0.4226$ the solid line thus gives the correct phase boundary between the pure states with $m_0 = 1$ and $m_0 = 2$, and the transition is discontinuous. For $\sigma a > 0.4226$, there is a competition between the pure state $\Psi = \Phi_0$ and the state $\Psi = c_0 \Phi_0 + c_2 \Phi_2 + c_4 \Phi_4$ identified in Sec. III (to the left of the dashed line). Comparison of the energy of the system in the rotating frame in these two states allows us to find the boundary between the two phases (dotted-dashed line). Since each state describes a local energy minimum, the transition between them is still discontinuous and of first order. Beyond $\sigma a = 0.4226$ the solid line is thus meaningless, and the dotted-dashed line determines the phase boundary. Note that the slope of the solid line is constant while that of the dotted-dashed line is not. This implies a discontinuity in the curvature at the joining of these curves. The use of a better variational wave function on the right of the solid curve indicates that the correct phase boundary must be pushed to the left. This is clearly seen in the results of Figs. 1, 2, 3, and 5.

As mentioned above, the triple point, i.e., the point where the three phases coexist [10], lies on the boundary between the states with $m_0 = 6$, $(m_1, m_0, m_2) = (0, 6, 12)$, and $(1, 7, 13)$. To determine its precise location we need to combine the calculations presented in Secs. III and IV. This triple point is found to be at $\Omega/\omega = 1.3633$ and $\sigma a = 2.1300$ and is indicated by the first circle in the phase diagram of Fig. 2.
V. UNIVERSALITY AND EXACTNESS OF THE PHASE BOUNDARY

The phase diagram of Fig. 2 is universal in the limit of weak coupling and small anharmonicity. More precisely, if given values of $\sigma a$ and $1 - \Omega/\omega$ locate a point on the phase boundary of Fig. 2 for a specific $\lambda$, the corresponding values become $(\sigma a)' = \beta \sigma a$ and $(1 - \Omega/\omega)' = \beta (1 - \Omega/\omega)$ for another $\lambda' = \beta \lambda$. As $\lambda$ changes, both axes scale by the same amount and in that sense the phase diagram of Fig. 2 is universal. Of course, this conclusion is valid only if the anharmonicity and the interaction are both sufficiently weak to justify our restriction to radial states with $n_r = 0$.

The method adopted in the present study of the continuous transitions is also exact in the limit of weak coupling and small anharmonicity since, infinitesimally close to the corresponding boundaries, one can neglect all terms with $m \neq m_0$ and $m \neq m_1$ in the expansion of Eq. (5). (In this regard, see also Sec. VI.) As one proceeds away from the phase boundary, additional angular momentum states must of course be included. With increasing coupling but arbitrarily close to the phase boundary, it may be necessary to consider states with $n_r \neq 0$. However, the restrictions on the angular momentum can be maintained. Thus, our approach can be used to explore the phase boundary for significantly larger values of $\Omega/\omega$ and $\sigma a$ by including additional radially excited states as needed. The phase boundary between multiply-quantized vortex states is also calculated exactly in the same limit close to the line of second-order transitions. On the other hand, the calculation of phase boundaries for the discontinuous transitions involving multiply-quantized vortices and the mixed phases (i.e., the case considered in Sec. IV) is only approximate since additional angular momentum states are in principle required for the description of the mixed phase.

VI. PHYSICAL INTERPRETATION OF THE RESULTS

It is useful to examine the spatial distribution of vortices and their multiplicity. The order parameter in the phases of multiple quantization is $\Psi = \Phi_{m_0}$, and just above the phase boundaries it has the form

$$\Psi = (c_{m_1} N_{m_1} z^{m_1} + c_{m_0} N_{m_0} z^{m_0} + c_{m_2} N_{m_2} z^{m_2}) e^{-|z|^2/2a_0^2},$$

(13)

where $m_1 < m_0 < m_2$ and $\tilde{z} = p e^{i \phi} = x + i y$. A numerical calculation reveals that close to the phase boundary and to leading order,

$$c_{m_1} \propto c_{m_2} \propto \delta^{1/2}, \quad c_{m_0} - 1 \propto \delta^{1/2},$$

(14)

where $\delta = \sigma a - (\sigma a)_c$. Here $(\sigma a)_c$ is the value of $\sigma a$ on the phase boundary for a fixed $\Omega$. It is straightforward to develop a power-series expansion in $\delta$ for all coefficients that are not strictly zero (as a consequence of angular momentum conservation) [18]. For example, the next coefficient $c_m$ contributing to the state of Eq. (13) is determined by the condition $c_{m_0}^2 c_m^2 \sim c_{m_1} c_{m_0} c_{m_2} c_m$ where $m = m_0 + m_2 - m_1$. This implies that

$$c_m \propto c_{m_1} c_{m_2} \propto \delta,$$

(15)

The next coefficient $c_{m'}$ is determined by $c_{m_0}^2 c_{m'}^2 \sim c_{m_2} c_{m_2} c_m c_m'$, where $m' = m_2 + m - m_0 = 2m_2 - m_1$, and thus

$$c_{m'} \propto c_{m_2} c_m \propto \delta^{3/2}.$$  

(16)

Evidently, the higher coefficients can be neglected at the phase boundary.

Returning to Eq. (13), $\Psi$ has $m_2$ nodes in the $xy$ plane, and each of them corresponds to a vortex state. However, only the $m_0$ of them are located in a region of non-negligible density since the density drops rapidly away from the center of the cloud due to the exponential factor. More precisely, because of the first term on the right of Eq. (13), at a small $\delta$ above the phase boundary there is still a multiply-quantized vortex state at the center of the cloud with $m_1$ units of circulation. In addition, there are $m_0 - m_1$ simple zeros or singly-quantized vortices surrounding it at a distance $R_1$ which is

$$R_1 \sim \frac{a_0}{m_1} \left[ \frac{\delta m_0}{m_1} \right]^{1/2(m_0 - m_1)}.$$  

(17)
The remaining vortices are located at a distance $R_2$

$$\frac{R_2}{\alpha_0} \sim \left( \frac{m_0!}{\delta m^2!} \right)^{1/[2(m_2-m_0)]}.$$  \hspace{1cm} (18)

For example, when the $m_0 = 8$ state becomes unstable to $m_1 = 2$, $m_0 = 8$, and $m_2 = 14$, there is a doubly-quantized vortex at the center of the gas, six more vortices around it at a distance of order $R_1/\alpha_0 \sim 2.3\delta^{1/2}$ and a final six at a distance of order $R_2/\alpha_0 \sim 3.4\delta^{1/2}$, where the density is essentially zero for sufficiently small $\delta$ (see the bottom graphs in Fig. 4). Since $m_1$ increases with increasing $m_0$, just above the phase boundary a multiply-quantized vortex state splits into a multiply-quantized vortex state with lower circulation, with the remaining (singly-quantized) vortices surrounding it. This is a mixed phase, and the hole which develops in the middle of the cloud is a multiply-quantized vortex state with a quantum number $m_1$, in agreement with Refs. [6,8]. It is worth mentioning that the maximum of the function $\rho|\Phi_m|^2$ occurs for $\rho/\alpha_0 = \sqrt{m+1}$, which gives the typical size of a vortex state with $m$ quanta.

Figure 4 shows contour plots of the density of the cloud for $m_0 = 6, 7$, and $8$, and summarizes the results mentioned above. These graphs all have distinct discrete rotational symmetries. Obviously, there cannot be a continuous transition from one discrete symmetry to another unless one passes through the cylindrically symmetric triple point. Thus, such transitions are in general of first order.

VII. EXACT TREATMENT OF THE SINGLE-PARTICLE ENERGIES

While our method treats the effect of the anharmonic term in the trapping potential perturbatively, it can easily be extended to higher values of $\lambda$ and/or $m_0$ by employing exact single-particle energies and wave functions which include the effects of the anharmonicity to all orders. In Fig. 5 we show the phase diagram that results from the use of such exact solutions for the non-interacting system (i.e., by solving the Schrödinger equation in the trapping potential $V(\rho)$ rather than treating the anharmonic term perturbatively.) The results are shown for $\lambda = 0.05$ and $m_0 \leq 8$. In this case the phase boundary has no simple scaling properties and depends on the specific choice of this parameter. Although it is tempting to regard this value of $\lambda$ as small, we find significant deviations from the results of Fig. 2. [Note, however, the quadratic dependence of the perturbative energy on $m$ in Eq. (7), which implies that perturbation theory becomes less reliable with increasing $m$. In this case the most unstable mode for $m_0 \leq 7$ is of the form $(m_1, m_0, m_2) = (0, m_0, 2m_0)$, while for $m_0 = 8$ it is the one with $(1, 8, 15)$. Therefore the triple point occurs between the phases with $m_0 = 7$, $(m_1, m_0, m_2) = (0, 7, 14)$, and $(1, 8, 15)$, and is shown inside the circle in Fig. 5.

VIII. NUMERICAL SOLUTIONS OF THE GROSS-PITAEVSKII EQUATION

In addition to the above variational calculations, we have also solved the Gross-Pitaevskii equation numerically. In order to locate the continuous phase transition lines to a few digits precision, we have used the following procedure. For given values of $\sigma \alpha$ and $\Omega/\omega$, we take as initial state the harmonic-oscillator eigenfunction $\Phi_m$ with the addition of small-amplitude random noise in the core region. (The appropriate value of $m_0$ is easily determined numerically provided that we are not close to the first-order transition and is consistent with the variational results for the values reported here.) This wave function is propagated in imaginary time subject to the Gross-Pitaevskii equation in a rotating frame, and the overlaps, $c_{m'}$, of the wave function with the harmonic-oscillator functions $\Phi_{m'}$ are monitored for a range of $m'$. A steady increase in $|c_{m'}|$ for some $m'$ indicates that the multiply-quantized vortex state is energetically unstable towards a splitting into several vortices. If, on the other hand, all $|c_{m'}|$ decrease with time, the multiply-quantized vortex is the energetically favorable configuration. This procedure makes it easy to locate the position of the second-order transition without actually having to compute the ground state for each point in phase space. The latter procedure, while straightforward, is ill-suited to the problem at hand because convergence towards the ground state is extremely slow for the weak couplings
considered here. In addition, the difference in energy between the many local energy minima is extremely small and requires exceedingly high precision. In contrast, the method we have used is a fast and reliable way to locate the continuous phase transition lines and represents a significant extension of the variational study.

The results of this numerical study are shown in Fig. 6. The anharmonicity is chosen as $\lambda = 0.005$. According to the universality shown in Sec. V, we need only rescale the axes of Fig. 2 in order to obtain the variational phase diagram for this new value of $\lambda$. Agreement with the variational study is good for this very weak anharmonicity, but the discrepancy grows rapidly with increasing $\lambda$. Exact wave functions have been computed for a few values of $\sigma a$ and $\Omega/\omega$, and their symmetries are in agreement with those shown in Fig. 4.

The first triple point occurs between $m_0 = 6$ and $m_0 = 7$, precisely as predicted by the variational method in Sec. III. The most unstable mode for $m_0 = 7$ thus involves the states with $(1, 7, 13)$. This is not inconsistent with the results reported in Sec. VII computed using exact anharmonic-oscillator eigenfunctions, because the value of $\lambda$ was different in the two cases. On the other hand, the present anharmonicity $\lambda = 0.005$ is still large enough to result in one important difference compared to Figs. 2-4. The triple point on the line of continuous transitions for $m_0 = 8$ is absent. Thus, the most unstable mode is the one with $(m_1, m_0, m_2) = (1, 8, 15)$ for all values of $\Omega$ considered. We find instabilities of the form $(m_1, m_0, m_2)$ with $m_1 \geq 2$ only for $m_0 \geq 9$. This again emphasizes that very weak anharmonicities are required to ensure the quantitative validity of the variational approach.

IX. SUMMARY

In summary, we have studied a rotating Bose-Einstein condensate with repulsive forces that is trapped in a quadratic-plus-quartic potential. As the rotational frequency of the cloud and the coupling between the atoms vary, the system exhibits three phases: a phase of multiple-quantization, a phase of singly-quantized vortices, and a mixed phase. Our calculated phase diagram turns out to be universal and partially exact in the limit of weak coupling and small anharmonicity.

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FIG. 6. Same graph as in Fig. 2, for $\lambda = 0.005$. The lines are identical to those in Fig. 2, except that the axes have been scaled according to the universality of the phase diagram. The crosses represent points of the continuous transitions computed numerically from the Gross-Pitaevskii equation (see Sec. VIII).

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