Some remarks on open analytic curves over non-archimedean fields

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Abstract

We study open analytic curves over non-archimedean fields and their formal models. In particular, we give a criterion, in terms of étale cohomology, when such a formal model is (almost) semistable.

Introduction

Let $X$ be a smooth analytic curve over a non-archimedean complete valued field $K$. A formal model of $X$ is a formal scheme $\mathcal{X}$ over the valuation ring of $K$ with generic fiber $X$. The purpose of this note is to formulate a criterion which is, in certain concrete situations, able to decide whether the model $\mathcal{X}$ is semistable.

The chief motivation to formulate such a criterion comes from the author’s work on the semistable reduction of Lubin-Tate spaces of dimension one [15]. There the analytic curve $X$ in question arises as a finite étale Galois cover of the open unit disk, $f : X \to Y := \{ y \in K \mid |y| < 1 \}$. Such a cover belongs to a class of analytic spaces which we call open analytic curves and which provides a non-archimedian analogue of open Riemann surfaces with finitely many holes. Coleman has studied this class of analytic spaces (which he calls wide open spaces), also in connection with the problem of semistable reduction ([7]). One difference to our approach is that in [7] the space $X$ is always considered as an open analytic subspace of an algebraic curve $C$ (whose semistable reduction one wants to analyze). For the applications we have in mind it is however important to consider $X$ as an object of its own, independent of any embedding into an algebraic curve. This does not result in any serious problems but, since open analytic curves are not quasi-compact, one has to be somewhat careful in applying results which rely on finiteness arguments. For instance, it has seemed safer to the author to always work over a discrete valued field $K$. Moreover, lacking adequate references it seemed necessary to

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reformulate and discuss many definitions which are well known in the algebraic context. This is done in Section 1.

In Section 2 we look at the $\ell$-adic cohomology of open analytic curves and at the vanishing cycles on the special fiber of formal models. Using the framework provided by the work of Berkovich ([2], [3]), we formulate a first version of our semistable reduction criterion. Given a formal model $\mathcal{X}$ of an open analytic curve $X$ with special fiber $\mathcal{X}_s$, we say that $\mathcal{X}$ is *almost semistable* if for every closed point $z \in \mathcal{X}_s$ the formal fiber

$$X_z := ]z[ \subset X$$

is an open analytic curve of genus zero, i.e. is isomorphic to the complement of finitely many closed disks lying inside an open disk. (Note that $\mathcal{X}$ is semistable if and only if $X_z$ is either an open disk or an open annulus, for all $z$.) Then we prove that $\mathcal{X}$ is almost semistable if the image of the natural map

$$H^1(\mathcal{X}_s) \to H^1(X)$$

is equal to the *cuspidal part* of $H^1(X)$, i.e. the image of cohomology with compact support.

One thing that should be mentioned is that the conclusion of our criterion is somewhat stronger than stated above. In addition to the assertion that $\mathcal{X}$ is almost semistable one concludes that $X$ has *tree-like reduction*. By this we mean that the special fiber of any semistable model of $X$ has a graph of components which is a tree. So our criterion would not work in a situation where this additional conclusion does not hold. Fortunately, the Lubin-Tate spaces studied in [15] do have tree-like reduction, and our method can be applied to them.

In Section 3 we give an ‘equivariant version’ of the above criterion. Let $f : X \to Y$ be a finite étale Galois cover of open analytic curves, with Galois group $G$. Let $\mathcal{Y}$ be a semistable model of $Y$ and let $\mathcal{X}$ be the normalization of $\mathcal{Y}$ in $X$. The question is now: is $\mathcal{X}$ (almost) semistable? Obviously, it suffices to verify the above criterion on the $\tau$-isotypical part of the cohomology of $X$, for every irreducible representation $\tau$ of $G$. The $\tau$-isotypical part of the cohomology of $X$ can be expressed in terms of the cohomology of the étale sheaf $\mathcal{F}_\tau$ on $Y$ associated to the Galois cover $f : X \to Y$ and the representation $\tau$. In this situation we prove (Proposition 3.7):

**Proposition 0.1** Suppose that for every irreducible $G$-representation $\tau$ there exists an affinoid subdomain $U \subset Y$ with the following properties:

(i) $U = \bigcup W \subset Y$ for an open subset $W$ of the special fiber of $\mathcal{Y}$, and

(ii) the sheaf $\mathcal{F}_\tau$ is ‘resolved’ over $U$ (see Definition 3.3).

Then the formal model $\mathcal{X}$ is almost semistable.

This equivariant version of our criterion, although it is essentially a reformulation of the original version, turns out to be very useful. A psychological
advantage is that one ‘can forget about the curve X’. In practice, one has to find for every representation \( \tau \) an affinoid \( U \) such that Condition (ii) holds. To do this there are several useful tools available, for instance Huber’s theory of Swan conductors [12]. The \( U \)’s that arise in this way determine the semistable model \( Y \) we should take. If \( Y \) is an open disk then this last step is trivial.

We end this paper with an extension of the above criterion which sometimes allows one to conclude that the model \( X \) is actually semistable (and not just almost semistable). Here we use in an essential way results and arguments from Raynaud’s paper [13].

1 Open analytic curves

1.1 The definition We fix, once and for all, a field \( K_0 \) which is complete with respect to a discrete non-archimedian valuation \(|\cdot|\), and whose residue field \( k \) is algebraically closed and of positive characteristic \( p > 0 \). We choose an algebraic closure \( K_{0}^{ac} \) of \( K_0 \) and extend the valuation \(|\cdot|\) to \( K_{0}^{ac} \).

Definition 1.1 An open analytic curve is given by a pair \((K,X)\), where \( K \subset K_{0}^{ac} \) is a finite extension of \( K_0 \) and \( X \) is a rigid analytic space over \( K \). We demand that \( X \) is isomorphic to \( C-D \), where \( C \) is the analytification of a smooth projective curve over \( K \) and \( D \subset C \) is an affinoid subdomain intersecting every connected component of \( C \).

A morphism between two open analytic curves \((K_1,X_1)\) and \((K_2,X_2)\) is an element of the direct limit

\[
\text{Hom}(X_1,X_2) := \lim_{\longrightarrow} \text{Hom}(X_1 \otimes K_3, X_2 \otimes K_3),
\]

where \( K_3 \subset K_{0}^{ac} \) ranges over all common finite extensions of \( K_1 \) and \( K_2 \).

Definition 1.1 corresponds to the definition of wide opens in [7]. However, our definition is more complicated because we insist on having a field of definition with a discrete valuation, whereas Coleman works over \( \hat{K}_{0}^{ac} \), the completion of \( K_{0}^{ac} \). Note that a morphism \( X_1 \otimes \hat{K}_{0}^{ac} \rightarrow X_2 \otimes \hat{K}_{0}^{ac} \) of rigid analytic spaces over \( \hat{K}_{0}^{ac} \) may not descend to an element of \( \text{Hom}(X_1,X_2) \), so this difference is more than just formal.

Most facts about open analytic curves that we are interested in only hold after replacing its field of definition by some finite extension (e.g. the existence of a semistable model). However, the exact choice of this extension will not be important for us. Therefore we will simply write \( X \) instead of \((K,X)\) to denote an open analytic curve. Whenever it is necessary to mention the field \( K \) (which we then call the field of definition of \( X \)) we will always assume that it is chosen ‘sufficiently large’. For instance, if we say that \( X \) is connected we actually mean that \( X \otimes K' \) is connected for every finite extension \( K'/K \).
Example 1.2  (i) An open disk is an open analytic curve isomorphic to the open unit disk
\[ D(0, 1) = \{ x \mid |x| < 1 \}. \]

(ii) An open annulus is an open analytic curve isomorphic to the standard annulus
\[ A(\epsilon, 1) = \{ x \mid \epsilon < |x| < 1 \}, \]
for some \( \epsilon \in |K^\times| \) with \( \epsilon < 1 \). The number \( \epsilon \) is easily seen to depend only on the isomorphism class of \( A(\epsilon, 1) \).

1.2 Formal and semistable models Let \( X \) be an open analytic curve, with field of definition \( K \). Let \( O \) denote the valuation ring of \( K \) and \( \wp \) the maximal ideal of \( O \). If \( Y \) is a (formal) scheme over \( O \) we shall write \( Y_s := Y \otimes k \) to denote its special fiber and \( Y_\eta := Y \otimes K \) to denote its generic fiber (whenever this makes sense). For a constructible subset \( Z \subseteq Y_s \) we write \( \mid Z \mid_Y \subseteq Y_\eta \) for the formal fiber of \( Z \) (which is an open rigid analytic subspace). By a curve we mean a morphism of schemes which is flat and of finite type and has geometric fibers of pure dimension one.

Definition 1.3 An algebraic model of \( X \) is a triple \((Y, Z, \varphi_Y)\), where \( Y \) is a curve over \( O \), \( Z \subseteq Y_s \) is a reduced closed subset and \( \varphi : X \overset{\sim}{\rightarrow} Z_{|Y} \) is an isomorphism of rigid analytic spaces over \( K \). The algebraic model \((Y, Z, \varphi_Y)\) is called good if \( Y \) is normal and \( Y_s \) is reduced. It is called minimal if it is good and if \( Z \) is purely of dimension zero. It is called semistable if \( Y_s \) is a semistable curve and \( Z \) is purely of dimension one.

Let \((Y, Z, \varphi_Y)\) be a semistable algebraic model and \( W \) an irreducible component of \( Y_s \) contained in \( Z \). We call \( W \) instable if it is isomorphic to a projective line and intersects other irreducible components of \( Y_s \) in at most two points. We call the algebraic model \((Y, Z, \varphi_Y)\) stable if there do not exist any instable components of \( Y_s \) contained in \( Z \).

A formal model of \( X \) is a pair \((X, \varphi_X)\), where \( X \) is a formal scheme over \( O \) and \( \varphi_X : X \overset{\sim}{\rightarrow} X_\eta \) is an isomorphism of rigid analytic spaces over \( K \), satisfying the following condition. There exists an algebraic model \((Y, Z, \varphi_Y)\) such that \( X \) is isomorphic to the formal completion of \( Y \) along \( Z \), and the isomorphism \( \varphi_X \) is induced from \( \varphi_Y \). The triple \((Y, Z, \varphi_Y)\) is called an algebraization of \((X, \varphi_X)\). A formal model \((X, \varphi_X)\) is said to be good (resp. minimal, resp. semistable, resp. stable) if it has an algebraization which is good (resp. minimal, resp. semistable, resp. stable).

Whenever this is unlikely to cause confusion, we will omit the isomorphisms \( \varphi_Y \) and \( \varphi_X \) from the notation.

Remark 1.4  (i) Let \( X \) be a formal model of \( X \). Then the formal scheme \( X \) is special in the sense of [3], §1, and therefore the generic fiber \( X_\eta \) is well defined. Furthermore, if \( (Y, Z) \) is an algebraization of \( X \) then \( Z \) can be identified with the closed subscheme of \( X_s \) corresponding to an ideal of
definition of the formal scheme $\mathcal{X}$, and is hence independent of $Y$. We call $Z$ the reduction of $\mathcal{X}$. We denote by
\[
\text{red}_\mathcal{X} : \mathcal{X} \otimes \mathcal{X}_q \to Z
\]
the reduction map.

(ii) Note that the formal scheme $\mathcal{X}$ is not topologically of finite type over $\mathcal{O}$. This corresponds to the fact that the rigid analytic space $\mathcal{X}$ is not quasi-compact. Note also that $Z \neq \mathcal{X}$ and that $\mathcal{O}_X \cdot q \subset \mathcal{O}_X$ is not an ideal of definition for $\mathcal{X}$.

(iii) The property of being a good (resp. minimal, semistable or stable) formal model is stable under finite extension of the base field $K$. More precisely, if $K'/K$ is a finite extension, $\mathcal{O}'$ the valuation ring of $K'$ and if $\mathcal{X}$ is good (resp. minimal, semistable or stable) then the formal model $\mathcal{X}' := \mathcal{X} \otimes \mathcal{O}'$ has the same property.

(iv) Let $Y$ be a proper curve over $\mathcal{O}$ and $Z \subset Y_s$ a reduced closed subset which has nontrivial intersection with every connected component of $Y_s$. Then the rigid space $\mathcal{X} := Y[Z]_Y$ is an open analytic curve and $(Y, Z)$ is an algebraic model. Indeed, after blowing down all irreducible components of $Y_s$ which do not meet $Z$, we may assume that the open subset $V := Y_s - Z$ is affine and hence $D := V[Y]$ is an affinoid subdomain of $Y_q$.

(v) Let $\mathcal{X}$ be a semistable model of $X$. Then there exists a semistable algebraization $(Y, Z)$ of $\mathcal{X}$ such that $V := Y_s - Z$ is isomorphic to a disjoint union of affine lines. This follows from a standard argument using formal patching, see e.g. [10]. As a consequence we can write $X = C - D$ where $C$ is a smooth projective curve over $K$ and $D \subset C$ is an affinoid subdomain isomorphic to a disjoint union of closed disks (set $C := Y_q$ and $D := V[Y]$).

Analogous to the semistable reduction theorem for smooth projective curves we have the following result.

**Proposition 1.5** Let $X$ be an open analytic curve.

(i) After enlarging the field of definition, if necessary, there exists a semistable formal model of $X$.

(ii) If no connected component of $X$ is a disk or an annulus then $X$ has a stable formal model.

**Proof:** It is no restriction to assume that $X$ is connected. By definition we can write $X = C - D$, where $C$ is a smooth projective curve over $K$ and $D$ is a nonempty affinoid subdomain. After enlarging $K$ we may assume that $C$ has a semistable model $Y$ over $\mathcal{O}$ ([8]). After an admissible blowup of $Y$ (which preserves semistability) we may further assume that the affinoid $D \subset C$ is equal to the formal fiber $V[Y]$ of an open subscheme $V \subset Y_s$ ([6]). Set $Z := Y_s - V$. 

Then $X = ]Z[Y$ and therefore $Z$ is connected. In case $Z$ consists of a single closed point, we do a further blowup to make sure that $Z$ is the union of irreducible components of $Y_s$. Now let $X'$ be the formal completion of $Y$ along $Z$. By construction $X'$ is a semistable formal model of $X$. This proves (i).

Let $(Y', Z')$ denote the algebraic model of $X$ obtained from blowing down all unstable components of $Z$. It is well known that the curve $Y'$ is still semistable. There are two possible cases to consider. The first case is that $Z'$ consists of a single point $z'$. Then $z'$ is either a smooth point or an ordinary double point of $Y_s'$. Therefore, $X'$ is either a disk or an annulus. In the second case, $Z'$ is a union of stable irreducible components of $Y_s'$. Therefore, $(Y', Z')$ is a stable algebraic model of $X$ and the formal completion of $Y'$ along $Z'$ is a stable formal model. This finishes the proof of (ii).

\begin{proof}
By Proposition 1.5 (i) there exists a semistable algebraic model $(Y, Z)$ of $X$. Let $(Y', Z')$ denote the algebraic model obtained from blowing down all irreducible components of $Z$ of dimension one. Then $(Y', Z')$ is a minimal algebraic model. This settles the existence part of (i). Let $U \subset Y'$ be an open affine subset containing $Z'$. Let $U$ denote the formal completion of $U$ along its special fiber. Since $U$ is normal, it is the canonical integral model of its generic fiber $U_{\eta}$. Moreover, $X \subset U_{\eta}$ is the disjoint union of the formal fibers of the points contained in $Z'$. In this situation, a result of Bosch [4] (Korollar 5.9 and Satz 6.1) implies that $X'$, the formal completion of $U$ in $Z'$ can be canonically identified with $\text{Spf} A$ where

$$A = \text{H}^0(X, \mathcal{O}_X)$$

is the ring of power bounded analytic functions on $X$. This ring is obviously independent of the choices we made and depends functorially on $X$. We conclude that the minimal formal model $X'$ is unique up to unique isomorphism. This finishes the proof of (i).

To prove (ii), let $X_1$ and $X_2$ be stable models of $X$. We have to show that there exists a unique isomorphism $X_1 \cong X_2$ of formal schemes over $O$ extending the identity on $X$. The uniqueness of such an isomorphism is obvious.

For $i = 1, 2$ choose a stable algebraization $(Y_i, Z_i)$ of the formal model $X_i$. Using standard techniques, one shows that we may choose the curve $Y_i$ to be stable. (This is not automatic, since Definition 1.3 requires stability only for the irreducible components contained in $Z_i$.) Let $(Y'_i, Z'_i)$ be the minimal algebraic model obtained by blowing down all irreducible components of $Z_i$ (as in the proof of (i)) and let $X'_i$ denote the resulting minimal formal model. By (i) we may identify $X'_1$ with $X'_2$. This means that we can also identify the stable model

\end{proof}
$X_2$ with the blowup of $X'_2$ at an admissible sheaf of ideals $\mathcal{J}$ on $X'_2$ with support in the point $z_1$. Let $\mathcal{J}$ denote the unique sheaf of ideals on $Y'_1$ with support in $z_1$ which gives rise to $\mathcal{J}$ after completion at $z_1$. Let $Y_3$ denote the blowup of $Y'_1$ in $\mathcal{J}$ and $Z_3 \subset Y_3$ the exceptional divisor. By construction, the formal completion of $Y_3$ along $Z_3$ can be identified with the formal model $X_2$. It is also clear that the curve $Y_3$ is stable. Now the uniqueness of the stable model for projective curves shows that there exists a unique isomorphism $Y_1 \cong Y_3$ extending the identity on the generic fiber. It induces the desired isomorphism $X_1 \cong X_2$ of formal models.

\[\blacksquare\]

Remark 1.7 The proof of Proposition 1.6 shows more generally that the minimal model of $X$ is ‘minimal among all good models’.

1.3 The ends Let $X$ be an open analytic curve. The following definitions are taken over word by word from [7].

Definition 1.8 An underlying affinoid is an affinoid subdomain $U \subset X$ such that $X - U$ is the disjoint union of annuli none of which is contained in an affinoid subdomain of $X$. An end of $X$ is an element of the inverse limit of the set of connected components of $X - U$, where $U$ ranges over all underlying affinoids. The set of all ends is denoted by $\partial X$.

By Remark 1.4 (v) we can write $X = C - D$ where $C$ is a smooth projective curve and $D = \bigcup_i D_i$ is a disjoint union of closed disks $D_i$. If $U \subset X$ is an underlying affinoid then $C - U$ is disjoint union of open disks $D'_i$ such that $D_i \subset D'_i$ and $D'_i - D_i$ is an open annulus representing an end of $X$. It follows that there is a natural bijection between the set of disks $D_i$ and the set of ends $\partial X$. See [7].

Let $(Y, Z)$ be a semistable algebraic model of $X$ and $X'$ the resulting formal model. We let $\partial Z \subset Z$ denote the subset where $Z$ intersects an irreducible component of $Y$, not belonging to $Z$ and call it the boundary of $Z$. The open subset $Z^o := Z - \partial Z$ is called the interior of $Z$. It is clear that the definition of $\partial Z$ and $Z^o$ depend only on the formal model $X'$ but not on the chosen algebraization.

Every $z \in \partial Z$ is a smooth point of $Z$. The formal fiber $A_z := |z|_{X \subset X}$ is an open annulus. Furthermore, $U := |Z^o|_{X \subset X}$ is an underlying affinoid for $X$. Hence we obtain a bijection between $\partial Z$ and $\partial X$. For every point $z \in \partial Z$ there is a unique closed formal subscheme of the special fiber $T_z \subset X'$ with support in $z$ and isomorphic to $\text{Spf} \ k[[t]]$. We call $T_z$ a virtual component of the reduction $Z$. If $\xi$ denotes the end of $X$ corresponding to $z$, we also write $T_\xi$ instead of $T_z$.

Let $X' \to \mathcal{X}' = \text{Spf} A$ be the natural map onto the minimal formal model which blows down $Z$ to a finite set of closed points (see the proof of Proposition 1.6). This map identifies the virtual components $T_\xi$ with the normalizations of the irreducible components of the scheme $\text{Spec} (A \otimes_k k)$. Therefore, Proposition 1.6 (i) implies that the scheme $T_\xi$ depends only, and in a functorial way, on the open analytic curve $X$ and the end $\xi$. In particular, any automorphism of $X$
which fixes the end $\xi$ induces an automorphism of $T_\xi$. We say that such an automorphism acts trivially on the end $\xi$ if it induces the identity on $T_\xi$.

2 Etale cohomology of open analytic curves

2.1 Let us fix, once and for all, a prime number $\ell$ which is prime to $p$, the characteristic of the residue field $k$ of $K_0$. We also fix an algebraic closure $\bar{Q}_\ell$ of $Q_\ell$. Given an open analytic curve $X$ with field of definition $K$, we define
\[
H^i(X) := \left( \lim_{\rightarrow \ell} H^i(X \otimes_K \hat{K}_0, \mathbb{Z}/\ell^r) \right) \otimes_{\mathbb{Z}} \bar{Q}_\ell.
\]
Here $H^i(\cdot, \mathbb{Z}/\ell^r)$ is the étale cohomology of rigid analytic spaces defined by Berkovich [2]. Similarly, we define cohomology with compact support $H^i_c(X)$ and, for every affinoid subdomain $U \subset X$, cohomology with support in $U$, $H^i_U(X)$. We define the cuspidal part of cohomology as
\[
H^i(X)^{\text{csp}} := \text{image of } H^i_c(X) \rightarrow H^i(X).
\]
It follows from general facts that all these cohomology groups are finite dimensional vector spaces over $\bar{Q}_\ell$. We also have a Poincaré duality isomorphism $H^i(X) \cong H^{2-i}(X)$.

By Remark 1.4 (v) we can write $X = C - D$, where $C$ is a smooth projective curve and $D = \bigcup_{\xi} D_\xi$ is a disjoint union of closed disks corresponding to the ends of $X$. This representation induces a long exact cohomology sequence
\[
\cdots \rightarrow H^1_c(X) \rightarrow H^1_c(C) \rightarrow \bigoplus_{\xi} H^1_c(D_\xi) \rightarrow \cdots \tag{1}
\]
Since $H^0_c(X) = H^1_c(D_\xi) = 0$ we obtain a short exact sequence
\[
0 \rightarrow B(X) \rightarrow H^1_c(X) \rightarrow H^1_c(C) \rightarrow 0, \tag{2}
\]
where the boundary module $B(X)$ is defined by the short exact sequence
\[
0 \rightarrow \bar{Q}_\ell^{\text{sa}(X)} \rightarrow \bar{Q}_\ell^{\partial X} \rightarrow B(X) \rightarrow 0.
\]
In particular, if $X$ is connected then $C$ is connected as well and we get
\[
\dim H^1_c(X) = 2g(C) + |\partial X| - 1.
\]
Taking the Poincaré dual of (1) we get the long exact sequence
\[
\cdots \rightarrow \bigoplus_{\xi} H^1(D_\xi) \rightarrow H^i(C) \rightarrow H^i(X) \rightarrow \cdots
\]
which induces a short exact sequence
\[
0 \rightarrow H^1(C) \rightarrow H^1(X) \rightarrow B(X)^* \rightarrow 0. \tag{3}
\]
Here $B(X)^*$ is the dual of $B(X)$. In particular, we obtain an isomorphism
\[
H^1(X)^{\text{csp}} \cong H^1(C).
\]
2.2 Let $X$ be an open analytic curve, and let $\mathcal{X}$ be a good formal model of $X$. Let $Z$ denote the reduction of $X$ with respect to $\mathcal{X}$. Let $f : \mathcal{X}' \to \mathcal{X}$ be an admissible blowup such that $\mathcal{X}'$ is a semistable model of $X$ (exists after enlarging the field of definition). Let $Z'$ be the reduction of $\mathcal{X}'$. This is a semistable curve over $k$. At each of the finitely many points $z \in Z$ where $f$ is not an isomorphism the inverse image $W_z := f^{-1}(z)$ is a connected union of irreducible components of $Z'$ and hence also a semistable curve over $k$.

Definition 2.1
(i) The model $\mathcal{X}$ is called almost semistable if the curves $W_z$ all have arithmetic genus zero.

(ii) The open analytic curve $X$ is said to have tree-like reduction if the graph of components of the semistable curve $Z'$ is a tree.

By [3] the model $X$ gives rise to a (derived) sheaf of vanishing cycles on $Z$, which we denote by $\mathbb{R}\psi_X$. The cohomology sheaves of $\mathbb{R}\psi_X$ are denoted by $R^i\psi_X$. By construction we have

$$(R^i\psi_X)_z = H^i(X_z),$$

where $X_z := \{z \in X \}$ is the formal fiber of a closed point $z \in Z$. (Since $X_z$ is again an open analytic curve, we see that the stalks of $R^i\psi_X$ are finite dimensional and that the passage from torsion coefficients to $\overline{\mathbb{Q}}_\ell$-coefficients is justified.) Since for any closed point $z \in Z$ the formal fiber $X_z$ is connected (see Remark 1.4 (iv)) we have $R^0\psi_X = \overline{\mathbb{Q}}_\ell$. Therefore, the spectral sequence $H^i(Z, R^j\psi_X) \Rightarrow H^{i+j}(X)$ gives rise to the exact sequence

$$0 \to H^1(Z) \to H^1(X) \to H^0(Z, R^1\psi_X) \xrightarrow{d} H^2(Z). \quad (4)$$

Proposition 2.2 The following two conditions are equivalent.

(i) The model $\mathcal{X}$ is almost semistable and $X$ has tree-like reduction.

(ii) The first map in the sequence (4) induces an isomorphism

$$H^1(Z) \cong H^1(X)^{\text{csp}}.$$  

Proof: Let $(Y', Z')$ be an algebraization of $\mathcal{X}'$ such that $Y'_s - Z'$ is a disjoint union of affine lines (see Remark 1.4 (v)). Then $X \subset Y'_s$ is the complement of closed disks and hence we have a natural isomorphism $H^1(X)^{\text{csp}} \cong H^1(Y'_s)$ (see the previous subsection). Since $Y'$ is semistable, the cospecialization map $H^1(Y_s) \to H^1(Y'_s)$ is an isomorphism if and only if the graph of components of $Y'_s$ is a tree. We conclude that the natural map $H^1(Z') \to H^1(X)$ induces an isomorphism $H^1(Z') \cong H^1(X)^{\text{csp}}$ if and only if $X$ has tree-like reduction.

By [3], Corollary 2.3 (ii), we have a natural isomorphism of derived sheaves

$$\mathbb{R}\psi_X \cong f_*\mathbb{R}\psi_{X'}. \quad (5)$$
In particular, we obtain a map between two exact sequences:

\[
\begin{array}{cccccc}
0 & \rightarrow & H^1(Z) & \rightarrow & H^1(X) & \rightarrow & H^0(Z, R^1\psi_X) \xrightarrow{d} H^2(Z) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^1(Z') & \rightarrow & H^1(X) & \rightarrow & H^0(Z', R^1\psi_{X'}) \xrightarrow{d} H^2(Z'). \\
\end{array}
\] (6)

In this diagram, the first, the second and the fourth vertical arrows are injective. The third vertical arrow is in general not injective; the isomorphism (5) induces an isomorphism

\[
\text{Ker}(H^0(Z, R^1\psi_X) \rightarrow H^0(Z', R^1\psi_{X'})) \cong \bigoplus_z H^1(W_z),
\] (7)

where \( z \in Z \) runs over the points where \( f \) is not an isomorphism. A simple diagram chase yields an isomorphism of (7) with the cokernel of the first vertical map in (6).

Now suppose that (ii) holds. Then the discussion in the first paragraph of this proof, together with the injectivity of \( H^1(Z) \rightarrow H^1(Z') \), shows that \( X \) has tree-like reduction. Furthermore, \( H^1(Z) \cong H^1(Z') \) and hence \( H^1(W_z) = 0 \) for all critical points \( z \). Since \( W_z \) is semistable, this means that all \( W_z \) have genus zero. We have proved that (ii) implies (i). The proof of the converse is similar.

\[\square\]

**Remark 2.3** Let \( X \) be a good formal model of \( X \) whose reduction \( Z \) is purely of dimension one. Then \( X \) is semistable if and only if for every closed point \( z \in Z \) we have

\[\dim(R^1\psi_X)_z \leq 1.\]

Indeed, the above condition implies that the formal fiber \( X_z \) is either a disk or an annulus.

### 3 Etale Galois covers

In this section we fix an open analytic curve \( Y \), a finite group \( G \) and an étale \( G \)-torsor \( f : X \rightarrow Y \). Let \( K \) denote the field of definition of \( Y \) (assumed to be sufficiently large). We assume that the auxiliary prime \( \ell \) chosen in the last section does not divide the order of \( G \).

#### 3.1 Algebraization

By Remark 1.4 (v) we can write \( Y = C - D \), where \( C \) is a smooth projective curve over \( K \) and \( D \) is an affinoid subdomain, isomorphic to a finite union of closed disks.

**Proposition 3.1** (i) After a finite extension of \( K \), the étale \( G \)-torsor \( f \) extends to a finite (possibly ramified) \( G \)-cover \( C' \rightarrow C \) of smooth projective curves over \( K \).
(ii) The rigid space $X$ is an open analytic curve.

**Proof:** Part (i) follows from a result of Garuti [11]. Part (ii) is an immediate consequence of (i), because the inverse image of the affinoid $D$ in $C'$ is again an affinoid.  

3.2 Formal models  A semistable (resp. a minimal) formal model of the $G$-torsor $f : X \to Y$ is given by a finite, $G$-invariant morphism of formal schemes $\mathcal{X} \to \mathcal{Y}$ extending $f$, where $\mathcal{X}$ and $\mathcal{Y}$ are semistable (resp. minimal) models of $X$ and $Y$.

We claim that there always exists a minimal and a semistable model of $f$. For the minimal model this is easy: the minimal model $X$ of $X$ exists and is unique (Proposition 1.6 (i)). Therefore, the $G$-action extends to $X$ and we can take $\mathcal{Y} := \mathcal{X}/G$. To obtain the semistable model, let us first assume that $X$ has a stable model $X$. Again we can use unicity to extend the $G$-action and set $\mathcal{Y} := \mathcal{X}/G$. That $\mathcal{Y}$ is semistable follows from [13], Proposition 5. If $X$ does not have a stable model then its connected components are open disks or annuli (Proposition 1.5). The claim in this case is left as an exercise.

Conversely, let us start with a good formal model $\mathcal{Y}$ of $Y$. We claim that there exists a unique formal model $\mathcal{X}$ which is normal and such that the $G$-torsor $f : X \to Y$ extends to a finite morphism $\mathcal{X} \to \mathcal{Y}$ and we have $\mathcal{Y} = \mathcal{X}/G$. Indeed, let $(Y, Z)$ be an algebraization of the formal model $\mathcal{Y}$ as in Remark 1.4 (v). By Proposition 3.1 (ii), the $G$-torsor $f$ extends to a finite cover $C' \to C = Y_\eta$ of smooth projective curves. Let $Y'$ be the normalization of $Y$ in $C'$ and let $Z' \subset Y'$ denote the inverse image of $Z \subset Y_\eta$. Then $(Y', Z')$ is an algebraic model of $X$ and gives rise to the desired formal model $\mathcal{X}$. To show that $\mathcal{X}$ is unique use [9], Appendix A.

In general, the formal model $\mathcal{X}$ will not be good because its special fiber may not be reduced. However, after replacing the field of definition $K$ by a finite extension, we may assume that $\mathcal{X}_s$ is reduced and so $\mathcal{X}$ is good. This follows easily from the Reduced Fiber Theorem of Grauert and Remmert ([5], §6.4.1). By Remark 1.4 (iii) the formation of $\mathcal{X}$ is then stable under further extension of $K$. Hence the definition of the normalization is compatible with our philosophy of keeping the field of definition $K$ variable and sufficiently large.

3.3 Decomposition and inertia groups  Let $U \subset Y$ be an affinoid subdomain. The canonical reduction of $U$ is an affine curve over $k$ which we denote by $\bar{U}$. By the Reduced Fiber Theorem [5] we may and will assume that $\bar{U}$ is reduced (after replacing the field of definition by a suitable finite extension). We say that $U$ has good (resp. irreducible) reduction if $\bar{U}$ is smooth over $k$ (resp. irreducible).

Let $U \subset Y$ be an affinoid with good and irreducible reduction. Choose a connected component $V$ of the inverse image $f^{-1}(U)$. Clearly, $V$ is an affinoid subdomain of $X$. We say that $U$ has good (resp. irreducible) reduction in $X$ if $V$ has good (resp. irreducible) reduction. We note that the canonical reduction
\( \bar{V} \) is connected because \( V \) is connected. Therefore, if \( U \) has good reduction in \( X \) then it also has irreducible reduction in \( X \).

Suppose that \( U \) has irreducible reduction in \( X \). The stabilizer in \( G \) of the connected component \( V \) is denoted by \( G(U) \) and is called the decomposition group of \( U \) (it is independent of the choice of \( V \), up to conjugation in \( G \)). The inertia group \( I(U) \triangleleft G(U) \) of \( U \) is defined as the kernel of the natural homomorphism
\[
G(U) \longrightarrow \text{Aut}_k(\bar{V}).
\]

**Proposition 3.2** Let \( U \subset Y \) be an affinoid with good and irreducible reduction. Let \( f : X \to Y \) be an \'{e}tale \( G \)-torsor.

(i) Suppose that \( U \) has irreducible reduction in \( X \). Choose a connected component \( V \subset f^{-1}(U) \) with stabilizer \( G(U) \) and set \( V' := V/I(U) \). Then the natural map
\[
\bar{V}' \to \bar{U}
\]
is an \'{e}tale Galois cover of smooth affine curves over \( k \), with Galois group \( G(U)/I(U) \). Furthermore, the natural map
\[
\bar{V} \to \bar{V}'
\]
is finite and radicial of degree \(|I(U)|\). Since \( \bar{V} \) is reduced it follows that \( I(U) \) is a \( p \)-group (where \( p \) is the characteristic of \( k \)).

(ii) Suppose that \( G \) is a \( p \)-group. Then \( U \) has irreducible reduction in \( X \).

**Proof:** Part (i) is a consequence of the Purity Theorem of Zariski-Nagata and the assumption that \( f \) is \'{e}tale. See e.g. [14], §2.4. Under the assumption that \( G \) is a \( p \)-group it is proved in [13] that the singularities of \( \bar{V} \) are unibranched. Therefore, connectedness of \( \bar{V} \) implies its irreducibility. \( \square \)

Let \( \xi \in \partial Y \) be an end of \( Y \). Choose an end \( \xi' \) of \( X \) lying above \( \xi \) and let \( G(\xi) \subset G \) denote the stabilizer of \( \xi' \) (which depends, up to conjugation in \( G \), only on \( \xi \)). We call \( G(\xi) \) the decomposition group of \( \xi \) in \( X \). The inertia group \( I(\xi) \triangleleft G(\xi) \) of \( \xi \) is the subgroup of elements which act trivially on \( \xi \) (see §1.3).

### 3.4 Vanishing cycle sheaves for nonconstant coefficients

Let \( \tau : G \to \text{GL}(W) \) be a representation of \( G \) on a finite dimensional \( \bar{Q}_\ell \)-vector space \( W \). Given a subgroup \( H \subset G \) which acts on a \( \bar{Q}_\ell \)-vector space \( W' \), we set \( W''[\tau] := \text{Hom}_H(W,W') \) (which subgroup \( H \) we mean should always be clear from the context). Extending this definition from \( \bar{Q}_\ell \)-vector spaces to sheaves, we obtain an exact functor \( \mathcal{F} \mapsto (f_*\mathcal{F})[\tau] \) from \( \bar{Q}_\ell \)-sheaves on \( X \) with \( G \)-action to \( \bar{Q}_\ell \)-sheaves on \( Y \). (To construct this functor, one has to choose a finite extension \( E/\bar{Q}_\ell \) and a projective \( \mathcal{O}_E[G] \)-module \( M \) such that \( W = M \otimes \bar{Q}_\ell \), and define everything first for \( \mathcal{O}_E/\ell^n \)-sheaves. Since we assume that \( \ell \) does not divide the order of \( G \), this poses no problem.) As a special case of this construction we set
\[
\mathcal{F}_\tau := (f_*\bar{Q}_\ell)[\tau].
\]
General arguments show that

\[ H^i(X, \overline{\mathbb{Q}}_\ell)[\tau] = H^i(Y, \mathcal{F}_\tau). \]

A similar equality holds for cohomology with support and for the cuspidal part

\[ H^i(X, \mathcal{F}_\tau)^{\text{cusp}} \quad (\text{which is defined as the image of the map } H^i_c(Y, \mathcal{F}_\tau) \rightarrow H^i(Y, \mathcal{F}_\tau)). \]

Let \( U \subset Y \) be an affinoid subdomain. By \([3]\) \( \mathcal{F}_\tau \) gives rise to a derived sheaf of vanishing cycles \( R\psi \mathcal{F}_\tau |_{\overline{U}} \) on the canonical reduction \( \overline{U} \) such that

\[ H^i_U(Y, \mathcal{F}_\tau) \cong H^i_c(\overline{U}, R\psi \mathcal{F}_\tau |_{\overline{U}}). \] (9)

In particular, we obtain an exact sequence

\[ 0 \rightarrow H^1_c(\overline{U}, R^0\psi \mathcal{F}_\tau |_{\overline{U}}) \rightarrow H^1_U(Y, \mathcal{F}_\tau) \rightarrow H^0_c(\overline{U}, R^1\psi \mathcal{F}_\tau |_{\overline{U}}) \rightarrow H^2_c(\overline{U}, R^0\psi \mathcal{F}_\tau |_{\overline{U}}). \] (10)

Let \( V \subset X \) be a connected component of \( f^{-1}(U) \). Let \( \varphi : \overline{V} \rightarrow \overline{U} \) denote the finite morphism induced by the natural map \( V \rightarrow U \). Using \([3]\), Corollary 2.3 (ii) and the fact that \( \varphi_* \) is exact, we obtain a canonical isomorphism

\[ \mathbb{R}\psi \mathcal{F}_\tau |_{\overline{U}} \cong \varphi_*(\mathbb{R}\psi \overline{\mathbb{Q}}_\ell |_{\overline{V}})[\tau]. \] (11)

In particular, we have a canonical isomorphism

\[ R^0\psi \mathcal{F}_\tau |_{\overline{U}} \cong (\varphi_* \overline{\mathbb{Q}}_\ell)[\tau]. \] (12)

**Definition 3.3** Let \( U \subset Y \) be an affinoid subdomain. We say that the sheaf \( \mathcal{F}_\tau \) is residual over \( U \) if the first map in (10) is an isomorphism,

\[ H^1_c(\overline{U}, R^0\psi \mathcal{F}_\tau |_{\overline{U}}) \cong H^1_U(Y, \mathcal{F}_\tau). \]

We say that \( \mathcal{F}_\tau \) is resolved over \( U \) if it is residual over \( U \) and the natural map

\[ H^1_U(Y, \mathcal{F}_\tau) \rightarrow H^1(Y, \mathcal{F}_\tau) \]

is surjective.

It is of course a nontrivial problem find an affinoid over which the sheaf \( \mathcal{F}_\tau \) is resolved. We limit the discussion of this problem to the following proposition which gives a criterion for \( \mathcal{F}_\tau \) to be residual, and we refer the reader to \([15]\) for concrete and nontrivial examples.

**Proposition 3.4** The sheaf \( \mathcal{F}_\tau \) is residual over \( U \) if either one of the following conditions holds.

(i) The affinoid \( U \) has good reduction in \( X \).

(ii) The affinoid \( U \) has irreducible reduction, and the restriction of \( \tau \) to the inertia group \( I(U) \subset G \) is trivial.
Proof: If $U$ has good reduction in $X$ then the vanishing cycle sheaves $R^i\psi F_\tau|_{\bar{U}}$ are zero for $i > 1$, by (11). Therefore, $F_\tau$ is residual over $U$.

Now assume that Condition (ii) holds. Let $V \subset X$ be a connected component of $f^{-1}(U)$ with decomposition group $G(U) \subset G$. It follows from [1], §5, that there exists an open analytic curve $Y_1 \subset Y$ containing $U$ such that $f^{-1}(Y_1)$ has a unique connected component $X_1 \subset X$ containing $V$. Set $X' := X_1/I(U)$ and $V' := V/I(U)$. Clearly, $f' : X' \to Y_1$ is an étale $G(U)/I(U)$-torsor. Our assumption says that the restriction of $\tau$ to $G(U)$ comes from a representation $\tau'$ of $G(U)/I(U)$. Hence the restriction of $F_\tau$ to $Y_1$ is isomorphic to the sheaf $F_{\tau'} := (f'|\mathbb{Q}_\ell)[\tau']$. By excision we obtain an isomorphism

$$H^1_U(Y,F_\tau) \cong H^1_U(Y_1,F_{\tau'}).$$

Furthermore, it follows from Proposition 3.2 (i) that $U$ has good reduction in $X'$. Therefore, we can use the first case of the proposition which is already proved. 

3.5 An equivariant criterion for almost semistability

Let $\mathcal{Y}$ be a semistable formal model of $Y$. We define the formal model $\mathcal{X}$ of $X$ as the normalization of $\mathcal{Y}$ in $X$ (see §3.2). We would like to have a criterion that ensures that $\mathcal{X}$ is semistable. We will denote the reduction of $\mathcal{X}$ (resp. of $\mathcal{Y}$) by $\bar{Z}$ (resp. $Z'$).

**Definition 3.5** Let $U \subset \mathcal{Y}$ be an affinoid subdomain. We say that $U$ is supported by the semistable model $\mathcal{Y}$ if there exists an open subset $W \subset Z'$ such that $U =]W[\mathcal{Y}$.

Suppose that $U$ is supported by $\mathcal{Y}$, and let $W \subset Z'$ be as in the definition. Let $\mathcal{U}$ be the canonical integral model of $U$ (with special fiber $\bar{U}$) and $W \subset \mathcal{Y}$ the open formal subscheme whose underlying topological space is $W$. Since $W$ is a normal model of the affinoid $U$, there exists a unique morphism of formal schemes $W \to \mathcal{U}$ extending the identity on the generic fiber $U$. In fact, the morphism $W \to \mathcal{U}$ is an admissible blowup. On the special fiber we obtain a proper and surjective morphism $W \to \bar{U}$ which is an isomorphism over some dense open subset of $\bar{U}$. See [6].

**Remark 3.6** The following facts are easy consequences of Definition 3.5 and the discussion following it.

(i) If $U$ has good reduction and is supported by $\mathcal{Y}$ then the morphism $W \to \bar{U}$ has a section. In particular, we have a canonical locally closed embedding $\bar{U} \hookrightarrow Z'$.

(ii) Suppose $U$ is supported by $\mathcal{Y}$. Let $V \subset X$ be a connected component of $f^{-1}(U)$. Then the affinoid $V$ is supported by the model $\mathcal{X}$.

(iii) Given a finite family of affinoids $U_i \subset \mathcal{Y}$ which meets every connected component of $\mathcal{Y}$, there exists a minimal semistable model $\mathcal{Y}$ of $Y$ which supports all $U_i$. 

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Proposition 3.7 Suppose that for every irreducible $G$-representation $\tau$ such that

$$H^1(Y, F_\tau)^{\text{csp}} \neq 0$$

there exists an affinoid $U_\tau \subset Y$ over which $F_\tau$ is resolved. Let $\mathcal{Y}$ be a semistable model of $Y$ which supports $U_\tau$ for every $\tau$. Let $X$ be the normalization of $\mathcal{Y}$ in $X$. Then $X$ is almost semistable. Moreover, $X$ has tree-like reduction.

**Proof:** Let $\mathcal{Y}$ and $X$ be the formal models from the statement of the proposition. Let $Z$ denote the reduction of $X$. By Proposition 2.2 it suffices to show that we have an isomorphism

$$H^1(Z) \cong H^1(X)^{\text{csp}}.$$  

Actually, since the natural map $H^1(Z) \to H^1(X)$ is injective, it suffices to show that $H^1(Z)$ maps onto $H^1(X)^{\text{csp}}$. The group $G$ acts on both these vector spaces, hence it suffices to show that for every irreducible $G$-representation $\tau$ the map $H^1(Z) \to H^1(X)$ induces a surjective map

$$H^1(Z)[\tau] \to H^1(X)^{\text{csp}}[\tau] = H^1(Y, F_\tau)^{\text{csp}}. \quad (13)$$

We may assume that the right hand side of (13) is not zero. Therefore, our hypothesis says that there exists an affinoid $U \subset Y$ which is supported by $\mathcal{Y}$ and over which the sheaf $F_\tau$ is resolved. Let $V \subset X$ be a connected component of $f^{-1}(U)$. By Remark 3.6 (ii) there exists an open subset $W \subset Z$ and a proper surjective map $W \to V$. We thus obtain natural maps

$$H^1(V) \to H^1(W) \to H^1(Z).$$

By (12), the composition of these two maps induces the left vertical arrow in the following diagram.

$$
\begin{array}{ccc}
H^1(U, R^0\psi F_\tau|_U) & \xrightarrow{\cong} & H^1(Y, F_\tau) \\
\downarrow & & \downarrow \\
H^1(Z)[\tau] & \longrightarrow & H^1(Y, F_\tau)^{\text{csp}}
\end{array}
\quad (14)
$$

The upper horizontal arrow is an isomorphism because $F_\tau$ is residual over $U$. The right vertical arrow is surjective because $F_\tau$ is resolved over $U$. It follows that the lower horizontal arrow is also surjective. This finishes the proof of the proposition. \hfill \Box

**Remark 3.8** The criterion given by Proposition 3.7 is sharp, in the following sense. Suppose that $X$ has tree-like reduction. Then there exists an affinoid $U \subset Y$ such that $F_\tau$ is resolved over $U$, for all $G$-representations $\tau$.

To construct $U$, let $X \to \mathcal{Y}$ be a semistable model of $f : X \to Y$ and let $Z$ denote the reduction of $X$. The complement

$$W := Z - \partial Z - Z^{\text{sing}}$$
is an affine and dense open subset. One checks that the natural map $H^1_c(W) \to H^1(Z)$ is surjective. Set $V := |W|_\mathcal{X}$ and set $U := f(V)$. By construction, $U$ is an affinoid over which $\mathcal{F}_\tau$ is resolved for every $G$-representation $\tau$.

It is clear that in practice this construction is not very useful to determine a stable model of $f$. The point of Proposition 3.7 is rather that it suffices to look for a suitable affinoid for each irreducible representation $\tau$ at a time. Also, such an affinoid may be much simpler, and neither include nor be included in the affinoid $U$ constructed above.

### 3.6 Étale covers of the disk

In this final subsection we assume that $Y$ is an open disk. In this case we can describe the semistable model $\mathcal{Y}$ in terms of a tree of disks. By a closed disk we shall mean an affinoid subdomain of $Y$ which is isomorphic (possibly after enlarging the field of definition) to the closed unit disk.

Let $S$ be a nonempty finite collection of closed disks. For every nonempty subset $T \subset S$ there exists a closed disk $D_T \subset Y$ which is minimal with the property that $D \subset D_T$ for every $D \in T$. We say that $S$ is closed if $D_T \in S$ for every nonempty subset $T \subset S$. It is easy to see that for every finite collection of closed disks $S$ there is a minimal finite collection of closed disks $\overline{S}$ which contains $S$ and is closed. We call $\overline{S}$ the closure of $S$.

Suppose that $S$ is closed. We will associate to $S$ a directed graph $\Gamma = \Gamma_S$, as follows. Write $S = \{D_v \mid v \in V\}$ for an index set $V$, and consider the elements of $V$ as vertices of $\Gamma$. We add to $\Gamma$ a distinguished vertex $v_0$, called the boundary. For every $v \in V$, the disk $D_v$ is either maximal among the disks in $S$, or there exists $w \in V$ such that the disk $D_w \in \overline{S}$ is minimal among all disks in $S$ strictly containing $D_v$. In the first case, we add to $\Gamma$ the edge $(v_0, v)$. In the second case, we add the edge $(w, v)$. It is clear that $\Gamma$ is a rooted and connected tree and that the boundary $v_0$ is the root. The datum $(\Gamma; D_v)$ is called a tree of disks in $Y$. One easily shows:

**Proposition 3.9** For every tree of disks $(\Gamma; D_v)$ there exists a semistable model $\mathcal{Y}$ of $Y$ which is minimal with the property that every disk $D_v$ is supported by $\mathcal{Y}$. Let $\mathcal{Z}'$ denote the reduction of $\mathcal{Y}$. Then the graph of components of the semistable curve $\mathcal{Z}'$ is naturally isomorphic to the tree $\Gamma$ (the boundary $v_0$ corresponds to the virtual component $T_\xi$, where $\xi$ is the unique end of $\mathcal{Y}$). Furthermore, every semistable model of $\mathcal{Y}$ arises in this way from a unique tree of disks.

Let us fix a tree of disks $(\Gamma; D_v)$, and let $\mathcal{Y}$ be the corresponding semistable model. Let $f : X \to Y$ be the étale $G$-cover fixed at the beginning of this section. Let $\mathcal{X}$ be the normalization of $\mathcal{Y}$ in $X$. We obtain the following reformulation of Proposition 3.7.

**Proposition 3.10** Suppose that for every irreducible $G$-representation $\tau$ such that

$$H^1(\mathcal{Y}, \mathcal{F}_\tau) \neq 0$$
there exists a set $V' \subset V$ of vertices of $\Gamma$ such that $F_\tau$ is resolved over the affinoid
\[ U := \bigcup_{v \in V'} D_v. \]
Then the formal model $X$ is almost semistable. Moreover, $X$ has tree-like reduction.

**Remark 3.11** In contrast to its ancestor, Proposition 3.7, this criterion is not sharp at all. In fact, given an étale $G$-Galois cover of the disk with tree-like reduction, there is no reason to expect that one can always resolve the sheaf $F_\tau$ over a union of closed disks. Nevertheless, this is true in important examples, for instance for the Lubin-Tate spaces studied in [15].

For the rest of this section we assume that the conclusion of Proposition 3.10 holds. We wish to give a further criterion which ensures that the model $X$ is semistable (and not just almost semistable).

We fix the following notation. Let $Z'$ denote the reduction of $Y$ and $Z$ the reduction of $X$. Let $e = (v_1, v_2)$ be an edge of the graph $\Gamma$. To $e$ corresponds a closed point $z' \in Z'$ such that $A_e := ]z'|_Y$ is an open annulus. If $v_1 \neq v_0$ then $z'$ is an ordinary double point of $Z'$ and $A_e = D_e - D_{v_2}$, where $D_e \subset D_{v_1}$ is the formal fiber containing $D_{v_2}$ (an open disk). If $v_1 = v_0$ then $z'$ is the unique element of the boundary $\partial Z'$ of $Z'$. In this case we have $A_e = Y - D_{v_2}$ and we set $D_e := Y$. For $i = 1, 2$ let $\xi_i'$ be the end of $A_e$ corresponding to the vertex $v_i$.

We choose a point $z \in Z$ lying over $z'$ and set $X_z := ]z|[X]$. Note that $X_z$ is a connected component of the inverse image of the annulus $A_e$. Choose, for $i = 1, 2$, an end $\xi_i \in \partial X_z$ lying over $\xi_i'$. Since the formal model $X$ is almost semistable, $X_z$ is an open analytic curve of genus zero. We want to show that $X_z$ is actually an open annulus. The last part of the following proposition gives a criterion when this is true.

**Proposition 3.12** Suppose that $G$ is a $p$-group. Then the following holds.

(i) The closed disks $D_v \subset Y$ have irreducible reduction in $X$. In particular, the decomposition groups $G(D_v) \subset G$ and inertia groups $I(D_v) \triangleleft G(D_v)$ are well defined.

(ii) Up to conjugation in $G$ we have
\[ G(D_{v_2}) = G(\xi_2) \subset I(\xi_1) = I(D_{v_1}). \]

(iii) The open curve $X_z$ is an open annulus if and only if
\[ G(D_{v_2}) = I(D_{v_1}). \tag{15} \]
In particular, if (15) holds for all edges $e = (v_1, v_2)$ then the formal model $X$ is semistable.
Proof: Part (i) is a special case of Proposition 3.2 (ii). Part (ii) follows from the following lemma, applied to the Galois cover \( C_e \to D_e \), where \( D_e \subset D_{v_1} \) is the residue class containing the disk \( D_{v_2} \) if \( v_1 \neq v_0 \) (and \( D_e := Y \) otherwise) and \( C_e \) is a connected component of \( f^{-1}(D_e) \). Part (iii) is now clear. \( \square \)

Lemma 3.13 Let \( G \) be a \( p \)-group, \( Y \) an open disk and \( f : X \to Y \) an étale \( G \)-torsor. Suppose that \( X \) is connected. Then the following holds.

(i) The open analytic curve \( X \) has a unique end.

(ii) Let \( D \subset Y \) be a closed disk. Set \( A := Y - D \) and let \( \xi_1, \xi_2 \) denote the two ends of the open annulus \( A \). We assume that \( \xi_1 \) corresponds to the unique end of \( Y \). Then we have (up to conjugation in \( G \)):

\[
G = I(\xi_1) \supset G(\xi_2) = G(D).
\]

Proof: (compare with [13]) Let \( D \subset Y \) be a closed disk and \( Y' \) the minimal semistable model of \( Y \) supporting \( D \). Let \( Z' \) denote the reduction of \( Y \) and \( z' \in \partial Z' \) the unique boundary point. Then \( Z' \cong \mathbb{P}^1_k \), and \( Z' - \{ z' \} \cong \mathbb{A}^1_k \) is the canonical reduction of the disk \( D \). Let \( X \) be the normalization of \( Y \) in \( Y' \) and \( Z \) the reduction of \( X \). Let \( C \subset f^{-1}(D) \) be a connected component. Then the canonical reduction \( \tilde{C} \) of the affinoid \( C \) can be identified with an open subset of \( Z \). By Proposition 3.2 the curve \( \tilde{C} \) is irreducible. Furthermore, the map \( \tilde{C} \to \tilde{D} = Z' - \{ z' \} \) factors as the composition of a finite radicial map \( \tilde{C} \to \tilde{C}' \) and an étale Galois cover \( \tilde{C}' \to \tilde{D} \cong \mathbb{A}^1_k \) whose Galois group is a \( p \)-group. A well known lemma says that such a cover of the affine line is totally ramified over infinity. (One uses the fact that any proper subgroup of a \( p \)-group is contained in a proper normal subgroup.) We conclude that there exists a unique point \( z \in Z \) which is mapped to \( z' \in Z' \) and lies in the closure of \( \tilde{C} \). Furthermore, there exists a unique branch of \( Z \) through the point \( z \) whose generic point lies on \( \tilde{C} \). Since the ends of \( f^{-1}(A) \) lying above \( \xi_2 \) are in natural bijection with the branches of \( Z \) through \( z \), the equality

\[
G(\xi_2) = G(D)
\]

follows immediately.

Let \( X' \to Y' \) be a semistable model of \( f : X \to Y \) and apply the previous argument to the largest disk \( D \) which is supported by the model \( Y' \). Then by construction, \( f^{-1}(D) \) is an underlying affinoid of \( X \). In particular, \( f^{-1}(D) \) is connected and \( f^{-1}(A) \) is a disjoint union of open annuli. Now \( (16) \) implies that \( f^{-1}(A) \) is connected and that

\[
G = G(\xi_1).
\]

Part (i) of the proposition follows. To finish the proof of (ii) we have to show that \( I(\xi_1) = G(\xi_1) \). After dividing out by \( I(\xi_1) \) (which is a normal subgroup of \( G = G(\xi_1) \)) we may assume that \( I(\xi_1) = 1 \). Applying purity of branch locus to the minimal formal model \( X'' \to Y'' \) \( = \text{Spf} \mathcal{O}[[t]] \) of \( f : X \to Y \), we conclude that \( G = 1 \). This finishes the proof of the proposition. \( \square \)
Corollary 3.14 Let $G$ be a finite $p$-group and $f: X \to Y = D(0,1)$ be an étale Galois cover of the open unit disk. Then

$$H^1(X) = H^1_c(X) = H^1(X)^{sp}.$$ 

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