ESTIMATION IN FUNCTIONAL REGRESSION FOR GENERAL EXPONENTIAL FAMILIES

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This paper studies a class of exponential family models whose canonical parameters are specified as linear functionals of an unknown infinite-dimensional slope function. The optimal minimax rates of convergence for slope function estimation are established. The estimators that achieve the optimal rates are constructed by constrained maximum likelihood estimation with parameters whose dimension grows with sample size. A change-of-measure argument, inspired by Le Cam’s theory of asymptotic equivalence, is used to eliminate the bias caused by the nonlinearity of exponential family models.

1. Introduction. There has been extensive exploratory and theoretical study of functional data analysis (FDA) over the past two decades. Two monographs by Ramsay and Silverman (2002, 2005) provide comprehensive discussions on the methods and applications.

Among many problems involving functional data, slope estimation in functional linear regression has received substantial attention in literature; for example, by Cardot, Ferraty and Sarda (2003), Li and Hsing (2007), and Hall and Horowitz (2007). In particular, Hall and Horowitz (2007) established minimax rates of convergence and proposed rate-optimal estimators based on spectral truncation (regression on functional principal components). They showed that the optimal rates depend on the smoothness of the slope function and the decay rate of the eigenvalues of the covariance kernel.

In this paper, we study optimal rates of convergence for slope estimation in functional generalized linear models, for which little theory is available. We introduce several new technical devices to overcome the problems caused by nonlinearity of the link function. To analyze our estimator, we establish a sharp approximation for maximum likelihood estimators for exponential families parametrized by linear functions of \(m\)-dimensional parameters, for an \(m\) that grows with sample size (see Lemma 1). We develop a change-of-measure argument—inspired by ideas from

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Le Cam’s theory of asymptotic equivalence of models—to eliminate the effect of bias terms caused by the nonlinearity of the link function (see Section 3.3 and 3.4).

More precisely, we consider problems where the observed data consist of independent, identically distributed pairs \((y_i, X_i)\) where each \(X_i\) is a Gaussian process indexed by a compact subinterval of the real line, which with no loss of generality we take to be \([0, 1]\). We denote the corresponding norm and inner product in the space \(L^2[0, 1]\) by \(\| \cdot \|\) and \(\langle \cdot, \cdot \rangle\).

We assume, for each \(i\), that the random variable \(y_i\) conditional on the process \(X_i\), follows a distribution \(Q_{\lambda_i}\), where \(\{Q_{\lambda} : \lambda \in \mathbb{R}\}\) is a one-parameter exponential family. We take parameter \(\lambda_i\) to be a linear functional of \(X_i\) of the form

\[
\lambda_i = a + \int_0^1 X_i(t) B(t) \, dt
\]

for an unknown constant \(a\) and an unknown \(B \in L^2[0, 1]\).

We focus on estimation of \(B\) using integrated squared error loss:

\[
L(B, \hat{B}_n) = \|B - \hat{B}_n\|^2 = \int_0^1 (B(t) - \hat{B}_n(t))^2 \, dt.
\]

Our models are indexed by parameters \(f = (K, a, \mu, B)\), where \(\mu\) is the mean and \(K\) is the covariance kernel of the Gaussian process. The universal constant \(\alpha\) controls the decay rate of eigenvalues of kernel \(K\) and \(\beta\) characterizes the ‘smoothness’ of the slope function \(B\). See Definition 1 (in Section 2) for the precise specification of the parameter set \(F = F(R, \alpha, \beta)\). The two main results are as follows.

**Theorem 1. (Minimax Upper Bound)** Under the assumptions stated in Section 2, there exists an estimating sequence of \(\hat{B}_n\)’s for which: for each \(\epsilon > 0\) there exists a finite constant \(C_\epsilon\) such that

\[
\sup_{f \in F} P_n, f \left\{ \|B - \hat{B}_n\|^2 > C_\epsilon n^{(1-2\beta)/(\alpha+2\beta)} \right\} < \epsilon \quad \text{for large enough } n.
\]

**Theorem 2. (Minimax Lower Bound)** Under the assumptions stated in Section 2,

\[
\liminf_{n \to \infty} n^{(2\beta-1)/(\alpha+2\beta)} \sup_{f \in F} P_n, f \|B - \hat{B}_n\|^2 > 0 \quad \text{for every estimator } \{\hat{B}_n\}.
\]

Two closely related works in functional data analysis are Cardot and Sarda (2005) and Müller and Stadtmüller (2005), which provided theory for the functional generalized linear model, including the rates of convergence for prediction in the random design case. However, the rate optimalities were not studied. In addition,
Müller and Stadtmüller (2005) established an upper bound for rates of convergence assuming the negligibility of the bias due to the approximation of the infinite-dimensional model by a sequence of finite-dimensional models, the issue we overcome by using a change-of-measure argument. In the functional linear regression setting, Cai and Hall (2006) and Crambes, Kneip and Sarda (2009) derived optimal rates of convergence for prediction in the fixed and random design cases. See also, Cardot, Mas and Sarda (2007) which derived a CLT for prediction in the fixed and random design cases and Cardot and Johannes (2010) which established a minimax optimal result for prediction at a random design using thresholding estimators. In a companion study to our paper, Dou (2010, Chapter 5) considers optimal prediction in functional generalized linear regressions with an application to the economic problem of predicting occurrence of recessions from the U.S. Treasury yield curve.

Our minimax upper bound result (Theorem 1) is proved in Section 3. The minimax lower bound result (Theorem 2) is established in Section 4. The proof of Theorem 1 depends on an approximation result (Lemma 1) for maximum likelihood estimators in exponential family models for parameters whose dimensions change with sample size. As an aid to the reader, we present our proof of Theorem 1 in two stages. In Section 3.3, we assume that both the mean $\mu$ and the covariance kernel $K$ are known. This allows us to emphasize the key ideas in our proofs without the many technical details that need to be handled when $\mu$ and $K$ are estimated in the natural way. Many of those details, as summarized in Lemma 5, involve the spectral theory of compact operators. We proceed in Section 3.4 to the case where $\mu$ and $K$ are estimated. The proofs for the lemmas are collected together in Section 5. Some of them invoke the perturbation-theoretic results collected in the supplemental Appendix.

2. Regularity conditions. Let $\{Q_\lambda : \lambda \in \mathbb{R}\}$ be a one-parameter exponential family,

\begin{equation}
\frac{dQ_\lambda}{dQ_0} = f_\lambda(y) := \exp(\lambda y - \psi(\lambda)) \quad \text{for all } \lambda \in \mathbb{R}.
\end{equation}

Necessarily $\psi(0) = 0$. Remember that $e^{\psi(\lambda)} = Q_0 e^{\lambda y}$ and that the distribution $Q_\lambda$ has mean $\psi(\lambda)$ and variance $\ddot{\psi}(\lambda)$.

**Remark.** We may assume that $\ddot{\psi}(\lambda) > 0$ for every real $\lambda$. Otherwise we would have $0 = \psi(\lambda_0) = \text{var}_{\lambda_0}(y) = Q_0 f_{\lambda_0}(y)(y - \psi(\lambda_0))^2$ for some $\lambda_0$, which would make $y = \psi(\lambda_0)$ for $Q_0$ almost all $y$ and $Q_\lambda \equiv Q_{\lambda_0}$ for every $\lambda$.

We assume:

(\dot{\psi}) For each $\epsilon > 0$ there exists a finite constant $C_\epsilon$ for which $\dot{\psi}(\lambda) \leq C_\epsilon \exp(\epsilon \lambda^2)$ for all $\lambda \in \mathbb{R}$. Equivalently, $\dot{\psi}(\lambda) \leq \exp \left( o(\lambda^2) \right)$ as $|\lambda| \to \infty$. 

There exists an increasing real function $G$ on $\mathbb{R}^+$ such that
\[
|\psi(\lambda + h)| \leq \psi(\lambda)G(|h|) \quad \text{for all } \lambda \text{ and } h.
\]

Without loss of generality we assume $G(0) \geq 1$.

As shown in Section 5.3, the assumption (\(\psi\)) implies that
\[
(3) \quad h^2(Q_\lambda, Q_{\lambda + \delta}) \leq \delta^2 \psi(\lambda) (1 + |\delta|) G(|\delta|) \quad \text{for all } \lambda, \delta \in \mathbb{R},
\]
which plays a key role in analyzing both upper and lower bounds.

We assume the observed data are iid pairs $(y_i, X_i)$ for $i = 1, \ldots, n$, where:

\begin{itemize}
  \item[(X)] Each $\{X_i(t) : 0 \leq t \leq 1\}$ is distributed like $\{X(t) : 0 \leq t \leq 1\}$, a Gaussian process with mean $\mu(t)$ and covariance kernel $K(s, t)$.
  \item[(Y)] $y_i | X_i \sim Q_{\lambda_i}$ with $\lambda_i = a + \langle X_i, B \rangle$ for an unknown $\{B(t) : 0 \leq t \leq 1\}$ in $L^2[0, 1]$ and $a \in \mathbb{R}$.
\end{itemize}

**Definition 1.** For real constants $\alpha > 1$ and $\beta > (\alpha + 3)/2$ and $R > 0$, define $\mathcal{F} = \mathcal{F}(R, \alpha, \beta)$ as the set of all $f = (K, a, \mu, \mathbb{B})$ that satisfy the following conditions.

\begin{itemize}
  \item[(K)] The covariance kernel is square integrable with respect to Lebesgue measure and has an eigenfunction expansion (as a compact operator on $L^2[0, 1]$)
  \[
  K(s, t) = \sum_{k \in \mathbb{N}} \theta_k \phi_k(s)\phi_k(t)
  \]
  where the eigenvalues $\theta_k$ are decreasing with $Rk^{-\alpha} \geq \theta_k \geq \theta_{k+1} + (\alpha/R)k^{-\alpha-1}$.
  \item[(a)] $|a| \leq R$
  \item[(\(\mu\))] $\|\mu\| \leq R$
  \item[(\(\mathbb{B}\))] $\mathbb{B}$ has an expansion $\mathbb{B}(t) = \sum_{k \in \mathbb{N}} b_k \phi_k(t)$ with $|b_k| \leq Rk^{-\beta}$, for the eigenfunctions defined by the kernel $K$.
\end{itemize}

**Remarks.** The awkward lower bound for $\theta_k$ in Assumption (K) implies, for all $k < j$,
\[
(4) \quad \theta_k - \theta_j \geq R^{-1} \int_k^j \alpha x^{-\alpha-1} \, dx = R^{-1} \left( k^{-\alpha} - j^{-\alpha} \right).
\]

If $K$ and $\mu$ were known, we would only need the lower bound $\theta_k \geq R^{-1}k^{-\alpha}$ and not the lower bound for $\theta_k - \theta_{k+1}$. As explained by Hall and Horowitz (2007, page 76), the stronger assumption is needed when one estimates the individual eigenfunctions of $K$. Note that the subset of $L^2[0, 1]$ in which $\mathbb{B}$ lies, denoted as $\mathcal{B}_K$, depends on $K$. We regard the need for the stronger assumption on the eigenvalues and the irksome Assumption (\(\mathbb{B}\)) as artifacts of the method of proof, but we have not yet succeeded in removing either assumption.
More formally, we write \( P_{\mu, K} \) for the distribution (a probability measure on the space \( L^2[0, 1] \)) of each Gaussian process \( X_i \). The joint distribution of \( X_1, \ldots, X_n \) is then \( P_{n, \mu, K} = P_{\mu, K}^n \). We identify the \( y_i \)'s with the coordinate maps on \( \mathbb{R}^n \) equipped with the product measure \( \mathbb{Q}_{n, a, B, X_1, \ldots, X_n} := \otimes_{i \leq n} \mathbb{Q}_{\lambda_i} \), which can also be thought of as the conditional joint distribution of \((y_1, \ldots, y_n)\) given \((X_1, \ldots, X_n)\). Thus the \( P_{n,f} \) in Theorems 1 and 2 can be rewritten as an iterated expectation,

\[
P_{n,f} = P_{n, \mu, K} \mathbb{Q}_{n, a, B, X_1, \ldots, X_n},
\]

the second expectation on the right-hand side averaging out over \( y_1, \ldots, y_n \) for given \( X_1, \ldots, X_n \), the first averaging out over \( X_1, \ldots, X_n \). To simplify notation, we will often abbreviate \( \mathbb{Q}_{n, a, B, X_1, \ldots, X_n} \) to \( \mathbb{Q}_{n, a, B} \).

### 3. Proof of Theorem 1.

The proof of Theorem 1 will be divided into two stages. In the first stage, we prove the theorem assuming that the covariance kernel \( K \) is known. This case is relatively simple and of course artificial, but it captures the essence of the idea of our proof. In the second stage where \( K \) is unknown, we shall show that using the natural estimate \( \widetilde{K} \) as in (5) will not affect the result achieved in the first stage. Lemma 5 is to control the gap between the two stages.

In Section 3.1 we introduce the methodology of constructing a sequence of estimators achieving the optimal rates of convergence. In Section 3.2 we state the technical lemmas which serve as building blocks for establishing the main theorems. Their proofs are postponed to the Section 5. In Section 3.3 we prove Theorem 1 assuming \( \mu \) and \( K \) are known, and then in Section 3.4 we complete the proof of Theorem 1 with unknown \( \mu \) and \( K \).

### 3.1. Methodology.

Under the assumptions (X) and (K) from Section 2, the process \( X_i \) admits the eigen decomposition:

\[
X_i - \mu = Z_i = \sum_{k \in \mathbb{N}} z_{i,k} \phi_k.
\]

The random variables \( z_{i,k} := \langle Z_i, \phi_k \rangle \) are independent with \( z_{i,k} \sim N(0, \theta_k) \).

Because \( \mu \) and \( K \) are unknown, we estimate them in the usual way: \( \tilde{\mu}_n(t) = \overline{X}(t) = n^{-1} \sum_{i \leq n} X_i(t) \) and

\[
\begin{align*}
\tilde{K}(s, t) &= (n - 1)^{-1} \sum_{i \leq n} (X_i(s) - \overline{X}(s)) (X_i(t) - \overline{X}(t)) \\
&= (n - 1)^{-1} \sum_{i \leq n} (Z_i(s) - \overline{Z}(s)) (Z_i(t) - \overline{Z}(t)),
\end{align*}
\]

which has spectral representation

\[
\tilde{K}(s, t) = \sum_{k \in \mathbb{N}} \tilde{\theta}_k \widetilde{\phi}_k(s) \widetilde{\phi}_k(t).
\]
with \( \tilde{\theta}_1 \geq \tilde{\theta}_2 \geq \cdots \geq \tilde{\theta}_{n-1} \geq 0 \). In fact we must have \( \tilde{\theta}_k = 0 \) for \( k \geq n \) because all the eigenfunctions \( \tilde{\phi}_k \) corresponding to nonzero \( \tilde{\theta}_k \)’s must lie in the \( n-1 \)-dimensional space spanned by \( \{Z_i - \bar{Z} : i = 1, 2, \ldots, n\} \).

Using the first \( N \) (as defined in (9)) principal components, we can approximate the original infinite-dimensional model by the following sequence of truncated finite-dimensional models:

\[
y_i | X_1, \cdots, X_n \sim Q_{\tilde{\lambda}_i}
\]

with

\[
\tilde{\lambda}_i = \tilde{b}_0 + \sum_{1 \leq j \leq N} \tilde{b}_j \tilde{z}_{i,j},
\]

where \( \tilde{b}_0 = a + \langle B, \bar{X} \rangle \), and \( \tilde{b}_j = \langle B, \tilde{\phi}_j \rangle \) for \( j \geq 1 \), and \( \tilde{z}_{i,j} = \langle X_i - \bar{X}, \tilde{\phi}_j \rangle \).

We estimate \( B \) by

\[
\hat{B} = \sum_{j \leq m} \hat{b}_j \hat{\phi}_j,
\]

where \( \hat{b}_0, \cdots, \hat{b}_N \) is the conditional MLE for the truncated model and \( m \leq N \). More precisely, \( \hat{b}_0, \cdots, \hat{b}_N \) is chosen to maximize the following conditional (on the \( X_i \)'s) log likelihood over \((g_0, g_1, \cdots, g_N)\) in \( \mathbb{R}^{N+1} \):

\[
\mathcal{L}_n(g_0, g_1, \cdots, g_N) = \sum_{i \leq n} y_i(g_0 + \sum_{j \leq N} g_j \tilde{z}_{i,j}) - \psi(g_0 + \sum_{j \leq N} g_j \tilde{z}_{i,j}),
\]

with

\[
m \asymp n^{1/(\alpha + 2\beta)}
\]

and

\[
N \sim n^\zeta \quad \text{with} \quad (2 + 2\alpha)^{-1} > \zeta > (\alpha + 2\beta - 1)^{-1}.
\]

Note that \( N \) is much larger than \( m \). Such a \( \zeta \) exists because the assumptions \( \alpha > 1 \) and \( \beta > (\alpha + 3)/2 \) imply \( \alpha + 2\beta - 1 > 2 + 2\alpha \).

3.2. Technical lemmas. We shall first introduce an approximation result for maximum likelihood estimators in exponential family models for parameters whose dimensions change with sample size. This lemma combines ideas from Portnoy (1988) and from Hjort and Pollard (1993). We write our results in a notation that makes the applications in Sections 3.3 and 3.4 more straightforward. The notational cost is that the parameters are indexed by \( \{0, 1, \ldots, N\} \). To avoid an excess of parentheses we write \( N+1 \) for \( N + 1 \). In the applications \( N \) changes with the sample size \( n \) and \( Q \) is replaced by \( Q_{n,a,B,N} \) or \( Q_{n,a,B} \). For each square matrix \( A \), the spectral norm is defined by \( \|A\|_2 := \sup_{\|v\|_2 \leq 1} |Av| \) where \( |v| \) denotes the \( l^2 \) norm of vector \( v \).
LEMMA 1. Let $Q_\lambda$ be the one-parameter exponential family distribution defined as in (2) and satisfying regularity condition ($\psi$). Suppose $\xi_1, \ldots, \xi_n$ are (non-random) vectors in $\mathbb{R}^{N^+}$. Suppose $Q = \otimes_{i \leq n} Q_{\lambda_i}$ with $\lambda_i = \xi_i' \gamma$ for a fixed $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_N) \in \mathbb{R}^{N^+}$. Under $Q$, the coordinate maps $y_1, \ldots, y_n$ are independent random variables with $y_i \sim Q_{\lambda_i}$.

The log-likelihood for fitting the model is

$$L_n(g) = \sum_{i \leq n} (\xi_i' g) y_i - \psi(\xi_i' g)$$

for $g \in \mathbb{R}^{N^+}$, which is maximized (over $\mathbb{R}^{N^+}$) at the MLE $\hat{g} = \hat{g}_n$. Suppose $\xi_i = D\eta_i$ for some nonsingular matrix $D$, so that

$$J_n = nDA_nD'$$

where $A_n := \frac{1}{n} \sum_{i \leq n} \eta_i\eta_i' \psi(\lambda_i)$.

If $B_n$ is another nonsingular matrix for which

(10) $$\|A_n - B_n\|_2 \leq (2\|B_n^{-1}\|_2)^{-1}$$

and if

(11) $$\max_{i \leq n} |\eta_i| \leq \frac{\epsilon \sqrt{n/N^+}}{G(1) \sqrt{32\|B_n^{-1}\|_2}}$$

for some $0 < \epsilon < 1$

then for each set of vectors $\kappa_0, \ldots, \kappa_M$ in $\mathbb{R}^{N^+}$ there is a set $\mathcal{Y}_{n,\epsilon}$ with $Q_{\mathcal{Y}_{n,\epsilon}} < 2\epsilon$ on which

$$\sum_{0 \leq j \leq M} |\kappa_j'(\hat{g} - \gamma)|^2 \leq \frac{6\|B_n^{-1}\|_2^2}{n\epsilon} \sum_{0 \leq j \leq M} |D^{-1} \kappa_j|^2.$$ 

The following approximation result for random matrices will be invoked in order to apply the Lemma 1 to show Theorem 1.

LEMMA 2. Suppose $\{\eta_{i,k} : i, k \geq 1\}$ are i.i.d. standard normal random variables. Let

(12) $$A_n = n^{-1} \sum_{i \leq n} \eta_i\eta_i' \psi(\gamma' D\eta_i),$$

where $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_N)'$, $\eta_i = (1, \eta_{i,1}, \ldots, \eta_{i,N})'$, and $D = \text{diag}(D_0, D_1, \ldots, D_N)$. Denote $B_n = PA_n$ and assume $\psi$ satisfies condition ($\ddot{\psi}$). If $\sum_{k \geq 1} D_k^2 \gamma_k^2 < \infty$ and $N = o\left(n^{-1/2}\right)$, it follows that $\|B_n^{-1}\|_2 = O_F(1)$ and $\mathbb{P}\|A_n - B_n\|^2_2 = o_F(1)$.

The following lemma establishes a bound on the Hellinger distance between members of an exponential family, which is the key to our change of measure argument. We write $h(P, Q)$ for the Hellinger distance.
LEMMA 3. Suppose \( \{Q_\lambda : \lambda \in \mathbb{R}\} \) is an exponential family defined as in (2) and satisfies regularity condition \((\psi)\). Then,

\[
h^2(Q_\lambda, Q_{\lambda + \delta}) \leq \delta^2 \hat{\psi}(\lambda) (1 + |\delta|) G(|\delta|) \quad \forall \ \lambda, \delta \in \mathbb{R}.
\]

The following lemma provides a maximal inequality for weighted-chi-square variables, which easily leads to maximal inequalities for Gaussian processes and multivariate normal vectors. These inequalities will be repeatedly invoked.

LEMMA 4. Suppose \( W_i = \sum_{k \in \mathbb{N}} \tau_{i,k} n_i^2 \) for \( i = 1, \ldots, n \), where the \( \eta_{i,k} \)'s are independent standard normals and the \( \tau_{i,k} \)'s are nonnegative constants with \( \infty > T := \max_{i \leq n} \sum_{k \in \mathbb{N}} \tau_{i,k} \). Then

\[
\mathbb{P}\{\max_{i \leq n} W_i > 4T(\log n + x)\} < 2e^{-x} \quad \text{for each} \ x \geq 0.
\]

When we want to indicate that a bound involving constants \( c, C, C_1, \ldots \) holds uniformly over all models indexed by a set of parameters \( \mathcal{F} \), we write \( c(\mathcal{F}), C(\mathcal{F}), C_1(\mathcal{F}), \ldots \). By the usual convention for eliminating subscripts, the values of the constants might change from one paragraph to the next: a constant \( C_1(\mathcal{F}) \) in one place needn't be the same as a constant \( C_1(\mathcal{F}) \) in another place. For sequences of constants \( c_n \) that might depend on \( \mathcal{F} \), we write \( c_n = O_F(1) \) and \( o_F(1) \) and so on to show that the asymptotic bounds hold uniformly over \( \mathcal{F} \).

LEMMA 5. Let \( \mathcal{X}_1, \ldots, \mathcal{X}_n \) be i.i.d. Gaussian processes satisfying \((X)\) and \((K)\). Let \( m \) and \( N \) be integers defined as in (8) and (9) respectively. Suppose \( H_p \) and \( H_{p} \) are orthogonal projections operators associated with \( \text{span}\{\phi_1, \ldots, \phi_p\} \) and \( \text{span}\{\tilde{\phi}_1, \ldots, \tilde{\phi}_p\} \). Define the matrix \( \tilde{S} := \text{diag}(\sigma_0, \ldots, \sigma_N) \) with \( \sigma_0 = 1 \) and \( \sigma_k = \text{sign}(\langle \phi_k, \tilde{\phi}_k \rangle) \) for \( k \geq 1 \). The key quantities are:

(i) \( \Delta := \tilde{K} - K \)

(ii) \( \tilde{D} = \text{diag}(1, \tilde{\theta}_1, \ldots, \tilde{\theta}_N)^{1/2} \)

(iii) \( \tilde{z}_i = (\tilde{z}_{i,1}, \ldots, \tilde{z}_{i,N})' \) where \( \tilde{z}_{i,k} = \langle \tilde{Z}_i, \tilde{\phi}_k \rangle \)

(iv) \( \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_N)' \) where \( \tilde{z}_k = \text{diag}(\tilde{\phi}_k) = n^{-1} \sum_{i \leq n} \tilde{z}_{i,k} \)

(v) \( \tilde{\xi}_i = \langle 1, \tilde{z}' - \tilde{z}' \rangle' \) and \( \tilde{\eta}_i = \tilde{D}^{-1} \tilde{\xi}_i \). We could define \( \tilde{\eta}_i = \tilde{D}^{-1} \tilde{\xi}_i \) but then we would need to show that \( \tilde{D}^{-1} \tilde{\xi}_i \approx D^{-1} \xi_i \). Our definition merely rearranges the approximation steps.

(vi) \( \tilde{\gamma}_i = (\tilde{\gamma}_0, \tilde{b}_1, \ldots, \tilde{b}_N)' \) where \( \tilde{B} = \sum_{k \in \mathbb{N}} \tilde{b}_k \tilde{\phi}_k \) and \( \tilde{\gamma}_0 := a + \langle \tilde{B}, \tilde{Z} \rangle \). [Note that \( \lambda_i = \tilde{\gamma}_0 + \langle \tilde{B}, \tilde{Z}_i - \tilde{\bar{Z}} \rangle \).]

(vii) \( \tilde{\lambda}_{i,N} = \tilde{\gamma}_0 + \langle \tilde{H}_N \tilde{B}, \tilde{Z}_i - \tilde{\bar{Z}} \rangle = \tilde{\xi}_i' \tilde{\gamma} \).

(viii) \( \tilde{A}_n = n^{-1} \sum_{i \leq n} \tilde{\eta}_i \tilde{b}_i' (\tilde{\lambda}_{i,N}) \)
For each $\epsilon > 0$ there exists a set $\mathcal{X}_{\epsilon,n}$, depending on $\mu$ and $K$, with
\[
\sup_F \mathbb{P}_{n,\mu,K} \mathcal{X}_{\epsilon,n}^c < \epsilon \quad \text{for all large enough } n
\]
and on which, for some constant $C_\epsilon$ that does not depend on $\mu$ or $K$,

(i) $\|\Delta\| \leq C_\epsilon n^{-1/2}$

(ii) $\max_{i \leq n} \|Z_i\| \leq C_\epsilon \sqrt{\log n}$ and $\|\mathbb{Z}\| \leq C_\epsilon n^{-1/2}$

(iii) $\|(\mathcal{H}_n - H_m)\mathbb{B}\|^2 = o_F(\rho_n)$

(iv) $\|(\mathcal{H}_N - H_N)\mathbb{B}\|^2 = O_F(n^{-1-\nu})$ for some $\nu > 0$ that depends only on $\alpha$ and $\beta$

(v) $\max_{i \leq n} |\eta_i|^2 = O_F(\sqrt{n}/N)$

(vi) $\|\mathcal{S}A_n \mathcal{S} - A_n\|_2 = o_F(1)$

3.3. Proof of Theorem 1 with known Gaussian distribution. Initially we suppose that $\mu$ and $K$ are known. Under $\mathbb{Q}_n = \mathbb{Q}_{n,\mu,\mathbb{B}}$, the $y_i$’s are independent, with $y_i \sim Q_\lambda$, and

$$\lambda_i = a + \langle \mathcal{X}_i, \mathbb{B} \rangle = b_0 + \sum_{k \in \mathbb{N}} z_{i,k} b_k \quad \text{where } b_0 = a + \langle \mu, \mathbb{B} \rangle.$$

Our task is to estimate the $b_k$’s with sufficient accuracy to be able to estimate $\mathbb{B}(t) = \sum_{k \in \mathbb{N}} b_k \phi_k(t)$ within an error of order $\rho_n = n^{(1-2\beta)/(\alpha + 2\beta)}$. In fact it will suffice to estimate the component $H_m \mathbb{B}$ of $\mathbb{B}$ in the subspace spanned by $\{\phi_1, \ldots, \phi_m\}$ with $m \asymp n^{1/(\alpha + 2\beta)}$ because

$$\|H_m \mathbb{B}\|^2 = \sum_{k > m} b_k^2 = O_F(m^{1-2\beta}) = O_F(\rho_n).$$

We might try to estimate the coefficients $(b_0, \ldots, b_m)$ by choosing $\hat{\gamma} = (\hat{\gamma}_0, \ldots, \hat{\gamma}_m)$ to maximize a conditional log likelihood over all $g$ in $\mathbb{R}^{n+1}$,

$$\sum_{i \leq n} y_i \lambda_{i,m} - \psi(\lambda_{i,m}) \quad \text{with } \lambda_{i,m} = g_0 + \sum_{1 \leq k \leq m} z_{i,k} g_k.$$

To this end we might try to appeal to Lemma 1 stated at the beginning of this Section, with $\kappa_j$ equal to the unit vector with a 1 in its $j$th position for $j \leq m$ and $\kappa_j = 0$ otherwise. That would give a bound for $\sum_{j \leq m} (\hat{\gamma}_j - \gamma_j)^2$. Unfortunately, we cannot directly invoke the Lemma with $N = m$ to estimate $\gamma = (b_0, b_1, \ldots, b_N)$ when

$$\mathbb{Q} = \mathbb{Q}_{n,\mu,\mathbb{B}} \quad \text{and} \quad D = \text{diag}(1, \theta_1, \ldots, \theta_N)^{1/2}$$

(14) $\xi_i = (1, z_{i,1}, \ldots, z_{i,N})$ and $\eta_i = (1, \eta_{i,1}, \ldots, \eta_{i,N}),$

because $\lambda_i \neq \xi_i \gamma$, a bias problem. Note that in this case $\eta_{i,j} = z_{i,j}/\sqrt{\theta_i}$ for all $i, j$ and hence the $\eta_{i,j}$’s are i.i.d. standard normal variables.
Remark. We could modify Lemma 1 to allow $\ell_i = \xi_i'\gamma + \text{bias}_i$, for a suitably small bias term, but at the cost of extra regularity conditions and a more delicate argument. The same difficulty arises whenever one investigates the asymptotics of maximum likelihood with the true distribution outside the model family.

Instead, we use a two-stage estimation procedure that eliminates the bias term by a change of measure conditional on the $X_i$'s. We shall present the proof in the following three steps.

Step 1. From the analysis above, one can see that the key in our proof is the change-of-measure argument and the application of Lemma 1. In this step, we shall construct a high probability set such that for each realization of the $X_i$'s on the set the assumptions of Lemma 1 are satisfied and the change-of-measure argument is ready to work.

Define $\xi_i$, $D$, and $\eta_i$ as in equation (14). Then we define matrix $A_n$ as in (12) and choose $B_n := \mathbb{P}_{n,\mu,K} A_n$. Define $X_{Z,n} := \{\max_{i \leq n} \|Z_i\|^2 \leq C_0 \log n\}$, $X_{\eta,n} := \{\max_{i \leq n} \|\eta_i\|^2 \leq C_0 N \log n\}$, and $X_{A,n} := \{\|A_n - B_n\|_2 \leq (2\|B_n^{-1}\|_2)^{-1}\}$

If we choose a large enough universal constant $C_0 = C_0(F)$, Lemma 4 ensures that $\mathbb{P}_{n,\mu,K} X_{Z,n} \leq 2/n$ and $\mathbb{P}_{n,\mu,K} X_{\eta,n} \leq 2/n$ by choosing $\tau_{i,k} = \theta_i$ and $\tau_{i,k} = \{i \leq N\}$ respectively for all $i,k$; and Lemma 2 shows that $\|B_n^{-1}\|_2 = O_F(1)$ and $\mathbb{P}_{n,\mu,K} \|A_n - B_n\|_2^2 = o_F(1)$, thus $\mathbb{P}_{n,\mu,K} X_{A,n} = o_F(1)$. And hence,

$$\mathbb{P}_{n,\mu,K} X_{\xi,n} \leq \mathbb{P}_{n,\mu,K} X_{Z,n} + \mathbb{P}_{n,\mu,K} X_{\eta,n} + \mathbb{P}_{n,\mu,K} X_{A,n} = o_F(1).$$

Step 2. Let us consider the approximate distribution

$$Q_{n,a,B} := \otimes_{i \leq n} Q_{\lambda_i,N} \quad \text{with} \quad \lambda_i := \xi_i'\gamma' = (b_0, b_1, \ldots, b_N).$$

In this step, we show that the divergence caused by replacing $Q_{n,a,B}$ by $Q_{n,a,B}$, $N$ is small enough that it will not compromise the asymptotic results. In replacing $Q_{n,a,B}$ by $Q_{n,a,B}$, $N$ we eliminate the bias problem but now we have to relate the probability bounds for $Q_{n,a,B}$ to bounds involving $Q_{n,a,B}$. A common control of this divergence is the total variation distance between $Q_{n,a,B}$ and $Q_{n,a,B}$. We shall show that there exists a sequence of nonnegative constants $c_n$ of order $o_F(\log n)$, such that

$$\|Q_{n,a,B} - Q_{n,a,B,N}\|_{TV}^2 \leq e^{2c_n} \sum_{i \leq n} |\lambda_i - \lambda_i|$$ on $X_n$. 


To establish inequality (19) we use the bound
\[ \|Q_{n,a,B} - Q_{n,a,B,N}\|_{TV}^2 \leq h^2(Q_{n,a,B}, Q_{n,a,B,N}) \leq \sum_{i \leq n} h^2(Q_{\lambda_i}, Q_{\lambda_i,N}) \]
By Lemma 3
\[ h^2(Q_{\lambda_i}, Q_{\lambda_i,N}) \leq \delta_i^2 \tilde{\psi}(\lambda_i) (1 + |\delta_i|) g(|\delta_i|) \]
where
\[ |\delta_i| = |\lambda_i - \lambda_{i,N}| = \langle Z_i, B \rangle - \langle H_N Z_i, B \rangle \]
\[ = |\langle Z_i, H_N^\perp B \rangle| \leq \|Z_i\| \|H_N^\perp\| \leq O_F \left( \sqrt{n^{1-2\beta}} \log n \right) = o_F(1) \]
Thus all the \((1 + |\delta_i|) g(|\delta_i|)\) factors can be bounded by a single \(O_F(1)\) term.

For \((a, B, \mu, K) \in F(R, \alpha, \beta)\) and with the \(\|Z_i\|^\gamma\) controlled by \(X_n\),
\[ |\lambda_i| \leq |a| + (\|\mu\| + \|Z_i\|) \|B\| \leq C_2 \sqrt{\log n} \]
for some constant \(C_2 = C_2(F)\). Assumption \((\tilde{\psi})\) then ensures that all the \(\tilde{\psi}(\lambda_i)\)
are bounded by a single \(\exp(o_F(\log n))\) term.

Step 3. On the set \(X_n\), we can apply Lemma 1 directly with \(Q = Q_{n,a,B,N}\),
because inequality (10) holds by construction and inequality (11) holds for large
enough \(n\) because
\[ \max_{i \leq n} |\eta_i|^2 \leq O_F(N \log n) = o_F(\sqrt{n}/N). \]

Estimate \(\gamma\) by the \(\hat{g} = (\hat{g}_0, \ldots, \hat{g}_N)\) defined in Lemma 1. Thus, the estimator
in Theorem 1 is \(\hat{B}_n = \sum_{1 \leq k \leq m} \hat{g}_k \phi_k\). For each realization of the \(X_i\)'s in \(X_n\),
Lemma 1 gives a set \(y_{m,e}\) with \(Q_{n,a,B,N} y_{m,e} < 2\epsilon\) on which
\[ \sum_{1 \leq k \leq m} |\hat{g}_k - \gamma_k|^2 = O_F \left( n^{-1} \sum_{1 \leq k \leq m} \theta_k^{-1} \right) = O_F(m^{1+\alpha}/n) = O_F(\rho_n), \]
which implies
\[ \|\hat{B}_n - B\|^2 = \sum_{1 \leq k \leq m} |\hat{g}_k - \gamma_k|^2 + \sum_{k > m} b_k^2 = O_F(\rho_n). \]
From the inequality (19) it follows, for a large enough constant \(C_\epsilon\), that
\[ P_{n,\mu,K}Q_{n,a,B}\{\|\hat{B}_n - B\|^2 > C_\epsilon \rho_n\} \]
\[ \leq P_{n,\mu,K}X_n^c + P_{n,\mu,K}X_n \left( \|Q_{n,a,B} - Q_{n,a,B,N}\|_{TV} + Q_{n,a,B,N} Y_{m,e} \right) \]
\[ \leq o_F(1) + 2\epsilon + c^n \left( \sum_{i \leq n} P_{n,\mu,K} |\lambda_i - \lambda_{i,N}|^2 \right)^{1/2}. \]
By construction, 
\[ \lambda_i - \lambda_{i,N} = \sum_{k > N} z_{i,k} b_k \]
with the \( z_{i,k} \)'s independent and \( z_{i,k} \sim N(0, \theta_k) \). Thus
\[ \sum_{i \leq n} \mathbb{P}_{n,\mu,K}[\lambda_i - \lambda_{i,N}]^2 \leq \mathbb{E} \sum_{k > N} \theta_k b_k^2 = O_F(nN^{1-\alpha-2\beta}) = o_F(e^{-2\alpha n}) \]
because \( \zeta > (\alpha + 2\beta - 1)^{-1} \). That is, we have an estimator that achieves the \( O_F(\rho_n) \) minimax rate.

3.4. Proof of Theorem 1 with unknown Gaussian distribution. As before, most of the analysis will be conditional on the \( \mathcal{X}_i \)'s lying in a set with high probability on which the various estimators and other random quantities are well behaved. Remember \( \mathcal{X}_{\epsilon,n} \) is the high probability set defined in Lemma 5. For the key quantities defined in Lemma 5, we shall keep their notations unchanged in this section for the purpose of making the application more straightforward.

As before, the component of \( B \) orthogonal to \( \text{span}\{\tilde{\phi}_1, \ldots, \tilde{\phi}_m\} \) causes no trouble because
\[ \left\| \mathbb{B} - B \right\|^2 = \sum_{1 \leq k \leq m} (\tilde{g}_k - \tilde{\gamma}_k)^2 + \left\| \tilde{H}_m \right\|^2 \]
and, by Lemma 5 part (iii),
\[ \left\| \tilde{H}_m \right\|^2 \leq 2\|H_m\|^2 + 2\|H_m - H_m\|^2 = O_F(\rho_n) \quad \text{on } \mathcal{X}_{\epsilon,n}. \]

To handle \( \sum_{1 \leq k \leq m} (\tilde{g}_k - \tilde{\gamma}_k)^2 \), invoke Lemma 1 for \( \mathcal{X}_i \)'s in \( \mathcal{X}_{\epsilon,n} \), with \( \eta_i \) replaced by \( \tilde{\eta}_i \) and \( A_n \) replaced by \( \tilde{A}_n \) and \( B_n \) replaced by \( \tilde{B}_n = \tilde{S}B_n\tilde{S} \), the same \( B_n \) and \( D \) as before, and \( \mathcal{Q} \) equal to
\[ \tilde{\mathcal{Q}}_{n,a,B,N} = \otimes_{i \leq n} \tilde{Q}_{\tilde{A}_{i,N}}. \]
to get a set \( \tilde{Y}_{m,\epsilon} \) with \( \tilde{\mathcal{Q}}_{n,a,B,N} \tilde{Y}_{m,\epsilon} < 2\epsilon \) on which \( \sum_{1 \leq k \leq m} (\tilde{g}_k - \tilde{\gamma}_k)^2 \). The conditions of Lemma 1 are satisfied on \( \mathcal{X}_{\epsilon,n} \), because of Lemma 5 part (v) and
\[ \| \tilde{A}_n - \tilde{B}_n \|_2 \leq \| \tilde{A}_n - \tilde{S}A_n\tilde{S} \|_2 + \| \tilde{S}A_n\tilde{S} - \tilde{S}B_n\tilde{S} \|_2 = o_F(1). \]

To complete the proof it suffices to show that \( \| \tilde{\mathcal{Q}}_{n,a,B,N} - \tilde{\mathcal{Q}}_{n,a,B,N} \|_{TV} \) tends to zero. First note that
\[ \tilde{\lambda}_{i,N} - \lambda_{i,N} = a + \langle B, \mathcal{X} \rangle + \langle \tilde{H}_N B, Z_i - \mathcal{Z} \rangle - a - \langle \mathcal{B}, \mu \rangle - \langle H_N B, Z_i \rangle \]
\[ = \langle \tilde{H}_N B, \mathcal{Z} \rangle - \langle H_N B, \mathcal{Z} \rangle + \langle H_N B, \mathcal{Z} \rangle + \langle \tilde{H}_N B - H_N B, Z_i \rangle \]
which implies that, on $\tilde{X}_{c,n}$,
\[
|\tilde{\lambda}_{i,N} - \lambda_{i,N}|^2 \leq 2|\langle H_{N}\beta, Z \rangle|^2 + 2\|H_N\beta - H_N\|^2 \left(\|Z_i\| + \|Z\|\right)^2 \\
\leq O_F(N^{1-2\beta})C_{\epsilon}n^{-1} + O_F(n^{-1-\nu})C_{\epsilon}^2 \left(n^{-1/2} + \sqrt{\log n}\right)^2 \\
= O_F(n^{-1-\nu}) \quad \text{for some } 0 < \nu' < \nu.
\]
(20)

Now argue as in the step 2 of the proof for the case of known $K$: on $\tilde{X}_{c,n}$,
\[
\|\tilde{Q}_{n,a,B,N} - Q_{n,a,B,N}\|_{TV}^2 \leq \sum_{i \leq n} h^2 \left(Q_{\lambda_{i,N}}, Q_{\lambda_{i,N}}\right) \\
\leq \exp(o_F(\log n)) \sum_{i \leq n} |\tilde{\lambda}_{i,N} - \lambda_{i,N}|^2 = o_F(1).
\]

Finish the argument as before, by splitting into contributions from $\tilde{X}_{c,n}$ and $\tilde{X}_{c,n} \cap \tilde{y}_{m,e}$ and $\tilde{X}_{c,n} \cap \tilde{y}_{m,e}$.

4. Proof of Theorem 2. We apply a slight variation on Assouad’s Lemma—combining ideas from Yu (1997) and from van der Vaart (1998, Section 24.3)—to establish the minimax lower bound result in Theorem 2.

We consider behavior only for $\mu = 0$ and $a = 0$, for a fixed $K$ with spectral decomposition $\sum_{j \in \mathbb{N}} \theta_j \phi_j \otimes \phi_j$. For simplicity we abbreviate $\mathbb{P}_{n,0,K}$ to $\mathbb{P}$. Let $J = \{m + 1, m + 2, \ldots, 2m\}$ and $\Gamma = \{0, 1\}^J$. Let $\beta_j = R_{j}^{-\beta}$. For each $\gamma$ in $\Gamma$ define $B_{\gamma} = \epsilon \sum_{j \in J} \gamma_j \beta_j \phi_j$, for a small $\epsilon > 0$ to be specified, and write $Q_{\gamma}$ for the product measure $\otimes_{i \leq n} Q_{\lambda_i(\gamma)}$ with
\[
\lambda_i(\gamma) = \langle B_{\gamma}, Z_i \rangle = \epsilon \sum_{j \in J} \gamma_j \beta_j z_{i,j}.
\]

For each $j$ let $\Gamma_j = \{\gamma \in \Gamma : \gamma_j = 1\}$ and let $\psi_j$ be the bijection on $\Gamma$ that flips the $j$th coordinate but leaves all other coordinates unchanged. Let $\pi$ be the uniform distribution on $\Gamma$, that is, $\pi_\gamma = 2^{-m}$ for each $\gamma$.

For each estimator $\hat{B} = \sum_{j \in \mathbb{N}} \hat{b}_j \phi_j$ we have $\|B_{\gamma} - \hat{B}\|^2 \geq \sum_{j \in J} \left(\gamma_j \beta_j - \hat{b}_j\right)^2$ and so
\[
\sup_{\mathbb{P}} \mathbb{P}_{n,f} \|B - \hat{B}\|^2 \geq \pi_\gamma \sum_{j \in J} \mathbb{P} Q_{\gamma} \left(\epsilon \gamma_j \beta_j - \hat{b}_j\right)^2 \\
= 2^{-m} \sum_{j \in J} \sum_{\gamma \in \Gamma_j} \mathbb{P} \left(Q_{\gamma} (\epsilon \beta_j - \hat{b}_j)^2 + Q_{\psi_j(\gamma)} (0 - \hat{b}_j)^2\right) \\
\geq 2^{-m} \sum_{j \in J} \sum_{\gamma \in \Gamma_j} \frac{1}{4}(\epsilon \beta_j)^2 \mathbb{P} \|Q_{\gamma} \wedge Q_{\psi_j(\gamma)}\|,
\]
(21)

the last lower bound coming from the fact that
\[
(\epsilon \beta_j - \hat{b}_j)^2 + (0 - \hat{b}_j)^2 \geq \frac{1}{4}(\epsilon \beta_j)^2 \quad \text{for all } \hat{b}_j.
\]
We assert that, if $\epsilon$ is chosen appropriately,

$$\min_{j, \gamma} \mathbb{P}\|Q_\gamma \wedge Q_{\psi_j(\gamma)}\| \text{ stays bounded away from zero as } n \to \infty,$$

which will ensure that the lower bound in (21) is eventually larger than a constant multiple of $\sum_{j \in J} \beta_j^2 \geq c \rho_n$ for some constant $c > 0$. The inequality in Theorem 2 will then follow.

To prove (22), consider a $\gamma$ in $\Gamma$ and the corresponding $\gamma' = \psi_j(\gamma)$. By virtue of the inequality

$$\|Q_\gamma \wedge Q_{\gamma'}\| = 1 - \|Q_\gamma - Q_{\gamma'}\|_{TV} \geq 1 - \left(2 \wedge \sum_{i \leq n} h^2(Q_{\lambda_i(\gamma)}, Q_{\lambda_i(\gamma')})\right)^{1/2},$$

it is enough to show that

$$\limsup_{n \to \infty} \max_{j, \gamma} \mathbb{P}\left(2 \wedge \sum_{i \leq n} h^2(Q_{\lambda_i(\gamma)}, Q_{\lambda_i(\gamma')})\right) < 1.$$

Define $X_n = \{\max_{i \leq n} \|Z_i\|^2 \leq C_0 \log n\}$. Based on Lemma 4, we know that $\mathbb{P}X^c_n = o(1)$ with the constant $C_0$ large enough. On $X_n$ we have

$$|\lambda_i(\gamma)|^2 \leq \sum_{j \in J} \beta_j^2 \|Z_i\|^2 = O(\rho_n) \log n = o(1)$$

and, by inequality (3),

$$h^2(Q_{\lambda_i(\gamma)}, Q_{\lambda_i(\gamma')}) \leq O(1)|\lambda_i(\gamma) - \lambda_i(\gamma')|^2 \leq \epsilon^2 O(1)\beta_j^2 z_{i,j}^2.$$

We deduce that

$$\mathbb{P}\left(2 \wedge \sum_{i \leq n} h^2(Q_{\lambda_i(\gamma)}, Q_{\lambda_i(\gamma')})\right) \leq 2\mathbb{P}X^c_n + \sum_{i \leq n} \epsilon^2 O(1)\beta_j^2 \mathbb{P}X_n z_{i,j}^2 \leq o(1) + \epsilon^2 O(1)n \beta_j^2 \theta_j.$$

The choice of $J$ makes $\beta_j^2 \theta_j \leq R^2 m^{-\alpha - 2\beta} \sim R^2 / n$. Assertion (23) follows.

5. Proof of technical lemmas.

5.1. Proof of Lemma 1. We need to first show the following lemma. Define

(i) $w_i := J_n^{-1/2} \xi_i$, an element of $\mathbb{R}^{N_+}$

(ii) $W_n = \sum_{i \leq n} w_i \left(y_i - \hat{\psi}(\lambda_i)\right)$, an element of $\mathbb{R}^{N_+}$

Notice that $QW_n = 0$ and $\text{var}_Q(W_n) = \sum_{i \leq n} w_i^2 \hat{\psi}(\lambda_i) = I_{N_+}$ and

$$Q|W_n|^2 = \text{trace} \left(\text{var}_Q(W_n)\right) = N_+.$$
LEMMA 6. Suppose \( 0 < \epsilon_1 \leq 1/2 \) and \( 0 < \epsilon_2 < 1 \) and
\[
\max_{i \leq n} |w_i| \leq \frac{\epsilon_1 \epsilon_2}{2G(1)N_+} \quad \text{with } G \text{ as in Assumption } (\psi) .
\]

Then \( \tilde{g} = \gamma + J_n^{-1/2}(W_n + r_n) \) with \( |r_n| \leq \epsilon_1 \) on the set \( \{|W_n| \leq \sqrt{N_+}/\epsilon_2\} \), which has \( \mathbb{Q} \)-probability greater than \( 1 - \epsilon_2 \).

PROOF. The equality \( \mathbb{Q}|W_n|^2 = N_+ \) and Tchebychev give
\[
\mathbb{Q}\{|W_n| > \sqrt{N_+}/\epsilon_2\} \leq \epsilon_2.
\]

Reparametrize by defining \( t = J_n^{1/2}(g - \gamma) \). The concave function
\[
\mathcal{L}_n(t) := L_n(\gamma + J_n^{-1/2}t) - L_n(\gamma) = \sum_{i \leq n} y_i w_i t + \psi(\lambda_i) - \psi(\lambda_i + w_i t)
\]
is maximized at \( \tilde{t}_n = J_n^{1/2}(\tilde{g} - \gamma) \). It has derivative
\[
\dot{\mathcal{L}}_n(t) = \sum_{i \leq n} w_i \left( y_i - \dot{\psi}(\lambda_i + w_i t) \right).
\]

For a fixed unit vector \( u \in \mathbb{R}^{N_+} \) and a fixed \( t \in \mathbb{R}^{N_+} \), consider the real-valued function of the real variable \( s \),
\[
H(s) := u'\dot{\mathcal{L}}_n(st) = \sum_{i \leq n} u' w_i \left( y_i - \dot{\psi}(\lambda_i + sw_i t) \right),
\]
which has derivatives
\[
\dot{H}(s) = -\sum_{i \leq n} (u' w_i)(w_i' t) \ddot{\psi}(\lambda_i + sw_i t) \\
\ddot{H}(s) = -\sum_{i \leq n} (u' w_i)(w_i' t)^2 \dddot{\psi}(\lambda_i + sw_i t).
\]

Notice that \( H(0) = u'W_n \) and \( \dot{H}(0) = -u' \sum_{i \leq n} w_i w_i' \ddot{\psi}(\lambda_i) t = -u' t \).

Write \( M_n \) for \( \max_{i \leq n} |w_i| \). By virtue of Assumption \( (\psi) \),
\[
|\dot{H}(s)| \leq \sum_{i \leq n} |u' w_i|(w_i' t)^2 \ddot{\psi}(\lambda_i) G \left(|sw_i' t|\right) \\
\leq M_n G(M_n|st|) t' \sum_{i \leq n} w_i w_i' \ddot{\psi}(\lambda_i) t \\
= M_n G(M_n|st|) |t|^2.
\]

By Taylor expansion, for some \( 0 < s^* < 1 \),
\[
|H(1) - H(0) - \dot{H}(0)| \leq \frac{1}{2}|\dddot{H}(s^*)| \leq \frac{1}{2} M_n G(M_n|t|) |t|^2.
\]
That is,
\begin{equation}
\left|u' \left( \tilde{\mathcal{L}}_n(t) - W_n + t \right) \right| \leq \frac{1}{2} M_n G (M_n |t|) \ |t|^2.
\end{equation}

Approximation (24) will control the behavior of \( \tilde{\mathcal{L}}(s) := \mathcal{L}_n(W_n + su) \), a concave function of the real argument \( s \), for each unit vector \( u \). By concavity, the derivative
\[
\dot{\tilde{\mathcal{L}}}(s) = u' \tilde{\mathcal{L}}_n(W_n + su) = -s + R(s)
\]
is a decreasing function of \( s \) with
\[
|R(s)| \leq \frac{1}{2} M_n G (M_n |W_n + su|) |W_n + su|^2
\]
On the set \{\(|W_n| \leq \sqrt{N_+/\epsilon_2}\)\} we have
\[
|W_n \pm \epsilon_1 u| \leq \sqrt{N_+/\epsilon_2} + \epsilon_1.
\]
Thus
\[
M_n |W_n \pm \epsilon_1 u| \leq \frac{\epsilon_1 \epsilon_2}{2G(1)N_+} \left( \sqrt{N_+/\epsilon_2} + \epsilon_1 \right) < 1,
\]
implies
\[
|R(\pm \epsilon_1)| \leq \frac{1}{2} M_n G(1) |W_n \pm \epsilon_1 u|^2
\leq \frac{\epsilon_1 \epsilon_2}{G(1)N_+} (N_+/\epsilon_2 + \epsilon_2^2)
\leq \epsilon_1 \left( 1 + \epsilon_1^2 \epsilon_2/N_+ \right) < \frac{5}{8} \epsilon_1.
\]
Deduce that
\[
\dot{\tilde{\mathcal{L}}}(-\epsilon_1) = \epsilon_1 + R(-\epsilon_1) \geq \frac{3}{8} \epsilon_1
\]
The concave function \( s \mapsto \mathcal{L}_n(W_n + su) \) must achieve its maximum for some \( s \) in the interval \([-\epsilon_1, \epsilon_1]\), for each unit vector \( u \). It follows that \(|\tilde{t}_n - W_n| \leq \epsilon_1\). \( \square \)

First we establish a bound on the spectral distance between \( A_n^{-1} \) and \( B_n^{-1} \). Define \( H = B_n^{-1} A_n - I \). Then\( \|H\|_2 \leq \|B_n^{-1}\|_2 \|A_n - B_n\|_2 \leq 1/2 \), which justifies the expansion
\[
\|A_n^{-1} - B_n^{-1}\|_2 = \| (I + H)^{-1} - I \ B_n^{-1}\|_2 \leq \sum_{j \geq 1} \|H\|_2^j \|B_n^{-1}\|_2 \leq \|B_n^{-1}\|_2.
\]
As a consequence, \( \|A_n^{-1}\|_2 \leq 2\|B_n^{-1}\|_2 \).
Choose \( \epsilon_1 = 1/2 \) and \( \epsilon_2 = \epsilon \) in Lemma 6. The bound on \( \max_i |\eta_i| \) gives the bound on \( \max_i |w_i| \) needed by the Lemma:
\[
n|w_i|^2 = \eta_i^2 D(J_n/n)^{-1} D\eta_i = \eta_i^2 A_n^{-1}\eta_i \leq \|A_n^{-1}\|_2 |\eta_i|^2.
\]

Define \( K_j := J_n^{-1/2} \kappa_j \), so that \( |\kappa'_j(\hat{g} - \gamma)|^2 \leq 2(K'_j W_n)^2 + 2(K'_j r_n)^2 \). By Cauchy-Schwarz,
\[
\sum_j (K'_j r_n)^2 \leq \sum_j |K'_j|^2 |r_n|^2 = U_\kappa |r_n|^2
\]
where
\[
U_\kappa := \sum_j \kappa'_j J_n^{-1/2} \kappa_j = \sum_j n^{-1}(D^{-1/2} \kappa_j)' A_n^{-1} D^{-1/2} \kappa_j \leq 2n^{-1} \|B_n^{-1}\|_2 \sum_j |D^{-1/2} \kappa_j|^2.
\]

For the contribution \( V_\kappa := \sum_j |K'_j W_n|^2 \) the Cauchy-Schwarz bound is too crude. Instead, notice that \( \mathbb{Q}V_\kappa = U_\kappa \), which ensures that the complement of the set
\[
Y_{n,\epsilon} := \{|W_n| \leq \sqrt{N_+}/\epsilon \} \cap \{V_\kappa \leq U_\kappa/\epsilon \}
\]
has \( \mathbb{Q} \) probability less than \( 2\epsilon \). On the set \( Y_{n,\epsilon} \),
\[
\sum_{0 \leq j \leq N} |\kappa'_j(\hat{g} - \gamma)|^2 \leq 2V_\kappa + 2U_\kappa |r_n|^2 \leq 3U_\kappa/\epsilon.
\]

The asserted bound follows.

5.2. Proof of Lemma 2. Throughout this subsection abbreviate \( \mathbb{P}_{n,\mu,K} \) to \( \mathbb{P} \).

The matrix \( A_n \) is an average of \( n \) independent random matrices each of which is distributed like \( \mathbb{N}N'\tilde{\psi}(\gamma'/DN) \), where \( N' = (N_0, N_1, \ldots, N_N) \) with \( N_0 \equiv 1 \) and the other \( N_j \)'s are independent \( N(0, 1) \)'s. Moreover, by rotational invariance of the spherical normal, we may assume with no loss of generality that \( \gamma'/DN = \tilde{a} + \kappa N_1 \), where
\[
\kappa^2 = \sum_{k=1}^N D_k^2 b_k^2 = O_F(1).
\]

Thus
\[
B_n = \mathbb{P}NN'\tilde{\psi}(\tilde{a} + \kappa N_1) = \text{diag}(F, r_0 I_{N-1})
\]
where
\[
r_j := \mathbb{P}N_j^2 \tilde{\psi}(\tilde{a} + \kappa N_1) \quad \text{and} \quad F = \begin{bmatrix} r_0 & r_1 \\ r_1 & r_2 \end{bmatrix}.
\]
The block diagonal form of $B_n$ simplifies calculation of spectral norms.

$$
\|B_n^{-1}\|_2 = \|\text{diag}(F^{-1}, r_0^{-1}I_{N-1})\|_2 \\
\leq \max(\|F^{-1}\|_2, \|r_0^{-1}I_{N-1}\|_2) \leq \max\left(\frac{r_0 + r_2}{r_0r_2 - r_1^2}, r_0^{-1}\right).
$$

Assumption ($\ddot{\psi}$) ensures that both $r_0$ and $r_2$ are $O_F(1)$.

Continuity and strict positivity of $\ddot{\psi}$, together with $\max(|\bar{a}|, \kappa) = O_F(1)$, ensure that $c_0 := \inf_{\bar{a}, \kappa} \inf_{|x| \leq 1} \ddot{\psi}(\bar{a} + \kappa x) > 0$. Thus

$$
\sqrt{2\pi r_0} \geq c_0 \int_{-1}^{+1} e^{-x^2/2} dx > 0
$$

Similarly

$$
\sqrt{2\pi(r_0r_2 - r_1^2)} = \sqrt{2\pi r_0} \mathbb{P} \ddot{\psi}(\bar{a} + \kappa N_1)(N_1 - r_1/r_0)^2 \\
\geq c_0 r_0 \int_{-1}^{+1} (x - r_1/r_0)^2 e^{-x^2/2} dx \geq c_0 r_0 \int_{-1}^{+1} x^2 e^{-x^2/2} dx.
$$

It follows that $\|B_n^{-1}\|_2 = O_F(1)$.

The random matrix $A_n - B_n$ is an average of $n$ independent random matrices each distributed like $NN' \ddot{\psi}(\bar{a} + \kappa N_1)$ minus its expected value. Thus

$$
\mathbb{P}\|A_n - B_n\|_2^2 \leq \mathbb{P}\|A_n - B_n\|_F^2 = n^{-1} \sum_{0 \leq j, k \leq N} \text{var}(N_j N_k \ddot{\psi}(\bar{a} + \kappa N_1)).
$$

Assumption ($\ddot{\psi}$) ensures that each summand is $O_F(1)$, which leaves us with a $O_F(N^2/n) = o_F(1)$ upper bound.

5.3. **Proof of Lemma 3.** Let us temporarily write $\lambda'$ for $\lambda + \delta$ and write $\overline{X}$ for $(\lambda + \lambda')/2 = \lambda + \delta/2$.

$$
1 - \frac{1}{2}h^2(Q_\lambda, Q_{\lambda'}) = \int \sqrt{f_\lambda(y)f_{\lambda'}(y)} \\
= \int \exp \left(\overline{\lambda}y - \frac{1}{2}\psi(\lambda) - \frac{1}{2}\psi(\lambda')\right) \\
= \exp \left(\psi(\overline{\lambda}) - \frac{1}{2}\psi(\lambda) - \frac{1}{2}\psi(\lambda')\right) \\
\geq 1 + \psi(\overline{\lambda}) - \frac{1}{2}\psi(\lambda) - \frac{1}{2}\psi(\lambda').
$$

That is,

$$
h^2(Q_\lambda, Q_{\lambda'}) \leq \psi(\lambda) + \psi(\lambda + \delta) - 2\psi(\lambda + \delta/2).
$$
By Taylor expansion in $\delta$ around 0, the right-hand side is less than
\[
\frac{1}{4} \delta^2 \dot{\psi}(\lambda) + \frac{1}{6} \delta^3 \left( \ddot{\psi}(\lambda + \delta^*) - \frac{1}{8} \ddot{\psi}(\lambda - \delta^*/2) \right)
\]
where $0 < |\delta^*| < |\delta|$. Invoke inequality twice to bound the coefficient of $\delta^3/6$ in absolute value by
\[
\dot{\psi}(\lambda) \left( G(|\delta|) + \frac{1}{8} G(|\delta|/2) \right) \leq \frac{9}{8} \ddot{\psi}(\lambda) G(|\delta|).
\]
The stated bound simplifies some unimportant constants.

5.4. Proof of Lemma 4. Without loss of generality, let us suppose $T = 1$. For $s = 1/4$, note that
\[
\mathbb{P} \exp(sW_i) = \prod_{k \in \mathbb{N}} (1 - 2s\tau_{i,k})^{-1/2} \leq \exp \left( \sum_{k \in \mathbb{N}} s\tau_{i,k} \right) \leq e^{1/4}
\]
by virtue of the inequality $-\log(1 - t) \leq 2t$ for $|t| \leq 1/2$. With the same $s$, it then follows that
\[
\mathbb{P}\{\max_{1 \leq n} W_i > 4(\log n + x)\} \leq \exp \left( -4s(\log n + x) \right) \mathbb{P} \exp(\max_{1 \leq n} sW_i)
\]
\[
\leq e^{-x} \frac{1}{n} \sum_{i \leq n} \mathbb{P} \exp(sW_i).
\]
The 2 is just a clean upper bound for $e^{1/4}$.

5.5. Proof of Lemma 5. We shall first show some preliminary results that will be used in the main proof throughout Sections 5.5.1 to 5.5.5. In this section, for notational simplicity, we write $\sum_{j}^*$ for $\sum_{j \neq k}$.

Many of the inequalities in this section involve sums of functions of the $\theta_j$'s. The following result will save us a lot of repetition. To simplify the notation, we drop the subscripts from $\mathbb{P}_{n,\mu,K}$.

LEMMA 7.

(i) For each $r \geq 1$ there is a constant $C_r = C_r(F)$ for which
\[
\kappa_k(r, \gamma) := \sum_{j \in \mathbb{N}} \{j \neq k\} \frac{j^{-\gamma}}{|\theta_j - \theta_k|^r} \leq \begin{cases} 
C_r \left( 1 + k^{r(1+\alpha) - \gamma} \right) & \text{if } r > 1 \\
C_1 \left( 1 + k^{1+\alpha - \gamma} \log k \right) & \text{if } r = 1 
\end{cases}
\]

(ii) For each $p$,
\[
\sum_{k \leq p} \sum_{j > p} \frac{k^{-\alpha} - 2^{3} j^{-\alpha}}{|\theta_k - \theta_j|^2} = O_F(p^{1-\alpha})
\]
PROOF. For (i), argue in the same way as Hall and Horowitz (2007, page 85),
using the lower bounds
\[
|\theta_j - \theta_k| \geq \begin{cases} 
  c_\alpha j^{-\alpha} & \text{if } j < k/2 \\
  c_\alpha |j - k|^{1-\alpha} & \text{if } k/2 \leq j \leq 2k \\
  c_\alpha k^{-\alpha} & \text{if } j > 2k
\end{cases}
\]
where \(c_\alpha\) is a positive constant.

For (ii), split the range of summation into two subsets: \(\{(k, j) : j > \max(p, 2k)\}\)
and \(\{(k, j) : p/2 < k \leq p < j \leq 2k\}\). The first subset contributes at most
\[
\sum_{k \leq p} k^{-\alpha - 2\beta} \sum_{j > \max(p, 2k)} j^{-\alpha} (c_\alpha k^{-\alpha})^{-2} = O_F(p^{1-\alpha})
\]
because \(\alpha - 2\beta < -3\). The second subset contributes at most
\[
\sum_{p/2 < k \leq p} k^{-\alpha - 2\beta} c_\alpha^{-2} k^{2\alpha + 2} \sum_{j > p} j^{-\alpha} (j - k)^{-2} = O_F \left( p^{2+\alpha - 2\beta} p^{-\alpha} O(1) \right),
\]
which is of order \(O_F(p^{-\alpha})\).

Now remember that
\[
\overline{Z}_i(t) - \overline{Z}(t) = \sum_{k \in \mathbb{N}} (\overline{z}_{i,k} - \overline{z}_k) \overline{\phi}_k(t)
\]
so that
\[
\tilde{\theta}_k \{ j = k \} = \int \int \tilde{K}(s, t) \tilde{\phi}_j(s) \tilde{\phi}_k(t) \, ds \, dt = (n - 1)^{-1} \sum_{i \leq n} (\overline{z}_{i,j} - \overline{z}_j)(\overline{z}_{i,k} - \overline{z}_k),
\]
which implies \((n - 1)^{-1} \sum_{i \leq n} \overline{z}_i \overline{z}_i' = \overline{D}^2\) and
\[
(n - 1)^{-1} \sum_{i \leq n} \overline{\eta}_i \overline{\eta}_i' = D^{-1} \overline{D}^2 D^{-1} := \text{diag}(1/\overline{\theta}_1, \ldots, 1/\overline{\theta}_N).
\]

We will analyze \(\tilde{K}\) by rewriting it using the eigenfunctions for \(K\). Remember
that \(z_{i,j} = \langle \overline{Z}_i, \phi_j \rangle\) and the standardized variables \(\eta_{i,j} = z_{i,j}/\sqrt{\overline{\theta}_j}\) are independent \(N(0, 1)\)'s. Define \(z_{j} = \langle \overline{Z}, \phi_j \rangle\) and \(\eta_{j} = n^{-1} \sum_{i \leq n} \eta_{i,j}\)
and
\[
E_{j,k} := (n - 1)^{-1} \sum_{i \leq n} (\eta_{i,j} - \eta_j)(\eta_{i,k} - \eta_k),
\]
the \((j, k)\)-element of a sample covariance matrix of i.i.d. \(N(0, I_N)\) random vectors. Then
\[
\overline{Z}_i(t) - \overline{Z}(t) = \sum_{j \in \mathbb{N}} (z_{i,j} - z_j) \phi_j(t) = \sum_{j \in \mathbb{N}} \sqrt{\overline{\theta}_j} (\eta_{i,j} - \eta_j) \phi_j(t)
\]
(26) \[ \tilde{K}(s,t) = \sum_{j,k \in \mathbb{N}} \tilde{K}_{j,k}(s) \phi_j(s) \phi_k(t) \quad \text{with} \quad \tilde{K}_{j,k} = \sqrt{\theta_j \theta_k} C_{j,k} \]

Moreover, as shown in Lemma 14 in the supplemental Appendix, the main contribution to\[ f_k = \sigma_k \phi_k - \phi_k \]
is

\[ \Lambda_k := \sum_{j \in \mathbb{N}} \Lambda_{k,j} \phi_j \quad \text{with} \quad \Lambda_{k,j} := \begin{cases} \sqrt{\theta_j \theta_k} C_{j,k} / (\theta_k - \theta_j) & \text{if } j \neq k \\ 0 & \text{if } j = k \end{cases} \]

Define\[ \epsilon_k := \min \{ |\theta_j - \theta_k| : j \neq k \}. \]

The following two lemmas related to perturbation theory for self-adjoint compact operators (cf. e.g. Birman and Solomjak, 1987; Bosq, 2000; Kato, 1995) are crucial in the development of Lemma 5. They are special cases of Lemma 13 and Lemma 15 in the Appendix under the general perturbation-theoretic framework. For Lemma 8, similar results were established by other authors see e.g. Hall and Hosseini-Nasab, 2006, equation 2.8 and Cai and Hall, 2006, Section 5.6. Lemma 9 extends the perturbation result for eigenprojections, obtained by Tyler (1981, Lemma 4.1), from the matrix case to the general operator case.

**Lemma 8.** If \( \epsilon_k > 5\|\Delta\| \), it follows that

\[ \|f_k\| \leq 3\|\Lambda_k\|. \]

Define \( H_J = \text{span}\{\phi_j : j \in J\} \) and \( \tilde{H}_J = \text{span}\{\tilde{\phi}_j : j \in J\} \) for \( J \subseteq \mathbb{N} \).

**Lemma 9.** If \( \min_{k \in J} \epsilon_k > 5\|\Delta\| \), then

\[ (\tilde{H}_J - H_J)B = \sum_{j \in J} \sum_{k \in J^c} \phi_j b_k (\Lambda_{j,k} + \Lambda_{k,j}) + e \]

where \( \|e\|^2 \) is bounded by a universal constant times \( R_1 + \|\Delta\|^2 R_2 \) with

\[ R_1 = \left( \sum_{k \in J} \|\Lambda_k\|^2 \right) \sum_{k \in J} \left( \sum_{j} \Lambda_{k,j} b_j \right)^2 \]

\[ R_2 = \sum_{k \in J} \|\Lambda_k\|^2 \left( \sum_{j} \frac{|b_j|}{|\theta_k - \theta_j|} \right)^2 + \left( \sum_{k \in J} \|\Lambda_k\| |b_k| \sum_{j} \frac{1}{|\theta_k - \theta_j|} \right)^2 + \sum_{k \in J} \|\Lambda_k\|^2 |b_k|^2 k^{2+2\alpha} \]
In fact, most of the inequalities that we need for analyzing the estimator \( \hat{B} \) defined in (6) - (9) come from simple moment bounds (Lemma 10) for the sample covariances \( C_{j,k} \) and the derived bounds (Lemma 11) for the \( \Lambda_k \)'s.

The distribution of \( C_{j,k} \) does not depend on the parameters of our model. Indeed, by the usual rotation of axes we can rewrite \((n − 1)C_{j,k}\) as \(U_j'U_k\), where \(U_1, U_2, \ldots\) are independent \(N(0, I_{n−1})\) random vectors. This representation gives some useful equalities and bounds.

**Lemma 10.** Uniformly over distinct \( j, k, \ell \),

(i) \( P(C_{j,j} = 1) \) and \( P((C_{j,j} − 1)^2 = 2(n − 1)^{-1}) \)

(ii) \( P(C_{j,k} = \Sigma_{j,k}C_{j,\ell} = 0) \)

(iii) \( P(C_{j,k}^2 = O(n^{-1})) \)

**Proof.** Assertion (i) is classical because \( |U_j|^2 \sim \chi^2_{n−1} \). For assertion (ii) use \( P(U_1'U_2 | U_2) = 0 \) and

\[ P(U_1'U_2U_3'U_4 | U_2) = \text{trace}(U_2U_3'P(U_4'U_1')) = 0. \]

For (iii) use \( P(U_1U_3') = I_{n−1} \) and

\[ P(U_1'U_2U_3'U_4 | U_2) = \text{trace}(U_2U_3'P(U_4'U_1')) = \text{trace}(U_2U_3') = |U_2|^2. \]

**Lemma 11.** Uniformly over distinct \( j, k, \ell \),

(i) \( P(\Lambda_{k,j} = \Sigma_{k,j}\Lambda_{k,\ell} = 0) \)

(ii) \( P(\Lambda_{k,j}^2 = O_F((n^{-1}k^{-\alpha}j^{-\alpha}(\theta_k - \theta_j)^{-2})) \)

(iii) \( P(\|\Lambda_k\|^2 = O_F(n^{-1}k^2)) \)

**Proof.** Assertions (i) and (ii) follow from Assertions (ii) and (iii) of Lemma 10. For (iii), note that

\[ P(\|\Lambda_k\|^2 = \sum_j \|\Lambda_{j,k}\|^2 = O_F(n^{-1}k^{-\alpha})k_k(2, \alpha) \)

To prove Lemma 5 we define \( \tilde{x}_{\epsilon,n} \) as an intersection of sets chosen to make the six assertions of the Lemma hold,

\[ \tilde{x}_{\epsilon,n} := \tilde{x}_{\Delta,n} \cap \tilde{x}_{\omega,n} \cap \tilde{x}_{\Lambda,n} \cap \tilde{x}_{\eta,n} \cap \tilde{x}_{A,n}. \]
where the complement of each of the five sets appearing on the right-hand side has probability less than $\epsilon/5$. More specifically, for a large enough constant $C_\epsilon$, we define

\[
\begin{align*}
\tilde{X}_{\Delta,n} &= \{ \|\Delta\| \leq C_\epsilon n^{-1/2} \} \\
\tilde{X}_{Z,n} &= \{ \max_{i \leq n} \|Z_i\|^2 \leq C_\epsilon \log n \text{ and } \|\tilde{Z}\| \leq C_\epsilon n^{-1/2} \} \\
\tilde{X}_{\eta,n} &= \{ \max_{i \leq n} |\eta_i|^2 \leq C_\epsilon N \log n \} \quad \text{as in Section 3.3} \\
\tilde{X}_{A,n} &= \{ \| \sum_{i \leq n} \tilde{m}_i \|_2 \leq C_\epsilon n \} 
\end{align*}
\]

The definition of $\tilde{X}_{A,n}$, in subsection 5.5.3, is slightly more complicated. It is defined by requiring various functions of the $\Lambda_k$’s to be smaller than $C_\epsilon$ times their expected values.

The set $\tilde{X}_{A,n}$ is almost redundant. From Definition 1 we know that

\[
\min_{1 \leq j < j' \leq N} |\theta_j - \theta_j'| \geq (\alpha/R) N^{-1-\alpha} \quad \text{and} \quad \min_{1 \leq j \leq N} \theta_j \geq R^{-1} N^{-\alpha}.
\]

The choice $N \sim n^\zeta$ with $\zeta < (2+2\alpha)^{-1}$ ensures that $n^{1/2} N^{-1-\alpha} \to \infty$. On $\tilde{X}_{\Delta,n}$ the spacing assumption used in Lemmas 8 and 9 holds for all $n$ large enough; all the bounds from those lemmas are available to us on $\tilde{X}_{\epsilon,n}$. In particular,

\[
\max_{j \leq N} |\tilde{\theta}_j / \theta_j - 1| \leq O_F(N^{\alpha}\|\Delta\|) = o_F(1).
\]

Equality (25) shows that $\tilde{X}_{A,n} \subseteq \tilde{X}_{\Delta,n}$ eventually if we make sure $C_\epsilon > 1$.

5.5.1. **Proof of Lemma 5 part (i)**. Observe that

\[
\begin{align*}
\mathbb{P}\|\Delta\|^2 &= \sum_{j,k} \mathbb{P}\left( \tilde{K}_{j,k} - \theta_j \{ j = k \} \right)^2 = \sum_{j,k} \theta_j \theta_k \mathbb{P}\left( \tilde{S}_{j,k} - \{ j = k \} \right)^2 \\
&\leq \sum_j \theta_j O_F(n^{-1}) + \sum_{j,k} \theta_j \theta_k O_F(n^{-2}) = O_F(n^{-1})
\end{align*}
\]

5.5.2. **Proof of Lemma 5 part (ii)**. As before, Lemma 4 controls $\max_{i \leq n} \|Z_i\|^2$. To control the $\tilde{Z}$ contribution, note that $n\|\tilde{Z}\|^2$ has the same distribution as $\|Z\|^2$, which has expected value $\sum_{j \leq n} \theta_j < \infty$.

5.5.3. **Proof of Lemma 5 parts (iii) and (iv)**. Calculate expected values for all the terms that appear in the bound of Lemma 9.

\[
\begin{align*}
\mathbb{P}_{n,\mu,K} \sum_{k \leq p} \sum_{j > p} \Lambda_{k,j} b_j + \mathbb{P}_{n,\mu,K} \sum_{j > p} \left( \sum_{k \leq p} \Lambda_{k,j} b_k \right)^2 \\
&= \sum_{k \leq p} \sum_{j > p} \mathbb{P}_{n,\mu,K} \Lambda_{k,j}^2 \left( b_j^2 + b_k^2 \right) \quad \text{by Lemma 11 part (i)} \\
&= O_F(n^{-1}) \sum_{k \leq p} \sum_{j > p} k^{-\alpha - 2\beta} j^{-\alpha} (\theta_k - \theta_j)^{-2} \\
&= O_F(n^{-1} p^{1-\alpha}) \quad \text{by Lemma 7} (27)
\end{align*}
\]
\[ \|\Delta\|^2 \mathbb{P}_{n,\mu,K} \sum_{k \leq p} b_k^2 \|\Lambda_k\|^2 k^{2+2\alpha} = O_F(n^{-1} \|\Delta\|^2) \sum_{k \leq p} k^{4+2\alpha-2\beta} \]
\[ = O_F(n^{-2}) \left( 1 + p^{5+2\alpha-2\beta} + \log p \right) \]

and
\[ \mathbb{P}_{n,\mu,K} \sum_{k \leq p} |b_k||\Lambda_k|^2 = O_F(n^{-1}) \sum_{k \leq p} k^{2-\beta} = O_F(n^{-1}) \left( 1 + p^{3-\beta} + \log p \right) \]
and
\[ \mathbb{P}_{n,\mu,K} \sum_{k \leq p} \|\Lambda_k\|^2 = O_F(n^{-1}p^3) \]
and
\[ \mathbb{P}_{n,\mu,K} \sum_{k \leq p} \left( \sum_j^* \Lambda_{kj} b_j \right)^2 = O_F(n^{-1}) \sum_{k \leq p} \sum_j^* k^{-\alpha} j^{-\alpha-2\beta} (\theta_k - \theta_j)^{-2} \]
(28)
\[ = O_F(n^{-1}) \quad \text{by Lemma 7} \]

and
\[ \|\Delta\|^2 \mathbb{P}_{n,\mu,K} \sum_{k \leq p} \|\Lambda_k\|^2 \left( \sum_j^* \frac{|b_j|}{|\theta_k - \theta_j|} \right)^2 \]
(29)
\[ = O_F(n^{-1} \|\Delta\|^2) \left( p^3 + p^{5+2\alpha-2\beta} \log^2 p \right) \]
and
\[ \sum_{k \leq p} b_k^2 \left( \sum_j^* \frac{1}{|\theta_k - \theta_j|} \right)^2 = O_F(1 + p^{3+2\alpha-2\beta} \log^2 p) \quad \text{by Lemma 7}. \]

For some constant \( C_\epsilon = C_\epsilon(F) \), on a set \( \mathcal{X}_{\Lambda,n} \) with \( \mathbb{P}_{n,\mu,K} \mathcal{X}_{\Lambda,n}^c < \epsilon \), each of the random quantities in the previous set of inequalities (for both \( p = m \) and \( p = N \)) is bounded by \( C_\epsilon \) times its \( \mathbb{P}_{n,\mu,K} \) expected value. By virtue of Lemma 11 part (iii), we may also assume that \( \|\Lambda_k\|^2 \leq C_\epsilon k^2/n \) on \( \mathcal{X}_{\Lambda,n} \).

From Lemma 9, it follows that on the set \( \mathcal{X}_{\Delta,n} \cap \mathcal{X}_{\Lambda,n} \), if \( p \leq N \),
\[ \| (\tilde{H}_p - H_p) \mathcal{B} \|^2 \]
\[ \leq O_F(n^{-1}p^{1-\alpha}) + O_F(n^{-2}) \left( 1 + p^{5+2\alpha-2\beta} + \log p + p^{6-\beta} + \log^2 p \right) \]
\[ + O_F(n^{-1}p^3) O_F(n^{-1}) + O_F(n^{-2}) \left( p^3 + p^{5+2\alpha-2\beta} \log^2 p \right) \]
\[ + O_F(n^{-2}p^3) O_F(1 + p^{3+2\alpha-2\beta} \log^2 p) \]
\[ = O_F(n^{-1}p^{1-\alpha}) \]

This inequality leads to the asserted conclusions when \( p = m \) or \( p = N \).
5.5.4. **Proof of Lemma 5 part (v).** By construction, \( \bar{\eta}_i = 1 \) for every \( i \) and, for \( j \geq 2 \),
\[
\sqrt{\theta_j \bar{\eta}_{i,j}} = (\bar{z}_{i,j} - \bar{z}_j) = \langle \bar{Z}_i - \bar{Z}, \phi_j \rangle
\]
Thus, for \( j \geq 2 \),
\[
\sigma_j \bar{\eta}_{i,j} = \theta_j^{-1/2} \langle \bar{Z}_i - \bar{Z}, \phi_j + f_j \rangle = \eta_{i,j} + \bar{\delta}_{i,j}
\]
with, due to Lemma 8,
\[
|\delta_{i,j}|^2 \leq \theta_j^{-1} (|\bar{Z}_i| + |\bar{Z}|)^2 \| f_j \|^2 \leq O_F \left( \frac{j^{2+\alpha} \log n}{n} \right) \quad \text{on } \bar{X}_{i,n}.
\]
In vector form,
\[
(31) \quad \bar{S} \bar{\eta}_i = \eta_i + \bar{\delta}_i \quad \text{with } |\bar{\delta}_i|^2 = O_F \left( \frac{N^{3+\alpha} \log n}{n} \right) \leq o_F(n/N^2) \quad \text{on } \bar{X}_{i,n}.
\]
It follows that
\[
\max_{i \leq n} |\bar{\eta}_i| = \max_{i \leq n} |\bar{S} \bar{\eta}_i| \leq \max_{i \leq n} |\eta_i| + O_F(\sqrt{n}/N) = O_F(\sqrt{n}/N) \quad \text{on } \bar{X}_{i,n}.
\]

5.5.5. **Proof of Lemma 5 part (vi).** From inequality (20) we know that
\[
\bar{\epsilon}_N := \max_{i \leq n} |\bar{\lambda}_i - \lambda_{i,N}| = O_F(n^{-(1+\nu')/2}) \quad \text{on } \bar{X}_{i,n}
\]
and from the Section 3.3 we have \( \max_{i \leq n} |\lambda_{i,N}| = O_F(\sqrt{n} \log n) \). Assumption (vi) in Section 2 and the Mean-Value theorem then give
\[
\max_{i \leq n} |\bar{\epsilon}_N(\bar{\lambda}_i) - \bar{\epsilon}_N(\lambda_{i,N})| \leq \bar{\epsilon}_N \bar{\epsilon}(\lambda_{i,N})G(\bar{\epsilon}_N) = o_F(1).
\]
If we replace \( \bar{\epsilon}_N(\lambda_{i,N}) \) in the definition of \( \bar{A}_n \) by \( L_i := \bar{\epsilon}_N(\lambda_{i,N}) \) we make a change \( \Gamma \) with
\[
||\Gamma||_2 \leq o_F(1) ||(n - 1)^{-1} \sum_{i \leq n} \bar{\eta}_i \bar{\eta}_i'||_2,
\]
which, by equality (25), is of order \( o_F(1) \) on \( \bar{X}_{i,n} \).

From Assumption (vi) we have \( c_n := \log \max_{i \leq n} L_i = o_F(\log n) \). Uniformly over all unit vectors \( u \) in \( \mathbb{R}^{N+1} \) we therefore have
\[
u' \bar{S} A_n \bar{S} u = o_F(1) + (n - 1)^{-1} \sum_{i \leq n} L_i u'(\eta_i + \bar{\delta}_i)(\eta_i + \bar{\delta}_i)'u
\]
\[
= o_F(1) + \left( 1 + O(n^{-1}) \right) u' A_n u + O_F \left( n^{-1} \right) \sum_{i \leq n} L_i \left( (u' \bar{\delta}_i)^2 + 2(u' \eta_i)(u' \bar{\delta}_i) \right)
\]
Rearrange then take a supremum over \( u \) to conclude that
\[
||\bar{S} A_n \bar{S} - A_n ||_2 \leq o_F(1) + O_F(e^{c_n}) \max_{i \leq n} \left( ||\bar{\delta}_i| + 2||\bar{\delta}_i|||\eta_i| \right)
\]
Representation (31) and the defining property of \( \bar{X}_{n,n} \) then ensures that the upper bound is of order \( o_F(1) \) on \( \bar{X}_{i,n} \).
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YU, B. (1997). Assouad, Fano, and Le Cam. In A Festschrift for Lucien Le Cam (D. Pollard, E. Torgersen and G. L. Yang, eds.) 423–435.
6. Appendix. In this section, we introduce some useful results in spectral theory and perturbation theory. Some of the results are well-established. We briefly review them for the purpose of easy reference. For example, the results for eigenvalues have become quite standard for decades (see, e.g. Dunford and Schwartz, 1988, Chapter VII.6). We derive a bound for the perturbation of eigenprojections (Lemma 15) which plays a key role in the slope function estimation problem. This bound is closely related Proposition 2 in Cardot, Mas and Sarda (2007), which was tailored to solve the prediction problem at a random design. However, the two results are different. A comparison between their result and our bound in Lemma 15 is discussed later following Lemma 15. We could not find the same (or stronger) bound explicitly in the existing perturbation literature.

The spectral theory and the perturbation theory in Hilbert spaces have been serving as powerful tools that allow statisticians to tackle the statistical approximation problems in an elegant way. From Lemma 12 to Lemma 14 we shall review the well-established perturbation-theoretic results for eigenvalues and eigenvectors of positive and self-adjoint compact operators respectively. Our main contribution of this section is to extend the perturbation result for eigenprojections, obtained by Tyler (1981, Lemma 4.1), from the matrix case to the general operator case. Our perturbation result for eigenprojections will be introduced in Lemma 15.

Suppose $T$ is a positive and self-adjoint compact operator in a Hilbert space $\mathcal{H}$. According to the spectral theory for positive and self-adjoint compact operators (see e.g. Birman and Solomjak, 1987, Page 209), the operator $T$ has a sequence of decreasing nonnegative eigenvalues $\{\theta_i\}$ and a sequence of corresponding eigenvectors $\{e_i\}$. That is, $Te_i = \theta_i e_i$ with $\theta_1 \geq \theta_2 \geq \cdots \geq 0$. Furthermore, $T$ has the spectral decomposition

$$T = \sum_{k \in \mathbb{N}} \theta_k e_k \otimes e_k$$

which converges in the operator norm.

In this section, the perturbation-theoretic results are the functional analysis results without involving randomness. The focus here is on the results for positive and self-adjoint compact operators, and for more general discussion on perturbation theory of linear operators please see, for example, Kato (1995). More precisely, let $\tilde{T}$ be another positive and self-adjoint compact operator in $\mathcal{H}$ with spectral decomposition

$$\tilde{T} = \sum_{k \in \mathbb{N}} \tilde{\theta}_k \tilde{e}_k \otimes \tilde{e}_k$$

The eigenprojection of the operator $\tilde{T}$ associated with eigenvalues $\tilde{\Theta}_J := \{\tilde{\theta}_j : j \in J\}$, denoted by $\tilde{H}_J$, is the orthogonal projection onto the eigenspace of $\tilde{T}$ associated with $\tilde{\Theta}_J$, that is, span$\{\tilde{e}_j : j \in J\}$. In fact, we have $\tilde{H}_J = \sum_{j \in J} \tilde{e}_j \otimes$
\(\tilde{e}_j\). Analogously, the eigenprojection \(H_J\) can be defined for the operator \(T\). We shall study how well the differences of \(\theta_j - \tilde{\theta}_j\), \(e_j - \tilde{e}_j\), and \(H_J - \tilde{H}_J\) can be controlled by \(\Delta = T - \tilde{T}\), given that \(\delta := ||\Delta||_2\) is small.

In statistical applications, the operator \(T\) is usually taken as unknown, while the operator \(\tilde{T}\) taken as the estimation of \(T\). Perturbation theory suggests that as long as \(\tilde{T}\) approximates \(T\) well, the eigen-elements of \(\tilde{T}\) can project the analogous eigen-elements of \(T\) well. This idea has been explored and utilized by Tyler (1981), Bosq (2000), Cai and Hall (2006), Hall and Horowitz (2007), and Cardot, Mas and Sarda (2007), among others. More interestingly, Hall and Hosseini-Nasab (2006) proposes a Taylor-expansion type of approximation of eigenvectors which is better adapted to the statistical approximation purposes.

In the application of Section 3.4, we draw probabilistic conclusions when \(\tilde{T}\) is random for the special case where \(T = \mathbb{K}\), the population covariance kernel, and \(\tilde{T} = \tilde{\mathbb{K}}\), the sample covariance kernel, both acting on \(\mathcal{H} = \mathcal{L}^2[0,1]\). The eigenvectors \(\{e_i\}\) and \(\{\tilde{e}_i\}\) will be principal components \(\{\phi_i\}\) and \(\{\tilde{\phi}_i\}\) respectively.

Before formally illustrating the perturbation-theoretic results in details, we shall introduce some necessary notations and basic mathematical relations here. Because \(\{e_j : j \in \mathbb{N}\}\) forms a complete orthonormal basis for the Hilbert space \(\mathcal{H}\), the operator \(\tilde{T}\) also has the following representation

\begin{equation}
\tilde{T} = \sum_{j,k \in \mathbb{N}} \tilde{T}_{j,k} e_j \otimes e_k
\end{equation}

which converges in the operator norm.

Note that \(\tilde{T}_{j,k} = \tilde{T}_{k,j}\) because \(\tilde{T}\) is self-adjoint. This representation gives

\[\Delta = \sum_{j,k \in \mathbb{N}} \left( \tilde{T}_{j,k} - \theta_j \{j = k\} \right) e_j \otimes e_k\]

and

\[\delta^2 = ||\Delta||^2 = \sup_{||x|| = 1} \langle x, \Delta x \rangle^2 \leq \sum_{j,k \in \mathbb{N}} \left( \tilde{T}_{j,k} - \theta_j \{j = k\} \right)^2\]

We also define

\begin{equation}
\Lambda_k := \sum_{j \in \mathbb{N}} \Lambda_{k,j} e_j \quad \text{with} \quad \Lambda_{k,j} := \begin{cases} \tilde{T}_{j,k}/(\theta_k - \theta_j) & \text{if } j \neq k \\ 0 & \text{if } j = k. \end{cases}
\end{equation}

Notice that \(\{\tilde{e}_k : k \in \mathbb{N}\}\) is also an orthonormal basis for \(\mathcal{H}\). Define \(\sigma_{j,k} := \langle e_j, \tilde{e}_k \rangle\). Then

\[e_j = \sum_{k \in \mathbb{N}} \sigma_{j,k} \tilde{e}_k \quad \text{and} \quad \tilde{e}_k = \sum_{j \in \mathbb{N}} \sigma_{j,k} e_j\]

and

\[\{ j = j' \} = \langle e_j, e_{j'} \rangle = \sum_{k \in \mathbb{N}} \sigma_{j,k} \sigma_{j',k}.\]
We cannot hope to find a useful bound on \( \| \tilde{e}_k - e_k \| \), because there is no way to decide which of \( \pm \tilde{e}_k \) should be approximating \( e_k \). However, we can bound \( \| f_k \| \), where

\[
(36) \quad f_k = \sigma_k \tilde{e}_k - e_k \quad \text{with } \sigma_k := \text{sign}(\sigma_{k,k}) := \begin{cases} +1 & \text{if } \sigma_{k,k} \geq 0 \\ -1 & \text{otherwise} \end{cases},
\]

which will be enough for our purposes.

To simplify notations, write \( \sum^*_j \) for \( \sum_{j \in \mathbb{N}} \{ j \neq k \} \) and \( \sum^*_k \) for \( \sum_{k \in \mathbb{N}} \{ k \neq j \} \) in this section.

The following lemma has been proved in multiple places. Here we stated it with a brief proof for the purpose of easy reference.

**Lemma 12.** Suppose \( T \) and \( \tilde{T} \) are two positive and self-adjoint operators with spectral decompositions \( (32) \) and \( (33) \), then it follows that

\[
(37) \quad |\theta_j - \tilde{\theta}_j| \leq \delta \quad \text{for all } j \in \mathbb{N}.
\]

**Proof.** The eigenvalues have a variational characterization; see Bosq (2000, Section 4.2) or Birman and Solomjak (1987, Chapter 9):

\[
(38) \quad \theta_j = \inf_{\dim(L) < j} \sup \{ \langle x, T x \rangle : x \perp L \text{ and } \|x\| = 1 \}.
\]

The first infimum runs over all subspaces \( L \) with dimension at most \( j - 1 \). (When \( j \) equals 1 the only such subspace is \( \emptyset \).) Both the infimum and the supremum are achieved: by \( L_{j-1} = \text{span} \{ e_i : 1 \leq i < j \} \) and \( x = e_j \). Similar assertions hold for \( \tilde{T} \) and its eigenvalues.

By the analog of \( (38) \) for \( \tilde{T} \),

\[
\tilde{\theta}_j \leq \sup \{ \langle x, \tilde{T} x \rangle : x \perp L_{j-1} \text{ and } \|x\| = 1 \} \\
\leq \sup \{ \langle x, T x \rangle + \delta : x \perp L_{j-1} \text{ and } \|x\| = 1 \} = \theta_j + \delta.
\]

Argue similarly with the roles of \( T \) and \( \tilde{T} \) reversed to conclude the result. \( \square \)

In order to approximate an eigenvector \( e_k \) reasonably well, we need to assume that the eigenvalue \( \theta_k \) is well separated from the other \( \theta_j \)'s, to avoid the problem that the eigenspace of \( \tilde{T} \) for the eigenvalue \( \tilde{\theta}_k \) might have dimension greater than one. More precisely, we consider a \( k \) for which

\[
(39) \quad \epsilon_k := \min \{ |\theta_j - \theta_k| : j \neq k \} > 5\delta,
\]

which implies

\[
|\tilde{\theta}_k - \theta_j| \geq |\theta_k - \theta_j| - \delta \geq \frac{4}{5} |\theta_k - \theta_j| \geq \frac{1}{5} \epsilon_k.
\]
The following lemmas provide approximative results for $f_k$ under the assumption that $\epsilon_k > 5\delta$. Similar results were established by other authors; see, for example, Hall and Hosseini-Nasab, 2006, equation 2.8 and Cai and Hall, 2006, Section 5.6.

**Lemma 13.** Suppose $T$ and $\tilde{T}$ are two positive and self-adjoint operators with spectral decompositions (32) and (33). The vectors $\{\Lambda_k\}$ and $\{f_k\}$ are defined as in (35) and (36) respectively. Then if $\epsilon_k > 5\delta$, it follows that

$$\|f_k\| \leq 3\|\Lambda_k\|.$$  

**Proof.** The starting point for our approximations is the equality

$$\langle \Delta \tilde{e}_k, e_j \rangle = \langle \tilde{T} \tilde{e}_k, e_j \rangle - \langle \tilde{e}_k, Te_j \rangle = (\tilde{\theta}_k - \theta_j)\sigma_{j,k}.$$  

For $j \neq k$ we then have

$$\frac{16}{25}(\theta_k - \theta_j)^2\sigma_{j,k} \leq \langle \sigma_k \Delta \tilde{e}_k, e_j \rangle^2 \leq 2\langle \Delta f_k, e_j \rangle^2 + 2\langle \Delta e_k, e_j \rangle^2,$$

which implies

$$\sigma_{j,k}^2 \leq \frac{25}{8}\frac{\langle \Delta f_k, e_j \rangle^2}{\epsilon_k^2} + 2\frac{\tilde{T}_{j,k}^2}{(\theta_k - \theta_j)^2} \quad \text{because } \langle Te_k, e_j \rangle = 0 \text{ for } j \neq k.$$  

The introduction of the $\sigma_k$ also ensures that

$$\|f_k\|^2 = \|e_k\|^2 - 2\sigma_k \langle e_k, \tilde{e}_k \rangle = 2 - 2|\sigma_k| \quad \text{because } |\sigma_k| \leq 1$$

$$= 2\sum_j^* \sigma_{j,k}^2$$

$$\leq \sum_j^* \frac{25}{4}\frac{\langle \Delta f_k, e_j \rangle^2}{\epsilon_k^2} + \frac{25}{4}\sum_j^* \frac{\tilde{T}_{j,k}^2}{(\theta_k - \theta_j)^2}.$$  

The first sum on the right-hand side is less than

$$\frac{25}{4}\frac{\|\Delta f_k\|^2}{\epsilon_k^2} \leq \delta^2\frac{\|f_k\|^2}{(4\delta^2)} = \|f_k\|^2/4.$$  

The second sum can be written as $25\|\Lambda_k\|^2/4$. Then,

$$\|f_k\|^2 \leq \frac{25}{3}\|\Lambda_k\|^2 < 9\|\Lambda_k\|^2.$$  

$\square$
LEMMA 14. Suppose $T$ and $\tilde{T}$ are two positive and self-adjoint operators with spectral decompositions (32) and (33). The vectors $\{\Lambda_k\}$ and $\{f_k\}$ are defined as in (35) and (36) respectively. Then if $\epsilon_k > 5\delta$, the corresponding operator $f_k$ has the representation:

$$f_k = \Lambda_k + r_k$$

with

$$\langle r_k, e_k \rangle = -\frac{1}{2} \|f_k\|^2 \quad \text{and} \quad |\langle r_k, e_j \rangle| \leq \frac{5\delta \|\Lambda_k\|}{|\theta_k - \theta_j|} \quad \forall \ j \neq k.$$

PROOF. Start once more from equality (40). For $j \neq k$,

$$\sigma_k \sigma_{j,k} = \sigma_k \langle \Delta \tilde{e}_k, e_j \rangle / (\tilde{\theta}_k - \theta_j)$$

$$= \langle \Delta (e_k + f_k), e_j \rangle / (\theta_k + \gamma_k - \theta_j) \quad \text{where} \ \gamma_k = \tilde{\theta}_k - \theta_k$$

$$= \Lambda_{k,j} \left(1 - \frac{\gamma_k}{\theta_j - \theta_k}\right)^{-1} + \frac{\langle \Delta f_k, e_j \rangle}{\tilde{\theta}_k - \theta_j} \quad \text{because} \ \langle Te_k, e_j \rangle = 0$$

(41)$$= \Lambda_{k,j} + r_{k,j} \quad \text{where} \ r_{k,j} := \frac{\tilde{\theta}_k - \theta_k}{\theta_j - \theta_k} \Lambda_{k,j} + \frac{\langle \Delta f_k, e_j \rangle}{\tilde{\theta}_k - \theta_j}.$$ 

The $r_{k,j}$'s are small:

$$|r_{k,j}| \leq \frac{5}{4} \left(\frac{\delta |\Lambda_{k,j}| + |\langle \Delta f_k, e_j \rangle|}{|\theta_k - \theta_j|}\right) \quad \text{for} \ j \neq k, \text{if} \ \epsilon_k > 5\delta$$

(42)$$\leq \frac{5\delta \|\Lambda_k\|}{|\theta_k - \theta_j|} \quad \text{by Lemma 13}.$$

Define $r_{k,k} = |\sigma_{k,k}| - 1 = -\frac{1}{2} \|f_k\|^2$ and $r_k = \sum_{j \in \mathbb{N}} r_{k,j} e_j$. Combine (35) and (41), we then have the representation:

(43)$$f_k = \sigma_k \tilde{e}_k - e_k = (\sigma_k \langle \tilde{e}_k, e_k \rangle - 1) e_k + \sum_{j}^* \sigma_k \sigma_{j,k} e_j = \Lambda_k + r_k.$$

In the rest of this section, we shall establish an approximation for $\tilde{H}_J \mathbb{B} - H_J \mathbb{B}$ for a $\mathbb{B} = \sum_{j} b_j e_j$ in $\mathcal{H}$, an extension of the finite-dimensional perturbation result Tyler (1981, Lemma 4.1) to the case of general infinite-dimensional operators.
The difference $\tilde{H}_J - H_J$ equals
\[
\sum_{k \in J} (\sigma_k e_k) \otimes (\sigma_k e_k) - e_k \otimes e_k \\
= \sum_{k \in J} \sigma_k e_k \otimes r_k + \sum_{k \in J} (e_k + f_k) \otimes \Lambda_k \\
+ \sum_{k \in J} ((e_k + \Lambda_k + r_k) \otimes e_k - e_k \otimes e_k) \\
= R_J + \sum_{k \in J} e_k \otimes \Lambda_k + \Lambda_k \otimes e_k
\]
(44)
where $R_J := \sum_{k \in J} \sigma_k e_k \otimes r_k + f_k \otimes \Lambda_k + r_k \otimes e_k$.

Self-adjointness of $T$ implies $\bar{T}_{j,k} = \bar{T}_{k,j}$ and hence $\Lambda_{j,k} = -\Lambda_{k,j}$. The anti-symmetry eliminates some terms from the main contribution to $\tilde{H}_J - H_J$:
\[
\sum_{k \in J} e_k \otimes \Lambda_k + \Lambda_k \otimes e_k = \sum_{k \in J} \sum_{j \in J^c} \Lambda_{k,j} (e_k \otimes e_j + e_j \otimes e_k).
\]
(45)
With this simplification we get the following representation for $(\tilde{H}_J - H_J)B$:
\[
(\tilde{H}_J - H_J)B = \sum_{j \in J} \sum_{k \in J^c} e_j b_k (\Lambda_{j,k} + \Lambda_{k,j}) + R_J B.
\]

For the three contributions to the bound for $\|R_J B\|^2$ we make repeated use of the inequalities, based on Lemma 13 and Lemma 14,
\[
|\langle r_k, x \rangle| \leq \frac{3}{2} \|\Lambda_k\| \|f_k\||x_k| + 5\delta \|\Lambda_k\| \sum_{j}^* \left| \frac{x_j}{\theta_k - \theta_j} \right|
\]
(46)
\[
\leq \frac{9}{2} \|\Lambda_k\|^2 |x_k| + 5\delta \|\Lambda_k\| \sum_{j}^* \left| \frac{x_j}{\theta_k - \theta_j} \right|
\]
(47)
which is valid whenever $\epsilon_k > 5\delta$. Combine (46) and the following well-known inequality (see e.g. Hall and Horowitz, 2007, Equ. 5.2):
\[
\|f_k\| \leq 2\sqrt{2} \delta \min\{\theta_{k-1} - \theta_k, \theta_k - \theta_{k+1}\}^{-1} = 2\sqrt{2} \delta_k^{-1},
\]
we get
\[
|\langle r_k, x \rangle| \leq C\delta \|\Lambda_k\| \left( \epsilon_k^{-1} |x_k| + \sum_{j}^* \left| \frac{x_j}{\theta_k - \theta_j} \right| \right).
\]

To avoid an unnecessary calculation of precise constants, we adopt the convention of the variable constant: we write $C$ for a universal constant whose value might change from one line to the next. The first two contributions are:
\[
\| \sum_{k \in J} \sigma_k e_k \langle r_k, B \rangle \|^2 = \sum_{k \in J} \langle r_k, B \rangle^2 \\
\leq C\delta^2 \sum_{k \in J} \|\Lambda_k\|^2 \left[ b_k^2 \epsilon_k^{-2} + \left( \sum_{j}^* \left| \frac{b_j}{\theta_k - \theta_j} \right| \right)^2 \right]
and
\[ \left\| \sum_{k \in J} f_k \langle \Lambda_k, b \rangle \right\|^2 \leq \left( \sum_{k \in J} \| f_k \| \| \langle \Lambda_k, b \rangle \| \right)^2 \leq C \left( \sum_{k \in J} \| \Lambda_k \|^2 \right) \sum_{k \in J} \left( \sum_{j}^* \Lambda_{k,j} b_j \right)^2. \]

For the third contribution, let \( x = \sum_j x_j e_j \) be an arbitrary unit vector in \( H \). Then
\[ \left( \sum_{k \in J} \langle r_k \otimes e_k b, x \rangle \right)^2 = \left( \sum_{k \in J} b_k \langle r_k, x \rangle \right)^2 \leq C \delta^2 \left[ \sum_{k \in J} |b_k| \| \Lambda_k \|^2 \left( |x_k| \epsilon_k^{-1} + \sum_j^* \frac{|x_j|}{|\theta_k - \theta_j|} \right) \right]^2 \]
(48)
\[ \leq C \delta^2 \sum_{k \in J} |b_k|^2 \| \Lambda_k \|^2 \epsilon_k^{-2} + C \delta^2 \left( \sum_{k \in J} \| \Lambda_k \| |b_k| \sum_j^* \frac{1}{|\theta_k - \theta_j|} \right)^2. \]
(49)
take the supremum over \( x \), which doesn’t even appear in the last line, to get the same bound for \( \left\| \sum_{k \in J} b_k r_k \right\|^2 \).

In sum, we can obtain the following lemma:

\text{L}E\text{M}E\text{M}A 15. \text{I}f \( \) \text{m}in_{k \in J} \epsilon_k > 5 \delta, \text{t}hen
\[ (\tilde{H}_J - H_J) b = \sum_{j \in J} \sum_{k \in J^*} e_{j,k} b \Lambda_{j,k} + R_J b \]
where \( R_J \) is defined in (44) and \( \| R_J b \|^2 \) is bounded by a universal constant times \( R_1 + \delta^2 R_2 \) with
\[ R_1 = \left( \sum_{k \in J} \| \Lambda_k \|^2 \right) \sum_{k \in J} \left( \sum_{j}^* \Lambda_{k,j} b_j \right)^2 \]
\[ R_2 = \sum_{k \in J} \| \Lambda_k \|^2 \left( \sum_{j}^* \frac{|b_j|}{|\theta_k - \theta_j|} \right)^2 + \left( \sum_{k \in J} \| \Lambda_k \| |b_k| \sum_{j}^* \frac{1}{|\theta_k - \theta_j|} \right)^2 \]
\[ + \sum_{k \in J} \| \Lambda_k \|^2 |b_k|^2 \epsilon_k^{-2} \]

This lemma is the keystone to establish the parts (iii) and (iv) in Lemma 5. It is similar to Proposition 2 in \text{C}ardot, \text{M}as and \text{S}arda (2007) in the sense that both deal with the approximation problems of eigenprojections. Particularly, we observe that the same trick of using anti-symmetry is applied to eliminate some terms from the main contributions to the approximation errors (see Equation (45) in this section and Equation (23) in \text{C}ardot, \text{M}as and \text{S}arda (2007)). However, the two results are
different. The other authors consider the bound for $\langle (\tilde{H}_J - H_J) B, X_{n+1} \rangle$ which is motivated by the prediction problem at a random design, whereas we establish a bound for $\| (\tilde{H}_J - H_J) B \|$ which is relevant to the slope function estimation problem. More precisely, the independent randomness of $X_{n+1}$ helps cancel off many cross-product terms and accelerates the decay rates of the summands in the expansion. See, for example, Equation (24) and (25) in Cardot, Mas and Sarda (2007). Due to the ‘smoothing’ effect of the independent random curve $X_{n+1}$, we cannot directly apply the convergence result in Cardot, Mas and Sarda (2007, Proposition 2) to our case. Besides, the bound in the lemma above is a pure mathematical perturbation-theoretic result not involving any randomness treatment, which we believe is a potentially more general result.