Duality Identities for Moduli Functions of Generalized Melvin Solutions Related to Classical Lie Algebras of Rank 4

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We consider generalized Melvin-like solutions associated with nonexceptional Lie algebras of rank 4 (namely, $A_4$, $B_4$, $C_4$, and $D_4$) corresponding to certain internal symmetries of the solutions. The system under consideration is a static cylindrically symmetric gravitational configuration in $D$ dimensions in presence of four Abelian 2-forms and four scalar fields. The solution is governed by four moduli functions $H_s(z)$ ($s = 1, \ldots, 4$) of squared radial coordinate $z = \rho^2$ obeying four differential equations of the Toda chain type. These functions turn out to be polynomials of powers $(n_1, n_2, n_3, n_4) = (4, 6, 6, 4), (8, 14, 18, 10), (7, 12, 15, 16), (6, 10, 6, 6)$ for Lie algebras $A_4, B_4, C_4$, and $D_4$, respectively. The asymptotic behaviour for the polynomials at large distances is governed by some integer-valued $4 \times 4$ matrix $\gamma$ connected in a certain way with the inverse Cartan matrix of the Lie algebra and (in $A_4$ case) the matrix representing a generator of the $\mathbb{Z}_2$ group of symmetry of the Dynkin diagram. The symmetry properties and duality identities for polynomials are obtained, as well as asymptotic relations for solutions at large distances. We also calculate 2-form flux integrals over 2-dimensional discs and corresponding Wilson loop factors over their boundaries.

1. Introduction

In this paper, we investigate properties of multidimensional generalization of Melvin’s solution [1], which was presented earlier in [2]. Originally, model from [2] contains metric, $n$ Abelian 2-forms and $l \geq n$ scalar fields. Here we consider a special solutions with $n = l = 4$, governed by a $4 \times 4$ Cartan matrix $(A_{ij})$ for simple nonexceptional Lie algebras of rank 4: $A_4, B_4, C_4, D_4$. The solutions from [2] are special case of the so-called generalized fluxbrane solutions from [3].

It is well known that the original Melvin’s solution in four dimensions describes the gravitational field of a magnetic flux tube. The multidimensional analog of such a flux tube, supported by a certain configuration of form fields, is referred to as a fluxbrane (a “thickened brane” of magnetic flux). The appearance of fluxbrane solutions was originally motivated by superstring/brane models and $M$-theory. For generalizations of the Melvin solution and fluxbrane solutions see [4–21] and references therein.

In [3] there were considered the generalized fluxbrane solutions which are described in terms of moduli functions $H_s(z) > 0$ defined on the interval $(0, +\infty)$, where $z = \rho^2$ and $\rho$ is a radial coordinate. Functions $H_s(z)$ obey $n$ nonlinear differential master equations of Toda-like type governed by some matrix $(A_{s\alpha})$, and the following boundary conditions are imposed: $H_s(+0) = 1, s = 1, \ldots, n$.

Here, as in [2], we assume that the matrix $(A_{s\alpha})$ is a Cartan matrix for some simple finite-dimensional Lie algebra $\mathfrak{g}$ of rank $n$ ($A_{s\alpha} = 2$ for all $s$). A conjecture was suggested in [3] that in this case the solutions to master equations with the above boundary conditions are polynomials of the form:

$$H_s(z) = 1 + \sum_{k=1}^{n_s} p_{ks}^{(k)} \rho^k,$$  

(1)
where $p_s^{(k)}$ are constants. Here $P_s^{(n)} \neq 0$ and

$$n_s = 2 \sum_{s=1}^{s} A^{(s)}_s,$$  \hspace{1cm} (2)

where we denote $(A^{(s)}_s) = (A_{s})^{-1}$. Integers $n_s$ are components of the twice dual Weyl vector in the basis of simple (co)roots [22].

Therefore, the functions $H_s$ (which may be called “fluxbrane polynomials”) define a special solution to open Toda chain equations [23, 24] corresponding to simple finite-dimensional Lie algebra $G$ [25]. In [2, 26] a program (in Maple) for calculation of these polynomials for classical series of Lie algebras ($A$, $B$, $C$, and $D$-series) was suggested. It was pointed out in [3] that the conjecture on polynomial structure $A_B$ of Lie algebras $(\{g\})$ for calculation of these polynomials for classical series (co)roots [22].

nents of the twice dual Weyl vector in the basis of simple and present asymptotic relations for the solutions. We also symmetry properties and duality relations for polynomials and geometric properties of the solution considered for case of rank-3 algebras.

2. The Setup and Generalized Melvin Solutions

Let us consider the following product manifold:

$$M = (0, +\infty) \times M_1 \times M_2,$$  \hspace{1cm} (3)

where $M_1 = S^1$ and $M_2$ is a $(D - 2)$-dimensional Ricci-flat manifold.

On this manifold, we define the action

$$S = \int d^Dx \sqrt{|g|} \left\{ R[g] - \delta_{ab} g^{MN} \partial_{\alpha} \phi^a \partial_\beta \phi^b \right\}$$  \hspace{1cm} (4)

$$- \frac{1}{2} \sum_{s=1}^{4} \exp \left\{ 2 \bar{\lambda}_s \phi \right\} \left( F^s \right)^2,$$

where $g = g_{MN}(x) dx^M \otimes dx^N$ is a metric on $M$, $\bar{\phi} = (\phi^a) \in \mathbb{R}^4$ is vector of scalar fields, $F^a = dA^a = (1/2) F^a_{MN} dx^M \wedge dx^N$ is a 2-form, and $\bar{\lambda}_s = (\lambda^a_s) \in \mathbb{R}^4$ is dilatonic coupling vector, $s = 1, \ldots, 4$; $a = 1, \ldots, 4$. Here we use the notations $|g| = |\det(g_{MN})|; (F^a)^2 \equiv F^a_{MN} F^a_{MN}, g^{MN} g^{MN}$. There is a family of exact cylindrically symmetric solutions to the field equations corresponding for the action (4) and depending on the radial coordinate $\rho$. The solution has the form [2]

$$g = \left( \frac{4}{s+1} \right)^{(D-2)/2} \cdot \left\{ d\rho \otimes d\rho + \left( \frac{4}{s+1} \right) H^2 \cdot \rho^2 d\phi \otimes d\phi + g^2 \right\}, \hspace{1cm} (5)$$

$$\exp (q^a) = \left( \frac{4}{s+1} \right) H^2 q^a_s,$$  \hspace{1cm} (6)

$$F^a = q_s \left( \frac{4}{s+1} \right) H_s^a \rho d\rho \wedge d\phi,$$  \hspace{1cm} (7)

$s, a = 1, \ldots, 4$, where $g^1 = d\phi \otimes d\phi$ is a metric on $M_1 = S^1$ and $g^2$ is a Ricci-flat metric of signature $(-, +, +)$ on $M_2$. Here $q_s \neq 0$ are integration constants ($q_s = -Q_s$ in notations of [2]).

For further convenience, let us denote $z = \rho^2$. As it was shown in earlier works, the functions $H_s(z) > 0$ obey the set of master equations

$$\frac{d}{dz} \left( \frac{z}{H_s} \frac{d}{dz} H_s \right) = P^4_{s=1} H_s^{A_s},$$  \hspace{1cm} (8)

with the boundary conditions

$$H_s(+0) = 1,$$  \hspace{1cm} (9)

where

$$P_s = \frac{1}{4} K_s^2,$$  \hspace{1cm} (10)

$s = 1, \ldots, 4$. The boundary condition (9) guarantees the absence of a conic singularity (for the metric (5)) for $\rho = +0$.

There are some relations for the parameters $h_{s}$:

$$h_{s} = K_{s}^{-1}, \hspace{1cm} K_{s} = B_{s0} > 0,$$  \hspace{1cm} (11)

where

$$B_{s} \equiv 1 + \frac{1}{2-D} + \bar{\lambda}_s \bar{\lambda}_s,$$  \hspace{1cm} (12)
In these relations, we have denoted

\[
(A_{s l}) = \left( \frac{2B_{sl}}{B_{ll}} \right) .
\]

The latter matrix is the so-called "quasi-Cartan" matrix. One can prove that if \((A_{s l})\) is a Cartan matrix for a certain simple Lie algebra \(\mathcal{G}\) of rank 4, there exists a set of vectors \(\vec{\lambda}_1, \ldots, \vec{\lambda}_4\) obeying (13). See also Remark 1 in the next section.

The solution considered can be understood as a special case of the fluxbrane solutions from [3, 19].

Therefore, here we investigate a multidimensional generalization of Melvin's solution [1] for the case of four scalar fields and four 2-forms. Note that the original Melvin's solution [1] for the case of four scalar functions and \(4\) electromagnetic 2-forms, \(M_1 = S^1\) \((0 < \phi < 2\pi)\), \(M_2 = \mathbb{R}^2\), and \(g^2 = -dt \otimes dt + dx \otimes dx\).

### 3. Solutions Related to Simple Classical Rank-4 Lie Algebras

In this section we consider the solutions associated with the simple nonexceptional Lie algebras \(\mathcal{G}\) of rank 4. This means that the matrix \(A = (A_{s l})\) coincides with one of the Cartan matrices

\[
(A_{s l}) = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{pmatrix},
\]

where \(h_i = K_i^{-1}\) and

\[
\vec{\lambda}_s \vec{\lambda}_l = \frac{1}{2} K_i A_{s l} - \frac{D-3}{D-2} \equiv G_{s l}
\]

for \(s, l = 1, 2, 3, 4\); (15) is a special case of (16).

From (12) and (13) it also follows that

\[
\frac{h_s}{h_i} \frac{K_i}{K_s} = \frac{B_{s l}}{B_{l l}} = \frac{B_{s l} B_{l l}}{B_{l s} B_{s l}} = \frac{A_{s l}}{A_{l s}}
\]

for any \(s \neq l\) obeying \(A_{s l}, A_{l s} \neq 0\). This implies

\[
K_1 = K_2 = K_3 = K,
\]

\[
K_4 = K, \frac{1}{2} K, 2K, K,
\]

or

\[
K_1 = K_2 = K_3 = h,
\]

\[
K_4 = h, 2h, \frac{1}{2} h, h
\]

\((h = K^{-1})\) for \(\mathcal{G} = A_4, B_4, C_4, D_4\), respectively.

**Remark 1.** For large enough \(K_1\) in (18) there exist vectors \(\vec{\lambda}_s\) obeying (16) (and hence (15)). Indeed, the matrix \((G_{s l})\) is positive definite if \(K_1 > K_2\), where \(K_2\) is some positive number. Hence there exists a matrix \(\Lambda\), such that \(\Lambda^T \Lambda = G\). We put \((A_{s l}) = (\lambda_s^l)\) and get the set of vectors obeying (16).

**Polynomials.** According to the polynomial conjecture, the set of moduli functions \(H_s(z), \ldots, H_4(z)\), obeying (8) and (9) with the Cartan matrix \(A = (A_{s l})\) from (14) are polynomials with powers \((n, n, n, n, n) = (4, 6, 4), (8, 14, 18, 10), (7, 12, 15, 16), (6, 10, 6, 6)\) calculated by using (2) for Lie algebras \(A_4, B_4, C_4, D_4\), respectively.

One can prove this conjecture by solving the system of nonlinear algebraic equations for the coefficients of these polynomials following from master equations (8). Below we present a list of the polynomials obtained by using appropriate MATHEMATICA algorithm. For convenience, we use the rescaled variables (as in [25]):

\[
P_s = \frac{P_s}{n_s}.
\]
$A_4$-Case. For the Lie algebra $A_4 \equiv sl(5)$ we have

\begin{align}
H_1 &= 1 + 4p_1z + 6p_1p_2z^2 + 4p_1p_2p_3z^3 \\
& \quad + p_1p_2p_3p_4z^4, \\
H_2 &= 1 + 6p_2z + (6p_1p_2 + 9p_3p_4)z^2 \\
& \quad + (16p_1p_2p_3 + 4p_3p_4)z^3 \\
& \quad + (6p_1p_2^2p_3 + 9p_1p_2p_3p_4)z^4 \\
& \quad + 6p_1p_2p_3p_4z^5 + p_1p_2^2p_3p_4z^6, \\
H_3 &= 1 + 6p_3z + (9p_2p_3 + 6p_4p_4)z^2 \\
& \quad + (4p_1p_2p_3 + 16p_3p_4)z^3 \\
& \quad + (9p_1p_2p_3p_4 + 6p_2p_3^2p_4)z^4 \\
& \quad + 6p_1p_2p_3^2p_4z^5 + p_1p_2^2p_3p_4^2z^6, \\
H_4 &= 1 + 4p_4z + 6p_3p_4z^2 + 4p_1p_2p_3z^3 \\
& \quad + p_1p_2p_3p_4z^4.
\end{align}

(21)

$B_4$-Case. For the Lie algebra $B_4 \equiv so(9)$ the fluxbrane polynomials are

\begin{align}
H_1 &= 1 + 8p_1z + 28p_1p_2z^2 + 56p_1p_2p_3z^3 \\
& \quad + 70p_1p_2p_3p_4z^4 + 56p_1p_2p_3p_4^2z^5 \\
& \quad + 28p_1p_2p_3^2p_4z^6 + 8p_1p_2p_3^2p_4^2z^7 + p_1p_2^2p_3^2p_4z^8, \\
H_2 &= 1 + 14p_2z + (28p_1p_4 + 63p_2p_4)z^2 \\
& \quad + (224p_1p_2p_3 + 140p_3p_4p_4)z^3 + (196p_1p_2^2p_3)z^4 \\
& \quad + 630p_1p_2p_3p_4 + 175p_2p_3^2p_4)z^4 + (980p_1p_2^2p_3p_4)z^5 \\
& \quad + 896p_1p_2p_3^2p_4 + 126p_2p_3^2p_4^2z^5 + (490p_1p_2^2p_3p_4)z^6 \\
& \quad + 1764p_1p_2^2p_3^2p_4 + 700p_1p_2^3p_3p_4 + (49p_1p_2^2p_3^2p_4)z^7 \\
& \quad + 3432p_1p_2^2p_3^2p_4^2 + 700p_1p_2^3p_3p_4 + 1764p_1p_2^3p_3^2p_4 \\
& \quad + 490p_1p_2^3p_3^2p_4 + 980p_1p_2^3p_3p_4^2)z^8 \\
& \quad + (126p_1p_2^3p_3^2p_4)z^9 + (126p_1p_2^3p_3^2p_4)z^10 \\
& \quad + (175p_1p_2^3p_3^2p_4 + 630p_1p_2p_3^3p_4^2)z^9 \\
& \quad + (175p_1^2p_2^3p_3^2p_4^2 + 630p_1p_2^3p_3^3p_4)z^10 \\
& \quad + 196p_1p_2^3p_3^3p_4^2)z^{10} + (140p_1p_2^3p_3^3p_4)z^{11} \\
& \quad + 224p_1p_2^3p_3^3p_4^2)z^{11} + (63p_1p_2^3p_3^3p_4)z^{12} \\
& \quad + 28p_1p_2^3p_3^3p_4^2)z^{12} + 14p_1^2p_2^3p_3^3p_4^2z^{13} \\
& \quad + p_1^2p_2^3p_3^3p_4^2z^{14},
\end{align}

(26)
\[ H_4 = 1 + 10p_1z + 45p_2p_3z^2 + (70p_2p_3p_4 + 50p_3^2)z^3 + (35p_1p_2p_3 + 175p_2^2p_3^2)z^4 + (126p_1p_2p_3p_4 + 126p_2^2p_3p_4^2)z^5 + (175p_1p_2p_3^2 + 35p_2^3p_3^3)z^6 + (50p_1^2p_2^2p_3^2p_4)z^7 + 70p_1p_2^2p_3p_4z^8 + 10p_1^2p_2^2p_3^2p_4z^9 \] + \left(90p_1^2p_2^2p_3^5p_4^5 + 63p_1^2p_2^2p_3^5p_4^6\right)z^{16} + 18p_1^2p_2^2p_3^5p_4^6z^{17} + p_1^2p_2^2p_3^5p_4^6z^{18}. \]

(27)

\[ C_4 \text{-Case. For the Lie algebra } C_4 \equiv sp(6) \text{ we get the following polynomials:} \]

\[ H_1 = 1 + 7p_1z + 21p_1p_2z^2 + 35p_1p_2p_3z^3 + 35p_1p_2p_3p_4z^4 + 21p_1p_2^2p_3^2z^5 + 7p_1^2p_2^2p_3^2p_4z^6 \]

(29)

\[ H_2 = 1 + 12p_2z + (21p_1 + 25p_2p_3)z^2 + (140p_1p_2p_3 + 80p_2p_3p_4)z^3 + (105p_1^2p_2^2p_3^2p_4)z^4 + (315p_1p_2p_3p_4 + 75p_2^2p_3^2p_4)z^5 + (420p_1p_2^2p_3^2p_4)z^6 + (336p_1^2p_2^2p_3^2p_4 + 56p_2^3p_3^3p_4)z^7 + 924p_1^2p_2^2p_3^2p_4z^8 + (36p_1^2p_2^3p_3^2p_4 + 336p_1^2p_2^3p_3^2p_4)z^9 + \left(75p_1^2p_2^3p_3^2p_4 + 315p_1^2p_2^3p_3^3p_4 + 105p_1^2p_2^3p_3^3p_4\right)z^{10} + (80p_1^2p_2^3p_3^3p_4)z^{11} + (140p_1^2p_2^3p_3^3p_4)z^{12} \]

(30)

\[ H_3 = 1 + 15p_3z + (45p_2p_3 + 60p_3p_4)z^2 + (35p_1p_2p_3 + 320p_2p_3p_4 + 100p_3^2p_4)z^3 + (315p_1p_2p_3p_4 + 1050p_2^3p_3^3p_4)z^4 + (1302p_1p_2p_3^2p_4 + 576p_2^3p_3^3p_4 + 1125p_2^3p_3^3p_4)z^5 + (1050p_1^2p_2^3p_3^3p_4 + 2240p_1p_2^3p_3^3p_4)z^6 + 1215p_1^2p_2^3p_3^3p_4z^7 + 500p_1^2p_2^3p_3^3p_4z^8 + (225p_1^2p_2^3p_3^3p_4)z^9 + 3990p_1^2p_2^3p_3^3p_4z^10 + 1260p_1p_2^3p_3^3p_4 + 960p_1^2p_2^3p_3^4)z^{11} + (960p_1^2p_2^3p_3^4)z^{12} + 3990p_1^2p_2^3p_3^4z^{13} + 1260p_1p_2^3p_3^4 + 960p_1^2p_2^3p_3^4z^{14} + 160p_1^3p_2^3p_3^4z^{15}. \]

(31)
Let us denote
\[ s = z^{2} + 10z + (15z)z^{4} + 6z_{p}^{2}p_{3}z_{p}z^{5} + p_{1}^{2}p_{2}p_{3}z_{p}z^{6}, \]
\[ H_{1} = 1 + 6p_{1}z + 15p_{1}p_{2}z^{2} + 10p_{1}p_{2}p_{3} \]
\[ + 10p_{1}p_{2}p_{4} (z^{3} + 15p_{1}p_{2}p_{3}p_{4}z^{4} + 6p_{1}^{2}p_{3}p_{4}z^{5} + p_{1}^{2}p_{2}p_{3}p_{4}z_{p}z^{6}), \]
\[ H_{2} = 1 + 10p_{2}z + (15p_{1} + 15p_{2}p_{3} + 15p_{2}p_{4}) z^{2} \]
\[ + (40p_{1}p_{2}p_{3} + 40p_{1}p_{2}p_{4} + 40p_{2}p_{3}p_{4}) z^{3} + (25p_{1}p_{2}p_{3} + 25p_{1}p_{2}p_{4} + 135p_{1}p_{2}p_{3}p_{4} \]
\[ + 25p_{2}p_{3}p_{4}) z^{4} + 25p_{2}p_{3}p_{4}z^{5} + (25p_{2}p_{3}p_{4}z_{p}z^{6}), \]
\[ H_{3} = 1 + 6p_{3}z + 15p_{2}p_{3}z^{2} + 10p_{1}p_{2}p_{3} \]
\[ + 10p_{1}p_{2}p_{3} (z^{3} + 15p_{1}p_{2}p_{3}p_{4}z^{4} + 6p_{1}^{2}p_{3}p_{4}z^{5} + p_{1}^{2}p_{2}p_{3}p_{4}z_{p}z^{6}), \]
\[ H_{4} = 1 + 6p_{4}z + 15p_{2}p_{4}z^{2} + (10p_{1}p_{2}p_{4} \]
\[ + 10p_{2}p_{3}p_{4} (z^{3} + 15p_{1}p_{2}p_{3}p_{4}z^{4} + 6p_{1}^{2}p_{3}p_{4}z^{5} + p_{1}^{2}p_{2}p_{3}p_{4}z_{p}z^{6}). \]

Let us denote
\[ H_{4} = H_{4}(z, (p_{i})) \equiv H_{4}(z, (p_{i})), \]
\[ (p_{i}) \equiv (p_{1}, p_{2}, p_{3}, p_{4}). \]

One can easily write down the asymptotic behaviour of the polynomials obtained:
\[ H_{4} = H_{4}(z, (p_{i})) \sim (\frac{4}{1} (p_{i})^{\nu}) z^{n} \equiv H_{4}^{nu}(z, (p_{i})), \]
\[ \text{as } z \to \infty, \]

where we introduced the integer-valued matrix \( \nu = (\nu^{j}) \)

having the form
\[ \nu = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \]
\[ \nu = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \end{pmatrix}, \]
\[ \nu = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}, \]
\[ \nu = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 4 & 6 \end{pmatrix}, \]
\[ \nu = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \end{pmatrix}, \]
\[ \nu = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 4 & 6 \end{pmatrix} \]

for Lie algebras \( A_{4}, B_{4}, C_{4}, D_{4} \), respectively. In these four cases there is a simple property
\[ \sum_{i=1}^{4} \nu^{j} = n_{i}, \quad s = 1, 2, 3, 4. \]

Note that for Lie algebras \( B_{4}, C_{4}, \) and \( D_{4} \) we have
\[ \nu = 2A^{-1}, \]
\[ \mathbb{G} = B_{4}, C_{4}, D_{4} \]

where \( A^{-1} \) is inverse Cartan matrix, whereas in the \( A_{4} \)-case the matrix \( \nu \) is related to the inverse Cartan matrix as follows:
\[ \nu = A^{-1} (I + P), \]
\[ \mathbb{G} = A_{4}. \]

Here \( I \) is \( 4 \times 4 \) identity matrix and
\[ P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

is a permutation matrix corresponding to the permutation \( \sigma \in S_{4} \) (\( S_{4} \) is symmetric group)
\[ \sigma : (1, 2, 3, 4) \mapsto (4, 3, 2, 1), \]

by the following relation \( P = (P_{0})^{\nu} = (\delta_{\nu}^{\nu_{0}}) \). Here \( \sigma \) is the generator of the group \( G = \{\sigma, \text{id}\} \) which is the group of symmetry of the Dynkin diagram for \( A_{4} \). \( G \) is isomorphic to the group \( Z_{2} \).

In case of \( D_{4} \) the group of symmetry of the Dynkin diagram \( G' \) is isomorphic to the symmetric group \( S_{4} \) acting on the set of three vertices \( 1, 3, 4 \) of the Dynkin diagram via their permutations. The existence of the above symmetry groups \( G \equiv \mathbb{Z}_{2} \) and \( G' \equiv S_{4} \) implies certain identity properties for the fluxbrane polynomials \( H_{4}(z) \).

Let us denote \( \tilde{p}_{i} = p_{0(i)} \) for the \( A_{4} \) case and \( \tilde{p}_{i} = p_{i} \) for \( B_{4}, C_{4}, \) and \( D_{4} \) cases \( (i = 1, 2, 3, 4) \). We call the ordered set \( (\tilde{p}_{i}) \) as dual one to the ordered set \( (p_{i}) \). It corresponds to the

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\[ \text{where} \quad H_{4} \text{as } z \to \infty, \]
action (trivial or nontrivial) of the group $\mathbb{Z}_2$ on vertices of the
Dynkin diagrams for above algebras.

Then we obtain the following identities which were
directly verified by using MATHEMATICA algorithms.

**Symmetry Relations**

**Proposition 2.** The fluxbrane polynomials obey for all $p_i$ and $z > 0$ the identities
\[
H_{s_i} (z, (p_i)) = H_s (z, (\hat{p}_i)) \quad \text{for } A_4 \text{ case,}
\]
\[
H_{s_i} (z, (p_i)) = H_s (z, (p_{s_i})) \quad \text{for } D_4 \text{ case,}
\]
for any $s_i \in S_5, s = 1, \ldots, 4$. We call relations (45) as symmetry ones.

**Duality Relations**

**Proposition 3.** The fluxbrane polynomials corresponding to Lie algebras $A_4, B_4, C_4,$ and $D_4$ obey for all $p_i > 0$ and $z > 0$ the identities
\[
H_s (z, (p_i)) = H^*_s (z, (\hat{p}_i)) H_s (z^{-1}, (\hat{p}_i^{-1})),
\]
\[s = 1, 2, 3, 4.
\]
We call relations (46) as duality ones.

**Fluxes.** Here we deal with an oriented 2-dimensional mani-
fold $\mathcal{M} = (0, R) \times S^1, R > 0$. One can define the flux integrals
over this manifold:
\[
\Phi'(R) = \int_{\mathcal{M}} F^s = 2\pi \int_0^R d\rho \rho B',
\]
where we denoted
\[
B' = q_s \prod_{i=1}^4 H_1^{-A_i}.
\]
It can be easily understood that total flux integrals $\Phi' =
\Phi'(+\infty)$ are convergent. Indeed, due to polynomial assump-
tion (1) we have
\[
H_s \sim C_s \rho^{2n_s}, \quad C_s = \prod_{i=1}^4 (p_i)^{y_i},
\]
as $\rho \to +\infty$. From (48), (49), and the equality $\sum_{i=1}^n A_i n_i = 2$, following from (2), we get
\[
B' \sim q_s C_s \rho^{-4}, \quad C^s = \prod_{i=1}^4 p_i^{-(\lambda_i)},
\]
and hence the integral (47) is convergent for any $s = 1, 2, 3, 4$.
Using (42) and (50) we have for the $A_4$-case
\[
C^s = \prod_{i=1}^4 \left(1 + p_i \right)^{4} = \prod_{i=1}^4 \left(1 - s_i - p_{s_i}^{-1} \right) = \prod_{i=1}^4 \left(1 - s_i - p_{s_i}^{-1} \right).
\]
Similarly, due to (41) and (50) we get for Lie algebras $B_4, C_4,$ and $D_4$
\[
C^s = p_{s_i}^{-2}.
\]
After that, we can calculate the fluxes $\Phi'(R)$. Using the master
equations (8) one can write
\[
\int_0^R d\rho \rho B' = q_s \frac{1}{2} \int_0^R dz \frac{d}{dz} \left( H_s \frac{d}{dz} H_s \right) = \frac{1}{2} q_s \rho_{\infty}^{-1} R^2 H_s^2 \left( R^2 \right)\,
\]
where $H_s' = dH_s/dz$. Thus, using (47) we easily obtain
\[
\Phi'(R) = \frac{4\pi q_s^{-1} h_s}{H_s^2 \left( R^2 \right)}.
\]
Note that the manifold $\mathcal{M}_s = (0, +\infty) \times S^1$ is isomorphic to
the manifold $\mathbb{R}^2 \setminus \{0\}$. Therefore, one can understand
the family of solutions under consideration as defined on the
manifold $\mathbb{R}^2 \times \mathbb{R}^2$, where coordinates $\rho, \phi$ are polar
coordinates in a domain of $\mathbb{R}^2$: $x = \rho \cos \phi$ and $y = \rho \sin \phi$,
where $x, y$ are standard coordinates of $\mathbb{R}^2$. It was shown in
[30] that there exist forms $A^s$ globally defined on $\mathbb{R}^2$ and
obeying the relation $F^s = dA^s$.

Now let us consider an 2-dimensional oriented mani-
fold (disk) $D_R = \{(x, y) : x^2 + y^2 \leq R^2\}$. Its boundary
$\partial D_R \equiv C_R = \{(x, y) : x^2 + y^2 = R^2\}$ is a circle of radius $R$, i.e., 1-dimensional oriented manifold with the orientation
inherited from that of $D_R$ obeying the relation $\int_{C_R} d\phi = 2\pi$.

The Stokes theorem yields in this case
\[
\Phi'(R) = \int_{C_R} F^s = \int_{D_R} dA^s = \int_{C_R} A^s.
\]
According to the definition of Abelian Wilson loop (factor), we have
\[
W^s(C_R) = \exp \left( i \int_{C_R} A^s \right) = \exp (i\Phi'(R)).
\]
Relations (1) and (54) imply (see (10))
\[
\Phi' = \Phi'(+\infty) = 4\pi n_i q_i^{-1} h_i, \quad s = 1, 2, 3, 4.
\]
Any (total) flux $\Phi'$ depends upon one integration constant $q_i \neq 0$, while the integrand form $F^s$ depends upon all constants: $q_1, q_2, q_3, q_4$. As a consequence, we obtain finite limits
\[
\lim_{R \to +\infty} W^s(C_R) = \exp (i\Phi').
\]
In the $A_4$-case we have
\[
(q_1 \Phi', q_2 \Phi^2, q_3 \Phi^3, q_4 \Phi^4) = 4\pi h (4, 6, 6, 4),
\]
where $h_1 = h_2 = h_3 = h_4 = h$. 
In the $B_4$-case we find
\[
(q_1\Phi^1, q_2\Phi^2, q_3\Phi^3, q_4\Phi^4)
\]
where $h_1 = h_2 = h_3 = h$, $h_4 = 2h$.

In the $C_4$-case we obtain
\[
(q_1\Phi^1, q_2\Phi^2, q_3\Phi^3, q_4\Phi^4)
\]
where $h_1 = h_2 = h_3 = h_4 = 2h$.

In the $D_4$-case we are led to relations
\[
(q_1\Phi^1, q_2\Phi^2, q_3\Phi^3, q_4\Phi^4)
\]
where $h_1 = h_2 = h_3 = h_4 = h$. (In all examples relations (19) are used.)

Note that, for $D = 4$ and $g^2 = -dt \otimes dt + dx \otimes dx$, $q_a$ coincides with the value of the $x$-component of the $a$-th magnetic field on the axis of symmetry, $s = 1, 2, 3, 4$.

**Asymptotic Relations.** Here we can write down the asymptotic relations for the solution under consideration as $\rho \rightarrow +\infty$:
\[
g_{as} = \left(\prod_{i=1}^{4} p_i^{a_i}\right)^{2/(D-2)} \rho^{2A} \left\{ \rho \rho \right\},
\]
where $\rho = \sqrt{R^2 + 1} \rho$ and $\rho = \sqrt{R^2 + 1} \rho$ are integration constants.

\[
a_s = 4 \prod_{i=1}^{4} p_i^{a_i},
\]
and in (65) we put $\theta = \sigma$ for $G = A_4$ and $\theta = \id$ for $G = B_4, C_4, D_4$.

In derivation of asymptotic relations, (40), (49), and (51) were used. We note that for $G = B_4, C_4, D_4$ the asymptotic value of form $F_{as}^a$ depends upon $q_a$, $s = 1, 2, 3, 4$, while in the $A_4$-case $F_{as}^a$ depends upon $q_1$ and $q_4$ for $s = 1, 4$, and $F_{as}^a$ depends upon $q_1$, $q_4$ for $s = 2, 3$.

We note also that by putting $q_1 = 0$ we get the Melvin-type solutions corresponding to Lie algebras $A_3, B_3, C_3$, and $A_3$, respectively, which were analyzed in [28]. (The case of the rank 2 Lie algebra $G_2$ [27] may be obtained for the $D_4$ case when $q_1 = q_3 = q_4$.)

**Dilatonic Black Holes.** Relations (constraints) on dilatonic coupling vectors (12), (13) appear also for dilatonic black hole (DBH) solutions which are defined on the manifold
\[
M = (R_0, +\infty) \times (M_0 = S^3) \times (M_1 = \mathbb{R}) \times M_2,
\]
where $R_0 = 2\mu > 0$ and $M_2$ is a Ricci-flat manifold. These DBH solutions on $M$ from (67) for the model under consideration may be extracted from general black brane solutions; see [21, 25, 31]. They read
\[
g = \left(\prod_{i=1}^{4} H_i^{2h_i/(D-2)}\right) \left\{ f^{-1} dR \otimes dR + R^2 g^0 \right\},
\]
where $f = 1 - 2\mu R^{-1}, g^0$ is the standard metric on $M_0 = S^3$, and $g^2$ is a Ricci-flat metric of signature $(+\ldots, +)$ on $M_2$. Here $Q_s \neq 0$ are integration constants (charges).

The functions $H_i = H_i(R) > 0$ obey the master equations
\[
R^2 \frac{d}{dR} \left( f \frac{R^2}{H_i} \frac{d}{dH_i} \right) = B_s \prod_{i=1}^{4} H_i^{-A_i},
\]
with the following boundary conditions on the horizon and at infinity imposed:
\[
H_i \left(R_0 + 0\right) = H_0 > 0,
\]
\[
H_i \left(+\infty\right) = 1,
\]
where
\[
B_s = -K_s Q_s^2,
\]
$s, a = 1, 2, 3, 4$, where $f = 1 - 2\mu R^{-1}, g^0$ is the standard metric on $M_0 = S^3$, and $g^2$ is a Ricci-flat metric of signature $(+\ldots, +)$ on $M_2$. Here $Q_s \neq 0$ are integration constants (charges).

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with the following boundary conditions on the horizon and at infinity imposed:
\[
H_i \left(R_0 + 0\right) = H_0 > 0,
\]
\[
H_i \left(+\infty\right) = 1,
\]
where
\[
B_s = -K_s Q_s^2,
\]
$s, a = 1, 2, 3, 4$, Here relations (11) are also valid. For Lie algebras of rank 4 the functions $H_i$ are polynomials with respect to $R^{-1}$, which may be obtained (at least for small enough $q_s$) from fluxbrane polynomials $H_i(z)$ presented in this paper. See [25].

**4. Conclusions**

In this paper, the generalized multidimensional family of Melvin-type solutions was considered corresponding to simple nonexceptional finite-dimensional Lie algebras of rank 4: $G = A_4, B_4, C_4, D_4$. Each solution of that family is governed by a set of 4 fluxbrane polynomials $H_i(z), s = 1, 2, 3, 4$. These polynomials define special solutions to open Toda chain equations corresponding to the Lie algebra $G$. 
The polynomials $H_s(z)$ depend also upon parameters $q_s$, which coincides for $D = 4$ (up to a sign) with the values of colored magnetic fields on the axis of symmetry.

We have found the symmetry relations and the duality identities for polynomials. These identities may be used in deriving $1/\rho$-expansion for solutions at large distances $\rho$, e.g., for asymptotic relations which are presented in the paper.

There were also calculated two-dimensional flux integrals $\Phi'(R) = \int_{D_2} P^s$ over a disc $D_2$ of radius $R$ and a corresponding Wilson loop factors $W^s(C_R)$ over a circle $C_R$. It turns out that each total flux $\Phi'(\infty)$ depends only upon one corresponding parameter $q^s$, whereas the integrand $F^s$ depends on all parameters $q^s, s = 1, 2, 3, 4$.

The case of exceptional Lie algebra $F_4$ will be considered in a separate publication.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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