High-accuracy finite element

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Abstract. The article deals with determining the rigidity matrix of a triangular finite element. Displacement vector components and displacement derivatives are used as nodal degrees of freedom. These finite elements have improved accuracy as they link with continuity conditions not only the displacement field but also deformation fields, which significantly simplifies nodal stress analysis and makes it possible to establish stress boundary conditions. The test calculations verified for a beam were provided.

Key words. Plane elasticity model, finite deformations, displacements, stresses, overhang beam calculations.

1. Introduction

At present a large number of diverse finite elements are used in the finite element method. Triangular finite elements should be considered exhibiting the most promise for membrane solutions of elasticity theory. Firstly, these finite elements provide a more flexible element discretization in the region under study. Secondly, the triangular region has certain advantages in terms of solving the two-dimensional interpolation problem.

Figure 1 features a high-accuracy finite element in which displacements and their derivatives are used as degrees of freedom. This finite element must possess improved accuracy, as it links with continuity conditions not only displacement fields but also deformation fields [1].

\[ \begin{align*}
V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \\
U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}
\end{align*} \]

**Figure 1.** High accuracy finite element

Taking the displacement field differential as a degree of freedom simplifies the nodal stress analysis. Genuinely, deformation tensor components and nodal stress tensor components are expressed with the first-order derivatives of the displacement field. For this very reason, there is an opportunity of setting the stress boundary conditions.
2. Displacement fields

Let us denote the triangle vertices (figure 2) by letters \( i, j \) and \( k \). For every triangular node there are coordinates \((X_j, Y_j), (X_k, Y_k)\) and \((X_i, Y_i)\).

![Figure 2.](image)

It is more natural to use \( L \)-coordinates for a triangular finite element. In this case the displacement field inside the finite element can be described by the pair of homogeneous cubic polynomials:

\[
\{u(L)\} = \{\alpha_{10}\} \{L_{10}\},
\]

where \( L_i, L_j, L_k \) are \( L \)-coordinates [2].

At this stage, the derivatives of displacements along triangle sides \( S_r, (r = i, j, k) \) will be used as degrees of freedom. To do this, it is necessary to differentiate the dependences (2).

\[
\frac{\partial \{u(L)\}}{\partial S_r} = \frac{\partial \{\alpha_{10}\}}{\partial S_r} \{L_{10}\} = \{\alpha_{10}\} \frac{\partial \{L_{10}\}}{\partial S_r}.
\]

Thus, the vector-valued function differentiation \( \{u(L)\} \) reduced to the differentiation of the vector-valued function \( \{L_{10}\} \). Let us find the corresponding derivatives. For which purpose, let us assign \( S_r \) coordinate starting in the least significant node to each side \( l_r (r = i, j, k) \) (figure 2).

Let us consider side \( jk \) of the triangle. The relationship between \( S_i \) coordinate and Cartesian system \( x, y \) is expressed by the obvious formulae:

\[
x = \frac{c_i}{l_i} S_i + X_j, \\
y = -\frac{b_j}{l_j} S_i + Y_j.
\]

Similar formulae can be written for other sides. In this case the derivative of a composite function with respect to coordinate \( S_r \ (r = i, j, k) \) will be written as follows:

\[
\frac{\partial \{L_{10}\}}{\partial S_r} = \frac{\partial \{L_{10}\}}{\partial L_i} \frac{\partial L_i}{\partial S_r} + \frac{\partial \{L_{10}\}}{\partial L_j} \frac{\partial L_j}{\partial S_r} + \frac{\partial \{L_{10}\}}{\partial L_k} \frac{\partial L_k}{\partial S_r}.
\]

Functions \( L_p (p = i, j, k) \) included within (5) are composite functions with respect to coordinates \( x, y \). Thus:
\[
\frac{\partial L_p}{\partial S_r} = \frac{\partial L_p}{\partial x} \frac{\partial x}{\partial S_r} + \frac{\partial L_p}{\partial y} \frac{\partial y}{\partial S_r}.
\]

Bearing in mind dependence (4)
\[
\frac{\partial L_p}{\partial S_r} = \frac{1}{2 \cdot \Delta \cdot l_r} \left( b_p c_r - b_r c_p \right).
\]

For a triangle the following correlations are known:
\[
c_r b_k - c_k b_r = c_r b_r - c_r b_j = c_j b_r - c_j b_k = a_i + a_j + a_k = 2 \Delta.
\]

Bearing in mind these correlations, for different combinations \( p \) and \( r \) from the set \( i, j, k \), we obtain:
\[
\frac{\partial L_p}{\partial S_r} = \begin{cases} 
L_{r}^{-1} & \text{where } r = i, p = k; \ r = j, p = i; \ r = k, p = j \\
-L_{r}^{-1} & \text{where } r = i, p = j; \ r = j, p = k; \ r = k, p = i. \\
0 & \text{where } r = p
\end{cases}
\]

Applying these dependences into (5), we obtain the derivatives of particular variables \( S_i, S_j, \) and \( S_k \):
\[
\frac{\partial [L_{10}]_{r}}{\partial S_j} = \frac{1}{l_i} \left( \frac{\partial [L_{10}]_{r}}{\partial L_i} + \frac{\partial [L_{10}]_{r}}{\partial L_j} \right),
\]
\[
\frac{\partial [L_{10}]_{r}}{\partial S_j} = \frac{1}{l_j} \left( \frac{\partial [L_{10}]_{r}}{\partial L_j} + \frac{\partial [L_{10}]_{r}}{\partial L_k} \right),
\]
\[
\frac{\partial [L_{10}]_{r}}{\partial S_j} = \frac{1}{l_k} \left( \frac{\partial [L_{10}]_{r}}{\partial L_k} + \frac{\partial [L_{10}]_{r}}{\partial L_j} \right).
\]

These dependences can be written as the product of some matrix \([T_r](r \in i, j, k)\), the structure of which is obvious, by vector \(\{L_{10}\}\):
\[
\frac{\partial [L_{10}]_{r}}{\partial S_j} = \frac{1}{l_r} [T_r] \{L_{10}\}, \text{ where } r \in i, j, k. \tag{6}
\]

Where:
\[
\{L_{10}\}^T = \{L_i^2, L_j^2, L_k^2, L_i L_j, L_i L_k, L_j L_k\}
\]

Applying (6) to (3), we obtain:
\[
\frac{\partial [u(L)]}{\partial S_r} = \frac{1}{l_r} [\alpha_{10}] [T_r] \{L_{10}\}, \text{ where } r \in i, j, k. \tag{7}
\]

3. Nodal displacement vectors

Let us introduce a nodal displacement vector. With this purpose, let us accomplish some transformations (7). Let us multiply (7) by the side length of a triangle \(l_r\). As a result, we will obtain:
\[
\frac{\partial [u(L)]}{\partial S_r} l_r = [\alpha_{10}] [T_r] \{L_{10}\}. \tag{8}
\]

Let us introduce the definition of function \([u(L)]\) along an arbitrary side of a triangle \(l_r\).
\[
[u(l)]_r = \frac{\partial [u(L)]}{\partial S_r} l_r.
\]

In this case, the dependence (8) takes the form:
\[
[u(l)]_r = [\alpha_{10}] [T_r] \{L_{10}\}. \tag{9}
\]
The new idea of derivative provides the independence of matrix elements $[\alpha_{ij}]$ from the triangle side dimension and makes dependence (9) universal for any finite element in the process of differentiation along any of its sides.

Now it is possible to define the nodal displacement values. Let us introduce two local vectors of nodal displacements with the elements ordered along the directions $x$ and $y$.

$$[U]_s^T = \{U_{i,j}, U_{i,k}, U_{j,i}, U_{j,k}, U_{k,i}, U_{k,j}\},$$

$$[V]_s^T = \{V_{i}, V_{i,j}, V_{i,k}, V_{j,i}, V_{j,k}, V_{k,i}, V_{k,j}\}.$$  

Where $s$ factor shows that the derivatives are taken along the corresponding sides of a triangle according to formula (9). Taken together, these three vectors form a global displacement vector for a triangular finite element with the components ordered in the directions $x$ and $y$.

$$[\bar{U}]_s^T = \{[U]_s^T, [V]_s^T\}.$$  

Let us also introduce three local vectors of nodal displacements for node $p$ of a triangular finite element

$$[U]_p^s = \{U_p, U_{p,j}, U_{p,k}, V_p, V_{p,i}, V_{p,k}\},$$  

where $p \in i, j, k; \ r \in i, j, k; \ t \in i, j, k.$ Taken together, these three vectors form a global vector of nodal displacements for a finite element whose components are ordered by nodes

$$[\bar{U}]_p^s = \{[U]_p^s, [U]_j^s, [U]_k^s\}.$$  

Let us relate vectors $[\bar{U}]_s^T$ and $[\bar{U}]_p^s$ by matrix $[E_i]$:

$$[\bar{U}]_s^T = [E_i][\bar{U}]_p^s.$$  

The structure of matrix $[E_i]$ is obvious.

The vector of nodal displacements with derivatives along the sides of a triangular finite element will be used for defining matrix elements $[\alpha_{ij}]$. However, it is impossible to use this vector for rigidity matrix and stress determination. In this instance a displacement vector is necessary, whose components will be derivatives with respect to $x$ and $y$ arguments. Let us introduce the following definition of derivatives of function $\{u(L)\}$ along coordinate axes $x$ and $y$.

$$\{u(L)\}_1 = \frac{\partial [u(L)]}{\partial x}, \ \ \ {u(L)}_2 = \frac{\partial [u(L)]}{\partial y}. \quad (10)$$

In accordance with (10) let us determine three local vectors of nodal displacements for $p$ node of a triangular finite element:

$$[U]_p^s = \{U_p, U_{p,i}, U_{p,j}, U_{p,k}, V_p, V_{p,i}, V_{p,j}, V_{p,k}\}, \quad (11)$$

where: $p \in i, j, k.$ $x$ factor in (11) shows that the derivatives are taken along coordinate axes $x$ and $y$.

From (8):

$$\{u(L)\}_r = \frac{\partial [u(L)]}{\partial S_r} L_r = L_r \left( \frac{\partial [u(L)]}{\partial x} \frac{\partial x}{\partial S_r} + \frac{\partial [u(L)]}{\partial y} \frac{\partial y}{\partial S_r} \right)$$

$$= L_r \left( \frac{\partial [u(L)]}{\partial x} c_r - \frac{\partial [u(L)]}{\partial y} b_r \right).$$

Now it is possible to find the relationship between vectors $[U]_p^s$ and $[U]_p^s$ by matrix $[M_{p,k}^s]$, so that:

$$[U]_p^s = [M_{p,k}^s] [U]_p^s \quad (12)$$
where:

$$[M_p]_{6x} = \begin{bmatrix} [M_i]_{6x} & [0_6] & [0_6] \\ [0_6] & [M_j]_{6x} & [0_6] \\ [0_6] & [0_6] & [M_k]_{6x} \end{bmatrix}.$$  

Here $[0_6]$ zero matrix has the size of $6 \times 6$.

$$[M_i]_{6x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_j - b_j & 0 \\ 0 & c_k - b_k & 0 \end{bmatrix}, \quad [M_j]_{6x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_k - b_k & 0 \\ 0 & c_j - b_j & 0 \end{bmatrix}, \quad [M_k]_{6x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_j - b_j & 0 \\ 0 & c_j - b_j & 0 \end{bmatrix}.$$  

Let us find $\alpha_i, \beta_i$ factors (where $i \in 1, 2, \ldots, 10$) of interpolation polynomials (1). These factors will be obtained from continuity conditions for displacements and derivatives along the triangle sides in its nodes. This procedure is well-known, therefore let us confine ourselves to the ultimate result.

First and foremost, let us write the interpolation polynomial (4) determining the displacement field $u(L)$, and, according to formula (12), let us find two interpolation polynomials describing the fields of derivatives of displacements inside the finite element. Thus, we will obtain:

$$\{u(L)\} = [\Omega(L)] \begin{bmatrix} Z \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} [U]_s \\ [V]_s \end{bmatrix}.$$  

Where $\{u(L)\} = \begin{bmatrix} u(L) \\ v(L) \end{bmatrix}$, $[\Omega(L)] = \begin{bmatrix} [l_{10}] & [0] & [0] \\ [0] & [l_{10}] & [0] \\ [0] & [0] & [l_{10}] \end{bmatrix}$, $\begin{bmatrix} [U]_s \\ [V]_s \end{bmatrix}$.

Here matrix $[Z]$ contains factors $\alpha_i, \beta_i$ and links the displacement vector components inside the finite element with nodal displacements.

Let us introduce the shape function matrix:

$$[N(L)] = [\Omega(L)] \begin{bmatrix} Z \\ 0 \\ 0 \end{bmatrix}.$$  

Then ultimately:

$$\{u(L)\} = [N(L)] \begin{bmatrix} [U]_s \\ [V]_s \end{bmatrix}.$$  

(13)

In order to simplify the computation of stress tensor, it is worthwhile to employ the nodal displacement vector $\begin{bmatrix} [U]_s \end{bmatrix}$, the elements of which are derivatives of displacements along coordinate axes $x$ and $y$. To do this, let us apply dependence (12) to (13):

$$\{u(L)\} = [N(L)] [M]_{6x} \begin{bmatrix} [U]_s \end{bmatrix}.$$  

(14)

4. Deformations and stresses

Let us introduce the deformation vector:

$$\{\varepsilon\}^T = \{\varepsilon_x, \varepsilon_y, \varepsilon_{xy}\} = \{\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22}\}.$$  

The Cauchy equations:

$$\{\varepsilon(L)\} = [S] \{u(L)\}.$$  

(15)

Let us apply (14) to (15):

$$\{\varepsilon(L)\} = [S] [N(L)] [M]_{6x} \begin{bmatrix} [U]_s \end{bmatrix}.$$  

The influence of matrix $[A]$ on matrix $[N(L)]$ will give rise to gradient matrix $[B(L)]$:

$$\{\varepsilon(L)\} = [B(L)] [M]_{6x} \begin{bmatrix} [U]_s \end{bmatrix}.$$  

(16)

Equation (16) provides the sought dependence between deformations and nodal replacements.

Let us introduce the stress vector:
Hooke’s law:

$$\{\sigma\}^T = \{\sigma_x, \sigma_y, \tau_{xy}\} = \{\sigma_{11}, \sigma_{12}, \sigma_{22}\}.$$  

where \([D]\) is the elasticity matrix having the size of \(3 \times 3\). The sought dependence between the stress tensor components and displacement vector can be obtained if equations (16) are applied to (17):

$$\{\sigma(L)\} = [D][B(L)][M]_{\kappa \lambda} \{U_p\}_\kappa^\prime.$$  

5. System of resolving finite element equations

Let us introduce the following notation:
- \(\Lambda\) – work of external forces
- \(\Pi\) – potential energy of deformed body
- \(\Delta\) – energy of external and internal forces system

To determine the resolving finite-element equations let us use the Lagrange formula:

$$\delta\Lambda = 0,$$

$$\Lambda = \Pi - \Delta.$$  

Potential energy is determined by the Clapeyron formula:

$$\Pi = \frac{1}{2} \int_{\Omega} \{\varepsilon\}^T \{\sigma\} dV.$$  

Let us find potential energy \(\Pi^\nu\) for a finite element with the number \(\nu\), applying equations (16) and (18) to (20):

$$\Pi^\nu = \frac{1}{2} \int_{\Omega} \{U_p\}_\kappa^\prime [M^\nu]_{\kappa \lambda} [B^\nu(L)]^T [D^\nu] [B^\nu(L)] [M^\nu]_{\lambda \xi} \{U_p\}_\xi^\prime dV.$$  

The work of external forces for node \(p\) of a finite element numbered \(\nu\):

$$A^\nu_p = [U_p]_{\kappa}^\prime [F_2]_\kappa \{P^\nu_p\},$$

By means of matrix \([F_2]\), let us introduce vector \([U_p]\) from (11) to dependence (21)

$$A^\nu_p = [U_p]_{\kappa}^\prime [F_2]_\kappa \{P^\nu_p\},$$

where:

$$[F_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$  

Then, for the finite element numbered \(\nu\):

$$A^\nu = \sum_{p=1}^{3} [U_p]_{\kappa}^\prime [F_2]_\kappa \{P^\nu_p\}.$$  

Summatng all the finite elements, we will obtain the energy of the system of external and internal forces (19):

$$\Lambda = \frac{1}{2} \sum_{\nu=1}^{E} \int_{\Omega} \{U_p\}_\kappa^\prime [M^\nu]_{\kappa \lambda} [B^\nu(L)]^T [D^\nu] [B^\nu(L)] [M^\nu]_{\lambda \xi} \{U_p\}_\xi^\prime dV - \sum_{\nu=1}^{E} \sum_{p=1}^{3} [U_p]_{\kappa}^\prime \{F_2\} \{P^\nu_p\}.$$  

Minimization of the obtained functional results in the system of resolving finite-element equations:

$$\sum_{\nu=1}^{E} \int_{\Omega} [M^\nu]_{\kappa \lambda} [B^\nu(L)]^T [D^\nu] [B^\nu(L)] [M^\nu]_{\lambda \xi} \{U_p\}_\xi^\prime dV = \sum_{\nu=1}^{E} \sum_{p=1}^{3} [F_2] \{P^\nu_p\}.$$  

6. Finite element rigidity matrix

The integral on the right side of (22) takes the form of rigidity matrix of finite element \([k]\). Assuming that the finite element width varies linearly, let us write:
\[
h(L) = h_i L_i + h_j L_j + h_k L_k.
\]

where: \( h_i, h_j, h_k \) – the finite element width in nodes \( x, i, j, k \) and assuming that the elements of matrix \( [M^v]_x \) are constant values:

\[
[k^v] = [M^v]_x \int_V [B^v(L)]^T [D^v] [B^v(L)] dV [M^v]_x, \tag{23}
\]

The integration in (23) is performed numerically, with 12 points [3].

7. Test calculation
Using the dependences obtained, the authors wrote the program for testing the finite element rigidity matrix and made calculations for a hammer beam displacements (Fig.3). The calculation is of purely mathematical character, therefore dimensions of quantity are not provided.

![Figure 3. Displacement model of a hammer beam for finite element testing](image)

Displacement calculations for point \( A \) of a beam (figure, 3), according to exact formulae, amounted to 2.744, but when calculated according to the finite element method, the displacement value amounted to 2.738. Thus, calculation error amounted to 0.22%.

8. Conclusions
In the work presented here the authors examined in detail how to determine the rigidity matrix of a high-accuracy finite element with six nodal degrees of freedom, which is necessary for membrane solutions within the framework of elasticity theory. The test calculation verified the The integration in (23) is performed numerically, with 12 points a tetrahedral finite element with twelve nodal degrees of freedom for solving multi-dimensional problems.

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