Example of shock wave in unstable medium: The focusing nonlinear Schrödinger equation

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Abstract
Dissipationless shock waves in modulational unstable one-dimensional medium are investigated on the simplest example of integrable focusing nonlinear Schrödinger (NS) equation. Our approach is based on the construction of special exact solution of the Whitham-NS system, which “partially saturates” the modulational instability.

1 Introduction
It is well known that focusing NS equation

\[ iu_t + u_{xx} + 2 |u|^2 u = 0, \]

may be rewritten as hydrodynamic type system

\[ f_t + 2(f \cdot v)_x = 0, \quad v_t + 2vv_x - 2f_x = \left(2f_{xx} + f_x^2 f^{-2}\right)_x / 4, \]

where

\[ u = \sqrt{f} \cdot \exp(i\varphi), \quad v = \varphi_x. \]

We consider (1)-(3) with the following step-like initial condition

\[ \sqrt{f(x, 0)} = \begin{cases} 2, & x < 0 \\ 1, & x \geq 0 \end{cases} \sim \begin{cases} 2a, & x < 0 \\ a, & x \geq 0 \end{cases}, \quad v(x, 0) = 0. \]

It is natural to call the solution \([f(x, t), v(x, t)]\) or \(u(x, t)\) of the problem (1)-(4) the shock wave solution.

The main purpose of this Letter is to describe the qualitative behavior of the shock wave at \(t \gg 1\). We note that this problem is much more difficult than the analogous problem for stable models [1,2].
The typical and simplest shock problem in modulational stable model was first studied by Gurevich and Pitaevskii [1] on the example of Korteweg-de Vries (KdV) equation. In [1] modulational theory developed by Whitham [3] was used. It is important that the Whitham system for KdV equation as well as for other stable models [3] is hyperbolic, which means that Whitham characteristic speeds are everywhere real: $\text{Im}(S_i) \equiv 0$. However for unstable models these speeds are generally speaking complex [3]: $\text{Im}(S_i) \neq 0$ (elliptic case). This complicates very much any analytical investigation of unstable models due to exponential growth of linear perturbation of the (locally) constant background. In particular there are obvious obstacles in studying of the shock problems in unstable systems, which are closely related to the classical problem of the modulational instability of monochromatic wave in (1) (see for example [4]).

In this Letter we demonstrate how to overcome these difficulties on the concrete example (1)-(4). We shall show that famous strategy [1] may be modified and successfully applied to (1)-(4). Our approach is based on the conception of "partially saturating modulational instability solutions" of elliptic Whitham-NS equations proposed in [5] for description of modulational instability in (1). The key role in our construction plays a special subclass of Whitham-NS system solutions with vanishing imaginary parts $\text{Im}(S_i) = 0$ of some (not all!) characteristic speeds (see (12), (17)).

2 One-phase NS equation solution and its Whitham-NS modulations.

It is well known that one-phase solution of (1), (2) has form
\[
    f_I(\theta) = f_3 + (f_1 - f_3)dn^2(\sqrt{f_1 - f_3} : \theta ; m), \quad m = (f_1 - f_2)/(f_1 - f_3),
\]
\[
    v_I(\theta) = U/2 - A/f(\theta); \quad \theta = x - Ut,
\]
where $f_1 \geq f \geq f_2 \geq f_3, A = \sqrt{-f_1f_2f_3} \geq 0, dn$ - is the Jacobi elliptic function. Elliptic spectral curve corresponding to (5), (6) has branching points $\lambda_i, i = 1, 2, 3, 4; \lambda_2 = \lambda_1^*, \lambda_4 = \lambda_3^*$ such that (c.f. [6])
\[
    \lambda_1 \equiv \alpha - i\gamma = U/4 - \sqrt{-f_3}/2 - i(\sqrt{f_1} + \sqrt{f_2})/2,
\]
\[
    \lambda_3 \equiv \beta - i\delta = U/4 + \sqrt{-f_3}/2 - i(\sqrt{f_1} - \sqrt{f_2})/2.
\]
Whitham-NS equations for (1), (2), (5)-(7) can be presented in diagonal form (c.f. [6])
\[
    d\lambda_i/dt + S_i(\lambda)d\lambda_i/dx = 0, \quad i = 1, 2, 3, 4,
\]
\[
    S_1 = U + 2\lambda_{12}/(1 - \mu\lambda_{32}/\lambda_{31}), \quad S_3 = U + 2\lambda_{34}/(1 - \mu\lambda_{14}/\lambda_{13}),
\]
\[
    S_2 = S_1^*, S_4 = S_3^*, \quad \mu \equiv E(m)/K(m),
\]
where $\lambda_{ij} \equiv \lambda_i - \lambda_j, K, E$ are the complete elliptic integral of the first and second kind respectively, $m = \lambda_{21}\lambda_{43}/\lambda_{32}\lambda_{14}$.
3 Solution of Whitham-NS system and long-time behavior of the NS-hydrodynamic shock wave.

We construct an approximate solution of the shock problem (1)-(4) for \( t \gg 1 \) as follows. In the "external" region on \((x, t)\) plane: \((x < 0, x > x^+(t))\), the solution is zero-phase one:

\[
\sqrt{f_0(x, t)} = \begin{cases} 
 2, & x < 0 \\
 1, & x \geq x^+(t) 
\end{cases}, \quad v_0(x, 0) = 0. \quad (10)
\]

In the oscillation region \( 0 \leq x \leq x^+(t) \) the "internal" solution is given by the one-phase solution (5), (6) with modulated due to (8), (9) parameters. In this region we use special solution of the Whitham-NS system (8), (9):

\[
\lambda_1 \equiv \text{const.}, \quad \text{Im}(S_3) = 0, \quad t \cdot \text{Re}(S_3) - x = g(\beta, \delta), \quad (11)
\]

where \(g(\beta, \delta)\) is an arbitrary smooth function of its variables, which is determined from the initial conditions.

Let us consider the simplest case \( g \equiv 0 \) (c.f. [1], [5,7]). From the initial condition (4) we get

\[
\gamma = 1, \quad \alpha = 0, \quad \iff \lambda_1 = -i. \quad (13)
\]

Proposition. The system (12), (13) with \( g \equiv 0 \) is compatible and has unique solution with \( \delta \geq 0, \beta \geq 0 \) in the "internal" region \( 0 \leq x \leq x^+(t) \). Near the boundary \( x^+ = x^+(t) \) the solution of the system (12) has the form

\[
x^+ = 4\sqrt{2}t, \quad x = x^+ - x', \quad 0 < x' \ll 1, \quad \beta \approx 1/\sqrt{2} - 7x'/48t, \quad \delta^2 \approx x'/2\sqrt{2}t. \quad (14)
\]

At the boundary \( x = x^+(t) \) the solution \((f_I, v_I)\) from (5), (6) is continuously glued with 1 from (4). If \((x/t) \to +0\) the points \((\lambda_3, \lambda_4)\) closely come to the points \((\lambda_1, \lambda_2)\). In this limit our solution \((f_I, v_I)\) from (5), (6) degenerates into the stationary soliton (breather), corresponding to \( \lambda_2 = \lambda_4 = i, \lambda_1 = \lambda_3 = -i \).

Remark 1. Due to Galilean invariance of (1) the above analysis may be easily extended to the case of the shock wave with more general initial conditions

\[
\sqrt{f_0(x, 0)} = \begin{cases} 
 2, & x < 0 \\
 1, & x \geq 0, \quad v(x, 0) = 2b. 
\end{cases} \quad (15)
\]

The corresponding changes in the formulae (11)-(14) are as follows: \( x^+ \to 4bt + x^+, \quad \lambda_1 \to b + \lambda_1, \quad \lambda_3 \to b + \lambda_3 \).

Remark 2. The mirror symmetry \( x \to -x \) allows to solve the shock problem symmetric to (4)

\[
\sqrt{f_0(x, 0)} = \begin{cases} 
 1, & x \leq 0 \\
 2, & x > 0, \quad v(x, 0) = 0. 
\end{cases} \quad (16)
\]
We do not go into the details, but just note that corresponding solution of Whitham-NS system (8) has the form

$$\lambda_3 \equiv \text{const.}, \quad \text{Im}(S_1) = 0, \quad t \cdot \text{Re}(S_1) - x = 0,$$

and the picture of the solution $u(x, t)$ at $t \gg 1$ is symmetric with respect to $x \rightarrow -x$ transformation to the above picture.

Remark 3. The problems (4), (15), (16) are of course the simplest examples of the shock problems in unstable model (1), (2). As we have shown above in these simplest cases the traditional ideology of [1,2] can be applied (after proper modification). For more general initial data it is necessary to invent new technique. For example, how to solve the following shock problem

$$u(x, 0) = \begin{cases} 
1, & x < 0 \\
0, & x \geq 0
\end{cases}$$

We hope to answer this question in the nearest future.

Remark 4. Note that the above shock wave process (1)-(14) has the same typical time $t \approx 1$ as ordinary modulational instability ($|\text{Im}(S_1)| \approx 1$ as $t = 0$).

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