An approach without using Hardy inequality for the linear heat equation with singular potential

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abstract. The aim of this paper is to employ a strategy known from fluid dynamics in order to provide results for the linear heat equation \( u_t - \Delta u - V(x)u = 0 \) in \( \mathbb{R}^n \) with singular potentials. We show well-posedness of solutions, without using Hardy inequality, in a framework based in the Fourier transform, namely \( PM^k \)-spaces. For arbitrary data \( u_0 \in PM^k \), the approach allows to compute an explicit smallness condition on \( V \) for global existence in the case of \( V \) with finitely many inverse square singularities. As a consequence, well-posedness of solutions is obtained for the case of the monopolar potential \( V(x) = \frac{\lambda}{|x|^2} \) with \( |\lambda| < \lambda_* = \frac{(n-2)^2}{4} \).

This threshold value is the same one obtained for the global well-posedness of \( L^2 \)-solutions by means of Hardy inequalities and energy estimates. Since there is no any inclusion relation between \( L^2 \) and \( PM^k \), our results indicate that \( \lambda_* \) is intrinsic of the PDE and independent of a particular approach. We also analyze the long-time behavior of solutions and show there are infinitely many possible asymptotics characterized by the cells of a disjoint partition of the initial data class \( PM^k \).

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1 Introduction and statements of results

We concern with the Cauchy problem for the liner heat equation

\[ u_t - \Delta u - V(x)u = 0 \quad \text{in} \quad \mathbb{R}^n \]
\[ u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^n, \tag{1.1} \]

where \( n \geq 3 \) and \( V(x) \) is a singular potential. Of particular interest are the negative power law ones which appear in a number of physical and mathematical contexts, such as molecular physics, non-relativistic quantum mechanics, quantum cosmology, linearized analysis of combustion models, and many others (see [15], [16], [18], [26], [28], [32] and references therein).

These potentials can be classified according to the number of singularities (poles), \( \sigma \)-degree of the singularity (order of the poles), dependence on directions (anisotropy) and decay at infinity.

A critical situation is when \( \sigma \) coincides with the order of the main part of the associated elliptic operator, e.g. \( \sigma = 2 \) for (1.1)-(1.2). This case presents further difficulties in its mathematical analysis due to the following features: \( V \) does not belong to Kato’s class and \( Vu \) cannot be handled as a lower order term. Examples of those are the inverse square (Hardy) potential

\[ V(x) = \frac{\lambda}{|x|^2}, \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{1.3} \]

and

\[ V(x) = \sum_{j=1}^{m} \frac{\lambda_j}{|x-x^j|^2} \quad \text{and} \quad V(x) = \sum_{j=1}^{m} \frac{(x-x^j).d^j}{|x-x^j|^3}, \tag{1.4} \]

where \( x^j = (x^j_1, x^j_2, ..., x^j_n) \in \mathbb{R}^n \) and \( d^j \in \mathbb{R}^n \) are given constant vectors. The potentials in (1.4) are called isotropic and anisotropic multipolar inverse square ones, respectively.

There is a well-known concept of criticality associated to the size of the parameter \( \lambda \) in (1.3) with respect to best constant in Hardy’s inequality \( \lambda_* = \frac{(n-2)^2}{4} \), which reads as

\[ \lambda_* \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \|\nabla u\|^2_{L^2(\mathbb{R}^n)}. \tag{1.5} \]

In fact, the celebrated work of Baras and Goldstein [5] established a threshold value that decides whether (or not) positive solutions in \( L^2(\mathbb{R}^n) \) exist. Precisely, if \( 0 \leq \lambda \leq \lambda_* \) then (1.1)-(1.2) is well-posedness in \( L^2(\mathbb{R}^n) \), and it is not well-posedness for \( \lambda > \lambda_* \). In the latter case, there is no nontrivial nonnegative solutions for \( u_0 \geq 0 \), while weak positive solutions do exist in the former one when \( u_0 \geq 0 \) and \( u_0 \not\equiv 0 \). Because of this dichotomy, the cases \( \lambda \in [0, \lambda_*) \), \( \lambda = \lambda_* \) and \( \lambda > \lambda_* \) are named respectively as sub-critical, critical and supercritical values for \( \lambda \) (see [23], [37] for a deeper discussion).

Over the last fifteen years, the results of [5] have motivated many works concerning heat equations with singular potentials. As well as [5], Hardy inequality and its versions play a crucial role in the results of the literature. In what follows, without making a complete list, we review some important works. In a smooth bounded domain \( \Omega \) and for general positive
singualr potential $V \in L^1_{loc}(\Omega)$, existence and non-existence results were proved in [7] via conditions on the infimum of the spectrum of the operator $\Delta - V$. Kombe [25] showed that the nonexistence result of positive solutions in [7] is not affected when the potential $V$ is perturbed by a highly oscillating singular sign-changing potential. Another extension for a parabolic equation with variable coefficients in the principal part can be found in [23]. The authors of [37] proved an improvement of the Hardy-Poincaré inequality in bounded domains and a weighted version of that inequality in $\mathbb{R}^n$. Afterwards they showed exponential stabilization towards a solution in separated variables in bounded domains, and polynomial stabilization towards a radially symmetric solution in self-similar variables. Using Hardy type inequalities of [37], comparison results were obtained in [13] for linear elliptic and parabolic equations with $V \in L^1_{loc}(\Omega)$ positive. Also, using some improved forms of Hardy-Poincaré inequalities and Carleman estimates, inverse source problems have been considered for (1.1) with $0 \leq \lambda \leq \lambda^*$ (see [36] and their references). Motivated by instantaneous blowing-up of non-negative $H^1_0(\Omega)$-solutions when $\lambda = \lambda^*$, the authors of [10] characterized some kinds of perturbations of the critical potential $\lambda \frac{|x|^2}{|x|^2}$ for obtaining existence and non-existence of $H^1$-solutions (see [21] for related results in the stationary case). Nonexistence for $\lambda > \lambda^*$ for a perturbation of (1.1) with $-\nabla \sigma \cdot \nabla u + \frac{\lambda}{|x|^2} u$ in place of $Vu$ was addressed in [22]. Results concerning existence, non-existence, Fujita exponent, self-similarity, bifurcations, instantaneous blow-up for perturbations of (1.1) by $u^p$ and $|\nabla u|^p$ can be found in [2], [3], [4], [11], [24], [30], [33] (see also [12]). Related linear and semilinear elliptic problems [1],[11],[13],[15],[16],[34] have been considered with results also presenting a dichotomy due to influence of Hardy potentials. See [17] for existence results for some semilinear elliptic equations with multipolar potentials (1.4) without employing Hardy inequalities.

From the above works, the use of Hardy inequality (1.5) imposes the $L^2$-framework for $u$ and the condition $0 \leq \lambda \leq \lambda^*$ for well-posedness of $L^2$-solutions. So, a natural question arises: does there exist a framework different from $L^2$ in which (1.1)-(1.2) with (1.3) is well-posedness for $0 \leq \lambda \leq \lambda^*$ or at least for $0 \leq \lambda < \lambda^*$? In this paper we give a positive response for this question by using $PM^k$-spaces and a strategy based on Fourier transform which does not use Hardy inequality (see Theorem 1.1). Since there is no any inclusion relation between $L^2$ and $PM^k$, and the corresponding techniques are of different natures, our results indicate that $\lambda^*$ is intrinsic of the PDE (1.1) and independent of a particular approach. Another goal is to obtain results for multipolar potentials like (1.4) providing explicit conditions on parameters of the potential for global well-posedness of solutions (see Corollary 1.2). Let us observe that, as well as in [37], we also consider sign-changing initial data, solutions and potentials.

From another viewpoint, parabolic regularization does not work for nontrivial solutions when $V$ is singular, and then solutions are not smooth in $\mathbb{R}^n$. This also motivates to analyze the well-posedness of (1.1) in a singular framework like $PM^k$ which contains functions that can be strongly rough and not to decay as $|x| \to \infty$. Indeed, Theorem 1.1 and Corollary 1.2 below show that the formal semigroup associated with (1.1) can be extended to the $PM^k$-framework for a wide class of singular potential, namely $V \in PM^{n-2}$ satisfying (1.12).

The authors of [37] showed that certain good solutions with positive data in the weighted space $L^2(\mathbb{R}^n, \exp(|x|^2/4)dx)$ converges in the norm $t^{1/2} \| \cdot \|_{L^2}$ for an explicitly given non-stationary solution (up to a constant) as $t \to \infty$. Here we obtain a partition of $PM^k$
into infinite pairwise disjoint subsets (induced by an equivalent relation) such that the long-time behavior of \( u \) depends on the subset that contains the data \( u_0 \) (see Theorem 1.4 and Remark 1.5). In particular, for \( u_0 \) belonging to certain special subsets, the solution converges towards a stationary state given explicitly. These results show that the asymptotic behavior of solutions in the \( PM^k \)-framework is more complex than that for \( L^2 \)-solutions, at least in an approximation \( o(1) \), as \( t \to \infty \).

The problem is formulated via a functional equation obtained by formally applying the Fourier transform in (1.1)-(1.2) and then using Duhamel principle. This approach has already been used in the context of fluid mechanics and semilinear parabolic equations without singular potentials (see e.g. [6], [8], [9], [27], [31]). Nonetheless, to our knowledge, the present paper seems to be its first application on a global existence problem for a parabolic PDE with an optimal threshold value explicitly characterized by another technique.

The authors of [19] studied the higher-order parabolic equation

\[
\frac{\partial u}{\partial t} + (-\Delta)^m u + \frac{\lambda}{|x|^\sigma} u = 0
\]

with critical singularity \( \sigma = 2m \) and \( n > 2m \). Among others, they extended Baras-Goldstein results by obtaining an explicit threshold value \( \lambda_m \) for well-posedness of \( L^2 \)-solutions, which is the so-called Hardy’s best constant of multiplicative inequalities involving \( V(x) = \lambda |x|^{-2m} \).

In what follows we describe precisely our results. For each \( k \geq 0 \), the space \( PM^k \) is defined

\[
PM^k \equiv \{ u \in S' : \hat{u} \in L^1_{\text{loc}}(\mathbb{R}^n), \text{ess sup}_{\xi \in \mathbb{R}^n} |\xi|^k |\hat{u}(\xi)| < \infty \},
\]

which is a Banach space with the norm

\[
\|u\|_{PM^k} \equiv \text{ess sup}_{\xi \in \mathbb{R}^n} |\xi|^k |\hat{u}(\xi)| < \infty.
\]

Here \( \hat{\cdot} \) stands for the Fourier transform in \( S'(\mathbb{R}^n) \) which is an extension of the classical definition \( \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx \) in \( S(\mathbb{R}^n) \).

The problem (1.1)-(1.2) is formally equivalent to functional equation

\[
u(t) = G(t)u_0 + L_V(u)(t),
\]

where the operators \( G(t)u_0 \) and \( L_V(u)(t) \) are defined via Fourier transform as

\[
\widehat{G(t)u_0}(\xi, t) = e^{-4\pi^2 |\xi|^2 t} \hat{u}_0
\]

\[
\widehat{L_V(u)}(\xi, t) = \int_0^t e^{-4\pi^2 |\xi|^2 (t-s)} \left( \hat{V} \ast \hat{u} \right)(\xi, s) ds.
\]

Notice that \( G(t) \) is a convolution operator with Gaussian kernel \( g(x, t) = (4\pi t)^{-\frac{n}{2}} e^{-|x|^2/4t} \) (heat semigroup) and if \( u \) and \( V \) are regular enough, then \( L_V(u)(t) = \int_0^t G(t-s)(V u)(s) ds \)
in the Bochner sense in $PM^k$. However, for general $u$ and $V$, the operator $L_V(u)$ cannot be understood in such a sense, and the integral with respect to $s$ in (1.10) should be meant in a weak-sense like e.g. [38] (see [6], [8]).

Global-in-time solutions $u(x,t)$ will be sought in the time-dependent Banach space

$$X_k = BC_w \left([0,\infty); PM^k\right)$$

with norm $\|u\|_{X_k} = \sup_{t \geq 0} \|u(\cdot,t)\|_{PM^k}$ . Here $BC_w$ stands for the set of bounded functions from a interval into a Banach space that are time-weakly continuous in the sense of $S'(\mathbb{R}^n)$ at each $t \geq 0$.

Our well-posedness result reads as follows.

**Theorem 1.1.** Suppose that $V \in PM^{n-2}$ and $u_0 \in PM^k$ with $2 < k < n$. Let $K(\theta_1, \theta_2, n) = (\nu_1^2 \nu_2 \nu_{n-1} \mu_2)/(\nu_1 + \nu_2 \nu_{n-1} \nu_{n-2})$, where $\nu_0 = \pi^{-\theta/2} \Gamma(\theta/2)$ and $\Gamma$ is the Gamma function.

1. (Existence and uniqueness) Let $C_{n-2,k} = \frac{K(2,n-k,n)}{4\pi^2}$ and assume that

$$\|V\|_{PM^{n-2}} < \frac{1}{C_{n-2,k}}.$$  (1.12)

Then the functional equation (1.8) has a unique solution $u$ in $X_k$.

2. (Hardy potential) For $V(x) = \frac{\lambda}{|x|^2}$, the condition on $V$ becomes equivalent to $|\lambda| < (k-2)(n-k)$. Notice that the maximum of $(k-2)(n-k)$ is $\lambda_s = \frac{(n-2)^2}{4}$ which is reached at $k = \frac{n+2}{2}$. Then the item (i) provides a global solution $u$ for (1.8) for all $u_0 \in PM^1 + \frac{\pi}{2}$ and $0 \leq |\lambda| < \lambda_s$.

3. (Continuous dependence) The data-solution map $(u_0, V) \rightarrow u$ is Lipschitz continuous from $PM^k \times PM^{n-2}$ to $X_k$. More precisely, if $u$ and $v$ are solutions obtained in item (i) corresponding to $u_0, V$ and $v_0, W$, respectively, then

$$\|u - v\|_{X_k} \leq \frac{1}{1 - C_{n-2,k} \|V\|_{PM^{n-2}}} \left(\|u_0 - v_0\|_{PM^k} + \frac{C_{n-2,k} \|v_0\|_{PM^k}}{1 - C_{n-2,k} \|W\|_{PM^{n-2}}} \|V - W\|_{PM^{n-2}}\right)$$

Theorem 1.1 can be applied for other types of singular potentials giving explicit conditions on the size of them.

**Corollary 1.2.** Under hypotheses of Theorem 1.1. Let $\beta(\cdot, \cdot)$ stands for the Beta function and $|\cdot|$ the sum norm in $\mathbb{R}^n$.

1. (Isotropic multipolar potential) Let $V(x) = \sum_{j=1}^m \frac{\lambda_j}{|x - x_j|^2}$ with $x_j = (x_{j1}, x_{j2}, \ldots, x_{jn})$ and $\lambda_j \in \mathbb{R}$. The condition (1.12) is verified for

$$\sum_{j=1}^m |\lambda_j| < (k-2)(n-k).$$  (1.13)

The better restriction in (1.13) holds for $k = \frac{n+2}{2}$. In this case we obtain

$$\sum_{j=1}^m |\lambda_j| < \lambda_s = \frac{(n-2)^2}{4}.$$  (1.14)
(ii) (Dipole potential) Let $V(x) = \frac{d \cdot x}{|x|^3}$ where $x = (x_1, x_2, ..., x_n)$ and $d = (d_1, d_2, ..., d_n)$. The condition (1.12) is satisfied for
\[ |d| < \frac{\pi}{(n-2)} \left( \frac{k-2}{\beta} \left( \frac{n-1}{2} \right) \right)^{\frac{1}{2}}. \] (1.15)
For $k = \frac{n+2}{2}$ one obtains
\[ |d| < \frac{\pi}{(n-2)} \frac{\lambda_*}{\beta} \left( \frac{n-1}{2} \right)^{\frac{1}{2}}, \] (1.16)
which corresponds to the maximum of the R.H.S. in (1.15).

(iii) (Anisotropic multipolar potential) Let $V(x) = \sum_{j=1}^{m} \frac{(x-x_j)^{d}}{|x-x_j|^{\beta}}$ with $x_j = (x_{j1}, x_{j2}, ..., x_{jn})$ and $d_j = (d_{j1}, d_{j2}, ..., d_{jn})$. The condition (1.12) is verified for
\[ \sum_{j=1}^{m} |d_j| < \frac{\pi}{(n-2)} \left( \frac{k-2}{\beta} \left( \frac{n-1}{2} \right) \right)^{\frac{1}{2}}. \] (1.17)
Similarly to item (ii), the maximum of the R.H.S. of (1.17) is achieved at $k = \frac{n+2}{2}$. In this case, we get
\[ \sum_{j=1}^{m} |d_j| < \frac{\pi}{(n-2)} \frac{\lambda_*}{\beta} \left( \frac{n-1}{2} \right)^{\frac{1}{2}}. \] (1.18)

Assuming a certain homogeneity on $u_0$ and $V$, then the solution $u$ is self-similar. The solution is positive when $V$ and $u_0$ are also. Depending on the radial symmetry of $u_0$ and $V$, we also investigate if $u$ is radially symmetric or not.

**Theorem 1.3.** Under the hypotheses of Theorem 1.1.

(i) (Self-similarity) Assume that $u_0$ and $V$ are homogeneous of degree $-(n-k)$ and $-2$, respectively. Then the solution $u$ satisfies
\[ u_{\lambda}(x,t) = \lambda^{n-k} u(\lambda x, \lambda^2 t), \]
for all $x \in \mathbb{R}^n$ and $t > 0$, i.e. $u$ is self-similar.

(ii) (Positivity) If $V, u_0 \geq 0$ (resp. $V \geq 0, u_0 \leq 0$) and $u_0 \neq 0$, then $u$ is positive (resp. negative).

(iii) (Radial symmetry) Let $V$ be radially symmetric. The solution $u$ obtained in Theorem 1.1 is radially symmetric for each $t > 0$ if and only if $u_0$ is radially symmetric.

For $V$ as in (1.3) and $0 \leq |\lambda| < \lambda_*$, the problem (1.1)-(1.2) (and then (1.8)) has the explicit stationary solutions
\[ \omega_1 = A_1 |x|^{\frac{m-1}{2}+l} \text{ and } \omega_2 = A_2 |x|^{\frac{m-1}{2}-l}, \] (1.19)
where $A_i$’s are arbitrary real constant and $l = \sqrt{\lambda - \lambda}$ (see [37]). A direct computation shows that the indexes
\[
k_1 = \frac{n + 2}{2} + l \quad \text{and} \quad k_2 = \frac{n + 2}{2} - l
\] (1.20)
are the unique ones such that $\omega_1 \in PM^{k_1}$ and $\omega_2 \in PM^{k_2}$. The uniqueness assertion in Theorem 1.1 says that the unique solution of (1.8) in $X_{k_i}$ with initial data $\omega_i$, is the corresponding stationary one.

In the next result we analyze the asymptotic behavior of solutions and give in particular a criterion for their converge towards a stationary state. Here we employ ideas introduced in [8].

**Theorem 1.4.** Assume the hypotheses of Theorem 1.1. Let $u$ and $v$ be two solutions for (1.8) in $X_{k_i}$ corresponding to the data $u_0$ and $v_0 \in PM^k$, respectively. We have that
\[
\lim_{t \to \infty} \|u(\cdot, t) - v(\cdot, t)\|_{PM^k} = 0 \quad (1.21)
\]
if and only if
\[
\lim_{t \to \infty} \|G(t)(u_0 - v_0)\|_{PM^k} = 0. \quad (1.22)
\]
The condition (1.22) is verified for $u_0 = v_0 + \varphi$ with $\varphi \in S(\mathbb{R}^n)$. In particular, if $V(x) = \frac{\lambda}{|x|}$ with $|\lambda| < (k_i - 2)(n - k_i)$, where $k_i$ is as in (1.20), and $u_0 = \omega_i + \varphi$ with $\varphi \in S(\mathbb{R}^n)$ then
\[
\lim_{t \to \infty} \|u(\cdot, t) - \omega_i\|_{PM^{k_i}} = 0, \quad (1.23)
\]
which shows an attractor-basin around each stationary solution $\omega_i$ in $PM^{k_i}$.

**Remark 1.5.** Let $V(x) = \lambda|x|^{-2}$ and $k$ be as in Theorem 1.1 and $u_0$ a non-radial homogeneous function of degree $-(n - k)$, for instance $u_0 = x_j|x|^{-(n-k+1)}$. It follows from Theorem 1.3 (i) that the corresponding solution $u$ is self-similar and a simple computation shows that it is not stationary. In the case $k = k_i$, the asymptotic behavior of $u$ is not described by the stationary solution $\omega_i$, because $\psi = u_0 - \omega_i \not\equiv 0$ is homogeneous of degree $-(n - k_i)$ and then
\[
\lim_{t \to \infty} \|G(t)\psi\|_{PM^{k_i}} = \|G(1)\psi\|_{PM^{k_i}} \neq 0.
\]
Instead, we obtain from (1.22) a basin of attraction around the self-similar solution $u$. More precisely, if $v_0 = u_0 + \varphi$ then the perturbed solution $v$ is attracted to the solution $u$ in the sense of (1.21). Indeed, (1.22) induces an equivalent relation in the set of initial data $PM^k$, that is, $u_0 \sim v_0$ if and only if we have (1.22).

The above considerations show a diversified asymptotic behavior of solutions in $PM^k$ with infinitely many possible asymptotics which are characterized by equivalent classes of initial data.

The outline of this paper is as follows. In Section 2.1 we recall some basic properties about Fourier transform and convolution operators useful to handle (1.8), and give estimates for the operators (1.9) and (1.10). Theorems 1.1 and 1.3 and Corollary 1.2 are proved in subsections 2.2, 2.3 and 2.4. Subsection 2.5 is devoted to proving Theorem 1.4.
2 Proof of the Results

2.1 Basic estimates in $PM^\alpha$

We start by recalling some results about convolution and Fourier transform of homogeneous functions which will be useful to perform estimates with explicit constants in $PM^k$-spaces (see [29] p. 124 and [35] p. 160, respectively).

**Lemma 2.1.** Let $0 < \theta_1 < n$, $0 < \theta_2 < n$ and $0 < \theta_1 + \theta_2 < n$. Then

$$\left( |x|^{\theta_1-n} * |x|^{\theta_2-n} \right)(y) = \int_{\mathbb{R}^n} |z|^{\theta_1-n} |y-z|^{\theta_2-n} dz = K(\theta_1, \theta_2, n)|y|^{\theta_1+\theta_2-n},$$

(2.1)

where $K(\theta_1, \theta_2, n) = (\nu_{\theta_1} \nu_{\theta_2} \nu_{n-\theta_1-\theta_2})/(\nu_{\theta_1+\theta_2} \nu_{n-\theta_1-\theta_2})$ and $\nu_\theta = \pi^{\theta/2}\Gamma(\theta/2)$.

**Lemma 2.2.** Suppose that $\alpha$ is a complex number such that $0 < Re(\alpha) < n$ and $P_t(x)$ is a harmonic polynomial on $\mathbb{R}^n$ homogeneous of degree $l$. If $K(x) = \frac{P_t(x)}{|x|^{n+\alpha}}$ then $\tilde{K}(\xi) = \gamma_{\ell, \alpha} \frac{P_\ell(\xi)}{|\xi|^{\ell+\alpha}}$, where

$$\gamma_{\ell, \alpha} = \frac{i^{-l} \pi^{(n/2)\alpha} \Gamma(\frac{l+\alpha}{2})}{\Gamma(\frac{\ell+\alpha}{2})}.\quad (2.2)$$

In the following we recall an estimate in $PM^k$-spaces for the heat semigroup $[1.9]$ (see e.g. [3]). The proof is included for the reader convenience.

**Lemma 2.3.** If $f \in PM^k$ with $k \geq 0$ then $G(t)f \in BC_w\left([0, \infty) ; PM^k\right)$ and

$$\sup_{t>0} \| G(t)f \|_{PM^k} \leq \| f \|_{PM^k}.$$  \quad (2.3)

**Proof.** We have that

$$|\xi|^k \left| \hat{G(t)f}(\xi) \right| \leq |\xi|^k \left| e^{-4\pi^2 t |\xi|^2} \hat{f}(\xi) \right| \leq |\xi|^k \left| \hat{f}(\xi) \right|,$$

(2.4)

which gives (2.3) after applying $ess \sup_{x \in \mathbb{R}^n}$ in both sides of (2.4).

\[\blacksquare\]

The next lemma gives an estimate for the linear operator $[1.10]$ in $PM^k$-spaces by explicitly the dependence of the norm of $V$ and giving an exact expression for the constant.

**Lemma 2.4.** Let $0 < b_1, b_2 < n$ be such that $n < b_1 + b_2 < 2n$. For $K(\theta_1, \theta_2, n)$ as in Lemma 2.1, we set

$$C_{b_1, b_2} = \frac{1}{4\pi^2} K(n-b_1, n-b_2, n).$$

(2.5)

Then $LV(u) \in BC_w\left([0, \infty) ; PM^b\right)$ with $b = b_1 + b_2 + 2 - n$ and

$$\sup_{t>0} \| LV(u)(t) \|_{PM^b} \leq C_{b_1, b_2} \| V \|_{PM^{b_1}} \sup_{t>0} \| u(\cdot, t) \|_{PM^{b_2}},$$

(2.6)

for all $V \in PM^{b_1}$ and $u \in X_{b_2}$.  

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Proof. Using Lemma 2.1 we obtain
\[ |\hat{V} * \hat{u}(\xi)| \leq \int_{\mathbb{R}^n} |\hat{V}(\xi - \eta)\hat{u}(\eta)| \, d\eta \]
\[ \leq \int_{\mathbb{R}^n} \frac{1}{|\xi - \eta|^{b_1}} \frac{1}{|\eta|^{b_2}} \|V\|_{P_{M^{b_1}}} \|u\|_{P_{M^{b_2}}} \, d\eta \]
\[ \leq K(n - b_1, n - b_2, n) \frac{1}{|\xi|^{b_1 + b_2 - n}} \|V\|_{P_{M^{b_1}}} \|u\|_{P_{M^{b_2}}} . \]
(2.7)

It follows from (1.10) and (2.7) that
\[ \left| \hat{L}_V(u)(\xi) \right| \leq \int_0^t e^{-4\pi^2(t-s)|\xi|^2} \left| \hat{V} * (\hat{u}(\xi, s)) \right| \, ds \]
\[ \leq \int_0^t e^{-4\pi^2(t-s)|\xi|^2} \frac{K(n - b_1, n - b_2, n)}{|\xi|^{b_1 + b_2 - n}} \|V\|_{P_{M^{b_1}}} \|u(\cdot, s)\|_{P_{M^{b_2}}} \, ds \]
\[ \leq \frac{K(n - b_1, n - b_2, n)}{|\xi|^{b_1 + b_2 - n}} \int_0^t e^{-4\pi^2(t-s)|\xi|^2} \, ds \sup_{t > 0} \|u(\cdot, t)\|_{P_{M^{b_2}}} \]
\[ \leq \frac{1}{4\pi^2} \frac{K(n - b_1, n - b_2, n)}{|\xi|^{b_1 + b_2 + 2 - n}} (1 - e^{-4\pi^2 t|\xi|^2}) \sup_{t > 0} \|u(\cdot, t)\|_{P_{M^{b_2}}} , \]
which yields the desired inequality.

The weak-time continuity in $S'(\mathbb{R}^n)$ is left to the reader (see e.g. [8] and [38]).

2.2 Proof of Theorem 1.1

Part (i): Let us set
\[ \tau = C_{n-2,k} \|V\|_{P_{M^{n-2}}} \]
Lemma 2.4 with $(b_1, b_2) = (n - 2, k)$ yields
\[ \|L_V(u)(t) - L_V(v)(t)\|_{X_k} = \sup_{t > 0} \|L_V(u - v)(t)\|_{P_{M^k}} \]
\[ \leq C_{n-2,k} \|V\|_{P_{M^{n-2}}} \|u - v\|_{X_k} , \]
(2.8)
and then
\[ \|L_V\|_{X_k \to X_k} \leq \tau < 1. \]

Also, Lemma 2.3 gives us
\[ \|G(t)u_0\|_{X_k} = \sup_{t > 0} \|G(t)u_0\|_{P_{M^k}} \leq \|u_0\|_{P_{M^k}} . \]
(2.9)

The estimate (2.8) with $v = 0$ and (2.9) imply that the operator $H : X_k \to X_k$ such that $H(u) = G(t)u_0 + L_V(u)(t)$ is well defined. Also, we have the estimate
\[ \|H(u) - H(v)\|_{X_k} \leq \|L_V(u)(t) - L_V(v)(t)\|_{X_k} \leq \tau \|u - v\|_{X_k} , \] for all $u, v \in X_k$, \quad (2.10)
which shows that $H$ is a contraction in $X_k$. Now the Banach fixed point theorem assures the existence of a unique solution $u \in X_k$ for (1.8).

Part (ii): From (2.5) and Lemma 2.1, we can compute $C_{n-2,k}$ explicitly as

$$C_{n-2,k} = \frac{K(2,n-k,n)}{4\pi^2} = \frac{\pi^{n/2} \Gamma(1) \Gamma\left(\frac{n-k}{2}\right) \Gamma\left(\frac{k-2}{2}\right)}{4\pi^2 \Gamma\left(\frac{n-2}{2}\right) \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{2+n-k}{2}\right)} = \frac{\pi^{n/2}(n-2)}{2\pi^2 \Gamma\left(\frac{n}{2}\right) (k-2)(n-k)}. \quad (2.11)$$

It follows from Theorem 1.1 (i) that

$$\|V\|_{PM^{n-2}} < \frac{1}{C_{n-2,k}} = \frac{2\pi^2 \Gamma\left(\frac{n}{2}\right) (k-2)(n-k)}{\pi^{n/2}(n-2)}. \quad (2.12)$$

For $V = \frac{\lambda}{|x|^n}$, applying Lemma 2.2 with $\alpha = n-2$ and $l = 0$, we obtain

$$\hat{V}(\xi) = \lambda \pi^{\frac{n-2}{2}} \Gamma\left(\frac{n-2}{2}\right) |\xi|^{2-n},$$

which gives $V \in PM^{n-2}$ with

$$\|V\|_{PM^{n-2}} = |\lambda| \pi^{\frac{n-2}{2}} \Gamma\left(\frac{n-2}{2}\right). \quad (2.13)$$

In view of (2.13), the condition (2.12) can be expressed by means of the size of $|\lambda|$ as

$$|\lambda| < \frac{1}{\pi^{2-\frac{n}{2}} \Gamma\left(\frac{n-2}{2}\right) C_{n-2,k}} = \frac{2\pi^2 \Gamma\left(\frac{n}{2}\right) (n-k)(k-2)}{\pi^2 \Gamma\left(\frac{n-2}{2}\right) (n-2)} = (n-k)(k-2). \quad (2.14)$$

Part (iii): Let $u$ and $v$ be two solutions obtained from item (i) with data $V, u_0$ and $W, v_0$, respectively. Firstly, using (2.10), we obtain

$$\|v\|_{X_k} = \|H(v)\|_{X_k} \leq \|G(t)v_0\|_{X_k} + \|L_W(v)\|_{X_k} \leq \|v_0\|_{PM^k} + C_{n-2,k} \|W\|_{PM^{n-2}} \|v\|_{X_k},$$

which implies

$$\|v\|_{X_k} \leq \frac{\|v_0\|_{PM^k}}{1 - C_{n-2,k} \|W\|_{PM^{n-2}}}. \quad (2.14)$$
Subtracting the respective equations verified by \( u, v \), and afterwards applying the norm \( \| \cdot \|_{X_k} \), we estimate
\[
\| u - v \|_{X_k} = \| G(t)(u_0 - v_0) + L_V(u - v) \|_{X_k} \\
\leq \| u_0 - v_0 \|_{PM^k} + C_{n-2,k} \| V \|_{PM^{n-2}} \| u - V \|_{X_k} \\
+ C_{n-2,k} \| V - W \|_{PM^{n-2}} \| V \|_{X_k}.
\]  
(2.15)
The estimates (2.14) and (2.15) lead us to
\[
\| u - v \|_{X_k} \leq \frac{1}{1 - C_{n-2,k} \| V \|_{PM^{n-2}}} \left( \| u_0 - v_0 \|_{PM^k} + \frac{C_{n-2,k} \| v_0 \|_{PM^k}}{1 - C_{n-2,k} \| W \|_{PM^{n-2}}} \| V - W \|_{PM^{n-2}} \right),
\]
as desired.

2.3 Proof of Corollary 1.2

Part (i): (Isotropic multipolar potential) Using the translation property of Fourier transform and Lemma 2.2, we obtain
\[
\hat{V}(\xi) = \sum_{j=1}^{m} \lambda_j \pi^{2-\frac{\alpha}{2}} \Gamma \left( \frac{n-2}{2} \right) e^{-2\pi i (x^j \cdot \xi)} |\xi|^{2-n},
\]
which gives \( V \in PM^{n-2} \) and
\[
\| V \|_{PM^{n-2}} \leq \pi^{2-\frac{\alpha}{2}} \Gamma \left( \frac{n-2}{2} \right) (\Sigma_{j=1}^{m} |\lambda_j|) < \frac{1}{C_{n-2,k}},
\]
provided that \( \Sigma_{j=1}^{m} |\lambda_j| < (n-k)(k-2) \).

Part (ii): (Dipole potential) Lemma 2.2 with \( \alpha = n-2 \) and \( l = 1 \) yields
\[
\hat{V}(\xi) = \left( \sum_{j=1}^{m} \frac{d_j x_j}{|x|^3} \right)^{\wedge} = \sum_{j=1}^{m} 2i^{-1} \pi^{\frac{3-n}{2}} \Gamma \left( \frac{n-1}{2} \right) \frac{d_j \xi_j}{|\xi|^{n-1}} \\
= 2i^{-1} \pi^{\frac{3-n}{2}} \Gamma \left( \frac{n-1}{2} \right) \frac{d \cdot \xi}{|\xi|^{n-1}}.
\]
It follows that \( V \in PM^{n-2} \) and
\[
\| V \|_{PM^{n-2}} \leq 2\pi^{\frac{3-n}{2}} \Gamma \left( \frac{n-1}{2} \right) |d|.
\]
Then, the condition (2.12) (i.e. (1.12)) is verified when
\[ |d| < \frac{1}{2\pi^{\frac{n-2}{2}}} \frac{2\pi^{2-n/2}(n-k)(k-2)\Gamma\left(\frac{n}{2}\right)}{(n-2)} \]
\[ = \frac{\pi^{1/2}(n-k)(k-2)\Gamma\left(\frac{n}{2}\right)}{(n-2)\Gamma\left(\frac{n-1}{2}\right)} \]
\[ = \frac{\pi(n-k)(k-2)}{(n-2)} \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)}. \]  
\( \text{(2.16)} \)

**Part (iii):** (Anisotropic multipolar potential) Computing the Fourier transform, it follows that
\[ \hat{V}(\xi) = \sum_{j=1}^{m} 2^{j-1} \pi^{\frac{n-2}{2}} \Gamma\left(\frac{n-1}{2}\right) e^{-2\pi i x^j \xi} \frac{\xi^j d^j}{|\xi|^{n-2}}, \]
and then, similarly to item (ii), the condition (1.12) for \( V \) in Theorem 1.1 is verified for
\[ \sum_{j=1}^{m} |d^j| < \frac{\pi(n-k)(k-2)}{(n-2)} \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)}. \]  

2.4 Proof of Theorem 1.3

**Part (i):** Let \( u \) be the solution given in Theorem 1.1 (i). It is not difficult to check that \( u(x, t) = \lambda^{n-k} u(\lambda x, \lambda^2 t) \in X_k \) also verifies (1.8) when \( u_0(x) \) and \( V(x) \) are homogeneous of degree \(-(n-k)\) and \(-2\), respectively. Now the uniqueness statement in Theorem 1.1 gives us \( u = u_\lambda \) for all \( \lambda > 0 \), as required.

**Part (ii):** We should prove only the positivity statement, because the proof of the one between brackets is similar. Recall first that \( F \in S'(\mathbb{R}^n) \) is said to be nonnegative (resp. nonpositive) if \( \langle F, \varphi \rangle \geq 0 \) (resp. \( \leq 0 \)), for all \( \varphi \geq 0 \) and \( \varphi \in S(\mathbb{R}^n) \). Also, \( F \) is positive (resp. negative) when \( \langle F, \varphi \rangle > 0 \) (resp. \( < 0 \)), for all \( \varphi > 0 \) and \( \varphi \in S(\mathbb{R}^n) \).

Note that \( u_1 = G(t) u_0 \) is a positive distribution in \( S'(\mathbb{R}^n) \), for \( t > 0 \), when \( u_0 \) is nonnegative and \( u_0 \not\equiv 0 \). Since the solution \( u \) has been obtained via Banach fixed point theorem, it is the limit of the Picard interaction
\[ u_1 = G(t) u_0 \quad \text{and} \quad u_{b+1} = u_1 + L_V(u_b), \quad b \in \mathbb{N}. \]  
\( \text{(2.17)} \)

An induction argument shows that all elements of (2.17) are positive distribution in \( S'(\mathbb{R}^n) \), for \( t > 0 \). Since \( u_b \to u \) in \( X_k \), we have that \( u_b \to u \) in \( S'(\mathbb{R}^n) \), for \( t > 0 \). It follows that \( u(\cdot, t) \) is a nonnegative distribution, for \( t > 0 \), because the convergence in \( S'(\mathbb{R}^n) \) preserves nonnegativity. As \( u_1 \) is positive and \( L_V(u) \) is nonnegative, it follows that
\[ \langle u(\cdot, t), \varphi \rangle = \langle u_1(\cdot, t), \varphi \rangle + \langle L_V(u)(t), \varphi \rangle \geq \langle u_1(\cdot, t), \varphi \rangle > 0, \quad \text{for } t > 0, \]
for all \( \varphi > 0 \) and \( \varphi \in S(\mathbb{R}^n) \).
Part (iii): Let $u_0$ and $V$ be radially symmetric. As the heat flow preserves radial symmetry, it follows that $u_1 = G(t)u_0$ is radially symmetric, for each fixed $t > 0$. Also, $L_V(u)$ is radially symmetric provided that $u$ is also radially symmetric. One can prove by induction that $\{u_k\}_{k \geq 1}$ (see (2.17)) is radially symmetric, for each fixed $t > 0$. Since $u_k \to u$ in $X_k$ and Fourier transform preserves radial symmetry, we get that $u$ is radially symmetric, for each fixed $t > 0$.

Assume now that $u_0$ is not radially symmetric and $V$ is radially symmetric. Suppose, to the contrary, that $u$ were radially symmetric, then $L_V(u)$ also would be radially symmetric. So, $G(t)u_0 = u - L_V(u)$ would be radially symmetric, which gives a contradiction because $(G(t)u_0)^\wedge = e^{-|\xi|^2 t} \hat{u}_0$ is radially symmetric if and only if $\hat{u}_0$ is radially symmetric.

2.5 Proof of Theorem 1.4

We only prove that (1.22) implies (1.21). The converse statement follows similarly and is left to the reader. Subtracting the equations satisfied by $u$ and $v$, and afterwards computing the $PM^k$-norm, we obtain

$$\|u(\cdot, t) - v(\cdot, t)\|_{PM^k} \leq \|G(t)(u_0 - v_0)\|_{PM^k} + J_1(t) + J_2(t)$$

(2.18)

where

$$J_1(t) = 4\pi^2 C_{n-2,k} \|V\|_{PM^{n-2}} \sup_{\xi \in \mathbb{R}^n} \int_0^\delta t |\xi|^2 e^{-4\pi^2 |\xi|^2 (t-s)} \|u(\cdot, s) - v(\cdot, s)\|_{PM^k} ds$$

$$J_2(t) = 4\pi^2 C_{n-2,k} \|V\|_{PM^{n-2}} \sup_{\xi \in \mathbb{R}^n} \int_0^t t |\xi|^2 e^{-4\pi^2 |\xi|^2 (t-s)} \|u(\cdot, s) - v(\cdot, s)\|_{PM^k} ds.$$

with $\delta > 0$ being a constant that will be chosen later. Using that

$$\sup_{\xi \in \mathbb{R}^n} t |\xi|^2 e^{-t(1-s)4\pi^2 |\xi|^2} = \frac{e^{-t}}{4\pi^2 (1-s)},$$

and the change $s = tz$ in $J_1(t)$, we estimate

$$J_1(t) \leq 4\pi^2 C_{n-2,k} \|V\|_{PM^{n-2}} \sup_{\xi \in \mathbb{R}^n} \int_0^\delta t |\xi|^2 e^{-t(1-s)4\pi^2 |\xi|^2} \|u(\cdot, ts) - v(\cdot, ts)\|_{PM^k} ds$$

$$\leq C \int_0^\delta (1-s)^{-1} \|u(\cdot, ts) - v(\cdot, ts)\|_{PM^k} ds.$$

(2.19)

The term $J_2(t)$ can be estimated directly by

$$J_2(t) \leq 4\pi^2 C_{n-2,k} \|V\|_{PM^{n-2}} \left( \sup_{\xi \in \mathbb{R}^n} \int_0^t |\xi|^2 e^{-t(1-s)4\pi^2 |\xi|^2} ds \right) \left( \sup_{\delta t \leq s < t} \|u(\cdot, s) - v(\cdot, s)\|_{PM^k} \right)$$

$$= C_{n-2,k} \|V\|_{PM^{n-2}} \sup_{\delta t < s < t} \|u(\cdot, s) - v(\cdot, s)\|_{PM^k},$$

(2.20)
because $\int_0^t |\xi|^2 e^{-\frac{(t-s)^4}{4\pi^2}} ds = \frac{1}{4\pi^2} \left( 1 - e^{-4\pi^2 (1-\delta) |\xi|^2} \right)$. Noting that
\[ \Gamma = \limsup_{t \to \infty} \|u(\cdot, t) - v(\cdot, t)\|_{P M k} \leq (\|u\|_{X_k} + \|v\|_{X_k}) < \infty, \]
we can calculate the superior limit in (2.18), and then use (2.19) and (2.20) in order to obtain
\[ \Gamma \leq \left( C \log \left( \frac{1}{1 - \delta} \right) + C_{n-2,k} \|V\|_{P M^{n-2}} \right) \Gamma = M \Gamma. \]
In view of $C_{n-2,k} \|V\|_{P M^{n-2}} < 1$ (see (1.12)), one can take $\delta > 0$ in such a way that $0 < M < 1$, and so $\Gamma = 0$, as required.

The further conclusions in the statement follow by employing (1.21) with $v(x, t) \equiv \omega_i$ and noting that $\lim_{t \to \infty} \|G(t)\varphi\|_{P M^k} = 0$ when $\varphi \in S(\mathbb{R}^n)$.

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