Helical spin texture in a thin film of superfluid $^3$He

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We consider a thin film of superfluid $^3$He under conditions that stabilize the A-phase. We show that in the presence of a uniform superflow and an external magnetic field perpendicular to the film, the spin degrees of freedom develop a nonuniform, helical texture. Our prediction is robust and relies solely on Galilei invariance and other symmetries of $^3$He, which induce a coupling of the orbital and spin degrees of freedom. The length scale of the helical order can be tuned by varying the velocity of the superflow and the magnetic field, and may be in reach of near-future experiments.

Introduction.—The experimental discovery of superfluidity in $^3$He [1] was a major breakthrough in low-temperature physics. The unconventional pairing of fermions in this system provided one of the first examples of topological quantum matter. The intricate symmetry-breaking patterns realized in $^3$He give rise to a number of unexpected phenomena which have kept both theorists and experimentalists busy for nearly a half century [2–4].

Recent advances in nanofabrication made it possible to study superfluidity experimentally under well-controlled conditions in $^3$He confined to two spatial dimensions [5, 6]. Two-dimensional confinement leads to a substantial modification of the phase diagram of superfluid $^3$He. In particular, at zero temperature it is the chiral A-phase that is energetically stabilized in a film with a thickness of the order of a few times the superfluid coherence length $\xi$ [7].

Motivated by these developments, we analyze in this Letter the spin dynamics in the A-phase of quasi-two-dimensional $^3$He at zero temperature. We use the effective field theory approach, based solely on symmetry and the low-energy degrees of freedom. Our main result is that the presence of a uniform superflow and a magnetic field $H \gtrsim 30$ G, perpendicular to the $^3$He film, induces a nonuniform, planar helical texture (see Fig. 1) in the ground state of the spin degrees of freedom. The pitch of the helical texture depends, apart from the macroscopic superflow velocity and the magnetic field, on a sole intrinsic observable: the phase velocity of spin waves. The pitch can be tuned by varying the former two macroscopic parameters, and within near-future experiments with superfluid $^3$He films, it may reach the centimeter range.

Owing to the rich structure of the order parameter, the precise form of the ground state of superfluid $^3$He usually depends on many factors, including geometrical constraints (boundary conditions), interaction with external fields, and last but not least, the weak dipole (spin-orbit) coupling between the spin and orbital degrees of freedom. This results in a large number of possible textures in superfluid $^3$He, depending on precise external conditions [2]. Thus, for instance, similar helical textures were previously predicted in bulk $^3$He [8] and in $^3$He confined to a nanotube [9]. Likewise, a periodic texture was predicted for the A-phase of $^3$He confined to a thin slab [10]. The textures proposed in Refs. [8, 10] depend crucially on the presence of the dipole interaction.

The texture found in this Letter is fundamentally different in that it does not rely on the presence of the dipole interaction. In contrast, it is a robust consequence of Galilei invariance and other symmetries of $^3$He. The only assumptions we make, that set constraints on possible experimental realization of this novel texture, are: (i) a slab geometry that stabilizes the A-phase, and (ii) a magnetic field strong enough to rotate the spins into the slab plane.

The plan of this Letter is as follows. First, we overview the essentials of quasi-two-dimensional $^3$He, including its symmetries and some basic order-of-magnitude estimates, relevant for its experimental realization. Next, we develop the low-energy effective field theory of the spin degrees of freedom in the A-phase, stressing the role of Galilei invariance. This is followed by a detailed derivation of the helical texture in the ground state. In the end, we discuss the excitation spectrum above the helical texture, and its possible signatures through nuclear magnetic resonance (NMR) spectroscopy. Some further technical details are presented in Ref. [11].

![Helical spin texture in a film of $^3$He-A. The orbital vector $\hat{l}$ is forced by surface interactions to be perpendicular to the film. The magnetic field $H$ is chosen to point in the same direction. The local, in-plane spin vector $\hat{d}$ varies along the superflow velocity $u$, but remains uniform in the transverse direction.](image-url)
Quasi-two-dimensional $^3$He.—Bulk $^3$He at zero temperature and low pressures features the isotropic B-phase. The ground state, however, changes when $^3$He is confined to a narrow slab. Weak-coupling theory predicts [12] that the A-phase is stabilized for slab thickness $D \lesssim 9\xi_0$, being separated from the B-phase by a stripe phase at $9\xi_0 \lesssim D \lesssim 13\xi_0$. While the question of the existence of the stripe phase remains unresolved by experiment, the stability of the A-phase in narrow slabs has been confirmed [6, 13]. Given that $\xi_0 \approx 70\text{ nm}$ for pressures below ca 2 bar [5], and that the dipole interaction only becomes important at length scales above the order of $10\ \mu\text{m}$ [4], the latter will play a negligible role in our analysis.

The order parameter of the A-phase of $^3$He has the structure
\[
\Delta_{ir} \propto \hat{d}_i (\hat{m}_r + i\hat{n}_r),
\]
where $\hat{d}$ is a unit vector in the spin space and $\hat{m}, \hat{n}$ are two orthogonal unit vectors in the orbital space. The three degrees of freedom contained in $\hat{m}, \hat{n}$ can be encoded in a single vector, $\vec{l} \equiv \hat{m} \times \hat{n}$, and an overall complex phase $\theta$. Boundary effects induce an aligning force on $\vec{l}$ that tries to orient it perpendicularly to the surface. In the quasi-two-dimensional regime of $^3$He confined to a narrow slab, the $\vec{l}$-vector will be completely oriented to the direction normal to the slab, and the only active orbital degree of freedom will thus be the superfluid phase $\theta$.

In the absence of other symmetry-breaking perturbations, it can be encoded in a single vector, $\hat{d} \equiv \hat{m} \times \hat{n}$, and an overall complex phase $\theta$. Boundary effects induce an aligning force on $\hat{d}$ that tries to orient it perpendicularly to the surface. In the quasi-two-dimensional regime of $^3$He confined to a narrow slab, the $\hat{d}$-vector will be completely oriented to the direction normal to the slab, and the only active orbital degree of freedom will thus be the superfluid phase $\theta$. The total of three degrees of freedom, contained in $\hat{d}$ and $\theta$, correspond to the symmetry-breaking pattern in the A-phase in two spatial dimensions,
\[
SU(2)_S \times SO(2)_L \times U(1)_\phi \to U(1)_S \times U(1)_\phi - L,
\]

where “S” and “L” refer respectively to spin and orbital symmetries and $U(1)_\phi$ stands for the particle number symmetry.

The dipole interaction breaks the independent spin and orbital symmetries down to the diagonal SO(2)$_L \times$U(1)$_\phi$ subgroup. In the absence of other symmetry-breaking perturbations, it aligns the $\hat{d}$-vector (anti)parallel to $\vec{l}$. To overcome this weak aligning force and make the spins oriented in the slab plane, we assume the presence of a magnetic field $\vec{H}$, perpendicular to the slab. The desired orientation of the $\hat{d}$-vector will be achieved provided $\vec{H}$ is stronger than the characteristic field of the dipole interaction, $H_0 \approx 30\text{ G}$ [2]. This is equivalent to the requirement that the Larmor frequency $\omega_L$ of spin precession in the magnetic field be larger than the so-called Leggett frequency $\Omega_L$. In the A-phase, we have $\Omega_L \approx 50\text{ KHz}$ [4]. On the other hand, the magnetic field should not be too strong so as not to distort significantly the order parameter. Taking the temperature scale of the order parameter as $T_\Delta \sim 1\text{ mK}$, we can estimate the corresponding critical angular frequency as $\hbar k_BT_\Delta / \hbar \approx 100\text{ MHz}$. The Larmor frequency typically used in current experiments, $f_L = \omega_L / (2\pi) \approx 1\text{ MHz}$ [5], satisfies with a good margin both bounds.

Finally, recall that the superfluid becomes unstable when the superflow velocity $\vec{u}$ exceeds the Landau critical velocity. For the A-phase of $^3$He, this is of the order of $u_{cr} \approx 5\text{ cm/s}$ [4]. The superflow velocity in actual narrow-slab experiments on $^3$He is typically much lower, in the sub-mm/s range.

Effective theory of spin dynamics.—The dynamics of the A-phase of quasi-two-dimensional $^3$He at low energies is dominated by the soft degrees of freedom corresponding to the symmetry-breaking pattern (2), that is, the variables $\hat{d}$ and $\theta$. In this Letter, we assume that the superflow, defined by its velocity $\vec{u} = \nabla \theta / m$, constitutes a fixed background for the spin dynamics. This is a reasonable assumption for $u \ll u_{cr}$, and can be justified formally using the power counting of the low-energy effective theory [14]. With this assumption, the low-energy spin dynamics can be fully captured by an effective theory for the vector $\hat{d}$ alone.

The effective theory must respect all the symmetries of the microscopic interactions among $^3$He atoms. The spacetime symmetries include space and time translations, Galilei invariance, spatial rotation invariance SO(2)$_L$, two-dimensional parity $P$ (under which $x \leftrightarrow y$) and time reversal $T$. The internal symmetries include the spin rotation invariance SU(2)$_S$ and the particle number symmetry U(1)$_\phi$.

Under an infinitesimal boost, $x' = x + vt$, the superfluid phase $\theta$ shifts as $\theta'(x') = \theta(x) + vD \cdot x$. Galilei invariance then requires that time derivatives of other, boost-invariant fields only enter the action through the “material derivative”, $\partial_t \equiv \partial_t + \vec{u} \cdot \nabla$. To the leading order in the derivative expansion, the effective spin Lagrangian density then reads [15]
\[
\mathcal{L} = \frac{1}{2} (D_i \hat{d} + u_i D_i \hat{d})^2 - \frac{c_s^2}{2} (D_i \hat{d})^2 + \mathcal{L}_{\text{dip}}.
\]

Here $c_s$ is the phase velocity of spin waves in the absence of background fields. The covariant derivative of the $\hat{d}$-vector is defined by
\[
D_\mu \hat{d} \equiv \partial_\mu \hat{d} + A_\mu \times \hat{d},
\]
where $A_\mu$ is the gauge field of the SU(2)$_S$ group. In presence of a magnetic field $\vec{H}$ and no other external fields, it takes the value $A_\mu = -\epsilon_{\mu\nu} \vec{H}$ [16] (the magnetic moment is absorbed in $\hat{d}$). Finally, the symmetry-breaking perturbation $\mathcal{L}_{\text{dip}}$ represents the dipole interaction,
\[
\mathcal{L}_{\text{dip}} = \frac{1}{2} D_i^2 (\hat{l} \cdot \hat{d})^2.
\]

We stress that the coupling to the magnetic field, defined by Eq. (4), is not a perturbation in the same sense as the dipole coupling. Namely, it is completely fixed by the SU(2)$_S$ invariance, and involves no new, a priori arbitrary, parameters.

In two spatial dimensions, the term $\epsilon_{\mu\nu} \hat{l} \cdot (D_\mu \hat{d} \times D_\nu \hat{d})$ is also consistent with the continuous symmetries of the system. This term is, however, prohibited by the discrete parity and time-reversal symmetries.

Our construction above is completely general and relies on the symmetries of the system only. Given a microscopic model of a thin film of $^3$He, on can alternatively derive the effective Lagrangian (3) by integrating out the fermionic degrees of freedom. Such an approach allows one to fix the spin wave velocity in terms of the parameters of the microscopic model. To complement our general construction presented here, we perform this calculation for the Bogoliubov-de-Gennes mean-field theory in Ref. [11].
Ground state texture.—We are now interested in the ground state of the system in the presence of a uniform background superfluid and an external magnetic field perpendicular to the film, see Fig. 1. To that end, we first compute the canonical Hamiltonian density, 
\[ \mathcal{H} = \frac{1}{2} \partial_t \mathbf{d} \cdot \partial_x \mathcal{L} / \partial (\partial_t \mathbf{d}) - \mathcal{L}, \]

\[ \mathcal{H} = \frac{1}{2} (\partial_t \mathbf{d})^2 - \frac{1}{2} (\mathbf{H} \times \mathbf{d} + u_r \partial_r \mathbf{d})^2 + \frac{\mathcal{L}^2}{2} (\partial_\mathbf{d} \hat{\mathbf{d}})^2 - \frac{1}{2} \Omega_\mathbf{d}^2 (\partial_\mathbf{d} \hat{\mathbf{d}})^2. \]

Given the way the temporal derivatives enter the Hamiltonian, the ground state of the system in the presence of a uniform background magnetic field is given by

\[ \mathcal{H} = \mathcal{H}_0 + \frac{1}{2} (1 + \frac{u}{\mathcal{L}^2}) (\mathbf{H} \cdot \mathbf{d})^2 - \frac{1}{2} \Omega_\mathbf{d}^2 (\partial_\mathbf{d} \hat{\mathbf{d}})^2 + \frac{\mathcal{L}^2}{2} (\partial_\mathbf{d} \hat{\mathbf{d}})^2, \]

where, without loss of generality, we chose the x-axis along the superflow. We also defined \( c_{\text{eff}}^2 \equiv c_s^2 - u^2 \). Note that in practice, the Landau critical velocity \( u_{\text{cr}} \) is much smaller than the spin-wave velocity \( c_s \), hence the coefficient \( c_{\text{eff}}^2 \) is always positive and approximately equal to \( c_s^2 \). Next, we combine the terms containing \( \partial_x \mathbf{d} \) and rewrite (\( \mathbf{H} \times \mathbf{d} \))^2 = \( \mathbf{H}^2 - (\mathbf{H} \cdot \mathbf{d})^2 \), which leads to

\[ \mathcal{H} = \mathcal{H}_0 + \frac{1}{2} \left( 1 + \frac{u}{\mathcal{L}^2} \right) (\mathbf{H} \cdot \mathbf{d})^2 - \frac{1}{2} \Omega_\mathbf{d}^2 (\partial_\mathbf{d} \hat{\mathbf{d}})^2 + \frac{\mathcal{L}^2}{2} (\partial_\mathbf{d} \hat{\mathbf{d}})^2, \]

where \( \mathcal{H}_0 \equiv -\frac{1}{2} (1 + \frac{u}{\mathcal{L}^2}) \mathbf{H}^2 \). It is now clear that for \( \mathbf{H} \parallel \hat{\mathbf{i}} \) and \( H > \Omega_\mathbf{d} \) (as assumed), the following conditions must be satisfied simultaneously in the state of lowest energy,

\[ \mathbf{H} \cdot \mathbf{d} = 0, \quad \partial_r \mathbf{d} = \frac{u}{\mathcal{L}^2} \mathbf{H} \times \mathbf{d}, \quad \partial_\mathbf{d} \hat{\mathbf{d}} = 0. \]

The unique solution up to an overall spin rotation is given by in-plane Larmor precession of the \( \mathbf{d} \)-vector with the coordinate \( x \) along the superflow playing the role of time, see Fig. 1.

The pitch of the helical texture follows from Eq. (9) and can be expressed in terms of easily measurable quantities as

\[ \lambda = \frac{1}{f_L} \frac{c_s^2 - u^2}{u}. \]

Assuming that \( u \ll c_s \) and approximating the spin-wave velocity by \( c_s \approx 20 \text{ m/s} \), we get a numerical estimate for the pitch in terms of the tunable parameters \( f_L \) and \( u \),

\[ \lambda \approx 40 \text{ cm} \times \left( \frac{f_L}{\text{MHz}} \frac{u}{\text{mm/s}} \right)^{-1}. \]

The pitch of the helical texture can be reduced to the centimeter range by a moderate increase of both \( f_L \) and \( u \) compared to values typical for current experiments. Even if the whole pitch turns out to be too long, it should still be possible to observe the effect through chirality of spin-spin correlations.

Let us now mention some theoretical aspects of the discovered helical texture. First of all, the derivation of the ground state was carried out in a fixed reference frame attached to the slab confining the \( ^3\text{He} \) sample; the parameter \( u \) measures the velocity of the superflow with respect to the slab. The same result can, however, be obtained in any other reference frame due to Galilei invariance; see Ref. [11] for details.

Second, the generation of dissipation-less spin currents has been of great theoretical as well as practical interest lately (see e.g. Ref. [18]), and the structure of the helical ground state might suggest that it carries such a current. However, a closer look reveals that this is not the case. Indeed, the Noether current of the SU(2) spin symmetry,

\[ \mathbf{j} = \frac{\partial \mathcal{L}}{\partial (\partial_\mathbf{d} \hat{\mathbf{d}})}, \]

only has a nonzero temporal component, \( \mathbf{H} c_s^2 / c_{\text{eff}}^2 \), indicating nonzero spin density, but vanishing spin current. This, at first surprising, result is reminiscent of the Bloch theorem [19].

Third, previous theoretical work [20] discovered that the effective theory of spin in a superfluid \(^3\text{He}-\text{A}\) film contains a topological Hopf term, responsible for the quantum statistics of skyrmions and quantized spin Hall effect. The Hopf term is defined by the Lagrangian

\[ \mathcal{L}_{\text{Hopf}} = \frac{1}{32\pi^2} \int d^2 x dt \epsilon^νλμνλ \mathbf{F}_{νλ}, \]

where \( \mathbf{F}_{νλ} = \partial_ν \mathbf{A}_λ - \partial_λ \mathbf{A}_ν \equiv \hat{\mathbf{d}} \cdot (\partial_ν \mathbf{d} \times \partial_λ \mathbf{d}) \) is an auxiliary composite gauge field. The Hopf term was not included in our effective theory, being formally of higher order in the derivative expansion. Moreover, our helical texture only varies in one spatial direction, hence it carries zero skyrmion number and the Hopf term accordingly vanishes.

Finally, the ground state can be found using the Hamiltonian (8) also for other orientations of the magnetic field than perpendicular to the slab. In the ideal limit of exact spin symmetry, \( \Omega_\mathbf{d} \rightarrow 0 \), the ground state will correspond to an analogous helical texture featuring spin precession around the \( \mathbf{H} \)-vector. A nonzero dipole coupling will in general lead to a distortion of the helix when \( \mathbf{H} \parallel \hat{\mathbf{i}} \).

Excitation spectrum.—The basic tool for identification of nonuniform textures in \(^3\text{He}\) is NMR [21]. To understand possible NMR signatures of our helical texture, we need to determine the excitation spectrum. To that end, we write the local spin vector in the ground state as

\[ \langle \hat{d}_1 \rangle = \cos \alpha x, \quad \langle \hat{d}_2 \rangle = \sin \alpha x, \quad \langle \hat{d}_3 \rangle = 0, \]

where \( \alpha \equiv u H / c_{\text{eff}}^2 \). Next, we introduce the “comoving” spin variable \( \mathbf{d}' \) through

\[ \hat{d}(r) = \begin{pmatrix} \cos \alpha x & -\sin \alpha x & 0 \\ \sin \alpha x & \cos \alpha x & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{d}'(r), \]

where \( \alpha \equiv u H / c_{\text{eff}}^2 \).
in which the ground state is trivial, \( \langle \hat{d}' \rangle = (1, 0, 0) \). Upon this redefinition, the Lagrangian (3) becomes, up to a constant,
\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \hat{d}' + w \partial_x \hat{d}')^2 - \frac{1}{2} v^2 (\partial_\mu \hat{d}')^2 + H \left( 1 + \frac{u^2}{c_{\text{eff}}^2} \right) (\hat{d}_1 \partial_\delta \hat{d}_2 - \hat{d}_2 \partial_\delta \hat{d}_1) - \frac{1}{2} \left[ H^2 \left( 1 + \frac{u^2}{c_{\text{eff}}^2} \right) - \Omega_2^2 \right] \hat{d}_3^2.
\]
(16)

Since the ground state is oriented in the \( \hat{d}_1 \) direction, the spectrum is determined by the part of the Lagrangian bilinear in \( \hat{d}_{2,3} \). The dispersion relations of the two modes, corresponding to \( \hat{d}_{2,3} \), can be read off the first and third line of Eq. (16),
\[
\omega_{2,3}(k) = u k_x + \sqrt{c_2^2 k_x^2 + \mu_2^2,3},
\]
(17)
where
\[
\mu_2 = 0, \quad \mu_3 = \sqrt{H^2 \left( 1 + \frac{u^2}{c_{\text{eff}}^2} \right) - \Omega_2^2}.
\]
(18)

Note that \( \hat{d}_2 \) remains gapless in spite of the presence of the external magnetic field and the dipole coupling. This reflects the exact \( U(1)_r \) symmetry corresponding to in-plane spin rotations, which is spontaneously broken in the ground state.

The characteristic frequency of collective spin oscillations, probed by NMR spectroscopy with a uniform magnetic field, corresponds to the spin-wave dispersion relation at \( k = 0 \), and is thus given directly by the \( \mu_{2,3} \) parameters. The tiny \( u \)-dependent shift of the resonance frequency of the \( \hat{d}_3 \) mode can in principle be used as evidence for our helical texture.

Conclusions.—Galilei invariance is known to impose powerful constraints on effective theories of nonrelativistic superfluids [14, 22]. In this Letter we argued that in case of a thin film of \( ^3\text{He}-\text{A} \), it inevitably leads to a coupling between superflow and spin degrees of freedom, an effect that could easily be overlooked by considering only the orbital and spin symmetries and their spontaneous breaking. Based on this observation, we predicted that the ground state of a superfluid film of \( ^3\text{He}-\text{A} \) in presence of a uniform superflow and an external magnetic field perpendicular to the film features a nonuniform, helical texture. The helix pitch depends only on the phase velocity of spin waves, the superflow velocity and the magnetic field, and can be tuned by varying the latter two.

In order to gain a better grasp on the phenomenological implications of our prediction, it would be highly desirable to study in detail the effects of nonzero temperature. On the one hand, this would help to clarify in what temperature range the helical texture represents the equilibrium state of a thin film of superfluid \( ^3\text{He}-\text{A} \). By the same token, it would be important to understand the role of thermal fluctuations in the equilibrium state. Finally, given the model-independent nature of the effective theory used here, it would be interesting to search for other systems where the combination of uniform external fields and Galilei invariance might lead to a nonuniform ground state.

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SUPPLEMENTAL MATERIAL: HELICAL SPIN TEXTURE IN A THIN FILM OF SUPERFLUID \(^3\)HE

MICROSCOPIC DERIVATION OF EFFECTIVE ACTION

Here the effective theory for the spin and superfluid degrees of freedom will be derived from a microscopic fermionic model. To that end, we will first specify the microscopic theory and make sure that it has the desired symmetries. Subsequently, we will integrate out the fermionic degrees of freedom to obtain the effective action. For simplicity, the dipole interaction will be neglected here.

Microscopic action and its symmetries

We consider an idealized theory of strictly two-dimensional \(^3\)He where the fermionic degrees of freedom are fully gapped in the \(A\)-phase. Without specifying a concrete microscopic interaction, we assume that the theory has been semi-bosonized. This leads to a Bogoliubov-de-Gennes-type theory that describes noninteracting fermions propagating on a background of collective pair fields. Following closely the notation introduced by Stone and Roy in Phys. Rev. D 89, 184511 (2004), we write the Euclidean Lagrangian of this microscopic mean-field theory as

\[
\mathcal{L} = \frac{1}{2} \Psi^\dagger (\partial_\tau + \hat{H}) \Psi, \quad \hat{H} \equiv \left( \frac{\hat{\mu}}{\hat{\Delta} + \hat{\mu}^T} \right), \quad (S1)
\]

where \(\Psi \equiv (\psi_\alpha \; \psi_\alpha^*)^T\) is the Nambu spinor with \(\alpha = \uparrow, \downarrow\). In addition,

\[
\hat{\mu} \equiv -\frac{1}{2m} (\nabla - i A - i B)^2 - (A_0 + B_0) \quad (S2)
\]

is the one-particle Hamiltonian. It will turn out convenient to couple the microscopic fermionic theory to a set of background gauge fields. Thus, \(A_\mu\) is the matrix-valued gauge field of the spin \(SU(2)_S\) group, whereas \(B_\mu\) is the gauge field of the \(U(1)_\phi\) symmetry.

The physical content of Eq. (S1) can be highlighted by disposing of the Nambu notation and rewriting the Lagrangian, up to a surface term, as

\[
\mathcal{L} = \psi^\dagger (\partial_\tau + \hat{h}) \psi + \frac{1}{2} (\psi^\dagger \hat{\Delta} \psi + \text{H.c.}). \quad (S3)
\]

The pairing field \(\hat{\Delta}\) must be antisymmetric as a consequence of the Pauli principle, and can be cast as

\[
\hat{\Delta} = \frac{\Delta}{2k_F} (\hat{P} \hat{\Sigma} e^{i\phi} - e^{i\phi} \hat{\Sigma} \hat{P}^T), \quad (S4)
\]

where

\[
\hat{\Sigma} \equiv i (\hat{d} \cdot \sigma) \sigma_2, \quad \hat{P} \equiv -i(D_x + iD_y). \quad (S5)
\]

Here \(k_F\) is the Fermi momentum, \(\Delta\) the gap parameter, \(\sigma\) the vector of Pauli matrices, and the covariant derivatives with spatial and temporal indices are defined as

\[
D \equiv \nabla - i(A + B) \equiv \nabla - iA, \quad (S6)
\]

Finally, we used the shorthand notation \(\Phi \equiv \frac{2}{\hbar} \theta\) for the collective field of the spontaneously broken \(U(1)_\phi\) symmetry. Note that our expression for \(\Delta\) differs somewhat from that of Stone and Roy. The form (S4) is necessary for maintaining the full gauge symmetry, as long as we wish to write the Lagrangian in terms of simple, covariant building blocks.

Let us now specify the symmetries of the Lagrangian. We will denote by \(U\) a generic element of the \(SU(2)_S \times U(1)_\phi\) gauge group. It can be decomposed as \(U = U_1 U_2 = U_2 U_1\), using the natural notation for \(U_1 \in U(1)_\phi\) and \(U_2 \in SU(2)_S\). The transformation rules for the fermions and the gauge field \(A_\mu\), then read

\[
\psi \rightarrow U \psi, \quad \mathcal{A} \rightarrow U \mathcal{A} U^{-1} + iU \nabla U^{-1}, \quad A_0 \rightarrow U A_0 U^{-1} - U \partial_\tau U^{-1}. \quad (S7)
\]

The second and third line summarize the usual transformation rule for a non-Abelian gauge field, modified owing to the fact that we work in Euclidean space. The transformation rules for the collective fields \(\hat{d}\) and \(\Phi\) read accordingly

\[
\hat{d} \cdot \sigma \rightarrow U_2 (\hat{d} \cdot \sigma) U_2^{-1}, \quad e^{i\phi} \rightarrow U_1 e^{i\phi} U_1 = U_2^T e^{i\phi} U_2^2. \quad (S8)
\]

The first line above implies

\[
\hat{\Sigma} \rightarrow U_2 \hat{\Sigma} U_2^T. \quad (S9)
\]

Since the covariant derivatives transform by construction covariantly, \(\hat{P} \rightarrow U \hat{P} U^{-1}\), one finds in the end that

\[
\hat{\Delta} \rightarrow U \hat{\Delta} U^T. \quad (S10)
\]

Based on Eqs. (S7), (S8) and (S10), we can conclude that the Lagrangian (S1) is gauge-invariant under transformations from the \(SU(2)_S \times U(1)_\phi\) group.

Effective action

By integrating out the fermions, we arrive at the effective action, given in Euclidean space by

\[
S_{\text{eff}} = -\frac{1}{2} Tr \log(\partial_\tau + \hat{H}) \equiv -\frac{1}{2} Tr \log \frac{\mathcal{S}}{\mathcal{S}^{-1}}. \quad (S11)
\]

This action is a functional of \(\Phi\), \(\hat{d}\) and \(A_\mu\), and inherits the gauge invariance of the microscopic action under a simultaneous gauge transformation of these fields. There is no anomaly
involved in integrating out the fermions, since the symmetry transformation of the fermion field $\Psi$ is realized by a unitary similarity transformation of the Bogoliubov-de-Gennes (BdG) operator $\partial_x + \hat{H}$, and thus does not affect its spectrum.

At this intermediate stage, it is convenient to use the gauge invariance of the effective action to remove the collective scalar fields. The variable $\hat{d}$ transforms in the vector, or adjoint, representation of $SU(2)_S$ and can be rotated to any fixed direction by a local $SU(2)_S$ transformation. In other words, there is a unitary matrix $V$ such that

$$\hat{d} \cdot \sigma = V \sigma_d V^{-1}, \quad \hat{\Sigma} = i V V^T. \quad (S12)$$

From Eqs. (S8) and (S9), we can see that both $\Phi$ and $\hat{d}$ can then be absorbed into a redefinition of the gauge field $A_\mu$ by choosing

$$U_1 = e^{-i \Phi/2}, \quad U_2 = V^{-1}. \quad (S13)$$

The effective action now depends solely on the composite gauge field, defined by Eq. (S7) with the above choice for $U_{1,2}$. In the following, this composite gauge field will be denoted by the same symbol $A_\mu$. Only at the very end of this section, we will restore the dependence of the action on the spin vector $\hat{d}$ and the phase $\Phi$.

To evaluate the effective action, we adopt a derivative expansion scheme. Since we are interested in the dynamics of small fluctuations of the spin degrees of freedom, we shall count each derivative of $d$ as order 1. At the same time, we allow for a finite uniform velocity of the superflow background. Hence, one derivative acting on $\Phi$ will count as order 0, and every other derivative acting on the same field as order 1. As a consequence, the fields $A_\mu$ and $B_\mu$ are of order 1 and 0, respectively. We shall evaluate the effective action (S11) to the leading order in both fields, which means order 2 for $A_\mu$ and order 0 for $B_\mu$. In this approximation, we can treat $A_\mu$ as a constant fixed background. We need to expand to second order in $A_\mu$, whereas $B_\mu$ has to be resummed to all orders.

To facilitate the Taylor expansion in the non-Abelian gauge field $A_\mu$, it is suitable to split the BdG operator into parts of order zero, one and two in $A_\mu$, $\mathcal{D}_0^{-1} = \mathcal{D}_{01}^{-1} + \mathcal{D}_{02}^{-1}$. Upon Fourier transforming to frequency $\omega$ and momentum $p$,

$$\mathcal{D}_0^{-1} = \begin{pmatrix} i \omega + \frac{\pi^2}{2m} - B_0 & \frac{i\Delta p_+}{\pi^2} & -i \frac{\Delta p_+}{2m} - B_0 \\ -i \frac{\Delta p_+}{\pi^2} & i \omega - \frac{\pi^2}{2m} - B_0 \end{pmatrix},$$

$$\mathcal{D}_1^{-1} = \begin{pmatrix} -i \frac{\pi}{2m} \mathbf{A} - A_0 & \frac{i\Delta p_+}{2m} (-A_+ + A_T^+) \\ -i \frac{\Delta p_+}{2m} (-A_+ + A_T^+) & i \frac{\Delta p_+}{2m} \mathbf{A}^T + A_0^T \end{pmatrix},$$

$$\mathcal{D}_2^{-1} = \begin{pmatrix} \frac{\mathbf{A}^2}{2m} & 0 \\ 0 & \frac{(A_T^+)^2}{2m} \end{pmatrix},$$

where we introduced the notation $\pi = p - B$, $\pi^\dagger = p + B$, $p_\pm = p_x \pm ip_y$, and similarly for other quantities. The zeroth, first and second-order piece of the action in the expansion in the $SU(2)_S$ gauge field now read

$$-S_{\text{eff}} = \frac{1}{2} \operatorname{Tr} \log \mathcal{D}_0^{-1} + \frac{1}{2} \operatorname{Tr}(\mathcal{D}_2 \mathcal{D}_0^{-1}) \quad (S15)$$

$$+ \frac{1}{4} \operatorname{Tr}(2\mathcal{D}_0 \mathcal{D}_2^{-1} - \mathcal{D}_0 \mathcal{D}_1^{-1} \mathcal{D}_2 \mathcal{D}_0^{-1}) + \cdots .$$

The propagator $\mathcal{D}_0$ is obtained by inverting the BdG operator $\mathcal{D}_0^{-1}$ and in momentum space takes the form

$$\mathcal{D}_0 = \frac{1}{\det} \begin{pmatrix} i \omega - \frac{\pi^2}{2m} + B_0 & -\frac{i\Delta p_+}{\pi^2} \\ -\frac{i\Delta p_+}{\pi^2} & i \omega + \frac{\pi^2}{2m} - B_0 \end{pmatrix}, \quad (S16)$$

$$\text{det} \equiv - \left( \omega + \frac{\mathbf{p} \cdot B}{m} \right)^2 - \left( \frac{p^2 + B^2}{2m} - B_0 \right)^2 - \frac{\Delta p^2}{k_F^2}. \quad (S17)$$

As a consistency check, note that the last expression implies that for $B = 0$ and $B_0 = \mu$, the well-known spectrum of fermion excitations in the mean-field approximation follows,

$$E(p) = \sqrt{\left( \frac{p^2}{2m} - \mu \right)^2 + \frac{\Delta^2 p^2}{k_F^2}}. \quad (S17)$$

The leading-order, pure superfluid part of the effective action is given by the first term in Eq. (S15). The corresponding effective Lagrangian reads

$$\mathcal{L}_{\text{eff}}^{(0)} = - \int \frac{d\omega d^2 p}{(2\pi)^3} \times \log \left[ \omega^2 + \left( \frac{p^2 + B^2}{2m} - B_0 \right)^2 + \frac{\Delta^2 p^2}{k_F^2} \right],$$

and upon frequency integration,

$$\mathcal{L}_{\text{eff}}^{(0)} = - \int \frac{d^2 p}{(2\pi)^2} \sqrt{\left( \frac{p^2 + B^2}{2m} - B_0 \right)^2 + \frac{\Delta^2 p^2}{k_F^2}}. \quad (S19)$$

The effective Lagrangian is a (nonlinear) function of the combination $B^2/2m - B_0$, as dictated by Galilei invariance.

The next-to-leading order of the effective action is given by the term quadratic in the $SU(2)_S$ gauge field $A_\mu$. A straightforward, if slightly tedious, manipulation leads to the following expression,
The coefficient $c_{\text{eff}}$ represents the effective action on the collective fields in the infrared cutoff.

In the present problem, the inverse size of the hard core of the system is considered, whereas the inverse of the size of the sample provides an interference term independent of the choice of normalization of the $SU(2)$ generators.

The frequency integration can easily be carried out analytically. The momentum integration is, however, potentially ultraviolet divergent and thus requires regularization. Here we will use dimensional regularization, modifying the integration region into a Euclidean space of dimension $d \equiv 2 - 2\epsilon$. Upon some manipulation, it can be shown that the second spin component of $A_\mu$ drops out of the action. (One arrives at the same conclusion if regularization with a hard cutoff $\Lambda$ is used instead and the limit $\Lambda \to \infty$ is taken.) Denoting the remaining matrix-valued components as $A_{\perp,\mu} = (A_{\perp,0}, A_{\perp})$, the effective Lagrangian takes the form

$$\mathcal{L}_{\text{eff}}^{\text{NLO}} = \frac{1}{4} \left\langle \left( \frac{2}{\det m} (A \cdot A) + \frac{1}{\det^2} \left[ 2(\alpha^2 + \beta^2) + \left( \frac{B \cdot A}{m} \right)^2 - \left( \frac{B \cdot A}{m} \right)^T \right] \right) \right\rangle. $$

The coefficients $c_{1,2}$ can be read off Eq. (S20). Upon frequency integration, they can be cast as

$$c_1 = \frac{1}{2m} \int \frac{d^4p}{(2\pi)^4} \frac{\gamma^2}{\sqrt{\beta^2 + \gamma^2} (\beta^2 + \gamma^2 + 2\epsilon)}, $$

$$c_2 = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \frac{\gamma^2}{(\beta^2 + \gamma^2)^{3/2}}. $$

The coefficient $c_2$ is well-defined through a convergent integral. The coefficient $c_1$, on the other hand, is given by a logarithmically divergent integral. To estimate such an integral in practice requires the knowledge of the ultraviolet and infrared momentum scales, where the integration is effectively cut off. In the present problem, the inverse size of the hard core of the interatomic potential can be taken as the ultraviolet cutoff, whereas the inverse of the size of the sample provides an infrared cutoff.

We are now in a position to restore the dependence of the effective action on the collective fields $d$ and $\theta$. Using Eqs. (S7), (S12) and (S13), it is straightforward to show that

$$\left\langle A_{\perp} \cdot A_{\perp} \right\rangle = \frac{1}{2} (D_\mu \hat{d})^2, $$

where the covariant derivative in the vector notation is given by $D_\mu \equiv \partial_\mu \hat{d} + A_\mu \times \hat{d}$. Likewise, it readily follows upon analytical continuation to real time that

$$\left\langle \left( \frac{B \cdot A}{m} \right)^2 \right\rangle = \frac{1}{2} \left[ \frac{1}{m} (\partial_\mu \phi - B_\mu) D_\mu \hat{d} \right]^2. $$

In the above expressions, $A_\mu$ and $B_\mu$ are not composite anymore, but rather denote the original external gauge fields of the $SU(2) \times U(1)$ group.

We have thus recovered the effective spin Lagrangian density from the main text, Eq. (3) therein (without the dipole term $\mathcal{L}_{\text{dip}}$). The phase velocity of the spin waves is determined by the parameters of the microscopic theory through

$$c_s^2 = -\frac{c_1}{c_2}. $$

GALILEI INVARIANCE OF THE HELICAL TEXTURE

Since we are discussing a superfluid system that does not require an underlying crystal lattice or substrate, the microscopic physics must be Galilei-invariant. One can thus ask the following question: how can we deduce the existence of the helical spin texture in the ground state in a reference frame where the background superflow vanishes?

First, the fact that the magnetic field is introduced through the temporal component of the $SU(2)$ gauge field implies that we have to use an unusual, so-called electric, limit of electromagnetism [Nuovo Cimento B 14, 217 (1973)] if we want the coupling to the background fields to maintain Galilei invariance. In this limit, the Maxwell equations miss the term that induces the Faraday effect (electromagnetic induction). The electromagnetic potentials $\varphi$ and $A$ transform under a Galilei boost with velocity $v$ as

$$\varphi' = \varphi, \quad A' = A - \epsilon_0 \mu_0 \varphi v. $$

Accordingly, the electric and magnetic fields $E$ and $B$ transform as

$$E' = E, \quad B' = B - \epsilon_0 \mu_0 v \times E. $$

The combination of a constant magnetic field and zero electric field, imposed on our system, is therefore invariant under the Galilei transformations in this limit.
Second, equilibrium properties of a many-body system are generally described by a density matrix that follows from the principle of maximum entropy. The principle in turn dictates that we have to correctly take into account all macroscopic constraints on the state of the system. In a system with macroscopic motion such as the background superflow, this means that we need to introduce a Lagrange multiplier for the momentum operator.

To carry out this procedure properly, we first have to rewrite the canonical Hamiltonian in terms of the canonical variables, that is, the field $\hat{d}$ and the associated canonical momentum,

$$\pi \equiv \frac{\partial L}{\partial (\partial_t \hat{d})} = \tilde{D}_t \hat{d},$$

which is itself invariant under Galilei boosts. The Hamiltonian, defined by Eq. (6) of the main text, is then rewritten as

$$\mathcal{H} = \frac{1}{2} \pi^2 - \pi \cdot (H \times \hat{d} + \frac{1}{m} \partial_t \tilde{\theta} \partial_t \hat{d}) + \frac{c_s^2}{2} (\partial_t \hat{d})^2 - \frac{1}{2} \Omega_L^2 (\hat{l} \cdot \hat{d})^2.$$

Next, we introduce the Lagrange multiplier $w_r$ for the operator of momentum density $\mathcal{P}_r$, given by the standard Noether expression

$$\mathcal{P}_r = - \frac{\partial L}{\partial (\partial_t \hat{d})} \cdot \partial_t \hat{d} = -\pi \cdot \partial_t \hat{d}.$$

The grandcanonical Hamiltonian $\mathcal{H}_w$ for the spin wave sector is then obtained from the canonical Hamiltonian (S29) by subtracting the term $w_r \mathcal{P}_r$,

$$\mathcal{H}_w = \mathcal{H} - w_r \mathcal{P}_r = \frac{1}{2} \pi^2 - \pi \cdot (H \times \hat{d} + \frac{1}{m} \partial_t \tilde{\theta} \partial_t \hat{d}) + \frac{c_s^2}{2} (\partial_t \hat{d})^2 - \frac{1}{2} \Omega_L^2 (\hat{l} \cdot \hat{d})^2,$$

where $\tilde{\theta} \equiv \theta - mw \cdot x$. Unlike the Hamiltonian $\mathcal{H}$, the grandcanonical Hamiltonian $\mathcal{H}_w$ is invariant under the simultaneous Galilei transformation of the coordinates and fields, whose infinitesimal form reads

$$x' = x + vt, \quad w' = w + v, \quad \theta'(x') = \theta(x) + mv \cdot x.$$

The many-body ground state of the system, which is determined by the absolute minimum of (the spatial integral of) $\mathcal{H}_w$, is therefore independent of the choice of reference frame, as it should. To proceed towards finding the ground state, all one has to do is to cast Eq. (S31) as

$$\mathcal{H}_w = \frac{1}{2} \left( \pi - H \times \hat{d} - \frac{1}{m} \partial_t \tilde{\theta} \partial_t \hat{d} \right)^2 - \frac{1}{2} \left( H \times \hat{d} + \frac{1}{m} \partial_t \tilde{\theta} \partial_t \hat{d} \right)^2 + \frac{c_s^2}{2} (\partial_t \hat{d})^2 - \frac{1}{2} \Omega_L^2 (\hat{l} \cdot \hat{d})^2,$$

and then follow the argument below Eq. (6) of the main text. We conclude that the helical texture, discovered in the main text in the frame where $w = 0$, can be obtained as well for instance in the frame where there is no background superflow. All that matters is the relative motion of the superfluid and the spin degrees of freedom.