Stability Criterion for Superfluidity in the light of Density Spectral Function

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The Landau’s criterion for superfluids gives the critical velocity where superfluidity disappears at a large velocity. Actually, in the presence of impurities, the dissipation appears at smaller velocity, where solitons or quantized vortices are emitted depending on the shape of an impurity potential barrier and the dimensionality of the system. This instability is categorized as the bifurcation. Therefore, in the present paper, we propose a universal stability criterion for superfluids that is applicable to Landau’s instability and also to soliton or vortex emission instability. For that, we study the local density spectral function $I_n(r,\omega)$ and the autocorrelation function $C_n(r,t)$ in uniform and inhomogeneous systems. According to results from the Bogoliubov theory and the Feynman’s single-mode approximation beyond the mean-field theory, we find a universal feature in the local density spectral function and in the autocorrelation function. When superfluids flow below a threshold, we find in the $d$-dimensional system, $I_n(r,\omega) \propto \omega^{d}$ holds in the low-energy regime and $C_n(r,t) \propto 1/t^{d+1}$ holds in the long-time regime. When superfluids flow with the critical current, on the other hand, we find $I_n(r,\omega) \propto \omega^{3}$ in the low-energy regime and $C_n(r,t) \propto 1/t^{d+1}$ in the long-time regime with $\beta < d$.

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I. INTRODUCTION

Dynamical properties of superfluids have attracted great attention \cite{1, 2}, although anomalies of their thermodynamic properties were first observed \cite{3,4}. The first measurement of the viscosity using a torsional oscillation did not give big differences between He I and He II \cite{5}, which are liquefied helium above and below the $\lambda$ transition temperature, respectively. However, anomalies were discovered in the thermal conductivity \cite{6}, and the viscosity measured in the Hagen-Poiseille flow which drops to an essentially unmeasurable value \cite{1, 2}.

Landau pointed out a relation between excitations and superfluidity \cite{7}. He showed that an ideal Bose gas does not show superfluidity on the basis of his stability criterion of superfluids, which is now called the Landau’s criterion, and extended the two fluid model \cite{8} to elementary excitations. His criterion was developed on the basis of the Galilean criterion, and extended the two fluid model \cite{8} to elementary excitations.

In the Landau’s paper \cite{7}, he carefully noted that “We leave aside the question as to whether superfluidity disappears at small velocities for some other reason.” Onsager and Feynman gave one of other mechanisms \cite{9, 10}: dissipation of superfluids could be caused by emissions of quantized vortices. Feynman also discussed that a low-lying excitation in a Bose liquid is phonon, through a relation between energy of elementary excitations and the static structure factor, i.e., $E_{h\mathbf{q}} = \hbar^2 q^2/[2mS(\mathbf{q})]$ \cite{11}.

The critical velocity of He II was measured by making use of negative ions as impurities drifted by the electric field \cite{12, 13}. The critical velocity in \cite{13} was reported as the phase velocity of elementary excitations at the roton minimum, whose order is 10m/sec. The critical velocity of the superfluid in the orifice geometry was also measured \cite{14, 15}, and discussed a relation with the phase slippage \cite{10}; a vortex passing through a superfluid makes the superfluid flow slower. The order of the critical velocity reported in \cite{14} is 10cm/sec, which is consistent with the Feynman’s criterion for vortex emission \cite{10}. This critical velocity is less than the Landau’s critical velocity.

The realization of the Bose-Einstein condensates (BECs) of alkali atomic gases in 1995 led chances to investigate superfluidity much more extensively \cite{16–21}. This is because ultracold atomic gases are dilute system, to which the mean-field theory is applicable. The Gross-Pitaevskii equation \cite{17, 18} for a weakly interacting Bose system is a nonlinear Schrödinger equation describing superfluid phenomena.

T. Frisch, Y. Pomeau, and S. Rica numerically investigated the superfluid flowing around a disk shaped obstacle \cite{22}, and found no drag force and no vortices when the superfluid velocity is less than a critical velocity $v_c$. When the superfluid velocity is larger than $v_c$, they observed drag force and vortex emissions. In their simula-
tion, the critical velocity \( v_c \) is \( v_c \simeq 0.4 c_s \), where \( c_s \) is the speed of Bogoliubov phonon corresponding to the Landau’s critical velocity. The critical velocity in the present case is less than the Landau’s critical velocity.

For a simpler case, Hakim studied the superfluidity flowing through a potential wall [24]. He also found steady non-dissipative flow states below a critical velocity \( v_c \), and dissipative flow states emitting soliton-like and phonon-like excitations above \( v_c \). This critical velocity \( v_c \) is also less than the Landau’s critical velocity \( v_{c, \text{Landau}} \). Since a stable branch with a stationary dissipative flow disappears with an unstable branch, he discussed this instability as a saddle node bifurcation. Instability of vortex emissions is also categorized as the saddle-node bifurcation [25, 26]. The dynamical scaling law for the vortex emission rate was reported, which is characteristic to this bifurcation [27].

Using the one-dimensional potential barrier, Pavloff reported a phase diagram of stationary solutions with and without dissipation, and non-stationary solutions [28]. When the potential barrier is repulsive, the critical velocity is less than the Landau’s critical velocity. When the potential is attractive, on the other hand, the critical velocity is equal to the Landau’s critical velocity. Other numerical works on the superfluid in two and three-dimensional systems flowing around an obstacle were also reported [29–31]. According to these papers, all the critical velocities are found to be less than the Landau’s critical velocity. Their values \( v_c \) depend on the dimensionality and parameters such as a size of an obstacle.

Experimental studies of the critical velocity in quantum gases were intensively done by the Ketterle’s group in earlier stages. In the experiment [32], the critical velocity \( v_c \) = 1.6mm/s was reported. The speed of the Bogoliubov phonon at the peak density \( c_s \) = 6.2mm/s is about 4 times greater than this critical velocity \( v_c \). In the experiment [33], the critical velocity \( v_c \) = (0.11 \pm 0.02) \( c_s \) for a high density (9 \times 10^{13} \text{cm}^{-3}), and the critical velocity \( v_c \) = (0.07 \pm 0.02) \( c_s \) for a low density (1.9 \times 10^{14} \text{cm}^{-3}) were reported. Interference patterns were also used to check the emission of vortices [33]. In this experiment, the critical velocity is \( v_c \sim 0.08 c_s \). Contrary to the above results where \( v_c < c_s \) was measured, the critical velocity close to the Landau’s critical velocity, \( v_c \simeq c_s \), was reported in the experiment where a stimulated Raman transition was used to produce moving atoms [34].

P. Engels and C. Atherton also studied the critical velocity using a BEC of 87Rb [35]. The critical velocity was reported as \( v_c = 0.3 \text{mm/s} \). The speed of the Bogoliubov phonon at the peak density is \( c_s = 3 \text{mm/s} \), so that the measured critical velocity is less than the Landau’s critical velocity. According to the theoretical prediction [28], for an attractive potential, the critical velocity is expected to be equal to the Landau’s critical velocity, i.e., the speed of the Bogoliubov phonon. However, in this experiment [35], soliton emissions were observed at 1.25mm/s, which is less than the Landau’s critical velocity. It is inconsistent with the theoretical study [28], but this reason was considered as an inhomogeneity of a trap potential [36].

As reviewed, the critical velocity has been intensively and extensively studied. From a series of data above, the results can be summarized as follows. When impurities have the same size as atoms, the critical velocity tends to be close to the velocity determined by the Landau’s criterion. When the impurity size becomes larger, the critical velocity is less than the Landau’s critical velocity.

A non-dissipative flow below a threshold is striking feature of superfluidity. It is interesting to study dissipation processes and to determine stable and unstable conditions of superfluidity. From earlier studies, we know two mechanisms of the dissipation in the superfluid. One is the anomaly of the excitation spectrum; that is, the Landau criterion applicable to homogeneous systems which gives the highest critical velocity. The other is emissions of phase defects, such as solitons and quantized vortices, whose critical velocity is less than the Landau’s critical velocity. According to numerical studies of the Gross-Pitaevskii equation, this mechanism is categorized as the saddle node bifurcation. It is natural to ask whether there is a unified criterion for superfluidity applicable to homogeneous and inhomogeneous systems. On the basis of these questions, we study the property of the superfluid at and below the threshold, and submit a stability criterion for superfluidity.

Our stability criterion presented in this paper is written in terms of the local density spectral function \( I_n(r, \omega) \) or the autocorrelation function of the density \( C_n(r, t) \). The Landau instability, which originates from anomaly of the energy spectrum, is involved in \( I_n(r, \omega) \) and \( C_n(r, t) \) through the density of states. On the other hand, these functions \( I_n(r, \omega) \) and \( C_n(r, t) \) involve anomaly of the density fluctuation through their matrix elements, when soliton or vortex emission instability occurs. Our findings are summarized as follows. Let \( J \) and \( J_c \) be the current of superflow and be its critical current. We find that the local density spectral function in the d-dimensional system is given by

\[
\lim_{\omega \to 0} I_n(r, \omega) \propto \begin{cases} \omega^\beta & (J = J_c) \\ \omega^d & (J < J_c) \end{cases}
\]  

and the autocorrelation function is given by

\[
\lim_{t \to \infty} C_n(r, t) \propto \begin{cases} 1/t^{\beta+1} & (J = J_c) \\ 1/t^{d+1} & (J < J_c) \end{cases}
\]

with \( \beta < d \). According to [24], the density fluctuation is found to be enhanced at \( J = J_c \) in the low-energy regime. The local density spectral function and the autocorrelation function can probe breakdown phenomena of superfluidity.

The present paper is a detailed report on our recent papers [37, 38], and its content is mainly in the PhD thesis of the first author [39]. This paper is organized as follows. Section 11 describes the mean-field theory of the condensed Bose-system. Section 11 serves as an introduction to the local density spectral function, which plays
II. MEAN-FIELD THEORY

We start with the mean-field theory of weakly interacting condensed Bose systems. An equation describing condensates (superfluids) is called the Gross-Pitaevskii equation [17, 18]; an equation describing excitations from condensates is called the Bogoliubov equation [19]. The Hamiltonian of the spinless Bose systems is given by

\[ \hat{H} = \int \frac{\hbar^2}{2m} \nabla \hat{\Psi} \nabla \hat{\Psi} + \hat{\Psi} \hat{V} \hat{\Psi}, \]

where \( \hat{\Psi} \) is the atomic mass, \( \hat{V} \) the external potential, and \( \hat{\Psi} \) the two-body interaction potential satisfying \( \hat{V}(\mathbf{r} - \mathbf{r}') = \hat{V}(\mathbf{r}' - \mathbf{r}). \) For weekly interacting dilute Bose gases, the two-body interaction potential is reduced to

\[ V(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}'), \quad g = \frac{4\pi \hbar^2 a_s}{m}. \]

The field operators \( \hat{\Psi}(\mathbf{r}) \) and \( \hat{\Psi}^\dagger(\mathbf{r}) \) satisfy the commutation relations

\[ [\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}'), \quad [\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}')] = [\hat{\Psi}^\dagger(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')] = 0. \]

Using the Heisenberg representation \( \hat{\Psi}(\mathbf{r}, t) = e^{\hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r})} e^{-\hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r})} \), the time dependence of the field operator is given by \( i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) = \hat{\Psi}(\mathbf{r}, t) \hat{H} \), which becomes

\[ i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) = \left[ \frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}, t), \]

In the mean-field theory, the condensate wave function \( \Psi_0(\mathbf{r}, t) \), which is the order parameter, is defined as

\[ \Psi_0(\mathbf{r}, t) = \langle \hat{\Psi}(\mathbf{r}, t) \rangle = \langle g; N - 1 | e^{\hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r})} e^{-\hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r})} | g; N \rangle, \]

where \( |g; N \rangle \) is the N-particle ground state. The order parameter is related to the condensate density \( n_0(\mathbf{r}, t) \), which is given by

\[ n_0(\mathbf{r}, t) = |\Psi_0(\mathbf{r}, t)|^2. \]

The equation of motion of the condensate wave function, which is called the Gross-Pitaevskii equation, is now given by

\[ i\hbar \frac{\partial}{\partial t} \Psi_0(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) + g|\Psi_0(\mathbf{r}, t)|^2 \right] \Psi_0(\mathbf{r}, t), \]

whose stationary solution is

\[ \Psi_0(\mathbf{r}, t) = \Psi_0(\mathbf{r}) e^{-i\mu t/\hbar}, \]

because

\[ \langle g; N - 1 | e^{i\hat{H} t/\hbar} \Psi_0(\mathbf{r}) e^{-i\hat{H} t/\hbar} | g; N \rangle = \Psi_0(\mathbf{r}) e^{-i(E_N - \mu) t/\hbar}. \]

Here, \( \hat{H}|g; N \rangle = E_N|g; N \rangle \) holds and the chemical potential \( \mu \) is given by \( \mu = E_N - E_{N-1} \). We thus get the stationary Gross-Pitaevskii equation

\[ \hat{\mathcal{H}} \Psi_0(\mathbf{r}) = 0, \]

where

\[ \hat{\mathcal{H}} = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) - \mu + g|\Psi_0(\mathbf{r})|^2. \]

Excitations from the condensate can be discussed through the following prescription. The field operator \( \hat{\Psi}(\mathbf{r}, t) \) splits into the condensate wave function \( \Psi_0(\mathbf{r}, t) \) and its fluctuation \( \delta \hat{\Psi}(\mathbf{r}, t) \), i.e.,

\[ \hat{\Psi}(\mathbf{r}, t) \simeq \Psi_0(\mathbf{r}, t) + \delta \hat{\Psi}(\mathbf{r}, t). \]

The equation of motion of \( \delta \hat{\Psi}(\mathbf{r}, t) \) in the first order of the fluctuation is given by

\[ i\hbar \frac{\partial}{\partial t} \delta \hat{\Psi}(\mathbf{r}, t) = [\hat{\mathcal{H}}_0 + \mu + g|\Psi_0(\mathbf{r}, t)|^2] \delta \hat{\Psi}(\mathbf{r}, t) + g\Psi_0^2(\mathbf{r}, t) \delta \hat{\Psi}(\mathbf{r}, t). \]

If we take the second order of \( \delta \hat{\Psi}(\mathbf{r}, t) \), the grand canonical Hamiltonian \( \hat{K} = \hat{\mathcal{H}} - \mu N \) becomes \( \hat{K} \simeq E_0 + \hat{K}_1 + \hat{K}_2 \), with

\[ E_0 = \int d\mathbf{r} \left[ \Psi_0^*(\mathbf{r}) \hat{K}_0 \Psi_0(\mathbf{r}) + \frac{g}{2} |\Psi_0(\mathbf{r})|^4 \right], \]

\[ \hat{K}_1 = \int d\mathbf{r} \left[ \delta \Psi^\dagger(\mathbf{r}) \hat{K}_0 \Psi_0(\mathbf{r}) + \text{c.c.} \right], \]

\[ \hat{K}_2 = \int d\mathbf{r} \left[ \frac{1}{2} \delta \Psi^\dagger(\mathbf{r}) \hat{K}_0 \delta \Psi(\mathbf{r}) + g |\Psi_0(\mathbf{r})|^2 \delta \Psi^\dagger(\mathbf{r}) \delta \Psi(\mathbf{r}) + \frac{g}{2} \Psi_0^2(\mathbf{r}) |\delta \Psi^\dagger(\mathbf{r})|^2 + \text{c.c.} \right]. \]
with
\[ \hat{K}_0 = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(r) - \mu. \] (23)

According to (10), we have \( \hat{K}_1 = 0 \).

For studying excitations, we introduce the Bogoliubov transformation,
\[ \delta \hat{\Psi}(r, t) = \sum_j \delta \hat{\Psi}(r, t; E_j)e^{-iut/\hbar}, \] (24)
with
\[ \delta \hat{\Psi}(r, t; E_j) = u_j(r)e^{-iE_jt/\hbar} \hat{a}_j - v_j^*(r)e^{iE_jt/\hbar} \hat{a}_j^\dagger. \] (25)

\( E_j \) is the eigenenergy of the state \( j \). The field operators \( \hat{a}_j \) and \( \hat{a}_j^\dagger \) satisfy the bosonic commutation relation
\[ [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = 0. \] (26)

According to (19), the Bogoliubov equation of \( u(r) \) and \( v(r) \) is
\[ \begin{pmatrix} \hat{H}_0 + g|\Psi_0(r)|^2 \\ -g\Psi_0^*(r) \end{pmatrix} \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = E \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}. \] (27)

Here, we omitted the index \( j \). According to the bosonic commutation relation, we get
\[ \sum_j [u_j(r)u_j^*(r') - v_j^*(r)v_j(r')] = \delta(r - r'), \] (28)
\[ \sum_j [u_j(r)v_j^*(r') - u_j(r')v_j^*(r')] = 0. \] (29)

For these equations, we used \( E_j \) to be real.

If \( (u, v) \) is a solution of the Bogoliubov equation with the energy \( E \), then \( (v^*, \hat{a}^\dagger, u \hat{a}) \) is also a solution with an eigenvalue \(-E^*\). This means that
\[ \delta \hat{\Psi}(r, t; -E^*) = [v_j^*(r)e^{-i(-E_j)t/\hbar}\hat{a}_j - u_j(r)e^{i(-E_j)t/\hbar}\hat{a}_j^\dagger]. \] (30)
\[ = -[u_j(r)e^{-iE_jt/\hbar}\hat{a}_j - v_j^*(r)e^{iE_jt/\hbar}\hat{a}_j^\dagger]. \] (31)
\[ = -\delta \hat{\Psi}(r, t; E_j). \] (32)

Since there is only the difference in the coefficient, we choose the state such that \( \text{Re}(E) > 0 \). The wave function now reads as
\[ \delta \hat{\Psi}(r, t) = \sum_{j \text{ s.t. } \text{Re}(E_j) > 0} \delta \hat{\Psi}(r, t; E_j)e^{-iut/\hbar}. \] (33)

For simplicity, we write the Bogoliubov equation in the following manner,
\[ E_j \begin{pmatrix} u_j(r) \\ v_j(r) \end{pmatrix} = \hat{L} \begin{pmatrix} u_j(r) \\ v_j(r) \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} \hat{L}_a & L_b \\ -L_b & -\hat{L}_a \end{pmatrix}, \] (34)
where
\[ \hat{L}_a = \hat{R}_0 + g|\Psi_0(r)|^2, \quad L_b = -g\Psi_0^2(r). \] (35)

Let us multiply (34) by \((u_k^*, \hat{a}_k, v_k)\) from the left side, and prepare the equation whose indices \( j \) and \( k \) are exchanged. Using \( \int d\vec{r} u_j^* \hat{L}_a v_k = \int d\vec{r} u_k \hat{L}_a u_j^* \), and subtracting the complex conjugate of one equation from the other equation, we get
\[ (E_j - E_k) \int d\vec{r} (u_j u_k^* - v_k v_j^*) = 0. \] (36)

Multiplying (34) by \((v_j, \hat{a}_j, u_j^*)\) from the left side, and using the same technique and also using \( \int d\vec{r} \hat{L}_a u_k v_j = \int d\vec{r} u_k \hat{L}_a v_j \), we get
\[ (E_j + E_k) \int d\vec{r} (u_j v_k - v_k u_j) = 0. \] (37)

If the system is stable, the eigenvalue is real, i.e., \( E_j \in \mathbb{R} \). In addition, if the system does not have degeneracies, we get
\[ \int d\vec{r} (u_k^* u_j - v_k^* v_j) = 0, \quad \text{for} \quad j \neq k. \] (38)

If we do not consider the zero-mode, we obtain
\[ \int d\vec{r} (u_k v_j - v_k u_j) = 0. \] (39)

Using these relations, (22) becomes
\[ \hat{K}_2 = \sum_{j \text{ s.t. } E_j > 0} E_j \hat{a}_j^\dagger \hat{a}_j \int d\vec{r} |u_j|^2 - |v_j|^2 \]
\[ - \frac{1}{2} \sum_{j \text{ s.t. } E_j > 0} E_j \int d\vec{r} |v_j|^2. \] (40)

We are considering \( E_i \in \mathbb{R} \) and \( E_i > 0 \), so that the normalization condition \( \int d\vec{r} |u_i|^2 - |v_i|^2 < 0 \) is unphysical, because the emission of excitations makes energy lower and the system is energetically unstable. Therefore, we take \( \int d\vec{r} |u_i|^2 - |v_i|^2 = 1 \). By combining this condition and (38), we obtain the orthogonality
\[ \int d\vec{r} (u_j^* u_j(r) - v_j^* v_j(r)) = \delta_{ij}. \] (41)

As a result, we end with
\[ \hat{K}_2 = \sum_{j \text{ s.t. } E_j > 0} E_j \hat{a}_j^\dagger \hat{a}_j - \frac{1}{2} \sum_{j \text{ s.t. } E_j > 0} E_j \int d\vec{r} |v_j(r)|^2. \] (42)

III. LOCAL SPECTRAL FUNCTION

The Landau’s criterion is applicable only to the uniform system because of its use of the Galilean transformation. The criterion using such an energy spectrum
has limitations, when we discuss the instability of the superfluid flowing through an obstacle. We expect that a response of density describes its rigidity of superfluid and characterizes the stability of the superfluid. One of candidates is the dynamic structure factor $S(q, \omega)$. But, it is inadequate because the translational invariance is broken in inhomogeneous systems. We here propose the use of the local density spectral function, which gives a litmus test to see if the superfluid is stable.

Let $|g\rangle$ be the ket vector of the ground state. We start with the correlation function

$$C_{AB}(x_1; x_2) = (g|\delta \hat{A}(x_1)\delta \hat{B}(x_2)|g),$$

where $x_1 = (r_1, t_1)$ and $x_2 = (r_2, t_2)$. $\delta \hat{A}(x)$ is an operator given by

$$\delta \hat{A}(x) = \hat{A}(x) - (g|\hat{A}(x)|g).$$

$\hat{A}(x)$ is an arbitrary operator with

$$\hat{A}(r, t) = e^{i\hat{H}t/\hbar} \hat{A}(r)e^{-i\hat{H}t/\hbar}.$$  (45)

$\delta \hat{B}(x)$ is defined in a way similar to $\delta \hat{A}(x)$. Let $|l\rangle$ be a ket vector of an excited state with an index $l$. States $|g\rangle$ and $|l\rangle$ respectively have energies $\hbar \omega_g$ and $\hbar \omega_l$, i.e.,

$$\hat{H}|g\rangle = \hbar \omega_g|g\rangle, \quad \hat{H}|l\rangle = \hbar \omega_l|l\rangle,$$

where $\hbar \omega_l > \hbar \omega_g$ holds. Using

$$e^{-i\hat{H}t/\hbar}|g\rangle = e^{-i\omega_g t}|g\rangle, \quad e^{-i\hat{H}t/\hbar}|l\rangle = e^{-i\omega_l t}|l\rangle,$$

and the completeness relation

$$\sum_l |l\rangle\langle l| = 1,$$

we get

$$C_{AB}(x_1; x_2) = \sum_l (g|\delta \hat{A}(r_1)|l\rangle\langle l|\delta \hat{B}(r_2)|g\rangle)e^{-i(\omega_l - \omega_g)(t_1 - t_2)}.$$  (49)

Using the Fourier transformation and the delta-function

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t},$$

we obtain the spectral function

$$\mathcal{I}_{AB}(r_1, r_2; \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-i\omega(t_2 - t_1)} C_{AB}(x_1; x_2) e^{-i\omega(t_2 - t_1)} = \sum_l (g|\delta \hat{A}(r_1)|l\rangle\langle l|\delta \hat{B}(r_2)|g\rangle \delta(\omega - \omega_l + \omega_g).$$

The local density spectral function $\mathcal{I}_n(r, \omega)$ is defined as

$$\mathcal{I}_n(r, \omega) = \mathcal{I}_{nn}(r, r; \omega) = \sum_l |(l|\delta \hat{n}(r)|g\rangle|^2 \delta(\omega - \omega_l + \omega_g),$$

where $\delta \hat{n}(r)$ is an operator of the density fluctuation. The local density spectral function $\mathcal{I}_n(r, \omega)$ can find where and how the superfluid breaks. This function is suited to studying stability of superfluids.

Let us introduce the symmetrized correlation function

$$C_{AB}^S(x_1; x_2) = \frac{1}{2} [\langle g|\delta \hat{A}(x_1)\delta \hat{B}(x_2)|g\rangle + \langle g|\delta \hat{B}(x_2)\delta \hat{A}(x_1)|g\rangle].$$

Using the local density spectral function $\mathcal{I}_n(r, \omega)$, we obtain the autocorrelation function $C_n(r, t)$

$$C_n(r, t) = C_{nn}^S(r, t; r, 0) = \int d\omega \mathcal{I}_n(r, \omega) \cos(\omega t).$$

Since the Landau instability originates from instability of the energy spectrum, the local density spectral function describes this instability as anomaly of the density of states in this function. For soliton or vortex emission instability, the local density spectral function will describe this instability as anomaly of the matrix element of density fluctuations in this function. Therefore, through the local density spectral function, we can probe both instabilities.

For the fluid composed of atoms with a common mass $m$, the mass density operator $\hat{\rho}(r)$ and the mass current density operator $\hat{j}(r)$ are given by

$$\hat{\rho}(r) = m \sum_i \delta(r_i - r),$$

$$\hat{j}(r) = \frac{1}{2} \sum_i [\hat{p}_i \delta(r_i - r) + \delta(r_i - r)\hat{p}_i],$$

where $\hat{p}_i$ is the momentum operator of $i$-th particle $\hat{p}_i = -i\hbar \nabla_i$. These operators satisfy

$$\hat{j}(r)\hat{\rho}(r') = \hat{\rho}(r')\hat{j}(r) = -i\hbar \hat{\rho}(r)[\nabla \delta(r - r')].$$

We now define the operator of the fluid velocity $\hat{\mathbf{v}}(r)$ as

$$\hat{j}(r)\hat{\rho}(r') = \hat{\rho}(r')\hat{j}(r) = -i\hbar \hat{\rho}(r)[\nabla \delta(r - r')].$$

$$\hat{\mathbf{v}}(r) = \frac{\hbar}{m} \hat{\mathbf{v}}(r)$$.
If we suppose that an arbitrary constant factor is zero, we finally obtain [41]
\[ \hat{\theta}(\mathbf{r})\hat{n}(\mathbf{r}') - \hat{n}(\mathbf{r}')\hat{\theta}(\mathbf{r}) = -i\delta(\mathbf{r} - \mathbf{r}'). \] (63)

Because we assume that the constant is zero and this velocity field operator is linked to the phase operator, we find that \( \hat{\theta}(\mathbf{r}) \) is not observable. Within the mean-field theory, the operators split into mean-values
\[ n_0(\mathbf{r}) = \langle g|\hat{n}(\mathbf{r})|g\rangle, \quad \theta_0(\mathbf{r}) = \langle g|\hat{\theta}(\mathbf{r})|g\rangle \] (64)
and their fluctuations \( \delta\hat{n}(\mathbf{r}) \) and \( \delta\hat{\theta}(\mathbf{r}) \), i.e.,
\[ \hat{n}(\mathbf{r}) = n_0(\mathbf{r}) + \delta\hat{n}(\mathbf{r}), \quad \hat{\theta}(\mathbf{r}) = \theta_0(\mathbf{r}) + \delta\hat{\theta}(\mathbf{r}). \] (65)

We now end with
\[ \delta\hat{\theta}(\mathbf{r})\delta\hat{n}(\mathbf{r}') - \delta\hat{n}(\mathbf{r}')\delta\hat{\theta}(\mathbf{r}) = -i\delta(\mathbf{r} - \mathbf{r}'). \] (66)

We now discuss the local density spectral function for the condensed Bose system, in terms of the Bogoliubov theory. The density operator \( \hat{n}(\mathbf{r}) \) in our case is given by
\[ \hat{n}(\mathbf{r}) = \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}). \] (67)
If we write \( \Psi_0(\mathbf{r}, t) \) in terms of its amplitude \( A(\mathbf{r}) \) and its phase \( \theta_0(\mathbf{r}) \), i.e.,
\[ \Psi_0(\mathbf{r}, t) = A(\mathbf{r})e^{i\theta_0(\mathbf{r})}e^{-i\mu t/\hbar}, \] (68)
and take the first order of the fluctuation \( \delta\Psi(\mathbf{r}, t) = \sum_{j,t, E_j > 0} \delta\hat{\Psi}(\mathbf{r}, t) E_j e^{-i\mu t/\hbar} \), the mean value of the density is the condensate density
\[ n_0(\mathbf{r}) = A^2(\mathbf{r}), \] (69)
and the operator of the density fluctuation is
\[ \delta\hat{n}(\mathbf{r}, t) = A(\mathbf{r}) \sum_j \left[ G_j(\mathbf{r}) e^{-iE_j t/\hbar} \hat{a}_j + G_j^*(\mathbf{r}) e^{iE_j t/\hbar} \hat{a}_j^\dagger \right], \] (70)
where
\[ G(\mathbf{r}) = u(\mathbf{r})e^{-i\theta_0(\mathbf{r})} - v(\mathbf{r})e^{i\theta_0(\mathbf{r})}. \] (71)
The mean-value of the velocity potential field is the phase of the condensate wave function. If we take
\[ S(\mathbf{r}) = u(\mathbf{r})e^{-i\theta_0(\mathbf{r})} + v(\mathbf{r})e^{i\theta_0(\mathbf{r})}, \] (72)
we find a relation
\[ \frac{1}{2} \sum_j \left[ S_j(\mathbf{r}) G_j^*(\mathbf{r}') + S_j^*(\mathbf{r}) G_j(\mathbf{r}') \right] = \delta(\mathbf{r} - \mathbf{r}'), \] (73)
by making use of [25]. From [28] and the commutation relation between \( \delta\hat{n}(\mathbf{r}) \) and \( \delta\hat{\theta}(\mathbf{r}) \), we find the velocity potential fluctuation operator, which is given by
\[ \delta\hat{\theta}(\mathbf{r}, t) = \frac{1}{2iA(\mathbf{r})} \sum_j \left[ S_j(\mathbf{r}) e^{-iE_j t/\hbar} \hat{a}_j + S_j^*(\mathbf{r}) e^{iE_j t/\hbar} \hat{a}_j^\dagger \right]. \] (74)
The density and phase operators are discussed in [42] for \( \theta_0(\mathbf{r}) = 0 \) and [70] and [71] are extensions of these operators to the case for the current carrying state. Relations between these fluctuations and \( (S, G) \), which are the non-quantized version, are discussed in [43, 44].

Using [70], the density spectral function is given by
\[ I_{nn}(\mathbf{r}_1, \mathbf{r}_2; \omega) = \sqrt{n_0(\mathbf{r}_1)n_0(\mathbf{r}_2)} \sum_{l} G_l(\mathbf{r}_1) G_l^*(\mathbf{r}_2) \delta(\omega - \omega_l + \omega_g). \] (75)
The equal point local density spectral function for \( \mathbf{r}_1 = \mathbf{r}_2 \) is
\[ I_{nn}(\mathbf{r}, \omega) = n_0(\mathbf{r}) \sum_{l} |G_l(\mathbf{r})|^2 \delta(\omega - \omega_l + \omega_g). \] (76)
We can also define the spectral function of the velocity field potential using [74], which is given by
\[ I_{\theta\theta}(\mathbf{r}_1, \mathbf{r}_2; \omega) = \frac{1}{4\sqrt{n_0(\mathbf{r}_1)n_0(\mathbf{r}_2)}} \sum_{l} S_l(\mathbf{r}_1) S_l^*(\mathbf{r}_2) \delta(\omega - \omega_l + \omega_g). \] (77)
We note that these velocity potential spectral functions have less physical meaning than the density one. Rather, the spectral functions of velocity and current density which are constructed from \( \nabla \delta\hat{\theta}(\mathbf{r}) \) are observable and have more meanings. We leave aside studies of these functions in this paper.

IV. SUPERFLUID FLOWING THROUGH ONE-DIMENSIONAL POTENTIAL

We assume that the external potential has only the \( x \)-dependence and the translational invariance holds in the \( y \) and \( z \)-directions. We also assume that the superfluid flows along the \( x \)-direction, i.e., \( \psi_0(x) = A(x) e^{i\theta_0(x)} \). In this case, the Gross-Pitaevskii equation becomes
\[ \hat{\mathcal{H}} A(x) = 0, \quad \hbar A^2(\mathbf{r}) \frac{d\theta_0(\mathbf{r})}{dx} = J, \] (79)
where
\[ \hat{\mathcal{H}} = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{J^2}{2mA^4(x)} + V_{\text{ext}}(x) - \mu + gA^2(x). \] (80)
At \( J \geq 0 \), supercurrent flows from \( x = -\infty \) to \( x = +\infty \). If an external potential \( V_{\text{ext}}(x) \) is localized around \( x = 0 \),
and $V_\text{ext}(|x| \to \infty) = 0$, the amplitude of the condensate wave function appears as $A(|x| \to \infty) = \sqrt{n_0}$, where $n_0$ is the condensate density at $|x| \to \infty$. In this case, according to (79), the phase $\theta_0(x)$ is given by

$$\theta_0(x) = \theta_0(0) + \frac{J}{\hbar n_0}x + \frac{J}{\hbar n_0} \int_0^x dx' \left( \frac{1}{A^2(x')} - \frac{1}{n_0} \right),$$

(81)

and the phase difference $\varphi$ is

$$\varphi = \frac{J}{\hbar n_0} \int_{-\infty}^\infty dx \left( \frac{1}{A^2(x)} - \frac{1}{n_0} \right).$$

(82)

The current can flow without dissipation, when the phase is twisted ($\varphi \neq 0$) and also when the current is less than the critical current $J_c$.

In Fig. 1 we plot the $J$-$\varphi$ relations in the delta-function potential barrier case. The current $J$ is an odd-function of the phase difference $\varphi$. For a repulsive barrier case, stable branches (thick lines) and unstable branches (thin lines) in (a) merge at the maximum value of the stable supercurrent (i.e., the critical current $J = J_c$) with $dJ/d\varphi = 0$. The value $J_c$ is less than the critical current of the Landau’s criterion $J = 1$ in the dimensionless form which is introduced below. On the other hand, according to Fig. 1(b), in an attractive potential case, the critical current $J_c$ is equal to the Landau’s critical current $J_c = 1$.

It is sometimes discussed that the critical velocity in an inhomogeneous system less than the Landau’s critical velocity is due to the local Landau’s criterion. Excitations could be emitted if the velocity of the superfluid exceeded the local Landau’s critical velocity determined by the local density. In the Bogoliubov theory, the Landau’s critical velocity is given by the speed of the Bogoliubov excitation $c_s = \sqrt{gn_0/m}$. If $c_s(|r|)$ had a position dependence through the density $n_0(|r|)$ in an inhomogeneous system, superfluidity would break at the position where the fluid speed $v(|r|) = |\hbar \nabla \theta_0(|r|)/m|$ satisfies $v(|r|) > c_s(|r|)$, according to the local Landau’s criterion.

This statement is totally wrong, however. The Landau’s criterion is applicable to the uniform system because it is based on the Galilean transformation. For making a more concrete example, we plot the velocity of the superfluid and the local sound velocity of the Bogoliubov phonon $c_s(|r|) = \sqrt{gn_0(|r|)/m}$ in Fig. 2. This is the result obtained from the Gross-Pitaevskii equation for the stable superfluid state with the supercurrent $J(< J_c)$ against the delta-function potential barrier. Although the state is stable since $J < J_c$, the speed of fluid $v(|r|)$ is larger than the local sound velocity of the Bogoliubov phonon $c_s(|r|) = \sqrt{gn_0(|r|)/m}$. This state might be regarded as an unstable state according to the local Landau’s criterion, but this is stable. As a result, the local Landau instability is not a mechanism of breakdown of stable superfluids in a non-uniform system.

We now discuss $(S,G)$. In the case we are considering,
This spectrum in the long wavelength regime becomes

\[ E \simeq (1 + J \cos \theta)k^{in} + \frac{1}{8}(k^{in})^3. \]  

(92)

At \( J < 1 \), the excitation in the low-energy regime is phonon. At \( J = 1 \) and \( \theta = \pi \), \( E \propto (k^{in})^3 \) holds.

Once we set an excitation energy \( E \) and an incident angle \( \theta \), we obtain \( k^{in} > 0 \) from

\( (k^{in})^4 + 4(1 - J^2 \cos^2 \theta)(k^{in})^2 

+ (8JE \cos \theta)k^{in} - 4E^2 = 0, \)  

(93)

which comes from (91). Since \( k^{in} \) is the modulus of \( k^{in} \), it is a positive and real solution to (93). We then get \( k_\perp = k^{in} \sin \theta \). Using these values and energy spectrum

\[ E = JK_\parallel + \sqrt{\frac{k_\parallel^2 + k_\perp^2}{2} \left( \frac{k_\parallel^2 + k_\perp^2}{2} + 2 \right)}, \]  

(94)

we have the following biquadratic equation of \( k_\parallel = k \cdot e_x \) with real coefficients:

\[ k_\parallel^4 + (2k_\perp^2 + 4 - 4J^2)k_\parallel^2 + 8JEk_\parallel \]

\[ + k_\perp^4 + 4k_\perp^2 - 4E^2 = 0. \]  

(95)

This equation has four solutions \( k_\parallel \). Two are real, one of which satisfies \( k_\parallel = k^{in} \cos \theta \). The other two are imaginary. \( \hat{u} \) and \( \hat{v} \) are given by

\[ \left( \begin{array}{c} \hat{u} \\ \hat{v} \end{array} \right) = N^{-1} \left( -E + \left( \frac{k_\parallel^2}{2} + k_\parallel J + \frac{k_\perp^2}{2} + 1 \right) \right), \]  

(96)

where the normalization coefficient is

\[ N = \sqrt{1 - |E - \left( \frac{k_\parallel^2}{2} + k_\parallel J + \frac{k_\perp^2}{2} + 1 \right)|^2}, \]  

(97)

so that (91) satisfies \(|\hat{u}|^2 - |\hat{v}|^2 = 1\).

On the other hand, \( S \) and \( G \) at \( |x| = \infty \) become

\[ \left( \begin{array}{c} S(x) \\ G(x) \end{array} \right) = \left( \begin{array}{c} \alpha_{k_\parallel} \\ \beta_{k_\parallel} \end{array} \right) e^{ik_\parallel x}, \]  

\[ \left( \begin{array}{c} \alpha_{k_\perp} \\ \beta_{k_\perp} \end{array} \right) = \frac{1}{\sqrt{\text{Re}[\mathcal{M}]}} \left( \begin{array}{c} 1 \\ \mathcal{M} \end{array} \right), \]  

(98)

where the normalization coefficient \( \mathcal{M} \) is

\[ \mathcal{M} = \frac{k_\perp^2 + k_\parallel^2}{2(E - JK_\parallel)}, \]  

(99)

so that (98) satisfies \((SG^{\ast} + S^{\ast}G)/2 = 1\). In the low-momentum regime, we find

\[ S \simeq \sqrt{\frac{2}{k}}, \quad G \simeq \sqrt{\frac{k}{2}}, \]  

(100)

with \( k = \sqrt{k_\parallel^2 + k_\perp^2} \).
V. LOCAL DENSITY SPECTRAL FUNCTION IN BOGOLIUBOV THEORY

When superfluid flows through an obstacle above a critical current $J_c$, excitations are emitted from the obstacle. In the one-dimensional potential case, solitons are emitted at $J > J_c$. In this section, we evaluate the density spectral function in the presence of the one-dimensional potential barrier.

Since the ground state energy is $\omega_k = 0$ and the translational invariance holds in $y$ and $z$-directions, the local density spectral function in the $d$-dimensional system is given by

$$I_n(x, \omega) = \sum_i M(x, E_i) \frac{1}{L^d} \delta(\omega - E_i),$$

where the squared matrix element $M(x, E_i)$ is

$$M(x, E_i) = \langle u_l(x) | \Psi_0(x) - v_l(x) | \Psi_0(x) \rangle^2 = L^d \langle u_l(x) | G_l(x) \rangle^2.$$  

(101)

(102)

(103)

Here, $L$ is the system size. To obtain $M(x, E_i)$, we solve Eq. (27) (or Eqs. (82) and (83)) with a periodic boundary condition

$$u_l(L/2) = u_l(-L/2), \quad \partial_x u_l(L/2) = \partial_x u_l(-L/2),$$

$$v_l(L/2) = v_l(-L/2), \quad \partial_x v_l(L/2) = \partial_x v_l(-L/2),$$

(104)

(105)

and the normalization condition

$$\int_{-L/2}^{L/2} dx [\langle u_l(x) \rangle^2 - \langle v_l(x) \rangle^2] = 1.$$  

(106)

To determine the spectral function, we need to calculate the spectral function in the thermodynamic limit. In general, it is hard to solve the Bogoliubov equation numerically for a large system $L \gg 1$, because solutions involve exponentially diverging components.

However, analytic solutions are known in the one-dimensional system with the $\delta$-function potential barrier $V_{\text{ext}}(x) = V_0 \delta(x)$ [46]. We consider this simple system. We use analytic solutions $u_{\pm}(x, k)$ and $v_{\pm}(x, k)$ in Eqs. (46) and (47) of Ref. [46], which are originally written as $u_n(x)$ and $v_n(x)$ in Ref. [46]. The indices $\pm$ denote the solutions for $x \geq 0$ and $x < 0$, respectively. Let $k_{\parallel}^{(1)}$ be a real solution satisfying $k_{\parallel}^{(1)} = k_{\parallel}^{\text{in}} \cos \theta$, and $k_{\parallel}^{(2)}$ be the other real solution of Eq. (45). In the one-dimensional case, we set $\theta = 0$. Let $k_{\perp}$ be imaginary solutions satisfying $\text{Im}(k_{\perp}) > 0$ and $\text{Im}(k_{\perp}) < 0$. Since the coefficients of equations are real, we find $(k_{\perp})^* = k_{\perp}$. The solutions are now given by

$$\begin{pmatrix} u_{\pm}(x) \\ v_{\pm}(x) \end{pmatrix} = \sum_{k=k_{\parallel}^{(1)}, k_{\parallel}^{(2)}, k_{\perp}} c_{\pm,k} \begin{pmatrix} u_{\pm}(x, k) \\ v_{\pm}(x, k) \end{pmatrix}.$$  

(107)

We here impose the boundary conditions at $x = 0$:

$$u_+(0) = u_-(0), \quad \partial_x u_+(0) - \partial_x u_-(0) = 2V_0 u_+(0),$$

$$v_+(0) = v_-(0), \quad \partial_x v_+(0) - \partial_x v_-(0) = 2V_0 v_+(0).$$  

(108)

(109)

We determine eight coefficients $c_{\pm,k}$ and eigenenergy $E_l$ through four conditions in Eqs. (108) and (109), four conditions in Eqs. (104) and (105), and a normalization condition $\int_{-L/2}^{L/2} dx |\langle u_l(x) \rangle|^2 = 1$. Since $u_{\pm}(x, k)$ and $v_{\pm}(x, k)$ are solutions of the Bogoliubov equation, $E_l$ satisfies the orthogonality when $E_l \neq E_i$, because of (35).

In Fig. 3, we plot the squared matrix element $M(x = 0, E_l)$ as a function of $E_l$. In the numerical calculation, we fix the current $J$, and we determine the system size $L$ to satisfy the periodic boundary condition of the condensate wave function at $x = \pm L/2$ with the given current $J$. The lowest excitation energy is $O(L^{-1})$. From the data, we find that the first excitation energy $E_{\text{ext}}$ at $J = 0$ is $E_{\text{ext}} \sim 2\pi/L$, $E_{\text{ext}}$ is a monotonically decreasing function of the current $J$. From the data, we find $E_{\text{ext}} \sim \pi/L$ at $J = J_c$. For example, in the system whose size is $L = 696.5 \cdot \cdot \cdot$, we find that $E_{\text{ext}} = 0.00903 \cdot \cdot \cdot$ at $J = 0$, which is close to $2\pi/L = 0.00902 \cdot \cdot \cdot$, and $E_{\text{ext}} = 0.00446 \cdot \cdot \cdot$ at $J = J_c$ which is close to $\pi/L = 0.00451$. Here, we used $V_{\text{ext}} = V_0 \delta(x)$ with $V_0 = 10$.

We find two types of excitations. The type-I excitation dominantly contributes the density fluctuation at $J \neq 0$ whose matrix element becomes larger for lower energies in particular at $J = J_c$. The type-II excitation gives smaller contributions to the density fluctuation than that of the type-I at $J \neq 0$, whose matrix element becomes smaller for lower energies at an arbitrary $J \leq J_c$. The 1st excitation is always the type-I. In the low-energy regime, we find the parity rule; the odd (even)-numbered excitations belong to the type-I (II). For higher energy regimes, it is hard to distinguish these two types of excitations. At $J = 0$, we cannot distinguish the type-I from type-II because of degeneracy.

When we plot the squared matrix element for several system sizes as shown in Fig. 3, we find that for the type-I excitation, it produces a smooth line in the low-energy regime, which is no longer dependent on the system size $L$. We thus introduce an interpolation function $\tilde{M}(x, \omega)$ that is a slowly-varying function of $\omega$ compared with the energy interval $\Delta E = |E_{l+1} - E_l|$ where $l \in \text{type-I}$. This function satisfies

$$|\partial \tilde{M}(x, \omega)/\partial \omega| \Delta E \ll |\tilde{M}(x, \omega)|,$$

(110)

and satisfies

$$\tilde{M}(x, E_l) = M(x, E_l), \quad (l \in \text{type-I}).$$  

(111)

$\tilde{M}(x, \omega)$ will trace the squared matrix element of the local density spectral function of the type-I excitation in the thermodynamic limit. Using this interpolation function $\tilde{M}(x, \omega)$, we end with

$$I_n(x, \omega) = \tilde{M}(x, \omega) D_{L,d=1}(\omega),$$  

(112)
where \( D \) for (a), (b), (c) is the density of states per volume for the type-I ( ), II ( ) dimensional system, \( E \) is the zero-energy mode is only the phase mode. In the critical current state at \( J = J_c \), however, \( G \) in the zero-energy limit appears, which is proportional to a derivative of the amplitude \( A(x) \) with respect to the phase difference \( \varphi \), i.e.,

\[
G_j(x) \propto \frac{a_e}{\sqrt{E_j}} \frac{\partial A(x)}{\partial \varphi},
\]

which characterizes the critical current state according to the \( J - \varphi \) relation as in Fig. 3. It will be discussed in detail in Appendix 4.

Equation (115) means that the density mode appears at \( J = J_c \) even in the zero-energy limit. Using these solutions, we obtain the squared matrix element as

\[
\hat{M}(x, \omega) \simeq \left\{ \begin{array}{ll}
\omega n_0(x)\left| \tilde{G}(x) \right|^2 & (J < J_c) \\
\omega^{-1} \left| \frac{\partial}{\partial x} n_0(x) \right|^2 & (J = J_c),
\end{array} \right.
\]

at low-energy regime up to a constant factor. We here used \( n_0(x) = A^2(x) \).

We now introduce the coarse-grained density of states,

\[
\tilde{D}_d(\omega) = \frac{1}{\delta} \int_{\omega - \delta/2}^{\omega + \delta/2} d\omega' D_{L,d}(\omega'),
\]

where \( \delta \) satisfies an arbitrary small value satisfying \( \Delta E < \delta \ll 1 \) for large \( L \). \( \tilde{D}_d(\omega) \) is a smooth function, and we regard it as the density of states in the thermodynamic limit. As a result, the local density spectral function in the low-energy regime for \( d = 1 \) is given by

\[
\tilde{M}_n(x, \omega) = \tilde{M}(x, \omega) \tilde{D}_d(\omega).
\]

In the thermodynamic limit, we approximate \( \tilde{D}_d(\omega) \) as

\[
\tilde{D}_d(\omega) = \int \frac{dk}{(2\pi)^d} \frac{1}{v_g(k)} \delta(k - k^{\text{th}}),
\]

where \( D_{L,d}(\omega) \) is the density of states per volume for the Bogoliubov excitations in the \( d \)-dimensional system, given by

\[
D_{L,d}(\omega) = \frac{1}{L^d} \sum_{i} \delta(\omega - E_i).
\]
where $v_g(k)$ is the group velocity

$$v_g(k) = \frac{\partial E}{\partial k} = J \cos \theta + \frac{k [k^2 + 2]}{2(\omega - J k \cos \theta)}. \quad (122)$$

Here, $k^{in}$ is a real and positive solution of

$$\omega = J k^{in} \cos \theta + \sqrt{\left(\frac{k^{in}}{2}\right)^2 + 2}. \quad (123)$$

We end with

$$\tilde{D}_d(\omega) = \begin{cases} \frac{1}{2\pi} v_g(k^{in}) & (d = 1) \\ \frac{2}{(2\pi)^2} \int_0^{\frac{\pi}{2}} d\theta v_g(k^{in}) & (d = 2) \\ \frac{2}{(2\pi)^3} \int_0^{\frac{\pi}{2}} d\theta \sin \theta (v_g(k^{in}))^2 & (d = 3). \end{cases} \quad (124)$$

When $J < 1$, the excitation is phonon, i.e., $k^{in} \propto \omega$, so that we get $\tilde{D}_d(\omega) \propto \omega^{-d-1}$.

For $d = 2$, let us classify the eigenstates $l$ by $\theta \in [-\pi/2, \pi/2]$, and use solutions of (105). We introduce infinitesimally small intervals $\Delta \theta_m = [m \Delta \theta, (m + 1) \Delta \theta]$ for $m \in [\pi/2, (N/2) - 1]$, where $1 \ll \pi/\Delta \theta \equiv N$ and $m \in \mathbb{Z}$. In this case, the eigenstate can be labeled as $E_l = E_{l,m}$. The density spectral function is given by

$$I_n(x, \omega) = \sum_{l,m} M(x, E_{l,m}) \frac{1}{L^2} \delta(\omega - E_{l,m}). \quad (125)$$

We can discuss the case for $d = 3$ in a similar way. Since $k_\perp = \mathcal{O}(E)$, the Bogoliubov equation (123) and (124) can be reduced into that for the one-dimensional case within $\mathcal{O}(E)$. The $\theta$-dependence is included only through $k_\parallel$, but in the low-energy regimes, the solution has the same structure as (114) at $J < J_c$ or (115) at $J = J_c$. As a result, the $\omega$-dependence of the squared matrix element is also the same as (117). The excitation is phonon at $J < 1$, so that the $\omega$-dependence of the remaining factor of $I_n$ is proportional to $\omega^{-d-1}$. We thus end with

$$I_n(x, \omega) \simeq \begin{cases} \omega^d n_0(x) |G(x)|^2 & (J < J_c) \\ \omega^{-d} \left| \partial_x n_0(x) \right|^2 & (J = J_c), \end{cases} \quad (126)$$

at low energy regime up to a constant factor.

We compare the local density spectral function, whose matrix element is constructed by the orthogonal set for the finite system with the periodic boundary condition, with that obtained from the boundary condition of the tunneling problem of excitations. The purpose of this comparison is to see whether the solutions of the tunneling problem reproduce low-energy behaviors obtained from the orthogonal basis for the finite system, because it is easier to use the tunneling solutions for evaluating the local density spectral function in the thermodynamic limit.

Suppose that the right (left)-moving excitation is incident on the barrier, and the left (right)-moving excitation is the reflection wave. In these cases, we consider the following solutions

$$S(x) = \begin{pmatrix} \alpha_{k_\parallel}^{(1)} \\ \beta_{k_\parallel}^{(1)} \end{pmatrix} e^{i k_\parallel x} + r \begin{pmatrix} \alpha_{k_\parallel}^{(2)} \\ \beta_{k_\parallel}^{(2)} \end{pmatrix} e^{i k_\parallel x},$$

$$= \begin{pmatrix} \alpha_{k_\parallel}^{+} \\ \beta_{k_\parallel}^{+} \end{pmatrix} e^{i k_\parallel x} \quad (x \to \mp \infty), \quad (127)$$

$$G(x) = \begin{pmatrix} t \alpha_{k_\parallel}^{(1)} \\ t \beta_{k_\parallel}^{(1)} \end{pmatrix} e^{i k_\parallel x} + b \begin{pmatrix} \alpha_{k_\parallel}^{+} \\ \beta_{k_\parallel}^{+} \end{pmatrix} e^{i k_\parallel x} \quad (x \to \pm \infty). \quad (128)$$

The terms with exp $(ik_\parallel x)$ are converging parts at $x \to \pm \infty$. The coefficients $t$, $r$, $a$ and $b$ are determined from (108) and (109) in the delta-function potential barrier case. Here, $t$ and $r$ are amplitude transmission and reflection coefficients, respectively. The tunneling problems of the Bogoliubov excitations have been investigated as the anomalous tunneling phenomena. These have been intensively and extensively studied for the current free state of scalar Bose-Einstein condensates (147) and of Bose-Einstein condensates with internal degrees of freedom (54, 57). For the current carrying state, we can find articles for scalar Bose-Einstein condensates (43, 44, 46) and for Bose-Einstein condensates with internal degrees of freedom (58, 60).

The state can be characterized by the incident momentum $k^{in}$. The local density spectral function in the
$d$-dimensional system becomes
\[
I_n(x, \omega) = n_0(x) \int \frac{dk}{(2\pi)^d} |G(x; k)|^2 \times \delta \left( \omega - J \cdot k - \sqrt{\frac{k^2}{2} \left( \frac{k^2}{2} + 2 \right)} \right).
\]
(129)

For evaluating this value, we solve the Bogoliubov equations and with the boundary conditions and with a given energy $E$ and a given incident angle $\theta$. According to Appendix A, this spectral function can be simplified as
\[
I_n(x, \omega) = n_0(x) \int \frac{dk}{(2\pi)^d} \frac{|G(x; k)|^2}{v_k(k)} \delta(k - k^\text{in}).
\]
(130)

Figure 4 plots the density spectral function at $x = 0$ which is obtained from the tunneling problem, and $M(x = 0, E)$ for $l \in \text{type-I}$, whose matrix element is obtained from the orthogonal basis. Only at $J = 0$, because of the degeneracy, the orthonormality is not guaranteed. For $D_{d=1}(\omega)$, we used (128). In the low-energy regime, the local density spectral function $I_n$ constructed from the tunneling solutions well reproduces the $\omega$-dependence of $I_n$ constructed from the orthogonal basis. On the basis of this fact, in the following, we use the solutions of the tunneling problem to calculate the local density spectral function effectively, and discuss the low-energy behaviors of the local density spectral function.

In Figs. 5 and 6, we plot the local density spectral function $I_n(x, \omega)$ for the 3-dimensional case in the presence of a repulsive potential barrier, and for the 1-dimensional case in the presence of an attractive potential barrier. According to both figures, anomaly occurs around the region where the density is minimum. The enhancement of the density fluctuation strongly suggests the breakdown of the stable superfluid. In the repulsive potential barrier case, the enhancement of the local density spectral function appears around the barrier, which is localized at $x = 0$. This enhancement arises as the current $J$ becomes close to $J_c$. For the attractive potential barrier case, the enhancement of the local density spectral function also arises as the current $J$ becomes close to $J_c$, but the region where it occurs is different. The enhancement appears far from the attractive potential, where the density is minimum and also uniform.

For seeing the energy dependence of $I_n(x, \omega)$, in Fig. 7 we plot the local density spectral function $I_n(x, \omega)$ at $x = 0$ for a repulsive potential barrier. We find not only the quantitative enhancement of the local density spectral function with $J \to J_c$ but also a qualitative change in it. The exponent of the local density spectral function in the low-energy regime is different between the state at $J = J_c$ and the other states at $J < J_c$. As shown in [120], in the $d$-dimensional system at $J < J_c$, the relation $I_n(x, \omega) \propto \omega^d$ holds. At $J = J_c$, on the other hand, the relation $I_n(x, \omega) \propto \omega^{d-2}$ holds. Anomaly of the local density spectral function for an attractive potential case originates essentially from the Landau’s instability. The exponent of the local density spectral function will be studied in [16].

At $J = J_c(<1)$ in a repulsive potential case, we can estimate a profile of the local density spectral function in the low-energy regime irrespective of shapes of the potential wall, because the result given by $\lim_{E \to 0} G(x) \propto A_\varphi(x)$ is irrespective of the shape of the potential barrier, where $A_\varphi(x) = \partial A(x)/\partial \varphi$. Let $\eta$ be
\[
\eta = \frac{\int_{-\infty}^{\infty} dx A(x) A_\varphi(x)}{\int_{-\infty}^{\infty} dx A^2(x) / A^\varphi(x)}.
\]
(131)

We consider a high-barrier regime which leads to $J_c \ll 1$. 

![FIG. 5: (color online) Local density spectral function $I_n(x, \omega)$ as functions of $\omega$ and $x$ in the 3-dimensional case, in the presence of a repulsive delta-function potential barrier $V_{\text{ext}}(x) = V_0 \delta(x)$ with $V_0 = 10$. In this case, the critical current $J_c$ is $J_c = 0.049753 \cdots$.](image)
FIG. 6: (color online) Local density spectral function $I_n(x, \omega)$ as functions of $\omega$ and $x$ in the 3-dimensional case, in the presence of an attractive delta-function potential barrier $V_{\text{ext}}(x) = V_0 \delta(x)$ with $V_0 = -2$. In this case, the critical current $J_c$ is equal to the Landau’s critical current $J_c = 1$.

We also assume $\eta \ll 1$, because $\eta = \mathcal{O}(J)$ as discussed in Appendix III. In this case, we have analytical expressions of the low-energy local density spectral function for the dimensionality $d$,

$$ I_n(\omega, x) \sim \frac{F_d}{\pi} \omega^{d-2} \left[ \partial_x n_0(x) \right]^2, \quad (132) $$

with

$$ F_d = \begin{cases} 
\frac{2J_c^2}{J_c^2 + \eta^2} & \text{for } (d = 1) \\
\frac{1 - \eta^2}{\sqrt{J_c^2 + \eta^2}} & \text{for } (d = 2) \\
\frac{1}{\pi} \left[ 1 - \frac{\eta}{J_c} \tan^{-1} \left( \frac{J_c}{\eta} \right) \right] & \text{for } (d = 3).
\end{cases} \quad (133) $$

Detailed derivations are shown in Appendix III. In Fig. 7 we plot the spatial dependence of the local density spectral function. The density fluctuation increases around the repulsive potential barrier as $J \rightarrow J_c$ when we fix $\omega$. We can see that the spatial dependence of the local density spectral function $I_n(x, \omega)$ is consistent with our analytical results shown in (132). If we make $\omega$ lower, the range where our analytical results agree with the numerical results becomes broader.

FIG. 7: (color online) Local density spectral function $I_n(x, \omega)$ at $x = 0$, in the presence of a repulsive delta-function potential barrier $V_{\text{ext}}(x) = V_0 \delta(x)$ with $V_0 = 10$. (a), (b) and (c) are for the 1, 2 and 3-dimensional systems, respectively. We used the set of the current $J = J_c (= 0.049753 \cdots, 0.04975, 0.0497, 0.049, 0.045, 0.04, 0.03, 0.02, 0.01$ and 0. Red and blue lines are respectively for $J = J_c$ and $J = 0$. We find that the functions are shifted from $J = 0$ to $J = J_c$ with the increase of the current $J$. 

We also assume $\eta \ll 1$, because $\eta = \mathcal{O}(J)$ as discussed in Appendix III. In this case, we have analytical expressions of the low-energy local density spectral function for the dimensionality $d$,
Before discussing the Landau instability, we discuss the relation with the saddle-node bifurcation. Hakim discussed the soliton instability as the saddle-node bifurcation, where the stable and unstable branches merge at the bifurcation point (the critical current) \( J_c \). Near the saddle node bifurcation point, a dynamical scaling relation can be found. For example, the emission rate \( \Gamma \) of the gray soliton is \( \Gamma \propto \sqrt{|V-V_c|} \), with \( V \) being the strength of the potential barrier and \( V_c \) being its critical strength. The bifurcation is written in parameter spaces, so that the relation does hold not only for the strength of the barrier but also do for the supercurrent \( J \). To see this, in Fig. 8 (a), we plot the frequency \( \omega_{\text{peak}} \) giving the peak of the local density spectral function at \( x=0 \) as a function of the scaling factor \( \sqrt{J_c-J} \) in the 1-dimensional system. The data are taken from the result in Fig. 7 (a). (b) The scaling function \( F_d(x, \omega^*) = \omega^{2-d} I_n(x, \omega) \) at \( x=0 \) as a function of the scaled energy (frequency) \( \omega^* = \omega/\sqrt{J_c-J} \), in 1, 2 and 3 dimensional systems. Each symbol represents data at \( J = 0.04975 \) (circle), 0.0497 (square) and 0.049 (triangle). The result (b) is referred from [24]. In both (a) and (b), we used the delta-function potential barrier \( V_{\text{ext}}(x) = V_0 \delta(x) \) with \( V_0 = 10 \), and its critical current is \( J_c = 0.049753 \cdots \).

![FIG. 8](image_url)

**FIG. 8:** (color online) The spatial dependence of the local density spectral function \( I_n(x, \omega) \) at \( \omega = 10^{-4} \) in the presence of a repulsive delta-function potential barrier \( V_{\text{ext}}(x) = V_0 \delta(x) \) with \( V_0 = 10 \). (a), (b) and (c) are for the 1, 2 and 3-dimensional systems, respectively. We used the set of the current \( J = J_c(=0.049753 \cdots, 0.04975, 0.049, 0.045, 0.04, 0.03, 0.02, 0.01 \) and 0. Red and blue lines are for \( J = J_c \) and \( J = 0 \). We find that the functions are shifted from \( J = 0 \) to \( J = J_c \) with the current \( J \) increasing. Red dotted lines are analytical results in [27].
scaling relation \( \omega^* = \omega / \sqrt{J_c - J} \), we obtain the scaling function \( F_d(x, \omega^*) \) as

\[
\mathcal{I}_n(x, \omega) = \omega^{d-2} F_d(x, \omega | J - J_c|^{-1/2}).
\]  

(134)

In Fig. 10(b), we plot this scaling function \( F_d(x, \omega^*) = \omega^{d-2} \mathcal{I}_n(x, \omega) \) at \( x = 0 \) as a function of the scaled energy (frequency) \( \omega^* = \omega / \sqrt{J_c - J} \), in 1, 2 and 3 dimensional systems. In each dimension, the local density spectral functions near the critical current collapse onto a single curve, which implies the dynamical scaling law.

VI. LOCAL DENSITY SPECTRAL FUNCTION IN BOGOLIUBOV THEORY: LANDAU INSTABILITY

Landau instability occurs when

\[
E_p + p \cdot v < 0,
\]

(135)

where \( v \) is the velocity of the fluid, and \( E_p \) is the energy spectrum when the supercurrent is absent. In the dimensionless form, the Bogoliubov spectrum in the presence of the supercurrent is

\[
E_k = v k \cos \theta + \sqrt{\frac{k^2}{2} \left( \frac{k^2}{2} + 2 \right)}.
\]

(136)

Here, \( \theta \) is the angle between the vector of the supercurrent velocity and the momentum vector. In the dimensionless form, we can replace the velocity \( v \) with the current \( J \) in the uniform system. The critical velocity is given by \( v_c = 1 \).

In the uniform system, the local density spectral function is linked to the Fourier transformation of the dynamic structure factor through

\[
\mathcal{I}_n(r_1, r_2; \omega) = \int \frac{dq}{(2\pi)^d} S(q, \omega) e^{i q \cdot (r_1 - r_2)},
\]

(137)

for the dimensionality \( d \). In this case, the equal point local density spectral function does not have the \( r \)-dependence, and is given by

\[
\mathcal{I}_n(\omega) = \mathcal{I}_n(r, r; \omega) = \int \frac{dq}{(2\pi)^d} S(q, \omega).
\]

(138)

In the Bogoliubov theory, the local density spectral function is

\[
\mathcal{I}_n(\omega) = \sum_l |G_l(r)|^2 \delta(\omega - E_l),
\]

(139)

with \( n_0 = 1 \).

We start with the stable superfluid state at \( v < v_c \). In the \( d \)-dimensional system, we have the following low-energy behaviors of the density spectral function

\[
\mathcal{I}_n(\omega) \sim \frac{\Gamma_d}{2\pi} \frac{d + v^2}{(1 - v^2)(d+3)/2} \omega^d,
\]

(140)

Detailed derivations are shown in Appendix E. Since the low-energy has phonon spectrum, so that the group velocity is constant. As a result, we can easily check \( \mathcal{I}_n \propto \omega^d \) in the low-energy regime.

On the other hand, when the system is in the superfluid state with the Landau’s critical velocity \( v = v_c(= 1) \), the
density spectral function is

\[ I_n(\omega) \simeq \int \frac{dk}{(2\pi)^3} \frac{k}{2} \left( \omega - k \sqrt{1 + \frac{k^2}{4} - k} \right). \]  

Let \( k_+(\omega) \) be the solution of the equation

\[ \omega = f_+(k) = \pm k + \sqrt{\frac{k^2}{2} + 2}. \]

Using these \( k_\pm(\omega) \), we get

\[ I_n(\omega) \simeq \begin{cases} \frac{1}{8\pi} \sum_{j=\pm} \frac{dk_j^2}{d\omega} & (d = 1) \\ \frac{1}{4\pi^2} \int_{k_+} k^2 dk \sqrt{f_+ - \omega} [\omega - f_-] & (d = 2) \\ \frac{1}{12\pi^2} \left[ k_0^2(\omega) - k_3^2(\omega) \right] & (d = 3). \end{cases} \]  

In the low-energy regime, we have \( k_+ \simeq \omega/2 \) and \( k_- \simeq 2\omega^{1/3} \). As a result, the low-energy behavior of the density spectral function for the dimensionality \( d \) is given by

\[ I_n(\omega) \simeq \frac{\Gamma_1'}{3\pi} \omega^{(2d-3)/3}, \]  

with

\[ (\Gamma_1', \Gamma_2', \Gamma_3') = \left( 1, \frac{2\sqrt{3}}{\pi}, \frac{1}{\pi} \right). \]

Detailed derivations are shown in Appendix E.

In Fig. 10, we plot the density spectral function \( I_n(\omega) \) in the uniform system using the Bogoliubov theory. We can see that the enhancement of the local density spectral function, and the qualitative change of the low-energy behavior at \( v = v_c \).

In a stable superfluid state \( v < v_c (= 1) \), the energy spectrum is phonon, i.e., \( E = (1 + v \cos \theta)k \). At \( v > v_c \), on the other hand, the energy spectrum can be negative, whose state is unstable because of the Landau’s criterion. In the critical current state at \( v = v_c (= 1) \), \( E \simeq k^3/8 \) holds for low \( k \) when \( \theta = \pi \), i.e., the direction of the momentum of the excitation is opposite to the direction of the supercurrent flow. The change of the energy spectrum from \( E \propto k \) to \( E \propto k^3 \) increases the density of states, so that the density spectral function is enhanced in the critical current state. This leads to the change of the exponent of \( \omega \) in the density spectral function.

VII. LANDAU INSTABILITY IN FEYNMAN’S SINGLE-MODE APPROXIMATION

Apart from the mean-field theory, we consider the local density spectral function in the uniform system. We here use the Feynman’s single-mode approximation [11]. The dynamic structure factor is given by

\[ S(q, \omega) = \sum_{l s.t. q(l) = q} |\langle l | \hat{n}_q | g \rangle|^2 \delta(\omega - \omega_l + \omega_g). \]  

Since

\[ \hat{n}_q = \frac{1}{\sqrt{\Omega}} \int dr \hat{n}(r)e^{-iq \cdot r}, \]  

with the system volume \( \Omega \), the density spectral function reads as

\[ I_n(\omega) = \sum_l \sum_q |\langle l | \hat{n}_q | g \rangle|^2 \delta(\omega - \omega_l + \omega_g) \]  

\[ = \frac{1}{\Omega} \sum_l \sum_q |\langle l | \hat{n}_q | g \rangle|^2 \delta(\omega - \omega_l + \omega_g). \]  

When we write the momentum in the state \( l \) as \( q(l) \), the density spectral function can be written in terms of the dynamic structure factor

\[ I_n(\omega) = \frac{1}{\Omega} \sum_q \sum_{l s.t. q(l) = q} |\langle l | \hat{n}_q | g \rangle|^2 \delta(\omega - \omega_l + \omega_g) \]  

\[ = \int \frac{dq}{(2\pi)^d} S(q, \omega), \]

where in the last equation we used

\[ \frac{1}{\Omega} \sum_q = \int \frac{dq}{(2\pi)^d}. \]

When the momentum in the ground state is zero, the relation between the energy of the elementary excitation \( E_q \) and the static structure factor \( S(q) \) is

\[ E_q = \frac{q^2}{2S(q)}, \quad S(q) = \sum_{l s.t. q(l) = q} |\langle l | \hat{n}_q | g \rangle|^2. \]

The static and dynamic structure factors are linked each other through

\[ S(q) = \int_0^\infty d\omega S(q, \omega). \]

The dynamic structure factor in the Feynman’s single-mode approximation is readily given by

\[ S(q, \omega) = \frac{q^2}{2E_q} \delta(\omega - E_q). \]

Even in the current flowing state, the strength of the dynamic structure factor is the same as that in the current free state because of the translational invariance. In the current carrying state flowing along the \( x \)-direction, we get

\[ S(q, \omega) = \frac{q^2}{2E_q} \delta(\omega - E_q - vq_x), \]
and end with
\[ I_n(\omega) = \int \frac{dq}{(2\pi)^d} \frac{q^2}{2E_q} \delta(\omega - E_q - vq_z). \]  
(158)

For low \( q = |q| \), we suppose that
\[ E_q \approx c_1 q + c_3 q^3 + O(q^5) \]  
(159)
holds, where \( c_1 \) and \( c_3 \) are positive coefficients. We now focus on the phonon regime, i.e., \( c_1 q \gg c_3 q^3 \). Let \( k_\pm(\omega) \) be solutions of
\[ \omega = E_q \pm vq = f_\pm(q). \]  
(160)

We end with
\[ I_n(\omega) \approx \begin{cases} 
\frac{1}{8\pi c_1} \sum_{j=\pm} \frac{dk_j^2(\omega)}{d\omega} & (d = 1) \\
\frac{1}{8\pi^2 c_1} \int_{k-} \frac{q^2 dq}{\sqrt{(f_+ - \omega)(\omega - f_-)}} & (d = 2) \\
\frac{1}{24\pi^2 v_c} \sum_{k-} (k_+^2 - k_+^3) & (d = 3).
\end{cases} \]  
(161)

When \( \omega_- \gg \omega \), where \( \omega_- = \sqrt{(c_1 - v)^3/c_3} \), we have
\[ I_n(\omega) \approx \frac{\Gamma_d}{2\pi c_1} \frac{d(e^2 + v^2)}{(v_1^2 - v^2)(d+3)/2} \omega^d, \]  
(162)
where \( \Gamma_d \) is the same as in (\ref{eq:gamma_d}). On the other hand, when \( \omega_- \ll \omega \ll \omega_+ \), where \( \omega_+ = \sqrt{c_1^2/c_3} \), we get
\[ I_n(\omega) \approx \frac{\Gamma_d'}{3\pi c_1} \frac{\omega^{(2d-3)/3}}{2^{(d+3)/2}v_1^{(d-1)/2}c_3^2(d+3)/6}. \]  
(163)
where \( \Gamma_d' \) is the same as in (\ref{eq:gamma_d}). Detailed derivations are shown in Appendix G. If we set \( c_1 = 1 \), \( c_3 = 1/8 \) and \( v = 1 \), we recover (\ref{eq:gamma_d}) and (\ref{eq:gamma_d'}). 

We finally discuss the local density spectral function for an ideal gas, with the energy spectrum
\[ E_k = \frac{\hbar^2 k^2}{2m}. \]  
(164)
where we used \( \delta_k |g; N \rangle = \delta_{k', 0} \sqrt{N} |g; N - 1 \rangle \). Therefore, the density spectral function is given by
\[ I_n(\omega) = \sum \frac{|\langle l; N|\hat{n}(0)|g; N \rangle|^2}{\Omega} \delta(\hbar\omega - E_l + E_n). \]  
(167)

Since the density of states \( D(\omega) \) is
\[ D(\omega) = \frac{d}{d\omega} \int d^d \omega \delta(\hbar\omega - E_k), \]  
(170)
we have \( I_n(\omega) = ND(\omega) \). This means that the density spectral function of the ideal Bose gas is proportional to the density of states. As a result, we end with
\[ I_n(\omega) = \frac{N}{\Omega} \frac{C_d m^d}{2^{(d+2)/2}\pi^{(d+1)/2}c_3^2(d+3)/6} \omega^{(d-2)/2}, \]  
(171)
in the \( d \)-dimensional system, where
\[ (C_1, C_2, C_3) = (2, 2\pi, 4\pi). \]  
(172)

VIII. STABILITY CRITERION

On the basis of all the results, we propose a stability criterion for superfluidity in the light of the density spectral function, which is applicable to the Landau instability and the instability due to the saddle-node bifurcation.

We first summarize results in the uniform systems examined in Secs. VIII and VII. For the stable superfluid in the 1,2 and 3-dimensional systems, exponents of the local density spectral function are respectively 1,2 and 3 with respect to the energy (frequency) in the low-energy regime. These exponents are equal to the dimensionality of the system. On the other hand, for the critical current state in the 1,2 and 3-dimensional systems, these are respectively -1/3, 1/3 and 1, which are less than the dimensionality of the system.

We also examined the local density spectral function in the presence of the potential wall in Sec. IX. For the stable superfluid in the 1,2 and 3-dimensional systems, exponents of this function are respectively 1, 2, and 3, with respect to the energy (frequency) in the low-energy regime. These exponents are equal to the dimensionality of the system. On the other hand, for the critical current state in the 1,2 and 3-dimensional systems, these are respectively -1/3, 1/3 and 1, which are less than the dimensionality of the system.

In both cases, the exponent is equal to the dimensionality of the system for the stable superfluid state. For the critical current state, however, it is less than the dimensionality and this leads to the enhancement of the local
density fluctuation in the low-energy regime. According to
to these results, we propose the stability criterion of super-
fluidity in terms of the local density spectral function.

This criterion states that the exponent of the local
density spectral function for the stable superfluid corre-
des to the dimensionality of the system, while at the
critical current, it becomes less than the dimensionality
of the system. This criterion can be written as

$$\lim_{\omega \to 0} I_n(r, \omega) \propto \begin{cases} \omega^\beta & (J = J_c) \\ \omega^d & (J < J_c) \end{cases}$$

(173)

with $\beta < d$.

We mainly developed the stability criterion from re-
results of weakly interacting condensed Bose systems.
However, we expect that this criterion will hold in gen-
eral case. Even in liquid helium, the static structure
factor $S(q)$ in the low-momentum regime is $S(q) \propto q$,
which leads $E_q \propto q$ through the Feynman theory $E_q =
q^2/(2mS(q))$ as in [11], where we set $\hbar = 1$. The dy-
namic structure factor within the Feynman’s single-mode
approximation given by $S(q, \omega) = S(q)\delta(\omega - E_q) \propto
q\delta(\omega - E_q)$ means that the density fluctuation disappar-
ishes in the low-energy limit. As examined in Sec. VII this
leads $I_n(\omega) \propto \omega^d$ in the $d$-dimensional system. When
the supercurrent flows in a stable state, the qualitative
behavior of $S(q, \omega)$ in the low-energy regime would not
change. If so, $I_n(\omega) \propto \omega^d$ will hold. The appearance of
the density fluctuation in the low-energy regime will lead
anomalies of the exponent with respect to $\omega$ in the local
density spectral function. In this case, its exponent will
be less than $d$.

According to the context of the Landau’s criterion, an
ideal Bose gas does not show superfluidity, i.e., the criti-
cal velocity is zero and the state without current is the
critical current state. As examined in Sec. VII, this
density spectral function of an ideal Bose gas has a factor
$\omega^{d-3}/2$. According to our criterion, the exponent is less
than the dimensionality $d$, so that an ideal Bose gas with-
out current is the critical current state, which is consist-
ent with the Landau’s criterion context.

The local density spectral function $I_n(r, t)$ is linked to
the autocorrelation function $C_n(r, t)$ according to [59].
Because an exponent of $\omega$ in the local density spectral
function changes in the low-energy regime at $J = J_c$, an
exponent of $t$ in the autocorrelation function also does
in the long-time regime at $J = J_c$. From the viewpoint
of dimensional analysis, the autocorrelation function at
large $t$ is given by

$$\lim_{t \to \infty} C_n(r, t) \propto \begin{cases} 1/t^{\beta+1} & (J = J_c) \\ 1/t^{d+1} & (J < J_c) \end{cases}$$

(174)

To see this explicitly, we evaluate the autocorrelation
function. For eliminating the high frequency behavior
we are not interested in, we introduce the coarse-grained
local density spectral function. The coarse-grained local
density fluctuation is given by

$$\delta\hat{n}_{CG}(r) = \int dr' f_a(r - r')\delta\hat{n}(r'),$$

(175)

FIG. 11: The coarse-grained autocorrelation function $C_{nCG}(x, t)$ at $x = 0$ in the 3-dimensional system. We used the one-dimensional Gaussian potential barrier $V_{ext}(x) = V_0\exp(-x^2/d^2)$ with $V_0 = 2$ and $d = 1$. The critical cur-
in in this case is $J_c = 0.05740\cdots$. We used (176) with $a = 1$.

where we take $\int dr f_a(r) = 1$, and $f_a(r) \simeq 0$ for $|r| \gg a$.
One of functions satisfying the above conditions is

$$f_a(r) = \frac{1}{\pi^{d/2}a^d} \exp\left(-|r|^2/a^2\right).$$

(176)

The coarse-grained local density spectral function
$I_{nCG}(r, \omega)$ and the coarse-grained autocorrelation
function $C_{nCG}(r, t)$ are respectively given by

$$I_{nCG}(r, \omega) = \sum_i \left|\langle \delta\hat{n}_{CG}(r)|g_i\rangle\right|^2 \delta(\omega - E_i + E_g),$$

(177)

$$C_{nCG}(r, t) = \int d\omega I_{nCG}(r, \omega) \cos(\omega t).$$

(178)

Details are shown in Appendix [11].

In Fig. 11 we plot the coarse-grained autocorrelation
function $C_{nCG}(x, t)$ at $x = 0$ in the 3-dimensional sys-
tem. We find that the long-time behaviors of the coarse-
grained autocorrelation function are different between the
critical current state at $J = J_c$ and the other state at
$J < J_c$. The long-time behavior at $J(= J_c)$ is $t^{-2}$
and that at $J(= J_c)$ is $t^{-4}$. This is consistent with our
criterion (173). This strongly suggests that the dynamical
behavior of the correlation function is important for
discussing stability of superfluid.

We briefly discuss a related issue. In the Tomonaga-
Luttinger liquids, the autocorrelation function is given by [61]

$$C_n(r, t) \sim \frac{A_0}{t^2} + \frac{A_1}{t^{2K}} + \frac{A_2}{t^{4K}} + \cdots.$$  

(179)

$A_{0,1,2}$ are coefficients, and $K$ is the Tomonaga-Luttinger
parameter. In the superconducting phase ($K > 1$),
$C_n(r, t) \propto 1/t^2$ holds for $t \to \infty$, whose exponent of $t^{-1}$ is
2. On the other hand, in the charge-density wave (CDW)
phase \( K < 1 \), \( C_n(r, t) \propto 1/t^{2K} \) holds for \( t \to \infty \), whose exponent of \( t^{-1} \) is \( 2K(< 2) \). In the one-dimensional system, the conductance in the superconducting phase is not infinity even when the small but non-zero voltage is applied \([61]\), so that this does not completely correspond to the superfluidity discussed in this paper. However, when we read the superconducting phase as the stable supercurrent state, and the CDW phase as the critical current state, the classification between the superconducting phase and the CDW phase is common to \([172]\).

**IX. CONCLUSIONS**

Superflow through defects without dissipation is one of the most interesting phenomena. The Landau’s criterion for superfluidity is developed from the consideration of the elementary excitation energy on the basis of the Galilean transformation. Other mechanism of dissipation is emissions of quantized vortices or solitons from an impurity potential when the supercurrent is above the critical current. Through numerical calculations, these instabilities were categorized as the saddle node bifurcation. So, we desire to understand stability of superfluidity in both cases in an equal manner.

By introducing the local density spectral function, we study both instabilities, i.e., Landau instability and soliton or vortex emission instability, using the Bogoliubov theory. Beyond this mean-field theory, the local density spectral function is also investigated in the Feynman’s single-mode approximation. On the basis of these results, we proposed a stability criterion for superfluidity in the light of the density spectral function.

This criterion states that if an exponent of this function in the low-energy regime is equal to the dimensionality of the system \( d \), i.e., \( \omega^d \), the superfluid state is stable; but if it is less than \( d \), it is in the critical current state, i.e., \( \omega^\beta \) with \( \beta < d \). Here, \( \omega \) is the energy (frequency). We can translate this criterion into the autocorrelation function language. The translated criterion states that if the \( t \)-dependence of this function in the long-time regime is equal to \( 1/t^{\beta+1} \), the superfluid state is stable; but if it shows \( 1/t^{\beta+1} \) with \( \beta < d \), it is in the critical current state.

We summarize interesting subjects for future studies. One of prospects is to apply the criterion to the super-solidity, in which translation invariance is broken. It is also natural to ask whether other spectral and correlation functions, such as the current-current correlation functions, show anomalous behaviors in low \( \omega \) or large \( t \), and to ask whether the transport coefficients and drag force are anomalous in the critical current state. It will be interesting to discuss the relation between the present criterion and the drag force studied in \([62]\). To study autocorrelation functions in the strongly interacting systems in the presence of the potential barrier beyond the mean-field theory is also interesting.

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**Appendix A: Condensate wave function in the presence of the delta-function potential**

In this appendix, we summarize the properties of the condensate wave function in the presence of the delta-function potential barrier \( V_{\text{ext}}(x) = V_0\delta(x) \).

For \( |x| > 0 \), we have the following Gross-Pitaevskii equations in terms of the amplitude \( A \) and the phase \( \theta \)

\[
-\frac{1}{2} \frac{d^2 A}{d x^2} + \frac{J^2}{2A^3} - \mu A + A^3 = 0, \quad \frac{d \theta}{d x} = \frac{J}{A^2}.
\]

Considering the equation at \( |x| \to \infty \), we get \( \mu = 1 + J^2/2 \). Then, we obtain

\[
-\frac{1}{2} \frac{d^2 A}{d x^2} + \frac{J^2}{2A^3} - \left(1 + \frac{J^2}{2}\right) A + A^3 = 0.
\]

When we multiply this equation by \( dA/dx \), we get

\[
-\frac{1}{4} \left( \frac{dA}{dx} \right)^2 - \frac{J^2}{4A^2} - \left(1 + \frac{J^2}{2}\right) \frac{1}{2} A^2 + \frac{1}{4} A^4 = C_A,
\]

where \( C_A \) is an arbitrary constant, but from the condition at \( |x| \to \infty \), \( C_A \) is found to be \( C_A = -(1/4 + J^2/2) \).

Using this constant, we can reduce \((A3)\) to

\[
\left( \frac{dA}{dx} \right)^2 = (1 - A^2)^2 \left(1 - \frac{J^2}{A^2}\right).
\]

When we use \( F = A^2 - J^2 \), we get

\[
\pm dx = \frac{d\sqrt{F}}{(\sqrt{F} + \sqrt{1 - J^2})(\sqrt{F} - \sqrt{1 - J^2})},
\]

which gives

\[
F = (1 - J^2) \left[ \frac{1 \pm e^{\mp 2\sqrt{1 - J^2}(x + x_0)}}{1 \mp e^{\mp 2\sqrt{1 - J^2}(x + x_0)}} \right]^2.
\]

We thus end with \( A^2 = J^2 + (1 - J^2) h^2 \sqrt{1 - J^2(x + x_0)} \).

Here, the function \( h(x) \) is \( h(x) = \tanh(x) \) or \( h(x) = \coth(x) \). The boundary condition chooses one or the
other. Since we are considering superfluids in the presence of the delta-function potential barrier, the amplitude is an even function of $x$. Therefore, the amplitude of the condensate wave function is given by

$$A^2(x) = J^2 + (1 - J^2)\hbar^2[\sqrt{1 - J^2(|x| + x_0)}].$$ \hspace{1cm} (A7)

The boundary condition at $x = 0$ reads

$$\left.\frac{d\Psi_0}{dx}\right|_{x=+0} - \left.\frac{d\Psi_0}{dx}\right|_{x=-0} = 2V_0\Psi_0(0).$$ \hspace{1cm} (A8)

Since $A(+0) = A(-0)$ and $\theta(+0) = \theta(-0)$, the boundary condition can be reduced to

$$\left.\frac{dA}{dx}\right|_{x=+0} - \left.\frac{dA}{dx}\right|_{x=-0} = 2V_0A(0),$$ \hspace{1cm} (A9)

where we used $\Psi_0 = Ae^{i\theta}$. An even parity condition of $A(x)$ gives

$$\frac{dA}{dx}\bigg|_{x=+0} = -\frac{dA}{dx}\bigg|_{x=-0}. \hspace{1cm} (A10)$$

Then, the boundary condition can be simplified as

$$\left.\frac{dA}{dx}\right|_{x=0} = V_0A(0).$$ \hspace{1cm} (A11)

When we give two parameters $V_0$ and $J$, we obtain $x_0$. Using $A(x)$ obtained already, we get the equation of the phase

$$\frac{d\theta}{dx} = \frac{J}{A^2} \hspace{1cm} (A12)$$

$$= J \left(1 + \frac{1 - A^2}{A^2}\right) \hspace{1cm} (A13)$$

$$= J + \frac{\text{sgn}(x)J\sqrt{1 - J^2h\left|\sqrt{1 - J^2(|x| + x_0)}\right|}}{J^2 + (1 - J^2)\hbar^2[\sqrt{1 - J^2(|x| + x_0)}]} \hspace{1cm} (A14)$$

where we used $dh(x)/dx = \text{sgn}(x)[1 - \hbar^2(x)]$. As a result, it can be reduced into

$$\frac{d\theta}{dx} = J + \text{sgn}(x)\frac{d}{dx}\left\{\tan^{-1}\left[\frac{\sqrt{A^2(x) - J^2}}{J}\right]\right\}, \hspace{1cm} (A15)$$

where we used

$$\sqrt{1 - J^2h\left|\sqrt{1 - J^2(|x| + x_0)}\right|} = \sqrt{A^2(x) - J^2}. \hspace{1cm} (A16)$$

By integrating over $x$, we end with

$$\theta(x) = \Theta_0 + Jx + \text{sgn}(x)\left\{\tan^{-1}\left[\frac{\sqrt{A^2(x) - J^2}}{J}\right] - \tan^{-1}\left[\frac{\sqrt{A_0^2 - J^2}}{J}\right]\right\}, \hspace{1cm} (A17)$$

where $\Theta_0 = \theta(x = 0)$. Since $\theta(x)$ is an odd function of $x$, the phase difference $\varphi$ is simply given by

$$\varphi = \lim_{x \to +\infty} \left[\theta(x) - \Theta_0 - Jx\right]. \hspace{1cm} (A18)$$

We thus end with

$$\varphi = 2\tan^{-1}\left(\frac{\sqrt{1 - J^2}}{J}\right) - \tan^{-1}\left(\frac{\sqrt{A_0^2 - J^2}}{J}\right),$$ \hspace{1cm} (A19)

where $A_0 = A(x)$. If we take

$$\theta(x) = Jx + \text{sgn}(x)\tan^{-1}\left[\frac{\sqrt{A^2(x) - J^2}}{J}\right],$$ \hspace{1cm} (A20)

we get

$$e^{i\theta} = e^{iJx}\frac{1}{A(x)}\left[J + \text{sgn}(x)\sqrt{A^2(x) - J^2}\right],$$ \hspace{1cm} (A21)

where we used $\exp[i\tan^{-1}(x)] = (1 + ix)/\sqrt{1 + x^2}$. As a result, the condensate wave function is given by

$$\Psi_0(x) = e^{iJx}\left[J + \text{sgn}(x)\sqrt{A^2(x) - J^2}\right]$$ \hspace{1cm} (A22)

The critical current for an attractive potential is $J_c = 1$. In this case, we get

$$\pm(x + x_0) = \int \frac{d\sqrt{F}}{F}. \hspace{1cm} (A23)$$

As a result, $F = (x + x_0) - x$ holds. Since $F = A^2 - 1$, we get $A = \sqrt{1 + (x + x_0)^2}$. We are considering an even parity solution, so that we end with

$$A(x) = \sqrt{1 + \frac{1}{(|x| + x_0)^2}}.$$ \hspace{1cm} (A24)

By using (A11) and

$$\frac{dA}{dx} = -\frac{1}{A} \frac{\text{sgn}(x)}{(|x| + x_0)^2},$$ \hspace{1cm} (A25)

we get an equation $x_0^2 + x_0 + V_0^{-1} = 0$, which gives $x_0$. The phase of the condensate wave function at $J_c = 1$ is obtained from

$$\frac{d\theta}{dx} = \frac{J_c}{A^2}, \hspace{1cm} (A26)$$

$$= J_c + J_c \frac{1 - A^2}{A^2}, \hspace{1cm} (A27)$$

$$= 1 - \frac{1}{(|x| + x_0)^2 + 1}. \hspace{1cm} (A28)$$

As a result, we get

$$\theta = x - \text{sgn}(x)[\tan^{-1}(|x| + x_0) - \tan^{-1}(x_0)],$$ \hspace{1cm} (A29)

where we take $\Theta_0 = 0$. The phase difference $\varphi$ is given by

$$\frac{\varphi}{2} = \lim_{x \to +\infty} \left[\theta(x) - x\right] = -\frac{\pi}{2} + \tan^{-1}(x_0).$$ \hspace{1cm} (A30)

As a result, we end with $\varphi = 2\tan^{-1}(x_0) - \pi$. 

Appendix B: Wave functions of critical current state in the presence of an impurity potential

In this section, we discuss the wave function of the Bogoliubov excitation in the critical current state. We review details about that on the basis of articles \[43\] and \[44\]. We start with \[83\] and \[84\]. Since we are considering the supercurrent through a repulsive potential barrier, we have \(J_c < 1\). In this case, the modulus of the incident momentum in the low-energy regime is linear in \(E\), so that we have \(k_{\text{in}} = \mathcal{O}(E)\) and also have \(k_1 = \mathcal{O}(E)\).

From that, when we expand \(S(x)\) and \(G(x)\) with respect to the energy \(E\)

\[
S(x) = \sum_{n=0}^{\infty} E^n S^{(n)}(x), \quad G(x) = \sum_{n=0}^{\infty} E^n G^{(n)}(x), \quad (B1)
\]

we get equations for \(n = 0\)

\[
\hat{H}S^{(0)}(x) - \frac{iJ_0}{A_0} d \frac{G^{(0)}(x)}{A_0} = 0, \quad (B2)
\]

\[
\hat{H} + 2A^2(x) G^{(0)}(x) - \frac{iJ_0}{A_0} d \frac{S^{(0)}(x)}{A_0} = 0. \quad (B3)
\]

For \(n = 1\), we also get

\[
\hat{H}S^{(1)}(x) - \frac{iJ_1}{A_1} d \frac{G^{(1)}(x)}{A_1} = G^{(0)}(x), \quad (B4)
\]

\[
\hat{H} + 2A^2(x) G^{(1)}(x) - \frac{iJ_1}{A_1} d \frac{S^{(1)}(x)}{A_1} = S^{(0)}(x). \quad (B5)
\]

The general solutions \(S_1\) and \(S_2\) obtained from \(\hat{H}S^{(0)}(x) = 0\) are given by

\[
S_1(x) = A(x), \quad S_2(x) = A(x) \int_{0}^{x} \frac{dx'}{A^2(x')}, \quad (B6)
\]

Here, the Wronskian \(\Delta_S\) is

\[
\Delta_S = S_1(dS_2/dx) - (dS_1/dx)S_2 = 1. \quad (B7)
\]

A particular solution \(f_S\) is

\[
f_S = -2 \left(-S_1 \int dx' \frac{F_S S_2}{\Delta_S} + S_2 \int dx' \frac{F_S S_1}{\Delta_S}\right), \quad (B8)
\]

with

\[
F_S = \frac{iJ_0}{A_0} d \frac{G^{(0)}(x)}{A_0}. \quad (B9)
\]

Here, the factor \(-2\) comes from \(-\frac{1}{2}d^2/dx^2\) in \(\hat{H}\). By simplifying \(f_S\), we finally get

\[
f_S(x) = -2iJ A(x) \int_{0}^{x} dx' \frac{G^{(0)}(x')}{A^2(x')}, \quad (B10)
\]

The solution \(S^{(0)}(x)\) is thus given by

\[
S^{(0)}(x) = C_1 S_1(x) + C_{11} S_2(x) + f_S(x), \quad (B11)
\]

where \(C_1\) and \(C_{11}\) are coefficients. Substituting this result into \(B3\), we get

\[
\left[\hat{H} + 2A^2(x) - 2 \frac{J^2}{A^4(x)}\right] G^{(0)} = C_{11} \frac{iJ}{A^3(x)}. \quad (B12)
\]

Let \(B(x) = G_1(x)\) be an even parity solution of the homogeneous equation of \(B11\) which is given by

\[
\left[\hat{H} + 2A^2(x) - 2 \frac{J^2}{A^4(x)}\right] G^{(0)} = 0. \quad (B13)
\]

The other solution is given by

\[
G_2(x) = B(x) \int_{0}^{x} \frac{dx'}{B^2(x')}, \quad (B14)
\]

and the Wronskian \(\Delta_G\) becomes unity i.e., \(\Delta_G = 1\). The particular solution is

\[
f_G = -2 \left(-G_1 \int dx' \frac{F_G G_2}{\Delta_G} + G_2 \int dx' \frac{F_G G_1}{\Delta_G}\right), \quad (B15)
\]

with

\[
A_3(x) = \int_{0}^{x} dx' \frac{B(x')}{A^3(x')} \quad (B16)
\]

As a result, we end with

\[
G^{(0)}(x) = C_{11} G_{11}(x) + C_{111} G_{111}(x) + C_{11V} G_{11V}(x), \quad (B17)
\]

with \(G_{11} = f_G/C_{11}, G_{111} = G_1 = B, \text{ and } G_{11V} = G_2\). Here, \(C_{11}\) and \(C_{11V}\) are coefficients. Substituting this solution into \(B10\), we get

\[
\left(\frac{S^{(0)}(x)}{G^{(0)}(x)}\right) = \sum_{j=1,II,III,IV} C_j \left(\frac{S_j(x)}{G_j(x)}\right), \quad (B18)
\]

with

\[
\left(\begin{array}{c}
S_1 \\
G_1 \\
S_{11} \\
G_{11} \\
S_{111} \\
G_{111} \\
S_{11V} \\
G_{11V}
\end{array}\right) = \left(\begin{array}{c}
A \\
0 \\
\hat{P}_A(1) - 2iJ \hat{P}_B(A_1/A) \\
-2iJ \hat{P}_B(A_3) \\
-2iJ \hat{P}_B(A_1/A) \\
\hat{P}_B(1)
\end{array}\right), \quad (B19)
\]

Here, we introduced

\[
\hat{P}_X(iY) = X(x) \int_{0}^{x} dx' \frac{Y(x')}{X^2(x')}. \quad (B20)
\]
Particular solutions for $n = 1$ are given by

$$S_p^{(1)}(x) = -2 \hat{P}_A \left( \int_0^x dx' A(x') G^{(0)}(x') \right) - 2iJ \hat{P}_A \left( \frac{G^{(1)}(x)}{A} \right),$$

$$G_p^{(1)}(x) = -2 \hat{P}_B \left( \int_0^x dx' B(x') \left[ S^{(0)}(x') - \frac{2iq}{A^2(x')} \int_0^{x'} dx'' A''(x'') G^{(0)}(x'') \right] \right).$$

(B24)  

(B25)

One of the character at the critical point is $\partial J / \partial \varphi = 0$. Using this, we get an equation of $\partial A(x) / \partial \varphi$, which is

$$\left[ \frac{\partial}{\partial x} + 2A^2(x) - \frac{2J^2}{A^4(x)} \right] \partial A(x) / \partial \varphi = J \frac{dJ}{d\varphi} \left[ A(x) - \frac{1}{A^3(x)} \right] = 0.$$  

(B26)

This equation is obtained from (79), where we take derivative with respect to $\varphi$. Comparing this equation with (103), we find

$$B(x) = \frac{\partial A(x)}{\partial \varphi} \equiv A_\varphi(x).$$

Since $A(x \to \pm \infty) = 1$, we find $B(x \to \pm \infty) = 0$. Indeed, since

$$\left( -\frac{1}{2} \frac{d^2}{dx^2} + 2 - 2J^2 \right) B(x) = 0,$$

we find that $B(x)$ at $|x| \gg 1$ is given by

$$B(x) = \beta e^{-\kappa |x|},$$

with $\beta$ being a constant and $\kappa = 2\sqrt{1 - J^2}$.

$(S_{1,1}^{(1)}, G_{1,1}^{(1)})$ converge at $|x| \to \infty$. On the other hand, $(S_{1,1}^{(1)}, G_{1,1}^{(1)})$ diverge at $|x| \to \infty$. Let us consider these diverging properties. For simplicity, we introduce

$$A_1(x) = \int_0^x dx' A(x') B(x').$$

(B30)

We also introduce the following quantities

$$\alpha_1 = A_1(\infty), \quad \alpha_3 = A_3(\infty), \quad \eta = \frac{\alpha_1}{\alpha_3}.$$  

(B31)

At $|x| \gg 1$, we have $A_j = \text{sgn}(x) \left[ \alpha_j - (\beta / \kappa) e^{-\kappa |x|} \right]$ for $j = 1$ and 3. Using these behaviors, we get

$$G_{11} \simeq -2iJ \beta e^{-\kappa |x|} \int_{\infty}^x dx \text{sgn}(x) \left[ \alpha_3 - (\beta / \kappa) e^{-\kappa |x|} \right] \sim -\frac{iJ \alpha_3}{\beta \kappa} e^{\kappa |x|},$$

$$S_{11} \simeq x + \gamma \text{sgn}(x) - 2iJ \int_{\infty}^x dx' \frac{iJ \alpha_3}{\beta \kappa} e^{\kappa |x'|} \sim -\frac{2J^2 \alpha_3}{\beta \kappa^2} e^{\kappa |x|} \text{sgn}(x),$$

(B32)  

(B33)  

(B34)  

(B35)

where we used $A(x) \int_{\infty}^x dx' A^{-2}(x') \sim x + \gamma \text{sgn}(x)$ at $|x| \to \infty$. The constant $\gamma$ is given by $\gamma = A(x) \int_{\infty}^x dx' \left[ A^{-2}(x') - 1 \right]$. We also have

$$\left( \frac{G_{11}}{S_{11}} \right) \simeq \left( \frac{2\kappa^2 \beta}{iJ \kappa} \right) e^{\kappa |x|}.$$  

(B36)

We here consider a set of particular solutions $(G_{p,1}^{(1)}, S_{p,1}^{(1)})$, where $p$ is suppose that $(S^{(0)}, G^{(0)})$ is given by $(S_1^{(1)}, G_1^{(1)})$. In this case, using (B24) and (B25), we get

$$G_{p,1}^{(1)}(x) = -2B(x) \int_0^x dx' A_1(x') \frac{A_1(x')}{B^2(x')}.$$  

(B37)

$$S_{p,1}^{(1)}(x) = -2iJA(x) \int_0^x dx' G_{p,1}^{(1)}(x') \frac{A_1(x')}{A^3(x')}.$$  

(B38)

From these equations, we end with

$$\left( \frac{G_{p,1}^{(1)}(x)}{S_{p,1}^{(1)}} \right) \simeq \left( -\frac{\alpha_1}{\kappa^2 \beta} e^{\kappa |x|} \right).$$  

(B39)

In the case where $(S^{(0)}, G^{(0)})$ is given by $(S_{11}^{(1)}, G_{11}^{(1)})$, a set of particular solutions $(G_{p,1}^{(1)}, S_{p,1}^{(1)})$ is given by

$$G_{p,11}^{(1)}(x) = 4iJ \hat{P}_B (A_1 A_3),$$  

$$S_{p,11}^{(1)}(x) = -2 \hat{P}_A (A_1) - 2iJ \hat{P}_A (G_{p,11}^{(1)}/A),$$

where we used (B24), (B25) and

$$B(x) \left[ -2iJ A(x) A_3(x) - \frac{2iJ}{A^3(x)} A_1(x) \right] = -2iJ \frac{d}{dx} [A_1(x) A_3(x)].$$

(B40)  

(B41)

These at $|x| \gg 1$ are given by

$$\left( \frac{G_{p,11}^{(1)}(x)}{S_{p,11}^{(1)}} \right) \simeq \left( \frac{2iJ \alpha_1 \alpha_3}{\beta \kappa} e^{\kappa |x|} \text{sgn}(x) \right).$$

(B42)

As a result, we end with

$$\left( \frac{G_{11}}{S_{11}} \right) = \eta \left( \frac{G_{11}}{S_{11}} \right) = \left( -\frac{\alpha_1}{\kappa^2 \beta} e^{\kappa |x|} \text{sgn}(x) \right),$$

(B43)

$$\left( \frac{G_{11}}{S_{11}} \right) \simeq 4iJ \alpha_3 \left( \frac{G_{11}}{S_{11}} \right) = \left( \frac{2iJ \alpha_1 \alpha_3}{\beta \kappa} e^{\kappa |x|} \text{sgn}(x) \right).$$

(B44)

This means that we can construct the converging solutions from the combination of diverging solutions.
(S_{I,IV}, G_{I,IV}) \) and \((S_{p,I,III}^{(1)}, G_{p,I,III}^{(1)})\). The converging solutions we are discussing are

\[
\begin{align*}
(S_{I}^{(1)}(x), G_{I}^{(1)}(x)) &= \left( S_{p,I}^{(1)}(x), G_{p,I}^{(1)}(x) \right) - \frac{\eta}{iJ} \left( S_{II}(x), G_{II}(x) \right), \quad (B47) \\
(S_{III}^{(1)}(x), G_{III}^{(1)}(x)) &= \left( S_{p,III}^{(1)}(x), G_{p,III}^{(1)}(x) \right) - 4iJ\alpha_3 \left( S_{IV}(x), G_{IV}(x) \right). \quad (B48)
\end{align*}
\]

By estimating these functions at \(|x| \gg 1\), we get

\[
S_{I}^{(1)}(x) \simeq -\frac{\eta}{iJ} [x + \gamma \text{sgn}(x) - \frac{iJ(1 - \eta)}{1 - J^2} [x + \nu \text{sgn}(x)],
\]

\[
G_{I}^{(1)}(x) \simeq \frac{1 - \eta}{2(1 - J^2)},
\]

with \(\nu = \int_0^\infty dx' \left[ A^{-3}(x') - 1 \right]\). On the other hand, we get

\[
S_{III}^{(1)}(x) \simeq \alpha_3 \left[ -2\eta(x + \lambda) - 2\eta \left( 1 - \eta \right) \frac{J^2}{1 - J^2} [x + \nu] \right].
\]

When we take

\[
\begin{align*}
(S_{I,III}^{\text{total}}(x), G_{I,III}^{\text{total}}(x)) &= \left( S_{I,III}^{(0)}(x), G_{I,III}^{(0)}(x) \right) + E \left( S_{I,III}^{(1)}(x), G_{I,III}^{(1)}(x) \right) + O(E^2), \quad (B53)
\end{align*}
\]

we find

\[
S_{I}^{\text{total}} = 1 + E \left[ \frac{J^2 - \eta}{iJ(1 - J^2)} x + \hat{\gamma} \text{sgn}(x) \right],
\]

with

\[
\hat{\gamma} = -\frac{1}{iJ} \left[ \eta \gamma - \frac{J^2(1 - \eta)}{1 - J^2} \nu \right].
\]

We also find

\[
\frac{S_{III}^{\text{total}}}{-2iJ\alpha_3} = \text{sgn}(x) + \frac{J^2 + \eta}{iJ(1 - J^2)} E|x| + \hat{\lambda} E,
\]

with

\[
\hat{\lambda} = \frac{1}{iJ} \left[ \eta \lambda + \frac{J^2(1 + \eta)}{1 - J^2} \nu \right].
\]

When we replace \(C_{III}\) by \(C_{III}/(-2iJ\alpha_3)\), because \(C_{III}\) is just a coefficient which we will determine later, we obtain

\[
\begin{align*}
\left( S(x), G(x) \right) &= C_I \left( S_{I,III}^{\text{total}}(x), G_{I,III}^{\text{total}}(x) \right) - C_{III} \frac{2iJ\alpha_3}{G_{III}^{\text{total}}(x)} \left( S_{III}^{\text{total}}(x), G_{III}^{\text{total}}(x) \right). \quad (B58)
\end{align*}
\]

Expanding \(C_{I,III}\) by energy \(E\), i.e., \(C_{I,III} = C_{I,III}^{(0)} + EC_{I,III}^{(1)} + O(E^2)\), we get the following behaviors at \(|x| \gg 1\),

\[
S = C_I^{(0)} + C_{III}^{(0)} \text{sgn}(x)
\]

\[
+ E[C_I^{(1)} + C_{III}^{(1)} \text{sgn}(x) + C_{III}^{(0)} \text{sgn}(x) + \lambda C_{III}^{(0)}]
\]

\[
+ E x[C_I^{(0)} J^2 - \eta \frac{iJ(1 - J^2)}{x} + C_{III}^{(0)} \alpha_3 J^2 + \eta \frac{iJ(1 - J^2)}{x} \text{sgn}(x)].
\]

\[
(B59)
\]

Since \(A_3\) in the critical current state is given by

\[
A_3(x) = \int_0^x dx' \frac{A_x(x')}{A^2(x')} = -\frac{1}{2} \frac{\partial}{\partial \varphi} \int_0^x dx' \left[ \frac{1}{A^2(x')} - 1 \right],
\]

and the phase difference is given by

\[
\varphi = \frac{J}{2} \int_0^\infty dx' \left[ \frac{1}{A^2(x')} - 1 \right],
\]

we get

\[
\alpha_3 = A_3(\infty) = -\frac{1}{2} \frac{\partial}{\partial \varphi} \left( \frac{\varphi}{2J} \right) = -\frac{1}{4J}.
\]

Therefore, the factor \(-2iJ\alpha_3\) can be reduced into \(-2iJ\alpha_3 = i/2\). We also find that \(\eta = O(J)\) holds.

So far, we have assumed that the wave function in the low-energy regime starts with \(O(1)\) with respect to the energy. However, according to \(B58\), it starts with \(O(k^{-1/2})\). Actually, \(S\) in the low-energy regime reads \(S \propto \sqrt{2/k}\). We find \(G \propto \sqrt{k/2}\), but this is true only for the uniform system. According to \(B58\), \(G(x)\) in the critical current state starts with the same order as \(S(x)\) with respect to \(E\). Therefore, when we calculate physical quantities, we must multiply \(B58\) by the factor \(\sqrt{2/k}\).

Since \(\lim_{E \to 0} G_{\text{total}}(x) = 0\) and \(\lim_{E \to 0} C_{I,III}^{\text{total}}(x) = B(x) = A_\varphi(x)\) hold at the critical current, \(G(x)\) in the low-energy limit is given by

\[
\lim_{E \to 0} G(x) \simeq \sqrt{\frac{2}{k}} \frac{C_{I,III}^{(0)}}{k} A_\varphi(x)
\]

\[
= -\frac{2\sqrt{2}}{\sqrt{k}} C_{I,III}^{(0)} A_\varphi(x),
\]

where the factor \(\sqrt{2/k}\) comes from the reason discussed already.

**Appendix C: Evaluation of local density spectral function in the Bogoliubov theory in the presence of an impurity potential**

In this section, we evaluate the local density spectral function using the tunneling solutions. The excited state
is characterized by the incident momentum $k_{\text{in}}$. The local density spectral function is given by

$$I_n(x, \omega) = n_0(x) \int \frac{dk}{(2\pi)^d} |G(x; k)|^2 \times \delta \left( \omega - J \cdot k - \sqrt{\frac{k^2}{2} \left( \frac{k^2}{2} + 2 \right)} \right).$$  \hfill (C1)

To obtain $G(x; k)$, we first solve the equation

$$\omega = Jk \cos \theta + \sqrt{\frac{k^2}{2} \left( \frac{k^2}{2} + 2 \right)},$$  \hfill (C2)

with a given excitation energy $\omega$ and a given incident angle $\theta$, which gives the zero of the equation in the delta-function in $I_n$. This equation can be recast into $|G|^2$, where the solution $k = k_{\text{in}}$ we are considering is the modulus of $k_{\text{in}}$, i.e., $k_{\text{in}} > 0$ and $k_{\text{in}} \in \mathbb{R}$. Using the incident angle $\theta$ and the solution $k_{\text{in}}$ we obtained, we set $k_{\perp} = k_{\text{in}} \sin \theta$. By solving

$$k_{\perp}^4 + (2k_{\perp}^2 + 4 - 4J^2)k_{\perp}^2 + 8EJk_{\perp} + k_{\perp}^4 + 4k_{\perp}^2 - 4\omega^2 = 0,$$  \hfill (C3)

we obtain four solutions of $k_{\perp}$. One comes from the energy spectrum

$$\omega = Jk_{\perp} + \sqrt{\frac{k_{\perp}^2 + k_{\perp}^2}{2} \left( \frac{k_{\perp}^2 + k_{\perp}^2}{2} + 2 \right)}.$$  \hfill (C4)

Four solutions are as follows. One is a real solution satisfying $k_{\perp}^{(1)} = k_{\text{in}} \cos \theta$. Another is the other real solution $k_{\perp}^{(2)}$. The others are two imaginary solutions $k_{\perp}^{+}$ and $k_{\perp}^{-}$, where $\text{Im}(k_{\perp}^{(1)}) > 0$ and $\text{Im}(k_{\perp}^{(2)}) < 0$. These are related through $(k_{\perp}^{(1)})^* = k_{\perp}^{(2)}$, because coefficients in (C3) are real.

By setting $E$ and $\theta$, and solving (C3) with the boundary conditions (127) and (128), we obtain coefficients $t, r, a$ and $b$ and also the profiles of $S(x; k_{\text{in}})$ and $G(x; k_{\text{in}})$. We have a remark with respect to the reflection. One of relations between solutions and coefficients with respect to (C3) is given by

$$k_{\perp}^{(1)}k_{\perp}^{(2)}k_{\perp}^{+}k_{\perp}^{-} = k_{\perp}^4 + 4k_{\perp}^2 - 4E^2.$$  \hfill (C5)

Since $(k_{\perp}^{(1)})^* = k_{\perp}^{(2)}$ holds, the left hand side of (C5) is $k_{\perp}^{(1)}k_{\perp}^{(2)}|k_{\perp}^{(2)}|^2$. When the real solutions $k_{\perp}^{(1)}$ and $k_{\perp}^{(2)}$ have the same sign, we have no reflection wave, i.e., the double refraction could occur. This condition can be reduced to

$$E < \sqrt{\frac{k_{\perp}^2}{2} \left( \frac{k_{\perp}^2}{2} + 2 \right)},$$  \hfill (C6)

We found that the region satisfying Eq. (C6) is very narrow, which exists around $\theta = \pi/2$. In this case, we change the boundary conditions from (127) and (128) to the proper ones

$$S(x) \quad \begin{cases} \frac{\alpha_{k_{\parallel}^{(1)}}}{\beta_{k_{\parallel}^{(1)}}} e^{ik_{\parallel}^{(1)}x} + a \frac{\alpha_{k_{\parallel}^{(2)}}}{\beta_{k_{\parallel}^{(2)}}} e^{ik_{\parallel}^{(2)}x} & (x \to \mp \infty), \\ \frac{\alpha_{k_{\parallel}^{(1)}}}{\beta_{k_{\parallel}^{(1)}}} e^{ik_{\parallel}^{(1)}x} + r \frac{\alpha_{k_{\parallel}^{(2)}}}{\beta_{k_{\parallel}^{(2)}}} e^{ik_{\parallel}^{(2)}x} & (x \to \pm \infty). \end{cases}$$  \hfill (C7)

In the one dimensional case, the density spectral function is given by

$$I_n(x) = \frac{n_0(x)}{2\pi} \int \frac{dk}{\theta = 0, \pi} \int_0^\infty \int \delta(k - k_{\text{in}}(\theta)) |G(x; k)|^2$$  \hfill (C9)

$$= \frac{n_0(x)}{2\pi} \sum_{\theta = 0, \pi} \frac{1}{v_g(k_{\text{in}})} |G(x; k_{\text{in}})|^2,$$  \hfill (C10)

where $k_{\text{in}}$ is a function of an incident angle $\theta$, i.e., $k_{\text{in}}(\theta)$. Here, $v_g(k_{\text{in}})$ is given by

$$v_g(k_{\text{in}}) = \frac{\partial E}{\partial k_{\text{in}}} = J \cos \theta + \frac{k_{\text{in}}(k_{\text{in}})^2 + 2}{2(\omega - Jk_{\text{in}} \cos \theta)}.$$  \hfill (C11)

The density spectral function in the two-dimensional case is given by

$$I_n(x) = \frac{n_0(x)}{(2\pi)^2} \int d\theta \int_0^\infty dk k \delta(k - k_{\text{in}}(\theta)) |G(x; k)|^2$$  \hfill (C12)

$$= \frac{2n_0(x)}{(2\pi)^2} \int d\theta \frac{k_{\text{in}}(\theta)}{v_g(k_{\text{in}})} |G(x; k_{\text{in}})|^2.$$  \hfill (C13)

The density spectral function in the three-dimensional case is given by

$$I_n(x) = \frac{n_0(x)}{(2\pi)^3} \int d\theta \sin \theta \int_0^\infty dk \times k^2 \frac{\partial \delta(k - k_{\text{in}}(\theta))}{\partial E/\partial k} |G(x; k)|^2$$  \hfill (C14)

$$= \frac{2\pi n_0(x)}{(2\pi)^3} \int d\theta \sin \theta \frac{k_{\text{in}}(\theta)^2}{v_g(k_{\text{in}})} |G(x; k_{\text{in}})|^2.$$  \hfill (C15)

The density spectral function in the $d$-dimensional system is thus summarized as

$$I_n(x, \omega) = n_0(x) \int \frac{dk}{(2\pi)^d} \frac{|G(x; k)|^2}{v_g(k)} \delta(k - k_{\text{in}}).$$  \hfill (C16)
Appendix D: Local density spectral function in the critical current state for soliton instability

We consider the local density spectral function in the critical current state in the presence of a repulsive potential barrier.

We start with the one-dimensional system. When the incident excitation is the right-moving one, we find

\[ k_1^{(1)} = \frac{E}{1 + J}, \quad k_2^{(1)} = \frac{E}{1 - J}. \]  

(D.1)

Expanding the boundary conditions with \( E \) and coefficients as \( t \approx t^{(0)} + Et^{(1)} + O(E^2) \), and \( r \approx r^{(0)} + Er^{(1)} + O(E^2) \), we get

\[ S(x) = \exp[ik_1^{(1)}x] + r \exp[ik_2^{(1)}x] \]

\[ \simeq 1 + r^{(0)} + Er^{(1)} + Ex \left[ \frac{i}{1 + J} + \frac{ir^{(0)}}{-1 + J} \right] \quad (x \ll -1), \]

(D.2)

\[ S(x) = t \exp[ik_1^{(1)}x] \]

\[ \simeq t^{(0)} + Et^{(1)} + Ex \left[ \frac{it^{(0)}}{1 + J} \right] \quad (x \gg 1). \]

(D.3)

Comparing coefficients in [B59] with those in the above equations, we end with

\[ \begin{pmatrix} t^{(0)} \\ r^{(0)} \end{pmatrix} = \begin{pmatrix} \frac{2J\eta}{\eta^2 + J^2} \\ \frac{J^2 - \eta^2}{\eta^2 + J^2} \end{pmatrix}, \quad \begin{pmatrix} C_1^{(0)} \\ C_III^{(0)} \end{pmatrix} = \begin{pmatrix} \frac{J(J + \eta)}{J^2 + \eta^2} \\ \frac{J^2 + \eta^2}{J(J + \eta)} \end{pmatrix}. \]

(D.4)

The coefficients in the case of the right-moving excitation \( C_{III,R}^{(0)} \), and the left-moving excitation \( C_{III,L}^{(0)} \) can be summarized as

\[ C_{III,L}^{(0)} = \frac{J(J - \eta)}{J^2 + \eta^2}, \quad C_{III,R}^{(0)} = \frac{J(J + \eta)}{J^2 + \eta^2}. \]

(D.13)

As a result, the local density spectral function in the one-dimensional system is

\[ \rho(\omega) = \frac{n_0(x)}{2\pi} \int_{-\infty}^{0} dk_x \left[ \frac{2}{|k_x|^2 - 2iJ\alpha} A_\varphi(x) \right]^2 \]

\[ \times \delta(\omega - |k_x| - Vk_x) \]

\[ + \frac{n_0(x)}{2\pi} \int_{0}^{\infty} dk_x \left[ \frac{2}{|k_x|^2 - 2iJ\alpha} A_\varphi(x) \right]^2 \]

\[ \times \delta(\omega - |k_x| - Vk_x) \]

\[ \times \frac{2J(J + \eta)}{|i(J^2 + \eta^2)|} \]

\[ \left( \varphi - \varphi_0(x) \right)^2, \]

(D.17)

where we used \( n_0(x) = A^2(x) \).

Now, we consider the two and three-dimensional systems. In the low-energy regime, the energy spectrum is given by

\[ E \simeq Jk^2 \cos \theta + k^2. \]

(D.18)

As a result, we get

\[ k^2 = \frac{E}{1 + J \cos \theta}, \quad k_\perp = \frac{E \cos \theta}{1 + J \cos \theta}, \]

(D.19)

and \( k_\perp = k \sin \theta \). In the low-energy regime, we also get

\[ E = Jk_\parallel + \sqrt{k_\parallel^2 + k_\perp^2}, \]

(D.20)

which can be reduced into

\[ k_\parallel^2 + \frac{2JE}{1 - J^2} k_\parallel + \frac{k_\perp^2 - E^2}{1 - J^2} = 0. \]

(D.21)

Finally, we obtain

\[ k_\parallel = -\frac{JE}{1 - J^2} \pm \frac{1}{1 - J^2} \sqrt{E^2 - (1 - J^2)k_\perp^2}. \]

(D.22)

When the potential barrier is strong, the critical current is small, i.e., \( J = J_c \ll 1 \). In this case, we get

\[ k_\parallel \simeq -JE \pm E \sqrt{1 - \sin^2 \theta} = -JE \pm E |\cos \theta|. \]

(D.23)

We also consider the low-energy regime, so that we take \( JE \ll 1 \). In this case, we end with

\[ k_\parallel \simeq \pm E |\cos \theta|. \]

(D.24)
The incident and reflection momenta $k_{||}^{(1)}$ and $k_{||}^{(2)}$ are
\[ k_{||}^{(1)} = +E \cos \theta, \quad k_{||}^{(2)} = -E \cos \theta. \] (D25)

Let $\alpha_{\pm}$ be $\pm \cos \theta$. When an incident excitation is right-moving one, i.e., $0 \leq \theta < \pi/2$, $S(x)$ reads as
\[
S(x) = \exp [ik_{||}^{(1)} x] + r \exp [ik_{||}^{(2)} x] \
\simeq 1 + r^{(0)} + Ex[i\alpha_+ + \alpha_- r^{(0)}] \quad (x \ll -1),
\]
\[
S(x) = t \exp [ik_{||}^{(1)} x] \
\simeq t^{(0)} + Er^{(1)} + t^{(0)} iE\alpha_+ x \quad (x \gg 1).\] (D29)

Comparing coefficients in (B59) with those in the above equations, we end with
\[
\begin{pmatrix}
(t^{(0)}) \\
(r^{(0)})
\end{pmatrix} = \begin{pmatrix}
\frac{-\eta(A_+ - A_-)}{A_+ A_- - \eta^2} \\
\frac{\eta^2 - A_+^2}{A_+ A_- - \eta^2}
\end{pmatrix},
\]
\[
\begin{pmatrix}
(C_{1}^{(0)}) \\
(C_{11}^{(0)})
\end{pmatrix} = \begin{pmatrix}
\frac{-\eta (A_+ + A_-)}{2(A_+ A_- - \eta^2)} \\
\frac{-\eta (A_+ - A_-)}{2(A_+ A_- - \eta^2)}
\end{pmatrix},
\]
with $A_{\pm} = J^2 + J(1 - J^2) \alpha_{\pm}$.

When an incident excitation is left-moving one, i.e., $\pi/2 < \theta \leq \pi$, $S(x)$ reads as
\[
S(x) = \exp [ik_{||}^{(1)} x] \
\simeq t^{(0)} + Er^{(1)} + t^{(0)} iE\alpha_+ x \quad (x \ll -1),
\]
\[
S(x) = \exp [ik_{||}^{(1)} x] + r \exp [ik_{||}^{(2)} x] \
\simeq 1 + r^{(0)} + Ex[i\alpha_+ + \alpha_- r^{(0)}] \quad (x \gg 1).\] (D35)

Comparing coefficients in (B59) with those in the above equations, we end with
\[
\begin{pmatrix}
(t^{(0)}) \\
(r^{(0)})
\end{pmatrix} = \begin{pmatrix}
\frac{-\eta(A_+ - A_-)}{A_+ A_- - \eta^2} \\
\frac{\eta^2 - A_+^2}{A_+ A_- - \eta^2}
\end{pmatrix},
\]
\[
\begin{pmatrix}
(C_{1}^{(0)}) \\
(C_{11}^{(0)})
\end{pmatrix} = \begin{pmatrix}
\frac{-\eta (A_+ + A_-)}{2(A_+ A_- - \eta^2)} \\
\frac{-\eta (A_+ - A_-)}{2(A_+ A_- - \eta^2)}
\end{pmatrix},
\]
\[ -A_- \simeq J \cos \theta. \]

As a result, we end with
\[
\begin{pmatrix}
(t^{(0)}) \\
(r^{(0)})
\end{pmatrix} = \begin{pmatrix}
\frac{2\eta J \cos \theta}{J^2 \cos^2 \theta + \eta^2} \\
\frac{J^2 \cos^2 \theta - \eta^2}{J^2 \cos^2 \theta + \eta^2}
\end{pmatrix},
\]
\[
\begin{pmatrix}
(C_{1}^{(0)}) \\
(C_{11}^{(0)})
\end{pmatrix} = \begin{pmatrix}
\frac{J \cos \theta (\eta + J \cos \theta)}{J^2 \cos^2 \theta + \eta^2} \\
\frac{J^2 \cos^2 \theta - \eta^2}{J^2 \cos^2 \theta + \eta^2}
\end{pmatrix},
\]
where the upper sign is for $0 \leq \theta < \pi/2$, and the lower sign for $\pi/2 \leq \theta \leq \pi$.

In the tunneling problem, transmission and reflection coefficients $T$ and $R$ are
\[
T = |t^{(0)}|^2 = \frac{4\eta J^2 \cos^2 \theta}{(J^2 \cos^2 \theta + \eta^2)^2},
\]
\[
R = 1 - |t^{(0)}|^2 = \frac{(J^2 \cos^2 \theta - \eta^2)^2}{(J^2 \cos^2 \theta + \eta^2)^2}.\] (D41)

We find the perfect transmission at $\cos^2 \theta = \pm \eta/J$.

Since the spectral function in the $d$-dimensional system is given by
\[
I_n(x, \omega) \simeq n_0(x) \int \frac{dk}{(2\pi)^d} |G^{(0)}(x; k)|^2 \delta(\omega - k - kJ \cos \theta),
\]
and we have
\[
|G^{(0)}(x; k)|^2 = 8A^2(x) \frac{1}{k^2} |C_{111}^{(0)}|^2,
\]
\[
I_n(x, \omega) \text{ can be reduced into}
\]
\[
I_n(x, \omega) \simeq 2(\partial_\omega n_0(x))^2 W(\omega),\] (D44)

where
\[
W(\omega) = \int \frac{dk}{(2\pi)^d} \frac{1}{k^2} |C_{111}^{(0)}|^2 \delta(\omega - k - kJ \cos \theta).
\] (D45)

Here, we used $n_0(x) = A^2(x)$.

In the 2-dimensional system, we get
\[
W(\omega) \simeq 2 \int_0^{\infty} \frac{dk}{(2\pi)^2} k \int_0^\pi d\theta \frac{1}{k^2} |C_{111}^{(0)}|^2 \times \frac{1}{1 + \frac{\omega}{1 - \frac{k}{J \cos \theta}}}
\]
\[
\simeq \frac{2}{(2\pi)^2} \int_0^\pi d\theta \frac{J^2 \cos^2 \theta (\eta - J \cos \theta)^2}{(J^2 \cos^2 \theta + \eta^2)^2}
\]
\[
= \frac{1}{2\pi} \left( 1 - \frac{\eta}{\sqrt{J^2 + \eta^2}} \right),
\]
where we used $J \ll 1$. As a result, we end with
\[
I_n(\omega) \simeq \frac{1}{\pi} \left( 1 - \frac{\eta}{\sqrt{J^2 + \eta^2}} \right) [\partial_\omega n_0(x)]^2.\] (D49)
In the 3-dimensional system, we get

$$ W(\omega) \simeq 2\pi \int_0^\infty \frac{dk}{(2\pi)^2} k^2 \int_0^\pi d\theta \sin \theta \frac{1}{k} |C_{III}^{(0)}|^2 $$

\[ \times \frac{\omega}{1 + J \cos \theta} \left( k - \frac{\omega}{1 + J \cos \theta} \right) \] (D50)

\[ \simeq \frac{2\pi}{(2\pi)^3} \omega \int_0^\pi d\theta \sin \theta \frac{J^2 \cos^2 \theta (\eta - J \cos \theta)^2}{(J^2 \cos^2 \theta + \eta^2)^2} \] (D51)

\[ = \frac{\omega}{2\pi^2} \left[ 1 - \frac{\eta}{J} \tan^{-1} \left( \frac{J}{\eta} \right) \right]. \] (D52)

As a result, we end with

$$ I_n(\omega) \simeq \frac{\omega}{\pi^2} \left[ 1 - \frac{\eta}{J} \tan^{-1} \left( \frac{J}{\eta} \right) \right] \left[ \partial_x n(x) \right]^2. \] (D53)

We thus obtain \[ \text{(E32)}. \]

**Appendix E: Density spectral function in the Bogoliubov theory for uniform system at } v < v_c \]

We evaluate the density spectral function for the stable superfluid \( v < v_c \) within the Bogoliubov theory in the uniform system. In the low-energy regime, we have \( |G|^2 = k/2 \), and the local density spectral function is given by

$$ I_n(\omega) \simeq \int_0^{\infty} \frac{dk}{(2\pi)^d} |G|^2 \delta(\omega - k - vk_x). \] (E1)

We first discuss the 1-dimensional system. The spectral function is reduced into

$$ I_n(\omega) \simeq \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{|k_x|}{2} \delta(\omega - |k_x| - vk_x) \] (E2)

\[ = \int_0^{\infty} \frac{dk_x}{2\pi} \frac{k_x}{2} \delta(\omega + k_x - vk_x) + \int_0^{\infty} \frac{dk_x}{2\pi} \frac{k_x}{2} \delta(\omega - k_x - vk_x) + \frac{\omega}{2\pi} \frac{1 + v^2}{(1 - v^2)^2}. \] (E3)

The spectral function in the 2-dimensional system is evaluated as follows.

$$ I_n(\omega) \simeq \int_0^{\infty} \frac{dk}{(2\pi)^2} \int_0^\pi d\theta \sin \theta \frac{1}{k} |C_{II}^{(0)}|^2 $$

\[ \times \frac{\omega^2}{8\pi^2} \left( 1 + J \cos \theta \right)^{-1} \left( \omega - \frac{\omega}{1 + J \cos \theta} \right) \] (E5)

\[ = \frac{\omega^2}{8\pi^2} \int_0^\pi d\theta \sin \theta \frac{1}{(1 + v \cos \theta)^3} \] (E6)

\[ = \frac{\omega^2}{8\pi} \frac{2 + v^2}{(1 - v^2)^{3/2}}. \] (E7)

The spectral function in the 3-dimensional system is evaluated as follows.

$$ I_n(\omega) \simeq \int_0^{\infty} \frac{dk}{(2\pi)^3} \int_0^\pi d\theta \sin \theta \frac{1}{k} |C_{III}^{(0)}|^2 $$

\[ \times \frac{\omega^3}{8\pi^3} \left( \omega - \frac{\omega}{1 + J \cos \theta} \right) \] (E8)

\[ = \frac{\omega^3}{8\pi^3} \frac{2 + v^2}{(1 - v^2)^3}. \] (E9)

\[ = \frac{\omega^3}{12\pi^2} \left( 1 + v \cos \theta \right)^{-1} \] (E10)

**Appendix F: Density spectral function in the Bogoliubov theory for the uniform system at } v = v_c \]

We evaluate the density spectral function in the critical current state \( v = v_c \) within the Bogoliubov theory in the uniform system. The local density spectral function is given by

$$ I_n(\omega) \simeq \int_0^{\infty} \frac{dk}{(2\pi)^d} |G|^2 \delta(\omega - E), \] (F1)

where the energy \( E \) is

$$ E = vk_x + \frac{k}{2} \sqrt{k^2 + 4}. \] (F2)

In the critical current state, we have \( v = 1(= v_c) \).

We first discuss the 1-dimensional system. Using \( |G|^2 = k/2 \) in the low-energy regime, we can reduce the spectral function into

$$ I_n(\omega) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{k_x}{2} \delta \left( \omega - k_x - \frac{k}{2} \sqrt{k^2 + 4} \right), \] (F3)

where \( k = |k_x| \). The equation in the delta-function is zero when \( \omega - (k/2) \sqrt{k^2 + 4}/k = \pm 1 \) holds. Let \( \pm f_{\pm}(\omega) \) be the solutions of \( \omega = f_{\pm}(k) \), with

$$ f_{\pm}(k) = \pm k + \frac{k}{2} \sqrt{k^2 + 4}. \] (F4)

Using these functions, we get

$$ I_n(\omega) = \sum_{j=\pm} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{|k_x|}{2} \delta(k_x - k_j(\omega)) \left| \frac{\partial f_j}{\partial k} \right|^{-1}_{k=k_j(\omega)} \] (F5)

\[ = \frac{1}{4\pi} \sum_{j=\pm} k_j(\omega) \left| \frac{\partial f_j}{\partial k} \right|^{-1}_{k=k_j(\omega)}, \] (F6)

\[ = \frac{1}{8\pi} \sum_{j=\pm} \frac{\partial k_j^2(\omega)}{\partial \omega}, \] (F7)

where we used

$$ \left| \frac{\partial f_{\pm}}{\partial k} \right|^{-1} = \frac{\partial k_{\pm}(\omega)}{\partial \omega}. \] (F8)
In the low-momentum regime, we get
\[ f_\pm(k) \simeq \pm k + k + \frac{1}{8}k^3. \] (F9)
Therefore, \( k_\pm \) are given by
\[ k_+ \simeq \frac{\omega}{2}, \quad k_- \simeq 2\omega^{1/3}. \] (F10)
As a result, in the low-energy regime \( \omega \ll 1 \), we end with
\[ \mathcal{I}_n(\omega) \simeq \frac{1}{3\pi} \omega^{-1/3}. \] (F11)

The spectral function in the 2-dimensional system is evaluated as follows
\[
\mathcal{I}_n(\omega) = 2 \int_0^\infty \frac{dk}{(2\pi)^2} \int_0^\pi d\theta \frac{k^2}{2} \times \delta \left( \omega - k \cos \theta - k \sqrt{k^2 + 4} \right). \quad \text{(F12)}
\]
The equation in the delta-function is zero when
\[ \frac{|\omega - (k/2)\sqrt{k^2 + 4}|}{k} = |\cos \theta| \leq 1. \] (F13)
This means that the range of \( k \) is \( k_+ \leq k \leq k_- \). Let \( \theta_0 \) be
\[ \cos \theta_0 = \frac{\omega}{k} \frac{1}{2}\sqrt{k^2 + 4}. \] (F14)
If we write
\[ f = \omega - k \cos \theta - k \sqrt{k^2 + 4}, \] (F15)
we get \( |\partial f/\partial \theta|_{\theta=\theta_0} = |k \sin \theta_0| \). As a result, the density spectral function is given by
\[
\mathcal{I}_n(\omega) = \int_0^\infty \frac{dk}{(2\pi)^2} \int_0^\pi d\theta \frac{k^2}{2} \left[ \frac{1}{|k \sin \theta_0|} \delta(\theta - \theta_0) \right] \quad \text{(F16)}
\]
\[
= \int_{k_+}^{k_-} \frac{dk}{(2\pi)^2} \frac{1}{k^2} \left[ \frac{1}{\sin \theta_0} \delta(\theta - \theta_0) \right] \quad \text{(F17)}
\]
\[
= \frac{1}{4\pi^2} \int_{k_+}^{k_-} \frac{dk}{k^2} \delta \left( \omega - f_+(k) \right) \quad \text{(F18)}
\]
The singularity in integrand exists at \( k \simeq k_+(\omega) \) and \( k \simeq k_- (\omega) \). We first consider the lower bound of the integration. When
\[ \omega = k_+ + \frac{k_+}{2} \sqrt{k_+^2 + 4}, \] (F19)
we get
\[ f_+(k) - \omega \simeq 2(k - k_+), \quad \omega - f_-(k) \simeq 2k_+. \] (F20)
where we used \( k \simeq k_+ \). As a result, the integrand of (F18) is reduced into
\[
\frac{k^2}{\sqrt{k^2 - (\omega - k \sqrt{k^2 + 4})}} \simeq \frac{k_+^{3/2}}{2\sqrt{k - k_1}}. \quad \text{(F21)}
\]
Introducing a proper cutoff \( \Lambda_\pm = O(\omega) \), we get
\[
\int_{k_+(\omega)}^{\Lambda_+} \frac{k_+^{3/2}dk}{2\sqrt{k - k_+}} \simeq \left[ k_+^{3/2} \sqrt{k - k_+} \right]_{k_+(\omega)}^{\Lambda_+} \simeq O(\omega^2). \quad \text{(F22)}
\]
We now consider the upper bound of the integration. When
\[ \omega = -k_+ - \frac{k_+}{2} \sqrt{k_+^2 + 4}, \] (F23)
we get
\[ f_+(k) - \omega \simeq 2k_-, \quad \omega - f_-(k) \simeq 3k_2^2 (k_- - k). \] (F24)
As a result, the integrand of (F18) is reduced into
\[
\frac{k^2}{\sqrt{k^2 - (\omega - k \sqrt{k_2^2 + 4})}} \simeq \frac{2}{\sqrt{3} \sqrt{k_-}} \frac{k_-}{k_- - k}. \quad \text{(F25)}
\]
Introducing a proper cutoff \( \Lambda_- = O(\omega^{1/3}) \), we get
\[
\int_{k_- (\omega)}^{\Lambda_-} \frac{k_-dk}{\sqrt{k_- (k_- - k)}} \simeq \sqrt{k_-} \left[ -2\sqrt{k_- - k} \right]_{k_- (\omega)}^{\Lambda_-} \simeq 2k_-, \quad \text{(F26)}
\]
where we used \( k_- = O(\omega^{1/3}) \).

We finally obtain the \( \omega \)-dependence of the density spectral function in the low-energy regime as
\[
\mathcal{I}_n(\omega) \simeq \frac{k_-}{\sqrt{3\pi^2}} \frac{1}{\omega^{1/3}}. \quad \text{(F28)}
\]

The spectral function in the 3-dimensional system is evaluated as follows
\[
\mathcal{I}_n(\omega) \simeq \frac{1}{(2\pi)^3} \int_0^\infty \frac{dk}{2\pi} \int_0^1 d(\cos \theta) \times \frac{k^3}{2} \delta \left( \omega - k \cos \theta - k \sqrt{k^2 + 4} \right) \quad \text{(F29)}
\]
\[
= \frac{1}{(2\pi)^3} \int_{k_+ (\omega)}^{k_- (\omega)} \frac{k^3}{2} \frac{1}{k} \quad \text{(F30)}
\]
\[
= \frac{1}{24\pi^2} (k^3 - k^3_+). \quad \text{(F31)}
\]
Since \( k_\pm \) in the low-energy regime are given in (F10), the low-energy behavior of \( \mathcal{I}_n(\omega) \) reads as
\[
\mathcal{I}_n(\omega) \simeq \frac{1}{3\pi^2} \omega. \quad \text{(F32)}
\]
Appendix G: Spectral functions in the Feynman’s single-mode approximation

We evaluate the local density spectral function in the $d$-dimensional system within the Feynman’s single-mode approximation

$$I_n(\omega) = \int \frac{dq}{(2\pi)^d} \frac{q^2}{2E_q} \delta(\omega - E_q - vq_x). \quad (G1)$$

For low $q = |q|$, we suppose that the energy spectrum is given by

$$E_q \approx c_1 q + c_3 q^3 + O(q^5), \quad (G2)$$

where $c_1$ and $c_3$ are positive coefficients.

We first evaluate the 1-dimensional system, where the spectral function is given by

$$I_n(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq_x \frac{q^2}{2E_{q_x}} \delta(\omega - E_{q_x} - vq_x). \quad (G3)$$

We use the same technique in Appendix E. Let $k_\pm(\omega)$ be solutions of

$$\omega = E_q \pm vq = f_\pm(q). \quad (G4)$$

In this case, we get

$$I_n(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq_x \sum_{j=\pm} \frac{q^2}{2E_{q_x}} \delta(q_x - jk_j) \left| \frac{\partial f_j}{\partial q} \right|^{-1}_{q=k_j}$$

$$= \frac{1}{4\pi} \sum_{j=\pm} \frac{k^2(\omega)}{E_{k_j}(\omega)} \frac{dk_j(\omega)}{d\omega}, \quad (G5)$$

where we used

$$\left| \frac{\partial f_j}{\partial q} \right|^{-1}_{q=k_j(\omega)} = \frac{dk_j(\omega)}{d\omega}. \quad (G6)$$

If we suppose $c_1 q \gg c_3 q^3$, we get

$$E_q = c_1 q, \quad k_+(\omega) \approx \frac{\omega}{c_1 + v}. \quad (G7)$$

In this case, we end with

$$I_n(\omega) = \frac{1}{8\pi c_1} \sum_{j=\pm} \frac{dk^2(\omega)}{d\omega}. \quad (G8)$$

When $(c_1 - v)q \ll c_3 q^3$, i.e., $q_- \ll q$ with $q_- = \sqrt{(c_1 - v)/c_3}$, we get

$$k_-(\omega) \approx \left( \frac{\omega}{c_3} \right)^{1/3}. \quad (G9)$$

The condition $q_- \ll q$ can be reduced to $\omega_- \ll \omega$ with $\omega_- = \sqrt{(c_1 - v)^3/c_3}$.

On the other hand, when $(c_1 - v)q \gg c_3 q^3$, i.e., $q_- \gg q$, we get

$$k_-(\omega) \approx \frac{\omega}{c_1 - v}. \quad (G10)$$

The condition $q_- \gg q$ can be reduced to $\omega_- \gg \omega$. As a result, when $\omega_- \gg \omega$, we get

$$I_n(\omega) = \frac{1}{8\pi c_1} \frac{d}{d\omega} \left[ \left( \frac{\omega}{c_1 - v} \right)^2 + \left( \frac{\omega}{c_1 + v} \right)^2 \right]$$

$$= \frac{\omega}{2\pi c_1 (c_1^2 + v^2)}. \quad (G11)$$

On the other hand, when $\omega_+ \ll \omega \ll \omega_+$, with $\omega_+ = \sqrt{c_1^2/c_3}$, we get

$$I_n(\omega) = \frac{1}{12\pi c_1} \frac{\omega^{1/3}}{c_3^{2/3}}. \quad (G12)$$

We evaluate the spectral function in the 2-dimensional system with

$$I_n(\omega) = \frac{1}{(2\pi)^2} \int dq_1 dq_2 \delta(\omega - E_{q_1} - vq_1 \cos \theta), \quad (G13)$$

The condition where the equation in the delta-function is zero is given by

$$\left| \frac{\omega - E_{q_1}}{vq_1} \right| \leq 1. \quad (G14)$$

The condition can be reduced into $k_+(\omega) \leq q \leq k_-(\omega)$. Then, we get

$$I_n(\omega) = \frac{2}{(2\pi)^2} \int_{k_+(\omega)}^{k_-(\omega)} dq_1 dq_2 \delta(\theta - \theta_0), \quad (G15)$$

where $\theta_0$ satisfies $\cos \theta_0 = (\omega - E_{q_1})/(vq_1 \sin \theta)$. As a result, we get

$$I_n(\omega) = \frac{1}{4\pi^2 c_1} \int_{k_+}^{k_-} dq_1 \frac{q^2}{\sqrt{(f_+ - \omega)(\omega - f_-)}}, \quad (G16)$$

where we used $vq \sin \theta_0 = \sqrt{(f_+ - \omega)(\omega - f_-)}$

When $\omega \ll \omega_-$, $k_\pm \approx \omega/(c_1 \pm v)$ holds. As a result, we get

$$I_n(\omega) = \frac{\omega^2}{4\pi^2 c_1} \int_{1/(c_1 - v)}^{1/(c_1 + v)} \frac{x^2 dx}{\sqrt{(c_1 + v)x - 1][1 - (c_1 - v)x]}}, \quad (G17)$$

$$= \frac{\omega^2}{8\pi c_1 (c_1^2 - v^2)^{3/2}}. \quad (G18)$$
When $\omega_- \ll \omega \ll \omega_+$, the main contribution to the integral comes from $q \simeq k_-(\omega)$. In this case, we get $\omega \simeq E_{k_- - vk_-}$, and

$$f_+ - \omega \simeq 2vk_-, \quad \omega - f_- \simeq \frac{\partial f_-}{\partial q} \bigg|_{k_-} (k_- - q). \quad (G22)$$

Introducing a proper cutoff $\Lambda = \mathcal{O}(\omega)$, and using

$$\left. \frac{\partial f_-}{\partial q} \right|_{k_-} = \frac{\partial \omega}{\partial k_-}, \quad (G23)$$

we obtain

$$\mathcal{I}_n(\omega) \simeq \frac{1}{4\pi^2 c_1} \int_{\Lambda}^{k_-} \frac{q^2 dq}{\sqrt{2vk_-}} \frac{\partial f_-}{\partial q} \bigg|_{k_-} (k_- - q) \quad (G24)$$

$$\simeq \frac{1}{4\pi^2 c_1 \sqrt{2v}} \int_{\Lambda}^{k_-} dq \sqrt{k_-/2} \frac{1}{\omega - \sqrt{k_- - \Lambda}} \quad (G25)$$

$$= \frac{1}{4\pi^2 c_1 \sqrt{2v}} \sqrt{k_-/2} \frac{1}{\omega - \sqrt{k_- - \Lambda}} \quad (G26)$$

$$\simeq \frac{1}{4\pi^2 c_1} \sqrt{2v} \sqrt{k_-/2}. \quad (G27)$$

Since $k_- \simeq (\omega/c_3)^{1/3}$, we end with

$$\mathcal{I}_n(\omega) \simeq \frac{1}{4\pi^2 c_1} \sqrt{\frac{2}{3} \omega^{1/3}}. \quad (G28)$$

We evaluate the spectral function in the 3-dimensional system, which is given by

$$\mathcal{I}_n(\omega) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dq^2 dq}{2E_q} \int_{-\pi}^{\pi} d(cos \theta)$$

$$\times \frac{q^2}{2E_q} \delta(\omega - E_q - vq \cos \theta) \quad (G29)$$

$$= \frac{1}{(2\pi)^2} \int_{k_+}^{k_-} dq \frac{q^2}{2c_1 v}$$

$$= \frac{1}{24\pi^2 c_1} (k_-^3 - k_+^3). \quad (G30)$$

When $\omega \ll \omega_-$, we find

$$\mathcal{I}_n(\omega) = \frac{1}{24\pi^2 c_1} \left[ \frac{\omega^3}{(c_1 - v)^3} - \frac{\omega^3}{(c_1 + v)^3} \right] \quad (G32)$$

$$= \frac{\omega^3}{12\pi^2 c_1} \frac{3c_1^2 + v^2}{(c_1^2 - v^2)^3}. \quad (G33)$$

When $\omega_- \ll \omega \ll \omega_+$, we find

$$\mathcal{I}_n(\omega) = \frac{1}{24\pi^2 c_1} \left[ \frac{\omega^3}{c_3} - \frac{\omega^3}{(c_1 + v)^3} \right] \quad (G34)$$

$$\simeq \frac{1}{24\pi^2 c_1} \frac{\omega}{c_3}. \quad (G35)$$

Appendix II: Coarse-grained local density spectral function

In this appendix, we discuss the coarse-grained local density spectral function. The coarse-grained density fluctuation is given by

$$\delta \hat{n}_{CG}(r) = \int dr' f_a(r - r') \delta \hat{n}(r'), \quad (H1)$$

where we take $\int dr' f_a(r) = 1$, and $f_a(r) \simeq 0$ for $|r| \gg a$. One of functions satisfying the above conditions is $\theta(r)$. Within the Bogoliubov approximation, the matrix element is given by

$$\langle |\delta \hat{n}_{CG}(r, t)| g \rangle = \langle \Psi_0(r) u_l(r) - \Psi_0(r) v_l(r) \rangle e^{i(E_l - E_a)t} \quad (H2)$$

$$= A(x) G_l(r) e^{i(E_l - E_a)t}. \quad (H3)$$

Therefore, the matrix element of the coarse-grained density fluctuation is

$$\langle |\delta \hat{n}_{CG}(r, t)| g \rangle = \int dr' f_a(r - r') A(x') G_l(r') e^{i(E_l - E_a)t}. \quad (H4)$$

When the $y$ and $z$-directions have the translational invariance, we get

$$\alpha_l(r) \equiv \int dr' f_a(r - r') A(x') G_l(r') \quad (H6)$$

$$= \frac{1}{\pi^{3/2} a^3} \int dr' e^{-\frac{(x - r')^2}{2a^2}} A(x') G_l(x) e^{ik_y y + ik_z z} \quad (H7)$$

$$= \frac{\sqrt{\pi}}{\pi a} \alpha_l(x) e^{ik_y y + ik_z z} e^{-a^2 k^2_1/4}, \quad (H8)$$

with

$$\alpha_l(x) \equiv \int_{-\infty}^{\infty} dx' e^{-\frac{(x - x')^2}{\sigma^2}} A(x') G_l(x'). \quad (H9)$$

Here, we used $k_\perp = \sqrt{k_y^2 + k_z^2}$ and

$$\int dy' e^{ik_y y'} e^{-\frac{(y - y')^2}{\sigma^2}} = \sqrt{\pi} e^{ik_y y} e^{-a^2 k^2_1/4}. \quad (H10)$$

As a result, from [53], we end with

$$\mathcal{I}_{n_{CG}}(r, \omega) \quad (H11)$$

$$= \sum_l \langle |\delta \hat{n}_{CG}(r, t)| g \rangle^2 \delta(\omega - E_l + E_g) \quad (H12)$$

$$= \frac{1}{\pi^2 a^2} \sum_l e^{-a^2 k^2_1/2} |\alpha_l(x)|^2 \delta(\omega - E_l + E_g). \quad (H13)$$

The autocorrelation function is now given by

$$C_{n_{CG}}(r, t) = \int d\omega \mathcal{I}_{n_{CG}}(r, \omega) \cos(\omega t) \quad (H14)$$

$$= \frac{1}{\pi^2 a^2} \sum_l e^{-a^2 k^2_1/2} |\alpha_l(x)|^2 \cos[(E_l - E_g)t]. \quad (H15)$$
In the uniform system, we have $\alpha_l(x) = \alpha_l e^{ik_l x}$. Using (H10), we get $|\tilde{\alpha}_l(x)|^2 = |\alpha_l|^2 \pi \alpha^2 e^{-a^2 k_l^2/2}$. As a result, we end with

$$I_{nCG}(\omega) = \sum_l e^{-a^2 k_l^2/2} |\alpha_l|^2 \delta(\omega - E_l + E_g), \quad \text{(H16)}$$

$$C_{nCG}(t) = \sum_l e^{-a^2 k_l^2/2} |\alpha_l|^2 \cos((E_l - E_g)t). \quad \text{(H17)}$$

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