Quanta of Geometry and Unification

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Abstract

This is a tribute to Abdus Salam’s memory whose insight and creative thinking set for me a role model to follow. In this contribution I show that the simple requirement of volume quantization in space-time (with Euclidean signature) uniquely determines the geometry to be that of a noncommutative space whose finite part is based on an algebra that leads to Pati-Salam grand unified models. The Standard Model corresponds to a special case where a mathematical constraint (order one condition) is satisfied. This provides evidence that Salam was a visionary who was generations ahead of his time.
1 Introduction

In 1973 I got a scholarship from government of Lebanon to pursue my graduate studies at Imperial College, London. Shortly after I arrived, I was walking through the corridor of the theoretical physics group, I saw the name Abdus Salam on a door. At that time my information about research in theoretical physics was zero, and since Salam is an Arabic name, and the prime minister in Lebanon at that time was also called Salam, I knocked at his door and asked him whether he is Lebanese. He laughed and explained to me that he is from Pakistan. He then asked me why I wanted to study theoretical physics. I said the reason is that I love mathematics. He smiled and told me that I am in the wrong department. In June 1974, having finished the Diploma exams I asked Salam to be my Ph.D. advisor and he immediately accepted and gave me two preprints to read and to chose one of them as my research topic. The first paper was with Strathdee [1] on the newly established field of supersymmetry (a word he coined), and the other is his paper with Pati [2] on the first Grand Unification model, now known as the Pati-Salam model. Few days later I came back and told Salam that I have chosen supersymmetry which I thought to be new and promising. Little I knew that the second project will come back to me forty years later from studying the geometric structure of space-time, as will be explained in what follows. In this respect, Salam was blessed with amazing foresight. I realized this early enough. In 1974 I got him a metallic Plaque with Arabic Calligraphy engraving from Koran which translates ”We opened for thee a manifest victory”. Indeed soon after the year 1974, the Salam’s path to victory and great fame has already started. His influence on me, although brief, was very strong. He was to me the role model of a dedicated scientist pioneering in helping fellow scientists, especially those who come from developing countries. For his insight and kindness I am eternally grateful.

2 Volume Quantization

The work I am reporting in this presentation is the result of a long-term collaboration with Alain Connes over a span of twenty years starting in 1996 [3] [4] [5] [6] [7] [8]. In the latest work on volume quantization we were joined by Slava Mukhanov [9] [10]. On the inner fluctuations of the Dirac operator over automorphisms of the noncommutative algebra times its opposite, we
Figure 1: We opened for thee a manifest victory

were joined by Walter von Suijlekom [11] [12] [13].

At very small distances, of the order of Planck length $1.6 \times 10^{-35}$\,m, we expect the nature of space-time to change. It is then natural to ask whether there is a fundamental unit of volume in terms of which the volume of space is quantized. The present volume of space is of the order of $10^{60}$ Planck units [5], but at the time of the big bang, this volume must have been much smaller. To explore this idea, we start with the observation that it is always possible to define a map $Y^A$, $A = 1, \cdots, n+1$, from an $n$-dimensional manifold $M_n$ to the $n$-sphere $S^n$ so that $Y^A Y^A = 1$ [14]. This map has a degree, the winding number over the sphere, which is an integer. This is defined in terms of an $n$-form $\omega_n$, the integral of which is a topological invariant

$$\omega_n = \frac{1}{n!} \epsilon_{A_1 A_2 \cdots A_{n+1}} Y^{A_1} dY^{A_2} \cdots dY^{A_{n+1}}$$

(1)

If we equate $\omega_n$ with the volume form

$$v_n = \sqrt{g} dx^1 \wedge dx^2 \cdots \wedge dx^n$$

(2)

so that

$$\int_{M_n} v_n = \int_{M_n} w_n = \text{deg } (Y) \in \mathbb{Z}$$

(3)

then the volume of the manifold $M_n$ will be quantized [14] and given by an integer multiple of the unit sphere in Planck units. This hypothesis, however,
has topological obstruction. For the equality to hold, the pullback of the
volume form is a four-form that does not vanish anywhere, and thus the
Jacobian of the map $Y^A$ does not vanish anywhere and is then a covering
of the sphere $S^n$. However, since the sphere is simply connected, the manifold
$M_n$ must be a covering of the sphere $S^n$ [22] [10]. This implies that the
manifold must be disconnected, each component being a sphere. This gives
a bubble picture of space, where every bubble of Planckian volume has the
topology of a sphere $S^n$. This is not an attractive picture because one would
have to invent a mechanism for condensation of the bubbles at lower energies.
To rescue this idea, we first rewrite the proposal (3) in a different form. Let
$D_0$ be the Dirac operator on $M_n$ given by

$$D_0 = \gamma^a e^\mu_a \left( \partial_\mu + \frac{1}{4} \omega^{bc}_\mu \gamma_{bc} \right)$$

(4)

where $e^\mu_a$ is the (inverse) vielbein and $\omega^{bc}_\mu$ is the spin-connection. Notice that
in momentum space, the Dirac operator could be identified with momenta $p_a$, Feynman slashed with the Clifford algebra spanned by $\gamma^a$. In analogy,
introduce a new Clifford algebra $\Gamma^A$ such that

$$\{ \Gamma_A, \Gamma_B \} = 2\kappa \delta_{AB}, \quad \kappa = \pm 1, \quad A = 1, \cdots, n+1$$

(5)

and slash the coordinates $Y^A$ with $\Gamma_A$

$$Y = Y^A \Gamma_A, \quad Y = Y^*, \quad Y^2 = 1$$

(6)

then a compact way of writing equation (3) is given by

$$\langle Y [D_0, Y]^n \rangle = \gamma, \quad n = \text{even}$$

(7)

where $\gamma$ is the chirality operator on $M_n$, $\gamma = \gamma_1 \cdots \gamma_n$ and $\langle \rangle$ denotes taking the trace over the Clifford algebra spanned by $\Gamma^A$. In this form, the
quantization condition is a generalization of the Heisenberg commutator for
momenta and coordinates $[p, x] = -i\hbar$. It is also identical to the Chern character formula in noncommutative geometry, which is a special case of the
orientability condition with idempotent elements. This suggests to consider
the above proposal for a noncommutative space defined by a spectral triple
$(\mathcal{A}, \mathcal{H}, D)$ together with reality operator $J$ and chirality $\gamma$ [15] [16] [17]. Here
$\mathcal{A}$ is an associative * algebra with involution and unit element, $\mathcal{H}$ a Hilbert
space, $D$ is a self-adjoint operator with bounded spectrum for $(D^2 + 1)^{-1}$. 

3
The chirality operator commutes with the algebra $\mathcal{A}$, $\gamma a = a\gamma$, $\forall a \in \mathcal{A}$. The following properties are assumed to hold

$$J^2 = \epsilon, \quad JD = e'DJ, \quad J\gamma = \epsilon''\gamma J, \quad \epsilon, e', \epsilon'' \in \{-1, 1\} \quad (8)$$

which defines a $KO$ dimension (mod 8) of the noncommutative space. As an example, for a Riemannian manifold $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(S)$, $D$ is the Dirac operator, $\gamma$ is the chirality, $J$ is the charge conjugation operator.

The operator $J$ sends the algebra $\mathcal{A}$ into its commutant

$$[a, Jb^*J^{-1}] = 0, \quad \forall a, b \in \mathcal{A} \quad (9)$$

where $Ja^*J^{-1} \in \mathcal{A}^o$, and thus the left action and the right action acting on elements of the Hilbert space commute.

Going back to the volume quantization condition, the slashed coordinates $Y$ when acted on with the $J$ become $Y' = JYJ^{-1}$ which commutes with it $[Y, Y'] = 0$. If $Y$ is slashed with the Clifford algebra $\kappa = 1$ so that $Y'$ will correspond to the Clifford algebra with $\kappa = -1$. It is essential to have a volume quantization condition involving both $Y$ and $Y'$. To do this, let $e = \frac{1}{2}(Y + 1)$ so that $e^2 = e$ and similarly $e' = \frac{1}{2}(Y' + 1)$ with $e'^2 = e'$. The product $E = ee'$ also satisfies $E^2 = E$ which implies that the composite coordinate $Z = 2E - 1$ satisfies $Z^2 = 1$. This suggests that we modify our volume quantization condition (7) to become

$$\langle Z [D_0, Z]^n \rangle = \gamma, \quad n = \text{even} \quad (10)$$

For dimensions $n = 2$ and $n = 4$ this relation splits into two pieces, one is a function of $Y$ and the other a function of $Y'$

$$\langle Y [D_0, Y]^n \rangle + \langle Y' [D_0, Y']^n \rangle = \gamma, \quad n = 2, 4 \quad (11)$$

For $n = 6$ there are mixing terms between $Y$ and $Y'$ and the relation (10) does not factorize. Thus the only realistic case where we take the volume quantization condition (10) to hold is for manifolds of dimension $n = 4$. In what follows we restrict our considerations to dimensions $n = 4$, where we will find out the special importance of the number 4.

### 3 Noncommutative space

From now on we specialize to dimension $n = 4$. In this case, (11) holds and with this condition, we prove that if $M_4$ is an oriented four-manifold
then a solution of the two sided equation \( (10) \) exists and is equivalent to the existence of the two maps \( Y \) and \( Y' \) : \( M_4 \to S^4 \) such that the sum of the two pullbacks \( Y^*(\omega) + Y'^*(\omega) \) does not vanish anywhere and

\[
\text{vol}(M) = ((\text{deg } Y) + (\text{deg } Y')) \text{vol}(S^4) \quad (12)
\]

The proof is difficult because in four dimensions, the kernel of the map \( Y \) is of codimension 2. Details are given in reference \([10]\). Fortunately, for this relation to hold, the only conditions on the manifold \( M_4 \) is the vanishing of the second Steifel-Whitney class \( w_2 \), which is automatically satisfied for spin manifolds, and that \( \text{vol}(M) \) should be larger than four units \([10]\). In this setting, the four dimensional manifold emerges as a composite of the inverse maps of the product of two spheres of Planck size. The manifold \( M_4 \) which is folded many times in the product, unfolds to macroscopic size. The two different spheres, associated with the two Clifford algebras can be considered as quanta of geometry which are the building blocks to generate an arbitrary oriented four dimensional spin-manifold. We can show that the manifold \( M_4 \), the two spheres with their maps \( Y, Y' \) and their associated Clifford algebras define a noncommutative space which is the basis of unification of all fundamental interactions, including gravity.

To study this noncommutative space, we first note that the Clifford algebras with \( \kappa = 1 \) and \( \kappa = -1 \) are given by \([18]\)

\[
\text{Cliff}(+,+,+,+,+) = M_2(\mathbb{H}) \quad (13)
\]

\[
\text{Cliff}(-,-,-,-,-) = M_4(\mathbb{C}) \quad (14)
\]

and thus the algebra \( A_F \) of the finite space is

\[
A_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C}) \quad (15)
\]

The finite Hilbert space \( H_F \) is then the basic representation \((4,4)\) where the the first 4 is acted on by the matrix elements \( M_2(\mathbb{H}) \) and the second 4 is acted on by the matrix elements \( M_4(\mathbb{C}) \). It is tantalizing to observe that the same finite algebra \([15]\) was obtained by classifying all finite algebras of \( KO \) dimension 6, required to avoid mirror fermions. The maps \( Y \) and \( Y' \) are functions of the coordinates \( x^\mu \). Since \( Y^2 = 1 \) composing words from the elements of the algebra \( M_2(\mathbb{H}) \) and \( Y \) of the form \( a_1 Y a_2 Y \cdots a_i Y, \ \forall i \), and similarly for \( Y' \) will generate the algebra

\[
A = C^\infty(M_4, M_2(\mathbb{H}) \oplus M_4(\mathbb{C})) \quad (16)
\]

\[
= C^\infty(M_4) \otimes (M_2(\mathbb{H}) \oplus M_4(\mathbb{C})) \quad (17)
\]
The associated Hilbert space is then

$$\mathcal{H} = L^2(S) \otimes \mathcal{H}_F$$

(18)

and the Dirac operator is

$$D = D_0 \otimes 1 + \gamma_5 \otimes D_F$$

(19)

where $D_F$ is a self-adjoint matrix operator acting on $\mathcal{H}_F$. The reality operator $J_F$ acting on the finite algebra $\mathcal{A}_F$ satisfies

$$J_F(x, y) = (y^*, x^*)$$

(20)

so that the reality operator $J$ will be

$$J = C \otimes J_F$$

(21)

where $C$ is the charge conjugation operator. Finally, the chirality operator $\gamma$ is given by

$$\gamma = \gamma_5 \otimes \gamma_F$$

(22)

One then finds that the finite noncommutative space $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ with $J_F$ and $\gamma_F$ have a KO dimension 6. Thus the KO dimension of the full noncommutative space is 10 [19] [20]. Elements of the Hilbert space $\mathcal{H}$ are of the form

$$\Psi = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}$$

(23)

which can be denoted by $\Psi_{\tilde{a}aI}$ where $\tilde{a} = 1, \cdots, 4$ transforms as a space-time spinor, $\alpha = 1, \cdots, 4$ transforms as a 4 under the action of $M_2(\mathbb{H})$ and $I = 1, \cdots, 4$ transforms as a 4 under the action of $M_4(\mathbb{C})$. Thus $\Psi$ represents a 16 space-time Dirac spinor and its conjugate. However, since the KO dimension of the full space is 10, this allows to impose the following two conditions (in the Lorentzian version) on $\Psi$, the chirality and the reality conditions

$$\gamma \Psi = \Psi, \quad J \Psi = \Psi$$

(24)

which implies that $\psi$ is a chiral 16 and $\psi^c$ is not an independent spinor, and is given by $C\bar{\psi}^T$. Now, the $\gamma_F$ chirality operator must commute with $\mathcal{A}_F$. If this operator is taken to act on the first algebra $M_2(\mathbb{H})$, this implies that
only the subalgebra $\mathbb{H}_R \oplus \mathbb{H}_L$ is preserved. The 16 spinor then transforms under the finite algebra

$$\mathcal{A}_F = (\mathbb{H}_R \oplus \mathbb{H}_L) \oplus M_4(\mathbb{C})$$  \hspace{1cm} (25)$$
as $(2_R, 1_L, 4) + (1_R, 2_L, 4)$. Elements of the Hilbert space do transform under automorphisms of the algebra $\mathcal{A} \otimes \mathcal{A}^\circ$ as

$$\Psi \to U\Psi, \quad U = u \hat{u}, \quad u \in \mathcal{A}, \quad \hat{u} = Ju^* J^{-1} \in \mathcal{A}^\circ$$  \hspace{1cm} (26)$$
The action of the Dirac operator $D$ on elements of the Hilbert space $\Psi$ does not transform covariantly under the automorphisms $U$. It can be made so by adding to $D$ a connection $A$ so that

$$DA = D + A$$  \hspace{1cm} (27)$$
such that

$$DA (U\Psi) = UD_{A^u} \Psi$$  \hspace{1cm} (28)$$
This fixes the connection $A$ to be given by

$$A = \sum a \hat{a} \left[ D, \hat{b} \right]$$  \hspace{1cm} (29)$$
which can be decomposed into three parts

$$A = A^{(1)} + JA^{(1)} J^{-1} + A^{(2)}$$  \hspace{1cm} (30)$$
where

$$A^{(1)} = \sum a [D, b]$$  \hspace{1cm} (31)$$
$$A^{(2)} = \sum \hat{a} \left[ A^{(1)}, \hat{b} \right]$$  \hspace{1cm} (32)$$
The connection $A$ transforms as

$$A^{(1)^u} = u A^{(1)^u} + u [D, u^*]$$  \hspace{1cm} (33)$$
$$A^{(2)^u} = \hat{u} A^{(1)^u} + \hat{u} [u [D, u^*], \hat{u}^*]$$  \hspace{1cm} (34)$$
In the special case where the order one condition is satisfied,

$$\left[ a, \left[ D, \hat{b} \right] \right] = 0, \quad \forall a, b \in \mathcal{A}$$  \hspace{1cm} (35)$$
this implies that

$$A^{(2)} = 0$$  \hspace{1cm} (36)$$

The condition (35) restricts the algebra \((\mathbb{H}_R \oplus \mathbb{H}_L) \oplus M_4(\mathbb{C})\) to the subalgebra

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L \oplus M_3(\mathbb{C})$$  \hspace{1cm} (37)$$

where the algebra \(\mathbb{C}\) is embedded in the diagonal part of \(\mathbb{H}_R \oplus M_4(\mathbb{C})\). This is the algebra that gives rise to the Standard Model.

In practical terms, the connection (30) can be calculated using simple matrix algebra. The results show that the elements of the connection along the three separate algebras \(\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})\) are tensored with the space-time \(\gamma^\mu\) and are the gauge fields of \(SU(2)_R\), \(SU(2)_L\), and \(SU(4)\) which are those of the Pati-Salam models. Components of the connection along the off-diagonal elements between the three different algebras are tensored with the space-time chirality \(\gamma_5\) and are the Higgs fields. Representations of the Higgs fields depend on the form of the initial finite space Dirac operator and fall into three different classes of Pati-Salam models. The first class have the Higgs fields in the \(SU(2)_R \times SU(2)_L \times SU(4)\) representations

$$\Sigma^b_{aI} = (2, 2, 1) + (1, 1, 1 + 15),$$  \hspace{1cm} (38)$$

$$H_{aIbJ} = (1, 1, 6) + (1, 3, 10),$$  \hspace{1cm} (39)$$

$$H_{aIbJ} = (1, 1, 6) + (3, 1, 10)$$  \hspace{1cm} (40)$$

For the second class we have the same fields as the first class with the restriction \(H_{aIbJ} = 0\). The third class is a special case of the second class, but where the fields \(\Sigma^b_{aI}\) and \(H_{aIbJ}\) are composites of more fundamental fields

$$H_{aIbJ} = \Delta_{aJ} \Delta_{bI}, \quad \Sigma^b_{aI} \sim \phi^b_a \Sigma^J_I$$  \hspace{1cm} (41)$$

where \(\Delta_{aJ}\) is in the \((2, 1, 4)\) representation, \(\phi^b_a\) is in the \((2, 2, 1)\) representation and \(\Sigma^J_I\) is in the \((1, 1, 1 + 15)\).

When the order one condition (35) on the algebra is satisfied, the algebra of the finite noncommutative space reduces to the subalgebra (37). Components of the connection along the separate three algebras are tensored with \(\gamma^\mu\) and are those of \(U(1)_Y \times SU(2) \times SU(3)\). The component of the connection along the off-diagonal elements of the algebras \(\mathbb{C}\) and \(\mathbb{H}\) tensored with \(\gamma_5\) is the Higgs doublet. There is also a singlet component, tensored with \(\gamma_5\).
and connects the right-handed neutrino to its conjugate. The 16 fermions have the representations under \( SU(2)_R \times SU(2)_L \times SU(4) \) given by

\[(2, 1, 4) + (1, 2, 4) \quad (42)\]

In the special case of the subalgebra is that of the SM, the fermion representation with respect to \( U(1)_Y \times SU(2) \times SU(3) \) becomes

\[(1, 1, 1) + (1', 1, 1) + (1, 1, 3) + (1', 1, 3) + (1, 2, 1) + (1, 2, 3) \quad (43)\]

These correspond to the particles, respectively, \( \nu_R, e_R, u_R, d_R, l_L, q_L \) where \( l_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \) is the lepton doublet and \( q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \) is the quark doublet.

### 4 Spectral Action

The Euclidean fermionic action, including all vertex interactions is extremely simple and is given by

\[(J \Psi, D_A \Psi) \quad (44)\]

where the path integral is equal to the Pfaffian of the operator \( D_A \), eliminating half of the degrees of freedom associated with mirror fermions \[4\]. In the Lorentzian form \( J \Psi = \Psi \) and half of the degrees of freedom are eliminated by the reality condition, showing the equivalence of both cases when the \( KO \) dimension of the space is 10 \[19\] \[20\]. The bosonic action which gives the dynamics of all the bosonic fields, including graviton, gauge fields and Higgs fields, is governed by the spectral action principle which states that the action depends only on the spectrum of the Dirac operator \( D_A \) given by its eigenvalues which are geometric invariants. The spectral action is given by

\[\text{Tr} \left[ f \left( \frac{D_A}{\Lambda} \right) \right] \quad (45)\]

where \( f \) is a positive function and \( \Lambda \) is a cut-off scale. At scales below \( \Lambda \) the function \( f \) could be expanded in a Laurent series in \( D_A \) thus reducing evaluating the spectral action \[45\] to that of calculating the heat-kernel coefficients, which are geometric invariants. Thus the spectral action, at energies lower than the cut-off scale is determined by the Seeley-de Witt invariants of the operator \( D_A \) with the coefficients in the expansion related
to the Mellin transform of the function \( f \). The result for the Standard Model, at unification scale is given by

\[
S_b = \frac{24}{\pi^2} f_4 \Lambda^4 \int d^4x \sqrt{g} \\
- \frac{2}{\pi^2} f_2 \Lambda^2 \int d^4x \sqrt{g} \left( R + \frac{1}{2} a H H + \frac{1}{4} c \sigma^2 \right) \\
+ \frac{1}{2\pi^2} f_0 \int d^4x \sqrt{g} \left[ \frac{1}{30} (-18 C_{\mu\nu\rho\sigma}^2 + 11 R^2 R^2) + \frac{5}{3} g_1^2 B_{\mu\nu}^2 + g_2^2 (W_{\mu\nu}^a)^2 + g_3^2 (V_{\mu\nu}^m)^2 \\
+ \frac{1}{6} a R H H + b (HH)^2 + a |\nabla_\mu H_\alpha|^2 + 2c HH \sigma^2 + \frac{1}{2} d \sigma^4 + \frac{1}{12} c R\sigma^2 + \frac{1}{2} c (\partial_\mu \sigma)^2 \right]
\]

where \( C_{\mu\nu\rho\sigma} \) is the conformal tensor, \( B_\mu, W^a_\mu, V^m_\mu \) are the gauge fields of \( U(1)_Y \times SU(2)_L \times SU(3) \), \( H \) is a Higgs doublet and \( \sigma \) is a singlet. The coefficients \( a, b, c, d, e \) are given in terms of the Yukawa couplings of the Higgs fields to the fermions. A similar calculation for the unbroken algebra \((25)\) will give the bosonic action of Pati-Salam models including all the gauge and Higgs interactions with their potential.

## 5 Conclusions

It is remarkable that the simple two sided relation \((10)\) leads to volume quantization of the four-dimensional Riemannian manifold with Euclidean signature. The manifold could be reconstructed as a composition of the pullback maps from two separate four spheres with coordinates defined over two Clifford algebras. The phase space of coordinates and Dirac operator defines a noncommutative space of \(KO\) dimension 10. The symmetries of the algebras defining the noncommutative space turn out to be those of \( SU(2)_R \times SU(2)_L \times SU(4) \) known as the Pati-Salam models. Connections along discrete directions are the Higgs fields. A special case of this configuration occurs when the order one condition \((35)\) is satisfied, reducing the finite algebra to the subalgebra given by \((37)\). The action has a very simple form given by a Dirac action for fermions and a spectral action for bosons. The 16 fermions (per family) are in the correct representations with respect to Pati-Salam symmetries or the SM symmetries. There are many consequences of the volume quantization condition which could be investigated. For example imposing the quantization condition through a Lagrange multiplier would
imply that the cosmological constant will arise as an integrating constant in the equations of motion. One can also look at the possibility that only the three volume (space-like) is quantized. This can be achieved provided that the four-dimensional manifold arise due to the motion of three dimensional hypersurfaces, which is equivalent to the $3+1$ splitting of a four-dimensional Lorentzian manifold. Then three dimensional space volume will be quantized, provided that the field $X$ that maps the real line have a gradient of unit norm $g^{\mu\nu}\partial_\mu X \partial_\nu X = 1$. It is known that this condition when satisfied gives a modified version of Einstein gravity with integrating functions giving rise to mimetic dark matter [21] [22]. All of this could be considered as a first step towards quantizing gravity. What is done here is the analogue of phase space quantization where the Dirac operator plays the role of momenta and the maps $Z$ play the role of coordinates. The next step would be to study consequences of this new idea on the quantization of fields which are functions of either coordinates $Z$ or Dirac operator $D_A$. All of this and other ideas will be presented elsewhere.

To conclude, the influence of Salam on my work which started with supersymmetry, supergravity and their applications continued to string theory, topological gravity and noncommutative geometry, cannot be underestimated. It is my honor and privilege to have followed his footsteps. It was my good fortune to have him as my thesis advisor at Imperial College and postdoctoral mentor at ICTP.

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