ON UNIFORM DEFINABILITY OF TYPES OVER FINITE SETS FOR NIP FORMULAS

SHLOMO ESHEL AND ITAY KAPLAN

Abstract. Combining two results from machine learning theory we prove that a formula is NIP if and only if it satisfies uniform definability of types over finite sets (UDTFS). This settles a conjecture of Laskowski.

1. Introduction

Let $L$ be any language and let $T$ be any $L$-theory. An $L$-formula $\varphi(x, y)$ has uniform definability of types over finite sets (UDTFS) in $T$ iff there is a formula $\psi(y, z)$ which uniformly (in any model of $T$) defines $\varphi$-types over finite sets of size $\geq 2$ (see Definition 3). If $\varphi$ has UDTFS, then for any finite $A \subseteq M^y \models T$, the number of $\varphi$-types over $A$ is bounded by $|A|^{|z|}$, which immediately implies that $\varphi$ has finite VC-dimension in $T$, i.e., $\varphi$ is NIP in $T$ (if $\varphi$ shatters a finite set $A$, then the number of $\varphi$-types over $A$ is exponential in $|A|$). This raises the question, asked by Laskowski, of whether these two notions (UDTFS and NIP) are equivalent. Note that in that case, this also implies the Sauer-Shelah lemma in the sense of counting types (see [She90, Chapter II, Theorem 4.10(4)]). See also the discussion in [LS13].

This question was first addressed in [JL10] where it was proved assuming that $T$ is weakly o-minimal. Later, [Gui12] extended this result to dp-minimal theories. Finally, in [CS15, Theorem 15] it was proved in the level of the theory $T$: a (complete) theory is NIP iff every formula has UDTFS. They actually proved something stronger: in NIP theories, every formula has uniform honest definitions. See Section 3.2 below for the definition.

The main theorem in this paper solves Laskowski’s question (and thus answers all the questions in the final paragraph of [Gui12]).

**Main Theorem 1.** The following are equivalent for an $L$-theory $T$ and an $L$-formula $\varphi(x, y)$.

1. $\varphi$ is NIP in $T$ (i.e., NIP in any completion of $T$).
2. $\varphi$ has UDTFS in $T$.

(see Theorem 13 below.)

The proof has two ingredients, both from machine learning theory.
The first is [MY16] which proves the existence of sample compression schemes for concept classes of finite VC-dimension \( d \) whose sizes are bounded in terms of \( d \) (answering a question of Littlestone and Warmuth). Roughly speaking, this result says that there is some number \( k \) depending only on \( d \) such that for any finite set of labeled examples (concepts), it is possible to recover our knowledge on that concept by considering a specific subset of size \( k \). We do not use the result but rather its proof, and most importantly the proof of Claim 3.1 from there, which we translate to our language.

The second ingredient is [CCT16] where an upper bound for the recursive teaching dimension (RTD) is given for concept classes of finite VC-dimension \( d \) (the bound in [CCT16] is exponential in \( d \) and was later improved to a quadratic bound in [HWL17]). Roughly speaking this means that there is some number \( t \) (depending only on \( d \)) such that every concept can be identified by at most \( t \) samples according to the recursive teaching model. See Fact [11] for a precise statement which follows by reading the definitions. This results translates in our language to the existence of \( \varphi \)-types which are isolated by their restriction to a set of bounded size (see Corollary [12]). This result (or rather, its proof) will be used in a forthcoming work with Martin Bays and Pierre Simon which deals with “compressible” types in NIP theories.

Despite the fact that our proof is based on these two results, we do not need to define any of the machine learning notions mentioned above so that the proof can be read by anyone with a basic understanding of model theory.

Remark 2. The first ingredient was known by experts for quite some time now (it was brought to our attention by Pierre Simon in 2015). It was known that it alone implies UDTFS assuming that the theory has definable Skolem functions (see Section 3.1.2).

We became aware of the second ingredient during the aforementioned work with Martin Bays and Pierre Simon, thanks to Nati Linial who answered our question about it.

The paper is organized as follows: Section 2 contains all the preliminaries and the proof of the main theorem. Section 3 contains some open questions.

2. Proof of the main theorem

Throughout fix a language \( L \); all formulas, theories and structures will be in \( L \).

**Definition 3.** Let \( M \) be a \( L \)-structure and let \( \varphi(x,y) \) be a formula. Suppose that \( p(x) \) is a \( \varphi \)-type over a set \( A \subseteq M^y \). We say that a formula \( \psi(y) \) (not necessarily over \( \emptyset \)) defines \( p \) if for all \( b \in A \) we have \( M \models \psi(b) \iff \varphi(x,b) \in p(x) \).

**Definition 4.** (UDTFS) Let \( T \) be an \( L \)-theory. We say that \( \varphi(x,y) \) have uniform definability of types over finite sets in \( T \) (UDTFS) if there exists a formula \( \psi(y,z) \) such that for every \( M \models T \)
and all finite sets $A \subseteq M^y$ with $|A| \geq 2$ the following holds: for every $p(x) \in S_\varphi(A)$ there exist $c \in A^x$ such that $\psi(y,c)$ defines $p(x)$.

**Definition 5.** (VC-dimension) Let $X$ be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$. Given $A \subseteq X$, we say that it is shattered by $\mathcal{F}$ if for every $S \subseteq A$ there is $F \in \mathcal{F}$ such that $F \cap A = S$. A family $\mathcal{F}$ is said be a VC-class if there is some $n < \omega$ such that no subset of $X$ of size $n$ is shattered by $\mathcal{F}$. In this case the VC-dimension of $\mathcal{F}$, that we will denote by $VC(\mathcal{F})$, is the smallest integer $n$ such that no subset of $X$ of size $n + 1$ is shattered by $\mathcal{F}$.

**Definition 6.** ($\varphi$ is NIP in $T$) Suppose $T$ is an $L$-theory and $\varphi(x,y)$ a formula. Say that $\varphi(x,y)$ is NIP in $T$ if for every $M \models T$, the family $\{\varphi(M,a) \mid a \in M^y\}$ is a VC-class. This is equivalent saying that $\varphi$ is NIP in any completion of $T$.

**Remark 7.** Note that $\varphi$ is NIP in $T$ iff there is a bound $n < \omega$ such that for every $M \models T$, the VC-dimension of the family $\{\varphi(M,a) \mid a \in M^y\}$ is bounded by $n$ (and this is first-order expressible). This follows by compactness. Denote the minimal such $n$ by $VC_T(\varphi)$ or just $VC(\varphi)$ if $T$ is clear from the context.

**Fact 8.** [Adl08] Remark 9 Suppose that $T$ is an $L$-theory. The family of formulas $\varphi(x,y)$ which are NIP in $T$ is closed under finite Boolean combinations. Moreover, (it follows from the proof that) if $\varphi_0, \varphi_1$ are such and $\psi$ is a Boolean combination of $\varphi_0, \varphi_1$, then $VC_T(\psi)$ is bounded in terms of $VC_T(\varphi_0), VC_T(\varphi_1)$.

**Fact 9.** [Adl08] Proposition 2]If $\varphi(x,y)$ is NIP in $T$ then so is $\varphi^{opp}(y,x)$ where $\varphi^{opp}$ is the formula $\varphi$ but with the partition of variables switched. As above, $VC_T(\varphi^{opp})$ is bounded in terms of $VC_T(\varphi)$, in fact $VC_T(\varphi^{opp}) < 2^{VC_T(\varphi)+1}$.

**Fact 10.** [Sim15] Corollary 6.9] (VC-theorem; the existence of $\epsilon$-approximations) For any $d < \omega$ and $0 < \epsilon$ there is some $N = N(d,\epsilon) < \omega$ such that for any finite set $X$, for any $C \subseteq \mathcal{P}(X)$ of VC-dimension $\leq d$ and every finite probability measure $\mu$ on $X$ there exists a multiset $Y \subseteq X$ of size $|Y| \leq N$ such that for all $s \in C$, $\left| \mu(s) - \frac{|\{a \in X \mid \varphi^{opp}(x,a) \in C\}|}{|Y|} \right| \leq \epsilon$.

The next fact is the “second ingredient” mentioned in the introduction. To see it, read [CCT10] Definition 1] and the preceding paragraph.

**Fact 11.** [CCT10] Theorem 3] [HWL17] Theorem 6] For all $n < \omega$ there is some $t = t(n)$ such that if $X$ is a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ is a family of VC-dimension $\leq n$, then there is some $F \in \mathcal{F}$ and $X_0 \subseteq X$ of size $|X_0| \leq t$ such that for all $F' \in \mathcal{F}$, $F = F'$ iff $X_0 \cap F' = X_0 \cap F$.

**Corollary 12.** Let $m,n < \omega$. Then there is some $k = k(n,m)$ such that if $T$ is an $L$-theory, $\varphi(x,y)$ is NIP in $T$ and $VC(\varphi) \leq n$ then for every $M \models T$ and finite $A \subseteq M^y$ the following holds:
If \( \chi(x) = \bigwedge_{i<m} \varphi(x,a_i)^{\varepsilon_i} \) where for every \( i < m, a_i \in A \) and \( \varepsilon_i < 2 \) (in general, \( \varphi^0 = \varphi \) and \( \varphi^1 = \neg \varphi \)) is consistent, then there is a type \( p_0 \in S_\varphi(A) \) and \( A_0 \subseteq A \) of size \( |A_0| \leq k \) such that \( p_0|_{A_0} \vdash p_0 \vdash \chi(x) \).

Proof. Let \( k = k(n,m) = t(2^{n+1} - 1) + m \) where \( t(n) \) is from Fact \([\text{II}]\). Let \( A' = A \setminus \{ a_i \mid i < m \} \), and consider the family \( F = \{ \varphi(b,A') \mid b \vdash \chi \} \). Then the VC-dimension of \( F \) is bounded by the dual VC-dimension of \( \varphi \) which is \( \leq 2^{n+1} - 1 \). By Fact \([\text{II}]\) there is some \( b \in M_x \) and some \( X_0 \subseteq A \) of size \( \leq t(2^{n+1} - 1) \) such that \( \chi(b) \) holds and for all \( b' \models \chi \), \( \varphi(b,A') = \varphi(b',A') \) iff \( \varphi(b,X_0) = \varphi(b',X_0) \). Let \( p_0 = \text{tp}_\varphi(b/A) \) and let \( A_0 = X_0 \cup \{ a_i \mid i < m \} \). Obviously \( p_0 \vdash \chi(x) \) and if \( b' \models p_0|_{A_0} \) then \( b' \models \chi \) and since \( \varphi(b,X_0) = \varphi(b',X_0) \), it follows that \( \varphi(b,A') = \varphi(b',A') \), so that \( \varphi(b,A) = \varphi(b',A') \), i.e., \( b' \models p_0 \).

The following theorem is the main result of this paper.

**Theorem 13.** Fix a theory \( T \) and a formula \( \varphi(x,y) \). Then the following are equivalent:

1. \( \varphi(x,y) \) is NIP in \( T \).
2. There is an integer \( K(\varphi) \) such that for every model \( M \models T \) and for every finite nonempty \( A \subseteq M^y \) and \( p \in S_\varphi(A) \), there is a formula \( \psi(y,z) \) where \( z = (z_i \mid i < K(\varphi)) \) which is a finite disjunction of formulas each of the form
   \[ \exists d_0,\ldots,d_{m-1} \bigwedge_{t \leq m} \bigwedge_{i \in s_t} \varphi(d_t,z_i)^{\varepsilon_{i,t}} \wedge \bigwedge_{t \in s} \varphi(d_t,y) \]
   where \( m \leq K(\varphi), s \subseteq m \) and for every \( t < m, s_t \subseteq K(\varphi) \) and \( \varepsilon_{i,t} < 2 \) for all \( i \in s_t \) such that \( \psi(y,a) \) define \( p \) for some \( a \in A^x \).
3. There is an integer \( K(\varphi) \) such that for every model \( M \models T \) and for every finite nonempty \( A \subseteq M^y \) and \( p \in S_\varphi(A) \), there is a formula \( \psi(y,z) \) where \( z = (z_i \mid i < K(\varphi)) \) which is a finite disjunction of formulas each of the form
   \[ \forall d_0,\ldots,d_{m-1} \bigwedge_{t \leq m} \bigwedge_{i \in s_t} \varphi(d_t,z_i)^{\varepsilon_{i,t}} \rightarrow \bigwedge_{t \in s} \varphi(d_t,y) \]
   where \( m \leq K(\varphi), s \subseteq m \) and for every \( t < m, s_t \subseteq K(\varphi) \) and \( \varepsilon_{i,t} < 2 \) for all \( i \in s_t \) such that \( \psi(y,a) \) define \( p \) for some \( a \in A^x \).
4. \( \varphi(x,y) \) have UDTFS in \( T \).

We start with a proof of (1) implies (2), (3), so assume (1). By Remark \([\text{II}]\) and (1), the VC-dimension of the family \( \{ \varphi(M,b) \mid b \in M \} \) is bounded by some constant integer \( VC(\varphi) \).

Towards a proof of (2) and (3), for the next few claims we fix \( M \models T \), a finite \( A \subseteq M^y \) and \( p \in S_\varphi(A) \). Fix also \( c \models p \).

We will produce integers \( N, J, k \) depending only on \( VC(\varphi) \) and not on \( M, A \) or \( p \). From these we will be able to construct the defining formula \( \psi \).
Definition 14. For $n < \omega$, we say that $f : A^n \times 2^n \to M$ is an $(n, \varphi)$-Skolem function if for every $(\overline{a}, \overline{c}) \in A^n \times 2^n$, if there is $b \in M^x$ such that $M \models \bigwedge_{i<n} \varphi(b, a_i)^{c_i}$ then $M \models \bigwedge_{i<n} \varphi(f(\overline{a}, \overline{c}), a_i)^{c_i}$. The function $f$ is a $\varphi$-Skolem function if it is an $(n, \varphi)$-Skolem function for some $n < \omega$.

For every $\overline{a} \in A^n$ and $b \in M^x$, let $\varepsilon(a, b) \in 2^n$ be the unique tuple $\varepsilon$ for which $M \models \bigwedge_{i<n} \varphi(b, a_i)^{c_i}$ (when $n = 1$ we write $\varepsilon(a, b)$).

Claim 15. There is some integer $N = N(VC(\varphi))$ such that for every finite probability measure $\mu$ on $A$, there is a tuple $\overline{a} \in A^{\leq N}$ such that for every $(\overline{a}, \varphi)$-Skolem function $f$ and for every $b \in M^x$ we have that $\mu(\{a \in A \mid \varphi(b, a) \leftrightarrow \varphi(f(\overline{a}, \varepsilon(b)), a)\}) \geq \frac{3}{4}$.

Proof. Consider the set $S = \{\varphi(c_1, A) \triangle \varphi(c_2, A) \mid c_1, c_2 \in M^x\} \subseteq \mathcal{P}(A)$.

By Fact 8, the formula $\psi(xx', y) = \varphi(x, y) \triangle \varphi(x', y)$ is NIP in $T$ and has a finite VC-dimension which is moreover bounded in terms of $VC(\varphi)$. By Fact 8 it follows that the same is true for the family $S$. Let $N$ be the number provided by the VC-theorem for this bound and $\varepsilon = \frac{1}{5}$. It then follows that for any $\mu$ as above there is a multiset $E \subseteq A$ with $|E| \leq N$ such that $|\mu(S) - \frac{|S \cap E|}{|E|}| \leq \frac{1}{3}$ for every $S \in S$. Let $\overline{a} = \langle a_i \mid i < n \rangle \in A^{\leq N}$ be a tuple listing $E$. Now, fix $b \in M^x$ and a $\varphi$-Skolem function $f$ as above. Let $S_0 = \varphi(b, A) \triangle \varphi(f(\overline{a}, \varepsilon(b)), A)$. Then for every $i < n$:

$$a_i \in \varphi(b, A) \iff M \models \varphi(h, a_i) \iff M \models \varphi(f(\overline{a}, \varepsilon(b)), A) \iff a_i \in \varphi(f(\overline{a}, \varepsilon(b)), A),$$

i.e., $E \cap S_0 = \emptyset$. Therefore $|\mu(S_0)| \leq \frac{1}{3}$ and we are done. \hfill \Box

The following claim is a translation of [MY16, Claim 3.1]. It can actually be deduced from there but for the completeness of the paper we chose to include the proof.

Claim 16. There is an integer $J = J(VC(\varphi))$ such that for every tuple of $\varphi$-Skolem functions $(f_n \mid n \leq N)$ where $f_n$ is an $(n, \varphi)$-Skolem function, there are tuples $\overline{a}_0, ..., \overline{a}_{m-1} \in A^{\leq N}$ for $m \leq J$ such that for every $a \in A$

$$\left| \left\{ t < m \mid M \models (\varphi(c, a) \leftrightarrow \varphi(f(\overline{a}_t, \varepsilon(c)), a)) \right\} \right| > \frac{m}{2}.$$
For any finite probability measure $\mu$ on $A$, treat $\mu$ as a distribution vector of length $|A|$. For $j < |H|$, let $v^j \in 2^{|H|}$ be such that $v^j_j = 1$ iff $j = j'$ and let $A_j = \{a_i \in A \mid B_{i,j} = 1\}$. We have that (when both $v^j$ and $\mu$ are treated as column vectors):

$$\mu^t B v^j = \mu \begin{pmatrix} B_j \end{pmatrix} = \sum_{i<|A|, B_{i,j}=1} \mu(a_i) = \sum_{a \in A_j} \mu(a) = \mu(\{a \in A \mid M \models (\varphi(c,a) \leftrightarrow \varphi(h_j,a))\}) \geq \frac{2}{3}.$$

In particular, we have $\min_{\mu \in \Delta^{|A|}} \max_{\nu \in \Delta^{|H|}} \mu^t B \nu \geq \frac{2}{3}$ where (in general) $\Delta^n$ is the set of all finite probability measures on the set $n = \{0, \ldots, n-1\}$.

**Fact 17.** (Von-Neumann’s minimax theorem) For every matrix $M \in \mathbb{R}^{n \times m}$

$$\min_{q \in \Delta^n} \max_{p \in \Delta^m} q^t M p = \max_{p \in \Delta^m} \min_{q \in \Delta^n} q^t M p.$$  

By Von-Neumann’s minimax theorem it follows that $\max_{\nu \in \Delta^{|H|}} \min_{\mu \in \Delta^{|A|}} \mu^t B \nu \geq \frac{2}{3}$. Therefore, there is some $\nu \in \Delta^{|H|}$ such that for every $\mu \in \Delta^{|A|}$, $\mu^t B \nu \geq \frac{2}{3}$. For $i < |A|$, let $u^i \in 2^{|A|}$ be such that $u^i_i = 1$ iff $i = i'$. In particular, we have that

$$(u^i)^t B \nu \geq \frac{2}{3} \Rightarrow \left( - B_i - \right) \nu \geq \frac{2}{3} \Rightarrow \sum_{h \in \widetilde{H}_i} \nu(h) \geq \frac{2}{3}$$

where $\widetilde{H}_i = \{h_j \in H \mid B_{i,j} = 1\}$. Thus, for every $a \in A$ we have

$$\nu(\{h_{\pi} \in H \mid M \models (\varphi(c,a) \leftrightarrow \varphi(h_{\pi},a))\}) \geq \frac{2}{3}.$$  

Consider the sets $S_\varepsilon = \{\varphi(H, a)^\varepsilon \mid a \in M^y\}$ for $\varepsilon < 2$. Recall that $S_0$ has VC-dimension which is bounded by $VC(\phi)$. By the choice of $J$ we can apply the VC-theorem on $S_0$, $\nu$ and $\epsilon = \frac{1}{8}$ and get that there is a multiset $F = \{h_{\pi} \mid i < m\} \subseteq H$ such that $m \leq J$ and $|\nu(\varphi(H, a)) - \frac{|F \cap \varphi(H, a)|}{|F|}| \leq \frac{1}{8}$ for every $a \in M^y$.

**Remark 18.** For every finite probability measure $p$ on a set $X$ and every finite multiset $F \subseteq X$ if $\left| p(Y) - \frac{|F \cap Y|}{|F|} \right| \leq \epsilon$ for some $Y \subseteq X$ and $\epsilon > 0$ then

$$\left| p(X \setminus Y) - \frac{|F \cap (X \setminus Y)|}{|F|} \right| \leq \epsilon.$$  

By Remark [18] it follows that for every $a \in M$

$$|\nu(\varphi(H, a)) - \frac{|F \setminus \varphi(H, a)|}{|F|}| \leq \frac{1}{8}.$$  

Now note that for every $a \in M^y, b \in M^x$ we have that $D_{a,b} \in S_0$ or $D_{a,b} \in S_1$, where $D_{a,b} = \{h_{\pi} \in H \mid M \models (\varphi(b,a) \leftrightarrow \varphi(h_{\pi},a))\}$ (if $M \models \varphi(b,a)$ then $D_{a,b} = \varphi(H, a)$ and $D_{a,b} = \neg \varphi(H, a)$ otherwise). So for every $D_{a,b}$ we have $|\nu(D_{a,b}) - \frac{|F \cap D_{a,b}|}{|F|}| \leq \frac{1}{8}$ which implies that
\[ \frac{|F \cap D_{a,c}|}{m} \geq \nu(D_{a,b}) - \frac{1}{5}. \] As \( \nu(D_{a,c}) \geq \frac{2}{3} \) for all \( a \in A \) (by the choice of \( \nu \)), it then follows that for every \( a \in A \) we have \( \frac{|F \cap D_{a,c}|}{m} \geq \frac{2}{3} - \frac{1}{5} > \frac{1}{3} \). Thus, for all \( a \in A \), \( |F \cap D_{a,c}| > \frac{m}{2} \), i.e., \( \{ t < m \mid M \models (\varphi(c, a) \leftrightarrow \varphi(h_{\pi, a})) \} > \frac{m}{2} \).

Now, let us finish the proof of (1) implies (2), (3) from Theorem [13]

**Proof of (1) implies (2),(3).** By Corollary [12] there is an integer \( k = k(VC(\varphi), N) \) such that for all \( n \leq N \) and \( (\pi, \tau) \in A^n \times 2^n \) if \( \chi(\pi, \tau)(x) = \bigwedge_{i < k} \varphi(x, a_i) \) is consistent then there are \( p_0(\pi, \tau) \in S_\varphi(A) \) and \( p_0(\pi, \tau), \ldots, p_{k-1}(\pi, \tau) \in A \) such that, letting \( A_0(\pi, \tau) = \{ a_i(\pi, \tau) \mid i < k \} \), we have that \( p_0(\pi, \tau) \vdash \chi(\pi, \tau)(x) \) and \( p_0(\pi, \tau) \models \models A_0 \vdash p_0(\pi, \tau) \). Now, for every such \( \pi, \tau \), define \( f^*_n(\pi, \tau) = h \) for some \( h \models p_0(\pi, \tau) \). It follows that for all \( n \leq N \), \( f^*_n \) is an \( (n, \varphi) \)-Skolem function.

By Claim [13] for every sequence of \( \varphi \)-Skolem functions \( \langle f_n \mid n \leq N \rangle \) (such that for all \( n \leq N \), \( f_n \) is an \( (n, \varphi) \)-Skolem function), there is some \( m \leq J \) and tuples \( \overline{a}_0, \ldots, \overline{a}_{m-1} \in A \) such that for every \( a \in A \)

\[ \{ t < m \mid \varphi(x, a) \in p \iff M \models \varphi(f^*_m(\overline{a}_t, \overline{a}_{\pi, c}), a) \} > \frac{m}{2} \]

In particular this true for \( \langle f^*_n \mid n \leq N \rangle \), hence we get that there are \( h_0, \ldots, h_{m-1} \in M^x \) (namely \( h_t = f^*_m(\overline{a}_t, \overline{a}_{\pi, c}) \)) such that:

- For every \( t < m \), \( tp_\varphi(h_t/A) \) is \( k \)-isolated, i.e., there are \( a_0^t, \ldots, a_{k-1}^t \in A \) such that for every \( h' \in M^x \) if for every \( i < k \), \( M \models \varphi(h_t, a_i^t) \leftrightarrow \varphi(h', a_i^t) \) then \( tp_\varphi(h_t/A) = tp_\varphi(h'/A) \), and
- We have that \( \{ t < m \mid \varphi(x, a) \in p \iff M \models \varphi(h_t, a) \} > \frac{m}{2} \).

We claim that the following formula (which is over \( A \))

\[ \psi(y) = \exists d_0, \ldots, d_{m-1} \bigwedge_{t < m} \bigwedge_{i < k} \varphi(d_t, a_i^t)^{(a_i^t, h_t)} \wedge \left( \{ t < m \mid \varphi(d_t, y) \} > \frac{m}{2} \right) \]

defines the type \( p \).

Indeed: first assume that \( \varphi(x, a) \in p \). Then by the second bullet we have that

\[ M \models \bigwedge_{t < m} \bigwedge_{i < k} \varphi(h_t, a_i^t)^{(a_i^t, h_t)} \wedge \left( \{ t < m \mid \varphi(h_t, a) \} > \frac{m}{2} \right). \]

And hence \( \psi(a) \) holds.

Now suppose \( M \models \psi(a) \) for \( a \in A \). Then there are \( h_0', \ldots, h_{m-1}' \) witnessing that \( \psi \) holds. But since \( h_i' \) agrees with \( h_i \) on \( a_0^t, \ldots, a_{k-1}^t \) for every \( t < m \), it follows by the first bullet that \( tp_\varphi(h_t/A) = tp_\varphi(h_t'/A) \) and in particular \( tp_\varphi(h_t/A) = tp_\varphi(h_t'/A) \) which implies that \( \{ t < m \mid \varphi(h_t, a) \} > \frac{m}{2} \). Towards contradiction, if \( \varphi(x, a) \not\in p \) then by the second bullet it follows that \( M \models \neg \varphi(h_t, a) \) for more than half the \( t \)'s but there are more than half the \( t \)'s for which \( M \models \varphi(h_t, a) \) — contradiction.
To see that (2) is holds just note that
\[
\psi(y) = \exists d_0, \ldots, d_{m-1} \bigwedge_{t < m} \bigwedge_{i < k} \varphi(d_t, a^t_i)^{\varepsilon(a^t_i, h_t)} \wedge \big( \{|t < m| \varphi(d_t, y)| > m/2 \} \big)
\]
is equivalent to
\[
\bigvee_{s \subseteq m, |s| > m/2} \exists d_0, \ldots, d_{m-1} \bigwedge_{t < m} \bigwedge_{i < k} \varphi(d_t, a^t_i)^{\varepsilon(a^t_i, h_t)} \wedge \bigwedge_{t \in s} \varphi(d_t, y).
\]
This proves (2), where \(K(\varphi)\) can be easily recovered from the proof using \(N(VC(\varphi))\), \(J(VC(\varphi))\) and \(k(VC(\varphi), N(VC(\varphi)))\) — a rough estimate is \(K(\varphi) \leq J(VC(\varphi)) \cdot k(VC(\varphi), N(VC(\varphi)))\) (assuming both are positive which we may).

In order to show that (3) holds note that for every \(a \in A\) and every \(s \subseteq m\) we have:
\[
M \models \exists d_0, \ldots, d_{m-1} \bigwedge_{t < m} \bigwedge_{i < k} \varphi(d_t, a^t_i)^{\varepsilon(a^t_i, h_t)} \wedge \bigwedge_{t \in s} \varphi(d_t, a)
\]
if and only if
\[
M \models \forall d_0, \ldots, d_{m-1} \bigwedge_{t < m} \bigwedge_{i < k} \varphi(d_t, a^t_i)^{\varepsilon(a^t_i, h_t)} \to \bigwedge_{t \in s} \varphi(d_t, a).
\]
Indeed, this follows easily from the first bullet as above (and the fact that for all \(t < m\) the formula \(\bigwedge_{i < k} \varphi(x, a^t_i)^{\varepsilon(a^t_i, h_t)}\) is consistent).

From the above we get that for every \(a \in A\) we have that \(M \models \psi(a) \iff \psi'(a)\) where
\[
\psi'(y) = \bigvee_{s \subseteq m, |s| > m/2} \forall d_0, \ldots, d_{m-1} \bigwedge_{t < m} \bigwedge_{i < k} \varphi(d_t, a^t_i)^{\varepsilon(a^t_i, h_t)} \to \bigwedge_{t \in s} \varphi(d_t, y)
\]
which gives (3). \(\square\)

Finally, let us finish the proof.

**Proof of Theorem 13.** Assuming either (2) or (3), it is clear that there are at most finitely many formulas of the forms described there (note that the number of disjuncts in the formula is bounded). Thus by coding finitely many formulas into one as in [Gui12, Lemma 2.5] we get (4).

(4) implies (1) easily follows from type-counting argument as mentioned in the introduction. From (4) it follows that for any finite set \(A \subseteq M^n\) for any \(M \models T\), \(|S_{\varphi}(A)| \leq |A|^{|z|}\) where \(\psi(y, z)\) uniformly defines \(\varphi\)-types. On the other hand, if the VC-dimension of \(\varphi\) is not bounded in \(T\), then we can find finite sets \(A\) as above such that \(|S_{\varphi}(A)|\) is exponential in \(|A|\), contradiction. \(\square\)

**Remark 19.** Note that from the proof of Theorem 13 one can extract an explicit bound for the length of the variable \(z\) in the defining formula \(\psi(y, z)\) that suits \(\varphi(x, y)\) (as in Definition 4) which depends only on \(VC(\varphi)\). An inspection of the proof can give a better description of the formula given in the formulation of Theorem 13. Any improvement on the bounds we used will automatically induce a shorter formula.
3. Open questions

3.1. NIP defining formula.

Question 20. Can we improve Theorem 13 as to ensure that the defining formula \( \psi(y, z) \) is itself NIP?

Let us justify this question in two instances, one is when \( \varphi \) is stable in \( T \), and the other is when \( T \) has definable Skolem functions.

3.1.1. Stable formulas. Suppose that \( T \) is a theory. As in Definition 10, a formula \( \varphi(x, y) \) is stable in \( T \) if it is stable in any completion of \( T \). If \( \varphi(x, y) \) is stable in \( T \) then a much stronger result than UDTFS holds: there is some formula \( \psi(y, z) \) such that for any \( M \models T \) and any \( A \subseteq M^y \), every \( \varphi \)-type \( p \in S_\varphi(A) \) is definable by a formula of the form \( \psi(y, a) \) for some \( a \in A^x \). This can be deduced using the 2-rank as in [She90, Chapter II, Theorem 2.12 (3)]. However, the formula that this proof gives involves quantifiers so it is not obviously stable. Using the apparatus of non-forking extensions one can overcome this as we now explain (this is probably well-known, but it is not stated explicitly like this as far as we know).

First, for every \( M \models T \), any \( \varphi \)-type over \( M \) is definable over \( M \) by some formula \( \psi(y, m) \) where \( \psi(y, z) \) is a Boolean combination of (positive) instances of \( \varphi^{opp} \), see e.g., [Pil96, Lemma 2.2(i)]. In particular, \( \psi(y, z) \) is itself a stable formula and moreover stable in \( T \) (see e.g., [Pil96, Lemma 2.1]).

If \( A \subseteq M \models T \) is any set and \( p \in S_\varphi(A) \) (in the notation of the previous section we should have written \( p \in S_\varphi(A^y) \), but we ignore this for now), then \( p \) has a non-forking extension \( q \in S_\varphi(M) \) which is definable over \( acl^q(A) \) via some formula \( \theta(y, a) \) (see e.g., [Pil96, Lemma 2.7]). By the first paragraph \( q \) is also definable over \( M \) via a Boolean combination of instances of \( \varphi \), so that \( \theta(y, a) \) is itself equivalent in \( M^q \) to a Boolean combination of instances of \( \varphi \).

Proposition 21. Suppose that \( M \models T \), \( \psi(x, y) \) and \( \theta(x, z) \) are formulas and \( a \in M^x, a' \in M^z \). Suppose that \( \psi(x, a) \) is equivalent to \( \theta(x, a') \) in \( M \) and that \( \theta(x, z) \) is stable in \( T \). Then there is a formula \( \psi'(x, y) \) such that \( \psi'(x, a) \) is equivalent to \( \psi(x, a) \) in \( M \) and \( \psi'(x, y) \) is stable in \( T \).

Proof. Let \( \tau(yz) = \forall x(\psi(x, y) \leftrightarrow \theta(x, z)) \) so that \( M \models \tau(aa') \). Let \( \psi'(x, y) = \psi(x, y) \land \exists z \tau(yz) \). Then \( \psi'(x, a) \) is equivalent to \( \psi(x, a) \) in \( M \) and \( \psi' \) is stable in \( T \): suppose towards contradiction that \( \langle b_i a_i | i < \omega \rangle \) is an indiscernible sequence which witnesses that \( \psi' \) has the order property \( (\psi'(b_i, a_j) \text{ iff } i \leq j) \) in some model \( N \models T \). Then, since e.g., \( \psi'(b_0, a_0) \) holds, by indiscernibility it follows that for every \( j < \omega \), there exists some \( a'_j \) such that \( \tau(a_j a'_j) \) holds. Hence we get that \( i \leq j \text{ iff } \theta(b_i, a'_j) \).

Continuing our discussion from above, from Proposition 21 it follows that \( q \) is definable over \( acl^q(A) \) via some formula \( \theta'(y, a) \) such that \( \theta'(y, z) \) is stable in \( T \).
Let $\chi(z,c) \in \text{tp}(a/A)$ be an algebraic formula of minimal (finite) size so that if $a' \models \chi$ then $a' \equiv_A a$. Let $\beta(y,c) = \exists z \chi(z,c) \land \theta'(y,z)$. Then $\beta(y,c)$ is equivalent (in $M$) to $\bigvee_{a' \models \chi(z,c)} \theta'(y,a')$ and $\bigvee_{i<m} \theta'(y,z_i)$ is stable in $T$ as a Boolean combination of stable (in $T$) formulas (where $m$ is the size of $\chi(M,c)$). Hence by Proposition 21 there is some formula $\beta'(y,w)$ such that $\beta'(y,c)$ defines $p$ and $\beta'(y,w)$ is stable in $T$.

Now, if we had started with $A \subseteq M^y$, then we could have let $A'$ be the set of all elements appearing in some tuple from $A$, and do the same process.

From all this we get that:

**Corollary 22.** If $\varphi(x,y)$ stable in $T$, $M \models T$, $A \subseteq M^y$ and $p \in S_{\varphi}(A)$ then $p$ is $A$-definable via a formula $\psi(y,z)$ which is stable in $T$.

And a uniform version:

**Corollary 23.** Suppose that $T$ is any $L$-theory and that $\varphi(x,y)$ is stable in $T$. Then there is a formula $\psi(y,z)$, stable in $T$, such that for all $M \models T$, all $A \subseteq M^y$ with $|A| \geq 2$ and all $p \in S_{\varphi}(A)$ there is some $a \in A^z$ such that $\psi(y,a)$ defines $p$.

**Proof.** By compactness, as in the proof of [Shelah 90, Chapter II, Theorem 2.12(3)].

First we show that (*) there are finitely many formulas, each stable in $T$, which work for every $p \in S_{\varphi}(A)$. Indeed, assume not and add a new predicate $P(y)$ to the language and consider the partial type $\Gamma(x)$ in the language $L \cup \{P\}$ consisting of formulas $\theta_{\psi}(x)$ where

$$\theta_{\psi}(x) = \neg \exists z \in P(\forall y \in P(\varphi(x,y) \leftrightarrow \psi(y,z)))$$

for every formula $\psi(y,z)$ which is stable in $T$ (where $z$ is a tuple of copies of $y$).

By our assumption towards contradiction, for all formulas $\psi_0(y,z_0), \ldots, \psi_{k-1}(y,z_{k-1})$ which are stable in $T$, there is some $M \models T$, $A \subseteq M^y$ and a type $p \in S_{\varphi}(A)$ which is not definable by any of the formulas $\psi_0, \ldots, \psi_{k-1}$. This means that $\Gamma(x)$ is consistent. By compactness there is a model $M \models T$, $A \subseteq M$ and $a \in M^z$ such that $\text{tp}_{\varphi}(a/A)$ is not definable by any stable in $T$ formula $\psi(y,z)$. But this contradicts Corollary 22. This shows (*).

Now using the standard coding trick as in [Gu12, Lemma 2.5] we can code finitely many formulas $\psi_0, \ldots, \psi_{k-1}$ into one formula $\psi$. We leave it to the reader to make sure that if all the formulas $\psi_0, \ldots, \psi_{k-1}$ are stable in $T$, then so is $\psi$. \hfill $\square$

3.1.2. **Having definable Skolem functions.**

**Proposition 24.** If $T$ is any theory with definable Skolem functions and $\varphi(x,y)$ is a NIP formula in $T$, then there is a formula $\psi(y,z)$ which uniformly defines $\varphi$-types over finite sets and such that $\psi(y,z)$ is itself NIP in $T$. 
Proof. By inspecting the proof of Theorem 13 (1) implies (2), one sees that in the case when \( T \) has definable Skolem functions, then we would not have to use Corollary 12 at all. Instead, we could define the \( \varphi \)-type \( p \in S_\varphi(A) \) by

\[
\varphi(x,a) \in p \iff \left\{ t < m \mid \varphi(\tilde{f}(\tilde{x},\tilde{a},\tilde{b}),a) > \frac{m}{2}, \right. \]

where \( f, \tilde{f} \) are the definable Skolem whose existence we assume (i.e., \( f_n, \tilde{f}(\tilde{x}) \) returns some element \( b \) satisfying \( \bigwedge_{i<n} \varphi(x,a_i) \) if such exists). In other words, the formula \( \psi(y,b) \) defining \( p \) is a Boolean combination of instances of formulas of the form \( \varphi(f(b),y) \) for \( b \) a tuple from \( A \) and \( f \) some definable function.

In general, when \( \varphi(x,y) \) is a NIP formula in \( T \) and \( f \) is some definable function, then \( \psi(z,y) = \varphi(f(z),y) \) is also NIP in \( T \) (if \( \{ \psi(M,b) \mid b \in M^y \} \) shatters some set \( A \subseteq M^z \) then \( \{ \varphi(M,b) \mid b \in M^y \} \) shatters the image of \( A \) under \( f \) which has the same size as \( A \). This means (by Fact 8) that \( \psi(y,z) \) is NIP in \( T \).

As in the stable case, coding finitely many NIP formulas into one gives a NIP formula, so we are done. \( \Box \)

3.2. Honest definitions.

**Definition 25.** [Sim15, Definition 3.16 and Remark 3.14] Work in some theory \( T \). Suppose that \( \varphi(x,y) \) is a formula, \( A \subseteq M^y \) some set and \( p \in S_\varphi(A) \). Say that a formula \( \psi(y,z) \) (over \( \emptyset \)) is an honest definition of \( p \) if for every finite \( A_0 \subseteq A \) there is some \( b \in A^z \) such that for all \( a \in A \), if \( \psi(a,b) \) holds then \( \varphi(x,a) \in p \) and for all \( a \in A_0 \) the other direction holds: if \( \varphi(x,a) \in p \) then \( \psi(a,b) \) holds.

It is proved in [Sim15, Theorem 6.16], [CSI15, Theorem 11] that if \( T \) is NIP then for every \( \varphi(x,y) \) there is a formula \( \psi(y,z) \) that serves as an honest definition for any type in \( S_\varphi(A) \) provided that \( |A| \geq 2 \).

**Question 26.** Is this true only assuming that \( \varphi(x,y) \) is NIP?

Note that for the proof in [Sim15, Theorem 6.16], [CSI15, Theorem 11], only a very mild assumption of NIP is actually needed. See [CSI15, Remark 16].

**References**

[Adl08] Hans Adler. An introduction to theories without the independence property. *Archive of Mathematical Logic*, 2008. accepted.

[CCT16] Xi Chen, Yu Cheng, and Bo Tang. On the recursive teaching dimension of VC classes. In Daniel D. Lee, Masashi Sugiyama, Ulrike von Luxburg, Isabelle Guyon, and Roman Garnett, editors, *Advances in Neural Information Processing Systems 29: Annual Conference on Neural Information Processing Systems 2016, December 5-10, 2016, Barcelona, Spain*, pages 2164–2171, 2016.
[CS15] Artem Chernikov and Pierre Simon. Externally definable sets and dependent pairs II. Trans. Amer. Math. Soc., 367(7):5217–5235, 2015.

[Gui12] Vincent Guingona. On uniform definability of types over finite sets. J. Symbolic Logic, 77(2):499–514, 2012.

[HWL17] Lunjia Hu, Ruihan Wu, Tianhong Li, and Liwei Wang. Quadratic upper bound for recursive teaching dimension of finite VC classes. CoRR, abs/1702.05677, 2017.

[JL10] H. R. Johnson and M. C. Laskowski. Compression schemes, stable definable families, and o-minimal structures. Discrete Comput. Geom., 43(4):914–926, 2010.

[LS13] Roi Livni and Pierre Simon. Honest compressions and their application to compression schemes. In Shai Shalev-Shwartz and Ingo Steinwart, editors, COLT 2013 - The 26th Annual Conference on Learning Theory, June 12–14, 2013, Princeton University, NJ, USA, volume 30 of JMLR Proceedings, pages 77–92. JMLR.org, 2013.

[MY16] Shay Moran and Amir Yehudayoff. Sample compression schemes for VC classes. J. ACM, 63(3):Art. 21, 10, 2016.

[Pil96] Anand Pillay. Geometric stability theory, volume 32 of Oxford Logic Guides. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[She90] Saharon Shelah. Classification theory and the number of nonisomorphic models, volume 92 of Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, second edition, 1990.

[Sim15] P. Simon. A Guide to NIP Theories. Lecture Notes in Logic. Cambridge University Press, 2015.

[vN28] J. v. Neumann. Zur Theorie der Gesellschaftsspiele. Math. Ann., 100(1):295–320, 1928.

Itay Kaplan, Einstein Institute of Mathematics, Hebrew University of Jerusalem, 91904, Jerusalem Israel.

E-mail address: kaplan@math.huji.ac.il

Shlomo Eshel, Einstein Institute of Mathematics, Hebrew University of Jerusalem, 91904, Jerusalem Israel.

E-mail address: Shlomo.Eshel@mail.huji.ac.il