ON TOTALLY RAMIFIED EXTENSIONS OF DISCRETE VALUED FIELDS

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Abstract. We give a simple characterization of the totally wild ramified valuations in a Galois extension of fields of characteristic $p$. This criterion involves the valuations of Artin-Schreier cosets of the $\mathbb{F}_p^r$-translation of a single element. We apply the criterion to construct some interesting examples.

1. Introduction

Let $F/E$ be a Galois extension of fields of characteristic $p$ of degree $q$, a power of $p$. This work gives a simple criterion that classifies the totally ramified discrete valuations of $F/E$.

The classical case where $F/E$ is a $p$-extension, hence generated by a root of an Artin-Schreier polynomial $X^p - X - a$ with $a \in E$, is well known: a discrete valuation $v$ of $E$ totally ramifies in $F$ if and only if the maximum of the valuation in the coset $a + E^p - E$ is negative, i.e., $m_{a,v} = \max\{v(b) \mid b \in a + E^p - E\} < 0$. A standard Frattini argument reduces the general case to finitely many $p$-extensions, or in other words to a criterion with finitely many elements. More precisely, there exist $a_1, \ldots, a_n \in E$ such that $v$ totally ramifies in $F$ if and only if $m_{a_i,v} < 0$ for all $i$ ($n$ being the minimal number of generators of the Frattini quotient).

The goal of this work is to simplify this criterion and show that there exists (a single) $a \in E\mathbb{F}_q$ such that $v$ totally ramifies in $F$ if and only if $m_{a,v} < 0$, for all $\gamma \in \mathbb{F}_q^\times$ (see Theorem 3.2).

We apply our criterion to construct somewhat surprising examples: Assume $\mathbb{F}_q \subseteq E$ and that $F/E$ is generated by a degree $q$ Artin-Schreier polynomial $X^q - X - a$, $a \in E$. For a discrete valuation $v$ of $E$ let $M_{a,v} = \max\{v(b) \mid b \in a + E^q - E\}$ be the maximum of the valuation of the $q$-Artin-Schreier coset of $a$. It is an easy exercise to show that if $M_{a,v} < 0$ and $\gcd(p, M_{a,v}) = 1$, then $v$ totally ramifies in $F$. So one might suspect that $M_{a,v}$ encodes the information whether $v$ totally ramifies in $F$ as in the case $q = p$. However this is false: We construct two extensions with the same $M_{a,v} < 0$. In the first example $v$ totally ramifies in $F$ although $p \mid M_{a,v}$. In the second example $v$ does not totally ramify although it does ramify in $F$.

Date: July 2, 2009.

2000 Mathematics Subject Classification. 12G10.

Key words and phrases. Ramification, Artin-Schreier.

The first author was partially supported by the Lady Davis fellowship trust, and the second author was partially supported by the Israel Science Foundation (Grant No. 343/07).
Notation. Let $F/E$ be a Galois extension of fields of characteristic $p$ of degree a power of $p$. We write $q = p^r$ for the degree $[F : E]$ of the extension. The symbol $v$ denotes a discrete valuation of $E$, and $w$ a valuation of $F$ lying above $v$. We denote by $\mathbb{F}_{p^r}$ the finite field with $p^r$ elements. Sometimes we identify $\mathbb{F}_{p^r}$ with its additive group. The multiplicative group of a field $K$ is denoted by $K^\times$.

For an element $a \in E$ and discrete valuation $v$ of $E$ we denote
\begin{equation}
(1) \quad m_{a,v} = m(a, E, v) = \max \{ v(b) \mid b \in a + E^p - E \}
\end{equation}
if the valuation set of the elements in the coset is bounded, and $m_{a,v} = \infty$ otherwise.

2. Classical Theory

Let us start this discussion by recalling the well known case $q = p$. In this case Artin-Schreier theory tells us that $F = E(\alpha)$, where $\alpha$ satisfies an equation $X^p - X = a$, for some $a \in E$. Furthermore, one can replace $\alpha$ with a solution of $X^p - X = b$, for any $b \in a + E^p - E$. We have the following classical result (cf. [3, Proposition III.7.8]).

**Theorem 2.1.** Assume $F = E(\alpha)$, for some $\alpha \in F$ satisfying an equation $X^p - X = a$, $a \in E$. Then the following conditions are equivalent for a discrete valuation $v$ of $E$.

(a) $v$ totally ramifies in $F$.
(b) there exists $b \in a + E^p - E$ such that $\gcd(p, v(b)) = 1$ and $v(b) < 0$.
(c) $m_{a,v} < 0$.

If these conditions hold, then $v(b) = m_{a,v}$, and in particular $v(b)$ is independent of the choice of $b$. Moreover, if $\beta$ is another Artin-Schreier generator, i.e., $F = E(\beta)$, and $\beta^p - \beta = a_\beta \in E$, then $m_{a_\beta,v} = m_{a,v}$.

We return to the case of an arbitrary $q = p^r$. Then a standard Frattini argument reduces the question of when a discrete valuation $v$ of $E$ totally ramifies in $F$ to extensions with $p$-elementary Galois group. Here a group $G$ is $p$-elementary if $G$ is abelian and of exponent $p$; equivalently $G \cong \mathbb{F}_q$. For the sake of completeness, we provide a formal proof of the reduction.

**Proposition 2.2.** There exists $\bar{F} \subseteq F$ such that $\text{Gal}(\bar{F}/E)$ is $p$-elementary and a discrete valuation $v$ of $E$ totally ramifies in $F$ if and only if $v$ totally ramifies in $\bar{F}$.

**Proof.** Prolong $v$ to a valuation $w$ of $F$. Let $G = \text{Gal}(F/E)$, let $\Phi = \Phi(G) = G^p[G, G]$ be the Frattini subgroup of $G$, and let $\bar{F} = F^\Phi$ be the fixed field of $\Phi$ in $F$. Let $\bar{w}$ be the restriction of $w$ to $\bar{F}$. Then $\text{Gal}(\bar{F}/E) \cong G/\Phi$ is $p$-elementary.

Let $I_{w/v}, I_{\bar{w}/v}$ be the inertia groups of $w/v, \bar{w}/v$, respectively. Let $r: \text{Gal}(F/E) \to \text{Gal}(\bar{F}/E)$ be the restriction map. Then $r(I_{w/v}) = I_{\bar{w}/v}$ [2, Proposition I.8.22]. This implies that $I_{w/v} = G$ if and only if $I_{\bar{w}/v} = r(I_{w/v}) = \text{Gal}(\bar{F}/E)$ (recall that a subgroup $H$ of a finite group $G$ satisfies $H\Phi(G) = G$ if and only if $H = G$).

**Remark 2.3.** The Frattini subgroup is the intersection of all maximal subgroups. Therefore $\bar{F}$, as its fixed field, is the compositum of all minimal sub-extensions of $F/E$.

Applying Theorem 2.1 for $\bar{F}$ gives the following
Corollary 2.4. Let $F/E$ be a Galois extension of degree $q = p^r$. Then there exist $a_1, \ldots, a_n \in E$ such that for any discrete valuation $v$ of $E$ we have $v$ totally ramifies in $F$ if and only if $m_{a_i, v} < 0$ for all $i$.

3. CRITERION FOR TOTAL RAMIFICATION USING ONE ELEMENT

In this section we strengthen Corollary 2.4 and prove that it suffices to take $\mathbb{F}_q^\times$-translation of a single element. For this we need the following lemma.

Lemma 3.1. Let $p$ be a prime and $q = p^r$ a power of $p$. Consider a tower of extensions $\mathbb{F}_q \subseteq E \subseteq F$ with $q = [F : E]$. Assume $F = E(x)$ for some $x \in F$ that satisfies $a = x^q - x \in E$. Then the family of fields generated over $E$ by roots of $X^q - X - \gamma a$, where $\gamma$ runs over $\mathbb{F}_q^\times$, coincides with the family of all minimal sub-extensions of $F/E$.

Proof. Since $X^q - X - a = \prod_{\alpha \in \mathbb{F}_q}(X - (x + \alpha))$, the extension $F/E$ is Galois. Let $G = \text{Gal}(F/E)$, then the map

$$\phi: \begin{cases} G \to \mathbb{F}_q \\ \sigma \mapsto \sigma(x) - x \end{cases}$$

is well defined. Moreover it is immediate to verify that $\phi$ is an isomorphism.

Let $C$ be the kernel of the trace map $\text{Tr}: \mathbb{F}_q \to \mathbb{F}_p$: $\text{Tr}(u) = u^{q^r - 1} + \cdots + u$. It is well known that $T$ is a non-trivial linear transformation [1] Theorem VI.5.2] over $\mathbb{F}_p$. This implies that $T$ is surjective, so $C$ is a hyper-space of $\mathbb{F}_q$ (as a vector space over $\mathbb{F}_p$).

The minimal sub-extensions of $F/E$ are the fixed fields of maximal subgroups of $\text{Gal}(F/E)$, which correspond to hyper-spaces of $\mathbb{F}_q$ via $\phi$. Let $C'$ be a hyper-space in $\mathbb{F}_q$. Then there exists an automorphism $M: \mathbb{F}_q \to \mathbb{F}_q$ under which $M(C') = C$. But $\text{Aut}(\mathbb{F}_q) = \mathbb{F}_q^\times$, so $M$ acts by multiplying by some $\gamma \in \mathbb{F}_q^\times$. Hence $\gamma C' = C$. Vice-versa, if $\gamma \in \mathbb{F}_q^\times$, then $\gamma^{-1}C$ is a hyper-space. Therefore, it suffices to show, for an arbitrary $\gamma \in \mathbb{F}_q^\times$, that the fixed field $F'$ of $H := \phi^{-1}(\gamma^{-1}C)$ in $F$ is generated by a root of $X^q - X - \gamma a$.

Let $y = \gamma x$. Then $F = E(y)$ and

$$y^q - y = \gamma^q x^q - \gamma x = \gamma(x^q - x) = \gamma a.$$

Let $z = y^{p^r - 1} + \cdots + y^p + y$. Then $z \notin E$, hence $[F : E(z)] \leq p^{r-1}$. We have

$$z^p - z = y^{p^r} + \cdots + y^{p^2} + y^p - (y^{p^r - 1} + \cdots + y^p + y) = y^q - y = \gamma a.$$ 

Thus $[E(z) : E] \leq p$, and we get $[F : E(z)] = p^{r-1}$. To complete the proof we need to show that $F' = E(z)$, so it suffices to show that $H$ fixes $z$. Indeed, let $\sigma \in H = \phi^{-1}(\gamma^{-1}C)$. Then $\beta := \sigma(y) - y = \gamma(\sigma(x) - x) = \gamma \phi(\sigma) \in C$. We have

$$\sigma(z) - z = \sigma(y^{p^r - 1} + \cdots + y^p + y) - (y^{p^r - 1} + \cdots + y^p + y) = (\sigma(y) - y)^{p^r - 1} + \cdots + (\sigma(y) - y)^p + (\sigma(y) - y) = \beta^{p^r - 1} + \cdots + \beta = \text{Tr}(\beta) = 0,$$

as needed. \qed
We are now ready for the main result that classifies totally ramified discrete valuations of Galois extensions in characteristic \( p \).

**Theorem 3.2.** Assume \( F/E \) is a Galois extension of fields of characteristic \( p \) of degree a power of \( p \) and with Galois group \( G \). Let \( d = d(G) \) be the minimal number of generators of \( G \) and let \( q = p^r \), for some \( r \geq d \) (e.g., \( q = [F : E] \)). Let \( F' = F \mathbb{F}_q \) and \( E' = E \mathbb{F}_q \). If \( v \) is a valuation of \( E \), we denote by \( v' \) its (unique) extension to \( E' \). Then there exists \( a \in E' \) such that for every discrete valuation \( v \) of \( E \) the following is equivalent.

(a) \( v \) totally ramifies in \( F' \).
(b) \( v' \) totally ramifies in \( F' \).
(c) \( m(\gamma a, E', v') < 0 \), for every \( \gamma \in \mathbb{F}_q^\times \).
(d) There exists \( b_\gamma \in \gamma a + (E')^p - E' \) such that \( \gcd(p, v'(b_\gamma)) = 1 \) and \( v'(b_\gamma) < 0 \), for every \( \gamma \in \mathbb{F}_q^\times \).

**Remark 3.3.** In the above conditions (c) and (d) it suffices that \( \gamma \) runs over representatives of \( \mathbb{F}_q^\times / \mathbb{F}_p^\times \).

**Proof.** Since finite fields admit only trivial valuations, we get that both \( F'/F \) and \( E'/E \) are unramified, so (a) and (b) are equivalent. Theorem 2.1 implies that (c) and (d) are equivalent. So it remains to prove that (a) and (c) are equivalent. For simplicity of notation, we replace \( F, E \) with \( F', E' \) and assume that \( \mathbb{F}_q \subseteq E \).

Let \( \bar{F} \subseteq F \) be the extension given in Proposition 2.2. Let \( d \) be the minimal number of generators of \( \text{Gal}(\bar{F}/E) \). Then \( \bar{q} = p^d = [\bar{F} : E] \) and \( d \leq d \). By Proposition 2.2 we may replace \( \bar{F} \) with \( F \), and assume that \( \text{Gal}(F/E) \cong \mathbb{F}_q \).

By Artin-Schreier Theory \( F = E(x) \), where \( x \) satisfies the equation \( x^q - x = a \), for some \( a \in E \). Lemma 3.1 implies that all the minimal sub-extensions of \( F/E \) are generated by roots of \( X^q - X - \gamma a \), where \( \gamma \) runs over \( \mathbb{F}_q^\times \). Note that \( v \) totally ramifies in \( F \) if and only if \( v \) totally ramifies in all the minimal sub-extensions of \( F/E \) (since if the inertia group is not the whole group, it fixes some minimal sub-extension, so \( v \) does not ramify in this sub-extension). This finishes the proof, since by Theorem 2.1 \( v \) totally ramifies in all the minimal sub-extensions of \( F/E \) if and only if \( m(\gamma a, E, v) < 0 \), for all \( \gamma \in \mathbb{F}_q^\times \). \( \square \)

4. An application

We come back to the case where \( \mathbb{F}_q \subseteq E \subseteq F \), and \( F/E \) is a Galois extension with Galois group isomorphic to \( \mathbb{F}_q \). By Artin-Schreier Theory \( F = E(x) \), where \( x \in F \) satisfies an equation \( X^q - X = a \), for some \( a \in E \). This \( a \) can be replaced by any element of the coset \( a + E^q - E \). If there exists \( b \in a + E^q - E \) such that \( v(b) < 0 \) and \( \gcd(q, v(b)) = 1 \), then \( v \) totally ramifies in \( F \). It is reasonable to suspect that the converse also holds, as in the case \( q = p \). We bring two interesting examples. The first is a totally ramified extension such that there exists no \( b \) as above. The other construction is of an extension which is not totally ramified, although Condition (c) of Theorem 3.2 holds for \( \gamma = 1 \).

**Proposition 4.1.** Let \( p \) be a prime, \( d \geq 1 \) prime to \( p \), \( q = p^r \), and let \( E = \mathbb{F}_q(t) \). Consider the \( t \)-adic valuation, i.e., \( v(t) = 1 \). Let \( \gamma \neq 1 \) be an element of \( \mathbb{F}_q \) with norm 1 (w.r.t. the
extension $\mathbb{F}_q / \mathbb{F}_p$). Consider an element
\[ a(t) = \frac{1}{td} - \frac{\gamma}{td} + f(t) \in E \]
and let $F = E(x)$, where $x$ satisfies $x^q - x = a$. Then
(a) If $d > 1$ and $f(t) = \frac{1}{t}$, then $\text{Gal}(F/E) \cong \mathbb{F}_q$, $v$ totally ramifies in $F$, but there is no $b \in a + E^q - E$ whose valuation is prime to $p$.
(b) If $f(t) = t$, then $\max\{v(b) \mid b \in a + E^q - E\} < 0$ but $v$ does not totally ramify in $F$.

Proof. Let $\delta \in \mathbb{F}_q^\times$. For $\epsilon \in \mathbb{F}_q$ with $\epsilon^p = \delta$ we set
\[ b_\delta(t) = \delta \alpha(t) - (\epsilon \frac{\epsilon}{td})^p + \epsilon \frac{\epsilon}{td} = \frac{\epsilon}{td} - \delta \gamma + \delta f(t). \]

Since $\gamma \neq 1$ has norm 1, $\gamma = \frac{\delta_0}{\delta^q}$, for some $\delta_0 \in \mathbb{F}_q$ (Hilbert 90).

Take $f(t) = \frac{1}{t}$. Then $v(b(t))$ is either $-d$ if $\epsilon \neq \delta \gamma$ or $-1$ if $\epsilon = \gamma \delta$, so $p \nmid v(b_\delta) < 0$. By Theorem 3.2, $v$ totally ramifies in $F$.

To this end assume there exists $b \in a + E^q - E$ with $p \nmid v(b) < 0$, and let $m = v(b)$.

By Lemma 3.1 the minimal sub-extensions of $F/E$ are generated by roots of $X^p - X - \delta b$, where $\delta \in \mathbb{F}_q^\times$. But $v(\delta b) = v(b)$, so $-d = m(b_\delta, E, v) = m(b, E, v) = m(b_\delta + E^q - E) = -1$ (Theorem 2.1). This contradiction implies that such $b$ does not exists, as needed for (a).

For (b) assume that $f(t) = t$, so $v(b_\delta) = v(f(t)) = 1$ by (2). So $v$ is not totally ramified in $F$ (Theorem 3.2). Assume there was $b \in a + E^q - E$ with $v(b) \geq 0$. Then all the minimal sub-extensions $F'$ of $F/E$ were generated by $X^p - X - \delta b$, where $\delta \in \mathbb{F}_q^\times$. But $v(\delta b) = v(b) \geq 0$, so all $F'$ are unramified (Theorem 2.1). This conclusion contradicts the fact that the extension generated by $X^p - X - b_\delta^q$ is ramified. So $\max\{v(b) \mid b \in a + E^q - E\} < 0$, as needed.

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