ON THE POSITIVE-DEFINITENESS OF AN ANISOTROPIC OPERATOR

CHARLES E. BAKER

Abstract. We study the positive-definiteness of a family of $L^2(\mathbb{R})$ integral operators with kernel $K_{t,a}(x, y) = (1 + (x - y)^2 + a(x^2 + y^2)t)^{-1}$, for $t > 0$ and $a > 0$. For $0 < t \leq 1$ and $a > 0$, the known theory of positive-definite kernels and conditionally negative-definite kernels confirms positive-definiteness. For $t > 1$ and $a$ sufficiently large, the integral operator is not positive-definite. For $t$ not an integer, but with integer odd part, the integral operator is not positive-definite.

1. Introduction

In this paper, we study the integral operators $K_{t,a}$ defined by

\[ K_{t,a}[c](x) = \int_{\mathbb{R}} K_{t,a}(x, y)c(y) \, dy, \quad c \in L^2(\mathbb{R}), \]

where

\[ K_{t,a}(x, y) = \frac{1}{\pi} \cdot \frac{1}{1 + (x - y)^2 + a(x^2 + y^2)t}, \quad t > 0, \ a > 0. \]

$K_{t,a}$ is a bounded, compact operator from $L^2(\mathbb{R})$ to itself for $t > 0$ and $a > 0$. This is a straightforward verification using the basic facts of compact operators and the Schur Test.

This operator (with $t = 2$) was considered by P. Krotkov and A. Chubukov in the papers [3], [4] as a part of their simplified model for high-temperature superconductivity. The asymptotics of the largest eigenvalue of this operator, around $a = 0$, were studied in the papers [6], [7], and [1]. Examining an open question stated in Section 6.2 of [7], we wish to determine for which $t > 0$ and $a > 0$ the operator $K_{t,a}$ is positive-definite; that is, we wish to determine for which $t > 0$ and $a > 0$ the inequality

\[ \langle K_{t,a}[c], c \rangle = \iint_{\mathbb{R} \times \mathbb{R}} K_{t,a}(x, y)c(x)\overline{c(y)} \, dy \, dx \geq 0 \]

holds for all $c \in L^2(\mathbb{R})$.

By the continuity of the kernel $K_{t,a}$ and the boundedness of the operator $K_{t,a}$, positive-definiteness of $K_{t,a}$ is equivalent to the statement that

\[ \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k K_{t,a}(x_j, x_k) \geq 0 \]

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for all \( n \in \mathbb{N} \), \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), and \( c = (c_1, \ldots, c_n) \in \mathbb{C}^n \).

More generally, for any nonempty set \( X \), a function \( K : X \times X \to \mathbb{C} \) is called a positive-definite kernel if
\[
\sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} K(x_j, x_k) \geq 0
\]
holds for all \( n \in \mathbb{N} \), \( x = (x_1, \ldots, x_n) \in X^n \), and \( c = (c_1, \ldots, c_n) \in \mathbb{C}^n \).

In this paper, we prove the following.

**Proposition 1.** For \( 0 < t \leq 1 \) and \( a > 0 \), \( K_{t,a} \) is a positive-definite kernel; hence, \( K_{t,a} \) is a positive-definite operator.

**Proposition 2.** If \( t > 1 \), and \( a > a_0(t) := \frac{2t^2-1 + 2t^2-t}{(2t-1)(2t-1)} \), then \( K_{t,a} \) is not a positive-definite kernel; hence, \( K_{t,a} \) is not a positive-definite operator.

**Theorem 3.** If \( t \in \bigcup_{k \in \mathbb{N}} (2k-1, 2k) \), and if \( a > a_0 \), then \( K_{t,a} \) is not a positive-definite kernel; hence, \( K_{t,a} \) is not a positive-definite operator.

We prove these statements in Sections 2, 3, and 4, respectively.

2. **Conditionally Negative-Definite Kernels: the case \( 0 < t \leq 1 \)**

**Definition 2.1.** Let \( X \) be a nonempty set. A function \( K : X \times X \to \mathbb{C} \) is a conditionally negative-definite kernel if it is Hermitian (that is, for all \( x \) and \( y \) in \( X \), \( K(y, x) = \overline{K(x, y)} \)) and satisfies
\[
\sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} K(x_j, x_k) \leq 0
\]
for all \( n \in \mathbb{N} \), \( n \geq 2 \), \( x = (x_1, \ldots, x_n) \in X^n \), and \( c = (c_1, \ldots, c_n) \in \mathbb{C}^n \) with \( \sum_{j=1}^n c_j = 0 \).

Our interest in conditionally negative-definite kernels stems from the following connection to positive-definite kernels. Let \( \mathbb{C}^+ := \{ \zeta \in \mathbb{C} : \text{Re} \zeta \geq 0 \} \).

**Proposition 2.2** ([2], p. 75). \( K : X \times X \to \mathbb{C}^+ \) is a conditionally negative-definite kernel if and only if, for all \( r > 0 \),
\[
\frac{1}{r + \text{Re} K(x, y)}
\]
is a positive-definite kernel.

Therefore, if \( (x - y)^2 + a(x^2 + y^2)^t \) is a conditionally negative-definite kernel for some \( t > 0 \) and \( a > 0 \), then by the \( r = 1 \) case of Proposition 2.2
\[
\frac{1}{1 + (x - y)^2 + a(x^2 + y^2)^t}
\]
is a positive-definite kernel, and hence \( K_{t,a} \) is a positive-definite kernel. To prove Proposition 1 it therefore suffices to demonstrate the following.

**Claim 2.3.** If \( 0 < t \leq 1 \) and \( a > 0 \), then \( (x - y)^2 + a(x^2 + y^2)^t \) is a conditionally negative-definite kernel.
To prove this, we use the following fact.

**Proposition 2.4** ([2], Corollary 3.2.10). If $K : X \times X \to \mathbb{C}$ is a conditionally negative-definite kernel and satisfies $K(x, x) \geq 0$ for all $x \in X$, then $K^t$ is a conditionally negative-definite kernel for all $t$ such that $0 < t \leq 1$.

**Proof of Claim 2.3.** First, $(x - y)^2$ is a conditionally negative-definite kernel on $\mathbb{R} \times \mathbb{R}$: see [2], Section 3.1.10. Therefore, fixing $t, 0 < t \leq 1$, and $a > 0$, it suffices to show that $(x^2 + y^2)^t$ is a conditionally negative-definite kernel on $\mathbb{R} \times \mathbb{R}$, since the class of conditionally negative-definite kernels is closed under addition and positive scalar multiplication ([2], 3.1.11).

Moreover, for any function $f : X \to \mathbb{C}$, $f(x) + \overline{f(y)}$ is a conditionally negative-definite kernel; see [2], Section 3.1.9. In particular, $x^2 + y^2$ is a conditionally negative-definite kernel on $\mathbb{R} \times \mathbb{R}$. We may invoke Proposition 2.4, with $K(x, y) = x^2 + y^2$, because $x^2 + y^2 \geq 0$ whenever $x \in \mathbb{R}$. Therefore, $(x^2 + y^2)^t$ is a conditionally negative-definite kernel for all $t, 0 < t \leq 1$, as required. □

3. A necessary condition: the case $t > 1$, $a$ large

The $n = 2$ case of [2] implies that for all $x, y \in \mathbb{R}$,

$$|K_{t,a}(x, y)|^2 \leq K_{t,a}(x, x)K_{t,a}(y, y)$$

(see [2], Section 3.1.8), or

$$\left(\frac{1}{\pi} \cdot \frac{1}{1 + (x - y)^2 + a(x^2 + y^2)^t}\right)^2 \leq \left(\frac{1}{\pi} \cdot \frac{1}{1 + (x - x)^2 + a(x^2 + x^2)^t}\right) \cdot \left(\frac{1}{\pi} \cdot \frac{1}{1 + (y - y)^2 + a(y^2 + y^2)^t}\right).$$

We may rewrite this inequality as

$$0 \leq (1 + (x - y)^2)^2 - 1$$

$$+ 2a((1 + (x - y)^2)(x^2 + y^2)^t - 2^{t-1}((x^2)^t + (y^2)^t))$$

$$+ a^2((x^2 + y^2)^{2t} - (4x^2y^2)^t).$$

We now prove Proposition 2 by refuting a special case of (6).

**Proof of Proposition 2** Suppose that for some $t > 1$ and $a > 0$, $K_{t,a}$ is positive-definite. (6), with $y = 0$, implies that

$$0 \leq (1 + x^2)^2 - 1 + 2a(x^2)^t((1 + x^2) - 2^{t-1}) + a^2(x^2)^{2t},$$

or, letting $z = x^2 \geq 0$,

$$0 \leq g(z; t, a) := (1 + z)^2 - 1 + 2az^t((1 + z) - 2^{t-1}) + a^2 z^{2t}.$$

Note that unless $(1 + z) < 2^{t-1}$, $g(z; t, a)$ must be nonnegative for nonnegative $z$. Note also that $g(0; t, a) = 0$. Therefore, we may assume that $0 < z < 2^{t-1} - 1$.

We minimize $g(z; t, a)$ in the variable $a$: the minimizing value of $a$ is

$$a = \tilde{a}(z) := \frac{2^{t-1} - (1 + z)}{z^t}, \quad \text{or} \quad az^t = (2^{t-1} - (1 + z)),$$
and the minimum value of $g(z; t, a)$ in $a$ is

$$g(z; t, \tilde{a}_t(z)) = (1 + z)^2 - 1 - (2^{t-1} - (1 + z))^2 = 2^t z - (2^{t-1} - 1)^2.$$ 

Therefore, we see that $g(z; t, \tilde{a}_t(z)) < 0$ if $0 < z < z_0 := \frac{(2^{t-1} - 1)^2}{2^t}$; this contradicts (7) for $t > 1$ and $a = \tilde{a}_t(z)$. We now determine the set of $a$ that can be written as $\tilde{a}_t(z)$ for $0 < z < z_0$.

**Lemma 3.1.** For $t > 1$, the range of $\tilde{a}_t$ on $(0, z_0)$ is $(\tilde{a}_t(z_0), \infty)$.

**Proof.** We first show that for $t > 1$, $\tilde{a}_t(z)$ is continuous and strictly decreasing for $z \in (0, z_0)$. Indeed, the continuity and differentiability of $\tilde{a}_t(z)$, $z > 0$, is evident from (8), and the derivative in $z$ is

$$\frac{d}{dz} \tilde{a}_t(z) = \frac{-z^t - (2^{t-1} - (1 + z)) t z^{t-1}}{z^{2t}},$$

which is negative if $0 < z < 2^{t-1} - 1$. Since $0 < z < z_0$, and

$$z_0 = (2^{t-1} - 1) \cdot \frac{(2^{t-1} - 1)}{2^t} < \frac{1}{2}(2^{t-1} - 1),$$

$(0, z_0) \subset (0, 2^{t-1} - 1)$, so $\tilde{a}_t(z)$ is continuous and strictly decreasing on $(0, z_0)$.

Moreover, for all $t > 1$, $\lim_{z \to 0^+} \tilde{a}_t(z) = \infty$. Therefore, as $z$ decreases from $z_0$ to 0, $\tilde{a}_t(z)$ increases continuously from $\tilde{a}_t(z_0)$ to $\infty$. $\square$

Calculating the bounding value of $a$, we determine that

$$a_0(t) := \tilde{a}_t(z_0) = \frac{2^{2t-1} + 2^{2t-2}}{(2^{t-1} - 1)^2},$$

so for all $t > 1$ and $a > a_0(t)$, $K_{t,a}$ is not positive-definite. $\square$

We note that the above argument only studies a very special case of the $n = 2$ condition for positive-definiteness. For $a \leq a_0(t)$, the positive-definiteness of $a$ is in general undetermined. We now proceed to rule out positive-definiteness for more $(t, a)$ pairs.

4. An asymptotics argument: the case $t \notin \mathbb{N}$, $|t|$ odd

4.1. **Rewriting (2) to permit an asymptotics argument.** To describe another obstruction to positive-definiteness, we adjust (2). First, note that since $K_{t,a}$ is a real-valued and symmetric kernel (i.e., $K_{t,a}(y, x) = K_{t,a}(x, y)$), in (2), we may take the $c_i$ to be real as well; see [2], Section 3.1.6.

Adding together the fractions $c_j c_k K_{t,a}(x_j, x_k) = \frac{c_j c_k}{1 + (x_j - x_k)^2 + a(x_j^2 + x_k^2)}$ in (2), we note that the resulting denominator is positive, since each summand’s denominator is positive. We thus see that $K_{t,a}$ is a positive-definite kernel if and only if for all $n \in \mathbb{N}$, for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and for all $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$,

$$\sum_{j=1}^n \sum_{k=1}^n c_j c_k \prod_{\substack{(p, q) \in \{1, 2, \ldots, n\}^2 \atop (p, q) \neq (j, k)}} \left(1 + (x_p - x_q)^2 + a(x_p^2 + x_q^2)^t\right) \geq 0.$$ (9)
To separate the terms by homogeneity, for any fixed $x > 0$, define $y_j$ by $x_j = y_j x$. By the squaring, in all terms we achieve $x^2 = z > 0$, so we see that we wish to study the positivity of

$$
\sum_{j=1}^{n} \sum_{k=1}^{n} \prod_{(p,q) \in \{1,2,\ldots,n\}^2} \left(1 + (y_p - y_q)^2z + a(y_p^2 + y_q^2)z^{r_p}\right).
$$

For fixed $n$ in $\mathbb{N}$, $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$, and $c = (c_1,\ldots,c_n) \in \mathbb{R}^n$, we therefore define a function from $z \in \mathbb{R}^+$ to $\mathbb{R}$,

$$
f(z; n, y, c)
$$

$$
:= \sum_{j=1}^{n} \sum_{k=1}^{n} \prod_{(p,q) \in \{1,2,\ldots,n\}^2} \left(1 + (y_p - y_q)^2z + a(y_p^2 + y_q^2)z^{r_p}\right),
$$

and the positive-definiteness of $K_{t,a}$ requires that this function is always nonnegative on $(0, \infty)$, for all choices of the parameters $n, y, \text{and } c$.

We see that after writing out the products and collecting terms with like powers of $z$, $f$ admits a representation as a finite sum of the form

$$
f(z; n, y, c) = \sum_{k=1}^{\nu(n)} b_k(n, y, c) \cdot z^{r_k},
$$

where $r_0 < r_1 < \cdots < r_{\nu(n)}$ is an enumeration of the distinct powers of $z$ in $f$. If we can find parameters $n, y, \text{and } c$ such that the term of smallest degree in $z$ has negative coefficient, then by elementary asymptotics, for $z$ positive and small, $f(z; n, y, c)$ will be negative, and so $K_{t,a}$ will not be positive-definite.

Therefore, we organize the remainder of our paper as follows. Fix $t \notin \mathbb{N}$. In Section 4.2, we will choose a set of parameters $n, y, \text{and } c$ such that all coefficients of terms of degree less than $t$ in $f$ are zero. In Section 4.3, we will show that for the same parameters, the coefficient of $z^t$ is nonzero, and if the integer part of $t$ is odd, it will be negative. In Section 4.4, we will complete the proof.

Before continuing, however, we note some conventions. The decomposition of $t$ into its integer and fractional parts will be denoted $t = T + \tau$, where $T = \lfloor t \rfloor$ is the greatest integer less than or equal to $t$, and $\tau \in [0,1)$ is the fractional part. The set $\{1,2,\ldots,n\}$ is denoted $\mathbf{N}$. For any set $Y$, the set of $p$-element subsets of $Y$ is denoted $\binom{Y}{p}$; if $Y \neq \emptyset$, $\binom{Y}{0} = \{\emptyset\}$, the set whose only element is the empty set. Finally, we denote the $n$th rising factorial of $\alpha \in \mathbb{R}$ as $(\alpha)_n = (\alpha)(\alpha+1)\ldots(\alpha+n-1)$, for $n \in \mathbb{N}$.

4.2. Removing integer powers of $z$ in $f$. For $t$ not in $\mathbb{N}$, we now study the coefficients of terms in $f$ of degree less than $t$. Since all terms in $f$ are products of terms of degree 0, 1, or $t$, any term of degree less than $t$ must be the product of degree 0 and 1 terms, and hence must be a nonnegative integer. Therefore, to study coefficients of terms of degree less than $t$ in $f$, we study the same powers in
the following polynomial in $z$,

\begin{equation}
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \left( \prod_{(p,q) \in \pi^2 \setminus \{(j,k)\}} (1 + (y_p - y_q)^2 z) \right),
\end{equation}

rather than the full expression in (11). We therefore will prove the following statement about the above polynomial.

**Claim 4.1.** Fix $n \in \mathbb{N}$, and fix $m \in \mathbb{N} \cup \{0\}$, $m \leq n^2 - 1$. If there exist $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ such that

\[ \sum_{j=1}^{n} c_j y_j^\ell = 0 \quad \text{for all } \ell \in \{0, 1, \ldots, m\}, \]

then the coefficient of $z^m$ in (13) is zero.

The small-$m$ cases of the above yield the following result, which we will use in the sequel.

**Corollary 4.2.** Fix $t > 0$, $t \not\in \mathbb{N}$, and write $t = T + \tau$ with $T \in \mathbb{N} \cup \{0\}$ and $\tau \in (0, 1)$. Also, fix $a > 0$. If for some $n \in \mathbb{N}$, there exist $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ such that

\begin{equation}
\sum_{j=1}^{n} c_j y_j^\ell = 0 \quad \text{for all } \ell \in \{0, 1, \ldots, T\},
\end{equation}

then the the coefficients of $z^0, z^1, \ldots, z^T$ in $f$ are all zero, and so the smallest-power term in $f$ is the $z^t$ term.

**Proof.** Our hypothesis is sufficient to use the cases $m = 0, 1, \ldots, T$ of Claim 4.1 so the $z^0, z^1, \ldots, z^T$ terms in (13) have zero coefficients. Since the coefficients of these powers are the same in (13) and (11), we see that these powers have zero coefficients in $f$. \hfill \square

To prove Claim 4.1 we first describe the $z^m$ term in (13). To create the $(j, k)$-th term’s contribution to the $z^m$ coefficient, we collect all products of $m$ distinctively-indexed terms of the form $(y_p - y_q)^2$, $(p, q) \in \pi^2 \setminus \{(j,k)\}$, add them together, and then multiply by $c_j c_k$. The total $z^m$ coefficient is therefore

\begin{equation}
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \left( \sum_{J \subseteq \pi^2 \setminus \{(j,k)\}} \prod_{(p,q) \in J} (y_p - y_q)^2 \right).
\end{equation}

We would like to make the summation in $J$ independent of $j$ and $k$, so that we can move terms out of the sum in $j$ and $k$. We get the following.
Lemma 4.3. For \( n \in \mathbb{N} \), for any \( \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n \), for any \( (j, k) \in \mathbb{N}^2 \), and for any \( m \in \mathbb{N} \cup \{0\} \), 0 \leq m \leq n^2 - 1, \nabla
\[
\left( \sum_{J \in \mathbb{N}^2 \setminus \{(j, k)\}} \prod_{(p, q) \in J} (y_p - y_q)^2 \right) \nabla
\]
\[
= \sum_{v=0}^{m} (-1)^v (y_j - y_k)^{2v} \sum_{J \in \mathbb{N}^2 \setminus \{(m-v)\}} \prod_{(p, q) \in J} (y_p - y_q)^2 . \nabla
\]

Proof. We prove the lemma by induction on \( m \). For \( m = 0 \), by the convention that the empty product is 1, both sides are equal to
\[
\left( \sum_{J} \prod_{(p, q) \in J} (y_p - y_q)^2 \right) = 1, \nabla
\]
so the initial case holds.

For the inductive argument, suppose that the statement is true for \( m = \mu \); we wish to prove it for \( m = \mu + 1 \). We start with the left-hand side of (16) for \( m = \mu + 1 \),
\[
\sum_{J \in \mathbb{N}^2 \setminus \{(\mu+1)\}} \prod_{(p, q) \in J} (y_p - y_q)^2 , \nabla
\]
and add and subtract terms so that, in the primary term, the sum in \( J \) is independent of \( j \) and \( k \). The only missing terms are \( (\mu + 1) \)-fold products with distinct indices, in which one index is \( (j, k) \); the other \( m \) terms must therefore be members of \( \mathbb{N}^2 \setminus \{(j, k)\} \). Thus, we have
\[
\left( \sum_{J \in \mathbb{N}^2 \setminus \{(j, k)\}} \prod_{(p, q) \in J} (y_p - y_q)^2 \right) = \left( \sum_{J \in \mathbb{N}^2 \setminus \{(\mu+1)\}} \prod_{(p, q) \in J} (y_p - y_q)^2 \right) - (y_j - y_k)^2 \left( \sum_{J \in \mathbb{N}^2 \setminus \{(\mu+1)\}} \prod_{(p, q) \in J} (y_p - y_q)^2 \right) . \nabla
\]

Yet the second term above is \( -(y_j - y_k)^2 \) times the left-hand side of (16) (for \( m = \mu \)), so by the inductive hypothesis, we have
\[
\left( \sum_{J \in \mathbb{N}^2 \setminus \{(\mu+1)\}} \prod_{(p, q) \in J} (y_p - y_q)^2 \right) \nabla
\]
\[
+ \sum_{v=0}^{\mu} (-1)^{v+1} (y_j - y_k)^{2(v+1)} \left( \sum_{J \in \mathbb{N}^2 \setminus \{(\mu)\}} \prod_{(p, q) \in J} (y_p - y_q)^2 \right) . \nabla
\]

Note that \( \left( \sum_{J \in \mathbb{N}^2 \setminus \{(\mu+1)\}} \prod_{(p, q) \in J} (y_p - y_q)^2 \right) \) becomes the \( v = -1 \) term of the series in
Then shifting the index by 1, we have finished the inductive step. \[\square\]

By Lemma 4.3, the coefficient of $z^m$ in (13), namely (15), becomes

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \left( \sum_{v=0}^{m} (-1)^v (y_j - y_k)^{2v} \right) \left( \prod_{J \in \binom{\mathbb{N}}{m-v}} (y_p - y_q)^2 \right)
\]

\[
= \sum_{v=0}^{m} \left( \sum_{J \in \binom{\mathbb{N}}{m-v}} \prod_{(p,q) \in J} (y_p - y_q)^2 \right) \sum_{j=1}^{n} \sum_{k=1}^{n} (-1)^v c_j c_k (y_j - y_k)^{2v}.
\]

Therefore, we see that the coefficient of $z^m$ in (13) is a linear combination of the terms $\sum_{j=1}^{n} \sum_{k=1}^{n} (-1)^v c_j c_k (y_j - y_k)^{2v}$. To control these terms, we use the following lemma.

**Lemma 4.4.** For $v \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, if $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ satisfy

\[
\sum_{j=1}^{n} c_j y_j^u = 0, \quad \text{for all } u \in \{0, 1, \ldots, v\},
\]

then

\[
(-1)^v \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j - y_k)^{2v} = 0.
\]

**Proof.** We use the binomial formula to expand the desired expression.

\[
(-1)^v \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j - y_k)^{2v}
= (-1)^v \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \sum_{w=0}^{2v} \binom{2v}{w} (y_j)^w (-y_k)^{2v-w}
= \sum_{w=0}^{2v} (-1)^{v+w} \binom{2v}{w} \left( \sum_{j=1}^{n} c_j y_j^w \right) \left( \sum_{k=1}^{n} c_k y_k^{2v-w} \right).
\]

If $w \leq v$, then by (20), $\sum_{j=0}^{n} c_j y_j^w = 0$, and the sum in $j$ becomes zero. Similarly, if $w \geq v$, then $2v - w \leq v$, so $\sum_{k=0}^{n} c_k y_k^{2v-w} = 0$, and the sum in $k$ becomes zero.

Therefore, all terms become zero. \[\square\]

We now are able to finish the proof of Claim 4.1.
Proof of Claim 4.1. Fix $n \in \mathbb{N}$, and let $m \in \mathbb{N} \cup \{0\}$, $0 \leq m \leq n^2 - 1$. Suppose that $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ satisfy the following statements:

$$\sum_{j=1}^{n} c_j y_j^\ell = 0 \text{ for all } \ell \in \{0, 1, \ldots, m\}. \tag{22}$$

By (19), the coefficient of $z^m$ in (13) is

$$\sum_{v=0}^{m} \left( \sum_{J \in \binom{n^2}{m-v}} \prod_{(p,q) \in J} (y_p - y_q)^2 \right) \sum_{j=1}^{n} \sum_{k=1}^{n} (-1)^v c_j c_k (y_j - y_k)^{2v}. \tag{13}$$

Yet by $v \leq m$, the statements in (22) assure us that

$$\sum_{j=1}^{n} c_j y_j^u = 0$$

for all $u \in \{0, 1, \ldots, v\}$. Invoking Lemma 4.4, we see that

$$(-1)^v \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j - y_k)^{2v} = 0,$$

and hence the $v$th term becomes zero. This works for all $v$, $0 \leq v \leq m$, so the coefficient of $z^m$ in (13) becomes zero. \hfill \Box

We now choose $n$, $y$, and $c$ satisfying the hypothesis of Claim 4.1 for arbitrarily large, but fixed, $m$. Although a dimension-counting argument would suffice, there is a simple choice coming from combinatorics.

Proposition 4.5 ([5], Exercise 1.2.4). Fix $m \in \mathbb{N} \cup \{0\}$, $n := m + 2$, and for $1 \leq j \leq n$, let $y_j := j - 1$ and $c_j := (-1)^{j-1} \binom{m+1}{j-1}$. Then for all $\ell \in \{0, 1, \ldots, m\}$,

$$\sum_{j=1}^{n} c_j y_j^\ell = 0. \tag{21}$$

(Notice that the indices are shifted from [5]).

This allows us to fulfill the conditions of Claim 4.1 for any $m$. In particular, to fulfill the conditions of Corollary 4.2 we use the following case of Proposition 4.5.

Corollary 4.6. A set of solutions to (14) is given by $n := T + 2$, $y = (0, 1, \ldots, T + 1)$, and $c = \left((-1)^{j-1} \binom{T+1}{j-1}\right)_{j=1}^{T+2}$. We will now show that the choices of $n$, $y$, and $c$ as above are sufficient to ensure that if $t \in \bigcup_{k \in \mathbb{N}} (2k - 1, 2k)$, then the $z^t$ coefficient is negative.

4.3. Analyzing the coefficient of $z^t$ in $f$. If $t \notin \mathbb{N}$, we note that the coefficient of $z^t$ in (12) comes solely from products of one degree-$t$ term and $n^2 - 2$ degree-0 terms in (11); no integer equals $t$, so no product of degree-0 and degree-1 terms works, and the only other combination of terms with nonzero, but small enough,
degree is one degree-$t$ term and $n^2 - 2$ degree-0 terms. Therefore, for all $t > 0$, $a > 0$, the coefficient of $z^t$ in (12) is

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \left( \sum_{(p,q) \in \Pi^2 \backslash \{(j,k)\}} a(y_p^2 + y_q^2)^t \right).$$

As before, we attempt to make the summation independent of $(j, k)$ by adding and subtracting the $(j, k)$-th term in the sum, so the coefficient of $z^t$ becomes

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a c_j c_k \left( \sum_{(p,q) \in \Pi^2} (y_p^2 + y_q^2)^t - (y_j^2 + y_k^2)^t \right) = a \left( \sum_{(p,q) \in \Pi^2} (y_p^2 + y_q^2)^t \right)^2 - a \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j^2 + y_k^2)^t.$$

Now, assume that $t \not\in \mathbb{N}$ and that the values $\{y_j\}_{j=1}^{n}$ and $\{c_j\}_{j=1}^{n}$ are chosen so that the conditions in (14) hold. Then in particular, $\sum_{j=1}^{n} c_j = 0$. Hence, the first term becomes zero, and we are left with

$$-a \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j^2 + y_k^2)^t.$$

Our objective will be to show the following.

**Claim 4.7.** Fix $t > 0$, $t \not\in \mathbb{N}$, and write $t = T + \tau$ with $T \in \mathbb{N} \cup \{0\}$ and $\tau \in (0, 1)$. Fix $a > 0$. For any $n \in \mathbb{N}$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ such that the conditions in (14) hold, (23)

$$-a \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j^2 + y_k^2)^t$$

is nonnegative if $T$ is even, and is nonpositive if $T$ is odd. Moreover, for any such $t$ and $a$, a choice of $n$, $y$ and $c$ can be found such that the conditions in (14) hold and, in addition, (23) is nonzero.

To proceed, we use a convenient integral representation for positive noninteger powers, such as the $t$-th power above.

**Lemma 4.8.** If $\text{Re } w > 0$ and $s > 0$, $s \not\in \mathbb{N}$, then letting $s = S + \sigma$, where $S \in \mathbb{N} \cup \{0\}$, $\sigma \in (0, 1)$,

$$w^s = h(w; s)$$

(24)

$$:= (-1)^S \frac{(\sigma)_{S+1}}{\Gamma(1-\sigma)} \int_0^\infty \left( \sum_{\ell=0}^{S} \frac{(-\lambda w)\ell}{\ell!} \right) - e^{-\lambda w} \frac{d\lambda}{\lambda^{S+1+\sigma}}.$$
Note that (24) also holds when \( w = 0 \), if we define \( 0^r = 0 \).

The basic idea of the formula is not original: for example, Berg, Christensen, and Ressel present the formula for \( 0 < s < 1 \) in the proof of [2], Proposition 3.2.10, and then proceed to derive the formula for \( 1 < s < 2 \) in the proof of [2], Proposition 3.2.11. (24) simply continues the pattern further. As such, we defer the details of the proof of Lemma 4.8 to Section 5, and proceed to demonstrate the validity of Claim 4.7, given the integral representation.

Proof of Claim 4.7 Let \( n \in \mathbb{N}, \ y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), and \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \) satisfy (14). Then applying Lemma 4.8 to (14) and letting \( B \) denote the formula for \( 0 < s < 1 \), \( w = y_j^2 + y_k^2 \), letting \( t = T + \tau \), and letting \( B(t) \) denote the positive constant \( \frac{(\tau)_{T+1}}{\Gamma(1-\tau)} \), we have that

\[
-a \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j^2 + y_k^2)^t
= (-1)^{T+1} a B(t) \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \cdot \int_0^\infty \left( \sum_{\ell=0}^{T} \frac{(-\lambda)^\ell (y_j^2 + y_k^2)^\ell}{\ell!} - e^{-\lambda(y_j^2 + y_k^2)} \right) \frac{d\lambda}{\lambda^{T+1+\tau}}.
\]

Interchanging the sum in \( j \) and \( k \) and the integral, we get

\[
(-1)^{T+1} a B(t) \left( \int_0^\infty \left( \sum_{\ell=0}^{T} \frac{(-\lambda)^\ell}{\ell!} \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j^2 + y_k^2)^\ell
- \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k e^{-\lambda(y_j^2 + y_k^2)} \right) \frac{d\lambda}{\lambda^{T+1+\tau}} \right).
\]

We now prove that if (14) holds, the term \( \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j^2 + y_k^2)^\ell \) zeroes out for all \( \ell \in \{0, 1, \ldots, T\} \), and hence the initial terms of the integrand become zero. We use the binomial formula to rewrite the sums, and we get, for the \( \ell \)-th double-sum,

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k (y_j^2 + y_k^2)^\ell
= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \sum_{u=0}^{\ell} \binom{\ell}{u} (y_j)^{2u} \cdot (y_k)^{2(\ell-u)}
= \sum_{u=0}^{\ell} \binom{\ell}{u} \left( \sum_{j=1}^{n} c_j y_j^{2u} \right) \left( \sum_{k=1}^{n} c_k y_k^{2(\ell-u)} \right).
\]

Since \( \ell \leq T \), either \( 2u \leq T \) or \( 2(\ell - u) \leq T \), since \( 2u + 2(\ell - u) = 2\ell \leq 2T \). Therefore, either the sum in \( j \) or the sum in \( k \) is 0, by (14). Therefore, the \( \ell \)-th term becomes zero. Since this works for all \( \ell \), \( 0 \leq \ell \leq T \), the initial terms of the integrand become zero.
Therefore, the expression in (23) becomes
\[
(-1)^{T+1} a B(t) \left( - \int_0^{\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k e^{-\lambda(y_j^2+y_k^2)} \frac{d\lambda}{\lambda^{T+1+\tau}} \right);
\]
after simplifying, we have
\[
(-1)^T a B(t) \left( \int_0^{\infty} \left( \sum_{j=1}^{n} c_j e^{-\lambda y_j^2} \right)^2 \frac{d\lambda}{\lambda^{T+1+\tau}} \right).
\]

Since all terms are real, the integrand is clearly nonnegative, and \( B(t) = \frac{(\tau)^{T+1}}{\Gamma(1 - \tau)} \) is positive. Therefore, the sign of the expression in (23) depends only on \((-1)^T\), so if \( T \) is even and nonnegative if \( T \) is odd.

To ensure that strict positivity in the integral can occur, we note that the integral is strictly positive unless for Lebesgue-a.e. \( \lambda > 0 \),
\[
\sum_{j=1}^{n} c_j e^{-\lambda y_j^2} = 0.
\]

By continuity in \( \lambda \), we may assert that the integral is strictly positive unless (25) holds for all \( \lambda > 0 \). Now, we do not immediately have that if (25) holds for all \( \lambda > 0 \), \( c_j = 0 \) for all \( j \); for example, take \( n = 2 \), \( y_2 = -y_1 \), \( c_2 = -c_1 \). The following statement, however, gives a sufficient condition to draw such a conclusion.

**Lemma 4.9.** If \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) is a set of distinct nonnegative real numbers, and if (25) holds for all \( \lambda > 0 \), then \( c_j = 0 \) for all \( j \).

**Proof.** Without loss of generality, we may assume that \( 0 \leq y_1 < y_2 < \cdots < y_n \). Suppose that (25) holds, but that not all \( c_j \) are zero; let \( j_0 \) be the smallest \( j \) such that \( c_j \neq 0 \). Then multiplying (23) by \( e^{\lambda y_{j_0}^2} / c_{j_0} \), we get that for all \( \lambda > 0 \),
\[
g(\lambda; y, c) := 1 + \sum_{j=j_0+1}^{n} c_j e^{-\lambda(y_j^2-y_{j_0}^2)} = 0.
\]
Yet by \( 0 \leq y_{j_0} < y_j \) for all \( j > j_0 \), \( \lim_{\lambda \to \infty} g(\lambda; y, c) = 1 \), but \( \lim_{\lambda \to \infty} 0 = 0 \). Contradiction. \( \square \)

In particular, then, for \( y \) and \( c \) chosen as Corollary 4.6 we have that \( y_j = j - 1 \), so the \( y_j \)'s are distinct nonnegative real numbers. By Lemma 4.9 \( \sum_{j=1}^{n} c_j e^{-\lambda y_j^2} \equiv 0 \) if and only if \( c_j = 0 \) for all \( j \), which does not hold for the choice of \( c_j \) as in Corollary 4.6. Therefore, for an appropriate choice of \( n \), \( \{y_j\}_{j=1}^{n} \), and \( \{c_j\}_{j=1}^{n} \), the integral is nonzero, so the expression in (23) is strictly negative or positive, depending on the parity of \( T \). \( \square \)
4.4. Conclusion. We now finish the proof of Theorem 3.

Proof of Theorem 3. Fix \( t \in \bigcup_{n \in \mathbb{N}} (2k - 1, 2k) \), and write \( t = T + \tau \), where \( T \in \mathbb{N} \) is odd, and \( \alpha \in (0, 1) \). Also, fix \( a > 0 \). By Corollary 4.2 and Corollary 4.6 for \( n = T + 2, y = (0, 1, \ldots, T + 1) \), and \( c = \left( (-1)^{j-1} \frac{T + 1}{j - 1} \right)_{j=1}^{T+2} \), all terms of degree less than \( t \) in (11) have zero coefficients. By Claim 4.7, the same choice of \( n, y, \) and \( c \) ensures that the coefficient of \( z^t \) in (11) is negative. Since all remaining terms are of degree larger than \( t \), we have that (11) is of the form

\[
f(z; n, y, c) = z^t (\kappa + \epsilon(z)),
\]

where \( \kappa < 0 \) is the \( z^t \) coefficient, and since \( r_m > t \) for all terms such that \( b_m \neq 0 \),

\[
\lim_{z \to 0} (\kappa + \epsilon(z)) = \kappa < 0.
\]

Thus, for sufficiently small positive \( z \), \( f \) is negative; hence, \( K_{t,a} \) is not positive-definite.

To finish this section, we note that we still have not solved the specific question raised in [7], namely the existence of negative eigenvalues for \( K_{t,a} \) in the case \( t = 2, a \) near 0. We therefore end with the following.

Open Problem. Determine whether or not \( K_{t,a} \) admits a negative eigenvalue for \( t = 2 \) and \( 0 < a \leq 12 \); \( a_0(2) \).

5. Proof of Lemma 4.8

To prove Lemma 4.8 we use the following \( w \)-dependent bound on the \( L^1 \)-norm of the integrand in \( h(w; s) \).

Lemma 5.1. If \( \Re(w) \geq 0, s > 0, s \notin \mathbb{N} \), then writing \( s = S + \sigma \), where \( S \in \mathbb{N} \cup \{0\} \) and \( \sigma \in (0, 1) \),

\[
\int_0^\infty \left| \sum_{\ell=0}^S \frac{(-\lambda w)\ell^s}{\ell!} \right| e^{-\lambda w} \frac{d\lambda}{\lambda^{S+1+\sigma}} \leq C(s) |w|^s,
\]

where \( C(s) \) is a constant depending on \( s \).

Proof. If \( w = 0 \), both sides of (26) are 0. So assume that \( w \neq 0 \). To take advantage of homogeneity, set \( \mu = |w| \lambda, d\mu = |w| d\lambda \). Letting \( \omega = \frac{w}{|w|} \in S^1 \cap \{ \zeta \in \mathbb{C} : \Re \zeta \geq 0 \} \), we get

\[
\int_0^\infty \left| \sum_{\ell=0}^S \frac{(-\mu w)\ell^s}{\ell!} \right| e^{-\mu w} \frac{d\mu}{|w|^{S+1+\sigma}}
\]

\[
= |w|^s \int_0^\infty \left| \sum_{\ell=0}^S \frac{(-\mu w)\ell^s}{\ell!} \right| e^{-\mu w} \frac{d\mu}{\mu^{S+1+\sigma}}.
\]

We have extracted the \( |w|^s \) term; now we need only show that for \( \omega \) in \( S^1 \cap \{ \Re \zeta \geq 0 \} \), the integral is bounded by a constant depending on \( s \) alone. We split the integral over \([0, \infty)\) into integrals over \([0, 1]\) and \([1, \infty)\). For the integral on
we recognize the initial sum as the first few terms of the Taylor series for $e^{-\mu \omega}$, and we get

$$\int_0^1 \left| \left( \sum_{\ell=0}^{S} \frac{(-\mu \omega)^\ell}{\ell!} \right) - e^{-\mu \omega} \right| \frac{d\mu}{\mu^{s+1+\sigma}}$$

$$= \int_0^1 \left| \left( \sum_{\ell=S+1}^{\infty} \frac{(-\mu \omega)^\ell}{\ell!} \right) \right| \frac{d\mu}{\mu^{s+1+\sigma}}$$

$$\leq \int_0^1 \sum_{\ell=S+1}^{\infty} \frac{1}{\ell!} \mu^{\ell-(S+1+\sigma)} d\mu$$

$$= \sum_{\ell=S+1}^{\infty} \frac{1}{(\ell-(S+\sigma)) \cdot \ell!}.$$

For the integral on $[1, \infty)$, we bound $e^{-\mu \omega}$ in absolute value by 1 (by $\Re \omega \geq 0$) and get

$$\int_1^{\infty} \left| \left( \sum_{\ell=0}^{S} \frac{(-\mu \omega)^\ell}{\ell!} \right) - e^{-\mu \omega} \right| \frac{1}{\mu^{s+1+\sigma}} d\mu$$

$$\leq \int_1^{\infty} \left( \sum_{\ell=0}^{S} \frac{1}{\ell!} \mu^{\ell-(S+1+\sigma)} + \mu^{-(S+1+\sigma)} \right) d\mu$$

$$= \left( \sum_{\ell=0}^{S} \frac{1}{(\ell-(S+\sigma)) \cdot \ell!} \right) + \frac{1}{(S+\sigma)}.$$

Altogether, then, the integral is bounded by

$$\left( \sum_{\ell=0}^{\infty} \frac{1}{(\ell-(S+\sigma)) \cdot \ell!} \right) + \frac{1}{S+\sigma}.$$

Moreover, for all $\ell \in \mathbb{N} \cup \{0\}$, $|\ell-(S+\sigma)| \geq \min(\sigma, 1-\sigma) =: C_1(\sigma)$; hence, the term in parentheses is bounded above by

$$(C_1(\sigma))^{-1} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} + \frac{1}{S+\sigma} = \frac{e}{C_1(\sigma)} + \frac{1}{s}.$$

Defining $C(s) := \frac{e}{C_1(\sigma)} + \frac{1}{s}$, the proof is complete. □

**Proof of Lemma 4.8.** Since any noninteger $s > 0$, can be written as $s = S + \sigma$ for some $S \in \mathbb{N} \cup \{0\}$ and $\sigma \in (0, 1)$, it suffices to prove for each $\sigma \in (0, 1)$ that $s = S + \sigma$ satisfies (24) for all $S \in \mathbb{N} \cup \{0\}$.

We therefore induct on $S$. We note that if $s = 0 + \sigma$, implying $0 < s < 1$, the fact is well known; see [2], p. 78.

Suppose that for some $S \in \mathbb{N} \cup \{0\}$, (24) is true for $s = p := S + \sigma$. We will prove (24) true for $s = q := (S + 1) + \sigma$. We first demonstrate that for $s = q$, both sides of (24) have the same derivative, namely $q \cdot h(w; p)$.
The derivative of \( w^q \) is \( qw^{q-1} = qw^p \). By the inductive hypothesis, \( qw^p = q \cdot h(w; p) \). On the other hand, the derivative of the integrand in \( h(w; q) \) is

\[
\frac{d}{dw} \left( \sum_{\ell=0}^{S+1} \frac{(-\lambda w)^\ell}{\ell!} - e^{-\lambda w} \right) = \frac{1}{\lambda^{S+2+\sigma}}.
\]

Reindexing by \( u = \ell - 1 \), we get

\[
\left( \sum_{u=0}^{S} \frac{(-\lambda w)^u}{u!} \right) - (-\lambda)^M e^{-\lambda w} \frac{1}{\lambda^{S+2+\sigma}}.
\]

The absolute value of this expression is the integrand in (20), with \( s = p \). Thus, by Lemma 5.1, we have \( L^1(\mathbb{R}^+, d\lambda) \) convergence, uniform in \( w \) for \( w \in \{ |\zeta| \leq M \} \cap \{ \Re \zeta \geq 0 \} \), for any \( M > 0 \). Hence, we can differentiate under the integral sign on the right-hand side, and we get

\[
\frac{d}{dw}(h(w; q)) = \frac{d}{dw} \left( \sum_{\ell=0}^{S+1} \frac{(-\lambda w)^\ell}{\ell!} - e^{-\lambda w} \right) = \frac{-d\lambda}{\lambda^{S+2+\sigma}}.
\]

By definition, this is \( (S + 1 + \sigma) \cdot h(w; p) = q \cdot h(w; p) \). Yet the derivative of \( w^q \) was \( Q \cdot h(w; p) \). So we see that \( w^q \) and \( h(w; q) \) have the same derivative. Hence, they are equal up to a constant on the domain of mutual definition: \( h(w; q) - w^q = D \) for all \( w \in \{ \Re \zeta > 0 \} \).

To show that \( D = 0 \), we note that as \( w \) approaches \( 0 \) along the positive real axis, \( w^q \rightarrow 0 \). More importantly, by Lemma 5.1, \( D \) as \( w \) approaches \( 0 \) along the positive real axis, \( h(w; q) \rightarrow 0 \) as well. Hence,

\[
D = \lim_{w \rightarrow 0^+} h(w; q) = 0.
\]

\[
\Box
\]

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Department of Mathematics, The Ohio State University, Columbus, OH 43210

E-mail address: baker.1656@osu.edu