One-dimensional Stark operators in the half-line

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Abstract

We obtain asymptotic formulas for the spectral data of perturbed Stark operators associated with the differential expression

$$\frac{d^2}{dx^2} + x + q(x), \quad x \in [0, \infty), \quad q \in L^1(0, \infty),$$

and having either Dirichlet or Neumann boundary condition at the origin.

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1 Introduction and statement of results

Self-adjoint operators of the form

$$\frac{d^2}{dx^2} + f(x) + q(x), \quad x \in (0, \infty),$$

with domain in $L^2(\mathbb{R}_+)$, occur naturally in the context of quantum-mechanical operators with spherical symmetry; here $q$ plays the role of a small perturbation of $f$ in some suitable sense. The spectral analysis of this kind of operators have attracted considerable attention for various choices of the dominant term $f$, usually in connection with well-known special functions. Most remarkable among them are the investigations concerning perturbed Bessel operators $[1,3,4,10,16–19]$ (corresponding to $f(x) = l(l + 1)x^{-2}, \ l \geq -1/2$), and perturbed harmonic oscillator in the half-line $[7,8]$ (in this case $f(x) = x^2$); the latter is closely related to the spectral analysis of perturbed harmonic oscillator in the whole real line $[5,6]$.

In this paper we consider self-adjoint operators associated with a differential expression of the form

$$\tau = -\frac{d^2}{dx^2} + x + q(x), \quad x \in [0, \infty),$$

acting in the space $L^2(\mathbb{R}_+)$, where $q$ is a real-valued function that lies in $L^1(\mathbb{R}_+)$. Self-adjoint operators are defined by adjoining to $\tau$ a standard boundary condition (see Section 2) at the left endpoint. For the sake of brevity, we only consider Dirichlet ($\varphi(0) = 0$) and Neumann
\((\varphi'(0) = 0)\) boundary conditions; let \(H^D\) and \(H^N\) denote the corresponding self-adjoint operators.

Clearly the unperturbed case \(q \equiv 0\) can be solved explicitly. For in this case a square-integrable solution to the associated eigenvalue problem is given by the Airy function of the first kind \(\text{Ai}(z)\) so

\[
\sigma(H_0^D) = \{-a_k\}_{k \in \mathbb{N}} \quad \text{and} \quad \sigma(H_0^N) = \{-a'_k\}_{k \in \mathbb{N}},
\]

where \(a_k\) and \(a'_k\) denote the zeros of \(\text{Ai}(z)\) and its derivative \(\text{Ai}'(z)\), respectively. The corresponding set of norming constants \(\{\nu^D_{0,k}\}_{k \in \mathbb{N}}\) and \(\{\nu^N_{0,k}\}_{k \in \mathbb{N}}\) are then given by

\[
\frac{1}{\nu^D_{0,k}} = \frac{\|\text{Ai}(\cdot + a_k)\|^2}{(\text{Ai}'(a_k))^2} = 1 \quad \text{and} \quad \frac{1}{\nu^N_{0,k}} = \frac{\|\text{Ai}(\cdot + a'_k)\|^2}{(\text{Ai}'(a'_k))^2} = -a'_k.
\]

The related results for arbitrary \(q\) are stated in Theorems 3.5, 3.6, 3.8 and 3.9. They can be summarized as follows:

**Theorem.** Suppose \(q \in L^1(\mathbb{R}_+)\). Then the eigenvalues and norming constants of \(H^D\), the operator associated with \(\tau\) and boundary condition \(\varphi(0) = 0\), satisfy

\[
\lambda^D_k = \left(\frac{3}{2} \pi (k - \frac{1}{4})\right)^{2/3} \left(1 + O(k^{-1})\right) \quad \text{and} \quad \frac{1}{\nu^D_k} = 1 + o(1)
\]

as \(k \to \infty\). Similarly, the eigenvalues and norming constants of \(H^N\) corresponding to the boundary condition \(\varphi'(0) = 0\) satisfy

\[
\lambda^N_k = \left(\frac{3}{2} \pi (k - \frac{3}{4})\right)^{2/3} \left(1 + O(k^{-1})\right) \quad \text{and} \quad \frac{1}{\nu^N_k} = \left(\frac{3}{2} \pi (k - \frac{3}{4})\right)^{2/3} (1 + o(1))
\]

as \(k \to \infty\).

The direct spectral problem for the one-dimensional Stark operator in the semi-axis, with Dirichlet boundary condition, has also been treated recently in [22], where the authors use transformation operator methods and their results are valid under the more restrictive assumption \(q \in C^1(0, \infty) \cap L^1(\mathbb{R}_+, x^4 dx)\), \(q(x) = o(x)\) as \(x \to \infty\). The corresponding inverse spectral problem is discussed in [20].

Finally, it is worth mentioning that one-dimensional Stark operators have been studied mostly when defined on the whole real line, see for instance [2, 11–13, 21, 25]. As it is well-known, Stark operators on the real line are characterized by the presence of resonances; see [14, 15] for some recent developments on this subject.

## 2 Preliminaries

In what follows, we consider the differential expression

\[
\tau = -\frac{d^2}{dx^2} + x + q(x), \quad x \in [0, \infty),
\]

where \(q \in L^1(\mathbb{R}_+)\) and it is real-valued.

By standard theory (see e.g. [26, Ch. 6]), \(\tau\) is in the limit-circle case at 0 and in the limit-point case at \(\infty\). Hence (the closure of) the minimal operator \(H'\) defined by \(\tau\) is symmetric and has deficiency indices \((1, 1)\). Also, there exists a solution \(\psi(z, x)\) to the eigenvalue equation \(\tau \varphi = z \varphi\), real entire as a function of \(z \in \mathbb{C}\) for every \(x \in [0, \infty)\), such that \(\psi(z, \cdot) \in L^2(\mathbb{R}_+)\) for
every $z \in \mathbb{C}$. This function is unique up to multiplication by a zero-free, real entire function of the spectral parameter $z$.

The self-adjoint extensions $H^\beta$ ($0 \leq \beta < \pi$) of $H'$ are determined by imposing the usual boundary condition at $x = 0$. Namely,

\[ D(H^\beta) = \left\{ \varphi \in L^2(\mathbb{R}_+) : \varphi, \varphi' \in AC_{\text{loc}}([0, \infty)), \quad \cos(\beta)\varphi(0) - \sin(\beta)\varphi'(0) = 0 \right\}, \quad H^\beta \varphi = \tau \varphi. \]

Since $x + q(x) \to \infty$ as $x \to \infty$, it follows that $\sigma(H^\beta)$ has only eigenvalues of multiplicity one, possibly with a finite number of them being negative. Moreover,

\[ \sigma(H^\beta) = \{ \lambda \in \mathbb{R} : \cos(\beta)\psi(\lambda, 0) - \sin(\beta)\psi'(\lambda, 0) = 0 \} \quad (0 \leq \beta < \pi). \]

We henceforth suppose $\sigma(H^\beta)$ is arranged as an increasing sequence, viz., $\sigma(H^\beta) = \{ \lambda_k^\beta \}_{k \in \mathbb{N}}$ with $\lambda_k^\beta < \lambda_{k+1}^\beta$.

In what follows we use the notation $' = \partial_x$ and $\cdot = \partial_z$. Along with the spectrum $\{ \lambda_k^\beta \}_{k \in \mathbb{N}}$ one has the corresponding set of norming constants $\{ \nu_k^\beta \}_{k \in \mathbb{N}}$. In terms of $\psi(z, x)$, the norming constants for Dirichlet ($\beta = 0$) and Neumann ($\beta = \pi/2$) boundary conditions are given by the formulas

\[ \frac{1}{\nu_k^D} = \frac{\psi(\lambda_k^D, 0)}{\psi'(\lambda_k^D, 0)} \quad \text{and} \quad \frac{1}{\nu_k^N} = \frac{\psi(\lambda_k^N, 0)}{\psi'(\lambda_k^N, 0)}, \]

respectively. The second part of these equations follows from the identity $W'(\eta, \dot{\eta}) = -\eta^2$, which is valid for any solution to $\tau \eta = z \eta$. We recall that the spectral data $\{ \{ \mu_k^\beta \}_{N \in \mathbb{N}}, \{ \nu_k^\beta \}_{k \in \mathbb{N}} \}$ are the poles and residues of the Weyl function associated with $H^\beta$, and they determine the potential $q$ by virtue of the Borg–Marchenko uniqueness theorem [9].

As mentioned in the Introduction, the unperturbed case $q = 0$ can be treated explicitly. A solution to the equation $-\varphi'' + (x - z)\varphi$, belonging to $L^2(\mathbb{R}_+)$, is

\[ \psi_0(z, x) = \sqrt{\pi} \text{Ai}(x - z), \]

where the factor $\sqrt{\pi}$ is included for convenience. It follows that

\[ \sigma(H_0^D) = \{-a_k\}_{k \in \mathbb{N}} \quad \text{and} \quad \sigma(H_0^N) = \{-a_k'\}_{k \in \mathbb{N}} \]

respectively, where the zeros of $\text{Ai}(z)$ and $\text{Ai}'(z)$ obey the asymptotic formulas

\[ -a_k = \left( \frac{3}{2} \pi (k - \frac{1}{4}) \right)^{2/3} \left( 1 + O(k^{-2}) \right) \quad (1) \]

and

\[ -a_k' = \left( \frac{3}{2} \pi (k - \frac{3}{4}) \right)^{2/3} \left( 1 + O(k^{-2}) \right) \]

as $k \to \infty$ (see [23, §9.9(iv)]).

**Lemma 2.1.** There exists a constant $C_0 > 0$ such that

\[ |\text{Ai}(z)| \leq C_0 \frac{g_A(z)}{1 + |z|^{1/4}} \quad \text{and} \quad |\text{Ai}'(z)| \leq C_0 (1 + |z|^{1/4}) g_A(z), \]

for all $z \in \mathbb{C}$, where $g_A(z) = \exp(-\frac{2}{3} \text{Re} z^{3/2})$.  

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Proof. Define \( \zeta = \frac{2}{3} z^{3/2} \) with branch cut along \( \mathbb{R}_- \). According to [23, §9.7(ii)], the function \( \text{Ai}(z) \) satisfies the asymptotic expansions

\[
\text{Ai}(z) = \frac{e^{-\zeta}}{2\sqrt{\pi} z^{1/4}} \left[ 1 + O(\zeta^{-1}) \right], \quad |\text{arg}(z)| \leq \pi - \delta, \tag{2}
\]

and

\[
\text{Ai}(-z) = \frac{1}{\sqrt{\pi} z^{1/4}} \left[ \sin(\zeta + \frac{\pi}{4}) + O\left(\zeta^{-1} e^{i\text{Im}\zeta}\right) \right], \quad |\text{arg}(z)| \leq \frac{2\pi}{3} - \delta, \tag{3}
\]
as \( |z| \to \infty \). These expansions are uniform for any given small \( \delta > 0 \) and \( |z| \geq 1 \). In what follows we set \( \delta = \pi/3 \). Since \( \text{Ai}(z) \) is an entire function, it follows that there exists \( C_0 > 0 \) such that

\[
|\text{Ai}(z)| \leq \frac{C_0}{1 + |z|^{1/4}} \times \begin{cases} \exp(-\frac{2}{3} \text{Re} z^{3/2}), \quad \text{arg}(z) \in [-\frac{2\pi}{3}, \frac{2\pi}{3}], \\ \exp(\frac{2}{3} \text{Im}(-z)^{3/2}), \quad \text{arg}(z) \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi]. \end{cases}
\]

Thus, the bound on \( \text{Ai}(z) \) follows after noticing that \( |\text{Im}(-z)^{3/2}| = |\text{Re} z^{3/2}| \) and \( |\text{Re} z^{3/2}| = -\text{Re} z^{3/2} \) if \( \text{arg}(z) \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi]. \) The bound on \( \text{Ai}'(z) \) follows an analogous argument so the details are omitted. \( \blacksquare \)

Lemma 2.1 clearly implies

\[
|\psi_0(z, x)| \leq C_0 e^{-\frac{2}{3} \text{Re}(x-z)^{3/2}} \] and \( |\psi_0'(z, x)| \leq C_0 (1 + |x-z|^{1/4}) e^{-\frac{2}{3} \text{Re}(x-z)^{3/2}} \tag{4}
\]

with \((x, z) \in \mathbb{R}_+ \times \mathbb{C}. \) Later we will make use of a linearly independent solution to \(-\varphi'' + (x-z)\varphi. \) An obvious choice is given by the Airy function of the second kind

\[
\theta_0(z, x) = \sqrt{\pi} \text{Bi}(x-z).
\]

However, it will be more convenient to use an independent solution of the form

\[
\theta_{\pm}(z, x) = \theta_0(z, x) \mp i\psi_0(z, x) = 2\sqrt{\pi} e^{\mp i\pi/6} \text{Ai}((x-z)e^{\mp i2\pi/3})
\]

(in the context of the present work any of these two functions is equally good). According to [23, §9.2(iv)], one has \( W(\psi_0(z), \theta_\pm(z)) \equiv 1 \). Moreover, since \( \text{Re}(z e^{\mp i2\pi/3})^{3/2} = -\text{Re} z^{3/2} \), we have the bounds

\[
|\theta_\pm(z, x)| \leq 2C_0 e^{\frac{2}{3} \text{Re}(x-z)^{3/2}} \quad \text{and} \quad |\theta_\pm'(z, x)| \leq 2C_0 (1 + |x-z|^{1/4}) e^{\frac{2}{3} \text{Re}(x-z)^{3/2}}. \tag{5}
\]

Lemma 2.2. The map \( x \mapsto g_A(x-z), x \in \mathbb{R}_+, \) is decreasing whenever \( z \in \mathbb{C} \setminus \mathbb{R}. \) If \( z \in \mathbb{R}, \) then \( g_A(x-z) \) is constant (equal to 1) for \( x \in [0, z] \) and decreasing for \( x \in (z, \infty). \)

Proof. Suppose \( z \in \mathbb{C}. \) A simple computation shows that, given \( x \in \mathbb{R}, \) there exists a unique \( \gamma \in (0, \pi) \) such that

\[
x - z = \frac{|\text{Im} z|}{\sin \gamma} e^{i\gamma}.
\]

Then,

\[
\text{Re}(x - z)^{3/2} = |\text{Im} z|^{3/2} \frac{\cos \frac{3\gamma}{2}}{(\sin \gamma)^{3/2}}.
\]

The right hand side of the last equation is decreasing as a function of \( \gamma. \) But the map \( x \mapsto \gamma \) is also decreasing so the map \( x \mapsto \text{Re}(x-z)^{3/2} \) is increasing. This in turn implies the assertion. Clearly, a similar reasoning works if \( z \in \mathbb{C}_+. \) The statement is obvious for \( z \in \mathbb{R}. \) \( \blacksquare \)
3 Main results

3.1 Adding a perturbation

We look for a solution to the eigenvalue equation $\tau \varphi = z \varphi$, with $q \in L^1(\mathbb{R}_+)$, that is real entire with respect to the spectral parameter $z \in \mathbb{C}$ and lies in $L^2(\mathbb{R}_+)$. To this end we introduce the auxiliary function

$$\omega(z) = \int_0^\infty \frac{|q(x)|}{\sqrt{1+|x-z|}} \, dx.$$ 

Clearly, $\omega(z)$ is well defined for all $z \in \mathbb{C}$. Moreover, $\omega(z)$ is well defined under the weaker assumption $q \in L^1(\mathbb{R}_+, (1+x)^{-1/2} \, dx)$. However, our hypothesis on $q$ gives us control on the decay of $\omega(z)$ as it is shown next.

**Lemma 3.1.** Assume $q \in L^1(\mathbb{R}_+)$. Then $\omega(z) \to 0$ as $z \to \infty$.

**Proof.** Given $\varepsilon > 0$, choose $x_+ > 0$ and $\mu_+ > x_+$ such that

$$\int_{x_+}^\infty |q(x)| \, dx < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{\sqrt{\mu_+ - x_+}} < \frac{\varepsilon}{2\|q\|_1}.$$ 

Suppose $|\text{Im}(z)| > \mu_+$. Then $|x-z| > \mu_+$ for any $x > 0$. Hence,

$$\frac{1}{\sqrt{1+|x-z|}} \leq \frac{1}{\sqrt{\mu_+ - x_+}} \int_{x_+}^x |q(x)| \, dx + \int_x^\infty |q(x)| \, dx < \varepsilon,$$

for all $x \in \mathbb{R}_+$, which in turn implies $\omega(q, z) < \varepsilon$. A similar reasoning applies when $|\text{Im}(z)| \leq \mu_+$ and $\text{Re}(z) < -\mu_+$. Finally, suppose that $|\text{Im}(z)| \leq \mu_+$ and $\text{Re}(z) > \mu_+$. Since $\omega(q, z) \leq \omega(q, \text{Re}(z))$, it suffices to consider $z = \mu \in \mathbb{R}$ with $\mu > \mu_+$. Then,

$$\omega(q, \mu) < \frac{1}{\sqrt{1+|\mu_+ - x_+|}} \int_{x_+}^\mu |q(x)| \, dx + \int_{x_+}^\infty |q(x)| \, dx < \varepsilon.$$ 

Thus, we have shown that $\omega(q, z) < \varepsilon$ whenever $|\text{Re}(z)| + |\text{Im}(z)| > \mu_+$. □

In what follows $C$ denotes a generic positive constant.

**Proposition 3.2.** Suppose $q \in L^1(\mathbb{R}_+, (1+x)^{-1/2} \, dx)$. Then, the eigenvalue equation $\tau \varphi = z \varphi$ admits a solution $\psi(z, x)$, real entire with respect to $z$, such that:

(i) $\psi(z, x)$ solves the Volterra integral equation

$$\psi(z, x) = \psi_0(z, x) - \int_x^\infty J_0(z, x, y) q(y) \psi(z, y) \, dy,$$ 

where

$$J_0(z, x, y) = \psi_0(z, y) \theta_0(z, x) - \psi_0(z, x) \theta_0(z, y),$$

and satisfies the estimates

$$|\psi(z, x)| \leq C e^{C \omega(z)} \frac{g_A(x-z)}{1+|x-z|^{1/4}} \quad \text{and} \quad |\psi(z, x) - \psi_0(z, x)| \leq C \omega(z) e^{C \omega(z)} \frac{g_A(x-z)}{1+|x-z|^{1/4}}.$$ 

(7)
(ii) Moreover, \( \psi'(z, x) \) obeys the equation

\[
\psi'(z, x) = \psi'_0(z, x) - \int_x^\infty \partial_x J_0(z, x, y)q(y)\psi(z, y)dy
\]

and satisfies the estimates

\[
|\psi'(z, x) - \psi'_0(z, x)| \leq C \omega(z)e^{C\omega(z)}(1 + |x - z|^{1/4})g_A(x - z).
\]

**Proof.** For \( n \in \mathbb{N} \) define

\[
\psi_n(z, x) = -\int_x^\infty J_0(z, x, y)q(y)\psi_{n-1}(z, y)dy.
\]

Then,

\[
|\psi_n(z, x)| \leq \int_x^\infty |J_0(z, x, y)||q(y)||\psi_{n-1}(z, y)|
dy.
\]

Next, we note that

\[
J_0(z, x, y) = \pm i[\psi_0(z, x)\theta_{\pm}(z, y) - \psi_0(z, y)\theta_{\pm}(z, x)]
\]

(the choice of sign is irrelevant). Then, recalling (4) and (5), (9) yields

\[
|\psi_n(z, x)| \leq 2C_0^2 \frac{g_A(x - z)}{1 + |x - z|^{1/4}} \int_x^\infty \frac{|q(y)|}{1 + |y - z|^{1/4}}g_{\#}(y - z)|\psi_{n-1}(z, y)|
dy + 2C_0^2 \frac{g_{\#}(x - z)}{1 + |x - z|^{1/4}} \int_x^\infty \frac{|q(y)|}{1 + |y - z|^{1/4}}g_A(y - z)|\psi_{n-1}(z, y)|
dy.
\]

where \( g_{\#}(z) := 1/g_A(z) \). We claim that every \( \psi_n(z, x) \) is real entire with respect to the spectral parameter and satisfies the estimate

\[
|\psi_n(z, x)| \leq \frac{4^n}{n!}c_0^{2n+1}\omega^n(z)\frac{g_A(x - z)}{1 + |x - z|^{1/4}}.
\]

From this it will follow that

\[
\psi(z, x) = \sum_{n=0}^\infty \psi_n(z, x)
\]

converges uniformly on compact subsets of \( \mathbb{C} \) to the solution with the desired properties.

First, consider \( n = 1 \). Then, we have

\[
|\psi_1(z, x)| \leq 2C_0^3 \frac{g_A(x - z)}{1 + |x - z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1 + |y - z|^{1/4})^2}g_{\#}(y - z)g_A(y - z)dy
\]

\[
+ 2C_0^3 \frac{g_{\#}(x - z)}{1 + |x - z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1 + |y - z|^{1/4})^2}(g_A(y - z))^2dy.
\]

Clearly, \( (g_Ag_{\#})(x - z) \equiv 1 \). Also, Lemma 2.2 implies \( g_A(y - z) \leq g_A(x - z) \) for all \( y \in [x, \infty) \). Hence,

\[
|\psi_1(z, x)| \leq 2C_0^3 \frac{g_A(x - z)}{1 + |x - z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1 + |y - z|^{1/4})^2}dy
\]

\[
+ 2C_0^3 \frac{(g_{\#}g_A)(x - z)}{1 + |x - z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1 + |y - z|^{1/4})^2}dy,
\]

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that is,
\[ |\psi_1(z, x)| \leq 4C_0^3 \frac{g_A(x - z)}{1 + |x - z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1 + |y - z|^{1/4})^2} dy. \]

For arbitrary \( n \in \mathbb{N} \) we use the identity
\[
\int_x^\infty \int_{y_1}^\infty \cdots \int_{y_{n-1}}^\infty \prod_{i=1}^n h(y_i) dy_1 \cdots dy_n = \frac{1}{n!} \left( \int_x^\infty h(y) dy \right)^n
\]
to obtain
\[
|\psi_n(z, x)| \leq \frac{4^n}{n!} C_0^{2n+1} \frac{g_A(x - z)}{1 + |x - z|^{1/4}} \left( \int_x^\infty \frac{|q(y)|}{(1 + |y - z|^{1/4})^2} dy \right)^n
\]
which in turn implies (10). Then (i) follows after a suitable choice for the constant \( C \).

Clearly, (7) implies that \( \psi(z, x) \) so constructed belongs to the domain of the maximal operator \( H \).

The asymptotic analysis of the norming constants depends also on the following estimates.

**Proposition 3.3.** Suppose \( q \in L^1(\mathbb{R}_+) \). Then, \( \psi(z, x) \) satisfies
\[
|\psi'(z, x) - \psi_0'(z, x)| \leq Ce^{C\|q\|} \left( (1 + |x - z|^{1/4}) \omega(z) + \frac{\|q\|}{1 + |x - z|^{1/4}} \right) g_A(x - z).
\]

Also,
\[
|\psi'(z, x) - \psi'_0(z, x)| \leq Ce^{C\|q\|} \left( (1 + |x - z|^{1/4}) \|q\|^2 + \frac{|x - z| \omega(z)}{1 + |x - z|^{1/4}} \right) g_A(x - z).
\]

**Proof.** From (6) we see that \( \psi(z, x) \) is a solution to the integral equation
\[
\psi(z, x) = \psi_0(z, x) - \int_x^\infty \partial_z J_0(z, x, y) q(y) \psi(z, y) dy - \int_x^\infty J_0(z, x, y) q(y) \psi(z, y) dy.
\]

Let \( \{\eta_k(z, x)\}_{k \in \mathbb{N}} \), be solutions to the recursive equation
\[
\eta_k(z, x) = -\int_x^\infty \partial_z J_0(z, x, y) q(y) \psi_{k-1}(z, y) dy - \int_x^\infty J_0(z, x, y) q(y) \eta_{k-1}(z, y),
\]
where \( \{\psi_k(z, x)\}_{k \in \mathbb{N}} \) are defined in the proof of Proposition 3.2 and \( \eta_0(z, x) := \psi_0(z, x) \). Using induction one can show that
\[
|\eta_k(z, x)| \leq \frac{4k C_0^{2k+1}}{k!} \left( (1 + |x - z|^{1/4}) \left( \int_x^\infty \frac{|q(y)|}{(1 + |y - z|^{1/4})^2} dy \right)^k + \frac{2k}{1 + |x - z|^{1/4}} \left( \int_x^\infty |q(y)| dy \right)^k \right) g_A(x - z),
\]

hence
\[
|\eta_k(z, x)| \leq \frac{4k C_0^{2k+1}}{k!} \left( (1 + |x - z|^{1/4}) \omega(z)^k + \frac{2k}{1 + |x - z|^{1/4}} \|q\|^k \right) g_A(x - z).
\]

It follows that
\[
\psi(z, x) = \sum_{k=0}^\infty \eta_k(z, x)
\]
(the convergence being uniform on compact subsets of \( \mathbb{C} \)) which in turn implies the assertion.

The proof of the second inequality follows from an analogous reasoning.
3.2 Dirichlet boundary condition

Define the contours
\[ E^m := \{ z \in \mathbb{C} : |\zeta| = (m + \frac{1}{4})\pi \}, \quad E_k := \{ z \in \mathbb{C} : |\zeta - (k - \frac{1}{4})\pi| = \frac{\pi}{2} \}, \quad m, k \in \mathbb{N}. \]

In view of (1), every \( E_k \) encloses one and only one zero of \( \text{Ai}(-\lambda) \), at least for \( k \) sufficiently large.

**Lemma 3.4.** There exists \( m_0, k_0 \in \mathbb{N} \) such that, for every \( m \geq m_0 \) and \( k \geq k_0 \), the following statement holds true:
\[
\frac{g_A(-z)}{1 + |z|^{1/4}} < 8\sqrt{\pi} |\text{Ai}(-z)|, \tag{15}
\]

whenever \( z \in E^m \) or \( z \in E_k \).

**Proof.** Let us begin by recalling (2) and (3) in more precise terms:
\[
\text{Ai}(z) = \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}} [1 + W_1(z)], \quad |\arg(z)| \leq \frac{2\pi}{3}, \quad |z| \geq 1, \tag{16}
\]
\[
\text{Ai}(-z) = \frac{1}{\sqrt{\pi}z^{1/4}} [\sin(\zeta + \frac{\pi}{4}) + W_2(z)], \quad |\arg(z)| \leq \frac{\pi}{3}, \quad |z| \geq 1, \tag{17}
\]

where the functions \( W_1(z) \) and \( W_2(z) \) satisfy
\[
\left| \frac{W_1(z)}{\zeta^{1/4}} \right| \leq D_1, \quad |\arg(z)| \leq \frac{2\pi}{3}, \quad |z| \geq 1, \quad \left| \frac{W_2(z)}{\zeta^{-1}e^{\text{Im}\zeta}} \right| \leq D_2, \quad |\arg(z)| \leq \frac{\pi}{3}, \quad |z| \geq 1. \tag{18}
\]

There exists \( k_0 \in \mathbb{N} \) such that, for all \( k \geq k_0 \), \( z \in E_k \) implies \( \text{Re} z \geq 1 \) and \( \arg(z) \in (-\frac{\pi}{3}, \frac{\pi}{3}) \) so \( \text{arg}(-z) \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi] \). Since in this case \( |\text{Im} z^{3/2}| = -\text{Re}(-z)^{3/2} \), one has
\[
\frac{g_A(-z)}{1 + |z|^{1/4}} = \frac{e^{\text{Im}\zeta}}{1 + |z|^{1/4}} \leq \frac{e^{\text{Im}(\zeta + \frac{\pi}{4})}}{|z|^{1/4}}
\]
for all \( z \in E_k \) and \( k \geq k_0 \). By a well-known result (see [24, Ch. 2, Lemma 1]),
\[
|w - n\pi| \geq \frac{\pi}{4} \implies e^{\text{Im} w} < 4 |\sin w|
\]
for all integer \( n \). Hence,
\[
\frac{g_A(-z)}{1 + |z|^{1/4}} < 4 \frac{|\sin(\zeta + \frac{\pi}{4})|}{|z|^{1/4}} \tag{19}
\]
for all \( z \in E_k \) and \( k \geq k_0 \). On the other hand, since \( |\sin(\zeta + \frac{\pi}{4})| \geq d > 0 \) for all \( z \in E_k \) and \( k \geq k_0 \), (17) implies
\[
|\text{Ai}(-z)| \geq \frac{|\sin(\zeta + \frac{\pi}{4})|}{\sqrt{\pi}|z|^{1/4}} \left| 1 - \frac{|W_2(z)|}{|\sin(\zeta + \frac{\pi}{4})|} \right|.
\]

However,
\[
\frac{|W_2(z)|}{|\sin(\zeta + \frac{\pi}{4})|} \leq \frac{e^{\text{Im}\zeta}}{|\zeta|} \frac{D_2}{d},
\]

8
and note that $|\text{Im } \zeta| \leq \pi/2$ if $z \in \mathcal{E}_k$. Thus, by increasing $k_0$ if necessary, we have

$$|\text{Ai}(-z)| \geq \frac{|\sin(\zeta + \frac{\pi}{4})|}{2\sqrt{\pi}|z|^{1/4}},$$

for all $z \in \mathcal{E}_k$ with $k \geq k_0$.

The proof concerning $\mathcal{E}_m$ is analogous: Suppose $m_0 = k_0$. Then, by the previous argument, (15) holds for $z \in \mathcal{E}_m$ within the sector $\arg(z) \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, for $m \geq m_0$. Within the sector $\arg(-z) \in [-\frac{2\pi}{3}, \frac{2\pi}{3}]$, we have ($\eta := \frac{2}{3}(-z)^{3/2}$)

$$\frac{g_A(-z)}{1 + |z|^{1/4}} \leq \frac{e^{-\text{Re } \eta}}{|z|^{1/4}},$$

and, due to (16),

$$|\text{Ai}(-z)| \geq \frac{e^{-\text{Re } \eta}}{2\sqrt{\pi}|z|^{1/4}|1 - |W_1(-z)||}.$$

Finally, using (18) —and increasing $m_0$ if required—, we have $1 - |W_1(-z)| \geq 1/4$ whenever $|z| \geq m_0$. ■

**Theorem 3.5.** Suppose $q \in L^1(\mathbb{R}_+)$. Then, the eigenvalues of $H^D$ satisfy

$$\lambda^D_k = \left(\frac{3}{2\pi}(k - \frac{1}{4})\right)^{2/3} \left(1 + O(k^{-1})\right), \quad k \to \infty.$$

**Proof.** Abbreviate

$$\psi_0(z) := \psi_0(z, 0), \quad \psi(z) := \psi(z, 0), \quad \mu_k := -a_k.$$

Since $\sup_{z \in \mathcal{C}} \omega(z) < \infty$, Proposition 3.2 yields

$$|\psi(z) - \psi_0(z)| \leq C\omega(z)\frac{g_A(-z)}{1 + |z|^{1/4}},$$

after redefining the constant $C$. Due to Lemma 3.1, there exists $k_1 \in \mathbb{N}$ such that $\omega(z) \leq (8C)^{-1}$ whenever $|z| \geq (\frac{3}{2}\pi(k_1 + \frac{1}{4}))^{2/3}$. Then, by Lemma 3.4, there exists $k_2 \geq k_1$ such that

$$|\psi(z) - \psi_0(z)| < |\psi_0(z)|$$

for all $z \in \mathcal{E}^{k_2}$; $k_2$ can be assumed large enough so $\mathcal{E}^{k_2}$ encloses all the (finitely many) negative zeros of $\psi(z)$. Increase $k_2$ (if necessary) to ensure that (20) holds true for $z$ on every contour $\mathcal{E}_n$ whenever $n \geq n_2$. Then, in view of Rouché's theorem, we obtain

$$\left|\frac{3}{2}(\lambda^D_k)^{3/2} - \frac{3}{2}(-a_k)^{3/2}\right| \leq \pi$$

for sufficiently large $k$, whence the asymptotics for the eigenvalues follows. ■

**Theorem 3.6.** Suppose $q \in L^1(\mathbb{R}_+)$. Then the Dirichlet norming constants $\nu^D_k$ satisfies

$$\frac{1}{\nu^D_k} = 1 + o(1)$$

as $k \to \infty$. 9
Proof. Abbreviate
\[
\Delta_1(\lambda) := \frac{\psi'(\lambda, 0) - \psi_0'(\lambda, 0)}{\sqrt{\pi} A'(-\lambda)}, \quad \Delta_2(\lambda) := \frac{\psi(\lambda, 0) - \psi_0(\lambda, 0)}{\sqrt{\pi} A'(-\lambda)}.
\]
It is straightforward to see that
\[
-\frac{\psi(\lambda_k^D, 0)}{\psi'(\lambda_k^D, 0)} = 1 - \frac{\Delta_1(\lambda_k^D)}{1 + \Delta_1(\lambda_k^D)} - \frac{\Delta_2(\lambda_k^D)}{1 + \Delta_1(\lambda_k^D)}
\]
so it suffices to show that
\[
\Delta_1(\lambda_k^D) \to 0 \quad \text{and} \quad \Delta_2(\lambda_k^D) \to 0
\]
as \(k \to \infty\).

From Theorem 3.5 we obtain \(\lambda_k^D = -a_k + O(k^{-1/3})\) thus
\[
\sqrt{\pi} A'(-\lambda_k^D) = (-1)^{k-1} \left(\frac{3}{2\pi}(k - \frac{1}{4})\right)^{1/6} (1 + o(1)) = (-1)^{k-1}(\lambda_k^D)^{1/4}(1 + o(1))
\]
as \(k \to \infty\). On the other hand, from (8), we have
\[
\left|\psi(\lambda_k^D, 0) - \psi_0(\lambda_k^D, 0)\right| \leq C\omega(\lambda_k^D)e^{C\omega(\lambda_k^D)}|\lambda_k^D|^{1/4}
\]
hence the assertion on \(\Delta_1(\lambda_k^D)\) holds true since \(\omega(\lambda_k^D) \to 0\) as \(k \to \infty\) due to Lemma 3.1. Finally, (13) implies the corresponding assertion on \(\Delta_2(\lambda_k^D)\).

### 3.3 Neumann boundary condition

The analysis of the asymptotic behavior of \(\sigma(H^N)\) does not differ much from the Dirichlet case. We start by defining the contours
\[
\mathcal{F}^m := \left\{ z \in \mathbb{C} : |\zeta| = (m - \frac{1}{4})\pi \right\}, \quad \mathcal{F}_k := \left\{ z \in \mathbb{C} : \left|\zeta - (k + \frac{1}{4})\pi\right| = \frac{\pi}{2} \right\}, \quad m, k \in \mathbb{N}.
\]

As expected, \(\mathcal{F}_k\) encloses exactly one zero of \(A'(\lambda)\) for sufficiently large values of \(k\).

**Lemma 3.7.** There exists \(m_0, k_0 \in \mathbb{N}\) such that, for every \(m \geq m_0\) and \(k \geq k_0\), the following statement holds true:
\[
(1 + |x - z|^{1/4})g_A(-z) < 16\sqrt{\pi}|A'(z)|,
\]
whenever \(z \in \mathcal{F}^m\) or \(z \in \mathcal{F}_k\).

The proof of this assertion is nearly identical to the proof of Lemma 3.4, except that it relies on the identities
\[
A'(z) = -z^{1/4} \frac{e^{-\zeta}}{2\sqrt{\pi}} \left[1 + W_3(z)\right], \quad |\arg(z)| \leq \frac{2\pi}{3}, \quad |z| \geq 1,
\]
\[
A'(z) = z^{1/4} \frac{\sin(\zeta - \frac{1}{4}) + W_4(z)}{\sqrt{\pi}}, \quad |\arg(z)| \leq \frac{\pi}{4}, \quad |z| \geq 1,
\]
where the functions \(W_3(z)\) and \(W_4(z)\) satisfy
\[
|W_3(z)| \leq D_1, \quad |\arg(z)| \leq \frac{2\pi}{3}, \quad |z| \geq 1,
\]
\[
|W_4(z)| \leq D_2, \quad |\arg(z)| \leq \frac{\pi}{4}, \quad |z| \geq 1.
\]
The details are therefore omitted.
Theorem 3.8. Suppose \( q \in L^1(\mathbb{R}_+) \). Then, the eigenvalues of \( H^N \) satisfy

\[
\lambda^N_k = \left( \frac{2}{3} \pi (k - \frac{3}{4}) \right)^{2/3} \left( 1 + O(k^{-1}) \right), \quad k \to \infty.
\]

Proof. Since it is similar to the proof of Theorem 3.5, we only hint at the main departure from it. Recalling that \( \sup_{z \in \mathbb{C}} \omega(z) < \infty \), (8) implies

\[
|\psi'(z) - \psi'_0(z)| \leq C \omega(z)(1 + |z|^{1/4})g_A(-z)
\]

for certain positive constant \( C \). Because of Lemma 3.7, there exists \( k_1 \in \mathbb{N} \) such that

\[
|\psi'(z) - \psi'_0(z)| < |\psi'_0(z)|
\]

for all \( z \in F^{k_1} \) and \( z \in F_k \) for every \( k > k_1 \), hence \( |(\lambda^N_k)^{3/2} - (-a_k^N)^{3/2}| < \pi \) for all \( k \) large enough. ■

Theorem 3.9. Suppose \( q \in L^1(\mathbb{R}_+) \). Then the Neumann norming constants \( \nu^N_k \) satisfies

\[
\frac{1}{\nu^N_k} = \left( \frac{2}{3} \pi (k - \frac{3}{4}) \right)^{2/3} (1 + o(1))
\]

as \( k \to \infty \).

Proof. The argument goes along the lines of the proof of Theorem 3.6. Define

\[
\Delta_3(\lambda) := \frac{\psi'(\lambda, 0) - \psi'_0(\lambda, 0)}{\sqrt{\pi \lambda} \text{Ai}(-\lambda)}, \quad \Delta_4(\lambda) := \frac{\psi(\lambda, 0) - \psi'_0(\lambda, 0)}{\sqrt{\pi} \text{Ai}(-\lambda)}.
\]

Then

\[
\frac{\psi'(\lambda^N_k, 0)}{\psi(\lambda^N_k, 0)} = \lambda^N_k \frac{1 + \Delta_3(\lambda^N_k)}{1 + \Delta_4(\lambda^N_k)}
\]

so we only need to prove that

\[
\Delta_3(\lambda^N_k) \to 0 \quad \text{and} \quad \Delta_4(\lambda^N_k) \to 0
\]

as \( k \to \infty \). But this follows from (7) and (14). ■

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