THE VAN KAMPEN OBSTRUCTION AND ITS RELATIVES

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Abstract. We review a cochain-free treatment of the classical van Kampen obstruction \( \vartheta \) to embeddability of an \( n \)-polyhedron into \( \mathbb{R}^{2n} \) and consider several analogues and generalizations of \( \vartheta \), including an extraordinary lift of \( \vartheta \) which in the manifold case has been studied by J.-P. Dax. The following results are obtained.

- The mod2 reduction of \( \vartheta \) is incomplete, which answers a question of Sarkaria.
- An odd-dimensional analogue of \( \vartheta \) is a complete obstruction to linkless embeddability (="intrinsic unlinking") of the given \( n \)-polyhedron in \( \mathbb{R}^{2n+1} \).
- A “blown up” 1-parameter version of \( \vartheta \) is a universal type 1 invariant of singular knots, i.e. knots in \( \mathbb{R}^3 \) with a finite number of rigid transverse double points. We use it to decide in simple homological terms when a given integer-valued type 1 invariant of singular knots admits an integral arrow diagram (=Polyak–Viro) formula.
- Settling a problem of Yashchenko in the metastable range, we obtain that every PL manifold \( N \), non-embeddable in a given \( \mathbb{R}^m \), \( m \geq \frac{3(n+1)}{2} \), contains a subset \( X \) such that no map \( N \to \mathbb{R}^m \) sends \( X \) and \( N \setminus X \) to disjoint sets.
- We elaborate on McCrory’s analysis of the Zeeman spectral sequence to geometrically characterize “\( k \)-co-connected and locally \( k \)-co-connected” polyhedra, which we embed in \( \mathbb{R}^{2n-k} \) for \( k < \frac{n-3}{2} \) extending the Penrose–Whitehead–Zeeman theorem.

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Dependence of sections. Dashed arrows stand for ‘consult as the need arises’.

1. INTRODUCTION

This paper is intended to be the first in a series devoted to extracting obstructions to embeddability and (especially) knot and link invariants from cohomological
invariants of maps between configuration spaces and their various compactifications. In the present paper we shall consciously limit ourselves to a few very basic situations, and merely try to put them “in order”.

After some algebraic preparation in §2 (which is mostly needed for the purposes of §5 and can be skipped on the first reading) we start in §3 by reviewing some known facts about the van Kampen obstruction \( \vartheta(X) \) in an invariant language. The only treatment of \( \vartheta(X) \) of this kind found by the author in the literature is \( \check{\text{S}} \)varc’s classical paper [Sch] on the sectional category of a fibration.\(^1\) A clear treatment of the mod2 reduction of \( \vartheta(X) \) in invariant terms has been given by Bestvina, Kapovich and Kleiner [BKK] (see also a refinement in [SV]).

As a byproduct, we reidentify the cohomology group containing \( \vartheta(X) \): it was correctly identified in the classical papers of Shapiro and Wu, but much of the modern literature, starting with [FKT] and including, for instance, [Va2], replicates a sign error, which instead of the order 2 element \( \vartheta(K_5) \), say, leads to a non-invariant element of infinite order.

Thus in §3 we recall:

- several definitions of the van Kampen obstruction \( \vartheta(X) \), which obstructs embeddability of the \( n \)-polyhedron \( X \) in \( \mathbb{R}^{2n} \);
- its 1-parameter version \( \vartheta(f,g) \), which is an obstruction to isotopy of the two embeddings \( f,g: X \hookrightarrow \mathbb{R}^{2n+1} \);
- why \( \vartheta(K^n) \) is nonzero, where \( K^n \) is either the \( n \)-skeleton of the \((2n+2)\)-simplex or the join of \( n+1 \) copies of the 3-point set;
- a proof that \( \vartheta(X) \) is complete when \( n > 2 \).

The remaining content of the paper is organized as follows.

**Van Kampen obstruction.** K. Sarkaria noticed that if \( X \) is a graph and the mod2 reduction of \( \vartheta(X) \) is zero, then \( \vartheta(X) \) is zero, and he asked whether this is true of every \( n \)-polyhedron \( X \) [Sa2] (see also [Sk2] concerning an error in [Sa2]). In Example 3.6 we show that this is not so for each \( n > 1 \).

In Theorem 4.4 we find that for each \( n \) (including \( n = 2 \)), \( \vartheta(X) \) is a complete obstruction to the existence of an embedding \( X \hookrightarrow \mathbb{R}^{2n+1} \) extending to a map of the cone \( f: CX \rightarrow \mathbb{R}^{2n+1} \) such that \( f^{-1}(X) = X \).

We also give a recipe for computation of \( H^{2n}(\bar{X}) \), where \( \bar{X} = (X \times X \setminus \Delta)/(\mathbb{Z}/2) \), the group that contains \( \vartheta(X) \) (Theorem 3.8). In particular, this provides an easily verifiable sufficient condition for the vanishing of \( \vartheta(X) \).

**Linkless embeddings.** An odd-dimensional analogue \( \eta(X) \) of \( \vartheta(X) \) turns out to be a complete obstruction to linkless embeddability of the \( n \)-polyhedron \( X \) into \( \mathbb{R}^{2n+1} \) (Theorem 4.2).\(^2\) We call an embedding \( g \) of an \( n \)-polyhedron \( X \) in \( \mathbb{R}^{2n+1} \) linkless if every two disjoint closed subpolyhedra of \( g(X) \) can be separated by an embedded \((m-1)\)-sphere; variants of this definition are discussed in §4. (Nonexistence of linkless embeddings is also known as “intrinsic linking” in the literature.)

In fact, the principal initial motivation of this paper, coming from [MS], was to find a method of “local” computation of \( \vartheta(X) \), which would take into account

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\(^1\)Beware that it occurs on page 69 of the 84-page paper among other applications. The author is grateful to M. Grant for calling his attention to this reference, which was overlooked in the initial version of the present paper (posted on the arXiv in 2006).

\(^2\)Added in v5: This is not quite true of \( \vartheta(X) \) as defined in the published version of this paper, but true of its modified version \( \vartheta'(X) \).
such information as $\vartheta(L)$ and $\eta(L)$, where $L$ runs over the links of all simplices of some triangulation of $X$. While this is not quite achieved at this point, some simple connections between embeddability and local linking phenomena are given by Lemma 3.7, which is folklore, and by Proposition 4.6. The latter is used to give yet another proof that the $n$-skeleton of the $(2n + 3)$-simplex admits no linkless embeddings in $\mathbb{R}^{2n+1}$ (Example 4.7) — based on the fact that the $(n + 1)$-skeleton of the $(2n + 4)$-simplex does not embed in $\mathbb{R}^{2n+2}$ even “modulo 2”.

**Type 1 invariants of singular knots in $\mathbb{R}^3$.** In the theory of finite type invariants of knots, the study of type $m$ invariants of knots (up to smooth isotopy) is often broken down in steps by integrating the successive Vassiliev derivatives of the invariant (see, in particular, the “actuality tables” in [Va1]). At the initial step, one looks at the $(m - 1)$th Vassiliev derivatives of type $m$ invariants of knots. These can be viewed as type 1 invariants of singular knots, that is, of smooth immersions $S^1 \hookrightarrow \mathbb{R}^3$ with prescribed double points, considered up to smooth regular homotopy with prescribed double points. (More detailed definitions are given in §5.) Notice that the smoothness of the regular homotopy guarantees that the germs of the four strands going out of each double point of the knot lie on an infinitesimal 2-disk in $\mathbb{R}^3$, and in particular their cyclic order on this disk does not change under the regular homotopy. Thus the relation of smooth regular homotopy with prescribed double points is strictly stronger than that of piecewise-smooth regular homotopy with prescribed double points. It is easy to exhibit specific singular knots that are distinct but equivalent up to piecewise-smooth regular homotopy.

We construct (or rather reconstruct from [Va2]) a configuration space of the singular knot, using blowups to keep track of the infinitesimal 2-disks at the vertices. The resulting “smooth” analogue $\zeta$ of the van Kampen graph planarity obstruction $\vartheta$ is found to be a complete obstruction to planarity of a singular knot, with its cyclic order of edges around each vertex (Corollary 5.13). The 1-parameter version of $\zeta$ is found to yield a universal type 1 invariant of singular knots (Corollary 5.3).

Next we study, prompted by Vassiliev’s work [Va2], the question of existence of arrow diagram formulas (also known as Polyak–Viro formulas or “combinatorial formulas”) with integer coefficients. In fact, §5 originated from the author’s extended review for Mathematical Reviews [M3], where the results of Vassiliev’s 50-page paper [Va2], with the omission of explicit calculations and of generalization to higher dimensions, were reproved in much less space and by elementary means, without using Vassiliev’s sophisticated machinery. In the present paper we elaborate on this development and fill in some details. In particular, we correct an inaccuracy in the description of the configuration space $\Theta$ given in [M3] and eliminate the usage of Vassiliev’s calculations, making our approach to his results self-contained.

More specifically, among other results, we decide in simple homological terms whether a given type 1 invariant of singular knots has an integral arrow diagram formula (Theorem 5.6) and express Vassiliev’s partial obstruction $\propto (v)$ to existence of an integral arrow diagram formula in terms of $\zeta$ (Theorem 5.10).

**Embeddings in the metastable range via generalized cohomology.** A homotopy-theoretic classification of embeddings (up to isotopy) of a compact polyhedron $X$ in $\mathbb{R}^m$ in the metastable range is provided by the well-known Haefliger–Weber Criterion 3.1. In §6, we combine it with some basic equivariant algebraic topology to obtain a more concrete classification in terms of generalized cohomology. Namely, the set of isotopy classes of embeddings $X \hookrightarrow \mathbb{R}^m$, if nonempty,
is in a natural bijection with an equivariant\(^3\) stable cohomotopy group of \(\tilde{X}_{+} := (X \times X \setminus \Delta) \sqcup \) basepoint (Theorem 6.6). We also define a complete obstruction \(\Theta^m(X)\) in generalized cohomology to embeddability of \(X\) in \(\mathbb{R}^m\) in the metastable range (Theorem 6.3). It is an element of an equivariant stable cohomotopy group of \(\tilde{X}_{+}\), and when \(m = 2n\), it is identified with \(\vartheta(X)\) by the Hurewicz isomorphism.

On the day of the final deadline for submission of the final version of this paper to the special issue of the journal, the author discovered a paper by J. P. Dax [Dax], where essentially the same results as those of §6 have been obtained in the case of embeddings of smooth manifolds. The restriction to manifolds is essential for Dax’s arguments, but modulo [We], [Har] and [BRS], they could have been easily modified (in fact, simplified) so as to apply to polyhedra. Regrettably Dax’s work was not as influential as Hatcher–Quinn’s [HQ; Theorem 2.3] (compare [KW; Corollary G] and [Kl]), where a closely related problem is approached with a slightly different philosophy, leading to a rather frighteningly looking answer. For certain manifolds, a geometric description of Dax’s group structure on the set of isotopy classes of embeddings in \(\mathbb{R}^m\) was found by A. Skopenkov [S2].

A controlled version \(\Theta(f)\) of \(\Theta(X)\) is a complete obstruction to \(C^0\)-approximability by embeddings of the map \(f : X \to \mathbb{R}^m\) in the metastable range (Criterion 7.1). \(\Theta(f)\) generalizes both Skopenkov’s cohomology obstruction [RS] and Akhmetiev’s obstruction in skew-framed bordism [Ah].

**Towards understanding the notion of embeddability.** An intriguing connection between embeddability and what apparently is some kind of local linking is given by Yashchenko’s PhD thesis along with our affirmative solution of the metastable case of his problem.

**Yashchenko’s Theorem 1.1** [see [Ya]]. *If a compactum \(N\) does not admit a map to \(\mathbb{R}^m\) with countable singular set, there exists a subset \(X \subset N\) such that no map \(f : N \to \mathbb{R}^m\) sends \(X\) and \(N \setminus X\) to disjoint sets.*

By the *singular set* of a (continuous) map \(f : N \to \mathbb{R}^m\) we mean \(S_f = \{x \in N \mid f^{-1}(f(x)) \neq \{x\}\}\).

**Yashchenko’s Problem 1.2** [Ya]. *If a smooth manifold \(N\) admits a map to \(\mathbb{R}^m\) with countable singular set, does it embed in \(\mathbb{R}^m\)?*

As Yashchenko notes in [Ya], the answer would be negative for any compact \(n\)-polyhedron, non-embeddable in \(\mathbb{R}^{2n}\).

Maps with countable singular sets abound. Whenever \(n < m\), one can easily construct a sequence of embeddings \(g_n : S^n \to \mathbb{R}^m\) that uniformly converge to a map \(f\) with countable dense \(S_f\) (compare [RR+]). It is equally easy to construct a sequence of immersions \(g_n : S^1 \hookrightarrow \mathbb{R}^2\) (with winding numbers \(w(g_n) = 2^n\), say) that uniformly converge to a map \(f : S^1 \to \mathbb{R}^2\) whose restriction to any open subset is non-approximable by embeddings, and whose \(S_f\) is the set of all dyadic rational points of \(S^1 = \mathbb{R}/\mathbb{Z}\). More generally, if two \((n-1)\)-spheres link nontrivially in \(S^{m-1}\), any embedding of an \(n\)-manifold \(N\) into \(\mathbb{R}^m\) can be modified by connect-summing with increasingly small copies of the suspension over the link into a map \(f : N \to \mathbb{R}^m\) whose restriction to any open subset is non-approximable by embeddings, and whose \(S_f\) is a countable dense set.

\(^3\)Equivariant generalized cohomology reduces to non-equivariant one by a standard procedure, which we will recall.
Using the controlled extraordinary van Kampen obstruction, we show that a map of a PL $n$-manifold in $\mathbb{R}^m$, $m \geq \frac{3(n+1)}{2}$, with a countable (possibly dense) singular set can be arbitrarily closely approximated by maps with finite singular sets (Theorem 7.2). In Theorem 7.4 we eliminate the finite singular set by a standard technique. These results combine with Yashchenko’s theorem to yield

**Corollary 1.3.** If a PL $n$-manifold $N$ does not embed in some $\mathbb{R}^m$, $m \geq \frac{3(n+1)}{2}$, then $N$ contains a subset $X$ such that no map $N \to \mathbb{R}^m$ sends $X$ and $N \setminus X$ to disjoint sets.

The trouble with this result is that it does not tell us the answer to

**Problem 1.4.** Can this $X$ be chosen to be a subpolyhedron?

**Polyhedra satisfying partial Alexander duality.** In the last section we generalize the Penrose–Whitehead–Zeeman theorem that $k$-connected PL $n$-manifolds embed in $\mathbb{R}^{2n-k}$ to $n$-polyhedra that might be termed ‘homologically $k$-co-connected and locally homologically $k$-co-connected’. We find that such polyhedra coincide with those whose subsets satisfy the first $k$ Alexander duality homomorphisms, and also (up to a shift by one dimension) with those homologically $k$-connected polyhedra whose $i$-cycles enjoy some kind of “homological transversality” with respect to subpolyhedra of codimension $< i$, for all $i < k$ (Theorem 8.3). This provides several related answers to a question of V. M. Buchstaber, who queried the author regarding generalizations of the Penrose–Whitehead–Zeeman theorem to polyhedra.

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### 2. A Glimpse of Underground Algebraic Topology

In this section we recall the notions of homological and cohomological transfers of a double covering and of the Euler class of the associated line bundle, and use them to give a geometric description of homomorphisms in the Smith sequences of a free action of $\mathbb{Z}/2$. A convenient language to discuss the transfers and the Euler class (which language in particular has a name for the natural generality where they are defined) is the geometric understanding of cohomology from the unique book of Buoncristiano, Rourke and Sanderson [BRS]. The reader is hereby warned that mainstream algebraic topology (as exemplified by J. F. Adams’ review of [BRS]) considers the ideas contained in this book to be dangerous (for one’s

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4Added in v5: I am also grateful to R. Nikkuni and V. Turchin for useful feedback.
“balance between the particular and the general”), and is openly uncomfortable with the fact that its publication could not have been prevented.

Local coefficient systems are discussed in the textbooks by Hilton–Wylie, Spanier and Hatcher; and in more detail in those by G. Whitehead and Davis–Kirk, not to mention sheaf theory texts such as Bredon’s [Bre].

**Pseudo-manifolds, local coefficients.** By an \( m \)-pseudo-manifold we shall understand a finite simplicial complex where all simplices have dimension \( \leq m \) and every \((m-1)\)-simplex is a facet of precisely two \( m \)-simplices. An \( m \)-pseudo-manifold with boundary is a pair of finite simplicial complexes \((M, \partial M)\) where \( \partial M \) is an \((m-1)\)-pseudo-manifold and the double \( M \times \{0\} \cup \partial M \times \{0\} = \partial M \times \{1\} \) is an \( m \)-pseudo-manifold.

An orientation of an \( n \)-simplex, \( n > 0 \) is an orbit of the alternating group \( A_{n+1} \) acting on the set of numberings of its vertices by \( 1, \ldots, n+1 \). By convention, the \( 0 \)-simplex has two orientations + and −. An oriented \( n \)-simplex \( \sigma \) that is a facet of an oriented \((n+1)\)-simplex \( \tau \) is its oriented facet if either \( n > 0 \) and the first \( n+1 \) vertices of some representative numbering of the vertices of \( \tau \) constitute a representative numbering of the vertices of \( \sigma \), or \( n = 0 \) and the orientation of \( \sigma \) equals the sign of the unique permutation converting a representative numbering of the vertices of \( \tau \) into one where \( \sigma \) is numbered 1. An orientation of an \( m \)-pseudo-manifold with boundary is a choice of orientations of its \( m \)- and \((m-1)\)-simplices such that each oriented \((m-1)\)-simplex is an oriented facet of precisely one oriented \( m \)-simplex.

A local coefficient system on a polyhedron \( P \) is a collection of abelian groups \( G_x \) indexed by \( x \in P \) along with isomorphisms \( h_l: G_{l(0)} \to G_{l(1)} \) for every path \( l: I \to P \) such that each \( h_l \) depends only on the homotopy class of \( l \) relative to its endpoints. (Here \( I = [0,1] \).) If \( p: Q \to P \) is a double covering, it determines an integral (i.e. with each \( G_x \simeq \mathbb{Z} \)) local coefficient system \( \mathbb{Z}_p \) on \( P \), called the orientation sheaf of \( p \), that associates the nontrivial automorphism of \( \mathbb{Z} \) precisely to those loops that do not lift to \( Q \). Clearly, every integral local coefficient system is of this type.

A simplicial map \( f \) of an \( m \)-pseudo-manifold \( M \) into a simplicial complex \( K \) is called a singular \( m \)-pseudo-manifold in \( K \). If \( \mathcal{O} \) is an integral local coefficient system on the underlying polyhedron \( P \) of \( K \), an \( \mathcal{O} \)-orientation of \( f \) is a choice of orientations of \( m \)- and \((m-1)\)-simplices of \( M \) such that a loop \( l \) in the dual 1-skeleton of \( M \) intersects an odd number of “disoriented” \((m-1)\)-simplices if and only if \( f(l) \) acts nontrivially on \( \mathbb{Z} \). (An \((m-1)\)-simplex is considered “disoriented” if it fails to be an oriented facet of precisely one oriented \( m \)-simplex.)

We define \( H_i(P; \mathbb{Z}/2) \) (resp. \( H_i(P; \mathcal{O}) \)), where \( P \) is a polyhedron (and \( \mathcal{O} \) an integral local coefficient system on \( P \)), to be the group of \( (\mathcal{O} \text{-oriented}) \) singular pseudo-bordism classes of \((\mathcal{O} \text{-oriented}) \) singular \( i \)-pseudo-manifolds in some fixed triangulation of \( P \). It is well-known, and easy to see that this is equivalent to the usual definition (compare [BRS], [Ah]), so in particular does not depend on the choice of a triangulation of \( P \). An \( m \)-pseudo-manifold \( M \) is \( \mathcal{O} \)-orientable iff \( H^m(M; \mathcal{O}) \) contains no torsion; in which case its \( \mathcal{O} \)-orientation is equivalent to a choice of a set of generators of \( H_m(M; \mathcal{O}) \) representable by cycles with disjoint supports.\(^5\)

\(^5\)Added in v5: Here the “cycles” are meant to be simplicial, and by the support of an \( m \)-cycle we mean a subset of the set of \( m \)-simplices of \( M \), rather than a subset of \( M \) itself.
Pseudo-comanifolds. We call a PL map $f: W \to P$ between polyhedra a $k$-pseudo-comanifold over a triangulation $K$ of $P$ if the preimage $f^{-1}(\sigma)$ of every $i$-simplex $\sigma$ of $K$ is triangulated by an orientable $(i-k)$-pseudo-manifold $M$ with boundary $\partial M$ such that $f^{-1}(\partial \sigma)$ coincides with the underlying polyhedron of $\partial M$. For instance, if $T$ is the three-page book $\{0\} \times \{1, 2, 3\} \times \mathbb{R}$, then every embedded 1-pseudo-comanifold $G$ in $T$ is a graph with vertices of degrees 1 and 3, where the degree 3 vertices coincide with the intersection points of $G$ and the binding $\{0\} \times \mathbb{R}$, and the degree 1 vertices coincide with the intersection points of $G$ and the page edges $\{1, 2, 3\} \times \mathbb{R}$. Thus embedded 1-pseudo-comanifolds in $T$ can be alternatively described as transversal point-inverses of maps $T \to S^1$ (using that they are always co-oriented, see below). Pseudo-comanifolds originate in [BRS], where they are called “mock bundles with codimension two singularities”. See also [Fe], [FS], [CG], [Gra], [De]. Using the Poincaré duality, pseudo-comanifolds over a triangulation $K$ of a polyhedron $P$ can also be viewed as pullbacks over $P$ of PL maps of pseudo-manifolds with boundary $(M, \partial M) \to (N, \partial N)$ that are transverse to $K$, where $N$ is a PL manifold neighborhood of a copy of $P$ in some Euclidean space.

If $f: W \to P$ is a $k$-pseudo-comanifold over a triangulation $K$ of $P$, a co-orientation of $f$ over a simplex $\sigma$ of $K$ is a choice of bijection $b_{\sigma}$ between the two orientations of $\sigma$ and the two orientations of $^{6}$ $f^{-1}(\sigma)$. If $(\sigma, o_{\sigma})$ is an oriented facet of $(\tau, o_{\tau})$, a co-orientation $b_{\sigma}$ of $f$ over $\sigma$ is said to cohere with a co-orientation $b_{\tau}$ of $f$ over $\tau$ if $(f^{-1}(\sigma), b_{\sigma}(o_{\sigma}))$ is an oriented submanifold of the oriented boundary of $(f^{-1}(\tau), b_{\tau}(o_{\tau}))$. A co-orientation of $f$ over $K$ is a coherent choice of its co-orientations over all simplices of $K$. If $\mathcal{O}$ is an integral local coefficient system on $P$, an $\mathcal{O}$-co-orientation of $f$ over $K$ is choice of its co-orientations $b_{\sigma}$ over all simplices $\sigma$ of $K$ such that for every sequence of simplices $\sigma = \sigma_1, \ldots, \sigma_k = \sigma$ of $K$ where for each $i$, either $\sigma_i$ is a facet of $\sigma_{i+1}$ or vice versa, the number of incoherences between consecutive $b_{\sigma_i}$ is odd if and only if the corresponding loop in the 1-skeleton of the barycentric subdivision of $K$ acts nontrivially on $\mathbb{Z}$.

We define $H^k(P; \mathbb{Z}/2)$ (resp. $H^k(P; \mathcal{O})$), where $P$ is a polyhedron (and $\mathcal{O}$ a local coefficient system on $P$), to be the group of $\mathcal{O}$-co-oriented $k$-pseudo-comanifolds over some fixed triangulation of $P$. It is easy to see that this is equivalent to the usual definition (cf. [BRS]) so in particular does not depend on the choice of a triangulation of $P$. If $W \subset P$ is an embedded $k$-pseudo-comanifold, it is $\mathcal{O}$-co-orientable iff $H_k(P, P \setminus W; \mathcal{O})$ contains no torsion; in which case its $\mathcal{O}$-co-orientation is equivalent to a choice of a set of generators of $H^k(P, P \setminus W; \mathcal{O})$ representable by cocycles with disjoint supports. Note that if $f: W \to P$ is a pseudo-comanifold, where $P$ is finite-dimensional, there exists an embedded pseudo-comanifold $\tilde{f}: W \to P \times \mathbb{R}^N$ for some $N$ that projects onto $f$.

Transfers, products, Euler class. Let $\varphi: P \to Q$ be a PL map between polyhedra and let $\mathcal{O}$ be a local coefficient system on $Q$. Suppose first that $\varphi$ is triangulated by a simplicial map $K \to L$. If $f$ is a singular $i$-pseudo-manifold in $K$, then $\varphi f$ is a singular $i$-pseudo-manifold in $L$, so $[f] \mapsto [\varphi f]$ defines the induced homomorphism $\varphi_*: H_i(P; \varphi^*\mathcal{O}) \to H_i(Q; \mathcal{O})$. On the other hand, if $f: W \to Q$ is an $\mathcal{O}$-co-oriented $i$-pseudo-comanifold over $L$, the pullback $\varphi^*(f): V \to P$ is a $\varphi^*\mathcal{O}$-co-oriented $i$-co pseudo-manifold over $K$, cf. [BRS; bottom of p. 23]. This defines the induced

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$^6$Added in v5: each connected component of
homomorphism $\varphi^*: H^i(Q; \mathcal{O}) \to H^i(P; \varphi^*\mathcal{O})$.

$$
\begin{array}{ccc}
V & \xrightarrow{\varphi^*(f)} & P \\
\downarrow & & \varphi \\
W & \xrightarrow{f} & Q
\end{array}
$$

Now suppose that $\varphi$ is an $\mathcal{F}$-co-oriented $k$-pseudo-comanifold over a triangulation $L'$ of $Q$, where $\mathcal{F}$ is a local coefficient system on $Q$, and the $\varphi$-preimages of simplices of $L'$ are subcomplexes of a triangulation $K'$ of $P$. If $f$ is an $\varphi^*\mathcal{O}$-co-oriented $i$-pseudo-comanifold over $K'$, then $\varphi f$ is an $\mathcal{O}$-co-oriented $(i+k)$-pseudo-comanifold over $L'$, so $[f] \mapsto [\varphi f]$ defines the transfer $\varphi^*: H^i(P; \varphi^*\mathcal{O}) \to H^{i+k}(Q; \mathcal{O} \otimes \mathcal{F})$. On the other hand, if $f: M \to L'$ is an $\mathcal{O} \otimes \mathcal{F}$-co-oriented $i$-pseudo-comanifold in $L'$, then the pullback $\varphi^*(f): N \to K'$ will be a singular $\varphi^*\mathcal{O}$-oriented $(i-k)$-pseudo-manifold in $K'$, cf. [BRS; II.1.2]. This defines the transfer $\varphi^*: H_i(Q; \mathcal{O} \otimes \mathcal{F}) \to H_{i-k}(P; \varphi^*\mathcal{O})$.

Using the techniques of [BRS], it is easy to show that the latter transfer is well-defined; for the former transfer and the induced maps this has been done in [BRS]. Note that passing between $L$ and $L'$ generally requires making perturbations, by the virtue of transversality (see [BRS]).

For the cup- and cap-products, we have $\alpha \smile [\varphi] = \varphi_i \varphi^*(\alpha)$ [BRS; II.2.2]; and $\beta \smile [\varphi] = \varphi_+ \varphi^!(\beta)$ [BRS; p. 29]. The Euler class $e(i)$ of an arbitrary $\mathcal{O}$-co-oriented $k$-pseudo-comanifold $i: B \to E$ is defined to be $i^*i_!([\text{id}_B]) \in H^k(B; \mathcal{O})$, where the co-oriented 0-comanifold $\text{id}_B$ represents a generator of $H^0(B)$. If $i$ is the zero cross-section of a $k$-vector bundle $\xi: E \to B$, then $e(i) \in H^k(B; \mathcal{O}_\xi)$, where $\mathcal{O}_\xi$ is the orientation sheaf of $\xi$, and we call $e(i)$ the Euler class of $\xi$ and denote it by $e(\xi)$. When $\xi$ is orientable, this is its usual Euler class; in general, its mod2 reduction is the Stiefel–Whitney class $w_k$ (see [BRS; p. 26]). By definition, $e(\xi)$ can be represented by a pseudo-comanifold that is the zero set $B \cap s(B)$ of a generic cross-section $s(B)$ of $\xi$.

**Euler class of a line bundle.** Let $p: E \to B$ be the double covering and $\lambda_p$ the associated line bundle. The Euler class $e(\lambda_p) \in H^1(B; \mathbb{Z}_p)$ is always an element of order two, since it is induced from $e(\lambda_q) \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_q) \simeq \mathbb{Z}/2$ by a classifying map of $\lambda_p$, which also induces $p$ from the double covering $q: S^\infty \to \mathbb{RP}^\infty$. Nevertheless, $e(\lambda_p)$ should not be confused with its mod2 reduction, the first Stiefel–Whitney class $w_1(\lambda_p) \in H^1(B; \mathbb{Z}/2)$. The latter is less informative in general. For instance, $w_1(\xi)^2 = 0 \neq e(\xi)^2$ for the double covering $\xi: \mathbb{RP}^3 \to L(4,1)$.

**Remarks.** (i) If $\lambda$ is a line bundle over a compact polyhedron $P$ and $w_1(\lambda) = 0$, then $e(\lambda) = 0$. For otherwise $e(\lambda)$ would be divisible by 2 and so would be in the Bockstein image of $H^0(P; \mathbb{Z}_2/4) = 0$. This explains how reduction of coefficients mod2 “does not lead to a loss of information” in [M2; remark after Lemma 6].

(ii) If $\lambda$ is a line bundle over a $d$-manifold $M$ and $w_1^d(\lambda) = 0$, then $e^d(\lambda) = 0$. Indeed, $H^d(M; \mathbb{Z}_2^\otimes d)$ is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$ (one summand for each compact component of $M$) and so does not contain elements of order 4. This fills a minor gap in [M2; proof of Theorem 5].

**Lemma 2.1.** If $i: B \hookrightarrow E$ is an embedded $\mathcal{O}$-co-oriented $k$-pseudo-comanifold, then $i^*i_!(\alpha) = \alpha \smile e(i)$ and $i^!i_*^!(\beta) = \beta \smile e(i)$. 
The author is grateful to E. Kudryavtseva for calling his attention to the problem of computation of the “reverse transfer compositions” $i^* i_t$ and $i^* i_s$. When $i$ is an immersion, Lemma 2.1 can be combined with Herbert’s formula (see [EG]).

**Proof.** By considering a regular neighborhood $R$ of $i(B)$ in $E$ we may assume that $i^*$ is an isomorphism and so $\alpha = i^*(\gamma)$ for some $\gamma$. Then we have

$$i^* i_t(\alpha) = i^* i_t(i^* \gamma \sim [id_B]) = i^*(\gamma \sim i_t[id_B]) = i^* \gamma \sim i^* i_t[id_B] = \alpha \sim e(i)$$

using the formula $p_1(p^* \xi \sim \zeta) = \xi \sim p_1 \zeta$ from [BRS; II.2.6]. The two homological analogues of this formula read:

$$p_* (p^! \xi \sim \zeta) = \xi \sim p_1 \zeta \quad (\ast)$$

and $p_* (\zeta \sim p^* \xi) = p_* \zeta \sim \xi$, the latter being the familiar functoriality of $\sim$-product. On the other hand, the functoriality of $\sim$-product is proved (from the definition $\alpha \sim [\varphi] = \varphi_1 \varphi^*(\alpha)$) by a cubical diagram, which in the case of $\sim$-product yields

$$p^!(\xi \sim \zeta) = p^! \xi \sim p^* \zeta. \quad (\ast\ast)$$

Using formulas (\ast) and (\ast\ast), we obtain: if $\beta = i^! (\delta)$ for some $\delta$, then

$$i^! i^*_s (\beta) = i^! i^*_s (i^! \delta \sim [id_B]) = i^! (\delta \sim i^*_t[id_B]) = i^! \delta \sim i^* i_t[id_B] = \beta \sim e(i).$$

It remains to observe that $i^!$ may be assumed to be an isomorphism (specifically, the Thom isomorphism) by considering the pair $(R, FrR)$. \hfill \Box

**Equivariant cohomology and Smith sequences.** Let $P$ be a polyhedron endowed with a free PL action of $\mathbb{Z}/2 = \langle t \mid t^2 \rangle$, and let $p: P \to P/t$ be the double covering. Let us fix a module $M$ over the group ring $\Lambda := \mathbb{Z}[\mathbb{Z}/2]$. Let $C_\ast$ be the simplicial chain complex of some equivariant triangulation of $P$, where simplices from the same orbit are oriented coherently. Then $C_\ast$ is a free $\Lambda$-module.

We recall that equivariant homology and cohomology groups are defined by $H_{\ast}^{\mathbb{Z}/2}(P; M) = H_\ast(C_\ast \otimes_\Lambda M)$ and $H^{\mathbb{Z}/2}_\ast(P; M) = H^\ast(\text{Hom}_\Lambda(C_\ast; M))$ (compare [Bro]).\footnote{Added in v5: See e.g. Hatcher’s textbook for a more thorough discussion.} Note that $C_\ast \otimes_\Lambda \Lambda \simeq C_\ast$, whereas $\text{Hom}_\Lambda(C_\ast; \Lambda)$ is isomorphic to the simplicial cochain complex $C^* := \text{Hom}_\mathbb{Z}(C_\ast; \mathbb{Z})$. Hence $H_{\ast}^{\mathbb{Z}/2}(P; \Lambda) \simeq H_\ast(P; \mathbb{Z})$ and $H^{\mathbb{Z}/2}_\ast(P; \Lambda) \simeq H^\ast(P; \mathbb{Z})$.

Let us consider the trivial $\Lambda$-module $\mathbb{Z}$, which is isomorphic to $\Lambda/(t - 1)$, and the augmentation ideal $I$ of $\Lambda$, which is isomorphic to $\Lambda/(t + 1)$. By definition, $\text{Hom}_\Lambda(C_\ast; \mathbb{Z})$ (resp. $\text{Hom}_\Lambda(C_\ast; I)$) is the subcomplex of $C^*$ consisting of all cochains $f$ satisfying $f(c) = f(tc)$ (resp. $f(c) = -f(tc)$). It follows that $H_{\ast}^{\mathbb{Z}/2}(P; \mathbb{Z}) \simeq H^\ast(P/t; \mathbb{Z})$ and $H^{\mathbb{Z}/2}_\ast(P; I) \simeq H^\ast(P/t; \mathbb{Z}_p)$. Similarly, $C_\ast \otimes_\Lambda \mathbb{Z}$ (resp. $C_\ast \otimes_\Lambda I$) is the quotient complex of $C_\ast$ by $c = tc$ (resp. $c = -tc$). Hence $H_{\ast}^{\mathbb{Z}/2}(P; \mathbb{Z}) \simeq H_\ast(P/t; \mathbb{Z})$ and $H^{\mathbb{Z}/2}_\ast(P; I) \simeq H^\ast(P/t; \mathbb{Z}_p)$.\footnote{More generally, every module $M$ over $\Lambda$ gives rise to a local coefficient system $\mathcal{F}_M$ on $P/t$ whose stalks are isomorphic to the underlying abelian group of $M$, and the action of $g \in \pi_1(P/t)$ on the stalks is given by the action of $G_\ast(g) \in \pi_1(\mathbb{R}P^\infty) \simeq \mathbb{Z}/2$, where $G: P/t \to \mathbb{R}P^\infty$ is a classifying map of $p$. We have $\mathcal{F}_{M \otimes N} \simeq \mathcal{F}_M \otimes \mathcal{F}_N$, $\mathcal{F}_{M \otimes N} \simeq \mathcal{F}_M \otimes \mathcal{F}_N$ and $H_{\ast}^{\mathbb{Z}/2}(P; M) \simeq H_\ast(P/t; \mathcal{F}_M)$. $H^{\mathbb{Z}/2}_\ast(P; M) \simeq H^\ast(P/t; \mathcal{F}_M)$. In particular, we obtain the Vietoris–Begle isomorphism for double coverings: $H_\ast(P/t; \mathcal{F}_\Lambda) \simeq H_\ast(P/t; \mathbb{Z})$, $H^\ast(P/t; \mathcal{F}_\Lambda) \simeq H^\ast(P/t; \mathbb{Z})$.}

\footnote{Added in v5: Here $M \otimes N$ denotes tensor product over $\mathbb{Z}$ with the diagonal action of $\mathbb{Z}/2$.}
The coefficient sequences \(0 \rightarrow I \rightarrow \Lambda \rightarrow \mathbb{Z} \rightarrow 0\) and \(0 \rightarrow \mathbb{Z} \rightarrow \Lambda \rightarrow I \rightarrow 0\) give rise to long exact sequences, which can be written as follows using the above identifications of groups and straightforward identifications of homomorphisms:

\[
\begin{align*}
\cdots & \rightarrow H^n(P/t; \mathbb{Z}_p) \xrightarrow{p^*} H^n(P) \xrightarrow{p^*} H^n(P/t; \mathbb{Z}_p) \xrightarrow{i_*i^*} \cdots \\
\cdots & \rightarrow H^n(P/t) \xrightarrow{p^*} H^n(P) \xrightarrow{p^*} H^n(P/t; \mathbb{Z}_p) \xrightarrow{i_*i^*} \cdots \\
\cdots & \rightarrow H_n(P/t; \mathbb{Z}_p) \xrightarrow{p^*} H_n(P) \xrightarrow{p^*} H_n(P/t; \mathbb{Z}_p) \xrightarrow{i_*i^*} \cdots \\
\cdots & \rightarrow H_n(P/t) \xrightarrow{p^*} H_n(P) \xrightarrow{p^*} H_n(P/t; \mathbb{Z}_p) \xrightarrow{i_*i^*} \cdots
\end{align*}
\]

(here \(i\) denotes the inclusion of \(P/t\) onto the zero cross-section of the line bundle \(\lambda_p\) associated with the double covering \(p: P \rightarrow P/t\); constant integer coefficients are omitted). These are known as the Smith sequences. They can also be identified with the exact sequences of the relative mapping cylinder of the double covering \(p\). By Lemma 2.1 the connecting homomorphisms in cohomology and homology can be described respectively as \(\sim\)– and \(\sim\)-multiplication by the Euler class \(e(\lambda_p)\).

**Lemma 2.2.** Let \(M = H_n(P)\). Then \(H^1(\mathbb{Z}/2; M) = 0\) (resp. \(H_1(\mathbb{Z}/2; M) = 0\)) iff each element of \(H_n(P/t; \mathbb{Z}_p)\) (resp. \(H_n(P/t)\)) is either divisible or annihilated by \(e(\lambda_p)\), and each element of \(H_n(P/t)\) (resp. \(H_n(P/t; \mathbb{Z}_p)\)) is either not divisible or not annihilated by \(e(\lambda_p)\). In addition, “\(H_n\)” may be replaced by “\(H^n\)” throughout.

This follows trivially from the definitions, which for convenience of the reader we do recall (compare [Bro]).

**Proof.** We recall that \(H_*(\mathbb{Z}/2; M) = \text{Tor}_*^\Lambda(\mathbb{Z}, M)\) and \(H^*(\mathbb{Z}/2; M) = \text{Ext}_\Lambda^*(\mathbb{Z}, M)\) are the derived functors of \(H_0(\mathbb{Z}/2; M) = \mathbb{Z} \otimes_\Lambda M\) and \(H^0(\mathbb{Z}/2; M) = \text{Hom}_\Lambda(\mathbb{Z}, M)\) viewed as functors of \(\mathbb{Z}\). Thus we start with a projective resolution of \(\mathbb{Z}\) over \(\Lambda = \mathbb{Z}[\mathbb{Z}/2]\), namely let us take the chain complex of the simplest \(\mathbb{Z}/2\)-equivariant cell complex structure on \(S^\infty\):

\[
\mathbb{Z} \xrightarrow{i} \Lambda \xrightarrow{1-t} \Lambda \xrightarrow{1+t} \Lambda \xrightarrow{1-t} \Lambda \xrightarrow{1+t} \cdots
\]

Now we drop the initial \(\mathbb{Z}\), apply the functor, and take homology; thus \(H_*(\mathbb{Z}/2; M)\) is the homology of the chain complex

\[
M \xrightarrow{1-t} M \xrightarrow{1+t} \cdots
\]

and \(H^*(\mathbb{Z}/2; M)\) is the homology of the cochain complex

\[
M \xrightarrow{1-t} M \xrightarrow{1+t} \cdots
\]

So \(H_1(\mathbb{Z}/2; M) = \ker(1-t)/\text{im}(1+t)\) and \(H^1(\mathbb{Z}/2; M) = \ker(1+t)/\text{im}(1-t)\).

Now let \(M = H_n(P)\), and consider \(H^1(\mathbb{Z}/2; M)\); the other cases are similar. We have \(1+t = p^!p_*(x)\) with coefficients in \(\mathbb{Z}\), and \(1-t = p^!p_*\) with coefficients in \(\mathbb{Z}_p\). Then \(x \in \ker(1+t) \setminus \text{im}(1-t)\) iff either \(y := p_*(x) \in H_n(P/t)\) is nonzero but lies in \(\ker p^!\), or \(x = p^!(z)\) for some \(z \in H_n(P/t; \mathbb{Z}_p)\) that does not lie in \(\text{im} p_*\). Hence \(\ker(1+t) \neq \text{im}(1-t)\) iff either some \(y \in H_n(P/t)\) is divisible and annihilated by \(e(\lambda_p)\) or some \(z \in H_n(P/t; \mathbb{Z}_p)\) is not divisible and not annihilated by \(e(\lambda_p)\). \(\square\)
Remark 2.3 (added in v5). The generalized Smith sequence, found in [Bre; §2.19 and §5.20], can be quickly obtained as follows. Note that it is a special case of the generalized Smith–Gysin sequence, found in [Bre; §4.12.3] and used e.g. in [M1*].

Let $M$ be a $\Lambda$-module and $M_0$ its underlying abelian group, and let $M'$ denote $M \otimes \mathbb{Z} I$ with the diagonal action of $\mathbb{Z}/2$. By a well-known lemma [Bro; III.5.7], $M \otimes \Lambda$ with the diagonal action of $\mathbb{Z}/2$ is isomorphic to the induced module $M_0 \otimes \Lambda$ via $m \otimes g \mapsto g^{-1}m \otimes g$. Then the same argument used above to construct the usual Smith sequences shows that the short exact sequence $0 \to M \to M \otimes \Lambda \to M' \to 0$ of $\Lambda$-modules gives rise to long exact sequences

$$
\cdots \to H^n(P; M_0) \to H^n(P/t; \mathcal{F}_{M'}) \to H^{n+1}(P/t; \mathcal{F}_M) \to \cdots
$$

$$
\cdots \to H_n(P; M_0) \to H_n(P/t; \mathcal{F}_{M'}) \to H_{n-1}(P/t; \mathcal{F}_M) \to \cdots
$$

These may be convenient, but do not really contain additional information: for $M = \mathbb{Z}$ and $M = I$ we get the usual Smith sequences constructed earlier in this section, and for $M = \Lambda$ the generalized Smith sequences split into short exact sequences, $0 \to H \to H \otimes \Lambda \to H \otimes I \to 0$, where $H = H^n(P)$ or $H_n(P)$. On the other hand, every $\Lambda$-module is a direct sum of $G_1$, $G_2 \otimes I$ and $G_3 \otimes \Lambda$ for some abelian groups $G_1$, $G_2$ and $G_3$ (see [CR; Theorem 74.3]).

3. The van Kampen obstruction

If $X$ is a polyhedron, the factor exchanging involution $(x, y) \mapsto (y, x)$ on $X \times X$ restricts to a free action of $\mathbb{Z}/2$ on the deleted product $X \times X \setminus \Delta$. We write $\tilde{X}$ for the $\mathbb{Z}/2$-space $(X \times X \setminus \Delta, \mathbb{Z}/2)$, and $\bar{X}$ for the quotient $\tilde{X}/(\mathbb{Z}/2)$. The $m$-sphere $S^m \subset \mathbb{R}^{m+1}$ endowed with the antipodal involution $x \mapsto -x$ will be denoted $S_m^-$. If $g: X \hookrightarrow \mathbb{R}^m$ is an embedding, its Gauss map $\tilde{g}: \tilde{X} \to S^{m-1}_-$ is defined by $(x, y) \mapsto \frac{g(x)-g(y)}{|g(x)-g(y)|}$. In other words, $\tilde{g}$ is the restriction $\tilde{X} \to \mathbb{R}^-_m$ of $g \times g$ followed by the obvious equivariant homotopy equivalences $\mathbb{R}^-_m \simeq \mathbb{R}^-_m \setminus \{0\} \simeq S^{m-1}_-$. The Gauss map descends to $\bar{g}: \bar{X} \to \mathbb{R}^{m-1}$, which assigns to an unordered pair $\{x, y\}$ of distinct points of $X$ the 1-subspace $\langle g(x) - g(y) \rangle$ of $\mathbb{R}^m$.

Sectional category. It is easy to see that the existence an equivariant map from a $\mathbb{Z}/2$-space $K$ into $S^{m-1}_-$ is equivalent to the existence of a cross-section of the sphere bundle $p_m: K \times_{\mathbb{Z}/2} S^{m-1}_- \to K/(\mathbb{Z}/2)$. This bundle is the Whitney join of $m$ copies of the double covering $p_1: K \to K/(\mathbb{Z}/2)$, and it follows that $p_m$ has a cross-section if and only if there exists a cover of $P$ by $m$ open sets $U_i$ such that $p_1$ has a cross-section over each $U_i$ (see [CF], [Sch], [Ja] or [CL+] for details). The largest integer $m$ that does not satisfy any of these three equivalent properties is called the sectional category $\text{secat} p_1$ (cf. [CL+]).

The largest integer $m$ such that a polyhedron $X$ does not embed in $\mathbb{R}^m$ is therefore bounded below by the sectional category of the double covering $\tilde{X} \to \bar{X}$. In the metastable range the two numbers are equal:

\[10\text{In older literature, “sectional category” used to mean } 1 + \text{secat } p_1, \text{ which is the least integer satisfying any of the three equivalent properties (cf. [Ja]). Originally, } 1 + \text{secat } p_1 \text{ was known as the Krasnosel’sky–Svarc genus of } p_1 \text{ [Sch] and secat } p_1 \text{ itself as Yang’s B-index or Conner–Floyd co-index of the } \mathbb{Z}/2\text{-space } K \text{ [CF]. Furthermore, secat } p_1 \text{ equals the category of a classifying map } K \to \mathbb{R}P^\infty \text{ of } p_1 \text{ (see [CL+]).}\]
Haefliger–Weber Criterion 3.1 [We] (see also [Ad]). Let $X$ be an $n$-polyhedron.

(a) Given a $\mathbb{Z}/2$-map $\Phi: \tilde{X} \to S^{n-1}$, if $m \geq \frac{3(n+1)}{2}$, there exists an embedding $g: X \hookrightarrow \mathbb{R}^m$; moreover, $\tilde{g}$ can be chosen $\mathbb{Z}/2$-homotopic to $\Phi$.

(b) If $m > \frac{3(n+1)}{2}$, two embeddings $f, g: X \hookrightarrow \mathbb{R}^m$ are ambient isotopic if and only if $\tilde{f}$ and $\tilde{g}$ are $\mathbb{Z}/2$-homotopic.

Proof of the case $m = 2n$ in (a). Fix some triangulation of $X$ and embed $X^{(n-1)}$ by general position. Then $\tilde{g}$, which is now defined on $X^{(n-1)}$, is equivariantly homotopic to the restriction of $\Phi$ since $S^{2n-1}$ is $(2n-2)$-connected. Let $\sigma_1, \sigma_2, \ldots$ be the $n$-simplices of $X$. Extend the embedding to $\sigma_1$ by general position. We want $\tilde{g}$, which is now also defined on $(\sigma_1 \times X^{n-1} \cup X^{n-1} \times \sigma_1) \setminus \Delta$, to be equivariantly homotopic to the restriction of $\Phi$. (Not just to prove the moreover part, but also to prepare for the following steps.) This is easy to achieve: if the two maps differ on $\sigma_1 \times \tau$ for some $(n-1)$-simplex $\tau$ by a degree $l$ map, pick a small $S^n$ winding around $\tau$ with linking number $l$, and amend $\sigma_1$ by taking a connected sum with this $S^n$ along a small tube.

Suppose that $Y := X^{(n-1)} \cup \sigma_1 \cup \cdots \cup \sigma_{n-1}$ is embedded via $g$, and that $\tilde{g}$ where defined is equivariantly homotopic to the restriction of $\Phi$. Let $Z$ be $Y$ minus the interior of the simplicial neighborhood of $\sigma_i$. Then the restriction of $\tilde{g}$ to $Z \times \partial \sigma_i$ extends to $Z \times \sigma_i$, and so is null-homotopic. On the other hand, this map is homotopic to the composition $Z \times \partial \sigma_i \xrightarrow{\psi \times \text{id}_{\sigma_i}} S^n \times \partial \sigma_i \xrightarrow{\chi|S^n \times \partial \sigma_i} S^{2n-1}$, where $\psi$ is the composition of the inclusion $Z \subset \mathbb{R}^{2n} \setminus \partial \sigma_i \subset S^{2n} \setminus \partial \sigma_i$ and a deformation retraction of $S^{2n} \setminus \partial \sigma_i$ onto a small $S^n$ winding around $\partial \sigma_i$ with linking number 1; and $\chi$ is the inclusion $S^n \sqcup \partial \sigma_i \hookrightarrow \mathbb{R}^{2n}$. Since $\chi|S^n \times \partial \sigma_i$ has degree one, $\psi \times \text{id}_{\sigma_i}$ must be trivial in $2n$-cohomology. Then $\psi: Z \to S^n$ is trivial in $n$-cohomology, so by the Hopf classification theorem, $\psi$ is null-homotopic. Hence the inclusion of $Z$ in $S^{2n} \setminus \partial \sigma_i$ is null homotopic.

If $H$ is a homeomorphism of $\mathbb{R}^{2n}$ fixing $\partial \sigma_i$ and sending $Z^{(n-1)}$ into a small ball $B^{2n}$, by dimensional reasons there is a null-homotopy $h_t$ of $\tilde{H}(Z)$ in $\mathbb{R}^{2n} \setminus \partial \sigma_i$ such that $Z^{(n-1)}$ stays in $B^{2n}$. If $D^n$ is a disk bounded by $\partial \sigma_i$ and disjoint from $B^{2n}$, the null-homotopy $H^{-1}h_t$ of $Z$ is such that $Z^{(n-1)}$ stays disjoint from $g(\sigma_i) := H^{-1}(D^n)$. Hence $g(\sigma_i)$ has zero intersection number with each $n$-simplex of $Z$.

Then $g(\sigma_i)$ can be made disjoint from each $n$-simplex $\tau$ of $Z$ by the Whitney trick. Specifically, every pair of double points between $\sigma_i$ and $\tau$ with opposite signs can be eliminated by the price of adding a 1-handle to $\sigma_i$, that is, cutting out $B^n \times S^0$ and gluing in $S^{n-1} \times B^1$. Since $n > 2$, every circle in the modified $\sigma_i$ bounds a 2-disc in $\mathbb{R}^{2n} \setminus Z$, hence all the 1-handles can be cancelled without introducing new self-intersections.

Finally, $\sigma_i$ can be made disjoint from every adjacent $n$-simplex $\tau$ of $Y$ by the Penrose–Whitehead–Zeeman trick. Specifically, if $g$ sends $p \in \sigma_i$ and $q \in \tau$ to the same point in $\mathbb{R}^{2n}$, let $J \subset \sigma_i \cup \tau$ be an arc with endpoints $p, q$ and with $J \cap \sigma_i \cap \tau$ consisting of precisely one point $v$. Then a small regular neighborhood of $g(J)$ in $\mathbb{R}^{2n}$ is a ball $B^{2n}$, and its preimage in $\sigma_i \cup Y$, which is a regular neighborhood of $J$ in $\sigma_i \cup Y$, is a cone $v \ast P$. Redefining $g$ on $v \ast P$ by mapping it conically into $B^{2n} \cong v \ast \partial B^{2n}$ eliminates the double point $g(p) = g(q)$. This yields an extension of $g$.

---

\[^{11}\text{Added in v5: Since } n > 2, \text{ the embedded 1-sphere } g(J) \text{ bounds an embedded 2-disk } D \text{ that meets } g(Y \cup \sigma_i) \text{ only } \partial D. \text{ Then a small regular neighborhood of } D \text{ in } \mathbb{R}^m \text{ is a ball } B^{2n}.\]
Van Kampen obstruction \cite{vK}, \cite{Sh}, \cite{Wu}. Let $G \colon X \to \mathbb{R}P^\infty$ be any map classifying the line bundle $\lambda$ associated to the double covering $\bar{X} \to \tilde{X}$. The van Kampen obstruction $\vartheta(X) \in H^{2n}(\tilde{X}; \mathbb{Z})$ is defined to be $G^*(\xi)$, where $\xi$ is the generator of $H^{2n}(\mathbb{R}P^\infty; \mathbb{Z}) \simeq \mathbb{Z}/2$. Since $G$ is unique up to homotopy, $\vartheta(X)$ is well-defined. Given an embedding $g \colon X \hookrightarrow \mathbb{R}^n$, we have $\vartheta(X) = \tilde{g}^*(\xi) = 0$. So $\vartheta(X)$ obstructs embeddability of $X$ into $\mathbb{R}^n$.

A useful interpretation of $\vartheta(X)$, due to Švarc \cite{Sch}, can be given in terms of

Cohomological sectional category. Recall from §2 that if $t$ is a free PL involution on a polyhedron $P$, the Euler class $e(\lambda_p)$ of the line bundle $\lambda_p$ associated with the double covering $p \colon P \to P/t$ is an element of $H^1(P/t; \mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the orientation sheaf of $p$. Note that $e(\lambda_p)$ may be defined as the image of the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_q) \simeq \mathbb{Z}/2$ under a classifying map of $\lambda_p$, where $q \colon S^\infty \to \mathbb{R}P^\infty$ is the double covering. Indeed, this generator is nothing but $e(\gamma)$, where $\gamma = \lambda_q$ is the tautological line bundle over $\mathbb{R}P^\infty$.

We denote the maximal $k$ such that $e(\lambda_p)^k \neq 0$ by $\text{secat}_\mathbb{Z}(P,t)$. (Note that $e(\lambda_p)^k \in H^k(P/t; \mathbb{Z}_p^\otimes k)$, where $\mathbb{Z}_p^\otimes k$ is isomorphic to $\mathbb{Z}_p$ when $k$ is odd and to $\mathbb{Z}$ when $k$ is even, cf. §2.) This number is also known as the Conner–Floyd cohomological co-index of $(P,t)$ over $\mathbb{Z}$ \cite{CF} or James’ “Euler index” of $p$ \cite{Ja}.

Now we are ready for Švarc’s intrinsic description of the van Kampen obstruction. Since $\xi = e(\gamma)^{2n}$, we have $\vartheta(X) = e(\lambda)^{2n} = e(2n\lambda)$, the Euler class of the orientable vector bundle $(2n)\lambda \colon \bar{X} \times_{\mathbb{Z}/2} \mathbb{R}^{2n-2} \to \bar{X}$. Thus $\vartheta(X) \neq 0$ if and only if $\text{secat}_\mathbb{Z}(\bar{X}) = 2n$. By similar arguments, if $X$ embeds in $\mathbb{R}^m$, then $\text{secat}_\mathbb{Z}(\tilde{X}) < m$.

**Theorem 3.2 (Shapiro \cite{Sh}, Wu \cite{Wu}).** Let $X$ be an $n$-polyhedron. If $\vartheta(X) = 0$ and $n > 2$, then $X$ embeds in $\mathbb{R}^{2n}$.

**Proof.** Suppose that $G^*(\xi) = 0$ for some map $\tilde{X} \to \mathbb{R}P^\infty$ classifying $\lambda$. We may assume that $G$ sends the $(2n-1)$-skeleton of $\tilde{X}$ into $\mathbb{R}P^{2n-1}$. Since $\pi_1(\mathbb{R}P^{2n-1})$ acts trivially on $\pi_{2n-1}(\mathbb{R}P^{2n-1})$, by definition $G^*(\xi)$ is the primary obstruction to the existence of a map $F \colon \tilde{X} \to \mathbb{R}P^{2n-1}$, coinciding with $G$ on $\bar{X}$. By the non-homotopically-simple obstruction theory (as described e.g. in the Hilton–Wylie textbook), such an $F$ exists; if $n > 1$, it still classifies $\lambda$. By the covering theory, $F$ lifts to an equivariant map $\tilde{X} \to S^{2n-1}_\mathbb{Z}$. Since $n > 2$, by the Haefliger–Weber Criterion 3.1(a), $X$ embeds into $\mathbb{R}^{2n}$. \(\square\)

**Alternative proof.** The existence of an equivariant map $\tilde{X} \to S^{2n-1}_\mathbb{Z}$ is obviously equivalent to the existence of a cross-section of the bundle $\tilde{X} \times_{\mathbb{Z}/2} S^{2n-1} \to \tilde{X}$ (with fiber $S^{2n-1}$). The primary obstruction to the latter is the Euler class of this bundle, which is well-known to be complete in this situation. But it is the same as the Euler class $e(2n\lambda)$, which as we have seen above coincides with $\vartheta(X)$. \(\square\)

When $n = 2$, $\vartheta(X)$ is incomplete \cite{FKT}. When $n = 1$, $\vartheta(X)$ is complete, due to the classical Kuratowski–Pontryagin Theorem: either $X$ embeds in $\mathbb{R}^2$ or it contains a copy of either $K_5$ (the 1-skeleton of the 4-simplex) or $K_{3,3}$ (the join of the 0-skeleta of 2 copies of the 2-simplex). Indeed, $\vartheta(K_5)$ and $\vartheta(K_{3,3})$ are non-zero by the following

\[\text{Added in v5: As I learned from S. Parsa, the same number is also known as the Smith index.}\]
Example 3.3. If $J$ is an $n$-dimensional join of $n_i$-skeleta of $(2n_t + 2)$-simplices, then $\vartheta(J) \neq 0$. Indeed, suppose that $\vartheta(J) = 0$. Then by the proof of Theorem 3.2, $\tilde{J}$ admits an equivariant map to $S^{2n-1}_2$. Hence so does the simplicial deleted product $\tilde{J}_s$, that is, the subpolyhedron of $\tilde{J}$ consisting of the products of all pairs of disjoint simplices of $J$.

Let us consider the simplicial deleted join $\tilde{J}_s$, that is, the subpolyhedron of the join $J \ast J$ consisting of the joins of all pairs of disjoint simplices of $J$. The suspension $\Sigma \tilde{J}_s$ is $\mathbb{Z}/2$-homeomorphic to $\tilde{J}_s/(J \ast \emptyset \cup \emptyset \ast J)$. Hence $\tilde{J}_s$ admits an equivariant map to $\Sigma \tilde{J}_s$, and therefore to $\Sigma S^{2n-1}_2 = S^{2n}_2$. But this cannot be by the Borsuk–Ulam Theorem, since $\tilde{J}_s$ is well-known to be equivariantly homeomorphic to $S^{2n+1}_2$ [F1], [F2], [Gr] (see also [Ro] and [dL]+[Mat]).

Remark. In what follows, we will freely use the well-known fact that for any polyhedron $X$ with a fixed triangulation, $\tilde{X}$ equivariantly deformation retracts onto $\tilde{X}_s$ (see [Hu] or [S1; 5.3.c]; beware that the proof given in [Sh] is incorrect).

1-Parameter van Kampen obstruction. Let $f, g : X \to \mathbb{R}^{2n+1}$ be embeddings. We define $\vartheta(f, g)$ to be the primary obstruction to the existence of a homotopy between $\tilde{f}, \tilde{g} : \tilde{X} \to \mathbb{R}^{2n}$. In more detail, let us pick a homotopy $h : \tilde{X} \times I \to \mathbb{R}^\infty$ between their compositions with the inclusion $\mathbb{R}P^{2n} \subset \mathbb{R}^\infty$. Let $\Psi$ be a generator of $H^{2n+1}(\mathbb{R}^\infty, \mathbb{R}^{2n}; Z_2) \simeq \mathbb{Z}$. Then $\vartheta(f, g)$ is defined to be the image of $h^*(\Psi)$ under the Thom isomorphism $H^{2n+1}(\tilde{X} \times I, \tilde{X} \times \partial I; Z_p) \simeq H^{2n}(\tilde{X}; Z_p)$, where $p : \tilde{X} \to \tilde{X}$ is the double covering.

Clearly, $\vartheta(f, g)$ is independent of the choice of $h$, and vanishes if $f$ and $g$ are isotopic. It follows from Criterion 3.1(b) that the converse holds when $n > 1$; this was first proved in [Wu]. When $n = 1$, the converse holds if $X \times I$ is allowed to be amended by taking connected sum with tori (attached to open 2-disks in $X \times I$) [Ta1]. Furthermore, when $n = 1$, it is shown in [ST] (see also [Ta1; 5.2]) that $\vartheta(f, g) = 0$ iff $\vartheta(f|_K, g|_K) = 0$ for each subgraph $\tilde{K}$ of $X$ homeomorphic to $K_5$, $K_{3,3}$ or $S^1 \sqcup S^1$.

The group $H^{2n}(\tilde{X}; Z_p)$ also contains $\tilde{f}^*(\Xi)$ and $\tilde{g}^*(\Xi)$, where $\Xi$ is a generator of $H^{2n}(\mathbb{R}P^{2n}; Z_2) \simeq \mathbb{Z}$. Up to a sign, we may assume that $\delta^*(\Xi) = 2\psi$. It then follows that $2\vartheta(f, g) = \tilde{g}^*(\Xi) - \tilde{f}^*(\Xi)$. When $n = 1$, there is no 2-torsion in $H^{2n}(\tilde{X}; Z_p)$ [Ta1], so in this case $\vartheta(f, g)$ reduces to $\tilde{g}^*(\Xi) - \tilde{f}^*(\Xi)$.

Yang index & unoriented van Kampen obstruction. If $t$ is a free involution on a polyhedron $P$ and $p : P \to P/t$ is the double covering, the first Stiefel–Whitney class $w_1(\lambda_p) \in H^1(\tilde{X}; Z_2)$ is generally less informative than $\epsilon(\lambda_p)$ (see §2).

The Yang index $\mathrm{secat}_{Z/2}(P, t)$ is the maximal $k$ such that $w_1(\lambda_p)^k \neq 0$ (see [CF]). Thus $\mathrm{secat}_{Z/2}(X) = 2n$ iff $\vartheta(X)$ has trivial mod 2 reduction. If $M$ is a closed manifold, $\mathrm{secat}_{Z/2}(M)$ equals the minimal $d$ such that the normal Stiefel–Whitney classes $\bar{w}_i(M) = 0$ for $i \geq d$ [Wu], [Sch; Ch. VII, §2] (see also [Wu; paper II]; for a geometric proof of the inequality $\mathrm{secat}_{Z/2}(M) \geq d$ see [Mc4]).

We note that if $\tilde{X}$ has Yang index $< 2n - 1$ then $\vartheta(\tilde{X}) = 0$. Indeed, since the generator $\xi \in H^{2n}(\mathbb{R}P^{2n}; Z)$ is of order two, it is the Bockstein image of the

\[\text{13}^\text{The construction of this homeomorphism deserves a thorough analysis, which in turn has many other interesting consequences (compare remark at the end of §4); this will be the subject of a forthcoming paper by the author. (Added in v5: arXiv:1103.5457.)}\]
generator $w_1(\gamma)^{2n-1} \in H^{2n-1}(\mathbb{R}P^{2n}; \mathbb{Z}/2)$. Hence $\vartheta(X)$ is the Bockstein image of $w_1(\lambda_p)^{2n-1}$. (This also yields an alternative intrinsic definition of $\vartheta(X)$.)

**Example 3.4.** Krushkal [Kr] relates the intersection pairing of a 4-thickening of a 2-polyhedron $X$ with the image of $\vartheta(X)$ in $H^4(\bar{X}; \mathbb{Q})$ under the homomorphism induced by the 2-covering $\bar{X} \to \bar{X}$. This image is identically trivial since $\vartheta(X)$ is of order two. In fact, since $\vartheta(X)$ is divisible by $c(\lambda)$, the image of $\vartheta(X)$ in the integral cohomology $H^4(\bar{X}; \mathbb{Z})$ is zero as well by the Smith sequence.

**Geometric definition of $\vartheta(X)$**. Pick an embedding $g$: $X \hookrightarrow \mathbb{R}^{2n+1}$. Consider the equivariant map $\tilde{g}: \bar{X} \to S^{2n}$, assigning to a pair $(x, y)$ of distinct points of $X$ the unit vector in the direction from $g(x)$ to $g(y)$. Then $\vartheta(X)$ can be identified with $\tilde{g}_{eq}(\xi_{eq}) \in H_{\mathbb{Z}/2}^{2n}(\bar{X}; \mathbb{Z})$, induced from the generator $\xi_{eq} \in H_{\mathbb{Z}/2}^{2n}(S^{2n}; \mathbb{Z}) \cong \mathbb{Z}/2$.

If $g$ is generic, it projects onto an immersion $f$: $X \hookrightarrow \mathbb{R}^{2n}$. We may assume that $f$ maps the vertices of some triangulation of $X$ in general position and extends linearly to each simplex. Consider the cochain $\vartheta_f$ on $\bar{X}$ assigning to a product $\sigma \times \tau$ of distinct oriented $n$-simplices of the triangulation of $X$ the algebraic number of their intersections in $\mathbb{R}^{2n}$ (i.e. 0 if they are mapped disjointly and 1 or $-1$ otherwise, according as the orientation of $\sigma \times \tau$ matches or not that of $\mathbb{R}^{2n}$). Then $\vartheta_f(\sigma \times \tau) = (-1)^n \vartheta_f(\sigma \times \tau)$, but also $t(\sigma \times \tau) = (-1)^n \vartheta_f(\sigma \times \tau)$ (as chains). Thus $\vartheta_f$ lies in the subcomplex $\text{Hom}_A(C_*(\bar{X}_s), \mathbb{Z})$ of $C_*(\bar{X}_s)$, and it is immediate that $\vartheta_f = \tilde{g}_{eq}^*(\xi_{eq})$, where $\xi_{eq}$ is a cocyclic representative of $\xi_{eq}$ with support in two antipodal $2n$-simplices of $S^{2n} \setminus S^{2n-1}$. In conclusion, $\vartheta(X) = [\vartheta_f]$.

Using the language of §2, the idea of construction of $\vartheta_f$ can be formulated in a more elegant way. Let $\Delta_f/t$ be the set of unordered pairs $\{x, y\}$ of points of $X$ such that $f(x) = f(y)$ and $x \neq y$. We have $\Delta_f/t = \tilde{g}_1^{-1}(pt)$, where $pt \in P^{2n}$ is the vertical direction of $\mathbb{R}^{2n+1}$. If $f$ is generic, $\Delta_f/t$ is an embedded $2n$-pseudo-comanifold in $\bar{X}$, which is co-oriented once $pt \in \mathbb{R}^{2n}$ is. Thus $[\Delta_f/t] = \tilde{g}_1^*([pt]) = \vartheta(X) \in H^{2n}(\bar{X})$.

**Example 3.5.** The $n$-skeleton $K$ of the $(2n + 2)$-simplex $\Delta^n \star \Delta^{n+1}$ is, apart from the simplex $\Delta^n$, already contained in the $2n$-sphere $\partial\Delta^n \star \partial\Delta^{n+1}$. This simplex can be mapped conewise onto $c \star \partial\Delta^n$, where $c$ is a point in the interior $U$ of some $n$-simplex in $\partial\Delta^{n+1}$. This yields a map $K \to \mathbb{R}^{2n}$ with one generic double point between non-adjacent simplices.

More generally, an $n$-dimensional join $J$ of $n_i$-skeleta of $(2n_i + 2)$-simplices $\Delta^{n_i+1} \star \Delta^{n_i}$ is, apart from the join of the simplices $\Delta^{n_i}$, already contained in the $2n$-sphere $(c \star \partial\Delta^{n+1}) \star \partial(c \star \Delta^{n+1})$ (since $\partial(A \star B)$ contains $A \star \partial B$). The remaining simplex can be mapped conewise onto $c \star \partial(c \star \Delta^{n+1})$, where $c$ is a point in the interior of some $n$-simplex in $(c \star \partial\Delta^{n+1})$. This again yields a map $J \to \mathbb{R}^{2n}$ with one generic double point between non-adjacent simplices.

**Example 3.6.** The mod2 reduction of $\vartheta(G)$ is a complete obstruction to embeddability of the graph $G$ into $\mathbb{R}^2$, since it is nonzero for $G = K_5$ and $K_{3,3}$ by Example 3.3. We will show that this is not the case in every other dimension $n > 1$.

In the notation of the preceding example, cut out a small $n$-ball from $U \setminus c$, and reattach this cell $B^n$ back to its former boundary in $K$ by some degree two map.

\footnote{This point was apparently missed in [FKT], leading to an incorrect identification of the group $H^{2n}(\bar{X}; \mathbb{Z})$ that contains $\vartheta(X)$ as $H^{2n}(\bar{X}; \mathbb{Z}_p)$ for odd $n$.}
Let \( X \) be an \( n \)-polyhedron with a fixed triangulation and such that \( \vartheta(X) \neq 0 \). Suppose that for some \( n \)-simplex \( \sigma \) of \( X \) we are given an embedding \( g: X \setminus \text{Int}\sigma \to \mathbb{R}^{2n} \), and let \( Z \) denote \( X \) minus the interior of the simplicial neighborhood of \( \sigma \). Then \( \text{lk} (g(\partial\sigma), g(Z)) \neq 0 \in H^n(Z) \).

This linking number is defined to be the image of a fixed generator under

\[
H^{2n-1}(S^{2n-1}) \xrightarrow{\tilde{g}^*} H^{2n-1}(\partial\sigma \times Z) \cong H^n(Z).
\]

If \( X = K \) and \( \sigma = \Delta^n \), then \( Z \) is the \( n \)-sphere \( \partial\Delta^{n+1} \) and we clearly get the usual linking number.

**Proof.** Suppose that the linking number is zero, and write \( Y = X \setminus \text{Int}\sigma \). Then the restriction of \( \tilde{g}: \tilde{Y} \to S^{2n-1} \) to \( \partial\sigma \times Z \) is null-homotopic by the Hopf classification theorem. Therefore \( \tilde{g} \) equivariantly extends to \( \tilde{Y} \cup \sigma \times Z \cup Z \times \sigma \), which contains the simplicial deleted product \( \tilde{X}_s \). Hence \( \vartheta(X) = 0 \), a contradiction. \( \square \)

**Remark.** The following explicit computation of \( \vartheta(X) \) has been proposed recently [MTW], based on the well-known fact that every set of at most \( 2n+1 \) distinct points on the “moment curve” \( \gamma(\mathbb{R}) \subset \mathbb{R}^{2n} \), \( \gamma(t) = (t, t^2, \ldots, t^{2n}) \), is linearly independent. Fixing some injection \( \varphi: X(0) \to \mathbb{R} \) of the 0-skeleton of some triangulation of \( X \) and extending \( X(0) \xrightarrow{\varphi} \mathbb{R} \to \mathbb{R}^{2n} \) linearly to the simplices of \( X \), one thus obtains a specific generic map \( f: X \to \mathbb{R}^{2n} \). It follows (see [MTW]) that \( \vartheta(X) = [\vartheta_f] \) is the class of the cochain \( c \in C_{Z/2}^n(\tilde{X}_s) \) defined by \( c(\sigma \times \tau) = 1 \) or \((-1)^n\) if the triple \((\mathbb{R}, \varphi(\sigma(0)), \varphi(\tau(0)))\) is homeomorphic to \((\mathbb{R}, 2[n], 2[n]+1)\) or \((\mathbb{R}, 2[n]+1, 2[n])\) respectively, where \([n] = \{0, 1, \ldots, n-1\} \), and by \( c(\sigma \times \tau) = 0 \) in all other cases. As remarked in [MTW], the same result can also be obtained directly from Švarc’s interpretation of \( \vartheta(X) \).

Of course, describing an explicit cocycle representing \( \vartheta(X) \) still leaves the question of deciding whether its class is trivial in \( H^{2n}(\tilde{X}) \).

**Theorem 3.8.** Let \( X \) be an \( n \)-polyhedron with a fixed triangulation and a choice of orientation of simplices. Let \( \Gamma \) be the graph with vertices corresponding to the \( n \)-simplices of \( X \) and edges to non-adjacent pairs of \( n \)-simplices. Then

\[
H^{2n}(\tilde{X}) \cong \text{coker} \left[ C_1(\Gamma; \mathbb{Z}) \xrightarrow{h} \bigoplus_{\sigma \in \Gamma(0)} H^n(X \setminus N\sigma) \right],
\]

where \( N \) stands for open simplicial neighborhood and \( h \) sends a non-adjacent pair \((\sigma^n, \tau^n)\) to the difference between \([\sigma] \in H^n(X \setminus N\tau)\) and \((-1)^n[\tau] \in H^n(X \setminus N\sigma)\).

It is convenient to think of each group \( H^n(X \setminus N\sigma) \) as being put into the vertex of \( \Gamma \) corresponding to \( \sigma \), so that we may regard \( h \) as the “boundary homomorphism”. (This can be made precise: \( \text{coker} \ h \) is the 0-homology of \( \Gamma \) with the pre-cosheaf coefficients.)
Example 3.9. If $X = K_5$, then $\Gamma$ is the Petersen graph $P$. If we orient the edges of $K_5$ according to a representation of $K_5$ as the union of two cycles of length 5, then 5 of the cycles of length 3 will be oriented coherently and the other 5 not. Let us fix the generator of each $H^1(\text{cycle of length 3})$ corresponding to the orientations of the 3 edges, if they are coherent, and to those of the 2 coherently oriented edges otherwise. This puts a $\mathbb{Z}$ with a preferred generator into each vertex of $P$. Each edge of $P$ corresponds to an isomorphism between the groups in its endpoints; due to the $(-1)^n$ multiplier, it identifies the preferred generators iff precisely one of the vertices corresponds to a coherently oriented cycle of length 3. This puts a sign onto each edge of $P$, and it is easy to check that there is a cycle in $P$ which carries an odd number of minus signs. Thus some preferred generator gets identified with negative itself, and we conclude that $H^2(\tilde{K}_5) \simeq \mathbb{Z}/2$.

Proof of Theorem 3.8. We have $H^{2n}(\bar{X}) \simeq H^{2n}(\bar{X}_s, \bar{Y}_s) \simeq H^{2n}_c(\bar{X}_s \setminus \bar{Y}_s)$, where $Y = X^{(n-1)}$, and $\bar{X}_s$ denotes $\bar{X}/t$. Now $\bar{X}_s \setminus \bar{Y}_s = \bigcup_{\sigma \in X} W_\sigma$, where $W_\sigma = [(\text{Int}\sigma) \times (X \setminus N\sigma) \cup (\text{Int}\sigma) \times (X \setminus N\sigma)]/t$. Since $W_\rho \cap W_\sigma \cap W_\tau = \emptyset$ for any pairwise distinct $\rho, \sigma, \tau$, we have the Mayer–Vietoris exact sequence

$$\bigoplus_{\{\sigma^n, \tau^n\}} H^{2n}_c(W_\sigma \cap W_\tau) \to \bigoplus_{\sigma^n} H^{2n}_c(W_\sigma) \to H^{2n}_c(\bar{X}_s \setminus \bar{Y}_s) \to 0.$$ 

We have $H^{2n}_2(W_\sigma) \simeq H^n(X \setminus N\sigma)$ by the Künneth formula, and $H^{2n}_c(W_\sigma \cap W_\tau) \simeq H^{2n}_c(\text{Int}\sigma \times \text{Int}\tau) \simeq \mathbb{Z}$ when $\sigma$ and $\tau$ are non-adjacent and $W_\sigma \cap W_\tau = \emptyset$ otherwise. Finally, given orientations on $\sigma$ and $\tau$ such that $[\tau] \in H^n(X \setminus N\sigma)$ and $[\sigma] \in H^n(X \setminus N\tau)$ coincide with some fixed generators, then

$$[\sigma \times \tau] \in H^{2n}_c((\text{Int}\sigma) \times (X \setminus N\sigma)) \simeq H^{2n}_c(W_\sigma)$$

and

$$[\tau \times \sigma] \in H^{2n}_c((\text{Int}\tau) \times (X \setminus N\tau)) \simeq H^{2n}_c(W_\tau)$$

may be assumed to coincide with fixed generators. At the same time their preimages

$$[\sigma \times \tau] \in H^{2n}_c(\text{Int}\sigma \times \text{Int}\tau) \simeq H^{2n}_c(W_\sigma \cap W_\tau)$$

and

$$[\tau \times \sigma] \in H^{2n}_c(\text{Int}\tau \times \text{Int}\sigma) \simeq H^{2n}_c(W_\sigma \cap W_\tau)$$

differ by $(-1)^n$ in the right hand group. $\Box$

The same argument works to prove

Addendum 3.10. Let $\bar{\Gamma}$ be the 2-cover of $\Gamma$ induced from the 2-covering $S^1 \to \mathbb{RP}^1$ by a map $f: \Gamma \to \mathbb{RP}^1$ such that $f^{-1}(\mathbb{RP}^0) = \Gamma^{(0)}$. Then

$$H^{2n}(\bar{X}) \simeq \coker \left[ C_1(\bar{\Gamma}; \mathbb{Z}) \overset{\bar{h}}{\to} \bigoplus_{\sigma \in \Gamma^{(0)}} H^n(X \setminus N\sigma) \right],$$

where $\bar{h}$ lies over $h$. Moreover, $H^{2n}(\bar{X}) \overset{p^*}{\to} H^{2n}(\bar{X})$ is given by $C_1(\bar{X}) \overset{\pi^*}{\to} C_1(\bar{X})$.

It follows that $\ker p^*$, which is a group containing $\vartheta(X)$, is generated by elements $a \in H^n(X \setminus N\sigma)$ in the vertices $\sigma \in \Gamma^{(0)}$ that are homologous (with pre-cosheaf coefficients) to $-a$ via loops in $\Gamma$ that do not lift to $\bar{\Gamma}$. Note that a (graph-theoretic) cycle in $\Gamma$ lifts to $\bar{\Gamma}$ iff it has an even length.

For further computations of cohomology of deleted products see [Bau], [Um], [Ni], [BF] and references there.
4. Linkless and panelled embeddings

An interesting theme emerged in combinatorial embedding theory in the early 1980s when J. H. Conway and C. Gordon, and independently H. Sachs observed that every embedding of the complete graph $K_6$ in $\mathbb{R}^3$ links some pair of disjoint (graph-theoretic) cycles of $K_6$ with an odd linking number. Amazingly, Robertson, Seymour and Thomas proved that if a graph $G$ embeds in $\mathbb{R}^3$ in such a way that no pair of disjoint cycles is linked with an odd linking number, then it also admits an embedding in $\mathbb{R}^3$ that is panelled (or “flat”) in the sense that the image of every cycle of $G$ bounds a disk in $\mathbb{R}^3$ whose interior is disjoint from the image of $G$ [RST]. Let us note that every panelled embedding is knotless, that is, contains no nontrivially knotted cycles.

Robertson, Seymour and Thomas also showed that if two panelled embeddings of a graph in $\mathbb{R}^3$ are inequivalent then they differ already on some subgraph that is isomorphic to a subdivision of one of the Kuratowski graphs $K_5$, $K_{3,3}$ [RST]. In particular, if $G$ is planar, then all its panelled embeddings in $\mathbb{R}^3$ are equivalent. Most famously, Robertson, Seymour and Thomas proved that a graph admits a panelled embedding in $\mathbb{R}^3$ if and only if it has no minor among the seven graphs known as the Petersen family [RST].

The “panelled” in the latter result can be replaced, as we just discussed, by “admitting an embedding that links no pair of disjoint cycles with an odd linking number” and also by any intermediate property, including the following one. We say that an embedding $g$ of an $n$-polyhedron $X$ in $\mathbb{R}^m$ is linkless if for every two disjoint closed subpolyhedra of $g(X)$, one is contained in an $m$-ball disjoint from the other one. Thus a linkless embedding of a graph in $\mathbb{R}^3$ need not be knotless, but it may not contain a non-split link of any number of cycles (such as the Whitehead link or the Borromean rings). Consequently, it may not contain a link of any number of cycles that is nontrivial up to PL isotopy.

**Lemma 4.1.** A panelled embedding of a graph in $\mathbb{R}^3$ is linkless.\(^{17}\)

*Proof. L*et $g: G \rightarrow \mathbb{R}^3$ be a panelled embedding. If $H$ is a subgraph of a graph $G$, by $\bar{H}$ we shall denote the union of all edges of $G$ disjoint from $H$. It suffices to prove that for each subgraph $H$ of $G$, there exists a 3-ball containing $g(H)$ and disjoint from $g(\bar{H})$. Let $T$ be a spanning forest of $H$ (i.e., a union of spanning trees of the components of $H$) and let $E_1, \ldots, E_r$ be the edges of $H$ that are not in $T$. Let $C_i$ be the unique graph-theoretic cycle in $T \cup E_i$. Note that each $A_{ij} := C_i \cap C_j$ is connected, for it is the intersection of $T \cap C_i$ and $T \cap C_j$ which are connected and whose union contains no cycles. Let $H^+$ be the 2-complex obtained from $H$ by glueing up $C_1, \ldots, C_r$ by 2-cells $D_1, \ldots, D_r$, then $H^+$ collapses onto $T$, which is collapsible onto finitely many points. Hence it suffices to prove that $g|_H$ extends to an embedding $h: H^+ \hookrightarrow \mathbb{R}^3 \setminus g(\bar{H})$, for a regular neighborhood of its image will then be a union of 3-balls, which can be connected by thin tubes to get the desired

\(^{15}\)This section was considerably altered in v5.

\(^{16}\)But not conversely: an embedding of a graph in $\mathbb{R}^3$ whose restriction to any disjoint union of cycles is a split link does not have to be linkless, see [FN; Figure 2.1 (2)].

\(^{17}\)The published version of the present paper also claimed a converse: a linkless, knotless embedding of a graph in $\mathbb{R}^3$ is panelled; the last step in the proof of this claim was erroneous. The author is grateful to R. Nikkuni for pointing out that this assertion is false, as shown, for instance, by Kinoshita's $\Theta$-curve [Ki; Figure 1], which is a knotless (and, trivially, linkless) embedding of $K_{3,2}$ in $\mathbb{R}^3$ that is not equivalent to the standard embedding, and consequently is not panelled.
3-ball. The existence of $h$ follows from a well-known lemma of Böhme and H. Saran (see [Bö] or [RST; (2.8)]); for convenience of the reader, we include a short proof.

Suppose that $g|_H$ has been extended to an embedding $h_i: H \cup D_1 \cup \cdots \cup D_i \hookrightarrow \mathbb{R}^3 \setminus H$. By the hypothesis, $g|_{C_i}$ extends to an embedding $d: D_i \hookrightarrow \mathbb{R}^3$ such that $d(D_i \setminus C_i)$ is disjoint from $g(G)$. We may assume that $d$ is transverse to each $h_i(D_j)$, relative to the boundary. Since $A_{ij}$ is connected, every connected component of $h_i(D_j \setminus A_{ij}) \cap d(D_i)$ is null-homologous as a locally finite cycle in $h_i(D_j \setminus A_{ij})$. Then at least one such component $S$ is innermost in $h_i(D_j \setminus A_{ij})$, that is, the unique graph-theoretic cycle $S'$ in $S \cup g(A_{ij})$ bounds a disk $D_S$ in $h_i(D_j)$ whose interior is disjoint from $d(D_i)$. Redefine $d$ by replacing the disk bounded by $S'$ in $d(D_i)$ with $D_S$, and pushing it slightly off $h_i$ (relative to $g(A_{ij})$). This decreases the number of components in $h_i(D_j \setminus A_{ij}) \cap d(D_i)$ at least by one, so if we continue this process, it will terminate with a new $d$ that combines with $h_i$ to yield the desired embedding $h_{i+1}$. □

Let us say that an embedding $g$ of an $n$-polyhedron $X$ into $\mathbb{R}^{2n+1}$ is $Y$-panelled, where $Y$ is an $(n+1)$-polyhedron, $Y \supseteq X$, if $g$ extends to a map $f: Y \to \mathbb{R}^{2n+1}$ such that $f^{-1}(X) = X$. A $Y$-panelled embedding of $X$ into $\mathbb{R}^{2n+1}$ yields an equivariant map $\tilde{X}^Y \to S^{2n}$, where $\tilde{X}^Y = (X \times Y \cup Y \times X) \setminus \Delta_X$. Let $\lambda_r$ be the line bundle associated with the double covering $r: \tilde{X}^Y \to \bar{X}^Y$, where $\bar{X}^Y = \tilde{X}^Y / t$. Thus

$$\eta(X,Y) := e(\lambda_r)^{2n+1} \in H^{2n+1}(\bar{X}^Y; \mathbb{Z}_p)$$

vanishes if $X$ admits a $Y$-panelled embedding into $\mathbb{R}^{2n+1}$.

Let us call a subpolyhedron $S$ of an $n$-polyhedron $X$ linkable if $S \times (X \setminus S)$ admits an essential map to $S^{2n}$ (or equivalently, $H^n(S) \otimes H^n(X \setminus S)$ is nonzero). Let us call a simplicial complex $K$ bounded if every linkable subcomplex of $K$ is homeomorphic to a quotient of a PL $n$-manifold $M$ with boundary by some identification on $\partial M$.

For $n > 1$ it is not hard to see that an embedding of $g$ of an $n$-polyhedron $X$ in $\mathbb{R}^{2n+1}$ is linkless if it is $X_+\text{-panelled}$, where $X_+$ is obtained from $X$ by glueing up by cones all linkable subpolyhedra of $X$, triangulated by subcomplexes of a fixed triangulation of $X$.\(^{18}\) It is also not hard to see that the converse holds if $n > 1$ and $X$ has a bounded triangulation.\(^{19}\) Thus for $X$ with a bounded triangulation and $n > 1$, $\eta(X) := \eta(X,X_+)\) vanishes if $X$ admits a linkless embedding in $\mathbb{R}^{2n+1}$.

**Theorem 4.2.** Let $X$ be an $n$-polyhedron and $Y$ an $(n+1)$-polyhedron containing $X$.

(a) Let $n > 1$. Then $X$ admits a $Y$-panelled embedding in $\mathbb{R}^{2n+1}$ if and only if $\eta(X,Y) = 0$.

\(^{18}\)Indeed, if $P$ and $Q$ are disjoint subcomplexes of $X$, then either $P \times Q$ admits no essential map to $S^{2n}$ or $g(P)$ is null-homotopic in the complement to $g(Q)$. In both cases $\tilde{g}|_{P \times Q}$ is null-homotopic, so by the Haefliger–Weber Criterion 3.1, $g|_{P \cup Q}$ is equivalent to the embedding $h: P \cup Q \hookrightarrow \mathbb{R}^{2n+1}$, obtained by combining $e_1g|_P$ and $e_2g|_Q$, where $e_1, e_2: B^{2n+1} \to \mathbb{R}^{2n+1}$ are embeddings with disjoint images.

\(^{19}\)Indeed, let $S$ be a linkable subcomplex of a bounded triangulation of $X$. Since $g$ is linkless, $g(S)$ lies in a ball $B$ disjoint from $g(\bar{S})$, where $\bar{S}$ is the union of all simplices disjoint from $S$. Then $g|_S$ extends to a map $f: CS \to B$, and since $S$ is the quotient of a manifold by some identification on the boundary, using the Penrose–Whitehead–Zeeman trick it is easy to achieve that $f^{-1}(S) = S$ (thus, $g|_S$ is $CS$-panelled) and that $f(CS)$ is disjoint from $g(X \setminus S)$.\)
(b) $X$ admits a linkless embedding in $\mathbb{R}^{2n+1}$ if $\eta(X) = 0$. If $X$ has a bounded triangulation, then the converse also holds.

Proof. (a). It follows from the proof of the Haefliger–Weber Criterion 3.1 that if $\bar{X}^Y$ admits an equivariant map to $S^{2n}$ and $n > 1$, then $X$ admits a $Y$-panelled embedding into $\mathbb{R}^{2n+1}$. □

(b). The case $n > 1$ follows from (a) and the preceding remarks. Suppose that $n = 1$. If $\eta(X) = 0$, the proof of (a) works to construct an embedding $X \hookrightarrow \mathbb{R}^3$ such that every pair of cycles has zero linking number. By [RST], this suffices to re-embed $X$ linklessly in $\mathbb{R}^3$. Conversely, if $X$ admits a linkless embedding in $\mathbb{R}^3$, then by [RST] it admits a panelled embedding $g$ in $\mathbb{R}^3$. If additionally $X$ has a bounded triangulation, then every linkable subcomplex of this triangulation is homeomorphic to $S^1$. Hence $g$ is $X_+\text{-panelled}$, and thus $\eta(X) = 0$. □

Example 4.3. Consider the complete graph $K_6$ as the 1-skeleton of the cone $CK_5$. Since $CK_5$ contains no pair of disjoint 2-simplices, every embedding of $K_6$ in $\mathbb{R}^3$ may be considered “$CK_5$-linkless”. However, $K_6$ admits no $CK_5$-panelled embedding in $\mathbb{R}^3$, since $\text{secat}_Z(CK_5) = 3$ (see Example 3.3).

Theorem 4.4. For $n > 0$, an $n$-polyhedron $X$ admits a $CX$-panelled embedding into $\mathbb{R}^{2n+1}$ if and only if $\vartheta(X) = 0$.

Note that the case $n = 2$ is not an exception here.

Proof. Note that $\bar{X}_s^Y := \bigcup\{A \times B, B \times A \mid A \in X, B \in Y, A \cap B = \emptyset\}$ is equivariantly homotopy equivalent to $\bar{X}^Y$. On the other hand $\bar{X}_s^{CX}$ coincides with the simplicial deleted product $CX_s$. By Lemma 4.5 below, for $n > 0$ there is an isomorphism $H_{2n+1}(CX_s; I) \simeq H_{2n+1}(\Sigma(\bar{X}_s); I)$ induced by an equivariant map, $I$ being the augmentation ideal of $\mathbb{Z}[\mathbb{Z}/2]$, so $\text{secat}_Z(CX_s) = \text{secat}_Z(\Sigma(\bar{X}_s))$. It is not hard to see that $\text{secat}_Z(\Sigma(\bar{X}_s)) = \text{secat}_Z(\bar{X}_s) + 1$, cf. [CF; (5.1)]. It follows that $\text{secat}_Z(\bar{X}) = \text{secat}_Z(\bar{X}^{CX}) - 1$. Hence $\vartheta(X) = 0$ iff $\bar{X}^{CX}$ admits an equivariant map to $S^{2n}$. This implies the “only if” part of the theorem.

The “if” part now follows from Theorem 4.2(a) when $n > 1$. For $n = 1$ we note that if $X$ contains a subgraph $K$ homeomorphic to either $K_5$ or $K_{3,3}$, then $\bar{X}_s$ contains $\bar{CK}_s$, which is equivariantly homeomorphic to $S^3$ (see Example 3.3). This contradicts the existence of an equivariant map $\bar{CX}_s \to S^2$. So $X$ embeds into $\mathbb{R}^2$ and $CX$ embeds into $\mathbb{R}^3$. □

Lemma 4.5. There exists an equivariant map $p: \bar{CX}_s \to \Sigma(\bar{X}_s)$ whose relative mapping cylinder has the same (generalized) equivariant cohomology as $(\Sigma X, \text{pt}) \times \mathbb{Z}/2$.

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20The published version of the present paper erroneously omitted this hypothesis. To see that it cannot be omitted, note that $\eta(K_5 \sqcup K_5) \neq 0$ by part (a) and Theorem 4.4 below. The author is grateful to V. Turchin for calling his attention to the proof of Theorem 4.2(b). A more satisfactory correction, imposing no restrictions on $X$, appears in a subsequent paper by the author, arXiv:1103.5457v2, where the proof of Theorem 4.6 shows that $X$ admits a linkless embedding in $\mathbb{R}^{2n+1}$ if and only if the following cohomology class $\eta'(X)$ vanishes. Fix some triangulation $K$ of $X$. If $S$ is triangulated by a subcomplex of $K$, let $\bar{S}$ be the union of all simplices of $K$ disjoint from $S$. Let $\bar{X}_s^+$ be obtained from $\bar{X}_s$ by glueing up by cones all products $S \times \bar{S}$ that admit an essential map into $S^{2n}$. Then $\eta'(X)$ is the $(2n + 1)$th power of the Euler class of the line bundle associated with the double cover $\bar{X}_s^+ \to \bar{X}_s^+$. 
Proof. We define \( p \) by sending \( X \times \{e\} \) and \( \{e\} \times X \) onto the suspension points and via the homeomorphism on their complement. The relative mapping cylinder of \( p \) equivariantly collapses onto \( (CX \cup_{\{e\} \times X} CX_s \cup_{X \times \{e\}} CX, CX_s) \), which is excision-equivalent to \( (CX \cup CX, X \cup X) \). \( \square \)

The following is a generalization of Lemma 3.7.

**Proposition 4.6.** Let \( K \) be a simplicial complex triangulating an \( n \)-polyhedron \( X \). Let \( S \) and \( L \) be the subpolyhedra of \( X \) triangulated by the star \( \text{st}(v, K) \) and the link \( \text{lk}(v, K) \) of some vertex \( v \) of \( K \). Set \( Y = (X \setminus S) \cup L \).

(a) If \( g: Y \hookrightarrow \mathbb{R}^{2n} \) is an embedding and \( (\bar{g}|_{L_{V}})^*: H^{2n-1}(\mathbb{R}P^{2n-1}) \to H^{2n-1}(\bar{L}_{V}) \) is the zero map, then \( \vartheta(X) = 0 \).

(b) If \( Y \) embeds into \( \mathbb{R}^{2n} \) and \( L \) admits a \( Y \)-panelled embedding into \( \mathbb{R}^{2n-1} \), then \( \vartheta(X) = 0 \) (mod 2).

**Proof.** (a). Since \( H^{2n-1}(\mathbb{R}P^{2n-1}) \) maps onto \( H^{2n}(\mathbb{R}P^\infty, \mathbb{R}P^{2n-1}) \), the latter has trivial image in \( H^{2n}(\bar{S}X, \bar{L}_{V}) \). This image is precisely the obstruction to extending \( \bar{g}|_{L_{V}} \) to \( \bar{X} \). Since \( \bar{Y} \cup \bar{S}X = \bar{X} \), we conclude that \( \vartheta(X) = 0 \). \( \square \)

(b). Given a \( g: Y \hookrightarrow \mathbb{R}^{2n} \), our \( (\bar{g}|_{L_{V}})^*: H^{2n-1}(\mathbb{R}P^{2n-1}; \mathbb{Z}/2) \to H^{2n-1}(\bar{L}_{V}; \mathbb{Z}/2) \) factors as \( H^{2n-1}(\bar{L}_{V}; \mathbb{Z}/2) \xrightarrow{G_{\bar{L}_{V}}} H^{2n-1}(\mathbb{R}P^\infty; \mathbb{Z}/2) \xrightarrow{\cong} H^{2n-1}(\mathbb{R}P^{2n-1}; \mathbb{Z}/2) \), where \( G: \bar{L}_{V} \to \mathbb{R}P^\infty \) is any classifying map. Hence modulo 2, \( (\bar{g}|_{L_{V}})^* \) does not depend on the choice of \( g \), and the proof of (a) applies. \( \square \)

**Example 4.7.** If \( X \) is the \( n \)-skeleton of \( \Delta^{2n+2} \), and \( v \) any its vertex, then \( L \) is the \( (n - 1) \)-skeleton of \( \Delta^{2n+1} \), and \( Y = L_{+} \), the \( n \)-skeleton of \( \Delta^{2n+1} \). Since \( \vartheta(X) \) is nonzero even modulo 2 (see §3), we conclude that \( L \) admits no linkless embedding into \( \mathbb{R}^{2n-1} \). This was originally proved in [LS; Corollary 1.1] and [Ta2] by different arguments.

**Remark.** Interestingly, an argument of converse type was found in [Sk1], where non-embeddability of a certain polyhedron is derived from non-existence of linkless embeddings of its links of vertices.

**Remark.** Let \( K \) be an \( n \)-dimensional simplicial complex. If \( S \) is a subcomplex of \( K \), let \( S' \) be the union of all simplices of \( K \) disjoint from \( S \). Consider all subcomplexes \( S_{1}, \ldots, S_{r} \) of \( K \) such that \( S_{i} \times S'_{i} \) admits an essential map to \( S^{2n} \). Let \( \hat{K} \) be obtained from the simplicial deleted product \( \hat{C}K_{s} \) by attaching \( CS_{i} \times CS'_{i} \) to each \( (CS_{i}) \times S'_{i} \cup S_{i} \times (CS'_{i}) \). If \( L \) is the \( n \)-skeleton of the \( (2n + 3) \)-simplex, then \( \hat{L} \) is \( \mathbb{Z}/2 \)-homeomorphic to \( S_{-}^{2n+2} \). If \( G \) is one of the seven graphs Petersen graphs, then \( G \) is \( \mathbb{Z}/2 \)-homeomorphic to \( S_{+}^{4} \) in six cases, and \( \mathbb{Z}/2 \)-homotopy equivalent but not homeomorphic to \( S_{-}^{4} \) in the remaining case. This provides yet another proof, in the spirit of Example 3.3, that \( L \) does not linklessly embed in \( \mathbb{R}^{2n+1} \) and \( G \) does not linklessly embed in \( \mathbb{R}^{3} \). The details will appear in a forthcoming paper of the author, where these homeomorphisms are unified in one general result.

5. Arrow diagram formulas for type 1 invariants of singular knots

Let us fix a closed 1-manifold \( M \). All results of this section are interesting already in the case \( M = S^{1} \). A chord diagram \( \Theta \) on \( M \) is a collection of \( m \) unordered pairs of distinct points of \( M \). Each pair is called a chord. When \( M = S^{1} \), \( \Theta \) is called reducible if it can be partitioned into two nonempty chord diagrams \( \Theta_{1}, \Theta_{2} \) such
that every chord of $\Theta_{1}$ is unlinked with every chord of $\Theta_{2}$. Associated to $\Theta$ is the equivalence relation $R_{\Theta}$ on $M$, where $x R_{\Theta} y$ iff $x = y$ or $(x, y) \in \Theta$. A $\Theta$-knot is a smooth immersion $\varphi: M \rightarrow \mathbb{R}^{3}$ that factors through an embedding of the graph $G := M / R_{\Theta}$ and has no self-tangencies at the double points of $\varphi$. We consider $\Theta$-knots up to $\Theta$-isotopy, that is smooth regular homotopy through $\Theta$-knots.

If $v$ is an invariant of $\Theta$-knots with values in an abelian group, the Vassiliev derivative $v'$ is an invariant of $\Theta^{+}$-knots for each chord diagram $\Theta^{+}$ obtained by adding one chord $\{x, y\}$ to $\Theta$. It is defined by $v'(k) = v(k_{+}) - v(k_{-})$, where $k_{+}$ and $k_{-}$ are $\Theta$-knots, $C^{1}$-close to the $\Theta^{+}$-knot $k$ and such that $(k_{+}(x), k_{+}(y), k_{+} - k_{+}(y))$ is a positive frame and $(k_{-}'(x), k_{-}'(y), k_{-} - k_{-}(y))$ is a negative frame. (“Positive” and “negative” clearly do not depend on the choice of the ordering of $x$ and $y$ and the orientation of $M$.) The invariant $v$ is a type $k$ invariant if the $(k + 1)$th derivative $v^{(k+1)}$ vanishes identically.

We will be concerned with type 1 invariants of $\Theta$-knots, which include (the restrictions of) the $m$th Vassiliev derivatives of all type $m + 1$ invariants of genuine knots. The derivative of a type 1 invariant $v$ of $\Theta$-knots may be regarded as a function $v'((x, y))$ of the extended chord diagram $\Theta^{+} = \Theta \cup \{(x, y)\}$.

**Configuration spaces $\tilde{\Theta}$ and $\Theta$.** For convenience, we shall assume hereafter that each component of $M$ contains at least one chord (the general case can be treated similarly). Then $G = M / R_{\Theta}$ is a graph, possibly with loops and multiple edges. Let $\tilde{\Theta}_{q} = G \times G \setminus \bigcup C \times C$, where $C$ runs over all open edges of $G$, and let $\tilde{\Theta}_{0} = \tilde{\Theta}_{q} / (\mathbb{Z}/2)$. On the other hand, let us fix some triangulations of $M$ and $G$ such that the quotient map $q: M \rightarrow G$ is simplicial. Let us consider $(q \times q)(\tilde{M}_{s})$, the image of the simplicial deleted product of $M$ in $G \times G$. Let $\tilde{\Theta}_{0}^{s} = (q \times q)(\tilde{M}_{s}) \cap \tilde{\Theta}_{0}$ and $\tilde{\Theta}_{0}^{s} = \tilde{\Theta}_{0}^{s} / (\mathbb{Z}/2)$.

Each of $\tilde{\Theta}_{0}$, $(q \times q)(\tilde{M}_{s})$ and $\tilde{\Theta}_{0}^{s}$ meets the diagonal of $G \times G$ precisely in points of the type $(w, w)$, where $w = q(x) = q(y)$ is a vertex of $G$ covered by the points of some chord $(x, y)$ of $\Theta$. For every such $(w, w)$, let us consider its links $L_{(w, w)}$ and $L_{(w, w)}^{s}$ respectively in $\tilde{\Theta}_{0}$ and in $\tilde{\Theta}_{0}^{s}$. Each of them is endowed with a free involution, induced from the factor-exchanging involution of $G \times G$. Clearly, $L_{(w, w)}^{s}$ is $\mathbb{Z}/2$-homeomorphic to the link of $(w, w)$ in $(q \times q)(\tilde{M}_{s})$.

Let us temporarily use the following notation: $[\pm]_1 = \{1, 3\}$, $[-]_2 = \{2, 4\}$ and $[\pm]_3 = [\pm]_1 \cup [-]_2$. The link of $w$ in $G$ is homeomorphic to $[\pm]_3$, consequently the link of $(w, w)$ in $G \times G$ is homeomorphic to $[\pm]_3 \ast [\pm]_3$. Then $L_{(w, w)}$ is $\mathbb{Z}/2$-homeomorphic to the simplicial deleted join $[\pm]_3$, and $L_{(w, w)}^{s}$ to its subset $[\pm]_3 \ast [-]_2 \cup [-]_2 \ast [\pm]_3$. Now the pair $([\pm]_3, [\pm]_3 \ast [-]_2 \cup [-]_2 \ast [\pm]_3)$ is in turn $\mathbb{Z}/2$-homeomorphic to (all edges of $I^{3}$, all horizontal edges of $I^{3}$), with involution induced from the antipodal involution on $\partial I^{3}$. The latter homeomorphism sends the first copy of $[\pm]_3$ to the vertices of some diagonal of the top face of the cube, the first copy of $[-]_2$ to the vertices of the perpendicular diagonal of the bottom face of the cube, the second copies of $[\pm]_3$ and $[-]_2$ symmetrically, and all edges linearly. Thus the star $(w, w) \ast L_{(w, w)}$ of $(w, w)$ in the second derived subdivision of $\tilde{\Theta}_{0}$ is homeomorphic to the cone over the 1-skeleton of the 3-cube, by a homeomorphism throwing $(w, w) \ast L_{(w, w)}^{s}$ to the cone over the boundaries of the two horizontal faces of the cube.

Let $\tilde{\Theta}$ be obtained from $\tilde{\Theta}_{0}$ by removing, for every vertex $w$ as above, the interior of the star $(w, w) \ast L_{(w, w)}$, identified with the cone over the 1-skeleton of the
cube, and replacing it with the 4 vertical faces of the cube. The factor exchanging involution on \( \Theta_0 \) carries over to a free involution on \( \Theta \) by considering the antipodal involution on the boundary of the cube. We set \( \Theta = \Theta / (\mathbb{Z}/2) \) and denote the double covering by \( p: \Theta \to \Theta \).

**Remark.** Let us discuss the geometry of \( \Theta^s \subset \Theta \) and \( \Theta^s \subset \Theta_0 \), which are similarly obtained from \( \Theta_0^s \subset \Theta_0 \). At each \((w, w)\), we have in \( \Theta^s \) a singularity of the type \( C(S^1 \sqcup S^1) \), which is resolved in the standard way, by gluing in the handle \( S^1 \times I \).

The involution on this handle is orientation-reversing, so the quotient is homeomorphic to the Möbius band. Thus \( \Theta^s \) is obtained from \( \Theta_0^s \) by blowing it up at all points of the type \{w, w\}.

Returning to \( \Theta \), notice that the open star of \{w, w\} in \( \Theta_0 \) is homeomorphic to

\[
\mathbb{R} \times \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{0\} \times [0, \infty) \cup \{0\} \times \mathbb{R} \times (-\infty, 0]
\]

where \( \mathbb{R} \times \mathbb{R} \times \{0\} \) is identified with the open star of \{w, w\} in \( \Theta_0^s \). Then \( \Theta \) is obtained from \( \Theta_0 \) by blowing up the \( \mathbb{R} \times \mathbb{R} \times \{0\} \) at the origin and then regluing \( \mathbb{R} \times \{0\} \times [0, \infty) \) and \( \{0\} \times \mathbb{R} \times (-\infty, 0] \) back along the images of \( \mathbb{R} \times \{0\} \times \{0\} \) and \( \{0\} \times \mathbb{R} \times \{0\} \) under the blowup, and repeating for each \{w, w\}.

It follows that \( \Theta \) is a simplicial analogue of the space \( B_2(\Theta) \) from [Va2], which collapses onto \( \Theta \). An attempt to define \( \Theta \) occurs in [M3], where \( \Theta^s \) was defined instead.

**Lemma 5.1.** Let \( v \) be an integer-valued type 1 invariant of \( \Theta \)-knots. Let \( z^\# = \sum v'(\{c, d\})C \times D \), where the sum runs over all products \( C \times D \) of components of \( M \setminus \Theta \) and each \( c \in C \) and \( d \in D \). Then \( z^\#_v \) is a skew-invariant 2-cycle in \( G \times G \), which lies in \( \Theta_0 \). Moreover, it “blows up” to a unique skew-invariant 2-cycle \( z_v \) in \( \Theta \), which may be identified with an element of \( H_2(\Theta; \mathbb{Z}_p) \).

By a skew-invariant element of a \( \mathbb{Z}[\mathbb{Z}/2] \)-module \( M \) we mean an \( x \in M \) such that \( t(x) = -x \), where \( t \) is the generator of \( \mathbb{Z}/2 \).

**Proof.** Since \( t(C \times D) = -D \times C \) as 2-chains, \( z^\#_v \) is a skew-invariant chain in \( \Theta_0 \).

An edge of the product cell structure on \( \Theta_0 \) is of the form \( p \times C \) or \( C \times p \), where \( p \) is a point of \( M/R_\Theta \) corresponding to a chord of \( \Theta \), and \( C \) is a 1-simplex of \( T_\Theta \). Such an edge is incident to four 2-cells in \( \Theta_0 \) of the form \( D \times C \) (respectively, \( C \times D \)) where \( D \) is a 1-simplex of \( T_\Theta \). Hence \( \partial z^\#_v = 0 \) is a reformulation of the well-known (and obvious) “four-term relation”.

The assertion that \( z^\#_v \) lies in \( \Theta_0 \) is a direct consequence of the well-known (and obvious) “one-term relation”, also known as the “framing independence relation”.

To prove the final assertion, let us consider \( h: H_2(\Theta; \mathbb{Z}_p) \to H_2(\Theta, \mu; \mathbb{Z}_p) \), where \( \mu \) is the union of the Möbius bands obtained upon the blowup of the chords. Since \( \mu \) is homotopically 1-dimensional, \( h \) is injective. Since \( H_1(\mu; \mathbb{Z}_p) = 0 \) as well\(^1\), it is an isomorphism. Now since \( (\Theta, \nu) \), where \( \nu \) is the preimage of \( \mu \), is 2-dimensional, its 2-dimensional cohomology classes are identified with its 2-cycles, and therefore the homomorphism \( \nu \) from \( H_2(\Theta, \mu; \mathbb{Z}_p) \cong H^2_2(\Theta, \nu; I) \) to \( H_2(\Theta, \nu) \) is an embedding onto the subgroup of skew-invariant elements (see §32). Let \( \Theta^+ \) be the “bordism” between \( \Theta_0 \) and \( \Theta \), obtained by adding the cone over each component of \( \nu \), and let \( \nu^+ \) be the union of these cones. Then \( H_2(\Theta, \nu) \cong H_2(\Theta^+, \nu^+) \) by excision, and at

\(^1\)This is the only point where the proof fails for \( \mathbb{Z}/2 \)-valued type 1 invariants.
Theorem 5.2. obstruction to homotopy of their Gauss maps $\bar{\Theta}$. Pick a generic homotopy $k: M/\Theta \to \mathbb{R}^3$ is a $\Theta$-knot, we may consider its Gauss maps $\tilde{k}: \tilde{\Theta} \to S^2$ and $k: \Theta \to \mathbb{R}P^2$. (The details of the construction are left as an exercise for the reader.) We define the parametric van Kampen obstruction $\bar{\Theta}$-isotopy between the $\bar{\Theta}$-knots $k_1, k_2$ occurring in $\tilde{\Theta}$ reader.) We define the parametric van Kampen obstruction $\bar{\Theta}$-isotopy between the $\bar{\Theta}$-knots $k_1, k_2$ occurring in $\tilde{\Theta}$ to existence of a $\bar{\Theta}$-isotopy between the $\bar{\Theta}$-knots $k_1, k_2$. Let $\Theta$ be the $\Theta$-universal type 1 invariant of $\Theta$-knots. Then $\zeta(k) = \zeta(k_0)$ is a universal type 1 invariant of $\Theta$-knots.

Proof. Pick a generic homotopy $h_t$ between $k_1$ and $k_2$. Let $k_i$ be the $\Theta \cup \{\{c_i, d_i\}\}$-knots occurring in $h_t$ and $\varepsilon_i$ be their signs corresponding to the increase of $t$. Then $v(k_1) - v(k_2) = \sum_i \varepsilon_i v'(k_i) = \sum_i \varepsilon_i v'(\{c_i, d_i\})$.

On the other hand, the Thom isomorphism $t: H^2(\bar{\Theta}; \mathbb{Z}_p) \simeq H^3(\bar{\Theta} \times I, \bar{\Theta} \times \partial I; \mathbb{Z}_p)$ identifies $\zeta(k_1, k_2)$ with $H^3(\chi)$, where $H: M/\Theta \to \mathbb{R}^4$ is a $\Theta$-isotopy projecting to $h$ and $\chi$ generates $H^3(\mathbb{R}P^\infty, \mathbb{R}P^2; \mathbb{Z}_p) \simeq \mathbb{Z}$. Clearly, $H^*(\chi) = \sum_i \varepsilon_i \omega(C_i, D_i)$, where $\omega(C_i, D_i)$ is the class of the cocycle assuming 1 on the skew-equivariant chain $C_i \times D_i + D_i \times C_i$ and 0 on all other basic skew-equivariant chains. $\square$

Corollary 5.3. Let $k_0$ be a fixed $\Theta$-knot. Then $u(k) := \zeta(k, k_0)$ is a universal type 1 invariant of $\Theta$-knots.

In more detail, $u$ takes values in $H^2(\bar{\Theta}; \mathbb{Z}_p)$ (which is free abelian, since $\bar{\Theta}$ is 2-dimensional) and every integer-valued type 1 invariant $v$ can be recovered from $u$. In fact, the same arguments work for invariants $v$ with values in any abelian group containing no elements of order 2. (Elements of order 2 do not fit in the proof of Lemma 5.1.)

Proof. By the proof of Theorem 5.2, $u(k_+) - u(k_-) = \zeta(k_+, k_-)$ is the class of the cocycle assuming 1 on the skew-invariant chain $C \times D + D \times C$, where $C$ and $D$ are the components of $M \setminus \bigcup \Theta$ being intersected, and 0 on all other basic skew-invariant chains. So $u$ is a type 1 invariant. By Theorem 5.2, any invariant $v$ of $\Theta$-knots is a function of $u$, namely $v(k) = v(k_0) + u(k)[z_v]$. Thus $u$ is universal. $\square$

Given a chord $\alpha$ of $\Theta$ and some choice of orientation on it (i.e. an ordering of its points), along with a fixed orientation of $M$, let $\alpha^+$ and $\alpha^-$ denote the images in $G$ of the oriented half-circles $a^+, a^- \subset M$ with $\partial a^+ = \partial a^- = \alpha$.

Corollary 5.4. (a) The group $\Gamma_1(\Theta)$ of integer-valued type 1 invariants of $\Theta$-knots modulo type 0 invariants is isomorphic to $H^2(\bar{\Theta}; \mathbb{Z}_p)$.

(b) [Va2; Theorem 1(b)] If $M = S^1$, $\Gamma_1(\Theta)$ is free abelian of rank $\binom{m}{2} + n$, where $m$ is the number chords and $n$ is the number of irreducible factors of $\Theta$.

Proof. (a). A map $\Gamma_1(\Theta) \to H^2(\bar{\Theta}; \mathbb{Z}_p)$ is given by Lemma 5.1; by construction, it is a homomorphism. By Theorem 5.2, this map has an inverse. $\square$

(b). By the proof of Lemma 5.1, $H_2(\bar{\Theta}; \mathbb{Z}_p)$ is isomorphic to the subgroup of skew-invariant elements of $H_2(\bar{\Theta})$. By the K"unneth formula, $H_2(G \times G)$ is freely generated by products of the type $[\alpha_i^+ \times \alpha_j^+]$, where $\alpha_1, \ldots, \alpha_m$ are all chords of $\Theta$ with arbitrary but fixed orientations, and $\alpha_0^+$ is the image of $[M]$ in $G$. The subgroup of skew-invariant elements of $H_2(G \times G)$ is freely generated by elements of the type $[\alpha_i^+ \times \alpha_i^+]$ and $[\alpha_i^+ \times \alpha_j^+] + [\alpha_j^+ \times \alpha_i^+]$ with $j > i$, and so is of rank $m + 1 + \binom{m+1}{2}$. 
Clearly, $H_2(G \times G, \tilde{\Theta}_0)$ is free abelian of rank $2m$, and all its elements are skew-invariant. It is easy to see that an edge $e$ of $G$ is such that $[e \times e] \in H_2(G \times G, \tilde{\Theta}_0)$ lies in the image of $H_2(G \times G)$ iff it is the unique common edge of some pair of cycles $a, b \in H_1(G)$.

The irreducible factors of $\Theta$ correspond to arcs $J_1, \ldots, J_n$ in $M$ with endpoints in $\bigcup \Theta$ such that each pair of arcs is either disjoint or one arc is contained in the interior of the other; the $i$-th irreducible factor of $\Theta$ consists of all chords with both endpoints in $J_i$ but not in any $J_j \subset J_i$. Clearly, an edge $e$ of $G$ is the unique common edge of some pair of cycles iff it is connecting chords from the same irreducible component. Let $G'$ be the graph with $n$ vertices, obtained from $G$ by contracting all such edges. If $G'$ has $k$ edges, $H_1(G')$ is of rank $k - n + 1$. Since no edge of $G'$ is the unique common edge of some pair of cycles, $H_1(G')$ has a basis where no two elements have a common edge. Let $N$ be the union of products of distinct edges of $G'$. Then $H_2(G' \times G', N)$ has rank $k$ and the image of $H_2(G' \times G')$ in $H_2(G' \times G', N)$ has rank $k - n + 1$. Hence coker $(H_2(G' \times G') \to H_2(G \times G, \tilde{\Theta}_0))$ has rank $n - 1$, and it follows that coker $(H_2(G \times G) \to H_2(G \times G, \tilde{\Theta}_0))$ also has rank $n - 1$. Hence $K := \text{im}(H_2(G \times G) \to H_2(G \times G, \tilde{\Theta}_0))$ has rank $2m - n + 1$.

The image of the skew-invariant subgroup of $H_2(G \times G)$ in $K$ contains $2K$, hence has the same rank $2m - n + 1$. Since $H_2(\tilde{\Theta}_0)$ maps injectively to $H_2(G \times G)$, we conclude that the skew-invariant subgroup of $H_2(\tilde{\Theta}_0)$ has rank $(m - 1) - m + n = (m + n) - n$. □

**Arrow diagram formulas.** A one-arrow diagram over $\Theta$ is an ordered pair $(C, D)$ of connected components of $M \setminus \bigcup \Theta$. It corresponds to the function $f_{(C,D)}$ on generic $\Theta$-knots, whose value on $k: M \leftrightarrow \mathbb{R}^3$ is the algebraic number of ways to draw an upward oriented vertical segment in $\mathbb{R}^3$ whose initial endpoint is in $k(C)$ and terminal in $k(D)$. Each such segment is counted with the sign of the frame, consisting of this oriented segment and the tangent vectors to $k$ at its endpoints. In addition, the unique arrowless diagram over $\Theta$ corresponds to the function assuming 1 on every $\Theta$-knot.

An arrow diagram formula for a type 1 invariant $v$ of $\Theta$-knots is a representation of $v$ as a sum $\sum_{(C,D)} f_{(C,D)} + \sum 1$ of the functions corresponding to one-arrow and arrowless diagrams.

**Corollary 5.5** [Va2; Theorem 5(1)]. Every integer-valued type 1 invariant $v$ of $\Theta$-knots admits an arrow diagram formula with half-integer coefficients.

**Proof.** Let $w = \sum_{(C,D)} v'([c,d])(C,D)$, where $c \in C, d \in D$. Clearly, $w' = 2w'$, as long as we know that $w$ is an invariant of $\Theta$-knots. To prove the latter, pick a suitable generator $\varphi$ of $H^2(S^2)$. Representing $\varphi$ by a cocycle with support in the north pole, we see that $w(k) = \tilde{k}^*([\varphi])(p'[z_v])$, which is an invariant of $k$. □

**Theorem 5.6.** An integer-valued type 1 invariant $v$ of $\Theta$-knots admits an integral arrow diagram formula if and only if $[z_v] \cup e(\lambda_p) = 0$, where $e(\lambda_p) \in H^1(\tilde{\Theta}; \mathbb{Z}_p)$ is the Euler class of the line bundle associated with the 2-covering $p: \tilde{\Theta} \to \Theta$.

**Proof.** From the Smith sequence of $p$, whose connecting homomorphism is $\cdot \cup e(\lambda_p)$, we have $[z_v] = p_*(c)$ for some $c \in H_2(\tilde{\Theta})$. Then $k \mapsto \tilde{k}^*([\varphi])(c)$ is an arrow diagram formula for $v$ (up to a type 0 invariant), since $\zeta(k_1, k_2)(p_*c) = p^*\zeta(k_1, k_2)(c) = [k_2^*([\varphi]) - k_1^*([\varphi])].$
Conversely, by definition, an integral arrow diagram for \( v \) is a 2-chain \( c \in C_2(\hat{\Theta}) \) such that \( v(k) = \tilde{k}^*(\varphi)(c) \) up to a type 0 invariant. Similarly to the proof of Lemma 5.1, \( \partial c = 0 \). Then by the preceding paragraph, \( \zeta(k_1, k_2)(p_\ast [c]) = \zeta(k_1, k_2)[z_v] \) for any \( \Theta \)-knots \( k_1, k_2 \). Since \( \zeta(k_1, k_2) \) can be an arbitrary skew-equivariant 2-cochain (see the proof of Corollary 5.3), \( p_\ast ([c]) = [z_v] \) and therefore \( [z_v] \sim e(\lambda_p) = 0 \). □

We call \( \Theta \) planar if there exists a \( \Theta \)-knot in the plane, \( k : M/\Theta \hookrightarrow \mathbb{R}^2 \subset \mathbb{R}^3 \).

**Lemma 5.7.** Let \( \mu \subset \hat{\Theta} \) be the union of the Möbius bands obtained upon the blowup of the chords of \( \Theta \).

(a) If \( M = S^1 \), then \( \sim e(\lambda_p) : H_2(\hat{\Theta}, \mu; \mathbb{Z}_p) \to H_1(\hat{\Theta}, \mu) \) is trivial.

(b) If \( M = S^1 \) and \( \Theta \) is irreducible, \( \ker[H_1(\hat{\Theta}) \to H_1(\hat{\Theta}, \mu)]/\text{(odd torsion)} \) is cyclic.

(c) If \( M = S^1 \) and \( \Theta \) is irreducible and planar, the cyclic group from (b) is isomorphic to \( \mathbb{Z} \), and consequently \( \sim e(\lambda_p) : H_2(\hat{\Theta}; \mathbb{Z}_p) \to H_1(\hat{\Theta}) \) is trivial.

Part (a) replaces [Va2; Lemma 4], which is easier due to the fact that genuine knots are replaced with long knots in [Va2]. The idea of proof of (b) is taken from [Va2; proof of Lemma 6].

**Proof.** (a). By the proof of Lemma 5.1, \( H_2(\hat{\Theta}, \nu) \simeq H_2(\bar{\Theta}_0) \) over \( \Lambda = \mathbb{Z}[\mathbb{Z}/2] \), where \( \nu \) is the preimage of \( \mu \) in \( \hat{\Theta} \). By the proof of Corollary 5.4(b), we have a short exact sequence of \( \Lambda \)-modules

\[
0 \to H_2(\hat{\Theta}_0) \to H_2(G \times G) \to K \to 0,
\]

where \( K \) is isomorphic to the direct sum of \( 2m - n + 1 \) copies of the augmentation ideal \( I = \ker(\Lambda \to \mathbb{Z}) \). The \( \mathbb{Z}/2 \)-invariant subgroup \( H^0(\mathbb{Z}/2; K) \) of \( K \) is trivial, and from the explicit description of generators given in the proof of 5.4(b), the diagonal subgroup \( D \) of \( H_2(G \times G) = H_1(G) \otimes H_1(G) \) embeds onto a subgroup \( e(D) \) of \( K \), moreover \( K/e(D) \) is isomorphic to the direct sum of \( m - n \) copies of \( I \). Hence \( H^0(\mathbb{Z}/2; K/e(D)) = 0 \) and so \( H^1(\mathbb{Z}/2; D) \to H^1(\mathbb{Z}/2; K) \) is injective. On the other hand, \( H^1(\mathbb{Z}/2; D) \to H^1(\mathbb{Z}/2; H_2(G \times G)) \) is surjective, since obviously \( H_2(G \times G)/D \) is free over \( \Lambda \). Hence \( H^1(\mathbb{Z}/2; H_2(G \times G)) \to H^1(\mathbb{Z}/2; K) \) is injective. Thus \( H^1(\mathbb{Z}/2; H_2(\hat{\Theta}_0)) \) is trivial. Consequently, \( H^1(\mathbb{Z}/2; H_2(\hat{\Theta}, \nu)) \) is trivial. The assertion now follows from Lemma 2.2. □

(b). The *intersection graph* of \( \Theta \) has a vertex for every chord of \( \Theta \), and two vertices are connected by an edge iff the corresponding chords are linked. It is well-known and easy to see that \( \Theta \) is irreducible iff its intersection graph is connected.

Let \( \alpha = \{x, y\} \) and \( \beta = \{x', y'\} \) be a pair of linked chords of \( \Theta \) with some orientations. Let us denote the union of the edges of \( \alpha^+ \) and \( \beta^+ \) by \( e_{\alpha^+ \beta^+} \). Then \( \alpha^+ \times \beta^+ - \beta^+ \times \alpha^+ \) is a \( \mathbb{Z}/2 \)-invariant cycle of \( G \times G \), which in fact lies in \( \hat{\Theta}_0 \).\(^{22}\) This cycle gives rise to an integral 2-chain \( R_{\alpha^+ \beta^+} \) in \( \hat{\Theta} \) with \( \partial R_{\alpha^+ \beta^+} = \pm c_{\alpha} \pm c_{\beta} + 2 \sum \pm c_{\gamma_i} \), where \( \gamma_i = \{x_i, y_i\} \) are chords of \( \Theta \) with \( (x_i, y_i) \) “lying” in the support of \( R_{\alpha^+ \beta^+} \), and \( c_{\alpha} \) denotes the central curve of the Möbius band corresponding to \( \alpha \) (with some choice of orientation). Then each \( \gamma_i \) is linked with either \( \alpha \) or \( \beta \) (or both), and if \( \gamma_i \) is linked with \( \alpha \) (resp. \( \beta \)), then \( e_{\alpha^+ \beta^+} \) contains \( e_{\gamma_i^+ \alpha^+} \) (resp.

\(^{22}\)Every such cycle reduced mod 2 gives rise to a \( \mathbb{Z}/2 \)-valued type 1 invariant that does not lift to an integral-valued one.
\(e_{\alpha^{+}+\beta^{+}}\) for a unique orientation of \(\gamma_i\). Inducting on the number of edges in \(e_{\alpha^{+}+\beta^{+}}\), we obtain an equation of the form \(k[c_{\alpha}] = l[c_{\beta}]\), where \(k\) and \(l\) are odd integers. Working modulo odd torsion, we may assume that \(\gcd(k,l) = 1\), and the assertion follows. □

**Remark.** It is easy to see that a choice of orientations of \(M\) and of a chord \(\alpha\) determines an orientation of \(c_{\alpha}\). Furthermore, if all \(\gamma_i\) are oriented as above, then it is not hard to see that all signs in the formula for \(\partial R_{\alpha^{+}+\beta^{+}}\) are positive. According to the assertion of [Va2; Lemma 6], it can be deduced from this that \(k, l = \pm 1\), in which case \(\ker[H_1(\Theta) \rightarrow H_1(\Theta, \mu)]\) contains no odd torsion.

(c) Clearly, \([c_{\alpha}] \sim e(\lambda_p) \neq 0\), where \(c_{\alpha}\) is the central curve of the M"obius band corresponding to \(\alpha\). Given a planar \(\Theta\)-knot \(k: G \hookrightarrow \mathbb{R}^2\), it follows that the Gauss map \(\tilde{k}\) sends \(c_{\alpha}\) with nonzero degree to \(\mathbb{RP}^1\). Hence no multiple of \([c_{\alpha}]\) can be trivial in \(H_1(\tilde{\Theta})\).

By (a), the image of \(H_2(\tilde{\Theta}; \mathbb{Z}_p) \xrightarrow{e(\lambda_p)} H_1(\tilde{\Theta})\) is contained in the kernel of \(H_1(\tilde{\Theta}) \rightarrow H_1(\Theta, \mu)\). The latter kernel contains no elements of order \(2\) by the above. However, \(2e(\lambda_p) = 0\) as \(\lambda_p\) is induced from the tautological line bundle over \(\mathbb{RP}^\infty\). □

**Corollary 5.8** [Va2; Theorem 5(2)]. If \(\Theta\) is planar, every integer-valued type \(1\) invariant of \(\Theta\)-knots admits an integral arrow diagram formula.

**Proof.** If \(\Theta\) has irreducible factors \(\Theta^i\), we can pick embeddings \(e_i: M/\Theta^i \rightarrow M/\Theta\) respecting the smooth structure at each crossing of \(M/\Theta^i\), and thus obtain embeddings \(e_{i*}: H_2(\tilde{\Theta}^i; \mathbb{Z}_p) \rightarrow H_2(\tilde{\Theta}; \mathbb{Z}_p)\). By the proof of Lemma 5.1, this yields embeddings \(E_i: H_2(\tilde{\Theta}^i; \mathbb{Z}_p) \rightarrow H_2(\tilde{\Theta}; \mathbb{Z}_p)\). By the proof of Corollary 5.4(b), \(\Gamma_1(\Theta)\) is generated by \(E_i(\Gamma_1(\Theta^i))\) and by the classes of \(v(k) = \text{lk}(k(\alpha_j^+,k(\beta_j^+)))\), where \((\alpha_j,\beta_j)\) runs over \(\Theta^k \times \Theta^l\) for all \((k,l)\) with \(k > l\), and each \(\alpha_j\) and \(\beta_j\) are oriented so that \(\alpha_j^+ \cap \beta_j^+ = \emptyset\). Obviously (see e.g. the proof of Theorem 5.6), every such \(v\) has an integral arrow diagram formula. On the other hand, by Theorem 5.6 and Lemma 5.7(c), each \(w \in \Gamma_1(\Theta^i) = H_2(\Theta^i; \mathbb{Z}_p)\) has an arrow diagram formula. Then again by Theorem 5.6, \(E_i(w)\) also has an arrow diagram formula. □

**Lemma 5.9.** (a) [Va2; Theorem 7(1)] If \(v\) is a type \(1\) invariant of \(\Theta\)-knots and \(R\) is a reflection of \(\mathbb{R}^3\), then \(v(k) + v(Rk)\) is a type \(0\) invariant of \(\Theta\)-knots.

(b) [Va2; Theorem 7(2)] Its parity \(\alpha (v)\) obstructs existence of an integral arrow diagram formula for the integer-valued invariant \(v\).

In particular, \(\alpha \ (v^{(m)})\) is an obstruction to existence of integral Polyak–Viro formulas for the type \(m + 1\) knot invariant \(v\).

**Proof.** (a). \(v'(k_s) = -v'(Rk_s)\) for every singular knot \(k_s\). □

(b). If the reflection \(R\) is taken to be in a vertical plane, \(A(Rk) = -A(k)\) for every one-arrow diagram function \(A\) (only the signs of the frames are changed). □

**Theorem 5.10.** If \(M = S^1\) and \(v\) is an integer-valued type \(1\) invariant of \(\Theta\)-knots, \(\alpha (v) = [z_v] \sim e(\lambda_p)^2\).

**Proof.** If \(h\) is an isotopy with values in \(\mathbb{R}^4\) between \(\Theta\)-knots \(k_1\) and \(k_2\), we have \(2\zeta(k_1,k_2) = 2t^{-1}h^*(\chi) = t^{-1}h^*(\delta^*\psi) = \tilde{k}_2^*(\psi) - \tilde{k}_1^*(\psi)\), where \(\psi = p_i\varphi\) generates
Let \( a, b : S^1 \to G \) be a pair of oriented cycles in \( M/R_\Theta \) with no common edges. A type one invariant \( v_{ab} \) of \( \Theta \)-knots \( k : M/R_\Theta \hookrightarrow \mathbb{R}^3 \) is given by the linking number between \( C^1 \)-close pushoffs \( ka_\ast \) and \( kb_\ast \) of \( ka \) and \( kb \), which are chosen as follows. In those common vertices of \( a(S^1) \) and \( b(S^1) \) where \( a \) and \( b \) have corners, we smoothen the corners of \( ka \) and \( kb \), and in each vertex \( a(x) = b(y) \) of transversal intersection between \( a \) and \( b \), we slide \( ka \) and \( kb \) away from each other so that \( (ka_\ast(x), kb_\ast(y), ka_\ast(x) - kb_\ast(y)) \) form a positive frame.

Let \( a, b : S^1 \to G \) be a pair of oriented graph-theoretic cycles in \( M/R_\Theta \) with disjoint edges. We call them a Manturov pair if they have precisely one transversal self-intersection. (They may have any number of common vertices where each has a corner.) Clearly, \( v_{ab}(Rk) = \pm 1 - v_{ab}(k) \), so \( \propto (v_{ab}) \) is nontrivial.

**Example 5.11.** Let \( \Theta_0 \) be the unique irreducible diagram with two chords on \( M = S^1 \). Both basic type 1 invariants of \( \Theta_0 \)-knots, which are yielded by the two possible choices of Manturov cycles, have nonzero \( \propto \). However the basic type 3 invariant \( v_3 \) of genuine knots has \( v'_3 \) equal to the sum of these two, so \( \propto (v'_3) = 0 \). In fact, \( v_3 \) is known to have an integral Polyak–Viro formula.

Remark. It is computed in [Va2; Example 5] that one of the three basic invariants of knots of type 4 satisfies \( \propto (v''_4) \neq 0 \). In fact, \( v''_4 = v_{ab} \) for a certain Manturov pair of cycles \( a, b \). Thus \( v''_4 \) admits no integral arrow diagram formula, and in particular \( v_4 \) admits no integral Polyak–Viro formula.

Obviously, if \( M/\Theta \) admits a pair of Manturov cycles, \( \Theta \) cannot be planar. Verifying Vassiliev’s earlier conjecture, Manturov proved the converse, by a nontrivial combinatorial argument.

**Manturov’s Criterion 5.12 [Man].** Let \( M = S^1 \). Then \( \Theta \) is non-planar if and only if \( M/R_\Theta \) contains a Manturov pair of cycles.

The van Kampen obstruction \( \zeta(\Theta) \in H^2(\tilde{\Theta}) \) to planarity of \( \Theta \) is induced from the generator of \( H^2(\mathbb{R}P^\infty) \) by a classifying map of the 2-covering \( p : \tilde{\Theta} \to \Theta \). Equivalently, \( \zeta(\Theta) = e(\lambda_p)^2 \), where \( e(\lambda_p) \in H^1(\Theta; \mathbb{Z}_p) \) is the Euler class of the line bundle associated with the 2-covering \( p : \tilde{\Theta} \to \Theta \).

**Corollary 5.13.** Let \( M = S^1 \). Then \( \Theta \) is planar iff \( \zeta(\Theta) = 0 \).

*Proof.* If \( k \) is a planar \( \Theta \)-knot, \( \tilde{k} : \tilde{\Theta} \to \mathbb{R}P^1 \subset \mathbb{R}P^\infty \) classifies \( p : \tilde{\Theta} \to \Theta \), hence \( \zeta(\Theta) = 0 \). If \( \Theta \) is non-planar, let \( v = v_{ab} \), where \( a, b \) is a Manturov pair of cycles. By Theorem 5.10 we have \( \propto (v) \neq 0 \), hence \( \zeta(\Theta) \neq 0 \). □
Corollary 5.14 [Va2; Theorem 6]. Let \( M = S^1 \). If \( \Theta \) is non-planar, there exists an integer-valued type 1 invariant \( v \) of \( \Theta \)-knots that admits no integral arrow diagram formula.

Proof. Let \( v = v_{ab} \), where \( a, b \) is a Manturov pair. Then \( \alpha(v) \neq 0 \). \( \square \)

Lemma 5.15. If \( M = S^1 \) and \( \Theta \) is irreducible and non-planar, the cyclic group from Lemma 5.7(b) is isomorphic to \( \mathbb{Z}/2 \).

Proof. By Manturov’s Criterion 5.12, there is a Manturov pair \( a, b \), and therefore a map \( \Theta^1 \to \Theta \), where \( \Theta^1 \) consists of a single chord \( \beta \) on \( S^1 \sqcup S^1 \), with endpoints at distinct components. The component of \( \Theta_{1s} \subset \Theta^1 \) containing the central curve \( c_\beta \) of the Möbius band corresponding to \( \beta \) is homeomorphic to the blowup of \( S^1 \times S^1 \) at a point. Hence \( 2c_\beta \) bounds a punctured torus and \( 2[c_\beta] = 0 \). Then \( 2[c_\alpha] = 0 \in H_1(\Theta) \), where \( \alpha \) is the transversal intersection of \( a \) and \( b \). \( \square \)

The following result was announced by Vassiliev [Va2; Theorem 7(3)], from whom the author learned that his proof of this fact was inadvertently omitted from [Va2]. The following proof was found by the author [M3].

Corollary 5.16. Suppose that \( M = S^1 \) and \( \Theta \) is irreducible. If \( v \) is an integer-valued type 1 invariant of \( \Theta \)-knots and \( \alpha(v) = 0 \in \mathbb{Z}/2 \), then \( v \) admits an integral arrow diagram formula.

Proof. By Theorem 5.10, \( [z_v] \wedge e(\lambda_p)^2 = 0 \). If \( \Theta \) is planar, we are done by Corollary 5.8. Otherwise Lemma 5.15 gives an explicit generator \( [c_\alpha] \) of \( H_1(\Theta) \cong \mathbb{Z}/2 \). Clearly, \( [c_\alpha] \wedge e(\lambda_p) \neq 0 \). Hence already \( [z_v] \wedge e(\lambda_p) = 0 \). By Theorem 5.6, \( v \) admits an integral arrow diagram formula. \( \square \)

6. Extraordinary van Kampen obstruction

Equivariant stable cohomotopy. Let \( K \) be a \( k \)-polyhedron, homotopy equivalent to a compact polyhedron. It is well known that if \( k \leq 2m - 2 \), the cohomotopy set \( \pi^m(K) := [K, S^m] \) is an abelian group and \( \Sigma: \pi^m(K) \to \pi^{m+1}(\Sigma K) \) is an isomorphism. Hence the stable cohomotopy group \( \omega^m(K) := [\Sigma^\infty K, S^{m+\infty}] = [S^\infty \wedge K, S^{m+\infty}] \) is well-defined, where \( \infty \) denotes a sufficiently large natural number, and \( m \) may be negative. These groups form a generalized cohomology theory.

Throughout this section, \( * \) will stand for the basepoint, and if \( K \) is an unpointed space, \( K_+ \) will denote the pointed space \( K \sqcup * \). We recall that the smash product of pointed spaces \( P \wedge Q = P \times Q/P \vee Q \). Note that when \( P \) and \( Q \) are compact and not of the form \( K_+ \), then \( P \wedge Q \) is the one-point compactification by \( * \) of \( (P \setminus *) \times (Q \setminus *) \).

Now suppose that \( P \) is pointed and \( G \) is a finite group acting on \( P \) and fixing the basepoint. We also assume that \( P \) is \( G \)-homotopy equivalent to a compact polyhedron. If \( V \) is a finite-dimensional \( \mathbb{R}G \)-module, let \( S^V \) be the one-point compactification of the Euclidean space \( V \) with the obvious action of \( G \). If \( G \) acts trivially on \( V \), one identifies \( V \) with the integer dim \( V \). The equivariant stable cohomotopy group

\[
\omega^V_{G}(P) := [S^{W + V_\infty} \wedge P, S^{W + V_\infty}^V]_G
\]

is well-defined, where \( V_\infty \) denotes a sufficiently large (with respect to the partial ordering with respect to inclusion) finite-dimensional \( \mathbb{R}G \)-submodule of the countable
direct sum \( \mathbb{R}G \oplus \mathbb{R}G \oplus \ldots \) [M+; p. 81] (see also [Ada], [Ko], [Hau]). As emphasized in [M+; IX.5 and XIII.1] and explained in [M+; pp. 101–102], one should not think of \( V, W \) and \( V_\infty \) as abstract \( \mathbb{R}G \)-modules but rather as specific submodules of \( \mathbb{R}G \oplus \mathbb{R}G \oplus \ldots \), in the same vein as \( \pi_1(X) \) is technically \( \pi_1(X,*) \) if we care not only for the groups but also for their homomorphisms. However for our purposes it will be safe enough to ignore this point, as is indeed done in later chapters of [M+], so that \( \omega^*_G(P) \) is effectively graded by the real representation ring of \( G \).

As explained in [M+; p. 100], one cannot take \( V_\infty \) to be any \( \mathbb{R}G \)-module of sufficiently large dimension, even in the case \( G = \mathbb{Z}/2 \). However:

**Theorem 6.1.** [Hau] (cf. [M+; IX.1.4], [Ada; 3.3]) \([P,Q]_G \rightarrow [S^V \wedge P,S^V \wedge Q]_G \) is surjective if \( \dim P_H \leq \nu_H \) for all \( H \subseteq G \) and injective if \( \dim P_H \leq \nu_H - 1 \) for all \( H \subseteq G \), where \( \nu_H \) is the maximal integer such that

1. either \( V^H = 0 \) or \( \pi_i(Q^H) = 0 \) for all \( i \) with \( 2i + 1 \leq \nu_H \); and
2. either \( V^K = V^H \) or \( \pi_i(Q^K) = 0 \) for all \( i \leq \nu_H \) and all \( K \subseteq H \).

Let \( kT + l \) denote \( \mathbb{R}^k \times \mathbb{R}^l \) with the involution \( (x,y) \mapsto (-x,y) \); every \( \mathbb{R}[\mathbb{Z}/2] \)-module is of this form.

**Corollary 6.2.** \([P,S^{mT}]_{/2} = \omega^m_{/2}(P) \) if \( \dim P \leq 2m - 2 \) and \( \dim P_{/2} \leq m - 2 \).

**Extraordinary van Kampen obstruction.** As in §3, by \( S^k \) we denote the \( k \)-sphere with the antipodal involution, i.e. the unit sphere in \((k+1)T\). For \( k \geq m \), we have that \( S^k \setminus S^{m-1} \) is \( \mathbb{Z}/2 \)-homeomorphic to \( S^{k-m} \times mT \). Shrinking to points \( S^{k-m} \) and each fiber \( S^{k-m} \times \{pt\} \), we get an equivariant map \( \rho^k_m : S^k_{/2} \rightarrow S^{mT} \).

For any \( k \)-polyhedron \( K \) with a free PL action of \( \mathbb{Z}/2 \) we have an equivariant map \( \varphi_K : K \rightarrow S^\infty \), which is unique up to equivariant homotopy. Here \( \infty \) may be thought of as a sufficiently large natural number (specifically, \( k + 1 \) will do). Given a compact \( n \)-polyhedron \( X \) and a positive integer \( m \geq n + 2 \), we let

\[ \Theta^m(X) := [\rho^\infty_m \varphi_X] \in [\tilde{X}+,S^{mT}]_{/2} = \omega^m_{/2}(\tilde{X}+) \]

**Remark.** The geometric description of ordinary cohomology given in §2 extends to an arbitrary equivariant (with respect to a finite group) generalized cohomology theory \([\mathcal{B},\mathcal{R},\mathcal{S}]\). In particular, \( \omega^m_{/2}(\tilde{X}+) \) is identified with the cobordism group of \( \mathbb{Z}^{\otimes m}_p \)-co-oriented \( m\lambda_p \)-framed \( m \)-comanifolds in \( \tilde{X} \), where “comanifold” is our synonym for “mock bundle” from \([\mathcal{B},\mathcal{R},\mathcal{S}]\), and an \( m\lambda_p \)-framing of an \( m \)-comanifold is an isomorphism of its stable normal block bundle with the sum of \( m \) copies of the line bundle \( \lambda_p \) (associated to the double covering \( p : \tilde{X} \rightarrow X \)).

23 Similarly to the geometric interpretation of \( \Theta(X) \), it is easy to see that if \( f : X \rightarrow \mathbb{R}^m \) is a generic PL map that lifts to an embedding \( g : X \hookrightarrow \mathbb{R}^{2n+1} \), then \( \Theta^m(X) \) is represented by the \( m \)-comanifold \( \Delta_f / t = \{ [x,y] \in \tilde{X} \mid f(x) = f(y) \} \) in \( \tilde{X} \), which is \( \mathbb{Z}^{\otimes m}_p \)-co-oriented and \( m \)-framed as the preimage of \( \mathbb{R}P^{2n-m} \) under \( \tilde{g} : \tilde{X} \rightarrow \mathbb{R}P^{2n} \). We also have \( \Theta^m(X) = E(\lambda_p)^m \), where the equivariant Euler class \( E(\lambda_p) = i^*i!(\text{id}_\tilde{X}) \) as in §2.

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23 Let us indicate a relation with a more traditional approach. Let us consider the cobordism group of \( \mathbb{Z}^{\otimes m}_p \)-co-oriented \( m\lambda \)-framed \( m \)-comanifolds in \( \tilde{X} \), where \( \lambda \) is not globally defined, but is a part of the data of the \( m\lambda \)-comanifold. It is well-known that this group is isomorphic to \( [S^{\infty} \times \tilde{X} ; S^{\infty} \times (\mathbb{R}P^{\infty}/\mathbb{R}P^{\nu-1})] \) (see [AE]). Here \( \mathbb{R}P^{\infty}/\mathbb{R}P^{\nu-1} \) is the Thom space of the bundle \( q \gamma \) over \( \mathbb{R}P^{\infty-q} \), so the point-inverse \( Q \) of the basepoint of this space is \( q\lambda \)-framed in \( \tilde{X} \), where \( \lambda \) is the pullback of \( \gamma \) under the map \( Q \rightarrow \mathbb{R}P^{\infty-q} \).
**Theorem 6.3.** Let $X$ be a compact $n$-polyhedron and $m \geq \frac{3(n+1)}{2}$. Then $X$ embeds in $\mathbb{R}^m$ if and only if $\Theta^m(X) = 0$.

This follows from the Haefliger–Weber Criterion 3.1 and

**Lemma 6.4.** Let $\mathbb{Z}/2$ act freely on a $k$-polyhedron $K$ and let $k \leq 2m - 3$. There exists an equivariant map $K \to S^{m-1}_K$ if and only if $\rho^\infty\varphi_K: K \to S^{mT}_K$ is equivariantly null-homotopic.

**Proof.** Let $S_m$ denote the “cosphere” $S^\infty_m/S^\infty_{m-1}$ with basepoint at the shrunk $S^{m-1}_m$. Then $\rho^\infty$ is the composition $S^\infty_m \xrightarrow{\rho} S_m \xrightarrow{p} S^{mT}_K$, where $q$ shrinks $S^{m-1}_m$ and $p$ shrinks all fibers $S^\infty_{m-1} \times \{pt\}$ of its complement $S^\infty_{m-1} \times mT$. All point-inverses of $p$ are weakly contractible, and it follows that $p$ is a fibration. Since $p$ is also equivariant, the given equivariant null-homotopy $K \times I \to S^{mT}_K$ lifts to an equivariant homotopy $F: K \times I \to S_m$, which has to be a null-homotopy. We are interested in equivariantly lifting it to $S^\infty_m$. It certainly does lift over $K \times 0$ and over $A := F^{-1}(S^\infty_{m-1})$, where $S^\infty_{m-1}$ is Hopf linked with the shrunk $S^{m-1}_m$. Here $A$ may be assumed to be a $(k + 1 - m)$-polyhedron (after a small perturbation of $F$). Moreover, if $U$ is a $\mathbb{Z}/2$-invariant tubular neighborhood of $S^\infty_{m-1}$ in the complement of the shrunk $S^{m-1}_m$, then $F$ also lifts over $A^+ := K \times 0 \cup F^{-1}(U)$. Next we would like to extend this partial lift $A^+ \to S^\infty_m$ over the shadow of $A$ in $K \times I$, which is a $(k - m + 2)$-polyhedron $B$. The obstructions to this relative equivariant lifting problem lie in the groups $H^i_{\mathbb{Z}/2}(B, A; \pi_{i-1}(S^\infty_{m-1}))$, which are all zero for $i \leq k - m + 2$ since $k - m + 2 \leq m - 1$ by the hypothesis. The obtained partial lift $B \cup A^+ \to S^\infty_m$ of $F$ sends the frontier of $B \cup A^+$ in $K \times I$ into the complement of $S^\infty_{m-1}$. Since $K \times I$ collapses onto $B \cup A^+$, this lift can be extended to an equivariant map $K \times I \to S^\infty_m$ sending the exterior of $B \cup A^+$ into the complement of $S^\infty_{m-1}$. In particular, we get an equivariant map of $K \times \{1\}$ into $S^\infty_m \setminus S^\infty_{m-1}$, which is equivariantly homotopy equivalent to $S^{m-1}_m$. □

**Remark.** By the covering theory, there exists an equivariant map $K \to S^{m-1}_K$ iff there exists a map $K/t \to \mathbb{R}^{Pm-1}$ whose composition with $\mathbb{R}^{Pm-1} \subset \mathbb{R}^{P\infty}$ is up to homotopy the classifying map $\psi_p$ of the line bundle associated to the 2-covering $p: K \to K/t$. As shown by the following example, this is not equivalent to null-homotopy of the composition $K/t \xrightarrow{\psi_p} \mathbb{R}^{P\infty} \to P_m$, where $P_m = \mathbb{R}^{P\infty}/\mathbb{R}^{Pm-1}$. The proof of Lemma 6.4 breaks down in this setting since $\pi_1(\mathbb{R}^{Pm-1})$ is nontrivial as opposed to $\pi_1(S^{m-1}_m)$.

**Example 6.5.** (P. M. Akhmetiev) Let $K/t$ be obtained from $\mathbb{R}P^{2n}$ by attaching a $(2n+1)$-cell along the composition $S^{2n} \to S^{2n} \to \mathbb{R}P^{2n}$ of a degree 2 map and the 2-covering. Let $p: K \to K/t$ be the universal covering. Then $\psi^p_p: H^{2n+1}(\mathbb{R}P^{2n}; \mathbb{Z}_p) \to H^{2n+1}(K/t; \mathbb{Z}_p)$ is easily seen to be the inclusion $\mathbb{Z}/2 \subset \mathbb{Z}/4$, whence $\psi_p$ is not homotopic to a map into $\mathbb{R}P^{2n}$. On the other hand, $K/t \xrightarrow{\psi_p} \mathbb{R}^{P\infty} \to P_{2n+1}$ may be assumed to shrink $\mathbb{R}P^{2n}$ to the basepoint. Since $\pi_{2n+1}(P_{2n+1}) \simeq H_{2n+1}(P_{2n+1}) \simeq \mathbb{Z}/2$, this composition is null-homotopic.

Nevertheless, $K \xrightarrow{\varphi_K} S^{\infty}_{2n+1} \to S_{2n+1}$ is not equivariantly null-homotopic since it sends $H^{2n+1}_{\mathbb{Z}/2}(S_{2n+1}; I) \simeq \mathbb{Z}$ onto the even subgroup of $H^{2n+1}_{\mathbb{Z}/2}(K; I) \simeq \mathbb{Z}/4$. This is not surprising as $S_{2n+1} \to P_{2n+1}$ is not a fibration.
\(\Theta^m(X)\) as cohomotopy Euler class. Let \(\mathbb{Z}/2\) act freely on a \(k\)-polyhedron \(K\) with quotient \(K/t\). Consider the vector bundle \(\xi: (K \times m\mathcal{T})/t \to K/t\). Let \(\xi^{-1}\) be a vector bundle over \(K/t\) such that \(\xi \oplus \xi^{-1}\) is a trivial bundle of dimension \(M\) say. Let \(\tilde{\xi}^{-1}\) be the pullback of \(\xi^{-1}\) over \(K\). Recall that the Thom space \(T\xi\) is the one-point compactification of the total space of \(\xi\). We have \(T\tilde{\xi}^{-1} \wedge S^{mT} = T(K \times M) = K_+ \wedge S^M\). Therefore

\[
\omega_{\mathbb{Z}/2}^{mT}(K_+) = [K_+ \wedge S^M, S^{mT + M}]_{\mathbb{Z}/2} \simeq [T\tilde{\xi}^{-1}, S^M]_{\mathbb{Z}/2} = \omega_{\mathbb{Z}/2}^M(T\tilde{\xi}^{-1}) \simeq \omega^M(T\xi^{-1}).
\]

(See also [Be] for a generalization of this isomorphism to the case of actions with fixed points.)

In particular, this isomorphism identifies the extraordinary van Kampen obstruction with an element of a non-equivariant stable cohomotopy group. This element is well-known in the literature as the cohomotopy Euler class of \(\xi\) (see [St]). It can alternatively be defined as the class of the composition \(T\xi^{-1} \circ T(\xi \oplus \xi^{-1}) = (K/t)_+ \wedge S^M \to S^M\). If \(\xi\) admits a nowhere vanishing cross-section, the inclusion \(K/t \subset T\xi\) is null-homotopic, hence so is the inclusion \(j\) in the above composition. The converse is known to hold when \(k \leq 2m - 3\) [Cr; Prop. 2.4] (see also [Mey] for a nontrivial modp generalization). This is in accordance with our Lemma 6.4, for it is easy to see that \(\xi\) admits a nowhere vanishing cross-section if and only if \(K\) admits an equivariant map to \(S^{m-1}\).

Hurewicz homomorphism. Let \(\mathbb{Z}/2\) act freely on a \(k\)-polyhedron \(K\) with quotient \(K/t\). Let us define \(h: \omega_{\mathbb{Z}/2}^{mT}(K_+) \to H_{\mathbb{Z}/2}^{m}(K; I^{\otimes m})\) by \(h([\varphi]) = \varphi^* (\xi^m)\), where \(\xi^m \in H_{\mathbb{Z}/2}^{m}(S^{mT}; \pi_m(S^{mT})) \simeq \mathbb{Z}/2\) is the generator. Clearly \(h(\Theta^m(X))\) is the \(m\)th power of the Euler class of the line bundle associated with the 2-covering \(\tilde{X} \to \tilde{X}\), in particular \(h(\Theta^{2n}(X)) = \vartheta(X)\), under the isomorphism \(H_{\mathbb{Z}/2}^{m}(\tilde{X}; I^{\otimes m}) \simeq H^m(\tilde{X}; \mathbb{Z}^{\otimes m})\).

One also has the natural transformation \(H: \omega_{\mathbb{Z}/2}^{mT}(K_+) \to H_{\mathbb{Z}/2}^{mT}(K_+)\) into the equivariant (representation graded) ordinary cohomology theory [M+], [Ko]. We claim that \(H = h\). Indeed, the above proof that \(\omega_{\mathbb{Z}/2}^{mT}(K_+) \simeq \omega^M(T\xi^{-1})\) works with any equivariant cohomology theory, in particular \(H_{\mathbb{Z}/2}^{mT}(K_+) \simeq H^M(T\xi^{-1})\). On the other hand, using a twisted Thom isomorphism, \(H^M(T\xi^{-1}) \simeq H^m(K/t; \mathbb{Z}^{\otimes m})\).

Classification of embeddings. An isotopy classification of embeddings in the metastable range is given by a generalized cohomotopy:

**Theorem 6.6.** If a compact \(n\)-polyhedron \(X\) embeds in \(\mathbb{R}^m\) and \(m > \frac{3(n+1)}{2}\), the set of isotopy classes of embeddings of \(X\) into \(\mathbb{R}^m\) is in bijection with \(\omega_{\mathbb{Z}/2}^{mT-1}(\tilde{X}_+)\).

Note that \(\omega_{\mathbb{Z}/2}^{mT-1}(\tilde{X}_+) \simeq \omega^M-1(T\xi^{-1})\) similarly to the above.

**Proof.** By Corollary 6.2, \([\tilde{X}_+ \wedge S^1, S^{mT}] \simeq \omega_{\mathbb{Z}/2}^{mT}(\tilde{X}_+ \wedge S^1) \simeq \omega_{\mathbb{Z}/2}^{mT}(\tilde{X}_+)\). There is an equivariant map \(\varphi: \tilde{X} \times I \to \tilde{X}_+ \wedge S^1\) sending \(\tilde{X} \times \partial I\) to the basepoint, where \(\mathbb{Z}/2\) acts trivially on \(I\). So any \(\alpha: \tilde{X}_+ \wedge S^1 \to S^{mT}\), precomposed with \(\varphi\), is an equivariant homotopy between two constant maps \(\tilde{X} \to \ast \subset S^{mT}\). Given an embedding \(g_0: X \hookrightarrow \mathbb{R}^m\), one of these constant maps lifts to \(\tilde{g}_0: \tilde{X} \to S^{m-1}\subset S^\infty\).

By Lemma 6.7(b) below, the null-homotopy \(\alpha\varphi\) of this constant map corresponds to an equivariant homotopy \(\tilde{X} \times I \to S^\infty\) between \(\tilde{g}_0\) and some map \(G: \tilde{X} \to S^{m-1}\).
By the Haefliger–Weber Criterion 3.1(a), $G$ is equivariantly homotopic within $S^{m-1}$ to $\tilde{g}_a$ for some embedding $g_\alpha : X \hookrightarrow \mathbb{R}^m$. If $\alpha'$ is homotopic to $\alpha$, the Haefliger–Weber Criterion 3.1(b) along with Lemma 6.7(b) imply that $g_{\alpha'}$ is isotopic to $g_\alpha$.

Conversely, given any embedding $g : X \hookrightarrow \mathbb{R}^m$, $\tilde{g}_0$ and $\tilde{g}$ are equivariantly homotopic with values in $S^\infty$. Projecting this homotopy to $S^{mT}$ we get a map $\bar{X} \times I \rightarrow S^{mT}$ that factors into the composition of $\varphi$ and a map $\alpha_g : \bar{X} \wedge S^1 \rightarrow S^{mT}$. If $g'$ is isotopic to $g$, then $\tilde{g}'$ and $\tilde{g}$ are equivariantly homotopic with values in $S^{m-1}$, therefore $\alpha_{g'}$ coincides with $\alpha_g$. □

If $f,g : K \rightarrow S^{m-1}$ are equivariant maps of an $k$-polyhedron with a free $\mathbb{Z}/2$ action, they are related by an equivariant homotopy $\varphi_{f,g} : K \times I \rightarrow S^\infty$, which is unique up to equivariant homotopy rel $K \times \partial I$.

**Lemma 6.7.** Let $\mathbb{Z}/2$ act freely on a $k$-polyhedron $K$ and trivially on $I = [0,1]$.

(a) Suppose $k \leq 2m - 4$. Equivariant maps $f,g : K \rightarrow S^{m-1}$ are equivariantly homotopic iff $\rho^\infty_m \varphi_{f,g} : K \times I \rightarrow S^{mT}$ is equivariantly null-homotopic rel $K \times \partial I$.

(b) Suppose $k \leq 2m - 3$. For any equivariant map $f : K \rightarrow S^{m-1}$ and any equivariant self-homotopy $H : K \times I \rightarrow S^{mT}$ of the constant map $K \rightarrow * \subset S^{mT}$ there exists an equivariant map $g : K \rightarrow S^{m-1}$ such that $\rho^\infty_m \varphi_{f,g}$ and $H$ are equivariantly homotopic rel $K \times \partial I$.

The proof is similar to that of Lemma 6.4.

**Theorem 6.8.** Let $X$ be a compact $n$-polyhedron, and let $S = N \setminus \Delta_X$, where $N$ is the second derived neighborhood of $\Delta_X$ in a $\mathbb{Z}/2$-invariant triangulation of $X \times X$.

If $m \geq \frac{3(n+1)}{2}$, then $X$ immerses in $\mathbb{R}^m$ if and only if the image of $\Theta^m(X)$ in $\omega^m_{\mathbb{Z}/2} (S_+)$ is trivial.

If $m > \frac{3(n+1)}{2}$ and $X$ immerses in $\mathbb{R}^m$, then the set of regular isotopy classes of immersions of $X$ into $\mathbb{R}^m$ is in bijection with $\omega^m_{\mathbb{Z}/2} (S_+)$.

The proof repeats those of Theorems 6.3 and 6.6, if instead of the Haefliger–Weber Criterion 3.1 one uses its analogue for immersions [Har; Theorem 2 and footnote on p. 3]. If $X$ is a closed smooth manifold, the dimensional restrictions in Theorem 6.8 can be weakened to $m \geq \frac{3n+1}{2}$ and $m > \frac{3n+1}{2}$, respectively, by using [HH] instead of [Har].

**Remark.** If $X$ is a closed smooth $n$-manifold and $S$ is as in Theorem 6.8, then

$$\text{secat}_{\mathbb{Z}/2} (S) = \text{secat}_{\mathbb{Z}/2} (\tilde{X}) - 1 = n + \nu (X),$$

where $\nu (X)$ is the largest number $k$ such that the normal Stiefel–Whitney class $w_k (X)$ does not vanish [CF; 6.6] (compare [Mc1], [De]), [Wu], [Wu'], [Sch; Ch. VII, §§2-3]. On the other hand, by a well-known result of Massey, $\nu (X) \leq n - \alpha (n)$, where $\alpha (n)$ is the number of ones in the dyadic expansion of $n$. In view of Theorems 6.3 and 6.8, it would be of great interest to find extraordinary (equivariant stable cohomotopy) versions of these results if they exist. (Note that the case of unoriented cobordism should be easy to check in view of the literature on tom Dieck operations and Conner–Floyd characteristic classes, see in particular [Mc1], [BRS], [De].) A well-known old conjecture says that $X$ always embeds in $\mathbb{R}^{2n - \alpha (n)+1}$ and immerses in $\mathbb{R}^{2n - \alpha (n)}$; a difficult proof of the immersion conjecture has been published by R. Cohen in mid-80s, while the embedding conjecture remains open.
Problem 6.9. Can the geometric interpretation of the extraordinary van Kampen obstruction be used to obtain a polyhedral version of Haefliger’s generalized Whitney trick, thus proving Criterion 3.1 without induction on simplices?

7. Embeddability versus disjoinability

If \( f : X \to \mathbb{R}^m \) is a (continuous) map, we write \( \Delta_f = \{(x,y) \in \tilde{X} \mid f(x) = f(y)\} \) and \( \Delta_f^j = \{(x,y) \in \tilde{X} \mid ||f(x) - f(y)|| < \varepsilon\} \).

Approximability Criterion 7.1. Let \( X \) be a compact \( n \)-polyhedron, suppose that \( m \geq \frac{3(n+1)}{2} \), and let \( f : X \to \mathbb{R}^m \) a continuous map.

(a) \( f \) is \( C^0 \)-approximable by embeddings iff \( \Theta(f) = 0 \in \lim_{\leftarrow} \omega_{\mathbb{Z}/2}^n(\tilde{X}_+, \tilde{X}_+ \setminus \Delta_f^j) \).

(b) For each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( f \) is \( C^0,\varepsilon \)-approximable by an embedding if \( \Theta_{\varepsilon}(f) = 0 \in \omega_{\mathbb{Z}/2}^n(\tilde{X}_+, \tilde{X}_+ \setminus \Delta_f^j) \) and only if \( \Theta_{\varepsilon}(f) = 0 \).

The definition of the controlled extraordinary van Kampen obstruction \( \Theta_{\varepsilon}(f) \) and the proof of (b) is analogous to [M2; §3] (see also [Ah]), taking into account the controlled version of the Haefliger–Weber Criterion [RS]. \( \hat{\Theta}(f) \) is the thread of the elements \( \Theta_{\varepsilon}(f) \) in the inverse limit (compare [M1]), so (a) is a special case of (b). Taking \( f \) to be a constant map, we recover Theorem 6.3.

Remark. Let \( x \) be an isolated double point \( f(p) = f(q) \) of a generic PL map \( f \) of an \( n \)-manifold into \( \mathbb{R}^m \) in the metastable range. At this double point, \( f \) looks locally like the cone on a link \( L_x : S^{n-1} \sqcup S^{n-1} \hookrightarrow S^{m-1} \). When \( m < 2n \), such a link may be non-trivial, even though each component is null-homologous in the complement to the other one. Therefore \( f \) may fail to be \( C^0 \)-approximable by embeddings, even though the (first) cohomological obstruction to such an approximation, that is the Hurewicz image of \( \Theta(f) \), vanishes (see [M1; Example 1]). Indeed the component \( \Theta_{\varepsilon}(f) \) of \( \Theta(f) \) in the direct summand \( \pi_2n(S^m) \simeq \omega_{\mathbb{Z}/2}^n((D^{2n} \times S^0)_+, (\partial D^{2n} \times S^0)_+) \) corresponding to \( x \) is nothing but the \( \alpha \)-invariant of the link map \( L_x \). Indeed, \( \Theta_{\varepsilon}(f) \) is the homotopy class of the composition \( (D^{2n}, \partial D^{2n}) \to (S^{\infty}, S^{m-1}) \to (S^{m}, \ast) \) which is the double suspension of the map \( S^{n-1} \land S^{n-1} \to S^{m-2} \) representing \( \alpha(L_x) \), cf. [MR].

Theorem 7.2. If \( m \geq \frac{3(n+1)}{2} \), any map of a compact PL \( n \)-manifold into \( \mathbb{R}^m \) with countable singular set is \( C^0 \)-approximable by PL maps with finite singular sets.

The original proof was significantly simplified by P. M. Akhmetiev, who explained to the author how to eliminate induction by countable infinite ordinals from that argument.

Proof. Without loss of generality \( m < 2n \). Let \( f : N \to \mathbb{R}^m \) be the map and \( \varepsilon > 0 \) the desired closeness. Let \( \delta = \delta(\varepsilon) \) be given by Criterion 7.1(b). Since \( \Delta_f \) is contained in \( S_f \times S_f \), it is countable, and since it is the preimage of \( \Delta_{\mathbb{R}^m} \), it is closed in \( \tilde{X} \). Let \( DN \) be an equivariant closed tubular \( \delta \)-neighborhood of the infinity (i.e., of the removed diagonal) in \( \tilde{X} \). By either proof of Lemma 7.3 below, we may assume that \( \partial DN \) is disjoint from \( \Delta_f \). By Lemma 7.3, \( (\Delta_f \setminus DN)/t \) is contained in the interior of a disjoint union \( U/t \) of finitely many small PL balls \( B_i \). For each \( i \), let \( \{x_i, y_i\} \) be some point in \( (\Delta_f/t) \cap B_i \). Let \( X \) be the quotient of \( N \) by the equivalence relation \( x_i \sim y_i \) for all \( i \), so that \( f \) factors as \( N \xrightarrow{p} X \xrightarrow{\bar{f}} \mathbb{R}^m \). Then \( \hat{X} \) is obtained from \( \hat{N} \) by removing the set \( P \) of all points \((x_i, y_i)\) and \((y_i, x_i)\) and identifying each.
\{x_i\} \times (N \setminus \{y_i\}) \times (N \setminus \{x_j\}) \times (N \setminus \{x_i\})$, as well as the symmetric sets, via a map $q: \tilde{N} \setminus P \to X$. The cone of $q$ collapses onto an $(n + 1)$-polyhedron, hence $q$ induces an isomorphism $\omega_{Z/\tilde{g}}^m(\tilde{N}_{+}(P, \tilde{N}_{+}(DN \cup U)) \to \omega_{Z/\tilde{g}}^m(\tilde{X}_{+}, \tilde{X}_{+}(q(DN \cup U \setminus P))$ since $m > n + 1$. But the former group is zero. Hence $\Theta_\delta(g) = 0$ so $g$ is $\varepsilon$-approximable by an embedding. Thus $f$ is $\varepsilon$-approximable by a map with finitely many double points. \hfill \Box

**Lemma 7.3.** A countable closed set $S$ in a PL manifold $M$ is contained in the interior of a disjoint union of arbitrarily small PL balls in $M$.

**First proof.** Let $K^{(m-1)}$ be the codimension one dual skeleton of a sufficiently fine triangulation $T$ of $M^m$. The interior of each simplex of $T$ of dimension $> 0$ contains precisely one vertex of $K^{(m-1)}$, and the position of $K^{(m-1)}$ is entirely determined by the positions of its vertices in the open simplices of $T$. Since each vertex can assume uncountably many positions, it follows by an easy induction on the dual skeleta $K^{(i)}$ that $K^{(m-1)}$ can be made disjoint from $S$ by an appropriate perturbation of these positions. Then the stars of vertices of $T$ in this new derived subdivision of $T$ contain $S$ in their interiors. They still do so after subtracting sufficiently small collars of the boundary, which makes them disjoint. \hfill \Box

**Second proof.** Since every perfect set is uncountable, $S \supset S' \supset \cdots \supset S^{(\alpha)} = \emptyset$ for some countable ordinal $\alpha$. Here $S^{(\kappa+1)} = [S^{(\kappa)}]'$ is the Cantor–Bendixon derivative, i.e. the set of all limit points of $S^{(\kappa)}$ and, whenever $\lambda$ is not of the form $\kappa + 1$, $S^{(\lambda)}$ is the intersection of $S^{(\kappa)}$’s for all $\kappa < \lambda$. Since $S$ is 0-dimensional, it is contained in a disjoint union of compact PL manifolds, so without loss of generality $M$ is compact. We may also assume that $S^{(\kappa)} \neq \emptyset$ for all $\kappa < \alpha$. Then by Cantor’s theorem (every decreasing chain of compact sets has non-empty intersection), $\alpha$ is of the form $\beta + 1$ for some $\beta$. Since $S^{(\beta)}$ is finite, it is contained in a disjoint union $V \subset M$ of arbitrarily small PL balls. On the other hand, $T = S \setminus S^{(\beta)}$ satisfies $T^{(\beta)} = \emptyset$, hence by the induction hypothesis is contained in a disjoint union $U \subset M \setminus S^{(\beta)}$ of arbitrarily small PL balls. Let $\varphi: M \to Q$ be the quotient of $M$ by each of the balls $U_0 \subset U$ that meet $\partial V$. Then $\varphi$ can be arbitrarily closely approximated by a homeomorphism $h$, and we may assume that $h(\partial V)$ is disjoint from both the finite set $\varphi(U_0)$ and the sufficiently distant set $\varphi(U \setminus U_0)$. Thus $U \cup \varphi^{-1}(h(V))$ is a disjoint union of PL balls containing $S$. \hfill \Box

**Theorem 7.4.** Suppose $m - n \geq 3$. Any PL $n$-manifold that admits a map to $\mathbb{R}^m$ with finite singular set embeds in $\mathbb{R}^m$.

**Proof.** Let $f: N^n \to \mathbb{R}^m$ be the map. Then $f$ is the composition of a PL map $\varphi$ onto a polyhedron $K^n$ and a topological embedding $\psi: K \hookrightarrow \mathbb{R}^m$. By the Chernavskij–Miller–Bryant Theorem [B1] $\psi$ is $C^0$-approximable by PL embeddings. Let $\chi$ be one, and consider the PL map $g: N \xrightarrow{\varphi} K \xrightarrow{\psi} \mathbb{R}^m$.

What follows is a simplified version the Penrose–Whitehead–Zeeman trick. We may assume without loss of generality that $n > 1$ and that $N$ is connected. Then the inclusion $S_0 \subset N$ extends to a PL embedding of the cone $i: c \ast S_0 \hookrightarrow N$. Since $m - n \geq 3$, by general position the inclusion of the polyhedron $Q := g(i(c \ast S_0))$ into $\mathbb{R}^m$ extends to a PL embedding of the cone $j: c \ast Q \hookrightarrow \mathbb{R}^m$, with image disjoint from $g(N) \setminus Q$. Let $D^n$ and $B^m$ be the second derived neighborhoods of $i(c \ast S_0)$ and $j(c \ast Q)$ in some triangulations of $N$ and $\mathbb{R}^m$ where $g$ is simplicial. Then $g^{-1}(Q) = P$, and since $i(c \ast S_0)$ and $j(c \ast Q)$ are collapsible, $D$ and $B$ are balls.
Viewing them as cones on their boundaries, define \( h : N \to \mathbb{R}^m \) to coincide with \( g \) outside \( D \) and to conewise extend \( g|_{\partial D} : \partial D \to \partial B \) on \( D \). Since the latter map is an embedding, so is \( h \). □

In conclusion, we show that the method used in the proof of Theorem 7.4 (the Penrose–Whitehead–Zeeman trick) gives an alternative proof for a weak form of Theorem 7.2. An advantage of this approach is that it extends to codimension 3.

**Theorem 7.5.** If \( m - n \geq 3 \), any map of a compact PL \( n \)-manifold into a PL \( m \)-manifold with countable closure of the singular set is \( C^0 \)-approximable by PL maps with finite singular sets.

**Proof.** Let \( f : N \to M \) be the given map and \( \varepsilon > 0 \) the desired closeness. The open manifold \( N \setminus \text{Cl}(S_f) \) can be represented as the union of a sequence \( K_0 \subset K_1 \subset K_2 \subset \ldots \) of compact manifolds with boundary. By the Chernavskij–Miller Theorem (see [B2]) \( f \) is \( \frac{\varepsilon}{2} \)-homotopic with support in \( K_2 \) to a map which still embeds \( K_2 \) and is PL on \( K_1 \). This map is \( \frac{\varepsilon}{2} \)-homotopic with support in \( K_3 \setminus K_0 \) to a map which still embeds \( K_3 \) and is PL on \( K_2 \). Proceeding in this fashion, in the end\(^{24} \) we will obtain that \( f \) is \( \frac{\varepsilon}{2} \)-homotopic rel \( \text{Cl}(S_f) \) to a map still denoted \( f \), which PL embeds \( N \setminus \text{Cl}(S_f) \).

By Lemma 7.3, \( f(\text{Cl}(S_f)) \) is contained in the interior of a disjoint union \( P_k \) of PL balls of diameters \( < \varepsilon/2 \) in \( M \). Then \( L := f^{-1}(P_k) \) is a compact PL manifold. By Lemma 7.3, the intersection of \( \text{Cl}(S_f) \) with each connected component of \( L \) is contained in a PL ball in this component. Let \( J_k \) be the union of all these PL balls. Then \( Z := f(\text{Cl}(N \setminus J_k)) \) is disjoint from \( f(\text{Cl}(S_f)) \). By Lemma 7.3, \( f(\text{Cl}(S_f)) \) is contained in the interior of a disjoint union \( P_{k-1} \) of PL balls in \( P_k \setminus Z \). By construction, \( f^{-1}(P_{k-1}) \subset J_k \). Proceeding in this fashion, we obtain \( J_i \)'s and \( P_i \)'s as in Lemma 7.6. Finally, if we choose them with something to spare (i.e. leaving sufficient margins), their inclusion properties will preserve for a sufficiently close PL approximation of \( f \). □

**Lemma 7.6.** Let \( f : N^n \to M^m \) be a PL map between PL manifolds, \( m - n \geq 3 \), and let \( k \) be such that \( m \geq (1 + \frac{1}{k})(n + 1) \). Suppose that \( S_f \subset J_0 \subset \ldots \subset J_k \subset M \), where each \( J_k \) is a disjoint union of PL \( n \)-balls and each \( f(J_i) \) is contained in a PL \( m \)-ball \( P_i \) such that for \( i < k \), \( f^{-1}(P_i) \subset J_{i+1} \). Then \( f \) is homotopic with support in \( J_k \) and with image of the support in \( P_k \) to a PL map \( g \) with finite \( S_g \).

**Proof.** The Penrose–Whitehead–Zeeman–Irwin trick [Z]. □

**Remark.** If \( f \) is a map of a compact PL \( n \)-manifold into a PL \( m \)-manifold, \( m - n \geq 3 \), such that both \( \text{Cl}(S_f) \) and \( f(\text{Cl}(S_f)) \) are \( k \)-dimensional and tame in the sense of Shtan’ko, then \( f \) is approximable by PL maps whose singular sets are \( k \)-polyhedra. This can be proved using the Homma–Bryant argument [B2] for the Chernavskij–Miller approximation theorem. We omit the details.

**Problem 7.7.** Let \( f \) be a codimension \( \geq 3 \) continuous map between compact PL manifolds with a \( k \)-dimensional \( S_f \). Is \( f \) approximable by PL maps \( g \) with \( k \)-dimensional \( S_g \)\

One might also wonder whether there exists a map \( f : S^3 \to \mathbb{R}^5 \) with \( S_f \) the Antoine necklace, which is non-approximable by maps with finite singular sets.

\(^{24}\) An extra care about the epsilonics in what follows would allow us to do with just one application of the Chernavskij–Miller Theorem here.
It should be mentioned here that every map \( f \) from a closed 2-manifold to a closed 3-manifold with 0-dimensional \( \mathrm{Cl}(S(f)) \) is approximable by embeddings [Bra]. It has been conjectured in mid-80s that this is the case more generally for every map \( f \) from a closed \((m - 1)\)-manifold to a closed \( m \)-manifold, \( m \geq 3 \), with 0-dimensional \( S(f) \) (see [RR+]).

8. Polyhedra whose subsets satisfy partial Alexander duality

**Lemma 8.1.** For every compact polyhedron \( X \) with \( H^i(\tilde{X}) = 0 \) for \( i \geq m \), there exists an equivariant map \( \tilde{X} \to S^{m-1} \).

**Proof (revised in v5)\(^{25}\).** Let \( G \) be a finitely generated abelian group. By the universal coefficients formula, \( H^i(\tilde{X}; G) = 0 \) for all \( i \geq m \). It follows from the two Smith sequences, by downward induction on \( i \), that \( H^i(\tilde{X}; G) = 0 = H^i(\tilde{X}; G \otimes \mathbb{Z}_p) \) for all \( i \geq m \), where \( \mathbb{Z}_p \) is the integral local coefficient system associated with the double covering \( p: \tilde{X} \to \tilde{X} \).

Let \( M \) be a finitely generated \( \Lambda \)-module, where \( \Lambda = \mathbb{Z}[\mathbb{Z}/2] \), and let \( I \) be the augmentation ideal of \( \Lambda \). By a trivial case of [CR; Theorem 74.3], which is easy to verify directly, \( M \cong G_1 \oplus (G_2 \otimes \mathbb{Z}_2 I) \oplus (G_3 \otimes \mathbb{Z}_2 \Lambda) \) for some finitely generated abelian groups \( G_1 \), \( G_2 \) and \( G_3 \). Since \( H^i_{\mathbb{Z}/2}(\tilde{X}; G_1) \cong H^i(\tilde{X}; G_1) \), \( H^i_{\mathbb{Z}/2}(\tilde{X}; G_2 \otimes I) \cong H^i(\tilde{X}; G_2 \otimes \mathbb{Z}_p) \) and \( H^i_{\mathbb{Z}/2}(\tilde{X}; G_3 \otimes \Lambda) \cong H^i(\tilde{X}; G_3) \) (see §2), we conclude that \( H^i_{\mathbb{Z}/2}(\tilde{X}; M) = 0 \) for all \( i \geq m \).

Let \( M_i \) be \( \pi_i(S^{m-1}) \) regarded as a \( \Lambda \)-module under the action induced by the antipodal involution on \( S^{m-1} \), and let \( \mathcal{F}_{M_i} \) be the corresponding local coefficient system on \( \tilde{X} \). Then \( H^{i+1}(\tilde{X}; \mathcal{F}_{M_i}) \cong H^{i+1}_{\mathbb{Z}/2}(\tilde{X}; M_i) = 0 \) for all \( i \geq m - 1 \). The assertion now follows by the standard non-homotopically-simple obstruction theory (as e.g. in the Hilton–Wylie textbook). \( \Box \)

**Theorem 8.2.** Every \( n \)-polyhedron \( X \) with \( H^{n-d}(X \setminus x) = 0 \) for each \( x \in X \) and \( d \leq k \), where \( k < \frac{n-3}{2} \), embeds in \( \mathbb{R}^{2n-k} \).

**Proof.** Let \( \mathcal{H}^{n-d}(\pi) \) be the Leray sheaf of the projection \( \pi: \tilde{X}_s \subset X \times X \to X \) [Bre]. Since the simplicial deleted product \( \tilde{X}_s \) is compact, \( \pi \) is closed, so the stalks \( \mathcal{H}^{n-d}(\pi)_x \cong H^{n-d}(X \setminus x) \). Consider the Leray spectral sequence [Bre]

\[
E_2^{pq} = H^q(X; \mathcal{H}^p(\pi)) \Rightarrow H^{p+q}(\tilde{X}_s).
\]

Since \( \mathcal{H}^{n-d}(\pi) = 0 \) for all \( d \leq k \), we get that \( H^{2n-d}(\tilde{X}_s) = 0 \) for all \( d \leq k \). \( \Box \)

V. M. Buchstaber asked the author in September 2005, whether there is a generalization to polyhedra of the classical Penrose–Whitehead–Zeeman Theorem that \( k \)-connected PL \( n \)-manifolds embed in \( \mathbb{R}^{2n-k} \) in the metastable range. (For a proof

\(^{25}\)A. Skopenkov pointed out that the published version of this proof was insufficient. That argument, which only showed that \( H^{i+1}(\tilde{X}; \pi_1(S^{m-1})) = 0 = H^{i+1}(\tilde{X}; \pi_1(S^{m-1}) \otimes \mathbb{Z}_p) \), is in fact well-known [HH; p. 237], [Ad; Lemma 7.3] and, as observed in [HH; p. 236], suffices to prove Lemma 8.1 under the (harmless) additional hypothesis \( \dim X < m \). Indeed, the antipodal action on \( \pi_1(S^{m-1}) \) is trivial when \( m - 1 \) is odd (because \( [-\mathrm{id}] = 0 \) in this case), and coincides with multiplication by \(-1\) when \( m - 1 \) is even and the suspension homomorphism \( \pi_{i-1}(S^{m-2}) \to \pi_i(S^{m-1}) \) is onto (because \([\mathrm{id}] + [-\mathrm{id}] = 0 \) in this case, and it is easy to see that \([g(S\Sigma f) + h(\Sigma f)] = ([g+h](\Sigma f)) \). An alternative proof of Lemma 8.1 (without the additional hypothesis) was found in [GS] using the generalized Smith sequence (see Remark 2.3).
of this theorem see [Z], which includes Irwin’s extension to codimension three.\footnote{Added in v5: Another proof, which also works for homologically $k$-connected manifolds, is given by the Haefliger–Weber Criterion 3.1 along with Lemma 8.1 and the Poincare duality $H^i(\hat{X}) \simeq H_{2n-i}(X \times X, \Delta X)$; cf. [Hae; p. 66], [We; Théorème 4], [Ad].} The author learned from A. Skopenkov that the extension of the Penrose–Whitehead–Zeeman Theorem to homologically $k$-connected manifolds has been long known to A. Haefliger and can be proved by the methods of [GS], but apparently has not been explicitly stated in the literature.

One answer to Buchstaber’s question is already given by Theorem 8.2. A minimal set of purely local and purely global conditions implying the hypothesis of Theorem 8.2 is as follows:

(i) $H^{n-d}(X) \simeq H_d(\text{pt})$ for $d \leq k$, and

(ii) $H^{n-d}(X, X \setminus x) \simeq H_d(\text{pt})$ for $d \leq k-1$ and each $x \in X$.

**Theorem 8.3.** Let $X$ be a compact $n$-polyhedron.

(a) The following are equivalent:

- conditions (i)$_k$ and (ii)$_{k-1}$;
- $\check{H}_i(X \setminus Y) \simeq H^{n-i-1}(Y)$ for $i \leq k-1$ and all $Y \subset X$.

(b) If $X$ satisfies conditions (i)$_0$ and (ii)$_0$, the following are equivalent:

- conditions (i)$_k$ and (ii)$_k$;
- $H_i(X) = 0$ for $i \leq k$, and $H_i(X, X \setminus Y) = 0$ for $i \leq k+1$ and all $Y \subset X$ with $\dim Y < n-i$;
- $H_i(X) = 0$ for $i \leq k$, and $X/Z$ is a homology manifold for some $(n-k-1)$-polyhedron $Z \subset X$ such that $H_i(X, X \setminus Y) = 0$ for $i \leq k+1$ and all $Y \subset Z$.

Notice that $H_i(X, X \setminus Y) = 0$, where $\dim Y < n-i$, may be viewed as a transversality-type condition: it says that $i$-pseudo-manifolds in $X$ can be made disjoint, by pseudo-bordism relative to the boundary, from the subpolyhedron $Y$ of codimension $> i$ (compare [Mc3], [RoS; Theorem 4.2]).

At the same time, the original Penrose–Whitehead–Zeeman argument can be seen to work to embed into $\mathbb{R}^{2n-k}$, $k < \frac{n-3}{2}$, every $k$-connected compact $n$-polyhedron $X$ that is a manifold away from a subpolyhedron $Z$ of dimension $< n-k-1$ such that $\pi_i(X, X \setminus Z) = 0$ for all $i \leq k+1$ (see [Z]).

In the case $k = 0$, Sarkaria [Sa1] noticed that the PWZ method also works to embed in $\mathbb{R}^{2n}$ every quotient $X$ of an $n$-manifold $M$, $n > 2$, by an identification on the boundary such that no two components of $M$ remain disjoint in $X$. Clearly, this includes all $n$-polyhedra satisfying the hypothesis of Theorem 8.2.

We now proceed to the proof of Theorem 8.3. It will be convenient to work in a slightly greater generality. To this end, the orientation sheaf $\mathcal{H}_n(X)$ of an $n$-polyhedron $X$ is defined by $U \mapsto H_n(X, X \setminus U)$ and has stalks $\mathcal{H}_n(X)_x = H_n(X, X \setminus x)$. The fundamental cosheaf $\mathcal{H}^n(X)$ is defined by $U \mapsto H^n(X, X \setminus U)$ and has stalks $\mathcal{H}^n(X)_x = H^n(X, X \setminus x)$. (It is an obvious feature of the top dimension that this presheaf, resp. precosheaf is indeed a sheaf, resp. cosheaf.) If $H^n(X, X \setminus x) \simeq \mathbb{Z}$ for each $x \in X$, the fundamental cosheaf is clearly locally constant. By the universal coefficients formula, the orientation sheaf $\mathcal{H}_n(X) = \text{Hom}(\mathcal{H}^n(X), \mathbb{Z})$. The notions of a cosheaf and cosheaf homology are defined by reversing all arrows in those of a sheaf and sheaf cohomology, cf. [Bre].
Proposition 8.4. Let $X$ be a compact $n$-polyhedron and $d$ a non-negative integer. The following assertions are equivalent:

1. $H^{n-i}(X, X \setminus x) = 0$ for all $x \in X$ and $1 \leq i \leq d$;
2. $H_i(X \setminus Z, X \setminus Y; \mathcal{H}^n(X))$ is isomorphic to $H^{n-i}(Y, Z)$ when $i \leq d$ and to its subgroup when $i = d + 1$, for all $Z \subset X$.
3. $H_i(X, X \setminus Y; \mathcal{H}^n(X)) = 0$ for $i \leq d + 1$ and all $Y \subset X$ with $\dim Y < n - i$.

The proof also shows that (2) is equivalent to its versions with $Z = \emptyset$ and with $Y = X$.

Proof. Clearly, (2) implies (1). Conversely, the Zeeman spectral sequence runs

$$E_{p,q}^2 = H_q(X \setminus Z, X \setminus Y; \mathcal{H}^p(X)) \Rightarrow H^{p-q}(Y, Z)$$

(see [Mc3] and [Bre; §VI.14 with $f = \text{id}]$). Here $\mathcal{H}^p(X)$ is the cosheaf generated [Bre] by the precosheaf $U \mapsto H^p(X, X \setminus U)$. The hypothesis implies that $E_{p,q}^\infty = E_{p,q}^2 = 0$ for $p - q \geq n - d$, $p \neq n$. So the edge homomorphism $e_i: H_i(X \setminus Z, X \setminus Y; \mathcal{H}^n(X)) \rightarrow H^{n-i}(Y, Z)$ is an isomorphism for each $i \leq d$. Moreover, all differentials $E_{p,q}^r \rightarrow E_{p+r-1,q+r}^r$ must be trivial for $p - q \geq n - d$ (cf. Figure 1 in [Mc3]). These include all differentials with target group $E_{p,q}^r$ where $p - q = n - d - 1$, hence $E_{p,q}^\infty = E_{p,q}^2$ for such $p, q$. Thus $e_{d+1}$ is injective.

Clearly, (2) implies (3), so to complete the proof it suffices to show that (3) implies (1). Let us fix some triangulation of $X$. Then (1) is equivalent to the assertion

(*) The link $L$ of every simplex $A^a$ of $X$ has trivial reduced cohomology groups in dimensions $l-1, \ldots, l-d$, where $l = \dim L = n - a - 1$.

Assume inductively that (*) holds for all simplices of dimension $> a$. The link in $L$ of a simplex $B^b$ of $L$ is the link in $X$ of the simplex $A \ast B$ of $X$. So $L$ itself satisfies condition (1). Hence by the above it also satisfies (2), in particular $H_i(L; \mathcal{H}^i(L)) \simeq H^{n-i}(L)$ for $i \leq d$. By Lemma 8.5 below, $H_i(L; \mathcal{H}^i(L)) \simeq H_{i+1}(X \setminus \partial A, X \setminus A; \mathcal{H}^n(X))$ for $i > 0$. If $a < n - i - 1$ and $i + 1 \leq d + 1$, the latter group is zero since $H_{i+1}(X, X \setminus A; \mathcal{H}^n(X)) = H_i(X, X \setminus \partial A; \mathcal{H}^n(X)) = 0$ by (3). \qed

Lemma 8.5. If $L'$ is the link of a simplex $A$ in a fixed triangulation of an $n$-polyhedron $X$, then $H_i(L; \mathcal{H}^i(L)) \simeq H_{i+1}(X \setminus \partial A, X \setminus A; \mathcal{H}^n(X))$ for $i > 0$.

This is obvious when $\mathcal{H}^n(X)$ is locally constant, but we need to be more careful in the general case (which is not used anywhere except Proposition 8.4).

Proof. Let $S$ be the open star $c \ast L \setminus L$ of $A$. Since $S \setminus c$ is homeomorphic to $L \times \mathbb{R}^1$, we have $H_i(L; \mathcal{H}^i(L)) \simeq H_i(S \setminus c; \mathcal{H}^{i+1}(S \setminus c))$. Now $H_i(S; \mathcal{H}^{i+1}(S))$ is isomorphic to $H_i(c \ast L; \mathcal{H}^{i+1}(cL \cup L \times I)|_{c \ast L})$. The chain complex $C_*(c \ast L; \mathcal{H}^{i+1}(cL \cup L \times I)|_{c \ast L})$ is, apart from an additional summand in dimension 0, the cone of the identity map on $C_*(L; \mathcal{H}^i(L))$. It follows that $H_i(S; \mathcal{H}^{i+1}(S))$ vanishes for $i > 0$, and therefore $H_i(S \setminus c; \mathcal{H}^{i+1}(S \setminus c)) \simeq H_{i+1}(S, S \setminus c; \mathcal{H}^{i+1}(S))$ for $i > 0$.

Let $V = A \setminus \partial A$ and $U = V \ast L \setminus L$. By considering the projection $p: U \rightarrow S$, we get $H_{i+1}(S, S \setminus c; \mathcal{H}^{i+1}(S)) \simeq H_{i+1}(U, U \setminus V; \mathcal{H}^n(U))$ due to $\mathcal{H}^n(U) = p^*\mathcal{H}^{i+1}(S)$. Finally $H_{i+1}(U, U \setminus V; \mathcal{H}^n(V)) \simeq H_{i+1}(X \setminus \partial A, X \setminus A; \mathcal{H}^n(X))$ by excision. \qed
Remark. The proof of Proposition 8.4 works to show that the obvious dual homological conditions (1′), (2′) and (3′) are mutually equivalent as well. For example, (2′) says:

(2′) \( H^i(X \setminus Z, X \setminus Y; \mathcal{H}_n(X)) \) is isomorphic to \( H_{n-i}(Y, Z) \) when \( i \leq d \) and to its subgroup when \( i = d + 1 \), for all closed subpolyhedra \( Z \subset Y \subset X \).

When \( \mathcal{H}^n(X) \) is locally constant, \( H_n(X; \mathcal{H}^n(X)) \) is isomorphic to \( \mathbb{Z} \) by (1′) \( \iff \) (2′) with \( d = 0 \), and the iso/monomorphism in (2′) can be identified (see [Mc3; 6.3]) as the cap product with a generator of this group.

It would be interesting to know if conditions (i) and (ii) can also be rethought from the viewpoint of the slant product (see [Mc2]).

Lemma 8.6. An \( n \)-polyhedron \( X \) satisfying condition (ii) \( k-1 \) is a homology manifold away from a subpolyhedron of dimension at most \( n - (2k + 1) \).

Proof. By the proof of (3) \( \Rightarrow \) (1) in Proposition 8.4, the link \( L \) of an \( (n - i - 1) \)-simplex \( \sigma \) of \( X \) has its top \( k \) cohomology (and hence also homology) groups, as well as the first \( k \) homology groups isomorphic to those of \( S^i \). Hence if \( i < 2k \), \( L \) has the same homology as the \( i \)-sphere, and so \( X \) is a homology manifold at the interior points of \( \sigma^{n-i-1} \).

□

Theorem 8.3 follows immediately from Proposition 8.4 and Lemma 8.6.

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