The Density of Tuples Restricted by Relatively $r$-Prime Conditions

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Abstract

In order to consider $j$-wise relative $r$-primality conditions that do not necessarily require all $j$-tuples of elements in a Dedekind domain to be relatively $r$-prime, we define the notion of $j$-wise relative $r$-primality with respect to a fixed $j$-uniform hypergraph $H$. This allows us to provide further generalisations to several results on natural densities not only for a ring of algebraic integers $O$, but also for the ring $\mathbb{F}_q[x]$.

1 Introduction

In 1976, Benkoski proved that the natural density of the set of relatively $r$-prime $m$-tuples of positive integers (with $rm > 1$) equals $1/\zeta(rm)$, where $\zeta$ is the Riemann zeta function [1]. This acted as a culmination of the work of Mertens [8], Lehmer [7], and Gegenbauer [3]. Thereafter, Tóth [14] and Hu [5] found the natural density of the set of $j$-wise relatively prime $m$-tuples of positive integers (where $j \leq m$). Extensions of these results have been made to ideals in a ring of algebraic integers $O$ by Sittinger [11, 13] and subsequently to elements in a ring of algebraic integers as well by Micheli [2] and Sittinger [12]. Moreover, Morrison [9] and Guo [4] gave analogous results for elements in $\mathbb{F}_q[x]$.

We can further generalise the notion of $j$-wise relatively primality by considering relative primality conditions that require some but not all $j$-tuples to be relatively prime. A first step in this direction was investigated by Hu [4], who used graphs to notate which pairs of integers are to be relatively prime.

Definition 1.1. Let $G$ be a simple undirected graph whose $m$ vertices are the positive integers $a_1, \ldots, a_m$. We say that the $m$-tuple $(a_1, \ldots, a_m)$ is $G$-wise relatively prime if $\gcd(a_i, a_j) = 1$ for all adjacent vertices $a_i$ and $a_j$.

In the case that $G = K_m$, a complete graph with $m$ vertices, we see that $m$ positive integers being $G$-wise relatively prime is the same as saying that these integers are pairwise relatively prime. If we let $i_k(G)$ denote the number of independent sets of $k$ vertices in $G$ (such a set has no two vertices adjacent
in $G$), then Hu [6] proved that the density of the set of $G$-wise relatively prime ordered $m$-tuples of positive integers equals
\[
\prod_p \left[ \sum_{k=0}^{m} i_k(G) \left( 1 - \frac{1}{p} \right)^{m-k} \left( \frac{1}{p} \right)^k \right],
\]
where the product is over all prime numbers.

We now extend the definition of $G$-wise relative primality to accommodate relative primality conditions on certain sets of $j$-tuples of elements from $m$ elements (where $2 \leq j \leq m$) not only from the ring of integers, but also from any Dedekind domain. First, we give a notion of $m$-tuples of elements from a Dedekind domain being relatively $r$-prime.

**Definition 1.2.** Let $D$ be a Dedekind domain. Fix $r, m \in \mathbb{N}$. We say that $\beta_1, \ldots, \beta_m \in D$ are relatively $r$-prime if $p^r \nmid \langle \beta_1, \ldots, \beta_m \rangle$ for any prime ideal $p \subseteq D$.

In order to properly generalise the notion of $G$-wise relative primality, we use the concept of a $j$-uniform hypergraph $H$, in which any edge connects exactly $j$ vertices.

**Definition 1.3.** Let $D$ be a Dedekind domain. Fix $r, j, m \in \mathbb{N}$ where $j \leq m$, and let $H$ be a simple undirected $j$-uniform hypergraph whose $m$ vertices are $\beta_1, \ldots, \beta_m \in D$. We say that $\beta_1, \ldots, \beta_m \in D$ are $H$-wise relatively $r$-prime if any $j$ adjacent vertices of $H$ are relatively $r$-prime.

A few remarks are now in order. First, although we state the definitions in this generality, we are in particular interested in the cases of a ring of algebraic integers as well the polynomial rings $\mathbb{F}_q[x]$. Next, suppose we take $D = \mathbb{Z}$, $j = 2$, and $r = 1$. Then our hypergraph is a graph $G$, and Definition 1.3 reduces to $m$ integers are $G$-wise relatively prime as defined in [6]. Moreover when $D = \mathcal{O}$ and $H = K^{(j)}_m$, the complete $j$-uniform hypergraph on $m$ vertices, this definition reduces to $m$ elements being $j$-wise relatively $r$-prime as defined in [12].

**Definition 1.4.** Given a $j$-uniform hypergraph $H$, we say that a subset $S$ of vertices from $H$ is an independent vertex set if $S$ does not contain any hyperedge of $H$. Moreover for any non-negative integer $k$, we let $i_k(H)$ denote the number of independent sets of $k$ vertices in $H$.

We now state the main results of this article, starting with the algebraic integer case.

**Theorem 1.5.** Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$, and let $K$ be an algebraic number field over $\mathbb{Q}$ with ring of integers $\mathcal{O}$. Then, the density of the set of $H$-wise relatively $r$-prime ordered $m$-tuples of elements in $\mathcal{O}$ equals
\[
\prod_p \left[ \sum_{k=0}^{m} i_k(H) \left( 1 - \frac{1}{\mathfrak{N}(p^r)} \right)^{m-k} \left( \frac{1}{\mathfrak{N}(p^r)} \right)^k \right],
\]
where the product is over all nonzero prime ideals in $\mathcal{O}$.
After setting up the pertinent notation in Section 2, we prove Theorem 1.5. Since the arithmetic in the rings $\mathbb{Z}$ and $\mathbb{F}_q[x]$ have striking similarities (for further details, see [10]), we would expect that we can derive a $H$-wise relatively $r$-prime density statement for $\mathbb{F}_q[x]$. In Section 3, we state and prove an analogue of Theorem 1.4 for the function field case $\mathbb{F}_q[x]$.

**Theorem 1.6.** Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$. Then the density of the set of $H$-wise relatively $r$-prime ordered $m$-tuples of polynomials in $\mathbb{F}_q[x]$ equals

$$\prod_{f \text{ irred.}} \left( \sum_{k=0}^{m} i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k \right),$$

where it is understood that the product is over all monic irreducible polynomials in $\mathbb{F}_q[x]$.

**Remark:** By noting that $\mathcal{N}(f) = |\mathbb{F}_q[x]/\langle f \rangle| = q^{\deg f}$, the analogy between this latter density statement and the one given in the algebraic number ring case is made clear.

## 2 Density of $H$-wise relatively $r$-prime elements in $\mathcal{O}$

Let $K$ be an algebraic number field of degree $n$ over $\mathbb{Q}$ with $\mathcal{O}$ as its ring of integers having integral basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$. As a way to generalise the notion of all positive integers less than or equal to some positive constant $M$, we define

$$\mathcal{O}_\mathcal{B}[M] = \left\{ \sum_{i=1}^{n} c_i \alpha_i : c_i \in [-M, M) \cap \mathbb{Z} \right\}.$$

The goal of this section is to derive a $H$-wise relatively prime density statement in $\mathcal{O}$ by using the methods developed by [2] and [12]. First, we define a notion of density for a subset $T$ of $\mathcal{O}^m$ that reduces to the classic notion of density over $\mathbb{Z}$ as follows.

**Definition 2.1.** Let $T \subseteq \mathcal{O}^m$ and fix an integral basis $\mathcal{B}$ of $\mathcal{O}$. The upper and lower densities of $T$ with respect to $\mathcal{B}$ are respectively defined as

$$\overline{D}_\mathcal{B}(T) = \limsup_{M \to \infty} \frac{|T \cap \mathcal{O}_\mathcal{B}[M]^m|}{|\mathcal{O}_\mathcal{B}[M]^m|} \quad \text{and} \quad \underline{D}_\mathcal{B}(T) = \liminf_{M \to \infty} \frac{|T \cap \mathcal{O}_\mathcal{B}[M]^m|}{|\mathcal{O}_\mathcal{B}[M]^m|}.$$

If $\overline{D}_\mathcal{B}(T) = \underline{D}_\mathcal{B}(T)$, we say that its common value is called the density of $T$ with respect to $\mathcal{B}$ and denote this as $D_\mathcal{B}(T)$. Whenever this density is independent of the chosen integral basis $\mathcal{B}$, we denote this density as $D(T)$.

Although the manner in which we cover $\mathcal{O}$ could potentially depend on the choice of the given integral basis $\mathcal{B}$, it is a direct corollary to Theorem 1.5 that
the density of the set of $H$-wise relatively $r$-prime elements in $\mathcal{O}$ is actually independent of the integral basis used.

For the remainder of this section, let $S$ be a finite set of rational primes, and fix positive integers $r, j, m$ such that $j \leq m$. Fix a $j$-uniform hypergraph $H$, and define $E_S$ to be the set of $m$-tuples $z = (z_1, \ldots, z_m)$ in $\mathcal{O}^m$ such that any ideal generated by $j$ entries of $z$ is $H$-wise relatively $r$-prime with respect to all $p | \langle p \rangle$ for each $p \in S$. That is, $E_S$ consists of the $H$-wise relatively $r$-prime $m$-tuples of algebraic integers from $\mathcal{O}$ with respect to $S$.

In order to aide us in analysing $E_S$, let

$$\pi : \mathcal{O}^m \to \left( \prod_{p(\langle p \rangle \in S)} \mathcal{O}/p^r \right)^m$$

be the surjective homomorphism induced by the family of natural projections

$$\pi_{p^r} : \mathcal{O} \to \mathcal{O}/p^r$$

for all $p | \langle p \rangle$ where $p \in S$.

From the definition of $H$-wise relative $r$-primality of algebraic integers, we immediately deduce the following lemma.

**Lemma 2.2.** For a given prime ideal $p | \langle p \rangle$ where $p \in S$ and $k \in \{1, 2, \ldots, m\}$, let $A_k(p)$ denote the set of elements in $(\mathcal{O}/p^r)^m$ where exactly $k$ of their $m$ components are 0, and these $k$ components form an independent vertex set in $H$. Then,

$$E_S = \pi^{-1}\left( \prod_{p \in S} \bigcup_{k=0}^{m} A_k(p) \right).$$

**Proposition 2.3.** Suppose that $p$ is a prime ideal in $\mathcal{O}$ that lies above a fixed rational prime $p$. If we fix $q \in \mathbb{N}$ and set $N = \prod_{p \in S} p^r$, then

$$|E_S \cap \mathcal{O}[qN]^m| = (2q)^m \prod_{p | \langle p \rangle} p^{rm(n-D_p)} \left[ \sum_{k=0}^{m} i_k(H)(n(p^r) - 1)^{m-k} n(p^r)^k \right].$$

**Proof:** We first examine the map $\pi$. For brevity, we set $R_p = \prod_{p | \langle p \rangle} \mathcal{O}/p^r$. Then we let $\pi_N$ denote the reduction modulo $N$ homomorphism, and $\psi = (\psi_p)_{p \in S}$ where $\psi_p : (\mathcal{O}/\langle p^r \rangle)^m \to R_p^m$ is the homomorphism induced by the projection maps $\mathcal{O}/\langle p^r \rangle \to R_p$. Finally, let $\bar{\psi}$ be its extension to $(\mathcal{O}/\langle N \rangle)^m$ (by applying the Chinese Remainder Theorem to the primes in $S$). These maps are related to each other through the following diagram

$$\begin{array}{ccc}
\mathcal{O}^m & \xrightarrow{\pi_N} & (\mathcal{O}/\langle N \rangle)^m \\
\downarrow \cong & & \downarrow \cong \\
(\prod_{p \in S} \mathcal{O}/\langle p^r \rangle)^m & \xrightarrow{\psi} & (\prod_{p \in S} R_p)^m
\end{array}$$
and it follows that $\pi = \psi \circ \pi_N$.

To prove this proposition, we start by examining $\psi^{-1}$. Since for each rational prime $p$ the mapping $\psi_p : (\mathcal{O}/(p^r))^m \to R_p^m$ is a surjective free $\mathbb{Z}_{p^r}$-module homomorphism, we have for all $y \in (\prod_{p \in S} R_p)^m$:

$$|\psi^{-1}(y)| = \prod_{p \in S} |\psi_p^{-1}(y_p)| = \prod_{p \in S} |\ker \psi_p| = \prod_{p \in S} p^{rm(n-D_p)}.$$

Next, we compute $|\pi^{-1}_N(z) \cap \mathcal{O}_S[qN]^m|$. Given $\mathfrak{f} = (\mathfrak{t}_1, \ldots, \mathfrak{t}_m) \in (\mathcal{O}/(N))^m$, observe that since $\mathcal{O}/(N)$ is a free $\mathbb{Z}_N$-module with basis $\{\pi(\alpha_1), \ldots, \pi(\alpha_n)\}$, there exist unique $c'_j \in [0, N) \cap \mathbb{Z}$ such that

$$\mathfrak{t}_j = \sum_{t=1}^n c'_t \pi(\alpha_t),$$

Then for $z = (z_1, \ldots, z_m) \in \mathcal{O}_S^m$, it follows that $\pi_N(z) = \mathfrak{f}$ if and only if

$$z_j = \sum_{t=1}^n (c'_t + l'_t N) \alpha_t$$

for some $l'_t \in \mathbb{Z}$. Moreover, since we need $l'_t \in [-q, q) \cap \mathbb{Z}$ for each pair of indices $j$ and $t$, we deduce that

$$|\pi^{-1}_N(z) \cap \mathcal{O}_S[qN]^m| = (2q)^{mn}.$$

We are ready to compute $|E_S \cap \mathcal{O}_S[qN]^m|$. By the definition of $A_k^{(p)}$, we have for any fixed $k$ and $p$:

$$|A_k^{(p)}| = i_k(H)(\mathfrak{N}(p^r) - 1)^{m-k} \mathfrak{N}(p^r)^k.$$

Since we know from the last lemma that $E_S = \pi^{-1}(J)$, where

$$J = \psi^{-1}\left( \prod_{p \in S} \bigcup_{k=0}^m A_k^{(p)} \right),$$

it immediately follows that

$$|J| = \prod_{p \in S} p^{rm(n-D_p)} \sum_{k=0}^m i_k(H)(\mathfrak{N}(p^r) - 1)^{m-k} \mathfrak{N}(p^r)^k.$$
Therefore, we conclude that

\[ |E_S \cap \mathcal{O}_S[qN]|^m = (2q)^{mn}|J| \]

\[ = (2q)^{mn} \prod_{p \mid (p^r)} p^{\mathfrak{m}(n-D_p)} \left[ \sum_{k=0}^{m} i_k(H) \left( \mathfrak{N}(p^r) - 1 \right)^{m-k} \mathfrak{N}(p^r)^k \right], \]

as desired. ■

We now compute the density of \( E_S \).

**Lemma 2.4.** Using the previous notation, we have for any integral basis \( \mathcal{B} \) of \( \mathcal{O} \),

\[ \mathbb{D}(E_S) = \mathbb{D}_\mathcal{B}(E_S) = \prod_{p \mid (p^r)} \left[ \sum_{k=0}^{m} i_k(H) \left( 1 - \frac{1}{\mathfrak{N}(p^r)} \right)^{m-k} \left( \frac{1}{\mathfrak{N}(p^r)} \right)^k \right]. \]

**Proof:** Define the sequence \( \{a_j\} \) by \( a_j = \frac{|E_S \cap \mathcal{O}_S[j]|^m}{|\mathcal{O}_S[j]|^m} \), and let \( D \) denote value of the density in question.

First, we consider the subsequence \( \{a_{qN}\} \) where \( N = \prod_{p \in \mathcal{S}} p^r \). We claim that this subsequence is constant. By the previous proposition along with the definitions for \( N \) and \( D_p \),

\[ a_{qN} = \frac{1}{(2qN)^{mn}} \prod_{p \mid (p^r)} p^{\mathfrak{m}(n-D_p)} \sum_{k=0}^{m} i_k(H) \left( \mathfrak{N}(p^r) - 1 \right)^{m-k} \mathfrak{N}(p^r)^k \]

\[ = \prod_{p \mid (p^r)} \left[ \sum_{k=0}^{m} i_k(H) \left( 1 - \frac{1}{\mathfrak{N}(p^r)} \right)^{m-k} \left( \frac{1}{\mathfrak{N}(p^r)} \right)^k \right]. \]

Hence, \( \{a_{qN}\} \) is a constant subsequence and converges to \( D \).

Next, we show that \( \{a_{c+qN}\} \) also converges to \( D \) for any \( c \in \{1, 2, \ldots, N-1\} \), we first find bounds for \( a_{c+qN} \). To this end, note that

\[ a_{qN} \left( \frac{2qN}{2c+2qN} \right)^{mn} \leq a_{c+qN} \leq a_{(q+1)N} \left( \frac{2(q+1)N}{2c+2qN} \right)^{mn}. \]

By letting \( q \to \infty \) and applying the Squeeze Theorem, we conclude that \( \{a_{c+qN}\} \) converges to \( D \) for any \( c \in \{1, 2, \ldots, N-1\} \). Finally, since \( \{a_{c+qN}\} \) converges to \( D \) for any \( c \in \{0, 1, \ldots, N-1\} \), we conclude that \( \{a_j\} \) converges to \( D \). ■

Note that the density in Lemma 2.4 is independent of the integral basis \( \mathcal{B} \) used. Now we are ready to establish to the main theorem of this section. For convenience, we restate it here before proving it.
Theorem 1.5. Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$, and let $K$ be an algebraic number field over $\mathbb{Q}$ with ring of integers $\mathcal{O}$. Then, the density of the set $E$ consisting of $H$-wise relatively $r$-prime ordered $m$-tuples of elements in $\mathcal{O}$ equals

$$\prod_p \left[ \sum_{k=0}^m i_k(H) \left(1 - \frac{1}{\Omega(p^r)} \right)^{m-k} \left( \frac{1}{\Omega(p^r)} \right)^k \right],$$

where the product is over all nonzero prime ideals in $\mathcal{O}$.

Proof: Fix $t \in \mathbb{N}$ and let $S_t$ denote the set of the first $t$ rational primes. For brevity, we write $E_t = E_{S_t}$. Since $E_t \supseteq E$, $D_B(\overline{E}) \leq D_B(E_t) = D(E)$. Observe that the last equality is due to the existence of $D_B(\overline{E})$. Letting $t \to \infty$, $D_B(\overline{E}) \leq \limsup M \to \infty \sum\limits_{CM^n \supseteq \mathcal{O}[M]} (p^r \cap \mathcal{O}[M])^m \cdot (2M)^{-mn}$.

It remains to show the opposite inequality. Noting that $D_B(E_t) - D_B(E_t \setminus E) \leq D_B(\overline{E})$, it suffices to show that $\lim_{t \to \infty} D_B(E_t \setminus E) = 0$.

To this end, we introduce the following notation. Let $p$ be a prime ideal in $\mathcal{O}$, $p_t$ be the $t$th rational prime, and $M$ be a positive integer.

1. We write $p \succ M$ iff $p$ lies over a rational prime greater than $M$.

2. We write $M \succ p$ iff the rational prime lying under $p$ is less than $M$.

Using this notation, we can write

$$E_t \setminus E \subseteq \bigcup_{p \succ p_t} \left( \prod_{j=1}^m p^r \right) \subseteq \mathcal{O}^m,$$

where it is understood that $\prod_{j=1}^m p^r$ is the subset of $\mathcal{O}^m$ such that each entry of the $m$-tuple is an element of $p^r$. Then, we see that

$$(E_t \setminus E) \cap \mathcal{O}_B[M]^m \subseteq \bigcup_{CM^n \supseteq \mathcal{O}[M]} \prod_{j=1}^m (p^r \cap \mathcal{O}_B[M])$$

for some constant $C > 0$ dependent only on $B$, and thus

$$\overline{D_B}(E_t \setminus E) \leq \limsup_{M \to \infty} \sum\limits_{CM^n \supseteq \mathcal{O}[M]} |(p^r \cap \mathcal{O}_B[M])^m| \cdot (2M)^{-mn}.$$

By [2] Proposition 13, there exist constants $c, d > 0$ independent of $M$ and $p$ such that

$$|(p^r \cap \mathcal{O}_B[M])^m| \leq \left( \frac{2M}{\Omega(p^r)} \right)^m + c \left( \frac{2M}{d \Omega(p^r)^{1/n} + 1} \right)^{mn-1}.$$
Using this bound along with the facts that \( \mathfrak{N}(p) \geq p \) for every \( p \) lying above a fixed rational prime \( p \), and at most \( n \) prime ideals lie above a fixed rational prime, we obtain

\[
\overline{\mathfrak{D}}_B(E_t \setminus E) \leq \limsup_{M \to \infty} \sum_{C, M^{\prime} > p > p_t} \left[ \frac{1}{\mathfrak{N}(p^{r^{i}_m})^m} + c \left( \frac{2M}{d \mathfrak{N}(p^{r^{i}_m})^{\frac{1}{n}}} + 1 \right)^{m_{n-1}} (2M)^{-mn} \right]
\]

\[
\leq \limsup_{M \to \infty} \sum_{C, M^{\prime} > p > p_t} \left[ \frac{n}{p^{r^{i}_m}} + c \left( \frac{2M}{d p^{r^{i}_m}/n} + 1 \right)^{m_{n-1}} (2M)^{-mn} \right].
\]

It remains to show that the right side goes to 0 as \( t \to \infty \). First, observe that for all sufficiently large \( M \), we have \( 2M/d p^{r^{i}_m}/n > 1 \) and thus

\[
\left( \frac{2M}{d p^{r^{i}_m}/n} + 1 \right)^{m_{n-1}} (2M)^{-mn} < \left( \frac{2}{d} \right)^{m_{n}} \cdot \frac{1}{p^{r^{i}_m}}.
\]

Then, by writing \( A = n + cn(2/d)^{m_{n}} \) which is a constant independent of \( M \) and \( p \), we deduce that

\[
\overline{\mathfrak{D}}_B(E_t \setminus E) \leq \limsup_{M \to \infty} \sum_{C, M^{\prime} > p > p_t} \frac{A}{p^{r^{i}_m}} \leq \sum_{k=p_t}^{\infty} \frac{A}{k^{r^{i}_m}}
\]

for all sufficiently large \( M \).

Finally since \( \sum_{k=1}^{\infty} \frac{1}{k^{r^{i}_m}} \) is convergent, we conclude that \( \overline{\mathfrak{D}}_B(E_t \setminus E) = 0 \).  

To conclude this section, we now state a corollary that indicates how this main result provides a generalisation of the work from [12].

**Corollary 2.5.** Fix \( r, j, m \in \mathbb{N} \) such that \( j \leq m \) and \( rm \geq 2 \), and let \( K \) be an algebraic number field over \( \mathbb{Q} \) with ring of integers \( \mathcal{O} \). Then the density of the set of \( j \)-wise relatively \( r \)-prime ordered \( m \)-tuples of elements in \( \mathcal{O} \) equals

\[
\prod_p \left[ \sum_{k=0}^{j-1} \binom{m}{k} \left( 1 - \frac{1}{\mathfrak{N}(p^{r^{i}_m})} \right)^{m-k} \left( \frac{1}{\mathfrak{N}(p^{r^{i}_m})} \right)^k \right].
\]

**Proof:** Take \( H = K^{(j)}_m \) as the hypergraph, and observe that

\[
i_k(H) = \begin{cases} \binom{m}{k} & \text{if } 0 \leq k \leq j - 1 \\ 0 & \text{otherwise}. \end{cases}
\]

Applying Theorem 1.5 immediately yields the desired result.  

8
3 Density of $H$-wise relatively $r$-prime elements in $\mathbb{F}_q[x]$

Let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ where $q = p^k$ for some prime $p$ and $k \in \mathbb{N}$. The goal of this section is to derive a $H$-wise density statement in $\mathbb{F}_q[x]$ by using methods developed in [4].

In order to define a suitable definition of density in $\mathbb{F}_q[x]$, we begin by giving an enumeration of the polynomials in $\mathbb{F}_q[x]$. Denoting the elements of $\mathbb{F}_q$ as $a_0 = 0, a_1, \ldots, a_{q-1}$, let $\Sigma$ be the set of all $(a_{d_0}, a_{d_1}, a_{d_2}, \ldots)$ whose entries are in $\mathbb{F}_q$ and $d_i = 0$ for all sufficiently large $i$. Then since non-negative integers have a unique expansion base $q$, where $q$ is a positive integer greater than 1, we have a bijection $\Phi: \Sigma \rightarrow \mathbb{Z}_{\geq 0}$ defined by

$$\Phi(a_{d_0}, a_{d_1}, \ldots) = \sum_{i=0}^{\infty} d_i q^i.$$ 

Using this bijection, we define for each $j \in \mathbb{Z}_{\geq 0}$

$$f_j(x) = \sum_{i=0}^{\infty} a_{d_i} x^i,$$ 

where $j = \phi(a_{d_0}, a_{d_1}, \ldots)$.

Note that $\mathbb{F}_q[x] = \{f_j(x) : j \in \mathbb{Z}_{\geq 0}\}$, thereby giving an ordering of the elements in $\mathbb{F}_q[x]$. Now, we are able to define a density in this ring.

**Definition 3.1.** Fix a positive integer $m \geq 2$, and let $\mathcal{M}_N$ be the subset of $(\mathbb{F}_q[x])^m$ consisting of $m$-tuples of elements in $\mathbb{F}_q[x]$ whose entries are taken from $\{f_0, f_1, \ldots, f_N\}$. For any subset $T \subseteq (\mathbb{F}_q[x])^m$, we define the **upper and lower densities of $T$** are respectively defined as

$$\overline{D}(T) = \limsup_{N \to \infty} \frac{|T \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \quad \text{and} \quad \underline{D}(T) = \liminf_{N \to \infty} \frac{|T \cap \mathcal{M}_N|}{|\mathcal{M}_N|}.$$ 

If $\overline{D}(T) = \underline{D}(T)$, we say that its common value is called the **density of $T$** and denote this as $D(T)$.

Let $S$ be a finite set of irreducible polynomials in $\mathbb{F}_q[x]$, and fix $r, j, m \in \mathbb{N}$ satisfying $j \leq m$. Fix a $j$-uniform hypergraph $H$, and let $E_S$ denote the set of $m$-tuples of polynomials from $\mathbb{F}_q[x]$ that are $H$-wise relatively $r$-prime with respect to all irreducible polynomials in $S$.

For the following lemma and proposition, let

$$\pi: (\mathbb{F}_q[x])^m \rightarrow \left( \prod_{f \in S} \mathbb{F}_q[x]/\langle f^r \rangle \right)^m$$

be the surjective homomorphism induced by the family of natural projections $\pi_{fr}: \mathbb{F}_q[x] \rightarrow \mathbb{F}_q[x]/\langle f^r \rangle$ for each $f \in S$. 

As in the algebraic integer case, the following lemma follows immediately from the definition of $H$-wise relative $r$-primality of elements in $\mathbb{F}_q[x]$.

**Lemma 3.2.** For a given irreducible polynomial $f \in S$, let $A_k^{(f)}$ denote the set of elements in $(\mathbb{F}_q[x]/\langle f^r \rangle)^m$ where exactly $k$ of their $m$ components are 0, and these $k$ components form an independent vertex set in $H$. Then,

$$E_S = \pi^{-1}\left( \prod_{f \in S} \bigcup_{k=0}^{m} A_k^{(f)} \right).$$

**Proposition 3.3.** Let $N = bq^{deg F} - 1$ where $b \in \mathbb{N}$, and $F = \prod_{f \in S} f^r$. Then,

$$|E_S \cap \mathcal{M}_N| = (bq^{deg F})^m \prod_{f \in S} q^{-rm \deg f} \cdot \sum_{k=0}^{m} i_k(H)(q^{r \deg f} - 1)^{m-k}(q^{r \deg f})^k.$$

**Proof:** Let $\pi_F$ denote the reduction modulo $F$ homomorphism, and let

$$\psi : (\mathbb{F}_q[x]/\langle F \rangle)^m \to \left( \prod_{f \in S} (\mathbb{F}_q[x]/\langle f^r \rangle) \right)^m \to \prod_{f \in S} (\mathbb{F}_q[x]/\langle f^r \rangle)^m,$$

where the first part of $\psi$ is induced by the Chinese Remainder Theorem and the second part is an obvious isomorphism of free $\mathbb{F}_q[x]$-modules.

Now we compute $|\pi_F^{-1}(h(x)) \cap \mathcal{M}_N|$. By the Division Algorithm, we have that

$$\{f_t(x)\}_{t=0}^{N} = \{f_s(x) \cdot x^{deg F} + f_t(x) \mid 0 \leq t \leq q^{deg F} - 1 \text{ and } 0 \leq s \leq b - 1\}.$$  

Then for any fixed $s \in \{0,1,\ldots,b-1\}$, the map $\pi_F$ restricted to

$$\{f_s(x) \cdot x^{deg F} + f_t(x)\}_{t=0}^{q^{deg F} - 1} \to \mathbb{F}_q[x]/\langle F \rangle$$

is one-to-one. Since $|\ker(\pi_F)| = b^m$, we conclude that $|\pi_F^{-1}(h(x)) \cap \mathcal{M}_N| = b^m$.

We are now ready to compute $|E_S \cap \mathcal{M}_N|$. We know that $E_S = \pi^{-1}(J)$, where

$$J = \psi^{-1}\left( \prod_{f \in S} \bigcup_{k=0}^{m} A_k^{(f)} \right).$$

Since for any fixed $k \in \{0,1,\ldots,m\}$ and $f \in S$ we have

$$|A_k^{(f)}| = i_k(H)(q^{r \deg f} - 1)^{m-k}(q^{r \deg f})^k,$$

we deduce that

$$|J| = q^{m \deg F} \prod_{f \in S} q^{-rm \deg f} \cdot \sum_{k=0}^{m} i_k(H)(q^{r \deg f} - 1)^{m-k}(q^{r \deg f})^k.$$  

Therefore,
\[ |E_S \cap \mathcal{M}_N| = b^m \cdot |J| \]
\[ = (b q^{\text{deg } F})^m \prod_{f \in S} q^{-rm \deg f} \sum_{k=0}^{m} i_k(H)(q^{r \deg f} - 1)^{m-k}(q^{r \deg f})^k. \]

We now find the density of \( E_S \).

**Lemma 3.4.** Using the notation from Proposition 3.3,
\[ D(E_S) = \prod_{f \in S} \left[ \sum_{k=0}^{m} i_k(H) \left( 1 - \frac{1}{q^{r \deg f}} \right)^{m-k} \left( \frac{1}{q^{r \deg f}} \right)^k \right]. \]

**Proof:** Let \( a_j = \frac{|E_S \cap M_j|}{|M_j|^2} \) and let \( D \) be the value of the density in question. For notational brevity, we let \( n = q^{\text{deg } F} \).

We first consider the subsequence \( \{a_{bn-1}\}_{b \in \mathbb{N}} \). By Proposition 3.3, we find that
\[ \frac{|E_S \cap \mathcal{M}_{bn-1}|}{|\mathcal{M}_{bn-1}|} = \prod_{f \in S} \left[ \sum_{k=0}^{m} i_k(H) \left( 1 - \frac{1}{q^{r \deg f}} \right)^{m-k} \left( \frac{1}{q^{r \deg f}} \right)^k \right]. \]
Hence, \( \{a_{bn-1}\} \) trivially converges to \( D \).

Next, we show \( \{a_{bn+c}\} \) converges to \( D \) as well for each \( c \in \{0, 1, \ldots, n-2\} \).

In a manner reminiscent of the proof to Lemma 2.4, we find that
\[ \left( \frac{bn}{bn+c+1} \right)^m a_{bn-1} \leq a_{bn+c} \leq \left( \frac{(b+1)n}{(b+1)n+c+1} \right)^m a_{(b+1)n-1}. \]
Letting \( b \to \infty \), the Squeeze Theorem implies that \( \{a_{bn+c}\} \) converges to \( D \) for each \( c \in \{0, 1, \ldots, n-2\} \). Finally, since \( \{a_{bn+c}\} \) converges to \( D \) for each \( c \in \{0, 1, \ldots, n-1\} \), we conclude that \( \{a_j\} \) converges to \( D \), as desired. \( \blacksquare \)

Now we are ready to state and prove the main theorem of this section.

**Theorem 1.6.** Fix \( r, j, m \in \mathbb{N} \) such that \( j \leq m \) and \( rm \geq 2 \). Then the density of the set of \( H \)-wise relatively \( r \)-prime ordered \( m \)-tuples of polynomials in \( \mathbb{F}_q[x] \) equals
\[ \prod_{f \text{ irreducible}} \left[ \sum_{k=0}^{m} i_k(H) \left( 1 - \frac{1}{q^{r \deg f}} \right)^{m-k} \left( \frac{1}{q^{r \deg f}} \right)^k \right], \]
where it is understood that the product is over all monic irreducible polynomials in \( \mathbb{F}_q[x] \).

**Proof:** Fix a monic irreducible polynomial \( f \in \mathbb{F}_q[x] \) and let \( K_f \) denote the set of ordered \( m \)-tuples \( (g_1, \ldots, g_m) \) such that \( f \) divides the gcd of \( k \) of the
entries from \((g_1, \ldots, g_m)\) whenever these \(k\) entries form an independent vertex set. Then by Lemma 3.4, we have

\[
\mathcal{D}(K_f) = 1 - \sum_{k=0}^{m} i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k.
\]

However for any \(x \in [0, 1]\), Bernoulli’s Inequality implies that

\[
\sum_{k=0}^{m} i_k(H)x^k(1-x)^{m-k} \geq (1-x)^m + mx(1-x)^{m-1}
\]

\[
= (1-x)^{m-1}(1+(m-1)x)
\]

\[
\geq (1-(m-1)x)(1+(m-1)x)
\]

\[
= 1 - (m-1)^2x^2.
\]

Therefore, letting \(x = q^{-\deg f}\) yields

\[
\mathcal{D}(K_f) \leq \left(\frac{m-1}{q^{r \deg f}}\right)^2.
\]

Next, let \(S_t\) be the set of monic irreducible polynomials of a degree greater or equal to \(t\) where \(t \in \mathbb{N}\), and set \(E_t = ES_t\). Moreover, let \(\hat{S}\) be the set of all monic irreducible polynomials in \(\mathbb{F}_q[x]\). Then,

\[
\mathbb{D}(E_t \setminus E) \leq \limsup_{N \to \infty} \frac{|\bigcup_{f \in \hat{S}\setminus S_t} K_f \cap M_N|}{|M_N|}
\]

\[
\leq \limsup_{N \to \infty} \frac{\sum_{f \in \hat{S}\setminus S_t} |K_f \cap M_N|}{|M_N|}
\]

\[
\leq \sum_{f \in \hat{S}\setminus S_t} \mathbb{D}(K_f).
\]

Since \(\mathbb{D}(K_f) = \mathcal{D}(K_f)\), we obtain

\[
\mathbb{D}(E_t \setminus E) \leq \sum_{f \in \hat{S}\setminus S_t} \mathcal{D}(K_f)
\]

\[
\leq \sum_{f \in \hat{S}\setminus S_t} \left(\frac{m-1}{q^{r \deg f}}\right)^2
\]

\[
= \sum_{j=t+1}^{\infty} \frac{(m-1)^2}{q^{2rj}} \cdot \varphi(j),
\]

where \(\varphi(j)\) denotes the number of monic irreducible polynomials of degree \(j\) in \(\mathbb{F}_q[x]\).
Since any irreducible polynomial over \( \mathbb{F}_q[x] \) with degree \( j \) divides \( x^{q^j} - x \) (which has no multiple roots), we have \( j \cdot \varphi(j) \leq q^j \). Therefore
\[
\overline{D}(E_t \setminus E) \leq \sum_{j=t+1}^{\infty} \frac{(m-1)^2}{j q^{(2r-1)j}} \leq \frac{(m-1)^2}{q^t (q-1)},
\]
in which the last inequality follows from
\[
\sum_{j=t+1}^{\infty} \frac{1}{j q^{(2r-1)j}} = \frac{1}{q^{(2r-1)(t+1)}} \cdot \sum_{j=0}^{\infty} \frac{1}{(j + t + 1) q^{(2r-1)j}} \leq \frac{1}{q^{(2r-1)(t+1)}} \cdot \sum_{j=0}^{\infty} \frac{1}{q^{(2r-1)j}} = \frac{1}{q^t (q-1)}.
\]

Next, since \( E \cap \mathcal{M}_N \subseteq E_t \cap \mathcal{M}_N \), it follows that
\[
\overline{D}(E) \leq \overline{D}(E_t) \leq D(E).
\]
Similarly, since \( E \cap \mathcal{M}_N = (E_t \cap \mathcal{M}_N) - (E_t \setminus \mathcal{M}_N) \), we obtain
\[
\overline{D}(E) \geq \overline{D}(E) - \overline{D}(E \setminus E_t) \geq D(E_t) - \frac{(m-1)^2}{q^t (q-1)}.
\]
Finally noting that \( D(E_t) \) exists, we conclude by letting \( t \to \infty \) that
\[
\overline{D}(E) = \lim_{t \to \infty} \overline{D}(E_t)
= \lim_{t \to \infty} \prod_{f \in S_t} \left[ \sum_{k=0}^{m} i_k(H) \left( 1 - \frac{1}{q^r \deg f} \right)^{m-k} \left( \frac{1}{q^r \deg f} \right)^k \right]
= \prod_{f \text{ irred.}} \left[ \sum_{k=0}^{m} i_k(H) \left( 1 - \frac{1}{q^r \deg f} \right)^{m-k} \left( \frac{1}{q^r \deg f} \right)^k \right],
\]
and this concludes the proof. \( \square \)

In a manner reminiscent of the previous section, we conclude by giving without proof the analogue of Corollary 2.5 for \( \mathbb{F}_q[x] \) as originally given in [4].

**Corollary 3.5.** Fix \( r, j, m \in \mathbb{N} \) such that \( j \leq m \) and \( rm \geq 2 \). Then the density of the set of \( j \)-wise relatively \( r \)-prime ordered \( m \)-tuples of elements in \( \mathbb{F}_q[x] \) equals
\[
\prod_{f \text{ irred.}} \left[ \sum_{k=0}^{j-1} \binom{m}{k} \left( 1 - \frac{1}{q^r \deg f} \right)^{m-k} \left( \frac{1}{q^r \deg f} \right)^k \right],
\]
where it is understood that the product is over all monic irreducible polynomials in \( \mathbb{F}_q[x] \).
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