LOWER ASSOUAD TYPE DIMENSIONS OF UNIFORMLY PERFECT SETS IN DOUBLING METRIC SPACE

HAIPENG CHEN, MIN WU, AND YUANYANG CHANG

Abstract. In this paper, we are concerned with the relationships among the lower Assouad type dimensions. For uniformly perfect sets in a doubling metric space, we obtain a variational result between two different but closely related lower Assouad spectra. As an application, we give an equivalent but more accessible definition of quasi-lower Assouad dimension.

Keywords Quasi-lower Assouad dimension, Lower Assouad spectrum, Lower Assouad dimension.

1. Introduction

The lower Assouad dimension, introduced by Larman [13, 14], is a tool to describe the local scaling properties of a set. Let $(X, d)$ be a metric space, for any non-empty set $E \subset X$, denote by $N_r(E)$ the smallest number of open balls of radius $r$ needed to cover $E$. Let $B(x, R)$ be the open ball centered at $x$ with radius $R$. The lower Assouad dimension is defined by

$$\dim_L E = \sup \left\{ s \geq 0 \mid \text{there exist constants } \rho, c > 0, \text{ such that for any } 0 < r < R < \rho \right. \left. \text{and any } x \in E, N_r(B(x, R) \cap E) \geq c \left( \frac{R}{r} \right)^s \right\}.$$ 

The lower Assouad dimension provides a rigorous gauge on how efficiently a set can be covered in those areas which are easiest to cover. Precisely, it tells the non-trivial minimal exponential growth rate of $N_r(B(x, R) \cap E)$ for two arbitrary scales $0 < r < R$. However, it indicates no information about which scales can witness this rate. To see how the gauge depends on the scales, Fraser and Yu [7] introduced the lower Assouad spectrum, which is a function of $\theta \in (0, 1)$ defined by

$$\dim^\theta_L E = \sup \left\{ s \geq 0 \mid \text{there exist constants } \rho, c > 0, \text{ such that for any } 0 < R < \rho \right. \left. \text{and any } x \in E, N_{R^{1/\theta}}(B(x, R) \cap E) \geq c \left( \frac{R}{R^{1/\theta}} \right)^s \right\}.$$ 

Thus, for each fixed $\theta \in (0, 1)$, we get a ‘restricted’ version of the lower Assouad dimension by letting the scales satisfy the relationship $r = R^{1/\theta}$. Then one can vary $\theta$ and obtain a spectrum of dimensions which gives finer scaling information on the local structure of a set. It turns out that the lower Assouad spectrum takes its values between the lower Assouad dimension and the lower box-counting dimension. One can refer to [5, 7, 8] for more details on properties of lower Assouad dimension and lower Assouad spectrum.
It is also natural to consider another ‘restricted’ version of lower Assouad dimension, which was introduced by Chen, Du and Wei [1]. For any fixed $\theta \in (0, 1)$, they defined

$$\dim^\theta_L E = \sup \left\{ s \geq 0 \mid \text{there exist constants } \rho, c > 0, \text{ such that for any } 0 < r \leq R^{1/\theta} < R < \rho \right. \left. \text{and any } x \in E, N_r(B(x, R) \cap E) \geq c \left( R/r \right)^s \right\}$$

by leaving ‘an exponential gap’ between $r$ and $R$. Note that $\dim^\theta_L E$ is monotonically decreasing as $\theta$ tends to 1. As a result, they defined the quasi-lower Assouad dimension

$$\dim_{qL} E = \lim_{\theta \to 1} \dim^\theta_L E.$$

to get a quasi-Lipschitz invariant. It should be noted that the lower Assouad spectrum $\dim^\theta_L E$ is not necessarily monotonic (see [7, Section 8]), thus $\dim^\theta_L E$ and $\dim_{qL} E$ are not necessarily equal.

We call the above four dimensions the lower Assouad type dimensions. In this paper, we are interested in the relationships among the lower Assouad type dimensions. We denote by $\dim_B E$ the lower box-counting dimension and refer the readers to [4, 17] for the definition. For totally bounded set $E \subset X$ and any $\theta \in (0, 1)$, combining the results of Fraser [5], Fraser and Yu [7] and Chen et al. [1], we have

$$\dim_L E \leq \dim_{qL} E \leq \dim^\theta_L E \leq \dim_{bL} E \leq \dim_B E.$$ 

Moreover, the lower Assouad type dimensions can give an insight into the fractal sets having a certain degree of homogeneity, like self-similar sets, self-affine sets, etc. More discussion of lower Assouad type dimensions of fractal sets can be found in [2, 5, 8, 9, 10].

Owing to the local nature of the definitions, the lower Assouad type dimensions have some strange properties. For instance, the sets containing isolated points have lower Assouad type dimensions zero, and they may take value zero for an open set in $\mathbb{R}$ [see [5, Example 2.5]]. However, Käenmäki et al. [11] proved that the lower Assouad dimension is strict positive if and only if the set is uniformly perfect. On the other hand, Luukkainen [15] showed that the lower Assouad type dimensions of a metric space is finite if it is doubling. To avoid some ‘strange’ sets whose lower Assouad type dimensions are 0 or $\infty$, we are mainly concerned with the uniformly perfect sets in a doubling metric space.

Our first result is stated as follows.

**Theorem 1.1.** Let $X$ be a doubling metric space and $E \subset X$ be a uniformly perfect set. Then for any $\theta \in (0, 1)$, we have

$$\dim^\theta_L E = \inf_{\theta < \theta' \leq 1} \dim_{\theta'}_L E.$$ 

From Theorem 1, we see that all the information of $\dim_{\theta}^L E$ are covered by the lower Assouad spectrum, hence we can study $\dim_{\theta}^L E$ by the lower Assouad spectrum. This result can be seen as a dual of [3, Theorem 2.1]. It follows from [7, Theorem 3.10] that the function $\dim_{\theta}^L E$ is continuous in $\theta \in (0, 1)$ and Lipschitz continuous on any subinterval $[a, b] \subset (0, 1)$. Comparatively, it is complicated to get the continuity of $\dim_{\theta}^L E$ from its definition directly. We can apply Theorem 1 together with [7, Theorem 3.10] to get the following result.

**Theorem 1.2.** Let $X$ be a doubling metric space and $E \subset X$ be a uniformly perfect set. Then $\dim_{\theta}^q L E$ is continuous in $\theta \in (0, 1)$.

Besides, it follows immediately from Theorem 1 that

$$\dim_{qL} E = \lim_{\theta \to 1} \inf_{0 < \theta' \leq \theta} \dim_{\theta'}_L E.$$
With further discussions on lower Assouad spectrum in Section 3, we give an equivalent definition of the quasi-lower Assouad dimension by virtue of the lower Assouad spectrum, which is more convenient for computation. Examples can be found in Section 6.

**Theorem 1.3.** Let $X$ be a doubling metric space and $E \subset X$ be a uniformly perfect set. Then
\[
\dim_{qL} E = \lim_{\theta \to 1} \dim_{\theta}^{E}. \]

In comparison, we do not know what the lower Assouad spectrum approaches to as $\theta$ tends to 0. Then it is natural to ask whether the lower Assouad dimension is strictly smaller than the quasi-lower Assouad dimension or the lower Assouad spectrum. We shall give some examples of Cantor cut-out sets in Section 6 to show that neither lower Assouad spectrum nor quasi-lower Assouad dimension can approach the lower Assouad dimension.

This paper is organized as follows. In Section 2, we discuss the basic properties of the numbers of ball covers and discrete subsets of a set. In Section 3, we study some further properties of lower Assouad spectrum. Section 4 is devoted to the proof of Theorem 1. In Section 5, we prove Theorem 2 and Theorem 3. In Section 6, we discuss the lower Assouad type dimensions of Cantor cut-out sets.

**2. Ball covers and Discrete subsets**

In this section, for a fixed $r > 0$, we discuss the relationship between $r$-ball covers and $r$-discrete subsets for a set $E$.

We first give some notations. A subset $F \subset E$ is said to be a $r$-discrete subset if for any $x, y \in F, x \neq y$, we have $d(x, y) \geq r$. $F$ is called a maximal $r$-discrete subset if $F$ is a $r$-discrete subset and for any $x \in E$, there exist $x_i \in F$ such that $d(x, x_i) < r$. We denote by $M_r(E)$ the largest number of points of $r$-discrete subsets of $E$, that is,
\[
M_r(E) = \sup\{\#F \mid F \text{ is a } r\text{-discrete subset of } E\}.
\]

Recall that $N_r(E)$ denotes the smallest number of open balls of radius $r$ needed to cover the set $E$, it is worth noting that both $M_r(E)$ and $N_r(E)$ could be $\infty$. Since $(X, d)$ is a doubling metric space, then for any bounded set $E \subset X$ and $r > 0$, both $M_r(E)$ and $N_r(E)$ are finite.

Clearly, for any set $E \subset X$ and any $r > 0$,
\[
(2.1) \quad M_{4r}(E) \leq N_r(E) \leq M_r(E).
\]

The right hand side of (2.1) is obvious, since let $\{x_n\}_{n \geq 1}$ be any maximal $r$-discrete subset of $E$, then $\{B(x_n, r)\}_{n \geq 1}$ is a $r$-ball cover of $E$. As for the left hand side of (2.1), let $\{y_n\}_{n \geq 1}$ be a $4r$-discrete subset of $E$, then the result follows from a fact that each ball $B(x, r)$ contains at most one point of $\{y_n\}_{n \geq 1}$.

From now on, we assume that $E \subset X$ is a subset of a doubling metric space $X$. For any ball $B(x, R)$ and $0 < r < R$, the first lemma gives an inequality between $N_r(B(x, R) \cap E)$ and $N_{4r}(B(x, R) \cap E)$.

**Lemma 2.1.** For any $x \in X$ and $0 < r < R$, we have
\[
N_r(B(x, R) \cap E) \leq N_{4r}(B(x, R) \cap E) \cdot \sup_{y \in X} N_r(B(y, 4r) \cap E).
\]

**Proof.** The case $4r \geq R$ is trivial. As for the case $4r < R$, let $\{B(x_i, 4r)\}_{i=1}^N$ be a $4r$-ball cover of $B(x, R) \cap E$. For any $1 \leq i \leq N$, let $\{B(x_{i,j}, r)\}_{j=1}^{n_i}$ be a $r$-ball cover of $B(x_i, 4r) \cap E$. Then we obtain that
\[
B(x, R) \cap E \subset \bigcup_{i=1}^N \bigcup_{j=1}^{n_i} B(x_{i,j}, r).
\]
Hence the result holds.

The second lemma concerns the relationship between $M_{r_1}(B(x, R) \cap E)$ and $M_{r_2}(B(x, R/4) \cap E)$ for any $0 < r_1 \leq \frac{R}{4}$ and $0 < r_2 \leq \frac{R}{4}$.

**Lemma 2.2.** For any $x \in E$ and $0 < r_1 \leq \frac{R}{4}, 0 < r_2 \leq \frac{R}{4}$, we have

$$M_{r_1}(B(x, R) \cap E) \geq M_{r_2}(B(x, R/4) \cap E) \cdot \inf_{y \in E} M_{r_1}(B(y, \frac{r_2}{4}) \cap E).$$

**Proof.** It follows from (2.1) that for any fixed $0 < r < R$, $M_{r}(B(x, R) \cap E)$ is uniformly bounded for any $x \in X$. Hence there exists a maximal $r_2$-discrete subset $\{x_1, x_2, \ldots, x_n\}$ of $B(x, \frac{R}{4}) \cap E$ such that $n_1 = M_{r_1}(B(x, \frac{R}{4}) \cap E)$. For any $1 \leq i \leq n_1$, denote by $\{y_1^{(i)}, y_2^{(i)}, \ldots, y_{m_i}^{(i)}\}$ a maximal $r_1$-discrete subset of $B(x_i, \frac{r}{4}) \cap E$. Then $\{y_j^{(i)}\}_{1 \leq j \leq m_i, 1 \leq i \leq n_1}$ is a $r_1$-discrete subset of $B(x, R) \cap E$. It follows from the definition of $M_{r_1}(B(x, R) \cap E)$ that

$$M_{r_1}(B(x, R) \cap E) \geq \sum_{i=1}^{n_1} m_i \cdot \inf_{y \in E} M_{r_1}(B(y, \frac{r_2}{4}) \cap E) \geq M_{r_1}(B(x, \frac{R}{4}) \cap E) \cdot \inf_{y \in E} M_{r_1}(B(y, \frac{r_2}{4}) \cap E).$$

□

For a fixed $\theta \in (0, 1)$ and any $R > 0$ with $4R^{\frac{1}{\theta}} < \frac{R}{4}$, the third lemma deals with the relationship between $M_{4R^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E)$ and $M_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E)$.

**Lemma 2.3.** For any $\theta \in (0, 1)$, there exist a constant $C(\theta) > 1$ such that for any $R > 0$ with $4R^{\frac{1}{\theta}} < \frac{R}{4}$, we have

$$M_{4R^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E) \geq C(\theta)^{-1} \cdot M_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E).$$

**Proof.** It follows from (2.1) that for any $x \in E$ and any $R > 0$ with $4R^{\frac{1}{\theta}} < \frac{R}{4}$, we have

$$M_{4R^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E) \geq N_{4R^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E).$$

For $N_{4R^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E)$, similar to the proof of Lemma 2.1, we can verify that

$$N_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E) \leq N_{4R^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E) \cdot \sup_{y \in X} N_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(y, 4R^{\frac{1}{\theta}}) \cap E).$$

Since $X$ is doubling, there exists a constant $C_1(\theta) > 1$ such that

$$\sup_{y \in X} N_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(y, 4R^{\frac{1}{\theta}}) \cap E) \leq C_1(\theta).$$

For $N_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E)$, similar to (2.2) and (2.3), we obtain

$$N_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E) \leq N_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(x, \frac{R}{4}) \cap E) \cdot \sup_{y \in X} N_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(y, \frac{R}{4}) \cap E),$$

and there exists a constant $C_2 > 1$ such that

$$\sup_{y \in X} N_{(\frac{R}{4})^{\frac{1}{\theta}}}(B(y, \frac{R}{4}) \cap E) \leq C_2.$$

Hence the result holds by combining (2.1)–(2.5). □
3. A property of lower Assouad spectrum

It was shown by Fraser and Yu [7] that the lower Assouad spectrum is continuous but not necessarily monotonic in $(0, 1)$. However, the following result indicates that the lower Assouad spectrum is ‘monotonic’ in some weak form. We always assume that $X$ is a doubling metric space, and $E \subset X$ is uniformly perfect.

**Proposition 3.1.** Let $(X, d)$ be a doubling metric space and $E \subset X$ be a uniformly perfect set. Then for any $0 < \theta_1 < \theta_2 < 1$, we have

$$\dim^\theta_L E \geq \left( \frac{\theta_2 - \theta_1}{1 - \theta_1} \right) \dim^\theta_2 L E + \left( \frac{1 - \theta_2}{1 - \theta_1} \right) \dim^\theta_1 L E.$$ 

**Proof.** For any fixed $0 < \theta_1 < \theta_2 < 1$ and for any $\varepsilon > 0$, it follows from the definition of $\dim^\theta_L E$ that there exist $\{x_i\}_{i=1}^\infty \subset E$ and $\{R_i\}_{i=1}^\infty$ satisfying $R_i \to 0$ as $i \to \infty$, such that for any $i \geq 1$, we have

$$N_{\frac{R_i}{R_i^{\theta}}} (B(x_i, R_i) \cap E) \leq \left( \frac{R_i}{R_i^{\theta}} \right)^{\dim^\theta_L E + \varepsilon}. $$

Then it follows from (2.1) and Lemma 2.2 that for sufficiently large $i$,

$$N_{\frac{R_i}{R_i^{\theta}}} (B(x_i, R_i) \cap E) \geq M_{\frac{R_i}{R_i^{\theta}}} (B(x_i, R_i) \cap E)$$

(3.1)

$$\geq M_{\frac{R_i}{R_i^{\theta}}} (B(x_i, \frac{R_i^\theta}{4}) \cap E) \cdot \inf_{y \in E} M_{\frac{R_i}{R_i^{\theta}}} (B(y, \frac{R_i^\theta}{4}) \cap E)$$

By Lemma 2.3 there exist a constant $C_0(\theta_1, \theta_2) > 0$ such that

$$\inf_{y \in E} M_{\frac{R_i}{R_i^{\theta}}} (B(y, \frac{R_i^\theta}{4}) \cap E) \geq C_0(\theta_1, \theta_2) \cdot \inf_{y \in E} M_{\frac{R_i}{R_i^{\theta}}} (B(y, \frac{R_i^\theta}{4}) \cap E)$$

(3.2)

$$\geq C_0(\theta_1, \theta_2) \cdot M_{\frac{R_i}{R_i^{\theta}}} (B(x_i, \frac{R_i^\theta}{4}) \cap E)$$

and

$$M_{\frac{R_i}{R_i^{\theta}}} (B(x_i, \frac{R_i^\theta}{4}) \cap E) \geq M_{\frac{R_i}{R_i^{\theta}}} (B(x_i, \frac{R_i^\theta}{4}) \cap E)$$

(3.3)

$$\geq C_0(\theta_1, \theta_2) \cdot M_{\frac{R_i}{R_i^{\theta}}} (B(x_i, \frac{R_i^\theta}{4}) \cap E).$$

Since

$$\frac{R_i^{\frac{\theta_2}{\theta_1}}}{\frac{R_i^\theta}{4}} = \left( \frac{R_i^{\frac{\theta_2}{\theta_1}}}{\frac{R_i^\theta}{4}} \right)^{\frac{\theta_2}{\theta_1}} = \left( \frac{R_i^\theta}{4} \right)^{\frac{\theta_2}{\theta_1}} \left( \frac{R_i^\theta}{4} \right)^{\frac{\theta_2 - \theta_1}{\theta_1}},$$

then it follows from (2.1), (3.1), (3.2), (3.3), and the definition of lower Assouad spectrum that there exists a constant $C(\theta_1, \theta_2) > 0$ such that for any sufficiently large $i$, we have

$$C(\theta_1, \theta_2) \cdot R_i^{(1 - \frac{\theta_2}{\theta_1})(\dim^\theta_2 L E - \varepsilon)} \cdot R_i^{(\frac{\theta_2}{\theta_1} - \frac{\theta_1}{\theta_1})(\dim^\theta_1 L E - \varepsilon)} \leq R_i^{(1 - \frac{\theta_2}{\theta_1})(\dim^\theta_1 L E + \varepsilon)}.$$

It implies that

$$\log C(\theta_1, \theta_2) + \left( \frac{\theta_2}{\theta_1} - 1 \right) \cdot (\dim^\theta_2 L E - \varepsilon) + \left( \frac{1}{\theta_1} - \frac{\theta_2}{\theta_1} \right) \cdot (\dim^\theta_1 L E - \varepsilon) \leq \left( \frac{1}{\theta_1} - 1 \right) \cdot (\dim^\theta_1 L E + \varepsilon).$$

Hence the result holds by taking $i \to \infty$ and then letting $\varepsilon \to 0$. □
Corollary 3.1. Let $E$ satisfy the assumptions of Proposition 3.1. For any $0 < \theta_1 < \theta_2 < \cdots < \theta_n < 1$, we have
\[
\dim_{\mathcal{L}}^\theta E \geq \left(\frac{1 - \theta_n}{1 - \theta_1}\right) \dim_{\mathcal{L}}^{\theta_1} E + \sum_{i=1}^{n-1} \left(\frac{\theta_{i+1} - \theta_i}{1 - \theta_1}\right) \dim_{\mathcal{L}}^{\frac{\theta_i}{1 - \theta_1}} E.
\]

Proof. This result directly comes from Proposition 3.1 by induction. \qed

Corollary 3.2. Let $E$ satisfy the assumptions of Proposition 3.1. For any $n \geq 1$ and any $0 < \theta < 1$, we have
\[
\dim_{\mathcal{L}}^\theta E \geq \dim_{\mathcal{L}}^\theta E.
\]

Proof. This result follows from Corollary 3.1 by taking $\theta_i = \frac{n-i+1}{n}$ for any $1 \leq i \leq n$. \qed

Remark. As Corollary 3.2 shows, there exist an increasing subsequence $\{\theta_n\}_{n=1}^\infty$ with $\theta_n \to 1$ as $n \to \infty$ such that $\{\dim_{\mathcal{L}}^\theta E\}_{n=1}^\infty$ is monotonically decreasing as $n \to \infty$.

4. Proof of Theorem 1.1

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. It follows immediately from the definitions of $\dim_{\mathcal{L}}^\theta E$ that for any $0 < \theta < 1$,
\[
\dim_{\mathcal{L}}^\theta E \leq \inf_{0 < \theta' \leq \theta} \dim_{\mathcal{L}}^{\theta'} E.
\]

Hence we only need to prove
\[
(4.1) \quad \inf_{0 < \theta' \leq \theta} \dim_{\mathcal{L}}^{\theta'} E \leq \dim_{\mathcal{L}}^\theta E.
\]

If (4.1) does not hold, then there exist $\theta \in (0, 1)$ such that
\[
(4.2) \quad \inf_{0 < \theta' \leq \theta} \dim_{\mathcal{L}}^{\theta'} E > \dim_{\mathcal{L}}^\theta E,
\]
hence there exist sufficient small $\varepsilon_0 > 0$ such that
\[
\inf_{0 < \theta' \leq \theta} \dim_{\mathcal{L}}^{\theta'} E > \dim_{\mathcal{L}}^\theta E + 3\varepsilon_0.
\]

Fix $\theta$ and denote $s = \dim_{\mathcal{L}}^\theta E > 0$. For any $0 < \varepsilon < \varepsilon_0$, by definition, there exist $\{(r_i, R_i)\}_{i=1}^\infty$ with $0 < r_i \leq R_i^{1/\theta} < R_i < 1$, $R_i \to 0$ as $i \to \infty$ and $\{x_i\}_{i=1}^\infty \subset E$ such that for any $i \geq 1$,
\[
(4.3) \quad N_{r_i}(B(x, R_i) \cap E) \leq \left(\frac{R_i}{r_i}\right)^{s+\varepsilon}.
\]

It is worth noting that $\frac{R_i}{r_i} \to \infty$ as $i \to \infty$. For each $i$, let $\theta_i$ be the root of $r_i = R_i^{1/\theta}$. It follows from $\{\theta_i\}_{i=1}^\infty \subset [0, \theta]$ that there exist a subsequence $\{\theta_{i_j}\}_{j=1}^\infty$ and a point $\theta_0 \in [0, \theta]$ such that $\theta_{i_j} \to \theta_0$ as $j \to \infty$. Without loss of generality, we may assume that $\theta_i \to \theta_0$ as $i \to \infty$. We may also assume that the sequence $\{\theta_{i_j}\}_{j=1}^\infty$ is either increasing or decreasing.

We divide the proof into three cases.

Case 1. Assume $\{\theta_{i_j}\}_{j=1}^\infty$ is monotonically increasing and $\theta_0 > 0$.

For any $i \geq 1$, we see that $R_i^{\theta_{i_j}} < R_i^{\theta}$ for any $0 < R < 1$. For any small $\delta > 0$, we have
\[
0 < \theta_0 - \delta < \delta \quad \text{for arbitrary large } i.
\]
Since for any $i \geq 1$,
\[
N_{R_i^{\theta_{i_j}}}(B(x, R_i) \cap E) \leq N_{R_i^{\theta_{i_j}}}(B(x, R_i) \cap E),
\]

\[
\text{(4.3)} \quad N_{r_i}(B(x, R_i) \cap E) \leq \left(\frac{R_i}{r_i}\right)^{s+\varepsilon}.
\]
then it follows from the definition of lower Assouad spectrum and (4.3) that

\[ R_i^{(1 - \frac{1}{\theta_i})(\dim_L^0 E - \varepsilon)} \leq R_i^{(1 - \frac{1}{\theta_i})(s + \varepsilon)}. \]

Taking logarithm on both sides, letting \( i \to \infty \) and then letting \( \varepsilon \to 0 \), we obtain

\[ \dim_L^0 E \leq s, \]

which contradicts with (4.2).

**Case 2.** Assume \( \{\theta_i\}_{i=1}^{\infty} \) is monotonically decreasing and \( \theta_0 > 0 \).

Without loss of generality, we suppose that \( \theta_i < 2\theta_0 \) for any \( i \geq 1 \). For any \( i \geq 1 \), we see that \( R_i^{\frac{1}{\theta_i}} > R_i^{\frac{1}{\theta_0}} \) for any \( 0 < R < 1 \). For any small \( \delta > 0 \), we have \( 0 < \theta_i - \theta_0 < \delta \) for arbitrary large \( i \).

Since \((X, d)\) is a doubling metric space, then

\[ \sup_{y \in X} N_{\frac{1}{R_i}}(B(y, R_i) \cap E) \leq C \left( \frac{R_i^{\frac{1}{\theta_0}}}{R_i^{\frac{1}{\theta_0}}} \right)^{\log_2 C}, \]

where \( C \) is the doubling constant.

Similar to the proof of Lemma [2.1] for any \( i \geq 1 \), we have

\[ N_{\frac{1}{R_i}}(B(x_1, R_i) \cap E) \leq N_{\frac{1}{R_i}}(B(x_1, R_i) \cap E) \cdot \sup_{y \in X} N_{\frac{1}{R_i}}(B(x, R_i) \cap E). \]

By (4.3), (4.4), (4.5) and the definition of lower Assouad spectrum, there exists a constant \( C' > 0 \) such that

\[ R_i^{(1 - \frac{1}{\theta_0})(\dim_L^0 E - \varepsilon)} \leq C' \cdot R_i^{(1 - \frac{1}{\theta_0})(s + \varepsilon)} \cdot R_i^{(\frac{1}{\theta_0} - \frac{1}{\theta_i}) \log_2 C}. \]

Therefore,

\[ \left( 1 - \frac{1}{\theta_0} \right) \cdot (\dim_L^0 E - \varepsilon) \geq \left( 1 - \frac{1}{\theta_i} \right) \cdot (s + \varepsilon) + \left( \frac{1}{\theta_i} - \frac{1}{\theta_0} \right) \cdot \log_2 C + \frac{\log_2 C'}{\log_2 R_i}. \]

Letting \( i \to \infty \) and then letting \( \varepsilon \to 0 \), we have \( \dim_L^0 E \leq s \). It also contradicts with (4.2).

**Case 3.** Assume \( \{\theta_i\}_{i=1}^{\infty} \) is monotonically decreasing and \( \theta_0 = 0 \).

Since \( E \) is doubling, we may assume that the doubling constant \( \log C > s + 2\varepsilon \). For any sufficiently small \( 0 < \varepsilon < \varepsilon_0 \), by (4.2), there exist \( \psi \in (0, \theta) \) such that

\[ \frac{\log \psi}{\log \theta} \notin \mathbb{Q} \]

and \( \min\{\dim_L^0 E, \dim_L^0 E\} > s + 3\varepsilon \). This implies that there exist constant \( \rho, c > 0 \) such that for any \( 0 < R < \rho \),

\[ N_{\frac{1}{R_i}}(B(x, R) \cap E) \geq c \left( \frac{R}{R_i^{\frac{1}{\theta_0}}} \right)^{s + 3\varepsilon}, \]

\[ N_{\frac{1}{R_i}}(B(x, R) \cap E) \geq c \left( \frac{R}{R_i^{\frac{1}{\theta_0}}} \right)^{s + 3\varepsilon}. \]

By the irrationality of \( \frac{\log \psi}{\log \theta} \) and applying Chebyshev Theorem (see [12] Theorem 24) in inhomogeneous Diophantine approximation, for any sufficient small \( \eta > 0 \) and for any \( \theta_i \), there exist \( m \in \mathbb{N}, n \in \mathbb{Z} \) related to \( \theta_i \) such that

\[ |\log \psi^m \theta^n - \log \theta_i| \leq \eta. \]
Taking $\eta > 0$ satisfying $\max\{e^\eta - 1, 1 - e^{-\eta}\} < \frac{\varepsilon}{4 \log C}$, we have

\[
\left| \frac{1}{\theta_i} - \frac{1}{\psi m \theta^n} \right| \leq \frac{\varepsilon}{(4 \log C) \cdot \theta_i}.
\]

We now give two Claims for the rest of the proof. We will repeatedly use (2.1) and the doubling property of $E$.

**Claim 1.** Fix $0 < \varepsilon < \varepsilon_0$. Let $\theta_i$, $m$, $n$ be as stated above. If $m, n \geq 1$, then for any $R > 0$ with $\max\{4R^{\frac{1}{m}}, 4R^{\frac{1}{n}}\} < \frac{R}{4}$ and any $x \in E$, we have

\[
N_{R^{\frac{1}{m}}} (B(x, R) \cap E) \geq \left( \frac{R}{R^{\frac{1}{m}}} \right)^{s+2\varepsilon}.
\]

**Proof of Claim 1.** By (2.1), we have

\[
N_{R^{\frac{1}{m}}} (B(x, R) \cap E) \geq M_{4R^{\frac{1}{m}}} (B(x, R) \cap E).
\]

It follows from Lemma 2.2 that for any $x \in E$ and sufficiently small $R > 0$, we have

\[
M_{4R^{\frac{1}{m}}} (B(x, R) \cap E) \geq \inf_{y_1 \in E} M_{4R^{\frac{1}{m}}} (B(y_1, \frac{R}{4}) \cap E) \cdot \inf_{y \in E} M_{4R^{\frac{1}{m}}} (B(y, \frac{R}{4}) \cap E).
\]

By combining (4.1), (4.2) and repeatedly using Lemma 2.2, we obtain

\[
N_{R^{\frac{1}{m}}} (B(x, R) \cap E) \geq \inf_{y_1 \in E} M_{4R^{\frac{1}{m}}} (B(y_1, \frac{R}{4}) \cap E) \cdot \inf_{y \in E} M_{4R^{\frac{1}{m}}} (B(y, \frac{R}{4}) \cap E) \cdot \ldots
\]

\[
\times \inf_{y_{m+n} \in E} M_{4R^{\frac{1}{m}}} (B(y_{m+n}, \frac{R^{\frac{1}{m}}}{4}) \cap E)
\]

Besides, since $E$ is doubling, then by Lemma 2.1, for any $x \in E$ and sufficiently small $R > 0$, we have

\[
\sup_{y \in E} N_{R^{\frac{1}{m}}} (B(y, 4R^{\frac{1}{m}}) \cap E) \cdot N_{R^{\frac{1}{m}}} (B(x, \frac{R}{4}) \cap E) \geq N_{R^{\frac{1}{m}}} (B(x, \frac{R}{4}) \cap E).
\]

Hence there exist a constant $C_1 > 0$ such that for any $x \in E$ and $R > 0$,

\[
N_{4R^{\frac{1}{m}}} (B(x, \frac{R}{4}) \cap E) \geq C_1 \cdot N_{R^{\frac{1}{m}}} (B(x, \frac{R}{4}) \cap E).
\]

By the assumption, for any $x \in E$ and sufficiently small $R > 0$, we have

\[
M_{4R^{\frac{1}{m}}} (B(x, \frac{R}{4}) \cap E) \geq N_{4R^{\frac{1}{m}}} (B(x, \frac{R}{4}) \cap E) \quad \text{ (by 2.1) }
\]

\[
\geq C_1 \cdot N_{R^{\frac{1}{m}}} (B(x, \frac{R}{4}) \cap E) \quad \text{ (by 4.10) }
\]

\[
\geq c \cdot C_1 \cdot \left( \frac{R}{R^{\frac{1}{m}}} \right)^{s+3\varepsilon} \quad \text{ (by definition) }
\]

\[
\geq c \cdot C_1 \cdot \left( \frac{R}{R^{\frac{1}{m}}} \right)^{s+3\varepsilon} \cdot \left( \frac{R}{R^{\frac{1}{m}}} \right)^{s+2\varepsilon} \cdot \left( \frac{R}{R^{\frac{1}{m}}} \right)^{s+2\varepsilon} \quad \text{ (since $R$ is arbitrary small) }
\]

\[
\geq \left( \frac{R}{R^{\frac{1}{m}}} \right)^{s+2\varepsilon},
\]
that is,

\[(4.11)\quad M_{4R^\#}(B(x, \frac{R}{4}) \cap E) \geq \left(\frac{R}{R^\#}\right)^{s+2\varepsilon}.
\]

Similarly, for any \( n \geq 1 \) and \( m \geq 1 \), for any \( x \in E \) and sufficiently small \( R > 0 \), we have

\[(4.12)\quad M_{4R^\#}(B(x, R^{\#^\#}) \cap E) \geq \left(\frac{R}{R^\#}\right)^{s+2\varepsilon}.
\]

Then (4.9), (4.11) together with (4.12) imply

\[N_{R^{\#^\#}}(B(x, R) \cap E) \geq \left(\frac{R}{R^\#}\right)^{s+2\varepsilon} \cdot \left(\frac{R^\#}{R^{\#^\#}}\right)^{s+2\varepsilon} \cdot \cdots \cdot \left(\frac{R^{\#^\#}}{R^{\#^\#}}\right)^{s+2\varepsilon} \geq \left(\frac{R}{R^{\#^\#}}\right)^{s+2\varepsilon}.
\]

The proof of Claim 1 is finished.

**Claim 2.** Fix \( 0 < \varepsilon < \varepsilon_0 \). Let \( \theta_i \leq \min\{\psi^3, (\frac{\log C}{\log \varepsilon} \cdot \frac{1}{\psi^2}) \} \) and \( m, n \) be as stated above. If \( m \geq 1 \), \( n \leq -1 \), then for any \( R > 0 \) with \( \max\{4R^\#, 4R^\#\} < \frac{R}{4} \) and any \( x \in E \), we have

\[N_{R^{\#^\#}}(B(x, R) \cap E) \geq \left(\frac{R}{R^\#}\right)^{s+2\varepsilon}.
\]

**Proof of Claim 2.** By (4.6), there exist \( 1 \leq l_0 < m \) such that

\[(4.13)\quad \frac{1}{\psi} \leq \frac{1}{\psi^l_0 \theta^n} \leq \frac{1}{\psi^2}.
\]

Hence

\[4R^{\#^\#} \leq 4R^\# < \frac{R}{4}
\]

and for any \( i \geq 1 \),

\[R^{\#^\#} \cdot \frac{1}{\psi^l_0 \theta^n} \leq R^\# \cdot \frac{1}{\psi} \leq \frac{1}{16}.
\]

By the assumption of \( \theta_i \) and (4.6), we obtain

\[(4.14)\quad \frac{\varepsilon}{2} \cdot \left(\frac{1}{\psi^m \theta^n} - 1\right) \geq \frac{\varepsilon}{2} \cdot \left(\frac{1}{\theta_i} \left(1 - \frac{\varepsilon}{4 \log C}\right) - 1\right) \geq \log_2 C \cdot \left(\frac{1}{\psi^2} \cdot \frac{1}{\theta_i}\right).
\]

It follows from (2.1) and Lemma 2.2 that

\[N_{R^{\#^\#}}(B(x, R) \cap E) \geq M_{4R^{\#^\#}}(B(x, R) \cap E) \quad \text{(by (2.1))}
\]

\[\geq M_{4R^{\#^\#}}(B(x, \frac{R}{4}) \cap E) \quad \text{(by Lemma 2.2)}
\]

\[\times \inf_{y \in E} M_{4R^{\#^\#}}(B(y, \frac{R}{4}) \cap E).
\]
For the lower bound of $M_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E)$, by (2.1) and the proof of Lemma 2.1, we have (4.16)
\[
M_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \leq N_{\frac{R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \quad \text{(by 2.1)}
\]
\[
\leq N_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \quad \text{(by the proof of Lemma 2.1)}
\]
\[
\times \sup_{y \in X} N_{\frac{R}{\psi(y)}}(B(y, 4R_{\psi(y)}) \cap E)
\]
\[
\leq M_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \cdot \sup_{y \in X} N_{\frac{R}{\psi(y)}}(B(y, 4R_{\psi(y)}) \cap E) \quad \text{(by 2.1)}.
\]

We now estimate $\sup_{y \in X} N_{\frac{R}{\psi(y)}}(B(y, 4R_{\psi(y)}) \cap E)$. Because $E$ is doubling and
\[
\frac{1}{\psi(y)} - \frac{1}{\psi(y_0)} \leq \frac{1}{\psi(y)} < \infty,
\]
we obtain
\[
\sup_{y \in X} N_{\frac{R}{\psi(y)}}(B(y, 4R_{\psi(y)}) \cap E) \leq C^{\log_2(4R_{\psi(y)} \cdot \frac{1}{\psi(y)})} \leq C^2 \cdot R\left(\frac{1}{\psi(y)} - \frac{1}{\psi}\right) \log_2 C.
\]

Now we arrive at
(4.17) \quad M_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \leq C^2 \cdot R\left(\frac{1}{\psi(y)} - \frac{1}{\psi}\right) \log_2 C \cdot M_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E).

Then it suffices to estimate the lower bound of $M_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E)$. Since $E$ is doubling and
\[
N_{\frac{R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \leq N_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \cdot \sup_{y \in X} N_{\frac{R}{\psi(y)}}(B(y, 4R_{\psi(y)}) \cap E),
\]
there exist a constant $C_2 > 0$ such that for any $x \in E$ and $R > 0$,
(4.18) \quad C_2^{-1} \cdot N_{\frac{R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \leq N_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E).

Combining (4.17) and (4.18), we have
(4.19) \quad M_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \geq (C^2 \cdot R\left(\frac{1}{\psi(y)} - \frac{1}{\psi}\right) \log_2 C)^{-1} M_{\frac{4R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \quad \text{(by 4.17)}
\]
\[
\geq (C^2 \cdot R\left(\frac{1}{\psi(y)} - \frac{1}{\psi}\right) \log_2 C)^{-1} N_{\frac{R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \quad \text{(by 2.1)}
\]
\[
\geq (C^2 \cdot R\left(\frac{1}{\psi(y)} - \frac{1}{\psi}\right) \log_2 C)^{-1} C_2^{-1} \cdot N_{\frac{R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E) \quad \text{(by 4.18)}.
\]

Now we estimate the lower bound of $N_{\frac{R}{\psi(y)}}(B(x, \frac{R}{4}) \cap E)$. Since $\dim_L E > s + 3\varepsilon$, then by Corollary 3.2
\[
\dim_L E \geq \dim_L E > s + 3\varepsilon.
\]
Hence it follows from the definition of \( \dim_L^\psi E \) and (4.19) that for sufficiently small \( R > 0 \)

\[
M_{4R \psi^{n+1} (B(x, R/4) \cap E)} \geq (C^2 \cdot R^{(\frac{1}{\psi^{n+1}} - \frac{1}{\psi^n}) \log_2 C})^{-1} C_2^{-1} N_{\psi^{n+1} (B(x, R/4) \cap E)} \quad \text{(by (4.19))}
\]

\[
\geq c \cdot (C^2 \cdot R^{(\frac{1}{\psi^{n+1}} - \frac{1}{\psi^n}) \log_2 C})^{-1} C_2^{-1} \cdot \left( \frac{R}{(\frac{1}{\psi^{n+1}})} \right)^{s+3c} \quad \text{(by definition of \( \dim_L^\psi E \))}
\]

\[
\geq \left( \frac{R}{(R_{\psi^{n+1}})} \right)^{s+\frac{s}{2c}} \cdot (R^{(\frac{1}{\psi^{n+1}} - \frac{1}{\psi^n}) \log_2 C})^{-1} \quad \text{(since \( R \) is sufficiently small)}
\]

\[
\geq \left( \frac{R}{(R_{\psi^{n+1}})} \right)^{s+\frac{s}{2c}} \cdot (R^{(\frac{1}{\psi^{n+1}} - \frac{1}{\psi^n}) \log_2 C})^{-1} \quad \text{(by (4.13))}
\]

which implies that

\[
M_{4R \psi^{n+1} (B(x, R/4) \cap E)} \geq \left( \frac{R}{(R_{\psi^{n+1}})} \right)^{s+\frac{s}{2c}} \cdot (R^{(\frac{1}{\psi^{n+1}} - \frac{1}{\psi^n}) \log_2 C})^{-1}.
\]

On the other hand, to estimate the lower bound of \( \inf_{y \in E} M_{4R \psi^{n+1} (B(y, R_{\psi^{n+1}}/4) \cap E)} \), like (4.19), it suffices to consider

\[
\inf_{y \in E} M_{4R \psi^{n+1} (B(y, R_{\psi^{n+1}}/4) \cap E)} \quad \text{for each} \quad 1 \leq i \leq m - l_0.
\]

Similar to the proof of (4.11), we have

\[
M_{4R \psi^{n+1} (B(y, R_{\psi^{n+1}}/4) \cap E)} \geq \left( \frac{R_{\psi^{n+1}}}{R} \right)^{s+\frac{s}{2c}}.
\]

By combining (4.11), (4.21), (4.22) and repeatedly using Lemma 2.2 we have

\[
N_{\psi^{n+1} (B(x, R) \cap E)} \geq \left( \frac{R}{(R_{\psi^{n+1}})} \right)^{s+\frac{s}{2c}} \cdot \left( \frac{R_{\psi^{n+1}}}{R} \right)^{s+\frac{s}{2c}} \cdot \left( \frac{R_{\psi^{n+1}}}{R_{\psi^{n+1}}/4} \right)^{s+\frac{s}{2c}} \cdot (R^{(\frac{1}{\psi^{n+1}} - \frac{1}{\psi^n}) \log_2 C})^{-1}
\]

\[
\geq \left( \frac{R}{(R_{\psi^{n+1}})} \right)^{s+2c} \cdot \left( \frac{R_{\psi^{n+1}}}{R_{\psi^{n+1}}/4} \right)^{s+2c} \cdot \left( \frac{R_{\psi^{n+1}}}{R_{\psi^{n+1}}/4} \right)^{s+2c} \cdot (R^{(\frac{1}{\psi^{n+1}} - \frac{1}{\psi^n}) \log_2 C})^{-1}
\]

\[
\geq \left( \frac{R}{(R_{\psi^{n+1}})} \right)^{s+2c} \cdot (R^{(\frac{1}{\psi^{n+1}} - \frac{1}{\psi^n}) \log_2 C})^{-1} \cdot \left( \frac{R}{(R_{\psi^{n+1}})} \right)^{s+2c} \quad \text{(by (4.11))}
\]

The proof of Claim 2 is finished.

We now continue the proof of Case 3.
If
\[ 0 < \frac{1}{\psi^m \theta^n} - \frac{1}{\theta_i} \leq \frac{\varepsilon}{(4 \log C) \cdot \theta_i}, \]
then for sufficiently large \( i \), we have
\[ N_{R_i^{\psi^m \theta^n}}(B(x_i, R_i) \cap E) \leq N_{R_i^{\psi^m \theta^n}}(B(x_i, R_i) \cap E) \cdot \sup_{y \in X} N_{R_i^{\psi^m \theta^n}}(B(y, R_i^\frac{1}{\theta_i}) \cap E). \]

Since \( E \) is doubling, we obtain
\[ \sup_{y \in X} N_{R_i^{\psi^m \theta^n}}(B(y, R_i^\frac{1}{\theta_i}) \cap E) \leq C \cdot \left( \frac{R_i^{\psi^m \theta^n}}{R_i^{\psi^m \theta^n}} \right)^{\log C}. \]

By (4.23), (4.24), (4.25) if \( m, n \geq 1 \), or by Claim 2, (4.23), (4.24), (4.25) if \( m \geq 1 \), \( n \leq -1 \), for sufficiently small \( R_i > 0 \), we obtain that
\[ \left( \frac{1}{\psi^m \theta^n} - 1 \right) \cdot (s + 2 \varepsilon) \leq \left( \frac{1}{\theta_i} - 1 \right) \cdot (s + \varepsilon) + \frac{\varepsilon}{4 \theta_i} + \frac{\log C}{-\log R_i}, \]
thus,
\[ (1 - \theta_i) \cdot (s + 2 \varepsilon) \leq (1 - \theta_i) \cdot (s + \varepsilon) + \frac{\varepsilon}{4} + \frac{\log C}{-\log R_i} \cdot \theta_i. \]

Letting \( i \to \infty \), we obtain that
\[ s + 2 \varepsilon < s + \frac{3}{2} \varepsilon, \]
which implies a contradiction.

Otherwise, if
\[ 0 < \frac{1}{\psi^m \theta^n} - \frac{1}{\theta_i} \leq \frac{\varepsilon}{(4 \log C) \cdot \theta_i}, \]
then for any \( i \geq 1 \), we have
\[ N_{R_i^{\psi^m \theta^n}}(B(x_i, R_i) \cap E) \leq N_{R_i^{\psi^m \theta^n}}(B(x_i, R_i) \cap E). \]

Similarly, by Claim 1 and (4.26) if \( m, n \geq 1 \), or by Claim 2 and (4.26) if \( m \geq 1 \), \( n \leq -1 \), for sufficiently small \( R_i > 0 \), we have
\[ \left( \frac{R_i}{R_i^{\psi^m \theta^n}} \right)^{s+2\varepsilon} \leq \left( \frac{R_i}{R_i^{\psi^m \theta^n}} \right)^{s+\varepsilon}, \]
thus,
\[ \left( \frac{1}{\psi^m \theta^n} - 1 \right) \cdot (s + 2 \varepsilon) \leq \left( \frac{1}{\theta_i} - 1 \right) \cdot (s + \varepsilon), \]
which implies that
\[ (1 - \theta_i) \cdot (s + 2 \varepsilon) \leq (1 - \theta_i) \cdot (s + \varepsilon) + \frac{s + 2 \varepsilon}{4 \log C} \varepsilon. \]
Letting \( i \to \infty \), we obtain that
\[
s + 2\varepsilon < s + \frac{5\varepsilon}{4},
\]
which implies a contradiction.

\[
\square
\]

### 5. Proofs of Theorem 1.2 and Theorem 1.3

#### 5.1. Proof of Theorem 1.2

Proof. For any \( 0 < \theta < 1 \), it follows from the continuity of \( \dim^\theta_L E \) that for any \( \varepsilon > 0 \) there exist \( \delta_0 > 0 \) such that for any \( \eta \in [\theta - \delta_0, \theta + \delta_0] \), we have
\[
\left| \dim^\theta_L E - \dim^\eta_L E \right| < \varepsilon.
\]

**Case 1.** If \( \inf_{0 < \theta' \leq \theta} \dim^\theta_L E = \dim^\theta_L E \) and \( \eta > \theta \), then we obtain that
\[
\inf_{0 < \theta' \leq \eta} \dim^\theta_L E = \min \left\{ \inf_{0 < \theta' \leq \theta} \dim^\theta_L E, \inf_{\theta < \theta' \leq \eta} \dim^\theta_L E \right\}.
\]

If \( \inf_{0 < \theta' \leq \eta} \dim^\theta_L E = \inf_{0 < \theta' \leq \theta} \dim^\theta_L E \), the result directly holds.

If \( \inf_{0 < \theta' \leq \eta} \dim^\theta_L E = \inf_{\theta < \theta' \leq \eta} \dim^\theta_L E \), by the continuity of \( \dim^\theta_L E \), for any \( \eta \in [\theta - \delta_0, \theta + \delta_0] \), we have
\[
\dim^\theta_L E - \varepsilon \leq \dim^\eta_L E \leq \dim^\theta_L E + \varepsilon.
\]

Hence
\[
\left| \inf_{\theta < \theta' \leq \eta} \dim^\theta_L E - \inf_{0 < \theta' \leq \theta} \dim^\theta_L E \right| < \varepsilon.
\]

The result holds.

**Case 2.** If \( \inf_{0 < \theta' \leq \theta} \dim^\theta_L E = \dim^\theta_L E \) and \( \eta < \theta \), then
\[
\inf_{0 < \theta' \leq \theta} \dim^\theta_L E \leq \inf_{0 < \theta' \leq \eta} \dim^\theta_L E
\]
\[
\leq \dim^\eta_L E \leq \dim^\theta_L E + \varepsilon
\]
\[
\leq \inf_{0 < \theta' \leq \theta} \dim^\theta_L E + \varepsilon.
\]

Hence
\[
\left| \inf_{0 < \theta' \leq \eta} \dim^\theta_L E - \inf_{0 < \theta' \leq \theta} \dim^\theta_L E \right| < \varepsilon.
\]

**Case 3.** If \( \inf_{0 < \theta' \leq \theta} \dim^\theta_L E < \dim^\theta_L E \), denote
\[
d = \left| \inf_{0 < \theta' \leq \theta} \dim^\theta_L E - \dim^\theta_L E \right|.
\]

For any \( 0 < \varepsilon < d/2 \), by the continuity of \( \dim^\theta_L E \), there exists \( \delta_0 > 0 \) such that for any \( \eta \in [\theta - \delta_0, \theta + \delta_0] \), we have
\[
\left| \dim^\eta_L E - \dim^\theta_L E \right| < \varepsilon.
\]

Therefore,
\[
\dim^\eta_L E \geq \dim^\theta_L E - \frac{d}{2} > \inf_{0 < \theta' \leq \theta} \dim^\theta_L E,
\]

hence
\[
\inf_{0 < \theta' \leq \eta} \dim^\theta_L E = \inf_{0 < \theta' \leq \theta} \dim^\theta_L E.
\]

Then the result also holds.

\[
\square
\]
5.2. **Proof of Theorem 1.3.** We first give a lemma which indicates that the limit of lower Assouad spectrum exist as $\theta \to 0$ and $\theta \to 1$.

**Lemma 5.1.** Let $X$ be a doubling metric space and $E \subset X$ be a uniformly perfect set. Then
\begin{align}
\lim_{\theta \to 0} \dim^{\theta} L E &= \lim_{\theta \to 0} \dim_{L}^{\theta} E, \\
\lim_{\theta \to 1} \dim^{\theta} L E &= \lim_{\theta \to 1} \dim_{L}^{\theta} E.
\end{align}

**Proof.** For (5.2), it suffices to prove
\[ \lim_{\theta \to 0} \dim^{\theta} L E \leq \lim_{\theta \to 0} \dim_{L}^{\theta} E. \]
Write $t = \lim_{\theta \to 0} \dim_{L}^{\theta} E$. For any $s < t$, it follows from the definition of limit superior and the continuity of $\dim_{L}^{\theta} E$ that there exist an interval $[a, b] \subset (0, 1)$ such that for any $\theta \in [a, b]$, we have
\[ \dim_{L}^{\theta} E > s. \]
Besides, it follows from Corollary 3.2 that for any $n \geq 1$ and any $\eta \in (0, 1)$, we have
\[ \dim_{L}^{\eta} E \leq \dim_{L}^{\eta n} E. \]
Since there exists $N > 0$ such that for any $n > N$, we have $a^n < b^{n+1}$, which yields that
\[ [a^n, b^n] \cap [a^{n+1}, b^{n+1}] \neq \emptyset. \]
As a result, there exist an interval $(0, x) \subset (0, 1)$ such that
\[ (0, x) \subset \bigcup_{n=1}^{\infty} [a^n, b^n]. \]
Hence by (5.4) and (5.5), for any $\theta \in (0, x)$, we have $\dim_{L}^{\theta} E > s$. Therefore,
\[ \lim_{\theta \to 0} \dim_{L}^{\theta} E \geq s. \]
For (5.3), it suffices to prove
\[ \lim_{\theta \to 1} \dim_{L}^{\theta} E \leq \lim_{\theta \to 1} \dim_{L}^{\theta} E. \]
Write $t = \lim_{\theta \to 1} \dim_{L}^{\theta} E$. For any $s > t$, it follows from the definition of limit inferior and the continuity of $\dim_{L}^{\theta} E$ that there exist an interval $[a, b] \subset (0, 1)$ such that for any $\theta \in [a, b]$, we have
\[ \dim_{L}^{\theta} E < s. \]
Besides, since there exist $N$ such that for any $n > N$, we have $^*\sqrt[a]{a} < \sqrt[b]{b}$, which yields that
\[ [^*\sqrt[a]{a}, \sqrt[b]{b}] \cap [^*\sqrt[a]{a}, \sqrt[a]{b}] \neq \emptyset. \]
As a consequence, there exist an interval $(x, 1) \subset (0, 1)$ such that
\[ \bigcup_{n=1}^{\infty} [^*\sqrt[a]{a}, \sqrt[b]{b}] \supset (x, 1). \]
By Corollary 3.2 and (5.6), for any $\theta \in (x, 1)$, we have $\dim_{L}^{\theta} E < s$. Hence,
\[ \lim_{\theta \to 1} \dim_{L}^{\theta} E < s. \]
\[ \square \]

We now give the proof of Theorem 1.3.
Proof of Theorem 1.3. It follows from Theorem 1.1 that
\[
\dim_{\eta L} E = \lim_{\theta \to 1} \dim_{\theta L} E = \lim_{\theta \to 1} \inf_{0 < \theta' \leq \theta} \dim_{\theta L} E.
\]
By (5.3), it suffices to prove
\[
(5.7) \quad \lim_{\theta \to 1} \dim_{\theta L} E \leq \lim_{\theta \to 1} \inf_{0 < \theta' \leq \theta} \dim_{\theta L} E \leq \lim_{\theta \to 1} \dim_{\theta L} E.
\]
Write \( t = \lim_{\theta \to 1} \dim_{\theta L} E \). For any \( s > t \), it follows from the definition of limit superior that there exist \( \theta_0 \in (0, 1) \) such that \( \dim_{\theta_0} E < s \). Hence for any \( \theta > \theta_0 \), we have
\[
\inf_{0 < \theta' \leq \theta} \dim_{\theta'} E \leq \dim_{\theta L} E \leq \lim_{\theta \to 1} \dim_{\theta L} E.
\]
Therefore,
\[
\lim_{\theta \to 1} \inf_{0 < \theta' \leq \theta} \dim_{\theta L} E \leq s.
\]
Since \( s \) is arbitrary, we get
\[
\lim_{\theta \to 1} \inf_{0 < \theta' \leq \theta} \dim_{\theta L} E \leq \lim_{\theta \to 1} \dim_{\theta L} E.
\]
For the other inequality, fix \( \theta \) and write \( l = \inf_{0 < \theta' \leq \theta} \dim_{\theta'} E \).

If there exist \( \theta_0 \in (0, \theta] \) such that \( l = \dim_{\theta_0} E \), then for any \( s > l \), it follows from the continuity of the lower Assouad spectrum at \( \theta_0 \) that there exist an interval \( [a, b] \subset (0, 1) \) such that for any \( \eta \in [a, b] \), we have
\[
\dim_{\eta L} E < s.
\]
Hence, in both cases, for any \( s > l \), there exist an interval \( [a, b] \subset (0, 1) \) such that for any \( \eta \in [a, b] \), we have
\[
\dim_{\eta L} E < s.
\]
Besides, since there exist \( N \) such that for any \( n > N \), we have \( n^{-1} \sqrt[3]{a} < \sqrt[3]{b} \), then we obtain that
\[
[\sqrt[3]{a}, \sqrt[3]{b}] \cap [n^{-1} \sqrt[3]{a}, n^{-1} \sqrt[3]{b}] = \emptyset.
\]
Hence, there exist an interval \( (x, 1) \subset (0, 1) \) such that
\[
\bigcup_{n=1}^{\infty} [\sqrt[3]{a}, \sqrt[3]{b}] \supset (x, 1).
\]
By Corollary 3.2 for any \( \theta \in (x, 1) \), we have \( \dim_{L} E < s \). Hence,
\[
\lim_{\theta \to 1} \dim_{\theta L} E < s.
\]
Since \( s > l \) is arbitrary, we obtain
\[
\lim_{\theta \to 1} \dim_{\theta L} E \leq \lim_{\theta \to 1} \inf_{0 < \theta' \leq \theta} \dim_{\theta L} E
\]
Then the result holds by letting \( \theta \to 1 \) on both sides.
Hence by (5.3) and (5.7), we have
\[
\dim_{\eta L} E = \lim_{\theta \to 1} \inf_{0 < \theta' \leq \theta} \dim_{\theta L} E = \lim_{\theta \to 1} \dim_{\theta L} E.
\]
Remark. Although the limit of lower Assouad spectrum as $\theta \to 0$ exists, we do not know what the lower Assouad spectrum approaches as $\theta \to 0$.

6. Example: Cantor cut-out sets

In this section, we discuss the lower Assouad type dimensions of Cantor cut-out sets to illustrate that the lower Assouad spectrum is helpful in the computation of quasi-lower Assouad dimension. We also give an example to show that neither lower Assouad spectrum nor quasi-lower Assouad dimension can approach the lower Assouad dimension.

We first recall the definition of Cantor cut-out sets, see [3, 10] for more details. For any set $E \subset \mathbb{R}$, we denote by $|E|$ the length of $E$. Let $a = \{a_n\}_{n=1}^{\infty}$ be a decreasing summable positive real sequence. Let $\{A_n\}_{n=1}^{\infty}$ be a family of disjoint open intervals in $\mathbb{R}$ with $|A_n| = a_n$. We call $\{a_n\}_{n=1}^{\infty}$ the gap sequence. Without loss of generality, we always assume that $\sum_{n=1}^{\infty} a_n = 1$. The Cantor cut-out set, denoted by $C_a$, is defined as follows.

**Step 1.** We remove $A_1$ from the interval $[0, 1]$, resulting in two closed intervals $I_1^{(1)}$ and $I_2^{(1)}$.

**Step 2.** We remove $A_2$ from $I_1^{(1)}$, resulting in two closed intervals $I_1^{(2)}$ and $I_2^{(2)}$; and remove $A_3$ from $I_2^{(1)}$, resulting in two closed intervals $I_3^{(2)}$ and $I_4^{(2)}$.

\[ \ldots \]

**Step $k+1$.** After $k$ steps, we obtain the closed intervals $I_1^{(k)}, \ldots, I_{2^k}^{(k)}$ contained in $[0, 1]$. For any $1 \leq j \leq 2^k$, we remove $A_{2^{k+1}}$ from $I_j^{(k)}$, obtaining two closed intervals $I_{2j-1}^{(k+1)}$ and $I_{2j}^{(k+1)}$.

Continuing the above steps, we obtain a class of closed intervals $\{I_j^{(k)}\}_{1 \leq j \leq 2^k, k \geq 1}$ and call them basic intervals. For any $k \geq 1$, we call $\{I_j^{(k)}\}_{1 \leq j \leq 2^k}$ the basic intervals of level $k$. Let

\[
C_a = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{2^k} I_j^{(k)}.
\]

We call $C_a$ the Cantor cut-out sets. Let

\[
s_n = \frac{1}{2^n} \sum_{i=2^n}^{\infty} a_i.
\]

Clearly,

\[
s_{n+1} \leq |I_j^{(n)}| \leq s_{n-1}
\]

for any $n \geq 1$ and for any $1 \leq j \leq 2^n$.

**Lemma 6.1 ([10]).** For $C_a$ and for any ball $B(x, r)$ centered at $x \in C_a$ with radius $s_{k+1} \leq r < s_k$, there exist at least one basic interval of level $k + 2$ contained in $B(x, r)$ and no more than four basic intervals of level $k - 1$ intersecting $B(x, r)$.

We now discuss the lower Assouad spectrum of Cantor cut-out sets. For any $k \geq 1$ and any $0 < \theta < 1$, we denote

\[
l(k, \theta) = \max\{n | s_{k+n} \geq s_k^\theta\}.
\]

We give some lemmas concerning $l(k, \theta)$.

**Lemma 6.2.** If $\lim_{k \to \infty} \frac{s_{k+1}}{s_k} > 0$, then for any $\theta \in (0, 1)$, we have

\[
\lim_{k \to \infty} \frac{\log s_{k+l(k, \theta)}}{\log s_k} = \frac{1}{\theta} - 1.
\]
Lemma 6.3. Assume \( \inf_{k \geq 1} \frac{s_{k+1}}{s_k} > 0 \). Then for any \( \theta \in (0, 1) \), there exist an integer \( N > 0 \) such that

1. if \( 0 < R < |E| \) satisfies \( s_{k+1} \leq R < s_k \), we have
   \[ s_{k+l(k, \theta)+N} \leq R^\frac{1}{\theta} < s_{k+l(k, \theta)}; \]

2. if \( 0 < R < |E| \) satisfies \( s_k \leq R < s_{k-1} \), we have
   \[ s_{k+l(k, \theta)+1} \leq R^\frac{1}{\theta} < s_{k+l(k, \theta)-N}. \]

We now give the lower Assouad spectrum formula and the quasi-lower Assouad dimension formula of Cantor cut-out sets.

Proposition 6.1. If \( \inf_{k \geq 1} \frac{s_{k+1}}{s_k} > 0 \), then for any \( \theta \in (0, 1) \),

\[
\dim^\theta_L C_a = \lim_{k \to \infty} \frac{l(k, \theta) \cdot \log 2}{(1 - \frac{1}{\theta}) \cdot \log s_k}.
\]

In particular,

\[
\dim_{qL} C_a = \lim_{\theta \to 1} \lim_{k \to \infty} \frac{l(k, \theta) \cdot \log 2}{(1 - \frac{1}{\theta}) \cdot \log s_k}.
\]

We leave the proofs of Lemma 6.3, Lemma 6.2 and Proposition 6.1 in Appendix.

Remark. The assumption \( \inf_{k \geq 1} \frac{s_{k+1}}{s_k} > 0 \) is essential, since it guarantees that \( C_a \) is uniformly perfect. See [10, 11] for more details.

Example 6.1. For any \( 0 < \alpha < \beta < \frac{\log 2}{\log 3} \), there exists a gap sequence \( a = \{a_n\}_{n=1}^\infty \) and a Cantor cut-out set \( C_a \), such that

\[
\dim_L C_a = \alpha < \dim_{qL} C_a = \beta.
\]

Proof. The construction is stated as follows, which is motivated from [10]. Let \( \{l_i\}_{i=1}^\infty \) be an integer sequence satisfying \( l_1 = 1, l_{i+1} > 2l_i + i, \lim_{i \to \infty} \frac{1}{l_i} = 0, \lim_{i \to \infty} \frac{1}{l_i} + \frac{1}{l_{i-1}} = 0 \) and \( \lim_{i \to \infty} \frac{l_{i+1} + l_{i-1}}{l_i} = 0 \). Let \( \{d_k\}_{k=1}^\infty \) be a sequence such that for each \( i \geq 1, \)

\[
d_k = \begin{cases} 1 - 2 \cdot 2^{-\frac{1}{3}}, & k \in \{l_i, \ldots, l_{i+1} - i - 1\}; \\ 1 - 2 \cdot 2^{-\frac{1}{3}}, & k \in \{l_i + 1 - i, \ldots, l_{i+1} - 1\}. \end{cases}
\]

Let \( g_1 = d_1 \) and \( g_k = \prod_{i=1}^{k-1} \left(1 - \frac{d_i}{2}\right) \cdot d_k \) for any \( k \geq 2 \). For any \( k \geq 1 \) and \( 2^{k-1} \leq n < 2^k \), let \( a_n = g_k \).

Hence we obtain that

\[
\frac{s_k}{s_{k-1}} = \begin{cases} 2^{-\frac{1}{3}}, & k \in \{l_i, \ldots, l_{i+1} - i - 1\}; \\ 2^{-\frac{1}{3}}, & k \in \{l_i + 1 - i, \ldots, l_{i+1} - 1\}. \end{cases}
\]

It is a complicated process to compute the quasi-lower Assouad dimension by definition in this example, but it is simple to compute it by the lower Assouad spectrum.

We first show that for any \( \theta \in (0, 1) \),

\[
\lim_{k \to \infty} \frac{1}{\beta \theta} \cdot \frac{l(k, \theta) \cdot \log 2}{-\log s_k} = \frac{1}{\theta} - 1.
\]

Indeed, for any \( \theta \in (0, 1) \) and any \( \varepsilon > 0 \), there exist constants \( I, K > 0 \) such that for any \( i \geq I \) and \( k \geq K \), we have

\[
\frac{1 + 2 + \cdots + i - 1}{l_i} < \varepsilon, \quad \frac{l_1 + \cdots + l_{i-1}}{l_i} < \varepsilon.
\]
and

\[ \left| \log \frac{s_{k+l(k, \theta)}}{s_k} - \left( \frac{1}{\theta} - 1 \right) \right| < \varepsilon. \tag{6.5} \]

For any \( k \geq \max \{ K, l_1 \} \), denote \( l_1(k), l_2(k) \) such that \( l_1(k) + l_2(k) = k \) and

\[ s_k = 2^{-\frac{1}{\beta} l_1(k) - \frac{1}{\alpha} l_2(k)}, \]

where

\[ l_1(k) = \# \left\{ 1 \leq i \leq k \mid \frac{s_i}{s_{i-1}} = 2^{-\frac{1}{\beta}} \right\}, \quad l_2(k) = \# \left\{ 1 \leq i \leq k \mid \frac{s_i}{s_{i-1}} = 2^{-\frac{1}{\alpha}} \right\}. \]

For any fixed \( k \), for \( l(k, \theta) \), we also denote \( \hat{l}_1(k, \theta), \hat{l}_2(k, \theta) \) such that \( l(k, \theta) = \hat{l}_1(k, \theta) + \hat{l}_2(k, \theta) \) and

\[ \frac{s_{k+l(k, \theta)}}{s_k} = 2^{-\frac{1}{\beta} \hat{l}_1(k, \theta) - \frac{1}{\alpha} \hat{l}_2(k, \theta)}, \]

where

\[ \hat{l}_1(k, \theta) = \# \left\{ 1 \leq i \leq l(k, \theta) \mid \frac{s_i}{s_{i-1}} = 2^{-\frac{1}{\beta}} \right\}, \quad \hat{l}_2(k, \theta) = \# \left\{ 1 \leq i \leq l(k, \theta) \mid \frac{s_i}{s_{i-1}} = 2^{-\frac{1}{\alpha}} \right\}. \]

Since for any \( k \geq \max \{ K, l_1 \} \),

\[ \frac{1}{\beta} \cdot \frac{l(k, \theta)}{\log s_k} \leq \frac{\log \frac{s_{k+l(k, \theta)}}{s_k}}{\log s_k} \leq \frac{1}{\alpha} \cdot \frac{l(k, \theta)}{\log s_k}, \]

\[ \frac{\alpha}{\beta} \cdot \frac{l(k, \theta)}{k} \leq \frac{\log \frac{s_{k+l(k, \theta)}}{s_k}}{\log s_k} \leq \frac{\beta}{\alpha} \cdot \frac{l(k, \theta)}{k}, \]

then for sufficiently large \( k \), we obtain

\[ (6.6) \quad \frac{\alpha}{2} \left( \frac{1}{\theta} - 1 \right) \leq \frac{l(k, \theta)}{\log s_k} \leq 2\beta \left( \frac{1}{\theta} - 1 \right), \quad \frac{\alpha}{2\beta} \left( \frac{1}{\theta} - 1 \right) \leq \frac{\log s_{k+l(k, \theta)}}{\log s_k} \leq \frac{2\beta}{\alpha} \left( \frac{1}{\theta} - 1 \right). \]

For any \( k \geq \max \{ K, l_1 \} \), there exists \( n(k) \) such that

\[ l_n(k) \leq k \leq l_{n(k)+1}. \]

Then we obtain that \( l(k, \theta) \leq l_{n(k)+2} - l_{n(k)} \), if not, then

\[ \frac{\log \frac{s_{k+l(k, \theta)}}{s_k}}{\log s_k} \geq \frac{1}{\alpha} \cdot \frac{l_{n(k)+2} - l_{n(k)}}{l_{n(k)+1}} \to \infty, \]

which contradicts with Lemma 6.2.

Hence we obtain that \( \hat{l}_1(k, \theta) \leq 2n(k) + 1 \). By 6.1 and (6.6), for any \( k \geq \max \{ K, l_1 \} \),

\[ -\left( \frac{1}{\beta} \cdot l(k, \theta) \right) \cdot \log 2 \leq \frac{\log \frac{s_{k+l(k, \theta)}}{s_k}}{\log s_k} \leq -\left( \frac{1}{\beta} \cdot l(k, \theta) \right) \cdot \log 2 \cdot (1 - \varepsilon) + \frac{1}{\alpha} \cdot l(n, k) \cdot \varepsilon \cdot \log 2. \]

Then by (6.6), we get

\[ (6.7) \quad \left| \frac{\log \frac{s_{k+l(k, \theta)}}{s_k}}{\log s_k} - \left( \frac{1}{\beta} \cdot l(k, \theta) \right) \cdot \log 2 \right| \leq \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \cdot l(k, \theta) \cdot \varepsilon \leq \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \cdot 2\beta \left( \frac{1}{\theta} - 1 \right) \cdot \varepsilon. \]

Therefore, 6.5 together with 6.7 imply 6.3.

By 6.3 and Proposition 6.1, we have

\[ \dim_q C_a = \lim_{\theta \to 1} \lim_{k \to \infty} \frac{l(k, \theta) \cdot \log 2}{(1 - \frac{1}{\theta}) \cdot \log s_k} = \beta. \]

By the lower Assouad dimension formula in \([10]\), we have 
\[
\dim_L C_a = \lim_{m \to \infty} \inf_{k \geq 1} \frac{m \log 2}{\log s_k}\frac{s_k}{s_{k+m}} = \alpha.
\]
Therefore, we obtain 
\[
\dim_L C_a = \alpha < \dim_q L C_a = \beta.
\]

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Appendix

Proof of Lemma 6.2. For any \(\theta \in (0, 1)\), by the definition of \(l(k, \theta)\), we have 
\[
s_{k+l(k,\theta)} < s_k < s_{k+l(k,\theta)+1}.
\]
Since \(\inf_{k \geq 1} \frac{s_{k+1}}{s_k} > 0\), there exist a constant \(c < 1\) such that 
\[
c \cdot s_{k+l(k,\theta)} < \frac{s_k}{s_k} < s_{k+l(k,\theta)}.<
\]
Then we obtain 
\[
c \cdot \frac{s_{k+l(k,\theta)}}{s_k} < \frac{s_k}{s_k} < s_{k+l(k,\theta)}.
\]
Taking logarithm on both sides and dividing \(\log s_k\), we have 
\[
\left(\frac{1}{\theta} - 1\right) - \frac{\log c}{\log s_k} < \frac{\log s_{k+l(k,\theta)}}{\log s_k} = \frac{1}{\theta} - 1.
\]
Then the result holds by letting \(k\) tend to infinity.

Proof of Lemma 6.3. We first give the proof of (6.1). By the definition of \(l(k, \theta)\), we have 
\[
R_{\frac{1}{\theta}} < s_{\frac{1}{\theta}} < s_{k+l(k,\theta)}.
\]
For the lower bound, for any \(k \geq 1\), it follows from \(\frac{s_{k+1}}{s_k} < \frac{1}{2}, c_* \triangleq \inf_{k \geq 1} \frac{s_{k+1}}{s_k} > 0\) and the definition of \(l(k, \theta)\) that there exist an integer \(N > 0\) such that for any \(k \geq 1\) 
\[
R_{\frac{1}{\theta}} \geq s_{k+1} = (c_* s_k) \geq c_* \cdot s_{k+l(k,\theta)} \geq s_{k+l(k,\theta)+N}.
\]

We now give the proof of (6.2). From the definition of \(l(k, \theta)\), we obtain 
\[
R_{\frac{1}{\theta}} \geq s_{\frac{1}{\theta}} \geq s_{k+l(k,\theta)+1}.
\]
For the upper bound, it follows from \(c_* \triangleq \inf_{k \geq 1} \frac{s_{k+1}}{s_k} > 0\) and the definition of \(l(k, \theta)\) that there exist an integer \(N > 0\) such that for any \(k \geq 1\) 
\[
R_{\frac{1}{\theta}} \leq s_{\frac{1}{\theta}} \leq (c_*^{-1} s_k) \leq c_*^{-1} \cdot s_{k+l(k,\theta)} \leq s_{k+l(k,\theta)-N}.
\]
Proof of Proposition 6.1. For any $s < \dim L C_a$, by the definition of $\dim L C_a$, there exist constants $\rho, c > 0$ such that for any $0 < R < \rho$ and $x \in C_a$, we have

$$N_R^\perp(B(x, R) \cap C_a) \geq c \cdot R^{(1 - \frac{1}{\theta})s}.$$ 

Let $K_0 = \max\{k | s_k \geq \rho\}$, then for any $0 < R < \rho$, there exist $k > K_0$ such that

$$s_{k+1} \leq R < s_k.$$ 

By (6.1), we have

$$s_k + l(k, \theta) + N \leq R < s_k + l(k, \theta).$$

Hence it follows from Lemma 6.1 that there exist at most 4 basic intervals of level $k - 1$ intersecting $B(x, R)$, and for any $B(y, R^\perp)$, it contains at least 1 basic interval of level $k + l(k, \theta) + N + 1$, which implies that

$$N_R^\perp(B(x, R) \cap C_a) \leq 4 \cdot 2^{k + l(k, \theta) + N + 1} \leq 2^{N + 4} \cdot 2^{l(k, \theta)},$$

hence there exist a constant $C > 0$ such that for any $k \geq 1$, we obtain

$$N_R^\perp(B(x, R) \cap C_a) \leq C \cdot 2^{l(k, \theta)}.$$ 

Hence

$$c \cdot s_k^{(1 - \frac{1}{\theta})s} \leq N_R^\perp(B(x, R) \cap C_a) \leq C \cdot 2^{l(k, \theta)}.$$ 

Taking logarithm on both sides, we have

$$\log \frac{c}{C} + (1 - \frac{1}{\theta}) \log s_k \cdot s \leq l(k, \theta) \cdot \log 2.$$ 

Since $s_k \to 0$, then for any $\varepsilon > 0$, there exist $K_1$ such that for any $k \geq K_1$, we have

$$\left|\log \frac{c}{C} + (1 - \frac{1}{\theta}) \log s_k \cdot s\right| < \varepsilon.$$ 

Choose $K = \max\{K_0, K_1\}$, then for any $k \geq K$, we get

$$l(k, \theta) \cdot \log 2 \left(1 - \frac{1}{\theta}\right) \cdot \log s_k \geq s - \varepsilon.$$ 

Since $\varepsilon$ is arbitrary,

$$\lim_{k \to \infty} \frac{l(k, \theta) \cdot \log 2}{\left(1 - \frac{1}{\theta}\right) \cdot \log s_k} \geq s.$$ 

Conversely, for any

$$s < \lim_{k \to \infty} \frac{l(k, \theta) \cdot \log 2}{\left(1 - \frac{1}{\theta}\right) \cdot \log s_k},$$

then there exist $K > 0$ such that for any $k \geq K$,

$$\frac{l(k, \theta) \cdot \log 2}{\left(1 - \frac{1}{\theta}\right) \cdot \log s_k} > s,$$

which implies that

$$2^{l(k, \theta)} \geq s_k^{(1 - \frac{1}{\theta})s}.$$ 

Choose $\rho = \frac{s_k + 2}{4}$, then for any $0 < R < \rho$, there exist $k \geq K + 1$ such that

$$s_k \leq R < s_{k-1}.$$ 

By (6.2), we have

$$s_{k+1} \leq R \leq s_{k+1} < s_k - N.$$
By Lemma 6.1, for any $B(x, R)$, it contains at least one basic interval of level $k + 1$, and for any $B(y, R^*)$, it intersects at most 4 basic intervals of level $k + l(k, \theta) - N - 1$. For sufficiently large $k$, by Lemma 6.2, we can show $l(k, \theta) > N + 2$, then

$$N_{R^*}(B(x, R) \cap C_a) \geq \frac{1}{4} \cdot 2^{k+l(k,\theta)-N-1-(k+1)} \geq 2^{-N-4} \cdot 2^{l(k,\theta)}.$$ 

Thus there exist a constant $C > 0$ such that for any $k \geq 1$, we have

$$N_{R^*}(B(x, R) \cap C_a) \geq C \cdot 2^{l(k,\theta)}.$$ 

Hence

$$N_{R^*}(B(x, R) \cap C_a) \geq C \cdot 2^{l(k,\theta)} \geq C \cdot s_k^{(1-\frac{1}{\theta})s} > C \cdot R^{(1-\frac{1}{\theta})s}.$$ 

By the definition of $\dim^\theta L C_a$, we have $\dim^\theta L C_a \geq s$.

The proof is complete. \qed

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E-mail address: hpchen0703@foxmail.com, wumin@scut.edu.cn, changyy@scut.edu.cn.

Department of Mathematics, South China University of Technology, GUANGZHOU, 510640, P. R. CHINA