A topological semigroup structure on the space of actions modulo weak equivalence.

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Abstract

We introduce a topology on the space of actions modulo weak equivalence finer than the one previously studied in the literature. We show that the product of actions is a continuous operation with respect to this topology, so that the space of actions modulo weak equivalence becomes a topological semigroup.

1 Introduction.

Let $\Gamma$ be a countable group and let $(X, \mu)$ be a standard probability space. All partitions considered in this note will be assumed to be measurable. If $a$ is a measure-preserving action of $\Gamma$ on $(X, \mu)$ and $\gamma \in \Gamma$ we write $\gamma a$ for the element of $\text{Aut}(X, \mu)$ corresponding to $\gamma$ under $a$. Let $A(\Gamma, X, \mu)$ be the space of measure-preserving actions of $\Gamma$ on $(X, \mu)$. We have the following basic definition, due to Kechris.

Definition 1. For actions $a, b \in A(\Gamma, X, \mu)$ we say that $a$ is weakly contained in $b$ if for every partition $(A_i)_{i=1}^n$ of $(X, \mu)$, finite set $F \subseteq \Gamma$ and $\epsilon > 0$ there is a partition $(B_i)_{i=1}^n$ of $(X, \mu)$ such that

$$|\mu(\gamma A_i \cap A_j) - \mu(\gamma B_i \cap B_j)| < \epsilon$$

for all $i, j \leq n$ and all $\gamma \in F$. We write $a \prec b$ to mean that $a$ is weakly contained in $b$. We say $a$ is weakly equivalent to $b$ if we have both $a \prec b$ and $b \prec a$. $\sim$ is an equivalence relation and we write $[a]$ for the weak equivalence class of $a$.

For more information on the space of actions and the relation of weak equivalence, we refer the reader to [3]. Let $A_{\sim}(\Gamma, X, \mu) = A(\Gamma, X, \mu)/\sim$ be the set of weak equivalence classes of actions. Freeness is invariant under weak equivalence, so the set $\text{FR}_{\sim}(\Gamma, X, \mu)$ of weak equivalence classes of free actions is a subset of $A_{\sim}(\Gamma, X, \mu)$.

Given $[a], [b] \in A_{\sim}(\Gamma, X, \mu)$ with representatives $a$ and $b$ consider the action $a \times b$ on $(X^2, \mu^2)$. We can choose an isomorphism of $(X^2, \mu^2)$ with $(X, \mu)$ and thereby regard $a \times b$ as an action on $(X, \mu)$. The weak equivalence class of the resulting action on $(X, \mu)$ does not depend on our choice of isomorphism, nor on the choice of representatives. So we have a well-defined binary operation $\times$ on $A_{\sim}(\Gamma, X, \mu)$. This is clearly associative and commutative. In Section 2 we introduce a new topology on $A_{\sim}(\Gamma, X, \mu)$ which is finer than the one studied in [1], [2] and [4]. We call this the fine topology. The goal of this note is to prove the following result.

Theorem 1. $\times$ is continuous with respect to the fine topology, so that in this topology $(A_{\sim}(\Gamma, X, \mu), \times)$ is a commutative topological semigroup.
In [4], Tucker-Drob shows that for any free action $a$ we have $a \times s_{\Gamma} \sim a$, where $s_{\Gamma}$ is the Bernoulli shift on $([0,1]^\Gamma,\lambda^\Gamma)$ with $\lambda$ being Lebesgue measure. Thus if we restrict attention to the free actions there is additional algebraic structure.

**Corollary 1.** With the fine topology, $(\text{FR}_{\sim}(\Gamma, X, \mu), \times)$ is a commutative topological monoid.

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## 2 Definition of the fine topology.

Fix an enumeration $\Gamma = (\gamma_s)_{s=1}^\infty$ of $\Gamma$. Given $a \in A(\Gamma, X, \mu)$, $t, k \in \mathbb{N}$ and a partition $A = (A_i)_{i=1}^k$ of $X$ into $k$ pieces let $M_{t,k}(a)$ be the point in $[0,1]^{t \times k \times k}$ whose $s, l, m$ coordinate is $\mu(\gamma_s^aA_l \cap A_m)$. Endow $[0,1]^{t \times k \times k}$ with the metric given by the sum of the distances between coordinates and let $d_H$ be the corresponding Hausdorff metric on the space of compact subsets of $[0,1]^{t \times k \times k}$. Let $C_{t,k}(a)$ be the closure of the set

$$\{ M_{t,k}(a) : A \text{ is a partition of } X \text{ into } k \text{ pieces } \}.$$  

We have $a \sim b$ if and only if $C_{t,k}(a) = C_{t,k}(b)$ for all $t, k$. Define a metric $d_f$ on $A_{\sim}(\Gamma, X, \mu)$ by

$$d_f([a],[b]) = \sum_{t=1}^{\infty} \frac{1}{2^t} \left( \sup_k d_H(C_{t,k}(a), C_{t,k}(b)) \right).$$

This is clearly finer than the topology on $A_{\sim}(\Gamma, X, \mu)$ discussed in the references.

**Definition 2.** The topology induced by $d_f$ is called the the **fine topology**.

We have $[a_n] \to [a]$ in the fine topology if and only if for every finite set $F \subseteq \Gamma$ and $\epsilon > 0$ there is $N$ so that when $n \geq N$, for every $k \in \mathbb{N}$ and every partition $(A_i)_{i=1}^k$ of $(X, \mu)$ there is a partition $(B_i)_{i=1}^k$ so that

$$\sum_{l,m=1}^{k} |\mu(\gamma^a A_l \cap A_m) - \mu(\gamma^a B_l \cap B_m)| < \epsilon$$

for all $\gamma \in F$ and $l, m \leq k$.

## 3 Proof of the theorem.

We begin by showing a simple arithmetic lemma.

**Lemma 1.** Suppose $I$ and $J$ are finite sets and $(a_i)_{i \in I}, (b_i)_{i \in I}, (c_j)_{j \in J}, (d_j)_{j \in J}$ are sequences of elements of $[0,1]$ with $\sum_{i \in I} a_i = 1$, $\sum_{j \in J} d_j = 1$, $\sum_{i \in I} |a_i - b_i| < \delta$ and $\sum_{j \in J} |c_j - d_j| < \delta$. Then $\sum_{(i,j) \in I \times J} |a_i c_j - b_i d_j| < 2\delta$. 

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Proof. Fix $i$. We have

$$\sum_{j \in J} |a_i c_j - b_i d_j| \leq \sum_{j \in J} (|a_i c_j - a_i d_j| + |d_j a_i - d_j b_i|)$$

$$= \sum_{j \in J} (a_i |c_j - d_j| + d_j |a_i - b_i|)$$

$$\leq \delta a_i + |a_i - b_i|.$$ 

Therefore

$$\sum_{(i,j) \in I \times J} |a_i c_j - b_i d_j| \leq \sum_{i \in I} (a_i \delta + |a_i - b_i|) \leq 2 \delta. \quad \square$$

We now give the main argument.

Proof of Theorem 1. Suppose $[a_n] \to [a]$ and $[b_n] \to [b]$ in the fine topology. Fix $\epsilon > 0$ and $t \in \mathbb{N}$. Let $N$ be large enough so that when $n \geq N$ we have

$$\max \left( \sup_k d_H (C_{t,k} (a_n), C_{t,k} (a)) , \sup_k d_H (C_{t,k} (b_n), C_{t,k} (b)) \right) < \frac{\epsilon}{4}. \quad (1)$$

Fix $n \geq N$. Let $k \in \mathbb{N}$ be arbitrary and consider a partition $\mathcal{A} = (A_i)_{i=1}^k$ of $X^2$ into $k$ pieces. Find partitions $(D_i^1)_{i=1}^p$ and $(D_i^2)_{i=1}^q$ of $X$ such that for each $l \leq k$ there are pairwise disjoint sets $I_l \subseteq p \times q$ such that if we write $D_l = \bigcup_{(i,j) \in I_l} E_1^i \times E_2^j$ then

$$\mu^2 (D_l \triangle A_l) < \frac{\epsilon}{4k^2}. \quad (2)$$

Write $(\gamma_s)_{s=1}^k = F$. By (1) we can find a partition $(E_i^1)_{i=1}^p$ of $X$ such that for all $\gamma \in F$ we have

$$\sum_{i,j=1}^p |\mu (\gamma^a D_i^1 \cap D_j^1) - \mu (\gamma^a E_i^1 \cap E_j^1)| < \frac{\epsilon}{4} \quad (3)$$

and a partition $(E_i^2)_{i=1}^q$ of $X$ such that for all $\gamma \in F$ we have

$$\sum_{i,j=1}^q |\mu (\gamma^b D_i^2 \cap D_j^2) - \mu (\gamma^b E_i^2 \cap E_j^2)| < \frac{\epsilon}{4}. \quad (4)$$

Define a partition $\mathcal{B} = (B_l)_{l=1}^k$ of $X^2$ by setting $B_l = \bigcup_{(i,j) \in I_l} E_1^i \times E_2^j$. For $\gamma \in F$ we now have
\[
\begin{align*}
\sum_{l,m=1}^{k} & \left| \mu^2(\gamma_{a \times b} D_{l} \cap D_{m}) - \mu^2(\gamma_{a \times b} B_{l} \cap B_{m}) \right| \\
= \sum_{l,m=1}^{k} & \left| \mu^2 \left( \gamma_{a \times b} \left( \bigcup_{(i_1,j_1) \in I_l} D_{i_1}^1 \times D_{j_1}^2 \right) \cap \left( \bigcup_{(i_2,j_2) \in I_m} D_{i_2}^1 \times D_{j_2}^2 \right) \right) \right| \\
& - \mu^2 \left( \gamma_{a \times b} \left( \bigcup_{(i_1,j_1) \in I_l} E_{i_1}^1 \times E_{j_1}^2 \right) \cap \left( \bigcup_{(i_2,j_2) \in I_m} E_{i_2}^1 \times E_{j_2}^2 \right) \right) \right| \\
= \sum_{l,m=1}^{k} & \left| \mu^2 \left( \left( \bigcup_{(i_1,j_1) \in I_l} \gamma_a D_{i_1}^1 \times \gamma_b D_{j_1}^2 \right) \cap \left( \bigcup_{(i_2,j_2) \in I_m} D_{i_2}^1 \times D_{j_2}^2 \right) \right) \right| \\
& - \mu^2 \left( \left( \bigcup_{(i_1,j_1) \in I_l} \gamma_{a \times b} E_{i_1}^1 \times E_{j_1}^2 \right) \cap \left( \bigcup_{(i_2,j_2) \in I_m} E_{i_2}^1 \times E_{j_2}^2 \right) \right) \right| \\
= \sum_{l,m=1}^{k} & \left| \mu^2 \left( \left( \bigcup_{(i_1,j_1,j_2) \in I_l \times I_m} \gamma_a D_{i_1}^1 \times \gamma_b D_{j_1}^2 \right) \cap \left( D_{i_2}^1 \times D_{j_2}^2 \right) \right) \right| \\
& - \mu^2 \left( \left( \bigcup_{(i_1,j_1,j_2) \in I_l \times I_m} \gamma_{a \times b} E_{i_1}^1 \times E_{j_1}^2 \right) \cap \left( E_{i_2}^1 \times E_{j_2}^2 \right) \right) \right| \\
\leq \sum_{l,m=1}^{k} & \sum_{(i_1,j_1,j_2) \in I_l \times I_m} \left| \mu \left( \gamma_a D_{i_1}^1 \cap D_{i_2}^1 \right) \mu \left( \gamma_b D_{j_1}^2 \cap D_{j_2}^2 \right) \right| - \mu \left( \gamma_{a \times b} E_{i_1}^1 \cap E_{i_2}^1 \right) \mu \left( \gamma_{b \times b} E_{j_1}^2 \cap E_{j_2}^2 \right) \right| \\
\leq & \sum_{(i_1,j_1,j_2) \in P \times q \times \bar{q}} \left| \mu \left( \gamma_a D_{i_1}^1 \cap D_{i_2}^1 \right) \mu \left( \gamma_b D_{j_1}^2 \cap D_{j_2}^2 \right) \right| - \mu \left( \gamma_{a \times b} E_{i_1}^1 \cap E_{i_2}^1 \right) \mu \left( \gamma_{b \times b} E_{j_1}^2 \cap E_{j_2}^2 \right) \right| \\
= & \sum_{(i_1,j_1,j_2) \in P \times q \times \bar{q}} \left| \mu \left( \gamma_a D_{i_1}^1 \cap D_{i_2}^1 \right) \mu \left( \gamma_b D_{j_1}^2 \cap D_{j_2}^2 \right) \right| - \mu \left( \gamma_{a \times b} E_{i_1}^1 \cap E_{i_2}^1 \right) \mu \left( \gamma_{b \times b} E_{j_1}^2 \cap E_{j_2}^2 \right) \right|. \tag{5}
\end{align*}
\]
Now (3) and (4) let us apply Lemma 1 with $I = p^2$, $J = q^2$ and $\delta = \frac{\epsilon}{4}$ to conclude that (5) $\leq \frac{\epsilon}{2}$. Note that for any three subsets $S_1, S_2, S_3$ of a probability space $(Y, \nu)$ we have

$$|\nu(S_1 \cap S_3) - \nu(S_2 \cap S_3)| = |\nu(S_1 \cap S_2 \cap S_3) + \nu((S_1 \setminus S_2) \cap S_3) - \nu(S_1 \cap S_2 \cap S_3) - \nu((S_2 \setminus S_1) \cap S_3)|$$

$$\leq \nu(S_1 \triangle S_2),$$

hence for any $l, m \leq k$ and any action $c \in A(\Gamma, X^2, \mu^2)$ we have

$$\left| \mu^2(\gamma^c A_l \cap A_m) - \mu^2(\gamma^c D_l \cap D_m) \right|$$

$$\leq \left| \mu^2(\gamma^c A_l \cap A_m) - \mu^2(\gamma^c D_l \cap A_m) \right| + \left| \mu^2(\gamma^c D_l \cap A_m) - \mu^2(\gamma^c D_l \cap D_m) \right|$$

$$\leq \mu^2(\gamma^c A_l \triangle \gamma^c D_l) + \mu^2(A_m \triangle D_m) \leq \frac{\epsilon}{2k^2},$$

where the last inequality follows from (2). Hence for all $\gamma \in F$,

$$\sum_{l,m=1}^{k} \left| \mu^2(\gamma^{a \times b} A_l \cap A_m) - \mu^2(\gamma^{a \times b} D_l \cap B_m) \right|$$

$$\leq \sum_{l,m=1}^{k} \left( \left| \mu^2(\gamma^a A_l \cap A_m) - \mu^2(\gamma^a D_l \cap D_m) \right| + \left| \mu^2(\gamma^{a \times b} D_l \cap D_m) - \mu^2(\gamma^{a \times b} B_l \cap B_m) \right| \right)$$

$$\leq \sum_{l,m=1}^{k} \left( \frac{\epsilon}{2k^2} + \left| \mu^2(\gamma^{a \times b} D_l \cap D_m) - \mu^2(\gamma^{a \times b} B_l \cap B_m) \right| \right)$$

$$\leq \frac{\epsilon}{2} + (5) \leq \epsilon.$$

Therefore $M_{t,k}^A(a \times b)$ is within $\epsilon$ of $M_{t,k}^B(a_m \times b_n)$ and we have shown that for all $k$, $C_{t,k}(a \times b)$ is contained in the ball of radius $\epsilon$ around $C_{t,k}(a_m \times b_n)$. A symmetric argument shows that if $n \geq N$ then for all $k$, $C_{t,k}(a_m \times b_n)$ is contained in the ball of radius $\epsilon$ around $C_{t,k}(a \times b)$ and thus the theorem is proved. \(\square\)

References

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