A REMARK ON PERIMETER-DIAMETER AND
PERIMETER-CIRCUMRADIUS INEQUALITIES UNDER
LATTICE CONSTRAINTS

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Abstract. In this note, we study several inequalities involving geo-
metric functionals for lattice point-free planar convex sets. We focus on
the previously not addressed cases perimeter–diameter and perimeter–
circumradius.

1. Introduction

Let $\mathcal{K}^2$ be the set of all planar closed convex sets and denote by $\mathbb{Z}^2$
the standard integer lattice in $\mathbb{R}^2$. Some $K \in \mathcal{K}^2$ is called lattice-free if
$\text{int} K \cap \mathbb{Z}^2 = \emptyset$, that is, the interior of $K$ does not contain any lattice point
of $\mathbb{Z}^2$.

The perimeter, diameter, circumradius, inradius, minimal width and the
area of a convex body $K \in \mathcal{K}^2$ are denoted by $p(K)$, $D(K)$, $R(K)$, $r(K)$,
$\omega(K)$ and $A(K)$, respectively. The study of optimal relations between two
of these functionals (for convex sets of arbitrary dimension) is a classical
problem in Convex Geometry (cf. [BF87, pp. 56–59]).

In the planar case, there is an extensive bibliography if one adds the
extra assumption that $K$ is lattice-free (cf. [CFG94, EGH89, GW93, Ham77,
HCS98, Sco88]). For this situation, Hillock & Scott [HS02] collected the
known best possible inequalities relating pairs of the six functionals above.

The only pairs that are missing in their list are $(p, D)$ and $(p, R)$. They
have not been addressed so far and are the subject of our interest. The fact
that lattice-freeness is not preserved by arbitrary scaling is usually reflected
in the non-homogeneity of the geometric inequalities that are derived. In
this spirit, we propose the study of sharp upper bounds for the non-negative
functionals $p(K) - 2D(K)$ and $p(K) - 4R(K)$, for lattice-free $K \in \mathcal{K}^2$. The
existence of such upper bounds is proven by

\[ p(K) - 4R(K) \leq p(K) - 2D(K) \leq 2.65, \]

which follows from $\sqrt{3}D(K)(p(K)-2D(K)) \leq 4A(K)$ (see [Kub24]) together
with $A(K) \leq \lambda D(K)$ (see [Sco74]), $\lambda \approx 1.144$.

We conjecture, however, that the following bounds are the best possible

\[ p(K) - 2D(K) \leq 1 + \frac{2}{\sqrt{3}} \approx 2.1547 \quad \text{and} \quad p(K) - 4R(K) \leq 2. \]

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The equilateral triangle of edge lengths $1 + 2/\sqrt{3}$ for the pair $(p, D)$ and the split $\{x \in \mathbb{R}^2 : 0 \leq x_2 \leq 1\}$ for the pair $(p, R)$ attain equality.

In the following, we prove our conjectured inequalities in various cases, and offer sharp bounds on some non-linear functionals related to these magnitudes. A general proof for (1.1) has to be left as an open problem.

For our first result, we need to recall the notion of an unconditional set: some $K \in \mathcal{K}^2$ that is symmetric with respect to the lines $z + \text{lin}\{e_1\}$ and $z + \text{lin}\{e_2\}$, for a suitable $z \in \mathbb{R}^2$.

**Theorem 1.1.** Let $K \in \mathcal{K}^2$ be lattice-free and unconditional. Then
\begin{equation}
(p(K) - 2D(K) = p(K) - 4R(K) \leq 2.
\end{equation}
The inequality is best possible.

Often one can apply appropriate Steiner symmetrizations to a general lattice-free $K$ to obtain a lattice-free unconditional set (cf. [Sco74]). Unfortunately, this method usually decreases the functional $p(K) - 4R(K)$ and hence is not applicable in our situation.

Our second result shows the validity of the first conjectured inequality in (1.1) for triangles.

**Theorem 1.2.** Let $T \in \mathcal{K}^2$ be a lattice-free triangle. Then
\begin{equation}
p(T) - 2D(T) \leq \frac{2}{\sqrt{3}} \left(1 + \frac{\omega(T)}{D(T)}\right).
\end{equation}
In particular, $p(T) - 2D(T) \leq 1 + 2/\sqrt{3}$, and equality holds in (1.3) if and only if $T$ is an equilateral triangle with edge lengths $1 + 2/\sqrt{3}$.

Note that the refined inequality (1.3) is specific to triangles and does not hold for general lattice-free convex sets.

Complementing the partial results above, we found the following sharp, yet weaker inequalities relating the magnitudes of interest.

**Theorem 1.3.** Let $K \in \mathcal{K}^2$ be lattice-free. Then
\begin{enumerate}
  \item $\frac{D(K) - 1}{D(K)}(p(K) - 2D(K)) < 2$,
  \item $\frac{2R(K) - 1}{2R(K)}(p(K) - 4R(K)) < 2$.
\end{enumerate}
None of the inequalities can be improved.

Observe that our conjectured bound for the pair $(p, D)$ in (1.1) is independent from inequality i) above, whereas the conjectured bound for $(p, R)$ would strengthen inequality ii) by $\frac{2R(K) - 1}{2R(K)}(p(K) - 4R(K)) \leq 2R(K) - 1 \cdot 2 < 2$.

2. Proofs of the inequalities

**Proof of Theorem 1.1.** First of all, since $K$ is unconditional we have $D(K) = 2R(K)$ and it suffices to show the inequality $p(K) - 4R(K) \leq 2$.

Let $z \in \mathbb{R}^2$ be the center of symmetry of $K$. Note, that $z$ lies in the interior of $K$ and is at the same time its circumcenter. As $\mathbb{Z}^2$ is symmetric with respect to the coordinate axes, we may assume that after suitable reflections and translations of $K$ its center $z$ is contained in $[0, 1/2]^2$.

Since $0 \notin \text{int}K$, there exists a supporting line $L$ of $K$ with $0 \in L$. We first suppose that $L \cap [0, 1]^2 \neq \{0\}$. Since $z \in [0, 1/2]^2$, it holds $d(z, L) = \ldots$
min_{y \in L} \|z - y\| \leq 1/2$, where $\| \cdot \|$ denotes the Euclidean norm. Due to the unconditionality of $K$, the symmetric line $L'$ to $L$ with respect to $z$ supports $K$ as well. Therefore, $K$ is contained in the strip determined by $L$ and $L'$ which has width at most 1, hence $r(K) \leq 1/2$. Using an inequality of Henk & Tsintsifas [HT94], we get $p(K) \leq 4R(K) + 4r(K) \leq 4R(K) + 2$, as desired.

We now consider the case $L \cap [0,1]^2 = \{0\}$. We shoot a ray from $z$ in direction $(-1,-1)$ and let $q \in L$ be the intersection point of this ray and $L$. Since $L$ has negative slope, $q_1 \geq 0$ if $z_1 \geq z_2$, and $q_2 \geq 0$ if $z_1 \leq z_2$. In both cases, it follows that $0 \neq \lambda = \|z - q\| \leq 1/\sqrt{2}$. Let $K' := (\lambda \sqrt{2})^{-1}(-q + K)$. The functionals $p$ and $R$ are homogeneous of degree 1, and so

$$p(K) - 4R(K) \leq (\lambda \sqrt{2})^{-1}(p(K) - 4R(K)) = p(K') - 4R(K').$$

We observe that $K'$ is unconditional with respect to $(\lambda \sqrt{2})^{-1}(-q + z) = (1/2, 1/2)$, and the line $(\lambda \sqrt{2})^{-1}(-q + L) = L$ supports $K'$. Moreover, the unconditionality of $K'$ implies that the lines $L_1, L_2$, and $L_3$ symmetric to $L$, with respect to $(1/2, 1/2)$, $(1/2, 1/2) + \text{lin}\{e_1\}$, and $(1/2, 1/2) + \text{lin}\{e_2\}$, respectively, support $K'$. Thus $K' \subseteq Q$, where $Q$ is the rhombus determined by these four lines and therefore $K'$ is lattice-free. By definition of the circumradius, we have $K' \subseteq (1/2, 1/2) + R(K')[−1,1]^2 = Q'$. Thus $K' \subseteq Q \cap Q'$ and hence $p(K') \leq p(Q \cap Q')$.

In the last step, we show

$$(2.1) \quad p(Q \cap Q') \leq p((1/2, 1/2) + [-R(K'), R(K')] \times [-1/2, 1/2])$$

$$= 4R(K') + 2,$$

which implies the desired inequality (see Figure 1). To this end, we remark that the four vertices of $Q$ cannot all lie in $\text{int}Q'$, as this would mean that $R(Q) < R(K') \leq R(Q)$, a contradiction. Thus, we assume without loss of generality that the two vertices of $Q$ that are contained in the line $(1/2, 1/2) + \text{lin}\{e_2\}$ lie outside of $\text{int}Q'$.

Let $N$ be the intersection point of $L$ and the boundary of $Q'$ with $N_1 \leq 0$, and let $M$ be the intersection point of $L$ with the boundary of $[0,1/2] \times [-R(K') + 1/2, 0]$ with $M_1 \geq 0$. Moreover, we define the following distances of segments in $Q$ and $Q'$ (see Figure 1):

$$a = \|M\|, A = |M_1|, B' = |M_2|, \quad \text{and} \quad b = \|N\|, B = |N_1|, C = |N_2|.$$

By the symmetry of $Q \cap Q'$, it is enough to prove $a + b \leq A + B + C$ in order to get (2.1). Using basic properties of homothetic triangles and Pythagoras’ theorem, we obtain

$$B' \leq B, \quad \frac{a}{A} = \frac{b}{B}, \quad b^2 = B^2 + C^2, \quad \frac{C}{B} = \frac{B'}{A}.$$

Writing $a = bA/B$ and $b = \sqrt{B^2 + C^2}$, the inequality $a + b \leq A + B + C$ becomes

$$\sqrt{B^2 + C^2} (A + B) \leq B(A + B + C).$$

Since $C = BB'/A$, this is equivalent to

$$\sqrt{A^2 + (B')^2} (A + B) \leq A^2 + AB + BB'.$$
Taking squares on both sides gives
\[ AB' + 2BB' \leq 2AB + 2B^2, \]
which follows from \( B' \leq B \). Therefore, inequality (2.1) holds and we have \( p(K) - 4R(K) \leq p(Q \cap Q') - 4R(K') \leq 2 \). \( \square \)

**Remark 2.1.** The first part of the above proof shows that, in general, if \( r(K) \leq 1/2 \) for some \( K \in K^2 \), then \( p(K) - 4R(K) \leq 2 \).

**Proof of Theorem 1.2.** We start by determining the scaling factor \( \lambda > 0 \) for which \( T' = \lambda T \) is such that the length of the segment \( T' \cap L \) is equal to 1, where \( L \) is the line that is parallel and at distance 1 to the longest edge \( e \) of \( T' \) and on the same side of \( e \) as the vertex of \( T' \) that is not contained in \( e \) (see Figure 2).

Since the diameter of \( T' \) is attained by its longest edge, we get from Thales’ Theorem that
\[ \frac{1}{\lambda \omega(T) - 1} = \frac{1}{\omega(T') - 1} = \frac{D(T')}{\omega(T')} = \frac{\lambda D(T)}{\omega(T)} = \frac{D(T)}{\omega(T)}, \]
and thus \( \lambda = (\omega(T) + D(T))/(\omega(T)D(T)) \). Scott [Scn78] showed that for lattice-free \( T \) it holds \( (\omega(T) - 1)(D(T) - 1) \leq 1 \). This is equivalent to \( \omega(T)D(T) \leq \omega(T) + D(T) \) and hence \( \lambda \geq 1 \). Therefore, we have \( p(T) - 2D(T) \leq \lambda(p(T) - 2D(T)) = p(T') - 2D(T') \) and we can restrict our attention to the triangle \( T' \).
Therefore, with vertices \((\frac{\ell}{1+\omega}, \frac{\ell+1}{1+\omega})\) the third vertex of \(T'\), and moreover \(\omega(T') = \frac{\ell+1}{1+\omega}\) and \(D(T') = \ell + r + 1\). The vertices \(\left(\frac{\ell}{1+\omega}, \frac{\ell+1}{1+\omega}\right)\) and \((r+1, 0)\) determine an edge of length at most \(D(T')\), and thus

\[
\ell + r + 1 \geq \left( r + 1 - \frac{\ell}{\ell + r} \right)^2 + \left( \frac{\ell + r + 1}{\ell + r} \right)^2.
\]

Taking squares and dividing by \((\ell + r + 1)^2\) we obtain \((\ell + r)^2 \geq r^2 + 1\), and hence \(\ell \geq \sqrt{r^2 + 1} - r\). Together with \(\ell \leq r\), this gives \(r \geq 1/\sqrt{3}\).

As \(p(T') - 2D(T')\) equals the sum of the short edges minus \(D(T')\), we get

\[
p(T') - 2D(T') = \frac{\ell + r + 1}{\ell + r} \left( \sqrt{r^2 + 1} - r + \sqrt{\ell^2 + 1 - \ell} \right)
= \omega(T') \left( \sqrt{r^2 + 1} - r + \sqrt{\ell^2 + 1 - \ell} \right).
\]

Since \(f(r) = \sqrt{r^2 + 1} - r\) is non-increasing and \(\ell \geq \sqrt{r^2 + 1} - r\), we get an upper bound on \(p(T') - 2D(T')\) by substituting \(\ell\) by \(\sqrt{r^2 + 1} - r\) as follows

\[
p(T') - 2D(T') \leq \omega(T') \sqrt{(\sqrt{r^2 + 1} - r)^2 + 1}.
\]

Now, we define \(g(r) = \sqrt{f(r)^2 + 1}\) and we compute that

\[
g'(r) = \left(\frac{\sqrt{r^2 + 1} - 1}{\sqrt{(r - \sqrt{r^2 + 1})^2 + 1}}\right) \leq 0.
\]

Therefore, \(g(r)\) is non-increasing as well, and by \(r \geq 1/\sqrt{3}\), we have \(g(1/\sqrt{3}) = 2/\sqrt{3}\). Using the formula for the scaling factor \(\lambda\), we arrive at

\[
\tag{2.2}
p(T) - 2D(T) \leq p(T') - 2D(T') \leq \frac{2}{\sqrt{3}}\omega(T')
= \frac{2}{\sqrt{3}} \lambda \omega(T) = \frac{2}{\sqrt{3}} \left(1 + \frac{\omega(T)}{D(T)}\right).
\]

It is easy to see that \(\omega(T) \leq \sqrt{3}/2 D(T)\), and hence \(p(T) - 2D(T) \leq 1 + 2/\sqrt{3}\).

Tracing back the inequalities, we see that equality holds in (2.2) if and only if \(\lambda = 1\) and \(\ell = r = 1/\sqrt{3}\). This means that \(T\) is similar to the triangle with vertices \((-1/\sqrt{3}, 0), (1 + 1/\sqrt{3}, 0),\) and \((1/2, 1 + 1/\sqrt{3}/2)\). This triangle is equilateral with edge lengths \(1 + 2/\sqrt{3}\). \(\square\)

**Proof of Theorem 1.3.** The claimed inequalities are direct consequences of \((2r(K) - 1)(D(K) - 1) < 1\) (see [AS96]), \((2r(K) - 1)(2R(K) - 1) < 1\) (see [SA99]), and \(p(K) \leq 2D(K) + 4r(K) \leq 4R(K) + 4r(K)\) (see [HT94]).

We may assume, that \(D(K) > 1\) and \(R(K) > \frac{1}{2}\), respectively, since i) and ii) are otherwise certainly true. Now, we have

\[
p(K) \leq 2D(K) + 4r(K) < 2D(K) + 2\frac{D(K)}{D(K) - 1},
\]
which shows i), and part ii) follows analogously from
\[
p(K) \leq 4R(K) + 4r(K) < 4R(K) + 2 - \frac{2R(K)}{2R(K) - 1}.
\]

Let’s see why the inequalities are tight. Let \(K_n = \text{conv}\{ (\pm n, 0), (\pm n, 1) \}\), for \(n \in \mathbb{N}\). Clearly, \(K_n\) is lattice-free, \(D(K_n) = 2R(K_n)\), and for \(n \to \infty\),
\[
\frac{D(K_n) - 1}{D(K_n)} \left( p(K_n) - 2D(K_n) \right) = \frac{2R(K_n) - 1}{2R(K_n)} \left( p(K_n) - 4R(K_n) \right)
= \frac{2}{2} \sqrt{n^2 + \frac{1}{4}} - 1
= \frac{2}{2} \sqrt{n^2 + \frac{1}{4}} \left( 4n + 2 - 4 \sqrt{n^2 + \frac{1}{4}} \right) \geq 2.
\]

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