Equivariant infinitesimal deformations of algebraic threefolds with an action of an algebraic torus of complexity 1

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Abstract

Let $X$ be a 3-dimensional affine variety with a faithful action of a 2-dimensional torus $T$. Then the space of infinitesimal deformations $T^1(X)$ is graded by the characters of $T$, and the zeroth graded component $T^1(X)_0$ consists of all equivariant infinitesimal deformations.

Suppose that using the construction of such varieties from [1], one can obtain $X$ from a proper polyhedral divisor $\mathcal{D}$ on $F^3$ such that the tail cone of (any of) the used polyhedra is pointed and full-dimensional, and all vertices of all polyhedra are lattice points. Then we compute $\dim T^1(X)_0$.

1 Introduction

1.1 T-varieties

As proved and explained in [1], normal affine varieties of dimension $d$ with a faithful action of a $k$-dimensional torus $T$ (which are called T-varieties in the sequel) are described by the following data:

1. A $(d - k)$-dimensional (not necessarily affine) variety $Y$.
2. A pointed cone $\sigma$ in the rational dual character lattice $N_\mathbb{Q} = \mathfrak{X}(T)^*$. 
3. A proper (see definition below) polyhedral divisor $\mathcal{D}$, i.e., an element of the group $\text{Pol}_\sigma(N) \otimes \mathbb{Q}$, where $\text{Pol}_\sigma(N)$ is the Grothendick construction for the semigroup of polyhedra in $N$ with the tail cone $\sigma$, and $\text{Div}_\mathbb{Q}$ is the group of $\mathbb{Q}$-divisors.

More exactly, consider the character lattice $M = \mathfrak{X}(T)$, the rational character lattice $M_\mathbb{Q} = M \otimes \mathbb{Q}$ and the cone $\sigma^\vee \subseteq M_\mathbb{Q}$. For every element $\chi \in \sigma^\vee \cap M$, $\mathcal{D}$ defines a rational divisor $\mathcal{D}(\chi)$ as follows. Notice that $\chi$ can be considered as a function on $N$. Let $\mathcal{D} = \sum a_i Z_i \otimes (\Delta_i - \Delta'_i)$, where $a_i \in \mathbb{Q}$, $Z_i$’s are irreducible hypersurfaces in $Y$, and $\Delta_i$’s and $\Delta'_i$’s are polyhedra with the tail cone $\sigma$. We put $\mathcal{D}(\chi) = \sum a_i (\min_{p \in \Delta_i} \chi(p) - \min_{p \in \Delta'_i} \chi(p)) D_i$.

Definition 1. A polyhedral divisor $\mathcal{D}$ is called proper, if

1. It can be written in the form $\mathcal{D} = \sum a_i Z_i \otimes \Delta_i$, where $a_i \in \mathbb{Q}$, $Z_i$’s are irreducible hypersurfaces in $Y$, and $\Delta_i$’s are polyhedra with the tail cone $\sigma$ and $a_i \geq 0$.
2. For every $\chi \in \sigma^\vee \cap M$, $\mathcal{D}(\chi)$ is semiample, and if $\chi$ is in the interior of $\sigma^\vee$, $\mathcal{D}(\chi)$ is big.

Now, notice that if $\chi, \chi' \in \sigma^\vee \cap M$, then $\mathcal{D}(\chi) + \mathcal{D}(\chi') = \mathcal{D}(\chi + \chi')$ is an effective divisor, so a product of (rational) functions from $\mathcal{O}(\mathcal{D}(\chi))$ and from $\mathcal{O}(\mathcal{D}(\chi'))$ is in $\mathcal{O}(\mathcal{D}(\chi + \chi'))$. So we have a graded algebra $A = \bigoplus_{\chi \in \sigma^\vee \cap M} \mathcal{O}(\mathcal{D}(\chi))$. If $\mathcal{D}$ is proper, this algebra is finitely generated. The T-variety in question is $X = \text{Spec} A$. If $\mathcal{D}$ is proper, $\dim X = d$. Notice also that if $\chi, \chi' \in \sigma^\vee \cap M$ are proportional, then $\mathcal{D}(\chi + \chi') = \mathcal{D}(\chi) + \mathcal{D}(\chi')$, and in general the function $\chi \mapsto \mathcal{D}(\chi)$ is piecewise-linear.

Within the construction of $A$ we use, the elements of $\mathcal{O}(\mathcal{D}(\chi))$ may be interpreted in two ways: they are rational functions on $Y$ and they are global algebraic functions on $X$. If $f \in \mathcal{O}(\mathcal{D}(\chi))$, we will write $\tilde{f}$ for a rational function on $Y$ and $f$ for a global function on $X$. 

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Proposition 1. (see [4, Theorem 3.1]) There exists a rational surjective map \( \pi : X \to Y \) such that for every degree \( \chi \in n^1 \cap M \), for every point \( x \in X \) such that \( \pi \) is defined at \( x \), and for every \( f, g \in \mathcal{O}(D(\chi)) \) the following conditions are equivalent:

1. \( \tilde{f}/\tilde{g} \) is defined at \( \pi(x) \) as a rational function.

2. \( \tilde{f}/\tilde{g} \) is defined at \( x \) as a rational function.

In this case, \( (\tilde{f}/\tilde{g})(\pi(x)) = (\tilde{f}/\tilde{g})(x) \).

We are going to study 3-dimensional varieties with an action of a 2-dimensional torus \( T \) defined as above with the 1-dimensional variety being \( \mathbb{P}^1 \) and the polyhedral divisor being of the form \( D = \sum p_i \odot \Delta_i \), where \( p_i \in \mathbb{P}^1 \) are points and \( \Delta_i \)'s are polyhedra with the tail cone \( \sigma \) and with all vertices in lattice points. In this case all divisors \( D(\chi) \) are integral, not rational. We suppose that \( \sigma \) is full-dimensional. The properness condition in this case means that the Minkowski sum of all polyhedra \( \Delta_i \) is strictly contained in \( \sigma \).

1.2 First order deformations

For a general reference on deformation theory, see [2].

In general, a deformation of a variety \( X \) with a scheme \( Z \) with a marked point \( z \in Z \) being the parameter space of the deformation is a flat morphism \( p : Y \to Z \), where \( Y \) is a scheme, together with an isomorphism \( \iota \) between \( X \) and \( p^{-1}(z) \). Two deformations \((p : Y \to Z, \iota : X \to p^{-1}(z)) \) and \((p' : Y' \to Z, \iota' : X' \to p'^{-1}(z)) \) with the same parameter space \( Z \) and the same marked point \( z \) are called equivalent if there exists an isomorphism \( q : Z \to Z' \) such that \( p = p'q \) and \( q|_{p^{-1}(z)} \iota = \iota' \). If \( Z \) is the double point, i.e. \( Z = \text{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2) \), and \( X \) is affine \((X = \text{Spec} A)\), then the set of all possible deformations is denoted by \( T^1(X) \), and one can define an \( A \)-module structure on it.

Namely, choose an embedding \( X \hookrightarrow \mathbb{C}^n \), then \( A \) can be written as \( A = \mathbb{C}[x_1, \ldots, x_n]/I \), where \( I \) is an ideal. Then \( I/I^2 \) is an \( A \)-module. Consider also the following \( \mathbb{C}[x_1, \ldots, x_n] \)-module \( \Theta = \text{Der} \mathbb{C}[x_1, \ldots, x_n] \): its elements are of the form \( \sum g_i \partial g_i/\partial x_i \), where \( g_i \in \mathbb{C}[x_1, \ldots, x_n] \). Every such differential operator defines an \( A \)-homomorphism between \( I/I^2 \) and \( A \): if \( g \in I \), then \( g/I^2 \in I/I^2 \) maps to \( \sum g_i \partial g_i/\partial x_i \in \Theta \). If \( g \in I^2 \), \( g = \sum g_i g_i' \), then \( \sum i,j \partial g_i g_j'/\partial x_i + \sum i,j \partial g_i g_j g_j'/\partial x_i = \sum i,j g_i g_j g_j'/\partial x_i \in I \), so the map is well-defined. If \( a \in \mathbb{C}[x_1, \ldots, x_n] \), \( a/I \in \Theta \), then \( \sum g_i \partial g_i/\partial x_i \in I \), so \( \sum g_i \partial g_i/\partial x_i = (a \sum g_i \partial g_i/\partial x_i)/I = (a \sum g_i \partial g_i/\partial x_i)/I + (g \sum g_i \partial g_i/\partial x_i)/I = (a \sum g_i \partial g_i/\partial x_i)/I \), and the map is \( A \)-linear. In fact we have defined a map \( \phi : \Theta \to \text{Hom}_\mathbb{C}(I/I^2, A) \).

Moreover, \( \sum g_i \partial g_i/\partial x_i \in I \Theta \), i.e. if all \( g_i \) are in \( I \), then \( \sum g_i \partial g_i/\partial x_i \in I \) for all \( g \in I \), so \( \phi \) is well-defined on \( \Theta/I \Theta \), which is an \( A \)-module. It is clear that \( \phi \) is \( A \)-linear. One can prove that \( T^1(X) \) can be identified with \( \text{coker} \phi \) so that these identifications for all affine varieties together have good category-theoretical properties. We will not need these properties explicitly, and we will use this identification as a definition of \( T^1(X) \).

If \( M \) is a lattice, \( A \) is \( M \)-graded, and the generators \( x_1, \ldots, x_n \) are homogeneous, then one can introduce an \( M \)-grading on \( \mathbb{C}[x_1, \ldots, x_n] \) as well. Then \( I \) becomes an \( M \)-graded ideal, and \( \Theta \) becomes an \( M \)-graded module with \( \deg(\partial/\partial x_i) = -\deg x_i \). The map \( \phi \) preserves this grading, so we can introduce a grading on \( T^1(X) \).

1.3 Schlessinger’s formula for \( T^1 \)

Extending Schlessinger’s result [3, Lemma 2], we prove the following theorem:

Theorem 1. Let \( X \) be an affine normal algebraic variety, and let \( U \) be a non-singular open subset of \( X \) such that \( \text{codim}_X(X \setminus U) \geq 2 \). Then \( T^1(X) \) can be computed as follows. Let \( \Theta_X \) denote the tangent sheaf on \( X \), and let \( \tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{C}[X] \) be a set of generators. Consider the following map \( \psi : \Theta \to \mathcal{O}_X^n \): it maps a (locally defined) vector field \( w \) to \( (\partial \tilde{x}_1(w), \ldots, \partial \tilde{x}_n(w)) \).

Then \( T^1(X) = \ker (H^1(U, \Theta_X) \xrightarrow{H^1(U, \mathcal{O}_X^n)} H^1(U, \mathcal{O}_X^n)) \) as \( \mathbb{C}[X] \)-modules.
1.4 Leray spectral sequence

We are going to use the following theorem:

**Theorem 2.** Let \( f : X \to Y \) be a morphism of algebraic varieties, and let \( F \) be a quasicoherent sheaf on \( X \). Then there exists a spectral sequence called Leray spectral sequence with the second sheet

\[
E_2^{p,q} = H^p(Y, R^q f_* F),
\]

where the corresponding differentials map \( E_2^{p,q} \) to \( E_2^{p+r,q-r+1} \), \( r \geq 2 \), that converges to \( H^{p+q}(X, F) \). Denote the corresponding filtration on \( H^{p+q}(X, F) \) by \( F_\bullet \).

The sheaves \( R^q f_* F \) can be considered as sheaves of \( f_* \mathcal{O}_X \)-modules, and \( H^p(Y, R^q f_* F) \) can be therefore considered as \( \mathcal{C}[X] \)-modules. In this sense, the isomorphism

\[
F_\bullet H^{p+q}(X, F)/F_{p+1} H^{p+q}(X, F) \cong E_\infty^{p,q}
\]
is an isomorphism of \( \mathcal{C}[X] \)-modules.

Here \( R^q f_* \) denotes the \( q \)th derived functors of the direct image functor in quasicoherent sheaf category (or shortly, "\( q \)th derived direct image")

Notice that if \( \dim Y = 1 \), then (since all sheaves \( R^q f_* F \) are coherent) \( H^p(Y, R^q f_* F) = 0 \) for \( p \geq 2 \) (and \( p < 0 \)), so all differentials vanish, \( E_2^{1,0} = E_\infty^{1,0} \), and we have a short exact sequence

\[
0 \to H^1(Y, R^{q-1} f_* F) \to H^q(X, F) \to H^q(Y, R^q f_* F) \to 0.
\]

1.5 Cech complexes cohomology

We need two more facts related to Cech complexes. The first proposition explains how to compute derived direct images using Cech resolutions.

Let \( F \) be a quasicoherent sheaf on a separated algebraic variety \( U \), and let \( \{U_i\}_{i=1}^n \) be an affine covering of \( U \). Consider the following sheaf Cech resolution of \( F \): it consists of sheaves \( \mathcal{F}_i \) on \( U \), \( i \geq 0 \), and

\[
\mathcal{F}_i = \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} \mathcal{F}_{a_1, a_2, \ldots, a_{i+1}},
\]

where if \( V \subseteq U \) is an open subset, then \( \Gamma(V, \mathcal{F}_{a_1, a_2, \ldots, a_{i+1}}) = \Gamma(V \cap U_{a_1} \cap \ldots \cap U_{a_{i+1}}, F) \). The differentials in the resolution are defined in the usual Cech sense: given a section

\[
(x_{a_1, a_2, \ldots, a_{i+1}})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} \in \Gamma(V, \mathcal{F}_{i-1}),
\]

the differential maps it to

\[
(y_{a_1, a_2, \ldots, a_{i+1}})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} \in \Gamma(V, \mathcal{F}_i),
\]

where

\[
y_{a_1, a_2, \ldots, a_{i+1}} = \sum_{j=1}^{i+1} (-1)^j (x_{a_1, a_2, \ldots, a_{j-1} a_{j+1} a_{j+2} \ldots a_{i+1}})_{V \cap U_{a_1} \cap \ldots \cap U_{a_{i+1}}}.\]

Notice that if we take the global sections of all \( \mathcal{F}_i \), we obtain a Cech complex of \( F \) in the "usual", non-sheaf sense.

Suppose we have a map \( f : U \to Y \), where \( Y \) is also a separated algebraic variety.

**Proposition 2.** [3, Proposition III.8.7] \( R^q f_*(\mathcal{F}) = H^q(f_*(\mathcal{F}_\bullet)) \), where \( H^\cdot \) is the \( \cdot \)th cohomology of the complex formed by \( f_*(\mathcal{F}_i) \) for \( i \geq 0 \), not the \( \cdot \)th cohomology of a particular sheaf. \( \square \)
The second fact gives an easier way to compute the first cohomology of complexes that "look like a Cech complex" under certain circumstances in any abelian category. Suppose that $C$ is an abelian category, let $A$ be an object, let $q \in \mathbb{N}$, and let for every $1 \leq i \leq q$ indices $a_i$ satisfying $1 \leq a_1 < \ldots < a_i \leq q$ a subobject of $A$ (i.e., an object together with a morphism $A_{a_1}, \ldots, a_i \rightarrow A$ whose kernel is zero). Suppose also that if $(a_j)_{j=1}^i$ is a subsequence of $(b_j)_{j=1}^{i+1}$, then $A_{a_1}, \ldots, a_i$ is a subobject of $A_{b_1}, \ldots, b_{i+1}$, and the embedding $A_{a_1}, \ldots, a_i \rightarrow A_{b_1}, \ldots, b_{i+1}$ commutes with the embeddings of these objects into $A$.

Now consider the following complex $B$:

$$B_i = \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} A_{a_1}, \ldots, a_{i+1}, i \geq 0.$$  

To define a differential $d: B_{i-1} \rightarrow B_i$, we use universal properties of two direct sums and define maps between direct summands of $B_{i-1}$ and of $B_i$ as follows: $d_{a_1}, \ldots, a_{i+1} = A_{a_1}, \ldots, a_i \rightarrow A_{b_1}, \ldots, b_{i+1}$ is zero if $(a_j)_{j=1}^i$ is not a subsequence of $(b_j)_{j=1}^{i+1}$. Otherwise, let $k$ be the index such that $a_j = b_k$ for $j < k$, and $a_j = b_{j+1}$ for $j \geq k$. If $k$ is even, we say that $d_{a_1}, \ldots, a_{i+1} = A_{a_1}, \ldots, a_i$ is the embedding $A_{a_1}, \ldots, a_i \rightarrow A_{b_1}, \ldots, b_{i+1}$ we chose before, and $k$ is odd, we say that $d_{a_1}, \ldots, a_{i+1} = A_{a_1}, \ldots, a_i$ is minus this embedding (the negation here comes from the abelian group structure on $\text{Hom}(A_{a_1}, \ldots, a_i, A_{b_1}, \ldots, b_{i+1})$).

We also need the following complex $B'$:

$$B'_i = \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} A/A_{a_1}, \ldots, a_i, i \geq 0.$$  

Here we allow $i = 0$, and we say that $A$ the empty sequence $= 0$, so $B'_0 = A$. Differentials are defined exactly as in $B$ using universal properties of direct sums: $d_{a_1}, \ldots, a_{i+1} = A_{a_1}, \ldots, a_i \rightarrow A_{b_1}, \ldots, b_i$ is zero if $(a_j)_{j=1}^{i-1}$ is not a subsequence of $(b_j)_{j=1}^{i+1}$. Otherwise, let $k$ be the index such that $a_j = b_k$ for $j < k$, and $a_j = b_{j+1}$ for $j \geq k$. We have two consecutive embeddings $A_{a_1}, \ldots, a_{i-1} \rightarrow A_{b_1}, \ldots, b_i \rightarrow A$, and universal properties of quotients define a map $A/A_{a_1}, \ldots, a_{i-1} \rightarrow A/A_{b_1}, \ldots, b_i$. We use this map for $d_{a_1}, \ldots, a_{i-1}, b_i$ if $k$ is even, and we use minus this map if $k$ is odd.

**Proposition 3.** For $i \geq 0$, $H^i(B'_\bullet) = H^i(B_\bullet)$. This isomorphism is functorial in $B$ and $B'$ if the embeddings $A_{a_1}, \ldots, a_i \rightarrow A$ are functorial in $A_{a_1}, \ldots, a_i$ and $A$.

**Corollary 1.** If $A_{i,k} = A$ for all $1 \leq j < k \leq q$, then $H^3(B'_\bullet) = (\bigoplus_{j=1}^q A/A_j)/A$, where $A$ is mapped to $\bigoplus_{j=1}^q A/A_j$ diagonally.

### 1.6 Notation and terminology

First, we need some notation for lattice polyhedra. Let $\Delta$ be a polyhedron with tail cone $\sigma$ and with all vertices in $N$, where $\dim \sigma = \dim N = 2$, and $\sigma$ is pointed. We denote the number of (finite) vertices of $\Delta$ by $v(\Delta)$ and we denote the vertices of $\Delta$ by $V_1(\Delta), \ldots, V_{v(\Delta)}(\Delta)$ so that pairs of consecutive vertices in this enumeration form the finite edges of $\Delta$. We denote the finite edge between $V_i(\Delta)$ and $V_{i+1}(\Delta)$ by $E_i(\Delta)$. We denote the infinite edge with the endpoint $V_i(\Delta)$ by $E_{\infty}(\Delta)$ and the infinite edge with the endpoint $V_{\infty}(\Delta)$ by $E_{\infty}(\Delta)$. For each vertex $V_i(\Delta)$ denote by $N(V_i(\Delta), \Delta)$ the subcone of $\sigma^\vee$ consisting of all $\chi \in \sigma^\vee$ such that $\chi(V_i(\Delta)) = \min a \in \Delta \chi(a)$. We call $N(V_i(\Delta), \Delta)$ the normal subcone of the vertex $V_i(\Delta)$. One checks easily that this is really a subcone, that $\sigma^\vee = \bigcup N(V_i(\Delta), \Delta)$, that the intersection of two such subcones is either a ray or the origin, and it is a ray if and only if the two corresponding vertices form an edge $E_i(\Delta)$. In the latter case this ray is exactly the set of all $\chi \in \sigma^\vee$ whose minimum on $\Delta$ is attained on $E_i(\Delta)$. We denote this ray by $N(E_i(\Delta), \Delta)$ and call it the normal ray of the edge $E_i(\Delta)$. Finally, we extend this notation for infinite edges of $\Delta$: we denote by $N(E_{\infty}(\Delta), \Delta)$ (resp. $N(E_{\infty}(\Delta), \Delta)$) the ray in $\Delta$ consisting of all $\chi \in \sigma^\vee$ whose minimum on $\Delta$ is attained on $E_{\infty}(\Delta)$ (resp. $E_{\infty}(\Delta)$). These two rays are in fact the two rays forming $\partial(\sigma^\vee)$, and they are also called the normal rays of the corresponding edges. We always choose the order on vertices of $\Delta$ so that $E_{\infty}(\Delta)$ is always the same one of the two rays forming $\partial(\sigma^\vee)$ (it must not depend on $\Delta$). This ray is denoted by $E_{\infty}(\sigma^\vee)$, and the other ray of $\partial(\sigma^\vee)$ is denoted by $E_1(\sigma^\vee)$. Denote the primitive lattice vectors on $E_j(\sigma^\vee)$ by $a_j$.
If $\rho$ is a ray in $M_Q$, we denote the primitive lattice vector on $\rho$ by $b(\rho)$. If $a$ is a vector or a segment in $N$, denote by $|a|$ the lattice length of $a$, i.e., the number of lattice points in $a$ including exactly one of the endpoints.

The notation listed below will be properly introduced later, we list it now to ease reading and navigation only, without going into details of the underlying notions. We are going to deal with a polyhedral divisor $D = \sum_{i=1}^n p_i \Delta_{p_i}$, where $p_i \in \mathbb{P}^1$ and $\Delta_{p_i}$ are polyhedra with all their vertices in a two-dimensional lattice $N$ and tail cone $\sigma$, which is pointed and two-dimensional. We shortly write $v_p$ instead of $v(\Delta_p)$ and $V_p, i$ instead of $V_i(\Delta_p)$ for $p \in \mathbb{P}^1$. At some point we will consider the Hilbert basis of $a^\vee$, and we will denote it by $\lambda_1, \ldots, \lambda_m$. We also need a notion of an essential special point, which is a point $p_i$ such that $\Delta_{p_i}$ is not a translation of $\sigma$. We denote the number of essential special points by $r$, and we denote all essential special points by $p_1', \ldots, p_r'$. We are going to study a $T$-variety $X$, and we will choose a set of generators of $\mathbb{C}[X]$. Since $X$ is a $T$-variety, $\mathbb{C}[X]$ is an $M$-graded algebra, and in fact we will choose homogeneous generators. More precisely, we will choose a linear space $O$ of the endpoints.

As we will usually use “standard” subscripts to enumerate different descriptions of the same vector field or function, we say explicitly which set of notation will be used later. Every time, when we consider a description of a vector field or of a function, we use, which vector field or function on $X$, subset $U$ an open subset of $U$ corresponds to one of the chosen open subsets of $U$, and we will denote it by $\lambda_1, \ldots, \lambda_m$. We also need a notion of an essential special point, which is a point $p_i$ such that $\Delta_{p_i}$ is not a translation of $\sigma$. We denote the number of essential special points by $r$, and we denote all essential special points by $p_1', \ldots, p_r'$. We are going to study a $T$-variety $X$, and we will choose a set of generators of $\mathbb{C}[X]$. Since $X$ is a $T$-variety, $\mathbb{C}[X]$ is an $M$-graded algebra, and in fact we will choose homogeneous generators. More precisely, we will choose a linear space $O$ of the endpoints.

At some point we will introduce a notion of the $U_i$-description of a vector field or of a function on an open subset of $X$. Such a description is a tuple of two functions and a vector field on an open subset of $\mathbb{P}^1$ in the case of a vector field on $X$ and is just a function on an open subset of $\mathbb{P}^1$ in the case of a function on $X$. There are many possible descriptions for a given vector field or a function on $X$, each one corresponds to one of the chosen open subsets $U_i$. Sometimes we will need many descriptions of a given vector field or of a given function on $X$ simultaneously. And sometimes we will simultaneously deal with descriptions of many different functions or vector fields. To distinguish between these situations clearly, we will usually use “standard” subscripts to enumerate different descriptions of the same vector field or function, for example:

$$(g_{i,1,1}, g_{i,1,2}, v_1, \ldots, g_{i,1,1}, g_{i,2,2}, v_2, \ldots, g_{i,q,1}, g_{i,q,2}, v_{q}).$$

Here $(g_{i,1,1}, g_{i,1,2}, v_1, \ldots, g_{i,1,1}, g_{i,2,2}, v_2, \ldots, g_{i,q,1}, g_{i,q,2}, v_{q})$ is the $U_i$-description of a vector field that does not depend on $i$. In one has several different vector fields and one description of each of them, we enumerate them using indices in brackets, for example:

$$(g[i,1], g[i,2], v[i,1], \ldots, g[i,1], g[i,2], v[i,2], \ldots, g[r], g[r], v[r]).$$

Here $(g[i,1], g[i,2], v[i])$ can be, for example, the $U_1$-description of a vector field $v[i]$ on $X$, and these vector fields may vary independently. These are only generic rules, they are stated here to demonstrate what kind of notation will be used later. Every time, when we consider a description of a vector field or of a function, we say explicitly which set $U_i$ we use, which vector field or function on $X$ we describe, and how we denote the description.

2 Proofs of preliminary results

Proof of Theorem 1 If $F$ is a coherent sheaf on $X$, denote $F^\vee = \text{Hom}_X(F, \mathcal{O}_X)$. First, we prove the following three lemmas, which extend Lemma 1 from [4].

Lemma 1. Let $X$ be a normal affine algebraic variety, $U$ be an open subset such that codim$_X(X \setminus U) \geq 2$, and $F$ be a free sheaf of finite rank on $X$. Then $H^0_{X \setminus U}(X, F) = H^1_{X \setminus U}(X, F) = 0$.

Proof. Write the long exact sequence for cohomology with support:

$$0 \to H^0_{X \setminus U}(X, F) \to H^0(X, F) \to H^0(U, F) \to H^1_{X \setminus U}(X, F) \to H^1(X, F) \to \ldots$$

$F$ is a free sheaf of finite rank, $X$ is normal, and codim$_X(X \setminus U) \geq 2$, therefore the restriction map $H^0(X, F) \to H^0(U, F)$ is an isomorphism. Hence, $H^0_{X \setminus U}(X, F) = 0$ and the map $H^1_{X \setminus U}(X, F) \to H^1(X, F)$ is an embedding. Since $X$ is affine, $H^1(X, F) = 0$, so $H^1_{X \setminus U}(X, F) = 0$. \qed
Lemma 3. Let $X$ be a normal affine algebraic variety, $U$ be an open subset such that $\text{codim}_X(X \setminus U) \geq 2$, and $\mathcal{F}$ be a coherent sheaf on $X$ such that there exists a coherent sheaf $\mathcal{G}$ on $X$ such that $\mathcal{F} = \mathcal{G}^\vee$. Then $H^0(X \setminus U)(X, \mathcal{F}) = 0$.

Proof. Since $\mathcal{G}$ is a coherent sheaf, there exists an exact sequence of coherent sheaves on $X$

$$0 \to \mathcal{G}' \to \mathcal{G}'' \to \mathcal{G} \to 0,$$

where $\mathcal{G}''$ is free. Since $\text{Hom}_X(\cdot, \mathcal{O}_X)$ and $H^0(X \setminus U)(X, \cdot)$ are left exact functors, the corresponding map

$$H^0(X \setminus U)(X, \mathcal{F}) \to H^0(X \setminus U)(X, \mathcal{G}^\vee)$$

is an embedding. $\mathcal{G}''$ is free and coherent, i.e. it is a free sheaf of finite rank, so $\mathcal{G}^\vee$ is also a free sheaf of finite rank. By Lemma 1 $H^0(X \setminus U)(X, \mathcal{G}^\vee) = 0$. Hence, $H^0(X \setminus U)(X, \mathcal{F}) = 0$. □

Lemma 3. Let $X$ be a normal affine algebraic variety, $U$ be an open subset such that $\text{codim}_X(X \setminus U) \geq 2$, and $\mathcal{F}$ be a coherent sheaf on $X$ such that there exists a coherent sheaf $\mathcal{G}$ on $X$ such that $\mathcal{F} = \text{Hom}_X(\mathcal{G}, \mathcal{O}_X)$. Then the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is an isomorphism.

Proof. Again write an exact sequence of coherent sheaves on $X$

$$0 \to \mathcal{G}' \to \mathcal{G}'' \to \mathcal{G} \to 0,$$

where $\mathcal{G}''$ is free. The dualization functor is left exact, so the corresponding map $\mathcal{F} \to \mathcal{G}''\vee$ is an embedding, and its cokernel (denote it by $\mathcal{Q}$) is a subsheaf of $\mathcal{G}^\vee$. By Lemma 2 $H^0(X \setminus U)(X, \mathcal{G}^\vee) = 0$. Since $\mathcal{Q}$ is a subsheaf of $\mathcal{G}^\vee$ and $H^0(X \setminus U)(X, \cdot)$ is a left exact functor, $H^0(X \setminus U)(X, \mathcal{Q}) = 0$. Again, since $\mathcal{G}''$ is free and coherent, $\mathcal{G}^\vee$ is a free sheaf of finite rank. By Lemma 1 $H^0(X \setminus U)(X, \mathcal{G}^\vee) = 0$. We have the following exact sequence of cohomology:

$$0 \to H^0(X \setminus U)(X, \mathcal{F}) \to H^0(X \setminus U)(X, \mathcal{G}''\vee) \to H^0(X \setminus U)(X, \mathcal{Q}) \to H^1(X \setminus U)(X, \mathcal{F}) \to H^1(X \setminus U)(X, \mathcal{G}^\vee) \to \ldots,$$

and we see that $H^1(X \setminus U)(X, \mathcal{F}) = 0$. By Lemma 2 $H^0(X \setminus U)(X, \mathcal{F}) = 0$. Now write the following long exact sequence:

$$0 \to H^0(X \setminus U)(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \to H^1(X \setminus U)(X, \mathcal{F}) \to \ldots$$

We see that the restriction map $H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F})$ is an isomorphism. □

Now we are ready to prove Theorem 1. Denote $A = \mathbb{C}[X]$. The generators $\bar{x}_1, \ldots, \bar{x}_n$ define an embedding $X \hookrightarrow \mathbb{C}^n = \text{Spec} \mathbb{C}[x_1, \ldots, x_n]$ and a morphism of algebras $\mathbb{C}[x_1, \ldots, x_n] \to A$ so that $x_i \mapsto \bar{x}_i$. Denote the kernel of this algebra morphism by $I$. As we have previously seen, $I/I^2$ is an $A$-module. Denote the corresponding sheaf on $X$ by $\mathcal{I}$. Observe that the $A$-module $\Theta/I\Theta$ introduced in the definition of $T^1(X)$ is isomorphic to the free $A$-module of rank $n$ as an $A$-module. The kernel of the map $\phi: \Theta/I\Theta \to \text{Hom}_A(I/I^2, A)$ consists of all $n$-tuples $(g_1, \ldots, g_n)$ of functions on $X$ such that for all $h \in I$ one has $\sum g_i \partial h/\partial x_i = 0$ in $A$ (to evaluate this expression, we take arbitrary representatives in the cosets corresponding to $g_i$ and to $h$ in $\mathbb{C}[x_1, \ldots, x_n]$ and in $I$, respectively, we have seen previously that its value in $A$ does not depend on this choice). In other words, the $n$-tuple $(g_1, \ldots, g_n)$ defines a tangent vector field to $X$. The embedding of the tangent bundle on $X$ into the rank $n$ trivial bundle on $X$ we have just obtained coincides with the map $\psi$ in the statement of Theorem 1. So, we have the following exact sequence of $A$-modules:

$$0 \to \Gamma(X, \Theta_X) \xrightarrow{\Gamma(\psi|_{\mathcal{I}})} A^n \to \text{Hom}_A(I/I^2, A) \to T^1(X) \to 0.$$

Since $X$ is affine, we also have an exact sequence of sheaves (denote the sheaf generated by the $A$-module $T^1(X)$ by $\mathcal{I}^1$):

$$0 \to \Theta_X \xrightarrow{\psi} \mathcal{O}_X^n \to \mathcal{I}^1 \to T^1 \to 0.$$
Denote the map between sheaves $O_X^\oplus$ and $\mathcal{I}^\vee$ by $\tilde{\phi}$. It is known that if $U' \subseteq X$ is smooth, then $\Gamma(U', T^1) = 0$, and we have the following exact sequence:

$$0 \to \Gamma(U', \Theta_X) \xrightarrow{\Gamma(\psi|_U)} \Gamma(U', O_X^\oplus) \xrightarrow{\Gamma(\tilde{\phi}|_U)} \Gamma(U', \mathcal{I}^\vee) \to 0.$$ 

In particular, this holds for affine sets $U'$ forming an affine cover of $U$. Therefore, we have the following exact sequence of sheaves on $U'$:

$$0 \to \Theta_X|_U \xrightarrow{\psi|_U} O_X^\oplus|_U \xrightarrow{\tilde{\phi}|_U} \mathcal{I}^\vee|_U \to 0,$$

and we can write the long exact sequence of cohomology:

$$0 \to H^0(U, \Theta_X) \xrightarrow{H^0(\psi|_U)} H^0(U, O_X^\oplus) \xrightarrow{H^0(\tilde{\phi}|_U)} H^1(U, \mathcal{I}^\vee) \to H^1(U, \Theta_X) \xrightarrow{H^1(\psi|_U)} H^1(U, O_X^\oplus) \to \ldots$$

Denote the map between $H^0(U, \mathcal{I}^\vee)$ and $H^1(U, \Theta_X)$ by $\delta$. We have $\ker H^1(\psi|_U) = \text{im} \delta = H^0(U, \mathcal{I}^\vee)/\ker \delta = H^0(U, \mathcal{I}^\vee)/\text{im} H^0(\tilde{\phi}|_U) = \text{coker} H^0(\tilde{\phi}|_U)$.

Recall the exact sequence of $A$-modules we started with:

$$0 \to \Gamma(X, \Theta_X) \xrightarrow{\Gamma(\psi)} A^\oplus \to \text{Hom}_A(I/I^2, A) \to T^1(X) \to 0.$$ 

We can write $A^\oplus$ as $\Gamma(X, O_X^\oplus)$. and $\text{Hom}_A(I/I^2, A)$ as $\Gamma(X, \mathcal{I}^\vee)$. Now we can apply Lemma 3. $\Theta_X$ is dual to $\Omega_X$, $O_X^\oplus$ is dual to itself, and $\mathcal{I}^\vee$ is dual to $\mathcal{I}$ by construction. So we can rewrite the exact sequence as follows:

$$0 \to H^0(U, \Theta_X) \xrightarrow{H^0(\psi|_U)} H^0(U, O_X^\oplus) \xrightarrow{H^0(\tilde{\phi}|_U)} H^1(U, \mathcal{I}^\vee) \to H^1(U, \Theta_X) \xrightarrow{H^1(\psi|_U)} H^1(U, O_X^\oplus) \to T^1(X),$$

and we see that $\text{coker} H^0(\tilde{\phi}|_U) = T^1(X)$. 

**Proof of Proposition 3** We are going to use Mitchell’s embedding theorem. Consider the abelian subcategory in $\mathcal{C}_0$ in $\mathcal{C}$ generated by $A$ and by all objects $A_{a_1, \ldots, a_i}$. This is a small category, therefore by Mitchell’s embedding theorem it is equivalent to an abelian subcategory in the category of left modules over a (not necessarily commutative) ring $R$. So, in what follows we suppose that $A$, all $A_{a_1, \ldots, a_i}$, and all $B_i$ and $B_i'$ are $R$-modules. Fix an index $i$ ($0 \leq i \leq q$). We are going to define a morphism $q: H^i(B'_i) \to H^i(B_i')$. $H^i(B'_i)$ is a quotient of a submodule in

$$B'_i = \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} A/A_{a_1, \ldots, a_i},$$

which is a quotient of

$$\bigoplus_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} A,$$

so each element of $H^i(B'_i)$ can be represented by a tuple

$$(x_{a_1, \ldots, a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q},$$

where $x_{a_1, \ldots, a_i} \in A$. We say that $q$ maps it to

$$(y_{a_1, \ldots, a_{i+1}})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} \in \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} A,$$

where

$$y_{a_1, \ldots, a_{i+1}} = \sum_{j=1}^{i+1} (-1)^j x_{a_1, \ldots, \hat{a}_j, \ldots, a_{i+1}}.$$
We still have to prove that \((y_{a_1,\ldots,a_{i+1}})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q}\) is actually an element of

\[
B_i = \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} A_{a_1,\ldots,a_{i+1}} \subseteq \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} A.
\]

Observe that \(q\) and \(d: B'_i \to B'_{i+1}\) are in fact defined by the same formula, the difference is that we apply it to elements of different subquotients of

\[
\bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} A \quad \text{and} \quad \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} A.
\]

Now we note that if \((x_{a_1,\ldots,a_i})\) represents an element of \(\ker(d: B'_i \to B'_{i+1})\), then \((y_{a_1,\ldots,a_{i+1}})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q}\) represents the zero element in

\[
B'_{i+1} = \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q} A/A_{a_1,\ldots,a_{i+1}},
\]

i. e. \(y_{a_1,\ldots,a_{i+1}} \in A_{a_1,\ldots,a_{i+1}}\) for all suitable sequences of indices \(a_1,\ldots,a_{i+1}\). Denote the composition of the canonical projection

\[
\bigoplus_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} A \to \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} A/A_{a_1,\ldots,a_i} = B'_i
\]

and the differential \(B'_i \to B'_{i+1}\) also by \(d: \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} A \to B'_{i+1}\). Then we can say that \(q\) really defines a morphism from \(\ker(d: \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} A \to B'_{i+1})\) to \(B_i\). Moreover, if we write the differential of \((y_{a_1,\ldots,a_{i+1}})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q}\) in \(B_{i+1}\) as a sequence of the form

\[
(x_{a_1,\ldots,a_{i+2}})_{1 \leq a_1 < a_2 < \ldots \leq q},
\]

then

\[
z_{a_1,\ldots,a_{i+2}} = \sum_{j=1}^{i+2} (-1)^j y_{a_1,\ldots,a_j,a_{i+2}} =
\]

\[
\sum_{j=1}^{i+2} (-1)^j \left( \sum_{k=1}^{j-1} (-1)^k x_{a_1,\ldots,a_k,a_{j},a_{i+2}} + \sum_{k=j+1}^{i+2} (-1)^{k-1} x_{a_1,\ldots,a_j,a_k,a_{i+2}} \right) = 0.
\]

In other words, \((y_{a_1,\ldots,a_{i+1}})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q}\) defines a morphism \(\ker(d: B_i \to B_{i+1})\). Hence, \(q\) defines a morphism from \(\ker(d: \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} A \to B'_{i+1})\) to \(\ker(d: B_i \to B_{i+1})\), and we can further reduce \(q\) to a morphism from \(\ker(d: \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} A \to B'_{i+1})\) to \(H^1(B_*)\).

Now we are ready to prove that \(q\) defines a morphism \(H^1(B'_*) \to H^1(B_*)\) well. First suppose that \((x_{a_1,\ldots,a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}\) defines the zero element in \(B'_i\), i. e. \(x_{a_1,\ldots,a_i} \in A_{a_1,\ldots,a_i}\) for all suitable sequences of indices \(a_1,\ldots,a_i\). This is essentially possible only for \(i > 0\), since for \(i = 0\) one would have \(x\) the empty sequence \(\in A_0\) the empty sequence = \(0\). But if \(i > 0\), then \((x_{a_1,\ldots,a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}\) also defines an element of \(B_{i-1}\), so \((y_{a_1,\ldots,a_{i+1}})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q}\) is in \(\ker(d: B_{i-1} \to B_i)\) and defines the zero element of \(H^1(B_*)\). Now let \((x_{a_1,\ldots,a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}\) be the sequence \((y_{a_1,\ldots,a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}\) defined by \(x_{a_1,\ldots,a_i} = x_{a_1,\ldots,a_i} + x_{a_1,\ldots,a_i} \in A_{a_1,\ldots,a_i}\), and there exists a sequence

\[
(z'_{a_1,\ldots,a_{i-1}})_{1 \leq a_1 < a_2 < \ldots < a_{i-1} \leq q} \in \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i-1} \leq q} A
\]

representing an element of \(B'_{i-1}\) such that

\[
x_{a_1,\ldots,a_i} = \sum_{j=1}^{i} (-1)^j z'_{a_1,\ldots,a_j,a_{i-1}}
\]
for all suitable sequences of indices. We already know that $q((x''_a, \ldots, a_j)_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}) = 0$ in $H^i(B_*)$. Denote $q((x'_a, \ldots, a_j)_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}) = (y'_a, \ldots, a_{i+1})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q}$, then

$$y'_a = \sum_{j=1}^{i+1} (-1)^j x'_a, \ldots, a_{j-1}, a_{j+1} = \sum_{j=1}^{i+1} (-1)^j \left( \sum_{k=1}^{j-1} (-1)^k z'_{a_1, \ldots, a_k, \ldots, a_{j-1}, a_{j+1}} + \sum_{k=j+1}^{i+2} (-1)^{k-1} z'_{a_1, \ldots, a_{j-1}, a_{j+1}} \right) = 0.$$

Therefore, $q$ really defines a morphism $H^i(B'_* \to H^i(B_*)$. Now we have to prove that it is injective and surjective. We keep using the notation we introduced before, i.e. $(x_a, \ldots, a_i)_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}$ represents an element of $H^i(B_*)$, and its image in $H^i(B'_*)$ is represented by $(y_a, \ldots, a_{i+1})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q}$. Suppose that $i > 0$ and $(y_a, \ldots, a_{i+1})_{1 \leq a_1 < a_2 < \ldots < a_{i+1} \leq q}$ represents the zero element of $H^i(B'_*)$, i.e. there exist $x_a, \ldots, a_i \in A_a, \ldots, a_i$ such that

$$y_a = \sum_{j=1}^{i+1} (-1)^j x'_a, \ldots, a_{j-1}, a_{j+1} = \sum_{j=1}^{i+1} (-1)^j x'_a, \ldots, a_{j-1}, a_{j+1}.$$

If $i = 1$, set $z_{a_1, \ldots, a_i} = -x_a + x'_a$. Otherwise set $z_{a_1, \ldots, a_i} = 0$ for all $1 < a_2 < \ldots < a_i \leq q$, and set $z_{a_1, \ldots, a_i} = -x_{a_1, a_2, \ldots, a_i} - z'_{a_1, a_2, \ldots, a_i}$ for all $1 < a_1 < \ldots < a_i \leq q$. Consider the class of $(z_{a_1, \ldots, a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}$ in $B'_i$ and let its differential be represented by $(x''_{a_1, \ldots, a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}$. Then

$$z''_{a_1, a_2, \ldots, a_i} = -x_{a_1, a_2, \ldots, a_i} + \sum_{j=2}^{i} (-1)^j z_{a_1, a_2, \ldots, a_j, \ldots, a_i} = x_{a_1, a_2, \ldots, a_i} - x'_{a_1, a_2, \ldots, a_i}$$

for all $1 < a_2 < \ldots < a_i \leq q$. (If $i = 1$, the summand with the summation sign in the formula above should be omitted.) If $1 < a_1 < \ldots < a_i \leq q$, then

$$x''_{a_1, a_2, \ldots, a_i} = \sum_{j=1}^{i} (-1)^j z_{a_1, a_2, \ldots, a_j} = \sum_{j=1}^{i} (-1)^j (-x_{a_1, a_2, \ldots, a_j} + x'_{a_1, a_2, \ldots, a_j}) =$$

$$= \sum_{j=1}^{i} (-1)^{i+1} x_{a_1, a_2, \ldots, a_j} = \sum_{j=1}^{i} (-1)^{i+1} x_{a_1, a_2, \ldots, a_j} = y_{a_1, a_2, \ldots, a_i} + x_{a_1, a_2, \ldots, a_i} = (y_{a_1, a_2, \ldots, a_i} + x'_{a_1, a_2, \ldots, a_i}) = x_{a_1, a_2, \ldots, a_i} - x'_{a_1, a_2, \ldots, a_i}.$$

Hence, for all sequences of indices $a_1, \ldots, a_i$ satisfying $1 \leq a_1 < \ldots < a_i \leq q$ we have $x''_{a_1, a_2, \ldots, a_i} = x_{a_1, a_2, \ldots, a_i} - x'_{a_1, a_2, \ldots, a_i}$. Since $x''_{a_1, a_2, \ldots, a_i} \in A_a, \ldots, a_i$, we see that $(x_{a_1, a_2, \ldots, a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}$ and $(x''_{a_1, a_2, \ldots, a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}$ represent the same element of $B'_i$. Therefore, $(x_{a_1, a_2, \ldots, a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}$ represents the zero element of $H^i(B'_*)$, and $q: H^i(B'_*) \to H^i(B_*)$ is injective.

Now we prove surjectivity similarly. Take an element of $H^i(B_*)$ represented by $(y_{a_1, a_2, \ldots, a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}$, where $y_{a_1, a_2, \ldots, a_i} \in A_a, a_2, \ldots, a_i$. Set $x_{a_1, a_2, \ldots, a_i} = -y_{a_1, a_2, \ldots, a_i}$. We would like to prove first that

$$(x_{a_1, a_2, \ldots, a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q} \in \bigoplus_{1 \leq a_1 < a_2 < \ldots < a_{i-1} \leq q} A$$

really defines an element of $H^i(B'_*)$, i.e. the element of $B'_i$ defined by $(x_{a_1, a_2, \ldots, a_i})_{1 \leq a_1 < a_2 < \ldots < a_i \leq q}$ is in the kernel of $d: B'_i \to B'_{i+1}$. Denote

$$y'_{a_1, a_2, \ldots, a_i} = \sum_{j=1}^{i+1} (-1)^j x_{a_1, a_2, \ldots, a_j}.$$
Then we have to prove that $y'_{a_1,\ldots,a_{i+1}} \in A_{a_1,\ldots,a_{i+1}}$. We can write

$$y'_{a_1,\ldots,a_{i+1}} = -\sum_{j=1}^{i+1} (-1)^j y_{1,a_1,\ldots,a_j,\ldots,a_{i+1}}.$$ 

Since $(y_{a_1,\ldots,a_{i+1}})_{1\leq a_1<a_2<\ldots<a_{i+1}\leq q}$ represents an element of $H^i(B_•)$, it is annihilated by $d: B_i \to B_{i+1}$, so

$$-y_{a_1,\ldots,a_{i+1}} + \sum_{j=1}^{i+1} (-1)^{j+1} y_{1,a_1,\ldots,a_j,\ldots,a_{i+1}} = 0.$$ 

We see that $y'_{a_1,\ldots,a_{i+1}} = y_{a_1,\ldots,a_{i+1}} \in A_{a_1,\ldots,a_{i+1}}$. Hence, $(x_{a_1,\ldots,a_i})_{1\leq a_1<a_2<\ldots<a_i\leq q}$ indeed defines an element of $H_i(B_•)$, and we can apply $q$ to this element. As we pointed out before, $q$ and $d: B_i' \to B_{i+1}'$ are defined by the same formulas, so we may use $(y'_{a_1,\ldots,a_{i+1}})$ to determine the image of $(x_{a_1,\ldots,a_i})$ under $q$. But we already know that $y'_{a_1,\ldots,a_{i+1}} = y_{a_1,\ldots,a_{i+1}}$, therefore $q: H^i(B_•) \to H^i(B_•)$ is surjective.

3 Formula for the graded component of $T^1$ of degree 0 in terms of sheaf cohomology

Let $\sigma \subset N_\mathbb{Q}$ be a pointed full-dimensional cone, $p_1,\ldots,p_r$ be points on $\mathbb{P}^1$, $\Delta_p \subset N_\mathbb{Q}$ be polyhedra whose vertices are lattice points and whose tail cones are all $\sigma$. Unlike what is assumed sometimes, we do not allow $\emptyset$ to appear among these polyhedra. These data define a polyhedral divisor $D = \sum_{i=1}^r \Delta_p_i \otimes p_i$ and a graded algebra $A = \bigoplus_{\chi \in \sigma \cap M} \mathcal{O}(\mathcal{D}(\chi))$. If $p \in \mathbb{P}^1$ does not coincide with any of the points $p_i$, we denote $\Delta_p = \sigma$. Suppose in the sequel that $D$ is proper, then $A$ defines a 3-dimensional variety $X = \text{Spec} A$ with an action of a 2-dimensional torus. We use the notation $\pi$ for the map from $X$ to $\mathbb{P}^1$ introduced in Proposition 4. It is known that all such varieties are normal.

In the sequel we will always keep in mind that very ample divisors on $\mathbb{P}^1$ are exactly the divisors of positive degree and principal divisors are exactly the divisors of degree zero. We call a point $p \in \mathbb{P}^1$ ordinary if it is not one of the points $p_i$, otherwise we call it special. We require that the sum $\sum \Delta_p_i \otimes p_i$ is finite, but we do not require that all summands are nontrivial, i.e. we allow summands of the form $\sigma \otimes p_i$, which are zeros in the polyhedral divisor group. We call such points $p_i$ special anyway, according to the definition above. So in fact the notions of a special point and an ordinary point depend on the choice of exact presentation $D = \sum \Delta_p_i \otimes p_i$, and we suppose that it is also fixed. If $\Delta_p = \sigma + a$ for some $a \in N$, (including $a = 0$), we call such $p_i$ a removable special point, otherwise we call $p_i$ an essential special point.

Fix a coordinate $t$ on $\mathbb{P}^1$, i.e. fix a rational function $t$ on $\mathbb{P}^1$ that has one pole of order 1 and one zero of order 1. Without loss of generality we may suppose that $t = 0$ and $t = \infty$ are ordinary points.

**Lemma 4.** Given two nonzero rational functions $f$ and $g$ on $\mathbb{P}^1$ such that $f/g$ has one zero and one pole, and both of them are of order one, there exist $a_1, b_1, a_2, b_2 \in \mathbb{C}$ such that $(a_1 f + b_1 g)/(a_2 f + b_2 g) = t$.

**Proof.** First, let us find $a_1', b_1', a_2', b_2' \in \mathbb{C}$ such that $(a_1' f + b_1' g)/(a_2' f + b_2' g)$ is regular at all points where $f$ is finite and has pole of order one at $t = \infty$. If $f/g = 0$ at $t = \infty$, then this zero is of order one, and $a_1' = 0, b_1' = 1, a_2' = 1, b_2' = 0$ yield the function $f/g$, which has pole of degree one at $\infty$. It has no other poles since they would be other zeros of $f/g$, so this function has the desired properties. Otherwise denote the value of $f/g$ at $t = \infty$ by $w_1$. Consider the following function: $g/f = w_1 = (g - w_1 f)/f$. Clearly, it has a zero at $t = \infty$. Observe that $g/f$ has exactly one pole of order one, namely, at the point where $f/g$ has zero of order one. Hence, $g/f + w_1$ has exactly one pole of order one. The sum of minus orders of all poles and of (plus) orders of all zeros of a rational function on $\mathbb{P}^1$ is zero. Thus, $g/f + w_1$ has exactly one zero, and this zero is of order one. But we already know one zero of $g/f + w_1$, namely, $t = \infty$. Therefore, this zero is of order one, and $f/(g - w_1 f)$ has exactly one pole, this pole is of order one and is at $t = \infty$.

Now we have a function $(a_1' f + b_1' g)/(a_2' f + b_2' g)$, which is regular at all points where $t$ takes finite value and has a pole of order one at $t = \infty$. Denote the value of this function at $t = 0$ by $w_2$. Consider
the following function: \((a'_1 f + b'_2 g)/(a'_2 f + b'_2 g) - w_2 = ((a'_1 - w_2 a'_2 f) + (b'_1 - w_2 b'_2 g))/(a'_2 f + b'_2 g)\). It has exactly one pole, this pole is at \(t = \infty\) and of order one, and it has a zero at \(t = 0\). If we divide this function by \(t\), the resulting function \(((a'_1 - w_2 a'_2 f) + (b'_1 - w_2 b'_2 g))/(t(a'_2 f + b'_2 g))\) has no poles on \(\mathbb{P}^1\), so it is a constant. Therefore, if we multiply \(((a'_1 - w_2 a'_2 f) + (b'_1 - w_2 b'_2 g))/(t(a'_2 f + b'_2 g))\) by the appropriate constant, it will be equal to \(t\).

\(\square\)

**Corollary 2.** For every divisor \(D\) on \(\mathbb{P}^1\) of positive degree and for every non-zero rational function \(f \in \mathcal{O}(D)\) there exist \(g \in \mathcal{O}(D)\) and \(a_1, b_1, a_2, b_2 \in \mathbb{C}\) such that \((a_1 f + b_1 g)/(a_2 f + b_2 g) = t\).

*Proof.* Since \(f \in \mathcal{O}(D)\), \(\text{div}(f) + D\) is an effective divisor. Write \(\text{div}(f) + D = \sum a'_i p'_i\), where \(a'_i \in \mathbb{Z}_{\geq 0}\), \(p'_i \in \mathbb{P}^1\). Since \(f\) is a rational function on \(\mathbb{P}^1\), \(\deg \text{div}(f) = 0\), and \(\sum a'_i = \deg \text{div}(f) + \deg D = \deg D > 0\).

There exists a point \(p'_i\) such that \(a'_i > 0\). Choose another point \(p''_i\), and consider the following divisor: \(D_1 = \sum a'_i p'_i - p'_i + p''_i\). This is an effective divisor since \(a'_i > 0\). Let \(y\) be a rational function on \(\mathbb{P}^1\) such that \(\text{div}(y) = -p'_i + p''_i\). Then \(D + \text{div}(y) = D_1 \geq 0\). Hence, \(g = f y \in \mathcal{O}(D)\), and we can apply Lemma \(\square\) to \(f\) and \(g\) since \(\text{div}(f/g) = \text{div}(1/y) = p'_i - p''_i\).

In particular, this holds for every divisor of the form \(\mathcal{D}(\chi)\), \(\chi \in \sigma' \cap M\) such that \(\dim \mathcal{O}(\mathcal{D}(\chi)) \geq 2\). Such a rational function \((a_1 f + b_1 g)/(a_2 f + b_2 g)\) can be considered as a (rational) map to \(\mathbb{P}^1\), and by Proposition \(\square\) it coincides with \(\pi\).

**Corollary 3.** Let \(x \in X\). If there exists a degree \(\chi \in \sigma' \cap M\) such that \(\dim \mathcal{O}(\mathcal{D}(\chi)) \geq 2\) and \(f \in \mathcal{O}(\mathcal{D}(\chi))\) such that \(\tilde{f}(x) \neq 0\), then \(\pi\) is defined at \(x\).

So we define an open subset \(U_0 \subseteq X\) as follows: it consists of all points \(x \in X\) such that there exists a degree \(\chi \in \sigma' \cap M\) such that \(\dim \mathcal{O}(\mathcal{D}(\chi)) \geq 2\) and there exists \(f \in \mathcal{O}(\mathcal{D}(\chi))\) such that \(\tilde{f}(x) \neq 0\). Corollary \(\square\) shows that \(\pi\) is defined on \(U_0\). In fact, \(\pi\) is not defined outside \(U_0\), but we will not need this.

Our next goal is to understand fibers of \(\pi\). First, consider an ordinary point \(p \in \mathbb{P}^1\). For every degree \(\chi \in \sigma' \cap M\), the sections of \(\mathcal{O}(\mathcal{D}(\chi))\) do not have poles at \(p\). For each \(\chi \in \sigma' \cap M\), choose a basis \(e_{p,\chi,1}, \ldots, e_{p,\chi,\dim \mathcal{O}(\mathcal{D}(\chi))}\) of \(\mathcal{O}(\mathcal{D}(\chi))\) such that \(e_{p,\chi,1}(p) = 1, e_{p,\chi,2}(p) = \ldots = e_{p,\chi,\dim \mathcal{O}(\mathcal{D}(\chi))}(p) = 0\). In particular, observe that for \(\chi = 0\) we have \(\mathcal{O}(\mathcal{D}(0)) = \mathcal{O}_{\mathbb{P}^1}\), and the only global functions of degree 0 are constants. The condition \(e_{p,\chi,1}(p) = 1\) guarantees in this case that \(e_{p,\chi,0,1} = 1\) and \(e_{p,0,1} \equiv 1\) everywhere.

By Proposition \(\square\) if \(\pi(x) = p\) and \(2 \leq i \leq \dim \mathcal{O}(\mathcal{D}(\chi))\), then \((e_{p,\chi,i}/e_{p,\chi,1})(x) = (e_{p,\chi,i}/e_{p,\chi,1})(p) = 0\), so \(e_{p,\chi,i}(x) = 0\) since \(e_{p,\chi,1}\) is a global function.

For every \(\chi, \chi' \in \sigma' \cap M\), \(a, a' \in \mathbb{Z}_{\geq 0}\), \((e_{p,\chi,1})^a(e_{p,\chi',1})^{a'}\) is an element of \(\mathcal{O}(\mathcal{D}(ax + a' \chi'))\), so it can be written as \((e_{p,\chi,1})^a(e_{p,\chi',1})^{a'} = \sum c_{i,\chi,\chi',a,a'} e_{p,ax+a' \chi',i}, c_{i,\chi,\chi',a,a'} \in \mathbb{C}\). This equality holds for rational functions on \(\mathbb{P}^1\), and evaluation at \(p\) shows that \(c_{1,\chi,\chi',a,a'} = 1\). The equality also holds for the corresponding global functions on \(X\).

These computations prove the following lemma:

**Lemma 5.** For every \(\chi, \chi' \in \sigma' \cap M\), \(a, a' \in \mathbb{Z}_{\geq 0}\) and for every \(x \in \pi^{-1}(p)\), \((e_{p,\chi,1}(x))^a(e_{p,\chi',1}(x))^{a'} = e_{p,ax+a' \chi',1}(x)\).

\(\square\)

**Proposition 4.** For a point \(x \in X\), \(x \in \pi^{-1}(p) \cup U_0\), there are at most three possibilities:

1. For every \(\chi \in \sigma' \cap M\), \(e_{p,\chi,1}(x) \neq 0\).

2. For every \(\chi \in E_0(\sigma'') \cap M\), \(e_{p,\chi,1}(x) \neq 0\), and \(e_{p,\chi,1}(x) = 0\) for all other \(\chi \in \sigma' \cap M\). This is possible if and only if \(\deg \mathcal{D}(a_0) > 0\).

3. For every \(\chi \in E_1(\sigma'') \cap M\), \(e_{p,\chi,1}(x) \neq 0\), and \(e_{p,\chi,1}(x) = 0\) for all other \(\chi \in \sigma' \cap M\). This is possible if and only if \(\deg \mathcal{D}(a_1) > 0\).

*Proof.* Until the end of the proof, denote the sublattice in \(M\) generated by \(a_0\) and \(a_1\) by \(M'\). First, consider a degree \(\chi' \in M'\). We know that if \(\chi' = a_0 a_0 + a_1 a_1\), then \(e_{p,\chi,1}(x) = (e_{p,a_0,1}(x))^{a_0}(e_{p,a_1,1}(x))^{a_1}\).

So there can be four possibilities:

1. \(e_{p,a_0,1}(x) \neq 0\) and \(e_{p,a_1,1}(x) \neq 0\). Then \(e_{p,\chi,1}(x) \neq 0\) for all \(\chi' \in \sigma' \cap M'\).

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2. $\epsilon_{p,\alpha,1}(x) \neq 0$, but $\epsilon_{p,\alpha,1}(x) = 0$. Then for all $\chi' \in \sigma^\vee \cap M'$ we have $\epsilon_{p,\chi',1}(x) \neq 0$ if and only if $\chi' \in E_0(\sigma^\vee)$.

3. $\epsilon_{p,\alpha,1}(x) = 0, \epsilon_{p,\alpha,1}(x) \neq 0$. Similarly, $\epsilon_{p,\chi',1}(x) \neq 0$ if and only if $\chi' \in E_1(\sigma^\vee)$.

4. $\epsilon_{p,\alpha,1}(x) = \epsilon_{p,\alpha,1}(x) = 0$. Then $\epsilon_{p,\chi',1}(x) = 0$ for all $\chi' \in \sigma^\vee \cap M'$ except $\chi' = 0$.

Since $M'$ is a sublattice of finite index in $M$ (recall that $\dim M = 2$), for every $\chi \in M$ there is $\chi' = \omega_0 \chi \in M'$, $\omega_0 \in \mathbb{N}$. We have $\epsilon_{p,\chi',1}(x) = (\epsilon_{p,\chi',1}(x))^0$, so $\epsilon_{p,\chi',1}(x) = 0$ if and only if $\epsilon_{p,\chi',1}(x) = 0$. Therefore, the classification above also works for $\chi \in M$:

1. $\epsilon_{p,\alpha,1}(x) \neq 0$ and $\epsilon_{p,\alpha,1}(x) \neq 0$. Then $\epsilon_{p,\chi',1}(x) \neq 0$ for all $\chi' \in \sigma^\vee \cap M$.

2. $\epsilon_{p,\alpha,1}(x) \neq 0$, but $\epsilon_{p,\alpha,1}(x) = 0$. Then for all $\chi' \in \sigma^\vee \cap M$ we have $\epsilon_{p,\chi',1}(x) \neq 0$ if and only if $\chi' \in E_0(\sigma^\vee)$.

3. $\epsilon_{p,\alpha,1}(x) = 0, \epsilon_{p,\alpha,1}(x) \neq 0$. Similarly, $\epsilon_{p,\chi',1}(x) \neq 0$ if and only if $\chi' \in E_1(\sigma^\vee)$.

4. $\epsilon_{p,\alpha,1}(x) = \epsilon_{p,\alpha,1}(x) = 0$. Then $\epsilon_{p,\chi',1}(x) = 0$ for all $\chi' \in \sigma^\vee \cap M$ except $\chi' = 0$.

Notice that case 4 is impossible in $U_0$, and case 2 (resp. 3) is possible if and only if there is a degree $\chi \in E_0(\sigma^\vee) \cap M$ (resp. $\chi \in E_1(\sigma^\vee) \cap M$) such that $\deg \mathcal{D}(\chi) > 0$. Now recall that $\mathcal{D}(\chi)$ becomes a linear function after a restriction to a line in $M$, so existence of such $\chi$ is equivalent to $\deg \mathcal{D}(\alpha_0) > 0$ (resp. $\deg \mathcal{D}(\alpha_1) > 0$).

This proposition can be reformulated without mentioning bases of $\mathcal{O}(\mathcal{D}(\chi))$ explicitly as follows:

**Corollary 4.** For each $x \in \pi^{-1}(p) \cap U_0$, there exists a subcone $\tau \subseteq \sigma^\vee$ such that if $\chi \in \sigma^\vee \cap M$ and $f \in \mathcal{O}(\mathcal{D}(\chi))$, then

$$\tilde{f}(x) \neq 0 \iff \chi \in \tau \text{ and } \text{ord}_p(f) = 0.$$

For the cone $\tau$ (which depends on $x$) there are at most three possibilities:

1. $\tau = \sigma^\vee$.

2. $\tau = E_0(\sigma^\vee)$. This is possible if and only if $\deg \mathcal{D}(\alpha_0) > 0$.

3. $\tau = E_1(\sigma^\vee)$. This is possible if and only if $\deg \mathcal{D}(\alpha_1) > 0$.

**Proof.** First, fix a degree $\chi \in \sigma^\vee \cap M$. Notice that if $f \in \mathcal{O}(\mathcal{D}(\chi))$, then $\text{ord}_p(f) = 0$ if and only if the decomposition of $f$ into a linear combination of functions $\epsilon_{p,\chi',1}(x)$ contains $\epsilon_{p,\chi',1}(x)$ with a nonzero coefficient.

Now fix a point $x \in \pi^{-1}(p) \cap U_0$. Recall that all functions $\epsilon_{p,\chi',1}(x)$ for $i > 1$ vanish on $\pi^{-1}(p) \cap U_0$. We see that $\epsilon_{p,\chi',1}(x) \neq 0$ if and only if $\tilde{f}(x) \neq 0$ for all $f \in \mathcal{O}(\mathcal{D}(\chi))$ such that $\text{ord}_p(f) = 0$. We also see that, independently of the value of $\epsilon_{p,\chi',1}(x)$, $\tilde{f}(x) = 0$ for all $f \in \mathcal{O}(\mathcal{D}(\chi))$ such that $\text{ord}_p(f) > 0$.

Following [2], denote the set of all points $x \in \pi^{-1}(p) \cap U_0$ such that case 4 holds by $\text{orb}(p, V_1(\sigma))$ (resp. $\text{orb}(p, E_0(\sigma))$, $\text{orb}(p, E_1(\sigma))$). In fact (see [5, 1]), these sets are orbits of the torus. Sometimes we can simply write $\text{orb}(p, 0)$ instead of $\text{orb}(p, V_1(\sigma))$.

Now we are going to understand the structure of a fiber $\pi^{-1}(p)$ over a special point $p = p_i$. The function $\chi \mapsto \min_{a \in \Delta_p} \chi(a)$ (which defines the coefficient for $p$ in $\mathcal{D}(\chi)$, denote it shortly by $\mathcal{D}_p(\chi)$) is piecewise linear. One checks easily that the maximal subcones of $\sigma^\vee$ where $\mathcal{D}(\chi)$ is linear are exactly the cones $N(V_j(\Delta_p), \Delta_p)$ ($1 \leq j \leq \nu(\Delta_p)$). In what follows, we write $V_p$ instead of $V(\Delta_p)$ and $V_{p,j}$ instead of $V_j(\Delta_p)$ for brevity. Observe that $V_p = 1$ if and only if $p$ is a removable special point.

This time we choose bases of $\mathcal{O}(\mathcal{D}(\chi))$ as follows: let $\epsilon_{p,\chi,1}, \ldots, \epsilon_{p,\chi,\dim \mathcal{O}(\mathcal{D}(\chi))}$ be a basis of $\mathcal{O}(\mathcal{D}(\chi))$ such that $\text{ord}_p(\epsilon_{p,\chi,1}) = -\mathcal{D}_p(\chi)$ and $\text{ord}_p(\epsilon_{p,\chi,1}) > -\mathcal{D}_p(\chi)$ for $i > 1$. Then functions $\epsilon_{p,\chi,1}, \epsilon_{p,\chi,i}$ for $i > 1$ are defined at $p$ and evaluate to $0$ there, so if $x \in \pi^{-1}(p)$, then by Proposition 1, $\epsilon_{p,\chi,1}(x) = 0$, and $\epsilon_{p,\chi,i}(x) = 0$ for $i > 1$. In this case we demand explicitly for $\chi = 0$ that $\tau_{p,0,1} = 1$ and $\epsilon_{p,0,1} = 1$ everywhere.

Now let $\chi, \chi' \in \sigma^\vee \cap M$, $a, a' \in \mathbb{Z}_{\geq 0}$, then $(\epsilon_{p,\chi,1})^a(\epsilon_{p,\chi',1})^{a'}$ is an element of $\mathcal{O}(\mathcal{D}(\alpha_1 + a'\chi'))$, so it can be written as

$$(\epsilon_{p,\chi,1})^a(\epsilon_{p,\chi',1})^{a'} = \sum_i c_{i,\chi',\alpha_1} \epsilon_{p,\alpha_1 + a'\chi'}.$$

$c_{i,\chi',\alpha_1} \in \mathbb{C}$. 

We have \( \text{ord}_p((p/\chi)^a \cdot p/\chi')^a' = -aD_p(\chi) - a'D_p(\chi') \), \( \text{ord}_p((p/\alpha + p/\chi')^a' > -D_p(\alpha + a') \) for \( i > 1 \). Therefore, \( c_{1,\chi',\alpha,a'} \neq 0 \) if and only if \( aD_p(\chi) + a'D_p(\chi') = D_p(\alpha + a') \) if and only if \( a = 0 \) or \( a' = 0 \) or \( \chi \) and \( \chi' \) are in the same subcone of \( \sigma' \) where \( D_p(\cdot) \) is linear, i.e. \( \chi, \chi' \in N(V_{p,j}, \Delta_p) \) for some \( j \).

These computations prove the following lemma:

**Lemma 6.** For every \( \chi, \chi' \in \sigma' \cap M, a, a' \in \mathbb{Z}_{\geq 0} \) and for every \( x \in \pi^{-1}(p) \), \((e_{p,\chi,1}(x))^a((e_{p,\chi',1}(x))^a' = c_{1,\chi',\alpha,a'}e_{p,\alpha + a'}(\chi',1)(x) \), where \( c_{1,\chi',\alpha,a'} \) depends on \( p \) and on the choice of \( e_{p,\chi,1} \) but not on \( n \). \( c_{1,\chi',\alpha,a'} \neq 0 \) if and only if \( a = 0 \) or \( a' = 0 \) or there exists a vertex \( V_{p,j} \) of \( \Delta_p \) such that \( \chi, \chi' \in N(V_{p,j}, \Delta_p) \).

**Corollary 5.** Let \( \chi, \chi' \in \sigma' \cap M, a, a' \in \mathbb{N}, x \in \pi^{-1}(p) \). Suppose that there exist no vertex \( V_{p,j} \) such that \( \chi, \chi' \in N(V_{p,j}, \Delta_p) \). Then for every \( f \in \mathcal{O}(D(\chi)), g \in \mathcal{O}(D(\chi')) \) we have \( \hat{f}(x)\hat{g}(x) = 0 \).

**Proposition 5.** Let \( x \in X \) be a point, \( x \in \pi^{-1}(p) \cup U_0 \). The set of degrees \( \chi \) such that \( e_{p,\chi,1}(x) \neq 0 \) can be the set of all lattice points in one of the following cones:

1. \( N(V_{p,j}, \Delta_p) \) for some \( j, 1 \leq j \leq v_p \).
2. \( N(E_{p,j}, \Delta_p) \) for some \( j, 0 < j < v_p \).
3. \( N(E_{p,j}, \Delta_p) \) for \( j = 0 \) or \( j = v_p \). This is possible if and only if \( \deg D(\chi_j) > 0 \).

**Proof.** Denote \( \chi_j = b(E_{p,j}) \) for \( 0 \leq j \leq v_p \). (In particular, we have \( \chi_0 = \alpha_0 \) and \( \chi_{v_p} = \alpha_1 \). Consider all indices \( j \) such that \( e_{p,\chi_j}(x) \neq 0 \). Since \( \chi_j \) is in \( N(V_{p,j'}, \Delta_p) \) only for \( j' = j \) or \( j' = j - 1 \), there can be at most two such indices \( j \), and if there are two of them, they should be two consecutive natural numbers.

Suppose first that \( e_{p,\chi_j}(x) \neq 0 \) and \( e_{p,\chi_j'}(x) \neq 0 \) for some \( j \). The argument is similar to the proof of Proposition 4. Namely, consider the sublattice in \( M \) generated by \( \chi_j - \chi_1 \). It is a sublattice of finite index, denote it by \( M' \). For every \( \chi' \in M' \), \( \chi' = \alpha_j + a' \chi_j \) we have \( c_{1,\chi_j,\alpha_j,a'}e_{p,\chi_j}(x) = (e_{p,\chi_j}(x))^a((e_{p,\chi_j}(x))^a' \neq 0, \text{ so } \hat{e}_{p,\chi_j}(x) \neq 0 \). For every \( \chi \in N(V_{p,j}, \Delta_p) \cap M \) there exists \( a'' \in \mathbb{N} \) such that \( a'' \chi \in M', \text{ so } e_{p,a'',\chi}(x) \neq 0 \). By lemma 6, \( (e_{p,\chi_j}(x))^a'' = c_{1,\chi_0,\alpha_0,a''}e_{p,\chi_0}(x) \), and \( c_{1,\chi_0,\alpha_0,a''} \neq 0 \), so \( e_{p,\chi_j}(x) \neq 0 \). Finally, for a degree \( \chi \in N(V_{p,j'}, \Delta_p) \) choose an arbitrary degree \( \chi' \) in the interior of \( \chi \in N(V_{p,j'}, \Delta_p) \cap M \). Then by Lemma 6, \( e_{p,\chi_j}(x)e_{p,\chi_j}(x) = 0 \), we already know that \( e_{p,\chi_j}(x) \neq 0 \), so \( e_{p,\chi_j}(x) = 0 \).

Now suppose that there exists a degree \( \chi \) such that \( e_{p,\chi_j}(x) \neq 0 \) and \( \chi \) is in the interior of a cone \( N(V_{p,j}, \Delta_p) \). Again denote the lattice generated by \( \chi_j - \chi_1 \) by \( M' \). There exists \( a'' \in \mathbb{N} \) such that \( \chi = \alpha'' \chi_j \in M' \). We have \( c_{1,\chi_j,\alpha',a''}e_{p,\chi}(x) = (e_{p,\chi}(x))^a'' \), so \( e_{p,\chi_j}(x) \neq 0 \). \( \chi' \) is also in the interior of \( N(V_{p,j}, \Delta_p) \), thus there exist \( a, a' \in \mathbb{N} \) such that \( a\chi_j + a' \chi' = \chi \). Again we have \( (e_{p,\chi_j}(x))^a((e_{p,\chi_j}(x))^a' = c_{1,\chi_j-1,\alpha,\alpha'}e_{p,\chi_j}(x), \text{ where } c_{1,\chi_j-1,\alpha,\alpha'} \neq 0, \text{ so } e_{p,\chi_j}(x) \neq 0 \) and \( e_{p,\chi_j}(x) \neq 0 \). Therefore, if there exists a degree \( \chi \) in the interior of a cone \( N(V_{p,j}, \Delta_p) \) such that \( e_{p,\chi_j}(x) \neq 0 \), then there are two indices \( j' \) such that \( e_{p,\chi_j}(x) \neq 0 \).

Now consider the case when there is only one \( j \) such that \( e_{p,\chi_j}(x) \neq 0 \). We already know that in this case for all degrees \( \chi \) from the interiors of the cones \( N(V_{p,j}, \Delta_p) \), we have \( e_{p,\chi_j}(x) = 0 \). So the only possible degrees \( \chi \) such that \( e_{p,\chi_j}(x) \neq 0 \) are multiples of \( \chi_j = b(E_{p,j}) \). And for these degrees we have \( c_{1,\chi_j,0,a,0}e_{p,\chi_j}(x) = (e_{p,\chi_j}(x))^a, \text{ so } e_{p,\chi_j}(x) \neq 0 \). Such \( x \) can be in \( U_0 \) only if \( \deg D(\chi_j) > 0 \). Properness guarantees this for \( 0 < j < v_p \), and for \( j = 0 \) or \( j = v_p \) we have to check this explicitly.

And again this proposition can be reformulated without referring to bases of \( \mathcal{O}(D(\chi)) \).

**Corollary 6.** For each \( x \in \pi^{-1}(p) \cap U_0 \), there exists there exists a subcone \( \tau \subseteq \sigma' \) such that if \( \chi \in \sigma' \cap M \) and \( f \in \mathcal{O}(D(\chi)) \), then \( \hat{f}(x) \neq 0 \Leftrightarrow \chi \in \tau \) and \( \text{ord}_p(\tau) = -D_p(\chi) \).

\( \tau \) can be one of the following cones:

1. The normal subcone \( N(V_{p,j}, \Delta_p) \) of a vertex \( V_{p,j} \) of \( \Delta_p \).
2. The normal subcone $\mathcal{N}(E_{p,j}, \Delta_p)$ of a finite edge $E_{p,j}$ ($0 < j < v_p$).

3. The normal subcone $\mathcal{N}(E_{p,j}, \Delta_p)$ of an infinite edge $E_{p,j}$ ($j = 0$ or $j = v_p$, respectively). This is possible if and only if $\deg D(\alpha_0) > 0$ or $\deg D(\alpha_1) > 0$, respectively.

**Proof.** The proof is very similar to the proof of Corollary 4. Again, we fix a degree $f$ notice that if $f \in O(D(\chi))$, then $\text{ord}_p(f) = -D_p(\chi)$ if and only if the decomposition of $f$ into a linear combination of functions $e_{p,\chi,1}$ contains $e_{p,\chi,1}$ with a nonzero coefficient. Fix a point $x \in \pi^{-1}(p) \cap U_0$. Again for all functions $e_{p,\chi,1}$, where $i > 1$, we have $e_{p,\chi,1}(x) = 0$. Therefore, $e_{p,\chi,1}(x) \neq 0$ if and only if $\tilde{f}(x) \neq 0$ for all $f \in O(D(\chi))$ such that $\text{ord}_p(f) = -D_p(\chi)$. And, independently of the value of $e_{p,\chi,1}(x)$, $\tilde{f}(x) = 0$ for all $f \in O(D(\chi))$ such that $\text{ord}_p(f) > -D_p(\chi)$.

And again, following [5], we denote the set of all points $x \in \pi^{-1}(p) \cap U_0$ such that case 1 (resp. case 2 or 3) holds by $\text{orb}(p, V_{p,j})$ (resp. by $\text{orb}(p, E_{p,j}(p))$). In fact (see [5], [11]), these sets are orbits of the torus.

We are going to use Theorem [1] Leray exact sequence for the map $\pi$ and Proposition [2] to compute $T^1(X)$. To do this, we need an open subset $U \subseteq X$ suitable for Theorem [1] and an affine covering of $U$. We first choose affine sets $U_i$ and then set $U = \bigcup U_i$.

To define a set $U_i$, we fix the following data:

1. a pair of degrees $(\beta_{i,1}, \beta_{i,2}) \in \sigma^\vee \cap M$ generating $M$ as a lattice and such that $\deg D(\beta_{i,1}) > 0$, $\deg D(\beta_{i,2}) > 0$, and $\beta_{i,2}$ is in the interior of $\sigma^\vee$,
2. two sections $h_{i,1} \in O(D(\beta_{i,1})), h_{i,2} \in O(D(\beta_{i,2})).$

Let $V_i \subseteq \mathbf{P}^1$ be an arbitrary open subset of the set of all points $p \in \mathbf{P}^1$ such that:

1. $\text{ord}_p(h_{i,1}) = -D_p(\beta_{i,1}), \text{ord}_p(h_{i,2}) = -D_p(\beta_{i,2})$ (in particular, if $p$ is an ordinary point, $\text{ord}_p(h_{i,1}) = \text{ord}_p(h_{i,2}) = 0$).
2. If $p$ is a special point and $\beta_{i,1}$ is in the interior of $\sigma^\vee$, then $\beta_{i,1}$ and $\beta_{i,2}$ are in the interior of the same normal subcone $\mathcal{N}(V_{p,j}, \Delta_p)$ of the same vertex $V_{p,j}$.
3. If $p$ is a special point and $\beta_{i,1} \in E_0(\sigma^\vee)$, then $\beta_{i,2}$ is in the interior of $\mathcal{N}(V_{p,0}, \Delta_p)$.
4. If $p$ is a special point and $\beta_{i,1} \in E_1(\sigma^\vee)$, then $\beta_{i,2}$ is in the interior of $\mathcal{N}(V_{p,v_p}, \Delta_p)$.

$U_i$ is defined to be the set of points $x \in U_0 \subseteq X$ such that:

1. $\pi(x) \in V_i$,
2. $\tilde{h}_{i,1}(x) \neq 0$,
3. if $\beta_{i,1}$ is in the interior of $\sigma^\vee$, then $\tilde{h}_{i,2}(x) \neq 0$.

We say that sets $U_i$ defined this way form a sufficient system if

1. for every ordinary point $p \in \mathbf{P}^1$ there exists $i$ such that $p \in V_i$,
2. for every special point $p \in \mathbf{P}^1$ and for every normal subcone $\mathcal{N}(V_{p,j}, \Delta_p)$ there exists an index $i$ such that $p \in V_i$ and $\beta_{i,1}, \beta_{i,2} \in \mathcal{N}(V_{p,j}, \Delta_p)$,
3. for every primitive degree $\chi \in \partial \sigma^\vee$ such that $\deg D(\chi) > 0$ and for every $p \in \mathbf{P}^1$ there exists an index $i$ such that $\beta_{i,1} = \chi$ and $p \in V_i$.

Clearly, sufficient systems exist. Fix one of them and set $U = \bigcup U_i$. Denote the number of sets $U_i$ in the sufficient system we chose by $q$.

We are going to prove that $\text{codim}_X(X \setminus U) \geq 2$, i.e. that $\text{dim}(X \setminus U) \leq 1$.

**Lemma 7.** $\text{dim}(X \setminus U_0) \leq 1$. 

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Proof. Let $x \in X \setminus U_0$. For every degree $\chi \in \sigma' \cap M$ such that $\deg D(\chi) > 0$, for every $f \in \mathcal{O}(D(\chi))$ we have $\tilde{f}(x) = 0$. $\deg D(\chi)$ can be zero only if $\chi \in \partial \sigma'$. If there are functions $f \in \mathcal{O}(D(\alpha_0)), g \in \mathcal{O}(D(\alpha_1))$ that do not vanish at $x$, then $fg \in \mathcal{O}(D(\alpha_0 + \alpha_1)), f(x)g(x) \neq 0$, but $\alpha_0 + \alpha_1 \notin \partial \sigma'$. So for at most one of the degrees $\alpha_0$ and $\alpha_1$, there are functions of this degree that vanish at $x$. Without loss of generality suppose that if $f \in \mathcal{O}(D(\alpha_0))$, then $\tilde{f}(x) = 0$. If $\deg D(\alpha_1) > 0$, then $\deg D(\chi) > 0$ for all multiples $\chi$ of $\alpha_1$, so for every such $\chi$ all functions of degree $\chi$ vanish at $x$. Otherwise $\dim \mathcal{O}(D(\chi)) = 1$ for every multiple $\chi$ of $\alpha_0$, and if $f \in \mathcal{O}(D(\alpha_0)), f \neq 0$, then $f^n$ generate $\mathcal{O}(D(\alpha_0))$ as a vector space, so all functions of degree $\alpha_0$ vanish at $x$. Summarizing, we conclude that if $\chi \in \mathcal{E}_0(\sigma') \cap M$, then all functions of degree $\chi$ vanish at $x$. Consequently, if $\deg D(\alpha_1) > 0$, then all functions of nonzero degree, i.e. all nonconstant functions on $X$ vanish at $x$. There exists only one such point $x$. Otherwise, if $f$ forms a basis of $\mathcal{O}(D(\alpha_1))$, then $f^n$ forms a basis of $\mathcal{O}(D(\alpha_0))$, so values of all functions of all degrees at $x$ are determined by $f(x)$. Therefore, such points $x$ form a 1-dimensional subset.

Now we are going to consider points from $U_0$.

Lemma 8. For every ordinary point $p \in \mathbb{P}^1$ we have $\pi^{-1}(p) \cap U_0 = \pi^{-1}(p) \cap U$.

Proof. Clearly, $\pi^{-1}(p) \cap U_0 \subseteq \pi^{-1}(p) \cap U$. To prove the other inclusion, we use the description of $\pi^{-1}(p) \cap U_0$ from Corollary 3. Recall that if $p \in V_i$ for some index $i$, then $\ord_p(h_{i,1}) = \ord_p(h_{i,2}) = 0$. If $x \in \text{orb}(p,0)$, then it is sufficient to take any index $i$ such that $p \in V_i$ (it exists by the definition of a sufficient system). Then by Corollary 3, $\tilde{h}_{i,1}(x) \neq 0$, $\tilde{h}_{i,2}(x) \neq 0$, and $x \in U_i$. If $x \notin \text{orb}(p,\mathcal{E}_0(\sigma))$, then $\deg D(\alpha_0) > 0$, and there exists an index $i$ such that $\alpha_0 = \beta_{i,1}$ and $p \in V_i$. Then $f_i$ is a function of degree $\alpha_0$, so Corollary 3 says that $\tilde{h}_{i,1}(x) \neq 0$, and, since $\deg D(\alpha_0) > 0$, this is enough for $x$ to be in $U_i$.

The case $x \in \text{orb}(p,\mathcal{E}_1(\sigma))$ can be considered similarly.

Now we are going to consider the fiber of $\pi$ over a special point $p \in \mathbb{P}^1$.

Lemma 9. Let $p \in \mathbb{P}^1$ be a special point. Then $\dim(\pi^{-1}(p) \cap (U_0 \setminus U)) \leq 1$.

Proof. We use the description of $\pi^{-1}(p) \cap U_0$ from Corollary 3. First, pick a vertex $V_{p,j}$ $(1 \leq j \leq v_p)$ and consider a point $x \in \text{orb}(p, V_{p,j})$. Since the system $\{U_i\}$ is sufficient, there exists $i$ such that $\beta_{i,1}, \beta_{i,2} \in N(V_{p,j}, \Delta_p)$ and $p \in V_i$. By the definition of $V_i$, $\ord_p(h_{i,1}) = D_p(\beta_{i,1})$ and $\ord_p(h_{i,2}) = D_p(\beta_{i,2})$, and by the definition of $\text{orb}(p, V_{p,j}), \tilde{h}_{i,1}(x) \neq 0$ and $\tilde{h}_{i,2}(x) \neq 0$. Hence, $x \in U_i$. Therefore, if $x \in \pi^{-1}(p) \cap U_0$, but $x \notin \pi^{-1}(p) \cap U$, then $x \in \text{orb}(p, \mathcal{E}_{p,j})$ for some (finite or infinite) edge $E_{p,j}$.

It is sufficient to prove that for each (finite or infinite) edge $E_{p,j}$, we have $\dim \text{orb}(p, E_{p,j}) \leq 1$. Denote $\chi = b(N(E_{p,j}, \Delta_p))$ and choose a basis $e_{p,x,1}, \ldots, e_{p,x,\dim \mathcal{O}(D(\chi))}$ of $\mathcal{O}(D(\chi))$ as previously, i.e. so that $\ord_{\alpha}(\tilde{h}_{p,\chi}) = -D_p(\chi), \ord_{\alpha}(\tilde{h}_{p,\chi}) > -D_p(\chi)$ for $1 \leq \dim \mathcal{O}(D(\chi))$. Consider a degree $\chi' = a\chi, a \in \mathbb{N}$. Choose a basis of $\mathcal{O}(D(\chi'))$ as follows. Its first element is $e_{p,x,1} = (e_{p,x,a})^a$, so we have $\ord_{\alpha}(\tilde{h}_{p,\chi'}) = -aD_p(\chi) = -D(\chi')$. All other elements of the basis, denoted by $e_{p,x,2}, \ldots, e_{p,x,\dim \mathcal{O}(D(\chi'))}$, satisfy $\ord_{\alpha}(\tilde{h}_{p,\chi'}) > -D(\chi')$. We have already seen for such a basis that $e_{p,x,\chi'}(x) = 0$ for all $x \in \pi^{-1}(p) \cap U_0, l > 1$. So again values of all functions of all degrees at $x$ are determined by $e_{p,x,1}(x)$, and $W_j$ is at most one-dimensional.

We are going to use $\{U_i\}$ to compute cohomology groups, so we are going to prove that all $U_i$ are affine. Fix an index $i$.

Lemma 10. Let $\chi \in \sigma' \cap M$ be a degree. Let $\chi = a_1\beta_{i,1} + a_2\beta_{i,2}, a_1, a_2 \in \mathbb{Z}$. Let $p \in V_i$. Then, independently of the signs of $a_1$ and $a_2$, $D_p(\chi) \leq a_1D_p(\beta_{i,1}) + a_2D_p(\beta_{i,2})$.

Proof. Recall that the function $\mathcal{D}_p(\cdot)$ is always linear on the cone spanned by $\beta_{i,1}$ and $\beta_{i,2}$ if $p \in V_i$. Hence, if $a_1 \geq 0$ and $a_2 \geq 0$, then $D_p(\chi) = a_1D_p(\beta_{i,1}) + a_2D_p(\beta_{i,2})$. If $a_1 < 0$ or $a_2 < 0$, in other words, if $\chi$ is not in the cone generated by $\beta_{i,1}$ and $\beta_{i,2}$, then, since $\mathcal{D}_p(\cdot)$ is a convex function, $\mathcal{D}_p(\chi) \leq a_1D_p(\beta_{i,1}) + a_2D_p(\beta_{i,2})$.

Lemma 11. $U_i$ is isomorphic to $V_i \times (\mathbb{C} \setminus 0) \times L$, where $L$ is isomorphic to $\mathbb{C}$ or $\mathbb{C} \setminus 0$. More exactly, $L = \mathbb{C}$ if and only if $\beta_{i,1} \in \partial \sigma'$, otherwise $L = \mathbb{C} \setminus 0$. $V_i$ is isomorphic to an open set in an affine line.

The isomorphism is given by $(\pi, h_{i,1}, h_{i,2})$. 15
Proof. We know that $V_t \subseteq \mathbf{P}^1$, and to prove that $V_t$ is isomorphic to an open subset in an affine line, it is sufficient to prove that $V_t$ cannot be equal to $\mathbf{P}^1$. Indeed, if $p \in V_t$, then in particular, $\text{ord}_p(h_{t,1}) = D_p(\beta_{t,1})$. If $V_t = \mathbf{P}^1$, this would mean that $\text{div}(h_{t,1}) = D(\beta_{t,1})$. But $\deg D(\beta_{t,1}) > 0$, and $\deg \text{div}(h_{t,1}) = 0$.

Consider the map $U_t \to V_t \times (\mathbb{C} \setminus 0) \times L$ given by $(x, h_{t,1}, h_{t,2})$ (recall that $h_{t,2} = 0$ is possible in $U_t$ if and only if $\beta_{t,1} \in \partial \Sigma$). To define its inverse, we need for every triple $(p, t_1, t_2)$, where $p \in V_t$, $t_1 \in \mathbb{C} \setminus 0$, $t_2 \in L$, define a point $x \in U_t$. To do this, we define a homomorphism $\mathbb{C}[X] \to \mathbb{C}$. We define it on each graded component of $\mathbb{C}[X]$.

Let $\chi \in \sigma' \cap M$ be a degree. Let $\chi = a_1 \beta_{1,1} + a_2 \beta_{1,2}$, $a_1, a_2 \in \mathbb{Z}$. By Lemma 10, $D_p(\chi) \leq a_1 D_p(\beta_{1,1}) + a_2 D_p(\beta_{1,2}) = \text{ord}_p(h_{t,1})^a_1 h_{t,2}^a_2$. Therefore, if $f \in \mathcal{O}(D(\chi))$, then $\text{ord}_p(f) \geq D_p(\chi) = \text{ord}_p(h_{t,1})^a_1 h_{t,2}^a_2$, and the rational function $f/\langle h_{t,1}, h_{t,2} \rangle$ is defined at $p$.

Now we define a map $\mathbb{C}[X] \to \mathbb{C}$ as follows: if $f \in \mathcal{O}(D(\chi))$, then $f \mapsto (\tilde{f}_1)^{a_1} (\tilde{f}_2)^{a_2} (f/\langle h_{t,1}, h_{t,2} \rangle)(p)$. Note that $a_2 < 0$ is possible if and only if $\beta_{t,1} \notin \partial \sigma'$, i.e., exactly if and only if $L = \mathbb{C} \setminus 0$. It is clear from the construction that this map is an algebraic homomorphism, so it defines a point $x \in X$. If we choose a set of homogeneous generators of $\mathbb{C}[X]$, we see that the values of these generators at $x$ depend algebraically on $p, t_1,$ and $t_2$, so we have defined an algebraic morphism $\varphi \colon V_t \times (\mathbb{C} \setminus 0) \times L \to X$.

Now we are going to prove that two morphisms we have defined are mutually inverse. Fix points $p \in V_t$, $t_1 \in \mathbb{C} \setminus 0$, and $t_2 \in L$, denote $x = \varphi(p, t_1, t_2)$. First, $x \in U_0$ since $\deg D(\beta_{t,1}) > 0$ and $\tilde{h}_{t,1}(x) = t_1 \tilde{h}_{t,1}(h_{t,1}, h_{t,2})(p) = t_1 \neq 0$. Now denote $\pi(x) = p'$. For every degree $\chi = a_1 \beta_{t,1} + a_2 \beta_{t,2}$ and for every pair of functions $f_1, f_2 \in \mathcal{O}(D(\chi))$, we have the following equalities of rational functions $(p'' \in V_t, t'_1 \in \mathbb{C} \setminus 0, t'_2 \in L$ are arbitrary points): $(\tilde{f}_1/\tilde{f}_2)(\varphi(p'', t'_1, t'_2)) = \langle \tilde{f}_1, \tilde{f}_2 \rangle(h_{t,1}^a_1 h_{t,2}^a_2) \langle f_1 f_2 \rangle(h_{t,1}^a_1 h_{t,2}^a_2)$. Choose a degree $\chi$ such that $\deg D(\chi) > 0$. By Corollary 2 there exist functions $f_1, f_2 \in \mathcal{O}(D(\chi))$ such that $\tilde{f}_1/\tilde{f}_2$ is defined at $p'$, and if $(\tilde{f}_1/\tilde{f}_2)(p'') = \langle f_1, f_2 \rangle(p'')$ for some $p'' \in \mathbf{P}^1$, then $p' = p''$. By Proposition 1, $\tilde{f}_1/\tilde{f}_2$ is defined at $x$, and $(\tilde{f}_1/\tilde{f}_2)(x) = \langle f_1, f_2 \rangle(p)$.

On the other hand, it follows from the computation above that $(\tilde{f}_1/\tilde{f}_2)(x) = \langle f_1, f_2 \rangle(\varphi(p, t_1, t_2)) = \langle f_1, f_2 \rangle(p)$, so $p' = p''$, and $\pi(x) = p$. We have already checked that $h_{t,1}(x) = t_1$, a similar computation shows that $h_{t,2}(x) = t_2$. The conditions from the definition of $U_t$ are therefore satisfied, and $x \in U_t$.

Finally, check that the other composition of morphisms $X \to V_t \times (\mathbb{C} \setminus 0) \times L \to X$ is also the identity morphism. To do this, fix a point $x \in U_t$, a degree $\chi = a_1 \beta_{t,1} + a_2 \beta_{t,2}$ and a function $f \in \mathcal{O}(D(\chi))$. We have the following equality of rational functions: $\tilde{f}_1/\tilde{f}_2 = \langle \tilde{f}_1, \tilde{f}_2 \rangle(\varphi(p, t_1, t_2)) = \langle f_1, f_2 \rangle(p)$, so $p' = p''$, and $\pi(x) = p$. We have already checked that $h_{t,1}(x) = t_1$, a similar computation shows that $h_{t,2}(x) = t_2$. The conditions from the definition of $U_t$ are therefore satisfied, and $x \in U_t$.

We will also need the fact that all intersections of $U_t$ are affine. Fix several indices $a_1, \ldots, a_k$.

**Lemma 12.** $U' = U_{a_1} \cap \cdots \cap U_{a_k}$ is isomorphic to $V' \times (\mathbb{C} \setminus 0) \times L'$, where $V'$ is an open subset of $V_{a_1}$, and $L'$ is isomorphic to $\mathbb{C}$ or $\mathbb{C} \setminus 0$. The isomorphism is given by $(\pi, h_{a_1,1}, h_{a_1,2})$ (this is exactly the restriction of the isomorphism from Lemma 7 to the subset $U' \subseteq U_{a_1}$).

In this case, $L' = \mathbb{C}$ if and only if $\beta_{a_1,1} = \cdots = \beta_{a_k,1} \in \partial \Sigma$.

Here the set of ordinary points in $V'$ is the set of ordinary points in $V_{a_1} \cap \cdots \cap V_{a_k}$. If $p \in \mathbf{P}^1$ is a special point, then $p \in V'$ if and only if $p \in V_{a_1} \cap \cdots \cap V_{a_k}$ and all degrees $\beta_{a_1,1}, \ldots, \beta_{a_k,1}, \beta_{a_1,2}, \ldots, \beta_{a_k,2}$ belong to the normal subcone of the same vertex of $\Delta_p$.

**Proof.** Consider a fiber $\pi^{-1}(p) \cap U'$, where $p \in V_{a_1}$. It is a subset of $\pi^{-1}(p) \cap U_{a_1}$, which is isomorphic to $(\mathbb{C} \setminus 0) \times L$ by Lemma 10. It is sufficient to prove that for each $p \in V_{a_1}$, in terms of this isomorphism, $\pi^{-1}(p) \cap U'$ is the empty set, or equals $(\mathbb{C} \setminus 0) \times L'$.

First, let $p \in V_{a_1}$ be an ordinary point. If there exists an index $i$ such that $p \notin V_{a_i}$, then $\pi^{-1}(p) \cap U' = \emptyset$. Otherwise, consider a point $x \in \pi^{-1}(p) \cap U_{a_i}$. There are two possibilities: either $h_{a_i,2}(x) \neq 0$ (in other words, the last coordinate of $x$ in terms of the isomorphism $U_i \cong V_i \times (\mathbb{C} \setminus 0) \times L$ from Lemma 10 is nonzero), or $\beta_{a_i,1} \in \partial \Sigma$ and $h_{a_i,2}(x) = 0$ (in other words, the last coordinate of $x$ in terms of the isomorphism from Lemma 10 is zero). If the first possibility takes place, then, by Corollary 3, $x \in U_{a_i}$ for all $i$. If the second possibility takes place, then it follows from Corollary 4 that $x \in U_{a_i}$ if and only
if $\beta_{a,1} \in \partial \sigma^\vee$ (i.e., we have no condition for $\tilde{h}_{a,1}(x)$, which is in fact zero since $\beta_{a,2}$ is in the interior of $\sigma^\vee$) and $\beta_{a,1} = \beta_{a,1}$ (otherwise $\tilde{h}_{a,1}(x) = 0$). This finishes the proof for an ordinary point.

Now let $p \in V_{a,1}$ be a special point. Again, if there exists an index $i$ such that $p \notin V_{a,i}$, then $\pi^{-1}(p) \cap U' = \emptyset$. Moreover, by Corollary 8, if there exist no vertex $V_{p,j}$ such that $\beta_{a,1} \in \mathcal{N}(V_{p,j}, \Delta_p)$ for all $i$, then $\pi^{-1}(p) \cap U' = \emptyset$ (recall that we require that $\beta_{a,1}$ is in the interior of the normal cone of a vertex of $\Delta_p$, unless $\beta_{a,1} \in \partial \sigma^\vee$, in the definition of $V_{a,i}$, so $\beta_{a,1}$ cannot be in the normal cones of two different vertices simultaneously). And again, if $p \in V_{a,i}$ for all $i$ and there exists a vertex $V_{p,j}$ such that $\beta_{a,1} \in \mathcal{N}(V_{p,j}, \Delta_p)$ for all $i$ (by the definition of $V_{a,i}$, this implies that $\beta_{a,2}$ is in the interior of $\mathcal{N}(V_{p,j}, \Delta_p)$ for all $i$), then there are two possibilities. Either $\tilde{h}_{a,1}(x) \neq 0$, (i.e. the last coordinate of $x$ is nonzero), or $\beta_{a,1} \in \partial \sigma^\vee$ and $\tilde{h}_{a,1}(x) = 0$, (i.e. the last coordinate of $x$ is zero). The rest of the proof repeats the proof for an ordinary point. Namely, if the first possibility holds, it follows from Corollary 8 that $x \in U_{a,i}$ for all $i$. If the second possibility holds, then, by Corollary 8: $x \in U_{a,i}$ if and only if $\beta_{a,1} \in \partial \sigma^\vee$ (i.e. we have no condition for $\tilde{h}_{a,1}(x)$, while $h_{a,1}(x) = 0$ since $\beta_{a,2}$ is in the interior of $\mathcal{N}(V_{p,j}, \Delta_p)$) and $\beta_{a,1} = \beta_{a,1}$ (this is a criterion for $h_{a,1}(x) \neq 0$, nevertheless, this condition can only be violated if $\sigma^\vee = \mathcal{N}(V_{p,j}, \Delta_p)$, i.e. $p$ is a removable special point).

We know that codim$_X(X \setminus U) \geq 2$, so Theorem 1 can be applied. To apply it, we need a set of generators of $\mathcal{C}[X]$. We choose it as follows. For each special point $p$, the cone $\sigma^\vee$ can be split into the union of normal cones of all vertices of $\Delta_p$. All intersections of these cones (for different special points) split $\sigma^\vee$ into a fan, which we call the total normal fan of $\mathcal{D}$. For each cone $\tau$ in this fan, the function $\mathcal{D}(\cdot)_{|\tau} : \tau \rightarrow \text{Div}(\mathbb{P}^1)$ is linear. Let $\lambda_1, \ldots, \lambda_m$ be the union of Hilbert bases of $\mathcal{D}(\tau)$. We now consider a possibly bigger set of degrees $\lambda_1, \ldots, \lambda_{m'}, \ldots, \lambda_m$ satisfying the following additional properties for each special point $p$.

1. For each (finite or infinite) edge $E_{p,j}$, $b(\mathcal{N}(E_{p,j}, \Delta_p)) \in \{\lambda_1, \ldots, \lambda_m\}$.
2. For each vertex $V_{p,j}$ there exists a degree $\chi \in \{\lambda_1, \ldots, \lambda_m\} \cap \mathcal{N}(V_{p,j}, \Delta_p)$ such that $\chi$ and $b(\mathcal{N}(V_{p,j}, \Delta_p))$ form a lattice basis of $M$.
3. For each vertex $V_{p,j}$ there exists a degree $\chi \in \{\lambda_1, \ldots, \lambda_m\} \cap \mathcal{N}(V_{p,j}, \Delta_p)$ such that $\chi$ and $b(\mathcal{N}(V_{p,j}, \Delta_p))$ form a lattice basis of $M$.

In fact, these properties are already satisfied by $\{\lambda_1, \ldots, \lambda_{m'}\}$, but we don’t need this fact and we will not prove it. For each $i, 1 \leq i \leq m$ let $x_{\lambda_i,j}$ (for $1 \leq i \leq m, 1 \leq j \leq \dim \mathcal{O}(\mathcal{D}(\lambda_i))$ be a basis of $\mathcal{O}(\mathcal{D}(\lambda_i))$.

**Lemma 13.** All $x_{\lambda_{m,j}}$ (for $1 \leq i \leq m, 1 \leq j \leq \dim \mathcal{O}(\mathcal{D}(\lambda_i))$) together generate $\mathcal{C}[X]$.

**Proof.** It is sufficient to prove that every homogeneous element of $\mathcal{C}[X]$ can be generated by $x_{\lambda_{m,j}}$. So, fix a degree $\chi$ in $\sigma^\vee \cap M$, and let $f \in \mathcal{O}(\mathcal{D}(\chi))$. If $\chi \in \{\lambda_1, \ldots, \lambda_m\}$, the claim is clear. Otherwise, choose a cone $\tau$ from the total normal fan so that $\chi \in \tau$. $\chi$ is not an element of the Hilbert basis of $\tau$, so there exist $\chi', \chi'' \in \tau \cap M$, $\chi' \neq 0, \chi'' \neq 0$, such that $\chi' + \chi'' = \chi$. Since $\mathcal{D}(\cdot) : \sigma^\vee \rightarrow \text{Div}(\mathbb{P}^1)$ becomes a linear function after being restricted to $\tau$, $\mathcal{D}(\chi) = \mathcal{D}(\chi') + \mathcal{D}(\chi'')$.

Let $r_1$ be the number of points $p \in \mathbb{P}^1$ that are either special or are zeros of $\mathcal{F}$. Denote zeros of $\mathcal{F}$ that are ordinary points by $p_{r+1}, \ldots, p_{r_1}$ (recall that we have $r$ special points $p_1, \ldots, p_r$). Consider the following $r_1$ integers: $a_i = \mathcal{D}(p_i) + \text{ord}_{p_i}(\mathcal{F})$. By the definition of $\mathcal{O}(\mathcal{D}(\chi))$, all these numbers are nonnegative integers. Also, $a_1 + \ldots + a_{r_1} = \mathcal{D}(p_1) + \ldots + \mathcal{D}(p_{r_1}) + \text{ord}_{p_1}(\mathcal{F}) + \ldots + \text{ord}_{p_{r_1}}(\mathcal{F}) = \text{deg} \mathcal{D}(\chi) + \text{deg} \text{div}(\mathcal{F}) = \text{deg} \mathcal{D}(\chi)$. Then it is possible to split each of these numbers into a sum $a_i = a'_i + a''_i$ of two nonnegative integers so that $a'_1 + \ldots + a'_{r_1} = \text{deg} \mathcal{D}(\chi')$ and $a''_1 + \ldots + a''_{r_1} = \text{deg} \mathcal{D}(\chi'')$ (recall that $\mathcal{D}(\chi) = \mathcal{D}(\chi') + \mathcal{D}(\chi'')$). Then $D_1 = (a'_1 - \mathcal{D}(p_1))p_1 + \ldots + (a'_{r_1} - \mathcal{D}(p_{r_1}))p_{r_1}$ and $D_2 = (a''_1 - \mathcal{D}(p_1))p_1 + \ldots + (a''_{r_1} - \mathcal{D}(p_{r_1}))p_{r_1}$ are divisors of degree 0, and $D_1 \geq -\mathcal{D}(\chi'), D_2 \geq -\mathcal{D}(\chi'')$. Therefore, there exist functions $f' \in \mathcal{O}(\mathcal{D}(\chi'))$ and $f'' \in \mathcal{O}(\mathcal{D}(\chi''))$ such that $\text{div}(\mathcal{F}) = D_1$ and $\text{div}(\mathcal{F}') = D_2$. Now, for every point $p_i$, we have the following: $\text{ord}_{p_i}(\mathcal{F}/\mathcal{F}') = a'_i - \mathcal{D}(p_i) = a''_i - \mathcal{D}(p_i)$ and $a_i = \mathcal{D}(p_i) = \text{ord}_{p_i}(\mathcal{F})$. Hence, $\mathcal{F}/\mathcal{F}'$ is a rational function on $\mathbb{P}^1$ that does not have zeros or poles, so it is a constant, and $\mathcal{F}$ is a multiple of $f'f''$.

Repeating this procedure by induction on $\chi \in \tau$, we can write $f$ as a product of functions whose degrees are in the set $\{\lambda_1, \ldots, \lambda_m\}$.
Now we construct a map $\psi$ required for Theorem 1 using these generators. Denote the total number of these generators by $s$. By Theorem 1 we have the following isomorphism of $\mathbb{C}[X]$-modules:

$$T^1(X) = \ker(H^1(U, \Theta_X) \xrightarrow{H^1(\psi|_U)} H^1(U, \mathcal{O}^{\geq s}_X)).$$

By Lemma 11 \{U_i\} form an affine covering of $U$, so it can be used to compute homology groups in this formula as Čech homology. Moreover, all conditions defining $U_i$ as subsets of $X$ are formulated in terms of fibers of $\pi$ and inequalities of the form $f \neq 0$, where $f$ is a homogeneous function. Since $\pi$ is $T$-invariant and the inequalities of form $f \neq 0$ are also invariant if $f$ is homogeneous, the sets $U_i$ are $T$-invariant. The sheaves involved in the formula above are the tangent bundle and the trivial bundle, so $T$ acts on the modules of their sections on $U_i$. Hence, these modules are $M$-graded. This enables us to introduce an $M$-grading on $H^1(U, \Theta_X)$ and on $H^1(U, \mathcal{O}^{\geq s}_X)$. The map $\psi$ is defined by $s$ maps $\Theta_X \to \mathcal{O}_X$, each of them corresponds to a generator $x_{\lambda,j,k}$ of degree $\lambda_j$. It maps the graded component of $\Gamma(U_i, \Theta_X)$ of degree $\chi \in M$ to the graded component of $\Gamma(U_i, \mathcal{O}^{\geq s}_X)$ of degree $\chi + \lambda_j$. Hence, $H^1(\psi|_U)$ maps different graded components of $H^1(U, \Theta_X)$ to different graded components of $H^1(U, \mathcal{O}^{\geq s}_X) = H^1(U, \mathcal{O}^{\geq s}_X)$, and $\ker H^1(\psi|_U)$ is a graded submodule in $H^1(U, \Theta_X)$. It follows from the proof of Theorem 1 that the isomorphism $T^1(X) = \ker H^1(\psi|_U)$ is an isomorphism of graded $\mathbb{C}[X]$-modules.

Now, we apply Leray exact sequence for the map $\pi: U \to \mathbb{P}^1$ and get the following short exact sequences of $\mathbb{C}[X]$-modules (note that Lemmas 5 and 6 guarantee that $\pi(U) = \mathbb{P}^1$):

$$0 \to H^1(\mathbb{P}^1, (\pi|_U)_*, (\Theta_X|_U)) \to H^1(U, \Theta_X) \to H^0(\mathbb{P}^1, R^1(\pi|_U)_*, (\Theta_X|_U)) \to 0$$

and

$$0 \to H^1(\mathbb{P}^1, (\pi|_U)_*, (\mathcal{O}^{\geq s}_X|_U)) \to H^1(U, \mathcal{O}^{\geq s}_X) \to H^0(\mathbb{P}^1, R^1(\pi|_U)_*, (\mathcal{O}^{\geq s}_X|_U)) \to 0.$$

Snake lemma yields the following exact sequence:

$$0 \to \ker \left( H^1(\mathbb{P}^1, (\pi|_U)_*, (\Theta_X|_U)) \xrightarrow{H^1(\psi|_U)} H^1(\mathbb{P}^1, (\pi|_U)_*, (\mathcal{O}^{\geq s}_X|_U)) \right) \to T^1(X)$$

and

$$0 \to \ker \left( H^0(\mathbb{P}^1, R^1(\pi|_U)_*, (\Theta_X|_U)) \xrightarrow{H^0(\psi|_U)} H^0(\mathbb{P}^1, R^1(\pi|_U)_*, (\mathcal{O}^{\geq s}_X|_U)) \right).$$

This is an isomorphism of $\mathbb{C}[X]$-modules, and it is possible to introduce $M$-grading on these modules. Indeed, in fact the sheaves $(\pi|_U)_*(\Theta_X|_U)$ and $(\pi|_U)_*(\mathcal{O}^{\geq s}_X|_U)$ are graded themselves, i.e. they are direct sums of their graded components in the category of sheaves of $\mathcal{O}_{\mathbb{P}^1}$-modules, since their sections on any open subset $V \subseteq \mathbb{P}^1$ are sections of the tangent bundle and of rank $s$ trivial bundle on a $T$-invariant subset $\pi^{-1}(V)$, and multiplication by functions from $\Gamma(V, \mathcal{O}_{\mathbb{P}^1})$ does not change the grading of a section. This is also true for the sheaves $R^1(\pi|_U)_*(\Theta_X|_U)$ and $R^1(\pi|_U)_*(\mathcal{O}^{\geq s}_X|_U)$ if we compute them using Proposition 2 with $\{U_i\}$ being the required affine covering of $U$ since in this case the module of sections of any sheaf in the complex on any open subset $V \subseteq \mathbb{P}^1$ is also a direct sum of modules of sections of the tangent bundle or of the trivial bundle on a $T$-invariant subset of $X$, and the differentials in the complex preserve this grading. So, again there is an $M$-grading on cohomology groups: on $H^1(\mathbb{P}^1, (\pi|_U)_*(\Theta_X|_U))$, on $H^0(\mathbb{P}^1, R^1(\pi|_U)_*(\Theta_X|_U))$, on $H^1(\mathbb{P}^1, (\pi|_U)_*(\mathcal{O}^{\geq s}_X|_U))$, and on $H^0(\mathbb{P}^1, R^1(\pi|_U)_*(\mathcal{O}^{\geq s}_X|_U))$. And again, the map $(\pi|_U)_*: (\Theta_X|_U) \to ((\pi|_U)_*(\Theta_X|_U))^{\geq s}$ is defined by $s$ maps $(\pi|_U)_*(\Theta_X|_U) \to ((\pi|_U)_*(\Theta_X|_U))^{\geq s}$, each of them corresponds to a generator $x_{\lambda,j,k}$ of degree $\lambda_j$. It maps the graded component of $(\pi|_U)_*(\Theta_X|_U)$ of degree $\chi \in M$ to graded components of $(\pi|_U)_*(\mathcal{O}^{\geq s}_X|_U)$ of degree $\chi + \lambda_j$. So, different graded components of $H^1(\mathbb{P}^1, (\pi|_U)_*(\Theta_X|_U)) \cong H^0(\mathbb{P}^1, R^1(\pi|_U)_*(\Theta_X|_U))$ are mapped by $H^1((\pi|_U)_*, \psi) \oplus H^0(R^1(\pi|_U)_*, \psi)$ to different graded components of $H^1(\mathbb{P}^1, (\pi|_U)_*(\mathcal{O}^{\geq s}_X|_U)) \cong H^0(\mathbb{P}^1, R^1(\pi|_U)_*(\mathcal{O}^{\geq s}_X|_U))$, and $\ker H^1((\pi|_U)_*, \psi) \oplus H^0(R^1(\pi|_U)_*, \psi)$ is an $M$-graded $\mathbb{C}[X]$-module. This grading coincides (in terms of the isomorphisms mentioned above) with gradings on $T^1(X)$ and on $\ker H^1((\pi|_U)_*, \psi)$. Now we are going to obtain a formula for the graded component of $T^1(X)$ of degree $0$. Denote it by $T^1(X)_0$. Denote also the graded component of $(\pi|_U)_*(\Theta_X)$ of degree $0$ by $G^{0\text{triv}}$, the graded component
of $R^1(\pi|_U)_\Theta_X$ of degree 0 by $G^\text{inv}$._1. We need graded components of $(\pi|_U)_\Theta_X$ and of $R^1(\pi|_U)_\Theta_X$ of different degrees, so for a degree $\chi$ denote by $G^\text{inv}_\chi$ the graded component of $(\pi|_U)_\Theta_X$ of degree $\chi$, and denote by $G^\text{inv}_X$ the graded component of $R^1(\pi|_U)_\Theta_X$ of degree $\chi$. The morphism $H^1((\pi|_U)_\psi)$ maps $H^1(\mathbb{P}^1, G^\text{inv}_1)$ to $H^1(\mathbb{P}^1, G^\text{inv}_5)$, where

$$G^\text{inv}_3 = \bigoplus_{i=1}^m \bigoplus_{j=1}^\dim \mathcal{O}(\mathcal{D}(\lambda_i))^i \bigoplus_{j=1}^\dim \mathcal{O}(\mathcal{D}(\lambda_j))_j.$$

The morphism $H^0(R^1(\pi|_U)_\psi)$ maps $H^0(\mathbb{P}^1, G^\text{inv}_4)$ to $H^0(\mathbb{P}^1, G^\text{inv}_8)$, where

$$G^\text{inv}_8 = \bigoplus_{i=1}^m \bigoplus_{j=1}^\dim \mathcal{O}(\mathcal{D}(\lambda_i))^i \bigoplus_{j=1}^\dim \mathcal{O}(\mathcal{D}(\lambda_j))_j.$$

So, the above exact sequence for $T^1(X)$ can be written in the graded form as follows:

**Proposition 6.** The following sequence is exact:

$$0 \to \ker \left( H^1(\mathbb{P}^1, G^\text{inv}_1) \to H^1(\mathbb{P}^1, G^\text{inv}_5) \right) \to T^1(X)_0 \to \ker \left( H^0(\mathbb{P}^1, G^\text{inv}_4) \to H^0(\mathbb{P}^1, G^\text{inv}_8) \right) \to \ker \left( H^1(\mathbb{P}^1, G^\text{inv}_1) \to H^1(\mathbb{P}^1, G^\text{inv}_5) \right).$$

Our next goal is to find expressions for the sheaves $G^\text{inv}_1, G^\text{inv}_4, G^\text{inv}_5,$ and $G^\text{inv}_8$ including only functions on $\mathbb{P}^1$ and the combinatorics of $\mathcal{D}$. Given an index $i$ and a point $p \in V_i$, Proposition 6 provides an isomorphism between $\pi^{-1}(p) \cap U_i$ and $\mathbb{C} \setminus \{0\} \times L$, where $L$ is $\mathbb{C} \setminus \{0\}$ or $\mathbb{C}$. Call the point identified by this isomorphism with $(1, 1) \in (\mathbb{C} \setminus \{0\}) \times L$ the canonical point in the fiber $\pi^{-1}(p)$ with respect to $U_i$. In other words, the canonical point in $\pi^{-1}(p)$ with respect to $U_i$ is the (unique) point $x \in \pi^{-1}(p) \cap U_i$ such that $h_{i,1}(x) = h_{i,3}(x) = 1$.

Fix an embedding $V_i \hookrightarrow \mathbb{C}$. As long as such an embedding is fixed, we identify each point of $p \in V_i$ with its coordinate $t_0 \in \mathbb{C}$. Denote the coordinates of a point $x \in U_i$ provided by the isomorphism $U_i \cong V_i \times (\mathbb{C} \setminus \{0\}) \times L$ by $t_0 \in V_i, t_1 \in \mathbb{C} \setminus \{0\}, t_2 \in L$.

We are going to study homogeneous vector fields of degree 0 (i.e., $T$-invariant vector fields) on open sets $U'_i \subset X$ of the form $V'_i \times (\mathbb{C} \setminus \{0\}) \times L' \subseteq U_i$, where $V'_i \subseteq V_i$ is an open subset, $L' \subseteq L$ is $\mathbb{C}$ or $(\mathbb{C} \setminus \{0\})$, $L$ is defined in Lemma 14 and $U'_i$ is embedded in $U_i$ as a subset of $V_i \times (\mathbb{C} \setminus \{0\}) \times L$ via isomorphism from Lemma 14.

**Lemma 14.** Let $V'_i \subseteq V_i$ be an open subset, $L' \subseteq L$ be an open subset that can be equal $\mathbb{C}$ or $(\mathbb{C} \setminus \{0\})$, $U'_i = V'_i \times (\mathbb{C} \setminus \{0\}) \times L' \subseteq U_i$. A homogeneous vector field of degree $\theta$ on $U'_i$ is uniquely determined by its values at canonical points in all fibers $\pi^{-1}(t_0)$ (for $t_0 \in V'_i$) with respect to $U_i$. These values can be arbitrary vectors depending algebraically on $t_0 \in V'_i$.

**Proof.** Let $w$ be a vector field of degree 0 on $U'_i$, and suppose that $w(t_0, 1, 1) = f_0(t_0)\partial/\partial t_0 + f_1(t_0)\partial/\partial t_1 + f_2(t_0)\partial/\partial t_2$, where $f_j$, $V'_i \to \mathbb{C}$ are algebraic functions. Since $M$ is the character lattice of $T$, and $\beta_{i,j}$ and $\beta_{i,2}$ form a basis of $M$, every pair $(t_1, t_2) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$ uniquely and algebraically determines an element $\tau \in T$ such that $\beta_{i,1}(\tau) = t_1, \beta_{i,2}(\tau) = t_2$. This element acts on $U'_i$, i.e., it defines an automorphism of $U'_i$, which we also denote by $\tau$. Recall that $t_j = h_{i,j}(t_0), j = 1, 2,$ and $h_{i,1}$ (resp. $h_{i,2}$) is a function of degree $\beta_{i,1}$ (resp. $\beta_{i,2}$), so $\tau(t_0, 1, 1) = (t_0, t_1, t_2)$ for every $t_0 \in V'_i$. By the definition of a $T$-invariant vector field, $w$ is a field of degree 0 if and only if $w(\tau x) = dw' w(x)$ for every $x \in U'_i, \tau' \in T$. 19
In particular, this holds for \( x = (t_0, 1, 1) \), \( \tau' = \tau \), so \( w \) is uniquely determined on \( V'_i \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \), which is at least an open subset in \( U'_i \), so it is determined uniquely on \( U'_i \).

We still have to check that if we start with arbitrary functions \( f_0, f_1, f_2 : V'_i \to \mathbb{C} \), the vector field on \( V'_i \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \) constructed this way can be extended to the whole \( U'_i \) if and only if \( f_0, f_1, f_2 \) satisfy the statement of the Lemma and that the resulting vector field on \( U'_i \) is \( T \)-invariant. To do this, let us first write the vector field we have constructed in terms of \( f_j \) and \( \partial/\partial t_j \). Take a point \( x = (t_0, t_1, t_2) \in V'_i \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \), \( t_0 = \pi(x) \). We have \( w(t_0, t_1, t_2) = d\pi w(t_0, 1, 1) = df_0(t_0)\partial/\partial t_0 + f_1(t_0)\partial/\partial t_1 + f_2(t_0)\partial/\partial t_2 \) = \( (f_0(t_0)\partial/\partial t_0 + t_1f_1(t_0)\partial/\partial t_1 + t_2f_2(t_0)\partial/\partial t_2) \). Clearly, functions of the form \( f_j(t_0)t_1^{a_1}t_2^{a_2} \) with \( a_1 \geq 0, a_2 \geq 0 \) can be extended to the whole \( U'_i \).

Observe that to check homogeneity, we have to check an equality of two vector fields for each \( \tau \in T \). This equality holds if it holds on an open subset in \( U'_i \), in particular, it is sufficient to check homogeneity of the resulting vector field on \( V'_i \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \). Take a point \( x = (t_0, t_1, t_2) \in V'_i \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \) and an element \( \tau' \in T \). Denote by \( \tau \in T \) the element of \( T \) such that \( \beta_{i,1}(\tau) = t_1, \beta_{i,2}(\tau) = t_2 \). We have \( w(\tau' x) = w(\tau^2(\tau(t_0, 1, 1))) = d(\tau^2\tau_1)w(t_0, 1, 1) = d\tau d\tau_1w(t_0, 1, 1) = d\tau w(t_0, t_1, t_2) \), and the vector field is \( T \)-invariant.

**Corollary 7.** A homogeneous vector field of degree 0 on \( U'_i \) is also uniquely determined by the following data:

1. The derivatives of \( \tilde{h}_{i,j} \) \((j = 1, 2)\) along \( w \) at canonical points, considered as two algebraic functions \( V_i \to \mathbb{C} \).

2. The vector field on \( V'_i \) obtained by applying \( d\tau \) to the values of \( w \) at canonical points, \( d(t_0, 1, 1)\pi w(t_0, 1, 1) \).

The vector field and two functions can be arbitrary algebraic.

**Proof.** Write \( w(t_0, 1, 1) = f_0(t_0)\partial/\partial t_0 + f_1(t_0)\partial/\partial t_1 + f_2(t_0)\partial/\partial t_2 \). Then \( d(t_0, 1, 1)\pi w(t_0, 1, 1) = f_0(t_0)\partial/\partial t_0, dh_{i,j}w(t_0, 1, 1) = f_j(t_0) \) \((j = 1, 2)\).

Note that these data (the image of a vector at a canonical point under \( d\tau \), the derivatives of functions along \( w \)) do not depend on the choice of an embedding \( V_i \to \mathbb{C} \). Given a vector field \( w \) of degree 0 on \( U'_i \), we call the data from Corollary 4 the \( U_i \)-description of \( w \).

Observe also that the operation of taking the \( U_i \)-description is compatible with replacing \( U'_i \) by a smaller subset \( U''_i \) of the same form, or, more precisely, we can say the following:

**Remark 1.** Let \( U''_i \subseteq U'_i \) be a subset of \( U'_i \) of the same form, i. e. let \( V''_i \subseteq V'_i \) be an open subset, let \( L''_i \subseteq L'_i \) be an open subset that can be equal \( \mathbb{C} \) or \( \mathbb{C} \setminus 0 \), and let \( U''_i = V''_i \times (\mathbb{C} \setminus 0) \times L''_i \) be embedded into \( V'_i \times (\mathbb{C} \setminus 0) \times L'_i = U'_i \) via the embeddings \( V''_i \subseteq V'_i \) and \( L''_i \subseteq L'_i \) above. Let \( w' \) be the restriction of \( w \) to \( U''_i \). Then the \( U''_i \)-description of \( w' \) consists of the restrictions from \( V''_i \) to \( V'_i \) of the vector field and two functions forming the \( U'_i \)-description of \( w \).

Choose two indices \( i \) and \( j \) \((1 \leq i, j \leq q)\). The following lemma relates the \( U_i \)-description with the \( U_j \)-description of a vector field \( w \) of degree 0. We need some more notation to formulate it. Let \( C^p_{i,j} \) be the \( 2 \times 2 \)-matrix with integer entries such that \( \beta_{i,1} = (C^p_{i,j})_{1,1}\beta_{1,1} + (C^p_{i,j})_{1,2}\beta_{1,2} \) \(, \beta_{j,2} = (C^p_{i,j})_{2,1}\beta_{1,1} + (C^p_{i,j})_{2,2}\beta_{1,2} \). Denote

\[
C_{i,j}(p) = \begin{pmatrix}
C^p_{i,j} & \frac{h_{i,1}(p)(C^p_{i,j})_{1,1}h_{i,2}(p)(C^p_{i,j})_{1,2}}{h_{i,1}(p)h_{i,2}(p)} \\
0 & 0
\end{pmatrix},
\]

where \( p \in \mathbb{P}^1 \) is an arbitrary point, and the first and the second entry in the third column are understood as rational covector fields on \( \mathbb{P}^1 \). In particular, if \( i = j \), \( C^p_{i,i} \) and \( C_{i,i} \) are unit matrices. By Lemma 12 \( U_i \cap U_j \) is isomorphic to \( V' \times (\mathbb{C} \setminus 0) \times L' \), where \( V' \) is an open subset of \( V_i \cap V_j \) and \( L' = \mathbb{C} \) or \( \mathbb{C} \setminus 0 \). This product is embedded into \( U_i \) via the isomorphism from Lemma 11.
Lemma 15. Let $V''$ be an open subset of $V'$, $L''$ be an open subset of $L'$, $L'' = C$ or $L'' = C \backslash 0$, and let $U'' = V'' \times (C \setminus 0) \times L''$ be embedded into $U_1 \cap U_2$ via the map from Lemma 12. Let $w$ be a vector field on $U''$ of degree 0, and let $g_{1,1}, g_{1,2}, v_{1}$ be the $U_1$-description of $w$, and $g_{1,1}, g_{1,2}, v_{j}$ be the $U_j$-description of $w$. Then for every $p \in V''$

\[
\begin{pmatrix}
  g_{1,1}(p) \\
g_{1,2}(p) \\
v_j(p)
\end{pmatrix} = C_{i,j}(p) \begin{pmatrix}
  g_{1,1}(p) \\
g_{1,2}(p) \\
v_i(p)
\end{pmatrix}.
\]

In particular, $v_i(p) = v_j(p)$.

Proof. It is sufficient to check this equality on an arbitrary open subset of $V''$, so let $p \in V''$ be an ordinary point. Let $x$ be the canonical point in $\pi^{-1}(p)$ with respect to $U_1$. It follows from the definition of the canonical point that $x \in U''$. Let $x'$ be the canonical point in $\pi^{-1}(p)$ with respect to $U_j$. By Corollary 4. $h_{1,1}(x') \neq 0$, $h_{1,2}(x') \neq 0$, so $x' \in U''$.

Let $\tau \in T$ be the element of $T$ such that $\beta_{1,1}(\tau) = h_{1,1}(x')$, $\beta_{1,2}(\tau) = h_{1,2}(x')$. It defines an automorphism of $U''$, and we also denote this automorphism by $\tau$. Then $h_{1,1}(\tau x) = h_{1,1}(x')$, $h_{1,2}(\tau x) = h_{1,2}(x')$, $\pi(\tau x) = \pi(x')$, so $\tau x = x'$.

Since $w$ is a vector field of degree 0, $w(x') = d_{x'} \tau w(x)$. Since $\pi = \pi x$, we have $d_{x'} \pi = d_{x'} \pi d_{x'} \tau = d_{x'} \pi d_{x'} \tau$, and $v_j(p) = d_{x'} \pi w(x') = (d_{x'} \pi)(d_{x'} \tau w(x)) = d_{x'} \tau w(x) = v_i(p)$.

Now we are going to compute $g_{1,1}(p) = d_{x'} h_{1,1}(x')$. Until the end of the proof, denote $a_{1,1} = (C_{1,1})_1,1$, $a_{1,2} = (C_{1,1})_1,2$, $a_{2,1} = (C_{1,2})_2,1$, and $a_{2,2} = (C_{1,2})_2,2$. We have

\[
\tilde{h}_{1,1} = \frac{\tilde{h}_{1,1}^{-a_{1,1}}}{\tilde{h}_{1,1}^{a_{1,1}}} \frac{\tilde{h}_{1,2}^{-a_{1,2}}}{\tilde{h}_{1,2}^{a_{1,2}}},
\]

and

\[
d_{x'} h_{1,1} = a_{1,1} d_{x'} \tilde{h}_{1,1} h_{1,2}(x')^{-a_{1,2}} h_{1,1}(x') a_{1,1}^{-1} d_{x'} h_{1,2} h_{1,2}(x')^{-a_{1,2}} + \frac{\tilde{h}_{1,1}(x')}{h_{1,1}(x') a_{1,1}^{-1} h_{1,2}(x') a_{1,2}} + \frac{\tilde{h}_{1,2}(x')}{h_{1,1}(x') a_{1,1} h_{1,2}(x') a_{1,2}} d_{x'} \left( \frac{\tilde{h}_{1,1}}{\tilde{h}_{1,1}^{a_{1,1}}} \frac{\tilde{h}_{1,2}^{a_{1,2}}}{\tilde{h}_{1,2}^{a_{1,2}}} \right).
\]

Taking into account that $h_{1,1}(x') = 1$, we get

\[
d_{x'} h_{1,1} = a_{1,1} d_{x'} h_{1,1} + a_{1,2} d_{x'} h_{1,2} h_{1,2}(x') a_{1,2} + \frac{h_{1,1}(x')}{h_{1,1}(x') a_{1,1} h_{1,2}(x') a_{1,2}} d_{x'} \left( \frac{\tilde{h}_{1,1}}{h_{1,1}^{a_{1,1}}} \frac{\tilde{h}_{1,2}^{a_{1,2}}}{h_{1,2}^{a_{1,2}}} \right).
\]

We are computing $d_{x'} \tilde{h}_{1,1} w(x')$. We have $d_{x'} \tilde{h}_{1,1} w(x') = d_{x'} h_{1,1} d_{x'} \tau w(x)$. Since $h_{1,1}$ is a homogeneous function of degree $\beta_{1,1}$, we have the following equality of maps $X \to \mathbb{C}$: $h_{1,1} \circ \tau = h_{1,1} = h_{1,1} \circ h_{1,1}$. So, $d_{x'} h_{1,1} d_{x'} \tau w(x) = h_{1,1}(x') d_{x'} h_{1,1}(x') = h_{1,1}(x') g_{1,1}(p)$. Similarly, $d_{x'} \tilde{h}_{1,1} w(x') = h_{1,1}(x') g_{1,2}(p)$.

Now we are going to deal with the last summand in the formula for $d_{x'} h_{1,1}$ above. Since $h_{1,1}$ and $\tilde{h}_{1,1}^{-a_{1,1}} h_{1,2}^{a_{1,2}}$ are functions of the same degree $\beta_{1,1}$, by Proposition 4. we have the following equalities of maps from the open subset where they are defined as regular functions, not only as rational functions, to $\mathbb{C}$:

\[
\frac{\tilde{h}_{1,1}}{h_{1,1}^{a_{1,1}}} = \frac{h_{1,1}}{h_{1,1}^{a_{1,1}}} \circ \pi \text{ and } \frac{\tilde{h}_{1,2}}{h_{1,2}^{a_{1,2}}} = \frac{h_{1,2}}{h_{1,2}^{a_{1,2}}} \circ \pi.
\]

As we already know, $\tilde{h}_{1,1}(x') \neq 0$, $h_{1,2}(x') \neq 0$. Also, $\tilde{h}_{1,1}(x') = 1$ by the definition of $x'$, so these maps are defined at $x'$, and we get

\[
\frac{\tilde{h}_{1,1}(x') a_{1,1}}{h_{1,1}(x')} d_{x'} \left( \frac{\tilde{h}_{1,1}}{h_{1,1}^{a_{1,1}}} \frac{\tilde{h}_{1,2}^{a_{1,2}}}{h_{1,2}^{a_{1,2}}} \right) w(x') = \frac{h_{1,1}(p) a_{1,1}}{h_{1,1}(p)} d_{p} \left( \frac{h_{1,1}}{h_{1,1}^{a_{1,1}}} \frac{h_{1,2}^{a_{1,2}}}{h_{1,2}^{a_{1,2}}} \right) d_{x'} \tau w(x') =
\]

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\[
\frac{h_{i,1}(p)^{a_{1,1}} h_{i,2}(p)^{a_{1,2}}}{h_{j,1}(p)} d_p \left( \frac{h_{j,1}}{h_{i,1}^{a_{1,1}} h_{i,2}^{a_{1,2}}} \right) v_j(p) = \frac{h_{i,1}(p)^{a_{1,1}} h_{i,2}(p)^{a_{1,2}}}{h_{j,1}(p)} d_p \left( \frac{h_{j,1}}{h_{i,1}^{a_{1,1}} h_{i,2}^{a_{1,2}}} \right) v_i(p).
\]

Finally, we get the following formula for \( g_{j,1}(p) \):

\[
g_{j,1}(p) = d_x \tilde{h}_{j,1} w(x') = 
\frac{a_{1,1} d_x \tilde{h}_{i,1} w(x')}{h_{i,1}(x')^{a_{1,1}}} + \frac{a_{1,2} d_x \tilde{h}_{i,2} w(x')}{h_{i,2}(x')^{a_{1,2}}} d_x \left( \frac{h_{j,1}}{h_{i,1}^{a_{1,1}} h_{i,2}^{a_{1,2}}} \right) w(x') = 
\frac{a_{1,1} \tilde{h}_{i,1}(x') g_{i,1}(p)}{h_{i,1}(x')^{a_{1,1}}} + \frac{a_{1,2} \tilde{h}_{i,2}(x') g_{i,2}(p)}{h_{i,2}(x')^{a_{1,2}}} + 
\frac{h_{i,1}(p)^{a_{1,1}} h_{i,2}(p)^{a_{1,2}}}{h_{j,1}(p)} d_p \left( \frac{h_{j,1}}{h_{i,1}^{a_{1,1}} h_{i,2}^{a_{1,2}}} \right) v_i(p) = 
\left( a_{1,1} g_{i,1}(p) + a_{1,2} g_{i,2}(p) + \frac{h_{i,1}(p)^{a_{1,1}} h_{i,2}(p)^{a_{1,2}}}{h_{j,1}(p)} d_p \left( \frac{h_{j,1}}{h_{i,1}^{a_{1,1}} h_{i,2}^{a_{1,2}}} \right) v_i(p) \right).
\]

Similarly,

\[
g_{j,2}(p) = \left( a_{2,1} g_{i,1}(p) + a_{2,2} g_{i,2}(p) + \frac{h_{i,1}(p)^{a_{2,1}} h_{i,2}(p)^{a_{2,2}}}{h_{j,2}(p)} d_p \left( \frac{h_{j,2}}{h_{i,1}^{a_{2,1}} h_{i,2}^{a_{2,2}}} \right) v_i(p) \right).
\]

Now we are ready to describe the sheaf \( G_{\text{inv}} \) only using functions on \( P^1 \) and the notion of a sufficient system of \( U_i \) (which uses only combinatorics of \( D \) and functions on \( P^1 \)). Namely, consider the following sheaf \( G_1 \). Let \( V \subseteq P^1 \) be an open subset. The space of sections \( \Gamma(V, G_1) \) is the space of sequences of length \( 2q + 1 \)

\[
(g_{1,1}, g_{1,2}, \ldots, g_{1,1}, g_{1,2}, \ldots, g_{q,1}, g_{q,2}, v),
\]

where \( g_{i,j} \in \Gamma(V \cap V_i, \mathcal{O}_{P^1}) \), \( v \in \Gamma(V, \mathcal{O}_{P^1}) \) satisfy the following condition: For every indices \( i, i' \):

\[
\begin{pmatrix}
g_{i',1}(p) \\
g_{i',2}(p) \\
v(p)
\end{pmatrix} = C_{i,i'} \begin{pmatrix}
g_{i,1}(p) \\
g_{i,2}(p) \\
v(p)
\end{pmatrix}
\]

Proposition 7. \( G_{\text{inv}} \) is isomorphic to \( G_1 \). For an open set \( V \subseteq P^1 \), the isomorphism maps a vector field \( w \) defined on \( \pi^{-1}(V) \cap U \) to the sequence

\[
(g_{1,1}, g_{1,2}, \ldots, g_{1,1}, g_{1,2}, \ldots, g_{q,1}, g_{q,2}, v),
\]

such that \( (g_{1,1}, g_{1,2}, \ldots, g_{q,1}, g_{q,2}, v) \) is the \( U_i \)-description of \( w \).

Proof. This is a direct consequence of Lemma 15, Lemma 11, and the definition of a pushforward of a sheaf.

The following three lemmas make it easier to construct sections of \( G_1 \) explicitly.

Lemma 16. All entries of \( C_{i,j} \) are regular at ordinary points \( p \) such that \( p \in V_i \cap V_j \).

Proof. For constant entries the claim is clear, and non-constant entries are logarithmic derivatives of functions

\[
\frac{h_{j,1}}{h_{i,1}^{a_{1,1}} h_{i,2}^{a_{1,2}}} \quad \text{ and } \quad \frac{h_{j,2}}{h_{i,1}^{a_{1,1}} h_{i,2}^{a_{1,2}}}.
\]
If \( p \) is an ordinary point and \( p \in V_i \cap V_j \), then, by the definition of \( V_i \) and of \( V_j \), \( \text{ord}_p h_{i,1} = \text{ord}_p h_{i,2} = \text{ord}_p h_{j,1} = \text{ord}_p h_{j,2} = 0 \). Hence, both functions

\[
\frac{h_{j,1}}{h_{i,1} (C_{i,j})_{1,1} h_{i,2} (C_{i,j})_{1,2}} \quad \text{and} \quad \frac{h_{j,2}}{h_{i,1} (C_{i,j})_{1,2} (C_{i,j})_{2,2}}
\]

are defined at \( p \) and do not vanish at \( p \), so their logarithmic derivatives are regular at \( p \).

**Lemma 17.** Let \( p \) be a special point, and let \( i \) and \( j \) be two indices such that \( p \in V_i \cap V_j \), and \( \beta_{i,1} \) and \( \beta_{j,1} \) belong to the normal vertex cones of two different vertices of \( \Delta_p \). Then each non-constant entry of \( C_{i,j} \) has pole of degree exactly 1 at \( p \).

**Proof.** We know that each of the degrees \( \beta_{i,1} \) and \( \beta_{j,2} \) belongs to the normal subcone of exactly one vertex of \( \Delta_p \), and this vertex is the same one for \( \beta_{i,1} \) and for \( \beta_{j,2} \). \( \beta_{i,1} \) belong to the normal subcone of a different vertex of \( \Delta_p \), which is also unique. Since \( D_p(\cdot) \) is a convex function, it cannot be linear on the union of these two subcones, and \( D_p(\beta_{j,1}) < (C_{i,j})_{1,1} D_p(\beta_{i,1}) + (C_{i,j})_{1,2} D_p(\beta_{i,2}) \). Therefore,

\[
\text{ord}_p \left( \frac{h_{j,1}}{h_{i,1} (C_{i,j})_{1,1} h_{i,2} (C_{i,j})_{1,2}} \right) = -D_p(\beta_{j,1}) + (C_{i,j})_{1,1} D_p(\beta_{i,1}) + (C_{i,j})_{1,2} D_p(\beta_{i,2}) > 0,
\]

and, by a property of logarithmic derivative,

\[
\text{ord}_p \left( \frac{h_{i,2}}{h_{i,1} (C_{i,j})_{1,2} (C_{i,j})_{2,2}} \right) = -1.
\]

The argument for the second non-constant entry of \( C_{i,j} \) is similar.

**Lemma 18.** For the matrices \( C^\circ \) and \( C^{\circ}\) \( i,k \) and \( C_{i,j} \) defined above, one has \( C^\circ_{i,k} = C^{\circ}_{j,k} C^{-1}_{i,j} \) and \( C_{i,k} = C_{j,k} C_{i,j} \) for every triple of indices \( (i,j,k) \).

**Proof.** By the definition of \( C^\circ_{i,j} \) and of \( C^\circ_{j,k} \), one has \( \beta_{j,1} = (C^\circ_{i,j})_{1,1} \beta_{i,1} + (C^\circ_{i,j})_{1,2} \beta_{i,2} = (C^\circ_{i,j})_{2,1} \beta_{i,1} + (C^\circ_{i,j})_{2,2} \beta_{i,2} \). \( \beta_{i,1} = (C^\circ_{j,k})_{1,1} \beta_{i,1} + (C^\circ_{j,k})_{1,2} \beta_{i,2} = (C^\circ_{j,k})_{2,1} \beta_{i,1} + (C^\circ_{j,k})_{2,2} \beta_{i,2} \). Hence, \( \beta_{i,1} = (C^\circ_{j,k})_{1,1} ((C^\circ_{i,j})_{1,1} \beta_{i,1} + (C^\circ_{i,j})_{1,2} \beta_{i,2}) + (C^\circ_{j,k})_{1,2} ((C^\circ_{i,j})_{2,1} \beta_{i,1} + (C^\circ_{i,j})_{2,2} \beta_{i,2}) = (C^\circ_{j,k})_{1,1} (C^\circ_{i,j})_{1,1} + (C^\circ_{j,k})_{1,2} (C^\circ_{i,j})_{1,2} \beta_{i,1} + (C^\circ_{j,k})_{2,1} (C^\circ_{i,j})_{2,1} \beta_{i,2} + (C^\circ_{j,k})_{2,2} (C^\circ_{i,j})_{2,2} \beta_{i,2} \). Similarly, \( \beta_{i,2} = (C^\circ_{j,k})_{2,1} ((C^\circ_{i,j})_{1,1} \beta_{i,1} + (C^\circ_{i,j})_{1,2} \beta_{i,2}) + (C^\circ_{j,k})_{2,2} ((C^\circ_{i,j})_{2,1} \beta_{i,1} + (C^\circ_{i,j})_{2,2} \beta_{i,2}) \) \( (C^\circ_{j,k})_{2,1} (C^\circ_{i,j})_{1,2} + (C^\circ_{j,k})_{2,2} (C^\circ_{i,j})_{2,2} \beta_{i,1} + (C^\circ_{j,k})_{2,2} (C^\circ_{i,j})_{2,2} \beta_{i,2} = (C^\circ_{j,k})_{2,1} (C^\circ_{i,j})_{1,1} + (C^\circ_{j,k})_{2,2} (C^\circ_{i,j})_{2,2} \beta_{i,2} \). Since \( \beta_{i,1} \) and \( \beta_{i,2} \) form a basis of \( M \), \( C^\circ_{i,k} = C^{\circ}_{j,k} C^{-1}_{i,j} \).

Now, to prove that \( C_{i,k} = C_{j,k} C_{i,j} \), it is sufficient to check that

\[
\frac{h_{i,1} (C^\circ_{i,j})_{1,1} h_{i,2} (C^\circ_{i,j})_{1,2}}{h_{i,1} (C^\circ_{i,j})_{2,1} h_{i,2} (C^\circ_{i,j})_{2,2}} \cdot \frac{h_{i,1} (C^\circ_{j,k})_{1,1} h_{i,2} (C^\circ_{j,k})_{1,2}}{h_{i,1} (C^\circ_{j,k})_{2,1} h_{i,2} (C^\circ_{j,k})_{2,2}} \cdot \frac{h_{i,1} (C^\circ_{i,j})_{1,1} h_{i,2} (C^\circ_{i,j})_{1,2}}{h_{i,1} (C^\circ_{i,j})_{2,1} h_{i,2} (C^\circ_{i,j})_{2,2}} = \frac{h_{i,1} (C^\circ_{j,k})_{1,1} h_{i,2} (C^\circ_{j,k})_{1,2}}{h_{i,1} (C^\circ_{j,k})_{2,1} h_{i,2} (C^\circ_{j,k})_{2,2}} \cdot \frac{h_{i,1} (C^\circ_{i,j})_{1,1} h_{i,2} (C^\circ_{i,j})_{1,2}}{h_{i,1} (C^\circ_{i,j})_{2,1} h_{i,2} (C^\circ_{i,j})_{2,2}} \cdot \frac{h_{i,1} (C^\circ_{i,j})_{1,1} h_{i,2} (C^\circ_{i,j})_{1,2}}{h_{i,1} (C^\circ_{i,j})_{2,1} h_{i,2} (C^\circ_{i,j})_{2,2}}
\]

By a property of logarithmic derivatives, if \( f_1, f_2 \) are (rational) functions,

\[
\frac{d(f_1^\circ f_2^\circ)}{f_1^\circ f_2^\circ} = a_1 \frac{df_1}{f_1} + a_2 \frac{df_2}{f_2}.
\]
Hence, the left-hand side of the equality we are proving can be written as
\[
\left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) \cdot (C_{i,k}^{\infty})_{1,1} + \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) \cdot (C_{i,k}^{\infty})_{1,2} = \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) \cdot C_{i,k}^{\infty} \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) - C_{i,k}^{\infty} \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) .
\]

Similarly, the right-hand side can be written as
\[
C_{j,k}^{\infty} \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) - C_{i,j}^{\infty} \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) + C_{j,k}^{\infty} \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) = \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) \cdot C_{j,k}^{\infty} \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) - C_{j,k}^{\infty} \left( \frac{dh_{1,1}}{h_{1,1}} - \frac{dh_{2,1}}{h_{2,1}} \right) .
\]

By taking into account that $C_{i,k}^{\infty} = C_{j,k}^{\infty} C_{i,j}^{\infty}$, we obtain the desired equality.

Now we are going to compute $G_4^{\text{inv}}$, using Proposition 2. We can use $\{U_i\}$ as an affine covering of $U$. We have to consider a complex of sheaves on $U$ that we temporarily denote by $F_*$. For an open subset $U' \subseteq U$, $\Gamma(U', F_0)$ consists of sequences $(w_1, \ldots, w_q)$, where $w_i$ is a vector field on $U_i \cap U'$, $\Gamma(U', F_1)$ consists of sequences $(w_{i,j})_{1 \leq i < j \leq q}$, where $w_{i,j}$ is a vector field on $U_i \cap U_j \cap U'$, and $\Gamma(U', F_2)$ consists of sequences $(w_{i,j,k})_{1 \leq i < j < k \leq q}$, where $w_{i,j,k}$ is a vector field on $U_i \cap U_j \cap U_k \cap U'$. Denote the graded components of degree 0 of the pushforwards of these sheaves by $G_2^{\text{inv}}$, $G_2^{\text{inv}}$, $G_2^{\text{inv}}$, respectively. Using Corollary 7 we get the following description of these sheaves:

Consider the following sheaves $G_2$, $G_2'$, and $G_2''$. For an open subset $V \subseteq P^1$, $\Gamma(V, G_2)$ consists of sequences
\[
(g[i], g[i], v[i], \ldots, g[i], v[i], v[i], \ldots, g[q], g[q], v[q]),
\]
where $g[i] \in \Gamma(V_i \cap V, \mathcal{O}_{P^1})$, $v[i] \in \Gamma(V_i \cap V, \mathcal{O}_{P^1})$. Then $G_2$ is isomorphic to $G_2^{\text{inv}}$, and the isomorphism maps a sequence of q vector fields $(w[i], \ldots, w[q])$ to the sequence
\[
(g[i], g[i], v[i], \ldots, g[i], v[i], v[i], \ldots, g[q], g[q], v[q]),
\]
where $g[i], v[i], g[i], v[i]$ form the $U_i$-description of $w[i]$.

\[
\Gamma(V, G_2')
\]
consists of sequences $(g[i], g[i], v[i], g[i], v[i], v[i], \ldots, g[i], v[i], v[i], \ldots, g[q], g[q], v[q])$, where $g[i], v[i] \in \Gamma(V_i \cap V, \mathcal{O}_{P^1})$, $v[i] \in \Gamma(V_i \cap V, \mathcal{O}_{P^1})$. Similarly, $G_2'$ is isomorphic to $G_2^{\text{inv}}$, and the isomorphism maps a sequence $(w[i], v[i])_{1 \leq i \leq q}$ to the sequence of vector fields on open subsets of $U \cap q^{-1}(V)$ defined on $U_i \cap U_j \cap U_k \cap V^{-1}(U)$. In fact, at this point we can choose arbitrarily whether this is the $U_i$-description or the $U_j$-description of $w[i], v[i]$, and we choose that this is the $U_i$-description, and not the $U_j$-description.)

Finally, $\Gamma(V, G_2'')$ consists of sequences $(g[i], j, k], g[i], j, k], v[i, j, k])_{1 \leq i < j \leq q}$, where $g[i], j, k], v[i, j, k] \in \Gamma(V_i \cap V_j \cap V_k \cap V, \mathcal{O}_{P^1})$, $v[i, j, k] \in \Gamma(V_i \cap V_j \cap V_k \cap V, \mathcal{O}_{P^1})$. The isomorphism between $G_2^{\text{inv}}$ and $G_2''$ is constructed similarly, and here we again say (we choose) that $g[i], j, k], v[i, j, k]$ is the $U_i$-description of a vector field on $U_i \cap U_j \cap U_k \cap q^{-1}(V)$, not its $U_j$- or $U_k$-description.

Let us compute the kernel $\ker(G_2' \to G_2'')$. Denote it by $G_3$. A kernel of a sheaf map can be computed on each open subset independently, and the map here comes from the standard Čech map $F_1 \to F_2$ via the pushforward and the isomorphisms $G_2^{\text{inv}} \cong G_2'$ and $G_2^{\text{inv}} \cong G_2''$ defined above. Summarizing these definitions (and choices between $U_i$-descriptions made there), we get the following formula for the map $G_2' \to G_2''$, where we have to calculate a $U_i$-description from a $U_j$-description once:
\[
\begin{pmatrix}
g[i, j, k_1](p) 
g[i, j, k_2](p) 
g[i, j, k_3](p)
\end{pmatrix} + C_{j,i}(p) \begin{pmatrix}
g[j, k_1](p) 
g[j, k_2](p) 
g[j, k_3](p)
\end{pmatrix} = \begin{pmatrix}
g[i, k_1](p) 
g[i, k_2](p) 
g[i, k_3](p)
\end{pmatrix}.
\]

So we get the following description of $G_3$. The space of sections of $G_3$ over an open subset $V \subseteq P^1$ is the space of sequences of length $3q(q - 1)/2$ of the form $(g[i], j, k_1, g[i], j, k_2, v[i, j, k])_{1 \leq i < j \leq q}$, where $g[i, j, k] \in$
have constructed can be written as follows:
\[
\begin{bmatrix}
g[i,j_1](p)
g[i,j_2](p) \\
v[i,j](p)
\end{bmatrix} + C_{j,i}(p) \begin{bmatrix}
g[j,k_1](p)
g[j,k_2](p) \\
v[j,k](p)
\end{bmatrix} - \begin{bmatrix}
g[i,k_1](p)
g[i,k_2](p) \\
v[i,k](p)
\end{bmatrix} = 0.
\]

Finally, by Proposition 2, \( G^\text{inv}_4 \) is isomorphic to \( G_4 = \text{coker}(G_2 \rightarrow G_3) \), where the map \( G_2 \rightarrow G_3 \) can be written as follows:
\[
\begin{bmatrix}
g[i,j_1](p)
g[i,j_2](p) \\
v[i,j](p)
\end{bmatrix} = \begin{bmatrix}
g[i',j_1](p)
g[i',j_2](p) \\
v[i',j](p)
\end{bmatrix} - C_{j,i}(p) \begin{bmatrix}
g[j',k_1](p)
g[j',k_2](p) \\
v[j',k](p)
\end{bmatrix}.
\]

The sheaves \( G_{5,\chi}^\text{inv} \) and \( G_{6,\chi}^\text{inv} \) can be computed similarly. We start with the following Lemma.

**Lemma 19.** Let \( V'_l \subseteq V' \) be an open subset, \( L' \subseteq L \) be an open subset that can be equal \( \mathbb{C} \) or \( (\mathbb{C} \setminus 0) \), \( U'_l = V'_l \times (\mathbb{C} \setminus 0) \times L' \subseteq U_l \). A homogeneous function of degree \( \chi \in M \) \( \chi = a_1 \beta_{l,1} + a_2 \beta_{l,2} \) on \( U'_l \) is uniquely determined by its values at canonical points in all fibers \( \pi^{-1}(t_0) \) (for \( t_0 \in V'_l \)) with respect to \( U_l \).

1. If \( L' = \mathbb{C} \setminus 0 \), these values can form an arbitrary function depending algebraically on \( p \) in \( V'_l \).
2. If \( L' = \mathbb{C} \) and \( a_2 < 0 \), these values must vanish. This is only possible if \( \chi \notin \sigma^\vee \).

**Proof.** The proof is similar to the proof of Lemma 19. Denote the coordinates of a point \( x \in U_l \) provided by the isomorphism \( U_l \cong V_l \times (\mathbb{C} \setminus 0) \times L \) by \( t_0 \in V_l, t_1 \in \mathbb{C} \setminus 0, t_2 \in L \). Let \( f \) be a function of degree \( \chi \) on \( U'_l \), and suppose that \( f(t_0,1,1) = f_0(t_0) \), where \( f_0 : V_l \rightarrow \mathbb{C} \) is an algebraic function. Fix a pair \( (t_1,t_2) \in (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \) and let \( \tau \in T \) be the element of \( T \) such that \( \beta_{1,1}(\tau) = t_1, \beta_{1,2}(\tau) = t_2 \). Denote by \( \tau \) the automorphism of \( U'_l \) provided by \( \tau \) as well. By the definition of a homogeneous function of degree \( \chi \), \( f(t_0,t_1,t_2) = f(\tau \cdot (t_0,1,1)) = \chi(\tau)f(t_0,1,1) = \chi(\tau)f_0(t_0) \), so \( f_0 \) determines \( f \) uniquely on \( V'_l \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \), which is at least an open subset in \( U'_l \).

We still have to check that if we start with an arbitrary functions \( f_0 : V_l \rightarrow \mathbb{C} \), the resulting function on \( V'_l \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \) can be extended to the whole \( U'_l \) if and only if \( a_2 < 0 \) or \( L' = \mathbb{C} \setminus 0 \) (in the last case there is nothing to extend) and that the resulting function on \( U'_l \) is homogeneous of degree \( \chi \). The function we have constructed can be written as follows: \( f(t_0,t_1,t_2) = \chi(\tau)f_0(t_0) = \beta_{1,1}(\tau)^{a_1}\beta_{1,2}(\tau)^{a_2} = t_1^{a_1}t_2^{a_2}f_0(t_0) \).

Recall that \( t_1 \) (resp. \( t_2 \)) is a function on \( X \) of degree \( \beta_{1,1} \) (resp. \( \beta_{1,2} \)), so this function is clearly homogeneous of degree \( a_1\beta_{1,1} + a_2\beta_{1,2} = \chi \) on \( V'_l \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \). If the function can be extended to the whole \( U'_l \), it remains homogeneous there since homogeneity means an equality of two functions for each element of \( T \), and this equality holds if it holds on an open subset.

If \( L' = \mathbb{C} \setminus 0 \), there is nothing to extend. If \( L' = \mathbb{C} \), \( f \) can be extended to \( U'_l \) if and only if \( a_2 > 0 \).

Finally, \( L' = \mathbb{C} \), then \( \beta_{1,1} \in \partial \sigma^\vee \), and \( a_2 < 0 \) in this case, then \( \chi \notin \sigma^\vee \). \( \square \)

Given a homogeneous function \( f \) of degree \( \chi \in M \) defined on a set \( U_l \) as described in Lemma 19, we call the function \( f_0 : V_l \rightarrow \mathbb{C} \) such that \( f_0(p) = f(x) \), where \( x \) is the canonical point in \( \pi^{-1}(p) \) with respect to \( U_l \) the \( U_l \)-description of \( f \). And again we can make a remark similar to Remark 1.

**Remark 2.** Let \( V''_l \subseteq V'_l \) and \( L'' \subseteq L' \) be open subset, and \( L'' = \mathbb{C} \) or \( L'' = \mathbb{C} \setminus 0 \). These embeddings give rise to an embedding of \( U''_l = V''_l \times (\mathbb{C} \setminus 0) \times L'' \) into \( V'_l \times (\mathbb{C} \setminus 0) \times L' \). Let \( f' \) be the restriction of \( \chi \) to \( U''_l \). Then the \( U_l \)-description of \( f' \) is the restriction of the \( U_l \)-description of \( f \) to \( V''_l \).
Remark 3. For every three indices \(i, j, k\) one has \(\mu_{i,j,k} = \mu_{i,j,k}\).

By Lemma 12, \(U_i \cap U_j\) can be written as \(V' \times (\mathbb{C} \setminus \{0\}) \times L'\), where \(V' \subseteq V_i \cap V_j\) is an open subset, and \(L'\) equals \(\mathbb{C}\) or \((\mathbb{C} \setminus \{0\})\). This product is embedded into \(U_i\) via the isomorphism from Lemma 11.

Lemma 20. Let \(V''\) be an open subset of \(V'\), \(L''\) be an open subset of \(L'\), \(L'' = \mathbb{C}\) or \(L'' = \mathbb{C} \setminus \{0\}\), and let \(U'' = V'' \times (\mathbb{C} \setminus \{0\}) \times L''\) be embedded into \(U_i \cap U_j\) via the map from Lemma 12.

Let \(f\) be a homogeneous function on \(V''\) of degree \(\chi\), and let \(g_i\) (resp. \(g_j\)) be the \(U_i\)-description (resp. \(U_j\)-description) of \(f\). Then for every \(p \in V''\):

\[
g_j(p) = \mu_{i,j,k}g_i(p).
\]

Proof. As in the proof of Lemma 13, it is sufficient to prove the equality for all ordinary points \(p \in V''\). So let \(p \in V''\) be an ordinary point and let \(x\) (resp. \(x'\)) be the canonical point in \(\pi^{-1}(p)\) with respect to \(U_i\) (resp. to \(U_j\)). It follows from Corollary 3 that \(\tilde{h}_{i,1}(x') \neq 0, \tilde{h}_{i,2}(x') \neq 0\), hence \(x' \in U''\).

Let \(\tau\) be the element of \(T\) such that \(\beta_{i,1}(\tau) = \tilde{h}_{i,1}(x'), \beta_{i,2}(\tau) = \tilde{h}_{i,2}(x')\). As usual, denote the corresponding automorphism of \(U''\) by \(\tau\) as well. Since \(\tilde{h}_{i,1}\) (resp. \(\tilde{h}_{i,2}\)) is a homogeneous function of degree \(\beta_{i,1}\) (resp. \(\beta_{i,2}\)), \(\tilde{h}_{i,1}(\tau x) = \tilde{h}_{i,1}(x'), \tilde{h}_{i,2}(\tau x) = \tilde{h}_{i,2}(x')\), so \(\tau x = x'\).

Choose \(a_1, a_2, a_1', a_2' \in \mathbb{Z}\) so that \(x = a_1 \beta_{i,1} + a_2 \beta_{i,2} = a_1' \beta_{i,1} + a_2' \beta_{i,2}\). Since \(f\) is a homogeneous function of degree \(\chi\), \(f(x') = f(\tau x) = \chi(\tau)f(x) = \beta_{i,1}(\tau)^{a_1} \beta_{i,2}(\tau)^{a_2} f(x) = \tilde{h}_{i,1}(x')^{a_1} \tilde{h}_{i,2}(x')^{a_2} f(x)\).

Recall that \(\tilde{h}_{j,1}(x') = \tilde{h}_{j,2}(x') = 1\). We have

\[
f(x') = \tilde{h}_{i,1}(x')^{a_1} \tilde{h}_{i,2}(x')^{a_2} f(x).
\]

Since the numerator and the denominator of this fraction are homogeneous functions of degree \(a_1 \beta_{i,1} + a_2 \beta_{i,2} = a_1' \beta_{i,1} + a_2' \beta_{i,2} = \chi\), by Proposition 1

\[
f(x') = \tilde{h}_{j,1}(x')^{a_1} \tilde{h}_{j,2}(x')^{a_2} f(x) = \mu_{i,j,k}(p) g_i(p).
\]

Lemma 20 enables us to formulate a description of \(\mathcal{G}_{\text{inv}}^{\text{ov}}\) similar to the description of \(\mathcal{G}_{\text{inv}}\) above. Namely, define a sheaf \(\mathcal{G}_{6,\chi}\) as follows: Let \(V \subseteq \mathbb{P}^1\) be an open subset. The space of sections \(\Gamma(V, \mathcal{G}_{6,\chi})\) is the space of sequences \((g_1, \ldots, g_q)\) of functions on \(V\) satisfying the following conditions:

1. \(g_i = \mu_{i,i',\chi} g_i\) for all indices \(i, i'\).

2. If \(\beta_{i,1} \in \partial\sigma^\vee\) and \(\chi\) can be written as \(\chi = a_1 \beta_{i,1} + a_2 \beta_{i,2}\) with \(a_2 < 0\), then \(g_i = 0\).

Lemma 21. \(\mathcal{G}_{6,\chi}^{\text{inv}}\) is isomorphic to \(\mathcal{G}_{6,\chi}\). If \(f\) is a function on \(\pi^{-1}(V) \cap U\) of degree \(\chi\), then the isomorphism maps it to \((g_1, \ldots, g_q)\), where \(g_i\) is the \(U_i\)-description of \(f\).

And again we can compute \(\mathcal{G}_{6,\chi}^{\text{inv}}\) using Proposition 1 with \(\{U_i\}\) being the required affine covering of \(U\). Again denote temporarily the complex of sheaves on \(U\) we have to consider in Proposition 2 by \(\mathcal{F}_{\bullet}\). Let \(U'\) be an open subset of \(U\). Then \(\Gamma(U', \mathcal{F}_0)\) consists of sequences \((f_1, \ldots, f_q)\), where \(f_i \in \Gamma(U \cap U' \cap U', \mathcal{O}_X)\), \(\Gamma(U', \mathcal{F}_1)\) consists of sequences \((f_{i,j})_{1 \leq i < j \leq q}\), where \(f_{i,j} \in \Gamma(U \cap U' \cap U' \cap U' \cap U', \mathcal{O}_X)\), and \(\Gamma(U', \mathcal{F}_2)\) consists of sequences \((f_{i,j,k})_{1 \leq i < j < k \leq q}\), where \(f_{i,j,k} \in \Gamma(U \cap U' \cap U' \cap U' \cap U' \cap U', \mathcal{O}_X)\). Denote the graded components of degree \(\chi\) of the pushforwards of these sheaves by \(\mathcal{G}_{6,\chi}^{\text{inv}}, \mathcal{G}_{6,\chi}^{\text{inv}}, \mathcal{G}_{6,\chi}^{\text{inv}}\), respectively. We get the following descriptions of these sheaves from Lemma 19.

Define sheaves \(\mathcal{G}_{6,\chi}, \mathcal{G}_{6,\chi}^{\text{ov}}, \mathcal{G}_{6,\chi}^{\text{ov}}\) as follows. Fix an open subset \(V' \subseteq \mathbb{P}^1\). Let \(\Gamma(V, \mathcal{G}_{6,\chi})\) be the space of sequences of the form \((g[1], \ldots, g[q])\), where \(g[i] \in \Gamma(V' \cap V, \mathcal{O}_V)\) and \(g[0] = 0\) if \(\beta_{i,1} \in \partial\sigma^\vee\) and \(\chi\) can be written as \(\chi = a_1 \beta_{i,1} + a_2 \beta_{i,2}\) with \(a_2 < 0\). Then \(\mathcal{G}_{6,\chi} \cong \mathcal{G}_{6,\chi}^{\text{ov}}\), and the isomorphism maps a sequence \((f[1], \ldots, f[q])\) of functions of degree \(\chi\) defined on open subsets of \(\pi^{-1}(V) \cap U\) to \((g[1], \ldots, g[q])\), where \(g[i]\) is the \(U_i\)-description of \(f[i]\).
Let $\Gamma(V, G_{6,\chi}^\nu)$ be the space of sequences $(g[i,j])_{1 \leq i < j \leq q}$, where $g[i,j] \in \Gamma(O_{\mathcal{P}_1}, V \cap V_i \cap V_j)$. These functions should be zero in some cases if $\beta_{i,j} = \beta_{j,i} \in \partial \sigma^\nu$ (see Lemma [12]). To define these cases, note first that if both pairs $(\beta_{i,j}, \beta_{j,i})$ and $(\beta_{j,i}, \beta_{i,j})$ form bases of $M$ and $\beta_{i,j} = \beta_{j,i}$, then, if we write $\chi = a_1 \beta_{i,j} + a_2 \beta_{j,i} = a_1 \beta_{j,i} + a_2 \beta_{i,j}$, we will get $\alpha = a_2$. So, the condition is: if $\chi = a_1 \beta_{i,j} + a_2 \beta_{j,i}$, where $a_2 < 0$, then $g[i,j] = 0$. Again, $G_{6,\chi}^\nu \cong G_{6,\chi}^\nu$, and the isomorphism maps a sequence $(f[i,j])_{1 \leq i < j \leq q}$ of functions of degree $\chi$ defined on open subsets of $\pi^{-1}(V) \cup U$ to the sequence $(g[i,j])_{1 \leq i < j \leq q}$ of functions on $V$ such that $g[i,j]$ is the $U_i$-description of $f_{i,j}$. (Again, we could choose the $U_j$-description here, as well, but we choose the $U_i$-description.)

Finally, let $\Gamma(V, G_{7,\chi}^\nu)$ be the space of sequences $(g[i,j,k])_{1 \leq i < j < k \leq q}$, where $g[i,j,k] \in \Gamma(V \cap V_i \cap V_j \cap V_k, O_{\mathcal{P}_1})$ and, as in the previous case, $g[i,j,k] = 0$ if $\beta_{i,j} = \beta_{j,k} = \beta_{i,k} \in \partial \sigma^\nu$ and $\chi$ can be written as $\chi = a_1 \beta_{i,j} + a_2 \beta_{j,k}$ or $a_2 < 0$. Then $G_{7,\chi}^\nu \cong G_{6,\chi}^\nu$, the isomorphism is constructed similarly, and again we say that $g[i,j,k]$ is the $U_i$-description of a function defined on $U_i \cap U_j \cap U_k \cap \pi^{-1}(V)$, not its $U_j$- or $U_k$-description.

Denote $G_{7,\chi}^\nu = \ker(G_{6,\chi}^\nu \to G_{6,\chi}^\nu)$, where the map $G_{6,\chi}^\nu \to G_{6,\chi}^\nu$ comes from the standard Cech map $F_2 \to F_1$ via the pushforward, then the restriction to the degree $\chi$, and then the isomorphisms $G_{6,\chi}^\nu \cong G_{6,\chi}^\nu$ and $G_{6,\chi}^\nu \cong G_{6,\chi}^\nu$ defined above. To compute a kernel of a map between sheaves, it is sufficient to compute the kernels of the corresponding maps between modules on each open subset. So let $V \subseteq \mathbb{P}^1$ be an open subset. Taking into account the choice of $U_i$-description in the definition of the isomorphisms $G_{6,\chi}^\nu \cong G_{6,\chi}^\nu$ and $G_{6,\chi}^\nu \cong G_{6,\chi}^\nu$, we see that the corresponding map $\Gamma(V, G_{6,\chi}^\nu) \to \Gamma(V, G_{7,\chi}^\nu)$ can be written as follows:

$$g[i,j,k] = g[i,j] + \mu_{i,j,\chi}g[j,k] - g[i,k],$$

and $\Gamma(V, G_{7,\chi}^\nu)$ is the space of sequences of the form $(g[i,j,k])_{1 \leq i < j < k \leq q}$, where $g[i,j,k] \in \Gamma(V \cap V_i \cap V_j \cap V_k, O_{\mathcal{P}_1})$ satisfies the following conditions:

1. $g[i,j] + \mu_{i,j,\chi}g[j,k] - g[i,k] = 0$ for all indices $i \leq j < k$.
2. If $\beta_{i,j} = \beta_{j,k} \in \partial \sigma^\nu$ and $\chi = a_1 \beta_{i,j} + a_2 \beta_{j,k}$, where $a_2 < 0$, then $g[i,j,k] = 0$.

Now, by Proposition [2], $G_{7,\chi}^\nu$ is isomorphic to $G_{8,\chi}^\nu = \ker(G_{6,\chi}^\nu \to G_{7,\chi}^\nu)$, where the map $G_{6,\chi}^\nu \to G_{7,\chi}^\nu$ can be written as follows: $g[i,j,k](p) = g[i,j](p) + \mu_{i,j,\chi}g[j,k](p)$. After we have defined the sheafs $G_{5,\chi}$ and $G_{8,\chi}$ isomorphic to $G_{7,\chi}^\nu$ and $G_{8,\chi}^\nu$ (respectively) for each degree $\chi$, we define

$$G_5 = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{\dim(O(D(\lambda)))} G_{5,\chi}, \quad G_8 = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{\dim(O(D(\lambda)))} G_{8,\chi}.$$

Proposition [3] involves maps $H^1((\pi(V), \psi)|_{G_{7,\chi}^\nu}): H^1(\mathbb{P}^1, G_{7,\chi}^\nu) \to H^1(\mathbb{P}^1, G_{7,\chi}^\nu)$ and $H^0((R^1(\pi(V), \psi)|_{G_{7,\chi}^\nu}): H^0(\mathbb{P}^1, G_{7,\chi}^\nu) \to H^0(\mathbb{P}^1, G_{7,\chi}^\nu)$. The isomorphisms $G_{7,\chi}^\nu \cong G_{5,\chi}$, $G_{7,\chi}^\nu \cong G_{4,\chi}$, and $G_{7,\chi}^\nu \cong G_{8,\chi}$ constructed above enable us to consider maps $H^1(\mathbb{P}^1, G_5) \to H^1(\mathbb{P}^1, G_4)$ and $H^0(\mathbb{P}^1, G_5) \to H^0(\mathbb{P}^1, G_4)$ instead. Denote them by $H^1((\pi(V), \psi)|_{G_{7,\chi}^\nu})$ and $H^0((R^1(\pi(V), \psi)|_{G_{7,\chi}^\nu})$, respectively. The following lemma establishes relations between $U_i$-descriptions of sections of $\Theta_X$ and their images under $\psi$, so it will help us to understand these maps.

**Lemma 22.** Let $V'$ be an open subset of $V$, $L'$ be an open subset of $L$, $L' = \mathbb{C}$ or $L' = \mathbb{C} \setminus 0$, and let $U' = V' \times (\mathbb{C} \setminus 0 \times \mathbb{C})$ embedded into $U_i$ via the map from Lemma [7].

Let $(g_{1,2}, v_i)$ be the $U_i$-description of a vector field $w$ defined on $V'$, $\chi = a_1 \beta_{1,2} + a_2 \beta_{2,1} \in \sigma^\nu \cap \mathbb{M}$ be a degree, $f \in O(D(\lambda))$. Then the $U_i$-description of $(d_f)w$ is

$$\overline{\int_{h_{i,1}}^{h_{i,2}} (a_1 g_{1,2} + a_2 g_{2,1}) + \overline{\int_{h_{i,1}}^{h_{i,2}} v_i}}.$$

**Proof.** The proof is similar to the proof of Lemma [15]. It is sufficient to prove the equality for an arbitrary open subset of $V'$, so let $p \in V'$ be an arbitrary point, and let $x$ be the canonical point in $\pi^{-1}(p)$ with respect to $U_i$. Denote by $h$ the $U_j$-description of the function $(df)w$. Then $h(p) = (d_s f)w(x)$. We have

$$d_s f = d_s \left( \frac{\tilde{f}}{h_{i,1}} \right) \left( \frac{\tilde{f}}{h_{i,2}} \right) = a_1 (d_s h_{i,1}) h_{i,2} (x) a_2 (x) h_{i,1} (x).$$

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equality of rational maps from \( \tilde{\text{ordinary point}} \). By the definition of a sufficient system, there exists an index 
\[ a \] 
does not depend on 
\[ f \] 
Therefore, 
\[ 1 \] 
First, let 
\[ g \] 
the conditions 
\[ g \] 
Finally, we get 
\[ 2 \] 
\[ 3 \] 
Summarizing, we have found an explicit description for the sheaves \( G_1, G_2, G_3, G_{\chi}, G_{\chi,\psi}, G_{7,\chi} \) and for the map \( \psi \), and all sheaves involved in Proposition \( \psi \) can be obtained from these sheaves by taking a cokernel of a map we have explicitly described and forming a direct sum. To continue, we prove first that \( G_{5,\chi} \) admits an easier description when \( \chi \in \sigma^\vee \cap M \).

**Lemma 23.** If \( \chi \in \sigma^\vee \cap M, \) then \( G_{5,\chi} \cong \mathcal{O}(\mathcal{D}(\chi)) \). Let \( \chi = a_{i_1} \beta_{i_1} + a_{i_2} \beta_{i_2} \). Then the isomorphism is given by 
\[ f \leftrightarrow \begin{pmatrix} f & \cdots & f & \cdots & f \\ h_{1_1}^{-a_{i_1}} & h_{1_2}^{-a_{i_2}} & \cdots & h_{2_1}^{-a_{i_1}} & h_{2_2}^{-a_{i_2}} \end{pmatrix}, \]
where \( f \in \Gamma(V, \mathcal{O}(\mathcal{D}(\chi))) \), \( V \subseteq \mathbb{P}^1 \) is an open subset.

**Proof.** First, let \( f \in \Gamma(V, \mathcal{O}(\mathcal{D}(\chi))) \) be a function. Then it is clear that \( g_i = f / (h_{1_1}^{-a_{i_1}} h_{1_2}^{-a_{i_2}}) \) satisfy the conditions \( g_i = \mu_{i,j,\chi} g_j \) from Lemma 22 by construction. The condition 2 from the definition of \( G_{5,\chi} \) is void since \( \chi \in \sigma^\vee \). We have to check that \( g_i \) are well-defined at points \( p \in V \cap V_i \). If \( p \in V \cap V_i \), then by Lemma 13 \( \mathcal{D}_p(\chi) \leq a_{i_1} \mathcal{D}_p(\beta_{i_1}) + a_{i_2} \mathcal{D}_p(\beta_{i_2}) \). By the definition of \( V_i \), \( \text{ord}_p(h_{1_1}) = -\mathcal{D}_p(\beta_{i_1}) \), \( \text{ord}_p(h_{1_2}) = -\mathcal{D}_p(\beta_{i_2}) \). Since \( f \in \Gamma(V, \mathcal{O}(\mathcal{D}(\chi))) \), \( \text{ord}_p(f) \geq -\mathcal{D}_p(\chi) \), so \( \text{ord}_p(f) \geq \text{ord}_p(h_{1_1}^{-a_{i_1}} h_{1_2}^{-a_{i_2}}) \), and \( g_i \) is well-defined on \( V \cap V_i \). Therefore, \( (g_1, \ldots, g_q) \) defines an element of \( G_{5,\chi} \).

Now, let \( (g_1, \ldots, g_q) \in \Gamma(V, G_{5,\chi}) \). The condition \( g_j = \mu_{i,j,\chi} g_i \) guarantees that \( f = g_i h_{1_1}^{-a_{i_1}} h_{1_2}^{-a_{i_2}} \) does not depend on \( i \) as a rational function. We have to check that \( f \in \Gamma(V, \mathcal{O}(\mathcal{D}(\chi))) \). Let \( p \in V \) be an ordinary point. By the definition of a sufficient system, there exists an index \( i \) such that \( p \in V_i \). Then \( g_i \) is well-defined as \( p \), and \( h_{1_1}^{-a_{i_1}} h_{1_2}^{-a_{i_2}} \) are defined at \( p \) since \( p \) is an ordinary point.

Now suppose that \( p \in V \) is a special point. Let \( V_{p,j} \) be a vertex such that \( \chi \in \mathcal{N}(V_{p,j}, \mathcal{D}_p) \). By the definition of a sufficient system, there exists an index \( i \) such that \( p \in V_i \) and \( \beta_{i_1}, \beta_{i_2} \in \mathcal{N}(V_{p,j}, \mathcal{D}_p) \). The function \( \mathcal{D}_p(\cdot) \) is linear on \( \mathcal{N}(V_{p,j}, \mathcal{D}_p) \) so \( \mathcal{D}_p(\chi) = a_{i_1} \mathcal{D}_p(\beta_{i_1}) + a_{i_2} \mathcal{D}_p(\beta_{i_2}) \). Then \( \text{ord}_p(f) = \text{ord}_p(g_j) + a_{i_1} \text{ord}_p(h_{1_1}) + a_{i_2} \text{ord}_p(h_{1_2}) \geq a_{i_1} \text{ord}_p(h_{1_1}) + a_{i_2} \text{ord}_p(h_{1_2}) = -a_{i_1} \mathcal{D}_p(\beta_{i_1}) - a_{i_2} \mathcal{D}_p(\beta_{i_2}) = -\mathcal{D}_p(\chi) \). Therefore, \( f \in \Gamma(V, \mathcal{O}(\mathcal{D}(\chi))) \).
Recall that
\[ \mathcal{G}_5 = \bigoplus_{i=1}^m \bigoplus_{j=1}^{\dim \mathcal{O}(\mathcal{D}(\lambda_i))} \mathcal{G}_{5,\lambda_i}, \]
where \( \lambda_i \) form the Hilbert basis of \( \sigma^\vee \cap M \), in particular, \( \lambda_i \in \sigma^\vee \cap M \). Therefore,
\[ \mathcal{G}_5 = \bigoplus_{i=1}^m \bigoplus_{j=1}^{\dim \mathcal{O}(\mathcal{D}(\lambda_i))} \mathcal{O}(\mathcal{D}(\lambda_i)). \]
In particular, \( \mathcal{D}(\lambda_i) \) are divisors of non-negative degree on \( \mathbb{P}^1 \), and \( H^1(\mathbb{P}^1, \mathcal{G}_5) = 0 \). Therefore,
\[ \ker(H^1(\mathbb{P}^1, \mathcal{G}_5)^{H^1((\mathcal{D}(\lambda_i))^{\vee})}) = 0, \ker(H^1(\mathbb{P}^1, \mathcal{G}_1)^{H^1((\mathcal{D}(\lambda_i))^{\vee})}) = H^1(\mathbb{P}^1, \mathcal{G}_1), \]
and the exact sequence from Proposition \( \blacklozenge \) can be written in the following form:

**Theorem 3.** The following sequence is exact:
\[ 0 \to H^1(\mathbb{P}^1, \mathcal{G}_1) \to T^1(X)_0 \to H^0(\mathbb{P}^1, \mathcal{G}_5)^{H^0((\mathcal{D}(\lambda_i))^{\vee})} \to H^0(\mathbb{P}^1, \mathcal{G}_5). \]

\[ \square \]

### 4 Combinatorial formula for the dimension of the graded component of \( T^1 \) of degree 0

Without loss of generality, in this section we will assume that there are at least two special points (we always can add trivial special points). We are going to construct a sufficient system of sets \( U_i \) more explicitly. Recall that we have chosen a coordinate \( t \) on \( \mathbb{P}^1 \) such that \( t = 0 \) and \( t = \infty \) are ordinary points.

**Lemma 24.** Let \( p \in \mathbb{P}^1 \) be a special point and let \( \chi \in \sigma^\vee \cap M \) be a degree. There exists a rational function \( f \in \mathcal{O}(\mathcal{D}(\chi)) \) such that \( \text{ord}_p(f) = -\mathcal{D}_p(\chi) \), and \( f \) does not have zeros or poles at ordinary points.

**Proof.** Recall that we have denoted all special points by \( p_1, \ldots, p_r \). Let \( p = p_i \). Denote \( a_i = \mathcal{D}_p(\chi) \). Since \( \deg \mathcal{D}(\chi) \geq 0 \), there exist \( a_1, \ldots, a_i-1, a_{i+1}, \ldots, a_r \in \mathbb{Z} \) such that \( a_1 + \ldots + a_r = 0 \) and \( a_j \leq \mathcal{D}_p(\chi) \) for \( 1 \leq j \leq r \). Denote the coordinate of \( p_i \) by \( t_j \) for \( 1 \leq j \leq r \). Consider the following function: \( f(t) = (t - t_1)^{-a_1} \cdots (t - t_i)^{-a_i} \). Since the sum of the exponents is zero, \( f \) is defined at \( t = \infty \) and \( f(\infty) = 1 \). Clearly, \( f \) has no zeros or poles at other ordinary points. At \( p \), we have \( \text{ord}_p(f) = -a_i = -\mathcal{D}_p(\chi) \), and at \( p_j \) (\( j \neq i \)), we have \( \text{ord}_{p_j}(f) = -a_j \geq -\mathcal{D}_p(\chi) \).

We are going to use a system of sets \( U_i \) indexed as follows. We have several (in fact, up to two) sets \( U_i \) for every pair \( (p, j) \), where \( p \in \mathbb{P}^1 \) is a special point, and \( j \) corresponds to a vertex \( V_{p,j} \) of \( \Delta_p \) (\( 1 \leq j \leq v_p \), we write \( (p, j) \) instead of \( (p, V_{p,j}) \) to simplify notation). Each of these sets \( U_i \) chosen for \( (p, j) \) corresponds to a face of \( \mathcal{N}(V_{p,j}, \Delta_p) \) (which can be \( \mathcal{N}(E_{p,j-1}, \Delta_p) \), \( \mathcal{N}(E_{p,j-1}, \Delta_p) \), or the interior of \( \mathcal{N}(V_{p,j}, \Delta_p) \)). Additionally, we will use one more set \( U_i \) corresponding to no special point.

More precisely, for every special point \( p \), for every vertex \( V_{p,j} \) of \( \Delta_p \) and for each of the two rays \( \mathcal{N}(E_{p,j-1}, \Delta_p) \) and \( \mathcal{N}(E_{p,j-1}, \Delta_p) \) forming \( \partial \mathcal{N}(V_{p,j}, \Delta_p) \) we choose a basis of \( M \) as follows. First, let \( \chi \in M \) be the lattice basis of the chosen ray. If \( \chi \notin \partial \sigma^\vee \), we do not choose a basis for this pair \( (p, j) \) and for this ray. If \( \deg \mathcal{D}(\chi) = 0 \), we do not choose a basis for this pair \( (p, j) \) and for this ray. Otherwise, we choose a basis \( \beta_{i,1}, \beta_{i,2} \) of \( M \) where \( \beta_{i,1} = \chi \) and \( \beta_{i,2} \) is a lattice point in the interior of \( \mathcal{N}(V_{p,j}, \Delta_p) \).

We choose a basis of \( M \) corresponding to a pair \( (p, j) \) and to the interior of \( \mathcal{N}(V_{p,j}, \Delta_p) \) only if at the previous step we finally did not choose any basis corresponding to the pair \( (p, j) \) and to one of the two rays \( \mathcal{N}(E_{p,j-1}, \Delta_p) \) and \( \mathcal{N}(E_{p,j-1}, \Delta_p) \) (for example, this can happen if \( \partial \sigma^\vee \cap \partial \mathcal{N}(V_{p,j}, \Delta_p) = 0 \)). In this case, we choose a basis \( \beta_{i,1}, \beta_{i,2} \) of \( M \) such that \( \beta_{i,1} \) and \( \beta_{i,2} \) are lattice points in the interior of \( \mathcal{N}(V_{p,j}, \Delta_p) \).
Observe that we chose exactly one or two bases for each pair \((p, j)\). We chose two bases if and only if \(p\) is a removable special point and \(\deg D(\alpha_0) > 0\) and \(\deg D(\alpha_1) > 0\).

Now for every chosen basis, we choose functions \(h_{i,1} \in \mathcal{O}(D(\beta_{i,1}))\) and \(h_{i,2} \in \mathcal{O}(D(\beta_{i,2}))\) satisfying the conditions of Lemma 24 for the corresponding special point \(p\) and the degree \(\beta_{i,1}\) or \(\beta_{i,2}\), respectively. Then we may set \(V_i\) to consist of all ordinary points and \(p\).

To define one more set \(U_i\), choose an arbitrary basis \(\beta_{i,1}, \beta_{i,2}\) of \(M\) such that \(\beta_{i,1}, \beta_{i,2}\) are in the interior of \(\sigma^V\), and choose functions \(h_{i,1} \in \mathcal{O}(D(\beta_{i,1}))\) and \(h_{i,2} \in \mathcal{O}(D(\beta_{i,2}))\) that do not have zeros or poles at ordinary points (such functions exist by Lemma 24). In this case, let \(V_i\) be the set of all ordinary points. We refer to the corresponding index \(i\) as to the last one.

All these data define a sufficient system of sets \(U_i\), which we are going to use to compute \(H^1(P^1, \mathcal{G}_1), H^0(P^1, \mathcal{G}_1),\) and \(H^0(P^1, \mathcal{G}_q)\). Note that if we remove the last set (the one that does not correspond to any special point), we will still get a sufficient system. We are going to use the old notation \(q\) for the number of sets \(U_i\) in the sufficient system we have just constructed. The subset of \(X\) defined above that does not correspond to any special point is denoted by \(U_q\).

We start with \(H^1(P^1, \mathcal{G}_1)\). To compute this space, we need an affine covering of \(P^1\). So, for each special point \(p \in P^1\), we denote by \(W_p\) the set consisting of all ordinary points of \(P^1\) and \(p\). Denote also the set of all ordinary points by \(W\). This really is an affine covering since we have at least two special points. We use Cech cohomology and Corollary 1 (note that if \(p \neq p'\) are special points, then \(W_p \cap W_{p'} = W\)). By Corollary 1:

\[
H^1(P^1, \mathcal{G}_1) = \left( \bigoplus_{\text{p special point}} \left( \Gamma(W, \mathcal{G}_1) / \Gamma(W_p, \mathcal{G}_1) \right) \right) / \left( \Gamma(W, \mathcal{G}_1) \right).
\]

For an essential special point \(p\), denote by \(G_{1,p,q}\) the space of triples \((g_{q,1}, g_{q,2}, v)\), where \(g_{q,1}, g_{q,2} \in \Gamma(W_p, \mathcal{G}_p), v \in \Gamma(W_p, \mathcal{G}_p),\) and \(v(p) = 0\). The last index \(q\) indicates that these triples will be considered as \(U_q\)-descriptions of vector fields on \(\pi^{-1}(W_p) \cap U\).

**Lemma 25.** Let \(p \in P^1\) be an essential special point. Then \(\Gamma(W_p, \mathcal{G}_1)\) can be identified with \(G_{1,p,q}\). The isomorphism here maps \((g_{q,1}, g_{q,2}, v) \in G_{1,p,q}\) to \((g_{1,1}, g_{1,2}, \ldots, g_{q,1}, g_{q,2}, v)\), where

\[
\begin{pmatrix} g_{1,1} \\ g_{1,2} \\ v \end{pmatrix} = C_{q,i} \begin{pmatrix} g_{q,1} \\ g_{q,2} \\ v \end{pmatrix}.
\]

**Proof.** First, we have to check that the \((2q + 1)\)-tuple obtained this way from an element of \(G_{1,p,q}\) really defines an element of \(\Gamma(W_p, \mathcal{G}_1)\). The equalities

\[
\begin{pmatrix} g_{1,1} \\ g_{1,2} \\ v \end{pmatrix} = C_{j,i} \begin{pmatrix} g_{j,1} \\ g_{j,2} \\ v \end{pmatrix}
\]

for arbitrary indices \(i, j\) follow from Lemma 13. All functions \(g_{i,1}\) and \(g_{i,2}\) are regular at ordinary points by Lemma 12. Let \(i\) be an index such that \(U_i\) corresponds to the special point \(p\) in the above construction. We have

\[
g_{i,1} = (C_{q,i}^0)_{1,1} g_{q,1} + (C_{q,i}^0)_{1,2} g_{q,2} + \frac{h_{q,1}}{h_{i,1}} (C_{q,i}^0)_{1,1} \frac{h_{q,2}}{h_{i,2}} (C_{q,i}^0)_{1,2} d \left( \frac{h_{i,1}}{h_{q,1}} (C_{q,i}^0)_{1,1} \frac{h_{i,2}}{h_{q,2}} (C_{q,i}^0)_{1,2} \right) v.
\]

The covector field in the last summand is a logarithmic derivative of a rational function on \(P^1\), so it cannot have a pole of order more than 1. Since \(v(p) = 0\), \(g_{i,1}\) is defined at \(p\). The argument for \(g_{i,2}\) is similar.

Clearly, this map from the space of triples to \(\Gamma(W_p, \mathcal{G}_1)\) is injective. To prove surjectivity, we have to check that if \((g_{i,1}, g_{i,2}, \ldots, g_{q,1}, g_{q,2}, v) \in \Gamma(W_p, \mathcal{G}_1)\), then \(v(p) = 0\) and \(g_{q,1}\) and \(g_{q,2}\) have no poles at
Lemma 26. Let $U_i$ and $U_j$ be two open subsets corresponding to the special point $p$ and two normal subcones of two different vertices of $\Delta_p$. If $v(p) \neq 0$, then by Lemma 17
\[
\text{ord}_p \left( \frac{h_{j,1}(C_{i,j}^1)_{1:2}}{h_{j,1}} \right) \left( \frac{h_{j,1}}{h_{i,1}(C_{i,j}^1)_{1:2}} \right)^d \left( \frac{h_{j,1}}{h_{i,1}(C_{i,j}^1)_{1:2}} \right) v = -1,
\]
and $g_{j,1}$, $g_{i,1}$ and $g_{i,2}$ cannot be defined at $p$ simultaneously. Therefore, $v(p) = 0$. Finally,
\[
g_{q,1} = (C_{i,q}^1)_{1:2} g_{i,1} + (C_{i,q}^1)_{1:2} 2g_{i,2} + \frac{h_{j,1}(C_{i,j}^1)_{1:2}}{h_{q,1}} d \left( \frac{h_{j,1}}{h_{i,1}(C_{i,j}^1)_{1:2}} \right) v.
\]
Again, covector field in the last summand here is a logarithmic derivative of a rational function on $P^1$, so it cannot have a pole of order more than 1. Since $v(p) = 0$, $g_{q,1}$ has no pole at $p$. Similarly, $g_{q,2}$ has no pole at $p$.

Now for an essential special point $p$, denote by $\nabla_{1,0,p}$ the space of triples of Laurent polynomials of the form $(a_{1,-1}(t-t_0)^{-1} + \ldots + a_{1,-n_1}(t-t_0)^{-n_1}, a_{2,-1}(t-t_0)^{-1} + \ldots + a_{2,-n_2}(t-t_0)^{-n_2}, b_0 + b_{-1}(t-t_0)^{-1} + \ldots + b_{-n_3}(t-t_0)^{-n_3})/\partial/\partial t$, where $t_0$ is the coordinate of $p$.

Lemma 26. If $p \in P^1$ is an essential special point, then $\nabla_{1,0,p}$ is isomorphic to $\Gamma(W, G_1)/\Gamma(W, G_1)$. The isomorphism here is the composition of the map $(g_{q,1}, g_{q,2}, v) \mapsto (g_{1,1}, g_{1,2}, \ldots, g_{q,1}, g_{q,2}, v) \in \Gamma(W, G_1)$, where
\[
\begin{pmatrix}
g_{1,1} \\
g_{1,2} \\
v
\end{pmatrix} = C_{i,q} \begin{pmatrix}
g_{q,1} \\
g_{q,2}
\end{pmatrix},
\]
and the canonical projection $\Gamma(W, G_1) \to \Gamma(W, G_1)/\Gamma(W, G_1)$.

If $(g_{1,1}^1, g_{1,2}^1, \ldots, g_{q,1}^1, g_{q,2}^1, v^1) \in \Gamma(W, G_1)$ is a section that belongs to the same coset in $\Gamma(W, G_1)/\Gamma(W, G_1)$ as the image of $(g_{q,1}, g_{q,2}, v) \in \nabla_{1,0,p}$ under the isomorphism above, then $g_{q,1}^1 - g_{q,1}$ and $g_{q,2}^1 - g_{q,2}$ are functions regular at $p$, and $v^1 - v$ is a vector field that vanishes at $p$.

Proof. The proof is similar to the proof of the previous lemma. Let $(g_{q,1}, g_{q,2}, v) \in \nabla_{1,0,p}$. Denote its image in $\Gamma(W, G_1)$ by $(g_{1,1}, g_{1,2}, \ldots, g_{q,1}, g_{q,2}, v)$. The functions $g_{q,1}$ and $g_{q,2}$ and the vector field $v$ have no poles except $p$, the entries of $C_{i,q}$ have no poles at ordinary points by Lemma 18 so $g_{1,1}, g_{1,2} \in \Gamma(W, \mathcal{O}_P)$. Therefore, $(g_{1,1}, g_{1,2}, \ldots, g_{q,1}, g_{q,2}, v)$ really defines an element of $\Gamma(W, G_1)$ since all necessary equations are satisfied by Lemma 18.

Now let $(g_{1,1}^1, g_{1,2}^1, \ldots, g_{q,1}^1, g_{q,2}^1, v^1) \in \Gamma(W, G_1)$ be a section. Let
\[
g_{q,1}^1 = \sum_{k=-n_1}^{\infty} a_{1,k}(t-t_0)^k, \quad g_{q,2}^1 = \sum_{k=-n_2}^{\infty} a_{2,k}(t-t_0)^k, \quad v^1 = \left( \sum_{k=-n_3}^{\infty} b_k(t-t_0)^k \right) \frac{\partial}{\partial t},
\]
be the Laurent series for $g_{q,1}, g_{q,2}$, and $v$, respectively (in the sense of complex analysis). Denote
\[
g_{q,1} = \sum_{k=-n_1}^{-1} a_{1,k}(t-t_0)^k, \quad g_{q,2} = \sum_{k=-n_2}^{-1} a_{2,k}(t-t_0)^k, \quad v = \left( \sum_{k=-n_3}^{0} b_k(t-t_0)^k \right) \frac{\partial}{\partial t}.
\]
These sums are finite, so they define algebraic rational functions and an algebraic rational vector field. Hence, $g_{q,1} - g_{q,1}^1$, $g_{q,2} - g_{q,2}^1$, and $v - v^1$ are also algebraic rational. They are defined at $p$ in complex-analytic sense, hence they have no poles at $p$ in algebraic sense. Note also that $(v - v^1)(p) = 0$. By Lemma 25, the triple $(g_{q,1} - g_{q,1}^1, g_{q,2} - g_{q,2}^1, v - v^1)$ defines an element of $\Gamma(W, G_1)$, so $(g_{1,1}, g_{1,2}, v)$ is equivalent to $(g_{q,1}^1, g_{q,2}^1, v^1)$ in $\Gamma(W, G_1)/\Gamma(W, G_1)$. But $(g_{q,1}, g_{q,2}, v) \in \nabla_{1,0,p}$, so the map from $\nabla_{1,0,p}$ to $\Gamma(W, G_1)/\Gamma(W, G_1)$ is surjective. The injectivity of the map $\nabla_{1,0,p} \to \Gamma(W, G_1)$ is clear, and it follows from Lemma 25 that the only triple that maps to $\Gamma(W, G_1)$ is $(0,0,0)$.
Let \( p \in \mathbb{P}^1 \) be a removable special point, and let \( U_i \) be a subset of \( X \) corresponding to \( p \). Denote by \( G_{1,p,i} \) the space of triples \((g_{1,1},g_{1,2},v)\), where \( g_{1,1},g_{1,2} \in \Gamma(W_p,\Theta_{\mathbb{P}^1}) \), \( v \in \Gamma(W_p,\Theta_{\mathbb{P}^1}) \), but this time it is not necessarily true that \( v(p) = 0 \). The last index \( i \) in the notation \( G_{1,p,i} \) indicates that these triples will be considered as \( U_i \)-descriptions of vector fields on \( \pi^{-1}(W_p) \cap U \).

**Lemma 27.** Let \( p \in \mathbb{P}^1 \) be a removable special point, and let \( U_i \) be a subset of \( X \) corresponding to \( p \). Then \( \Gamma(W_p,G_1) \) can be identified with \( G_{1,p,i} \).

The isomorphism here maps \((g_{1,1},g_{1,2},v)\) to \((g_{1,1},g_{1,2},\ldots,g_{q,1},g_{q,2},v)\), where

\[
\begin{pmatrix}
g_{j,1} \\
g_{j,2} \\
v
\end{pmatrix} = C_{i,j} \begin{pmatrix}
g_{i,1} \\
g_{i,2} \\
v
\end{pmatrix}.
\]

**Proof.** The proof is similar to the proofs of two previous lemmas. All necessary linear equations in the definition of \( G_1 \) are satisfied by Lemma 18. We only have to check that if \( U_j \) is another subset of \( X \) corresponding to \( p \), then \( g_{j,1} \) and \( g_{j,2} \) do not have poles at \( p \). The only entries of \( C_{i,j} \) that could have poles at \( p \) are

\[
\frac{h_{i,1}}{h_{j,1}} g^*(\frac{c_{i,j}}{1})_{1,1} \quad \text{and} \quad \frac{h_{i,1}}{h_{j,1}} g^*(\frac{c_{i,j}}{1})_{2,2},
\]

and

\[
\frac{h_{i,2}}{h_{j,2}} g^*(\frac{c_{i,j}}{1})_{2,1} \quad \text{and} \quad \frac{h_{i,2}}{h_{j,2}} g^*(\frac{c_{i,j}}{1})_{1,2},
\]

Consider the first one of them, the second one is considered similarly. We have \( \text{ord}_p(h_{i,1}) = -D_p(\beta_{i,1}) \), \( \text{ord}_p(h_{i,2}) = -D_p(\beta_{i,2}) \), \( \text{ord}_p(h_{j,1}) = -D_p(\beta_{j,1}) \). Since \( p \) is a removable special point and \( \beta_{j,1} = (c^*_{i,j})_{1,1}\beta_{i,1} + (c^*_{i,j})_{1,2}\beta_{i,2} \), \( D_p(\beta_{j,1}) = (c^*_{i,j})_{1,1}D_p(\beta_{i,1}) + (c^*_{i,j})_{1,2}D_p(\beta_{i,2}) \), and

\[
\text{ord}_p \left( \frac{h_{j,1}}{h_{i,1}} g^*(c^*_{i,j})_{1,1} \right) = 0.
\]

Therefore, the logarithmic derivative of this function does not have a zero or a pole at \( p \), and \( g_{j,1} \) is well-defined at \( p \). The argument for \( g_{j,2} \) is similar.

The injectivity of the map from \( G_{1,p,i} \) to \( \Gamma(W_p,G_1) \) is again clear since a \((2q+1)\)-tuple defines the zero section only if all entries are zeros, and the surjectivity is also clear this time in since every section from \( \Gamma(W_p,G_1) \), \( g_{1,1},g_{1,2},v \) should be well-defined at \( p \).

Note that in this lemma, we use an affine open set \( U_i \), which depends on \( p \), and in fact used the \( U_i \)-description of a vector field, while in Lemma 25 we used \( U_q \), which did not depend on \( p \), and used the \( U_q \)-description. However, in the next lemma, we are going to use \( U_q \) again.

For a removable special point \( p \), denote by \( \nabla_{1,0,p} \) the space of triples of Laurent polynomials of the form \( (a_{1,1}(t-t_0)^{-1} + \ldots + a_{1,n_1}(t-t_0)^{-n_1}, a_{2,1}(t-t_0)^{-1} + \ldots + a_{2,n_2}(t-t_0)^{-n_2}, \ldots) \), \( t_0 \) is the coordinate of \( p \).

**Lemma 28.** If \( p \in \mathbb{P}^1 \) is a removable special point, the space \( \Gamma(W,G_1)/\Gamma(W_p,G_1) \) can be identified with \( \nabla_{1,0,p} \).

More exactly, these three Laurent polynomials are three last entries in a \((2q+1)\)-tuple defining an element of \( \Gamma(W,G_1) \), which in turn defines a coset in \( \Gamma(W,G_1)/\Gamma(W_p,G_1) \). In other words, the isomorphism is the composition of the map \((g_{1,1},g_{1,2},v) \mapsto (g_{1,1},g_{1,2},\ldots,g_{q,1},g_{q,2},v) \in \Gamma(W,G_1) \), where

\[
\begin{pmatrix}
g_{i,1} \\
g_{i,2} \\
v
\end{pmatrix} = C_{i,q} \begin{pmatrix}
g_{q,1} \\
g_{q,2} \\
v
\end{pmatrix},
\]

and the canonical projection \( \Gamma(W,G_1) \to \Gamma(W,G_1)/\Gamma(W_p,G_1) \).

The vector field here always differs from the last entry of any element of \( \Gamma(W,G_1) \) from the same coset in \( \Gamma(W,G_1)/\Gamma(W_p,G_1) \) by a vector field that has no pole at \( p \). This is true for the two functions if the vector field is zero in both representatives of the coset.
Proof. First, if \((g_{q,1}, g_{q,2}, v) \mapsto (g_{1,1}, g_{1,2}, \ldots, g_{q,1}, g_{q,2}, v)\), then \(g_{1,1}, g_{1,2}, v\) have no poles outside \(p\), entries of \(C_{1,j}\) have no poles at ordinary points, and all necessary equations are satisfied by Lemma 13 so these functions and this vector field really define an element of \(\Gamma(W, G_{1})\), and hence an element of \(\Gamma(W, G_{1})/\Gamma(W, G_{1})\).

The proof of injectivity is quite easy. If \((g_{q,1}, g_{q,2}, v) \mapsto (g_{1,1}, g_{1,2}, \ldots, g_{q,1}, g_{q,2}, v) \in \Gamma(W, G_{1})\), then \(v = 0\) since otherwise it has pole at \(p\). But then we can choose an open set \(U_i\) corresponding to \(p\) and write

\[
\begin{pmatrix} g_{q,1} \\ g_{q,2} \end{pmatrix} = C_{q,x}^i \begin{pmatrix} g_{1,1} \\ g_{1,2} \end{pmatrix}.
\]

The matrix \(C_{q,x}^i\) has only constant entries, so if \(g_{1,1}\) and \(g_{1,2}\) are regular at \(p\), then \(g_{q,1}\) and \(g_{q,2}\) are regular at \(p\) as well. But then \(g_{q,1} = g_{q,2} = 0\).

Now we prove surjectivity. Let \((g_{1,1}', g_{1,2}', \ldots, g_{q,1}', g_{q,2}', v') \in \Gamma(W, G_{1})\) be a section. Choose an index \(i\) such that \(U_i\) corresponds to \(p\) and write complex-analytic Laurent series:

\[
g_{i,1}' = \sum_{k=-n_1}^{\infty} a_{1,k}' (t-t_0)^k, \quad g_{i,2}' = \sum_{k=-n_2}^{\infty} a_{2,k}' (t-t_0)^k, \quad v' = \left( \sum_{k=-n_3}^{\infty} b_{k}' (t-t_0)^k \right) \frac{\partial}{\partial t'},
\]

Set

\[
g_{i,1}'' = \sum_{k=-n_1}^{-1} a_{1,k}' (t-t_0)^k, \quad g_{i,2}'' = \sum_{k=-n_2}^{-1} a_{2,k}' (t-t_0)^k, \quad v'' = \left( \sum_{k=-n_3}^{-1} b_{k}' (t-t_0)^k \right) \frac{\partial}{\partial t'},
\]

and

\[
\begin{pmatrix} g_{i,1}' \\ g_{i,2}' \\ v' \end{pmatrix} = C_{i,j} \begin{pmatrix} g_{i,1}'' \\ g_{i,2}'' \end{pmatrix}
\]

for all \(1 \leq j \leq q\). Observe that \(g_{i,1}' - g_{i,1}'' \), \(g_{i,2}' - g_{i,2}'' \) and \(v' - v''\) are well-defined at \(p\), so \((g_{i,1}' - g_{i,1}'' - g_{i,2}' - g_{i,2}'', v' - v'') \in G_{1,p,i}\). The image of this element of \(G_{1,p,i}\) under the isomorphism from Lemma 22 equals \((g_{i,1}' - g_{i,1}'' - g_{i,2}' - g_{i,2}'', \ldots, g_{q,1}' - g_{q,1}'' - g_{q,2}' - g_{q,2}'', v' - v'') \in \Gamma(W, G_{1});\) hence, by Remark 11 \((g_{i,1}' - g_{i,1}'' - g_{i,2}' - g_{i,2}'', \ldots, g_{q,1}' - g_{q,1}'' - g_{q,2}' - g_{q,2}'', v' - v'') \in \Gamma(W, G_{1})\) defines the zero coset in \(\Gamma(W, G_{1})/\Gamma(W, G_{1})\). It is sufficient to prove that \((g_{1,1}', g_{1,2}', \ldots, g_{q,1}', g_{q,2}', v'')\) is in the image of the morphism \(\nabla_{1,0,p} \to \Gamma(W, G_{1})/\Gamma(W, G_{1})\).

Now write

\[
g_{q,1}'' = \sum_{k=-n_1}^{\infty} a_{1,k}'' (t-t_0)^k, \quad g_{q,2}'' = \sum_{k=-n_2}^{\infty} a_{2,k}'' (t-t_0)^k
\]

(without loss of generality, we may suppose that \(n_1\) and \(n_2\) did not change, we may add more zeros in the negative part of Laurent series) and recall that

\[
v'' = \left( \sum_{k=-n_3}^{-1} b_{k}' (t-t_0)^k \right) \frac{\partial}{\partial t'}
\]

Set

\[
g_{q,1} = \sum_{k=-n_1}^{-1} a_{1,k}'' (t-t_0)^k, \quad g_{q,2} = \sum_{k=-n_2}^{-1} a_{2,k}'' (t-t_0)^k, \quad v = v'',
\]

and

\[
\begin{pmatrix} g_{j,1} \\ g_{j,2} \\ v \end{pmatrix} = C_{q,j} \begin{pmatrix} g_{q,1} \\ g_{q,2} \\ v'' \end{pmatrix}
\]

for all \(1 \leq j \leq q\). Then \((g_{q,1}, g_{q,2}, v) \in \nabla_{1,0,p}\). Since \(v = v''\), we have

\[
\begin{pmatrix} g_{1,1} \\ g_{1,2} \\ v \end{pmatrix} - \begin{pmatrix} g_{q,1}'' \\ g_{q,2}'' \end{pmatrix} = C_{q,j} \begin{pmatrix} g_{q,1} - g_{q,1}' \\ g_{q,2} - g_{q,2}' \end{pmatrix},
\]

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and
\[
\left( \begin{array}{c} g_{i,1} \\ g_{i,2} \end{array} \right) - \left( \begin{array}{c} g_{i,1}' \\ g_{i,2}' \end{array} \right) = C_{q,i} \left( \begin{array}{c} g_{q,1} - g_{q,1}' \\ g_{q,2} - g_{q,2}' \end{array} \right).
\]
Hence, \( g_{i,1}' - g_{i,1} \) and \( g_{i,2}' - g_{i,2} \) are regular at \( p \), \((g_{i,1}' - g_{i,1}, g_{i,2}' - g_{i,2}, v'' - v) \in G_{1,p,i}, \) and the isomorphism from Lemma 27 maps this triple to \((g_{i,1}' - g_{i,1}, g_{i,2}' - g_{i,2}, v'' - v) \in \Gamma(W_p, G_1).\) Therefore, \((g_{i,1}'', g_{i,2}'', \ldots, g_{q,1}'', g_{q,2}'', v''')\) defines the same coset in \( \Gamma(W, G_1) / \Gamma(W_p, G_1) \) as \((g_{i,1}, g_{i,2}, \ldots, g_{q,1}, g_{q,2}, v), \) which is the image of \((g_{i,1}, g_{i,2}, v) \in \nabla_{1,0,p}.\)

During the proof of surjectivity, we have changed the vector field from \( v' \) to \( v'' = v, \) and we chose \( v''' \) so that \( v' - v''' \) is regular at \( p. \) If we started with \( v' = 0, \) then \( v''' = 0 \) as well. In this case
\[
\left( \begin{array}{c} g_{q,1}'' \\ g_{q,2}'' \end{array} \right) = C_{r,q} \left( \begin{array}{c} g_{q,1}' - g_{q,1}'' \\ g_{q,2}' - g_{q,2}'' \end{array} \right).
\]
\( g_{q,1}' - g_{q,1}'' \) and \( g_{q,2}' - g_{q,2}'' \) are regular at \( p \) by construction, all entries in \( C_{r,q} \) are constants, so \( g_{q,1}' = g_{q,1}'' \) and \( g_{q,2}' = g_{q,2}'' \) are regular at \( p. \) Recall also that \( g_{q,1} - g_{q,1}' \) and \( g_{q,2} - g_{q,2}' \) are regular at \( p \) by construction, so finally we see that \( g_{q,1} - g_{q,1}' \) and \( g_{q,2} - g_{q,2}' \) are regular at \( p \) if \( v' = 0. \)

Note that in this lemma, we do not claim (and this is not true in general) that if \((g_{1,1}'', g_{1,2}'', \ldots, g_{q,1}'', g_{q,2}'', v'')\) is any representative of the coset in \( \Gamma(W, G_1) / \Gamma(W_p, G_1) \) defined by three Laurent polynomials in lemma, then, for example, the difference between the first of these Laurent polynomials and \( g_{q,1}' \) is regular at \( p. \) We only claim that this is true if \( v' = 0 \) and the third Laurent polynomial is also 0, and we also claim that independently of \( v' \), there always exists such a representative in the coset.

Let \( t_1, \ldots, t_r \) be the coordinates of all special points \( p_1, \ldots, p_r, \) respectively. Using Lemmas 26 and 28 we identify the direct sum
\[
\bigoplus_{i=1}^{r} \Gamma(W, G_1) / \Gamma(W_p, G_1)
\]
with the space
\[
\bigoplus_{i=1}^{r} \nabla_{1,0,p_i}
\]
of \( 3r \)-tuples of Laurent polynomials of a certain form, where the first three polynomials correspond to \( p_1, \) the second three polynomials correspond to \( p_2, \) etc.

**Lemma 29.** Let
\[
(g[1]_1, g[1]_2, v[1], \ldots, g[r]_1, g[r]_2, v[r])
\]
be an element of \( \bigoplus_{i=1}^{r} \nabla_{1,0,p_i}. \) Then there exists another element
\[
(g[1]'_1, g[1]'_2, v'[1], \ldots, g[r]'_1, g[r]'_2, v[r]') \in \bigoplus_{i=1}^{r} \nabla_{1,0,p_i}
\]
such that these two elements represent the same class in \( \bigoplus_{i=1}^{r} \Gamma(W, G_1) / \Gamma(W_p, G_1) \), and \( v[i]' \) is a vector field regular at \( p_i \) for all \( i. \)

**Proof.** All \( v[i]' \)'s can be written using Laurent polynomials as follows: \( v[i] = (b_{-1,i}(t-t_i)^{-1} + \cdots + b_{-k,i}(t-t_i)^{-k}) \partial / \partial t \) or \( v[i] = v[i] = (b_{0,i} + b_{-1,i}(t-t_i)^{-1} + \cdots + b_{-k,i}(t-t_i)^{-k}) \partial / \partial t, \) the exact form depends on whether \( p_i \) is a removable special point or an essential special point. Denote \( v[i]' = (b_{-1,i}(t-t_i)^{-1} + \cdots + b_{-k,i}(t-t_i)^{-k}) \partial / \partial t (\text{if } p_i \text{ is removable, then } v[i] = v[i]''). \) This vector field is regular at all points of \( \mathbb{P}^1 \) except \( p_i \) (including \( t = \infty, \) where it has a zero of order 3). Then \( v''' = v[1]'' + \cdots + v[r]''' \in \Gamma(W, G_1), \) and we can construct an element of \( \Gamma(W, G_1) \) similarly to what we did in previous proofs: we set \( g_{q,1}' = g_{q,2}' = 0, \) and
\[
\left( \begin{array}{c} g_{q,1}'' \\ g_{q,2}'' \\ v'''' \end{array} \right) = C_{q,i} \left( \begin{array}{c} g_{q,1}' \\ g_{q,2}' \\ v''' \end{array} \right).
\]
By Lemma [16] all entries in $C_{q,i}$ are regular at ordinary points, and $(g_{q,1}', g_{q,2}', \ldots, g_{q,1}'', g_{q,2}'', v'') \in \Gamma(W, G_1)$. By Lemmas [26] and [28] this section defines elements of $\nabla_{1,0,p_i}$ of the form $(g[i]'', g[i]'', v[i]'')$, where $v[i]''' - v''$ is regular at $p_i$. Recall that $v[i]'''$ is regular at $p_i$ if $i \neq j$, so $v[i]''' - v[j]'''$ is regular at $p_i$ as well. Also, $v[i]''' - v[i]'''$ is regular at $p_i$, so $v[i]''' - v[i]'''$ is regular at $p_i$. Finally, we set $g[i]' = g[i] - g[i]'''$, $g[i]'_2 = g[i]_2 - g[i]'''_2$, and $v[i]' = v[i] - v[i]'''$. Since $(g[i]'', g[i]'', v[i]'')$, $(g[i]'', g[i]'', v[i]'')$, $(g[i]'', g[i]'', v[i]'')$ define an element of the zero coset in $(\bigoplus_{i=1}^r \Gamma(W, G_1)/\Gamma(W, G_1))$, and $(g[1]'', g[1]'', v[1]'')$ defines the same coset as $(g[1]'', g[1]'', v[1]'')$ in $(\bigoplus_{i=1}^r \Gamma(W, G_1)/\Gamma(W, G_1))$. \hfill \Box

Lemma 30. Suppose that

$$(g[1], g[1], v[1], \ldots, g[r], g[r], v[r]) \in \bigoplus_{i=1}^r \nabla_{1,0,p_i}$$

and

$$(g[1]'', g[1]'', v[1]'', \ldots, g[r]'', g[r]'', v[r]'') \in \bigoplus_{i=1}^r \nabla_{1,0,p_i}$$

define the same class in $(\bigoplus_{i=1}^r \Gamma(W, G_1)/\Gamma(W, G_1))/\Gamma(W, G_1)$, and for every $i$, $v[i]$ and $v[i]'$ are regular at $p_i$. Then there exists a globally defined vector field $v \in \Gamma(P^1, \Theta_{P_1})$ such that $v(p_i) = v[i](p_i) - v[i]'(p_i)$ if $p_i$ is an essential special point.

And vice versa, if

$$(g[1], g[1], v[1], \ldots, g[r], g[r], v[r]) \in \bigoplus_{i=1}^r \nabla_{1,0,p_i}$$

is such that every $v[i]$ is regular at $p_i$, and $v \in \Gamma(P^1, \Theta_{P_1})$ is a globally defined vector field, then there exists

$$(g[1]'', g[1]'', v[1]'', \ldots, g[r]'', g[r]'', v[r]'') \in \bigoplus_{i=1}^r \nabla_{1,0,p_i}$$

equivalent to $(g[1], g[1], v[1], \ldots, g[r], g[r], v[r])$ in $(\bigoplus_{i=1}^r \Gamma(W, G_1)/\Gamma(W, G_1))/\Gamma(W, G_1)$ and such that $v[i]'$ is regular at $p_i$ for every $i$. Here $v[i](p_i) - v[i]'(p_i) = v(p_i)$ for all $i$ such that $p_i$ is an essential special point.

Proof. The first statement follows easily from Lemmas [26] and [28]. Namely, all triples $(g[i] - g[i]'', g[i]' - g[i]'', v[i] - v[i]''')$ define the same section from $\Gamma(W, G_1)$ in the sense of Lemmas [26] and [28] applied at $p_i$. This element of $\Gamma(W, G_1)$ can be written as $(g[1]'', g[1]'', g[1], g[1], v'')$. Let us prove that $v$ is the desired vector field. We know that $v$ is defined at all ordinary points. If $p_i$ is a removable special point, then by Lemma [26] $v[i]' - v[i]'''$, $v[i] - v[i]''''$ is regular at $p_i$, but we already know that $v[i]' - v[i]'''$ is regular at $p_i$, so $v$ is regular at $p_i$. If $p_i$ is an essential special point, then by Lemma [26] $v[i]' - v[i]'''' - v[i]''' = v(p_i)$ and $v[i]' - v[i]'''' = v(p_i)$ equals 0 there. Hence, $v$ is defined at $p_i$. And $v[i]'(p_i) - v[i]''''(p_i) = v(p_i)$.

The proof of the second statement is similar to the proof of the previous lemma. Namely, we start with $g_{q,1}' = g_{q,2}' = 0$ and construct a section $(g[1]'', g[1]'', \ldots, g[r]'', g[r]'', v'') \in \Gamma(W, G_1)$ via

$$
\begin{pmatrix}
g[1]'' \\
g[1]'_2 \\
v
\end{pmatrix} = C_{q,i} \begin{pmatrix}
g_{q,1}' \\
g_{q,2}' \\
v
\end{pmatrix}.
$$

This section defines elements of $\Gamma(W, G_1)/\Gamma(W, G_1)$, and the isomorphisms from Lemmas [26] and [28] map them to $(g[i]'', g[i]'', v[i]'')$. Both Lemmas says that $v[i]'''' - v$ is defined at $p_i$, and, since $v$ is defined globally, $v[i]''''$ is defined at $p_i$. So we can set $g[1]' = g[1]' + g[1]''''$, $g[2]' = g[2]' + g[2]''''$, and $v[i]' = v[i]' + v[i]''''$. If $p_i$ is an essential special point, Lemma [26] says that $v[i]''''(p_i) = v(p_i)$, so $v[i]'(p_i) - v[i]''''(p_i) = v(p_i)$.

Lemma 31. Let

$$(g[1], g[1], v[1], \ldots, g[r], g[r], v[r]) \in \bigoplus_{i=1}^r \nabla_{1,0,p_i}$$

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Then this element of \( \bigoplus_{i=1}^{r} \nabla_{1,0,p_i} \) and
\[
(0,0,v[1],\ldots,0,0,v[r])
\]
define the same class in \( \left( \bigoplus_{i=1}^{r} \Gamma(W,G_i) / \Gamma(W,\mathbb{R}_i) \right) / \Gamma(W,G_i) \).

Proof. The proof is similar to the proof of Lemma 29. Since all \( g[i] \) here are Laurent polynomials of the form \( a_{i,j} - (t-t_i)^{-1} + \ldots + a_{i,j} - (t-t_i)^{-n} \) (we do not mean here that \( a_{i,j} - n \neq 0 \), so we can use the same \( n \) for all polynomials), they have no poles except \( p_i \), and functions \( g'_{q,1} = g[1] + \ldots + g'[r] \) and \( g'_{q,2} = g[1] + \ldots + g'[r] \) have no poles at ordinary points. Using these functions, we can construct a section \( (g'_{1,1}, g'_{1,2}, \ldots, g'_{q,1}, g'_{q,2}, 0) \in \Gamma(W,G_i) \) as in proofs of previous lemmas, namely
\[
\begin{pmatrix}
g'_{1,1} \\
g'_{1,2} \\
0
\end{pmatrix}
= C_{q,1} \begin{pmatrix}
g_{q,1} \\
g_{q,2} \\
0
\end{pmatrix},
\]
or, in other words,
\[
\begin{pmatrix}
g'_{1,1} \\
g'_{1,2}
\end{pmatrix}
= C_{q,1} \begin{pmatrix}
g_{q,1} \\
g_{q,2}
\end{pmatrix}.
\]
Since all entries in \( C_{q,1} \) are constants, all functions \( g'_{i,j} \) are defined on \( W \), and they define an element of \( \Gamma(W,G_i) \).

A function \( g[i] \) or \( g[i] \) that has pole at \( p_i \) only if \( i = j \). Hence, the class of \( (g'_{1,1}, g'_{1,2}, \ldots, g'_{q,1}, g'_{q,2}, 0) \) in \( \Gamma(W,G_i) / \Gamma(W,p_i,G_i) \) is mapped by the isomorphism from Lemma 29 or 28 to \( (g[i], g[i], 0) \). Therefore, \( (g[1], g[1], 0, \ldots, g[r], g[r], 0) \) defines the zero coset in \( \left( \bigoplus_{i=1}^{r} \Gamma(W,G_i) / \Gamma(W,p_i,G_i) \right) / \Gamma(W,G_i) \), and \( (g[1], g[1], 0, \ldots, g[r], g[r], 0) \) and \( (0, 0, 0, \ldots, 0, v[r]) \) define the same coset in \( \left( \bigoplus_{i=1}^{r} \Gamma(W,G_i) / \Gamma(W,p_i,G_i) \right) / \Gamma(W,G_i) \).

Denote now by \( r' \) the number of essential special points. Denote these special points by \( p'_1, \ldots, p'_{r'} \).

Lemma 32. If \( r' \geq 3 \), then every globally defined vector field on \( \mathbb{P}^1 \) is uniquely determined by its values at \( p'_1, \ldots, p'_{r'} \). If \( r' \leq 3 \), then for every set of tangent vectors at \( p'_1, \ldots, p'_{r'} \) there exists a globally defined vector field that takes these values at these points.

Proof. Every globally defined vector field on \( \mathbb{P}^1 \) can be written as \( (a_0 + a_1 t + a_2 t^2) \partial/\partial t \). (If the polynomial here is of higher degree, the vector field has a pole at infinity.) A polynomial of degree 2 is completely determined by its values at least three points (if there are more than three points, these values cannot be arbitrary, but a polynomial of degree two is still unique if it exists). A polynomial of degree 2 can take arbitrary prescribed values at at most three points.

Proposition 8. If \( r' \leq 3 \), then \( H^1(\mathbb{P}^1, G_i) = 0 \).

If \( r' \geq 3 \), then there exists a vector space \( \nabla_{1,1} \) of dimension \( r' \) and an embedding \( \Gamma(\mathbb{P}^1, \Theta_{\mathbb{P}^1}) \hookrightarrow \nabla_{1,1} \) such that \( H^1(\mathbb{P}^1, G_i) \cong \nabla_{1,1} / \Gamma(\mathbb{P}^1, \Theta_{\mathbb{P}^1}) \). Therefore, \( \dim H^1(\mathbb{P}^1, G_i) = r' - 3 \) in this case.

Proof. By applying first Lemma 29 and then Lemma 31 to an element of \( \bigoplus_{i=1}^{r} \nabla_{1,0,p_i} \), we can get another element of \( \bigoplus_{i=1}^{r} \nabla_{1,0,p_i} \) equivalent to the original element of \( \bigoplus_{i=1}^{r} \nabla_{1,0,p_i} \) in \( \left( \bigoplus_{i=1}^{r} \Gamma(W,G_i) / \Gamma(W,p_i,G_i) \right) / \Gamma(W,G_i) \). Here, \( v[i] \) are Laurent polynomials regular at \( p_i \), i.e. they don’t have non-zero coefficients at negative degrees. But Lemmas 29 and 31 describe exact form of these polynomials, and the highest possible degree of a non-zero term is 0 if \( p_i \) is an essential special point, and \( -1 \) if it is removable. We conclude that if \( p_i \) is a removable special point, then \( v[i] = 0 \). Otherwise, \( v[i] \) is a vector field of the form \( a \partial/\partial t \) (\( a \in \mathbb{C} \)), which is completely determined by its value at \( p_i \).

Therefore, we have constructed a surjective linear map from
\[
\nabla_{1,1} \cong \bigoplus_{i=1}^{r'} \Theta_{\mathbb{P}^1, p'_i}
\]
to $H^i(P^1, \mathcal{G}_i)$. Denote this map by $\zeta$. $\Gamma(P^1, \mathcal{O}_P)$ can be mapped to $\nabla_{i,1}$ via evaluation of a vector field at points $p_1, \ldots, p_r$. Denote this map by $\xi$. Let us prove that $\ker \zeta = \ker (\Gamma(P^1, \mathcal{O}_P))$. First, if $v$ is a globally defined vector field, by the second part of Lemma 30, there exists $(g_1)[1], [g_2][1], [v_1][1], \ldots, [g_3][1], [v_3][1]$ equivalent to 0 in $(\bigoplus_{i=1}^3) \Gamma(W, \mathcal{G}_i)/\Gamma(W, \mathcal{G}_i)$ and such that $v[i][1]$ is defined at $p_i$ and $v[i][1](p_i) = v(p_i)$ for all essential special points $p_i$. We have already seen, $v[i][1] = 0$ if $p_i$ is removable. By Lemma 31 $(g_1)[1], [g_2][1], [v_1][1], \ldots, [g_3][1], [v_3][1]$ is equivalent to $(0, 0, 0, \ldots, 0, 0, [v][1])$, so $\ker (\Gamma(P^1, \mathcal{O}_P)) \subseteq \ker \zeta$. On the other hand, if $(0, 0, [v_1][1], \ldots, 0, 0, [v][1]) \in \ker \zeta$, then by the first part of Lemma 30 there exists a vector field $v \in \Gamma(P^1, \mathcal{O}_P)$ such that $v[i](p_i) = v(p_i)$ for all essential special points $p_i$. This means that $\ker \zeta \subseteq \ker (\Gamma(P^1, \mathcal{O}_P))$, and we finally conclude that $\ker \zeta = \ker (\Gamma(P^1, \mathcal{O}_P))$.

Now, by Lemma 32, $\xi$ is surjective if $r \geq 3$, and $\zeta$ is injective if $r \geq 3$, and the claim follows.

Now we continue with $\ker H^0(P^1, \mathcal{G}_j) \to H^0(P^1, \mathcal{G}_8)$. Recall that $\mathcal{G}_4$ (resp. $\mathcal{G}_8$) is the first cohomology of the complex $\mathcal{G}_2 \to \mathcal{G}_2' \to \mathcal{G}_8''$ (resp. $\mathcal{G}_6 \to \mathcal{G}_6' \to \mathcal{G}_8''$). The maps between $\mathcal{G}_2'$ and $\mathcal{G}_6$ can also be written as the cohomology in the middle of a map between these two complexes. Here $\mathcal{G}_2$ can be written as a direct sum of sheaves, each of them corresponds to an open subset $U_i$, namely, its sections over an open set $V \subseteq P^1$ are the $U_i$-descriptions of homogeneous vector fields of degree 0 defined on $\pi^{-1}(V) \cap U_i$. Denote this direct summand by $\mathcal{G}_2_i$. The sheaves $\mathcal{G}_2'$ and $\mathcal{G}_6'$ can be decomposed into direct sums similarly, and each direct summand corresponds to two or three of the sets $U_i$, respectively. Denote these direct summands by $\mathcal{G}_2_i,j$ and by $\mathcal{G}_2_i,j,k$ respectively. Similarly, we can define decompositions $\mathcal{G}_8''' = \bigoplus \mathcal{G}_2_i,j,k$, $\mathcal{G}_6''' = \bigoplus \mathcal{G}_2_i,j,k$, $\mathcal{G}_6'' = \bigoplus \mathcal{G}_6_i,j,k$, and $\mathcal{G}_8'' = \bigoplus \mathcal{G}_6_i,j,k$.

The maps $\mathcal{G}_2' \to \mathcal{G}_6$, $\mathcal{G}_2' \to \mathcal{G}_6'$, $\mathcal{G}_2' \to \mathcal{G}_6''$ map each of these direct summands in $\mathcal{G}_2'$, $\mathcal{G}_2'$, $\mathcal{G}_8''$ to the corresponding direct summand in $\mathcal{G}_6$, $\mathcal{G}_6'$, $\mathcal{G}_6''$, respectively.

Our next goal is to simplify the expressions for $\mathcal{G}_4$ and $\mathcal{G}_8$ we have now. For this goal, it will be more convenient to work with the "invariant" versions of the sheaves, i.e. with $\mathcal{G}_2''$, $\mathcal{G}_2''$, $\mathcal{G}_2''$, $\mathcal{G}_2''$, $\mathcal{G}_2''$, and $\mathcal{G}_2'''$, which do not involve any $U_i$-descriptions explicitly. By Lemma 32 $U_q \subseteq U_i$ is a dense open subset for all $i$. Hence, each of the sheaves $\mathcal{G}_2''$, $\mathcal{G}_2''$, and $\mathcal{G}_2''$ can be embedded into the following sheaf that we denote by $\mathcal{G}_2''$: $\Gamma(V, \mathcal{G}_2''')$ is the space of $T$-invariant vector fields on $\pi^{-1}(V) \cap U_q$. Similarly, each of sheaves $\mathcal{G}_6''$, $\mathcal{G}_6''$, and $\mathcal{G}_6''$ can be embedded into the following sheaf $\mathcal{G}_6'''$, $\Gamma(V, \mathcal{G}_6''')$ is the space of sequences of length $\sum_{i,j} \dim \mathcal{O}(D(\lambda_j))$ of functions of degree $\lambda_j$ defined on $\pi^{-1}(V) \cap U_q$. Therefore, we can apply Proposition 3 to compute $\mathcal{G}_4''$ and $\mathcal{G}_8''$. Then by Proposition 3 we have the following formulas for $\mathcal{G}_4''$ and $\mathcal{G}_8''$:

$$\mathcal{G}_4'' = \ker \left( \bigoplus_{i=1}^q \mathcal{G}_2''' / \mathcal{G}_2''' \to \bigoplus_{1 \leq i < j \leq q} \left( \mathcal{G}_2''' / \mathcal{G}_2''' \right) \right) / \mathcal{G}_2''',$$

$$\mathcal{G}_8'' = \ker \left( \bigoplus_{i=1}^q \mathcal{G}_6''' / \mathcal{G}_6''' \to \bigoplus_{1 \leq i < j \leq q} \left( \mathcal{G}_6''' / \mathcal{G}_6''' \right) \right) / \mathcal{G}_6'''.$$

And again, the map $\mathcal{G}_4'' = \mathcal{G}_4''$ maps each direct summand of $\mathcal{G}_4''$ to the corresponding direct summand of $\mathcal{G}_8''$. Corollary 1 cannot be applied here directly because it is not always true that $\mathcal{G}_6'' = \mathcal{G}_6''$. However, we can prove the following two lemmas. Recall that by Lemma 11 $U_i$ is isomorphic to $V_i \times (\mathbb{C} \setminus \{0\}) \times L$, where $L$ is isomorphic to $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$.

**Lemma 33.** Let $V'_i \subseteq V_i$ be an open subset. The space of $T$-invariant vector fields defined on $V'_i \times (\mathbb{C} \setminus \{0\}) \times L$ and on $V'_i \times (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$ coincide, in other words, the restriction homomorphism from

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the space of $T$-invariant vector fields on $V' \times (\mathbb{C} \setminus 0) \times L$ to the space of $T$-invariant vector fields on $V' \times \{(0)\} \times (\mathbb{C} \setminus 0)$ is in fact an isomorphism. This is also true for functions of degree $\chi$ instead of vector fields of degree 0, if $\chi \in \sigma'$.

**Proof.** The claim for vector fields follows directly from Corollary 7, namely, the description of the space of vector fields there does not depend on whether $L' = \mathbb{C}$ or $L' = \mathbb{C} \setminus 0$ (in terms of the notation used in Corollary 7). For functions of degree $\chi$, Lemma 19 gives the same description for $L' = \mathbb{C}$ and for $L' = \mathbb{C} \setminus 0$, if $\chi \in \sigma' \cap M$.

**Lemma 34.** The embeddings $G^\text{inv}_{2,i,j} \to G^\text{inv}_2$ and $G^\text{inv}_{6,i,j} \to G^\text{inv}_6$ are isomorphisms for $1 \leq i < j \leq q$, except for the following case: both indices $i$ and $j$ correspond to the same removable special point $p$. In this case, the embeddings $G^\text{inv}_{2,i} \to G^\text{inv}_{2,i,j}$, $G^\text{inv}_{2,j} \to G^\text{inv}_{2,i,j}$, $G^\text{inv}_{6,i} \to G^\text{inv}_{6,i,j}$, $G^\text{inv}_{6,j} \to G^\text{inv}_{6,i,j}$ are isomorphisms.

**Proof.** If $U_i$ and $U_j$ correspond to different special points, then $V_i \cap V_j = W$, and by Lemma 12 $U_i \cap U_j = W \times (\mathbb{C} \setminus 0) \times L$, where $L$ is isomorphic to $\mathbb{C}$ or $\mathbb{C} \setminus 0$. If $U_i$ and $U_j$ correspond to the same essential special point $p$, then they must correspond to the normal subcones of different vertices of $\Delta_p$, so Lemma 11 says that $U_i \cap U_j$ is isomorphic to $W \times (\mathbb{C} \setminus 0) \times L$ again. If $L = \mathbb{C} \setminus 0$, then $U_i \cap U_j \cap U_k = W \times (\mathbb{C} \setminus 0) \times L$ as well, and the isomorphism here, as well as in the equality $U_i \cap U_j = W \times (\mathbb{C} \setminus 0) \times L$, is given by the isomorphism defined by Lemma 11 for $U_i$, so $U_i \cap U_j \cap U_k = U_i \cap U_j \cap U_k = U_i \cap U_j$. We already know that $U_q \subseteq U_i$, $U_q \subseteq U_j$, so $U_q = U_i \cap U_j$ if $L = \mathbb{C} \setminus 0$. If $L = \mathbb{C}$, then $U_i \cap U_j = W \times (\mathbb{C} \setminus 0) \times \mathbb{C}$ and $U_i \cap U_j \subseteq W \cap L = U_i \cap U_j$ and by Lemma 12, $U_i \cap U_j \cap U_k = U_i \cap U_j \cap U_k = U_i \cap U_j$, where the isomorphism in both equalities is given by the isomorphism defined by Lemma 11 for $U_i$. Let $V \subseteq P^1$ be an open subset. Now it follows from Lemma 33 that we always have $\Gamma(V,G^\text{inv}_{2,i,j}) = \Gamma(V,G^\text{inv}_{2,i})$ and $\Gamma(V,G^\text{inv}_{6,i,j}) = \Gamma(V,G^\text{inv}_{6,i})$ if $U_i$ and $U_j$ correspond to different special points or $U_i$ and $U_j$ correspond to the same essential special point $p$.

Suppose now that both $U_i$ and $U_j$ correspond to the same removable special point $p$. Let us prove that the embeddings $G^\text{inv}_{2,i} \to G^\text{inv}_{2,i,j}$ and $G^\text{inv}_{6,i} \to G^\text{inv}_{6,i,j}$ are isomorphisms, the situation for $G^\text{inv}_{2,j} \to G^\text{inv}_{2,i,j}$ and $G^\text{inv}_{6,j} \to G^\text{inv}_{6,i,j}$ is completely symmetric. We have $\beta_{1,1}, \beta_{1,1} \in \partial \sigma'$, but $\beta_{1,1} \neq \beta_{2,1}$, so by Lemmas 11 and 12 $U_1 = V_1 \times (\mathbb{C} \setminus 0) \times \mathbb{C}$, $U_2 \cap U_j = V_1 \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0)$, and the isomorphism in the second equality is a restriction of the isomorphism in the first equality. The claim again follows from Lemma 33.

Since kernels of sheaf maps can be computed on each open subset independently, Lemma 34 implies that

$$\ker \left( \bigoplus_{i=1}^{q} (G^\text{inv}_{2,i} / G^\text{inv}_{2,i,j}) \to \bigoplus_{1 \leq i < j \leq q} (G^\text{inv}_{2,i} / G^\text{inv}_{2,i,j}) \right)$$

can be computed as follows. Its sections over an open subset $V \subseteq P^1$ are sequences of the form $(w_1, \ldots, w_q) \in \bigoplus_{i=1}^{q} \Gamma(V,G^\text{inv}_{2,i} / G^\text{inv}_{2,i,j})$ satisfying the following conditions: if indices $i$ and $j$ correspond to the same removable special point $p$, then $w_i = w_j$. So we can do the following. For each removable special point $p$, if there are two indices $i$ and $j$ corresponding to $p$, choose one of them and call it excessive. Then the kernel is isomorphic to the following sheaf:

$$\bigoplus_{1 \leq i \leq q} (G^\text{inv}_{2,i} / G^\text{inv}_{2,i,j}).$$

Similarly,

$$\ker \left( \bigoplus_{i=1}^{q} (G^\text{inv}_{6,i} / G^\text{inv}_{6,i,j}) \to \bigoplus_{1 \leq i < j \leq q} (G^\text{inv}_{6,i} / G^\text{inv}_{6,i,j}) \right) \cong \bigoplus_{1 \leq i \leq q} (G^\text{inv}_{6,i} / G^\text{inv}_{6,i,j}).$$

So we get the following formulas for $G^\text{inv}_4$ and $G^\text{inv}_8$:

$$G^\text{inv}_4 \cong \left( \bigoplus_{1 \leq i \leq q} (G^\text{inv}_{2,i} / G^\text{inv}_{2,i,j}) \right) / G^\text{inv}_{2,1}.$$
\[ G_{\text{inv}}^{\text{inv}} \cong \left( \bigoplus_{1 \leq i \leq q} (G_{0}^{\text{inv}} / G_{0,i}^{\text{inv}}) \right) / G_{0}^{\text{inv}}. \]

Sections of quotients of sheaves can only be computed directly on affine subsets. To compute the space of global sections on \( P^{1} \), we should first compute sections for an affine covering of \( P^{1} \), then global sections are tuples of local sections that coincide on the intersections of the sets from the affine covering. We already have an affine covering of \( P^{1} \), namely, we have sets \( W_{p} \). Recall that \( W_{p} \cap W_{p'} = W \) for every pair of special points \( p \neq p' \).

**Lemma 35.** Let \( V \subseteq P^{1} \) be an open subset and \( p \in P^{1} \) be a special point such that \( V \cap W_{p} = W \). Let an index \( i \) correspond to \( p \). Then \( \Gamma(V, G_{\text{inv}}^{2}) = \Gamma(V, G_{\text{inv}}^{2,i}) \) and \( \Gamma(V, G_{\text{inv}}^{6}) = \Gamma(V, G_{\text{inv}}^{6,i}) \).

**Proof.** \( \Gamma(V, G_{\text{inv}}^{2}) \) (resp. \( \Gamma(V, G_{\text{inv}}^{6}) \)) is the space of \( T \)-invariant vector fields (resp. sequences of functions of certain degrees) defined on \( \pi^{-1}(V) \cap U_{i} = \pi^{-1}(V) \cap (W_{p} \times (\mathbb{C} \setminus 0) \times L) = (V \cap W_{p}) \times (\mathbb{C} \setminus 0) \times L = W \times (\mathbb{C} \setminus 0) \times L \), where \( L = \mathbb{C} \) or \( L = \mathbb{C} \setminus 0 \). By Lemma 53 these spaces are isomorphic to the spaces of (respectively) vector fields and sequences of functions of certain degrees defined on \( W \times (\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0) \subseteq U_{i} \), where the embedding is given by the isomorphism for \( U_{i} \) in Lemma 11. On the other hand, by Lemma 12 \( U_{i} \cap U_{q} \) is also isomorphic to \( U_{i} \cap U_{q} \), and the isomorphism here is also the restriction of the isomorphism in Lemma 11 for \( U_{i} \) to \( U_{i} \cap U_{q} \). Therefore, in fact we have proved that the restriction of spaces of \( T \)-invariant vector fields and of functions of the required degrees from \( \pi^{-1}(V) \cap U_{i} \) to \( U_{i} \cap U_{q} \) are isomorphisms. But \( U_{i} \cap U_{q} = \pi^{-1}(V) \cap U_{q} \), and \( \Gamma(V, G_{\text{inv}}^{2}) \) (resp. \( \Gamma(V, G_{\text{inv}}^{6}) \)) is the space of \( T \)-invariant vector fields (resp. sequences of functions of the required degrees) defined on \( \pi^{-1}(V) \cap U_{q} \).

**Corollary 8.** \( \Gamma(W, G_{\text{inv}}^{2}) = 0 \), \( \Gamma(W, G_{\text{inv}}^{6}) = 0 \).

**Proof.** \( W \cap W_{p} = W \) for all special points \( p \), so all direct summands of the from \( G_{\text{inv}}^{2} / G_{2,i}^{\text{inv}} \) and \( G_{\text{inv}}^{6} / G_{6,i}^{\text{inv}} \) in the formulas above vanish.

This corollary enables us to omit the condition that sections of \( G_{\text{inv}}^{2} \) (or of \( G_{\text{inv}}^{6} \)) over different sets \( W_{p} \) should coincide on intersections to form a global section. Therefore,

\[ \Gamma(P^{1}, G_{\text{inv}}^{2}) \cong \bigoplus_{p \text{ special point}} \left( \bigoplus_{1 \leq i \leq q} \left( \Gamma(W_{p}, G_{\text{inv}}^{2}) / \Gamma(W_{p}, G_{2,i}^{\text{inv}}) \right) / \Gamma(W_{p}, G_{2}^{\text{inv}}) \right), \]

\[ \Gamma(P^{1}, G_{\text{inv}}^{6}) \cong \bigoplus_{p \text{ special point}} \left( \bigoplus_{1 \leq i \leq q} \left( \Gamma(W_{p}, G_{\text{inv}}^{6}) / \Gamma(W_{p}, G_{6,i}^{\text{inv}}) \right) / \Gamma(W_{p}, G_{6}^{\text{inv}}) \right). \]

This formulas can be simplified more. Namely, recall that every set \( V_{i} \) equals \( W_{p} \) or \( W \). If \( p \) is a special point, and \( V_{i} = W \) or \( V_{i} = W_{p'} \), where \( p' \neq p \), then \( V_{i} \cap W_{p} = W \), and by Lemma 55 \( \Gamma(W_{p}, G_{\text{inv}}^{2}) / \Gamma(W_{p}, G_{2,i}^{\text{inv}}) = 0 \) and \( \Gamma(W_{p}, G_{\text{inv}}^{6}) / \Gamma(W_{p}, G_{6,i}^{\text{inv}}) = 0 \). So, we can write global sections of \( G_{\text{inv}}^{2} \) and \( G_{\text{inv}}^{6} \) as follows:

\[ \Gamma(P^{1}, G_{\text{inv}}^{2}) \cong \bigoplus_{p \text{ special point}} \left( \bigoplus_{1 \leq i \leq q} \left( \Gamma(W_{p}, G_{\text{inv}}^{2}) / \Gamma(W_{p}, G_{2,i}^{\text{inv}}) \right) / \Gamma(W_{p}, G_{2}^{\text{inv}}) \right), \]

\[ \Gamma(P^{1}, G_{\text{inv}}^{6}) \cong \bigoplus_{p \text{ special point}} \left( \bigoplus_{1 \leq i \leq q} \left( \Gamma(W_{p}, G_{\text{inv}}^{6}) / \Gamma(W_{p}, G_{6,i}^{\text{inv}}) \right) / \Gamma(W_{p}, G_{6}^{\text{inv}}) \right). \]
Now each sheaf $\mathcal{G}^{\text{inv}}_{2,j}$ and $\mathcal{G}^{\text{inv}}_{6,i}$ occurs only once in these summations. Each direct summand in the first direct sum in the formula for $\Gamma(P^1, \mathcal{G}^{\text{inv}}_4)$ is mapped to the corresponding direct summand of $\Gamma(P^1, \mathcal{G}^{\text{inv}}_8)$, so we have proved the following lemma:

**Lemma 36.**

\[
\ker(\Gamma(P^1, \mathcal{G}^{\text{inv}}_4) \to \Gamma(P^1, \mathcal{G}^{\text{inv}}_8)) \cong \bigoplus_{p \text{ special point}} \ker \left( \bigoplus_{1 \leq i \leq q} \left( \Gamma(W_p, \mathcal{G}^{\text{inv}}_2)/\Gamma(W_p, \mathcal{G}^{\text{inv}}_{6,i}) \right) \biggr/ \Gamma(W_p, \mathcal{G}^{\text{inv}}_2) \right).
\]

Fix a special point $p$, let $t_p \in \mathbb{C}$ be its coordinate. Our next goal is to compute the kernel

\[
\ker \left( \bigoplus_{1 \leq i \leq q} \left( \Gamma(W_p, \mathcal{G}^{\text{inv}}_2)/\Gamma(W_p, \mathcal{G}^{\text{inv}}_{6,i}) \right) \biggr/ \Gamma(W_p, \mathcal{G}^{\text{inv}}_2) \right).
\]

Recall first that if $p$ is a removable special point, then there exists only one non-excessive index $i$ corresponding to $p$. But then each direct sum in the formula above contains only one summand, and $(\Gamma(W_p, \mathcal{G}^{\text{inv}}_2)/\Gamma(W_p, \mathcal{G}^{\text{inv}}_{6,i}))/\Gamma(W_p, \mathcal{G}^{\text{inv}}_{6,i}) = 0$. So in the sequel we suppose that $p$ is an essential special point. Then there are no excessive indices corresponding to $p$. Moreover, in this case we chose exactly one set $U_i$ for each pair $(p,j)$, where $1 \leq j \leq v_p$. In other words, these sets $U_i$ (and the summands in each of the direct sums in the formula above) are in bijection with the vertices $V_{p,j}$ of $\Delta_p$. For each pair $(p,j)$, denote the index $i$ such that $U_i$ corresponds to $(p,j)$ by $i_{p,j}$. So, now we are computing the kernel

\[
\ker \left( \bigoplus_{1 \leq i \leq q} \left( \Gamma(W_p, \mathcal{G}^{\text{inv}}_2)/\Gamma(W_p, \mathcal{G}^{\text{inv}}_{6,i}) \right) \biggr/ \Gamma(W_p, \mathcal{G}^{\text{inv}}_2) \right).
\]

Fix an index $j$, $1 \leq j \leq v_p$. Now we come back to using $U_i$-descriptions, namely, We are going to use $U_{i_{p,j}}$-descriptions to compute $\Gamma(W_p, \mathcal{G}^{\text{inv}}_2)/\Gamma(W_p, \mathcal{G}^{\text{inv}}_{2,i})$ and $\Gamma(W_p, \mathcal{G}^{\text{inv}}_6)/\Gamma(W_p, \mathcal{G}^{\text{inv}}_{6,i})$. $\Gamma(W_p, \mathcal{G}^{\text{inv}}_{2,i})$ is the space of $T$-invariant vector fields defined on $U_{i_{p,j}}$, and, by Corollary 6, they are determined by triples of a vector field and two functions defined on $W_p$, which form $\Gamma(W_p, \mathcal{G}^{\text{inv}}_{2,i})$. We shortly write $\mathcal{G}^{\text{inv}}_{2,p,j} = \Gamma(W_p, \mathcal{G}^{\text{inv}}_{2,i})$. $\Gamma(W_p, \mathcal{G}^{\text{inv}}_6)$ is the space of $T$-invariant vector fields defined on $U_q = \pi^{-1}(W_p) \cap U_q$, and by Lemma 12 and by Corollary 10 they are determined by triples of a vector field and two functions defined on $W$. Denote this space of triples of a vector field and two functions defined on $W$
Consider complex-analytic Laurent series:

\[ \Gamma(G_{2,p,j}^{op}) \] and the natural projection
\[ \psi : \Gamma(G_{2,p,j}^{inv}) \to \Gamma(G_{2,p,j}^{inv}) \] after applying these isomorphisms becomes the restriction of vector fields and functions from \( W_p \) to \( W \).

Similarly, \( \Gamma(W_p, G_{6,1,p}) \) is the space of sequences of functions of certain degrees from \( \sigma^1 \cap M \) defined on \( U_{1,p,j} \). Lemma \[ \text{for } i = 1 \] identifies this space with the space of sequences of functions defined on \( U_p \) (each function is identified with its \( U_{1,p,j} \)-description), denote this space of sequences of functions by \( G_{6,1,p} \). \( \Gamma(W_p, G_{6,1,p}^{inv}) \) is the space of sequences of functions of the same degrees, but they are defined on \( U_q = \pi^{-1}(W_p) \cap U_q \). Again, Lemma \[ \text{for } i = 1 \] identifies this space with the space of sequences of functions defined on \( W \) (again, each function is identified with its \( U_{1,p,j} \)-description, not with its \( U_q \)-description).

Denote this space of sequences of functions by \( G_{6,1,p}^{op} \), and denote this isomorphism between \( G_{6,1,p}^{op} \) and \( \Gamma(W_p, G_{6,1,p}^{inv}) \) by \( \kappa_{6,1,p,j} \). And again, despite the spaces themselves do not depend on \( p \) and \( j \), the isomorphism is based on \( U_{1,p,j} \)-descriptions and depends on \( p \) and \( j \). By Remark \[ \text{the embedding} \] \( \Gamma(W_p, G_{6,1,p}) \to \Gamma(W_p, G_{6,1,p}^{inv}) \) after these identifications becomes the restriction of functions from \( W_p \) to \( W \).

Finally, the formula in Lemma \[ \text{for different indices } i_{p,j} \] defines morphisms \( \Gamma(W_p, G_{2,i_{p,j}}^{inv}) \to \Gamma(G_{2,p,j}^{op}) \). Denote the corresponding morphisms between \( G_{2,p,j}^{op} \) and \( G_{2,i_{p,j}}^{op} \) by \( \psi_{p,j} \). Denote also the map \( \bigoplus_{j=1}^{n} G_{2,i_{p,j}}^{op} \to \bigoplus_{j=1}^{n} G_{2,p,j}^{op} \) formed by maps \( \psi_{p,j} \) for all \( j \) (\( 1 \leq j \leq v_p \)) by \( \psi_p \). It follows from functoriality of the isomorphism in Proposition \[ \text{that } \psi_p \] induces the morphism in question between

\[
\left( \bigoplus_{j=1}^{v_p} \left( \Gamma(W_p, G_{2,i_{p,j}}^{inv})/\Gamma(G_{2,p,j}^{inv}) \right) \right) / \Gamma(W_p, G_{2,i_{p,j}}^{inv})
\]

and

\[
\left( \bigoplus_{j=1}^{v_p} \left( \Gamma(W_p, G_{6,1,p}^{inv})/\Gamma(W_p, G_{6,1,p}^{inv}) \right) \right) / \Gamma(W_p, G_{6,1,p}^{inv}).
\]

**Lemma 37.** Let \( p \) be a special point, \( j \) be an index, \( 1 \leq j \leq v_p \), \( t_p \in \mathbb{C} \) be the coordinate of \( p \).

The composition of the restriction of \( \kappa_{2,p,j} \) to the space of triples of the form \( (a_1, a_2, a_3) \) with \( a_2 = \sum_{k=-n_2}^{n_1} a_{l,k} (t-t_p)^k \), \( l = 1, 2 \), \( v = \sum_{k=-n_3}^{n_1} b_k (t-t_p)^k \partial/\partial t \) and the natural projection \( \Gamma(W_p, G_2^{inv}) \to \Gamma(W_p, G_2^{inv})/\Gamma(W_p, G_2^{inv}) \) is an isomorphism.

**Proof.** The proof is similar to the proof of Lemma \[ \text{Namely, let } g_1, g_2 \in \Gamma(W, O_{p_1}), v \in \Gamma(W, \Theta_{p_1}) \] Consider complex-analytic Laurent series:

\[
g_1 = \sum_{k=-n_2}^{\infty} a_{l,k} (t-t_p)^k, \quad (l = 1, 2), \quad v = \sum_{k=-n_3}^{\infty} b_k (t-t_p)^k \frac{\partial}{\partial t}
\]

Set

\[
g'_1 = \sum_{k=-n_2}^{-1} a_{l,k} (t-t_p)^k, \quad (l = 1, 2), \quad v' = \sum_{k=-n_3}^{-1} b_k (t-t_p)^k \frac{\partial}{\partial t}
\]

These functions and this vector field are algebraic since the sums are finite. The functions have zero of degree at least 1 at \( \infty \), the vector field has zero of degree at least 3 at \( \infty \), so \( g_1, g_2 \in \Gamma(W, O_{p_1}) \) and \( v' \in \Gamma(W, \Theta_{p_1}) \). Hence, \( g_1 - g_2, g_1 - g_2 \in \Gamma(W, O_{p_1}) \) and \( v' - v \in \Gamma(W, \Theta_{p_1}) \), but \( \kappa_{2,p,j} (g_1, g_2, v) \) and \( \kappa_{2,p,j} (g_1', g_2', v') \) define the same element of \( \Gamma(W_p, G_2^{inv})/\Gamma(W_p, G_2^{inv}) \) and the composition of the restriction of \( \kappa_{2,p,j} \) and the natural projection under consideration is surjective. Injectivity is also clear since if a Laurent polynomial of the considered form has no pole at \( p \), then it is zero.

Note that despite the proof is similar to the proof of Lemma \[ \text{Laurent polynomials here and in Lemma \[ have different meanings: here they from the } U_{1,p,j} \text{-description of a vector field on } U_q \text{, while in Lemma \[ they formed the } U_q \text{-description of a vector field on } U_q. \]
Denote the direct sum of maps $\kappa_{2,p,j}$, which maps $\bigoplus_{j=1}^{\nu_p} G_{2,p,j}^{G_{2,p,j}}$ to $\bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_{2,p,j}^{\text{inv}})$, by $\kappa_{2,p}$. Denote by $\nabla_{2,0}$ the subspace of $\bigoplus_{j=1}^{\nu_p} G_{2,p,j}^{G_{2,p,j}}$ formed by the $3\nu_p$-tuples of the form

$$(0, 0, 0, g[2j], g[2j+1], v[2], \ldots, g[v_p], g[v_p+1], v[v_p]),$$

where

$$g[j] = \sum_{k=-n_{j,l}}^{-1} a_{j,l,k}(t-t_p)^k, \quad v[j] = \left( \sum_{k=-n_{j,3}}^{-1} b_{j,k}(t-t_p)^k \right) \frac{\partial}{\partial t}.$$ 

**Lemma 38.** The restriction of the composition of $\kappa_{2,p}$ and the natural projection

$$\nabla_{2,0} \rightarrow \bigoplus_{j=1}^{\nu_p} \left( \Gamma(W_p, G_{2,p,j}^{\text{inv}}) / (\Gamma(W_p, G_{2,p,j}^{\text{inv}})) \right) / \Gamma(W_p, G_{2,p,j}^{\text{inv}})$$

to $\nabla_{2,0}$ is surjective. Its kernel is one-dimensional.

**Proof.** To prove surjectivity, consider a $3\nu_p$-tuple $(g[1], g[1], v[1], \ldots, g[v_p], g[v_p], v[v_p]) \in \bigoplus_{j=1}^{\nu_p} G_{2,p,j}^{G_{2,p,j}}$. For every $j$, $1 \leq j \leq \nu_p$, set

$$\left( g_{i,j}, g_{j,j}, v_{j,j} \right) = C_{i,j}, i, j, 1.$$ 

By Lemma [16] these functions and vector fields are regular on $W$. By Lemma [15] $\kappa_{2,p,j}(g''_{i,j}, g_{j,j}, v_{j,j}) = \kappa_{2,p,j}(g[1], g[1], v[1])$. Hence, $\kappa_{2,p,j}(g[1], g[1], v[1], \ldots, g[v_p], g[v_p], v[v_p])$ and $\kappa_{2,p,j}(g[1], g[1], v[1], \ldots, g[v_p], g[v_p], v[v_p], v''[j])$ define the same coset in $(\bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_{2,p,j}^{\text{inv}})) / \Gamma(W_p, G_{2,p,j}^{\text{inv}})$. Observe that $g[1] = g''_{i,j}, g[1] = g''_{i,j},$ and $v[1] = v''_j$. Now, by Lemma [37] every triple $(g[j], g[j], v[j], v[j]) \in G_{2,p,j}^{G_{2,p,j}}$ can be replaced with $(g[j], g[j], v[j]) \in G_{2,p,j}^{G_{2,p,j}}$, where

$$g[j] = \sum_{k=-n_{j,3}}^{-1} a_{j,l,k}(t-t_p)^k, \quad v[j] = \left( \sum_{k=-n_{j,3}}^{-1} b_{j,k}(t-t_p)^k \right) \frac{\partial}{\partial t},$$

so that $\kappa_{2,p,j}(g[j], g[j], v[j])$ and $\kappa_{2,p,j}(g[j], g[j], v[j])$ define the same coset in $\Gamma(W_p, G_{2,p,j}^{\text{inv}}) / \Gamma(W_p, G_{2,p,j}^{\text{inv}})$. Hence, $\kappa_{2,p}(0, 0, 0, g[2], g[2], v[2], \ldots, g[v_p], g[v_p], v[v_p])$ and $\kappa_{2,p}(g[1], g[1], v[1], \ldots, g[v_p], g[v_p], v[v_p])$ define the same element of $(\bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_{2,p,j}^{\text{inv}})) / \Gamma(W_p, G_{2,p,j}^{\text{inv}})$, and $(0, 0, 0, g[2], g[2], v[2], \ldots, g[v_p], g[v_p], v[v_p]) \in \nabla_{2,0}$, therefore, the restriction of the composition to $\nabla_{2,0}$ is surjective.

Now suppose that $(0, 0, 0, g[2], g[2], v[2], \ldots, g[v_p], g[v_p], v[v_p]) \in \nabla_{2,0}$ and $\kappa_{2,p}(0, 0, 0, g[2], g[2], v[2], \ldots, g[v_p], g[v_p], v[v_p])$ defines the zero coset in $(\bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_{2,p,j}^{\text{inv}})) / \Gamma(W_p, G_{2,p,j}^{\text{inv}})$. For simplicity of notation, denote $g[1] = g[1]$ and $v[1] = 0$. Then there exist $(g[j], g[j], v'[j]) \in G_{2,p,j}$ and $(g''_{i,j}, g''_{i,j}, v''[j]) \in G_{2,p,j}^{G_{2,p,j}}$ such that $g[j] = g[j]' + g''_{i,j}, v'[j] = v'[j]' + v''[j]$ and $\kappa_{2,p,j}(g[j], g[j], v'[j]) = \kappa_{2,p,j}(g''_{i,j}, g''_{i,j}, v''[j]) = \ldots = \kappa_{2,p,v_p}(g''_{i,v_p}, g''_{i,v_p}, v''[v_p])$. By Lemma [17] this means that

$$\left( g_{i,j}', g_{j,j}' \right) = C_{i,j}, i, j, 1.$$ 

In particular, $v'' = v''[j]'$ and all functions $g''_{i,j}$ and all vector fields $v''[j]'$ are determined by $(g''_{i,j}, g''_{i,j}, v''[j])$. On the other hand, the conditions $g[j] = g[j]' + g''_{i,j}, v'[j] = v[j]' + v''[j]'$ for $j = 1$ mean that $g[1]' = -g''_{i,j}, v[1]' = -v''[j]'$. Therefore, $g''_{i,j}, g''_{i,j}$, and $v''[j]'$ are regular at $p$. By Lemma [14] $\text{ord}_p(g''_{i,j}) \geq -1$ for $t = 1, 2,$
1 ≤ j ≤ v_p. Now it follows from the definition of \( \nabla_{2,0} \) that \( g[j]_l = a_{-1,j,l}(t - t_p)^{-1} \) for some \( a_{-1,j,l} \in \mathbb{C} \), and \( v[j]_l = 0 \). Moreover, it follows from a consideration of Laurent series of \( v'_i \), of entries of \( C_{b_i,b_i,1} \), and of \( g[j]_l \) that all numbers \( a_{-1,j,l} \) are uniquely determined by the value of \( v'_i \) at \( p \), which is an element of a one-dimensional space (the tangent space of \( \mathbb{P}^1 \) at \( p \)). Therefore, the kernel of the composition of \( \kappa_{2,p} \) and the projection is at most one-dimensional.

Now let us prove that the kernel contains a nonzero element. Set \( g'_{1,1} = g''_{1,2} = 0 \), \( v'_i = \partial/\partial t \), and

\[
\left( \begin{array}{c} g''_{1,1} \\ g''_{1,1} \\ v''_i \\ v''_i \\ \end{array} \right) = C_{b_i,j,b_i,1} \left( \begin{array}{c} g''_{1,1} \\ g''_{1,1} \\ v''_i \\ v''_i \end{array} \right).
\]

By Lemma 16 all functions \( g''_{j,l} \) are regular on \( W \). By Lemma 15 \( \kappa_{2,p,j}(g''_{j,i},g''_{j,2},v''_i) = \kappa_{2,p,j}(g''_{j,i},g''_{j,2},v''_i) \) for \( 1 \leq j \leq v_p \) and \( \kappa_{2,p}(g''_{j,i},g''_{j,2},v''_i) \) is the zero coset in \( (\bigoplus_{j=1}^{v_p}(\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})/\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})))\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}}) \). By Lemma 17 \( \text{ord}_p(g''_{j,i}) = -1 \) for \( 2 \leq j \leq v_p \), \( l = 1,2 \). So we can write \( g''_{j,i} = g[j]_i \), where \( g[j]_i = a_{-1,j,l}(t - t_p)^{-1} \) for \( a_{-1,j,l} \in \mathbb{C} \), and for \( j \geq 2 \) we have \( a_{-1,j,l} \neq 0 \) and \( g[j]_l \) is regular at \( p \) (and hence on \( W_p \)). By the definition of matrices \( C_{b_i,j,b_i,1} \), \( v''_i = v''_i \), and we can set \( v'_i = v''_i \), \( v''_i = 0 \). Then \( v[j]_i \in \Gamma(W_p,\mathcal{O}_{2,p}) \) and \( \kappa_{2,p,j}(g[j]_i,v[j]_i,v[j]_i) \) defines the zero coset in \( \Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})/\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}}) \). By construction, \( g = (g[1],g[1],v[1],\ldots,g[v_{-1}],v[v_p],v[v_p]) \in \nabla_{2,0} \). Since \( a_{-1,j,l} \neq 0 \) for all \( 2 \leq j \leq v_p \) and \( l = 1,2 \), \( g \) is an element of the kernel of the composition of \( \kappa_{2,p} \) and the projection from \( \bigoplus_{j=1}^{v_p} \Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}}) \) to \( \bigoplus_{j=1}^{v_p}(\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})/\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})) \).

Now we are going to use the map \( \psi_{p,j}: \mathcal{G}_{2,p,j}^{\text{op}} \to \mathcal{G}_{6,j}^{\text{op}} \) we have introduced before using Lemma 22. Each of the functions it computes is the \( U_{b,j} \)-description of a function of a degree \( \chi \) on \( U_q(\chi = \lambda_1,\ldots,\lambda_m) \), and exactly \( \dim \mathcal{O}(\mathcal{D}(\chi)) \) of these functions are of this degree. Denote the morphism \( \mathcal{G}_{2,p,j}^{\text{op}} \to \Gamma(W,\mathcal{O}_{2,p}) \) computing the \( k \)th of the functions of degree \( \chi \) by \( \psi_{p,j,\chi,k} \). Denote also by \( \kappa_{b,j} \) the direct sum of morphisms \( \kappa_{6,p,j} \), which maps \( \bigoplus_{j=1}^{v_p} \mathcal{G}_{6,j}^{\text{op}} \) to \( \bigoplus_{j=1}^{v_p}(\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})/\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})) \). We are computing the kernel of the map

\[
\varphi_p \left( \bigoplus_{j=1}^{v_p}(\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})/\Gamma(W_p,\mathcal{G}_{6,b_i,j,\chi})) \right) / \Gamma(W_p,\mathcal{G}_{6}^{\text{inv}})
\]

induced by \( \psi_p \), so by Lemma 33 it is sufficient to consider the restriction of \( \psi_p \) to \( \nabla_{2,0} \). Then \( \kappa_{2,p} \) maps the kernel of the composition of \( \kappa_{6,p} \circ \psi_p^{\text{op}} \) and the natural projection

\[
\varphi_p \left( \bigoplus_{j=1}^{v_p}(\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})/\Gamma(W_p,\mathcal{G}_{6,b_i,j,\chi})) \right) / \Gamma(W_p,\mathcal{G}_{6}^{\text{inv}})
\]

to

\[
\ker \left( \bigoplus_{j=1}^{v_p}(\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})/\Gamma(W_p,\mathcal{G}_{6,b_i,j,\chi})) \right) / \Gamma(W_p,\mathcal{G}_{6}^{\text{inv}})
\]

surjectively, and the kernel of this composition contains the one-dimensional kernel \( \kappa_{2,p}^{\text{op}} \nabla_{2,0} \) since \( \psi_p \) induces the map

\[
\varphi_p \left( \bigoplus_{j=1}^{v_p}(\Gamma(W_p,\mathcal{G}_{2,p}^{\text{inv}})/\Gamma(W_p,\mathcal{G}_{6,b_i,j,\chi})) \right) / \Gamma(W_p,\mathcal{G}_{6}^{\text{inv}})
\]
\[
\left( \bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_6^{\text{inv}}) / \Gamma(W_p, G_{6,p,j}) \right) / \Gamma(W_p, G_6^{\text{inv}})
\]
via \(\kappa_{2,p}\) and \(\kappa_{6,p}\). So we have to find the preimage
\[
\nabla_{2,0} \cap \psi_p^{-1} \left( \kappa_{6,p}^{-1} \left( \bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_{6,p,j}) \right) + \Gamma(W_p, G_6^{\text{inv}}) \right) \).
\]
This is illustrated by the following commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_6^{\text{inv}}) & \xrightarrow{\text{canonical projection}} & \bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_6^{\text{inv}}) \\
\nabla_{2,0} \cap \psi_p^{-1} \Gamma(W_p, G_{6,p,j}) & & \psi_p \cap \nabla_{2,0} \cap \psi_p^{-1} \Gamma(W_p, G_{6,p,j}) \\
\dim \ker = 1 & & \kappa_{6,p} \cap \nabla_{2,0} \cap \psi_p^{-1} \Gamma(W_p, G_{6,p,j})
\end{array}
\]

**Lemma 39.** Let \(g = (0, 0, 0, g[2]_1, g[2]_2, v[2], \ldots, g[v_p]_1, g[v_p]_2, v[v_p]) \in \nabla_{2,0} \). Suppose that \(\kappa_{6,p}(\psi_p(g)) \in \left( \bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_{6,p,j}) \right) + \Gamma(W_p, G_6^{\text{inv}}) \), where the last summand is embedded into \(\bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_{6,p,j})\) diagonally. Pick two vertices \(V_{p,j_1}\) and \(V_{p,j_2}\) of \(\Delta_p\) and denote \(i_1 = 1\), \(i_2 = 1\). Also choose \(\chi \in \{\lambda_1, \ldots, \lambda_m\}\) and an index \(k (1 \leq k \leq \dim \mathcal{O}(\chi))\).

Then it is possible to write \(\psi_{p,j_1,\chi,k}(g) - \mu_{2,1,\chi,2,2,k}(g)\) as \(f[j_1]_{X,k} - \mu_{2,1,1,\chi,j_2}\), where \(f_{j_1,\chi,k} \in \Gamma(W_p, \mathcal{O}_{P,j_1})\) for \(j = j_1, j_2\).

**Proof.** Since \(\kappa_{6,p}(\psi_p(g)) \in \left( \bigoplus_{j=1}^{\nu_p} \Gamma(W_p, G_{6,p,j}) \right) + \Gamma(W_p, G_6^{\text{inv}})\), \(\psi_p(g)\) can be written as \(f + f'\), where

\[
f = (f[j]_{X,k})_{1 \leq j \leq \nu_p, \chi \in \{\lambda_1, \ldots, \lambda_m\}, 1 \leq k \leq \dim \mathcal{O}(\chi)} \in \bigoplus_{j=1}^{\nu_p} G_{6,p,j},
\]

\[
f' = (f'_{j',X,k})_{1 \leq j' \leq \nu_p, \chi \in \{\lambda_1, \ldots, \lambda_m\}, 1 \leq k \leq \dim \mathcal{O}(\chi)} \in \bigoplus_{j=1}^{\nu_p} G_{6,p,j},
\]

and

\[
\kappa_{6,p}((f[j]_{X,k})_{\chi \in \{\lambda_1, \ldots, \lambda_m\}, 1 \leq k \leq \dim \mathcal{O}(\chi)})
\]
does not depend on \(j\). By the definition of \(G_{6,p,j}\), \(f[j]_{X,k} \in \Gamma(W_p, \mathcal{O}_{P,j})\) for all \(j\). It also follows from Lemma 20 that \(f'_{j',X,k} = \mu_{2,1,1,\chi,j_2\chi,k}\). Thus, \(\psi_{p,j_1,\chi,k}(g) - \mu_{2,1,\chi,j_2,j_1\chi}(g) = (f[j_1]_{X,k} + f'_{j_1,X,k}) - \mu_{2,1,\chi,j_2,j_1\chi}\).

**Corollary 9.** If the hypothesis of Lemma 39 holds, then \(\ord_p(\psi_{p,j_1,\chi,k}(g) - \mu_{2,1,\chi,j_2,j_1\chi}(g)) \geq \min(0, \ord_p(\mu_{2,1,\chi}))\).

**Corollary 10.** Suppose that the hypothesis of Lemma 39 holds. Let \(\chi = a_{1,1}\beta_{1,1} + a_{1,2}\beta_{1,2} = a_{2,1}\beta_{2,1} + a_{2,2}\beta_{1,2}\), and let \(f \in \mathcal{O}(\Delta(\chi))\). Then

\[
\ord_p \left( \frac{\gamma}{h_{1,1}} + \frac{\gamma}{h_{1,2}} \right) v_{j_1} - \mu_{2,1,\chi} \gamma \left( \frac{\gamma}{h_{1,1}} + \frac{\gamma}{h_{1,2}} \right) v_{j_2} \right) \geq \min(0, \ord_p(\mu_{2,1,\chi})).
\]
Proof. Observe that the function under the ord sign in the left-hand side of the inequality is linear in \( f \), and the right-hand side does not depend on \( f \), so it is sufficient to prove the inequality for all functions \( f \) forming a basis of \( \mathcal{O}(\mathcal{D}(\chi)) \). For example, we can use the functions of degree \( \chi \) among the generators of \( \mathbb{C}[X] \) we have chosen to define the map \( \psi \) for Theorem [1]. Recall that we have denoted these generators by \( x_{\chi,1}, \ldots, x_{\chi, \text{dim} \mathcal{O}(\mathcal{D}(\chi))} \) and that they form a basis of \( \mathcal{O}(\mathcal{D}(\chi)) \). So, set \( f = x_{\chi,k} \). By Lemma 22.

\[
\psi_{p,j_1, \chi, k}(g) - \mu_{i_2, i_1, \chi} \psi_{p,j_2, \chi, k}(g) = \frac{x_{\chi, k}}{h_{i_1,1}} \frac{x_{\chi, k}}{h_{i_1,2}} (a_{1,1} g[j_1] + a_{1,2} g[j_2]) + d \left( \frac{x_{\chi, k}}{h_{i_1,1}} \frac{x_{\chi, k}}{h_{i_1,2}} \right) v_{j_1} - \mu_{i_2, i_1, \chi} \frac{x_{\chi, k}}{h_{i_1,1}} \frac{x_{\chi, k}}{h_{i_1,2}} (a_{1,1} g[j_1] + a_{1,2} g[j_2]) + d \left( \frac{x_{\chi, k}}{h_{i_1,1}} \frac{x_{\chi, k}}{h_{i_1,2}} \right) v_{j_2} = \]

\[
\frac{x_{\chi, k}}{h_{i_1,1}} \frac{x_{\chi, k}}{h_{i_1,2}} (a_{1,1} g[j_1] + a_{1,2} g[j_1] - a_{1,2} g[j_1] - a_{1,2} g[j_2]) + d \left( \frac{x_{\chi, k}}{h_{i_1,1}} \frac{x_{\chi, k}}{h_{i_1,2}} \right) v_{j_1} - \mu_{i_2, i_1, \chi} d \left( \frac{x_{\chi, k}}{h_{i_1,1}} \frac{x_{\chi, k}}{h_{i_1,2}} \right) v_{j_2} = \]

and the claim follows from Corollary 9. \( \square \)

Now we need more information about the behavior of \( \text{ord}_p(\mu_{p,j_2, i_{p,j_1}, \chi}) \) depending on \( j_1, j_2, \chi \). Here we perform arithmetic actions on vertices of \( \Delta_p \), they are understood as arithmetic actions in \( N \).

Lemma 40. For each degree \( \chi \) and for any two vertices \( V_{p,j_1}, V_{p,j_2} \) \((1 \leq j_1, j_2 \leq v_p)\) one has \( \text{ord}_p(\mu_{p,j_2, i_{p,j_1}, \chi}) = \chi(V_{p,j_1} - V_{p,j_2}) \).

Proof. Again denote \( i_1 = i_{p,j_1}, i_2 = i_{p,j_2} \). We chose \( h_{i_1,1}, h_{i_1,2}, h_{i_1,1}, h_{i_1,2} \) so that \( \text{ord}_p(h_{i_1,1}) = -D_p(\beta_{1,1}), \text{ord}_p(h_{i_1,2}) = -D_p(\beta_{1,2}) \), \( \text{ord}_p(h_{i_1,1}) = -D_p(\beta_{1,1}), \text{ord}_p(h_{i_1,2}) = -D_p(\beta_{1,2}) \). Write \( \chi \) as \( \chi = a_{1,1} \beta_{1,1} + a_{1,2} \beta_{1,2} = a_{2,1} \beta_{1,1} + a_{2,2} \beta_{1,2} \), where \( a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2} \in \mathbb{Z} \). By the definition of \( \mu_{i_2, i_1, \chi} \), we have

\[
\text{ord}_p(\mu_{i_2, i_1, \chi}) = \text{ord}_p \left( \frac{h_{i_1,1}^{-2} h_{i_1,2}^{-2}}{h_{i_1,1} h_{i_1,2}} \right) = \]

\[
a_{1,1} D_p(\beta_{1,1}) + a_{2,1} D_p(\beta_{1,2}) = \text{ord}_p(\mu_{i_2, i_1, \chi}) = \chi(V_{p,j_1} - V_{p,j_2}). \]

\( \square \)

Lemma 41. Let \( E_{p,j} \) be a finite edge of \( \Delta_p \) \((1 \leq j < v_p)\), let \( \chi = \theta(N(E_{p,j}, \Delta_p)) \). Choose \( \chi' \in \mathbb{N}(V_{p,j}, M) \) so that \( \chi \) and \( \chi' \) form a lattice basis of \( M \). Then \( \chi(V_{p,j} - V_{p,j+1}) = 0 \) and \( \chi'(V_{p,j} - V_{p,j+1}) = 0 \).

Proof. Since \( \chi \in \mathbb{N}(E_{p,j}, \Delta_p) \), the minimum \( \text{min}_a \in \Delta_p(\chi(a)) \) is attained at both \( a = V_{p,j} \) and \( a = V_{p,j+1} \), so \( \chi(V_{p,j}) = \chi(V_{p,j+1}) \), \( \chi(V_{p,j} - V_{p,j+1}) = 0 \), and \( \chi((1/E_{p,j}))(V_{p,j} - V_{p,j+1}) = 0 \). It follows from the definition of \( E_{p,j} \) that \((1/E_{p,j}))(V_{p,j} - V_{p,j+1}) \) is a primitive lattice vector. Hence, elements of \( M \) can take arbitrary values at it. Since \( \chi \) and \( \chi' \) form a lattice basis of \( M \) and \( \chi((1/E_{p,j}))(V_{p,j} - V_{p,j+1}) = 0 \), we conclude that \( \chi'(1/E_{p,j}))(V_{p,j} - V_{p,j+1}) = \pm 1 \), and \( \chi'(V_{p,j} - V_{p,j+1}) = \pm 1(\theta_p) \). But the minimum \( \text{min}_a \in \Delta_p \chi'(a) \) is attained at \( V_{p,j} \) since \( \chi' \in \mathbb{N}(V_{p,j}, \Delta_p), \) so \( \chi(V_{p,j} - V_{p,j+1}) = -1 \). \( \square \)
Lemma 43. Let \( V_{p,j} \) be a vertex of \( \Delta_p \), let \( E_{p,j} \) be a finite edge of \( \Delta_p \) (1 \( \leq j < v_p \)), and suppose that \( j_1 \leq j_2 \). Pick a degree \( \chi'' \in N(V_{p,j_1}, \Delta_p) \). Suppose that \( \chi'' \notin N(V_{p,j_2+1}, \Delta_p) \). (Note that the contrary is possible since we allow \( j_1 = j_2 \).)

Then \( \chi''(V_{p,j_2} - V_{p,j_2+1}) \leq -|E_{p,j_2}| \).

Proof. Let \( \chi = b(N(E_{p,j_2}, \Delta_p)) \). Let \( \chi' \in N(V_{p,j_2}, \Delta_p) \) be a degree such that \( \chi \) and \( \chi' \) form a lattice basis of \( M \). Then \( \chi(V_{p,j_2+1} - V_{p,j_2}) = 0 \) and \( \chi'((V_{p,j_2+1} - V_{p,j_2}) = -|E_{p,j_2}| \).

The following lemma can be proved completely similarly to Lemma 41, one only has to interchange \( V_{p,j_2+1} \) and \( V_{p,j_2} \).

Lemma 44. Let \( V_{p,j_2} \) be a vertex of \( \Delta_p \), let \( E_{p,j} \) be a finite edge of \( \Delta_p \) (1 \( \leq j < v_p \)), and suppose that \( j_1 > j_2 + 1 \). Pick a degree \( \chi'' \in N(V_{p,j_1}, \Delta_p) \). Suppose that \( \chi'' \notin N(V_{p,j_2}, \Delta_p) \). Then \( \chi''(V_{p,j_2} - V_{p,j_2+1}) \leq -|E_{p,j_2}| \).

Proof. Fix an index \( j, 2 \leq j \leq v_p \). For simplicity of notation, denote \( g[1] = g[1] = 0, v[1] = 0 \). Set \( \chi = b(N(E_{p,j_1}, \Delta_p)) \). It follows from the choice of the degrees \( \lambda_1, \ldots, \lambda_m \) above that \( \chi \in \{ \lambda_1, \ldots, \lambda_m \} \).

Choose integers \( a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2} \in \mathbb{Z} \) such that \( \chi = a_{1,1} \beta_{j-1} + a_{1,2} \beta_j = a_{2,1} \beta_{j+1} + a_{2,2} \beta_{j+2} \).

By Lemma 24 there exists a function \( f \in O(\Delta(\chi)) \) defined at all ordinary points such that \( \text{ord}_p(\overline{f}) = -D_p(\chi) \). \( \chi \) is in the interior of \( \sigma' \), so \( \text{deg}(\Delta(\chi)) > 0 \), while \( \text{deg}(\partial(\chi)) = 0 \). Hence, there exists a point \( p_2 \in \mathbb{P}^1 \) such that \( \text{ord}_p(f) > -D_p(\chi) \). Choose a rational function \( f' \) on \( \mathbb{P}^1 \) that has exactly one zero of order one at \( p_2 \) and exactly one pole of order one at \( p' \). Then \( f' f \in O(D(\Delta(\chi))) \). Note also that \( df' \) is regular at \( p \) and \( d_p f' \neq 0 \). Set \( f'' = (1 + f') f \in O(D(\Delta(\chi))) \). Then \( f'' \) is also defined at all ordinary points, and \( \text{ord}_p(\overline{f''}) = -D_p(\chi) \).

Since \( D_p(\cdot) \) is linear on \( N(V_{p,j_1}, \Delta_p) \), \( D_p(\chi) = a_{1,1} D_p(\beta_{j-1}) + a_{1,2} D_p(\beta_{j+1}) + a_{2,1} D_p(\beta_{j+1}) + a_{2,2} D_p(\beta_{j+2}) \). According to the choice of the functions \( h_{1,i}, h_{2,i} \) for all indices \( i \), we have \( -D_p(\chi) = a_{1,1} \text{ord}_p(h_{p,j_1-1}) + a_{2,2} \text{ord}_p(h_{p,j_2+2}) \). Denote \( i_1 = i_{p,j_1-1}, i_2 = i_{p,j_2} \). We have

\[
\text{ord}_p\left(\frac{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}}{f^{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}}}\right) = \text{ord}_p\left(\frac{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}}{f^{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}}}\right) = 0,
\]

and it follows from Corollary 11 that

\[
\text{ord}_p \left( (a_{1,1} g[j - 1] + a_{1,2} g[j - 1]) - a_{2,1} g[j - 2] - a_{2,2} g[j] + \frac{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}}{f^{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}}} d \left( \frac{7}{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}} v[j] \right) \right) \geq 0
\]

and

\[
\text{ord}_p \left( (a_{1,1} g[j - 1] + a_{1,2} g[j - 1]) - a_{2,1} g[j - 2] - a_{2,2} g[j] + \frac{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}}{f^{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}}} d \left( \frac{7}{h_{1,1}^{a_{1,1}} h_{1,2}^{a_{1,2}}} v[j] \right) \right) \geq 0
\]

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Now we can rewrite the following function is regular at $d$:

$$-\frac{\mu_{i_2,i_1,\chi}}{\mu_{i_2,i_1}d} \left( \frac{\nabla}{\mu_{i_2,i_1}} \right) v[j] \geq \min(0, \text{ord}_{p}(\mu_{i_2,i_1,\chi})).$$

By Lemma 11, $\text{ord}_{p}(\mu_{i_2,i_1,\chi}) = 0$. Hence, these two functions under the ord signs are regular at $p$. Subtract the expressions under the ord signs and substitute the definition of $\mu_{i_2,i_1,\chi}$. We see that the following function is regular at $p$:

$$\frac{h_{i_1}}{h_{i_2}} d \left( \frac{\nabla}{h_{i_1}} \right) v[j - 1]$$

When we deal with elements of $\mathcal{O}(\Delta)$, we can rewrite this as:

$$\frac{\nabla}{f} d \left( \frac{\nabla}{f} \right) v[j] = -\frac{\nabla}{f} d \left( \frac{\nabla}{f} \right) v[j] - \frac{\nabla}{f} d \left( \frac{\nabla}{f} \right) v[j] = \frac{\nabla}{f} d \left( \frac{\nabla}{f} \right) v[j] - \frac{\nabla}{f} d \left( \frac{\nabla}{f} \right) v[j].$$

Now we can rewrite $d(\nabla/f)$ as $d(\nabla/f)/f = df'/f$. As we noted before, $df'$ does not have a zero or a pole at $p$. We have chosen $f$ and $f''$ so that $\text{ord}_{p}(\nabla/f) = \text{ord}_{p}(\nabla/f'')$, hence $\nabla/f''$ does not have a zero or a pole at $p$ either. We conclude that $v[j - 1] - v[j]$ is regular at $p$.

Now recall that $v[1] = 0$, therefore $v[j]$ is regular at $p$ for every $j$. Finally, it follows from the definition of $\nabla_{2,0}$ that if $v[j]$ is regular at $p$, then $v[j] = 0$.



Now we can reformulate Corollary 11 as follows:

**Corollary 11.** Let $g = (0, 0, 0, g[2]_1, g[2]_2, \ldots, g[v_p], 0) \in \nabla_{2,0}$. Suppose that $\kappa_{2,0}(\psi_g) \in \left( \bigoplus_{i=1}^{v_p} \Gamma(V_p, G_{i,2}) \right) + \Gamma(V_p, G_{i,2}) p$. Pick two vertices $V_p,j$ and $V_p,j_2$ of $\Delta_p$ and denote $i_1 = i_p,j_2$, $i_2 = i_p,j_2$. Also choose $\chi \in \{\lambda_1, \ldots, \lambda_m\}$ and write it as $\chi = a_1\beta_{i_1} + a_2\beta_{i_2}$. Let $f \in \mathcal{O}(\Delta(\chi))$ be an arbitrary function.

Then

$$\text{ord}_{p} \left( \frac{\nabla}{h_{i_1}} \right) (a_1 g[j]-1 + a_2 g[j]2 - a_2 g[j]2 - a_2 g[j]2) \geq \min(0, \text{ord}_{p}(\mu_{i_2,i_1,\chi})).$$

When we deal with elements of $\nabla_{2,0}$ such that all vector fields $v[j]$ are zeros, it is more convenient to use $U_{g}$-descriptions instead of $U_{p,j}$-descriptions. So denote by $\nabla_{2,1}$ the space of $2v_p$-tuples of the form $g[j] = \sum_{k=-n_{ij}}^{a_{ij}} a_{ij} (t - t_p)^k$. 



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Denote by $\rho: \nabla_{2,1} \to \nabla_{2,0}$ the map that computes $U_{i_p}$-descriptions out of $U_q$-descriptions, i.e. $\rho(0,0,g[2]_1,g[2]_2,\ldots, g[v_{p,1}], g[v_{p,2}]) = (0,0,0, g[2]_1, g[2]_2, 0, \ldots, g[v_{p,1}], g[v_{p,2}], 0)$, where

$$ \begin{pmatrix} g[j'_1]_1 \\ g[j'_2]_1 \\ 0 \end{pmatrix} = C_{q,i_p,j} \begin{pmatrix} g[j]_1 \\ g[j]_2 \\ 0 \end{pmatrix}.$$ 

In other words,

$$ \begin{pmatrix} g[j'_1] \\ g[j'_2] \end{pmatrix} = C_{q,i_p} \begin{pmatrix} g[j]_1 \\ g[j]_2 \end{pmatrix}.$$ 

Clearly, $\rho$ is injective. It also follows from Lemma 15 that $\rho(\nabla_{2,1})$ contains $\nabla_{2,0} \cap \psi_p^{-1}(\kappa_{\rho,p}^{-1}(\bigoplus_{i=1}^{\nu_p} \Gamma(W_p,G_0^{\text{inv}})))$. So now we are going to find the following preimage:

$$ \psi_p^{-1}(\kappa_{\rho,p}^{-1}(\bigoplus_{i=1}^{\nu_p} \Gamma(W_p,G_0^{\text{inv}})))) \cap (\bigoplus_{i=1}^{\nu_p} \Gamma(W_p,G_0^{\text{inv}})).$$

**Lemma 46.** Let $g = (0,0,g[2]_1,g[2]_2,\ldots, g[v_{p,1}], g[v_{p,2}]) \in \nabla_{2,1}$ be such that $\kappa_{\rho,p}(\psi_p(g)) \subseteq (\bigoplus_{i=1}^{\nu_p} \Gamma(W_p,G_0^{\text{inv}}))$. Pick two vertices $V_{p,j_1}$ and $V_{p,j_2}$ of $D_p$, choose $\chi \in (\lambda_1, \ldots, \lambda_m) \cap N(V_{p,j_1}, \Delta_p)$, and write it as $\chi = b_1 \beta_{q_1} + b_2 \beta_{q_2}$.

Then

$$ \text{ord}_p(b_1(g[j'_1] - g[j'_2]) + b_2(g[j]_1 - g[j]_2)) = \chi(V_{p,j_1} - V_{p,j_2}).$$

**Proof.** Denote $i_1 = i_{p,j_1}$, $i_2 = i_{p,j_2}$. Recall that by the definition of $C_0^{i_1}$, $\beta_{q_1} = (C_0^{i_1})_{11} \beta_{1,1} + (C_0^{i_1})_{12} \beta_{1,2}$ and $\beta_{q_2} = (C_0^{i_2})_{21} \beta_{2,1} + (C_0^{i_2})_{22} \beta_{2,2}$. Hence, $\chi = b_1 \beta_{q_1} + b_2 \beta_{q_2} = b_1((C_0^{i_1})_{11} \beta_{1,1} + (C_0^{i_1})_{12} \beta_{1,2}) + b_2((C_0^{i_2})_{21} \beta_{2,1} + (C_0^{i_2})_{22} \beta_{2,2})$. Denote $a_{1,1} = b_1((C_0^{i_1})_{11} + b_2((C_0^{i_2})_{11} + b_2((C_0^{i_2})_{22} \beta_{2,2})$. Then

$$ \chi = a_{1,1} \beta_{1,1} + a_{2,2} \beta_{2,2}.$$ 

Similarly, $\chi = a_{1,2} \beta_{1,2} + a_{2,2} \beta_{2,2}$, where $a_{2,2} = b_1((C_0^{i_1})_{11} + b_2((C_0^{i_2})_{12} \beta_{1,2}) + b_2((C_0^{i_2})_{22} \beta_{2,2})$. Observe that the definitions of $a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}$ can be written using matrices as follows:

$$ \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ a_{1,1} & a_{1,2} \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Denote

$$ \begin{pmatrix} g[j'_1] \\ g[j'_2] \\ g[j]_1 \\ g[j]_2 \end{pmatrix} = C_{q,i} \begin{pmatrix} g[j]_1 \\ g[j]_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g[j'_1] \\ g[j'_2] \end{pmatrix} = C_{q,i} \begin{pmatrix} g[j]_1 \\ g[j]_2 \end{pmatrix}.$$ 

Then by Lemma 15

$$ \begin{pmatrix} g[j'_1] \\ g[j'_2] \end{pmatrix} = C_{q,i} \begin{pmatrix} g[j]_1 \\ g[j]_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g[j'_1] \\ g[j'_2] \end{pmatrix} = C_{q,i} \begin{pmatrix} g[j]_1 \\ g[j]_2 \end{pmatrix}.$$ 

We can write

$$ b_1(g[j'_1] - g[j'_2]) + b_2(g[j]_1 - g[j]_2) = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} g[j'_1] - g[j'_2] \\ g[j]_1 - g[j]_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} g[j]_1 \\ g[j]_2 \end{pmatrix}.$$ 

By Lemma 23 there exists $f \in O(D(\chi))$ such that $\text{ord}_p(f) = -D_p(\chi)$. Since $\beta_{q_{1,1}, \beta_{q_{1,2}}, \chi} \in N(V_{p,j_1}, \Delta_p)$, $D_p(\chi)$ is linear on $N(V_{p,j_1}, \Delta_p)$, and $\chi = a_{1,1} \beta_{1,1} + a_{2,2} \beta_{2,2}$, we can write $D_p(\chi) = a_{1,1} D_p(\beta_{1,1}) + a_{2,2} D_p(\beta_{2,2})$. We chose $h_{i,1}$ and $h_{i,2}$ so that $\text{ord}_p(h_{i,1}) = -D_p(\beta_{i,1})$, $\text{ord}_p(h_{i,2}) = -D_p(\beta_{i,2})$. Therefore,

$$ \text{ord}_p \begin{pmatrix} f(h_{i,1}) \\ f(h_{i,2}) \end{pmatrix} = 0.$$ 

By Lemma 10 $\text{ord}_p(\mu_{i_2,j_1,\chi}) = (V_{p,j_2})$. Since $\chi \in N(V_{p,j_1}, \Delta_p)$, $V_{p,j_2}$ is a point where $\chi$ attains its minimum on $\Delta_p$. Hence, $\text{ord}_p(\mu_{i_2,j_1,\chi}) \leq 0$.
Now we are ready to formulate an exact description for \( \rho^{-1}(\psi^{-1}(\kappa_0^{-1}(\bigvee_{j=1}^p \Gamma(W_\mathbf{p}, G_6^{\text{inv}}))) + \Gamma(W_\mathbf{p}, G_6^{\text{inv}}))) \). Let \( \nabla_{2,2} \subseteq \nabla_{2,1} \) be the space of \( 2\mathbf{p}\)-tuples of the form \((g[1], g[1], g[2], g[2], g[2], \ldots, g[\mathbf{p}], g[\mathbf{p}])\) such that

1. \( g[j][k] \) is a Laurent polynomial in \((t - t_p)\) with no terms of nonnegative degree.
2. \( g[1][1] = g[1][2] = 0. \)
3. If \( a_1 \beta_{q,1} + a_2 \beta_{q,2} = \mathbf{b}(N(\mathbf{E}_p, j, \Delta_p)) \), where \( \mathbf{E}_p, j \) is a finite edge, then \( a_1(g[j][1] - g[j+1][1]) + a_2(g[j][2] - g[j+1][2]) = 0. \)
4. \( \text{ord}_p(g[j][1] - g[j+1][1]) \geq -|\mathbf{E}_p, j|, \text{ord}_p(g[j][2] - g[j+1][2]) \geq -|\mathbf{E}_p, j| \) for all finite edges \( \mathbf{E}_p, j \) \((1 \leq j < \mathbf{p}). \)

**Proposition 9.** \( \nabla_{2,2} = \rho^{-1}(\psi^{-1}(\kappa_0^{-1}(\bigvee_{j=1}^p \Gamma(W_\mathbf{p}, G_6^{\text{inv}}))) + \Gamma(W_\mathbf{p}, G_6^{\text{inv}}))) \).

**Proof.** The inclusion \( \nabla_{2,2} \supseteq \rho^{-1}(\psi^{-1}(\kappa_0^{-1}(\bigvee_{j=1}^p \Gamma(W_\mathbf{p}, G_6^{\text{inv}}))) + \Gamma(W_\mathbf{p}, G_6^{\text{inv}}))) \) follows easily from Lemmas I1 and II1. Namely, let \( g = (g[1], g[1][2], g[2], \ldots, g[\mathbf{p}], g[\mathbf{p}]) \in \rho^{-1}(\psi^{-1}(\kappa_0^{-1}(\bigvee_{j=1}^p \Gamma(W_\mathbf{p}, G_6^{\text{inv}}))) + \Gamma(W_\mathbf{p}, G_6^{\text{inv}}))) \). Properties II1 and III1 in the definition of \( \nabla_{2,1} \) follow from the definition of \( \nabla_{2,1} \). Fix a finite edge \( \mathbf{E}_p, j \), \( 1 \leq j < \mathbf{p}. \) Let \( \chi = \mathbf{b}(N(\mathbf{E}_p, j, \Delta_p)) \), write \( \chi = a_1 \beta_{q,1} + a_2 \beta_{q,2}. \) According to our choice of the set \( \{\lambda_1, \ldots, \lambda_m\} \), \( \chi \in \{\lambda_1, \ldots, \lambda_m\}. \) There also exists a degree \( \chi' \in \{\lambda_1, \ldots, \lambda_m\} \) such that \( \chi' \in N(\mathbf{E}_p, j, \Delta_p) \) and \( \chi' \) form a basis of \( M. \) Write \( \chi' = a_1 \beta_{q,1} + a_2 \beta_{q,2}. \) By Lemma III1, \( \text{ord}_p(a_1(g[j][1] - g[j+1][1]) + a_2(g[j][2] - g[j+1][2])) \geq \chi(\mathbf{V}_p, \mathbf{V}_p, j+1). \) By Lemma II1, \( \chi(\mathbf{V}_p, \mathbf{V}_p, j+1) = 0, \) in other words, \( a_1(g[j][1] - g[j+1][1]) + a_2(g[j][2] - g[j+1][2]) = 0. \) for a degree zero polynomial \( p. \) On the other hand, it is a Laurent polynomial whose terms of nonnegative degree are zero, so \( a_1(g[j][1] - g[j+1][1]) + a_2(g[j][2] - g[j+1][2]) = 0. \) Now, using Lemmas I1 and II1, again, we see that \( \text{ord}_p(a_1(g[j][1] - g[j+1][1]) + a_2(g[j][2] - g[j+1][2])) \geq -\chi(\mathbf{E}_p, j). \) Since \( \beta_{q,1}, \beta_{q,2} \) form a basis of \( M, \) \( a_1 \beta_{q,1} + a_2 \beta_{q,2} \) and \( \chi = a_1 \beta_{q,1} + a_2 \beta_{q,2} \) also form a basis of \( M, \) \((a_1, a_2)\) and \((a'_1, a'_2)\) form a basis of \( \mathbb{E}^2. \) We know that \( \text{ord}_p(a_1(g[j][1] - g[j+1][1]) + a_2(g[j][2] - g[j+1][2])) \geq -\chi(\mathbf{E}_p, j) \) and \( \text{ord}_p(a_1(g[j][1] - g[j+1][1]) + a_2(g[j][2] - g[j+1][2])) \geq -\chi(\mathbf{E}_p, j) \) and \( \text{ord}_p(a_1(g[j][1] - g[j+1][1]) + a_2(g[j][2] - g[j+1][2])) \geq -\chi(\mathbf{E}_p, j) \). So, the conditions II and III from the definition of \( \nabla_{2,2} \) hold, and \( g \in \nabla_{2,2}. \)

Now we are going to prove the other inclusion. Let \( g = (g[1], g[1][2], g[2], \ldots, g[\mathbf{p}], g[\mathbf{p}]) \in \nabla_{2,2}. \) We have to write \( \psi_\mathbf{p}(\rho(g)) \) as \( f + f', \) where

\[
f = (f[j][\chi][k])_{1 \leq j \leq \psi_\mathbf{p}, \chi \in \{\lambda_1, \ldots, \lambda_m\}, 1 \leq k \leq \dim \mathcal{O}(\mathcal{D}(\chi))} \bigoplus \mathbb{C}\mathbf{G}_6^p, j,
\]

\[
f' = (f'_j[j][\chi][k])_{1 \leq j \leq \psi_\mathbf{p}, \chi \in \{\lambda_1, \ldots, \lambda_m\}, 1 \leq k \leq \dim \mathcal{O}(\mathcal{D}(\chi))} \bigoplus \mathbb{C}\mathbf{G}_6^{\text{op}}, j,
\]

and

\[
\kappa_6, p, j, \chi \bigoplus \mathbb{C}\mathbf{G}_6^{\text{op}}, j
\]

does not depend on \( j. \) In other words, we have to find functions \( f[j][\chi][k] \) regular at \( p \) and functions \( f'_j[j][\chi][k] \) such that (see the definition of \( \kappa_6, p, j, \chi \bigoplus \mathbb{C}\mathbf{G}_6^{\text{op}}, j \))

\[
f_j[j][\chi][k] = \mu_{j, 2, j} f'_j[j][\chi][k] \text{ for each } j, k. \]

These conditions can be verified for different degrees \( \chi \) and different indices \( k \) independently, so fix a degree \( \chi = a_1 \beta_{q,1} + a_2 \beta_{q,2} \) and a generator \( x_{\chi, k} \) until the end of the proof.

The map \( \psip \) uses \( \mathcal{U}_{p, j, \mathbf{p}} \)-descriptions of functions and of vector fields on \( \mathcal{U}_q, \) but it follows from the definitions of \( \psi_\mathbf{p}, \) of \( \mathcal{U}_{p, j, \mathbf{p}} \)-description and of \( \mathcal{U}_q \)-description that instead of computing the \( (j, \chi, k) \)th component of \( \psi_p(\rho(g)) \) using \( \psi_\mathbf{p} \) and \( \rho \), we can first compute the \( \mathcal{U}_q \)-description of the derivative of \( x_{\chi, k} \) along the vector field on \( \mathcal{U}_q \) whose \( \mathcal{U}_q \)-description is \( (g[j][1], g[j][2], 0) \), and then use \( \mu_{q, \mathbf{p}, j, \chi, k} \) to compute the \( \mathcal{U}_{p, j, \mathbf{p}} \)-description of the function on \( \mathcal{U}_q \) whose \( \mathcal{U}_q \)-description we obtain this way. So, consider the \( \mathcal{U}_q \)-descriptions of the functions on \( \mathcal{U}_q \) whose \( \mathcal{U}_{p, j, \mathbf{p}} \)-descriptions are functions \( f[j][\chi][k] \) and \( f'_j[j][\chi][k] \) we are looking for. Denote these \( \mathcal{U}_q \)-descriptions by \( f[j][\chi][k]^{\text{op}} \) and \( f'_j[j][\chi][k]^{\text{op}} \), respectively (we do not use indices \( \chi \) and \( k \) here, because they are already fixed until the end of the proof, and we do not mean that these functions are the same for different \( \chi \) and \( k \)). In other words, \( f[j][\chi][k] = \mu_{q, \mathbf{p}, j, \chi} f[j][\chi][k]^{\text{op}} \) and \( f'_j[j][\chi][k] = \mu_{q, \mathbf{p}, j, \chi} f'_j[j][\chi][k]^{\text{op}}. \)
terms of these functions, we need to meet the following conditions: first, $\mu_{q_p,j_1,\chi} f[j]''$ should be regular at $p$ for each $j$, and second, $\mu_{q_p,j_1,\chi} f[j]'''$ and $\mu_{q_p,j_2,\chi} f[j]'''$ should be the $U_{q_1}^{-1}$ and $U_{q_2}^{-1}$-descriptions (respectively) of the same function defined on $U_q$. These conditions can be reformulated as follows: the inequality $\text{ord}_p(\mu_{q_p,j_1,\chi} f[j]'') \geq 0$ should hold, and all functions $f[j]'', f[j]'''$ should be the same function $f''$, which should not depend on $j$.

Let $j_1$ be the maximal index such that $\chi \in N(V_{p,j_1}, \Delta_p)$. (The convention that we take the maximal index is nontrivial if $\chi \in N(E_{p,j_1-1}, \Delta_p)$.) Fix this index $j_1$ until the end of the proof. Set

$$f'' = \frac{X_{\chi,k}}{h_{q_1} h_{q_2}} (a_1 g[j_1] + a_2 g[j_2]),$$

and for each $j_2$ ($1 \leq j_2 \leq v_p$) set

$$f[j_2]' = \frac{X_{\chi,k}}{h_{q_1} h_{q_2}} (a_1 g[j_2] + a_2 g[j_2] - f'').$$

Observe that $f[j_1]'' = 0$. By Lemma 22, $f[j_2]' + f''$ is the $U_q$-description of the derivative of $X_{\chi,k}$ along the vector field whose $U_q$-description is $(g[j_2], g[j_2], 0)$. It is sufficient to prove that $\text{ord}_p(\mu_{q_p,j_2,\chi} f[j_2]'') \geq 0$. We can write this function as follows (suppose that $\chi = b_1, b_2 b_{p,j_1}, 1 + b_{1,2} b_{p,j_2}, 2 + 2 b_{2,2} b_{p,j_2}, 2$):

$$\mu_{q_p,j_2,\chi} f[j_2]' = \frac{X_{\chi,k}}{h_{p,j_1} h_{p,j_2} h_{q_1} h_{q_2}} (a_1 (g[j_2] - g[j_2]) + a_2 (g[j_2] - g[j_2]) =$$

$$\frac{X_{\chi,k}}{h_{p,j_1} h_{p,j_2} h_{q_1} h_{q_2}} (a_1 (g[j_2] - g[j_2]) + a_2 (g[j_2] - g[j_2]) =$$

$$\frac{X_{\chi,k}}{h_{p,j_1} h_{p,j_2} h_{q_1} h_{q_2}} (a_1 (g[j_2] - g[j_2]) + a_2 (g[j_2] - g[j_2]) =$$

$$\frac{X_{\chi,k}}{h_{p,j_1} h_{p,j_2} h_{q_1} h_{q_2}} (a_1 (g[j_2] - g[j_2]) + a_2 (g[j_2] - g[j_2]) =$$

Since $X_{\chi,k} \in O(D(\chi))$, $\text{ord}_p(X_{\chi,k}) \geq -D_p(\chi)$. We chose $h_{p,j_1,1}$ and $h_{p,j_2,2}$ so that $\text{ord}_p(h_{p,j_1,1}) = -D_p(\beta_{p,j_1,2})$. We know that $\chi = b_1, b_2 b_{p,j_1}, 1 + b_{1,2} b_{p,j_2}, 2, \chi, b_{p,j_1}, 1, \beta_{p,j_2}, 2 \in N(V_{p,j_1}, \Delta_p)$, and $D_p(\cdot)$ is linear on $N(V_{p,j_1}, \Delta_p)$, therefore $\text{ord}_p(h_{p,j_1,1} h_{p,j_2,2}) = -b_1, D_p(\beta_{p,j_1,2} - b_1, D_p(\beta_{p,j_2,2} = -D_p(\chi)$. Hence,

$$\text{ord}_p\left(\frac{X_{\chi,k}}{h_{p,j_1,1} h_{p,j_2,2}}\right) \geq 0.$$

So, now we are done for $j_2 = j_1$. Otherwise, we have to consider two cases: $j_2 > j_1$ and $j_2 < j_1$. Suppose first that $j_2 > j_1$. Then $\mu_{q_p,j_2,\chi} f[j_2] = \chi (a_1 (g[j_2] - g[j_2]) + a_2 (g[j_2] - g[j_2]) = \mu_{q_p,j_2,\chi}$, $\chi (a_1 (g[j_2] - g[j_2]) + a_2 (g[j_2] - g[j_2]) + a_1 (g[j_2] - g[j_2]) + a_2 (g[j_2] - g[j_2]) = \chi (V_{p,j_1} - V_{p,j_2})$. By Lemma 22, $\text{ord}_p(\mu_{q_p,j_2,\chi}) = -D_p(\chi)$. Since $\chi \notin N(V_{p,j_1}, \Delta_p)$ for all $j > j_1$, by Lemma 22 we have $\text{ord}_p(\mu_{q_p,j_2,\chi}) = D_p(\chi)$. This sum contains at least one summand since $j_2 > j_1$. By the definition of $V_{p,j_2}$, $\text{ord}_p(a_1 (g[j_2] - g[j_2]) + a_2 (g[j_2] - g[j_2]) + a_1 (g[j_2] - g[j_2]) + a_2 (g[j_2] - g[j_2]) = \text{ord}_p(\mu_{q_p,j_2,\chi}) = -D_p(\chi)$.

Now consider the case $j_2 < j_1$. This time we are going to consider indices smaller than $j_1$, and it is possible that $\chi \in N(V_{p,j_1}, \Delta_p)$ for some $j < j_1$, namely for $j = j_1 - 1$ (and this is the
only possibility). So, we have to consider two cases: \( \chi \notin \mathcal{N}(V_{p,j-1}, \Delta_p) \) and \( \chi \in \mathcal{N}(V_{p,j-1}, \Delta_p) \). Suppose first that \( \chi \notin \mathcal{N}(V_{p,j-1}, \Delta_p) \). Then we can again write\( \mu_{p,j-1,j',\chi}(a_1(g[j_1]_1 - g[j_1]_1) + a_2(g[j_2]_2 - g[j_2]_2)) = \mu_{p,j-1,j',\chi}(a_1(g[j_1]_1 - g[j_1]_1) + a_2(g[j_2]_2 - g[j_2]_2)) + \ldots + a_1(g[j_1]_1 - g[j_1]_1) + a_2(g[j_2]_2 - g[j_2]_2) \) for all \( j < j_1 \), we can apply Lemmas\( \text{[70]} \) and\( \text{[71]} \) We see that \( \text{ord}_{p}(\mu_{p,j-1,j',\chi}) = \chi(V_{p,j-1} - V_{p,j-1}) + \ldots + \chi(V_{p,j} - V_{p,j-1}) \geq [E_{p,j} - 1] + \ldots + [E_{p,j} - 1] \). And again, by the definition of \( \nabla_{2,2}, \text{ord}_{p}(a_1(g[j_1]_1 - g[j_1]_1) + a_2(g[j_2]_2 - g[j_2]_2)) = \ldots + a_1(g[j_1]_1 - g[j_1]_1) + a_2(g[j_2]_2 - g[j_2]_2) \). Hence, \( \chi(V_{p,j-1} - V_{p,j-1}) + \ldots + \chi(V_{p,j} - V_{p,j-1}) \geq [E_{p,j} - 1] + \ldots + [E_{p,j} - 1] \). Therefore, \( \text{ord}_{p}(\mu_{p,j-1,j',\chi}) \geq [E_{p,j} - 1] + \ldots + [E_{p,j} - 1] \). Finally, consider the case when \( j_2 < j_1 \) and \( \chi \notin \mathcal{N}(V_{p,j-1}, \Delta_p) \), and property\( \text{[8]} \) in the definition of \( \nabla_{2,2} \) guarantees that \( a_1(g[j_1]_1 - g[j_1]_1) + a_2(g[j_2]_2 - g[j_2]_2) = 0 \). It also follows from Lemmas\( \text{[70]} \) and\( \text{[71]} \) that \( \text{ord}_{p}(\mu_{p,j-1,j',\chi}) = \chi(V_{p,j-1} - V_{p,j-1}) = 0 \). We conclude that \( \text{ord}_{p}(\mu_{p,j-1,j',\chi}) \geq [E_{p,j} - 1] + \ldots + [E_{p,j} - 1] \). The order of the other multiplier can be rewritten as \( \text{ord}_{p}(a_1(g[j_1]_1 - g[j_1]_1) + a_2(g[j_2]_2 - g[j_2]_2)) = \ldots + a_1(g[j_1]_1 - g[j_1]_1) + a_2(g[j_2]_2 - g[j_2]_2) \). Hence, \( \chi(V_{p,j-1} - V_{p,j-1}) \geq [E_{p,j} - 1] + \ldots + [E_{p,j} - 1] \). Again we see that \( \text{ord}_{p}(\mu_{p,j-1,j',\chi}) = \chi(V_{p,j-1} - V_{p,j-1}) \) is regular at \( p \).
5 Connections between the graded component of degree 0 of \( T^1(X) \) and graded components of \( T^1 \) of toric varieties

Given an affine toric 3-dimensional variety \( X \), one can restrict the space the action of the 3-dimensional torus to a 2-dimensional subtorus, and consider \( X \) as a 3-dimensional \( T \)-variety with an action of a 2-dimensional torus. Toric varieties are parametrized by pointed cones of the same dimension, and \( T \)-varieties are parametrized by polyhedral divisors as described in the Introduction. These two parametrizations are related via the following toric downgrade procedure.

Let \( X \) be an affine toric 3-dimensional variety defined by a pointed cone \( \tau \in N_\mathbb{Q} \) is parametrized by primitive vectors \( \chi \in M \). Fix one of them until the end of this section, denote it by \( \chi_0 \). We are going to consider the action of \( T = \ker \chi_0 \) on \( X \). To describe this action by a polyhedral divisor, choose a line \( N' \subset N \) complementary to \( N = \ker \chi_0 \). Consider also the projection from \( N \) to \( N' \cap N \subset N' \cap N' \subset N \). It maps each face of \( \tau \) surjectively onto a cone in \( N' \cap N \). Then the variety \( Y \), where the polyhedral divisor will be constructed, is defined by the coarsest fan in \( N' \cap N \) containing all these cones. It can be \( \mathbb{P}^1 \), \( \mathbb{C} \), or \( \mathbb{C}^* \), depending on whether the image of \( \tau \) is the whole line, a half-line, or a point, respectively. We are interested in the case \( Y = \mathbb{P}^1 \), so suppose in the sequel that it holds. It takes place if and only if \( N' \cap N \) separates \( \tau \) into two nonempty cones.

To construct the polyhedral divisor itself, recall that the two half-lines of \( N' \cap N \) correspond to the two fixed points of a torus acting on \( \mathbb{P}^1 \), which we can denote by 0 and \( \infty \). More exactly, let 0 (resp. \( \infty \)) correspond to the half-line \( \{ \chi_0 > 0 \} \) (resp. \( \{ \chi_0 < 0 \} \)). Then the polyhedral divisor contains nontrivial polyhedra at 0 and at \( \infty \), and the polyhedron at 0 (resp. \( \infty \)) is the projection of \( \tau \cap \{ \chi_0 = 0 \} \) (resp. of \( \tau \cap \{ \chi_0 = -1 \} \)) to \( N' \cap N \). As previously, denote these polyhedra by \( \Delta_0 \) and \( \Delta_{\infty} \). The tail cone of both of these polyhedra is \( \sigma = \tau \cap N' \cap N \). We only considered the cases when it was full-dimensional, and, together with the requirement \( Y = \mathbb{P}^1 \), this means that \( \sigma \) is full-dimensional.

The last requirement we had says that all vertices of \( \Delta_0 \) and \( \Delta_{\infty} \) have to be lattice points. Since \( \chi_0 \) is a primitive vector, \( \tau' \cap \{ \chi_0 = 1 \} \) and \( \tau' \cap \{ \chi_0 = -1 \} \) are lattice points, so the projections of the planes \( \{ \chi_0 = 1 \} \) and \( \{ \chi_0 = -1 \} \) onto \( N' \cap N \) along \( N' \cap N \) map lattice points to lattice points. Hence, the last condition we should impose says that if a one-dimensional face of \( \tau \) intersects one of the planes \( \{ \chi_0 = 1 \} \) and \( \{ \chi_0 = -1 \} \), then the intersection point is a lattice point.

Now we need some notation and terminology. Call an edge of \( \tau \) positive (resp. nonnegative, negative, nonpositive) if \( \chi_0 \) takes positive (resp. nonnegative, negative, nonpositive) values on this edge (except the origin). Call an edge of \( \tau \) orthogonal if \( \chi_0 \) takes only zero values in this edge. Call a facet of \( \tau \) positive (resp. negative) if \( \chi_0 \) takes only positive (resp. only negative) values on the interior of this facet. Denote the edges of \( \tau \) by \( E_i \) (resp. \( E_{i+1} \)) and the facets of \( \tau \) by \( F_i \) (resp. \( F_{i+1} \)). Sometimes we can write \( \chi_0 \) instead of \( \chi_0 \). We also require that \( \tau \) has a full-dimensional tail cone \( \sigma \). The notation \( \tau \) means that \( \sigma \) is full-dimensional.

It is also convenient to introduce some notation for positive and negative edges separately. Denote the number of positive edges by \( e^+(\tau) \). Denote the positive edges themselves by \( E^+_i(\tau) \). Here the edges are enumerated in the same order as when we enumerated all edges, i.e., \( E^+_i(\tau) = E_{i+1}(\tau) \) for \( 1 \leq i \leq e^+(\tau) \). Similarly, denote the number of negative edges by \( e^-\tau \) and denote the negative edges themselves by \( E^-_i(\tau) \). This time we reverse the order that we used when we enumerated all edges together. In other words, if \( E^-_i(\tau) = E_{i-1}(\tau) \) for some \( i \) (which can equal \( e^-\tau \) or \( e^-\tau + 1 \), then \( E^-_j(\tau) = E_{j-1}(\tau) \) for \( 1 \leq j \leq e^-\tau \). The notation \( E^+_i(\tau) \) may look a bit redundant, but it is convenient to have uniform notation for positive and negative edges.

Now let us introduce notation for positive and negative facets. Denote the facet whose boundary is \( E^-_i(\tau) \cup E^-_{i+1}(\tau) \) by \( F^-_i(\tau) \). Again we have \( F^-_i(\tau) = F_i(\tau) \) for \( 1 \leq i \leq e^-\tau \). Extend this notation as follows. First, set \( F^+_0(\tau) = F_0(\tau) \) and \( F^+_{e^+}(\tau) = F_{e^+(\tau)}(\tau) \). If \( E^+_i(\tau) = E_{i-1}(\tau) \), denote \( F^+_i(\tau) = F^-_i(\tau) \) and \( F^+_{i+1}(\tau) = F^-_{i+1}(\tau) \).
\[ F_{\epsilon^-(\tau)}(\tau) = F_{i-\epsilon^-(\tau)}(\tau). \]

**Remark 4.** This notation agrees with the notation for edges and vertices of \( \Delta_\tau \) we introduced in the beginning. Namely, within the notation that we introduced now, we have \( \epsilon^+(\tau) = v(\Delta_0) \), \( \epsilon^-(\tau) = E_i^\ast(\tau) \cap \{\chi_0 = 1\} \) for \( 1 \leq i \leq \epsilon^+(\tau) \), \( E_i(\Delta_0) = F_i^\ast(\tau) \cap \{\chi_0 = 1\} \) for \( 0 \leq i \leq \epsilon^+(\tau) \), \( \epsilon^-(\tau) = v(\Delta_\infty) \), \( \epsilon^+(\tau) = E_i^\ast(\tau) \cap \{\chi_0 = -1\} \) for \( 1 \leq i \leq \epsilon^+(\tau) \), and \( \epsilon^-(\tau) = v(\Delta_\infty) \), \( E_i^\ast(\tau) \cap \{\chi_0 = -1\} \) for \( 0 \leq i \leq \epsilon^+(\tau) \).

The faces of the cone \( \tau' \) dual to \( \tau \) put be set into bijection with the faces of \( \tau \). Namely, each face \( \tau' \) of \( \tau \) defines a face of \( \tau' \) consisting of all \( a \in \tau' \) such that \( a(\tau') = 0 \). We call this face of \( \tau' \) the **normal face** of \( \tau' \) and denote it by \( N(\tau', \tau) \). Clearly, the normal faces of edges are facets and vice versa.

A formula for the graded components of the first-order deformation space of a toric variety was given in [6]. To formulate it, we need to quote also some notation from [6]. (We slightly change the letters we use there to avoid confusion.) First, let \( \Lambda_1, \ldots, \Lambda_m \) be the Hilbert basis of \( \tau' \). If \( \tau' \) is an edge of \( \tau \), and \( \chi \in \tilde{M} \) is a degree, denote
\[
\Lambda^\chi_{\tau'} = \{ \tilde{\lambda}_i | \tilde{\lambda}_i(b(\tau')) < \chi(b(\tau')) \}.
\]

Now, if \( \tau' \) is a facet of \( \tau \), we set
\[
\Lambda^\chi_{\tau'} = \bigcap_{\tau'' \text{ is an edge of } \tau} \Lambda^\chi_{\tau''},
\]

and for the origin (which is also a facet of \( \tau \)) we set
\[
\Lambda^\chi_0 = \bigcup_{\tau' \text{ is an edge of } \tau} \Lambda^\chi_{\tau'}.
\]

Finally, we set
\[
\Lambda^{\chi,i} = \bigoplus_{\tau' \text{ is a face of } \tau} \text{Span}_{\tilde{M}}(\Lambda^\chi_{\tau'})
\]

for \( i = 0, 1, 2 \). Here \( \text{Span}_{\tilde{M}} \) denotes the sublattice of \( \tilde{M} \) generated by the subset of \( \tilde{M} \) under the \( \text{Span}_{\tilde{M}} \) sign. In the sequel we will also use notation \( \text{Span}_Q \) for the \( Q \)-linear subspace of \( \tilde{M}_Q = \tilde{M} \otimes \mathbb{Q} \) generated by a set of elements of \( \tilde{M} \) or of \( \tilde{M}_Q \). generated Consider the complex
\[
(\Lambda^{\chi,0} \otimes \mathbb{C})^* \to (\Lambda^{\chi,1} \otimes \mathbb{C})^* \to (\Lambda^{\chi,2} \otimes \mathbb{C})^*,
\]

where the maps are standard Cech differentials. Denote the graded component of \( T^1(X) \) of degree \( \chi \) by \( T^1_{\chi}(X) \).

**Theorem 5.** [6] Theorem 2.1]
\[
T^1_{\chi}(X) \cong H^1\left( (\Lambda^\chi \otimes \mathbb{C})^* \right).
\]

Our goal for this section is to deduce Theorem 4 in the case of toric \( X \) from Theorem 5. It is known [CITATION NEEDED] that the 0th graded component of \( X \) considered as a \( T \)-variety is isomorphic to
\[
\bigoplus_{\alpha \in \mathbb{Z}} T^1_{\alpha \chi}(X),
\]

where the degrees are understood with respect to the action of the three-dimensional torus. So, in the sequel we will study the spaces \( T^1_{\chi}(X) \), where \( \chi \) is a multiple of \( \chi_0 \).

**Lemma 47.** Let \( \chi \) be a multiple of \( \chi_0 \) and \( \tau' \) be an edge of \( \tau \). Then \( \Lambda^\chi_{\tau'} = \emptyset \) if one of the following conditions holds:

1. \( \chi = 0 \).
2. \( \tau' \) is an orthogonal edge.
3. \( \chi = a\chi_0 \), where \( a > 0 \), and \( \tau' \) is a negative edge.

4. \( \chi = a\chi_0 \), where \( a < 0 \), and \( \tau' \) is a positive edge.

**Proof.** Choose a Hilbert basis element \( \tilde{\lambda}_i \), where \( 1 \leq i \leq \tilde{m} \). Since \( \tilde{\lambda}_i \in \tau' \), we have \( \tilde{\lambda}_i(b(\tau')) \geq 0 \). On the other hand, \( \chi(b(\tau')) = 0 \) if case [1] or [2] from the above classification holds. If case [3] or [4] takes place, then \( \chi(b(\tau')) < 0 \). Hence, \( \tilde{\lambda}_i(b(\tau')) \geq \chi(b(\tau')) \), and \( \tilde{\lambda}_i \notin \Lambda_{\gamma}^{\tau} \).

**Corollary 12.** If \( \chi = 0 \), then \( \Lambda^{\chi,1} = 0 \) and \( T_0(\chi) = 0 \).

**Lemma 48.** If \( \tau' \) is a positive (resp. negative) edge of \( \tau \), then \( \chi_0(b(\tau')) \) equals \( 1 \) (resp. \( -1 \)).

**Proof.** If \( \tau' \) is a positive edge, denote \( a = \tau' \cap [\chi_0 = 1] \). If \( \tau' \) is a negative edge, denote \( a = \tau' \cap [\chi_0 = -1] \). Recall that one of the requirements we have imposed on \( \tau \) says that the planes \( \chi_0 = 1 \) and \( \chi_0 = -1 \) intersect edges of \( \tau \) at lattice points (otherwise the polyhedral divisor we obtain from \( \tau \) does not consist of lattice polyhedra), so \( a \) is a lattice point, and hence \( a \) is a multiple of \( b(\tau') \). On the other hand, if \( a \neq b(\tau') \), then \( \chi_0(b(\tau')) \) cannot be an integer. So, \( a = b(\tau) \), and \( \chi_0(b(\tau')) = 1 \) (resp. \( \chi_0(b(\tau')) = -1 \)) if \( \tau' \) is a positive (resp. negative) edge.

**Lemma 49.** If \( \tau' \) is a positive (resp. negative) edge of \( \tau \), and \( \chi = \chi_0 \) (resp. \( \chi = -\chi_0 \)), then \( \text{Span}_Q(\Lambda_{\chi}^{\tau}) = \text{Span}_Q(\mathcal{N}(\tau', \tau)) \) and \( \dim \text{Span}_Q(\Lambda_{\chi}^{\tau}) = 2 \).

**Proof.** Without loss of generality, suppose that \( \tau' \) is a positive edge and \( \chi = \chi_0 \) (the other case can be considered completely analogously). Then by Lemma [13], \( \chi(b(\tau')) = 1 \). So, if \( \tilde{\lambda}_i \notin \mathcal{N}(\tau', \tau) \), then \( \tilde{\lambda}_i(b(\tau')) > 0 \), so \( \tilde{\lambda}_i(b(\tau')) \geq 1 \) (this is an integer number), and \( \tilde{\lambda}_i(b(\tau')) \geq \chi(b(\tau')) \). Hence, \( \tilde{\lambda}_i \notin \Lambda_{\chi_0}^{\tau} \). On the other hand, if \( \tilde{\lambda}_i \in \mathcal{N}(\tau', \tau) \), then \( \tilde{\lambda}_i(b(\tau')) = 0 \), and \( \tilde{\lambda}_i(b(\tau')) < \chi(b(\tau')) \). Hence, \( \tilde{\lambda}_i \in \Lambda_{\chi_0}^{\tau} \).

Therefore, \( \Lambda_{\chi_0}^{\tau} \) is the intersection of the Hilbert basis of \( \tau' \cap M \) and the normal facet of \( \tau' \), which is the Hilbert basis of \( \mathcal{N}(\tau', \tau) \cap M \). In particular, \( \Lambda_{\chi_0}^{\tau} \) generates \( \text{Span}_Q(\mathcal{N}(\tau', \tau)) \) as a \( Q \)-vector space.

**Lemma 50.** If \( \tau' \) is a positive (resp. negative) edge of \( \tau \), and \( \chi = a\chi_0 \), where \( a \geq 2 \) (resp. \( a \leq -2 \)), then \( \text{Span}_Q(\Lambda_{\chi}^{\tau}) = \tilde{M}_Q \).

**Proof.** Again, without loss of generality we may suppose that \( \tau' \) is a positive edge and \( a \geq 2 \), the other case is completely similar.

First, let us prove that there exists a degree \( \chi' \in \tau' \cap \tilde{M} \) such that \( \chi'(b(\tau')) = 1 \). This is done by a standard continuity argument. Namely, consider a lattice point \( \chi'' \) in the relative interior of \( \mathcal{N}(\tau', \tau) \). Consider also a line \( \chi'' + Q\chi_0 \). This line cannot be contained in the plane containing \( \mathcal{N}(\tau', \tau) \) since \( \chi_0(b(\tau')) \neq 0 \). So, the intersection of this line and this plane is exactly \( \chi'' \), and \( \mathcal{N}(\tau', \tau) \) splits the line \( \chi'' + Q\chi_0 \) into two rays, and one of these rays passes through the interior of \( \tau' \). Since \( \chi''(b(\tau')) = 0 \) and \( \chi_0(b(\tau')) > 0 \), the ray passing through the interior of \( \tau' \) cannot be \( \chi'' + Q\geq\chi_0 \), and it must be \( \chi'' + Q<\chi_0 \). Hence, if \( b \in \mathbb{N} \) is large enough, \( \chi'' + (1/b)\chi_0 \in \tau' \). Then \( b\chi'' + \chi_0 \in \tau' \), but \( b\chi'' + \chi_0 \) is a lattice point, and \( (b\chi'' + \chi_0)(b(\tau')) = 1 \), so we can take \( \chi' = b\chi'' + \chi_0 \).

Since all \( \tilde{\lambda}_i \) form the Hilbert basis of \( \tau' \cap \tilde{M} \), \( \chi' \) can be written as a positive integer linear combination of \( \lambda_i \). Since \( \lambda_i(b(\tau')) \geq 0 \), there exists \( \lambda_i \) such that \( \lambda_i(b(\tau')) = 1 \).

As we have already noted previously, the set of all \( \tilde{\lambda}_i \) such that \( \tilde{\lambda}_i(b(\tau')) = 0 \) form the Hilbert basis of \( \mathcal{N}(\tau', \tau) \cap \tilde{M} \), therefore they generate \( \text{Span}_Q(\mathcal{N}(\tau', \tau)) \) as a \( Q \)-vector space. Clearly, all these \( \tilde{\lambda}_i \) are in \( \Lambda_{\chi}^{\tau} \). Together they generate a 2-dimensional vector space, so if we add one more vector, which is outside \( \text{Span}_Q(\mathcal{N}(\tau', \tau)) \), all vectors together will generate a bigger vector space, but then this space must be \( \tilde{M}_Q \) since \( \dim \tilde{M}_Q = 3 \). But we already know that there exists a \( \tilde{\lambda}_i \in \Lambda_{\chi}^{\tau} \) such that \( \tilde{\lambda}_i(b(\tau')) = 1 \). By the definition of \( \mathcal{N}(\tau', \tau) \), all vectors from \( \text{Span}_Q(\mathcal{N}(\tau', \tau)) \) vanish on \( b(\tau') \), so this \( \tilde{\lambda}_i \) cannot be in \( \text{Span}_Q(\mathcal{N}(\tau', \tau)) \). Therefore, \( \text{Span}_Q(\Lambda_{\chi}^{\tau}) = \tilde{M}_Q \).

**Corollary 13.** If \( \chi = a\chi_0 \), \( a \in \mathbb{Z} \), \( a \neq 0 \), then \( \Lambda^{\chi,1} \otimes \mathbb{C} \) can be written as follows:
Lemma 51. If $P$ (resp. negative) edge. These are exactly the facets we have denoted by
Corollary 14. Since in this case $a \leq 0$, then it is sufficient to consider only the facets of

These lemmas also enable us to describe $\Lambda^{x,0}$ explicitly:

Corollary 14. If $\chi = a \chi_0$, $a \in \mathbb{Z}$, $a \neq 0$, then $\Lambda^{x,0} \otimes \mathbb{C}$ can be written as follows:

1. If $a = 1$, then
   
   $\Lambda^{x,0} \otimes \mathbb{C} = \text{Span}_q \left( \bigcup_{i=1}^{e_+(\tau)} N(E^+_i(\tau), \tau) \right) \otimes \mathbb{C}$.

2. If $a \geq 2$, then
   
   $\Lambda^{x,0} \otimes \mathbb{C} = M_q \otimes \mathbb{C}$.

3. If $a = -1$, then
   
   $\Lambda^{x,0} \otimes \mathbb{C} = \text{Span}_q \left( \bigcup_{i=1}^{e_-(\tau)} N(E^-_i(\tau), \tau) \right) \otimes \mathbb{C}$.

4. If $a \leq -2$, then
   
   $\Lambda^{x,0} \otimes \mathbb{C} = M_q \otimes \mathbb{C}$.

Now we have to find $\ker((\Lambda^{x,1} \otimes \mathbb{C})^* \to (\Lambda^{x,2} \otimes \mathbb{C})^*))$, where $\chi$ is a multiple of $\chi_0$. To compute this kernel, we need some information about $\Lambda^{x,2}$. First, let us make the following observation. An element of $(\Lambda^{x,2} \otimes \mathbb{C})^*$ can be written as a sequence $(a_1, \ldots, a_{e(\tau)})$, where $a_i \in (\text{Span}_q(M^\tau_i(\tau)) \otimes \mathbb{C})^*$. In particular, the image of an element of $(\Lambda^{x,1} \otimes \mathbb{C})^*$ can be written in this form. Consider an entry $a_i$ such that $\partial \mathbf{F}_i(\tau)$ consists only of edges such that $\text{Span}_q(M^\tau_i(\tau)) = 0$. Observe that in this case $a_i = 0$, since in this case $a_i$ is the difference of two elements of two vector spaces, and each of these vector spaces has dimension 0. So, it is sufficient to consider only the facets whose boundary contains at least one edge $\tau'$ such that $\text{Span}_q(M^\tau_i(\tau')) \neq 0$. Using Corollary 13, we can say that if $\chi = a \chi_0$, where $a > 0$ (resp. $a < 0$), then it is sufficient to consider only the facets of $\tau$ whose boundary contains at least one positive (resp. negative) edge. These are exactly the facets we have denoted by $\mathbf{F}^+_i(\tau), \ldots, \mathbf{F}^-_i(\tau)$ (resp. by $\mathbf{F}^+_0(\tau), \ldots, \mathbf{F}^-_0(\tau)$).

Lemma 51. If $\chi = a \chi_0$, where $a > 0$, then $\text{Span}_q(M^\tau_i(\tau)) = 0$ for $i = 0$ and $i = 0(\tau)$. 
Proof. Let us consider the case \( i = 0 \), the other case is completely similar. By the definition of \( \mathbf{F}^\tau_\chi \), its boundary consists of \( \mathbf{E}^\tau_\chi \), which is a positive edge, and another edge \( \tau' \), which is nonpositive. Hence, by Lemma 54, \( \Lambda^\chi_\tau = \emptyset \), so \( \Lambda^\chi_{\mathbf{F}^\tau_\chi} = \emptyset \cap \Lambda^\chi_{\mathbf{E}^\tau_\chi} = \emptyset \) as well, and \( \text{Span}_Q(\Lambda^\chi_{\mathbf{F}^\tau_\chi}) = 0 \). \( \square \)

Lemma 52. If \( \chi = a \chi_0 \), where \( a < 0 \), then \( \text{Span}_Q(\Lambda^\chi_{\mathbf{F}^\tau_\chi}) = 0 \) for \( i = 0 \) and \( i = \mathbf{e}^{-}(\tau) \).

Proof. The proof here is again completely similar to the proof of the previous lemma, but this time we present it to ease reading. Let us consider the case \( i = \mathbf{e}^{-}(\tau) \), the other case is completely similar. By the definition of \( \mathbf{F}^\tau_{\mathbf{e}^{-}(\tau)} \), its boundary consists of \( \mathbf{E}^\tau_{\mathbf{e}^{-}(\tau)} \), which is a negative edge, and another edge \( \tau' \), which is nonnegative. Hence, by Lemma 54, \( \Lambda^\chi_\tau = \emptyset \), so \( \Lambda^\chi_{\mathbf{F}^\tau_{\mathbf{e}^{-}(\tau)}} = \emptyset \cap \Lambda^\chi_{\mathbf{E}^\tau_{\mathbf{e}^{-}(\tau)}} = \emptyset \) as well, and \( \text{Span}_Q(\Lambda^\chi_{\mathbf{F}^\tau_{\mathbf{e}^{-}(\tau)}}) = 0 \). \( \square \)

To understand the behavior of \( \Lambda^\chi_{\mathbf{F}^\tau_\chi} \), where \( 1 \leq i \leq \mathbf{e}^{\tau}(\tau) - 1 \), (resp. \( \Lambda^\chi_{\mathbf{F}^\tau_{\mathbf{e}^{-}(\tau)}} \), where \( 1 \leq i \leq \mathbf{e}^{\tau}(\tau) - 1 \)) for degrees \( \chi = a \chi_0 \) with \( a > 0 \) (resp. \( a < 0 \)), we start with the following lemma.

Lemma 53. Let \( \mathcal{N} \) be a two-dimensional lattice, and let \( \mathcal{M} \) be its dual lattice. Let \( a_1, a_2 \in \mathcal{N} \) and \( \chi \in \mathcal{M} \) be such that \( \chi(a_1) = \chi(a_2) = 1 \) and \( a_1 \neq a_2 \). Then \( a_1 \) and \( a_2 \) generate \( \mathcal{N} \otimes_{\mathbb{Z}} \mathbb{Q} \) as a \( \mathbb{Q} \)-vector space.

Denote the primitive lattice point on the ray \( \{ \chi' \in \mathcal{M} : \chi'(a_1) > 0, \chi'(a_2) = 0 \} \) by \( \chi_1 \). Similarly, denote by \( \chi_2 \) the primitive lattice point on the ray \( \{ \chi' \in \mathcal{M} : \chi'(a_1) = 0, \chi'(a_2) > 0 \} \). Then \( \chi_1(a_1) = \chi_2(a_2) = |a_1 - a_2| \). The sets

\[
\mathcal{X}_{\chi,a_1,a_2,b}\{\chi' \in \mathcal{M} : \chi'(a_1) \geq 0, \chi'(a_2) \geq 0, \chi'(a_1) < b, \chi'(a_2) < b \}
\]

for \( b \in \mathbb{N} \) behave as follows:

1. If \( 0 < b \leq |a_1 - a_2| \), then \( \mathcal{X}_{\chi,a_1,a_2,b} \) is the set of all \( \chi' \) of the form \( \chi' = b' \chi \), \( 0 \leq b' < b \).
2. If \( b > |a_1 - a_2| \), then \( \mathcal{X}_{\chi,a_1,a_2,b} \) contains \( \chi_1 \) and \( \chi_2 \).

Proof. Consider the \( \mathbb{Q} \)-linear span of \( a_1 \) and \( a_2 \) in \( \mathcal{N} \otimes_{\mathbb{Z}} \mathbb{Q} \). Since \( \chi(a_1) \neq 0 \) and \( \chi(a_2) \neq 0 \), this linear span can be one-dimensional only if \( a_1 \) and \( a_2 \) are a \( \mathbb{Q} \)-multiple of an \( \mathbb{Q} \)-multiple of each other. But in this case, since \( \chi(a_1) = \chi(a_2) = 0 \), \( a_1 \) and \( a_2 \) must coincide, which is a contradiction.

Denote \( k = |a_1 - a_2| \) and denote \( a^* = (1 - 1/k)a_1 + (1/k)a_2 \). Then \( a^* \in \mathcal{N} \), and \( a^* - a_1 \) is a primitive lattice vector. Hence, there exists a function \( \chi'' \in \mathcal{M} \) such that \( \chi''(a^* - a_1) = 1 \). Since \( \chi(a_1) = \chi(a_2) = 1 \), we also have \( \chi(a') = 1 \). Consider the following functions \( \chi''''(i = 1, 2) : \chi''' = \chi'' - \chi''(a') \). We have \( \chi'''(a_1) = \chi''(a_1) - \chi''(a_1)(\chi(a_1) = 0, so \( \chi'''' = \) a multiple of \( \chi_2 \) and \( \chi'''' = \) a multiple of \( \chi_1 \), since \( \chi_1 \) and \( \chi_2 \) are linear vectors on the corresponding rays.

We also have \( \chi''(a_2) = \chi''(a_1) + \chi''(a_2 - a_1) = k \chi''(a^* - a_1) = k(\chi''(a^* - a_1) - \chi''(a_1)(\chi(a_1) = 1, \chi''(a^* - a_1)) = k(1 - \chi''(a_1)(1 - 1)) = k\). And \( \chi''(a_2 - a_1) = k\chi''(a^* - a_1) = -k(\chi''(a^* - a_1) - \chi''(a_1))(\chi(a_1) = 0) \). On the other hand, \( \chi_2(a_2) = \chi_2(a_1) + \chi_2(a_2 - a_1) = k\chi_2(a^* - a_1) = -k(\chi_2(a^* - a_1) - \chi_2(a_1))(\chi(a_1) = 0) \). Hence, \( \chi_1(a_1) \) is a multiple of \( k = \chi''''(a_1) \) and \( \chi_2(a_2) \) is a multiple of \( k = \chi''''(a_2) \). Recall that \( \chi'''' = \) a multiple of \( \chi_2 \) and \( \chi'''' = \) a multiple of \( \chi_1 \). Summarizing, we conclude that \( \chi_1 = \pm \chi'''' \) and \( \chi_2 = \pm \chi'''' \). But then \( \chi_1(a_1) = \pm \chi_2''''(a_1) = \pm k \) and \( \chi_2(a_2) = \pm \chi_2''''(a_2) = \pm k \). Since \( \chi_1(a_1) > 0 \) and \( \chi_2(a_2) > 0 \), by the definitions of \( \chi_1 \) and \( \chi_2 \), we have \( \chi_1(a_1) = \chi_2(a_2) = k \).

Now fix some \( b \in \mathbb{N} \) and consider the set

\[
\mathcal{X}_{\chi,a_1,a_2,b}\{\chi' \in \mathcal{M} : \chi'(a_1) \geq 0, \chi'(a_2) \geq 0, \chi'(a_1) < b, \chi'(a_2) < b \}
\]

If \( b > |a_1 - a_2| \), then it is already clear that \( \mathcal{X}_{\chi,a_1,a_2,b} \) contains \( \chi_1 \) and \( \chi_2 \) since \( \chi_1(a_1) = |a_1 - a_2| \), \( \chi_1(a_2) = 0 \), \( \chi_2(a_2) = 0 \), and \( \chi_2(a_2) = |a_1 - a_2| \). So suppose that \( b \leq |a_1 - a_2| \). In this case it is also clear that \( b' \chi \in \mathcal{X}_{\chi,a_1,a_2,b} \) for \( 0 \leq b' < b \) since \( \chi(a_1) = \chi(a_2) = 1 \).

Suppose that \( \chi'' \in \mathcal{X}_{\chi,a_1,a_2,b} \). Without loss of generality, \( \chi'(a_1) \geq \chi'(a_2) \). Consider \( \chi'' = \chi''(a_2) \). We have \( \chi''(a_1) = \chi''(a_2) \chi(a_1) = \chi''(a_2) - \chi''(a_2) \geq 0 \) and \( \chi''(a_2) = \chi''(a_2) - \chi''(a_2) \chi(a_1) = 0 \). So, \( \chi'' = \) a lattice point on the (closed) ray \( \{ \chi'' \in \mathcal{M} : \chi''(a_1) \geq 0, \chi''(a_2) = 0 \} \). But we already

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know that the primitive lattice vector on this ray is $\chi_1$, so $\chi''$ is a (possibly zero) integer multiple of $\chi_1$. If $\chi'(a_1) > \chi'(a_2)$, then $\chi'' \neq 0$, and we have a contradiction with $\chi_1(a_1) = |a_1 - a_2|$ since $\chi'(a_1) < b \leq |a_1 - a_2|$, $\chi'(a_2) \geq 0$, and $\chi''(a_1) = \chi'(a_1) - \chi'(a_2)$. If $\chi'(a_1) = \chi'(a_2)$, then we see that $\chi'$ and $\chi'(a_1)\chi$ take the same values on $a_1$ and $a_2$. Since $a_1$ and $a_2$ $Q$-generate $N \otimes Q$, we can conclude that $\chi' = \chi'(a_1)\chi$ as desired.

**Lemma 54.** Let $E_j$ be facet of $\tau$, and let $E_{j_2}$ be an edge of $\tau$ on the boundary of $F_i$. Let $E_{j_2}$ be the other edge on the boundary of $F_i$. Suppose that we have a degree $\chi \in \text{Span}_Q(N(E_{j_2}(\tau),\tau)) \cap M$ such that $\chi(b(E_{j_2}(\tau))) > 0$.

Then there exists $a \in \mathbb{N}$ such that $\chi + ab\mathcal{N}(N(E_{j_2}(\tau),\tau)) \in \mathcal{N}(E_{j_2}(\tau),\tau)$.

**Proof.** Let $F_i$ be the facet of $\tau$ such that $\partial N(E_{j_2}(\tau),\tau) = N(F_i,\tau) \cup N(F_i,\tau)$. In other words, $F_i$ and $F_k$ are the two facets whose boundary contains $E_{j_2}(\tau)$. Then $\mathcal{N}(F_i,\tau)$ and $\mathcal{N}(F_k,\tau)$ is determined inside $\text{Span}_Q(N(E_{j_2}(\tau),\tau))$ by two inequalities corresponding to $N(F_i,\tau)$ and $N(F_k,\tau)$. For an inequality corresponding to $\mathcal{N}(F_i,\tau)$, we can take the restriction to $\text{Span}_Q(N(E_{j_2}(\tau),\tau))$ of the inequality in the definition of $\tau''$ corresponding to the other facet of $\tau'$ whose boundary contains $N(F_i,\tau)$. This other facet is $\mathcal{N}(E_{j_2}(\tau),\tau)$, and the corresponding inequality says that if $\chi' \in \mathcal{N}(E_{j_2}(\tau),\tau)$, then $\chi'$ takes nonnegative values on $E_{j_2}(\tau)$, in other words, $\chi(b(E_{j_2}(\tau))) \geq 0$.

Similarly, for an inequality corresponding to $\mathcal{N}(F_k,\tau)$, we can take the restriction to $\text{Span}_Q(N(E_{j_2}(\tau),\tau))$ of the inequality corresponding to the facet of $\tau''$ different from $\mathcal{N}(E_{j_2}(\tau),\tau)$ and whose boundary contains $N(F_k,\tau)$. This facet is the normal facet of the edge on the boundary of $F_k(\tau)$ different from $E_{j_2}(\tau)$. Denote it by $E_{j_2}(\tau)$ so that $\partial F_k(\tau) = E_{j_2}(\tau) \cup E_{j_2}(\tau)$. Then the inequality corresponding to $\mathcal{N}(E_{j_2}(\tau),\tau)$ in the definition of $\tau''$ says that if $\chi' \in \tau''$, then $\chi'$ takes nonnegative values on $E_{j_2}(\tau)$, in other words, $\chi(b(E_{j_2}(\tau))) \geq 0$. Therefore, $\mathcal{N}(E_{j_2}(\tau),\tau)$ is determined inside $\text{Span}_Q(N(E_{j_2}(\tau),\tau))$ by the restrictions to $\text{Span}_Q(N(E_{j_2}(\tau),\tau))$ of the inequalities $\chi'(b(E_{j_2}(\tau))) \geq 0$ and $\chi'(b(E_{j_2}(\tau))) > 0$ for $\chi' \in \mathcal{M}_Q$.

Therefore, if $\chi'(b(E_{j_2}(\tau))) \geq 0$, then we can take $a = 0$. Suppose that $\chi'(b(E_{j_2}(\tau))) < 0$.

We chose $F_i$ so that $\mathcal{N}(F_i,\tau) \neq \mathcal{N}(F_k,\tau)$, and we also know that $\mathcal{N}(E_{j_2}(\tau),\tau) \cap \mathcal{N}(E_{j_2}(\tau),\tau)$ is $\mathcal{N}(F_i,\tau)$, so $\mathcal{N}(F_i,\tau) \not\mathcal{N}(F_k,\tau)$. Hence, $\mathcal{N}(F_i,\tau) \mathcal{N}(F_k,\tau) > 0$. Then there exists $a \in \mathbb{N}$ such that $\mathcal{N}(\mathcal{N}(F_i,\tau),\tau) \mathcal{N}(E_{j_2}(\tau)) > 0$. We have $a \mathcal{N}(\mathcal{N}(F_i,\tau),\tau) > 0$. We also have $\mathcal{N}(\mathcal{N}(F_i,\tau),\tau) \mathcal{N}(E_{j_2}(\tau)) > 0$ by assumption, and $\mathcal{N}(\mathcal{N}(F_i,\tau),\tau) + \chi(\mathcal{N}(E_{j_2}(\tau))) > 0$.

**Lemma 55.** Let $F_i(\tau)$ (resp. $F_i(\tau)$), where $1 \leq i \leq e(\tau) - 1$ (resp. $1 \leq i \leq e(-\tau) - 1$), be a facet of $\tau$. Then $b(\mathcal{N}(F_i(\tau),\tau)) + \chi(\mathcal{N}(F_i(\tau),\tau)) = e(\tau)$ (resp. $b(\mathcal{N}(F_i(\tau),\tau)) = e(\tau)$).

**Proof.** Since $b(\mathcal{N}(F_i(\tau),\tau)) = e(\tau)$ (resp. $b(\mathcal{N}(F_i(\tau),\tau)) = e(\tau)$), it takes nonnegative values on the edges of $\tau$. Since $\partial F_i(\tau) = E_{j_2}(\tau) \cup E_{j_2}(\tau)$ (resp. $\partial F_i(\tau) = E_{j_2}(\tau) \cup E_{j_2}(\tau)$), the only two edges of $\tau$ where $b(\mathcal{N}(F_i(\tau),\tau)) = e(\tau)$ (resp. $\mathcal{N}(F_i(\tau),\tau)$) are $E_{j_2}(\tau)$ and $E_{j_2}(\tau)$ (resp. $E_{j_2}(\tau)$ and $E_{j_2}(\tau)$). But both of these edges are positive (resp. negative), so if $E_{j_2}(\tau)$ is one of these two edges, then $\chi(\mathcal{N}(E_{j_2}(\tau))) = 1$. Hence, $b(\mathcal{N}(F_i(\tau),\tau)) \chi(\mathcal{N}(E_{j_2}(\tau))) = 1$. Observe the $-\text{sign}$ in front of $\chi_0$ for $E_{j_2}(\tau)$, then $E_{j_2}(\tau) = E_{j_2}(\tau)$ (resp. $E_{j_2}(\tau) = E_{j_2}(\tau) = E_{j_2}(\tau)$).

Now suppose that $E_{j_2}(\tau)$ is another edge, i.e. $E_{j_2}(\tau) \not\mathcal{N}(F_i(\tau))$ (resp. $E_{j_2}(\tau) \not\mathcal{N}(F_i(\tau))$). Then $b(\mathcal{N}(F_i(\tau),\tau))b(E_{j_2}(\tau)) > 0$ (resp. $b(\mathcal{N}(F_i(\tau),\tau))b(E_{j_2}(\tau)) > 0$), and, since $b(\mathcal{N}(F_i(\tau),\tau))$ (resp. $b(\mathcal{N}(F_i(\tau),\tau))$) and $b(E_{j_2}(\tau))$ are lattice points, we have $b(\mathcal{N}(F_i(\tau),\tau))b(E_{j_2}(\tau)) \geq 1$ (resp. $b(\mathcal{N}(F_i(\tau),\tau))b(E_{j_2}(\tau)) \geq 1$). Now recall that if an edge of $\tau$ intersects one of the planes $[x_0 = 1]$ and $[x_0 = -1]$, then the intersection point is a lattice point. This lattice point must be the primitive lattice vector on this edge, otherwise $\chi_0$ would have taken a noninteger value at the primitive lattice vector. Therefore, if $E_{j_2}(\tau)$ intersects one of the planes $[x_0 = 1]$ and $[x_0 = -1]$, then $\chi_0(b(E_{j_2}(\tau)))$ can only equal 1 or -1. If $E_{j_2}(\tau)$ intersects none of these planes, then $\chi_0$ vanishes on $E_{j_2}(\tau)$ everywhere, in particular $\chi_0(b(E_{j_2}(\tau))) = 0$. Therefore, in all cases we have $|\chi_0(b(E_{j_2}(\tau)))| \leq 1$. But then $b(\mathcal{N}(F_i(\tau),\tau),\tau) \chi_0(b(E_{j_2}(\tau))) \geq 0$ (resp. $b(\mathcal{N}(F_i(\tau),\tau)) - \chi_0(b(E_{j_2}(\tau))) \geq 0$, this time the sign in front of $\chi_0$ does not matter).
Summarizing, we see that if $E_j(\tau)$ is an arbitrary edge of $\tau$, then $(b(\mathcal{N}(F_j^+(\tau), \tau) + \chi_0) (b(E_j(\tau))) \geq 0$ (resp. $(b(\mathcal{N}(F_j^-(\tau), \tau)) + \chi_0)(b(E_j(\tau))) \geq 0$). Therefore, $(b(\mathcal{N}(F_j^+(\tau), \tau)) + \chi_0 \in \tau^\vee$ (resp. $(b(\mathcal{N}(F_j^-(\tau), \tau)) - \chi_0 \in \tau^\vee$).

\begin{proof}
Let $F_j^+(\tau)$ (resp. $F_j^-(\tau)$), where $1 \leq i \leq e^+(\tau) - 1$ (resp. $1 \leq i \leq e^- (\tau) - 1$), be a facet of $\tau$. Let $\chi = b\chi_0$ (resp. $\chi = -b\chi_0$), where $b \in \mathbb{N}.$

1. If $b = 1$, then $\operatorname{Span}_Q(\Lambda^X_{F_j^+(\tau)}) = \operatorname{Span}_Q(\mathcal{N}(F_j^+(\tau), \tau))$ (resp. $\operatorname{Span}_Q(\Lambda^X_{F_j^-(\tau)}) = \operatorname{Span}_Q(\mathcal{N}(F_j^-(\tau), \tau))$).

2. If $|F_j^+(\tau) \cap [\chi_0 = 1]| \geq 2$ (resp. $|F_j^-(\tau) \cap [\chi_0 = -1]| \geq 2$) and $2 \leq b \leq |F_j^+(\tau) \cap [\chi_0 = 1]|$ (resp. $2 \leq b \leq |F_j^-(\tau) \cap [\chi_0 = -1]|$), then $\operatorname{Span}_Q(\Lambda^X_{F_j^+(\tau)}) = \operatorname{Span}_Q(\chi_0, \mathcal{N}(F_j^+(\tau), \tau))$ (resp. $\operatorname{Span}_Q(\Lambda^X_{F_j^-(\tau)}) = \operatorname{Span}_Q(\chi_0, \mathcal{N}(F_j^-(\tau), \tau))$).

3. If $b > |F_j^+(\tau) \cap [\chi_0 = 1]|$ (resp. $b > |F_j^-(\tau) \cap [\chi_0 = -1]|$), then $\operatorname{Span}_Q(\Lambda^X_{F_j^+(\tau)}) = \tilde{M}_Q$.

\end{proof}

Proof. Again, the positive and the negative cases here are completely similar. This time let us consider the negative case.

Consider the lattices $\mathcal{M} = \mathcal{M}/(\mathcal{M} \cap \operatorname{Span}_Q(\mathcal{N}(F_j^+(\tau), \tau)))$ and $\mathcal{N} = \tilde{N} \cap \operatorname{Span}_Q(F_j^+(\tau))$. By the definition of $\mathcal{N}(F_j^+(\tau), \tau)$, a function from $\mathcal{M}$ vanishes on the whole $\mathcal{N}$ (which is a saturated sublattice of $\tilde{N}$ by construction) if and only if this function is contained in $\mathcal{M} \cap \operatorname{Span}_Q(\mathcal{N}(F_j^+(\tau), \tau))$. Therefore, $\tilde{M}$ is the dual lattice of $\mathcal{N}$, and the values of elements of $\mathcal{M}$ at points from $\mathcal{N}$ are well-defined. We denote the class of a function $\chi' \in \mathcal{M}$ by $\chi'$.

Denote $a_1 = b(E_j^+(\tau))$, $a_2 = b(E_j^-(\tau))$. Recall that if $\mathcal{N}(F_j^+(\tau)) = E_j^+(\tau) \cup E_j^-(\tau)$, so $a_1, a_2 \in \mathcal{N}$. We have already seen that $\chi_0(a_1) = \chi_0(a_2) = -1$ and that $a_1 = E_j^+(\tau) \cap [\chi_0 = -1], a_2 = E_j^-(\tau) \cap [\chi_0 = -1]$. So, $a_1, a_2,$ and $\chi_0(a_1)$ satisfy the hypothesis of Lemma 538 and $[a_1 - a_2] = [F_j^-(\tau) \cap [\chi_0 = -1]]$. Consider the set $\Lambda^X_{F_j^+(\tau)} \cap [a_1 - a_2]$ from Lemma 538. It follows directly from the definitions of $\Lambda^X_{F_j^+(\tau)}$ and $\Lambda^X_{F_j^-(\tau)}$ that the image of $\Lambda^X_{F_j^+(\tau)} \cap [a_1 - a_2]$ under the canonical projection $\tilde{M} \to \mathcal{M}$ is contained in $\Lambda^X_{F_j^-(\tau)} \cap [a_1 - a_2]$. Moreover, if $\hat{\lambda}_j$ is an element of the Hilbert basis of $\tau^\vee$ such that $\chi' = \chi_j(a_1) < b = -b \cdot (-1) = -\chi_0(a_1) = \chi(a_1)$, then $\hat{\lambda}_j \in \Lambda^X_{F_j^+(\tau)}$. Similarly, $\hat{\lambda}_j(a_2) < b = (-b) \cdot (-1) = -\chi_0(a_2) = \chi(a_2)$, so $\hat{\lambda}_j \in \Lambda^X_{F_j^-(\tau)}$.

Consider the case $b = 1$. Then by Lemma 533, $\Lambda^X_{F_j^+(\tau)} \cap [a_1 - a_2] = \{0\}$, and all elements of $\Lambda^X_{F_j^+(\tau)}$ are in $\ker(\tilde{M} \to \mathcal{M}) = \mathcal{M} \cap \operatorname{Span}_Q(N(F_j^-(\tau), \tau))$. On the other hand, since $N(F_j^-(\tau), \tau)$ is a face of $\tau^\vee$, $b(N(F_j^-(\tau), \tau)) = \Lambda_j$ for some $j$. As we have seen previously, this means that $\hat{\lambda}_j \in \Lambda^X_{F_j^+(\tau)}$. Hence, $\hat{\lambda}_j \in \Lambda^X_{F_j^+(\tau)} \cap \Lambda^X_{F_j^-(\tau)} = \Lambda^X_{F_j^+(\tau)}$.

Now suppose that $|a_1 - a_2| \geq 2$ and $2 \leq b \leq |a_1 - a_2|$. Then by Lemma 533, $\Lambda^X_{F_j^+(\tau)} \cap [a_1 - a_2] \subseteq \mathcal{N}$ in the line generated by $-\chi_0$. Hence, $\Lambda^X_{F_j^+(\tau)}$ is contained in the plane generated by $\mathcal{N}(F_j^+(\tau), \tau)$ and $-\chi_0$. On the other hand, we already know that $b(N(F_j^-(\tau), \tau))$ is an element of the Hilbert basis of $\tau^\vee$, and, since $\chi''(a_1) = \chi''(a_2) = 1$, the elements of the Hilbert basis present in this combination may only take values $0$ or $1$ at $a_1$ and $a_2$ (in arbitrary order). But if there exists $\lambda_k$ such that $\lambda_k(a_1) = 1$ and $\lambda_k(a_2) = 0$, then $\lambda_k \in \Lambda^X_{F_j^-(\tau)}$, and this is a contradiction with Lemma 533. Similarly, one cannot have $\lambda_k(a_1) = 0$ and $\lambda_k(a_2) = 1$. Hence, there exist an element $\hat{\lambda}_k$ of the Hilbert basis such that $\hat{\lambda}_k(a_1) = \hat{\lambda}_k(a_2) = 1$. By Lemma 533, $a_1$ and $a_2$ $Q$-generate $\mathcal{N}_Q \oplus Q$. Therefore, elements of $\mathcal{M}$ are determined by their values at $a_1$ and $a_2$, and $\lambda_k \in \Lambda^X_{F_j^+(\tau)}$. We already know that this means that $\hat{\lambda}_k \in \Lambda^X_{F_j^+(\tau)}$.

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\[ \bar{\lambda}_k = -\lambda_0, -\chi_0 - \bar{\lambda}_k \in \text{Span}_Q(\mathcal{N}(F^-_i(\tau), \tau)) \text{ and } \bar{\lambda}_k \text{ and } \mathcal{N}(F^-_i(\tau), \tau) \text{ together } Q\text{-generate the same plane as } -\chi_0 \text{ and } \mathcal{N}(F^-_i(\tau), \tau) \text{ } Q\text{-generate, i.e. they } Q\text{-generate } \text{Span}_Q(\chi_0, \mathcal{N}(F^-_i(\tau), \tau)). \text{ Therefore,}
\]
\[ \text{Span}_Q(\Lambda^\chi_{F^-_i(\tau)}) = \text{Span}_Q(\chi_0, \mathcal{N}(F^-_i(\tau), \tau)). \]

Finally, let us consider the case \( b > |a_1 - a_2| \). By Lemma \[ \text{there exist } \chi_1, \chi_2 \in \mathcal{M}_{-\chi_0,a_1,a_2,b} \text{ such that } \chi_1(a_1) > 0, \chi_1(a_2) = 0, \chi_2(a_1) = 0, \text{ and } \chi_2(a_2) > 0. \text{ Pick arbitrary } \chi'_1, \chi'_2 \in \mathcal{M} \text{ such that } \chi'_1 = \chi_1 \text{ and } \chi'_2 = \chi_2. \text{ We have } \chi'_1(a_1) > 0, \chi'_1(a_2) = 0, \chi'_2(a_1) = 0, \text{ and } \chi'_2(a_2) > 0, \text{ so, by the definitions of } \mathcal{N}(E^-_i(\tau), \tau) \text{ and of } \mathcal{N}(E^-_{i+1}(\tau), \tau), \text{ we have } \chi'_1 \in \mathcal{N}(E^-_{i+1}(\tau), \tau) \text{ and } \chi'_2 \in \mathcal{N}(E^-_i(\tau), \tau). \text{ Therefore, we can apply Lemma \[ \text{52 to the facet } F^-_i(\tau) \text{ of } \tau, \text{ to the edge } E^-_{i+1}(\tau) \text{ of } \tau, \text{ and to the degree } \chi'_1 \text{ and find another degree } \chi'' \text{ such that } \chi''(a_1) = \chi''(a_2) = 0, \chi''(a_1) = \chi''(a_2) = 0, \text{ and } \chi''(a_2) > 0, \text{ so, by the definitions of } \mathcal{N}(E^-_i(\tau), \tau) \text{ and of } \mathcal{N}(E^-_{i+1}(\tau), \tau), \text{ we have } \chi''_1 \in \mathcal{N}(E^-_{i+1}(\tau), \tau) \text{ and } \chi''_2 \in \mathcal{N}(E^-_i(\tau), \tau). \text{ Similarly, by Lemma \[ \text{applied to } F^-_i(\tau), \text{ to } E^-_i(\tau), \text{ and to } \chi'_1 \text{, there exists a degree } \chi'' \in \mathcal{N}(E^-_i(\tau), \tau) \text{ such that } \chi''(a_1) = \chi''(a_2) = 0, \text{ and } \chi''(a_2) > 0. \text{ Decompose } \chi'' \text{ into a positive integer linear combination of } \lambda_j. \text{ The elements } \lambda_j \text{ of the Hilbert basis occurring in this decomposition satisfy } \lambda_j(a_2) = 0 \text{ and } \lambda_j(a_1) > 0. \text{ Similarly, there exists } \lambda_k \text{ satisfying } \lambda_k(a_1) = 0 \text{ and } \lambda_k(a_2) > 0. \text{ We can write this as } \lambda_j(a_1) = (-b) \cdot (-1) = -b\lambda_0(a_1) = \chi(a_1) \text{ and } \lambda_k(a_2) = (-b) \cdot (-1) = -b\lambda_0(a_2) = \chi(a_2), \text{ so } \lambda_j \in \Lambda^\chi_{F^-_i(\tau)} \text{ and } \lambda_k \in \Lambda^\chi_{F^-_{i+1}(\tau)}. \text{ Finally, as we have already seen previously, } b(\mathcal{N}(F^-_i(\tau), \tau)) \text{ is an element of the Hilbert basis, its class in } \mathcal{M} \text{ is } 0 \in \mathcal{M}_{-\chi_0,a_1,a_2,b}, \text{ so } b(\mathcal{N}(F^-_i(\tau), \tau)) \in \Lambda^\chi_{F^-_i(\tau)}. \]

Now we claim that \( \lambda_j, \lambda_k, \) and \( b(\mathcal{N}(F^-_i(\tau), \tau)) \) \( Q\)-generate \( M_Q. \) Indeed, \( \lambda_j(a_1) \neq 0, \) while \( b(\mathcal{N}(F^-_i(\tau), \tau))(a_1) = 0 \) by the definition of \( \mathcal{N}(F^-_i(\tau), \tau). \) Hence, \( \lambda_j \) and \( b(\mathcal{N}(F^-_i(\tau), \tau)) \) are linearly independent and \( Q\)-generate \( \text{Span}_Q(\mathcal{N}(E^-_{i+1}(\tau), \tau)). \) Similarly, \( \lambda_k \) and \( b(\mathcal{N}(F^-_{i+1}(\tau), \tau)) \) \( Q\)-generate \( \text{Span}_Q(\mathcal{N}(E^-_i(\tau), \tau)). \) The linear span of these two planes can be two-dimensional only if these two planes coincide, but \( \mathcal{N}(F^-_i(\tau), \tau) \) and \( \mathcal{N}(E^-_{i+1}(\tau), \tau) \) are two different facets of \( \tau', \) so \( \text{Span}_Q(\lambda_j, \lambda_k, b(\mathcal{N}(F^-_i(\tau), \tau))) = M_Q, \) and \( \text{Span}_Q(\Lambda^\chi_{F^-_i(\tau)}) = M_Q. \)

**Corollary 15.** If \( \chi = \chi_0 \) (resp. \( \chi = -\chi_0 \)), then \( \text{ker}((\Lambda^{\chi_0} \otimes \mathbb{C})^*) \rightarrow (\Lambda^{\chi_0} \otimes \mathbb{C})^* \) equals the space of sequences of the form \( (g_1, \ldots, g_{e^+(\tau)}) \) (resp. \( (g_1, \ldots, g_{e^-(\tau)}) \)), where \( g_i \) is a linear function on \( \text{Span}_Q(\mathcal{N}(E^-_i(\tau), \tau)) \otimes \mathbb{C} \) (resp. on \( \text{Span}_Q(\mathcal{N}(E^-_{i+1}(\tau), \tau)) \otimes \mathbb{C} \)), and where

\[ g_i|_{\text{Span}_Q(\mathcal{N}(F^-_i(\tau), \tau)) \otimes \mathbb{C}} = g_{i+1}|_{\text{Span}_Q(\mathcal{N}(F^-_{i+1}(\tau), \tau)) \otimes \mathbb{C}} \]

for \( 1 \leq i < e^+(\tau) \) (resp.

\[ g_i|_{\text{Span}_Q(\mathcal{N}(F^-_i(\tau), \tau)) \otimes \mathbb{C}} = g_{i+1}|_{\text{Span}_Q(\mathcal{N}(F^-_{i+1}(\tau), \tau)) \otimes \mathbb{C}} \]

for \( 1 \leq i < e^-(\tau) \).

**Proof.** The claim follows directly from Corollary \[ \text{13, Lemma \[ \text{51, Lemma \[ \text{52 and Proposition \[ \text{10} \). }}\]

**Corollary 16.** If \( \chi = a\chi_0 \) (resp. \( \chi = -a\chi_0 \)), where \( a \in \mathbb{N}, a \geq 2, \) then \( \text{ker}((\Lambda^{a\chi_0} \otimes \mathbb{C})^* \rightarrow (\Lambda^{a\chi_0} \otimes \mathbb{C})^*) \) equals the space of sequences of the form \( (g_1, \ldots, g_{e^+(\tau)}) \) (resp. \( (g_1, \ldots, g_{e^-(\tau)}) \)), where \( g_i \) are linear functions on \( M_Q \otimes \mathbb{C} \) satisfying the following conditions for \( 1 \leq i < e^+(\tau) \) (resp. for \( 1 \leq i < e^-(\tau))): \]

1. If \( b \leq |F^-_i(\tau) \cap [\chi_0 = 1]| \) (resp. \( b \leq |F^-_i(\tau) \cap [\chi_0 = -1]| \)), then

\[ g_i|_{\text{Span}_Q(\mathcal{N}(F^-_i(\tau), \tau)) \otimes \mathbb{C}} = g_{i+1}|_{\text{Span}_Q(\mathcal{N}(F^-_{i+1}(\tau), \tau)) \otimes \mathbb{C}} \]

(resp.

\[ g_i|_{\text{Span}_Q(\mathcal{N}(F^-_{i+1}(\tau), \tau)) \otimes \mathbb{C}} = g_{i+1}|_{\text{Span}_Q(\mathcal{N}(F^-_i(\tau), \tau)) \otimes \mathbb{C}} \]

).
2. If \( b > |F^+_i(\tau) \cap \{\chi_0 = 1\}| \) (resp. \( b > |F^-_i(\tau) \cap \{\chi_0 = -1\}| \)), then \( g_i = g_{i+1} \).

**Proof.** The claim follows directly from Corollary 13, Lemma 51, Lemma 52, and Proposition 10.

Now we construct a less invariant, but more explicit vector space isomorphic to \( \ker((\Lambda^{x^1} \otimes \mathbb{C})^* \to (\Lambda^{x^2} \otimes \mathbb{C})^*) \). Namely, denote by \( \mathcal{V}_{3,1,1} \) (resp. by \( \mathcal{V}_{3,1,-1} \)) the space of sequences of the form \((g_0', \ldots, g'_{e^+(\tau)})\) (resp. \((g_0', \ldots, g'_{e^-(\tau)})\)), where \( g_i' \) is a linear function on \( \text{Span}_Q(\mathcal{N}(F^+_i(\tau), \tau)) \otimes \mathbb{C} \) (resp. on \( \text{Span}_Q(\mathcal{N}(F^-_i(\tau), \tau)) \otimes \mathbb{C} \)).

1. \( g_i' \) is a linear function on \( \tilde{M}_Q \otimes \mathbb{C} \).

2. If \( 1 < i \leq e^+(\tau) \) (resp. \( 1 < i \leq e^-(\tau) \)) and \( a \leq |E_{i-1}(\Delta_0)| \) (resp. \( a \leq |E_{i-1}(\Delta_{\infty})| \)), then \( g_i' \) is a linear function on \( (\tilde{M}_Q \otimes \mathbb{C})/(\text{Span}_Q(\chi_0, \mathcal{N}(F^-_{i-1}(\tau), \tau)) \otimes \mathbb{C}) \) (resp. on \( (\tilde{M}_Q \otimes \mathbb{C})/(\text{Span}_Q(\chi_0, \mathcal{N}(F^+_{i-1}(\tau), \tau)) \otimes \mathbb{C}) \)).

3. If \( 1 < i \leq e^+(\tau) \) (resp. \( 1 < i \leq e^-(\tau) \)) and \( a > |E_{i-1}(\Delta_0)| \) (resp. \( a > |E_{i-1}(\Delta_{\infty})| \)), then \( g_i' = 0 \).

**Lemma 56.** If \( \chi = \chi_0 \) (resp. \( \chi = -\chi_0 \)), then \( \ker((\Lambda^{x^1} \otimes \mathbb{C})^* \to (\Lambda^{x^2} \otimes \mathbb{C})^*) \) is isomorphic to \( \mathcal{V}_{3,1,1} \) (resp. to \( \mathcal{V}_{3,1,-1} \)). After this identification, the map \( (\Lambda^{x^0} \otimes \mathbb{C})^* \to (\Lambda^{x^1} \otimes \mathbb{C})^* \) (in fact, the map \( (\Lambda^{x^0} \otimes \mathbb{C})^* \to \ker((\Lambda^{x^1} \otimes \mathbb{C})^* \to (\Lambda^{x^2} \otimes \mathbb{C})^*)) \) becomes the following map: it maps \( g \in (\Lambda^{x^0} \otimes \mathbb{C})^* \) to the sequence of restrictions of \( g \) to the lines \( \text{Span}_Q(\mathcal{N}(F^+_i(\tau), \tau)) \otimes \mathbb{C} \) for \( 0 \leq i \leq e^+(\tau) \) (resp. \( \text{Span}_Q(\mathcal{N}(F^-_i(\tau), \tau)) \otimes \mathbb{C} \) for \( 0 \leq i \leq e^-(\tau) \)).

**Proof.** Again, the positive and the negative cases are completely analogous, so we consider only one of them, for example, the case \( \chi = -\chi_0 \).

First, we should note that a function from \( (\Lambda^{x^0} \otimes \mathbb{C})^* \) is really defined on all lines \( \text{Span}_Q(\mathcal{N}(F^+_i(\tau), \tau)) \otimes \mathbb{C} \) (and the restriction mentioned in the statement of the Lemma really exists) by Lemma 13 since each normal cone \( \mathcal{N}(F^+_i(\tau), \tau) \) (for \( 0 \leq i \leq e^+(\tau) \)) is contained in (the boundary of) a cone \( \mathcal{N}(E^+_j(\tau), \tau) \) for some \( j \), \( 1 \leq j \leq e^+(\tau) \).

The isomorphism is constructed as follows. Given a sequence

\[
(g_1, \ldots, g_{e^-(\tau)}) \in \ker((\Lambda^{x^1} \otimes \mathbb{C})^* \to (\Lambda^{x^2} \otimes \mathbb{C})^*),
\]

we set

\[
g_i' = g_i |_{\text{Span}_Q(\mathcal{N}(F^+_i(\tau), \tau)) \otimes \mathbb{C}}
\]

and

\[
g_i' = g_i |_{\text{Span}_Q(\mathcal{N}(F^-_i(\tau), \tau)) \otimes \mathbb{C}}
\]

for \( 0 < i \leq e^+(\tau) \) and say that \( (g_1, \ldots, g_{e^-(\tau)}) \mapsto (g_0', \ldots, g_{e^-(\tau)}') \). Observe that by Corollary 15 we also have

\[
g_{i-1}' = g_i |_{\text{Span}_Q(\mathcal{N}(F^-_{i-1}(\tau), \tau)) \otimes \mathbb{C}}
\]

for \( 0 < i \leq e^-(\tau) \). Since \( \text{Span}_Q(\mathcal{N}(E^-_i(\tau), \tau)) \otimes \mathbb{C} \) is a two-dimensional space, and \( \text{Span}_Q(\mathcal{N}(F^-_{i-1}(\tau), \tau)) \otimes \mathbb{C} \) and \( \text{Span}_Q(\mathcal{N}(F^+_i(\tau), \tau)) \otimes \mathbb{C} \) are its noncoinciding one-dimensional subspaces, a linear function on \( \text{Span}_Q(\mathcal{N}(E^-_i(\tau), \tau)) \otimes \mathbb{C} \) is uniquely determined by its restrictions to \( \text{Span}_Q(\mathcal{N}(F^-_i(\tau), \tau)) \otimes \mathbb{C} \) and \( \text{Span}_Q(\mathcal{N}(E^-_i(\tau), \tau)) \otimes \mathbb{C} \), and these restrictions can be arbitrary linear functions. Therefore, the map we have constructed is really an isomorphism. The correctness of the explicit description of the map \( (\Lambda^{x^0} \otimes \mathbb{C})^* \to \mathcal{V}_{3,1,-1} \) in the statement of the lemma follows directly from the definition of the map \( (\Lambda^{x^0} \otimes \mathbb{C})^* \to (\Lambda^{x^1} \otimes \mathbb{C})^* \) and of the isomorphism between \( \ker((\Lambda^{x^1} \otimes \mathbb{C})^* \to (\Lambda^{x^2} \otimes \mathbb{C})^*) \) and \( \mathcal{V}_{3,1,-1} \).

**Lemma 57.** If \( \chi = a \chi_0 \) (resp. \( \chi = -a \chi_0 \)), where \( a \in \mathbb{N}, a \geq 2 \), then \( \ker((\Lambda^{x^1} \otimes \mathbb{C})^* \to (\Lambda^{x^2} \otimes \mathbb{C})^*) \) is isomorphic to \( \mathcal{V}_{3,1,a} \) (resp. to \( \mathcal{V}_{3,1,-a} \)). After this identification, the map \( (\Lambda^{x^0} \otimes \mathbb{C})^* \to (\Lambda^{x^1} \otimes \mathbb{C})^* \) becomes the following map: it maps \( g \in (\Lambda^{x^0} \otimes \mathbb{C})^* = (\tilde{M} \otimes \mathbb{C})^* \) to \( (g, 0, \ldots, 0) \).
Proof. This time let us consider the case $\chi = \chi_0$, the other case is completely similar.

First, let us construct a map from $\ker((\Lambda^{1,0}\otimes\mathbb{C})^* \to (\Lambda^{1,2}\otimes\mathbb{C})^*)$ to $\nabla_{3,1,0}$. Given a sequence 

$$(g_1, \ldots, g_{e^+(\tau)}) \in \ker((\Lambda^{1,0}\otimes\mathbb{C})^* \to (\Lambda^{1,2}\otimes\mathbb{C})^*),$$

we set 

$$g_i' = g_i$$

and 

$$g_i' = g_i - g_{i-1}$$

for $1 < i \leq e^+(\tau)$. By Corollary 18, 

$$g_i|_{\text{Span}_\mathbb{Q}(\chi_i, N(F_{e_i}^\tau(\tau), \tau)) \otimes \mathbb{C}} = g_i - g_{i-1}|_{\text{Span}_\mathbb{Q}(\chi_i, N(F_{e_i}^\tau(\tau), \tau)) \otimes \mathbb{C}}$$

if $a \leq |F_{e_i}^\tau(\tau) \cap [\chi_0 = 1]|$, and $g_i = g_i-1$ if $a > |F_{e_i}^\tau(\tau) \cap [\chi_0 = 1]|$. (Here $1 < i \leq e^+(\tau)$). Recall that $E_i(\Delta_0) = F_i^\tau(\tau) \cap [\chi_0 = 1]$. So, we can say that 

$$g_i'|_{\text{Span}_\mathbb{Q}(\chi_i, N(F_{e_i}^\tau(\tau), \tau)) \otimes \mathbb{C}} = (g_i - g_{i-1})|_{\text{Span}_\mathbb{Q}(\chi_i, N(F_{e_i}^\tau(\tau), \tau)) \otimes \mathbb{C}} = 0$$

if $a \leq |E_{e_i} - (\Delta_0)|$, and $g_i' = g_i - g_{i-1} = 0$ if $a > |E_{e_i} - (\Delta_0)|$. Therefore, $(g_1', \ldots, g_{e^+(\tau)})$ really defines an element of $\nabla_{3,1,0}$, and we say that $(g_1, \ldots, g_{e^+(\tau)})$ becomes $(g_1', \ldots, g_{e^+(\tau)})$.

The inverse map can be constructed by induction on $i$. Let $(g_1', \ldots, g_{e^+(\tau)}) \in \nabla_{3,1,0}$. First, set $g_1 = g_1'$. Now suppose that we already have $g_i = g_i'$. If $a > |E_{e_i} - (\Delta_0)|$, set $g_i = g_i'$. Otherwise, $g_i'$ is a linear function on $(\tilde{M}_Q \otimes \mathbb{C})/(\text{Span}_\mathbb{Q}(\chi_i, N(F_{e_i}^\tau(\tau), \tau)) \otimes \mathbb{C})$. It gives rise to a function on $\tilde{M}_Q \otimes \mathbb{C}$, which vanishes on $\text{Span}_\mathbb{Q}(\chi_i, N(F_{e_i}^\tau(\tau), \tau)) \otimes \mathbb{C}$ and which we also denote by $g_i'$. Set $g_i = g_i + g_i'$. Then 

$$(g_i - g_{i-1})|_{\text{Span}_\mathbb{Q}(\chi_i, N(F_{e_i}^\tau(\tau), \tau)) \otimes \mathbb{C}} = 0.$$ 

Now we have a sequence $(g_1, \ldots, g_{e^+(\tau)})$ of functions on $\tilde{M}_Q \otimes \mathbb{C}$, and by Corollary 18, $(g_1, \ldots, g_{e^+(\tau)}) \in \ker((\Lambda^{1,0}\otimes\mathbb{C})^* \to (\Lambda^{1,2}\otimes\mathbb{C})^*)$. So, we have constructed a map $\nabla_{3,1,0} \to \ker((\Lambda^{1,0}\otimes\mathbb{C})^* \to (\Lambda^{1,2}\otimes\mathbb{C})^*)$. It is clear from the construction that the two maps we have are mutually inverse.

By Corollary 18, $(\Lambda^{1,0}\otimes\mathbb{C})^* = (\tilde{M}_Q \otimes \mathbb{C})^*$. Again, it is clear from the definition of the map $(\Lambda^{1,0}\otimes\mathbb{C})^* \to (\Lambda^{1,2}\otimes\mathbb{C})^*$ and from the construction of the isomorphism between $\ker((\Lambda^{1,0}\otimes\mathbb{C})^* \to (\Lambda^{1,2}\otimes\mathbb{C})^*)$ and $\nabla_{3,1,0}$ that after this identification $\ker((\Lambda^{1,0}\otimes\mathbb{C})^* \to (\Lambda^{1,2}\otimes\mathbb{C})^*) \cong \nabla_{3,1,0}$ the map $(\Lambda^{1,0}\otimes\mathbb{C})^* \to (\Lambda^{1,2}\otimes\mathbb{C})^*$ becomes the map 

$$g \in (\tilde{M}_Q \otimes \mathbb{C})^* \mapsto (g(0, \ldots, 0) \in \nabla_{3,1,0}).$$

$\square$

Corollary 17. If $\chi = \chi_0$ (resp. $\chi = -\chi_0$) and $e^+(\tau) = 1$ (resp. $e^+(\tau) = 1$), then $\dim T_{-\chi}^1(X) = 0$.

If $\chi = \chi_0$ (resp. $\chi = -\chi_0$) and $e^+(\tau) \geq 2$ (resp. $e^+(\tau) \geq 2$), then $\dim T_{+\chi}^1(X) = e^+(\tau) - 2$ (resp. $\dim T_{-\chi}^1(X) = e^+(\tau) - 2$).

Proof. We consider the case $\chi = \chi_0$, the other case is completely similar. Note that $\dim \text{Span}_\mathbb{Q}(N(F_{e_i}^\tau(\tau), \tau)) \otimes \mathbb{C} = 1$, so $\nabla_{3,1,1} = e^+(\tau) + 1$. Also note that it follows from the description of $\text{Span}_M(\Lambda^{1,0}\otimes\mathbb{C})$ in Corollary 18 and from Lemma 55 that the map $(\Lambda^{1,0}\otimes\mathbb{C})^* \to \nabla_{3,1,1}$ is in fact an embedding, so $\dim T_{-\chi}^1(X) = \dim \nabla_{3,1,1} = \dim(\Lambda^{1,0}\otimes\mathbb{C})^*$. Now, since $\ker((\Lambda^{1,0}\otimes\mathbb{C})^*)$ for different $i$ are different edges of $\tau^\nu$, we have $\dim(\Lambda^{1,0}\otimes\mathbb{C})^* = \min(3, e^+(\tau) + 1)$. Thus, $\dim(\Lambda^{1,0}\otimes\mathbb{C})^* = 2$ if $e^+(\tau) = 1$ and $\dim(\Lambda^{1,0}\otimes\mathbb{C})^* = 3$ if $e^+(\tau) \geq 2$. Finally, we have $\dim \nabla_{3,1,1} = 1 + 1 - 2 = 0$ if $e^+(\tau) = 1$ and $\dim \nabla_{3,1,1} = e^+(\tau) + 1 - 3 = e^+(\tau) - 2$ if $e^+(\tau) \geq 2$.

$\square$

Corollary 18. If $\chi = a\chi_0$ (resp. $\chi = -a\chi_0$), where $a \in \mathbb{N}$, $a \geq 2$, then $\dim T_{-\chi}^1(X)$ equals the number of indices $i$ such that $1 \leq i < e^+(\tau)$ (resp. $1 \leq i < e^-(\tau)$) and $a \leq |E_i(\Delta_0)|$ (resp. $a \leq |E_i(\Delta_{\infty})|$).

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Proof. This follows directly from the definition of $\nabla_{3,1,\alpha}$ and Lemma 57.

Now it is already easy to deduce Theorem 3 in the case when $X$ is in fact toric from Theorem 5. First, let us compute the sum

$$ \sum_{a=2}^{\infty} \dim T_{-a\chi_0}(X). $$

By Corollary 13, this sum can be decomposed into $e^+(\tau) - 1 = v(\Delta_0) - 1$ sums (indexed by $i = 1, \ldots, e^+(\tau) - 1$), and each of these sums contributes 1 for $2 \leq a \leq |E_i(\Delta_0)|$ and 0 for larger values of $a$. Therefore, the $i$th of these sums equals $|E_i(\Delta_0)| - 1$, and we have

$$ \sum_{a=2}^{\infty} \dim T_{-a\chi_0}(X) = v(\Delta_0) - 1 \sum_{i=1}^{e^+(\tau) - 1} (|E_i(\Delta_0)| - 1). $$

Observe that this sum vanishes if $v(\Delta_0) = 1$ (i.e. if $0 \in \mathbb{P}^1$ is a removable special point). Similarly,

$$ \sum_{a=-2}^{-\infty} \dim T_{-a\chi_0}(X) = v(\Delta_\infty) - 1 \sum_{i=1}^{e^+(\tau) - 1} (|E_i(\Delta_\infty)| - 1). $$

And again, this sum vanishes if $v(\Delta_\infty) = 1$, i.e. if $\infty \in \mathbb{P}^1$ is a removable special point. Now, by Corollary 15, $\dim T_{-\chi_0}(X) = 0$ if $0 \in \mathbb{P}^1$ is a removable special point, and $\dim T_{-\chi_0}(X) = v(\Delta_0) - 2$ otherwise. Similarly, $\dim T_{-\chi_0}(X) = 0$ if $\infty \in \mathbb{P}^1$ is a removable special point, $\dim T_{-\chi_0}(X) = v(\Delta_\infty) - 2$ otherwise. Hence, if $0 \in \mathbb{P}^1$ is a removable special point, then

$$ \sum_{a=1}^{\infty} \dim T_{-a\chi_0}(X) = 0, $$

and if $0 \in \mathbb{P}^1$ is an essential special point, then

$$ \sum_{a=1}^{\infty} \dim T_{-a\chi_0}(X) = v(\Delta_0) - 2 + v(\Delta_0) - 1 \sum_{i=1}^{e^+(\tau) - 1} (|E_i(\Delta_0)| - 1) $$

$$ = -1 + v(\Delta_0) - 1 + \sum_{i=1}^{v(\Delta_0) - 1} (|E_i(\Delta_0)| - 1) $$

Similarly, if $\infty \in \mathbb{P}^1$ is a removable special point, then

$$ \sum_{a=-1}^{-\infty} \dim T_{-a\chi_0}(X) = 0, $$

and if $\infty \in \mathbb{P}^1$ is an essential special point, then

$$ \sum_{a=-1}^{-\infty} \dim T_{-a\chi_0}(X) = -1 + \sum_{i=1}^{v(\Delta_\infty) - 1} (|E_i(\Delta_\infty)|). $$

Finally, recall that by Corollary 12, $\dim T_{0 \in \widetilde{M}}(X) = 0$, and we get the formula from Theorem 3.

References

[1] K. Altmann, J. Hausen, Polyhedral divisors and algebraic torus actions, Math. Ann. 334 (2006), no. 3, pp. 557–607.
[2] R. Hartshorne, *Deformation theory*, GTM 257, Springer Verlag, 2010.

[3] R. Hartshorne, *Algebraic Geometry*, GTM 52, Springer Verlag, 1977.

[4] M. Schlessinger, *Rigidity of Quotient Singularities*, Inventiones math. 14, 1971, pp. 17–26.

[5] K. Altmann, N. O. Ilten, L. Petersen. H. Süß, R. Vollmert, *The geometry of T-varieties*, Contributions to algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 17–69.

[6] K. Altmann, *One parameter families containing three-dimensional toric-Gorenstein singularities*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000, pp. 21–50.