THE SECOND COHOMOLOGY OF SMALL IRREDUCIBLE
MODULES FOR SIMPLE ALGEBRAIC GROUPS

GEORGE J. MCNINCH

Abstract. Let $G$ be a simple, simply connected and connected algebraic
group over an algebraically closed field of characteristic $p > 0$, and let $V$
be a rational $G$-module such that $\dim V \leq p$. According to a result of Jantzen,$V$ is completely reducible, and $H^1(G, V) = 0$. In this paper we show that
$H^2(G, V) = 0$ unless some composition factor of $V$ is a non-trivial Frobenius
twist of the adjoint representation of $G$.

1. Introduction

Let $G$ be a quasisimple, connected, and simply connected algebraic group over
the algebraically closed field $k$ of characteristic $p > 0$. By a $G$-module $V$, we
always understand a rational $G$-module (one given by a morphism of algebraic
groups $G \to \text{GL}(V)$). In this paper, we study the cohomology of a $G$-module $V
such that $\dim V \leq p$. By results of Jantzen [Jan96] one knows that $V$ is semisimple
and that $H^1(G, V) = 0$.

Recall that the Lie algebra $\mathfrak{g}$ of $G$ is a $G$-module via the adjoint action. Our
main result is:

**Theorem A.** Let $V$ be a $G$-module with $\dim V \leq p$. Then $H^2(G, V) \neq 0$ if and
only if $V$ has a composition factor isomorphic with a Frobenius twist $\mathfrak{g}^{[d]}$ of $\mathfrak{g}$ for
some $d \geq 1$.

Differentiating the representation of $G$ on $V$ gives a representation for the Lie
algebra $\mathfrak{g}$ on $V$. Assume that $V^g = 0$. Then the theorem says that $H^2(G, V) = 0$. For
$V$ of this sort, the vanishing of $H^2$ is a consequence of the linkage principle
for $G$ together with results in section 2 which give estimates for the dimensions of
Weyl modules whose high weights are simultaneously in the low alcove and in the
orbit $W_p \cdot 0$. In fact, the same argument shows that $H^i(G, V) = 0$ for all $i \geq 1$; see
Proposition 5.2. It was pointed out to me that an earlier version of this manuscript
contained an overly complicated proof of this observation.

The crucial case for Theorem A is when $V$ is simple, non-trivial and $V^g = V$.
There is a unique $d \geq 1$ such that the “Frobenius untwist” $V^{[-d]}$ is a $G$-module on
which $\mathfrak{g}$ acts non-trivially. We have already seen that $H^i(G, V^{[-d]}) = 0$ for $i = 1, 2$, so
Theorem A follows from the following two results (see [5,3]). [We denote by $h$ the
Coxeter number of the group $G$.]

**Theorem B.** Suppose that $p \geq h$ and that $W$ is a $G$-module for which $H^i(G, W) = 0
for i = 1, 2$. Then $H^2(G, W^{[d]}) \simeq \text{Hom}_G(\mathfrak{g}, W)$ for all $d \geq 1$.

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Theorem C. If \( p > h \), \( \dim H^2(G, g^{[d]}) = 1 \) for all \( d \geq 1 \). For any \( p \), there is a \( d_0 \geq 1 \) so that \( H^2(G, g^{[d]}) \neq 0 \) for all \( d \geq d_0 \).

Theorem B is proved in \( \text{[2.3]} \); it immediately implies the first assertion of Theorem C (see \( \text{[2.3]} \)). We give a proof the second assertion of Theorem C in section \( \text{5.6} \).

We end the paper by applying the results of section 2 to calculations of cohomology groups \( H^i(G_1, L) \), where \( G_1 \) is the Frobenius kernel, and \( L \) is a simple \( G_1 \) module with \( \dim L \leq p \); see Proposition \( \text{[6]} \).

We conclude this introduction by remarking that the result of Jantzen \( \text{[Jan96]} \) cited above is one of several recent results studying the semisimplicity of low dimensional representations of groups in characteristic \( p \). See \( \text{[Ser94, McN98, McN99, Gur99, McN00]} \) for related work.

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2. Root systems

2.1. We denote by \( R \) an indecomposable root system in its weight lattice \( X \) with simple roots \( S \subset R^+ \). For each \( \alpha \in S \), there is a fundamental dominant weight \( \varpi_\alpha \in X \); the \( \varpi_\alpha \) form a \( \mathbb{Z} \) basis of \( X \).

We write \( \alpha_0 \) for the dominant short root, and \( \tilde{\alpha} \) for the dominant long root in \( R \) (these coincide in case there is only one root length).

The Coxeter number of \( R \) is given by

\[
  h - 1 = \sup_{\alpha \in R^+} \{ \langle \rho, \alpha^\vee \rangle \} = \langle \rho, \alpha_0^\vee \rangle.
\]

For \( m \in \mathbb{Z} \) and \( \alpha \in R \), let \( s_{\alpha,m} \) denote the affine reflection of \( X_\mathbb{R} = X \otimes_\mathbb{Z} \mathbb{R} \) in the hyperplane \( H_{\alpha,m} = \{ x \in X_\mathbb{R} : \langle x, \alpha^\vee \rangle = m \} \).

Let \( l > h \) be an integer. The affine Weyl group \( \tilde{W}_l \) is the group of affine transformations of \( X_\mathbb{R} \) generated by all \( s_{\alpha,n} \) for \( n \in \mathbb{Z} \). According to \( \text{[Bou72, ch. VI, \S 2.1, Prop. 1]} \) \( \tilde{W}_l \) is isomorphic to the semidirect product of \( W \) (the finite Weyl group) with \( l\mathbb{Z}R \). The normalizer of \( \tilde{W}_l \) in the full affine transformation group of \( X_\mathbb{R} \) contains all translations by \( lX \), hence \( \tilde{W}_l \) is a normal subgroup of \( \tilde{W} \), the semidirect product of \( W \) and \( lX \). Moreover, \( \tilde{W}_l/W_l \simeq lX/l\mathbb{Z}R \simeq X/\mathbb{Z}R \) is the fundamental group of \( R \), which we will denote by \( \pi \).

Let \( \rho = \frac{1}{2} \sum_{\alpha \in S} \alpha \). We always consider the dot action of \( \tilde{W}_l \) (also of \( W \) and \( W_l \)) on \( X \): for \( w \in \tilde{W}_l \) and \( \lambda \in X \), this is given by \( w \cdot \lambda = w(\lambda + \rho) - \rho \).

The subset \( C_l \) of \( X_\mathbb{R} \) given by

\[
  C_l = \{ \lambda \in X_\mathbb{R} : 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l \text{ for each } \alpha \in R^+ \}
\]

is a fundamental domain for the dot action of \( W_l \) on \( X \); its conjugates under \( \tilde{W}_l \) are known as alcoves, and \( C_l \) is the lowest alcove. Since \( \tilde{W}_l \) normalizes \( W_l \), \( \tilde{W}_l \) permutes the alcoves. Since \( \tilde{W}_l \) normalizes \( W_l \), \( \tilde{W}_l \) permutes the alcoves simply transitively, one deduces that \( \tilde{W}_l \) is the semidirect product of \( \Omega \) and \( W_l \). Thus \( \Omega \simeq \tilde{W}_l/W_l \simeq \pi \).
2.2. Let $I$ index the simple roots $S = \{\alpha_i\}$, write $\alpha_0 = \sum_{i \in I} n_i \alpha_i$, and put $J = \{i \in I \mid n_i = 1\}$. A dominant weight $0 \neq \varpi \in X$ is minuscule if whenever $\lambda \leq \varpi$ and $\lambda$ is a dominant weight, then $\varpi = \lambda$. According to [Bou72, Ch. VI, exer. 23,24], $\varpi$ is minuscule just in case $\varpi = \varpi_i$ for some $i \in J$.

For $\ell \geq 1$ and equality if and only if $\ell > h$. Let $w_{0i} \in W_i$ be the unique element which makes all positive roots in $R_i$ negative.

For $x \in X$, let $t(x)$ denote the affine translation by $x$; for $i \in J$, let $\gamma_i = t(l \varpi_i)w_{0i} \in \hat{W}_i$. Note that $\gamma_i$ represents $\varpi_i \in X/ZR \cong lX/lZR \cong \hat{W}_i/W_i$.

Applying [Bou72, ch. VI, §2.2 Prop. 6 and Cor.] one obtains:

**Proposition.** (a) Each non-0 coset of $ZR$ in $X$ is uniquely represented by a minuscule weight. In particular, $|\pi| = |J| + 1$.

(c) The non-identity elements of $\Omega$ are precisely the $\gamma_i$ for $i \in J$. We have

$$\hat{W}_i \bullet 0 \cap C_i = \{0\} \cup \{\gamma_i \bullet 0 = (l-h)\varpi_i \mid i \in J\}$$

2.3. For a dominant weight $\lambda$, let

$$d(\lambda) = \prod_{\alpha > 0} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

be the value of Weyl's degree formula at $\lambda$.

**Proposition.** Let $\lambda = (l-h)\varpi_i$ for some $i \in J$.

(a) $d(\lambda) \geq \binom{l-1}{l-h}$, with equality if and only if $h-1 = \ell(w_{0i}w_i)$.

(b) If $1 - h \geq 2$ and $h \geq 3$, then $d(\lambda) > l$.

**Proof.** For $1 \leq k \leq h-1$, let $e(k)$ be the number of $\alpha \in R_i^+ \setminus R_i^+$ with $\langle \rho, \alpha^\vee \rangle = k$. The argument in the remark on p. 520-521 of [Ser94] (following Prop. 6) shows that $e(k) \geq 1$ for each $1 \leq k \leq h-1$. Thus, we have

$$d(\lambda) = \prod_{k=1}^{h-1} \frac{l-h+k}{k} \geq \prod_{k=1}^{h-1} \frac{l-h}{k} = \binom{l-1}{l-h}.$$ 

If $\ell(w_{0i}w_i) = |R_i^+| - |R_i^+| = h-1$, then $e(k) = 1$ for each $1 \leq k \leq h-1$ and equality holds. This proves (a).

For (b), note that under the given hypothesis we have $l \geq 5$. Since $\binom{l-1}{l-h} \geq \binom{l-2}{l-2} > l$ for all such $l$, (b) follows immediately.

**Remark.** Using the table in the proof of Proposition 2.4 below, it is straightforward to verify that equality holds in (a) if and only if either $R = A_r$ and $i \in \{1, r\}$ or $R = C_r$ and $i = 1$. (Since $B_2 = C_2$, the latter case includes $B_2$ and $i = 2$.)
2.4. In the following, let me emphasize the standing assumption $l > h$.

**Proposition.** If $0 \neq \lambda \in \hat{W}_1 \cap C$ and $d(\lambda) < l$ then $d(\lambda) = l - 1$ and $(R, \lambda)$ is listed in the following table. If the rank of $R$ is $\geq 2$, then $l = h + 1$.

| $R$   | $l$ | $\lambda$   |
|-------|-----|-------------|
| $A_1$ | any | $(l - 2)\varpi_1$ |
| $A_{l-2}$ | $\varpi_1, \varpi_{l-2}$ |
| $B_2$ | $l = 5$ | $\varpi_2$ |
| $C_{(l-1)/2}$ | $l$ odd | $\varpi_1$ |

**Proof.** The rank 1 situation leads to the item listed in the table. When the rank is at least 2, one applies Proposition 2.3 to obtain $l = h + 1$, whence $\lambda = \varpi_i$ for some $i \in J$; i.e. $\lambda$ is minuscule.

We handle the minuscule cases by classification. For each indecomposable root system $R$ for which $J \neq \emptyset$, we list in the following table the Coxeter number, the set $J$, and the value $d(\varpi_i)$ for each $i \in J$. The simple roots are indexed as in the tables in [Bou72, Planche I-X]; the data recorded here, with the exception of the values $d(\varpi_i)$, may be verified by inspecting those tables as well. The values $d(\varpi_i)$ are well known (and can anyway be computed from the formula, or by representation theoretic arguments).

| Type of $R$ | $h$ | $J$ | $d(\varpi_i)$, $i \in J$ |
|-------------|-----|-----|-------------------|
| $A_r$       | $r + 1$ | $\{1, 2, \ldots, r\}$ | $\binom{r + 1}{2}$ |
| $B_r, r \geq 2$ | $2r$ | $\{r\}$ | $2^r$ |
| $C_r, r \geq 2$ | $2r$ | $\{1\}$ | $2r$ |
| $D_r, r \geq 4$ | $2r - 2$ | $\{1, r - 1, r\}$ | $2r, 2^{r-1}, 2^{r-1}$ respectively |
| $E_6$       | 12  | $\{1, 6\}$ | 27, 27 |
| $E_7$       | 18  | $\{7\}$ | 56 |

From this table, one can list all pairs $(R, \lambda)$ for which $R$ has Coxeter number $l - 1$ and $\lambda$ is minuscule. It is a simple matter to see that $d(\lambda) < l$ only when $(R, \lambda)$ is as claimed.

3. The algebraic groups

3.1. Let $k$ be an algebraically closed field of characteristic $p > 0$, and let $G$ be a connected, simply connected semisimple algebraic $k$-group. The non-0 weights of a maximal torus $T \leq G$ on $g = \text{Lie}(G)$ form an indecomposable root system $R$ of rank $r = \dim T$ in the character group $X = X^*(T)$. Since $G$ is simply connected, $X$ identifies with the full weight lattice of $R$ as in section 2. We fix a choice of simple roots $S$ and positive roots $R^+$. The dominant weights are denoted $X^+$. The group $G$ is assumed to be quasisimple; i.e. the root system $R$ is indecomposable.

3.2. For each dominant weight $\lambda \in X^+$, the space of global sections of the corresponding line bundle on the flag variety affords an indecomposable rational $G$-module $H^0(\lambda)$ with simple socle. The modules $L(\lambda) = \text{soc} H^0(\lambda)$ comprise all of the simple rational modules for $G$ (and are pairwise non-isomorphic).

The character of each $H^0(\lambda)$ is the same as in characteristic 0; hence in particular $\dim_k H^0(\lambda)$ is given by the Weyl degree formula, whose value at $\lambda$ we denote $d(\lambda)$ as in 2.3.
3.3. Any dominant $\lambda$ may be written as a finite sum $\sum_{i \geq 0} p^i \lambda_i$ with each $\lambda_i$ a restricted weight. Recall that a dominant weight $\mu$ if $\langle \mu, \alpha^\vee \rangle < p$ for all simple roots $\alpha$. Steinberg’s tensor product theorem says:

$$L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes L(\lambda_2)^{[2]} \otimes \cdots$$

where for a $G$-module $V$, $V^{[m]}$ standards for the $m$-th Frobenius twist of $V$.

For $d \geq 1$, let $G_d$ be the $d$-th Frobenius kernel of $G$. Let $V$ be a rational $G$-module and $m \geq 1$. If there is a rational $G$ module $W$ with $W^{[m]} \simeq V$, we regard $W$ as the Frobenius untwist $W = V^{-[m]}$ of $V$. Now regard $V$ as a module for $G_d$. Since $G_d$ is a normal subgroup scheme, $G$ acts on $V^{G_d}$; since $G_d$ acts trivially on this $G$-module, there is an untwisted rational $G$-module $(V^{G_d})^{-[d]}$. It follows that there is an untwist $H^i(G_d, V)^{-[d]}$ for all $i \geq 0$.

Consider now two $G$-modules $V_1$ and $V_2$, and form $W = V_1 \otimes V_2^{[d]}$. The Frobenius kernel $G_d$ acts trivially on $V_2^{[d]}$, so that

$$(1) \quad H^i(G_d, W)^{-[d]} \simeq H^i(G_d, V_1)^{-[d]} \otimes V_2$$

as $G$-modules for every $i \geq 0$.

3.4. Let $W_p \leq \overline{W}_p$ be as in section 2 (for $l = p$), let $C = C_p \cap X^+$ denote the dominant weights in the lowest alcove, and let $\overline{C} = \overline{C}_p \cap X^+$ ($\overline{C}_p$ is the closure in $X_\mathbb{R}$).

**Proposition.** Let $\lambda \in X^+$.

(a) If $H^i(G, L(\lambda)) \neq 0$ for some $i \geq 0$, then $\lambda \in W_p \bullet 0$.

(b) If $H^i(G_1, L(\lambda)) \neq 0$ for some $i \geq 0$, then $\lambda \in \overline{W}_p \bullet 0$.

(c) $H^i(G, H^0(\lambda)) = 0$ for all $i > 0$.

(d) If $\lambda \in \overline{C}$, then $L(\lambda) = H^0(\lambda)$; in particular, $\dim L(\lambda) = d(\lambda)$.

**Proof.** (a) follows from the linkage principle for $G$ [Jan87, Cor. II.6.17], and (b) from the linkage principle for $G_1$ [Jan87, Lemma II.9.16]. (c) follows from [Jan87, II.4.12]. (d) follows from [Jan87, II.6.13, II.5.10].

4. THE LIE ALGEBRA AND THE COHOMOLOGY OF $G_1$

We want to describe explicitly the cohomology $H^*(G_1, k)$ in degree $\leq 2$. For this, we need some information on the Lie algebra $\mathfrak{g}$.

4.1. Recall that the prime $p$ is bad [= not good] for the indecomposable root system $R$ if one of the following holds: $p = 2$ and $R$ is not of type $A_r$; $p = 3$ and $R$ is of type $G_2, F_4$, or $E_7$; $p = 5$ and $R$ is of type $E_8$.

The prime $p$ is very good if it is not bad, and in case $R = A_r$, if also $p$ does not divide $r + 1$.

Application of the summary in [Hum95, 0.13] yields:

**Lemma A.** Assume that $p$ is very good. Then $\mathfrak{g}$ is a simple Lie algebra. The adjoint $G$-module is simple, self-dual, and isomorphic with $L(\hat{\alpha})$ where $\hat{\alpha}$ is the dominant long root.

Notice that if $p > h$, then $p$ is very good.

**Lemma B.** Assume that $p \geq h$. If $W$ is any $G$-module, then $\text{Hom}_{G}(\mathfrak{g}, W^{[d]}) = 0$ for $d \geq 1$. 
Proof. When \( p > h \) this follows since \( \mathfrak{g} \) is a simple \( \mathfrak{g} \)-module with restricted highest weight. When \( p = h \), we have \( R = A_{p-1} \). Since \( G \) is simply connected, we have \( \mathfrak{g} = \mathfrak{s}t_p \). Thus \( \mathfrak{g} \) is an indecomposable \( \mathfrak{g} \)-module with unique simple quotient \( L(\hat{\chi}) \), and the lemma follows. \( \square \)

4.2. Let \( B \) be a Borel subgroup of \( G \), and let \( \mathfrak{u} \) be the nilradical of \( \text{Lie}(B) \). Regarding \( \mathfrak{u}^* \) as a \( B \)-module, we get a vector bundle on \( G/B \) which we also write as \( \mathfrak{u}^* \). According to \( [\text{AJ84}, 3.8] \), the formal character of the \( G \)-module \( H^0(G/B, \mathfrak{u}^*) \) is \( \chi(\hat{\alpha}) = \text{ch}(\mathfrak{u}^*) \).

Let \( \mathcal{N} \subset \mathfrak{g} \) be the nilpotent cone. There is by \( [\text{AJ84}, 3.9] \) an injective homomorphism of graded algebras \( k[\mathcal{N}] \to H^0(G/B, \mathfrak{u}^*) \).

**Lemma.** For simply connected, quasisimple algebraic groups \( G \), \( \mathfrak{g}^* \simeq k[\mathcal{N}]_1 \simeq H^0(G/B, \mathfrak{u}^*) \).

**Proof.** Let \( I(\mathcal{N}) \triangleleft k[\mathfrak{g}] = S\mathfrak{g}^* \) be the (homogeneous) defining ideal of the variety \( \mathcal{N} \). We need to show that \( I(\mathcal{N}) = 0 \). If not, then \( \mathcal{N} \subset V \subset \mathfrak{g} \) for some proper \( G \)-submodule \( V \). A look at the summary in \( [\text{Hum95}, 0.13] \) shows that, since \( G \) is simply connected, the only \( G \)-submodules of \( \mathfrak{g} \) have dimension 0 or 1. On the other hand, by \( [\text{Hum95}, \text{Theorem 6.19}] \), the variety \( \mathcal{N} \) has codimension \( \text{rank}(G) \) in \( \mathfrak{g} \) and so clearly can’t be contained in a 1 dimensional linear subspace! \( \square \)

**Remarks.**

1. Here is a fancier result which implies the lemma if we assume that the prime \( p \) is good for \( G \). Since \( G \) is simply connected and \( p \) is good, the Springer resolution

\[
\varphi : \tilde{\mathcal{N}} = G \times_B \mathfrak{u} \to \mathcal{N}
\]

given by \( (g, X) \mapsto \text{Ad}(g)(X) \) is a desingularization, hence in particular a birational map; see \( [\text{Hum95}, \text{Theorem 6.3 and Theorem 6.20}] \). Again since \( G \) is simply connected and \( p \) is good, the variety \( \tilde{\mathcal{N}} \) is normal (\( [\text{Hum95}, \text{Theorem 4.24}] \)). Standard arguments then yield an isomorphism of graded algebras \( k[\tilde{\mathcal{N}}] \overset{\varphi^*}{\to} \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}) \). Finally, the projection \( \tilde{\mathcal{N}} \to G/B \) is an affine morphism, so that \( \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}) = H^0(G/B, \mathfrak{u}^*) \) as a graded algebra.

2. On the other hand, if \( G = PGL_r \), and \( p \mid r \), one can find a linear form on \( \mathfrak{g} \) that vanishes on \( \mathcal{N} \), hence there can be no isomorphism \( k[\mathcal{N}]_1 \to H^0(G/B, \mathfrak{u}^*) \) (compare formal characters). So the lemma can fail when \( G \) is not simply connected. [Note that \( \varphi \) is not birational in this example. One can show that there is a \( G_{sc} \)-isomorphism \( \psi : \tilde{\mathcal{N}}_{sc} \to \tilde{\mathcal{N}} \) (using some obvious notations). We get therefore a commuting diagram:

\[
\begin{array}{ccc}
\tilde{\mathcal{N}} & \xrightarrow{\varphi_{sc}} & \tilde{\mathcal{N}}_{sc} \\
\gamma \downarrow & & \downarrow \\
\mathcal{N} & &
\end{array}
\]

The map \( \varphi_{sc} \circ \psi^{-1} \) is birational. Since \( \gamma^* k(\mathcal{N}) \subset k(\mathcal{N}_{sc}) \) is a proper purely inseparable extension, so too is \( \varphi^* k(\mathcal{N}) \subset k(\mathcal{N}) \).

**Proposition.**

1. If \( p \neq 2 \) or if \( R \) is not of type \( C_r \), then \( H^1(G_1, k) = 0 \).

2. Assume that \( p \geq h \). Then \( H^2(G_1, k)[-1] \simeq \mathfrak{g}^* \) as \( G \)-modules.
Proof. For (1) see \[Jan87\], Lemma II.12.1. For (2), first suppose that $p > h$. By \[AJ84\] 3.7.3.9, there is a $G$-equivariant isomorphism of graded rings $k[N]' \cong H^*(G_1,k)[-1]$ where $k[N]'$ is again the graded coordinate ring of $N$, but with the linear functions on $g$ given degree 2. The claim now follows from the lemma.

When $p = h$, apply \[AJ84\] Cor. 6.3] to see that $H^2(G_1,k)[-1] \cong H^0(G/B,u^*)$; the claim follows again from the lemma in this case.

\[\Box\]

5. Low dimensional modules for $G$

5.1. We recall first some facts about low dimensional modules established in \[Jan96\] and \[Ser94\].

**Proposition.** Let $L$ be a simple non-trivial restricted $G$ module with highest weight $\lambda$. Suppose that $\dim L \leq p$.

(a) $\lambda \in \bar{C}$.

(b) $\lambda \in C$ if and only if $\dim_k L < p$.

(c) $h \leq p$. If moreover $\dim L < p$, then $h < p$.

(d) If $R$ is not of type $A$ and $\dim L = p$, then $h < p$. If $p = h$ and $\dim L = p$, then $R = A_{p-1}$ and $\lambda = w_i$ with $i \in \{1, p-1\}$.

**Proof.** (a) follows from \[Jan96\], Lemma 1.4], and (b) from \[Jan96\], 1.6], see also \[Ser94\]. For (c), note first that (a) implies $\dim L = d(\lambda)$ by Proposition 5.4(d). If $\lambda \in C \setminus C$, then (a) and (b) imply that $\dim L = p$, whence $p = h$ follows from Weyl’s degree formula. (c) now follows since $C$ is empty if $p < h$ and $C = \{0\}$ if $p = h$.

In \[Jan96\], 1.6], Jantzen made a list of all simple restricted modules for $G$ with dimension $p$. Inspecting that list yields (d).

5.2. Vanishing results when $g$ acts non-trivially. Let $L$ be a simple module for $G$.

**Proposition.** If $G_1$ (equivalently, $g$) acts non-trivially on $L$ and $\dim L \leq p$, then $H^i(G,L) = 0$ for all $i \geq 0$.

**Proof.** Write the highest weight of $L$ as $\lambda = \mu_1 + \mu_2$ with $\mu_1$ restricted. Since $L^g = 0$, we have $\mu_1 \neq 0$. Since $p \geq \dim L \geq \dim L(\mu_1)$, Proposition 5.1 implies that $\mu_1 \in \bar{C}$ and that $h \leq p$. We have in particular that $L(\mu_1) = H^0(\mu_1)$, hence the proposition will follow from Proposition 5.4 if we show that $\mu_2$ is 0.

If $\dim L = p$, Steinberg’s tensor product theorem gives $\mu_2 = 0$. If $\dim L < p$ then 5.3 shows that $p < h$ and $\mu_1 \in C$. If $H^i(G,L) \neq 0$ for some $i$, then $\lambda \in W_p \cdot 0$ by the linkage principle, whence $\mu_1 \in W \cdot 0 + pX = \hat{W}_p \cdot 0$. Now Proposition 2.4 applies; it shows that $\dim L(\mu_1) = p - 1$ whence we have $\mu_2 = 0$ by another application of Steinberg’s theorem.

5.3. Second cohomology. Here we prove our main tool for describing second cohomology; first we require the following:

**Lemma.** Let $E_2^{p,q} \Rightarrow H^{p+q}$ be a convergent, first quadrant spectral sequence.

1. If $E_2^{0,1} = E_2^{1,1} = E_2^{0,2} = 0$, then $H^2 \cong E_2^{2,0}$

2. If $E_2^{1,0} = E_2^{1,1} = E_2^{2,0} = 0$, then $H^2 \cong E_2^{0,2}$.
The second cohomology of small modules.

5.4. Theorem. Suppose that $E_{\infty}^{2,0} \simeq E_{2}^{2,0}$; first note that $E_{3}^{2,0}$ is the cohomology of the sequence

$$E_{2}^{0,1} \to E_{2}^{2,0} \to E_{2}^{-1,1}$$

from which we get $E_{3}^{2,0} \simeq E_{2}^{2,0}$. For any first quadrant spectral sequence one has (by similar reasoning) that $E_{a}^{2,0} \simeq E_{a+1}^{2,0}$ for $a > 2$, so we get the desired isomorphism.

Proof. We verify (1), the argument for (2) is the same. We must show that $E_{1}^{2,0} \simeq E_{2}^{2,0}$, first note that $E_{3}^{2,0}$ is the cohomology of the sequence

$$E_{2}^{0,1} \to E_{2}^{2,0} \to E_{2}^{-1,1}$$

Thus $E_{1}^{2,0} = 1$, by Lemma 4.2(2) shows now that $H^2(G,V[d]) \simeq H^2(G,V[d-1])$. We get now (1) by induction on $d$. Lemma 4.2(2) shows now that $H^2(G_1,k) \simeq g^*$. Thus, the only possible non-0 $E_2$ terms of total degree 2 are

$$E_2^{0,2} = H^0(G,g^* \otimes V[d-1]) = \operatorname{Hom}_G(g,V[d-1])$$

$$E_2^{2,0} = H^2(G,V[d-1]).$$

For $d > 1$, we apply Lemma B to see that $E_2^{0,2} = 0$ whence $H^2(G,V[d]) \simeq H^2(G,V[d-1])$ by part (1) of the lemma; thus (2) will follow provided it holds for $d = 1$. In that case, we have $E_2^{0,2} = 0$ by assumption, and the result just proved in part (1) shows that $E_2^{1,0} = 0$. Thus part (2) of the lemma applies; it shows that $H^2(G,V[1]) \simeq E_2^{0,2} = \operatorname{Hom}_G(g,V)$ as desired.

5.4. The second cohomology of small modules. Let $L = L(\lambda)$ be a simple $G$-module, and suppose that $\dim L \leq p$. Proposition 5.2 showed that the vanishing of cohomology for $L$ is a consequence of the linkage principle when $\lambda \notin pX$. However, if $\lambda \notin p\mathbb{Z}/p$ $\lambda$ is linked to 0, so the linkage principle does not yield vanishing. The following result shows that, despite the linkage of $\lambda$ and 0 in this case, the second cohomology is usually 0.

Theorem. Let $L$ be a simple $G$-module with $\dim L \leq p$. If $H^2(G,L) \neq 0$, then $L \simeq g[d]$ for some $d \geq 1$.

Proof. Let $L'$ be such that $L \simeq (L')[d]$ for $d \geq 0$, and such that $g$ acts non-trivially on $L'$. We have by 5.1 that $p \geq h$. Also, we have by Proposition 5.2 that $H^i(G,L') = 0$ for $i \geq 1$. If $d = 0$, we are done. If $d > 0$, then Theorem 5.3 applies, and we get that

$$H^2(G,L) \simeq \operatorname{Hom}_G(g,L').$$
We get by Proposition 3.4 that $p > h$ unless $R = A_{p-1}$ and $L' = L(\omega_i)$ with $i \in \{1, p-1\}$. If $p > h$, then $g$ is a simple $G$-module by Lemma 3.4. So if $\text{Hom}_G(g, L') \neq 0$ then $L' \cong g$ whence $L \cong g^{[d]}$ as claimed.

In the remaining case, one must just note that weight considerations yield $\text{Hom}_G(g, L(\omega_i)) = 0$ for $i = 1, p-1$, whence $H^2(G, L) = 0$. \hfill \Box

5.5. The second cohomology of twists of the adjoint module. The first assertion of Theorem 3 of the introduction follows from the following:

**Proposition.** Assume that $p > h$. Then $H^1(G, g^{[d]}) = 0$ and $H^2(G, g^{[d]}) \simeq \text{End}_G(g)$ has dimension 1 for $d \geq 1$.

**Proof.** Since $p > h$, Lemma 4.1 shows that $g$ is the simple module with highest weight $\tilde{\alpha}$. It follows that $g = H^0(\tilde{\alpha})$, and thus that $H^i(G, g) = 0$ for $i \geq 1$ by Proposition 5.4. The proposition now follows from Theorem 5.3. \hfill \Box

**Remark.** Note that $\dim g > h$ (in fact, $\dim g = (h + 1)r$ where $r$ is the rank of $G$). So we get also: if $\dim g \leq p$, then $\dim H^2(G, g^{[d]}) = 1$ for $d \geq 1$.

5.6. A second proof. Here we give a second proof of the non-vanishing of $H^2$ for twists of the adjoint module; the result proved here verifies the remaining assertion of Theorem 3 of the introduction. We have included the argument since it offers some “explanation” for the non-vanishing.

The group $G$ arises by base change from a split reductive group scheme $G$ over $\mathbb{Z}$. Let $\mathbb{Z}_p$ be the complete ring of $p$-adic integers, and let $\mathbb{Q}_p$ be its field of quotients. For any finite field extension $F$ of $\mathbb{Q}_p$, let $\mathfrak{o}$ denote the integers in $F$. The residue field $\mathfrak{o}/m$ may be identified with the extension $\mathbb{F}_q$ of $\mathbb{F}_p$.

Let $K$ denote the group of points $G(\mathfrak{o})$ regarded as a subgroup of $G(F)$. Since $G$ is smooth, the reduction homomorphism $K \to G(\mathbb{F}_q)$ is surjective (see [Tit79, 3.4.4]).

For $n \geq 1$, let $K_n \subset K$ be the kernel of the map $K \to G(\mathfrak{o}/m^n)$. Note that $K/K_1 = G(\mathbb{F}_q)$ acts by conjugation on each quotient $K_n/K_{n+1}$.

**Proposition.** (a) There is for each $m \geq 1$ a canonical isomorphism $K_m/K_{m+1} \cong g_{\mathbb{F}_q}$ as representations for $G(\mathbb{F}_q)$, where $g_{\mathbb{F}_q}$ is the Lie algebra of $G_{\mathbb{F}_q}$.

(b) If $H^2(G(\mathbb{F}_q), g_{\mathbb{F}_q}) = 0$, the exact sequence of groups

$$1 \to K_1 \to K \to G(\mathbb{F}_q) \to 1$$

splits.

(c) There is a $p$-power $q_0$, depending only on the root system $R$ of $G$, such that $H^2(G(\mathbb{F}_q), g_{\mathbb{F}_q}) \neq 0$ whenever $q \geq q_0$.

(d) There is an integer $a_0 \geq 1$ such that $H^2(G, g^{[a]}) \neq 0$ whenever $a \geq a_0$.

**Proof.** (a) Follows from [DG76, II.§4.3]. (b) Since $K_1$ is a pro-$p$ group [PR94, Lemma 3.8], this follows from [Ser67, Lemma 3].

(c) Choose a $\mathbb{Q}_p$ vectorspace $V$ and a non-trivial faithful $\mathbb{Q}_p$-rational representation $G_{\mathbb{Q}_p} \to \text{GL}(V)$. For each extension $F$ of $\mathbb{Q}_p$ with integers $\mathfrak{o}$, the group $K = G(\mathfrak{o})$ is a subgroup of (the group of $F$-points of) $\text{GL}(V_F)$. If $H^2(G(\mathbb{F}_q), g_{\mathbb{F}_q}) = 0$, the sequence in (b) is split and $V_F$ is a non-trivial $F[G(\mathbb{F}_q)]$-module.

Since $F$ has characteristic 0, it is well known that the minimal dimension of a non-trivial $F[G(\mathbb{F}_q)]$ module is bounded below by the value $f(q)$ of a polynomial $f \in \mathbb{Q}[x]$, depending only on $G$, for which $f(q) \to \infty$ as $q \to \infty$. We may choose $q_0$ such that $f(q) > \dim_{\mathbb{Q}_p} V$ for each $q > q_0$, and (c) follows at once.
(d) now follows from (c) and [CPSvdK77 Cor. 6.9].

6. SMALL SIMPLE MODULES FOR $G_1$

Combining results of [KLT99] with the results recorded in 2.4, we obtain some explicit results on $G_1$ cohomology of low dimensional simple modules:

**Proposition.** Let $L$ be a non-trivial simple $G_1$ module with dim $\leq p$. Assume for some $i \geq 0$ that $H^i(G_1, L) \neq 0$. Then dim $L = p-1$. Moreover, there is a quadruple $(R, \lambda, i(0), V)$ in the following table for which $R$ is the root system of $G$, $\lambda$ the high weight of $L$, $i \geq i(0)$ and $H^{i(0)}(G_1, L)[-1] \simeq V$ as $G$-modules.

| $R$       | $\lambda$ | $i(0)$ | $H^{i(0)}(G_1, L)[-1]$ |
|-----------|-----------|--------|------------------------|
| $A_1$     | $(p-2)\varpi_1$ | 1      | $L(\varpi_1)$          |
| $A_{p-2}$ | $\varpi_1, \varpi_{p-2}$ | $p-2$  | $L(\lambda)$          |
| $C_{(p-1)/2}$ | $p$ odd | $\varpi_1$ | $p-2$ | $L(\lambda)$ |

**Proof.** By [Jan84, Prop. II.3.14], $L = \text{res}_{G_1}^G L(\lambda)$ for some restricted dominant weight $0 \neq \lambda$. Thus $L(\lambda)$ is a restricted, simple $G$ module with dimension $\leq p$. It follows from Proposition 5.1 that $h \leq p$, that $\lambda \in \mathcal{C}$, and that $L = H^0(\lambda)$ as modules for $G$.

Suppose that $H^i(G_1, L) \neq 0$ for some $i$. By the linkage principle for $G_1$ (Proposition 3.3(b)), we must have $\lambda \in \mathcal{W}_p \bullet 0$, hence $\lambda \in \mathcal{C}$. This implies that $h < p$. Proposition 2.3 shows that $\lambda = (p-h)\varpi_i = w_0 w_i \bullet 0 + p \varpi_i$ for some $i \in J$, and Proposition 2.4 yields dim $L = p-1$ and lists the possible pairs $(R, \lambda)$.

For $h < p$, Kumar, Lauritzen and Thomsen [KLT99, Theorem 8] have extended a result of Andersen and Jantzen [AJ84, 3.7]; this result implies in particular that the minimal degree for which $H^*(G_1, L)$ is non-0 is $\ell(w_0 w_i)$, and that $H^{\ell(w_0 w_i)}(G_1, L)[-1] \simeq H^0(\varpi_i)$.

It is straightforward to compute for each pair $(R, \lambda)$ the length $\ell(w_0 w_i)$; one gets in this way the result.

**Remark.** The Theorem implies the fact (used by Jantzen in the proof of [Jan96, Lemma 1.7]) that $H^1(G_1, L) = 0$ for all simple $G_1$ modules $L$ with dim $L \leq p$. The argument used by Jantzen there relied on the calculations of $H^1$ carried out in [Jan91].

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E-mail address: McNinch.1@nd.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556 USA