STRONG SHIFT EQUIVALENCE IN THE $C^*$-ALGEBRAIC SETTING: GRAPHS AND $C^*$-CORRESPONDENCES

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Abstract. We discuss strong shift equivalence, which has been used to characterize conjugacy of edge shifts, and its application to $C^*$-algebras of graphs and Cuntz-Pimsner algebras.

1. Background and Motivation: Edge Shifts of Finite Graphs

A directed graph (hereafter simply called a graph) is a quadruple $E = (E^0, E^1, r, s)$ consisting of a countable set of vertices $E^0$, a countable set of edges $E^1$, and functions $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge. We say a graph is finite if $E^0$ and $E^1$ are finite sets. If $E$ is a finite graph, then one may create a (two-sided) shift space $X_E$ with shift map $\sigma_E : X_E \to X_E$ defined by

$$X_E := \{(e_i)_{i \in \mathbb{Z}} : e_i \in E^1 \text{ for all } i \in \mathbb{Z} \text{ and } r(e_i) = s(e_{i+1})\}$$

and $\sigma_E((e_i)_{i \in \mathbb{Z}}) = (e_{i+1})_{i \in \mathbb{Z}}$. The shift spaces that arise in this way are called edge shifts. We refer the reader to [12] for more about shift spaces and edge shifts.

For a given graph $E$, the vertex matrix $A_E$ is the non-negative $E^0 \times E^0$ matrix defined by

$$A_E(v, w) = \#\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}.$$ 

Definition 1.1. Let $A$ and $B$ be finite, square, non-negative, integer matrices. We say that $A$ and $B$ are elementary strong shift equivalent if there exist non-negative, integer matrices $R$ and $S$ such that $A = RS$ and $B = SR$. (Note that $A$ and $B$ need not have the same size.)

Elementary strong shift equivalence is a relation that is reflexive and symmetric, but not transitive. Therefore, we consider the equivalence relation it generates:

Definition 1.2. We say that two finite, square, non-negative, integer matrices $A$ and $B$ are strong shift equivalent if there exists a finite sequence...
In 1973, R. F. Williams proved a remarkable theorem \cite{Williams}, which states that if $E$ and $F$ are finite graphs, then the edge shifts $(X_E, \sigma_E)$ and $(X_F, \sigma_F)$ are conjugate (i.e. there exists a homeomorphism $h : X_E \to X_F$ with $\sigma_F \circ h = h \circ \sigma_E$) if and only if the edge matrices $A_E$ and $A_F$ are strong shift equivalent. Thus the conjugacy class of an edge shift $(X_E, \sigma_E)$ is completely characterized by the strong shift equivalence class of its vertex matrix $A_E$.

2. Strong Shift Equivalence and Graph $C^*$-algebras

If $E = (E^0, E^1, r, s)$ is a (not necessarily finite) graph, then the graph $C^*$-algebra $C^*(E)$ is the universal $C^*$-algebra generated by a collection of mutually orthogonal projections \( \{p_v : v \in E^0\} \) together with a collection of partial isometries \( \{s_e : e \in E^1\} \) with mutually orthogonal range projections that satisfy the Cuntz-Krieger relations:

\begin{enumerate}
  \item $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$
  \item $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$
  \item $p_v = \sum_{e \in E^1 : s(e) = v} s_e s_e^*$ for all $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$.
\end{enumerate}

We use the conventions established in \cite{Tomforde} for graph $C^*$-algebras. We also refer the reader to \cite{Williams} for a more comprehensive treatment of graph $C^*$-algebra theory — although we warn the reader that the direction of the arrows in \cite{Williams} is “opposite” of what is used in \cite{Tomforde} and of what is used here.

Definition 2.1. A graph $E = (E^0, E^1, r, s)$ is said to be regular if each vertex emits a finite and nonzero number of edges; i.e. $0 < |s^{-1}(v)| < \infty$ for all $v \in E^0$. A (possibly infinite) matrix $A$ with entries is said to be regular if every row of $A$ contains a finite and nonzero number of positive entries. Note that a graph $E$ is regular if and only if its vertex matrix $A_E$ is regular.

We now extend the definition of strong shift equivalent to regular matrices.

Definition 2.2. Let $A$ and $B$ be regular, square, non-negative, integer matrices. We say $A$ and $B$ are elementary strong shift equivalent if there exist non-negative, integer matrices $R$ and $S$ such that $A = RS$ and $B = SR$. (Note that in order for $RS$ and $SR$ to be regular it is necessary that $R$ and $S$ be regular.) We say that $A$ and $B$ are strong shift equivalent if there exists a finite sequence $C_1, C_2, \ldots, C_n$ of regular, square, non-negative, integer matrices with $C_1 = A$, $C_n = B$, and $C_i$ elementary strong shift equivalent to $C_{i+1}$ for $i = 1, \ldots, n - 1$.

When $E$ is a finite graph with no sinks, the $C^*$-algebra $C^*(E)$ is intimately related to the edge shift $(X_E, \sigma_E)$. We have already described how in this case the strong shift equivalence class of $A_E$ determines the conjugacy class of $(X_E, \sigma_E)$. As we shall see, this has implications for the $C^*$-algebra $C^*(E)$. In particular, if two graphs have vertex matrices that are strong shift
STRONG SHIFT EQUIVALENCE

equivalent, then their associated graph $C^*$-algebras are Morita equivalent. Interestingly, this result holds also for infinite graphs that are regular.

**Theorem 2.3.** Let $E$ and $F$ be regular graphs. If the vertex matrices $A_E$ and $A_F$ are strong shift equivalent, then $C^*(E)$ and $C^*(F)$ are Morita equivalent.

This theorem was proven for Cuntz-Krieger algebras (which correspond to $C^*$-algebras of finite graphs with no sinks) by Cuntz and Krieger in [4, Theorem 3.8]. The theorem was proven for $C^*$-algebras of regular graphs by Bates in [1, Theorem 5.2] and by Drinen and Sieben in [5, Proposition 7.2].

**Sketch of Proof:** Since Morita equivalence is an equivalence relation, it suffices to verify the claim when $A_E$ and $A_F$ are elementary strong shift equivalent. Suppose that $A_E = RS$ and $A_F = SR$. We may then form a bipartite graph $G_{R,S}$ that has vertices $E_0 \sqcup F_0$, and for each $v \in E_0$ and $w \in F_0$ there are $R(v,w)$ edges from $v$ to $w$ and $S(w,v)$ edges from $w$ to $v$. Because $A_E = RS$, the paths of length two beginning in $E_0$ form a copy of $E$ in $G_{R,S}$, and because $A_F = SR$, the paths of length two beginning in $F_0$ form a copy of $F$ in $G_{R,S}$. One can then use the Gauge-Invariant Uniqueness Theorem [3, Theorem 2.1] to prove that $C^*(G_{R,S})$ contains subalgebras isomorphic to of $C^*(E)$ and $C^*(F)$, and furthermore one can show that these subalgebras are complementary full corners of $C^*(G_{R,S})$ determined by the projections $P = \sum_{v \in E_0} P_v$ and $Q = \sum_{v \in F_0} P_v$. (We mention that if these sums are infinite, one can show that they converge to a projection in the multiplier algebra.) Thus $C^*(E)$ and $C^*(F)$ are Morita equivalent.

**Example 2.4.** The techniques of this proof are perhaps best illustrated with an example. Suppose $E$ and $F$ are the graphs

$$
E = \begin{array}{ccc}
& v & w \\
\alpha & b & \gamma \\
& c & & \beta \\
\end{array}
E = \begin{array}{ccc}
& x & y \\
\alpha & e & f \\
& d & & \gamma \\
\end{array}
$$

Then we see that $A_E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_F = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ are elementary strong shift equivalent by taking $R = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. The bipartite graph $G_{R,S}$ is then equal to

$$
G_{R,S} = \begin{array}{ccc}
v & x & w \\
\beta & \alpha & \delta \\
\gamma & \epsilon & \zeta \\
\end{array}
$$

We see that the paths of length two in $G_{R,S}$ beginning in $E_0$ form a copy of $E$. (The edge $a$ corresponds to the path $\beta \alpha$, the edge $b$ corresponds to the path $\gamma \delta$, and the edge $c$ corresponds to the path $\epsilon \zeta$.) Similarly, the paths of length two in $G_{R,S}$ beginning in $F_0$ form a copy of $F$. Also, $C^*(E)$ and
$C^*(F)$ are isomorphic to complementary full corners of $C^*(G_{R,S})$. In fact, if \{s_e, p_v\} is a generating Cuntz-Krieger $E$-family for $C^*(E)$ and if \{S_e, P_v\} is a generating Cuntz-Krieger $G$-family for $C^*(G_{R,S})$, then the $*$-homomorphism that identifies $C^*(E)$ with a full corner of $C^*(G_{R,S})$ maps

\[
p_v \mapsto P_v, \quad p_w \mapsto P_w, \quad s_a \mapsto S_\beta S_{\alpha}, \quad s_b \mapsto S_\gamma S_\delta, \quad \text{and} \quad s_c \mapsto S_\zeta S_{\varepsilon}.
\]

Furthermore, this subalgebra is equal to the corner determined by $P := P_v + P_w$, and this corner is full since any hereditary subset of $G_{R,S}$ containing $E^0$ must contain all the vertices of $G_{R,S}$. Similarly, one can see that $C^*(F)$ is isomorphic to the full corner determined by $Q := P_x + P_y + P_z$, and these corners are complementary since $P + Q = 1$.

We mention that there are counterexamples to Theorem 2.3 if one of the graphs contains a vertex that emits either no edges or infinitely many edges. Thus the regularity of the graphs is necessary.

3. Strong Shift Equivalence and Cuntz-Pimsner Algebras

In this section we shall discuss a notion of strong shift equivalence for $C^*$-correspondences, and discuss how the strong shift equivalence class of a (essential, regular) $C^*$-correspondence determines the Morita equivalence class of the associated Cuntz-Pimsner algebra. We will begin by giving some basic definitions and establishing notation. Afterward we shall discuss how graph $C^*$-algebras can be realized as Cuntz-Pimsner algebras, and use this to motivate a definition of strong shift equivalence for $C^*$-correspondences. We will then conclude with a description of how the proof of Theorem 2.3 can be generalized to the Cuntz-Pimsner setting.

3.1. $C^*$-correspondences. If $X$ is a right Hilbert $A$-module we let $\mathcal{L}(X)$ denote the $C^*$-algebra of adjointable operators on $X$, and we let $\mathcal{K}(X)$ denote the closed two-sided ideal of compact operators given by

\[
\mathcal{K}(X) := \overline{\text{span}} \{\Theta^X_{\xi,\eta} : \xi,\eta \in X\}
\]

where $\Theta^X_{\xi,\eta}$ is defined by $\Theta^X_{\xi,\eta}(\zeta) := \langle \eta, \zeta \rangle_A$. When no confusion arises we shall often omit the superscript and write $\Theta_{\xi,\eta}$ in place of $\Theta^X_{\xi,\eta}$.

**Definition 3.1.** If $A$ and $B$ are $C^*$-algebras, then a $C^*$-correspondence from $A$ to $B$ is a right Hilbert $B$-module $X$ together with a $*$-homomorphism $\phi_X : A \to \mathcal{L}(X)$. We consider $\phi_X$ as giving a left action of $A$ on $X$ by setting $a \cdot x := \phi_X(a)x$. When $X$ is a $C^*$-correspondence from $A$ to $B$ we will sometimes write $A \cdot X_B$ to keep track of the $C^*$-algebras. If $A = B$ we refer to $X$ as a $C^*$-correspondence over $A$.

**Definition 3.2.** A $C^*$-correspondence $X$ from $A$ to $B$ is said to be essential if $\overline{\text{span}} \{\phi_X(a)x : a \in A \text{ and } x \in X\} = X$. A $C^*$-correspondence is said to be regular if $\phi_X$ is injective and $\phi_X(A) \subseteq \mathcal{K}(X)$. 
Definition 3.3. If $X$ is a $C^*$-correspondence over $A$, then a representation of $X$ into a $C^*$-algebra $B$ is a pair $(t, \pi)$ consisting of a linear map $t : X \to B$ and a $*$-homomorphism $\pi : A \to B$ satisfying

(i) $t(\xi)^*t(\eta) = \pi((\xi, \eta)_X)$
(ii) $t(\phi_X(a)\xi) = \pi(a)t(\xi)$
(iii) $t(\xi a) = t(\xi)\pi(a)$

for all $\xi, \eta \in X$ and $a \in A$. We often write $(t, \pi) : (X, A) \to B$ in this situation.

If $(t, \pi) : (X, A) \to B$ is a representation of $X$ into a $C^*$-algebra $B$, we let $C^*(t, \pi)$ denote the $C^*$-subalgebra of $B$ generated by $t(X) \cup \pi(A)$.

Definition 3.4 (The Toeplitz Algebra of a $C^*$-correspondence). Given a $C^*$-correspondence $X$ over a $C^*$-algebra $A$, there is a $C^*$-algebra $T_X$ and a representation $(\overline{t}_X, \overline{\pi}_X) : (X, A) \to T_X$ that is universal in the following sense:

1. $T_X$ is generated as a $C^*$-algebra by $\overline{t}_X(X) \cup \overline{\pi}_X(A)$; and
2. Given any representation $(t, \pi) : (X, A) \to B$ of $X$ into a $C^*$-algebra $B$, there exists a $*$-homomorphism of $\rho(t, \pi) : T_X \to B$, such that $t = \rho(t, \pi) \circ \overline{t}_X$ and $\pi = \rho(t, \pi) \circ \overline{\pi}_X$.

The $C^*$-algebra $T_X$ and the representation $(\overline{t}_X, \overline{\pi}_X)$ exist (see [7], for example) and are unique up to an obvious notion of isomorphism. We call $T_X$ the Toeplitz algebra of the $C^*$-correspondence $X$, and we call $(\overline{t}_X, \overline{\pi}_X)$ a universal representation of $X$ in $T_X$.

The Toeplitz algebra is a very natural $C^*$-algebra associated with a $C^*$-correspondence; however, in many practical situations it is too large. The appropriate $C^*$-algebra to associate with a $C^*$-correspondence is called the Cuntz-Pimsner algebra. It turns out that the Cuntz-Pimsner algebra is a quotient of the Toeplitz algebra, and it can be defined in this way. However, we will instead define it in terms of its universal property, which involves coisometric representations.

Definition 3.5. For a representation $(t, \pi) : (X, A) \to B$ of $X$ into a $C^*$-algebra $B$ there exists a $*$-homomorphism $\psi_t : \mathcal{K}(X) \to B$ with the property that

$$\psi_t(\Theta_{\xi, \eta}) = t(\xi)t(\eta)^*.$$ 

See [16] p. 202], [8] Lemma 2.2], and [7] Remark 1.7] for details on the existence of this $*$-homomorphism. (We warn the reader that our map $\psi_t$ is denoted by $\pi^{(1)}$ in much of the literature, and by $\rho^{(t, \pi)} = \rho^{(\psi, \pi)}$ in [7]. We have chosen to use $\psi_t$ because the map depends only on $t$ and not on $\pi$.)

Definition 3.6. For an ideal $I$ in a $C^*$-algebra $A$ we define

$$I^\perp := \{a \in A : ab = 0 \text{ for all } b \in I\}.$$
If $X$ is a $C^*$-correspondence over $A$, we define an ideal $J(X)$ of $A$ by $J(X) := \phi_X^{-1}(\ker(X))$. We also define an ideal $J_X$ of $A$ by

$$J_X := J(X) \cap (\ker \phi_X)^\perp.$$ 

Note that $J_X = J(X)$ when $\phi_X$ is injective, and that $J_X$ is the maximal ideal on which the restriction of $\phi$ is an injection into $\ker(X)$.

**Definition 3.7.** If $X$ is a $C^*$-correspondence over $A$, then a representation $(t, \pi) : (X, A) \to B$ of $X$ into a $C^*$-algebra $B$ is said to be coisometric if

$$\psi_t(\phi_X(a)) = \pi(a) \quad \text{for all } a \in J_X,$$

**Definition 3.8 (The Cuntz-Pimsner Algebra of a $C^*$-correspondence).** Given a $C^*$-correspondence $X$ over a $C^*$-algebra $A$, there is a $C^*$-algebra $O_X$ and a coisometric representation $(t, \pi_X) : (X, A) \to O_X$ that is universal in the following sense:

1. $O_X$ is generated as a $C^*$-algebra by $t_X(X) \cup \pi_X(A)$; and
2. Given any coisometric representation $(t, \pi) : (X, A) \to B$ of $X$ into a $C^*$-algebra $B$, there exists a $*$-homomorphism of $\rho(t,\pi) : O_X \to B$, such that $t = \rho(t,\pi) \circ t_X$ and $\pi = \rho(t,\pi) \circ \pi_X$.

The $C^*$-algebra $O_X$ and the representation $(t_X, \pi_X)$ exist (see [9 §4]) and are unique up to an obvious notion of isomorphism. We call $O_X$ the Cuntz-Pimsner algebra of the $C^*$-correspondence $X$, and we call $(t_X, \pi_X)$ a universal coisometric representation of $X$ in $O_X$. The universal property of $T_X$ allows one to see that there is an epimorphism $\rho : T_X \to O_X$ and thus $O_X$ is isomorphic to a quotient of $T_X$.

### 3.2. Viewing Graph $C^*$-algebras as Cuntz-Pimsner algebras.

Given a graph $E = (E^0, E^1, r, s)$ one may define a $C^*$-correspondence $X(E)$ over $A := C_0(E^0)$ by letting

$$X(E) := \{ x : E^1 \to \mathbb{C} : \text{the function } v \mapsto \sum_{\{ f \in E^1 : r(f) = v \}} |x(f)|^2 \text{ is in } C_0(E^0) \}.$$ 

and giving $X(E)$ the operations

$$(x \cdot a)(f) := x(f)a(r(f)) \quad \text{for } f \in E^1$$

$$(x, y)_{X(E)}(v) := \sum_{\{ f \in E^1 : r(f) = v \}} \overline{x(f)} y(f) \quad \text{for } v \in E^0$$

$$(a \cdot x)(f) := a(s(f))x(f) \quad \text{for } f \in E^1.$$ 

We call $X(E)$ the graph $C^*$-correspondence associated to $E$, and it is a fact that $O_{X(E)} \cong C^*(E)$ [7 Proposition 4.4].

In particular, if we let $P_v := \pi_{X(E)}(\delta_v)$ and $S_e := t_{X(E)}(\delta_e)$, where $\delta_v$ and $\delta_e$ denote point masses, then $\{ P_v, S_e : v \in E^0, e \in E^1 \}$ is a collection of projections and partial isometries that satisfy the Cuntz-Krieger relations and generate $O_{X(E)}$. Furthermore, the graph $E$ is regular if and only if the $C^*$-correspondence $X(E)$ is regular. Also, $X(E)$ is always essential.
Thus the graph C*-algebra may be thought of as the Cuntz-Pimsner algebra associated to the graph C*-correspondence. We refer the reader to §3 of [15] for a more detailed discussion and analysis of graph C*-correspondences.

We shall now use the graph C*-correspondence to generalize the notion of strong shift equivalence to essential, regular C*-correspondences. Suppose that E and F are regular graphs and that there are non-negative, integer matrices R and S with $A_E = RS$ and $A_F = SR$. Then R is an $E^0 \times F^0$ matrix, and we may create a bipartite graph $G_R$ by defining $G^1_R := E^0 \sqcup F^0$ and for $v \in E^0$ and $w \in F^0$ we draw $R(v, w)$ edges from $v$ to $w$. For this graph we may construct a C*-correspondence $X_R$ from $A := C_0(E^0)$ to $B := C_0(F^0)$ by setting

$$X_R := \{ x : G^1_R \to \mathbb{C} : \text{the function } v \mapsto \sum_{f \in G^1_R : r(f) = v} |x(f)|^2 \text{ is in } C_0(F^0) \}.$$ 

and giving $X_R$ the operations

$$(x \cdot b)(f) := x(f)b(r(f)) \quad \text{for } f \in G^1_R,$$

$$(x,y)_{X_R}(w) := \sum_{f \in G^1_R : r(f) = w} x(f)y(f) \quad \text{for } w \in F^0,$$

$$(a \cdot x)(f) := a(s(f))x(f) \quad \text{for } f \in G^1_R.$$ 

In a similar way, we define $G_S$ and a C*-correspondence $X_S$ from $B := C_0(F^0)$ to $A := C_0(E^0)$.

**Example 3.9.** If $R$ and $S$ are the matrices in Example 2.4 then $G_R$ and $G_S$ are the following graphs:

$$G_R:\begin{array}{c}
v \\ w \\ \searrow\\ \downarrow \\
\downarrow \downarrow \\
\downarrow \downarrow \\
z\\v \\
x\\x \\
x
\end{array}$$

$$G_S:\begin{array}{c}
v \\ w \\ \searrow\\ \downarrow \\
\downarrow \\
y\\y \\
z\\z
\end{array}$$

The fact that $A_E = RS$ shows that any edge $e \in E^1$ corresponds to a path of length two $\alpha \beta$ with $\alpha \in G^1_R$ and $\beta \in G^1_S$. Furthermore, this allows one to prove that $X(E) \cong X(G_R) \otimes_B X(G_S)$ via an isomorphism that sends $\delta_\alpha \mapsto \delta_\alpha \otimes \delta_\beta$. In a similar way we see that $A_F = SR$ implies that $X(F) \cong X(G_S) \otimes_A X(G_R)$.

This motivates the following definition.

**Definition 3.10.** Let $X$ be an essential, regular C*-correspondence over $A$, and let $Y$ be an essential, regular C*-correspondence over $B$. We say $X$ and $Y$ are **elementary strong shift equivalent** if there exists a C*-correspondence $R$ from $A$ to $B$, and a C*-correspondence $S$ from $B$ to $A$ such that $X = R \otimes_A S$ and $Y = S \otimes_B R$. (We mention that if $X$ and $Y$ are essential and regular, then it follows from [14] Corollary 3.11] and [14] Lemma 3.1.3]
that $R$ and $S$ may be chosen essential and regular.) We say that $X$ and $Y$ are strong shift equivalent if there exists a finite sequence $C_1, C_2, \ldots C_n$ of essential, regular $C^*$-correspondences with $C_1 = X$, $C_n = Y$, and $C_i$ elementary strong shift equivalent to $C_{i+1}$ for $i = 1, \ldots n-1$.

We then have the following generalization of Theorem 2.3.

**Theorem 3.11.** Let $X$ be an essential, regular $C^*$-correspondence over $A$, and let $Y$ be an essential, regular $C^*$-correspondence over $B$. If $X$ and $Y$ are strong shift equivalent, then $\mathcal{O}_X$ and $\mathcal{O}_Y$ are Morita equivalent.

This result appears as Theorem 3.14 of [14], and a full proof can be found there. Here we shall give a sketch of the proof with references to the appropriate lemmas of [14].

**Sketch of Proof:** Since Morita equivalence is an equivalence relation, it suffices to verify the claim when $X$ and $Y$ are elementary strong shift equivalent. Suppose that $X = R \otimes_A S$ and $Y = S \otimes_B R$. In analogy with the proof of Theorem 2.3 we create a $C^*$-correspondence $Z$ over $A \oplus B$, called the bipartite inflation of $S$ by $R$. We let $Z := S \oplus R$, and give $Z$ the structure of a right Hilbert $A \oplus B$-module via

$$(s, r) \cdot (a, b) := (sa, rb) \quad \text{and} \quad \langle (r_1, s_1), (r_2, s_2) \rangle := \langle (r_1, s_1), (r_2, s_2) \rangle$$

and we make $Z$ a $C^*$-correspondence over $A \oplus B$ by defining the left action as

$$\phi_{A \oplus B}(a, b)(s, r) := (\phi_B(b)s, \phi_A(a)r).$$

The way to verify the claim is then to show that $\mathcal{O}_X$ and $\mathcal{O}_Y$ are isomorphic to complementary full corners of $\mathcal{O}_Z$. We shall outline how this is done:

Let $(t_Z, \pi_Z) : (Z, A \oplus B) \to \mathcal{O}_Z$ be a universal coisometric representation of $Z$ into $\mathcal{O}_Z$. We then define a representation $(t, \pi) : (X, A) \to \mathcal{O}_Z$ by setting

$$t(r \otimes s) := t_Z(0, r)t_Z(s, 0) \quad \text{and} \quad \pi(a) := \pi_Z(a, 0).$$

It is proven in [14, Lemma 3.8] that $(t, \pi)$ is a coisometric representation, and therefore induces a homomorphism $\rho_{(t, \pi)} : \mathcal{O}_X \to \mathcal{O}_Z$. An application of the Gauge-Invariant Uniqueness Theorem allows one to show that $\rho_{(t, \pi)}$ is injective and $\mathcal{O}_X$ is isomorphic to the subalgebra $C^*(t, \pi) := \text{im} \rho_{(t, \pi)}$ of $\mathcal{O}_Z$. Similarly, one can define $(t', \pi') : (Y, B) \to \mathcal{O}_Z$ and it is shown in [14, Lemma 3.9] that $(t', \pi')$ is a coisometric representation and that the induced homomorphism $\rho_{(t', \pi')} : \mathcal{O}_Y \to \mathcal{O}_Z$ is injective, so that $\mathcal{O}_Y$ is isomorphic to the subalgebra $C^*(t', \pi') := \text{im} \rho_{(t', \pi')}$ of $\mathcal{O}_Z$.

Finally, we let $\{e_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for $A$, and we let $\{f_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for $B$. Since $X$ and $Y$ are essential, it follows that $Z$ is essential, and thus $P := \lim_\lambda \pi_Z(e_\lambda, 0)$ and $Q := \lim_\lambda \pi(0, f_\lambda)$ converge to projections in the multiplier algebra of $\mathcal{O}_Z$. Furthermore, the following hold:
The fact that $X$ and $Y$ are essential and regular implies that $P\mathcal{O}_ZP = C^*(t,\pi)$ and $Q\mathcal{O}_ZQ = C^*(t',\pi')$. (See the proof of [14 Theorem 3.14].)

The fact that $X$ and $Y$ are regular implies that the corners $P\mathcal{O}_ZP$ and $Q\mathcal{O}_ZQ$ are full. (See the proof of [14 Theorem 3.14].)

$P + Q = 1$ (see [14 Lemma 3.12]).

Thus $\mathcal{O}_X$ and $\mathcal{O}_Y$ are isomorphic to complementary full corners of $\mathcal{O}_Z$.

4. Concluding Remarks

4.1. Non-essential $C^*$-correspondences. In Theorem 3.11 we required that the $C^*$-correspondences in question be essential and regular. (Also, since all graph $C^*$-correspondences are essential, the $C^*$-correspondences covered by Theorem 2.5 are both essential and regular.) As pointed out at the end of §2, the regularity condition is a necessary hypothesis, and there are known counterexamples if it is removed. On the other hand, it is currently unknown whether the essential condition is necessary in Theorem 3.11. Nonetheless, it has been shown in [14] that the condition that the $C^*$-correspondence is essential may be replaced by the condition that the underlying $C^*$-algebra is unital; that is:

**Theorem 4.1** (Theorem 4.3 of [14]). Let $X$ be a regular $C^*$-correspondence over a $C^*$-algebra $A$, and let $Y$ be a regular $C^*$-correspondence over a $C^*$-algebra $B$. Suppose that either $X$ is essential or $A$ is unital. Also suppose that either $Y$ is essential or $B$ is unital. If $X$ is elementary strong shift equivalent to $Y$, then $\mathcal{O}_X$ is Morita equivalent to $\mathcal{O}_Y$.

It is currently unknown whether the condition that the underlying $C^*$-algebra is unital can be removed (or weakened) in the above theorem.

4.2. Morita equivalence at other levels. Suppose that $X$ is an essential, regular $C^*$-correspondence over $A$ and that $Y$ is an essential, regular $C^*$-correspondence over $B$. There are three natural questions that one can ask:

**Question 1**: If $X$ and $Y$ are elementary strong shift equivalent, then is it necessarily the case that $X$ and $Y$ are Morita equivalent as $C^*$-correspondences (as defined in [13])?

**Question 2**: If $X$ and $Y$ are elementary strong shift equivalent, then is it necessarily the case that the Toeplitz algebras $\mathcal{T}_X$ and $\mathcal{T}_Y$ are Morita equivalent?

**Question 3**: If $X$ and $Y$ are elementary strong shift equivalent, then is it necessarily the case that the Cuntz-Pimsner algebras $\mathcal{O}_X$ and $\mathcal{O}_Y$ are Morita equivalent?

We have seen that Theorem 3.11 provides an affirmative answer to Question 3. In addition, we mention that an affirmative answer to Question 1
implies an affirmative answer to Question 2. Furthermore, since the Cuntz-Pimsner algebra is a quotient of the Toeplitz algebra, we see that if $\mathcal{T}_X$ is Morita equivalent to $\mathcal{T}_Y$, and if the Morita equivalence takes the appropriate ideal in $\mathcal{T}_X$ to the appropriate ideal in $\mathcal{T}_Y$ via the Rieffel correspondence, then $\mathcal{O}_X$ is Morita equivalent to $\mathcal{O}_Y$.

Originally it was hoped that one could prove an affirmative answer to Question 1, and then use this result to obtain affirmative answers to Question 2 and Question 3 as described in the previous paragraph. Surprisingly, however, it has been shown that this will not work. In the example described in [14, §5.2] it is shown that there are $C^*$-correspondences $X$ and $Y$ (in fact, both can be chosen to be graph $C^*$-correspondences) with the property that the Toeplitz algebra $\mathcal{T}_X$ is not Morita equivalent to the Toeplitz algebra $\mathcal{T}_Y$. It, of course, also follows that the $C^*$-correspondences $X$ and $Y$ are not Morita equivalent. Consequently this example provides a counterexample to Question 1 and Question 2, and it shows that strong shift equivalence of $C^*$-correspondences implies Morita equivalence at the level of Cuntz-Pimsner algebras, but not at the level of $C^*$-correspondences or at the level of Toeplitz algebras.

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