Universal properties of superconformal OPEs for 1/2 BPS operators in 3 ≤ D ≤ 6

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Abstract. We give a general analysis of OPEs of 1/2 BPS superfield operators for the D = 3, 4, 5, 6 superconformal algebras OSp(8/4, R), PSU(2, 2/4), F4 and OSp(8*/4) which underlie maximal AdS supergravity in 4 ≤ D + 1 ≤ 7.

The corresponding three-point functions can be formally factorized in a way similar to the decomposition of a generic superconformal UIR into a product of supersingletons. This allows for a simple derivation of branching rules for primary superfields. The operators of protected conformal dimension which may appear in the OPE are classified and are shown to be either 1/2 or 1/4 BPS, or semishort. As an application, we discuss the ‘non-renormalization’ of extremal n-point correlators.

1. Introduction

Maximal AdS supergravities in four, five, six and seven dimensions are dual to superconformal field theories on the world volume of M2, D3, D4/D5 and M5 branes, respectively [1]. Although only the D3 brane dynamics has a perturbative description in the superconformal regime, some general properties of abstract superconformal field theories can be obtained by using the BPS nature of a certain class of superconformal primary operators and the model-independent nature of OPEs (for reviews see, e.g., [2]–[4]).

Superconformal algebras satisfying the Haag–Lopuszanski–Sohnius theorem [5] exist only for D ≤ 6 [6]. The maximal ones for D = 3, 4, 5 and 6 are OSp(8/4, R), PSU(2, 2/4), F4 and OSp(8*/4), respectively.

In the classification of UIRs of superconformal algebras an important role is played by the representations with ‘quantized’ conformal dimension, since in the quantum field theory

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framework they correspond to operators with ‘protected’ scaling dimension and therefore imply ‘non-renormalization theorems’ at the quantum level.

There are different classes of operators with protected dimension in supersymmetric field theories. One of them is the BPS operators corresponding to different fractions of preserved supersymmetry. In the superfield language these operators are the natural generalization of the ‘chiral’ superfields of $N = 1$, $D = 4$ SUSY theories [7, 8]. For $N$-extended theories, independence of a certain subset of the Grassmann coordinates in superspace corresponds to a certain fraction of preserved supersymmetry. These operators, like $N = 1$ chiral superfields, form a ring under multiplication. Their natural description is ‘harmonic superspace’ [9, 10] and the corresponding BPS superfields are called Grassmann analytic [11]. A well-known example [12]–[14] of $1/2$ BPS operators (which depend only on half of the $\theta$s) in $N = 4$, $D = 4$ super-Yang–Mills theory are the operators corresponding to Kaluza–Klein excitations of type IIB SUGRA on $AdS_5 \times S^5$ [15, 16].

However, it is known from explicit calculations in $N = 4$, $D = 4$ SYM and in $D = 5$ AdS SUGRA [17]–[22] that there exist other ‘protected operators’ which do not correspond to any BPS states. They form another class of operators with protected dimension called ‘current-like’ or ‘semishort’ [23, 8]. The corresponding superfields satisfy differential constraints in superspace which imply that certain terms in the $\theta$ expansion are missing (but not entire $\theta$s, as in the BPS case). Only in very particular cases do these constraints affect the spacetime dependence and the superfields contain true spacetime currents.

The protected supermultiplet found in [17]–[19, 21, 22] was identified as a semishort multiplet in [24, 25]. Its lowest component is a scalar of conformal dimension 4 in the 20 of $SU(4)$. Other examples of semishort superfields are certain K–K excitations of type IIB SUGRA on $AdS_5 \times T_{11}$ [26] studied in [27]. In fact, the K–K excitations of the graviton, while being $1/2$ BPS states in the $N = 8$, $D = 5$ AdS supergravity, are semishort multiplets in the dual $N = 1$, $D = 4$ superconformal field theory formulated in [28].

In this paper we give a unified discussion of all superconformal field theories with maximal supersymmetry in $3 \leq D \leq 6$, dual to AdS maximal supergravities in $4 \leq D + 1 \leq 7$. We study the general branching rules (or equivalently, the OPE) of two $1/2$ BPS states into a third, a priori arbitrary state. To this end we examine the three-point functions that two $1/2$ BPS operators can form with any other operator. Such three-point functions are uniquely determined by conformal supersymmetry. In addition, imposing the conditions of BPS shortness at two of the points leads to selection rules on the operator at the third point. In this way we find out, in all different channels specified by the R symmetry quantum numbers, which are the allowed types of operator that can appear in such OPEs.

This analysis clarifies the possibility of having operators with ‘anomalous dimension’ in certain channels (such as the Konishi multiplet) and the occurrence of only operators with ‘protected dimension’ in other channels. These results find applications, for example, in the proof of the non-renormalization of the so-called ‘extremal’ higher $n$-point functions of $1/2$ BPS operators [29, 30].

The method we propose here differs from earlier approaches to the same problem in $D = 4$ [20, 24, 25, 31] and in $D = 6$ [32] in the sense that it avoids studying the detailed structure of the superspace three-point functions. Instead, we put the emphasis on the purely group-theoretic aspects of the problem. We efficiently exploit a newly established formal factorization property of the three-point functions. It reflects the possibility [7, 8, 33]–[35] of realizing the generic superconformal UIRs as composite operators made out of one basic constituent, the

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so-called ‘supersingleton’ (or massless supermultiplet) [36]–[39]. This description of the three-point functions results in a unified treatment of all theories where the supersingleton constituents are identified with the microscopic ‘degrees of freedom’ of the brane world volume dynamics†.

We show that the selection rules or, to put it differently, the ‘protection mechanism’ for certain channels in the OPE of two 1/2 BPS operators has a very simple origin, which can be illustrated by the following example from ordinary conformal field theory [40, 41]. Consider the three-point function of two scalar fields with a rank \( s \) symmetric tensor field:

\[
\langle \phi_A(1)\phi_B(2)j^{\{\mu_1\cdots\mu_s\}}(3) \rangle. 
\]

These fields have conformal dimensions \( \ell_A, \ell_B \) and \( \ell \), respectively. Now, suppose that the scalars are massless (‘singletons’),

\[
\square\phi_A = \square\phi_B = 0. 
\]

These equations are conformally invariant only if the scalars have the canonical dimension \( \ell_A = \ell_B = (D-2)/2 \). However, this is not all: a direct calculation shows that the condition (1.2) also fixes the dimension of the tensor \( j^{\{\mu_1\cdots\mu_s\}} \) at its canonical value \( \ell = s + D - 2 \) and, moreover, forces this tensor to be conserved. Thus, imposing a condition on the operators at points 1 and 2 of the three-point function can have the effect of ‘protecting’ the operator at point 3.

A similar phenomenon takes place with the three-point functions involving two 1/2 BPS short multiplets. We write them down formally in a factorized form in which the R symmetry quantum numbers at the third point are associated with a BPS factor whereas the spin and the (possibly anomalous) dimension are carried by a singlet factor. Depending on the choice of the R symmetry irrep at point 3, this singlet factor turns out to be either trivial, or of the type (1.1), (1.2) above, or unconstrained. Correspondingly, we find the following selection rules for the third operator: it is either BPS short, or semishort, or unconstrained. Thus, in the first two cases the operator at point 3 is ‘protected’ while in the third case it is ‘unprotected’.

The paper is organized as follows. In section 2 we describe the supersingleton degrees of freedom (brane supercoordinates) for all these theories and their relation to supermultiplet shortening. We also explain how various superconformal UIRs, including different kinds of BPS short and semishort multiplet, can be obtained as composite operators made out of supersingletons. In section 3 we give a unified treatment of the three-point functions involving two 1/2 BPS operators by writing them down in the factorized form described above and deriving the selection rules for the third operator. The results are interpreted in section 4, where particular attention is paid to the occurrence and the role of the semishort protected multiplets. In section 5 we give an application of these results in the form of a general non-renormalization theorem for ‘extremal’ \( n \)-point correlators of 1/2 BPS multiplets.

2. Supersingletons

2.1. Standard description of the supersingletons

Supersingletons are massless representations of the \( D \)-dimensional superconformal algebra or, equivalently, superfields satisfying conformally covariant massless field equations. It is well
known [42]–[44] that there exist only a finite number of them in $D = 3, 5$ while there are infinite sets in $D = 4, 6$. Here we restrict ourselves to the so-called 1/2 BPS supersingletons, which have been identified with the basic brane degrees of freedom in the context of the AdS/CFT correspondence. These supersingletons are ‘ultrashort’ supermultiplets with maximal spin $s_{\text{max}} = 1/2$ in $D = 3, 5$ and $s_{\text{max}} = 1$ in $D = 4, 6$. Such supermultiplets can be characterized by the quantum numbers of their lowest component: vanishing Lorentz spin, canonical conformal dimension of a massless scalar $(D - 2)/2$ and R symmetry irrep with Dynkin labels (DL) according to the following list:

\begin{align}
D = 3 & : [0010] \quad 8_s \text{ of } SO(8) \\
D = 4 & : [010] \quad 6 \text{ of } SU(4) \\
D = 5 & : [1] \quad 2 \text{ of } SU(2) \\
D = 6 & : [01] \quad 5 \text{ of } USp(4).
\end{align}

(2.1)

Note that in the case $D = 3$ there are two inequivalent choices of the basic supersingleton, for example $[0010] (8_s)$ and $[0001] (8_c)$, and a third one, $[0100] (8_v)$, related to the first two by $SO(8)$ triality. Here we restrict ourselves to the OPE of two supersingletons of the same type $8_s$.

These supersingletons can be described in terms of scalar superfields carrying external R symmetry indices according to (2.1) and satisfying the following on-shell constraints [45]–[49]:

\begin{align}
D = 3 & : D^i_{\alpha} W_a = \frac{1}{8} (\Gamma^i \bar{\Gamma}^j)_{ab} D^j_{\alpha} W_b \\
D = 4 & : D^{(k}_{\alpha} W^{[i]}_{j]} = 0, \quad \bar{D}^{[k}_{\alpha} W^{i]}_{j]} = 0 \\
D = 5 & : D^{(k}_{\alpha} W^i_{j]} = 0 \\
D = 6 & : D^{(k}_{\alpha} W^{[i]}_{j]} = 0.
\end{align}

(2.2)

Here the indices $i, j, k$ belong to the fundamental representation (or its complex conjugate) of the R symmetry groups, except for the case $D = 3$, where $i$ is an $8_v$ index and $a, b$ are $8_s$ indices; $\Gamma^i$ denote the gamma matrices of $SO(8)$. The symbols $(), []$ and $\{}\$ mean symmetrization, antisymmetrization and traceless part, respectively. These constraints eliminate most of the components of the superfields and put the remaining ones on the massless shell.

### 2.2. Supersingletons as 1/2 BPS short superfields

Harmonic superspace allows us to write down all these supersingletons as 1/2 BPS short (or Grassmann analytic) superfields depending only on half of the Grassmann variables. To this end we introduce harmonic variables $u$ on the coset $R/H$, where $R$ is the R symmetry group and $H = [U(1)]^{\text{rank}(R)}$ is its maximal torus. With their help we can covariantly project $R$ indices onto $H$ ones. Then suitable projections of the original massless superfields become Grassmann analytic objects, i.e. superfields which depend only on half of the full set of Grassmann coordinates. The most convenient way is to always choose the projection of the external $R$ indices onto what corresponds to the highest-weight state (HWS) of the representation. The details can be found in [8] but they are not essential for the argument we present here. Let us just list these Grassmann analytic superfields:

\begin{align}
D = 3 & : W^{(+)(+)}(x, \theta^{++}, \theta^{(++)}, \theta^{[+]}\{\pm\}, u) \\
D = 4 & : W^{(12}(x, \theta_3, \theta_4, \bar{\theta}_1, \bar{\theta}_2, u) \\
D = 5 & : W^{1}(x, \theta^1, u) \\
D = 6 & : W^{12}(x, \theta^1, \theta^2, u).
\end{align}

(2.3)
In the case $D = 3$ the four sets of $U(1)$ charges are denoted by $\pm(\pm)[\pm\{\pm\}]$; in the other cases it is more convenient to use the individual projections of the indices of the fundamental representation ($i = 1, 2$ for $SU(2)$ and $i = 1, 2, 3, 4$ for $SU(4)$ and $USp(4)$) to label the different states of an $R$ symmetry irrep.

These superfields are in general functions of the harmonic variables having infinite expansions on the harmonic coset $R/H$. The condition which cuts these expansions down to polynomials in the harmonics $u$ is the condition of $R$ symmetry irreducibility. It takes the familiar form of the definition of a HWS:

$$D_u^\uparrow W = 0.$$  \tag{2.4}$$

Here $D_u^\uparrow$ denotes the set of raising (step-up or creation) operators of the group $R$ realized in the form of covariant harmonic derivatives on the coset $R/H$. If one uses a complex parametrization of the coset, conditions (2.4) become covariant Cauchy–Riemann conditions (harmonic analyticity [50]). The combination of Grassmann analyticity (2.3) with irreducibility under the $R$ symmetry group (2.4) is equivalent to the original formulation (2.2) of the supersingletons.

Supersingletons are ‘ultrashort’ superfields in the sense that their $\theta$ expansion contains just a few massless fields. Here we show only the bosonic content of the G- and H-analytic superfields (2.3):

$$\begin{align*}
D = 3 & : W^{+(+)[+]} = \phi_a(x)u_a^{+(+)[+] + \text{derivative terms}} \\
D = 4 & : W^{12} = \phi^{ij}(x)u_i^1 u_j^2 + (\theta_3\sigma^{\mu\nu}\theta_4 + \theta^1\sigma^{\mu\nu}\bar{\theta}^2)F_{\mu\nu}(x) + \text{d.t.} \\
D = 5 & : W^1 = \phi^i(x)u_i^1 + \text{d.t.} \\
D = 6 & : W^{12} = \phi^{ij}(x)u_i^1 u_j^2 + \theta^1\gamma^{\mu\nu\lambda}\theta^2 F_{\mu\nu\lambda}(x) + \text{d.t.}
\end{align*}$$

The massless scalar fields $\phi$ belong to the $R$ symmetry irreps listed in (2.1); the on-shell two-form and three-form field strengths $F$ are singlets.

Concluding this section we mention that (2.3) is not the only possible realization of the supersingletons as Grassmann analytic superfields. Instead of projecting the external $R$ symmetry indices onto the HWS, we could take any other state and accordingly choose the half of the $\theta$s that the superfield depends on, for example

$$\begin{align*}
D = 3 & : W^{+(+)[-]}(x, \theta^{++}, \theta^{(+)+}, \theta^{[-]}\{\pm\}, u) \\
D = 4 & : W^{13}(x, \theta_2, \theta_4, \bar{\theta}^1, \bar{\theta}^3, u) \\
D = 6 & : W^{13}(x, \theta^1, \theta^3, u)
\end{align*}$$

(there is no need to do this in the case $D = 5$). Unlike the superfields (2.3), the new ones are not harmonic analytic (since they are not HWS), i.e. they are not annihilated by all of the raising operators (harmonic derivatives). Instead, they are related to the HWS (2.3) by the action of some of the raising operators:

$$D_3^{+++}W^{+(+)[-]} = W^{+(+)[+]}, \quad D_3^2 W^{13} = W^{12}.$$  \tag{2.7}$$

The use of this alternative realization is explained in the next section.

2.3. BPS short multiplets as products of supersingletons

An important advantage of the description of the supersingletons as Grassmann and harmonic analytic superfields is the possibility of obtaining new BPS short objects by simply multiplying

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the basic supersingletons [8]. The reason is that analytic objects form a ring structure, i.e. a set closed under multiplication.

Thus, any power $[W]^k$ is automatically Grassmann analytic, i.e. depends on the same half of the Grassmann variables; recall (2.3). Further, since the supersingleton $W$ is the HWS of the R symmetry irrep with DL listed in (2.1), the power $[W]^k$ satisfies the constraint which defines it as the HWS of one of the following irreps:

$$
\begin{align*}
D = 3 & : [00k0] \text{ of } SO(8) \\
D = 4 & : [0k0] \text{ of } SU(4) \\
D = 5 & : [k] \text{ of } SU(2) \\
D = 6 & : [0k] \text{ of } USp(4).
\end{align*}
$$

(2.8)

This way of obtaining the 1/2 BPS operators as composite objects makes clear the important fact that the implications of the BPS shortness conditions depend on the quantum numbers of the superfield. Consider the 1/2 BPS short superfields

$$
\text{BPS}^{(k)}_{1/2} : \begin{cases}
D = 3 : \mathcal{W}^{[00k0]}(x, \theta^{++}, \theta^{(++)}, \theta^{[\pm]}(\pm), u) & \leftrightarrow [W^{(+) [+]}]^k \\
D = 4 : \mathcal{W}^{[0k0]}(x, \theta_3, \theta_4, \bar{\theta}^1, \bar{\theta}^2, u) & \leftrightarrow [W^{[12]}]^k \\
D = 5 : \mathcal{W}^{[k]}(x, \theta^1, u) & \leftrightarrow [W^1]^k \\
D = 6 : \mathcal{W}^{[0k]}(x, \theta^1, \theta^2, u) & \leftrightarrow [W^{[12]}]^k
\end{cases}
$$

(2.9)

where in the left column we have indicated the R symmetry DL instead of the $U(1)$ charges. These superfields satisfy the same conditions of BPS shortness (i.e. of Grassmann and harmonic analyticity), but their component content strongly depends on the value of $k$. In the case of the supersingleton ($k = 1$) we have seen that the combination of the two conditions puts the superfield on the massless shell. An even stronger constraint is obtained for $k = 0$: a singlet analytic object can only be a constant, as follows from the obvious property $(W)^0 = 1$. However, for $k \geq 2$ the constraints become much weaker. In particular, the first component of the superfield, a scalar of dimension $k(D - 2)/2$, satisfies no constraint whatsoever. Indeed, if we realize the 1/2 BPS superfield as a composite operator $(W)^k$ for $k \geq 2$, we see that the first component is a generic scalar composite made out of the massless scalars $\phi$ from (2.5). This crucial distinction among the cases $k = 0, 1$ and $k \geq 2$ is at the origin of the selection rules for the three-point functions which are derived in section 3.

Another possibility of obtaining BPS short composite operators is to multiply together two different realizations of the basic supersingleton. For instance, the product of Grassmann analytic superfields of the types (2.3) and (2.6), or of any of their powers, is a superfield which does not depend on 1/4 of the full set of $\theta$s. According to the AdS terminology, such operators are called ‘1/4 BPS short’:

$$
\text{BPS}^{(jp)}_{1/4} : \begin{cases}
D = 3 : \mathcal{W}^{[0jp0]}(x, \theta^{++}, \theta^{(++)}, \theta^{[\pm]}(\pm), u) & \leftrightarrow [W^{(+) [+]}]^p + [W^{(+) [-]}]^j \\
D = 4 : \mathcal{W}^{[jp0]}(x, \theta_2, \theta_3, \theta_4, \bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3, u) & \leftrightarrow [W^{[12]}]^p + [W^{[13]}]^j \\
D = 6 : \mathcal{W}^{[2jp]}(x, \theta^1, \theta^2, \theta^3, u) & \leftrightarrow [W^{[12]}]^p + [W^{[13]}]^j
\end{cases}
$$

(2.10)

† It should be mentioned that for $k = 2$ in $D = 4, 5, 6$ and for $k = 2, 3$ in $D = 3$ some of the higher components of this composite superfield are conserved vectors or tensors. The best-known example is the $N = 4, D = 4$ SYM stress-tensor multiplet which is described by the supersingleton bilinear $W^{12}W^{12}$.
This time, since the factor of the type (2.6) is not harmonic analytic, the product does not automatically define a HWS of a new R symmetry irrep. This can be achieved by imposing further irreducibility conditions. For example, in the case \( D = 3 \) the harmonic condition (recall (2.7))

\[
D^{[++]([W^{+(+)]p+j}[W^{+(+)}])} = 0
\]

(2.11)
defines the HWS of the irrep \([0jp0]\) of \( SO(8) \). Similarly, in the case \( D = 4 \) (or \( D = 6 \)) the condition

\[
D^2_j([W^{12}]p+j[W^{13}]) = 0
\]

(2.12)
turns the product into the HWS of the irrep \([jpj]\) of \( SU(4) \) (or \([2jp]\) of \( USp(4) \)).

We remark that equation (2.10) does not exhaust the list of composite BPS objects obtained by multiplying various realizations of the basic supersingleton \([8\]). For example, in \( D = 3 \) one could have \( 1/8 \) and \( 3/8 \), in \( D = 4 \) \( 1/8 \) BPS multiplets, etc. We do not consider them here because they are associated with R symmetry irreps different from those appearing in the OPE of two \( 1/2 \) BPS operators (see equation (3.2)).

2.4. Semishort multiplets

In what follows the so-called ‘semishort’ (or ‘current-like’) multiplets will play an important role. In this section we give a brief summary of the origin of such multiplets as limiting cases or as isolated points in the series of UIRs of the superconformal algebras. We also give their realization as composite operators made out of supersingletons which satisfy some ‘current-like’ superspace constraints.†

The semishort multiplets are to some extent the analogues of the ‘conserved’ tensor representations of the ordinary conformal group \( SO(D,2) \). It is well known that the rank \( s \) symmetric traceless tensor field \( j^{\{\mu_1...\mu_s\}}(x) \) of the so-called ‘canonical’ dimension \( \ell = s + D - 2 \) forms a reducible but indecomposable representation of the conformal group \( SO(D,2) \) \([41\]

This means that its divergence \( \partial_{\mu_1}j^{\{\mu_1...\mu_s\}} \) transforms covariantly and can be set to zero. The resulting ‘transverse’ tensor is already irreducible.‡

The conserved tensors can be viewed as a limiting case of the generic series of UIRs of \( SO(D,2) \). A generic conformal operator carrying conformal dimension \( \ell \) and Lorentz spin \( s \) can be written down in the following composite form:

\[
O_{\ell s} = j^{\{\mu_1...\mu_s\}} \phi^\delta.
\]

(2.13)

Here \( j^{\{\mu_1...\mu_s\}} \) is a conserved tensor and \( \phi \) is a massless scalar (‘singleton’) field:

\[
\partial_{\mu_1}j^{\{\mu_1...\mu_s\}} = 0, \quad \ell_j = s + D - 2; \quad \Box \phi = 0, \quad \ell_\phi = (D - 2)/2.
\]

(2.14)

Note that \( j^{\{\mu_1...\mu_s\}} \) can itself be represented as a composite operator made out of singletons, for example for \( s = 1 \)

\[
\partial_{\mu_1}j^{\{\mu_1\}} = 1(\phi \partial^\mu \phi' - \phi' \partial^\mu \phi)
\]

(2.15)

† Note that if the supersingletons carry a colour index \( N_c \), under which the composite is a singlet, there are in principle different operators with the same spin and R symmetry quantum numbers \([14, 21, 25, 51\]

‡ The representation is indecomposable because the ‘longitudinal’ part cannot be projected out by a local conformal operator.
where $\phi'$ is another copy of the singleton. The parameter $\delta$ in equation (2.13) can take non-integer values, $\delta \geq 0$ (for $s > 0$) or $\delta \geq 1$ (for $s = 0$). This accounts for the possible ‘anomalous’ dimension of the operator $O_{\ell s}$ subject to the unitarity bound $\ell \geq s + D - 2$ (for $s > 0$) or $\ell \geq (D - 2)/2$ (for $s = 0$). From the ‘composite’ form (2.13) it is clear that the unitarity bound is saturated if $\delta = 0$, $s > 0$ (no massless scalar appears) or if $\delta = 1$ and $s = 0$. Thus, the conserved tensor is at the threshold of the continuous series of UIRs represented by the composite operators (2.13).

A similar phenomenon takes place in the classification of the superconformal UIRs. Let us first recall some of the known series of UIRs [52]–[54]. We restrict ourselves to those which can possibly form a three-point function with two 1/2 BPS UIRs. They must carry Lorentz indices corresponding to a symmetric traceless tensor of rank $s$ and R symmetry quantum numbers which are listed in (3.2).

**OSp(8/4) ($D = 3$).** The Lorentz quantum number (spin) is an integer $J = s$ and we are dealing with $SO(8)$ representations of the type $[0a_2a_30]$. There exist two series of UIRs:

(A) $\ell \geq 1 + s + a_2 + \frac{1}{2}a_3,$

(B) $s = 0, \quad \ell = a_2 + \frac{1}{2}a_3.$

(2.16)

The discrete series (B) contains the BPS multiplets.

**PSU(2,2/4) ($D = 4$).** We consider Lorentz spins $J_1 = J_2 = s/2$ and $SU(4)$ representations of the type $[a_1a_2a_1]$. Two of the three existing series of UIRs are relevant in this case:

(A) $\ell \geq 2 + s + 2a_1 + a_2,$

(C) $s = 0, \quad \ell = 2a_1 + a_2.$

(2.17)

The discrete series (C) contains the BPS multiplets.

**F_4 ($D = 5$).** We consider Lorentz spins $J_1 = 0$, $J_2 = s$ and $SU(2)$ representations $[a_1]$. There exist three series of UIRs:

(A) $\ell \geq 4 + s + \frac{3}{2}a_1,$

(B) $\ell = 3 + s + \frac{3}{2}a_1,$

(C) $s = 0, \quad \ell = \frac{3}{2}a_1.$

(2.18)

Series (B) is an ‘isolated’ series and series (C) contains the BPS multiplets.

**OSp(8'/4) ($D = 6$).** We consider Lorentz spins $J_1 = 0$, $J_2 = s$, $J_3 = 0$ and $USp(4)$ representations of the type $[a_1a_2]$ (with even $a_1$). There exist four series of UIRs:

(A) $\ell \geq 6 + s + a_2 + 2(a_1 + a_2),$

(B) $\ell = 4 + s + 2(a_1 + a_2),$

(C) $s = 0, \quad \ell = 2 + 2(a_1 + a_2),$

(D) $s = 0, \quad \ell = 2(a_1 + a_2).$

(2.19)

Series (B) and (C) are isolated and series (D) contains the BPS multiplets.

In close analogy with the factorization (2.13) of the conformal UIRs in terms of singletons, we can write down an operator $O_{\ell s}^{[a_1]}$ belonging to a generic superconformal UIR labelled by its
conformal dimension $\ell$, spin $s$ and R symmetry $a_i$ as a formal product of three factors [8]:

$$O_{ls}^{[a_i]} = J^{[\mu_1\cdots\mu_s]} \Phi^s \text{ BPS}^{[a_i]}. \quad (2.20)$$

The first factor accounts for the Lorentz spin, the second for the conformal dimension and the third for the R symmetry labels of the composite operator $O_{ls}^{[a_i]}$.

Each of these factors can in turn be viewed as a ‘fake composite’ operator obtained from the basic supersingletons. Thus, the spin factor has the form of a bilinear composite of dimension $s + D - 2$, for example for $s = 1$

$$D = 3 : J^\mu = (\gamma^\mu)_{\alpha\beta} D^\alpha_\alpha W_\alpha (\Gamma^{i\bar{r}ij})_{ab} D^\beta_\beta W^i_b + 32i(W_a^i \partial^\mu W^i_a - W_a^i \partial^\mu W^i_a)$$

and similarly for $s > 1$ (see, e.g., [46]). Note that these composites satisfy superspace ‘conservation conditions’ following from the massless superfield equations (2.2). Using spinor notation, they can be written down as follows:

$$D = 3 : D^{[\mu_1 \cdots \mu_s]} J^{[\alpha_1 \cdots \alpha_s]} = 0$$

These superspace constraints imply spacetime conservation conditions on the components of $J^{[\mu_1 \cdots \mu_s]}$, including the lowest component $j^{[\mu_1 \cdots \mu_s]} = J^{[\mu_1 \cdots \mu_s]}(\theta = 0)$, which has canonical dimension $s + D - 2$. For this reason we call these composites ‘supercurrents’.

Next, the scalar factor $\Phi$ in equation (2.20) is another bilinear composite made out of the basic supersingleton, $W_a W_a$ ($D = 3$), $\epsilon^{ijkl} W_i W_j W_k (D = 4, 6)$ and $\epsilon^{ij} W_i W^i_j (D = 5)$. It also satisfies a superspace conservation constraint whose explicit form we do need here. As a result, the ‘supercurrent’ $\Phi$ contains conserved tensors, including a vector at the level $\theta^a \gamma^a \theta$. The power $\delta$ of this scalar factor accounts for the possible anomalous dimension of the composite operator $O_{ls}^{[a_i]}$. By choosing the appropriate values of $\delta$ we can reproduce both the continuous and isolated series of UIRs (setting $s = \delta = 0$ gives the BPS series).

Finally, the BPS factor in equation (2.20) is made out of the Grassmann analytic supersingletons as explained in section 2.3.

Now, the formal factorization (2.20) allows us to explain the origin of the semishort multiplets as limiting cases of the generic series of UIRs. The idea is to keep just one ‘supercurrent’ factor in (2.20) as well as the BPS factor. Thus, we either set $s > 0$ and $\delta = 0$:

$$S^{[\mu_1 \cdots \mu_s]} [a_i] = J^{[\mu_1 \cdots \mu_s]} \text{ BPS}^{[a_i]} \quad (2.23)$$

or $s = 0$ and $\delta = 1$:

$$S^{[a_i]} = \Phi \text{ BPS}^{[a_i]} \quad (2.24)$$

In both cases the conformal dimension of $S$ is ‘quantized’ (fixed):

$$D = 3 : \ell = s + 1 + a_2 + \frac{1}{2}a_3$$

$$D = 4 : \ell = s + 2 + 2a_1 + a_2$$

$$D = 5 : \ell = s + 3 + \frac{3}{2}a_1$$

$$D = 6 : \ell = s + 4 + 2(a_1 + a_2). \quad (2.25)$$

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According to the classification of UIRs given above, these values correspond to the saturated unitarity bound of the continuous series (A) for \( D = 3, 4 \) or to the isolated series (B) for \( D = 5, 6 \).

The defining property of the semishort superfields is that they satisfy some superspace constraints obtained as the intersection of the supercurrent constraints (2.22) (or of their analogues for the scalar supercurrent \( \Phi \)) and the Grassmann analyticity constraints on the BPS factor. For example, in the case of \( 1/4 \) BPS shortening (see (2.10)) only the following projections of equations (2.22) hold:

\[
\begin{align*}
D = 3: & \quad D^{\alpha_1 \ldots \alpha_s} \left[ \sum_{k=0}^{n} \sum_{j=0}^{n-k} [0, j, m + n - 2k - 2j, 0] \right] = 0 \\
D = 4: & \quad D^{\alpha_1 \ldots \alpha_s} \left[ \sum_{k=0}^{n} \sum_{j=0}^{n-k} [j, m + n - 2k - 2j, j] \right] = 0 \\
D = 5: & \quad \epsilon_{\gamma\alpha_1 \beta_1} D_{\gamma \alpha_1 \ldots \alpha_s} \left[ \sum_{k=0}^{n} [m + n - 2k] \right] = 0 \\
D = 6: & \quad \epsilon_{\gamma\alpha_1 \beta_1} D_{\gamma \alpha_1 \ldots \alpha_s, \beta_1 \ldots \beta_s} = 0.
\end{align*}
\] (2.26)

These constraints are significantly weaker in the sense that the corresponding ‘current-like’ superfield does not contain any conserved tensor components. Without going into the details of the \( \theta \) expansion, this is quite clear from the factorized form of the ‘current-like’ operators, which is at least trilinear in the basic supersingletons. The role of the constraints now is to simply eliminate some components from the \( \theta \) expansion (but not entire projections of \( \theta \)'s, as in the BPS case). Thus, the ‘current-like’ multiplets do not reach the maximal spin of the generic superfield of the same type, and for this reason we call them ‘semishort’.

3. Selection rules for three-point functions involving two \( 1/2 \) BPS operators

In this section we address the main subject of the present paper. The OPE of two \( 1/2 \) BPS operators is determined by the three-point functions of the following type:

\[
\langle \text{BPS}_{1/2}^{(m)} (1) \text{BPS}_{1/2}^{(n)} (2) \text{O}_{\ell s}^{(jk)} (3) \rangle.
\] (3.1)

Here \( \text{BPS}_{1/2}^{(m)} \) and \( \text{BPS}_{1/2}^{(n)} \) denote two \( 1/2 \) BPS short operators described in section 2.3. The third operator in equation (3.1) is characterized by the quantum numbers of its lowest (\( \theta_3 = 0 \) component) ‘superconformal primary field’. These are conformal dimension \( \ell \) (a priori arbitrary), Lorentz spin \( s \) (meaning that the component is a symmetric traceless rank \( s \) tensor) and an \( R \) symmetry irrep labelled by a pair of integers \( (jk) \). The latter appears in the tensor product \( (m) \otimes (n) \) of two of the irreps listed in (2.8) (we assume that \( m \geq n \)):

\[
(m) \otimes (n) = \begin{cases} 
D = 3: & \bigoplus_{k=0}^{n} \bigoplus_{j=0}^{n-k} [0, j, m + n - 2k - 2j, 0] \\
D = 4: & \bigoplus_{k=0}^{n} \bigoplus_{j=0}^{n-k} [j, m + n - 2k - 2j, j] \\
D = 5: & \bigoplus_{k=0}^{n} [m + n - 2k] \\
D = 6: & \bigoplus_{k=0}^{n} \bigoplus_{j=0}^{n-k} [2j, m + n - 2k - 2j].
\end{cases}
\] (3.2)

In what follows we show that the few rather elementary facts about supersingletons and their products we have presented in section 2 are sufficient to explain the selection rules on the operator \( \text{O}_{\ell s}^{(jk)} \) in (3.1). Although we will be discussing three-point functions in superspace,
we will hardly need to go into any details of their $\theta$ dependence. The examination of the lowest ($\theta_1 = \theta_2 = \theta_3 = 0$) component will give us all the necessary information. The reason for this is the remarkable property of the three-point functions (3.1) that they are uniquely fixed by conformal supersymmetry. Indeed, the superfunction (3.1) depends on half of the Grassmann variables at points 1 and 2 and on a full set of such variables at point 3. Thus, the total number of odd variables exactly matches the number of supersymmetries (Poincaré $Q$ plus special conformal $S$). Therefore there exist no nilpotent superconformal invariants made out of the $\theta$s and the complete $\theta_{1,2,3}$ expansion of (3.1) is determined by its lowest component. The latter is the three-point function of two scalars and one tensor field, and is fixed by conformal invariance up to an overall factor.

Before proceeding, we would like to compare the method we follow here with earlier approaches [20, 24, 25, 32, 31]. There the origin of the selection rules was related to the requirement of absence of harmonic singularities (harmonic analyticity) at the higher levels of the $\theta$ expansion of the three-point function. This is certainly an equivalent explanation; however, here we insist upon the fact that harmonic analyticity is nothing but the coordinate expression of R symmetry irreducibility. Thus, by just analysing the occurrence of the different R symmetry irreps in conjunction with our knowledge of the supermultiplet structure, we are able to derive the same selection rules without inspecting the actual harmonic or Grassmann coordinate dependence.

3.1. Factorization of the three-point functions

The crucial observation is that the lowest component of the three-point function (3.1) can be factorized as follows:

$$\langle \text{BPS}^{(m)}_{1/2}(1) \cdot \text{BPS}^{(n)}_{1/2}(2) \cdot \mathcal{O}_{\ell s}^{(jk)}(3) \rangle_{\theta_{1,2,3}=0}$$

$$= \left( \frac{m}{n} \right) \mathcal{O}_{\ell s}^{(jk)} \otimes (k) \mathcal{O}_{ds} \otimes (n-k) (j)$$

$$= \left( \frac{m}{n} \right) \mathcal{O}_{ds} \otimes (j)$$

(3.3)

where the new ‘fake operator’ $\mathcal{O}_{ds}$ is an R symmetry singlet, but it carries spin $s$ and dimension

$$d = \ell - \frac{D-2}{2} (m + n - 2k).$$

(3.4)

The two factors in the rhs of (3.3) have the following structure. The factor carrying the spin at point 3 is made out of the three-point conformal vector

$$Y^\mu = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$$

(3.5)

and of the supersingleton two-point function $\langle W(1)W(2)\rangle_{\theta_{1,2}=0}$. The latter is completely determined by just R symmetry, translation and dilatation invariance and is given by

$$\langle W(1)W(2)\rangle_{\theta_{1,2}=0} = \frac{(12)}{(x_{12}^2)^{D-2}}.$$
Here (12) symbolizes the irreducible harmonic structure which carries the quantum numbers of a HWS of the R symmetry group corresponding to the basic supersingleton:

\[
\begin{align*}
D = 3 & : \quad (u_1)^+_{a} (u_2)^+_{a} \\
D = 4, 6 & : \quad (u_1)^2_{j} e^{ijkl} (u_2)^l_{k} \\
D = 5 & : \quad (u_1)^1_{i} e^{ij} (u_2)^1_{j}.
\end{align*}
\]

Thus, the complete first factor in the rhs of (3.3) has the form

\[
\mathcal{O}_{ds} = \left( \frac{(12)}{(x_{12}^2)^{D/2}} \right)^k (Y^2)^{\frac{D-2}{2}} Y^{\{\mu_1 \ldots \mu_s\}}.
\]

The second factor in the rhs of (3.3) accounts for the R symmetry quantum numbers at point 3. It is entirely made out of supersingleton two-point functions:

\[
\left( \frac{(13)}{(x_{13}^2)^{D/2}} \right)^{m-j-k} \left( \frac{(23)}{(x_{23}^2)^{D/2}} \right)^{n-j-k}
\]

\[
\times \left[ \frac{(13)}{(x_{13}^2)^{D/2}} \frac{(23^-)}{(x_{23}^2)^{D/2}} - 1 \leftrightarrow \frac{2}{2} \frac{2}{2} \right]^j.
\]

In order to reproduce the structure of the R symmetry representations at point 3 listed in equation (3.2) we have to use both types of supersingleton, according to equation (2.10). Thus, while the harmonic factors (13), (23) originate from two-point functions involving only HWS (recall (3.6)), the factor (23^-) (and similarly (13^-)) comes from a two-point function of the mixed type, i.e. where the supersingleton at point 3 is not the HWS of the corresponding representation:

\[
\langle W(2)W^-(3) \rangle_{\theta=0} = \left( \frac{(23^-)}{(x_{23}^2)^{D/2}} \right);
\]

\[
(23^-) \Leftrightarrow \begin{cases} 
D = 3 & : \quad (u_2)^+_{a} (u_3)^+_{a} \\
D = 4, 6 & : \quad (u_2)^2_{j} e^{ijkl} (u_3)^l_{k} 
\end{cases}
\]

(no such factors are needed in the case $D = 5$). The irreducibility conditions at point 3, equations (2.11) or (2.12), are then automatically satisfied, given the fact that the raising operators act on the harmonics as follows:

\[
D^{++} u^+_{a} = u^+_{a}, \quad D^{23}_{u_1^3} u^3_{u_1^3} = u^2_{u_1^3}
\]

and the antisymmetry of the factor $[\cdots]^j$ in equation (3.9). (Note that the first factor (3.8), being an R symmetry singlet at point 3, does not depend on the harmonics at that point.)
3.2. BPS shortness and selection rules

The form of the lowest component of the three-point function that we found in the preceding subsection satisfies the basic requirements of conformal covariance and R symmetry irreducibility. The construction we presented clearly shows that this form is unique (up to an overall constant factor). Next, we have to extend this lowest component to a full superspace three-point function. According to the counting argument from the beginning of this section, this superextension is also unique (if it exists).

One way to proceed would be to use various techniques to construct the superconformal three-point covariant starting from its first component [31, 55]–[58]. This results in rather complicated expressions which are not so easy to analyse. Here we present a different approach based on the factorized form (3.3) which directly leads to the conditions for existence of such a superextension.

The origin of possible constraints on the three-point functions is in the fact that the operators at points 1 and 2 are \(\frac{1}{2}\) BPS short. The second factor (3.9) from equation (3.3) can immediately be extended to a superfunction having the required properties at points 1 and 2. Indeed, each factor in equation (3.9) is the lowest component of a two-point function of supersingletons, so the obvious superextension of equation (3.9) is

\[
\begin{align*}
(W(1)W(3))^{m-j-k} & (W(2)W(3))^{n-j-k} \\
& \times \left[ \langle W(1)W(3) \rangle \langle W(2)W^{-}(3) \rangle - 1 \leftrightarrow 2 \right]^j.
\end{align*}
\] (3.12)

Here \(W^{-}(3)\) denotes the alternative realization (2.6) of the supersingleton.

As explained in section 2, BPS shortness is equivalent to analyticity, i.e. it is a multiplicative property. The product (3.12) of BPS objects automatically has the required properties at points 1 and 2. So, we need to concentrate on the first factor (3.8). We can formally treat this factor as the lowest component of another, ‘fake’ three-point function involving two \(\frac{1}{2}\) BPS short objects of identical labels \((k)\) at points 1 and 2 and an R symmetry singlet, \textit{a priori} long multiplet at point 3. Such a three-point function is also uniquely fixed by conformal supersymmetry, if it exists. To find out under what conditions this lowest component can be extended to a complete superfunction, let us restrict our attention to point 1. Then we can view the expression (3.8) as the lowest component of a \(\frac{1}{2}\) BPS short composite of the type \([W]^k\). Now, according to the discussion of section 2, there are only two cases where this lowest component must satisfy constraints following from the conditions of BPS shortness. These cases are the following.

(i) If \(k = 0\) the only possible \(\frac{1}{2}\) BPS operator is the identity, so the three-point function (3.8) must be trivial. This implies \(s = 0\) and \(d = 0\) or, equivalently,

\[
\text{Case (i)} : \quad k = 0, \quad s = 0, \quad \ell = \frac{D-2}{2}(m+n).
\] (3.13)

(ii) If \(k = 1\) we are dealing with the lowest component of a supersingleton, i.e. with a massless scalar at point 1. Consequently, we must require

\[
k = 1 \quad \Rightarrow \quad \square_1 \left( \frac{x_{12}^2}{2} \frac{D-4}{4} (Y^2) \frac{D-6}{6} Y^{(\mu_1} \ldots Y^{\mu_s)} \right) = 0.
\] (3.14)
This equation is conformally covariant by construction, which allows us to go to a special coordinate frame where it becomes very simple. Multiplying it by \((x_{23}^2)^{\frac{D-2}{2}}\) (this does not affect the differential operator \(\Box\)) and using translations and conformal boosts to set \(x_3 = 0, x_2 = \infty\), we obtain (denoting \(x_1 \equiv x\))

\[
\Box[(x^2)^{-\frac{D+2}{2}} x^{\{m_1 \ldots x^m\}}] = 0.
\]

(3.15)

It is easy to see that this equation admits two solutions. The first solution is \(d = -s\) or, equivalently,

\[
\ell = -s + \frac{D - 2}{2}(m + n - 2).
\]

(3.16)

In order for this solution to not violate the unitarity bounds on the superconformal representation at point 3 of section 2.4, we must set \(s = 0\), which results in

Case (ii.a) : \(k = 1, s = 0, \ell = \frac{D - 2}{2}(m + n - 2).\)

(3.17)

The second solution to equation (3.15) is \(d = s + D - 2\), which gives rise to

Case (ii.b) : \(k = 1, \ell = s + \frac{D - 2}{2}(m + n)\)

(3.18)

for arbitrary \(s\).

We stress the fact that although the constraints above have been derived as necessary conditions, they are also sufficient for the existence of the corresponding three-point functions. The argument is the same as stating that given a massless scalar in the appropriate \(R\) symmetry representation, conformal supersymmetry enables us to reconstruct the entire supersingleton multiplet whose lowest component this scalar is. Similarly, when \(k \geq 2\) we are dealing with the situation in which conformal supersymmetry restores the full 1/2 BPS composite operator \([W]^k\) starting from an *unconstrained* scalar lowest component. Hence, for \(k \geq 2\) we do not find any restrictions on the quantum numbers \(\ell, s\) at point 3, apart from the unitarity bounds from section (2.4). This is true even in the particular case \(k = 2\) when the bilinear \([W]^2\) is a supercurrent multiplet: the required conserved vector (tensor) components will be automatically created as conformal supersymmetry descendents of the lowest, unconstrained scalar component.

### 4. Interpretation of the results. Protected operators

In this section we show that the three cases (3.13), (3.17) and (3.18) correspond to *protected operators* at point 3, namely, to 1/2 or 1/4 BPS short operators in cases (i) and (ii.a) and to semishort operators in case (ii.b).

**Case (i).** The simplest situation occurs when \(k = 0\). Then the factor (3.8) in equation (3.3) becomes trivial and the entire three-point function is reduced to the second factor (3.9), which is just a product of two-point functions of supersingletons (see equation (3.12)):

\[
\langle \text{BPS}_{1/2}(1) \text{BPS}_{1/2}(2) \text{BPS}(3) \rangle = \langle W(1)W(3) \rangle^{m-j} \langle W(2)W(3) \rangle^{n-j} \left[\langle W(1)W(3) \rangle \langle W(2)W(3) \rangle - 1 \leftrightarrow 2 \right]^j.
\]

(4.1)
Note the absence of a two-point function connecting points 1 and 2. Now we can identify the operator at point 3 with a composite BPS operator (recall equation (2.10)).

\[
\text{BPS}_{1/2}(3) = [W(3)]^{m+n} \quad \text{if} \quad j = 0 \\
\text{BPS}_{1/4}(3) = [W(3)]^{m+n-j} [W^-(3)]^j \quad \text{if} \quad j \neq 0.
\]

**(Case (ii.a).)** In this case the three-point function still factorizes into two-point functions of supersingletons:

\[
\langle \text{BPS}^{(m)}_{1/2}(1) \text{BPS}^{(n)}_{1/2}(2) \text{BPS}(3) \rangle \\
= \langle W(1)W(2) \rangle \langle W(1)W(3) \rangle^{m-j-1} \langle W(2)W(3) \rangle^{n-j-1} \\
\times \left[ \langle W(1)W(3) \rangle \langle W(2)W^-(3) \rangle - 1 \leftrightarrow 2 \right]^2
\]

and the operator at point 3 is

\[
\text{BPS}_{1/2}(3) = [W(3)]^{m+n-2} \quad \text{if} \quad j = 0 \\
\text{BPS}_{1/4}(3) = [W(3)]^{m+n-2-j} [W^-(3)]^j \quad \text{if} \quad j \neq 0.
\]

**(Case (ii.b).)** In this case the three-point function does not factorize into two-point functions. Instead, we remark that the conformal dimension of the operator \(O_{\ell s}^{(j)}\) has the right quantum numbers is not yet sufficient to claim that it is indeed semishort. A simple counterexample illustrating this point is the three-point function of two scalars of different dimensions \(\ell_1,2\) and a vector of canonical dimension \(\ell_j = D - 1\):

\[
\langle \phi_{\ell_1}(1) \phi_{\ell_2}(2) j^{\mu}(3) \rangle = \langle x_{12}^{\ell_1} x_{23}^{\ell_2} \rangle \langle x_{13}^{\ell_2} x_{13}^{\ell_1} \rangle \langle x_{23}^{\ell_1} x_{23}^{\ell_2} \rangle \langle x_{12}^{\ell_1} - x_{23}^{\ell_2} \rangle.
\]

A direct calculation shows that the vector at point 3 is conserved if and only if the two scalars have the same dimension, \(\ell_1 = \ell_2\). The same is true if we replace the vector \(j^{\mu}\) by any symmetric traceless tensor \(j^{(\mu_1 \cdots \mu_s)}\) of canonical dimension \(\ell = s + D - 2\).

Thus, we need to provide additional evidence that the operator \(O_{\ell s}^{(j)}\) is semishort. In earlier publications [24, 32] we have done this by restoring the \(\theta_3\) dependence of the three-point function and then showing directly that the superspace constraints of the type (2.26) are satisfied. Here we present a much simpler argument based, once again, on the factorization (3.3) of the three-point function, on the one hand, and on the composite form of the operators (2.20), on the other hand.

We already know that the operator \(O_{\ell s}^{(j)}\) can be factorized as follows:

\[
O_{\ell s}^{(j)} = O_{ds} \text{ BPS}^{(j)}
\]

where the BPS factor \(\text{BPS}^{(j)}\) is given in equation (3.12). Then we should expect that the remaining singlet factor \(O_{ds}\) can be identified with a ‘supercurrent’ \(J^{(\mu_1 \cdots \mu_s)}\) if \(s > 0\), as in (2.23) (or with a scalar ‘supercurrent’ \(\Phi\) if \(s = 0\), as in (2.24)). That this is the case is evident from the bosonic example above. Indeed, now the first factor (3.8) of the lowest
component (3.3) corresponds to the three-point function of two scalars of equal dimensions \(\ell_1 = \ell_2 = (D - 2)/2\) and a tensor of canonical dimension \(\ell = s + D - 2\). Consequently, this tensor must be conserved. Since it is the first component† of a supermultiplet, its conservation implies that the operator \(O_{ds}\) does satisfy a ‘supercurrent’ constraint of the type (2.22),

\[
O_{ds} \rightarrow J^{[\mu_1 \cdots \mu_s]}.
\]

Thus, we can interpret (3.8) as the lowest component of a ‘fake’ three-point function,

\[
\langle O_{+D-2,s} \rangle = \langle \text{BPS}^{(1)}_{1/2}(1) \text{BPS}^{(1)}_{1/2}(2) J^{[\mu_1 \cdots \mu_s]}(3) \rangle_{\theta_{1,2,3}=0}.
\]

Finally, the presence of the factor \(\text{BPS}^{(j_1)}\) weakens the constraint on the composite operator \(O_{ts}^{(j_1)}\) and turns it into a semishort operator

\[
O_{ts}^{(j_1)} \rightarrow S^{[\mu_1 \cdots \mu_s]}(j_1) = J^{[\mu_1 \cdots \mu_s]} \text{BPS}^{(j_1)}
\]

which proves (4.5).

It is important to realize that the above factorization is only formal, it just helps us investigate the supermultiplet structure. In fact, the singlet factor \(O_{ds}\) which looks like a ‘supercurrent’ is not the true operator at point 3. The full operator \(O_{ts}^{(j_1)}\) is only semishort and not a supercurrent (and, consequently, does not contain conserved tensor components). The reason is that in the case \(k = 1\) the BPS factor \(\text{BPS}^{(j_1)}\) is always present. Indeed, from (3.2) it follows that in order not to have a BPS factor the operator \(O_{ts}^{(j_1)}\) must be an R symmetry singlet. Thus, we should set \(j = 0\) and \(m + n = 2\), which implies \(m = n = 1\). However, this corresponds to putting just one supersingleton at points 1 and 2, which is not a gauge-invariant object and thus is not of physical relevance.

The same argument shows that if \(k \geq 2\) one can have a situation where the BPS factor is absent and the operator \(O_{ts}^{(jk)}\) is a true ‘supercurrent’. Going back to the generic factorized form (2.20), we see that this may happen if \(j = 0, m + n = 2k \geq 4\) and if we choose to set \(s > 0, \delta = 0\) or \(s = 0, \delta = 1\). A well-known example is the Konishi multiplet in \(N = 4, D = 4\) SYM which appears in the OPE of two stress-tensor multiplets and corresponds to \(m = n = k = 2, j = s = 0\). In the free field theory it is known to satisfy a superspace constraint and to contain conserved tensor components. However, in the presence of interactions the Konishi multiplet acquires an anomalous dimension \([59]\) and thus ceases to be a ‘supercurrent’. Further examples of operators which have anomalous dimension are the higher-spin and R symmetry singlet multiplets \((m = n = k = 2, j = 0\) and \(s > 0\)) considered in \([60]\). These operators again reduce to ‘supercurrents’ in the free field theory.

The above discussion clearly shows the key difference between the cases \(k = 0, 1\) and \(k \geq 2\). In the cases \(k = 0, 1\) the conformal dimension at point 3 is fixed by the branching rules and thus \(O\) necessarily becomes BPS or semishort. In the case \(k \geq 2\) there is no reason to maintain the conformal dimension at one of these fixed values, so \(O\) may be a BPS, a semishort

† In the case \(s = 0\) the first conserved component of the supercurrent \(\Phi\) is not the lowest component of the supermultiplet, but this does not affect the argument.
or a generic long multiplet. It follows that for \( k = 0, 1 \) any operator \( O^{(jk)}_l \) appearing in the OPE of two 1/2 BPS operators is \textit{protected} by the superconformal kinematics whereas for \( k \geq 2 \) it is \textit{unprotected}, i.e. its conformal dimension is determined by the dynamics of the theory.

5. Extremal correlators

One of the possible applications of the branching rules that we have found is the proof that a certain class of \( n \)-point correlation functions of 1/2 BPS operators

\[
\langle W^{m_1}(1) W^{m_2}(2) \cdots W^{m_n}(n) \rangle,
\]

remain non-renormalized in the interacting theory. According to the terminology introduced in \[29\] they are called ‘extremal’ if

\[
m_1 = \sum_{i=2}^{n} m_i.
\]

Using AdS supergravity considerations, in \[29, 30\] it was shown that the extremal correlators are not renormalized and factorize into products of two-point functions. Here we give a simple explanation of this fact from the CFT point of view based on our results on the three-point functions and the related OPEs of 1/2 BPS operators. The same argument has already been presented in \[32\] for the case \( D = 6 \).

For simplicity we restrict ourselves to four-point extremal correlators. They can be represented as the convolution of two OPEs:

\[
\langle W^{m_1}(1) W^{m_2}(2) W^{m_3}(3) W^{m_4}(4) \rangle = \sum \int_{5,5'} \langle W^{m_1}(1) W^{m_2}(2) O(5) \rangle \langle O(5) O(5') \rangle^{-1} \langle O(5') W^{m_3}(3) W^{m_4}(4) \rangle
\]

where the sum goes over all possible operators which appear in the intersection of the two OPEs. Due to the orthogonality of different operators the inverse two-point function \( \langle O(5) O(5') \rangle^{-1} \) only exists if \( O(5) \) and \( O(5') \) are identical. To find out their spectrum, we first examine the R symmetry quantum numbers. From (3.2) we see that they are given by a pair of integers:

\[
O(5) : j, m_1 + m_2 - 2j' - 2k, \quad 0 \leq k \leq m_2, \quad 0 \leq j \leq m_2 - k
\]

\[
O(5') : j', m_3 + m_4 - 2j' - 2k', \quad 0 \leq k' \leq m_4, \quad 0 \leq j' \leq m_4 - k'
\]

where we have assumed \( m_3 \geq m_4 \). Since in the extremal case \( m_1 = m_2 + m_3 + m_4 \) (recall (5.2)), the intersection is given by the following conditions:

\[
j = j', \quad 0 \leq k' = k - m_2 \leq 0,
\]

whose only solution is

\[
k = m_2 \Rightarrow j = j' = 0, \quad k' = 0.
\]

Further, from (4.2) we deduce that \( k' = 0 \) and \( j' = 0 \) imply that \( O(5') \), and by orthogonality \( O(5) \) must be identical 1/2 BPS operators,

\[
O = W^{m_3 + m_4}.
\]

† A different proof in the case \( D = 4 \), based on a direct analysis of the \( n \)-point superconformal covariants, was given in \[61\]. See also \[31\] for a recent argument.

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Finally, in this particular case the three-point functions in (5.3) degenerate into products of two two-point functions (recall (4.1)), so (5.3) becomes

\[ \langle W^{m_1}(1) W^{m_2}(2) W^{m_3}(3) W^{m_4}(4) \rangle = \int_5 \langle W(1) W(5') \rangle^{m_2} \int_5 \langle W(5) W(5') \rangle^{m_3} \langle W(5') W(4) \rangle^{m_4} \]

\[ = \langle W(1) W(2) \rangle^{m_2} \langle W(1) W(3) \rangle^{m_3} \langle W(1) W(4) \rangle^{m_4} . \]

This clearly shows that the extremal four-point correlator factorizes into a product of two-point functions. In other words, it always takes its free (Born approximation) form, so it remains non-renormalized.

The generalization of the above result to an arbitrary number of points is straightforward and it follows the \( D = 6 \) pattern exhibited in [32]. Further, the argument concerning the non-renormalization of ‘next-to-extremal’ \([62, 63]\) \( D = 6 \) correlators (i.e. those for which \( m_1 = \sum_{i=2}^{n} m_i - 2 \)) presented in [32] applies to the cases \( D = 3, 4, 5 \) as well.

Notice that three-point functions of protected operators in \( D = 4 \), other than \( 1/2 \) BPS, have recently been proved not to suffer from renormalization [31, 64].

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