Energy in the Einstein-Aether Theory

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We investigate the energy of a theory with a unit vector field (the “aether”) coupled to gravity. Both the Weinberg and Einstein type energy-momentum pseudotensors are employed. In the linearized theory we find expressions for the energy density of the 5 wave modes. The requirement that the modes have positive energy is then used to constrain the theory. In the fully non-linear theory we compute the total energy of an asymptotically flat spacetime. The resulting energy expression is modified by the presence of the aether due to the non-zero value of the unit vector at infinity and its $1/r$ falloff. The question of non-linear energy positivity is also discussed, but not resolved.

I. INTRODUCTION

Although Lorentz invariance has been a key feature of theoretical physics for a century, recently there have been a number of reasons for questioning whether it holds at all energy scales. For example, some possible quantum gravity effects hint that it may not be a fundamental symmetry [1]. Thus, it is useful to construct effective, low energy symmetry breaking models in the regimes of the Standard Model and General Relativity (GR). The new effects that appear can then be studied in a familiar context and compared with observations. In the flat spacetime background used in the Standard Model, Lorentz invariance can be broken by background tensor fields [2]. However, when we attempt to couple such fields to gravity, they will also break the general covariance of GR, which we regard as fundamental. In order to bypass this problem, it is straightforward to consider these fields as dynamical quantities along with the metric. In this paper, the source of the Lorentz violation (LV) will be modeled as a unit timelike vector $u^a$. The unit timelike restriction preserves the well-tested $SO(3)$ group of rotations while enforcing the breaking of boost symmetry at every point in the curved spacetime. Therefore, the vector $u^a$ can be said to act as an “aether”.

Similar work was initiated in the early 1970’s by Will, Nordtvedt and Hellings [3, 4, 5] who studied a vector-tensor model without the constraint in the context of alternative theories of gravity. For a review of more recent work on the subject, including aspects of observational constraints, waves, cosmology, and black holes, see [6] and the references therein. Following these authors, we will refer to this theory as the “Einstein-Aether” theory. One important open question is whether the Einstein-Aether theory is energetically viable and stable. The Will-Nordtvedt-Hellings models, for example, are unstable because fluctuations of the unconstrained vector can be either timelike or spacelike, allowing ghost configurations and energy of arbitrary sign [2].

Energy in a field theory is defined as the the value of the Hamiltonian, which acts as the generator of time translations. Although in diffeomorphism invariant theories there is generally no preferred notion of time (and thus energy), in asymptotically flat spacetimes one can naturally define the ADM and Bondi energies associated with asymptotic time translations at spatial and null infinity respectively. The ADM and Bondi definitions for GR have also been shown to satisfy positive energy theorems [8].

In this paper we examine energy in the Einstein-Aether theory. Since it has proven difficult to directly construct the Hamiltonian for the theory, we instead consider the pseudotensor method of studying gravitational energy. Such an approach was first taken in a similar context by Lee, Lightman, and Ni [9], who derived pseudotensors for the unconstrained vector-tensor models but did not evaluate them on solutions. Despite the non-covariance of pseudotensors, it is known that they give well-defined results for the spatially averaged energy carried by waves in linearized theory and the total energy of asymptotically flat spacetimes. In gravitational wave physics they provide a simple and straightforward method for calculating averaged energy densities and the energy-momentum flux radiated away from sources. In addition, Chang, Nester, and Chen [10] have shown that the superpotential associated with every pseudotensor corresponds to a (albeit non-covariant) quasi-local Hamiltonian boundary term.

We first discuss the Einstein-Aether theory and then motivate and construct its modified Weinberg pseudotensor expression. As the calculational and consistency check we also use a Lagrangian based method to derive the modified Einstein “canonical” pseudotensor and its associated superpotential. We then apply these expressions to solutions in both the linear and non-linear regimes. In the linearized theory we find that the Einstein and Weinberg prescriptions give the same energy densities for the plane wave modes derived in [11]. Restricting these densities to be positive yields constraints on the model in terms of the coefficients of the aether part of the action. These constraints are also compared to results obtained in the limit where the metric and aether decouple [12]. In the full non-linear theory the Einstein-Aether superpotential is used to obtain the total energy for an asymptotically flat spacetime. This result agrees

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with \[13\], where the total energy in the Einstein-Aether theory is derived via the covariant Noether charge formalism. We conclude with a discussion of the status of positive energy in the non-linear regime and prospects for a positive energy theorem.

II. EINSTEIN-AETHER ACTION

We can model gravity with a dynamical preferred frame using a timelike unit vector \(u^a\). This vector field breaks local Lorentz invariance spontaneously in every configuration leaving behind the 3-D rotation group as the residual symmetry. The unit norm condition is required to avoid ghosts and also desirable because we regard the norm as extra information beyond what is necessary to determine the preferred frame. Taking an effective field theory point of view we can consider the action as a derivative expansion, subject to diffeomorphism symmetry. The result, up to two or fewer derivatives, is

\[
S = \frac{1}{16\pi G} \int \sqrt{-g} \; L_{ae} \; d^4x \tag{1}
\]

where

\[
L_{ae} = -R - K_{mn}^a \nabla_a u^m \nabla_b u^n - \lambda(g_{ab} u^a u^b - 1). \tag{2}
\]

The "kinetic" term \(K_{mn}^a\) is defined as

\[
K_{mn}^a = c_1 g_{ab} g_{mn} + c_2 \delta_m^a \delta_n^b + c_3 \delta_n^a \delta_m^b + c_4 u^a u^b g_{mn} \tag{3}
\]

and \(\lambda\) is a Lagrange multiplier enforcing the unit timelike constraint. \(R\) is the familiar Ricci scalar and the coefficients \(c_i\) in \(K_{mn}^a\) are dimensionless constants. Note that a term proportional to \(R_{ab} u^a u^b\) is not explicitly included as it comes about as a combination of the \(c_2\) and \(c_3\) terms in \([1]\). The metric signature is \((+---)\) and the units are chosen so that the speed of light defined by the metric \(g_{ab}\) is unity.

The field equations from varying the action in \([1]\) together with a matter action with respect to \(a^{ab}\) and \(u^a\) are given by

\[
G_{ab} = T^{(u)}_{ab} + 8\pi G T^M_{ab} \tag{4}
\]

\[
\nabla_a J^m_b - c_4 u_a \nabla_m u^b = \lambda u_m, \tag{5}
\]

\[
g_{ab} u^a u^b = 1. \tag{6}
\]

where

\[
J^m_a = K_{mn}^a \nabla_b u^n \tag{7}
\]

and

\[
\dot{u}_a = u^b \nabla_b u_a. \tag{8}
\]

Here we assume that there are no aether-matter couplings in the matter action. The aether stress tensor is given by

\[
T^{(u)}_{ab} = \nabla_m (J^m_{(a} u_{b)}) - J^m_{(a} u_{b)} - J^m_{(ab)} u^m + c_1 \left[(\nabla_m u_a)(\nabla^m u_b) - (\nabla_a u_m)(\nabla^b u^m)\right] + c_4 u_a u_b + \left[\nu_n \nabla_m J^{mn} - c_4 \dot{u}^2\right] u_a u_b - \frac{1}{2} \nu g_{ab}, \tag{9}
\]

where \(\nu = -K_{mn}^a \nabla_a u^m \nabla_b u^n\) and \(\dot{u}^2 = \dot{u}_a \dot{u}^a\). The Lagrange multiplier \(\lambda\) has been eliminated from \([9]\) by solving for it via the contraction of the aether field \([5]\) with \(u^a\). As we will see below, the the form of the aether stress tensor and Einstein-Aether Lagrangian will be important tools in derivation of the modified Weinberg and Einstein pseudotensors.

III. WEINBERG PSEUDOTENSOR

Weinberg’s pseudotensor construction \([15]\) is based on the “field theoretic” approach to GR that treats gravity as a spin-2 field on a flat background spacetime. Using Greek indices to represent coordinate indices, we begin by writing the metric in coordinates such that \(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\), where \(\eta_{\mu\nu}\) is the flat Minkowski metric and \(h_{\mu\nu}\) is an symmetric tensor field with the asymptotic conditions \(h_{\mu\nu} \sim O(1/r), \partial_\mu h_{\nu\sigma} \sim O(1/r^2), \partial_\tau \partial_\sigma h_{\mu\nu} \sim O(1/r^3)\). The Einstein tensor can be expanded into a series of parts linear, quadratic, and higher order in the field variable \(h_{\mu\nu}\). Following Ch. 20 of Misner, Thorne, and Wheeler \([16]\) the non-linear corrections to the Einstein tensor are defined as follows

\[
16\pi G \; t_{\mu\nu} = 2G^{(1)}_{\mu\nu} - 2G_{\mu\nu}, \tag{10}
\]

where \(G^{(1)}_{\mu\nu}\) and \(G_{\mu\nu}\) are the linearized and full non-linear Einstein tensors respectively. Note that this splitting is non-unique because it depends on the coordinate system. Since the linearized Einstein tensor is symmetric and satisfies a linearized Bianchi identity \(\partial^{\nu} G^{(1)}_{\mu\nu} = 0\), it can be rewritten in superpotential form

\[
2G^{(1)}_{\mu\nu} = H_{\mu\nu\alpha\beta} \; a^{\alpha\beta} \tag{11}
\]

where \(H_{\mu\nu\alpha\beta}\) has the symmetries of the Riemann tensor \(H_{\mu\nu\alpha\beta} = H_{\mu[\nu][\alpha\beta]} = H_{\alpha\beta\mu\nu}\) (see, for example [17]). Using \([10]\) and \([11]\) the full Einstein equation becomes

\[
H_{\mu\nu\alpha\beta} \; a^{\alpha\beta} = 16\pi G \; (t_{\mu\nu} + T_{\mu\nu}). \tag{12}
\]

Due to the symmetries of \(H_{\mu\nu\alpha\beta}\), this implies that \(\partial^{\nu} (t_{\mu\nu} + T_{\mu\nu}) = 0\). Therefore the integral of \(t_{00} + T_{00}\) over a spacelike slice is a conserved quantity. This conserved quantity

\[
\int (t_{00} + T_{00}) \; d^3x = \frac{1}{16\pi G} \int H_{\mu\nu\alpha\beta} \; a^{\alpha\beta} \; d^3x = \frac{1}{16\pi G} \int H_{\alpha\beta\nu\mu} \; a^{\alpha\beta} \; d^3x. \tag{13}
\]
where $n^\beta$ is the unit normal to the surface at spatial infinity, is in fact the total energy, with $t_{00}$ acting as the energy density of the gravitational field alone. To sharpen this point, consider the case where the gravitational field is weak everywhere, allowing use of the linearized theory. The leftmost member of (13) then gives the total matter energy, which in this case is the total energy. The rightmost member is insensitive to the interior volume, so replacement by arbitrary sources and strong fields in the interior will not affect the identification of (13) as the total energy.

The extension to the Einstein-Aether theory is straightforward. The metric field equations (9) take the form

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - T^{(u)}_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (14)$$

In addition to the metric, we now decompose the aether into background and dynamical part by writing $u^\mu = u^\mu + v^\mu$. Unlike normal matter fields the aether stress $T^{(u)}_{\mu\nu}$ contains linear pieces in the perturbation $v^\mu$ due to the fact that the aether does not vanish in the background (since it is always a unit vector). These linear terms will modify the Weinberg pseudotensor and superpotential. Performing the split of the modified Einstein tensor $\tilde{G}_{\mu\nu}$ as in (10) we find

$$16\pi G \left( \tilde{\Gamma}^{(1)}_{\mu\nu} - 2\tilde{G}_{\mu\nu} \right) \quad (15)$$

where $\tilde{G}^{(1)}_{\mu\nu} = G^{(1)}_{\mu\nu} - T^{(1)}(u)_{\mu\nu}$. $\tilde{G}_{\mu\nu}$ satisfies a Bianchi identity $\nabla^\mu \tilde{G}_{\mu\nu} = 0$ if the aether is uncoupled to the matter and if the aether field equation (5) is satisfied. Therefore in the linearized case we can write

$$2\tilde{G}^{(1)}_{\mu\nu} = H_{\mu\nu\alpha\beta} \beta$$

along with

$$H_{\mu\nu\alpha\beta} = 16\pi G (\tilde{\Gamma}^{(1)}_{\mu\nu} + T^{(u)}_{\mu\nu}). \quad (17)$$

By the same reasoning as before we could conclude that the total energy is given by

$$E = \frac{1}{16\pi G} \int \tilde{h}_{\alpha0\beta}\alpha^\beta \nu^\alpha dx. \quad (18)$$

However, unlike the GR case (12), it is not clear whether the new Weinberg superpotential $H_{\mu\nu\alpha\beta} \alpha$ can be expressed as a local function of the fields $h_{ab}$ and $u^a$ (21). On the other hand, the pseudotensor $\tilde{\Gamma}^{(1)}_{\mu\nu}$ can be calculated directly via the non-linear pieces of $T^{(u)}_{\mu\nu}$ and $G_{\mu\nu}$ in $\tilde{G}_{\mu\nu}$. This will be used to compute the linearized wave energy densities. Evaluation of the total energy as a surface integral at spatial infinity requires a locally defined superpotential. Since we do not have knowledge of the aether corrections to Weinberg superpotential we shall instead consider the Einstein superpotential, which can be derived directly from the form of the Lagrangian. The Einstein formulation of gravitational energy-momentum will also provide a consistency check when we evaluate the energy density of the linearized plane wave modes.

IV. EINSTEIN “CANONICAL” PSEUDOTENSOR

The gravitational energy pseudotensor originally derived by Einstein in 1916 shortly after his discovery of the field equations of GR is closely related to the familiar canonical stress tensor of matter fields in flat spacetime. In order to derive the corresponding expression for the Einstein-Aether theory, we use a Lagrangian approach based upon the famous work of Noether relating symmetries to conservation laws. In flat spacetime, invariance of a Lagrangian under global space and time translations is associated with the conservation of energy-momentum expressed by the conservation of the canonical stress tensor

$$T_\nu^\mu = \frac{\delta L}{\delta (\partial_\nu \psi)} \partial_\mu \psi - \delta_\mu^\nu L, \quad (19)$$

where $L = L(\psi, \partial \psi)$ and $\psi$ represents a general collection of fields with indices suppressed. In the case of local symmetries, such as the diffeomorphism invariance of the Einstein-Aether theory, the situation is more complex. In the Appendix we review a general formalism due to Julia and Silva (18) for constructing Noether currents and superpotentials and apply it to the Einstein-Aether theory. Using these results we show that the pseudotensor and superpotential have the following general form

$$t_\nu^\mu = \frac{\sqrt{-g}}{16\pi G} \left( \frac{\partial L}{\partial (\partial_\mu g_{\alpha\beta})} \partial_\nu g_{\alpha\beta} \right.$$  

$$+ \frac{\partial L}{\partial (\partial_\mu u_{\alpha})} \partial_\nu u_{\alpha} - \delta_\mu^\nu L) \right) \quad (20)$$

$$U_\nu^\gamma = \frac{\sqrt{-g}}{16\pi G} \left( \frac{\partial L}{\partial (\partial_\nu g_{\alpha\beta})} (\tilde{G}_{\gamma}^{\nu\beta} + \delta_\gamma^\nu g_{\nu\alpha}) \right.$$  

$$- \frac{\partial L}{\partial (\partial_\mu u_{\alpha})} \delta_\nu^\gamma \delta_\alpha^\mu \right) \quad (21)$$

where $L$ is the Lagrangian

$$L = -g^{\alpha\beta} (\Gamma^\gamma_{\alpha\delta} \Gamma^\delta_{\gamma\beta} - \Gamma^\gamma_{\gamma\delta} \Gamma^\delta_{\alpha\beta})$$  

$$- K_{\mu\nu} \nabla_\alpha u^\alpha \nabla_\beta u^\nu - \lambda (g_{\mu\nu} u^\mu u^\nu - 1). \quad (22)$$

Note that we have eliminated a surface term in the Einstein-Hilbert Lagrangian, replacing the Ricci scalar $R$ with the Einstein-Schrodinger “Γ2” action, which depends only on the metric and its first derivatives. When evaluated on-shell the pseudotensor and superpotential obey the following relations

$$\partial_\mu t_\nu^\mu = 0 \quad (23)$$

$$t_\nu^\mu = -\partial_\gamma U_\nu^\gamma \mu. \quad (24)$$

To account for the presence of any non-aether matter sources one only has to make the replacement $t_\nu^\mu \rightarrow t_\nu^\mu + T_\nu^\mu$ in (23) and (24). Like the Weinberg construction, the pseudotensor $t_\nu^\mu$ is a conserved quantity and is related to the divergence of a superpotential.
The contributions from the pure GR $\Gamma^2$ Lagrangian are the Einstein pseudotensor

$$
einstein_{\rho}^\mu = \frac{\sqrt{-g}}{16\pi G} \left( \delta^\mu_\rho (\Gamma^\gamma_\beta \Gamma^\beta_\alpha - \Gamma^\alpha_\beta \Gamma^\beta_\gamma) g^{\gamma\delta} + \Gamma^\gamma_\rho \Gamma^\beta_\gamma g_{\mu\alpha} - \Gamma^\gamma_\beta \Gamma^\gamma_\alpha g^{\mu\alpha} + \Gamma^\alpha_\beta \Gamma^\mu_\gamma g^{\beta\gamma} + \Gamma^\mu_\rho \Gamma^\beta_\gamma g^{\alpha\gamma} - 2 \Gamma^\mu_\alpha \Gamma^\beta_\gamma g^{\rho\gamma} \right)$$

(25)

and the von Freud superpotential (see, e.g. [19])

$$f U^\alpha_\beta = \frac{1}{16\pi G} \frac{1}{\sqrt{-g}} g^\gamma_\beta \partial_\gamma \left\{ (-g)(g^{\lambda\gamma} g^{\mu\gamma} - g^{\sigma\gamma} g^{\lambda\gamma}) \right\}.$$  

(26)

To compute the additional aether modifications, we use the relation

$$\frac{\partial L}{\partial (\partial_\nu g_{\alpha\beta})} = \frac{1}{2} (g^{\alpha\nu} \delta^\beta_\delta + g^{\alpha\delta} \delta^\beta_\nu - g^{\mu\nu} \delta^\alpha_\delta) \frac{\partial L}{\partial (\Gamma^\nu_\gamma)}$$

(27)

and $L_u = K \eta^\alpha_{\mu\nu} \nabla_\alpha \nu \mu \nabla_\beta u_\nu$ in (20) and (21) since when evaluated on solutions any terms related to the unit constraint will vanish. We find the pseudotensor

$$x t^\nu_\lambda = \frac{1}{16\pi G} (2 \sqrt{-g} J^\mu_\rho \nabla_\nu u^\rho - \sqrt{-g} \left\{ (J^\lambda_\beta + J^\beta_\lambda) u^\alpha - (J^\alpha_\beta + J^\beta_\alpha) u^\lambda + (J^{\alpha\lambda} - J^{\lambda\alpha}) u_\beta \right\} \Gamma^\beta_\alpha \nu + \delta^\nu_\lambda \sqrt{-g} L_u \right).$$

(28)

and the superpotential

$$x U^\alpha_\beta = \frac{1}{16\pi G} \sqrt{-g} \left\{ (J^\lambda_\beta + J^\beta_\lambda) u^\alpha - (J^\alpha_\beta + J^\beta_\alpha) u^\lambda + (J^{\alpha\lambda} - J^{\lambda\alpha}) u_\beta \right\} (29)$$

where $J^\alpha_\beta$ is defined in (4). The above decompositions of (20) and (21) into GR and aether pieces do not satisfy (23) independently. A key requirement when evaluating these pseudotensorial expressions is that the metric must be written in a coordinate system where the connection coefficients vanish like $O(1/r)$ or faster in the asymptotic limit. If the coordinate system is not chosen properly then these expressions will yield incorrect energies and momenta [22]. This condition was not well understood in the early literature on gravitational energy-momentum, but can now be explained using an analysis of the boundary terms and conditions in an action. See the Appendix for further details.

V. ENERGY IN LINEARIZED THEORY

Equipped with the modified Einstein and Weinberg pseudotensors we can now calculate the energy density of the linearized plane wave solutions to the Einstein-Aether theory. The plane wave solutions in the absence of matter are found by linearizing the field equations above, (31)-(34), with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $u^\mu = v^\mu + \nu^\mu$. This gives

$$\partial_\alpha J^\alpha_\beta = \lambda \omega_\beta$$

(30)

$$G^{(1)}_\alpha_\beta = T^{(1)}_\alpha_\beta$$

(31)

$$v^0 = -\frac{1}{2} h_{00}. \tag{32}$$

Cartesian coordinates are used in the flat background, $\eta_{\mu\nu} = (1, -1, -1, -1)$ and $u^0 = (1, 0, 0, 0)$. Since the background value of the Lagrange multiplier vanishes, $\lambda$ in (30) represents a perturbation. The superscript (1) represents quantities written to first order in the perturbation. Jacobson and Mattingly [11] then proceed to analyze these equations using the gauge choice

$$h_{0i} = 0 \tag{33}$$

$$v_{i, i} = 0 \tag{34}$$

which they prove to be accessible. Inserting plane wave solutions

$$h_{\mu\nu} = \epsilon^{\mu\nu} e^{ik_z x^z} \tag{35}$$

$$v^\mu = \epsilon^\mu e^{ik_z x^z} \tag{36}$$

into the equations of motion, imposing the 4 gauge conditions (33)-(34), and choosing coordinates such that the wave-vector is $(k_0, 0, 0, k_z)$ (travelling in the $z$ direction), it is found [11] that the mode polarizations and speeds are completely determined. The result is a total of 5 wave modes falling into spin 2, spin 1, and spin 0 types as shown in Table II [22]. The notation $I$ in subscript refers to the transverse components of the metric and aether while $c_{11} = c_1 + c_4$, etc. The 2 spin-2 TT metric modes look exactly like the usual GR case, except for the modification of the speed. The 2 spin-1 transverse aether modes and 1 spin-0 trace mode are new modes coming from the constrained aether, which is characterized by 3 degrees of freedom.

In order to determine the energy, note that in the absence of matter the Weinberg prescription [13] reduces to

$$E = \frac{1}{16\pi G} \int \tilde{H}_{00,0,0} \tilde{a} c_1^2 d^2 x = \int \tilde{t}_{00} d^3 x, \tag{37}$$

which clearly produces infinite total energy for plane wave modes. One could reformulate the problem in terms of wavepackets with the appropriate asymptotic fall-off conditions, but a far more direct approach is to simply evaluate the plane wave energy density $\tilde{t}_{00}$. This quantity is meaningless at a point for plane waves, but the average over a cycle is well-defined. Consider a large, but finite region with nearly plane waves. There are “surface effects”, but the the contribution to $\int \tilde{t}_{00} d^3 x$ is dominated by the volume. Thus, $\tilde{t}_{00}$ gives an effective energy density.

The 3 general classes of modes were analyzed separately using the Riemann tensor package [20] in Maple.
The package allows the user to enter the components of the metric and aether vector, calculate curvature tensors, and to define new tensors involving both ordinary and covariant derivatives. In this case we entered the linearized metric and aether, where $h_{\mu\nu}$ and $v^\mu$ take the plane wave forms. A polarization was written as

$$A \exp ik_3(z - st) + \overline{A} \exp(-ik_3(z - st))$$

(38)

where $s$ are speeds shown in Table I and $A$ is a complex-valued function. Using this metric we calculated the explicit form of the Weinberg pseudotensor $\tilde{t}_{00}$ up to quadratic order. Higher order terms will be small in the linearized theory and oscillatory terms proportional to $\tilde{A}^2$ and $A^2$ can be neglected in the usual time averaging process. These energy densities were then compared with the modified Einstein pseudotensor $\eta_{00}$ from [25] and [28] again up to quadratic order in the perturbations. Note that while (23) holds at quadratic order when the linearized equations of motion are imposed, (24) does not. Therefore, one must use the modified Einstein pseudotensor directly to compute the energy densities. The results of the Weinberg and Einstein prescriptions agreed and are displayed below:

$$\mathcal{E}_{\text{spin-2}} = \frac{1}{8\pi G} k_2^2 |A|^2$$

(39)

$$\mathcal{E}_{\text{spin-1}} = \frac{1}{8\pi G} k_2^2 |A|^2 c_1^2 + 2c_1 - c_1^2 - c_3$$

(40)

$$\mathcal{E}_{\text{spin-0}} = \frac{1}{8\pi G} k_2^2 |A|^2 c_1(2 - c_1)$$

(41)

These results have been independently verified in [21] using the Noether charge method and a decomposition of $v^a$ into irreducible pieces. The lack of $c_i$ dependence in (39) and the simplicity of (40)-(41) is striking considering the complicated form of the pseudotensor expressions. The energy of the spin-2 mode is positive definite, like pure GR, while for the other 2 modes the sign of the energy density depends upon a combination of $c_1$, $c_3$, and $c_4$. Note that when the $c_i$’s are zero (39) and (41) are zero as expected. This set of results for the coefficients also holds for exponentially growing modes (i.e. when $s^2 < 0$)

$$A \cos(kz + \varphi) \exp(kst)$$

(42)

when we average over the spatial oscillations. Restricting $s^2 > 0$ in Table I to eliminate the unstable modes and enforcing positivity in the energy densities in (39)-(41) restricts the $c_i$ values in the Einstein-Aether theory.

In [12], Lim worked in the limit where the aether and metric perturbations decouple, with the aether propagating in flat spacetime. Mathematically this amounts to tuning $c_i, G \to 0$ while holding the ratio $c_i/G$ fixed in the action (1). If we then expand the metric as $g = \eta + \sqrt{G} h$ and take the limit, the action reduces to that of linearized gravity plus aether terms coupled only to $\eta_{ab}$. In this limit the linearized constraint reduces to $v^0 = 0$ and we can decompose $v^i$ into spin-0 and spin-1 parts via $v^i = \partial^i S + N^i$ where $N^i = 0$. By examining the Hamiltonian of these modes they found $c_1 > 0$ for positivity in both cases, neglecting $c_4$. We can make contact with this result simply by examining in the small c1 limit of the wave solutions. The trace and transverse aether energy waves then correspond to the flat spacetime spin-0 and spin-1 modes. To lowest order in $c_i/G$ we find that

$$c_1 > 0$$

(43)

$$c_1 > 0$$

(44)

for positive energy densities of the spin-0 and spin-1 modes respectively. Restoring $c_4$ in the flat spacetime analysis yields complete agreement. Note that for small $c_i$ the $s^2 > 0$ criteria for stable, non-exponentially growing modes reduce to $c_1/c_4 \geq 0$ for the spin 1 aether-metric mode and $c_1$ $c_4$ $\geq 0$ for the spin 0 trace mode. Thus, modes with positive energy are stable if $c_{123} > 0$.

VI. NON-LINEAR ENERGY

In this section we will attempt to extend the criteria for positive energy from linearized theory into the nonlinear regime. As a first step, let us consider the total energy of an asymptotically flat spacetime in the full nonlinear theory. Integrating (23) over a spacelike slice in the presence of non-aether matter gives the total energy

$$\int T_{\text{eff}}^\text{tot} = \int \partial_\lambda U^\text{tot} = \int_\infty \text{tot} U^{0,\lambda}_0 h_\lambda dS$$

(45)

where $\text{tot} U = \eta U + v U$ are the aether and von-Freud superpotentials, (29) and (26) and $T_{\text{eff}} = t + T$ is total matter and gravitational energy-momentum. The problem now is to calculate the superpotentials for the asymptotically flat solutions to the Einstein-Aether theory. We will use Cartesian coordinates throughout since these have

| Mode   | Squared Speed $s^2$ | Polarisations |
|--------|---------------------|---------------|
| spin-2 | $1/(1 - c_1)$      | $h_{12} = h_{22}$ |
| spin-1 | $(c_1 - c_1^2 + 2c_3)/c_1$ | $h_{13} = [c_1/(1 - c_1)] \eta v$ |
| spin-0 | $c_{123}(2 - c_1)/c_1(1 - c_1)(2 + c_1 + 3c_2)$ | $h_{00} = -2c_0$, $h_{11} = h_{22} = -c_{14}v_0$, $h_{33} = [2c_{14}(c_2 + 1)/c_{123}]v_0$ |

TABLE I: Wave Mode Speeds and Polarizations
the required asymptotic behavior discussed at the end of Section [V]. Therefore, the surface element is \( dS = r^2 d\Omega^2 \) and the unit normal is \((\sqrt{2}, x/r, y/r, z/r)\) where \( r = \sqrt{x^2 + y^2 + z^2} \). For asymptotically flat boundary conditions we will assume that as \( r \to \infty \)

\[
\begin{align*}
g_{\mu\nu} &= \eta_{\mu\nu} + O(1/r) + \cdots \quad (46) \\
u^\mu &= \eta^{\mu} + O(1/r) + \cdots \quad (47)
\end{align*}
\]

where \( \eta^{\mu} = (1, 0, 0, 0) \) with respect to the Minkowski metric \( \eta_{\mu\nu} = (1, -1, -1, -1) \). Equation (45) will only be affected by terms in the metric and aether up to \( O(1/r) \). Using the analysis of the Newtonian limit [22] and applying the unit constraint, we find that far from the source in any asymptotically flat solution

\[
\begin{align*}
g_{00} &= 1 - \frac{r_0}{r} + \cdots \quad (48) \\
g_{ij} &= -1 - \frac{r_0}{r} + \cdots \quad (49) \\
g_{0i} &= O(1/r^2) + \cdots \quad (50) \\
\eta_{ij} &= \frac{1}{2} \frac{r_0}{r} + \cdots \quad (51) \\
u^i &= O(1/r^2) + \cdots. \quad (52)
\end{align*}
\]

The constant value at infinity and \( 1/r \) fall-off term in the aether are due to the unit timelike constraint. Thus, unlike ordinary fields, the aether will contribute to the energy expression directly. Inserting (18), (52) into the von-Freud superpotential [26] and aether superpotential [20] yields the usual ‘ADM mass’ of GR

\[
E_{GR} = \frac{1}{16\pi G} \int_{\infty} (g_{jk,k} - g_{kk,j}) n_j d^2S = \frac{r_0}{2G} \quad (53)
\]

and the aether modification

\[
E_{ae} = \frac{c_{14}}{8\pi G} \int_{\infty} \partial^i u^i n_4 d^2S = -\frac{c_{14}}{2G} \frac{r_0}{2} \quad (54)
\]

Combining, we find

\[
E_{tot} = \frac{r_0}{2G} (1 - \frac{c_{14}}{2}). \quad (55)
\]

This shows that the aether contribution effectively renormalizes the \( r_0/2G \) value we usually find for the total energy of an asymptotically flat spacetime in GR. This renormalization can also be understood as a rescaling of Newton’s constant of the form \( G_N = G/(1-c_{14}/2) \), which agrees with the result of [22].

Equation (55) implies that if \( c_{14} < 2 \) then the total energy of the Einstein-Aether theory is positive if the ADM mass \( r_0/2G \) is positive. However, the positive energy theorem for GR [8] requires a stress-tensor that satisfies the dominant energy condition. The aether stress-tensor [9] does not appear to generally satisfy this condition, so proof of total positive energy remains elusive. For some speculative thoughts on modifying the positive energy theorem, see [6].

Despite these difficulties, there are special cases of the non-linear theory that are simple enough for calculations of the energy, yet still give important results. One sector of interest is the non-linear decoupled limit. As discussed above in Section [V] this limit allows one to essentially replace \( g_{ab} \) with the flat Minkowski metric \( \eta_{ab} \) in the aether parts of (11). One significant example is \( c_2 = c_3 = c_4 = 0, c_1 \neq 0 \) theory. In this case the Lagrangian density for the aether is

\[
L = c_1 \eta^{ab} \eta_{mn} \partial_a u^m \partial_b u^n + \lambda (u^2 - 1) \quad (56)
\]

This corresponds to a nonlinear sigma model on the unit hyperboloid, which has a stress tensor satisfying the dominant energy condition. A simple way to see this is to note that the derivatives of the individual scalar components \( u^\mu \) and are contracted with \( \eta_{\mu\nu} \), which is positive definite on the unit hyperboloid. Returning to the linearized plane wave energy densities of Section [V] we see that in this special case of (55), if \( 0 < c_1 < 1 \) energy is positive in both the linearized and decoupled non-linear regimes of the theory.

Another important application of the decoupling limit relevant for our analysis of energy is the work of Clayton [23]. Clayton examined the Maxwell-like simplified theory where \( c_1 = -c_3, c_2 = c_4 = 0 \) in the decoupled version of non-linear Lagrangian (11), yielding

\[
L = \int d^3x \left\{ \frac{1}{4} (\partial_i u_i - \partial_0 u_0)^2 - \frac{3}{2} F_{ij}^2 + \frac{1}{4} \lambda (u_0^2 - \vec{u}^2 - 1) \right\}. \quad (57)
\]

where \( F_{\mu\nu} = \partial_\mu u_\nu - \partial_\nu u_\mu \). The standard calculation of the Hamiltonian and the constraint equations then produces the following on-shell value for the Hamiltonian

\[
H = \int d^3x \left\{ \frac{1}{2} \vec{P}^2 + P_i \partial_i u_0 + \frac{1}{2} (F_{ij})^2 \right\}. \quad (58)
\]

Unlike the electromagnetic case, the second term cannot be turned into a total divergence since now \( \nabla \cdot \vec{P} = -\lambda u_0 \) on-shell. This implies that for some solutions the value of the Hamiltonian is negative. For example, as initial data choose \( u_0 \) to be the gradient of a scalar field and \( P_i = -\partial_i u_0 \). Evaluating the Hamiltonian then yields

\[
E = -1/2 (\partial_0 u_0)^2, \quad (59)
\]

which can be made arbitrarily negative by an appropriate choice of \( u_0 \).

Moreover, as Clayton points out, the negative energies are not restricted to this special case. In particular, allowing \( c_2 \neq 0 \) does not affect the \( \vec{P} \cdot \vec{u} \) term in the Hamiltonian and even produces additional questionable terms. The indefinite nature of the decoupled Hamiltonian contrasts with with the wave energy densities of Section [V] which clearly can be made positive definite in the Maxwell-like case. The key point is that the wave results are in the linearized theory and associated with quadratic parts of the Hamiltonian, while the indefinite
terms appear at higher orders. For example, the linearized constraint equation $\rho^0 = 0$ eliminates $\mathbf{P} \cdot \partial u_0$ from the Maxwell-like Hamiltonian and forces the $u_0$ in (59) to be quadratic or higher in the perturbation. Thus, the indefinite pieces begin to appear at quartic order in the Hamiltonian. This indefiniteness at higher orders implies that the decoupled, linearized results of Lin and the “coupled”, linearized analysis of this paper generally do not detect possible energies of arbitrary sign in the fully non-linear decoupled Einstein-Aether theory.

VII. DISCUSSION

In this paper we have derived two energy-momentum pseudotensor expressions for the Einstein-Aether theory and used them to compute the energy densities of weak gravitational waves and the total energy of an asymptotically flat solution. The constraints of Section V show that a sector of this LV model satisfies the important theoretical condition of positive energy in the linearized case. However, a remaining open question is whether the energy remains positive when we consider the full non-linear theory. We have argued that in the decoupled limit the $c_1 \neq 0$ non-linear sigma model is immune to the sickness of energies of indefinite sign. However, other special cases of the coupling constants yield negative energy solutions even when the linearized theory has positive energy. A complete answer to the question of positivity of energy in the non-linear theory is not yet in hand.

The recipe for the Weinberg pseudotensor discussed in Section III and the Einstein superpotential and pseudotensor derived in Section IV also has applications in studying the emission of gravitational-aether radiation from astrophysical sources. In [21] the analog of the quadrupole formula (which also involves monopole and dipole moments), is obtained using a pseudotensor expression derived from the related Noether charge approach. This expression is then used to track radiative energy losses and study constraints on the model.

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Appendix

Background

In this appendix we will derive the Einstein pseudotensor and superpotential using the Noether current formalism of Julia and Silva [18] applied to Lagrangians that depend on the fields and their first and second derivatives. We can write a variation in the Lagrangian as

$$\delta L = \frac{\partial L}{\partial \delta \psi} + \frac{\partial L}{\partial \delta (\partial \psi)} \delta \psi + \frac{\partial L}{\partial \delta (\partial \psi)} \delta (\partial \psi)$$

(A.1)

and then integrate by parts to isolate the equations of motion $E$ and a symplectic current $\theta^\mu$,

$$\delta L = E \delta \psi + \partial_\mu \theta^\mu, \quad (A.2)$$

where

$$\partial_\mu \theta^\mu = \partial_\mu \left( \frac{\partial L}{\partial \delta \psi} \delta \psi - \frac{\partial L}{\partial \delta (\partial \psi)} \delta (\partial \psi) \right)$$

(A.3)

If the action associated with $L$ is invariant under a continuous transformation of the fields, $\delta L = \partial_\mu S^\mu$. Thus, we have the equation

$$\partial_\mu (S^\mu - \theta^\mu) = E \delta \psi. \quad (A.4)$$

This identifies the on-shell ($E = 0$) conserved Noether current,

$$J^\mu = \theta^\mu - S^\mu = \frac{\partial L}{\partial \delta \psi} \delta \psi - \frac{\partial L}{\partial \delta (\partial \psi)} \delta (\partial \psi)$$

$$+ \frac{\partial L}{\partial \delta (\partial \psi)} \delta (\partial \psi) - S^\mu. \quad (A.5)$$

We now want to consider a gauge transformation of the fields that involves derivatives of the generator $\xi^A(x)$. Here we will focus on the special case restricting attention to only the first derivative. Following the analysis and notation of [13] we parameterize the gauge transformation as

$$\delta \psi = \xi^A \Delta_A + (\partial_\nu \xi^A) \Delta^\nu_A \quad (A.6)$$

where $A$ is an internal or spacetime index and $\Delta$ is a transformation matrix. The quantity $S^\mu$ can be expressed similarly as

$$S^\mu = \xi^A \Sigma^\mu_A + (\partial_\nu \xi^A) \Sigma^\mu_A + (\partial_\nu \partial_\tau \xi^A) \Sigma^B_\nu \Delta^\tau_A \quad (A.7)$$

Inserting these forms into (A.5) and combining terms, we find that on-shell

$$\partial_\mu \left( \xi^A J^\mu_A + \partial_\nu \xi^A U^\nu_A + \partial_\nu \partial_\tau \xi^A U^{\nu \tau}_A \right) = 0, \quad (A.8)$$
where

\[ J^\mu_A = \left( \frac{\partial L}{\partial (\partial_\nu \psi)} - \partial_\nu \left( \frac{\partial L}{\partial (\partial_\mu \partial_\nu \psi)} \right) \right) \Delta_A + \frac{\partial L}{\partial (\partial_\mu \partial_\tau \psi)} \partial_\tau \Delta_A - \Sigma^\mu_A \] (A.9)

\[ U^{\mu\nu}_A = \left( \frac{\partial L}{\partial (\partial_\nu \psi)} - \partial_\nu \left( \frac{\partial L}{\partial (\partial_\mu \partial_\nu \psi)} \right) \right) \Delta^\nu_A + \frac{\partial L}{\partial (\partial_\mu \partial_\tau \psi)} \partial_\tau \Delta^\nu_A - \Sigma^{\mu\nu}_A \] (A.10)

\[ V^{\mu(\tau\nu)}_A = \frac{\partial L}{\partial (\partial_\mu \partial_\tau \partial_\nu \psi)} \Delta^\nu_A - \Sigma^{\mu(\tau\nu)}_A. \] (A.11)

Since \( \xi^A \) and its derivatives should be arbitrary and independent, this single equation decomposes into 4 equations

\[ \partial_\mu J^\mu_A \approx 0 \] (A.12)

\[ J^\mu_A + \partial_\nu U^{\nu\mu}_A \approx 0 \] (A.13)

\[ U^{\mu(\nu\tau)}_A + \partial_\tau V^{\mu(\nu\tau)}_A = 0 \] (A.14)

\[ V^{\mu(\tau\nu)}_A = 0. \] (A.15)

The first two equations hold on-shell, while the last two are identities (since there are no second or third derivatives of \( \xi^A \) on the right hand side of (A.10)).

The gauge symmetry implies that \( J^\mu_A \) is conserved and equal to the divergence of the superpotential \( U^{\mu}_A \). Since \( \xi^A = \xi^A(x) \), the Noether current \( J^\mu \) will now be parameter dependent. Let us consider a one parameter subgroup of the local gauge or diffeomorphism symmetry where \( \xi^A \) has the decomposition,

\[ \xi^A(x) = \epsilon(x) \xi^A_0, \] (A.16)

and \( \xi^A_0 \) is fixed. Inserting this form into (A.10) produces

\[ \xi^A_0 J^\mu_A + \partial_\nu (\xi^A_0 U^{\nu\mu}_A) = 0, \] (A.17)

with \( J^\mu_0 = \xi^A_0 J^\mu_A \). The conserved charge is

\[ Q = \int J^\mu_0 d^4x = \int \xi^A_0 U^{\nu\mu}_A n_\nu d^2x. \] (A.18)

\( Q \) depends on the choice of \( \xi^A_0 \) and can be expressed in terms of the gauge fields up using (A.10). If \( \xi^A_0 \) is an asymptotic translation in an asymptotically flat spacetime, then the conserved charge will be a total energy or momentum. Using the variational principle \( \delta S = 0 \Rightarrow \) equations of motion), we will show in the next section that the choice of \( \xi^A_0 \) is subject to certain boundary conditions at infinity. We now apply this type of Lagrangian analysis to the Einstein-Aether theory.

**Application**

Assume that the Lagrangian density \( \mathcal{L}(\psi, \partial \psi, \partial^2 \psi) \) is invariant under diffeomorphisms and is a combination of a scalar density \( \tilde{L} \) and a total divergence \( \partial_\mu W^\mu \),

\[ \mathcal{L}(\psi, \partial \psi, \partial^2 \psi) = \tilde{L}(\psi, \partial \psi, \partial^2 \psi) + \partial_\mu [W^\mu(\psi, \partial \psi)]. \] (A.19)

If \( W^\mu \) is a vector density then the total divergence is a scalar density, but we allow for a non-covariant total divergence. For a variation that is an infinitesimal diffeomorphism generated by a vector field \( \xi^\nu \), we have \( \delta \tilde{L} = \partial_\mu (\xi^\mu \tilde{L}) \) since \( \tilde{L} \) is a scalar density and \( \delta (\partial_\mu W^\mu) = \partial_\mu (\delta W^\mu) \). Therefore the surface term \( S^\mu \) in (A.4) has the form

\[ S^\mu = \xi^\mu \tilde{L} + \delta W^\mu. \] (A.20)

Now consider the Einstein-Hilbert action

\[ S_{EH} = \int \sqrt{-g} R \] (A.21)

of pure GR. The Ricci scalar \( R \) has a dependence on second derivatives of the metric. In light of this, Einstein exploited a property of the Hilbert action that allows it to be separated into a bulk and a surface term

\[ \int \sqrt{-g} R \ d^4x = \int \sqrt{-g} L_{bulk} + \partial_\mu V^\mu d^4x. \] (A.22)

This decomposition of the Ricci scalar takes the following form

\[ L_{bulk} = g^{\alpha\beta} \left\{ \Gamma_{\alpha\beta}^{\gamma} \Gamma_{\gamma}^{\delta} - \Gamma_{\gamma\beta}^{\delta} \Gamma_{\gamma}^{\alpha} \right\} \] (A.23)

\[ V^\mu = \sqrt{-g} \left\{ \Gamma_{\alpha\beta}^{\mu} g^{\alpha\beta} - \Gamma_{\alpha\beta}^{\mu} g^{\alpha\beta} \right\}. \] (A.24)

where \( \Gamma \) is the Levi-Civita connection. One can eliminate the total divergence by adding its negative to the Einstein-Hilbert action

\[ \int \sqrt{-g} L_{bulk} = \int \sqrt{-g} R \ d^4x - \partial_\mu V^\mu d^4x. \] (A.25)

The elimination does not affect the equations of motion and is consistent with the general action (A.19) with \( \tilde{L} = \sqrt{-g} R \) and \( W^\mu = -V^\mu \). The result of this is a loss of diffeomorphism invariance since the remaining \( L_{bulk} \) in the “\( \Gamma^2 \)” action is not a scalar. We have allowed for this possibility with the non-covariant \( \partial_\mu W^\mu \) term in (A.19).

With \( \mathcal{L}(\psi, \partial \psi, \partial^2 \psi) \) the bulk part of the Einstein-Hilbert action plus the aether terms in (1), we arrive at the Einstein-Aether form of (A.4) on shell

\[ \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu g_{\alpha\beta})} \delta g_{\alpha\beta} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu u^\alpha)} \delta u^\alpha - S^\mu \right) = 0. \] (A.26)

The second derivative terms in (A.25) vanish in this case. Under a diffeomorphism generated by a vector field \( \xi^\nu \) the variation of the metric and the aether is simply the Lie derivative, \( \delta g_{\alpha\beta} = \xi^\nu \partial_\nu g_{\alpha\beta} \) and \( \delta u^\alpha = \xi^\nu \partial_\nu u^\alpha - u^\nu \partial_\nu \xi^\nu \alpha \). It follows from (A.20) that
\[ S^\mu = \xi^\mu L_{\text{bulk}} - \sqrt{-g} (\partial_t \partial_\xi \xi^\mu g^{\nu\tau} - \partial_\nu \partial_\tau \xi^\nu g^{\mu\beta}) \]. Inserting these forms into (A.26) produces

\[ t_\nu^\mu = \sqrt{-g} \left( \frac{\partial L}{\partial (\partial_\mu g_{\alpha\beta})} \partial_\nu g_{\alpha\beta} + \frac{\partial L}{\partial (\partial_\nu u^\alpha)} \partial_\nu u^\alpha \right) - \delta_\nu^\mu \]  

\[ U_\nu^{\mu\gamma} = \sqrt{-g} \left( \frac{\partial L}{\partial (\partial_\mu g_{\alpha\beta})} (\delta_\nu^\alpha g_{\beta\gamma} + \delta_\nu^\beta g_{\alpha\gamma}) - \frac{\partial L}{\partial (\partial_\nu u^\alpha)} \delta_\nu^\mu u^\gamma \right) \]  

\[ V_\nu^{\mu(\gamma\lambda)} = \sqrt{-g} \left( \delta_\mu^\gamma g_{\lambda\nu} - \frac{1}{2} \delta_\gamma^\nu g \lambda^\mu - \frac{1}{2} \delta_\lambda^\nu g \mu^\gamma \right) \]  

as coefficients of \( \xi^\mu \), \( \partial \xi^\mu \) and \( \partial^2 \xi^\mu \) respectively. \( t_\nu^\mu \) and \( U_\nu^{\mu\gamma} \), and \( V_\nu^{\mu(\gamma\lambda)} \) are the analogs of \( J_\nu^A \), \( U_\nu^{\mu\rho} \), \( V_\nu^{\mu(\gamma\lambda)} \) in (A.9)–(A.11). The resulting equations due to the arbitrariness and independence of the derivatives of \( \xi^\mu \) are

\[ \partial_\mu t_\nu^\mu \approx 0 \]  

\[ t_\nu^\mu \approx -\partial_\nu U_\nu^{\gamma\mu} \]  

\[ U_\nu^{\gamma\mu} + \partial_\nu V_\nu^{\lambda(\gamma\nu)} = 0 \]  

\[ V_\nu^{\gamma\lambda\mu} = 0 \]  

Following (A.16), we can keep the \( \xi^\mu \) vector fixed (and determine it later for each conserved charge) by choosing \( \xi^\nu = c(x) \xi_0^\nu \). The main result, as before, is

\[ \xi_0^\nu t_\nu^\mu = -\partial_\gamma (\xi_0^\nu U_\nu^{\gamma\mu}) \]  

showing that a Noether charge is again obtained as a surface term. Einstein effectively chose the \( \xi_0^\nu \) vector to be a constant in [A.34], reducing the pseudotensor to a form consistent with the flat spacetime canonical stress tensor [A.19]. However, this choice is not inconsequential. The variational principle for the \( \Gamma^2 \) action requires the vanishing of the surface term in the asymptotic region,

\[ \int_{S_\infty} \frac{\partial L}{\partial (\partial_\mu g_{\alpha\beta})} \delta g_{\alpha\beta} + \frac{\partial L}{\partial (\partial_\mu u^\alpha)} \delta u^\alpha. \]  

Therefore we have Dirichlet boundary conditions \( \delta g_{\alpha\beta} = 0 \) and \( \delta u^\alpha = 0 \) on the metric and the aether at infinity. Inserting the Lie derivatives for the variations above, we see that \( r \to \infty \)

\[ \nabla (\alpha \xi_0^\beta) \to 0 \]  

\[ \mathcal{L}_\xi u^\alpha \to 0. \]  

Since \( \xi^\nu \) has been chosen to be constant everywhere and \( u^\alpha \) is asymptotically constant, the connection coefficients must vanish as one approaches spatial infinity. Thus, one must compute the pseudotensor and superpotential in a coordinate system where the connection vanishes asymptotically as \( O(1/r) \) or faster.

[1] see for example the following and references therein: Proceedings of the Second Meeting on CPT and Lorentz Symmetry, Bloomington, Indiana, 2001, edited by V. A. Kostelecky (World Scientific, Singapore, 2002); G. Amelino-Camelia, arXiv:gr-qc/0309054, N. E. Mavromatos, arXiv:gr-qc/0407005, T. Jacobson, S. Liberati and D. Mattingly, arXiv:hep-ph/0407370.

[2] see for example, V. A. Kostelecky, Phys. Rev. D 69 105009 (2004) arXiv:hep-th/0312310.

[3] C. M. Will and K. Nordvedt, Jr., Astrophys. J. 177, 757 (1972).

[4] K. Nordvedt, Jr. and C. M. Will, Astrophys. J. 177, 775 (1972).

[5] R. W. Hellings and K. Nordvedt, Jr., Phys. Rev. D 7, 3593 (1973).

[6] C. Eling, T. Jacobson, D. Mattingly, “Einstein-aether theory”, in Deserfest, eds. J. Liu, K. Stelle, and R. P. Woodard (World Scientific, 2006) arXiv:gr-qc/0410001.

[7] see for example, J. W. Elliott, G. D. Moore, and H. Stoica, JHEP 0508 (2005) 066 arXiv:hep-ph/0505211.

[8] E. Witten, Comm. Math. Phys. 80, 381 (1981); R. Schoen and S.-T. Yau, Phys. Rev. Lett. 48, 369 (1982).

[9] D. L. Lee, A. P. Lightman, and W.-T. Ni, Phys. Rev. D 10, 1685 (1974).

[10] C. C. Chang, J. M. Nester, and C. M. Chen, Phys. Rev. Lett. 83 1897-1901 (1999) arXiv:gr-qc/9809040

[11] T. Jacobson and D. Mattingly, Phys. Rev. D 70 024003 (2004) arXiv:gr-qc/0402005.

[12] E. A. Lim, Phys. Rev. D 71 063504 (2005) arXiv:astro-ph/0407437.

[13] B. Z. Foster, Phys. Rev. D 73 024005 (2006) arXiv:gr-qc/0509121.

[14] C. Eling and T. Jacobson, Phys. Rev D 69, 064005 (2004) arXiv:gr-qc/0310044.

[15] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972).

[16] C. W. Misner, K. Thorne, and J. A. Wheeler, Gravitation (Freeman, San Francisco, 1973).

[17] R. M. Wald, General Relativity, (University of Chicago Press, 1984), Ch.4 Problem 5.

[18] B. Julia and S. Silva, Class. Quant. Grav., 15, 2173-2215 (1998) arXiv:gr-qc/9804029.

[19] Ph. Freud, Ann. of Math. 40 417 (1938).

[20] R. Portugal and S. Sautu, Computer Physics Communications, 105, 233 (1997). Also see http://www.cbpf.br/~portugal/Riemann.html

[21] B. Z. Foster, “Radiation damping in Einstein-aether theory,” arXiv:gr-qc/0602004.

[22] S. M. Carroll and E. A. Lim, Phys. Rev. D 70 123525 (2004) arXiv:hep-th/0407149.

[23] M. A. Clayton, arXiv:gr-qc/0104103
[24] The author thanks an anonymous referee for pointing out this fact.

[25] For example, if one uses the Schwarzschild metric in spherical polar coordinates $ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2d\Omega^2$, the von-Freud superpotential will yield an incorrect total energy. After re-expressing the metric in Cartesian coordinates $(t, x, y, z)$, the pseudotensor expression gives $E = M$.

[26] Unlike (for example) the Lorentz gauge in GR, the residual gauge of (33)-(34) is not compatible with the equations of motion (30)-(32) so it is not clear how to fix the remaining gauge in a simple way. However, it is possible to argue for the existence of 5 wave modes by counting gauge inequivalent degrees of freedom. Consider a theory with $N$ field variables and $M$ gauge symmetries. One can always use the $M$ constraint equations to solve for the $M$ variables whose time derivative does not appear in the equations of motion. This reduces the number of degrees of freedom to $N - M$. Then the remaining $M$ gauge functions can be used to further reduce to $N - 2M$. In the case of the aether theory the metric has 10 degrees of freedom and the constrained vector has 3, making 13 field variables. There are 4 diffeomorphism symmetries, and $13 - 2x4 = 5$. 