Stability of mixed Nash equilibria in symmetric quantum games

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Abstract

In bi-matrix games the Bishop-Cannings theorem of the classical evolutionary game theory does not permit pure evolutionarily stable strategies (ESSs) when a mixed ESS exists. We find the necessary form of two-qubit initial quantum states when a switch-over to a quantum version of the game also changes the evolutionary stability of a mixed symmetric Nash equilibrium.

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Key words: Quantum games, Evolutionarily Stable Strategies (ESSs), Mixed strategies

1 Introduction

Quantum game theory has gained considerable interest recently. In a pioneering work, Meyer presented the idea of playing a quantum form of a sequential game by unitary manipulation of a qubit. A measurement of the final quantum state of the qubit gives the payoffs to the players. Eisert, Wilkens, and Lewenstein, while focussing on the concept of Nash equilibrium (NE), from noncooperative game theory, extended the famous game of prisoner’s dilemma to quantum domain. Using a maximally entangled two-qubit initial quantum state, they showed that the dilemma can be made to disappear when players have access to a particular set of unitary operators. Also the classical game can be reproduced as a subset. Later Marinatto and Weber followed a different approach and studied the game of battle of sexes in quantum settings, showing that the introduction of entangled strategies leads to a unique solution of this game. Moreover, they showed that in their scheme the classical game corresponds to an unentangled initial quantum state.

An important question in quantum game theory is to draw a comparison with the corresponding classical version of the game. In classical game theory there is well developed mathematical formalism to study the evolutionary
dynamics of a population consisting of interacting individuals \[5, 6\]. It is interesting to investigate and extend this formalism to quantum domain, and also to compare the predictions of classical and quantum game-theoretical models of evolution. In other words, how the established evolutionary concepts of mathematical biology, based on classical game-theoretical modeling, are modified by the introduction of Hilbert space? In our earlier papers \[7, 8, 9, 10\] we explored the relevance of the concept of evolutionary stability in quantum game theory.

In evolutionary game theory, an \textit{evolutionarily stable strategy} (ESS) \[11\] is a well known concept describing the stable states of a population resulting from dynamics of evolution. ESSs are known to be symmetric Nash equilibria robust against small mutations \[12\]. We explored how a strategy, being an ESS in classical version of the game, performs if the game is played in quantum settings. Playing a game in Marinatto and Weber’s scheme \[4\] with particular choice of initial quantum state, i.e., \(|\psi_{\text{ini}}\rangle = c_{11} |1, 1\rangle + c_{22} |2, 2\rangle\), where 1 and 2 represent the classical pure strategies and \(|c_{11}|^2 + |c_{22}|^2 = 1\) with \(c_{11}, c_{22} \in \mathbb{C}\), we showed that evolutionary stability of a pure strategy in a symmetric quantum form of a bi-matrix game can be changed by a control on the parameters of the initial quantum state \[7, 8\]. However, with an initial state in this form, the evolutionary stability of a mixed NE cannot be changed for two-player games but it becomes possible when the number of players is increased from two to three \[9\]. In these considerations the corresponding symmetric NE remain intact both in the quantum and classical versions of the game.

In evolutionary game theory mixed strategies play a significant role. The well-known \textit{Bishop-Cannings theorem} (BCT) \[13\] describes an interesting property of mixed ESSs in symmetric bi-matrix games. It is useful to introduce the concept of \textit{support} of an ESS to understand more easily the BCT \[15, 14\]. Suppose a strategy vector \(p = (p_i)\) is an ESS. Its support \(S(p)\) is the set \(S(p) = \{i : p_i > 0\}\). Thus the support of \(p\) is the set of pure strategies that can be played by a \(p\)-player. BCT states that if \(p\) is an ESS with support \(I\) and \(r \neq p\) is an ESS with support \(J\), then \(I \nsubseteq J\). For bi-matrix games the BCT shows that \textit{no pure strategy can be evolutionary stable when a mixed ESS exists} \[14\]. Naturally one, then, asks about the classical pure ESSs when a switch-over to a quantum form of a classical symmetric bi-matrix game also gives evolutionary stability to a mixed symmetric NE.

In present paper, following an approach developed for the quantum version of the rock-scissor-paper (RSP) game \[10\], we consider a general form of a two-qubit initial quantum state. Our results show that for this form of initial quantum state, the corresponding quantum version of a bi-matrix game can give evolutionary stability to a mixed NE, when classically it is not stable. It is interesting to observe that by ensuring evolutionary stability to a mixed NE in a quantum form of the game, the BCT forces out the pure ESSs present in classical form of the game.
2 Evolutionary stability of a mixed NE

In a classical symmetric bi-matrix game, played in an evolutionary set-up involving a population, all the members of the population are indistinguishable and each individual is equally likely to face each other. In such a set-up one assumes that individuals interact only in pair-wise encounters. Suppose that the finite set of pure strategies \( \{1, 2, ..., n\} \) is available to each player. In one pair-wise encounter let a player \( A \) receives a reward \( a_{ij} \) by playing strategy \( i \) against another player \( B \) playing strategy \( j \). In symmetric situation the player \( B \), then, gets \( a_{ji} \) as a reward. The value \( a_{ij} \) is an element in the \( n \times n \) payoff matrix \( M \). We assume that the players also have an option to play a mixed strategy. It means he/she plays the strategy \( i \) with probability \( p_i \) for all \( i = 1, 2, ..., n \). A strategy vector \( p \), with components \( p_i \), represents the mixed strategy played by the player. In standard notation an average, or expected, payoff for player \( A \), playing strategy \( p \), against player \( B \) playing \( q \), is written as \( P(p, q) \)

\[
P(p, q) = \sum a_{ij} p_i q_j = p^T M q
\]

where \( T \) is for transpose. Suppose that the strategy \( p \) is played by almost all the members of the population, the rest of population forms a small mutant group constituting a fraction \( \epsilon \) of the total population playing \( q \). \( p \) is said to be evolutionary stable against \( q \) if

\[
P(p, (1 - \epsilon)p + \epsilon q) > P(q, (1 - \epsilon)p + \epsilon q)
\]

for all sufficiently small \( \epsilon \). Thus \( p \) does better against the mean population strategy than \( q \) does. The condition (2) implies that either (i) \( P(p, p) > P(q, p) \) or (ii) \( P(p, p) = P(q, p) \) and \( P(p, q) > P(q, q) \). The vector \( p \) is said to be evolutionary stable against \( q \) if \( p \) is evolutionary stable against all \( q \neq p \).

The payoff to a player in the quantum version of rock-scissors-paper (RSP) game [10] can also be written in similar form to (1), provided the matrix \( M \) is replaced with a matrix corresponding to the quantum version of the game. In RSP each player has access to three pure strategies, represented by 1, 2, and 3, and the quantum version of the game is given by the following matrix, with the players recognized as Alice and Bob

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & (\alpha_{11}, \alpha_{12}) & (\alpha_{13}, \alpha_{31}) \\
2 & (\alpha_{21}, \alpha_{12}) & (\alpha_{23}, \alpha_{32}) \\
3 & (\alpha_{31}, \alpha_{13}) & (\alpha_{32}, \alpha_{33}) \\
\end{array}
\]

where, for example, \((\alpha_{23}, \alpha_{32})\) means that Alice and Bob get \( \alpha_{23} \) and \( \alpha_{32} \), respectively, when Alice plays the strategy 2 and Bob plays 3. In quantum version of the game the players apply unitary operators \( I, C, \) and \( D \) on an initial quantum state defined as follows [4, 10]:

\[
\text{Initial state: } |\psi\rangle = \langle \alpha_1 | I | 0 \rangle + \langle \alpha_2 | C \rangle + \langle \alpha_3 | D \rangle
\]
Suppose Alice applies the operators \( C \) where

\[
C |1\rangle = |1\rangle \quad C |2\rangle = |2\rangle \quad C |3\rangle = |3\rangle
\]

and (1 − \( \omega \)).

Similarly Bob applies the operators \( D \) with probabilities \( q \), \( q_1 \), and \( (1 − q − q_1) \) respectively, on the initial quantum state \( |\psi_{ini}\rangle \) where

\[
|\psi_{ini}\rangle = \sum_{i,j=1,2,3} c_{ij} |i, j\rangle \quad \text{where} \quad \sum_{i,j=1,2,3} |c_{ij}|^2 = 1
\]

The payoff to Alice who plays the strategy \( p \) (where \( p^T = [1 − p − p_1 \quad p_1 \quad p] \)) against Bob who plays the strategy \( q \) (where \( q^T = [1 − q − q_1 \quad q_1 \quad q] \)) can be written as

\[
P_A(p, q) = p^T \omega q
\]

where the matrix \( \omega \) is given by

\[
\omega = \begin{pmatrix}
\omega_{11} & \omega_{12} & \omega_{13} \\
\omega_{21} & \omega_{22} & \omega_{23} \\
\omega_{31} & \omega_{32} & \omega_{33}
\end{pmatrix}
\]

and the elements of \( \omega \) are given by following matrix equation

\[
\begin{pmatrix}
\omega_{11} & \omega_{12} & \omega_{13} \\
\omega_{21} & \omega_{22} & \omega_{23} \\
\omega_{31} & \omega_{32} & \omega_{33}
\end{pmatrix}
= \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix} \times
\begin{pmatrix}
|c_{11}|^2 & |c_{12}|^2 & |c_{13}|^2 \\
|c_{21}|^2 & |c_{22}|^2 & |c_{23}|^2 \\
|c_{31}|^2 & |c_{32}|^2 & |c_{33}|^2
\end{pmatrix}
\]

The above matrix (7) reduces to its classical form of Eq. (3), by making the initial state unentangled i.e., \(|c_{11}|^2 = 1\).

In a symmetric game the exchange of strategies by Alice and Bob also exchanges their respective payoffs. The concept of an ESS was originally defined
for symmetric games where a player’s payoff is given by his strategy and his identity does not affect it \[11\]. It is seen that the quantum game corresponding to the matrix \((3)\), when played using the initial quantum state of eq. \((5)\), becomes symmetric when

\[|c_{ij}|^2 = |c_{ji}|^2 \quad \text{for} \quad i \neq j \]  

(9)

Here the two-player quantum game, with three pure strategies, has a form similar to a classical matrix game. The payoff matrix of the classical game is, however, replaced now with its quantum version \((7)\). Also the matrix \((7)\) now involves the coefficients \(c_{ij}\) of the initial quantum state \((5)\).

To reduce the above mathematical formalism to two-players, two-strategy quantum game let us fix \(p_1 = q_1 = 0\), i.e., both players do not use the operator \(D\) at all, and apply only the operators \(C\) and \(I\), with classical probabilities, on the initial quantum state. Payoff to the player who plays the strategy vector \(p\) (where \(p^T = [1 - p \quad p]\)) against the player playing the strategy vector \(q\) (where \(q^T = [1 - q \quad q]\)) can again be written as \(P(p, q) = p^T \omega q\). Nevertheless, \(\omega\) is now reduced to its simpler form given as

\[\omega = \begin{pmatrix} \omega_{11} & \omega_{13} \\ \omega_{31} & \omega_{33} \end{pmatrix} \]  

(10)

where the elements of the matrix are

\[
\begin{pmatrix}
\omega_{11} \\
\omega_{13} \\
\omega_{31} \\
\omega_{33}
\end{pmatrix} =

\begin{pmatrix}
|c_{11}|^2 & |c_{13}|^2 & |c_{31}|^2 & |c_{33}|^2 \\
|c_{31}|^2 & |c_{11}|^2 & |c_{33}|^2 & |c_{13}|^2 \\
|c_{33}|^2 & |c_{31}|^2 & |c_{13}|^2 & |c_{11}|^2 \\
|c_{13}|^2 & |c_{33}|^2 & |c_{11}|^2 & |c_{31}|^2
\end{pmatrix}
\begin{pmatrix}
\alpha_{11} \\
\alpha_{13} \\
\alpha_{31} \\
\alpha_{33}
\end{pmatrix}
\]  

(11)

It is now a bi-matrix game played with the initial quantum state \((5)\). The available pure strategies are now 1 and 3 only and the terms with subscripts containing 2 disappear. Take \(x = 1 - p\) and \(y = 1 - q\), so that \(x\) and \(y\) are probabilities with which players apply identity operator on the initial state \(|\psi_{\text{ini}}\rangle\). The strategy vectors \(p\) and \(q\) can then be represented only by the numbers \(x\) and \(y\), respectively. Payoff to a \(x\)-player against a \(y\)-player is obtained as

\[
P(x, y) = p^T \omega q = x \{\omega_{11}y + \omega_{13}(1 - y)\} + (1 - x) \{\omega_{31}y + \omega_{33}(1 - y)\}.
\]  

(12)

Suppose \((x^*, x^*)\) is a Nash equilibrium, i.e.,

\[
P(x^*, x^*) - P(x, x^*)
= \quad (x^* - x) \{x^*(\omega_{11} - \omega_{13} - \omega_{31} + \omega_{33}) + (\omega_{13} - \omega_{33})\} \geq 0
\]  

(13)
for all $x \in [0,1]$. The mixed strategy $x^* = x_q^* = \frac{ω_{11} - ω_{13} - ω_{31} + ω_{33}}{ω_{11} - ω_{13} - ω_{31} + ω_{33}}$ makes the payoff difference $P(x^*, x^*) - P(x, x^*)$ identically zero. The subscript $q$ is for ‘quantum’. Let $Δx = x^* - x$ then

$$P(x_q^*, x) - P(x, x) = -(Δx)^2 \{ω_{11} - ω_{13} - ω_{31} + ω_{33}\}$$  (14)

Now $x_q^*$ is an ESS if $P(x_q^*, x) - P(x, x) > 0$ for all $x \neq x_q^*$.  which leads to the requirement $(ω_{11} - ω_{31} - ω_{13} + ω_{33}) < 0$.

The classical game corresponds when $|c_{11}|^2 = 1$ and it gives $ω_{11} = α_{11}$, $ω_{13} = α_{13}$, $ω_{31} = α_{31}$, and $ω_{33} = α_{33}$, in accordance with the Eq. (8). In case $(α_{11} - α_{13} - α_{31} + α_{33}) > 0$, the mixed NE of a classical game, i.e.,

$$x^* = x_q^* = \frac{α_{33} - α_{13}}{α_{11} - α_{31} - α_{13} + α_{33}} = \frac{ω_{33} - ω_{13}}{ω_{11} - ω_{31} - ω_{13} + ω_{33}}$$  (15)

saying that the classical NE $x_q^*$ is also a NE in quantum form of the game. One notices from the matrix in the Eq. (11)

$$\begin{pmatrix}
ω_{11} - ω_{31} - ω_{13} + ω_{33} \\
(α_{11} - α_{13} - α_{31} + α_{33})(|c_{11}|^2 - |c_{13}|^2 - |c_{31}|^2 + |c_{33}|^2)
\end{pmatrix} \begin{pmatrix}
α_{33} - α_{13} \\
α_{11} - α_{31} - α_{13} + α_{33}
\end{pmatrix} = \begin{pmatrix}
ω_{33} - ω_{13} \\
ω_{11} - ω_{31} - ω_{13} + ω_{33}
\end{pmatrix}$$  (16)

and

$$ω_{33} - ω_{13} = |c_{11}|^2 (α_{33} - α_{13}) + |c_{13}|^2 (α_{31} - α_{11}) + |c_{31}|^2 (α_{13} - α_{33}) + |c_{33}|^2 (α_{11} - α_{31})$$  (17)

Now a substitution from Eqs. (16,17) into the Eq. (15) gives $α_{33} - α_{13} = α_{11} - α_{31}$, and this leads to $x_q^* = x_q^* = \frac{1}{2}$ Therefore, the mixed strategy $x^* = \frac{1}{2}$, remain a NE in both classical and a quantum form of the game. Consider now this mixed NE for a classical game with $(α_{11} - α_{13} - α_{31} + α_{33}) > 0$ – showing that it is not an ESS. The above Eq. (10) shows an interesting possibility that it is still possible to have $(ω_{11} - ω_{31} - ω_{13} + ω_{33}) < 0$ if

$$|(c_{11}|^2 + |c_{33}|^2) < |c_{13}|^2 + |c_{31}|^2$$  (18)

In other words, now the evolutionary stability of a mixed strategy –which is a NE in both classical and quantum versions of the game– changes when the game switches-over between its two forms. To have a symmetric game in its quantum form one also needs $|c_{13}|^2 = |c_{31}|^2$ and the inequality (18) reduces to $|c_{11}|^2 + |c_{33}|^2 < |c_{13}|^2 + |c_{31}|^2$.

Therefore, a quantum version of a symmetric bi-matrix classical game of the matrix

$$\begin{pmatrix}
(α_{11}, α_{11}) & (α_{13}, α_{31}) \\
(α_{31}, α_{13}) & (α_{33}, α_{33})
\end{pmatrix}$$  (19)
can be played by players having two unitary operators and a general two-qubit quantum state of the form

$$|\psi_{ini}\rangle = \sum_{i,j=1,3} c_{ij} |ij\rangle$$

(20)

where \(\sum_{i,j=1,3} |c_{ij}|^2 = 1\). In case \(\alpha_{33} - \alpha_{13} = \alpha_{11} - \alpha_{31}\) the mixed strategy \(x^* = \frac{1}{2}\) is not an ESS in the classical game if \((\alpha_{33} - \alpha_{13}) > 0\). Nevertheless, the strategy \(x^* = \frac{1}{2}\) becomes an ESS when \(|c_{11}|^2 + |c_{33}|^2 < |c_{13}|^2 + |c_{31}|^2\). In case \((\alpha_{33} - \alpha_{13}) < 0\) the strategy \(x^* = \frac{1}{2}\) is an ESS classically but does not remain if \(|c_{11}|^2 + |c_{33}|^2 < |c_{13}|^2 + |c_{31}|^2\). Now suppose \(|c_{13}|^2 = |c_{31}|^2 = 0\). Then the Eq. (16) reduces to

$$\left(\omega_{11} - \omega_{13} - \omega_{31} + \omega_{33}\right) = (\alpha_{11} - \alpha_{13} - \alpha_{31} + \alpha_{33})$$

(21)

One observes from the above equation that if a quantum game is played by following simple form of the initial quantum state

$$|\psi_{ini}\rangle = c_{11} |11\rangle + c_{33} |33\rangle$$

(22)

it is not possible to influence the evolutionary stability of a mixed NE, as it is concluded in our earlier work \[8, 9\].

3 Summary

Mixed ESSs appear in many games of interest that are played in the natural world. The examples of the Rock-Scissors-Paper (RSP) and the Hawks and Doves games are well known from evolutionary game theory. In evolutionary game theory the Bishop-Cannings theorem does not permit pure ESSs when a mixed ESS exists in a bi-matrix game. In earlier work \[7, 8, 9, 10\] we showed that it is possible to change evolutionary stability of a pure symmetric NE with a control of the parameters \(c_{11}\) and \(c_{22}\) when the game is played with an initial two-qubit quantum state of the form \(|\psi_{ini}\rangle = c_{11} |1, 1\rangle + c_{22} |2, 2\rangle\) where \(|c_{11}|^2 + |c_{22}|^2 = 1\). However, evolutionary stability of a mixed symmetric NE cannot be changed with such a control. In this paper, following the approach developed for the quantum version of the rock-scissors-paper (RSP) game \[10\], we allowed the game to be played with a general form of a two-qubit initial quantum state. With this state it becomes possible to change evolutionary stability of a mixed NE. For a bi-matrix game we worked out a symmetric mixed NE that remains intact in both the classical and quantum versions of the game. For this mixed NE we, then, found conditions making it possible that evolutionary stability of a mixed symmetric NE changes with a switch-over of the game between its two forms, one classical and the other quantum.
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