Statistics of fluctuations for two types of crossover: from ballistic to diffusive regime and from orthogonal to unitary ensemble

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I. INTRODUCTION

When electromagnetic wave propagates in a random medium and undergoes multiple scattering the density of energy in a given point (intensity) \( I = \langle E^2 \rangle \) is a strongly fluctuating quantity. In a diffusive regime these fluctuations are traditionally described by Rayleigh statistics. The distribution function \( P_R(I) \) is a simple negative exponential:

\[
P_R(I) = \frac{1}{\langle I \rangle} \exp \left( -\frac{I}{\langle I \rangle} \right), \tag{1}
\]

which corresponds to the equation for the moments:

\[
\langle I^n \rangle = n! \langle I \rangle^n. \tag{2}
\]

In Eqs. (1) and (2) the averaging is with respect to all macroscopically equivalent but microscopically different scatterers configurations. By diffusive we mean the regime when it is possible to discard the "coherent" component of the field, that is

\[
\langle E \rangle = 0. \tag{3}
\]

This assumption is true for the case \( r/\ell \gg 1 \), where \( r \) is a distance from the source to the observation point, and \( \ell \) is a mean free path. The aim of the present publication is to get the distribution function for arbitrary relation between \( r \) and \( \ell \), when generally speaking the "coherent" component of the field should also be taken into account.

Let us remind how the issue of intensity statistics is treated in the framework of the traditional diagrammatic technique. For the sake of definiteness consider the case when the point source is situated at the origin; the intensity is measured in the point \( r \). In the diagrammatic representation \( \langle I \rangle \) is given by the diagrams with a pair of wave propagators \( G_{0r}^R \) and \( G_{0r}^A \), summed with respect to all possible interactions with the scatterers. The \( n \)-th moment \( \langle I^n \rangle \) is given by the set of diagrams with \( n \) propagators \( G_{0r}^R \) and \( n \) propagators \( G_{0r}^A \). For us is important the following property of diagrams: if the diagram consists from several disconnected parts then the contribution of that diagram is equal to the product of contributions of disconnected parts, so all the moments can be expressed at least in principle through the contributions of connected diagrams only. To get Rayleigh statistics one should ignore all connected diagrams save those consisting from a pair of propagators (advanced and retarded).

Previously we have formulated a perturbation theory which systematically takes all connected diagrams into account. It is connected diagrams with more than two propagators which give the deviation from the simple Rayleigh statistics for large values of intensity (for the tail of the distribution function). In this paper we would be interested not in the tail but in the "body" of the distribution function. (Because the deviation appears for the values of intensity which are inversely proportional to \( r \) this limitation looks quite natural in the regime considered). Hence we would discard connected diagrams with more than two propagators. On the other hand previously we ignored the diagrams which contain isolated (dressed) propagators. The ground for this is that such diagrams give contribution of the order of \( \exp(-r/\ell) \), and in the fully developed diffusive regime one can ignore them. Here we should generally speaking take them into account.

II. CROSSOVER BETWEEN DIFFUSIVE AND BALLISTIC REGIME

For \( \langle I^n \rangle \) we consider only diagrams which consists of isolated (dressed) propagators and pairs of propagators (advanced and retarded one). The sum of all diagrams can be written down in the following way:
\[ \langle I^n \rangle = \sum_{m=0}^{n} P(n, m) A^m B^{n-m}, \quad (4) \]

where \( A = |< E >|^2, \ B = |< I > - |< E >|^2, \) and \( P(n, m) \) is the coefficient of purely combinatorial origin:

\[ P(n, m) = \frac{(n!)^2}{(m!)^2 (n-m)!}. \quad (5) \]

The distribution function \( P(I) \) is connected with its moments by usual way:

\[ P(I) = \int_{-\infty}^{\infty} \exp(i\xi I) \sum_{n=0}^{\infty} \frac{(-i\xi I)^n}{n!} < I^n > \frac{d\xi}{2\pi}. \quad (6) \]

Substituting the Eq. (4) into the Eq. (6) and changing the order of summation, which is possible due to absolute convergence of the series, we get:

\[ P(I) = \int_{-\infty}^{\infty} \exp(i\xi I) \sum_{m=0}^{\infty} \frac{(-i\xi A)^m}{(m!)^2} \times \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} (-i\xi B)^{n-m} \frac{d\xi}{2\pi}. \quad (7) \]

Taking into account that for \( n \geq m \)

\[ \frac{n!}{(n-m)!} X^{n-m} = \frac{d^m}{dX^m} X^n, \quad (8) \]

we finally obtain the distribution function in the form:

\[ P(I) = \int_{-\infty}^{\infty} \exp \left( i\xi I - \frac{i\xi a}{1 + i\xi(1-a)} \right) \frac{d\xi}{2\pi}. \quad (9) \]

(The intensity is measured in the units of \( < I > \) and parameter \( a = A/(A + B) = |< E >|^2/|< I >, \) is introduced). It is obvious that for \( a = 0 \) (purely diffusive transport), the Eq. (8) coincides with Eq. (4), so this parameter gives the degree of deviation from Rayleigh statistics. In the opposite limiting case \( a = 1 \) Eq. (8) gives \( P(I) = 0 (I - 1). \) This limiting case means that there are no fluctuations, which can occur only when there is no scattering at all, in other words in the ballistic regime.

For intermediate value of \( a \) the distribution function can easily be calculated numerically. This behavior is represented on Fig. 1. We see that even for \( a \ll 1 \) (almost diffusive transport) taking into account of the “coherent” component of the field drastically changes the distribution function for small \( I; \) in particular for \( I = 0 \) we get: \( P \approx 1/2. \) Also interesting are the predicted oscillations of the distribution function.

III. CROSSOVER BETWEEN ORTHOGONAL AND UNITARY ENSEMBLE

Now we want to consider a second problem, which though physically different from the considered above is as we would see very similar mathematically. It’s the problem of statistics of fluctuations of wave functions of chaotic electrons in a quantum dot in an arbitrary magnetic field. This problem has been a subject of several studies. The distribution function \( P_e(I) \) of a single state local electron density \( I = |\psi(r)|^2 \) was shown to be for unitary ensemble (strong magnetic field) a simple negative exponent (like in Eq. (10)), and for orthogonal ensemble (no magnetic field)

\[ P_o(I) = \sqrt{\langle I \rangle \exp \left( -\frac{I}{2\langle I \rangle} \right)} . \quad (10) \]

In the paper the distribution function was calculated for the case of arbitrary magnetic field using the supersymmetry technique. Let us see what the simple method gives in application to this problem. Both limiting cases can be easily obtained using the method of moments, if we postulate for the density moment

\[ < I^n > = < \psi...\psi \psi^*...\psi^* > \quad (11) \]

the existence of Wick theorem. Then we obtain

\[ < I^n > = n! < \psi \psi^* >^n, \quad (12) \]

where the multiplier \( n! \) is simply the number of possible pairings between amplitudes, and hence Rayleigh statistics. For orthogonal ensemble the wavefunction is real, that is there is no difference between \( \psi \) and \( \psi^* \) any two propagators can be coupled and instead of Eq. (12) we get:

\[ < \psi...\psi > = (2n-1)!! < \psi \psi^* >, \quad (13) \]

which gives us Eq. (10). (This statistics, by the way, is also known in the theory of propagation of classical waves in a disordered medium; Eq. (10) gives distribution function for the case of acoustic waves). This gives us the idea how the general issue of electron density distribution function should be addressed. We should take into account both the averages \( < \psi \psi^* > \) and \( < \psi \psi > \) (not supposing of course them to be equal). Then taking into account all possible ways of coupling and using simple combinatorics we get:

\[ < I^n > = \sum_{m=0}^{[n/2]} P(n, m) < \psi \psi >^m < \psi \psi^* >^{n-m}, \quad (14) \]

where \( P(n, m) \) is the coefficient of purely combinatorial origin:

\[ P(n, m) = \frac{(n!)^2}{(m!)^2 (n-2m)! 2^{2m}}. \quad (15) \]

Substituting Eq. (4) into Eq. (6) and changing the order of summation we get:
\[ P_e(I) = \int_{-\infty}^{\infty} \exp(i\xi I) \sum_{m=0}^{\infty} \frac{(-i\xi |\langle \psi\psi \rangle|)^m}{2^{2m}(m!)^2} \]
\[ \times \sum_{n=2m}^{\infty} \frac{n!}{(n-2m)!} (-i\xi |\langle \psi^*\psi \rangle|)^{n-m} \frac{d\xi}{2\pi} \] (16)

Using Eq. (8) we obtain:

\[ P_e(I) = \int_{-\infty}^{\infty} \exp (i\xi I) \sqrt{1 + 2i\xi - \xi^2 Y/(1+Y)} \frac{d\xi}{2\pi} \] (17)

where the density is measured in the units of \( < I > \) and \( Y = \langle \psi^*\psi \rangle / |\langle \psi\psi \rangle| - 1 \) is the parameter which shows the character of the ensemble (for orthogonal ensemble \( Y = 0 \) and for unitary \( Y = \infty \)). It is obvious that for \( Y = \infty \) we get Eq. (1) and for \( Y = 0 \) we get Eq. (10).

For intermediate value of \( Y \) the distribution function can easily be calculated numerically. To compare our result with the results of Ref. 7, on Fig. 2 we plotted the distribution \( \varphi(\tau) = 2\tau P_e(\tau^2) \) as function of \( X = \sqrt{Y} \) (looking at \( I \to 0 \ Y \to 0 \) asymptotics of Eq. (17) we see that apart from numerical coefficient very close to 1 our parameter \( X \) is equal to that introduced Ref. 7). Comparing our Fig. 2 with Fig. 1 of Ref. 7 we see that they look very much alike, though analytical expression for distribution function we have got differs from that of Ref. 7.

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1 J. W. Goodman, *Statistical Optics* (J. Wiley, New York, NY, 1985).
2 B. Shapiro, Phys. Rev. Lett. 57, 2168 (1986).
3 E. Kogan, M. Kaveh, R. Baumgartner, and R. Berkovits, Phys. Rev. B 48, 9404 (1993); Physica A 200, 469 (1993).
4 F. Wegner, Z. Phys. 36, 207 (1980).
5 B. L. Altshuler and V. N. Prigodin, Zh. Eksp. Teor. Phys. 95, 348 (1989) [Sov. Phys. JETP 68, 198 (1989)].
6 K. B. Efetov and V. N. Prigodin, Phys. Rev. Lett, 70, 1315 (1993).
7 V. I. Fal’ko and K. B. Efetov, Phys. Rev. B 50, 11 267 (1994).

**FIG. 1.** Distribution function for: (1) \( a = .1 \) (solid line), (2) \( a = .5 \) (dashed line), (3) \( a = .9 \) (dot-dashed line).

**FIG. 2.** The distribution function \( \varphi(\tau) \) of the local amplitude \( \tau \) shown in the crossover regime for different values of the parameter \( X \).
