Certain Class of Bi-Bazilevič Functions with Bounded Boundary Rotation Involving Sălăgean Operator

MOHAMED KAMAL AOUF AND TAMER SEOUDY*

ABSTRACT. In the present paper, we consider certain classes of bi-univalent Bazilevič functions with bounded boundary rotation involving Sălăgean operator to obtain the estimates of their second and third coefficients. Further, certain special cases are also indicated. Some interesting remarks about the results presented here are also discussed.

Keywords: Analytic function, bi-univalent, Bazilevič functions, Sălăgean operator.

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1. INTRODUCTION

Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}$. Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $\mathbb{U}$. It is well known that every function $f \in S$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left( |w| < r_0(f) ; r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^2 - 5a_2a_3 + a_4) w^4 + ... .$$

A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples in the class $\Sigma$ (see [26]).

For functions $f \in A$, Sălăgean [27] (see also [4] and [28]) defined the linear operator $D^m : A \to A \ (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, 3, \ldots\})$ as follows:

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = D f(z) = z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n.$$
and (in general)

\[ D^m f(z) = D \left( D^{m-1} f(z) \right) = z + \sum_{n=2}^{\infty} n^m a_n z^n. \]

From (1.3), we can easily deduce that

\[ D^{m+1} f(z) = z (D^m f(z))^{'}. \]

Let \( P_\alpha^\lambda \) be the class of analytic functions \( p(z) \) in \( \mathbb{U} \) normalized by \( p(0) = 1 \) and satisfying

\[ \int_0^{2\pi} \left| \Re \left\{ \frac{e^{i\lambda} p(z)}{1 - \alpha} \right\} \right| d\theta \leq k\pi \cos \lambda, \]

where \( z = re^{i\theta}, 0 \leq r < 1, |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1 \) and \( k \geq 2 \). The class \( P_\alpha^\lambda \) was introduced and studied by Moulis [16] (see also Aouf [3] and Noor et al. [21]). We note that

(i) \( P_0^0(0) = P_k \), is the class of functions have their real parts bounded in the mean on \( \mathbb{U} \), introduced by Robertson [25] and studied Pinchuk [24];

(ii) \( P_\alpha^\lambda(0) = P_\lambda \), is the class of functions introduced by Robertson [25] and he derived a variational formula for functions in this class;

(iii) \( P_\alpha^\lambda(0) = P_k(\alpha) \), is the class of functions introduced by Padmanabhan and Parvatham [23] (see also Umarani and Aouf [31]);

(iv) \( P_2^\alpha(0) = \mathbb{P} \), is the class of functions with positive real part of order \( \alpha, 0 \leq \alpha < 1 \);

(v) \( P_2^0(0) = \mathbb{P} \), is the class of functions having positive real part for \( z \in \mathbb{U} \).

Using Salăgean operator \( D^m \) and the class \( \mathcal{P}_k \), we now introduce the following subclass of Bi-Bazilevič analytic functions of the class \( \Sigma \) as follows:

**Definition 1.1.** A function \( f \in \Sigma \) is said to be in the class \( \mathcal{B}_{\alpha}^\lambda(\gamma, \delta, b; k) \) if it satisfies the following subordination condition:

\[ 1 + \frac{1}{b} \left[ (1 - \gamma) \left( \frac{D^m f(z)}{z} \right)^\delta + \gamma \frac{D^{m+1} f(z)}{D^m f(z)} \left( \frac{D^m f(z)}{z} \right)^\delta - 1 \right] \in \mathcal{P}_k \]

and

\[ 1 + \frac{1}{b} \left[ (1 - \gamma) \left( \frac{D^m g(w)}{w} \right)^\delta + \gamma \frac{D^{m+1} g(w)}{D^m g(w)} \left( \frac{D^m g(w)}{w} \right)^\delta - 1 \right] \in \mathcal{P}_k, \]

where \( g = f^{-1}, b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \gamma, \delta \in \mathbb{C}, m \in \mathbb{N}_0, k \geq 2 \) and all powers are understood as principle values.

Taking additional choices of \( m, \gamma, \delta, k \) and \( b \), the class \( \mathcal{B}_{\alpha}^\lambda(\gamma, \delta, b; k) \) reduces to the following subclasses of \( \Sigma \):

(i) \( \mathcal{B}_\Sigma^0(\gamma, \delta, 1; k) = \mathcal{B}_\Sigma(\gamma, \delta; k) \)

\[ = \left\{ f \in \Sigma : (1 - \gamma) \left( \frac{f(z)}{z} \right)^\delta + \gamma \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\delta \in \mathcal{P}_k \text{ and } (1 - \gamma) \left( \frac{g(w)}{w} \right)^\delta + \gamma \frac{wg'(w)}{g(w)} \left( \frac{g(w)}{w} \right)^\delta \in \mathcal{P}_k \right\} ; \]
(ii) $B^0_\Sigma (\gamma, \delta, 1 - \eta; 2) = B_\Sigma (\gamma, \delta, \eta) (0 \leq \eta < 1)$ (see [15] for $f \in \mathcal{A}$)(see also [29])

$$f \in \Sigma : \Re \left\{ (1 - \gamma) \left( \frac{f(z)}{z} \right)^\delta + \gamma \frac{zf'(z)}{f(z)} \right\} > \eta$$

and $\Re \left\{ (1 - \gamma) \left( \frac{g(w)}{w} \right)^\delta + \gamma \frac{wg'(w)}{g(w)} \right\} > \eta$.

(iii) $B^0_\Sigma (\gamma, 1, 1; k) = B_\Sigma (\gamma; k)$

$$f \in \Sigma : (1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) \in \mathcal{P}_k$$

and $\Re \left\{ (1 - \gamma) \frac{g(w)}{w} + \gamma g'(w) \right\} > \eta$.

(iv) $B^0_\Sigma (\gamma, 1, 1 - \eta; 2) = B_\Sigma (\gamma, \eta) (0 \leq \eta < 1)$ (see [10] for $f \in \mathcal{A}$)

$$f \in \Sigma : \Re \left\{ (1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) \right\} > \eta$$

and $\Re \left\{ (1 - \gamma) \frac{g(w)}{w} + \gamma g'(w) \right\} > \eta$.

(v) $B^0_\Sigma (1, \delta, 1 - \eta; 2) = B_\Sigma (\delta, \eta) (0 \leq \eta < 1)$ (see [22] for $f \in \mathcal{A}$)

$$f \in \Sigma : \Re \left\{ \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\delta \right\} > \eta$$

and $\Re \left\{ \frac{wg'(w)}{g(w)} \left( \frac{g(w)}{w} \right)^\delta \right\} > \eta$.

(vi) $B^0_\Sigma (1, 0, b; k) = S_\Sigma (b; k)$ (see Nasr and Aouf [20] for $f \in \mathcal{A}$)

$$f \in \Sigma : 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \in \mathcal{P}_k$$

and $1 + \frac{1}{b} \left( \frac{wg'(w)}{g(w)} - 1 \right) \in \mathcal{P}_k$.

(vii) $B^0_\Sigma (1, 0, b; 2) = S_\Sigma (b)$ (see Nasr and Aouf [19] for $f \in \mathcal{A}$) (see also [5])

$$f \in \Sigma : \Re \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0$$

and $\Re \left\{ 1 + \frac{1}{b} \left( \frac{wg'(w)}{g(w)} - 1 \right) \right\} > 0$.

(viii) $B^1_\Sigma (1, 0, b; k) = C_\Sigma (b; k)$ (see Nasr and Aouf [20] for $f \in \mathcal{A}$)

$$f \in \Sigma : 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \in \mathcal{P}_k$$

and $1 + \frac{1}{b} \left( \frac{wg''(w)}{g'(w)} \right) \in \mathcal{P}_k$.

(ix) $B^1_\Sigma (1, 0, b; 2) = C_\Sigma (b)$ (see Nasr and Aouf [18] for $f \in \mathcal{A}$) (see also [5])

$$f \in \Sigma : \Re \left\{ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > 0$$

and $\Re \left\{ 1 + \frac{1}{b} \left( \frac{wg''(w)}{g'(w)} \right) \right\} > 0$.

(x) $B^0_\Sigma (1, 0, 1; k) = S_\Sigma (k)$ (see Pinchuk [24] for $f \in \mathcal{A}$)

$$f \in \Sigma : \frac{zf'(z)}{f(z)} \in \mathcal{P}_k$$

and $\frac{wg'(w)}{g(w)} \in \mathcal{P}_k$. 
(xi) $B^1_\Sigma(1,0,1;k) = C^1_\Sigma(k)$ (see Pinchuk [24] for $f \in \mathcal{A}$)

$$\left\{ f \in \Sigma : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_k \text{ and } 1 + \frac{wg''(w)}{g'(w)} \in \mathcal{P}_k \right\};$$

(xii) $B^0_\Sigma(1,0,1-\eta;2) = S_\Sigma(\eta)$ ($0 \leq \eta < 1$) (see [9] and [30])

$$\left\{ f \in \Sigma : \Re \left( \frac{zf'(z)}{f(z)} \right) > \eta \text{ and } \Re \left( \frac{wg'(w)}{g(w)} \right) > \eta \right\};$$

(xiii) $B^1_\Sigma(1,0,1-\eta;2) = C_\Sigma(\eta)$ ($0 \leq \eta < 1$) (see [9] and [30])

$$\left\{ f \in \Sigma : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \eta \text{ and } \Re \left( 1 + \frac{wg''(w)}{g'(w)} \right) > \eta \right\};$$

(xiv) $B^0_\Sigma(\gamma,\delta,(1-\alpha)e^{-i\lambda}\cos\lambda;k) = B_\Sigma(\gamma,\delta,\alpha,\lambda;k)$ \((|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1)\)

$$\left\{ f \in \Sigma : \frac{e^{i\lambda}(1-\gamma)(f(z))^{\delta} + \gamma zf'(z)}{(1-\alpha)\cos\lambda} \in \mathcal{P}_k \right\}$$

or

$$\left\{ f \in \Sigma : (1-\gamma) \left( \frac{f(z)}{z} \right)^{\delta} + \gamma \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^{\delta} \in \mathcal{P}_k^{\alpha} \right\};$$

(xv) $B^0_\Sigma(1,0,be^{-i\lambda}\cos\lambda;2) = S^1_\Sigma(b)$ \((|\lambda| < \frac{\pi}{2}, b \in \mathbb{C}^*)\) (see Al-Oboudi and Haidan [2] for $f \in \mathcal{A}$)

$$\left\{ f \in \Sigma : \Re \left\{ 1 + \frac{e^{i\lambda}}{b\cos\lambda} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \right\};$$

and

$$\left\{ 1 + \frac{e^{i\lambda}}{b\cos\lambda} \left( \frac{wg'(w)}{g(w)} - 1 \right) > 0 \right\};$$

(xvi) $B^1_\Sigma(1,0,be^{-i\lambda}\cos\lambda;2) = C^1_\Sigma(b)$ \((|\lambda| < \frac{\pi}{2}, b \in \mathbb{C}^*)\) (see Al-Oboudi and Haidan [2] for $f \in \mathcal{A}$)

$$\left\{ f \in \Sigma : \Re \left\{ 1 + \frac{e^{i\lambda}}{b\cos\lambda} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \right\};$$

and

$$\left\{ 1 + \frac{e^{i\lambda}}{b\cos\lambda} \left( \frac{wg''(w)}{g'(w)} \right) > 0 \right\};$$
(xvii) \( B^0_{\Sigma} (1, 0, (1 - \alpha) e^{-i\lambda} \cos \lambda; k) = \mathcal{B}_{\alpha} (k) \left( \left| \lambda \right| < \frac{\pi}{2}, 0 \leq \alpha < 1 \right) \)

\[
\begin{align*}
&= \left\{ f \in \Sigma : \frac{e^{i\lambda} z f'(z) - \alpha \cos \lambda - i \sin \lambda}{f(z) (1 - \alpha) \cos \lambda} \in \mathcal{P}_{k} \\
&\quad \text{and} \quad \frac{e^{i\lambda} \frac{w g'(w)}{g(w)} - \alpha \cos \lambda - i \sin \lambda}{(1 - \alpha) \cos \lambda} \in \mathcal{P}_{k} \right\}
\end{align*}
\]

or

\[
\begin{align*}
&= \left\{ f \in \Sigma : \frac{zf'(z)}{f(z)} \in \mathcal{P}_{k}^\lambda (\alpha) \quad \text{and} \quad \frac{wg'(w)}{g(w)} \in \mathcal{P}_{k}^\lambda (\alpha) \right\};
\end{align*}
\]

(xviii) \( B^1_{\Sigma} (1, 0, (1 - \alpha) e^{-i\lambda} \cos \lambda; k) = \mathcal{C}_{\alpha} (k) \left( \left| \lambda \right| < \frac{\pi}{2}, 0 \leq \alpha < 1 \right) \)

\[
\begin{align*}
&= \left\{ f \in \Sigma : \frac{e^{i\lambda} \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \alpha \cos \lambda - i \sin \lambda}{(1 - \alpha) \cos \lambda} \in \mathcal{P}_{k} \\
&\quad \text{and} \quad \frac{e^{i\lambda} \left( 1 + \frac{w g''(w)}{g'(w)} \right) - \alpha \cos \lambda - i \sin \lambda}{(1 - \alpha) \cos \lambda} \in \mathcal{P}_{k} \right\}
\end{align*}
\]

or

\[
\begin{align*}
&= \left\{ f \in \Sigma : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_{k}^\lambda (\alpha) \quad \text{and} \quad 1 + \frac{wg''(w)}{g'(w)} \in \mathcal{P}_{k}^\lambda (\alpha) \right\}.
\end{align*}
\]

In order to establish our main results, we need the following lemma:

**Lemma 1.1.** [3, Theorem 5 with \( p = 1 \)] If \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}_{k}^\lambda (\alpha) \) in \( \mathbb{U} \), then

\[
|c_n| \leq (1 - \alpha) k \cos \lambda \quad (n \in \mathbb{N}).
\]

The result is sharp. Equality is attained for the odd coefficients and even coefficients, respectively, for the functions

\[
p_1 (z) = 1 + (1 - \alpha) \cos \lambda e^{-i\lambda} \left[ \left( \frac{k+2}{4} \right) \left( \frac{1-z}{1+z} \right) - \left( \frac{k-2}{4} \right) \left( \frac{1+z}{1-z} \right) - 1 \right],
\]

\[
p_2 (z) = 1 + (1 - \alpha) \cos \lambda e^{-i\lambda} \left[ \left( \frac{k+2}{4} \right) \left( \frac{1-z^2}{1+z^2} \right) - \left( \frac{k-2}{4} \right) \left( \frac{1+z^2}{1-z^2} \right) - 1 \right].
\]

**Remark 1.1.** For \( \lambda = \alpha = 0 \) in Lemma 1.1, we obtain the result obtained by Goswami et al. [11] for the class \( \mathcal{P}_{k} \).
Lewin [13] defined the class of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [9] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced the concept of bi-starlike functions and the bi-convex functions, and obtained estimates for the initial coefficients. Recently, Srivastava et al. [26], Ali et al. [1], Frasin and Aouf [10], Goyal and Goswami [12] and many others (see [6], [7], [8], [14], [17] and [32]) have introduced and investigated subclasses of bi-univalent functions and obtained non-sharp bounds for the initial coefficients.

In the present paper, we estimates on the coefficients for second and third coefficients for the functions in the subclass $B^m_{\Sigma_k}(\gamma, \delta, b; k)$ and its special subclasses.

2. MAIN RESULTS

Unless otherwise mentioned, we assume throughout this paper that $g = f^{-1}, b \in \mathbb{C}^*, \gamma, \delta \in \mathbb{C}, k \geq 2, m \in \mathbb{N}_0$ and all powers are understood as principle values.

Theorem 2.1. Let $f(z)$ given by (1.1) belongs to the class $B^m_{\Sigma_k}(\gamma, \delta, b; k)$ with $\delta \neq 1 - \frac{3^m}{2^{2m-1}}, \delta \neq -\gamma$ and $\delta \neq -2\gamma$, then

$$|a_2| \leq \min \left\{ \sqrt{\frac{|b|}{(\delta - 1)2^{2m-1} + 3^m|\delta + 2\gamma|}}, \frac{|b|}{2^m|\delta + \gamma|} \right\}$$

and

$$|a_3| \leq \frac{|b| k}{3^m|\delta + 2\gamma|} \min \left\{ 1 + \frac{3^m}{(\delta - 1)2^{2m-1} + 3^m|\delta - 1||b|k}, 1 + \frac{|\delta + 2\gamma||1 - \delta||b|k}{2|\delta + \gamma|^2} + \frac{3^m|\delta + 2\gamma||b|k}{2^{2m-1}|\delta + \gamma|^2} \right\}.$$  

Proof. If $f(z) \in B^m_{\Sigma_k}(\gamma, \delta, b; k)$, according to the Definition 1.1, we have

$$1 + \frac{1}{b} \left( 1 - \gamma \right) \left( \frac{D^m f(z)}{z} \right)^\delta + \gamma \frac{D^{m+1} f(z)}{D^m f(z)} \left( \frac{D^m f(z)}{z} \right)^\delta - 1 = p(z)$$

and

$$1 + \frac{1}{b} \left( 1 - \gamma \right) \left( \frac{D^m g(w)}{w} \right)^\delta + \gamma \frac{D^{m+1} g(w)}{D^m g(w)} \left( \frac{D^m g(w)}{w} \right)^\delta - 1 = q(w),$$

where $p(z), q(w) \in \mathcal{P}_k$ and $g = f^{-1}$. Using the fact that the functions $p(z)$ and $q(w)$ have the following Taylor expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + ...$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + ....$$

Since

$$1 + \frac{1}{b} \left( 1 - \gamma \right) \left( \frac{D^m f(z)}{z} \right)^\delta + \gamma \frac{D^{m+1} f(z)}{D^m f(z)} \left( \frac{D^m f(z)}{z} \right)^\delta - 1 = 1 + \left( \frac{\delta + \gamma}{b} \right) 2^m a_2 z + \left( \frac{\delta + 2\gamma}{b} \right) \left[ 3^m a_3 + \frac{\delta - 1}{2} 2^{2m} a_2^2 \right] z^2 + ...$$
and according to (1.2), we have

\[
1 + \frac{1}{b} \left[ (1 - \gamma) \left( \frac{D^m g(w)}{w} \right)^\delta + \gamma \frac{D^{m+1} g(w)}{D^m g(w)} \left( \frac{D^m g(w)}{w} \right)^\delta - 1 \right]
\]

(2.16)

\[
= 1 - \left( \frac{\delta + \gamma}{b} \right) 2^m a_2 w + \left( \frac{\delta + 2 \gamma}{b} \right) \left[ (2 a_2^2 - a_3) 3^m + \frac{\delta - 1}{2} 2^{2m} a_2^2 \right] w^2 + ...
\]

from (2.13) and (2.14), combined with (2.15) and (2.16), it follows that

(2.17) \quad p_1 = \left( \frac{\delta + \gamma}{b} \right) 2^m a_2,

(2.18) \quad p_2 = \left( \frac{\delta + 2 \gamma}{b} \right) \left[ 3^m a_3 + \frac{\delta - 1}{2} 2^{2m} a_2^2 \right],

(2.19) \quad q_1 = - \left( \frac{\delta + \gamma}{b} \right) 2^m a_2,

(2.20) \quad q_2 = \left( \frac{\delta + 2 \gamma}{b} \right) \left[ (2 a_2^2 - a_3) 3^m + \frac{\delta - 1}{2} 2^{2m} a_2^2 \right].

Now, from (2.18) and (2.20), we deduce that

(2.21) \quad a_2^2 = \frac{b (p_2 + q_2)}{[(\delta - 1) 2^{2m} + (2) 3^m] (\delta + 2 \gamma)}

and

(2.22) \quad a_3 - a_2^2 = \frac{b (p_2 - q_2)}{2 (\delta + 2 \gamma) 3^m}.

Using (2.21) in (2.22), we obtain

(2.23) \quad a_3 = \frac{b}{\delta + 2 \gamma} \left[ \frac{p_2 - q_2}{(2) 3^m} + \frac{p_2 + q_2}{(\delta - 1) 2^{2m} + (2) 3^m} \right].

From (2.17) and (2.18), we get

(2.24) \quad a_3 = \frac{b}{3^m (\delta + 2 \gamma)} \left[ p_2 + \frac{(\delta + 2 \gamma) (1 - \delta) p_1^2 b}{2 (\delta + \gamma)^2} \right],

while from (2.19) and (2.20), we deduce that

(2.25) \quad a_3 = \frac{b}{3^m (\delta + 2 \gamma)} \left[ -q_2 + \frac{(\delta + 2 \gamma) (\delta - 1) bq_1^2}{2 (\delta + \gamma)^2} + \frac{2 (\delta + 2 \gamma) 3^m bq_1^2}{2^{2m} (\delta + \gamma)^2} \right].

Combining (2.17) and (2.21) for the computation of the upper-bound of \(|a_2|\), and (2.23), (2.24) and (2.25) for the computation of \(|a_3|\), by using Lemma 1.1 (with \(\alpha = \lambda = 0\)), we easily find the estimates of Theorem 2.1. This completes the proof of Theorem 2.1. \(\square\)

Taking \(m = 0\) and \(b = 1\) in Theorem 2.1, we obtain the following result for the functions belonging to the class \(B_\Sigma (\gamma, \delta; k)\).

**Corollary 2.1.** Let \(f(z)\) given by (1.1) belongs to the class \(B_\Sigma (\gamma, \delta; k)\) with \(\delta \neq -1, \delta \neq -\gamma\) and \(\delta \neq -2\gamma\), then

\[
|a_2| \leq \min \left\{ \sqrt{\frac{2k}{|\delta + 1| (\delta + 2 \gamma)}}, \frac{k}{|\delta + \gamma|} \right\}
\]
and
\[ |a_3| \leq \frac{k}{|\delta + 2\gamma|} \min \left\{ 1 + \frac{1}{|\delta + 1|}; 1 + \frac{|\delta + 2\gamma| |1 - \delta| k}{2 |\delta + \gamma|^2}; 1 + \frac{|\delta + 2\gamma| |\delta + 3| k}{2 |\delta + \gamma|^2} \right\}. \]

Taking \( m = 0, b = 1 - \eta (0 \leq \eta < 1) \) and \( k = 2 \) in Theorem 2.1, we obtain the following result for the functions belonging to the class \( B_\Sigma (\gamma, \delta, \eta) \).

**Corollary 2.2.** Let \( f(z) \) given by \( (1.1) \) belongs to the class \( B_\Sigma (\gamma, \delta, \eta) \) with \( 0 \leq \eta < 1, \delta \neq -1, \delta \neq -\gamma \) and \( \delta \neq -2\gamma \), then
\[ |a_2| \leq \min \left\{ \sqrt{\frac{4 (1 - \eta)}{|\delta + 1|}}; \frac{2 (1 - \eta)}{|\delta + \gamma|} \right\} \]

and
\[ |a_3| \leq \frac{2 (1 - \eta)}{|\delta + 2\gamma|} \min \left\{ 1 + \frac{1}{|\delta + 1|}; 1 + \frac{|\delta + 2\gamma| |1 - \delta| (1 - \eta)}{|\delta + \gamma|^2}; 1 + \frac{|\delta + 2\gamma| |\delta + 3| (1 - \eta)}{|\delta + \gamma|^2} \right\}. \]

Taking \( m = 0, \delta = 1, b = 1 - \eta (0 \leq \eta < 1) \) and \( k = 2 \) in Theorem 2.1, we obtain the following corollary which improves the result of Frasin and Aouf [10, Theorem 3.2].

**Corollary 2.3.** Let \( f(z) \) given by \( (1.1) \) belongs to the class \( B_\Sigma (\gamma, \eta) \) with \( 0 \leq \eta < 1, \gamma \neq -1 \) and \( \gamma \neq -\frac{1}{2} \), then
\[ |a_2| \leq \min \left\{ \sqrt{\frac{2 (1 - \eta)}{|2\gamma + 1|}}; \frac{2 (1 - \eta)}{|\gamma + 1|} \right\} \]

and
\[ |a_3| \leq \frac{2 (1 - \eta)}{|2\gamma + 1|} \min \left\{ 2, 1 + \frac{4 |2\gamma + 1| (1 - \eta)}{|\gamma + 1|^2} \right\}. \]

Taking \( m = 0, \gamma = 1, b = 1 - \eta (0 \leq \eta < 1) \) and \( k = 2 \) in Theorem 2.1, we obtain the following result for the functions belonging to the class \( B_\Sigma (\delta, \eta) \).

**Corollary 2.4.** Let \( f(z) \) given by \( (1.1) \) belongs to the class \( B_\Sigma (\delta, \eta) \) with \( \delta \neq -1 \) and \( \delta \neq -2 \), then
\[ |a_2| \leq \min \left\{ \sqrt{\frac{4 (1 - \eta)}{|\delta + 1|}}; \frac{2 (1 - \eta)}{|\delta + 2|} \right\} \]

and
\[ |a_3| \leq \frac{2 (1 - \eta)}{|\delta + 2|} \min \left\{ 1 + \frac{1}{|\delta + 1|}; 1 + \frac{|\delta + 2| |1 - \delta| (1 - \eta)}{|\delta + 1|^2}; 1 + \frac{|\delta + 2| |\delta + 3| (1 - \eta)}{|\delta + 1|^2} \right\}. \]

Taking \( \delta = m = 0, \gamma = 1 \) and \( k = 2 \) in Theorem 2.1, we obtain the following result for the functions belonging to the class \( S_\Sigma (b) \).

**Corollary 2.5.** Let \( f(z) \) given by \( (1.1) \) belongs to the class \( S_\Sigma (b) \), then
\[ |a_2| \leq \min \left\{ \sqrt{2|b|}; 2|b| \right\} \]

and
\[ |a_3| \leq |b| \min \{3, 1 + 2|b| \}. \]

Taking \( \delta = 0, m = 1, \gamma = 1 \) and \( k = 2 \) in Theorem 2.1, we obtain the following result for the functions belonging to the class \( C_\Sigma (b) \).
Corollary 2.6. Let \( f(z) \) given by (1.1) belongs to the class \( C_{\Sigma} (b) \), then
\[
|a_2| \leq \min \left\{ \sqrt{|b|}, |b| \right\}
\]
and
\[
|a_3| \leq \frac{|b|}{3} \min \{4, 1 + 2|b|\}.
\]

Taking \( m = 0 \) and \( b = (1 - \alpha) e^{-i\lambda} \cos \lambda \left( |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1 \right) \) in Theorem 2.1, we obtain the following result for the functions belonging to the class \( B_{\Sigma} (\gamma, \delta, \alpha, \lambda; k) \).

Corollary 2.7. Let \( f(z) \) given by (1.1) belongs to the class \( B_{\Sigma} (\gamma, \delta, \alpha, \lambda; k) \) with \( \delta \neq -1, \delta \neq -\gamma \) and \( \delta \neq -2\gamma \), then
\[
|a_2| \leq \min \left\{ \sqrt{\frac{2k(1 - \alpha) \cos \lambda}{|\delta + 1||\delta + 2\gamma|}}, \frac{k(1 - \alpha) \cos \lambda}{|\delta + \gamma|} \right\}
\]
and
\[
|a_3| \leq \frac{k(1 - \alpha) \cos \lambda}{|\delta + 2\gamma|} \min \left\{ 1 + \frac{2}{|\delta + 1|}, 1 + \frac{|\delta + 2\gamma|}{|\delta + \gamma|}, 1 + \frac{|\delta + 2\gamma|}{|\delta + 2\gamma|} \right\}.
\]

Taking \( m = \delta = 0, \gamma = 1, k = 2 \) and \( b \to be^{-i\lambda} \cos \lambda \left( |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1 \right) \) in Theorem 2.1, we obtain the following result for the functions belonging to the class \( S_{\Sigma}^\lambda (b) \).

Corollary 2.8. Let \( f(z) \) given by (1.1) belongs to the class \( S_{\Sigma}^\lambda (b) \), then
\[
|a_2| \leq \min \left\{ \sqrt{2|b| \cos \lambda}, 2|b| \cos \lambda \right\}
\]
and
\[
|a_3| \leq |b| \cos \lambda \min \{3, 1 + 2|b| \cos \lambda\}.
\]

Taking \( m = \gamma = 1, \delta = 0, k = 2 \) and \( b \to be^{-i\lambda} \cos \lambda \left( |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1 \right) \) in Theorem 2.1, we obtain the following result for the functions belonging to the class \( C_{\Sigma}^\lambda (b) \).

Corollary 2.9. Let \( f(z) \) given by (1.1) belongs to the class \( C_{\Sigma}^\lambda (b) \), then
\[
|a_2| \leq \min \left\{ \sqrt{|b| \cos \lambda}, |b| \cos \lambda \right\}
\]
and
\[
|a_3| \leq \frac{|b| \cos \lambda}{3} \min \{4, 1 + 2|b| \cos \lambda\}.
\]

Taking \( \delta = m = 0, \gamma = 1 \) and \( b = (1 - \alpha) e^{-i\lambda} \cos \lambda \left( |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1 \right) \) in Theorem 2.1, we obtain the following result for the functions belonging to the class \( S_{\alpha}^\lambda (k) \).

Corollary 2.10. Let \( f(z) \) given by (1.1) belongs to the class \( S_{\alpha}^\lambda (k) \left( |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1 \right) \), then
\[
|a_2| \leq \min \left\{ \sqrt{k(1 - \alpha) \cos \lambda}, k(1 - \alpha) \cos \lambda \right\}
\]
and
\[
|a_3| \leq \frac{k(1 - \alpha) \cos \lambda}{2} \min \{3, 1 + k(1 - \alpha) \cos \lambda\}.
\]

Taking \( \delta = 0, \gamma = m = 1 \) and \( b = (1 - \alpha) e^{-i\lambda} \cos \lambda \left( |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1 \right) \) in Theorem 2.1, we obtain the following result for the functions belonging to the class \( C_{\alpha}^\lambda (k) \).
Corollary 2.11. Let \( f(z) \) given by (1.1) belongs to the class \( C_{\alpha}^{k}(\mu) \) \(((|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1)\), then

\[
|a_2| \leq \min \left\{ \frac{k(1-\alpha)\cos\lambda}{2}, \frac{k(1-\alpha)\cos\lambda}{2} \right\}
\]

and

\[
|a_3| \leq \frac{k(1-\alpha)\cos\lambda}{6} \min \{4, 1 + k(1-\alpha)\cos\lambda\}.
\]

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MOHAMED KAMAL AOUF  
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,  
MANSOURA UNIVERSITY, MANSOURA 35516, EGYPT  
ORCID: 0000-0001-9398-4042  
E-mail address: mkaouf127@yahoo.com

TAMER SEoudy  
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,  
FAYOUM UNIVERSITY, FAYOUM 63514, EGYPT  
DEPARTMENT OF MATHEMATICS, JAMOUM UNIVERSITY COLLEGE,  
UMM AL-QURA UNIVERSITY, MAKKAH, SAUDI ARABIA  
ORCID: 0000-0001-6427-6960  
E-mail address: tms00@fayoum.edu.eg, tmsaman@uqu.edu.sa