Quantum theory of massless \((p, 0)\)-forms

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ABSTRACT: We describe the quantum theory of massless \((p, 0)\)-forms that satisfy a suitable holomorphic generalization of the free Maxwell equations on Kähler spaces. These equations arise by first-quantizing a spinning particle with a U(1)-extended local supersymmetry on the worldline. Dirac quantization of the spinning particle produces a physical Hilbert space made up of \((p, 0)\)-forms that satisfy holomorphic Maxwell equations coupled to the background Kähler geometry, containing in particular a charge that measures the amount of coupling to the U(1) part of the U(d) holonomy group of the \(d\)-dimensional Kähler space. The relevant differential operators appearing in these equations are a twisted exterior holomorphic derivative \(\partial_q\) and its hermitian conjugate \(\partial_q^\dagger\) (twisted Dolbeault operators with charge \(q\)). The particle model is used to obtain a worldline representation of the one-loop effective action of the \((p, 0)\)-forms. This representation allows to compute the first few heat kernel coefficients contained in the local expansion of the effective action and to derive duality relations between \((p, 0)\) and \((d - p - 2, 0)\)-forms that include a topological mismatch appearing at one-loop.

KEYWORDS: Sigma Models, Duality in Gauge Field Theories
1 Introduction

In this paper we wish to describe the quantization of \((p,0)\)-form gauge fields \(A\), defined on Kähler spaces, which satisfy a holomorphic generalizations of the free Maxwell equations

\[
\partial_q^* F = 0, \quad F = \partial_q A
\]  

(1.1)

where the twisted exterior holomorphic derivative \(\partial_q = \partial + q\Gamma\) contains a coupling to the \(U(1)\) part of the \(U(d)\) holonomy group of the \(d\)-dimensional Kähler space \((\Gamma \equiv \Gamma_\mu dx^\mu = \Gamma_{\mu\nu} dx^\mu \text{ with } x^\mu \text{ complex coordinates})\) and is a nilpotent operator \((\partial_q^2 = 0)\). It is the natural generalization on Kähler manifolds of the standard quantum theory of differential \(p\)-forms \(A\) that satisfy the Maxwell equation \(d^* dA = 0\) and enjoy a gauge invariance of the form \(\delta A = d\lambda\) where \(\lambda\) is a \((p-1)\)-form.

We are going to use a worldline approach in which the physical degrees of freedom of the \((p,0)\) gauge field are carried by a spinning particle with a \(U(1)\)-extended local supersymmetry on the worldline. This approach parallels the one used in [1, 2] for standard differential \(p\)-forms, which allowed to derive quite elegantly exact duality relations, compute heat kernel coefficients, and calculate the one-loop contribution to the graviton self-energy (the two-point function of the \(p\)-form stress tensors). In that case, some of those results had already been obtained previously using standard QFT tools, which include the correct way of covariantly gauge fixing the \(p\)-form gauge symmetries [3, 4], and the derivation of topological mismatches between the unregulated effective actions of dual forms [5, 6]. In
the present case we proceed directly by employing a worldline representation, and use it to study the one-loop effective action as function of the background metric, compute the first few heat kernel coefficients that characterize it, and derive exact duality relations between the effective actions of \((p, 0)\) and \((d - p - 2, 0)\)-forms.

The spinning particle that we use to treat the \((p, 0)\)-form gauge fields is a \(U(1)\) spinning particle, by which we mean a particle model that contains a \(U(1)\)-extended local supersymmetry on the worldline. The corresponding supersymmetry charges \(Q\) and \(\bar{Q}\) are realized quantum mechanically by twisted Dolbeault operators \(\partial_q\) and \(\partial_q^\dagger\) acting on the Hilbert space of the \((p, 0)\)-forms with any allowed \(p\). This model was derived sometimes ago in [7] to describe the so-called topological B model in a simple setting, and then generalized in [8] to a class of \(U(N)\) spinning particles that have been used in [9] to derive higher spin equations on complex manifolds. For \(N = 1\) those equations reduce precisely to the ones that are analyzed in the present paper. At the ungauged level, i.e. when supersymmetry is kept only as a rigid symmetry, one obtains a related sigma model that has been used recently in [10, 11] to study the twisted Dolbeault complex and related index theorems.

We present our material in the following way. We start with section 2 describing the canonical quantization of the \(U(1)\) spinning particle in flat complex space. This allows to introduce in a simple context the holomorphic equations briefly presented above. In section 3 we consider a generic Kähler space, and discuss canonical quantization of the spinning particle, paying attention to the ordering ambiguities that allow the introduction of a free coupling to the \(U(1)\) part of the holonomy group of the background Kähler space. We describe how the supercharges of the model realize the twisted Dolbeault operators \(\partial_q\) and \(\partial_q^\dagger\), with \(q\) indicating the free coupling constant just mentioned. Then we consider the ungauged model (rigid susy), and use operatorial methods to compute perturbatively the transition amplitude as well as path integral methods to obtain the Dirac index \((q = \frac{1}{2})\) and its twisted versions \((q \neq \frac{1}{2})\). In section 4 we consider the gauged model, i.e. the complete \(U(1)\) spinning particle, to give a worldline representation of the one-loop effective action of the \((p, 0)\)-form gauge fields, and use it to compute the first few heat kernel coefficients characterizing the effective action. This provides the quantization with worldline methods of the gauge invariant field equations \(\partial_q^\dagger \partial_q A = 0\). As a side result, we present the heat kernel coefficients for the ungauged model as well, that corresponds to the worldline quantization of the field equations \((\partial_q^\dagger \partial_q + \partial_q \partial_q^\dagger)B = 0\), which do not carry any gauge invariance. In section 5 we discuss various dualities and derive topological mismatches appearing at one-loop, checking them versus the explicit results found in the preceding section. Presenting the (unregulated) effective action of a \((p, 0)\)-form gauge field with a \(U(1)\) charge \(q\) in the form of an integral over proper time of a corresponding density, \(Z_p(q) = \int \frac{d\beta}{\pi} Z_p(q, \beta)\), we find a duality between a \((p, 0)\)-form and a \((d - p - 2, 0)\)-form described by the following relation

\[
Z_p(q, \beta) = Z_{d-p-2}(\frac{1}{2} - q, \beta) + (-1)^p Z_{d-1}(\frac{1}{2} - q, \beta) + (-1)^p (p + 1) \text{ind}(\mathcal{D}_{q-1/4}) \tag{1.2}
\]

where \(Z_{d-1}(\frac{1}{2} - q, \beta)\) is a purely topological contribution (no propagating degrees of freedom are associated to a \((d - 1, 0)\)-form for \(d > 1\)) that can be related to the analytic torsion,
and \( \text{ind}(\mathcal{D}_{q-1/4}) \) is the index of the (twisted for \( q \neq \frac{1}{4} \)) Dirac operator. Finally, we present our conclusions and perspectives in section 6.

2 Free particle and canonical quantization

In this section we review the free U(1) spinning particle and its Dirac quantization to describe in the simple context of \( \mathbb{C}^d \), the flat complex space, how the Maxwell equations for a \((p,0)\)-form emerge naturally from first-quantization. The particle system of interest is constructed by first considering a supersymmetric particle that produces a Hilbert space \( \mathcal{H} \) formed by the sum of all \((p,0)\)-forms with any allowed \( p \),

\[
\mathcal{H} = \bigoplus_{p=0}^{d} \Lambda^{p,0}(\mathbb{C}^d)
\]

where \( \Lambda^{p,q} \) indicates the space of \((p,q)\)-forms. This mechanical model contains conserved supercharges \( Q \) and \( \bar{Q} \) that are realized on the Hilbert space by the Dolbeault operator \( \bar{\partial} \) and its hermitian conjugate \( \bar{\partial}^\dagger \). It is seen that the supercharges belong to a multiplet of conserved charges containing the hamiltonian \( H \) and a U(1) charge \( J \) as well. Altogether these charges satisfy a U(1)-extended supersymmetry algebra. Gauging all of them produces the action of the U(1) spinning particle that leads to the quantum theory of a \((p,0)\)-form obeying the Maxwell equations in (1.1). The details are as follows.

We consider a particle moving in flat complex space \( \mathbb{C}^d \), described by the complex coordinates \((x^\mu, \bar{x}\bar{\mu})\) with \( \mu = 1,..,d \). The particle carries additional degrees of freedom associated to the Grassmann variable \( \psi^\mu \) and its complex conjugate \( \bar{\psi}^\bar{\mu} \). Indices are lowered and raised with the flat metric \( \delta_{\mu\bar{\nu}} \) and its inverse.

\[\text{We often use a redundant notation by indicating complex conjugate variables by using a bar on both the variable itself and its indices, such as } \bar{x}^\mu, \bar{p}_\mu \text{ or } \bar{\partial}_\mu. \text{ This allows for a quick interpretation of various formulas, containing for example } \bar{p}^\mu = g^{\bar{\nu}\bar{\mu}} \bar{p}_{\bar{\nu}} \text{ or similar tensors with upper indices. No confusion should arise whenever we use such a redundant notation.}\]
By considering \((x^\mu, \bar{x}^\rho, \psi^\mu)\) as coordinates and \((p_\mu, \bar{p}_\rho, \bar{\psi}_\mu)\) as momenta, one may realize the latter as differential operators with respect to the former,

\[ p_\mu = -i \partial_\mu, \quad \bar{p}_\rho = -i \bar{\partial}_\rho, \quad \bar{\psi}_\mu = \frac{\partial}{\partial \bar{\psi}_\mu} \]  

(2.5)

(we use left derivative for Grassmann variables), so that a generic wave function \(\phi(x, \bar{x}, \psi)\) has a finite expansion with respect to the Grassmann variables \(\psi^\mu\), and contains all possible differential \((p, 0)\)-forms up to \(p = d\)

\[ \phi(x, \bar{x}, \psi) = F(x, \bar{x}) + F_\mu(x, \bar{x})\psi^\mu + \frac{1}{2} F_{\mu\nu}(x, \bar{x})\psi^\mu\psi^\nu + \ldots + \frac{1}{d!} F_{\mu_1\ldots\mu_d}(x, \bar{x})\psi^{\mu_1}\ldots\psi^{\mu_d} \]  

(2.6)

There are a total of \(2d\) independent components, which equals the number of the independent components of a Dirac fermion. This is not a coincidence, as it is known that on Kähler manifolds the space of all \((p, 0)\)-forms is equivalent to the Hilbert space of a Dirac fermion, see appendix B. The Hilbert space metric is the one that emerges naturally by considering coherent states for worldline fermions, and takes the following schematic form

\[ \langle \chi | \phi \rangle = \int dx d\bar{x} d\psi d\bar{\psi} \ e^{\bar{\psi}\psi} \chi(x, \bar{x}, \psi) \phi(x, \bar{x}, \psi) \]  

(2.7)

so that \(\bar{x}^\rho\) is the hermitian conjugate of \(x^\mu\), \(\bar{p}_\rho\) is the hermitian conjugate of \(p_\mu\), and \(\bar{\psi}_\mu\) is the hermitian conjugate of \(\psi^\mu\) (note that in flat space \(\bar{\psi}_\mu = \bar{\psi}_\mu\)).

On the Hilbert space thus constructed the quantized conserved charges are represented by differential operators. In particular, the operator \(iQ = \psi^\mu \partial_\mu\) naturally acts as the Dolbeault operator \(\partial = dx^\mu \wedge \partial_\mu\) on \((p, 0)\)-forms. Similarly \(i\bar{Q} = \bar{\partial}^{\mu} \frac{\partial}{\partial \bar{x}^{\mu}}\) corresponds, up to a sign, to its adjoint \(\partial^\dagger\) acting on \((p, 0)\)-forms. The Hamiltonian is given by the laplacian \(H = -\bar{\partial}^{\mu} \partial_\mu\). Finally, the U(1) charge operator \(J = \psi^\mu \frac{\partial}{\partial \bar{x}^{\mu}}\) counts the rank \(p\) of a \((p, 0)\)-form, up to a normal ordering ambiguity that we shall discuss in a moment. The \(U(1)\)-extended supersymmetry algebra satisfied by these operators is easily computed and reads

\[ \{Q, \bar{Q}\} = H, \quad [J, Q] = Q, \quad [J, \bar{Q}] = -\bar{Q} \]  

(2.8)

while other (anti-)commutators vanish.

The U(1) spinning particle we shall consider is obtained by gauging all of the symmetries generated by the charges in (2.3). The emerging model has a \(U(1)\)-extended local supersymmetry on the worldline, and it is characterized by the phase space action

\[ S = \int dt \left[ p_\mu \dot{x}^\mu + \bar{p}_\rho \dot{\bar{x}}^\rho + i\bar{\psi}_\mu \dot{\psi}^\mu - eH - i\bar{\chi}Q - i\bar{\chi}\bar{Q} + a(J - s) \right] \]  

(2.9)

where \(G \equiv (e, \chi, \bar{\chi}, a)\) are the worldline gauge fields that make local the symmetries generated by the constraints \(T \equiv (H, Q, \bar{Q}, J - s)\). The coupling \(s\) in (2.9) is a Chern-Simons coupling (note that its redefinition can take into account different ordering prescriptions that may be chosen when constructing the operator \(J\) in canonical quantization). It is crucial for obtaining quantum mechanically a non-empty model, and for this purpose it must be quantized to integer values. In a Dirac quantization scheme, one can gauge-fix the
worldline gauge fields to predetermined values, and require the constraints to annihilate physical states: \( T|\phi_{\text{phys}}\rangle = 0 \). The constraint \( J = s = 0 \) selects \((s,0)\)-forms
\[
\phi_{\text{phys}}(x, \bar{x}, \psi) = \frac{1}{s!} F_{\mu_1..\mu_s}(x, \bar{x}) \psi^{\mu_1..\mu_s}
\] (2.10)
so that the model may be non-empty if the coupling \( s \) is an integer with values \( 0 \leq s \leq d \).

For convenience we set \( s \equiv p + 1 \), so that the \( J \) constraint selects the \((p + 1, 0)\)-form
\[
\phi_{\text{phys}}(x, \bar{x}, \psi) = F^{\mu_1..\mu_{p+1}}
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Finally, the \( \bar{Q} \) constraint gives the remaining Maxwell equation
\[
\partial^\dagger F^{(p+1,0)} = 0
\] (2.12)
that reads as \( \partial^\dagger \partial A_{\mu_0} = 0 \) in terms of the gauge potential.

In components, the equations of motion of the field strength take the form
\[
\partial_{[\mu} F_{\mu_1..\mu_{p+1]} = 0 , \quad \bar{\partial}^\mu_1 F_{\mu_1..\mu_{p+1}} = 0
\] (2.13)
and are expressed in terms of the gauge potential by
\[
F_{\mu_1..\mu_{p+1}} = \partial_{\mu_1} A_{\mu_2..\mu_{p+1}} \pm \text{cyclic permutations}
\] (2.14)
with square brackets indicating weighted antisymmetrization. These equations are invariant under the gauge transformations \( \delta A_{\mu_0} = \partial \lambda_{(p-1,0)} \), i.e.
\[
\delta A_{\mu_1..\mu_p} = \partial_{\mu_1} \lambda_{\mu_2..\mu_p} \pm \text{cyclic permutations} .
\] (2.15)
In particular, for \( p = 1 \) one obtains the simple holomorphic Maxwell equations
\[
\bar{\partial}^\mu F_{\mu\nu} = 0 , \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}
\] (2.16)
with gauge symmetry \( \delta A_{\mu} = \partial_{\mu} \lambda \).

Of course, different models can be obtained by gauging different subgroups of the U(1) extended supermultiplet of charges. In particular, if one decides to gauge only the hamiltonian \( H \) and the real linear combination of the supercharges \( Q + \bar{Q} \), one obtains a first quantized description of a massless Dirac field. In fact, on Kähler manifolds the Hilbert space of a fermion corresponds to the collection of all \((p,0)\)-forms, and the Dirac operator corresponds to the real supercharge \( Q + \bar{Q} \sim \partial + \partial^\dagger \) (although on curved Kähler manifolds this happens only when the Dolbeault operator acquires a specific coupling to the U(1) part of the holonomy group, as discussed in appendix B). Thus, a massless Dirac field in first quantization is obtained by quantizing the worldline action
\[
S = \int dt \left[ p_{\mu} \dot{x}^\mu + \bar{p}_{\dot{\mu}} \dot{\tilde{x}}^{\dot{\mu}} + i \bar{\psi}_{\mu} \dot{\psi}^{\mu} - e H - i \chi (Q + \bar{Q}) \right]
\] (2.17)
where \( \chi \) is a real worldline gravitino.
3 Coupling to gravity, transition amplitude, and the Dirac index

We are now going to consider the coupling to an arbitrary background Kähler metric. It is useful to start with the ungauged version of the particle, which provides us with a nonlinear sigma model that contains already all operators of interest. As a preparation for subsequent applications, we evaluate its quantum mechanical transition amplitude and compute the Dirac index by considering its partition function with periodic boundary conditions. The notations employed are listed in appendix A.

A simple way to introduce couplings to the background Kähler metric, while maintaining the U(1)-extended supersymmetry, is to consider the covariantization of the symmetry charges $J, Q, \bar{Q}$, and then imposing the susy algebra to obtain the correct hamiltonian $H$. We consider the Grassmann variables $\psi^\mu$ and $\bar{\psi}^\mu$ as tensors transforming under holomorphic change of coordinates according to the position of their indices. Then the classical charge $J_{cl} = \psi^\mu \bar{\psi}_\mu$ is already covariant (a scalar). As for the susy charges, it is convenient to substitute the momenta $(p^\mu, \bar{p}^\mu)$ there contained by “covariant” momenta $(\pi^\mu, \bar{\pi}^\mu)$ defined by

$$\pi^\mu = p^\mu + i \Gamma^\lambda_{\mu\nu} \psi^\nu \bar{\psi}_\lambda,$$

$$\bar{\pi}^\mu = \bar{p}^\mu$$

that indeed are characterized by a Poisson bracket proportional to the curvature tensor

$$\{\pi^\mu, \bar{\pi}^\nu\}_{PB} = i R^\lambda_{\mu\nu\sigma} \psi^\sigma \bar{\psi}_\lambda.$$

Thus one obtains

$$Q_{cl} = \psi^\mu \pi^\mu = \psi^\mu (p^\mu + i \Gamma^\lambda_{\mu\nu} \psi^\nu \bar{\psi}_\lambda) = \psi^\mu p^\mu,$$

$$\bar{Q}_{cl} = \bar{\psi}^\mu \bar{\pi}^\mu = \bar{\psi}^\mu \bar{p}^\mu .$$

Thanks to the anticommuting character of the Grassmann variables, the term with the Christoffel connection vanishes in $Q_{cl}$, and the curved Kähler metric appears only in $\bar{Q}_{cl}$. Now one can compute their Poisson bracket, and check that the U(1)-extended supersymmetry algebra is realized with the classical hamiltonian

$$H_{cl} = g^{\mu\nu} \bar{p}_\nu (p^\mu + i \Gamma^\lambda_{\mu\sigma} \psi^\sigma \bar{\psi}_\lambda) .$$

With this $H_{cl}$ the phase space action for the searched for covariant model reads

$$S_{ph} = \int dt \left[ p^\mu \dot{x}^\mu + \bar{p}^\mu \dot{x}_\mu + i \bar{\psi}^\mu \dot{\psi}_\mu - H_{cl} \right] .$$

Eliminating the momenta $(p, \bar{p})$ one obtains the corresponding nonlinear sigma model in configuration space

$$S_{con} = \int dt \left[ g_{\mu\nu} \dot{x}^\mu \dot{x}_\nu + i \bar{\psi}_\mu D_t \psi^\mu \right]$$

where the covariant time derivative is given by $D_t \psi^\mu = \dot{\psi}^\mu + \dot{\bar{\psi}}^\mu \Gamma^\lambda_{\nu\lambda} \psi^\lambda$. This action is real up to boundary terms. Of course, one could have proceeded differently, covariantizing the configuration space action first and casting it in hamiltonian form afterwards.
Now, we may study canonical quantization. As outlined in the flat space case, canonical quantization produces an Hilbert space formed by the space of all \((p,0)\)-forms living on the Kähler manifold \(M\), that is \(H = \bigoplus_{p=0}^{d} \Lambda^{p,0}(M)\). One may again expect the susy charges \(Q\) and \(\bar{Q}\) to be represented by the Dolbeault operators \(\partial\) and \(\partial^{\dagger}\), and the real charge \(Q + \bar{Q}\) by the Dirac operator \(\gamma^\mu D_\mu + \gamma^\bar{\mu} D_{\bar{\mu}}\). This is correct on manifolds of \(SU(d)\) holonomy, where the Dirac operator indeed satisfies \(\gamma^\mu D_\mu + \gamma^\bar{\mu} D_{\bar{\mu}} \sim \partial + \partial^{\dagger}\). However, on generic Kähler manifolds of \(U(d)\) holonomy, one finds a nontrivial coupling to the \(U(1)\) part of the \(U(d) = U(1) \times SU(d)\) connection. This is required by the couplings of the Dirac operator, see appendix B. Therefore, let us analyze in more details the operatorial realization of the susy charges in terms of differential operators to appreciate how the ordering ambiguities leave enough room for the emergence of an additional free coupling to the \(U(1)\) part of the connection. This coupling is fixed if one wants to reproduce the Dirac operator, otherwise it can be considered arbitrary if one wishes to consider more general (covariant) models.

The commutation relations between the basic dynamical variables are as in (2.4), however the construction of composite operators may suffer from ordering ambiguities. The latter can be resolved partially by (i) requiring covariance under holomorphic change of coordinates and (ii) imposing the correct hermiticity properties that arise from the analogous properties under complex conjugation of the classical model. As we shall see this leaves the possibility of having a free \(U(1)\) charge in the quantum model. Generically on Kähler manifolds there is no need to introduce flat indices, and we will proceed that way as much as we can. The \(U(1)\) R-charge \(J\) is quadratic, and suffers only of a quite mild ordering ambiguity upon quantization. Having in mind path integral calculations, where ordering ambiguities take the form of different regularizations of the path integral, we choose an ordering that is naturally related to the way we regulate and compute the path integral. This corresponds to the antisymmetrization of the quadratic fermionic term

\[
J_{cl} = \psi^\mu \bar{\psi}_\mu \quad \rightarrow \quad J = \frac{1}{2}(\psi^\mu \bar{\psi}_\mu - \bar{\psi}_\mu \psi^\mu) = \psi^\mu \bar{\psi}_\mu - d^2. \tag{3.7}
\]

As already mentioned, different orderings can be taken into account by a redefinition of the Chern-Simons coupling of the \(U(1)\) spinning particle. In particular, choosing the value \(s \equiv p + 1 - \frac{d}{2}\) in the covariant version of (2.9) (so that \(J - s = \psi^\mu \bar{\psi}_\mu - (p + 1)\) as an operator) allows to project onto the sector of the Hilbert space containing \((p + 1,0)\)-forms only. The covariance of this operator is manifest.

A bit more subtle is the construction of the covariant supercharges. It is useful to start again from covariant momenta, as past experience with the standard spinning particle on riemannian manifolds indicates. In this case (as opposite to the riemannian case) covariance is not enough to fix all ordering ambiguities, and one finds a nontrivial coupling to the \(U(1)\) part of the holonomy

\[
\pi_\mu = p_\mu + i \Gamma^\lambda_{\mu \nu} \psi^\nu \bar{\psi}_\lambda \quad \rightarrow \quad \pi_\mu = p_\mu + i \Gamma^\lambda_{\mu \nu} \psi^\nu \bar{\psi}_\lambda - iq \Gamma_{\mu} \bar{\pi}_{\bar{\mu}} \quad \bar{\pi}_{\bar{\mu}} = \bar{p}_{\bar{\mu}} + iq \bar{\Gamma}_{\bar{\mu}} \tag{3.8}
\]

where on the left hand side we have listed the classical expressions, and on the right hand side the quantum expressions. A different ordering of the term with the fermionic operators
can be compensated by a redefinition of the charge \( q \). With the chosen ordering convention the charge \( q \) measures precisely the extra coupling to the U(1) part of the connection. The quantum covariant momenta are hermitian conjugate to each other when using the covariant version of the inner product in (2.7), namely

\[
\langle \chi | \phi \rangle = \int dx d\bar{x} \, g \, d\psi d\bar{\psi} \, e^{\bar{\psi} \chi} \, \chi(x, \bar{x}, \psi) \, \phi(x, \bar{x}, \psi)
\]

(3.9)

where \( g = \text{det} g_{\mu\nu} \). Note that with this inner product the hermiticity property of the momentum reads: \( p_\mu^q = \bar{p}_\mu + ig_{\lambda\mu} g^{\nu\rho} \Gamma_{\lambda\mu}^{\rho} \psi^\lambda \bar{\psi}_{\nu} \).

At this point one is ready to recognize the quantum version of the supersymmetric charges

\[
Q_{cl} = \psi^\mu \pi_\mu \quad \rightarrow \quad Q = \psi^\mu g^{1/2} \pi_\mu g^{-1/2} = \psi^\mu g^{1/2} \left( p_\mu - iq \Gamma_\mu \right) g^{-1/2}
\]

\[
\bar{Q}_{cl} = \bar{\psi}_\mu g^{\mu\nu} \bar{\pi}_\nu \quad \rightarrow \quad \bar{Q} = \bar{\psi}_\mu g^{\mu\nu} g^{1/2} \bar{\pi}_\nu g^{-1/2} = \bar{\psi}_\mu g^{\mu\nu} g^{1/2} \left( \bar{p}_\nu + iq \bar{\Gamma}_\nu \right) g^{-1/2} .
\]

(3.10)

The powers of \( g \) are required to obtain the correct hermiticity properties. Again, the Christoffel connection drops out from the supercharge \( Q \), as in the classical case. As the \( \psi \)'s can be represented by the coordinate basis of the (1, 0)-forms, \( \psi^\mu = dx^\mu \), while their momenta as formal derivatives thereof, \( \psi_\mu = \frac{\partial}{\partial (dx^\mu)} \), we recognize that the supercharge \( Q \) is represented by the Dolbeault operator twisted by the U(1) connection

\[
iQ = i\psi^\mu \pi_\mu = \partial_q \equiv \partial + q \Gamma \, ,
\]

(3.11)

where \( \Gamma = \Gamma_\mu dx^\mu = \Gamma_{\mu\nu} dx^\mu \) is the U(1) connection form, and obeys \( \partial_q^2 = 0 \). Conversely, the charge \( Q \) is given by a twisted divergence

\[
i\bar{Q} = i\bar{\psi}_\mu g^{\mu\nu} \bar{\pi}_\nu = -\partial_{\bar{q}}^\dagger \equiv \frac{\partial}{\partial (dx^\mu)} g^{\mu\nu} (\partial_\nu - q \bar{\Gamma}_\nu) .
\]

(3.12)

Thus, the quantum supercharges are conjugates under the adjoint operation, \( Q^\dagger = \bar{Q} \), and define a self adjoint hamiltonian

\[
H_q = \{ Q, \bar{Q} \} = \partial_q \partial_{\bar{q}}^\dagger + \partial_{\bar{q}} \partial_q^\dagger
\]

\[
= \frac{1}{2} g^{\mu\nu} g^{1/2} (\pi_\mu \pi_\nu + \bar{\pi}_\nu \bar{\pi}_\mu) g^{-1/2} + \frac{1}{2} (1 - 4q) R^\mu_\nu \psi^\nu \psi_\mu + q R
\]

\[
= -\frac{1}{2} \nabla_q^2 + \frac{1}{2} (1 - 4q) R^\mu_\nu \, dx^\nu \frac{\partial}{\partial (dx^\mu)} + q R ,
\]

(3.13)

where the laplacian dressed with the U(1) charge \( q \) reads

\[
\nabla_q^2 \equiv g^{\mu\nu} \left[ (\nabla_\mu + q \Gamma_\mu)(\nabla_\nu - q \bar{\Gamma}_\nu) + (\nabla_\nu - q \bar{\Gamma}_\nu)(\nabla_\mu + q \Gamma_\mu) \right] .
\]

Let us notice that for the choice \( q = \frac{1}{4} \) the coupling to the Ricci tensor disappears, and the hamiltonian reduces to the square of the Dirac operator, as outlined in appendix B, \( H_{1/4} = \frac{1}{2} g^{1/2} \pi_{sym} g^{-1/2} + \frac{1}{4} R \).
By means of the differential operators just introduced the Maxwell-like equations for the \((p+1,0)\) curvature form read as
\[
\partial_q F_{(p+1,0)} = 0, \quad \partial_q^\dagger F_{(p+1,0)} = 0.
\] (3.14)

As in flat space, the first one can be integrated by introducing a \((p,0)\)-form gauge field:
\[
F_{(p+1,0)} = \partial_q A_{(p,0)},
\]
defined up to gauge transformations \(\delta A_{(p,0)} = \partial_q \lambda_{(p-1,0)}\). The field equations then read \(\partial_q^\dagger \partial_q A_{(p,0)} = 0\), and are a natural generalization of Maxwell’s equations. If desired, one may extract the laplacian \(\nabla_q^2\) and cast them in the alternative form
\[
\left(-\frac{1}{2}\nabla_q^2 + qR\right)A_{(p,0)} + \frac{p}{2}(1 - 4q)\text{Ric} \cdot A_{(p,0)} - \partial_q \partial_q^\dagger A_{(p,0)} = 0,
\] (3.15)
with \(\text{Ric} \cdot A_{(p,0)} \equiv R^\lambda_{\mu\nu2...\mu_p}dx^\mu_1 ∧ ... ∧ dx^\mu_p\).

At the present stage, it is useful to study the transition amplitude associated to the quantum hamiltonian (3.13), as it will be of primary importance in the set up of the correct path integral that is needed in subsequent applications, such as the evaluation of the effective action of the \((p,0)\)-form gauge fields. One can evaluate the matrix element of the euclidean evolution operator between position eigenstates and coherent states for fermionic variables,
\[
\langle x\tilde{\eta} | e^{-\beta H_q} | x\xi \rangle,
\]
as a perturbative expansion in \(\beta\). As usual, the calculation can be performed either by operatorial or functional methods. The operatorial computation, that makes use of the fundamental (anti)-commutation relations, is more involved, but it gives a completely non-ambiguous result for the transition amplitude and can be used as a bench mark for setting up the path integral. Following the same computational method illustrated in [12, 13] for generic curved spaces, and in [14] for models on Kähler manifolds, we find the transition amplitude associated to the hamiltonian (3.13), up to first order in \(\beta\). We restrict ourselves to the computation at coincident points, which is enough for our purposes, and find
\[
\langle x\tilde{\eta} | e^{-\beta H_q} | x\xi \rangle = (2\pi\beta)^{-d}e^{\tilde{\eta}\xi}\left\{1 + \beta\left[(q - \frac{1}{3})R + \frac{1}{2}(4q - 1)R_{\mu\nu} \xi^\mu \tilde{\eta}^\nu\right] + \mathcal{O}(\beta^2)\right\}. \tag{3.16}
\]

Let us now turn to the functional computation. The classical hamiltonian corresponding to (3.13) is given by eq. (3.4), and produces the configuration space action (3.6). If we perform the path integral quantization by using the action (3.6), and regulate it to maintain covariance, it is natural to expect that a well defined quantum charge for the U(1) subgroup of the holonomy group will be reproduced. In order to keep room for an arbitrary charge \(q\), we dress the path integral action with a “gauge field” counterterm proportional to an extra coupling \(q_1\)
\[
S = \int dt \left[ g_{\mu\nu}\ddot{x}^\mu \ddot{x}^\nu + i\tilde{\psi}_\mu D_t \psi^\mu + iq_1 \ddot{x}^\mu \Gamma_\mu - i\dot{q}_1 \ddot{x}^\bar{\mu} \bar{\Gamma}_{\bar{\mu}} + 2q_1 R^\mu_{\nu\rho} \psi^\nu \psi^\rho \right], \tag{3.17}
\]
whose structure is dictated by reality of the action and supersymmetry at the classical level. At this juncture we can evaluate the transition amplitude \(\langle x\tilde{\eta} | e^{-\beta H_q} | x\xi \rangle\) by means of a functional integral suitably regulated (we use TS regularization, which generically
requires only covariant counterterms on Kähler manifolds, but MR and DR could be used as well, see \[13, 15\]) giving at order $\beta$

$$\langle \bar{\eta} \bar{\xi} \rangle = (2\pi\beta)^{-d} \bar{\eta} \bar{\xi} \left\{ 1 + \beta \left[ (q_1 - \frac{1}{12}) R + 2q_1 R_{\mu \bar{\nu}} \xi^\mu \bar{\eta}^\bar{\nu} \right] + \mathcal{O}(\beta^2) \right\}.$$  \hspace{1cm} (3.18)

By comparing the two results (3.16) and (3.18) we can exploit the relation among the true quantum charge $q$ and the counterterm one $q_1$. The path integral with action (3.6) without extra charges ($q_1 = 0$) reproduces $q = \frac{1}{4}$, and more generally it follows that $q_1 = q - \frac{1}{4}$. This allows to keep control on the precise $U(1)$ couplings of the model in all the subsequent applications.

We end up this section with a review of the calculation of the Witten index identified by the present supersymmetric sigma model, as it will enter subsequent analyses. It yields the topological index of the (twisted) Dirac operator on Kähler manifolds. The basics of this calculation were originally presented in \[16, 17\], and analyzed more recently in \[10, 11\]. The connection between index theorems and supersymmetric quantum mechanics makes use of the concept of the Witten index, defined as $\text{Tr}(-1)^F$, where $F$ is the fermion number and the trace is over the quantum mechanical Hilbert space. Standard reasonings show that the Witten index counts the number of bosonic zero energy states minus the number of fermionic zero energy states \[18\]. It is a topological invariant that computes the index of the differential operator representing the hermitian supercharge $Q + \bar{Q}$. For the value $q = \frac{1}{4}$, that we analyze first, it realizes the Dirac operator $\bar{\psi} \sim \bar{\partial}_\tau + \frac{1}{4},$ see appendix B. In the Hilbert space of the particle system, bosonic states are given by $(p, 0)$-forms with even $p$, and fermionic states by forms with odd $p$. They correspond to positive chirality and negative chirality spinors, respectively. Thus for our quantum mechanical model the Witten index reduces to the Dirac index. Being a topological invariant it can be regulated as $\text{Tr}(-1)^F = \int_D D_x D\psi e^{-S}$, where $H$ is the hamiltonian, and computed for small $\beta$ using its path integral representation

$$\text{ind}(\bar{D}) = \text{Tr}(-1)^F e^{-\beta H} = \int_D D_x D\psi e^{-S}$$ \hspace{1cm} (3.19)

where the subscript $P$ indicates periodic boundary conditions for bosonic and fermionic fields, and $S$ is the Wick rotated version of the action in (3.6), namely

$$S = \int_0^\beta d\tau \left[ g_{\mu \bar{\nu}} \bar{\psi}^\mu \dot{x}^\bar{\nu} + \bar{\psi}_\mu D_\tau \psi^\mu \right].$$ \hspace{1cm} (3.20)

The pure Dirac case is given by $q = \frac{1}{4}$, and thus $q_1 = 0$, so that we disregard the counterterms inserted in (3.17). To calculate (3.19) one expands all periodic fields in Fourier series with frequencies $\frac{2\pi n}{\beta}$. For small $\beta$ the zero modes dominate, and one only needs to take care of the semiclassical corrections due to a bosonic determinant. It is useful to use Riemann normal coordinates adapted to the Kähler structure, scale suitably the fermionic zero mode by $\beta^{-\frac{1}{2}}$, and obtain

$$\text{ind}(\bar{D}) = \int \frac{d^d x_0 d^d \bar{x}_0 d^d \psi_0 d^d \bar{\psi}_0}{(2\pi)^d} \left[ \frac{\text{Det}'(-\partial_\tau^2 + R \partial_\tau)}{\text{Det}'(-\partial_\tau^2)} \right]^{-1}$$ \hspace{1cm} (3.21)
where $\text{Det}'$ indicates a functional determinant on the space of periodic fields orthogonal to the zero modes, the subscript 0 indicates zero modes, and $R = R^\mu{}_{\nu\lambda\sigma} \bar{\psi}_0^\lambda \psi_0^\sigma$ describes a matrix valued two-form evaluated at the point $(x_0, \bar{x}_0)$. Now one can compute the functional determinant and express it in terms of a standard $d \times d$ determinant of a matrix given by a function of $R$

$$\frac{\text{Det}'(-\partial_\tau^2 + R \partial_\tau)}{\text{Det}'(-\partial_\tau^2)} = \det \left( \frac{\sinh \frac{R}{2}}{\sinh \frac{R}{4\pi i}} \right).$$  \hfill (3.22)

Berezin integration over the Grassmann variables extracts from the expansion of the determinant the contribution of the top $2d$-form only. Thus one can reabsorb the measure factor\(^2\) into the determinant and present the final answer as

$$\text{ind}(\mathcal{D}) = \int_M \det \left( \frac{R}{4\pi i} \sinh \frac{R}{4\pi i} \right)$$ \hfill (3.23)

where now $R = R^\mu{}_{\nu\lambda\sigma} d\bar{x}^\lambda dx^\sigma$.

As just mentioned, for a given Kähler manifold $M$ only the top form coming from the expansion of the determinant contributes. Since the determinant of an even function of $R$ has an expansion in terms of $R^{2d}$, the index is nonvanishing only for manifolds of even complex dimensions. The first example is for $d = 2$, where the above formula gives

$$\text{ind}(\mathcal{D}) = \int_M \text{tr} R^2 = \frac{1}{96\pi^2} \int_M d\bar{x}^1 d\bar{x}^2 dx^1 dx^2 g \left( R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - R_{\mu\nu} R^{\mu\nu} \right).$$  \hfill (3.24)

In general we are interested in keeping an arbitrary U(1) charge in the twisted Dolbeault operators $\partial_q$ and $\partial_{\tilde{q}}$. For $q \neq \frac{1}{4}$ this identifies a sort of twisted Dirac operator, which we denote by $\mathcal{D}_q$ (so that $\mathcal{D}_0 = \mathcal{D}$). To compute its index we have to dress the previous computation by considering the counterterms proportional to $q_1$ in (3.17), which under Wick rotation produce

$$\Delta S = \int_0^\beta d\tau \left[ q_1 \bar{\psi}_{\mu} \Gamma_\mu - q_1 \bar{\psi}_{\mu} \Gamma_\mu - 2q_1 R_{\mu\nu} \psi_{\nu} \bar{\psi}_{\mu} \right].$$ \hfill (3.25)

Suitably rescaling the quantum fields as described above, one recognizes that only the last term may contribute through its leading expansion around the zero modes. This appears in the exponential of the path integral as

$$e^{2q_1 R_{\mu\nu}(x_0, \bar{x}_0) \psi_{\mu}^0 \bar{\psi}_{\mu}^0}$$ \hfill (3.26)

which must be inserted inside the integral of eq. (3.21). The final formula for the twisted Dirac operator is then

$$\text{ind}(\mathcal{D}_q) = \int_M \exp \left( \frac{q_1 \mathcal{F}}{\pi i} \right) \det \left( \frac{R}{4\pi i} \sinh \frac{R}{4\pi i} \right)$$ \hfill (3.27)

with $\mathcal{F} = R_{\mu\nu} dx^\mu d\bar{x}^\nu$. In $d = 2$ it produces the following extra contribution

$$\frac{q_1^2}{2\pi^2} \int_M d\bar{x}^1 d\bar{x}^2 dx^1 dx^2 g \left( R^2 - R_{\mu\nu} R^{\mu\nu} \right).$$ \hfill (3.28)

that added to (3.24) gives the index of the twisted Dirac operator $\text{ind}(\mathcal{D}_q)$.

\(^2\)This is $(2\pi i)^d$ when taking into account the choice of a suitable orientation and the factors of $i$ present in the measure (A.3).
4 Effective action of quantized \((p,0)\)-forms

We are now ready to come to the main part of the paper, discuss the quantization of \((p,0)\)-forms and compute the corresponding effective actions using worldline methods.

To start with, we aim at obtaining a useful worldline representation of the one-loop effective action in an arbitrary Kähler background. The effective action may be depicted by the sum of all Feynman diagrams of the form shown in figure 1, where a quantum \((p,0)\)-form gauge field circulates in the loop and external lines represent the curved background.

\[
\begin{align*}
\int_0^1 d\tau \left[ e^{-1} g_{\mu\bar{\nu}} \left( \dot{x}^\mu - \dot{\bar{x}}^\mu \right) \left( \dot{x}^\bar{\nu} - \dot{\bar{x}}^\bar{\nu} \right) + \bar{\psi}_\mu \left[ D_\tau + i a \right] \psi^\mu + i a \right] \\
+ q_1 \int_0^1 d\tau \left[ \dot{x}^\mu \Gamma_\mu - \dot{\bar{x}}^\bar{\mu} \bar{\Gamma}_{\bar{\mu}} - 2e R^\mu_\mu \psi^\mu \bar{\psi}_\mu \right],
\end{align*}
\]

where we recall that a counterterm proportional to \(q_1 \equiv q - \frac{1}{4}\) is needed in order to reproduce a quantum coupling \(q\) to the \(U(1)\) part of the connection. We denote the basic dynamical variables by \(X = (x^\mu, \bar{x}^\mu, \psi^\mu, \bar{\psi}_\mu)\) and \(G = (e, \chi, \bar{\chi}, a)\). Of course \(\bar{\psi}^\bar{\nu} = g^{\mu\bar{\nu}} \bar{\psi}_\mu\), while the covariant time derivative is given by \(D_\tau \psi^\mu = \dot{\psi}^\mu + \dot{x}^\nu \Gamma^\mu_{\nu\lambda} \psi^\lambda\). Note that along
with the Wick rotation $t \rightarrow -i\tau$, we have rotated also the gauge field $a \rightarrow ia$ to keep the U(1) gauge group compact.

Quantization of this spinning particle model on a circle parametrized by $\tau \in [0, 1]$ gives the partition function for the $(p, 0)$-form gauge field coupled to the metric of the curved Kähler space

$$Z[g] \propto \int \frac{Dx'DG}{\text{Vol(Gauge)}} e^{-S[X,G]} \quad (4.2)$$

and visually corresponds to figure 1. A point worth stressing again is that we regulate the path integral and related functional determinants so that they correspond to a graded-symmetric operatorial ordering of the current $J$, namely $J = \frac{1}{2} (\psi^\mu \tilde{\psi}_\mu - \tilde{\psi}_\mu \psi^\mu) = \psi^\mu \tilde{\psi}_\mu - \frac{d}{2}$, an ordering that is responsible, for example, to the standard fermionic zero point energy. Then the projection onto the physical field strenght $F_{(p+1, 0)}$ is obtained by using the Chern-Simons coupling $s \equiv p + 1 - \frac{d}{2}$ (so that $J - s = \psi \tilde{\psi} - (p + 1)$ as an operator).

Using the standard Fadeev-Popov procedure to get rid of gauge redundancy, we fix the gauge fields to the constant values $\tilde{\chi} = \tilde{G} = (\beta, 0, 0, \phi)$, and are left with modular integrations over $\beta$ and $\phi$, with the following one-loop measure that was carefully studied in [1]

$$Z[g] \propto \int_0^{2\pi} \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \left( 2 \cos \frac{\phi}{2} \right)^{-2} \int_P Dx D\tilde{x} \int_A D\tilde{\psi} D\psi e^{-S[X,G]} \quad (4.3)$$

with $S[X,G]$ denoting the gauge fixed action, i.e. eq. (4.1) evaluated at $G = \tilde{G}$. The subscript P and A denote periodic and antiperiodic boundary conditions, respectively. The integral over $\beta$ is the usual proper time integral with the well known one-loop measure, while the factor $\left( 2 \cos \frac{\phi}{2} \right)^{-2}$ is the Fadeev-Popov determinant of the bosonic superghosts associated to $\chi$ and $\bar{\chi}$. We denote with $Dx$ the general coordinate invariant measure, i.e. $Dx D\tilde{x} \sim \prod_{\tau = 0}^1 d^4 x(\tau) d^4 \tilde{x}(\tau) g(x(\tau), \tilde{x}(\tau))$, with $g = \text{det} g_{\mu\nu}$, while $D\psi \sim \prod_{\tau = 0}^1 d^4 \psi(\tau)$ is the simple translational invariant measure.\footnote{Note that, since $\psi$'s are spacetime vectors, while $\tilde{\psi}$'s are covectors, one has $D\tilde{\psi} D\psi = D\tilde{\psi} D\psi$.} This formula gives the worldline representation of the effective action of the $(p, 0)$-form gauge field.

For computational purposes, it is useful to manipulate it a bit further. The path integral over loops, i.e. over coordinate fields with periodic boundary conditions, can be done in several ways [19]. Here we choose to fall back on quantum fields with Dirichlet boundary conditions. Thus, we pick an arbitrary $x_0$ as a base-point for our loops. The path integral then factorizes as $\int_P Dx D\tilde{x} = \int d^4 x_0 d^4 \tilde{x}_0 g(x_0) \int_{x(0) = x_0} Dx D\tilde{x}$. It is possible then to perform background-quantum fluctuations splitting as $x^\mu(\tau) = x_0^\mu + q^\mu(\tau)$, with $q^\mu(0) = q^\mu(1) = 0$. Clearly the $x$ path integral becomes $\int_D Dq D\tilde{q}$, where $D$ stands for Dirichlet boundary conditions, i.e. fields are taken to vanish at boundaries. The next step is that of getting rid of the field dependent measure $Dq D\tilde{q}$. Following the trick of [20, 21] we exponentiate the $g$ factors with a path integral over fermionic complex ghosts $b^\mu$ and $\bar{c}^\mu$: $Dq D\tilde{q} = Dq D\tilde{q} \int D\bar{b} Dc e^{-S_{cb}}$. At this stage the gauge fixed action $S_{gf} \equiv S[X,G]$ plus
the ghost action for the path integral measure \( S_{gh} \) take the following form\(^4\)

\[
S_{gd} + S_{gh} = \frac{1}{\beta} \int_0^1 d\tau \left[ g_{\mu\nu}(q^{\mu}\dot{q}^{\nu} + b^{\mu}\tilde{c}^{\nu}) + \bar{\psi}_\mu(D_\tau + i\phi)\psi^{\mu} \\
+ \beta q_1(q^{\mu}\Gamma_\mu - \dot{q}^{\mu}\Gamma_{\dot{\mu}} - 2R^{\mu}_\nu \psi^{\nu} \bar{\psi}_\mu + i\phi \right].
\]

(4.4)

In order to perform perturbative calculations we expand all background fields around the fixed point \( x_0 \). The action written above splits into a quadratic part \( S_q \) and an interaction part. We denote as \( x \) the quantum average weighted with the free path integral: \( \langle \bullet \rangle = \frac{1}{e^{-S_2}} \int \bullet e^{-S_2} \). The partition function (4.3) now reads

\[
Z \propto \int_0^1 \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \left( 2 \cos \frac{\phi}{2} \right)^{d-2} e^{-i\phi} \int \frac{d^d x_0 d^d \bar{x}_0}{(2\pi\beta)^d} g(x_0)(e^{-S_{int}}),
\]

(4.5)

where \((2 \cos \frac{\phi}{2})^{d}(2\pi\beta)^{-d}\) is the usual free path integral normalization, and the interaction part is

\[
S_{int} = \frac{1}{\beta} \int_0^1 d\tau \left[ (g_{\mu\nu}(x_0 + q) - g_{\mu\nu}(x_0)) (q^{\mu}\dot{q}^{\nu} + b^{\mu}\tilde{c}^{\nu}) + q^{\mu}\Gamma^{\mu}_{\nu\lambda}(x_0 + q)\bar{\psi}_\nu\psi^{\lambda} \\
+ \beta q_1(q^{\mu}\Gamma_\mu(x_0 + q) - \dot{q}^{\mu}\Gamma_{\dot{\mu}}(x_0 + q) - 2R^{\mu}_\nu(x_0 + q) \psi^{\nu} \bar{\psi}_\mu) \right].
\]

(4.6)

For our computation we can choose any coordinate system so, in order to be able to reconstruct covariance, and at the same time to maintain holomorphic coordinates, we use Kähler normal coordinates (see [22], for example) centered at \( x_0 \). Denoting with \( S_n \) the part of \( S_{int} \) containing \( n \)-fields vertices (or less, but producing a result of the same order in \( \beta \)), it results that, in Kähler normal coordinates, the only terms giving non vanishing contribution up to order \( \beta^2 \) are the following ones

\[
S_4 = \frac{1}{\beta} \int_0^1 d\tau \left[ R^{\mu\nu\lambda\delta}_{\rho\sigma\tau\kappa}(q^{\mu}\dot{q}^{\nu} + b^{\mu}\tilde{c}^{\nu}) + R^{\kappa\nu\lambda\sigma}_{\rho\sigma\kappa\mu} q^{\mu}\dot{q}^{\nu} \bar{\psi}_\lambda\psi^{\sigma} \\
+ q_1 \int_0^1 d\tau \left[ R^{\mu\nu}(q^{\mu}\dot{q}^{\nu} - \dot{q}^{\mu}\tilde{c}^{\nu}) - 2R^{\nu}_\mu \psi^{\nu} \bar{\psi}_\mu \right],
\]

\[
S_6 = \frac{1}{\beta} \int_0^1 d\tau \left[ \frac{1}{4} \left( \nabla_\sigma \nabla_\lambda R^{\mu\nu\rho\sigma}_{\rho\sigma\nu\mu} \right) + 3R^{\sigma}_{\rho\lambda\rho\nu\mu} q^{\lambda}\bar{\psi}^{\sigma} q^{\mu}\psi^{\nu} \bar{\psi}_\lambda \\
- \frac{1}{2} \left[ \nabla_\sigma \nabla_\lambda R^{\kappa\nu}_{\rho\sigma\nu\mu} + R^{\sigma}_{\rho\lambda\rho\mu} R^{\lambda\nu}_{\kappa\nu\mu} \right] q^{\sigma}\bar{q}^{\nu} \bar{\psi}_\lambda \psi^{\nu} \\
- q_1 \int_0^1 d\tau \left[ \frac{1}{2} \left( \nabla_\sigma \nabla_\lambda R^{\mu\nu}_{\rho\sigma\nu\mu} + R^{\sigma}_{\rho\lambda\rho\mu} R^{\lambda\nu}_{\kappa\nu\mu} \right) q^{\sigma}\bar{q}^{\nu} \bar{\psi}_\lambda \psi^{\nu} \\
- \frac{1}{2} \left( \nabla_\sigma \nabla_\lambda R^{\mu\nu}_{\rho\sigma\nu\mu} + R^{\sigma}_{\rho\lambda\rho\mu} R^{\lambda\nu}_{\kappa\nu\mu} \right) q^{\sigma}\bar{q}^{\nu} \bar{\psi}_\lambda \psi^{\nu} + 2\n\right].
\]

(4.7)

where all tensors are calculated at \( x_0 \) and round brackets denote weighted symmetrization, separately among holomorphic and (anti-) holomorphic indices, i.e. \( A_{(\mu_1...\mu_n,\nu_1...\nu_m)} \equiv \]

\[^4\text{We rescaled fermions by } \psi \to \frac{1}{\gamma} \psi \text{ in order to extract a common } \beta \text{ as loop counting parameter.}\]
\[ A(\mu_1, \mu_2, \nu_1, \nu_2) \] From the quadratic action \( S_2 = \frac{1}{2} \int [g_{\mu\nu}(x_0)(\partial^\mu \bar{\phi} + b^\mu \phi^0 + \bar{\psi}_\mu (\partial_\tau + i\phi) \psi^\mu] \)

one extracts the following two point functions

\[
\langle q^\mu(\tau) q^\nu(\sigma) \rangle = -\beta g^{\mu\nu}(x_0) \Delta(\tau, \sigma), \quad \langle b^\mu(\tau) b^\nu(\sigma) \rangle = -\beta g^{\mu\nu}(x_0) \delta(\tau, \sigma) \tag{4.8}
\]

where the propagators in the continuum limit read

\[
\Delta(\tau, \sigma) = \sigma(\tau - \sigma) \theta(\tau - \sigma) + \tau(\sigma - \tau) \theta(\sigma - \tau),
\]

\[
\Delta_f(x, \phi) = \frac{e^{-i\phi x}}{2 \cos \frac{\phi}{2}} \left[ e^{i\frac{\phi}{2} \theta(x)} - e^{-i\frac{\phi}{2} \theta(-x)} \right] \tag{4.9}
\]

with \( \theta(x) \) the step function and \( \delta(\tau, \sigma) \) the Dirac delta acting on functions vanishing at the boundaries. We note that in performing perturbative calculations one encounters products and derivatives of such distributions, that are ill defined. To resolve this ambiguity we use Time Slicing (TS) regularization \([15, 23, 24]\), that gives well-known prescriptions on how to handle such products of distributions and necessitates no counterterms (the standard TS counterterm vanish on Kähler manifolds). The rules are as follows: when computing the handle such products of distributions and necessitates no counterterms (the standard TS Time Slicing (TS) regularization \([15, 23, 24]\), that gives well-known prescriptions on how to

and derivatives of such distributions, that are ill defined. To resolve this ambiguity we use

Looking at (4.8) we immediately see that each piece \( S_n \) of \( S_{\text{int}} \) gives a contribution of order \( \beta^{n/2-1} \). Therefore, our quantum average can be written explicitly as

\[
\langle e^{-S_{\text{int}}} \rangle = 1 - \langle S_4 \rangle - \langle S_6 \rangle + \frac{1}{2} \langle S_4^2 \rangle + O(\beta^3). \tag{4.10}
\]

Using the expressions given in (4.7) and TS prescriptions in calculating Feynman diagrams, one finally obtains

\[
\langle e^{-S_{\text{int}}} \rangle = 1 + \beta \left( \frac{iq_1 \tan \frac{\phi}{2} - \frac{1}{12}}{2} \right) R + \beta^2 \left\{ \left( \frac{1}{180} - \frac{1}{96} \cos^2 \frac{\phi}{2} \right) R_{\mu\nu\lambda\sigma} R^\mu R^\nu R^\lambda R^\sigma \right. \\
+ \left[ - \frac{19}{1440} \frac{q_1^2}{6} + \left( \frac{1}{160} + \frac{q_1^2}{2} \right) \cos^{-2} \frac{\phi}{2} + \frac{iq_1}{12} \tan \frac{\phi}{2} \right] R_{\mu\nu} R_{\mu\nu} \\
+ \left( \frac{1}{288} + \frac{q_1^2}{12} \cos^{-2} \frac{\phi}{2} - \frac{iq_1}{12} \tan \frac{\phi}{2} \right) R^2 \\
+ \left( - \frac{1}{240} + \frac{iq_1}{12} \tan \frac{\phi}{2} \right) \nabla^2 R \right\}, \tag{4.11}
\]

where \( \nabla^2 R = 2g^{\mu\nu} \partial_\mu \partial_\nu R \).

Plugging this result into the partition function (4.3) one faces the task of performing

the \( \phi \) integral, taking care of the possible pole arising at \( \phi = \pi \). Switching to the Wilson loop variable \( w = e^{i\phi} \) one has a contour integral on the unit circle surrounding the origin, with a possible pole on the integration path at \( w = -1 \). Its presence is related to topological mismatches, affecting duality relations, that we are going to investigate in the next section.

We need a prescription to deal with this additional pole, and the correct one turns out to be to slightly deform our path in a way that excludes the pole, as shown in figure 2. We
The regulated contour $\gamma^-$ that excludes the pole at $w = -1$. Call this regulated contour $\gamma^-$. The correctness of this choice is confirmed by checking the result for a scalar field, that indeed comes out correctly only by using the aforementioned prescription.

The additional pole at $w = -1$ shows up already at order $\beta^2$ for $d < 4$, while for $d \geq 4$ it appears at higher orders in $\beta$. For this reason we present the results separately for $d \geq 4$ and for lower dimensions, recalling that $(p, 0)$-forms propagate only for $0 \leq p \leq d - 2$. First of all, let us parametrize the structure of the first heat kernel coefficients as follows

$$Z \propto \int_0^{\infty} \frac{d\beta}{\beta} \int d^d x_0 d^d \bar{x}_0 (2\pi \beta)^d g(x_0) \left\{ v_1 + v_2 \beta R + \beta^2 \left[ v_3 R_{\mu\nu\lambda\bar{\lambda}} R^\mu^\nu^\lambda^\bar{\lambda} + v_4 R_{\mu\bar{\nu}} R^\mu R_{\bar{\nu}} + v_5 R^2 + v_6 \nabla^2 R \right] \right\}. \tag{4.12}$$

Let us recall that the first coefficient $v_1$ in (4.12) represents the number of physical degrees of freedom, and will be zero when considering the contributions to the effective action of non-propagating fields. We may now list the coefficients of a gauge $(p, 0)$-form with charge $q$ in the format: $A_p^{(q)} \rightarrow (v_1; v_2; v_3; v_4; v_5; v_6)$, where the $v_i$ are the coefficients appearing in eq. (4.12).

Let us start by giving the Seeley-DeWitt coefficients for a $(p, 0)$-form in $d \geq 4$

$$A_p^{(q)} \rightarrow \left( \frac{d - 2}{p} \right) \times \left( 1; \frac{1}{6} - \frac{p}{2(d - 2)} - \frac{d - 2 - 2p}{d - 2}; \frac{1}{180} - \frac{p(d - p - 2)}{24(d - 2)(d - 3)}; \right.$$

$$- \frac{1}{360} + \frac{p(3d - 4p - 5)}{24(d - 2)(d - 3)} + \frac{q p(6d - 5d + 9)}{6 (d - 2)(d - 3)} + \frac{q^2}{6} \left[ \frac{12p(d - p - 2)}{(d - 2)(d - 3)} - 1 \right];$$

$$\frac{1}{72} + \frac{p(3p - 2d + 3)}{24(d - 2)(d - 3)} - \frac{q}{6} \left[ \frac{p(6p - 5d + 9)}{(d - 2)(d - 3)} + 1 \right] + \frac{q^2}{2} \left[ 1 - \frac{4p(d - p - 2)}{(d - 2)(d - 3)} \right];$$

$$\frac{1}{60} - \frac{p}{24(d - 2)} - \frac{q}{12} \left[ \frac{d - 2 - 2p}{d - 2} \right]. \tag{4.13}$$

These are the coefficients for a gauge $(p, 0)$-form coupled to the U(1) part of the connection.
via a charge $q$, obeying $\partial_q^{\dagger} \partial_q A_p = 0$. They are invariant under the exchange $p \leftrightarrow (d - p - 2)$ and $q \leftrightarrow \frac{1}{2} - q$, as it is obvious if one rewrites them in terms of $q_1 = q - \frac{1}{4}$, the duality being $q_1 \leftrightarrow -q_1$. This hints indeed towards a duality between $(p,0)$ and $(d-p-2,0)$-forms $A_p^{(q)} \leftrightarrow A_{d-p-2}^{(1/2-q)}$, that will be investigated in the next section.

We can immediately check that the result (4.13) correctly reproduces the known coefficients for a scalar field: setting $p = 0$ one gets

$$A_0^{(q)} \to \left(1; \frac{1}{6} - q; \frac{1}{180}; -\frac{1}{360} - \frac{q^2}{6}; \frac{1}{72} - \frac{q}{6} + \frac{q^2}{2}; \frac{1}{60} - \frac{q}{12} \right),$$  

(4.14)

which coincide with the standard results\(^5\) once one turns off the charge $q$.

Let us examine a bit closer what happens in lower dimensions. In $d = 3$ complex dimensions, only scalars and one-forms propagate. The formula (4.13), that is ill-defined for generic $p$ at $d = 3$, has indeed a smooth limit for $p = 0$, that reads

$$d = 3, \quad p = 0, 1$$

$$A_p^{(q)} \to \left(1; \frac{1}{6} - \frac{p}{2} + q(2p - 1); \frac{1}{180} - \frac{p}{24}; -\frac{1}{360} + \frac{p}{8} - \frac{5}{6}qp + \frac{q^2}{6} (12p - 1); \frac{1}{72} - \frac{p}{12} + \frac{q}{6}(5p - 1) + \frac{q^2}{2} (1 - 4p); \frac{1}{60} - \frac{p}{24} + \frac{q}{12} (2p - 1) \right).$$  

(4.15)

In $d = 3$ zero-forms are expected to be dual to one-forms, but (4.15) is not invariant under the exchange $p \leftrightarrow 1 - p$ and $q \leftrightarrow \frac{1}{2} - q$. In fact, in $d = 3$ the mismatches that are discussed in the next section appear already at order $\beta^2$. For $p > 1$ the heat kernel coefficients are not zero in $d = 3$, even though nothing propagates, and give just a topological contribution that will be exploited when addressing exact dualities.

A similar reasoning holds in $d = 2$: now only scalars propagate, and equation (4.13) has a smooth $d = 2$ limit for $p = 0$, yielding the known result (4.14).

Let us also discuss briefly the case of $d = 1$, that is somewhat degenerate. The expansion of the generic wave function (2.6) suggests as possible models those related to $p = -1$ and $p = 0$, as now one can write $\phi(x, \bar{x}, \psi) = F_0(x, \bar{x}) + F_1(x, \bar{x})\psi$. For each of them one of the susy constraint equations collapse to an identity, and the remaining one corresponds to $\partial_q F_0 = 0$ and $\partial_q F_1 = 0$. In both cases one cannot legally introduce a gauge potential $A_p$. Nevertheless the path integral computes their effective action, showing that for $p = -1$ (i.e. $F_0$) the model is empty, while for $p = 0$ (i.e. $F_1$) one obtains again the values of a scalar field as in eq. (4.14).

As another interesting application of our $U(1)$ spinning particle, we can choose not to gauge the $U(1)$ part of the first class algebra, i.e. $J - s$. Then, we do not have a modular integration over $\phi$ any more, and the result for this new model is obtained for free by setting $\phi = 0$ in (4.11). It corresponds to the quantum theory of the sum of all $(p,0)$-forms $F_p$ with dynamics dictated by the Maxwell equations. We know that this system is equivalent, on Kähler manifolds, to a Dirac spinor; hence its effective action must be proportional to

---

\(^5\)See appendix A to compare our conventions on curvatures with the standard riemannian ones.
the one-loop effective action of a Dirac field. In fact, the path integral over the complex gravitino present in (4.2) can at most change the overall normalization of the partition function if compared with the path integral over the real gravitino needed for the Dirac field, recall eq. (2.17). Indeed, one may check that fixing suitably the overall normalization, one recovers the heat kernel coefficients of a Dirac spinor. In order to do so, we recall from previous sections that the sum $\partial_q \bar{\partial}_q^\dagger$ is equivalent to the Dirac operator only for $q = \frac{1}{2}$, that is $q_1 = 0$. In terms of $q_1$, the heat kernel coefficients of the $U(1)$-ungauged model read

$$
\Psi^{(n)} \to 2^d \left( 1; -\frac{1}{12}; -\frac{7}{1440}; -\frac{1}{360}; +\frac{q_1^2}{3}; \frac{1}{288}; -\frac{1}{240} \right),
$$

(4.16)

that indeed agree at $q_1 = 0$ with the standard results for a Dirac fermion, compare for example with [25, 26].

Finally, one might wish not to gauge the two supersymmetries at all, but gauge the $U(1)$ charge instead. This produce the effective action of a single $(p,0)$-form $B_p$, now with dynamics dictated by the hamiltonian $H_q$ only, namely a $(p,0)$-form without any gauge invariance but with dynamical equation $(q \partial_q \bar{\partial}_q^\dagger + \partial_q \partial_q^\dagger) B_p = 0$. To achieve this, we only need to drop from (4.3) the Faddeev-Popov determinant $\left(2 \cos \frac{\phi}{2}\right)^{-2}$ due to the gauge fixing of the gravitini, fix the Chern-Simons coupling $s = p - \frac{d}{2}$, and obtain the following coefficients for the “non gauge” $(p,0)$-form $B_p$ with charge $q$

$$
B_p^{(q)} \to \left( \frac{d}{p} \right) \times \left( 1; \frac{1}{6} - \frac{p}{2d} + q \left[ \frac{2p}{d} - 1 \right]; \frac{1}{180} - \frac{p(d-p)}{24d(d-1)}; -\frac{1}{360} + \frac{p(3d-4p+1)}{24d(d-1)} + \frac{q}{6} \frac{p(6p-5d-1)}{d(d-1)} + \frac{q^2}{6} \frac{12p(d-p)}{d(d-1)} - 1; \frac{1}{72} + \frac{p(3p-2d-1)}{24d(d-1)} - \frac{q}{6} \left[ \frac{p(6p-5d-1)}{d(d-1)} + 1 \right] + \frac{q^2}{2} \left[ 1 - \frac{4p(d-p)}{d(d-1)} \right]; \frac{1}{60} - \frac{p}{24d} + \frac{q}{12} \left[ \frac{2p}{d} - 1 \right] \right).
$$

(4.17)

This formula is valid for $d > 1$. We notice that no additional pole arises at $w = -1$, and that the result (4.17) is invariant under the simultaneous exchange $p \leftrightarrow d - p$ and $q \leftrightarrow \frac{1}{2} - q$.

This points towards a duality between $(p,0)$ and $(d-p,0)$ “non gauge” differential forms. In the special case of $d = 1$ the Riemann and Ricci tensor are not independent from the scalar curvature, so that it is enough to list the coefficients for the $(p,0)$-forms in the format $(v_1; v_2; \bar{v} \equiv v_3 + v_4 + v_5; v_6)$ as $R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} = R_{\mu\nu} R^{\mu\nu} = R^2$. The two possibilities are for $p = 0, 1$ and one gets

$$
B_p^{(q)} \to \left( 1; \frac{1}{6} - q + \frac{p}{2}(4q - 1); \frac{1}{60} - \frac{q}{6} + \frac{q^2}{3} - \frac{1}{60} - \frac{q}{12} + \frac{p}{24}(4q - 1) \right)
$$

(4.18)

which signals a duality between $p = 0$ and $p = 1$. 

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5 Dualities

We now wish to discuss in more depth the issue of duality, as emerged “experimentally” from the results of the last section. Here we prove exact relations between dual formulations.

It is useful to start with the classical particle action given in (4.1), which is characterized by the Chern-Simons coupling $s$ and the U(1) charge $q_1 \equiv q - \frac{1}{4}$. One may begin by noticing that the model with couplings $(-s, -q_1)$ is equivalent to the model with couplings $(s, q_1)$. In fact, one obtains the latter from the former by a suitable transformation of the dynamical variables: one needs to change the sign of the U(1) gauge field $a \rightarrow -a$ (to bring the coupling $-s$ back to the value $+s$), exchange $\psi \leftrightarrow \bar{\psi}$ (to bring the couplings of the gauge field $a$ to the fermions back to its original form, which contains a covariant derivative of the form $\partial_\tau + ia$), and then exchange $x \leftrightarrow \bar{x}$ together with $\chi \leftrightarrow \bar{\chi}$ (to reinstate the correct overall $q_1$ coupling and achieve at the same time full equivalence with the $(s, q_1)$ model). Thus, one verifies that this change of variables relates the model with couplings $(-s, -q_1)$ to the one with couplings $(s, q_1)$. At the quantum level the equivalence between the two models corresponds to a duality between different forms.

To discuss the latter is useful to switch to an operatorial picture and cast the effective action (4.3) as follows

\[ Z_p(q) \propto \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \left( \frac{2\cos \phi}{2} \right)^{-2} \int_p D\bar{x} D\bar{x} \int_A D\bar{\psi} D\psi e^{-S[X,\tilde{G}]} \]  

(5.1)

\[ = \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \left( \frac{2\cos \phi}{2} \right)^{-2} \Tr \left[ e^{i\phi(J-s)} e^{-\beta H_q} \right] \]  

(5.2)

\[ = \int_0^\infty \frac{d\beta}{\beta} \oint_{|w|=1} \frac{dw}{2\pi i w} \frac{w}{(1+w)^2} \Tr \left[ w^{J-s} e^{-\beta H_q} \right] \]  

(5.3)

\[ = \int_0^\infty \frac{d\beta}{\beta} \oint_{|w|=1} \frac{dw}{2\pi i w} \frac{w}{(1+w)^2} \Tr \left[ w^{F-(p+1)} e^{-\beta H_q} \right] . \]  

(5.4)

where we have used different notations to be able to underline various properties. The passage from (5.1) to (5.2) corresponds to the equivalence between path integrals and operatorial quantization, and $J$ and $H_q$ are the corresponding quantum operators described in section 3. In (5.3) we have employed the Wilson loop variable $w = e^{i\phi}$, and the contour integral is along the unit circle $|w| = 1$, regulated as discussed in the last section by excluding the pole at $w = -1$. In the last expression, eq. (5.4), we have made explicit the fermion number operator $F = \psi \bar{\psi}$, as used in the Dirac index computation. As $J = \frac{1}{2} (\psi \bar{\psi} - \bar{\psi} \psi) = \psi \bar{\psi} - \frac{d}{2}$ and $s = p + 1 - \frac{d}{2}$, one obtains that $J - s = F - (p+1)$, which achieves the projection to the $(p+1,0)$-form sector. In this last expression we have defined for convenience the “effective action density in proper time” $Z_p(\beta, q)$ for the $(p,0)$-form gauge field with charge $q$.

Let us now analyze these formulas in various cases:
where \( t_n(\beta, q) \) indicates the contribution arising from the trace restricted to the Hilbert space sector with fermion number \( F = n \). No poles are present along the contour \(|w| = 1\), that we indicate with \( \gamma \), and the integral extracts from the pole at \( w = 0 \) the contribution \( t_p(\beta, q) \) due to a \( p \)-form. It corresponds to the quantum theory of a \((p,0)\)-form effective actions, with \( 0 \leq p \leq d \), and with \( \partial_\bar{q} \) replaced by \( \partial_{\frac{1}{2}-q} \). Indeed, recalling that now \( s \equiv p - \frac{d}{2} \) and \( q_1 \equiv q - \frac{1}{4} \), one may compute

\[
Z_{d-p}^{\text{ungauged}}(\beta, \frac{1}{2} - q) = \oint_{\gamma} \frac{dw}{2\pi i w} \text{Tr} \left[ w^{J+s} e^{-\beta H_{1/2-q}} \right]
\]

\[
= \oint_{\gamma} \frac{dw'}{2\pi i w'} \text{Tr} \left[ w'^{(-J+s)} e^{-\beta H_q} \right]
\]

\[
= \oint_{\gamma} \frac{dw}{2\pi i w} \text{Tr} \left[ w^{J-s} e^{-\beta H_q} \right] = Z_p^{\text{ungauged}}(\beta, q)
\]

where we have first written down the definition of the effective action density for the model with couplings \((-s,-q_1)\), corresponding to \( Z_{d-p}^{\text{ungauged}}(\beta, \frac{1}{2} - q) \). Then we changed \( J \to -J \) and \( H_{1/2-q} \to H_q \), corresponding to \( q_1 \to -q_1 \), to take into account the exchanged role of \((x, \psi)\) and \((\bar{x}, \bar{\psi})\), and used \( w \to w' = \frac{1}{w} \) to take into account the sign change of the gauge field \( \phi \to -\phi \). Finally, a change of variables to the original coordinate \( w = \frac{1}{w} \) shows that this expression coincides with the one corresponding to the couplings \((s, q_1)\). This proves a duality between \((p,0)\)-form and \((d-p,0)\)-form at the quantum level, namely \( t_p(\beta, q) = t_{d-p}(\beta, \frac{1}{2} - q) \).

To check duality in our previous examples, it may be easier to rewrite the heat kernel coefficients in terms of the parameter \( q_1 \equiv q - \frac{1}{4} \). For \( d > 1 \) they read

\[
B_p^{(q)} \to \binom{d}{p} \times \left( 1; -\frac{1}{12} + q_1 \frac{2(p-d)}{d}; \frac{1}{180} - \frac{p(d-p)}{24d(d-1)}; \right.
\]

\[
\left. -\frac{19}{1440} + \frac{p(d-p)}{24d(d-1)} + q_1 \frac{(2p-d)}{12d} + \frac{q_1^2}{6} \frac{12p(d-p)}{d(d-1)} - 1 \right); \quad (5.7)
\]

\[
\frac{1}{288} - \frac{q_1(2p-d)}{12d} + \frac{q_1^2}{2} \left( 1 - \frac{4p(d-p)}{d(d-1)} \right); \quad -\frac{1}{240} + \frac{q_1(2p-d)}{12d}
\]
and for $d = 1$, recalling the special format $(v_1; v_2; \bar{v} = v_3 + v_4 + v_5; v_6)$, they read

$$ B_p^{(q)} \to \left( 1; -\frac{1}{12} + q_1(2p - 1); -\frac{1}{240} + \frac{q_2^2}{3}, -\frac{1}{240} + \frac{q_1}{12}(2p - 1) \right) . \quad (5.8) $$

At the classical geometrical level, this duality can be understood as follows. It is well-known that a $(p, q)$-form is Hodge dual to a $(d - q, d - p)$-form, which in turn is related to a $(d - p, d - q)$-form by complex conjugation. Thus a $(p, 0)$-form is certainly related to a $(d - p, d)$-form. Now, on a non-compact, topologically trivial Kähler manifold one may split the volume form in chiral components using the vielbein field

$$ g\epsilon_{\mu_1 \ldots \mu_d \bar{\nu}_1 \ldots \bar{\nu}_d} = e\epsilon_{\mu_1 \ldots \mu_d} \bar{\epsilon}_{\bar{\nu}_1 \ldots \bar{\nu}_d} \quad (5.9) $$

and use the tensor $e\epsilon_{\mu_1 \ldots \mu_d}$ to dualize the $(d - p, d)$-form to a $(d - p, 0)$-form. The correct U(1) charge assignments are seen to emerge as well, when taking care of the U(1) charge $e\epsilon_{\mu}$ and use the tensor $e\epsilon_{\mu}$ to integrate over the coordinates to the contour $\gamma$ of integration variables $w$. As already discussed, the correct prescription is to exclude the pole indicating a small contour encircling the pole at $w = -1$. This reproduces, in particular, the correct scalar result at $p = 0$. Duality is again obtained by $(s, q_1) \to (-s, -q_1)$, with $s = p + 1 - \frac{d}{2}$. Calculating as above we obtain

$$ Z_{d - p - 2}(\beta, \frac{1}{2} - q) = \int_{\gamma^-} dw \frac{w}{2\pi iw (1 + w)^2} \text{Tr} [w^{J + s} e^{-\beta H_{1/2 - s}}] $$

$$ = \int_{\gamma^-} dw' \frac{w'}{2\pi iw' (1 + w')^2} \text{Tr} [w'^{(-J + s)} e^{-\beta H_s}] $$

$$ = \int_{\gamma^+} dw \frac{w}{2\pi iw (1 + w)^2} \text{Tr} [w^{J - s} e^{-\beta H_s}] $$

$$ = \left( \int_{\gamma^-} + \int_{\gamma^0} \right) dw \frac{w}{2\pi iw (1 + w)^2} \text{Tr} [w^{J - s} e^{-\beta H_s}] $$

$$ = Z_p(\beta, q) + Z_{p, \text{top}}(\beta, q) . \quad (5.10) $$

Again, we have first written down the definition of the partition function at the values $(-s, -q_1)$, then used the change of variables for the dynamical fields (the fields integrated over in the path integral) to relate the model to its dual, thus obtaining the second line above, where in particular $w' = e^{-i\phi}$ takes into account the sign change of the worldline U(1) gauge field. To better interpret the resulting expression we performed a change of integration variables $w' \to w = \frac{w'}{w}$, which maps the regulated contour $\gamma^-$ in the $w'$ coordinates to the contour $\gamma^+$ in the $w$ coordinates, as shown in figure 3.

As $\gamma^+ = \gamma^- + \gamma^0$, with $\gamma^0$ indicating a small contour encircling the pole at $w = -1$, we recognize the partition function for the gauged $(s, q_1)$ model plus a “topological” contribution $Z_{p, \text{top}}(\beta, q)$ arising from the contour integral around $\gamma^0$. To appreciate the significance of the latter term, let us analyze it further by evaluating the integral on $\gamma^0$ using the residue
Theorem

\[ Z_{p \rightarrow (p+1)}^{\text{top}}(\beta, q) = \frac{1}{2\pi i w} \frac{1}{(1+w)^2} \text{Tr} [w^{-s} e^{-\beta H_q}] = \frac{d}{dw} \text{Tr} [w^{-(p+1)} e^{-\beta H_q}] \bigg|_{w=-1} \]

\[ = \text{Tr} [(F - (p+1))(-1)^{F - p} e^{-\beta H_q}] \]

\[ = (-1)^p \text{Tr} [\underbrace{F(-1)^{F - \beta H_q}}_{-Z_{d-1}(\beta, q)}] - (p+1)(-1)^p \text{Tr} [(-1)^{\beta H_q}] \bigg|_{w=-1} \]

\[ = Z_{d-1}(\beta, \frac{1}{2} - q) + (-1)^p Z_{d-1}(\beta, \frac{1}{2} - q) + (-1)^p (p+1) \text{ind}(D_{q-1/4}) \] (5.11)

The second identification in the last line in terms of the Dirac index is obvious form the discussion in section 3, while the first one is proved in appendix C, where it is shown that it is related to the analytic torsion of the complex manifold.

Putting all things together we obtain the following duality relation

\[ Z_p(\beta, q) = Z_{d-p-2}(\beta, \frac{1}{2} - q) + (-1)^p Z_{d-1}(\beta, \frac{1}{2} - q) + (-1)^p (p+1) \text{ind}(D_{q-1/4}) \] (5.12)

where we recall that the term due to a \((d-1,0)\)-form is purely topological and carries no degrees of freedom in \(d > 1\).

Having found the exact duality relation (5.12), we may try to check it on some examples. To do so we rewrite the Seeley-DeWitt coefficients (4.13) for gauge \((p,0)\)-forms in terms of the parameter \(q_1 = q - \frac{1}{4}\), since the duality relations are most apparent in terms of \(q_1\) rather than \(q\). As in the previous section we use the format \(A^p_q \rightarrow (v_1; v_2; v_3; v_4; v_5; v_6)\)
to present the coefficients; hence we have, for a gauge \((p, 0)\)-form in \(d > 3\)

\[
d > 3, \quad 0 \leq p \leq d - 2
\]

\[
A_p^{(q)} \to \left( \frac{d - 2}{p} \right) \times \left( 1; -\frac{1}{12} - q_1 \frac{d - 2 - 2p}{d - 2}; \frac{1}{180} - \frac{p(d - p - 2)}{24(d - 2)(d - 3)}; \right.
\]

\[
- \frac{19}{1440} q_1^2 + (1 + 48q_1^2) \frac{p(d - p - 2)}{24(d - 2)(d - 3)} - \frac{q_1^2}{12} \frac{d - 2 - 2p}{d - 2};
\]

\[
\frac{1}{288} + \frac{q_1^2}{2} - 2q_1^2 \frac{p(d - p - 2)}{(d - 2)(d - 3)} + \frac{q_1}{12} \frac{d - 2 - 2p}{d - 2};
\]

\[
\frac{1}{240} - \frac{q_1}{12} \frac{d - 2 - 2p}{d - 2} \right).
\]

By noticing that, under \(p \leftrightarrow d - p - 2\), the number \((d - 2 - 2p)\) goes into minus itself, it is immediate to see that \((5.13)\) is invariant under the simultaneous exchange of \(p \leftrightarrow d - p - 2\) and \(q_1 \leftrightarrow -q_1\), representing the duality between \(A_p^{(q)}\) and \(A_{d-p-2}^{(1/2-q)}\). The duality, as expected, does not show any topological mismatch up to order \(\beta^2\) in \(d > 3\).

On the other hand, the topological contributions are visible at order \(\beta^2\) for \(d \leq 3\). In \(d = 3\), the coefficients for the propagating 0 and 1-forms read, in terms of \(q_1\),

\[
d = 3, \quad p = 0, 1
\]

\[
A_p^{(q)} \to \left( 1; -\frac{1}{12} + q_1(2p - 1); \frac{1}{180} - \frac{p}{24}; \frac{19}{1440} - \frac{q_1^2}{6} + (1 + 48q_1^2) \frac{p}{24} + \frac{q_1}{12} (2p - 1); \right.
\]

\[
\frac{1}{288} + \frac{q_1^2}{2} - 2q_1^2 \frac{p}{(d - 2)(d - 3)} + \frac{q_1}{12} (2p - 1); - \frac{1}{240} + \frac{q_1}{12} (2p - 1) \right).
\]

(5.14)

As one can see they are not invariant under the exchange \(p \leftrightarrow 1 - p\) and \(q_1 \leftrightarrow -q_1\), the difference being due to topological terms. To check \((5.12)\), we compute the \(v_i\) coefficients for the topological \(A_2\) form

\[
d = 3, \quad p = 2, \quad A_2^{(q)} \to \left( 0; 0; \frac{1}{24}; -\frac{1}{24} - 2q_1^2; 2q_1^2; 0 \right)
\]

(5.15)

and can verify successfully, up to order \(\beta^2\), the validity of the \(d = 3\) relation

\[
Z_0(\beta, q) = Z_1(\beta, \frac{1}{2} - q) + Z_2(\beta, \frac{1}{2} - q) + \text{ind}(\mathcal{D}_{q-1/4})
\]

(5.16)

as the Dirac index contributes only at order \(\beta^3\) (and gives a \(\beta\)-independent term when inserted in eq. \((4.12)\)).

A second nontrivial check of our duality relations may be obtained in two complex dimensions, where the zero form is almost selfdual

\[
Z_0(\beta, q) = Z_0(\beta, \frac{1}{2} - q) + Z_1(\beta, \frac{1}{2} - q) + \text{ind}(\mathcal{D}_{q-1/4})
\]

(5.17)
This relation can be successfully verified by using the scalar field coefficients, that can be computed directly in $d = 2$ from the general result (4.11), and seen to agree with those obtained by setting $p = 0$ in (5.14),

$$d = 2, \quad p = 0$$

$$A_0^{(q)} \rightarrow \left(1; -\frac{1}{12} - q_1; \frac{1}{180}; -\frac{19}{1440} - \frac{q_1^2}{6} - \frac{q_1}{12}; \frac{1}{288} + \frac{q_1^2}{2} + \frac{q_1}{12}; \frac{1}{240} - \frac{q_1}{12}\right) \quad (5.18)$$

together with the non propagating $A_1$ form that produces the coefficients

$$d = 2, \quad p = 1, \quad A_1^{(q)} \rightarrow \left(0; 2q_1; -\frac{1}{24}; \frac{1}{24} + 2q_1^2 + \frac{q_1}{6}; -2q_1^3 - \frac{q_1}{6}; \frac{q_1}{6}\right) \quad (5.19)$$

and the twisted Dirac index of section 3 that gives

$$d = 2, \quad \text{ind}(D_{q-1/4}) \rightarrow \left(0; 0; -\frac{1}{24}; \frac{1}{24} + 2q_1^2; -2q_1^3; 0\right). \quad (5.20)$$

Finally, we may have a look also at the somewhat degenerate case of $d = 1$. Considering that the model at $p = -1$ is empty, the duality relation for $p = 0$ collapses to

$$Z_0(\beta, q) - Z_0(\beta, \frac{1}{2} - q) = \text{ind}(D_{q-1/4}) \quad (5.21)$$

that is indeed verified, after taking care of the $d = 1$ relation between the Ricci tensor and the scalar curvature, and considering that the integral of a total derivative term may be dropped. Note that, for $d = 1$, the $p = 0$ form is not topological, but carries one degree of freedom. This is consistent with the results in appendix C.

### 6 Conclusions

We have described the quantum theory of massless $(p, 0)$-form gauge fields, as well as massless $(p, 0)$-form fields without gauge symmetries, using a worldline approach. The worldline description uses a supersymmetric nonlinear sigma model, whose backbone is the basis for proving index theorems on complex manifolds [16, 17] with the physical methods of supersymmetric quantum mechanics [18, 27, 28]. As in that case the physical motivations for studying such models are rather indirect, as a direct spacetime interpretation is prevented by the complex nature of the target space which allows only an even number of time directions. Nevertheless complex manifolds find many useful applications in the context of string theory and/or supersymmetric theories. From a different perspective they offer a useful playground to test methods and ideas of quantum field theory, such as the worldline approach to theories in a curved background [29]. In particular, we have studied the effective action of massless $(p, 0)$-forms on curved Kähler manifolds, and discovered exact duality relations. The calculation of several heat kernel coefficients has been presented as well.

As possible extensions of the present work one might push the calculation of the heat kernel coefficients up to order $\beta^3$, dressing up the bosonic calculation of [30] with fermionic contributions, or study the duality relations on spaces with nontrivial topology. Also, it could be interesting to use similar methods to study the quantum theory of $(p, q)$-forms as well as the higher spin gauge fields introduced in [9] on a class of complex manifolds.
### A Notations and conventions

Kähler manifolds can be seen as a subclass of Riemannian manifolds with additional structures. We list here the conventions employed and some useful formulas for Kähler geometry, indicating occasionally their rewriting in real coordinates, as used in Riemannian geometry.

A metric is specified by
\[
 ds^2 = G_{MN} dX^M dX^N = 2g_{\mu\bar{\nu}} dx^\mu d\bar{x}^\bar{\nu} \tag{A.1}
\]
and the integration measure for manifolds of real dimension \( D = 2d \) is given by
\[
 d\mu = \sqrt{\text{det} G_{MN}} d^D X = \det g_{\mu\bar{\nu}} d^d x d^d \bar{x} \tag{A.2}
\]
with the notation
\[
 d^d x d^d \bar{x} \equiv i^d \prod_{\mu=1}^d dx^\mu \wedge d\bar{x}^\bar{\mu} . \tag{A.3}
\]
For simplicity we also use the notation \( g \equiv \det g_{\mu\bar{\nu}} \). On flat manifolds one may use cartesian coordinates for which \( G_{MN} = \delta_{MN} \) and \( g_{\mu\bar{\nu}} = \delta_{\mu\bar{\nu}} \). One can relate real and complex coordinates by
\[
 x^\mu = \frac{1}{\sqrt{2}} (X^{2\mu-1} + iX^{2\mu}) , \quad \bar{x}^\bar{\mu} = \frac{1}{\sqrt{2}} (X^{2\mu-1} - iX^{2\mu}) , \quad \mu = 1, \ldots, d \tag{A.4}
\]
though other choices are also possible, of course.

We now list our conventions for connections and curvatures on Kähler spaces. In holomorphic coordinates the non-vanishing Christoffel symbols are given, in terms of the metric, by
\[
 \Gamma^\mu_{\nu\lambda} = g^{\mu\bar{\mu}} \partial_\nu g_{\lambda\bar{\mu}} , \quad \bar{\Gamma}^{\bar{\mu}}_{\bar{\nu}\bar{\lambda}} = g^{\mu\bar{\mu}} \partial_{\bar{\nu}} g_{\lambda\mu} , \tag{A.5}
\]
and we shall denote their traces as
\[
 \Gamma_{\mu} \equiv \Gamma^\nu_{\mu\nu} = \partial_\mu \ln g , \quad \bar{\Gamma}_{\bar{\mu}} \equiv \bar{\Gamma}^{\bar{\nu}}_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\nu}} \ln g . \tag{A.6}
\]
The non-zero components of the Riemann curvature read
\[
 R^\mu_{\nu\sigma\lambda} = \partial_\sigma \Gamma^\mu_{\nu\lambda} , \quad R^{\bar{\mu}}_{\bar{\nu}\sigma\lambda} = \partial_\sigma \bar{\Gamma}^{\bar{\mu}}_{\bar{\nu}\bar{\lambda}} , \tag{A.7}
\]
while the Ricci tensor and the curvature scalar can be expressed as
\[
 R_{\mu\bar{\nu}} = - R^\lambda_{\lambda\nu\mu} = - \partial_\mu \bar{\Gamma}_{\bar{\nu}} = - \partial_{\bar{\nu}} \Gamma_\mu = - \partial_\mu \partial_{\bar{\nu}} \ln g , \quad R = g^{\mu\bar{\nu}} R_{\mu\bar{\nu}} . \tag{A.8}
\]
With our conventions, common in complex geometry, the curvature scalar is one half of the usual riemannian one: \( R = \frac{1}{2} R_{(G)} \equiv \frac{1}{2} R_{MM} \).

Let us now introduce vielbeins and spin connections, that are not used in the main text but are employed in appendix B to study the Dirac operator. In holomorphic coordinates
the vielbein $e_M^A$ splits as $(e_\mu^a, e_{\bar{\mu}}^\bar{a})$. The metric is given by $g_{\mu\bar{\nu}} = e_\mu^a e_{\bar{\nu}}^\bar{a} \delta_{ab}$. We denote the vielbein determinants by

$$\det(e_\mu^a) \equiv e, \quad \det(e_{\bar{\mu}}^{\bar{a}}) \equiv \bar{e},$$

so that $g = e\bar{e}$. Imposing the vielbein postulate $\nabla_M e_N^A = 0$ we find for the $U(d)$ spin connection

$$\omega_{\mu ab} = -e_\ell^b \partial_\mu e_{\rho a}, \quad \omega_{\bar{\mu} \bar{a}\bar{b}} = e_{\ell}^a \partial_{\bar{\mu}} e_{\rho \bar{b}},$$

while for its $U(1)$ parts we get

$$\omega_\mu \equiv \omega_{\mu ab} \delta^{ab} = -\partial_\mu \ln e, \quad \bar{\omega}_{\bar{\mu}} \equiv \omega_{\mu \bar{a} \bar{b}} \delta^{\bar{a}\bar{b}} = \partial_{\bar{\mu}} \ln e.$$  

The Christoffel symbols are related to the spin connection via

$$\Gamma^\lambda_{\nu\lambda} = e^\lambda_a \left( \partial_\mu e^a_\nu + \omega_{\mu ab} e^b_\nu \right),$$

$$\Gamma_\mu = -2\omega_\mu + \partial_\mu \ln e, \quad \bar{\Gamma}_{\bar{\mu}} = 2\bar{\omega}_{\bar{\mu}} - \partial_{\bar{\mu}} \ln e.$$  

Finally, in order to easily compare the Seeley-DeWitt coefficients computed in the present paper with the literature, we list the quadratic terms in curvatures as they appear in riemannian or Kähler notations

$$R(G) \equiv g^{MN} R_{MN} = 2R, \quad R_{MN} R^{MN} = 2R_{\mu\bar{\nu}} R^{\mu\bar{\nu}}, \quad R_{MNRS} R^{MNRS} = 4R_{\mu\bar{\nu}\rho\bar{\sigma}} R^{\mu\bar{\nu}\rho\bar{\sigma}}.$$  

### B  Dirac operator on Kähler manifolds

On Kähler manifolds the space of Dirac spinors is equivalent to the space of $(p, 0)$-forms with any allowed $p$, see for example [31]. Here we review this decomposition and study the Dirac operator.

On real manifolds admitting spinors it is natural to define the Dirac equation using the spin connections $\omega_M^{AB}$, which is the $SO(D)$ connection that keeps the vielbein $e_M^A$ covariantly constant

$$\nabla_M e_N^A = \partial_M e_N^A - \Gamma^L_{MN} e^A_L + \omega_M^{AB} e_{NB} = 0.$$  

The Dirac operator $\slashed{D}$ is defined using the Dirac gamma matrices $\gamma^A$, which satisfy the usual Clifford algebra $\{\gamma^A, \gamma^B\} = 2\eta^{AB}$,

$$\slashed{D} = \gamma^A e_A^M D_M = \gamma^A e_A^M \left( \partial_M + \frac{1}{4} \omega_{MBC} \gamma^B \gamma^C \right).$$  

On Kähler manifolds one may use complex coordinates, so that curved indices split as $M \to (\mu, \bar{\mu})$, and similarly flat indices $A \to (a, \bar{a})$. Thus the Dirac operator splits as

$$\slashed{D} = \gamma^\mu D_\mu + \gamma^{\bar{\mu}} D_{\bar{\mu}}.$$
where $\gamma^\mu = e_a^\mu \gamma^a$, $\gamma^{\bar{\mu}} = e_a^{\bar{\mu}} \gamma^a$, and the covariant derivatives as

$$
D_\mu = \partial_\mu + \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b - \frac{1}{2} \omega^{\mu b} \delta^{ab}, \quad \omega_\mu \equiv \omega_{\mu ab} \delta^{ab}
$$

(B.4)

which shows how a precise coupling to the U(1) part of the spin connection emerges by reducing the $SO(2d)$ connection to the $U(d)$ connection of Kähler manifolds. To compare with the main text it is useful to rewrite these formulas using the spinor variables $\psi$’s with flat tangent space indices. They are related to the gamma matrices by

$$
\begin{aligned}
\bar{\phi}^a &= \frac{\sqrt{2}}{2} \psi^a, \\
\bar{\psi}^a &= \frac{\sqrt{2}}{2} \psi^a.
\end{aligned}
$$

Then the covariant derivatives take the form

$$
D_\mu = \partial_\mu + \frac{1}{2} \omega_{\mu ab} \psi^a \bar{\psi}^b - \frac{1}{2} \omega^{\mu b} \bar{\psi}^b, \quad D_{\bar{\mu}} = \partial_{\bar{\mu}} + \omega_{\bar{\mu} ab} \psi^a \bar{\psi}^b - \frac{1}{2} \omega^{\bar{\mu} b} \bar{\psi}^b. 
$$

(B.5)

Let us now review the construction of the spinor space, i.e. the representation space of the gamma matrices. Using the spinor variables which satisfy

$$
\{ \psi^a, \bar{\psi}_b \} = \delta^a_b
$$

one may construct the fermionic Fock space, using $\psi^a$ as creation and $\bar{\psi}_a$ as destruction operators. Thus, just as in the expansion of eq. (2.6), a generic spinor takes the form

$$
\phi(x, \bar{x}, \psi) = F(x, \bar{x}) + F_a(x, \bar{x}) \psi^a + \frac{1}{2} F_{a1a2}(x, \bar{x}) \psi^a \psi^{a2} + \ldots + \frac{1}{d!} F_{a1 \ldots a_d}(x, \bar{x}) \psi^{a1} \ldots \psi^{a_d}. 
$$

(B.6)

This shows that locally a spinor field is equivalent to the complete set of $(p, 0)$-forms.

The operators $\psi^a D_\mu$ and $\bar{\psi}^b D_{\bar{\mu}}$, obviously related to those appearing in (B.3), act on these forms as Dolbeault operators twisted by the U(1) part of the spin connection. In fact, using the vielbein to convert to tensors with curved indices one finds

$$
\psi^a D_\mu \phi(x, \bar{x}, \psi) = \left( \partial_\mu - \frac{1}{2} \omega_\mu \right) F \psi^a + \frac{1}{2} \left( \partial_\mu - \frac{1}{2} \omega_\mu \right) F_\nu - \left( \partial_\nu - \frac{1}{2} \omega_\nu \right) F_\mu \psi^a \psi^\nu + \ldots 
$$

(B.7)

and

$$
\bar{\psi}^b D_{\bar{\mu}} \phi(x, \bar{x}, \psi) = \left( g^{\mu \bar{\rho}} \left( \partial_{\bar{\rho}} - \frac{1}{2} \omega_{\bar{\rho}} \right) F_{\mu} \right) \psi^a + \left( g^{\mu \bar{\rho}} \left( \partial_{\bar{\rho}} - \frac{1}{2} \omega_{\bar{\rho}} \right) F_{\mu \lambda} \right) \psi^\lambda + \ldots 
$$

(B.8)

which contain a precise U(1) charge. Considering that in our conventions

$$
\begin{aligned}
\omega_\mu &= -\partial_\mu \ln \bar{e}, \quad \omega_{\bar{\mu}} = \partial_{\bar{\mu}} \ln e, \quad \Gamma_\mu = \partial_\mu \ln g, \quad \bar{\Gamma}_{\bar{\mu}} = \partial_{\bar{\mu}} \ln g
\end{aligned}
$$

(B.9)

with $g = \det g_{\mu \bar{\nu}}$, $e = \det e^a_{\mu}$, and $\bar{e} = \det e^{\bar{a}}_{\bar{\mu}}$, so that $g = e \bar{e}$, one finds

$$
\Gamma_\mu = \partial_\mu \ln g = \partial_\mu \ln e + \partial_\mu \ln \bar{e} = -2\omega_\mu + \partial_\mu \ln \frac{e}{\bar{e}}
$$

(B.10)
together with its complex conjugate expression $\bar{\Gamma}_\mu = 2\bar{\omega}_\mu + \partial_\mu \ln \frac{e}{\bar{e}}$. These formulas allow to switch to the Christoffel connection and obtain

\[
\psi^\mu D_\mu \phi(x, \bar{x}, \psi) = \left( \left( \frac{e}{\bar{e}} \right)^{1/2} \left( \partial_\mu + \frac{1}{4} \Gamma_\mu \right) \left( \frac{\bar{e}}{e} \right)^{1/2} F_\mu \right) \psi^\mu + \frac{1}{2} \left( \left( \frac{e}{\bar{e}} \right)^{1/2} \left( \partial_\mu + \frac{1}{4} \Gamma_\mu \right) \left( \frac{\bar{e}}{e} \right)^{1/2} F_\nu - \left( \frac{e}{\bar{e}} \right)^{1/2} \left( \partial_\nu + \frac{1}{4} \Gamma_\nu \right) \left( \frac{\bar{e}}{e} \right)^{1/2} F_\mu \right) \psi^\mu \psi^\nu + \cdots \tag{B.11}
\]

and

\[
\bar{\psi}^\dagger D_\dagger \bar{\phi}(x, \bar{x}, \bar{\psi}) = \left( \left( \frac{\bar{e}}{e} \right)^{1/2} \left( \partial_\dagger + \frac{1}{4} \bar{\Gamma}_\dagger \right) \left( \frac{e}{\bar{e}} \right)^{1/2} F_\dagger \right) \bar{\psi}^\dagger + \frac{1}{2} \left( \left( \frac{\bar{e}}{e} \right)^{1/2} \left( \partial_\dagger + \frac{1}{4} \bar{\Gamma}_\dagger \right) \left( \frac{e}{\bar{e}} \right)^{1/2} F_\dagger - \left( \frac{\bar{e}}{e} \right)^{1/2} \left( \partial_\mu + \frac{1}{4} \bar{\Gamma}_\mu \right) \left( \frac{e}{\bar{e}} \right)^{1/2} F_\dagger \right) \bar{\psi}^\dagger + \cdots . \tag{B.12}
\]

The $U(1)$ phase $(\frac{\bar{e}}{e})^{1/2}$ can be locally eliminated by redefining the fields (or choosing a Lorentz gauge for which $e = \bar{e}$), so that one may use tensor fields with curved indices and Christoffel connections only. This proves that the Dirac operator is related to the twisted Dolbeault operators with Christoffel connections only. This proves that the Dirac operator is related to the twisted Dolbeault operators with $U(1)$ charge $q = \frac{1}{4}$, as used in the main text. This assertion is certainly true locally, i.e. in a coordinate patch. As we do not address topological issues, apart form the use of topological densities as found in the duality relations, this suffices for the purposes of the present paper.

We end this appendix by reporting the $U(1)$ charges of the chiral epsilon tensors that arise when splitting the volume form in chiral components using the vielbein

\[
g \epsilon_{\mu_1 \ldots \mu_d \nu_1 \ldots \nu_d} = \epsilon \epsilon_{\mu_1 \ldots \mu_d} \bar{\epsilon} \epsilon_{\nu_1 \ldots \nu_d} . \tag{B.13}
\]

Passing to flat tangent space indices, one may compute their covariant derivative, which includes the spin connection, and check that they satisfy

\[
\nabla_\mu \epsilon_{a_1 \ldots a_d} = \omega_\mu \epsilon_{a_1 \ldots a_d} , \quad \nabla_\bar{\mu} \bar{\epsilon}_{a_1 \ldots a_d} = \bar{\omega}_{\bar{\mu}} \bar{\epsilon}_{a_1 \ldots a_d} \\
\nabla_\mu \bar{\epsilon}_{\bar{a}_1 \ldots \bar{a}_d} = -\omega_\mu \bar{\epsilon}_{\bar{a}_1 \ldots \bar{a}_d} , \quad \nabla_{\bar{\mu}} \epsilon_{\bar{a}_1 \ldots \bar{a}_d} = -\bar{\omega}_{\bar{\mu}} \epsilon_{\bar{a}_1 \ldots \bar{a}_d} \tag{B.14}
\]

as only the $U(1)$ subgroup of the $U(d)$ holonomy group does not leave the epsilon tensors invariant.

\section{Topological $(d - 1, 0)$-form and analytic torsion}

In order to find the effective action for the topological $(d - 1, 0)$-form in (5.11), it is useful to analyze the relations among the effective actions of gauge $(p, 0)$-forms and “non gauge” forms. As we have seen, they are produced by our spinning particle model with gauged or ungauged supersymmetry, respectively. In this appendix, we will denote with $Z^A_p(q)$ the effective action for a gauge $(p, 0)$-form with field strength $F_{p+1} = \partial_q A_p$, and we will refer to $Z^B_p(q)$ as to the effective action of a “non gauge” $(p, 0)$-form obeying $(\partial_q \partial^q + \partial^q \partial_q) B_p = 0$. We will extend these notations to the effective action densities as well.
In the computation of the Seeley-DeWitt coefficients of $Z^A_p(q)$ to all orders one encounters two kinds of modular integrals, which we shall denote $I_n(d, p)$ and $J_n(d, p)$

\[ I_n(d, p) = \int_0^{2\pi} \frac{d\phi}{2\pi} \left( 2 \cos \frac{\phi}{2} \right)^{d-2} e^{-i\phi} \left( \cos \frac{\phi}{2} \right)^{-2(n-1)} \]

\[ = 2^{2n-2} \int_{\gamma_-} \frac{dw}{2\pi i w} \frac{(w + 1)^{d-2n}}{w^{p+1-n}} \]

\[ J_n(d, p) = \int_0^{2\pi} \frac{d\phi}{2\pi} \left( 2 \cos \frac{\phi}{2} \right)^{d-2} e^{-i\phi} \left( \cos \frac{\phi}{2} \right)^{-2(n-1)} \left( i \tan \frac{\phi}{2} \right) \]

\[ = 2^{2n-2} \int_{\gamma_-} \frac{dw}{2\pi i w} \frac{(w + 1)^{d-2n-1}}{w^{p+1-n}} (w - 1) , \]

where $s = p + 1 - \frac{d}{2}$, $n \geq 1$, and $w = e^{i\phi}$ is the Wilson loop variable. Since the regulated contour $\gamma_-$ excludes the pole at $w = -1$, the integrals are easily computed by the residue at $w = 0$, so that

\[ I_n(d, p) = \left\{ \begin{array}{ll}
2^{2n-2} \frac{d^{p+1-n}}{dw^{p+1-n}} (1 + w)^{d-2n} |_{w=0} & p \geq n - 1 \\
0 & p < n - 1
\end{array} \right. \]

\[ J_n(d, p) = \left\{ \begin{array}{ll}
2^{2n-2} \frac{d^{p+1-n}}{dw^{p+1-n}} [(1 + w)^{d-2n-1} (w - 1)] |_{w=0} & p \geq n - 1 \\
0 & p < n - 1
\end{array} \right. \]

If one wants, instead, to compute the effective action $Z^B_p(q)$ for the ungauged model, the very same heat kernel coefficients will be multiplied by different modular integrals $\tilde{I}_n(d, p)$ and $\tilde{J}_n(d, p)$, that differ from (C.1) by the replacement $(2 \cos \frac{\phi}{2})^{d-2} \to (2 \cos \frac{\phi}{2})^d$ and $s = p + 1 - \frac{d}{2} \to s = p - \frac{d}{2}$. This gives the simple identification

\[ \tilde{I}_n(d, p) = 4I_{n-1}(d, p - 1) , \quad \tilde{J}_n(d, p) = 4J_{n-1}(d, p - 1) . \]

Following [1], we are now ready to prove the main result that will be used in deriving (5.12). For $p \geq n - 1$ we have:

\[ I_n(d, p) = \left( \frac{2^{2n-2}}{(p + 1 - n)!} \frac{d^{p+1-n}}{dw^{p+1-n}} (1 + w)^{d-2n} \right) |_{w=0} \]

\[ = \left( \frac{2^{2n-2}}{(p + 1 - n)!} \frac{d^{p+1-n}}{dw^{p+1-n}} [(1 + w)^{d-2n+2} (1 + w)^{-2}] \right) |_{w=0} \]

\[ = 2^{2n-2} \sum_{k=0}^{p+1-n} \left( \frac{\partial_{w}^{p+1-n-k} (1 + w)^{d-2n+2} (1 + w)^{-2}}{(p + 1 - n - k)!} \right) |_{w=0} \]

\[ = \sum_{k=0}^{p+1-n} 4I_{n-1}(d, p - 1 - k)(-1)^k(k + 1) . \]

Since $I_{n-1}(d, p - 1 - k)$ is zero for $k > p + 1 - n$, we can extend the sum in the above formula up to $k = p$, and recalling that $\tilde{I}_n(d, p) = 4I_{n-1}(d, p - 1)$ one gets

\[ I_n(d, p) = \sum_{k=0}^{p} (-1)^k(k + 1)\tilde{I}_n(d, p - k) . \]
It is straightforward to see that an analogous formula holds as well for $J_n(d, p)$. Hence, since it holds order by order for every modular integral, we can conclude that it is valid for the whole effective actions, namely

$$Z^A_p(q) = \sum_{k=0}^p (-1)^k (k + 1) Z^B_{p-k}(q).$$  \hspace{1cm} (C.5)$$

We have now all the ingredients to evaluate $\text{Tr}[-1^F F e^{-\beta H_q}]$ in (5.11)

$$\text{Tr}\left[(-1)^F F e^{-\beta H_q}\right] = \sum_{n=0}^d (-1)^n n t_n(\beta, q) = \sum_{n=1}^d (-1)^n n t_n(\beta, q)$$

$$= - \sum_{n=0}^{d-1} (-1)^n (n + 1) t_{n+1}(\beta, q)$$

$$= - \sum_{n=0}^{d-1} (-1)^n (n + 1) t_{d-1-n}(\beta, \frac{1}{2} - q)$$

$$= - \sum_{n=0}^{d-1} (-1)^n (n + 1) Z^B_{d-1-n}(\beta, \frac{1}{2} - q) = - Z^A_{d-1}(\beta, \frac{1}{2} - q).$$ \hspace{1cm} (C.6)

To derive this relation we first shifted the summation variable $n$, then we used the duality for the ungauged model: $t_p(\beta, q) = t_{d-p}(\beta, \frac{1}{2} - q)$, and finally the relation (C.5) to recognize the effective action for the $(d-1, 0)$-form. For $d > 1$ this form does not carry any degree of freedom and it is purely topological. It can be related to the analytic torsion introduced in [32] for complex manifolds: exponentiating the effective action in (C.6) one obtains the product of determinants of the Dolbeault laplacians with the correct powers, as seen from the expressions in the first line of eq. (C.6), which defines the analytic torsion.
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