MODULAR KOSZUL DUALITY FOR SOERGEL BIMODULES

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ABSTRACT. We generalize the modular Koszul duality of Achar–Riche [AR16b] to the setting of Soergel bimodules associated to any finite Coxeter system. The key new tools are a functorial monodromy action and wall-crossing functors in the mixed modular derived category of ibid. In characteristic 0, this duality together with Soergel’s conjecture (proved by Elias–Williamson [EW14]) imply that our Soergel-theoretic graded category $O$ is Koszul self-dual, generalizing the result of Beilinson–Ginzburg–Soergel [Soe90, BGS96].

1. Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Fix a Borel subalgebra $\mathfrak{b}$ containing a Cartan subalgebra $\mathfrak{h}$, and consider the principal block $O_0$ of the associated BGG category $O$, i.e. the block containing the trivial representation. As is well known, $O_0$ is a finite-length abelian category with enough projectives. It therefore has a minimal projective generator $P$, so that $O_0$ is equivalent to the category of finitely-generated modules over

$$A := \text{End}_{O_0}(P)^{\text{opp}}.$$  

Beilinson–Ginzburg–Soergel [BGS96] showed that this ring admits a Koszul grading: a positive grading $A = \bigoplus_{i \geq 0} A_i$ with $A_0$ semisimple, and such that the left $A$-module $A_0 = A/A_{>0}$ admits a graded projective resolution $P^* \to A_0$ where $P^i$ is generated in degree $i$. The existence of a Koszul grading on the algebra controlling $O_0$ is a deep fact closely related to the Kazhdan–Lusztig conjecture.

For a Koszul ring $B = \bigoplus_{i \geq 0} B_i$, its Koszul dual ring is $E(B) := \text{Ext}^*_{B_{\text{mod}}}(B_0)$, where Ext is taken in ungraded $B$-modules. It was moreover shown in [Soe90, BGS96] that $A$ is Koszul self-dual: $A \cong E(A)$ as graded algebras. Thus the Koszul grading on $A$ reveals a hidden self-duality of $O_0$.

The Weyl group $W$ of $\mathfrak{g}$ acts on the Cartan subalgebra $\mathfrak{h}$. By work of Soergel, the algebra $A$ only depends on this $W$-representation. The theory of Soergel bimodules [Soe92, Soe00, Soe07] allows one more generally to define a graded algebra $A_{W,\mathfrak{h}}$ for any suitable “reflection faithful realization” $\mathfrak{h}$ of an arbitrary Coxeter system $W$. See §2 for background on Soergel bimodules as well as the definition of $A_{W,\mathfrak{h}}$.

For Soergel bimodules, one still has an analogue of the Kazhdan–Lusztig conjecture known as Soergel’s conjecture. Elias–Williamson suggested [EW16, Remark 3.4] that there should also be a rich Koszul duality in this generality. In this paper, we realize this vision for finite Coxeter systems.

The following theorem is a consequence of our main result together with the Soergel’s conjecture proved by Elias–Williamson [EW14].

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Theorem 1.1. For any finite (not necessarily crystallographic) Coxeter group $W$ and its geometric representation $h_{\text{geom}}$, the graded algebra $A_{W,h_{\text{geom}}}$ is Koszul self-dual.

1.1. Mixed modular derived category. The Koszul duality of [BGS96] is a derived equivalence involving $O_0^{\text{gr}}$ (“graded category $O_0$”), a graded version of $O_0$. As a first step towards Theorem 1.1, we defined in [Mak] a Soergel-theoretic analogue of $O_0^{\text{gr}}$ for a general pair $(W,h)$ using the mixed modular derived category formalism of [AR16b].

As explained in [BGS96], the grading on $O_0$ comes from mixed geometry; $O_0^{\text{gr}}$ may be identified with a certain category of mixed $\ell$-adic perverse sheaves on the flag variety defined over a finite field. In [AR16b], Achar–Riche introduced a new approach to defining the mixed category based on the parity complexes of [JMVL14]. This “mixed (modular) derived category” has a notion of weights and Tate twist, and for positive characteristic coefficients, may serve as a replacement for mixed $\ell$-adic sheaves. In particular, for a connected complex reductive group $G$ with a Borel subgroup $B$, [AR16b] defined and studied the categories of $B$-constructible “mixed complexes” and “mixed perverse sheaves” on the flag variety $G/B$ with coefficients in a field $k$:

$$D^\text{mix}_{(B)}(G/B, k) \supset P^\text{mix}_{(B)}(G/B, k).$$

Parity complexes on the flag variety are related to Soergel (bi)modules (see §2.3). Motivated by this connection, in [Mak] we adapted the Achar–Riche mixed derived category to the setting of Soergel bimodules associated to $(W,h)$, not necessarily arising from complex reductive groups; this is our analogue of $O_0^{\text{gr}}$. We will recall the facts we need from this framework in §2.5.

1.2. Modular Koszul duality. Geometrically, the Koszul duality of $O_0$ is a derived equivalence relating mixed sheaves on Langlands dual flag varieties. Achar–Riche [AR16b] proved an analogous equivalence for the mixed modular derived category, which they call “modular Koszul duality,” although it may not involve any Koszul algebra.

Our main result (Theorem 2.8) is an analogous result in the setting of Soergel theory for any finite Coxeter system. It should be thought of as a derived equivalence relating mixed modular sheaves on possibly non-existent Langlands dual flag varieties.

Our key new tools are a monodromy action and wall-crossing functors in our analogue of the mixed derived category. These tools allow us to imitate the strategy of the characteristic-zero Koszul duality of Bezrukavnikov–Yun [BY13]. In particular, even in the geometric setting of [AR16b], our approach gives a new proof of modular Koszul duality that is independent of [BY13] and of the Kazhdan–Lusztig conjecture.

In [AR13], Achar–Riche identified $P^\text{mix}_{(B)}(G/B, \mathbb{C})$ with the $O_0^{\text{gr}}$ of [BGS96]. Via this identification, our approach also gives a new proof of the classical Koszul duality of Soergel [BGS96].

Remark 1.2. In [Mak], we crucially used the Braden–MacPherson and Fiebig theory of moment graph sheaves. However, once the framework is set up, all results we

\footnote{This, however, relies on the Koszulity of $O_0^{\text{gr}}$, hence on some form of the Kazhdan–Lusztig conjecture.}
quote from ibid. are exact analogues of those of [AR16b], which in turn are analogues of well-known results in characteristic 0. The constructions we introduce are already new for $D^\text{mix}_B(G/B, \mathbb{C})$, and should be accessible to readers who prefer to think in this setting.

1.3. Related work. For finite dihedral groups, the Koszulity and Koszul self-duality of $A_{W, h_{\text{geom}}}$ was proved earlier by explicit methods by Sauerwein [Sau18].

After an early draft of this article had been written in 2015, the author learned of a project by Achar, Riche, and Williamson that contained constructions similar to those of this article. Our joint work [AMRW] (in particular) clarifies and extends the constructions in §4–5.

1.4. Contents. In §2, we recall some background on Soergel (bi)modules and the mixed modular derived category and state the main result. After some preliminaries in §3, we introduce the key new constructions in §4 (monodromy action) and §5 (wall-crossing functors). The main result is proved in §6.

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2. Background and main result

2.1. Background on the Hecke algebras. Let $(W, S)$ be a Coxeter system. We always tacitly assume that $|S| < \infty$. We denote by $e$ the identity element, $\ell: W \to \mathbb{Z}_{\geq 0}$ the length function, and $\leq$ the Bruhat order. An expression is a word $w = s_1 \cdots s_k$ in $S$. We write $w = s_1 \cdots s_k$ for the corresponding element in $W$.

We follow Soergel’s normalization for the Hecke algebra; see [Soe90] for details. The Hecke algebra $H_W$ is the algebra with free $\mathbb{Z}[v, v^{-1}]$-basis $\{H_w\}_{w \in W}$ and multiplication

$$H_w H_s = \begin{cases} H_{ws} & \text{if } ws > w, \\ (v^{-1} - v)H_w + H_{ws} & \text{if } ws < w. \end{cases}$$

This algebra has another basis, the Kazhdan–Lusztig basis $\{H_w\}_{w \in W}$. In our normalization, $H_e = H_e + v H_e$ for $s \in S$. For any expression $w = s_1 \cdots s_k$, we set $H_w := H_{s_1} \cdots H_{s_k}$. 
2.2. Background on Soergel (bi)modules. A realization of \((W,S)\) over a commutative ring \(k\) [Eli16, EW16] is a triple

\[
(h, \{\alpha_s^\vee\}_{s \in S} \subset h, \{\alpha_s\}_{s \in S} \subset h^*),
\]

where \(h\) is a finite-rank free \(k\)-module and \(h^\ast = \text{Hom}_k(h,k)\), such that: \(\langle \alpha_s^\vee, \alpha_s \rangle = 2\) for all \(s \in S\); the assignment \(s(v) := v - \langle v, \alpha_s \rangle \alpha_s^\vee\) for \(s \in S, v \in h\) defines a representation of \(W\) on \(h\); and a technical condition [EW16 (3.3)] that will always holds in our setting. We often simply speak of a realization \(h\).

We assume that \(k\) is a field of characteristic not equal to 2, and that the \(W\)-representation \(h\) is reflection faithful in the sense of [Soe07, Definition 1.5] \(\bullet\ h\) is faithful, and for all \(w \in W\), the fixed subspace \(h^w\) has codimension 1 if and only if \(w\) is a reflection in \(W\), i.e. a conjugate of an element of \(S\). These conditions guarantee that Soergel’s theory in [Soe07] is available.

We call the data \((W,h)\) a reflection faithful realization. This is the starting data for our categories.

Remark 2.1. Reflection faithfulness is a serious condition. Note that an infinite Coxeter system admits no faithful representation over a finite field or its algebraic closure. For an irreducible finite Coxeter system, the only reflection faithful realization \(h\) is \(\mathbb{C}\). For an irreducible finite Coxeter system, the only reflection faithful representations over \(\mathbb{R}\) are Galois conjugates of the geometric representation (see [MM10, Theorem 1.2] for a more general statement).

Example 2.2. Let \(G\) be a connected reductive group, and choose a Borel subgroup \(B\) containing a maximal torus \(T\). The associated Weyl group \(W\) has the natural structure of a Coxeter system. Let \(X_\ast(T)\) be group of cocharacters of \(T\). Then for any field \(k\), the base change \(k \otimes \mathbb{Z} \otimes X_\ast(T)\) has the natural structure of a realization of \(W\). In this way, the triple \((G,B,T)\) gives rise to a realization \((W,h)\), called the Cartan realization, over any field \(k\). Libedinsky shows in [Lab15, Appendix] that the Cartan realization is reflection faithful if \(\text{char } k \notin \{2,3\}\).

Let \(R = \text{Sym}_k(h^\ast)\) be the symmetric algebra, viewed as a \((\mathbb{Z})\)-graded algebra with \(\text{deg } h^\ast = 2\). Let \(R\text{-gmod-R}\) be the category of \((\mathbb{Z})\)-graded \(R\)-bimodules and graded \(R\)-bimodule homomorphisms (of degree 0). This category has a grading shift autoequivalence \(\{1\}\), defined on \(M = \bigoplus_{i \in \mathbb{Z}} M_i\) by \((M\{n\})_i = M_{n+i}\). It is also naturally monoidal with the product \(\otimes_R\), which we often omit from the notation.

The \(W\)-action on \(h\) induces a \(W\)-action on \(R\). For \(s \in S\), let \(R^s\) denote the ring of \(s\)-invariants in \(R\). Define the graded \(R\)-bimodule

\[
B_s := R \otimes_{R^s} R\{1\}
\]

and the endofunctor

\[
\theta_s = B_s \otimes_R (-) : R\text{-gmod-R} \to R\text{-gmod-R}.
\]

For any expression \(w = s_1 \cdots s_k\), define the Bott-Samelson bimodule

\[
B_w := \theta_{s_1} \cdots \theta_{s_k}(R) = B_{s_1} \cdots B_{s_k},
\]

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\(^2\)Soergel assumed in addition that \(k\) is infinite. However, as has been noted for example in [Rel §1.3], this assumption is not necessary and was only imposed in [Soe07] in order to identify \(R = \text{Sym}_k(h^\ast)\) with the ring of regular functions on \(h\).
where $R$ is the regular $R$-bimodule. (Note that $B_\emptyset = R$ for the empty expression $\emptyset$.) Let $SBim_{BS}(W, h)$ be the smallest strictly full subcategory of $R$-$gmod$-$R$ containing all Bott–Samelson bimodules and closed under $\oplus$ and $\{n\}$. Let $SBim(W, h)$ be the Karoubi envelope of $SBim_{BS}(W, h)$, which we always identify with a strictly full subcategory of $R$-$gmod$-$R$:

\begin{equation}
(2.1) \quad SBim_{BS}(W, h) \subset SBim(W, h) \subset R$-$gmod$-$R$.
\end{equation}

Each of these categories is monoidal under $\otimes_R$ with unit object $R$. The objects of $SBim(W, h)$ are called Soergel bimodules. Soergel [Soe07] proved the following classification of the indecomposable objects.

**Proposition 2.3.** For each $w \in W$, there is an object $B_w \in SBim(W, h)$, characterized up to isomorphism by the following property: for any reduced expression $\underline{w}$ of $w$, $B_{\underline{w}}$ is the unique indecomposable direct summand of $B_w$ that does not occur as a direct summand of $B_{\underline{x}}$ for any expression $\underline{x}$ with $l(\underline{x}) < l(\underline{w})$.

The set $\{B_w\}_{w \in W}$ is a complete list of isomorphism classes of indecomposable Soergel bimodules up to shift. Every object of $SBim(W, h)$ is isomorphic to a finite direct sum of shifts of various $B_w$, and such a decomposition is unique in the obvious sense.

The split Grothendieck group $[SBim(W, h)]$ of $SBim(W, h)$ is a $\mathbb{Z}[v, v^{-1}]$-algebra under $v[M] = [M\{1\}]$ and $[M][N] = [M \otimes_R N]$. Soergel’s categorification theorem [Soe07] states that the assignment $h_s \mapsto [B_s]$ for $s \in S$ determines a $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism $\mathcal{H}_W \simeq [SBim(W, h)]$. Let $\text{ch} : [SBim(W, h)] \xrightarrow{\sim} \mathcal{H}_W$ denote the inverse isomorphism.

**Definition 2.4.** We say that the realization $(W, h)$ satisfies *Soergel’s conjecture* \(^3\) if $\text{ch}([B_w]) = H_w$ for all $w \in W$.

This is a Soergel-theoretic analogue of the Kazhdan–Lusztig conjecture.

Let $gmod$-$R$ denote the category of graded right $R$-modules and graded $R$-module homomorphisms, and let $\{1\}$ denote as before the grading shift down. Let $\mathfrak{m} \subset R$ denote the augmentation ideal, i.e. the graded ideal of $R$ generated by $\mathfrak{h}^*$. Consider the functor

$$R$-$gmod$-$R \rightarrow gmod$-$R : M \mapsto k \otimes_R M,$$

where $k := R/\mathfrak{m}$. That is, $k \otimes_R M$ is obtained from $M$ by killing the image of positive degree elements of $R$ acting on the left. Now, define the categories

$SBim_{BS}(W, h) \subset SBim(W, h) \subset gmod$-$R$.

The objects of $SBim(W, h)$ are called (right) *Soergel modules*. The modules $\overline{B}_w := k \otimes_R B_w$ remain indecomposable (as follows from Proposition 3.2 below) and pairwise distinct. Thus $\overline{B}_w$ is again characterized as the “largest” direct summand of a Bott–Samelson module $\overline{B}_w := k \otimes_R B_w$, and we have a classification theorem entirely analogous to the case of bimodules. We similarly define Bott–Samelson and Soergel modules in $R$-$gmod$ by reducing the right $R$-action.

We can now define the graded algebra $A_{W, h}$ from the introduction.

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\(^3\)Soergel only conjectured this result for specific realizations in characteristic 0.
Definition 2.5. Let $A_{W, \mathfrak{h}}$ be the graded endomorphism algebra

$$A_{W, \mathfrak{h}} := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{gmod-R}}(\mathfrak{k} \otimes_R B, k \otimes_R B \langle n \rangle),$$

where $B := \bigoplus_{w \in W} B_w \in \text{SBim}(W, \mathfrak{h})$.

2.3. Relation to parity sheaves. Parity complexes were introduced by Juteau–Mautner–Williamson [JMW14]. In this subsection, we recall some well-known results (essentially due to Soergel [Soe00]) relating parity complexes on flag varieties and Soergel (bi)modules. Since we never actually use parity complexes, we will be brief; their importance for us is purely as motivation.

Let $G$ be a connected complex reductive group. Choose a Borel subgroup $B$ containing a maximal torus $T$. Then $B$ act by left multiplication on the flag variety $G/B$; the $B$-orbits are the Schubert cells $X_w$, $w \in W$, where $W$ is the Weyl group. Let $k$ be a field, and let $D_B^b(G/B, k)$ (resp. $D_{(B)}^b(G/B, k)$) denote the $B$-equivariant (resp. $B$-constructible) derived category of sheaves of $k$-vector spaces on $G/B$. Denote by $\{1\}$ the cohomological shift (usually denoted by $[1]$).

Let $\text{Parity}_B(G/B, k) \subset D_B^b(G/B, k)$ be the full additive subcategory of $B$-equivariant parity complexes. This category is stable under $\{1\}$, and indecomposable objects up to shift and isomorphism are indexed by $W$. More specifically, for $w \in W$, the indecomposable parity complex $\mathcal{E}_w$, called parity sheaf, is characterized by having support $X_w$ and the normalization $\mathcal{E}_w|_{X_w} \cong k_{X_w}(\ell(w))$.

The category $D_B^b(G/B, k)$ is monoidal under $B$-convolution $*$ and unit object $\mathcal{E}_e = \delta$, the skyscraper at the point stratum $X_e$, and $\text{Parity}_B(G/B, k)$ is a monoidal subcategory. For any expression $w = s_1 \cdots s_k$, define the Bott–Samelson parity complex

$$\mathcal{E}_w := \mathcal{E}_{s_1} * \cdots * \mathcal{E}_{s_k}.$$  

(For the empty expression, $\mathcal{E}_\emptyset = \mathcal{E}_e = \delta$.) Then $\mathcal{E}_w$ also admits a “Bott–Samelson-type” characterization: for any reduced expression $\underline{w}$ of $w$, it is the unique direct summand of $\mathcal{E}_w$ that does not appear as a direct summand of $\mathcal{E}_{\underline{w}}$ for any expression $\underline{w}$ with $\ell(\underline{w}) < \ell(w)$.

Let $\text{Parity}_{BS}^B(G/B, k)$ be the smallest strictly full subcategory of $D_B^b(G/B, k)$ containing all Bott–Samelson complexes and closed under $\oplus$ and $\{1\}$. By the preceding discussion, its Karoubi envelope can be identified with $\text{Parity}_B(G/B, k)$. Thus we have categories

$$\text{Parity}_{BS}^B(G/B, k) \subset \text{Parity}_B(G/B, k) \subset D_B^b(G/B, k),$$

each monoidal under $*$ with unit object $\delta$.

Let

$$\text{Parity}_{BS}^B(B)(G/B, k) \subset \text{Parity}_B(B)(G/B, k) \subset D_{(B)}^b(G/B, k)$$

be the essential images of the categories in (2.2) under the forgetful functor

$$\text{For}: D_B^b(G/B, k) \rightarrow D_{(B)}^b(G/B, k).$$

Then $\text{Parity}_{BS}^B(B)(G/B, k)$ agrees with the category of $B$-constructible parity complexes on $G/B$. Each $\mathcal{E}_w := \text{For}(\mathcal{E}_w)$ remains indecomposable and again admits two characterizations: by a support condition and a normalization, and as the “largest” direct summand of the Bott–Samelson parity complex $\mathcal{E}_w := \text{For}(\mathcal{E}_w)$. We also have a classification theorem entirely analogous to the $B$-equivariant case.
The connection to Soergel (bi)modules is as follows. Let \((W, h)\) be the Cartan realization over \(k\) associated to \((G, B, T)\) (see Example 2.2). Via the Borel isomorphism \(H^\bullet_B(pt, k) \cong R = \text{Sym}_k(h^*)\), total hypercohomology can be viewed as functors
\[
\mathbb{H}_B^\bullet : D_B^b(G/B, k) \to R\text{-}\text{gmod}\text{-}R, \quad \mathbb{H}^\bullet : D_{B(B)}^b(G/B, k) \to \text{gmod}\text{-}R
\]
intertwining \(\{1\}\). If the characteristic of \(k\) is good for \(G\) and moreover not equal to 2, then it can be deduced from [AR16a, §4] that these functors restrict to equivalences
\[
(\text{Parity}_B(G/B, k), *) \simeq (\text{SBim}(W, h), \otimes_R), \quad \text{Parity}_{(B)}(G/B, k) \simeq \text{SBim}(W, h)
\]
sending \(E_w \mapsto B_w, E_w \mapsto B_w\) and \(E_w \mapsto B_w\), \(E_w \mapsto B_w\), and both intertwining \(\{1\}\). These equivalences intertwine \(\text{For}\) with \(k \otimes_R (-)\). Moreover, the first equivalence is monoidal and makes the second equivalence into an equivalence of right module categories.

2.4. Geometric notation for Soergel (bi)modules. Let \((W, h)\) be a reflection faithful realization (over a field \(k\)). Our point of view, motivated by the discussion in §2.3, is that a general \((W, h)\) should still be thought of as arising from a triple \((G, B, T)\), which however may not actually exist (e.g. if \(W\) is not crystallographic). Soergel and Bott–Samelson bimodules (resp. modules) are then \(B\)-equivariant (resp. \(B\)-constructible) parity complexes and Bott–Samelson parity complexes, with \(k\)-coefficients, on the possibly non-existent flag variety \(G/B\).

Accordingly, having fixed the realization \((W, h)\), we denote the category of Soergel bimodules \(\text{SBim}(W, h)\) by
\[
\text{Parity}(B \backslash G/B, k) \subset R\text{-}\text{gmod}\text{-}R,
\]
and the categories of right and left Soergel modules by
\[
\text{Parity}(U \backslash G/B, k) \subset \text{gmod}\text{-}R, \quad \text{Parity}(B \backslash G/U, k) \subset R\text{-}\text{gmod}.
\]
We will often omit the coefficients \(k\) from the notation. We use similar geometric notation for Bott–Samelson (bi)modules. Moreover, we sometimes write * instead of \(\otimes_R\) and speak of forgetful functors
\[
\text{For} = (-) \otimes_R k : \text{Parity}(B \backslash G/B, k) \to \text{Parity}(U \backslash G/B, k),
\]
\[
\text{For} = k \otimes_R (-) : \text{Parity}(B \backslash G/B, k) \to \text{Parity}(U \backslash G/B, k).
\]
We also sometimes write \(E_w\) to mean any of
\[
B_w \in \text{Parity}(B \backslash G/B), \quad k \otimes_R B_w \in \text{Parity}(U \backslash G/B), \quad B_w \otimes_R k \in \text{Parity}(B \backslash G/U),
\]
and write \(\delta\) for the “skyscraper” \(E_e\).

Remark 2.6. We stress that \(B \backslash G/B, U \backslash G/B, B \backslash G/U\) are purely notational device used to emphasize the analogy with geometry. However, the constructions introduced in §§4–5 of this paper also apply to actual parity complexes in place of Soergel (bi)modules. In that case, the proof of the main theorem in [AR16b] gives a new proof of the modular Koszul duality of [AR16a, Theorem 5.4].
2.5. Background on the mixed modular derived category. In this subsection, we recall the results of [Mak] adapting the formalism of [AR16b] to the setting of Soergel (bi)modules.

Let \((W, h)\) be a reflection faithful realization. We define the associated equivariant and constructible mixed derived category by

\[
D_{\text{mix}}(B \backslash G/B) := k^b \text{Parity}(B \backslash G/B), \quad D_{\text{mix}}(U \backslash G/B) := k^b \text{Parity}(U \backslash G/B).
\]

Each category has an induced internal grading shift \(\{1\}\) and a new cohomological shift \([1]\). Define the Tate twist \(\langle 1 \rangle = [1](-1)\). The forgetful functor induces an exact (i.e. triangulated) functor

\[
\text{For: } D_{\text{mix}}(B \backslash G/B) \to D_{\text{mix}}(U \backslash G/B).
\]

We think of these categories as the \(B\)-equivariant and \(B\)-constructible mixed derived categories of a possibly non-existent \(G/B\).

The Braden–MacPherson [BM01] and Fiebig [Fie08b, Fie08a] theory of moment graph sheaves allows us to take this point of view more seriously: it provides a notion of “strata” and “support,” so that indecomposable Soergel bimodules \(E_w\) (in the guise of so-called Braden–MacPherson sheaves) may be characterized by a support condition and a normalization analogous to those for parity sheaves.

In [Mak], we used this theory to define a recollement structure on \(D_{\text{mix}}(B \backslash G/B)\) and \(D_{\text{mix}}(U \backslash G/B)\), allowing one to speak of the “standard” and “costandard” sheaves

\[
\Delta_w := "i_w X_w \{t(w)\}, " \quad \nabla_w := "i_w X_w \{t(w)\}"
\]

for each \(w \in W\), where \(i_w\) is the “inclusion of the Schubert cell \(i_w : X_w \to G/B\).”

We also defined a “perverse” t-structure with heart\(^4\)

\[
P_{\text{mix}}(B \backslash G/B) \subset D_{\text{mix}}(B \backslash G/B), \quad P_{\text{mix}}(U \backslash G/B) \subset D_{\text{mix}}(U \backslash G/B),
\]

consisting of mixed perverse sheaves, each stable under Tate twist and having simple objects \(\{IC_w\}_{w \in W}\) up to Tate twist and isomorphism.

One of the main results in [Mak] is that the pair \((P_{\text{mix}}(U \backslash G/B), \{1\})\) has the natural structure of a graded highest weight category\(^5\) indexed by \((W, \leq)\) with standard (resp. costandard) objects \(\Delta_w\) (resp. \(\nabla_w\)). As in any graded highest weight category, one may then speak of the full additive subcategory

\[
\text{Tilt}_{\text{mix}}(U \backslash G/B) \subset P_{\text{mix}}(U \backslash G/B),
\]

stable under Tate twist, of tilting objects. The indecomposable tilting objects are \(\{T_w\}_{w \in W}\) up to Tate twist and isomorphism, where \(T_w\) is characterized by a support condition and a normalization (see [AR16b] Proposition A.4).

When the index set is finite, graded highest weight categories have enough projectives, and enough projectives in the additional presence of a duality functor. Thus for finite \(W\), \(P_{\text{mix}}(U \backslash G/B)\) contains the usual collection of objects

\[
\Delta_w, \quad \nabla_w, \quad IC_w, \quad P_w, \quad I_w, \quad T_w \quad \text{for } w \in W,
\]

where \(P_w\) (resp. \(I_w\)) denotes the projective cover (resp. injective hull) of \(IC_w\). The category \(P_{\text{mix}}(U \backslash G/B)\) is our Soergel-theoretic analogue of \(O^gr_{0}\). However,

\(^4\)In [Mak], these categories were denoted by \(P_{\text{mix}}(B) \subset D_{\text{mix}}(B)\) and \(P_{\text{mix}}(B) \subset D_{\text{mix}}(B)\), where \(B\) is the Bruhat moment graph associated to \((W, h)\).

\(^5\)The notion of a graded highest weight category is as in [AR16b] Definition A.1] (who instead use the term “graded quasihereditary”), except that we do not require the index set to be finite.
in general the parity objects $E_w$, viewed as a complex supported in cohomological degree 0, need not be perverse.

Consider the objects

$$\mathcal{P} := \bigoplus_{w \in W} \mathcal{P}_w, \quad \mathcal{E} := \bigoplus_{w \in W} \mathcal{E}_w$$

in $D^{\text{mix}}(U\backslash G/B)$, and define the graded algebras

$$A^{\text{proj}}_{W, h} := \left( \bigoplus_{n \in \mathbb{Z}} \text{Hom}(\mathcal{P}, \mathcal{P}(n)) \right)^{\text{opp}}, \quad A^{\text{parity}}_{W, h} := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(\mathcal{E}, \mathcal{E}(n)) = A_{W, h}.$$

Note that $A^{\text{proj}}_{W, h}$ is the graded algebra controlling $P^{\text{mix}}(U\backslash G/B)$, while $A_{W, h}$ agrees with the graded algebra from Definition 2.5. The following result was proved in [Mak] as a consequence of the fact that Soergel’s conjecture implies the isomorphism $E_w \cong IC_w$ for all $w \in W$.

**Proposition 2.7.** If $W$ is finite and $(W, h)$ satisfies Soergel’s conjecture, then $A^{\text{proj}}_{W, h}$ and $A^{\text{parity}}_{W, h}$ are Koszul, and Koszul dual to each other.

Given a realization $(h, \{a_s^\vee\}, \{a_s\})$, we have the dual realization $(h^*, \{\alpha_s\}, \{\alpha_s^\vee\})$. We say that $h$ is self-dual if it is isomorphic to $h^*$ as a realization in the obvious sense. Assume that $h^*$ is also reflection faithful. Let $R^\vee = \text{Sym}_k(h)$, graded with $\text{deg } h = 2$. Again motivated by geometry, we view the realization $(W, h^*)$ as arising from the triple $(G^\vee, B^\vee, T^\vee)$ Langlands dual to $(G, B, T)$. We again use the geometric notation of [2.4] for Soergel-theoretic notions associated to $(W, h^*)$. For instance,

$$\text{Parity}(B^\vee\backslash G^\vee/U^\vee) \subset R^\vee\text{-gmod}$$

denotes the category of left Soergel modules associated to $(W, h^*)$, with indecomposable objects $E_w$ “generated” by the endofunctors

$$\theta_s := B^\vee_s \otimes_{R^\vee} (-) : R^\vee\text{-gmod} \to R^\vee\text{-gmod}, \quad B_s^\vee := (R^\vee) \otimes_{(R^\vee)} R^\vee \{1\}, \quad s \in S$$

from $E_s^\vee$. Repeating the constructions of [Mak] but with left instead of and right Soergel modules, we obtain in particular the categories

$$\text{Tilt}^{\text{mix}}(B^\vee\backslash G^\vee/U^\vee) \subset P^{\text{mix}}(B^\vee\backslash G^\vee/U^\vee) \subset D^{\text{mix}}(B^\vee\backslash G^\vee/U^\vee)$$

and may speak of the objects $\Delta_w^\vee, \nabla_w^\vee, \mathcal{T}_w^\vee$ for $w \in W$.

2.6. **Statements.** Let $(W, S)$ be a finite Coxeter system, $k$ a field of characteristic not equal to 2, and $h$ a realization of $(W, S)$ over $k$. Assume that both $h$ and $h^*$ are reflection faithful, so that all categories of the preceding subsections are defined.

Our main result, to be proved in [6.3] is a Soergel-theoretic analogue of modular Koszul duality [ARI16b, Theorem 5.4].

**Theorem 2.8.** There exists a triangulated equivalence

$$\kappa : D^{\text{mix}}(U\backslash G/B, k) \to D^{\text{mix}}(B^\vee\backslash G^\vee/U^\vee, k)$$

satisfying $\kappa \circ [1] \cong [1] \circ \kappa$, $\kappa \circ [1] \cong [1] \circ \kappa$, $\kappa \circ [1] \cong [1] \circ \kappa$, and

$$\kappa(\Delta_w) \cong \Delta_w^\vee, \quad \kappa(\nabla_w) \cong \nabla_w^\vee, \quad \kappa(\mathcal{T}_w) \cong \mathcal{E}_w^\vee, \quad \kappa(\mathcal{E}_w) \cong \mathcal{T}_w^\vee.$$

We note the following immediate consequences. First, we obtain the following equivalence by composing $\kappa$ with the Ringel self-duality of $D^{\text{mix}}(U\backslash G/B, k)$ (proved in [Mak] by imitating [ARI16b Proposition 4.11]).
Corollary 2.9. There exists a triangulated equivalence 
\[ \kappa': D^{\text{mix}}(U \g B, k) \sim D^{\text{mix}}(U^\vee \g B^\vee, k) \]

satisfying \( \kappa' \circ [1] \cong [1] \circ \kappa' \), \( \kappa' \circ (1) \cong (1) \circ \kappa' \), and \( \kappa' \circ (1) \cong (1) \circ \kappa' \), and \( \kappa'(C) \) is graded.

From this equivalence and Proposition 2.7, we deduce the following statement about graded algebras.

Theorem 2.10. Suppose that \( W \) is finite and \((W, \mathfrak{h})\) satisfies Soergel’s conjecture. Then \( A_{W, h}^{\text{proj}} \) is Koszul, and \( E(A_{W, h}^{\text{proj}}) \cong A_{W, h}^{\text{proj}} \). In particular, if \( \mathfrak{h} \) is self-dual, then \( A_{W, h}^{\text{proj}} \) is Koszul self-dual.

The geometric representation \( h_{\text{geom}} \) over \( \mathbb{R} \) is self-dual. Moreover, Soergel’s conjecture for \( h_{\text{geom}} \) is a theorem for arbitrary Coxeter systems due to Elias–Williamson [EW14]. As a result, we obtain a uniform, purely algebraic proof of Theorem 1.1, the Koszul self-duality of \( A_{W, h_{\text{geom}}} \) for all finite \( W \).

2.7. Structure of the proof. Our proof of Theorem 2.8 follows an established pattern; we imitate in particular the proof of the “self-duality” in the work of Bezrukavnikov–Yun [BY13]. In short, the goal is to invent functors \( \xi_s \) for \( s \in S \) and \( V \) as in the diagram

\[
\begin{array}{ccc}
\xi_s \ & \ & \theta_s \\
\downarrow \ & \ & \downarrow \\
D^{\text{mix}}(U \g B) \ & \sim \ & \mathcal{R}^V -\mathrm{mod}
\end{array}
\]

satisfying the following properties: (1) \( \{\xi_s\}_{s \in S} \) “generate” \( \text{Tilt}^{\text{mix}}(U \g B) \) from the smallest tilting object \( T_e = \delta \) in the same way that the endofunctors \( \{\theta_s\}_{s \in S} \) “generate” \( \text{Parity}(B^\vee \g B^{\vee}/U^{\vee}) \) from the smallest parity object \( E_e = k \); (2) \( \forall (T_e) \cong E_e \); (3) \( \forall \circ \xi_s \cong \theta_s \circ \forall \) for all \( s \in S \). The monodromy action in \( D^{\text{mix}}(U \g B) \), constructed in [4], will play a key role in defining both \( \xi_s \) and \( \forall \).

3. Preliminaries

In this section, let \( k \) be a commutative ring. By “grading,” we always mean a \( \mathbb{Z} \)-grading.

3.1. Graded modules and graded categories. A graded \( k \)-linear additive category will mean for us a pair \((C, \{1\})\) consisting of a \( k \)-linear additive category \( C \) and an autoequivalence \( \{1\} \), called (grading) shift. For \( M, N \in C \), define the graded Hom

\[ \text{HOM}_C(M, N) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_C(M, N\{n\}), \]

a graded \( k \)-module. For \( L, M, N \in C \), the obvious induced composition

\[ (-) \circ (-) : \text{HOM}_C(M, N) \times \text{HOM}_C(L, M) \to \text{HOM}_C(L, N), \]

is graded \( k \)-bilinear. The notation \( f : M \to N \) will be reserved for an actual morphism of \( C \). Thus \( f : M \to N\{n\} \) denotes an element \( f \in \text{HOM}_C(M, N) \) of degree \( n \).

Let \( A \) be a graded \( k \)-algebra. We call a graded \( k \)-linear additive category \((C, \{1\})\) graded \( A \)-linear if its graded Homs are equipped with the structure of a graded \( A \)-module in such a way that composition is graded \( A \)-bilinear.
Example 3.1. Let $A$-$gmod$ denote the category of graded $A$-modules and graded (degree 0) $A$-module homomorphisms. For a graded $A$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$, define its grading shift $M(n) = M_{i+n}$. Then $(A$-$gmod, \{1\})$ is a graded $k$-linear additive category. If $A$ is moreover commutative, then $(A$-$gmod, \{1\})$ is graded $A$-linear.

An additive functor between graded $k$-linear (resp. graded $A$-linear) categories is called graded $k$-linear (resp. graded $A$-linear) if it intertwines the shifts on the nose and the induced maps of graded Hom are graded $k$-linear (resp. $A$-linear).

3.2. Further background on Soergel (bi)modules and mixed derived category. Let $(W, h)$ be a reflection faithful realization.

3.2.1. A little Soergel diagrammatics. Given $M, N \in R$-$gmod$-$R$, we write $\text{Hom}(M, N)$ for the space of degree 0 $R$-bimodule homomorphisms, and $\text{HOM}(M, N) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(M, N\{n\}) \in R$-$gmod$-$R$ for the graded Hom.

For each $s \in S$, define the following graded $R$-bimodule homomorphisms:
\[
\begin{align*}
\uparrow &: B_s \to R\{1\}: f \otimes g \mapsto f g \\
\downarrow &: R \to B_s\{1\}: f \mapsto f \left( \frac{\alpha_s}{2} \otimes 1 + 1 \otimes \frac{\alpha_s}{2} \right) \\
\downarrow \uparrow &= \downarrow \circ \uparrow &: B_s \to B_s\{2\}, \quad := \text{id}_{B_s}: B_s \to B_s
\end{align*}
\]
Each diagram is “$s$-colored,” but we will always have a fixed $s$ in mind.

These diagrams are borrowed from [EK10, Eli16, EW16], but we treat them as symbols like any other (rather than embedded graphs up to isotopy). However, the topology of the diagram reminds us whether the $R$-action can be moved from right to left: we have
\[
\begin{align*}
\uparrow f &= f \uparrow, & \downarrow f &= f \downarrow, & \downarrow \uparrow f &= f \downarrow \uparrow & \text{for all } f \in R, \\
\lambda &= s(\lambda), & \left( \alpha_s^y, \lambda \right) & \in \text{Hom}(B_s, B_s\{2\}) & \text{for all } \lambda \in h^*. 
\end{align*}
\]

3.2.2. Soergel Hom formula and equivariant formality. On the Hecke algebra $\mathcal{H}_W$ (see [2.1]), define the $\mathbb{Z}[v, v^{-1}]$-bilinear pairing
\[
\langle -, \rangle: \mathcal{H}_W \times \mathcal{H}_W \to \mathbb{Z}[v, v^{-1}]
\]
determined by $\langle H_x, H_y \rangle = \delta_{xy}$. The graded dimension of a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is
\[
gdim V := \sum_{n \in \mathbb{Z}} (\dim V_n) v^n \in \mathbb{Z}[v, v^{-1}].
\]

The following “equivariant formality” statement will play an important role throughout this paper.

Proposition 3.2. For any $M, N \in \text{Parity}(B\backslash G/B)$, the graded Hom $\text{HOM}(M, N)$ is graded free as a left $R$-module. If moreover $W$ is finite, then the natural map
\[
\mathbb{k} \otimes_R \text{HOM}_{R$-$gmod$-$R}(M, N) \to \text{HOM}_{\text{gmod}$-$R}(\mathbb{k} \otimes_R M, \mathbb{k} \otimes_R N)
\]
induced by the functor $\mathbb{k} \otimes_R (-): R$-$gmod$-$R \to \text{gmod}$-$R$ is an isomorphism.
Proof. The first statement is part of [Soe07, Theorem 5.15]. Soergel originally proved the second statement for the geometric representation of finite Weyl groups [Soe92, Theorem 2], and more recently for reflection faithful realizations (see [Ric, Proposition 1.13]). □

We will also need the Soergel Hom formula [Soe07, Theorem 5.15]: given expressions $x, y$, we have
\[ \text{gdim}(k \otimes_R \text{HOM}(B_x, B_y)) = \langle H_x, H_y \rangle. \]

For finite $W$, we deduce by Proposition 3.2 that
\[ (3.3) \quad \text{gdim} \text{HOM}_{R\text{-gmod}}(k \otimes B_x, k \otimes B_y) = \langle H_x, H_y \rangle. \]

3.2.3. Singular Soergel bimodules. The entire story in §2.5 generalizes to singular Soergel theory, or at least to the subregular case. Namely, for every $s \in S$, there are categories
\[ \text{Parity}(B \setminus G/P^s) = \text{SBim}^s(W, h) \subset R\text{-gmod}-R^s, \]
\[ \text{Parity}(U \setminus G/P^s) = \text{SBim}^s(W, h) \subset g\text{-mod}-R^s, \]
of singular Soergel (bi)modules [Wil11], with indecomposable objects indexed by the coset space $W/\{e, s\}$. The geometric notation here is motivated by the fact that, when $(W, h)$ arises from a complex reductive group $G$, these categories are related to parity complexes on minimal partial flag varieties $G/P^s$ in the same way as in §2.3.

As in the regular case, one can use moment graphs to define the categories
\[ P^{\text{mix}}(U \setminus G/P^s) \subset D^{\text{mix}}(U \setminus G/P^s), \quad P^{\text{mix}}(B \setminus G/P^s) \subset D^{\text{mix}}(B \setminus G/P^s). \]

Note that $D^{\text{mix}}(B \setminus G/P^s)$ is a left module category for $D^{\text{mix}}(B \setminus G/B)$ via $\ast$. There are also exact functors
\[ \pi^{\ast} : D^{\text{mix}}(B \setminus G/P^s) \to D^{\text{mix}}(B \setminus G/B), \quad \pi^s : D^{\text{mix}}(B \setminus G/B) \to D^{\text{mix}}(B \setminus G/P^s), \]
and their constructible versions. We noted in [Mak] that these constructions satisfy various properties one expects from geometry, with mostly the same proofs as in [AR16b].

3.3. Sign convention in homological algebra. This technical subsection can safely be skipped on a first reading.

Let $\mathcal{A}$ be an additive category. Denote by $\text{Ch}^b \mathcal{A}$ (resp. $K^b \mathcal{A}$) the category of bounded complexes in $\mathcal{A}$ (resp. bounded homotopy category). The usual convention in homological algebra defines a shift functor $\Sigma$ (usually denoted by $[1]$) introducing a sign in the differential, then defines a triangulated structure on the category with shift $(K^b \mathcal{A}, \Sigma)$.

For certain computations in §4.5 it will be more convenient to use a different shift $\Sigma_\ell$ introducing no sign; the cone will also receive a different sign. Proposition 3.3 below assures that this is an inessential choice of convention. We first recall the usual triangulated structure on $(K^b \mathcal{A}, \Sigma_\ell)$, being careful to note the dependence on $\Sigma_\ell$. The shift $\Sigma_\ell$ on $\text{Ch}^b \mathcal{A}$ and $K^b \mathcal{A}$ is defined by
\[ (\Sigma_\ell A)^i = A^{i+1}, \quad d^{i+1}_{\Sigma_\ell A} = -d_A^i. \]
Given a map of complexes \( f: A \rightarrow B \), one defines the left cone \( C_\ell(f) \) to be the complex
\[
C_\ell(f)^i = A^{i+1} \oplus B^i, \quad d^i_{C_\ell(f)} = \begin{bmatrix} -d^{i+1}_A & f^{i+1} \\ f^i & d^i_B \end{bmatrix}.
\]
One also associates to \( f \) the left standard triangle
\[
S_\ell(f): A \xrightarrow{f} B \xrightarrow{\alpha(f)} C_\ell(f) \xrightarrow{\beta(f)} \Sigma_{\ell} A
\]
in \((\Ch^b A, \Sigma_\ell)\), where \( \alpha(f) \) and \( \beta(f) \) are inclusion and projection. A triangle in \((K^b A, \Sigma_\ell)\) is left distinguished if it is isomorphic to the image of some left standard triangle. One then shows that the collection of left distinguished triangles satisfies the axioms of distinguished triangles, hence defines a triangulated structure on \((K^b A, \Sigma_\ell)\).

Define a new shift \( \Sigma_r \) on \( \Ch^b A \) and \( K^b A \) by
\[
(\Sigma_r A)^i = A^{i+1}, \quad d^i_{\Sigma_r A} = d^{i+1}_A.
\]
For a map of complexes \( f: A \rightarrow B \), define the right cone \( C_r(f) \) by
\[
C_r(f)^i = A^{i+1} \oplus B^i, \quad d^i_{C_r(f)} = \begin{bmatrix} d^{i+1}_A & (-1)^i f^{i+1} \\ (-1)^i f^i & d^i_B \end{bmatrix},
\]
and the right standard triangle
\[
S_r(f): A \xrightarrow{f} B \xrightarrow{\alpha_r(f)} C_r(f) \xrightarrow{\beta_r(f)} \Sigma_r A,
\]
where \( \alpha_r(f) \) and \( \beta_r(f) \) are again inclusion and projection (involving no sign). A triangle in \((K^b A, \Sigma_r)\) is called right distinguished if it is isomorphic to the image of some right standard triangle.

**Proposition 3.3.** The collection of right distinguished triangles defines a triangulated structure on \((K^b A, \Sigma_r)\). Moreover, there is a natural isomorphism \( \eta: \Sigma_\ell \rightarrow \Sigma_r \) such that the pair \((\id_{K^b A}, \eta)\) defines a triangulated equivalence \((K^b A, \Sigma_\ell) \cong (K^b A, \Sigma_r)\).

**Proof.** For any complex \( A \), there is an isomorphism of complexes \( \eta_A: \Sigma_\ell A \rightarrow \Sigma_r A \) defined by
\[
\eta_A^i = (-1)^i = (-1)^i \id_{A^{i+1}}.
\]
This defines a natural isomorphism \( \eta: \Sigma_\ell \rightarrow \Sigma_r \). For any map of complexes \( f: A \rightarrow B \), there is an isomorphism of complexes \( \gamma_f: C_\ell(f) \rightarrow C_r(f) \) defined by
\[
\gamma_f^i = \begin{bmatrix} (-1)^i & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-1)^i \id_{A^{i+1}} \\ \id_{B^{i+1}} \end{bmatrix}.
\]
We have the following commutative diagram in \( \Ch^b A \):
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\quad \begin{array}{ccc}
C_\ell(f) & \xrightarrow{\alpha(f)} & C_r(f) \\
\downarrow & & \downarrow \\
\Sigma_\ell A & \xrightarrow{\beta(f)} & \Sigma_r A
\end{array}
\quad \begin{array}{ccc}
\Sigma_\ell A & \xrightarrow{} & \Sigma_r A \\
\downarrow & & \downarrow \\
\Sigma_\ell A & \xrightarrow{} & \Sigma_r A
\end{array}
\]
It follows that the isomorphism \((\id_{K^b A}, \eta): (K^b A, \Sigma_\ell) \rightarrow (K^b A, \Sigma_r)\) of categories with shift identifies the collection of left distinguished triangles with the collection of right distinguished triangles. The result now follows since the former defines a triangulated structure on \((K^b A, \Sigma_\ell)\). \( \square \)
Remark 3.4. Let $\mathcal{A} = \mathcal{k}\text{-mod}$, the category of $\mathcal{k}$-modules. In this case, the shift $\Sigma_r$ (resp. $\Sigma_\ell$) can be identified with the endofunctor that tensors on the left (resp. right) with the complex $\mathcal{k}[1]$ concentrated in degree $-1$. (The differential of a tensor product of complexes is determined by the graded Leibniz rule, with the differential viewed as acting on the left.) This explains the left/right terminology.

We use the “right” sign convention throughout this paper. We write $A[1]$ for $\Sigma_r(A)$ and drop the subscript $r$ and the adjective “right.”

4. Monodromy action

Let $(W, h)$ be a reflection faithful realization (as always, over a field $\mathcal{k}$ of characteristic not equal to 2). In this section, we define the monodromy action in $D^{\text{mix}}(U \setminus G/B)$.

4.1. Idea. To motivate the homological algebra that follows, we first explain the rough idea of the construction.

Let us briefly recall the monodromy action in geometry. Let $G, B, T, U$ be as before, but now defined over a finite field $\mathbb{F}_q$. One can then consider $D^b_m(B \setminus G/B)$ (resp. $D^b_m(U \setminus G/B)$), the derived category of $B$-equivariant (resp. $B$-constructible) mixed $\mathbb{Q}_r$-complexes on $G/B$ as in [BBDS2]. These categories are equipped with an autoequivalence (1) called Tate twist $\mathbb{H}$. Let

$$F \in D^b_m(B \setminus G/B) \to D^b_m(U \setminus G/B)$$

be pullback under the natural projection $\pi : U \setminus G/B \to B \setminus G/B$. Let $X_\pi(T)$ be the group of cocharacters of $T$. As explained by Bezrukavnikov–Yun [BY13 §A.1], the construction of [Ver83]§5 applied to the $T$-torsor $\pi$ produces a functorial (log of) monodromy action of $\mathfrak{h} = \mathcal{Q}_r \otimes \mathbb{Z} X_\pi(T)$ on any $F \in D^b_m(U \setminus G/B)$: for any $X \in \mathfrak{h}$, there are morphisms $\mu_{F,X} : F \to F(2)$ in $D^b_m(U \setminus G/B)$, functorial in $F$, such that $F$ admits a $B$-equivariant lift if and only if it has trivial monodromy:

$$\mu_{F,X} \neq 0 \quad \text{for all } X \in \mathfrak{h}. \tag{4.1}$$

Now, return to our Soergel-theoretic setup: let $(W, h)$ be a reflection faithful realization. Our goal is to produce an analogous action in $D^{\text{mix}}(U \setminus G/B)$. An object of $D^{\text{mix}}(U \setminus G/B)$ is a complex $(F, d_F)$ in $\text{Parity}(U \setminus G/B)$:

$$\cdots \to F^i \xrightarrow{d_F} F^{i+1} \xrightarrow{d_F} F^{i+2} \to \cdots.$$ 

Since For: $\text{Parity}(B \setminus G/B) \to \text{Parity}(U \setminus G/B)$ is essentially surjective and full, we may (arbitrarily) lift each term and each differential to obtain a “pre-complex” $(\tilde{F}, \tilde{d}_F)$ in $\text{Parity}(B \setminus G/B)$:

$$\cdots \to \tilde{F}^i \xrightarrow{\tilde{d}_F} \tilde{F}^{i+1} \xrightarrow{\tilde{d}_F} \tilde{F}^{i+2} \to \cdots.$$ 

Note that $\tilde{d}_F^{i+1} \circ \tilde{d}_F^i$ may not be 0, but by Proposition 3.2 it lies in $\mathfrak{m}\text{HOM}(\tilde{F}^i, \tilde{F}^{i+2})$, where $\mathfrak{m}$ denotes the augmentation ideal of $R$. We say that $\tilde{F}$ is a “pseudo complex” in $\text{Parity}(B \setminus G/B)$. Thus, any $F$ admits a pseudo complex lift $\tilde{F}$, and

$$\tilde{F} \in D^{\text{mix}}(B \setminus G/B) \iff \tilde{d}_F^{i+1} \circ \tilde{d}_F^i = 0 \quad \text{for all } i \in \mathbb{Z}. \tag{4.2}$$

6More precisely, $(1) := (-\frac{1}{2})$, where we fix a square root of $q$ in $\mathcal{Q}_r$ to make sense of the half Tate twist $(\frac{1}{2})$. 
Comparing (4.1) with (4.2) suggests that monodromy in $D^{\text{mix}}(U \setminus G/B)$ should measure the failure of $\tilde{d}_\mathcal{F} \circ \tilde{d}_\mathcal{F}$ to vanish. We illustrate this in an example.

**Example 4.1.** Let $s \in S$. We noted in [Mak] that the indecomposable tilting object $T_s \in D^{\text{mix}}(U \setminus G/B)$ is the image of the three-term complex

$$T_s = (R\{-1\} \rightarrow B_s \rightarrow R\{1\}),$$

where $(T_s)^0 = B_s$ and $(-) = k \otimes_R (-)$. Since $\uparrow \circ \downarrow = \alpha_s \text{id}_R$ lies in $\mathfrak{m} \text{END}(R)$, this really is a complex in $\text{Parity}(U \setminus G/B)$.

This complex does not admit a lift to any complex in $\text{Parity}(B \setminus G/B)$. To compute the monodromy action, instead lift it to the pseudo complex

$$\tilde{T}_s = (R\{-1\} \rightarrow B_s \rightarrow R\{1\}),$$

and consider the “morphism of pseudo complexes” $\tilde{d}_\mathcal{T}_s \circ \tilde{d}_\mathcal{T}_s : \tilde{T}_s \rightarrow \tilde{T}_s[2]$:

$$R\{-1\} \rightarrow B_s \rightarrow R\{1\} \xrightarrow{\alpha_s} R\{-1\} \rightarrow B_s \rightarrow R\{1\}.$$

Here, the rows depict $\tilde{T}_s[2]$ and $\tilde{T}_s$, and the vertical arrow is the component $(\tilde{T}_s)^0 \rightarrow (\tilde{T}_s[2])^0$ of $\tilde{d}_\mathcal{T} \circ \tilde{d}_\mathcal{T}$. From this, there is a natural way to cook up a morphism $\mathcal{T}_s \rightarrow \mathcal{T}_s(2) = \mathcal{T}_s[2]\{-2\}$: “dividing through” by $\alpha_s$ decreases the internal (bimodule) degree by 2, and we get

$$R\{-3\} \rightarrow B_s\{-2\} \rightarrow R\{-1\} \xrightarrow{\text{id}} R\{-1\} \rightarrow B_s \rightarrow R\{1\}.$$

This will be the monodromy morphism $\mu_{\mathcal{T},X} : \mathcal{T}_s \rightarrow \mathcal{T}_s(2)$, where $X \in \mathfrak{h}$ is any element satisfying $\langle X, \alpha_s \rangle = 1$.

**4.2. Statement.** The following ad hoc definition will be useful for us.

**Definition 4.2.** An $\mathfrak{h}$-monodromic triple $(\mathcal{C}_\mathcal{T}, \mathcal{C}_{(T)}, \text{For})$ consists of a graded $R$-linear category $(\mathcal{C}_\mathcal{T}, \{1\})$, a graded $k$-linear category $(\mathcal{C}_{(T)}, \{1\})$, and a graded $k$-linear functor $\text{For} : \mathcal{C}_\mathcal{T} \rightarrow \mathcal{C}_{(T)}$, satisfying the following “equivariant formality” properties: for $\mathcal{F}, \mathcal{G} \in \mathcal{C}_T$,

(\text{EF1}) $\text{HOM}_{\mathcal{C}_\mathcal{T}}(\mathcal{F}, \mathcal{G})$ is a graded free $R$-module;

(\text{EF2}) the natural map

$$k \otimes_R \text{HOM}_{\mathcal{C}_\mathcal{T}}(\mathcal{F}, \mathcal{G}) \rightarrow \text{HOM}_{\mathcal{C}_{(T)}}(\text{For}\mathcal{F}, \text{For}\mathcal{G})$$

is an isomorphism.

The bounded homotopy category $K^b\mathcal{C}_{(T)}$ has an induced shift $\{1\}$ and a new cohomological shift $[1]$. Define the Tate twist $\langle 1 \rangle := [1\{-1\}]$. For a $k$-linear category
with shift \((D, \Sigma)\), its graded center \(Z(D, \Sigma)\) is the graded \(k\)-algebra with degree \(n\) part:

\[
Z(D, \Sigma)_n = \{\alpha : \text{id}_D \to \Sigma^n | \alpha_\Sigma = (-1)^n \Sigma \alpha\}.
\]

The following is the main result of this section. Recall that \(R^\vee = \text{Sym}_k(h)\), graded with degree \(h = 2\). Let \(m^\vee\) be its augmentation ideal.

**Proposition 4.3.** Let \((C_T, C(T), \text{For})\) be an \(h\)-monodromic triple. There exists a graded \(k\)-algebra map

\[
\mu : R^\vee \to Z(K^b C(T), \langle 1 \rangle) : f \mapsto \mu_{-f},
\]

called the monodromy action, with the following property. For any \(F \in K^b C(T)\), denote by

\[
\mu_F : R^\vee \to \bigoplus_{n \in \mathbb{Z}} \text{Hom}(F,F\langle n \rangle) : f \mapsto \mu_F f
\]

the induced \(k\)-algebra map. If \(F\) lies in the essential image of the induced functor \(\text{For} : K^b C_T \to K^b C(T)\), then \(\mu_F(m^\vee) = \{0\}\), or equivalently, \(\mu_F(h) = \{0\}\).

The construction of \(\mu\) will occupy the rest of this section.

**Remark 4.4.** The intuition for Definition 4.2 is that \(h = k \otimes X_\ast(T)\) for an algebraic torus \(T\), \(C_T\) (resp. \(C(T)\)) consists of \(T\)-equivariant (resp. \(T\)-monodromic) \(k\)-sheaves on a \(T\)-space, and \(\text{For}\) forgets the \(T\)-equivariance. Proposition 4.3 should then be thought of as constructing a functorial \(T\)-monodromy action.

**Remark 4.5.** As pointed out to the author by G. Dhillon, our monodromy action is a special case of the “cohomology operators” of [Gul74, Eis80, AS98]. Indeed, the nonstandard homological algebra introduced below is not needed to define our monodromy action. However, we will use the entire setup below to define the wall-crossing functors in [5].

4.3. **Categories of pseudo complexes.** Let \(A\) be a graded \(k\)-algebra, and let \((\mathcal{C}, \{1\})\) be a graded \(A\)-linear category. A graded object in \(\mathcal{C}\) is a sequence \(\mathcal{F} = (\mathcal{F}^i)_{i \in \mathbb{Z}}\) of objects in \(\mathcal{C}\). Given graded objects \(\mathcal{F}\) and \(\mathcal{G}\) in \(\mathcal{C}\), define \(\text{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})\), a graded object in \(A\text{-gmod}\), by

\[
\text{Hom}_{\mathcal{C}}^n(\mathcal{F}, \mathcal{G}) = \prod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(\mathcal{F}^i, \mathcal{G}^{i+n}).
\]

As a \(k\)-module, \(\text{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})\) is bigraded: degree \(m\) elements of the graded \(A\)-module \(\text{Hom}_{\mathcal{C}}^n(\mathcal{F}, \mathcal{G})\) are given bidegree \((n, m)\). For graded objects \(\mathcal{F}, \mathcal{G}, \mathcal{H}\) in \(\mathcal{C}\), the obvious induced composition

\[
(-) \circ (-) : \text{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{H}) \times \text{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \to \text{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{H})
\]

is bigraded and \(A\)-bilinear.

A (bounded) pre-complex \((\mathcal{F}, d_\mathcal{F})\) in \(\mathcal{C}\) consists of a graded object \(\mathcal{F} = (\mathcal{F}^i)\) in \(\mathcal{C}\), with \(\mathcal{F}^i = 0\) for all but finitely many \(i\), together with a degree \((1, 0)\) element \(d_\mathcal{F} \in \text{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})\), called the pre-differential. In components, this consists of morphisms \(d_\mathcal{F} : \mathcal{F}^i \to \mathcal{F}^{i+1}\) in \(\mathcal{C}\).\footnote{Our results only involve the even degree elements of the graded center, so the sign \((-1)^n\) disappears.}
Given pre-complexes $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{C}$, we make $\mathcal{HOM}(\mathcal{F}, \mathcal{G})$ into a pre-complex in $\mathcal{A}\text{-}g\text{mod}$ using the pre-differential

$$df = d_{\mathcal{G}} \circ f - (-1)^n f \circ d_{\mathcal{F}} \quad \text{for } f \in \mathcal{HOM}^n(\mathcal{F}, \mathcal{G}).$$

This is the \textit{graded Hom pre-complex} $(\mathcal{HOM}(\mathcal{F}, \mathcal{G}), d)$. Restricting to elements whose second degree in the bidegree is 0, we get the (non-graded) \textit{Hom pre-complex} $(\text{Hom}(\mathcal{F}, \mathcal{G}), d)$, a pre-complex in $\mathcal{K}\text{-mod}$.

Now let $(\mathcal{C}_T, \mathcal{C}^{[T]}, \text{For})$ be an $\mathfrak{h}$-monodromic triple. Henceforth, we will drop the subscript $\mathcal{C}_T$ from the various Homs.

**Definition 4.6.** A \textit{pseudo complex} $(\mathcal{F}, d_{\mathcal{F}})$ (in $\mathcal{C}_T$) is a pre-complex in $\mathcal{C}_T$ satisfying

$$d_{\mathcal{F}} \circ d_{\mathcal{F}} \in \mathfrak{m}\mathcal{HOM}^2(\mathcal{F}, \mathcal{F}).$$

We call $d_{\mathcal{F}}$ a \textit{pseudo differential}. In components, this consists of morphisms $d_{\mathcal{F}}^i : F^i \to F^{i+1}$ in $\mathcal{C}_T$ satisfying $d_{\mathcal{F}}^{i+1} \circ d_{\mathcal{F}}^i \in \mathfrak{m}\text{HOM}(F^i, F^{i+2})$.

Let $\mathcal{F}, \mathcal{G}$ be pseudo complexes. Then the graded Hom pre-complex $(\mathcal{HOM}(\mathcal{F}, \mathcal{G}), d)$ has the property that for any $n \in \mathbb{Z}$, $d_{\mathcal{F}}^{i+1} \circ d_{\mathcal{F}}^i$ maps $\mathcal{HOM}^n(\mathcal{F}, \mathcal{G})$ into $\mathfrak{m}\mathcal{HOM}^{n+2}(\mathcal{F}, \mathcal{G})$. Applying $\mathbb{K} \otimes_R (\_ \_ \_)$ termwise yields an honest complex $(\mathcal{HOM}(\mathcal{F}, \mathcal{G}), \mathcal{G})$ in $\mathcal{K}\text{-}g\text{mod}$. Thus we have the following commutative diagram, where the vertical arrows are the natural quotient maps:

$$
\begin{array}{cccccccc}
\cdots & \to & \mathcal{HOM}^{-1}(\mathcal{F}, \mathcal{G}) & \overset{d^{-1}}{\to} & \mathcal{HOM}^0(\mathcal{F}, \mathcal{G}) & \overset{d^0}{\to} & \mathcal{HOM}^1(\mathcal{F}, \mathcal{G}) & \overset{d^1}{\to} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \mathcal{HOM}^{-1}(\mathcal{F}, \mathcal{G}) & \overset{d^{-1}}{\to} & \mathcal{HOM}^0(\mathcal{F}, \mathcal{G}) & \overset{d^0}{\to} & \mathcal{HOM}^1(\mathcal{F}, \mathcal{G}) & \overset{d^1}{\to} & \cdots
\end{array}
$$

We now define the following graded $\mathbb{R}$-linear categories and functors:

$$P\text{Ch}^b\mathcal{C}_T \overset{P}{\longrightarrow} \mathcal{Q}\text{Ch}^b\mathcal{C}_T \overset{Q}{\longrightarrow} \mathcal{Q}\text{K}^b\mathcal{C}_T.$$

The objects in each category are pseudo complexes. Grading shift $\{1\}$ is induced from that of $\mathcal{C}_T$. For pseudo complexes $\mathcal{F}$ and $\mathcal{G}$, the graded Homs are defined by

$$\mathcal{HOM}_{P\text{Ch}^b\mathcal{C}_T}(\mathcal{F}, \mathcal{G}) = (d^0)^{-1}(\mathfrak{m}\mathcal{HOM}^1(\mathcal{F}, \mathcal{G})),$$

$$\mathcal{HOM}_{P\text{Ch}^b\mathcal{C}_T}(\mathcal{F}, \mathcal{G}) = \ker(\mathfrak{m}\mathcal{HOM}^0(\mathcal{F}, \mathcal{G})),$$

$$\mathcal{HOM}_{P\text{K}^b\mathcal{C}_T}(\mathcal{F}, \mathcal{G}) = \ker(\mathfrak{m}\mathcal{HOM}^{i+1}(\mathcal{F}, \mathcal{G}))/\im(\mathfrak{m}\mathcal{HOM}^i(\mathcal{F}, \mathcal{G})).$$

The functors $P, Q$ are the identity map on objects and natural quotient maps on morphisms.

**Definition 4.7.** Morphisms in $P\text{Ch}^b\mathcal{C}_T$ are called \textit{pseudo maps}. Thus, a pseudo map $\varphi : \mathcal{F} \to \mathcal{G}$ is an element of $\mathcal{HOM}^0(\mathcal{F}, \mathcal{G})$ satisfying

$$d_\varphi \in \mathfrak{m}\mathcal{HOM}^1(\mathcal{F}, \mathcal{G})$$

In components, this consists of morphisms $\varphi^i : F^i \to G^i$ in $\mathcal{C}_T$ satisfying

$$d_{\mathcal{G}}^i \circ \varphi^i - \varphi^{i+1} \circ d_{\mathcal{F}}^{i+1} \in \mathfrak{m}\text{HOM}(F^{i+1}, G^{i+1}).$$

If $(\mathcal{F}, d_{\mathcal{F}})$ is a pseudo complex in $\mathcal{C}_T$, then $(\text{EF2})$ implies that $(\text{For} \mathcal{F}, \text{For} d_{\mathcal{F}})$ is a complex in $\mathcal{C}_{(T)}$. Moreover, for pseudo complexes $\mathcal{F}$ and $\mathcal{G}$, the isomorphism of $(\text{EF2})$ induces an identification of complexes

$$(\mathcal{HOM}(\mathcal{F}, \mathcal{G}), 0) \cong (\mathcal{HOM}(\text{For} \mathcal{F}, \text{For} \mathcal{G}), d).$$
The latter complex is the graded version of the Hom complex used in ordinary homological algebra to define $\text{Ch}^b_{C(T)}$ and $K^b_{C(T)}$. Hence this identification induces equivalences $R_{\text{Ch}}: \mathcal{P} \text{Ch}^b_{C(T)} \to \text{Ch}^b_{C(T)}$ and $R_K: \mathcal{P} K^b_{C(T)} \to K^b_{C(T)}$.

The situation is summarized in the following diagram:

$$
\begin{array}{ccc}
\text{PCh}^b_{C(T)} & \xrightarrow{P} & \mathcal{P} \text{Ch}^b_{C(T)} & \xrightarrow{Q} & \mathcal{P} K^b_{C(T)} \\
\downarrow R_{\text{Ch}} & & \downarrow R_K & & \\
\text{Ch}^b_{C(T)} & \xrightarrow{q} & K^b_{C(T)}
\end{array}
$$

Each category is graded $R$-linear with grading shift $\{1\}$, and has an additional autoequivalence, the homological shift $[1]$. Define the functors are graded $R$-linear and commute with $\{1\}, [1], (1)$ on the nose.

4.4. Triangulated structure on $\mathcal{P} K^b_{C(T)}$. Let $\varphi: \mathcal{F} \to \mathcal{G}$ be a pseudo map of pseudo complexes. Define the cone pseudo complex $C(\varphi)$ and standard triangle

$$
\mathcal{P} S(\varphi): \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\alpha(\varphi)} C(\varphi) \xrightarrow{\beta(\varphi)} \mathcal{F} \left[1\right]
$$

in $\text{PCh}^b_{C(T)}$ by the same formula as for complexes.

Since the standard triangles in $\text{Ch}^b_{C(T)}$ are precisely triangles of the form $R_{\text{Ch}} P(\mathcal{P} S(\varphi))$, distinguished triangles in $K^b_{C(T)}$ are triangles that are isomorphic to some $q R_{\text{Ch}} P(\mathcal{P} S(\varphi))$. Define a triangle $T$ in $\mathcal{P} K^b_{C(T)}$ to be distinguished if $T \cong Q P(\mathcal{P} S(\varphi))$ for some $\varphi$. This is equivalent to $R_K(T) \cong R_K Q P(S(\varphi)) = q R_{\text{Ch}} P(S(\varphi))$, i.e. $R_K(T)$ is isomorphic to a standard triangle in $K^b_{C(T)}$. From this description, we deduce the following result.

**Lemma 4.8.** The definitions above define a triangulated structure on $\mathcal{P} K^b_{C(T)}$ making $R_K$ exact.

4.5. Construction of monodromy. Let $M$ be a graded free $R$-module. For any $X \in \mathfrak{h}$, define a graded $k$-module map

$$
\Phi_{M,X}: \mathfrak{m} M \to \mathcal{O} M \{ -2 \}
$$

as follows. If $X = 0$, set $\Phi_{M,X} = 0$. Otherwise, extend $X$ to a basis $\{X = X_1, X_2, \ldots, X_r\}$ of $\mathfrak{h}$, with dual basis $\{X_1^*, \ldots, X_r^*\}$ of $\mathfrak{h}^*$. Any element $m \in \mathfrak{m} M$ can be written as $m = X_1^* m_1 + \cdots + X_r^* m_r$ for some $m_i \in M$. Although the elements $m_i$ are not unique, $M$ being graded free ensures that their classes $\mathfrak{m} m_i \in \mathcal{O} M$ are well-defined; set $\Phi_{M,X}(m) = \mathfrak{m} m_i$. It is easy to see that this does not depend on the choice of basis.

The following result is straightforward.

**Lemma 4.9.** Let $M, N$ be graded free $R$-modules.

1. $\Phi_{M,X}$ is linear in $X$.
2. For any $m \in M$, $\beta \in \mathfrak{h}^*$, and $X \in \mathfrak{h}$, we have

$$
\Phi_{M,X}(\beta m) = (X, \beta) m.
$$

3. For any graded $R$-linear map $f: M \to N$ and $X \in \mathfrak{h}$, we have

$$
\Phi_{N,X} \circ f \mid_{\mathfrak{m} M} = \Phi_{M,X} \circ f |_{\mathfrak{m} M}.
$$

We apply this lemma to $\mathcal{H} \text{OM}(\mathcal{F}, \mathcal{G})$, which is graded free by (EF1).

**Lemma 4.10.** Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be pre-complexes in $C_T$, and let $X \in \mathfrak{h}$.
Lemma 4.10. □

Proof. Each equation is straightforward to prove from the definitions and pseudo map $\mu$.

For all $X$, we can define, for any $\phi \in \text{Hom}(\mathcal{F}, \mathcal{G})$, we can define, for any $X$.

By (4.3) (resp. (4.4)), we can define, for any $\phi \in \text{Hom}(\mathcal{F}, \mathcal{G})$, then

$\Phi_{\text{Hom}(\mathcal{F}, \mathcal{G})}(\psi \circ \phi) = \Phi_{\text{Hom}(\mathcal{G}, \mathcal{H})}(\psi) \circ 0_{\phi}$.

If $\phi \in \text{Hom}(\mathcal{F}, \mathcal{G})$ and $\psi \in \text{Hom}(\mathcal{G}, \mathcal{H})$, then

$\Phi_{\text{Hom}(\mathcal{F}, \mathcal{H})}(\psi \circ \phi) = 0_{\psi} \circ \Phi_{\text{Hom}(\mathcal{F}, \mathcal{G})}(\phi)$.

Proof. (1) Apply Lemma 4.9 to $d: \text{Hom}(\mathcal{F}, \mathcal{G}) \to \text{Hom}(\mathcal{F}, \mathcal{G})$.

(2) For the first equality, apply Lemma 4.9 to $-\circ \phi: \text{Hom}(\mathcal{G}, \mathcal{H}) \to \text{Hom}(\mathcal{F}, \mathcal{H})$. The second equality is similar.

Let $(\mathcal{F}, d_{\mathcal{F}})$ be a pseudo complex, and $\varphi: \mathcal{F} \to \mathcal{G}$ a pseudo map of pseudo complexes. By 4.3 (resp. 4.4), we can define, for any $X \in \mathfrak{h}$,

$\mu_{\mathcal{F}, X} = \Phi_{\text{Hom}(\mathcal{F}, \mathcal{G})}(d_{\mathcal{F}} \circ d_{\mathcal{F}})$ (resp. $\nu_{\varphi, X} = \Phi_{\text{Hom}(\mathcal{F}, \mathcal{G})}((d_{\mathcal{F}}))$).

This is a degree $-2$ element in $\beta \text{Hom}(\mathcal{F}, \mathcal{F})$ (resp. $\beta \text{Hom}(\mathcal{F}, \mathcal{G})$) that measures the failure of $\mathcal{F}$ to be a complex (resp. the failure of $\varphi$ to be a map of complexes) “in the $X$ direction.”

Lemma 4.11. Let $\varphi: \mathcal{F} \to \mathcal{G}$ and $\psi: \mathcal{G} \to \mathcal{H}$ be pseudo maps of pseudo complexes. For all $X \in \mathfrak{h}$, we have

$0d(\mu_{\mathcal{F}, X}) = 0$,

$0d(\nu_{\varphi, X}) = 0_{\varphi} \circ 0_{\mu_{\mathcal{F}, X}}$,

$\nu_{\psi \circ \varphi, X} = \nu_{\psi, X} \circ 0_{\psi} \circ \nu_{\varphi, X}$.

Proof. Each equation is straightforward to prove from the definitions and Lemma 4.10.

We view $\mu_{\mathcal{F}, X}$ as a degree 0 element of $\beta \text{Hom}(\mathcal{F}, \mathcal{F})$ via the natural identification of complexes

$\beta \text{Hom}(\mathcal{F}, \mathcal{F}), \beta d) \cong (\beta \text{Hom}(\mathcal{F}, \mathcal{F}, \mathcal{F}), \beta d - 2) \cong (\beta \text{Hom}(\mathcal{F}, \mathcal{F}), \beta d - 2)$.

By Lemma 4.11, $\mu_{\mathcal{F}, X}$ defines a morphism $\mathcal{F} \to \mathcal{F}$ in both $\beta \text{PCh}^b \mathcal{C}_T$ and $\beta \text{PCh}^b \mathcal{C}_T$. We similarly view $\nu_{\varphi, X}$ as a degree 0 element of $\beta \text{Hom}(\mathcal{F}, \mathcal{G}, \mathcal{F})$.

Lemma 4.12. Any $\mu_{\mathcal{F}, X}, X \in \mathfrak{h}$, is a degree 2 element of $Z(\beta \text{PCh}^b \mathcal{C}_T, \{1\})$.

Proof. It is clear from the construction that $\mu_{\mathcal{F}, X}$ commutes with (1). For any pseudo map $\varphi: \mathcal{F} \to \mathcal{G}$, we must show that

$0\varphi(2) \circ \mu_{\mathcal{F}, X} = \mu_{\mathcal{G}, X} \circ 0_{\varphi}$

in $\beta \text{PCh}^b \mathcal{C}_T$. This follows from Lemma 4.11, which translates to

$0d(\nu_{\varphi, X}) = 0_{\varphi} \circ 0_{\mu_{\mathcal{F}, X}} \in \beta \text{Hom}(\mathcal{F}, \mathcal{G}, \mathcal{F})$. □
Proof of Proposition 4.3. Since $\mu_{F,X}$ is linear in $X$, the map $X \mapsto \mu_{-,X}$ extends uniquely to a graded $k$-algebra map

$$\mu: R^i \to Z(PK^bC_T, \langle 1 \rangle): f \mapsto \mu_{-,f}.$$ 

An equivalence of categories with shift induces an isomorphism of their graded centers. Hence, using $R_K$, we obtain the desired map $\mu$. The last statement is clear from the construction. $\square$

5. WALL-CROSSING FUNCTORS

Let $(W, h)$ be a reflection faithful realization, and fix $s \in S$. In this section, we will construct an endofunctor $\xi_s$ of $D^{mix}(U\backslash G/B)$. These functors will restrict to exact functors on $\mathcal{P}^{mix}(U\backslash G/B)$, where they correspond to the wall-crossing functors on $\mathcal{O}_\mathfrak{g}^s$.

5.1. Idea. Return for a moment to the geometric setup of 4.1. Let $D^b(U\backslash G/U)$ be the derived category of $U$-equivariant mixed complexes on the enhanced flag variety $G/U$. Bezrukavnikov–Yun [BY13] defined a category $\widehat{D}^b(U\backslash G\backslash U, \langle \cdot \rangle)$, monoidal under $U$-convolution $*$, as a certain (“free-monodromic”) completion of a full subcategory of $D^b(U\backslash G/U)$. This category $\widehat{D}^b(U\backslash G\backslash U, \langle \cdot \rangle)$ acts by $*$ on the left of $D^b(U\backslash G/B)$ and contains the “free-monodromic” tilting sheaf $\widehat{T}_s$, giving the endofunctor

$$\xi_s = \widehat{T}_s^U (-): D^b(U\backslash G/B) \to D^b(U\backslash G/B).$$

This is the functor we wish to imitate in our Soergel-theoretic setting.

Although we do not have an analogue of $\widehat{D}^b(U\backslash G\backslash U, \langle \cdot \rangle)$, we can start to guess the definition of $\xi_s$, as follows. First, since proper pushforward along the natural projection $U\backslash G/U \to U\backslash G/B$ sends $\widehat{T}_s$ to $T_s$, the following diagram commutes up to natural isomorphism by proper base change:

$$\begin{array}{ccc}
D^b(U\backslash G/B) & \xrightarrow{\xi_s = \widehat{T}_s^U (-)} & D^b(U\backslash G/B) \\
\text{For} & & \\

D^b(B\backslash G/B) & \xrightarrow{T_s^B (-)} & D^b(U\backslash G/B).
\end{array}$$

(5.1)

Now, back in our setting, we do have an analogue of the functor $T_s^B (-)$:

$$T_s \otimes_R (-): D^{mix}(B\backslash G/B) \to D^{mix}(U\backslash G/B).$$

From here on, we omit $\otimes_R$ from the notation. Explicitly, this functor sends $\mathcal{F}$ in $D^{mix}(B\backslash G/B)$ to the complex $T_s \mathcal{F}$ is given by

$$(T_s \mathcal{F})^i = \mathcal{F}^{i-1} \{1\} \oplus B_s \mathcal{F}^i \oplus \mathcal{F}^{i+1} \{-1\},$$

and the morphism $\varphi: \mathcal{F} \to \mathcal{G}$ to $1_{T_s} \varphi: T_s \mathcal{F} \to T_s \mathcal{G}$, given by

$$(1_{T_s} \varphi)^i = \text{diag}(\varphi^{-1}, \varphi, \varphi^{i+1}): (T_s \mathcal{F})^i \to (T_s \mathcal{G})^i.$$

Here, each matrix entry is viewed modulo $m$, i.e. as morphisms in $\text{Parity}(U\backslash G/B)$. 
By analogy with \([3,2]\), the desired functor \(\xi_s\) should extend this to \(D^{mix}(U\backslash G/B)\), i.e. to a map of complexes \(\varphi: F \to G\) in \(\text{Parity}(U\backslash G/B)\). There are two difficulties. First, since \(d_F\) and \(\varphi^i\) are only defined modulo \(m\) (on the left!), \(d_F^i\) and \(\varphi^i\) are not well-defined modulo \(m\). We solve this by lifting these to actual bimodule maps, i.e. work with a pseudo map \(\varphi: F \to G\) of pseudo complexes. Second, for a pseudo complex \(F\) (resp. pseudo map \(\varphi: F \to G\)), the pre-complex \(T_sF\) (resp. pre-complex map \(1_{T_s}\varphi\)) is in general not a pseudo complex (resp. pseudo map); see Figure 1. Indeed, \((3,2)\) implies that for the component

\[
(d_F \circ d_F): B_sF \to B_sF \quad \text{(resp. } d_\varphi: B_sF \to B_sG)\]

of \(d_{T_s}F \circ d_{T_s}F\) (resp. \(d(1_{T_s}\varphi)\)) to vanish modulo \(m\), \(d_F \circ d_F\) (resp. \(d_\varphi\)) must vanish modulo \(m^2\).
The following key computation tells us how to proceed. Choose a basis \( \{X_1, \ldots, X_r\} \) of \( \mathfrak{h} \), and write
\[
d_F \circ d_F = \sum X_j^* \tilde{\mu}_{F,X_j}
\]
for some \( \tilde{\mu}_{F,X_j} \in \text{Hom}^2(F,F) \), so that \( \tilde{\mu}_{F,X_j} \) lifts \( \mu_{F,X_j} \). Then by (5.2) and the linearity of \( \mu_{F,-} \),
\[
(5.2) \quad \left| (d_F \circ d_F) = \sum X_j^* \tilde{\mu}_{F,X_j} \right| = \sum (\alpha_s^\vee, X_j^*) \mu_{F,X_j} = \sum \mu_{F,\alpha^\vee}
\]
in \( \text{Hom}^2(F,F) \). Therefore, we may add correction components \( - \mu_{F,\alpha} \) (or \( - \mu_{F,\alpha} \)) as in Figure 2 to turn \( \mathcal{T}_s \mathcal{F} \) into an actual complex in \( \text{Parity}(U\backslash G/B) \).
(5.3) The following key computation tells us how to proceed. Choose a basis \( \{X_1, \ldots, X_r\} \) of \( \mathfrak{h} \), and write
\[
(5.3) \quad \left| d\varphi = \sum X_j^* \tilde{\nu}_{\varphi,X_j} \right| = \sum (\alpha_s^\vee, X_j^*) \nu_{\varphi,X_j} = \sum \nu_{\varphi,\alpha^\vee}
\]
in \( \text{Hom}^1(F,G) \) suggests adding correction components \( - \nu_{\varphi,\alpha} \) (or \( - \nu_{\varphi,\alpha} \)) to turn \( \mathcal{T}_s \varphi \) into an actual complex of complexes in \( \text{Parity}(U\backslash G/B) \).

5.2. Statement. Let \( C_{s,T} \subset \text{Parity}(B\backslash G/B) \) be the full subcategory consisting of \( R\{n\} \) and \( B_s \{n\} \) for \( n \in \mathbb{Z} \). Let \( (C_T, C_{(T)}, \text{For}) \) be an \( \mathfrak{h} \)-monodromic triple. Suppose that we have a bifunctor
\[
(\cdot) * (\cdot): C_{s,T} \times C_T \to C_T
\]
such that the induced maps on graded Hom are graded, \( R \)-linear on the left, and \( R \)-middle-linear. These assumptions ensure that we have an induced exact functor \( \mathcal{T}_s * (\cdot): K^bC_T \to K^bC_{(T)} \) as in (5.1) and that the key computations (5.2) and (5.3) still hold.

The following is the main result of this section.

**Proposition 5.1.** In the situation above, there exists a functor \( \xi_s: K^bC_{(T)} \to K^bC_{(T)} \), defined up to natural isomorphism, with the following properties:

1. \( \xi_s \circ \{1\} = \{1\} \circ \xi_s \) and \( \xi_s \circ [1] = [1] \circ \xi_s \).
2. \( \xi_s \) is exact.
3. There is a natural isomorphism \( \mathcal{T}_s * (\cdot) \cong \xi_s \circ \text{For} \).
4. Let \( F \in K^bC_{(T)} \) and \( f \in (R^*)^s \), homogeneous of degree \( d \). Then
\[
\xi_s \mu_{F,f} = \mu_{\xi_s F,f} = \xi_s F \to \xi_s F(d).
\]
5. Let \( (C_T', C_{(T)}', \text{For}) \) be another \( \mathfrak{h} \)-monodromic triple equipped with a bifunctor
\[
(\cdot) * (\cdot): C_{s,T} \times C_T' \to C_T' \text{ as above. Let } \xi'_s: K^bC_{(T)}' \to K^bC_{(T)}' \text{ be the resulting endofunctor. Let } F: C_T \to C_T' \text{ be a graded } R \text{-linear functor, inducing a functor } F: K^bC_T \to K^bC_{(T)}'. \text{ If } F \text{ intertwines } * \text{ and } *' \text{ up to natural isomorphism, then } F \circ \xi_s \cong \xi'_s \circ F.
\]

We construct \( \xi_s \) in (5.3). The proof of Proposition 5.1 occupies (5.4).

**Remark 5.2.** We insist on equality, not just natural isomorphism, in (1) to simplify certain computations with \( \xi_s \). In particular, the exactness of \( \xi_s \) is proved by checking directly that it sends a standard triangle to a distinguished triangle. The sign convention in (5.3) is chosen to simplify this computation.
5.3. Construction of $\xi_s$. We will proceed as follows (see diagram (5.4) below): define a functor $\xi_s: \text{PCh}^b T \rightarrow \text{Ch}^b C(T)$ by explicitly modifying $T_s * (-)$ as in (5.1); show that $q \circ \xi_s$ factors (uniquely) through a functor $\theta(\xi_s): \text{PCh}^b T \rightarrow \text{Ch}^b C(T)$; finally, choose a quasi-inverse $R_K^{-1}$ commuting with $\{1\}$ and $[1]$ on the nose, and set $\xi_s = \theta(\xi_s) \circ R_K^{-1}$.

\[
\begin{align*}
\text{PCh}^b T &\xrightarrow{F} \text{Ch}^b C(T) \xrightarrow{q} \text{Ch}^b C(T) \\
\text{PCh}^b T &\xrightarrow{\xi_s} \text{Ch}^b C(T) \xrightarrow{q} \text{Ch}^b C(T)
\end{align*}
\]

Remark 5.3. Recall from (5.1) that we could add the correction components to $T_s * (-)$ in two ways. Here, we work with one of these ways. Both choices lead to naturally isomorphic $\theta(\xi_s)$, hence naturally isomorphic $\xi_s$.

Step 1: Define $\xi_s$. As in (5.1) we omit $*$ from the notation. Define $\xi_s$ on objects by

\[
(\xi_s, F)^i = F^{i-1}\{1\} \oplus B_s F^{i} \oplus F^{i+1}\{1\}, \quad d_{\xi_s, F} = \begin{bmatrix} -d_{F}^{i-1} & 1_F & -d_{F}^{i+1} \\ d_{F}^{i} & 1_F & -d_{F}^{i+1} \\ \mu_{F, \alpha} & -d_{F}^{i+1} +1 \\ \mu_{F, \alpha} & -d_{F}^{i+1} +1 \\ \mu_{F, \alpha} & -d_{F}^{i+1} +1 \end{bmatrix},
\]

and on morphisms by

\[
(\xi_s, \varphi)^i = \begin{bmatrix} \varphi^{i-1} \\ \varphi^i \\ \varphi^{i+1} \end{bmatrix},
\]

where all matrix entries are to be viewed as a morphism in $C(T)$ (e.g. the top left entry of $(\xi_s, F)^i$ is in $\text{HOM}(F^{i-1}\{1\}, F^i\{1\})$). This is illustrated in Figure 3.
To check that this is a map of complexes in $\mathcal{C}_T$, we compute (now omitting indices)

$$d_{\xi_s, G} \circ (\xi_s \phi) = \begin{pmatrix} -d_G & 1_G \\ -\phi \mu_G & -d_G \end{pmatrix} \begin{pmatrix} \phi \\ -\nu_{\phi, \alpha_G} \phi \end{pmatrix}$$

and

$$(\xi_s \phi) \circ d_{\xi_s, F} = \begin{pmatrix} \phi \\ -\nu_{\phi, \alpha_F} \phi \\ -\phi \mu_F & -d_F \end{pmatrix} \begin{pmatrix} -d_F & 1_F \\ -\phi \mu_F, \alpha_F & -d_F \end{pmatrix} \begin{pmatrix} \phi \\ -\nu_{\phi, \alpha_F} \phi \\ -\phi \mu_F, \alpha_F \phi \end{pmatrix}.$$

The $(1, 1)$ and $(3, 3)$ entries of these two matrices agree because $\phi$ is a pseudo map. The $(2, 2)$ entries agree by Lemma 4.11(2). The $(3, 2)$ entries agree by Lemma 4.11(3).

Given pseudo maps $\phi: F \to G$ and $\psi: G \to H$, a similar direct computation using Lemma 4.11(3) shows that $\xi_s \psi \circ \xi_s \phi = \xi_s (\psi \circ \phi)$. Thus $\xi_s$ is a functor.

Step 2: $q \circ \xi_s$ factors through $P$. Given $\psi \in \text{HOM}^0(F, G)$ of degree $-2$ and $\beta \in m$, we must show that the map $(\xi_s)(\beta \psi): \xi_s F \to \xi_s G$ of complexes in $\mathcal{C}_T$ is nullhomotopic. We claim that a homotopy is given by

$$h^i = \begin{pmatrix} \phi \circ d_F \end{pmatrix} : (\xi_s F)^i \to (\xi_s G)^{i-1},$$

where $\phi \circ d_F: B_s F^i \to R^i G^i$ is in the $(3, 2)$ entry (see Figure 4).
Indeed, by direct computation,

\[(dh)^i = d_{E_{ij}}^{-1} \circ h^i + h^{i+1} \circ d_{E_{ij}}^i = \begin{bmatrix} (\alpha', \beta)^i \\ (\alpha', \beta)(d_{E_{ij}}^i \circ h^i - h^{i+1} \circ d_{E_{ij}}^i) \end{bmatrix}.\]

By \(3.2\), \(\langle \alpha', \beta \rangle^i \equiv \beta^i \mod m, \) and byLemma \(1.10(2),\)

\[
\nu_{\beta \gamma, \alpha'} = \Phi_{HOM^0(F, G), \alpha'}(\beta(d)) = \Phi_{HOM^0(F, G), \alpha'}(\beta(d)) = \langle \alpha', \beta \rangle(d) = \langle \alpha', \beta \rangle(d_{E_{ij}} \circ \psi - \psi \circ d_{E_{ij}}). \]

Thus \((dh)^i = (\xi_{\alpha'}(\beta \psi))^i\), as desired.

**Step 3:** \(q \circ \xi_{\alpha'}\) factors through \(Q \circ P.\) Since the quotient map \(HOM^{-1}(F, G) \to 0HOM^{-1}(F, G)\) is surjective, it remains to show that for any \(h \in HOM^{-1}(F, G),\) the map \(\xi_{\alpha'}(dh): \xi_{\alpha'}F \to \xi_{\alpha'}G\) of complexes in \(C_{(T)}\) is nullhomotopic. We claim that a homotopy is given by

\[
H^i = \begin{bmatrix} -h^{i-1} \\ h^i \\ -h^{i+1} \end{bmatrix} : (\xi_{\alpha'}F)^i \to (\xi_{\alpha'}G)^{i-1}.
\]

Indeed,

\[
(dH)^i = d_{E_{ij}}^{-1} \circ h^i + H^{i+1} \circ d_{E_{ij}}^i = \begin{bmatrix} (dh)^i-1 \\ (dh)^i \\ -(\mu_{E_{ij}}^{-1} \circ h^i - d_{E_{ij}}^i) \end{bmatrix}, \]

and by Lemma \(1.10(2),\)

\[
\mu_{dh, \alpha'} = \Phi_{\alpha'}(dh) = \Phi_{\alpha'}(d_{E_{ij}} \circ \psi - \psi \circ \alpha_{\gamma}) = \mu_{E_{ij}, \alpha'}(d_{E_{ij}} \circ \psi - \psi \circ \alpha_{\gamma}), \]

so \((dH)^i = (\xi_{\alpha'}(dh))^i.\)

This concludes the construction of \(\xi_{\alpha'}.\)

**5.4. Proof of Proposition 5.1.**

(1) Since \(P, Q, q,\) and \(R^{-1}_K\) commute with shifts on the nose, it suffices to prove the claim for \(\xi_{\alpha'}.\) This is a direct computation.

(2) By the definition of the triangulated structure on \(PK^0 C_{(T)}\) and the construction of \(\xi_{\alpha'},\) it suffices to show that \(q \circ \xi_{\alpha'}\) sends a standard triangle to a distinguished triangle. In fact, given a pseudomap \(\phi: F \to G,\) we claim that there is an isomorphism

\[
\begin{array}{cccc}
\xi_{\alpha'}F & \xi_{\alpha'}G & \xi_{\alpha'}C(\phi) & \xi_{\alpha'}F[1] \\
\xi_{\alpha'}F & \xi_{\alpha'}G & \xi_{\alpha'}C(\phi) & \xi_{\alpha'}F[1] \\
\end{array}
\]

of triangles even in \(Ch^b C_{(T)}\). Here, \(\gamma\) is given by the evident isomorphism between

\[
(\xi_{\alpha'}(\phi))^i = (\phi(i)^{-1} \{1\} \oplus C(\phi)^i_i) \oplus C(\phi)_{i+1} \{1\} \oplus B_s(\phi)_{i+2} \oplus B_s(\phi)_{i+1} \{1\}
\]

""
and

\[ C(\xi_s) = (\xi_s F)^{i+1} + (\xi_s G)^i \]

\[ = (F^i \{1\} \oplus B F^{i+1} \oplus F^{i+2} \{-1\}) \oplus (G^i \{1\} \oplus B G^i \oplus G^{i+1} \{-1\}). \]

The only claim that is not clear is that \( \gamma \) is a map of complexes. A direct computation writing out \( d_C^{i+1}(\xi_s) \) and \( d_C^i(\xi_s \gamma) \) as 6-by-6 matrices, together with the following lemma, shows that they are indeed identified by \( \gamma^i \).

**Lemma 5.4.** Let \( \varphi: F \to G \) be a pseudo map of pseudo complexes. Then

\[ \mu^i_{C(\varphi), X} = \begin{bmatrix} \mu^i_{F, X} + (1)^i \mu^i_{G, X} \\ 0 \end{bmatrix}, \quad \nu_{\alpha(\varphi), X} = 0, \quad \nu_{\beta(\varphi), X} = 0 \]

for all \( X \in \mathfrak{h} \) and \( i \in \mathbb{Z} \).

**Proof.** The first relation follows from calculating \( d_{C(\varphi)} \circ d_{C(\varphi)} \). The second and third relations hold because \( \alpha(\varphi) \) and \( \beta(\varphi) \) commute on the nose (not just modulo \( m \)) with the pseudo differentials. \( \square \)

(3) This is clear from the construction.

(4) Choose a basis \( \{X_1, \ldots, X_r\} \) of \( \mathfrak{h} \) with \( X_1 = \alpha_Y^X, s(X_i) = X_i \) for \( i > 1 \). Then

\[ R^n = k[\langle \alpha_Y^X \rangle], \]

so it suffices to consider \( f = \langle \alpha_Y^X \rangle \) and \( f = X \in \mathfrak{h} \) with \( \langle \alpha_s, X \rangle = 0 \). For these cases, it is straightforward to verify the statement directly from the definitions.

(5) Let \( G \) be a pseudo complex. Choose a basis \( X_1, \ldots, X_r \) of \( \mathfrak{h} \), and write \( d_G \circ d_G = \sum_j X_j^i \mu_G, X_j \). Since \( F \) is \( R \)-linear, \( F(d_G \circ d_G) = \sum_j X_j^i F(\mu_G, X_j) \). It follows that \( \mu_{F \circ G, X} = F(\mu_G, X): FG \to FG \{2\} \) for any \( X \in \mathfrak{h} \). The rest of the argument is straightforward.

This concludes the proof of Proposition 5.1

### 5.5. Generating tilting objects

Let \( s, t \in S \). Applying Proposition 5.1 with \( C_T = \text{Parity}(B \setminus G / B) \) and \( C_T = \text{Parity}(B \setminus G / P^t) \), we obtain exact functors

\[ \xi_s: D^{\text{mix}}(U \setminus G / B) \to D^{\text{mix}}(U \setminus G / B), \quad \xi_s: D^{\text{mix}}(U \setminus G / P^t) \to D^{\text{mix}}(U \setminus G / P^t), \]

which we call wall-crossing functors. It follows follow from Proposition 5.1(5) that

\[ (5.5) \quad \xi_s \circ \pi_i^* \cong \pi_i^* \circ \xi_s, \quad \xi_s \circ \pi_t^* \cong \pi_t^* \circ \xi_s, \]

and that for \( F \in D^{\text{mix}}(U \setminus G / B) \) and \( G \in D^{\text{mix}}(B \setminus G / B) \), we have

\[ (5.6) \quad (\xi_s F) \ast G \cong \xi_s (F \ast G). \]

The following result is an analogue of the mixed version of [AR16a, Lemma 5.21], and is proved in the same way.

**Lemma 5.5.**

1. For all \( w \in W \), \( \xi_s \Delta_w \) is perverse. It admits a standard filtration with associated graded \( \Delta_{sw} \oplus \Delta_w \{1\} \) if \( sw > w \) and \( \Delta_{sw} \oplus \Delta_w \{-1\} \) if \( sw < w \).
2. For all \( w \in W \), \( \xi_s \nabla_w \) is perverse. It admits a standard filtration with associated graded \( \nabla_{sw} \oplus \nabla_w \{-1\} \) if \( sw > w \) and \( \nabla_{sw} \oplus \nabla_w \{1\} \) if \( sw < w \).
For any expression \( w = s_1 \ldots s_k \), define the Bott–Samelson tilting object
\[
T_w := \xi_{s_1} \cdots \xi_{s_k}(\delta).
\]
Lemma 5.5 shows that \( T_w \in \text{Tit}^{\text{mix}}(U \setminus G/B) \), and also implies the following Bott–Samelson characterization of indecomposable tilting objects, analogous to the one for Soergel modules (Proposition 2.3).

**Proposition 5.6.** For any reduced expression \( w \) of \( w \in W \), \( T_w \) can be identified with the unique indecomposable direct summand of \( T_w \) that does not occur as a direct summand of \( T_w \) for any expression \( w \) with \( \ell(w) < \ell(w) \).

We also need a tilting analogue of the Soergel Hom formula (3.3). In general, for a graded highest weight category \( (A, (1)) \) indexed by \((\mathcal{S}, \leq)\), let \( \mathcal{F}_\Delta \) (resp. \( \mathcal{F}_\nabla \)) denote the full subcategory consisting of standardly (resp. costandardly) filtered objects. For \( X \in \mathcal{F}_\Delta \), let \( \langle X : \Delta_s(n) \rangle \) denote the multiplicity of \( \Delta_s(n) \) in any standard filtration of \( X \). For \( Y \in \mathcal{F}_\nabla \), similarly write \( \langle Y : \nabla_s(n) \rangle \). Let \( X \in \mathcal{F}_\Delta \) and \( Y \in \mathcal{F}_\nabla \). Since \( \text{Ext}^k(\Delta_s, \nabla_t(n)) = 0 \) for any \( s, t \in \mathcal{S}, n \in \mathbb{Z}, k > 0 \), we get
\[
\dim \text{Hom}_A(X, Y) = \sum_{s \in \mathcal{S}, n \in \mathbb{Z}} \langle X : \Delta_s(n) \rangle \langle Y : \nabla_s(n) \rangle
\]
by inducting on the length of a standard (resp. costandard) filtration of \( X \) (resp. \( Y \)). It follows that
\[
\text{gdim} \bigoplus_{n \in \mathbb{Z}} \text{Hom}_A(X, Y(n)) = \sum_{s \in \mathcal{S}, n_1, n_2 \in \mathbb{Z}} \langle X : \Delta_s(n_1) \rangle \langle Y : \nabla_s(n_2) \rangle v^{n_1-n_2}.
\]

Now consider \( A = P^{\text{mix}}(U \setminus G/B) \). Recall the Hecke algebra \( \mathcal{H}_W \) (see 2.4 for our normalization) and the pairing \((-, -)\) (see 3.2). Define
\[
\text{ch}_\Delta : \text{Ob}(\mathcal{F}_\Delta) \to \mathcal{H}_W : F \mapsto \sum_{w \in W, n \in \mathbb{Z}} \langle F : \Delta_w(n) \rangle v^n H_w,
\]
\[
\text{ch}_\nabla : \text{Ob}(\mathcal{F}_\nabla) \to \mathcal{H}_W : G \mapsto \sum_{w \in W, n \in \mathbb{Z}} \langle G : \nabla_w(n) \rangle v^{-n} H_w.
\]
Then (5.7) may be restated as follows: for \( F \in \mathcal{F}_\Delta \) and \( G \in \mathcal{F}_\nabla \), we have
\[
\text{gdim} \bigoplus_{n \in \mathbb{Z}} \text{Hom}(F, G(n)) = \langle \text{ch}_\Delta(F), \text{ch}_\nabla(G) \rangle.
\]

Lemma 5.5 implies that each \( \xi_s \) restricts to an endofunctor on \( \mathcal{F}_\Delta \) (resp. \( \mathcal{F}_\nabla \)), and for \( F \in \mathcal{F}_\Delta \) and \( G \in \mathcal{F}_\nabla \), we have
\[
\text{ch}_\Delta(\xi_s F) = H_s \text{ch}_\Delta(F), \quad \text{ch}_\nabla(\xi_s G) = H_s \text{ch}_\nabla(G).
\]
Given expressions \( x, y \), it follows from (5.8) and (5.9) that
\[
\text{gdim} \bigoplus_{n \in \mathbb{Z}} \text{Hom}(T_x, T_y(n)) = \langle \text{ch}_\Delta(T_x), \text{ch}_\nabla(T_y) \rangle = \langle H_{w_x}, H_{w_y} \rangle.
\]

6. Koszul Duality

Let \((W, h)\) be a reflection faithful realization. In this section, we assume in addition that \( W \) is finite. Denote the longest element of \( W \) by \( w_0 \).
6.1. Preliminaries. We collect a few results about $\mathbf{P}_{\text{mix}}(U \setminus G/B)$.

As in [AR16b], the following result may be proved by imitating the argument of [BBM04, §2.1] or [BY13, Lemma 4.4.7].

Lemma 6.1 (cf. [AR16b], Lemma 4.9). Let $\omega \in W$.

(1) There exists an embedding $\delta(-\ell(\omega)) \hookrightarrow \Delta_{\omega}$ whose cokernel has no composition factor of the form $\delta(n)$.

(2) There exists a surjection $\nabla_{\omega} \twoheadrightarrow \delta(\ell(\omega))$ whose kernel has no composition factor of the form $\delta(n)$.

Fix once and for all a projective cover $\pi : \mathcal{P}_e \rightarrow \delta$ of the skyscraper.

Lemma 6.2. Let $\omega \in W$. We have

$$(\mathcal{P}_e : \Delta_{\omega}(n)) = \begin{cases} 1 & \text{if } n = -\ell(\omega); \\ 0 & \text{otherwise}. \end{cases}$$

Proof. This follows from graded BGG reciprocity [AR16b, Theorem A.3] and Lemma 6.1(2). $\square$

Lemma 6.3. Let $s \in S$. Then $[\xi_s \mathcal{P}_e (-1) : \delta] = 1$.

Proof. Use Lemma 6.2 and Lemma 5.5 to find the associated graded of the standard filtration of $\xi_s \mathcal{P}_e (-1)$, then use Lemma 6.1. $\square$

6.2. $\mathbb{V}$ functor. Define $\mathbb{V}$ as the composition

$$
\mathbf{D}_{\text{mix}}(U \setminus G/B) \boxplus \mathbf{Hom} \mathbf{mod}(\mathcal{P}_e, (-)(n)) \xrightarrow{\oplus \mathbf{Hom}(\mathcal{P}_e, (\mathcal{P}_e)(n))} \mu_{\mathcal{P}_e} : R^\mathbb{V} \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathbf{Hom}(\mathcal{P}_e, (\mathcal{P}_e)(n))
$$

where $\mu_{\mathcal{P}_e}$ denotes pullback under the monodromy map $\mu_{\mathcal{P}_e} : R^\mathbb{V} \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathbf{Hom}(\mathcal{P}_e, (\mathcal{P}_e)(n))$ constructed in [31]. Recall the functors $\theta_s := B^s \otimes R^\mathbb{V} (-) : R^\mathbb{V} \mathbf{mod} \rightarrow R^\mathbb{V} \mathbf{mod}$, $B^s := (R^\mathbb{V}) \otimes (R^\mathbb{V})^* : R^\mathbb{V} \{1\}$ “generating” Parity$(B^\mathbb{V} \setminus G^\mathbb{V} / U^\mathbb{V})$.

Proposition 6.4. The $\mathbb{V}$ functor is cohomological (transforms a distinguished triangle into a long exact sequence) and satisfies $\mathbb{V} \circ (1) = \{1\} \circ \mathbb{V}$. Moreover, for any $s \in S$, there is a natural isomorphism $\theta_s \circ \mathbb{V} \cong \mathbb{V} \circ \xi_s$.

The following proof uses subregular Soergel theory (see [31, 2.3]).

Proof. The first two claims are clear from the construction. For the last claim, let $s \in S$. Fix a nonzero morphism $\text{can}_s : \mathcal{P}_e \rightarrow \xi_s \mathcal{P}_e (-1)$ in $\mathbf{P}_{\text{mix}}(U \setminus G/B)$, unique up to scalar by Lemma 6.3. For each $\mathcal{F} \in \mathbf{D}_{\text{mix}}(U \setminus G/B)$, define the graded $k$-linear map

$$
\gamma_\mathcal{F} : \mathbb{V}(\mathcal{F})\{1\} \rightarrow \mathbb{V}(\xi_s \mathcal{F})
$$
on homogeneous elements by

$$(\mathcal{P}_e \xrightarrow{\text{can}_s} \mathcal{F}(n + 1)) \rightarrow (\mathcal{P}_e \xrightarrow{\text{can}_s} \xi_s \mathcal{P}_e (-1) \xrightarrow{\xi_s \mathcal{F}(-1)} \xi_s \mathcal{F}(n)).$$
It follows from Lemma 4.12 and Proposition 5.3 that $\gamma'_s$ is $(R^\vee)^s$-linear, so it induces a natural transformation

$$\gamma: \theta_s \circ V \to V \circ \xi_s.$$  

We claim that this is an equivalence. Since $V$ is cohomological and $\theta_s$ is exact (because $R^\vee$ is free over $(R^\vee)^s$), $\theta_s \circ V$ is cohomological. Similarly, since $\xi_s$ is exact, $V \circ \xi_s$ is cohomological. Thus by the five lemma, it suffices to show that

$$\gamma_{IC_w}: R^\vee \otimes (R^\vee)^s \circ V(\mathcal{I}(\mathcal{C}_w)^1) \to V(\mathcal{I}(\mathcal{C}_w)^1)$$

is an isomorphism for all $w \in W$.

First suppose $w \neq e$, so $V(\mathcal{I}(\mathcal{C}_w)) = 0$. We claim that $V(\mathcal{I}(\mathcal{C}_w)) = 0$. There is some $t \in S$ (which may be $s$) with $wt < w$, and $\mathcal{I}(\mathcal{C}_w) \cong \pi^{ts} \mathcal{I}(\mathcal{C}_w^\vee 1)$, where $\pi$ is the image of $w$ in $W/\{1,t\}$. So by (5.5), $\mathcal{I}(\mathcal{C}_w) \cong \xi_s \pi^{ts} \mathcal{I}(\mathcal{C}_w^\vee 1) \cong \pi^{ts} \xi_s \mathcal{I}(\mathcal{C}_w^\vee 1)$. Since $\pi^{ts} \{1\}$ is perverse $t$-exact, this shows that no twist of $\delta$ can appear as a composition factor of $\xi_s \mathcal{I}(\mathcal{C}_w)$, and the claim follows.

Now let $w = e$. Then as graded $k$-vector spaces, both sides of (6.1) are isomorphic to $k \{1\} \oplus k\{-1\}$. From the monodromy of $\xi s = T_s$, we know that the action of $\alpha_\xi^s$ on $V(\xi, \delta)$ maps the degree $-1$ part isomorphically to the degree 1 part. Hence to show that $\gamma_s$ is an isomorphism, it is enough to check it in degree $-1$, i.e. that $\gamma_s(1 \otimes \pi) = \xi_s \pi(-1) \circ \text{can}_s$ is an epimorphism, so the map

$$\xi_s \pi(-1) \circ -: \text{Hom}(P_e, \xi_s P_e(-1)) \to \text{Hom}(P_e, \xi_s \delta(-1))$$

is surjective. But the right hand side is one-dimensional, and so is the left hand side by Lemma 6.3 so this is an isomorphism. In particular, $\xi_s \pi(-1) \circ \text{can}_s \neq 0$. □

6.3. Proof of the main result. We begin with the following analogue of [AR16h, Proposition 5.3].

**Proposition 6.5.** The $V$ functor restricts to an equivalence of additive categories

$$\nu: \text{Tilt}^\text{mix}(U \setminus G/B) \to \text{Parity}(B^\vee \setminus G^\vee / U^\vee)$$

satisfying $\nu \circ (1) \cong \{1\} \circ \nu$ and $\nu(T_w) \cong E_{w}^\vee$.

**Proof.** The claim about the interaction with shifts follows from the corresponding claim in Proposition 6.4. The last claim in Proposition 6.4 implies that $V(\mathcal{T}_w) \cong E_{w}^\vee$ for any expression $w$. It then follows from the Bott–Samelson characterization of indecomposable parity complexes (Proposition 2.3) and indecomposable tilting objects (Proposition 5.6) that $V$ restricts to a functor $\nu$ as claimed, and that $\nu(T_w) \cong E_{w}^\vee$.

By the argument of [BBM04, §2.1], it follows from Lemma 6.1 that $\nu$ is faithful. It therefore suffices to compare dimensions of Hom spaces between Bott–Samelson objects. These agree by the Soergel Hom formula (3.3) and its tilting analogue (5.10). □

We are now ready to prove our main result.

**Proof of Theorem 2.8.** Since $P^\text{mix}(U \setminus G/B)$ is graded highest weight, the natural functors

$$(6.2) \quad K^b \text{Tilt}^\text{mix}(U \setminus G/B) \to D^b P^\text{mix}(U \setminus G/B) \to D^\text{mix}(U \setminus G/B)$$
are equivalences by [AR16a, Lemma B.5] (cf. [AR16b, Lemma 3.15]). Define $\kappa$ as the composition
\[
D^{\text{mix}}(U \setminus G/B) = K^b\text{Parity}(U \setminus G/B) \xrightarrow{K^b\nu} K^b\text{Tilt}^{\text{mix}}(B^\vee \setminus G^\vee / U^\vee) \xrightarrow{6.2} D^{\text{mix}}(B^\vee \setminus G^\vee / U^\vee).
\]
The claims about the interaction with shifts are clear. Proposition 6.5 implies $\kappa(T^v_w) \cong \mathcal{E}^v_w$. This is enough to determine $\kappa(\Delta^v_w)$ and $\kappa(\nabla^v_w)$ as in the proof of [AR16b, Lemma 5.2]. It remains to show that $\kappa(\mathcal{E}^v_w) \cong T^v_w$. Consider the functor $\kappa^v : D^{\text{mix}}(B^\vee \setminus G^\vee / U^\vee) \xrightarrow{\sim} D^{\text{mix}}(U \setminus G/B)$ defined in the same way for $G^\vee$, so all but the last claim is also known for $\kappa^v$. Since $T^v_w$ is a successive extension of various $\Delta^v_x(n)$ (resp. $\nabla^v_x(n)$), we may apply $\kappa \circ \kappa^v$ to conclude the same for $\kappa(\mathcal{E}^v_w)$. Hence $\kappa(\mathcal{E}^v_w)$ is perverse. Repeating this argument, we deduce that $\kappa(\mathcal{E}^v_w)$ is tilting. Since $\mathcal{E}^v_w$ is indecomposable, so is $\kappa(\mathcal{E}^v_w)$. By inducting on $w$ as in the argument of [AR16b, Lemma 5.2], we see that the support condition and the normalization of $\mathcal{E}^v_w$ implies the same for $\kappa(\mathcal{E}^v_w)$. These conditions characterize $T^v_w$. □

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