Existence of families of Galois representations and new cases of the Fontaine-Mazur conjecture

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Abstract

In a previous article, we have proved a result asserting the existence of a compatible family of Galois representations containing a given crystalline irreducible odd two-dimensional representation. We apply this result to establish new cases of the Fontaine-Mazur conjecture, namely, an irreducible Barsotti-Tate $\lambda$-adic 2-dimensional Galois representation unramified at 3 and such that the traces $a_p$ of the images of Frobenii verify $Q(\{a_p^2\}) = Q$ always comes from an abelian variety. We also show the non-existence of irreducible Barsotti-Tate 2-dimensional Galois representations of conductor 1 and apply this to the irreducibility of Galois representations on level 1 genus 2 Siegel cusp forms.

1 Existence of families

The following is a slight generalization of a result proved in [D3], which follows from the results and techniques in [T1], [T2]:

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Theorem 1.1 Let $q$ be an odd prime, and $Q$ a place above $q$. Let $\sigma_Q$ be a two dimensional odd irreducible $Q$-adic Galois representation (of the absolute Galois group of $Q$, continuous) ramified only at $q$ and at a finite set of primes $S$. Assume that $\sigma_Q$ is crystalline at $q$, with Hodge-Tate weights $\{0, w\}$ ($w$ odd). Assume also that $q \geq 2w + 1$. Then, there exists a compatible family of Galois representations $\{\sigma_\lambda\}$ containing $\sigma_Q$, such that for every $\ell \not\in S$, $\lambda | \ell$, the representation $\sigma_\lambda$ is unramified outside $\{\ell\} \cup S$ and is crystalline at $\ell$ with Hodge-Tate weights $\{0, w\}$. Moreover, the family $\{\sigma_\lambda\}$ is strictly compatible (see [T1] for the definition) and all its members are irreducible.

Remark: the proof given in [D3] applies in this generality, just observe that for the case of $w = 1$, in general when Taylor proves that the restriction of $\sigma_Q$ to some totally real field $F$ will correspond to a representation attached to a Hilbert modular form $h$, this is not enough to conclude that the representation is motivic (the construction of Blasius-Rogawski does not apply in some cases), thus is not enough in general to prove the Fontaine-Mazur conjecture, but in any case there is a strictly compatible family of Galois representations attached to $h$ (by previous results of Taylor) and so the argument in [D3] (descent of this family to a compatible family of $G_Q$-representations) can be applied also in this case. The family obtained will be strictly compatible, as follows from the results in [T1]. All representations in the family are irreducible because when restricted to the Galois group of $F$ they agree with the modular Galois representations attached to $h$, which are irreducible (because $h$ is cuspidal).

2 Fontaine-Mazur for “projectively rational” Barsotti-Tate representations

Now suppose that we are given a representation $\sigma_Q$ as in theorem 1.1 with $w = 1$ and $\det(\sigma_Q) = \chi$. Then, we will prove the following:

Theorem 2.1 Assume that $\sigma_Q$ verifies also the following two conditions:
1) If $q \neq 3$, then $3 \not\in S$ ($\sigma_Q$ unramified at 3).
2) The traces $\{a_p\}$ of the images of Frobenii, for every $p \neq q, p \not\in S$ verify: $a_p^2 \in \mathbb{Z}$.

Then $\sigma_Q$ can be attached to an abelian variety $A$ (of course, $A$ is of $GL_2$-type).
Remark: In particular, in case all the traces are integers, this result shows that a Galois representation that “looks like” the one attached to an elliptic curve with good reduction at 3 does indeed come from such an elliptic curve.

Remark: A similar result is proved in [T1] without restriction on the traces but (sticking to the case \( w = 1 \)) with the extra assumption that there is a prime \( u \in S \) such that the restriction of \( \sigma_Q \) to the decomposition group \( D_u \) is of a particular type (corresponding to discrete series under the local Langlands correspondence). Thus, this result does not apply, for example, if \( \sigma_Q \) has conductor 1 or is semistable (i.e., unipotent) locally at every prime of \( S \). Therefore, in the semistable case, the results of [T1] are not enough to prove the Fontaine-Mazur conjecture for \( \sigma_Q \).

Proof: The proof follows from the combination of theorem [T1] with modularity results la Wiles. We know that there exists a strictly compatible family \( \{ \sigma_\lambda \} \) containing \( \sigma_Q \). Take \( t \mid 3 \) and consider the Galois representation \( \sigma_t \) (if \( q = 3 \), just take \( t = Q \)). As \( 3 \notin S \), we know that \( \sigma_t \) is crystalline at \( t \), and has Hodge-Tate weights \( \{0, 1\} \). Following the initial idea of Wiles (see also [D1]) we know, from condition 2) in the theorem (via results of Langlands and Tunnell), that the residual representation \( \overline{\sigma}_t \) will be either modular or reducible. The information we have on \( \sigma_t \) is enough then to conclude, via a combination of modularity results la Taylor-Wiles and Skinner-Wiles, that \( \sigma_t \) is modular. This is a non-trivial assertion, but this is done in [D1] in exactly the same situation! So we conclude that the family \( \{ \sigma_\lambda \} \) is modular, and from \( w = 1 \) we easily check that it will be attached to a weight 2 cusp form \( f \). This proves that \( \sigma_Q \) can be attached to the abelian variety \( A_f \).

Remark: It follows from condition 2) in the theorem that the variety \( A_f \) will have a large endomorphism algebra (cf. [R]).

3 Non-existence of Barsotti-Tate representations of conductor 1

In this section we will prove the following result:
Theorem 3.1 There are no irreducible Barsotti-Tate 2-dimensional Galois representations of conductor 1 and odd residual characteristic.

Remark: By Barsotti-Tate we mean crystalline Galois representations as $\sigma_Q$ in theorem 1.1 with $w = 1$.

Proof: As in the previous section, $q$ is odd and $\sigma_Q$ has $w = 1$. Now there is no restriction in the field of coefficients, but $S$ is empty. If $q \neq 3$, once again we use the results in section 1 to construct a strictly compatible family $\{\sigma_\lambda\}$ containing $\sigma_Q$. We consider again $\sigma_t$ for $t \mid 3$ (just take $t = Q$ if $q = 3$). From strict compatibility, this $t$-adic Galois representation will be unramified outside 3. But such a representation can not exist, as was proved in [D2]. Let us briefly recall the argument for the convenience of the reader: the residual representation $\bar{\sigma}_t$ has coefficient in a finite field of characteristic 3 and is unramified outside 3. Then, a result of Serre tells us that it must be reducible. An application of results of Skinner-Wiles (that can be applied because $\sigma_t$ is Barsotti-Tate) shows that $\sigma_t$ is modular, but this is a contradiction because it must correspond to a level 1, weight 2, cuspidal modular form.

Corollary 3.2 Let $f$ be a genus 2, level 1, cuspidal Siegel modular form (Hecke eigenform) having multiplicity one and weight $k > 3$. Suppose that $f$ is not a Maass spezialform. Then, for every odd prime $\ell$, $\lambda \mid \ell$ in $E =$ field generated by the eigenvalues of $f$, the Galois representation $\rho_{f,\lambda}$ attached to $f$ is absolutely irreducible. In particular, this representation can be defined over $E_\lambda$.

Proof: In [D2] it is shown that the only possible reducible case is: $\rho_{f,\lambda} \cong \sigma_{1,\lambda} \oplus \sigma_{2,\lambda}$ where one of the two (necessarily irreducible) components, say $\sigma_{2,\lambda}$, is crystalline, with Hodge-Tate weights $\{k - 2, k - 1\}$, and has conductor 1. But theorem 3.1 says that such a representation (after twisting by $\lambda^{2-k}$) can not exist. This proves the corollary.

Remark: This improves the main result of [D3], for the level 1 case: “uniformity of reducibility” now does not make sense, because all representations will be irreducible.

In the semistable case, the main theorem of [D3] can be extended, just by applying theorem 1.1 to be valid for every prime:
Proposition 3.3 Let \( \{\rho_\lambda\} \) be a compatible family of 4-dimensional, pure, symplectic, Galois representations, with finite ramification set \( S \), semistable (at every place in \( S \)) in the sense of \([D3]\) and such that for every \( \ell \not\in S \), \( \lambda | \ell \), \( \rho_\lambda \) is crystalline with Hodge-Tate weights \( \{0, k-2, k-1, 2k-3\} \). Then, if for some \( q > 2, q \not\in S \), \( Q \mid q \), the representation \( \rho_Q \) is reducible, all the representations in the family are reducible.

Proof: It is known (cf. \([D2]\), \([D3]\)) that the only possible reducible case is: \( \rho_Q \cong \sigma_1,Q \oplus \sigma_2,Q \), where the two components are irreducible odd two-dimensional Galois representations of the same determinant, one of them having Hodge-Tate weights \( \{0, 2k-3\} \), the other having weights \( \{k-2, k-1\} \).

If \( q \geq 4k-5 \), an application of theorem \([D1]\) to both components (for one of them, you may twist by a power of \( \chi \) before applying the theorem, and then untwist to obtain the desired family) proves the result, because from compatibility and Cebotarev density theorem it is clear that the families \( \{\sigma_{1,\lambda}\} \) and \( \{\sigma_{2,\lambda}\} \) verify: \( \rho_\lambda \cong \sigma_{1,\lambda} \oplus \sigma_{2,\lambda} \) (up to semisimplification) for every \( \lambda \).

If \( 2 < q < 4k-5 \), then we can only apply (with the twist and untwist trick) theorem \([D2]\) to one of the components, say \( \sigma_2,Q \in \{\sigma_{2,\lambda}\} \). The techniques of \([D3]\) apply precisely in this situation: it is enough (because we are assuming semistability) to have the “existence of a family” result for one component, to conclude reducibility of \( \{\rho_\lambda\} \) for almost every prime But this implies that we can take now a second prime \( r \) as large as we want \( (r \geq 4k-5, r \not\in S) \), and \( R \mid r \) such that \( \rho_R \) is reducible: \( \rho_R \cong \sigma_{1,R} \oplus \sigma_{2,R} \), and now because \( r \) is sufficiently large we can apply theorem \([D1]\) also to the other component, and conclude as before that the whole family \( \{\rho_\lambda\} \) is reducible.

Remark: When we assume that \( \rho_Q \) is reducible, reducibility must necessarily “occur over the field of coefficients” (the field generated by the traces of the images of Frobenii), i.e., the field of coefficients of \( \rho_Q \) must contain the fields of coefficients of its irreducible components (this is required in the proof above, because we have used the results of \([D3]\)). That reducibility always occurs over the field of coefficients was proved in \([D2]\) for \( q \geq 4k-5 \) but the proof holds for every odd prime \( q \): if we assume that this is not the case, following the arguments in \([D2]\), we conclude that the field of coefficients of \( \sigma_{2,Q} \) is an infinite extension of that of \( \rho_Q \) (to orient the reader, let us point out that this result strongly depends on the fact that \( \rho_Q \) has four different Hodge-Tate weights, and that when reducibility is not over the fields of coefficients this forces the images of the representations \( \{\rho_\lambda\} \) to be
“generically large”, cf. [D2], [D3]), but this contradicts the fact that $\sigma_{2,Q}$ (after twisting) is potentially modular (thus, the field of coefficients of $\rho_{Q|G_F}$ is a number field for some totally real number field $F$, and a fortiori the field of coefficients of $\rho_Q$ is also a finite extension of $\mathbb{Q}$).

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