Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations

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The Hamiltonian theory of isomonodromy equations for meromorphic connections with irregular singularities on algebraic curves is constructed. An explicit formula for the symplectic structure on the space of monodromy and Stokes matrices is obtained. The Whitham equations for the isomonodromy equations are derived. It is shown that they provide a flat connection on the space of the spectral curves of the Hitchin systems.

Abstract

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1 Introduction

The goal of this paper is multi-fold. Our first objective is to construct isomonodromy equations for meromorphic connections with irregular and regular singularities on algebraic curves. The isomonodromy equations for linear systems with irregular singularities on rational curves generalizing Schlesinger’s equations [1] were introduced by Jimbo, Miwa and Ueno [2]. A particular case of these equations was considered earlier by Flashka and Newell [3] in connection with a theory of self-similar solutions of the mKdV equation. Fuchsian systems on higher genus Riemann surfaces were considered in [4]. The case of linear systems with one irregular singularity on an elliptic curve in [5]. The recent burst of interest to the isomonodromy equations for linear systems with regular singularities on higher genus Riemann surfaces is due to their connections with the classical limit of Knizhnik-Zamolodchikov-Bernard equations for correlation functions of the Wess-Zumino-Witten-Novikov theory. For the case of rational and elliptic curves these connections were revealed in [6, 7, 8]. General case was considered in [9], where more complete list of references can be found.

The conventional modern approach to a theory of the isomonodromy equations is based on their representation in a form of compatible non-autonomous Hamiltonian systems that can be identified as the Hamiltonian reduction of some free Hamiltonian theory. This approach presents an almost exhaustive geometric description of the system but it requires solving the corresponding moment map equations in order to get an explicit form of the equations or their Lax representation. The moment map equations are differential equations on the algebraic curve. They have been solved explicitly only in very few cases [9].

As in [1, 2], the starting point of our approach is the Lax representation of the isomonodromy equations. In the next section the space of meromorphic connections on stable, rank \( r \), and degree \( rg \) holomorphic vector bundles on an algebraic curve \( \Gamma \) with the poles divisor \( D = \sum_m (h_m+1)P_m \) is identified with orbits \( \mathcal{A}^D/SL_r \) of the adjoint action of \( SL_r \) on a certain subspace \( \mathcal{A}^D \) of meromorphic matrix-valued differentials on \( \Gamma \). A characteristic property of \( \tilde{L} \in \mathcal{A}^D \) is that all its additional singularities at points \( \gamma_s \notin D \) have the form \( \frac{d}{d\Phi} \Phi^{-1} \), where \( \Phi \) is holomorphic. We show that an open set of \( \mathcal{A}^D \), corresponding to the case when all the additional poles of \( \tilde{L} \) are simple, is parametrized by the data

\[
\tilde{L}_m, (\gamma_s, \kappa_s), \quad L_{s0} = \beta_s \alpha_s^T,
\]

\[
\sum_{s=0}^{rg} L_{s0} + \sum_m \text{res}_{P_m} \tilde{L}_m = 0,
\]

where \( \tilde{L}_m \) is the singular part of \( \tilde{L} \) at \( P_m \), \( (\gamma_s, \kappa_s) \) is a point of the bundle of scalar affine connections on \( \Gamma \), and \( L_{s0} \) is a rank 1 matrix such that \( \text{Tr} L_{s0} = 1 \). We identify matrices \( L_{s0} \) with pairs of \( r \)-dimensional vectors \( \alpha_s = (\alpha_s^i) \), \( \beta_s = (\beta_s^i) \), considered modulo transformation \( \alpha_s \rightarrow \lambda_s \alpha_s, \beta_s \rightarrow \lambda_s^{-1} \beta_s \), and such that \( (\alpha_s^T \beta_s) = 1 \).

From the definition of \( \tilde{L} \in \mathcal{A}^D \) it follows that the equation

\[
d\Psi = \tilde{L}\Psi
\]

has a multi-valued holomorphic solution on \( \Gamma \setminus D \). Let us fix a point \( Q \in \Gamma \). Then, the analytical continuation of \( \Psi \), normalized by the condition \( \Psi(Q) = 1 \), defines a representation
of the fundamental group $\pi_1(\Gamma \setminus D; Q) \hookrightarrow GL_r$. The Stokes matrices and the so-called exponents at the irregular singularities $P_m$, $h_m > 0$ can be defined purely locally, as in the case of genus $g = 0$, if a local coordinate in the neighborhood of $P_m$ is fixed.

The Stokes data and the exponents at $P_m$ depend only on the $h_m$-jet of the local coordinate, and therefore, we identify the space of the isomonodromy deformations of the linear system (1.2) with the moduli space $\mathcal{M}_{g,1}(h)$ of smooth genus $g$ algebraic curves with a puncture $Q$, and with fixed $h_m$-jets of local coordinate at punctures $P_m$. Here and below the isomonodromy deformations are the ones which preserve the monodromy representation, the Stokes matrices, and the exponents. For brevity, all these data we simply call monodromy data.

It is necessary to emphasize that for $g = 0$ our definition of the deformation space is equivalent to the traditional one. According to [2, 3], the isomonodr omy deformations of $\tilde{L}$ are parameterized by the positions of poles and the exponents at the irregular singularities. In this setting the local coordinates at the poles are always fixed, and are defined by the global coordinate on a complex plane. It is easy to show that the deformations of the exponents that correspond to gauge invariant equations for $\tilde{L}$ can be identified with deformations corresponding to a change of the local coordinate.

A change of normalization point $Q$, and the gauge transformation of $\tilde{L}' = g\tilde{L}g^{-1}$, $g \in GL_r$ correspond to conjugation of the monodromy data by a constant matrix. Hence, the space of the isomonodromy deformations of the meromorphic connections $\mathcal{A}^D/SL_r$ is the moduli space $\mathcal{M}_g(h)$ of curves with $h_m$-jets of local coordinates at $P_m$. For the connections with regular singularities $h_m \equiv 0$ at $N$ points the deformation space is just $\mathcal{M}_{g,N}$. The space $\mathcal{A}(h)$ of the all admissible meromorphic differentials with fixed multiplicities $(h_m + 1)$ at the punctures, and the corresponding factor-space of meromorphic connections we consider as bundles

$$\mathcal{A}(h) \hookrightarrow \mathcal{M}_{g,1}(h), \quad \mathcal{A}(h)/SL_r \hookrightarrow \mathcal{M}_g(h).$$

In section 4 the Lax representation for a full hierarchy of the isomonodromy equations is derived. We show that the Lax equations are equivalent to a system of well-defined compatible evolution equations on the space of dynamical variables, which are the parameters (1.1). Naively, the Lax representation

$$\partial_{t_a} \tilde{L} = [M_a, \tilde{L}] - dM_a,$$  

of the isomonodromy equations is just the coordinate-dependent way of saying that $M_a$ are the coefficients of a flat connection on the space of linear systems (1.2) defined by the monodromy data. In order to make sense out of (1.4), it is necessary first to express $M_a$ as a function of $\tilde{L}$, and to show then, that the Lax equation is equivalent to a well-defined system of differential equations for $\tilde{L}$. A’priory the last statement is not obvious, because (1.4) has to be fulfilled identically on $\Gamma$, and the space of $\tilde{L}$ is finite-dimensional. For example, for $g > 0$ it is impossible to define the isomonodromy deformations for matrix-valued differentials with poles only at $D$. The presence of extra poles $\gamma_s$, which become dynamical variables, is a key element, which allows us to overcome that difficulty in defining the isomonodromy equations on higher genus algebraic curves. The very same idea was used in our earlier work.
where an explicit parameterization of the Hitchin systems was obtained, and where infinite-dimensional field generalizations of the Hitchin systems were proposed.

In Section 5 we show that the approach to the Hamiltonian theory of soliton equations proposed in is also applicable to the case of isomonodromy deformations. The key element of this approach is a definition of the universal two-form which is expressed in terms of the Lax operator and its eigenvectors. The proof that the contraction of this form by the vector-field defined by a Lax equation is an exact one-form is very general and does not rely on any specific form of the Lax operator. It provides a direct way to show that the Lax equations are Hamiltonian on suitable subspaces, and at the same time allows to identify the corresponding Hamiltonians.

It turns out that the universal two-form on a space of meromorphic connections is defined identically to the case of isospectral equations if we replace eigenvectors by a solution of equation (1.2). More precisely, let be a subspace of with fixed exponents at the punctures, and let be the formal local solutions of (1.2) at (see (3.6) below). Then the formula

\[
\omega = -\frac{1}{2} \sum_{s=1}^{rg} \text{res}_{\gamma_s} \text{Tr} (\psi^{-1} \delta \tilde{L} \wedge \delta \psi) - \frac{1}{2} \sum_{P_m} \text{res}_{P_m} \text{Tr} (\psi^{-1}_m \delta \tilde{L} \wedge \delta \psi_m)
\]

(1.5)
defines a closed, non-degenerate form on the factor space . The Lax equations restricted to descend to a system of commuting flows which are Hamiltonian with respect to the symplectic structure defined by .

We show that in terms of the parameters can be written as

\[
\omega = \sum_{s=1}^{rg} \left( \delta \kappa_s \wedge \delta z_s + \sum_{i=1}^{r} \delta \beta_i^s \wedge \delta \alpha_i^s \right) + \sum_{m} \omega_m,
\]

(1.6)

where is the canonical symplectic structure on an orbit of the adjoint action of the group of invertible formal holomorphic matrix functions on the space of singular parts of meromorphic matrix differentials in a formal disc with the pole of order . (A set of orbits corresponds to the set of fixed exponents.)

A remarkable property of the symplectic structure for isospectral equations, defined in terms of the Lax operator is that it provides, under quite general circumstances, a straightforward way of construction of action-angle type variables (see examples in [13]-[18]). In section 6 we show that in the case of the isomonodromy equations almost the same arguments lead to an expression of the symplectic form in terms of the monodromy data.

For example, the monodromy data corresponding to a meromorphic connection on an elliptic curve are just a pair of matrices and , considered modulo common conjugation. The monodromy matrix around the puncture is equal to

\[
J = B^{-1}A^{-1}BA.
\]

(1.7)

Symplectic leaves are defined by a choice of the orbit for . Therefore, they can be seen as level sets of the invariants . We show that the symplectic form on , defined by \( \omega \)
is equal to the restriction onto $\mathcal{P}$ of the two-form
\[\chi(A, B) = \text{Tr} \left[ B^{-1} \delta B \wedge \delta AA^{-1} - A^{-1} \delta A \wedge \delta BB^{-1} + \delta JJ^{-1} \wedge B^{-1}A^{-1}\delta(AB) \right]. \tag{1.8}\]

The expression for $\omega$ on symplectic leaves of the space of conjugacy classes of representation of the fundamental group of genus $g$ Riemann surface with one puncture is given by the formula (3.3). In a different form this result was obtained in [19]. An $r$-matrix representation of the Poisson structure on the space of flat connections on Riemann surfaces with boundaries was found in [20].

To the best of the author’s knowledge, the general closed expression for the symplectic structure on orbits of the adjoint action of $\text{SL}_r$ on the space of monodromy matrices $A_i, B_i$ and Stokes matrices, given by the Theorem 6.1 is new. Even in the genus 0 case, the Poisson structure on the space of Stokes matrices corresponding to meromorphic connections with one irregular singularity of order 2 and one regular singularity was found only recently [21]. The Poisson structure was identified with that of the Poisson-Lie group $G^*$ dual to $G = \text{GL}_r$. In [22] this result was generalized for $G$-valued Stokes matrices for arbitrary simple Lie group, and very interesting connections with a theory of Weyl quantum groups was found. The Poisson structure on the space of Stokes matrices corresponding to skew-symmetric meromorphic connection with one regular and one irregular order 2 singularity was found earlier in [23]. The Poisson structure for $(2 \times 2)$ Stokes matrices corresponding to meromorphic connections with one irregular singularity of order 4 was obtained in [3].

To some extent our main result of Section 6 is preliminary. The general expression for $\omega$ in terms of the monodromy data came out of the blue, as a result of straightforward computations. It seems urgent to find its interpretation in terms of the Poisson-Lie group theory.

The last goal of this paper is to establish connections between solutions of the isomonodromy equations on algebraic curves and solutions of the Hitchin systems. It is well-known that solutions of the Schlesinger equations after proper rescaling can be treated as “modulation” of solutions of the Garnier system [24]. An attempt to revisit this connection in light of the Whitham theory [25, 26, 27, 28] was made in [29], but the heuristic arguments used in [29] do not allow to derive the modulation equations in a closed form.

The problem which we address in Section 7 is as follows. The space of meromorphic $\varepsilon$-connection with fixed multiplicities $h = \{h_m\}$ of poles is the space of orbits of the adjoint action of $\text{SL}_r$ on the space $\mathcal{A}_\varepsilon(h)$ of meromorphic differentials $\tilde{L}_\varepsilon$ such that $\varepsilon^{-1}L_\varepsilon \in \mathcal{A}(h)$. They are parametrized by the data (1.1) such that $\text{Tr} L_{s0} = \varepsilon$. The family of meromorphic $\varepsilon$-connections defined for $\varepsilon \neq 0$ extends to a smooth family over the whole disc. The central fiber over $\varepsilon = 0$ parametrizes the space $\mathcal{L}$ of Lax matrices on algebraic curves introduced in [11]. The orbits of the adjoint action of $\text{SL}_r$ on a subspace of $\mathcal{L}$, corresponding to a fixed algebraic curve $\Gamma$, and fixed singular parts of the eigenvalues of $\tilde{L}_m$ can be identified with the phase space of the generalized Hitchin system.

In order to get a smooth at $\varepsilon = 0$ family of the isomonodromy equations for $\varepsilon$-connections, it is necessary to rescale the coordinates $T_a$ on $\mathcal{M}_{q=1}(h)$. More precisely, if we define the coordinates $t_a = \varepsilon^{-1}T_a$, then the deformations of $\tilde{L}_\varepsilon$ that preserve the monodromy data
associated with a solution of the equation
\[ \varepsilon d\psi = \tilde{L}_\varepsilon \psi \] (1.9)
are described by the equations
\[ \partial_a \tilde{L}_\varepsilon - \varepsilon dM_a + [\tilde{L}_\varepsilon, M_a] = 0. \] (1.10)

The equations (1.10) are Hamiltonian and the corresponding Hamiltonians do converge to certain quadratic Hamiltonians of the Hitchin system, as \( \varepsilon \to 0 \). Therefore, locally solutions of (1.10) converge to the solutions of the Hitchin system. At the same time a global behaviour of solutions of the isomonodromy and isospectral flows is quite different. The monodromy data which are preserved by (1.10) vanish in the limit \( \varepsilon \to 0 \). The space of integrals of the Hitchin system can be regarded as a space \( \mathcal{S} \) of the so-called spectral curves. It is of dimension which is only half of the dimension the monodromy data. For \( \tilde{L}_0 \in \mathcal{L} = \mathcal{A}_0(h) \) the time-independent spectral curve is defined by the characteristic equation
\[ \det (\tilde{k} - \tilde{L}) = 0. \] (1.11)

The spectral curve \( \hat{\Gamma} \) is \( r \)-fold branch cover of the initial algebraic curve \( \Gamma \). The equations of motion for the Hitchin system are linearized on the Jacobian of \( \hat{\Gamma} \).

In Section 7 we apply ideas of the multi-scale perturbation theory to the construction of asymptotic solutions of the isomonodromy equations using solutions of the Hitchin system. In this approach the leading term of the approximation describes the motion which is to first order the original fast motion on the Jacobian, combined with a slow drift on the moduli space of the spectral curves. We obtain an explicit form of the Whitham equations describing that slow drift. They imply that the real part of the periods of the differential \( \tilde{k} \) on \( \hat{\Gamma} \) are preserved along the slow drift. We would like to emphasize that the correspondence
\[ \hat{\Gamma} \in \mathcal{S} \mapsto \text{Re} \int_c \tilde{k}, \quad c \in H_1(\hat{\Gamma}) \] (1.12)
define a flat real connection on the moduli space of the spectral curves, considered as a bundle over \( \mathcal{M}_g(h) \). To some extend our result provides evidence that this connection is a residual of the flat connection on \( \mathcal{A}_c(h) \) defined by the monodromy data in the limit \( \varepsilon \to 0 \). It would be quite interesting to find a more geometrical interpretation of that residual correspondence.

## 2 Meromorphic connections

Let \( V \) be a stable, rank \( r \), and degree \( rg \) holomorphic vector bundle on a smooth genus \( g \) algebraic curve \( \Gamma \). Then the dimension of the space of its holomorphic sections is \( r = \dim H^0(\Gamma, V) \). Let \( \sigma_1, \ldots, \sigma_r \) be a basis of this space. The vectors \( \sigma_i(\gamma) \) are linear independent at the fiber of \( V \) over a generic point \( \gamma \in \Gamma \), and are linearly dependent
\[ \sum_{i=1}^r \alpha_i^s \sigma_i(\gamma_s) = 0 \] (2.1)
at zeros $\gamma_s$ of the corresponding section of the determinant bundle associated to $V$. For a generic $V$ these zeros are simple, i.e. the number of distinct points $\gamma_s$ is equal to $rg = \deg V$, and the vectors $\alpha_s = (\alpha^j_s)$ of linear dependence (2.1) are uniquely defined up to a multiplication. A change of the basis $\sigma_i$ corresponds to a linear transformation of $\alpha'_s = g^T \alpha_s$. Hence, an open set $\mathcal{M} \subset \tilde{\mathcal{M}}$ of the moduli space of vector bundles is parameterized by points of the factor-space

$$\mathcal{M} = \mathcal{M}_0 / SL_r, \quad \mathcal{M}_0 \subset S^{rg} (\Gamma \times CP^{r-1}),$$

(2.2)

where $SL_r$ acts diagonally on the symmetric power of $CP^{r-1}$. In [30, 31] the parameters $(\gamma_s, \alpha_s)$ were called Tyurin parameters.

Let $(\gamma, \alpha) = \{\gamma_s, \alpha_s\}$ be a point of the symmetric product $X = S^{rg} (\Gamma \times CP^{r-1})$. Throughout the paper it is assumed that the points $\gamma_s \in \Gamma$ are distinct, $\gamma_s \neq \gamma_k$. The vector bundle $V_{\gamma, \alpha}$ corresponding to $(\gamma, \alpha)$ under the inverse to the Tyurin map is described in terms of Hecke modification of the trivial bundle. In this description the space of local sections of the vector bundle $V_{\gamma, \alpha}$ is identified with the space $F_s$ of meromorphic (row)vector-functions in the neighborhood of $\gamma_s$ that have simple pole at $\gamma_s$ of the form

$$f^T(z) = \frac{\lambda_s \alpha^T_s}{z - z(\gamma_s)} + O(1), \quad \lambda_s \in C.$$

(2.3)

Our next goal is to describe in similar terms the space of meromorphic connections on $V_{\gamma, \alpha}$. Let $D = \sum_m (h_m + 1) P_m$ be an effective divisor on $\Gamma$ that does not intersect with $\gamma$. Then we define the space $A^D_{\gamma, \alpha}$ of meromorphic matrix valued differentials $\tilde{L} = L(z)dz$ on $\Gamma$ such that:

1. $\tilde{L}$ is holomorphic except at the points $\gamma_s$, where it has at most simple poles, and at the points $P_m$ of $D$, where it has poles of degree not greater than $(h_m + 1)$;

2. the singular term of the expansion

$$\tilde{L} = \left( \frac{L_{s0}}{z - z_s} + L_{s1} + L_{s2}(z - z_s) + O((z - z_s)^2) \right) dz, \quad z_s = z(\gamma_s),$$

(2.4)

is a rank 1 matrix of the form

$$L_{s0} = \beta_s \alpha^T_s \iff L^{ij}_{s0} = b^i_s \alpha^j_s,$$

(2.5)

where $\beta_s$ is a vector. Trace of the residue of $\tilde{L}$ at $\gamma_s$ equals 1:

$$\text{res}_{\gamma_s} \text{Tr} \tilde{L} = 1 \iff \alpha^T_s \beta_s = \text{tr} L_{s0} = 1;$$

(2.6)

3. $\alpha^T_s$ is a left eigenvector of the matrix $L_{s1}$, i.e.

$$\alpha^T_s L_{s1} = \alpha^T_s \kappa_s;$$

(2.7)

where $\kappa_s$ is a scalar.

Note that the condition (3) is well-defined, although the expansion (2.3) by itself does depend on the choice of a local coordinate $z$ in the neighborhood of $\gamma_s$. Under a change of local coordinate $w = w(z)$ the eigenvalue $\kappa_s$ in (2.7) gets transformed to $\kappa'_s$, where

$$\kappa_s = \kappa'_s w'(z_s) - \frac{w''(z_s)}{2w'(z_s)}.$$ 

(2.8)
Therefore, the pair \((\gamma_s, \kappa_s)\) is a well-defined point of a total space of the bundle \(C^{\text{aff}}(\Gamma)\) of scalar affine connections on \(\Gamma\).

Sum of all the residues of a meromorphic differential equals zero. Therefore,

\[
\sum_{P_m \in D} \text{res}_{P_m} \text{Tr} \, \tilde{L} = -rg. \tag{2.9}
\]

Hence, in what follows we always assume that \(\deg D = N > 0\). The Riemann-Roch theorem implies that for a generic degree \(N\) divisor \(D\) and a generic set of Tyurin parameters \((\gamma, \alpha)\) the space \(A^D_{\gamma, \alpha}\) is of dimension

\[
\dim A^D_{\gamma, \alpha} = r^2(N + rg + g - 1) - r^2g(r - 1) - rg(r - 1) = r^2(N + g - 1). \tag{2.10}
\]

The first term is the dimension of the space of meromorphic differentials on \(\Gamma\) with the pole divisor \(D + \gamma\). The consecutive terms count the numbers of the constraints \((2.4-2.7)\). A key characterization of these constraints is as follows.

**Lemma 2.1** A meromorphic matrix-function \(L\) in the neighborhood \(U\) of \(\gamma_s\) with a pole at \(\gamma_s\) satisfies the constraints \((2.4-2.7)\) if and only if it has the form

\[
\tilde{L} = d\Phi_s(z)\Phi_s^{-1}(z) + \Phi_s(z)\tilde{L}_s(z)\Phi_s^{-1}(z), \tag{2.11}
\]

where \(\tilde{L}_s\) and \(\Phi_s\) are holomorphic in \(U\), and \(\det \Phi_s\) has at most simple zero at \(\gamma_s\).

The proof is almost identical to that of the Lemma 2.1 in [11].

The constraints \((2.4-2.7)\) imply that the space \(\mathcal{F}_s\) is invariant under the adjoint action of the operator \((\partial_z - L)\), i.e.

\[
f^T \in \mathcal{F}_s \mapsto f^T(\partial_z - L) = -\left(\partial_z f^T + f^T L\right) \in \mathcal{F}_s. \tag{2.12}
\]

Therefore, for generic set of the Tyurin parameters \((\gamma, \alpha)\) the factor-space \(A^D_{\gamma, \alpha}/\text{SL}_r\) corresponding to the gauge transformations

\[
\tilde{L} \rightarrow g\tilde{L}g^{-1}, \quad g \in \text{SL}_r, \tag{2.13}
\]

can be identified with the space of meromorphic connections on \(V_{\gamma, \alpha}\) that have poles at \(P_m\) of degree not greater that \(h_m + 1\).

The explicit parameterization of an open set of the phase space of the Hitchin systems proposed in [11] can be easily extended to the case under consideration. Consider first an open set of the Tyurin parameters such that the dimension of the space \(\mathcal{F}_{\gamma, \alpha}\) of meromorphic (row)vector-functions on \(\Gamma\) with simple poles at \(\gamma_s\) of the form \((2.3)\) equals \(r\). Then, as shown in [11] the matrix \(\alpha^i_s\) is of rank \(r\). We call \((\gamma, \alpha)\) a non-special set of the Tyurin parameters if additionally they satisfy the constraint: there is a subset of \((r + 1)\) indices \(s_1, \ldots, s_{r+1}\) such that all minors of \((r + 1) \times r\) matrix \(\alpha^i_{s_j}\) are non-degenerate. The action of the gauge group on the space of non-special sets of the Tyurin parameters \(\mathcal{M}_0\) is free. We also assume that the corresponding points \(\gamma_s\) do not coincide with the points \(P_m\).

By definition, the singular part \(\tilde{L}_m\) of a meromorphic differential \(\tilde{L}\) is an equivalence class of meromorphic differentials in the neighborhood of \(P_m\) considered modulo holomorphic differentials.
Lemma 2.2 Let $\mathcal{A}^D$ be an affine bundle over $\mathcal{M}_0$ with fibers $\mathcal{A}_{\gamma,\alpha}^D$. Then the map

$$\tilde{L} \in \mathcal{A}^D \mapsto \{\alpha_s, \beta_s, \gamma_s, \kappa_s, \tilde{L}_m\}, \quad (2.14)$$

is a bijective correspondence between points of the bundle $\mathcal{A}^D$ over $\mathcal{M}_0$ and sets of the data $(\alpha_s, \beta_s)$ subject to the constraints $(\alpha_s^T \beta_s) = 1$, and

$$\sum_{s=1}^{rg} \beta_s \alpha_s^T + \sum_{P_m \in D'} \text{res}_{P_m} \tilde{L}_m = 0, \quad (2.15)$$

modulo gauge transformations

$$\alpha_s \rightarrow \lambda_s \alpha_s, \quad \beta_s \rightarrow \lambda_s^{-1} \beta_s. \quad (2.16)$$

Recall that we consider the pairs $(\gamma_s, \kappa_s)$ as points of the bundle $C^{aff}(\Gamma)$.

Example. Let $\Gamma$ be a hyperelliptic curve defined by the equation

$$y^2 = R(x) = x^{2g+1} + \sum_{i=0}^{2g} u_i x^i, \quad (2.17)$$

Parametrization of connections on $\Gamma$ with simple pole at the infinity is almost identical to parameterization of the Hitchin systems on $\Gamma$ proposed in [11]. A set of points $\gamma_s$ on $\Gamma$ is a set of pairs $(y_s, x_s)$, such that

$$y_s^2 = R(x_s). \quad (2.18)$$

A meromorphic differential on $\Gamma$ with residues $(\beta_s \alpha_s^T)$ at $\gamma_s$ and a simple pole at the infinity has the form

$$L \frac{dx}{2y} = \left( \sum_{i=0}^{g-1} L_i x^i + \sum_{s=1}^{rg} (\beta_s \alpha_s^T) \frac{y+y_s}{x-x_s} \right) \frac{dx}{2y}, \quad (2.19)$$

where $L_i$ is a set of arbitrary matrices. The constraints (2.17) are a system of linear equations defining $L_i$:

$$\sum_{i=0}^{g} \alpha_n^T L_i x^i_k + \sum_{s \neq n} (\alpha_n^T \beta_s) \alpha_s^T \frac{y_n+y_s}{x_n-x_s} = \kappa_n \alpha_n^T, \quad n = 1, \ldots, rg, \quad (2.20)$$

in terms of data $\{\gamma_s, \kappa_s, \alpha_s, \beta_s\}$, where $(\alpha_s, \beta_s)$ are arbitrary vectors such that $\alpha_s^T \beta_s = 1$.

For $g > 1$, the correspondence (2.14) descends to a system of local coordinates on $\mathcal{A}^D/SL_r$. Consider the open set of $\mathcal{M}_0$ such that the vectors $\alpha_j, \; j = 1, \ldots, r$, are linearly independent and all the coefficients of an expansion of $\alpha_{r+1}$ in this basis do not vanish

$$\alpha_{r+1} = \sum_{s=1}^{r} c_j \alpha_j, \quad c_j \neq 0. \quad (2.21)$$

Then for each point of this open set there exists a unique matrix $W \in GL_r$, such that $\alpha_j^T W$ is proportional to the basis vector $e_j$ with the coordinates $e_j^i = \delta_j^i$, and $\alpha_{r+1}^T W$ is proportional to the vector $e_0 = \sum_j e_j$. Using the global gauge transformation defined by $W$

$$b_s = W^{-1} \beta_s, \quad a_s = W^T \alpha_s, \quad (2.22)$$
and the part of local transformations

\[ a_s \rightarrow \lambda_s a_s; \quad a_s \rightarrow \lambda_s^{-1} b_s, \]  

(2.23)

for \( s = 1, \ldots, r + 1 \), we obtain that on the open set of \( \mathcal{M}_0 \) each equivalence class has representation of the form \((a_s, b_s)\) such that

\[ a_i = e_i, \quad i = 1, \ldots, r; \quad a_{r+1} = e_0. \]  

(2.24)

This representation is unique up to local transformations (2.23) for \( s = r + 2, \ldots, rg \).

In the gauge (2.24) equation (2.15) can be easily solved for \( b_1, \ldots, b_{r+1} \). Using (2.24), we get

\[ b_j^i + b_{r+1}^i = - \sum_{s=r+2}^g b_s^i a_s^j - \sum_m \text{res} \tilde{L}_{ij}^m. \]  

(2.25)

The condition \( a_j^T b_j = 1 \) for \( a_j = e_j \) implies \( b_j^i = 1 \). Hence,

\[ b_{r+1}^i = -1 - \sum_{s=r+2}^g b_s^i a_s^j - \sum_m \text{res} \tilde{L}_{ij}^m. \]  

(2.26)

Note, that the constraint (2.23) implies \( a_{r+1}^T b_{r+1} = 1 \).

Sets of vectors \( a_*, b_*; a_*^T b_* = 1, r + 1 < s \leq rg \), modulo the transformations (2.23), and points \( \{\gamma_s, \kappa_s\} \in S^{rg}(C^\text{aff}(\Gamma)) \), and sets \( \tilde{L}_m \), satisfying (2.9), provide a parametrization of an open set of the bundle \( \mathcal{A}^D/SL_r \) over \( \mathcal{M} = \mathcal{M}_0/SL_r \). The dimension of this bundle equals

\[ \dim \mathcal{A}^D/SL_r = r^2(N + 2g - 2) + 1. \]  

(2.27)

In the same way, taking various subsets of \((r+1)\) indices we obtain charts of local coordinates which cover \( \mathcal{A}^D/SL_r \).

### 3 Monodromy data

Our next goal is to introduce monodromy data corresponding to \( \tilde{L} \in \mathcal{A}^D_{\gamma,\alpha} \) along identical lines to their definition in the zero genus case. From Lemma 2.1 it follows that the equation

\[ d\Psi = \tilde{L}\Psi \]  

(3.1)

has multi-valued holomorphic solutions on \( \Gamma \setminus \{P_m\} \), \( P_m \in D \). Let \( Q \) be a point on \( \Gamma \). Then the normalization

\[ \Psi(Q) = 1 \]  

(3.2)

defines \( \Psi \) uniquely in the neighborhood of \( Q \). Analytic continuation of \( \Psi \) along cycles in \( \Gamma \setminus \{P_m\} \) defines the monodromy representation

\[ \mu : \pi_1(\Gamma \setminus \{P_m\}; Q) \mapsto GL_r. \]  

(3.3)

It is well-known, that for connections with simple poles the correspondence \( \tilde{L} \rightarrow \mu \) is an injection, and that the inverse map is defined on an open set of the space of representations. For connections with poles of higher order additional so-called Stokes data are needed. Their construction is local, and we mainly follow here [32].
Lemma 3.1 Let \( L \) be a formal Laurent series
\[
\tilde{L} = \sum_{i=-h}^{\infty} L_i w^{s-1} dw
\] (3.4)
such that the leading coefficient has the form
\[
L_{-h} = \Phi K \Phi^{-1}, \quad K = \text{diag} \left( k_1, \ldots, k_r \right), \quad \begin{cases} 
  k_i - k_j \neq 0, & h > 0, \\
  k_i - k_j \notin \mathbb{Z}, & h = 0, \quad i \neq j.
\end{cases}
\] (3.5)

Then the equation (3.1) has a unique formal solution
\[
\psi = \Phi \left( 1 + \sum_{s=1}^{\infty} \xi_s w^s \right) \exp \left( \sum_{i=-h}^{\infty} K_i \int w^{i-1} dw \right),
\] (3.6)
where \( K_i \) are diagonal matrices, \( K_{-h} = K \), and the matrices \( \xi_s \) have zero diagonals, \( \xi_{ss} = 0 \).

Substitution of (3.6) into (3.1) gives a system of equations, which for \( h > 0 \) have the form
\[
K_s + [K, \xi_{s+h}] = R(\xi_1, \ldots, \xi_{s+h-1}, K_{-h+1}, \ldots, K_s), \quad s > -h.
\] (3.7)

They recursively determine the off-diagonal part of \( \xi_s \) and the diagonal matrix \( K_s \). In the similar way \( \psi \) is constructed for \( h = 0 \).

Suppose now that (3.4) is the Laurent expansion of a meromorphic differential in a punctured disk \( U \) which is holomorphic in \( \hat{U} = U \setminus 0 \). Let \( V \) be sector of \( \hat{U} \) that for any pair \((i, j)\) contains only one ray, such that
\[
\text{Re} \ (k_i - k_j) w^{-h} = 0.
\] (3.8)

Then there exists a holomorphic in \( V \) solution \( \Psi_V \) of (3.1) such that the formal solution (3.6) is an asymptotic series for \( \Psi_V \). The asymptotic is uniform in any closed subsector of \( V \).

The punctured disk can be covered by a set of sectors \( V_1, \ldots, V_{2h+1} \) which satisfy the constraint described above, and such that the sectors \( V_\nu \) and \( V_{\nu+1} \) do intersect. On their intersection the solutions \( \Psi_\nu = \Psi_{V_\nu} \) and \( \Psi_{\nu+1} = \Psi_{V_{\nu+1}} \) satisfy the relation
\[
\Psi_{\nu+1} = \Psi_\nu S_\nu, \quad \nu = 1, \ldots, 2h.
\] (3.9)

The Stokes’ matrices \( S_\nu \) are constant matrices. For each \( S_\nu \) there exists a unique permutation under which \( S_\nu \) gets transformed to an upper triangular matrix with the diagonal elements equal 1.

The last property of the Stokes’ matrices follows from more precise statement which we will use in Section 6. Namely, if \( w \) tends to 0 in the intersection of \( V_\nu \) and \( V_{\nu+1} \), then the following limit exists and equals
\[
\lim_{w \to 0} \exp(Kw^{-h}) S_\nu \exp(-Kw^{-h}) = 1.
\] (3.10)
For any pair \((i \neq j)\) the left hand side of (3.8) has a definite sign in \(V_\nu \cap V_{\nu+1}\). Therefore, if this sign is positive, then (3.10) implies \(S^{ij}_\nu = 0\).

Let us fix a local coordinate \(w_m\) in the neighborhood of \(P_m\), \(w_m(P_m) = 0\), and paths \(c_m\) connecting \(Q\) with \(P_m\). In the neighborhood of \(P_m\) we also fix a set of sectors \(V^{(m)}_{\nu}\) described above, and always assume that the path \(c_m\) in the neighborhood of \(P_m\) belongs to the first sector \(V^{(m)}_1\). Then the Laurent expansion of \(\tilde{L} \in \mathcal{A}^D\) at \(P_m\) in this coordinate, defines the diagonal matrices \(K^{(m)}_i\), the Stokes’ matrices \(S^{(m)}_\nu\), and the transition matrix \(G_m\), which connects \(\Psi\) and \(\Psi_1^{(m)}\)

\[
\Psi = \Psi_1^{(m)} G_m. \tag{3.11}
\]

In each of the sectors \(V^{(m)}_{\nu}\) we have

\[
\Psi = \Psi_{\nu}^{(m)} g^{(m)}_{\nu}, \tag{3.12}
\]

where

\[
g^{(m)}_1 = G_m, \quad g^{(m)}_{\nu+1} = \left(S^{(m)}_1 S^{(m)}_2 \cdots S^{(m)}_{\nu}\right)^{-1} G_m, \quad \nu = 1, \ldots, 2h_m. \tag{3.13}
\]

The monodromy \(\mu_m\) around \(P_m\) is equal to

\[
\mu_m = \left(g^{(m)}_1\right)^{-1} e^{2\pi i K^{(m)}_0} g^{(m)}_{2h_m+1} = G_m^{-1} e^{2\pi i K^{(m)}_0} \left(S^{(m)}_1 S^{(m)}_2 \cdots S^{(m)}_{2h_m}\right)^{-1} G_m. \tag{3.14}
\]

If we choose a basis of \(a_j, b_j\) cycles on \(\Gamma\) with the canonical matrix of intersections, then we denote the monodromy matrices along the cycles by \(A_j, B_j, j = 1, \ldots, g\).

**Lemma 3.2** The correspondence

\[
\tilde{L} \in \mathcal{A}^D \mapsto \{K^{(m)}_i, S^{(m)}_{\nu}, G_m, A_j, B_j\}, \tag{3.15}
\]

\[-h_m \leq i \leq 0, \quad \nu = 1, \ldots, 2h_m, \tag{3.16}\]

where the transition and the Stokes matrices are considered modulo transformations

\[
G_m \mapsto W_m G_m, \quad S^{(m)}_\nu \mapsto W_m S^{(m)}_\nu W_m^{-1}, \quad W_m = \text{diag} (W_m,i) \tag{3.17}
\]

is an injection.

**Important remark.** The definition of the full set of the Stokes’ data requires a choice of the local coordinate in the neighborhood of the puncture. But the data \((3.13)\) depend only on the \(h_m\)-jets of the local coordinates, because it contains only diagonal matrices \(K_i\) with indices \(i \leq 0\). We define an \(h\)-jet \([w]_h\) to be an equivalence class of \(w\), with \(w'\) and \(w\) equivalent if

\[
w' = w + O(w^{h+1}). \tag{3.18}
\]

**Proof.** Suppose that \(\tilde{L}\) and \(\tilde{L}_1\) have the same data \((3.13)\) modulo \((3.17)\). Then solutions \(\Psi\) and \(\Psi_1\) of the corresponding systems \((3.1)\) have the same monodromy along each cycle on \(\Gamma \setminus \{P_m\}\). Therefore, \(\phi = \Psi_1 \Psi^{-1}\) is a single-valued meromorphic matrix function on \(\Gamma \setminus \{P_m\}\). From \((3.0)\) and \((3.14)\) it follows that \(\phi\) is bounded in the neighborhood of \(P_m\). Hence, \(\phi\) is a meromorphic function on \(\Gamma\) and is holomorphic at the points \(P_m\). The function \(\Psi\) is invertible.
everywhere except at the the poles $\gamma_s$ of $\bar{L}$. The equation (3.1) implies that vector rows of the residue of $\Psi_1\Psi^{-1}$ at $\gamma_s$ has the form (2.3). The assumption that $(\gamma, \alpha)$ are non-special Tyurin parameters implies that $\phi$ is a constant matrix. Then, from the normalization (3.2) it follows that $\Psi_1 = \Psi$, and $\bar{L} = \bar{L}_1$.

Simple counting shows that $\mathcal{A}_D$ and the space of data (3.15) modulo transformations (3.17) have the same dimension. Therefore, the map (3.15) is a bijective correspondence of $\mathcal{A}_D$ and an open set of the data.

4 Isomonodromy deformations.

Our next goal is to construct differential equations describing deformations of $L \in \mathcal{A}_D(\Gamma)$, which preserve a full set of the data (3.15). For brevity we call them isomonodromy deformations. As it was mentioned above, in order to define the data (3.15) it is necessary to fix a normalization point $Q \in \Gamma$, a basis of $a_i, b_i$-cycles, paths $c_m$ connecting $Q$ with $P_m$, and a set of $h_m$-jets of local coordinates in the neighborhoods of the punctures $P_m$.

Let $h = \{h_m, \sum_m (h_m + 1) = N\}$ be a set of nonnegative integers. Then we denote the moduli space of smooth genus $g$ algebraic curves with a puncture $Q \in \Gamma$, and fixed $h_m$-jets of local coordinates $w_m$ in the neighborhoods of punctures $P_m$ by $\mathcal{M}_{g,1}(h)$. The space $\mathcal{A}(h)$ of admissible meromorphic differentials on algebraic curves with fixed multiplicities $(h_m + 1)$ of the poles can be seen as a total space of the bundle

$$\mathcal{A}(h) \longrightarrow \mathcal{M}_{g,1}(h) = \{\Gamma, P_m, [w_m], Q\}$$

with fibers $\mathcal{A}_D(\Gamma)$, $D = \sum_m (h_m + 1)P_m$. Here and below $[w_m]$ stands for the $h_m$-jet of $w_m$. The space $\mathcal{M}_{g,1}(h)$ is of dimension

$$\dim \mathcal{M}_g(h) = 3g - 2 + N.$$  (4.2)

An explicit form of the isomonodromy equations depends on a choice of coordinates on $\mathcal{M}_{g,1}(h)$. Their Lax representation requires in addition some sort of connection on the universal curve $\mathcal{N}_g(h)$ which is a total space of the bundle

$$\mathcal{N}_g(h) \longrightarrow \mathcal{M}_{g,1}(h)$$

which fibers $\Gamma$.

The following construction solves two problems simultaneously. It goes back to a theory of the Whitham equations [26, 27]. Details can be found in [13, 14]. First of all, locally we can replace the moduli space of algebraic curves by the Teichmuller space of marked algebraic curves, i.e. smooth algebraic curves with fixed basis of $(a_i, b_i)$-cycles, and paths $c_m$ between $Q$ and punctures, which do not intersect cycles. Let us fix a set of integers $r_m, \sum_m r_m = 0$. Then for any set of local coordinates $w_m$ at $P_m$, there is a unique meromorphic differential $dE$ which in the neighborhood of $P_m$ has the form

$$dE = d\left(w_m^{-h_m} + r_m \log w_m + O(w_m)\right),$$  (4.4)
and normalized by the condition
\[ \oint_{a_i} dE = 0. \] (4.5)

The differential \( dE \) depends only on the \( h_m \)-jets of the local coordinates \( w_m \). The zero divisor of \( dE \) has degree \( 2g - 2 + N \). Let \( \mathcal{M}_{g,1}^0(h) \) be an open set of \( \mathcal{M}_{g,1}(h) \) such that the corresponding differential \( dE \) has simple zeros \( q_s \neq Q \)
\[ dE(q_k) = 0, \quad k = 1, \ldots, 2g - 2 + N. \] (4.6)

The Abelian integral
\[ E(q) = \int^q_Q dE, \] (4.7)
is single valued on the cover \( \Gamma^* \) of \( \Gamma \setminus \{P_m\} \) generated by shifts along \( b_i \)-cycles and shifts along cycles \( c'_m \) around the punctures \( P_m \). The curve \( \Gamma \) with cuts along \( a_i \)-cycles and paths \( c_m \) we regard as a marked sheet of \( \Gamma^* \). The critical values
\[ T_k = E(q_k), \] (4.8)
of \( E \) on this sheet, and the \( b \)-periods of \( dE \)
\[ T_{b_i} = \oint_{b_i} dE, \] (4.9)
provide a system of local coordinates on \( \mathcal{M}_{g,1}^0(h) \) (see details in [13]). The Abelian integral \( E \) defines a local coordinate on \( \Gamma^* \) everywhere except at the preimages \( \hat{q}_s \) of the critical points \( q_s \). Therefore, \( (E, T_k, T_{b_i}) \) can be seen as a system of local coordinates on an open set of a total space of the bundle \( \mathcal{N}_{g,1}^*(h) \) over \( \mathcal{M}_{g,1}(h) \) with fibers \( \Gamma^* \).

Let \( \tilde{L}(\tau) \in \mathcal{A}(h) \) be a one-parametric family of admissible differentials. Its projection under (4.1) defines a path \( T_a(\tau) \) in \( \mathcal{M}_{g,1}(h) \). Here and below \( \{a\} \) stands for the both types of indices, i.e. \( T_a = \{T_k, T_{b_i}\} \). The family \( \tilde{L}(\tau) \) we regard as a family of one-forms
\[ \tilde{L}(\tau) = L(E; T_a(\tau))dE, \] (4.10)
where \( L \) is a function of the variable \( E \) on \( \Gamma^*(\tau) \), which is meromorphic everywhere except at \( \hat{q}_s(\tau) \), and such that
\[ L(E + T_{b_j}; T_a) = L(E; T_a), \quad L(E + 2\pi ir_m; T_a) = L(E; T_a). \] (4.11)

In the same way the corresponding solution \( \Psi \) of equation (3.1) can be seen as a multi-valued function \( \Psi(E; T_a) \) of the variable \( E \) which is holomorphic everywhere except at \( \hat{q}_s \) and the preimages \( \hat{P}_m \) on \( \hat{\Gamma} \) of the punctures \( P_m \), and such that
\[ \Psi(E + T_{b_j}; T_a) = \Psi(E; T_a)B_j, \quad \Psi(E + 2\pi ir_m; T_a) = \Psi(E; T_a)\mu_m. \] (4.12)
Lemma 4.1 A one parametric family of meromorphic connections $\tilde{L}(\tau) \in \mathcal{A}_{\gamma(\tau),\alpha(\tau)}^{D(\tau)}(\Gamma(\tau))$ is an isomonodromy family if and only if the logarithmic derivative of the corresponding solution $\Psi$ of (3.1)

$$M(E, \tau) = \partial_{\tau} \Psi(E, \tau)\Psi^{-1}(E, \tau)$$

as a function of $E$ is single valued on $\Gamma^*(\tau)$, (i) equals zero at $Q$, and is holomorphic everywhere except at the points $\tilde{\gamma}_s$, $\tilde{q}_k$ where it has at most simple poles, (ii) the vector rows of $M$ in the neighborhood of $\tilde{\gamma}_s$ have the form (2.3), (iii) the singular part of $M$ at $\tilde{q}_k(\tau)$ equals

$$M(E, \tau) = -\partial_{\tau} E(\tilde{q}_k) L(E, \tau) + O(1), \ E \to E(\tilde{q}_k),$$

(iv) $M$ satisfy the following monodromy properties

$$M(E + T_b; T_a) = M(E; T_a) - (\partial_{\tau} T_b) L(E; T_a),$$

$$M(E + 2\pi i \tau; T_a) = M(E; T_a).$$

Proof. The same arguments, as in the proof of the Lemma 3.2, show that if the Stokes data do not depend on $\tau$, then $M$ is holomorphic at the punctures $\tilde{P}_m$. The matrix $M$ is single-valued on $\Gamma^*$ because monodromies $A_j$ also do not depend on $\tau$. Unlike the previous case, $M$ is single-valued only on $\Gamma^*$, and acquires additional poles at $\tilde{q}_k$, because $E$ is multivalued on $\Gamma$, and is not a local coordinate at the critical points $\tilde{q}_k$.

At the points $\tilde{q}_k = \tilde{q}_k(\tau)$ a local coordinate is $(E - E(\tilde{q}_k))^{1/2}$. Recall, that $E(\tilde{q}_k)$ equals $T_k$ plus an integer linear combination of $T_b_j$ which depends on the branch of $E$ corresponding to $\tilde{q}_k$. The matrix function $\Psi$ is holomorphic in the neighborhood of $\tilde{q}_k$. Therefore, its expansion at $\tilde{q}_k$ has the form

$$\Psi = \phi_0(\tau) + \phi_1(\tau)(E - E(\tilde{q}_k))^{1/2} + O(E - E(\tilde{q}_k)), \tilde{q}_k = \tilde{q}_k(\tau).$$

Then

$$M = -\partial_{\tau} E(\tilde{q}_k) \frac{\phi_1\phi_0^{-1}}{2\sqrt{E - E(\tilde{q}_k)}} + O(1).$$

The logarithmic differential of $\Psi$ has the form

$$d\Psi\Psi^{-1} = LdE = \frac{\phi_1\phi_0^{-1}dE}{2\sqrt{E - E(\tilde{q}_k)}} + O(1)dE.$$ 

Equations (4.18, 4.19) imply (4.14). Equations (4.13, 4.16) directly follow from (4.12, 3.1), and the Lemma is proved.

Let us now introduce basic functions $M_\alpha$ corresponding to the isomonodromy deformations along the coordinates $T_a$. Simple dimension counting proves the following statement.

Lemma 4.2 If $(\gamma, \alpha)$ is a non-special set of the Tyurin parameters, then for each $\tilde{L} \in \mathcal{A}_{\gamma,\alpha}(\Gamma)$ there is a unique meromorphic function $M_k$ on $\Gamma$ such that: (i) $M_k$ is holomorphic everywhere except at the points $\gamma_s$, and at the point $q_k$, (ii) the vector rows of $M_k$ at $\gamma_s$ have the form (2.3), (iii) at the point $q_k$ the singular part of $M_k$ has the form $M_k = -L + O(1)$, (iv) $M(Q) = 0$. 

Let us denote $\Gamma$ with a cut along the $a_i$ cycle by $\Gamma_i^*$.

**Lemma 4.3** If $(\gamma, \alpha)$ is a non-special set of the Tyurin parameters, then for each $\tilde{L} \in \mathcal{A}_{\gamma, \alpha}^D(\Gamma)$ there is a unique function $M_{b_i}$ on $\Gamma_i^*$ such that: (i) $M_{b_i}$ is holomorphic everywhere except at the points $\gamma_s$, where the vector rows of $M_k$ have the form (2.3), (ii) $M_{b_i}$ can be extended as a continuous function on the closure of $\Gamma_i^*$, and its boundary values $M_{b_i}^\pm$ on the two sides of the cut satisfy the relation $M_{b_i}^+ - M_{b_i}^- = -L$, (iii) $M_{b_i}(Q) = 0$.

A meromorphic matrix function on $\Gamma_i^*$ which satisfies the boundary condition (ii) can be represented as the Cauchy type integral over the cycle. The difference of any two such functions is a meromorphic function on $\Gamma$. Therefore, once again a proof of the existence and uniqueness of the function with prescribed analytical properties is reduced to the Riemann-Roch theorem.

If we keep the same notations for pull back of $M_a$ on $\Gamma^*$, then the logarithmic derivative $M = \partial_\tau \Psi \Psi^{-1}$ in the Lemma 4.1 can be written as

$$M = \sum_a (\partial_\tau T_a) M_a.$$  \hspace{1cm} (4.20)

Now we are in position to define a hierarchy of differential equations which describe the isomonodromy deformations. In the neighborhood of $\gamma_s$ the Laurent expansion of $L = \tilde{L}/dE$ and $M_a$ have the form

$$L = \frac{\beta_s \alpha^T_s}{E - e_s} + L_{s1} + L_{s2}(E - e_s) + O((E - e_s)^2), \quad e_s = E(\gamma_s),$$

$$M_a = \frac{m^a_s \alpha^T_s}{E - e_s} + M^a_{s1} + M^a_{s2}(E - e_s) + O((E - e_s)^2),$$ \hspace{1cm} (4.21) (4.22)

where $m^a_s$ are vectors.

**Theorem 4.1** The Lax equations

$$\partial_a \tilde{L} - dM_a + [\tilde{L}, M_a] = 0,$$ \hspace{1cm} (4.23)

define a hierarchy of commuting flows on $\mathcal{A}(h)$ which preserve extended set of monodromy data $\mathcal{B}, \mathcal{T}$. They are equivalent to the equations

$$\partial_a e_s = -\alpha^T_s m^a_s, \quad e_s = E(\gamma_s),$$

$$\partial_a \alpha^T_s = -\alpha^T_s M^a_{s1} - \lambda_s \alpha^T_s,$$ \hspace{1cm} (4.24) (4.25)

$$\partial_a \beta_s = M^a_{s1} \beta_s - (L_{s1} - \kappa_s) m^a_s + \lambda_s \beta_s,$$ \hspace{1cm} (4.26)

$$\partial_a \kappa_s = \alpha^T_s (M^a_{s2} - L_{s2}) \beta_s,$$ \hspace{1cm} (4.27)

$$\partial_a \tilde{L}_m = [M_a, \tilde{L}_m]_+.$$ \hspace{1cm} (4.28)

where $\lambda_s$ are scalar functions, and $[M_a, \tilde{L}_m]_+$ denotes the singular part of $[M_a, \tilde{L}_m]$ at $P_m$.  

Note that if $h_m = 0$, then the right hand side of (4.28) is just $[M_a(q_k), \tilde{L}_m]$. 

**Proof.** First, let us show that the left hand side of (4.23), which we denote by $\phi$, is a single-valued meromorphic function on $\Gamma$ which is holomorphic everywhere except at the points $\gamma_s$ and $P_m$. Indeed, for $M_a = M_k$ it is single-valued by the definition of $M_k$, but may have a pole at $q_k$. Taking derivative of the Laurent expansion of $L$ at $q_k$, we obtain that $\partial_a L$ acquires pole at $q_k$ of the form $\partial_a L = -dL/dE + O(1)$. Hence, the singular part of $\partial_a L$ is just $-dL$, which cancels with the singular part of $dM_k$. From (4.28) it follows that $[M_k, \tilde{L}]$ is regular at $q_k$. Almost identical arguments show that for $M_a = M_b$, the matrix differentials $dM_a$ and $\partial_a L$ have the same monodromy properties along the cycle $b_i$, and therefore, $\phi$ is single-valued on $\Gamma$.

Equations (4.24-4.26) and (4.28) are equivalent to the condition that $\phi$ is a holomorphic matrix differential on $\Gamma$. Then equation (4.27) is equivalent to the condition $\alpha^T \phi(\gamma_s) = 0$. That gives us a system of $r^2g$ linear equations for $\phi$. As shown in [11], for non-special sets of the Tyurin parameters these equation are linear independent, and therefore, imply $\phi = 0$.

The matrix functions $M_a$ are uniquely defined by $\tilde{L}$. Hence equations (4.24-4.28) is a closed system of differential equations on the space of parameters $(\epsilon_s, \kappa_s, \alpha_s, \beta_s, L_m)$. Compatibility of the equations for different indices $a$ is equivalent to the equation 

$$\partial_a M_b - \partial_b M_a + [M_b, M_a] = 0.$$  \hspace{1cm} (4.29)

In order to prove (4.29) we first check that the left hand side of the equation is single valued meromorphic matrix function which is holomorphic everywhere except at $\gamma_s$. Then, from equations (4.27-4.26) it follows that this function has at $\gamma_s$ at most simple pole of the form (2.3). For non-special sets of the Tyurin parameters the last condition implies that the left hand side of (4.29) is a constant matrix function on $\Gamma$. It equals zero due to normalization $M_a(0) = 0$. Recall, that at the marked point $E(Q) = 0$.

From the Lax representation of equations (4.24-4.28) it follows that, if $\tilde{L}$ is a solution of these equations and $\Psi$ is the normalized solution of (3.1), then $\partial_a \Psi \Psi^{-1} = M_a$. The Lemma 4.1 implies the isomonodromy property of the flow, and the Theorem is proved.

**Example 1.** The Schlesinger equations [1] 

$$\partial_i A_j = \frac{[A_i, A_j]}{t_i - t_j}, \ i \neq j,$$  \hspace{1cm} (4.30)

$$\partial_i A_i = \sum_{j \neq i} \frac{[A_j, A_i]}{t_i - t_j}$$  \hspace{1cm} (4.31)

describe the isomonodromy deformations of a meromorphic connection with regular singularities 

$$Ldz = \sum_i \frac{A_i}{z - t_i}dz$$  \hspace{1cm} (4.32)
on the rational curve. In the conventional approach the coordinates $t_i$ of the punctures on the complex plane are considered as coordinates on the space of rational curve with punctures.
In our approach, which also works for higher genus case, we use the function $E = \sum_i \ln(z-t_i)$ for parameterization of points of the complex plane. Critical values $T_k(t)$ of $E$

\[ T_k = \sum_i \ln(q_k - t_i), \quad (4.33) \]

locally define $t_i$ uniquely up to a common shift $t_i \rightarrow t_i + c$. The critical points $q_k$ are roots of the equation

\[ E'(q_k) = \sum_i \frac{1}{q_k - t_i} = 0, \quad E'(z) = \partial_z E(z). \quad (4.34) \]

Note that (4.34) implies

\[ E'(z) = \sum_i \frac{1}{z - t_i} = \frac{\prod_k(z-q_k)}{\prod_i(z-t_i)}. \quad (4.35) \]

From (4.34) it follows that

\[ \partial_i T_k = -\frac{1}{q_k - t_i}. \quad (4.36) \]

According to the Lemma 4.2, the matrix $M_k(z)$ corresponding to the variable $T_k$ has the only pole at $q_k$ which coincides with the singular part of $-L/E_z$. Hence

\[ M_k = -\frac{\text{res}_{q_k}(L/E')}{z-q_k} = -\frac{1}{z-q_k} \left( \sum_i \frac{A_i}{q_k - t_i} \right) \frac{\prod_j(q_k - t_j)}{\prod_{s \neq k}(q_k - q_s)} \quad (4.37) \]

From equation (4.28) we obtain that ismonodromy deformations of $L$ with respect to new coordinates have the form

\[ \partial_{T_k} A_j = -\frac{1}{t_j - q_k} \left( \sum_i \frac{[A_i, A_j]}{q_k - t_i} \right) \frac{\prod_j(q_k - t_j)}{\prod_{s \neq k}(q_k - q_s)} \quad (4.38) \]

It is instructive to check directly that equations (4.38) are equivalent to (4.30). For $i \neq j$ we have

\[ \partial_i A_j = \sum_k (\partial_i T_k) \partial_{T_k} A_j = -\sum_k \text{res}_{z=q_k} \left[ L(z), A_j \right] \frac{\prod_{s \neq i,j}(z-t_s)}{\prod_k(z-q_k)} \quad (4.39) \]

The expression in right hand side of (4.39) has poles at $t_i$ and $q_k$. Hence, $\partial_i A_j$ equals the residue of this expression at $z = t_j$.

\[ \partial_i A_j = [A_i, A_j] \frac{\prod_{s \neq i,j}(t_i-t_s)}{\prod_k(t_i-q_k)} \quad (4.40) \]

Equation (4.35) implies

\[ 1 = \text{res}_{t_i} E'(z) = \frac{\prod_k(t_i-q_k)}{\prod_{s \neq i}(t_i-t_s)}. \quad (4.41) \]

Therefore, the last factor in (4.40) equals $1/(t_i-t_j)$, and we obtain equation (4.30). Equation (4.34) can be replaced by the equation $\sum_i \partial_i A_j = 0$. Therefore, it is enough to check that $\sum_i \partial_i T_k = 0$. The last equation follows from (4.34), and (4.36).
Example 2. The Painleve-II equation

\[ u_{xx} - xu - 2u^3 = \nu \]  

(4.42)

describes an isomonodromy deformation of the rational connection

\[ L = Az^2 + Bz + C + Dz^{-1}, \]  

(4.43)

where

\[ A = -4i\sigma_3, \quad B = -4u\sigma_2, \quad C = -(2iu^2 + x)\sigma_3 - 2u_x \sigma_1, \quad D = \nu \sigma_2, \]  

(4.44)

and \( \sigma_i \) are the Pauli matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(4.45)

We would like to stress once again, that the conventional definition of the isomonodromy deformations of rational connections with irregular singularities are ones which preserve monodromy, transition, and the Stokes matrices. Exponents \( K_i \) are considered as parameters of the deformation (see [3, 2]).

In this example we show that the same equations can be seen as equations describing deformations over the space of jets of local coordinate, which preserve the full set of data (3.15), including exponents. Let us consider the isomonodromy deformation of \( L \) corresponding to the deformation of \( z \) defined by the function

\[ E(z) = \frac{4}{3}z^3 + xz + \ln z. \]  

(4.46)

The critical points \( q_k \) are roots of the equation

\[ 4q_k^2 + x + q_k^{-1} = 0 \implies E'(z) = 4z^{-1} \prod_{k=1}^{3}(z - q_k) \]  

(4.47)

As above, the Lax matrix \( M_k \) corresponding to the coordinate \( T_k = E(q_k) \) equals

\[ M_k = -\frac{\text{res}_{q_k}(L/E')}{z - q_k} \]  

(4.48)

As is (3.16), we obtain that \( \partial_x T_k = q_k \). Therefore, if \( \Psi \) is a solution of (3.1), then

\[ \partial_x \Psi(E) = M\Psi(E), \quad M = -\sum_k \frac{q_k \text{res}_{q_k}(L/E')}{z - q_k}. \]  

(4.49)

In our notation we skip indication on an explicit dependence of functions on \( x \) but keep track of the variable which is considered fixed with respect to \( x \).

The matrix \( M \) in (4.49) equals

\[ M = -\sum_k \text{res}_{q_k} F(q) = \text{res}_x F(q) + \text{res}_\infty F(q), \quad F = \frac{qL(q)}{E'(q)(z - q)}. \]  

(4.50)
The residue at \( q = z \) equals
\[
\text{res}_\infty F(q) = -zL(z)/E'(z).
\] (4.51)

Expansion of \( F(q) \) at \( q = \infty \) has the form
\[
F = -\frac{1}{4} \left( Aq^2 + Bq + C + Dq^{-1} \right) \left( \sum_{s=0}^{\infty} zq^{-1} \right)^{-2} \left( 1 + O(q^{-2}) \right).
\] (4.52)

Therefore,
\[
\text{res}_z F(q) = \frac{1}{4} (Az + B).
\] (4.53)

Derivatives with fixed values of \( E \) and \( z \) are related to each other by the chain rule
\[
\partial_x \Psi(x, z) = \partial_x \Psi(x, E(z)) + \frac{d\Psi}{dE} \partial_x E(x, z) = \partial_x \Psi(x, E(z)) + \frac{L(x, z)}{E'(x, z)} z
\] (4.54)

Equations (4.49-4.54) imply
\[
\partial_x \Psi(x, z) = (Az + B)\Psi(x, z).
\] (4.55)

The compatibility condition of (3.1) and (4.55) gives the well-know Lax representation for (4.42).

5 Hamiltonian approach

In this section we show that the general algebraic approach to the Hamiltonian theory of the Lax equations proposed in [13, 14, 15] is also applicable to the isomonodromy equations. Since the arguments here are very close to the ones in the author’s earlier work [11], except for slight modifications, we shall be brief.

The entries of \( \tilde{L} \in \mathcal{A}(h) \) can be regarded as functions on \( \mathcal{A}(h) \) with values in the space of meromorphic differentials on \( \Gamma \). Therefore, \( \tilde{L} \) by itself can be seen as matrix-valued function and its external derivative \( \delta \tilde{L} \) as a matrix-valued one-form on \( \mathcal{A}(h) \). The formal solutions \( \psi_m \) of the form (3.6) corresponding to an expansion of \( L \) at the punctures \( P_m \) can also be regarded as matrix function on \( \mathcal{A}(h) \) defined modulo permutation of the columns and the transformation
\[
\psi'_m = \psi_m f_m,
\] (5.1)
where \( f_m \) is a diagonal matrix. Hence, its differential \( \delta \psi_m \) is one-form on \( \mathcal{A}^D \) with values in the space of formal series of the form (3.6). In the same way we consider the differentials \( \delta K_i^{(m)} \) of the exponents in (3.6).

Let \( \mathcal{P}_0 \) be a subspace of \( \mathcal{A}(h) \) such that restriction to \( \mathcal{P}_0 \) of the differentials \( \delta K_i^{(m)} \) of the exponents in (3.6) vanishes for \( i \leq 0 \), i.e.
\[
\delta K_i^{(m)}|_{\mathcal{P}_0} = 0, \ i \leq 0.
\] (5.2)
In other words, $P_0$ is a subspace of $\mathcal{A}(h)$ such that for $L \in P_0$ the singular parts $L_m$ of $L$ at the punctures are points of a fixed set of orbits $O_m$ of the adjoint action of $GL^+(w)$ on on the space of equivalence classes of meromorphic differentials at $P_m$, modulo holomorphic differentials. Here $GL^+(w)$ is the group of invertible, holomorphic in the neighborhood of $P_m$ matrix functions.

We define a scalar valued two-form on $P_0$ by the the formula

$$\omega = -\frac{1}{2} \left( \sum_{s=1}^{rg} \text{res}_{\gamma_s} \bar{\Omega} + \sum_{P_m} \text{res}_{P_m} \bar{\Omega} \right),$$

where

$$\bar{\Omega} = \text{Tr} \left( \psi^{-1} \delta \bar{L} \wedge \delta \psi \right),$$

and $\psi$ in the neighborhood of $\gamma_s$ is a solution of (3.1), and in the neighborhood of the puncture $\psi = \psi_m$ is the formal solution (3.6).

Let us check that $\omega$ is well-defined. Indeed, if $\psi' = \psi g$ is another solution of (3.1) in the neighborhood of $\gamma_s$, then

$$\bar{\Omega}' = \bar{\Omega} + \text{Tr} \left[ (\psi^{-1} \delta \bar{L} \psi) \wedge \delta gg^{-1} \right].$$

Taking external derivative of (3.1) we obtain the equalities

$$\delta d\psi = \delta \bar{L} \psi + \bar{L} \delta \psi, \quad -\delta d\psi^{-1} = \psi^{-1} \delta \bar{L} + \delta \psi^{-1} \bar{L}$$

They imply

$$\psi^{-1} \delta \bar{L} \psi = d \left( \psi^{-1} \delta \psi \right)$$

Therefore,

$$\bar{\Omega}' = \bar{\Omega} + \text{Tr} \left[ d(\psi^{-1} \delta \psi) \wedge \delta gg^{-1} \right].$$

The matrix $g$ is a constant matrix in the neighborhood of $\gamma_s$. Therefore, the second term in (5.5) is a full differential of a meromorphic function and does not contribute to the residue.

Consider now the residues of $\bar{\Omega}$ at the puncture $P_m$. Essential singularities of $\psi_m$ and $\psi_m^{-1}$ cancel each other. Therefore, $\bar{\Omega}$ in the neighborhood of $P_m$ is a formal meromorphic differential, and its residue at $P_m$ is well-defined. It does not depend on a permutation of columns of $\psi_m$. Under the transformation (5.1) it gets additional term

$$\text{Tr} \left[ (\psi_m^{-1} \delta \bar{L} \psi_m) \wedge \delta f_m f_m^{-1} \right] = \text{Tr} \left[ d(\psi_m^{-1} \delta \psi_m) \wedge \delta f_m f_m^{-1} \right].$$

The matrix $f_m$ is diagonal, and therefore commutes with $K^{(m)}$. The constraints (5.2) imply (5.9) is a holomorphic differential, and therefore has zero residue.

**Theorem 5.1** The two-form $\omega$ defined by (5.3) is gauge invariant and descends to a closed, non-degenerate form on $P = P_0/SL_r$. Under the correspondence (2.14) it takes the form

$$\omega = \sum_{s=1}^{rg} \left( \delta \kappa_s \wedge \delta z_s + \sum_{i=1}^{r} \delta \beta_s^i \wedge \delta \alpha_s^i \right) + \sum_{m} \omega_m$$

(5.10)
where $\omega_m$ is the canonical symplectic structure on an orbit $\tilde{O}_m$.

The isomonodromy equations (4.23) are Hamiltonian with respect to symplectic structure defined by $\omega$. The Hamiltonians $H_a$ are equal

$$H_k = -\frac{1}{2} \text{res}_{q_s} \text{Tr} \left( \tilde{L}^2 / dE \right), \quad H_k = -\frac{1}{2} \oint_{a_t} \text{Tr} \left( \tilde{L}^2 / dE \right)$$

(5.11)

Recall, that the tangent space to $\tilde{O}_m$ at $\tilde{L}_m$ is isomorphic to $\mathfrak{sl}^+ + \mathfrak{r} / \mathfrak{sl}^+ + \mathfrak{r}(\tilde{L}_m)$, where $\mathfrak{sl}^+ + \mathfrak{r}(\tilde{L}_m)$ is the subalgebra of traceless matrix functions $\xi$ which are holomorphic in the neighborhood of $P_m$, and such that $[\tilde{L}_m, \xi]$ is holomorphic at $P_m$. The symplectic structure on $\tilde{O}_m$ is defined by the formula (see details in [11])

$$\omega_m = \text{res}_{P_m} \text{Tr} \left( \tilde{L}_m [\xi, \eta] \right).$$

(5.12)

**Proof.** It is easy to check directly, that under the gauge transformation

$$L' = g^{-1} L g, \quad \psi' = g^{-1} \psi$$

(5.13)

$\tilde{\Omega}$ gets transformed to $\tilde{\Omega}' = \tilde{\Omega} + F$, where

$$F = \text{Tr} \left( \psi^{-1} [\tilde{L}, \delta h] \wedge \delta \psi - [\tilde{L}, \delta h] \wedge \delta h - \delta \tilde{L} \wedge \delta h \right).$$

(5.14)

Using (5.6), we obtain

$$\text{Tr} \left( \psi^{-1} [\tilde{L}, \delta h] \wedge \delta \psi \right) = \text{Tr} \left( \delta h \wedge \delta \tilde{L} - d \left( \psi^{-1} \delta h \wedge \delta \psi \right) \right).$$

(5.15)

The last term in (5.15) is holomorphic in the neighborhoods of $\gamma_s$ and $P_m$. The rest of $F$ is a global meromorphic differential on $\Gamma$ with the only poles at $\gamma_s$ and $P_m$. Therefore, sum of all the residues of $F$ vanishes. Hence, $\omega$ is gauge invariant. Arguments needed to complete a proof of (5.10) are identical to the ones in the proof of the Theorem 4.1 in [11]. From (5.10) it follows that $\omega$ is closed. It descends to a non-degenerate form on $P_m$, because $\omega_m$ is non-degenerate on $\tilde{O}_m$, and the first term in (5.10) equals

$$\omega_0 = \sum_{s=1}^{rg} \delta \kappa_s \wedge \delta z_s + \sum_{s=r+1}^{rg} \delta b_s^T \wedge \delta a_s, \quad g > 1,$$

(5.16)

where $a_s, b_s$ are local coordinates on $\mathcal{M}$ defined by (2.22).

Our next goal is to show that the isomonodromy equations are Hamiltonian with respect to the symplectic form $\omega$. By definition a vector field $\partial_a$ on a symplectic manifold is Hamiltonian, if the contraction $i_{\partial_a} \omega(X) = \omega(X, \partial_a)$ of the symplectic form is an exact one-form $dH_a(X)$. The function $H_a$ is the Hamiltonian corresponding to the vector field $\partial_a$.

For each $\tilde{L} \in \mathcal{A}_\gamma^D$, let us define meromorphic differentials $d\Omega_a = d\Omega_a(L)$. The differential $d\Omega_k$ is a unique meromorphic differential on $\Gamma$ that has pole of the form

$$d\Omega_k = -dL + 0(1), \quad \tilde{L} = L dE,$$

(5.17)
at \( q_k \), and is holomorphic everywhere else, and satisfy the equations
\[
\alpha_s^T d\Omega_a(\gamma_s) = 0. \tag{5.18}
\]
The differential \( d\Omega_b \) is a unique holomorphic differential on \( \Gamma^*_i \) which satisfies (5.18), and is continuous on the closure of \( \Gamma^*_i \). Its boundary values on two sides of the cut along \( a_i \)-cycle satisfy the relation
\[
d\Omega^+_b - d\Omega^-_b = -dL. \tag{5.19}
\]

**Lemma 5.1** Evaluations of the one-forms \( \delta L \) and \( \delta \psi \) on the vector field \( \partial_a \) defined by the Lax equation (4.23) equal
\[
\delta \tilde{L}(\partial_a) = \partial_a L - d\Omega_a = [M_a, \tilde{L}] + dM_a - d\Omega_a, \tag{5.20}
\]
\[
\delta \psi(\partial_a) = M_a \psi + \phi_a, \tag{5.21}
\]
where \( \phi_a \) is a solution of the equation
\[
d\phi_a = \tilde{L} \phi_a - d\Omega_a \psi. \tag{5.22}
\]

**Proof.** The right hand sides of (5.20, 5.21) are not equal to the derivatives of \( \tilde{L} \) and \( \psi \), because by definition \( \delta \) is the external differential on a fiber \( A^D(\Gamma) \), but not on a total space of the bundle \( A(h) \). In other words, if \( I_k \) are coordinates on the space \( A^D(\Gamma) \) on a fixed curve with punctures, then
\[
\delta \tilde{L} = \left( \partial \tilde{L}/\partial I_k \right) \delta I_k \mapsto \delta \tilde{L}(\partial_a) = \left( \partial \tilde{L}/\partial I_k \right) \partial_a I_k. \tag{5.23}
\]
The data (2.14) are coordinates on \( A^D(\Gamma) \). From equations (4.24-4.28) it follows that the difference \( \Phi \) of the both sides of (5.20) is a holomorphic differential on \( \Gamma \) such that \( \alpha_s^T \Phi(\gamma_s) = 0 \). For non-special sets of the Tyurin parameters the last equation implies \( \Phi \equiv 0 \). Evaluation of (5.6) at \( \partial_a \), and equation (5.20) directly imply (5.21).

From (5.20, 5.22) it follows that the evaluation \( \tilde{\Omega}(\partial_a) \) of the matrix valued two-form \( \tilde{\Omega} \) given by (5.4) equals
\[
\tilde{\Omega}(\partial_a) = \text{Tr} \left( \psi^{-1} \delta L(M_a \psi + \phi_a) - \psi^{-1} [M_a, \tilde{L}] \delta \psi \right). \tag{5.24}
\]
From (5.6) and (5.7) it follows that
\[
\tilde{\Omega}(\partial_a) = \text{Tr} \left( M_a \delta \tilde{L} + \delta L M_a - \psi^{-1} d\Omega_a \delta \psi - d(\psi^{-1} M_a \delta \psi) - d(\delta \psi^{-1} \phi_a) \right). \tag{5.25}
\]
The last two terms in (5.25) are differentials of meromorphic functions in the neighborhoods of \( \gamma_s \) and \( P_m \). Therefore, their residues at these points equal zero. From (5.18) it follows that the third term is holomorphic at \( \gamma_s \). It is holomorphic also at \( P_m \). For \( M_k \) the first two terms are meromorphic on \( \Gamma \) with poles at \( \gamma_s, P_m \) and with a pole at the critical point \( q_k \). Hence,
\[
i_{\partial_a} \omega = \frac{1}{2} \text{res}_{q_k} \text{Tr} \left( \delta \tilde{L} M_k + M_k \delta \tilde{L} \right). \tag{5.26}
\]
By definition, the matrix $M_k$ in the neighborhood of $q_k$ has the form $M_k = -\bar{L}/dE + O(1)$. That implies (5.11) for $T_a = T_{k}$. In the similar way we prove that $\omega(\partial T_{h_i})$ equals to the external differential of $H_{b_i}$, and therefore, the Theorem is proved.

The basic flows constructed above, easy allow to describe isomonodromy equations, corresponding to various subspaces of $\mathcal{M}_{g,1}(h)$, and to various changes of coordinates. Let $T_a = T_a(\tau)$ depend on a variable $\tau$, and let $z = z(E, T_a)$ be a local coordinate along $\Gamma(T_a(\tau))$. Then the matrix function $M$ which define a isomonodromic deformation of $\bar{L}$ in the $\tau$-direction equals

$$M = \sum_a (\partial_\tau T_a) M_a(z) + \frac{\bar{L}}{dE} \partial_\tau E(z) \tag{5.27}$$

Let us consider the following instructive example.

**Isomonodromy equations on a fixed algebraic curve.** A variation of the coordinates $T_a$ introduced above changes simultaneously a curve, punctures and jets of local coordinates. In these coordinates it is hard to identify variations that preserve $\Gamma$. For such deformations it is more convenient to use more traditional setting.

If $z = z_m$ be a local coordinate on $\Gamma$ in an open domain $U_m$, then the variables $t_m = z(P_m)$ are local coordinates on the space of punctures $P_m \in U_m$. Let $\bar{L} \in \mathcal{A}_{\gamma,\alpha}(\Gamma)$ be an admissible meromorphic differential on $\Gamma$ with with regular singularities at $P_m$, i.e. in $U_m$ it has the form

$$\bar{L} = \left( \frac{L_m}{z - t_m} + O(1) \right) dz, \tag{5.28}$$

and corresponds to a non-special set of the Tyurin parameters $(\gamma, \alpha)$.

From (5.27) it follows that $M^{(m)}$ corresponding to the coordinate $t_m$ can be defined as a unique meromorphic matrix function on $\Gamma$ such that: (i) $M^{(m)}$ is holomorphic on $\Gamma$ everywhere except at $\gamma_s$ and the point $P_m$; (ii) the rows of $M^{(m)}$ at $\gamma_s$ have the form (2.3); (iii) in the neighborhood of $P_m$ the matrix $M^{(m)}$ has the form

$$M^{(m)} = -\frac{L_m}{z - t_m} + O(1), \tag{5.29}$$

and normalized by the condition $M^{(m)}(Q) = 0$.

**Corollary 5.1** The Lax equations

$$\partial_{t_m} \bar{L} - dM^{(m)} + [\bar{L}, M^{(m)}] = 0, \tag{5.30}$$

describe isomonodromy deformations of $\bar{L}$ with respect to the variables $t_m$. They descend to the Hamiltonian equations on $\mathcal{P}$ with the Hamiltonians

$$H^{(m)} = -\frac{1}{2} \text{res}_{P_m} \text{Tr} \left( \bar{L}^2/dz \right). \tag{5.31}$$
A proof of the last statement is almost identical to that in the Theorem 5.1. The differential $d\Omega_s$ in (5.24) has to be changed to the differential $d\Omega^{(m)}$. The later has pole the only pole at $P_m$, where

$$\Omega^{(m)} = -\frac{L_m}{z - t_m} + 0(1)$$  \hspace{1cm} (5.32)

It is normalized by the same condition (5.18). As a result of that change the only term in (5.23) which has nontrivial sum of the residues at $\gamma_s$ and $P_m$, is the third term. It has nontrivial residue at $P_m$ which can be easy found using (5.1).

**Elliptic Schlesinger equations.** Let $Ldz$ be a meromorphic connection on an elliptic curve $\Gamma = C/\{2n\omega_1, 2m\omega_2\}$ with simple poles at punctures $z = t_m$. In this example we denote the parameters $\gamma_s$ and $\kappa_s$ by $q_s$ and $p_s$, respectively.

In the gauge $\alpha_s = e_s$, $e_s^j = \delta^j_s$ the $j$-th column of the matrix $L^{ij}$ has poles only at the points $q_j$ and the punctures $t_m$. Equation (2.7) implies $L^{ij}(q_j) = 0$, $i \neq j$. From equations (2.5,2.7) it follows that $L^{ij}$ at $q_j$ has the expansion $L^{ij}(z) = (z - q_j)^{-1} + p_j + O(z - q_j)$. An elliptic function with these properties is uniquely defined by its residues $L^{ij}_{in}$ at the punctures $t_m$, and can be written in terms of the Weierstrass $\zeta$-function as follows

$$L^{ii}(z) = p_i + \sum_m L^{ii}_m (\zeta(z - t_m) - \zeta(z - q_i) - \zeta(q_i - t_m)), \sum_m L^{ii}_m = -1,$$  \hspace{1cm} (5.33)

$$L^{ij}(z) = \sum_m L^{ij}_m (\zeta(z - t_m) - \zeta(z - q_j) - \zeta(q_i - t_m) + \zeta(q_i - q_j)), \ i \neq j$$  \hspace{1cm} (5.34)

The Poisson brackets are defined by the standard formulae

$$\{p_i, q_j\} = \delta_{ij}, \ \{L^{ij}_m, L^{is}_k\} = \delta_{mk} \left(-\delta_{jl}L^{is}_m + \delta_{is}L^{lj}_m\right)$$  \hspace{1cm} (5.35)

The elliptic Schlesinger equations are generated by the Hamiltonians

$$H^{(m)} = -\sum_i p_i L^{ii}_m + \sum_i \sum_{k \neq m} L^{ii}_m L^{ii}_k (\zeta(t_m - t_k) - \zeta(t_m - q_i) - \zeta(q_i - t_k))$$

$$-\sum_{i \neq j} L^{ij}_m L^{ij}_m (\zeta(q_j - q_i) - \zeta(t_m - q_i) - \zeta(q_i - t_m))$$

$$-\sum_{k \neq m} L^{ij}_m L^{ij}_k (\zeta(t_m - t_k) - \zeta(t_m - q_i) - \zeta(q_i - t_k) - \zeta(q_j - t_k) - \zeta(q_i - q_j)).$$  \hspace{1cm} (5.36)

**Example 3.** As an example of the isomonodromy equations corresponding to deformations of algebraic curves, we consider a meromorphic connection on an elliptic curve $\Gamma = C/\{n, m\}$ with one puncture, which without loss of generality we put at $z = 0$. That example in the framework of the Hamiltonian reduction approach was considered in [9].

We use the same gauge as in the previous example. Let us assume that the residue of $\bar{L}$ at $z = 0$ has the form $-(1 + h) + f$, where $1 + h$ is a scalar matrix, and $f$ is a matrix of rank one: $f^{ij} = a^i b^j$. As it was mentioned above, the equations $\alpha_s = e_s$ fix the gauge up to transformations by diagonal matrices. We can use these transformation to make $a^i = b^i$. The corresponding momentum is given then by the collection $(a^i)^2$ and we fix it to the values
(a^i)^2 = h. Then, using the same arguments as before, we obtain that the matrix $L$ can be written as

$$L^{ij} = \hbar \frac{\sigma(z + q_i - q_j) \sigma(z - q_i) \sigma(q_j)}{\sigma(z) \sigma(z - q_i) \sigma(q_i - q_j)} \sigma(q_j), \quad i \neq j;$$

$$L^{ii} = p_i + \zeta(z - q_i) - \zeta(z) + \zeta(q_i), \quad (5.37)$$

where $\sigma(z) = \sigma(z|1, \tau)$ is the Weierstrass $\sigma$-function.

According to the Theorem 5.1, the isomonodromy deformation of $L$ with respect to the module $\tau$ of the elliptic curve is generated by the Hamiltonian

$$H = -\frac{1}{2} \int_0^1 \text{Tr} \ L^2 dz \quad (5.38)$$

The addition formula for the $\sigma$ function implies

$$\int_0^1 L^{ij} L^{ji} dz = h^2 \int_0^1 (\varphi(z) - \varphi(q_i - q_j)) dz = h^2 (2\eta_1 - \varphi(q_i - q_j)), \quad i \neq j. \quad (5.39)$$

Here and below $\eta_1 = \zeta(1/2), \eta_2 = \zeta(\tau/2).$ The formula

$$(\zeta(z - q_i) - \zeta(z) + \zeta(q_i))^2 = \varphi(z - q_i) + \varphi(z) + \varphi(q_i), \quad (5.40)$$

and the monodromy property $\sigma(z + 1) = -\sigma(z) e^{2\pi i(z - 1/2)}$ of the $\sigma$-function imply

$$\int_0^1 (L^{ii}) dz = p_i^2 + \varphi(q_i) + 2\eta_1 + 2p_i(\zeta(q_i) - 2\eta_1 q_i). \quad (5.41)$$

The $(p, q)$-independent term in $H$ which is proportional to $\eta_1(\tau)$ does not effect the equations of motion. Therefore, the Hamiltonian generating the isomonodromy equations for $p_i = p_i(\tau), q_i = q_i(\tau)$ equals

$$-4\pi i H = \sum_i \left( p_n^2 + 2p_n(\zeta(q_n) - 2\eta_1 q_n) + \varphi(q_n) \right) - h^2 \sum_{n \neq m} \varphi(q_n - q_m). \quad (5.42)$$

The equations of motion are

$$q_{n, \tau} = -\frac{1}{2\pi i} (p_n + \zeta(q_n) - 2\eta_1 q_n), \quad (5.43)$$

$$p_{n, \tau} = \frac{1}{4\pi i} \left( -2p_n(\varphi(q_n) + 2\eta_1) + \varphi'(q_n) - h^2 \sum_{n \neq m} \varphi'(q_n - q_m) \right). \quad (5.44)$$

Equation (5.43) implies

$$q_{n, \tau\tau} = -\frac{1}{2\pi i} \left( p_{n, \tau} - q_{n, \tau} (\varphi(q_n) + 2\eta_1) + \chi(q_n) \right), \quad (5.45)$$

where

$$\chi(z) = \chi(z; \tau) = \partial_{\tau} (\zeta(z|1, \tau) - 2\eta_1(\tau) z)) \quad (5.46)$$

26
The function $\xi = \zeta(z|1, \tau) - 2\eta_1(\tau)z$ has the following monodromy properties $\xi(z + 1) = \xi(z)$, $\xi(z + \tau) = \xi(z) - 2\pi i$. Therefore, $\chi(z)$ is an entire function of $z$ such that $\chi(z + 1) = \chi(z)$, $\chi(z + \tau) = \chi(z) - \partial_z \xi(z) = \chi(z) + (\wp(z) + 2\eta_1)$. These analytic properties imply the following expression for $\chi$ in terms of the Weierstrass functions:

$$\chi(z) = -\frac{1}{4\pi i} \left(2(\zeta(z) - 2\eta_1 z)(\wp(z) + 2\eta_1) + \wp'(z)\right).$$

(5.47)

From (5.43) to (5.47) we get

$$q_{n,\tau\tau} = -\frac{h}{8\pi^2} \sum_{n\neq m} \wp'(q_n - q_m|1, \tau).$$

(5.48)

For $r = 2$ equation (5.48) for the variable $u = q_1 - q_2$ is a particular case of the Painleve VI equation (see details in [9] and [33]). It is to be said, that although equations (5.48) do coincide with one which were obtained in [9], the Hamiltonian (5.42) has intriguing new form.

### 6 Canonical transformations

In the previous section the symplectic form $\omega$, initially defined by the formula (5.3), was then expressed in terms of the dynamical variables (2.14). As a result it was identified with the canonical symplectic structure on the space of meromorphic connections. The main goal of this section is to express $\omega$ in terms of the monodromy data (3.15).

Note first, that sum in (5.3) is taken over all the poles of $\tilde{L}$. It is not equal to zero, because the solutions of (3.1), used in (5.3,5.4), are formal local solutions in the neighborhoods of the punctures. Consider now the differential $\Omega_0$ given by same formula, as $\Omega$ in (5.4), i.e.

$$\Omega_0 = \text{Tr} \left( \Psi^{-1} \delta \tilde{L} \wedge \delta \Psi \right),$$

(6.1)

but where $\Psi$ is a (global) multi-valued holomorphic solution of (3.1) on $\Gamma \setminus \{P_m\}$. The differential $\Omega_0$ is single-valued on $\Gamma$ with cuts along $(a_k, b_k)$-cycles and paths $c_m$ between the marked point $Q$ and the punctures $P_m$. Therefore,

$$\sum_{s=1}^{\emptyset} \text{res}_{\gamma_s} \Omega_0 = \frac{1}{2\pi i} \oint_{\mathcal{C}} \Omega_0 = \frac{1}{2\pi i} \oint_{\mathcal{C}} \Omega_0,$$

(6.2)

where $\mathcal{C} = \prod_{k=1}^{\emptyset} (a_k b_k^{-1} a_k^{-1} b_k^{-1})$, and $\mathcal{C} = \prod_{m} C_m$ are loops in $\Gamma \setminus \{P_m\}$ (see fig.1).
If $\Psi(Q) = 1$ at the initial point, then the monodromy of $\Psi$ along the loop $aba^{-1}b^{-1}$ is equal to
\[ J(A, B) = B^{-1}A^{-1}BA, \] (6.3)

where $A, B$ are the monodromies corresponding to $a$- and $b$-cycles. The monodromy of $\Psi$ along $b$ segment of the loop is $A^{-1}BA = BJ$. From (5.8) it follows that the sum of integrals of $\Omega_0$ along the $a$ and $a^{-1}$ segments of the loop is equal to
\[ I_1 = -\text{Tr} \left( A^{-1}\delta A \wedge \delta(BJ)J^{-1}B^{-1} \right) \]
\[ = -\text{Tr} \left[ A^{-1}\delta A \wedge \delta BB^{-1} + B^{-1}A^{-1}(\delta A)B \wedge \delta JJ^{-1} \right]. \] (6.4)

The monodromy of $\Psi$ along the $a^{-1}$-segment of the loop is $A^{-1}J$. Therefore, sum of the integrals of $\Omega_0$ along $b$ and $b^{-1}$ segments of the loop is equal to
\[ I_2 = -\text{Tr} \left[ (A^{-1}B^{-1}\delta(BA) - A^{-1}\delta A) \wedge \delta(A^{-1}J)J^{-1}A \right] \]
\[ = -\text{Tr} \left[ -B^{-1}\delta B \wedge \delta AA^{-1} + B^{-1}\delta B \wedge \delta JJ^{-1} \right]. \] (6.5)

The sum $\chi = I_1 + I_2$ equals
\[ \chi(A, B) = \text{Tr} \left[ B^{-1}\delta B \wedge \delta AA^{-1} - A^{-1}\delta A \wedge \delta BB^{-1} + \delta JJ^{-1} \wedge B^{-1}A^{-1}\delta(AB) \right]. \] (6.6)

Due to analytical continuation, the solution $\Psi$ on the segment of the loop $\mathcal{L}$ differs from the normalized solution $\Psi_0$ used in the previous formulae by the factor
\[ H_1 = 1; \quad H_k = J_{k-1}J_{k-2} \cdots J_1, \quad k > 1; \quad J_s = J(A_s, B_s). \] (6.7)

From (5.8) it follows that the integral of $\Omega_0$ over $(a_kb_k^{-1}a_k^{-1}b_k^{-1})$ under the transformation $\Psi = \Psi_0H_k$ gets additional term
\[ \text{Tr} \left( J_k^{-1}\delta J_k \wedge \delta H_kH_k^{-1} \right). \] (6.8)
Let us denote the integral of $\Omega_0$ over $\mathcal{L}$ by

$$\omega_1(A, B) := \oint_{\mathcal{L}} \Omega_0 = \sum_{k=1}^{g} \left[ \chi(A_k, B_k) + \text{Tr} \left( J_k^{-1} \delta J_k \wedge \delta H_k H_k^{-1} \right) \right].$$

(6.9)

It is a two-form on the space of sets of matrices $A = \{A_k\}$, $B = \{B_k\}$.

Next we compute the integral of $\Omega_0$ along the cycle $C_m$, which goes along one side of the cut $c_m$, then goes around $P_m$ along a small circle $c'_m$, and finally goes back along the other side of $c_m$ (see fig.1).

Consider first the integral of $\Omega_0$ around the puncture. We split the circle $c'$ into $2h + 1$ arcs $c_\nu$ which lies in the sectors $V_\nu$ (here and below we skip for brevity the index $m$ of the puncture). Recall, that in each of the sectors the formal solution $\psi$ of (3.11) given by the Lemma 3.1 is an asymptotic series for the holomorphic function $\Psi_\nu = \Psi g_\nu^{-1}$. Let $\Omega_\nu$ be given by the same formula as for $\Omega_0$ with $\Psi$ replaced by $\Psi_\nu$. Then,

$$\int_{c_\nu} \Omega_0 = \int_{c_\nu} \Omega_\nu + \text{Tr} \left[ \int_{c_\nu} d \left( \Psi^{-1}_\nu \delta \Psi_\nu \right) \wedge \delta g_\nu g_\nu^{-1} \right]$$

(6.10)

The form $\tilde{\Omega}$ defined by (5.4), where $\psi$ is the formal solution (3.1) gives an asymptotic series for $\Omega_\nu$ in $V_\nu$. Therefore, as $c'$ shrinks to the puncture

$$\lim_{c' \to P} \sum_\nu \int_{c_\nu} \Omega_\nu = (2\pi i) \res_P \tilde{\Omega}.$$ 

(6.11)

A sum of the second terms in (6.10) equals

$$\text{Tr} \left( \Psi_{2h+1}^{-1}(p) \delta \Psi_{2h+1}(p) \wedge \delta g_{2h+1} g_{2h+1}^{-1} - \Psi_1^{-1}(p) \delta \Psi_1(p) \wedge \delta g_1 g_1^{-1} \right) +$$

$$\sum_{\nu=1}^{2h} \text{Tr} \left( \Psi_\nu^{-1}(p_\nu) \delta \Psi_\nu(p_\nu) \wedge \delta g_\nu g_\nu^{-1} - \Psi^{-1}_{\nu+1}(p_\nu) \delta \Psi_{\nu+1}(p_\nu) \wedge \delta g_{\nu+1} g_{\nu+1}^{-1} \right),$$

(6.12)

where $p_\nu \in V_\nu \cap V_{\nu+1}$ is common edge point of the arcs $c_\nu$ and $c_{\nu+1}$. The point $p$ is the intersection point of the cut $c$ and the circle $c'$. We assume that the cut tends to the puncture in the intersection $V_1 \cap V_{2h+1}$.

The matrices $\Psi_{2h+1}$ and $\Psi_1$ are connected by the relation $\Psi_{2h+1} = \Psi_1 e^{2\pi i K_0}$. Recall, that the monodromy $\mu$ along the whole path $C$ is $\mu = g_1^{-1} e^{2\pi i K_0} g_{2h+1}$. Therefore, the first two terms in (6.12) give

$$\text{Tr} \left[ \Psi_1^{-1} \delta \Psi_1 \wedge \left( e^{2\pi i K_0} \delta g_{2h+1} g_{2h+1} e^{-2\pi i K_0} - \delta g_1 g_1^{-1} \right) \right] = \text{Tr} \left( \Psi_1^{-1} \delta \Psi_1 \wedge g_1 \delta \mu \mu^{-1} g_1^{-1} \right).$$

(6.13)

Boundary values of $\Psi$ on the two sides of the cut $c$ between $Q$ and $P$ satisfy the relation $\Psi^+ = \Psi^- \mu$. Therefore, sum of the integrals of $\Omega_0$ along the first and the last segments of the path $C$ equals

$$- \text{Tr} \left( \Psi^{-1}(p) \delta \Psi(p \wedge \delta \mu \mu^{-1}) \right) = \text{Tr} \left( \delta \mu \mu^{-1} \wedge g_1^{-1} \delta g_1 + \delta \mu \mu^{-1} \wedge g_1^{-1} \Psi_1^{-1}(p) \delta \Psi_1(p) g_1 \right).$$

(6.14)
Here we use the relation \( \Psi(p) = \Psi_1(p)g_1 \). The sum of (6.13) and (6.14) is equal to

\[
I_3 = \text{Tr} \left( \delta \mu^{-1} \wedge g_1^{-1} \delta g_1 \right).
\] (6.15)

Recall, that \( \Psi_{\nu+1} = \Psi_{\nu}S_{\nu} \), where the Stokes matrix \( S_{\nu} \) equals \( S_{\nu} = g_{\nu}g_{\nu+1}^{-1} \). Therefore, the terms of the sum in (6.12) are equal to

\[
\text{Tr} \left[ -S_{\nu}^{-1} \delta S_{\nu} \wedge \delta g_{\nu+1}g_{\nu}^{-1} + \Psi_{\nu+1}^{-1}(g_{\nu}) \delta \Psi_{\nu} \wedge \left( \delta g_{\nu}g_{\nu}^{-1} - S_{\nu} \delta g_{\nu+1}g_{\nu}^{-1} S_{\nu}^{-1} \right) \right] = \text{Tr} \left( -\delta S_{\nu}S_{\nu}^{-1} \wedge \delta g_{\nu}g_{\nu}^{-1} + \Psi_{\nu+1}^{-1}\delta \Psi_{\nu} \wedge \delta S_{\nu}S_{\nu}^{-1} \right).
\]

In the sector \( V_\nu \), we have \( \Psi_{\nu}e^{-Kw^{-h}} = O(1) \), where \( K \) is the leading exponent \( K_{-h} \). Therefore, (3.10) implies

\[
\lim_{\nu \to P} \text{Tr} \left( \Psi_{\nu+1}^{-1}(g_{\nu}) \delta \Psi_{\nu} \wedge \delta S_{\nu}S_{\nu}^{-1} \right) = 0.
\] (6.16)

Hence, the second term in (6.12) tends to

\[
I_4 = \sum_{\nu=1}^{2h} \text{Tr} \left( -\delta S_{\nu}S_{\nu}^{-1} \wedge \delta g_{\nu}g_{\nu}^{-1} \right),
\] (6.17)
as \( c' \) shrinks to \( P \). Let us denote the sum of (6.15) and (6.17) by

\[
\sigma(S,G,K_0) = \text{Tr} \left( \delta \mu^{-1} \wedge G^{-1}\delta G - \sum_{\nu=1}^{2h} \delta S_{\nu}S_{\nu}^{-1} \wedge \delta g_{\nu}g_{\nu}^{-1} \right),
\] (6.18)

where \( S = \{S_{\nu}\} \), and matrices \( \mu, g_{\nu} \) are given by (3.13, 3.14). The integral of \( \Omega_0 \) along \( C \) equals \( \sigma(S,G,K_0) \) under the assumption that \( \Psi = \Psi_0 \), where \( \Psi_0 = 1 \) at the initial point of the cycle. Due to analytical continuation along the path \( C = \prod_m C_m \) the initial value for the cycle \( C_m \) equals

\[
F_1 = 1; \quad F_m = \mu_{m-1}\mu_{m-2} \cdots \mu_1, \quad m > 1.
\] (6.19)

From (5.8) it follows that the integral of \( \Omega_0 \) along the segment \( C_m \) of the path \( C \) under the transformation \( \Psi = \Psi_0F_m \) acquires additional term

\[
\text{Tr} \left( \mu_m^{-1}\delta \mu_m \wedge \delta F_m F_m^{-1} \right).
\] (6.20)

Let us define a family of two-forms on the space of sets of matrices \( S = \{S^{(m)}\} \), \( G = \{G^{(m)}\} \) parametrized by a set \( K_0 = \{K_0^{(m)}\} \) of diagonal matrices:

\[
\omega_2(S,G | K_0) := \oint_C \Omega_0 - 2\pi i \sum_m \text{res}_{p_m} \Omega = \sum_m \left[ \sigma(S^{(m)}, G_m, K_0^{(m)}) + \text{Tr} \left( \mu_m^{-1}\delta \mu_m \wedge \delta F_m F_m^{-1} \right) \right].
\] (6.21)

Summarizing we obtain the following statement.
Theorem 6.1  The symplectic form $\omega$ defined by (5.3) is equal to
\[
\omega = \frac{1}{4\pi i} \left[ \omega_2(S, G | K_0) - \omega_1(A, B) \right],
\]
where $\omega_1$ and $\omega_2$ are given by (6.9) and (6.21), respectively.

It would be quite interesting to check directly that the formula (6.22) defines a symplectic structure on orbits of the adjoint action of $SL_r$ on the space of the sets $(A, B, S, G)$, of matrices, which satisfy the only relation
\[
\prod_k (B_k^{-1} A_k^{-1} B_k A_k) = \prod_m \mu_m.
\]
The factors in (6.23) are ordered such that indices increase from right to left.

Example. Let us consider the case of meromorphic connections on the rational curve with one irregular singularity of order 2 and one regular singularity. Without loss of generality we assume that $\tilde{L} = Ldz$ has irregular singularity at $z = 0$ and regular singularity at $z = \infty$, i.e.
\[
L = l_1 z^{-2} + l_0 z^{-1}.
\]
Let us fix a gauge in which $l_1 = K_0^\infty$ is a diagonal matrix. Then the monodromy matrix at the infinity is $\mu_\infty = \exp(2\pi i K_0^\infty)$. Recall, that we always assume that the exponents are fixed. The monodromy data at $z = 0$ are two Stokes matrices $S_1, S_2$, transition matrix $G$, and the exponents $K_1, K_0$. The monodromy matrix at $z = 0$ equals
\[
\mu_0 = G^{-1} e^{2\pi i K_0} S_2^{-1} S_1^{-1} G = \mu_\infty^{-1}.
\]
The transition matrices to the first and the second sectors at $z = 0$ equal
\[
g_1 = G, \quad g_2 = S_1^{-1} G
\]
Substitution of (6.26) in (6.28) implies
\[
4\pi i \omega = -\text{Tr} \left( \delta S_1 S_1^{-1} \wedge \delta GG^{-1} + \delta S_2 S_2^{-1} \wedge \delta(S_1^{-1} G) G^{-1} S_1 \right).
\]
Using skew-symmetry of the wedge product and (6.23) we can rewrite the last term as
\[
\text{Tr} \left( \delta S_2 S_2^{-1} \wedge \delta(S_1^{-1} G) G^{-1} S_1 \right) = \text{Tr} \left( S_2^{-1} \delta S_2 \wedge \delta(S_1^{-1} G) G^{-1} S_1 S_2 \right) = \text{Tr} \left( e^{2\pi i K_0} S_2^{-1} \delta S_2 e^{-2\pi i K_0} \wedge \delta GG^{-1} \right).
\]
Hence,
\[
\omega = -\frac{1}{4\pi i} \text{Tr} \left[ (\delta S_1 S_1^{-1} + e^{2\pi i K_0} S_2^{-1} \delta S_2 e^{-2\pi i K_0}) \wedge \delta GG^{-1} \right].
\]
The formula (6.29) after change of notations $G = C, S_1 = b_\gamma^{-1}, S_2 e^{-2\pi i K_0} = b_-$ coincides (up to a factor 2) with the formula (14) in [22], where the symplectic structure on the space of monodromy data for the linear system (6.24) was identified with the symplectic structure of the group $G^*$ dual to $G = GL_r$. 

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7 The Whitham equations

It is well-known that the family of flat $\varepsilon \neq 0$-connections on holomorphic vector bundles over an algebraic curve $\Gamma$ with punctures extends to a smooth family over the whole $\varepsilon$-plane. The central fiber over $\varepsilon = 0$ is identified with the cotangent bundle to the moduli space of holomorphic vector bundles on $\Gamma$. The correspondence (2.14) makes these statements transparent.

The space of meromorphic $\varepsilon$-connections with fixed multiplicities $h = \{h_m\}$ of poles is the factor space $\mathcal{A}_\varepsilon(h)/SL_r$ of the space of meromorphic differentials $\tilde{L}_\varepsilon$ such that $\varepsilon^{-1}L_\varepsilon \in \mathcal{A}(h)$. A meromorphic differential $\tilde{L}_\varepsilon \in \mathcal{A}_\varepsilon(h)$ satisfies the constraints (2.4, 2.5, 2.7), and the condition

$$\text{res}_{\gamma_s} \text{Tr} \tilde{L} = (\alpha_s^T \beta_s) = \varepsilon. \quad (7.1)$$

The characteristic property of meromorphic $\varepsilon$-connections is that in the neighborhood of the points $\gamma_s$ they have the form

$$\tilde{L} = \varepsilon d\Phi_s(z)\Phi_s^{-1}(z) + \Phi_s(z)\tilde{L}_s(z)\Phi_s^{-1}(z), \quad (7.2)$$

where $L_s$ and $\Phi_s$ are holomorphic at $\gamma_s$, and $\det \Psi_s$ has at most simple zero at $\gamma_s$. In the earlier work of the author [11] the space $\mathcal{A}_0(h)$ was called the space of Lax matrices, and orbits of the adjoint action of $SL_r$ on subspaces of $\mathcal{A}_0(h)$ with fixed singular parts of the eigenvalues were identified with phase spaces of the Hitchin systems.

The space $\mathcal{A}_\varepsilon(h)$ is a bundle over the moduli space $\mathcal{M}_{g,1}(h)$. Let $\tilde{L}_\varepsilon(\tau)$ be a one-parametric deformation of $\tilde{L}_\varepsilon$ which preserves the full set of monodromy data (3.15) associated with a holomorphic solution of the equation

$$\varepsilon d\Psi = \tilde{L}\Psi, \quad (7.3)$$

Along identical lines to the proof of Lemma 4.1 it can be shown that the singularities of $M = \partial_\varepsilon \Psi \Psi^{-1}$ are of the form $\varepsilon^{-1}\tilde{L}_\varepsilon/d\tilde{E}$. Therefore, in order to get a smooth at $\varepsilon = 0$ family of the isomonodromy equations, it is necessary to make a proper rescaling of coordinates on $\mathcal{M}_{g,1}(h)$. Namely, if we introduce the fast coordinates $t_a = e^{-1}T_a$, then the isomonodromy equations are equivalent to the Lax equations

$$\partial_{t_a} \tilde{L}_\varepsilon - \varepsilon dM_a + [\tilde{L}_\varepsilon, M_a] = 0, \quad (7.4)$$

where matrices $M_a = M_a(\tilde{L}_\varepsilon)$ are defined by the same analytical properties as above in Section 4. Moreover, the corresponding Hamiltonians are given by the same formulae (5.11).

Remark. Here and below we use the coordinates $T_a$ on $\mathcal{M}_{g,1}$ introduced in Section 4, but mainly our arguments do not rely on any specific choice of the coordinates.

As follows from [11], equations (7.4) for $\varepsilon = 0$

$$\partial_{t_a} \tilde{L}_0 = [M_a, \tilde{L}_0] \quad (7.5)$$

coincide with the Lax equations for commuting flows of the Hitchin system corresponding to the second order Hamiltonians given by the same formula (5.11). Equations (7.5) describe
**isospectral** deformations. If $\tilde{L}_0 \in A_0(h)$ is a solution of (7.3), then the spectral curve $\tilde{\Gamma}$ of $\tilde{L}_0$ defined by the characteristic equation

$$\det(\tilde{k} - \tilde{L}_0) = \tilde{k}^r + \sum_i u_i \tilde{k}^i = 0,$$  \hspace{1cm} (7.6)$$
is **time-independent.** The spectral transform identifies $A_0(h)$ with the Jacobian bundle over the moduli space $S$ of the spectral curves. The fiber of this bundle over $\tilde{\Gamma}$ is the Jacobian $J(\tilde{\Gamma})$. The bijective correspondence between $A(h)$ and the Jacobian bundle over $S$ can be seen as a parametrization of $A_0(h)$ in the form $\tilde{L}_0 = \tilde{L}_0(\phi|I)$. Here and below we regard $\tilde{L}_0(\phi|I)$ as an abelian function of the variable $\phi \in J(\tilde{\Gamma})$ depending on $I \in S$. The function $\tilde{L}_0(\phi|I)$ takes values in the space of meromorphic matrix differentials on $\Gamma$. The equations of motions are linearized on the Jacobian of the spectral curve, and therefore, the general solution of (7.4) can be represented in the form $\tilde{L}_0 = \tilde{L}_0(Ut|I)$, where $Ut = \sum_a U_a t_a$, and $U_a = U_a(I)$ are constant vectors depending on $I$ (see details in [11]).

The main goal of this section is to apply ideas of the Whitham averaging method to construct asymptotic solutions of the isomonodromy equations (7.4)

$$\tilde{L}_\varepsilon = \tilde{L}_0 + \varepsilon \tilde{L}_1 + \varepsilon^2 \tilde{L}_2 + \cdots, \quad M_\varepsilon = M_0 + \varepsilon M_1 + \varepsilon^2 M_2 + \cdots,$$ \hspace{1cm} (7.7)$$
where the leading terms have the form

$$\tilde{L}_0(\varepsilon^{-1} S(T)|I(T)), \quad M_0 = (\varepsilon^{-1} S(T)|I(T)),$$ \hspace{1cm} (7.8)$$
and $T = \varepsilon t$ are **slow** variables. If the vector-function function $S(T)$ satisfies the equation

$$\partial_T S(T) = U(I(T)) = U(T), \text{ i.e. } S(T) = \int^T U(T)dT,$$ \hspace{1cm} (7.9)$$
then the leading term of (7.7) satisfies the original equation up to first order one in $\varepsilon$. All the other terms of the asymptotic series are obtained from the non-homogeneous linear equations with a homogeneous part which is just the linearization of the original non-linear equation on the background of the exact solution $\tilde{L}_0$. In general, the asymptotic series becomes unreliable on scales of the original variables $t$ of order $\varepsilon^{-1}$. In order to have a reliable approximation, one needs to require a special dependence of the parameters $I(T)$. Geometrically, we note that $\varepsilon^{-1} S(T)$ agrees to first order with $Ut$, and $t$ is the fast variables. Thus $\tilde{L}_0(\varepsilon^{-1} S(T)|I(T))$ describes a motion which is to first order the original **fast** periodic motion on the Jacobian, combined with a slow drift on the moduli space of exact solutions. The equations which describe this drift are in general called **Whitham equations**, although there is no systematic scheme to obtain them.

Below we follow lines of the scheme proposed in [20], where the Whitham equations for general $(2 + 1)$ integrable soliton systems were derived. First, we introduce sets of Abelian differentials $dv^r_c, dv^i_a$ on spectral curves $\tilde{\Gamma}$. The differentials $dv^r_c, dv^i_k$ are real normalized, i.e. their periods are pure imaginary

$$Re \oint_c dv^r_a = Re \oint_c dv^i_a = 0, \quad c \in H^1(\tilde{\Gamma}).$$ \hspace{1cm} (7.10)
For indices \(a = k\) the corresponding differentials have pole only at the preimages \(q_k^j\) on \(\hat{\Gamma}\) of the point \(q_k\), where
\[
dv_r^k = dk_j(z) + O(1), \quad dv_k^i = idk_j(z) + O(1),
\]
and \(k_j(z)\) is the corresponding branch of the eigenvalue of \(\bar{L}(z)/dE\), i.e. the corresponding root of the equation
\[
\det(k - \bar{L}_0/dE) = 0.
\]
For indices \(a = b_j\) the corresponding differentials are holomorphic on \(\hat{\Gamma}\), with cuts along all the preimages \(a_j^l \in H^1(\hat{\Gamma})\) of the cycle \(a_j \in H^1(\Gamma)\), and their boundary values on two sides of the cut \(a_j^l\) satisfy the relation
\[
(dv^r_{b_j})^+ - (dv^r_{b_j})^- = dk_l, \quad (dv^i_{b_j})^+ - (dv^i_{b_j})^- = idk_l
\]
Here, as before, \(k_l\) is the corresponding eigenvalue of \(\bar{L}/dE\). Note, that \(dv^r_a + idv^i_a\) is a holomorphic differential.

**Theorem 7.1** A necessary condition for the existence of the asymptotic solution (7.7) of the isomonodromy equation (7.4) with the leading term (7.8) and with bounded first correction term \(\bar{L}_1\) is the equations
\[
\partial X_a \bar{k} = -dv^r_a, \quad \partial Y_a \bar{k} = -dv^i_a.
\]
where \(X_a\) and \(Y_a\) are the real and the imaginary parts of the slow variable \(T_a = X_a + iY_a\).

Along lines of (26) it can be shown that equations (7.14) are generating form of the equations on the space \(S\) of the spectral curves (see details in [13, 14, 27, 28]).

Equations (7.14) can be written in the form
\[
\partial_{T_a} \bar{k} = -dv_a,
\]
where
\[
\partial_{T_a} = \frac{1}{2} \left( \frac{\partial}{\partial x_a} - i \frac{\partial}{\partial y} \right), \quad dv_a = \frac{1}{2}(dv^r_a - idv^i_a).
\]

**Remark.** The equation (7.13) is a particular case of the exact solutions of the universal Whitham hierarchy. It is connected with a theory of WDVV equations and the Seiberg-Witten theory of \(N = 2\) supersymmetric gauge models (see [13, 14, 27, 28]).

**Corollary 7.1** The real parts of the periods of the differential \(\bar{k}\) over the spectral curve are integrals of the Whitham equations. The correspondence
\[
\bar{L}_0 \mapsto \oint_c K, \quad c \in H^1(\hat{\Gamma})
\]
defines a flat connection on the bundle \(S\) over \(\mathcal{M}_g(h)\)
Proof of the Theorem. Substitution of the series (7.7) into (7.4) gives non-homogeneous linear equation for the first order terms

$$
\partial_t \tilde{L}_1 - dM_1 + [\tilde{L}_0, M_1] + [\tilde{L}_1, M_0] = dM_0 - \partial_T \tilde{L}_0 - t \sum_{i=1}^{2g} (\partial_T U_i) \partial_{\phi_i} \tilde{L}_0,
$$

(7.18)

where $\phi_i$ are coordinates on fibers of the Jacobian bundle. Here and below we skip for brevity index $a$, i.e. $t = t_a$, and $T = T_a$. Let $\psi$ and $\psi^*$ be solutions of the adjoint systems of equations

$$
\tilde{L}_0 \psi = \tilde{k} \psi, \quad \partial_t \psi = M_0 \psi,
$$

(7.19)

$$
\psi^* \tilde{L}_0 = \tilde{k} \psi^*, \quad \partial_t \psi^* = -\psi^* M_0.
$$

(7.20)

Here $\psi^*$ is a vector-row, normalized by the condition $\psi^* \psi = 1$. From (7.18, 7.19, 7.20) it follows that

$$
\partial_t (\psi^* L_1 \psi) = -\psi^* \left( \partial_T \tilde{L}_0 - dM_0 - t \sum_{i=1}^{2g} (\partial_T U_i) \partial_{\phi_i} \tilde{L}_0 \right) \psi.
$$

(7.21)

From the equations

$$
\psi^*(\delta \tilde{L}_0 - \delta \tilde{k}) \psi = -\psi^*(\tilde{L}_0 - \tilde{k}) \delta \psi = 0,
$$

(7.22)

and the normalization $\psi^* \psi = 1$ it follows that

$$
\psi^* \delta \tilde{L}_0 \psi = \delta \tilde{k}.
$$

(7.23)

Variations of $\tilde{L}_0$ with respect to the variables $\phi_i$ preserve the spectral curve, i.e. for such variations $\delta \tilde{k} = 0$. Hence,

$$
\psi^* \left( \partial_{\phi_i} \tilde{L}_0 \right) \psi = 0.
$$

(7.24)

Equation (7.23) also implies

$$
\partial_T \tilde{k} = \psi^* (\partial_T \tilde{L}) \psi.
$$

(7.25)

Equations (7.19, 7.20) imply also

$$
\psi^*(dM_0) \psi = -\psi^* M_0 d\psi + \psi^* (\partial_t d\psi) = \partial_t (\psi^* d\psi).
$$

(7.26)

Hence, (7.21) can be written as

$$
\partial_t (\psi^* L_1 \psi) + \partial_T \tilde{k} - \partial_t (\psi^* d\psi) = 0.
$$

(7.27)

In [11] it was shown that the solution $\psi$ of equations (7.19) is the conventional Baker-Akhiezer function on $\hat{\Gamma}$. Therefore, it can be written explicitly in terms of the Riemann theta-functions of the spectral curve. In [23] the original formulae were adapted for the averaging procedure. In order to complete a proof of the Theorem, we do not need these formulae in full. Let us present necessary facts.
The function $\psi = \psi(x, y; P)$, and the dual Baker-Akhiezer function $\psi^*$ considered as functions of the real variables $x, y$ can be represented in the form

$$
\psi(x, y; P) = \Phi(xU^r + yU^i + \zeta; P) \exp \left( - \int_P x dv^r + y dv^i \right), \quad (7.28)
$$

$$
\psi^*(x, y; P) = \Phi^*(xU^i + yU^r + \zeta; P) \exp \left( \int_P x dv^r + y dv^i \right), \quad (7.29)
$$

where $U^r, U^i$ are real $2g$-dimensional vectors, and for each $P \in \hat{\Gamma}$ the functions $\Phi(\zeta; P)$ and $\Phi^*(\zeta; P)$ as functions of $2g$ real variables $\zeta = (\zeta_1, \ldots, \zeta_{2g})$ have the following monodromy properties

$$
\Phi(\zeta + e_i; P) = w_i \Phi(\zeta), \quad \Phi^*(\zeta + e_i; P) = w_i^{-1} \Phi^*(\zeta; P), \quad |w_i| = 1. \quad (7.30)
$$

where $e_i$ are the basis vectors of $R^{2g}$.

The functions $\tilde{L}_0, \psi$ and $\psi^*$ as functions of the complex variable $t = x + iy$ are meromorphic functions. Therefore, if $L_1$ is uniformly bounded outside of some neighborhood of the singularity locus, then the average value $<\partial_t(\psi L_1 \psi)>$ in $t$ of the first term in (7.27) equals zero

$$
<f(t)> = \lim_{\Lambda_i \to \infty} \Lambda_i^{-1} \int_0^{\Lambda_i} f dt \quad (7.31)
$$

It is necessary to make a few remarks to clarify averaging procedure. First of all, we assume that 0 and $\Lambda_i$ are not in the locus. The integral is taken along the path in the complex plane of the variable $t$ which do not intersect singularities.

As follows from (7.28-7.30) the average value of the last term in (7.27) does exist but depends on the direction in $t$-plane. If we consider $t$, as a real variable then this average equals $-dv^r$. For $t = iy$ it equals $-dv^i$, and therefore, the Theorem is proved.

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