Small systems and regulator dependence in relativistic hydrodynamics

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Abstract

Consistent theories of hydrodynamics necessarily include nonhydrodynamic modes, which can be viewed as a regulator necessary to ensure causality. Under many circumstances the choice of regulator is not relevant, but this is not always the case. In particular, for sufficiently small systems (such as those arising in pA or pp collisions) such dependence may be inevitable. We address this issue in the context of the modern version of Müller-Israel-Stewart theory of relativistic hydrodynamics. In this case, by demanding that the nonhydrodynamic modes be subdominant, we find that regulator dependence becomes inevitable only for multiplicities $dN/dY$ of the order of a few. This conclusion supports earlier studies based on hydrodynamic simulations of small systems, at the same time providing a simple physical picture of how hydrodynamics can be reliable even in such seemingly extreme conditions.

1 Introduction

The modern version of Müller-Israel-Stewart theory (MIS) \cite{1,2}, which will be referred to as BRSSS \cite{3}, is the basic phenomenological tool for understanding the dynamics of quark-gluon plasma (QGP) produced in heavy ion (AA) collisions. It has recently been found that this theory of relativistic hydrodynamics works remarkably well also in the case of other processes such as...
pA or even pp collisions [4,5], which lead to smaller drops of plasma. This raises the question of why hydrodynamics applies here, and where the limit of its applicability lies [6–9]. The aim of this note is to address this question in the context of recent advances in our understanding of relativistic hydrodynamics.

The key point is that the factor which signals the emergence of hydrodynamic behaviour in a microscopic theory such as QCD is the decay of nonhydrodynamic modes. This point has frequently been emphasized in the context of holographic studies of $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma [10–12], but it is valid generally. In particular, it is valid within for BRSSS theory, which incorporates a particular nonhydrodynamic sector needed for its self-consistency [13].

Unlike nonrelativistic Navier-Stokes theory, its direct relativistic generalisation [14] is not consistent, because it is not causal [15–17]. The only known way to achieve causality is to include additional modes beyond the basic hydrodynamic variables (the energy density and fluid velocity). The simplest example where this works is MIS theory, which adds a single purely damped nonhydrodynamic degree of freedom. This mode should be thought of as a regulator, ensuring that the speed of propagation does not exceed the speed of light. Indeed, the speed of propagation of linear perturbations

$$v = \frac{1}{\sqrt{3}} \sqrt{1 + \frac{4\eta/s}{T\tau_{\Pi}}} ,$$

(1)

where $\eta$ is the shear viscosity, $s$ is the entropy density, $T$ is the effective temperature and $\tau_{\Pi}$ is the relaxation time associated with the nonhydrodynamic mode. This formula (which follows from the sound channel dispersion relation given in Eq. (4) below) implies that as long as the relaxation time is sufficiently large

$$T\tau_{\Pi} > 2\eta/s$$

(2)

there is no transluminal signal propagation. This is clearly not the case if one tries to eliminate the relaxation time by taking it to vanish.

The presence of nonhydrodynamic modes is therefore essential for the consistency of the hydrodynamic description in the relativistic case. The success of relativistic hydrodynamics in

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1 The formula Eq. (1) pertains specifically to the sound channel.

2 Only the conformal case will be discussed explicitly.
describing the dynamics of QGP created in AA collisions can be ascribed to the exponential decay of these modes, which leads to the fast emergence of quasuniversal, attractor behaviour of this system \[13\].

The nonhydrodynamic modes act as a regulator which cannot be removed, but whose effects may or may not be practically significant in the regime of interest. It is important to understand when the effects of nonhydrodynamic modes may be ignored, otherwise one may be studying the physics of the regulator rather than universal hydrodynamic behaviour. In particular, in the case of a small system it may happen that the nonhydrodynamic modes do not have time to decay, and hydrodynamic simulations become sensitive to the choice of the nonhydrodynamic sector — that is, to the choice of regulator\[^{3}\] If this happens, it may be necessary to compare different regulators. Examples of hydrodynamic theories with a qualitatively different nonhydrodynamic sector were discussed in Ref. \[18\].

The statement that hydro works for small systems \[^{5}\] means specifically that parameters of BRSSS theory (or some other variant of MIS theory) can be fitted to describe the data. The point we are making here is that in some situations this becomes a test of the nonhydrodynamic sector of this theory rather than of hydrodynamics. This is problematic if one wishes to regard BRSSS as an effective description of QCD plasma. Implications of this are further discussed in Sec. \[4\].

The question of the domain of validity of hydrodynamics applied to small systems was considered recently in Refs. \[8,9\], which considered the dependence on the magnitude of second order terms in the gradient expansion as a measure of systematic error. In cases where this error becomes significant, the authors concluded that hydrodynamics ceases to be useful. From a theoretical standpoint it is the decay of the nonhydro modes and not the size of gradient corrections which sets the domain of validity of hydrodynamics. However, within BRSSS theory the parameter which governs the decay of the nonhydro modes, the relaxation time, is also responsible for some of the second order terms — indeed, from a modern perspective \[^{3}\] the MIS relaxation time is just one of a number of second order transport coefficients. This is clearly the

\[^{3}\]One can think of the regulator sector as an analogue of the notion of a “UV-completion”, which arose in the context of effective field theories.
appropriate view when discussing the gradient expansion generated from the hydro equations of motion. However when solving the equations numerically, the relaxation time should be regarded as a regularization parameter. If the results depend significantly on the value taken for this parameter, one may infer that one is not really testing the hydrodynamic sector, but rather the physics of the regulator itself. Here we follow this logic directly: by comparing the decay rates of the hydro and nonhydro sectors. This leads to a straightforward analytic argument which results in a simple inequality, whose violation indicates that nonhydrodynamic modes are not subdominant. This inequality can be phrased either in terms of the size and temperature of the system, or in terms of the final multiplicity measured. When expressed in terms of local effective temperature and size the inequality is in fact more general than the context of small systems; it is a bound on the size of features (such as spikes in the energy density), whose violation implies regulator dependence.

2 Dispersion relations in BRSSS theory

The BRSSS theory of relativistic hydrodynamics [3] is a generalization of the original MIS theory [1,2], which includes the full set of transport coefficients allowed by Lorentz and conformal symmetry (the latter assumption was relaxed in Ref. [19]). The spectrum of linearized perturbations around equilibrium reveals two types of behavior: hydrodynamic modes whose frequency vanishes with the wave vector, as well as nonhydrodynamic modes whose frequency approaches a nonzero value at $k \equiv |k| = 0$. The imaginary parts of these frequencies determine the decay rates. Formally, at $k \approx 0$ the hydro modes are long lived, while the nonhydro modes decay exponentially.

The dispersion relations for BRSSS theory, assuming solutions of the form

$$\delta T \sim \exp(-i(\omega t - k \cdot x)), \quad \delta u^\mu \sim \exp(-i(\omega t - k \cdot x)) \quad (3)$$

have been worked out in Ref. [3]. In the sound channel we have

$$\omega^3 + \frac{i}{\tau_\Pi} \omega^2 - \frac{k^2}{3} \left(1 + 4 \frac{\eta/s}{T \tau_\Pi}\right) \omega - \frac{ik^2}{3\tau_\Pi} = 0 \quad (4)$$

An analogous argument can be carried out in the shear channel and leads to identical conclusions.
For small $k$ one finds a pair of hydrodynamic modes (whose frequency tends to zero with $k$)

$$
\omega_{H}^{(\pm)} = \pm \frac{k}{\sqrt{3}} - \frac{2i}{3T} \frac{\eta}{s} k^2 + \ldots
$$

and a nonhydrodynamic mode

$$
\omega_{NH} = -i \left( \frac{1}{\tau_\Pi} - \frac{4}{3T} \frac{\eta}{s} k^2 \right) + \ldots
$$

The dominant mode at long wavelengths is the one whose imaginary part is largest (least negative). One usually assumes that the hydrodynamic modes dominate, but (as seen in Fig. 1)

![Figure 1: The hydrodynamic mode (blue) and the nonhydrodynamic mode (red, dashed) cross at the value of $k$ given in Eq. (7). The plot was made taking $\tau_\Pi T = 2\eta/s$ and $\eta/s = 1/4\pi$.](image)

this is true only for $k < K$, where

$$
K \approx \frac{T}{\sqrt{2(\tau_\Pi)(\eta/s)}} .
$$

This follows by taking the approximate solutions given in Eq. (5) and (6). If exact solutions of Eq. (4) were used, the curves in Fig. 1 would not actually cross, but approach each other to coincide asymptotically for large $k$. The adopted procedure is an estimate of the scale at which the hydro modes cease to dominate.
This can be read as follows \cite{20}: if a drop of plasma has spacial extent $R$, then for nonhydrodynamic modes to be subdominant one would like $R > 2\pi/K$. This way one gets the condition

$$RT > 2\pi \sqrt{2(T\tau\Pi)(\eta/s)}.$$  \hspace{1cm} (8)

Note that in a conformal theory $T\tau\Pi$ as well as $\eta/s$ are dimensionless constants.

The implication of this mode crossing\footnote{In the context of QGP created in colliders, $R$ should be identified roughly with the transverse size of the plasma drop. This size varies very slowly in comparison with the rate of longitudinal expansion.} is that once the inequality Eq. (8) is violated, one is no longer testing hydrodynamics, but rather the particular theory of the nonhydrodynamic sector implicit in BRSSS theory; in other words, one is testing the regulator. In the latter case one should really compare the regulator implicit in BRSSS theory with alternatives, such as (for example) those discussed in Ref. \cite{18}.

The bound Eq. (8) is intuitively very clear and entirely consistent with the idea that hydrodynamics may work well even in small systems as long as they are strongly coupled, since for such systems one expects both $\eta/s$ and the relaxation time to be small. It is interesting to examine just how small the RHS of Eq. (8) can be. The magnitude of the relaxation time is bounded below by causality, as in Eq. (2). There appears to be no firm bound for $\eta/s$, but if we take the Kovtun-Son-Starinets\footnote{Similar mode-crossing phenomena have recently been discussed (in different contexts) by Janik et al. \cite{21,22}, Romatschke \cite{23} and Grozdanov et al. \cite{24}. In the last two references, the hydro modes disappear altogether rather than become subdominant.} value $\eta/s = 1/4\pi$ as a reasonable estimate, we find the simple result

$$RT > 1.$$  \hspace{1cm} (9)

This inequality is reminiscent of the condition that the system size should exceed the mean free path (set by the inverse of the temperature), but we have obtained it here without any reference to a particle description, which may or may not exist in a given situation. The number appearing on the RHS of Eq. (9) is the smallest sensible minimum, which is attained using parameter values suggested by the AdS/CFT description of strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In reality, the values of the relaxation time and shear viscosity may be larger, which would imply that the nonhydrodynamic sector becomes important already on larger scales.
The fact that the applicability of conformal hydrodynamics (in one sense or another) is
governed by the magnitude of the quantity $RT$ has been emphasized already in Refs. [6,7]. It is
also amusing to note that our findings are consistent with Chesler’s observations [26,27] made
in the context of AdS/CFT simulations [28]. He found that the exact, numerically calculated
energy-momentum tensor can be well approximated by hydrodynamics down to drop sizes of
order $RT \approx 1$ or even somewhat less. The analysis presented here is very different, as it refers
only to the effective, hydrodynamic description, but it is perhaps not so surprising that the
same answer appears, since hydrodynamics is a very general framework, which clearly includes
systems which are strongly coupled and whose typical excitations do not have a quasiparticle
interpretation.

3 Relation to observables

The limit Eq. (9) can be translated into an explicit estimate of the minimum entropy per unit of
rapidity $Y$ below which one can expect regulator independence. First note that if one neglects
the transverse expansion of the plasma drop, and follows essentially the Bjorken model [29] (see
e.g. Ref. [30]) one has

$$\frac{dS}{dY} = \pi R^2 \tau_H s ,$$

where $s$ is the entropy density and $\tau_H$ is the earliest time when hydrodynamics could be applicable
(the “hydrodynamization time”). Numerical studies of thermalization based on the AdS/CFT
correspondence [31–33] indicate that $w_H \equiv \tau_H T(\tau_H)$ varies in the approximate range 0.3 – 0.9
(depending on initial conditions). These results apply directly not to QCD, but to $\mathcal{N} = 4$
supersymmetric Yang-Mills theory, but we will take them to be a reasonable indication of the
real-world situation.

To estimate the entropy density appearing in Eq. (10) one can take the expression for a gas
of free gas of quarks and gluons corrected by a factor of $3/4$ to account for strong interactions.
This factor can be motivated by recalling that in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory the
ratio of energy density at strong coupling to the energy density at zero coupling is $3/4$. This
way one obtains the estimate
\[ s = \frac{19}{12} \pi^2 T^3, \]  
which is numerically very close to the result of lattice calculations at temperatures above the deconfinement transition.

Combining the arguments outlined above one arrives at the conclusion that
\[ \frac{dS}{dY} = \frac{19}{12} \pi^3 w_H (R_H T_H)^2, \]  
where the subscripts indicate evaluation at \( \tau = \tau_H \). This formula can be used to translate the bound Eq. (9) into the statement that regulator dependence is inevitable if the entropy per unit of rapidity is less than \((dS/dY)_{MIN} \approx 25\).

Finally, using the approximate connection between entropy and charged particle multiplicity (see e.g. Refs. [30,35,36])
\[ \frac{dS}{dY} \approx 7.5 \frac{dN}{dY}, \]  
one finds
\[ \left( \frac{dN}{dY} \right)_{MIN} \approx 3. \]  
This result is at least qualitatively consistent with the studies of Refs. [8,9], which, as recalled in Sec. I, used a different, but related criterion for estimating the limits of applicability of BRSSS hydrodynamics.

It is important to remember that to arrive at Eq. (14) we assumed essentially the smallest possible values for \( \eta/s \) and the relaxation time as well as a number of other reasonable, but not iron-clad estimates. However, the main point here is not the particular number appearing in Eq. (14), but the observation that a simple physical argument concerning the relative importance of hydro and nonhydro modes leads to an inequality of this kind, with the right hand side of Eq. (14) of order 1.

4 Summary and conclusions

From the perspective of fundamental theory, the reason why hydrodynamics describes the late time behaviour of QGP studied experimentally at RHIC and the LHC is that it is an
effective description of the late-time behaviour of QCD. Our ignorance of QCD in this regime is encapsulated in the values of the hydrodynamic transport coefficients which at this time cannot be calculated \textit{ab initio} and are treated as phenomenological parameters. In principle however such a matching is possible. Furthermore, in practice only very few of these parameters are quantitatively relevant.

As reviewed in the Introduction, relativistic hydrodynamics necessarily includes nonhydrodynamic modes which act as a regulator necessary for causality. The underlying, microscopic theory, such as QCD, also has some spectrum of nonhydro modes, but it is very difficult to make any plausible statements about them. It is thus reasonable to focus on phenomena which do not depend quantitatively on the details of the regulator. The use of BRSSS theory implicitly assumes this. As discussed in Sec. 1, experimental studies of small systems arising in pA and pp collisions make it necessary to actually test whether this assumption is realistic.

Here the issue was addressed directly by estimating the length scale on which nonhydrodynamic modes decay at a rate comparable to the decay rate of hydro modes. This leads to the inequality Eq. (8) which clearly shows that for strongly coupled systems hydrodynamic behaviour can dominate despite small size. This bound can be reformulated as a lower bound on the multiplicity, supporting the conclusion of hydro simulations analysed in Refs. 8,9. As discussed there, this seemingly low bound is not at all absurd in the context of strongly coupled systems. The basic reason for this may be summarized by saying that the hydrodynamic description is invoked at a stage of evolution where QGP may not even be amenable to a quasiparticle description, whereas the bound on multiplicity pertains to the final state after the system has hadronized.

What is to be done in situations when Eq. (14) is violated? Since in such cases BRSSS theory significantly depends on its implicit regulator sector, one has to conclude that one is in fact testing this sector directly. If BRSSS theory is found to work well in such circumstances, one should view this as consistent with the possibility that the leading nonhydrodynamic mode of QCD is purely damped. Otherwise one should consider using a hydrodynamic theory with a

\footnote{In particular, it is hard to attempt to match the nonhydrodynamic sector of QCD with an effective hydrodynamic description.}
different nonhydro sector, such as the theories proposed in [18]. In the end this may also not work unless leading nonhydrodynamic modes are well isolated from higher ones. Whether this is true in QCD is not known at this time.

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