THE PERFORMANCE OF QUADRATIC MODELS FOR COMPLEX PHASE RETRIEVAL

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Abstract. The aim of this paper is to study the performance of the amplitude-based model \( \hat{x} \in \arg\min_{x \in \mathbb{C}^d} \sum_{j=1}^{m} (|\langle a_j, x \rangle| - b_j)^2 \), where \( b_j = |\langle a_j, x_0 \rangle| + \eta_j \) and \( x_0 \in \mathbb{C}^d \) is a target signal. The model is raised in phase retrieval and one has developed many efficient algorithms to solve it. However, there are very few results about the estimation performance in complex case. We show that \( \min_{\theta \in [0, 2\pi]} \| \hat{x} - \exp(i\theta) \cdot x_0 \|_2 \lesssim \| \eta \|_2 \sqrt{\frac{\log d}{m}} \) holds with high probability provided the measurement vectors \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \), are complex Gaussian random vectors and \( m \gtrsim d \). Here \( \eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m \) is the noise vector without any assumption on the distribution. Furthermore, we prove that the reconstruction error is sharp. For the case where the target signal \( x_0 \in \mathbb{C}^d \) is sparse, we establish a similar result for the nonlinear constrained LASSO. This paper presents the first theoretical guarantee on quadratic models for complex phase retrieval. To accomplish this, we leverage a strong version of restricted isometry property for low-rank matrices.

1. Introduction

1.1. Problem setup. The aim of phase retrieval is to recover a target signal \( x_0 \in \mathbb{F}^d \) (\( \mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \} \)) up to a unimodular constant from noise corrupted phaseless measurements \( b_j, j = 1, \ldots, m \), that is,

\[
b_j := |\langle a_j, x_0 \rangle| + \eta_j, \quad j = 1, \ldots, m,
\]

where \( a_j \in \mathbb{F}^n \) are the measurement vectors and \( \eta := (\eta_1, \ldots, \eta_m)^T \in \mathbb{R}^m \) is a noise vector. The phase retrieval problem is raised in various applications, such as X-ray crystallography [15], quantum mechanics [7], optics and astronomy [8] and so on. In the noiseless case, it is shown that \( m \geq 2d - 1 \) (resp. \( m \geq 4d - 4 \)) generic measurements are sufficient to exactly recover \( x_0 \in \mathbb{F}^d \) up to a unimodular constant for \( \mathbb{F} = \mathbb{R} \) (resp. \( \mathbb{F} = \mathbb{C} \)) [11, 21, 6]. An intuitive method to estimate \( x_0 \) is to solve the following nonlinear least square model:

\[
\arg\min_{x \in \mathbb{F}^d} \sum_{j=1}^{m} (|\langle a_j, x \rangle| - b_j)^2.
\]
For the case where the target signal $x_0$ is sparse, the following constrained nonlinear LASSO model can be employed to recover $x_0$:

\begin{equation}
\text{argmin}_{x \in \mathbb{F}^d} \sum_{j=1}^{m} (|\langle a_j, x \rangle| - b_j)^2 \quad \text{s.t. } \|x\|_1 \leq R,
\end{equation}

where $R \in \mathbb{R}$ is a parameter related with the sparsity level. Naturally, one is interested in knowing that how well can one recover $x_0$ by solving (1.1) and (1.2)? For the case $\mathbb{F} = \mathbb{R}$, the performances of (1.1) and (1.2) are studied in [12]. It was shown that if $a_j, j = 1, \ldots, m$, are independent real Gaussian random vectors, the solution $\hat{x}$ to (1.1) satisfies

\begin{equation}
\min\{\|\hat{x} - x_0\|_2, \|\hat{x} + x_0\|_2\} \lesssim \frac{\|\eta\|_2}{\sqrt{m}}
\end{equation}

provided $m \gtrsim d$. One also shows that the reconstruction error is sharp [12]. A similar result is obtained for (1.2) provided $m \gtrsim k \log(ed/k)$, where $k := \|x_0\|_0$ [12]. However, the methods employed in [12] heavily depend on $x_0$ being real. More specifically, the modulo phase in real case is only $\pm 1$-sign, but, in complex case, it becomes the complex unit circle, which is an uncountable set. That is the main reason why it is highly nontrivial to extend the results in [12] to complex signals. The aim of this paper is to overcome the problem, presenting the performance of the amplitude-based models for complex signals.

1.2. Related work. The investigations of models for phase retrieval are divided into two cases: one for real signals and the other for complex ones. We next introduce the known results for these two cases.

1.2.1. Real signals. Most of the results about the performance of models for phase retrieval focus on real signals, i.e., $\mathbb{F} = \mathbb{R}$. As mentioned before, if $a_j, j = 1, \ldots, m$, are independent real Gaussian random vectors, the solution $\hat{x}$ to (1.1) satisfies

\begin{equation}
\min\{\|\hat{x} - x_0\|_2, \|\hat{x} + x_0\|_2\} \lesssim \frac{\|\eta\|_2}{\sqrt{m}}
\end{equation}

provided $m \gtrsim d$ [12]. It is also shown that the bound $\|\eta\|_2/\sqrt{m}$ is tight up to a constant [12]. In fact, there exist algorithms which can almost achieve the performance upper bound under mild conditions about $\eta$. One of the oldest algorithms to solve the amplitude-based model (1.1) is the alternating projection algorithm which is raised by Fienup and Gerchberg [8, 11]. However, the performance of the alternating projection algorithm is only investigated for the noiseless case, i.e., $\eta = 0$ [10, 18]. For the noise case, some generalized gradient-type algorithms, such as Amplitude Flow (AF) and Truncated Amplitude Flow (TAF), are discussed in [23, 13, 9]. Particularly, when $\|\eta\|_\infty \lesssim \|x_0\|_2$ and $a_j, j = 1, \ldots, m$, are independent real Gaussian random vectors with $m \gtrsim d$, the
sequence \( \{ z^{(k)} \}_{k \geq 1} \), which is generated by AF and TAF, obeys
\[
\min\{ \| z^{(k)} + x_0 \|_2, \| z^{(k)} - x_0 \|_2 \} \lesssim \frac{\| \eta \|_2}{\sqrt{m}} + \delta^k \| x_0 \|_2
\]
for some \( \delta \in (0, 1) \) (see [23, 19]). A simple observation is that the right hand side of (1.4) tends to reach the performance upper bound \( \| \eta \|_2 / \sqrt{m} \), when \( k \) tends to infinity.

In [5], Chen and Candès designed the Truncated Wirtinger Flow (TWF) algorithm for solving the following model:
\[
\arg\min_{x \in \mathbb{R}^d} - \sum_{j=1}^{m} (y_j \log(|\langle a_j, x \rangle|^2) - |\langle a_j, x \rangle|^2).
\]
They show that if \( \| \eta \|_{\infty} \lesssim \| x_0 \|_2^2 \) and \( a_j, j = 1, \ldots, m, \) are real Gaussian random vectors with \( m \gtrsim d \), the sequence \( \{ z^{(k)} \}_{k \geq 1} \) generated by the TWF satisfies
\[
\min\{ \| z^{(k)} + x_0 \|_2, \| z^{(k)} - x_0 \|_2 \} \lesssim \frac{\| \eta \|_2}{\sqrt{m \| x_0 \|_2}} + \delta^k \| x_0 \|_2
\]
for some \( \delta \in (0, 1) \).

1.2.2. Complex signals. In this subsection, we introduce the known results for the recovery performance of phase retrieval for complex signals. To our knowledge, all the historical works are based on the intensity-based observations, i.e.,
\[
y_j = |\langle a_j, x_0 \rangle|^2 + \eta_j, \quad j = 1, \ldots, m.
\]
To estimate \( x_0 \in \mathbb{C}^d \) from (1.5), both convex and nonconvex methods are developed. The convex method applies the “lifting” technique, that is, lifting the signal vector \( x \) into rank-one matrix \( X = xx^* \). Then the quadratic measurements are linearized and convex relaxation is shown in [2, 4]. In [2], Candès and Li considered the model
\[
\arg\min_{X \in \mathbb{C}^{d \times d}} \sum_{j=1}^{m} |a_j^* X a_j - y_j| \quad \text{s.t.} \quad X \succeq 0.
\]
They proved that the solution \( \hat{X} \) to (1.6) satisfies
\[
\| \hat{X} - x_0x_0^* \|_F \lesssim \frac{\| \eta \|_1}{m},
\]
provided \( a_j, j = 1, \ldots, m, \) are complex Gaussian random vectors with \( m \gtrsim d \). In [2], it is also shown that
\[
\min_{\theta \in [0, 2\pi]} \| \hat{x} - \exp(i\theta)x_0 \|_2 \lesssim \min \left\{ \| x_0 \|_2, \frac{\| \eta \|_1}{m \| x_0 \|_2} \right\},
\]
where \( \hat{x} \) is the largest eigenvector corresponding to the largest eigenvalue of \( \hat{X} \).
Since the parameter space in convex relaxation is usually much larger than that of the original problem, a renewed interest is focused directly on optimizing nonconvex formulation:

\[(1.7) \quad \arg\min_{x \in \mathbb{C}^d} \sum_{j=1}^{m} \left( |\langle a_j, x \rangle|^2 - y_j \right)^2.\]

For the noiseless case, Candès, Li and Soltanolkotabi developed Wirtinger Flow method (WF) for solving \((1.7)\) [9]. For the noise case, Huang and Xu studied the performance of \((1.7)\) showing that the solution \(\hat{x} \in \mathbb{C}^d\) to \((1.7)\) satisfies

\[\min_{\theta \in [0, 2\pi]} \|\hat{x} - \exp(i\theta)x_0\|_2 \lesssim \min \left\{ \frac{\sqrt{\|\eta\|_2}}{m^{1/4}}, \frac{\|\eta\|_2}{\|x_0\|_2\sqrt{m}} \right\},\]

provided \(m \gtrsim d\) [13].

1.2.3. Sparsity-based models. The recovery of sparse signals \(x_0 \in \mathbb{R}^d\) from phaseless measurements is an active topic recently, which is referred to as the compressive phase retrieval problem [14][10][20]. In this subsection, we assume that \(\|x_0\|_0 \leq k\). The performance of the \(\ell_1\)-model for recovering sparse signals \(x_0\) is studied in [10][20][22]:

\[(1.8) \quad \arg\min_{x \in \mathbb{R}^d} \|x\|_1 \quad \text{s.t.} \quad \left( \sum_{j=1}^{m} |\langle a_j, x \rangle - b_j| \right)^{1/2} \leq \epsilon.\]

Particularly, if \(\{a_j\}_{j=1}^{m}\) are real Gaussian random vectors and \(b_j = |\langle a_j, x_0 \rangle| + \eta_j, j = 1, \ldots, m\) with \(m \gtrsim k \log(ed/k)\), the solution \(\hat{x}\) to \((1.8)\) satisfies \(\min\{\|\hat{x} - x_0\|, \|\hat{x} + x_0\|\} \lesssim \epsilon [20]\). A similar result is obtained for complex signals in [22].

The \(\ell_1\)-model \((1.8)\) requires some prior knowledge about the noise level. The LASSO model \((1.2)\) is often employed to estimate \(x_0\) without using any prior information about the noise. For the real case, the estimation performance of \((1.2)\) is studied in [14], showing that \(\min\{\|\hat{x} - x_0\|_2, \|\hat{x} + x_0\|_2\} \lesssim \|\eta\|_2/\sqrt{m}\). Here, \(\hat{x}\) is a solution to \((1.2)\) with \(R := \|x_0\|_1\) and \(a_j, j = 1, \ldots, m\), are real Gaussian random vectors with \(m \gtrsim k \log(ed/k)\). The method employed in [12] heavily depends on \(\text{sign}(\langle a_j, x \rangle) \in \{-1, 1\}\). Hence, one needs employing novel ideas to extend the result to complex signals.

1.3. Notations. For matrix \(X \in \mathbb{C}^{d_1 \times d_2}\), we use \(X_{S,T}\) to denote the submatrix of \(X\) with the rows indexed in \(S\) and columns indexed in \(T\). We use \(X_{l,:}\) and \(X_{:,j}\) to denote the \(l\)-th row and the \(j\)-th column of \(X\), respectively. For any \(X, Y \in \mathbb{C}^{d_1 \times d_2}\), set \(\langle X, Y \rangle := \text{Tr}(X^*Y)\). For \(x \in \mathbb{C}\), we use \(R(x)\) and \(I(x)\) to denote the real and complex parts of \(x\), respectively. Set \(\|X\|_1 := \sum_{j,l} \sqrt{R(X_{j,l})^2 + I(X_{j,l})^2}, \|X\|_F := \sqrt{\sum_{j,l} (R(X_{j,l})^2 + I(X_{j,l})^2)}\) and \(\|X\|_{1,2} := \sum_j \|X_{:,j}\|_2\). We use...
\|X\|_{0,2} \) to denote the number of non-zero columns in \( X \). We use \( A \gtrsim B \) to denote \( A \geq cB \), where \( c \) is some positive absolute constant. The notation \( \lesssim \) can be defined similarly. For convenience, we use \( C, c \) and their superscript (subscript) forms to denote universal constants whose values vary with different contexts.

1.4. Our contribution. As mentioned before, in [12], Huang and Xu studied the reconstruction errors of \( (1.1) \) and \( (1.2) \) for \( F = \mathbb{R} \). The following theorem establishes the performance of the amplitude-based model for complex signals. We would like to mention that the proof strategy of Theorem 1.1 is quite different with that in [12].

**Theorem 1.1.** Assume that \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \), are i.i.d. complex Gaussian random vectors, i.e., \( a_j \sim \mathcal{N}(0, \frac{1}{2}I_{d \times d}) + \mathcal{N}(0, \frac{1}{2}I_{d \times d})i \). Assume that \( m \gtrsim d \). Then the following holds with probability at least \( 1 - 2 \exp(-c_0 m/4) \): for any \( x_0 \in \mathbb{C}^d \) and \( \eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m \), the solution \( \hat{x} \) to \( (1.1) \) with \( F = \mathbb{C} \) and \( b_j = |\langle a_j, x_0 \rangle| + \eta_j, j = 1, \ldots, m \), satisfies

\[
\min_{\theta \in [0, 2\pi]} \| \hat{x} - \exp(i\theta)x_0 \|_2 \lesssim \frac{\| \eta \|_2}{\sqrt{m}}.
\]

Here, \( c_0 \) is a positive absolute constant.

The following theorem presents a lower bound for the estimation error for any fixed \( x_0 \), which implies that reconstruction error presented in Theorem 1.1 is tight.

**Theorem 1.2.** Assume that \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \), are i.i.d. complex Gaussian random vectors, i.e., \( a_j \sim \mathcal{N}(0, \frac{1}{2}I_{d \times d}) + \mathcal{N}(0, \frac{1}{2}I_{d \times d})i \). Assume that \( \eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m \) satisfies

\[
(1.9) \quad \left| \sum_{j=1}^{m} \eta_j \right| \geq C \sqrt{m} \| \eta \|_2,
\]

for some absolute constant \( C \in (0, 1) \). For any fixed \( x_0 \in \mathbb{C}^d \) satisfying \( \| x_0 \|_2 \gtrsim \| \eta \|_2 / \sqrt{m} \), the following holds with probability at least \( 1 - \exp(-cd) \): the solution \( \hat{x} \) to \( (1.1) \) with \( F = \mathbb{C} \) and \( b_j := |\langle a_j, x_0 \rangle| + \eta_j, j = 1, \ldots, m \), satisfies

\[
\min_{\theta \in [0, 2\pi]} \| \hat{x} - \exp(i\theta)x_0 \|_2 \gtrsim \frac{\| \eta \|_2}{\sqrt{m}},
\]

provided \( m \gtrsim d \). Here, \( c \) is a positive absolute constant.

**Remark 1.3.** Theorem 1.2 requires that \( \eta \) satisfies \( (1.9) \). In fact, there exist a lot of \( \eta \in \mathbb{R}^m \) which satisfy \( (1.9) \). For instance, if \( \eta_j \overset{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2) \) with \( \mu \geq 0 \), then we have \( \sum_{j=1}^{m} \eta_j \sim \mathcal{N}(\mu m, \sigma^2 m) \).
To guarantee \( \eta \) is a random vectors, i.e., Theorem 1.4.\[ \]

model (1.2) is presented in the following theorem.

\[ C(1.11) \]

which implies

\[ \sum_{j=1}^{m} \eta_j - \mu m \geq \sigma t_1 \leq 2 \exp \left( -\frac{c_1 t_1^2}{m} \right) \]

and

\[ \sum_{j=1}^{m} \eta_j - \mu m \geq \sigma^2 t_2 \leq 2 \exp \left( -\frac{c_2 t_2}{m} \right). \]

Taking \( t_1 = c_1 m \) and \( t_2 = c_2 m \) in (1.10), we obtain that the following holds with probability at least \( 1 - 2 \exp(-c_1 m) - 2 \exp(-c_2 m) \):

\[ (\mu - c_1 \sigma)m \leq \sum_{j=1}^{m} \eta_j \leq (\mu + c_1 \sigma)m \quad \text{and} \quad \sigma^2(1 - c_2)m \leq \sum_{j=1}^{m} (\eta_j - \mu)^2 \leq \sigma^2(1 + c_2)m, \]

which implies

\[ \sum_{j=1}^{m} \eta_j^2 \leq 2 \mu \sum_{j=1}^{m} \eta_j + (-\mu^2 + \sigma^2(1 + c_2)m \leq 2\mu(\mu + c_1 \sigma)m + (-\mu^2 + \sigma^2(1 + c_2)m.m. \]

To guarantee \( \eta \) satisfy \( |\sum_{j=1}^{m} \eta_j| \geq C\sqrt{m}\|\eta\|_2 \), it is enough to require that

\[ C\sqrt{2\mu(\mu + c_1 \sigma) - \mu^2 + \sigma^2(1 + c_2)} \leq \mu - c_1 \sigma, \]

when \( c_1 < \mu/\sigma \), which is equivalent to

\[ C \leq \frac{\mu - c_1 \sigma}{\sqrt{2\mu(\mu + c_1 \sigma) - \mu^2 + \sigma^2(1 + c_2)}}. \]

We can choose \( \sigma > 0 \) is small enough, say \( \sigma = \sigma_0 \), so that (1.11) holds. Hence, if \( \eta_j \sim \mathcal{N}(\mu, \sigma_0^2) \) with \( \mu \geq 0 \), then (1.11) holds with high probability.

We next turn to the complex sparse phase retrieval problem. The vector \( x_0 \in \mathbb{C}^d \) is called \( k \)-sparse, if there are at most \( k \) nonzero elements in \( x_0 \). The performance of constrained LASSO-type model (1.2) is presented in the following theorem.

**Theorem 1.4.** Assume that \( a_j \in \mathbb{C}^d \), \( j = 1, \ldots, m \), are independently taken as complex Gaussian random vectors, i.e., \( a_j \sim \mathcal{N}(0, \frac{1}{2}I_{d \times d}) + \mathcal{N}(0, \frac{1}{2}I_{d \times d})i \). Assume that \( m \geq k \log(ed/k) \). Then the following holds with probability at least \( 1 - 2 \exp(-c_0 m/4) \): for any \( k \)-sparse vector \( x_0 \in \mathbb{C}^d \) and \( \eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m \), the solution \( \hat{x} \) to (1.2) with \( F = C \), \( R = \|x_0\|_1 \) and \( b_j := |\langle a_j, x_0 \rangle| + \eta_j \), \( j = 1, \ldots, m \) satisfies

\[ \min_{\theta \in [0, 2\pi]} \|\hat{x} - \exp(i\theta)x_0\|_2 \lesssim \frac{\|\eta\|_2}{\sqrt{m}}. \]

Here, \( c_0 \) is the same constant as that in Theorem 1.1.
2. Proof of Theorem 1.1

Assume that \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \). Let \( \mathcal{A} : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^m \) be a linear operator corresponding to \( \{a_j\}_{j=1}^m \), which is defined as

\[
\mathcal{A}(X) = (a_1^* X a_1, \ldots, a_m^* X a_m),
\]

where \( X \in \mathbb{C}^{d \times d} \). Furthermore, for \( I \subset \{1, \ldots, m\} \), let \( \mathcal{A}_I : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{|I|} \) be a linear operator corresponding to \( \{a_j\}_{j \in I} \) which is defined as

\[
\mathcal{A}_I(X) = (a_j^* X a_j)_{j \in I}.
\]

We first introduce the following results, which are helpful for the proofs.

**Lemma 2.1.** [22, Lemma 3.2] If \( x, y \in \mathbb{C}^d \), and \( \langle x, y \rangle \geq 0 \), then

\[
\|xx^* - yy^*\|_F^2 \geq \frac{1}{2} \cdot \|x\|_2^2 \cdot \|x - y\|_2^2.
\]

**Remark 2.2.** Suppose that \( x, y \in \mathbb{C}^d \) satisfy \( \langle x, y \rangle \geq 0 \). According to Lemma 2.1, we have

\[
\|xx^* - yy^*\|_F^2 \geq \frac{1}{2} \cdot \|x\|_2^2 \cdot \|x - y\|_2^2 \quad \text{and} \quad \|xx^* - yy^*\|_F^2 \geq \frac{1}{2} \cdot \|y\|_2^2 \cdot \|x - y\|_2^2,
\]

which implies that

\[
(2.2) \quad \|xx^* - yy^*\|_F^2 \geq \frac{1}{4} \left( \|x\|_2^2 + \|y\|_2^2 \right) \|xx^* - yy^*\|_F^2 \geq \frac{1}{8} (\|x\|_2 + \|y\|_2)^2 \|xx^* - yy^*\|_F^2.
\]

**Theorem 2.3.** [22, Theorem 1.2] Assume that \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \), are independently taken as complex Gaussian random vectors, i.e., \( a_j \sim \mathcal{N}(0, \frac{1}{2} I_{d \times d}) + \mathcal{N}(0, \frac{1}{2} I_{d \times d})i \). Assume that \( \mathcal{A} : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^m \) is the linear operator corresponding to \( \{a_j\}_{j=1}^m \), which is defined in (2.1). If \( m \geq k \log(ed/k) \), with probability at least \( 1 - 2 \exp(-c_0 m) \), \( \mathcal{A} \) satisfies the restricted isometry property on the order of \( (2, k) \), i.e.,

\[
C_- \|X\|_F \leq \frac{1}{m} \|\mathcal{A}(X)\|_1 \leq C_+ \|X\|_F,
\]

for all Hermitian \( X \in \mathbb{C}^{d \times d} \) with \( \text{rank}(X) \leq 2 \) and \( \|X\|_{0,2} \leq k \) (also \( \|X^*\|_{0,2} \leq k \)). Here, \( c_0, C_-, C_+ \) are positive absolute constants.

The following theorem is a strong version of Theorem 2.3.

**Theorem 2.4.** Assume that \( a_j, j = 1, \ldots, m \), are independently taken as complex Gaussian random vectors, i.e., \( a_j \sim \mathcal{N}(0, \frac{1}{2} I_{d \times d}) + \mathcal{N}(0, \frac{1}{2} I_{d \times d})i \). There exists a universal constant \( \beta_0 > 0 \) such that, with probability at least \( 1 - 2 \exp(-c_0 m/4) \), the following holds:

\[
(2.3) \quad C_- (1 - \beta_0) \|X\|_F \leq \frac{1}{m} \|\mathcal{A}_I(X)\|_1 \leq C_+ \|X\|_F, \quad \text{for any } I \subset \{1, \ldots, m\}, \text{ with } |I| \geq (1 - \beta_0)m,
\]
for all Hermitian $X \in \mathbb{C}^{d \times d}$ with rank$(X) \leq 2$ and \( \|X\|_{0,2} \leq k \) (also \( \|X^*\|_{0,2} \leq k \)), provided
\[
m \geq \frac{1}{1 - \beta_0} k \log(\frac{ed}{k}).
\]

Here, $c_0$, $C_-$, $C_+$ are positive constants presented in Theorem 2.3.

**Proof.** According to Theorem 2.3, for any fixed $I_0 \subset \{1, \ldots, m\}$ with \( \#I_0 \geq (1 - \beta_0)m \),
\[
C_-(1 - \beta_0) \|X\|_F \leq \frac{1}{\#I_0} \|A_{I_0}(X)\|_1 \leq C_+ \|X\|_F
\]
holds with probability at least \( 1 - 2 \exp(-c_0(1 - \beta_0)m) \), provided \( (1 - \beta_0)m \gtrsim k \log(\frac{ed}{k}) \).

A simple observation is that
\[
C_-(1 - \beta_0) \|X\|_F \leq \frac{1}{m} \|A_{I_0}(X)\|_1 = \frac{\#I_0}{m} \frac{1}{\#I_0} \|A_{I_0}(X)\|_1 \leq C_+ \|X\|_F.
\]

Then (2.3) holds with probability at least
\[
1 - 2 \sum_{\#I \geq (1 - \beta_0)m} \left( \frac{m}{\# I} \right) \exp(-c_0(1 - \beta_0)m) \geq 1 - 2 \sum_{\#I \geq (1 - \beta_0)m} \left( \frac{m}{(1 - \beta_0)m} \right) \exp(-c_0(1 - \beta_0)m)
\]
\[
= 1 - 2(\beta_0 m + 1) \left( \frac{m}{(1 - \beta_0)m} \right) \exp(-c_0(1 - \beta_0)m)
\]
\[
\geq 1 - 2 \exp(\beta_0 m + \beta_0 \log(\frac{e}{\beta_0})m - c_0(1 - \beta_0)m).
\]

The last inequality follows from \( 1 + \beta_0 m \leq \exp(\beta_0 m) \), and
\[
\left( \frac{m}{(1 - \beta_0)m} \right) = \left( \frac{m}{\beta_0 m} \right) \frac{1}{\beta_0} \leq \frac{e}{\beta_0} \exp(\beta_0 \log(\frac{e}{\beta_0})m).
\]

We can take $\beta_0 > 0$ small enough so that
\[
(2.4) \quad \beta_0 \log(\frac{e}{\beta_0}) \leq \frac{c_0}{4}, \quad \beta_0 \leq \frac{c_0}{4}, \quad \text{and} \quad \beta_0 \leq \frac{1}{4}.
\]

Hence, the inequality (2.3) holds with probability at least
\[
1 - 2 \exp(\beta_0 m + \beta_0 \log(\frac{e}{\beta_0})m - c_0(1 - \beta_0)m) \geq 1 - 2 \exp(-c_0 m/4).
\]

We arrive at the conclusion. \[\square\]

**Remark 2.5.** Theorem 2.4 claims that there exists $\beta_0 > 0$ so that (2.3) holds. In fact, according to (2.4), we can take $\beta_0 = \min(\frac{c_0}{4}, \exp(-\frac{c_0}{4}), \frac{1}{4})$, where $c_0$ is defined in Theorem 2.3. If we take $k = d$ in Theorem 2.4, then (2.3) holds with high probability provided $m \gtrsim d$.

The following lemma plays a key role in the proof of Theorem 1.1. We postpone its proof to the end of this section.
Lemma 2.6. Assume that \(a_j, j = 1, \ldots, m\), are independently taken as complex Gaussian random vectors, i.e., \(a_j \sim N(0, \frac{1}{2}I_{d \times d})\). For any \(x_0 \in \mathbb{C}^d\) and \(\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m\), we set
\[
b_j := |\langle a_j, x_0 \rangle| + \eta_j, \quad j = 1, \ldots, m.
\]
We assume that \(\tilde{x} \in \mathbb{C}^d\) satisfies
\[
\sum_{j=1}^{m}(\langle a_j, \tilde{x} \rangle - b_j)^2 \leq \sum_{j=1}^{m}(\langle a_j, x_0 \rangle - b_j)^2.
\]
If \(m \geq d\), the following holds with probability at least \(1 - 2 \exp(-c_0m/4)\):
\[
\|A_I(\tilde{x}^* - x_0x_0^*)\|_1 \leq \sqrt{m}\|\eta\|_2(\|\tilde{x}\|_2 + \|x_0\|_2).
\]
Here, \(c_0\) is the positive constant which is defined in Theorem 2.3. The index set \(I := I_{\tilde{x}, x_0} \subset \{1, \ldots, m\}\) is chosen using the following way. Set \(\xi_j := |\langle a_j, \tilde{x} \rangle| + |\langle a_j, x_0 \rangle|\) and assume that \(\xi_{j_1} \leq \xi_{j_2} \leq \cdots \leq \xi_{j_m}\). Take \(I = \{j_t \mid t \leq (1 - \beta_0)m\}\), where \(\beta_0 > 0\) is defined in Theorem 2.4.

We next present the proof of Theorem 1.1.

Proof of Theorem 1.1. If \(\tilde{x} = x_0 = 0\), the conclusion holds. We next assume that either \(\tilde{x} \neq 0\) or \(x_0 \neq 0\). According to (2.2), we have
\[
\frac{\sqrt{2}}{4} (\|\tilde{x}\|_2 + \|x_0\|_2) \inf_{\theta \in [0, 2\pi]} \|\tilde{x} - \exp(i\theta)x_0\|_2 \leq \|\tilde{x}^* - x_0x_0^*\|_F.
\]
We assume that \(I \subset \{1, \ldots, m\}\) is the set defined in Lemma 2.6. Since \(\tilde{x}\) is a solution to (1.1), we have
\[
\sum_{j=1}^{m}(\langle a_j, \tilde{x} \rangle - b_j)^2 \leq \sum_{j=1}^{m}(\langle a_j, x_0 \rangle - b_j)^2.
\]
Combining (2.3), Lemma 2.6 and Theorem 2.4 with \(k = d\), we obtain that
\[
\frac{\sqrt{2}}{4} (\|\tilde{x}\|_2 + \|x_0\|_2) \inf_{\theta \in [0, 2\pi]} \|\tilde{x} - \exp(i\theta)x_0\|_2 \leq C_- \|\tilde{x}^* - x_0x_0^*\|_F
\]
\[
\leq \frac{1}{(1 - \beta_0)m} \|A_I(\tilde{x}^* - x_0x_0^*)\|_1 \leq \frac{1}{\sqrt{m}} \|\eta\|_2 \|\tilde{x}\|_2 + \|x_0\|_2,
\]
which implies the conclusion.

Proof of Lemma 2.4. If \(\tilde{x} = x_0 = 0\), then the conclusion holds. We next assume that either \(\tilde{x} \neq 0\) or \(x_0 \neq 0\). According to the definition of \(\tilde{x}\), we have
\[
\sum_{j=1}^{m}(\langle a_j, \tilde{x} \rangle - b_j) \leq \sqrt{m} \sqrt{\sum_{j=1}^{m}(\langle a_j, \tilde{x} \rangle - b_j)^2} \leq \sqrt{m} \sqrt{\sum_{j=1}^{m}(\langle a_j, x_0 \rangle - b_j)^2} \leq \sqrt{m}\|\eta\|_2.
\]
Furthermore, we have

\[
\sum_{j=1}^{m} | |\langle a_j, \tilde{x} \rangle| - b_j | = \sum_{j=1}^{m} | |\langle a_j, \tilde{x} \rangle| - |\langle a_j, x_0 \rangle| - \eta_j |
\]

(2.8)

\[
\geq \sum_{j=1}^{m} \frac{| \langle a_j, \tilde{x} \rangle | - \eta_j (| \langle a_j, \tilde{x} \rangle| + | \langle a_j, x_0 \rangle|) }{ | \langle a_j, \tilde{x} \rangle| + | \langle a_j, x_0 \rangle| } \geq \sum_{j \in I} \frac{ | | a_j a_j^*, \tilde{x} x_0^* - x_0 x_0^* | | - | \eta_j | (| \langle a_j, \tilde{x} \rangle| + | \langle a_j, x_0 \rangle|) }{ | \langle a_j, \tilde{x} \rangle| + | \langle a_j, x_0 \rangle| }.
\]

Combining (2.7) and (2.8), we obtain that

(2.9)

\[
\sqrt{m} \eta_2 \geq \sum_{j \in I} \frac{ | | a_j a_j^*, \tilde{x} x_0^* - x_0 x_0^* | | - | \eta_j | (| \langle a_j, \tilde{x} \rangle| + | \langle a_j, x_0 \rangle|) }{ | \langle a_j, \tilde{x} \rangle| + | \langle a_j, x_0 \rangle| }.
\]

Then we have

(2.10)

\[
\sqrt{m} \eta_2 \geq \frac{ \sum_{j \in I} | | a_j a_j^*, \tilde{x} x_0^* - x_0 x_0^* | | - | \eta_j | (| \langle a_j, \tilde{x} \rangle| + | \langle a_j, x_0 \rangle|) }{ \sqrt{2C_{\beta_6} (\| x_0 \|_2 + \| \tilde{x} \|_2) } } \geq \frac{ \sum_{j \in I} | | a_j a_j^*, \tilde{x} x_0^* - x_0 x_0^* | | - \sum_{j \in I} \eta_j (| \langle a_j, \tilde{x} \rangle| + | \langle a_j, x_0 \rangle|) }{ \sqrt{2C_{\beta_6} (\| x_0 \|_2 + \| \tilde{x} \|_2) } },
\]

where \( C_{\beta_6} := \frac{\beta_6}{\beta_0} \) and the second inequality in (2.10) follows from

(2.11)

\[
| \langle a_j, x_0 \rangle | + | \langle a_j, \tilde{x} \rangle | \leq \sqrt{2C_{\beta_6} (\| x_0 \|_2 + \| \tilde{x} \|_2) }, \quad \text{for } j \in I.
\]

We postpone the argument of (2.11) until the end of this proof. We next employ (2.10) to show that

(2.12)

\[
| A_t (\tilde{x} x_0^* - x_0 x_0^*) |_1 \lesssim \sqrt{m} | \eta_2 (\| \tilde{x} \|_2 + \| x_0 \|_2) |
\]

under the following two cases.

**Case 1:** We assume that the following holds:

(2.13)

\[
\sum_{j \in I} | | a_j a_j^*, \tilde{x} x_0^* - x_0 x_0^* | | \geq 4 \sum_{j \in I} \eta_j (| \langle a_j, \tilde{x} \rangle| + | \langle a_j, x_0 \rangle|).
\]
Then we have

\((2.14)\)

\[
\left( \sum_{j \in I} |\langle a_j, a_j^*, \tilde{x}\tilde{x}^* - x_0x_0^* \rangle| - \sum_{j \in I} |\eta_j||\langle a_j, \tilde{x}\rangle| + |\langle a_j, x_0 \rangle| \right)^2 \\
= \left( \sum_{j \in I} |\langle a_j, a_j^*, \tilde{x}\tilde{x}^* - x_0x_0^* \rangle| \right)^2 + \left( \sum_{j \in I} |\eta_j||\langle a_j, \tilde{x}\rangle| + |\langle a_j, x_0 \rangle| \right)^2 \\
- 2 \left( \sum_{j \in I} |\langle a_j, a_j^*, \tilde{x}\tilde{x}^* - x_0x_0^* \rangle| \right) \left( \sum_{j \in I} |\eta_j||\langle a_j, \tilde{x}\rangle| + |\langle a_j, x_0 \rangle| \right) \\
\geq \left( \sum_{j \in I} |\langle a_j, a_j^*, \tilde{x}\tilde{x}^* - x_0x_0^* \rangle| \right)^2 - 2 \left( \sum_{j \in I} |\langle a_j, a_j^*, \tilde{x}\tilde{x}^* - x_0x_0^* \rangle| \right) \left( \sum_{j \in I} |\eta_j||\langle a_j, \tilde{x}\rangle| + |\langle a_j, x_0 \rangle| \right) \\
\geq \left( \sum_{j \in I} |\langle a_j, a_j^*, \tilde{x}\tilde{x}^* - x_0x_0^* \rangle| \right)^2 - \frac{1}{2} \left( \sum_{j \in I} |\langle a_j, a_j^*, \tilde{x}\tilde{x}^* - x_0x_0^* \rangle| \right)^2 \\
= \frac{1}{2} \left( \sum_{j \in I} |\langle a_j, a_j^*, \tilde{x}\tilde{x}^* - x_0x_0^* \rangle| \right)^2 = \frac{1}{2} \|A_I(\tilde{x}\tilde{x}^* - x_0x_0^*)\|_1^2,
\]

where the second inequality follows from \((2.13)\). Combining \((2.14)\) and \((2.10)\), we obtain that

\[
\|A_I(\tilde{x}\tilde{x}^* - x_0x_0^*)\|_1 \leq 2 \sqrt{C_{\beta_0}} \sqrt{m} \eta_2 (\|\tilde{x}\|_2 + \|x_0\|_2),
\]

which implies the conclusion.

**Case 2:** We next consider the case where

\[(2.15)\]

\[
\sum_{j \in I} |\langle a_j, a_j^*, \tilde{x}\tilde{x}^* - x_0x_0^* \rangle| < 4 \sum_{j \in I} |\eta_j||\langle a_j, \tilde{x}\rangle| + |\langle a_j, x_0 \rangle|.
\]

According to \((2.15)\), we have

\[
\|A_I(\tilde{x}\tilde{x}^* - x_0x_0^*)\|_1 \leq 4 \|\eta\|_2 \sqrt{\sum_{j \in I} (|\langle a_j, \tilde{x}\rangle| + |\langle a_j, x_0 \rangle|)^2} \\
\leq 4 \sqrt{2C_{\beta_0}} \sqrt{m} \|\eta\|_2 (\|x_0\|_2 + \|\tilde{x}\|_2).
\]

The last inequality above follows from \((2.11)\). We arrive at the conclusion.
It remains to prove (2.11). A simple observation is that, for \( j \in I \),
\[
(|\langle a_j, x_0 \rangle| + |\langle a_j, \tilde{x} \rangle|)^2 \leq 2|\langle a_j, x_0 \rangle|^2 + 2|\langle a_j, \tilde{x} \rangle|^2
\]
\[
\leq \frac{2}{\beta_0 m} \sum_{t \in \{1, \ldots, m\} \setminus t} (|\langle a_t, x_0 \rangle|^2 + |\langle a_t, \tilde{x} \rangle|^2)
\]
\[
= \frac{2}{\beta_0 m} \|A_{\{1, \ldots, m\} \setminus t}(x_0 x_0^* + \tilde{x} \tilde{x}^*)\|_1
\]
\[
\leq \frac{2}{\beta_0} \cdot \frac{1}{m} \|A(x_0 x_0^* + \tilde{x} \tilde{x}^*)\|_1
\]
\[
\overset{(a)}{=} \frac{2C_\delta}{\beta_0} \|x_0 x_0^* + \tilde{x} \tilde{x}^*\|_F
\]
\[
\leq 2C_\delta \|x_0\|_2^2 + \|\tilde{x}\|_2^2 \leq 2C_\delta (\|x_0\|_2 + \|\tilde{x}\|_2)^2,
\]
which implies (2.11). Here, the inequality (a) follows from Theorem [2.3]. \( \square \)

3. PROOF OF THEOREM 1.2

Before presenting the proof of Theorem 1.2, we introduce the following lemmas.

Lemma 3.1. [17] Theorem 2.6.3 (Hoeffding-type inequality) Let \( x_1, \ldots, x_N \) be independent centered sub-Gaussian random variables, and let \( K = \max_i \|x_i\|_{\psi_2} \). Then, for every \( a = (a_1, \ldots, a_N) \in \mathbb{R}^N \) and every \( t \geq 0 \), we have
\[
\mathbb{P} \left\{ \left| \sum_{i=1}^N a_i x_i \right| \geq t \right\} \leq 2 \exp \left( - \frac{c' t^2}{K^2 \|a\|_2^2} \right),
\]
where \( c' > 0 \) is an absolute constant. The sub-Gaussian norm of \( x \), denoted as \( \|x\|_{\psi_2} \), is defined as \( \|x\|_{\psi_2} := \inf\{t > 0 : \mathbb{E} \exp(x^2/t^2) \leq 2\} \).

Theorem 3.2. [17] Theorem 4.4.5 (Norm of matrices with sub-Gaussian entries) Let \( A \) be an \( m \times d \) random matrix whose entries \( A_{j,i} \) are independent, mean zero, sub-Gaussian random variables. Then, for any \( t > 0 \), the following holds with probability at least \( 1 - 2 \exp(-t^2) \):
\[
\|A\| \leq C' K (\sqrt{m} + \sqrt{d} + t).
\]
Here \( K = \max_{j,i} \|A_{j,i}\|_{\psi_2} \) and \( C' > 0 \) is an absolute constant.

Lemma 3.3. Assume that \( a_j, j = 1, \ldots, m \), are i.i.d. complex Gaussian random vectors, i.e., \( a_j \sim \mathcal{N}(0, \frac{1}{2} I_{d \times d}) + \mathcal{N}(0, \frac{1}{2} I_{d \times d})i \). Assume that \( \eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m \) which satisfies \( |\sum_{j=1}^m \eta_j| \geq C \sqrt{m} \|\eta\|_2 \) with \( C \in (0, 1) \). Then the following holds with probability at least \( 1 - 4 \exp(-cd) \):
\[
\left| \sum_{j=1}^m \eta_j |\langle a_j, x \rangle| \right| \geq \sqrt{m} \|\eta\|_2 \|x\|_2 \quad \text{for any} \ x \in \mathbb{C}^d,
\]
provided $m \geq d$. Here, $c > 0$ is an absolute constant.

**Proof.** First of all, we consider an upper bound of $\left| \sum_{j=1}^{m} |\eta_j| \cdot |\langle a_j, x \rangle| \right|$. We can obtain that the following holds for any $x \in \mathbb{C}^d$ with probability at least $1 - 2 \exp(-d)$:

\[
\left| \sum_{j=1}^{m} |\eta_j| \cdot |\langle a_j, x \rangle| \right| \leq \|\eta\|_2 \cdot \left( \sum_{j=1}^{m} |\langle a_j, x \rangle|^2 \right)^{1/2} \leq 3 \cdot C_\psi \cdot C' \cdot \sqrt{m} \|\eta\|_2 \|x\|_2,
\]

where $C' > 0$ is the positive absolute constant defined in Theorem 3.2. Here, the second inequality follows from Theorem 3.2 with $t = \sqrt{d}$ and $m \geq C_2 d$ for some $C_2 \geq 1$, which will be chosen later. $C_\psi$ is an absolute constant denoted as the sub-Gaussian norm of complex Gaussian random variable drawn from $\mathcal{N}(0, \frac{1}{2}) + \mathcal{N}(0, \frac{1}{2})i$ [17, Example 2.5.8].

Then we begin to consider the lower bound of $\left| \sum_{j=1}^{m} |\eta_j| \cdot |\langle a_j, x \rangle| \right|$. For any fixed $x \in \mathbb{C}^d$ with $\|x\|_2 = 1$, we have $\langle a_j, x \rangle \sim \mathcal{N}(0, \frac{1}{2}) + \mathcal{N}(0, \frac{1}{2})i$. Assume that $\xi_1, \xi_2 \sim \mathcal{N}(0, 1)$. Then

\[
\mathbb{E}[|\langle a_j, x \rangle|] = \frac{\sqrt{2}}{2} \mathbb{E}[|\xi_1 + \xi_2 i|] = \frac{\sqrt{2}}{2} \mathbb{E}\sqrt{\xi_1^2 + \xi_2^2} = \frac{\sqrt{2}}{2} \cdot 1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{\xi_1^2 + \xi_2^2} \exp \left( -\frac{\xi_1^2 + \xi_2^2}{2} \right) d\xi_1 d\xi_2 = \frac{\sqrt{\pi}}{2},
\]

Noting that $|\langle a_j, x \rangle|, j = 1, \ldots, m$, are independent sub-Gaussian random variables with $\|\langle a_j, x \rangle\|_{\psi_2} \leq C_{\psi_2}$, we can use Lemma 3.1 to obtain that

\[
\mathbb{P} \left\{ \left| \sum_{j=1}^{m} |\eta_j| \cdot |\langle a_j, x \rangle| \right| - \frac{\sqrt{2}}{2} \sum_{j=1}^{m} |\eta_j| \geq t \right\} \leq 2 \exp \left( -\frac{c t^2}{C_{\psi_2}^2 \|\eta\|_2^2} \right), \quad \text{for any } t \geq 0.
\]

Taking $t = C_1 \sqrt{d} \|\eta\|_2$ in (3.2), we obtain that the following holds with probability at least $1 - 2 \exp(-c' \cdot C_1^2 / C_{\psi_2}^2 d)$:

\[
\left| \sum_{j=1}^{m} |\eta_j| \cdot |\langle a_j, x \rangle| \right| \geq \frac{\sqrt{2}}{2} \left| \sum_{j=1}^{m} |\eta_j| - C_1 \sqrt{d} \|\eta\|_2 \right| \geq \left( \frac{\sqrt{\pi}}{2} C \sqrt{m} - C_1 \sqrt{d} \right) \|\eta\|_2 \geq \left( \frac{\sqrt{\pi}}{2} C - \frac{C_1}{\sqrt{C_2}} \right) \sqrt{m} \|\eta\|_2.
\]

Here, we use $\sum_{j=1}^{m} |\eta_j| \geq C \sqrt{m} \|\eta\|_2$ and $m \geq C_2 d$. The constant $C_1 > 0$ will be chosen later.

Assume that $\mathcal{N}$ is an $\varepsilon$-net of the unit complex sphere in $\mathbb{C}^d$ and hence the covering number $\#\mathcal{N} \leq (1 + \frac{\varepsilon}{2})^{2d}$. By the union bound, we obtain that

\[
\left| \sum_{j=1}^{m} |\eta_j| \cdot |\langle a_j, x \rangle| \right| \geq \left( \frac{\sqrt{\pi}}{2} C - \frac{C_1}{\sqrt{C_2}} \right) \sqrt{m} \|\eta\|_2 \quad \text{for any } x \in \mathcal{N}.
\]
holds with probability at least \(1 - 2\exp\left(-((c' \cdot C_1^2/C_2^2) - 2\log(1 + 2/\epsilon))d\right)\). For any \(x' \in \mathbb{C}^d\) with \(\|x'\|_2 = 1\), there exists some \(x \in \mathcal{N}\) such that \(\|x' - x\|_2 \leq \epsilon\). Then

\[
(3.5) \quad \left| \sum_{j=1}^{m} \eta_j \langle a_j, x' \rangle \right| - \left| \sum_{j=1}^{m} \eta_j \langle a_j, x \rangle \right| \leq \left| \sum_{j=1}^{m} \eta_j \cdot \|a_j, x' - x\|_2 = \|x' - x\|_2 \sum_{j=1}^{m} \eta_j \cdot \left\langle a_j, \frac{x' - x}{\|x' - x\|_2} \right\rangle \right| 
\leq 3 \cdot \epsilon \cdot C \cdot C' \cdot \sqrt{m}\eta\|_2.
\]

The last line above is based on (3.1). Combining (3.4) and (3.5), we obtain that

\[
(3.6) \quad \min_{\|x\|_2 = 1} \left| \sum_{j=1}^{m} \eta_j \langle a_j, x \rangle \right| \geq \left( \frac{\sqrt{\pi}}{2} \right) C - \frac{C_1}{\sqrt{C_2}} \sqrt{m}\eta\|_2 - 3 \cdot \epsilon \cdot C \cdot C' \cdot \sqrt{m}\eta\|_2 = \left( \frac{\sqrt{\pi}}{2} C - \frac{C_1}{\sqrt{C_2}} - 3 \cdot \epsilon \cdot C \cdot C' \cdot \sqrt{m}\eta\|_2, \right.
\]

holds with probability at least \(1 - 2\exp\left(-((c' \cdot C_1^2/C_2^2) - 2\log(1 + 2/\epsilon))d\right)\) \(-\exp(-d)\). Therefore, if \(\epsilon\) satisfies \(\sqrt{\pi}/8 \geq \epsilon \cdot C \cdot C' > 4\log(1 + 2/\epsilon)\) and \(C_2\) satisfies \(\frac{C_1}{\sqrt{C_2}} \leq \frac{\pi}{4} \), we can obtain that

\[
\min_{\|x\|_2 = 1} \left| \sum_{j=1}^{m} \eta_j \langle a_j, x \rangle \right| \geq \frac{\sqrt{\pi}}{8} C \sqrt{m}\eta\|_2.
\]

holds with probability at least \(1 - 2\exp\left(-\frac{1}{2}(c' \cdot C_1^2/C_2^2) d\right) - 2\exp(-d) \geq 1 - 4\exp(-cd),\) where \(c = \max\{1, \frac{1}{2}(c' \cdot C_1^2/C_2^2)\}\).

\[
\text{Lemma 3.4. Assume that } a_j, j = 1, \ldots, m, \text{ are i.i.d. complex Gaussian random vectors, i.e., } a_j \sim \mathcal{N}(0, \frac{1}{2}I_{d \times d}) + \mathcal{N}(0, \frac{1}{2}I_{d \times d})i. \text{ Assume that } m \geq d. \text{ Then the following holds with probability at least } 1 - 2\exp(-d):\]

\[
\sum_{j=1}^{m} |\langle a_j, u \rangle| \cdot |\langle a_j, v \rangle| \lesssim m\|u\|_2\|v\|_2, \quad \text{for any } u, v \in \mathbb{C}^d.
\]

**Proof.** Taking \(t = \sqrt{d}\) in Theorem 3.2, we can obtain that

\[
\sum_{j=1}^{m} |\langle a_j, u \rangle| \cdot |\langle a_j, v \rangle| \leq \left( \sum_{j=1}^{m} |\langle a_j, u \rangle|^2 \right)^{1/2} \left( \sum_{j=1}^{m} |\langle a_j, v \rangle|^2 \right)^{1/2} \lesssim m\|u\|_2\|v\|_2,
\]

holds with probability at least \(1 - 2\exp(-d)\). Here, we use \(m \geq d\).

We next present the proof of Theorem 1.2.

**Proof of Theorem 1.2** Recall that \(\tilde{x}\) is a solution to \(\arg\min_{x \in \mathbb{C}^d} \|a_j, x\| - b_j^2\). For the case where \(\tilde{x} = 0\), the conclusion follows from \(\|x_0\|_2 \gtrsim \frac{\|a\|_2}{\sqrt{m}}\). We next assume that \(\tilde{x} \neq 0\).
We claim that
\[ \sum_{j=1}^{m} |\langle a_j, x \rangle| \leq \sum_{j=1}^{m} (|\langle a_j, x \rangle| - b_j)^2 = 0 \]
and postpone its argument until the end of the proof. Substituting $b_j = |\langle a_j, x_0 \rangle| + \eta_j$, $j = 1, \ldots, m$ into (3.11), we obtain that
\[ \sum_{j=1}^{m} \eta_j |\langle a_j, \hat{x} \rangle| = \sum_{j=1}^{m} (|\langle a_j, \hat{x} \rangle|^2 - |\langle a_j, x_0 \rangle| \cdot |\langle a_j, \hat{x} \rangle|), \]
which implies
\[ \sum_{j=1}^{m} \eta_j |\langle a_j, \hat{x} \rangle| \leq \sum_{j=1}^{m} |\langle a_j, \hat{x} \rangle| \cdot |\langle a_j, x_0 \rangle| \leq \sum_{j=1}^{m} |\langle a_j, \hat{x} \rangle| \cdot |\langle a_j, \hat{x} - \exp(i\theta)x_0 \rangle| \]
for any $\theta \in [0, 2\pi)$. We use (3.10) to obtain that
\[ \sqrt{m} \| \eta \|_2 \| \hat{x} \|_2 \leq \sum_{j=1}^{m} \eta_j |\langle a_j, \hat{x} \rangle| \leq \sum_{j=1}^{m} |\langle a_j, \hat{x} \rangle| \cdot |\langle a_j, \hat{x} - \exp(i\theta)x_0 \rangle| \leq m \| \hat{x} \|_2 \| \hat{x} - \exp(i\theta)x_0 \|_2, \]
for any $\theta \in [0, 2\pi)$, where the inequalities (a) and (b) follow from Lemma 3.3 and Lemma 3.4 respectively. According to (3.9), we have
\[ \frac{\| \eta \|_2}{\sqrt{m}} \leq \min_{\theta \in [0, 2\pi)} \| \hat{x} - \exp(i\theta)x_0 \|_2, \]
which leads to the conclusion.

It remains to prove (3.9). A simple observation is that
\[ \sum_{j=1}^{m} (|\langle a_j, x \rangle| - b_j)^2 = \sum_{j=1}^{m} (\sqrt{a_{j,1}^Tx_1 + a_{j,2}^Tx_2}^2 + (a_{j,1}^T x_2 - a_{j,2}^T x_1)^2 - b_j)^2 =: g(x_1, x_2), \]
where $x_1$ and $x_2$ are the real and imaginary parts of $x$, i.e., $x = x_1 + x_2i$. Similarly, we use $a_{j,1}$ and $a_{j,2}$ to denote the real and imaginary parts of $a_j$, i.e., $a_j = a_{j,1} + a_{j,2}i$. The sub-gradient set of $g(x_1, x_2)$ at $(x_1, x_2)$ is
\[ \partial g(x_1, x_2) := \left( \frac{\partial g(x_1, x_2)}{\partial x_1}, \frac{\partial g(x_1, x_2)}{\partial x_2} \right) = \left\{ \begin{array}{l} 2 \sum_{j=1}^{m} (|\langle a_j, x \rangle| - b_j)(z_{j,1}a_{j,1} - z_{j,2}a_{j,2}), \\ 2 \sum_{j=1}^{m} (|\langle a_j, x \rangle| - b_j)(z_{j,1}a_{j,2} + z_{j,2}a_{j,1}) \end{array} \right\}, \]
where $z_{j,1}$ and $z_{j,2}$ are defined in (3.10) or (3.11). Here,
\[ z_{j,1} := \frac{a_{j,1}^Tx_1 + a_{j,2}^Tx_2}{|\langle a_j, x \rangle|}, \quad z_{j,2} := \frac{a_{j,1}^Tx_2 - a_{j,2}^Tx_1}{|\langle a_j, x \rangle|}, \quad \text{if } |\langle a_j, x \rangle| \neq 0. \]
Otherwise, we require $(z_{j,1}, z_{j,2}) \in \mathbb{R}^2$ satisfies
\[ z_{j,1}^2 + z_{j,2}^2 \leq 1, \quad \text{if } |\langle a_j, x \rangle| = 0. \]
Recall that $\hat{x} = \hat{x}_1 + \hat{x}_2$ is a solution to (1.1). Then we have $0 \in \partial g(x_1, x_2)|_{x_1 = \hat{x}_1, x_2 = \hat{x}_2}$.

Therefore, denote specific $\hat{z}_{j,1}$ and $\hat{z}_{j,2}$, $j = 1, \ldots, m$, such that

$$0 = 2 \sum_{j=1}^{m} (\langle a_j, \hat{x} \rangle - b_j) (\hat{z}_{j,1} a_j - \hat{z}_{j,2} a_j) =: \hat{g}_1,$$

and

$$0 = 2 \sum_{j=1}^{m} (\langle a_j, \hat{x} \rangle - b_j) (\hat{z}_{j,1} a_j + \hat{z}_{j,2} a_j) =: \hat{g}_2.$$

Then by direct calculation, we have

$$0 = (\hat{g}_1 + \hat{g}_2 i, \hat{x}_1 + \hat{x}_2 i) = 2 \sum_{j=1}^{m} (\langle a_j, \hat{x} \rangle - b_j) \langle a_j, \hat{x} \rangle.$$

\[\square\]

4. Proof of Theorem 1.4

**Lemma 4.1.** Assume that $x_0 \in \mathbb{C}^d$ is $k$-sparse. Assume that $a_j \in \mathbb{C}^d$ and $b_j = |\langle a_j, x_0 \rangle| + \eta_j$, $j = 1, \ldots, m$. We assume that $\hat{x} \in \mathbb{C}^d$ satisfies $\sum_{j=1}^{m} (|\langle a_j, \hat{x} \rangle| - b_j)^2 \leq \sum_{j=1}^{m} (|\langle a_j, x_0 \rangle| - b_j)^2$ and $\|\hat{x}\|_1 \leq \|x_0\|_1$. If $m \geq k \log(ed/k)$ then the following holds with probability at least $1 - 2 \exp(-c_0 m/4)$:

$$\|A_I(\hat{x}\hat{x}^* - x_0 x_0^*)\|_1 \lesssim \sqrt{m}\|\eta\|_2 (\|\hat{x}\|_2 + \|x_0\|_2).$$

Here, $c_0$ is the positive constant defined in Theorem 2.4. The index set $I := I_{\hat{x}, x_0} \subset \{1, \ldots, m\}$ is chosen using the following way. Set $\xi_j := |\langle a_j, \hat{x} \rangle| + |\langle a_j, x_0 \rangle|$ and assume that $\xi_{j_1} \leq \xi_{j_2} \leq \cdots \leq \xi_{j_m}$. Take $I = \{j_t \mid t \leq (1 - \beta_0)m\}$, where $\beta_0$ is defined in Theorem 2.4.

**Proof.** Using a similar argument for (4.4), we obtain that

$$\sum_{j \in I} \frac{|\langle a_j a_j^*, \hat{x}\hat{x}^* - x_0 x_0^* \rangle| - |\eta_j|(|\langle a_j, \hat{x} \rangle| + |\langle a_j, x_0 \rangle|)}{|\langle a_j, \hat{x} \rangle| + |\langle a_j, x_0 \rangle|} \leq \sum_{j=1}^{m} |\langle a_j, \hat{x} \rangle| - b_j \leq \sqrt{m} \|\eta\|_2.$$

We claim that

$$|\langle a_j, x_0 \rangle| + |\langle a_j, \hat{x} \rangle| \leq 2C_{\beta_0} (\|x_0\|_2 + \|\hat{x}\|_2)^2,$$

where $C_{\beta_0} := (2 + 4\sqrt{2}) C_{\beta_0}$. Combining (4.2) and (4.3), we can obtain that

$$\frac{\sum_{j \in I} |\langle a_j a_j^*, \hat{x}\hat{x}^* - x_0 x_0^* \rangle| - \sum_{j \in I} |\eta_j|(|\langle a_j, \hat{x} \rangle| + |\langle a_j, x_0 \rangle|)}{\sqrt{2C_{\beta_0} (\|x_0\|_2 + \|\hat{x}\|_2)^2}} \leq \frac{\sqrt{m}}{2C_{\beta_0} (\|x_0\|_2 + \|\hat{x}\|_2)^2}.$$

We can employ (4.3) to obtain that

$$\|A_I(\hat{x}\hat{x}^* - x_0 x_0^*)\|_1 \lesssim \sqrt{m}\|\eta\|_2 (\|\hat{x}\|_2 + \|x_0\|_2).$$
Here, we use the same argument as in the proof of Lemma 2.6 and we omit the details. It remains to prove (4.3). Set

\[ Z := X_0 + \tilde{X} = x_0 x_0^* + \tilde{x} \tilde{x}^*, \quad \text{with} \quad X_0 := x_0 x_0^* \quad \text{and} \quad \tilde{X} := \tilde{x} \tilde{x}^*. \]

Set \( S_0 := \text{supp}(x_0) \subset \{1, \ldots, d\} \). Set \( S_1 \) as the index set which contains the indices of the \( k \) largest elements of \( \tilde{x}_{S_0} \) in magnitude, and \( S_2 \) contains the indices of the next \( k \) largest elements, and so on.

For simplicity, we set \( S_{01} := S_0 \cup S_1 \) and \( \tilde{Z} := Z_{S_{01}, S_{01}} \). Then, for any \( j \in I \), we have

\[
(4.5) \quad \frac{1}{\beta_0 m} \| A_{\{1, \ldots, m\} \setminus I} (\tilde{Z}) \|_1 \leq \frac{1}{\beta_0 m} \| A(\tilde{Z}) \|_1 \leq \frac{C_+}{\beta_0} \| \tilde{Z} \|_F \leq \frac{C_+}{\beta_0} (\| x_0 \|_2^2 + \| \tilde{x} \|_2^2).
\]

We also have

\[
(4.6) \quad \frac{1}{\beta_0 m} \| A_{\{1, \ldots, m\} \setminus I} (Z - \tilde{Z}) \|_1 \leq \frac{1}{\beta_0 m} \| A(Z_{S_0, S_{01}} + Z_{S_{01}, S_0}) \|_1 + \frac{1}{\beta_0 m} \| A(Z_{S_1, S_{01}} + Z_{S_{01}, S_1}) \|_1 \leq \frac{1}{\beta_0 m} \| A(Z_{S_{01}, S_{01}}) \|_1.
\]

We claim that

\[
(4.7) \quad \frac{1}{\beta_0 m} \| A(Z_{S_0, S_{01}} + Z_{S_{01}, S_0}) \|_1 \leq \frac{\sqrt{2} C_+}{\beta_0} (\| x_0 \|_2^2 + \| \tilde{x} \|_2^2), \quad \text{for } l \in \{0, 1\},
\]

and

\[
(4.8) \quad \frac{1}{\beta_0 m} \| A(Z_{S_1, S_{01}} + Z_{S_{01}, S_1}) \|_1 \leq \frac{\sqrt{2} C_+}{\beta_0} (\| x_0 \|_2^2 + \| \tilde{x} \|_2^2).
\]

Combining (4.3), (4.6), (4.7), (4.8) and (4.9), we obtain that

\[
(4.9) \quad (\| a_j, x_0 \| + |\langle a_j, \tilde{x} \rangle|)^2 \leq 2 \left( 2 + 4 \sqrt{2} \right) \frac{C_+}{\beta_0} (\| x_0 \|_2^2 + \| \tilde{x} \|_2^2).
\]
It remains to prove (4.8) and (4.9). We firstly consider (4.8). According to Theorem 2.4, for \( l \in \{0,1\} \), we have
\[
\frac{1}{\beta_0 m} \|A(Z_{S_l}S_{l_{01}} + Z_{S_{l_{01}}}S_l)\|_1 \leq \sum_{j \geq 2} \frac{1}{\beta_0 m} \|A(Z_{S_l}S_j + Z_{S_j}S_l)\|_1 \leq \sum_{j \geq 2} \frac{C_+}{\beta_0} \|Z_{S_l}S_j + Z_{S_j}S_l\|_F
\]
for \( (a) \) and
\[
\leq \frac{C_+}{\beta_0} \sum_{j \geq 2} (\|\tilde{x}_{S_l}\|_F + \|\tilde{x}_{S_j}\|_F) = \frac{2C_+}{\beta_0} \sum_{j \geq 2} \|\tilde{x}_{S_l}\|_2 \|\tilde{x}_{S_l}\|_2
\]
for \( (b) \), where
\[
\|\tilde{x}_{S_l}\|_2 \cdot \sum_{j \geq 2} \|\tilde{x}_{S_l}\|_2 \leq \frac{1}{\sqrt{k}} \|\tilde{x}_{S_{l_{01}}}\|_1 \|\tilde{x}_{S_{l_{01}}}\|_2 \leq \|\tilde{x}_{S_{l_{01}}}\|_2 \|\tilde{x}_{S_{l_{01}}} - x_0\|_2.
\]
The last inequality in (4.11) is based on \( \|\tilde{x}\|_1 \leq \|x_0\|_1 \), which leads to
\[
\|\tilde{x}_{S_{l_{01}}}\|_1 \leq \|x_0\|_1 - \|\tilde{x}_{S_{l_{01}}} - x_0\|_1 \leq \sqrt{k} \|\tilde{x}_{S_{l_{01}}} - x_0\|_2 \leq \sqrt{k} \|\tilde{x}_{S_{l_{01}}} - x_0\|_2.
\]
We next turn to (4.9). We have
\[
\frac{1}{\beta_0 m} \|A(Z_{S_{l_{01}}}S_{l_{01}})\|_1 \leq \frac{1}{\beta_0 m} \sum_{l_{02}, l \geq 2} \|A(Z_{S_l}S_j) + A(Z_{S_j}S_l)\|_1
\]
\[
\leq \sum_{l_{02}, l \geq 2} \frac{C_+}{\beta_0} \|\tilde{x}_{S_l}\|_2 \|\tilde{x}_{S_j}\|_2 = \frac{C_+}{\beta_0} \left( \sum_{l_{02}} \|\tilde{x}_{S_l}\|_2 \right)^2
\]
for \( (c) \) and
\[
\leq \frac{C_+}{\beta_0} \cdot \frac{1}{k} \|\tilde{x}_{S_{l_{01}}}\|_2^2 = \frac{C_+}{\beta_0} \cdot \frac{1}{k} \|Z_{S_{l_{01}}}S_{l_{01}}\|_1
\]
for \( (d) \), where
\[
\|\tilde{x}_{S_{l_{01}}}\|_2 \leq \|\tilde{x}_{S_{l_{01}}} - 1/\sqrt{k}\), for \( l \geq 2 \) and the inequality \( (d) \) is based on \( \|Z_{S_{l_{01}}}S_{l_{01}}\|_1 \leq \|Z - Z_{S_{l_{01}}}S_{l_{01}}\|_1 \leq \|Z_{S_{l_{01}}}S_{l_{01}}\|_1 \). Indeed, according to \( \|\tilde{X}\|_1 \leq \|X_0\|_1 \), we have
\[
\|Z - Z_{S_{l_{01}}}S_{l_{01}}\|_1 = \|\tilde{X} - \tilde{x}_{S_{l_{01}}}S_{l_{01}}\|_1 \leq \|X_0\|_1 - \|\tilde{x}_{S_{l_{01}}}S_{l_{01}}\|_1 \leq \|X_0 + \tilde{x}_{S_{l_{01}}}S_{l_{01}}\|_1 = \|Z_{S_{l_{01}}}S_{l_{01}}\|_1.
\]
We next prove Theorem 1.4. Though the proof is quite similar as that in [22, Theorem 1.3], we present it here for the completeness and convenience of the readers.

**Proof of Theorem 1.4.** Noting that \( \| \hat{x} \|_1 \leq \| x_0 \|_1 \), we obtain that \( \| \hat{X} \|_1 \leq \| X_0 \|_1 \) where \( X_0 := x_0x_0^* \) and \( \hat{X} := \hat{x}\hat{x}^* \). Set \( H := \hat{X} - X_0 = \hat{x}\hat{x}^* - x_0x_0^* \) and \( T_0 := \text{supp}(x_0) \). Set \( T_1 \) as the index set which contains the indices of the \( a \cdot k \) largest elements of \( \hat{x}T_0^c \) in magnitude, and \( T_2 \) contains the indices of the next \( a \cdot k \) largest elements, and so on. Here, we require that the absolute constant \( a > 0 \) satisfies

\[
C_- (1 - \beta_0) - \frac{4C_+}{\sqrt{a}} - \frac{C_+}{a} > 0, \tag{4.12}
\]

where \( C_- \), \( C_+ \), and \( \beta_0 \) are defined in Theorem 2.4. Set \( T_{01} := T_0 \cup T_1 \) and \( \tilde{H} := HT_{01}^cT_{01} \). We claim that

\[
\| \hat{x}\hat{x}^* - x_0x_0^* \|_F = \| H \|_F \leq \| \tilde{H} \|_F + \| H - \tilde{H} \|_F \leq \left( \frac{1}{a} + \frac{4}{\sqrt{a}} + 1 \right) \| \tilde{H} \|_F \tag{4.13}
\]

Furthermore, according to (2.2), we have

\[
\sqrt{\frac{3}{4}} \left( \| \hat{x} \|_2 + \| x_0 \|_2 \right) \inf_{\theta \in [0, 2\pi)} \| \hat{x} - \exp(i\theta)x_0 \|_2 \leq \| \hat{x}\hat{x}^* - x_0x_0^* \|_F. \tag{4.14}
\]

Combining (4.13) and (4.14), we obtain that

\[
\min_{\theta \in [0, 2\pi)} \| \hat{x} - \exp(i\theta)x_0 \|_2 \lesssim \frac{1}{\sqrt{m}}\| \eta \|_2,
\]

which leads to the conclusion.

We next turn to prove (4.13). The second inequality and third inequality in (4.13) follow from

\[
\| H - \tilde{H} \|_F \leq \left( \frac{1}{a} + \frac{4}{\sqrt{a}} \right) \| \tilde{H} \|_F, \tag{4.15}
\]

and

\[
\| \tilde{H} \|_F \leq \frac{1}{C_- (1 - \beta_0) - \frac{4C_+}{\sqrt{a}} - \frac{C_+}{a}} \frac{2\| \eta \|_2 (\| \hat{x} \|_2 + \| x_0 \|_2)}{\sqrt{m}}, \tag{4.16}
\]

respectively. To this end, it is enough to prove (4.14) and (4.16).

**Step 1:** The proof of (4.15).
A simple observation is that
\[
\|H - \tilde{H}\|_F \leq \sum_{l \geq 2, j \geq 2} \|H_{T_l, T_j}\|_F + \sum_{l=0, 1} \sum_{j \geq 2} \|H_{T_l, T_j}\|_F + \sum_{j=0, 1} \sum_{l \geq 2} \|H_{T_l, T_j}\|_F
\]
\[
= \sum_{l \geq 2, j \geq 2} \|H_{T_l, T_j}\|_F + 2 \sum_{l=0, 1} \sum_{j \geq 2} \|H_{T_l, T_j}\|_F.
\]
(4.17)

On one hand, we have
\[
\sum_{l \geq 2, j \geq 2} \|H_{T_l, T_j}\|_F = \sum_{l \geq 2, j \geq 2} \|\tilde{\mathbf{x}}_{T_l}\|_2 \cdot \|\tilde{\mathbf{x}}_{T_j}\|_2 = \left( \sum_{l \geq 2} \|\tilde{\mathbf{x}}_{T_l}\|_2^2 \right)^2 \leq \frac{1}{ak} \|\tilde{\mathbf{x}}_{T_2}\|_2^2,
\]
(4.18)

according to \(\|\tilde{\mathbf{x}}_{T_l}\|_2 \leq \|\tilde{\mathbf{x}}_{T_{l-1}}\|_1/\sqrt{ak} \) \((l \geq 2)\), and
\[
\|H_{T_s, T_0}\|_1 \leq \|H - H_{T_0, T_0}\|_1 = \|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_{T_0, T_0}\|_1 \leq \|X_0\|_1 - \|\tilde{\mathbf{x}}_{T_0, T_0}\|_1 \leq \|X_0 - \tilde{\mathbf{x}}_{T_0, T_0}\|_1 = \|H_{T_0, T_0}\|_1.
\]
(4.19)

On the other hand, for \(l \in \{0, 1\}\), we have
\[
\sum_{j \geq 2} \|H_{T_l, T_j}\|_F = \|\tilde{\mathbf{x}}_{T_l}\|_2 \cdot \sum_{j \geq 2} \|\tilde{\mathbf{x}}_{T_j}\|_2 \leq \frac{1}{\sqrt{ak}} \|\tilde{\mathbf{x}}_{T_0}\|_1 \cdot \|\tilde{\mathbf{x}}_{T_2}\|_2 \leq \frac{1}{\sqrt{a}} \|\tilde{\mathbf{x}}_{T_0}\|_2 \cdot \|\tilde{\mathbf{x}}_{T_0} - \mathbf{x}_0\|_2.
\]
(4.20)

The last inequality in (4.20) is based on
\[
\|\tilde{\mathbf{x}}_{T_0}\|_1 \leq \|\mathbf{x}_0\|_1 - \|\tilde{\mathbf{x}}_{T_0}\|_1 \leq \|\tilde{\mathbf{x}}_{T_0} - \mathbf{x}_0\|_1 \leq \sqrt{k} \|\tilde{\mathbf{x}}_{T_0} - \mathbf{x}_0\|_2 \leq \sqrt{2k} \|\tilde{\mathbf{x}}_{T_0} - \mathbf{x}_0\|_2.
\]

Substituting (4.18) and (4.20) into (4.17), we can obtain that
\[
\|H - \tilde{H}\|_F \leq \sum_{l \geq 2, j \geq 2} \|H_{T_l, T_j}\|_F + \sum_{l=0, 1} \sum_{j \geq 2} \|H_{T_l, T_j}\|_F + \sum_{j=0, 1} \sum_{l \geq 2} \|H_{T_l, T_j}\|_F
\]
\[
\leq \frac{1}{a} \|H\|_F + \frac{2 \sqrt{2}}{\sqrt{a}} \|\tilde{\mathbf{x}}_{T_0}\|_2 \|\tilde{\mathbf{x}}_{T_0} - \mathbf{x}_0\|_2 \leq \left( \frac{1}{a} + \frac{4}{\sqrt{a}} \right) \|\tilde{H}\|_F.
\]
(4.21)

We arrive at (4.16).

**Step 2: The proof of (4.16).** According to Lemma 4.1, we have
\[
\frac{1}{m} \|A_I(\tilde{\mathbf{x}} - \mathbf{x}_0^*)\|_1 \leq \frac{1}{\sqrt{m}} \|\eta\|_2 (\|\tilde{\mathbf{x}}\|_2 + \|\mathbf{x}_0\|_2),
\]
approved \(m \geq k \log(\text{ed}/k)\), which implies
\[
\frac{\|\eta\|_2 (\|\tilde{\mathbf{x}}\|_2 + \|\mathbf{x}_0\|_2)}{\sqrt{m}} \geq \frac{1}{m} \|A_I(\tilde{H})\|_1 \geq \frac{1}{m} \|A_I(H)\|_1 - \frac{1}{m} \|A_I(H - \tilde{H})\|_1.
\]
(4.22)

In order to get a lower bound of \(\frac{1}{m} \|A_I(\tilde{H})\|_1 - \frac{1}{m} \|A_I(H - \tilde{H})\|_1\), we bound \(\frac{1}{m} \|A_I(\tilde{H})\|_1\) from below and \(\frac{1}{m} \|A_I(H - \tilde{H})\|_1\) from above. As \(\text{rank}(\tilde{H}) \leq 2\) and \(\tilde{H}_{0,2} \leq (a + 1)k\), we use Theorem 2.4 on the order of \((2, (a + 1)k)\) to obtain that
\[
\frac{1}{m} \|A_I(\tilde{H})\|_1 \geq C_1 - (1 - \beta_0) \|\tilde{H}\|_F.
\]
(4.23)
Since $H - \tilde{H} = (H_{T_0}, x_{T_0}^0 + H_{T_0}, t_0) + (H_{T_1}, x_{T_1}^1 + H_{T_0}, t_1) + H_{T_0}, t_0^1$, we have

\[(4.25)\]
\[
\frac{1}{m} \| A_I(H - \tilde{H}) \|_1 \leq \frac{1}{m} \| A_I(H_{T_0}, t_0^0 + H_{T_0}, t_0^0) \|_1 + \frac{1}{m} \| A_I(H_{T_1}, t_1^1 + H_{T_0}, t_1^1) \|_1 + \frac{1}{m} \| A_I(H_{T_0}, t_0^0) \|_1.
\]

According to Theorem 2.4 on the order of $(2, 2a_k)$, for $l \in \{0, 1\}$, we have

\[(4.26)\]
\[
\frac{1}{m} \| A_I(H_{T_0}, t_0^0 + H_{T_0}, t_0^0) \|_1 \leq \sum_{j \geq 2} \frac{1}{m} \| A_I(H_{T_0}, t_0^0 + H_{T_0}, t_0^0) \|_1 \leq \sum_{j \geq 2} C_+ \| H_{T_1}, t_1 + H_{T_1}, t_1 \|_F
\]
\[
\leq C_+ \sum_{j \geq 2} (\| \mathbf{x}_{T_1} \|_F + \| \hat{\mathbf{x}}_{T_1} \|_F) = 2C_+ \| \mathbf{x}_{T_1} \|_2 \sum_{j \geq 2} \| \hat{\mathbf{x}}_{T_1} \|_2
\]
\[
\leq \frac{2C_+}{\sqrt{a}} \| \hat{\mathbf{x}}_{T_1} \|_2 \cdot \| \hat{\mathbf{x}}_{T_0} - \mathbf{x}_0 \|_2,
\]

where the third line above is obtained as in (4.20).

To bound $\frac{1}{m} \| A_I(H_{T_0}, t_0^0) \|_1$, we have

\[(4.27)\]
\[
\frac{1}{m} \| A_I(H_{T_0}, t_0^0) \|_1 \leq \frac{1}{m} \sum_{l \geq 2} \| A_I(H_{T_0}, t_0^0) \|_1 \leq \sum_{l \geq 2} C_+ \| H_{T_0}, t_0 \|_F
\]
\[
= \frac{C_+}{a_k} \| H_{T_0}, t_0 \|_1 \leq \frac{C_+}{a_k} \| H_{T_0}, t_0 \|_F \leq \frac{C_+}{a_k} \| \tilde{H} \|_F.
\]

The third line above is based on (4.19). Now combining (4.26) and (4.27), we obtain that

\[(4.28)\]
\[
\frac{1}{m} \| A_I(H - \tilde{H}) \|_1 \leq \frac{1}{m} \| A_I(H_{T_1}, t_1^1 + H_{T_1}, t_1^1) \|_1 + \frac{1}{m} \| A_I(H_{T_0}, t_0^0) \|_1 + \frac{1}{m} \| A_I(H_{T_0}, t_0^0) \|_1
\]
\[
\leq \frac{2C_+}{\sqrt{a}} \| \hat{\mathbf{x}}_{T_0} \|_2 \| \hat{\mathbf{x}}_{T_0} - \mathbf{x}_0 \|_2 + \frac{2C_+}{\sqrt{a}} \| \hat{\mathbf{x}}_{T_1} \|_2 \| \hat{\mathbf{x}}_{T_0} - \mathbf{x}_0 \|_2 + \frac{C_+}{a} \| \tilde{H} \|_F
\]
\[
\leq \frac{2\sqrt{2}C_+}{\sqrt{a}} \| \hat{\mathbf{x}}_{T_0} \|_2 \| \hat{\mathbf{x}}_{T_0} - \mathbf{x}_0 \|_2 + \frac{C_+}{a} \| \tilde{H} \|_F
\]
\[
\leq C_+ \left( \frac{4}{\sqrt{a}} + \frac{1}{a} \right) \| \tilde{H} \|_F.
\]

Putting (4.24) and (4.28) into (4.23), we obtain that

\[
\frac{2\| \eta \|_2 (\| \hat{\mathbf{x}} \|_2 + \| \mathbf{x}_0 \|_2)}{\sqrt{m}} \geq \frac{1}{m} \| A_I(\tilde{H}) \|_1 - \frac{1}{m} \| A_I(H - \tilde{H}) \|_1
\]
\[
\geq C_- (1 - \beta_0) \| \tilde{H} \|_F - C_+ \left( \frac{4}{\sqrt{a}} + \frac{1}{a} \right) \| \tilde{H} \|_F = \left( C_- (1 - \beta_0) - \frac{4C_+}{\sqrt{a}} - \frac{C_+}{a} \right) \| \tilde{H} \|_F,
\]

which implies (4.16). \qed

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