Near-MDS codes from elliptic curves

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Abstract
We provide a geometric construction of \([n, 9, n − 9]_q\) near-MDS codes arising from elliptic curves with \(n \mathbb{F}_q\)-rational points. Furthermore, we show that in some cases these codes cannot be extended to longer near-MDS codes.

Keywords Linear code · Near-MDS code · Elliptic curve

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1 Introduction

Maximum distance separable (for short MDS) codes are the best linear \([n, k, d]_q\) codes as they meet the Singleton bound, that is, \(n = d + k - 1\). The non-negative integer \(s(C) := n - k + 1 - d\) is said to be the Singleton defect of the code \(C\). Thus, the Singleton defect of an MDS code is zero.

A linear code \(C\) is defined to be a near-MDS (for short NMDS) code if \(s(C) = s(C^\perp) = 1\) where \(C^\perp\) is the dual code of \(C\). Hence, a NMDS \([n, k]\) code has minimum distance \(n - k\).

NMDS codes were introduced by Dodunekov and Landjev [4] with the aim of constructing good linear codes by slightly weakening the restrictions in the definition of an MDS code. NMDS codes have similar properties to MDS codes. Some non-binary linear codes such as...
the ternary Golay codes, the quaternary quadratic residue [11, 6, 5]_4-code, and the quaternary extended quadratic residue [12, 6, 6]_4-code are notable examples of NMDS codes; see [11].

The geometrical counterpart of an NMDS code is an n-track in a Galois space which is a set of n points in an N-dimensional Galois space such that every N of them are linearly independent but some N + 1 of them, see [3]. If every N + 2 points of the n-track generate the whole space then the n × (N + 1) matrix whose columns are homogeneous coordinates of the n-track points is a generator matrix of an NMDS code. The n-track is complete, i.e. maximal with respect to set theoretical inclusion, if and only if the code is not extendable.

Let N_q denote the maximum number of \( \mathbb{F}_q \)-rational points on an elliptic curve defined over \( \mathbb{F}_q \); it is well-known that, by Hasse theorem, \(|N_q - (q + 1)| \leq 2\sqrt{q}\).

NMDS codes of length up to \( N_q \) may be constructed from elliptic curves. An interesting question is whether there exist NMDS codes of length greater than \( N_q \). Constructions of NMDS codes from elliptic curves are found in [1,2,7] where results both from combinatorics and algebraic geometry are used.

Here we provide a geometric construction of 9 dimensional NMDS codes using an algebraic curve of order 9 in \( \text{PG}(9, q) \) which arises from a non-singular cubic curve \( E : f(X, Y, Z) = 0 \) of PG(2, q) via the (modified) Veronese embedding:

\[
\nu_3^2: (X:Y:Z) \mapsto (f(X, Y, Z) : X^2Y : X^2Z : XY^2 : XZ^2 : Y^3 : Y^2Z : YZ^2 : Z^3).
\]

We also show that certain codes from elliptic curves are not extendible to longer NMDS codes. The proof depends on some results on the number of \( \mathbb{F}_q \)-rational lines through a given point \( P \) that meet a plane elliptic curve in exactly three \( \mathbb{F}_q \)-rational points and on some computations carried out with the aid of GAP [13].

2 Preliminaries

The following definitions of an NMDS code of length \( n \) and dimension \( k \) over a finite field \( \mathbb{F}_q \) are equivalent to that given in the Introduction; see [5].

**Definition 1** A linear \([n, k]\) code over \( \mathbb{F}_q \) is NMDS if any of its generator matrices, say \( G \), satisfies the following conditions:

(i) any \( k - 1 \) columns of \( G \) are linearly independent;
(ii) \( G \) contains \( k \) linearly dependent columns;
(iii) any \( k + 1 \) columns of \( G \) have full rank.

**Definition 2** A linear \([n, k]\) code over \( \mathbb{F}_q \) is NMDS if any of its parity check matrices, say \( H \), satisfies the following conditions:

(i) any \( n - k - 1 \) columns of \( H \) are linearly independent;
(ii) \( H \) contains \( n - k \) linearly dependent columns;
(iii) any \( n - k + 1 \) columns of \( H \) have full rank.

From a geometric point of view, a NMDS \([n, k]\) code \( \mathcal{C} \) over \( \mathbb{F}_q \) can be regarded as a projective system (i.e. a distinguished point set) \( \mathcal{C} \) in a projective space \( \text{PG}(k - 1, q) \); see [14] for more details.

**Definition 3** A subset \( \mathcal{C} \subseteq \text{PG}(k - 1, q) \) is an \((n; k, k - 2)\)-set in \( \text{PG}(k - 1, \mathbb{F}_q) \) if it satisfies the following conditions:
Theorem 2.1

Let \( \Sigma_1 \) be any cubic curve; it would be suitable as first element of the basis \( B \), not necessarily limited to \( q \) points, where \( PG \) is the projective space \( PG(3, q) \). To set-theoretical inclusion.

(i) every \( k - 1 \) points in \( C \) span a hyperplane of \( PG(k - 1, q) \);
(ii) there exists a hyperplane of \( PG(k - 1, q) \) containing exactly \( k \) points of \( C \);
(iii) every \( k + 1 \) points of \( C \) generate the whole \( PG(k - 1, q) \).

Definition 4

An \( (n; k, k - 2) \)-set in \( PG(k - 1, F_q) \) is complete if it is maximal with respect to set-theoretical inclusion.

Thus, in this setting, an NMDS \([n, k] \) code over \( \mathbb{F}_q \) is an \( (n; k, k - 2) \)-set in \( PG(k - 1, \mathbb{F}_q) \).

Given an integer \( \nu \geq 1 \) and a prime power \( q = p^h \), consider the set \( C^\nu \) of all the curves of degree \( \nu \) contained in the projective plane \( PG(2, q) \) over a finite field \( \mathbb{F}_q \). Since any curve \( C \in C^\nu \) is uniquely determined by \( m + 1 = \binom{\nu + 2}{2} \) parameters in \( \mathbb{F}_q \), that is, the coefficients of its equation

\[
a_0 Z^\nu + (a_1 X + a_2 Y) Z^{\nu-1} + (a_3 X^2 + a_4 XY + a_5 Y^2) Z^{\nu-2} + \cdots
\]

\[
+ (a_m X^\nu + a_m-1 X^{\nu-1} Y + \cdots + a_1 X Y^{\nu-1} + a_0 Y^\nu) = 0,
\]

and the curve is unchanged if these parameters are multiplied by a common factor, then \( C^\nu \) can be regarded as a projective space \( PG(m, q) \) with homogeneous coordinates \( (a_0:a_1: \cdots :a_m) \).

We may also denote a curve \( C \) by using its defining polynomial.

The following result—which is an implicit formulation of the famous Cayley-Bacharach theorem—will be useful later; see [6].

Theorem 2.1

Let \( \mathcal{C} \) and \( \mathcal{C} \) be two distinct cubic curves meeting in a set \( \mathcal{S} \) consisting of 9 points (counted with multiplicities). If \( \mathcal{D} \subset PG(2, q) \) is any cubic curve containing all but one point of \( \mathcal{S} \), then \( \mathcal{C} \cap \mathcal{D} \neq \emptyset \).

3 Lifting point sets

The space \( C^3 \) consisting of all the cubics in \( PG(2, q) \) has projective dimension 9, hence 10 independent cubic curves are required to generate it. Let \( \mathcal{E} \) be a non-singular cubic curve of equation \( f(X, Y, Z) = 0 \) over \( \mathbb{F}_q \). A suitable basis \( B \) for \( C^3 \), containing \( \mathcal{E} \), can be written by using the following polynomials:

\[
\mathcal{B} = \{ f(X, Y, Z), X^2 Y, X^2 Z, XY^2, X Y Z, X Z^2, Y^3, Y^2 Z, Y Z^2, Z^3 \},
\]

where \( f(X, Y, Z) \) is required to contain the term \( X^3 \). In fact, the defining polynomial of any cubic curve would be suitable as first element of the basis \( B \), as long as it contains the monomial \( X^3 \); nevertheless, the choice of an elliptic curve is motivated by the fact that, unlike the case of genus 0, the number of \( \mathbb{F}_q \)-rational points of a carefully chosen elliptic curve is not necessarily limited to \( q + 1 \).

We consider the following embedding of the points of \( PG(2, q) \) onto \( PG(9, q) \) with projective coordinates \( (X_0; X_1; X_2; X_3; X_4; X_5; X_6; X_7; X_8; X_9) \) by means of the mapping \( v_3^2 : PG(2, q) \to PG(9, q) \) (1) which is a Veronese embedding of degree 3. Let \( \mathcal{V}_3 \) be the image of \( v_3^2 \); clearly \( \mathcal{V}_3 \) is (projective equivalent to) the cubic Veronese surface.

More in detail, the points of the curve \( \mathcal{E} \) are mapped onto a curve \( \Gamma \) of \( PG(9, q) \) with the same number \( n \) of \( \mathbb{F}_q \)-rational points as \( \mathcal{E} \). Also, \( \Gamma \) is the complete intersection of \( \mathcal{V}_3 \) with the hyperplane \( \Sigma \cong PG(8, q) \) of equation \( X_0 = 0 \). Since for every cubic curve \( C \) of equation \( g(X, Y, Z) = 0 \) in \( PG(2, q) \), the defining polynomial is a linear combination of the elements of \( B \), that is,
\[ g(X, Y, Z) = \lambda_0 f(X, Y, Z) + \lambda_1 Y^3 + \lambda_2 XZ^2 + \lambda_3 YZ^2 + \lambda_4 X^2 Z + \lambda_5 Y^2 Z + \lambda_6 XYZ + \lambda_7 X^2 Y + \lambda_8 XY^2 + \lambda_9 Z^3, \]

it turns out that \( v_3^2(\mathcal{C}) \) is the complete intersection of \( \mathcal{V}_3 \) with the hyperplane \( \Pi \subset \text{PG}(9, q) \) of equation

\[ \sum_{i=0}^{9} \lambda_i X_i = 0, \tag{2} \]

which is distinct from \( \Sigma \). Thus, every cubic curve \( \mathcal{C} : g(X, Y, Z) = 0 \) of \( \text{PG}(2, q) \) corresponds to a hyperplane of Eq. (2). Back to \( \text{PG}(2, q) \), the set \((v_3^2)^{-1}(\Pi \cap \mathcal{V}_3)\) corresponds to a unique cubic curve \( \mathcal{C} \) distinct from \( \mathcal{E} \), and, clearly, \((v_3^2)^{-1}(\Pi \cap \Gamma)\) corresponds to \( \mathcal{E} \cap \mathcal{C} \).

**Theorem 3.1** Suppose that \( \mathcal{E} \) has \( n \geq 9 \) points. Then the point set \( \Gamma \) is an \( (n; 9, 7) \)-set in \( \Sigma = \text{PG}(8, q) \).

**Proof** To prove the theorem it suffices to consider the mutual position of cubic curves in \( \text{PG}(2, q) \).

(i) Take eight distinct points \( P_1, \ldots, P_8 \in \Gamma \) and consider the corresponding distinct points \( Q_1, \ldots, Q_8 \in \mathcal{E} \), with \( Q_i = (v_3^2)^{-1}(P_i) \). Suppose that there is a \( t \)-dimensional net with \( t \geq 2 \), say \( \mathcal{F} \), consisting of cubics through \( Q_1, \ldots, Q_8 \). Then, from Theorem 2.1 there is a ninth point \( Q_9 \in \mathcal{E} \) such that the points \( Q_1, \ldots, Q_9 \) are in the support of \( \mathcal{F} \). This implies that every further point \( Q_{10} \in \mathcal{E} \setminus \{Q_1, \ldots, Q_9\} \) yields a \((t-1)\)-dimensional net consisting of cubics through \( Q_1, \ldots, Q_9 \) which are distinct from \( \mathcal{E} \) and have ten points in common with it, contradicting Bézout’s theorem. Hence, \( \mathcal{F} \) must be a pencil of cubic curves in \( \text{PG}(2, q) \) including \( \mathcal{E} \) and passing through \( Q_1, \ldots, Q_8 \). Back to \( \text{PG}(9, q) \), we observe that \( \mathcal{F} \) corresponds to a pencil of hyperplanes of \( \text{PG}(9, q) \) which meet in a unique \( 7 \)-dimensional subspace \( \Delta \) such that \( \{P_1, \ldots, P_8\} \subset (\Gamma \cap \Delta) \), that is, \( P_1, \ldots, P_8 \) generate the hyperplane \( \Delta \) of \( \Sigma \).

(ii) From Theorem 2.1, there is a further point \( Q_9 \in \text{PG}(2, q) \) which belongs to the intersection of \( \mathcal{E} \) and all the other cubics of the above pencil \( \mathcal{F} \). This proves that the previous subspace \( \Delta \) meets \( \Gamma \) in \( P_1, \ldots, P_8, P_9 = v_3^2(Q_9) \).

(iii) Let \( \Pi \) be a hyperplane of \( \text{PG}(9, q) \) different from \( \Sigma \). Put \( \mathcal{E}' = (v_3^2)^{-1}(\Pi) \). From Bézout’s theorem we know that \(|\mathcal{E} \cap \mathcal{E}'| \leq 9 \), therefore any hyperplane of \( \text{PG}(9, q) \) has at most 9 points in common with \( \Gamma \). Hence, \( \Gamma \) is a curve of order 9, therefore 10 points of \( \Gamma \) generate the whole \( \Sigma \).

The claim follows. \( \square \)

**Remark** The code associated to \( \Gamma \) can also be interpreted as an AG-code, see [14]. Indeed, Theorem 3.1 is a consequence of [14, Theorem 4.4.19]. However, our proof does not use the Riemman-Roch Theorem.

### 4 Some complete NMDS codes

In this section we provide some examples of complete NMDS codes in the set of codes constructed above by lifting the elliptic curve \( \mathcal{E} \) in the case when the base field is large enough.

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By Definition 4, the algebraic curve $\Gamma = v_2(\mathcal{E})$ provides a complete NMDS code, that is a complete $(n; 9, 7)$-set of $\text{PG}(8, q)$, if and only if for any $Q \in \Sigma$ there exists at least one hyperplane $\Pi$ of $\Sigma$ with $Q \in \Pi$ meeting $\Gamma$ in 9 points.

**Definition 5** We call a point $Q \in \Sigma$ special for $\Gamma$ if for all hyperplanes $\Pi$ of $\Sigma$ through $Q$ we have $|\Pi \cap \Gamma| < 9$.

For a point $Q$ to be special means that there is a system of cubic curves satisfying one linear constraint such that each element $\mathcal{E}$ of this system has intersection multiplicity with $\mathcal{E}$ at least 2 in at least one point or meets $\mathcal{E}$ in some non-$\mathbb{F}_q$-rational point.

We expect that for large $q$ special points, if they exist at all, are very few. So we propose the following conjecture.

**Conjecture 1** Suppose $q \geq 121$ to be such that $2, 3 \nmid q$. Then there are no special points for $\Gamma$.

In order to verify Conjecture 1, we performed some computer searches for some values of $q$. For $q \in \{7, 11, 13\}$ we executed a (non-trivial) exhaustive search. For $q \geq 121$ we provide an argument showing that there cannot be too many special points, if they exist at all. We leave the solution of the problem and its generalization to a future work.

### 4.1 Search for small $q$

Recall that any 8 distinct points of $\mathbb{F}_3$ are linearly independent; see [9].

For small values of $q$ it is possible to perform an exhaustive search, adopting the following procedure:

1. Let $\Gamma = v_2(\mathcal{E})$ be the embedding of $\mathcal{E}$;
2. for any set of 9 points of $\Gamma$, consider the matrix containing their components; let $\mathfrak{S}$ be the list of such matrices having rank 8. In particular, each element of $\mathfrak{S}$ corresponds to a hyperplane meeting $\Gamma$ in 9 points. We call such hyperplanes good.
3. For each matrix $H \in \mathfrak{S}$, let $H'$ be a column vector spanning the kernel of $H$. In particular, we have that a row vector $v$ belongs to the span of the rows of $H$ if and only if $vH' = 0$.
4. Consider the linear code $C$ with parameters $|\mathfrak{S}|, 9$ whose generator matrix $G$ consists of all columns of the form $H'$ as $H$ varies in $\mathfrak{S}$. A point $P$ represented by a vector $v$ can be added to $\Gamma$ if, and only if, $P$ does not belong to any of the hyperplanes represented by the columns of $G$; in other words $P$ can be added to $\Gamma$ if and only if the word $PG$ corresponding to $P$ does not contain any 0-component.

Using the above argument, we can state the following.

**Theorem 4.1** The $(n;9,7)$-set $\Gamma$ is complete if and only if the code $C$ with generator matrix $G$ constructed above does not contain any word of maximum weight $n$.

Clearly, it is not restrictive to replace the code $C$ with a code $C'$ equivalent to $C$. In particular, if we transform its generator matrix $G$ to row-reduced echelon form, we see that no point with at least a 0 component can give a word of $C'$ of weight $n$; this allows to exclude from the search all points whose transforms (under the operations yielding the reduction of $C$) lie on the coordinate hyperplanes.

We now limit ourselves to the odd order case with $q$ not divisible by 3. Then any elliptic curve $\mathcal{E}$ of $\text{PG}(2, q)$ admits an equation in canonical Weierstrass form

$$Y^2 = X^3 + aX + b.$$
with \( a, b \in \mathbb{F}_q \) such that \(-16(4a^3 + 27b^2) \neq 0\); see [12].

**Remark** Good hyperplanes correspond to linear systems of cubic curves cutting \( E \) in 9 points; by [10, Theorem 43], we see that the number of such hyperplanes is approximately \( \frac{q^7}{7} \).

We leave to a future work to determine exactly what sets of 9 distinct points of a given elliptic curve \( E \) might arise as intersection divisor with another curve, in other terms to determine what the good hyperplanes are.

Our Conjecture 1 can be restated by saying that the union of all good hyperplanes for \( E \) is \( \text{PG}(8, q) \) for \( q \) sufficiently large.

We can now apply the aforementioned strategy for all possible values of \( a, b \) yielding elliptic curves. This leads to the following.

**Theorem 4.2** Suppose \( q \in \{7, 11, 13\} \). Then, the lifted \((n; 9, 7)\)-set \( \Gamma \) in \( \text{PG}(8, q) \) is complete if and only if \( n = |E| \geq 15 \). In particular, for \( q = 7 \) the lifted set \( \Gamma \) is never complete.

### 4.2 Properties for large \( q \)

We now provide an argument to prove that there might not be too many special points. This makes it possible to verify for several values of \( q \) that the \((n; 9, 7)\)-set \( \Gamma \) in \( \Sigma = \text{PG}(8, q) \) is complete and gives evidence supporting Conjecture 1.

As in the previous section, the projective plane \( \text{PG}(2, q) \) is assumed to be of order \( q \) odd and not divisible by 3. Furthermore we suppose \( q \geq 121 \). Let \( j(E) \) be the \( j \)-invariant of \( E \), that is the six cross-ratios of the four tangents from a point of \( E \) to other points of \( E \). We limit ourselves to the case \( j(E) \neq 0 \), see [8, Theorem 11.15].

We will use the following result which is a direct consequence of [7, Lemma 3.2].

**Lemma 4.3** Let \( q \geq 121 \) and consider an elliptic cubic \( E(\mathbb{F}_q) \) with \( j(E) \neq 0 \). Then there are at least 7 trisecant \( \mathbb{F}_q \)-rational lines through any given \( \mathbb{F}_q \)-rational point.

Up to a change of projective reference, we can assume without loss of generality that the curve \( E \) in \( \text{PG}(2, q) \) is met by the reducible cubic \( XYZ = 0 \) in 9 distinct \( \mathbb{F}_q \)-rational points.

**Lemma 4.4** Under the assumption \( q \geq 121 \) any special point \( Q \in \Sigma \) has to be a point \( Q = (0, q_1, q_2, \ldots, q_9) \in \Sigma \setminus \Gamma \) such that \([q_1, q_3, q_4], [q_4, q_7, q_8] \in E\) and one of the following conditions holds

- \( q_1, q_7 = 0; q_3, q_4, q_8 \neq 0; \)
- \( q_1, q_5 = 0; q_3, q_4, q_7 \neq 0; \)
- \( q_3, q_7 = 0; q_1, q_4, q_8 \neq 0; \)
- \( q_5, q_8 = 0; q_1, q_4, q_7 \neq 0. \)

**Proof** Let \( Q = (0, q_1, q_2, \ldots, q_9) \in \Sigma \). If \( Q \in \Gamma \), then \( Q \) is not special; indeed, if \( Q \in \Gamma \), then \( Q = v_3^2(P) \) with \( P \in E \). Consider a reducible cubic curve \( C \) in \( \text{PG}(2, q) \), union of 3 lines \( \ell, m, r \) with \( P \in \ell \setminus \{ m \cup r \} \) and such that \( |(\ell \cup m \cup r) \cap E| = 9 \). Such a curve if \( |E| > 9 \) is guaranteed to exist by Lemma 4.3 and it corresponds to a hyperplane of \( \text{PG}(9, q) \) through \( Q \) meeting \( \Gamma \) in 9 distinct points. So \( Q \) is not special.

Now consider a cubic curve \( C \) in \( \text{PG}(2, q) \) with equation of the form

\[
YZ(\alpha X + \beta Y + \gamma Z) = 0,
\]  

(3)
and a cubic curve \( \mathcal{C}' \) with equation of type

\[
XY(aX + bY + cZ) = 0.
\]

(4)

Via the Veronese embedding \( v_2^2 \), \( \mathcal{C} \) corresponds to the hyperplane of equation \( \alpha X_4 + \beta X_7 + \gamma X_8 = 0 \), whereas \( \mathcal{C}' \) corresponds to the hyperplane \( aX_1 + bX_3 + cX_4 = 0 \).

For any \( Q \in \Sigma \setminus \Gamma \) write \( P_Q := [q_4, q_7, q_8] \) and \( P'_Q := [q_1, q_3, q_4] \) \( \in PG(2, q) \).

If \( P_Q \notin \mathcal{E} \), by Lemma 4.3 there are at least 7 lines through \( P_Q \) meeting \( \mathcal{E} \) in 3 distinct points; in particular there is at least one line of equation \( \alpha X + \beta Y + \gamma Z = 0 \) through \( P_Q \) meeting \( \mathcal{E} \setminus \{(Y = 0) \cup (Z = 0)\} \) in 3 distinct points. Consequently the cubic \( \mathcal{C}' : YZ(\alpha X + \beta Y + \gamma Z) = 0 \) corresponds to a hyperplane \( \Pi \) of \( PG(9, q) \) through \( Q \), meeting \( \Gamma \) in 9 distinct points and we are done.

If \( P_Q \in \mathcal{E} \) but \( P'_Q \notin \mathcal{E} \), repeating the same argument starting from a cubic \( \mathcal{C}' \) with Eq. (4), we see that \( Q \) is not special.

Thus, we suppose \( P_Q, P'_Q \in \mathcal{E} \) and distinguish several cases:

1. If \( q_4 = 0 \), then the cubic \( \mathcal{C} \) of equation \( XYZ = 0 \) corresponds to the hyperplane \( X_4 = 0 \) passing through \( Q \) with 9 intersections with \( \Gamma \).

2. If \( q_4 \neq 0 \) and \( q_7 = q_8 = 0 \), then \( P_Q = [1, 0, 0] \notin \mathcal{E} \), which is excluded.

3. If \( q_4 \neq 0 \) and \( q_1 = q_3 = 0 \), then \( P'_Q = [0, 0, 1] \notin \mathcal{E} \), which is excluded.

4. Let \( q_4 \neq 0 \) with \( q_7 \neq 0 \) and \( q_8 \neq 0 \), then \( P_Q \) is not on \( (Y = 0) \cup (Z = 0) \) in \( PG(2, q) \).

Then, from Lemma 4.3 there are at least 7 lines in \( PG(2, q) \) through \( P_Q \) which are 3-secants to \( \mathcal{E} \). Since \( \mathcal{E} \) has 6 points on the union of the lines \((Y = 0) \) and \((Z = 0) \), there is at least one line through \( P_Q \) with equation: \( \alpha_1 X + \beta_1 Y + \gamma_1 Z = 0 \) meeting \( \mathcal{E} \) in 3 points none of which is on \( (Y = 0) \) and \((Z = 0) \). So, the hyperplane of \( PG(9, q) \) through \( Q \), corresponding to the cubic \( \mathcal{C}' : YZ(\alpha_1 X + \beta_1 Y + \gamma_1 Z) = 0 \) meets \( \Gamma \) in 9 points.

5. Let \( q_4 \neq 0 \), \( q_7 \neq 0 \) and \( q_8 = 0 \) (or, equivalently, \( q_4 \neq 0 \), \( q_7 = 0 \) and \( q_8 \neq 0 \)).

Using an argument similar to that of point 4, but starting from a cubic \( \mathcal{C}' \) through \( P'_Q \) with equation of the form (4), it turns out that if \( q_1 \neq 0 \) and \( q_3 \neq 0 \) then the points \( Q(0, q_1, q_2, \ldots, q_7, 0, q_9) \) (or \( Q(0, q_1, \ldots, q_6, 0, q_8, q_9) \)) are not special.

Thus, our lemma follows. \( \square \)

Remark Let \( Q = (0, q_1, \ldots, q_9) \in \Sigma \) such that \( Q \) is not ruled out as special point in Lemma 4.4. For instance, suppose \( q_8 = 0 \) and either \( q_1 = 0 \) or \( q_3 = 0 \) with \([q_1, q_3, q_4] \in \mathcal{E} \).

So, take \( P(a, 0, 1) \in PG(2, q) \setminus \mathcal{E} \) and consider a cubic \( \mathcal{C} \) with equation: \( Y(Y - m_1 X + am_1 Z)(Y - m_2 X + am_2 Z) = 0 \) passing through \( P \) meeting \( \mathcal{E} \) in 9 distinct points. Then, \( \mathcal{C} \) corresponds to the hyperplane \( \pi : m_1 m_2 X_1 - (m_1 + m_2)X_3 - 2am_1 m_2 X_4 + X_6 + a(m_1 + m_2)X_7 + a^2 m_1 m_2 X_8 = 0 \) which passes through \( Q \) if and only if

\[
m_1 m_2 q_1 - (m_1 + m_2)q_3 - 2am_1 m_2 q_4 + q_6 + a(m_1 + m_2)q_7 = 0.
\]

(5)

In particular, if we can determine \( m_1, m_2 \) and \( a \) such that (5) is satisfied, then the point \( Q \) is not special.

A similar argument applies when \( q_7 = 0 \).

Let now \( q \equiv 1 \mod 3 \) and \( \omega \) be a root of \( T^2 + T + 1 = 0 \). Consider a non-singular plane cubic curve \( \mathcal{E} \) over \( \mathbb{F}_q \) with canonical equation:

\[
X^3 + Y^3 + Z^3 - 3cXYZ = 0,
\]

where \( c \neq \infty, 1, \omega, \omega^2 \).
If \( c = 1 + \sqrt{3} \), then the elliptic curve \( \mathcal{E} \) is harmonic, that is, \( j(\mathcal{E}) \neq 0 \), see [8, Lemma 11.47]. Using Remark 4.2 and the symmetry \( Y \leftrightarrow Z \) of the curve \( \mathcal{E} \) it is possible to test for the completeness of \( \nu_2^2(\mathcal{E}) \). With the aid of GAP [13], we see that for \( q = 121 \) we obtain a curve with \( n = 144 \) rational points, for \( q = 157, 169 \) we obtain curves with \( n = 180 \) rational points whereas for \( q = 179 \) we get a curve with \( n = 180 \) points and in each case the \( n \) rational points define a complete NMDS code.

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