Field-theoretic treatment of mixed neutrinos in a neutrino and matter background

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Abstract

We use the method of finite temperature field theory to examine the propagation of mixed neutrinos through dense media, putting the emphasis in those situations in which the neutrinos themselves are in the background. The evolution equation for the flavor amplitudes is deduced, and the expressions for the corresponding hamiltonian matrix are given explicitly. We find that, in order to include the non-linear effects due to the $\nu$-$\nu$ interactions, the neutrino propagator that must be used in the calculation of the neutrino self-energy diagrams that contain neutrinos in the internal lines is the propagator for the
neutrino modes in the medium instead of the thermal free-field propa-
gator. We also show how the absorptive contributions are included
in terms of a non-hermitian part of the hamiltonian matrix, which
we indicate how it is calculated. Our treatment provides a consistent
generalization of a method that has been successfully applied to the
study of neutrino oscillations in matter.

1 Introduction and Summary

Finite Temperature Field Theory (FTFT) provides a very useful framework
to study the behavior of elementary particles, including neutrinos, in the
presence of a thermal background\cite{1}. A quantity of fundamental interest
in this formalism is the self-energy of a particle, from which the dispersion
relation of the modes that propagate in the medium are obtained. In the
case of neutrinos, the density and temperature dependent contributions to
the self-energy can be expressed as an effective potential for each neutrino
flavor that, added to the vacuum kinetic energy, yield the Hamiltonian ma-
trix that governs the neutrino oscillations in matter\cite{2, 3}. In normal matter
(electrons and nucleons) the universality of the neutral-current interactions
of the neutrinos implies that their contribution to the effective potential is the
same for all flavors and hence irrelevant as far as oscillations are concerned.
However, in environments like the early universe or the core of a supernova,
where the neutrinos represent an appreciable fraction of the total density,
the contributions to the potential energy arising from the $\nu-\nu$ scattering are
not in general proportional to the unit matrix and should be included in the
analysis of the resonant flavor transformations.

The calculation of the contribution from the background neutrinos to
the effective potential presents a new challenge because now the background
depends itself on the flavor amplitudes, whose time evolution is in turn deter-
mined by the oscillation mechanism. Thus the problem becomes a non-linear
one that must be solved in a self-consistent manner. Several studies of the
astrophysical and cosmological implications of neutrino oscillations under these
conditions have been carried out during the last years\cite{4, 5, 6}. These works
left out the off-diagonal contributions of the $\nu - \nu$ scattering to the flavor
energy matrix. The existence of these extra terms was first recognized by
Pantaleone\cite{7} and later on have been included in certain numerical analysis
of neutrinos oscillations.

Most of the extensive studies of neutrino oscillations in matter and the MSW solution to the solar neutrino problem are based on the Wolfenstein equation for the flavor amplitudes of a single-particle wave function. On the other hand, according to the prevailing attitude in the literature on the subject, a consistent description of neutrino oscillations that includes the effects of their mutual interactions and the noncoherent interactions with the background is possible only within the density matrix formalism. The works cited in Ref. give a general treatment of relativistic mixed neutrinos along this line which include the neutrino self-interactions, and have been recently applied to examine the matter-enhanced flavor transformations in supernovae.

While it is widely accepted that FTFT is an efficient method to compute the background effects that are incorporated as a potential energy in the Wolfenstein equation, a description based on this approach when neutrinos compose the background is lacking or incomplete. The present work fills this gap. By working within the formalism of the FTFT, we derive the evolution equation for the flavor amplitudes of a neutrino that propagates in a medium containing several species of massive neutrinos. Explicit formulas for the energy matrix are given in a familiar form that make them amenable for application to concrete situations. In order to bring out the salient features without introducing unnecessary complications, we work in detail the case of two Dirac neutrinos in the Standard Model, but our approach makes evident the path to treat more general cases, such as including additional families and Majorana neutrinos.

One of our objectives is to show that FTFT, which is simpler to use than other approaches, can be consistently applied to examine the type of problem we have referred to. The procedure is to calculate the self-energy matrix for mixed neutrinos, which leads to the effective Dirac equation obeyed by the modes that propagate in the medium. We obtain the Hamiltonian matrix for the flavor evolution equation directly from the condition that determines the energy-momentum relation for the neutrino modes. A particular new feature that arises when neutrinos appear in the background is that the neutrino densities are modulated by the oscillation mechanism itself, but on the other hand the standard FTFT propagators do not take this into account. Therefore an essential part of the problem is to find the appropriate neutrino propagator to be used in the self-energy diagrams that contain neutrinos.
in the internal lines. This aspect of the problem is considered in Section 3, where we show that this is not the standard free-particle thermal propagator, but rather the one for the effective field constructed out of the one-particle wave functions of the propagating modes. The use of this propagator is a key ingredient in our treatment. We also show how to include the damping effects by means of a non-hermitian part in the Hamiltonian, which is determined in our scheme from the absorptive part of the self-energy. In Section 4 we prove that the non-hermitian part of the Hamiltonian can be expressed in terms of the total rates for the emission and absorption processes of the mixed neutrinos by the background. There we also discuss some subtleties that arise if these rates are determined in the traditional way, in terms of probability amplitudes, when neutrinos compose the background and we explain how they are overcome in our treatment.

2 Normal Matter

The dispersion relations and wave functions of the neutrino modes that propagate through a medium are determined from the linear part of the effective field equation. In momentum space it takes the form

\[(k - m - \Sigma_{eff}) \psi = 0, \tag{1}\]

where \(k_\mu\) is the momentum vector and \(m\) is the diagonal matrix whose elements are the neutrino masses in vacuum. In the basis of the mass eigenstates \(\nu_i\), the background contributions to the neutrino self-energy \(\Sigma_{eff}\) is a non-diagonal matrix which can be written as

\[(\Sigma_{eff})_{ij} = (a_{Lij}k + b_{Lij}\gamma_5 + c_{Lij})L + (a_{Rij}k + b_{Rij}\gamma_5 + c_{Rij})R, \tag{2}\]

where \(u^\mu\) is the velocity four-vector of the medium and \(L, R = (1 \mp \gamma^5)/2\). In the rest frame of the medium

\[u^\mu = (1, 0),\]

and the components of \(k^\mu\) are given by

\[k^\mu = (\omega, \vec{\kappa}).\]
In general, the elements of the matrices $a_{L,R}$, $b_{L,R}$ and $c_{L,R}$ in Eq. (2) depend on $\omega$ and $\vec{\kappa}$, but in a isotropic medium they are functions only of the Lorentz scalars
\[
\omega = k \cdot u, \\
\kappa = \sqrt{\omega^2 - k^2},
\]
with $\kappa = |\vec{\kappa}|$. In what follows, the arguments indicating the dependence on these variables will be generally omitted, but they will be explicitly included when needed for clarity.

The background is taken to consist of nucleons, electrons and neutrinos, and their respective antiparticles. We assume that the temperature is low enough so that the contributions of the heavier charged leptons can be neglected. The relevant piece of the weak interaction Lagrangian is
\[
L_{\text{int}} = -\frac{g}{2\cos\theta_W}Z^\mu \left[ \sum_i \overline{\nu}_{Li} \gamma_\mu \nu_{Li} + \sum_{f=e,n,p} \overline{\nu}_f \gamma_\mu (X_f + Y_f \gamma^5) f \right] \\
- \frac{g}{\sqrt{2}} W^\mu \sum_i U_{ei} \overline{\nu}_{Li} \gamma_\mu \nu_{Li},
\]
where the index $i$ runs over all the neutrino species. The $\nu_{Li}$ are the left-handed components of the neutrino fields with a definite mass, and $U$ is the unitary matrix that relates them with the neutrino flavor fields $\nu_{L\alpha} = \sum_i U_{\alpha i} \nu_{Li}$ ($\alpha = e, \mu, \tau$). The coefficients $X_f$ and $Y_f$ are
\[
X_e = -\frac{1}{2} + 2 \sin^2 \theta_W, \\
Y_e = \frac{1}{2},
\]
for the electron and
\[
X_p = \frac{1}{2} - 2 \sin^2 \theta_W, \\
Y_n = -X_n = -Y_p = \frac{1}{2},
\]
for the nucleons. In addition,
\[
m_Z \cos \theta_W = m_W, \\
\frac{g^2}{4m_W^2} = \sqrt{2}G_F.
\]
The calculation of the self-energy proceeds as in the vacuum, but with the free propagators for the internal lines replaced by their thermal generalizations. Since we are assuming that there are no $W$ and $Z$ bosons in the medium, their propagators are the same as in the vacuum.

Let us first discuss the simplest case of a homogeneous background that does not contain neutrinos. For Dirac neutrinos propagating in such a background, the result of the one-loop calculation of the self-energy, up to terms of order $g^2/m_W^2$, is\cite{17}

\begin{align}
  a_{L,Rij} &= c_{L,Rij} = 0, \\
  b_{Rij} &= b_{ij} = 0, \\
  b_{Lij} &= b_{ij} = \sqrt{2} G_F \left[ U_{e_i}^* U_{e_j} (n_e - n_{\bar{e}}) + \delta_{ij} Q_Z \right], \quad (8)
\end{align}

where

\begin{equation}
  Q_Z = \sum_{f=e,n,p} X_f (n_f - n_{\bar{f}}), \quad (9)
\end{equation}

is the average $Z$-charge of the medium. In these formulas, $n_f$ and $n_{\bar{f}}$ denote the total number densities of the particles (electrons, neutrons and protons) and the antiparticles, respectively.

In terms of the left- and right-handed components of $\psi$ Eq. (11) becomes

\begin{align}
  A_{\mu L} \psi_L - m \psi_R &= 0, \\
  A_{\mu R} \psi_R - m \psi_L &= 0, \quad (10)
\end{align}

where

\begin{align}
  A_{\mu L} &= \frac{1}{2} \left( k_{\mu} - b_{\mu} \right), \\
  A_{\mu R} &= \frac{1}{2} \left( k_{\mu} - b_{\mu} \right). \quad (11)
\end{align}

In writing these equations we have used the particular results given in Eq. (8). In a more general case $A_{\mu L,R} = (1 - a_{L,R})k_{\mu} - b_{L,R} u_{\mu}$ and the matrix $m$ in Eq. (10) has to be replaced by $m + c_{R,L}$ in the first and in the second equation, respectively.

We use the Weyl representation of the gamma matrices and put

\begin{align}
  \psi_R &= \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \\
  \psi_L &= \begin{pmatrix} 0 \\ \eta \end{pmatrix}. \quad (12)
\end{align}
The equations to be solved then become
\[
\begin{align*}
(A^0_L + \vec{\sigma} \cdot \vec{A}_L) \eta - m \xi &= 0, \\
(A^0_R - \vec{\sigma} \cdot \vec{A}_R) \xi - m \eta &= 0.
\end{align*}
\tag{13}
\]

Since \(\vec{A}_L\) and \(\vec{A}_R\) are both proportional to \(\vec{\kappa}\), these equations can have non-trivial solutions only if \(\xi\) and \(\eta\) are proportional to the same spinor \(\phi_\lambda\) with definite helicity, defined by
\[
(\vec{\sigma} \cdot \hat{k}) \phi_\lambda = \lambda \phi_\lambda,
\tag{14}
\]
with \(\lambda = \pm 1\). Therefore, we write
\[
\begin{align*}
\xi &= y \phi_\lambda, \\
\eta &= x \phi_\lambda,
\end{align*}
\tag{15}
\]
where \(y\) and \(x\) are spinors in the basis of the \(\nu_i\). For a given helicity \(\lambda\), the equations for \(y\) and \(x\) that follow from Eq. (13) are
\[
\begin{align*}
A^\lambda_L x - my &= 0, \\
A^{(-\lambda)}_R y - mx &= 0,
\end{align*}
\tag{16}
\]
where
\[
\begin{align*}
A^\lambda_L &= \omega + \lambda \kappa - b, \\
A^{(-\lambda)}_R &= \omega + \lambda \kappa.
\end{align*}
\tag{17}
\]

Using the second equation in (16) to eliminate \(y\) from the first one, the equations for \(y\) and \(x\) become
\[
\begin{align*}
\left[ A^{(-\lambda)}_R m^{-1} A^\lambda_L - m \right] x &= 0, \\
y &= \frac{1}{A^{(-\lambda)}_R} mx,
\end{align*}
\tag{18}
\]
which have solutions only if
\[
\text{Det} \left( A^{(-\lambda)}_R m^{-1} A^\lambda_L - m \right) = 0.
\tag{19}
\]
The roots of this equation determine the dispersion relations of the propagating modes. The complete characterization of the different solutions requires, in addition, the determination of the wave function for each mode. For these purposes it is instructive to notice that for one (unmixed) Dirac neutrino, Eq. (19) reduces to

\begin{align}
(\omega - \kappa - b)(\omega + \kappa) &= m^2, \quad (20) \\
(\omega + \kappa - b)(\omega - \kappa) &= m^2, \quad (21)
\end{align}

where the expression for \( b \) is obtained from Eq. (8) by making obvious simplifications. For example, for an electron neutrino in normal matter, \( b = \sqrt{2}G_F(n_e + \sum_f X_f n_f) \). Each one of Eqs. (20) and (21) yields two solutions: one with a positive real part of \( \omega \) and another with a negative real part. The four solutions correspond to the positive and negative helicity states of the neutrino and the antineutrino. In particular, the dispersion relation for the left-handed neutrino corresponds to the positive solution of Eq. (20):

\[
\omega = \left[ (\kappa + \frac{b}{2})^2 + m^2 \right]^{1/2} + \frac{b}{2}. \quad (22)
\]

For future purposes it is useful to notice here that, for relativistic neutrinos, the effect on the dispersion relations of a non-zero value of the coefficients \( a_L \) and \( a_R \) is negligible. To see this notice, for example, that if \( a_L \neq 0 \) Eq. (20) is modified to

\[
[(1 - a_L)(\omega - \kappa) - b](\omega + \kappa) = m^2. \quad (23)
\]

For \( \omega \approx \kappa \), the corrections to the solution given in Eq. (22) are either of higher order in \( g^2 \) or of order \( m^2 g^2 / \kappa \) and we ignore them.

Based on these remarks, let us now return to Eq. (19) and look for solutions with \( \lambda = -1 \). As already mentioned, for clarity we consider the case of mixing between only two Dirac neutrinos. The condition that determines the energy-momentum relation is given explicitly by

\[
\left[ \omega^2 - \kappa^2 - (\omega + \kappa)b_{11} - m_1^2 \right] \left[ \omega^2 - \kappa^2 - (\omega + \kappa)b_{22} - m_2^2 \right] - (\omega + \kappa)^2 b_{12} b_{21} = 0, \quad (24)
\]

where

\[
b_{11} = b_e \cos^2 \theta + \sqrt{2}G_F Q_Z
\]
\[ b_{22} = b_e \sin^2 \theta + \sqrt{2} G_F Q_Z \]
\[ b_{12} = b_{21} = b_e \sin \theta \cos \theta. \]  
(25)

Here \( \theta \) is the vacuum mixing angle and
\[ b_e = \sqrt{2} G_F (n_e - \bar{n}_e). \]  
(26)

Eq. (24) has in general two positive and two negative solutions, the latter of which correspond to the antineutrinos. We seek approximate solutions to this equation in the relativistic limit, which is the adequate regime for neutrino oscillations. We do it first for neutrinos (positive solutions), and afterwards apply the same considerations to the antineutrinos. Dividing Eq. (24) by \((\omega + \kappa)^2\) and using the approximation
\[ \frac{m_i^2}{\omega + \kappa} \approx \frac{m_i^2}{2\kappa}, \]  
(27)
which is valid for \(\kappa \gg m_{1,2}, b_{ij}\), Eq. (24) reduces to
\[ (\omega - \kappa - b_{11} - \frac{m_1^2}{2\kappa}) (\omega - \kappa - b_{22} - \frac{m_2^2}{2\kappa}) - b_{12} b_{21} = 0. \]  
(28)

with solutions
\[ \omega_{1,2} = \kappa + \frac{m_1^2 + m_2^2}{4\kappa} + \frac{b_{11} + b_{22}}{2} \pm \frac{1}{2} \left[ \left( b_{11} - b_{22} + \frac{m_1^2 - m_2^2}{2\kappa} \right)^2 + 4b_{12}b_{21} \right]^{1/2}. \]  
(29)

With each dispersion relation \(\omega_{1,2}\) there is associated a two-component vector \(x = e_{1,2}\), which is obtained by solving
\[ \left[ A_R^{(+)} m^{-1} A_L^{(-)} - m \right]_{\omega = \omega_{1,2}} e_{1,2} = 0. \]  
(30)

The corresponding \(y\)-vector is given by
\[ y_{1,2} = \frac{m}{A_R^{(+)}_{\omega = \omega_{1,2}}} e_{1,2}, \]  
(31)

and together with Eq. (30), it determine the wave function of the propagating mode. Clearly, for relativistic neutrinos, \(y_{1,2} \approx (m/\kappa) e_{1,2}\).
Using the same approximation of (27), Eq. (30) can be recast in the form

$$H e_i = \omega_i e_i,$$

(32)

where

$$H = \kappa + \frac{m^2}{2\kappa} + b(\kappa, \bar{\kappa}).$$

(33)

For the two-flavor mixing case we are considering,

$$H = \kappa + \sqrt{2}G_F Q_Z + \frac{1}{2\kappa} \left( \begin{array}{cc} m_1^2 & 0 \\ 0 & m_2^2 \end{array} \right) + U^\dagger \left( \begin{array}{cc} b_e & 0 \\ 0 & 0 \end{array} \right) U.$$  

(34)

with

$$U = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right).$$  

(35)

The corresponding Hamiltonian in the flavor basis is

$$H_{flav} = U H U^\dagger,$$

(36)

and after subtracting an irrelevant term proportional to the unit matrix, it can be written in the form

$$H_{flav} = \frac{1}{2} \left( \begin{array}{cc} -\Delta_0 \cos 2\theta + b_e & \Delta_0 \sin 2\theta \\ \Delta_0 \sin 2\theta & \Delta_0 \cos 2\theta - b_e \end{array} \right),$$

(37)

where

$$\Delta_0 = \frac{m_2^2 - m_1^2}{2\kappa}.$$  

(38)

This is the energy matrix traditionally used in the studies of neutrino oscillations inside the Sun. Its eigenvalues are simply $\pm \frac{1}{2}(\omega_2 - \omega_1)$, with

$$\omega_2 - \omega_1 = \sqrt{(\Delta_0 \cos 2\theta - b_e)^2 + \Delta_0^2 \sin^2 2\theta}.$$  

(39)

The explicit solutions of Eq. (30) in the flavor basis are readily found to be

$$e_1 = \left( \begin{array}{c} \cos \theta_m \\ -\sin \theta_m \end{array} \right),$$

$$e_2 = \left( \begin{array}{c} \sin \theta_m \\ \cos \theta_m \end{array} \right),$$

(40)
where $\theta_m$ is the mixing angle in the medium given by

$$
\sin 2\theta_m = \frac{\Delta_0 \sin 2\theta}{\sqrt{(\Delta_0 \cos 2\theta - b_e)^2 + \Delta_0^2 \sin^2 2\theta}}.
$$  (41)

Written as in Eq. (32), the dispersion relations $\omega_{1,2}$ are simply the eigenvalues of the Hamiltonian $H$ with the $e_i$ being the corresponding eigenvectors. The physical interpretation that emerges is the following. The Dirac wave function for a relativistic left-handed neutrino with momentum $\vec{\kappa}$ that propagates through a medium with a definite dispersion relation $\omega_i$ is

$$
\psi(x) \simeq \psi_L(x)
= e^{i\vec{\kappa} \cdot \vec{x}} \begin{pmatrix} 0 \\ \phi_- \end{pmatrix} e^{-i\omega_i t} e_i,
$$

(42)

where we neglect $\psi_R$, which is of order $m/2\kappa$ as compared to the left-handed component. If the neutrino is not initially in a state corresponding to one of the eigenmodes, then the wave function is

$$
\psi_L = e^{i\vec{\kappa} \cdot \vec{x}} \begin{pmatrix} 0 \\ \phi_- \end{pmatrix} \chi(t),
$$

(43)

where $\chi(t)$ is the general solution of

$$
i \frac{d\chi}{dt} = H\chi,
$$

(44)

that satisfies the specified boundary condition for $\chi(0)$, and can be written as

$$
\chi(t) = \sum_{i=1,2} (e_i^\dagger \chi(0)) e^{-i\omega_i t} e_i.
$$

(45)

Since neutrinos are produced in states of definite flavor, in neutrino oscillations problems the initial conditions are commonly specified by giving the flavor components of the initial state. Thus, if we denote

$$
\chi(0) = \begin{pmatrix} a_e \\ a_\mu \end{pmatrix},
$$

(46)

in the flavor basis, then at a later time $t$

$$
\chi(t) = \chi^{(e)}(t)a_e + \chi^{(\mu)}(t)a_\mu,
$$

(47)
where $\chi^{(e,\mu)}(t)$ are the solutions to Eq. (44) with the initial conditions
\[
\chi^{(e)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\chi^{(\mu)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\] (48)

Physically, $a_{e,\mu}$ is the amplitude for finding the neutrino initially in the state corresponding to $|\nu_{e,\mu}\rangle$. If we write
\[
\chi(t) = \begin{pmatrix} \chi_e(t) \\ \chi_\mu(t) \end{pmatrix}
\] (49)
then
\[
\chi_\alpha(t) = a_e \chi_\alpha^{(e)}(t) + a_\mu \chi_\alpha^{(\mu)}(t), \quad (\alpha = e, \mu)
\] is the amplitude for finding the neutrino in the corresponding flavor state at subsequent times, with a similar interpretation for the components of $\chi(t)$ in other basis.

Associated with the wave functions there is a quantum field
\[
\Psi^{(\nu)}_L(x) = \int \frac{d^3\kappa}{(2\pi)^3} e^{i\kappa \cdot \vec{x}} \begin{pmatrix} 0 \\ \phi_- \end{pmatrix} (\chi^{(e)}(t)a_e(\kappa) + \chi^{(\mu)}(t)a_\mu(\kappa)),
\] (50)
where $\chi^{(e,\mu)}$ are the functions defined above, while the $a_{e,\mu}(\kappa)$ are now interpreted as the annihilation operators of left-handed neutrinos of definite flavor with momentum $\kappa$.

Up to now we have restricted ourselves to the positive frequency solutions of Eq. (19) with $\lambda = -1$. In order to include the right-handed antineutrinos we need to consider the negative solutions with $\lambda = +1$. Writing the negative solutions as $\omega = -\overline{\omega}$, then $\overline{\omega}_{1,2}$ are given by the same formulas as in Eq. (29) but with the sign of the coefficients $b_{ij}$ reversed:
\[
\overline{\omega}_{1,2} = \kappa + \frac{m_1^2 + m_2^2}{4\kappa} - \frac{b_{11} + b_{22}}{2} \pm \frac{1}{2} \left[ \left( b_{22} - b_{11} + \frac{m_1^2 - m_2^2}{2\kappa} \right)^2 + 4b_{12}b_{21} \right]^{1/2}.
\] (51)

The equations corresponding to Eq. (50) for the two-component vectors $x = \overline{\omega}_{1,2}$ are
\[
\left[ A_R^{(-)} m^{-1} A_L^{(+)} - m \right] \big|_{\omega = -\overline{\omega}_{1,2}} \overline{\omega}_{1,2} = 0.
\] (52)
Similarly to Eq. (32), they can be recast in the form

$$H_{-} = \kappa + \frac{m^2}{2\kappa} - b(-\kappa, \vec{\kappa}).$$

(54)

In particular, $H_-$ differs from Eq. (34) only in the sign of the $Q_Z$ and $b_e$ terms. In analogy with the positive frequency field $\Psi_L^{(\nu)}$, we built a negative frequency field

$$\Psi_L^{(\bar{\nu})}(x) = \int \frac{d^3\kappa}{(2\pi)^3} e^{-i\vec{\kappa} \cdot \vec{x}} \begin{pmatrix} 0 \\ i\sigma_2\phi^*_+ \end{pmatrix} (\chi^{(\bar{\nu})}(t)a_{\bar{\nu}}(\vec{\kappa}) + \chi^{(\bar{\nu})}(t)a_{\bar{\nu}}(\vec{\kappa}))^*,$$

(55)

where the $a_{\bar{\nu}}(\vec{\kappa})$ represent the annihilation operators of (right-handed) antineutrinos of definite flavor, with momentum $\vec{\kappa}$. The $\chi^{(\bar{\nu})}$ are solutions of

$$i\frac{d\chi}{dt} = \mathcal{H}\chi,$$

(56)

that satisfy

$$\chi^{(\bar{\nu})}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\chi^{(\bar{\nu})}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

(57)

with

$$\mathcal{H} \equiv H_*(-\vec{\kappa}),$$

$$= \kappa + \frac{m^2}{2\kappa} - b^L_{\bar{\nu}}(-\kappa, -\vec{\kappa}),$$

$$= \kappa - \sqrt{2}G_FQ_Z + \frac{1}{2\kappa} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} - U^T \begin{pmatrix} b_e & 0 \\ 0 & 0 \end{pmatrix} U^*. $$

(58)

The corresponding matrix in the flavor basis is

$$\mathcal{H}_{flav} = U^*\mathcal{H}U^T$$

$$= \frac{1}{2} \begin{pmatrix} -\Delta_0 \cos 2\theta - b_e & \Delta_0 \sin 2\theta \\ \Delta_0 \sin 2\theta & \Delta_0 \cos 2\theta + b_e \end{pmatrix},$$

(59)

13
where we have again discarded a multiple of the identity. Two comments are in order. Although we are considering a homogeneous medium, in some of the previous expressions we have retained the dependence in $\vec{\kappa}$ to make more obvious their generalization to the anisotropic situation. In addition, for the case of mixing between two Dirac neutrinos, the matrix $U$ is real (see Eq. (35)), but in the case of Majorana neutrinos three Dirac neutrinos, this is not necessarily true; for that reason, in Eq. (59) we have not replaced $U^*$ by $U$.

Neglecting the right-handed components then, the complete effective neutrino field is

$$ \Psi(x) = \Psi^{(\nu)}_{L}(x) + \Psi^{(\bar{\nu})}_{L}(x). $$

(60)

The wave functions used to define $\Psi(x)$ must include appropriate normalization factors $N_{\kappa}$. They can be calculated in terms of the self-energy[18] and, in general, differ from the normalizations in the vacuum. With the coefficients as given in Eq. (8), $N_{\kappa} = 1$ and therefore, if we take the creation and annihilation operators normalized such that

$$ \{a_{\alpha'}^{*}(\vec{\kappa'}), a_{\alpha}(\vec{\kappa})\} = (2\pi)^{3}\delta^{(3)}(\vec{\kappa'} - \vec{\kappa})\delta_{\alpha',\alpha}, $$

(61)

then the spin wave functions $\phi_{\lambda}$ have to be chosen to satisfy

$$ \phi^{\dagger}\phi = 1. $$

(62)

So far we have assumed that the background is homogeneous. Now suppose that the matter density is not constant. Then along the neutrino path the densities of the different particles in the background become functions of the time $t$. The crucial step that we take is to postulate that, in such situation, Eq. (43) and (44) remain valid and determine the spinor wave functions of mixed left-handed neutrinos with momentum $\vec{\kappa}$. Consequently, Eq. (31) is assumed to give the associated effective field also for an inhomogeneous medium. Of course, in that case Eq. (43) no longer holds, and an appropriate solution of Eq. (44) must be found. Identical considerations applies to the (right-handed) antineutrinos. Our Eq. (44), with the (time-dependent) Hamiltonian expressed in terms of $b_\nu(t) = \sqrt{2}G_Fn_\nu(t)$, is precisely the Wolfenstein equation that determines the flavor neutrino evolution through ordinary matter [3].
3 Neutrino background

As illustrated in the previous section, our formalism reproduces the standard formulas that have been utilized in the numerous studies of the MSW effect when it is applied to a medium made of electrons and nucleons. In what follows we show how to extend this field-theoretic approach to take into account the new features that appear when several species of neutrinos are also part of the background. In this situation, the neutrino self-energy receives additional contributions from the diagrams depicted in Fig. 1, which involve neutrinos propagating in the internal lines of the loop. These contributions are given by

\[ -i \Sigma_{ij}^{(\nu)}(k) = \left(-i \frac{g}{2 \cos \theta_W}\right)^2 \int \frac{d^4 k'}{(2\pi)^4} \gamma^\mu L i \Delta^{(Z)}(k' - k) i S_F^{(\nu)}(k')_{ij} \gamma^\nu L , \]

\[ -i \Sigma_{ij}^{(\nu)}(k) = \delta_{ij} \left(-i \frac{g}{2 \cos \theta_W}\right)^2 \gamma^\mu L \left(\frac{ig_{\mu\nu}}{m_Z^2}\right) \times (-1)^{\text{Tr}} \int \frac{d^4 k'}{(2\pi)^4} \left(\sum_l i S_F^{(\nu)}(k')_{il}\right) \gamma^\nu L , \]

where the trace is taken over the Dirac indices. \( S_F^{(\nu)}(k') \) is the thermal neutrino propagator defined by

\[ i \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} S_F(k)_{ij} = \langle T \Psi_i(x) \bar{\Psi}_j(y) \rangle , \]

where \( T \) stands for the time-ordered product of the fields. The angle brackets in Eq. (65) indicate a thermal average, defined by

\[ \langle O \rangle = \frac{1}{Z} \sum_i \langle i | \rho | i \rangle , \]

for any operator \( O \), where

\[ \rho = e^{-\beta H + \sum_A \alpha_A Q_A} , \]

and

\[ Z = \sum_i \langle i | \rho | i \rangle . \]
In Eq. (67) $\beta$ is the inverse temperature, $\mathcal{H}$ is the Hamiltonian of the system, the $Q_A$ are the (conserved) charges that commute with $\mathcal{H}$, and $\alpha_A$ are the chemical potentials that characterize the composition of the background.

In the ordinary perturbative applications of FTFT, the right-hand side of Eq. (65) is evaluated in terms of the free fields. Then, a straightforward calculation of the thermal averages of the products of creation and annihilation operators yields the canonical formulas for $S_F(k)$ in terms of the particle number densities. In particular, the propagators for the electron and the nucleons calculated that way, were used to compute the contributions to the neutrino self-energy given in Eq. (8). However, for the problem at hand we cannot use the thermal free neutrino propagator to calculate the additional contributions to the self-energy introduced by the neutrino mutual interactions. The reason is that the neutrino densities are modulated by the oscillation amplitudes, which evolve according to the equations we have obtained (Eqs. (44) and (53)), and the thermal free neutrino propagator does not incorporate this effect.

Our proposal in this work is to use, instead of the thermal propagator of the neutrino free-field, the propagator that corresponds to the effective neutrino field defined in Eq. (60). The calculation of this propagator proceeds as in the free field case, simply by substituting the expansions in Eqs. (50) and (56) into (65). However, as will be seen shortly, for our purpose here, it is not necessary to compute it explicitly. The result of a straightforward calculation, whose details we give below, are

$$
\Sigma^{(\nu\nu)}(k)_{ij} = \sqrt{2} G_F (\gamma^\mu L) (j^{(\nu)}_\mu + j^{(\overline{\nu})}_\mu)_{ij},
$$

$$
\Sigma'(\nu\nu)(k)_{ij} = \delta_{ij} \sqrt{2} G_F (\gamma^\mu L) \int \frac{d^3k'}{(2\pi)^3} n'_\mu \left[ f_{\nu e} - f_{\overline{\nu} e} + (e \rightarrow \mu) \right],
$$

where

$$
\begin{align*}
   j^{(\nu)}_\mu &= \int \frac{d^3k'}{(2\pi)^3} n'_\mu \left[ \chi^{(e)} \chi^{(e)\dagger} f_{\nu e} + (e \rightarrow \mu) \right], \\
   j^{(\overline{\nu})}_\mu &= -\int \frac{d^3k'}{(2\pi)^3} n'_\mu \left[ (\chi^{(\overline{\nu})} \chi^{(\overline{\nu})\dagger})^* f_{\overline{\nu} e} + (e \rightarrow \mu) \right].
\end{align*}
$$

The vector $n'^\mu$ has the components

$$
n'^\mu = (1, \hat{k}'),
$$
in the rest frame of the medium, and $f_{\nu e, \mu}$ ($f_{\nu e, \mu}$) are the momentum distributions, normalized such that

$$
n_n^{0 \nu e, \mu} = \int \frac{d^3 k'}{(2\pi)^3} f_{\nu e, \mu},$$

$$
n_n^{0 \nu e, \mu} = \int \frac{d^3 k'}{(2\pi)^3} f_{\nu e, \mu},$$

(73)

are the number densities of the flavor neutrinos (antineutrinos) at the initial time.

Let us first derive the result for $\Sigma^{(\nu \nu)}$. Using the local approximation for the $Z$-boson propagator,

$$
\Delta^{(Z)}_{\mu \nu} \simeq \frac{g_{\mu \nu}}{m_Z^2},
$$

(74)

and the Fierz-like identity

$$
\gamma^\alpha L A \gamma^\alpha L = -(\text{Tr} A \gamma^\alpha L) \gamma^\alpha L,
$$

which is valid for any $4 \times 4$ matrix $A$, we have

$$
\Sigma^{(\nu \nu)}_{ij}(k) = -i \left( \frac{g^2_{\mu \nu}}{4m_W^2} \right) (\gamma^\mu L) \int \frac{d^4 k'}{(2\pi)^4} \text{Tr}(S_F(k')_{ij} \gamma^\mu L).
$$

(75)

As in Eq. (64), the trace is with respect the Dirac indices. While we can calculate the propagator using the definition in Eq. (65) and then substitute the result in the above formula, it is more expedient to observe that

$$
- i \int \frac{d^4 k'}{(2\pi)^4} \text{Tr}(S_F(k')_{ij} \gamma^\mu L) = \langle N \overline{\Psi}_j(x) \gamma_\mu \Psi_i(x) \rangle,
$$

(76)

which results immediately from Eq. (65), by taking the limit $x \to y$. In the last equation $N$ stands for the normal-ordered product. The quantity in the right-hand side is easily computed by inserting the plane wave expansions given in Eqs. (50) and (53). The statistical averages are evaluated by means of the formulas

$$
\langle a_{\alpha'}^\ast (\vec{k}') a_\alpha(\vec{k}) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}' - \vec{k}) \delta_{\alpha' \alpha} f_{\nu e},
$$

$$
\langle a_{\alpha'}^\ast (\vec{k}') a_\alpha(\vec{k}) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}' - \vec{k}) \delta_{\alpha' \alpha} f_{\nu e},
$$

(77)
where \( \alpha = e, \mu \). In this manner we obtain
\[
\langle N \Psi_j(x) \gamma_\mu \Psi_i(x) \rangle = (j^{(\nu)}_\mu + j^{(\bar{\nu})}_\mu)_{ji},
\]
and using Eqs. (78) and (76) in (75) we finally arrive at Eq. (69). For \( \Sigma^{(\nu\nu)} \), by a similar procedure it follows that
\[
\Sigma^{(\nu\nu)}(k)_{ij} = \delta_{ij} \frac{g^2}{4 m_W^2} \sum_l (j^{(\nu)}_\mu + j^{(\bar{\nu})}_\mu)_l.
\]
(79)
Since the Hamiltonian matrix is hermitian,
\[
\chi^{(e,\mu)\dagger} \chi^{(e,\mu)} = 1,
\]
(80)
and
\[
\sum_l (j^{(\nu)}_\mu + j^{(\bar{\nu})}_\mu)_l = \int \frac{d^3 \kappa'}{(2\pi)^3} n_\mu [f_{\nu_e} - f_{\bar{\nu}_e} + (e \rightarrow \mu)],
\]
(81)
which gives Eq. (70).
\( \Sigma^{(\nu\nu)} \) can be decomposed as in Eq. (2),
\[
\Sigma^{(\nu\nu)} = (a^{(\nu\nu)} \hat{k} + b^{(\nu\nu)} \hat{\mu}) L,
\]
(82)
with
\[
a^{(\nu\nu)} = \sqrt{2} G_F \int \frac{d^3 \kappa'}{(2\pi)^3} \left( \frac{\hat{k}' \cdot \hat{\kappa}}{\kappa} \right)
\times \left[ \chi^{(e)} \chi^{(e)\dagger} f_{\nu_e} - (\chi^{(\bar{\nu})} \chi^{(\bar{\nu})\dagger})^* f_{\bar{\nu}_e} + (e \rightarrow \mu) \right],
\]
(83)
As remarked below Eq. (22), the coefficient \( a^{(\nu\nu)} \) has a negligible effect on the dispersion relation and therefore we do not consider it further. From Eq. (83) and the results of Ref. [18], it can also be verified that the wave function normalization to which we alluded before remains equal to one. On the contrary, \( b^{(\nu\nu)} \) produces a new significant contribution that modifies the Hamiltonian \( H \) given in Section 2. Since, for relativistic neutrinos (antineutrinos), \( \omega \approx \kappa \)
the above formula for \( b^{(\nu \nu)} \) translates into the following additional term to the matrices exhibited in Eqs. (34) and (58):

\[
H^{(\nu \nu)} = b^{(\nu \nu)}(\kappa, \vec{\kappa}) ,
\]

\[
\overline{H}^{(\nu \nu)} = -b^{(\nu \nu)*}(-\kappa, -\vec{\kappa}) = -H^{(\nu \nu)*} .
\]

The additional terms produced by \( \Sigma^{(\nu \nu)} \) can be obtained in similar fashion; they are proportional to the identity matrix \( I \) and are given by

\[
H^{(\nu \nu)} = -H^{(\nu \nu)*} = \sqrt{2} G_F Q^{(\nu \nu)} Z_I ,
\]

(84)

To summarize, the propagation of mixed neutrinos through a background that includes neutrinos is described by the effective neutrino field \( \Psi(x) \) given in Eq. (60). The time evolution of the amplitudes \( \chi^{(\alpha)}(t) \) and \( \chi^{(\mu)}(t) \) that appear in the Fourier expansion of the field is governed by

\[
i d\chi^{(e,\mu)} = (H^{(\text{matter})} + H^{(\nu \nu)} + H^{(\nu \nu)*}) \chi^{(e,\mu)} ,
\]

\[
i d\chi^{(\nu \mu)} = (\overline{H}^{(\text{matter})} + \overline{H}^{(\nu \nu)} + \overline{H}^{(\nu \nu)*}) \chi^{(\nu \mu)} ,
\]

(87)

where \( H^{(\text{matter})} \) and \( \overline{H}^{(\text{matter})} \) stand for the normal-matter contributions determined in the previous section (Eqs. (34) and (58)), with the label \textit{matter} added to single them out. Discarding a term proportional to the identity matrix, in the flavor basis the above equations can be put in the form

\[
i d\chi^{(e,\mu)} = \frac{1}{2} \left( \begin{array}{cc}
- \Delta_0 \cos 2\theta + b_e + 2h_{e\mu} & \Delta_0 \sin 2\theta + 2h_{e\mu} \\
\Delta_0 \sin 2\theta + 2h_{e\mu} & \Delta_0 \cos 2\theta - b_e + 2h_{\mu\mu}
\end{array} \right) \chi^{(e,\mu)} ,
\]

\[
i d\chi^{(\nu \mu)} = \frac{1}{2} \left( \begin{array}{cc}
- \Delta_0 \cos 2\theta - b_e - 2h_{e\mu} & \Delta_0 \sin 2\theta - 2h_{e\mu} \\
\Delta_0 \sin 2\theta - 2h_{e\mu} & \Delta_0 \cos 2\theta + b_e - 2h_{\mu\mu}
\end{array} \right) \chi^{(\nu \mu)} ,
\]

(88)

where \( h_{\alpha \alpha'} \) are the elements of

\[
h \equiv H^{(\nu \nu)} - \frac{1}{2} \text{Tr} H^{(\nu \nu)} ,
\]

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and are explicitly given by

$$h_{ee} = -h_{\mu\mu} = \sqrt{2G_F} \int \frac{d^3 \kappa'}{(2\pi)^3} \left(1 - \hat{\kappa}' \cdot \hat{\kappa}\right)$$

$$\times \left[ \frac{1}{2} \sum_{\alpha=e,\mu} \left[ \left| \chi^{(\alpha)}_e \right|^2 - \left| \chi^{(\alpha)}_\mu \right|^2 \right] f_{\nu_\alpha} - \left( \left| \chi^{(\overline{\alpha})}_{\overline{e}} \right|^2 - \left| \chi^{(\overline{\alpha})}_{\overline{\mu}} \right|^2 \right) f_{\overline{\nu_\alpha}} \right],$$

(89)

$$h_{e\mu} = h^{*}_{\mu e} = \sqrt{2G_F} \int \frac{d^3 \kappa'}{(2\pi)^3} \left(1 - \hat{\kappa}' \cdot \hat{\kappa}\right)$$

$$\times \sum_{\alpha=e,\mu} \left[ \left( \chi^{(\alpha)}_e \chi^{(\alpha)*}_{\mu} \right) f_{\nu_\alpha} - \left( \chi^{(\overline{\alpha})}_{\overline{e}} \chi^{(\overline{\alpha})*}_{\overline{\mu}} \right) f_{\overline{\nu_\alpha}} \right].$$

(90)

In the present case, in which we are neglecting absorptive effects, we can use the probability conservation conditions $|\chi^{(\alpha)}|^2 = 1$ and $|\chi^{(\overline{\alpha})}|^2 = 1$ in Eq. (89) and get

$$h_{ee} = -h_{\mu\mu} = \sqrt{2G_F} \int \frac{d^3 \kappa'}{(2\pi)^3} \left(1 - \hat{\kappa}' \cdot \hat{\kappa}\right)$$

$$\times \left[ \frac{1}{2} \sum_{\alpha=e,\mu} \left[ \left| \chi^{(\alpha)}_e \right|^2 - \frac{1}{2} \right] f_{\nu_\alpha} - \left( \left| \chi^{(\overline{\alpha})}_{\overline{e}} \right|^2 - \frac{1}{2} \right) f_{\overline{\nu_\alpha}} \right].$$

(91)

The set of (non-linear) equations in (88) (or (87)), subject to the initial conditions given in Eq. (48) and (57), are the ones that must be solved in the application to an actual physical problem\[19\]. They represent the generalization of the Wolfenstein equation to the physical situation under consideration. In our notation, $h_{e,\mu}$ and $h_{\mu,e}$ account for the nondiagonal contributions to the potential energy in the flavor basis, that arise from the $\nu\nu$ interactions\[7\].

In deriving the previous results no assumption has been made on the characteristics of the neutrino background. If the medium is isotropic, then the quantities between brackets in, for example, Eq. (83) can not depend on the angular integration variables. In such a case, the integral involving the factor $\hat{\kappa}' \cdot \hat{\kappa}$ vanishes and the coefficients in the self-energy decomposition reduce to

$$a^{(\nu\nu)} = 0,$$

$$b^{(\nu\nu)} = \sqrt{2G_F} \int \frac{d^3 \kappa'}{(2\pi)^3} \left[ \chi^{(e)}_e \chi^{(e)*}_{\nu_\nu} f_{\nu_e} - \left( \chi^{(\overline{\nu})}_{\overline{e}} \chi^{(\overline{\nu})*}_{\overline{\nu_\nu}} \right) f_{\overline{\nu_e}} + (e \rightarrow \mu) \right],$$

(92)
with similar simplifications in Eqs. (89) and (90).

Usually, the quantity of interest in applications is the density of electron neutrinos at a given time. In our language it is simply

\[
n_{\nu_e} = \langle \Psi^{(e)\dagger}_L(x) \Psi^{(e)}_L(x) \rangle,
\]

(93)

which, upon substituting the expansion given in Eq. (50) and evaluating the statistical averages by means of Eq. (77), yields

\[
n_{\nu_e} = \int \frac{d^3\kappa'}{(2\pi)^3} \left[ |\chi^{(e)}_e|^2 f_{\nu_e} + |\chi^{(\mu)}_\mu|^2 f_{\nu_\mu} \right].
\]

(94)

The density of neutrinos of other types can be computed in identical manner in terms of the initial distributions and the flavor amplitudes. For example,

\[
n_{\nu_\mu} = \langle \Psi^{(\mu)\dagger}_L(x) \Psi^{(\mu)}_L(x) \rangle
\]

\[
= \int \frac{d^3\kappa'}{(2\pi)^3} \left[ |\chi^{(e)}_e|^2 f_{\nu_e} + |\chi^{(\mu)}_\mu|^2 f_{\nu_\mu} \right],
\]

(95)

with analogous expressions for antineutrinos. At \( t = 0 \), these formulas for \( n_{\nu_e} \) and \( n_{\nu_\mu} \) reduce to their appropriate initial values defined in Eq. (73).

At each time \( t \), the Hamiltonian matrix in Eq. (88) can be diagonalized by the unitary transformation \( U_m(t) \)

\[
U_m(t) = \begin{pmatrix} e^{i\varphi_m/2} & 0 \\ 0 & e^{-i\varphi_m/2} \end{pmatrix} \begin{pmatrix} \cos \theta_m & \sin \theta_m \\ -\sin \theta_m & \cos \theta_m \end{pmatrix},
\]

(96)

with the time-dependent mixing angles \( \theta_m \) and \( \varphi_m \) defined by

\[
\sin 2\theta_m = \frac{\Delta_0 \sin 2\theta + 2h_{e\mu}}{\sqrt{(\Delta_0 \cos 2\theta - b_e - 2h_{ee})^2 + (\Delta_0 \sin 2\theta + 2h_{e\mu})^2}},
\]

(97)

\[
\varphi_m = \arg[\Delta_0 \sin 2\theta + 2h_{e\mu}].
\]

(98)

Under the present conditions, the difference between the (instantaneous) energy eigenvalues is given by

\[
\omega_2 - \omega_1 = \sqrt{(\Delta_0 \cos 2\theta - b_e - 2h_{ee})^2 + (\Delta_0 \sin 2\theta + 2h_{e\mu})^2}.
\]

(99)
As is evident from Eq. (97) the neutrino mixing angle in matter is modified by their mutual interactions, and the resonance condition \((\sin 2\theta_m = 1)\) becomes

\[
\Delta_0 \cos 2\theta = b_e + 2h_{ee}.
\]

Moreover, the off-diagonal elements of the Hamiltonian are in general complex, and the phase \(\varphi_m\) will be present in contrast with the ordinary-matter situation. Similarly, a unitary matrix \(U_m(t)\) can be introduced for antineutrinos; the respective formulas for the angles \(\bar{\theta}_m\) and \(\bar{\varphi}_m\) are obtained by replacing \(b_e, h_{ee}\), and \(h_{e\mu}\) by their negatives in Eqs. (97) and (98).

4 Absorptive effects

In the previous sections we have assumed that the corrections of order \(g^2/m_W^4\) to the real part of the self-energy are not important and in the same spirit we have discarded the imaginary part of the self-energy. This is normally justified since the corrections of order \(g^2/m_W^2\), which are proportional to the particle-antiparticle asymmetries, give the dominant contributions. However, this is not true for a CP-symmetric plasma like the early Universe, and in such situations the real and the imaginary part of the neutrino potential become comparable [20]. Our purpose now is to complement the formalism by examining the effects induced when the imaginary part of \(\Sigma_{eff}\) is not negligible.

The damping effects, which are due to the incoherent interactions of the neutrinos with the particles in the background, are described through an antihermitian part of the Hamiltonian that governs the evolution of the flavor amplitudes. In our treatment, the damping terms in the Hamiltonian are determined from the imaginary part of \(\Sigma_{eff}\) in a way that we now explain.

We have learned that the complete Hamiltonian for neutrinos is

\[
H = \kappa + \frac{m^2}{2\kappa} + b(\kappa, \kappa),
\]

where \(b(\omega, \kappa)\) stands for the coefficient of \(\hat{\psi}L\psi\) in the decomposition of \(\Sigma_{eff}\) in Eq. (2). In terms of the vector \(n^\mu\) defined in Eq. (72), we have

\[
b(\kappa, \kappa) = \frac{1}{2} \left( \text{Tr} L\frac{\partial}{\partial \varphi} \Sigma_{eff} \right)_{\omega = \kappa}
\]
\[
\frac{1}{2\kappa} \bar{\pi}^0_L \Sigma_{\text{eff}}(\kappa, \vec{\kappa}) u^0_L, \tag{102}
\]
where, in the second equality, we have introduced the massless vacuum Dirac spinors \( u^0_L \) normalized such that
\[
u^0_L \bar{\nu}^0_L = \kappa L \phi. \tag{103}
\]
If \( \Sigma_{\text{eff}} \) develops an absorptive part, so that
\[
\gamma^0 \Sigma_{\text{eff}}^\dagger \gamma^0 \neq \Sigma_{\text{eff}}, \tag{104}
\]
the matrix \( b \) develops a non-hermitian term. Thus, if we write
\[
\Sigma_{\text{eff}} = \Sigma_r + i\Sigma_i \tag{105}
\]
\[
b = b_r + ib_i \tag{106}
\]
where \( b_{r,i} \) are hermitian matrices and
\[
\Sigma_r = \frac{1}{2}(\Sigma_{\text{eff}} + \gamma^0 \Sigma_{\text{eff}}^\dagger \gamma^0),
\]
\[
\Sigma_i = \frac{1}{2i}(\Sigma_{\text{eff}} - \gamma^0 \Sigma_{\text{eff}}^\dagger \gamma^0) \tag{107}
\]
are the dispersive and absorptive part of the self-energy, then
\[
b_i = \frac{1}{2} \left( \text{Tr} L \phi \Sigma_i \right)_{\omega = \kappa}
\]
\[
= \frac{1}{2\kappa} \bar{\pi}^0_L \Sigma_i(\kappa, \vec{\kappa}) u^0_L. \tag{108}
\]
This formula allows us to calculate the absorptive terms in the evolution equation for the amplitudes \( \chi^{(a)} \), from an explicit calculation of the absorptive part of the self-energy. However, using a very general argument, we will now show that the matrix \( b_i \) is simply related to the total rates for decay and inverse decays of the neutrinos.

The starting point of our discussion is the formal definition of the elements of the self-energy matrix in the real-time formulation of FTFT
\[
i \Sigma_{21}(z - y)_{a\beta,ij} = -\langle \eta_{a,i}(z) \bar{\eta}_{\beta,j}(y) \rangle,
\]
\[
i \Sigma_{12}(z - y)_{a\beta,ij} = \langle \bar{\eta}_{\beta,j}(y) \eta_{a,i}(z) \rangle,
\]
\[
-\Sigma_{11}(z - y) = \Sigma_{21}(z - y) \theta(z^0 - y^0) + \Sigma_{12}(z - y) \theta(y^0 - z^0),
\]
\[
-\Sigma_{22}(z - y) = \Sigma_{21}(z - y) \theta(y^0 - z^0) + \Sigma_{12}(z - y) \theta(z^0 - y^0). \tag{109}
\]
where \( \alpha, \beta \) are Dirac indices and \( i, j \) are indices labeling the neutrino mass eigenfields. \( \eta_i \) and \( \overline{\eta}_i \) are the neutrino source fields, and in terms of them the interaction Lagrangian is

\[
L_{\text{int}} = \overline{\nu}_L \eta + \nu_L \eta.
\]  

In terms of the elements \( \Sigma_{ab} \) the physical self-energy is given by\[21\]

\[
\Sigma_{\text{eff}} = \Sigma_{11} + \Sigma_{12},
\] (111)

which from Eq. (109) is immediately recognized to be the retarded self-energy. Decomposing \( \Sigma_{11} \) in analogy to Eqs. (105) and (107), Eq. (111) is equivalent to

\[
\begin{align*}
\Sigma_r(k) &= \Sigma_{11r}, \\
\Sigma_i(k) &= \Sigma_{11i} - i\Sigma_{12}(k).
\end{align*}
\] (112)

On the other hand, using the integral representation of the step function, it follows from Eq. (109)

\[
\Sigma_{11i}(k) = \frac{i}{2} [\Sigma_{21}(k) + \Sigma_{12}(k)],
\] (113)

which implies that \( \Sigma_i \), determined from Eq. (112), can be equivalently computed using

\[
\Sigma_i(k) = \frac{i}{2} [\Sigma_{21}(k) - \Sigma_{12}(k)].
\] (114)

Then from Eq. (108)

\[
b_i = \frac{i}{4k} \overline{u}_L^0 [\Sigma_{21}(k) - \Sigma_{12}(k)] u_L^0,
\] (115)

which is a useful formula because the matrix elements of \( \Sigma_{21} \) and \( \Sigma_{12} \) are related to the rates for emission and absorption of the neutrino. To see this, we insert a complete set of states between the field operators \( \eta \) and \( \overline{\eta} \) in Eq. (109), giving

\[
i \Sigma_{21}(z - y)_{\alpha\beta} = -\frac{1}{Z} \sum_{n,m} e^{-i(q_m - q_n) \cdot (z - y)} \times \\
\langle n | \eta(0) | m \rangle \langle m | \overline{\eta}(0) | n \rangle Z_n,
\] (116)
where $Z$ is defined in Eq. (68),

$$Z_n = \langle n | \rho | n \rangle = e^{-\beta q_n \cdot u + \alpha_n},$$

and $\alpha_n$ is the eigenvalue of the operator $\hat{\alpha} = \sum_A \alpha_A Q_A$ corresponding to the state $|n\rangle$; i.e., $\hat{\alpha}|n\rangle = \alpha_n|n\rangle$. From Eq. (116) we immediately obtain

$$i\vec{\mu}_L \Sigma_{21}(\kappa, \vec{\kappa}) u^0_L = -2\kappa \Gamma_D$$

where

$$\Gamma_D = \frac{1}{Z} \frac{1}{2\kappa} \sum_{n,m} \langle n | \vec{\mu}_L^0 \eta(0) | m \rangle \langle m | \overline{\eta}(0) u^0_L | n \rangle (2\pi)^4 \delta^{(4)}(k + q_n - q_m) Z_n. \quad (119)$$

In particular, the diagonal elements $(\Gamma_D)_{ii}$ are the total rates for the processes $n + \nu_i \rightarrow m$, averaged over all possible initial states. The off-diagonal elements have a similar structure but they involve the product of the amplitudes for two different neutrino species. In similar fashion,

$$i\vec{\mu}_L^0 \Sigma_{12}(\kappa, \vec{\kappa}) u^0_L = 2\kappa \Gamma_I$$

where $\Gamma_I$ is the matrix

$$\Gamma_I = \frac{1}{Z} \frac{1}{2\kappa} \sum_{n,m} \langle n | \vec{\mu}_L^0 \eta(0) | m \rangle \langle m | \overline{\eta}(0) u^0_L | n \rangle (2\pi)^4 \delta^{(4)}(k + q_n - q_m) Z_m, \quad (121)$$

whose diagonal elements are the total rates for the inverse processes $m \rightarrow n + \nu_i$, averaged over the initial states. Then using the results of Eqs. (118) and (120) in Eq. (113) we finally arrive at

$$b_i = -\frac{\Gamma}{2}, \quad (122)$$

with

$$\Gamma = \Gamma_D + \Gamma_I. \quad (123)$$

In situations of exact equilibrium, the $\nu_{Li}$ share the same chemical potential $\alpha_\nu$. In that case, since $\eta$ has the same quantum numbers as the neutrino field, then

$$\langle n | \overline{\eta}(p) \eta(0) | m \rangle \neq 0$$
only for those states such that $\alpha_m = \alpha_n + \alpha_\nu$. Thus, in Eq. (121) we can replace

$$Z_m = e^{-\beta \kappa + \alpha_\nu} Z_n,$$

which yields

$$\Gamma = (1 + e^{-x}) \Gamma_D,$$

with $x = \beta \kappa - \alpha_\nu$. However in situations that do not correspond to exact equilibrium the formula in Eq. (123) is the appropriate one.

Then in summary, if we denote by $H_r$ the energy matrix displayed in Eq. (87)

$$H_r = H^{(\text{matter})} + H^{(\nu \nu)} + H'^{(\nu \nu)},$$

then the complete Hamiltonian describing the attenuation effects induced by the incoherent neutrino interactions is

$$H = H_r - i \frac{\Gamma}{2}. \quad (127)$$

We remark that, since the total probability is not conserved any more, the replacements

$$|\chi^{(\alpha, \pi)}_{\mu}|^2 = 1 - |\chi^{(\alpha, \pi)}_{e}|^2$$

made in Eq. (91) are not valid, and the correct formulas for $h_{ee}$ and $h_{\mu\mu}$ are those given in Eq. (89) in this case.

For the antineutrinos, the expression equivalent to Eq. (101) is

$$\overline{H} = \kappa + \frac{m^2}{2\kappa} - b^* (-\kappa, -\overline{\kappa}), \quad (128)$$

which determines the evolution of the amplitudes $\chi^{(\overline{\pi})}$. Since

$$b^* (-\kappa, -\overline{\kappa}) = \frac{1}{2} \left( \text{Tr} \ L \bar{\Pi} \Sigma_{eff} (-\kappa, -\overline{\kappa}) \right)$$

$$= \frac{1}{2\kappa} \bar{\Pi}_L \Sigma_{eff} (-\kappa, -\overline{\kappa}) u_L^0, \quad (129)$$

it follows that the complete Hamiltonian becomes

$$\overline{H} = \overline{H}_r - i \frac{\Gamma}{2}, \quad (130)$$
where
\[
\Pi_r = \Pi^{\text{matter}} + \Pi^{(\nu\nu)} + \Pi^{(\nu\nu)},
\]
\[
\Gamma = \Gamma_D + \Gamma_I.
\]  
(131)

\(\Gamma_{D,I}\) are given by the same formulas for \(\Gamma_{D,I}\) in Eqs. (119) and (121), but with the substitution \(k \rightarrow -k\) in the delta function. By crossing then, it follows that the diagonal elements of \(\Gamma_D\) and \(\Gamma_I\) are the rates for the processes \(n \rightarrow m + \nu_i\) and \(m + \bar{\nu}_i \rightarrow n\), respectively [22].

There is one final comment we wish to make. The precise identification that we have made between the anti-hermitian part of the Hamiltonian and the total rate for decays and inverse decays is intuitively appealing and useful for practical purposes. While ordinarily that allows us to determine the anti-hermitian part of the Hamiltonian by computing the relevant rates, there is one situation in which it does not. If the background contains neutrinos, the rates for processes involving the background neutrinos depends on their number densities which, as already remarked, are modulated by the oscillation mechanism. It is then not obvious how to calculate the rates for such processes by the traditional way of calculating transition probability amplitudes. However, as a byproduct of our formalism we obtain the following very precise procedure. We calculate the 21 and 12 elements of the self-energy matrix using the FTFT Feynman rules and then from Eqs. (119) and (121) obtain \(\Gamma_{D,I}\). The dependence on the neutrino number densities shows up in the diagrams in which background neutrinos propagate in the internal lines of the loops. Thus, by using in those diagrams the same effective neutrino propagator that was introduced in Section 3 to calculate the effects of the neutrino background on the dispersive terms, the effect of the modulation of the number densities by the oscillating amplitudes is properly taken into account in this case also.
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[19] The formulas given in Eq. (57) are equivalent to the ones of Ref. [14], which were written by adapting the general (density-matrix) approach of Ref. [12] to the problem of neutrino propagation and transformations in a supernova.

[20] Within the framework of the FTFT, the real contributions of order $g^2/m_W^4$ arise from the momentum-dependent terms of the boson propagators in the neutrino self-energy diagrams [17]. In order to calculate them for a neutrino background, we have to go back to Eq. (63) and keep the next terms in the momentum expansion of the $Z$-boson propagator.

[21] A careful discussion of the justification and implications of this definition is given by J. C. D’Olivo and J. F. Nieves, “Coulomb and Covariant Gauges in Finite Temperature QED”, preprint LTP-043-UPR, July 1994.

[22] Notice that for massless particles the vacuum spinors $u^0_L$ and $v^0_L$ coincide.
Figure Captions

Fig. 1. Neutrino background contribution to the self-energy.
This figure "fig1-1.png" is available in "png" format from:

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