SPHERICAL FUNCTIONS ON SPHERICAL VARIETIES

YIANNIS SAKELLARIDIS

Abstract. Let $X = H\backslash G$ be a spherical variety for a split reductive group $G$ over the integers $\mathfrak{o}$ of a $p$-adic field $k$, and $K = G(\mathfrak{o})$ a hyperspecial maximal compact subgroup of $G = G(k)$. We compute eigenfunctions ("spherical functions") on $X = X(k)$ under the action of the unramified (or spherical) Hecke algebra of $G$, generalizing many classical results of "Casselman-Shalika" type. Under some additional assumptions on $X$ we also prove a variant of the formula which involves a certain $L$-value, and we present several applications such as: (1) a statement on "good test vectors" (namely, that an $H$-invariant functional on an irreducible unramified representation $\pi$ is always non-zero on $\pi^K$), (2) the unramified Plancherel formula for $X$, including a formula for the "Tamagawa measure" of $X(\mathfrak{o})$, and (3) a computation of the most continuous part of $H$-period integrals of principal Eisenstein series.

Contents

1. Introduction 2

Part 1. The sum formula
2. Invariants and orbits 7
3. Intertwining operators 7
4. Construction of a $K$-invariant vector 10
5. Functional equations 23

Part 2. The formula with $L$-values
6. Simple spherical reflections 38
7. The formula 38

Part 3. Applications
8. The Hecke module structure, multiplicity one and good test vectors. 60
9. Unramified Plancherel formula 61
10. Periods of Eisenstein series. 64
References 68

2000 Mathematics Subject Classification. 22E50 (Primary); 11S40, 43A90, 14M17 (Secondary).

Key words and phrases. Spherical varieties, distinguished representations, spherical functions, Plancherel formula, period integrals.
1. Introduction

1.1. The problem. Let $G$ be a split reductive group over the ring of integers $\mathfrak{o}$ of a local non-archimedean field $k$ in characteristic zero (with residual degree $q$, a power of the prime number $p$) and let $X$ be a homogeneous spherical scheme for $G$ over $\mathfrak{o}$ (i.e. a $G$-scheme on which the Borel subgroup $B$ has an open orbit – this includes, but is not limited to, symmetric spaces). Then $X = H \backslash G$, where $H$ is the stabilizer of an $\mathfrak{o}$-point $x_0$ in the open Borel orbit ($HB$ is open in $G$). We make throughout certain assumptions on $X$ (cf. §1.7), including that $X$ is quasi-affine, which cause no serious harm to generality. We denote by regular font the corresponding sets of $k$-points of each of the groups. We also denote $K = G(\mathfrak{o})$ – it is a (hyperspecial) maximal compact subgroup. To avoid complicated formulas, we will assume that $k$ is unramified over the field $\mathbb{Q}_p$; the modifications needed to remove this assumption are trivial.

We consider the right regular representation of $G$ on $V = C^\infty(X)$ (if $G$ has a unique orbit on $X$, this is just the induced representation $\text{Ind}_H^G(1)$). The unramified Hecke algebra $\mathcal{H}(G,K)$ acts on $V$, and the goal of this article is to compute an explicit formula for its eigenvectors, expressed in terms of the geometry of $X$. More generally, if $H$ contains the unipotent radical $U_P$ of a proper parabolic, $\Lambda : U_P \to \mathbb{G}_a$ is a homomorphism which is fixed under $H$-conjugation and $\psi : k \to \mathbb{C}^\times$ is a character, then we can also consider the space $V = C^\infty(X, L_\Psi)$, where $\Psi = \psi \circ \Lambda$ considered as a character of $H$ and $L_\Psi$ is the complex line bundle defined by it. (This space is just $\text{Ind}_H^G(\Psi)$ if $G$ has unique orbit on $X$.) For an explanation of how this case can be understood geometrically, cf. [Sa08, §5.5]. Though most of the results carry over verbatim to this “twisted” case, for simplicity we do not discuss it in the introduction.

This is a problem of both harmonic-analytic and arithmetic interest, and there is a long history of particular examples which have been computed [Ma71, Ca80, CS80, HS80, Hi99, Hi05, KMS03, Of04, Sa06], to mention just a few. In the first part of this paper, we compute a very general formula, covering all previously established cases (when the group $G$ is split) and many more. Besides its case-specific arithmetic applications, which have been the motivation for most of the literature, the computation of such a formula is relevant to the $H$-period integrals of principal Eisenstein series (which we undertake in §10), and conjecturally also to the $H$-period integrals of other automorphic forms, via the local Plancherel formula (which we develop in §9). These applications, presented in the third part of the paper, require an improved version of the general formula, which we develop in the second part and which, in particular, involves a certain local $L$-value of the unramified representation in question.

Before we proceed to a more detailed description of the results, let us put this work in a more general context, under the perspective of automorphic forms. The study of period integrals of automorphic forms (a major stream of which is related to the relative trace formula of Jacquet – see [Lap06] for a presentation) has revealed relationships between the non-vanishing of certain period integrals of automorphic forms (typically, over spherical subgroups) and functorial lifts, on one hand, and between the values of these period integrals and $L$-functions or special values thereof.

---

1In the literature, the term “spherical” is often used for “unramified”, but to avoid confusion I will reserve this term for the notion of spherical varieties.
on the other. Though no general theory or conjectures exist to describe these phenomena, it has recently started to become clear that a general and systematic approach should be possible. The “dual group” attached by Gaitsgory and Nadler [GN], in the context of the Geometric Langlands program, to any spherical variety $X$ was shown in [Sa08] to be related to unramified representations in the spectrum of $X$ and is, more generally, conjectured in [SV] to be describing $X$-distinguished representations, in a certain sense. Moreover, under certain assumptions period integrals are (roughly speaking) conjectured to be “Eulerian” with local Euler factors equal to “the $H$-invariant functionals which appear in the local Plancherel formula for $X$”. The last step is to relate these $H$-invariant functionals to special values of $L$-functions, and this is part of what we accomplish here, when the local group is split and the representation is unramified.

1.2. The general formula. To formulate the general formula, we will for simplicity assume in the introduction that $B = B(k)$ has a unique open orbit on $X$. This includes, in particular, the multiplicity-free case, i.e. the case when the eigenspaces for the unramified Hecke algebra are one-dimensional. Then orbits of $K$ on $X$ are in bijection $x_\lambda \leftrightarrow \lambda$ with anti-dominant, in a suitable sense, cocharacters into $A_X$, where $A_X$ is a quotient of a maximal torus $A$ of $G$ identified with the $A$-orbit of a chosen point $x_0 \in X(o)$.

Let $A^*$ denote the “Langlands dual” complex torus of $A$ (the torus of unramified complex characters of $A$) and let $A^*_X$ denote the “Langlands dual” of $A_X$. Naturally (under the assumptions of the introduction) $A^*_X \subset A^*$. There is a surjective map $A^* \to \text{spec}_M \mathcal{H}(G, K)$ with generic fiber finite of order $|W|$, where $W$ is the Weyl group of $G$. The support of $V^K$ as a module for $\mathcal{H}(G, K)$ coincides with the image of a translate $\delta^\natural_{(X)} A^*_X$ of $A^*_X$, see [Sa08]. The torus $A^*_X$ (and its translate $\delta^\natural_{(X)} A^*_X$) carries a natural action of a finite reflection group $W_X \subset W$, called the “little Weyl group” of $X$, and the map $\delta^\natural_{(X)} A^*_X \to \text{spec}_M \mathcal{H}(G, K)$ factors through the topological quotient $\delta^\natural_{(X)} A^*_X / W_X$. We can now formulate our general formula:

1.2.1. Theorem (cf. Theorems 4.2.2, 5.2.1, see also (6.2)). For almost all points $\eta \in \text{spec}_M \mathcal{H}(G, K)$ in the support of $V^K$, the corresponding eigenspace in $V^K$ admits a basis consisting of the functions:

\[
\Omega_{\delta^\natural_{(X)} A^*_X}(x_\lambda) = e^{-\lambda(\delta^\natural_{P(X)} \chi)} \sum_{w \in W_X} B_w(\chi) e^{\lambda(w \chi)}
\]

where: $\delta^\natural_{(X)} \chi \in \delta^\natural_{(X)} A^*_X$ ranges over a set of representatives for the elements of $\delta^\natural_{(X)} A^*_X / W_X$ which map to $\eta \in \text{spec}_M \mathcal{H}(G, K)$ and the coefficients $B_w(\chi)$ are certain cocycles: $W_X \to C(A^*_X)$ for the computation of which we give an explicit algorithm in terms of the geometry of $X$. For the definition of $\delta_{(X)}, \delta_{P(X)}$ cf. §2.1.

The cocycles $B_w$ have the following conceptual meaning: It was proven in [Sa08] that certain natural morphisms: $S_\chi : C_c^\infty(X) \to I(\chi)$ satisfy proportionality relations: $T_w \circ S_\chi \sim S_{w \chi}$ for all $w \in W_X$, where $I(\chi) = \text{Ind}^G_B(\chi \delta^\natural)$ is an unramified principal series representation and $T_w$ are intertwining operators between principal series. Then the $B_w$ are a standard term times the coefficients of proportionality.

\footnote{Part of what we show in the third part – under some additional assumptions – is that this is equivalent to the weaker condition that generic eigenspaces be one-dimensional.}
when \( T_w \) are normalized in a “good” way. This “good” normalization comes from the theory of equivariant Fourier transform on the basic affine space \( U \backslash G \) [BK98], which we review in §3.8. Use of the equivariant Fourier transform allows us to avoid complicated formulas and mysterious cancellations which are abundant in the relevant literature.

The method employed here is based on the basic idea of Casselman and Shalika [Ca80, CS80], namely that instead of computing an unramified eigenfunction one should first compute an Iwahori-invariant function coming from a vector of small support in the principal series, and then use the fact that the space of Iwahori-invariant vectors has dimension equal to the number of intertwining operators between principal series. The representation-theoretic results of [Sa08] allow us to reduce the computation to the case of \( SL_2 \), and there is only a small number of possibilities to consider there.

### 1.3. The formula with \( L \)-values.

Because of its inductive, algorithmic definition, the general formula may be difficult to use in applications. Therefore, in the second part of the paper we transform it (under additional assumptions, which are in particular satisfied by many affine homogeneous spherical varieties) to a more useful one, expressed in terms of the “dual group” \( \tilde{G}_X \) of \( X \) – more precisely the root datum of this group (cf. §6.1). This should be, conjecturally, isogenous to the subgroup \( G \) attached to \( X \) by Gaitsgory and Nadler [GN].

#### 1.3.1. Theorem (cf. Theorem 7.2.1).

(Under additional assumptions.) We have:

\[
\Omega_{\delta_{\chi}^\beta}^{\tilde{\delta}_{\chi}^\beta}(x_\lambda) = \delta_{P(X)}^{-\frac{1}{2}}(x_\lambda) \prod_{\Theta^+}(1 - \sigma_\theta q^{-r_\theta} s_\lambda(\chi))
\]

where \( s_\lambda = \sum_{w \in W_{\chi}} (-1)^w e^{-w_\beta + w_\lambda} \prod_{\gamma > 0}(1 - e^{\gamma})^{-1} \) is the Schur polynomial indexed by lowest weight (that is, if \( \lambda \) is anti-dominant then \( s_\lambda \) is the character of the irreducible representation of \( \tilde{G}_X \) with lowest weight \( \lambda \)) and \( T_\theta \) denotes the formal operator: \( T_\theta s_\lambda = s_{\theta + \lambda} \).

Here

\[
\beta(\chi) := \prod_{\gamma \in \Phi_X^+(1 - e^{\gamma})} \prod_{(\hat{\theta}, \sigma_\theta, r_\theta) \in \Theta}(1 - \sigma_\theta q^{-r_\theta} e^{\theta}(\chi))^{-1},
\]

and the triples \( (\hat{\theta}, \sigma_\theta, r_\theta) \in \Theta^+ \) consisting of a weight of \( A_X^* \), a sign and a positive half-integer are given explicitly in terms of the geometry of \( X \).

If \( X \) is affine then, in particular:

\[
\Omega_{\delta_{\chi}^\beta}^{\tilde{\delta}_{\chi}^\beta}(x_0) = c\beta(\chi)
\]

with the constant:

\[
c = \beta(\delta_{P(X)}^{\tilde{\delta}_{P(X)}^{-1}}).
\]

Notice that the value of \( \Omega_{\delta_{\chi}^\beta}^{\tilde{\delta}_{\chi}^\beta} \) at \( x_0 \) is equal to “half” a local \( \Omega \)-value for \( \tilde{G}_X \), more precisely \( L_X := c^2 \beta(\chi)\beta(\chi^{-1}) \) is a quotient of \( \Omega \)-values for \( \tilde{G}_X \). As we will explain below, this \( \Omega \)-value is related to the Plancherel measure for the unramified spectrum of \( X \), and should (conjecturally) be related to \( H \)-period integrals of automorphic forms. It would certainly be desirable to have a more natural, geometric
understanding of this $L$-value, rather than the combinatorial one which we provide here.

The weights $\theta$ which appear in this formula are related to the colors of the spherical variety, in other words the irreducible $B$-invariant divisors on $X$. Each of them induces a $B$-invariant valuation on $k(X)$ which, by restriction to $B$-eigenfunctions, defines a weight of $A^*_X$. Roughly speaking, in the affine case the weights $\theta$ contained in the above formula are those $W_X$-translates of the weights defined by colors which are contained in the positive span of colors.

1.4. Representation-theoretic results. A benefit of the last formula is that it allows us to draw certain representation-theoretic conclusions, in combination with the results of [Sa08]. More precisely:

1.4.1. Theorem (Multiplicity-freeness, “good test vectors”, cf. §8.). (Under the assumptions of the Theorem 1.3.1, with $X$ affine.) Assume that for a generic point in the support of $V^K$ the corresponding eigenspace is one-dimensional. Then $V^K$ is a principal module over $\mathcal{H}(G,K)$, generated by the characteristic function of $X(\mathfrak{o})$. In particular, if $\pi$ is an irreducible, unramified representation of $G$ and $L$ is a non-zero $H$-invariant functional on $\pi$, then $L$ is nonzero on $\pi^K$.

The last statement in the theorem gives a conceptual answer to non-vanishing questions related to Eulerian period integrals of automorphic forms; namely, it proves that if $\pi = \otimes_v \pi_v$ is an irreducible representation of the points of a group $G$ over the adele ring $A$ of a global field $F$, and $H$ is a spherical subgroup over $F$ such that $(H \backslash G)_F$, satisfies the assumptions of the above theorem for almost every place $v$, then $\text{Hom}_{H_v}(\pi_v, 1) \neq 0$ for every place $v$ implies $\text{Hom}_{H(\mathcal{A}_F)}(\pi, 1) \neq 0$.

1.5. Unramified Plancherel formula. Another application which we present is the Plancherel formula for the unramified part of the spectrum of $X$, i.e. for $L^2(X)^K$. More precisely, keeping the assumptions of Theorem 1.3.1 with $X$ affine, we normalize the invariant measure on $X$ such that $\text{Vol}(x_0J) = 1$, where $J$ denotes the Iwahori subgroup of $G$. We denote by $A^*_X$ the maximal compact subgroup of $A^*_X$ (the subgroup of unitary characters.) Then the Plancherel formula restricted to $C^\infty_c(X)^K$ reads:

1.5.1. Theorem (cf. Theorem 9.0.1). For every $\Phi \in C^\infty_c(X)^K$ we have:

$$\|\Phi\|^2 = \frac{1}{Q \cdot |W_X|} \iint_{A^*_X} \left| \left< \Phi, \Omega_{\chi}^{\pi} \right> \right|^2 d\chi, \quad \|\Phi\|^2 = \frac{1}{Q \cdot |W_X|} \iint_{A^*_X} \left| \left< \Phi, \Omega_{\chi}^{\pi} \right> \right|^2 d\chi,$$

where $d\chi$ is probability Haar measure on $A^*_X$ and

$$Q = \frac{\text{Vol}(K)}{\text{Vol}(Jw_1J)} = \prod_{\alpha \in \Phi^+} \frac{1 - q^{-1-\langle \alpha, \rho \rangle}}{1 - q^{-\langle \alpha, \rho \rangle}}.$$

Equivalently, if the eigenfunctions $\Omega_{\chi}^{\pi} \chi$ were re-normalized to have value 1 at $x_0$, then the corresponding Plancherel measure on $\delta_{\chi}(X)^K/W_X$ would be given by $Q^{-1} L_X(\chi) d\chi$.

This theorem also leads to a computation of $\text{Vol}(X(\mathfrak{o}))$ (Theorem 9.0.3). This volume is essentially (up to a factor $(1-q^{-1})^{\text{rk}(A^*_X)}$) the local “Tamagawa” volume, by which we mean the volume with respect to an integral, residually non-vanishing invariant volume form.
1.6. Periods of automorphic forms. Finally, as a direct application of our formula we compute in §10 the “most continuous part” of \( H \)-period integrals of principal Eisenstein series, when \( H \backslash G \) is a spherical subgroup of a split group defined over a global field \( F \) (and locally satisfying the assumptions of Theorem 1.3.1) and show that it is given, essentially, by the “half \( L \)-value” \( L^+_X \) of Theorem 1.3.1. More precisely, we consider a pseudo-Eisenstein series on \( G(F) \backslash G(A_F) \), which can be analyzed as an integral of Eisenstein series \( E(f_X,g) \), and show (for simplicity: if the Eisenstein series are induced from characters trivial on the maximal compact subgroup of \( B(A_F) \)):

1.6.1. Theorem (cf. Theorem 10.0.2). The period integral of:

\[
\sum_{\gamma \in B(F) \backslash G(F)} \Phi(\gamma g) = \int_{\exp(\kappa+i\alpha_\gamma^*)} E(f_X,g) d\chi
\]

over \( H(F) \backslash H(A_F) \) is equal to:

\[
\int_{\exp(\kappa+i\alpha^X)} \left( \frac{L^+_X}{\Delta_{Y,Tam}(X)} \right)^* \sum_{[W/W(\chi)]} \left( \Delta^Y_{\chi,v}(f_{\chi,v}) \right) d\chi
\]

plus terms which depend on the restriction of \( f_X \), as a function of \( \chi \), to a subvariety of smaller dimension.

For the notation, see §10. The same \( L \)-values should show up in the \( H \)-period integral of cusp forms, according to an idea of A. Venkatesh which we formulate as an (almost) precise conjecture in [SV].

1.7. Assumptions and notation. All our schemes, subgroups etc. are over \( \mathfrak{o} \) unless otherwise specified. The group \( G \) is assumed reductive, connected and split over \( \mathfrak{o} \). For simplicity, we also assume that the derived group of \( G \) is simply connected. We fix throughout a Borel subgroup \( B \) with unipotent radical \( U \) and a maximal split torus \( A \subset B \). The choice of \( A \) will be explained in §2.1. The group \( A(\mathfrak{o}) \) will also be denoted by \( A_0 \), and similarly \( B_0 = B(\mathfrak{o}) \), \( U_0 = U(\mathfrak{o}) \). The complex torus of unramified characters of \( A \) is denoted by \( A^* \). The Weyl group is denoted by \( W \), roots are generally denoted by small Greek letters \( \alpha, \beta \) etc. and the corresponding co-roots by \( \check{\alpha}, \check{\beta} \) etc. Similarly, \( \rho \) denotes half the sum of positive roots and \( \check{\rho} \) denotes half the sum of positive co-roots. We will be using exponential notation for characters of tori, i.e. \( e^\alpha \). The real part \( \Re(\chi) \) of a character \( \chi = e^\theta \) of a torus is, by definition, the real part of \( \theta \). We denote by \( \Phi, \Phi^+ \) and \( \Delta \) the sets of roots, positive roots, and simple positive roots, respectively (for our choice of Borel). The parabolic corresponding to a set of simple positive roots \( \{ \alpha, \beta, \ldots \} \) will be denoted by \( P_{\alpha\beta\ldots} \). We denote by \( G_m, G_a \) the multiplicative, resp. additive group and by \( \mathbb{A}^n \) the \( n \)-dimensional affine space, sometimes with an index denoting chosen coordinates for this space. For a subgroup \( M \) we will denote by \( \mathcal{Z}(M) \) its center, by \( \mathcal{N}(M) \) its normalizer, by \( \mathcal{R}(M) \) its radical, by \( U_M \) its unipotent radical and by \( \mathfrak{o}_M \) its modular character (the quotient of a right- by a left-invariant volume form), considered as a character of \( \mathcal{N}(M) \). We denote by \( \delta_M \) the absolute value of \( \mathfrak{o}_M \), and \( \delta_B \) will simply be denoted by \( \delta \). We denote \( K = G(\mathfrak{o}) \), and \( J \) the standard Iwahori subgroup (the inverse image of \( B(F_q) \) under the reduction map \( K \to G(F_q) \)).
We are given a homogeneous spherical scheme $X$ over $\mathfrak{o}$ such that the action map $G \times X \to X$ is a smooth morphism of $\mathfrak{o}$-schemes. We denote by $X$ the open Borel orbit. The quotient of a scheme $V$ (resp. a topological space $V$) by the action of a group scheme $\Gamma$ (resp. a topological group $\Gamma$), whenever it exists (in the sense of geometric quotient for schemes, or topological quotient in the topological setting), will be denoted by $V/\Gamma$ (resp. $V/\Gamma$). Thus, the reader should not be confused by expressions of the form $G/\Gamma$, where $G$ is a group and $\Gamma$ is another group acting on it, though not a subgroup of $G$. We fix throughout a complex character $\psi$ of the additive group $k$, whose conductor is equal to the ring of integers $\mathfrak{o}$; this character will be used for our Fourier transforms, but also when we consider line bundles $\mathcal{L}_\psi$ as described in §1.1.

As mentioned above, $X$ is assumed to be quasi-affine, and such that $X$ admits a $G$-eigenmeasure and $\tilde{X}$ admits a $B$-invariant measure. This can always be achieved by a trivial modification, see [Sa08, §3.8]. We also assume that $X$ satisfies Axioms 2.4.1 and 2.4.2, which is always the case at almost every place if $X$ and $G$ are defined over a global field (with $G$ split).

1.8. Acknowledgements. I would like to thank Joseph Bernstein and Akshay Venkatesh for very useful and motivating discussions. The first part of this work was conducted when I was supported by the Marie Curie Research Training Network in Arithmetic Algebraic Geometry, EU Contract MRTN-CT-2003-504917.

Part 1. The sum formula

2. Invariants and orbits

2.1. Invariants associated to spherical varieties. We start with the description of certain invariants associated to a spherical variety. For a variety $Y$ with a $B$-action, we denote by $k(Y)^{(B)}$ the group of non-zero rational $B$-eigenfunctions on $Y$ and by $\mathcal{X}(Y)$ the group of $B$-eigencharacters on $k(Y)^{(B)}$. If $Y$ has a dense $B$-orbit, then we have a short exact sequence: $0 \to k^* \to k(Y)^{(B)} \to \mathcal{X}(Y) \to 0$. The rank of a $B$-variety $Y$ is, by definition, the rank of $\mathcal{X}(Y)$.

Given a spherical variety $X$ with open Borel orbit $X$, the associated parabolic is the standard parabolic $P(X) := \{ p \in G|X \cdot p = X \}$. We will denote by $\Delta(X)$ the set of simple roots to which it corresponds. We make once and for all a choice of a point $x_0 \in X(\mathfrak{o})$ and let $H$ denote its stabilizer; hence $X = H\backslash G$ and $HB$ is open in $G$. Pick $f \in k[X]$ such that the set-theoretic zero locus of $f$ is $X \setminus X$. Its differential $df$ at $1 \in G$ defines an element in the coadjoint representation of $G$, and the centralizer $L(X)$ of $df$ is a Levi subgroup of $P(X)$. The absolute value of the modular character of $B \cap L(X)$ will be denoted by $\delta_X$. We fix a maximal torus $A \subset B$ with $A \subset L(X)$ and define $A_X$ to be the torus: $A_X := L(X)/(L(X) \cap H)$. We may consider $A_X$ as a subset of $X$ by identifying it with the orbit of $L(X)$ (or $A$) through $x_0$. Let $A_X$ be the coweight lattice of $A_X$; it can be naturally identified with $\text{Hom}(\mathcal{X}(X), \mathbb{Z})$. Let $\Lambda^+_X$ denote the monoid of $G$-invariant ($\mathbb{Z}$-valued, trivial on $k$) valuations on $k(X)$; it can be considered as a submonoid of $\Lambda_X$ by restriction to $k(X)^{(B)}$. (Indeed, no non-trivial $G$-invariant valuation vanishes on $k(X)^{(B)}$, cf. [Kn91, Corollary 1.8].) Let $Q = \text{Hom}(\mathcal{X}(X), \mathbb{Q})$, and let $\mathcal{V} \subset Q$ be the cone spanned by $\Lambda^+_X$ in $Q$. It is known that $\mathcal{V}$ contains the image of the negative Weyl chamber, and in many cases (notably, symmetric varieties) coincides with it [Kn91,
§5]. We have a natural bijection: $\Lambda_X \simeq A_X(k)/A_X(o)$, induced from $\lambda \mapsto e^{\lambda}(\varpi)$. Elements of $A_X$ which are mapped to elements of $A_X^+$ under this bijection will be called “X-anti-dominant”, and the set of those will be denoted by $A_X^+$.  

2.2. Knop’s action. F. Knop has defined an action of the Weyl group of $G$ on the set of Borel orbits of maximal rank (cf. [Kn95]). We review it briefly: If $Y$ is a $B$-orbit and $\alpha$ is a simple root, then $YP_\alpha/R(P_\alpha)$ is a homogeneous spherical variety for $P_\alpha/R(P_\alpha) \simeq PGL_2$, hence has one of the following types: Type $G$ $(PGL_2 \backslash PGL_2)$, type $U$ $(SU \backslash PGL_2$ where $U$ a maximal unipotent subgroup and $S \subset N(U))$, type $T$ $(T \backslash PGL_2$, where $T$ is a non-trivial torus) or type $N$ $(N(T) \backslash PGL_2)$. We say, correspondingly, that the pair $(Y, \alpha)$ is of type $G$, $U$, $T$ or $N$. Since we are working over a non-algebraically closed field, in case $T$ we will distinguish two sub-cases, called “split” and “non-split”, according to whether $T$ has the corresponding property. By definition, the simple reflection $w_\alpha$ fixes the orbit of maximal rank in $YP_\alpha$, unless it is of type $U$, in which case there are two orbits of maximal rank and $w_\alpha$ interchanges them. The action defined this way is transitive on the set of Borel orbits of maximal rank (which includes the open $B$-orbit). The stabilizer of the open orbit is $W_{(X)} := W_X \times WP(X)$, where $W_X$ is a canonical subgroup of $W$ called the “little Weyl group” of $X$. The group $W_{(X)}$ normalizes $\mathcal{X}(X)$ and acts on it through the quotient $W_X$. It is known that the dual action of $W_X$ on $\mathcal{Q}$ is faithful, generated by reflections, and admits the cone $\mathcal{V}$ defined above as a fundamental domain. (More on this action will be recalled in §6.1.)

Since we are also discussing non-trivial line bundles $L_\varphi$ as were described in §1.1, there is also a fifth case, called $(U, \psi)$. For a discussion of how this fits into the same setting (more precisely, into Knop’s extension of his action to non-spherical varieties) we refer the reader to [Sa08, §5.5]. We caution the reader that, while the little Weyl group is still well-defined in this case, it does not coincide with the little Weyl group of the spherical variety. In this case the action of Knop is not transitive on all orbits of maximal rank, but only on some, called “admissible”.

2.3. Brion’s description of simple spherical reflections. Consider the labelled graph $\mathcal{S}$ (to be called “Knop’s graph”) whose vertices are the $B$-orbits of maximal rank, and where between any two vertices corresponding to orbits $Y$ and $Z$ there is an edge labelled by a simple root $\alpha$ if $w_\alpha Y = Z$. Let $Y$ be an orbit of maximal rank. Following Brion [Br01], for every path $\gamma$ from $Y$ to $X$ we denote by $w(\gamma)$ the Weyl group element corresponding to the path, having the property that $w(\gamma)Y = X$, and by $W(Y)$ the set of all $w(\gamma)$, where $\gamma$ is a path between $Y$ and $X$ of length equal to the codimension of $Y$. Let also $\mathcal{S}(Y)$ denote the set of such paths. Then we have:

2.3.1. Proposition ([Br01], Propositions 2 and 4). Given $w, w' \in W(Y)$ there is a series $w = w_0, w_1, \ldots, w_n = w'$ in $W(Y)$ such that any consecutive $w_i, w_{i+1}$ can be written as $w_i = uw_\alpha v, w_{i+1} = uw_\beta v$ with $l(w_i) = l(w_{i+1}) = l(u) + l(v) + 1$ and $\alpha, \beta$ two mutually orthogonal simple roots.

Based on this, Brion proves:

2.3.2. Theorem ([Br01], Theorem 4). A set of generators of $W_{(X)}$ consists of the elements $w_\alpha, \alpha \in \Delta(X)$, and elements $w$ with the following property: There is a decomposition $w = w_1^{-1}w_2w_1$ such that:
• $w_1 \hat{X} = Y$ with $\text{codim}(Y) = l(w_1)$ (i.e. $w_1^{-1} \in W(Y)$.)

• $w_2$ is either of the following two:
  1. equal to $w_\alpha$, where $\alpha$ is a simple root such that $(Y, \alpha)$ is of type $T$ or $N$, or
  2. equal to $w_\alpha w_\beta$, where $\alpha, \beta$ are two orthogonal simple roots which both lower $Y$ to the same orbit $Y'$.

(This description does not apply to the case of a non-trivial $\mathcal{L}_\Psi$ and the extension of Knop’s action there.)

It follows from a theorem that we will recall later (Theorem 3.7.1) that the generators $w$ of this theorem which are not of the form $w_\alpha, \alpha \in \Delta(X)$ all belong to the little Weyl group $W_X$. We will also see in §6.1 that the “canonical” generators for $W_X$, the simple reflections corresponding to “spherical roots”, admit a description of this form.

2.4. Hyperspecial and Iwahori orbits. As mentioned, we assume that $G$ possesses a smooth model over $\mathfrak{o}$, so that $K = G(\mathfrak{o})$ is a hyperspecial maximal compact subgroup. We also set $J$ for the Iwahori subgroup of $K$, that is, the inverse image of the Borel subgroup under the map $K \to G(F_q)$. We will assume throughout the following axioms pertaining to $K$- and $J$-orbits on $X$.

2.4.1. Axiom. The set $A^+_X$ contains a complete set of representatives for $K$-orbits on $X$; elements of $A^+_X$ which map to distinct elements of $\Lambda^+_X$ belong to different $K$-orbits.

2.4.2. Axiom. For $x \in A^+_X$ we have $xJ \subset xB(\mathfrak{o})$.

The first axiom generalizes the Iwasawa (for $X = U \setminus G$) and Cartan (for $X = G'$, $G = G' \times G'$) decompositions. It was proven by Luna and Vust [LV83] in the case $\mathfrak{o} = \mathbb{C}[[t]]$. An alternate proof was given by Gaitsgory and Nadler in [GN], which can be adapted, with minor modifications, to the $p$-adic case. The adaptation requires the existence and a certain property, over $\mathfrak{o}$, of certain compactifications of the spherical variety which, if the spherical variety is defined and the group is split over a global field, are satisfied at almost every place. The second axiom also follows from the existence and local structure of such compactifications. I hope to include a complete presentation of this in future work; for now, the reader may take the above as axioms and verify them in every particular case. In the case of a symmetric variety, alternate proofs of similar statements have recently been presented by Benoist and Oh [BO07], Delorme and Sécherre [DS].

Remark. From the first axiom it follows that we have surjective maps: $A^+_X/A_0 \to X/K \to \Lambda^+_X$, but in general, unlike the case $\mathfrak{o} = \mathbb{C}[[t]]$, these maps are not bijective. An example is: $G = \mathbb{G}_m$, $X = \{ \pm 1 \} \setminus G$ where the first map is bijective but not the second (since the residue field has non-trivial square classes), and another is $X = \mathcal{O}_2 \setminus \mathbb{GL}_2$, the space of non-degenerate quadratic forms (assume that the residue characteristic is not two), where the first map is also not bijective since the set $A_X(\mathfrak{o})/A(\mathfrak{o})$ has four elements, but there are only two classes of non-degenerate quadratic forms over the residue field, and hence these four elements are contained in only two orbits of $K$ on $X$. (We certainly expect that the second map is bijective when $H$ is connected.)
3. INTERTWINING OPERATORS

3.1. Overview. The goal of this paper is to compute eigenfunctions of the Hecke algebra on $C^\infty(X)$ or, more generally, on $C^\infty(X, L_\Psi)$, when $L_\Psi$ is as in §1.1. We let $I(\chi) = \text{Ind}_{B}^{G}(\chi \delta^\sharp)$, a principal series representation. Then a Hecke eigenfunction on $C^\infty(X)$ is the image of a $K$-invariant vector $\phi_{K,\chi} \in I(\chi)$ via an intertwining operator $I(\chi) \to C^\infty(X)$ (for some $\chi$).

The details of the present section are quite technical, therefore we give here an overview of its contents. We will introduce a $G$-eigenmeasure $|\omega_X|$ on $X$ to set up a duality:

$$C^\infty_c(X) \otimes \nu \otimes C^\infty(X) \to \mathbb{C}$$

(3.1)

(where $\nu$ is the character of $|\omega_X|$). As notation suggests, $\omega_X$ is the absolute value of a volume eigen-form on $X$, the eigencharacter of which will be denoted by $n$ (hence $\nu = |n|$).

For every $B$-orbit $Y$ on $X$ (assuming for now that $Y = Y(k)$ is a single $B$-orbit) we will introduce morphisms:

$$S_Y^X : C^\infty(X) \to I(\chi)$$

given (in some domain of convergence for $\chi$) by the formula:

$$S_Y^X(\phi)(1) = \int_Y \phi \mu_Y^X$$

where $\mu_Y^X$ is a suitable $B$-eigenmeasure on $Y$. For $Y = \tilde{X}$ we will sometimes omit the exponent $Y$ from the notation.

We view $I(\chi)$ as a subspace of the space of smooth, tempered generalized functions on $U \setminus G$, and through the duality (3.1) we have an adjoint for $S_{Y-1,\nu-1}^X$ (again, by fixing an invariant measure on $U \setminus G$):

$$\Delta_Y^X : \mathcal{S}(U \setminus G) \to C^\infty(X).$$

The space $\mathcal{S}(U \setminus G)$ is a larger space than $C^\infty_c(U \setminus G)$ suitable for the theory of Fourier transforms, called the Schwartz space of $U \setminus G^{\text{aff}}$. (The exponent aff denotes the affine closure of $U \setminus G$; however we will allow ourselves to abuse language and notation, write $\mathcal{S}(U \setminus G)$ and say “Schwartz space of $U \setminus G$”.)

We will show that $\Delta_Y^X$ can be expressed (in a suitable domain of convergence for $\chi$) by a formula:

$$\Delta_Y^X(\Phi)(\xi) = \int_{U \setminus G} \Phi \mu_Y^X$$

where the point of evaluation $\xi$ belongs to $Y$ and $\mu_Y^X$ is a suitable $A \times G_\xi$-eigenmeasure on $U \setminus G$. (Notice that $U \setminus G$ carries a natural action of $A \times G$.)

While our goal is to compute $\Delta_Y^X(\Phi_K)$, with $\Phi_K \in \mathcal{S}(X)^K$, this cannot be done directly. Rather, we first compute $\Delta_Y^X(\Phi_J)$, where $\Phi_J$ is a suitable function of small support in $\mathcal{S}(U \setminus G)^J$, and explain the steps needed to deduce from this

---

3The case of $\mathcal{L}_\Psi$ will generally be suppressed from our notation, except when it needs to be treated separately. Unless otherwise stated, all our statements hold – with obvious modifications – in that case, as well.
the computation of $\Delta^Y_\chi(\Phi_K)$. These steps will be undertaken in the following two
sections.

The biggest part of this section is devoted in choosing eigenmeasures in compatible ways and rigorously proving the formula for $\Delta^Y_\chi$. A major complication that arises is that $Y = Y(k)$ will, in general, consist of several $B$-orbits – in this case the morphisms $\Delta^Y_\chi$ are not uniquely defined, and we must instead introduce variants, denoted $\Delta^Y_\chi$. These are defined by suitable eigenmeasures on $Y$, and the correct way to choose eigenmeasures is that these be absolute values of differential
eigenforms. Then we need to compare differential forms on different $B$-orbits with each other, or with differential forms on $U \backslash G$. The most technical parts of this section can be skipped at first reading.

3.2. Morphisms. Let $Y$ be a $B$-orbit on $X$ with a distinguished $k$-point $y_0$ (we will later
discuss how to choose $y_0$), and let $A_Y$ denote the quotient of $A$ by the stabilizer
modulo $U$ of $y_0$. Fix a top-degree $B$-eigenform $\omega_Y$ on $Y$, whose character we will
denote by $\chi$, and let $\chi'$ vary over all eigencharacters of $A_Y$ which are unramified
when restricted to the image of $A \to A_Y$. Through our choice of $y_0$ we get an
identification: $Y/U \simeq A_Y$, and we consider $\chi'$ as a function on $Y$. In [Sa08] we
defined a morphism $S^Y_\chi : C^\infty_c(X) \to I(\chi)$ which, composed with evaluation at “1”
is given by the rational continuation$^4$ of the integral:

$$\text{ev}_1 \circ S^Y_\chi(\phi) = \int_Y \phi(y) \chi'^{-1}(y) |\omega_Y|(y).$$

3.2.1. Definition. An eigenmeasure on $Y$ of the form $\chi'^{-1}|\omega_Y|$ will be called
pseudo-rational.

We explain now what the index $\tilde{\chi}$ which appears in the notation is: Consider
the map: $A(k) \to A_Y(k)$ and let $R$ denote the preimage of $A_Y$ (k). The expression
(3.2) provides a complex character $\tilde{\chi}$ of $R$ defined as: $\tilde{\chi} = \delta^{-\frac{1}{2}} \cdot \chi' \cdot |c|^{-1}$. (Notice
that unramified characters such as $\delta^{-\frac{1}{2}}$ and $|c|^{-1}$ make sense on the whole of
$A(k)$ – the reason is that there are canonical algebraic characters, of which these
unramified characters are a power of the absolute value.) The characters obtained
this way are a torsor for the group of characters of $A_Y$ which are unramified on
the image of $A$. If $\chi$ denotes the restriction of $\tilde{\chi}$ to $A$, the above expression (3.2)
defines a $(B, \chi^\delta \tilde{\chi})$-equivariant functional on $C^\infty_c(X)$, and by Frobenius reciprocity
a morphism: $C^\infty_c(X) \to I(\chi)$. When no confusion arises, we will denote by $S^Y_\tilde{\chi}$
both the functional and the corresponding morphism into $I(\chi)$.

For the principal series $I(\chi)$ to admit such a morphism, the character $\chi$ must satisfy the condition:

$$\chi^{-1} \delta^\chi |_{B_{\nu}} = \delta_{B_{\nu}}$$

for a (any) point $y \in Y$. By abuse of language we will often say that “$\tilde{\chi}$ is a character
of $A_Y$ which extends $\chi$”, although neither $\tilde{\chi}$ nor $\chi$ are, in general, characters of $A_Y$.

$^4$It was shown that this integral converges and represents a rational function for a certain
domain of the parameter $\chi$. We will discuss again domains of convergence in §7.3.
Notice that until now our definition of the family $S_\chi^Y$ depends on the choice of $y_0$ and of $\omega_Y$. This was enough for the purposes of [Sa08], but here we need to be more careful and to normalize the morphisms $S_\chi^Y$ in a precise way.

### 3.3. Compatible choices of eigenforms and $B_0$-orbits

Our goal now is to normalize the morphisms $S_\chi^Y$ of (3.2) in compatible ways, for all $B$-orbits $Y$ of maximal rank. At the same time, we will also define compatible isomorphisms: $Y/U \simeq A_Y$, up to multiplication by $A_0$.

#### 3.3.1. Lemma
Given a $B$-orbit $Y$ with $k$-points, the following data are equivalent:

1. A trivialization of the $A_Y$-torsor $Y/U$, i.e. an isomorphism: $Y/U \simeq A_Y$.
2. A splitting of the short exact sequence:
   \[ 1 \to k^* \to k(Y)^{(B)} \to \mathcal{X}(Y) \to 1. \]
3. A family $\{\omega_\epsilon\}_\epsilon$ of non-zero $k$-rational $B$-eigen-volume forms on $Y$, determined up to a common multiple, where $\epsilon$ ranges over all possible characters of such eigenforms and the $\omega_\epsilon$ have the property that their quotients (which are $B$-eigenfunctions) all are equal to 1 at the same $k$-point of $Y$.

**Proof.** The implication 1 $\Rightarrow$ 2 is obvious. For the converse, we notice that $k(Y)^{(B)} = k[Y]^{(B)}$ and $k[Y/U] = \oplus_{\epsilon \in \chi(Y)} k[Y]_\epsilon$, where $k[Y]_\epsilon$ denotes the corresponding 1-dimensional subspace. For any splitting $\chi \mapsto e_\chi \in k[Y]_\chi$ the maps $e_\chi \mapsto 1 \in k$ extend uniquely to a homomorphism of algebras: $k[Y/U] \to k$, i.e. a $k$-point on $Y/U$.

For the third condition we notice that there exists an eigen-volume form on $Y$ with character $\epsilon$ if and only if $\epsilon|_{B_y} = \epsilon|_{B_x}^{-1}|_{B_y}$, and that every algebraic character of $B_y$ (where $y \in Y(k)$) extends to a character of $B$. In particular, there exists an eigen-volume form. Multiplying such a form by elements of $\mathcal{X}(Y)$, considered as eigenfunctions on $Y$ via an identification $Y/U \simeq A_Y$, we get a family $\{\omega_\epsilon\}_\epsilon$ with the stated property, which depends only up to a common scalar on the form chosen originally. Vice versa, the quotients of elements in such a family are eigenfunctions distinguishing a unique point in $Y/U(k)$. \qed

A variant of this lemma is:

#### 3.3.2. Lemma
Given a $B$-orbit $Y$, the following data are equivalent:

1. An isomorphism: $Y/A_0U \simeq A_Y/A_0$.
2. A multiplicatively-closed family of non-zero $A_Y$-eigenfunctions on $Y/U$, one for each character of $A_Y$ which is unramified in the image of $A \to A_Y$.
3. A family of non-zero pseudo-rational $B$-eigenmeasures on $Y$, determined up to a common multiple, which have the property that their quotients by any fixed element of the family form a family of eigenfunctions as in 2.

Notice also that choosing an isomorphism: $A_Y/A_0 \to Y/A_0U$ with $1 \mapsto$ (the coset of an element of $Y(\mathfrak{o})$) is equivalent to choosing a $B_0$-orbit on $Y(\mathfrak{o})$.

### 3.3.3. Generalities on differential forms

Let $G_1 \supset G_2 \supset G_3$ be algebraic groups. The following is the algebro-geometric version of the factorization of an integral:

\[
\int_{G_3 \backslash G_1} f(g)dg = \int_{G_2 \backslash G_1} \int_{G_3 \backslash G_2} f(hx)dhdx
\]
(where $dg, dh, dx$ are suitable eigenmeasures), which will be useful when the corresponding sets of $k$-points do not surject on the $k$-points of the quotient.

Let $q: V \rightarrow Y$ be a smooth morphism of schemes. Then, by definition, the sheaf of relative differentials $\Omega_{V/Y}$ is locally free on $V$. Now assume in addition that $Y$ (and hence also $V$) is smooth over spec($k$), and let $\Lambda^{\text{top}}$ denote the top-degree exterior powers (i.e. determinants) of the vector bundles $\Omega_V, \Omega_Y, \Omega_{V/Y}$. From the short exact sequence: $0 \rightarrow \Omega_Y \xrightarrow{\text{pr}} \Omega_V \rightarrow \Omega_{V/Y} \rightarrow 0$ we get a canonical isomorphism of line bundles on $V$: $\Lambda^{\text{top}}(\Omega_V) = \Lambda^{\text{top}}(\Omega_{V/Y}) \otimes q^* \Lambda^{\text{top}}(\Omega_Y)$. If $\omega_1$ denotes a section of $\Lambda^{\text{top}}(\Omega_V)$ and $\omega_2$ denotes a section of $\Lambda^{\text{top}}(\Omega_Y)$ the corresponding section $\omega_3$ of $\Lambda^{\text{top}}(\Omega_V)$ will be denoted (by a slight abuse of notation) by $\omega_1 \wedge q^*(\omega_2)$ and will be called a factorization of $\omega_3$ with respect to the morphism $q: V \rightarrow Y$.

Now we return to the situation of $G_1 \supset G_2 \supset G_3$. Let $\omega$ be a $G_1$-eigen-volume form on $G_3 \backslash G_1$. For a non-zero such form to exist, the character $c$ of $\omega$ must satisfy: $\zeta G_2 = \zeta G_3 \mid G_2$. Let $\omega'$ be the character $\omega G_2 \mid G_2$ of $G_2$. Any character $\omega'$ of $G_2$ defines a line bundle over $G_2 \backslash G_1$, equipped with a trivialization of its pull-back to $G_1$; by definition, the sections over an open $\mathcal{Y} \subset G_2 \backslash G_1$ with preimage $V \subset G_1$ are sections $f$ of $\mathcal{O}_V$ such that $f(g_2 g) = \omega'(g_2) f(g)$ for any $g_2 \in G_2$. Let $\mathcal{L}$ be the line bundle defined by $\omega'$. Then $\omega$ admits a factorization: $\omega_1 \wedge q^*(\omega_2)$ with respect to $q: G_3 \backslash G_1 \rightarrow G_2 \backslash G_1$ and $\mathcal{L}$, where $\omega_1 = \Gamma(G_3 \backslash G_1, \Lambda^{\text{top}}(\Omega_{G_3 \backslash G_1}) \otimes q^* \mathcal{L}^{-1})$ and $\omega_2 = \Gamma(G_2 \backslash G_1, \Lambda^{\text{top}}(\Omega_{G_2 \backslash G_1}) \otimes \mathcal{L})$ are eigenforms for $G_1$ and $G_2$, with eigencharacters $c$ and $\omega'$, respectively. For every point $x \in G_3 \backslash G_1$, the form $\omega_1$ gives rise to a volume form on the $G_2$-orbit of $x$: First, the choice of $x$ gives rise to a trivialization of $q^* \mathcal{L}^{-1}$ over its $G_2$-orbit, and then the pull-back of $\omega_1$ via the inclusion $xG_2 \rightarrow G_3 \backslash G_1$ is a volume form on $xG_2$. Therefore, the form $\omega_1$ will be called a “volume form along $G_2$-orbits”.

### 3.3.4. The case of two adjacent orbits.

Now let $Y$ and $Z$ be two $B$-orbits on $X$ with $Z \subset Y$, $\alpha$ a simple root joining them of type $U$ under Knop’s action. Assume that we are given a family of non-zero pseudo-rational $B$-eigenmeasures on $Y$ in Lemma 3.3.2. Assume moreover that the distinguished $A_0U$ orbit on $Y$ is that of a point $y_0 \in Y(\alpha)$. We will show how to associate to this family a similar family of eigenforms on $Z$, uniquely up to the action of $A_0$.

Pick a representative $\tilde{w}_\alpha$ for $w_\alpha$ in $\mathcal{N}(A)(\alpha)$; then as schemes: $Z \times U_\alpha \simeq Y$ under the map $(z, u) \mapsto z\tilde{w}_\alpha u$. Thus, to a $B$-eigen-volume form $\omega_Z$ on $Z$ (with eigencharacter $c$) we can associate the volume form $\omega_Y = \omega_Z \wedge dx$ on $Y$, where $dx$ denotes the obvious eigenform on $U_\alpha$ induced through an $\alpha$-isomorphism: $U_\alpha \simeq \mathfrak{h}_\alpha$. Then $\omega_Y$ is also an eigenform for $B$ with eigencharacter equal to $e^{-\alpha} \cdot w_\alpha c$. Notice that the corresponding measure $|\omega_Y|$ does not depend on the choices made. Hence, for any family of pseudo-rational eigenmeasures on $Y$ as in Lemma 3.3.2 (3), we get a family of pseudo-rational eigenmeasures on $Z$.

This process, in particular, gives rise by Lemma 3.3.2 to compatible choices of $B_0$-orbits on $Y(\alpha)$ and $Z(\alpha)$ which we explicate here: Let $y_0B_0$ be a given $B_0$-orbit on $Y(\alpha)$, consider the quotient $YF_\alpha \rightarrow YP_\alpha / U_{P_\alpha}$ and denote by $\tilde{y}_0B_0$ the image of $y_0B_0$. The $L_\alpha(\alpha)$-orbit of that is isomorphic to $FU_\alpha(\alpha) \backslash L_\alpha(\alpha)$, where $F \subset B_\alpha(\alpha)$ a finite subgroup. Let $J_\alpha$ be the Iwahori subgroup of $L_\alpha$ with respect to the Borel $L_\alpha \cap B$. By the Iwahori-Bruhat decomposition, $L_\alpha(\alpha) = J_\alpha \cup I_\alpha w_\alpha I_\alpha = J_\alpha \cup U_\alpha(\alpha)w_\alpha B_\alpha(\alpha)$, the group $J_\alpha$ has two orbits in $\tilde{y}_0L_\alpha(\alpha)$: one of them is the
$B_{\alpha}(o)$-orbit of $y_0$ and the other intersects the smaller $B_{\alpha}$-orbit in a unique $B_{\alpha}(o)$-orbit $z_0B_0$. Each fiber of the quotient: $YP_{\alpha} \rightarrow YP_{\alpha}/U_{P_{\alpha}}$ being homogeneous under the unipotent group $U_{P_{\alpha}}$, the preimage of $z_0B_0$ intersects $Z(o)$ in a unique $B_0$-orbit $z_0B_0$.

3.3.5. Lemma. Let $Y$, $Z$ be two orbits joined by a root $\alpha$ under Knop’s action, and assume that they are equipped with compatible families of pseudo-rational $B$-eigenmeasures as above. Use these eigenmeasures to define the morphisms $S_{\chi}^Y$ of (3.2). Let $T_{w_{\alpha}} : I(\chi) \rightarrow I(^{w_\alpha}\chi)$ denote the standard intertwining operator between principal series:

$$T_{w_{\alpha}}(f)(g) = \int_{U_{\alpha}} f(\tilde{w}_{\alpha}u)gdu$$

defined with some preimage $\tilde{w}_{\alpha} \in \mathcal{N}(A)(o)$ of $w_{\alpha}$, and with measure $dx$ on $U_{\alpha} \simeq A^1_x$ (with the latter isomorphism over $o$). Then:

$$T_{w_{\alpha}} \circ S_{\chi}^Z = S_{^{w_{\alpha}}\chi}^Y.$$  

This is just a strengthened version of a special case of [Sa08, Theorem 5.2.1].

Proof. We know already from [Sa08, Theorem 5.2.1] that one side has to be a rational multiple of the other. To compute the proportionality constant, it is enough to do it when the morphisms of both sides – viewed as functionals by Frobenius reciprocity, are applied to functions with support in $Z \cup Y$. For those, one writes down explicitly the integral expressions for both sides, when $\tilde{\chi}$ is in the region of convergence of $T_{w_{\alpha}}$.  

Now consider Knop’s graph $\mathcal{G}$, which was defined in §2.3. We fix a class of pseudo-rational eigenmeasures, to be called “standard”, on $\tilde{X}$, with the property that their quotients are equal to 1 on our distinguished point $x_0$. This condition determines them up to a common multiple, and the precise normalization will be discussed in a later chapter. Choosing a path $\gamma$ in $\mathcal{G}$ joining $\tilde{X}$ to $Y$, the above process induces a similar class of eigenmeasures on $\tilde{Y}$ (and hence also a $B_{\alpha}$-orbit $y_0B_0 \subset Y(o)$) which, a priori, depends on the choice of path $\gamma$. Our goal is to show:

3.3.6. Proposition. The induced class of eigenmeasures on $Y$ (and therefore also the orbit $y_0B_0$) does not depend on the choice of path $\gamma$.

Proof. First, we claim that the induced class of eigenmeasures on $Y$ does not depend on $\gamma$ itself but only, possibly, on $w(\gamma)$. For this, it is enough to show that for two paths $\gamma_1, \gamma_2 \in \mathcal{G}(Y)$ with $w(\gamma_1) = w(\gamma_2) = w$ the families of morphisms $S_{\chi}^Y$ defined with the eigenmeasures obtained through each of the two paths coincide. But this follows from Lemma 3.3.5: Indeed, the lemma implies that $T_{w} \circ S_{\chi}^Y = S_{^{w}\chi}^X$ for both families $S_{\chi}^Y$, and therefore $S_{\chi}^Y$ is the same rational multiple of $T_{w^{-1}} \circ S_{^{w^{-1}}\chi}^X$ for both families.

Now, using Proposition 2.3.1, it is sufficient to prove the following: If $\alpha, \beta$ are two orthogonal simple roots which raise an orbit $Z$ to an orbit $Y$ then the identifications defined by $\alpha$ and $\beta$ coincide. We can substitute the variety $YP_{\alpha\beta}$ by the variety $YP_{\alpha\beta}/U_{P_{\alpha\beta}}$. The $L_{\alpha\beta}$-orbit of a $p$-point on the latter is isomorphic to one of the varieties $SL_2$, $PGL_2$, acted upon by left and right multiplication (under an isomorphism $L_{\alpha\beta} \simeq SL_2$). We are thus reduced to the case $Y =$ the open Bruhat cell and $Z =$ the closed Bruhat cell in $SL_2$ (or $PGL_2$). Then the
isomorphisms $\mathbf{Z} \times U_\alpha \simeq \mathbf{Y}$ and $\mathbf{Z} \times U_\beta \simeq \mathbf{Y}$, used in §3.3.4 to define compatible choices of eigenmeasures, correspond to nothing else than the decomposition of the open Bruhat cell as a product $U_\mathbf{w}B$ or $BwU$, respectively, and hence it is immediate to see that given a class of eigenforms on $B$ of the form of Lemma 3.3.2, the induced class of eigenforms on $BwB$ does not depend on which decomposition we use.

This proposition allows us to define the notion of “standard” eigenmeasures on each orbit $Y$ of maximal rank, which depends only on the choice of standard eigenmeasures on $X$.

3.4. Adjoint. Having chosen a point $x_0 \in \tilde{X}(\mathfrak{o})$, we normalize our family of standard pseudo-rational eigenmeasures on $\tilde{X}$ by requiring that $\mu(x_0J)=1$, where $J$ is the Iwahori subgroup. Recall that $x_0J \subset x_0B_0$ by Axiom 2.4.2, therefore such a normalization is possible, since all eigenmeasures in the family differ by unramified $B$-eigenfunctions. So we have:

3.4.1. Definition. The pseudo-rational $B$-eigenmeasures on $\tilde{X}$ normalized according to the property $\mu(x_0J)=1$ will be called standard. The corresponding eigenmeasures on any orbit $Y$ of maximal rank, obtained via Proposition 3.3.6, will also be called standard.

From now on we will only be using the morphisms $S^Y_X$ defined as in (3.2) with the use of standard eigenforms and eigenmeasures.

Recall that we are assuming the existence of a $G$-eigenmeasure on $X$ with unramified eigencharacter $\nu$. This measure is the absolute value of a volume eigenform $\omega_X$ on $\mathbf{X}$, which from now on we will also assume to be in the class of standard eigenforms. Multiplication by this measure yields an identification:

$$C_c^\infty(X) \otimes \nu \simeq M_c^\infty(X).$$

We can now, via (3.4), modify $S^Y_{X^{-1}I^{-1}}$ into a morphism:

$$\tilde{S}^Y_{X^{-1}I^{-1}} : M_c^\infty(X) \simeq C_c^\infty(X) \otimes \nu \xrightarrow{S^Y_{X^{-1}I^{-1}}} I(\chi^{-1}\nu^{-1}) \otimes \nu \simeq I(\chi^{-1})$$

via the canonical isomorphism: $I(\chi^{-1}\nu^{-1}) \otimes \nu \simeq I(\chi^{-1})$. From now on we drop the notation $\tilde{S}^Y_{X^{-1}I^{-1}}$, and we will denote $S^Y_{X^{-1}I^{-1}}$ by $S^Y_{X^{-1}I^{-1}}$. It should be clear from the context whether we are referring to a morphism from $C_c^\infty(X)$ to $I(\chi^{-1}\nu^{-1})$ or a morphism from $M_c^\infty(X)$ to $I(\chi^{-1})$.

Remark. In fact, it is more natural to introduce the morphism $\tilde{S}_{X^{-1}I^{-1}}$ (recall that we have $Y = \mathbf{X}$ when the exponent $Y$ is omitted) directly as a morphism: $M_c^\infty(X) \rightarrow I(\chi^{-1})$, namely, integration of any measure against the character $\delta^{-\frac{3}{2}}\chi$, considered as a function on $X$ via the choice of the point $x_0$. The reference to $C_c^\infty(X)$ is artificial and less canonical, since it depends on the choice of $|\omega_X|$. However, to handle the sheaf $M_c^\infty$ on $X$ and describe its restrictions to the smaller orbits, it is convenient to work with functions instead of measures, since restrictions of functions make sense as functions.

We will denote by

$$\Delta_X^Y : I(\chi) \rightarrow C_c^\infty(X)$$
the morphism adjoint to $S_Y^{\chi^{-1},v-1}$. (Here we are using the standard pairing between $I(\chi)$ and $I(\chi^{-1})$: $(f_1,f_2) \mapsto \int_K f_1(k)f_2(k)dk$ with $\text{Vol}(K) = 1$.) Notice that for $Y = \hat{X}$, $\Delta \hat{X}$ is defined whenever $\chi \in \delta^{\frac{1}{2}}A_X$. By our assumption that $\hat{X}$ carries a $B$-invariant measure, we have $\delta|_{B_y} = \delta_{B_y}$ for a point $y$ on the open orbit. It is known [BLV86, Théorème 3.4] that the unipotent radical of $B_y$ is equal to $L(X)\cap U$; therefore, $\delta_{B_y} = \delta_{(X)}|_{B_y}$ and hence:

$$\delta_{p(X)} \in A_{\hat{X}}$$

and the condition for the open orbit can be written:

$$\chi \in \delta^{\frac{1}{2}}_{(X)}A_{\hat{X}}.$$  

In general, the condition (see (3.3)) is:

$$\chi_{\nu}\delta^{\frac{1}{2}}|_{B_y} = \delta_{B_y}$$

for any point $y \in Y$.

We want to obtain an explicit formula for $\text{ev}_{\xi} \circ \Delta Y$, where $\xi$ is some point of $Y$. Assume for now that a point $\xi$ has been chosen, and denote by $G_\xi$ its stabilizer.

We fix volume eigen-forms on the spaces $X, G, U \backslash G$ as follows:

- As mentioned above, the volume form $\omega_X$ on $X$ will be standard, hence normalized by the condition: $|\omega_X(x_0J)| = 1$.
- On $U \backslash G$, we fix the invariant volume form $\omega_{U \backslash G}$ such that $|\omega_{U \backslash G}|(U \backslash U K) = 1$.
- On $G$ we fix the invariant volume form $\omega_G$ such that $|\omega_G|(K) = 1$.

These induce differential forms $\omega_\xi, \omega_U$ “along” $G_\xi$-orbits, respectively $U$-orbits, on $G$ (cf. §3.3.3) which factorize $\omega_G$ with respect to the orbit maps:

$$o_\xi : G \ni g \mapsto \xi g \in X$$

and

$$o_U : G \ni g \mapsto Ug \in U \backslash G.$$ 

In other words, in the notation of §3.3.3 we have:

$$\omega_G := n_\xi^{-1}\omega_\xi \land o_\xi^*\omega_X = \omega_U \land o_U^*(\omega_{U \backslash G}).$$

(This statement is essentially just saying that for a compactly supported function $\phi$ on $G$ we have:

$$\int_X \int_{G_x} (\nu^{-1}\phi)(hx) = \int_{U \backslash G} \int_{U \backslash U} \phi(ug)$$

with respect to the corresponding measures.)

Finally, recall that we have a standard top $B$-eigenform $\omega_Y$ on $Y$, involved in the definition of $S_Y^{\chi}$. We use $\omega_\xi$ to define compatible eigenforms $\omega_{\hat{Y}}, \omega_{\hat{Y}}$ on $\hat{Y} := BG_\xi \subset G$ and $\hat{Y} := U \backslash BG_\xi \subset U \backslash G$; explicitly, these forms satisfy:

$$\omega_{\hat{Y}} := \iota^*(\omega_\xi \land o_\xi^*(\omega_Y)) = \omega_{U \backslash U} \land o_U^*(\omega_{\hat{Y}})$$

where $\iota : G \rightarrow G$ is the map $g \mapsto g^{-1}$. Notice that $\omega_{\hat{Y}}$, resp. $\omega_{\hat{Y}}$, is a $B \times G_\xi$-eigenform, resp. an $A \times G_\xi$-eigenform, with eigencharacter equal to $\delta_{G_\xi}$ times the character of $\omega_Y$.

For reasons that will become apparent later, we would like to think of $I(\chi^{-1})$ as living in $C^\infty(U \backslash G)$ (more precisely, for almost every $\chi$, in $C^\infty_{\text{temp}}(U \backslash G)$, the space
of tempered, smooth generalized functions on $U \setminus G$, which will be defined later), and hence think of $\Delta^Y_\chi$ as a morphism:

$$M^\infty_c(U \setminus G) \rightarrow I(\chi) \rightarrow C^\infty_{\text{temp}}(X).$$

Identifying $M^\infty_c(U \setminus G)$ and $C^\infty_c(U \setminus G)$ via the invariant measure that we fixed before we get a morphism:

$$C^\infty_c(U \setminus G) \rightarrow I(\chi) \rightarrow C^\infty_{\text{temp}}(X),$$

still to be denoted by $\Delta^Y_\chi$. We notice that the first arrow of this morphism can be described as the integral:

$$\Phi \mapsto \int_A \Phi(Ua) \chi^{-1} \delta^{-\frac{1}{2}}(a) da$$

with measure 1 on $A_0$.

The following is an explicit description of $\Delta^Y_\chi$:

3.4.2. Lemma. For $\phi \in C^\infty_c(U \setminus G)$, $\xi \in Y$ and $\chi$ in a suitable region of convergence we have:

$$\text{ev}_\xi \circ \Delta^Y_\chi(\phi) = \int_Y \phi(y) \nu^{-1} \chi'(y) |\omega_{\hat{Y}}|(y),$$

where $\chi'$ is the character of $A_Y$ such that $\nu^{-1} \chi'|e|^{-1} = \delta \frac{1}{2} \chi$, where $e$ is the eigencharacter of $\omega_{\hat{Y}}$ under the $A$-action.

Here we regard $\nu$ as a function on $U \setminus G$, since it is $U$-invariant on $G$. The $B$-eigenfunction $\chi'$ on $Y$, pulled back to $\hat{Y}$, can be considered as an $A \times G_\xi$-eigenfunction on $\hat{Y} \subset U \setminus G$ with trivial eigencharacter for $G_\xi$. Notice that the eigencharacter of $\omega_{\hat{Y}}$ for the action of $G_\xi$ is $\delta^{-1}_{G_\xi}$, and $\nu|_{G_\xi} = \delta_{G_\xi}$, therefore the stated functional is indeed $G_\xi$-invariant.

Proof. We notice that with the chosen normalizations we have a commutative diagram of $G$-equivariant maps:

$$\begin{array}{ccc}
C^\infty_c(G) \otimes \nu & \xrightarrow{o_\xi^*} & C^\infty_c(X) \otimes \nu \\
\downarrow & & \downarrow \\
M^\infty_c(G) & \xrightarrow{o_\xi^*} & M^\infty_c(X)
\end{array}$$

where: the first vertical map is $\Phi \mapsto \Phi(g)\nu(g)|\omega_G|(g)$, the first horizontal map is integration over the fibers of $o_\xi$ with respect to $\omega_\xi$, the second horizontal map is push-forward of measures and the second vertical map is the one described previously, i.e. $\phi \mapsto \phi|\omega_X|$. A similar diagram holds for $o_U$ (where the push-forward $o_U^*$ of functions is with respect to $|\omega_U|$) without tensoring with $\nu$ on the upper row.

Let $\Phi_1, \Phi_2 \in C^\infty_c(G)$ with images $o_\xi(\Phi_1) = \phi_1 \in C^\infty_c(X)$, $o_U^*(\Phi_2) = \phi_2 \in C^\infty_c(U \setminus G)$. (In particular, $\phi_1$ is supported in $G_\xi \setminus G \subset X$.) Recall that, by definition
\[ S^Y_{X^{-1},\nu^{-1}}(\phi_1|\omega_X) = \nu \cdot S^Y_{X^{-1},\nu^{-1}}(\phi_1). \]

We compute:

\[
\langle \phi_1|\omega_X, \Delta^Y \phi_2 \rangle_X = \langle S^Y_{X^{-1},\nu^{-1}}(\phi_1|\omega_X), \phi_2 \rangle_{U\backslash G} =
\]

\[
= \int_{U\backslash G} \left( \nu(g) \int_Y \phi_1(yg) \chi^\nu(y)|\omega_Y|(g) \right) \phi_2(g)|\omega_{U\backslash G}|(g) =
\]

\[
= \int_G \nu(g) \int_Y \Phi_1(z^{-1}g) \Phi_2(g) \chi^\nu(z^{-1})|\omega_Y|(z)|\omega_G|(g)
\]

(since \( \int_{\tilde{Y}} \Phi_1(z^{-1})|\omega_{\tilde{Y}}|(z) = \int_Y \int_{G_{\xi}} \Phi_1(hy)|\omega_\xi|(h)|\omega_Y|(y) \) by definition)

\[
= \int_G \int_{\tilde{Y}} \nu(zg) \Phi_1(g) \Phi_2(zg) \chi^{\nu|\omega_\xi|(h)}|\omega_Y|(z)|\omega_G|(g) =
\]

\[
= \int_G \nu(g) \Phi_1(g) \left( \int_{\tilde{Y}} \nu(y) \phi_2(yg) \chi^{\nu|\omega_\xi|(h)}|\omega_Y|(y) \right) |\omega_G|(g) =
\]

\[
= \int_{G_{t}\backslash G} \left( \int_{\tilde{Y}} \phi_2(yg) \nu \chi^{\nu|\omega_\xi|(h)}|\omega_Y|(y) \right) \phi_1(g)|\omega_X|(g).
\]

To prove the statement on eigencharacters recall that, if we denote by \( \zeta_1 \) the eigencharacter of \( \omega_Y \) then \( \chi^{\nu|\omega_Y|}(\zeta_1) = \chi^\nu \delta_{\zeta_1} \) from the definition of \( S^Y_{X^{-1},\nu^{-1}} \). Given the factorization \( \omega_{\tilde{Y}} = \omega_Y \wedge \sigma_\nu^k(\omega_{\tilde{Y}}) \) and the fact that \( \omega_Y \) has a \( B \)-eigencharacter equal to \( \delta_{B} \), it follows that \( \zeta_1^{-1} = \) the \( B \)-eigencharacter of \( \omega_{\tilde{Y}} \) is equal to \( \delta_{B} \). \( \square \)

3.5. **Type N geometry.** Before we proceed, we must discuss the case of a type \( N \) reflection where additional complications arise and we will need to define other intertwining operators. Let \( Y \) be a Borel orbit of maximal rank. Based on Proposition 3.3.6 we have a distinguished \( B_0 \)-orbit \( y_0B_0 \) on \( Y \) (more precisely, on \( Y(0) \)), depending, of course, on the \( B_0 \)-orbit of the distinguished point \( x_0 \in X(0) \). Recall that this allows us to identify the geometric quotient \( Y/U \) with the quotient \( A_Y \) of \( A \), in a unique way up to translations by \( A_0 \). We have: \( A_Y = A/A_y \), where \( A_y \) is the stabilizer in \( B \), modulo \( U \), of any point \( y \in Y \).

Let \( \alpha \) be a simple root such that \( (Y, \alpha) \) is of type \( N \). We consider the quotient map: \( YP_\alpha \to YP_\alpha/\mathcal{R}(P_\alpha) \cong N(T) \backslash PGL_2 \) and let \( B_2 \) denote the corresponding Borel subgroup of \( PGL_2 \). (As usual, the above isomorphisms are unique up to integral automorphisms.)

The space \( X_2 := N(T) \backslash PGL_2 \) is the space of non-degenerate quadratic forms modulo homotheties. The stabilizer in \( B_2 \) (the Borel subgroup of \( PGL_2 \)) of a point on the open \( B_2 \)-orbit is \( \{ \pm I \} \). Therefore, the open \( B_2 \)-orbits are naturally a torsor for \( H^1(k, \{ \pm I \}) \cong k^x/(k^x)^2 \). The same group parametrizes the isomorphism classes of tori over \( k \), and it is easy to see that each open \( B_2 \)-orbit belongs to a different \( PGL_2 \)-orbit, with all of the possible isomorphism classes of tori appearing as the connected components of stabilizers in the different \( PGL_2 \)-orbits.

The quotient map: \( YP_\alpha \to YP_\alpha/\mathcal{R}(P_\alpha) \cong N(T) \backslash PGL_2 \) induces a map from the set of open \( B \)-orbits on \( (YP_\alpha)(k) \) to the set of open \( B_2 \)-orbits on \( X_2 \). The latter comes with a distinguished point (the image of \( y_0B \)) and the “kernel” of the composite map: \( Y/U = A_Y \to \{ B_2 \text{-orbits on } X_2 \} \) will be denoted by \( A_{Y,\alpha} \).

The word kernel is justified by the fact that the latter set is a trivialized torsor for \( H^1(k, \{ \pm I \}) \) and the last map is a group homomorphism; in particular, \( A_{Y,\alpha} \)
is a subgroup of \( A_Y \). Notice that this definition is specific to the root \( \alpha \) and this dependence will be a serious obstacle in obtaining a useful explicit formula in those cases.

Let \( \zeta \) be a coset of \( A_{Y,\alpha} \) in \( A_Y \). The points of \( Y \) mapping to \( \zeta \) are those belonging to the preimage of a single open \( B_2 \)-orbit on \( X_2 \); their set will be denoted by \( Y_{\zeta} \). Since the connected component of the isotropy subgroup in the \( \text{PGL}_2 \)-orbit of that \( B_2 \)-orbit is a torus, we can attach certain invariants to the coset \( \zeta \). More precisely, we let \( D(\zeta) \) be the discriminant of the splitting field of that torus. It is an element of \( \mathfrak{a} \setminus \mathfrak{p}^2 \), well-defined up to the action of \((o^\infty)^2\). According as \( D(\zeta) \in o^\infty \) or not we will say that “\( \zeta \) corresponds to an integral, resp. non-integral torus”. This is equivalent to \( \zeta \) containing (resp. not containing) \( o \)-points. If \( \zeta \) corresponds to an integral torus, we will say that it “corresponds to a split, resp. non-split, torus” according as \( D(\zeta) \in (o^\infty)^2 \) or not.

### 3.6. Spaces of morphisms and their bases

Let \( S^Y_X \) denote the space of morphisms \( C^\infty_c(X) \to I(\chi) \) spanned by the operators \( S^Y_X \chi \) with \( \tilde{\chi} \) extending \( \chi \). Likewise, let \( \Delta^Y_X \) be the span of the \( \Delta^Y_X \). The standard bases we will be using for these spaces are the bases of \( S^Y_X \) and \( \Delta^Y_X \). (And when we will be expressing linear maps between such spaces as matrices using this basis, unless otherwise stated.)

Unfortunately, in certain cases it is necessary to use a different basis: Let \((Y, \alpha)\) be of type \( N \). We defined above a subgroup \( A_{Y,\alpha} \) of \( A_Y \). Now we define the following basis of \( S^Y_X \): It will be indexed by the set of data \((\tilde{\chi}, \zeta)\) where \( \tilde{\chi} \) runs over representatives for equivalence classes of extensions\(^5\) of \( \chi \) to \( A_Y \), two of them considered equivalent if they have the same restriction to \( A_{Y,\alpha} \); and \( \zeta \) is a coset of \( A_{Y,\alpha} \) in \( A_Y \). Let \( Y_{\zeta} \) denote the set of points on \( Y \) which map to \( \zeta \) under \( Y \to Y/U \cong A_Y \). The corresponding operator \( S^{\tilde{\chi}-1,\nu-1,\zeta} \) will be given by the formula:

\[
e v_1 \circ S^{\tilde{\chi},\zeta} \phi(y) = \int_{Y_{\zeta}} \phi(y) \tilde{\chi}^{-1}(y) |\omega_\chi(y)|.
\]

In other words, the integral is the same as for \( S^Y_X \), except that integration is restricted to the subset \( Y_{\zeta} \). The adjoint of \( S^{\tilde{\chi}-1,\nu-1,\zeta} \) will be denoted by \( \Delta^{Y,\chi}_{Y_{\zeta}} \).

### 3.7. Composing intertwining operators

A main result of [Sa08] was:

#### 3.7.1. Theorem

For an orbit \( Y \) of maximal rank and for almost all \( \chi \in \text{Adm}_Y \), the space \( S^Y_X \) is mapped under composition with \( T_w \) into the space \( S^w_Y \). More precisely for \( w = w_\alpha \) a simple reflection, the composition is given as follows:

- **If** \((Y, \alpha)\) **is of type** \( G \) **then** \( T_w S^Y_X = 0 \).
- **If** \((Y, \alpha)\) **is of type** \( U \) **or** \((U, \psi)\) **or** \( T \) **then** \( T_w S^Y_X \sim S^{w,Y}_X \).
- **If** \((Y, \alpha)\) **is of type** \( N \) **then** \( T_w S^Y_X \sim S^{w,Y}_{\tilde{\chi},\zeta} \) **where** \( \tilde{\chi}, \zeta \) **are as at the end of the previous paragraph.**

Here the symbol \( \sim \) denotes equality up to a non-zero rational function of \( \chi \).

By passing to adjoints, it immediately follows that:

---

\(^5\)Recall that the phrase “extensions to \( A_Y \)” is by abuse of language, since \( \tilde{\chi} \) is not a character of \( A_Y \), cf. §3.2.
3.7.2. Corollary. For an orbit $Y$ of maximal rank and for almost all $\chi$ the space $\Delta^Y_\chi$ is mapped under composition with $T_w$ into the space $\Delta^w Y_\chi$. More precisely for $w = w_\alpha$ a simple reflection, the composition is given as follows:

- If $(Y, \alpha)$ is of type $G$ then $T_w \Delta^Y_\chi = 0$.
- If $(Y, \alpha)$ is of type $U$ or $(U, \psi)$ or $T$ then $T_w \Delta^Y_\chi \sim \Delta^w Y_\chi$.
- If $(Y, \alpha)$ is of type $N$ then $T_w \Delta^Y_\chi \sim \Delta^w Y_{\tilde{\chi}, \zeta}$ where $\tilde{\chi}, \zeta$ are as in §3.6.

(Notice that $T_{w^{-1}}$ is, up to a rational multiple, adjoint to $T_w$.)

In particular, for $Y = X$ (where we omit the superscript $Y$ from the notation) we get that $T_w \Delta^X_\chi$ is not identically zero if and only if $w \in [W/W_{P(\chi)}]$ (coset representatives of minimal length), that $T_w \Delta^X_\chi = \Delta^X_\chi$ if and only if $w \in W_X$ (for generic $\chi$) and that for almost all points in the support of $C^\infty_c(X)^K$ as an $I(G, K)$-module, the space of eigenvectors is spanned by the $K$-invariant vectors in the images of $\Delta^X_\chi$ (for suitable $\chi$), since for almost all points all other intertwining operators can be obtained by composing elements of $\Delta^X_\chi$ by some $T_w$. (Knop’s action is transitive on the set of orbits of maximal rank.) Therefore, the goal of computing eigenvectors of the unramified Hecke algebra now becomes:

Let $\phi_{K, \chi}$ denote a $K$-invariant vector in $I(\chi)$. Compute its image under $\Delta^X_\tilde{\chi}$, for every $\tilde{\chi}$.

A priori, this will provide us with all the eigenvectors for almost every point in the support. In §8 we will examine, among other things, to what extent this gives a complete answer for every point.

3.8. Schwartz space and Fourier transforms. We now explain a precise choice of intertwining operators which will make our computations easier and more transparent. In fact, we change the setting slightly and replace unramified principal series by functions on $U \backslash G$. We recall that there exists a notion of Schwartz space and Fourier transform for this space [BK98]:

If $G = SL_2$ then it is simply the space of compactly supported, locally constant functions on $k^2 \supset k^2 \setminus \{0\} \simeq U \backslash G$ with the equivariant Fourier transform. (The Fourier transform obtained by identifying $k^2$ with its dual via a non-degenerate symplectic form.) Since our spaces have models over $\mathfrak{o}$, we can impose some canonical conditions on our Fourier transforms, namely: The measure on $k^2$ is such that $\mathfrak{o}^2$ has measure 1 (this is the correct choice only under our simplifying assumption that $k$ is unramified over $\mathbb{Q}_p$); we use a symplectic form which in some basis for $\mathfrak{o}^2$ has the form: $\omega((x, y), (v, w)) = xw - yv$; and we use an additive character $\psi$ on $k$ whose conductor is the ring of integers $\mathfrak{o}$. (We choose and fix such a character.)

Using this data, Fourier transform on $k^2$ is given by the formula:

\[
(F f)(v, w) = \int_{k^2} f(x, y) \psi(-xv + yw) dxdy.
\]

In general, one defines for every simple reflection $w_\alpha$ in the Weyl group an equivariant Fourier transform: $F_{w_\alpha} : L^2(U \backslash G) \to L^2(U \backslash G)$ as follows: Let $P_\alpha$ be the parabolic of semisimple rank one corresponding to the simple root $\alpha$. Then we have $[P_\alpha, P_\alpha]/U_{P_\alpha} \simeq SL_2$ and $U/[P_\alpha, P_\alpha] \simeq k^2 \setminus \{0\}$. Notice that the integral structure on $G$ defines an integral structure on $k^2 \setminus \{0\}$, therefore after fixing the former isomorphism the latter is determined up to multiplication by an element of $\mathfrak{o}^\times$. Then one defines $F_{w_\alpha}$ to be the unique equivariant transform which is equal to the $SL_2$-equivariant Fourier transform on the restriction of a continuous function
in $L^2(U\backslash G)$ to $U\backslash [P_\alpha, P_\alpha]$. It is known that the equivariant Fourier transforms compose as follows: $\mathcal{F}_w \circ \mathcal{F}_{w'} = \mathcal{F}_{ww'}$, hence they define a unitary action of the Weyl group on $L^2(U\backslash G)$. Braverman and Kazhdan define the Schwartz space to be the smallest subspace which contains $C_c^\infty(U\backslash G)$ and is closed under all Fourier transforms. The Schwartz space comes with a canonical $K$-invariant vector $c_0$, which is stable under all Fourier transforms and whose restriction to the $[L_\alpha, L_\alpha]$-orbit of $U \cdot 1$, for each simple root $\alpha$, is equal, under the above identifications, to the characteristic function of $\sigma^2$ in $k^2$.

We define the action of $A$ on $L^2(U\backslash G)$ as follows, so that it is unitary:

$$(L_a f)(Ug) = \delta^{-\frac{1}{2}}(a)f(Uag).$$

This way, we have $\mathcal{F}_w(L_a f) = L_w \mathcal{F}_w f$.

We can extend Fourier transforms uniquely to tempered distributions (i.e., distributions on the Schwartz space) and tempered generalized functions. For almost every $\chi$, the elements of the principal series $I(\chi)$ are tempered, and by the equivariance properties we have $F_\chi : I(\chi) \rightarrow I(\chi'^{-1})$.

Equivalently, consider the surjective map: $\mathcal{S}(U\backslash G) \rightarrow I(\chi)$ given by the rational continuation of the integral:

$$f \mapsto \int_A L_a f \chi^{-1}(a)da.$$ 

(We normalize $\text{Vol}(A_0) = 1$.) The Fourier transforms and the intertwining operators between the $I(\chi)$'s fit into the commutative diagram:

$$
\begin{array}{ccc}
\mathcal{S}(U\backslash G) & \xrightarrow{f} & I(\chi) \\
\mathcal{F}_w & \downarrow & \downarrow T_w \\
\mathcal{S}(U\backslash G) & \xrightarrow{f} & I(\chi'^{-1})
\end{array}
$$

for $T_w$ suitably normalized. The operators $T_w$ normalized in this way are the operators $F_w$. Notice that the $F_w$'s satisfy: $F_w = F_{w^{-1}}$ (recall that the smooth dual of $I(\chi)$ is $I(\chi^{-1})$).

The reason for working with Schwartz space and Fourier transforms, instead of principal series and their intertwining operators, is that computations will be much easier there; in particular, we will be able to avoid all the mysterious cancellations and simplifications that one discovers a posteriori in all other variants of the Casselman-Shalika method.

3.9. An Iwahori-invariant vector of small support. The starting point for (all variants of) the Casselman-Shalika method is the observation that it is easy to compute the value of the functional $\Delta^Y_X$ on translates of an Iwahori-invariant vector of small support. More precisely, let $\Phi_j \in \mathcal{S}(U\backslash G)$ denote the characteristic function of $U \cdot J$.

**Lemma.** For $x \in A_{X}^\circ$ we have: $\Delta^Y_X(\Phi_j)(x) = \begin{cases} 
0 & \text{if } Y \neq \hat{X} \\
\text{Vol}(J) \cdot \hat{x} \delta^{-\frac{1}{2}}(x) & \text{otherwise}.
\end{cases}$

**Proof.**

$$
\Delta^Y_X(\Phi_j)(x) = \frac{\langle \Delta^Y_X(\Phi_j), 1_{x,j} | \omega_X \rangle_X}{|\omega_X|(xJ)} = \frac{\langle \Phi_j, S^Y_{\hat{x}^{-1}}(1_{x,j} | \omega_X \rangle)}{|\omega_X|(xJ)} =
$$
By Axiom 2.4.2 we have: \( xJ \subset xB_0 \). Therefore, from the definition of \( S^Y_{\hat{\chi}^{-1} \nu^{-1}} \) (3.2) we get: 
\[
\nu(x) ev_{U_1} S^Y_{\hat{\chi}^{-1} \nu^{-1}}(1_xJ) = |\omega_{\hat{\chi}}(xJ)\hat{\chi}^{-\frac{1}{2}}(x) |.
\]

\[\square\]

3.10. Functional equations. Using Fourier transforms, and the fact that \( F^*_{w^{-1}} = F_w \), Corollary 3.7.2 can be restated as follows:

For every \( w \in W \) there is a rational family of linear operators: \( b^Y_w(\chi) : \Delta^Y_{\hat{\chi}} \rightarrow \Delta^w_{\hat{\chi}} \) such that \( F_w \Delta = b^Y_w(\chi) \Delta \) for \( \Delta \in \Delta^Y_{\hat{\chi}} \). Moreover, for \( Y = \hat{X} \) the operator \( b^Y_w(\chi) \) is non-zero if and only if \( w \in [W/W_{\hat{X}}] \), and for \( w \) a simple reflection we have explicit bases which make the matrix of \( b^Y_w(\chi) \) diagonal. Since \( F_{w_1} \circ F_{w_2} = F_{w_1 w_2} \), the \( b^Y_w(\chi) \) satisfy the cocycle relations:

\[
(3.10) \quad b^Y_{w_1 w_2}(\chi) = b^{w_2 Y}_{w_1}(\chi) b^Y_{w_2}(\chi).
\]

As a matter of notation, we will be identifying \( b^Y_w(\chi) \) with its matrix in the bases \( [\Delta^Y_{\hat{\chi}}], [\Delta^w_{\hat{\chi}}] \).

Suppose that we knew the matrices \( b^Y_w(\chi) \) (recall that when we ignore the orbit as a superscript we mean \( \hat{X} \)). Then it would be easy to compute values of \( \Delta_{\hat{\chi}} \) applied to every vector of the form \( F_{w^{-1}} \Phi_J \) as follows:

\[
|\Delta_{\hat{\chi}}(F_{w^{-1}} \Phi_J)(x)|_{\hat{\chi}} = |F_w \Delta_{\hat{\chi}}(\Phi_J)(x)|_{\hat{\chi}} = b_{w}(\chi) \cdot |\Delta^{w \hat{\chi}}_{\hat{\chi}}(\Phi_J)(x)|_{\hat{\chi}}.
\]

This vanishes unless \( w \hat{X} = \hat{X} \), i.e. unless \( w \in W_{\hat{X}} \), in which case from the previous lemma we get:

\[
(3.11) \quad |\Delta_{\hat{\chi}}(F_{w^{-1}} \Phi_J)(x)|_{\hat{\chi}} = b_{w}(\chi) \cdot |\text{Vol}\(J\)| \cdot \(w \hat{\chi}^{-\frac{1}{2}}(x)\)|_{\hat{\chi}}.
\]

Therefore, the computation of the Hecke eigenvectors will be straightforward if we can perform the following two steps:

1. Compute the functional equations \( F_w |\Delta^Y_{\hat{\chi}}|_{\hat{\chi}} = b^Y_w(\chi) |\Delta^{w \hat{\chi}}|_{\hat{\chi}} \).

2. Express a non-zero \( K \)-invariant Schwartz function as a linear combination of the functions \( F_{w^{-1}} \Phi_J, w \in W \).

Remark. The second step is independent of the particular case that we are examining (i.e. independent of \( H, \Psi \)), and it suffices to do it once and for all. It would therefore be possible to retrieve it, for example, from the work of Casselman on zonal spherical functions. However, we construct a \( K \)-invariant vector in an independent way, which in the sequel will help us avoid all the mysterious cancellations which are usually found in the literature and will lead us directly to simple formul\ae. Notice also that the requirement that a linear combination of the \( F_{w^{-1}} \Phi_J \)'s be non-zero is automatically satisfied for any non-trivial linear combination. This follows from (3.11) (applied to any choice of \( H \)) and the linear independence of the characters \( w \chi \), for \( \chi \) in general position.
4. Construction of a $K$-invariant vector

4.1. The case of $\text{SL}_2$. In this section we perform the second step of §3.10.

We start by considering the case of $\text{SL}_2$. Then $\Phi_J = 1_{p \times o} = 1_{\mathbb{P} \times o} - 1_{\mathbb{P} \times p}$. Since $1_{\mathbb{P} \times p}$ is $K$-invariant we see that $\Phi_J \equiv 1_{\mathbb{P} \times o}$ up to $K$-invariants and the equivariant Fourier transform of $\Phi_J$ is, up to $K$-invariants, equal to

$$\mathcal{F}_{1_{\mathbb{P} \times o}} \equiv q^{-1}1_{\mathbb{P} \times p}^{-1} = L\left(\frac{-1}{o-1}\right)1_{\mathbb{P} \times o}.$$  

Hence

$$L\left(\frac{-1}{o-1}\right)\Phi_J - \mathcal{F}\Phi_J \equiv L\left(\frac{-1}{o-1}\right)1_{\mathbb{P} \times o} - \mathcal{F}_{1_{\mathbb{P} \times o}} \equiv 0$$ 

up to $K$-invariants; equivalently, the vector:

$$(4.1) \quad \Phi_J - L\left(\frac{-1}{o-1}\right)\mathcal{F}\Phi_J$$ 

is $K$-invariant.

4.2. The general case. We now return to an arbitrary $G$. To simplify notation, for each co-root $\alpha$ and each $\Phi \in \mathcal{S}(U \setminus G)$ we will denote $L_{e^\alpha}(\chi)\Phi$ by $e^\alpha \Phi$. This is consistent with the fact that the image of $L_{e^\alpha}(\chi)\Phi$ in $I(\chi)$ under the natural morphism (integration with respect to the $A$-action against the character $\chi^{-1}$) is $e^\alpha(\chi) := \chi(e^\alpha(\chi))$ times the image of $\Phi$.

For each simple root $\alpha$ let $K_\alpha \subset K$ denote the inverse image, through the reduction map, of $\mathbb{P}_\alpha(F_q)$. Evidently, $K$ is generated by all the $K_\alpha$ since they contain the Iwahori and representatives for the simple reflections in the Weyl group. Using the $\text{SL}_2$ case that we examined above, it is now easy to show:

4.2.1. Lemma. The vector $\Phi_K := \text{Vol}(J)^{-1} \sum_{\omega \in W} \prod_{\alpha > 0, \omega^{-1} \alpha < 0} (-1)e^{\alpha}\mathcal{F}_\omega \Phi_J$ is $K$-invariant.

Proof. It suffices to show that for any simple positive root $\beta$ it is $K_\beta$-invariant.

For the minimal parabolic $P_\beta$ we denote by $W_\beta$ the Weyl group of its Levi, and by $[W/W_\beta]$ the canonical set of representatives of $W/W_\beta$ cosets consisting of representatives of minimal length. Notice that if $w = w'w_\beta$, with $w' \in [W/W_\beta]$, then $\{\alpha|\alpha > 0, \omega^{-1} \alpha < 0\} = \{\alpha|\alpha > 0, \omega'^{-1} \alpha < 0\} \cup \{\omega'\beta\}$. Hence:

$$\text{Vol}(J)\Phi_K = \sum_{w \in [W/W_\beta]} \prod_{\alpha > 0, \omega^{-1} \alpha < 0} (-1)e^{\alpha} \left(\mathcal{F}_w \Phi_J - e^{w_\beta}\mathcal{F}_{w_\beta}\Phi_J\right) =$$ 

$$= \sum_{w \in [W/W_\beta]} \prod_{\alpha > 0, \omega^{-1} \alpha < 0} (-1)e^{\beta}\mathcal{F}_w \left(\Phi_J - e^{\beta}\Phi_J\right)$$

and the latter is $K_\beta$-invariant. \hfill \square

Notice that we can write: $\prod_{\alpha > 0, \omega^{-1} \alpha < 0} e^{\alpha}(\chi) = e^{\beta}(\chi)$ Combining this with (3.11) we get:

4.2.2. Theorem. For almost every $\chi$, a basis of eigenvectors of $\mathcal{H}(G, K)$ on $C^\infty(X)$ with eigencharacter corresponding to $\chi$ is given by the formula:

$$(4.2) \quad [\Omega_{\chi}(x)]_\chi = [\Delta_{\chi}(\Phi_K)(x)]_\chi = e^{\beta}(\chi)\delta^{-\frac{1}{2}}(x) \sum_{w \in W_X} \sigma(w)e^{-\beta}(\chi)\mathcal{F}_w(\chi) \cdot [w_{\chi}(x)]_\chi$$
for $x \in A_X^\chi$, where $\sigma(w)$ is the sign of $w$ as an element of $W$ and the matrices $\Omega_w(\chi)$ are given by the functional equations:

\[ F_w[\Delta \chi] = \Omega_w(\chi)[\Delta \chi]. \]

Remarks.  
(1) There is some abuse of notation here (which we will consistently practice!), in that the expression $^w \chi \delta^{-\frac{1}{2}}(x)$ should be taken as a whole and not as a product of two factors, since $^w \chi$ is not, in general, a character of $A_X$.

(2) In the case that $(Y, (X, \alpha)$ is never of type $N$ (for $Y$ of maximal rank and $\alpha$ a simple root) then we actually know from Corollary 3.7.2 that the matrices $\Omega_w(\chi)$ are diagonal. If in addition $\chi$ is unramified we may write: $\tilde{\chi}(x\chi) = e^{-\chi}(\chi)$ where $x\chi = e^{-\chi}(x\chi) \in A_X$ for any uniformizer $x$, and the formula becomes more pleasant:

\[ \Omega_{\chi}(x\chi) = e^{\hat{\gamma}}(\chi)e^{-\chi}(\chi) \sum_{w \in W \chi} \sigma(w)b_w(\chi)e^{-\hat{\gamma} + \hat{\gamma}}(\chi) \]

where the $b_w(\chi)$ are now scalars given by the functional equations: $F_w \Delta \chi = b_w(\chi)\Delta \chi$.

The following will be useful later:

4.2.3. Lemma. We have $\Omega_{\chi}(-\chi/2)(x) = 1$ for every $x$.

Proof. By our assumption that $X$ carries an eigen-volume form, and by the form of the morphisms $S_{\chi}$, it follows that $S_{\nu^{-1}}$ furnishes a morphism: $C^\infty_{\nu^{-1}}(X) \to \nu^{-1}$ or, tensoring by $\nu$, a morphism: $M^\infty_{\nu}(X) \to 1$ (the trivial representation). Therefore its dual, as a morphism: $I(\chi) \to C^\infty(X)$, factors through the trivial representation. Moreover, the image of $\Phi_K$ via the morphism $S(U \backslash G) \to I(\chi)$ is finite at $\delta_{\chi}$ (see (4.5) below). Therefore, $\Omega_{\chi}(-\chi)$ is a constant function, and by its form (4.2) it has to be equal to 1. $\square$

Remark. The lemma does not apply to the case of $C^\infty(X, L_\Psi)$, with non-trivial $\Psi$, since there is no $G$-eigenmeasure valued in $L_\Psi$ in that case.

4.3. Normalization. It is good to keep track of how our eigenfunctions are normalized, so we collect here the properties of $\Omega_{\chi} = \Delta \chi(\Phi_K)$:

1. Iwahori normalization: Let $\phi_{J, X} \in I(\chi)$ be the vector with $\phi_{J, X}|_K = 1_J$.
   Then $\phi_{J, X}$ is the image of $\Phi_J$ and $\Delta \chi(\phi_{J, X}) = Vol(J) = \frac{1}{|KJ|}$.

2. “Spherical vector” normalization: Let $\phi_{K, X} \in I(\chi)$ be the unramified vector with $\phi_{K, X}(bk) = \chi(\delta_{\chi})(b)$ for every $b \in B$. To determine the value of $\Delta \chi(\phi_{K, X})$ we must determine its relation with the image of $\Phi_K$. Let $\int_X$ denote the natural morphism: $S(U \backslash G) \to I(\chi)$. We claim:

\[ \phi_{K, X} = Q^{-1} \prod_{\alpha > 0} \frac{1 - q^{-1}e^{\alpha}}{1 - e^{\alpha}}(\chi) \int_X \Phi_K \]

where $Q = \frac{Vol(K)}{Vol(J)} = \prod_{\alpha > 0} \frac{1 - q^{-1}e^{\alpha}}{1 - e^{\alpha}}(\delta_{\chi})$. To prove this we need a result which we will prove later, together with a result of Casselman and Shalika. Let $\Delta \chi$ refer to the morphism corresponding to the Whittaker model, i.e.
4.2.1 continuation of the integral:

On the other hand, we have

\[ (\tilde{T}(4.7) \text{operators} F_c(4.6) \]

At this point we can also discuss the relationship between the intertwining operators

3. "Basic vector" normalization: In their study \[ \text{[CS80, Theorem 5.4]} \] Braverman and Kazhdan introduce a "basic vector" \( c_0 \in \mathcal{S}(U\backslash G) \) which generates \( \mathcal{S}(U\backslash G) \) over \( \mathcal{H}(A, A_0) \otimes \mathcal{H}(G, K) \). It is related to \( 1_{UK} \) as follows:

\[ 1_{UK} = \prod_{\alpha > 0} (1 - q^{-1} e^{\alpha}) c_0 \]

where \( e^{\alpha} \) denotes the corresponding element of \( \mathcal{H}(A, A_0) \) (acting via the normalized left A-action on \( U\backslash G \)) (see \[ \text{[BK98, eq. (3.15), (3.22)]} \]). Therefore

\[ \int_{\chi} c_0 = \prod_{\alpha > 0} (1 - q^{-1} e^{\alpha}) (\chi) \phi_K \]

and from our previous computation:

\[ c_0 = \frac{Q^{-1} \Phi_0}{\prod_{\alpha > 0} (1 - e^{\alpha})}. \]

Notice that, indeed, using the expression of Lemma 4.2.1, one immediately verifies that \( c_0 \) is invariant under Fourier transforms.

At this point we can also discuss the relationship between the intertwining operators \( F_w \) and the "classical" intertwining operators \( T_w \) defined as the rational continuation of the integral:

\[ T_w(\phi) = \int_{U \cap w^{-1}U \cap U} \phi(\check{w}u) du \]

(where \( \check{w} \in N(A)(\mathfrak{o}) \) is a representative for \( w \)). By \[ \text{[Ca80, Theorem 3.1]} \] one has:

\[ T_w \phi_{K, \chi} = \prod_{\alpha > 0, w \alpha < 0} \frac{1 - q^{-1} e^{\alpha}}{1 - e^{\alpha}} (\chi) \phi_{K, w \chi}. \]

On the other hand, we have

\[ F_w(\phi_{K, \chi}) = \prod_{\alpha > 0, w \alpha < 0} \frac{1 - q^{-1} e^{\alpha}}{1 - q^{-1} e^{-\alpha}} (\chi) \phi_{K, w \chi}. \]
and therefore:

\[
F_w = \prod_{\alpha > 0, w_{\alpha} < 0} \frac{1 - \ell^\alpha}{1 - q^{-1} e^{-\alpha}}(\chi)T_w
\]

5. Functional equations

5.1. Reduction to rank one. In this section we compute the proportionality factors in the functional equations:

\[
F_w[\Delta^Y_\chi]_\chi = \Delta^Y_\chi w[\Delta^Y_\chi]_\chi
\]
or equivalently:

\[
F_w[S^Y_{\chi^{-1}nu^{-1}}]_\chi = \Delta^Y_\chi w[S^Y_{\chi^{-1}nu^{-1}}]_\chi
\]

when \( w = w_\alpha \) is a simple reflection and \( Y \) is an orbit of maximal rank. For general \( w \), the functional equations follow from the cocycle relations (3.10). Assume \((Y, \alpha)\) not of type \( G \). We have already exhibited basis elements of \( \Delta^Y_\chi, \Delta^w_\chi \) which map to a multiple of each other under Fourier transform. As a matter of notation, the (scalar) quotient of \( F_w \Delta^Y_\chi \) by \( \Delta^Y_\chi \), in cases \( U \), \((U, \psi)\) and \( T \) will be denoted by \( b^Y_\chi(\chi) \), and the quotient of \( F_w \Delta^Y_\chi \) by \( \Delta^W_\chi \) in case \( N \) will be denoted by \( b^Y_\chi(\chi) \).

Hence, it is enough to compute these scalar quotients for all simple reflections \( w = w_\alpha \).

Remark. Though our notation below is adapted to the cases \( U \) and \( T \), the same discussion holds with obvious modifications for cases \( N \) and \( (U, \psi) \).

To reduce to the case of \( SL_2 \), we proceed as follows:

We can compose \( S^Y_{\chi^{-1}nu^{-1}} \) with the “restriction” morphism to get a sequence of \( P_\alpha \)-morphisms:

\[
C_c^\infty(X) \xrightarrow{S^Y_{\chi^{-1}nu^{-1}}} C_c^\infty(U\backslash G) \xrightarrow{Res} C_c^\infty(U\backslash P_\alpha). \]

We temporarily denote by \( S^Y_{\chi^{-1}nu^{-1}} \) the composition. (The dependence on \( \alpha \) has been suppressed from the notation.)

We have a commutative diagram of \( P_\alpha \)-morphisms:

\[
\begin{array}{ccc}
C_c^\infty(U\backslash G) & \xrightarrow{Res} & C_c^\infty(U\backslash P_\alpha) \\
\downarrow \mathcal{F}_{w_\alpha} & & \downarrow \mathcal{F}_{w_\alpha} \\
C_c^\infty(U\backslash P_\alpha) & \xrightarrow{Res} & C_c^\infty(U\backslash P_\alpha) \\
\end{array}
\]

(5.1)

Therefore the morphisms \( S^Y_{\chi^{-1}nu^{-1}} \) satisfy the same functional equations with respect to \( w_\alpha \) as the morphisms \( S^Y_{\chi^{-1}nu^{-1}} \).

Clearly, from the definition of \( S^Y_{\chi^{-1}nu^{-1}} \), the morphism \( S^Y_{\chi^{-1}nu^{-1}} \) factors through restriction of elements of \( C_c^\infty(X) \) to \( YP_\alpha \). To compute the scalar proportionality factors, it is enough to restrict our \( P_\alpha \)-morphisms to the \( P_\alpha \)-stable subspace of \( C_c^\infty(X) \) of those elements whose restriction to \( YP_\alpha \) is supported in \( YP_\alpha \). For those, we can factorize the morphism as:

\[
S^Y_{\chi^{-1}nu^{-1}} \otimes 1 : C_c^\infty(YP_\alpha) \otimes \nu \to C_c^\infty((H_\alpha \backslash L_\alpha)(k), \delta_{G\cap U\backslash P_\alpha \backslash U\backslash P_\alpha}) \otimes \nu \to \text{Ind}^{L_\alpha}_{B_\alpha}(\chi^{-1}nu^{-1}\delta^\frac{\lambda}{2}) \otimes \nu \simeq \text{Ind}^{L_\alpha}_{B_\alpha}(\chi^{-1}\delta^\frac{\lambda}{2}) \subset C_c^\infty(U\alpha \backslash L_\alpha)
\]
according to the decomposition of eigenforms as in §3.3. Here the notation is as follows: \( \xi \) is a point on \( Y P_{\alpha} \) (to be chosen more carefully later), and \( H_{\alpha} \) denotes the image of \( G_{\xi} \cap P_{\alpha} \) in \( L_{\alpha} = P_{\alpha}/U_{P_{\alpha}} \). The first arrow is “integration along \( U_{P_{\alpha}} \)-orbits”.

We denote the second arrow in the above sequence by \( S_{\chi^{-1}1,\nu^{-1}}^{Y,\alpha} \), fix invariant measures\(^6\) on \( U_{\alpha} \setminus L_{\alpha} \) and \( (H_{\alpha} \setminus L_{\alpha})(k) \) (this admits an invariant measure by inspection of the quasi-affine \( \text{PGL}_2 \)-spherical varieties), and denote by \( \Delta_{\chi,\alpha}^{Y} \) the adjoint:

\[
\Delta_{\chi,\alpha}^{Y} : S(U_{\alpha} \setminus L_{\alpha}) \rightarrow C^\infty((H_{\alpha} \setminus L_{\alpha})(k), \delta_{G_{\xi} \cap U_{P_{\alpha}}}^{-1}) \otimes \nu^{-1} \xrightarrow{\sim} C^\infty((H_{\alpha} \setminus L_{\alpha})(k), \delta_{G_{\xi} \cap U_{P_{\alpha}}}^{-1}).
\]

Notice that this factors through the dual of \( \text{Ind}_{B_{\alpha}}^{L_{\alpha}}(\chi^{-1}1,\nu^{-1}) \), which is canonically identified with \( \text{Ind}_{B_{\alpha}}^{L_{\alpha}}(\chi e^{\alpha}1,\nu^{-1}) \). The \( L_{\alpha} \)-morphisms \( \Delta_{\chi,\alpha}^{Y} \) satisfy the same functional equations as the morphisms \( \Delta_{\chi}^{Y} \). Let \( \tilde{Y}_{\alpha} := U_{\alpha} \setminus B_{\alpha} H_{\alpha} \subset U_{\alpha} \setminus L_{\alpha} \). In analogy with Lemma 3.4.2, we have the following result (notice that the line bundle \( \mathcal{L}_{\delta_{G_{\xi} \cap U_{P_{\alpha}}}^{-1}} \) over \( H_{\alpha} \setminus L_{\alpha}(k) \) comes with a canonical trivialization of its pull-back to \( L_{\alpha} \), therefore it makes sense to “evaluate at 1”):

5.1.1. Lemma. For \( \phi \in C_{c}^\infty(U_{\alpha} \setminus L_{\alpha}) \) and \( \chi \) in a suitable region of convergence we have:

\[
ev_{1} \circ \Delta_{\chi,\alpha}^{Y}(\phi) = \int_{\tilde{Y}_{\alpha}} \phi(y) \nu^{-1}(y) |\omega_{Y_{\alpha}}(\cdot)|^{1}(y)
\]

for a suitable \( A \times H_{\alpha} \)-eigenform \( \omega_{Y_{\alpha}} \) on \( Y_{\alpha} \) with eigencharacter:

\[
c_{\alpha} \times \delta_{G_{\xi} \cap U_{P_{\alpha}}}^{-1} \ni
\]

(unnormlized action of \( A \)) where \( c_{\alpha} \) and \( \chi' \) satisfy: \( \nu^{-1}(c_{\alpha})^{-1} = e^{\alpha}1,1 \).

For \( (Y, \alpha) \) of type \( N \) the same holds for \( \nu \circ \Delta_{\chi,\alpha}^{Y} \), provided that \( \xi \in Y_{\xi} \).

Proof. The proof is completely analogous to that of Lemma 3.4.2. The only thing to notice is that in case \( N \), if we consider the image of \( \tilde{Y}_{\alpha} = (B_{\alpha} H_{\alpha})(k) \) in \( Y \) under the action map \( g \mapsto \xi \cdot g^{-1} \), it belongs to \( Y_{\alpha} \).

For the computation, we pick the point \( \xi \) as follows: In cases \( U, (U, \psi) \) and \( T \) we choose \( \xi \in Y_{\xi} \), and more precisely we choose \( \xi \) in the distinguished \( B_{\alpha} \)-orbit (cf. Proposition 3.3.6). The choice of \( \xi \) in case \( N \) will be discussed in §5.8. (We caution the reader that in case \( N \) it will not be possible, in general, to choose \( \xi \) in the distinguished \( B_{\alpha} \)-orbit on \( Y \) (cf. §5.8), and for that reason the function \( \nu(\chi')^{-1}(y) \) of Lemma 5.1.1 on \( \tilde{Y}_{\alpha} \) will not, in general, be equal to one at \( U_{\alpha}1 \).) To simplify notation, we temporarily denote the distribution \( \nu \circ \Delta_{\chi,\alpha}^{Y} \) on \( U_{\alpha} \setminus L_{\alpha} \) (respectively \( \nu \circ \Delta_{\chi,\alpha}^{Y} \) in case \( N \)) by \( \Delta_{\chi}^{Y} \) (resp. \( \Delta_{\chi}^{Y} \)). Hence:

\[
\Delta_{\chi}^{Y} : S(U_{\alpha} \setminus L_{\alpha}) \rightarrow \mathbb{C}
\]

is an \( A \times H_{\alpha} \)-eigendistribution, with eigencharacter (considering the unnormalized action of \( A \)):

\[
\chi^{-1} e^{-\alpha}1,1 \times \delta_{G_{\xi} \cap U_{P_{\alpha}}}^{-1} \nu.
\]

To complete the reduction to \( \text{SL}_2 \), it suffices to consider the factorization: \( L_{\alpha} = Z_{\alpha} L'_{\alpha} \) (where \( Z_{\alpha} \) is the centre of \( L_{\alpha} \)) and to notice that the distribution \( \Delta_{\chi}^{Y} \) is

---

\(^6\)The measures have to be chosen in compatible ways, as we did in §3.4.
smooth along $Z_\alpha$-orbits with respect to Haar measure on $Z_\alpha$. More precisely, we can pick a small subgroup-neighborhood $N$ of $1 \in Z_\alpha$ such that:

- A neighborhood of $U_\alpha \setminus L'_\alpha$ is isomorphic to $U_\alpha \setminus L'_\alpha \times N$ under the action map.
- The restriction of $\Delta^2_\chi$ to this neighborhood can be written as: Haar measure on $N \times$ a distribution $\Delta^2_\chi$ on $U_\alpha \setminus L'_\alpha$.

Since the kernel of Fourier transform is supported on $U_\alpha \setminus L'_\alpha \times U_\alpha \setminus L'_\alpha$, it is enough for the functional equations to compute the Fourier transform of $\Delta^2_\chi$ (having fixed a Haar measure on $N$ which is independent of $\chi$). We are free to pick the Haar measure on $N$ (and hence vary $\Delta^2_\chi$ up to a constant which is independent of $\chi$), but we need to make the same choices when relating the distributions $\Delta^2_\chi$ coming from two orbits $Y, Z$ of maximal rank in the same $P_\alpha$-orbit. In particular, the corresponding distributions $\Delta^2_\chi$ will be related to each other in the way discussed in §3.3.4.

5.2. The result. The computation has now been reduced to the case of $\text{SL}_2$. To present the functional equations, we need to introduce some extra language: If $\lambda : \mathbb{G}_m \to A_Y$ is a cocharacter, its image is an algebraic subgroup $M_\lambda$ of $A_Y$ we say that a character $\tilde{\chi}$ of $A_Y$ is $\lambda$-unramified if it is unramified on $M_\lambda$.

Finally, in the split case $T$ we introduce some extra data coming from the geometry of the spherical variety. More precisely, $YP_\alpha$ is the union of $Y$ and two $k$-rational divisors $D, D'$. They define valuations $\nu_D, \nu_{D'} : k(Y) \to \mathbb{Z}$ which, restricted to $k(Y)^{(B)}$ define homomorphisms: $\mathcal{X}(Y) \to \mathbb{Z}$. These, in turn, can be identified with coweights $\tilde{\nu}_D, \tilde{\nu}_{D'}$ into $A_Y$. It is known that $\tilde{\nu}_D = -w_\alpha \tilde{\nu}_{D'}$ and that $\tilde{\nu}_D + \tilde{\nu}_{D'} = \tilde{\alpha}$ as elements of $\mathcal{X}(Y)^*$. (As we will see later, the image of $\tilde{\nu}_D$ in $A_Y$ coincides with the image the stabilizer in $B$ modulo $U$ of a point on $D$.)

We are ready now to state the functional equations for $\Delta^2_\chi$ (resp. $\Delta^2_\tilde{\chi}$) and for $w = w_\alpha$, a simple reflection. The data that we need is: The type of $(\chi, \alpha)$ $(U, (U, \psi), T$ split, $T$ non-split or $N)$; in case $T$ split, the coweights $\tilde{\nu}_D, \tilde{\nu}_{D'}$ (as we saw, knowledge of one implies knowledge of the other) and the modular character $\delta(U_{\alpha, \psi})$, as a character of the stabilizer of $\xi$ in $P_\alpha$; in case $N$, whether the coset $\zeta$ corresponds to an integral/non-integral, split/non-split torus.

Finally, we need to know the character $\tilde{\chi}$. We remind that $\tilde{\chi}$ is not really a character of $A_Y$ but rather a character of its preimage in $A(k)$ which satisfies the condition:

$$\tilde{\chi} \nu D^*_\chi|_{B_\xi} = \delta B_\xi.$$ 

In any case, for a co-root $\tilde{\alpha}$ the quantity $e^{\tilde{\alpha} \langle \tilde{\chi} \rangle}$ makes sense. It also makes sense in case $T$ to consider the expression $e^{\tilde{\nu}_D (\tilde{\chi} \nu D^*_\chi)_{(U_{\alpha, \psi})}}$, as we will explain in §5.6.

5.2.1. Theorem. In cases $U, (U, \psi)$ and $T$ the functional equations read as follows:

Case U: We have

$$\mathcal{F}_{w_\alpha} \Delta^2_\chi = \Delta^2_{w_\alpha \chi} \begin{cases} -e^{-\tilde{\alpha} \frac{1 - e^{-\alpha}}{1 - e^{\alpha}}} (\chi) & \text{if } \alpha \text{ lowers } Y \\
-e^{-\tilde{\alpha} \frac{1 - e^{\alpha}}{1 - e^{-\alpha}}} (\chi) & \text{if } \alpha \text{ raises } Y. \end{cases}$$

Case $U, \psi$: We have

$$\mathcal{F}_{w_\alpha} \Delta^2_\chi = \Delta^2_{w_\alpha \chi}.$$
Case T, split: We have
\[ \mathcal{F}_{w \alpha} \Delta^Y = \Delta^Y_{w \alpha \hat{\chi}} \cdot (1 - q^{-1}e^{-\tilde{\chi}(f)})(1 - q^{-1}e^{-\tilde{\chi}(g)}) \left( \hat{\chi}(\psi) \delta_{(U_{w \alpha})}^{-\alpha} \right) \Delta^{w \alpha}_{\hat{\chi}} \]
if \( \hat{\chi} \) is \( \tilde{\chi} \)-unramified, and
\[ \mathcal{F}_{w \alpha} \Delta^Y = \Delta^Y_{w \alpha \hat{\chi}} \cdot e^{-m\tilde{\alpha}(\chi)} \]
if it is \( \tilde{\chi} \)-ramified of conductor \( p^m \).

Case T, non-split: We have
\[ \mathcal{F}_{w \alpha} \Delta^Y = \Delta^Y_{w \alpha \hat{\chi}} \cdot \frac{1 - q^{-1}e^{-\tilde{\alpha}(\chi)}}{1 - q^{-1}e^{-\tilde{\alpha}(\chi)}} \]
if \( \hat{\chi} \) is \( \tilde{\alpha} \)-unramified, and
\[ \mathcal{F}_{w \alpha} \Delta^Y = \Delta^Y_{w \alpha \hat{\chi}} \cdot e^{-\tilde{\alpha}(\chi)} \]
otherwise.

Case N: Here we have four cases, according as the coset \( \zeta \) corresponds to a split or non-split, integral or non-integral torus, and in each of these cases we have to distinguish between \( \hat{\chi} \) being \( \tilde{\alpha} \)-ramified or not. The functional equations for this case are given in \( \S 5.8 \).

5.3. Preliminaries of the computation. We are going to compute the proportionality constant case-by-case. We will use many times the following fact: Consider usual Fourier transform of functions and distributions on \( k \), with respect to our fixed character \( \psi \) and the measure \( dx \) normalized as: \( dx(o) = 1 \). The Fourier transform of a function \( f \) on \( k \) will be denoted by \( \hat{f} \). By Tate’s thesis, the Fourier transform of the distribution \( \chi(x)d^x \) (where \( d^x \) denotes some fixed multiplicative Haar measure on \( k^\times \)) is equal to:
\[ \frac{1 - q^{-1} \chi^{-1}(\omega)}{1 - \chi(\omega)} \chi^{-1}(x)|x|d^x x = \frac{1 - q^{s-1}}{1 - q^{-s}}|x|^{-s}d^x x \]
if \( \chi(x) = |x|^s \) is unramified, and:
\[ q^{-m}\tau(\chi) \cdot \chi^{-1}(x)|x|d^x x \]
if \( \chi \) is a ramified character of conductor \( p^m \), where \( \tau(\chi) \) is the Gauss sum:
\[ \tau(\chi) = \sum_{\epsilon \in o^\times / (1+p^m)} \chi(\omega^m \epsilon)\psi(\omega^{-m} \epsilon). \]

Indeed, we know a priori from equivarience properties that the Fourier transform of \( \chi(x)d^x \) must be a multiple of \( \chi^{-1}(x)|x|d^x x \), and that multiple can be computed from the formula: \( \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \). Tate computes: \( \langle 1_{w}, \chi d^x \rangle =: \zeta(1_{w}, \chi) \| \bullet \| = (1 - \chi(\omega))^{-1} \) if \( \chi \) is unramified. If \( \chi = \eta \cdot | \bullet |^s \) is ramified, with \( \eta \) a unitary character, we compute the zeta integral of \( f_m := \psi(x)1_{p^m} \) whose Fourier transform is \( \hat{f}_m = q^{m}1_{1+p^m} \) and we get:
\[ \langle f_m, \chi d^x \rangle = \zeta(f_m, \eta) \| \bullet \|^s \rangle = q^{m}\tau(\eta) \text{Vol}(p^m) \]
and \( \langle \hat{f}_m, \chi^{-1} \bullet |d^x \rangle = \zeta(f_m, \eta^{-1} \bullet |^{-s+1}) = q^{m} \text{Vol}(p^m) \).

Notice that here we are using our simplifying assumption that \( k \) is unramified over \( \mathbb{Q}_p \); these equations would have to be modified otherwise.
5.4. **Case U.** Let us identify $L'_{\alpha}$ with $SL_2$ and $U_{\alpha}\backslash L'_\alpha$ with $A^2$ over $\mathfrak{o}$. The distribution $\Delta_\chi^2$ (defined at the end of §5.1) is here an $A_2 \times U_{\alpha}$-eigendistribution. Based on Lemma 5.1.1, depending on whether $Y$ is raised or lowered by $\alpha$, the distribution $\Delta_\chi^2$ on $U_{\alpha}\backslash L'_\alpha$ is, up to an integral change of coordinates and up to a common scalar multiple one of the following two:

$$\Delta_{1,\chi} = |x|^{s_1} dx dy \quad \text{and} \quad \Delta_{2,\chi} = |y|^{s_2} \delta_0(x) dy$$

where $s_1, s_2 \in \mathbb{C}$ is such that $q^{s_1+1} = q^{s_2} = e^{-\hat{\alpha}}(\chi \nu) = e^{-\hat{\alpha}}(\chi)$.

Now we see immediately that

$$F_{w_\alpha} \Delta_{1,\chi} = \frac{1 - q^{s_1}}{1 - q^{-s_1-1}} \Delta_{2,\omega^\alpha \chi} = \frac{1 - q^{-1} e^{-\hat{\alpha}}(\chi)}{1 - e^{\hat{\alpha}}(\chi)} \Delta_{2,\omega^\alpha \chi}$$

and

$$F_{w_\alpha} \Delta_{2,\chi} = \frac{1 - q^{s_2}}{1 - q^{-s_2-1}} \Delta_{1,\omega^\alpha \chi} = \frac{1 - e^{-\hat{\alpha}}(\chi)}{1 - q^{-1} e^{\hat{\alpha}}(\chi)} \Delta_{1,\omega^\alpha \chi}$$

Hence, for $(\alpha, Y)$ of type $U$, we have:

$$b^Y_{w_\alpha}(\chi) = \begin{cases} -e^{-\hat{\alpha}} \frac{1 - q^{-1} e^{-\hat{\alpha}}}{1 - e^{-\hat{\alpha}}} (\chi) \text{ if } \alpha \text{ lowers } Y \\ -e^{-\hat{\alpha}} \frac{1 - e^{\hat{\alpha}}}{1 - q^{-1} e^{\hat{\alpha}}} (\chi) \text{ if } \alpha \text{ raises } Y. \end{cases}$$

5.4.1. **Example.** Let $G' = G \times G$, $H = G^{\text{diag}}$, and consider the spherical variety $X = G \backslash G'$ of $G'$ with action $(g_1, g_2) \cdot x = g_1^{-1} x g_2$. Let us choose a Borel subgroup $B' = B^- \times B$, so that $HB'$ is open. The little Weyl group is generated by $w_\alpha w_\check{\alpha}$, where $-\check{\alpha}$ is a simple root of $B^-$ in the first copy of $G$ and $\check{\alpha}$ the corresponding root of $B$ in the second copy. The admissible characters $\chi'$ are of the form: $\chi^{-1} \otimes \chi$. (We put $\chi^{-1}$ in the first copy so that the eigenfunction $\Omega_{\chi^{-1} \otimes \chi}$ is proportional to the matrix coefficient $\langle \pi(g)v, \check{v}\rangle$ with $\pi = I(\chi)$ and $v, \check{v}$ the unramified vectors.) Notice that $w_\alpha$ lowers the open orbit to some orbit $Y_\alpha$, which $w_\check{\alpha}$ then raises back to the open orbit. Hence we compute: $e^{-\hat{\alpha}} e^{\check{\alpha}} b_{w_\alpha w_\check{\alpha}}(\chi') = \frac{1 - q^{-1} e^{-\hat{\alpha}}}{1 - e^{-\hat{\alpha}}} (\chi') \frac{1 - e^{\check{\alpha}}}{1 - q^{-1} e^{\check{\alpha}}} (\chi^{-1})$ and the formula reads:

$$\Omega_{\chi}(g_\lambda) = e^{-\check{\lambda}} (\delta^2) \sum_{W} \prod_{\check{\alpha} > 0, \alpha < 0} \frac{1 - q^{-1} e^{-\hat{\alpha}}}{1 - e^{-\hat{\alpha}}} \frac{1 - e^{\check{\alpha}}}{1 - q^{-1} e^{\check{\alpha}}} (\chi) e^\check{\lambda}(w_\check{\alpha} \chi)$$

which up to a factor which is independent of $\check{\lambda}$ is equal to:

$$e^{-\check{\lambda}} (\delta^2) \sum_{w \in W} \prod_{\check{\alpha} > 0} \frac{1 - q^{-1} e^{\hat{\alpha}}}{1 - e^{\hat{\alpha}}} e^{\check{\lambda}(w_\check{\alpha} \chi)}.$$

This is, of course, Macdonald’s formula for zonal spherical functions which was reproven by Casselman in [Ca80, Theorem 4.2].

5.5. **Case $U_\psi$:** Here the distribution $\Delta_\chi^2$ has eigencharacter $\psi$ under the action of $U$. Under the same identifications as above we have, up to an integral change of coordinates:

$$\Delta_\chi^2 = |x|^{s_1} \psi^{-1} \left( \frac{y}{x} \right) dx dy$$

where $s \in \mathbb{C}$ is such that $q^{s_1+1} = e^{-\hat{\alpha}}(\chi)$.

---

7To compare with the result in loc.cit., recall that our $\check{\lambda}$ is anti-dominant.
Here we see that
\[ F_{w_\alpha} |x|^s \psi^{-1} \left( \frac{y}{x} \right) dx dy = |x|^{-s-2} \psi^{-1} \left( \frac{y}{x} \right) dx dy \]
in other words:
\[ F_{w_\alpha} \Delta^2_\chi = \Delta^2_{\chi_\alpha} \]
and hence:
\[ (5.3) \]
\[ \delta_{w_\alpha}(\chi) = 1 \]

5.5.1. Example. Let \( H = U \) with the standard Whittaker character. Then \( W_X = W \), all simple roots are of type \((U, \psi)\), and we have:
\[ \Omega_\chi(g_\chi) = e^{-\tilde{\lambda}}(\delta_{\omega_\chi}) e^\delta(\chi) \sum_W \sigma(w) e^{-\tilde{\rho} + \tilde{\lambda}}(w_\chi). \]
This\(^7\) is the Shintani-Casselman-Shalika formula \([CS80]\).  

5.5.2. Example. Let \( G = \text{GL}_{2n} \) and let \( H \) be a conjugate of the Shalika subgroup such that \( HB \) is open, equipped with the Shalika character \( \Psi \). The Shalika subgroup of is the semidirect product of \( \text{GL}_n \), embedded diagonally in the \( \text{GL}_n \times \text{GL}_n \) parabolic, with the unipotent radical of this parabolic; the Shalika character is the complex character \((g, u) \mapsto \psi(\text{tr} u)\) of this subgroup. In this case, if we enumerate \( w_1, \ldots, w_{2n-1} \) the simple reflections in the Weyl group of \( G \), the little Weyl group is generated by the elements \( w_i w_{2n-i} \) (\( 1 \leq i \leq n-1 \)) and \( w_n \), hence can be identified with the Weyl group of the subgroup \( \text{Sp}_{2n}(\mathbb{C}) \) of the dual group \( \hat{G} = \text{GL}_{2n}(\mathbb{C}) \). The character \( \chi \) is of the form \( \chi_0 \otimes w_0 \chi_0^{-1} \), where \( \chi_0 \) is a character of the maximal torus of \( \text{GL}_n \), and \( w_0 \) denotes the longest Weyl element of \( \text{GL}_n \). The action of the element \( w_i w_{2n-1} \) on Knop’s graph is by lowering \( X \) to some orbit \( Y \), and then raising \( Y \) back to the open orbit, so the corresponding factor is:
\[ e^\alpha e^{\tilde{\alpha}}(\chi) b_{w_\alpha w_i}(\chi) = \sum_{X} \prod_{\alpha \in \Phi^+_{\text{sp}_{2n}}: \alpha > 0, \omega \alpha < 0} \frac{1 - q^{-1} e^{-\alpha}}{1 - e^{-\alpha}} \frac{1 - e^{\tilde{\alpha}}}{1 - q^{-1} e^{\tilde{\alpha}}} \chi(\cdot) \]
where \( \alpha, \tilde{\alpha} \) are the roots corresponding to \( w_1, w_{2n-i} \), and we have used the fact that \( \chi \) is of the aforementioned form. On the other hand, the pair \((X, w_n)\) is of type \((U, \psi)\), hence the corresponding factor is \( e^{\tilde{\alpha}}(\chi) b_{\alpha}(\chi) = \sum_{X} \prod_{\alpha \in \Phi^+_{\text{sp}_{2n}}: \alpha > 0, \omega \alpha < 0} \frac{1 - e^{\tilde{\alpha}}}{1 - e^{-\alpha}} \chi(\cdot) \]
Notice that \( \chi \) can be considered as an element of the maximal torus of \( \text{Sp}_{2n}(\mathbb{C}) \subset \hat{G} \), and the coroots \( \tilde{\alpha} \) which appear in the above factors are the roots of \( \text{Sp}_{2n}(\mathbb{C}) \). Hence the unramified Shalika function is given by:
\[ \Omega_\chi(g_\chi) = e^{-\tilde{\lambda}}(\delta_{\omega}) \sum_{W_X} \prod_{\alpha \in \Phi^+_{\text{sp}_{2n}}: \alpha > 0, \omega \alpha < 0} \frac{1 - q^{-1} e^{-\alpha}}{1 - e^{-\alpha}} \frac{1 - e^{\tilde{\alpha}}}{1 - q^{-1} e^{\tilde{\alpha}}} \chi(\cdot) \]
which, up to a factor independent of \( \tilde{\lambda} \), is:
\[ e^{-\tilde{\lambda}}(\delta_{\omega}) \sum_{W_X} \sigma(w) \prod_{\alpha \in \Phi^+_{\text{sp}_{2n}}} (1 - q^{-1} e^{\tilde{\alpha}}) e^{-\tilde{\rho} + \tilde{\lambda}}(w_\chi) \]
(σ(u) denoting the sign of u as an element of W), the formula of [Sa06, Theorem 2.1].

The symbols \( \Phi^S_{Sp_{2n}} \) and \( \Phi^L_{Sp_{2n}} \) denote the sets of short and long roots of \( Sp_{2n} \), respectively.

### 5.6. Case \( T \), split

The goal of the discussion that follows is to identify the distribution \( \Delta^2_\chi \) in case \( T \), split. The following is an easy fact:

#### 5.6.1. Lemma

In case \( T \), the group \( A \cap H_\alpha \) is central in \( L_\alpha \).

**Proof.** Its image in \( L_\alpha / R(L_\alpha) \simeq PGL_2 \) is trivial. □

This implies that the antidiagonal copy: \( g \mapsto (g^{-1}, g) \) of this group is the kernel of the morphism: \( A \times H_\alpha \to \text{Aut}(U_\alpha \backslash L_\alpha) \). Denote the quotient by \( \overline{A \times H_\alpha} \), i.e:

\[
\overline{A \times H_\alpha} := (A \times H_\alpha)/(A \cap H_\alpha)^{\text{diag}} \hookrightarrow \text{Aut}(U_\alpha \backslash L_\alpha).
\]

Let \( \bar{\nu}_D, \bar{\nu}_{D'} : G_m \to A_Y \) denote the cocharacters defined in §5.2. We will prove:

#### 5.6.2. Proposition

1. There is a natural way to identify \( \bar{\nu}_D, \bar{\nu}_{D'} \) with cocharacters \( \hat{\nu}_D, \hat{\nu}_{D'} \) into \( \overline{A \times H_\alpha} \) such that:
   
   i) Their compositions with the map \( \overline{A \times H_\alpha} \to A_Y \) are equal to \( \bar{\nu}_D, \bar{\nu}_{D'} \), respectively.
   
   ii) Their images are automorphisms which preserve the subvariety \( U_\alpha \backslash L_\alpha \).
   
   iii) Under an \( \alpha \)-isomorphism: \( U_\alpha \backslash L_\alpha \simeq k^2_{x,y} \) and up to an integral change of coordinates, the automorphisms \( \bar{\nu}_D(a), \bar{\nu}_{D'}(a) \) restrict to \( U_\alpha \backslash L_\alpha \) as:

   \( (x,y) \mapsto (ax, y) \) and \( (x,y) \mapsto (x, ay) \).

2. The distribution \( \Delta^2_\chi \) is an eigendistribution for \( \bar{\nu}_D(k^\times), \bar{\nu}_{D'}(k^\times) \), with eigen-character equal to the restriction of \( \chi^{-1} e^{-a} \delta^{\frac{1}{2}} \otimes \delta_G \| U_{P_{\alpha}} \setminus U_{P_{\alpha}} \nu \) to their image.

Notice that the statement about the eigencharacter makes sense, because we have \( \chi^{-1} e^{-a} \delta^{\frac{1}{2}} = \delta_G \| U_{P_{\alpha}} \setminus U_{P_{\alpha}} \nu \) on \( A \cap H_\alpha \). Indeed, the right hand side is equal to \( \delta_{U_{P_{\alpha}}} \delta^{\frac{1}{2}}(U_{P_{\alpha}}) \xi \) and in this case we have: \( (U_{P_{\alpha}}) \xi = U_{\xi} \). Therefore, the equality follows from (3.7). By a slight abuse of notation, we will be denoting the pull-back of this character to \( k^\times \) by \( \chi^{-1} \nu^{-1} \delta^{\frac{1}{2}}(U_{P_{\alpha}}) \xi \circ e^{\nu D} \) (and similarly for \( D' \)).

**Proof.** Let \( z \in D \) be any point. Consider the series of quotients: \( B \to A \to A_Y \).

Let \( B' \subset B, A' \subset A \) be the preimages of \( \bar{\nu}_D(G_m) \). We claim:

The subgroup \( B' \) coincides with the stabilizer of the \( U \)-orbit of \( z \).

An element \( b \in B \) is in the stabilizer of \( zU \) if and only if \( z \cdot b \) cannot be separated by a regular \( B \)-eigenfunction on \( D \). The pair \( (Y, \alpha) \) being of type \( T \), every \( B \)-eigenfunction on \( D \) extends to a \( B \)-eigenfunction on \( YP_{\alpha} \). By definition of \( \bar{\nu}_D \), the eigencharacters of eigenfunctions which restrict non-trivially to \( D \) are precisely those which are orthogonal to \( \bar{\nu}_D \). This proves the claim.

Under the identification: \( YP_{\alpha} \simeq (P_{\alpha})_\xi \P_{\alpha} \) (following from a choice of point \( \xi \)), we may let \( \hat{D} \) denote the orbit of \( A \times H_\alpha \) on \( U \P_{\alpha} = U \backslash L_\alpha \) corresponding to the \( B \)-orbit \( D \) on \( YP_{\alpha} \). Then the above claim translates to the following:

For every \( \hat{z} \in \hat{D} \) the stabilizer of \( z \) in \( A \times H_\alpha \) is isomorphic to \( A' \).

---

8 Again, here, one needs to substitute \( \hat{\lambda} \) by \( w_{\lambda} \hat{\lambda} \) to arrive at the formula of loc.cit. Notice that here \( e^{-\lambda} = e^{w_0 \lambda} \) where \( w_0 \) is the longest element of the Weyl group, so the signs of the coweights inside of the \( W_X \)-sum will be inverted.
Clearly, this isomorphism does not depend on the choice of $z$. Since the antidiagonal copy of $A \cap H$ acts trivially, we get:

For every $\tilde{z} \in \tilde{D}$ the stabilizer of $z$ in $\tilde{A} \times H$ is canonically isomorphic to $\tilde{v}_D(G_m)$.

Hence the cocharacter $\tilde{v}_D$ of the proposition.

We have a morphism: $U_\alpha \setminus L_\alpha \to L'_\alpha \setminus L_\alpha$. Therefore the stabilizer, in $A \times L_\alpha$, of any point on $U_\alpha \setminus L'_\alpha$ stabilizes $U_\alpha \setminus L_\alpha$, as well. Hence claim (i).

Identify $U_\alpha \setminus L'_\alpha \cong A^2_{x,y} \setminus \{0\}$. For a suitable choice of $\alpha$-isomorphisms, we can identify the non-open orbits of stabilizers $\tilde{A} \times H$ of $U_\alpha \setminus L_\alpha$ with the two coordinate axes. Let $f$ be a $B$-eigenfunction on $H_\alpha \setminus L_\alpha$ with eigencharacter $e_\alpha$. Equivalently, $f \circ \iota$ (where $\iota$ denotes inversion in the group) can be thought of as an $A \times H_\alpha$-eigenfunction on $U_\alpha \setminus L_\alpha$ with eigencharacter $e_\alpha \times 1$. From the definitions, $f \circ \tilde{v}_D(a) = af$. Since $\tilde{v}_D(a)$ stabilizes the points of a coordinate axis and preserves the other axis, it follows that $\tilde{v}_D(a)$ is the automorphism: $(x,y) \mapsto (ax,y)$ or $(x,y) \mapsto (y,ax)$.

The statement about the eigencharacter follows from the eigencharacter of $\Delta^Y_{x,\alpha}$, cf. Lemma 5.1.1.

Therefore, up to an integral change of variables we have on $A^2_{x,y}$:

$$\Delta^2_{x,y} = \eta_D(x)\eta_D'(y)d^xzd^y$$

where $\eta_D = \tilde{\chi}\nu\delta^\frac{1}{2}\delta^{-1}_{(U_{\nu_\alpha})_\xi} \circ \tilde{v}_D$, and similarly for $\eta_D'$. Its Fourier transform is, therefore, as follows:

(1) If $\tilde{\chi}$ is $\tilde{v}_D$-unramified (equivalently, since $\tilde{v}_D + \tilde{v}_D' = \tilde{\alpha}$, if it is $\tilde{v}_D'$-unramified):

$$F_{w_\alpha}\Delta^2_{\tilde{\chi}} = \frac{(1 - q^{-1}e^{-\tilde{v}_D})(1 - q^{-1}e^{-\tilde{v}_D'})}{(1 - e^{\tilde{v}_D})(1 - e^{\tilde{v}_D'})}(\tilde{\chi}\nu\delta^\frac{1}{2}\delta^{-1}_{(U_{\nu_\alpha})_\xi})\Delta^2_{\tilde{w}_\alpha \tilde{\chi}}$$

Consequently:

$$\begin{align*}
\Delta^2_{\tilde{w}_\alpha(\tilde{\chi})} &= \frac{(1 - q^{-1}e^{-\tilde{v}_D})(1 - q^{-1}e^{-\tilde{v}_D'})}{(1 - e^{\tilde{v}_D})(1 - e^{\tilde{v}_D'})}(\tilde{\chi}\nu\delta^\frac{1}{2}\delta^{-1}_{(U_{\nu_\alpha})_\xi}).
\end{align*}
$$

Remark. The vanishing of the terms $(1 - e^{\tilde{v}_D})(\tilde{\chi}\nu\delta^\frac{1}{2}\delta^{-1}_{(U_{\nu_\alpha})_\xi})$ and $(1 - e^{\tilde{v}_D'})(\tilde{\chi}\nu\delta^\frac{1}{2}\delta^{-1}_{(U_{\nu_\alpha})_\xi})$ in the denominator is precisely the condition for the orbit of $D$ (resp. $D'$) to support a morphism from $I(\tilde{\chi})$, in other words a condition for $\Delta^2_{\tilde{\chi}}$ (resp. $\Delta^2_{\tilde{\chi}'}$) to be defined. This, of course, was expected -- see [Ga99] or [Sa08, §4.6].

(2) If $\tilde{\chi}$ is $\tilde{v}_D$-ramified:

$$F_{w_\alpha}\Delta^2_{\tilde{\chi}} = q^{-2m}\tau(\eta_D)\tau(\eta_D')\eta_D(-1)\Delta^2_{\tilde{w}_\alpha \tilde{\chi}}.$$

We can simplify the above expression as follows: From the fact that $\hat{f}(x) = f(-x)$ and the formula $\eta(x)d^x x = q^{-m}\tau(\eta)\eta^{-1}(x)|x|d^x x$ for a ramified character $\eta$ of conductor $p^m$ it follows that $\tau(\eta)\tau(\eta^{-1}) = \eta(-1)q^m$. Applying this to $\eta = \eta_D$ and taking into account that $\eta_D' = \tilde{\chi}\nu\delta^\frac{1}{2}\delta^{-1}_{(U_{\nu_\alpha})_\xi} \circ \tilde{v}_D$.
\[ e^{\hat{\alpha}} \cdot \eta_D^{-1} \Rightarrow \tau(\eta_D^\prime) = \chi \nu \hat{\delta} \delta_{(\hat{\nu} \alpha)} \cdot e^{\hat{\alpha}}(\overline{\omega}^{-m}) \cdot \tau(\eta^{-1}) = q^m e^{-m \hat{\alpha}}(\chi) \tau(\eta^{-1}) \]
we get:
\[
F_{w_\alpha} \Delta^2_\hat{\chi} = e^{-m \hat{\alpha}}(\chi) \Delta^2_{w_\alpha \hat{\chi}}.
\]

Hence:
\[
| \mathcal{B}^Y_{w_\alpha}(\hat{\chi}) | = e^{-m \hat{\alpha}}(\chi).
\]

\[ (5.5) \]

5.6.3. Example. Let \( G = (\text{PGL}_2)^3 \) and \( H \) a \( G(\mathfrak{o}) \)-conjugate of the diagonal copy of \( \text{PGL}_2 \) such that \( HB \) is open. For instance, if we take for \( B \) three copies of the group of upper triangular matrices, then \( H \) can be the diagonal copy of \( \text{PGL}_2 \) conjugated by the element \(( \frac{1}{1}, \frac{1}{1}, \frac{1}{1} ) \). Here the little Weyl group is the whole Weyl group and \( A'_\chi = A^* \). Let \( \alpha_1, \alpha_2, \alpha_3 \) denote the simple positive roots in each copy respectively. Then we easily see that \( (X, \alpha_i) \) is of type \( T \) and that there are three orbits \( D_1, D_2, D_3 \) of codimension one, with \( \hat{\nu}_D_i = \frac{1}{2} (-\alpha_i + \sum_{j \neq i} \alpha_j) \) and the \( D_j, j \neq i \) being the two orbits in the \( P_{\alpha_i} \)-orbit of \( X \). Moreover, \( \delta_{B_\xi} \) is trivial. Therefore:
\[
b_{\alpha_i}(\chi) = \frac{(1 - q^{-\frac{1}{2}} e^{\alpha_i / 2}) (1 - q^{-\frac{1}{2}} e^{-\alpha_i / 2})}{(1 - q^{-\frac{1}{2}} e^{\alpha_i / 2}) (1 - q^{-\frac{1}{2}} e^{-\alpha_i / 2})} \]
and, up to a factor independent of \( \chi \) (see also Example 7.2.4), \( \Omega_\chi(\chi_\alpha) \) is equal to:
\[
e^{-\hat{\delta} \frac{1}{2} \sum \sigma(w) (1 - q^{-\frac{1}{2}} e^{\alpha_i / 2}) (1 - q^{-\frac{1}{2}} e^{-\alpha_i / 2})}.
\]

\[ (1 - q^{-\frac{1}{2}} e^{\alpha_i / 2}) (1 - q^{-\frac{1}{2}} e^{-\alpha_i / 2}) e^{-\hat{\delta} \lambda} w \chi). \]

5.7. Case T non-split: The analysis here simpler than in the split case, because all one-dimensional non-split tori of \( L_\alpha \) are contained in \( L_\alpha' \), and therefore \( H_\alpha \cap L_\alpha' \) is a spherical subgroup of \( L_\alpha' \). Under an \( \mathfrak{o} \)-identification of \( U_\alpha \backslash L_\alpha' \) with \( \mathbb{A}_{xy}^\ast \setminus \{ 0 \} \), the group \( H_\alpha \cap L_\alpha' \) is the special orthogonal group of a non-degenerate, non-split binary quadratic form \( Q \), and therefore:
\[
\Delta^2_\hat{\chi} = \eta(Q(x, y)) dxdy
\]
where \( \eta = \hat{\chi} \hat{\delta} \circ e^{\frac{\hat{\delta}}{2}} \). By our assumption that the action of the group on \( X \) is smooth over \( \mathfrak{o} \), up to an integral change of coordinates we have \( Q(x, y) = ax^2 + \kappa y^2 \), where \( -\kappa \) is a non-square unit element. We can write \( \eta = \eta_2 | \mathfrak{o} | | \mathfrak{z} \) where \( q^{s+1} = e^{-\alpha}(\chi) \) and \( \eta_2 \) is either the trivial character or a ramified quadratic character.

A fact which will make our computations easier is that if \( -\kappa \) is not a square then \( |x^2 + \kappa y^2| = \max \{|x^2|, |\kappa y^2|\} \). Since, moreover, \( \kappa \) is a unit, the value of \( \eta_2(x^2 + \kappa y^2) \) only depends on the restriction of \( \eta_2 \) on units. We will compute the constant in the functional equations through the Parseval formula:
\[
\langle \Delta^2_\hat{\chi}, f \rangle = \langle \mathcal{F} \Delta^2_\hat{\chi}, \mathcal{F} f \rangle
\]
with \( f = \mathcal{F} f = 1_{x,y} \).

A basic relation which will be used repeatedly is
\[
\int_{|x| < |y| \leq 1} |y|^s dxdy = \frac{q^{-1}(1 - q^{-1})}{1 - q^{-s-2}}.
\]
\( \tilde{\chi} \) is \( \frac{q}{2} \)-unramified. We compute:

\[
\langle \Delta_{\tilde{\chi}}^2, 1_{\sigma^2} \rangle = \int_{a^2} |x^2 + \kappa y^2|^2 \, dx \, dy = \int_{|x|<|y| \leq 1} |y|^s \, dx \, dy + \int_{|y|<|x| \leq 1} |x|^s \, dx \, dy + \int_{|x|=|y| \leq 1} |x|^s \, dx \, dy + \int_{|x|=|y| \leq 1} |y|^s \, dx \, dy
\]

Similarly, for \( \Delta_{\nu, \tilde{\chi}}^2 = |x^2 + \kappa y^2|^{-\frac{2}{q} - 1} \) we will get:

\[
\langle \Delta_{\nu, \tilde{\chi}}^2, 1_{\sigma^2} \rangle = \frac{1 - q^{-2}}{1 - q^s}
\]

and the quotient of the two is:

\[
b_{\nu, \tilde{\chi}}^Y = \frac{1 - q^{-1}e^{-\tilde{\alpha}(\chi)}}{1 - q^{-1}e^{\tilde{\alpha}(\chi)}}.
\]

Hence:

\[
(5.6)
\]

\[
b_{\nu, \tilde{\chi}}^Y = \frac{1 - q^{-1}e^{-\tilde{\alpha}(\chi)}}{1 - q^{-1}e^{\tilde{\alpha}(\chi)}}.
\]

\( \tilde{\chi} \) is \( \frac{q}{2} \)-ramified. It is an easy exercise in harmonic analysis and Gauss sums to verify the following identity:

\[
\int_{p^{\times}} \eta_2(\kappa + x^2) = \begin{cases} 
0, & \text{if } -1 \in (k^\times)^2 \\
-2q^{-1}, & \text{otherwise.}
\end{cases}
\]

(Recall that \(-\kappa \) is not a square.)

Now we compute:

\[
\langle \Delta_{\chi}^2, 1_{\sigma^2} \rangle = \int_{a^2} \eta_2(x^2 + \kappa y^2)|x^2 + \kappa y^2|^2 \, dx \, dy
\]

\[
= \eta_2(\kappa) \int_{|x|<|y| \leq 1} |y|^s \, dx \, dy + \int_{|y|<|x| \leq 1} |x|^s \, dx \, dy + \int_{|x|=|y| \leq 1} \eta_2(\kappa)|y|^s \, dx \, dy
\]

\[
= (1 - \eta_2(-1)) \frac{q^{-1}(1 - q^{-1})}{1 - q^{-s-2}} + \int_{\sigma^s} |y|^{s+1} \, dy \int_{\sigma^s} \eta_2(x^2 + \kappa y^2) =
\]

\[
= (1 - \eta_2(-1)) \frac{q^{-1}(1 - q^{-1})}{1 - q^{-s-2}} + \begin{cases} 
0, & \text{if } -1 \in (k^\times)^2 \\
-2q^{-1} \frac{1 - q^{-1}}{1 - q^{-s-2}}, & \text{otherwise.}
\end{cases}
\]

Hence the value of the distribution \( \Delta_{\chi}^2 \) on a \( K \)-invariant function is zero. (There is no contradiction here; this just says that the corresponding eigenfunction \( \Omega_{\chi} \) on \( C^\infty(\mathbf{T} \setminus \mathbf{SL}_2) \) has value zero at the chosen point – not that it vanishes identically.)

We will use instead the function \( \Phi_J = 1_{p^{\times} \sigma^s} \) whose Fourier transform has been computed in §4.1:

\[
\Delta_{\chi}^2(\Phi_J) = \int_{p^{\times} \sigma^s} \eta_2(x^2 + \kappa y^2)|x^2 + \kappa y^2|^2 \, dx \, dy = q^{-1}(1 - q^{-1})\eta_2(\kappa)
\]

and we have \( F(\Phi_J) \equiv L_{\tilde{\alpha}(\pi-1)} \Phi_J \) modulo \( K \)-invariants, whence by the vanishing of \( \Delta_{\chi}^2(1_{\sigma^2}) \):

\[
b_{\nu, \chi}(1)(1 - q^{-1})\eta_2(\kappa) = b_{\nu, \chi}(\tilde{\alpha}(\chi)) \langle \Delta_{\chi}^2, 1_{\sigma^2} \rangle = \langle F \Delta_{\chi}^2, \Phi_J \rangle =
\]

\[
= \langle \Delta_{\chi}^2, F \Phi_J \rangle = \langle \Delta_{\tilde{\alpha}(\pi-1)} \Phi_J \rangle = e^{-\tilde{\alpha}(\chi)}(1 - q^{-1})\eta_2(\kappa).
\]
Therefore:

\[ \beta_{\omega, \chi}(\check{\chi}) = e^{-\alpha}(\chi). \]

**Remark.** Notice that in the functional equations for the case \( T \) non-split all that matters is whether \( \check{\chi} \) is a ramified extension of \( \chi \) and not which unramified/ramified extension it is (for instance, the cocharacter \( e^{\Phi} \) does not appear). This is no coincidence: In this case the space \( (T \setminus SL_2)(k) \) has two \( SL_2 \)-orbits (isomorphic to each other), and therefore if instead of the distribution \( \mathcal{S}_{\chi} \) we used an integral over all open \( B_2 \)-orbits, we considered the restriction of that distribution to a single \( SL_2 \)-orbit, then its Fourier transform should also be supported on the same \( SL_2 \)-orbit. But these distributions which are supported on one \( SL_2 \)-orbit do not “see” the extension of \( \chi \) to the whole \( A_Y \), but only to a subgroup of it, and one can check that that subgroup can detect ramification (it contains \( A_Y(\sigma) \)) but not the precise extension of \( \chi \).

5.8. **Case N.** Notice that for an one-dimensional torus \( T \) which is a spherical subgroup of \( L_\alpha \), if \( N_{L_\alpha}(T) \neq Z_{L_\alpha}(T) \) then \( T \subset L'_\alpha \), and therefore in case \( N \) the variety \( H^0_H \) is of type \( T \) associated cocharacters \( \check{\chi} = \frac{\Phi}{2} \) if \( H^0_H \) is split.

To compute the functional equations for \( \Delta_{\check{\chi}, \zeta} \) (§5.3), we first describe how to choose the point \( \xi \in Y_\zeta \). Notice that \( Y_\zeta \) may not contain integral points, therefore \( H^0_H \) cannot always be assumed to be a smooth subgroup scheme over \( \sigma \) – hence, not all of the cases which we will encounter in case \( N \) have already been encountered in case \( T \).

First of all, let \( \xi_0 \in Y(\sigma) \) be a point in the “standard” \( B_0 \)-orbit on \( Y(\sigma) \), and \( \zeta_0 \) the corresponding coset of \( A_{Y, \alpha} \). We identify \( L_\alpha/R(L_\alpha) \) with \( PGL_2 \) and use \( \xi_0 \) to define a map: \( YP_\alpha \to X_2 := N(T) \setminus PGL_2 \) (for an integral torus \( T \) corresponding to the class of \( \xi_0 \)). The latter is the space of binary quadratic forms modulo homotheties. Without loss of generality, \( \xi_0 \) maps to the quadratic form \( x^2 + ay^2 \) for some \( a \in -D(\zeta_0) \subset \sigma^\times \). We choose \( \xi \in Y_\zeta \) such that under the above map \( \xi \) maps to the homothety class of the quadratic form \( x^2 + by^2 \) with \( b \in -D(\zeta) \). (Recall from §3.5 that the discriminant \( D(\zeta) \) is by its definition an element of \( \sigma \setminus \pi^2 \).

From the definition of \( \Delta_{\check{\chi}, \zeta} := ev_1 \circ \Delta_{\check{\chi}, \zeta}^Y \) and Lemma 5.1.1 we deduce that the distribution \( \Delta_{\check{\chi}, \zeta}^2 \) on \( U_\alpha \setminus L'_\alpha \) is given by:

\[ \Delta_{\check{\chi}, \zeta}^2 = \check{\chi}(\xi)\eta(Q(x, y))dxdy \]

where, \( \eta = \chi^2 \cdot e^\Phi \) and \( Q \) is the quadratic form stabilized by \( H^0_H \) and \( \check{\chi} ' \) is the character of Lemma 5.1.1. (The factor \( \check{\chi}(\xi) \) is non-canonical, as is the distribution \( \Delta_{\check{\chi}, \zeta}^2 \); cf. §5.1: what is important is how this factor varies as \( \check{\chi} \) varies.) By our choice of point \( \xi \), up to an integral change of coordinates and up to a constant we have \( Q(x, y) = x^2 + \pi y^2 \), where \( \pi = -D(\zeta) \).

To determine the functional equations for \( \Delta_{\check{\chi}, \zeta}^2 \), if \( \zeta \) corresponds to an integral torus, we can use our computations for the corresponding cases of type \( T \), with \( v_D = \frac{\Phi}{2} \). However, notice that now there is an extra factor of

\[ \frac{\check{\chi}(\xi)}{e^{-\alpha \cdot \omega_0} \check{\chi}(\xi)} = \check{\chi} \circ e^{-\Phi} \left( \frac{D(\zeta)}{D(\xi_0)} \right). \]

In the above, we took into account that if \( \check{\chi}' \) is the character used in the expression (5.8) for \( \Delta_{\check{\chi}, \zeta}^2 \), then the character in the expression for \( \Delta_{\check{\chi}, \zeta}^2 \) is \( e^{-\alpha \cdot \omega_0} \check{\chi}' \). But
the quotient \( \frac{\chi'(\xi)}{\kappa^\alpha(\chi') \chi(\xi)} \) can be written as \( \frac{\xi}{\kappa^\alpha(x)} \) (which in this case is a character of \( A_Y \)). By the definition of \( \xi \), if \( \xi \) denotes its image in \( A_Y \) we get:

\[
\frac{\chi}{\kappa^\alpha}(\xi) = \chi\left(\frac{\xi}{\kappa^\alpha(\xi)}\right) = \chi \circ e^{-\frac{\bar{\xi}}{\alpha}} \left(\frac{b}{a}\right) = \tilde{\chi} \circ e^{-\frac{\bar{\xi}}{\alpha}} \left(\frac{D(\xi)}{D(\xi_0)}\right).
\]

Hence, if \( \zeta \) corresponds to a split integral torus and \( \tilde{\chi} \) is \( \frac{\alpha}{2} \)-unramified then

\[
(5.9) \quad b_{\kappa^\alpha}(\tilde{\chi}, \zeta) = \left(1 - q^{-\frac{\alpha}{2}} e^{-\frac{\bar{\xi}}{\alpha}}(\tilde{\chi})\right)^2
\]

if \( \zeta \) corresponds to a split integral torus and \( \tilde{\chi} \) is \( \frac{\alpha}{2} \)-ramified then

\[
(5.10) \quad b^Y_{\kappa^\alpha}(\tilde{\chi}, \zeta) = \tilde{\chi} \circ e^{-\frac{\bar{\xi}}{\alpha}} \left(\frac{D(\xi)}{D(\xi_0)}\right) e^{-\bar{\xi}}(\chi).
\]

if \( \zeta \) corresponds to a non-split integral torus and \( \tilde{\chi} \) is \( \frac{\alpha}{2} \)-unramified then

\[
(5.11) \quad b^Y_{\kappa^\alpha}(\tilde{\chi}, \zeta) = \frac{1 - q^{-1} e^{-\bar{\xi}}(\chi)}{1 - q^{-1} e^{-\bar{\xi}}(\chi)}.
\]

while if \( \zeta \) corresponds to a non-split integral torus and \( \tilde{\chi} \) is \( \frac{\alpha}{2} \)-ramified then

\[
(5.12) \quad b^Y_{\kappa^\alpha}(\tilde{\chi}, \zeta) = \tilde{\chi} \circ e^{-\frac{\bar{\xi}}{\alpha}} \left(\frac{D(\xi)}{D(\xi_0)}\right) e^{-\bar{\xi}}(\chi).
\]

There remains to compute the equations for the case that \( \zeta \) does not correspond to an integral torus. Here we will denote \( b \) by \( \varpi \) to remind of the fact that \( \varpi = -D(\zeta) \) will be a uniformizing element in \( k \), uniquely defined up to multiplication by \( (a^\alpha)^2 \). Hence \( Q(x, y) = x^2 + \varpi y^2 \).

Now we notice the following: The form \( Q(x, y) \) takes values not in the whole of \( k \) but only in \( \{1, \varpi \} \cdot (a^\alpha)^2 \). This implies that the values of \( \eta Q(x, y) \) will be the same for two characters \( \eta \) agreeing on \( \{1, \varpi \} \cdot (a^\alpha)^2 \), and since \( \eta^2 \) is unramified there exists an unramified such character. To compute the Fourier transforms, we may therefore without loss of generality assume that \( \eta \) is unramified. The reader should compare this with Remark 5.7: Here it will not matter whether \( \tilde{\chi} \) is ramified or not, but only the value of its pull-back via \( e^{\frac{\bar{\xi}}{\alpha}} \) at \( \varpi = -D(\zeta) \).

Hence, we may write \( \eta(x) = | \cdot |^s \tilde{\eta} \) where \( q^s \tilde{\eta} = \tilde{\chi} \delta^{-\frac{s}{2}} \circ e^{-\frac{\bar{\xi}}{\alpha}}(-D(\zeta)) = q^{-\frac{s}{2}} \tilde{\chi} \circ e^{-\frac{\bar{\xi}}{\alpha}}(-D(\zeta)) \). Now we compute:

\[
\langle \Delta^s_{\kappa^\alpha}, 1_s \rangle = \tilde{\chi}'(\xi) \int_{a^s} |x^2 + \varpi y^2|^s \tilde{\eta} dxdy =
\]

\[
= \tilde{\chi} \nu \delta^{-\frac{s}{2}}(\xi) \left( \int_{|x| < |y| \leq 1} q^{-\frac{s}{2}} |y|^s dxdy + \int_{|y| < |x| \leq 1} |x|^s dxdy + \int_{|x| = |y| \leq 1} |x|^s dxdy \right) =
\]

\[
= \tilde{\chi} \nu \delta^{-\frac{s}{2}}(\xi) \left( (1 + q^{-\frac{s}{2}}) \frac{q^{-1}(1 - q^{-1})}{1 - q^{-s-2}} + \frac{(1 - q^{-1})^2}{1 - q^{-s-2}} \right) =
\]

\[
= \tilde{\chi} \nu \delta^{-\frac{s}{2}}(\xi) \left( \frac{1 - q^{-1}}{1 - q^{-s-2}} \right) = \tilde{\chi} \nu \delta^{-\frac{s}{2}}(\xi) \frac{1 - q^{-1}}{1 - q^{-s-2}}.
\]
Similarly, the integral of $\Delta^2_{\alpha,\xi} = e^{-\alpha} \cdot w_\alpha \chi'((\xi)) x^2 + w y^2 |-|^2^{-1} dxdy$:

$$\langle \Delta^2_{\alpha,\xi}, 1_{\sigma^2} \rangle = e^{-\alpha} \cdot w_\alpha \chi'((\xi)) \frac{1 - q^{-1}}{1 - q^2}.$$ 

The quotient of the two is:

$$b_{w_\alpha}^Y(\tilde{x}, \xi) = \tilde{x} \circ e^{-\frac{\alpha}{2}} \left( \frac{D(\xi)}{D(\zeta_0)} \right) \frac{1 - q^{\frac{1}{2}}}{1 - q^{-2}}.$$ 

Therefore:

$$b_{w_\alpha}^Y(\tilde{x}, \xi) = \tilde{x} \circ e^{-\frac{\alpha}{2}} \left( \frac{D(\xi)}{D(\zeta_0)} \right) \frac{1 - q^{-\frac{1}{2}} \tilde{x} \circ e^{-\frac{\alpha}{2}} (-D(\xi))}{1 - q^{-2} \tilde{x} \circ e^{-\frac{\alpha}{2}} (-D(\xi))}.$$ 

(5.13)

Part 2. The formula with $L$-values

6. Simple spherical reflections

From now on we assume that $X$ is a spherical variety such that there are no reflections of type $N$, in other words there does not exist an orbit $Y$ of maximal rank and a simple root $\alpha$ such that $(Y, \alpha)$ is of type $N$. Moreover, the character $\delta_{(x)}^{\pm} \chi$ of $A_X$ is assumed to be unramified. In the case of a parabolically induced $X$ with a non-trivial character $\Psi = \psi \circ \Lambda$, we assume that $\Lambda$ (or $\Psi$) is generic: that means that $\Lambda$ belongs to the open orbit of the corresponding Levi on the additive characters of the unipotent radical. (These are the most interesting cases.)

6.1. The root system of a spherical variety. Up to now we have developed an algorithm for calculating the constants in formula (4.2) of Theorem 4.2.2 by calculating the functional equations for all orbits $Y$ of maximal rank and simple roots $\alpha$ of $G$. However, the formula really depends only on the functional equations for $\Delta_X$, the morphism defined by the open $B$-orbit. Those are parametrized by the little Weyl group $W_X$.

It is known that the faithful action of $W_X$ on $\mathcal{L}(X) \otimes \mathbb{Q}$ (equivalently on $\mathbb{Q} = \text{Hom}(\mathcal{L}(X), \mathbb{Q})$) is generated by reflections, and the cone $\mathcal{V} \subset \mathbb{Q}$ of invariant valuations is a fundamental domain for its action on $\mathbb{Q}$. In fact $W_X$ is the Weyl group of a root system, and the rank of this root system is called the rank of the spherical variety $X$. (We will use this term only when it is clear that we are not referring to the other notion of rank – namely, the rank of $X$ as a $B$-variety.) This root system is defined as follows [Kn96]: One considers in $\mathcal{L}(A_X) \otimes \mathbb{Q}$ the cone (negative) dual to $\mathcal{V}$, i.e. the set $\{ \chi \in \mathcal{L}(X) \otimes \mathbb{Q} | \langle v, \chi \rangle \leq 0 \text{ for every } v \in \mathcal{V} \}$. This is a strictly convex cone, and the simple roots of the root system are the generators for the intersections of $\mathcal{L}(X)$ with its extremal rays. It seems that this root system is not quite the correct one for the purposes of representation theory, therefore we describe in [SV] (and recall below) a variant $\Phi_X$ of that, on the same vector space and with the same Weyl group, but with roots of different length. The dual root system $\Phi_X$ is expected to be the root system of the dual group $\tilde{G}$ that Gaitsgory and Nadler [GN] have associated to the spherical variety. In [SV] we show that if there are no reflections of type $N$ then the data $(\mathcal{L}(X)^*, \tilde{\Phi}_X, \mathcal{L}(X), \Phi_X)$ form a root datum and hence define uniquely up to isomorphism a complex group $\tilde{G}_X$ with a canonical maximal torus $A^*_X$ and root system $\Phi_X$: it will be called “the dual group of $X$".
For the case of $C^\infty(X, L_\Psi)$, we explained in [Sa08] how to associate similar invariants based on Knop’s definitions for non-spherical varieties. Notice the following:

6.1.1. **Lemma.** Let $\Lambda : U_{P^-} \to G_\alpha$ be an additive character of the unipotent radical of a parabolic $P^-$, normalized by a spherical subgroup $M$ of the Levi $L$. We assume that $P^-$ is opposite to a standard parabolic and $M \cdot (L \cap B)$ is open in $L$. Then $\Lambda$ is completely determined by its restriction to the simple root subgroups $U_{-\alpha}$, $\alpha \in \Delta \setminus \Delta_L$. A generic character is non-trivial on all those subgroups.

**Proof.** For the first statement, it suffices to show that if $\Lambda$ is zero on all those simple root subgroups, then it is trivial. Given that $[U_{-\alpha}, U_{\beta}] = 1$ for all distinct $\alpha, \beta \in \Delta$, the subgroups $U_{-\alpha}$, $\alpha \in \Delta \setminus \Delta_L$, are normalized by $B_L := L \cap B$. In combination with the fact that $M$ stabilizes $\Lambda$, $bm \cdot \Lambda$ is zero on those subgroups, for every $b \in B_L$, $m \in M$. But $B_L M$ is open in $L$, hence, $l \cdot \Lambda$ is zero on all those subgroups. The only character with this property is the trivial one.

For the second claim, the same argument shows that if $\Lambda$ is zero on one of the $U_{-\alpha}$’s, then the same is true for $l \cdot \Lambda$, for every $l \in L$. Such a $\Lambda$ cannot be generic. □

Here is a description of the simple roots of $\Phi_X$: First of all, among the elements of $W_X$ the reflections $w_\gamma \in W_X$ corresponding to simple roots are completely determined by the requirement that the corresponding negative Weyl chamber be $\mathcal{V}$, which contains the image of the negative Weyl chamber of $G$. Recall Brion’s description of generators for $W(X)$ (Theorem 2.3.2). We argue in [SV] that this description applies in particular to simple reflections $w_\gamma \in W_X$. Hence, to repeat, adding the case of a non-trivial $L_\Psi$, every $w = w_\gamma$ can be written as $w = w_1^{-1}w_2w_1$ where:

- $w_1 \hat{X}$: $Y$ with codim$(Y) = l(w_1)$. (In the notation of 2.3, $w_1^{-1} \in W(Y)$.)
- $w_2$ is either of the following three:
  - (1) equal to $w_\alpha$, where $\alpha$ is a simple root such that $(Y, \alpha)$ is of type $T$ (or $N$, if we allow type $N$ reflections), or
  - (2) equal to $w_\alpha w_\beta$, where $\alpha, \beta$ are two orthogonal simple roots which both lower $Y$ to the same orbit $Y'$, or
  - (3) equal to $w_\alpha$, where $\alpha$ is a simple root such that $(Y, \alpha)$ is of type $(U, \psi)$.
  (By our assumption that $\Psi$ be generic and Lemma 6.1.1, in this case $w_1$ is trivial.)

For any path $g$ in Knop’s diagram that corresponds to such a description of $w_\gamma$, we will say that $g$ leads or corresponds to $w_\gamma$. Depending on which of the enumerated cases above appears:

- (1) The weight $\gamma = w_1^{-1} \alpha$ will be called a spherical root of type $T$ (resp. $N$).
- (2) The weight $\gamma = w_1^{-1}(\alpha + \beta)$ will be called a spherical root of type $G$.
- (3) The weight $\gamma = \alpha = w_1^{-1} \alpha$ will be called a spherical root of type $(U, \psi)$.

We show in [SV] that the resulting root system $\Phi_X$, which is a modification of that defined by Knop, is well-defined and does not depend on choices. The set of simple spherical roots will be denoted by $\Delta_X$ (not to be confused with $\Delta(X)$), and the set of positive spherical roots by $\Phi_X^+$. The rank of this root system is called

---

9The name “type $G$” originates from the fact that this is the spherical root for the variety $\text{SL}_2 \setminus \text{SL}_2 \times \text{SL}_2$. This shouldn’t cause any confusion with “reflections of type $G$”, since the latter do not generate any spherical roots.
The values of the cocycles for simple spherical roots. For the statement of the results it is convenient to introduce the constants $B_w$ (for $w \in W_X$), defined as:

\begin{equation}
B_w(\hat{\chi}) = \prod_{\alpha \notin \Phi^+, w\alpha < 0} (-e^\alpha(\chi))b_w(\delta_{\chi(X)}\hat{\chi})
\end{equation}

The $B_w$ are cocycles: $W_X \to \mathbb{C}(A^*_X)$. Using those, Theorem 4.2.2 becomes:

\begin{equation}
\Omega_{\delta_{\chi(X)}\hat{\chi}}(x) = \frac{\delta_{\chi(X)}\hat{\chi}}{\delta_{\chi(X)}\hat{\chi}}(x) \sum_{w \in W_X} B_w(\hat{\chi}) w(\hat{\chi})(x).
\end{equation}

Our goal at this point is to compute $B_w$ for $w = w_\gamma$, where $w_\gamma$ is a simple spherical root. We are aiming at the following:

6.2.1. Statement. Let $g$ be a path in Knop’s diagram corresponding to the simple reflection $w_\gamma$. Let $\hat{\theta}$ be the valuation induced by the codimension-one orbit in the path, or if $X$ is the only node in the path, i.e. $\gamma$ is in $\Delta$ and hence a spherical root of type $T$, let $\hat{\theta}$ be the valuation induced by a color in $X\mathbb{P}_\gamma$. In any case, $\hat{\theta}$ is viewed as an element of $\text{Hom}(X(X), \mathbb{Z})$. Then, according to the type of the root $\gamma$ we have:

- If $\gamma$ is of type $G$, then $\hat{\theta} = \hat{\gamma}$ and:

\begin{equation}
B_{w_\gamma}(\hat{\chi}) = -e^{\hat{\gamma}} \frac{1 - q^{-\langle \gamma, \rho_P(X) \rangle} e^{-\hat{\gamma}}}{1 - q^{-\langle \gamma, \rho_P(X) \rangle} e^{\hat{\gamma}}}(\chi).
\end{equation}

- If $\gamma$ is of type $T$ split, then $\langle \hat{\theta}, \gamma \rangle = 1$ and:

\begin{equation}
B_{w_\gamma}(\hat{\chi}) = -e^{\hat{\theta}} \frac{1 - q^{-\langle \hat{\theta}, \rho_P(X) \rangle} e^{-\theta}}{1 - q^{-\langle \hat{\theta}, \rho_P(X) \rangle} e^{\theta}} - q^{\langle \hat{\theta}, \rho_P(X) \rangle} - (\hat{\theta}, \gamma) e^{w_\gamma, \theta}(\hat{\chi}).
\end{equation}

- If $\gamma$ is of type $T$ non-split, then $\hat{\theta} = \frac{\hat{\gamma}}{2}$ and:

\begin{equation}
B_{w_\gamma}(\hat{\chi}) = -e^{\hat{\theta}} \frac{1 - q^{-\langle \hat{\theta}, \rho_P(X) \rangle} e^{-\theta}}{1 + q^{\langle \hat{\theta}, \rho_P(X) \rangle} - (\hat{\theta}, \gamma) e^{w_\gamma, \theta}} + q^{\langle \hat{\theta}, \rho_P(X) \rangle} - (\hat{\theta}, \gamma) e^{-w_\gamma, \theta}(\hat{\chi}).
\end{equation}
If \( \gamma \) is of type \((U, \psi)\), then
\[
B_{w_\gamma}(\tilde{\chi}) = -e^{\check{\gamma}}(\chi).
\]
In case \( T \), the exponents \(-\langle \check{\theta}, \rho_{P(X)} \rangle\) and \(-\langle \check{\rho}, \gamma \rangle\) are equal, unless possibly if \( \check{\theta} = \frac{\check{\gamma}}{2} \). In any case, these exponents are negative half-integers.

Remarks.  
(1) Recall that \( \delta P(X) \in A^*_X \) (3.5) and therefore the exponential of \( \langle \check{\gamma}, \rho_{P(X)} \rangle \) makes sense.  
(2) A priori it is not clear that being of type \( T \) split or non-split is a well-defined property of \( \gamma \), independent of the path. A posteriori, this is true if the above statement is true.

We can only prove this statement by reducing to certain low-rank spherical varieties, and we have performed the corresponding computation only for classical groups:

6.2.2. Theorem. Let \( X \) be a spherical variety without type-\( N \) roots and such that for every \( \gamma \in \Delta_X \) the Levi \( L_\gamma \) is a classical group. Then \( X \) satisfies Statement 6.2.1.

6.3. Glueing paths from simple varieties. Our goal for the rest of this section is to prove Theorem 6.2.2. Let \( \gamma \in \Delta_X \), and choose a path \( g \) in Knop’s graph \( \mathcal{G} \) which corresponds to \( w_\gamma \). Let \( Z \neq X \) be a node in the path \( g \) and let \( e_1, e_2 \) be edges of the path having \( Z \) as one of their endpoints, and such that \( e_1 \) raises \( Z \). (Hence, \( e_2 \) either lowers \( Z \) or goes from \( Z \) to itself, in which case \( Z \) is the orbit of smallest dimension in the path.) Let \( \delta, \epsilon \) be the simple roots of \( G \) labelling \( e_1 \) and \( e_2 \), and consider the spherical variety \( ZP_{\delta\epsilon}/R(P_{\delta\epsilon}) \) for the group \( L_{\delta\epsilon} \). If \( G_1, G_2, \ldots \) are reductive groups of rank two, and \( X_1, X_2, \ldots \) spherical varieties thereof, we will say that the path \( g \) is “glued” from \( X_1, X_2, \ldots \) if all spherical varieties \( ZP_{\delta\epsilon}/R(P_{\delta\epsilon}) \) obtained this way are contained in the list of \( X_1, X_2, \ldots \). As a matter of language, if the path begins and ends at \( X \) then the path \( g \) is glued from any list of such spherical varieties.

We have:

6.3.1. Proposition. Assume that there is a path \( g \) corresponding to \( w_\gamma \) which over the algebraic closure is glued out of the following varieties:

- \( \text{PGL}_2 \setminus \text{PGL}_2 \times \text{PGL}_2, \text{GL}_2 \setminus \text{SL}_3 \),
- \( \text{Sp}_2 \times \text{Sp}_2 \setminus \text{Sp}_4, (\text{G}_m \times \text{G}_\alpha) \times \text{Sp}_2 \setminus \text{Sp}_4, \text{G}_m \times \text{Sp}_2 \setminus \text{Sp}_4 \),
- \( P_\alpha \setminus \text{SL}_3 \) (where \( \alpha \) is a simple root) and
- \( P_\alpha \setminus \text{Sp}_4 \) (where \( \alpha \) is the long simple root).

Then \( g \) satisfies Statement 6.2.1, except possibly for the last assertion (on the exponents).

Remark. The first two rows in the list consist of all spherical varieties without spherical roots of type \( N \), for simply connected groups of rank two other than \( G_2 \), and with the property that both simple roots are in the support of a spherical root (see Theorem 6.10.1 below). Therefore, unless \( X \) is the only node of the path \( g \), the bottom part of the path will necessarily consist of one of these varieties. Hence, if the group has no \( G_2 \)-factors, the content of the assumption is contained in the next lines, where one restricts the horospherical varieties which can glue the path.
Moreover:

6.3.2. Proposition. If \( \gamma \in \Delta_X \) with \( L_{\text{supp}(\gamma)} \) a classical group then for any path \( g \) corresponding to \( w_\gamma \) there is a path \( g' \) corresponding to \( w_\gamma \) such that \( g \) and \( g' \) have the same first edge and \( g' \) satisfies the assumptions of Proposition 6.3.1.

Moreover, in case \( T \) the exponents \( \langle \theta, \rho(\gamma) \rangle \) and \( \langle \theta, \rho(\gamma) \rangle - \langle \beta, \gamma \rangle \) are equal, unless possibly if \( \theta = \frac{\gamma}{2} \). In any case, these exponents are negative half-integers.

Since Statement 6.2.1 only depends on the first edge of \( g \), this proves Theorem 6.2.2. The goal of the rest of this section is to prove Propositions 6.3.1 and 6.3.2.

6.4. Computation for the simple varieties. Proposition 6.3.1 will follow from the following:

6.4.1. Proposition. Assume that the path \( g \) in Knop’s graph, corresponding to \( w_\gamma \) for a simple spherical root \( \gamma \), satisfies the assumptions of Proposition 6.3.1. Let \( Z \) be a node of \( g \) other than the lowest one (in particular, we assume that \( X \) is not the only node of \( g \)), \( w \) the element of the Weyl group corresponding to the path below \( Z \). Hence \( w = w_1^{-1}w_2w_1 \), where \( w_1 \) lowers \( Z \) to an orbit \( Y \) and \( w_2 = w_\beta w_2 \) or \( w_2 = w_\beta \), according as \( \gamma \) is of type \( G \) or \( T \). Let \( \alpha \) be the label of an edge in \( g \) which lowers \( Z \). Let \( \beta h_w^Z(\chi) = \prod_{e \in \delta^+, \omega \in 0} (\chi e(\chi)) h_w^Z(\chi) \). Then:

- If the root \( \gamma \) is of type \( G \) then

\[
\beta h_w^Z(\chi) = -e^\gamma(x) \frac{1 - q^{-1}e^{-\alpha}}{1 - q^{-1}e^{-\omega\alpha}}(x)
\]

where \( \gamma' = w_1^{-1}\beta \). Moreover, we have that \( \delta|_{\chi(\mathbf{Z})} = \delta'|_{\chi(\mathbf{Z})} \). (Recall that as an element of \( \chi(\mathbf{Z})^\ast \), this does not depend on the choice between \( \beta \) and \( \tilde{\beta} \), and similarly \( e^\gamma(x) \) does not depend on this choice for characters \( \chi \) of \( A \) for which \( \Delta_x \) is defined.)

- If the root \( \gamma \) is of type \( T \) split, then

\[
\beta h_w^Z(\chi) = -e^\gamma(x) \frac{1 - q^{-1}e^{-\alpha}}{1 - q^{-1}e^{-\rho(\gamma)\alpha}} \frac{1 - q^{-1}(\rho, \gamma)\omega\alpha}{1 - q^{-1}e^{-\omega\alpha}}(x)
\]

where \( \gamma' = w_1^{-1}\beta \). Moreover, in this case \( \gamma' \in \chi(\mathbf{Z}) \) and \( \langle \alpha, \gamma' \rangle = 1 \).

- If the root \( \gamma \) is of type \( T \) non-split, then

\[
\beta h_w^Z(\chi) = -e^\gamma(x) \frac{1 - q^{-1}e^{-\alpha}}{1 + q^{-1}(\rho, \gamma)\alpha} \frac{1 + q^{-1}(\rho, \gamma)\omega\alpha}{1 - q^{-1}e^{-\omega\alpha}}(x)
\]

where \( \gamma' = w_1^{-1}\beta \). Moreover, in this case \( \gamma' \in \chi(\mathbf{Z}) \) and \( \delta|_{\chi(\mathbf{Z})} = \frac{\gamma'}{2}. \)

Let us first see why this proves Proposition 6.3.1:

Proof of Proposition 6.3.1. First of all, if \( (X, \alpha) \) is of type \( T \) split, then setting \( \delta_{|U_{\rho_{\alpha}}} = \delta(\chi) \) and substituting \( \chi' \) by \( \delta_{|U_{\rho_{\alpha}}} \chi' \) in formula (5.4) we get:

\[
e^{\delta(\chi')\chi'} = e^{\delta(\chi')\chi'}
\]
and similarly for \( \hat{v}_D' \). Given the property that \( B_{w_\alpha}(\delta_{P(X)}^\perp) = 0 \) (by Lemma 4.2.3)\(^{10}\), it follows that \( e^{\hat{v}_D'}(\delta_{P(X)}) = q^{-1} \) for a suitable naming of \( D, D' \), and the fact that \( \hat{v}_D' = \hat{v}_D - \hat{v}_D' \) and \( e^{\hat{v}_D'}(\delta_{P(X)}) = e^{\hat{v}_D}(\delta) = q^{-2} \) implies that the same is true for \( \hat{v}_D' \) in place of \( \hat{v}_D \). This verifies (6.4) in this case.

Similarly, if \((X, \alpha)\) is of type \( T \) non-split, then (6.5) follows from (5.6) and if \((X, \alpha)\) is of type \((U, \psi)\) then (6.6) follows from (5.3).

In all other cases, if \( \alpha \) is the label of the first edge in the path \( g \) then \((X, \alpha)\) is of type \( U \) and the last proposition applies. Notice that \( \mathcal{X}(X) \perp \Delta(X) \), therefore \( e^\gamma(\chi_{\delta_{P(X)}^\perp}) = e^\gamma(\chi) \). From the fact that \( \delta_{P(X)} \in A_X \), we get in case of a root of type \( G \) that: \( e^{\alpha}(\delta_{X}^\perp) = e^{\hat{\alpha}}(\delta_{P(X)}^\perp) = q^{-1}e^\gamma(\delta_{P(X)}^\perp) \) and this proves (6.3). Finally, (6.4) and (6.5) follow from (6.8), resp. (6.9), because \( \hat{\theta} = \hat{\alpha} \) and \( e^{\alpha}(\delta_{X}^\perp)_{P(X)} = q^{-1} \). \( \square \)

Proof of Proposition 6.4.1. We prove it by induction on the dimension of \( Z \). We start with the case \( G \). First of all, if \( Z \) is equal to \( Y \) and \( \alpha = \beta, \hat{\beta} \) are the two orthogonal roots lowering \( Z \) to another orbit then we have computed in Example 5.4.1 that in this case \( \tilde{\gamma}' = \tilde{\alpha} \) and:

\[
\hat{q}_Z(Z, \tilde{\chi}) = -e^{\tilde{\alpha}} \frac{1 - q^{-1}e^{-\tilde{\alpha}}}{1 - q^{-1}e^{\tilde{\alpha}}} (\chi).
\]

Now assume that \( Z \) is lowered by \( \alpha \) to some vertex \( V \) of \( g \), and let \( \beta \) be an edge lowering \( V \). By the induction hypothesis,

\[
\hat{q}_V(w_{\alpha, w_{\alpha}}^{(w_{\alpha}, \chi)} = -e^{w_{\alpha} \tilde{\gamma}} \frac{1 - q^{-1}e^{-\tilde{\alpha}}}{1 - q^{-1}e^{\tilde{\alpha}}} (\chi).
\]

Using our functional equations for roots of type \( U \), we compute:

\[
\hat{q}_Z(Z, \tilde{\chi}) = -e^{\tilde{\alpha}} \frac{1 - q^{-1}e^{-\tilde{\alpha}}}{1 - e^{-\tilde{\alpha}}} (\chi) \cdot \frac{1 - q^{-1}e^{-w_{\alpha}, \beta}}{1 - q^{-1}e^{w_{\alpha}, \beta}} (\chi) \cdot \frac{1 - e^{-w_{\alpha}}}{1 - q^{-1}e^{-w_{\alpha}} (\chi)}.
\]

To complete the proof in this case, it suffices to prove that if \( \mathcal{V}P_{\alpha, \beta}/\mathcal{R}(P_{\alpha}, \beta) \) is among the list of varieties in the assumptions of the proposition, then:

\[
e^{-\tilde{\alpha}}(\chi) = q^{-1}e^{-w_{\alpha}, \beta}(\chi)
\]

This will be proved via a case-by-case computation below. Finally, since by the induction assumption we have \( \tilde{\beta}|_{\mathcal{X}(V)} = w_{\alpha}^{-1}|_{\mathcal{X}(V)} \) and \( \mathcal{X}(Z) = w_{\alpha}|_{\mathcal{X}(V)} \) it follows from (6.10) that \( \tilde{\alpha}|_{\mathcal{X}(Z)} = \tilde{\gamma}^{-1}|_{\mathcal{X}(Z)} \).

In cases \( T \) split and non-split, we will see in the case-by-case analysis that the statement of Proposition 6.4.1 is correct when \( Z \) is the node \( V \) of \( g \) which is immediately higher than the lowest node. Now, if the hypothesis is satisfied by a node \( V \) with \( \beta \) a root lowering \( V \) and \( \alpha \) a root raising it to a node \( Z \), then (in the case \( T \) split, the non-split case being completely analogous):

\[
\hat{q}_V(Z, \tilde{\chi}) = \frac{1 - q^{-1}e^{-\tilde{\alpha}}}{1 - e^{-\tilde{\alpha}}} (\chi) \cdot \frac{1 - q^{-1}e^{-\tilde{\alpha}}}{1 - q^{-1}e^{-w_{\alpha} \gamma}} e^{\beta} \cdot \frac{1 - q^{-1}e^{-\tilde{\alpha}}}{1 - q^{-1}e^{-w_{\alpha} \gamma}} e^{\beta} \cdot \frac{1 - q^{-1}e^{-w_{\alpha}}}{1 - q^{-1}e^{-w_{\alpha}} (\chi)}.
\]

\(^{10}\)In case we are considering a line bundle \( L_\Psi \) over \( X \), we still have \( B_{w_\alpha}(\delta_{P(X)}^\perp) = 0 \) by using, for instance, the variety \( X \) without the character \( \Psi \).
Once again, it suffices to prove (6.10). Indeed, then the factor \(1 - e^{-\tilde{\alpha}(\chi)}\) cancels \(1 - q^{-1}e^{-\tilde{\beta}(w_0\chi)}\). Moreover, since \(w_\alpha w_\gamma' \in \mathcal{X}(V)\) we get \(\langle \gamma', \tilde{\alpha} \rangle = \langle \gamma', w_\alpha \gamma', \tilde{\beta} \rangle = 1\) by (6.10) and the induction hypothesis. By (6.10) we also have \(e^{-\tilde{\alpha}(w_0^{-1}\chi)} = q^{-1}e^{-w_0\tilde{\beta}(w_0^{-1}\chi)}\) and therefore the factor \(1 - e^{-w_0\tilde{\alpha}(\chi)}\) cancels the factor \(1 - q^{-1}e^{-(w_\alpha w_0)\tilde{\beta}(w_\alpha \chi)}\). Finally, we have

\[
q^{-1}\langle w_\alpha \gamma', \tilde{\beta} \rangle e^{\tilde{\alpha}(w_\alpha \chi)} = q^{-1}\langle \gamma', \tilde{\alpha} \rangle q^{\langle \gamma', \tilde{\alpha} \rangle} e^{w_0 \tilde{\beta}(\chi)} = q^{-1}\langle \gamma', \tilde{\alpha} \rangle e^{\tilde{\alpha}(\chi)}
\]

since \(\gamma' \in \mathcal{X}(Z)\) and hence \(\langle \tilde{\alpha}, \gamma' \rangle = \langle \frac{\gamma}{2}, \gamma \rangle = 1\). This completes, up to the case-by-case analysis that follows, the proof of the proposition. \(\square\)

Now we study one-by-one the “simple” varieties of Proposition 6.3.1, in order to establish the remaining points of its proof. More precisely, let \(Z\) be an orbit in a given path \(g\) and \(\alpha, \beta\) two simple roots such that \(ZP_{\alpha, \beta}/R(U_{P_{\alpha, \beta}})\) is one of the varieties listed in Proposition 6.3.1. Let \(H_{\alpha, \beta}\) be the stabilizer mod \(U_{P_{\alpha, \beta}}\) of a point on \(ZP_{\alpha, \beta}\) (hence, by definition, \(H_{\alpha, \beta}\) is a subgroup of the reductive quotient \(L_{\alpha, \beta}\), and set \(H' = [H_{\alpha, \beta}, H_{\beta, \alpha}]\) and \(G' = [L_{\alpha, \beta}, L_{\alpha, \beta}]\). We can prove the remaining statements of Proposition 6.4.1 by looking at the variety \(H' \setminus G'\); the latter may not be spherical for \(G'\), in which case we will treat it as a spherical variety for \(G' \times \) the connected component of \(N_{\alpha, \beta}(H_{\alpha, \beta}) \cap G'\).

Recall that the formulas of section 5 (and in particular (5.4)) do not depend just on the space \(H' \setminus G'\) but also on some modular characters of subgroups of \(P_{\alpha, \beta}\). Fortunately, however, it will turn out that all coweights which appear in our computation are in the span of the coroots of \(G'\), thus rendering irrelevant the difference between those subgroups of \(P_{\alpha, \beta}\) and their image in \(L_{\alpha, \beta}\). Indeed, we have:

\textbf{Remark.} Assume that \((Z, \alpha)\) is of type \(T\) split, with \(D, D'\) the divisors of smaller rank in \(ZP_{\alpha}\). While \(\tilde{v}_D\) is an element of \(\mathcal{X}(Z)^*\), it will be convenient in the computations which follow to choose a lift of \(\tilde{v}_D\) to \(\text{Hom}(\mathcal{X}(A), \mathbb{Q})\) – to be denoted again by \(\tilde{v}_D\) (and similarly for \(\tilde{v}_{D'}\)). This will not affect the computation of \(e^\tilde{v}_D(\check{\chi}P_0\delta^\frac{1}{2}(U_{P_{\alpha, \beta}})\xi)\) of (5.4) since, as we already remarked, \(\check{\chi}P_0\delta^\frac{1}{2}(U_{P_{\alpha, \beta}})\xi \in A^*_Z\), but it allow us to split it into a product of factors, for instance we will be able to evaluate \(e^\tilde{v}_D(\delta^\frac{1}{2}(U_{P_{\alpha, \beta}})\xi)\).

\textbf{6.5.} \(SL_2 \setminus SL_3\). Below is Knop’s diagram for the orbits of maximal rank:

\begin{center}
\includegraphics[width=0.5\textwidth]{diagram}
\end{center}

We will use the path on the left to compute the coefficient \(q^\beta_{w_\alpha}(\chi)\). It is easy to compute that for the orbits \(D, D'\) corresponding to the bottom-left reflection of type \(T\) we have: \(\tilde{v}_D = -\tilde{\alpha}_1, \tilde{v}_D = \tilde{\alpha}_1 + \tilde{\alpha}_2, e^{\tilde{v}_D}(\delta^\frac{1}{2}\delta^\frac{1}{2}(U_{P_{\alpha, \beta}})\xi) = q^\frac{1}{2}, e^{\tilde{v}_{D'}}(\delta^\frac{1}{2}\delta^\frac{1}{2}(U_{P_{\alpha, \beta}})\xi) = 1\). Notice that, because \(\tilde{v}_D \neq \tilde{v}_{D'}\), the spherical root of this variety is necessarily of type \(T\) split. Therefore, we get:

\[
q^\beta_{w_\alpha}(\chi) = \frac{1 - q^{-1}e^{-\tilde{\alpha}_1}}{1 - e^{-\tilde{\alpha}_1}}(\chi).
\]
This agrees with the statement of Proposition 6.4.1.

6.6. \( \text{Sp}_2 \setminus \text{Sp}_4 \). This is a spherical variety for the group \( G_m \times \text{Sp}_4 \). (Of course, \( \text{Sp}_2 = \text{SL}_2 \), but we write \( \text{Sp}_2 \) to emphasize the way it embeds — namely, into the stabilizer of a 2-dimensional symplectic subspace.) Here is Knop’s diagram for orbits of maximal rank:

\[
\begin{array}{ccc}
\text{Z} & \overset{\beta,T}{\longrightarrow} & \text{Sp}^1 \\
\text{\alpha,U} & \downarrow & \\
\text{\beta,U} & \overset{\beta,T}{\longrightarrow} & \text{Z} \\
\end{array}
\]

where \( \alpha \) is the long root and \( \beta \) is the short one.

We are interested in the spherical root \( \alpha + \beta \), since the spherical root \( \beta \) belongs to the type \( T \setminus \text{SL}_2 \). One computes that if \( D, D' \) are the orbits of co-rank one corresponding to the reflection of type \( T \) at the bottom, then \( \hat{v}_D = -\hat{\alpha}, \hat{v}_{D'} = \hat{\alpha} + \hat{\beta}, \)
\[
e^{\hat{\beta}}(\delta U) = q^{-\frac{1}{2}}, \quad e^{\hat{\beta}'}(\delta U) = q^{\frac{1}{2}}.
\]
Again, for the same reason as above, the spherical root is necessarily of type \( T \) split. Therefore:

\[
\begin{align*}
\chi_{\text{Sp}^1}(\chi) &= \frac{1 - q^{-1} e^{-\hat{\alpha}}}{1 - e^{-\hat{\alpha}}}(\chi), \\
\chi_{\text{Sp}^1}(\chi) &= \frac{1 - q^{-1} e^{-\hat{\alpha} - \hat{\beta}}}{1 - e^{-\hat{\alpha} + \hat{\beta}}}(\chi) .
\end{align*}
\]

which agrees with Proposition 6.4.1.

6.7. \( G_a \times \text{Sp}_2 \setminus \text{Sp}_4 \). Here the subgroup \( G_a \) is the unipotent radical of the “second” \( \text{Sp}_2 \) inside \( \text{Sp}_4 \). Knop’s diagram is as follows in this case:

\[
\begin{array}{ccc}
\text{Z} & \overset{\beta,T}{\longrightarrow} & \text{Sp}^1 \\
\text{\alpha,U} & \downarrow & \\
\text{\beta,U} & \overset{\beta,T}{\longrightarrow} & \text{Sp}^1 \\
\text{\beta,T} & \overset{\alpha,U}{\longrightarrow} & \text{Z} \\
\end{array}
\]

where \( \alpha \) is the long root and \( \beta \) is the short one.
The left-most path is identical to the one of the previous example, including the same values for \( v_D, e^b_{D'} (\delta^{\cdot} \delta_{U(p_{a+2})}^{-1}) \) etc., whence again:

\[
\nu_{w}^Z(\chi) = -e^{2\hat{a} + \hat{b}} \frac{1 - q^{-1}e^{-\hat{a}}} {1 - q^{-1}e^{\hat{a}}} \frac{1 - q^{-1}e^{-\hat{a} - \hat{b}}} {1 - q^{-1}e^{\hat{a} + \hat{b}}} (\chi).
\]

The path on the right does not correspond to a simple reflection \( w_\gamma \), because it defines a decomposition of \( w_\gamma \) which is longer than the length of \( w_\gamma \).

6.8. \( \Sp_2 \times \Sp_2 \setminus \Sp_4 \). Here we have two cases over \( k \), namely:

- the variety \( \Sp_2 \times \Sp_2 \setminus \Sp_4 \) and
- the variety \( \left( \Res_{E/k} \Sp_2 \right) \setminus \Sp_4 \) where \( E/k \) is a quadratic field extension and \( \Res_{E/k} \) denotes restriction of scalars.

Here is Knop’s diagram for orbits of maximal rank:

\[ \begin{array}{c}
\alpha, U \\
\beta, T \\
\beta, G \\
\end{array} \]

where \( \alpha \) is the long root and \( \beta \) is the short one. The reflection of type \( T \) at the bottom is of split type in the first case and of non-split type in the second.

Again, for the divisors of co-rank one at the bottom we have: \( v_D = -\hat{a}, v_{D'} = \hat{a} + \hat{b}, e^b_{D'} (\delta^{\cdot} \delta_{U(p_{a+2})}^{-1}) = q^{-\frac{1}{2}}, e^b_{D'} (\delta^{\cdot} \delta_{U(p_{a+2})}^{-1}) = q^{\frac{1}{2}} \). Therefore, as before we have in the split case:

\[
\nu_{w}^Z(\chi) = -e^{2\hat{a} + \hat{b}} \frac{1 - q^{-1}e^{-\hat{a}}} {1 - q^{-1}e^{\hat{a}}} \frac{1 - q^{-1}e^{-\hat{a} - \hat{b}}} {1 - q^{-1}e^{\hat{a} + \hat{b}}} (\chi).
\]

which agrees with Proposition 6.4.1.

Notice that here \( \chi = e^c_\gamma \chi' \) with \( \chi' \in A^*_2 \) and \( \hat{a} = \hat{a} + \frac{\hat{c}}{2} \) on \( \mathcal{X}(\mathbb{Z}) \), therefore the above can also be written:

\[
\nu_{w}^Z(\chi') = -e^{c'}_\gamma \frac{1 - q^{-\frac{1}{2}}e^{-\frac{\gamma'}{2}}} {1 - q^{-\frac{1}{2}}e^{\frac{\gamma'}{2}}} \frac{1 - q^{-\frac{1}{2}}e^{-\frac{\gamma'}{2}}} {1 - q^{-\frac{1}{2}}e^{\frac{\gamma'}{2}}} (\chi').
\]

Similarly, in the non-split case we will have:

\[
\nu_{w}^Z(\chi') = -e^{c'}_\gamma \frac{1 - q^{-\frac{1}{2}}e^{-\frac{\gamma'}{2}}} {1 - q^{-\frac{1}{2}}e^{\frac{\gamma'}{2}}} \frac{1 + q^{-\frac{1}{2}}e^{-\frac{\gamma'}{2}}} {1 + q^{-\frac{1}{2}}e^{\frac{\gamma'}{2}}} (\chi').
\]

6.9. Horospherical varieties. For the varieties \( [P_{\alpha}, P_{\alpha}] \setminus \SL_3 \) or \( [P_{\alpha}, P_{\alpha}] \setminus \Sp_4 \) of Proposition 6.3.1, the graphs look as follows:
The validity of (6.10) at every intermediate node is easily verified.

6.10. **Spherical varieties of rank at most two.** We now come to the proof of Proposition 6.3.2. The starting point is classification of wonderful varieties of rank one by Akhiezer [Ak83] and of rank two by Wasserman [Was96].

6.10.1. **Theorem.** Below is a complete list of spherical varieties \( X' = H' \backslash G' \) over \( \overline{k} \) with the following properties:

- \( G' \) is semisimple, simply connected, and \( H' \) is equal to the connected component of its normalizer.
- There is a spherical root \( \gamma \) whose support is the set of simple roots of \( G' \).
- There are no spherical roots of type \( N \).

(For simplicity and since everything is over the algebraic closure, we do not use boldface letters in what follows. We also write \( k^n \) to denote an \( n \)-dimensional vector space.)
(In the case of $G' = \text{Spin}_{2n}$ and $H' = \text{Spin}_{2n-1}$, $n = 4$, this includes the "non-obvious" embedding of $\text{Spin}_7$ obtained from the "obvious" one by applying the outer automorphism of $\text{Spin}_8$.)

Proof. If we add the extra condition "the rank of $X'$ is at most two", then it follows by inspection of the tables of [Was96]. As we explain in [SV], for any spherical variety $X$ and any spherical root $\gamma$ of it, the spherical variety $X_\gamma$ is of rank at most two. Therefore, the condition on the rank of $X'$ being at most two is vacuous. □

Let $\gamma$ denote the spherical root in each of the above whose support is the set of all simple roots of $G'$. To prove Proposition 6.3.2, it suffices to check that for each of the varieties above (where $G'$ is a classical group) and for every path corresponding to $\gamma$ there is a path leading to $\gamma$ with the same first edge which is glued out of the spherical varieties of Proposition 6.3.1. We notice that the spherical roots for the examples below can be read off from the tables of [Was96], and that for a quasi-affine spherical variety $X$ the set $\Delta(X)$ consists precisely of the simple roots of $G$ which are orthogonal to $X(X)$ [Kn94, Lemma 3.1]. There is nothing to check for $G_m \backslash \text{SL}_2$ and $\text{SL}_2 \backslash \text{SL}_2 \times \text{SL}_2$.

6.11. $\text{GL}_n \backslash \text{SL}_{n+1}$, $n \geq 2$. The spherical root here is $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, we have $\Delta(X) = \{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\}$ and a path corresponding to it is the following:

\[
\begin{array}{c}
\bullet \\
\alpha_2, U & \alpha_n, U
\end{array}
\]

$\alpha_n, U$

\[
\begin{array}{c}
\bullet \\
\alpha_1, U & \alpha_n, U
\end{array}
\]

$\alpha_n, U$

\[
\begin{array}{c}
\bullet \\
\gamma
\end{array}
\]

where the arrows not shown are reflections of type $G$.

Clearly, this path is glued from $\text{SL}_2 \backslash \text{SL}_3$ (at the bottom) and $P_\alpha \backslash \text{SL}_3$. There is also a similar path starting with $\alpha_n$. We mention that here we have:

\[
B_{\omega_1}(\chi) = -e^{\gamma} \left( \frac{1 - q^{-\bar{\gamma}}e^{-\bar{\alpha}_1}}{1 - q^{-\bar{\gamma}}e^{\bar{\alpha}_1}} \right) \left( \frac{1 - q^{-\bar{\gamma}}e^{-\bar{\alpha}_n}}{1 - q^{-\bar{\gamma}}e^{\bar{\alpha}_n}} \right) (\chi).
\]

6.12. $\text{Sp}_4 \backslash \text{SL}_4$. Here $\Delta(X) = \{\alpha_1, \alpha_3\}$ and the spherical root is $\gamma = \alpha_1 + 2\alpha_2 + \alpha_3$. Knop’s diagram is as follows:
Therefore, this path is glued from varieties of the form $\text{PGL}_2 \setminus \text{PGL}_2 \times \text{PGL}_2$ and $P_{\alpha} \setminus \text{SL}_3$. Here we have:

\begin{equation}
B_{w_{\gamma}}(\chi) = -e^{\gamma} \frac{1 - q^{-2} e^{-\gamma}}{1 - q^{-2} e^{\gamma}}(\chi).
\end{equation}

6.13. $\text{Spin}_{2n} \setminus \text{Spin}_{2n+1}$. We denote by $\alpha_1, \ldots, \alpha_{n-1}$ the long roots and by $\alpha_n$ the short one. The spherical root is $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $\Delta(X) = \{\alpha_2, \ldots, \alpha_n\}$. A path to the spherical root is the following:

\begin{equation}
B_{w_{\gamma}}(\chi) = -e^{\gamma} \frac{1 - q^{-n} e^{-\frac{\gamma}{2}}(1 - q^{-\frac{1}{2} e^{-\frac{\gamma}{4}}})}{1 - q^{-n} e^{\frac{\gamma}{2}}(1 - q^{-\frac{1}{2} e^{\frac{\gamma}{4}}})}(\chi).
\end{equation}

6.14. $(\text{SL}_n \times \mathbb{C}_m) \ltimes \wedge^2 k^n \setminus \text{Spin}_{2n+1}$. With notation as above, the spherical root is the same as above (namely, $\gamma = \alpha_1 + \cdots + \alpha_n$) and $\Delta(X) = \{\alpha_2, \ldots, \alpha_{n-1}\}$. A path to the spherical root is the following:
It is glued from \((\mathbb{G}_m \ltimes \mathbb{G}_a) \times \text{Sp}_2 \backslash \text{Sp}_4\) (at the bottom) and \(\text{P}_\alpha \setminus \text{SL}_3\). The edge \(\alpha_n\) on the top-right cannot lead to \(w_\gamma\) since \(\alpha_n\) is orthogonal to \(\gamma\) and hence the length of any such path would be larger than the length of \(w_\gamma\).

Here we have:

\[
B_{w_\gamma}(\chi) = -e^{\frac{1}{2}} \left(1 - q^{-\frac{1}{2}e^{-\hat{\alpha}_1}}\right) \left(1 - q^{-\frac{1}{2}e^{-\hat{\alpha}_1-\hat{\alpha}_n}}\right) \left(1 - q^{-\frac{1}{2}e^{\hat{\alpha}_1}}\right) \left(1 - q^{-\frac{1}{2}e^{\hat{\alpha}_1+\hat{\alpha}_n}}\right) (\chi).
\]

6.15. \(\text{G}_2 \backslash \text{Spin}_7\). The spherical root is \(\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3\), and \(\Delta(X) = \{\alpha_1, \alpha_2\}\). Knop’s diagram looks as follows:

It is glued from the varieties \(\text{P}_\alpha \setminus \text{Sp}_4\) (where \(\alpha\) denotes the long root), \(\text{P}_\alpha \setminus \text{SL}_3\) and \(\text{PGL}_2 \setminus \text{PGL}_2 \times \text{PGL}_2\). We have:

\[
B_{w_\gamma}(\chi) = -e^{\frac{1}{2}} \frac{1 - q^{-3}e^{-\gamma}}{1 - q^{-3}e^{-\gamma}} (\chi).
\]

6.16. \(\text{Sp}_2 \times \text{Sp}_{2n-2} \backslash \text{Sp}_{2n}\). Denote by \(\alpha_1, \ldots, \alpha_{n-1}\) the short roots and by \(\alpha_n\) the long one. Then the spherical root is \(\gamma = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n\), and \(\Delta(X) = \{\alpha_1, \alpha_3, \alpha_4, \ldots, \alpha_n\}\). A path for the spherical root is the following:
It is glued from $\text{GL}_2 \backslash \text{SL}_3$ (at the bottom), $\text{P}_\alpha \backslash \text{Sp}_4$ (where $\alpha$ is the long root) and $\text{P}_\alpha \backslash \text{SL}_3$, unless $n = 2$, in which case it coincides with $\text{Sp}_2 \times \text{Sp}_2 \backslash \text{Sp}_4$. We have here:

$$B_{w_\gamma}(\chi) = -e^\gamma \frac{(1 - q^{-n+\frac{3}{2}} e^{-\frac{\gamma}{2}})(1 - q^{-n+\frac{3}{2}} e^{-\frac{\gamma}{2}})}{(1 - q^{-n+\frac{3}{2}} e^\gamma)(1 - q^{-n+\frac{3}{2}} e^\gamma)} (\chi).$$

6.17. $(G_m \ltimes k) \times \text{Sp}_{2n-2} \backslash \text{Sp}_{2n}$. The spherical root is the same as above, and $\Delta(X) = \{\alpha_3, \alpha_4, \ldots, \alpha_n\}$. A path to the root is the following:

which, again, is glued from $\text{GL}_2 \backslash \text{SL}_3$ (at the bottom), $\text{P}_\alpha \backslash \text{Sp}_4$ (where $\alpha$ is the long root) and $\text{P}_\alpha \backslash \text{SL}_3$ (unless $n = 2$, a case which was treated before). The edge labelled $\alpha_1$ on the top right cannot lead to $w_\gamma$, since $\alpha_1$ is orthogonal to $\gamma$. 
Here we have:

\[ B_{w_\gamma}(\chi) = -e^{\gamma} \frac{(1 - q^{-n+1} e^{-\tilde{a}_2})(1 - q^{-n+1} e^{-\tilde{a}_1 - \tilde{a}_2})}{(1 - q^{-n+1} e^{\tilde{a}_2})(1 - q^{-n+1} e^{\tilde{a}_1 + \tilde{a}_2})} (\chi). \]

6.18. Spin\(\mathbb{Z}_{2n-1} \setminus \text{Spin}_{2n}\). Denote the simple roots by \(\alpha_1, \alpha_2, \ldots, \alpha_n\), where \(\alpha_{n-2}\) is the root neighboring with three others (\(\alpha_{n-3}, \alpha_{n-1}\) and \(\alpha_n\)). Assume \(n \geq 3\), the case \(n = 3\) having been treated in §6.11. The spherical root is \(\gamma = 2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n\), and \(\Delta(X) = \{\alpha_2, \ldots, \alpha_n\}\). Knop’s graph looks as follows (omitting type \(G\) reflections):

It is glued from \(\text{PGL}_2 \setminus \text{PGL}_2 \times \text{PGL}_2\) and \(\text{P} \alpha \setminus \text{SL}_3\). We have:

\[ B_{w_\gamma}(\chi) = -e^{\gamma} \frac{1 - q^{-n+1} e^{-\gamma}}{1 - q^{-n+1} e^{\gamma}} (\chi). \]

6.19. Spin\(\mathbb{Z}_{7} \setminus \text{Spin}_9\). There are 2 spherical roots, \(\gamma = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\) and \(\alpha_2 + 2\alpha_3 + 3\alpha_4\) but, as in §6.6, we are interested only in \(\gamma\). Here \(\Delta(X) = \{\alpha_2, \alpha_3\}\).

A path corresponding to \(\gamma\) is the following:

where we have only drawn the arrows of interest.

It is glued from \((\mathbb{G}_m \times \mathbb{G}_a) \times \text{Sp}_2 \setminus \text{Sp}_4\) (at the bottom) and \(\text{P} \alpha \setminus \text{SL}_3\). The arrow labelled \(\alpha_4\) on the top cannot lead to \(w_\gamma\) since \(\alpha_4\) is orthogonal to \(w_\gamma\). We have:

\[ B_{w_\gamma}(\chi) = -e^{\gamma} \frac{(1 - q^{-2} e^{-\tilde{a}_1})(1 - q^{-2} e^{-\tilde{a}_1 - \tilde{a}_4})}{(1 - q^{-2} e^{\tilde{a}_1})(1 - q^{-2} e^{\tilde{a}_1 + \tilde{a}_4})} (\chi). \]
6.20. $G_m \times \text{Sp}_{2n-2} \setminus \text{Sp}_{2n}$. There are 2 spherical roots, $\alpha_1$ and $\gamma = \alpha_1 + 2 \alpha_2 + \cdots + 2 \alpha_{n-1} + \alpha_n$ but, as in §6.6, we are interested only in $\gamma$. We have $\Delta(X) = \{\alpha_3, \alpha_4, \ldots, \alpha_n\}$. A path corresponding to $\gamma$ is the following:

\[ B_{w_\gamma}(\chi) = -e^{\hat{\gamma}} \frac{(1 - q^{-n+1} e^{-\hat{\alpha}_2})(1 - q^{-n+1} e^{-\hat{\alpha}_1 - \hat{\alpha}_2})}{(1 - q^{-n+1} e^{\hat{\alpha}_2})(1 - q^{-n+1} e^{\hat{\alpha}_1 + \hat{\alpha}_2})} \chi. \]

7. **The formula**

The goal of this section is to develop a more useful and explicit formula, in which the cocycles of our previous formula have been substituted by simpler expressions, and where a certain quotient of $L$-values plays a distinguished role. In order to do this, we need to make a combinatorial assumption on the colors of a spherical variety, which is very easy to check in each particular case, but we do not know how to prove. This assumption is certainly not true in all cases, but we expect it to be true in the case of affine homogeneous spherical varieties.

7.1. **Relevant and virtual colors.** Assume that all spherical roots of $X$ satisfy Statement 6.2.1. We call a color of $X$ relevant if it lies at the beginning of a path corresponding to $w_\gamma$, for $\gamma$ a simple spherical root.

7.1.1. **Lemma.** If $X$ is affine, then all colors are relevant.

**Proof.** We give just a sketch of the proof, since it will not be used later: Let $D$ be an irrelevant color with valuation $\theta$, and let $\alpha$ be a simple root raising it to $X$. If $(X, \alpha)$ were of type $T$ then $D$ would be relevant, so $(X, \alpha)$ is of type $U$, and hence $\theta = \hat{\alpha}$. If $\alpha \in \text{supp}(\gamma)$ for some spherical root $\gamma$, then $X_\gamma$ is of rank at most two and one sees by a case-by-case analysis of spherical varieties of rank at most two that $\hat{\alpha} \perp \gamma$. On the other hand, if $\alpha \notin \text{supp}(\gamma)$ then $\langle \hat{\alpha}, \gamma \rangle \leq 0$. Hence $\langle \hat{\alpha}, \gamma \rangle \leq 0$ for all $\gamma \in \Delta_X$, which means that $\theta = \hat{\alpha} \in V$. But if $X$ is affine homogeneous then [Kn91,
We now make some remarks assuming the validity of Statement 6.2.1:

- A color $D$ with corresponding valuation $\theta$ is in the beginning of a path leading to a simple spherical reflection $w$, if and only if $\langle \theta, \gamma \rangle > 0$. In this case, we will say that $D$ belongs to the spherical root $\gamma$.
- If a color belongs to a spherical root of type $G$, type $T$ non-split, or type $(U, \psi)$, then it belongs only to that spherical root. Therefore, one can define the type of a relevant color as being one among the following: $G$, $T$ split, $T$ non-split or $(U, \psi)$.

We now define the set of virtual weighted colors as follows: It consists of triples $(\tilde{\theta}, \sigma, r)$ where $\tilde{\theta}$ is an element of $\mathcal{X}(X)^*$, $\sigma = \pm$ is a sign and $r \in \frac{1}{2} \mathbb{Z}$; for every relevant color $D$ with valuation $\theta$, the triple $(\tilde{\theta}, +, (\tilde{\theta}, \rho_P(X)))$ is a weighted color; if $\gamma$ is a simple spherical root of type $T$ split and the color $D$ (with valuation $\tilde{\theta}$) belongs to $\gamma$ then the triple $(-w_\gamma \tilde{\theta}, +, (\tilde{\theta}, \rho_P(X)))$ is a virtual weighted color; if it coincides with $(\tilde{\theta}, +, (\tilde{\theta}, \rho_P(X)))$ it is counted with multiplicity two; finally, if $\gamma$ is a simple spherical root of type $T$ non-split and the color $D$ (with valuation $\tilde{\theta}$) belongs to $\gamma$ then the triple $(-w_\gamma \tilde{\theta}, -, (\tilde{\theta}, \rho_P(X)))$ is also a virtual weighted color. No other triples are virtual weighted colors.

We clarify the following fact on multiplicities: To every simple spherical root $\gamma$ of type $T$ there are two “belonging” virtual weighted colors $(\tilde{\theta}, \sigma, r)$, characterized by the fact that $\langle \tilde{\theta}, \gamma \rangle > 0$. If in case $T$ non-split there are two “real” belonging colors (necessarily with the same valuation $\tilde{\theta} = \frac{1}{2} \gamma$), then one is taken as a virtual weighted color with positive sign and the other is taken with negative sign. In case $T$ split there may be two identical virtual weighted colors, or there may be virtual weighted colors that belong to more than one simple spherical root. Not all virtual weighted colors come from real colors, see for instance the examples of §6.6 and §6.8.

The group $W_X$ acts on triples $(\tilde{\theta}, \sigma, r)$ as above by acting on the first component. We let $\Theta$ denote the minimal set of such triples such that:

- $\Theta$ contains all virtual weighted colors.
- If $(\tilde{\theta}, \sigma, r) \in \Theta$ then $(-\tilde{\theta}, \sigma, r) \in \Theta$ (and with the same multiplicity).

By abuse of notation, we will sometimes write $\tilde{\theta} \in \Theta$ and mean a triple as above, in which case we will denote by $\sigma_{\tilde{\theta}}$, $r_{\tilde{\theta}}$ the corresponding sign and constant. We denote by $\Theta^+$ the set of $\tilde{\theta} \in \Theta$ such that $\langle \tilde{\theta}, \eta \rangle \geq 0$ for every $\eta \in \mathcal{X}(X)$ which appears in $k[X]^{(B)}$. Equivalently, since the regular $B$-eigenfunctions are precisely those rational $B$-eigenfunctions which don’t blow up on any of the colors, $\Theta^+$ consists of all $\tilde{\theta} \in \Theta$ such that $\tilde{\theta}$ lies in the cone generated by the valuations defined by all colors. (This cone is strictly convex since $X$ is assumed quasi-affine.) We also write $\tilde{\theta} > 0$ if $\tilde{\theta} \in \Theta^+$, and $\tilde{\theta} < 0$ if $-\tilde{\theta} \in \Theta^+$.

Our main assumption regarding the set $\Theta$ is the following:

7.1.2. Statement. For every $(\tilde{\theta}, \sigma, r) \in \Theta$ we either have $\tilde{\theta} > 0$ or $-\tilde{\theta} < 0$. For every $\gamma \in \Delta_X$ the set $\{(\tilde{\theta}, \sigma, r) \in \Theta^+| w_\gamma \tilde{\theta} < 0\}$ consists precisely of the virtual weighted colors belonging to $\gamma$. 

§6] the cone generated by valuations of colors is separated from $V$ by a hyperplane, a contradiction. 

\end{proof}
This assumption is certainly not true for every spherical variety. However, I conjecture that it is true if \( X = H \setminus G \) is (homogeneous and) affine, i.e. \( H \) is reductive. The benefit of this statement is that it is very easy to check in each particular case; its drawback is that it gives no insight into what it might mean. A geometric understanding of this statement may provide a better understanding of the nature of the \( L \)-functions which are about to appear. At the end of this section we will discuss an analytic criterion which would – via a representation-theoretic argument – imply this assumption, and which may be a way to understand it. (Although at this point it seems at least equally hard to prove.) But first we will see what this assumption implies.

7.2. The formula. The final form of our formula is the following:

7.2.1. Theorem. Assume that all simple spherical roots of \( X \) satisfy Statement 6.2.1 and that the set \( \Theta \) satisfies Statement 7.1.2. Then:

\[
\Omega_{\tilde{\lambda}}(x_0) = \omega(\tilde{\chi}) \beta(\tilde{\chi})
\]

where

\[
\beta(\tilde{\chi}) := \prod_{\gamma \in \Phi^+} (1 - e^{\gamma}(1 - \sigma_\theta q^{-r_s} e^{\theta})) \omega(\tilde{\chi}),
\]

and \( \omega \in \mathbb{C}[A^*_X]^{W_X} \). If \( X \) is affine, then \( \omega \) is a constant \( c \) equal to:

\[
c = \beta(\delta_{P(X)})^{-1}.
\]

In either case:

\[
\Omega_{\tilde{\lambda}}(x_\lambda) \beta(\tilde{\chi}) = \delta_{P(X)}(x_\lambda) \prod_{\Theta^+} (1 - \sigma_\theta q^{-r_s} T_\theta) s_\lambda(\tilde{\chi})
\]

where \( s_\lambda = \sum_{w \in W_X} (-1)^w e^{\rho - \rho + w_\lambda} \prod_{\alpha > 0} (1 - e^{\alpha}) \) is the Schur polynomial indexed by lowest weight (that is, if \( \lambda \) is anti-dominant then \( s_\lambda \) is the character of the irreducible representation of \( \tilde{G}_X \) with lowest weight \( \tilde{\lambda} \)) and \( T_\theta \) denotes the formal operator:

\[
T_\theta s_\lambda = s_{\theta + \lambda}.
\]

Remark. In the case of a pair \((X, \mathcal{L}_\Psi)\), where \( X \) is parabolically induced from an affine spherical variety and \( \Psi \) is a generic character, then it should be considered that we are in the same case as that of an “affine variety”. The statements above pertaining to affine varieties hold, except for the one about the value of the constant \( c \), due to failure of Lemma 4.2.3.

The proof of the theorem will be completely combinatorial. The only geometric information used is the following fact (cf. [Kn91, §6]):

The spherical variety \( X = H \setminus G \) is affine (equivalently, \( H \) is reductive) if and only if \( V \) and \{\( \check{\nu}_D \mid \) \( D \) a color\} are separated by a hyperplane. It is quasi-affine if and only if \{\( \check{\nu}_D \mid \) \( D \) a color\} spans a strictly convex cone and does not contain zero.
Proof. The first step is to show that $B_w(\bar{\chi}) = \frac{\beta(\bar{\chi})}{\beta(w, \bar{\chi})}$. Since the $B_w$ satisfy cocycle relations, it is enough to show that for $w = w_\gamma$, where $\gamma \in \Delta_X$. We compute:

$$\frac{\beta(\bar{\chi})}{\beta(w, \bar{\chi})} = -\epsilon^\ell(\bar{\chi}) \cdot \prod_{\theta > 0, w, \theta < 0} \frac{1 - \sigma_g q^{-r_\theta} e^{-\hat{\delta}}}{1 - \sigma_g q^{-r_\theta} e^{\hat{\delta}}}.$$ 

By Theorem 7.1.2 and Statement 6.2.1, for $w = w_\gamma$ this is equal to $B_{w_\gamma}(\bar{\chi})$. Hence,

$$\Omega_{\frac{\hat{\beta}(\bar{\chi})}{\beta(\bar{\chi})}}(x_\lambda) = \Omega_{\frac{\hat{\beta}(w, \bar{\chi})}{\beta(w, \bar{\chi})}}(x_\lambda) = \delta_{P(\lambda)}(x_\lambda) \sum_{w \in W_X} \frac{1}{\beta(w, \bar{\chi})} \chi(x_\lambda).$$

This proves (7.4), which in particular implies that $\Omega_{\frac{\hat{\beta}(w, \bar{\chi})}{\beta(w, \bar{\chi})}}$ is regular in $\bar{\chi}$, for every $\bar{\lambda}$. In particular, for $\bar{\lambda} = 0$ we have $\Omega_{\frac{\hat{\beta}(w, \bar{\chi})}{\beta(w, \bar{\chi})}}(x_0) = \omega(\bar{\chi})$ where $\omega$ is a $W_X$-invariant regular function of $\bar{\chi}$. Our proof will be complete if we show that in the affine case $\omega$ is a constant; the statement about the precise value of $e$ follows from the fact that $\Omega_{\frac{\hat{\beta}}{\beta}} = 1$ (Lemma 4.2.3).

We define two partial orders on the set of weights on $A^*_X$: We will write that $\bar{\lambda} \succ_1 \hat{\mu}$ if $\bar{\lambda} - \hat{\mu}$ is in the non-negative span $\mathcal{R}$ of $\hat{\Phi}^+_X$; and that $\bar{\lambda} \succ_2 \hat{\mu}$ if $\bar{\lambda} - \hat{\mu}$ is in the non-negative span $T$ of the valuations induced by all colors (i.e. the cone dual to the cone of characters of $B$ on $k[X]^{|B|}$). Since $T \supset \mathcal{R}$ (see the proof of Theorem 7.1.2), the second order is weaker than the first. Now we have:

7.2.2. Lemma. If $\omega$ is a non-constant, $(W_X)$-symmetric polynomial on $A^*_X$ and $\bar{\lambda}$ is a minimal weight appearing with non-zero coefficient in $\omega$, then $\bar{\lambda}$ is also a minimal weight, appearing with the same coefficient, in $\omega \cdot \prod_{\gamma \in \hat{\Phi}^+_X} (1 - e^{\gamma})$.

The validity of this lemma is obvious, since “minimal weight” means minimal for the $\succ_1$ ordering and all weights of $\prod_{\gamma \in \hat{\Phi}^+_X} (1 - e^{\gamma})$ are $\succ_1 0$, with strict inequality except for the summand which is equal to the constant 1.

Given this lemma and (7.4), it suffices to prove that $\sum_{w \in W_X} (-1)^w e^{\hat{\delta} - w \hat{\rho} + w \lambda}$ does not contain any anti-dominant weights when $\hat{\lambda} = \sum_{\delta \in I} \hat{\theta}$ and $I$ is any subset of $\Theta^+$. Here we use the fact that $\mathcal{V}$ and $\{ \tilde{e}_D | D$ a color $\}$ are separated by a hyperplane. By Proposition 7.1.2 and Statement 6.2.1, this means that:

$\mathcal{V}$ and $\Theta^+$ are separated by a hyperplane or, equivalently, that $\mathcal{V}$ and $T$ intersect trivially. Since we want to prove that for every $w \in W_X, I \subset \Theta^+$ the weight $\hat{\rho} - w \hat{\rho} + w \sum_{(\delta, r) \in I} \hat{\theta}$ is not anti-dominant, it suffices to show that:

$$\hat{\rho} - w \hat{\rho} + w \sum_{(\delta, r) \in I} \hat{\theta} \succ_2 0.$$ 

For simplicity of notation, in what follows, we denote by $|I|$ the sum $\sum_{(\delta, r) \in I} \hat{\theta}$, for any $I \subset \Theta$.

Given that $|\Theta^+| - w|\Theta^+| = \hat{\rho} - w \hat{\rho}$, we can write (7.5) as:

$$|\Theta^+| - w(|\Theta^+| - |I|) \succ_2 0.$$
Let $\Theta_1 = w\Theta^+ \cap \Theta^+$ and $\Theta_2 = w\Theta^+ \setminus \Theta_1 \subset -\Theta^+$. Let $I_1 = w(\Theta^+ \setminus I) \cap \Theta_1$ and $I_2 = w(\Theta^+ \setminus I) \cap \Theta_2$. Then (7.5) can be written:

$$|\Theta^+| - |I_1| + |I_2| \geq 2 \iff |\Theta^+ \setminus I_1| + |I_2| \geq 2.$$ 

Since both $\Theta^+ \setminus I_1$ and $-I_2$ belong to $\Theta^+$, this holds! \qed

**Remark.** With minor additions, this proof actually shows that:

$$\prod_{\Theta^+} (1 - \sigma \theta q^{-r_{\theta}} T_{\theta}) s_{\lambda}^{\chi}(\check{\chi}) - \sum_{W_X} e^{w_{\lambda} - 2} \lambda,$$

where, for a symmetric polynomial $p$, the expression $p \geq 2 \lambda$ means that all weights of $p$ are $\geq 2 \lambda$; equivalently, since $\geq 2$ is weaker than $> 1$, that all anti-dominant weights of $p$ are $> 2 \lambda$.

The theorem leads us to the following definition:

**7.2.3. Definition.** We denote by $L_X^2$ the function

$$\check{\chi} \mapsto c(\check{\chi}) = c \prod_{\theta \in \Theta^+} \frac{(1 - e^\gamma)}{(1 - \sigma q^{-r_{\theta}} e^\theta)} \chi(\check{\chi})$$

on $A_X^\lambda$.

We denote by $L_X$ and call “the $L$-function of $X$” the function

$$L_X^\frac{1}{2} L_X^\frac{1}{2} (\check{\chi})^{-1} = e^2 \prod e_{\theta}^{\frac{1}{2}} e_{\theta}^{-\frac{1}{2}} (1 - e^\gamma) \prod_{\theta \in \Theta^+} (1 - \sigma q^{-r_{\theta}} e^\theta),$$

which is $W_X$-invariant on $A_X^\lambda$ and hence can be thought of as a conjugation-invariant function on $G_X$.

**Remark.** The importance of this definition lies in the (mostly conjectural) relationship between period integrals of automorphic forms and the value of $\Omega_{\frac{1}{2}}$ in the affine, multiplicity-free case, see section 10.

**7.2.4. Example.** Let $X = \mathrm{PGL}_2 \setminus (\mathrm{PGL}_2)^3$, the variety of Example 5.6.3. Here the valuations induced by the colors are $\frac{\alpha_1 + \alpha_2}{2} - \frac{\alpha_3}{2}, \frac{\alpha_1 - \alpha_2 + \alpha_3}{2}, -\frac{\alpha_1 + \alpha_2 + \alpha_3}{2}$ and the set $\Theta^+$ contains those and also the co-weight $\frac{\alpha_1 + \alpha_2 + \alpha_3}{2}$ (for simplicity, since $\sigma \theta = +$ and $r_\theta = \frac{1}{2}$ for all $\theta$, we only write the value of $\hat{\theta}$). Clearly, $\Theta = \Theta^+ + (-\Theta^+)$ is $W = W_X$-invariant, and therefore the final formula of Example 5.6.3 holds. In this example, $L_X$ is up to zeta-factors equal to the the quotient of the tensor product $L$-function at $\frac{1}{2}$ by the adjoint $L$-function at $0$.

**7.2.5. Example.** This example is related to the period integral proposed by Gross and Prasad, cf. [GP92]. Let $X = \mathrm{SO}_n \setminus \mathrm{SO}_n \times \mathrm{SO}_{n+1}$ (the stabilizer embedded diagonally in the product of the two groups). Assume all groups are split. One can check that in this case $G_X = G$, and that the set $\Theta^+$ satisfies Statement 7.1.2 and is equal to the set of all non-trivial weights $\hat{\theta}$ of the tensor-product representation of $G_X$ such that $\langle \hat{\theta}, \rho \rangle > 0$ (where $\rho$ denotes, as usual, the half-sum of positive roots of $G$, and all $\sigma \rho = +$ and $r_\rho = \frac{1}{2}$). Hence, the corresponding $L$-value $L_X$.
is, up to zeta factors, equal to:

\[
\frac{L(\pi_1 \otimes \pi_2, \frac{1}{2})}{L(\pi_1, \text{Ad}, 0)L(\pi_2, \text{Ad}, 0)}
\]

where we have decomposed an unramified representation \( \pi \) of \( G \) as \( \pi_1 \otimes \pi_2 \) according to the decomposition \( G = \text{SO}_n \times \text{SO}_{n+1} \). In this case our calculation is similar to that performed (in greater generality) by Ichino and Ikeda [II].

7.2.6. Example. Let \( X = GL_n \setminus \text{SO}_{2n+1} \), where \( n \) is an even number. Here we have \( \hat{G}_X = \text{Sp}_n \times \text{Sp}_n \subset \text{Sp}_{2n} = \hat{G} \) and the valuations corresponding to colors are all simple short roots of \( \hat{G} \) (with \( \sigma_\theta = +, r_\theta = 1 \)), as well as half the long root of \( \hat{G} \) (with \( \sigma_\theta = +, r_\theta = \frac{1}{2} \)). Therefore, \( L_X \) (as a conjugation-invariant function on \( \hat{G}_X \)) is up to zeta factors equal to:

\[
\frac{L(\pi_1, \frac{1}{2})L(\pi_1 \otimes \pi_2, 1)}{L(\pi_1, \text{Ad}, 0)L(\pi_2, \text{Ad}, 0)}.
\]

7.3. An analytic criterion. Notice that, since the image of \( S_{\chi^{-1}} \) lies in the space of a representation induced from a character of \( P(X) \), the morphism \( \Delta_\chi \) (its adjoint) factors through the quotient: \( C_\infty^c(U \setminus G) \rightarrow C_\infty^c([P(X), P(X)] \setminus G) \). In fact, a completely analogous theory of Schwartz space and Fourier transforms exists for the space \( [P, P] \setminus G \), for any parabolic subgroup \( P \) (cf. [BK02]), and therefore \( \Delta_\chi \) can be viewed as a rational family of morphisms:

\[
S([P(X), P(X)] \setminus G) \rightarrow C_\infty^\ast(X).
\]

Set \( P = P(X) \), \( M = P/[P, P] \). As in Lemma 3.4.2, \( \Delta_{\chi^{-1}} : S([P, P] \setminus G) \rightarrow C_\infty^\ast(X) \) is defined by an integral of the form:

\[
ev_{\text{H1}} \circ \Delta_\chi(\Phi) = \int_{[P, P] \setminus G} \Phi(x)f_{\chi^{-1}}(x)dx
\]

where \( f_{\chi^{-1}} \) is a (pseudo-rational) eigenfunction for \( M \times H \) with eigencharacter \( \chi^{-1} \times 1 \) with respect to the normalized \( M \)-action (defined by \( (L_m \Phi)(x) = \delta_p \frac{1}{2}(m)\Phi(mx \cdot x) \)). Since the elements of \( S([P, P] \setminus G) \) are \( L^2 \), supported in a compact subset of \( [P, P] \setminus G \), and hence also \( L^1 \), it is clear that this integral converges if \( f_{\chi^{-1}} \) is locally bounded. This condition can be expressed in terms of the “valuations” of the pseudo-rational function \( f_{\chi^{-1}} \circ \iota \) (where \( \iota \) denotes the map \( g \mapsto g^{-1} \) on \( G \)) on the colors of \( X \):

7.3.1. Lemma. The function \( f_{\chi^{-1}} \) is locally bounded on \( [P, P] \setminus G \) if and only if

\[
\langle \hat{\iota}_D, \mathcal{R}(\chi) - \rho_{P(X)} \rangle \geq 0 \text{ for all colors } D \text{ of } X.
\]

Proof. The absolute value of \( f_{\chi^{-1}} \) can be expressed as a product of factors of the form \( |g_i|^{s_i} \) where \( s_i \in \mathbb{R} \) and \( g_i \in k([P, P] \setminus G)^{(M) \times H} = k(X)^{(P) \circ \iota} \) (where \( \iota \) denotes the map \( g \mapsto g^{-1} \) in \( G \)). Such a function \( |g_i| \) is bounded if and only if \( g_i \) is regular, which happens if and only if \( \langle \hat{\iota}_D, -\eta_i \rangle \geq 0 \) for all colors \( D \) of \( X \), where \( \eta_i \) is the character of \( g_i \) (in additive notation). More generally, the product of \( |g_i|^{s_i} \) is bounded if and only if \( \sum_i s_i \langle \hat{\iota}_D, -\eta_i \rangle \geq 0 \). But the latter is, by definition, equal to \( \langle \hat{\iota}_D, \mathcal{R}(\chi) - \rho_{P(X)} \rangle \). (Recall that if \( \chi^{-1} \) is the eigencharacter of \( f_{\chi^{-1}} \) with respect then \( \hat{\iota}_D + \hat{\iota}_{D'} = \hat{\beta} \). By these rules the calculation of one \( \hat{\iota}_D \) implies many others, in many cases (such as the present one) all of them.
7.3.2. **Corollary.** The integral expression for \( \Delta \frac{1}{\delta(X)^\chi} \) converges if \( \Re(\chi) \) lies in \( \rho_{P(\chi)} + T^\vee \), where \( T \) denotes the cone spanned by all valuations \( \bar{v}_D \) induced by colors of \( X \) and \( T^\vee \subset X^*(X) \otimes \mathbb{R} \) denotes its dual cone.

We will now see that extending this result to a larger domain, the interior \( (T^\vee)^\circ \) of the whole cone \( T^\vee \), would imply Statement 7.1.2.

7.3.3. **Proposition.** If all spherical roots \( \gamma \) of \( X \) satisfy Statement 6.2.1, and the integral expression for \( \Delta \frac{1}{\delta(X)^\chi} : S([P(\chi), P(\chi)]) \to C^\infty(X) \) converges for all \( \chi \in (T^\vee)^\circ \) then Statement 7.1.2 is true for \( X \).

**Proof.** Since the \( B_w \) are cocycles: \( W_X \to C(A_X^\ast) \), by Hilbert’s Theorem 90 there exists a \( \beta \in C(A_X^\ast) \) such that \( B_w(\chi) = \frac{\beta(\chi)}{2^\sigma \cdot \rho(\chi)} \). This \( \beta \) is uniquely defined up to a \( W_X \)-invariant function, and similarly the divisors of any two such \( \beta \)'s differ by a \( W_X \)-invariant divisor. Let \( B \) be the unique divisor on \( A_X^\ast \) with the following properties:

- \(-B\) is an effective divisor.
- There is no effective \( W_X \)-invariant divisor \( D \) such that \(-B-D\) is effective.
- There is a \( W_X \)-invariant divisor \( D \) such that \( B+D\) is the divisor of \( \beta \).

In particular, \( B \) has the following property: \( B - w^{-1}B = [B_w] \) for every \( w \in W_X \), where \([B_w]\) denotes the divisor of \( B_w \).

We will say that an effective divisor \( D \) “appears” in \( B \) if \(-B-D\) is effective. Given the form of \([B_w]\), we deduce that \( B \) is a \( (\text{negative}) \) linear combination of irreducible divisors of the form \([1 + q r^e \cdot e^\theta]\). Indeed, if any irreducible divisor which is not of this form appears, its whole \( W_X \)-orbit will have to appear because otherwise it will be an irreducible component of \( B - w^{-1}B \) for some \( w \). But the whole \( W_X \)-orbit appearing contradicts the axioms for \( B \). Notice that for a divisor of the form \([1 + q r^e \cdot e^\theta]\) with \( r > 0 \) the half-line of \( \bar{\theta} \) in \( \text{Hom}(\mathcal{A}(X), \mathbb{R}) \) is uniquely defined. Thus, we will call such a divisor positive if \( \langle \bar{\theta}, \eta \rangle \geq 0 \) for all eigencharacters \( \eta \) of \( B \) on \([k[X]_B]\). We claim:

Only positive divisors appear in \( B \).

The poles of \( B_w(\chi) \) are the same as those of \( b_w(\delta^\frac{1}{\delta(X)^\gamma}) \). From the functional equation: \( \Delta \frac{1}{\delta(X)^\gamma} \circ F_{w^{-1}} = b_w(\delta^\frac{1}{\delta(X)^\gamma}) \Delta \frac{1}{\delta(X)^\gamma} \), where \( F_{w^{-1}} \) denotes the \( w^{-1} \)-Fourier transform on \( S([P(\chi), P(\chi)]) \) and the fact that the morphisms \( \Delta \frac{1}{\delta(X)^\gamma} \) are never identically zero we deduce that a divisor \( D \) of \( B_w \) must be a pole of \( \Delta \frac{1}{\delta(X)^\gamma} \). Therefore, it cannot intersect the domain of convergence of \( \Delta \frac{1}{\delta(X)^\gamma} \), which implies that, under our assumption on the domain of convergence, the polar divisor cannot be negative.

Now, it is clear that for every simple spherical root \( \gamma \) the divisors \([1 - \sigma q^{-r} e^{\bar{\theta}}]\), where \( (\bar{\theta}, \sigma, r) \) ranges over all weighted virtual colors which belong to \( \gamma \), appear in \( B \), since they are pole of \( B_w \). We actually claim:
For every \((\tilde{\theta}, \sigma, r)\) in the \(W_X\)-orbit of weighted virtual colors which belong to \(\gamma\) the triple \((\tilde{\theta}, \sigma, r)\) or the triple \((-\tilde{\theta}, \sigma, r)\) appears in \(\mathcal{B}\). Moreover, the multiplicity of its appearance is constant for all elements of the \(W_X\)-orbit.

In combination with the previous claim, this completely identifies the divisor \(\mathcal{B}\) as the divisor of all positive \(W_X\)-translates of divisors of the form \([1 - \sigma q^{-r} e^\theta]\) (with the correct multiplicities) where \((\tilde{\theta}, \sigma, r)\) is a weighted virtual color. Moreover, it proves that for every such \(W_X\)-translate we have \(\tilde{\theta} > 0\) or \(-\tilde{\theta} > 0\).

To prove this claim, assume otherwise. Then there exists a \(\gamma \in \Delta_X\) and such a pair of opposite triples \((\pm \tilde{\theta}, \sigma, r)\) such that neither of the two triples in the pair appears in \(\mathcal{B}\), while one of the triples \((\pm \tilde{\theta}, \sigma, r)\) appears. But then \([1 - \sigma q^{-r} e^{\pm \theta}]\) is 

\[\begin{aligned}
\text{In particular:} \quad &
\text{From now on we let } X \text{ be a homogeneous, affine spherical variety which satisfies the conclusion of Theorem 6.2.2 and Statement 7.1.2. Assume also that } X \text{ has a unique } B\text{-orbit. In particular, the strong form of the Cartan decomposition holds:}
\end{aligned}\]

\[X/K \simeq \Lambda_X^+ \text{ and } A_X^+ \subset A^+.\]

Let \(H_X^*\) be the quotient of the “spherical Hecke algebra” \(H(G, K)\) which acts on \(C_c^\infty(X)^K\). Then, under the Satake isomorphism, the maximal spectrum of \(H_X\) is equal to the image of \(A_X^+\) in \(A^*/W\). It was proven in [Sa08] that \(C_c^\infty(X)^K\) is torsion-free over \(H_X\). Here we determine precisely the \(H(G, K)\)-structure:

8.0.4. **Theorem.** There is an isomorphism: \(C_c^\infty(X)^K \cong \mathbb{C}[\delta^\frac{1}{2} (X)] A_X^+ \big|_{W_X} \), compatible with the \(H(G, K)\)-structure and the Satake isomorphism. Under this isomorphism the element \(1_{X(o)}\) is mapped to the constant 1.

**Remark.** This proves the conjecture of [Sa08, §6.3] under the assumptions of the present section.

\[\begin{aligned}
\text{Proof.} \quad &
\text{Let } P_{A,} \text{ be the polynomial on } \delta^\frac{1}{2} (X), A_X^+ \text{ defined as } P_{A,} (\chi \delta^\frac{1}{2} (X)) = \frac{\Omega_{\lambda}(\chi \delta^\frac{1}{2} (X))}{L^\lambda(\chi)},\quad \text{(In particular, } P_0 = 1.) \text{ Then the map } 1_{X,K} \mapsto P_{A,} \text{ is } H(G, K)-\text{equivariant with respect to the Satake isomorphism } H(G, K) = \mathbb{C}[A^*]_W. \text{ Since all } P_{A,} \text{ are } W_X-\text{invariant polynomials and generate } \mathbb{C}[\delta^\frac{1}{2} (X)] A_X^+ \big|_{W_X}, \text{ the result follows.}\n\end{aligned}\]

In particular:

8.0.5. **Corollary.** If \(\dim \text{Hom}_{H(G,K)}(C_c^\infty(X)^K, \mathbb{C}_\eta) = 1 \) for a generic point \(\eta \in \text{spec}_M H_X\), then \(\dim \text{Hom}_{H(G,K)}(C_c^\infty(X)^K, \mathbb{C}_\eta) = 1 \) for every \(\eta \in \text{spec}_M H_X\) and \(\dim \text{Hom}_G(\pi, C_c^\infty(X)) \leq 1 \) for every irreducible unramified representation \(\pi\).
Besides, the previous theorem has the following corollary:

8.0.6. Theorem (Good test vectors). Assume that the unramified spectrum of $X$ is multiplicity-free (which, by the above corollary, is equivalent to saying that it is multiplicity-free for a generic representation in the spectrum). Let $\pi$ be an irreducible unramified representation and $L : \pi \to \mathbb{C}$ a non-zero $H$-invariant functional. Then $L|_{\pi K} \neq 0$.

Proof. We denote by $\Lambda : \pi \to C^0(X)$ the morphism defined from $L$ by Frobenius reciprocity. Let $v \in \pi^K$. Since $v$ generates $\pi$ and $L$ is non-zero, $\Lambda(v) \neq 0$. In other words, $\langle 1_{x_0K}, \Lambda(v) \rangle \neq 0$ for some $\lambda \in \Lambda^X_v$. There is $h \in \mathcal{H}(G, K)$ (whose image in $\mathcal{H}_X$ corresponds to the polynomial $P_\chi$) such that $1_{x_0K} = h * 1_{x_0K}$. Then:

$$\langle 1_{x_0K}, \Lambda(v) \rangle = \langle 1_{x_0K}, \Lambda(h^*v) \rangle = \eta_v(h^*) \langle 1_{x_0K}, \Lambda(v) \rangle = \eta_v(h^*)L(v)$$

where $\eta_v$ is the character of $\mathcal{H}(G, K)$ corresponding to $\pi$. Therefore $L(v) \neq 0$. \qed

8.0.7. Corollary. Let $H \subset G$ be reductive groups defined over a global field $F$, with $H$ a spherical subgroup of $G$. Assume that at almost every place $v$ the corresponding group $G_v$ is split and the variety $\mathcal{H}_v G_v$ satisfies the hypotheses of Theorem 8.0.6. Denote by $A_F$ the ring of adeles of $F$, and let $\pi \simeq \otimes_v \pi_v$ be a representation of $G(A_F)$. If $\text{Hom}_{\mathcal{H}_v}(\pi, \mathbb{C}) \neq 0$ for every $v$, then $\text{Hom}_{\mathcal{H}(A_F)}(\pi, \mathbb{C}) \neq 0$.

9. Unramified Plancherel formula

We continue to make the same assumptions on $X$ as in the previous section. Since $H$ is reductive, we may and will assume that the $G$-eigenform $\omega_X$ on $X$ is $G$-invariant, and hence defines a $G$-invariant measure $|\omega_X|$ on $X$. Remember that we have normalized that measure so that $|\omega_X|(x_0 J) = 1$. We will compute the Plancherel formula for unramified functions on $X$. More precisely, it is known that there exists a (n essentially unique) decomposition of $L^2(X)$ as a direct integral of irreducible, unitary representations. Specializing to $K$-invariant elements of $L^2(X)$, we get for every $\Phi \in C_c^\infty(X)^K$ a formula for $\|\Phi\|_{L^2(X)}^2$ as an integral of $|\langle \Phi, \Omega \rangle|^2$ where $\Omega$ ranges over $\mathcal{H}(G, K)$-eigenfunctions belonging to unitary representations, for every $\Phi \in C_c^\infty(X)^K$. We will compute this formula. As an application, we will compute the volume of $X(\phi)$ (essentially, its “Tamagawa volume”).

Since we are keeping the assumptions of the previous section, every Hecke eigenfunction is a multiple of

$$\Omega_X^\chi := \frac{\Omega_X}{L_X^\chi(\chi)}$$

for some $\chi \in \delta_{(X)}^+ A_X^\chi$. Notice that $\Omega_X(x_0) = 1$ and $\Omega_W^\chi = \Omega_X^\chi$ for every $w \in W_X$. Therefore, the Plancherel formula will have the form:

$$\|\Phi\|^2 = \int_{\delta_{(X)}^+ A_X^\chi / W_X} |\langle \Phi, \Omega_X^\chi \rangle|^2 d\mu(\chi)$$

for every $\Phi \in C_c^\infty(X)^K$ and a unique positive measure on $\delta_{(X)}^+ A_X^\chi / W_X$ (supported, of course, on the set of points belonging to unitary representations) which, for brevity, when normalized this way, will be called the Plancherel measure for $X$. 
9.0.1. **Theorem.** The Plancherel measure for the unramified spectrum of $X$ is supported on $\delta_{(X)}^{\frac{1}{2}}A_X^{*,1}/W_X$, where $A_X^{*,1}$ denotes the maximal compact subgroup of $A_X^*$. For every $\Phi \in C_c^\infty (X)^K$ we have:

\[
\|\Phi\|^2 = \frac{1}{Q \cdot |W_X|} \int_{A_X^{*,1}} \left| \left< \Phi, \Omega_{\delta_{(X)}^{\frac{1}{2}}\chi} \right> \right|^2 L_X(\chi) d\chi,
\]

where $d\chi$ is probability Haar measure on $A_X^{*,1}$ and

\[
Q = \frac{\text{Vol}(K)}{\text{Vol}(J\omega_J)} = \prod_{\alpha \in \Phi^+} \frac{1-q^{-1}\langle \alpha, \rho \rangle}{1-q^{-\langle \alpha, \rho \rangle}}.
\]

**Remark.** With respect to our original eigenfunctions $\Omega_{\chi}$ we have:

\[
\|\Phi\|^2 = \frac{1}{Q \cdot |W_X|} \int_{A_X^{*,1}} \left| \left< \Phi, \Omega_{\delta_{(X)}^{\frac{1}{2}}\chi} \right> \right|^2 d\chi.
\]

**Proof.** As in the previous section, let $P_{\lambda}$ be the polynomial on $\delta_{(X)}^{\frac{1}{2}}A_X^{*}/W_X$ defined as $P_{\lambda}(\chi) = \Omega_{\chi}(x_\lambda)$. The fact that $\left< 1_{x_\lambda K}, 1_{x_\lambda K} \right>_{L^2(X)} = 0$ for $\lambda \neq 0$ implies, via the abstract Plancherel formula (9.1), that

\[
\int_{\delta_{(X)}^{\frac{1}{2}}A_X^{*}/W_X} P_{\lambda}(\chi)d\mu(\chi) = 0
\]

for every $\lambda \neq 0$.

Recall that the $P_{\lambda}$ span the space $\mathcal{P}$ of polynomials on $\delta_{(X)}^{\frac{1}{2}}A_X^{*}/W_X$. Given a linear functional on this space of polynomials, there exists at most one real-valued measure of bounded support on $\delta_{(X)}^{\frac{1}{2}}A_X^{*}/W_X$ which represents this functional. (Indeed, this follows from the density of the functions of the form $\Re P \Im P$, $P \in \mathcal{P}$, in the space of continuous functions on any compact domain.) We will show in a combinatorial way that $\left< 1_{x_\lambda K}, 1_{x_\lambda K} \right> = 0$ for $\lambda \neq 0$, where $[,]$ denotes the hermitian inner product defined by the right hand side of (9.2) (or (9.3)). It will then follow that Plancherel measure is a multiple of the measure of (9.2).

9.0.2. **Lemma.** For every $\lambda \neq 0$ we have: $\left< 1_{x_\lambda K}, 1_{x_\lambda K} \right> = 0$, where $[,]$ denotes the hermitian inner product defined by the right hand side of (9.2).

It suffices to show that $\int_{A_X^{*,1}} P_{\lambda}(\delta_{(X)}^{\frac{1}{2}}\chi)L_X(\chi)d\chi = 0$ for $\lambda \neq 0$. The value of this integral is equal to the constant term of the Laurent series expansion of the rational function $P_{\lambda}(\delta_{(X)}^{\frac{1}{2}}\chi)L_X(\chi)$ (see, for instance, [Ma01]). We have:

\[
P_{\lambda}(\delta_{(X)}^{\frac{1}{2}}\chi)L_X(\chi) = c \sum_{\gamma \in \Phi_X} \frac{1-e^{\gamma}}{1-e^{-\gamma}} \sum_{\theta \in \Theta} \frac{1-e^{\gamma}}{1-e^{-\gamma}} \sum_{\delta \in \Theta} \frac{1-e^{\gamma}}{1-e^{-\gamma}} e^{\lambda(w_\chi)} =
\]

\[
= c \sum_{\gamma \in \Phi_X} \frac{1-e^{\gamma}}{1-e^{-\gamma}} e^{\lambda(w_\chi)}.
\]
In the notation of the proof of Theorem 7.2.1, all weights in the Laurent expansion of
\[ \prod_{\gamma \in \mathcal{W}} \frac{(1-e^{-\gamma})}{\prod_{\delta \in \Theta} (1-\sigma_\delta q^{-r_\delta e_\delta})} e^{\lambda} \] are \( \prec_2 \lambda \). Therefore, 0 is a weight of the above expression only if \( 0 \in w(\lambda-T) \) for some \( w \in W_X \), equivalently only if \( \lambda \in T \iff \lambda = 0 \). This proves the lemma.

Hence the Plancherel formula has to be a multiple of the right-hand-side of (9.2). On the other hand, a method of Bernstein (discussed in [SV]) implies that the right-hand-side of (9.2) is precisely the most continuous part of the Plancherel formula. The theorem follows.

Theorem 9.0.3. The measure of \( X(\mathfrak{o}) \) is:

\[ |\omega_X|(X(\mathfrak{o})) = Q \cdot c^{-1} = Q \frac{\prod_{\gamma \in \mathcal{W}} (1-e^{-\gamma})}{\prod_{\delta \in \Theta} (1-\sigma_\delta q^{-r_\delta e_\delta})} (\delta_{P(X)}^\mathfrak{o}). \]

Proof. We have:

\[ |\omega_X|(X(\mathfrak{o})) = \|1_{X(\mathfrak{o})}\|^2 = \frac{1}{Q \cdot |W_X|} \int_{A_X^{-1}} \left| \left< 1_{X(\mathfrak{o})}, \Omega_{\mathfrak{o}, \chi} \right> \right|^2 L_X(\chi) d\chi = \]

\[ = \frac{1}{Q \cdot |W_X|} \int_{A_X^{-1}} |P_0(\delta_{P(X)}^\mathfrak{o})\chi|^2 (|\omega_X|(X(\mathfrak{o})))^2 L_X(\chi) d\chi \Rightarrow \]

\[ |\omega_X|(X(\mathfrak{o})) = Q \cdot |W_X| \left( \int_{A_X^{-1}} L_X(\chi) d\chi \right)^{-1} \]

As before, the integral of \( L_X(\chi) = P_0(\delta_{P(X)}^\mathfrak{o}) L_X(\chi) \) is equal to the constant term in the Laurent expansion of:

\[ c \sum_{\gamma \in \mathcal{W}} \frac{\prod_{\gamma \in \mathcal{W}} (1-e^{-\gamma})}{\prod_{\delta \in \Theta} (1-\sigma_\delta q^{-r_\delta e_\delta})} \]

which in this case is equal to \( c \cdot |W_X| \). Therefore:

\[ |\omega_X|(X(\mathfrak{o})) = Q \cdot c^{-1} = Q \cdot \frac{\prod_{\gamma \in \mathcal{W}} (1-e^{-\gamma})}{\prod_{\delta \in \Theta} (1-\sigma_\delta q^{-r_\delta e_\delta})} (\delta_{P(X)}^\mathfrak{o}). \]

Remark. It is natural to call the measure \( (1-q^{-1})^{\text{rk}(A_X)} \cdot |\omega_X| \) the Tamagawa measure on \( X \). Indeed, any globally defined invariant volume form on \( X \) will induce, for almost every place, the same measure on \( X \) as \( q^*(a^{-1} da) \wedge du \) (where \( q : X \to A_X \) is the natural projection based at the point \( x_0 \)). With respect to the latter, we have \( \text{Vol}(x_0 J = x_0 B_0) = a^{-1} da(A_Xx_0) = (1-q^{-1})^{\text{rk}(A_X)} \), therefore the “Tamagawa measure” \( d^{\text{Tam}, X} \) is equal to \( (1-q^{-1})^{\text{rk}(A_X)} |\omega_X| \).

Therefore we have shown:

\[ d^{\text{Tam}, X}(X(\mathfrak{o})) = Q \cdot (1-q^{-1})^{\text{rk}(A_X)} \cdot \frac{\prod_{\gamma \in \mathcal{W}} (1-e^{-\gamma})}{\prod_{\delta \in \Theta} (1-\sigma_\delta q^{-r_\delta e_\delta})} (\delta_{P(X)}^\mathfrak{o}). \]

\[ \text{[12]While everything up to this point can be rephrased for the case of non-trivial line bundles } L_\phi, \text{ clearly Theorem 9.0.3 does not make sense in that case.} \]
10. Periods of Eisenstein series.

Let now \( F \) be a number field and \( G \) a split connected reductive group defined over the ring of integers of \( F \). We use the same notation: \( B, A, \) etc. as above and assume that \( B(F) \) has a single orbit on \( X(F) \). We also assume that for almost all completions \( F_v \) of \( F \) the variety \( X_F \), satisfies the assumptions of the previous two sections. We will denote by \( A(\mathbb{A}_F)^1 \) (and similarly for other groups than \( A \)) the intersection of the kernels of all homomorphisms: \( A(\mathbb{A}_F) \rightarrow \mathbb{G}_{m}(\mathbb{A}_F) \rightarrow \mathbb{R}^\times_+ \) where the first arrow denotes an algebraic character and the second denotes the absolute value. Hence, \( A(\mathbb{A}_F)/A(\mathbb{A}_F)^1 \simeq (\mathbb{R}^\times_+)^{r_k(A)} \). For every place \( v \) we denote the group \( G(F_v) \) by \( G_v \), the group \( G(\mathfrak{o}_v) \) (if \( v < \infty \)) by \( K_v \), etc.

Any idele class character of \( A(\mathbb{A}_F) \) can be twisted by characters of the group \( A(\mathbb{A}_F)/A(\mathbb{A}_F)^1 \), and thus lives in a \( r_k(A) \)-dimensional complex manifold of characters. Let \( \omega \) denote such a family. For \( \chi \in \omega \) we denote by \( I(\chi) = \text{Ind}_{B(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\chi\delta^2) \) the normalized principal series\(^{13} \) of \( G(\mathbb{A}_F) \), considered as a holomorphic family of vector spaces. For a meromorphic family of sections \( \chi \mapsto f_\chi \in I(\chi) \) we have the principal Eisenstein series defined by the convergent sum:

\[
E(f_\chi, g) = \sum_{\gamma \in B(F)G(F)} f_\chi(\gamma g)
\]

if \( \langle \alpha, \Re(\chi) \rangle \gg 0 \) for all \( \alpha \in \Delta \), and by meromorphic continuation to the whole \( \omega \).

Let \( H \) be a spherical subgroup of \( G \) over \( F \). We would like to compute the period integral \( \int_{H(F) \backslash H(\mathbb{A}_F)} E(f_\chi, h)dh \). Of course, this integral may not be convergent, therefore we have to make sense of it as a distribution on the Eisenstein spectrum of \( G \). Our goal is to compute the most continuous part of this distribution. For simplicity of notation, we will discuss only the case where \( \omega \) consists of the characters of \( A(\mathbb{A}_F)/A(\mathbb{A}_F)^1 \); however, the argument and the result hold for any family of idele class characters, with the only modification that the summation in (10.5) is only over orbits of maximal rank for which \( \omega \) is trivial on \( B(\mathbb{A}_F)^1 \) (in the notation that follows).

In what follows, we use Tamagawa measures for all groups. It is convenient to do formal computations with measures given by differential forms, without convergence factors, and interpret the non-convergent formal products in the end as special valued of zeta functions or residues thereof (if \( \zeta(1) \) appears, where \( \zeta \) denotes the Dedekind zeta function of \( F \)). Let \( \Phi \in \text{c-Ind}_{\mathbb{A}(\mathbb{A}_F) \cup (\mathbb{A}(\mathbb{A}_F)^1) \backslash A(\mathbb{A}_F)^1}(1) \) (where \( \text{c-Ind} \) denotes compact induction), and write \( \Phi \) in terms of its Mellin transform with respect to the left-\( A(\mathbb{A}_F) \)-action:

\[
\Phi(g) = \int_{A(\mathbb{A}_F)/A(\mathbb{A}_F)^1} f_\chi \delta^{-\frac{1}{2}}(g) d\chi
\]

where \( f_\chi \in I(\chi) \) and \( d\chi \) is Haar measure on the unitary dual of \( A(\mathbb{A}_F)/A(\mathbb{A}_F)^1 \). Note that the unitary dual of \( A(\mathbb{A}_F)/A(\mathbb{A}_F)^1 \) can naturally be identified with the imaginary points \( \mathfrak{a}_5^\infty \) of the Lie algebra of the dual torus, via an isomorphism which we will denote by \( \exp \), i.e. \( A(\mathbb{A}_F)/A(\mathbb{A}_F)^1 = \exp(\mathfrak{a}_5^\infty) \). Notice also the shift \( \delta^{-\frac{1}{2}} \) in the above formula because \( f_\chi \) has been defined with respect to the normalized action of \( A(\mathbb{A}_F) \). However, by our assumption that \( \Phi \) is compactly supported

\(^{13}\)Our convention for the archimedean places will be that \( \text{Ind} \) denotes the \( K_\infty \)-finite vectors in that representation, where \( K_\infty \) is a maximal compact subgroup of \( G_\infty \).
modulo $A(\mathbb{A}_F)^1U(\mathbb{A}_F)$, its Mellin transform is entire in $\chi$ and hence we can shift the contour of integration and write:

$$\Phi(g) = \int_{\exp(\kappa + i a^*_\gamma)} f_\chi(g)d\chi$$

for any $\kappa \in a^*_{\gamma}$. In particular, we can shift to the domain of convergence of the Eisenstein sum (10.1) and then we will have:

$$\sum_{\gamma \in B(F)\backslash G(F)} \Phi(\gamma g) = \int_{\exp(\kappa + i a^*_\gamma)} E(f_\chi, g)d\chi,$$

a function of rapid decay on the automorphic quotient $G(F)\backslash G(\mathbb{A}_F)$ (cf. [MW94, II.1.11]).

We integrate over $H(F)\backslash H(\mathbb{A}_F)$:

$$\int_{H(F)\backslash H(\mathbb{A}_F)} \sum_{\gamma \in B(F)\backslash G(F)} \Phi(\gamma h)dh =$$

$$= \sum_{\xi \in [B(F)\backslash G(F)/H(F)]} \int_{H_\xi(F)\backslash H(\mathbb{A}_F)} \int_{H_\xi(F)\backslash H(\mathbb{A}_F)} \int_{\exp(\kappa + i a^*_\gamma)} f_\chi(\xi ah)d\chi da dh.$$

(10.2)

Here $[B(F)\backslash G(F)/H(F)]$ denotes a (finite) set of representatives in $G(F)$ for the $(B(F), H(F))$-double cosets, and $H_\xi := H \cap \xi^{-1}B\xi$. Similarly we will denote: $B\xi := B \cap \xi H\xi^{-1}$, and we will let $Y$ denote the $B$-orbit of $\xi H$ on $G/H$. Here the measure $da$ is a (Tamagawa) $H_\xi(\mathbb{A}_F)$-eigenmeasure, with eigencharacter the inverse of the modular character of $H_\xi$, so that the integral over $H$ admits a factorization as above. We will denote this character by $\eta_\xi$, and its $\xi$-conjugate – which is a character of $B\xi(\mathbb{A}_F)$ – by $\eta_Y$. Hence the two inner integrals are valued in the line bundle over $H_\xi(\mathbb{A}_F)\backslash H(\mathbb{A}_F)$ defined by $\eta_\xi^{-1}$, and $dh$ is an $H(\mathbb{A}_F)$-invariant measure valued in the dual of that line bundle.

For a fixed $h \in H(\mathbb{A}_F)$ the two inner integrals:

$$\int_{H_\xi(F)\backslash H_\xi(\mathbb{A}_F)} \int_{\exp(\kappa + i a^*_\gamma)} f_\chi(\xi ah)d\chi da = \int_{B\xi(F)\backslash B_\xi(\mathbb{A}_F)} \int_{\exp(\kappa + i a^*_\gamma)} f_\chi(a\xi h)d\chi da$$

are equal to:

$$\text{Vol}(B_\xi(F)\backslash B_\xi(\mathbb{A}_F)) \cdot \int_{B_\xi(\mathbb{A}_F)\backslash B_\xi(\mathbb{A}_F)} \int_{\exp(\kappa + i a^*_\gamma)} f_\chi(\xi h)d\chi da.$$
is deep in the region where the morphisms $\Delta_{X,v}^Y$ introduced in the present paper\(^{14}\) (but now with an index $v$ to indicate the place of $F$) are convergent. Returning to (10.2), we can interchange the order of integration to express the contribution of the orbit $Y$ as:

$$\int_{\exp(\kappa_Y + ia_{Y,v}^\chi)} \int_{H_{\xi}(A_F) \backslash H(A_F)} f_\chi(\xi h) dh \, d\chi$$

and the new inner integral is equal to $\prod_v \Delta_{X,v}^{Y_{\text{Tam}}}$ (where the exponent $\text{Tam}$ stands to show that we are using Tamagawa measures, rather than the measures used throughout the paper). Interchanging the order of summation is justified as follows: The function

$$\Phi'(a) := \exp(\kappa_Y)(a) \int_{H_{\xi}(A_F) \backslash H(A_F)} \Phi(a^\chi) dh,$$

on $A_Y(\hat A_F)$ is a Schwartz–Harish-Chandra function if $\kappa_Y$ is sufficiently deep in the domain of convergence of the morphisms $\Delta_{X,v}^Y$. Therefore, its value at 1 (which is represented by the iterated integral above before changing the order of integration) is equal to the integral of its Mellin transforms. But those are given by the double integral of $\Phi \in C^\infty_{\text{c}}(U(\hat A_F)A(\hat A_F)\backslash G(\hat A_F))$ over the corresponding orbit of the group $A(\hat A_F)/A(\hat A_F)^1 \times H(A_F)$ and against a character of that subgroup: for characters of the form $\exp(\kappa_Y + ia_{Y,v}^\chi)$ similar considerations as in the local case imply that this double integral is absolutely convergent, and therefore equal to:

$$\int_{H_{\xi}(A_F) \backslash H(A_F)} f_\chi(\xi h) dh.$$
Theorem. \( (10.4) \)

Let \( B^{-1}B \times B Y \to X \) be birational [Br01, Lemma 5] that whenever \( \Delta_{X,v} \) and \( T_v \)
(acting on \( I(\chi) \)) converge simultaneously, \( \Delta'_{X,v} \) also converges. Therefore, if we substitute \( \chi \) by \( \kappa \), the domain of integration now can be taken to be \( \exp(\kappa + ia_X \in \mathbb{R}) \),
where \( \kappa \in \rho(\chi) + a_H \in \mathbb{R} \) is deep in the domain of convergence of the integral for \( \Delta_{X,v} \).

The Eisenstein sum also converges in that region.

We discuss how \( \Delta^{\text{Tam}}_{X,v} \) and \( \Delta_{X,v} \) are related:

10.0.1. Lemma. We have:

(10.3) \[
\Delta^{\text{Tam}}_{X,v} = Q_v \Delta_{X,v}.
\]

Proof. As we did in the case of \( \Delta_{X,v} \), we compose the morphism \( \Delta^{\text{Tam}}_{X,v} \) with the
map \( S(U \backslash G) \to I(\chi) \) (integration against an \( A \)-eigenmeasure with \( da(A_0) = 1 \))
to get a morphism: \( S(U \backslash G) \to C^\infty(X) \). Notice that \( da = (1 - q^{-1})^{-\text{rk}(A)}d^{\text{Tam}}a \).
Therefore, the integral expression for \( \Delta^{\text{Tam}}_{X,v} \) on \( U \backslash G \) is an integral, over the open \( A \times H \)-orbit, against \( (1 - q^{-1})^{-\text{rk}(A)} \) a Tamagawa eigenmeasure. On the other hand, the morphisms \( \Delta_{X,v} \) were defined using an \( A \times H \)-eigenfunction times the
\( G \)-invariant measure \( dx \) on \( U \backslash G \) such that: \( dx(U \backslash UK) = 1 \). Therefore, if \( dx^{\text{Tam}} \)
denotes “Tamagawa” measure on \( U \backslash G \) then we have:

\[
\Delta^{\text{Tam}}_{X,v} = (1 - q^{-1})^{-\text{rk}(A)} \frac{dx^{\text{Tam}}}{dx} \Delta_{X,v}.
\]

We compute: \( dx^{\text{Tam}}(U \backslash U w I J) = dx^{\text{Tam}}(B_0) = (1 - q^{-1})^{\text{rk}(A)} \). Therefore:

\[
\Delta^{\text{Tam}}_{X,v} = \frac{1}{dx(U \backslash U w I J)} \Delta_{X,v} = Q_v \Delta_{X,v}.
\]

Therefore:

\[
\Delta^{\text{Tam}}_{X,v}(f^{0}_{X,v}) = Q_v \Delta_{X,v}(f^{0}_{X,v}) = \prod_{\alpha > 0} \frac{1 - q^{-1} e^{\alpha}}{1 - e^{\alpha} - (\chi)} \Omega_{X,v}(x_0) =
\]

\[
= \prod_{\alpha > 0} \frac{1 - q^{-1} e^{\alpha}}{1 - e^{\alpha} - (\chi)} \cdot L^{\text{Tam}}_{X,v}
\]

Recall that \( L^{\text{Tam}}_{X,v}(\chi) = c_v \cdot \beta_v(\chi) \), where \( c_v \) is a quotient of products of local values
for the Dedekind zeta function of \( F \) and \( \beta_v \) is a quotient of products of Dirichlet \( L \)-values
which depend on \( \chi \). If we consider the product \( \prod_{v \notin S} c_v \) it may not converge in general. However, we can make sense of it by considering the leading term of its Laurent expansion, when considered as a specialization of a product/quotient
of translates of \( \varepsilon^S \). We will denote this number by \( (\varepsilon^S)^* \). The standard definition of Tamagawa measures implies that when computing the product over all \( v \notin S \) in the expression above, we should use \( (\varepsilon^S)^* \) wherever \( \varepsilon^S \) formally appears in the
product. Similarly, we will denote: \( \left( L^{\text{Tam}}_{X,v} \right)^* = (\varepsilon^S)^* \cdot \beta_v(\chi) \). Therefore we get:

10.0.2. Theorem. The period integral of:

(10.4) \[
\sum_{\gamma \in B(F) \backslash G(F)} \Phi(\gamma g) = \int_{\exp(\kappa + ia_X \in \mathbb{R})} E(f_{X,v}, g) d\chi
\]
over $H(F) \backslash H(\mathbb{A}_F)$ is equal to:

\begin{equation}
\int_{\exp(\kappa + i \alpha_X)} (L^S_X(\chi))^* \sum_{[W/W(\chi)]} \left( \sum_{v \in S} \Delta^Y_{w,v} \Delta X,v (f_{w,v}) \right) d\chi
\end{equation}

plus terms which depend on the restriction of $f_X$, as a function of $\chi$, to a subvariety of smaller dimension. Here $[W/W(\chi)]$ denotes a set of representatives of minimal length for $W/W(\chi)$-cosets, $\kappa \in \rho(\chi) + a_X^C$ is deep in the domain of convergence of $\Delta_X$, and $f_X$ the Mellin transform of $\Phi$ with respect to the normalized left $A(\mathbb{A}_F)/A(\mathbb{A}_F)$-action, is assumed to be factorizable with factors $f_{0,X,v}$ for $v \notin S$.

Remarks. (1) It appears as if the above expression depends on the choice of representatives $w \in W(Y) -1$. However, the factors $c_{\omega}^S(\chi)$ for $\chi \in A_X^0$, do not depend on the choice $w$. This is easily seen in the case of $\text{PGL}_2(\text{PGL}_2 \times \text{PGL}_2)$, and the general case can be reduced to that by Brion’s analysis of Knop’s graph (cf. Proposition 2.3.1).

(2) For the special case of the spherical variety $H \backslash (H \times H)$, where all $B$-orbits are of maximal rank and hence the above formula is precise, compare with the calculation of the scalar product of two pseudo-Eisenstein series in $[\text{MW94, II.2.1]}$.

(3) One can continue along these lines and give a new argument for the Tamagawa number of $H$, which would specialize to the argument of $[\text{Lan66}]$ in the group case; however, since the setup for this argument is much more general, we will present it in separate work.

What about period integrals of cusp forms? Based on the work of Waldspurger $[\text{Wald85}]$, Ichino and Ikeda $[\text{II}]$, we formulate in $[\text{SV}]$ a conjecture which links the period integral of cusp forms to the local Plancherel formula. More precisely, the “canonical” global functional: “$H$-period on $\pi$ times $H$-period on $\tilde{\pi}$” (where $\pi$ denotes the space of a cuspidal automorphic representation and $\tilde{\pi}$ its dual) is conjectured to be related to the Plancherel measure on $X$ and hence, through our computation of $\text{§9}$, to $L_X$.

References

[Ac83] D. Akhiezer, Equivariant completions of homogeneous algebraic varieties by homogeneous divisors. Ann. Global Anal. Geom. 1 (1983), no. 1, 49–78.
[BO07] Y. Benoist and H. Oh Polar decomposition for $p$-adic symmetric spaces. Int. Math. Res. Not. 2007, no. 24, Art. ID rnm121, 20 pp.
[BK98] A. Braverman and D. Kazhdan, On the Schwartz space of the basic affine space, Selecta Math. (N.S.) 5 (1999), no. 1, 1–28.
[BK02] A. Braverman and D. Kazhdan, Normalized intertwining operators and nilpotent elements in the Langlands dual group. Mosc. Math. J. 2 (2002), no. 3, 533–553.
[Br01] M. Brion, On orbit closures of spherical subgroups in flag varieties, Comment. Math. Helv. 76 (2001) 263299.
[BLV86] M. Brion, D. Luna, and Th. Vust, Espaces homogènes sphériques. Invent. Math. 84 (1986), no. 3, 617–632.
[Ca80] W. Casselman, The unramified principal series of $p$-adic groups. I. The spherical function. Compositio Math. 40 (1980), no. 3, 387–406.
[CS80] W. Casselman and J. Shalika, The unramified principal series of $p$-adic groups. II. The Whittaker function. Compositio Math. 41 (1980), no. 2, 207–231.
[DS] P. Delorme and V. Sécherre, An analogue of the Cartan decomposition for $p$-adic reductive symmetric spaces. Preprint, 2006, math.RT/0612545.
[Ga99] P. Garrett, *Euler factorization of global integrals*. Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), 35–101, Proc. Sympos. Pure Math., 66, Part 2, Amer. Math. Soc., Providence, RI, 1999.

[GN] D. Gaitsgory and D. Nadler, *Spherical varieties and Langlands duality*. Preprint, 2006, math.RT/0611323. To appear in Moscow Math. J., special issue in honor of Pierre Deligne.

[GP92] B. H. Gross and D. Prasad, *On the decomposition of a representation of SOₙ* when restricted to SOₙ₊₁, Canad. J. Math. 44 (1992), 974–1002.

[Hi99] Y. Hironaka, *Spherical functions and local densities on Hermitian forms*. J. Math. Soc. Japan 51 (1999), no. 3, 553–581.

[Hi05] Y. Hironaka, *Spherical functions on $Sp_2$ as a spherical homogeneous $Sp_2 	imes (Sp_1)^2$-space*. J. Number Theory 112 (2005), no. 2, 238–286.

[HS88] Y. Hironaka and F. Sato, *Spherical functions and local densities of alternating forms*. Amer. J. Math. 110 (1988), no. 3, 473–512.

[HS00] Y. Hironaka and F. Sato, *Local densities of representations of quadratic forms over $p$-adic integers (the non-dyadic case)*. J. Number Theory 83 (2000), no. 1, 106–136.

[I] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*. Preprint.

[KMS03] S.-i. Kato, A. Murase and T. Sugano, *Whittaker-Shintani functions for orthogonal groups*. Tohoku Math. J. (2) 55 (2003), no. 1, 1–64.

[Kn91] F. Knop, *The Luna-Vust theory of spherical embeddings*. Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 225–249, Manoj Prakashan, Madras, 1991.

[Kn94] F. Knop, *The asymptotic behavior of invariant collective motion*. Invent. Math. 116 (1994), no. 1-3, 309–328.

[Kn95] F. Knop, *On the set of orbits for a Borel subgroup*. Comment. Math. Helv. 70 (1995), no. 2, 285–309.

[Kn96] F. Knop, *Automorphisms, root systems, and compactifications of homogeneous varieties*. J. Amer. Math. Soc. 9 (1996), no. 1, 153–174.

[Ko88] R.E. Kottwitz, *Tamagawa numbers*. Ann. of Math., 127 (1988) pp. 629–646.

[Lan66] R.P. Langlands, *The Volume of the Fundamental Domain for Some Arithmetical Subgroups of Chevalley Groups*. Proc. Sym. Pure Math., 9, AMS, Providence, 1966, 143–148.

[Lap06] E. Lapid, *The relative trace formula and its applications*. Automorphic Forms and Automorphic $L$-Functions (Kyoto, 2005), RIMS Kokyuroku Vol. 1468 (2006) pp. 76–87

[Lu01] D. Luna, *Variétés sphériques de type A*. Publ. Math. Inst. Hautes tudes Sci. no. 94 (2001), 161–226.

[LV83] D. Luna, T. Vust, *Plongements d’espaces homogènes*. Comment. Math. Helv. 58 (1983), no. 2, 186–245.

[Ma71] I.G. Macdonald, *Spherical Functions on a Group of $p$-adic Type*. Publications of the Ramanujan Institute, no. 2. Ramanujan Institute, Centre for Advanced Study in Mathematics, University of Madras, Madras, 1971.

[Ma01] I.G. Macdonald, *Orthogonal polynomials associated with root systems*. Sém. Lothar. Combin. 45 (2000/01), Art. B45a, 40 pp. (electronic).

[MW94] C. Moeglin and J.-L. Waldspurger, *Décomposition spectrale et séries d’Eisenstein. Une paraphrase de l’Écriture*. Progress in Mathematics, 113. Birkhäuser Verlag, Basel, 1994.

[Of04] O. Offen, *Relative spherical functions on $p$-adic symmetric spaces (three cases)*. Pacific J. Math. 215 (2004), no. 1, 97–149.

[Sa06] Y. Sakellaridis, *A Casselman-Shalika formula for the Shalika model of GLₙ*. Canad. J. Math. 58 (2006), no. 5, 1095–1120.

[Sa08] Y. Sakellaridis, *On the unramified spectrum of spherical varieties over $p$-adic fields*. Compositio Math. 144 (2008), no. 4, 978–1016.

[SV] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties*. In preparation.

[Wa85] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions $L$ automorphes en leur centre de symétrie*. Compositio Math. 54 (1985), 173–242.

[Wa96] B. Wasserman, *Wonderful varieties of rank two*. Transform. Groups 1 (1996), no. 4, 375–403.
E-mail address: yiannis@math.toronto.edu

Department of Mathematics, University of Toronto, 40 St George Street, Toronto, ON M5S2E4, Canada