An asymptotic analysis for an integrable variant of the Lotka–Volterra prey–predator model via a determinant expansion technique

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Abstract: The Hankel determinant appears in representations of solutions to several integrable systems. An asymptotic expansion of the Hankel determinant thus plays a key role in the investigation of asymptotic analysis of such integrable systems. This paper presents an asymptotic expansion formula of a certain Casorati determinant as an extension of the Hankel case. This Casorati determinant is then shown to be associated with the solution to the discrete hungry Lotka–Volterra (dhLV) system, which is an integrable variant of the famous prey–predator model in mathematical biology. Finally, the asymptotic behavior of the dhLV system is clarified using the expansion formula for the Casorati determinant.

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1. Introduction

Integrable systems are often classified as nonlinear dynamical systems whose solutions can be explicitly expressed. Such an integrable system is the Toda equation which describes the current–voltage function in an electric circuit. A time discretization, called the discrete Toda equation (Hirota, 1981), is simply equal to the recursion formula of the qd algorithm for computing eigenvalues of a symmetric tridiagonal matrix (Henrici, 1988; Rutishauser, 1990) and singular values of a bidiagonal matrix (Parlett, 1995).

Another commonly investigated integrable system is the integrable Lotka–Volterra (LV) system, which is a prey–predator model in mathematical biology (Yamazaki, 1987). The discrete LV (dLV) system was shown in Iwasaki and Nakamura (2002) to be applicable to computing for bidiagonal
singular values. The hungry Lotka–Volterra (hLV) system is a variant that captures a more complicated prey–predator relationship in comparison with the original LV system (Bogoyavlensky, 1988; Itoh, 1987). Time discretization of this system leads to the discrete hungry Lotka–Volterra (dhLV) system. It was shown in Fukuda, Ishiwata, Iwasaki, and Nakamura (2009), Fukuda, Ishiwata, Yamamoto, Iwasaki, and Nakamura (2013), Yamamoto, Fukuda, Iwasaki, Ishiwata, and Nakamura (2010) that the dhLV system can generate LR matrix transformations for computing eigenvalues of a banded totally nonnegative (TN) matrix whose minors are all nonnegative.

The determinant solutions to both the discrete Toda equation and the dLV system can be expressed using the Hankel determinant,

\[ H^{(n)}_0 = 1, \quad H_j^{(n)} = \begin{vmatrix} a^{(n)}_0 & a^{(n+1)}_1 & \cdots & a^{(n+j-1)}_j \\ a^{(n+1)}_0 & a^{(n+2)}_1 & \cdots & a^{(n+j)}_j \\ \vdots & \vdots & \ddots & \vdots \\ a^{(n+j-1)}_0 & a^{(n+j)}_1 & \cdots & a^{(n+2j-2)}_j \end{vmatrix}, \quad j = 1, 2, \ldots \tag{1.1} \]

where \( j \) and \( n \) correspond to the discrete spatial and discrete time variables, respectively (Tsujimoto, 2001). Here, the formal power series \( f(z) = \sum_{n=0}^{\infty} a^{(n)} z^n \) associated with \( H_j^{(n)} \) is assumed to be analytic at \( z = 0 \) and meromorphic in the disk \( D = \{ z \mid |z| < \zeta \} \). The finite or infinite types of poles \( u_1^{-1}, u_2^{-1}, \ldots \) of \( f(z) \) are ordered such that \( 0 < |u_1^{-1}| < |u_2^{-1}| < \cdots < \zeta \). Then, there exists a nonzero constant \( c \), independent of \( n \) such that, for some \( \rho \) satisfying \( |u_j| > \rho > |u_{j+1}| \),

\[ H_j^{(n)} = c_j u_1 u_2 \cdots u_j^n \left( 1 + O\left( \left( \frac{\rho}{|u_j|} \right)^n \right) \right) \tag{1.2} \]

as \( n \to \infty \) (Henrici, 1988). The asymptotic expansion (1.2) as \( n \to \infty \) enables the asymptotic analysis of the discrete Toda equation and the dLV system as in Henrici (1988), Rutishauser (1990) and in Iwasaki and Nakamura (2002), respectively.

A generalization of the Hankel determinant \( H_j^{(n)} \) is given in the below determinant of a nonsymmetric square matrix of order \( j \),

\[ C^{(m)}_{i,j}, \quad C^{(m)}_{i,0} = 1, \quad C^{(m)}_{i,j} = \begin{vmatrix} a^{(m)}_{i,j} & a^{(m)}_{i,j+1} & \cdots & a^{(m)}_{i,j-1} \\ a^{(m+1)}_{i,j} & a^{(m+1)}_{i,j+1} & \cdots & a^{(m+1)}_{i,j-1} \\ \vdots & \vdots & \ddots & \vdots \\ a^{(m+j-1)}_{i,j} & a^{(m+j-1)}_{i,j+1} & \cdots & a^{(m+j-1)}_{i,j-1} \end{vmatrix}, \quad i = 0, 1, \ldots, \quad j = 1, 2, \ldots \tag{1.3} \]

which is called the Casorati determinant or Casoratian. The Casorati determinant is useful in the theory of difference equations, particularly in mathematical physics, and plays a role similar to the Wronskian in the theory of differential equations (Vein & Dale, 1999). No one wonder here that the formal power series \( f_i(z) = \sum_{n=0}^{\infty} a^{(m)}_{i,n} z^n \) is associated with the Casorati determinant \( C^{(m)}_{i,j} \) for each \( i \). The formal power series \( f_i(z) \) differs from \( f(z) \) in that not only the superscript, but also the subscript, appears in the coefficients.

To the best of our knowledge, from the viewpoint of the formal power series \( f_i(z) \), the asymptotic analysis for the Casorati determinant \( C^{(m)}_{i,j} \) has not yet been discussed in the literature. The first purpose of this paper is to present an asymptotic expansion of the Casorati determinant \( C^{(m)}_{i,j} \) as \( n \to \infty \). The asymptotic behavior of the dhLV system was discussed in Fukuda et al. (2009, 2013) in the case where the discretization parameter \( \delta^{(m)} \) is restricted to be positive. However, it was suggested in Yamamoto et al. (2010) that the choice \( \delta^{(m)} < 0 \) in the dhLV system yields a convergence acceleration of the LR transformations. The discrete time evolution in the dhLV system with \( \delta^{(m)} < 0 \) corresponds to a reverse of the continuous-time evolution in the hLV system. It is interesting to note that such artificial dynamics are useful for computing eigenvalues of a TN matrix. The second purpose of this paper is to provide an asymptotic analysis for the dhLV system without being limited by the sign of \( \delta^{(m)} \).
The remainder of this paper is organized as follows. In Section 2, we first observe that the entries in $C_{ij}^{(n)}$ can be expressed using poles of $f_i(z)$. We then give an asymptotic expansion of the Casorati determinant in terms of the poles of $f_i(z)$ as $n \to \infty$ by expanding the theorem analyticity for the Hankel determinant given in Henrici (1988). In Section 3, we find the determinant solution to the dhLV system through relating the dhLV system to a three-term recursion formula. With the help of the resulting theorem for the Casorati determinant $C_{ij}^{(n)}$, we explain in Section 4 that the determinant solution to the dhLV system can be rewritten using the Casorati determinant $C_{ij}^{(n)}$, and we clarify the asymptotic behavior of the solution to the dhLV system. Finally, we give concluding remarks in Section 5.

2. An asymptotic expansion of the Casorati determinant

In this section, we first give an expression of the entries of the Casorati determinant $C_{ij}^{(n)}$ in terms of poles of the formal power series $f_i(z)$ associated with $C_{ij}^{(n)}$. Referring to the theorem on analyticity for the Hankel determinant given in Henrici (1988), we present an asymptotic expansion of the Casorati determinant $C_{ij}^{(n)}$ as $n \to \infty$ using the poles of $f_i(z)$. We also describe the case where some restriction is imposed on the poles of $f_i(z)$.

Let $f_i(z) = \sum_{n=0}^{\infty} a_i^{(n)} z^n$, which is the formal power series associated with $C_{ij}^{(n)}$ for $i = 0, 1, \ldots$, be analytic at $z = 0$ and meromorphic in the disk $D = \{z \mid |z| < \rho \}$. Moreover, let $r_i^1, r_i^2, \ldots$, denote the poles of $f_i(z)$ such that $|r_i^1| < |r_i^2| < \cdots < \rho$. By extracting the principal parts in $f_i(z)$, we derive

$$f_i(z) = \frac{a_{i,1}}{r_i^1 - z} + \frac{a_{i,2}}{r_i^2 - z} + \cdots + \frac{a_{i,p}}{r_i^p - z} + \sum_{n=0}^{\infty} b_i^{(n)} z^n$$

(2.1)

where $p$ is an arbitrary positive integer, $a_{i,1}, a_{i,2}, \ldots, a_{i,p}$ are some nonzero constants, and $b_i^{(n)}$, which contains the terms with respect to $r_i^1, r_i^2, \ldots, r_i^p$, satisfies

$$|b_i^{(n)}| \leq \mu_i \rho^i$$

(2.2)

for some nonzero positive constants $\mu_i$ and $\rho_i$ with $|r_i^1| < \rho_i < |r_i^p|$. The proof of (2.2) is given in Henrici (1988) utilizing the Cauchy coefficient estimate. We now give a lemma for an expression of $a_i^{(n)}$ appearing in $f_i(z) = \sum_{n=0}^{\infty} a_i^{(n)} z^n$.

LEMMA 2.1 Let us assume that the poles $r_i^1, r_i^2, \ldots, r_i^p$ of $f_i(z)$ are not multiple. Then, $a_i^{(n)}$ can be expressed using $r_i^1, r_i^2, \ldots, r_i^p$ as

$$a_i^{(n)} = \sum_{\ell \geq 1} c_i^{(\ell)} r_i^{-(\ell)} + b_i^{(n)}$$

(2.3)

where $c_{i,1}, c_{i,2}, \ldots, c_{i,p}$ are some nonzero constants.

Proof The crucial element is the replacement $a_{i,1} = c_{i,1} r_i^{-1}, a_{i,2} = c_{i,2} r_i^{-2}, \ldots, a_{i,p} = c_{i,p} r_i^{-p}$ in (2.1), namely,

$$f_i(z) = \frac{c_{i,1} r_i^{-1}}{r_i - z} + \frac{c_{i,2} r_i^{-2}}{r_i^2 - z} + \cdots + \frac{c_{i,p} r_i^{-p}}{r_i^p - z} + \sum_{n=0}^{\infty} b_i^{(n)} z^n$$

(2.4)

Since each $c_{i,p} r_i^{-p}/(r_i^{-p} - z)$ in (2.4) can be regarded as the summation of a geometric series, we obtain

$$f_i(z) = \sum_{n=0}^{\infty} c_{i,1} r_i^n z^n + \sum_{n=0}^{\infty} c_{i,2} r_i^2 z^n + \cdots + \sum_{n=0}^{\infty} c_{i,p} r_i^p z^n + \sum_{n=0}^{\infty} b_i^{(n)} z^n$$

$$= \sum_{n=0}^{\infty} \left( \sum_{\ell} c_{i,\ell} r_i^{-\ell} \right) + b_i^{(n)} z^n$$
which implies (2.3).

Similarly to the asymptotic expansion as \( n \to \infty \) of the Hankel determinant \( H_f^{(n)} \) given in Henrici (1988), we have the following theorem for the Casorati determinant \( C_{i,j}^{(n)} \).

**THEOREM 2.2**  Let us assume that the poles \( r_i^{-1}, r_i^{-1}, \ldots, r_i^{-1} \) of \( f_i(z) \) are not multiple. Then there exists some constant \( c_i, m_{e_1}, e_2, \ldots, e_j \) independently of \( n \) such that, as \( n \to \infty \),

\[
C_{i,j}^{(n)} = \sum_{\sigma} C_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} \left( 1 + \sum_{j=1}^{n} O \left( \left( \frac{\rho_i + \ell}{|r_{i+1} - r_i|} \right)^n \right) \right)
\]

where \( \sigma \) denotes the mapping from \( \{ e_1, e_2, \ldots, e_j \} \) to \( \{ 1, 2, \ldots, p \} \) and \( \rho_i + \ell \) is some constant such that \( |r_{i+\ell} - 1, p+1| < \rho_i + \ell < |r_{i+\ell} - 1, p| \).

**Proof**  By applying Lemma 2.1 and the addition formula of determinants to the Casorati determinant \( C_{i,j}^{(n)} \), we derive

\[
C_{i,j}^{(n)} = \sum_{\sigma} D_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} + \sum_{\sigma} D_{\sigma, e_1, e_2, \ldots, e_j}^{(n)}
\]

where in the first summation

\[
D_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} = \left| \begin{array}{cccc}
C_{i, e_1} & C_{i, e_2}^{\sigma, e_1} & \cdots & C_{i, e_j}^{\sigma, e_1} \\
C_{i+1, e_1} & C_{i+1, e_2}^{\sigma, e_1} & \cdots & C_{i+1, e_j}^{\sigma, e_1} \\
\vdots & \vdots & \ddots & \vdots \\
C_{i, e_1} & C_{i, e_2}^{\sigma, e_1} & \cdots & C_{i, e_j}^{\sigma, e_1} \\
\end{array} \right|
\]

and \( D_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} \) in the second summation denotes a determinant of the same form as \( D_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} \) except that the \( \ell \)th column is replaced with \( b_{\ell} : = (b_{\ell, i+1, e_1}, b_{\ell, i+1, e_2}, \ldots, b_{\ell, i+1, e_j})^T \) for at least one of \( \ell \).

Evaluating the first summation in (2.6), we obtain

\[
\sum_{\sigma} D_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} = \sum_{\sigma} C_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} \left( r_{i+1}, 1, e_1, \ldots, e_j \right)^n
\]

where

\[
C_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} = \left| \begin{array}{cccc}
C_{i, e_1} & C_{i, e_2} & \cdots & C_{i, e_j} \\
C_{i+1, e_1} & C_{i+1, e_2} & \cdots & C_{i+1, e_j} \\
\vdots & \vdots & \ddots & \vdots \\
C_{i, e_1} & C_{i, e_2} & \cdots & C_{i, e_j} \\
\end{array} \right|
\]

To estimate the second summation in (2.6), for example, we consider the case where the 1st column is replaced with \( b_1 \). It immediately follows from (2.2) that

\[
\left| \begin{array}{cccc}
b_{1, i+1, e_1} & C_{i, e_2} & \cdots & C_{i, e_j} \\
b_{1, i+1, e_1}^{\sigma, e_1} & C_{i+1, e_2} & \cdots & C_{i+1, e_j}^{\sigma, e_1} \\
\vdots & \vdots & \ddots & \vdots \\
b_{1, i+1, e_1}^{\sigma, e_1} & C_{i, e_2} & \cdots & C_{i, e_j}^{\sigma, e_1} \\
\end{array} \right| = O \left( \left( \frac{\rho_i + \ell}{|r_{i+1} - 1, e_1|} \right)^n \right)
\]

It is also easy to check \( O \left( (r_{i, e_1}, f_{i, e_2}, \ldots, f_{i+1, e_1}, r_{i+1, e_2}, \ldots, r_{i+1, e_j})^n \right) \) if the \( \ell \)th column is replaced with \( b_{\ell} \). Similarly, by examining the case where some columns are replaced with some of \( b_1, b_2, \ldots, b_p \), we can see that

\[
\sum_{\sigma} D_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} = \sum_{\sigma} C_{\sigma, e_1, e_2, \ldots, e_j}^{(n)} \left( r_{i+1}, 1, e_1, \ldots, e_j \right)^n \sum_{\rho_i + \ell - 1} O \left( \left( \frac{\rho_i + \ell}{|r_{i+1} - 1, e_1|} \right)^n \right)
\]

(2.9)
Thus, from (2.7)–(2.9), we obtain (2.5)

Now, let us consider the restriction \( r_{i, 1} = r_1, r_{i, 2} = r_2, \ldots, r_{i, j} = r_j \) in \( f(z) \). Then, by replacing \( r_{i, \ell} \) with \( r_{\ell} \) in (2.3), we easily obtain

\[
a_i^{(n)} = \sum_{\ell=1}^p c_{i, \ell} r_{\ell}^n + b_j^{(n)} \tag{2.10}
\]

As a specialization of Theorem 2.2, we derive the following theorem for an asymptotic expansion of the Casorati determinant \( C_{i,j}^{(n)} \) with restricted \( a_i^{(n)} \) as \( n \to \infty \).

**Theorem 2.3** Let us assume that the poles \( r_1^{-1}, r_2^{-1}, \ldots, r_j^{-1} \) of \( f(z) \) are not multiple. Then there exists some constant \( c_{i,j} \neq 0 \) independently of \( n \) such that, for \( |r_{j+1}| < \rho_i < |r_j| \) as \( n \to \infty \),

\[
C_{i,j}^{(n)} = c_{i,j} (r_1 f_2 \cdots r_j)^n \left( 1 + \sum_{\ell=1}^j O\left( \left( \frac{\rho_\ell}{|r_j|}\right)^n \right) \right) \tag{2.11}
\]

**Proof** Replacing \( r_{i, 1} = r_1, r_{i, 2} = r_2, \ldots, r_{i, \rho} = r_{p} \) in (2.8) gives

\[
c_{r, \pi(k_1, k_2, \ldots, k_j)} = c_{i_1, j_1} c_{i_2, j_2} \cdots c_{i_{j-1}, j_{j-1}} \left| \begin{array}{cccc}
r_{i_1} & r_{j_2} & \cdots & r_{j_j} \\
\vdots & \ddots & \vdots \\\n\rho_1 & \rho_2 & \cdots & \rho_j
\end{array} \right| \tag{2.12}
\]

Thus, by taking into account that \( c_{r, \pi(k_1, k_2, \ldots, k_j)} \neq 0 \) only in the case where \( k_1, k_2, \ldots, k_j \) are distinct to each other, we can simplify (2.7) as

\[
\sum_{\pi} D_{i, \pi(k_1, k_2, \ldots, k_j)}^{(n)} = (r_1 f_2 \cdots r_j)^n \sum_{\pi} c_{\pi(k_1, k_2, \ldots, k_j)} \left| \begin{array}{cccc}
r_{k_1} & r_{k_2} & \cdots & r_{k_j} \\
\vdots & \ddots & \vdots \\\n\rho_1 & \rho_2 & \cdots & \rho_j
\end{array} \right| \tag{2.13}
\]

where \( \pi \) denotes the bijection from \( \{k_1, k_2, \ldots, k_j\} \) to \( \{1, 2, \ldots, j\} \). It is noted here that the bijection \( \pi \) is equal to the mapping \( \varphi \) with \( p = j \). Moreover, there exists a constant \( \rho_j \) which is not equal to one in Theorem 2.2, such that \( |r_{j+1}| < \rho_i < |r_j| \). This is because \( \rho_i \) and \( \rho_{j+1} \) do not always satisfy \( \rho_i = \rho_{j+1} \) even if \( r_{i, 1} = r_1, r_{i, 2} = r_2, \ldots, r_{i, j} = r_j \) in Theorem 2.2. Thus, (2.9) becomes

\[
\sum_{\ell=1}^j O\left( \left( r_1 f_2 \cdots f_{j-1} \rho_{i, \ell}^{-1}\right)^n \right) \tag{2.14}
\]

Therefore, from (2.13) and (2.14), we obtain (2.11).

Theorem 2.3 covers an asymptotic expansion of the Hankel determinant \( H_{i,j}^{(n)} \). Theorems 2.2 and 2.3 should be useful for the asymptotic analysis of dynamical systems with solutions expressed in terms of the Casorati determinant \( C_{i,j}^{(n)} \).

**3. The dhLV system and its determinant solution**

In this section, similarly to work in Tsujimoto and Kondo (2000), Spiridonov and Zhedanov (1997), we derive the dhLV system from a three-term recursion formula, and then clarify the determinant expression of an auxiliary variable in the solution to the dhLV system through investigating the three-term recursion formula.
Let us consider a three-term recursion formula with respect to the polynomials $T_k^{(n)}(x), \ldots$ at the discrete time $n$,

\[
\begin{align*}
T_k^{(n+1)}(x) &= xT_k^{(n)}(x) - V_k^{(n)}T_{k-M}^{(n)}(x), \quad k = M, M + 1, \ldots, \\
T_0^{(n)}(x) &= 1, \quad T_1^{(n)}(x) = x, \quad \ldots, \quad T_M^{(n)}(x) = x^M
\end{align*}
\]  

(3.1)

where $M$ is a positive integer and $V_0^{(n)}, V_1^{(n)}, \ldots$ do not depend on $x$. Accordingly, $T_0^{(n)}(x), T_1^{(n)}(x), \ldots$, are all monic. Moreover, let us prepare a time evolution from $n$ to $n + 1$,

\[
T_k^{(n+1)}(x) = \frac{1}{x^{M+1} - (\delta^{(n)})^M+1} \left( T_k^{(n)}(x) - V_k^{(n)}T_k^{(n)}(x) \right)
\]  

(3.2)

where $V_k^{(n)} := T_k^{(n)}/(\delta^{(n)})^{M+1}$. Then, by replacing $n$ with $n + 1$ in (3.1) and using (3.2), we obtain

\[
T_k^{(n)}(x) - V_k^{(n)}T_{k+1}^{(n)}(x) = x \left( T_k^{(n)}(x) + V_k^{(n)}T_k^{(n)}(x) \right) - V_k^{(n+1)} \left( T_{k+1}^{(n)}(x) - V_{k-M}^{(n)}T_{k-M}^{(n)}(x) \right)
\]  

(3.3)

By using (3.1) again for deleting except for terms with respect to $T_{k+1}^{(n)}(x)$ and $T_{k-M}^{(n)}(x)$ in (3.3), we derive

\[
\left( V_k^{(n)} + V_{k+1}^{(n)} - V_{k-M}^{(n)} \right) T_k^{(n)}(x) = \left( V_k^{(n)} - V_{k-M}^{(n)} \right) T_k^{(n)}(x)
\]

Thus, it is observed that

\[
V_k^{(n)} + V_{k+1}^{(n)} = V_k^{(n)} + V_{k+1}^{(n)}
\]  

(3.4)

\[
V_k^{(n)}T_k^{(n)} = V_k^{(n+1)}V_{k-M}^{(n)}
\]  

(3.5)

Let us introduce a new variable $u_k^{(n)}$ such that

\[
u_k^{(n)} = \prod_{j=1}^{M} \left( \delta^{(n)} + u_{k-j-M}^{(n)} \right)
\]  

(3.6)

\[
V_k^{(n)} = -\prod_{j=0}^{M} \left( \delta^{(n)} + u_{k-j}^{(n)} \right)
\]  

(3.7)

Then, it follows from (3.5)–(3.7) that

\[
u_k^{(n+1)} = u_k^{(n)} \prod_{j=1}^{M} \left( \delta^{(n)} + u_{k-j+1}^{(n)} \right)
\]  

(3.8)

Moreover, from (3.6) and (3.8), we see that

\[
u_k^{(n+1)} - \nu_k^{(n)} = \prod_{j=0}^{M} \left( \delta^{(n)} + u_{k+j}^{(n)} \right) - \prod_{j=0}^{M} \left( \delta^{(n)} + u_{k+j+1}^{(n)} \right)
\]  

(3.9)

It is obvious from (3.7) that the right-hand side of (3.9) is equal to $V_k^{(n+1)}$. This implies that $V_k^{(n+1)}$ in (3.8) also satisfies (3.4). Consequently, by combining (3.6) and (3.8), noting that

\[
\prod_{j=1}^{M} \left( \delta^{(n)} + u_{k+1-j}^{(n)} \right) = \prod_{j=1}^{M} \left( \delta^{(n)} + u_{k-M+j}^{(n)} \right)
\]

and replacing $k-M$ with $k$, we have the discrete system

\[
u_k^{(n+1+1)} = u_k^{(n)} \prod_{j=1}^{M} \left( \delta^{(n)} + u_{k+j}^{(n)} \right)
\]  

(3.10)

Equation (3.10) can be regarded as a discretization of the hLV system which differs from the simple LV system in that more than one food exists for each species. Thus, (3.10) is the dhLV system and $M$
corresponds to the number of the species of foods for each species. Clearly, from the definition, (3.10) with \( M = 1 \) is simply equal to the dLV system. The dLV system (3.10) is essentially equal to the dLV system in Fukuda et al. (2009),

\[
\mathbf{u}_{k}^{(n+1)} - \sum_{j=1}^{M} \mathbf{u}_{k-j}^{(n+1)} = \mathbf{u}_{k}^{(n)} - \sum_{j=1}^{M} \mathbf{u}_{k-j}^{(n)}
\]

(3.11)

This is because (3.11) is derived by replacing \( \mathbf{u}_{k}^{(n)} \) with \( [1/(\delta^{(n)})_{M}] \mathbf{u}_{k}^{(n)} \) and \( 1/(\delta^{(n)})_{M+1} \) with \( \delta^{(n)} \) for \( n = 0, 1, \ldots \) in (3.10).

Let \( \mathbf{T}_{0}(x), \mathbf{T}_{1}(x), \ldots \) be polynomials satisfying a three-term recursion formula,

\[
\begin{align*}
\mathbf{T}_{k+1}(x) &= \mathbf{T}_{k}(x) - \mathbf{T}_{k-M}(x), \\
\mathbf{T}_{0}(x) &= 1,
\end{align*}
\]

(3.12)

where \( \mathbf{T}_{k}(x) \), \( \mathbf{T}_{k+1}(x) \), \( \mathbf{T}_{k-M}(x) \), \( \mathbf{T}_{0}(x) \) are also all non-negative. Moreover, let us introduce a linear functional (form) \( \mathcal{L}^{(n)} \),

\[
\mathcal{L}^{(n)}[\mathbf{T}_{k}(x)] = \int_{\mathbb{R}} \mathbf{T}_{k}(x)\mathbf{T}_{\ell}(x)\omega^{(n)}(x)dx
\]

(3.13)

where \( \omega^{(n)}(x) \) is a weight function. The linear functional \( \mathcal{L}^{(n)} \) with \( M = 1 \) is equivalent to that in Chihara (1978). Further, \( \mathcal{L}^{(n)} \) with arbitrary \( M \) is a specialization of a linear functional appearing in Maeda, Miki, and Tsujimoto (2013). Since it follows from (3.1), (3.12), and (3.13) that \( \mathcal{L}^{(n)}[\mathbf{T}_{k}(x)] \) is expressed as

\[
\mathbf{T}_{k}(x) = \mathbf{T}_{0}(x) + \cdots + \mathbf{T}_{M}(x)
\]

(3.14)

Let \( \mu^{(n)}_{k} = \mathcal{L}^{(n)}[x^{k}] \) for \( k = 0, 1, \ldots \). From (3.12), it turns out that \( \mathbf{T}_{k}(x) \) is expressed as

\[
\mathbf{T}_{k}(x) = \mu^{(n)}_{k,0}h^{(n)}_{k,0} + \cdots + \mu^{(n)}_{k,k}h^{(n)}_{k,k} + \mu^{(n)}_{k,k+1} + \mu^{(n)}_{k,k-1} + \mu^{(n)}_{k-1,k} + \mu^{(n)}_{k+1,k}
\]

(3.15)

Thus, it follows that

\[
\begin{align*}
\mathcal{L}^{(n)}[\mathbf{T}_{k}(x)] &= \mu^{(n)}_{k,0}h^{(n)}_{k,0} + \cdots + \mu^{(n)}_{k,k}h^{(n)}_{k,k} + \mu^{(n)}_{k,k+1} + \mu^{(n)}_{k,k-1} + \mu^{(n)}_{k-1,k} + \mu^{(n)}_{k+1,k} \\
\mathcal{L}^{(n)}[\mathbf{T}_{k-1}(x)] &= \mu^{(n)}_{k,0}h^{(n)}_{k-1,0} + \cdots + \mu^{(n)}_{k,k}h^{(n)}_{k,k-1} + \mu^{(n)}_{k,k+1} + \mu^{(n)}_{k,k-1} + \mu^{(n)}_{k+1,k} \\
\mathcal{L}^{(n)}[\mathbf{T}_{k+1}(x)] &= \mu^{(n)}_{k,0}h^{(n)}_{k+1,0} + \cdots + \mu^{(n)}_{k,k}h^{(n)}_{k,k+1} + \mu^{(n)}_{k,k+1} + \mu^{(n)}_{k,k-1} + \mu^{(n)}_{k+1,k}
\end{align*}
\]

By combining the above with (3.13), we derive a system of linear equations

\[
\begin{pmatrix}
\mu^{(n)}_{0} & \cdots & \mu^{(n)}_{k-1} & \mu^{(n)}_{k} \\
\vdots & \ddots & \vdots & \vdots \\
\mu^{(n)}_{k-1,0} & \cdots & \mu^{(n)}_{k-1,M} & \mu^{(n)}_{k-1,M+1} \\
\mu^{(n)}_{k,0} & \cdots & \mu^{(n)}_{k,M} & \mu^{(n)}_{k+1}
\end{pmatrix}
\begin{pmatrix}
\mathbf{s}^{(n)}_{k,0} \\
\vdots \\
\mathbf{s}^{(n)}_{k,k-1} \\
\mathbf{s}^{(n)}_{k,k-1}
\end{pmatrix}
\begin{pmatrix}
\mu^{(n)}_{0} \\
\vdots \\
\mu^{(n)}_{k-1} \\
\mu^{(n)}_{k}
\end{pmatrix}
\begin{pmatrix}
\mathbf{h}^{(n)}_{k}
\end{pmatrix}
\]
Since \( s_{k,0}^{(n)}, \ldots, s_{k,k-1}^{(n)} \) are uniquely determined, the coefficient matrix in (3.15) is nonsingular. This suggests that (3.15) can be transformed into

\[
\begin{pmatrix}
  s_{k,0}^{(n)} \\
  \vdots \\
  s_{k,k-1}^{(n)} \\
  1
\end{pmatrix} = \frac{1}{\hat{x}_{k+1}^{(n)}}
\begin{pmatrix}
  \mu_0^{(n)} & \cdots & \mu_{k-1}^{(n)} & \mu_k^{(n)} \\
  \vdots & \ddots & \vdots & \vdots \\
  \mu_{k-1,M}^{(n)} & \cdots & \mu_{k-1,M+1}^{(n)} & \mu_{k-1,(M+1)}^{(n)} \\
  \mu_{k,M}^{(n)} & \cdots & \mu_{k,M+1}^{(n)} & \mu_{k,(M+1)}^{(n)}
\end{pmatrix}
\begin{pmatrix}
  0 \\
  \vdots \\
  0 \\
  h_k^{(n)}
\end{pmatrix}
\]

where the hat denotes cofactors of the coefficient matrix in (3.15) and

\[
\hat{x}_{k+1}^{(n)} := \begin{vmatrix}
  \mu_0^{(n)} & \cdots & \mu_{k-1}^{(n)} & \mu_k^{(n)} \\
  \vdots & \ddots & \vdots & \vdots \\
  \mu_{k-1,M}^{(n)} & \cdots & \mu_{k-1,M+1}^{(n)} & \mu_{k-1,(M+1)}^{(n)} \\
  \mu_{k,M}^{(n)} & \cdots & \mu_{k,M+1}^{(n)} & \mu_{k,(M+1)}^{(n)}
\end{vmatrix}
\]

It is of significance to note that \( \mu_{k,(M+1)}^{(n)} = \hat{x}_k^{(n)} \). Thus, by examining the last row for both sides of (3.16), we find

\[
h_k^{(n)} = \frac{\hat{x}_{k+1}^{(n)}}{\hat{x}_k^{(n)}}
\]

Equations (3.14) and (3.18) therefore lead to

\[
\nu_k^{(n)} = \frac{\hat{x}_{k+1}^{(n)}}{\hat{x}_k^{(n)}} \frac{\hat{x}_k^{(n)}}{\hat{x}_{k-M+1}^{(n)}}
\]

Since we can easily obtain the solution to the dhLV system (3.10), by combining (3.6) with (3.19), the determinant expression of \( \nu_k^{(n)} \) is important for the asymptotic analysis of the dhLV system (3.10) in the next section.

Let us define the time evolution of the linear functional from \( \mathcal{L}^{(n)} \) to \( \mathcal{L}^{(n+1)} \) by

\[
\mathcal{L}^{(n+1)}[P(x)] = \mathcal{L}^{(n)} \left[ (x^{M+1}) - (\hat{\nu}_k^{(n)})^{M+1} P(x) \right]
\]

where \( P(x) \) is an arbitrary polynomial. Then, it is easy to check that \( T_k^{(n+1)}(x^M) \) and \( \hat{T}_k^{(n)}(x) \) are orthogonal to each other with respect to \( \mathcal{L}^{(n+1)} \). Equation (3.20) yields a time evolution with respect to \( \mu \)'s,

\[
\mu_k^{(n+1)} = \mu_{k,(M+1)}^{(n)} - (\hat{\nu}_k^{(n)})^{M+1} \mu_k^{(n)}
\]

Noting (3.1) and (3.12), we find that \( \mathcal{L}^{(n)}[T_k^{(n)}(x^M)\hat{T}_k^{(n)}(x)] \) with \( k = \ell \) can be expressed as the linear combination of \( \mu_0^{(n)}, \mu_{M+1}^{(n)}, \ldots, \mu_{k,(M+1)}^{(n)} \). Thus, by combining it with (3.13), we derive

\[
\mu_{j,(M+1)}^{(n)} \neq 0, \quad j = 0, 1, \ldots, k
\]

Similarly, in the case where \( \mathcal{L}^{(n)}[T_k^{(n)}(x^M)\hat{T}_k^{(n)}(x)] \) with \( k \neq \ell \), we have

\[
\mu_{i,(M+1)}^{(n)} = 0, \quad i = 1, 2, \ldots, M, \quad j = 0, 1, \ldots, k
\]

Taking into account that the sequence \( \{\mu_{j,(M+1)}^{(n)}\}_{n=0}^{\infty} \) with (3.21) is a specialization of the sequence \( \{\alpha_j^{(n)}\}_{n=0}^{\infty} \) appearing in the previous section, we may replace \( \mu_{j,(M+1)}^{(n)} \) with \( \alpha_j^{(n)} \) in the following discussion. Thus, we can rewrite \( \hat{x}_k^{(n)} \) as
\[ \tau_{0}^{(n)} = 1, \quad \tau_{ij}^{(n)} = \begin{bmatrix} \tau_{0,M}^{(n)} & \tau_{1,M}^{(n)} & \cdots & \tau_{j-1,M}^{(n)} \\ \tau_{M,M}^{(n)} & \tau_{M+1,M}^{(n)} & \cdots & \tau_{Mj-1,M}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{j-1,M}^{(n)} & \tau_{j-1,M+1}^{(n)} & \cdots & \tau_{j(M+1)-1,M}^{(n)} \end{bmatrix}, \quad i = 1, 2, \ldots, M \]

where \( \tau_{s,t}^{(n)} = \text{diag}(a_{s}^{(n)}, a_{s+1}^{(n)}, \ldots, a_{s+t}^{(n)}) \) is an \((t+1)\)-by-\((t+1)\) diagonal matrix with the relationship concerning the evolution from \(n\) to \(n+1\),

\[
\alpha_{k}^{(n+1)} = \alpha_{k+M}^{(n)} - (\delta^{(n)})^{M+1} \alpha_{k}^{(n)}
\]

### 4. Asymptotic analysis of the dhLV system

This section begins by explaining that the auxiliary variable in the dhLV system can be rewritten in terms of the Casorati determinant. By using Theorem 2.2, we clarify the asymptotic behavior of the dhLV variables as \(n \to \infty\).

The 1st, 2nd, \(\ldots\), \((j-1)\)th row and column blocks in \(\tau_{ij}^{(n)}\), are \(M\)-by-\(M\) matrices, but the \(j\)th row and column blocks are \((i-1)\)-by-\((i-1)\) matrices. The following lemma gives the representation of \(V_{k}^{(n)}\) in terms of the \(C_{i,j}^{(n)}\) appearing in Section 1.

**Lemma 4.1** The auxiliary variable \(V_{s}^{(n)}\) is expressed as

\[
V_{ij}^{(n)} = \frac{C_{i,j}^{(n)} - C_{i+1,j}^{(n)}}{C_{i,j}^{(n+1)}}, \quad i = 0, 1, \ldots, M - 1, \quad j = 1, 2, \ldots, m - 1
\]

\[
V_{i,j}^{(n)} = \frac{C_{i+1,j}^{(n)} - C_{0,j}^{(n)}}{C_{i,j}^{(n+1)}}, \quad j = 0, 1, \ldots, m - 1
\]

**Proof** Let us introduce a new determinant of a square matrix of order \(j\),

\[
G_{i,j}^{(n)} = 1, \quad G_{i,j}^{(n)} = \begin{bmatrix} a_{i}^{(n)} & a_{i+1}^{(n)} & \cdots & a_{i+j-1}^{(n)} \\ a_{i+M}^{(n)} & a_{i+M+1}^{(n)} & \cdots & a_{i+Mj-1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i+Mj-1}^{(n)} & a_{i+Mj-1+1}^{(n)} & \cdots & a_{i+Mj(j-1)+1}^{(n)} \end{bmatrix}, \quad j = 1, 2, \ldots
\]

We begin by showing that \(\tau_{ij}^{(n)}\) can be transformed into a block diagonal determinant with respect to \(G_{i,j}^{(n)}\). By interchanging the 2nd, 3rd, \(\ldots\), \(j\)th rows and columns with the \([1+(M+1)\)th, \([1+2(M+1)\)th, \(\ldots\), \([1+(j-1)(M+1)\)th rows and columns in \(\tau_{ij}^{(n)}\), we observe that the same form of \(G_{i,j}^{(n)}\) appears in the 1st diagonal block of \(\tau_{ij}^{(n)}\). The entries in the 1st, 2nd, \(\ldots\), \(j\)th rows and columns in \(\tau_{ij}^{(n)}\) are simultaneously all 0, except for those in the diagonal block section. Permutations similar to the above provide the forms of \(G_{i,j}^{(n)}\), \(G_{i,j}^{(n)}\), \(\ldots\), \(G_{i,j}^{(n)}\) as the 2nd, 3rd, \(\ldots\), \((M+1)\)th blocks in \(\tau_{ij}^{(n)}\). Thus, \(\tau_{ij}^{(n)}\) can be expressed in terms of \(G_{0,j}^{(n)}\), \(G_{1,j}^{(n)}\), \(\ldots\), \(G_{M,j}^{(n)}\) as

\[
\tau_{ij}^{(n)}(M+1) = \prod_{r=0}^{M} G_{r,j}^{(n)}
\]
Similarly, \( r_{i,j}^{(n+1)} \) can be transformed into the determinant of a block diagonal matrix whose \( M + 1 \) blocks are \( A_{ij}^{(n+1)} \), \( B_{ij}^{(n+1)} \), \( C_{ij}^{(n+1)} \), and \( D_{ij}^{(n+1)} \). Thus, it follows that

\[
T^{(n+1)}_{i,j} = \left( \prod_{r=0}^{n-1} G^{(n)}_{r, j+1} \right) \left( \prod_{r=0}^{M} G^{(n)}_{r, j} \right)
\]  

(4.5)

The cases where \( k = i + j(M+1) \) and \( k = M + j(M + 1) \) in (3.19) become

\[
V^{(n)}_{i,j} = \frac{G^{(n)}_{i+1,j+1} - G^{(n)}_{i+1,j}}{G_{i,j}}
\]

(4.6)

By combining them with (4.4) and (4.5), we obtain

\[
V^{(n)}_{i,j} = \frac{G^{(n)}_{i,j+1} - G^{(n)}_{i,j}}{G_{i,j+1}}
\]

(4.7)

The entries in the \( j \)th row of \( G^{(n)}_{i,j} \) are given by the linear combination

\[
a^{(n)}_{i,j} = a^{(n+1)}_{i,j} + a^{(n+1)}_{i,j} \quad \text{for} \quad e = 0, 1, \ldots, j - 1.
\]

By multiplying the \( j - 1 \)th row by \(-a^{(n)}_{i,j} \) and then adding it to the \( j \)th row, we get row \( a^{(n+1)}_{i,j} \) as the new \( j \)th row. Similarly, for the \( j \)th, \( j - 1 \)th, \( j - 2 \)th, \( \ldots \), \( 2 \)nd rows, it follows that

\[
G^{(n)}_{i,j} = \begin{vmatrix}
\alpha_{i+1}^{(n)} & \ldots & \alpha_{i+1}^{(n)} \\
\vdots & \ddots & \vdots \\
\alpha^{(n+1)}_{i, M+1} & \ldots & \alpha^{(n+1)}_{i, M+1}
\end{vmatrix}
\]

It is worth noting here that the subscript \( M \) can be regarded as transformed into the superscript 1. Thus, \( G_{i,j}^{(n)} \) in (4.3) is equal to the Casorati determinant \( C_{i,j}^{(n)} \) in (1.3). Then, by accounting for it in (4.6) and (4.7), we have (4.1) and (4.2).

Lemma 4.1 with Theorem 2.2 leads to the following theorem for asymptotic behavior of \( V_{i,j}^{(n)} \) as \( n \to \infty \).

**Theorem 4.2** The auxiliary variable \( V_{i,j}^{(n)} \) converges to some constant \( \xi \) as \( n \to \infty \).

**Proof** Let \( \sigma' \) be the mapping from \( \{ \kappa_1, \kappa_2, \ldots, \kappa \} \) to \( \{ \kappa'_1, \kappa'_2, \ldots, \kappa'_p \} \) where \( \kappa_1, \kappa_2, \ldots, \kappa_p \) are positive integers such that \( r_{i+1}^\prime \leq r_{i+2}^\prime \leq \ldots \leq r_{i+1}^\prime \leq r_{i+1}^\prime \). Then, it follows from Theorem 2.2 that

\[
\lim_{n \to \infty} \frac{C_{i,j}^{(n)}}{\left( r_{i+1}^\prime \right)^n} = \lim_{n \to \infty} \left\{ c_i, \sigma'(e_1, \ldots, e_p) \left( 1 + \sum_{r=1}^q O\left( \rho_{i,r}^{-1} \left( r_{i,r}^\prime \right)^n \right) \right) + \sum_{\sigma' \neq \sigma} \left\{ c_i, \sigma'(e_1, \ldots, e_p) \left( 1 + \sum_{r=1}^q O\left( \rho_{i,r}^{-1} \left( r_{i,r}^\prime \right)^n \right) \right) \right\} = c_i, \sigma'(e_1, \ldots, e_p)
\]

(4.8)
It is of significance to note the relationship between \( f(z) \) and \( f_{i,M}(z) \) is derived from (3.24),

\[
f_{i,M}(z) = \frac{1 + (\delta^{(n)}(M+1)z) f(z) - a^{(n)}}{z}
\]

(4.9)

Equation (4.9) implies that the poles of \( f_i(z) \) and \( f_{i,M}(z) \) are equal to each other, namely, \( r_{i,1} = r_{i,M,1} \), \( r_{i,2} = r_{i,M,2} \), \( \ldots \). Thus, by combining them with Theorem 2.2, we derive

\[
\lim_{n \to \infty} \frac{C_{i,j,M,j}}{\left( r_{i,k}v_{i+1,k} \ldots v_{i+1,j-1} \right)^n} = \lim_{n \to \infty} \frac{C_{i,M,j}}{\left( r_{i,k}v_{i+1,k} \ldots v_{i+1,j-1} \right)^n} = \left( C_{i,M,j} / \left( r_{i,k}v_{i+1,k} \ldots v_{i+1,j-1} \right)^n \right)
\]

(4.10)

Since (4.8) and (4.10) imply that \( C_{n,j} / C_{M,j} \to C_0, 0, v_{n,1} \ldots v_{n,j} / C_0, 0, v_{M,1} \ldots v_{M,j} \) as \( n \to \infty \), we can conclude that \( V_{M,j,M+1} \to \zeta = (C_0, 0, v_{n,1} \ldots v_{n,j} / C_0, 0, v_{M,1} \ldots v_{M,j}) \) as \( n \to \infty \).

By considering the positivity of \( v_{n,j,M+1} \), \( v_{1,j,M+1} \ldots v_{M-1,j,M+1} \) we derive the following theorem for the asymptotic behavior of \( V_{j,M+1} \).

**THEOREM 4.3** Let us assume that \( V_{n,j,M+1} > 0, V_{n,1,j,M+1} > 0, \ldots, V_{M-1,j,M+1} > 0 \) for \( n = 0, 1, \ldots \). Then \( V_{n,j,M+1} \) converge to 0 as \( n \to \infty \).

**Proof** From the Jacobi determinant identity (Hirota, 2003), it follows that

\[
C_{i,j+1}^{(n+1)} C_{i+1,j-1}^{(n+1)} = C_{i,j}^{(n+1)} C_{i,j}^{(n+1)}
\]

(4.11)

Equation (4.11) allows us to simplify \( \sum_{i=0}^{M-1} \nu_{i,j,M+1} \) as

\[
\sum_{i=0}^{M-1} \nu_{i,j,M+1} = \sum_{i=0}^{M-1} \left( C_{i,j+1}^{(n+1)} - C_{i,j}^{(n+1)} \right)
\]

(4.12)

From (4.8), we derive

\[
\lim_{n \to \infty} \frac{C_{i,j}^{(n+1)}}{C_{i,j}^{(n)}} = \frac{r_{i,k}v_{i+1,k} \ldots v_{i+1,j-1}}{r_{i,k}v_{i+1,k} \ldots v_{i+1,j-1}}
\]

(4.13)

Thus, by combining (4.13) and \( r_{i,k} = r_{0,k} \ldots r_{M+1,k} = r_{1,k} \ldots r_{M,j-1,k} = r_{j-1,k} \) with (4.12), we have

\[
\lim_{n \to \infty} \sum_{i=0}^{M-1} \nu_{i,j,M+1} = 0
\]

(4.14)

Therefore, by taking into account that \( \nu_{j,M+1} > 0, \nu_{1,j,M+1} > 0, \ldots, \nu_{M-1,j,M+1} > 0 \) in (4.14), we find that \( \nu_{j,M+1} \to 0, \nu_{1,j,M+1} \to 0, \ldots, \nu_{M-1,j,M+1} \to 0 \) as \( n \to \infty \).

By recalling the relationship of the dhLV variable \( u_k^{(n)} \) to the auxiliary variable \( v_k^{(n)} \) in (3.6), we have the following theorem concerning an asymptotic convergence of \( u_k^{(n)} \) as \( n \to \infty \).

**THEOREM 4.4** As \( n \to \infty \), the dhLV variable \( u_{j,M+1}^{(n)} \) converges to some nonzero constant \( c_j \) and \( u_{1,j,M+1}^{(n)} \ldots u_{M,j,M+1}^{(n)} \to 0 \), provided that \( \delta^{(n)} \) satisfy \( u_{k,n}^{(n)} \prod_{j=1}^{M}(\delta^{(n)} + u_{k,j,n}^{(n)}) > 0 \) for \( n = 0, 1, \ldots \) and the limit of \( \delta^{(n)} \) as \( n \to \infty \) exists.
Figure 1. A graph of the discrete time \( n \) (x-axis) and the value of \( |u^{(n)}_k| \) (y-axis) in the dhLV system (3.10) with \( M = 3 \) and \( m = 3 \). Cross: \( \delta^{(n)} = 1 \), Circle: \( \delta^{(n)} = -0.069 \).

Proof. The proof is given by induction for \( j \). Without loss of generality, let us assume that \( \lim_{n \to \infty} \delta^{(n)} = \delta \) where \( \delta \) denotes some constant. From (3.6), it holds that

\[
u_k^{(n)} = \frac{\nu_{k,M}^{(n)}}{\prod_{\ell=1}^M (\delta^{(n)} + u_{k,\ell}^{(n)})}
\]

(4.15)

By taking the limit as \( n \to \infty \) of both sides of (4.15) with \( k = 0 \) and using \( v_{k,M}^{(n)} \to \bar{\nu}_0 \) as \( n \to \infty \), we obtain

\[
\lim_{n \to \infty} \nu_0^{(n)} = \bar{\nu}_0
\]

(4.16)

where \( \bar{\nu}_0 = \nu_0 / \delta^M \). By considering Theorem 4.2 with (4.16) in the case where \( k = 1, 2, \ldots, M \) in (4.15), we successively check that \( u_{1}^{(n)} \to 0, u_{2}^{(n)} \to 0, \ldots, u_{M}^{(n)} \to 0 \) as \( n \to \infty \).

Let us assume that \( u_{j+1}^{(n)} = \nu_j \) and \( u_{j}^{(n)} \to 0, u_{j+1}^{(n)} \to 0, \ldots, u_{M}^{(n)} \to 0 \) as \( n \to \infty \). Equation (4.15) with \( k = j + 1 \) becomes

\[
u_{j+1}^{(n)} = \frac{\nu_{j+1,M+1}^{(n)}}{\prod_{\ell=1}^M (\delta^{(n)} + u_{j+1,\ell}^{(n)})}
\]

(4.17)

It is clear that the denominator on the right-hand side of (4.17) converges to \( \delta^M \) as \( n \to \infty \) under this assumption. By combining it with \( v_{j+1}^{(n)} \to \bar{\nu}_{j+1} \) as \( n \to \infty \), we observe that \( u_{j+1}^{(n)} \to \bar{\nu}_{j+1} / \delta^M \) as \( n \to \infty \). Moreover, it follows that

\[
\lim_{n \to \infty} \nu_{j+1}^{(n)} = \lim_{n \to \infty} \frac{\nu_{j+1,M+1}^{(n)}}{\prod_{\ell=1}^M (\delta^{(n)} + u_{j+1,\ell}^{(n)})} = 0, \quad i = 0, 1, \ldots, M - 1
\]

(4.18)

since \( \prod_{\ell=1}^M (\delta^{(n)} + u_{j+1,\ell}^{(n)}) \to \delta^{M-1} (\delta + \bar{\nu}_{j+1}) \) and \( v_{j+1,M+1}^{(n)} \to 0 \) as \( n \to \infty \).

The convergence theorem concerning the dhLV system (3.10) in Fukuda et al. (2009) is restricted to the case where the dhLV variable \( u^{(n)}_k \) is positive and the discretization parameter \( \delta^{(n)} \) is fixed positive for every \( n \). Theorem 4.4 claims that the \( j(M+1) \)th species survives and the \( 1 + j(M+1) \)th, \( 2 + j(M+1) \)th, \ldots, \( M + j(M+1) \)th species vanish as \( n \to \infty \) even in the case where \( \delta^{(n)} \) is a changeable negative for each \( n \). Although the case of negative \( u^{(n)}_k \) is not longer recognized as a valid biological model, we note that the convergence is not different from the positive case if the values of \( \delta^{(n)} \) are suitable for \( n = 0, 1, \ldots \)
To observe the asymptotic convergence numerically, we consider two cases where $\delta^n = 1$ and $\delta^n = -0.069$ in the dhLV system (3.10). The initial values are set as $u_k^{(0)} = (\delta^n)^m / \prod_{i=1}^M (\delta^n + u_{k-i}^{(0)})$ for $k = 0, 1, \ldots, 8$ in the dhLV system (3.10) with $M = 3$ and $m = 3$. Figure 1 shows the behavior of $u_0^n$ for $n = 0, 1, \ldots, 50$ in the case where $\delta^n = 1$ and $\delta^n = -0.069$. This figure demonstrates that $u_0^n$ tends to 0 as $n$ grows larger even if $\delta^n < 0$. We also see that the case where $\delta^n = -0.069$ has a superior convergence speed in comparison with the case where $\delta^n = 1$. Similarly, the asymptotic behavior of $u_0^n, u_1^n, \ldots, u_8^n$ can be seen to follow Theorem 4.4.

5. Concluding remarks
In this paper, we associated a formal power series $f(z) = \sum_{n=0}^\infty a_n z^n$ with the Casorati determinant $C_{n_1, n_2}^{(m)}$ and gave asymptotic expansions of the Casorati determinants as $n \to \infty$ in Theorems 2.2 and 2.3. By making use of Theorem 2.2, we then clarified the asymptotic behavior of the dhLV variables as $n \to \infty$ in Theorem 4.4.

Theorems 2.2 and 2.3 may contribute to asymptotic analysis for other discrete integrable systems. One possible application is the discrete hungry Toda (dhToda) equation derived from the numbered box and ball system through inverse ultra-discretization (Tokihiro, Nagai, & Satsuma, 1999). The dhToda equation has a relationship of variables to the dhLV system whose solution is given in the Casorati determinant (Fukuda, Yamamoto, Iwasaki, Ishiwata, & Nakamura, 2011). The Casorati determinant directly appears in, for example, the solution to the discrete Darboux–Pöschl–Teller equation which is a discretization of a dynamical system concerning a special class of potentials for the 1-dimensional Schrödinger equation (Gaillard & Matveev, 2009).

It was proved in Fukuda et al. (2013) that the dhLV system (3.10) with a fixed positive $\delta^m$ is associated with the LR transformation for a TN matrix. The paper (Yamamoto et al., 2010) also suggested that the dhLV system (3.10) with changeable negative $\delta^m$ generates the shifted LR transformation for a TN matrix. Eigenvalues of an $m$-by-$m$ TN matrix correspond to the constants $\hat{c}_1 = \delta^m \hat{c}_2, \ldots, \hat{c}_m = \delta^m \hat{c}_m$ in Theorem 4.4. Theorems 4.2–4.4 will be useful for investigating the convergence of the sequence of the shifted LR transformations based on the dhLV system (3.10) in the changeable negative case of $\delta^n$.

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References
Bogoyavlensky, O. I. (1988). Integrable discretizations of the KdV equation. Physics Letters A, 134, 34–38.
Chihara, T. S. (1978). An introduction to orthogonal polynomials. New York, NY: Golden and Breach Science Publisher.
Fukuda, A., Ishiwata, E., Iwasaki, M., & Nakamura, Y. (2009). The discrete hungry Lotka–Volterra system and a new algorithm for computing matrix eigenvalues. Inverse Problems, 25, 015007.
Fukuda, A., Ishiwata, E., Yamamoto, Y., Iwasaki, M., & Nakamura, Y. (2013). Integrable discrete hungry
system and their related matrix eigenvalues. *Annali di Matematica Pura ed Applicata*, 192, 423–445.
Fukuda, A., Yamamoto, Y., Iwasaki, M., Ishiwata, E., & Nakamura, Y. (2011). A Bäcklund transformation between two integrable discrete hungry systems. *Physics Letters A*, 375, 303–308.
Gaïlard, P., & Matveev, V. B. (2009). Wronskian and Casorati determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions. *Journal of Physics A: Mathematical and Theoretical*, 42, 404009.
Henrici, P. (1988). *Applied and computational complex analysis* (Vol. 1). New York, NY: Wiley.
Hirota, R. (1981). Discrete analogue of a generalized Toda equation. *Journal of the Physical Society of Japan*, 50, 3785–3791.
Hirota, R. (2003). Determinant and Pfaffians. *Sūrikaisekikenkyūsho Kōkyūroku*, 1302, 220–242.
Itoh, Y. (1987). Integrals of a Lotka–Volterra system of odd number of variables. *Progress of Theoretical Physics*, 78, 507–510.
Iwasaki, M., & Nakamura, Y. (2002). On the convergence of a solution of the discrete Lotka–Volterra system. *Inverse Problems*, 18, 1569–1578.
Maeda, K., Miki, H., & Tsujimoto, S. (2013). From orthogonal polynomials to integrable systems [in Japanese]. *Transactions of the Japan Society for Industrial and Applied Mathematics*, 23, 341–380.
Parlett, B. N. (1995). The new qd algorithm. *Acta Numerica*, 4, 459–491.
Rutishauser, H. (1990). Lectures on numerical mathematics. Boston: Birkhäuser.
Spiridonov, V., & Zhedanov, A. (1997). Discrete-time Volterra chain and classical orthogonal polynomials. *Journal of Physics A: Mathematical and General*, 30, 8727–8737.
Tokihiro, T., Nagai, A., & Satsuma, J. (1999). Proof of solitonical nature of box and ball systems by means of inverse ultra-discretization. *Inverse Problems*, 15, 1639–1662.
Tsujimoto, S., & Kondo, K. (2000). Molecule solutions to discrete equations and orthogonal polynomials [in Japanese]. *Sūrikaisekikenkyūsho Kōkyūroku*, 1170, 1–8.
Tsujimoto, S., Nakamura, Y., & Iwasaki, M. (2001). The discrete Lotka–Volterra system computes singular values. *Inverse Problems*, 17, 53–58.
Vein, R., & Dale, P. (1999). *Determinants and their applications in mathematical physics* (Applied mathematical sciences, Vol. 134). New York, NY: Springer.
Yamamoto, Y., Fukuda, A., Iwasaki, M., Ishiwata, E., & Nakamura, Y. (2010). On a variable transformation between two integrable systems: The discrete hungry Toda equation and the discrete hungry Lotka–Volterra system. *AIP Conference Proceedings*, 1281, 2045–2048.
Yamazaki, S. (1987). On the system of non-linear differential equations $\ddot{y}_k = y_k (y_k + 1 - y_{k-1})$. *Journal of Physics A: Mathematical and General*, 20, 6237–6241.