Abstract. We present a self-contained proof of a strong version of van der Waerden’s Theorem. By using translation invariant filters that are maximal with respect to inclusion, a simple inductive argument shows the existence of “piecewise syndetically”—many monochromatic arithmetic progressions of any length \(k\) in every finite coloring of the natural numbers. All the constructions presented are constructive in nature, in the sense that the involved maximal filters are defined by recurrence on suitable countable algebras of sets. No use of the axiom of choice or of Zorn’s Lemma is needed.

Introduction

The importance of maximal objects in mathematics is well-known, starting from the fundamental examples of maximal ideals in algebra, and of ultrafilters in certain areas of topology and of Ramsey theory. In this paper we focus on maximal filters on suitable countable algebras of sets which are stable under translations. By using such maximal objects, along with ultrafilters extending it, we give a proof of a strong version of the following classical result in Ramsey theory:

\textbf{Theorem} (van der Waerden - 1927) \textit{In every finite partition} \(\mathbb{N} = C_1 \cup \ldots \cup C_r\) \textit{there exists a piece} \(C = C_i\) \textit{that contains arbitrarily long arithmetic progressions, that is, for every} \(k\) \textit{there exists a progression} \(x + y, x + 2y, \ldots, x + ky \in C\).

In fact, we will prove the existence of “piecewise syndetically”—many monochromatic arithmetic progressions of any length \(k\).

Usually, van der Waerden’s Theorem is proved either by double induction using elaborated, but elementary, combinatorial arguments in the style of the original proof [5], or by using properties of the smallest ideal \(K(\beta\mathbb{N}, \oplus)\) in the algebra of ultrafilters (see [4, Ch.14]; see also [2, 1] for stronger versions). In our proof, for any given piecewise set, we restrict to a suitable countable algebra of sets, and explicitly construct by recursion a maximal translation invariant filter, and then an ultrafilter extending it. The desired result is finally obtained by a short proof by induction, that is essentially a simplified version of an argument that was used in [3] in the framework of the compact right-topological semigroup \((\beta\mathbb{N}, \oplus)\). It is worth remarking that, contrarily to the usual ultrafilter proof, we make no explicit use of the algebra in the space of ultrafilters; in fact, we make no use of the axiom of choice nor of Zorn’s Lemma.

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1. Preliminary notions

\( \mathbb{N} \) denotes the set of positive integers, and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) the set of non-negative integers. For \( A \subseteq \mathbb{N} \) and \( n \in \mathbb{N}_0 \), the leftward shift of \( A \) by \( n \) is the set:

\[
A - n := \{ m \in \mathbb{N} \mid n + m \in A \}
\]

Elemental notions in combinatorics of numbers that we will use in this paper are those of thick set, syndetic set, and piecewise syndetic set. For completeness, let us recall them here.

A set \( A \subseteq \mathbb{N} \) is thick if it includes arbitrarily long intervals. Equivalently, \( A \) is thick if every finite set \( F = \{n_1, \ldots, n_k\} \subseteq \mathbb{N} \) has a rightward shift included in \( A \), that is, there exists \( x \) such that

\[
x + F := \{x + n_1, \ldots, x + n_k\} \subseteq A.
\]

Notice that such an \( x \) can be picked in \( A \). In terms of intersections, the property of thickness of \( A \) can be rephrased by saying that the family \( \{A - n \mid n \in \mathbb{N}_0\} \) has the finite intersection property (FIP for short), that is, \( \bigcap_{i=1}^{n_k} (A - n_i) \neq \emptyset \) for any \( n_1, \ldots, n_k \).

A set \( A \subseteq \mathbb{N} \) is syndetic if it has "bounded gaps", that is, there exists \( k \in \mathbb{N} \) such that \( A \) meets every interval of length \( k \). Equivalently, \( A \) is syndetic if a finite number of leftward shifts of \( A \) covers all the natural numbers, that is, \( \mathbb{N} = \bigcup_{i=1}^{n_k} (A - n_i) \) for suitable \( n_1, \ldots, n_k \in \mathbb{N}_0 \).

A set is piecewise syndetic if it is the intersection of a thick set with a syndetic set. Equivalently, \( A \) is piecewise syndetic if a finite number of leftward shifts cover a thick set, that is, \( \bigcup_{i=1}^{n_k} (A - n_i) \) is thick for suitable \( n_1, \ldots, n_k \in \mathbb{N}_0 \).

Notice that the families of thick, syndetic, and piecewise syndetic sets are all invariant with respect to shifts. A well-known relevant property of piecewise syndetic sets that is neither satisfied by thick sets nor by syndetic sets, is the Ramsey property below. For the sake of completeness, we include here a proof.

**Proposition 1.1.** In every finite partition \( A = C_1 \cup \ldots \cup C_r \) of a piecewise syndetic set \( A \), one of the pieces \( C_i \) is piecewise syndetic.

**Proof.** For simplicity, let us say that an interval \( I \) is \( k \)-good for the set \( B \) if for every sub-interval \( J \subseteq I \) of length \( k \) one has \( J \cap B \neq \emptyset \). By the hypothesis of piecewise syndeticity of \( A \), there exists \( k \in \mathbb{N} \) and a sequence of intervals \( \langle I_n \mid n \in \mathbb{N} \rangle \) with increasing length such that every \( I_n \) is \( k \)-good for \( A \). It is enough to consider the case when \( A = C_1 \cup C_2 \) is partitioned into two pieces, because the general case \( r \geq 2 \) will then follow by induction. We distinguish two cases.

Case # 1: There exists \( h \) and there exist infinitely many intervals \( I_n \) that are \( h \)-good for \( C_1 \). In this case \( C_1 \) is piecewise syndetic.

Case # 2: For every \( h \), there are only finitely many intervals \( I_n \) that are \( h \)-good for \( C_1 \). So, for every \( h \) we can pick an interval \( I_{n_h} \) of length \( h \) that is not \( h \)-good. Let \( J_h \subseteq I_{n_h} \) be a sub-interval of length \( h \) such that \( J_h \cap C_1 = \emptyset \). The sequence of intervals \( \langle J_h \mid h \in \mathbb{N} \rangle \) shows that \( C_2 \) is piecewise syndetic. Indeed, given \( h \), for every sub-interval \( J \subseteq J_h \) of length \( k \) we have that \( J \cap C_1 \subseteq J_h \cap C_1 = \emptyset \); and so \( J \cap C_2 = J \cap A \neq \emptyset \), since \( J \subseteq I_{n_h} \) and \( I_{n_h} \) is \( k \)-good for \( A \). \( \square \)
2. Maximal filters

An algebra of sets on \( N \) is a family of subsets of \( N \) that contains \( N \) and is closed under finite unions, finite intersections, and complements. Given a countable family \( \mathcal{A} \), one can explicitly construct the (countable) algebra of sets generated by \( \mathcal{A} \), that is, the smallest algebra that contains \( \mathcal{A} \).

A filter on an algebra of sets \( \mathcal{B} \) is a nonempty family \( \mathcal{F} \subseteq \mathcal{B} \) such that:

- \( \mathcal{F} \) is closed under finite intersections, that is, \( A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F} \);
- \( \mathcal{F} \) is closed under supersets, that is, if \( B \in \mathcal{B} \) and \( B \supseteq A \in \mathcal{F} \) then \( B \in \mathcal{F} \).

Every family \( \mathcal{G} \subseteq \mathcal{B} \) with the finite intersection property (FIP for short) generates a filter \( \langle \mathcal{G} \rangle \), namely

\[
\langle \mathcal{G} \rangle := \{ B \in \mathcal{B} \mid B \supseteq A_1 \cap \ldots \cap A_k \text{ where } A_i \in \mathcal{G} \}.
\]

An ultrafilter \( \mathcal{U} \) on the algebra of sets \( \mathcal{B} \) is a filter with the additional property that if \( A \in \mathcal{U} \) whenever \( A \in \mathcal{B} \) and the complement \( A^c \notin \mathcal{U} \). Equivalently, an ultrafilter is a filter with the Ramsey property: If \( A_1 \cup \ldots \cup A_k \in \mathcal{U} \) where all sets \( A_j \in \mathcal{B} \), then \( A_j \in \mathcal{U} \) for some \( j \). Ultrafilters can also be characterized as the filters that are maximal under inclusion and so, by a straightforward application of Zorn’s Lemma, it is proved that every filter can be extended to an ultrafilter.

The main ingredient in our proof of van der Waerden’s Theorem are those filters that are stable under shifts.

**Definition 2.1.** \( \mathcal{F} \) is a translation invariant filter (TIF for short) if \( \mathcal{F} \) is a filter with the additional property that if \( A \in \mathcal{F} \) then also \( A - n \in \mathcal{F} \) (and hence, \( A - n \in \mathcal{F} \) for all \( n \in \mathbb{N}_0 \)).

The notions of TIF and thick set are closely related.

**Proposition 2.2.** A set \( A \) is thick if and only if it belongs to a TIF \( \mathcal{F} \).

**Proof.** Recall that \( A \) is thick if and only if the family \( \mathcal{G} = \{ A - n \mid n \in \mathbb{N}_0 \} \) has the FIP. It is easily verified that the generated filter \( \langle \mathcal{G} \rangle \) is translation invariant.

Conversely, assume that \( A \in \mathcal{F} \) for some TIF \( \mathcal{F} \). Then trivially the family \( \mathcal{G} = \{ A - n \mid n \in \mathbb{N}_0 \} \) has the FIP because \( \mathcal{G} \subseteq \mathcal{F} \).

Similarly to ultrafilters, by a straightforward application of Zorn’s Lemma it can be shown that every TIF can be extended to a maximal TIF. However, in the countable case, recursive constructions suffice to produce both ultrafilters and maximal TIFs, which are thus obtained in a constructive manner, without any use of the axiom of choice.

**Proposition 2.3.** Let \( \mathcal{B} = \{ B_n \mid n \in \mathbb{N} \} \) be a countable algebra of sets.

\[\text{1 Let } \langle A_0 \mid n \in \mathbb{N} \rangle \text{ be an enumeration of the elements of } \mathcal{A}, \text{ and let } \langle F_n \mid n \in \mathbb{N} \rangle \text{ be an enumeration of the nonempty finite sets of natural numbers (e.g., if } \sum_{k=0}^{\infty} a_{nk} 2^{k-1} \text{ is written in binary expansion where } a_{nk} \in \{0, 1\}, \text{ let } F_n := \{ k \mid a_{nk} = 1 \}) \text{. Then it is easily verified that the following family } \mathcal{B} \text{ is the algebra generated by } \mathcal{A}, \text{ where we denoted } A^1 = A \text{ and } A^{-1} = A^c: \]

\[
\mathcal{B} := \left\{ \bigcup_{i=1}^t \left( \bigcap_{k \in F_{n_i}} A_{n_i}^{\sigma_{n_i}(k)} \right) \mid n_1, \ldots, n_t \in \mathbb{N}, \sigma_i : F_{n_i} \rightarrow \{-1, 1\} \right\}.
\]
(1) Given a family $\mathcal{G} \subseteq \mathcal{B}$ with the FIP, define $\mathcal{G}_0 = \mathcal{G}$; $\mathcal{G}_{n+1} = \mathcal{G}_n \cup \{B_n\}$ in case $B_n \cap A \neq \emptyset$ for every $A \in \mathcal{G}_n$; and $\mathcal{G}_{n+1} = \mathcal{G}_n$ otherwise. Then $\mathcal{U} := \bigcup_n \mathcal{G}_n$ is an ultrafilter that extends $\mathcal{G}$.

(2) Given a family $\mathcal{G} \subseteq \mathcal{B}$ with the FIP and such that $A \in \mathcal{G} \Rightarrow A - 1 \in \mathcal{G}$, define $\mathcal{G}_0 = \mathcal{G}$; $\mathcal{G}_{n+1} = \mathcal{G}_n \cup \{B_n - k \mid k \in \mathbb{N}_0\}$ in case that union has the FIP; and $\mathcal{G}_{n+1} = \mathcal{G}_n$ otherwise. Then $\mathcal{M} := \bigcup_n \mathcal{G}_n$ is a maximal TIF that extends $\mathcal{G}$.

Proof. (1). By the definition, it is clear that all families $\mathcal{G}_n$ have the FIP, and so also $\mathcal{U}$ has the FIP. Now assume by contradiction that $A \in \mathcal{B}$ is such that both $A, A^c \notin \mathcal{U}$. If $A = B_n$ and $A^c = B_m$, then, by the definition of $\mathcal{U}$, there exist $U \in \mathcal{G}_m$ and $U' \in \mathcal{G}_m$ such that $A \cap U = A^c \cap U' = \emptyset$. But then $U \cap U' \subseteq (A \cap U) \cup (A^c \cap U') = \emptyset$, against the FIP of $\mathcal{U}$. Finally, if $B \supseteq A$ where $B \in \mathcal{B}$ and $A \in \mathcal{U}$ then $B \in \mathcal{U}$, as otherwise, by what just proved, $B^c \notin \mathcal{U}$, and hence $\emptyset = B^c \cap A \in \mathcal{U}$, a contradiction.

(2). By induction, it directly follows from the definition that all families $\mathcal{G}_n$ have the FIP and satisfy the property “$A \in \mathcal{G}_n \Rightarrow A - 1 \in \mathcal{G}_n$”; so, the same properties hold for $\mathcal{M}$. If $B \supseteq A$ where $B \in \mathcal{B}$ and $A \in \mathcal{M}$, say $A \in \mathcal{G}_n$, then $B \in \mathcal{G}_{n+1}$ because trivially the FIP of $\mathcal{G}_n \cup \{A - n \mid n \in \mathbb{N}_0\}$ implies the FIP of $\mathcal{G}_n \cup \{B - n \mid n \in \mathbb{N}_0\}$. This shows that $\mathcal{M}$ is a TIF. As for the maximality, let $\mathcal{M}' \supseteq \mathcal{M}$ be a TIF. Given $A \in \mathcal{M}'$, pick $n$ with $A = B_n$. The family $\mathcal{G}_n \cup \{A - n \mid n \in \mathbb{N}_0\}$ has the FIP, since it is included in the filter $\mathcal{M}'$, and so $A \in \mathcal{G}_{n+1}$. This shows that $\mathcal{M}' \subseteq \mathcal{M}$, and hence the two TIFs are equal. □

Two properties of maximal TIFs that will be relevant to our purposes are the following.

Proposition 2.4. Let $\mathcal{U}$ be an ultrafilter that includes a maximal TIF $\mathcal{M}$. Then:

(1) Every $B \in \mathcal{U}$ is piecewise syndetic.

(2) For every $B \in \mathcal{U}$, the set $\mathcal{B}_U := \{n \in \mathbb{N} \mid B - n \in \mathcal{U}\}$ is syndetic.

Proof. Notice first that for every $B \in \mathcal{U}$ there exist $n_1, \ldots, n_k$ such that the union $\bigcup_{i=1}^k (B - n_i) \in \mathcal{M}$. Indeed, if $\Lambda := \{B - n \mid n \in \mathbb{N}_0\}$ then the union $\mathcal{M} \cup \Lambda$ does not have the FIP, as otherwise $\mathcal{M} \cup \Lambda$ would generate a TIF that properly extends $\mathcal{M}$, against the maximality. So, there exist $A \in \mathcal{M}$ and $n_1, \ldots, n_k$ such that $A \cap \bigcap_{i=1}^k (B - n_i) = \emptyset$. But then $\bigcup_{i=1}^k (B - n_i) \in \mathcal{M}$, as a superset of $A \in \mathcal{M}$.

(1). Pick a finite union of shifts $\bigcup_{i=1}^k (B - n_i) \in \mathcal{M}$. By Proposition 2.2, that union is thick as an element of a TIF, and hence $B$ is piecewise syndetic.

(2). As above, pick a finite union of shifts $\bigcup_{i=1}^k (B - n_i) \in \mathcal{M}$. By translation invariance, for every $m \in \mathbb{N}$ one has that $\bigcup_{i=1}^k (B - n_i - m) \in \mathcal{M} \subseteq \mathcal{U}$ and, by the Ramsey property of ultrafilters, there exists $i$ such that $B - n_i - m \in \mathcal{U}$, that is, $m \in \mathcal{B}_U - n_i$. This shows that $\mathbb{N} = \bigcup_{i=1}^k (\mathcal{B}_U - n_i)$ is a finite union of shifts of $\mathcal{B}_U$, and hence $\mathcal{B}_U$ is syndetic. □

3. A STRONG VERSION OF VAN DER WAERDEN’S THEOREM

The following property of piecewise syndetic sets was proved by exploiting the properties of ultrafilters in the smallest ideal of the right-topological semigroup $(\beta\mathbb{N}, +)$ (see [2][1]).

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2 We remark that in general the set $\mathcal{B}_U$ does not belong to the algebra of sets $\mathcal{B}$. 
Theorem 3.1. Let $A$ be a piecewise syndetic set. Then for every $k \in \mathbb{N}$, the set $AP_k(A) := \{ x \in A \mid \exists y \in \mathbb{N} \text{ s.t. } x + iy \in A \text{ for } i = 1, \ldots, k \}$ is piecewise syndetic.

Notice that, as a straight consequence, one obtains the following strong version of van der Waerden’s Theorem.

Corollary 3.2. In every finite partition $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there exists a piece $C \subset C_i$ such that, for every $k \in \mathbb{N}$, the set $AP_k(C)$ is piecewise syndetic.

Proof. By the Ramsey property of piecewise syndetic sets (see Proposition 1.1), we can pick a color $C_i$ which is piecewise syndetic. \hfill \Box

In this section we will give a new proof of the above theorem which only relies on the existence of a maximal TIF $M$ that contains $A$, and of an ultrafilter $U \supseteq M$.

Proof of Theorem 3.1. Let $B$ be the countable algebra of sets generated by the family $\{ A - n \mid n \in \mathbb{N}_0 \}$. By the property of piecewise syndeticity, a finite union of shifts $T = \bigcup_{i=1}^{k} (A - n_i)$ is thick. Then the family $G := \{ T - n \mid n \in \mathbb{N}_0 \} \subseteq B$ has the FIP, and since $G$ trivially satisfies the shift invariance property “$A \in G \Rightarrow A - 1 \in G$”, by Proposition 2.3 we can pick a maximal TIF $M$ on $B$ with $M \supseteq G$, and an ultrafilter $U$ on $B$ with $U \supseteq M$. The desired result is a consequence of the following general property.

Claim. Let $U$ be an ultrafilter that extends a maximal TIF. If a shift $B - \ell \in U$ for some $\ell \in \mathbb{N}_0$, then $B_U - \ell$ contains arbitrarily long arithmetic progressions.

Indeed, since $T = \bigcup_{i=1}^{k} (A - n_i) \in F \subseteq U$, there exists $n_j$ such that $A - n_j \in U$. Then for every $k \in \mathbb{N}$ there exists an arithmetic progression $\ell + iy \in A_{\ell} - n_j$ for $i = 0, 1, \ldots, k$. But then $\bigcap_{i=0}^{k} (A - n_j - \ell - iy) \subseteq AP_k(A) - n_j - \ell \in U$. In consequence, $AP_k(A)$ is piecewise syndetic by Proposition 2.4 as a shift of a member of $U$.

We are left to prove the Claim. We proceed by induction on $k$, and prove that if $B - \ell \in U$ for some $\ell \in \mathbb{N}_0$, then $B_U - \ell$ contains a $k$-term arithmetic progression.

If $B - \ell \in U$, then the set $(B - \ell)_U = B_U - \ell$ is syndetic by Proposition 2.4. In particular, $B_U - \ell \neq \emptyset$, and this proves the induction base $k = 1$.

Let us turn to the inductive step $k + 1$, and assume that $B - \ell_0 \in U$. By syndeticity of $B_U - \ell_0$, there exists a finite $F \subset \mathbb{N}_0$ such that for every $n \in \mathbb{N}$ there exists $x \in F$ with $\ell_0 + n + x \in B_U$. For convenience, let us assume that $0 \in F$. By the inductive hypothesis, there exist $\ell_1 \in \mathbb{N}_0$ and $y_1 \in \mathbb{N}$ such that $\ell_1 + iy_1 \in B_U - \ell_0$ for $i = 1, \ldots, k$, that is, $\ell_0 + \ell_1 + x_0 + iy_1 \in B_U$ where $x_0 = 0 \in F$. Pick $x_1 \in F$ with $\ell_0 + \ell_1 + x_1 \in B_U$. If $x_1 = x_0$ then we already found a $(k+1)$-term arithmetic progression in $B_U$, as desired. Otherwise, let us consider the intersection $B_1 = (B - x_1) \cap \bigcap_{i=1}^{k} (B - x_0 - iy_1)$.

Since $\ell_0 + \ell_1 + x_1 \in B_U$ and $\ell_0 + \ell_1 + x_0 + iy_1 \in B_U$, the shift $B_1 - \ell_0 - \ell_1 \in U$ and so, by the inductive hypothesis, there exist $\ell_2 \in \mathbb{N}_0$ and $y_2 \in \mathbb{N}$ such that $\ell_2 + iy_2 \in (B_1)_U - \ell_0 - \ell_1$ for $i = 1, \ldots, k$. In consequence, $\ell_0 + \ell_1 + \ell_2 + x_0 + iy_1 + (y_1 + y_2) \in B_U$ and $\ell_0 + \ell_1 + \ell_2 + x_1 + iy_2 \in B_U$ for every $i = 1, \ldots, k$. Pick $x_2 \in F$ with

\[ \text{This inductive construction uses a simplified version of an argument in [3].} \]
\[ \ell_0 + \ell_1 + \ell_2 + x_2 \in B_\mathcal{U}. \] Notice that if \( x_2 = x_0 \) or \( x_2 = x_1 \) then we have a \((k + 1)\)-term arithmetic progression in \( B_\mathcal{U} \). Otherwise, let us consider the intersection

\[ B_2 = (B - x_2) \cap \bigcap_{i=1}^{k} (B - x_1 - iy_2) \cap \bigcap_{i=1}^{k} (B - x_0 - i(y_1 + y_2)). \]

Since \( B_2 - \ell_0 - \ell_1 - \ell_2 \in \mathcal{U} \), by the inductive hypothesis we can find an arithmetic progression in \( B_\mathcal{U} - \ell_0 - \ell_1 - \ell_2 \) of length \( k \). We iterate the procedure. As the set \( F \) is finite, after finitely many steps we will find elements \( x_n = x_m \) where \( n > m \), and finally obtain the following arithmetic progression of length \( k + 1 \):

\[ \ell_0 + \ell_1 + \ldots + \ell_n + x_n + i(y_{m+1} + \ldots + y_n) \quad i = 0, 1, \ldots, k. \]

\[ \square \]

4. TIFs and left ideals in the space of ultrafilters

The usual ultrafilter proof of van der Waerden’s Theorem (see [14, §14.1]) is grounded on the existence of minimal ultrafilters, that is, on ultrafilters that belong to minimal left ideals of the compact right-topological semigroup \((\beta \mathbb{N}, \oplus)\). In this final section, we show how (maximal) translation invariant filters are in fact related to minimal left ideals of \((\beta \mathbb{N}, \oplus)\). Let us recall here the involved notions.

The space \( \beta \mathbb{N} \) is the topological space of all ultrafilters \( \mathcal{U} \) over the full algebra of sets \( \mathcal{B} = \mathcal{P}(\mathbb{N}) \) where a base of (cl)open sets is given by the family \( \{\mathcal{O}_A \mid A \subseteq \mathbb{N}\} \), with \( \mathcal{O}_A := \{U \in \beta \mathbb{N} \mid A \subseteq U\} \). The space \( \beta \mathbb{N} \) is Hausdorff and compact, and coincides with the Stone-Čech compactification of the discrete space \( \mathbb{N} \).

The pseudosum \( \mathcal{U} \oplus \mathcal{V} \) of ultrafilters \( \mathcal{U}, \mathcal{V} \in \beta \mathbb{N} \) is defined by letting:

\[ A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \in \mathbb{N} \mid A - n \in \mathcal{V}\} \in \mathcal{U}. \]

The operation \( \oplus \) is associative (but not commutative), and for every \( \mathcal{V} \), the map \( \mathcal{U} \mapsto \mathcal{U} \oplus \mathcal{V} \) is continuous. This makes \((\beta \mathbb{N}, \oplus)\) a right-topological semigroup.

A left ideal \( L \subseteq \beta \mathbb{N} \) is a nonempty set such that \( \mathcal{V} \in L \) implies \( \mathcal{U} \oplus \mathcal{V} \in L \) for all \( \mathcal{U} \in \beta \mathbb{N} \). Right ideals are defined similarly. Left ideals that are minimal with respect to inclusion are particularly relevant objects, as they satisfy special properties. For instance, their union \( K(\beta \mathbb{N}, \oplus) \) is shown to be the smallest bilater ideal (i.e. both a left and a right ideal). Moreover, all ultrafilters \( \mathcal{U} \) in \( K(\beta \mathbb{N}, \oplus) \), named minimal ultrafilters, have the property that every set \( A \in \mathcal{U} \) includes arbitrarily long arithmetic progressions.

It is well-known that there are natural correspondences between families with the finite intersection property on the full algebra \( \mathcal{P}(\mathbb{N}) \), and closed nonempty subsets of \( \beta \mathbb{N} \). Indeed, the following properties are directly verified from the definitions.

- If \( \mathcal{G} \subseteq \mathcal{P}(\mathbb{N}) \) is a family with the FIP then \( \mathcal{C}(\mathcal{G}) := \{\mathcal{V} \in \beta \mathbb{N} \mid \mathcal{V} \supseteq \mathcal{G}\} \) is a nonempty closed subspace.
- If \( X \subseteq \beta \mathbb{N} \) is nonempty then \( \mathcal{F}(X) := \bigcap \{\mathcal{V} \mid \mathcal{V} \subseteq X\} \) is a filter on \( \mathcal{P}(\mathbb{N}) \).
- \( \mathcal{C}(\mathcal{F}) = \overline{X} \) for every nonempty \( X \subseteq \beta \mathbb{N} \).
- \( \mathcal{F}(\mathcal{C}(\mathcal{F})) = \mathcal{F} \) for every filter \( \mathcal{F} \) on \( \mathcal{P}(\mathbb{N}) \).

**Proposition 4.1.** Let \( \mathcal{B} = \mathcal{P}(\mathbb{N}) \) be the full algebra of all subsets of \( \mathbb{N} \). If \( \mathcal{F} \) is a TIF then \( \mathcal{C}(\mathcal{F}) \) is a closed left ideal of \((\beta \mathbb{N}, \oplus)\); and conversely, if \( L \) is a left ideal of \((\beta \mathbb{N}, \oplus)\) then \( \mathcal{G}(L) \) is a TIF. Moreover, \( \mathcal{M} \) is a maximal TIF if and only if \( \mathcal{C}(\mathcal{M}) \)

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4 For all notions and basic results on the space of ultrafilters and on its algebraic structure, including properties of the smallest ideal \( K(\beta \mathbb{N}, \oplus) \), we refer the reader to the book [3].
is a minimal left ideal of $\beta \mathbb{N}$; and $L$ is a minimal left ideal of $\beta \mathbb{N}$ if and only if $\mathfrak{g}(L)$ is a maximal TIF.

Proof. Let $V \in \mathcal{C}(F)$ and let $U \in \beta \mathbb{N}$ be any ultrafilter. For every $A \in F$, by translation invariance we know that $A - n \in F$ for all $n$, and so $\{n \mid A - n \in V\} = \mathbb{N} \in U$. This shows that $A \in U \oplus V$. As this is true for every $A \in F$, we conclude that $U \oplus V \in \mathcal{C}(F)$, and so $\mathcal{C}(F)$ is a closed left ideal.

Now let $A \in \mathfrak{g}(L)$ where $L$ is a left ideal. For every $V \in L$, we have that $U_1 \oplus V \in L$, where $U_1 := \{B \subseteq \mathbb{N} \mid 1 \in B\}$ is the principal ultrafilter generated by 1. Then $A \in U_1 \oplus V$, which is equivalent to $A - 1 \in V$. As this holds for every $V \in L$, we have proved that $A - 1 \in \mathfrak{g}(L)$, as desired.

Let $F$ be a TIF. If the left ideal $\mathcal{C}(F)$ is not minimal, pick a minimal $L \subset \mathcal{C}(F)$. Then $F \subset \mathfrak{g}(L)$, and hence $F$ is not maximal. Indeed, $L \subset \mathcal{C}(F) \Rightarrow \mathfrak{g}(L) \supset \mathcal{C}(\mathcal{C}(F)) = F$; moreover, $F \neq \mathfrak{g}(L)$, as otherwise $\mathcal{C}(F) = \mathcal{C}(\mathfrak{g}(L)) = L = L$, against our assumptions. (Recall that a minimal left ideal $L$ is necessarily closed because, by minimality, $L = \beta \mathbb{N} \oplus V := \{U \oplus V \mid U \in \beta \mathbb{N}\}$ for every given $V \in L$, and $\beta \mathbb{N} \oplus V$ is closed as the image of the compact Hausdorff space $\beta \mathbb{N}$ under the continuous function $U \mapsto U \oplus V$.) In a similar way, one shows the converse implication: If the TIF $F$ is not maximal then the left ideal $\mathcal{C}(F)$ is not minimal. In consequence, $L = \mathcal{C}(\mathfrak{g}(L))$ is minimal if and only if $\mathfrak{g}(L)$ is maximal, and also the last equivalence follows.

As a straight consequence, we obtain the desired characterization.

Corollary 4.2. Let $B = \mathcal{P}(\mathbb{N})$ be the full algebra of all subsets of $\mathbb{N}$. An ultrafilter $U$ includes a maximal TIF if and only if $U$ belongs to the smallest ideal $K(\beta \mathbb{N}, \oplus)$.

Proof. Recall that $U \in K(\beta \mathbb{N}, \oplus)$ if and only if $U$ belongs to some minimal left ideal. Now let $U \supseteq \mathcal{M}$ where $\mathcal{M}$ is a maximal TIF. Since $\mathcal{M} = \mathfrak{g}(\mathcal{C}(\mathcal{M}))$, we have that $U \in \mathcal{C}(\mathcal{M})$, where $\mathcal{C}(\mathcal{M})$ is a minimal left ideal. Conversely, let $U \in L$ where $L$ is a minimal left ideal. Then $\mathfrak{g}(L)$ is a maximal TIF and $U \supseteq \mathfrak{g}(L)$, since $U \in L = \mathcal{C}(\mathfrak{g}(L))$.

Remark 4.3. One can generalize the contents of this paper from the natural numbers to arbitrary countable semigroups $(S, \cdot)$. Indeed, the notion of translation invariant filter also makes sense in that more general framework. Precisely, if $B$ is an algebra of subsets of $S$, one defines a TIF on $B$ as a filter $F$ with the additional property that $A \in F \Rightarrow s^{-1} A := \{ t \in S \mid s \cdot t \in A \} \in F$ for every $s \in S$. Then, by the same arguments as used in this paper, one can prove that Theorem 3.1 holds, provided one adopts a suitable generalization of the notion of piecewise syndetic set.

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5 Actually, our techniques also apply for uncountable semigroups, but in that case one needs Zorn’s Lemma to prove the existence of maximal TIFs and of ultrafilters.

6 In an arbitrary semigroup $(S, \cdot)$, one defines a subset $T \subseteq S$ to be thick if for every finite $F$ there exists $x \in S$ with $F \cdot x := \{ y \cdot x \mid y \in F \} \subseteq T$; a set $A \subseteq S$ is syndetic if a suitable finite union $\bigcup_{i=1}^k s_i^{-1} A = S$ covers the whole semigroup; and finally a set $A \subseteq S$ is piecewise syndetic if a suitable finite union $\bigcup_{i=1}^k s_i^{-1} A$ is thick (see [3] §4.4 and §4.5]).
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