Weyl double copy and massless free-fields in curved spacetimes

Shanzhong Han∗

The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
E-mail: shanzhong.han@nbi.ku.dk

Received 14 April 2022, revised 12 September 2022
Accepted for publication 30 September 2022
Published 24 October 2022

Abstract

In spinor formalism, since any massless free-field spinor with spin higher than 1/2 can be constructed with spin-1/2 spinors (Dirac–Weyl (DW) spinors) and scalars, we introduce a map between Weyl fields and DW fields. We determine the corresponding DW spinors in a given empty spacetime. Regarding them as basic units, other higher spin massless free-field spinors are then identified. Along this way, we find some hidden fundamental features related to these fields. In particular, for non-twisting vacuum Petrov type N solutions, we show that all higher spin massless free-field spinors can be constructed with one type of DW spinor and the zeroth copy. Furthermore, we systematically rebuild the Weyl double copy for non-twisting vacuum type N and vacuum type D solutions. Moreover, we show that the zeroth copy not only connects the gravity fields with a single copy but also connects the degenerate Maxwell fields with the DW fields in the curved spacetime, both for type N and type D cases. Besides, we extend the study to non-twisting vacuum type III solutions. We find a particular DW scalar independent of the proposed map and whose square is proportional to the Weyl scalar. A degenerate Maxwell field and an auxiliary scalar field are then identified. Both of them play similar roles as the Weyl double copy. The result further inspires us that there is a deep connection between gravity theory and gauge theory.

Keywords: Weyl double copy, massless free-fields, vacuum type N solutions, vacuum type D solutions, vacuum type III solutions, gauge theory, spinor formalism

Contents

1. Introduction 2

∗Author to whom any correspondence should be addressed.
1. Introduction

In recent years, the attempt to look for the connection between gravity theory and quantum theory has been actively investigated. As is known, Yang–Mills gauge theory is by far the most successful theory to describe the micro world. On the other hand, given the experimental confirmation of gravitational waves [1, 2], Einstein’s gravity theory is further confirmed as the most promising theory to describe the macro-scale Universe. Therefore, it is significant to explore the relationship between these two theories. Much work has been devoted to this study. One such attempt is the double copy. It started from the research of perturbative scattering amplitudes [3–6]; with the help of exact gravity solutions, the study was extended into the classical context [7]. There are two classes of double copies: Kerr–Schild double copy [7–19] and Weyl double copy [20–28]. The latter covers a broader range of spacetimes so we will focus on the Weyl double copy in this paper.

In spinor formalism, the Weyl double copy is written as

\[ \Psi_{ABCD} = \Phi_{(AB}\Phi_{CD)}S, \]  

which maps a Weyl spinor \( \Psi_{ABCD} \) (a vacuum gravity field) into a single copy \( \Phi_{AB} \) (a Maxwell field which satisfies the Maxwell equation in Minkowski spacetime) and a zeroth copy \( S \) (a scalar field which satisfies the wave equation in Minkowski spacetime).

Decades ago, some works [29, 30] had already given us the prediction about the Weyl double copy. In reference [29], given a Weyl spinor of a vacuum type D solution on a dyad \((o_A, \iota_A)\), \( \Psi_{ABCD} = \psi o_A o_B o_C o_D \), Walker and Penrose showed that there exists a Killing spinor of valence two \( \chi_{BC} = \psi^{-1/3} o_B o_C \), which satisfies the twistor equation \( \nabla^A (A\chi_{BC}) = 0 \). Based on this, the same authors, together with Hughston and Sommers, proposed [30] that in any vacuum type D spacetime with a Weyl spinor \( \Psi_{ABCD} \), one can construct a test electromagnetic field, such that \( \Phi_{AB} = \psi^{2/3} o_A o_B \). This work discovered an intriguing relation between gravity and Maxwell

\[1\text{ It might be more enlightening if we write it in form of equation (1.1), the difference is the background that the Maxwell and scalar fields are living in is a curved spacetime instead of a Minkowski spacetime.} \]
fields in curved spacetimes. Combined with the fact that the Maxwell field is the simplest solution of the gauge theory—the case of the group \( U(1) \), the Weyl double copy relation appears to be more essential between gravity theory and gauge theory. It was proposed for the first time for vacuum type D solutions [20]. Then the Weyl double copy was proven to work also for non-twisting vacuum type N solutions [22]. For the type III case, using the twistor theory, the study only showed it holds at the linearised level [23, 24]. On the other hand, the asymptotic behaviours of the Weyl double copy have been discussed in recent works [26, 27], which state that the Weyl double copy holds asymptotically for the algebraically general solutions by using the peeling property of the Weyl scalars [31, 32]. More recently, reference [28] studied the Weyl double copy for general type D spacetimes with external sources but without a cosmological constant and introduced an extended Weyl double copy. However, Weyl double copy for a general spacetime, even for vacuum spacetime with a cosmological constant, is still unknown. More generally speaking, our understanding of the connection between gravity theory and gauge theory remains to be improved. Hopefully, there are many exciting and promising roads awaiting us. In particular, although the Weyl double copy prescription does not capture the current double copy interpretation of twisting type N solution, which might require a more general and complicated prescription, the curved double copy indeed holds in this case. This fact leads us to consider that it might be helpful to study first the map between a gravity field and a test Maxwell field in the curved spacetime. Or, it might be worth exploring first the features of spin-\( n/2 \) (\( n = 0, 1, 2, 3 \)) massless free fields that live in the curved spacetime. Then the curvature information will be reflected by these lower spin fields. By probing the features of these fields, it would be easier to look for those curvature-independent fields, such as pure Maxwell field\(^2\). In this paper, regarding spin-1/2 massless free-field spinors—Dirac–Weyl (DW) spinors, as basic units, we not only identify the DW spinors but also construct higher massless free-field spinors following the proposed map. Then, one will see that the relations between gravity fields and Maxwell fields in curved spacetime found in [22, 30] are a particular case in the present work; more fundamental properties of these fields are revealed. Especially, a natural map similar to the Weyl double copy from gravity fields to pure Maxwell fields in type III spacetime is proposed with the aid of a scalar field.

The structure of our paper is as follows. In section 2, we give a brief review of spinor algebra and the massless free-field spinors. Section 3 identifies the DW spinors in vacuum type N, type D, and type III spacetimes, respectively. Regarding them as basic units, we analyze the properties of different spin massless free-fields, especially the DW and Maxwell fields. Then we systematically reconstruct the Weyl double copy for non-twisting vacuum type N and vacuum type D spacetimes. A new property of the zeroth copy is discovered after that. Following this way, a degenerate electromagnetic field that lives in Minkowski spacetime and an associated scalar field are obtained from the vacuum type III solutions. The discussion and conclusions are given in section 4.

2. Massless free-fields in spinor formalism

Since massless free-field equations have a simple form in spinor formalism, before going on, we shall give a short introduction to spinor algebra; for more details, one may refer to references [33, 34].

\(^2\) Where ‘pure Maxwell fields’ means that the Maxwell fields are living in Minkowski spacetime as the special solutions of gauge theory, so they are totally independent of the gravity theory.
First, let us consider an arbitrary vector $V$ on the basis $e_i$,

$$V = V^0 e_0 + V^1 e_1 + V^2 e_2 + V^3 e_3. \quad (2.1)$$

We transfer it to a $2 \times 2$ Hermitian matrix

$$H = V^0 \sigma_0 + V^1 \sigma_j = \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}, \quad (2.2)$$

where $\sigma_0$ is a $2 \times 2$ unit matrix, and $\sigma_j$ ($j = 1, 2, 3$) are Pauli spin matrices. By introducing a pair of complex numbers $(\xi, \eta)$ as follows,

$$V^0 = \frac{1}{\sqrt{2}}(\xi \bar{\xi} + \eta \bar{\eta}), \quad V^1 = \frac{1}{\sqrt{2}}(\xi \bar{\eta} + \eta \bar{\xi}), \quad V^2 = \frac{1}{\sqrt{2}}(\xi \bar{\eta} - \eta \bar{\xi}), \quad V^3 = \frac{1}{\sqrt{2}}(\xi \bar{\xi} - \eta \bar{\eta}), \quad (2.3)$$

where the bar denotes the operation of complex conjugation, equation (2.2) can be translated into a new form

$$\frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix} = \begin{pmatrix} \xi \bar{\xi} & \xi \bar{\eta} \\ \eta \bar{\xi} & \eta \bar{\eta} \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \xi & \eta \end{pmatrix}. \quad (2.4)$$

On the other hand, there is a complex linear transformation of pair $(\xi, \eta)^T$:

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad (2.5)$$

where the hat denotes the new quantity after the transformation. If we impose the condition $\det A = 1$, it corresponds to the spin transformation, all of such matrices form the group $SL(2, \mathbb{C})$. When a spin transformation is applied, equation (2.4) becomes

$$\begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix} A^\dagger = \begin{pmatrix} V^0 + \hat{V}^3 & V^1 + i\hat{V}^2 \\ V^1 - i\hat{V}^2 & V^0 - \hat{V}^3 \end{pmatrix} = \hat{H}, \quad (2.6)$$

where the dagger denotes the operation of conjugate transpose. It is worth noting that due to the condition $\det A = 1$, the determinant of the Hermitian matrix remains invariant,

$$\det \hat{H} = \det H = (V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2. \quad (2.7)$$

In other words, the norm of vector $V$ is invariant under the transformation. Therefore every matrix element $A$ of the group $SL(2, \mathbb{C})$ defines a restricted Lorentz transformation. As is known, $SL(2, \mathbb{C})$ is homomorphic to Lorentz group$^3$. Furthermore, since $SL(2, \mathbb{C})$ is isomorphic to the symplectic group $Sp(2, \mathbb{C})$, it is natural to introduce the two-dimensional symplectic vector space (spin-space) over $\mathbb{C}$. A tensor is defined in spinor form such that

$^3$Note $SO(3,1)$ is not isomorphic to $SL(2, \mathbb{C})$, not all spinors have a tensor counterpart. In general, tensors can be regarded as special cases of spinors.
Now, let us focus on spin-space. Any vectors ξA can be expanded on a spinor dyad (ν, ι),

\[ ξ^A = ξ^0 ν^A + ξ^1 ι^A ⇔ ξ^A = \begin{pmatrix} ξ^0 \\ ξ^1 \end{pmatrix}, \]

(2.10)

where \( ν^A \) can be any non-zero vector, and another vector \( ι^A \) is imposed to satisfy \( \{ ν, ι \} = 1 \). Then one may find

\[ ν^A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad ι^A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

(2.11)

In addition, the symplectic structure implies that the inner product of two arbitrary vectors satisfies

\[ \{ ξ, η \} = ε_{AB} η^A ξ^B = -\{ η, ξ \}, \]

(2.12)

where \( ε_{AB} \) plays a role analogous to the metric tensor; nevertheless, it is anti-symmetric

\[ ε_{AB} = -ε_{BA}. \]

(2.13)

Then normalization condition reads

\[ \{ ν, ι \} = ε_{AB} ν^A ι^B = -\{ ι, ν \} = -ε_{AB} ν^A ι^B = 1, \]

\[ \{ ν, ν \} = ε_{AB} ν^A ν^B = 0, \quad \{ ι, ι \} = ε_{AB} ι^A ι^B = 0, \]

(2.14)

where it is easy to see

\[ ε_{AB} = ε^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

(2.15)

and

\[ ε_{AB} = 2 ν_{[A}, ε^{AB} = 2 ι^{[A}, ε_{AB} = 2 ι^{A][B]. \]

(2.16)

In spinor algebra, the complex conjugate space is anti-isomorphic with the spin-space.
The rule of raising and lowering indices is as follows
\[
\varepsilon^{AB} \xi_B = \xi^A, \quad \xi^A \varepsilon_{AB} = \xi_B.
\] (2.17)

The relations above also hold in the complex conjugate space.

In addition, the null tetrad can be written in terms of the spinor bases
\[
\ell^a = \partial^A \bar{\mathcal{O}}_a, \quad n^a = \partial^A \mathcal{O}_a, \quad m^a = \partial^A \bar{\mathcal{O}}_a, \quad \bar{m}^a = \partial^A \mathcal{O}_a,
\]
\[
\ell_a = \mathcal{O}_A \bar{\mathcal{O}}_a, \quad n_a = \mathcal{O}_A \mathcal{O}_a, \quad m_a = \mathcal{O}_A \bar{\mathcal{O}}_a, \quad \bar{m}_a = \mathcal{O}_A \mathcal{O}_a,
\] (2.18)

where real null vectors \( \ell \) and \( n \) satisfy \( \ell^2 = n^2 = 0, \ell \cdot n = 1 \), complex null vectors \( m \) and \( \bar{m} \) satisfy \( m^2 = \bar{m}^2 = 0, \) \( m \cdot \bar{m} = -1 \), furthermore, \( \ell \cdot m = n \cdot \bar{m} = \ell \cdot \bar{m} = n \cdot m = 0 \). The definitions of spin coefficients in this paper are consistent with appendix B of reference [34] and reference [35]. They are listed as follows
\[
\kappa^* = m^a \bar{m}^b \nabla_b \ell_a, \quad \pi^* = n^a \bar{n}^b \nabla_b \ell_a, \quad \epsilon^* = \frac{1}{2} (n^a \bar{n}^b \nabla_b \ell_a + m^a \bar{m}^b \nabla_b m_a),
\]
\[
\tau^* = m^a \bar{m}^b \nabla_b \ell_a, \quad \nu^* = n^a \bar{n}^b \nabla_b \ell_a, \quad \gamma^* = \frac{1}{2} (n^a \bar{n}^b \nabla_b \ell_a + m^a \bar{m}^b \nabla_b m_a),
\]
\[
\sigma^* = m^a \bar{m}^b \nabla_b \ell_a, \quad \mu^* = n^a \bar{n}^b \nabla_b \ell_a, \quad \beta^* = \frac{1}{2} (n^a \bar{n}^b \nabla_b \ell_a + m^a \bar{m}^b \nabla_b m_a),
\]
\[
\rho^* = m^a \bar{m}^b \nabla_b \ell_a, \quad \lambda^* = n^a \bar{n}^b \nabla_b \ell_a, \quad \alpha^* = \frac{1}{2} (n^a \bar{n}^b \nabla_b \ell_a + m^a \bar{m}^b \nabla_b m_a).
\] (2.19)

To distinguish from other symbols, we use the star to mark the spin coefficients.

We have given a short introduction to spinor algebra. Now, let us turn to massless free-fields (source-free). We shall list the corresponding spinors without proof, one may refer to section 5.7 of reference [33] for more details.

Given a symmetric spinor with \( n \) indexes \( S_{A_1 A_2 \ldots A_n} \), spin-\( n/2 \) massless free-field equations are translated into a simple form
\[
\nabla^A_1 \cdots \nabla^A_n S_{A_1 A_2 \ldots A_n} = 0.
\] (2.20)

When \( n = 4 \), the spinor \( S \) refers to a Weyl spinor \( \Psi_{ABCD} \) translated from the Weyl tensor\(^5\)
\[
C_{abcd} = C_{A_1 A_2 A_3 A_4} = \Psi_{ABCD} \varepsilon^{AB} \varepsilon^{CD} + \Psi_{A_1 B_1 C_1 D_1} \varepsilon^{A_1 B_1} \varepsilon^{C_1 D_1}.
\] (2.21)

Following the vacuum Einstein’s field equation, equation (2.20) in this case represents the Bianchi identity (with or without a cosmological constant)
\[
\nabla^A_1 \Psi_{ABCD} = 0.
\] (2.22)

\(^5\) As an exception, \( \varepsilon^{AB} \) is usually abbreviated as \( \varepsilon_{CD} \), that is also applied for raised index version. In addition, one may realize that the first identity of equation (2.21) is a general tensor—spinor identity with abstract indices, we do not need to use the Infeld—van der Waerden symbols here.
According to Petrov classification, there are five different types of solutions:

\[ I: \Psi_{ABCD} \sim \bar{\alpha} (\bar{\alpha} \bar{D}) \]
\[ II: \Psi_{ABCD} \sim \bar{\alpha} (\bar{\alpha} \bar{B} C \bar{D}) \]
\[ III: \Psi_{ABCD} \sim \bar{\alpha} (\bar{\alpha} \bar{B} C \bar{D}) + 4 \psi_{0} \]
\[ IV: \Psi_{ABCD} \sim \bar{\alpha} (\bar{\alpha} \bar{B} C \bar{D}) + 4 \psi_{0} \]
\[ N: \Psi_{ABCD} \sim \bar{\alpha} (\bar{\alpha} \bar{B} C \bar{D}) + 4 \psi_{0} \]

where \( \alpha, \beta, \gamma \) and \( \delta \) are four different non-proportional and non-vanishing spinors. The tilde is used to distinguish them from the spin coefficients. In addition, with the help of Newman–Penrose formalism, the Weyl tensor is reduced to five independent complex scalars.

\[ \psi_{0} = C_{abcd} \epsilon^{a} \epsilon^{b} \epsilon^{c} \epsilon^{d} = \Psi_{ABCD} \theta^{A} \theta^{B} \theta^{C} \theta^{D} \]
\[ \psi_{1} = C_{abcd} \epsilon^{a} \epsilon^{b} \epsilon^{c} \epsilon^{d} = \Psi_{ABCD} \theta^{A} \theta^{B} \theta^{C} \theta^{D} \]
\[ \psi_{2} = C_{abcd} \epsilon^{a} \epsilon^{b} \epsilon^{c} \epsilon^{d} = \Psi_{ABCD} \theta^{A} \theta^{B} \theta^{C} \theta^{D} \]
\[ \psi_{3} = C_{abcd} \epsilon^{a} \epsilon^{b} \epsilon^{c} \epsilon^{d} = \Psi_{ABCD} \theta^{A} \theta^{B} \theta^{C} \theta^{D} \]
\[ \psi_{4} = C_{abcd} \epsilon^{a} \epsilon^{b} \epsilon^{c} \epsilon^{d} = \Psi_{ABCD} \theta^{A} \theta^{B} \theta^{C} \theta^{D} \]

The second set of equalities is obtained from equations (2.18) and (2.21). Then the Weyl spinor can be expanded in a general form

\[ \Psi_{ABCD} = \psi_{0} (\psi_{1} \psi_{2} \psi_{3} \psi_{4}) - 4 \psi_{0} \psi_{1} \psi_{2} \psi_{3} + 6 \psi_{0} \psi_{1} \psi_{2} \psi_{3} + 4 \psi_{0} \psi_{1} \psi_{2} \psi_{3} + 6 \psi_{0} \psi_{1} \psi_{2} \psi_{3} + 4 \psi_{0} \psi_{1} \psi_{2} \psi_{3} \]

When \( n = 2 \), the spinor \( S \) refers to an electromagnetic spinor \( \Phi_{AB} \) translated from the Maxwell tensor

\[ F_{\alpha \beta} = F_{AA'B'B'} = \Phi_{AB} \epsilon_{A'B'} + \Phi_{A'B'} \epsilon_{AB} \]

Equation (2.20) in this case represents the source-free Maxwell equation

\[ \nabla^{A'A'} \Phi_{AB} = 0 \]

In analogy to the Weyl spinor, there are two different types of Maxwell spinors:

\[ I: \Phi_{AB} \sim \bar{\alpha} (\bar{A} \bar{B}) \]
\[ N: \Phi_{AB} \sim \bar{\alpha} \bar{A} \bar{B} \]

where \( \bar{\alpha} \bar{A} \) and \( \bar{\delta} \bar{A} \) are two non-proportional spinors. We also call type \( N \) Maxwell spinor as degenerate Maxwell spinor. Because the corresponding electric fields \( E \) and magnetic fields \( B \) are of the same magnitude and they are perpendicular; namely, \( |E|^2 - |B|^2 = 0, B \cdot E = 0 \). In addition, for later convenience, we define three typical Maxwell spinors as follows.

Type I:
\[ \Phi_{AB}^{(1)} = \phi_{1} (\psi_{1} \psi_{2} \psi_{3} \psi_{4}) \]
Type N:

\[ \Phi_{AB}^{(0)} = \phi_0 \alpha_A \iota_B, \]  \hspace{1cm} (2.30)

\[ \Phi_{AB}^{(1)} = \phi_1 o_A o_B, \]  \hspace{1cm} (2.31)

where the coefficients \( \phi_1, \phi_0 \) and \( \phi_2 \) are called Maxwell scalars. They are expanded in three different ways in the spin space. Substituting equation (2.29) into equation (2.27), then multiplying \( o_B \) and \( \iota_B \) one equation (2.27), respectively, we obtain two dyad components of the field equation,

\[ o_A \nabla^{AA'} \log \phi_1 - 2 o_A o_B \nabla^{AA'} o_B = 0, \]  \hspace{1cm} (2.32)

\[ \iota_A \nabla^{AA'} \log \phi_1 + 2 o_A o_B \nabla^{AA'} \iota_B = 0. \]  \hspace{1cm} (2.33)

Analogously, from equations (2.30) and (2.31) we arrive at

\[ \iota_A \nabla^{AA'} \log \phi_0 - 2 o_A o_B \nabla^{AA'} o_B = 0, \]  \hspace{1cm} (2.34)

\[ o_A \nabla^{AA'} \log \phi_2 + 2 o_A o_B \nabla^{AA'} o_B = 0. \]  \hspace{1cm} (2.35)

Recalling equation (2.26), the tensor forms of the above three Maxwell spinors read

\[ F_{ab}^{(0)} = 2 \phi_0 \delta_{a[n} m_{b]} + 2 \bar{\phi}_0 \delta_{a[m} n_{b]}, \]  \hspace{1cm} (2.36)

\[ F_{ab}^{(1)} = 2 \phi_1 \left( \ell_{[a} n_{b]} + \bar{m}_{[a} m_{b]} \right) + 2 \bar{\phi}_1 \left( \ell_{[a} m_{b]} + m_{[a} \bar{m}_{b]} \right), \]  \hspace{1cm} (2.37)

\[ F_{ab}^{(2)} = 2 \phi_2 \ell_{[a} m_{b]} + 2 \bar{\phi}_2 \ell_{[a} \bar{m}_{b]}. \]  \hspace{1cm} (2.38)

When \( n = 1 \), the spinor \( S \) refers to a DW spinor \( \xi_A \) translated from the DW tensor

\[ P_{ab} = \xi_A \xi_B \epsilon_{AB}. \]  \hspace{1cm} (2.39)

Equation (2.20) in this case represents the DW equation

\[ \nabla^{AA'} \xi_A = 0. \]  \hspace{1cm} (2.40)

The tensor form is given by

\[ P_{ab} \nabla \iota \tau^P_c + P_{ad} \nabla \iota \tau^P_b = 0. \]  \hspace{1cm} (2.41)

On the spinor dyad \((o, i)\), clearly, there are only two types of DW spinors:

\[ \xi_A = \xi o_A, \]  \hspace{1cm} (2.42a)

\[ \eta_A = \eta o_A, \]  \hspace{1cm} (2.42b)

where \( \xi \) and \( \eta \) are called DW scalars. Substitution of the above equations into equation (2.40) yields

\[ o_A \nabla^{AA'} \log \xi + o_A o_B \nabla^{AA'} o_B - \iota_A o_B \nabla^{AA'} o_B = 0, \]  \hspace{1cm} (2.43)

\[ \iota_A \nabla^{AA'} \log \eta - \iota_A o_B \nabla^{AA'} o_B = 0. \]  \hspace{1cm} (2.44)
Throughout this paper, one will find that equations (2.32)–(2.35) and (2.43)–(2.44), and the given Bianchi identities are the basic equations for our calculation. It is worthwhile to mention that Dirac’s equation is just a pair of coupled DW equations with a source

\[
\begin{align*}
\nabla^A \xi_A = & \ \mu \eta_A, \\
\nabla^A \bar{\eta}_A = & \ \mu \xi_A,
\end{align*}
\]

(2.45)

where \( \mu \) is a real constant related to the mass of the spinor. The tensor version is written as

\[
\begin{align*}
P_{ab} \nabla_d P^d_c + P_{ad} \nabla_d P^d_b = & \ -2 \mu P_{ab} C_c, \\
Q_{ab} \nabla_d Q^d_c + Q_{ad} \nabla_d Q^d_b = & \ -2 \mu Q_{ab} C_c,
\end{align*}
\]

(2.46)

where \( C_a = \xi_{A} \bar{\eta}_A \), and the field \( Q_{ab} \) written in terms of another spin-1/2 spinor \( \eta \) reads

\[
Q_{ab} = \eta_A \bar{\eta}_B \varepsilon_{A'B'}.
\]

(2.47)

Although we will not use Dirac’s equation in this paper, the above formulas might be useful for studying the double copy in non-vacuum spacetimes in the future.

For spin-3/2 massless free-fields equation, the field equation is given by

\[
\nabla^{AA'} \Omega_{ABC} = 0.
\]

(2.48)

One may refer to section 5.8 of reference [33] for more details about \( \Omega_{ABC} \), and we will instead pay more attention to the other three massless free-fields in the following.

3. From gravity fields to lower spin massless free-fields

Inspired by the Weyl double copy relation equation (1.1), with the fact that any massless free-field spinor with spin higher than \( \frac{1}{2} \) can be constructed by scalar fields and DW spinors, we introduce a general map between vacuum gravity fields and DW fields\(^6\) in the curved spacetime

\[
\Psi_{ABCD} = \frac{\xi_{(A} \eta_{B} \sigma_{C} \chi_{D)}}{S_{14}}.
\]

(3.1)

Notably, the above four DW spinors could be identical depending on which type of spacetime we are considering, and \( S_{ij} \) is a scalar field connecting spin-i/2 spinors with spin-j/2 spinors e.g. \( i = 1, j = 4 \) here. Then, with equations (2.22) and (2.40), it is easy to identify what kind of DW fields can exist for a specific curved spacetime. Furthermore, if we regard DW spinors as basic units, other higher spin massless free fields in the curved spacetime are able to be constructed as well. For example, we have

\[
\Phi_{AB} = \frac{\xi_{(A} \eta_{B)}}{S_{12}}
\]

(3.2)

\(^6\) All of the lower spin massless free-field \( (i = 1, 2, 3) \) considered in this paper is assumed to be a test field, which will not curve the spacetime.
and $\Omega_{ABC} = \frac{\xi_{A} \eta_{B} \zeta_{C}}{S_{13}}$. Especially, with respect to three Maxwell spinors $\Phi^{(0)}_{AB}$, $\Phi^{(1)}_{AB}$, and $\Phi^{(2)}_{AB}$, we define the associated scalars $S_{12}$ as follows

$$
\Phi^{(0)}_{AB} = \frac{\eta_{A} \eta_{B}}{S_{12}}, \quad \Phi^{(1)}_{AB} = \frac{\xi_{A} \eta_{B}}{S_{12}^{1}}, \quad \Phi^{(2)}_{AB} = \frac{\xi_{A} \xi_{B}}{S_{12}^{2}}.
$$

(3.3)

where $\xi_{A} = \xi_{oA}$, $\eta_{A} = \eta_{iA}$. Based on equations (3.1) and (3.2), it is natural to lead to a map connecting gravity fields with Maxwell fields in the curved spacetime

$$
\Psi_{ABCD} = \Phi_{(AB} \Theta_{CD)}
$$

(3.4)

where $\Theta_{CD}$ is also a Maxwell spinor $\Theta_{CD} = \zeta_{C} \chi_{D}$ with another scalar field $S'_{12}$, as long as

$$
S_{24} = \frac{S_{14}}{S_{12} S'_{12}}.
$$

(3.5)

Notably, equation (3.4) admits a similar form to the Weyl double copy equation (1.1). In fact, the curved double copy for type N spacetimes and the specific relation, $\Phi_{AB} = \psi^{2/3} o_{(A(AB)}$ we mentioned in section 1, are just particular cases of the present work. Along the above method, one may ask about other situations. Is there a special relationship between different auxiliary scalar fields $S_{ij}$? What kind of spin-$i/2$ massless free-fields can exist in a specific spacetime? Can we directly map a gravity field to a DW field that is living in flat space? What about mapping to Maxwell fields for type III spacetimes? We shall answer these questions in the following.

3.1. Vacuum type N solutions

For a vacuum type N spacetime, the Weyl tensor has only one non-vanishing component $\psi_{4}$. Combining with equation (2.25), the Weyl spinor reads $\Psi_{ABCD} = \Psi_{4} o_{A} o_{B} o_{C} o_{D}$ with $\Psi_{4} = \psi_{4}$. According to equation (3.1), there is only one type of DW spinor $\xi_{A}$ along the basis $o$ in the spacetime, which follows equation (2.42a). In other words, one can always find a special DW spinor, such that

$$
\Psi_{ABCD} = \frac{\xi_{A} \xi_{B} \xi_{C} \xi_{D}}{S_{14}}.
$$

(3.6)

In this case, due to the symmetry property of $\Psi_{ABCD}$, the Bianchi identity equation (2.22) are expanded into two non-trivial dyad components

$$
o_{A} \nabla^{AA'} \log \Psi_{4} + 4 o_{A} o_{B} \nabla^{AA'} o_{B} - \xi_{A} o_{B} \nabla^{AA'} o_{B} = 0,
$$

(3.7)

$$
o_{A} o_{B} \nabla^{AA'} o_{B} = 0.
$$

(3.8)

Based on the Goldberg–Sachs theorem [36], the congruence formed by the principal null-direction $\ell$ for algebraically specially spacetime (e.g. type N spacetime here) must be geodesic $\kappa^{*} = 0$ and shear-free $\sigma^{*} = 0$, the second equation should hold automatically. Thus, only equation (3.7) is left. Making use of equations (2.43), (3.6) and (3.7), it is not hard to identify the DW spinor $\xi_{A}$

Before doing that, let us first pay attention to constructing the Maxwell spinor; then, the problem will be resolved automatically.

It is worthwhile pointing out here that equation (2.43) exactly verifies the statement of reference [22]—the coefficient of the middle term of the left side of the equation is the rank of the corresponding spinor.
Since there is only one type of DW spinor $\xi_A$ in the spacetime, the unique formula of the Maxwell spinor is given by

$$\Phi^{(2)}_{AB} = \frac{\xi^2}{S_{12}^{(2)}} o_A o_B = \phi_2 o_A o_B \Leftrightarrow \phi_2 = \frac{\xi^2}{S_{12}^{(2)}}. \tag{3.9}$$

According to equation (2.35), one can see that there is only one independent dyad component of the field equation. Substituting equation (3.9) into equation (2.35), one observes that the scalar field $S_{12}^{(2)}$ satisfies

$$o_A \nabla^{AA'} \log S_{12}^{(2)} - \iota_A o_B \nabla^{A'B'} o_B = 0. \tag{3.10}$$

It can be identified by solving equations

$$\ell \cdot \nabla \log S_{12}^{(2)} - \rho^* = 0, \quad m \cdot \nabla \log S_{12}^{(2)} - \tau^* = 0. \tag{3.11}$$

This is an interesting result, since the scalar $S_{12}^{(2)}$ shares the same equation with the scalar $S_{24}$ discovered by reference [22]. Furthermore, assuming spin-$3/2$ massless free-field spinor are constructed by DW spinors as follows

$$\Omega_{ABC} = \xi_A \xi_B \xi_C S_{13}^{(2)} = \omega o_A o_B o_C. \tag{3.12}$$

Setting $S_{13} = (S_{12}^{(2)})^3$ (it will be soon clear why we do this), combining equations (2.40) and (2.48), one will get equation (3.10) again. Therefore, for vacuum type N spacetimes, the connection between Weyl spinors and other lower spin massless free-field spinors can be summarized as follows

$$\Psi_4 = \xi^4 (S_{12}^{(2)})^3, \quad \omega = \frac{\xi^3}{(S_{12}^{(2)})^2}, \quad \phi_2 = \frac{\xi^2}{(S_{12}^{(2)})}. \tag{3.13}$$

Clearly, the curved double copy is covered by the above relations. In addition, according to equation (2.39), the DW tensor on the null tetrad reads

$$P_{ab} = 2\xi^2 \xi_{[a} m_{b]} = 2S_{12}^{(2)} \sqrt{\Psi_4 S_{12}^{(2)} \xi_{[a} m_{b]}}. \tag{3.14}$$

Using equation (2.41), it is easy to verify whether the DW fields depend on the curvature or not.

Next, we shall show several specific investigations on exact vacuum type N solutions.

3.1.1. Kundt solutions. Firstly, let us focus on non-diverging solutions ($\rho^* = 0$), usually, called Kundt solutions [37]. There are two classes of Kundt solutions. One of them is plane-fronted wave with parallel propagation, called pp waves, the metric reads

$$ds^2 = 2 du dv + H du - 2 \, dz \, d\bar{z}, \tag{3.15}$$

where $H(u, z, \bar{z}) = f(u, z) + \tilde{f}(u, \bar{z})$ with a general function $f$. Choosing a null tetrad

$$\ell = \partial_u, \quad n = \partial_{\mu} - H \partial_{\bar{\nu}}, \quad m = \partial_z, \tag{3.16}$$

8 Different from the original work, to keep consistent with the notion of this paper, here we use $S_{24}$ to denote the harmonic scalar field of the curved double copy, instead of $S$. 
one can find \( \tau^* = 0 \), and \( \Psi_4 = -\partial^2 \bar{f}(u, \bar{z}) \). Solving equation (3.11) we have \( S^{(2)}_{12} = \{ \mathcal{G} \}(u, \bar{z}) \), which is an arbitrary function of \( u \) and \( \bar{z} \). Therefore, from equation (3.13) the DW scalar is solved by

\[
\xi^2 = \sqrt{-\partial^2 \bar{f}(u, \bar{z})\mathcal{G}^3(u, \bar{z})}.
\] (3.17)

Clearly, due to the appearance of \( \{ \mathcal{G} \}(u, \bar{z}) \), \( \xi(u, \bar{z}) \) can be any function of \( u \) and \( \bar{z} \). Moreover, turning to tensor version, one observes

\[
2\ell_{(u,m)} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\] (3.18)

the only non-vanishing components of \( P_{ab} \) are \( P_{ac} = -P_{ca} = -\xi^2(u, \bar{z}) \). Particularly, it is simple to check that this satisfies the DW equation not only in curved spacetime but also in Minkowski spacetime where we just need to set \( H = f = 0 \) in the metric.

The degenerate Maxwell scalar is then given by

\[
\phi_2 = \sqrt{-\partial^2 \bar{f}(u, \bar{z})}\{ \mathcal{G} \}(u, \bar{z});
\] (3.19)

this is nothing but the result of reference [22], which admits the Weyl double copy.

Another class is given by

\[
d\sigma^2 = 2\, du (dv + W\, dz + \bar{W}\, d\bar{z} + H\, du) - 2\, dz d\bar{z},
\]

\[
W(v, z, \bar{z}) = -\frac{2v}{(z + \bar{z})}, \quad H(u, v, z, \bar{z}) = \left[ f(u, z) + \bar{f}(u, \bar{z}) \right](z + \bar{z}) - \frac{v^2}{(z + \bar{z})^2},
\] (3.20)

where \( f(u, z) \) is an arbitrary function. A null tetrad is chosen as follows

\[
\ell = \partial_v, \quad n = \partial_u - (H + W\bar{W})\partial_v + W\partial_z + W\partial_{\bar{z}}, \quad m = \partial_z.
\] (3.21)

Then one obtains

\[
\tau^* = -\frac{1}{z + \bar{z}}, \quad \Psi_4 = -(z + \bar{z})\partial^2_{\bar{z}} \bar{f}, \quad S_{12} = \frac{\zeta(u, \bar{z})}{z + \bar{z}}.
\] (3.22)

where \( \zeta(u, \bar{z}) \) is an arbitrary function. So, DW scalar is given by

\[
\xi^2 = \sqrt{-\partial^2 \bar{f}(u, \bar{z})\zeta^3(u, \bar{z})}.
\] (3.23)

In this case

\[
2\ell_{(u,m)} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\] (3.24)

it is easy to check that the corresponding DW equation also holds in Minkowski spacetime. As we can see from equations (3.23) and (3.24); the DW field is curvature-independent. More importantly, the degenerate Maxwell spinor is given by

\[
\phi_2 = \sqrt{-\zeta(u, \bar{z})\partial^2_{\bar{z}} \bar{f}},
\] (3.25)
which is consistent with the result of reference [22] and will lead to the double copy.

3.1.2. Robinson–Trautman solutions. The solutions of general vacuum type N spacetimes admitting a geodesic, shear-free, non-twisting but diverging null congruence are given by Robinson and Trautman [37, 38]

\[
\text{ds}^2 = H \, du^2 + 2 \, du \, dr - \frac{2r^2}{P^2} \, dz \, \bar{z},
\]

\[
H(u, r, z, \bar{z}) = k - 2r \partial_u \log P, \quad (k = 0, \pm 1)
\]

\[
k = 2P^2 \partial_z \bar{z} \log P(u, z, \bar{z}).
\]

Choosing a null tetrad as follows

\[
\ell = \partial_r, \quad n = \partial_u - \frac{1}{2} H \partial_r, \quad m = -\frac{P}{r} \partial_z,
\]

one obtains \( \rho^* = -1/r, \tau^* = 0, \) and

\[
\Psi_4 = \frac{P^2}{r} \partial_u \left( \frac{\partial^2 P}{P} \right), \quad S_{12} = \frac{\mathcal{R}(u, \bar{z})}{r},
\]

where \( \mathcal{R}(u, \bar{z}) \) is an arbitrary function. Since

\[
2P(\ell[m_n]) = \begin{pmatrix}
0 & 0 & 0 & r \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-r & 0 & 0 & 0
\end{pmatrix},
\]

which is the same as the Kundt cases; there is only one independent component. The DW spinor is represented by

\[
\xi^2 = \frac{P}{r} \sqrt{\mathcal{R}^3(u, \bar{z}) \partial_u \left( \frac{\partial^2 P}{P} \right)}.
\]

The information of the structure function \( P(u, z, \bar{z}) \) cannot be canceled by the function \( \mathcal{R}(u, \bar{z}) \). And it is easy to check that the DW field does not satisfy its field equation in Minkowski spacetime. While for the degenerate Maxwell scalar

\[
\phi_2 = \frac{P}{r} \sqrt{\mathcal{R}(u, \bar{z}) \partial_u \left( \frac{\partial^2 P}{P} \right)},
\]

as expected, it is consistent with the result of reference [22] and leads to the double copy relation.

In summary, we rebuild the Weyl double copy with the help of DW field for non-twisting type N solutions. In addition, we find that only for the Kundt class, we can obtain a DW field such that it satisfies its field equation in Minkowski spacetime.

3.2. Vacuum type D solutions

For vacuum type D spacetimes, according to equation (2.25), the Weyl tensor has only one non-vanishing component \( \psi_2 \). In this case, the Weyl spinor is reduced to
\[ \Psi_{ABCD} = \Psi_{20} o_{A}o_{B} l_{C}l_{D}, \]  
(3.32)

where we let \( \Psi_{2} = 6 \psi_{2} \). The map equation (3.1) is chosen as

\[ \Psi_{ABCD} = \frac{\xi(A\xi B\eta C\eta D)}{S_{14}}, \]  
(3.33)

where the Weyl spinor is constructed by two mutually orthogonal DW spinors given by equation (2.42) with the condition \( \xi = \eta \). Expanding the above equation on the spin bases, we obtain a scalar identity

\[ \Psi_{2} = \xi_{4} S_{14}, \]  
(3.34)

Following the Goldberg–Sachs theorem, the congruences are formed by two principal null-directions for type D spacetimes, namely, \( \ell \) and \( n \), and they should be geodesic and shear-free, i.e. \( \kappa^{*} = \sigma^{*} = \nu^{*} = \lambda^{*} = 0 \). So, non-trivial dyad components of the Bianchi identity equation (2.22) are given by

\[ o_{A} \nabla^{AA'} \log(\Psi_{2}) - 3 i_{A} o^{B} \nabla^{AA'} o_{B} = 0, \]  
(3.35)

\[ i_{A} \nabla^{AA'} \log(\Psi_{2}) + 3 o_{A} i^{B} \nabla^{AA'} i_{B} = 0. \]  
(3.36)

In analogy to the case of type N, combining equations (2.43), (3.34) and (3.35) we have

\[ o_{A} \nabla^{AA'} \log S_{14} + 4 o_{A} i^{B} \nabla^{AA'} o_{B} - i_{A} o^{B} \nabla^{AA'} o_{B} = 0. \]  
(3.37)

Similarly, from equations (2.44), (3.34) and (3.36) we have

\[ i_{A} \nabla^{AA'} \log S_{14} - 4 i_{A} o^{B} \nabla^{AA'} i_{B} + o_{A} i^{B} \nabla^{AA'} i_{B} = 0. \]  
(3.38)

Multiplying \( o_{A'} \) and \( i_{A'} \) on the above equations, respectively, \( S_{14} \) is solved by

\[ \ell \cdot \nabla \log S_{14} + 4 \epsilon^{*} - \rho^{*} = 0, \quad m \cdot \nabla \log S_{14} + 4 \beta^{*} - \tau^{*} = 0, \]  
\[ \bar{m} \cdot \nabla \log S_{14} - 4 \alpha^{*} + \pi^{*} = 0, \quad n \cdot \nabla \log S_{14} - 4 \gamma^{*} + \mu^{*} = 0. \]  
(3.39)

This is an overdetermined system since there is only one unknown quantity, and one will soon see that \( S_{14} \) satisfies its integrability condition, so we can always find its solution. Once \( S_{14} \) is solved, the DW scalars will then be identified. Different from the case of type N, since there are two types of DW spinors for vacuum type D spacetime, we can find two types of Maxwell fields in the curved spacetime. One is degenerate, the simplest forms are

\[ \Phi_{AB}^{(0)} = \frac{\xi_{12}^{2}}{S_{14}^{12}} l_{A}l_{B} = \phi_{0} l_{A}l_{B} \iff \phi_{0} = \frac{\xi_{12}^{2}}{S_{14}^{12}}, \]  
(3.40)

\[ \Phi_{AB}^{(2)} = \frac{\xi_{12}^{2}}{S_{14}^{12}} o_{A}o_{B} = \phi_{2} o_{A}o_{B} \iff \phi_{2} = \frac{\xi_{12}^{2}}{S_{14}^{12}}, \]  
(3.41)

The other one is non-degenerate, the simplest form reads
Correspondingly, we can build two different maps between gravity fields and Maxwell fields in the curved spacetime starting from the relation equation (3.34)

\[\Psi_2 = \frac{\phi_1^2}{S_{12}^{(0)}} \rightarrow \Psi_{ABCD} = \frac{\Phi^{(0)}_{AB} \Phi^{(2)}_{CD}}{S_{12}^{(0)}}, \quad (3.43)\]

\[\Psi_2 = \frac{\phi_1^2}{S_{12}^{(0)}} \rightarrow \Psi_{ABCD} = \frac{\Phi^{(1)}_{AB} \Phi^{(1)}_{CD}}{S_{12}^{(1)}}, \quad (3.44)\]

where upper index \((i, j)\) refers to the case of mixed Maxwell scalars \(\phi_i \phi_j\). One can see that the first case involves two degenerate Maxwell spinors which might lead to mixed double copy; while for the second case, it corresponds to the classical Weyl double copy [20], for which \(S_{12}^{(1)} = (\phi_1)^{1/2} = (\Psi_2)^{1/3}\). We will restudy this along a new way with the help of DW spinors. And we will also check whether the mixed double copy of the first case hold or not in Minkowski spacetime. The main point, in the following, is looking for source-independent Maxwell fields.

Firstly, let us focus on degenerate Maxwell spinors. Clearly, once DW spinors are identified, to obtain the Maxwell spinor, the only work left for us is to identify \(S_{12}\). Combining equations (2.34) and (3.40), \(S_{12}^{(0)}\) can be solved by

\[\bar{m} \cdot \nabla \log S_{12}^{(0)} + \pi^* = 0, \quad n \cdot \nabla \log S_{12}^{(0)} + \mu^* = 0. \quad (3.45)\]

Analogously, \(S_{12}^{(2)}\) can be solved as well; the corresponding equations have been shown in equation (3.10) for the type N case. Since the equation of \(S_{12}^{(2)}\) is independent of the Petrov type of spacetime. To avoid redundancy, we will not show this equation again.

For the second case, substitution of equations (2.43) and (3.42) into equation (2.32) yields

\[\alpha A \nabla^2 \log S_{12}^{(1)} + 2 \alpha A' B \nabla^2 \nabla B' = 0. \quad (3.46)\]

Similarly, substitution of equations (2.44) and (3.42) into equation (2.33) yields

\[\tau A \nabla^2 \log S_{12}^{(1)} - 2 \tau A' B \nabla^2 \nabla B' = 0. \quad (3.47)\]

Multiplying \(\bar{m}'\), \(\bar{\tau}'\) respectively, \(S_{12}^{(1)}\) is solved by

\[\ell \cdot \nabla \log S_{12}^{(1)} + 2 \epsilon^* = 0, \quad m \cdot \nabla \log S_{12}^{(1)} + 2 \beta^* = 0,
\]

\[\bar{m} \cdot \nabla \log S_{12}^{(1)} - 2 \alpha^* = 0, \quad n \cdot \nabla \log S_{12}^{(1)} - 2 \gamma^* = 0. \quad (3.48)\]

This is also an overdetermined system, it is easy to check that the integrability condition holds in this case, and the solution can always be solved.

Therefore, once \(S_{12}\) is fixed, we can identify the DW fields; the Maxwell fields’ property then will be revealed by \(S_{12}\). If one is instead interested in spin-3/2 massless free-fields, \(S_{13}\) then will be the key point. Thus, it is a good starting point to identify first the DW fields when investigating the higher spin massless free-fields. Then, the rest information of the target fields will be encoded in the associated scalar fields. Once DW fields and the associated scalar fields are identified, the property of the target fields should be clear. Using this method we can now
look for source-independent electromagnetic fields for vacuum type D solutions. More illustration on verifying the Weyl double copy relation will be given with the help of the modified Plebański–Demiański metric.

3.2.1. Modified vacuum Plebaski Demiański metric. The Plebański–Demiański metric gives a complete family of type D spacetimes [39, 40], the original line element reads

$$ds^2 = \frac{1}{(1 - \hat{p}^2)^2} \left[ \frac{Q(d\tau - \hat{p}^2 d\sigma)^2}{r^2 + \hat{p}^2} - \frac{\mathcal{P}(d\hat{\tau} + \hat{p}^2 d\hat{\sigma})^2}{r^2 + \hat{p}^2} - \frac{\hat{p}^2 + \hat{p}^2}{\hat{p}} dp^2 - \frac{\hat{p}^2 + \hat{p}^2}{Q} d\hat{p}^2 \right],$$

(3.49)

where

$$\mathcal{P} = k' + 2N' \hat{p} - \epsilon' \hat{p}^2 + 2M' \hat{p}^3 - \left(k' + \epsilon^2 + g^2 + \Lambda/3\right) \hat{p}^4,$$

$$Q = \left(k' + \epsilon^2 + g^2\right) - 2M' \hat{r} + \epsilon' \hat{r}^2 - 2N' \hat{r}^3 - (k' + \Lambda/3) \hat{r}^4.$$

(3.50)

It includes seven free real parameters, $M'$, $N'$, $\epsilon'$, $g'$, $k'$, and $\Lambda$. Besides the cosmological constant $\Lambda$, $M'$ is the mass parameter, $N'$ is related to the NUT parameter, $\epsilon'$ and $g'$ are the electric and magnetic charges, and $\epsilon'$ and $k'$ are related to the angular momentum per unit mass and the acceleration. Considering this metric cannot give an obvious physical interpretation; for example, it is not apparent that this line element does include the well-know $n$ Kerr metric, the NUT solution or C-metric, etc, we rescale the coordinates

$$\hat{p} = \sqrt{\alpha \omega} p, \quad \hat{r} = \sqrt{\frac{\alpha}{\omega}} r, \quad \hat{\sigma} = \sqrt{\frac{\omega}{\alpha}} \sigma, \quad \hat{\tau} = \sqrt{\frac{\omega}{\alpha}} \tau,$$

(3.51)

and the parameters

$$M' + iN' = \left(\frac{\alpha}{\omega}\right)^{3/2} (M + iN), \quad \epsilon' = \frac{\alpha}{\omega} \epsilon, \quad k' = \alpha^2 k.$$

(3.52)

A modified metric then is given by [40, 41]

$$ds^2 = \frac{1}{(1 - \alpha p r)^2} \left[ \frac{Q}{r^2 + \omega^2 p^2} (d\tau - \omega p^2 d\sigma)^2 - \frac{P}{r^2 + \omega^2 p^2} (\omega d\tau + r^2 d\sigma)^2 - \frac{r^2 + \omega^2 p^2}{p} dp^2 - \frac{r^2 + \omega^2 p^2}{Q} d\hat{p}^2 \right],$$

(3.53)

where

$$P = P(p) = k + 2\omega^{-1} Np - \epsilon p^2 + 2\alpha M p^3 - \alpha^2 \omega^2 kr^4,$$

$$Q = Q(r) = \omega^2 k - 2Mr + \epsilon r^2 - 2\alpha \omega^{-1} N \hat{r}^3 - \alpha^2 kr^4.$$

(3.54)

Since we only consider vacuum type D solutions, $\epsilon'$, $g'$ and $\Lambda$ are set to be vanishing here. In addition, it is worthwhile to mention here that this modified metric does not include a non-singular NUT solution. In practice, to get a metric to cover all of the cases, we still need to do a coordinate transformation: $p = \frac{b}{\omega} + \frac{a}{\omega} \hat{p}$, $\tau = t - \frac{\alpha + \omega}{\alpha} \phi$, and $\sigma = -\frac{\alpha}{\omega} \phi$ where new parameters $a$ and $b$ usually correspond to a rotation parameter and a NUT parameter, respectively.
However, considering the modified metric has a simple form and already covers the accelerating and rotating black hole solutions with the NUT parameter, we will use it as an example to investigate the double copy in this paper. Choosing a null tetrad

\[ \ell^\mu = \frac{(1 - \alpha pr)}{\sqrt{2(r^2 + \omega^2 p^2)}} \left[ \frac{1}{\sqrt{Q}} (r^2 \partial_r - \omega \partial_\omega) - \sqrt{Q} \partial_\tau \right], \]

\[ n^\mu = \frac{(1 - \alpha pr)}{\sqrt{2(r^2 + \omega^2 p^2)}} \left[ \frac{1}{\sqrt{Q}} (r^2 \partial_r - \omega \partial_\omega) + \sqrt{Q} \partial_\tau \right], \]

\[ m^\mu = \frac{(1 - \alpha pr)}{\sqrt{2(r^2 + \omega^2 p^2)}} \left[ -\frac{1}{\sqrt{P}} (\omega r^2 \partial_r + \partial_\sigma) + i \sqrt{P} \partial_\rho \right], \]

we have

\[ \rho' = \mu' = \frac{1 + i \alpha \omega r^2}{\sqrt{2(r + i \omega p)}} \sqrt{\frac{Q(r)}{r^2 + \omega^2 p^2}}, \]

\[ \tau^* = \pi^* = \frac{\omega - i \alpha r^2}{\sqrt{2(r + i \omega p)}} \sqrt{\frac{P(p)}{r^2 + \omega^2 p^2}}. \]

\[ e^* = \gamma^* = \frac{1}{4} \left[ \frac{2(1 - \alpha pr)}{r + i \omega p} - 2 \omega p - (1 - \alpha pr) \frac{\partial Q}{Q} \right] \sqrt{\frac{Q(r)}{r^2 + \omega^2 p^2}}, \]

\[ \alpha^* = \beta^* = \frac{1}{4} \left[ \frac{2 \omega(1 - \alpha pr)}{r + i \omega p} + 2 i \alpha r + i(1 - \alpha pr) \frac{\partial P}{P} \right] \sqrt{\frac{P(p)}{r^2 + \omega^2 p^2}}. \]

The Weyl scalar is given by

\[ \Psi_2 = 6 \psi_2 = \frac{6(M + i N)(1 - \alpha pr)^3}{(r + i \omega p)^3}, \]

which, as we can see, is independent of coordinates \( \tau \) and \( \sigma \). Plugging equations \((3.55)\) and \((3.56)\) into equation \((3.39)\) we have

\[ L_c + L_r \partial_r \log S_{14} + L_\omega \partial_\omega \log S_{14} + L_\sigma \partial_\sigma \log S_{14} = 0, \]

\[ M_c + M_r \partial_r \log S_{14} + M_\omega \partial_\omega \log S_{14} + M_\sigma \partial_\sigma \log S_{14} = 0, \]

\[ -M_c + M_r \partial_r \log S_{14} + M_\omega \partial_\omega \log S_{14} - M_\sigma \partial_\sigma \log S_{14} = 0, \]

\[ -L_c + L_r \partial_r \log S_{14} + L_\omega \partial_\omega \log S_{14} - L_\sigma \partial_\sigma \log S_{14} = 0, \]

\[ \log S_{14} = 0. \]
where

\[ L_c = \frac{Q(r)[-i + \alpha p(-3\omega p + i4r)] + (\omega p - ir)(\alpha pr - 1)Q'(r)}{(\omega p - ir)\sqrt{2Q(\omega^2 p^2 + r^2)}}. \]

\[ L_r = \frac{r^2(1 - \alpha pr)}{\sqrt{2Q(\omega^2 p^2 + r^2)}}, \]

\[ L_σ = -\frac{\omega(1 - \alpha pr)}{\sqrt{2Q(\omega^2 p^2 + r^2)}}, \]

\[ L_τ = -\frac{\sqrt{Q}(1 - \alpha pr)}{\sqrt{2(\omega^2 p^2 + r^2)}}, \]

\[ M_τ = \frac{P[3\alpha pr^2 + i\omega(4\alpha pr - 1)] - i(\omega p - ir)(\alpha pr - 1)P'(p)}{(\omega p - ir)\sqrt{2P(\omega^2 p^2 + r^2)}}, \]

\[ M_σ = \frac{-\omega p^2(1 - \alpha pr)}{\sqrt{2P(\omega^2 p^2 + r^2)}}, \]

\[ M_ρ = \frac{-(1 - \alpha pr)}{\sqrt{2P(\omega^2 p^2 + r^2)}}, \]

\[ M_ρ = \frac{i\sqrt{P}(1 - \alpha pr)}{\sqrt{2(\omega^2 p^2 + r^2)}}. \]

From equation (3.58), one can find that \( S_{14} \) is independent of the coordinates \( τ \) and \( σ \), which is the same as the Weyl scalar \( Ψ_2 \). The integrability condition of the above equations is then given by

\[ \partial_r \left( \frac{M_τ}{M_ρ} \right) = \partial_ρ \left( \frac{L_c}{L_r} \right). \] (3.60)

One can check that this condition does hold and we arrive at

\[ \log S_{14} = i \arctan \frac{-\omega p}{r} - \log \frac{P(p)Q(r)}{C_1(1 - \alpha pr)^3\sqrt{r^2 + \omega^2 p^2}}, \] (3.61)

where \( C_1 \) is an arbitrary constant of integration. Using the identity

\[ \arctan(z) = -\frac{i}{2} \log \left( \frac{i - z}{i + z} \right), \quad z ∈ \mathbb{C}, \] (3.62)

we obtain

\[ S_{14} = C_1 \left( \frac{1 - \alpha pr)^3(r + iω p)}{P(p)Q(r)} \right). \] (3.63)

The square of the coefficient of the DW spinor \( ξ_A \) reads

\[ ξ^2 = \sqrt{S_{14}\Psi_2} = \frac{6C_1(M + iN)(1 - \alpha pr)^3}{P(p)Q(r)} \left( \frac{1}{r + iω p} \right), \] (3.64)

which is the coefficient of the Maxwell spinor as well as the DW tensor equation (3.14). It is simple to check that the DW equation in the curved spacetime indeed holds in this case.
Moreover, one can see that the pre-factor \((M + iN)\) related to the source can be absorbed by \(\mathcal{C}_1\), so we may pay attention to the rest term. Here, \(P(p)Q(r)\) in the denominator is the only term related to the source, thus it is a crucial point when we look for a source-independent Maxwell field.

On the one hand, the degenerate Maxwell spinor \(\Phi^{(0)}_{AB}\) can now be identified once scalar field \(S^{(0)}_{12}\) is fixed. Following equation (3.45), we arrive at

\[
S^{(0)}_{12} = \mathcal{C}_2 \frac{1 - \alpha pr}{r + i \omega p} \sim \frac{1 - \alpha pr}{r + i \omega p},
\]

where \(\mathcal{C}_2\) is an arbitrary constant of integration. Analogously, the scalar field \(S^{(2)}_{12}\) is shown as

\[
S^{(2)}_{12} = S^{(0)}_{12},
\]

up to a constant. Notably, they are independent of the source, so \(P(p)Q(r)\) term of \(\xi^2\) will stay when mapped to the Maxwell field scalar, it is simple to verify that we cannot get a source-independent degenerate Maxwell field. In addition, according to equation (3.5), the scalar \(S^{(0,2)}_{24}\) is given by

\[
S^{(0,2)}_{24} \sim \frac{(1 - \alpha pr)(r + i \omega p)^3}{P(p)Q(r)},
\]

one can see that it is not equal to \(S^{(0)}_{12}\).

On the other hand, for non-degenerate Maxwell spinor \(\Phi^{(1)}_{AB}\), similar to the case of \(S_{14}\), from equation (3.48) one can find that \(S^{(1)}_{12}\) is independent of the coordinates \(\tau\) and \(\sigma\), and the integrability condition holds. With the aid of equation (3.62), we then have

\[
S^{(1)}_{12} = \mathcal{C}_3 \left(1 - \alpha pr\right) (r + i \omega p) \sqrt{P(p)Q(r)},
\]

as one can see, which depends on the source because of the appearance of the term \(P(p)Q(r)\). Furthermore, one can see that the \(P(p)Q(r)\) term of \(\xi^2\) in equation (3.64) will be cancel led out by \(S^{(1)}_{12}\) when mapped to \(\phi_1\) of equation (3.42). Therefore, we will get a source-independent non-degenerate Maxwell field\(^9\). Besides, \(S^{(1,1)}_{24}\) is given by

\[
S^{(1,1)}_{24} = \frac{S^{(1)}_{12}}{S^{(1)}_{12}^2} = \frac{\mathcal{C}_1}{(\mathcal{C}_3)^2} \frac{1 - \alpha pr}{r + i \omega p} \sim \frac{1 - \alpha pr}{r + i \omega p}.
\]

Interestingly, this scalar satisfies the wave equation even in Minkowski spacetime \((M = N = 0)\). So we discover a map from a vacuum gravity field to a source-independent Maxwell field. This means that the background of the Maxwell field can be flat. One may already realize that this is nothing but the Weyl double copy.

So far, all auxiliary scalar fields connecting Weyl, Maxwell and DW fields are identified. The maps among different spin massless-free fields are summarized as follows

\[
\Psi_2 = \frac{\xi^4}{(S^{(2)}_{12})^2 S^{(0,2)}_{24}} = \frac{\xi^4}{(S^{(1,2)}_{12})^2 S^{(1,1)}_{24}};
\]

\[
S^{(0)}_{12} = S^{(2)}_{12} = S^{(1,1)}_{24} = (\phi_1)^{1/2} = (\Psi_2)^{1/3}.
\]

\(^9\) It is easy to check that \(l_0 m_{01} + m_0 m_{01}\) and \(l_0 m_{01} + m_0 m_{01}\) are all independent of the source, from the tensor form equation (2.37) one can see that the Maxwell scalar \(\phi_1\) is the only physical quantity that could be affected by the source.
Compared with the case of vacuum type N solutions equation (3.13), $S_{24}$ is not equal to $S_{12}$ anymore regarding the above two cases of the first line of equation (3.70). While, unexpectedly, one can see that $S_{(0)12}$ and $S_{(2)12}$ are equal to $S_{(1,1)24}$ up to a constant. That means, the zeroth copy not only connects the vacuum gravity fields with single copy but also connects degenerate electromagnetic fields with DW fields in the curved spacetime for non-twisting vacuum type N solutions and vacuum type D solutions. The success of mapping gravity fields to the single and zeroth copies by using the DW spinors encourages us to extend the study to non-twisting vacuum type III solutions.

3.3. Vacuum type III solutions

For vacuum type III solutions, $\psi_0 = \psi_1 = \psi_2 = 0$. By making a null rotation about null vector $\ell$ [35],

$$\ell \rightarrow \ell, \quad n \rightarrow n + A^* m + A m + A A^* \ell, \quad m \rightarrow m + A \ell, \quad \bar{m} \rightarrow \bar{m} + A^* \ell,$$

(3.71)

where $A^*$ is the complex conjugate of a complex number $A$, then the Weyl scalars transform like

$$\begin{align*}
\psi_0 &\rightarrow \psi_0, \quad \psi_1 \rightarrow \psi_1 + A^* \psi_0, \quad \psi_2 \rightarrow \psi_2 + 2A^* \psi_1 + (A^*)^2 \psi_0, \\
\psi_3 &\rightarrow \psi_3 + 3A^* \psi_2 + 3(A^*)^2 \psi_1 + (A^*)^3 \psi_0, \\
\psi_4 &\rightarrow \psi_4 + 4A^* \psi_3 + 6(A^*)^2 \psi_2 + 4(A^*)^3 \psi_1 + (A^*)^4 \psi_0.
\end{align*}$$

(3.72)

Clearly, we can let $\psi_4$ vanish without changing other three Weyl scalars by requiring

$$A^* = -\frac{\psi_4}{4\psi_3},$$

(3.73)

and $\psi_3$ will be only non-vanishing Weyl scalar. The spinor form thus reduces to

$$\Psi_{ABCD} = \Psi_3 o_A o_B o_C o_D,$$

(3.74)

where we set $\Psi_3 = -4\psi_3$. Based on this, equation (3.1) can be written in the form

$$\Psi_{ABCD} = \frac{\xi_A \xi_B \xi_C \eta_D}{S_{14}},$$

(3.75)

where $\xi_A = \xi o_A$ and $\eta_A = \eta o_A$, or in scalar form

$$\Psi_3 = \frac{\xi^3 \eta}{S_{14}}.$$

(3.76)

According to equation (2.22), two independent dyad components of the Bianchi identity read

$$o_A \nabla^{AA'} \log \Psi_3 + 2 \nabla^{AA'} o_A = 0,$$

(3.77)

$$t_A \nabla^{AA'} \log \Psi_3 + 4 o_A t_B \nabla^{AA'} t_B + 2 t_A o_B \nabla^{AA'} t_B = 0.$$

(3.78)
Since
\[ o_A \nabla A' B O_B - \epsilon_A o_B \nabla A' o_B = (o_A B - \epsilon_A o_B) \nabla A' o_B = \epsilon_A' \nabla A' o_B = \nabla A'( \epsilon_A B o_B) = \nabla A' o_B. \] (3.79)

Equation (2.43) can be rewritten as
\[ o_A \nabla A' \log \xi + \nabla A' o_A = 0. \] (3.80)

Combining equations (3.77) and (3.80), we have
\[ \Psi_3 = \mathcal{C} \xi^2, \] (3.81)
where \( \mathcal{C} \) is a non-vanishing constant of integration. This result is even independent of our assumption equation (3.75). In order to keep the total spin invariant for the above equation\(^{10}\), the constant \( \mathcal{C} \) here should correspond to a field with a total spin of 1. However, we need to point out that it is not necessary to require it to be a source-free Maxwell scalar. This point can also be verified from equation (3.75) or equation (3.76), then one observes
\[ \mathcal{C} = \frac{\xi \eta}{S_{14}}; \] (3.82)

this is nothing but a constraint equation about spinor \( \xi_A \) and spinor \( \eta_{\lambda} \). Furthermore, one can see that \( \mathcal{C} \) indeed corresponds to a field with a total spin of 1 in view of the right side of equation (3.82). Yet there is no reason to require \( S_{14} = S_{12} \), \( \mathcal{C} \) does not have to be a source-free Maxwell scalar. We will come back to talk more about this in the discussions. Regarding spinor \( \xi_A \), according to equation (3.3), we can construct a degenerate Maxwell spinor \( \Phi^{(2)}_{AB} \) with it. Repeating the same calculation as the case of type N, \( S^{(2)}_{12} \) can be solved, and then the Maxwell field will be identified.

As for another DW spinor \( \eta_A = \eta_{\lambda A} \), the equation of motion is given by
\[ \iota_A \nabla A' \log \left( \frac{\Psi_3}{\eta} \right) + 3 o_A \nabla A' t_B + 3 \epsilon_A o_B \nabla A' t_B = 0 \] (3.83)

following equations (2.44) and (3.78). The tensor version then reads
\[ m \cdot \nabla \log \left( \frac{\Psi_3}{\eta} \right) + 3 \pi^* + 3 \alpha^* = 0, \quad n \cdot \nabla \log \left( \frac{\Psi_3}{\eta} \right) + 3 \mu^* + 3 \gamma^* = 0. \] (3.84)

Recalling the type N and type D cases, the DW scalars mapped from the gravity fields all depend on the same coordinates as the Weyl scalars, we expect \( \eta \) also behaves like that and we are in fact only interested in this case in the present work. However, one will see that its solution is also related to the other coordinates unless we impose an extra condition. Therefore, generally, there is no trivial relationship between the Weyl scalar \( \Psi_3 \) and the DW scalar \( \eta \), we thus shall pay more attention to the DW tensor \( \xi_A \) in this section.

Further investigation on exact non-twisting vacuum type III solutions is given in the following.

\(^{10}\) We thank Ricardo Monteiro for bringing this up.
3.3.1. **Kundt solutions.** There are two kinds of Kundt solutions for the type III case, the metric in general is given by [37]

\[
ds^2 = 2 du (H du + dv + W dz + \bar{W} d\bar{z}) - 2 dz d\bar{z},
\]

(3.85)

with a real function \(H\) and a complex function \(W\).

For the case of \(W_v = 0\),

\[
W = W(u, \bar{z}), \quad H = \frac{1}{2} (W_{\bar{z}} + W_{\bar{z}} v + H^0),
\]

(3.86)

\[H^0_{\bar{z}} - \Re \left[ W^2 + WW_{\bar{z}} + W_{\bar{z}} \right] = 0.\]

We choose a null tetrad

\[
\ell = \partial_u, \quad n = \partial_u - (H + WW) \partial_v + W \partial_{\bar{z}} + W \partial_{\bar{z}}, \quad m = \partial_{\bar{z}},
\]

(3.87)

The Weyl scalars in this case are given by

\[
\psi_3' = -\frac{1}{2} \partial^2_{\bar{z}} W(u, \bar{z}), \quad \psi_4' = -\bar{W}(u, \bar{z}) \partial^2_{\bar{z}} W(u, \bar{z}) - \frac{1}{2} v \partial^2_{\bar{z}} W(u, \bar{z}) - \partial^2_{\bar{z}} H^0 (u, \bar{z}).
\]

(3.88)

Note \(\partial^2_{\bar{z}} W(u, \bar{z}) \neq 0\) here, otherwise the metric reduces to type N solution. By making a null rotation with the help of equation (3.73), the only non-vanishing Weyl scalar left is [42]

\[
\Psi_3 = -4 \psi_3 = 2 \partial^2_{\bar{z}} W(u, \bar{z}).
\]

(3.89)

From equation (3.81), one of the corresponding DW fields is given by

\[
\xi^2 = \frac{2}{C} \partial^2_{\bar{z}} W(u, \bar{z}).
\]

(3.90)

In addition, some spin coefficients are given by

\[
\rho^* = \tau^* = \alpha = 0,
\]

\[
\mu^* = \frac{[\partial^2_{\bar{z}} W (2 W \partial^2_{\bar{z}} W + v \partial^2_{\bar{z}} W + 2 W \partial_z^2 H^0) + 8 \partial^2_{\bar{z}} W \partial_z^2 W - \partial_{\bar{z}} \partial_z^2 H^0]}{16 \partial^2_{\bar{z}} W \partial^2_{\bar{z}} W},
\]

\[
\pi^* = -\frac{\partial^2_{\bar{z}} W}{4 \partial^2_{\bar{z}} W},
\]

\[
\gamma^* = \frac{1}{2} \partial_{\bar{z}} W.
\]

(3.91)

solving equation (3.11) we have \(\partial_u S_{12}^{(2)} = \partial_{\bar{z}} S_{12}^{(2)} = 0\). \(S_{12}^{(2)}\) is independent of \(v\) and \(z\), namely, it can be an arbitrary function of \(u\) and \(\bar{z}\),

\[
S_{12}^{(2)} = S_{12}^{(2)}(u, \bar{z}).
\]

(3.92)

And, it is easy to check that \(S_{12}^{(2)}\) satisfies the wave equation even in the flat spacetime. Then the degenerate Maxwell scalar is given by
\[ \phi_2 = \frac{\xi^2}{S_{12}^{(2)}} = \frac{2}{C} \frac{\partial^2 \bar{z} W(u, \bar{z})}{S_{12}^{(2)}(u, \bar{z})} \]  

Combining the fact

\[ 2\ell_{[w m_b]} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\ell_{[w m_b]} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  

one can find that this degenerate Maxwell field also satisfies the field equation in Minkowski spacetime, for which we may let \( W = H^0 = 0 \) in the metric. Recalling the relationship equation (3.81), the Weyl scalar can be written as

\[ \Psi_3 = C S_{12}^{(2)} \phi_2. \]  

Besides, we also probe another DW spinor’s form. Keeping consistent with the Weyl scalar \( \Psi_3 \), we are only interested in the solution that \( \eta \) is independent of the coordinates \( v \) and \( z \). In this case, by solving equation (3.84) we obtain

\[ \partial_u \log \left( \frac{\Psi_3}{\eta} \right) = M, \quad \partial_{\bar{z}} \log \left( \frac{\Psi_3}{\eta} \right) = N, \]  

where

\[ M = -\frac{3(4\partial_\bar{z} W \partial^2 \bar{z} W + W \partial^3 \bar{z} W - 2\partial_z \partial^2 \bar{z} H_0)}{4\partial^2 \bar{z} W}, \]  

\[ N = \frac{3\partial \bar{z} W}{4\partial^2 \bar{z} W}. \]  

The integrability condition is given by \( \partial_z M = \partial_u N \). Clearly, to have a solution we have to impose one more condition, \( \partial^2 \bar{z}^2 H^0 = 0 \). In general, however, there is no solution which depends on the same coordinates as the Weyl scalar, and there is no trivial relation between DW scalar \( \eta_A \) and Weyl scalar \( \Psi_3 \). In the following, we will focus on the DW tensor \( \xi_A \).

For the case of \( W_{v} \neq 0 \),

\[ W = W^0(u, \bar{z}) - \frac{2v}{z + \bar{z}}, \quad H = H^0 + v \frac{W^0 + \bar{W}^0}{z + \bar{z}} - \frac{v^2}{(z + \bar{z})^2}. \]  

We choose a null tetrad

\[ \ell = \partial_v, \quad n = \partial_u - (H + W \bar{W}) \partial_z + \bar{W} \partial_{\bar{z}} + W \partial_z, \quad m = \partial_{\bar{z}}. \]  

By doing a null rotation with equation (3.73), the Weyl scalar is given by [42]

\[ \Psi_3 = -4\psi_3 = 4 \frac{\partial \bar{z} W^0(u, \bar{z})}{z + \bar{z}}. \]
Correspondingly, we arrive at
\[ \xi^2 = \frac{1}{\mathcal{C}} \Psi_3 = \frac{4}{\mathcal{C}} \frac{\partial \mathcal{W}_0(u, z)}{z + \bar{z}}. \] (3.101)

The spin coefficients \( \rho^* \) and \( \tau^* \) are given by
\[ \rho^* = 0, \quad \tau^* = \frac{1}{z + \bar{z}}. \] (3.102)

Following equation (3.11), the auxiliary scalar field is solved by
\[ S^{(2)}_{12} = \frac{\mathcal{V}(u, z)}{z + \bar{z}}, \] (3.103)
where function \( \mathcal{V}(u, z) \) is arbitrary. One can check that \( S^{(2)}_{12} \) satisfies the wave equation even in the flat spacetime. Moreover, we have
\[ \phi_2 = \frac{4}{\mathcal{C}} \frac{\partial \mathcal{W}_0(u, z)}{\mathcal{V}(u, z)}. \] (3.104)

Since
\[ 2 \ell_{[\omega m]} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad 2 \ell_{[\omega m]} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \] (3.105)

similar to the case of \( W_v = 0 \), one can show that the Maxwell field also satisfies its field equation in Minkowski space. The Weyl scalar is written as
\[ \Psi_3 = \mathcal{C} S^{(2)}_{12} \phi_2. \] (3.106)

### 3.3.2. Robinson–Trautman solutions.

The vacuum solution of diverging non-twisting type III case is given by [37, 38],
\[ ds^2 = du(H du + 2 dr) - \frac{2r^2}{P^2(u, z, \bar{z})} \, dz \, d\bar{z}, \]
\[ \Delta \log P = \mathcal{K} = -3 \left[ f(u, z) + \bar{f}(u, \bar{z}) \right], \quad f \neq 0, \] (3.107)
\[ H = \Delta \log P - 2r \partial_u \log P, \quad \Delta \equiv 2P^3 \partial_z \partial_{\bar{z}}. \]

where the structure function \( f(u, z) \) is complex.

Choosing a null tetrad
\[ \ell = \partial_r, \quad n = \partial_u = \frac{H}{2} \partial_r, \quad m = -\frac{P}{r} \partial_z, \] (3.108)

the non-vanishing Weyl scalars read
\[ \psi_3' = \frac{3P \partial_u \bar{f}}{2r^2}, \]
\[ \psi_4' = \frac{3P^2 \partial_z \bar{f} - 2r \partial_u \bar{f} \partial_u \partial_z P + 2P(3\partial_z \partial_{\bar{z}} \bar{f} + r \partial_u \partial_z P)}{2r^2}. \] (3.109)
Same to the case of Kundt class of the last section, by doing a null rotation with equation (3.73), the only non-vanishing Weyl scalar reads
\[ \Psi_3 = -4\psi_3 = -4\psi_3' = -\frac{6P\partial_\ell f}{r^2}. \] (3.110)

In the new null tetrad, according to equation (3.81), one of the DW fields mapping from the gravity side is given by
\[ \xi^2 = \frac{6P\partial_\ell f}{C \cdot r^2}. \] (3.111)

The spin coefficients \(\rho^*\) and \(\tau^*\) are solved by
\[
\rho^* = \frac{-1}{r}, \\
\tau^* = \frac{3P^2\partial_\ell f - 2r\partial_\ell P\partial_\ell P + 2P(3\partial_\ell f\partial_\ell P + r\partial_\ell \partial_\ell P)}{12Pr\partial_\ell f}.
\] (3.112)

Making use of equation (3.11), we find \(S^{(2)}_{\ell_2}\) has to satisfy
\[
\frac{1}{r} + \frac{\partial_\ell S^{(2)}_{\ell_2}}{S^{(2)}_{\ell_2}} = 0, \quad \partial_\bar{z} S^{(2)}_{\ell_2} = 0.
\] (3.113)

Therefore, we arrive at a general solution
\[ S^{(2)}_{\ell_2} = \frac{\lambda(u, \bar{z})}{r}, \] (3.114)
where \(\lambda(u, \bar{z})\) is an arbitrary function. From equation (3.41) the degenerate Maxwell scalar reads
\[ \phi_2 = \frac{\xi^2}{S^{(2)}_{\ell_2}} = -\frac{6P\partial_\ell f(u, \bar{z})}{C \cdot r\lambda(u, \bar{z})}. \] (3.115)

Going to the tensor version equation (2.38), we have
\[
2\frac{P}{r}t_{[\ell a]mb} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad 2\frac{P}{r}t_{[\ell a]mb} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\] (3.116)

Clearly, only the \([u\bar{z}]\) and \([\bar{u}z]\) components are non-vanishing. Similar to the case of type N, it is easy to show that this field satisfies the field equation in Minkowski spacetime. In addition, combining equations (3.81) and (3.115), we have
\[ \Psi_3 = C^{(2)}S^{(2)}_{\ell_2}\phi_2, \] (3.117)
where the scalar field \(S^{(2)}_{\ell_2}\) satisfies the wave equation not only in this curved spacetime but also in Minkowski spacetime.

Therefore, with the help of the DW spinors, we have successfully proved that there indeed exists a natural map between pure Maxwell fields and gravity fields for non-twisting vacuum type III spacetimes. Moreover, we found that the auxiliary scalar field, connecting the DW field with the degenerate electromagnetic field in the curved spacetime, plays a similar role to the zeroth copy.
4. Discussion and conclusions

In this paper, based on the fact that any massless free-field spinors with spin higher than 1/2 can be constructed with DW spinors (spin-1/2) and scalar fields, we introduced a map between vacuum gravity fields and DW fields in spin-space. The form of associated DW spinors are identified. Regarding these DW spinors as basic units, we investigated the other higher spin massless-free fields, especially the Maxwell fields, and showed some hidden fundamental features among these fields.

In particular, for Petrov type N solutions, inspired by the work [22], we found that only one type of DW spinor exists in the curved spacetime; combining with the zeroth copy, the DW spinor can construct any other higher spin massless free-fields. Following this, we studied the Petrov type D solutions. In this situation, there are two types of DW spinors in the curved spacetime. Unlike the case of type N, we found that \( S_{24} \) (a scalar field connecting a Maxwell field with a gravity field) is not equal to \( S_{12} \) (a scalar field connecting a DW field with a Maxwell field) anymore for each case. While, there remains an interesting relation, \( S_{12}^{(0)} = S_{12}^{(2)} = S_{24}^{(1,1)} \), the scalar fields connecting the DW fields with the degenerate electromagnetic fields are equal to the zeroth copy up to a constant. In general, by using the DW spinors and the auxiliary scalar fields, we systematically rebuilt the Weyl double copy for non-twisting vacuum type N and vacuum type D solutions in this paper. Our results are consistent with previous work [20, 22]. Moreover, we showed that the zeroth copy not only connects the gravity fields with the single copy but also connects DW fields with those degenerate electromagnetic fields living in the curved spacetime.

We also investigated the case of non-twisting vacuum type III solutions. Independent of the proposed map, we found that the square of a DW scalar is just proportional to the Weyl scalar \( \Psi_3 \). Such an interesting result produces a natural relationship between the gravity fields and the Maxwell fields in the flat spacetime, which is summarized as \( \Psi_3 = C S_{12}^{(2)} \phi_2 \), where \( S_{12}^{(2)} \) and \( \phi_2 \) correspond to a scalar field and a degenerate Maxwell field, respectively. Interestingly, both of them not only satisfy their field equation in curved spacetime but also in Minkowski spacetime. As an auxiliary scalar field associated with the degenerate electromagnetic field, it is not surprising that \( S_{12}^{(2)} \) plays a role similar to the zeroth copy considering our discovery in the cases of type N and type D solutions. However, why this scalar can play such an important role in connecting gravity theory with gauge theory is still unclear. On the whole, with the help of the chosen DW spinors, we systematically show that there indeed exists a deep connection between gravity theory and gauge theory by investigating non-twisting vacuum type N, III and vacuum type D solutions. The Weyl double copy proposed before is covered in the present work.

Next, it would be fascinating to study the case in non-vacuum spacetime using Dirac equation equation (2.45) instead of DW equation equation (2.40). The situation could be viewed as turning from a DW equation to DW equations with a source. In addition, so far, all of the works related to the Weyl double copy only focus on classical gravity solutions without a cosmological constant. Along the road of this work, it would be interesting to show a specific situation about the Weyl double copy for asymptotically (anti-)de Sitter spacetimes. In fact, we found that the Weyl double copy, in general, satisfies conformally invariant field equations even in conformally flat space times, which is consistent with the result of twistorial version of Weyl double copy [23]. Progress on this has been shown in another work [43].

In the end, we have to point out that although we have shown a natural map for type III cases between gravity fields and the Maxwell fields living in Minkowski spacetime, we did not prove if type III spacetime admits the classical Weyl double copy prescription. In terms of Kundt class with \( W_{\mu} = 0 \), we only shown that the DW scalar \( \eta \) does not depends on the
same coordinates as the Weyl scalar, unless we impose one more condition—\(H_{zz\bar{z}\bar{z}}^0 = 0\). If the Weyl double copy prescription does exist for vacuum type III solutions, the possible way to show it may start from regarding \(S^{(2)}_{12}\) as the zeroth copy. Then, it would be interesting to probe the physical meaning of the constant \(C\), since it corresponds to a field with a total spin of 1. Alternatively, we may need to extend the Weyl double copy to a more general form to cover even the twisting case. All in all, to get full knowledge about the relation between gravity theory and gauge theory, there is still a long way to go. We hope this paper provides new insights for a better understanding of double copy and the connection between gravity theory and gauge theory.

Acknowledgments

The author would like to thank Andrés Luna, Niels A Obers and Xin Qian for helpful discussions. The author also would like to thank Ricardo Monteiro and Niels A Obers for instructive comments on the manuscript. The author thanks the Theoretical Particle Physics and Cosmology section at the Niels Bohr Institute for support. This work is also financially supported by the China Scholarship Council.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

Shanzhong Han  https://orcid.org/0000-0002-4142-266X

References

[1] Aasi J et al (LIGO Scientific Collaboration) 2015 Advanced LIGO Class. Quantum Grav. 32 074001
[2] Abbott B P et al (LIGO Scientific Collaboration) 2017 Exploring the sensitivity of next generation gravitational wave detectors Class. Quantum Grav. 34 044001
[3] Kawai H, Lewellen D and Tye S-H 1986 A relation between tree amplitudes of closed and open strings Nucl. Phys. B 269 1–23
[4] Bern Z, Carrasco J J M and Johansson H 2008 New relations for gauge-theory amplitudes Phys. Rev. D 78 085011
[5] Bern Z, Carrasco J J M and Johansson H 2010 Perturbative quantum gravity as a double copy of gauge theory Phys. Rev. Lett. 105 061602
[6] Bern Z, Carrasco J J, Chiodaroli M, Johansson H and Roiban R 2019 The duality between color and kinematics and its applications (arXiv:1909.01358)
[7] Monteiro R, O’Connell D and White C D 2014 Black holes and the double copy J. High Energy Phys. JHEP12(2014)056
[8] Berman D S, Chacón E, Luna A and White C D 2019 The self-dual classical double copy, and the Eguchi–Hanson instanton J. High Energy Phys. JHEP01(2019)107
[9] Luna A, Monteiro R, O’Connell D and White C D 2015 The classical double copy for Taub-NUT spacetime Phys. Lett. B 750 272–7
[10] Ridgway A K and Wise M B 2016 Static spherically symmetric Kerr–Schild metrics and implications for the classical double copy Phys. Rev. D 94 044023
[11] White C D 2016 Exact solutions for the biadjoint scalar field Phys. Lett. B 763 365–9.
[12] Adamo T, Casali E, Mason L and Nekovar S 2018 Scattering on plane waves and the double copy Class. Quantum Grav. 35 015004
[13] De Smet P-J and White C D 2017 Extended solutions for the biadjoint scalar field Phys. Lett. B 775 163–7
[14] Bahjat-Abbas N, Luna A and White C D 2017 The Kerr–Schild double copy in curved spacetime J. High Energy Phys. JHEP12(2017)004
[15] Carrillo-González M, Penco R and Trodden M 2018 The classical double copy in maximally symmetric spacetimes J. High Energy Phys. JHEP04(2018)028
[16] Idelton A 2018 Screw-symmetric gravitational waves: a double copy of the vortex Phys. Lett. B 782 22–7
[17] Lee K 2018 Kerr–Schild double field theory and classical double copy J. High Energy Phys. JHEP01(2018)027
[18] Gürses M and Tekin B 2018 Classical double copy: Kerr–Schild–Kundt metrics from Yang–Mills theory Phys. Rev. D 98 126017
[19] Eloy G, Farnsworth K, Graesser M L and Herczeg G 2020 The Newman–Penrose map and the classical double copy J. High Energy Phys. JHEP12(2020)121
[20] Luna A, Monteiro R, Nicholson I and O’Connell D 2019 Type D spacetimes and the Weyl double copy Class. Quantum Grav. 36 065003
[21] Keeler C, Manton T and Monga N 2020 From Navier–Stokes to Maxwell via Einstein J. High Energy Phys. JHEP08(2020)147
[22] Godazgar H, Godazgar M, Monteiro R, Veiga D and Pope C N 2021 Weyl double copy for gravitational waves Phys. Rev. Lett. 126 101103
[23] White C D 2021 Twistorial foundation for the classical double copy Phys. Rev. Lett. 126 061602
[24] Chacón E, Nagy S and White C D 2021 The Weyl double copy from twistor space J. High Energy Phys. JHEP05(2021)123
[25] Chacón E, Luna A and White C D 2021 The double copy of the multipole expansion (arXiv:2108.07702)
[26] Adamo T and Kol U 2022 Classical double copy at null infinity Class. Quantum Grav. 39 105007
[27] Godazgar H, Godazgar M, Monteiro R, Veiga D and Pope C N 2021 Asymptotic Weyl double copy J. High Energy Phys. JHEP11(2021)126
[28] Easson D A, Manton T and Svesko A 2021 Sources in the Weyl double copy Phys. Rev. Lett. 127 271101
[29] Walker M and Penrose R 1970 On quadratic first integrals of the geodesic equations for type [22] spacetimes Commun. Math. Phys. 18 265–74
[30] Hughston L P, Penrose R, Sommers P and Walker M 1972 On a quadratic first integral for the charged particle orbits in the charged Kerr solution Commun. Math. Phys. 27 303–8
[31] Newman E and Penrose R 1962 An approach to gravitational radiation by a method of spin coefficients J. Math. Phys. 3 566–78
[32] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)
[33] Penrose R and Rindler W 1987 Spinors and Space-Time (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)
[34] Stewart J 1994 Advanced General Relativity (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press) https://doi.org/10.1017/CBO9780511564048
[35] Chandrasekhar S 1983 The Mathematical Theory of Black Holes (New York: Oxford University Press)
[36] Goldberg J and Sachs R 2009 Republication of: a theorem on Petrov types Gen. Relativ. Gravit. 41 433–44
[37] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2009 Exact Solutions of Einstein’s Field Equations (Cambridge: Cambridge University Press)
[38] Robinson I and Trautman A 1962 Some spherical gravitational waves in general relativity Proc. R. Soc. A 265 463–73
[39] Plebanski J and Demianski M 1976 Rotating, charged, and uniformly accelerating mass in general relativity Ann. Phys., NY 98 98–127
[40] Griffiths J B and Podolský J 2006 A new look at the Plebański–Demiański family of solutions Int. J. Mod. Phys. D \textbf{15} 335–69

[41] Plebański J F 1975 A class of solutions of Einstein–Maxwell equations Ann. Phys., NY \textbf{90} 196–255

[42] Pravda V, Pravdov A, Coley A and Milson R 2002 All spacetimes with vanishing curvature invariants Class. Quantum Grav. \textbf{19} 6213–36

[43] Han S 2022 The Weyl double copy in maximally symmetric spacetimes (arXiv:2205.08654)