A Discrete-Time, Time-Delayed Lur’e Model with Biased Self-Excited Oscillations

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Abstract—Self-excited systems arise in many applications, such as biochemical systems, mechanical systems with fluid-structure interaction, and fuel-driven systems with combustion dynamics. This paper presents a Lur’e model that exhibits biased self-excited oscillations under constant inputs. The model involves asymptotically stable linear dynamics, time delay, a washout filter, and a saturation nonlinearity. For all sufficiently large scalings of the loop transfer function, these components cause divergence under small signal levels and decay under large signal amplitudes, thus producing an oscillatory response. A bias-generation mechanism is used to specify the mean of the oscillation. The main contribution of the paper is a detailed analysis of a discrete-time version of this model.

I. INTRODUCTION

A self-excited system has the property that the input is constant but the response is oscillatory. Self-excited systems arise in numerous applications, such as biochemical systems, fluid-structure interaction, and combustion. The classical example of a self-excited system is the van der Pol oscillator, which has two states whose asymptotic response converges to a limit cycle. A self-excited system, however, may have an arbitrary number of states and need not possess a limit cycle. Overviews of self-excited systems are given in [1], [2], and applications to chemical and biochemical systems are discussed in [3]–[5]. Self-excited thermoacoustic oscillation in combustors is discussed in [6]–[8]. Self-excited oscillations of a tropical ocean-atmosphere system are discussed in [9]. Fluid-structure interaction and its role in aircraft wing flutter is discussed in [10]–[13]. Wind-induced self-excited motion and its role in the Tacoma Bridge collapse is discussed in [14].

Models of self-excited systems are typically derived in terms of the relevant physics of the application. From a systems perspective, the main interest is in understanding the features of the components of the system that give rise to self-sustained oscillations. Understanding these mechanisms can illuminate the relevant physics in specific domains and provide unity across various domains.

A unifying model for self-excited systems is a feedback loop involving linear and nonlinear elements; systems of this type are called Lur’e systems. Lur’e systems have been widely studied in the classical literature on stability theory [15]. Within the context of self-excited systems, Lur’e systems under various assumptions are considered in [2], [16]–[24]. Application to thermoacoustic oscillation in combustors is considered in [25]. Self-oscillating discrete-time systems are considered in [26]–[29].

Roughly speaking, self-excited oscillations arise from a combination of stabilizing and destabilizing effects. Destabilization at small signal levels causes the response to grow from the vicinity of an equilibrium, whereas stabilization at large signal levels causes the response to decay from large signal levels. In particular, negative damping at low signal levels and positive damping at high signal levels is the mechanism that gives rise to a limit cycle in the van der Pol oscillator [30, pp. 103–107]. Note that, although systems with limit-cycle oscillations are self-excited, the converse need not be true since the response of a self-excited system may oscillate without the trajectory reaching a limit cycle. Alternative mechanisms exist, however; for example, time delays are destabilizing, and Lur’e models with time delay have been extensively considered as models of self-excited systems [31].

The present paper considers a time-delayed Lur’e (TDL) model that exhibits self-excited oscillations. This model, which is illustrated in Figure 1, incorporates the following components:

i) Asymptotically stable linear dynamics.
ii) Time delay.
iii) A washout (that is, highpass) filter.
iv) A continuous, bounded nonlinearity $\mathcal{N}: \mathbb{R} \rightarrow \mathbb{R}$ that satisfies $\mathcal{N}(0) = 0$, is either nondecreasing or nonincreasing, and changes sign (positive to negative or vice versa) at the origin.
v) A bias-generation mechanism, which produces an offset in the oscillatory response that depends on the value of the constant external input.

A notable feature of this model is that self-oscillations are guaranteed to exist for asymptotically stable dynamics that are not necessarily passive as in [32]. We note that washout filters are used in [33] to achieve stabilization, whereas, in the present paper, they are used to create self-oscillations.

For this time-delay Lur’e model, the time-delay provides the...
oscillatory response of the model will be shown. The response include the following: 1) Lemma 2.1 and

IV extends the Lur’e model to include a bias-generation
response for sufficiently large values of the loop gain. Section
Lur’e model is shown to have an asymptotically oscillatory
totically stable. Section III extends the problem in Section
α of values of

The analysis and examples in the paper focus on a discrete-
time version of the time-delayed Lur’e model with the standard
saturation function. This setting simplifies the analysis of
solutions as well as the numerical simulations.

The contents of the paper are as follows. Section II considers
a discrete-time linear feedback model and analyzes the range of
values of α for which the closed-loop model is asympto-
tically stable. Section III extends the problem in Section
by including a saturation nonlinearity. This discrete-time
Lur’e model is shown to have an asymptotically oscillatory
response for sufficiently large values of the loop gain. Section
IV extends the Lur’e model to include a bias-generation
mechanism.

Preliminary results relating to the present paper appear in
[34]. Key differences between [34] and the present paper
include the following: 1) Lemma 2.1 and v of Theorem 2.2 are
not given in [34]; 2) due to limited space, no proofs are given
in [34]; and 3) the present paper includes several examples
that do not appear in [34].

Define \( \mathbb{Z} \triangleq \{ \ldots, -1, 0, 1, \ldots \} \), \( \mathbb{N} \triangleq \{ 0, 1, 2, \ldots \} \), and
\( \mathbb{P} \triangleq \{ 1, 2, \ldots \} \). For all polynomials \( p \), \( \text{spr}(p) \) denotes the max-
imum magnitude of all elements of \( \text{roots}(p) \). For all nonzero
\( z = x + jy \in \mathbb{C} \), where \( x \) and \( y \) are real, \( \arg z = \text{atan2}(y, x) \in (-\pi, \pi) \) denotes the principal angle of \( z \). Let \( P \triangleq N/D \) be a
transfer function with no zeros on the unit circle, where \( N \) and
\( D \) are coprime, \( n \triangleq \text{deg}N \) and \( m \triangleq \text{deg}D \). For all \( \theta \in [0, \pi] \),
writing \( P(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=1}^{m}(z-z_i)}{\sum_{i=1}^{n}(z-p_i)} \), where \( K \) is a nonzero
real number, \( \angle P(e^{j\theta}) \in \mathbb{R} \) denotes the unwrapped phase angle
of \( P \) evaluated at \( \theta \in (-\pi, \pi) \), such that

\[
\angle P(e^{i\theta}) = \sum_{i=1}^{m} \arg(e^{i\theta} - z_i) - \sum_{j=1}^{n} \arg(e^{i\theta} - p_j).
\]

Unlike \( \theta \mapsto \arg P(e^{i\theta}) \), which may be discontinuous on \([0, \pi]\), the function \( \theta \mapsto \angle P(e^{i\theta}) \) is \( C^1 \) on \([0, \pi]\). In addition, for all \( \theta \in [0, \pi] \), there exists \( r_\theta \in \mathbb{Z} \) such that \( \angle P(e^{i\theta}) = \arg P(e^{i\theta}) + 2\pi r_\theta \).

II. TIME-DELAYED LINEAR FEEDBACK MODEL

In this section we consider the discrete-time, time-delayed Lur’ë model shown in Figure 5, where \( \alpha \in \mathbb{R} \), \( G \) is a strictly proper asymptotically stable SISO transfer function with no zeros on the unit circle, \( G_d(z) = 1/z^d \) is a \( d \)-step delay, where \( d \in \mathbb{N} \), and \( W(z) = (z-1)/z \) is a washout (that is, highpass) filter. Let \( G = N/D \), where the polynomials \( N \) and \( D \) are coprime, \( D \) is monic, \( n \triangleq \deg D \), and \( m \triangleq \deg N \).

Let \((A,B,C,0)\) be a minimal realization of \( G \) whose internal state at step \( k \) is \( x_k \in \mathbb{R}^n \). Furthermore, consider the realization \((N_d,e_d,d,e_{1,d},0)\) of \( G_d \) with internal state \( x_{d,k} \in \mathbb{R}^d \), where \( N_d \) is the standard \( d \times d \) nilpotent matrix and \( e_{i,d} \) is the \( i \)-th column of the \( d \times d \) identity matrix \( I_d \). Finally, let \((0,1,-1,1)\) be a realization of \( W \) with internal state \( x_{t,k} \in \mathbb{R} \), and let \( \alpha \) be a real number that scales \( G \).

Then, the discrete-time, time-delayed linear feedback model shown in Figure 5 has the closed-loop dynamics

\[
\begin{bmatrix}
    x_{k+1} \\
    x_{d,k+1} \\
    x_{t,k+1}
\end{bmatrix}
=
\begin{bmatrix}
    A & \alpha & B^T \\
    0 & e_{d,d} & 0 \\
    0 & e_{1,d} & 0
\end{bmatrix}
\begin{bmatrix}
    x_k \\
    x_{d,k} \\
    x_{t,k}
\end{bmatrix}
= A_{d,1} x_k + B_{d,1} y_k,
\]

with output

\[
y_k = C x_k
\]

and internal signals

\[
y_{d,k} = e_{1,d}^T x_{d,k},
y_{t,k} = -x_{t,k} + y_{d,k}.
\]

For all \( d \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \), define

\[
L_{d,\alpha}(z) = \alpha G(z) W(z) G_d(z) = \frac{\alpha(z-1)N(z)}{z^{d+1}D(z)}.
\]

Furthermore, for all \( d \in \mathbb{N} \), define \( L_d = L_{d,1} = GWG_d \). Finally, for all \( d \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \), note that the closed-loop transfer function of the time-delayed linear feedback model is given by

\[
\frac{L_{d,\alpha}}{1-L_{d,\alpha}} = \frac{\alpha(z-1)N(z)}{p_{d,\alpha}(z)},
\]

where

\[
p_{d,\alpha}(z) = z^{d+1}D(z) - \alpha(z-1)N(z).
\]

Note that, for all \( \alpha \in \mathbb{R} \), \( 1 \) is not a root of \( p_{d,\alpha} \).

The following lemma is needed for the proof of Theorem 2.2.

**Lemma 2.1:** Let \( p \) and \( q \) be monic polynomials with real coefficients, assume that \( \deg q < \deg p \), assume that all of the roots of \( p \) are in the open unit disk, and, for all \( \alpha \in \mathbb{R} \), define \( p_\alpha = p + \alpha q \). Then, there exist \( \alpha_n < 0, \alpha_p > 0, \) and \( \delta > 0 \) such that \( \text{spr}(p_\alpha) = \text{spr}(p_{\alpha_p}) = 1 \), for all \( \alpha \in (\alpha_n, \alpha_n + \delta) \cup (\alpha_p - \delta, \alpha_p) \), \( \text{spr}(p_\alpha) < 1 \), and, for all \( \alpha \in (\alpha_n - \delta, \alpha_n) \cup (\alpha_p, \alpha_p + \delta) \), \( \text{spr}(p_\alpha) > 1 \).

**Proof.** Let \( k \) be the smallest positive integer such that

\[
h(x) = x^k [p(x)q(1/x) - q(x)p(1/x)]
\]

is a polynomial, and define

\[
\mathcal{Z} = \{ z \in \mathbb{C} : |z| = 1, h(z) = 0, \text{ and } q(z) \neq 0 \}.
\]

Note that \( \mathcal{Z} \) has at most \( \deg h \) elements. Furthermore, since \( h(z) = 0 \) for all \( z \in \mathcal{Z} \) and \( p \) and \( q \) have real coefficients, it follows that, for all \( z \in \mathbb{C} \),

\[
\frac{p(z)}{q(z)} = \frac{p(z)}{q(z)} = \frac{p(1/z)}{q(1/z)} = \frac{p(z)}{q(z)}
\]

which implies that, for all \( z \in \mathcal{Z} \), \( p(z)/q(z) \in \mathbb{R} \).

Next, define \( \mathcal{A} = \{ -p(z)/q(z) : z \in \mathcal{Z} \} \), and, in the case where \( \mathcal{A} \) is not empty, let \( \mathcal{A} = \{ \alpha_1, \ldots, \alpha_m \} \), where \( m \leq \deg h \) and \( \alpha_1 < \cdots < \alpha_m \). Note that, since, for all \( z \in \mathcal{Z} \), \( p(z) \neq 0 \), it follows that \( 0 \notin \mathcal{A} \). Now, let \( \alpha \) be a real number that is not contained in \( \mathcal{A} \), and suppose that \( \text{spr}(p_\alpha) = 1 \). Then, there exists \( z_\alpha \in \mathcal{C} \) such that \( p_\alpha(z_\alpha) = 0 \) and \( |z_\alpha| = 1 \).

To show that \( q(z_\alpha) \neq 0 \), suppose that \( q(z_\alpha) = 0 \). Then, since \( p_\alpha(z_\alpha) = 0 \), it follows that \( p(z_\alpha) = 0 \), which, since all of the roots of \( p \) are in the open unit disk, implies that \( \text{spr}(p_\alpha) < 1 \), which is a contradiction. Hence, \( q(z_\alpha) \neq 0 \).

Next, to show that \( \text{spr}(p_{\alpha}) \neq 1 \), note that \( p_\alpha(z_\alpha) = 0 \) implies

\[
0 = p_\alpha(z_\alpha) = p(z_\alpha) + \alpha q(z_\alpha) = p(z_\alpha) + \alpha q(z_\alpha) = p(z_\alpha) + \alpha q(z_\alpha) = p(1/z_\alpha) + \alpha q(1/z_\alpha).
\]

Since, in addition, \( \alpha = -p(z_\alpha)/q(z_\alpha) \), it follows from (8) that

\[
p(1/z_\alpha) - (p(z_\alpha)/q(z_\alpha)q(1/z_\alpha)) = 0.
\]

Now, multiplying both sides of (9) by \( q(z_\alpha) \) implies

\[
q(z_\alpha)p(1/z_\alpha) - q(z_\alpha)q(1/z_\alpha) = -h(z_\alpha)/z_\alpha^k = 0,
\]
and thus $h(z_{\alpha}) = 0$. Hence, $z_{\alpha} \in \mathbb{Z}$, and thus $\alpha = -p(z_{\alpha})/q(z_{\alpha}) \in \mathcal{A}$, which is a contradiction. Therefore, for all $\alpha \notin \mathcal{A}$, $\text{spr}(p_{\alpha}) \neq 1$.

Next, let $j \in \{0, 1, \ldots, m\}$, and define $I_j \equiv (\alpha_j, \alpha_{j+1})$, where $\alpha_0 \equiv -\infty$ and $\alpha_{m+1} \equiv \infty$. For all $\alpha \in I_j$, it follows from the continuity of $\alpha \mapsto \text{spr}(p_{\alpha})$ that either $\text{spr}(p_{\alpha}) < 1$ or $\text{spr}(p_{\alpha}) > 1$.

Next, write

$$p(x) = a_0 x^n + \cdots + a_0, \quad q(x) = b_0 x^d + \cdots + b_0$$

such that $b_d \neq 0$, $a_\alpha 

0$, and $d < n$, and let $z_{1, \alpha}, \ldots, z_{n, \alpha}$ be the roots of $p_{\alpha}$. Then, the coefficient of $x^d$ in $p_{\alpha}$ is related to the roots of $p_{\alpha}$ by

$$a_d + \alpha b_d = a_n (-1)^{n-d} \prod_{j \in B} z_{j, \alpha},$$

where the sum is taken over all $(n-d)$ subsets $B$ of $\{1, \ldots, n\}$ with $n - d$ elements. It thus follows from (10) that

$$|a_d + \alpha b_d| \leq |a_n| \left( \frac{n}{n-d} \right) \text{spr}(p_{\alpha})^{n-d},$$

which implies that

$$\lim_{\alpha \to -\infty} \text{spr}(p_{\alpha}) = \lim_{\alpha \to \infty} \text{spr}(p_{\alpha}) = \infty.$$ 

Hence, for all $\alpha \in I_0 \cup I_m$, $\text{spr}(p_{\alpha}) > 1$.

Next, since $\text{spr}(p_0) = \text{spr}(p) < 1$, $0 \notin \mathcal{A}$, and, for all $\alpha \in I_0 \cup I_m$, $\text{spr}(p_{\alpha}) > 1$, it follows that there exists a unique $j_0 \in \{1, \ldots, m-1\}$ such that $0 \in I_{j_0}$. Hence, for all $\alpha \in I_{j_0}$, $\text{spr}(p_{\alpha}) < 1$. Now, define

$$j_n \equiv \min\{j \in \{1, \ldots, m-1\} : \text{spr}(p_{\alpha}) < 1\} \quad \text{for all } \alpha \in I_j,$$

$$j_p \equiv \max\{j \in \{1, \ldots, m-1\} : \text{spr}(p_{\alpha}) < 1\} \quad \text{for all } \alpha \in I_j.$$ 

Then, for all $\alpha \in I_{j_0} \cup I_{j_p}$, $\text{spr}(p_{\alpha}) < 1$ and, for all $\alpha \in I_{j_n-1} \cup I_{j_p+1}$, $\text{spr}(p_{\alpha}) > 1$, and thus it follows from the continuity of $\text{spr}$ and the intermediate value theorem that $\text{spr}(p_{\alpha_{j_0}}) = \text{spr}(p_{\alpha_{j_0+1}}) = 1$. Furthermore, since $j_n \leq j_0 \leq j_p$, it follows that $\alpha_{j_n} < 0$ and $\alpha_{j_n+1} > 0$. Hence, defining $a_n \equiv \alpha_{j_n}$ and $a_p \equiv \alpha_{j_{n+1}}$, which, as an aside, shows that $\mathcal{A}$ has at least two elements, it follows that $\alpha_0 < 0$, $\alpha_p > 0$, and $\text{spr}(p_{\alpha_0}) = \text{spr}(p_{a_0}) = 1$, and, furthermore, there exists $\delta > 0$ such that, for all $\alpha \in (a_\alpha, a_\alpha + \delta) \cup (a_p - \delta, a_p)$, $\text{spr}(p_{\alpha}) < 1$, and, for all $\alpha \in (a_\alpha - \delta, a_\alpha) \cup (a_p, a_p + \delta)$, $\text{spr}(p_{\alpha}) > 1$, which completes the proof.

The following result shows that, for sufficiently large values of the delay $d$, the linear closed-loop system is not asymptotically stable outside of a bounded interval of values of $\alpha$. This result also shows that, for asymptotically large $d$, this range of values of $\alpha$ is finite and symmetric.

**Theorem 2.2:** The following statements hold:

i) For all $d \in \mathbb{N}$, there exist $\alpha_{d,0}, \alpha_d, \alpha_{d,1} > 0$ such that $\alpha_{d,0} < \alpha_d < \alpha_{d,1}$, $\text{spr}(p_{d,\alpha_0}) = 1$, for all $\alpha \in (\alpha_{d,0}, \alpha_d)$, $\text{spr}(p_{d,\alpha}) < 1$, and, for all $\alpha \in (\alpha_d, \alpha_{d,1})$, $\text{spr}(p_{d,\alpha}) > 1$.

ii) For all $d \in \mathbb{N}$, there exist $\alpha_{d,0}, \alpha_d, \alpha_{d,1} < 0$ such that $\alpha_{d,0} < \alpha_d < \alpha_{d,1}$, $\text{spr}(p_{d,\alpha_1}) = 1$, for all $\alpha \in (\alpha_{d,0}, \alpha_d)$, $\text{spr}(p_{d,\alpha}) < 1$, and, for all $\alpha \in (\alpha_d, \alpha_{d,1})$, $\text{spr}(p_{d,\alpha}) > 1$.

Furthermore, there exists $\tilde{d} \in \mathbb{N}$ such that the following statements hold:

iii) For all $d > \tilde{d}$ and $\theta \in (0, \pi)$, $L_d(e^{\theta}) \neq 0$ and

$$\frac{d}{d\theta} L_d(e^{\theta}) < 0.$$ 

(11)

iv) For all $d > \tilde{d}$, there exist $\alpha_{d,1} < 0$ and $\alpha_{d,1} > 0$ such that $p_{d,\alpha}$ is asymptotically stable if and only if $\alpha \in (\alpha_{d,1}, \alpha_{d,1})$, and $p_{d,\alpha}$ is not asymptotically stable if and only if $\alpha \in (-\infty, \alpha_{d,1}] \cup [\alpha_{d,1}, \infty)$.

v) Define

$$\alpha_\infty \equiv \min_{\theta \in (0, \pi)} \left| \frac{D(e^{\theta})}{(e^{\theta} - 1)N(e^{\theta})} \right|.$$ 

Then, $\alpha_\infty > 0$, for all $d > \tilde{d}$, $\alpha_\infty \leq \min\{-\alpha_{d,1}, \alpha_{d,1}\}$, and

$$\lim_{d \to \infty} -\alpha_{d,1} = \lim_{d \to \infty} \alpha_{d,1} = \alpha_\infty.$$ 

(12)

**Proof.** i) and ii) follow from Lemma 2.1. To prove iii), note that, for all $\theta \in (0, \pi)$, $G(e^{\theta}) \neq 0$, $W(e^{\theta}) \neq 0$, and $G_d(e^{\theta}) \neq 0$, and thus $L_d(e^{\theta}) \neq 0$. Next, let $\theta \in (0, \pi]$, and note that

$$\sin \theta \quad = \quad \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} - (\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2})} = \frac{\cos \frac{\theta}{2}}{-\sin \frac{\theta}{2}} = \frac{\sin \frac{\pi}{2} + \theta}{\cos \frac{\pi}{2} + \theta},$$

which implies

$$\arg(\cos \theta - 1 + j \sin \theta) = \arg(\cos \frac{\pi + \theta}{2} + j \sin \frac{\pi + \theta}{2}).$$

Hence

$$\angle W(e^{\theta}) = \arg(e^{\theta} - 1) - \arg(e^{\theta}) = \arg(\cos \theta - 1 + j \sin \theta) - \theta = \arg(\cos \frac{\pi + \theta}{2} + j \sin \frac{\pi + \theta}{2}) - \theta = \frac{\pi + \theta}{2} - \theta = \frac{\pi}{2} + \theta.$$ 

(13)

Next, letting $\tilde{d} \in \mathbb{N}$, it follows from (14) that

$$\angle L_d(e^{\theta}) = \angle G(e^{\theta}) + \angle W(e^{\theta}) + \angle G_d(e^{\theta}) = \angle G(e^{\theta}) + \frac{\pi}{2} - \frac{\theta}{2} - \tilde{d} \theta = \angle G(e^{\theta}) + \frac{\pi}{2} - (\tilde{d} + \frac{1}{2}) \theta.$$ 

(15)

Now, let $\tilde{d} \in \mathbb{N}$ satisfy

$$\max_{\theta \in [0, \pi]} \frac{d}{d\theta} \angle G(e^{\theta}) \leq \tilde{d} + \frac{1}{2}.$$
Therefore, for all $\theta \in (0, \pi]$ and $d > \bar{d}$,
\[
\frac{d}{d\theta} \angle L_d(e^{i\theta}) = \frac{d}{d\theta} \angle G(e^{i\theta}) - d - \frac{1}{2} \\
\leq \max_{\theta \in (0, \pi]} \frac{d}{d\theta} \angle G(e^{i\theta}) - d - \frac{1}{2} \\
\leq d + \frac{1}{2} - d - \frac{1}{2} < 0.
\]

To prove iv), note that iii) implies that, for all $d > \bar{d}$, $\angle L_d(e^{i\theta})$ is a decreasing function of $\theta$ on $(0, \pi]$. Hence, for all $\alpha > 0$, all crossings of the positive real axis by the Nyquist plot of $L_{d,\alpha}(e^{i\theta}) = \alpha L_d(e^{i\theta})$ as $\theta$ increases over the interval $(-\pi, \pi]$ occur from the first quadrant to the fourth quadrant. Next, note that, for all $d > \bar{d}$ and $\theta \in (0, \pi]$, $|L_{d,\alpha}(e^{i\theta})| = |\alpha| |L_d(e^{i\theta})|$ is a decreasing function of $\alpha$ on $(0, \infty)$, and that, for all $d > \bar{d}$ and $\alpha > 0$, since all of the poles of $L_{d,\alpha}$ are in the open unit disk, it follows that $\text{spr}(p_{d,\alpha}) > 1$ if and only if the number of clockwise encirclements of $1 + 0j$ of the Nyquist plot of $L_{d,\alpha}(e^{i\theta})$ over $\theta \in (-\pi, \pi]$ is at least one. Therefore, for all $d > \bar{d}$ and $\alpha_0, \alpha_1 > 0$ such that $\text{spr}(p_{d,\alpha_0}) > 1$ and $\alpha_0 > \alpha_1$, the Nyquist plot of $L_{d,\alpha_1}(e^{i\theta})$ over $\theta \in (-\pi, \pi]$ has at least one clockwise encirclement of $1 + 0j$. Furthermore, for all $d > \bar{d}$ and $\alpha_0, \alpha_1 > 0$ such that $\text{spr}(p_{d,\alpha_0}) < 1$ and $\alpha_1 < \alpha_0$, the Nyquist plot of $L_{d,\alpha_1}(e^{i\theta})$ over $\theta \in (-\pi, \pi]$ has zero encirclements of $1 + 0j$. Hence, i) implies that there exists a unique $\alpha_{d,u} > 0$ such that $\text{spr}(p_{d,\alpha_{d,u}}) = 1$, for all $\alpha \in [0, \alpha_{d,u})$, $\text{spr}(p_{d,\alpha}) < 1$, and, for all $\alpha \in [\alpha_{d,u}, \infty)$, $\text{spr}(p_{d,\alpha}) \geq 1$. Similarly, ii) implies that there exists a unique $\alpha_{d,l} < 0$ such that $\text{spr}(p_{d,\alpha_{d,l}}) = 1$, for all $\alpha \in (\alpha_{d,l}, 0]$, $\text{spr}(p_{d,\alpha}) < 1$ and, for all $\alpha \in (-\infty, \alpha_{d,l})$, $\text{spr}(p_{d,\alpha}) > 1$. Hence, iv) holds.

To prove v), let $\alpha \in \mathbb{R}$ and $d \geq \bar{d}$. Note that roots($p_{d,\alpha}$) has at most $n + d + 1$ elements and that $\lambda \in \text{roots}(p_{d,\alpha})$ if and only if $L_{d,\alpha}(\lambda) = 1$. Now, let $\lambda = \rho e^{i\theta} \in \text{roots}(p_{d,\alpha})$, where $\rho \in [0, \infty)$ and $\theta \in (-\pi, \pi]$. Writing $G(z) = \frac{K}{\prod_{k=1}^{n} |\lambda - z_k|}$, it follows from $L_{d,\alpha}(\lambda) = 1$ that
\[
|\alpha| = \frac{|\lambda|^{d+1} |\prod_{k=1}^{n} |\lambda - p_k|}{|K||\lambda - 1| |\prod_{k=1}^{n} |\lambda - z_k||} \\
= \rho^{d+n-m} |\prod_{k=1}^{n} |\lambda - p_k| |K||\lambda - 1| |\prod_{k=1}^{n} |\lambda - z_k||}
\]
\[
\geq \max_{\theta \in (0, \pi]} \frac{d}{d\theta} \angle G(e^{i\theta}) - d - \frac{1}{2} \\
\leq d + \frac{1}{2} - d - \frac{1}{2} < 0.
\]

Since $g$ is continuous and $\lim_{\theta_0} g(\theta) = \infty$, it follows that $g$ has a global minimizer. Hence, define the set of minimizers of $g$ by
\[
\mathcal{M} \triangleq \{ \theta \in (0, \pi] : g(\theta) = \alpha_{\infty} \},
\]
where
\[
\alpha_{\infty} = \min_{\theta \in (0, \pi]} g(\theta) \\
= \min_{\theta \in (0, \pi]} \frac{\prod_{k=1}^{n} |\lambda - p_k|}{|K||\lambda - 1| |\prod_{k=1}^{n} |\lambda - z_k||} \\
= \min_{\theta \in (0, \pi]} \left| \frac{D(e^{i\theta})}{(e^{i\theta} - 1)N(e^{i\theta})} \right|.
\]

Hence, the minimum in (12) exists, is positive, and is independent of $d$. Furthermore, for all $\alpha \in \mathcal{A}_{d,+}$, $\alpha_{\infty} \leq g(\theta_{\alpha}) = \alpha$, and thus $\alpha_{\infty} \leq \min \mathcal{A}_{d,+}$.

Next, we show that there exists $\eta_0 \in \mathbb{Z}$ such that $\angle L_{d}(1 + 2\eta_0 \pi \in (0, \pi]$ and that, for all $d \geq \bar{d}$, there exists $\eta_0 \in \mathbb{Z}$ such that $\angle L_{d}(e^{i\theta}) + 2\eta_0 \pi \in [-\pi, 0]$. We now consider the case where $G(1) > 0$; the case $G(1) < 0$ is addressed by the case where $\alpha < 0$. Let $\eta_0 \in \mathbb{Z}$ satisfy $\angle G(1 + 2\eta_0 \pi = 0$, and note that (15) implies that, for all $d \geq \bar{d}$, $\angle L_{d}(1) \triangleq \lim_{\theta_0 \to 0} \angle L_{d}(e^{i\theta}) = \frac{\pi}{2} + \angle G(1)$. It thus follows that
\[
\angle L_{d}(1 + 2\eta_0 \pi = \frac{\pi}{2} \in (0, \pi].
\]
Thus, since $\theta$ is a theorem that, for all $d, \alpha$, $\pi$.

Then, let $\theta_r, d, \alpha, \pi$. (1) + 2(\theta_r, d, \alpha, \pi). (28) implies that $\theta_r, d, \alpha, \pi$. (34) that, for all $\theta \in (0, \pi]$, there exists $d_{\theta, r} \geq d$ such that, for all $d \geq d_{\theta, r}$,

Hence, for all $\theta \in (0, \pi)$,

$$-2\pi - 2(\angle G(\theta_{r, d}, +) + 2\pi r_m) + \pi - \theta \leq \theta_{r, d} - \theta,$$

which implies that there exists $d_{\theta, r} \geq d$ such that, for all $d \geq d_{\theta, r}$,

$$\theta_{r, d} > \theta.$$ (36) Furthermore, (35) implies that

$$\pi \geq \lim_{d \to \infty} \theta_{r, d} \geq \pi - \lim_{d \to \infty} \frac{2\pi - 2(\angle G(\theta_{r, d}, +) + 2\pi r_m)}{2d + 1} = \pi.$$ (37)

Hence,

$$\lim_{d \to \infty} \theta_{r, d} = \pi.$$ (38)

Next, let $\theta_r \in M$. We first consider the case where $\theta_r \in (0, \pi)$. It follows from (34) and (36) with $\theta = \theta_r$ that, for all $d \geq \max\{d_{\theta, r}, d_{\theta, r, r}\}$, there exists $r \in \{r_0 + 1, \ldots, r_d\}$ such that $\theta_{r-1, d} \leq \theta \leq \theta_{r, d}$.
It follows from (32) and (38) that, for all \( \varepsilon > 0 \), there exists \( d_{\theta_{\infty},m} \geq \max\{d_{\theta_{\infty},1}, d_{\theta_{\infty},r}\} \) such that, for all \( d \geq d_{\theta_{\infty},m} \), there exists \( r \in \{r_0 + 1, \ldots, r_d\} \) such that
\[
0 \leq \theta_{\infty} - \theta_{r-1,d} \leq \theta_{r,d} - \theta_{r-1,d} < \varepsilon
\]  
and
\[
0 \leq \theta_{r,d} - \theta_{\infty} \leq \theta_{r,d} - \theta_{r-1,d} < \varepsilon.
\]  
Now, for all \( d \geq 0 \), define
\[
\psi_{\theta_{\infty},d} = \arg\min_{\theta \in \mathbb{C}_{d+1}} |\theta_{\infty} - \theta| \in (0, \pi].
\]
It follows from (38) that, for all \( d \geq \max\{d_{\theta_{\infty},1}, d_{\theta_{\infty},r}\} \), there exists \( r \in \{r_0 + 1, \ldots, r_d\} \) such that
\[
\psi_{\theta_{\infty},d} \in (\theta_{r-1,d}, \theta_{r,d}),
\]
and thus, for all \( \varepsilon > 0 \), (39) and (40) imply
\[
|\psi_{\theta_{\infty},d} - \theta_{\infty}| \in |\theta_{\infty} - \theta_{r-1,d}, \theta_{r,d} - \theta_{\infty}| < \varepsilon.
\]
Hence,
\[
\lim_{d \to \infty} \psi_{\theta_{\infty},d} = \theta_{\infty} \in (0, \pi).
\]  
In the case where \( \theta_{\infty} = \pi \), (37) implies
\[
\lim_{d \to \infty} \psi_{\pi,d} = \pi.
\]  
Hence, (41) and (42) imply
\[
\lim_{d \to \infty} \psi_{\theta_{\infty},d} = \theta_{\infty} \in (0, \pi].
\]  
Since (17) implies that, for all \( d \geq \tilde{d} \), \( d_{\alpha,1} = \min_{\theta \in \mathbb{C}_{\tilde{d}+1}} g(\theta) \), and, for all \( \theta_{\infty} \in M, \alpha_{\infty} = g(\theta_{\infty}) \), it follows from (43) that
\[
\lim_{d \to \infty} d_{\alpha,1} = \alpha_{\infty}.
\]  
Similarly, in the case where \( \alpha < 0 \),
\[
\lim_{d \to \infty} -\alpha_{d,1} = \alpha_{\infty}.
\]  
Finally, (44) and (45) imply
\[
\lim_{d \to \infty} -\alpha_{d,1} = \lim_{d \to \infty} d_{\alpha,1} = \min_{\theta \in (0,\pi]} \left| \frac{D(e^\theta)}{(e^\theta - 1)N(e^\theta)} \right|.
\]

**Proposition 2.3:** Let \( \alpha \in \mathbb{R}, d \geq 0, \text{ and } \theta \in (0, \pi] \), and assume that \( p_{d,\theta}(e^\theta) = 0 \). Then,
\[
\alpha = \frac{e^{i(d+1)\theta}}{(e^\theta - 1)G(e^\theta)}.
\]  
Furthermore, writing \( G^{-1}(e^\theta) = a + bj \), where \( a, b \in \mathbb{R} \), it follows that
\[
b = -a \frac{\sin \theta - \sin(d + 1)\theta}{\cos \theta - \cos(d + 1)\theta}
\]  
and
\[
\alpha = \frac{a}{\cos \theta - \cos(d + 1)\theta}.
\]

**Proof.** (46) follows from \( p_{d,\alpha}(e^\theta) = 0 \). Furthermore, (46) implies that
\[
\alpha = \frac{(\cos \theta - \cos(d + 1)\theta) + j(\sin \theta - \sin(d + 1)\theta)G^{-1}(e^\theta)}{2 - 2\cos(\theta)}
\]  
and thus
\[
\alpha = \frac{f + jg}{2 - 2\cos \theta},
\]
where
\[
f \triangleq a[\cos \theta - \cos(d + 1)\theta] - b[\sin \theta - \sin(d + 1)\theta],
\]
\[
g \triangleq b[\cos \theta - \cos(d + 1)\theta] + a[\sin \theta - \sin(d + 1)\theta].
\]
Since \( \alpha \) is real, (50) implies that \( g = 0 \), and thus (52) implies (47). Next, combining (47) with (51) yields
\[
f = a\frac{(\cos \theta - \cos(d + 1)\theta)^2 + \sin \theta - \sin(d + 1)\theta)^2}{\cos \theta - \cos(d + 1)\theta} = a\frac{2 - 2\cos \theta (\cos(d + 1)\theta - 2\sin \theta \sin(d + 1)\theta)}{\cos \theta - \cos(d + 1)\theta} = a\frac{2 - 2\cos \theta}{\cos \theta - \cos(d + 1)\theta}.
\]
Finally, combining \( g = 0 \) and (53) with (50) yields (48). \( \square \)

**Example 2.4:** Let \( G(z) = \frac{2z}{z^2 + p}, \text{ where } p \in (-1, 1), \text{ and } \) \( e^\theta \neq 1, \text{ where } \theta \in (0, \pi], \text{ be a root of } p_{d,\alpha} \text{ on the unit circle.} \)

Writing \( G^{-1}(e^\theta) = a + bj \), it follows that \( a = \cos \theta + p \) and \( b = \sin \theta \)., and (46) and (48) have the form
\[
\alpha(\theta) = \frac{e^{i(d+2)\theta} + pe^{i(d+1)\theta}}{e^\theta - 1} = \frac{\cos \theta + p}{\cos \theta - \cos(d + 1)\theta},
\]
which implies
\[
|\alpha(\theta)| = \sqrt{\frac{p^2 + 2p\cos \theta + 1}{2 - 2\cos \theta}}.
\]  
Furthermore, it follows from (47) that
\[
\sin(d + 2)\theta = (1 - p)\sin(d + 1)\theta + p\sin d\theta.
\]  
Since \( L_d \) has \( d + 2 \) poles in the open unit disk and one zero at 1, it follows that there exist exactly \( d + 1 \) distinct values \( \theta_1, \ldots, \theta_{d+1} \) of \( \theta \in [0, \pi) \) that satisfy (56). The corresponding values of \( \alpha(\theta_i) \) are given by
\[
\alpha(\theta_i) = \frac{\cos \theta_i + p}{\cos \theta_i - \cos(d + 1)\theta_i} = \frac{-\cos(d + 2)\theta_i + (1 - p)\cos(d + 1)\theta_i + p\cos d\theta_i}{2 - 2\cos \theta_i}.
\]  
Next, (v) in Theorem 2.2 and (55) imply that
\[
\alpha_{\infty} = \min_{\theta \in (0, \pi]} \frac{|e^\theta + p|}{|e^\theta - 1|} = \min_{\theta \in (0, \pi]} \sqrt{\frac{p^2 + 2p\cos \theta + 1}{2 - 2\cos \theta}}.
\]
Hence, it follows from (55) and (58) that
\[
\alpha_{\infty} = \min_{\theta \in (0, \pi]} |\alpha(\theta)|.
\]
Letting $\theta^* \in (0, \pi]$ be a minimizer of (55), it follows that
\[
\frac{d|\alpha|}{d\theta} \bigg|_{\theta=\theta^*} = -\frac{1}{2|\alpha(\theta^*)|} \frac{\sin \theta^2 (2p^2 + 4p + 2)}{(2 - 2 \cos \theta^2)^2} = 0,
\]
which implies that $\theta^* = \pi$. Hence, (58) implies
\[
\alpha_{\infty} = \frac{1 - p}{2} \in (0, 1).
\]

For $p = \frac{1}{2}$, $d = 6$, and $d = 7$, Figure 6 shows $\alpha(\theta)$ and $|\alpha(\theta_i)|$ versus $\theta_i$. Note that, for both values of $d$, the minimum value of $|\alpha(\theta_i)|$ is $\alpha_{\infty} = \frac{1}{2}$, as stated by (61), which occurs at $\theta = \pi$. Finally, Figure 7 shows $\alpha_{d,1}$ and $\alpha_{d,u}$ versus $d$ for $p = 0.5$, which indicates that $\lim_{d \to \infty} -\alpha_{d,1} = \lim_{d \to \infty} \alpha_{d,u} = \alpha_{\infty}$, as stated in (13).

**Special case:** For $p = 0$, (55) becomes
\[
|\alpha(\theta)| = \frac{1}{\sqrt{2 - 2 \cos \theta}},
\]
and (56) becomes
\[
\sin (d + 1)\theta = \sin (d + 2)\theta.
\]
Note that, for all $i \in \mathbb{Z}$, $\sin(d+1)\theta = \sin[(2i+1)\pi -(d+1)\theta]$. Therefore, (63) holds if and only if $\theta = \frac{2k+1}{2d+3} \pi$. Hence, $\theta \in [0, \pi]$ satisfies (63) if and only if there exists $i \in \{0, \ldots, d+1\}$ such that $\theta_k = \frac{(2i+1)\pi}{2d+3}$. For these $d + 2$ values of $\theta$, (57) implies that the corresponding values of $\alpha(\theta)$ are given by
\[
\alpha(\theta_i) = \frac{\cos \theta_i}{\cos d\theta_i - \cos(d + 1)\theta_i} = \frac{\cos(d + 1)\theta_i - \cos(d + 2)\theta_i}{2 - 2 \cos \theta_i}.
\]

Next, it can be shown that, for all $i \in \{1, \ldots, d\}$, $\alpha(\theta_i)\alpha(\theta_{i+1}) < 0$. Note that $\theta_{d+1} = \pi$ and $\alpha(\theta_{d+1}) = (-1)^{d+1} \frac{1}{2}$. Hence, $|\alpha(\theta_{d+1})| = \frac{1}{2}$. Furthermore, in the case where $d$ is even, $\alpha_{d,1} = \alpha(\theta_{d+1}) = -\frac{1}{2} < 0$ and $\alpha_{d,u} = \alpha_{d,u} > \frac{1}{2} > 0$, whereas, in the case where $d$ is odd, $\alpha_{d,1} = \alpha(\theta_{d+1}) = -\frac{1}{2} < 0$ and $\alpha_{d,u} = \alpha_{d,u} > \frac{1}{2} > 0$. In addition, although $\lim_{d \to \infty} \alpha(\theta_d)$ does not exist, it follows from (62) that $\lim_{d \to \infty} |\alpha(\theta_d)| = \lim_{d \to \infty} \sqrt{2 - 2 \cos \left(\frac{2d+1}{2d+3}\pi\right)} = \frac{1}{2}$, which confirms (13) and (61). For $d = 10$ and $d = 11$, Figure 8 shows $\alpha(\theta)$ and $|\alpha(\theta_i)|$ versus $\theta_i$. Note that, for both values of $d$, the minimum value of $|\alpha(\theta_i)|$ is $\frac{1}{2}$, which occurs at $\theta = \pi$. Finally, Figure 9 shows $\alpha_{d,1}$ and $\alpha_{d,u}$ versus $d$, which indicates that $\lim_{d \to \infty} \alpha_{d,1} = -\frac{1}{2}$ and $\lim_{d \to \infty} \alpha_{d,u} = \frac{1}{2}$.

**Example 2.5:** Let
\[
G(z) = \frac{N(z)}{D(z)} = \frac{z + 0.2 \pm 0.79}{z(z - 0.25 \pm 0.95)}.
\]
Figure 10 shows that, for all $d \geq 1$, there exists $\alpha_{d,1} < 0$ such that $p_{d,\alpha}$, if and only if $\alpha \in (\alpha_{d,1}, 0]$ as stated in $iv$) from Theorem 2.2. Furthermore, define

$$\alpha_{uc}(\theta) = \frac{D(e^{\theta})}{(1 - e^{\theta}) N(e^{\theta})}$$

such that $\alpha_{\infty} = \min_{\theta \in [0, \pi]} \alpha_{uc}(\theta)$. Figure 11 shows that $\alpha_{uc}$ has a minimum at $\theta \approx 0.4180\pi$, which implies that $\alpha_{\infty} \approx 0.0313$. Finally, Figure 12 shows $\alpha_{d,1}$ and $\alpha_{d,u}$ versus $d \geq 1$, which shows that $\lim_{d \to \infty} \alpha_{d,1} = -\alpha_{\infty}$ and $\lim_{d \to \infty} \alpha_{d,u} = \alpha_{\infty}$, as stated in $iv$) from Theorem 2.2.

To analyze the self-oscillating behavior of the time-delayed Lur’e model, we replace the saturation nonlinearity by its describing function. Describing functions are used to characterize self-excited oscillations in [2] Section 5.4] and [15, pp. 293–294]. The describing function $\Psi_\delta(e)$ for $\text{sat}_\delta$ for a sinusoidal input with amplitude $\varepsilon > 0$ is given by

$$\Psi_\delta(e) = \frac{2}{\pi} \left[ \sin^{-1} \left( \frac{\varepsilon}{\delta} \right) + \left( \frac{\varepsilon}{\delta} \right) \sqrt{1 - \left( \frac{\varepsilon}{\delta} \right)^2} \right], \quad \text{if } \varepsilon > \delta,$$

otherwise.

Note that, for $\varepsilon > \delta$, the function $\Psi_\delta$ confined to $(\delta, \infty)$ with codomain $(0, 1)$ is decreasing, one-to-one, and onto. Let $p_{d,\alpha,e}$ be the characteristic polynomial of the linearized time-delay Lur’e model, such that

$$p_{d,\alpha,e}(z) \triangleq z^{d+1} D(z) - \alpha \Psi_\delta(e)(z - 1) N(z).$$

For all $\varepsilon_1 > 0$, $\varepsilon_u > 0$, $\theta_1 \in \mathbb{R}$, and $\theta_u \in \mathbb{R}$ such that $\varepsilon_1 < \varepsilon_u$ and $\theta_1 < \theta_u$, define the rectangle

$$\Gamma_{\theta_1, \theta_u, \varepsilon_1, \varepsilon_u} \triangleq \{ (\theta, \varepsilon) : \theta_1 < \theta < \theta_u \text{ and } \varepsilon_1 < \varepsilon < \varepsilon_u \}$$

**Lemma 3.1:** Let $\alpha \in \mathbb{R}$, and let $\theta_0 \in \Theta$ be such that $\text{sign} \alpha_0 = \text{sign} \alpha$ and $|\alpha_0| < |\alpha|$, where $\alpha_0 \triangleq \alpha(\theta_0)$, and let $d > \bar{d}$. Then, the following statements hold:

i) There exist $\varepsilon_0 > 0$, $\theta_1 > 0$, $\theta_u > 0$, $\varepsilon_1 > 0$, and $\varepsilon_u > 0$ such that $\varepsilon_1 < \varepsilon_u$, $\theta_1 < \theta_u$, $(\theta_0, \varepsilon_0) \in \Gamma_{\theta_1, \theta_u, \varepsilon_1, \varepsilon_u}$, and, in the rectangle $\Gamma_{\theta_1, \theta_u, \varepsilon_1, \varepsilon_u}$, $\theta$, $e = (\theta_0, \varepsilon_0)$ is the unique solution of $p_{d,\alpha,e}(e^{\theta}) = 0$.

ii)\[iidi\]

$$\frac{d}{d\varepsilon} \Psi_\delta(e) \bigg|_{\varepsilon = \varepsilon_0} \neq 0.$$

iii)\[iiidi\]

$$\frac{d}{d\theta} \text{Im}[L_{d}(e^{\theta})] \bigg|_{\theta = \theta_0} \neq 0.$$ 

**Proof:** To prove i), note that, for $\text{sign} \alpha_0 = \text{sign} \alpha$ and $|\alpha_0| > |\alpha|$, there exists $\varepsilon_0 > \delta$ such that $\alpha_0 = \Psi_\delta(\varepsilon_0)$. Therefore, $p_{d,\alpha,e}(e^{\theta_0}) = p_{d,\alpha,e}(e^{\theta_u}) = 0$. Furthermore, there exists a rectangle $\Gamma_{\theta_1, \theta_u, \varepsilon_1, \varepsilon_u}$, where $\theta_1 > 0$, $\theta_u > 0$, $\varepsilon_1 > 0$, $\varepsilon_u > 0$, $\varepsilon_1 < \varepsilon_u$ and $\theta_1 < \theta_u$, such that $(\theta_0, \varepsilon_0) \in \Gamma_{\theta_1, \theta_u, \varepsilon_1, \varepsilon_u}$.
For Figure 14(a) shows that, for $k > s$, the saturated signal $v_y$ is periodic. Furthermore, Figure 14(b) shows how the saturation function is replaced by an odd sigmoidal nonlinear function $\Theta$ with $\theta = 0$ and thus $\Theta$ is an odd function. It follows from (73) and (74) that $\Theta$ and $\Theta$ are odd functions.

7.4 in [15, pp. 293, 294] are satisfied. It thus follows that the response of $\alpha$ is asymptotically periodic.

To prove ii), note that

$$\frac{d}{dz} \Psi_4(e^z) \bigg|_{e^z=\alpha} = -\frac{4\sqrt{\alpha^2 - d^2}}{\pi\varepsilon_0} < 0.$$

To prove iii), writing $G(e^\theta) = a_n - j b_n$, where $a_n = \alpha d$ and $b_n = \frac{b}{\alpha d}$. Then,

$$L_d(e^\theta) = G(e^\theta)G_d(e^\theta)W(e^\theta)$$

$$= (a_n - j b_n)(e^{-j\theta} - e^{-j(d+1)\theta})$$

$$= (a_n(\cos d\theta - \cos (d+1)\theta) - b_n(\sin d\theta - \sin (d+1)\theta))$$

$$+ j(-a_n(\sin d\theta - \sin (d+1)\theta) - b_n(\cos d\theta - \cos (d+1)\theta))$$

$$= \frac{f}{a^2 + b^2} - j \frac{g}{a^2 + b^2}.$$

Since $\theta_0 \in \Theta$ and $\alpha(\theta_0) \neq 0$, it follows from (72) and (73) with $\theta = \theta_0$ that $g = 0$ and thus

$$Re[L_d(e^{\theta_0})] \neq 0, \quad Im[L_d(e^{\theta_0})] = 0.$$ (73)

Furthermore, differentiating $\angle L_d(e^\theta)$ with respect to $\theta$ yields

$$\frac{d}{d\theta} \angle L_d(e^\theta) = \frac{\frac{d}{d\theta} \text{atan} \left( \frac{\text{Re}[L_d(e^\theta)]}{\text{Im}[L_d(e^\theta)]} \right)}{L_d(e^\theta)} - \frac{\text{Im}[L_d(e^\theta)] \frac{d}{d\theta} \text{Re}[L_d(e^\theta)] - \text{Re}[L_d(e^\theta)] \frac{d}{d\theta} \text{Im}[L_d(e^\theta)]}{L_d(e^\theta)}.$$

It follows from (73) and (74) that

$$\frac{d}{d\theta} \angle L_d(e^\theta) \bigg|_{\theta=\theta_0} = \frac{\text{Re}[L_d(e^{\theta_0})] \frac{d}{d\theta} \text{Im}[L_d(e^\theta)] \bigg|_{\theta=\theta_0}}{L_d(e^{\theta_0})}.$$

It follows from (11) that, for all $d > \bar{d}$, $\frac{d}{d\theta} \angle L_d(e^\theta) \bigg|_{\theta=\theta_0} < 0$. Hence, it follows from (75) that $\frac{d}{d\theta} \text{Im}[L_d(e^\theta)] \bigg|_{\theta=\theta_0} \neq 0$. \hfill □

Theorem 3.2: Consider the discrete-time time-delayed Lur’e model in Figure 13. Assume that $x_0 \neq 0$, and let $\alpha \in (-\infty, \alpha_d, 1) \cup (\alpha_d, \infty)$. Then, there exists a nonconstant periodic function $\tau : \mathbb{N} \to \mathbb{R}$ such that $\lim_{k \to \infty} |y_k - \tau_k| = 0$.

Proof. Lemma 3.1 implies that the assumptions of Theorem 7.4 in [15, pp. 293, 294] are satisfied. It thus follows that the response is asymptotically periodic. \hfill □

It can be seen that Theorem 3.2 holds in the case where the saturation function is replaced by an odd sigmoidal nonlinearity such as atan or tanh.

Example 3.3: Let $G(z) = 1/z$. Figure 14 shows the transient response and asymptotic oscillatory response for $\alpha = 1.1$, $d = 0$, and $\delta = 1$ along with plot of $y_{t,k}$ and $y_{f,k}$. Figure 14(a) shows that, for $k > 80$, $y_k$ is a nonconstant periodic function. Furthermore, Figure 14(b) shows how the saturation nonlinearity acts upon $y_{f,k}$, which results in the saturated signal $v_{r,k} \in [-\delta, \delta]$. Note that $v_{r,k}$ and $y_{f,k}$ are also nonconstant periodic functions for $k > 80$.

Figure 15 shows $\alpha(\theta_i)$ versus $\theta_i$ for $d = 0$ and $d = 1$. For $\alpha = 0.6$, only in the case $d = 1$ has $\alpha(\theta_i)$ such that $\text{sign}(\alpha(\theta_i)) = \text{sign}(\alpha)$ and $|\alpha(\theta_i)| < |\alpha|$. For $\alpha = 1.1$, both models meet the conditions for $\alpha$.

Figure 16 shows the response of $y_k$ for $d = 1$ and all possible pairs of $d = 0, 1$ and $\alpha = 0.6, 1.1$. For $\alpha = 0.6$, only the model with $d = 1$ yields a limit cycle. For $\alpha = 1.1$, both models yield oscillations. This follows from the conditions for $\alpha$ stated in the previous paragraph and in Lemma 3.1.

Finally, Figure 17 shows the magnitude of the frequency response for models with $\alpha = 1.1$, $\delta = 1$, and $d = 0, 1$. Note that the frequencies corresponding to the magnitude peaks are similar to the values of $\theta_i$ shown in Figure 15 such that $\text{sign}(\alpha(\theta_i)) = \text{sign}(\alpha)$ and $|\alpha(\theta_i)| < |\alpha|$.

\hfill \diamond

IV. TIME-DELAYED LUR’E MODEL WITH BIAS GENERATION

We now modify the discrete-time time-delay Lur’e model by including the bias-generation mechanism shown in Figure...
Fig. 16: Example 3.3: Response $y_k$ of the TDL model for $d = 0, 1$ and $\alpha = 0.6, 1.1$ with $\delta = 1$.

The corresponding closed-loop dynamics are thus given by

$$
\begin{bmatrix}
  x_{k+1} \\
  x_{d,k+1} \\
  x_{f,k+1}
\end{bmatrix} = 
\begin{bmatrix}
  A & 0 & 0 \\
  e_{d,d} C & N_d & 0 \\
  0 & e^T_{1,d} & 0
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  x_{d,k} \\
  x_{f,k}
\end{bmatrix} + 
\begin{bmatrix}
  B \\
  0 \\
  0
\end{bmatrix} v_{b,k},
$$

(76)

with $y_k$, $y_{d,k}$, and $y_{f,k}$ given by (2), (3), and (4), respectively, where $\beta$ is a constant,

$$v_{b,k} = (\beta + v_{l,k})v_k,
$$

(77)

and $v_{l,k} = \text{sat}_\delta(y_{f,k})$. Note that the constant $\alpha$ is now omitted. Instead, the constant input $v$ is injected multiplicatively inside the loop, thus playing the role of $\alpha$. This feature allows the offset of the oscillation to depend on the external input. The resulting bias $\bar{y}$ of the periodic response is thus given by

$$\bar{y} = v \beta G(1).
$$

(78)

**Example 4.1:** Let $G(z) = 1/z$, $d = 0$, $\beta = 2.5$, $v = 1.1$, and $\delta = 1$. Figure 19(a) shows that the output $y_k$ is oscillatory with offset $\bar{y} = v \beta G(1) = 2.75$. Figure 19(b) shows $v_{l,k}$ and $y_{l,k}$. Note that, as in Example 3.3, despite the offset $\bar{y}$ of $y_k$, the signals $y_{l,k}$ and $v_{l,k}$ oscillate without an offset.

Finally, Figure 19(c) shows the magnitude of the frequency response for $y_k - \bar{y}$. Note that the peak is located near the same frequency as in Example 3.3 and thus the oscillation frequency remains the same with the addition of the bias-generation mechanism.

**Example 4.2:** Let

$$G(z) = \frac{z - 0.75e^{\pm j5\pi/6}}{(z - 0.9e^{\pm j\pi/6})(z - 0.9e^{\pm j5\pi/12})},$$

$d = 4$, $\beta = 15$, $v = 1$, and $\delta = 1$. Figure 20(a) shows that the output $y_k$ is oscillatory with offset $\bar{y} = v \beta G(1) = 5.751$. Figure 20(b) shows that $v_{l,k}$ and $y_{l,k}$ have an oscillatory response without an offset, as in previous cases. Finally, Figure 20(c) shows the magnitude of the frequency response for $y_k - \bar{y}$.

Fig. 17: Example 3.3: Frequency response of $y_k$ for $d = 0, 1$ with $\alpha = 1.1$ and $\delta = 1$. Note that, for $d = 0$, the peak is located at $\theta_1$, whereas, for $d = 1$, the peak is located at $\theta_3$.

The corresponding closed-loop dynamics are thus given by

$$
\begin{bmatrix}
  x_{k+1} \\
  x_{d,k+1} \\
  x_{f,k+1}
\end{bmatrix} = 
\begin{bmatrix}
  A & 0 & 0 \\
  e_{d,d} C & N_d & 0 \\
  0 & e^T_{1,d} & 0
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  x_{d,k} \\
  x_{f,k}
\end{bmatrix} + 
\begin{bmatrix}
  B \\
  0 \\
  0
\end{bmatrix} v_{b,k},
$$

(76)

with $y_k$, $y_{d,k}$, and $y_{f,k}$ given by (2), (3), and (4), respectively, where $\beta$ is a constant,

$$v_{b,k} = (\beta + v_{l,k})v_k,
$$

(77)

and $v_{l,k} = \text{sat}_\delta(y_{f,k})$. Note that the constant $\alpha$ is now omitted. Instead, the constant input $v$ is injected multiplicatively inside the loop, thus playing the role of $\alpha$. This feature allows the offset of the oscillation to depend on the external input. The resulting bias $\bar{y}$ of the periodic response is thus given by

$$\bar{y} = v \beta G(1).
$$

(78)

**Example 4.1:** Let $G(z) = 1/z$, $d = 0$, $\beta = 2.5$, $v = 1.1$, and $\delta = 1$. Figure 19(a) shows that the output $y_k$ is oscillatory with offset $\bar{y} = v \beta G(1) = 2.75$. Figure 19(b) shows $v_{l,k}$ and $y_{l,k}$. Note that, as in Example 3.3, despite the offset $\bar{y}$ of $y_k$, the signals $y_{l,k}$ and $v_{l,k}$ oscillate without an offset.

Finally, Figure 19(c) shows the magnitude of the frequency response for $y_k - \bar{y}$. Note that the peak is located near the same frequency as in Example 3.3 and thus the oscillation frequency remains the same with the addition of the bias-generation mechanism.

**Example 4.2:** Let

$$G(z) = \frac{z - 0.75e^{\pm j5\pi/6}}{(z - 0.9e^{\pm j\pi/6})(z - 0.9e^{\pm j5\pi/12})},$$

$d = 4$, $\beta = 15$, $v = 1$, and $\delta = 1$. Figure 20(a) shows that the output $y_k$ is oscillatory with offset $\bar{y} = v \beta G(1) = 5.751$. Figure 20(b) shows that $v_{l,k}$ and $y_{l,k}$ have an oscillatory response without an offset, as in previous cases. Finally, Figure 20(c) shows the magnitude of the frequency response for $y_k - \bar{y}$.

Fig. 18: Discrete-time time-delayed Lur’e model with constant input $v$ and bias generation.

Fig. 19: Example 4.1: For $v = 1.1$, $\beta = 2.5$, $d = 0$, and $\delta = 1$, (a) shows $y_k$ and the offset $\bar{y}$, (b) shows $v_{l,k}$ and $y_{l,k}$, and (c) shows the frequency response of $y_k - \bar{y}$.

Fig. 20: Example 4.2: For $v = 1.1$, $\beta = 2.5$, $d = 0$, and $\delta = 1$, (a) shows $y_k$ and the offset $\bar{y}$, (b) shows $v_{l,k}$ and $y_{l,k}$, and (c) shows the frequency response of $y_k - \bar{y}$. 
V. CONCLUSIONS AND FUTURE EXTENSIONS

This paper presented and analyzed a discrete-time Lur’e model that exhibits self-excited oscillations. This model involves an asymptotically stable linear system, a time delay, a washout filter, and a saturation nonlinearity. It was shown that, for sufficiently large loop gains, the response converges to a periodic signal, and thus the system has self-excited oscillations. A bias-generation mechanism provides an input-dependent oscillation offset. The amplitude and spectral content of the oscillation were analyzed in terms of the components of the model.

An immediate extension of this work is to consider the case where $G$ has zeros on the unit circle. The main results of this paper appear to be valid for this case, although the proofs are more intricate. Extension to sigmoidal nonlinearities such as atanh and tanh as well as relay nonlinearities is of interest. In addition, continuous-time, time-delay Lur’e models described by $i\nu$ in Section I are of interest. Finally, future work will use this discrete-time self-excited model for system identification and adaptive stabilization.

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