D=10 SUPERSYMMETRIC

CHERN-SIMONS GAUGE THEORY

R. Kallosh

Physics Department, Stanford University
Stanford CA 94305

ABSTRACT

The Chern-Simons ten-dimensional manifestly supersymmetric non-Abelian gauge theory is presented by performing the second quantization of the superparticle theory.

The equation of motion is $F = (d + A)^2 = 0$, where $d$ is the nilpotent fermionic BRST operator of the first quantized theory and $A$ is the anti-commuting connection for the gauge group. This equation can be derived as the condition of the gauge independence of the first quantized theory in a background field $A$, or from the string field theory Lagrangian of the Chern-Simons type. The trivial solutions of the cohomology are the gauge symmetries, the

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2 On leave of absence from: Lebedev Physical Institute, Moscow, 117924, USSR
non-trivial solution is given by the D=10 superspace, describing
the super Yang-Mills theory on shell.

1 Introduction

The ten-dimensional supersymmetric gauge theory has many interesting prop-
erties. It is the largest globally supersymmetric gauge theory. Its spectrum
coincides with the spectrum of (non-gravitational) zero modes of heterotic
string theory. The result of its dimensional reduction to four dimensions
(N=4 supersymmetric Yang-Mills theory) is the first example of a finite four-
dimensional quantum field theory. Understanding the ten-dimensional super-
symmetric gauge theory would be an important step toward understanding
the geometry of superstring theory and would give insights into the structure
of the string field theory.

However, despite many attempts, the ten-dimensional supersymmetric
Yang-Mills theory (SYM) was not formulated in a manifestly supersymmet-
ic, Lorentz invariant and gauge invariant way. This is a major challenge to
those who are looking for the manifestly supersymmetric formulation of the
string theory.

The on shell SYM theory is known in the superspace \((X^\mu, \theta^\alpha)\), where \(\theta^\alpha\)
is a ten-dimensional 16-component Majorana-Weyl spinor. The geometry
of this superspace has been constructed by Nilsson [1]. The gauge connec-
tions \(A_A = (A_\mu, A_\alpha)\) depend on the coordinates of the superspace and take
values in some gauge group \(G\). The gauge covariant derivatives of the ten-
dimensional superspace \(\nabla_A = (\nabla_\mu, \nabla_\alpha)\) transform homogeneously under the
Yang-Mills transformation, \(\nabla'_A = e^A \nabla_A e^{-A}\), where \(\exp A(X, \theta)\) is an
element of the gauge group \(G\). The anti-commutator of spinorial derivatives
\(\{\nabla_\alpha, \nabla_\beta\} = 2\nabla_\alpha \beta + F_{\alpha \beta}\) defines the vector derivative \(\nabla_\alpha \beta = \gamma^\mu_{\alpha \beta} \nabla_\mu\) in case
that the spinorial curvature is constrained to be equal to zero,

\[ F_{\alpha \beta} = 0 \ . \]  \hspace{1cm} (1)

This is the Nilsson’s constraint for ten-dimensional SYM. From this con-
straint and from Bianchi identities one can derive the manifestly supersym-
metric gauge covariant Dirac equation for a ten-dimensional Majorana-Weyl spinorial superfield $\Psi^\alpha(z)$, where

$$\gamma^\mu_{\alpha\beta} \Psi^\beta(X, \theta) = [\nabla_\alpha, \nabla^\mu].$$

(2)

The superfield Dirac equation contains all equations of motion of the ten-dimensional supersymmetric Yang-Mills theory. They can be obtained from the superfield equation by expanding in $\theta$ at $\theta = 0$. The superfield Dirac equation is supersymmetric, Lorentz covariant and gauge covariant. However, this is not enough for construction of the corresponding quantum field theory; one still needs to have a manifestly supersymmetric, Lorentz covariant and gauge covariant Lagrangian.

The purpose of this talk is to present the formulation of the geometry and the gauge symmetries of the ten-dimensional supersymmetric non-Abelian gauge theory starting with the first quantized superparticle theory. The theory which we obtain has a Chern-Simons (CS) structure; it has manifest supersymmetry, Lorentz symmetry and gauge symmetry. The geometric equation of motion reproduces the on shell superspace describing this theory.

This theory will be constructed by using the first quantized superparticle theory [2] – [4]. The second quantization will be performed in accordance with the general program developed for string field theory [5]. The way towards a consistent first quantization of manifestly supersymmetric particle theory was long and difficult [6]. Recently two groups claimed to have found a consistent quantization of the superparticle theory, [2] – [4] and [7]. Different versions of the theory have been used by these two groups. In what follows we are going to use our version [3]. This theory of superparticle has the property that ten-dimensional $N = 1$ supersymmetry is realized in a form of a broken twisted supersymmetry $N = 2(p+1)$, where $p = 0, 1, \ldots, \infty$. In the light-cone gauge it coincides with that of the Brink-Schwarz superparticle.

The paper is organized as follows: In Sec.2 we present the general first quantized path integral in an arbitrary background, by introducing the gauge covariant BRST operator $\Omega(P, Q) + A(Q)$. We derive the condition of the quantum gauge independence of this functional, which turns out to be the
nilpotency condition of the covariant BRST operator. This condition, which can be written as an equation of motion of the Chern-Simons type, is equivalent to the condition of flatness of the target space. In Sec.3 we describe the D=10 superspace, which is suggested by the classical action of the superparticle [3]. This superspace has twisted $N = 2(p + 1)$ supersymmetry. We introduce the analogs of chiral (antichiral) subspaces. The chiral (antichiral) superfields realize the representations of $N = 1$ supersymmetry, since chiral (antichiral) constraints do not commute with all supersymmetries except one. In Sec.4 we describe the first quantized superparticle theory. We first consider the background functional with the unconstrained connection and the set of non-Abelian gauge symmetries. The flatness condition of the target space geometry will be shown to provide the gauge independence of the first quantized theory. In Sec.5 the Chern-Simons type geometry, the Lagrangian (and the problems in defining the measure of integration), its gauge symmetries and the equation of motion of the manifestly super-Poincare invariant ten-dimensional gauge theory are described. In Sec.6 the classical equation of motion is solved. The solution is given in terms of $N = 1$ Majorana-Weyl superfield, satisfying the manifestly supersymmetric gauge covariant Dirac equation.

2 Gauge covariant BRST operator

Consider a class of gauge theories for which the classical Hamiltonian is proportional to the constraints. The path integral for the gauge-fixed action of such theories can be presented in the form given by Batalin, Fradkin, Vilkovisky [8].

$$Z_{\Psi} = \int dPdQ \exp \left(i \int d\tau (P \dot{Q} - H_{\Psi}) \right),$$

(3)

where

$$H_{\Psi} = -\{\Psi, \Omega\}.$$  

(4)

In (4) $\Psi(P, Q)$ is the gauge fermion defining the gauge-fixing condition and $\Omega$ is the nilpotent BRST operator,

$$\Omega = \Omega(P, Q); \; \Omega^2 = 0.$$  

(5)

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The set of coordinates entering the gauge-fixed action is denoted by $Q$ and
the set of canonically conjugate momenta by $P$. Some of these coordinates
are original classical fields, some of them are the so-called ghost fields.

The gauge covariant BRST operator can be introduced into this path
integral by adding to the free BRST operator $\Omega(P,Q)$ an anticommuting
connection field $A(Q)$, which is Lie algebra valued. The background functional takes the form

$$Z^A_\psi = \int dPdQ \exp \left( i \int d\tau (P\dot{Q} + \{\Psi, \Omega(P,Q) + A(Q)\}) \right).$$

(6)

When the connection field $A$ is unconstrained, the background functional can
be shown to depend on the gauge fixing function $\Psi$. The gauge symmetry
transformation of the connection field $A$ can be absorbed into the canonical
change of variables and of the gauge fixing function.

$$(\Omega + A)\Lambda = e^\Lambda(\Omega + A)e^{-\Lambda},$$

(7)

$$\Psi^\Lambda = e^\Lambda \Psi e^{-\Lambda}.$$  

The condition of the independence of the background functional (6) on the
gauge fixing function $\Psi$ can be derived by performing the infinitesimal change
of integration variables given by (this is the generalization of BFV theorem
[8])

$$Q' = Q + [Q, \Omega + A]\chi,$$

$$P' = P + [P, \Omega + A]\chi,$$

(8)

where

$$\chi = i \int d\tau (\Psi' - \Psi).$$

(9)

The change in the measure of integration produces the term

$$\exp \left( -i \int \{\Omega + A, \Psi' - \Psi\}d\tau \right).$$

(10)

It follows that

$$\frac{\partial Z^A_\psi}{\partial \psi} = \langle [(\Omega + A)^2, \Psi] \rangle,$$

(11)
where the brackets \(<\>\) denote the functional integration with the weight given in eq. (3). Since the gauge-fixing function is arbitrary we may conclude that the nilpotency of the gauge covariant BRST operator

\[(\Omega + A)^2 = 0\]  \hspace{1cm} (12)

is the necessary and sufficient condition for the gauge independence of the background functional (3). Equation (12) is also known to be the fundamental equation of the string field theory [5], following from the string field theory Lagrangian.

\[L_{SFT} = Tr.A^\dagger(\frac{1}{2}\Omega \ast A + \frac{1}{3} \ast A \ast A)\]  \hspace{1cm} (13)

In the second quantized theory the canonical momenta are realized as derivatives over the coordinate, \(P = \frac{\partial}{\partial Q}\). We denote by \(d\) the differential operator, related to the free BRST operator as

\[d\left(\frac{\partial}{\partial Q}, Q\right) = \Omega(\frac{\partial}{\partial Q} , Q)\]  \hspace{1cm} (14)

and the covariant differential operator, related to gauge covariant BRST operator introduced above, as

\[D = d + A\]  \hspace{1cm} (15)

The consistency condition of the first quantized theory, or the classical equation of the second quantized theory take the geometrical form of the Chern-Simons type equation for the curvature of the target space:

\[F = D^2 = (d + A)^2 = 0\]  \hspace{1cm} (16)

### 3 D=10 superspace and chiral superfields

According to the superparticle theory [3], the flat D=10 superspace can be characterized by the classical coordinates \(X^\mu, \theta^\alpha_p\), where \(X^\mu\) is a D=10 vector and \(\theta^\alpha_p\) are D=10 anticommuting Majorana-Weyl spinors of positive chirality, \(\alpha = 1, \ldots, 16; p = 0, 1, \ldots\). Our notations and conventions are those of
The important ones are the following. The spinors of positive chirality have spinorial indices $\alpha$ up, the ones of negative chirality have them down. The $\gamma$-matrices have their two indices both up or both down, and they are symmetric.

The supersymmetry charges and covariant derivatives are defined as follows.

$$ q_\alpha^p = \frac{\partial}{\partial \theta_p^\alpha} + (\varnothing \theta_p)^\alpha, $$

$$ d_\alpha^p = \frac{\partial}{\partial \theta_p^\alpha} - (\varnothing \theta_p)^\alpha. $$

These charges and covariant derivatives form the twisted $N = 2(p + 1)$ supersymmetry algebra,

$$ \{ q_\alpha^p , q_\beta^l \} = +2 \varnothing_{\alpha\beta} \delta^{pl}, $$

$$ \{ d_\alpha^p , d_\beta^l \} = -2 \varnothing_{\alpha\beta} \delta^{pl}, $$

$$ \{ d^p , q^l \} = 0. $$

From the charges of the twisted $N = 2(p + 1)$ supersymmetry algebra given in eqs. (18) we build two combinations,

$$ F_\alpha^p = \frac{1}{2}(q_\alpha^{p+1} + d_\alpha^p), $$

$$ \tilde{F}_\alpha^p = \frac{1}{2}(q_\alpha^{p+1} - d_\alpha^p). $$

They form the following algebra:

$$ \{ F_\alpha^p , F_\beta^l \} = 0, $$

$$ \{ \tilde{F}_\alpha^p , \tilde{F}_\beta^l \} = 0, $$

$$ \{ F_\alpha^p , \tilde{F}_\beta^l \} = \varnothing_{\alpha\beta} \delta^{pl}. $$

This algebra has Abelian subalgebras which remind the Abelian subalgebras in D=4 where chiral and antichiral subalgebras exist,

$$ \{ q^A , q^B \} = 0, $$

$$ \{ q^A , \tilde{q}^B \} = 0, $$

$$ \{ q^A , \tilde{q}^B \} = 2 \varnothing^{AB}. $$


A, Ā being 2-component spinor indices. The treatment of D=10 flat super-space (20) and gauge theories in this superspace will have some analogies and some differences with 4D flat superspace (21) and gauge theories in it.

The irreducible representation of the algebra (20) is realized by putting the constraint which is analogous to the chiral constraint in D=4. We can put the constraint (we will call it chiral constraint by analogy)

\[ \tilde{F}_p^\alpha \Phi(X^\mu, \theta_1) = (q^{p+1} - d^p)_\alpha \Phi(X^\mu, \theta_1) = 0. \] (22)

The corresponding superfield can be real (as different from the chiral 4D superfield), since the constraint is real. The solution can be written in the formal way as

\[ \Phi(X^\mu, \theta_1) = \prod \tilde{F}_p^\alpha \Psi(X^\mu, \theta_1), \] (23)

where \( \Psi(X^\mu, \theta_0) \) is an unconstrained superfield and the product includes all \( \tilde{F}_p^\alpha \). We will call the superfield satisfying the constraint (22) the chiral superfield. From all the charges of the twisted \( N = 2(p + 1) \) supersymmetry algebra given in eq. (18) only

\[ q^0 = \frac{\partial}{\partial \theta_0} + \partial \theta_0 \] (24)

commutes with the constraint \( \tilde{F}_p^\alpha \). Therefore the chiral superfield has the trivial dependence on \( \theta_{p+1} \), even though it depends on all \( \theta_p, \ p = 0, 1, ... \) and not only on \( \theta_0 \) as the the normal \( N = 1 \) superfield in \( z = (X, \theta_0) \) superspace does. The solution to the chirality constraint can be presented by expanding the superfield in powers of \( \theta_1, \theta_2, ... \), where the coefficients of the expansion are normal \( N = 1 \) superfields,

\[ \Phi(X^\mu, \theta_p) = \Phi(X^\mu, \theta_0) + \Phi^1_\alpha(X^\mu, \theta_0)\theta^\alpha_1 + \Phi^{11}_{\alpha \beta}(X^\mu, \theta_0)\theta^\alpha_1\theta^\beta_1 + ... . \] (25)

The chiral constraint allows to express all components of this superfield through the first one and its covariant derivatives. For example, the first in the set of constraints (22) is

\[ \left( \frac{\partial}{\partial \theta^0_\alpha} - (\partial \theta^0_\alpha) \right) \Phi = \left( \frac{\partial}{\partial \theta^0_1} + (\partial \theta^0_1) \right) \Phi(X^\mu, \theta_p) . \] (26)
When applied at $\theta_1 = 0$, it gives
\begin{equation}
\Phi^1_{\alpha}(X^\mu, \theta_0) = d^0_\alpha \Phi(X^\mu, \theta_0) .
\end{equation}

Note, that $d^0_\alpha$ are the covariant derivatives for $N = 1$ superfield $\Phi(X^\mu, \theta_0)$.

Alternatively, one may choose the constraint
\begin{equation}
F^p_\alpha \bar{\Phi}(X^\mu, \theta_l) = (q^{p+1} + d^p)_\alpha \bar{\Phi}(X^\mu, \theta_l) = 0 .
\end{equation}
We will call this constraint and the corresponding superfield antichiral, by analogy with D=4 superspace.

To describe the gauge theory, one proceeds in a standard way of introducing connections for each direction in flat superspace,
\begin{align*}
F^p_\alpha &= \frac{1}{2}(Q^{p+1}_\alpha + D^p_\alpha) = \frac{1}{2}(q^{p+1}_\alpha + A^{p+1}_\alpha + d^p_\alpha + A^p_\alpha) , \\
\bar{F}^p_\alpha &= \frac{1}{2}(Q^{p+1}_\alpha - D^p_\alpha) = \frac{1}{2}(q^{p+1}_\alpha + A^{p+1}_\alpha - d^p_\alpha - A^p_\alpha) .
\end{align*}

One can rewrite this equations as follows:
\begin{align*}
D^p_\alpha &= \frac{\partial}{\partial \theta^p_\alpha} - \partial \theta^p_\alpha + A^p_\alpha , \\
Q^{p+1}_\alpha &= \frac{\partial}{\partial \theta^{p+1}_\alpha} + \partial \theta^{p+1}_\alpha + A^{p+1}_\alpha .
\end{align*}

The gauge symmetry is introduced into the space by choosing the parameter of the gauge transformation to be a covariantly chiral super field,
\begin{equation}
\bar{F}^p_\alpha \Lambda(X^\mu, \theta_l) = 0 .
\end{equation}
The integrability of this constraint requires the vanishing of the curvature,
\begin{equation}
\{ \bar{F}^p_\alpha, \bar{F}^q_\beta \} = \bar{F}^{p,q}_{\alpha \beta} = 0 .
\end{equation}
One can choose the basis in which covariant chiral derivatives are free and
\begin{align*}
\bar{F}^p_\alpha &= \tilde{F}^p_\alpha , \\
A^{p+1}_\alpha - A^{p+1}_\alpha &= 0 , \\
F^p_\alpha &= F^p_\alpha + A^p_\alpha(z) ,
\end{align*}
where $A^p_\alpha (z)$ is an unconstrained superfield. It is possible to impose the set of constraints on torsion and curvature in the classical superspace $z = (X^\mu, \theta_\rho)$ and to solve them to describe the on shell geometry. In addition to eq. (32), we impose the conventional constraint by expressing the vector connection through the spinor one:

$$\frac{1}{2}(\{\tilde{F}_{\alpha}^p, F_{\beta}^q\} + \{\tilde{F}_{\delta}^q, F_{\alpha}^p\}) = \gamma^\mu_{\alpha\beta} \nabla_\mu \delta^{p,q}. \quad (34)$$

The on shell constraint which is added to the geometry is

$$\{F_{\alpha}^p, F_{\beta}^q\} = 0. \quad (35)$$

The solution of this last constraint requires the connection on shell to be of the form

$$A^p_\alpha (z) = e^V (F_{\alpha}^p e^{-V}), \quad (36)$$

where $V(z)$ is an unconstrained superfield. It is necessary to solve Bianchi identities in the presence of all above mentioned constraints. We have found that the solution requires the particular properties of spinor-vector curvature,

$$[F_{\alpha}^p, \nabla^\alpha \gamma] = \Psi^\gamma. \quad (37)$$

This spinorial superfield must be covariantly antichiral and satisfy gauge covariant Dirac equation. Thus the geometry of the superspace described above is an alternative description of the on shell $D = 10$ supersymmetric Yang-Mills theory. The difference with the usual $(X, \theta_0)$ superspace is the fact that it is related to the first quantized superparticle action and that the geometry will be shown to be a solution to the nilpotency condition of the gauge covariant BRST operator.

4 First quantized superparticle

The classical action of the superparticle [3] is

$$S^{cl} = P \dot{X} + \sum_{p=0}^{\infty} \{\lambda^p \dot{\theta}_p + T_a \psi^a\}, \quad (38)$$
where \( \psi^a = \{g, \zeta^p, \eta_p\} \) are the Lagrange multipliers to the constraints. The first-class constraints are

\[
T_a = \left\{ \frac{1}{2} P^2, K_p, F^p \right\}.
\]

Here we are using the following notations:

\[
K_p = (d^p + q^p) P, \\
F^p = \frac{1}{2} (d^p + q^{p+1}).
\]

These constraints generate local symmetries of the action: reparametrization, \( \kappa \)- and \( \xi \)-symmetries, respectively. Note, that the commutator of the fermionic \( \kappa \)- and \( \xi \)-symmetries has reparametrization in the right-hand side,

\[
[K_q, F^p] = -\frac{1}{2} P^2 (\delta^p_q - \delta^p_{q+1}).
\]

In this sense the fermionic symmetries of the world-line are quite fundamental and can be considered as the “square root” of the Virasoro constraint. The phase space of the first quantized superparticle theory \cite{2}–\cite{4} in addition to classical coordinates \((X^\mu, \theta^a_p)\) and their canonical momenta contains a set of ghost fields and their canonical momenta. They are related to the local symmetries of the action.

The gauge-fixed action in arbitrary gauge, including the light-cone one and the free covariant one, has the following form \cite{3}:

\[
\mathcal{L}_{gf} = P \dot{X} + b \dot{c} + \sum_{q=0}^\infty \sum_{p=q}^\infty \left[ \lambda^{p,q} \dot{\theta}_{p,q} + \lambda_{p+1,q+1} \dot{\bar{\theta}}^{p+1,q+1} \right] + \{\Omega_{BRST}, \Psi\},
\]

where \( \Omega_{BRST} \) is a nilpotent ghost number +1 BRST operator, which will be presented below, and \( \Psi \) is an arbitrary gauge fixing function of ghost number \(-1\).

In the gauge fixed-action (42) \( b, c \) is a couple of reparametrization ghosts. The spinors (classical and ghost together) form a spinorial multiplet of \( OSp(9.1/4) \).

\[\text{The second ilk action in ref. 7 can be obtained from the action 42} \text{ at } \zeta^p = 0, p = 1, 2, \ldots. \text{ This leads also to a different set of ghosts and different BRST operator.}\]
supergroup, consisting of spinors of alternating Grassmann parity and chirality [\(9\) - \(11\)].

We are using the notations of [4], where the spinors of \(OSp(9.1/4)\) are presented using labels of SU(2) representations, e.g. \(\theta(j,m)\), and/or by using their original labels as they come from superstring quantization, where \(\theta_{p,q} = \theta(j = p+q/2, m = p-q/2)\) and \(\tilde{\theta}^{p+1,q+1} = \theta(j = p+q+1/2, m = q-p-1/2)\), \(p \geq q \geq 0\). The SU(2) label \(m\) is related both to ghost number and to conformal charge in string theory.

The nilpotent ghost number +1 BRST operator, which was found in [2], is given, according to [3] – [4], by the following expression:

\[
\Omega = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \lambda_{p+1,q} \left( P \theta_{p+1,q} - 2 \tilde{\theta}_{p+2,q} \right) + \sum_{q=1}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p+1,q} \left( \tilde{P} \tilde{\theta}^{p,q} - 2 \tilde{\theta}^{p-1,q} \right) + \sum_{p=1}^{\infty} \lambda_{p+1,p} \left( P \tilde{\theta}^{pp} + \left( \tilde{\lambda}^{p+1,p} + \lambda^{p,p-1} \right) - 2b (\theta_{p,p} - \tilde{\theta}_{p-1,p-1}) \right) + \sum_{p=0}^{\infty} \lambda_{p+1,p+1} \left( \tilde{\lambda}^{p+1,p+1} - P \theta_{p+1,p+1} + 2 \theta_{p+2,p+1} b + \tilde{\lambda}^{p,p} + P \theta_{p,p} - 2 \theta_{p+1,p} \right). \quad (43)
\]

The BRST operator commutes only with \(N = 1\) supersymmetry and does not commute with all of those extra supersymmetries which are not the symmetries of the physical spectrum.
It was shown in [2] that the cohomology of the nilpotent operator (44) is given by the 8+8 supersymmetric Yang-Mills multiplet.

In what follows we will use a version of the BRST operator related to the one in eq. (44) by canonical transformation
\[ \Omega' = e^\Phi \Omega e^{-\Phi}, \] (45)
where
\[ \Phi = b c_{p+1} \tilde{F}_p^\alpha, \] (46)
and
\[ \tilde{F}_p^\alpha = \frac{1}{2} (d^p - q^{p+1}). \] (47)

Consider now the functional given in eq. (6). The set of classical and quantum coordinates of the superparticle is defined above, the free BRST operator is given in eq. (45). The gauge covariant BRST operator will be introduced in a way that the connection field \( A(Q) \) can be absorbed in the change of variables in the path integral if the connection is a pure gauge only. Together with our choice of the BRST operator (45) we suggest a particular choice of coordinates \( Q \):
\[ Q = \{Z, Y\}, \] (48)
where
\[ Z = \{X^\mu, \theta^\alpha_p, b, c_{p+1}^\alpha \equiv \lambda_{p+1,p+1}\}, \]
\[ Y = \{\bar{\lambda}^{p+1,q}, \bar{\theta}^{p+2,q+1}\}, \quad p \geq q \geq 0. \] (49)

With this choice of coordinates the free BRST operator \( d \) is
\[ d = \frac{1}{2} P^2 \frac{\partial}{\partial b} + c_{p+1}^\alpha F_p^\alpha + b c_{p+1}^\alpha c_{q+1}^\beta \{\tilde{F}_p^\alpha, F_q^\beta\} + d_Y \frac{\partial}{\partial Y}. \] (50)

In the general case the consistency condition for the background field \( A(Q) \) is given by the nilpotency condition of the covariant BRST operator and the gauge symmetry of the target space is defined by the transformation
\[ d + A' = e^A (d + A) e^{-A}, \] (51)

\[ ^4 I am grateful to W. Siegel for suggestion to represent the BRST operator, which we have used for the second quantization, in this particular form. \]
where \( d \) is given above and the parameter is some general function of coordinates \( Q \). One has to be much more specific if one wants to build not only the geometry of the background of the first quantized theory but also to understand the way to perform the second quantization of the superparticle theory.

5 Non-Abelian D=10 Chern-Simons theory

The quadratic second quantized action for the superparticle was analysed in [12]. This action describes the Abelian supersymmetric gauge theory in D=10, where it is a free theory, since the Majorana spinors do not interact with the Abelian vector field.

\[
S = \frac{1}{2} \int dQA(Q)d\left(\frac{\partial}{\partial Q}, Q\right)A(Q).
\]

Here \( d \) is the BRST operator given in eq. (50) and \( A \) is the anticommuting connection. With our choice of coordinates and momenta all terms in \( d \) are at least linear in derivatives for the BRST operator [3] – [4] of the superparticle action [3]. This means that \( d \) is a differential operator and the rules of partial integration can be applied.

This action avoids the old problem of \( D = 10 \) supersymmetry. There exists the no-go theorem which says that one cannot construct a supersymmetric Lagrangian for this theory if it contains a finite number of auxiliary fields. In our case the off-shell connection \( A \) is a general superfield in (\( X, \theta_p \)) space containing an infinite number of components.

There is a problem, however, in defining the measure of integration over the coordinates of the superparticle in (52), since they include commuting spinors of a given ghost number. We hope that this problem will be solved and the action (52) will be used for the construction of the off-shell theory. This would be necessary for consistency of the second quantized theory (52) and the first quantized superparticle theory. Indeed, if the measure of integration in (52) is properly defined, then the variation of this action with respect to the field \( A \) leads to the same CS equation as the one derived from
the condition of consistency of the first quantized superparticle theory in the background field $A$.

The equation of motion following from (52) is

$$d\left(\frac{\partial}{\partial Q}, Q\right)A(Q) = 0.$$  \hspace{1cm} (53)

The action (52) and the equation of motion (53), have a gauge symmetry under the following transformation of a connection field $A(Q)$,

$$\delta A(Q) = d\left(\frac{\partial}{\partial Q}, Q\right)\Lambda(Q),$$  \hspace{1cm} (54)

since $d$ is a fermionic nilpotent operator, satisfying the equation

$$d^2 = \frac{1}{2}\{d, d\} = 0.$$  \hspace{1cm} (55)

It was shown in \[2\] that it is sufficient to consider the case $P^2 = 0$, when looking for the non-trivial solution of the cohomology. In addition to that we were looking for the solutions of the eq. (53) which has the property of being independent on the coordinates, denoted by $Y$, i.e. on $(\lambda^{p+1,q}, \theta^{p+2, q+1})$. It has been found under these conditions that the solution of equation (53) reproduces $D = 10$ supersymmetric QED.

The complications which arise in the non-Abelian case are related to the fact that the full BRST operator in equation (50) has some terms which are quadratic or even cubic in derivatives\[5\]. If we are going to use the string field type action

$$S = \frac{1}{2} \int dZ (Ad_1 A + \frac{2}{3} A^3),$$  \hspace{1cm} (56)

we are allowed to use only the part of $d$ which is linear in derivatives to be able to prove the non-Abelian gauge symmetry of the action (56). Also we need to consider a smaller set of coordinates, which includes only $Z$-coordinates, see eq. (49). Fortunately, as it follows from the analysis of the Abelian theory, the corresponding part of $d$

$$d_1 = c^\alpha_{p+1} F^p_\alpha + b c^\alpha_{p+1} c^\beta_{q+1} \{F^p_\alpha, F^q_\beta\}$$  \hspace{1cm} (57)

\[5\]This observation has been made by M. Peskin.
is indeed responsible for the non-trivial solution to cohomology. The equation of motion following from the action (56), (57) has the CS form

\[ F_1 = D_1^2 = (d_1 + A(Z))^2 = 0. \]  

Thus, from the second quantized theory (58) we have derived (at least formally until the integration over the coordinates of the superparticle will be better understood) the same equation which in a more general form with a full \( d \) and with an arbitrary choice of all coordinates versus momenta has been derived from the consistency of the first quantized superparticle theory in a background field \( A \).

6 Solution to CS equation of motion

The cohomology analysis of the free BRST operator performed in [2] has shown that the only non-trivial solution describes the \( D = 10 \) supersymmetric multiplet. Now we are going to demonstrate that the solution of the simple CS equation \( D^2 = 0 \) (16), which is the condition of the nilpotency of the gauge covariant BRST operator \( D \), gives a full non-linear description of the ten-dimensional supersymmetric Yang-Mills theory.

The solution we are looking for will not depend on \( Y \). Therefore, to find a non-trivial solution it will be sufficient to solve a simpler equation (58).

To solve this equation, we consider the target superspace

\[ Z = \{ X^\mu, \theta^\alpha_p, b, c^\alpha_{p+1} \}, \quad p = 0, 1, \ldots, \infty. \]  

In addition to classical coordinates \( z = (X^\mu, \theta^\alpha) \), it has also the anticommuting reparametrization antighost \( b \), with the ghost charge \(-1\), and commuting \( \xi \)-symmetry ghosts \( \lambda^\alpha_{p+1,p+1} \equiv c^\alpha_{p+1} \), each with the ghost charge \(+1\).

The anticommuting connection is \( A = A_0 + bA_1 \), where \( A_0 \) has ghost number \(+1\) and therefore is linear in \( c^\alpha_{p+1} \) and \( A_1 \) has ghost number \(+2\) and is quadratic in \( c^\alpha_{p+1} \).

\[ A(Z) = c^\alpha_{p+1}A^\alpha_0(z) + bc^\alpha_{p+1}c^\beta_{q+1}A^\beta_{pq}(z). \]  

16
$A^p_α(z)$ and $A^p_{αβ}(z)$ are two arbitrary independent functions on classical superspace $z$. The covariant derivative is

$$D_1 = c^α_p F^p_α + bc^α_p c^β_{q+1}(\{\tilde{F}^p_α, F^q_β\} + A^p_{αβ}) .$$

Under the gauge transformations it transforms as follows:

$$D'_1 = e^Λ D_1 e^{-Λ} ,$$

where the parameter of the gauge transformation $Λ$ has zero ghost charge and has the form

$$Λ(Z) = Λ_0(z) + bc^α_p c^β_{q+1} A^p_α(z) .$$

We can partially use the gauge freedom of $D_1$ by expressing $A_1$ through $A_0$ in the solutions,

$$A_1 = c^α_p \tilde{F}^p_α A_0 .$$

In this gauge our covariant derivative takes the form

$$D_1 = c^α_p F^p_α + bc^α_p c^β_{q+1} \{\tilde{F}^p_α, F^q_β\} .$$

The remaining gauge symmetry is given by eq. (62) with the only remaining parameter of transformation $Λ_0$ being a chiral superfield, satisfying the equation

$$\tilde{F}^p_α Λ_0 = 0 .$$

The CS equation in these gauge is

$$F_1 = D^2_1 = c^α_{p+1} c^β_{q+1}\{F^p_α, F^q_β\} + bc^α_p c^β_{q+1} c^γ_{l+1} [F^p_α, \{\tilde{F}^q_β, F^l_γ\}] = 0 .$$

We have found that the solution of this equation coincides with the solution of Bianchi identities in the classical superspace with the constraints, as given in Sec.3. In particular, we have to require that in this gauge

$$F^p_α = e^V(z) F^p_α e^{-V(z)} ,$$

where $V$ is a general function of $z$. In addition to that the gauge covariant curvature in the spinor-vector direction build up from $V$ according to equation (37) must satisfy a list of properties. In particular it is a covariantly antichiral superfield, for which the dependence on $θ_1, θ_2, ...$ is defined by the
dependence on $\theta_0$. This kind of a superfield is very close to the normal $N = 1$ superfield, depending only on $\theta_0$. This superfield satisfies the Dirac equation and describes the on shell content of the non-Abelian supersymmetric gauge theory in $D = 10$:

$$\gamma^\mu \nabla_\mu \Psi(X, \theta_p) = 0,$$

$$\mathcal{F}_{\alpha}^\beta \Psi^\beta(X, \theta_q) = 0,$$

$$\tilde{F}_{\alpha}^\beta \Psi^\beta(X, \theta_q) = \gamma^\mu_{\alpha} \nabla^\beta F_{\mu\nu}.$$  \hspace{1cm} (69)

Thus, the nilpotent gauge covariant derivative $D_1$ in this gauge is

$$D_1 = e^{V(z)} c^\alpha_{p+1} F_{\alpha}^\beta e^{-V(z)} + b c^\alpha_{p+1} c^\beta_{q+1} \nabla_{\alpha\beta} \delta_{p,q},$$  \hspace{1cm} (70)

where the general superfield $V$ defines the covariantly antichiral superfield $\Psi$ according to eq. (37) and $\Psi$ has the properties presented above. The dependence on $V$ and specifically on gauge covariant combination $\Psi$ cannot be gauged away in this gauge, since the only remaining symmetry is the one with the chiral parameter. In this sense it is a non-trivial solution to the cohomology equation which describes ten-dimensional supersymmetry.

We hope that this construction can be generalized to incorporate D=10 supergravity interacting with the supersymmetric Yang-Mills system. This would be an important step towards understanding the superstring geometry.

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