We investigate whether the generalized second law is valid, using two dimensional black hole spacetime, irrespective of models. A time derivative form of the generalized second law is formulated and it is shown that the law might become invalid. The way to resolve this difficulty is also presented and discussed.

I. INTRODUCTION

One of the most interesting developments in black hole physics is a discovery of the analogy between certain laws of black hole mechanics and the ordinary laws of thermodynamics [1]. According to this analogy, Bekenstein [2] introduced the concept of the black hole entropy as a quantity proportional to the surface area of the black hole (the proportionality coefficient was fixed by Hawking’s discovery of black hole radiance [3] later) and conjectured that the total entropy never decreases in any process, where the total entropy is the sum of the black hole entropy and the ordinary thermodynamic entropy of the matter outside the black hole. This is known as the generalized second law (GSL) of thermodynamics and it is important to check the validity of this conjecture because the validity strongly supports that the ordinary laws of thermodynamics can apply to a self-gravitating quantum system containing a black hole. Especially, it strongly suggests the notion that $A/4$ ($A$ is the surface area of the black hole) truly represents the physical entropy of the black hole.

In order to interpret $A/4$ as the black hole entropy, it would be necessary to derive $S_{BH} = A/4$ from a statistical mechanical calculation by counting the number of internal states of the black hole. The microscopic derivation of the black hole entropy along this line achieved some results in the recent progress in superstring theory [4]. However, general arguments for the validity of the second law of thermodynamics for ordinary systems are based on notions of the “fraction of time” a system spends in a given macroscopic state. Since the nature of time in general relativity is drastically different from that in nongravitational physics, it is not clear how the GSL will arise even if $A/4$ represents a measure of the number of internal states of the black hole. Therefore, it is important to examine the validity of the GSL by itself, in order to understand the connection between quantum theory, gravitation and thermodynamics further.

Historically, gedanken experiments have been done to test the validity of the GSL. The most famous one is that in which a box filled with matter is lowered to near the black hole and then dropped in [2]. Classically, a violation of the GSL can be achieved if one lowers the box sufficiently close to the horizon. However, when the quantum effects are properly taken into account, it was shown by Unruh and Wald [5] that the GSL always holds in this process.

On the other hand, there are some people who tried to prove the GSL under several assumptions for more general situations. Frolov and Page [6] proved the GSL for an eternal black hole by assuming that (i) the process in the investigation is quasistationary which means that the change in the black hole geometry are sufficiently small compared with the corresponding background quantities, (ii) the state of matter fields on the past horizon $\mathcal{H}^-$ is a thermal state with the Hawking temperature, (iii) initial set of radiation modes on the past horizon $\mathcal{H}^-$ and that on the past null infinity $I^-$ are quantum mechanically uncorrelated, and (iv) the Hilbert space and the Hamiltonian of modes at $\mathcal{H}^+$ are identical to those of modes at $\mathcal{H}^-$. But these assumptions are questionable for the black hole formed by a gravitational collapse. That is, the assumptions (iii) and (iv) break down due to the correlation between modes at $\mathcal{H}^-$ and modes at $I^-$ located after the horizon formation and the violation of time reversal symmetry, respectively. So we think that their proofs should be improved to realistic black holes formed by gravitational collapse.

The GSL for the black hole formed by gravitational collapse was studied by Sorkin [7] and Mukohyama [8], making use of the nondecreasing function in a Markov process. The proof finally come to showing that the matter fields in the black hole background have a stationary canonical distribution with its temperature equal to that of the black hole and the canonical partition function remains a constant.

But there are several problems in their proofs. Mukohyama showed that the canonical distribution with temperature equal to the black hole is stationary by calculating the transition matrix between states at the future null infinity $I^+$. However, the transition matrix does not satisfy the nondecreasing function in a Markov process. Moreover, the transition matrix is not a simple function of the black hole geometry. Therefore, it is important to improve their proofs to realistic black holes formed by gravitational collapse.

We investigate whether the generalized second law is valid, using two dimensional black hole spacetime, irrespective of models. A time derivative form of the generalized second law is formulated and it is shown that the law might become invalid. The way to resolve this difficulty is also presented and discussed.
and states at the portion of the past null infinity $I^-$ after the formation of the event horizon $H^+$. But, in collapsing cases, the assumption (i) can not be justified in general. By contrast, since Sorkin argued any process occurring between two adjacent time slices, the assumption (i) is valid. He concluded that the canonical distribution of matter fields are stationary because the Hamiltonian does not change between the two adjacent time slices, thanks to the time translation invariance in the background. Although he assumed implicitly the existence of the Killing time slices that do not go through the bifurcate point, and derived his result, there would not exist such Killing time slices that he had taken. If we take the Killing time slices, there is no energy flux across the event horizon. It means that we cannot see evaporating black hole by the Killing time slices.

In order to satisfy the assumption (i), we consider the infinitesimal time development of total entropy in two dimensional theories of gravity. Although we think two dimensional spacetime, it is worth to investigate the GSL in two dimensional black hole spacetime if there also exists the same black hole physics as those in four dimensional one (causal structure, Hawking radiation and so on). Because, in this case, we can expect that the essential point of the four dimensional physics would not be lost.

Using the Russo-Susskind-Thorlacius (RST) model [9], Fiola, et.al. discussed the infinitesimal time development of total entropy and showed that the GSL in the model is valid under suitable conditions. Although their investigation is beyond the quasistationary approximation and takes account of quantum-mechanical back-reaction effects, their argument is restricted to the very special (RST) model and it is too hasty in concluding that the GSL generally holds even in two dimensional spacetime. Because if “black hole entropy” truly represents the physical entropy of a black hole, it would be necessary to confirm the validity of the GSL for the more general models which possess the black hole mechanics.

The purpose of this paper is to investigate the GSL in any two dimensional black hole spacetime with the first law of black hole mechanics, irrespective of models. In fact, the existence of the first law is guaranteed for the wide class of gravitational theories by using the Noether charge method [10].

First, we write the change in total entropy between two adjacent time slices in terms of quantities of matter fields, using the assumption (i) and the first law of black hole mechanics. Thus, our task is to calculate the energy-momentum tensor and the entanglement entropy of matter fields. These are obtained easily for conformal fields in two dimensional spacetime. After these calculations, we will demonstrate that the GSL does not always hold for conformal vacuum states in a two dimensional black hole for two reasons. The first is that the GSL is violated by the decrease of the entanglement entropy of the field associated with the decrease of the size of the accessible region. But it might be possible to subtract this term by some physical procedure and define the new entropy. The second is that the GSL for the new entropy would be violated for some class of the vacuum states. It might suggest that even the GSL for the new entropy does not hold as long as there does not exist a physical reason that exclude these vacuum states. In this sense, there seem to exist two difficulties to rescue the GSL.

This paper is organized as follows. In Sec. II we present two dimensional black holes that we will consider in this paper and formulate a time derivative form of the GSL. Since we express the change in total entropy in terms of physical quantities of matter fields, our task is to calculate a time evolution of matter fields in a fixed black hole background. In Sec. III we calculate the change in energy of matter fields for general conformal vacuum states. In Sec. IV the entanglement entropy is obtained. By way of illustration, these results are applied to the typical two vacuum states: the Hartle-Hawking state and the Unruh one. Then in Sec. V it is shown that the time derivative form of the GSL does not always hold for the general situations. We will also give a physical interpretation of our result. Sec. VI is devoted to summary and discussion about our results. In particular, we propose a new entropy and argue the validity of the GSL for this quantity.

II. TWO DIMENSIONAL BLACK HOLES AND THE GSL

A. Two dimensional black holes

Four dimensional gravitational theories have many degrees of freedom and inherent complexity. So it would be useful to consider a toy model in which greater analytic control is possible. In our analysis, we consider any two dimensional theories of gravity which satisfy the following two assumptions; (1) the theory allows a stationary black hole solution, and (2) there exist black hole physics similar to those in four dimensional gravitational theories.

Since we want to examine the validity of the GSL in the two dimensional eternal black hole background, we assume by assumption (1) that the spacetime possesses an event horizon and a timelike Killing vector. Since we can always take spacelike hypersurface which is orthogonal to the orbits of the isometry in two dimensional spacetime, this means that the theory has a static black hole solution as
\[ ds^2 = -\xi^2(r) dt^2 + \frac{dr^2}{\xi^2(r)} \]
\[ = -\xi^2 \ dx^+ \ dx^- , \]  
(2.1)  
(2.2)

where \( \xi^\mu = (\partial/\partial t)^\mu \) is the timelike Killing vector which is normalized s.t. \( \xi^2 \to 1 \) as \( r \to \infty \), \( x^\pm = t \pm r_* \) and \( r_* = \int dr/\xi^2 \), respectively. The position of the horizon \( \mathcal{H} \) is specified by \( r \) with \( \xi(r) = 0 \) and the surface gravity of the black hole is given by \( \kappa = \partial_+ \ln \xi^2 |_{\mathcal{H}} \).

By assumption (2), we require that the black hole satisfies the first law of black hole mechanics. This is necessary to formulate the GSL in the form of the next subsection and to keep the essential features of four dimensional black hole physics in our toy model.

In fact, Wald [10] derived a first law of black hole mechanics for any diffeomorphism invariant gravitational theories in any dimensions relied on the Noether charge associated with the diffeomorphism invariance of the action \( [1] \). His technique is a quite general approach for a stationary black holes with a Killing horizon, and reproduces a known result for Einstein gravity with ordinary matter actions. Since gravitational theories are generally defined from a diffeomorphism invariant action, this assumption seems to hold naturally for a very broad class of gravitational theories.

One evidence to justify these assumption is the existence of an interesting toy model which satisfy these assumptions. It is known that the CGHS model [12] has a static black hole solution which evaporates by the Hawking effect, semiclassically. Moreover, the thermodynamical nature of this solution had been investigated by Frolov [13] and shown that black holes in the CGHS model also satisfy the three laws (including the first law of black hole mechanics) similar to "standard" four dimensional black hole physics.

Therefore, it is quite reasonable to think that these two assumptions hold for a wide class of gravitational theories.

**B. The GSL under the quasistationary approximation**

Before examining whether the GSL holds, we formulate a precise statement of the GSL in the quasistationary approximation.

As stated in the introduction, we foliate our black hole spacetime by spacelike time slices that are across the event horizon and do not cross one another at the horizon. We take two adjacent time slices among them to consider a quasistationary process (to justify the assumption (i) in the introduction) and consider the change in total entropy between two time slices.

Under the situation satisfying the assumption (i), by making use of the first law of black hole mechanics \( (\Delta S_{BH} = \Delta E_{BH}/T_{BH} \equiv \beta_{BH} \Delta E_{BH}) \), we can rewrite the change in total entropy in terms of quantities of matter fields alone,

\[ \Delta S_{total} = \Delta S_M + \Delta S_{BH} \]
\[ = \Delta S_M + \beta_{BH} \Delta E_{BH} \]
\[ = \Delta S_M - \beta_{BH} \Delta E_M , \]  
(2.3)  
(2.4)  
(2.5)

where we used the energy conservation law \( (\Delta E_{BH} = -\Delta E_M) \) in the last line.

Thus, our task is to calculate the change in energy and entropy of matter fields between two adjacent time slices in the black hole background. We will use the entanglement entropy of matter fields outside the horizon as the quantity \( S_M \). These are obtained easily for massless conformal fields in two dimensional spacetime.

Further if we define the free energy of matter fields \( F_M \) by \( F_M \equiv E_M - \beta_{BH} S_M \), we can write

\[ \Delta S_{total} = -\beta_{BH} \Delta F_M , \]  
(2.6)

and to prove the GSL in the quasistationary approximation is equivalent to show that the free energy \( F_M \) is a monotonically decreasing function of time. So we will be concerned with examining the change in free energy of matter fields immersed in the black hole background as a heat bath.

---

1 In the Euclidean method, we can get the Hawking temperature \( T_{BH} = \kappa/2\pi \) by requiring the nonsingularity of the Euclidean metric \( [1] \). Thus, we regard the quantity \( \kappa/2\pi \) as the Hawking temperature in the Noether charge method.
III. THE CALCULATION OF THE ENERGY-MOMENTUM TENSOR

For two dimensional conformal fields, we can obtain the energy-momentum tensor $T_{\mu \nu}$ by using its transformation law under conformal transformations and the trace anomaly formula, which is for a scalar field,

$$T = \frac{R}{24\pi}.$$  \hspace{1cm} (3.1)

When the spacetime metric in interest is given by

$$ds^2 = -\hat{\Omega}^2 d\hat{U} d\hat{V},$$

$T_{\mu \nu}$ for a conformal scalar field is written as

$$T_{\mu \nu}[g_{ab}] = T_{\mu \nu}[\eta_{ab}] + \theta_{\mu \nu} + \frac{T[g_{ab}]}{2} g_{\mu \nu},$$ \hspace{1cm} (3.3)

where the null coordinate $(\hat{U}, \hat{V})$ is given by $\hat{U} = \hat{U}(x^-), \hat{V} = \hat{V}(x^+)$, and $g_{ab} = \hat{\Omega}^2 \eta_{ab}$, respectively. Note that if we take the conformal vacuum, then the first term on the R.H.S. of Eq. (3.3) vanishes.

Now we apply this result to a two dimensional black hole in the previous section. We assume that we take the conformal vacuum associated with the $\hat{U}$ and $\hat{V}$. Due to the existence of the timelike Killing vector $\xi^\mu = (\partial/\partial t)^\mu$, the quantity $E_\Omega = -\int_\Sigma d\Sigma^\nu T_{\mu \nu} \xi^\mu$ is a function of the boundary of $\Sigma$. We put the inner boundary $P_0 = (x_0^+, x_0^-)$ of $\Sigma$ on the future horizon $H^+$ and fix the outer boundary $P_1 = (x_1^+, x_1^-)$ apart from the black hole. And then, we consider the change in $E_\Omega(x_0^+)$ by moving the inner boundary $P_0$ along $H^+$. This is given by

$$\frac{dE_\Omega(x_0^+)}{dx_0^+} = -T_{++}|_{H^+}.$$ \hspace{1cm} (3.5)

These relations are sketched in Fig.1.

![Figure 1](image_url)

**FIG. 1.** This figure shows schematic explanation of situation considered here, and the relation between the quantities that is necessary to calculate the change in energy between two spacelike time slices $\Sigma_0$ and $\Sigma_1$. 

4
Then, we can rewrite \( T_{++} \) by using the relations Eq.(3.2) as
\[
T_{++} = (\partial_+ V)^2 T_{V \dot{V}} = - (\partial_+ \dot{V})^2 \frac{\tilde{\Omega} \Omega^2}{12 \pi} \Omega^{-1}
\]
and can express the change in \( E_\Omega(x^+_0) \) in terms of the conformal factor \( \tilde{\Omega} \):
\[
\frac{dE_\Omega(x^+_0)}{dx^+} = - T_{++} |_{H^+} = -1 \frac{1}{12 \pi} \left[ \frac{\partial_+^2 \tilde{\Omega}}{\Omega} \bigg|_{H^+} - \frac{\partial_+ \tilde{\Omega}}{\Omega} \bigg|_{H^+} \right].
\]

Next, we apply the above result to the typical two vacuum states; the Hartle-Hawking one (\( HH \)) and the Unruh one (\( U \)).

For simplicity, we consider the case in which the horizons are not degenerate. If there exist several horizons, we take the most outer one and proceed to our arguments. In this case, we can rewrite the metric as follows;
\[
ds^2 = - \Omega_{HH}^2 dUdV = - \Omega_U^2 dU dx^+ ,
\]
\[
\Omega_{HH}^2 = \frac{\kappa^2}{\kappa^2 |U|} ,
\]
\[
\Omega_U^2 \equiv \Omega_{HH}^2 dV \bigg|_{dx^+} = \Omega_{HH}^2 e^{\kappa x^+} ,
\]
where the null coordinate \((U, V)\) is a Kruskal like one which is regular at the horizon \( H^+ \) given by \(|U| = e^{-\kappa x^-}/\kappa \) and \(|V| = e^{\kappa x^+}/\kappa \).

The Hartle-Hawking state represents a state in equilibrium with the black hole and is uniquely characterized by its global nonsingularity and its isometry invariance under the Killing time. We substitute \( \tilde{\Omega} = \Omega_{HH} \) in Eq.(3.10) and use \((U, V)\) for \((\tilde{U}, \tilde{V})\), respectively. Using the relation \( dV_0 = \exp(\kappa x^+_0) dx^+_0 \) and the fact that \( H^+ \) is a Killing horizon \((\partial_+ \kappa = 0)\), we obtain the change in energy for the Hartle-Hawking state as
\[
\frac{dE_{HH}(x^+_0)}{dx^+_0} = - T_{++} |_{H^+} = 0 .
\]
This result is expected one from the fact that the energy flow lines of the Hartle-Hawking state are along the orbits of the Killing vector because the Hartle-Hawking state is stationary state with respect to the Killing time.

On the other hand, the Unruh state represents, on the eternal black hole, a state which is in a gravitationally collapsing spacetime. Substituting \( \tilde{\Omega} = \Omega_U \), and using \((U, x^+)\) for \((\tilde{U}, \tilde{V})\), respectively, we get \( T_{++} |_{H^+} = - \kappa^2/(48 \pi) \) at the future event horizon, and obtain the result
\[
\frac{dE_{U}(x^+_0)}{dx^+_0} = + \frac{\kappa^2}{48 \pi} = + \frac{\pi}{12} (T_{BH})^2 ,
\]
where we used the fact that the Hawking temperature is given by \( T_{BH} = \frac{\kappa}{2 \pi} \). This represents the energy density of a right-moving one dimensional massless gas with the temperature \( T_{BH} \) and can be interpreted as the outgoing energy flux due to the Hawking radiation (Appendix A).

**IV. THE CALCULATION OF THE ENTANGLEMENT ENTROPY**

In this section, we will calculate the entanglement entropy \( S_M \) for the state of the fields outside the event horizon. The concept of the quantum entanglement entropy is associated with the notion of coarse graining associated with a division of the Hilbert space of a composite system. The division may be introduced by dividing the whole degrees of
freedom into accessible (system in interest) and inaccessible ones (environment). For instance, in a spacetime with black holes, it is natural to take degrees of freedom outside the event horizons as the accessible ones. The density matrix appropriate to the system in interest is obtained by tracing the whole density matrix $\hat{\rho}_{\text{whole}}$ over the environment

$$\hat{\rho}_{\text{sys}} = \text{Tr}_{\text{env}}(\hat{\rho}_{\text{whole}}).$$

(4.1)

This reduced density matrix no longer describes a pure state generally, even though the whole system is pure.

And then, we define the entropy of the system in interest as

$$S_{\text{sys}} = -\text{Tr}_{\text{sys}}[\hat{\rho}_{\text{sys}} \ln \hat{\rho}_{\text{sys}}].$$

(4.2)

The quantity $S_{\text{sys}}$ describes correlations between the system in interest and the environment, and measures the information which is lost by tracing over the environment. Note that when the whole system is in a pure state, there is a symmetry with respect to an exchange of the system in interest for the environment; the two density matrices obtained by tracing over accessible degrees of freedom or inaccessible ones give the same entropy [16]. When the whole system is in a mixed state, this symmetry hold no longer generally.

We are interested in the entanglement entropy of a local quantum field associated with the division of degrees of freedom by partitioning a time slice $\Sigma$ into accessible region $D$ and inaccessible one $\Sigma - D$. Then, the entanglement entropy is invariant for local deformations of the time slice which keep the boundary $\partial D$ fixed in the spacetime. This fact follows the unitary evolution of the whole system and the local causality because, in this case, the unitary evolution operator for the whole becomes a product of two commuting unitary operators, each of which is the evolution operator associated with the deformation of the time slice in the accessible or the inaccessible region. In this sense, the entanglement entropy of any local field is a quantity connected with the boundary of the accessible region.

### A. Entanglement entropy in flat spacetime

First, we consider the Minkowski vacuum of a massless conformal scalar field in flat two dimensional spacetime and calculate the entanglement entropy.

The metric of the flat spacetime is written by

$$ds^2 = -dUdV.$$  

(4.3)

We want to compute the entanglement entropy of the Minkowski vacuum when the accessible region is given by the interval between the point (inner boundary) $P_0 = (U_0, V_0)$ and the point (outer boundary) $P_1 = (U_1, V_1)$.

To proceed this calculation, we need to expand the Minkowski vacuum by the states which live in the Hilbert space associated with the accessible region and the ones which live in the Hilbert space associated with the inaccessible region. Such decomposition can be achieved by introducing the Rindler chart such that the point $P_0$ corresponds to the bifurcate point. This is given by Fiola, et.al. [17], following Unruh’s calculation [18].

Then, we can derive the entanglement entropy by standard procedure using Eq.(4.1) and Eq.(4.2). The result is given by [17]

$$S = \frac{1}{6} \left[ (\ln D - \ln \epsilon_0) + (\ln D - \ln \epsilon_1) \right],$$

(4.4)

where $D$, $\epsilon_0$ and $\epsilon_1$ are the size of the accessible region defined by $D^2 = |(V_1 - V_0)(U_1 - U_0)|$, the short distance cutoff at $P_0$ and $P_1$, respectively.

Next, we examine the change of entropy between two adjacent time slices. We consider the case that the outer boundary $P_1$ is fixed and the inner boundary $P_0$ moves along the null line $U_0 = \text{constant}$. Then, we get

$$\frac{dS}{dV_0 \mid U_0 = \text{const.}} = -\frac{1}{6} \frac{1}{V_1 - V_0} < 0,$$

(4.5)

where the proper lengths $\epsilon_0$ and $\epsilon_1$ are fixed.

This decrease of the entanglement entropy can be understood as a result of the decrease of the size of the accessible region (see Sec.\textsuperscript{5} for the more intuitive explanation of this result.)
B. Entanglement entropy in curved spacetime

In this subsection, we will generalize the result in the previous subsection to the case of curved spacetime. Since any two dimensional spacetime is conformally flat, we can write the metric as

$$ds^2 = -\Omega^2 d\hat{U} d\hat{V} = -\hat{\Omega}^2 d\hat{s}^2. \quad (4.6)$$

When we take conformal vacuum associated with the $\hat{U}$ and $\hat{V}$, as the quantum state, the expression Eq.(4.4) can be used as it is, because the spacetime with the metric $d\hat{s}^2$ is flat. Assuming that the accessible region is the interval between $P_0 = (\hat{U}_0, \hat{V}_0)$ and $P_1 = (\hat{U}_1, \hat{V}_1)$, the entanglement entropy is given by

$$S_\Omega = \frac{1}{6} \ln \left| \left( \hat{V}_1 - \hat{V}_0 \right) \left( \hat{U}_1 - \hat{U}_0 \right) \right| \frac{\epsilon_0 \epsilon_1}{\epsilon_0 \epsilon_1} \quad (4.7)$$

$$= \frac{1}{6} \ln \Omega_0 + \frac{1}{6} \ln \Omega_1 + \frac{1}{6} \ln \left| \left( \hat{V}_1 - \hat{V}_0 \right) \left( \hat{U}_1 - \hat{U}_0 \right) \right| - \frac{1}{3} \ln \epsilon, \quad (4.8)$$

where in the second line, we rewrite the short distance cutoffs $\epsilon_0$ and $\epsilon_1$ in the unphysical spacetime with the metric $d\hat{s}^2$ in terms of proper lengths, that is $\epsilon_i = \hat{\Omega}_i \hat{\epsilon}_i$ ($i = 0, 1$) and set $\epsilon_0 = \epsilon_1 = \epsilon$.

Thus, the change in entropy as we move the inner boundary $P_0$ along the future horizon $\mathcal{H}^+$ with the proper length $\epsilon$ and the outer boundary $P_1$ fixed is given by

$$\frac{dS_\Omega}{dx_0^+} = \frac{1}{6} \frac{\partial_x \hat{\Omega}}{\hat{\Omega}} x^+ - \frac{1}{6} \frac{\partial_x \hat{V}_0}{\hat{V}_1 - \hat{V}_0}. \quad (4.9)$$

We apply the above result Eq.(4.9) to the Hartle-Hawking state and the Unruh one.

For the Hartle-Hawking state, we substitute $\hat{\Omega} = \Omega_{HH}$ in Eq.(4.9) and use the relation $d\hat{V}_0 = \kappa V_0 dx_0^+$, we obtain the result

$$\frac{dS_{HH}}{dx_0^+} = -\frac{1}{6} \frac{\kappa V_0}{\hat{V}_1 - \hat{V}_0} < 0. \quad (4.10)$$

Against our intuition, this result shows that the entropy $S_{HH}$ decreases as time elapses, because of the decrease of the size of the accessible region.

For the Unruh state, substituting $\hat{\Omega} = \Omega_U$ in Eq.(4.9) and using $x^+$ for $\hat{V}$, we obtain

$$\frac{dS_U}{dx_0^+} = \frac{\kappa}{12} - \frac{1}{6} \frac{1}{\hat{x}_1^+ - \hat{x}_0^+} \quad (4.11)$$

$$= \frac{\pi}{6} T_{BH} - \frac{1}{6} \frac{1}{\hat{x}_1^+ - \hat{x}_0^+}. \quad (4.12)$$

As the interpretation of Eq.(3.15), the first term in Eq.(4.12) can be understood as the entropy density of right moving one dimensional massless gas with temperature $T_{BH}$ and can be interpreted as the entropy production rate due to the Hawking radiation (Appendix A). Since the second term gives a negative contribution, the R.H.S. of Eq.(4.12) can not have a definite sign. In fact, this term grows without bound as the time slice approaches to the null one.

Note that the first term in Eq.(4.12) which has a natural interpretation as the Hawking radiation appears by fixing the proper short distance cutoff $\epsilon$, not but fixing the cutoff $\hat{\epsilon}$ in the unphysical spacetime.

V. PHYSICAL INTERPRETATION OF OUR RESULT

We summarize the results for the typical two vacuum states in terms of the change in free energy. Using the relations $\beta_{BH} = 2\pi/\kappa$ and the definition $F_M = E_M - \beta_{BH}^{-1} S_M$, the change in free energy is given by:

$$\frac{dF_{HH}}{dx_0^+} = \frac{dE_{HH}}{dx_0^+} - \frac{\kappa}{2\pi} \frac{dS_{HH}}{dx_0^+} = + \frac{\kappa^2}{24\pi} \frac{V_0}{\hat{V}_1 - \hat{V}_0} > 0 \quad (5.1)$$

for the Hartle-Hawking state,
\[
\frac{dF_U}{dx_0} - \frac{\kappa}{2\pi} \frac{dS_U}{dx_0} = \frac{\kappa^2}{24\pi x_1^+} - \frac{\kappa}{48\pi} \frac{1}{x_1^- - x_0^+}
\]

(5.2)

for the Unruh state.

Where, if we take the limit that our accessible region extends to the spatial infinity, i.e., \(V_1, x_1^+ \to \infty\), we obtain the desired result for the GSL. However, provided that we hold the accessible region finite, the above results show that the free energy (total entropy) does not necessarily decrease (increase) and the time derivative form of the GSL does not always hold for two dimensional eternal black hole background. Note that the second term in Eq.(5.2) can grow unboundedly as time evolves. We can recognize that the violation of the GSL is caused by the decrease of the entanglement entropy of fields associated with the decrease of the size of the accessible region. The change in the entanglement entropy comes from two parts: one is associated with ultraviolet divergent term which represents short distance correlations between the modes near the horizon; and the other is associated with the size of the accessible region which contains long distance correlations between the modes far inside and far outside the horizon (Hereafter, we call this term infrared divergent term.). The ultraviolet divergent term can be interpreted as the entropy production due to the Hawking radiation for the Unruh state and it can be thought to give non-negative contribution to the change in total entropy. On the other hand, the infrared divergent term gives negative contribution due to the decrease of the size outside the event horizon and it is just this behavior that causes the violation of the GSL.

When the outer boundary exists at a finite distance, we do not observe the whole external region outside the horizon. Therefore one might think that the violation of the GSL is brought about by interaction with the field degrees of freedom in the rest external region that we do not observe. That is, one suspect that the composite system composed of the black hole and the field degrees of freedom in the accessible region might not be isolated. However, since we fix the outer boundary, there is no exchange of heat, work and so on, by the interaction between the composite system and the rest one. Therefore, the composite system is isolated in substance and our results Eqs.(5.1) and (5.2) point to the violation of the GSL.

Fiola, et.al. [17] considered the GSL for a black hole formed by gravitational collapse using the RST model [9] and concluded that the GSL holds in the RST model under suitable conditions. Note that our conclusion is different from theirs in spite of the fact that the entanglement entropy of fields are used as the quantity \(S_M\) for both cases. Of course, there are various differences between these works. Especially, a main difference is the contribution of the third term in Eq.(4.8) (the last term in their Eq.(75)) to the expression \(dS_M/dx_0^+\). That is, the behavior of the infrared divergent term aforementioned: the contribution in their case is always positive, while one in ours is negative.

The origin which causes this difference comes from the existence of the reflecting boundary, i.e., the difference of the boundary conditions. In the RST model, they need to impose reflecting boundary condition at "central point", beyond where the dilaton becomes an imaginary value. Therefore, their quantum state of the scalar field has correlation between the right moving modes and the left moving ones. In eternal black hole background, this corresponds to the case in which we make initial state to have correlation between the modes on \(\mathcal{H}^-\) and ones on \(\mathcal{I}^-\), while the states in our investigation have no correlation between them. This brings about the difference between the dependence of \(dS_M/dx_0^+\) on \(P_0\) in each case. (So if we impose the suitable boundary condition, we could reproduce the result corresponding to theirs.)

Next, we will give a more intuitive explanation of this difference by using quantum correlation between two wave packets of matter fields.

We first consider the case for an eternal black hole. It is one example of the spacetime without a reflecting boundary. We take the conformal vacuum states and consider two adjacent time slices \(\Sigma_1\) and \(\Sigma_2\) and notice one of the most correlated pairs \(\langle A, \bar{A}\rangle\) of the left-moving modes that are specified by the equal null distance \(\Delta V\) from \(V = 0\) (Fig. 2).

---

2 See [18] for the example of the Minkowski spacetime. The same argument can be applied to the black hole by choosing suitable basis: we can make the most entangled pairs located at the equal null distance from \(v = v_0\) (or \(V = 0\)) for a black hole with a boundary (or without a boundary).
FIG. 2. The most correlated pair \((A, \bar{A})\) on eternal black hole background. For an observer that locates at outside of the horizon, the contribution of this pair to the entropy decreases as time evolves and cross section \(\Sigma_t \cap \mathcal{H}^+\) move forward along \(\mathcal{H}^+\).

While we can access one of them \((A)\) on \(\Sigma_1\), we can access neither of them on \(\Sigma_2\) due to the existence of the horizon \(\mathcal{H}^+\). Thus, the contribution of this pair to the entanglement entropy exists on \(\Sigma_1\) and vanishes on \(\Sigma_2\). The same argument can also be applied to all the other pairs. Since we move the inner boundary \(P_0\) of \(\Sigma\) along the future horizon \(\mathcal{H}^+\), this fact reflects as a decrease of entanglement entropy as time evolves. Note that right-moving mode does not influence the change in entanglement entropy. After all, the decrease of the entanglement entropy for an eternal black hole can also be understood by the decrease of the number of the pairs which contributes to the entanglement entropy.

Subsequently, we apply the above arguments to the black hole formed by gravitational collapse with a reflecting boundary \(\text{III}\). Note that, different from the no boundary case, the correlation between the right moving modes and the left moving ones is induced by the existence of the boundary in this case. This produces an important difference from no boundary case.

We introduce the null coordinates \((u, v)\) and suppose that the formation of the event horizon \(\mathcal{H}^+\) is at \(v = v_0\). We notice one of the most correlated pairs \((A, \bar{A})\) of the ingoing modes that are specified by the equal null distance \(\Delta v\) from \(v = v_0\), and consider two adjacent time slices \(\Sigma_1\) and \(\Sigma_2\) (Fig.3).

FIG. 3. The most correlated pair \((A, \bar{A})\) on black hole formed by gravitational collapse with a boundary. For this case, the contribution of this pair to the entropy increases as time evolves.

\(^3\)Note that we can consider the black hole formed by gravitational collapse without a reflecting boundary: The shock wave solution in CGHS model, for example.
Since we can access both modes \((A, \bar{A})\) on \(\Sigma_1\), the contribution of this pair to the entanglement entropy is zero. But on \(\Sigma_2\), since we can access only one of them \((\bar{A})\), nonzero contribution is produced. As the same argument can be repeated for all the other pairs, the entropy is increasing as time evolves in the collapsing model with a boundary.

After all, we can understand the behavior of the infrared divergent term intuitively: it gives a positive contribution to the change in entropy in the cases with a boundary, and gives a negative contribution in the ones without a boundary. In other words, we can say that our work gives the situation where the entanglement entropy increase (or decrease) in time.

VI. SUMMARY AND DISCUSSION

In this paper, we examined the validity of the GSL for a black hole in two dimensional gravitational theories under the quasistationary approximation. In order to satisfy the quasistationary approximation, we considered the infinitesimal time development of the total entropy of the black hole and the field degrees of freedom outside the horizon. Our approach can be applied to test the validity of the GSL in any two dimensional stationary black hole spacetime which possesses the first law of black hole mechanics, irrespective of models.

Making use of the fact that the change in total entropy is equal to minus the change in free energy of the fields outside the horizon under the quasistationary approximation, we calculated the change in free energy for the conformal vacuum states. In particular, we applied the result to the Hartle-Hawking state and the Unruh one in the eternal black hole background. And then, we showed the differential form of the GSL to be invalid when our accessible region is finite and to be valid for infinite accessible region.

We recognized that the origin of the violation of the GSL is the decrease of the entanglement entropy of fields associated with the decrease of the size of the accessible region. However, the behavior of this term is something curious in our intuitive terms. Because it is usually thought that the total entropy does not change for the Hartle-Hawking state in eternal black hole background, which describes the thermal equilibrium state between the black hole and the surrounding matter field. So, it is natural to expect that the entropy production for the Hartle-Hawking state does not occur.

Myers \[20\] and Hirata, et.al. \[21\] examined the validity of the GSL by using the Noether charge method and taking into account 1-loop quantum back-reaction and showed that it is satisfied for both the RST model and the wide class of the CGHS model. In their analysis, the third term in Eq.(4.8) does not appear. This might suggest that the infrared divergent term could be dropped by some physical reasons, though, further argument about excluding this term is necessary.

Anyway, it is necessary to subtract this term and define the total entropy to rescue the GSL. Therefore, from now on, we suppose that this subtraction is performed systematically, and argue the GSL for this new quantity. That is, we define the new entropy by

\[
S^*_{\hat{\Omega}} = S_{\hat{\Omega}} - \frac{1}{6} \ln \left( \frac{\hat{D}}{\epsilon} \right),
\]

where \(\hat{D}\) is the size of the accessible region in the unphysical spacetime in which the conformal vacuum is defined and \(\epsilon\) is a proper length of short distance cutoff. Then,

\[
\frac{dS^*_{\hat{\Omega}}}{dx_0^+} = \frac{1}{6} \frac{\partial_{+} \hat{\Omega}}{\hat{\Omega}} \bigg|_{H^+},
\]

and we obtain the final result

\[
\frac{dF^*_{\hat{\Omega}}}{dx_0^+} = -\frac{\lambda}{2\pi} \frac{dS^*_{\hat{\Omega}}}{dx_0^+} + \frac{dE_{\hat{\Omega}}}{dx_0^+}
\]

\[
= -\frac{1}{12\pi} \frac{\partial^2 \hat{\Omega}}{\hat{\Omega}} \bigg|_{H^+}
\]

Since we consider the massless field, the term proportional to \(\ln \hat{D}\) appears in the result (6.3). However, if we consider the massive field, this term would not appear (inverse of the fixed mass \(1/m\) of the field enters into the expression instead of \(\hat{D}\)).
Therefore, the validity of the GSL depends only on the signature of the $\partial_+^2 \hat{\Omega}$ at the future horizon $\mathcal{H}^+$. If we apply this result to the typical two states aforementioned, we obtain the desired result: $dF_+ = 0$ for the Hartle-Hawking state and $dF_+ < 0$ for the Unruh state. However, since we can choose the vacuum state as we like (in other words, we can perform conformal transformation freely), we can violate the GSL by choosing the suitable conformal vacuum which satisfy $\partial_+^2 \hat{\Omega}|_{\mathcal{H}^+} < 0$ even if we can subtract the infrared divergent term by some physical procedure successfully.

Now we consider whether the violation of the GSL occurs or not for the wide range in the following sense: the state that violate the GSL is not a special one; and the violation occurs sufficiently long time. In our case, there seem to exist a lot of states which satisfy both of these conditions. Of course, since the vacuum states that we prepare should be a physically reasonable ones, there would be some requirement from physical nature. For example, the expectation value of the energy-momentum tensor should be finite at $\mathcal{H}^+$. But, in practice, this condition can not impose any condition on $\partial_+^2 \hat{\Omega}|_{\mathcal{H}^+}$, and we can not remove the cases that violate the GSL by this criterion. Further, noting that since $\hat{\Omega}$ is a function of the spacetime point, it is possible to violate the GSL during sufficiently long time interval, by choosing a suitable form of the function.

One of such examples is the case $\hat{\Omega}^2 = \xi^2 / |\kappa| U \cosh (\kappa x^+)$). In this case, the behavior of the energy-momentum tensor is regular at $\mathcal{H}^+$ and $\mathcal{I}^\pm$, so it seems to be a physically reasonable state. And then, it approaches the Hartle-Hawking state asymptotically at $\mathcal{I}^+$ (as $x^+ \to \infty$) while keeping $\partial_+^2 \hat{\Omega}|_{\mathcal{H}^+} < 0$. Thus, the time duration during which the violation of the GSL occurs can be taken arbitrary long. It would not seem that this is a special case, because we can find a lot of examples which give a similar behavior as this one.

Therefore, it seems that the violation of the GSL occurs for a rather wide range of vacuum states unless there exists a “selection rule” that all physically acceptable states should satisfy the condition $\partial_+^2 \hat{\Omega}|_{\mathcal{H}^+} > 0$. Then, considering that our analysis is independent of models, we would have only two choice to resolve the violation of the GSL. One is that the states which satisfy $\partial_+^2 \hat{\Omega}|_{\mathcal{H}^+} < 0$ like an above example should be excluded by some physical requirement that we can not find now. It means that all the physically reasonable states are restricted to ones that satisfy the condition $\partial_+^2 \hat{\Omega}|_{\mathcal{H}^+} > 0$. The other is that we improve the entropy formula $S^*$ further. That is, the violation of the GSL was thought to be caused by the wrong definition of the entropy.

In either cases, further investigation would be necessary and resolution of the difficulty must be brought over the future works.

ACKNOWLEDGMENTS

The authors would like to thank M. Shibao for valuable comments and stimulating discussions. We also appreciate Professor A. Hosoya for continuous encouragement.

APPENDIX A:

We can interpret the results, Eqs.(4.15) and (4.12) as the energy and entropy production rate due to the Hawking radiation.

An observer at the null infinity observes quanta distributed per mode and per unit time by

$$ < n_\omega > = \frac{\Gamma_\omega}{\exp(\omega/ T_{BH}) - 1} , $$

(A1)

in a two dimensional black hole background with the temperature $T_{BH}$. The graybody factor for an massless conformal scalar field is $\Gamma_\omega = 1$ because of no scattering.

Therefore, the energy production rate is given by

$$ \frac{dE_{\text{rad}}}{dt} = \frac{1}{2\pi} \int_0^\infty d\omega \omega \frac{\Gamma_\omega}{\exp(\omega/ T_{BH}) - 1} . $$

(A2)

Assuming the canonical distribution, the entropy per mode is given by

$$ S_\omega = (1+ < n_\omega >) \ln(1+ < n_\omega >) - < n_\omega > \ln < n_\omega > , $$

(A3)

and then, the entropy production rate of the emitted radiation is given by
\[
\frac{dS_{\text{rad}}}{dt} = \frac{1}{2\pi} \int_0^\infty d\omega \left[ (1+ < n_\omega >) \ln(1+ < n_\omega >) - < n_\omega > \ln < n_\omega > \right].
\]  

(A4)

The above quantities become for a massless conformal scalar field (\(\Gamma_\omega = 1\)) as

\[
\frac{dE_{\text{rad}}}{dt} = \frac{T_{BH}^2}{2\pi} \int_0^\infty dx \frac{x}{e^x - 1} = \frac{\pi}{12} T_{BH}^2,
\]

(A5)

\[
\frac{dS_{\text{rad}}}{dt} = \frac{T_{BH}}{2\pi} \int_0^\infty dx \left[ \frac{x}{e^x - 1} - \ln(1-e^{-x}) \right] = \frac{\pi}{6} T_{BH}.
\]

(A6)

These coincide with Eq.(3.13) and the first term in Eq.(4.12).

---

[1] J.M. Bardeen, B. Carter and S.W. Hawking, Comm.Math.Phys. 31, 161 (1973). 
[2] J.D. Bekenstein, Phys.Rev. D7, 2333 (1973) ; ibid. D9, 3292 (1974). 
[3] S.W. Hawking, Comm.Math.Phys. 43, 199 (1975). 
[4] A. Strominger and C. Vafa, Phys.Lett. B379, 99 (1996) ; G. Horowitz, “The Origin of Black Hole Entropy in String Theory”, gr-qc/9604051. 
[5] W.G. Unruh and R.M. Wald, Phys.Rev. D25, 942 (1982) ; ibid. D27, 2271 (1983). 
[6] V.P. Frolov and D.N. Page, Phys.Rev.Lett. 71, 3902 (1993). 
[7] R.D. Sorkin, Phys.Rev.Lett. 56, 1885 (1986). 
[8] S. Mukohyama, Phys.Rev. D56, 2192 (1997). 
[9] J.G. Russo, L.Susskind, and L. Thorlacius, Phys.Rev. D46, 3444 (1992) ; ibid. 47, 533 (1993). 
[10] R.M. Wald, Phy.Rev. D48, R3427 (1993) ; V.Iyer and R.M. Wald, Phys.Rev. D50, 846 (1994). 
[11] G.W. Gibbons and S.W. Hawking, Phys.Rev. D15, 2752 (1977). 
[12] C.G. Callan, S.B. Giddings, J.A. Harvey, and A. Strominger, Phys. Rev. D45, R1005 (1992). 
[13] V.P. Frolov, Phys. Rev. D46, 5383 (1992). 
[14] See, for example, N.D. Birrell and P.C.W. Davies, in Quantum Fields in Curved Space, (Cambridge University Press, Cambridge, 1982). 
[15] J.B. Hartle and S.W. Hawking, Phys. Rev. D13, 2188 (1976). 
[16] H.Araki and E.Lieb, Comm.Math.Phys. 18, 160 (1970). 
[17] T.M. Fiola, J. Preskill, A.Strominger, and S.P. Trivedi, Phys. Rev. D50, 3987 (1994). 
[18] W.G. Unruh, Phys.Rev. D14, 870 (1976). 
[19] S. Takagi, Prog. Theor. Phys. Suppl. 88 (1986). 
[20] Myers, Phys. Rev. D50, 6412 (1994). 
[21] Hirata, Fujiwara and Soda, Phys. Lett. B378, 68 (1996).