A problem of Wang on Davenport constant for the
multiplicative semigroup of the quotient ring of $\mathbb{F}_2[x]$

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Abstract

Let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field $\mathbb{F}_q$, and let $f$ be a polynomial of $\mathbb{F}_q[x]$. Let $R = \frac{\mathbb{F}_q[x]}{(f)}$ be a quotient ring of $\mathbb{F}_q[x]$ with $0 \neq R \neq \mathbb{F}_q[x]$. Let $S_R$ be the multiplicative semigroup of the ring $R$, and let $U(S_R)$ be the group of units of $S_R$. The Davenport constant $D(S_R)$ of the multiplicative semigroup $S_R$ is the least positive integer $\ell$ such that for any $\ell$ polynomials $g_1, g_2, \ldots, g_\ell \in \mathbb{F}_q[x]$, there exists a subset $I \subseteq [1, \ell]$ with

$$\prod_{i \in I} g_i \equiv \prod_{i=1}^{\ell} g_i \pmod{f}.$$ 

In this manuscript, we proved that for the case of $q = 2$,

$$D(U(S_R)) \leq D(S_R) \leq D(U(S_R)) + \delta_f,$$

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where

$$\delta_f = \begin{cases} 
0 & \text{if } \gcd(x \ast (x + 1_{\mathbb{F}_2}), f) = 1_{\mathbb{F}_2} \\
1 & \text{if } \gcd(x \ast (x + 1_{\mathbb{F}_2}), f) \in \{x, x + 1_{\mathbb{F}_2}\} \\
2 & \text{if } \gcd(x \ast (x + 1_{\mathbb{F}_2}), f) = x \ast (x + 1_{\mathbb{F}_2})
\end{cases}$$

which partially answered an open problem of Wang on Davenport constant for the multiplicative semigroup of \( \mathbb{F}_q[x] \) (G.Q. Wang, *Davenport constant for semigroups II*, Journal of Number Theory, 155 (2015) 124–134).

**Key Words**: Zero-sum; Davenport constant; Multiplicative semigroups; Polynomial rings

## 1 Introduction

The additive properties of sequences in abelian groups have been widely studied in the field of Zero-sum Theory (see [3] for a survey), since H. Davenport [2] in 1966 and K. Rogers [5] in 1963 independently proposed one combinatorial invariant, denoted \( D(G) \), for any finite abelian group \( G \), which is defined as the smallest \( \ell \in \mathbb{N} \) such that every sequence \( T \) of terms from the group \( G \) of length at least \( \ell \) contains a nonempty subsequence \( T' \) with sum of all terms from \( T' \) being equal to the identity element of the group \( G \). The Davenport constant is a central concept of zero-sum theory and has been investigated by many researchers in the scope of finite abelian groups.

In 2008, Gao and Wang [9] formulated the definition of Davenport constant for commutative semigroups, and made several related additive researches (see [1, 6–8]).

**Definition A.** [9] Let \( S \) be a commutative semigroup (not necessary finite). Let \( T \) be a sequence of terms from the semigroup \( S \). We call \( T \) reducible if \( T \) contains a proper subsequence \( T' \) \( (T' \neq T) \) such that the sum of all terms of \( T' \) equals the sum of all terms of \( T \). Define the Davenport constant of the semigroup \( S \), denoted \( D(S) \), to be the smallest \( \ell \in \mathbb{N} \cup \{\infty\} \) such that every sequence \( T \) of length at least \( \ell \) of terms from \( S \) is reducible.

Before then, starting from the research of Factorization Theory in Algebra, A. Geroldinger and F. Halter-Koch in 2006 have formulated another closely related definition, \( d(S) \), for any commutative semigroup \( S \), which is called the small Davenport constant.

**Definition B.** (Definition 2.8.12 in [4]) For a commutative semigroup \( S \), let \( d(S) \) denote the smallest \( \ell \in \mathbb{N}_0 \cup \{\infty\} \) with the following property:
For any $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in S$ there exists a subset $I \subset [1, m]$ such that $|I| \leq \ell$ and
\[
\sum_{i=1}^{m} a_i = \sum_{i \in I} a_i.
\]

The following connection between the (large) Davenport constant $D(S)$ and the small Davenport constant $d(S)$ was also obtained for any commutative semigroup $S$.

**Proposition C.** ([7]) Let $S$ be a commutative semigroup. Then $D(S)$ is finite if and only if $d(S)$ is finite. Moreover, in case that $D(S)$ is finite, we have
\[
D(S) = d(S) + 1.
\]

Very recently, Wang in 2015 obtained the following result on Davenport constant for the multiplicative semigroup associated with polynomial rings $\mathbb{F}_q[x]$.

**Proposition D.** ([6]) Let $q > 2$ be a prime power, and let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field $\mathbb{F}_q$. Let $R$ be a quotient ring of $\mathbb{F}_q[x]$ with $0 \neq R \neq \mathbb{F}_q[x]$. Then
\[
D(S_R) = D(U(S_R)),
\]
where $S_R$ denotes the multiplicative semigroup of the ring $R$, and $U(S_R)$ denotes the group of units in $S_R$.

However, for the case of $q = 2$, Wang proposed it as an open problem.

**Problem E.** (see concluding remarks in [6]) Let $R$ be a quotient ring of $\mathbb{F}_2[x]$ with $0 \neq R \neq \mathbb{F}_2[x]$. Determine $D(S_R) - D(U(S_R))$.

In this manuscript, we considered this open problem. By using the method employed by Wang, we obtained the following result, which is a partial solution of Problem E.

**Theorem 1.1.** Let $\mathbb{F}_2[x]$ be the ring of polynomials over the finite field $\mathbb{F}_2$, and let $R = \frac{\mathbb{F}_2[x]}{(f)}$ be a quotient ring of $\mathbb{F}_2[x]$ where $f \in \mathbb{F}_2[x]$ and $0 \neq R \neq \mathbb{F}_2[x]$. Then
\[
D(U(S_R)) \leq D(S_R) \leq D(U(S_R)) + \delta_f,
\]
where
\[
\delta_f = \begin{cases} 
0 & \text{if } \gcd(x \ast (x + 1_{\mathbb{F}_2}), f) = 1_{\mathbb{F}_2}; \\
1 & \text{if } \gcd(x \ast (x + 1_{\mathbb{F}_2}), f) \in \{x, x + 1_{\mathbb{F}_2}\}; \\
2 & \text{if } \gcd(x \ast (x + 1_{\mathbb{F}_2}), f) = x \ast (x + 1_{\mathbb{F}_2}).
\end{cases}
\]
The proof of Theorem 1.1

The notations and terminologies used here are consistent to ones used in [1, 6–8]. For the reader’s convenience, we need to give some necessary ones.

Let $S$ be a finite commutative semigroup. The operation on $S$ is denoted by $\cdot$. The identity element of $S$, denoted $0_S$ (if exists), is the unique element $e$ of $S$ such that $e + a = a$ for every $a \in S$. If $S$ has an identity element $0_S$, let

$$U(S) = \{a \in S : a + a' = 0_S \text{ for some } a' \in S\}$$

be the group of units of $S$. For any element $c \in S$, let

$$St(c) = \{a \in U(S) : a + c = c\}$$

denote the stabilizer of $c$ in the group $U(S)$. The Green’s preorder of the semigroup $S$, denoted $\leq_H$, is defined by

$$a \leq_H b \iff a = b \text{ or } a = b + c$$

for some $c \in S$. The Green’s congruence of $S$, denoted $\mathcal{H}$, is defined by:

$$a \mathcal{H} b \iff a \leq_H b \text{ and } b \leq_H a.$$

We write $a <_H b$ to mean that $a \leq_H b$ but $a \mathcal{H} b$ does not hold.

The sequence $T$ of terms from the semigroups $S$ is denoted by

$$T = a_1 a_2 \cdots a_\ell = \bigcup_{a \in S} a^{[v_a(T)]},$$

where $[v_a(T)]$ means that the element $a$ occurs $v_a(T)$ times in the sequence $T$. By $\cdot$ we denote the operation to join sequences. By $|T|$ we denote the length of the sequence, i.e.,

$$|T| = \sum_{a \in S} v_a(T) = \ell.$$

Let $T_1, T_2$ be two sequences of terms from the semigroups $S$. We call $T_2$ a subsequence of $T_1$ if

$$v_a(T_2) \leq v_a(T_1)$$

for every element $a \in S$, denoted by

$$T_2 \mid T_1.$$
In particular, if \( T_2 \neq T_1 \), we call \( T_2 \) a proper subsequence of \( T_1 \), and write
\[
T_3 = T_1 T_2^{[-1]}
\]
to mean the unique subsequence of \( T_1 \) with \( T_2 \cdot T_3 = T_1 \). Let
\[
\sigma(T) = a_1 + a_2 + \cdots + a_\ell
\]
be the sum of all terms of the sequence \( T \). By \( \varepsilon \) we denote the empty sequence. If \( S \) has an identity element \( 0_S \), we allow \( T = \varepsilon \) and adopt the convention that \( \sigma(\varepsilon) = 0_S \). We say that \( T \) is **reducible** if \( \sigma(T') = \sigma(T) \) for some proper subsequence \( T' \) of \( T \) (note that, \( T' \) is probably the empty sequence \( \varepsilon \) if \( S \) has the identity element \( 0_S \) and \( \sigma(T) = 0_S \)). Otherwise, we call \( T \) **irreducible**.

Throughout this paper, we shall always denote
\[
R = \mathbb{F}_2[x]/(f)
\]
to be the quotient ring of \( \mathbb{F}_2[x] \) modulo some nonconstant polynomial \( f \in \mathbb{F}_2[x] \), where
\[
f = f_1^{n_1} \ast f_2^{n_2} \ast \cdots \ast f_r^{n_r},
\]
(1)
such that \( f_1, f_2, \ldots, f_r \) are pairwise non-associate irreducible polynomials of \( \mathbb{F}_2[x] \) with
\[
f_1 = x, \quad f_2 = x + 1_{\mathbb{F}_2},
\]
\[
n_1 \geq 0, \quad n_2 \geq 0, \quad n_3, n_4, \ldots, n_r \geq 1.
\]
Let \( S_R \) be the multiplicative semigroup of the ring \( R \). For any element \( a \in S_R \), let \( \theta_a \in \mathbb{F}_2[x] \) be the unique polynomial corresponding to the element \( a \) with the least degree, i.e.,
\[
\overline{\theta_a} = \theta_a + (f)
\]
is the corresponding form of \( a \) in the quotient ring \( R \) with
\[
\deg(\theta_a) \leq \deg(f) - 1.
\]
By \( \gcd(\theta_a, f) \) we denote the greatest common divisor of the two polynomials \( \theta_a \) and \( f \) in \( \mathbb{F}_2[x] \) (**the unique polynomial with the greatest degree which divides both \( \theta_a \) and \( f \)**), in particular, by (1), we have
\[
\gcd(\theta_a, f) = f_1^{\alpha_1} \ast f_2^{n_2} \ast \cdots \ast f_r^{\alpha_r}
\]
(2)
where \( \alpha_i \in [0, n_i] \) for each \( i \in [1, r] \).
For any polynomial $g$ and any irreducible polynomial $h$ of $F_2[x]$, let $\text{pot}_h(g)$ be the largest integer $k$ such that $h^k | g$. Then in (2), $\alpha_i = \text{pot}_{f_i}(\gcd(\theta_a, f))$ for each $i \in [1, r]$. It is easy to observe that for any two elements $a, b \in S_R$,

$$\gcd(\theta_a, f) = \gcd(\theta_b, f)$$

if and only if

$$\text{pot}_{f_i}(\gcd(\theta_a, f)) = \text{pot}_{f_i}(\gcd(\theta_b, f))$$

for each $i \in [1, r]$.

To prove Theorem 1.1, we still need some lemmas.

**Lemma 2.1.** ([4], Lemma 6.1.3) Let $G$ be a finite abelian group, and let $H$ be a subgroup of $G$. Then, $D(G) \geq D(G/H) + D(H) - 1$.

**Lemma 2.2.** (see [9], Proposition 1.2) Let $S$ be a finite commutative semigroup with an identity. Then $D(U(S)) \leq D(S)$.

**Lemma 2.3.** Let $a$ and $b$ be two elements of $S_R$ with $a \leq_H b$. Let $\alpha_i = \text{pot}_{f_i}(\gcd(\theta_a, f))$ and $\beta_i = \text{pot}_{f_i}(\gcd(\theta_b, f))$ for each $i \in [1, r]$. Then,

(i). $\text{St}(b) \subseteq \text{St}(a)$ and $\beta_i \leq \alpha_i$ for each $i \in [1, r]$, in particular, if $a \leq_H b$ then $\text{St}(b) = \text{St}(a)$ and $\beta_i = \alpha_i$ for each $i \in [1, r]$;

(ii). if $\beta_i = \alpha_i$ for each $i \in [1, r]$, then $a \leq_H b$;

(iii). If $a \not\leq_H b$ and $(\alpha_1 - \beta_1)(2n_1 - 1 - \alpha_1 - \beta_1) + (\alpha_2 - \beta_2)(2n_2 - 1 - \alpha_2 - \beta_2) + \sum_{i=3}^{r}(\alpha_i - \beta_i) > 0$, then $\text{St}(b) \not\subseteq \text{St}(a)$.

**Proof of Lemma 2.3.** Note first that $a \leq_H b$ implies that $\alpha_i \geq \beta_i$ for each $i \in [1, r]$.

(i). Since $S_R$ has the identity element $0_{S_R}$, it follows from $a \leq_H b$ that

$$a = b + c \quad \text{for some } c \in S_R.$$

It follows that

$$\gcd(\theta_b, f) | \gcd(\theta_b \ast \theta_c, f) = \gcd(\theta_a, f),$$

equivalently, $\beta_i \leq \alpha_i$ for each $i \in [1, r]$. 

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Take an arbitrary element \( d \in \text{St}(b) \). Then \( d + a = d + (b + c) = (d + b) + c = b + c = a \), and so \( d \in \text{St}(a) \). It follows that
\[
\text{St}(b) \subseteq \text{St}(a).
\]

If \( a \trianglerighteq b \), i.e., \( a \leq_H b \) and \( b \leq_H a \), then \( \text{St}(b) = \text{St}(a) \) and \( \beta_i = \alpha_i \) for each \( i \in [1, r] \) follows readily. This proves Conclusion (i).

(ii). Assume \( \beta_i = \alpha_i \) for each \( i \in [1, r] \), that is,
\[
\gcd(\theta_b, f) = \gcd(\theta_a, f).
\]
It follows that there exist polynomials \( h, h' \in \mathbb{F}_q[x] \) such that
\[
\theta_a * h \equiv \theta_b \pmod{f}
\]
and
\[
\theta_b * h' \equiv \theta_a \pmod{f}.
\]
It follows that \( b \leq_H a \) and \( a \leq_H b \), i.e.,
\[
a \trianglerighteq b,
\]
and Conclusion (ii) is proved.

(iii). Now assume
\[
a <_H b
\]
and
\[
\sum_{i=3}^r (\alpha_i - \beta_i) + (\alpha_1 - \beta_1)(2n_1 - 1 - \alpha_1 - \beta_1) + (\alpha_2 - \beta_2)(2n_2 - 1 - \alpha_2 - \beta_2) > 0.
\]
It is sufficient to find some element \( d \in U(S_R) \) such that \( d \in \text{St}(a) \setminus \text{St}(b) \). We shall distinguish two cases.

**Case 1.** \( \sum_{i=3}^r (\alpha_i - \beta_i) > 0. \)

Then there exists some \( i \in [3, r] \) such that \( \alpha_i > \beta_i \), say
\[
\alpha_3 > \beta_3.
\]
Take a polynomial
\[
h = \frac{f}{f_3^m}.
\]

We show that
\[ \gcd(h + 1_{\mathbb{F}_2}, f) = 1_{\mathbb{F}_2} \] (6)
or
\[ \gcd(x \cdot h + 1_{\mathbb{F}_2}, f) = 1_{\mathbb{F}_2}. \] (7)

Suppose to the contrary that \( \gcd(h + 1_{\mathbb{F}_2}, f) \neq 1_{\mathbb{F}_2} \) and \( \gcd(x \cdot h + 1_{\mathbb{F}_2}, f) \neq 1_{\mathbb{F}_2} \). By (1) and (5), we have that \( f_i \nmid \gcd(h + 1_{\mathbb{F}_2}, f) \) and \( f_i \nmid \gcd(x \cdot h + 1_{\mathbb{F}_2}, f) \) for each \( i \in [1, r] \setminus \{3\} \). This implies that \( f_3 \mid (h + 1_{\mathbb{F}_2}) \) and \( f_3 \mid (x \cdot h + 1_{\mathbb{F}_2}) \), and thus, \( f_3 \mid x \cdot (h + 1_{\mathbb{F}_2}) - (x \cdot h + 1_{\mathbb{F}_2}) = x + 1_{\mathbb{F}_2} \), which is absurd. This proves that (6) or (7) holds.

Take an element \( d \in S_R \) with
\[ \theta_d \equiv h + 1_{\mathbb{F}_2} \pmod{f} \]
or
\[ \theta_d \equiv x \cdot h + 1_{\mathbb{F}_2} \pmod{f} \]
according to (6) or (7) holds respectively. It follows that
\[ d \in U(S_R), \]
and follows from (4) and (5) that
\[ \theta_a \ast \theta_d \equiv \theta_a \pmod{f} \]
and
\[ \theta_b \ast \theta_d \neq \theta_b \pmod{f}. \]
That is, \( d \in \text{St}(a) \setminus \text{St}(b) \), which implies
\[ \text{St}(b) \subset \text{St}(a). \]

Case 2. \((\alpha_1 - \beta_1)(2n_1 - 1 - \alpha_1 - \beta_1) > 0\) or \((\alpha_2 - \beta_2)(2n_2 - 1 - \alpha_2 - \beta_2) > 0.\)

Say
\[ (\alpha_1 - \beta_1)(2n_1 - 1 - \alpha_1 - \beta_1) > 0. \]

It follows that
\[ \alpha_1 > \beta_1 \] (8)
and
\[ n_1 > \beta_1 + 1. \] (9)
Take an polynomial

\[ h = \frac{f}{\beta_1 + 1}. \]  \quad (10)

Combined with (9) and (10), we conclude that

\[ \gcd(h + 1_{\mathbb{F}_2}, f) = 1_{\mathbb{F}_2}. \]

Take an element \( d \in S_R \) with

\[ \theta_d \equiv h + 1_{\mathbb{F}_2} \pmod{f}. \]

It follows that

\[ d \in U(S_R), \]

and follows from (8) and (10) that

\[ \theta_a \ast \theta_d \equiv \theta_a \pmod{f} \]

and

\[ \theta_b \ast \theta_d \neq \theta_b \pmod{f}. \]

That is, \( d \in St(a) \setminus St(b) \) which implies

\[ St(b) \subsetneq St(a). \]

This proves Lemma 2.3. \( \square \)

Now we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 2.2, it suffices to show that \( D(S_R) \leq D(U(S_R)) + \delta_f \). Let \( T = a_1 a_2 \cdots a_\ell \) be an arbitrary sequence of terms from \( S_R \) of length

\[ \ell = D(U(S_R)) + \delta_f. \]  \quad (11)

We shall prove that \( T \) contains a proper subsequence \( T' \) with \( \sigma(T') = \sigma(T) \).

Take a shortest subsequence \( V \) of \( T \) such that

\[ \sigma(V) \triangleq \sigma(T). \]  \quad (12)

We may assume without loss of generality that

\[ V = a_1 \cdot a_2 \cdot \ldots \cdot a_t \] where \( t \in [0, \ell]. \)
By the minimality of $|V|$, we derive that

$$0_{S_k} \rightarrow_{\mathcal{H}} a_1 \rightarrow_{\mathcal{H}} (a_1 + a_2) \rightarrow_{\mathcal{H}} \cdots \rightarrow_{\mathcal{H}} \sum_{i=1}^{t} a_i.$$

Denote

$$K_0 = \{0_{S_k}\}$$

and

$$K_i = \text{St}(\sum_{j=1}^{i} a_j) \quad \text{for each} \quad i \in [1, t].$$

Note that $K_i$ is a subgroup of $U(S_R)$ for each $i \in [1, t]$. Moreover, since $\text{St}(0_{S_k}) = K_0$, it follows from Conclusion (i) of Lemma 2.3 that

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_t,$$

and moreover, by applying Lemma 2.3 we conclude that there exists a subset $M$ of $[0, t - 1]$ with

$$|M| \geq t - \delta_f$$

such that

$$K_i \not\subseteq K_{i+1} \quad \text{for each} \quad i \in M.$$

For $i \in M$, since $U(S_R) \cong \frac{U(S_R)}{K_{i+1}/K_i}$ and $D(K_{i+1}/K_i) \geq 2$, it follows from Lemma 2.1 that

$$D(U(S_R)/K_{i+1}) = D\left(\frac{U(S_R)}{K_{i+1}/K_i}\right) \leq D(U(S_R)/K_i) - (D(K_{i+1}/K_i) - 1) \leq D(U(S_R)/K_i) - 1.$$  \hfill (14)

Combined with (11), (13) and (14), we conclude that

$$1 \leq D(U(S_R)/K_i) \leq D(U(S_R)/K_0) - |M| \leq D(U(S_R)) - (t - \delta_f) = (\ell - \delta_f) - (t - \delta_f) \leq \ell - t \leq |TV^{(-1)}|. \hfill (15)$$

By Conclusion (i) of Lemma 2.3 and (12), we have

$$\text{pot}_f(\gcd(\theta_{\sigma(V)}, f)) = \text{pot}_f(\gcd(\theta_{\sigma(T)}, f))$$

for each $i \in [1, r]$. Let

$$\mathcal{J} = \{ j \in [1, r] : f_j^{a_j} \mid \theta_{\sigma(T)} \}. $$
By (16), we have that
\[ f_i \nmid \theta_a \quad \text{for each term } a \text{ of } TV^{[-1]} \text{ and each } i \in [1, r] \setminus J, \quad (17) \]
and that
\[ f_j^{n_j} \mid \theta_{\sigma(V)} \quad \text{for each } j \in J. \quad (18) \]
For each term \( a \) of \( TV^{[-1]} \), let \( \tilde{a} \) be the element of \( S_R \) such that
\[ \theta_{\tilde{a}} \equiv \theta_a \pmod{f_i} \quad \text{for each } i \in [1, r] \setminus J, \quad (19) \]
and
\[ \theta_{\tilde{a}} \equiv 1_{\mathbb{F}_2} \pmod{f_j} \quad \text{for each } j \in J. \quad (20) \]
By (17), (19) and (20), we conclude that \( \gcd(\theta_{\tilde{a}}, f) = 1_{\mathbb{F}_2} \), i.e.,
\[ \tilde{a} \in U(S_R) \quad \text{for each term } a \text{ of } TV^{[-1]}. \quad (21) \]
By (18) and (19), we conclude that
\[ \sigma(V) + \tilde{a} = \sigma(V) + a \quad \text{for each term } a \text{ of } TV^{[-1]}. \quad (22) \]
By (15) and (21), we have that \( \prod_{a \in TV^{-1}} \tilde{a} \) is a nonempty sequence of elements in \( U(S_R) \) of length \( |\prod_{a \in TV^{-1}} \tilde{a}| = |TV^{[-1]}| \geq D(U(S_R) \setminus K_t) \). It follows that there exists a nonempty subsequence \( W \mid TV^{[-1]} \) such that
\[ \sigma(\prod_{a \in W} \tilde{a}) \in K_t \]
which implies
\[ \sigma(V) + \sigma(\prod_{a \in W} \tilde{a}) = \sigma(V). \quad (23) \]
By (22) and (23), we conclude that
\[ \sigma(T) = \sigma(TW^{[-1]}V^{[-1]}) + (\sigma(V) + \sigma(W)) = \sigma(TW^{[-1]}V^{[-1]}) + (\sigma(V) + \sigma(\prod_{a \in W} \tilde{a})) = \sigma(TW^{[-1]}V^{[-1]}) + \sigma(V) = \sigma(TW^{[-1]}), \]
and \( T' = TW^{[-1]} \) is the desired proper subsequence of \( T \). This completes the proof of the theorem. \( \square \)
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References

[1] S.D. Adhikari, W.D. Gao and G.Q. Wang, Erdős-Ginzburg-Ziv theorem for finite commutative semigroups, Semigroup Forum, 88 (2014) 555–568.

[2] H. Davenport, Proceedings of the Midwestern conference on group theory and number theory, Ohio State University, April 1966.

[3] W.D. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups: a survey, Expo. Math., 24 (2006) 337–369.

[4] A. Geroldinger and F. Halter-Koch, Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory, Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.

[5] K. Rogers, A Combinatorial problem in Abelian groups, Proc. Cambridge Phil. Soc., 59 (1963) 559–562.

[6] G.Q. Wang, Davenport constant for semigroups II, Journal of Number Theory, 155 (2015) 124–134.

[7] G.Q. Wang, Additively irreducible sequences in commutative semigroups, arXiv:1504.06818.

[8] G.Q. Wang, Structure of the largest idempotent-free sequences in finite semigroups, arXiv:1405.6278.

[9] G.Q. Wang and W.D. Gao, Davenport constant for semigroups, Semigroup Forum, 76 (2008) 234–238.