New Class of Solvable and Integrable Many-Body Models

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Abstract

Integrability conditions for systems of bosons or fermions with seniority conserving hamiltonians are derived. The conditions are shown to be invariant under a large class of transformations of the interaction matrix elements. Previously published integrable models are shown to satisfy these conditions and the existence of a new class of integrable models is demonstrated. The number of free parameters in the interaction in these models equals the number of single particle levels plus 3. Equations for the energy eigenstates and eigenvalues are derived and the eigenvalues of the complete set of two-body integrals of the motion are given for the new class. Some two-body correlations in these eigenstates are derived from the integrals of the motion.

I. INTRODUCTION

Shortly after the development of the BCS theory of superconductivity,[1] the Hamiltonian and the associated computational techniques were adapted to create the pairing model of nuclear structure. This model has been successful in qualitatively accounting for nuclear properties such as the energy gap in the single-particle spectrum of even-even nuclei, odd-even mass differences, and the moments of inertia of deformed nuclei in spite of the fact that the methods used by BCS do not apply to finite systems since they violate number conservation. A great deal of effort was invested into attempts to restore number conservation to the theory. At about the same time, an exact analytical expression[2, 3] for the eigenstates and eigenvalues of the pairing Hamiltonian with constant interaction matrix elements was published and largely ignored by the nuclear physics community. Shortly thereafter the corresponding many-boson problem was also solved[4]. More recently, the
attention of the condensed matter community has been directed towards understanding pairing correlations in finite systems by a series of experiments on metallic grains with a size of a few nanometers\cite{5, 6} and the exact solution has been extensively utilized in studies of these systems\cite{5}. A separate and important recent development has been the demonstration that the pairing Hamiltonian, with constant interaction matrix elements, is an integrable system.\cite{7} This system is then both solvable and integrable. Another important result is the demonstration of a class of solvable and integrable models involving both pairing forces and an interaction term quadratic in the particle number operators\cite{8, 9}. This paper draws much of its inspiration from these works and extends their results.

The structure of this paper is as follows. We start in Sec. 2 with a definition of the class of Hamiltonians that we will consider. They are the most general, seniority-conserving Hamiltonians with two-body forces, which describe systems of fermions or bosons. These Hamiltonians have a diagonal single-particle kinetic energy plus a pairing term and a term quadratic in the particle number operators with arbitrary matrix elements in all terms. We then consider a set of operators that have the same structure as the Hamiltonian and derive a set of conditions on the matrix elements in the members of the set that must be satisfied if these operators are required to commute with the Hamiltonian. We obtain an explicit solution for the matrix elements in the two-body part of these constants of the motion in terms of the matrix elements in the Hamiltonian and a set of conditions which the Hamiltonian matrix elements must satisfy. Solutions of these conditions yield integrable Hamiltonians. We show that, due to a symmetry of the conditions, each particular solution gives rise to a large family of solutions. The number of free parameters in each family is greater than the number of single-particle energy levels. We also show that previously published\cite{8, 9} solvable and integrable models satisfy our conditions. By generating a new particular solution of the conditions, we generate a whole new family of integrable models. In Sec. 3, we show that members of this new family of integrable models are also solvable by giving analytical forms for all their eigenstates and eigenvalues. In Sec. 4, we calculate the occupations of the single-particle levels and some two-body correlation functions in these eigenstates.
II. HAMILTONIAN AND CONSTANTS OF THE MOTION

We consider the class of seniority conserving Hamiltonians for Fermi or Bose systems that can be written in the following form $H = T + U + V$, where

$$T = \sum_i 2\epsilon_i n_i$$  \hspace{1cm} (2.1)

$$U = -\frac{s g}{4} \sum_{i,j} u_{i,j} (\Omega_i + 2 s n_i) (\Omega_j + 2 s n_j)$$

$$V = \frac{g}{2} \sum_{i,j} v_{i,j} (b^\dagger_i b_j + b^\dagger_j b_i)$$

In these expressions, $i$ labels a single-particle energy-level of energy $\epsilon_i$ and $g$ is an interaction strength. The interaction matrices $u$ and $v$ are real and symmetric. The statistics factor $s = -1 (+1)$ for Fermi (Bose) statistics. The operators $n$ and $b$ are defined by

Fermions:

The degeneracy of level $i$ is $2\Omega_i$ with the factor 2 coming from time-reversal degeneracy and $\alpha = \pm 1, \cdots, \pm \Omega_i$ with

$$n_i = \frac{1}{2} \sum_{\alpha=1}^{\Omega_i} (a^\dagger_{i,\alpha} a_{i,\alpha} + a^\dagger_{i,-\alpha} a_{i,-\alpha})$$ \hspace{1cm} (2.2)

$$b_i = \sum_{\alpha=1}^{\Omega_i} a_{i,-\alpha} a_{i,\alpha}$$

Bosons:

There are 2 possibilities for bosons. In the first case, the single-particle states are not time-reversal eigenstates and the above definition applies. In the second case, the single-particle states are time-reversal eigenstates as in the $k = 0$ state or the bound states of a potential. We then define the operators

$$n_i = \frac{1}{2} \sum_{\alpha=1}^{\Omega_i} a^\dagger_{i,\alpha} a_{i,\alpha}$$ \hspace{1cm} (2.3)

$$b_i = \frac{1}{\sqrt{2}} \sum_{\alpha=1}^{\Omega_i} a_{i,\alpha} a_{i,\alpha}$$
The single-particle operators $a$ satisfy (Fermi) Bose (anti)commutation relations which can be used to derive the commutation rules for the $n$ and $b$ operators

\[
\begin{align*}
[n_i, b_j^\dagger] &= \delta_{ij} b_j^\dagger \\
[b_i, b_j^\dagger] &= \delta_{ij} (\Omega_j + 2s n_j)
\end{align*}
\] (2.4)

The effects of degeneracy are expressed explicitly in the presence of the $\Omega_j$ in the commutation relations and the effects of statistics in the presence of the $s$ in $U$ and in the factors $(\Omega_j + 2s n_j)$ in the Hamiltonian and the commutation relations. The seniority operators are defined by

\[
\nu_i = \sum_{\alpha=1}^{\Omega_i} \left( a_{i,\alpha}^\dagger a_{i,\alpha} - a_{i,-\alpha}^\dagger a_{i,-\alpha} \right)
\]

and they commute with the Hamiltonian. They count the number of unpaired particles in each single particle level. The form of the interaction in (2.1) is chosen so as to mimic that of a Heisenberg XXZ spin chain model after a quasispin substitution for the fermion pair operators.

The eigenstates of such Hamiltonians factor into a product of a state of the unpaired particles times a state of the paired particles. The energy of the state is the sum of the energies of the unpaired particles, which interact only through $U$, plus that of the paired particles. The influence of the unpaired particles on the paired particles comes about through a modification of the degeneracies $\Omega_i$ - blocking in the case of fermions and antiblocking for bosons. In what follows, we only consider the seniority-zero paired particles and assume that the degeneracies are effective degeneracies which depend upon which state of the system is being considered.

Following Ref. 7, we look for two-body constants of the motion $K$ that have a form similar to that of the Hamiltonian, $K = X + Y + Z$, with
\[ X = \sum_j 2x_j n_j \quad \text{(2.5)} \]

\[ Y = -\frac{sg}{4} \sum_{j,k} y_{j,k} (\Omega_j + 2s n_j) (\Omega_k + 2s n_k) \]

\[ Z = \frac{g}{2} \sum_{j,k} z_{j,k} \left( b_j^\dagger b_k + b_k^\dagger b_j \right) \]

where \( y \) and \( z \) are real and symmetric matrices to be determined in terms of \( x, \epsilon, u, \) and \( v \) so that \( K \) commutes with the Hamiltonian. Since \( x \) is a set of free parameters, equal in number to the number of degrees of freedom of the system, the solution of the resulting equations will provide a complete set of constants of the motion specified by a choice of \( x \) and it will be shown that these constants, with different choices for \( x \), commute with each other.

The calculation of the commutator \([H, K]\) follows from (2.1), (2.4), and (2.5) and some tedious algebra. The result is

\[ [H, K] = g \sum_{k,l} \left\{ -(\xi_k - \xi_l) v_{k,l} + (e_k - e_l) z_{k,l} \right\} \left( b_k^\dagger b_l - b_l^\dagger b_k \right) \]

\[ + \frac{g^2}{2} \sum_{j,k,l} [v_{j,k} (y_{j,k} - y_{j,l}) - (u_{j,k} - u_{j,l}) z_{j,l} + (v_{j,k} \delta_{j,l} - v_{j,l} \delta_{j,k})] \left[ b_k^\dagger (\Omega_j + 2s n_j) b_l - b_l^\dagger (\Omega_j + 2s n_j) b_k \right] \]

where

\[ \xi_k = x_k - \frac{1}{2} s g y_{k,k} \quad \text{(2.7)} \]

\[ e_k = \epsilon_k - \frac{1}{2} s g u_{k,k} \]

Setting the commutator equal to zero yields the equations

\[ z_{k,l} = \frac{(\xi_k - \xi_l)}{(e_k - e_l)} v_{k,l}, \quad k \neq l \quad \text{(2.8)} \]

and
\[
v_{k,l} (y_{j,k} - y_{j,l}) - (u_{j,k} - u_{j,l}) z_{k,l} + (v_{j,k} z_{j,l} - v_{j,l} z_{j,k}) = 0 \quad , \quad k \neq l
\] (2.9)

We assume that \( v \neq 0 \) and first consider the case in which \( u = 0 \). Then (2.9), with \( j \to k \), is

\[
v_{k,l} (y_{k,k} - y_{k,l}) + v_{k,k} z_{k,l} = 0 \quad , \quad k \neq l
\] (2.10)

Without loss of generality, we can require

\[
y_{k,k} = z_{k,k} = 0
\] (2.11)

Then

\[
y_{k,l} = \frac{(x_k - x_l)}{(e_k - e_l)} v_{k,l} \quad , \quad k \neq l
\] (2.12)

which requires that \( v_{k,k} \) be a constant which can be taken to be one. Equation (2.9), with \( j \), \( k \) and \( l \) distinct, and, using (2.8) and (2.12), becomes

\[
(v_{k,l} - v_{j,k} v_{j,l}) \left( \frac{(x_j - x_k)}{(e_j - e_k)} - \frac{(x_j - x_l)}{(e_j - e_l)} \right) = 0 \quad , \quad j \neq k \neq l \neq j
\]

which requires \( v_{j,k} = 1 \) and is the simple pairing Hamiltonian with constant interaction matrix elements. This is the result of Ref. 7.

We now assume that both \( u \neq 0 \) and \( v \neq 0 \). We can solve (2.9), which is antisymmetric in \( k \) and \( l \), for \( y \) by setting \( j = k \) which gives

\[
v_{k,l} (y_{k,k} - z_{k,l}) - v_{k,l} y_{k,l} + (u_{k,l} + v_{k,k} - u_{k,k}) z_{k,l} = 0 \quad , \quad k \neq l
\] (2.13)

Without loss of generality, we can require

\[
y_{k,k} = z_{k,k} = v_{k,k} - u_{k,k} = 0
\] (2.14)
Then, using (2.8), we have

\[ y_{k,l} = \frac{(x_k - x_l)}{(e_k - e_l)} u_{k,l} , \ k \neq l \]  

(2.15)

Equation (2.9), with \( j \), \( k \) and \( l \) distinct, and using (2.8) and (2.15) provides an integrability condition on \( u \) and \( v \). This condition is linear in the \( x' \)'s and, equating their coefficients to zero, leads to the single equation

\[
(x - y) u(x, z) v(y, z) - (x - z) u(x, y) v(y, z) + (y - z) v(x, y) v(x, z) = 0 \ , \ x \neq y \neq z \neq x
\]

(2.16)

where we have introduced the notation

\[
u_{j,k} = u(e_j, e_k)
\]

(2.17)

\[
v_{j,k} = v(e_j, e_k)
\]

and write \((x, y, z)\) for \((e_j, e_k, e_l)\). Note that this equation is independent of particle statistics and single-particle degeneracies.

Eq. (2.16) has a curious symmetry that allows the generation of a whole class of solutions from any particular solution. Assume that we have a particular solution of (2.16), \(u(x, y)\) and \(v(x, y)\), then there will exist a class of solutions generated from this particular solution of the form

\[
u'(x,y) = \frac{(x - y)}{f(x) - f(y)} u[f(x), f(y)]
\]

(2.18)

\[
v'(x,y) = \frac{(x - y)}{f(x) - f(y)} v[f(x), f(y)]
\]

where \(f\) is some well behaved function of its argument. This result follows from the fact that the form of Eq. (2.16) is invariant under the transformation

\[
u u' , v \rightarrow v' , x \rightarrow f
\]

(2.19)
Note that iteration of the transformation (2.18) is equivalent to a redefinition of the function \( f \) so that each class of solutions is closed under this operation. If we require the functions \( f \) to be monotonic with single-valued inverses, then the transformations have a group structure. In the following we will look for particular solutions of (2.16) which will provide the basis for classes of solutions generated by the transformations (2.18). The members of each class are specified by the set of numbers \( f_i = f(e_i) \) so that the number of free parameters in a class is equal to the number of single-particle energy levels plus the number of free parameters in the particular solution.

We first show that recently published\[8, 9\] results are solutions of (2.16). The first class of solutions is based upon the simplest solution of (2.16): \( u = v = 1 \). With this solution, \( U \) is a function of the total number of particles, and the Hamiltonian is equivalent to the simple pairing model Hamiltonian with constant interaction matrix elements. The members of this class, after the transformation (2.18), are the descendents of this Hamiltonian with

\[
\frac{u(x, y) - v(x, y)}{f(x) - f(y)} = \frac{x - y}{f(x) - f(y)}
\]

(2.20)

This class can also be obtained directly from (2.16) by taking \( u = v \) and dividing (2.16) by \( v(x, y) v(x, z) v(y, z) \) to get the equation

\[
\frac{(x - y)}{v(x, y)} - \frac{(x - z)}{v(x, z)} + \frac{(y - z)}{v(y, z)} = 0
\]

(2.21)

which is linear in \( 1/v \). The solutions of this equation are \((x - y)/v(x, y) = f(x) - f(y)\) which is (2.20). A second class of solutions is obtained by writing

\[
u(x, y) = v(x, y) w(x, y)
\]

and obtaining from (2.16)

\[
\frac{(x - y)}{v(x, y)} w(x, z) - \frac{(x - z)}{v(x, z)} w(x, y) + \frac{(y - z)}{v(y, z)} = 0
\]

(2.22)

Solutions of this equation are\[8, 9\]
\[ v(x, y) = \frac{q(x - y)}{\text{Sinh}[q(x - y)]} \]  
\[ w(x, y) = \text{Cosh}[q(x - y)] \]  
\[ u(x, y) = \frac{q(x - y) \text{Cosh}[q(x - y)]}{\text{Sinh}[q(x - y)]} \]  

Note that the first class of solutions is just the \( q \to 0 \) limit of the second.

A third and new class of solutions is obtained by writing

\[ v(x, y) = \gamma(x) \gamma(y) \]  
(2.24)

where \( \gamma^2(x) \) is a polynomial in \( x \) and \( u(x, y) \) is a symmetric polynomial in \( x \) and \( y \).

Substituting this form into (2.16) gives the equation

\[
(x - y) u(x, z) - (x - z) u(x, y) + (y - z) \gamma^2(x) = 0
\]  
(2.25)

which is linear in \( u \) and \( \gamma^2 \). The solution of this equation is

\[
u(x, y) = 1 + \gamma_1 (x + y) + \gamma_2 xy
\]  
(2.26)

\[
\gamma(x) = (1 + 2 \gamma_1 x + \gamma_2 x^2)^{1/2}
\]

with \( \gamma_1 \) and \( \gamma_2 \) free parameters. Note that before the transformation (2.19), the first two terms in \( u \) can be ignored since they result in terms in the Hamiltonian that are either constants, proportional to the total number of particles, or a modification of the single-particle spectrum. However, they are important after the transformation. Note also that this solution reverts back to the first one for the case \( \gamma_2 = \gamma_1^2 \). The case in which there is no constant term in \( u \) can be handled by making the replacements \( g \to g\gamma_0 \), \( \gamma_i \to \gamma_i/\gamma_0 \), \( i = 1, 2 \), and then taking the limit \( \gamma_0 \to 0 \). The net effect of which is the replacement of \( 1 \) in (2.26) by \( 0 \).

Further study of Eq. (2.16) has not produced any other simple solutions that are not the result of symmetry transformations applied to the solutions (2.20), (2.23), and (2.26). In the next sections, we consider the eigenstates of the integrable model (2.26) after a symmetry transformation.
III. EIGENSTATES

It is a semiempirical theorem that the seniority-zero eigenstates of integrable Hamiltonians of the form (2.1), have the form of a product of correlated pairs. This is true for the states of the Hamiltonians with interaction matrix elements given by (2.20) or (2.23) and we now show, by construction, that it is also true for the solution (2.26).

Assuming that the eigenstates can be taken as a product of correlated pairs, we write, for a typical eigenstate of 2N particles,

\[ |\psi\rangle = \prod_{r=1}^{N} B_r^\dagger |0\rangle \]  \hspace{1cm} (3.1)

where

\[ B_r^\dagger = \sum_j \phi_r(e_j) b_j^\dagger \]  \hspace{1cm} (3.2)

Then, commuting the Hamiltonian past the product of \( B^\dagger \)'s, we have

\[
\left( H - 2 \sum_{p=1}^{N} E_p - E^{(0)} \right) |\psi\rangle = \sum_{p=1}^{N} \left( \prod_{r=1,(r\neq p)}^{N} B_r^\dagger \right) \left\{ [H, B_p^\dagger] - 2E_p B_p^\dagger \right\} |0\rangle \\
+ \frac{1}{2} \sum_{p,q=1(p\neq q)}^{N'} \left( \prod_{r=1,(r\neq p,q)}^{N} B_r^\dagger \right) \left\{ [H, B_p^\dagger] , B_q^\dagger \right\} |0\rangle
\]  \hspace{1cm} (3.3)

where we have written the energy of the vacuum state as

\[ E^{(0)} = -\frac{s_4}{4} \sum_{i,j} u_{i,j} \Omega_i \Omega_j \]  \hspace{1cm} (3.4)

and where the parameters \( 2E_p \) are pair-energies to be determined. The energy of the state will be given in terms of these parameters. The commutators can be calculated from (2.4) and (3.2) with the results

\[
\left\{ [H, B_p^\dagger] - 2E_p B_p^\dagger \right\} |0\rangle = 2 \sum_j \left[ \left( e_j - \frac{g}{2} \sum_k u_{j,k} \Omega_k - E_p \right) \phi_p(e_j) + \frac{g}{2} \sum_k v_{j,k} \Omega_k \phi_p(e_k) \right] b_j^\dagger |0\rangle
\]  \hspace{1cm} (3.5)
\[
[H, B_p^\dagger, B_q^\dagger] = -gs \sum_{j,k} \{ u_{j,k} [\phi_p(e_j) \phi_q(e_k) + \phi_p(e_k) \phi_q(e_j)] - v_{j,k} [\phi_p(e_j) \phi_q(e_j) + \phi_p(e_k) \phi_q(e_k)] \} b_j^\dagger b_k^\dagger
\]  

(3.6)

For the new set of integrable models, we take \( u \) and \( v \) to be given by (2.26). Then, after the symmetry transformation (2.18),

\[
\begin{align*}
  u_{j,k} &= u(e_j, e_k) \rightarrow \frac{(e_j - e_k)}{(f_j - f_k)} u(f_j, f_k) \\
  v_{j,k} &= v(e_j, e_k) = \gamma e_j \gamma e_k \rightarrow \frac{(e_j - e_k)}{(f_j - f_k)} \gamma f_j \gamma f_k \\
  \phi_p(e_k) &= \frac{\gamma e_k}{e_k - \omega_p} \rightarrow \phi_p(f_k) = \frac{\gamma f_k}{f_k - \omega_p}
\end{align*}
\]

with \( \omega_p \) a parameter to be determined. For (3.5), we then have

\[
\{ [H, B_p^\dagger] - 2E_p B_p^\dagger \} |0\rangle = 2 \sum_j \left[ \left( e_j - \frac{g}{2} \sum_k \Omega_k \frac{(e_j - e_k)}{(f_j - f_k)} \left[ u(f_j, f_k) - \gamma^2 f_k \right] - E_p \right) \\
+ \frac{g}{2} \sum_k \Omega_k (e_j - e_k) \frac{\gamma^2 (f_k)}{(f_k - \omega_p)} \phi_p(f_j) b_j^\dagger \right] |0\rangle
\]

(3.8)

\[
= 2 \sum_j \left[ e_j + \frac{g}{2} \sum_k \Omega_k (e_j - e_k) \frac{u(\omega_p, f_k)}{(f_k - \omega_p)} - E_p \right] \phi_p(e_j) b_j^\dagger |0\rangle
\]

and for (3.6), after algebraic simplification,

\[
[H, B_p^\dagger, B_q^\dagger] = -gs \frac{u(\omega_p, \omega_q)}{(\omega_p - \omega_q)} \sum_{j,k} (e_j - e_k) [\phi_p(f_j) \phi_q(f_k) - \phi_p(f_k) \phi_q(f_j)] b_j^\dagger b_k^\dagger
\]

(3.9)

\[
= -2gs \frac{u(\omega_p, \omega_q)}{(\omega_p - \omega_q)} \sum_j e_j [\phi_p(f_j) B_q^\dagger - \phi_q(f_j) B_p^\dagger] b_j^\dagger
\]

Substituting these results into (3.3) yields
\begin{equation}
\left( H - 2 \sum_{p=1}^{N} E_p - E^{(0)} \right) |\psi\rangle
\end{equation}

\begin{equation}
= 2 \sum_{p=1}^{N} \left( \prod_{r=1,(r\neq p)}^{N} B_r^{\dagger} \right) \sum_{j} \left[ \lambda_j \left( 1 + \frac{g}{2} \sum_{k} \Omega_k \frac{u(\omega_p, f_k)}{(f_k - \omega_p)} - gs \sum_{q=1(q\neq p)}^{N'} \frac{u(\omega_p, \omega_q)}{\omega_p - \omega_q} \right) \\
- \frac{g}{2} \sum_{k} \Omega_k e_k \frac{u(\omega_p, f_k)}{(f_k - \omega_p)} - E_p \phi_p (f_j) b_j^\dagger \right] |0\rangle
\end{equation}

Setting this equal to zero yields the equations

\begin{equation}
\left[ 1 + \frac{g}{2} \sum_{k} \Omega_k \frac{u(\omega_p, f_k)}{(f_k - \omega_p)} + gs \sum_{q=1(q\neq p)}^{N'} \frac{u(\omega_p, \omega_q)}{\omega_p - \omega_q} \right] = 0, \quad p = 1, \ldots, N
\end{equation}

\begin{equation}
E_p = -\frac{g}{2} \sum_{k} \Omega_k e_k \frac{u(\omega_p, f_k)}{(f_k - \omega_p)}
\end{equation}

The energy of the state is then given by

\begin{equation}
E = E^{(0)} + 2 \sum_{p=1}^{N} E_p
\end{equation}

\begin{equation}
= E^{(0)} - g \sum_{p=1}^{N} \sum_{k} \Omega_k e_k \frac{u(\omega_p, f_k)}{(f_k - \omega_p)}
\end{equation}

For \( f_k = e_k \), this expression simplifies to

\begin{equation}
E = E^{(0)} + 2 \sum_{p=1}^{N} \omega_p - gs \sum_{p,q=1(q\neq p)}^{N} u(\omega_p, \omega_q) - g \sum_{p=1}^{N} \sum_{k} \Omega_k u(\omega_p, e_k)
\end{equation}

The equations (3.11) can be transformed into a form that allows a two-dimensional electrostatic analogy for fermions with \( s = -1 \) Some algebraic manipulation of (3.11) yields the equations

\begin{equation}
- \frac{2 + g [\Omega + 2s(N - 1)] (\gamma_1 + \gamma_2 \omega_p)}{2g\gamma^2(\omega_p)} - \frac{1}{2} \sum_{k} \frac{\Omega_k}{(f_k - \omega_p)} - s \sum_{q=1(q\neq p)}^{N'} \frac{1}{(\omega_q - \omega_p)} = 0, \quad p = 1, \ldots, N
\end{equation}
For fermions, these equations can be interpreted as the equations for the unstable equilibrium positions of $N$ unit line charges at the locations $\omega_p$ in the complex plane interacting with charges $-\Omega_k/2$ at the locations $f_k$ and two charges located at the zeros of $\gamma^2(\omega)$.

In this analogy, the statistical repulsion between fermions is mapped onto the electrostatic repulsion of like charges in two dimensions. Such equations have been shown to lead to the BCS equations plus corrections in a carefully defined $N \to \infty$ limit.\[10\] Eq. (3.14) then shows that the energy of the states has terms proportional to the dipole and quadrupole moments of the free charge distribution. For bosons, we have a statistical attraction between particles and the electrostatic analogy does not work. Such equations have been shown to yield a generalized Bose condensation into the lowest two single-particle levels for a repulsive interaction in the $N \to \infty$ limit.\[4\] The Bogoliubov approximation, modified to accommodate this generalized condensation, then gives good results in this limit.

The same techniques can be used to calculate the eigenvalues of the integrals $K$.

We consider the solution (2.26) after a symmetry transformation. We then have

$$K \langle \psi | = \left[ K^{(0)} - g \sum_{p=1}^{N} \sum_{k} \Omega_k x_k \frac{u(f_k,\omega_p)}{(f_k - \omega_p)} \right] \langle \psi |$$  

(3.16)

where

$$K^{(0)} = -\frac{sg}{4} \sum_{j,k} \Omega_j \Omega_k y_{j,k}$$

(3.17)

If we make the choice $x_k \to \delta_{i,k}$ and $K \to K_i$, then $K_i$ and its eigenvalues, $k_i$, are

$$K_i = 2 \ n_i + g \ \sum_{k(k \neq i)} \frac{1}{(f_i - f_k)} \left[ v(f_i, f_k) \left( b_i^\dagger b_k + b_k^\dagger b_i \right) - \frac{s}{2} u(f_i, f_k) (\Omega_i + 2 s n_i) (\Omega_k + 2 s n_k) \right]$$

(3.18)

$$k_i = g \left[ K_i^{(0)} - \Omega_i \sum_{p=1}^{N} \frac{u(f_i,\omega_p)}{(f_i - \omega_p)} \right]$$

$$K_i^{(0)} = -\frac{s}{2} \Omega_i \ \sum_{k(k \neq i)} \Omega_k \frac{u(f_i, f_k)}{(f_i - f_k)}$$

This result will be used in the next section to calculate the occupations of the single-particle levels and some two-body correlation functions in these eigenstates.
IV. OCCUPATIONS AND CORRELATIONS

We can use the eigenvalues of the Hamiltonian and the $K_i$s and the Hellman-Feynman\[14\] theorem to calculate certain two-body correlations in the eigenstates presented in the previous section. The Hamiltonian and constants of the motion are functions of the parameters in the potential, $g$, $\gamma_1$, and $\gamma_2$ as well as the single-particle spectrum $e_j$, and its transform $f_j = f(e_j)$. Since the function $f$ is unspecified, we can treat $e_j$ and $f_j$ as independent parameters. In this picture the $\omega_p$ are functions of $g$ and the $f_j$, through (3.11), and do not depend upon the $e_j$. Furthermore, comparing (3.13) with (3.18) yields the result

$$k_i - K^{(0)} = \frac{\partial}{\partial e_i} (E - E^{(0)}) \quad (4.1)$$

In order to calculate the mean occupations of the single-particle levels, we use (3.18) to write

$$n_i = \frac{1}{2} \left( K_i - g \frac{\partial K_i}{\partial g} \right) \quad (4.2)$$

The Hellman-Feynman theorem then yields

$$\langle \psi | n_i | \psi \rangle = \frac{1}{2} \left( k_i - g \frac{\partial k_i}{\partial g} \right) \quad (4.3)$$

$$= \Omega_i \sum_{p=1}^{N} \frac{\gamma_2^2 (f_i)}{(f_i - \omega_p)^2} R_p$$

$$R_p = \frac{g^2}{2} \left( \frac{\partial \omega_p}{\partial g} \right)$$

The $R_p$ satisfy the linear set of equations

$$\left[ \sum_k \frac{\gamma_2^2 (f_k)}{(f_k - \omega_p)^2} + 2s \sum_{q=1(q\neq p)}^{N'} \frac{\gamma_2^2 (\omega_q)}{(\omega_p - \omega_q)^2} \right] R_p = 2s \sum_{q=1(q\neq p)}^{N'} \frac{\gamma_2^2 (\omega_q)}{(\omega_p - \omega_q)^2} R_q = 1 \quad (4.4)$$

obtained by differentiation of (3.11) with respect to $g$.

The Hellman-Feynman theorem can also be used to calculate the expectation values of the following operators.
\[
\frac{\partial K_i}{\partial f_j} = g \frac{1}{(f_i - f_j)^2} \frac{\gamma(f_i)}{\gamma(f_j)} K_{i,j}, \quad i \neq j
\]  

(4.5)

\[
K_{i,j} = \left\{ u(f_i, f_j) \left( b_i^\dagger b_j + b_j^\dagger b_i \right) - \frac{s}{2} \frac{\gamma(f_i)}{\gamma(f_j)} \left( \Omega_i + 2s n_i \right) \left( \Omega_j + 2s n_j \right) \right\}
\]

or

\[
K_{i,j} = \left( f_i - f_j \right)^2 \frac{\gamma(f_j)}{\gamma(f_i)} \left[ \frac{\gamma(f_j)}{\gamma(f_i)} \frac{\partial K_i}{\partial f_j} + \frac{\gamma(f_i)}{\gamma(f_j)} \frac{\partial K_i}{\partial f_i} \right]
\]  

(4.6)

The expectation value of $\frac{\partial K_i}{\partial f_j}$ is given by

\[
\langle \psi | \frac{\partial K_i}{\partial f_j} | \psi \rangle = \frac{\partial k_i}{\partial f_j}
\]  

(4.7)

and

\[
\langle \psi | K_{i,j} | \psi \rangle = \left( f_i - f_j \right)^2 \frac{\gamma(f_j)}{\gamma(f_i)} \left[ \frac{\gamma(f_j)}{\gamma(f_i)} \frac{\partial k_i}{\partial f_j} + \frac{\gamma(f_i)}{\gamma(f_j)} \frac{\partial k_i}{\partial f_i} \right]
\]  

(4.8)

\[
= -\Omega_i \Omega_j \gamma(f_i) \gamma(f_j) \left( \frac{s}{2} + \frac{(f_i - f_j)^2}{4} \sum_{p=1}^{N} \gamma^2(\omega_p) \left( \frac{R_{p,j}}{(f_i - \omega_p)^2} + \frac{R_{p,i}}{(f_j - \omega_p)^2} \right) \right)
\]

where we have written

\[
\frac{\partial \omega_p}{\partial f_j} = \Omega_j R_{p,j}
\]  

(4.9)

and where the $R_{p,j}$ satisfy the linear set of equations (4.4), with the right-hand-side replaced by $\gamma^2(\omega_p) / (f_j - \omega_p)^2$, obtained by differentiating (3.11),

V. CONCLUSIONS

We have derived a general set of conditions for the integrability of the most general seniority conserving Hamiltonian with two-body forces for bosons or fermions. They have been used to obtain a new class of integrable models. We have shown that the eigenvalues and eigenstates of the Hamiltonian and the associated integrals of the motion can be obtained
by the solution of a fairly simple set of coupled algebraic equations. While this solvability
depends upon the integrability conditions in a rather complicated way, we have not been
able to construct a general result that integrability implies solvability.

The results of this paper can be generalized to other systems such as a system of nu-
cleons with an isospin-independent $J = 0$ pairing in jj-coupling[11] or a spin- and isospin-
independent $L = 0$ pairing in ls-coupling.[12] Another system to which these methods can be
applied is the system of fermions with an equal mix of pairing in the $^1S_0$ and $^3P_0$ channels.[13]

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