FOURIER UNIFORMITY ON SUBSPACES

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ABSTRACT. Let $F$ be a fixed finite field, and let $A \subset F^n$. It is a well-known fact that there is a subspace $V \leq F^n$, codim $V \ll \delta$, and an $x$, such that $A$ is $\delta$-uniform when restricted to $x + V$ (that is, all non-trivial Fourier coefficients of $A$ restricted to $x + V$ have magnitude at most $\delta$). We show that if $F = F_2$ then it is possible to take $x = 0$; that is, $A$ is $\delta$-uniform on a subspace $V \leq F^n$. We give an example to show that this is not necessarily possible when $F = F_3$.

1. INTRODUCTION

Let $F$ be a fixed finite field of prime order $p$, and consider the vector space $F^n$. We identify $F^n$ with its own dual via the dot product, and in this way define the Fourier transform of a function $f : F^n \to \mathbb{C}$ by

$$\hat{f}(r) := \mathbb{E}_{x \in F^n} f(x) e_p(-r \cdot x),$$

where $e_p(t) := e^{2\pi it/p}$ and $r$ takes values in $F^n$. When $F = F_2$, we have $e_p(t) = (-1)^t$. For any non-empty set $S \subset G$ we write $\mu_S$ for the uniform probability measure induced on $S$, that is the probability measure assigning mass $|S|^{-1}$ to each $s \in S$.

Definition 1.1. Suppose that $A \subset F^n$ is a set and that $V \leq F^n$ is a subspace. Let $x \in F^n$. Then we say that $A$ is $\varepsilon$-uniform on the coset $x + V$ if

$$\sup_{r \notin V^\perp} |(1_A \mu_{x+V})(r)| \leq \varepsilon.$$

The following fact, proven by a “density increment argument” is well-known in the additive combinatorics literature and is implicit, for example, in the work of Meshulam [Mes95].

Theorem 1.1. Suppose that $A \subset F^n$ is a set. Then there is a subspace $V \leq F^n$, codim $V \ll \varepsilon^{-1}$, and an $x \in F^n$ such that $A$ is $\varepsilon$-uniform on the coset $x + V$.

Our aim in this note is to prove the following.

Theorem 1.2. Suppose that $A \subset F_2^n$ is a set. Then there is a subspace $V \leq F^n$, codim $V \ll \varepsilon$, such that $A$ is $\varepsilon$-uniform on $V$.

Remarks. Sadly, the implied constant in $\varepsilon$ is atrocious, being a tower of towers of height $O(\varepsilon^{-1})$. It would be interesting to get a better bound. Note that it is quite permissible for $A \cap V$ to be empty, and indeed this is generally unavoidable, as the example $A = \{x : x_1 = 1\}$ shows.

In Section 2 we give a simple example to show that this statement is not true when $F_2$ is replaced by $F_3$.
2. AN EXAMPLE OVER $\mathbb{F}_3$

In this section we give a simple example to show that the analogue of Theorem 1.2 is false over $\mathbb{F}_3$ (similar examples may be constructed over other prime fields). The example comes from the literature on Rado’s theorem over finite fields, in particular from [BDH92]. Indeed, if Theorem 1.2 had been true over $\mathbb{F}_3$ it would have implied that every homogeneous equation in three or more variables is partition regular in $\mathbb{F}_3^N$, a result which is known to be false.

**Theorem 2.1.** There is a set $A \subset \mathbb{F}_3^3$ such that for any subspace $V \leq \mathbb{F}_3^n$ of positive dimension we have

$$\sup_{r \notin V^\perp} \left| (1_A d \mu_V) \wedge (r) \right| \geq \frac{\sqrt{3}}{6}.$$  

**Proof.** Take $A = \{ x \in \mathbb{F}_3^n : x_1 = \cdots = x_i = 0, x_{i+1} = 1 \}$. Let $V \leq \mathbb{F}_3^n$ be a subspace of positive dimension, and let $j \in [n]$ be minimal such that $v_j \neq 0$ for at least one $v \in V$. Of course, we then have $x_1 = \cdots = x_{j-1} = 0$ for all $x \in V$. It follows that

(2.1) \hspace{1cm} \{ x \in V : x_j = 1 \} \subset A,

whilst

(2.2) \hspace{1cm} \{ x \in V : x_j = 2 \} \cap A = \emptyset.

Take $r \in \mathbb{F}_3^n$ to have $r_1 = \cdots = r_{j-1}, r_{j+1}, \ldots, r_n = 0$ and $r_j = 1$. Then $r \notin V^\perp$, since $r \cdot v \neq 0$. Furthermore, a short computation using (2.1) and (2.2) gives

$$\text{Im}\left((1_A d \mu_V) \wedge (r)\right) = -\frac{\sqrt{3}}{2} \mu_V(\{ x \in V : x_j = 1 \}) = -\frac{\sqrt{3}}{6},$$

the second equality being a consequence of the fact that the map $V \to \mathbb{F}_3$ given by $x \mapsto x_j$ is linear and nontrivial. \hfill \square

3. A RAMSEY RESULT FOR ALMOST COLOURINGS

By a $(1 - \delta)$-almost $r$-colouring of a set $X$, we mean a map $c : \tilde{X} \to [r]$ where $|X \setminus \tilde{X}| \leq \delta|X|$.

**Proposition 3.1.** Let $r, d$ be integers. Then there is an $\eta(r, d) > 0$ such that the following is true. If $n \geq n_0(r, d)$ is sufficiently large, $\eta \in [0, \eta(r, d)]$, and if we have a $(1 - \eta)$-almost $r$-colouring of $\mathbb{F}_2^n$, then there are linearly independent $x_1, \ldots, x_d$ such that all of the sums $\sum_{i \in I} x_i, I \subset [d], I \neq \emptyset$, are the same colour.

**Proof.** In the case $\eta = 0$ (that is, genuine $r$-colourings rather than almost-colourings) this follows quickly from a well-known theorem of Graham and Rothschild [GR69, Corollary 1] (for a short proof see [NR83]). Indeed, our colouring of $\mathbb{F}_2^n$ induces a colouring of the power set $\mathcal{P}([n])$ via the usual identification of these two sets by characteristic functions. In the power set $\mathcal{P}([n])$, the theorem of Graham and Rothschild guarantees that if $n$ is sufficiently large then there are disjoint subsets $S_1, \ldots, S_d \subset [n]$ such that every nontrivial union $\bigcup_{i \in I} S_i, I \subset [d], I \neq \emptyset$, is the same colour. These sets pull back under the identification to give $x_1, \ldots, x_d$ with the claimed property.

We may deduce the stronger result claimed (that is, with $\eta > 0$) by a simple averaging argument. Let $m = m(r, d)$ be a value of $n$ for which the result is true with $\eta = 0$. Now take $\eta(r, d) := 2^{-m-1}$, and suppose $\eta \in [0, \eta(r, d)]$ and we have a $(1 - \eta)$-almost colouring of $\mathbb{F}_2^n$. If $n \geq m + 3$, this induces a
Hence from (4.3) we have as desired. Now suppose that by (4.1). Summing over all \( \mu \)
which (after redefining \( \mu = 2^{d-m} \)), this is in fact a full colouring of \( V \setminus \{0\} \). The result follows by the choice of \( m \). \( \square \)

4. Proof of the Main Theorem

Suppose that \( A \subset F_2^n \) is a set and we are given a parameter \( \varepsilon \in (0, 1] \). Choose integers \( d, r \) with \( 2^d \sim r \sim \frac{1}{\varepsilon} \), and let \( \eta = \eta(r, d) \) be the parameter whose existence is guaranteed by Proposition 3.1. By the “arithmetic regularity lemma” in this context \([\text{Gre05b}] \), there is some subspace \( W \subseteq F_2^n \) for which we get an induced \( (1 - 2\eta(r, d)) \)-almost colouring of \( F_2^n \). However, since \( 1 - 2\eta(r, d) \geq 1 - 2^{-m} > 1 - \frac{1}{W(0)} \), this is in fact a full colouring of \( V \times (0) \). The result follows by the choice of \( m \). \( \square \)

\footnotesize{

\( \begin{align*}
(1 - 2\eta(r, d)) \text{-almost colouring of } F_2^n \setminus \{0\}. \text{ Now } F_2^n \setminus \{0\} \text{ is uniformly covered by sets } V \setminus \{0\}, \text{ where } V \text{ ranges over all } m \text{-dimensional subspaces of } F_2^n. \text{ Therefore, by the pigeonhole principle, there is some } V \text{ for which we get an induced } (1 - 2\eta(r, d)) \text{-almost colouring of } V \setminus \{0\}. \text{ However, since } 1 - 2\eta(r, d) \geq 1 - 2^{-m} > 1 - \frac{1}{W(0)} \text{, this is in fact a full colouring of } V \setminus \{0\}. \text{ The result follows by the choice of } m. \end{align*} \)

\( \begin{align*}
\text{Suppose that } A \subset F_2^n \text{ is a set and we are given a parameter } \varepsilon \in (0, 1]. \text{ Choose integers } d, r \text{ with } 2^d \sim r \sim \frac{1}{\varepsilon}, \text{ and let } \eta = \eta(r, d) \text{ be the parameter whose existence is guaranteed by Proposition 3.1. By the “arithmetic regularity lemma” in this context } [\text{Gre05b}], \text{ there is some subspace } W \subseteq F_2^n, \text{ } n_0(r, d) \leq \text{codim } W \ll \varepsilon 1, \text{ such that }
\end{align*} \)

\( \begin{align*}
(4.1) \quad \sup_{r \in W^\perp} |(1_A \mu x + W)^\wedge(r)| \leq \varepsilon
\end{align*} \)

for a proportion at least \( 1 - \eta \) of all \( x \in F_2^n \). For notational simplicity, put \( X := F_2^n \) and change basis so that \( W = \{x_i\} \times F_2^{n-m} \). Let \( \tilde{X} \subset X \) be the set of all \( x \in F_2^n \) for which (4.1) holds. Thus \( |	ilde{X}| \geq (1 - \eta)|X| \). Define an \( (r + 1) \)-colouring \( c : \tilde{X} \to \{0, 1, \ldots, r\} \) (and hence a \( (1 - \eta) \)-almost \( (r + 1) \)-colouring of \( X \)) by defining \( c(x) := \lfloor rE_{x + W} 1_A \rfloor \). That is, \( c(x) = j \) if the density of \( A \) on \( x + V \) lies in the range \( \left[ \frac{j}{r}, \frac{j+1}{r} \right) \).

By Proposition 3.1 we may find linearly independent \( x_1, \ldots, x_d \in F_2^n \) and \( j \in \{0, 1, \ldots, r\} \) such that \( c(\sum_{i \in I} x_i) = j \) for all \( I \subset [d], I \neq \emptyset \).

Set \( V := W + \langle x_1, \ldots, x_d \rangle \). We claim that
\( \begin{align*}
(4.2) \quad \sup_{r \in V^\perp} |(1_A \mu V)^\wedge(r)| = O(\varepsilon),
\end{align*} \)

which (after redefining \( \varepsilon \) to \( \varepsilon/C \)) implies our main theorem. Suppose first that \( r \notin W^\perp \). Note that
\( \begin{align*}
(4.3) \quad \mu V = 2^{-d} \sum_{I \subset [d]} \mu W + \sum_{i \in I} x_i.
\end{align*} \)

If \( I \neq \emptyset \) we have \( \sum_{i \in I} x_i \in \tilde{X}, \text{ and so in this case }
\end{align*} \)

\( \begin{align*}
|(1_A \mu \sum_{i \in I} x_i + W)^\wedge(r)| \leq \varepsilon
\end{align*} \)

by (4.1). Summing over all \( I \neq \emptyset, \text{ and handling the case } I = \emptyset \) trivially, we have
\( \begin{align*}
|(1_A \mu V)^\wedge(r)| \leq 2^{-d} + \varepsilon = O(\varepsilon),
\end{align*} \)

as desired. Now suppose that \( r \in W^\perp \setminus V^\perp \). In this case
\( \begin{align*}
(1_A \mu x + W)^\wedge(r) = (-1)^r E_{x + W} 1_A.
\end{align*} \)

Hence from (4.3) we have
\( \begin{align*}
(1_A \mu V)^\wedge(r) = 2^{-d} \sum_{I \subset [d]} (-1)^r \sum_{i \in I} x_i \mathbb{E}_{\sum_{i \in I} x_i} 1_A.
\end{align*} \)

\footnotesize

\( ^1 \text{Strictly speaking, the argument as presented in } [\text{Gre05b}] \text{ does not guarantee a lower bound on codim } W. \text{ However, this may very easily be arranged with a trivial modification of the proof, for example by foliating } F_2^n \text{ into cosets of some arbitrary subspace of codimension } n_0(r, d) \text{ and then running the energy increment argument as in that paper.} \)
By construction,

\[
\frac{j}{r} \leq \mathbb{E}\sum_{i \in I} x_i + W 1_A < \frac{j + 1}{r}
\]

whenever \( I \neq \emptyset \). It follows that

\[
(1_A \mu_V) ^ \wedge (r) = \frac{j}{r} 2^{-d} \sum_{I \subseteq [d]} (-1)^r \sum_{i \in I} x_i + O(2^{-d} + \frac{1}{r}).
\]

However

\[
\sum_{I \subseteq [d]} (-1)^r \sum_{i \in I} x_i = \prod_{i=1}^d (1 + (-1)^r x_i),
\]

and at least one of the factors here vanishes since \( r \notin V ^ \perp \). Hence

\[
(1_A \mu_V) ^ \wedge (r) = O(2^{-d} + \frac{1}{r}) = O(\varepsilon),
\]

and the proof is complete.

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