Anisotropic Combined Dry Friction in Problems of Pneumatics’ Dynamics

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Abstract

Background Traditionally used shimmy models assume vanishing slip at each point of a roadwheel contact spot, at the same time the observed instable wheel motion observed at unsteady rolling stages with significant sliding could not be described by traditional theories. Thus, a principally new model of the wheel shimmy is required to be consistent with such regimes.

Purpose A new model of a rolling wheel accounting for the dry friction under the conditions of combined kinematics (i.e. simultaneous sliding, spinning and rolling) as well as for contact pressure distributions in pneumatic tires could become a background for various engineering methods of prediction of the shimmy appearing at unsteady rolling regimes.

Methods The proposed model of a pneumatic wheel’s motion is based on the improved dry friction theory accounting for the friction anisotropy; the Coulomb law is assumed for each contact spot’s point where the summary slip velocity is resulted by the simultaneous sliding and spinning, and the contact pressure could be obtained from the finite element simulation of quasistatic tire deforming.

Results The theory of the coupled dry friction is improved by accounting for the anisotropy of the friction factor represented in a form of the second-rank tensor. The general formulation for the resultant vector of dry friction forces, dry friction torque and the rolling friction couple are obtained. The contact pressure for a typical tire is computed numerically, its polynomial interpolation is introduced, and the coefficients of the approximate model of the dry friction are obtained.

Conclusion The constructed model allows an efficient accounting for the dry friction effects on the rolling stability as well as the real contact pressure distribution; thus, it can be interpreted as the second approximation improving the model of the shimmy of quasi-rigid wheel.

Keywords Dry friction · Combined kinematics · Friction tensor · Pneumatic tires

Introduction

The phenomenon of the shimmy of wheels of aircraft landing gears and automotive vehicles is well known but not completely studied. Indeed, the traditionally used model of a shimmy proposed by Keldysh [11] is based on the following assumptions: (a) the sliding in each point of the road–wheel contact area vanishes and (b) the motion instability is induced essentially by the elastic forces appearing in deformed tires. Such an approach results the very simple model with reduced number of degrees of freedom, does not require complex numerical methods and remains still useful in the engineering practice [31]. At the same time, the instable motion of various wheels is observed at the stages of non-steady rolling with significant longitudinal sliding in the contact spot; moreover, the spinning can appear in case of the disturbed motion [32]. It is clear that the Keldysh assumptions are
inconsistent with such a regime, and the dry friction effects on the motion stability cannot be studied using the traditional shimmy theory [31]. On the other hand, the combined kinematics of relative motion in the contact area requires the qualitative improvement of dry friction theories as it was shown by Constenou [8] and Erismann [10]; Coulomb’s dry friction law is unable to describe a wide range of phenomena due to the coupling of sliding and spinning as well as some other effects [28]. This theory was further improved by many researchers (e.g., see [6, 7, 21, 23, 24, 27]).

The rigid wheel shimmy is one of such phenomena; it is induced only by the coupling of the dry friction forces and torque [1, 36]. This effect was found using the new dry friction model so-called “poly-component” or “multi-component” [34, 35] and being the essential improvement of previous theories of Contenou [8], Erismann [10], and other authors. The multi-component theory uses the local formulation of the Coulomb law for each small element of the contact spot where the corresponding summary sliding velocity is resulted by the longitudinal slip and the spinning. Therefore, the resultant dry friction force vector and moment are obtained as a result of integration over the contact area; both are depending as well on the slip velocity as on the angular spin velocity.

The exact integral formulation for the resultant force vector and couple was obtained in [34] assuming the contact area be small and consequently circular and the contact pressure distribution be Hertzian. These formulae are too complex to use them in the engineering analysis of dynamics of systems with the dry friction; it was shown nevertheless that their linear-fractional approximation is quite sufficient in most dynamical problems [35]. The Padé approximations were later replaced by the new type of the approximate formulae: \( \mathbf{F} \approx \mathbf{F}_0 + \sqrt{\mathbf{v}^2 + a \mathbf{a}^2} \) (e.g., see [13]). Such results were obtained for some other particular cases of the contact pressure distributions [14].

The quasi-rigid wheel theory [13, 35] was applied to investigate the shimmy of landing gears [2–4, 32]. It seems to be a good first approximation; nevertheless, this simplest model seems to be a good second approximation for a wheel under combined rolling, sliding, and spin, and can be applied to study the shimmy initiation conditions.

On the other hand, the dry friction anisotropy was introduced as a generalized Coulomb law by many authors (e.g., see [9, 22, 29, 30, 37]), the tensor coefficient of the dry friction was introduced, and the effect of the anisotropy on the dynamics of the material point on the plane was investigated in details [37]. It has to be noted nevertheless that all the mentioned models generalize the classical Coulomb law in case of the simple motion and have nothing to do with the combined kinematics. Here, the basics of the improved theory of coupled dry friction are presented, the dry friction anisotropy is introduced, and the formulae for the resultant vector and couple of the dry friction are constructed. Such a model seems to be a good second approximation for a wheel under combined rolling, sliding, and spin, and can be applied to study the shimmy initiation conditions.

**Model of The Combined Rolling, Sliding, and Spin of The Rigid Body Based On The Coupled Dry Friction Theory**

**On The Anisotropic Dry Friction**

In case of the frictional anisotropy induced by the structure of interacting bodies and/or texture of the contacted surface, the Amonton–Coulomb dry friction law can be written in the following formulation [9, 22, 30, 37]:

\[
\mathbf{F} = -|N| \frac{\mathbf{f} \cdot \mathbf{v}}{|\mathbf{v}|} \quad (\mathbf{v} \neq 0).
\]

(1)

Here, the second-rank tensor \( \mathbf{f} = f_{\alpha \beta} \delta^\alpha_\beta \) is the dry friction coefficient; \( \delta^\alpha_\beta \) are base vectors of some frame \( Ox^1 x^2 \) on the plane of interaction of the contacting bodies, \( \mathbf{v} = v^a \delta^a \) is the vector of the relative sliding velocity, and the symbol “\( \cdot \)” denotes the scalar product. In general, \( f_{\alpha \beta} \neq f_{\beta \alpha} \) (e.g., see [22, 29]).

Let us consider the dry friction tensor \( \mathbf{f} \) be positively defined:

\[
v_1^T \cdot \mathbf{f}_S \cdot v_1 > 0 \quad \forall v_1(q) : \quad \mathbf{q} \in \Omega \subset R^n, \quad \mathbf{f}_S = \frac{1}{2} (\mathbf{f} + \mathbf{f}^T),
\]

where \( \Omega \) is the configuration space of the considered mechanical system, and \( \mathbf{q} \) is the vector of generalized coordinates. Thus, the linear mapping of the set of unit vectors \( v_1 \) into the set of dry friction force vectors \( \mathbf{F} \) (1) is defined by the linear operator \( \mathbf{f} \) (1). As usually, the unit vector \( \mathbf{v}_1 = v/|v| \) is the relative sliding director.

The general cohesion condition can be represented in the formulation [37]:

\[
I_2(\mathbf{f}) \neq 0 : \quad I_2(\mathbf{f}) \left\{ I_2(\mathbf{f}) |\mathbf{F}|^2 \right. \\
+ \left. |f_1(\mathbf{f}) \mathbf{f}_S + \mathbf{f}_S^T \cdot \mathbf{f} | : \quad \mathbf{F} \otimes \mathbf{F} \right\} \leq |N|^2,
\]

(2)
where \( I_1(f) \) and \( I_2(f) \) are invariants of dry friction tensor \( f \), and the symbols \( "\otimes" \) and \( "\cdot" \) denote the tensor product and the double convolution of the tensor product, correspondingly. The physical components \( f_{\alpha \beta} \) of the tensor \( f \) can be defined experimentally on the groundwork of specific tests. For instance the concept proposed by Zmitrowiecz [37] can be used; the dry friction forces are measured directly for two non-collinear sliding directions. In general, these ones have not to coincide with the base vectors \( \mathbf{e}_a \) of the main frame \( OX^1\times x^2 \).

The Local Model of The Anisotropic Dry Friction

In general, the plane-parallel relative motion, i.e., the simultaneous sliding and spinning of the rigid bodies with the finite contact spot \( S \) requires the qualitative improvement of the Amonton–Coulomb dry friction law [1, 10, 13, 14, 34–36]. The first generalized formulation accounting for the spin in the formulae for the dry friction force was proposed by Erissmann [10] and Contensou [8]; it was shown that the resultant vector of dry friction forces depends significantly on the spin. Zhuravlev [1, 34, 35] and Kireenkov [13, 14, 35] have proposed the further improvement of this concept; finally, the multi-component dry friction theory was formulated for a general case of combined slip, spin, and rolling [1]. The aim of this theory consists in the differential formulation of the Coulomb law as a local model of the friction interaction in each point of the contact area \( S \):

\[
\forall M \in S \quad \tau = -\sigma_v \left[ \frac{f \cdot v_S}{|v_S|} \right] (v_S \neq 0),
\]

(3)

\[ v_S = v_0 - R\omega_s \times \mathbf{e}_3 + \omega_s \times \mathbf{r}_r. \]

(4)

Here, \( v_S \) denotes the summary velocity of the relative slip in the arbitrary point \( M \in S \), \( v_0(q) \) is the longitudinal absolute velocity, \( \omega_s(q) \) is the angular velocity of rolling, \( \omega_r(q) \) is the angular velocity of spinning, \( R(M) \) is the curvature radius of the rolling body calculated in the point \( M \), \( r_r(M) \) is the vector radius of the point \( M \in S \) in the plane of contact, \( \mathbf{e}_3 \) denotes the normal unit vector of the contact plane, \( \tau \) is the frictional tangential stress in the contact area \( S \), and \( \sigma_v \) denotes the normal contact pressure. Thus, the cohesion condition (2) can be formulated locally in the point \( M \in S \) as follows:

\[
I_2^{-2}(f) \left\{ I_2^2(f)|\tau|^2 + \left[ I_1(f)v_S + f^T \cdot f \right] : \tau \otimes \tau \right\} \leq |\sigma_v|^2.
\]

(5)

Let us consider the combined kinematics, i.e., the simultaneous slip, spin, and rolling (4). Therefore, taking into account the dry friction anisotropy, we obtain the following formula for the tangential stress (3):

\[
\tau_1 = -|\sigma_v| \left[ v_0 + \omega_s \times \mathbf{r}_r \right]^{-1} f \cdot \left( v_0 - R\omega_s \times \mathbf{e}_3 + \omega_s \times \mathbf{r}_r \right),
\]

(6)

where the normal pressure accounting for the rolling effect is represented by the linear approximation [15]:

\[
\sigma_v = \sigma_0 \left[ 1 + \left( \frac{\mathbf{r}_r \times \mathbf{h} \cdot \omega_r}{|\omega_r|} \right) \cdot \mathbf{e}_3 \right].
\]

(7)

Here, \( \sigma_0 = \sigma_v (\omega_r = 0) \), and \( \mathbf{h} = h_{\alpha \beta} \mathbf{e}_\alpha \mathbf{e}_\beta \) is the "rolling friction tensor" for the anisotropic elastic body; we assume it being homogeneous and positively defined:

\[
\mathbf{h} \neq h(M); \quad \forall \omega_r = \omega_r(q); \quad \omega_r \times \mathbf{h} \cdot \omega_r > 0.
\]

(8)

The global normal contact stress \( \sigma_0 \) is determined by the solution of the appropriate contact problem of elasticity theory in quasi-static statement. Thus, the model (6)–(8) allows one to use the static solution as a first approximation to model the coupled rolling and sliding friction, and the complex modeling of the dynamics of contact interaction is not required.

Now, accounting the rolling effect on the contact pressure on the basis of the formula (7) and using the formula (6) for the tangential contact stress that accounts by-turn both sliding (this term is denoted as \( \tau_1 \)) and spinning (this one is denoted as \( \tau_1 \)), we obtain finally the local model of the anisotropic dry friction in case of the combined kinematics:

\[
\tau = \tau_1 + \tau_2;
\]

\[
\tau_1 = -|\sigma_0| \left[ 1 + \left( \frac{\mathbf{r}_r \times \mathbf{h} \cdot \omega_r}{|\omega_r|} \right) \cdot \mathbf{e}_3 \right] \frac{f \cdot \left( v_0 - R\omega_s \times \mathbf{e}_3 \right)}{|v_0 - R\omega_s \times \mathbf{e}_3 + \omega_s \times \mathbf{r}_r|};
\]

\[
\tau_2 = -|\sigma_0| \left[ 1 + \left( \frac{\mathbf{h} \times \omega_r}{|\omega_r|} \right) \cdot \mathbf{e}_3 \right] \frac{f \cdot (\omega_r \times \mathbf{r}_r)}{|v_0 - R\omega_s \times \mathbf{e}_3 + \omega_s \times \mathbf{r}_r|};
\]

The Global Model of The Anisotropic Dry Friction Under Combined Kinematics

The dynamic interaction of the slightly deformed rigid body with the rough support plane is defined by the normal reaction \( \mathbf{N} \), the resultant vector of tangent forces \( \mathbf{T} \), the anti-rolling couple \( \mathbf{M}_r \), and the dry friction torque \( \mathbf{M}_f \). These quantities are obtained by integration of the normal contact stress (7) as well as the summary tangential stress (9) over the contact area \( S \). Taking into account the dry friction anisotropy, we can represent the appropriate integral relationships formulated in [12] as follows:

\[
\mathbf{N} = \int_S \sigma_v \mathbf{e}_3 dS = \int_S \sigma_0 \left[ \mathbf{e}_3 + \frac{\mathbf{r}_r \times (\mathbf{h} \cdot \omega_r)}{|\omega_r|} \right] dS;
\]

(10)

\[
\mathbf{M}_f = \int_S \sigma_v \mathbf{r}_r \times \mathbf{e}_3 dS = \int_S \sigma_0 \left[ \mathbf{e}_3 + \frac{\mathbf{r}_r \times (\mathbf{h} \cdot \omega_r)}{|\omega_r|} \right] dS;
\]

(11)
Here, the symbol "∧" denotes the exterior product. The homogeneity of the tensor (8) and the formulae (14)–(19) lead to the vanishing as well normal reaction \( N_1 \) as the rolling initiation moment \( M_i^0 \) in the frame \( O_x^g \) attached to the center of the figure \( S \); therefore, we have the following formulae for the normal reaction and anti-rolling couple:

\[
N = N_0 e_3; \quad M = -J_\sigma \cdot h \cdot (\omega / |\omega|).
\]

The resultant vector of the anisotropic dry friction (12) under combined kinematics can be now expressed through the following terms:

\[
T = \sum_{k=1}^{4} T_k,
\]

\[
T_1 = -\int_S \sigma_0 \cdot \frac{f \cdot v_s}{|v_s + \omega_s \times r_s|} dS;
\]

\[
T_2 = -\int_S \sigma_0 \cdot \frac{r_s \times (h \cdot \omega_s)}{|\omega_s|} \cdot e_3 \frac{f \cdot v_s}{|v_s + \omega_s \times r_s|} dS;
\]

\[
T_3 = -\int_S \sigma_0 \cdot \frac{f \cdot (\omega_s \times r_s)}{|v_s + \omega_s \times r_s|} dS;
\]

\[
T_4 = -\int_S \sigma_0 \cdot \frac{r_s \times (h \cdot \omega_s)}{|\omega_s|} \cdot e_3 \frac{f \cdot (\omega_s \times r_s)}{|v_s + \omega_s \times r_s|} dS.
\]

Here, (21) is the static dry friction force, (22) is the supplementary quasi-static dry friction force resulted by the rolling effect, (23) is the supplementary dry friction force due to the spin, and the term (24) denotes the variation of the dry friction force due to the coupling of the rolling and spinning of the body.

The torque of the anisotropic dry friction under combined kinematics (13) is also represented as a sum of four terms:

\[
M = \sum_{k=1}^{4} M_k,
\]

\[
M_1 = -\int_S \sigma_0 \cdot \frac{r \times (f \cdot v_s)}{|v_s + \omega_s \times r_s|} dS.
\]
\[
M_2 = - \int_S \sigma_0 \left[ \frac{\mathbf{r} \times (\mathbf{h} \cdot \mathbf{w})}{|\mathbf{w}|} \right] \cdot \mathbf{e}_3 \frac{\mathbf{r} \times \left( \mathbf{f} \cdot \mathbf{v}_S \right)}{|\mathbf{v}_S + \mathbf{w} \times \mathbf{r}_r|} \, dS; \quad (26)
\]

\[
M_3 = - \int_S \frac{\mathbf{r} \times \left[ \mathbf{f} \cdot (\mathbf{w} \times \mathbf{r}_r) \right]}{|\mathbf{v}_S + \mathbf{w} \times \mathbf{r}_r|} \, dS; \quad (27)
\]

\[
M_4 = - \int_S \sigma_0 \left[ \frac{\mathbf{r} \times (\mathbf{h} \cdot \mathbf{w})}{|\mathbf{w}|} \right] \cdot \mathbf{e}_3 \frac{\mathbf{r} \times \left( \mathbf{f} \cdot (\mathbf{w} \times \mathbf{r}_r) \right)}{|\mathbf{v}_S + \mathbf{w} \times \mathbf{r}_r|} \, dS, \quad (28)
\]

Here, \((25)\) is the dry friction torque under the pure sliding that is resulted by the coupling of the slip and spin, the formula \((26)\) defines the supplementary quasi-static dry friction torque resulted by the rolling effect on the slip, the ”proper static” dry friction torque is defined by the formula \((27)\) and its variation due to the coupling of the rolling and spinning is given by the term \((28)\). The vector \(\mathbf{v}_S\) denotes the summary velocity in the point \(M \in S\):

\[
\mathbf{v}_S = \mathbf{v}_0 - R \mathbf{w} \times \mathbf{e}_3; \quad |\mathbf{v}_S|^2 = v_s^2 + 2 \mathbf{v}_S \cdot (\mathbf{w} \times \mathbf{r}_r) + (\mathbf{w} \times \mathbf{r}_r) \cdot (\mathbf{w} \times \mathbf{r}_r). \quad (29)
\]

Thus, the invariant formulation for the coupled dry friction theory is obtained. The constructed model describes the coupling effects under combined sliding, spinning, and rolling of the deformed rigid body with the finite area of contact with the support plane. The invariant relationships \((10)\)–\((13)\) can be rewritten in the coordinate form by the appropriate choice of the main frame.

**Approximate Model of The Rolling Wheel**

Let us consider the orthotropic dry friction given by the tensor \(\mathbf{f} = f \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix}, \quad f \neq 0, \quad \kappa \neq 0. \quad (30)\)

Here, \(f\) and \(\kappa f\) are the principal components of the tensor \(\mathbf{f}\). Let us introduce the frame \(\mathbf{O}xy\) attached to the centroid of the contact spot \(S\); the corresponding base vectors \(\mathbf{e}_1\) and \(\mathbf{e}_2\) are collinear to the principal directions of the tensor \(\mathbf{f}\). The static contact pressure symmetry, \(\sigma_0(x,y) = \sigma_0(\pm x, \pm y)\), as well as the rolling friction isotropy are assumed.

Let us consider the motion defined by the longitudinal velocity \(\mathbf{v}_0 = \mathbf{v}^e_1\) along the axis \(O\xi\) of the global rest frame (for more details, see \([16]\)), the rolling angular velocity \(\mathbf{w} = -\Omega \mathbf{e}_2\), and the spinning velocity \(\mathbf{v}_1\). Such a situation corresponds to the rolling of the wheel with the thread characterized by the friction factors \(f\) along the tread and \(\kappa f\) across it. Let us also consider here axially symmetric contact sites with the characteristic size of the area of contact \(R\) for instance, the diameter of the corresponded set at the plane \((x,y)\) (Fig. 1).

In the presence of motion, the tangent stresses appeared that lead to distortion in the symmetric diagram of the normal contact stresses distribution. The directions of the relative sliding (axis \(\xi\)) and rolling (axis \(x\)) are coincided in the case of ideal motion of aviation tyre, but from the shimmy phenomenon, they differ on angle \(\varphi\) (Fig. 1).

Let us suppose that the shifting value of the gravity center of the contact spot relatively of the geometric center is described by the vector \(\mathbf{d}\), the module of which was calculated in \([16, 33]\).

Formula \((7)\) in this case can be rewriting in simplest form:

\[
\sigma(x,y) = \sigma_0(x,y) (1 + d_x x + d_y y), \quad (31)
\]

where \(d_x\) and \(d_y\) are projections of the vector \(\mathbf{d}\) on axes \(x\) and \(y\), correspondingly.

The resultant force vector can be represented as \(\mathbf{T} = T_1 \mathbf{e}_1 + T_2 \mathbf{e}_2\), so that \(T_1\) is the longitudinal and \(T_2\) is the lateral friction force. It was shown \([1, 12–14, 34, 35]\) that the last one is due to the coupling effects.

The quantities \((21)\)–\((28)\) can now be simplified as it was implemented in \([12–14, 19]\).

The orthogonal transformation of rotation on the angle \(\varphi\):

\[
\xi = x \cos \varphi - y \sin \varphi, \quad \eta = x \sin \varphi + y \cos \varphi
\]

enables to transit in formulae \((21)\)–\((28)\) to integration over area with fixed boundaries \([12, 25, 26]\). After the aforementioned change of variables, the corresponding expressions can be significantly simplified with the aid of well known fact that integrals will be zero in the case of a symmetric domain of integration and oddness of the integrand function by one of the arguments and, in result of the change of variables \((6)\), they have form:

![Fig. 1 Kinematics inside of contact spot](image-url)
These functions were investigated in details for Hertz type of the normal contact stresses distributions for elliptic contact spots in [25, 26] and for arbitrary symmetric forms of the contact spots and symmetric distributions of the normal contact stresses in [12].

However, the integral relationships (32) are too complex to apply them to the analysis of dynamics of real systems, while their linear-fractional approximations are adequately accurate and simple in the same time [1, 12–14, 35]. Let us sketch below only the main results; for more details concerning the construction of the approximate formulae, see [12, 19]. As a result, we have the following formulae:

\[
F_{\parallel} = \frac{F_0 v}{\sqrt{v^2 + au^2}}, \quad F_{\perp} = \frac{\mu_0 k F_0 v u^2}{\sqrt{(v^2 + bo b^2)}}, \quad M_v = \frac{M_0 u}{\sqrt{u^2 + m v^2}}
\]

(33)

Here, \(u = \omega_0 R\), \(F_0\) is the longitudinal resting friction force, but \(M_0\) is the resting friction torque. If the normal contact stresses distribution in the absence of motion is defined by analytical or numerical functions [16, 19], then coefficients of the model (33) are calculated by the following formulæ:

\[
F_0 = f \int_G \sigma_0(x,y) dxdy,
\]

\[
M_0 = f(1 + k) \int_G \sigma_0(x,y) \sqrt{x^2 + y^2} dxdy,
\]

\[
\sqrt{m} = \frac{1}{M_0} \int_G \sigma_0(x,y)(x^2 \cos^2 \varphi + y^2 \sin^2 \varphi) dxdy
\]

\[
\sqrt{a} = \frac{1}{F_0} \int_G \sigma_0(x,y)(x^2 \cos^2 \varphi + y^2 \sin^2 \varphi) dxdy
\]

\[
\sqrt{b} = \frac{1}{F_0} \int_G \sigma_0(x,y)(x^2 - y^2) \sin 2\varphi dxdy
\]

\[
(32)
\]

Accounting for The Contact Pressure Distribution

The contact pressure distribution close to the real one as well as the contact spot main axes can be obtained on the basis of the finite element model of a tire; several levels of accuracy of the model can be considered. The first and simplest one consists in the numerical simulation of the quasi-static nonlinear deforming of a tire using the plane elasticity problem statement for the tire cross section [2]. The contact pressure distribution computed for several levels of the vertical load or the radial deformation of the tire is then interpolated analytically [17–20].

Because the contact pressure distribution \(\sigma_0(x,y)\) is assumed to be symmetric, then, in the first approximation for purposes of numerical simulation, it can be presented in the following form:

\[
\sigma_0(x,y) = \sigma^j_0(x) \sigma^j_0(y),
\]

where each function \(\sigma^j_0, j = 1, 2\) is assumed to be symmetric:

\[
\sigma^j_0(r) = \sigma^j_0(-r), \quad r = \begin{cases} x/a, & j = 1 \\ y/b, & j = 2 \end{cases},
\]

where \(a\) and \(b\) are main semi-axes of the contact spot ellipse. Further, due to independence of the simulation procedure for each of the coordinates, to reduce the number of designations, the superscripts for the functions \(\sigma^j_0, j = 1, 2\) will be omitted.

The polynomial interpolation of the finite element solution \(\sigma_0\) is constructed as follows:

\[
\sigma_0(r) \approx \sigma^k_0 p_k(r), \quad r \in [-1, 1], \quad k = 0, 2 \ldots N \in \mathbb{N},
\]

where \(p_k(r)\) are Legendre polynomials. The factors \(\sigma^k_0 \in \mathbb{R}\) are obtained from the solution of the quadratic programming problem:

\[
G_{km} \sigma^k_0 \sigma^m_0 \rightarrow \min, \quad G_{km} = 2 \delta_{km}/(2k + 1)
\]

with the following restrictions:
where the first one corresponds to the lower approximation of the FEM solution, the second one corresponds to the condition \( \sigma_{0,i} = 0 \), and the third one guarantees the equivalence of the normal reaction following from the interpolation and the load applied to the FE model. The typical contact pressure and its approximation \((N = 10)\) are shown on the Fig. 2.

The high level of vertical loads results the O-shaped area of contact, where the contact pressure vanishes near the center of a spot. In such a situation, the higher-order approximation may be required. On the other hand, the strongly deformed state of a tire leads to the accounting for the tangential deformation and the elastic forces in the pneumatics; therefore, this model become combined with the Keldysh concept.

**Conclusion**

The theory of the coupled dry friction under combined kinematics is improved by accounting for the anisotropy of the dry friction coefficient represented in a form of the second-rank tensor. The general invariant formulation for the resultant vector of dry friction forces as well as for the dry friction couple and the rolling friction torque are obtained in case of the simultaneous sliding, spinning, and rolling of the rigid body. The contact pressure for a tire can be obtained from the numerical simulation on the groundwork of the finite element modeling; thus, the polynomial interpolation of such a solution is introduced, and the coefficients of the approximate model of the dry friction arte obtained. The constructed model being applied to the rolling wheel can be interpreted as the second approximation improving the model of the shimmy of quasi-rigid wheel.

**References**

1. Andronov VV, Zhuravlev VP (2010) Dry friction in problems of mechanics (in Russian). NITs Reg. Khaot. Dinam, Moscow-Izhevsk. ISBN 978-5-93972-856-0

2. Bernikova N, Zagordan AA, Zhavoronok SI (2012) Main landing gears shimmy models based on poly-component dry friction. In: Ogden RW, Holzapfel GA (eds) Proceedings of the 8th European solid mechanics conference (ESMC-2012), Graz, Austria. ISBN 978-3-85125-223-1

3. Bernikova NS, Stepanov EV, Zagordan AA, Zhavoronok SI (2013) Modelling of main landing gears shimmy and shimmy-like vibrations on the basis of the multi-component anisotropic dry friction theory. In: Dimitrovova Z, de Almeida J, Gonalves R (eds) Proceedings of the 11th international conference on vibration problems (ICOPV-2013). AMPTAC, Lisbon, Portugal, p 290. ISBN 978-989-96264-4-7

4. Bernikova NS, Zagordan AA, Zhavoronok SI (2014) Landing gears shimmy models based on the combined anisotropic dry friction theory. In: Proceedings of the 8th European nonlinear dynamics conference (ENOC-2014), Vienna, Austria. ISBN 978-3-200-03433-4

5. Bogoslovskii SE, Kuryumov NN (2015) Numerical solution a problem of contact pneumatic truck tire with road surface. Proc. PSU Tech Sci 8(2):138–147

6. Borisov AV, Erdakova NN, Ivanova TB, Mamaev IS (2014) The dynamics of a body with an axisymmetric base sliding on a rough plane. Regul Chaot Dyn 19:607. https://doi.org/10.1134/S1560354714060021

7. Borisov AV, Karavaev YL, Mamaev IS, Erdakova NN, Ivanova TB, Tarasov VV (2015) Experimental investigation of the motion of a body with an axisymmetric base sliding on a rough plane. Regul Chaot Dyn 20:518. https://doi.org/10.1134/S1560354715050020

8. Contensou P (1963) Couplage entre frottement de glissement et frottement de pivotement dans la theorie de la toupie. In: Ziegler H (eds) Kreiselprobleme / Gyrodynamics. International Union of Theoretical and Applied Mechanics. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-662-12200-6_15. Print ISBN 978-3-662-12201-3. Online ISBN 978-3-662-12200-6

9. Dmitriev N, Silantyeva O (2016) Terminal motion of a thin elliptical plate over a horizontal plane with orthotropic friction. Vestn St Petersb Univ Math 46(1):92–98

10. Erismann T (1954) Theorie und anwendungen des echten kugelgetriebes. ZAMP 5:355–388

11. Keldysh MV (1985) Selected works, mechanics. Nauka, Moscow

12. Kireenkov AA (2005) Three-dimensional model of combined dry friction and its application in non-holonomic mechanics. In: Proc. 5th European Nonlinear Dynamics Conf. ENOC-2005. Eindhoven, Netherlands

13. Kireenkov AA (2008) Coupled models of sliding and rolling friction. Dokl Akad Nauk 419(6):759–762

14. Kireenkov AA (2011) Coupled model of sliding and spinning friction. Dokl Akad Nauk 441(6):750–755

15. Kireenkov AA (2015) Further development of the theory of multicomponent dry friction. In: COUPLED PROBLEMS 2015. Proceedings of the 6th International Conference on Coupled Problems in Science and Engineering, Venice, Italy, pp. 203–209

16. Kireenkov AA (2017) Improved theory of the combined dry friction in problems of aviation pneumatics’ dynamics. In: Proceedings of the 7th International Conference on Coupled Problems in Science and Engineering, COUPLED PROBLEMS 2017, pp. 1293–1298

17. Kireenkov AA (2018) Improved friction model of the aviation tyre contact with the landing strip. IFAC-PapersOnLine 51(2):890–894. https://doi.org/10.1016/j.ifacol.2018.04.027

18. Kireenkov AA, Nushtaev DV, Zhavoronok SI (2018) A new approximate model of tyre accounting for both deformed state and dry friction forces in the contact spot on the background of the coupled model. MATEC Web Conf 211:08003. https://doi.org/10.1051/matecconf/201821108003

19. Kireenkov AA, Zhavoronok SI (2017) Coupled dry friction models in problems of aviation pneumatics’ dynamics. Int J Mech Sci 127:198–203. https://doi.org/10.1016/j.ijmecsci.2017.02.004

20. Kireenkov AA, Zhavoronok SI (2018) Implementation of analytical models of the anisotropic combined dry friction in problems of pneumatics’ dynamics. MATEC Web Conf 211:08004. https://doi.org/10.1051/matecconf/201821108004

21. Kosenko I, Aleksandrov E (2009) Implementation of the contensou-erismann model of friction in frame of the hertz
contact problem on modelica. In: 7th Modelica Conf. https://doi.org/10.3384/ecp09430006

22. Kozlov VV (2010) Lagrangian mechanics and dry friction. Nonlinear Dyn 6(4):855–868 (in Russian)

23. Kudra G, Awrejcewicz J (2011) Tangens hyperbolicus approximations of the spatial model of friction coupled with rolling resistance. Int J Bifurc Chaos 21:2905–2917. https://doi.org/10.1142/S0218127411030222

24. Kudra G, Awrejcewicz J (2012) Bifurcational dynamics of a two-dimensional stick-slip system. Differ Equ Dyn Syst 20:301–322. https://doi.org/10.1007/s12591-012-0104-z

25. Kudra G, Awrejcewicz J (2013) Approximate modelling of resulting dry friction forces and rolling resistance for elliptic contact shape. Eur J Mech A/Solids 42:358–375

26. Kudra G, Awrejcewicz J (2014) Mathematical modelling and simulation of the bifurcational wobblestone dynamics. Discont Nonlinear Complex 3(2):123–132

27. Leine R, Glocker C (2003) A set-valued force law for spatial coulomb–contensou friction. Eur J Mech A/Solids 22(2):193–216. https://doi.org/10.1016/S0997-7538(03)00025-1

28. Ramodanov SM, Kireenkov AA (2017) Controllability of a rigid body in a perfect fluid in the presence of friction. In: Proceedings of the 7th International Conference on Coupled Problems in Science and Engineering, COUPLED PROBLEMS 2017, pp. 179–184

29. Silantyeva O, Dmitriev N (2016) Dynamics of bodies under symmetric and asymmetric orthotropic friction forces. CRC Press, Boca Raton, pp 511–515

30. Vil’ke VG (2008) Anisotropic dry friction and unilateral non-holonomic constraints. Mech Solids 72:3–8

31. Zagordan AA (2011) Current state of the wheel shimmy theory. J Trudy MAI Mosc Aviat Inst 47:11 (in Russian)

32. Zagordan AA, Zhavoronok SL (2011) Main landing gear shimmy investigation using dry friction multicomponent model. Nelin Mir 10:646–656 (in Russian)

33. Zhavoronok SL, Kireenkov AA (2017) On the effect of the anisotropic dry friction and the deformed state of tires on the shimmy initiation. In: Proceedings of the 7th International Conference on Coupled Problems in Science and Engineering, COUPLED PROBLEMS 2017, pp. 216–226

34. Zhuravlev VP (1998) On the model of dry friction in the problem of rolling of rigid bodies. J Appl Math Mech 62(5):762–767

35. Zhuravlev VP, Kireenkov AA (2005) Padé expansions in the two-dimensional model of coulomb friction. Mech Solids 40(2):1–10

36. Zhuravlev VP, Klimov DM (2010) Theory of the shimmy phenomenon. Mech Solids 45(3):324–330

37. Zmitrowiecz A (1989) Mathematical descriptions of anisotropic friction. Int J Sol Struct 25(8):837–862

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