STONE-TYPE DUALITIES FOR SEPARATION LOGICS

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ABSTRACT. Stone-type duality theorems, which relate algebraic and relational/topological models, are important tools in logic because — in addition to elegant abstraction — they strengthen soundness and completeness to a categorical equivalence, yielding a framework through which both algebraic and topological methods can be brought to bear on a logic. We give a systematic treatment of Stone-type duality for the structures that interpret bunched logics, starting with the weakest systems, recovering the familiar BI and Boolean BI, and extending to both classical and intuitionistic Separation Logic. The structures we consider are the most general of all known existing algebraic approaches to Separation Logic and thus encompass them all. We demonstrate the uniformity of this analysis by additionally capturing the bunched logics obtained by extending BI and BBI with multiplicative connectives corresponding to disjunction, negation and falsum: De Morgan BI, Classical BI, and the sub-classical family of logics extending Bi-intuitionistic (B)BI. We additionally recover soundness and completeness theorems for the specific truth-functional models of these logics as presented in the literature, with new results given for DMBI, the sub-classical logics extending BiBI and a new bunched logic, CKBI, inspired by the interpretation of Concurrent Separation Logic in concurrent Kleene algebra. This approach synthesises a variety of techniques from modal, substructural and categorical logic and contextualizes the ‘resource semantics’ interpretation underpinning Separation Logic amongst them. As a consequence, theory from those fields — as well as algebraic and topological methods — can be applied to both Separation Logic and the systems of bunched logics it is built upon. Conversely, the notion of indexed frame (generalizing the standard memory model of Separation Logic) and its associated completeness proof can easily be adapted to other non-classical predicate logics.

1. INTRODUCTION

1.1. Background. Bunched logics, beginning with O’Hearn and Pym’s BI [60], have proved to be exceptionally useful tools in modelling and reasoning about computational and information-theoretic phenomena such as resources, the structure of complex systems, and access control [19, 20, 32]. Perhaps the most striking example is Separation Logic [62, 67]
(via Pointer Logic [44]), a specific theory of first-order (Boolean) BI with primitives for mutable data structures. Other examples include layered graph logics [19, 20, 32], modal and epistemic systems [27, 38], and Hennessy–Milner-style process logics that have applications in security [20] and systems modelling [2, 21].

The weakest bunched systems are the so-called layered graph logics [19, 32]. These logics have a multiplicative conjunction that is neither associative nor commutative, together with its associated implications, and additives that may be classical or intuitionistic. These systems can be used to describe the decomposition of directed graphs into layers (see Fig 1), with applications such as complex systems modelling (e.g., [19, 32]) and issues in security concerning the relationship of policies and the systems to which they are intended to apply (e.g., [20, 32]). Strengthening the multiplicative conjunction to be associative and commutative and adding a multiplicative unit yields BI, for intuitionistic additives, and Boolean BI (BBI), for classical additives. Further extensions include additive and multiplicative modalities and, with the addition of parametrization of modalities on actions, Hennessy–Milner-style process logics [21, 2]. Yet further extensions include additive and multiplicative epistemic modalities [38], with applications in security modelling.

All of the applications of bunched logics to reasoning about computational and information-theoretic phenomena essentially rely on the interpretation of the truth-functional models of these systems known as resource semantics. Truth-functional models of bunched logics are, essentially, constructed from pre- or partially ordered partial monoids [40] which, in resource semantics, are interpreted as describing how resource-elements can be combined (monoid composition) and compared (order). The program logic known as Separation Logic [44, 62, 67] is a specific theory of first-order bunched logic based on the partial monoid of elements of the heap (with the order being simply equality). Separation Logic has found industrial-strength application to static analysis through Facebook’s Infer tool (fbinfer.com).

Stone’s representation theorem for Boolean algebras [64] establishes that every Boolean algebra is isomorphic to a field of sets. Specifically, every Boolean algebra $A$ is isomorphic to the algebra of clopen subsets of its associated Stone space $S(A)$. This result generalizes to a family of Stone-type duality theorems which establish equivalences between certain categories of topological spaces and categories of partially ordered sets. From the logical point of view, Stone-type dualities strengthen the semantic equivalence of truth-functional (such as BI’s resource semantics or Kripke’s semantics for intuitionistic logic) and algebraic (such as BI algebras or Heyting algebras) models to a dual equivalence of categories. This is useful for a number of reasons: on the one hand, it provides a theoretically convenient abstract characterization of semantic interpretations and, on the other, it provides a systematic
approach to soundness and completeness theorems, via the close relationship between the algebraic structures and Hilbert-type proof systems. Beyond this, Stone-type dualities set up a framework through which techniques from both algebra and topology can be brought to bear on a logic.

1.2. Contributions. In this paper, we give a systematic account of resource semantics via a family of Stone-type duality theorems that encompass the range of systems from the layered graph logics, via $\text{BI}$ and $\text{BBI}$, to Separation Logic. Our analysis is extended to bunched logics with additional multiplicative connectives [10, 11, 14] and — through straightforward combination with the analogous results from the modal logic literature — can be given for the modal and epistemic systems extending ($\text{B}$)$\text{BI}$ [26, 27, 38]. As corollaries we retrieve the soundness and completeness of the standard truth-functional models in the literature as well as several new ones.

Soundness and completeness theorems for bunched logics and their extensions tend to be proved through labelled tableaux countermodel procedures [27, 38, 40, 53] that must be specified on a logic-by-logic basis, or by lengthy translations into auxiliary modal logics axiomatized by Sahlqvist formulae [11, 14, 16]. A notable exception to this (and precursor of the completeness result for ($\text{B}$)$\text{BI}$ given in the present work) is [39]. We predict our framework will increase the ease with which completeness theorems can be proved for future bunched logics, as the family of duality theorems can be extended in a modular fashion.

Of particular interest here are bunched logics with intuitionistic additives. In translations into Sahlqvist-axiomatized modal logics, some connectives must be converted into their ‘diamond-like’ De Morgan dual and back. For example, magic wand $\rightarrow$ must be encoded as septraction — that is, $\phi \rightarrow \triangledown \psi := \neg(\neg\phi \rightarrow \neg\psi)$ — in a logically equivalent Sahlqvist-axiomatized modal logic. This is not possible on weaker-than-Boolean bases, which necessarily lack negation. Our direct proof method side steps this issue and as a result we are able to give the first completeness proofs for the intuitionistic variants of bunched logics that were not amenable to the proof method used for their Boolean counterparts. We further demonstrate the viability of this proof technique for bunched logics by specifying a new logic $\text{CKBI}$, derived from the interpretation of Concurrent Separation Logic in concurrent Kleene algebra, and prove it sound and complete via duality. More generally, the notion of indexed frame (generalizing the standard model of Separation Logic) and its associated completeness proof can easily be adapted to other non-classical predicate logics.

All of the structures given in existing algebraic and relational approaches to Separation Logic — including [12], [17], [29], [33], and [34] — are instances of the structures utilized in the present work. Thus these approaches are all proved sound with respect to the standard semantics on store-heap pairs by the results of this paper. In particular, we strengthen Biering et al.’s [5] interpretation of Separation Logic in BI hyperdoctrines to a dual equivalence of categories. To do so we synthesise a variety of related work from modal [48, 66], relevant [1], substructural [6] and categorical logic [24], and thus mathematically some of what follows may be familiar, even if the application to Separation Logic is not. Much of the theory these areas enjoy is produced by way of algebraic and topological arguments. We hope that by recontextualizing the resource semantics of bunched logics in this way similar theory can be given for both Separation Logic and its underlying systems.
1.3. Structure. The paper proceeds as described below. In Section 2, we introduce the core bunched logics: the weakest systems LGL and ILGL, the resource logics BI and BBI, and finally the program logic Separation Logic (in both classical and intuitionistic form). In Section 3, we define the algebraic and topological structures suitable for interpreting (I)LGL and give representation and duality theorems relating them. In Section 4 these results are extended to the logics of bunched implications, (B)BI. In Section 5, we consider categorical structure appropriate for giving algebraic and truth-functional semantics for first-order (B)BI (FO(B)BI). We recall how FO(B)BI can be interpreted on (B)BI hyperdoctrines and define new structures called indexed (B)BI frames. Crucially, we show that the standard models of Separation Logic are instantiations of an indexed frames. We show that there is a dual adjunction between (B)BI hyperdoctrines and indexed (B)BI frames given by the semantics and extend (B)BI duality to obtain a dual equivalence of categories. In Section 6 we extend the theory of the previous sections to a variety of bunched logics extended with additional multiplicative connectives. In doing so we give new completeness theorems for DMBI and the logics extending BiBI, and specify a new logic CKBI. In Section 7, we consider possibilities for further work as a result of the duality theorems.

The present work is an extended and expanded version of a conference paper presented at MFPS XXXIII [30]. New to this version are the corresponding results for the intuitionistic variants of the logics considered there, as well as the extension to bunched logics with additional multiplicatives. The soundness and completeness results for DMBI and the various extensions of BiBI are, to our knowledge, new: the same is true for CKBI, including the specification of the logic itself. Proofs have been included wherever possible.

2. Preliminaries

2.1. Layered Graph Logics. We begin by presenting the layered graph logics LGL [19] and ILGL [32]. First, we give a formal, graph-theoretic definition of layered graph that, we claim, captures the concept as used in modelling complex systems [19, 20, 32]. Informally, two layers in a directed graph are connected by a specified set of edges, each element of which starts in the upper layer and ends in the lower layer.

Given a directed graph, \( G \), we refer to its vertex set and its edge set by \( V(G) \) and \( E(G) \) respectively, while its set of subgraphs is denoted \( Sg(G) \), with \( H \subseteq G \) iff \( H \in Sg(G) \). For a distinguished edge set \( \mathcal{E} \subseteq E(G) \), the reachability relation \( \sim_{\mathcal{E}} \) on \( Sg(G) \) is defined \( H \sim_{\mathcal{E}} K \) iff a vertex of \( K \) can be reached from a vertex of \( H \) by an \( \mathcal{E} \)-edge. This generates a partial composition \( \circ_{\mathcal{E}} \) on subgraphs, with \( H \circ_{\mathcal{E}} K \downarrow \) (where \( \downarrow \) denotes definedness) iff \( V(H) \cap V(K) = \emptyset, H \sim_{\mathcal{E}} K \) and \( K \not\sim_{\mathcal{E}} H \). Output is given by the graph union of the two subgraphs and the \( \mathcal{E} \)-edges between them. We say \( G \) is a layered graph (with respect to \( \mathcal{E} \)) if there exist \( H, K \) such that \( H \circ_{\mathcal{E}} K \downarrow \) and \( G = H \circ_{\mathcal{E}} K \) (see Fig 1). Layering is evidently neither commutative nor associative.

Let Prop be a set of atomic propositions, ranged over by \( p \). The set of all formulae of LGL and ILGL is generated by the following grammar:

\[
\phi ::= p \mid \top \mid \bot \mid \phi \land \phi \mid \phi \lor \phi \mid \phi \rightarrow \phi \mid \phi \rhd \phi \mid \phi \rhd \phi \mid \phi \cdot \phi.
\]

The connectives above are the standard logical connectives, together with a (non-commutative and non-associative) multiplicative conjunction, \( \cdot \), and its associated implications \( \rhd \) and \( \rhd \), in the spirit of the Lambek calculus [51, 52]. We define \( \neg \phi \) as \( \phi \rightarrow \bot \). Hilbert-type
systems for the logics are given in Fig 2: \text{ILGL} is specified by rules 1–15, whilst \text{LGL} is specified by rules 0–15.

\text{LGL} and \text{ILGL} are interpreted on scaffolds: structures \( \mathcal{X} = (\mathcal{G}, \mathcal{E}, \mathcal{X}, \preceq) \) where \( \mathcal{G} \) is a directed graph, \( \mathcal{E} \) is a distinguished edge set, \( \mathcal{X} \subseteq S_{\mathcal{G}}(\mathcal{G}) \) is such that – if \( \mathcal{H} @_{\mathcal{E}} \mathcal{K} \downarrow \), then \( \mathcal{H} @_{\mathcal{E}} \mathcal{K} \preceq \mathcal{G} \), \( \mathcal{H} \vdash \phi \) and \( \mathcal{K} \vdash \psi \).

To model the logic soundly in the intuitionistic case, valuations \( \mathcal{V} : \text{Prop} \rightarrow \mathcal{P}(\mathcal{X}) \) (where \( \mathcal{P}(\mathcal{X}) \) is the power set of \( \mathcal{X} \)) must be persistent: for all \( \mathcal{G}, \mathcal{H} \in \mathcal{X} \), if \( \mathcal{G} \in \mathcal{V}(p) \) and \( \mathcal{G} \preceq \mathcal{H} \), then \( \mathcal{H} \in \mathcal{V}(p) \). This has a spatial interpretation when we consider the order to be the subgraph relation: if \( p \) designates that a resource is located in the region \( \mathcal{G} \), it should also hold of the region \( \mathcal{H} \) containing \( \mathcal{G} \).

Given a scaffold \( \mathcal{X} \) and a persistent valuation \( \mathcal{V} \) the satisfaction relation \( \models \) for \text{ILGL} is inductively defined in Fig 3; the particular case where \( \preceq \) is equality yields \text{LGL}. As is necessary for a sound interpretation of intuitionistic logic, persistence extends to all formulas with respect to the semantics of Fig 3: for all \( \phi, \mathcal{G}, \mathcal{H} \), if \( \mathcal{G} \models \phi \) and \( \mathcal{G} \preceq \mathcal{H} \) then \( \mathcal{H} \models \phi \).

| Rule | Premises | Conclusion |
|------|----------|------------|
| 0.   | \( \neg \phi \vdash \phi \) | \( \neg \phi \vdash \phi \) |
| 1.   | \( \phi \vdash \phi \)   | \( \phi \vdash \phi \)   |
| 2.   | \( \phi \vdash \top \)    | \( \phi \vdash \top \)    |
| 3.   | \( \bot \vdash \phi \)    | \( \bot \vdash \phi \)    |
| 4.   | \( \eta \vdash \phi \quad \eta \vdash \psi \) | \( \eta \vdash \phi \land \psi \) |
| 5.   | \( \eta \vdash \psi \)    | \( \eta \vdash \phi \land \psi \) |
| 6.   | \( \phi \vdash \psi \)    | \( \eta \vdash \phi \land \psi \) |
| 7.   | \( \eta \vdash \phi \land \psi \) | \( \eta \vdash \psi \lor \phi \land \psi \) |
| 8.   | \( \eta \vdash \psi \lor \phi \land \psi \) | \( \phi \vdash \psi \lor \phi \land \psi \) |
| 9.   | \( \eta \vdash \phi \rightarrow \psi \) | \( \eta \vdash \phi \rightarrow \psi \) |
| 10.  | \( \eta \vdash \phi \rightarrow \psi \) | \( \eta \vdash \phi \rightarrow \psi \) |
| 11.  | \( \xi \vdash \phi \quad \eta \vdash \psi \) | \( \xi \vdash \phi \rightarrow \psi \) |
| 12.  | \( \eta \vdash \phi \rightarrow \psi \) | \( \xi \vdash \phi \rightarrow \psi \) |
| 13.  | \( \xi \vdash \phi \rightarrow \psi \) | \( \xi \vdash \phi \rightarrow \psi \) |
| 14.  | \( \xi \vdash \phi \rightarrow \psi \) | \( \xi \vdash \phi \rightarrow \psi \) |
| 15.  | \( \xi \vdash \phi \rightarrow \psi \) | \( \xi \vdash \phi \rightarrow \psi \) |

Figure 2: Hilbert systems for \( (I)LGL \): \text{ILGL} is given by 1 - 15, \text{LGL} by 0–15.

\( G \models p \) iff \( G \in \mathcal{V}(p) \)

\( G \models \phi \land \psi \) iff \( G \models \phi \) and \( G \models \psi \)

\( G \models \phi \rightarrow \psi \) iff for all \( \mathcal{H} \succ \mathcal{G} \), \( \mathcal{H} \models \phi \) implies \( \mathcal{H} \models \psi \)

\( G \models \phi \land \psi \) iff there exists \( \mathcal{H}, \mathcal{K} \) such that \( \mathcal{H} @_{\mathcal{E}} \mathcal{K} \downarrow \), \( \mathcal{H} @_{\mathcal{E}} \mathcal{K} \preceq \mathcal{G} \), \( \mathcal{H} \models \phi \) and \( \mathcal{K} \models \psi \)

\( G \models \phi \rightarrow \psi \) iff for all \( \mathcal{H}, \mathcal{K} \) such that \( \mathcal{H} @_{\mathcal{E}} \mathcal{K} \downarrow \) and \( \mathcal{G} \preceq \mathcal{H} \), \( \mathcal{H} \models \phi \) implies \( \mathcal{H} @_{\mathcal{E}} \mathcal{K} \models \psi \)

\( G \models \phi \land \psi \) iff for all \( \mathcal{H}, \mathcal{K} \) such that \( \mathcal{K} @_{\mathcal{E}} \mathcal{H} \downarrow \) and \( \mathcal{G} \preceq \mathcal{H} \), \( \mathcal{H} \models \phi \) implies \( \mathcal{K} @_{\mathcal{E}} \mathcal{H} \models \psi \)

Figure 3: Satisfaction on layered graphs for \( (I)LGL \). \text{LGL} is the case where \( \preceq \) is =.
2.2. BI and Boolean BI. Next we present the resource logics BI and BBI [60]. Let Prop be a set of atomic propositions, ranged over by $p$. The set of all formulae of (B)BI is generated by the following grammar:

$$
\phi ::= p \mid \top \mid \bot \mid I \mid \phi \land \psi \mid \phi \lor \psi \mid \phi \rightarrow \psi \mid \phi \ast \psi \mid \phi \dashast \psi.
$$

Once again we have the standard logical connectives, this time joined by multiplicative conjunction $\ast$ and implication $\dashast$, as well as a constant $I$. By extending rules 1–10 of Fig 2 with the rules of Fig 4 we obtain a system for BI; extending rules 0–10 of Fig 2 with the rules of Fig 4 yields a system for BBI. These rules enforce commutativity and associativity of the multiplicative conjunction $\ast$, the adjointness between $\ast$ and its associated implication $\dashast$, and the fact that $I$ is a unit for $\ast$.

Models of these logics arise from the analysis of the abstract notion of resource first outlined in the original work on BI [60]. There, the principal properties were determined to be composability and comparison, formalized by a commutative and associative composition $\circ$ with unit $e$, and a pre- or partial order $\preceq$. The interplay between composition and comparison in this analysis is formalized by a bifunctorality condition: if $a \preceq a'$ and $b \preceq b'$ then $a \circ a' \preceq b \circ b'$. Structures $M = (M, \preceq, \circ, e)$ satisfying these conditions are known as resource monoids.

(B)BI is interpreted on ordered monoidal structures that generalize the notion of resource monoid in various ways. For example, $\circ$ can be generalized to be non-deterministic and/or partial, the unit $e$ can be generalized to a set of units $E$, and the bifunctorality condition can be generalized to a number of different coherence conditions between $\circ$ and $\preceq$. The majority of these choices define the same notion of validity, although there are some sharp boundaries (an analysis of some of these issues can be found in [54]), and all of them can be grouped under the name resource semantics. As an example we give a class of models that corresponds closely to the standard intuitionistic model of Separation Logic. This class will be further generalized in Section 4 for the purposes of the duality theorem.
A monoidal frame is a structure $\mathcal{X} = (X, \preceq, o, E)$ s.t. $\preceq$ is a preorder, $o : X^2 \to P(X)$ is commutative operation satisfying non-deterministic associativity
\[ \forall x, y, z, t, s : s \in x \circ y \text{ and } t \in s \circ z \text{ implies } \exists s'(s' \in y \circ z \text{ and } t \in x \circ s), \]
and $E$ is a set of units satisfying, for all $x, y \in X$, the conditions (Weak) $\exists e \in E(x \in x \circ e)$; (Contr) $x \in y \circ e \wedge e \in E \to y \preceq x$; and (Up) $e \in E \land e \preceq e' \to e' \in E$. It is downwards-closed if, whenever $x \in y \circ z$, $y' \preceq y$ and $z' \preceq z$, there exists $x' \preceq x$ such that $x' \in y' \circ z'$. Conversely it is upwards-closed if, whenever $x \in y \circ z$ and $x \preceq x'$, there exists $y' \succeq y$ and $z' \succeq z$ such that $x' \in y' \circ z'$.

The notions of upwards and downwards closure are properties that the majority of models of Separation Logic satisfy. Cao et al. [18] show that when a model satisfies both upwards and downwards closure, the semantic clauses of $*$ and $\rightarrow$ can be given identically in the classical and intuitionistic cases.

Given an upwards and downwards closed monoidal frame (UDMF) $\mathcal{X}$ and a persistent valuation $\mathcal{V} : \text{Prop} \to P(X)$, the satisfaction relation $\models$ is inductively defined in Fig 5. Once again, BBI is obtained as the particular case where the order $\preceq$ is equality $=$. As with ILGL, persistence extends to all formulas of BI, with upwards and downwards closure ensuring that this is the case for formulas of the form $\phi \ast \psi$ and $\phi \Rightarrow \psi$. Validity and entailment are defined in much the same way as the case for (I)LGL.

### 2.3. Separation Logic

Separation Logic [56], introduced by Ishtiaq and O’Hearn [44], and Reynolds [62], is an extension of Hoare’s program logic which addresses reasoning about programs that access and mutate data structures. The usual presentation of Separation Logic is based on Hoare triples — for reasoning about the state of imperative programs — of the form $\{ \phi \} C \{ \psi \}$, where $C$ is a program command, $\phi$ is pre-condition for $C$, and $\psi$ is a post-condition for $C$. The formulas $\phi$ and $\psi$ are given by the following grammar:
\[
\phi ::= E = E' \mid E \leftrightarrow F \mid \top \mid \bot \mid \text{Emp} \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi \ast \phi \mid \phi \Rightarrow \phi \mid \exists v. \phi \mid \forall v. \phi.
\]

Reynolds’ programming language is a simple language of commands with a Lisp-like set-up for creating and accessing cons cells: $C ::= x := E \mid x := E.i \mid E.i := E' \mid x := \text{cons}(E_1, E_2) \mid \ldots$. Here the expressions $E$ of the language are built up using booleans, variables, etc., cons cells, and atomic expressions. Separation Logic thus facilitates verification procedures for programs that alter the heap.

A key feature of Separation Logic is the local reasoning provided by the so-called Frame Rule,
\[
\begin{align*}
\{ \phi \} C \{ \psi \} \\
\{ \phi \ast \chi \} C \{ \psi \ast \chi \}
\end{align*}
\]
where $\chi$ does not include any free variables modified by the program $C$. Static analysis procedures based on the Frame Rule form the basis of Facebook’s Infer tool (fbinfer.com) that is deployed in its code production. The decomposition of the analysis that is facilitated by the Frame Rule is critical to the practical deployability of Infer.

Separation Logic can usefully and safely be seen (see [67] for the details) as a presentation of (B)BI Pointer Logic [44]. The semantics of (B)BI Pointer Logic, a theory of (first-order) (B)BI, is an instance of (B)BI’s resource semantics in which the monoid of resources is constructed from the program’s heap. In detail, this model has two components, the store and the heap. The store is a partial function mapping from variables to values $a \in \text{Val}$, such as integers, and the heap is a partial function from natural numbers to values. In logic,
the store is often called the valuation, and the heap is a possible world. In programming languages, the store is sometimes called the environment. Within this set-up, the atomic formulae of (B)BI Pointer Logic include equality between expressions, \( E = E' \), and, crucially, the points-to predicate, \( E \mapsto F \).

We use the following additional notation: \( \text{dom}(h) \) denotes the domain of definition of a heap \( h \) and \( \text{dom}(s) \) is the domain of a store \( s \); \( h \# h' \) denotes that \( \text{dom}(h) \cap \text{dom}(h') = \emptyset \); \( h \cdot h' \) denotes the union of functions with disjoint domains, which is undefined if the domains overlap; \( h \sqsubseteq h' \) denotes that the graph of \( h \) is a subgraph of \( h' \); \( [] \) denotes the empty heap that is nowhere defined; \( [f \mid v \mapsto a] \) is the partial function that is equal to \( f \) except that \( v \) maps to \( a \); expressions \( E \) are built up from variables and constants, and so determine denotations \( \{E\} s \in \text{Val} \).

With this basic data we can define the satisfaction relations for BI and BBI pointer logic. BI pointer logic is given by extending the intuitionistic semantic clause for \( \mapsto \),

\[
s, h \vdash E \mapsto F \text{ iff } \{E\} s \in \text{dom}(h) \text{ and } h(\{E\} s) = \{F\} s,
\]

with those in Fig 6; similarly, BBI pointer logic is given by extending the classical semantic clause for \( \mapsto \),

\[
s, h \vdash E \mapsto F \text{ iff } \{E\} s = \text{dom}(h) \text{ and } h(\{E\} s) = \{F\} s,
\]

with those in Fig 6, where \( \sqsubseteq \) is replaced for \( = \) in the clauses for \( \mapsto \) and Emp. The judgement, \( s, h \vdash \phi \), says that the assertion \( \phi \) holds for a given store and heap, assuming that the free variables of \( \phi \) are contained in the domain of \( s \). The classical interpretation of \( \mapsto \) requires \( E \) to be the only active address in the current heap, whereas the intuitionistic interpretation is the weaker judgement that \( E \) is at least one of the active addresses in the current heap.

The technical reason for this difference is the requirement of persistence with respect to the heap extension ordering \( \sqsubseteq \) for soundness of the intuitionistic semantics. This leads to some quirks in the intuitionistic model: for example, the multiplicative unit Emp collapses to \( \top \) as every heap extends the empty heap. Some aspects remain the same: as heaps are upwards and downwards closed (in the sense defined in Section 2.2) the semantic clauses for \( * \) and \( \rightarrow \) can be given identically. Most importantly, in both cases descriptions of larger heaps can be built up using \( * \), and this corresponds to the local reasoning provided by the Frame Rule.

When Separation Logic is given as BI pointer logic it is known as Intuitionistic Separation Logic, whereas the formulation as BBI pointer logic is known as Classical Separation Logic. Although both were defined in the paper introducing Separation Logic as a theory of bunched

| \( s, h \vdash E = E' \text{ iff } \{E\} s = \{E'\} s \) | \( s, h \vdash \top \) | \( s, h \not\vdash \bot \) |
| \( s, h \vdash \phi \land \psi \text{ iff } s, h \vdash \phi \) and \( s, h \vdash \psi \) | \( s, h \vdash \phi \lor \psi \text{ iff } s, h \vdash \phi \) or \( s, h \vdash \psi \) |
| \( s, h \vdash \phi \rightarrow \psi \text{ iff all } h' \sqsupseteq h, h' \sqsubseteq \phi \implies h' \sqsubseteq \psi \) s, h \vdash \text{Emp if } h \sqsubseteq [] \) |
| \( s, h \vdash \phi \ast \psi \text{ iff there exists } h', h'' \text{ st. } h \# h', h = h' \cdot h'', s, h' \sqsubseteq \phi \) and \( s, h'' \sqsubseteq \psi \) |
| \( s, h \vdash \phi \rightarrow \psi \text{ iff for all } h' \text{ such that } h \# h', s, h' \sqsubseteq \phi \implies s, h \cdot h' \sqsubseteq \psi \) |
| \( s, h \vdash \exists v. \phi \text{ iff there exists } a \in \text{Val}, [s \mid v \mapsto a], h \vdash \phi \) |
| \( s, h \vdash \forall v. \phi \text{ iff for all } a \in \text{Val}, [s \mid v \mapsto a], h \vdash \phi \) |

Figure 6: Satisfaction for (B)BI Pointer Logic. The BBI variant replaces \( \sqsubseteq \) with \( = \).
logic [44], the classical variant has taken precedence in both theoretical work and practical implementations. A key exception to this is the higher-order Concurrent Separation Logic framework IRIS, which is based on Intuitionistic Separation Logic [49]. IRIS requires an underlying intuitionistic logic as it utilizes the ‘later’ modality $\triangleright$ [55], which collapses to triviality ($\triangleright \phi \leftrightarrow \top$ for all $\phi$) in the presence of the law of excluded middle.

3. Layered Graph Logics

3.1. Algebra and Frames for (I)LGL. We begin our analysis with the weakest systems, LGL and ILGL. Each of the logics we consider can be obtained by extending the basic structures associated with these logics and so we are able to systematically extend the theory in each case by accounting for just the extensions to the structure. First, we consider lattice-based algebras suitable for interpreting (I)LGL.

**Definition 3.1 (ILGL Algebra).**
(1) An ILGL algebra is an algebra $A = (A, \land, \lor, \rightarrow, \top, \bot, \cdot, \cdot\cdot, \cdot\cdot\cdot)$ such that $(A, \land, \lor, \rightarrow, \top, \bot)$ is a Heyting algebra and $\cdot, \cdot\cdot, \cdot\cdot\cdot$ are binary operations on $A$ satisfying, for all $a, b, c \in A$,

$$a \cdot b \leq c \text{ iff } a \leq b \rightarrow c \text{ iff } b \leq a \cdot c.$$ 

(2) A LGL algebra is an algebra $A = (A, \land, \lor, \neg, \top, \bot, \cdot, \cdot\cdot, \cdot\cdot\cdot)$ such that $(A, \land, \lor, \neg, \top, \bot)$ is a Boolean algebra and $\cdot, \cdot\cdot, \cdot\cdot\cdot$ are binary operations on $A$ satisfying, for all $a, b, c \in A$,

$$a \cdot b \leq c \text{ iff } a \leq b \rightarrow c \text{ iff } b \leq a \cdot c.$$ 

Residuation of $\cdot, \cdot\cdot$ and $\cdot\cdot\cdot$ with respect to the underlying lattice order entails a number of useful properties that are utilized in what follows.

**Proposition 3.2 (cf. [46]).** Let $A$ be an (I)LGL algebra. Then, for all $a, b, a', b' \in A$ and $X, Y \subseteq A$, we have the following:

1. If $a \leq a'$ and $b \leq b'$ then $a \cdot b \leq a' \cdot b'$;
2. If $\bigvee X$ and $\bigvee Y$ exist then $\bigvee_{x \in X, y \in Y} x \cdot y$ exists and $(\bigvee X) \cdot (\bigvee Y) = \bigvee_{x \in X, y \in Y} x \cdot y$;
3. If $a = \bot$ or $b = \bot$ then $a \cdot b = \bot$;
4. If $\bigvee X$ exists then for any $z \in A$, $\bigwedge_{x \in X} (x \rightarrow z)$ and $\bigwedge_{x \in X} (x \cdot z)$ exist with

$$\bigwedge_{x \in X} (x \rightarrow z) = (\bigvee X) \rightarrow z \text{ and } \bigwedge_{x \in X} (x \cdot z) = (\bigvee X) \cdot z;$$
5. If $\bigwedge X$ exists then for any $z \in A$, $\bigvee_{x \in X} (z \rightarrow x)$ and $\bigvee_{x \in X} (z \cdot x)$ exist with

$$\bigvee_{x \in X} (z \rightarrow x) = z \rightarrow (\bigvee X) \text{ and } \bigvee_{x \in X} (z \cdot x) = z \cdot (\bigvee X);$$

and
6. $a \rightarrow \top = a \cdot \top = \bot \rightarrow a = \bot \cdot a = \top$. 

$\square$
Interpretations of (I)LGL on an (I)LGL algebra work as follows: let $V : \text{Prop} \to A$ be an assignment of propositional variables to elements of the algebra; this is uniquely extended to an interpretation $[-]$ of every ILGL formula by induction, with $[p] = V(p)$, $[\top] = \top$ and $[\bot] = \bot$ as base cases:

$$[\phi \land \psi] = [\phi] \land [\psi], \quad [\phi \lor \psi] = [\phi] \lor [\psi], \quad [\phi \rightarrow \psi] = [\phi] \rightarrow [\psi].$$

In the case of LGL, the interpretation is the same except that $[\phi \rightarrow \psi] = \neg [\phi] \lor [\psi]$. An interpretation $[-]$ on an (I)LGL algebra $\mathbb{A}$ satisfies $\phi$ iff $[\phi] = \top$; $\phi$ is valid on (I)LGL algebras iff $\phi$ is satisfied by every interpretation $[-]$ on every ILGL algebra $\mathbb{A}$. By constructing a Lindenbaum-Tarski algebra from the Hilbert systems of Fig 2 we obtain the following soundness and completeness theorem for (I)LGL algebras.

**Theorem 3.3 (Algebraic Soundness & Completeness).** For all (I)LGL formulas $\phi$, $\psi$, $\phi \vdash \psi$ is provable in (I)LGL iff, for all (I)LGL algebras $\mathbb{A}$ and all interpretations $[-]$ on $\mathbb{A}$, $[\phi] \leq [\psi]$.

Next we generalize the intended model of the logics to a class of relational structures.

**Definition 3.4 ((I)LGL Frame).** An ILGL frame is a triple $\mathcal{X} = (X, \preceq, \circ)$ where $X$ is a set, $\preceq$ a preorder on $X$ and $\circ : X^2 \to \mathcal{P}(X)$ a binary operation. An LGL frame is an ILGL frame for which the order $\preceq$ is equality $=.$

Here $\circ$ is a generalization of Section 2.1’s $\oplus$. We can therefore reconfigure the semantics given in Figure 3 to give a satisfaction relation on (I)LGL frames by straightforward substitutions of $\circ$. Partiality is encoded by the fact $\circ$ has codomain $\mathcal{P}(X)$ and thus $x \circ y = \emptyset$ holds when $\circ$ is undefined for $x, y$. However, $\circ$ is strictly more general as it does not necessarily satisfy the partial determinism that holds of scaffolds: $G = H \circ \varepsilon K$ and $G' = H \circ \varepsilon K$ implies $G = G'$. As $\circ$ is nondeterministic, it can equivalently be seen as a ternary relation, though we maintain the partial function notation to emphasise its interpretation as a composition operator.

(I)LGL frames can be equipped a notion of morphism to obtain categories ILGLFr and LGLFr. As in modal logic, if there is a surjective (I)LGL morphism $g : \mathcal{X} \to \mathcal{X}'$ then any (I)LGL formula valid in $\mathcal{X}$ is also valid in $\mathcal{X}'$.

**Definition 3.5 (ILGL Morphism).** Given ILGL frames $\mathcal{X}$ and $\mathcal{X}'$, an ILGL morphism is a map $g : X \to X'$ satisfying

1. $x \preceq y$ implies $g(x) \preceq' g(y)$,
2. $g(x) \preceq' y'$ implies there exists $y \in X$ s.t. $x \preceq y$ and $g(y) = y'$,
3. $x \in y \circ z$ implies $g(x) \in g(y) \circ' g(z)$,
4. $w' \preceq' g(x)$ and $w' \in y' \circ' z'$ implies there exists $w, y, z \in X$ s.t. $w \preceq x$, $w \in y \circ z$,
5. $y' \preceq' g(y)$ and $z' \preceq' g(z)$,
6. $g(x) \preceq' w'$ and $z' \in w' \circ' y'$ implies there exists $w, y, z \in X$ s.t. $x \preceq w$, $z \in w \circ y$,
7. $y' \preceq' g(y)$ and $g(z) \preceq' z'$, and
8. $g(x) \preceq' w'$ and $z' \in y' \circ' w'$ implies there exists $w, y, z \in X$ s.t. $x \preceq w$, $z \in y \circ w$,
9. $y' \preceq' g(y)$ and $g(z) \preceq' z'$.

For LGL, replacing $\preceq$ with $=$ collapses the above definition to a simpler and more familiar notion of morphism.

**Definition 3.6 (LGL Morphism (cf. [13])).** Given LGL frames $\mathcal{X}$ and $\mathcal{X}'$, a LGL morphism is a map $g : \mathcal{X} \to \mathcal{X}'$ satisfying
(1) $x \in y \circ z$ implies $g(x) \in g(y) \circ' g(z)$,

(2) $g(x) \in y' \circ' z'$ implies there exists $y, z \in X$ s.t. $x \in y \circ z$, $g(y) = y'$ and $g(z) = z'$,

(3) $z' \in g(x) \circ' y'$ implies there exists $y, z \in X$ s.t. $z \in x \circ y$, $g(y) = y'$ and $g(z) = z'$, and

(4) $z' \in y' \circ' g(x)$ implies there exists $y, z \in X$ s.t. $z \in y \circ x$, $g(y) = y'$ and $g(z) = z'$. □

3.2. Duality for (I)LGL. We now give representation and duality theorems for ILGL algebras. As a corollary we obtain the equivalence of the relational semantics to the algebraic semantics, as well as its completeness with respect to the Hilbert system of Fig 2. We first define two transformations that underpin a dual adjunction between the category of (I)LGL algebras and the category of (I)LGL frames.

**Definition 3.7 (ILGL Complex Algebra).** Given an ILGL frame $X'$, the complex algebra of $X$ is given by $Com^{ILGL}(X) = (\mathcal{P}(X), \cap, \cup, \Rightarrow, X, \emptyset, \bullet_X, \bullet' - \bullet_X)$ where

- $\mathcal{P}(X) = \{ A \subseteq X \mid$ if $a \in A$ and $a \not\leq b$ then $b \not\in A\}$
- $A \Rightarrow B = \{ x \mid$ if $x \not\leq x'$ and $x' \in A$ then $x' \not\in B\}$
- $A \bullet_X B = \{ x \mid$ there exists $w, y, z$ s.t. $w \not\leq x$, $w \in y \circ z$, $y \in A$ and $z \in B\}$
- $A \bullet' - \bullet_X B = \{ x \mid$ for all $w, y, z$, if $x \not\leq w$, $z \in w \circ y$ and $y \in A$ then $z \in B\}$
- $A \bullet - \bullet_X B = \{ x \mid$ for all $w, y, z$, if $x \not\leq w$, $z \in y \circ w$ and $y \in A$ then $z \in B\}$.

Each operator here maps upwards-closed sets to upwards-closed sets so this is well defined. By substituting $\not\leq$ for $\Rightarrow$, the above definition collapses to one suitable for LGL.

**Definition 3.8 (LGL Complex Algebra).** Given a LGL frame $X$, the complex algebra of $X$ is given by $Com^{GL}(X) = (\mathcal{P}(X), \cap, \cup, \setminus, X, \emptyset, \bullet_X, \bullet' - \bullet_X)$, where $\bullet_X$, $\bullet' - \bullet_X$, and $\bullet - \bullet_X$ are defined as follows:

- $A \bullet_X B = \{ x \mid$ there exists $y \in A, z \in B$ s.t. $x \in y \circ z\}$
- $A \bullet' - \bullet_X B = \{ x \mid$ for all $y, z \in X, z \in x \circ y$ and $y \in A$ implies $z \in B\}$
- $A \bullet - \bullet_X B = \{ x \mid$ for all $y, z \in X, z \in y \circ x$ and $y \in A$ implies $z \in B\}$.

The residuated structure in each case is then easy to verify.

**Lemma 3.9.** Given an (I)LGL frame $X$, $Com^{(I)LGL}(X)$ is an (I)LGL algebra. □

Any valuation $\mathcal{V}$ on an (I)LGL frame $X$ generates an interpretation $[[\mathcal{V}^{-1}]_\mathcal{V}$ on the complex algebra $Com^{(I)LGL}(X)$. A straightforward inductive argument then shows satisfiability coincides on these two models.

**Proposition 3.10.** For any relational (I)LGL model $(X, \mathcal{V})$, $x \Vdash \phi$ iff $x \in [[\phi]]_\mathcal{V}$. □

Conversely we can create ILGL frames from ILGL algebras. We first recall some definitions. A filter on a bounded distributive lattice $\mathcal{A}$ is a non-empty set $F \subseteq A$ such that, for all $x, y \in A$, (i) $x \in F$ and $x \leq y$ implies $y \in F$, and (ii) $x, y \in F$ implies $x \wedge y \in F$. It is a proper filter if it additionally satisfies (iii) $\perp \not\in F$. It is prime if in addition it satisfies (iv) $x \vee y \in F$ implies $x \in F$ or $y \in F$. If $\mathcal{A}$ is a Boolean algebra and $F$ a proper filter, (iv) is equivalent to (v) $x \in F$ or $\neg x \in F$ and (vi) $F$ is a maximal proper filter with respect to $\subseteq$. The dual notion of a filter is an ideal, with proper and prime ideals defined as one would expect. Importantly, if $I$ is a prime ideal then the complement $\overline{I}$ is a prime filter.
Definition 3.11 (Prime Filter (I)LGL Frame). Given an ILGL algebra \( \mathcal{A} \), the prime filter frame of \( \mathcal{A} \) is given by \( Pr^{(I)LGL}(\mathcal{A}) = (Pr(A), \subseteq, \circ_{\mathcal{A}}) \) where
\[
F \circ_{\mathcal{A}} F' = \{ F'' \mid \forall a \in F, \forall b \in F' : a \cdot b \in F'' \}.
\]
For a LGL algebra \( \mathcal{A} \), the prime filter frame is simply \( Pr^{LGL}(\mathcal{A}) = (Pr(A), \circ_{\mathcal{A}}) \).

Lemma 3.12. Given an (I)LGL algebra \( \mathcal{A} \), \( Pr^{(I)LGL}(\mathcal{A}) \) is an ILGL frame.

In analogy with Stone’s representation theorem for Boolean algebras, we can give a representation theorem for (I)LGL algebras using these constructions. In particular, for ILGL algebras this extends the representation theorem for Heyting algebras [36], whereas for LGL algebras this extends Stone’s theorem. These results are closely related to various representation theorems for algebras with operators (e.g., [48, 43]). The key difference is the use of a single operation \( \circ \) for the operator \( \bullet \) and its non-operator adjoints \( \to \) and \( \gets \). The derived structure required to take care of these adjoints was not investigated in the frameworks of Jonsson-Tarski or Goldblatt but has been in the context of gaggle theory [6, 35]. There the result for LGL algebras can be found as a particular case of that for Boolean gaggles ([6], Theorem 1.4.16).

Theorem 3.13 (Representation Theorem for (I)LGL Algebras). Every (I)LGL algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given an (I)LGL algebra \( \mathcal{A} \), the map \( \theta_{\mathcal{A}} : \mathcal{A} \to Com^{(I)LGL}(Pr^{(I)LGL}(\mathcal{A})) \) defined \( \theta_{\mathcal{A}}(a) = \{ F \in Pr^{(I)LGL}(\mathcal{A}) \mid a \in F \} \) is an embedding.

Proof. We prove the theorem for ILGL algebras as the case for LGL is then an immediate corollary obtained by substituting \( \leq \) for \( = \) throughout. That \( \theta_{\mathcal{A}} \) is injective and respects the Heyting algebra operations is simply the representation theorem for Heyting algebras. It thus remains to show \( \theta_{\mathcal{A}} \) respects the operations \( \bullet, \cdot \) and \( \cdot \circ \). We address \( \to \) and leave the (similar) remaining cases to the reader.

We first note the simple boundary cases: for all \( a, b \in A \) we trivially have that
\[
\theta_{\mathcal{A}}(a \cdot \top) = \theta_{\mathcal{A}}(a) \cdot_{Pr(\mathcal{A})} \theta_{\mathcal{A}}(\top) \quad \text{and} \quad \theta_{\mathcal{A}}(\bot \to b) = \theta_{\mathcal{A}}(\bot) \cdot_{Pr(\mathcal{A})} \theta_{\mathcal{A}}(b) \quad \text{by Proposition 3.2 (6)}.
\]
Hence it is sufficient to consider \( a \to b \) where \( a \neq \bot \) and \( b \neq \top \). First suppose \( a \to b \neq \bot \). For the inclusion \( \theta_{\mathcal{A}}(a \to b) \subseteq \theta_{\mathcal{A}}(a) \cdot_{Pr(\mathcal{A})} \theta_{\mathcal{A}}(b) \), assume \( a \to b \in F \) with \( F_0, F_1, F_2 \) such that \( F \subseteq F_0 \), \( F_2 \in F_0 \circ_{\mathcal{A}} F_1 \) and \( a \in F_1 \). Then \( (a \to b) \cdot a \in F_2 \) and so \( b \in F_2 \) since residuation entails \( (a \to b) \cdot b \leq b \) and \( F_2 \) is upwards closed. Hence \( F \in \theta_{\mathcal{A}}(a) \cdot_{Pr(\mathcal{A})} \theta_{\mathcal{A}}(b) \).

Conversely, assume \( a \cdot b \notin F \). We show prime filters \( F_0 \) and \( F_1 \) exist such that \( F_2 \in F \circ_{\mathcal{A}} F_1 \) with \( a \in F_1 \) but \( b \notin F_2 \). Consider the sets \( f_1 = \{ x \mid a \leq x \} \) and \( i_2 = \{ x \mid x \leq b \} \). As \( a \neq \bot \) and \( b \neq \top \) these define a proper filter and a proper ideal respectively. Abusing notation, we can see \( f_2 \in F \circ_{\mathcal{A}} f_1 \), where \( f_2 \) is the complement of \( i_2 \): if \( x \in F \) and \( y \geq a \) then \( x \cdot y \notin b \), otherwise by residuation and monotonicity of \( \cdot \) we would have \( x \leq a \to b \in F \), a contradiction. Clearly \( a \in F_1 \) and \( b \in i_2 \). By Zorn’s Lemma, we can extend \( f_1 \) and \( i_2 \) to a prime filter \( F_1 \) and prime ideal \( I_2 \) with these properties (to show that these are prime requires the properties of Proposition 3.2: see [31]). Taking \( F_2 \) to be the complement of \( I_2 \), itself a prime filter, we have witnesses to the fact \( F \notin \theta_{\mathcal{A}}(a) \cdot_{Pr(\mathcal{A})} \theta_{\mathcal{A}}(b) \). This also gives us the required inclusion in the case \( a \to b = \bot \) and so we are done.

That \( \theta_{\mathcal{A}} \) is an embedding immediately gives us an analogous result to Prop 3.10.

Corollary 3.14. For all (I)LGL algebras \( \mathcal{A} \): given an interpretation \([-]\), the valuation \( V_{[-]}(p) = \theta_{\mathcal{A}}([p]) \) on \( Pr^{(I)LGL}(\mathcal{A}) \) is such that \( [\phi] \in F \iff F \models \phi \).
To strengthen the representation theorem to a dual adjunction we define maps $\eta_X$ for each (I)LGL frame $X$ by $\eta_X(x) = \{ A \in Com^{(I)LGL}(X) \mid x \in A \}$. By extending $Pr^{(I)LGL}$ and $Com^{(I)LGL}$ to contravariant functors by setting $Pr^{(I)LGL}(f) = f^{-1}$ and $Com^{(I)LGL}(g) = g^{-1}$ — that these define the appropriate morphisms is straightforward but tedious — we can see that $\theta_1$ and $\kappa_1$ in fact define natural transformations that form the dual adjunction.

**Theorem 3.15.** The functors $Pr^{(I)LGL}$ and $Com^{(I)LGL}$ and the natural transformations $\theta$ and $\eta$ form a dual adjunction of categories between (I)LGLAlg and (I)LGLFr.

**Corollary 3.16** (Relational Soundness and Completeness). For all formulas $\phi, \psi$ of (I)LGL: $\phi \vdash \psi$ is provable in (I)LGL iff $\phi \models \psi$ in the relational semantics.

This dual adjunction can be specialized to a dual equivalence of categories by introducing topology to (I)LGL frames. We first define ILGL spaces. This definition (necessarily) extends that of the topological duals of Heyting algebra given by Esakia duality [37]. The coherence conditions on the composition $\circ$ are inspired by those found on the topological duals of gaggles [6].

**Definition 3.17** (ILGL Space). An **ILGL space** is a structure $X = (X, \mathcal{O}, \preceq, \circ)$ such that:

1. $(X, \mathcal{O}, \preceq)$ is an Esakia space [37];
2. $(X, \preceq, \circ)$ is an ILGL frame;
3. The upwards-closed clopen sets of $(X, \mathcal{O}, \preceq)$ are closed under $\bullet_X, -\bullet_X, \bullet_X$;
4. If $x \notin y \circ z$ then there exist upwards-closed clopen sets $C_1, C_2$ such that $y \in C_1, z \in C_2$ and $x \notin C_1 \bullet_X C_2$.

A morphism of ILGL spaces is a continuous ILGL morphism, yielding a category ILGLSp.

Once again, substituting $\preceq$ for $=$ in the definition of ILGL space obtains the topological duals for LGL algebras.

**Definition 3.18** (LGL Space). An **LGL space** is a structure $X = (X, \mathcal{O}, \circ)$ such that:

1. $(X, \mathcal{O})$ is an Stone space [64];
2. $(X, \circ)$ is an LGL frame;
3. The clopen sets of $(X, \mathcal{O}, \preceq)$ are closed under $\bullet_X, -\bullet_X, -\bullet_X$;
4. If $x \notin y \circ z$ then there exist clopen sets $C_1, C_2$ such that $y \in C_1, z \in C_2$ and $x \notin C_1 \bullet_X C_2$.

A morphism of LGL spaces is a continuous LGL morphism, yielding a category ILGLSp.

We now adapt $Pr^{(I)LGL}, Com^{(I)LGL}, \theta$ and $\eta$ for these subcategories. First, it is straightforward to augment the prime filter frame with the topological structure required to make it an (I)LGL space. We define a subbase by $S = \{ \theta_\mathcal{A}(a) \mid a \in A \} \cup \{ \overline{\theta_\mathcal{A}(a)} \mid a \in A \}$ where $\overline{\theta_\mathcal{A}(a)}$ denotes the set complement. This generates a topology $\mathcal{O}_\mathcal{A}$ and we can define $PrSp^{(I)LGL} : ILGLAlg \to ILGLSp$ by $PrSp^{(I)LGL}(\mathcal{A}) = (Pr(A), \mathcal{O}_\mathcal{A}, \subseteq, \circ_\mathcal{A})$ and $PrSp^{(I)LGL}(f) = f^{-1}$.

In the case of LGL, the sets $\theta_\mathcal{A}(a)$ are redundant as for every prime filter $F$ on a Boolean algebra $\mathcal{A}$, $a \in A$ implies $a \in F$ or $-a \in F$. Hence $\mathcal{B} = \{ \theta_\mathcal{A}(a) \mid a \in A \}$ defines a base for a topology $\mathcal{O}_\mathcal{A}$ and we can set $PrSp^{LGL}(\mathcal{A}) = (Pr(A), \mathcal{O}_\mathcal{A}, 0_\mathcal{A})$ and $PrSp^{LGL}(f) = f^{-1}$.

Conversely, given an LGL space $X$ we now take the set of upwards-closed clopen sets $\mathcal{C}L_{\preceq}(X)$ as the carrier of an ILGL algebra, together with the operations of the ILGL complex algebra; ILGL space property (3) ensures this is well defined. Hence we take $Clop^{ILGL} : ILGLSp \to ILGLAlg$ to be $Clop^{ILGL}(X) = (\mathcal{C}L_{\preceq}(X), \cap, \cup, \Rightarrow, X, \emptyset, -\bullet_X, -\bullet_X)$ and $Clop^{ILGL}(g) = g^{-1}$. For LGL we simply take the clopen sets of the underlying topological
space \( \mathcal{L}(X) \) together with the operations of the LGL complex algebra. Hence \( \text{Clap}^{\text{ILGL}}(X) = (\mathcal{L}(X), \cap, \cup, \setminus, X, 0, \bullet, \rightarrow, I) \) and \( \text{Clap}^{\text{ILGL}}(g) = g^{-1} \).

The well-definedness of these functors can be seen by straightforwardly combining Stone (Esakia) duality for Boolean gaggles ([6], Theorem 9.2.22). thus obtain the duality theorem. For LGL algebras this is also obtainable as a specific case of Bimbó & Dunn’s duality theorem for Boolean gaggles ([6], Theorem 9.2.22).

**Theorem 3.19** (Duality Theorem for (I)LGL). The categories of (I)LGL algebras and (I)LGL spaces are dually equivalent.

4. The Logics of Bunched Implications

4.1. Algebra and Frames for (B)BI. We now extend the results of the previous section to the logics BI and BBI by systematically extending the structures defined for LGL and ILGL. We begin once again with the lattice-based algebras that interpret the logics.

**Definition 4.1** ((B)BI Algebra).

1. A **BI algebra** is an ILGL algebra \( A \) extended with a constant \( I \) such that \( (A, \bullet, I) \) is a commutative monoid.
2. A **BBI algebra** is a LGL algebra \( A \) extended with a constant \( I \) such that \( (A, \bullet, I) \) is a commutative monoid.

Thus BI algebras are the subclass of ILGL algebras with a commutative and associative \( \bullet \) that has unit \( I \), and likewise for BBI algebras with respect to LGL algebras. In particular, commutativity of \( \bullet \), together with residuation, causes \( \rightarrow = \bullet \). Interpretations on a BI and BBI algebras are given in much the same way as ILGL and LGL, except \( * \) and \( \ast \) are interpreted by \( \bullet \) and \( \rightarrow - [\phi * \psi] = [\phi] \bullet [\psi] \) and \( [\phi \ast \psi] = [\phi] \rightarrow [\psi] \) — and \( [I] = I \). A soundness and completeness theorem for algebraic interpretations is proved in the same fashion as Theorem 3.3.

**Theorem 4.2** (Algebraic Soundness & Completeness). For (B)BI formulas \( \phi, \psi, \phi \vdash \psi \) is provable in (B)BI iff for all (B)BI algebras \( A \) and all interpretations \([\_]\) on \( A \), \([\phi]\) \( \leq \) \([\psi]\).

Next we define a class of relational models appropriate for interpreting (B)BI. The outermost universal quantification in each frame condition is left implicit for readability.

**Definition 4.3** (BI Frame). A **BI Frame** is a tuple \( X = (X, \leq, \circ, E) \) such that \( \leq \) a preorder on \( X, \circ : X^2 \rightarrow \mathcal{P}(X) \) is a binary operation, and \( E \subseteq X \) a set, satisfying

\[
\begin{align*}
\text{(Comm)} & \quad z \in x \circ y \rightarrow z \in y \circ x & \quad \text{(Up)} & \quad e \in E \land e \not\leq e' \rightarrow e' \in E \\
\text{(Weak)} & \quad \exists e \in E( x \in x \circ e ) & \quad \text{ (Contr)} & \quad x \in y \circ e \land e \in E \rightarrow y \leq x \\
\text{(Assoc)} & \quad t' \supset t \in x \circ y \land w \in t' \circ z \rightarrow \exists s, s', w'(s' \supset s \in y \circ z \land w \supset w' \in x \circ s').
\end{align*}
\]
The notion of BI frame and its associated semantics can be defined in many ways, and indeed those given in Section 2.2 can suffice for a duality theorem if one defines the natural transformations and morphisms slightly differently. These choices are driven by the requirements that the model validates the associativity axiom and that the semantic clauses for * and → satisfy persistence. This requires careful interplay between the definition of (Assoc), the conditions relating ∘ and ≼, and the semantic clauses for * and →. One such solution is that given in the original papers on BI [60], in which ∘ is a deterministic, associative function that is bifunctorial with respect to the order. This is complicated somewhat when ∘ is non-deterministic and partial, however. A general analysis of the choices available for such a ∘ when the definition of the associativity-like condition it satisfies is kept fixed can be found in [18].

Here, we use a definition that enables a uniform extension of the structures and theorems relating to ILGL. In particular, we use a more general than usual formulation of (Assoc) that allows us to directly extend the results of Section 3. This solution places no coherence conditions on ∙ and ≼ and thus requires what Cao et al. [18] call the “strong semantics” for * and → to maintain persistence. These clauses are those given for ILGL’s multiplicatives. Explicitly, we have

\[ x \vdash \phi \cdot \psi \iff \text{there exists } x', y, z \text{ s.t. } x' \leq x, x' \in y \circ z, y \vdash \phi \text{ and } z \vdash \psi \]
\[ x \vdash \phi \leftarrow \psi \iff \text{for all } x', y, z \text{ s.t. } x \leq x' \text{ and } z \in x' \circ y, y \vdash \phi \text{ implies } z \vdash \psi. \]

This can be seen as a strict generalization of the model given in Section 2.2, as witnessed by the fact that the definition of associativity given there implies (Assoc) when ∘ is additionally upwards and downwards closed.

**Proposition 4.4.** Every upwards and downwards closed monoidal frame is a BI frame. \(\square\)

Further, the respective semantic clauses for * and → are equivalent when the underlying model is upwards and downwards closed [18]. The converse does not hold: not every BI frame is upwards or downwards closed. However, every BI frame generates a upwards and downwards closed monoidal frame with an equivalent satisfaction relation. Given a BI frame \(X = (X, \preceq, \circ, E)\), define its upwards and downwards closure by \(X^{\uparrow \downarrow} = (X, \preceq, \circ^{\uparrow \downarrow}, E)\) where \(x \in y \circ^{\uparrow \downarrow} z\) if there exist \(x', y', z'\) such that \(x' \preceq x, y \preceq y', z \preceq z'\) and \(x' \in y \circ z\). By taking care over the respective associativity properties of each frame, this can easily be seen to be an upwards and downwards closed monoidal frame. Letting \(\models_{\text{strong}}\) denote the satisfaction relation defined in Fig 5, we have the following result.

**Proposition 4.5.** For all BI frames \(X\), persistent valuations \(V\) and formulas \(\phi\) of BI, \(X, x \models X^{\uparrow \downarrow}, x \models_{\text{strong}} \phi.\) \(\square\)

So, as far as the logic is concerned, these choices are academic: they all collapse to the same notion of validity. For models satisfying one or both of upwards and downwards closure this is implicit in the preservation results of Cao et al. [18]. Our analysis of the remaining class of models that lack both conditions on ∘ and ≼ completes this picture.

**Definition 4.6.** A BBI frame \(X\) is a triple \(X = (X, \circ, E)\), such that \(\circ : X^2 \to \mathcal{P}(X)\) is a binary operation and \(E \subseteq X\), satisfying

1. \(\text{(Comm)} \quad z \in x \circ y \rightarrow z \in y \circ x \) (Assoc) \(t \in x \circ y \land w \in t \circ z \rightarrow \exists!(s \in y \circ z \land w \in x \circ s)\)
2. \(\text{(Weak)} \quad \exists e \in E(x \in x \circ e)\) (Contr) \(x \in y \circ e \land e \in E \rightarrow y = x.\) \(\square\)
We note that when \( \preceq \) is substituted for \( = \), all sound choices of the \( \text{(Assoc)} \) axiom that were possible for BI frames collapse to the axiom given here, while every coherence condition on \( \circ \) and \( \preceq \) becomes trivial. Thus, in comparison to BI, there are far fewer choices to be made about BBI models and so this definition is more familiar, appearing in the literature in precisely the same form as BBI frames [13] and non-deterministic monoids [54] and slightly modified as multi-unit separation algebras [33] and relational frames [39].

The key difference with the latter definitions is that multi-unit separation algebras are cancellative — \( z \in x \circ y \) and \( z \in x \circ y' \) implies \( y = y' \) — and relational frames have a single unit. BBI frames do not require cancellativity and have multiple units. This difference is crucial for the present work as the duality theorems do not hold when we restrict to frames satisfying either of these properties. This is witnessed by the fact that BBI is not expressive enough to distinguish between cancellative/non-cancellative models and single unit/multi-unit models [13], all of which define the same notion of validity [54]. An interpretation of the duality theorem might thus be that BBI frames are the most general relational structures that soundly and completely interpret BBI.

**Definition 4.7** ((B)BI Morphism). Given (B)BI frames \( \mathcal{X} \) and \( \mathcal{X}' \), a (B)BI morphism is an ILGL (LGL) morphism \( g : \mathcal{X} \to \mathcal{X}' \) that additionally satisfies (7) \( e \in E \) iff \( g(e) \in E' \).

BI frames together with BI morphisms form a category BIFr, itself a subcategory of ILGLFr; likewise, BBI frames and BBI morphisms form the category BBIFr, a subcategory of LGLFr. Note that commutativity of \( \circ \) collapses the final conditions in the definition of (I)LGL morphism when defined on BBI (BI) frames. As in the case for the layered graph logics, surjective (B)BI morphisms preserve validity in models.

We now relate the two categories of (B)BI model via a dual adjunction.

**Definition 4.8** ((B)BI Complex Algebra). Given a (B)BI frame \( \mathcal{X} \), the complex algebra of \( \mathcal{X} \), \( \text{Com}^{(B)BI}(\mathcal{X}) \) is given by extending \( \text{Com}^{ILGL}(\mathcal{X}) \) (\( \text{Com}^{LGL}(\mathcal{X}) \)) with the unit set of \( \mathcal{X}, E \).

**Lemma 4.9.** Given a (B)BI frame \( \mathcal{X} \), \( \text{Com}^{(b)BI}(\mathcal{X}) \) is a (B)BI algebra.

**Proof.** We give the argument for BI frames; the case for BBI frames is obtained from (Com).

Let \( E \in \mathcal{P}_\prec(\mathcal{X}) \) by virtue of (Up). Commutativity is easily derived from the property (Comm). Furthermore, the inclusion \( A \subseteq A \circ E \) follows immediately from (Weak). Slightly more involved is \( A \circ E \subseteq A \circ E \). Then there exists \( x', y, e \) such that \( x' \preceq x \) with \( x' \in y \circ e \). By (Contr) it follows that \( y \preceq x' \), so by transitivity \( y \preceq x \). By upwards-closure of \( x, A \subseteq A \).

We finally come to associativity. We only need to verify the inclusion \( (A \circ (B \circ C)) \subseteq A \circ (B \circ C) \) because of commutativity of \( \circ \). Let \( a \in (A \circ (B \circ C)) \). Then there exists \( w, t', z \) such that \( a \supseteq w \in t' \circ z \) with \( t' \in B \circ C \). This entails the existence of \( x, y, t \) such that \( t' \supseteq t \circ y \) with \( x \in A \) and \( y \in B \). By (Assoc) we thus have \( s, s', w' \) with \( s' \supseteq s \in y \circ z \) and \( w \supseteq w' \in x \circ s' \). Hence \( s' \in B \circ C \) and, because \( w' \preceq w \preceq a \), \( a \in (A \circ (B \circ C)) \).

As a special case of the analogous result for (I)LGL, we obtain a correspondence between satisfiability on a frame and its complex algebra when the algebraic interpretation is generated by the valuation on the frame.

**Proposition 4.10.** For any (B)BI frame \( \mathcal{X} \) and valuation \( \mathcal{V} \), \( x \models \phi \) iff \( x \in [\phi]_\mathcal{V} \).
In the other direction we transform BI algebras into BI frames.

**Definition 4.11** (Prime Filter (B)BI Frame). Given a (B)BI algebra $\mathcal{A}$, the prime filter frame of $\mathcal{A}$, $\text{Pr}^{(B)\text{BI}}(\mathcal{A})$, is given by extending $\text{Pr}^{LGL}(\mathcal{A}) \cup \{a \in \mathcal{A}^{LGL} \mid a \not\in F\}$ with $E_{\mathcal{A}} = \{F \in \text{Pr}(A) \mid I \subseteq F\}$. 

That the prime filter frame of a (B)BI algebra is a (B)BI frame follows an argument essentially given in the completeness theorem for the relational semantics of BBI [39].

**Lemma 4.12.** Given a (B)BI algebra $\mathcal{A}$, the prime filter frame $\text{Pr}^{(B)\text{BI}}(\mathcal{A})$ is a (B)BI frame.

*Proof.* In both cases, that $\circ_{\mathcal{A}}$ satisfies (Comm) can be read off of the definition of $\circ_{\mathcal{A}}$ and the commutativity of $\ast$. For BI algebras, $E_{\mathcal{A}}$ satisfying (Up) is trivial. We are left to prove (Assoc), (Weak) and (Contr). Our argument works for both BI and BBI algebras except for a final step on (Contr).

For (Assoc), suppose $F_{t'} \supseteq F_t \circ F_y$ and $F_w \subseteq F_{t'} \circ F_z$. Consider the set 

$$f_s = \{a \in A \mid \exists b \in F_y, c \in F_z : a \geq b \ast c\}.$$ 

We show that $f_s$ is a proper filter. First suppose $\perp \in f_s$. Then there exist $b \in F_y$, $c \in F_z$ such that $b \ast c = \perp$. However, if this was the case, for arbitrary $a \in F_x$ we would have $a \ast b \in F_{t'} \subseteq F_{t''}$, and so $(a \ast b) \ast c = a \ast (b \ast c) = a \ast \perp = \perp \in F_w$, contradicting that $F_w$ is a prime filter. It is easy to see that $f_s$ is upwards-closed so it remains to verify that it is closed under meets. If $a, a' \in f_s$ then there are $b, b' \in F_y$ and $c, c' \in F_z$ such that $a \geq b \ast c$ and $a' \geq b' \ast c'$. We have $b \land b' \in F_y$ and $c \land c' \in F_z$ since they are filters, and, using the monotonicity of $\ast$ and the fact that $\land$ gives greatest lower bounds, it can be shown that $(b \land b') \ast (c \land c') \leq (b \ast c) \land (b' \ast c') \leq a \land a'$, so $a \land a' \in f_s$.

Abusing notation, it’s easy to see that $f_s \subseteq F_y \circ_i F_z$. We also have $F_w \subseteq F_x \circ_i f_s$. Let $a \in F_x$ and $a' \in f_s$. Then there exist $b \in F_y$ and $c \in F_z$ such that $a' \geq b \ast c$ and $a \ast a' \geq a \ast (b \ast c) = (a \ast b) \ast c$. We have $a \ast b \in F_{t'} \subseteq F_t$ and so $(a \ast b) \ast c \in F_w$. Since $F_s$ is a proper filter, we can use Zorn’s Lemma to extend $f_s$ to a prime filter $F_s$ satisfying these properties. Then we have $F_s \supseteq F_x \circ_i F_y \circ_i F_z$ and $F_w \subseteq F_x \circ_i F_y \circ_i F_z$ as required.

For (Weak), consider the set $f_c = \{a \in A \mid a \ast I \geq 1\}$. We have $F \subseteq F \circ_i f_c$, as, for $a \in F$ and $b \geq I$, we have $a \ast b \geq a \ast I = a$ so by upwards closure of $F$, $a \ast b \in F$. Using Zorn’s Lemma we can extend $f_c$ to a prime filter $F_c$ satisfying this property, and so for all $F \in \text{Pr}(A)$ we have $F_c \subseteq E_{\mathcal{A}}$ such that $F \subseteq F \circ_i F_c$ as required.

For (Contr) note that if $F_w \supseteq F_y \circ F_z$ and $F_z \subseteq E_{\mathcal{A}}$ we have $I \subseteq F_y$ and so for all $a \in F_y$, $a \ast I = a \in F_x$. Hence $F_y \subseteq F_x$. This suffices for BI: however, for BBI there is one final step. By invoking the maximality of prime filters on Boolean algebras we may conclude $F_y = F_x$, thus satisfying the BBI frame condition (Contr).

The representation theorem for (B)BI algebras now follows immediately from the analogous result for ILGL algebras and the fact that $\theta_{\mathcal{A}}(I) = E_{\mathcal{A}}$. 

**Theorem 4.13** (Representation Theorem for (B)BI Algebras). Every (B)BI algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given a (B)BI algebra $\mathcal{A}$, the map $\theta_{\mathcal{A}} : \mathcal{A} \rightarrow \text{Com}^{(B)\text{BI}}(\text{Pr}^{(B)\text{BI}}(\mathcal{A}))$ defined $\theta_{\mathcal{A}}(a) = \{F \in \text{Pr}^{(B)\text{BI}}(\mathcal{A}) \mid a \in F\}$ is an embedding.

**Corollary 4.14.** For all (B)BI algebras $\mathcal{A}$, given an interpretation $[-]$, the valuation $V_{[-]}(p) = \theta_{\mathcal{A}}([p])$ on $\text{Pr}^{(B)\text{BI}}(\mathcal{A})$ is such that $[\phi] \in F$ iff $F \models \phi$. 

□
Once again $Pr^{(B)BI}$ and $Com^{(B)BI}$ can be made into functors by setting $Pr^{(B)BI}(f) = f^{-1}$ and $Com^{(B)BI}(g) = g^{-1}$. Then the natural transformation $\eta(\_)$ defined for (I)LGL also suffices for the dual adjunction for BI (BBI) when restricted to the appropriate domain.

**Theorem 4.15.** The functors $Pr^{(B)BI}$ and $Com^{(B)BI}$ and the natural transformations $\theta$ and $\eta$ form a dual adjunction of categories between (B)BIAlg and (B)BIFr. □

**Corollary 4.16.** For all formulas $\phi$, $\psi$ of BI, $\phi \vdash \psi$ is provable in BIH iff $\phi \models \psi$ in the relational semantics. □

To obtain a dual equivalence of categories we add topological structure to BI frames. This can be achieved by straightforwardly extending (I)LGL spaces with BBI (BI) frame structure, as well as a coherence condition for the unit set $E$.

**Definition 4.17** (BI Space). A BI space is a structure $X = (X, \mathcal{O}, \lhd, \circ, E)$ such that

1. $(X, \mathcal{O}, \lhd, \circ)$ is an ILGL space,
2. $(X, E)$ is a BI frame, and
3. $E$ is clopen in $(X, \mathcal{O})$.

A morphism of BI spaces is a continuous BI morphism, yielding a category BISp. □

**Definition 4.18** (BBI Space). A BBI space is a structure $X = (X, \mathcal{O}, \circ, E)$ such that

1. $(X, \mathcal{O}, \circ)$ is an LGL space,
2. $(X, E)$ is a BBI frame, and
3. $E$ is clopen in $(X, \mathcal{O})$.

A morphism of BBI spaces is a continuous BBI morphism, yielding a category BBISp. □

The duality theorems for BI and BBI follow essentially immediately from that for ILGL and LGL. The only additional structure that needs to be taken care of is the constant $I$ and the unit set $E$. We define the functors and natural isomorphisms explicitly for their use in the Separation Logic duality. Hence for BI we have $PrSp^{\text{BI}} : \text{BIAlg} \to \text{BISp}$ defined by $PrSp^{\text{BI}}(A) = (Pr(A), \mathcal{O}_A, \subseteq, \circ_A, E_A)$ (where $\mathcal{O}_A$ is as defined for ILGL) and once again $PrSp^{\text{BI}}(f) = f^{-1}$; correspondingly, $Clop^{\text{BI}} : \text{BISp} \to \text{BIAlg}$ is given by $Clop^{\text{BI}}(X) = (\mathcal{C}\mathcal{L}(X), \cap, \cup, \Rightarrow, X, \emptyset, \bullet_X, \rightarrow_X, E)$ (where $\bullet_X, \rightarrow_X$ are the ILGL complex algebra operations) and, as in the case for ILGL, $Clop^{\text{BI}}(g) = g^{-1}$; $\theta$ and $\eta$ are given precisely as they are in ILGL duality, relativized to BIAlg and BISp.

Similarly, for BBI we have $PrSp^{\text{BBI}}(A) = (Pr(A), \mathcal{O}_A, \circ_A, E_A)$ (where $\mathcal{O}_A$ is as defined for LGL) and $PrSp^{\text{BBI}}(f) = f^{-1}$; $Clop^{\text{BBI}}(X) = (\mathcal{C}\mathcal{L}(X), \cap, \cup, \setminus, X, \emptyset, \bullet_X, \rightarrow_X, E)$ (where $\bullet_X, \rightarrow_X$ are the LGL complex algebra operations) and $Clop^{\text{BBI}}(g) = g^{-1}$; $\theta$ and $\eta$ are given precisely as they are in LGL duality, relativized to BBIAlg and BBISp.

That $E_A$ is clopen in each instance can be seen by the fact that $E = \theta_A(I)$. By Esakia duality, every set of the form $\theta_A(a)$ for some $a \in A$ is an upwards-closed clopen set of the prime filter space of $A$. Similarly, the clopen sets of the prime filter space of a Boolean algebra $A$ are the sets $\theta_A(a)$ for $a \in A$ by Stone duality, so the analogous property for BBI spaces holds too. It is also easy to see that $\eta$ is isomorphic with respect to $E$. The duality theorems thus obtain.

**Theorem 4.19** (Duality Theorem for (B)BI). The categories of (B)BI algebras and (B)BI spaces are dually equivalent. □
5. Separation Logic

5.1. Hyperdoctrines and Indexed Frames for Separation Logic. We now extend the duality theorems for BI and BBI algebras to the algebraic and relational structures suitable for interpreting Separation Logic. First, we must consider first-order (B)BI \( (\text{FO}(B)\text{BI}) \). Hilbert-type proof systems \( \text{FO}(B)\text{BI}_H \) are obtained by extending those given for \( (B)\text{BI} \) in Section 2 with the usual rules for quantifiers (see, e.g., [65]). Second, to give the semantics for the additional structure of \( \text{FO}(B)\text{BI} \), we must expand our definitions from the propositional case with category-theoretic structure. As these semantic structures support it, we consider a many-sorted first-order logic. We start on the algebraic side with BI hyperdoctrines.

**Definition 5.1** ((B)BI Hyperdoctrine (cf. [5])). A \( (B)\text{BI} \) hyperdoctrine is a tuple
\[
(\mathcal{P} : C^{\text{op}} \to \text{Poset}, (= X)_{X \in \text{Ob}(C)}, (\exists X_{\Gamma}, \forall X_{\Gamma})_{\Gamma, X \in \text{Ob}(C)})
\]
such that:

1. \( C \) is a category with finite products;
2. \( \mathcal{P} : C^{\text{op}} \to \text{Poset} \) is a functor such that, for each object \( X \) in \( C \), \( \mathcal{P}(X) \) is a \((B)\text{BI}\) algebra, and, for each morphism \( f \) in \( C \), \( \mathcal{P}(f) \) is a \((B)\text{BI}\) algebra homomorphism;
3. For each object \( X \) in \( C \) and each diagonal morphism \( \Delta X : X \to X \times X \) in \( C \), the element \( =_X \in \mathcal{P}(X \times X) \) is adjoint at \( \top_{\mathcal{P}(X)} \). That is, for all \( a \in \mathcal{P}(X \times X) \),
\[
\top_{\mathcal{P}(X)} \leq \mathcal{P}(\Delta X)(a) \iff =_X \leq a;
\]
4. For each pair of objects \( \Gamma, X \) in \( C \) and each projection \( \pi_{\Gamma,X} : \Gamma \times X \to \Gamma \) in \( C \), \( \exists X_{\Gamma} \) and \( \forall X_{\Gamma} \) are left and right adjoint to \( \mathcal{P}(\pi_{\Gamma,X}) \). That is, they are monotone maps
\[
\exists X_{\Gamma}(a) \leq b \iff a \leq \mathcal{P}(\pi_{\Gamma,X})(b) \quad \text{and} \quad \mathcal{P}(\pi_{\Gamma,X})(b) \leq a \iff b \leq \forall X_{\Gamma}(a).
\]

This assignment of adjoints is additionally natural in \( \Gamma \): given a morphism \( s : \Gamma \to \Gamma' \), the following diagrams commute:
\[
\begin{array}{ccc}
\mathcal{P}(\Gamma' \times X)^{\mathcal{P}(s \times \text{id}_X)} & \xrightarrow{\mathcal{P}(\pi_{\Gamma'X})} & \mathcal{P}(\Gamma \times X) \\
\mathcal{P}(\Gamma') & \xrightarrow{\mathcal{P}(s)} & \mathcal{P}(\Gamma) \\
\end{array}
\]

(B)BI hyperdoctrines were first formulated by Biering et al. [5] to prove the existence of models of higher-order variants of Separation Logic. There it was shown that the standard model of Separation Logic could be seen as a BBI hyperdoctrine, and thus safely extended with additional structure in the domain \( C^{\text{op}} \) to directly define higher-order constructs like lists, trees, finite sets and relations inside the logic. The present work strengthens this result to a dual equivalence of categories. Other algebraic models of Separation Logic, like those based on Boolean quantales [29] or formal power series [34], can be seen as particular instantiations of BBI hyperdoctrines.

To specify an interpretation \([-\cdot-\cdot]\) of \( \text{FOBI} \) in a BI hyperdoctrine \( \mathcal{P} \) we assign each type \( X \) an object \([X]\) of \( C \), and for each context \( \Gamma = \{ v_1 : X_1, \ldots, v_n : X_n \} \) we have \([\Gamma] = [X_1] \times \cdots \times [X_n] \). Each function symbol \( f : X_1 \times \cdots \times X_n \to X \) is assigned a morphism...
A standard Lindenbaum-Tarski style construction is sufficient to prove soundness and

For all objects \( \Gamma \) consider (typed) equality and universal quantification. They also appear to have some

Coumans \([24]\) as a topological dual for Boolean hyperdoctrines. In contrast to the duality

\(\exists v : X, \phi = \exists [X]_{\Gamma_1}([\phi])\)  
\(\forall v : X, \phi = [\forall [X]_{\Gamma_1}([\phi])\).

Substitution of terms is given by \([\phi(t/x)] = \prod([t])([\phi])\). \(\phi\) is satisfied by an interpretation \([-]\) if \([\phi] = \prod([\Gamma])\). \(\phi\) is valid if it is satisfied by all interpretations. For \(\text{FO(BI)}\) on

BBI hyperdoctrines everything works the same way except \([\phi \to \psi] = \neg \prod([\Gamma])\) \(\lor \prod([\Gamma])\) \(\psi\).

A standard Lindenbaum-Tarski style construction is sufficient to prove soundness and
completeness in both cases.

\[\text{Theorem 5.2.} \quad [61, 5] \quad \text{For all \(\text{FO(}\text{B})\text{BI}\) formulae } \phi, \psi \text{ in context } \Gamma, \phi \vdash \Gamma \psi \text{ is provable in } \text{FO(BI)} \text{ iff, for all \(\text{(B)BI}\) hyperdoctrines } \mathbb{P} \text{ and all interpretations } [-], [\phi] \leq \prod([\Gamma])\) \(\psi\). \]

For the other side of the dual adjunction, we introduce new structures: \textit{indexed (B)BI frames}. This definition is adapted from the notion of indexed Stone space presented by

Coumans \([24]\) as a topological dual for Boolean hyperdoctrines. In contrast to the duality

presented there, we prove the duality for the more general intuitionistic case and additionally

counter (typed) equality and universal quantification. They also appear to have some

relation to a more general formulation of Shirasu’s metaframes \([63]\), another type of indexed frame introduced to interpret predicate superintuitionistic and modal logics, but we defer

an investigation of this connection to another occasion.

\[\text{Definition 5.3.} \quad \text{An \textit{indexed (B)BI frame} is a functor } \mathcal{R} : C \to \text{BIFr} \text{ such that} \]

(1) \(C\) is a category with finite products;

(2) For all objects \(\Gamma, \Gamma'\) and \(X\) in \(C\), all morphisms \(s : \Gamma \to \Gamma'\) and all product projections \(\pi_{\Gamma, X}\), for the following commutative square

\[
\begin{array}{ccc}
\mathcal{R}(\Gamma \times X) & \xrightarrow{\mathcal{R}(\pi_{\Gamma, X})} & \mathcal{R}(\Gamma) \\
\downarrow & & \downarrow \mathcal{R}(s) \\
\mathcal{R}(\Gamma' \times X) & \xrightarrow{\mathcal{R}(\pi'_{\Gamma', X})} & \mathcal{R}(\Gamma')
\end{array}
\]

(a) (for indexed BI frames) the (Pseudo Epi) property holds: \(\mathcal{R}(\pi_{\Gamma', X})(y) \preceq \mathcal{R}(s)(x)\) implies there exists \(z\) such that: \(\mathcal{R}(\pi_{\Gamma, X})(z) \preceq x\) and \(y \preceq \mathcal{R}(s \times id_X)(z)\);

(b) (for indexed BBI frames) the quasi-pullback property holds: the induced map \(\mathcal{R}(\Gamma \times X) \to \mathcal{R}(\Gamma) \times_{\mathcal{R}(\Gamma')} \mathcal{R}(\Gamma' \times X)\) is an epimorphism.

Given an arbitrary indexed BI frame \(\mathcal{R} : C \to \text{(B)BIFr}\) and an object \(X\) we denote the

BI frame at \(X\) by \(\mathcal{R}(X) = (\mathcal{R}(X), \preceq_{\mathcal{R}(X)}, \circ_{\mathcal{R}(X)}, E_{\mathcal{R}(X)})\). Analogously, we denote the BBI frame at \(X\) by \(\mathcal{R}(X) = (\mathcal{R}(X), \circ_{\mathcal{R}(X)}, E_{\mathcal{R}(X)})\) in the case of an indexed BBI frame. \(\square\)
\[
x,[-] \models \Gamma \quad \text{if} \quad x \in [P]
\]
\[
x,[-] \models \Gamma 
\]
\[
x,[-] \models t = x \quad \text{iff} \quad t \in \mathcal{R}(\Delta_{[X_1]})) \quad x,[-] \models \Gamma
\]
\[
x,[-] \models \phi \land \psi \quad \text{iff} \quad x,[-] \models \Gamma \phi \text{ and } x,[-] \models \Gamma \psi
\]
\[
x,[-] \models \phi \lor \psi \quad \text{iff} \quad x,[-] \models \Gamma \phi \text{ or } x,[-] \models \Gamma \psi
\]
\[
x,[-] \models \phi \rightarrow \psi \quad \text{iff} \quad x,[-] \models \Gamma \phi \text{ implies } x',[-] \models \Gamma \psi
\]
\[
x,[-] \models \Gamma \quad \text{if} \quad x \in E_R(\Gamma)
\]
\[
x,[-] \models \phi \ast \psi \quad \text{iff} \quad x,[-] \models \Gamma \phi \text{ implies } \exists z \in R(\Delta_{[X_1]})) \text{ with } z,[-] \models \Gamma \psi
\]
\[
x,[-] \models \exists v_{n+1} : \phi \quad \text{iff} \quad x,[-] \models \Gamma \exists v_{n+1} : \phi
\]
\[
x,[-] \models \forall v_{n+1} : \phi \quad \text{iff} \quad x,[-] \models \Gamma \forall v_{n+1} : \phi
\]

Figure 7: Satisfaction on indexed (B)BI frames for \(\text{FO}(B)BI\). \(\text{FOBBI}\) replaces \(\leq\) with =.

A truth-functional semantics can be given for \(\text{FO}(B)BI\) on indexed (B)BI frames. For \(\text{FOBBI}\), an interpretation \([-]\) is given in precisely the same way as for BI hyperdoctrines, except that the key-difference is that each \(m\)-ary predicate symbol \(P\) of type \(X_1, \ldots, X_m\) is assigned to an upwards closed subset \([P] \in \mathcal{P}(\mathcal{R}(\Delta_{[X_1]} \times \cdots \times [X_m]))\). Similarly, an interpretation \([-]\) for \(\text{FOBBI}\) is given in the same way as it is for BBI hyperdoctrines, except that, for every \(m\)-ary predicate symbol \(P\) of type \(X_1, \ldots, X_m, P\) is assigned to a subset \([P] \in \mathcal{P}(\mathcal{R}(\Delta_{[X_1]} \times \cdots \times [X_m]))\). For formulas \(\phi\) of \(\text{FO}(B)BI\) in context \(\Gamma\) with \(x \in \mathcal{R}(\Delta_{[\Gamma]})\) the satisfaction relation \(\models \Gamma\) is inductively defined in Fig 7. Then, \(\mathcal{R}(\Delta_{[X_1]}) = \{y \mid \exists z(\mathcal{R}(\Delta_{[X_1]})) = y\}\). We note that bound variables are renamed to be fresh throughout, in an order determined by quantifier depth.

The familiar persistence property of propositional intuitionistic logics also holds for satisfaction on indexed BI frames. For atomic predicate formulas this is by design, with the assignment of predicate symbols to upwards closed subsets akin to a persistent valuation. For formulas of the form \(t = x \quad t'\) this follows from the fact that \(\mathcal{R}(\Delta_{[X_1]})\) is a BI morphism and hence order preserving. The rest of the clauses follow by an inductive argument, the most involved of which is for formulas of the form \(\exists v_{n+1} : \phi\). Suppose \(x \models \mathcal{R}(\pi_{[\Gamma][X_1]})\) and \(y \models \mathcal{R}(\Delta_{[\Gamma]})\) \(y\). Then there is \(x'\) such that \(\mathcal{R}(\pi_{[\Gamma][X_1]}))(x') = y \models \mathcal{R}(\Delta_{[\Gamma]})\) \(y\) and \(x',[-] \models \Gamma \cup \{v_{n+1} : \phi\}\). Since \(\mathcal{R}(\pi_{[\Gamma][X_1]})\) is a BI morphism it satisfies property (2) of Definition 3.5: there exists \(y'\) such that \(y' \models \mathcal{R}(\Delta_{[\Gamma]})\) \(y'\) and \(\mathcal{R}(\pi_{[\Gamma][X_1]}))(y') = y\). By inductive hypothesis, \(y',[-] \models \Gamma \cup \{v_{n+1} : \phi\}\) and so \(y,[-] \models \Gamma \cup \{v_{n+1} : \phi\}\).

5.2. Pointer Logic as an Indexed Frame. Although at first sight it may not seem so, indexed frames and the semantics based upon them are a generalization of the standard store–heap semantics of Separation Logic.

Consider the BI frame \(\text{Heap}^{\text{hi}} = (H, \cup, \sqsubseteq, H)\), where \(H\) is the set of heaps, \(\sqsubseteq\) is heap extension, and \(\cup\) is defined by \(h_2 \sqsubseteq h_1\) iff \(h_0 \neq h_1\) and \(h_0 \cdot h_1 = h_2\). This is the BI frame corresponding to the partial monoid of heaps. We define an indexed BI frame \(\text{Store}^{\text{hi}} : \text{Set} \rightarrow \text{BIFr}\) on objects by \(\text{Store}^{\text{hi}}(X) = (X \times H, \cup, \sqsubseteq, X \times H)\), where
(x_2, h_2) ∈ (x_0, h_0) ⊨ X (x_1, h_1) iff x_0 = x_1 = x_2 and h_2 ∈ h_0 \uplus h_1, and (x_0, h_0) ⊨ X (x_1, h_1) iff x_0 = x_1 and h_0 ⊨ h_1. On morphisms, set Store^\text{BBI}(f : X \to Y)(x, h) = (f(x), h). It is straightforward to see this defines a functor: for arbitrary X, Store(X) inherits the BI frame properties from Heap and for arbitrary f : X \to Y, Store(f) is trivially a BI morphism as it is identity on the structure that determines the back and forth conditions. The property (Pseudo Epi) is also trivially satisfied so this defines an indexed BI frame.

For BBI pointer logic, we instead start with the BBI frame Heap^\text{BBI} = (\mathcal{H}, \uplus, \{\}) where \{\} is the empty heap. Then Store^\text{BBI} is defined in essentially the same way, with Store^\text{BBI}(X) = (X \times H, \uplus_X, X \times \{\}) and Store^\text{BBI}(f)(x, h) = (f(x), h). This defines an indexed BBI frame.

We now describe the interpretations \([-\cdot-]\) on Store^\text{BBI} that yield the standard models of Separation Logic. We have one type \text{Val} and we set \([\text{Val}] = \mathbb{Z}\), with the arithmetic operations \([+], [-] : [\text{Val}]^2 \to \text{Val}\) defined as one would expect. Term morphisms \([t] : [\text{Val}]^n \to [\text{Val}]\) in context \(\Gamma = \{v_1, \ldots v_n\}\) are then defined as usual, with each constant \(n\) assigned the morphism \([n] : [\Gamma] \to \{\ast\} \to^n [\text{Val}]\). As one would expect, the key difference between the two interpretations is in the interpretation of the points-to predicate. For Intuitionistic Separation Logic, the points-to predicate \(\mapsto\) is assigned

\[\mapsto = \{(a, a'), h) \mid a \in \text{dom}(h) \text{ and } h(a) = a'\} \in \mathcal{P}^\subseteq_{[\text{Val}]^2}(\text{Store}^\text{BBI}([\text{Val}]^2)).\]

This set is clearly upwards closed with respect to the order \(\subseteq_{[\text{Val}]^2}\) so this is a well-defined interpretation. For Classical Separation Logic, \(\mapsto\) is instead assigned

\[\mapsto = \{(a, a'), h) \mid \{a\} \in \text{dom}(h) \text{ and } h(a) = a'\} \in \mathcal{P}(\text{Store}^\text{BBI}([\text{Val}]^2)).\]

In the indexed (B)BI frame Store^\text{BBI} : Set \to \text{ResFr} with the interpretations just defined, a store is represented as an \(n\)-place vector of values over \([\text{Val}]\). That is, the store \(s = \{(v_1, a_1), \ldots, (v_n, a_n)\}\) is given by the element \((a_1, \ldots, a_n) \in [\text{Val}]^n\). By a simple inductive argument we have the following result:

**Theorem 5.4.** For all formulas \(\phi\) of (B)BI pointer logic, all stores \(s = \{(v_1, a_1), \ldots, (v_n, a_n)\}\) and all heaps \(h, s, h \vdash \phi\) iff \((a_1, \ldots, a_n), h), [-\cdot-] \vdash^\Gamma \phi.\]

After verifying that terms are evaluated to the same elements as the standard model in both representations, the equivalence of the clauses for atomic formulas can be computed directly. That the clauses for the multiplicatives \(*\) and \(+\ast\) are equivalent is a consequence of the upwards and downwards closure of heap composition with respect to heap extension, as discussed in Section 4. Finally, the equivalence of the quantifier clauses is down to the representation of stores as vectors and the action of the product projections under the functor Store. The notions of indexed (B)BI frame and its associated semantics are therefore a natural generalization of the standard Separation Logic model.

5.3. **Duality for (B)BI Hyperdoctrines.** We now extend the results given for (B)BI algebras to (B)BI hyperdoctrines. For such results to make sense, both (B)BI hyperdoctrines and indexed (B)BI frames need to be equipped with a notion of morphism to form categories. Our definition of hyperdoctrine morphism adapts that for coherent hyperdoctrines [25].

**Definition 5.5** ((B)BI Hyperdoctrine Morphism). Given a pair of (B)BI hyperdoctrines \(\mathcal{P} : \text{C}^\text{op} \to \text{Poset}\) and \(\mathcal{P}' : \text{D}^\text{op} \to \text{Poset}\), a \((B)BI\) hyperdoctrine morphism \((K, \tau) : \mathcal{P} \to \mathcal{P}'\) is a pair \((K, \tau)\) satisfying the following properties:
(1) $K : C \to D$ is a finite product preserving functor;
(2) $\tau : P \to P' \circ K$ is a natural transformation;
(3) For all objects $X$ in $C$: $\tau_{X \times X}(=x) = ='_K(x)$;
(4) For all objects $\Gamma$ and $X$ in $C$, the following squares commute:

$$
\begin{align*}
\exists x & \downarrow \quad \exists x' & \forall x & \downarrow \forall x' \\
\mathbb{P}(\Gamma \times X) & \xrightarrow{\tau \times X} \mathbb{P}'(K(\Gamma) \times K(X)) & \mathbb{P}(\Gamma \times X) & \xrightarrow{\tau \times X} \mathbb{P}'(K(\Gamma) \times K(X)) \\
\mathbb{P}(\Gamma) & \xrightarrow{\tau} \mathbb{P}'(K(\Gamma)) & \mathbb{P}(\Gamma) & \xrightarrow{\tau} \mathbb{P}'(K(\Gamma))
\end{align*}
$$

The composition of BI hyperdoctrine morphisms $(K, \tau) : \mathbb{P} \to \mathbb{P}'$ and $(K', \tau') : \mathbb{P}' \to \mathbb{P}''$ is given by $(K' \circ K, \tau'_K(-) \circ \tau)$. This yields a category BIhyp.

For indexed (B)BI frames the definition splits into two because of the weakening of equality to a preorder on the intuitionistic side. It is straightforward to show that the notion of indexed BI frame morphism collapses to that for indexed BBI frames when the preorders $\preceq$ are substituted for $=$.

**Definition 5.6** (Indexed BI Frame Morphism). Given indexed BI frames $\mathcal{R} : C \to \text{BIFr}$ and $\mathcal{R}' : D \to \text{BIFr}$, an *indexed BI frame morphism* $(L, \lambda) : \mathcal{R} \to \mathcal{R}'$ is a pair $(L, \lambda)$ such that:

1. $L : D \to C$ is a finite product preserving functor;
2. $\lambda : \mathcal{R} \circ L \to \mathcal{R}'$ is a natural transformation;
3. (Lift Property) If there exists $x$ and $y$ such that $\mathcal{R}'(\Delta_X)(y) \preceq x$, then there exists $y'$ such that $\mathcal{R}(\Delta_L(\lambda)(y')) \preceq x$;
4. (Morphism Pseudo Epi) If there exists $x$ and $y$ with $\mathcal{R}'(\pi_{1,X})(x) \preceq \lambda(y)$, then there exists $z$ such that $x \preceq \lambda_{\times X}(z)$ and $\mathcal{R}(\pi_{1,\Gamma}, L(X))(z) \preceq y$.

The composition of indexed BI frame morphisms $(L', \lambda') : \mathcal{R}' \to \mathcal{R}''$ and $(L, \lambda) : \mathcal{R} \to \mathcal{R}'$ is given by $(L \circ L', \lambda' \circ \lambda_L(-))$. This yields a category IndBIFr.

**Definition 5.7** (Indexed BBI Frame Morphism). For indexed BBI frames $\mathcal{R} : C \to \text{BBIFr}$ and $\mathcal{R}' : D \to \text{BBIFr}$, an *indexed BBI frame morphism* $(L, \lambda) : \mathcal{R} \to \mathcal{R}'$ is a pair $(L, \lambda)$ satisfying (1) and (2) of the previous definition as well as

1. (Lift Property') if there exist $x$ and $y$ such that $\lambda_{\times X}(x) = \mathcal{R}'(\Delta_X)(y)$, then there exists $y'$ such that $\mathcal{R}(\Delta_L(\lambda)(y')) = x$,
2. (Quasi-Pullback) for all objects $\Gamma$ and $X$ in $C$, the following square is a quasi-pullback:

$$
\begin{array}{ccc}
\mathcal{R}(L(\Gamma) \times L(X)) & \xrightarrow{\lambda_{\times X}} & \mathcal{R}(\Gamma \times X) \\
\downarrow & & \downarrow \\
\mathcal{R}(\pi_{1,\Gamma}, L(X)) & \xrightarrow{\mathcal{R}(\pi_{1,\Gamma}, X)} & \mathcal{R}'(\pi_{1,\Gamma}, X) \\
\mathcal{R}(L(\Gamma)) & \xrightarrow{\lambda_{\Gamma \times X}} & \mathcal{R}(\Gamma)
\end{array}
$$

The composition of indexed BBI frame morphisms $(L', \lambda') : \mathcal{R}' \to \mathcal{R}''$ and $(L, \lambda) : \mathcal{R} \to \mathcal{R}'$ is given by $(L \circ L', \lambda' \circ \lambda_L(-))$. This yields a category Ind(B)BIFr.

We can now show that the ‘algebraic’ and ‘relational’ semantics of FO(B)BI generate a dual adjunction between these categories. First we define structures that correspond to the complex algebras and prime filter frames of Section 3, and show these transformations underpin functors. To obtain complex hyperdoctrines, we straightforwardly compose an indexed frame with the appropriate complex algebra functor from (B)BI.
3.5 — it can be shown that \( \mathcal{R} \) can be obtained from hyperdoctrines by composing with the prime filter functors \( \mathcal{R} \).

We concentrate on the verifications relating to \( \mathcal{R} \). Using the fact that \( \mathcal{R}(\pi_{\Gamma,X}) \) is a BI morphism — specifically property (2) of Definition 3.5 — it can be shown that \( \mathcal{R}(\pi_{\Gamma,X}) \) is in fact identical to \( \mathcal{R}(\pi_{\Gamma,X}) \), so this is indeed the case. We use its presentation as \( \mathcal{R}(\pi_{\Gamma,X}) \) as it simplifies some proofs that follow.

**Definition 5.9** (Complex BBI Hyperdoctrine). Given an indexed BI frame \( \mathcal{R} : C \to \text{BIFr} \), the complex hyperdoctrine of \( \mathcal{R} \), is given by \( \text{Com}^{\text{BI}}(\mathcal{R}(-)) : C^{\text{op}} \to \text{BIAlg} \), together with \( \text{Ran}(\mathcal{R}(\Delta_X)) = \{ x \mid \text{there exists} y \in A : \mathcal{R}(\pi_{\Gamma,X})(y) \leq x \} \) and \( \mathcal{R}(\pi_{\Gamma,X})_{s}(A) = \{ x \mid \text{for all} y, \text{if} x \leq \mathcal{R}(\pi_{\Gamma,X})(y) \text{then} y \in A \} \).

**Lemma 5.10.** Given an indexed \((B)\text{BI} \) frame \( \mathcal{R} : C \to \text{(B)BIFr} \), the complex hyperdoctrine \( \text{Com}^{(B)\text{BI}}(\mathcal{R}(-)) \) is a \((B)\text{BI} \) hyperdoctrine.

**Proof.** We concentrate on the verifications relating to \( \mathcal{R}(\pi_{\Gamma,X}) \) and \( \mathcal{R}(\pi_{\Gamma,X})_{s} \), of a BI complex hyperdoctrine. It is straightforward to see these map upwards-closed sets to upwards-closed sets and are monotone with respect to the subset ordering \( \subseteq \). The adjointness properties follow from the definitions so it just remains to prove naturality.

We give the case for \( \exists X_{\Gamma} \). Given a morphism \( s : \Gamma \to \Gamma' \) in \( C \) and an element \( A \in \text{Com}^{\text{BI}}(\mathcal{R}(\Gamma' \times X)) \), we must show \( \mathcal{R}(\pi_{\Gamma,X})_{s}(\mathcal{R}(s \times X)^{-1}(A)) = \mathcal{R}(s)^{-1}(\mathcal{R}(\pi_{\Gamma',X})_{s}(A)) \).

Suppose \( x \in \mathcal{R}(\pi_{\Gamma,X})_{s}(\mathcal{R}(s \times X)^{-1}(A)) \): then there exists \( y \in \mathcal{R}(\pi_{\Gamma,X})_{s}(A) \). By upwards-closure of \( A \), \( \mathcal{R}(s \times X)(z) \in A \). Hence \( x \in \mathcal{R}(s)^{-1}(\mathcal{R}(\pi_{\Gamma',X})_{s}(A)) \), as required.

Conversely, suppose \( x \in \mathcal{R}(s)^{-1}(\mathcal{R}(\pi_{\Gamma',X})_{s}(A)) \). Then there exists \( y \in A \) such that \( \mathcal{R}(\pi_{\Gamma,X})(y) \leq \mathcal{R}(s)(x) \). Then by (Psuedo Epi), there exists \( z \) such that \( \mathcal{R}(\pi_{\Gamma,X})(z) \leq x \) and \( y \leq \mathcal{R}(s \times X)(z) \). By upwards-closure of \( A \), \( \mathcal{R}(s \times X)(z) \in A \). Hence we have \( x \in \mathcal{R}(\pi_{\Gamma',X})_{s}(\mathcal{R}(s \times X)^{-1}(A)) \), as required.

The proof for BBI complex hyperdoctrines follows immediately by substituting every instance of \( \leq \) with \( = \) in the above argument, where the quasi pullback property allows us to assume the existence of \( z \) such that \( \mathcal{R}(\pi_{\Gamma,X})(z) = x \) and \( y = \mathcal{R}(s \times X)(z) \) from \( \mathcal{R}(\pi_{\Gamma',X})(y) = \mathcal{R}(s)(x) \).

**Lemma 5.11.** Given a morphism of indexed \((B)\text{BI} \) frames \((L, \lambda) : \mathcal{R} \to \mathcal{R}' \), \((L, \lambda^{-1}) : \text{Com}^{(B)\text{BI}}(\mathcal{R}'(-)) \to \text{Com}^{(B)\text{BI}}(\mathcal{R}(-)) \) is a morphism of \((B)\text{BI} \) hyperdoctrines.

Hence we obtain functors \( \tilde{\mathcal{F}} : (B)\text{BIFr} \to (B)\text{BIHyp} \) where \( \tilde{\mathcal{F}}(\mathcal{R}) \) is the complex hyperdoctrine for objects \( \mathcal{R} \) and \( \tilde{\mathcal{F}}(L, \lambda) = (L, \lambda^{-1}) \) for morphisms \((L, \lambda) \). In similar fashion, we can obtain indexed frames from hyperdoctrines by composing with the prime filter functors of \((B)\text{BI} \).
Definition 5.12 (Indexed Prime Filter (B)BI Frame). Given a (B)BI hyperdoctrine \( P \), the indexed prime filter frame is given by \( Pr_{(B)BI}(P(-)) \). □

Lemma 5.13. Given a (B)BI hyperdoctrine \( P : C^{op} \to Poset \) the indexed prime filter frame \( Pr_{(B)BI}(P(-)) \) is an indexed (B)BI frame. □

Proof. We first show the (Pseudo Epi) property is satisfied when \( P \) is a BI hyperdoctrine. Assume we have objects \( \Gamma, \Gamma' \) and \( X \) in \( C \) and a morphism \( s : \Gamma \to \Gamma' \). Let prime filters \( F_0 \) and \( F_1 \) be such that \( P(\pi_{\Gamma', X})^{-1}(F_1) \subseteq P(s)^{-1}(F_0) \).

Consider the filter \( f_2 = \{ x \mid \exists a \in F_1, x \geq P(s \times id_X)(a) \} \). Suppose for contradiction that \( f_2 \) is not proper. Then there exists \( a \in F_1 \) such that \( P(s \times id_X)(a) \neq \perp \). By adjointness, \( \exists X_{\Gamma}(\perp) = \perp \), so \( P(s)(\exists X_{\Gamma}(a)) = \exists X_{\Gamma}(P(s \times id_X)(a)) = \perp \) by naturality. This entails \( \exists X_{\Gamma'}(a) \not\in P(s)^{-1}(F_0) \) so \( \exists X_{\Gamma'}(a) \not\in P(\pi_{\Gamma', X})^{-1}(F_1) \) by assumption. However, by adjointness and filterhood, \( P(\pi_{\Gamma', X})(\exists X_{\Gamma'}(a)) \in F_1 \), a contradiction.

Clearly \( P(s \times id_X)^{-1}(f_2) \supseteq F_1 \). To see that the other required inclusion holds, suppose \( a \in P(\pi_{\Gamma', X})^{-1}(f_2) \). Then there exists \( b \in F_1 \) such that \( P(s \times id_X)(b) \leq P(\pi_{\Gamma', X})(a) \). By adjointness \( \exists X_{\Gamma}(P(s \times id_X)(b)) \leq a \) and so by naturality \( P(s)(\exists X_{\Gamma}(b)) \leq a \). Since \( P(\pi_{\Gamma', X})(\exists X_{\Gamma'}(b)) \in F_1 \), we have \( \exists X_{\Gamma'}(b) \in P(\pi_{\Gamma', X})^{-1}(F_1) \subseteq P(s)^{-1}(F_0) \). Thus by filterhood, \( a \in F_0 \). Since \( f_2 \) is a filter, using Zorn’s Lemma, it can be extended to a prime filter \( F_2 \) satisfying these properties.

For \( P \) a BBI hyperdoctrine, we instead start with the assumption of prime filters \( F_0 \) and \( F_1 \) such that \( P(\pi_{\Gamma', X})^{-1}(F_1) = P(s)^{-1}(F_0) \). This is sufficient to once again prove the existence of a prime filter \( F_2 \) satisfying \( P(s \times id_X)^{-1}(F_2) \supseteq F_1 \) and \( P(\pi_{\Gamma', X})^{-1}(F_2) \subseteq F_0 \). However, maximality of prime filters on Boolean algebras collapses the inclusions to equalities — \( P(s \times id_X)^{-1}(F_2) = F_1 \) and \( P(\pi_{\Gamma', X})^{-1}(F_2) = F_0 \) — so the quasi pullback property holds. □

Lemma 5.14. Given a morphism of (B)BI hyperdoctrines \( (K, \tau) : P \to P' \), \( (K, \tau^{-1}) : Pr(P'(-)) \Rightarrow Pr(P(-)) \) is a morphism of indexed (B)BI frames. □

This gives us the corresponding functors \( \mathfrak{G} : (B)BIHyp \to (B)BIFr \), where \( \mathfrak{G}(P) \) is the indexed prime filter frame \( Pr_{(B)BI}(P(-)) \) for objects \( P \) and \( \mathfrak{G}(K, \tau) = (K, \tau^{-1}) \) for morphisms \( (K, \tau) \).

We can finally define natural transformations \( \Theta : Id_{(B)BIHyp} \Rightarrow \mathfrak{G} \), \( \mathfrak{H} : Id_{Ind(B)BIFr} \Rightarrow \mathfrak{Gf} \) straightforwardly from those given for (B)BI: \( \Theta_{P} = (Id_{C}, \theta_{P(-)}) \) and \( \mathfrak{H}_{\mathcal{R}} = (Id_{C}, \eta_{\mathcal{R}(-)}) \). The property of being a natural transformation is inherited from \( \theta \) and \( \eta \), so the last non-trivial verification before stating the dual adjunction theorem is that these define morphisms of the right sort at each \( P \) and \( \mathcal{R} \).

Theorem 5.16. The functors \( \mathfrak{f} \) and \( \mathfrak{G} \) and the natural transformations \( \Theta \) and \( \mathfrak{H} \) form a dual adjunction of categories between (B)BIHyp and Ind(B)BIFr.

We now make the connection to the respective semantics clear. Given any interpretation on an indexed BI frame \([-]_\cdot \), we automatically have a interpretation for the complex hyperdoctrine as predicate symbols are interpreted as upwards-closed subsets; that is,
elements of complex algebras of BI frames. The same is true for indexed BBI frames as the interpretation of predicate symbols are subsets and thus elements of complex algebras of BBI frames. A simple inductive argument shows that satisfaction coincides for these models.

**Proposition 5.17.** Given an indexed (B)BI frame $\mathcal{R}$ and an interpretation $[-]$, for all FO(B)BI formulas $\phi$ in context $\Gamma$ and $x \in \mathcal{R}(\lceil \Gamma \rceil)$, $x, [\cdot] \models^F \phi$ iff $x \in [\phi]$. \hfill $\square$

Conversely, given an interpretation $[-]$ on a (B)BI hyperdoctrine, an interpretation $[\cdot]$ can be defined by setting $[\mathcal{P}] = \theta_{\mathcal{P}(\lceil X_1, \ldots, X_m \rceil)(\lceil \mathcal{P} \rceil)}$ for each predicate symbol of type $X_1, \ldots, X_m$. The following proposition is a corollary of the dual adjunction.

**Proposition 5.18.** Given a (B)BI hyperdoctrine $\mathcal{P}$ and an interpretation $[-]$, for all FO(B)BI formulas $\phi$ in context $\Gamma$ and prime filters $F$ of $\mathcal{P}(\lceil \Gamma \rceil)$, $[\phi] \in F$ iff $F, [-] \models^F \phi$. \hfill $\square$

**Corollary 5.19** (Relational Soundness and Completeness). For all FO(B)BI formulas $\phi$, $\psi$ in context $\Gamma$, $\phi \vdash^F \psi$ is provable in FO(B)BI if $\phi \models^F \psi$.

To obtain a dual equivalence of categories, we must move from indexed (B)BI frames to indexed (B)BI spaces.

**Definition 5.20** (Indexed (B)BI Space). An indexed (B)BI space is a functor $\mathcal{R} : C \to \text{(B)BISp}$ such that

1. $U \circ \mathcal{R} : C \to \text{(B)BIFr}$ is an indexed (B)BI frame, where $U : \text{(B)BISp} \to \text{(B)BIFr}$ is the functor that forgets topological structure.
2. For each object $X$ in $C$, $\text{Ran}(\mathcal{R}(\Delta_X)))$ is clopen;
3. For each pair of objects $\Gamma$ and $X$ in $C$, $\mathcal{R}(\pi_{\Gamma, X})^{\ast}$ and $\mathcal{R}(\pi_{\Gamma, X})_{\ast}$ map
   a) (for indexed BI spaces) upwards-closed clopen sets to upwards-closed clopen sets.
   b) (for indexed BBI spaces) clopen sets to clopen sets. \hfill $\square$

Indexed (B)BI space morphisms are given identically to indexed (B)BI frame morphisms (relativized to the category (B)BISp) and this yields a category Ind(B)BISp. By uniformly substituting the functors and natural transformations of the (B)BI dual adjunction with those of the (B)BI duality in the definitions of the (B)BI hyperdoctrine dual adjunction, we obtain the maps required for the duality theorem.

Given an indexed (B)BI space $\mathcal{R}$, $\mathcal{F}(\mathcal{R})$ is the (B)BI hyperdoctrine given by the functor $\text{Clop}^{\text{Bis}}(\mathcal{R}(\cdot))$ together with $\text{Ran}(\mathcal{R}(\Delta_X))$ as $=_{\mathcal{X}}$, $\mathcal{R}(\pi_{\Gamma, X})^{\ast}$ as $\exists_{\mathcal{X}}$ and $\mathcal{R}(\pi_{\Gamma, X})_{\ast}$ as $\forall_{\mathcal{X}}$. That is well defined is immediate from the coherence conditions (2) and (3) in the definition of indexed (B)BI space, together with Lemma 5.10. $\mathcal{F}(L, \lambda) = (L, \lambda^{-1})$ as before.

Given a (B)BI hyperdoctrine $\mathcal{P}$, $\mathcal{G}(\mathcal{P}) = P_{\mathcal{F}}S_{\mathcal{P}_{\mathcal{B}}}^{\text{Bis}}(\mathcal{P}(\cdot))$. That this satisfies properties (2) and (3) in the definition of indexed space can be derived from the fact that — because of Stone (Esakia) duality — the (upwards-closed) clopen sets of the prime filter space of $A$ are all of the form $\theta_{\mathcal{A}}(a)$ for $a \in A$. It is then straightforward to use the adjointness properties of a (B)BI hyperdoctrine to show that $\text{Ran}(\mathcal{P}(\Delta_X)^{\ast}) = \theta_{\mathcal{P}(X \times X)}(=_{\mathcal{X}})$, $(\mathcal{P}(\pi_{\Gamma, X})^{\ast})^{\ast}(h_{\mathcal{P}(\Gamma \times X)}(a)) = \theta_{\mathcal{P}(\Gamma \times X)}(\exists_{\mathcal{X}}(a))$ and $(\mathcal{P}(\pi_{\Gamma, X})^{\ast})_{\ast}(h_{\mathcal{P}(\Gamma \times X)}(a)) = \theta_{\mathcal{P}(\Gamma)}(\forall_{\mathcal{X}}(a))$. $\mathcal{G}(K, \tau) = (K, \tau^{-1})$ once again.

Finally, $\Theta$ and $H$ are defined as in the dual adjunction except that $\theta$ and $\eta$ are taken from (B)BI duality instead. Since these new $\Theta$ and $H$ inherit the property of being natural isomorphisms from $\theta$ and $\eta$, we obtain FO(B)BI duality.

**Theorem 5.21** (Duality Theorem for FO(B)BI). The categories of (B)BI hyperdoctrines and indexed (B)BI spaces are dually equivalent. \hfill $\square$
Given that the structures we have defined comprise the most general classes of bunched logic model, to what extent can we restrict the dual adjunctions and dualities to a subclass of model that correspond specifically to the standard model of Separation Logic? There are a number of properties that the (classical) memory model satisfies implicitly, given as follows by Brotherston & Villard [13]:

- **Partial deterministic:** \( w, w' \in w_1 \circ w_2 \) implies \( w = w' \)
- **Cancellativity:** \( w \circ w_1 \cap w \circ w_2 \) implies \( w_1 = w_2 \)
- **Indivisible Units:** \( (w \circ w') \cap E \neq \emptyset \)
- **Disjointness:** \( w \circ w \neq \emptyset \) implies \( w \in E \)
- **Divisibility:** for every \( w \notin E \), there are \( w_1, w_2 \notin E \) such that \( w \in w_1 \circ w_2 \)
- **Cross Split:** whenever \( (t \circ u) \cap (v \circ w) \neq \emptyset \), there exist \( tv, tw, uw, v \in tv \circ uw \) such that \( t \in tw, u \in uo, v \in uv \circ uw \).

Here we do not consider their property Single Unit as it is only satisfied by the propositional heap model, and not by the predicate store-heap models. Brotherston & Villard show that while Divisibility and Indivisible Units are definable in BBI, Partial Deterministic, Cancellativity and Disjointness are not, with cross split conjectured to be similarly undefinable.

It is straightforward to restrict the dual adjunction and duality theorems to the algebras satisfying axioms corresponding to Indivisible Units \( I \land (a \ast b) \leq a \) and Divisibility \( -I \leq -I \ast -I \). However, in the other cases the undefinability results preclude this possibility: no algebraic axiom can possibly pick out these classes of model.

The extent to which the remaining properties define different notions of validity has been partially investigated by Larchey-Wendling & Galmiche [54]. In particular, they show that the formula \( \mathcal{I} \ast \mathcal{I} \to \mathcal{I} \), where \( \mathcal{I} = - (\top \to -I) \), distinguishes partial deterministic models from non-deterministic models. In summary, this situation isn't totally benign.

Brotherston & Villard’s solution is to give a conservative extension of BBI in the spirit of hybrid logic [3] called HyBBI. The additional expressivity of nominals and satisfaction operators allows the logic to pick out specific states of the model, making the axiomatization of the remaining properties possible. Thus, in order to precisely capture the concrete model of Separation Logic we would have to extend the techniques of the present work to hybrid extensions of bunched logics. This would also enable us to connect our work to the extensive tableau proof theory given for bunched logics [40, 53, 27, 38] via the close connection between hybrid extensions and labelled proof systems. Although duality theorems for axiomatic extensions of hybrid logic with one unary modality have been given [23], the generalization required to achieve such a result for hybrid bunched logics is beyond the scope of this paper. We defer such an investigation to another occasion.

6. Multiplicative Extensions

To conclude, we adapt the results of Section 4 to a variety of logics that extend BI and BBI. Applications of these logics include reasoning about deny-guarantee permissions, concurrency and — via the interpretation of heap intersection operations — complex resource sharing. We concentrate on three: the classical bunched logics that extend (B)BI with a De Morgan negation, facilitating the definition of multiplicatives corresponding to disjunction and falsum [11]; the family of sub-classical bunched logics that sit intermediate between (B)BI and the
classical bunched logics; and finally CKBI, a new logic suggested by algebraic interpretations of (a basic version of) Concurrent Separation Logic [59].

For reasons of space we only sketch arguments and do not give full details of the topological structures that allow us to restrict the dual adjunctions to dual equivalences of categories: it is, however, straightforward to do so by a careful choice of topological coherence conditions for the additional frame operations. We leave it to the reader to fill in these gaps.

6.1. Classical Bunched Logics. Brotherston & Calcagno [11] introduces the logic CBI that extends BBI with a De Morgan negation $\sim$. By substituting $*$ and $I$ into the De Morgan laws relating $\land$ to $\lor$ and $\top$ to $\bot$, this yields multiplicative connectives corresponding to disjunction and falsum. Although it is shown that the Separation Logic heap model is not a model of CBI, a number of interesting applications are suggested ranging from deny-guarantee permissions to regular languages. Brotherston [10] also gives a display calculus for DMBI, the apparent intuitionistic variant of CBI that instead extends BI, but no Kripke semantics or completeness proof can be found in the literature.

We give the syntax and Hilbert-type proof theory of the logics by defining the algebras they determine.

**Definition 6.1 (DMBI/CBI Algebra).** A DMBI (CBI) algebra is a (B)BI algebra $A$ extended with an operation $\sim: A \rightarrow A$ satisfying, for all $a \in A$, $\sim a = a \rightarrow \sim I$ and $\sim \sim a = a$.

On such an algebra we can define a multiplicative falsum $\bot^* = \sim I$ and a multiplicative disjunction $a \lor b = \sim (a \bullet \sim b)$. $\sim$ also makes $(A, \land, \lor, \sim, \bot)$ a De Morgan algebra: the fact that $\rightarrow$ converts joins in its left argument into meets entails
\[
\sim (a \lor b) = (a \lor b) \rightarrow \bot^* = (a \rightarrow \bot^*) \land (b \rightarrow \bot^*) = \sim a \land \sim b.
\]
It then follows that $a \leq b$ iff $\sim b \leq \sim a$, and thus also $\sim (a \land b) = \sim a \lor \sim b$. Hence $\sim$ is a dual automorphism on the underlying bounded distributive lattice of $A$. It is also straightforward to derive the following property from the residuation property of $\bullet$ and the equation $\sim a = a \rightarrow \sim I$.

**Proposition 6.2.** Given a DMBI (CBI) algebra $A$, for all $a, b, c \in A$,
\[a \bullet b \leq c \text{ iff } b \bullet \sim c \leq \sim a \text{ iff } a \bullet \sim c \leq \sim b.\]

**Definition 6.3 (DMBI/CBI Frame).** A DMBI frame is a BI frame extended with an operation $n: X \rightarrow X$ satisfying
\[
\begin{align*}
\text{(Dual)} & \quad x \preceq y \rightarrow n(y) \preceq n(x) \\
\text{(Invol)} & \quad n(n(x)) = x \\
\text{(Compat)} & \quad z \in x \circ y \leftrightarrow n(x) \in y \circ n(z) \leftrightarrow n(y) \in x \circ n(z).
\end{align*}
\]
A CBI frame is a BBI frame extended with an operation $n: X \rightarrow X$ satisfying (Invol) and (Compat).
The definition of CBI frame here looks different to the notion given by Brotherston & Calcagno [11] but is equivalent. There, a (multi-unit) CBI model is a tuple \((X, \circ, E, n, \infty)\) such that \((X, \circ, E)\) is a BBI frame, with \(n : X \to X\) and \(\infty \subseteq X\) satisfying, for all \(x \in X\), \(n(x)\) is the unique element such that \(\infty \cap (n(x) \circ x) \neq \emptyset\). (Invol) and (Compat) are then proved as consequences of this definition in Proposition 2.3 (1) and (3). As they discuss, the choice of \(\infty\) is fixed by the choice of \(n\), and it can easily be seen that defining \(\infty = \{n(e) \mid e \in E\}\) on our CBI frames yields their CBI models. We choose our presentation as it simplifies proofs.

The semantics for DMBI (CBI) is given by extending the semantics on (B)BI frames for (B)BI by the clause

\[ x \models \sim \phi \text{ iff } n(x) \not\models \phi. \]

Semantic clauses can then straightforwardly be given for the derived operations \(\bar{\vee}\) and \(\bot\) using the clauses for \(\sim, \ast\) and I. In the case of DMBI, persistence for formulas of the form \(\sim \phi\) is guaranteed by the frame property (Dual), and thus also for formulas of the form \(\phi \bar{\vee} \psi\) and \(\bot\). It is then simple to define complex algebras for DMBI and CBI frames.

**Definition 6.4** (DMBI/CBI Complex Algebra). Given a DMBI (CBI) frame \(\mathcal{X}\), the complex algebra \(\text{Com}^{\text{DMBI}}(\mathcal{X}) (\text{Com}^{\text{CBI}}(\mathcal{X}))\) is given by extending \(\text{Com}^{(B)BI}(\mathcal{X})\) with the operation \(\sim\), defined \(\sim_{\mathcal{X}}(A) = \{x \mid n(x) \not\in A\}\).

**Lemma 6.5.** Given a DMBI (CBI) frame \(\mathcal{X}\), \(\text{Com}^{\text{DMBI}}(\mathcal{X}) (\text{Com}^{\text{CBI}}(\mathcal{X}))\) is a DMBI (CBI) algebra.

**Proof.** We give the case for DMBI: the CBI case follows as a corollary. First note that (Dual) ensures \(\sim\) maps upwards-closed sets to upwards-closed sets and thus \(\text{Com}^{\text{DMBI}}(\mathcal{X})\).

That \(\sim_{\mathcal{X}}(A) = A\) is trivial so we finish by verifying \(\sim_{\mathcal{X}}(A) = A \implies \sim_{\mathcal{X}} E\). First assume \(a \in \sim_{\mathcal{X}}(A)\) and let \(a \leq a'\) with \(e \in a' \circ b\) and \(b \in A\): we need to show \(n(e) \not\in E\) so that \(a \in A \implies \sim_{\mathcal{X}}(E)\).

In the other direction, assume \(a \in A \implies \sim_{\mathcal{X}}(E)\). By the BI frame property (Weak), there exists \(e \in E\) such that \(a \in a \circ e\). By (Compat), \(n(e) \in a \circ n(a)\). Hence if \(n(a) \in A\), we would have \(n(e) \in \sim_{\mathcal{X}}(E) = \{x \mid n(x) \not\in E\}\), a contradiction since \(n(n(e)) = e\). Hence \(n(a) \not\in A\) so \(a \in \sim(A)\).

We can also use \(\sim\) to define an \(n_{A}\) for the prime filter frame of a DMBI (CBI) algebra. For a set \(X \subseteq A\), let \(\sim X = \{\sim x \mid x \in X\}\) and \(\overline{X} = \{x \mid x \not\in X\}\).

**Definition 6.6** (Prime Filter DMBI/CBI Frame). Given a DMBI (CBI) algebra \(A\), the prime filter frame \(Pr^{\text{DMBI}}(A) (Pr^{\text{CBI}}(A))\) is given by extending the prime filter frame \(Pr^{(B)BI}(A)\) with the operation \(n_{A}\), defined \(n_{A}(F) = \overline{F}\).

That this is well defined follows from the fact \(\sim\) is a dual automorphism on the underlying bounded distributive lattice: this entails that, given a prime filter \(F\), \(\sim F\) is a prime ideal, and thus \(\overline{F}\) is a prime filter.

**Lemma 6.7.** Given a DMBI (CBI) algebra \(A\), \(Pr^{\text{DMBI}}(A) (Pr^{\text{CBI}}(A))\) is a DMBI (CBI) frame.

**Proof.** Once again we restrict to the case for DMBI as the CBI case follows as a corollary. It is trivial that \(n_{A}\) satisfies (Dual) and (Invol). We give a characteristic portion of (Compat):
Let $\mathbb{A} = (A, \wedge, \vee, \neg, 0, 1)$ be a basic B(B)I algebra. Then, for all $a, b, a', b' \in A$ and $X, Y \subseteq A$, we have the following:

1. If $a \leq a'$ and $b \leq b'$ then $a \forall b \leq a' \forall b'$;
2. If $\bigwedge X$ and $\bigvee Y$ exist then $\bigvee_{x \in X, y \in Y} x \forall y$ exists and $(\bigwedge X) \forall (\bigvee Y) = \bigwedge_{x \in X, y \in Y} x \forall y$;
3. If $a = \top$ or $b = \top$ then $a \forall b = \top$;
4. If $\bigwedge X$ exists then for any $z \in A$: $\bigvee_{x \in X} (x \forall z)$ exists with $\bigvee_{x \in X} (x \forall z) = (\bigwedge X) \forall z$;
5. If $\bigvee X$ exists then for any $z \in A$: $\bigvee_{x \in X} (z \forall x)$ exists with $\bigvee_{x \in X} (z \forall x) = z \forall (\bigvee X)$;
6. $a \forall \top = \bot \forall a = \bot$.

In the context of $\mathbb{A} = (A, \wedge, \vee, \neg, 0, 1)$, the DMBI (CBI) morphisms require the DMBI (CBI) morphisms to additionally respect $\sim a$ that, for all $a, b, c \in A$, $\sim a \leq \sim b$ and $\sim c$. This entails, for any $a_0 \in F_0$, that $a_0 \leq \sim(a_1 \bullet a_2)$, and (since $\sim$ is a dual automorphism on the underlying lattice), $a_1 \bullet a_2 \not\leq \sim a_0$. By Proposition 6.2 it follows that, for all $a_0 \in F_0$, $\sim a_1 \in \sim F_1$ and $a_2 \in \sim F_2$, $a_0 \bullet a_2 \not\leq \sim a_1$. Thus, $a_0 \bullet a_2 \in \sim F_1$ would imply $a_0 \bullet a_2 \not\leq \sim(a_0 \bullet a_2) = a_0 \bullet a_2$ so $a_0 \bullet a_2 \not\leq \sim F_1$ and thus $a_0 \bullet a_2 \in \sim F_1$, as required.

Next, assuming the right hand side, we can similarly reason that for $\sim a_0 \in \sim F_0, a_1 \in F_1$ and $a_2 \in \sim n_\mathbb{A}(F_2)$, $a_1 \bullet a_2 \not\leq \sim a_0$. Hence $a_1 \bullet a_2 \not\leq \sim F_0$ and so $a_1 \bullet a_2 \in \sim n_\mathbb{A}(F_0)$ as required.\qed

Extending the notion of (B)BI morphism to DMBI (CBI) is straightforward: we just require the DMBI (CBI) morphisms to additionally respect $n$: $g(n(x)) = n'(g(x))$. Then we can adapt the functors and natural transformations of the (B)BI dual adjunction to this case: the functor action on morphisms gives the inverse image maps, and $\theta$ and $\eta$ are defined in the same way.

**Theorem 6.8** (Soundness and Completeness for DMBI and CBI). The functors $Pr^{DMBI}$ ($Pr^{CBI}$), $Com^{DMBI}$ ($Com^{CBI}$) and natural transformations $\theta$ and $\eta$ form a dual adjunction of categories between $DMBI\text{Alg}$ ($CBI\text{Alg}$) and $DMBIFr$ ($CBIFr$). \qed

Duality follows by taking the obvious extension of (B)BI space with the DMBI (CBI) frame structure, satisfying the coherence condition that $n$ maps clopen sets to clopen sets.

### 6.2. Sub-Classical Bunched Logics

Brotherston & Villard [14] introduces a family of logics extending BBI that they call sub-classical bunched logics, as they lie intermediate between BBI and CBI. As heaps equipped with intersection operations are models of the logics, they are of clear interest to the Separation Logic community, with verification of algorithms involving complex resource sharing suggested as a natural application. Basic Bi-intuitionistic Boolean Bunched logic (BiBBI) is defined to be the multiplicative extension of BBI that drops all De Morgan laws between multiplicative conjunction, disjunction, truth and falsum. Thus in BiBBI these connectives can no longer be defined in terms of each other (as they were for DMBI and CBI) and must be given as primitives. A number of these correspondences can then be added as axioms without the logic collapsing into CBI. We show that our framework captures BiBBI and its axiomatic extensions, as well as the evident intuitionistic variant BiBII and the intermediate logics weaker than DMBI.

**Definition 6.9** (Basic Bi(B)I Algebra). A basic Bi(B)I algebra $\mathbb{A}$ is a (B)I algebra extended with operations $\forall$, $\exists$, and a constant $\top$, such that $\forall$ is commutative and, for all $a, b, c \in A$, $a \leq b \forall c$ if $a \forall b \leq c$.

Residuation of $\forall$ and $\exists$ guarantees that $\forall$ is order-preserving in each argument, as well as a number of other properties dual to those of Proposition 3.2.

**Proposition 6.10.** Let $\mathbb{A}$ be a basic Bi(B)I algebra. Then, for all $a, b, a', b' \in A$ and $X, Y \subseteq A$, we have the following:

1. If $a \leq a'$ and $b \leq b'$ then $a \forall b \leq a' \forall b'$;
2. If $\forall X$ and $\exists Y$ exist then $\exists_{x \in X, y \in Y} x \forall y$ exists and $(\forall X) \forall (\exists Y) = \forall_{x \in X, y \in Y} x \forall y$;
3. If $a = \top$ or $b = \top$ then $a \forall b = \top$;
4. If $\forall X$ exists then for any $z \in A$: $\exists_{x \in X} (x \forall z)$ exists with $\exists_{x \in X} (x \forall z) = (\forall X) \forall z$;
5. If $\exists X$ exists then for any $z \in A$: $\forall_{x \in X} (z \forall x)$ exists with $\forall_{x \in X} (z \forall x) = \forall (\exists X)$; and
6. $a \forall \top = \bot \forall a = \bot$. \qed
\begin{align*}
  r \models \perp^* & \text{ iff } r \notin U \\
  r \models \phi \lor \psi & \text{ iff for all } s,t,u, r \preceq s \in t \lor u \text{ implies } t \models \phi \text{ or } u \models \psi \\
  r \models \phi^* \psi & \text{ iff there exist } s,t,u \text{ such that } r \succcurlyeq s, u \in t \lor s, u \models \phi \text{ and } t \not\models \psi
\end{align*}

Figure 8: Satisfaction for \( \text{Bi(B)}\text{BI} \). \( \text{BiBBI} \) is the case where \( \preceq \) is \( = \).

| Property | Axiom | Frame Correspondent |
|----------|-------|---------------------|
| Associativity | \( a \lor (b \lor c) \leq (a \lor b) \lor c \) | \( t' \preceq t \in x \lor y \land w \in t' \lor z \rightarrow \exists s, s', w'(s' \preceq s \in y \lor z \land w \preceq w' \in x \lor s') \) |
| Unit Weakening | \( a \leq a \lor \perp^* \) | \( u \in U \land x \in y \lor u \rightarrow x \preceq y \) |
| Unit Contraction | \( a \lor \perp^* \leq a \) | \( \exists u \in U(w \in w \lor u) \) |
| Contraction | \( a \lor a \leq a \) | \( x \in x \lor x \) |
| Weak Distributivity | \( a \bullet (b \lor c) \leq (a \bullet b) \lor c \) | \( t' \succcurlyeq t \in x_1 \lor x_2 \land t' \preceq t'' \in y_1 \lor y_2 \rightarrow \exists w(y_1 \in x_1 \lor w \land x_2 \in w \lor y_2) \) |

Figure 9: \( \text{Bi(B)}\text{BI} \) properties and axioms (cf. [14]). The \( \text{BiBBI} \) variants replace \( \preceq \) with \( = \).

**Definition 6.11 (Basic \( \text{Bi(B)}\text{BI} \) Frame).** A basic \( \text{BiB} \text{I} \) frame is a given by a tuple \( \mathcal{X} = (X, \preceq, \circ, E, \lor, U) \) such that \( (X, \preceq, \circ, E) \) is a BI frame, \( \lor : X^2 \rightarrow \mathcal{P}(X) \) and \( U \subseteq X \), satisfying:

- (Comm) \( z \in x \lor y \rightarrow z \in y \lor x \); (Up) \( u \notin U \land u \preceq w' \rightarrow u' \notin U \).

A basic \( \text{BiBBI} \) frame (cf. [14]) is a tuple \( \mathcal{X} = (X, \circ, E, \lor, U) \), such that \( (X, \circ, E) \) is a BBI frame, \( \lor : X^2 \rightarrow \mathcal{P}(X) \) satisfies (Comm) and \( U \subseteq X \). \( \square \)

The semantics for \( \text{Bi(B)}\text{BI} \) extends that of \( \text{(B)}\text{BI} \): the additional semantic clauses are given in Fig 8. It can be shown that for \( \text{BiB} \text{I} \) persistence extends to all formulas, as one would expect. \( \text{Bi(B)}\text{BI} \) is extended with axioms corresponding to properties of \( \text{Bi(B)}\text{BI} \) frames. These are given in Fig 9, with the axioms presented as inequalities on \( \text{Bi(B)}\text{BI} \) algebras.

Let \( \Sigma \) denote any combination of properties from Figure 9. By \( \text{Bi(B)}\text{BIAlg}_\Sigma \) we denote the category of basic \( \text{Bi(B)}\text{BI} \) algebras satisfying the axioms in \( \Sigma \); likewise, by \( \text{Bi(B)}\text{BIFr}_\Sigma \) we denote the category of \( \text{Bi(B)}\text{BI} \) frames satisfying the corresponding frame properties in \( \Sigma \), where \( \text{Bi(B)}\text{BI} \) morphisms are defined by appropriately extending \( \text{(B)}\text{BI} \) morphisms. We will show that for any \( \Sigma \), a dual adjunction exists between \( \text{Bi(B)}\text{BIAlg}_\Sigma \) and \( \text{Bi(B)}\text{BIFr}_\Sigma \); that is, \( \text{Bi(B)}\text{BI} + \Sigma \) is sound and complete for its Kripke semantics.

**Definition 6.12 (Bi(B)BI Complex Algebra).** Given a \( \text{Bi(B)}\text{BI} \) frame \( \mathcal{X} \) the complex algebra of \( \mathcal{X} \), \( \text{Com}^{\text{Bi(B)}\text{BI}}(\mathcal{X}) \), is given by extending the \( \text{Com}^{\text{(B)}\text{BI}}(\mathcal{X}) \) with the following:

(a) For \( \text{BiB} \text{I} \): \( \forall_{\mathcal{X}}, \forall_{\mathcal{X}}, \perp_{\mathcal{X}} \), defined as

\[
\begin{align*}
A \forall_{\mathcal{X}} B & = \{ x \mid \text{for all } s,t,u, x \preceq s \in t \lor u \text{ implies } t \in A \text{ or } u \in B \} \\
A \forall_{\mathcal{X}} B & = \{ x \mid \text{there exists } s,t,u \text{ s.t. } x \succcurlyeq s, u \in t \lor s, u \in A \text{ and } t \notin B \} \\
\perp_{\mathcal{X}} & = \{ x \mid x \notin U \};
\end{align*}
\]

(b) For \( \text{BiBBI} \): \( \forall_{\mathcal{X}}, \forall_{\mathcal{X}}, \perp_{\mathcal{X}} \) as above, with \( \preceq \) substituted for \( = \) throughout the definitions. \( \square \)
Lemma 6.13.

(1) Given a basic Bi(B)BI frame $\mathcal{X}$, $\text{Com}^{\text{Bi}(B)\text{BI}}(\mathcal{X})$ is a basic Bi(B)BI algebra.

(2) If $\mathcal{X}$ satisfies any frame property of Figure 9, $\text{Com}^{\text{Bi}(B)\text{BI}}(\mathcal{X})$ satisfies the corresponding axiom.

Proof. (1) is straightforward. For (2) we focus on the case of weak distributivity for BiBI frames, which collapses to the BiBBI variant when $\leq$ is $\equiv$. Let $t' \in A \cdot X (B \Updownarrow X C)$. Then $t' \succ t \in x_1 \circ x_2$ for some $x_1 \in A$ and $x_2 \in B \Updownarrow X C$. Suppose $t' \succeq t'' \in y_1 \Downarrow y_2$. We must show $y_1 \in A \cdot Y B$ or $y_2 \in C$. Suppose $y_2 \notin C$. By the weak distributivity frame property, there exists $w$ such that $y_1 \in x_1 \circ w$ and $x_2 \in w \Downarrow y_2$. Since $y_2 \notin C$ and $x_2 \in B \Updownarrow X C$ it follows that $w \in B$. Hence $y_1 \in A \cdot Y B$ as required, and so $t' \in (A \cdot X B) \Updownarrow X C$. \hfill $\Box$

Definition 6.14 (Prime Filter Bi(B)BI Frame). Given a basic Bi(B)BI algebra $\mathbb{A}$, the prime filter frame of $\mathbb{A}$, $Pr^{\text{Bi}(B)\text{BI}}(\mathbb{A})$, is given by extending $Pr^{\text{Bi}(B)\text{BI}}(\mathbb{A})$ with the operation $\Downarrow_X$, defined

$$F \Downarrow_X F' = \{F'' \mid \forall a, b \in A : a \Downarrow b \in F'' \text{ implies } a \in F \text{ or } b \in F'\}$$

and the set $U_\mathbb{A} = \{F \mid \nabla F \notin F\}$. \hfill $\Box$

Lemma 6.15.

(1) Given a basic Bi(B)BI algebra $\mathbb{A}$, $Pr^{\text{Bi}(B)\text{BI}}(\mathbb{A})$ is a basic Bi(B)BI frame.

(2) If $\mathbb{A}$ satisfies any axiom of Figure 9, $Pr^{\text{Bi}(B)\text{BI}}(\mathbb{A})$ satisfies the corresponding frame property.

Proof. Once again we restrict ourselves to the non-trivial (2). We focus once more on the weak distributivity property for BiBI. Suppose $F'' \supseteq F_i \in F_{x_1} \circ_X F_{x_2}$ and $F_i \subseteq F'' \in F_{y_1} \Downarrow_X F_{y_2}$. Consider the set $f_w = \{b \mid \exists d \notin F_{y_2} (b \Downarrow d \in F_{x_2})\}$.

Note $f_w$ is a proper filter. First, it is upwards-closed since $\Downarrow$ is order preserving. To see that it is closed under meets, suppose $b, b' \in f_w$. Then there exist $d, d' \notin F_{y_2}$ such that $b \Downarrow d, b' \Downarrow d' \in F_{x_2}$. WOLOG we may assume $d = d'$. $F_{y_2}$ is prime so $d \Downarrow d' \notin F_{y_2}$, and since $\Downarrow$ is order preserving we have $b \Downarrow (d \Downarrow d'), b' \Downarrow (d \Downarrow d') \in F_{x_2}$. Now by Proposition 6.9 $(b \Downarrow b') \Downarrow d = (b \Downarrow d) \Downarrow (b' \Downarrow d) \in F_{x_2}$, so $b \Downarrow b' \in f_w$. Finally, suppose for contradiction that $\bot \in f_w$. Then there exists $d \notin F_{x_2}$ such that $\bot \Downarrow d \in F_{x_2}$. Letting $a \in F_{x_2}$ be arbitrary, by weak distributivity we have $a \Downarrow (\bot \Downarrow d) \leq (a \Downarrow \bot) \Downarrow d = \bot \Downarrow d \in F_i \subseteq F'$. Thus $\bot \Downarrow d \in F_i \subseteq F'' \Downarrow_X F_{y_2}$ but $\bot \notin F_{y_1}$ and $d \notin F_{y_2}$, contradicting that $F'' \in F_{y_1} \Downarrow_X F_{y_2}$.

Now $F_{y_1} \in F_{x_1} \circ_X f_w$; let $a \in F_{x_1}$ and $b \in f_w$. Then there exists $d \notin F_{y_2}$ such that $b \Downarrow d \in F_{x_2}$. Hence $a \Downarrow (b \Downarrow d) \leq (a \Downarrow b) \Downarrow d \in F_i \subseteq F' \Downarrow_X F_{y_2}$, and since $d \notin F_{y_2}$ it must be that $a \downarrow b \in F_{y_1}$. Also we have $F_{x_2} \in f_w \Downarrow_X F_{y_2}$: let $b \Downarrow c \in F_{x_2}$ and suppose $c \notin F_{y_2}$. Then $b \in f_w$. Using Zorn’s lemma we can extend $f_w$ to a prime filter $F_w$ with these properties, and this suffices to show that the weak distributivity frame property is satisfied. \hfill $\Box$

The remaining structure for the dual adjunction is given in the same way as it was for (B)BI. We obtain functors by setting $Pr^{\text{Bi}(B)\text{BI}}(f) = f^{-1}$ and $\text{Com}^{\text{Bi}(B)\text{BI}}(g) = g^{-1}$ and we have natural transformations $\theta_A(a) = \{F \mid a \in F\}$ and $\eta_X = \{A \in \text{Com}^{\text{Bi}(B)\text{BI}}(X) \mid x \in A\}$. That $\theta$ respects the extra operations is verified by proving the existence of appropriate prime filters using Zorn’s lemma and the properties of basic BiBBI algebras given in Proposition 6.9.

Theorem 6.16 (Soundness and Completeness for Bi(B)BI + $\Sigma$). For any collection of Bi(B)BI properties $\Sigma$, $Pr^{\text{Bi}(B)\text{BI}}$, $\text{Com}^{\text{Bi}(B)\text{BI}}$, $\theta$ and $\eta$ form a dual adjunction of categories between Bi(B)BIAlg$_\Sigma$ and Bi(B)BIFr$_\Sigma$. \hfill $\Box$
Frame: \[
\frac{\{p\}c\{q\}}{\{p \circ r\}c\{q \circ r\}}
\]

Concurrency: \[
\frac{\{p_1\}c_1\{q_1\}, \{p_2\}c_2\{q_2\}}{\{p_1 \circ p_2\}c_1 || c_2\{q_1 \circ q_2\}}
\]

Skip: \[
\frac{\{p\}skip\{p\}}{}\]

Seq: \[
\frac{\{p\}c_1\{q\}}{\{p\}c_1; c_2\{r\}}
\]

NonDet: \[
\frac{\{p\}c_1\{q\}, \{p\}c_2\{q\}}{\{p\}c_1 + c_2\{q\}}
\]

Iterate: \[
\frac{\{p\}c\{p\}}{\{p\}Iterate(c)\{p\}}
\]

Disjunction: \[
\frac{\{p_i\}c\{q\}, \text{ all } i \in I \quad \{\bigvee_{i \in I} p\}c\{q\} \quad \{\bigvee_{i \in I} q\}c\{q\}}{\quad \{\bigvee_{i \in I} q'\}c\{q'\}}
\]

Figure 10: Rules for $\text{ASL}^-$. 

We note that the completeness result for the logics $\text{BiBI} + \Sigma$ is new. Indeed, it would not be possible to adapt the argument Brotherston & Villard give for $\text{BiBBI}$ [14] directly as it relies on a translation into a Sahqlvist-axiomatized modal logic that uses Boolean negation in an essential way. There, the weak distributivity property is particularly difficult to deal with, requiring translation into a significantly more complicated frame property that is equivalent in the auxiliary modal logic. In contrast, our proof is direct, and — with the groundwork of $(B)BI$ duality done — efficient.

6.3. Concurrent Kleene Algebra and Concurrent Separation Logic. We conclude with a tentative application of our framework to Concurrent Separation Logic (CSL) [9, 58]. Without an account of the semantics of the programming language it is not immediately obvious how our duality theoretic approach can be extended to CSL, which requires strictly more structure than just the heap model of $\text{FOBBI}$. In any case, algebraic models of a basic version of CSL, $\text{ASL}^-$, have been given that connect the logic to concurrent Kleene algebra [59]. The proof rules for $\text{ASL}^-$ are given in Figure 10.

Definition 6.17 (Concurrent Kleene Algebra (cf. [59])).

(1) A concurrent monoid $(M, \leq, ||, ;, \text{skip})$ is a partial order $(M, \leq)$, together with two monoids $(M, ||, \text{skip})$ (with $||$ commutative) and $(M, ;, \text{skip})$ satisfying the exchange law

\[
(p || r); (q || s) \leq (p; q) || (r; s).
\]

It is complete if $(M, \leq)$ is a complete lattice.

(2) A concurrent Kleene algebra (CKA) is a complete concurrent monoid where $||$ and $;$ preserve joins in both arguments.

(3) A weak CKA is a complete concurrent monoid together with a subset $A \subseteq M$ (the assertions of the algebra) such that i) $\text{skip} \in A$; ii) $A$ is closed under $||$ and all joins; iii) $||$ restricted to $A$ preserves all joins in both arguments; iv) for each $a \in A$, $a; (-) : M \to M$ preserves all joins; and v) for each $m \in M$, $(-); m : A \to M$ preserves all joins.

(4) A CKA or weak CKA is Boolean if the underlying lattice is a Boolean algebra and intuitionistic if the underlying lattice is a Heyting algebra. 

\[\square\]
O’Hearn et al. show that \textbf{ASL} is sound and complete for weak CKAs when Hoare triples \{p\}$c\{q\}$ are interpreted as inequalities \(p; c \leq q\) (where \(p, q \in A\) and \(c \in M\)) and \(*\) is interpreted as \(\parallel\) restricted to \(A\). This is achieved via the construction of a predicate transformer model over \textbf{ASL} propositions. They also show that a trace model of \textbf{ASL} generates a Boolean CKA.

Elsewhere, O’Hearn \cite{OHearn} suggests that the structures involved could be used as inspiration for a bunched logic extending \textbf{BBI}. We define such a logic and call it \textit{Concurrent Kleene BI} or \textbf{CKBI}. We leave the apparent intuitionistic variant extending \textbf{BI} to another occasion. The syntax and Hilbert-style proof theory of the logic is specified by the following CKA-like algebra.

\begin{definition}[\textbf{CKBI Algebra}] A \textbf{CKBI} algebra is an algebra
\[ A = (A, \land, \lor, \neg, \top, \bot, \parallel, \; ; \neg, ;, \neg, \text{skip}) \]
such that \((A, \land, \lor, \neg, \top, \bot, \parallel, \; ; \neg, ;, \neg, \text{skip})\) is a \textbf{BBI} algebra, \((A, ;, \text{skip})\) is a monoid, \(\parallel\) and \(;\) satisfy the exchange law and for all \(a, b, c \in A\), \(a; b \leq c\) iff \(a \leq b; c\) iff \(b \leq a; c\). \hfill \Box
\end{definition}

A \textbf{CKBI} algebra is essentially a Boolean CKA extended with the multiplicative implications associated with \(\parallel\) and \(;\). The properties of Proposition 3.2 ensure that \(\parallel\) and \(;\) are join preserving, and the properties of a \textbf{BBI} algebra make \((A, \parallel, \text{skip})\) a commutative monoid. One difference is we do not require \textbf{CKBI} algebras to be complete. This is mainly for the sake of uniformity, though we note that — by the apparent representation theorem for \textbf{CKBI} algebras — every \textbf{CKBI} algebra can be embedded in a complete \textbf{CKBI} algebra.

Kripke models for \textbf{CKBI} are given on extended \textbf{BBI} frames that satisfy a first-order correspondent to the exchange law, enforcing a relationship between the parallel and sequential compositions.

\begin{definition}[\textbf{CKBI Frame}] A \textbf{CKBI} frame \(X\) is a tuple \(X = (X, \circ, \triangleright, E)\) such that
\begin{enumerate}
  \item \((X, \circ, E)\) is a \textbf{BBI} frame,
  \item \((X, \triangleright, E)\) is a non-commutative \textbf{BBI} frame; that is, it satisfies
    \begin{enumerate}
      \item \textbf{(Assoc)} \(\exists t(t \in x \circ y \land w \in t \circ z) \leftrightarrow \exists s(s \in y \circ z \land w \in x \circ s)\)
      \item \textbf{(Contr)} \(x \in (y \circ e) \cup (e \circ y) \land e \in E \rightarrow y = x\)
      \item \textbf{(Weak)} \(\exists e \in E(x \in (x \circ e) \land (e \circ x))\), and
    \end{enumerate}
  \item \(\circ\) and \(\triangleright\) satisfy
    \begin{enumerate}
      \item \textbf{(Ex)} \(wy \in w \circ y \land xz \in x \circ z \land t \in wy \triangleright xz \rightarrow \exists wx, yz(wx \in w \triangleright x \land yz \in y \triangleright z \land t \in wx \circ yz)\). \hfill \Box
    \end{enumerate}
\end{enumerate}
\end{definition}

The traces model of \textbf{ASL} can be seen as (the complex algebra of) a \textbf{CKBI} frame, where \(\circ\) is interleaving, \(\triangleright\) is concatenation and \(E\) is the singleton set containing the empty trace. Another example is given by pomsets \cite{O'Hearn}, with \(\circ\) given by the parallel pomset composition, \(\triangleright\) the series pomset composition, and \(E\) the singleton set containing the empty pomset.

The semantics of \textbf{CKBI} on \textbf{CKBI} frames straightforwardly extends that of \textbf{BBI} by those in Figure 11, with the clauses for \(;\), \(\neg;\); and \(\neg\) following those for \textbf{LGL}'s multiplicatives.

It is straightforward to adapt the results of Section 4 to these structures. The main additional verification is the correspondence between the frame condition (Ex) and the exchange law. Let \(A\) be a \textbf{CKBI} algebra. Then the prime filter frame of \(A\), \(P_{\text{CKBI}}(A)\), is given by extending the prime filter frame of the underlying \textbf{BBI} algebra with the operation \(\triangleright\), defined \(F_{\triangleright} = \{F'' | \forall a \in F, \forall b \in F : a; b \in F''\}\). In the other direction, the complex
algebra of a CKBI frame $\mathcal{X}$, $Com^{\text{CKBI}}(\mathcal{X})$, is given by extending the complex algebra of the underlying BBI frame with the operation $A \bowtie_{\mathcal{X}} B = \{ z \mid \exists x \in A, y \in B (z \in x \bowtie y) \}$ and its associated adjoints.

Lemma 6.20.  

1. Given a CKBI algebra $\mathbb{A}$, the prime filter frame $P_{\text{CKBI}}(\mathbb{A})$ is a CKBI frame.  
2. Given a CKBI frame $\mathcal{X}$, the complex algebra $Com^{\text{CKBI}}(\mathcal{X})$ is a CKBI algebra.

Proof. We concentrate on the correspondence between the exchange law and (Ex).

1. Suppose we have prime filters of $\mathbb{A}$ satisfying $F_{wy} \in F_w \circ \bigtriangleup F_y$, $F_{xz} \in F_x \circ \bigtriangleup F_z$ and $F_t \in F_{wy} \circ \bigtriangleup F_{xz}$. Consider the sets $f_{ux} = \{ c \mid \exists a \in F_w, b \in F_x (a; b \leq c) \}$ and $f_{yz} = \{ c \mid \exists a \in F_y, b \in F_y (a; b \leq c) \}$. Both sets are obviously upwards-closed and closed under meets, as order preservation of $\circ \bigtriangleup$ gives that $a; b \leq c$ and $a'; b' \leq c'$ implies $(a \wedge a'); (b \wedge b') \leq c \wedge c'$. Hence $f_{ux}$ and $f_{yz}$ are filters. They are also proper: suppose $\bot \in f_{ux}$. Then there exists $a \in F_w$ and $b \in F_x$ such that $a; b = \bot$. Let $c \in F_y$ and $d \in F_z$ be arbitrary. By assumption we have $a \parallel c \in F_{wy}$, $b \parallel d \in F_{xz}$ and so $(a \parallel c); (b \parallel d) \in F_t$. By the exchange law and upwards closure of prime filters, $(a; b) \parallel (c; d) \in F_t$: but $a; b = \bot$, so $(a; b) \parallel (c; d) = \bot \in F_t$, a contradiction.

Clearly $f_{ux} \in F_w \bowtie F_x$ and $f_{yz} \in F_y \bowtie F_z$. Further, $F_t \in f_{ux} \circ \bigtriangleup f_{yz}$: let $c \geq a; b$ and $c' \geq a'; b'$ for $a \in F_w$, $b \in F_x$, $a' \in F_y$ and $b' \in F_z$. By order preservation of $\parallel \bowtie \circ \bigtriangleup$ and the exchange law, $(a \parallel a'); (b \parallel b') \leq (a \parallel b); (b \parallel a) \leq c \bowtie c'$. Since $(a \parallel a'); (b \parallel b') \in F_t$, by upwards closure it follows that $c \bowtie c' \in F_t$. We can thus use Zorn’s lemma to obtain prime filters $F_{ux}$ and $F_{yz}$ that extend $f_{ux}$ and $f_{yz}$ and satisfy these properties.

2. Suppose $t \in (A \parallel_{\mathcal{X}} C)_{\mathcal{X}} (B \parallel_{\mathcal{X}} D)$. Then there exist $w, x, y, z, wy, xz$ such that $wy \in w \circ y$, $xz \in x \circ z$ and $t \in wy \bowtie xz$. The frame property (Ex) then ensures the existence of witnesses to the fact that $t \in (A \bowtie_{\mathcal{X}} B) \bowtie_{\mathcal{X}} (C \bowtie_{\mathcal{X}} D)$. □

Once again the dual adjunction for BBI adapts straightforwardly to this extension by making $P_{\text{CKBI}}$ and $Com^{\text{CKBI}}$ functors in the usual fashion and considering the natural transformations $\theta$ and $\eta$.

Theorem 6.21 (Soundness and Completeness of CKBI). The functors $P_{\text{CKBI}}$ and $Com^{\text{CKBI}}$ and natural transformations $\theta$ and $\eta$ form a dual adjunction between the categories $\text{CKBIAlg}$ and $\text{CKBIFr}$. □

We remain agnostic about the extent to which CKBI can be used to reason about CSL. The fact that it essentially supplies a Kripke semantics formulation of CKAs suggests that it may have uses as a logic for reasoning about concurrency more generally. We defer a thorough investigation of these ideas to another occasion.
7. Conclusions and Further Work

We have given a systematic treatment of Stone-type duality for the structures that interpret bunched logics, starting with the weakest systems, recovering the familiar BI and BBI, and concluding with both the classical and intuitionistic variants of Separation Logic. Our results encompass all the known existing algebraic approaches to Separation Logic and prove them sound with respect to the standard store-heap semantics. As corollaries, we uniformly recover soundness and completeness theorems for the systems we consider. These results are extended to the bunched logics with additional multiplicatives corresponding to negation, disjunction and falsum — DMBI, CBI and the full range of sub-classical bunched logics — as well as CKBI, a new logic inspired by the algebraic structures that interpret (a basic version of) Concurrent Separation Logic. It is straightforward to use similar methods — in combination with the analogous results from the modal logic literature — to produce the same results for modal extensions of bunched logics [26, 27, 38]. We believe this treatment will simplify completeness arguments for future bunched logics by providing a modular framework within which existing results can be extended. This is demonstrated with our results on DMBI, BiBI and CKBI. More generally, the notion of indexed frame and its associated completeness argument can easily be adapted for a wide range of non-classical predicate logics.

We identify three areas of interest for further work. First, extending our approach to account for the operational semantics of program execution given by Hoare triples. As a consequence, we aim to interpret computational approaches to the Frame Rule such as bi-abduction [15] within our semantics and investigate if algebraic or topological methods can be brought to bear on these important aspects for implementations of Separation Logic. We believe the evident extension of our framework with Brink & Rewitzky’s [8] duality-theoretic approach to Hoare logic could facilitate this. Such a set up should straightforwardly adapt to non-standard Separation Logics tailored for specific tasks (of which there are now many) which all essentially instantiate an indexed (B)BI frame structure. A related approach would be to use the dual adjunctions of the present work to generate state-and effect triangles for Separation Logic, which would enable us to investigate healthiness for pointer manipulating programs [42].

A second area of investigation concerns coalgebraic generalizations of Separation Logic. Previous work has given a sound and complete coalgebraic semantics for (B)BI [28] in line with coalgebraic generalizations of modal logic, but we are interested in another direction: the use of coalgebras as a mathematical foundation for transition systems. Separation Logic has shown the power of resource semantics for modelling real world phenomena when extended with suitable dynamics. Another approach utilizing process algebras generated by resource semantics has shown one way in which this idea can be applied to a more general class of distributed systems [2, 21]. We believe an analysis of coalgebras definable over the category of (B)BI frames would provide a general mathematical foundation for both these instances and more. A well developed line of work in coalgebraic logic has produced general machinery for generating sound and complete coalgebraic logics from dual adjunctions under favourable conditions [45, 50]. We thus believe such an analysis, combined with the dual adjunctions given in the present work, would yield a powerful framework for modelling, specifying and reasoning about resource-sensitive transition systems, with obvious wide applicability.

A third line of research would be the expansion of the results here to give a general treatment of categorical structures for non-classical predicate logics. The results given on
(B)BI hyperdoctrines straightforwardly apply to any hyperdoctrine with target algebras that have a duality theorem and include the Heyting or Boolean connectives. To what extent can this treatment generalize existing semantic approaches to non-classical predicate logics that fit this criteria? For example, the various categorical semantics given for predicate modal logic on sheaves [4], metaframes [63] and modal hyperdoctrines [7]. Our results could also be generalized in two ways. First, for hyperdoctrines with weaker-than-Heyting target algebras and their corresponding dual indexed frames, allowing us to encompass predicate substructural logics, predicate relevant logics and predicate positive logics. Second, to more exotic notions of quantification. As one example, further extending our framework to encompass the breadth of the bunched logic literature would require an account of multiplicative quantification, an area which has only partially been explored algebraically. Collinson et al. [22] define category theoretic structures that combine hyperdoctrines with monoidal categories to give semantics to a bunched polymorphic lambda calculus. There, the right adjoint of \( \mathbb{P}(\pi_{\Gamma,X}) \) must satisfy a compatibility condition with the monoidal structure in order to interpret multiplicative universal quantification in the calculus, and — with some subtle technical tweaks and additional structure — it is possible to give left adjoints that can interpret multiplicative existential quantification. Is there a unification of this approach with the present work that adapts these ideas to the more general logical setting? We believe our framework provides the mathematical foundation to explore these ideas.

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