DETECTING PERIODIC ORBITS IN SOME 3D CHAOTIC QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

TIAGO DE CARVALHO
Departamento de Matemática, Faculdade de Ciências, UNESP
Av. Eng. Luiz Edmundo Carrijo Coube 14-01, CEP 17033-360, Bauru, SP, Brazil

RODRIGO DONIZETE EÚZÉBIO*
Departamento de Matemática, IMECC–UNICAMP
R. Sérgio Buarque de Holanda, 651, CEP 13083–970, Campinas, SP, Brazil

JAUME LLIBRE
Departament de Matemàtiques, Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Catalonia, Spain

DURVAL JOSÉ TONON
Universidade Federal de Goiás, IME
CEP 74001-970, Caixa Postal 131, Goiânia, Goiás, Brazil

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Abstract. Using the averaging theory we study the periodic solutions and their linear stability of the 3–dimensional chaotic quadratic polynomial differential systems without equilibria studied in [3]. All these differential systems depend only on one–parameter.

1. Introduction. After the computation of the equilibrium points of a differential system, the more interesting orbits are the periodic ones, and the study of their stability provides information of the dynamics in their neighborhood, mainly when the system under study models a real problem coming from biology, physics, engineering, etc. But in general the study of the periodic solutions of a differential system is not an easy task because while the study of the dynamics around an equilibrium point is a local problem, the study of the dynamics around a periodic orbit is a global problem.

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* Corresponding author: Rodrigo Donizete Euzébio.
In this paper our goal is to apply the averaging theory for studying analytically the existence of periodic orbits in some families of differential systems which for some values of the parameters exhibit chaos.

More precisely, we study the existence of periodic orbits in some quadratic polynomial differential systems in $\mathbb{R}^3$ provided by Jafari, Sprott and Golpayegani in [3]. In that paper the authors exhibit a list of seventeen 3-dimensional differential systems with quadratic nonlinearities, that they denote by $NE_i$, $i = 1, \ldots, 17$, all of them depending on one parameter $a$, and they provide values of $a$ in order to obtain chaotic behavior when the corresponding system has no equilibria. Our goal is to apply the averaging theory in order to study the periodic orbits of such differential systems when the parameter $a$ is sufficiently small. In other words we provide sufficient conditions in order that some of the systems $NE_i$, $i = 1, \ldots, 17$ possess periodic motion.

We note that the averaging theory does not provide information for all the differential systems from the list of [3]. For applying the averaging theory we need:

(I) To write the differential system into the normal form of the averaging theory, which is a periodic non-autonomous differential system depending on a small parameter.

(II) To average the dominant terms with respect to the mentioned small parameter of the system written into the normal form of the averaging theory.

(III) To compute the zeros of the nonlinear system $f(y) = 0$ given by the averaged function $f(y)$ obtained in (II).

(IV) To check that the zeros of the system $f(y) = 0$ satisfy that the Jacobian of $f$ in such zeros is not zero.

For each of such zeros satisfying (IV) the original differential system has a periodic solution. For more details on the averaging theory see the appendix. So the averaging theory does not work when we cannot do some of the previous four steps. Then, for the differential systems of the list of [3] which do not appear in the following some of the mentioned steps cannot be done. In short, the systems of such a list for which this theory provides information are

\begin{align*}
\text{(NE}_1) & \quad \dot{x} = y, \\
& \quad \dot{y} = -x - yz, \\
& \quad \dot{z} = y^2 - a.
\end{align*}

\begin{align*}
\text{(NE}_2) & \quad \dot{x} = -y, \\
& \quad \dot{y} = x + z, \\
& \quad \dot{z} = 2y^2 + xz - a.
\end{align*}

\begin{align*}
\text{(NE}_3) & \quad \dot{x} = y, \\
& \quad \dot{y} = z, \\
& \quad \dot{z} = -y - \frac{1}{10} x^2 + \frac{11}{10} xz + a.
\end{align*}

\begin{align*}
\text{(NE}_4) & \quad \dot{x} = -\frac{1}{10} y + a, \\
& \quad \dot{y} = x + z, \\
& \quad \dot{z} = xz - 3y.
\end{align*}

\begin{align*}
\text{(NE}_6) & \quad \dot{x} = y, \\
& \quad \dot{y} = z, \\
& \quad \dot{z} = -y - xz - yz - a.
\end{align*}
where we consider $a = O(\varepsilon^2)$, more specifically, we take $a = \varepsilon^2 b$, with $\varepsilon$ a small parameter. Indeed, in our results we obtain periodic orbits for the systems listed previously when the small parameter $\varepsilon$ goes to zero.

Systems $(NE_1)$, $(NE_2)$ and $(NE_3)$ are modified versions of the Sprott A system, Wei system and Wang-Chen system, respectively (for a detailed analyzes see [3]). In [9] the authors verify the existence of periodic orbits when $a = 1$ coming from infinity for a modified version of $(NE_1)$. However such periodic orbit does not coincide with the periodic orbit that we have found, because our periodic orbit does not come from infinity (see Theorem 1.1). Other interesting dynamical behaviors of related differential systems with the ones studied here can be found in [2, 4, 5, 6, 8, 9].

In what follows we present our main result.

**Theorem 1.1.** Assume $a = \varepsilon^2 b > 0$. Then each one of the systems $(NE_1)$, $(NE_2)$, $(NE_3)$, $(NE_4)$, $(NE_5)$, $(NE_7)$, $(NE_8)$, $(NE_9)$ and $(NE_{11})$ presents at least one periodic solution of the form

(i) $x(t, \varepsilon) = \varepsilon \sqrt{2b} \cos t + O(\varepsilon^2)$, \quad $y(t, \varepsilon) = -\varepsilon \sqrt{2b} \sin t + O(\varepsilon^2)$, \quad $z(t, \varepsilon) = O(\varepsilon^2)$;

(ii) $x(t, \varepsilon) = \varepsilon \sqrt{b} \cos t + O(\varepsilon^2)$, \quad $y(t, \varepsilon) = \varepsilon \sqrt{b} \sin t + O(\varepsilon^2)$, \quad $z(t, \varepsilon) = O(\varepsilon^2)$;

(iii) $x(t, \varepsilon) = -\varepsilon \sqrt{2b} \cos t + O(\varepsilon^2)$, \quad $y(t, \varepsilon) = \varepsilon \sqrt{2b} \sin t + O(\varepsilon^2)$, \quad $z(t, \varepsilon) = \varepsilon \sqrt{2b} \cos t + O(\varepsilon^2)$;

(iv) $x(t, \varepsilon) = -\varepsilon \sqrt{2b} \sin \left( \sqrt{\frac{11}{10}} t \right) + O(\varepsilon^2)$, \quad $y(t, \varepsilon) = 2\varepsilon \sqrt{155} \cos \left( \sqrt{\frac{11}{10}} t \right) + O(\varepsilon^2)$, \quad $z(t, \varepsilon) = -30\varepsilon \sqrt{2b} \sin \left( \sqrt{\frac{11}{10}} t \right) + O(\varepsilon^2)$;

(vi) $x(t, \varepsilon) = \varepsilon \sqrt{2b} \cos t + O(\varepsilon^2)$, \quad $y(t, \varepsilon) = \varepsilon \sqrt{2b} \cos t + O(\varepsilon^2)$, \quad $z(t, \varepsilon) = -\varepsilon \sqrt{2b} \sin t + O(\varepsilon^2)$;

(vii) $x(t, \varepsilon) = \varepsilon \sqrt{\frac{30}{2}} \cos t + O(\varepsilon^2)$, \quad $y(t, \varepsilon) = -\varepsilon \sqrt{\frac{30}{2}} \sin t + O(\varepsilon^2)$, \quad $z(t, \varepsilon) = O(\varepsilon^2)$;

(viii) $x(t, \varepsilon) = 2\varepsilon b \cos t + O(\varepsilon^2)$, \quad $y(t, \varepsilon) = -2\varepsilon \sqrt{5} \sin t + O(\varepsilon^2)$, \quad $z(t, \varepsilon) = O(\varepsilon^2)$;
Now we write this differential system in cylindrical coordinates to zero faster than \( \varepsilon \) are obtained for the original systems (\( \text{NE} \)) we want to study into the normal form of the averaging method. The results \( \varepsilon \) parameter \( \text{NE} \) in this way we can write the original differential system whose periodic orbits \( \text{NE} \) and \( \text{NE} \) are separated by the statements of Theorem 1. We start proving statement \( \text{NE} \) of Theorem 1 in subsection 2.1 and then, in subsection 2.2, we only provide the complete proofs of statements \( \text{NE} \) and \( \text{NE} \) are unstable (saddle type).

Theorem 1.1 is proved in section 2.

**Remark 1.** It is easy to check that the periodic solutions described in Theorem 1.1 exist when the corresponding differential system have no equilibria, except for system \( \text{NE} \) and \( \text{NE} \) that when the periodic solution exists there are also equilibria.

2. Proof of Theorem 1.1. In this section we prove our main result. The proofs are separated by the statements of Theorem 1.1. We start proving statement \( \text{NE} \) of Theorem 1 in subsection 2.1 and then, in subsection 2.2, we only provide the periodic solution of items \( \text{NE} \) and \( \text{NE} \), because their proofs are analogous to the proof of statement \( \text{NE} \). The proof of statement \( \text{NE} \) is given in subsection 2.3, and we do not provide the complete proofs of statements \( \text{NE} \), \( \text{NE} \) and \( \text{NE} \) in subsection 2.4 because are similar to the proof of statement \( \text{NE} \).

Moreover, in every proof we shall consider \( a = \varepsilon^2 b \), for indicating that \( a \) goes to zero faster than \( \varepsilon \). Note that the averaging method is performed with a small parameter \( \varepsilon \). We also consider a rescale of variables using again the small parameter \( \varepsilon \), in this way we can write the original differential system whose periodic orbits we want to study into the normal form of the averaging method. The results are obtained for the original systems \( \text{NE}_k \) for \( k = 1, 2, 3, 4, 6, 7, 8, 9, 11 \). Similar rescaling were used in [1].

Now we shall prove the results.

2.1. Proof of statement \( \text{NE} \) of Theorem 1.1. Consider \( a = \varepsilon^2 b \) in system \( \text{NE}_k \). Since the main tool for proving our results is the averaging theory, we need to transform the differential system \( \text{NE}_k \) into the differential system of the normal form (16) for applying the averaging theory, see the Appendix. Thus, first we rescale the variables as follows \( (x, y, z) = (\varepsilon \bar{x}, \varepsilon \bar{y}, \varepsilon \bar{z}) \), then system \( \text{NE}_k \) becomes

\[
\begin{align*}
\dot{\bar{x}} &= \bar{y}, \\
\dot{\bar{y}} &= -\bar{x} - \varepsilon \bar{y} \bar{z}, \\
\dot{\bar{z}} &= \varepsilon \left(-b + \frac{1}{2} \bar{x}^2 + \bar{y} \bar{z}\right). 
\end{align*}
\]

(1)

Now we write this differential system in cylindrical coordinates \( \bar{x} = r \cos \theta, \bar{y} = r \sin \theta, \) and \( \bar{z} = z, \) and we get the differential system

\[
\begin{align*}
\dot{r} &= -\varepsilon rz \sin^2 \theta, \\
\dot{\theta} &= -1 - \varepsilon z \cos \theta \sin \theta, \\
\dot{z} &= -\varepsilon \frac{1}{2} (2b - r^2 \cos^2 \theta - 2r^2 \cos \theta \sin \theta). 
\end{align*}
\]

(2)

Now we take as new independent variable in the differential system (2) the variable \( \theta \) and this system can be written as

\[
\begin{align*}
r' &= \varepsilon rz \sin^2 \theta + O(\varepsilon^2), \\
z' &= \varepsilon \frac{1}{2} (2b - r^2 \cos^2 \theta - 2r^2 \cos \theta \sin \theta) + O(\varepsilon^2). 
\end{align*}
\]

(3)
Here the prime denotes derivative with respect to the variable $\theta$.

Note that the differential system (3) is written into the normal form (16) for applying the averaging theory, see for more details the Appendix. Moreover, it is non-autonomous and $2\pi$-periodic. Now, using the notation of the Appendix we have

$$x = (r, z), \quad t = \theta, \quad F_1(t, x) = F_1(\theta, r, z),$$

where

$$F_1(r, \theta, z) = \left( \begin{array}{c} F_{11}(r, \theta, z) \\ F_{12}(r, \theta, z) \end{array} \right) = \left( \begin{array}{c} rz \sin^2 \theta \\ \frac{1}{2} (2b - r^2 \cos^2 \theta - 2r^2 \cos \theta \sin \theta) \end{array} \right).$$

Now we consider the averaging function (18) of the Appendix

$$f(r, z) = \left( \begin{array}{c} f_1(r, z) \\ f_2(r, z) \end{array} \right) = \left( \begin{array}{c} \frac{1}{2\pi} \int_0^{2\pi} F_{11}(r, \theta, z) d\theta \\ \frac{1}{2\pi} \int_0^{2\pi} F_{12}(r, \theta, z) d\theta \end{array} \right) = \left( \begin{array}{c} \frac{r z}{2} \\ \frac{4b - r^2}{4} \end{array} \right).$$

The averaged function $f(r, z)$ has a unique zero with $r > 0$, namely $\tilde{z} = (2\sqrt{b}, 0)$ if $b > 0$. The Jacobian (3.1) of the function $f$ at $\tilde{z}$ is $b$.

On the other hand, the eigenvalues associated to the zero $\tilde{z}$ are $-\sqrt{-b}$ and $\sqrt{-b}$.

Therefore according to statement (b) of Theorem 3.1 the periodic solution of system (3) is linearly stable.

Now we go back through the changes of variables in order to estimate in the initial variables $(x, y, z)$ how is the periodic orbit that we have obtained if $b > 0$.

According with statement (a) of Theorem 3.1 the periodic solution of system (3) associated to the zero $\tilde{z}$ is of the form

$$r(\theta, \varepsilon) = 2\sqrt{b} + O(\varepsilon), \quad z(\theta, \varepsilon) = O(\varepsilon).$$

and in the differential system (2) becomes

$$r(t, \varepsilon) = 2\sqrt{b} + O(\varepsilon), \quad \theta(t, \varepsilon) = -t + O(\varepsilon), \quad z(t, \varepsilon) = O(\varepsilon).$$

This periodic solution in the differential system (1) writes

$$\begin{array}{l}
\pi(t, \varepsilon) = 2\sqrt{b} \cos t + O(\varepsilon),
\gamma(t, \varepsilon) = -2\sqrt{b} \sin t + O(\varepsilon),
\tau(t, \varepsilon) = O(\varepsilon).
\end{array}$$

Finally for the differential system ($NE_8$) this last periodic solution writes

$$x(t, \varepsilon) = \varepsilon 2\sqrt{b} \cos t + O(\varepsilon^2), \quad y(t, \varepsilon) = -\varepsilon 2\sqrt{b} \sin t + O(\varepsilon^2), \quad z(t, \varepsilon) = O(\varepsilon^2).$$

Therefore we have proved item (viii) of Theorem 1.1.

2.2. **Proof of statements (i) and (ix) of Theorem 1.1.** Doing exactly the same computations than in the previous subsection for the systems ($NE_1$) and ($NE_9$) we get the periodic solutions of statements (i) and (ix) of Theorem 1.1.

2.3. **Proof of statement (ii) of Theorem 1.1.** Again we take $a = \varepsilon^2 b$. The proof of this statement is slightly different from the previous proofs because we must perform a change of variables in order to write the linear part of system ($NE_2$) with $\varepsilon = 0$ in its real Jordan normal form, and another change in order to be in the assumptions for applying the averaging theory.
First, as in the proof of statement (viii), we rescale the variables as follows 
\((x,y,z) = (\varepsilon \bar{x}, \varepsilon \bar{y}, \varepsilon \bar{z})\), then system \((NE_2)\) becomes
\[
\dot{\bar{x}} = -\bar{y}, \quad \dot{\bar{y}} = \bar{x} + \bar{z}, \quad \dot{\bar{z}} = \varepsilon (2\bar{y}^2 + \bar{y} \bar{z} - b).
\]
(4)
The linear part of this system at the origin when \(\varepsilon = 0\) is
\[
M = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]
(5)
Doing the change of variables
\[
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix},
\]
we write the linear part (5) in its real Jordan normal form, and we get the differential system
\[
\dot{u} = -v + \varepsilon (-b + 2v^2 + uw - w^2), \\
\dot{v} = u, \\
\dot{w} = \varepsilon (-b + 2v^2 + uw - w^2).
\]
(6)
We write this differential system in cylindrical coordinates
\(u = r \cos \theta, \ v = r \sin \theta, \ \text{and} \ w = w\), and we get the differential system
\[
\dot{r} = \varepsilon \cos \theta (-b - w^2 + rw \cos \theta + 2r^2 \sin^2 \theta), \\
\dot{\theta} = 1 + \varepsilon \frac{\sin \theta}{r} (b - 2r^2 \sin^2 \theta + w^2 - rw \cos \theta), \\
\dot{w} = \varepsilon (-b - w^2 + rw \cos \theta + 2r^2 \sin^2 \theta).
\]
(7)
Now taking \(\theta\) as the new independent variable in the differential system it becomes
\[
r' = \varepsilon \cos \theta (-b - w^2 + rw \cos \theta + 2r^2 \sin^2 \theta) + O(\varepsilon^2), \\
w' = \varepsilon (-b - w^2 + rw \cos \theta + 2r^2 \sin^2 \theta) + O(\varepsilon^2) \\
       = \varepsilon F_{12}(\theta, r, w) + O(\varepsilon^2).
\]
(8)
Note that system (8) is written into the standard form (16) for applying the averaging method, it is a non–autonomous \(2\pi\)–periodic differential system. So, using the same notation than in the Appendix we get the averaged function (18) of the Appendix is
\[
f(r, w) = \begin{pmatrix}
f_1(r, w) \\
f_2(r, w)
\end{pmatrix} = \begin{pmatrix}
\int_0^{2\pi} F_{11}(r, \theta, w) d\theta \\
\int_0^{2\pi} F_{12}(r, \theta, w) d\theta
\end{pmatrix},
\]
where
\[f_1(r, w) = \frac{rw}{2} \quad \text{and} \quad f_2(r, w) = -b + r^2 - w^2.\]
Consequently the zero of the averaging function \(f(r, w)\) satisfying \(r > 0\) is \((\sqrt{b}, 0)\) with \(b > 0\). Since the determinant of Jacobian matrix of the function \(f\) at this
zero is \(-b \neq 0\), by Theorem 3.1 this zero provides, for \(\varepsilon \neq 0\) sufficiently small, the following periodic solution of system (8)

\[
\begin{align*}
r(\theta, \varepsilon) &= \sqrt{b} + O(\varepsilon), \\
\theta(\varepsilon) &= O(\varepsilon).
\end{align*}
\]

The periodic solution (9) in the differential system (7) becomes

\[
\begin{align*}
r(t, \varepsilon) &= \sqrt{b} + O(\varepsilon), \\
\theta(t, \varepsilon) &= t + O(\varepsilon), \\
w(t, \varepsilon) &= O(\varepsilon).
\end{align*}
\]

Consequently the periodic solution in coordinates \((u, v, w)\) for system (6) writes

\[
\begin{align*}
u(t, \varepsilon) &= \sqrt{b} \cos t + O(\varepsilon), \\
v(t, \varepsilon) &= \sqrt{b} \sin t + O(\varepsilon), \\
w(t, \varepsilon) &= O(\varepsilon).
\end{align*}
\]

In the coordinates \((x, y, z)\) of system (4) the periodic solution becomes

\[
\begin{align*}
x(t, \varepsilon) &= \varepsilon \sqrt{b} \cos t + O(\varepsilon^2), \\
y(t, \varepsilon) &= \varepsilon \sqrt{b} \sin t + O(\varepsilon^2), \\
z(t, \varepsilon) &= O(\varepsilon^2).
\end{align*}
\]

2.4. **Proof of statements (iii), (iv), (vi) and (vii) of Theorem 1.1.** Repeating the proof of statement (ii) for the systems \((NE_3), (NE_6)\) and \((NE_7)\) we get for them the periodic solutions which appear in the statements (iii), (iv), (vi) and (vii) of Theorem 1.1.

2.5. **Proof of statement (xi) of Theorem 1.1.** We consider \(a = \varepsilon^2 b\). The proof of this statement again is slightly different from the previous proofs because additionally to do a change of variables for writing the linear part of system \((NE_{11})\) with \(\varepsilon = 0\) in its real Jordan normal form, we need an additional special change of variables in order to get the normal form (16) for applying the averaging theory.

As in the previous proofs we rescale the variables as follows \((x, y, z) = (\varepsilon \overline{x}, \varepsilon \overline{y}, \varepsilon \overline{z})\), then system \((NE_{11})\) becomes

\[
\begin{align*}
\dot{\overline{x}} &= \overline{y}, \\
\dot{\overline{y}} &= -\overline{x} + \overline{z}, \\
\dot{\overline{z}} &= \overline{x} + \varepsilon \left(-2\overline{xy} + \frac{9}{5} \overline{x} \overline{z} - b\right).
\end{align*}
\]

The linear part of this system at the origin when \(\varepsilon = 0\) is

\[
M = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

Doing the change of variables

\[
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix} = \begin{pmatrix}
1 & 0 & -\frac{1}{2} \\
0 & -1 & \frac{1}{2} \\
0 & 0 & -1
\end{pmatrix} \begin{pmatrix}
\overline{x} \\
\overline{y} \\
\overline{z}
\end{pmatrix},
\]
we write the linear part (11) in its real Jordan normal form, and we get the differential system
\[
\begin{align*}
\dot{u} &= -v + \varepsilon \frac{1}{10} (5b - 10uw - 14uw + 5vw + 7w^2), \\
\dot{v} &= u - \varepsilon \frac{1}{10} (-5b + 10uw + 14uw - 5vw - 7w^2), \\
\dot{w} &= w + \varepsilon \frac{1}{5} (5b - 10uw - 14uw + 5vw + 7w^2).
\end{align*}
\] (12)

We write this differential system in cylindrical coordinates \(u = r \cos \theta, \ v = r \sin \theta,\) and \(w = w,\) and we get the differential system
\[
\begin{align*}
\dot{r} &= \varepsilon \frac{1}{10} (\sin \theta - \cos \theta) \left( -5b + 10r^2 \sin \theta \cos \theta - 5rw \sin \theta + 14rw \cos \theta - 7w^2 \right), \\
\dot{\theta} &= 1 + \varepsilon \frac{1}{10r} (\sin \theta + \cos \theta) \left( -5b + 10r^2 \sin \theta \cos \theta - 5rw \sin \theta + 14rw \cos \theta - 7w^2 \right), \\
\dot{w} &= w + \varepsilon \frac{1}{5} (5b - 10r^2 \sin \theta \cos \theta + 5rw \sin \theta - 14rw \cos \theta + 7w^2).\end{align*}
\] (13)

Now taking \(\theta\) as the new independent variable in the differential system it becomes
\[
\begin{align*}
r' &= \varepsilon \frac{1}{10} (\sin \theta - \cos \theta) \left( -5b + 10r^2 \sin \theta \cos \theta - 5rw \sin \theta + 14rw \cos \theta - 7w^2 \right) + O(\varepsilon^2), \\
w' &= w + \varepsilon \frac{1}{10r} (-5b + 10r^2 \sin \theta \cos \theta - 5rw \sin \theta + 14rw \cos \theta - 7w^2) \\
&\quad + (2r + (\cos \theta + \sin \theta)w) + O(\varepsilon^2).\end{align*}
\] (14)

Finally doing the change of variables \(R = r\) and \(W = e^\theta w\) system (14) writes
\[
\begin{align*}
R' &= \varepsilon \frac{1}{10} (\sin \theta - \cos \theta) \left( -5b + 10R^2 \sin \theta \cos \theta - 5e^\theta RW \sin \theta + 14e^\theta RW \cos \theta - 7e^{2\theta}W^2 \right) + O(\varepsilon^2) \\
&= \varepsilon F_{11}(\theta, r, w) + O(\varepsilon^2), \\
W' &= \varepsilon \frac{1}{10R} e^{-\theta} \left( 2R + e^\theta W (\cos \theta + \sin \theta) \right) (5b + 7e^{2\theta}W^2 \\
&\quad + 5e^\theta RW \sin \theta - 2R \cos \theta) (7e^\theta W + 5R \sin \theta) + O(\varepsilon^2) \\
&= \varepsilon F_{12}(\theta, r, w) + O(\varepsilon^2).\end{align*}
\] (15)

Note that system (15) is written into the standard form (16) for applying the averaging method, it is a non-autonomous \(2\pi\)-periodic differential system. So, using the same notation than in the Appendix we get that the averaged function (18) of the Appendix is \(f(R, W) = (f_1(R, W), f_2(R, W))\) where
\[
\begin{align*}
f_1(R, W) &= \frac{(e^{2\pi} - 1)W (21 (1 + e^{2\pi}) W - 71R)}{100\pi}, \\
f_2(R, W) &= \frac{e^{-2\pi} (e^{2\pi} - 1)}{100\pi R} (50bR - 20R^3 + 47e^{2\pi}RW^2 + 7 (e^{2\pi} + e^{4\pi}) W^3).\end{align*}
\]
The averaging function $f(R, W)$ has two zeros, namely
\[
\left( \sqrt{\frac{5b}{2}}, 0 \right) \quad \text{and} \quad \left( \frac{105\sqrt{b} \cosh \pi}{\sqrt{2205 \cosh(2\pi) - 253943/6}}, \frac{355 e^{-\pi \sqrt{b}}}{\sqrt{8820 \cosh(2\pi) - 507886/3}} \right),
\]
with $b > 0$. The second zero provides a periodic solution for the differential system (15), but when we go back to system (14) such solution is not more periodic due to the change of variables where appears $e^{\theta}$. But the first zero provides a periodic solution also for system (14) due to the fact that in this zero $W = 0$ and cancels the function $e^{\theta}$ which appears in the change of variables for passing from system (15) to system (14). Of course, the absolute value determinant of the Jacobian matrix of the function $f(R, W)$ at both zeros is
\[
\frac{71b \sinh^2 \frac{\pi}{25}}{25\pi^2} \neq 0.
\]
Moreover, the eigenvalues of the Jacobian matrix at the first zero are
\[
\pm e^{-\pi \sqrt{71b} (e^{2\pi} - 1)} \frac{10}{10\pi}.
\]
Then the periodic orbit associated to this zero is unstable of type saddle.

Now we go back through the changes of variables in order to estimate the coordinates of the periodic solution in the initial variables $(x, y, z)$. According to statement (a) of Theorem 3.1 the periodic solution of system (15) associated to the first zero is of the form
\[
R(\theta, \varepsilon) = \sqrt{\frac{5b}{2}} + O(\varepsilon), \quad W(\theta, \varepsilon) = O(\varepsilon).
\]
This periodic solution in system (14) writes
\[
r(\theta, \varepsilon) = \sqrt{\frac{5b}{2}} + O(\varepsilon), \quad w(\theta, \varepsilon) = O(\varepsilon),
\]
and in the differential system (13) becomes
\[
r(t, \varepsilon) = \sqrt{\frac{5b}{2}} + O(\varepsilon), \quad \theta(t, \varepsilon) = t + O(\varepsilon), \quad w(t, \varepsilon) = O(\varepsilon).
\]
This periodic solution in the differential system (12) writes
\[
\begin{align*}
u(t, \varepsilon) &= \sqrt{\frac{5b}{2}} \cos t + O(\varepsilon), \\
v(t, \varepsilon) &= \sqrt{\frac{5b}{2}} \sin t + O(\varepsilon), \\
w(t, \varepsilon) &= O(\varepsilon),
\end{align*}
\]
and in the differential system (10) becomes
\[
\begin{align*}
\pi(t, \varepsilon) &= \sqrt{\frac{5b}{2}} \cos t + O(\varepsilon), \\
\gamma(t, \varepsilon) &= -\sqrt{\frac{5b}{2}} \sin t + O(\varepsilon), \\
\pi(t, \varepsilon) &= O(\varepsilon).
\end{align*}
\]
Finally undoing the re-scaling we get the periodic solution for the differential system (NE11) which writes
\[
\begin{align*}
x(t, \varepsilon) &= \varepsilon \sqrt{\frac{5b}{2}} \cos t + O(\varepsilon^2), \\
y(t, \varepsilon) &= \varepsilon \sqrt{\frac{5b}{2}} \sin t + O(\varepsilon^2), \\
z(t, \varepsilon) &= O(\varepsilon^2).
\end{align*}
\]
3. Conclusion. By using averaging theory we have shown the existence of periodic solutions for some chaotic systems from the list presented by Jafari Sprott and Golpayegani in [3], namely, for systems \((NE_1), (NE_2), (NE_3), (NE_4), (NE_6), (NE_7), (NE_8), (NE_9)\) and \((NE_{11})\). Moreover, we have characterized the linear stability of all the periodic solutions that we found. The approach involves the replacement of the parameter \(a\) and the variables \(x, y\) and \(z\) for new ones depending on a small parameter \(\varepsilon\), and then we can apply the first order averaging method.

Appendix: The averaging theory of first order. Consider the differential system
\[
\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon), \quad x(0) = x_0
\]  
with \(x \in D\), where \(D\) is an open subset of \(\mathbb{R}^n\), \(t \geq 0\). We also assume that the functions \(F_1(t, x)\) and \(F_2(t, x, \varepsilon)\) are \(T\)-periodic in \(t\). We define in \(D\) the averaged differential system
\[
\dot{y} = \varepsilon f(y), \quad y(0) = x_0,
\]
where
\[
f(y) = \frac{1}{T} \int_0^T F_1(t, y) dt.
\]
The next result shows that under the convenient hypotheses the equilibria of the averaged system will provide \(T\)-periodic solutions of system (16).

Theorem 3.1. Consider the two initial value problems (16) and (17). Suppose that

(i) the functions \(F_1, \partial F_1/\partial x, \partial^2 F_1/\partial x^2, F_2\) and \(\partial F_2/\partial x\) are defined, continuous and bounded by a constant independent of \(\varepsilon\) in \([0, \infty) \times D\) and \(\varepsilon \in (0, \varepsilon_0]\);

(ii) the functions \(F_1\) and \(F_2\) are \(T\)-periodic in \(t\) (\(T\) independent of \(\varepsilon\)).

Then the following statements hold.

(a) If \(p\) is an equilibrium point of the averaged system (17) satisfying
\[
\det \left( \frac{\partial f}{\partial y} \right)_{y=p} \neq 0,
\]
then there is a \(T\)-periodic solution \(x(t, \varepsilon)\) of system (16) such that \(x(0, \varepsilon) \to p\) as \(\varepsilon \to 0\).

(b) The kind of linear stability or instability of the periodic solution \(x(t, \varepsilon)\) coincides with the kind of stability or instability of the equilibrium point \(p\) of the averaged system (17). The equilibrium point \(p\) has the kind of stability behavior of the Poincaré map associated to the periodic solution \(x(t, \varepsilon)\).

For a proof of Theorem 3.1, see sections 6.3 and 11.8 of Verhulst [10].

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E-mail address: tcarvalho@fc.unesp.br
E-mail address: euzebio@ime.unicamp.br
E-mail address: jllibre@mat.uab.cat
E-mail address: djtonon@ufg.br