Continuum field theory of string-like objects.
Dislocations and superconducting vortices

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To the memory of Ekkehart Kröner

Summary: Dense distributions of string-like objects in material media are considered in terms of continuum field theory. The strings are assumed to carry a quantized abelian topological charge, such as the Burgers vector of dislocations in solids, or magnetic flux of supercurrent vortices in type-II superconductors. Within this common framework the facts known from dislocation theory can be extended, in appropriately modified forms, to other physical contexts. In particular, the concept of incompatible distortions is transplanted into the theory of type-II superconductivity. The compatibility law for type-II superconductors is derived in terms of differential forms. As a result, one obtains an inhomogeneous generalization of the classic London’s equation.

Keywords: linear defects, dislocations, incompatible distortion, type-II superconductivity, magnetic flux lines

1 Introduction

The heuristics of the present paper can be conveniently developed by starting from the striking analogy between dislocations in solids and magnetic vortices in superconductors. While the former determine the behaviour of materials in plastic deformation processes, the latter are responsible for magnetic and transport phenomena in type-II superconductors. The physical contexts in the situations are indeed very different. Nevertheless, the analogy is far from being superficial. It suffices to note common fundamental features such as the distinguished linear structure, quantized dynamical quantities concentrated on and conserved along the lines, as well as large amount of irreversibility in their dynamical behaviour.

In most situations the number of vortices in a macroscopic superconducting specimen is very large, what enables an effective description of vortex networks with the aid of continuum field theories [1-3, 5–6, 8–11, 18].

In the present paper we shall concentrate our attention on the compatibility laws which govern the behaviour of the continuum fields describing the macroscopic quantities. We adopt the following terminological convention. Whenever

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more precise language concerning type II superconductivity is necessary we shall make use of the term \textit{fluxon} in place of informal terms like \textit{flux line} or \textit{magnetic vortex}. It, however, should be contrasted with the term \textit{fluxoid}, for which we adopt London’s definition [10].

Throughout the paper, a theoretical description of a string system will be termed \textit{macroscopic} if it deals with collective quantities without paying attention to individual lines. The description which refers to individual strings, but not to individual atoms, will be called \textit{mesoscopic}; the term \textit{microscopic} will be reserved for the atomistic description.

2 \textbf{Analogy with dislocation networks}

In general terms, the relation between magnetic flux lines and macroscopic magnetic properties of superconductors is similar to the relation between dislocation lines and the macroscopic mechanical properties of solids.

Later on we shall find limitations of this analogy. For the present purposes it turns out to be useful, and we shall try to get some profits from it.

The continuum theory of distributed dislocations has been developed by Kröner in his classic work [13]. The theory has been generalized to moving dislocations by Kosevitch [12]. Modern geometric formulations of continuum theory of dislocations have been given by Kröner [14, 15], Mistura [16], and Trzósowski [17]. The dynamics of networks of string-like objects has been considered recently by Rogula & Sztyren [19, 20].

A single dislocation line $L$ is characterized by its Burgers vector $b_L$, which is topologically quantized and conserved along the line. On the other hand, a single magnetic flux line is characterized by associated magnetic flux $\Phi_L$, which also is topologically quantized and conserved along the line. An immediately visible difference between the two cases is the vectorial nature of $b_L$ and the pseudoscalar nature of $\Phi_L$. This difference, however, although important in many respects, does not break the analogy.

The presence of dislocation lines in a material continuum results in a distortion field which is well defined in the region of \textquotedblleft good material\textquotedblright{} – outside the
dislocation cores. The good material is, however, multi-connected and the dislocations give rise to the distortion field which cannot be represented with the aid of a globally defined single–valued displacement field. One of the possibilities here is to make use of multi–valued displacement fields. Another possibility consists in the following. Instead of a multi–valued displacement field \( u^i \), one considers a single–valued differential form \( du^i \) defined by the relation

\[
\oint_C du^i = \dot{b}^i[C],
\]

(1)

where the right–hand side denotes the total Burgers vector of the dislocation lines encircled by the contour \( C \). The entire contour \( C \) is confined to the good material and is otherwise arbitrary.

3 Macroscopic averages in string systems

To describe effectively the material media which contain a very large amount of string–like objects one has the need for a fully macroscopic theory which operates with quantities describing large collections of such objects without paying much attention to an individual string.

3.1 String density and string current

The placement and movement of an individual line segment at the mesoscopic level can be conveniently described with the aid of the following quantities, valid generally for networks of moving lines \( L = L(t) \)

\[
\alpha^i(x \mid L) = \int_L \delta(x - x') dx'
\]

(2)

and

\[
t^k(x, t \mid L(t)) = \int_{L(t)} \delta(x - x') \epsilon_{kjl} v^j(x') dx'
\]

(3)

where \( v^j(x') \) denotes the velocity of the line element at \( x' \). Only the components of velocity perpendicular to the line element contribute to the string current (3).

Note that the above defined quantities, which may be interpreted as the oriented string density and oriented string current, depend solely on the orientation and geometry of the lines, and are independent of any physical characteristics of the strings. For the string density and string current we obtain the following balance equations

\[
\frac{\partial \alpha^i}{\partial t} + \epsilon^{ijk} t_{k,j} = 0, \quad \alpha^i_{,i} = 0.
\]

(4)
It is expedient to notice that, in the case of point-like particles, the analogue of the first of the above equations expresses the conservation of the number of particles.

For networks of dislocations in crystals and magnetic flux lines in superconductors, the related quantities are defined in the following subsections.

### 3.2 Dislocation networks

Notation: $b^i_L$ represents the Burgers vector associated with the line segment $L$.

\[
\alpha^{ij}(x \mid L) = b^i_L \alpha^j(x \mid L),
\]

\[
\alpha^{ij}_{meso}(x) = \sum_L \alpha^{ij}(x \mid L),
\]

\[
\alpha^{ij}(x) = \langle \alpha^{ij}_{meso}(x) \rangle_{av},
\]

\[
J^i_k_{\text{meso}}(x) = \sum_{L(t)} b^i_L \lambda_k(x, t \mid L(t)),
\]

\[
J^i_k(x, t) = \langle J^i_k_{\text{meso}}(x, t) \rangle_{av}.
\]

The dislocation density $\alpha^{ij}$ and the dislocation current $J^i_k$ satisfy the equations

\[
\frac{\partial}{\partial t} \alpha^{ij} + \epsilon^{ijk} \frac{\partial}{\partial x^l} J^i_k = 0, \quad \alpha^{ij} = 0
\]

which express the Burgers vector conservation laws.

### 3.3 Magnetic flux networks

Notation: $\Phi_L$ and $\Phi_0$ represent the quantized flux associated with the line segment $L$ and the elementary flux quantum, respectively. Typically $\Phi_L = \pm \Phi_0$.

\[
\Theta^i(x \mid L) = \Phi_L \alpha^i(x \mid L),
\]

\[
\Theta^i_{meso}(x) = \sum_L \Theta^i(x \mid L),
\]

\[
\Theta^i(x) = \langle \Theta^i_{meso}(x) \rangle_{av},
\]

\[
K^i_{\text{meso}}(x) = \sum_{L(t)} \Phi_L \lambda_i(x, t \mid L(t)),
\]

\[
K^i(x, t) = \langle K^i_{\text{meso}}(x, t) \rangle_{av}.
\]

The density $\Theta_i$ and the current $K_i$ satisfy the conservation equations

\[
\frac{\partial}{\partial t} \Theta^i + \epsilon^{ij} \frac{\partial}{\partial x^j} K_j = 0, \quad \Theta^i = 0
\]

4
which are the analogues of eqns. (10). The quantity $\Theta^i$ represents the contribution of the magnetic flux lines (fluxons) to the macroscopic induction field $B^i$; therefore it will be called the (oriented) fluxon density. The quantity $K_j$ represents the macroscopic fluxon current density resulting from the motion of fluxons. On the other hand, the flux current produces some electric field, coupled to the moving magnetic flux through Maxwell equations. In consequence, we arrive at the following interpretation

$$B^{str} = \Theta, \quad E^{str} = \frac{1}{c}K,$$

where $B^{str}$ and $E^{str}$ stand for the contribution of strings to the electromagnetic fields.

Several approaches to the continuum description of superconductors containing dense distributions of vortex lines are known from the literature; see e.g. London & London [9, 10], Laue [8], Zhou [11], Abrikosov [1, 2], Anthony & Seeger [3], Chapman et al. [5, 6], Rogula [18]. Some of them can be derived with the aid of particular assumptions concerning the macroscopic averages. Note that, in general, due to statistical correlations between mesoscopic string configurations and their motions, the inequality

$$\langle \alpha^j v^l \rangle \neq \langle \alpha^j \rangle \langle v^l \rangle$$

(18)

holds. There are, however, important special cases, such as coherent fluxon flow, or else uncorrelated randomness of $\alpha$ and $v$, when both sides of the formula (18) are identical to a good degree of approximation, and the corresponding assumption of the vortex-density model [6] is justified. The fluxon current $K$ and the fluxon density $\Theta$ are then related through the macroscopic average of the fluxon’s velocity by the equation

$$K + v \wedge \Theta = 0,$$

valid in this special case.

4 The compatibility equations

4.1 The distortion field and dislocation density

The presence of continuously distributed dislocation lines in a material continuum causes an incompatible macroscopic distortion of the medium. The situation may be briefly sketched as follows. While the multi-valued mesoscopic displacement fields, mentioned in Section 2, are rather hard to be averaged, and would lead to ill-defined macroscopic displacement fields, the single-valued differential forms allow for unequivocal averages. Therefore, to describe incompatible distortion fields at the macroscopic level, instead of ill-defined displacement
field \( u^i \) one considers the well-definition differential form \( du^i \) undestood as the macroscopic average of the mesoscopic differential form \((1)\). The corresponding continuum field equations, expressed in terms of the material distortion \( \beta^i_j \) and the dislocation density \( \alpha^{il} \), take the form

\[
\epsilon^{lkj} \beta^i_{j,k} = \alpha^{il}, \quad \alpha^{il} = 0,
\]

and can be obtained from the expression

\[
du^i(x, t) = \beta^i_j(x, t) dx^j
\]

by taking into account the equations \((1)\) and \((7)\). Keeping that in mind, we will search for an analogue of the above relations in the framework of superconductivity.

### 4.2 The fluxoid density and superconducting vortices

Let us consider the expression

\[
j^k = \frac{e^*}{2im^*}(\bar{\psi}\psi_{,k} - \bar{\psi}_{,k}\psi) - \frac{e^{*2}}{m^*c} |\psi|^2 A_k.
\]

for the supercurrent density (for simplicity we start from an isotropic superconductor) with the following notation: \( \psi = \psi(x, t) \) represents the Ginzburg-Landau order parameter, \( \bar{\psi} \) its complex conjugate, \( A_k = A_k(x, t) \) the electromagnetic vector potential, \( e^* \) and \( m^* \) denote the electric charge and the effective mass of a Cooper pair, respectively. We substitute

\[
\psi = |\psi| z,
\]

where \(|z| = 1\) so that \( z = z(x, t) \) equals the phase factor of the order parameter field. The eqn. \((22)\) may then be rewritten as

\[
j^k = \frac{e^*}{m^*} |\psi|^2 \left( \frac{\hbar}{2t}(\bar{z}\psi_{,k} - \bar{\psi}_{,k}z) - \frac{e^*}{c} A_k \right).
\]

Now, let us introduce the following differential form:

\[
d\phi \overset{\text{def}}{=} -i\bar{z}dz = i dz\bar{z},
\]

which represents the phase differential. With the aid of this form we can write

\[
j^k dx^k = \frac{e^*}{m^*} |\psi|^2 \left( \hbar d\phi + \frac{e^*}{c} A_k dx^k \right)
\]

or, taking into account that in the superconducting region outside the vortices \( \psi \neq 0 \),

\[
-\frac{\hbar c}{e^*} d\phi = (A_k + \frac{m^*c}{e^*} |\psi|^2)dx^k.
\]
The right-hand side of the above equation represents the fluxoid density. After integration over an appropriately smooth surface $S$, we obtain

$$-\frac{\hbar c}{e^*} \oint_C d\phi = \Phi[C], \quad (28)$$

where $\Phi[C]$ equals the total fluxoid encircled by the contour $C = \partial S$,

$$\Phi[C] = \int_S \Theta^i ds_i. \quad (29)$$

In consequence, passing to the differential relation, we obtain the analogue of eqn. (20)

$$\epsilon^{ilk}(A_k + \frac{m^* c}{e^*} \frac{j_k}{|\psi|^2})_{,l} = \Theta^i, \quad \Theta^i_{,l} = 0 \quad (30)$$

or equivalently

$$B^i + \epsilon^{ilk}(\frac{m^* c}{e^* n_s} j_k)_{,l} = \Theta^i, \quad (31)$$

with $n_s = |\psi|^2$. The left-hand side of this equation corresponds to the gauge invariant Londons' equation

$$B^i + \epsilon^{ilk}(\frac{m c}{e^* n_s} j_k)_{,l} = 0, \quad (32)$$

with the charge $e$ and mass $m$ replaced by the Cooper pair charge $e^*$ and effective mass $m^*$, respectively, and with $|\psi|^2$ interpreted as $n_s$. Hence one can see that the incompatibility of the superconducting order parameter due to continuously distributed flux lines modifies Londons' relation (32) between the magnetic induction field and the supercurrent density.

Due to the electromagnetic character of the quantities involved, an electric counterpart of eqn. (31) has also to be valid. In fact, taking into account the flux flow balance (16) and making use of the Maxwell equation

$$\epsilon^{ijk} E_{k,j} + \frac{1}{c^2} \frac{\partial B^i}{\partial t} = 0, \quad (33)$$

we obtain

$$\epsilon^{ilm}(\frac{\partial}{\partial t}(\frac{m^*}{e^* n_s} j_m) + \frac{1}{c} K_m - E_m)_{,l} = 0. \quad (34)$$

The above equation can be conveniently integrated to the form

$$E_i = -\chi_{,i} + \frac{\partial}{\partial t}(\frac{m^*}{e^* n_s} j_i) + \frac{1}{c} K_i, \quad (35)$$

Note, however, that the integrated relation (35) becomes algebraically independent of the equations derived up to this point. This is due to the new gradient
term $\chi_i$ which depends on specific features of the configuration under consideration.

The equations given so far in the present section are valid literally for isotropic superconductors. The generalization to anisotropic superconductors can be, however, performed in a straightforward manner by applying the routine procedure based on substitution of the effective mass tensor $m_{ij}^*$ in place of the scalar mass $m^*$. The tensor $m_{ij}^*$ is real and symmetric by definition, and positive-definite by assumption; in particular, it has a well defined inverse which can be substituted to eqn. (22). As a result, the equations (31) take the form

$$B_i + \frac{4\pi}{c} \epsilon^{ij} (\lambda^2_{jk})_{,i} = \Theta_i,$$  \hspace{1cm} (36)

and

$$E_i = -\chi_i + \frac{4\pi}{c} \frac{\partial}{\partial t} (\lambda^2_{jk})_{,i} + \frac{1}{c} K_i.$$ \hspace{1cm} (37)

The tensor

$$\lambda^2_{jk} = \frac{c^2}{4\pi \kappa^2_n} m_{jk}^*$$ \hspace{1cm} (38)

generalizes the (squared) Londons' penetration depth $\lambda^2_L$.

5 Conclusion

The heuristic formulation of the analogy between dislocations in solids and flux lines in superconductors can now be stated in a more precise way: the diagram given in the Introduction should be complemented by the following scheme:

- distortion field → fluxoid density
- dislocation density → fluxon density
- dislocation current → fluxon current.

Due to the presence of extra quantities – the fluxon density and the fluxon current – the above given macroscopic equations, even when augmented with material constitutive relations, are incomplete. They can be completed by a variety of mathematical models stated in terms of the fields $\Theta_i$ and $K_i$ and taking into account the string kinetics. In this way one can, for instance, reproduce the behaviour of superconductors described by critical state models, such as
Bean’s [4] or Kim & Stephen’s [7] ones. On the other hand, by making use of the above stated analogy, one can adapt a selection of models conceived in the theory of dislocations in order to describe the irreversible behaviour of dense string systems.

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