GLOBAL SYMMETRY-BREAKING BIFURCATIONS OF CRITICAL ORBITS OF INVARIANT FUNCTIONALS

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Abstract. In this article we present a method of study of a global symmetry-breaking bifurcation of critical orbits of invariant functionals. As a topological tool we use the degree for equivariant gradient maps. We underline that many known results on bifurcations of non-radial solutions of elliptic PDE’s from the families of radial ones are consequences of our theory.

1. Introduction. Let us consider the following equation

$$\begin{cases} -\Delta u = f(u, \lambda) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a ball or annulus and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The phenomenon of symmetry-breaking of solutions of problem (1.1) has been studied by many authors under various assumptions on $f$, see for instance [4]-[7], [15]-[20], [22]-[24], [26, 29], [27]-[35]. Of course this list is far from being complete. At the beginning the authors prove the existence of radially non-degenerate families of radial solutions of problem (1.1). The radial non-degeneracy excludes a bifurcation of radial solutions of problem (1.1) from these families. Next they show a change of the Morse index computed along these families. After restriction to the subspace fixed by the group $O(n-1)$ they obtain a change of the Morse index by an odd number, which implies a change of the Leray-Schauder degree. Finally a change of the Leray-Schauder degree implies a global bifurcation of non-radial solutions of problem (1.1). Another possible approach is to apply the Crandall-Rabinowitz bifurcation theorem or reduce the problem to a finite-dimensional one and apply...
elements of equivariant bifurcation theory like equivariant Conley index, see [32, 33], or equivariant branching lemma, see [16].

In this article we develop another approach. Namely, we consider a family of $G$-invariant $C^2$-functionals $\Phi : H \times \mathbb{R} \to \mathbb{R}$, where $G$ is a compact Lie group and $H$ is an orthogonal representation of $G$. Assuming that $\{0\} \times \mathbb{R} \subset (\nabla u \Phi)^{-1}(0)$ we study bifurcations of solutions of equation

$$\nabla u \Phi(u, \lambda) = 0, \quad (1.2)$$

from the line of trivial solutions $\{0\} \times \mathbb{R}$, whose isotropy group is different from $G$. To prove our abstract results we use the degree for equivariant gradient maps, see [3]–[2], [11, 30] i.e. a topological invariant which is suitable for study of homotopy classes of equivariant gradient maps.

To study bifurcations of non-radial solutions of problem (1.1) from the family of radial ones in our abstract approach we put $H = H^1_0(\Omega)$ and $G = \text{SO}(n)$. Problem (1.1) is $\text{SO}(n)$-invariant and possesses, under some growth conditions, variational structure i.e. its solutions one can consider as critical $\text{SO}(n)$-orbits of an $\text{SO}(n)$-invariant functional $\Phi : H^1_0(\Omega) \times \mathbb{R} \to \mathbb{R}$. Note that radial solutions of problem (1.1) correspond to the solutions of equation (1.2) whose isotropy group equal $\text{SO}(n)$ i.e. solutions fixed by the group $\text{SO}(n)$.

Our main contribution in this article are abstract symmetry-breaking bifurcation theorems for solutions of equation (1.2). We underline that we prove simultaneously the existence of symmetry-breaking and global bifurcation phenomenon of solutions of problem (1.2).

We claim that most of results on bifurcations of non-radial solutions of elliptic PDE’s from families of radial ones proved in the cited articles follow from Theorem 3.3, see the discussion in the last section.

After this introduction our article is organized as follows.

In Section 2 we derive formulas for indices of isolated critical points of invariant functionals. Namely, we consider a $G$-invariant functional $\Phi : \mathbb{H} \to \mathbb{R}$ of the class $C^2$, where $\mathbb{H}$ is a separable Hilbert space which is an orthogonal representation of a compact Lie group $G$. We assume that $0 \in \mathbb{H}$ is an isolated critical point of $\Phi$ (we allow also degenerate critical points!) and compute an index of this point in terms of the degree for $G$-equivariant gradient maps $\nabla_{G-\text{deg}((\cdot, \cdot))} \in U(G)$, see [3, 11, 14, 30] for a definition and properties of this degree. In other words we compute $\nabla_{G-\text{deg}((\nabla \Phi, B_\alpha(\mathbb{H})))} \in U(G)$, where $\alpha > 0$ is sufficiently small, $B_\alpha(\mathbb{H})$ is an open ball in $\mathbb{H}$ of radius $\alpha$ centered at $0 \in \mathbb{H}$ and $(U(G), +, \ast)$ is the Euler ring of $G$ with the unit denoted by $I$, see [8, 9] for a definition and properties of $U(G)$. The main results of this section are Theorems 2.1, 2.3 and Corollaries 1, 2. In Theorem 2.1 we assume that the second derivative $\nabla^2 \Phi(0)$ is an isomorphism. Whereas in Theorem 2.3 and Corollaries 1, 2 we consider the degenerate case. We apply these theorems in the next section to prove sufficient conditions for the existence of a global symmetry-breaking bifurcation of solutions of problem (1.2).

The main result of Section 3 is Theorem 3.3. To prove this theorem we combine Theorem 3.2 with some results of Section 2. Theorem 3.3 is very important from the point of view of applications. We underline that assumptions of this theorem are relatively easy to verify i.e. this verification is reduced to some reasonings in representation theory of $G$. Moreover, since $G$ is connected, we have reduced the computations in the Euler ring $U(G)$ to equivalent but much simpler computations in the Euler ring $U(T)$, where $T \subset G$ is a maximal torus, see Remark 2. These
In this section we compute indices of isolated critical points of $G$-invariant functionals in terms of the degree for equivariant gradient maps. Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space, which is an orthogonal representation of a compact Lie group $G$. Fix $u_0 \in \mathbb{H}$ and define an isotropy group $G_{u_0}$ of $u_0$ by $G_{u_0} = \{ g \in G : gu_0 = u_0 \}$ and the orbit $G(u_0)$ through $u_0$ by $G(u_0) = \{ gu : g \in G \}$. Set $\mathbb{H}_1^G = \{ u \in \mathbb{H}_1 : G = G_k \} = \{ u \in \mathbb{H}_1 : gu = u \forall g \in G \} = \{ u \in \mathbb{H} : \langle u, v \rangle = 0 \forall v \in \mathbb{H}_1 \}$ for subrepresentations $\mathbb{H}_0 \subset \mathbb{H}_1 \subset \mathbb{H}$.

Two finite-dimensional orthogonal $G$-representations $\mathcal{V}, \mathcal{V}'$ are said to be equivalent if there is a $G$-equivariant linear isomorphism $Q : \mathcal{V} \to \mathcal{V}'$. A subset $\Omega \subset \mathbb{H}$ is called $G$-invariant if condition $u_0 \in \Omega$ implies $G(u_0) \subset \Omega$. Let $D(\mathcal{V})$ denote the closed disc of radius 1 in $\mathcal{V}$ and $S(\mathcal{V}) = \partial D(\mathcal{V})$. Set $S' = D(\mathcal{V})/S(\mathcal{V})$. Denote by $C^k_G(\mathbb{H}, \mathbb{R}), k \in \mathbb{N}$, the space of $G$-invariant functionals of the class $C^k$ i.e. functionals $\Phi : \mathbb{H} \to \mathbb{R}$ of the class $C^k$ satisfying condition $\Phi(gu) = \Phi(u)$ for every $g \in G$ and $u \in \mathbb{H}$. Moreover, let $C^k_G(\mathbb{H}, \mathbb{R})$ denote the space of $G$-equivariant operators of the class $C^{k-1}$ i.e. operators $\Psi : \mathbb{H} \to \mathbb{H}$ of the class $C^{k-1}$ such that $\Psi(gu) = g\Psi(u)$ for every $g \in G$ and $u \in \mathbb{H}$. It is a known fact that if $\Phi \in C^k_G(\mathbb{H}, \mathbb{R})$ then $\nabla \Phi = C^k_G(\mathbb{H}, \mathbb{R})$, where $\nabla \Phi$ is the gradient of $\Phi$. It is clear that if $\nabla \Phi(0) = 0$ then the gradient $\nabla \Phi$ vanishes on the orbit $G(u_0)$ i.e. $G(u_0) \subset (\nabla \Phi)^{-1}(0)$. In this article $\deg_{\mathcal{V}B}, \deg_{\mathcal{V}LS}$ stand for the Brouwer and the Leray-Schauder degree, respectively. Let $(U(G), +, \ast)$ be the Euler ring of $G$ with the unit denoted by $\mathbb{1}$, see [8, 9] for a definition and properties of the Euler ring of a compact Lie group $G$.

Denote by $C^k_G(\mathbb{H}, \mathbb{R})$ the set of functionals $\Phi \in C^k_G(\mathbb{H}, \mathbb{R})$ whose gradient $\nabla \Phi \in C^k_G(\mathbb{H}, \mathbb{R})$ satisfies the following assumptions:

(a1) $\nabla \Phi = I_d - \nabla \eta$, where $\nabla \eta : \mathbb{H} \to \mathbb{H}$ is a compact $G$-equivariant operator,

(a2) $\nabla \Phi(0) = 0$.

Fix $\Phi \in C^k_G(\mathbb{H}, \mathbb{R})$ satisfying assumption (a1) and choose an open bounded and $G$-invariant subset $\Omega \subset \mathbb{H}$ such that $(\nabla \Phi)^{-1}(0) \cap \partial \Omega = \emptyset$. Let $\nabla_G(\deg(\nabla \Phi, \Omega)) \in U(G)$ be an infinite-dimensional generalization of the degree for $G$-equivariant gradient maps due to G&ebaa, see [11, 14, 21, 30].

In this section we compute the index of an isolated critical point $0 \in \mathbb{H}$ of a functional $\Phi \in C^2_G(\mathbb{H}, \mathbb{R})$ using the degree for $G$-equivariant gradient maps. In other words we compute $\nabla_G(\deg(\nabla \Phi, B_\alpha(\mathbb{H}))) \in U(G)$, where $B_\alpha(\mathbb{H}) = \{ u \in \mathbb{H} : \| u \| < \alpha \}$, $D_\alpha = cl(B_\alpha(\mathbb{H}))$ and $(\nabla \Phi)^{-1}(0) \cap D_\alpha(\mathbb{H}) = \emptyset$.

More precisely, we consider the following four cases

1. $\nabla^2 \Phi(0)$ is an isomorphism, see Theorem 2.1,
2. $\nabla^2 \Phi(0)$ is not an isomorphism, see Theorem 2.3,
3. $\nabla^2 \Phi(0)$ is not an isomorphism and $\ker \nabla^2 \Phi(0) \subset \mathbb{H}^G$, see Corollary 1,
4. $\nabla^2 \Phi(0)$ is not an isomorphism and $\ker \nabla^2 \Phi(0) \cap \mathbb{H}^G = \emptyset$, see Corollary 2.
Throughout this section we denote by $\mathbb{H}^-$ the maximal subrepresentation of $\mathbb{H}$ on which the $G$-equivariant self-adjoint operator $\nabla^2 \Phi(0) = \text{Id} - \nabla^2 \eta(0)$ is negatively defined i.e. $\mathbb{H}^- = \bigoplus_{\lambda_i \in \sigma(\nabla^2 \eta(0)) \cap (1, +\infty)} \mathbb{H}(\lambda_i)$, where $\sigma(\nabla^2 \eta(0))$ is the spectrum of the self-adjoint, compact and $G$-equivariant operator $\nabla^2 \eta(0)$ and $\mathbb{H}(\lambda_i)$ is the eigenspace of $\nabla^2 \eta(0)$ corresponding to the eigenvalue $\lambda_i \in \sigma(\nabla^2 \eta(0))$. We point out that $\mathbb{H}(\lambda_i)$ is a finite-dimensional orthogonal representation of $G$. Set $\mathbb{H}_0 = \ker \nabla^2 \Phi(0) = \mathbb{H}(1)$, $\mathbb{H}_1 = \text{im} \nabla^2 \Phi(0)$ and $\mathbb{H}^+ = \text{cl}(\bigoplus_{\lambda_i \in \sigma(\nabla^2 \eta(0)) \cap (-\infty, +1)} \mathbb{H}(\lambda_i))$. It is clear that $\mathbb{H}_0, \mathbb{H}_1, \mathbb{H}^-, \mathbb{H}^+ \subset \mathbb{H}$ are orthogonal subrepresentations of $G$, $\dim \mathbb{H}_0, \dim \mathbb{H}^- < \infty$, and $\dim \mathbb{H}^+ = \infty$. Since $\nabla^2 \Phi(0)$ is self-adjoint, $\mathbb{H} = \mathbb{H}_0 \oplus \mathbb{H}_1 = \mathbb{H}_0 \oplus (\mathbb{H}^- \oplus \mathbb{H}^+)$. To simplify notations set $L = \nabla^2 \Phi(0)|_{\text{im} \nabla^2 \Phi(0)}$ and $L^\pm = L_{|\mathbb{H}^\pm}$.

In the theorem below we consider the simplest case. Namely, we compute the index of a non-degenerate critical point of $\Phi \in C^2_G(\mathbb{H}, \mathbb{R})$.

**Theorem 2.1.** Fix $\Phi \in C^2_G(\mathbb{H}, \mathbb{R})$ such that $0 \in \mathbb{H}$ is a non-degenerate critical point of $\Phi$. Then $\nabla_G\text{-deg}(\nabla \Phi, B_\alpha(\mathbb{H})) = \nabla_G\text{-deg}(-\text{Id}, B_\alpha(\mathbb{H}^-)) \in U(G)$ for $\alpha > 0$ sufficiently small. If moreover $\nabla^2 \Phi(0)$ is positively defined, then $\nabla_G\text{-deg}(\nabla \Phi, B_\alpha(\mathbb{H})) = I \in U(G)$.

**Proof.** Since $\nabla^2 \Phi(0)$ is an isomorphism, $\mathbb{H} = \mathbb{H}^- \oplus \mathbb{H}^+$ and $0 \in \mathbb{H}$ is an isolated critical point of $\Phi$ i.e. there is $\alpha > 0$ such that $(\nabla \Phi)^{-1}(0) \cap D_\alpha(\mathbb{H}) = \{0\}$. Consequently, taking into account the properties of the degree for equivariant gradient maps, see [11, 14, 21, 30, 31], we express the degree $\nabla_G\text{-deg}(\nabla \Phi, B_\alpha(\mathbb{H})) \in U(G)$ as a degree of a finite-dimensional map. To be more precise, applying the homotopy invariance of the degree we obtain

$$
\nabla_G\text{-deg}(\nabla \Phi, B_\alpha(\mathbb{H})) = \nabla_G\text{-deg}(\nabla^2 \Phi(0), B_\alpha(\mathbb{H})) =
$$

$$
= \nabla_G\text{-deg}((L^-, L^+), B_\alpha(\mathbb{H}^-) \times B_\alpha(\mathbb{H}^+)) =
$$

$$
= \nabla_G\text{-deg}((-\text{Id}, \text{Id}), B_\alpha(\mathbb{H}^-) \times B_\alpha(\mathbb{H}^+)) = (I).
$$

Next using the equality $\nabla_G\text{-deg}(\text{Id}, B_\alpha(\mathbb{H}^+)) = I \in U(G)$ and the product formula for the degree we obtain

$$(I) = \nabla_G\text{-deg}(-\text{Id}, B_\alpha(\mathbb{H}^-)) \ast \nabla_G\text{-deg}(\text{Id}, B_\alpha(\mathbb{H}^+)) = \nabla_G\text{-deg}(-\text{Id}, B_\alpha(\mathbb{H}^-)).$$

If additionally $L$ is positively definite then $\mathbb{H} = \mathbb{H}^+, L = L^+ = \nabla^2 \Phi(0)$ and consequently

$$
\nabla_G\text{-deg}(\nabla \Phi, B_\alpha(\mathbb{H})) = \nabla_G\text{-deg}(\nabla^2 \Phi(0), B_\alpha(\mathbb{H})) =
$$

$$
= \nabla_G\text{-deg}(L^+, B_\alpha(\mathbb{H}^+)) = \nabla_G\text{-deg}(\text{Id}, B_\alpha(\mathbb{H}^+)) = I \in U(G),
$$

which completes the proof. \qed

From now on we consider a degenerate isolated critical point $0 \in \mathbb{H}$ of the functional $\Phi \in C^2_G(\mathbb{H}, \mathbb{R})$.

Note that $(\nabla \Phi)^G = (\nabla(\Phi|_{\mathbb{H}^G})) \in C^1(\mathbb{H}^G, \mathbb{H}^G)$ and put $L = (L^G, L^+) : H^G_1 \oplus H^G_2 \to H^G_1 \oplus (H_1 \oplus H^G_1)$.

The principal significance of the lemma below is that it allows one to choose, in the homotopy class of equivariant gradient maps of $\nabla \Phi$, a product map whose degree is relatively easy to compute. We will use this lemma in the proofs of all the theorems of this subsection. A proof of this lemma one can find in [10].
Lemma 2.2. Let $\Phi \in C^2_G(\mathbb{H}, \mathbb{R})$ be such that $0 \in \mathbb{H}$ is an isolated degenerate critical point of $\Phi$. Then there are $\alpha > 0$, a potential $\phi \in C^1_G(\ker \nabla^2 \Phi(0), \mathbb{R})$ and a family of functionals $\Phi_t \in C^2_G(\mathbb{H}, \mathbb{R}), t \in [0, 1]$, such that

1. $(\nabla \Phi_t)^{-1}(0) \cap D_\alpha(\mathbb{H}) = \{0\}$ for every $t \in [0, 1]$.
2. $\nabla \Phi_0 = \nabla \Phi$.
3. $\nabla \Phi_1 = (\nabla \phi, \nabla^2 \Phi(0) \mid_{\ker \nabla^2 \Phi(0)}) : D_\alpha(\ker \nabla^2 \Phi(0)) \times D_\alpha(\im \nabla^2 \Phi(0)) \to \ker \nabla^2 \Phi(0) \oplus \im \nabla^2 \Phi(0)$.

In the theorem below we compute the index of a degenerate isolated critical point of a $G$-invariant functional $\Phi \in C^2_G(\mathbb{H}, \mathbb{R})$, using the degree for $G$-equivariant gradient maps. This is the most general theorem of this section i.e. we do not assume any restrictions on the location of $\ker \nabla^2 \Phi(0)$ with respect to the set of fixed points $\mathbb{H}^G$.

Theorem 2.3. Fix $\Phi \in C^2_G(\mathbb{H}, \mathbb{R})$ such that $0 \in \mathbb{H}$ is an isolated degenerate critical point of $\Phi$. Then there is $\alpha > 0$ and a potential $\phi \in C^1_G(\ker \nabla^2 \Phi(0), \mathbb{R})$ such that

$$\nabla G, -\text{deg}(\nabla \phi, B_\alpha(\mathbb{H})) = \nabla G, -\text{deg}(\ker \nabla^2 \Phi(0)) \ast \nabla G, -\text{deg}(-\text{Id}, B_\alpha(\mathbb{H}^+))$$

If moreover $\nabla^2 \Phi(0) \mid_{\ker \nabla^2 \Phi(0)}$ is positively defined then

$$\nabla G, -\text{deg}(\nabla \phi, B_\alpha(\mathbb{H})) = \nabla G, -\text{deg}(\ker \nabla^2 \Phi(0)) \mid U(G).$$

Proof. Combining the Splitting Lemma 2.2 with the product formula for the degree, see [30], we obtain the following

$$\nabla G, -\text{deg}(\nabla \phi, B_\alpha(\mathbb{H})) = \nabla G, -\text{deg}(\ker \nabla^2 \Phi(0)) \ast \nabla G, -\text{deg}(-\text{Id}, B_\alpha(\mathbb{H}^+)).$$

Finally, by equality $\nabla G, -\text{deg}(\text{Id}, B_\alpha(\mathbb{H}^+)) = \mathbb{I} \in U(G)$ we obtain

$$(I) = \nabla G, -\text{deg}(\ker \nabla^2 \Phi(0)) \ast \nabla G, -\text{deg}(-\text{Id}, B_\alpha(\mathbb{H}^+)).$$

If additionally $L$ is positively defined then $\mathbb{H}^1 = \mathbb{H}^+$ and $L = L^+$. Consequently, once more applying the equality $\nabla G, -\text{deg}(\text{Id}, B_\alpha(\mathbb{H}^+)) = \mathbb{I} \in U(G)$ we obtain that

$$(I) = \nabla G, -\text{deg}(\ker \nabla^2 \Phi(0)) \ast \nabla G, -\text{deg}(-\text{Id}, B_\alpha(\mathbb{H}^+))) = \nabla G, -\text{deg}(\ker \nabla^2 \Phi(0), \mathbb{H}^+)) = \nabla G, -\text{deg}(\ker \nabla^2 \Phi(0), \mathbb{H}^+),$$

which completes the proof. \qed

From now on deg stands for the Brouwer degree if $\dim \mathbb{H}^G < \infty$ and for the Leray-Schauder degree otherwise. In the corollary below we assume that $\ker \nabla^2 \Phi(0)$ consists only of fixed points of the action of the group $G$. This assumption allows us to express the index of the critical point $0 \in \mathbb{H}$ of the functional $\Phi$ in terms of the degrees of linear maps.

Corollary 1. Under the assumptions of Theorem 2.3, if moreover $\ker \nabla^2 \Phi(0) \subset \mathbb{H}^G$ then

$$\nabla G, -\text{deg}(\nabla \phi, B_\alpha(\mathbb{H})) = \text{deg}(\nabla \phi, B_\alpha(\mathbb{H}^G), 0) \ast \nabla G, -\text{deg}(-\text{Id}, B_\alpha(\mathbb{H}^+) \cap \mathbb{H}^-)) \in U(G).$$

Moreover, if $\nabla^2 \Phi(0) \mid_{\ker \nabla^2 \Phi(0)}$ is positively defined then

$$\nabla G, -\text{deg}(\nabla \phi, B_\alpha(\mathbb{H})) = \text{deg}(\nabla \phi, B_\alpha(\mathbb{H}^G), 0) \ast \mathbb{I} \in U(G).$$
Proof. Taking into account that $\mathbb{H}_0 \subset \mathbb{H}^G$, by Lemma 2.2 and the homotopy invariance of the degree, see [30], we obtain

$$
\nabla_G\deg((\nabla \Phi, B_\alpha(\mathbb{H}))) = \nabla_G\deg((\nabla \phi, L), B_\alpha(\mathbb{H}_0) \times B_\alpha(\mathbb{H}_1)) =
\nabla_G\deg((\nabla \phi, L^G, L^\perp), B_\alpha(\mathbb{H}_0) \times B_\alpha(\mathbb{H}^G) \times B_\alpha(\mathbb{H}_1 \cap \mathbb{H}_1^\perp)) =
\nabla_G\deg((\nabla \phi, L^G, L^\perp), B_\alpha(\mathbb{H}^G) \times B_\alpha(\mathbb{H}_1 \cap \mathbb{H}_1^\perp)) = (I).
$$

Consequently by the product formula for the degree, see [30], we obtain

$$(I) = \nabla_G\deg((\nabla \phi, L^G), B_\alpha(\mathbb{H}^G)) \ast \nabla_G\deg(L^\perp, B_\alpha(\mathbb{H}_1 \cap \mathbb{H}_1^\perp)) = (II).$$

Finally, by the properties of the degree we obtain

$$(II) = \deg((\nabla \Phi)^G, B_\alpha(\mathbb{H}^G), 0) \cdot \nabla_G\deg(L^\perp, B_\alpha(\mathbb{H}_1 \cap \mathbb{H}_1^\perp)) =
\deg((\nabla \Phi)^G, B_\alpha(\mathbb{H}^G), 0) \cdot \nabla_G\deg(-\text{Id}, B_\alpha((\mathbb{H}^G)^\perp \cap \mathbb{H}^-)).$$

which completes the proof.

If $L$ is positively defined then $\mathbb{H}_1 = \mathbb{H}^+, L = L^+$ and consequently

$$(II) = \deg((\nabla \Phi)^G, B_\alpha(\mathbb{H}^G), 0) \cdot \nabla_G\deg(L^\perp, B_\alpha(\mathbb{H}_1 \cap \mathbb{H}_1^\perp)) =
\deg((\nabla \Phi)^G, B_\alpha(\mathbb{H}^G), 0) \cdot \nabla_G\deg(-\text{Id}, B_\alpha((\mathbb{H}^G)^\perp \cap \mathbb{H}^-)) = (\text{deg}((\nabla \Phi)^G, B_\alpha(\mathbb{H}^G), 0) \cdot 1).$$

\[\square\]

In the following corollary, contrary to Corollary 1, we assume that $\ker \nabla^2 \Phi(0) \setminus \{0\}$ is the fixed point free subset.

**Corollary 2.** Under the assumptions of theorem 2.3, if moreover $\ker \nabla^2 \Phi(0) \cap \mathbb{H}^G = \{0\}$ then

$$\nabla_G\deg(\nabla \Phi, B_\alpha(\mathbb{H})) =
\deg((\nabla \Phi)^G, B_\alpha(\mathbb{H}^G), 0) \cdot \nabla_G\deg(\nabla \phi, B_\alpha(\ker \nabla^2 \Phi(0))) *
\ast \nabla_G\deg(-\text{Id}, B_\alpha((\mathbb{H}^G)^\perp \cap \mathbb{H}^-)) \in U(G).$$

If moreover $\nabla^2 \Phi(0)_{|\text{im} \nabla^2 \Phi(0)}$ is positively defined then

$$\nabla_G\deg(\nabla \Phi, B_\alpha(\mathbb{H})) = \nabla_G\deg(\nabla \phi, B_\alpha(\ker \nabla^2 \Phi(0))) \in U(G).$$

**Proof.** Since $\mathbb{H}_0 \cap \mathbb{H}^G = \{0\}$, applying Lemma 2.2 and the homotopy invariance of the degree, see [30], we obtain

$$\nabla_G\deg(\nabla \Phi, B_\alpha(\mathbb{H})) = \nabla_G\deg((\nabla \phi, L), B_\alpha(\mathbb{H}_0) \times B_\alpha(\mathbb{H}_1)) =
\nabla_G\deg((\nabla \phi, (L^+, L^-)), B_\alpha(\mathbb{H}_0) \times B_\alpha(\mathbb{H}^+) \times B_\alpha(\mathbb{H}^-)) = (I).
$$

Consequently by the product formula for the degree, see [30], we obtain

$$(I) = \nabla_G\deg((\nabla \phi, B_\alpha(\mathbb{H})), 0) \ast \nabla_G\deg((\text{Id}, -\text{Id}), B_\alpha(\mathbb{H}^+) \times B_\alpha(\mathbb{H}^-)) =
\nabla_G\deg((\nabla \phi, B_\alpha(\mathbb{H})), 0) \ast \nabla_G\deg(-\text{Id}, B_\alpha(\mathbb{H}^-)) =
\nabla_G\deg((\nabla \phi, B_\alpha(\mathbb{H})), 0) \ast \nabla_G\deg(-\text{Id}, B_\alpha(\mathbb{H}^+ \cap \mathbb{H}^G)) *
\ast \nabla_G\deg(-\text{Id}, B_\alpha(\mathbb{H}^- \cap (\mathbb{H}^G)^\perp)) =
\deg((\nabla \Phi)^G, B_\alpha(\mathbb{H}^G), 0) \ast \nabla_G\deg(\nabla \phi, B_\alpha(\mathbb{H})), 0) \ast \nabla_G\deg(-\text{Id}, B_\alpha(\mathbb{H}^+ \cap (\mathbb{H}^G)^\perp))$$

which completes the proof.

If additionally $L$ is positively defined then $\mathbb{H}_1 = \mathbb{H}^+, L = L^+$ and consequently

$$\nabla_G\deg(\nabla \Phi, B_\alpha(\mathbb{H})) = \nabla_G\deg(\nabla \phi, B_\alpha(\mathbb{H})).$$

\[\square\]
3. Symmetry-breaking bifurcation of critical orbits. In this section we have proved the main abstract symmetry-breaking theorems of our article. Namely, we have formulated sufficient conditions for the existence of a global symmetry-breaking bifurcation of critical orbits of invariant functionals.

Let $H$ be an orthogonal representation of a compact Lie group $G$ and $\Omega \subset H$ be an open and $G$-invariant neighborhood of $0 \in H$. Consider an open $G$-invariant subset $\Omega \times \mathbb{R} \subset H \times \mathbb{R}$ with an action defined by $g(u, \lambda) = (gu, \lambda)$ for every $(u, \lambda) \in \Omega \times \mathbb{R}$ and $g \in G$. Fix $\lambda^\pm \in \mathbb{R} \cup \{ \pm \infty \}$ and denote by $C^k_G(\Omega \times (\lambda^-, \lambda^+), \mathbb{R})$ the class of invariant functionals $\Phi$ such that the gradient of $\Phi$ with respect to the first coordinate $\nabla_u \Phi \in C^k(\Omega \times (\lambda^-, \lambda^+), \mathbb{R})$ is of the form compact perturbation of the identity i.e. $\nabla_u \Phi = Id - \nabla_u \eta$, where $\nabla_u \eta \in C^k(\Omega \times (\lambda^-, \lambda^+), \mathbb{R})$ is compact and $\nabla_u \Phi(0, \lambda) = 0$ for all $\lambda \in (\lambda^-, \lambda^+)$. Fix $\Phi \in C^k_G(\Omega \times (\lambda^-, \lambda^+), \mathbb{R})$ such that $\nabla_u \Phi(0, \lambda) = 0$ for all $\lambda \in (\lambda^-, \lambda^+)$. Let $T = \{(0) \times (\lambda^-, \lambda^+) \subset \Omega \times (\lambda^-, \lambda^+)\}$ be called a trivial family of solutions of equation

$$\nabla_u \Phi(u, \lambda) = 0. \quad (3.1)$$

Define the set of non-trivial solutions of equation (3.1) by $\mathcal{N} = \{(u, \lambda) \in \Omega \times (\lambda^-, \lambda^+) : \nabla_u \Phi(u, \lambda) = 0 \text{ and } u \neq 0\}$ i.e. $\mathcal{N} = (\nabla_u \Phi)^{-1}(0) \setminus T$. Fix $(0, \lambda_0) \in T$ and denote by $C(\lambda_0) \subset \Omega \times (\lambda^-, \lambda^+)$ a connected component of $\text{cl}(\mathcal{N})$ such that $(0, \lambda_0) \in C(\lambda_0)$.

Let $\Phi^G = \Phi|_{\Omega^G \times (\lambda^-, \lambda^+)} \in C^1(\Omega^G \times (\lambda^-, \lambda^+), \mathbb{R})$ and consider the following equation

$$\nabla_u \Phi^G(u, \lambda) = 0. \quad (3.2)$$

Note that $T \subset \Omega^G \times (\lambda^-, \lambda^+)$ and that $(\nabla_u \Phi^G)^{-1}(0) = (\nabla_u \Phi)^{-1}(0) \cap \Omega^G$.

**Definition 3.1.** A point $(0, \lambda_0) \in T$ is said to be a local bifurcation point of solutions of equation (3.1), if $(0, \lambda_0) \in \text{cl}(\mathcal{N})$. The set of local bifurcation points will be denoted by $\text{BLF}$. A point $(0, \lambda_0) \in T$ is said to be a global bifurcation point of solutions of equation (3.1), if either $C(\lambda_0) \cap (T \setminus \{(0, \lambda_0)\}) \neq \emptyset$ or $C(\lambda_0)$ is not compact in $\Omega \times (\lambda^-, \lambda^+)$. The set of global bifurcation points will be denoted by $\text{GLOB}$. The sets of local (global) bifurcation points of solutions of equation (3.2) will be denoted by $\text{BLF}^G$ ($\text{GLOB}^G$). A point $(0, \lambda_0) \in T$ is said to be a local symmetry-breaking bifurcation point of solutions of equation (3.1) provided that $(0, \lambda_0) \in \text{BLF} \setminus \text{BLF}^G$. A point $(0, \lambda_0) \in T$ is said to be a global symmetry-breaking bifurcation point of solutions of equation (3.1) provided that $(0, \lambda_0) \in \text{GLOB} \setminus \text{GLOB}^G$.

In other words a point $(0, \lambda_0) \in T$ is a local symmetry-breaking bifurcation point of solutions of equation (3.1), provided that in a sufficiently small neighborhood of $(0, \lambda_0) \in \Omega \times (\lambda^-, \lambda^+)$ the isotropy group of every non-trivial solution of equation (3.1) is different from $G$. Directly from the above definitions it follows that $\text{GLOB}^G \subset \text{GLOB} \subset \text{BLF}$ and $\text{GLOB}^G \subset \text{BLF}^G \subset \text{BLF}$.

Our aim is to study sufficient conditions for the existence of global symmetry-breaking bifurcation points of solutions of equation (3.1).

A change of any reasonable degree theory along the family of trivial solutions implies a global bifurcation of non-trivial solutions from this family. In the theorem below we have formulated a sufficient condition for the existence of a global symmetry-breaking bifurcation from the family $T$. Since the proof of this theorem is standard, we omit it.

**Theorem 3.2.** Fix $\Phi \in C^k_G(\Omega \times (\lambda^-, \lambda^+), \mathbb{R})$. Assume that $\text{BLF}^G = \emptyset$ and that there are $\lambda^- < \lambda' < \lambda'' < \lambda^+$ such that
Note that conditions (A1), (A2) result from the following assumption

\[ \begin{align*}
\ker & \text{ from the interval } (3.1) \\
\text{Then there is a global symmetry-breaking bifurcation of solutions of equation (3.1)} \\
\text{from the interval } \{0\} \times (\lambda', \lambda'').
\end{align*} \]

Fix \( \Phi \in C^2_G(\Omega \times (\lambda^-, \lambda^+), \mathbb{R}) \) such that

\[ \begin{align*}
\text{(A1) } & \mathcal{B}_G \cap (\{0\} \times (\lambda^-, \lambda^+)) = \emptyset, \\
\text{(A2) } & \deg_{\mathcal{G}_H}(\nabla u \Phi(\cdot, \lambda)^G, B_{\alpha}(\mathbb{H})^G), 0) \neq 0, \text{ for all } \lambda \in (\lambda^-, \lambda^+) \text{ and sufficiently small } \alpha > 0, \text{ where } \Phi(\cdot, \lambda)^G = \Phi(\cdot, \lambda)|_{\mathbb{H}^G}.
\end{align*} \]

Note that conditions (A1), (A2) result from the following assumption

\( \text{(A)} \) ker \( \nabla^2 u \Phi(0, \lambda) \cap \mathbb{H}^G = \{0\} \) for all \( \lambda \in (\lambda^-, \lambda^+) \).

It is clear that assumption (A) is equivalent to the condition ker \( \nabla^2 u \Phi(0, \lambda)^G = \{0\} \) for all \( \lambda \in (\lambda^-, \lambda^+) \) i.e. \( \nabla^2 u \Phi(0, \lambda)^G \) is an isomorphism for all \( \lambda \in (\lambda^-, \lambda^+) \). Assumption (A) is stronger than assumptions (A1), (A2) but in real applications it is easier to verify this assumption than conditions (A1), (A2). In fact, since \( \nabla^2 u \Phi(0, \lambda)^G \) is an isomorphism, condition (A1) is fulfilled. Additionally, since \( \nabla^2 u \Phi(0, \lambda)^G \) is an isomorphism, condition (A2) is fulfilled because \( \deg_{\mathcal{G}_H}(\nabla u \Phi(\cdot, \lambda)^G, B_{\alpha}(\mathbb{H}^G), 0) = \deg_{\mathcal{G}_H}(\nabla^2 u \Phi(0, \lambda)^G, B_{\alpha}(\mathbb{H}^G), 0) = \pm 1. \)

The theorem below provides a criterion for the existence of a global symmetry-breaking bifurcation of critical orbits of invariant functionals. It will prove extremely useful in the study of symmetry breaking of non-radial solutions of elliptic differential equations considered on a ball or an annulus in \( \mathbb{R}^n \) with the group of symmetry \( SO(n) \).

**Theorem 3.3.** Let \( G \) be a connected compact Lie group. Fix \( \Phi \in C^2_G(\Omega \times (\lambda^-, \lambda^+), \mathbb{R}) \) satisfying assumption (A) and choose \( \lambda^- < \lambda' < \lambda'' < \lambda^+ \). Suppose that

1. \( \nabla^2 u \Phi(0, \lambda', \lambda'') \) are isomorphisms,
2. \( \mathbb{H}^- \neq_G \mathbb{H}'' \), where \( \mathbb{H}^- \) is a direct sum of eigenspaces of \( \nabla^2 u \Phi(0, \lambda') \) corresponding to its negative eigenvalues for \( \nu \in \{', ''\} \).

Then there is a global symmetry-breaking bifurcation of solutions of equation (3.1) from the interval \( \{0\} \times (\lambda', \lambda'') \) i.e. \( \mathcal{GL}_G \setminus \mathcal{GL}_G^G \neq \emptyset \).

**Proof.** Let us first observe that since \( \nabla^2 u \Phi(0, \lambda'), \nabla^2 u \Phi(0, \lambda'') \) are isomorphisms, \( (0, \lambda'), (0, \lambda'') \notin \mathcal{B}_G \). Additionally, from assumption (A) it follows that \( \mathcal{B}_G^G = \emptyset \). The basic idea of the proof is to apply Theorem 3.2. By Theorem 2.1 for \( \nu \in \{', ''\} \) we obtain

\[ \nabla_G^{-\deg}(\nabla u \Phi(\cdot, \lambda'), B_{\alpha}(\mathbb{H})) = \nabla_G^{-\deg}(\nabla u \Phi(\cdot, \lambda''), B_{\alpha}(\mathbb{H})) \in U(G) \] (3.3)

or

\[ \nabla_G^{-\deg}(\nabla u \Phi(\cdot, \lambda'), B_{\alpha}(\mathbb{H})) = \mathbb{I} \in U(G). \]

We claim that from assumption (A) it follows that \( \dim(\mathbb{H}'^-)^G = \dim(\mathbb{H}''^-)^G \). Suppose contrary to our claim that \( \dim(\mathbb{H}'^-)^G \neq \dim(\mathbb{H}''^-)^G \). Then applying the Conley index defined in [12] one can prove that there is a local bifurcation of solutions of equation (3.2) from the interval \( \{0\} \times (\lambda', \lambda'') \) i.e. \( \mathcal{B}_G^G \cap (\{0\} \times (\lambda', \lambda'')) \neq \emptyset \), which contradicts assumption (A1).

1. **Case** \( \dim \mathbb{H}'^- < \dim \mathbb{H}''^- \). Since \( \dim(\mathbb{H}'^-)^G = \dim(\mathbb{H}''^-)^G \), \( \mathbb{H}''^- \neq_G \mathbb{H}'^- \oplus \mathbb{R}[2k] \), for all \( k \in \mathbb{N} \), where \( \mathbb{R}[2k] \) is a 2k-dimensional trivial representation of \( G \). Thus combining formula (3.3) with Theorem 3.1 of [31] (or Lemma 3.4 of [21] ) we
obtain $\nabla_G$-$\text{deg}(\nabla_u \Phi(\cdot, \lambda), B_\alpha(\mathbb{H})) \neq \nabla_G$-$\text{deg}(\nabla_u \Phi(\cdot, \lambda''), B_\alpha(\mathbb{H}))$. The rest of the proof is a direct consequence of Theorem 3.2.

(2) Case $\dim \mathbb{H}'' > \dim \mathbb{H}'$. Since $\mathbb{H}' \neq_G \mathbb{H}''$, combining formula (3.3) with Theorem 3.2 of [31] (or Lemma 3.4 of [21]) we obtain $\nabla_G$-$\text{deg}(\nabla_u \Phi(\cdot, \lambda'), B_\alpha(\mathbb{H})) \neq \nabla_G$-$\text{deg}(\nabla_u \Phi(\cdot, \lambda''), B_\alpha(\mathbb{H}))$. The rest of the proof is a direct consequence of Theorem 3.2.

\hfill $\Box$

Remark 1. Note that if in the above theorem $\dim \mathbb{H}'' > \dim \mathbb{H}'$ then applying a version of the Conley index defined in [12] one can prove that there is a local symmetry-breaking bifurcation of solutions of equation (3.1) from the interval $\{0\} \times (\lambda', \lambda'')$ i.e. $\mathcal{BIF} \setminus \mathcal{BIF}^G \neq \emptyset$.

Remark 2. Let $T \subset G$ be the maximal torus of a connected compact Lie group $G$. Computations in the ring $U(T)$ are much simpler than that in the ring $U(G)$, see [21]. Note that we can treat $G$-representations $\mathbb{H}_\nu -$, $\nu \in \{', ''\}$, as orthogonal $T$-representations, with the induced $T$-action. Moreover, it is easier to verify condition $\mathbb{H}' \neq_T \mathbb{H}''$ than condition $\mathbb{H}' \neq_G \mathbb{H}''$, see [21]. To prove the above theorem we have shown that

$$\nabla_G$-$\text{deg}(\nabla_u \Phi(\cdot, \lambda'), B_\alpha(\mathbb{H})) = \nabla_G$-$\text{deg}(-\text{Id}, B_\alpha(\mathbb{H}')) \neq \nabla_G$-$\text{deg}(-\text{Id}, B_\alpha(\mathbb{H}'')) = \nabla_G$-$\text{deg}(\nabla_u \Phi(\cdot, \lambda''), B_\alpha(\mathbb{H})).$$

It is known that $\nabla_G$-$\text{deg}(-\text{Id}, B_\alpha(\mathbb{H}')) = \chi_G(S^{\mathbb{H}'}), \chi_G$ see [11], where $\chi_G$ is the $G$-equivariant Euler characteristic, see [8, 9]. In other words, to prove the above theorem we have shown that $\chi_G(S^{\mathbb{H}'}) \neq \chi_G(S^{\mathbb{H}''})$. Since $G$ is connected and $\dim(\mathbb{H}'')^G = \dim(\mathbb{H}'')^G$, applying Theorems 3.1, 3.2 of [31] we obtain that $\mathbb{H}' \neq_G \mathbb{H}''$ iff $\chi_G(S^{\mathbb{H}'}) \neq \chi_G(S^{\mathbb{H}''})$. Finally, combining Theorem 3.3 of [21], Corollary 3.1 of [31] with the above we obtain that $\mathbb{H}' \neq_G \mathbb{H}''$ iff $\mathbb{H}' \neq_T \mathbb{H}''$. In other words one can reduce the reasoning in the above theorem to a $T$-equivariant problem.

4. Remarks on symmetry-breaking of solutions of elliptic PDE’s. In this section we have shown that some of well known results concerning symmetry-breaking of solutions of elliptic differential equations are consequences of Theorem 3.3.

Below we show that some of well known results due to Ramaswamy and Srikanth follow from our results. We begin our discussion with a result due to Ramaswamy and Srikanth [27]. Consider a problem

$$\begin{cases} -\Delta u = u^p - \lambda & \text{in } B^n \\ u = 0 & \text{on } S^{n-1} \end{cases}, \quad (4.1)$$

where $1 < p < (n+2)/(n-2)$.

We treat the Sobolev space $H^1_0(B^n)$ as an orthogonal representation of $SO(n)$ with an $SO(n)$-action defined by $(gu)(x) = u(g^{-1}x)$ for every $u \in H^1_0(B^n)$ and $g \in SO(n)$.

Solutions of problem (4.1) are in one-to-one correspondence with the critical points of a functional $\Psi \in C^2_{SO(n)}(H^1_0(B^n) \times \mathbb{R}, \mathbb{R})$ defined by

$$\Psi(u, \lambda) = \frac{1}{2} \int_{B^n} ||\nabla u||^2 \, dx - \int_{B^n} \frac{1}{p+1} u^{p+1} - \lambda u \, dx,$$

i.e. $\Psi(u, \lambda) = \frac{1}{2}||u||^2_{H^1_0(B^n)} - \eta(u, \lambda)$, where
1. \( \nabla_u \eta : H_0^1(B^n) \times \mathbb{R} \to H_0^1(B^n) \) is compact and \( SO(n) \)-equivariant,
2. \( \nabla_u^2 \Psi(u, \lambda) = \text{Id} - \nabla_u^2 \eta(u, \lambda) \).

It is known, see [27], that there are \( s^- < 0 < s_0 < s^+, \epsilon > 0 \) and a \( C^1 \)-map \( \gamma : (s^-, s^+) \to H_0^1(B^n) \times \mathbb{R} \) such that

(a1) \( \gamma(s) = (u(s), \lambda(s)) \) is a solution of (4.1),
(a2) \( u(s) \) is positive on \( B^n \), (and radial by results of [13]),
(a3) ker \( \nabla_u^2 \Psi(\gamma(s)) \cap H_0^1(B^n) \otimes SO(n) = \{0\} \), (radial non-degeneracy),
(a4) \( [s_0 - \epsilon, s_0 + \epsilon] \subset (s^-, s^+) \),
(a5) ker \( \nabla_u^2 \Psi(\gamma(s)) = \{0\} \) for \( s \in (s_0 - \epsilon, s_0) \cup (s_0, s_0 + \epsilon) \) (non-degeneracy),
(a6) ker \( \nabla_u^2 \Psi(\gamma(s_0)) \) is an orthogonal \( n \)-dimensional \( SO(n) \)-representation.

Define a functional \( \Phi \) of the class \( C^2 \) by

\[
\Phi(u, s) = \Psi(u + u(s), \lambda(s)).
\]

(4.2)

Since \( u(s) \) is positive, applying results of [13] we obtain that \( u(s) \) is radial. That is why the functional \( \Phi \) is \( SO(n) \)-invariant i.e. \( \Phi \in C^2_{SO(n)}(H_0^1(B^n) \times (s^-, s^+), \mathbb{R}) \).

The following lemma will play a crucial role in the proof of a global symmetry-breaking of solutions of problem (4.1).

**Lemma 4.1.** Let \( \Phi \in C^2_{SO(n)}(H_0^1(B^n) \times (s^-, s^+), \mathbb{R}) \) be a functional defined by formula (4.2). Then for all \( s' \in (s_0 - \epsilon, s_0) \), \( s'' \in (s_0, s_0 + \epsilon) \) and \( \phi \in \ker \nabla_u^2 \Phi(0, s_0) \setminus \{0\} \) the following inequality holds:

\[
\langle \nabla_u^2 \Phi(0, s'), \phi \rangle_{H_0^1(B^n)} + \langle \nabla_u^2 \Phi(0, s''), \phi \rangle_{H_0^1(B^n)} < 0.
\]

**Proof.** Note that \( \nabla_u^2 \Phi(0, s) = \nabla_u^2 \Psi(\gamma(s)) \). That is why

\[
\ker \nabla_u^2 \Phi(0, s_0) = \ker \nabla_u^2 \Psi(\gamma(s_0)).
\]

It is clear that for \( \phi \in \ker \nabla_u^2 \Phi(0, s_0) \) and \( s \in (s^-, s^+) \) the following equalities hold true

\[
\langle \nabla_u^2 \Phi(0, s'), \phi \rangle_{H_0^1(B^n)} = \langle \nabla_u^2 \Psi(\gamma(s)) \phi, \phi \rangle_{H_0^1(B^n)} =
\]

\[
\langle (\text{Id} - \nabla_u^2 \eta(u(s), \lambda(s))) \phi, \phi \rangle_{H_0^1(B^n)} = \| \phi \|^2_{H_0^1(B^n)} - \int_{B^n} pu(s)^{p-1} \phi^2 \, dx.
\]

Fix \( \phi \in \ker \nabla_u^2 \Phi(0, s_0) \setminus \{0\} \), define a function \( \omega_\phi : (s^-, s^+) \to \mathbb{R} \) by

\[
\omega_\phi(s) = \| \phi \|^2_{H_0^1(B^n)} - \int_{B^n} pu(s)^{p-1} \phi^2 \, dx
\]

and note that \( \omega_\phi(s_0) = - \int_{B^n} p(p-1) u(s_0) u'(s_0) u''(s_0) \phi^2 \, dx \). It is known that there is \( \phi_0 \in \ker \nabla_u^2 \Phi(0, s_0) \setminus \{0\} \) such that \( \omega_\phi(s_0) \neq 0 \), see non-degeneracy condition (iii) on page 844 of [27]. Hence \( \omega_\phi(s) \) changes sign at \( s_0 \) and there is \( \epsilon > 0 \) such that for every \( s' \in (s_0 - \epsilon, s_0), s'' \in (s_0, s_0 + \epsilon) \) we have \( \omega_\phi(s') \cdot \omega_\phi(s'') < 0 \). In other words we have proved that

\[
\langle \nabla_u^2 \Phi(0, s'), \phi_0 \rangle_{H_0^1(B^n)} \cdot \langle \nabla_u^2 \Phi(0, s''), \phi_0 \rangle_{H_0^1(B^n)} < 0.
\]

Since \( \ker \nabla_u^2 \Phi(0, s_0) \) is an orthogonal \( n \)-dimensional representation of \( SO(n) \), the \( SO(n) \)-action on the space \( \ker \nabla_u^2 \Phi(0, s_0) \) is transitive i.e. for every \( \phi \in \ker \nabla_u^2 \Phi(0, s_0) \setminus \{0\} \) the orbit \( SO(n) \phi \) is the sphere in \( \ker \nabla_u^2 \Phi(0, s_0) \) of radius \( \| \phi \|_{H_0^1(B^n)} \). Fix \( \phi \in \ker \nabla_u^2 \Phi(0, s_0) \setminus \{0\} \) and note that since the \( SO(n) \)-action on \( \ker \nabla_u^2 \Phi(0, s_0) \) is transitive and orthogonal, there is \( g \in SO(n) \) such that \( \| \phi \|_{H_0^1(B^n)}(g \phi_0) = \phi \). Consequently we obtain

\[
\langle \nabla_u^2 \Phi(0, s'), \phi \rangle_{H_0^1(B^n)} \cdot \langle \nabla_u^2 \Phi(0, s''), \phi \rangle_{H_0^1(B^n)} =
\]

\[
= \| \phi \|^4_{H_0^1(B^n)} \cdot \langle \nabla_u^2 \Phi(0, s') g \phi_0, g \phi_0 \rangle_{H_0^1(B^n)} \cdot \langle \nabla_u^2 \Phi(0, s'') g \phi_0, g \phi_0 \rangle_{H_0^1(B^n)} =
\]

\[
= \langle \nabla_u^2 \Phi(0, s'), \phi \rangle_{H_0^1(B^n)} \cdot \langle \nabla_u^2 \Phi(0, s''), \phi \rangle_{H_0^1(B^n)} < 0.
\]
There is also a direct consequence of Theorem 3.3.

\[ u, \lambda \]

\[ \Psi \]

\[ \text{Proof.} \]

\[ \Phi \]

\[ \text{Let } \Phi \]

\[ \text{condition. It is well known, see [25, 34], that there are a continuous map} \]

\[ \text{we have} \]

\[ n \]

\[ \nabla \]

\[ \text{equation} \]

\[ \nabla \]

\[ \text{it follows that a change of the Morse index of} \]

\[ \nabla \]

\[ \Phi(0, s) \]

\[ \gamma \]

\[ \xi \]

\[ (b1) \]

\[ (b2) \]

\[ (b3) \]

\[ (b4) \]

\[ \text{we have} \]

\[ (b1) \]

\[ (b2) \]

\[ (b3) \]

\[ (b4) \]

\[ \text{The rest of the proof is a direct consequence of Theorem 3.3.} \]

\[ \square \]

\[ \text{The following theorem is due to Ramaswamy and Srikanth [27]. We show that} \]

\[ \text{it is also a direct consequence of Theorem 3.3.} \]

\[ \textbf{Theorem 4.2.} \text{ There is} \ s_0 \in (s^-, s^+) \text{ such that} \]

\[ \gamma(s_0) = (u(s_0), \lambda(s_0)) \in H_0^1(B^n) \times \mathbb{R} \text{ is a global bifurcation point of non-radial solutions of problem (4.1).} \]

\[ \text{Proof.} \] \[ \text{Let } \Phi \in C^2_{SO(n)}(H_0^1(B^n) \times (s^-, s^+), \mathbb{R}) \text{ be a functional defined by (4.2).} \]

\[ \text{The study of solutions of problem (4.1) is equivalent to the study of solutions of equation} \]

\[ \text{What is left is to show that} \]

\[ \text{In order to prove this theorem it is enough to show that the functional } \Phi \]

\[ \text{In fact, if} \]

\[ \text{From Lemma 4.1} \]

\[ \text{it follows that a change of the Morse index of} \]

\[ \text{We treat the Sobolev space} \]

\[ \text{Solutions of problem (4.3) are in one-to-one correspondence with the critical points of a functional } \Psi \in C^2_{SO(n)}(H_0^1(\Omega), \mathbb{R}) \text{ defined by} \]

\[ \Psi(u, \lambda) = \frac{1}{2} \int_{\Omega} \| \nabla u \|^2 \, dx - \int_{\Omega} \frac{1}{p+1} u^{p+1} - \frac{\lambda}{2} u^2 \, dx, \]

\[ \text{i.e. } \Psi(u, \lambda) = \frac{1}{2} \| u \|^2_{H_0^1(\Omega_x)} - \eta(u, \lambda), \text{ where} \]

\[ 1. \nabla_u \eta : H_0^1(\Omega_x) \times \mathbb{R} \rightarrow H_0^1(\Omega_x) \text{ is a compact } SO(n) \text{-equivariant operator,} \]

\[ 2. \nabla_u^2 \Psi(u, \lambda) = Id - \nabla_u^2 \eta(u, \lambda). \]

\[ \lambda_1 \in \sigma(-\Delta, \Omega_x) \text{ be the first eigenvalue of } -\Delta \text{ with the Dirichlet boundary condition.} \]

\[ \text{It is well known, see [25, 34], that there are a continuous map } \gamma : \]

\[ (-\infty, \lambda_1) \rightarrow H_0^1(\Omega_x) \times \mathbb{R} \text{ and } \epsilon_0 > 0 \text{ such that for all } 0 < \epsilon < \epsilon_0 \text{ and } \lambda \in (-\infty, \lambda_1) \]

\[ \text{we have} \]

\[ (b1) \]

\[ (b2) \]

\[ (b3) \]

\[ (b4) \]

\[ \text{The following theorem is due to Srikanth [34]. It is also a consequence of our results.} \]
Theorem 4.3. There is \( \lambda_0 \in (0, \lambda_1) \) such that \( \gamma(\lambda_0) = (\mu(\lambda_0), \lambda_0) \in H_0^1(\Omega) \times \mathbb{R} \) is a global bifurcation point of non-radial solutions of problem (4.3).

Proof. Define a functional \( \Phi \) of the class \( C^2 \) by \( \Phi(u, \lambda) = \Psi(u + u(\lambda)). \) Since \( u(\lambda) \) is radial, the functional \( \Phi \) is \( SO_n \)-invariant i.e. \( \Phi \in C^2_{SO_n}(H_0^1(B^n) \times (-\infty, \lambda_1), \mathbb{R}) \).

It is easy to check that the study of solutions of equation \( \nabla_u \Psi(u, \lambda) = 0 \) is equivalent to the study of solutions of problem (4.3). That is why from now on we consider equation \( \nabla_u \Phi(u, \lambda) = 0. \)

Note that there is \( \lambda_0 \in (0, \lambda_1) \) such that \( \ker \nabla_u^2 \Phi(0, \lambda_0) = \ker \nabla_u^2 \Psi(\gamma(\lambda_0)) \) is an orthogonal \( n \)-dimensional representation of the group \( SO(n) \), see Step 3. of [34]. Moreover, there is \( \delta > 0 \) such that for every \( \lambda' \in (\lambda_0 - \delta, \lambda_0), \lambda'' \in (\lambda_0, \lambda_0 + \delta) \) we have

1. solutions \( (0, \lambda'), (0, \lambda'') \) are non-degenerate, i.e. \( \ker \nabla_u^2 \Phi(0, \lambda') = \{0\} \) for \( \nu \in \{0, \lambda''\} \).
2. Morse indices of solutions \( (0, \lambda'), (0, \lambda'') \) are different (the difference is at least \( n \)) i.e. \( m^-(\nabla_u^2 \Phi(0, \lambda')) \neq m^-(\nabla_u^2 \Phi(0, \lambda'')) \).

Taking into account properties (b1)-(b4) and the above, we can easily verify assumptions of Theorem 3.3, which completes the proof. \( \square \)

Remark 3. One can prove the symmetry breaking results due to Cerami [4] applying Theorem 3.3. The non-degeneracy condition considered by Cerami forces a change of the Morse index along the family of radial solutions. Using the techniques of equivariant degree theory the results due to Pacard [26] have been improved in [29]. Whereas results due to Smoller and Wassermann have been improved in [31]. Finally we would like to point out that the results due to Dancer [7] do not follow from our theory. Dancer has used a subtle reasoning on cones to prove his results.

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