1. Introduction

1.1. Motivation. In 1954, Eichler discovered the first instance of the link between zeta functions of Shimura varieties and automorphic L-functions. Shortly thereafter, Shimura extended Eichler results to compute the zeta functions of quaternionic curves. Their work was based on the congruence relation, known now as the Eichler–Shimura relation, which played an important role in the theory of arithmetic of elliptic curves and modular forms. Later on, in the 70s, Langlands launched a program that aims to generalize the previous work to compute zeta functions attached to all Shimura varieties. Gradually a conjecture generalizing the Eichler–Shimura relation has emerged and was formulated by Balsisus and Rogawski [BR94, §6]. We give its statement below after setting some background.

Let $G$ be a connected, reductive group defined over $\mathbb{Q}$ and let $S = \text{Res}_{C/R} G_{m,C}$. Suppose we have a homomorphism of algebraic $\mathbb{R}$-groups $S \to G_{\mathbb{R}}$, which satisfies the axioms of Deligne [Mil17,
Definition 5.5. Let $K$ be an open compact subgroup of $G(\mathbb{A}_f)$ of the form $\prod_{v<\infty} K_v$, where $K_v \subset G(\mathbb{Q}_v)$ and $K_v$ is hyperspecial for almost all the finite places $v$. This gives rise to the Shimura variety $Sh_K(G, \mathcal{X})$ with reflex field $E$ and whose complex points are

$$Sh_K(G, \mathcal{X})(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{X} \times G(\mathbb{A}_f)/K.$$

Assume that $K$ is sufficiently small, so that $Sh_K(G, \mathcal{X})$ is a smooth. We fix a prime $p$ over which $G$ is unramified and the level structure $K$ has the form $K^p K_p$ with $K_p$ hyperspecial. For each prime ideal $p$ of $E$ lying over $p$, Blasius and Rogawski have defined a polynomial $H_p \in \mathcal{H}(G(\mathbb{Q}_p)/K_p, \mathbb{Q})[X]$, and they conjectured that:

**Conjecture 1.1.1** (Blasius–Rogawski). Let $\ell$ be a prime $\neq p$ (i) The Shimura variety $Sh_K(G, \mathcal{X})$ has good reduction at $p$ (in some sense); and (ii) we have

$$H_p(\text{Fr}_p) = 0$$

This conjecture was proved by Ihara (extending cases treated by Eichler and Shimura) for Shimura curves. The first statement has been established for Shimura varieties of abelian type by Kisin [Kis09, Kis10] and the second part was proved by: Büttel for certain orthogonal groups [Bül97], Wedhorn [Wed00] in the PEL case for groups that are split over $\mathbb{Q}$, Büttel–Wedhorn for the unitary case of signature $(n-1, 1)$ with $n$ even [BW06], Koskivirta for a unitary similitude group of signature $(n-1, 1)$ over $\mathbb{Q}$ when $p$ is inert in the reflex field and $n$ odd [Kos13] and finally H. Li showed recently the conjecture for simple GSpin Shimura varieties [Li18].

In all these cases for which the conjecture is known, the authors prove a slightly stronger version of it where the desired annihilation is taking place in a “geometric” ring of correspondences or cycles in characteristic $p$. Assume that $Sh_K(G, \mathcal{X})$ is of Hodge-type and let $\mathcal{S}_K$ be its integral model over $\mathcal{O}_{E_p}$. This scheme has an interpretation as a moduli space of abelian schemes with additional structures. Following Chai–Faltings [FC90], Moonen defines in [Moo04] a stack $p - \text{Isog}$ over $\mathcal{O}_{E_p}$, parametrizing $p$-isogenies between two points of $\mathcal{S}_K$. It has two natural projections to the isogeny to its target and source, the subalgebra generated by the irreducible components. Consider the $\mathbb{Q}$-algebra of cycles $\mathbb{Q}[p - \text{Isog} \times E]$ and $\mathbb{Q}[p - \text{Isog} \times k_{\mathcal{O}_{E_p}}]$ where $k_{\mathcal{O}_{E_p}}$ is the residue field of $\mathcal{O}_{E_p}$, here multiplication is defined by composition of isogenies. Define $p - \text{Isog}^{\text{ord}} \times k_{\mathcal{O}_{E_p}}$ as the preimage of the $\mu$-ordinary locus of the special fiber of the $\mathcal{S}_K$, under the source projection. We get a diagram of $\mathbb{Q}$-algebra homomorphisms

$$\begin{array}{ccc}
\mathcal{H}(G(\mathbb{Q}_p)/K_p, \mathbb{Q}) & \xrightarrow{h} & \mathbb{Q}[p - \text{Isog} \times E] \\
\downarrow \mathcal{S}_M^G & & \downarrow \sigma \\
\mathcal{H}(M(\mathbb{Q}_p)/K_p \cap M(\mathbb{Q}_p), \mathbb{Q}) & \xrightarrow{h} & \mathbb{Q}[p - \text{Isog}^{\text{ord}} \times k_{\mathcal{O}_{E_p}}]
\end{array}$$

where the big square is commutative, $M$ is the centralizer of the norm of the dominant coweight $\mu$ given by the Shimura datum, the homomorphism $\mathcal{S}_M^G$ is the untwisted Satake transform, $\sigma$ is the specialization map of cycles, the map ord intersects a cycle with the ordinary $\mu$-locus while cl is the map sending a cycle to its closure. There is a natural Frobenius section of the source projection, mapping an abelian variety to its Frobenius isogeny, which produces a closed subscheme $F$ of $p - \text{Isog} \times k_{\mathcal{O}_{E_p}}$. 
Conjecture 1.1.2. The cycle $F$ is a root of the polynomial
$$\sigma \circ h(H_p)(X) \in \mathbb{Q}[p - \text{Isog} \times k_{O_{E_F}}][X].$$

 Functorial properties of cohomology shows that Conjecture 1.1.2 implies Conjecture 1.1.1. Most known cases of Conjecture 1.1.2 are obtained by proving first the conjecture on the generically ordinary $p$-isogenies. This reduces to Bültel’s group theoretic result which says that we have an annihilation

$$(*) \quad H_p(\mu) = 0 \text{ in the } \mathbb{Q}\text{-algebra } \mathcal{H}(M(Q_p)//K_p \cap M(Q_p), \mathbb{Q}).$$

Now, If the ordinary locus $p - \text{Isog}_{\text{ord}} \times k_{O_{E_F}}$ is dense in $p - \text{Isog} \times k_{O_{E_F}}$, then Bültel’s argument is sufficient to prove the full congruence conjecture. This is the cases studied by Chai-Faltings, Bültel, Wedhorn and Bültel–Wedhorn.

We have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{H}_K(\mathbb{Q}) & \xrightarrow{\mathcal{S}_{\mathcal{O}}} & \mathcal{H}(M(Q_p)//K_p \cap M(Q_p), \mathbb{Q}) \\
\downarrow & & \downarrow \\
\text{End}_{\mathbb{Z}[G]} \mathbb{Q}[G/K] & \xrightarrow{\mathcal{S}_{\mathcal{O}[G]}} & \text{End}_{\mathbb{Z}[P]} \mathbb{Q}[G/K] \\
\end{array}
$$

Our main results (Theorem 6.0.4) shows in particular that Bültel’s relation $(*)$ lifts naturally to an analogous relation

$$(†) \quad H_p(u_\mu) = 0 \in \text{End}_{\mathbb{Q}[P(Q_p)]} \mathbb{Q}[G(Q_p)/K_p]$$

where $u_\mu$ is the U-operator attached to $\sigma^\mu$ [Bou21d] and $P$ is the largest parabolic subgroup of $G$ relative to which $\mu$ is dominant. For applications, a key advantage of the latter relations (upon Bültel’s) is that while $\mathcal{H}(M(Q_p)//K_p \cap M(Q_p), \mathbb{Q})$ still had to be made acting on various spaces, the non-commutative ring $\text{End}_{\mathbb{Z}[P]}(\mathbb{Q}[G(Q_p)/K_p])$ already acts (faithfully and by definition) on the ubiquitous space $\mathbb{Q}[G(Q_p)/K_p]$.

In a work in progress the author is tackling (using † instead) a generalization of Conjecture 1.1.2 for abelian-type Shimura varieties [Bou21c].

1.2. Main result. Let $F$ be a finite extension of $\mathbb{Q}_p$ for some prime $p$, $\mathcal{O}_F$ its ring of integers, $\mathbb{Q}$ a fixed uniformizer in $\mathcal{O}_F$ and $k_F$ the residue field of $F$ of size $q$. For every scheme $X$ over $\text{Spec} \mathcal{O}_F$, we set $X_{k(F)} := X \times_{\text{Spec} \mathcal{O}_F} \text{Spec} k(F)$ for the special fiber.

Let $G/F$ be an unramified reductive group, $S$ a maximal $F$-split subtorus of $G$ and $A$ the apartment attached to $S$ in the extended Bruhat–Tits building of $G$, together with a fixed origin a hyperspecial point $a_o \in A$. Let $T$ be the centralizer of $S$, which is a maximal $F$-torus in $G$, $B = T \cdot U^+$ a Borel subgroup with unipotent radical $U^+$ and $W = N_G(S)(F)/T(F)$ be the Weyl group.

Let $K$ be a hyperspecial maximal open compact subgroup of $G$ attached $a_o$. Bruhat and Tits attach to $a_o$ a reductive $\mathcal{O}_F$-model $G$ of $G$. Let $K$ be the corresponding parahoric subgroup, i.e. $G(\mathcal{O}_F)$. This also applies to the reductive group $T$ and $a_o$, we get then a reductive $\mathcal{O}_F$-model $\mathcal{T}$ of $\mathcal{T}$. Let $I$ be the Iwahori subgroup that is defined by

$$I = \{ g \in G(\mathcal{O}_F) : \text{red}(g) \in B(k_F) \}.$$ 

For any algebraic $F$-groups $\mathcal{H}$ (bold style), we denote its group of $F$-points by the ordinary capital letter $H = H(F)$.
Let $\nu'_N : N \to (X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}) \rtimes W$ be the map characterized by

$$\nu'_N(\varpi^\lambda) = \lambda.$$ 

Note that $\nu'_N = -\nu_N$, where $\nu_N$ is the Bruhat–Tits translation homomorphism. Set $T_1 := T(O_F) = \ker \nu_N = \ker \kappa_T$, where $\kappa_T$ is the Kottwitz homomorphism. We embed $X_*(S)$ into $T$ (using $\nu'_N$) by identifying $\lambda \in X_*(S)$ with $\varpi^\lambda := \lambda(\varpi)$. Using this identification, we have

$$\Lambda_T := T/T_1 \simeq X_*(T)_F \simeq X_*(S).$$

Set $\Phi^+$ for the set of $B$-positive roots, the one that appears in $\text{Lie}(B)$, or equivalently if it takes positive values on the vectorial chamber $C^-$ opposite to $C^+$; where $C^+$ is the vectorial chamber corresponding to $B^2$.

We say that $\lambda \in X_*(S)$ is $B$-dominant if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$. Let $\mathcal{C} \subset \mathcal{A}_{\text{ext}}$ denotes the closed vectorial chamber corresponding to the Borel $B$ in the extended apartment attached to $S$. Thus, an element $t = \varpi^\lambda$ for $\lambda \in X_*(S)$ is antidominant if and only if $\lambda \in X_*(S) \cap \mathcal{C}$, if and only if $\lambda$ is $B$-dominant, since $\langle \nu'_N(t), \alpha \rangle = (\lambda, \alpha) \leq 0, \forall \alpha \in \Phi^+$. Write $\Lambda_T^-$ for the set of antidominant elements in $\Lambda_T$.

For any extension $E$ of $F$, let $\mathcal{M}(E)$ be the set of $G(E)$-conjugacy classes of (algebraic group) cocharacters $G_{m,E} \to G_E$. By [Kot84a, Lemma 1.1.3], the canonical surjective morphism $X_*(S) \to \mathcal{M}(F)$ yields the following identification

$$X_*(S)/W(G, S) \simeq \mathcal{M}(F) \simeq \mathcal{M}(\mathcal{T})^{\text{Gal}(\mathcal{T}/F)} \simeq (X_*(T)/W(G_{\mathcal{T}}, T))^{\text{Gal}(\mathcal{T}/F)}.$$ 

In addition, using the Cartan decomposition one gets another identification identification

$$\mathcal{M}(F) \simeq K\backslash G/K,$$

given by $[\lambda] \mapsto K \varpi^\lambda K.$

Let $\epsilon \in \mathcal{M}(\mathcal{T})$ and $F(\epsilon) \subset F^{\text{un}}$ its field of definition. Set $d = [F(\epsilon) : F]$. Let $\mu \in \text{Norm}_{F(\epsilon)/F} \epsilon$ be the cocharacter of $T$ which is $B$-dominant, i.e. $\varpi^\mu$ is antidominant. Let $P$ be the largest parabolic subgroup of $G$ relative to which $\mu$ is dominant, $L$ is a Levi factor of $P$ (which is also the centralizer of $\mu$ in $G$) and $U^+_P$ the unipotent radical of $P$.

In [Bon21d], to any element $t \in \Lambda_T^-$ is attached an operator $u_t \in \text{End}_{\mathcal{Z}[B]} \mathcal{Z}[G/K]$ characterized by sending the trivial class $K$ to $\sum_{u \in U^+_P} u t K$ (and extended $B$-equivariantly to $\mathcal{Z}[G/K]$).

The main result of the paper (which generalizes [BBJ18, Lemma 3.3]) is:

**Theorem 1.2.1 (Seed relation).** The operator $u_{\varpi^\mu} \in U$ is a right root of the Hecke polynomial $H_{G,c}$ in $\text{End}_{\mathcal{Z}[P]} \mathcal{Z}[q^{\mathbb{Z}}][G/K]$.

**Remark 1.2.2.** The minimal polynomial of $u_{\varpi^\mu}$ has actually its coefficients in the integral Hecke algebra $\text{End}_{\mathcal{Z}[G]} \mathcal{Z}[G/K]$.

**Remark 1.2.3.** This relation has another application; in [Bon21b] we construct a tame norm compatible system of special cycles in a (product of) unitary Shimura variety.

**Remark 1.2.4.** A very interesting and surprising aspect of this work is that in order to establish formulas relating the two non-commuting commutative subrings, $U$ and $H_K(G)$, of the Hecke algebra $\text{End}_{\mathcal{Z}[B]}(\mathcal{Z}[q^{\mathbb{Z}}][G/K])$ one has to embed them both in yet another noncommutative ring (the Iwahori–Hecke algebra $H_I(\mathcal{Z}[q^{\mathbb{Z}}])$), where they actually do commute!

1Note that in this unramified case, $T$ splits over the completion of $F^{\text{un}}$ denoted previously by $L$. Thus, the Kottwitz homomorphism takes the simpler form $\kappa_T : T(L) \to X_*(T)$.

2Given $B$, the chamber $C^+$ is the unique vectorial chamber with apex $a_0$ for which $T_1 U^+$ is the union of the fixators of all quartiers $a + C^+$ with $a \in \mathcal{A}$.
1.3. Acknowledgements. This work is based on results proved in the author’s EPFL 2019 thesis. Therefore, I am very thankful to my adviser D. Jetchev. I am also indebted and very grateful to C. Cornut for his comments and reading. I would also like to thank T. Wedhorn; the discerning reader will no doubt notice the importance of his paper [Wed00].

2. Langlands dual group

Let \( \Gamma_{un} = \text{Gal}(F^{un}/F) \simeq \text{Gal}(\mathbb{F}_p/F) \). As before, we let \( \sigma \in \Gamma_{un} \) be the arithmetic Frobenius of \( F \). The group \( G \) split over \( F^{un} \) [GD70, XXVI 7.15]. We consider a Langlands dual group of \( G \) with respect to \( \Gamma_{un} \). This group sits in the following short exact sequence

\[
1 \longrightarrow \hat{G} \longrightarrow \hat{L}G \longrightarrow \Gamma_{un} \longrightarrow 1,
\]

and every choice of épingle \((\hat{B}, \hat{T}, (e_\alpha))^3\) yields a splitting of the above exact sequence. We fix a \( \Gamma_{un} \)-invariant épingle \([\text{Kot84b, } \S 1]\) thus \( \hat{L}G = \hat{G} \rtimes \Gamma_{un} \).

The \( \Gamma_{un} \)-equivariant isomorphism \( X_*(T) \simeq X^*(\hat{T}) \) induces a canonical identification between the \( \Gamma_{un} \)-groups \( W(G_{\mathbb{F}_p}, T) \) and the Weyl group \( W(\hat{G}, \hat{T}) \) and an identification between the \( X_*(S) = X_*(T)_F \) and \( X^*(S) \). The inclusion \( S \hookrightarrow T \) gives an embedding \( X_*(S) \hookrightarrow X_*(T) \), which yields a short exact sequence

\[
1 \longrightarrow \hat{T}^{1-\sigma} \longrightarrow \hat{T} \longrightarrow \hat{S} \longrightarrow 1,
\]

showing that \( \hat{S} \simeq \hat{T}/(1-\sigma)\hat{T} \). Therefore,

\[
\hat{T} = \text{Spec}(\mathbb{C}[X^*(\hat{T})]) = \text{Spec}(\mathbb{C}[X_*(T)]),
\]

\[
\hat{S} = \text{Spec}(\mathbb{C}[X_*(S)]) = \text{Spec}(\mathbb{C}[\Lambda_T]) = \text{Spec}(\mathbb{C}[[T(F)], C)).
\]

In particular, \( \hat{S}(\mathbb{C}) = \text{Hom}(X_*(T)_F, \mathbb{C}^*) \). The above fixed canonical identification \( W(G_{\mathbb{F}_p}, T) \simeq W(\hat{G}, \hat{T}) \), lets \( W(G, S) \) operates on \( \hat{S} \) by duality. The space \( \hat{S}/W(G, S) \) has the structure of a smooth affine \( \mathbb{C} \)-scheme whose coordinate ring is \( \mathbb{C}[X_*(S)]^{W(G, S)} \):

\[
\hat{S}/W(G, S) = \text{Spec} \left( \mathbb{C}[X_*(S)]^{W(G, S)} \right) = \text{Spec} \left( \mathbb{C}[\Lambda_T]^{W(G, S)} \right).
\]

Using the untwisted Satake isomorphism of [Bou21a, Theorem 5.2.1] we obtain

\[
(2.1) \quad \hat{S}/W(G, S) = \text{Spec} \left( \mathcal{H}_K(\mathbb{C}) \right).
\]

3. Unramified representations and unramified \( L \)-parameters

Let \( W_F \subset \Gamma_{un} \) whose elements induce an integral power of the Frobenius automorphism \( \sigma: x \mapsto x^q \) on the algebraic closure of the residue field. The valuation \( \text{val}: W_F \rightarrow \mathbb{Z} \) sends an element \( \psi \in W_F \) to the power of \( \sigma \) it induces, e.g. \( \text{val}(\sigma) = 1 \). Define the ”Weyl form” of the Langlands group to be \( L_{\mathbb{F}_p}G := \hat{G} \rtimes W_F \subset \hat{G}L \). The isomorphism \( \mathbb{Z} \rightarrow W_F \) given by \( 1 \mapsto \sigma \) defines a semidirect product \( \hat{G} \rtimes \mathbb{Z} \) and we get a homomorphism

\[
L_{\mathbb{F}_p}G \rightarrow \hat{G} \rtimes \mathbb{Z}.
\]

Definition 3.0.1. An unramified \( L \)-parameter is a homomorphism \( \phi: W_F \rightarrow L_{\mathbb{F}_p}G \) that verifies the following properties:

\[
(1) \quad \text{The composition } W_F \xrightarrow{\phi} L_{\mathbb{F}_p}G \longrightarrow W_F \text{ is the identity.}
\]

\[
(2) \quad \text{For any } w \in W_F, \phi(w) \text{ is semisimple.}
\]

\[
\text{Here, for each simple root } \alpha \text{ of } \hat{T}, e_\alpha \text{ is a nonzero element of the root vector space } \text{Lie}(\hat{G})_\alpha.
\]
(3) The composition \( \mathcal{W}_F \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\mathcal{L}} \hat{\mathcal{G}} \rtimes \mathbb{Z} \) factors through \( \text{val} \).

Set \( \Phi_{un}(\mathcal{G}) \) for the set of equivalence\(^4\) classes of unramified \( L \)-parameters.

The set of \( L \)-parameters is in bijection with the set of semisimple elements of the form \( g \rtimes \sigma \in \hat{\mathcal{G}} \).

Therefore, \( \Phi_{un}(\mathcal{G}) \) identifies with the set of semisimple elements of \( \hat{\mathcal{G}} \) modulo \( \sigma \)-conjugation.

**Definition 3.0.2.** An unramified representation of \( \mathcal{G}(F) \) is a homomorphism of groups \( \pi : \mathcal{G}(F) \to \text{GL}(V) \) where \( V \) is a \( \mathbb{C} \)-vector space verifying the following conditions:

1. \( \pi \) is irreducible.
2. The stabilizer of any vector \( v \in V \) is an open subgroup of \( \mathcal{G}(F) \).
3. For any open subgroup \( O \subset \mathcal{G}(F) \), the vector subspace \( V^O \) of \( O \)-fixed vectors is finite dimensional.
4. The subspace \( V^K \) is nonzero.

Set \( \Pi_{un}(\mathcal{G}) \) for the set of equivalence\(^5\) classes of unramified representations of \( \mathcal{G}(F) \).

**Proposition 3.0.3.** There is a natural bijection

\[
\Phi_{un}(\mathcal{G}) \simeq \hat{\mathcal{S}}(\mathbb{C})/W(\mathcal{G}, S) \simeq \Pi_{un}(\mathcal{G}).
\]

**Proof.** In the proof of [BR94, Proposition 1.12.1], one shows first the above proposition for the torus \( T \):

\[
\Phi_{un}(T) \simeq \hat{\mathcal{S}}(\mathbb{C}) \simeq \Pi_{un}(T),
\]
then deduce it for \( \mathcal{G} \) using [Bor79, Proposition 6.7]. \( \square \)

Combining Proposition 3.0.3 and (2.1) yields

\[
(3.1) \quad \Phi_{un}(\mathcal{G}) \simeq \text{Spec}(\mathcal{H}_K(\mathbb{C})).
\]

**Remark 3.0.4.** The above proposition gives an alternative characterization of the untwisted Satake homomorphism. Consider the following injective homomorphism

\[
\mathcal{H}_K(\mathbb{C}) \xrightarrow{} \{ \Pi_{un}(\mathcal{G}) \to \mathbb{C} \}
\]

\[
h_g = 1_{KgK} \xrightarrow{} (\pi \to \text{Tr}(\pi(h_g)|_{V^K})),
\]

where, \( V \) is given a structure of a left \( \mathcal{H}_K(\mathbb{C}) \)-module defined by \( f \cdot v \) for \( f \in \mathcal{H}_K(\mathbb{C}) \) and \( v \in V \) by the formula

\[
f \cdot v = \int_G f(g)(\pi(g) \cdot v)d\mu_K(g).
\]

By Proposition 3.0.3 we get the following commutative diagram

\[
\begin{array}{c}
\mathcal{H}_K(\mathbb{C}) & \xrightarrow{S^G_f} & C_c(\Lambda_T, \mathbb{C}) \\
\text{\( \simeq \)} & & \text{\( \simeq \)} \\
C[\Pi_{un}(\mathcal{G})] & \xrightarrow{\simeq} & C[[\Pi_{un}(\mathcal{T})]^{W(\mathcal{G}, S)}} \xrightarrow{\simeq} C[[\Pi_{un}(\mathcal{T})]].
\end{array}
\]

\(^4\)Two \( L \)-parameters are equivalent if they are \( \hat{\mathcal{G}}(\mathbb{C}) \)-conjugate.

\(^5\)Two representations \((\pi_1, V_1)\) and \((\pi_2, V_2)\) are equivalent if there exists an isomorphism \( V_1 \to V_2 \) sending \( \pi_1 \) to \( \pi_2 \).
4. The Hecke polynomial

Let $\epsilon \in \mathcal{M}(F)$ and $\mu_\epsilon \in X_*(T)$ be the unique $B_P$-dominant cocharacter of $T_{\overline{F}}$. Both, $\epsilon$ and $\mu_\epsilon$ have the same field of definition, a finite unramified extension $F(\epsilon) \subset F_{un}$ of $F$. Set $d = [F(\epsilon) : F]$ and let

$$\text{Norm}_{F(\epsilon)/F} \epsilon := \prod_{\tau \in \text{Gal}(F(\epsilon)/F)} \tau(\mu_\epsilon) \in \mathcal{M}(F)$$

be the norm of $\epsilon^d$. We may assume that for some representative of the conjugacy class $\text{Norm}_{F(\epsilon)/F} \epsilon$ takes values in the torus $T$ (and hence for all). The conjugacy class $\epsilon \in \mathcal{M}(F(\epsilon))$ determines a Weyl orbit of a character of $\hat{T}$, in which there is a unique $\hat{\mu}_\epsilon \in X^*(\hat{T})$ that is dominant with respect to the Borel subgroup $B$.

Let $(r_\epsilon, V)$ be a representation of $L(G_{F(\epsilon)})$ (unique up to isomorphism) satisfying the conditions:

- The restriction of $r_\epsilon$ to $\hat{G}$ is irreducible with highest weight $\hat{\mu}_\epsilon$.
- For any admissible invariant splitting of $L(G_{F(\epsilon)})$ the subgroup $\Gamma_{un}^d$ of $L(G_{F(\epsilon)})$ acts trivially on the highest weight space of $r_\epsilon$.

Fix an invariant admissible splitting $L(G_{F(\epsilon)}) = \hat{G} \rtimes \Gamma_{un}^d$.

**Definition 4.0.1** (The Hecke polynomial). For every $\hat{g} \in \hat{G}$, consider the following polynomial:

$$P_{G, \epsilon}(X) = \det \left( X - q^{d(\mu_\epsilon, r_\epsilon)} ((\hat{g} \rtimes \sigma)^d) \right).$$

By varying $\hat{g}$, the coefficients of $P_{G, \epsilon}$ are viewed as elements of the algebra of regular functions of $\Phi_{un}(G)$. Let $H_{G, \epsilon} \in \mathcal{H}_K(\mathbb{C})[X]$ be the Hecke polynomial corresponding to $P_{G, \epsilon}$ via (3.1) (compare with [BR94, §6]).

5. Explicit twisted Satake transform

Let $\mu \in \text{Norm}_{F(\epsilon)/F} \epsilon$ be the cocharacter of $T$ which is $B$-dominant, i.e. $\varpi^\mu$ is antidominant. Let $L$ be the centralizer of $\mu$ in $G$. Let $P$ be the largest parabolic subgroup of $G$ relative to which $\mu$ is dominant, $L$ is a Levi factor of $P$ and $U_P^+ \subset U^+$ the unipotent radical of $P$. By definition we have $T \subset L$ and $U_P^+ \subset U^+$. Set $K = L \cap K$ for any $L \in \{ P, L, U_P^+ \}$. Denote by $f_{[\mu]} = 1_{K \varpi^\nu K} \in \mathcal{H}_K(R)$ (resp. $g_{[\mu]} = 1_{\varpi P K_{L}} \in C_c(L/\varpi K_{L}, R)$, resp. $h_{[\mu]} = 1_{\varpi T_1} \in C_c(T/\varpi T_1, R) \simeq R[\Lambda_T]$ the characteristic function of the double coset corresponding to $[\mu]$. Let $p: G_{sc} \to G$ be the simply connected covering of the derived group of $G$ and let $S_{sc}$ be the unique maximal $F$-split torus of $G_{sc}$ such that $p(S_{sc}) \subset S$. The map $p$ defines a homomorphism from $X_*(S_{sc})$ to $X_*(S)$. We are interested in the set

$$\Sigma_F(\mu) = \{ \nu \in X_*(S) : \mu - \nu \in \text{Im}(X_*(S_{sc})) \} \text{ and } \nu \preceq \lambda \text{ for all } \nu \in W(G, S) \}.$$ 

**Remark 5.0.1.** The above $W$-invariant sets of weights plays a prominent role in representation theory and they are called "saturated sets of weights". Moreover, we have (see [Kot84a, §2.3], [Hum72, 13.4 Exercise] and Bourbaki’s [Bou68, Chapter VI, Exercises of §1 and §2]) that

$$\Sigma_F(\mu) = \bigsqcup_{\lambda \in X_*(S) \cap \overline{C}} W\lambda$$

where $\preceq$ denotes\footnote{It is straightforward that the conjugacy class $\text{Norm}_{F(\epsilon)/F} \epsilon$ does not depend on the choice of the representative $\mu_\epsilon$.} the partial order on $X_*(S) \cap \overline{C}$ defined by

$$\lambda \preceq \nu \Leftrightarrow \nu - \lambda = \sum n_\alpha \alpha^\vee, n_\alpha \in \mathbb{Z}_{\geq 0}.$$ 

\footnote{Compare with [Bou21a, Definition 5.2.4]}
Moreover, when \( \mu \) is minuscule then \( \Sigma_F(\mu) = W(G,S)\mu \) [Bou21a, Remark 5.2.8].

We have the following explicit description of the twisted Satake homomorphism

**Proposition 5.0.2.** Write

\[
\hat{S}_T^G(f_{|\mu}) = \sum_{\nu \in \Sigma_F(\mu)} c(\nu) \cdot 1_{w^\nu T_1} \in C_c(T \sslash T_1, \mathbb{Z}),
\]

and the coefficients \( \{c(\nu)\} \) are positive powers of \( q \) and verifies

\[
c(w^\nu) = q^{(\delta, w(\nu))}c(\nu) \text{ for all } w \in W(G,S), \text{ with } c(\mu) = 1.
\]

**Proof.** This is a particular case of [Bou21a, Theorem 5.2.1 & Theorem 5.3.1]. The twisted Satake isomorphism ensures that \( \hat{S}_T^G(f_{|\mu}) \in C_c(T \sslash T_1, \mathbb{Z})^W \) where \( W \) denotes the Weyl group with its twisted dot-action (See [Bou21a, §3.16]). This shows that \( c(\nu)q^{(\delta, w(\nu))} = c(w(\mu))q^{(\delta, w(\nu))} \) for all \( w \in W(G,S) \). The coefficient \( c(\mu) = 1 \) is obtained by [Kot84a, Lemma 2.3.7 (b)] using [Bou21a, Remark 5.2.9].

The fact that \( c(\nu) > 0 \) if and only \( \nu \in \Sigma_F(\mu) \) is well known in this unramified case; it follows by [Kot84a, Lemma 2.3.7 (a)] for the "only if" and [Rap00] for the "if".

6. Seed relations and U-operators

Using the fixed épalingue, we can consider a \( \Gamma_{un} \)-equivariant embedding \( L \mathbf{T} = \widehat{T} \times \Gamma_{un} \hookrightarrow L \mathbf{G} \). The composition

\[
L(T_{F(c)}) \hookrightarrow L(G_{F(c)}) \overset{r_{\varepsilon}}{\longrightarrow} \text{GL}(V) \overset{P_{G,c}}{\longrightarrow} C[X],
\]

is independent of all fixed choices. The restriction of \( r_\varepsilon \) to \( \widehat{T} \) yields a weight space decomposition

\[
V = \bigoplus_{\lambda \in \Sigma_E(\varepsilon)} V_{\lambda}.
\]

We have

\[
S_T^G(P_{G,c}) = \det \left( X - q^{d(\mu, \rho)} r_{\varepsilon}|_{L(T_{F(c)})} (\overline{t} \times \sigma)^d \right) \in C[\Phi_{un}(T)]^{W(G,S)}.
\]

Define the twisted restriction of \( r_{\varepsilon} \) to be the morphism of schemes

\[
r_T: L(T_{F(c)}) = \widehat{T} \times \Gamma_{un}^{d} \rightarrow \text{GL}(V)
\]

given on \( \mathbb{C} \)-points by

\[
r_T(1 \times \sigma^d) = r_{\varepsilon}(1 \times \sigma^d) \text{ and } r_T(\overline{t} \times 1) \cdot v_{\lambda} = q^{-\langle \rho, \lambda \rangle} \lambda(\overline{t}) \cdot v_{\lambda}
\]

for \( v_{\lambda} \in V_{\lambda} \) for all \( \lambda \in \Sigma(\varepsilon) \). The homomorphism \( r_T \) is not a homomorphism of groups but maps conjugacy classes to conjugacy classes and it is defined to ensure, using [Bou21a, Remark 5.2.9] and (6.1), that

\[
\hat{S}_T^G(P_{G,c}) = \eta_B \circ S_T^G(P_{G,c})
\]

\[
= \det \left( X - q^{-d(\mu, \rho)} r_T (\overline{t} \times \sigma)^d \right) \in C[\Phi_{un}(T)].
\]
Remark 6.0.1. Note that our choice of the twisted representation $r_T$ depends crucially on the normalization of the isomorphism $X_*(S) \cong \Lambda_T$. We have adopted the following isomorphism $\lambda \mapsto \varpi^\lambda$. Using [Bou21a, Remark 5.2.9] and $\delta_B(\varpi^\lambda)^{1/2} = q^{-(\lambda, \rho)}$, we see that

$$X_*(S) \otimes \mathbb{C} \xrightarrow{\eta: \lambda \mapsto q^{-(\lambda, \rho)}} X_*(S) \otimes \mathbb{C} \cong \Lambda_T \otimes \mathbb{C} \xrightarrow{\eta: tT \mapsto \delta(t)^{1/2} tT} \Lambda_T \otimes \mathbb{C}.$$ 

As opposed to [Wed00, Proposition 2.7], we insist on the fact that we do not assume $\mu$ to be minuscule in the following proposition.

Proposition 6.0.2. (1) Let $S^{F(\epsilon)} \subset \mathbf{T}$ denotes the maximal split torus of $G_{F(\epsilon)}$ containing the image of $\mu_\epsilon$, let $\overline{c}_{F(\epsilon)} \subset \mathcal{B}(G_{F(\epsilon)}, F(\epsilon))_{\text{ext}}$ be the closed vectorial chamber corresponding to the Borel $B_{F(\epsilon)}$. We have

$$\deg(H_{G, \epsilon}) \geq \sum_{\lambda \in X_*(S^{F(\epsilon)}) \cap \overline{c}_{F(\epsilon)}: \lambda \preceq \mu_\epsilon} \#(W(G, S^{F(\epsilon)}) \lambda) = \#(\Sigma_{F(\epsilon)}(\mu_\epsilon)).$$

(2) The twisted restriction $r_T$ of $r_\epsilon$ to $L(T_{F(\epsilon)})$ is isomorphic to a direct sum

$$V = \bigoplus_{\Sigma_{F(\epsilon)}(\mu_\epsilon)} V_{\lambda}$$

where, $V_{w(\bar{\mu})}$ is one-dimensional with generator $v_\lambda$ for any $w \in W$, such that

$$r_T(l \otimes \sigma^d, w(\bar{\mu}))) = q^{-(\mu, w(\mu))} w(\bar{\mu})(l) \cdot v_{w(\bar{\mu})}.$$ 

Proof. We will just imitate the proof of [Wed00, (2) Proposition 2.7] without requiring $\mu$ to be minuscule.

(1) Fix a Borel pair $(\mathbf{T}, B)$ of $\mathbf{G}$ and let $\tilde{\mu}_\epsilon$ be the dominant character of $\mathbf{T}$ corresponding to the conjugacy classe $c$. By definition of the Hecke polynomial, its degree is the dimension of the representation $r_\epsilon$ which is irreducible with highest weight $\tilde{\mu}_\epsilon$ as a representation of $\mathbf{G}$. By remark 5.0.1, the only weights of $r_\epsilon$ are the elements $\bigcup_{\lambda \in X_*(\mathbf{T}_{\text{dom}})} W((\mathbf{G}, \mathbf{T})\lambda)$ where the disjoint union is taken over dominant wights $\lambda \preceq \tilde{\mu}_\epsilon$ (here $\preceq$ is the usual partial order on dominant weights $X^*(\mathbf{T}_{\text{dom}})$). By definition of the dual group, we then have

$$\bigcup_{\lambda \in X_*(\mathbf{T}_{\text{dom}}): \tilde{\lambda} \preceq \tilde{\mu}_\epsilon} W((\mathbf{G}, \mathbf{T})\lambda) = \bigcup_{\lambda \in X_*(S^{F(\epsilon)}) \cap \overline{c}_{F(\epsilon)}: \lambda \preceq \mu_\epsilon} W(G_{F(\epsilon)}, S^{F(\epsilon)})\lambda = \Sigma_{F(\epsilon)}(\mu_\epsilon).$$

(2) The twisted restriction $r_T$ of $r_\epsilon$ to $L(T_{F(\epsilon)})$ is isomorphic to a direct sum

$$V = \bigcup_{\tilde{\lambda} \in X^*(\mathbf{T}_{\text{dom}}): \tilde{\lambda} \preceq \tilde{\mu}_\epsilon} V_{\tilde{\lambda}}.$$
and the highest weight space $V_{\tilde{\mu}}$ is one-dimensional\textsuperscript{8} with generator $v_{\tilde{\mu}}$. Accordingly, $V_{\tilde{\lambda}}$ is one-dimensional for any $\tilde{\lambda} \in W(\tilde{G}, \tilde{T})\tilde{\mu}$. The conjugacy class $\mathfrak{c}$ being defined over $F(\mathfrak{c})$, we see that $\langle \sigma^n \rangle$ stabilizes $W(\tilde{G}, \tilde{T})\tilde{\mu}$.

Choose for each classe $Z \in W(\tilde{G}, \tilde{T})\tilde{\mu}/\langle \sigma^d \rangle$ a representative $\tilde{\lambda}_Z \in Z$ and a vector $v_{\tilde{\lambda}_Z} \in V_{\tilde{\lambda}_Z}$. Define

$$v_{\sigma^r(\tilde{\lambda}_Z)} := r_1(1 \times \sigma^r) \cdot v_{\tilde{\lambda}_Z}, \quad \text{for } 1 \leq r < r_Z := \min\{s : \sigma^{sd}\tilde{\lambda}_Z = \tilde{\lambda}_Z\}.$$ 

Therefore, taking $r = -1$ gives

$$r_T(\hat{t} \times \sigma^d) \cdot v_{\sigma^{d(r-1)}(\tilde{\lambda}_Z)} = r_T(\hat{t} \times 1) \cdot v_{\tilde{\lambda}_Z} = q^{-\langle \rho, \lambda \rangle} \tilde{\lambda}_Z(\hat{t}) \cdot v_{\tilde{\lambda}_Z}.$$ \hfill $\square$

**Lemma 6.0.3.** We have $(\tilde{S}_{\sigma T} H_{G, \mathfrak{c}})(\mu) = 0$ in $C_c(T \parallel T_1, R)$.

**Proof.** The conjugacy classe $[\mu]$ (resp. $\mathfrak{c}$) gave rise to a dominant character $\hat{\mu}$ (resp. $\tilde{\mu}$) of $T$ and $\tilde{\mu} = \hat{\mu} \sigma(\hat{\mu}) \cdots \sigma^{d-1}(\hat{\mu})$.

To prove the lemma, it suffices to show that

$$\det \left(X - q^{d(\mu, \sigma \mathfrak{t})} r_T |_{V_{\hat{\mu}}}(\sigma \times \hat{t}^d)\right) \in C[\Phi_{un}(T)||X|$$

has $\hat{\mu}(\hat{t})$ as a root for all $\hat{t} \in \hat{T}$. Identify $\Phi_{un}(T)$ with the set of $\sigma$-conjugacy classes $\{\hat{t}\}$ of elements $\hat{t} \in T(\mathbb{C})$. For any $v \in V_{\hat{\mu}}$, we have

$$q^{d(\mu, \sigma \mathfrak{t})} r_T((\sigma \times \hat{t}^d) \cdot v = q^{d(\mu, \sigma \mathfrak{t})} r_T(\sigma^d \times (\hat{t} \sigma(\hat{t}) \cdots \sigma^{d-1}(\hat{t}))) \cdot v \quad \text{(Prop. 6.0.2)}$$

$$= \hat{\mu}(\hat{t}) \sigma(\hat{\mu})(\hat{t}) \cdots \sigma^{d-1}(\hat{\mu})(\hat{t}) \cdot v$$

$$= \tilde{\mu}(\hat{t}) \cdot v.$$ \hfill $\square$

We will show now the main theorems of the paper:

**Theorem 6.0.4** (Seed relation). The operator $u_{\mathfrak{w}^n} \in \mathcal{U}$ is a right root of the Hecke polynomial $H_{G, \mathfrak{c}}$ in the non-commutatif $R$-algebra $\text{End}_P(C_c(G/K, R))$.

**Proof.** Under the identifications $\Lambda_T \simeq X_*(T)_F \simeq X^*(\hat{T})_F$ the element $\mathfrak{w}^n T_1 \in \Lambda_T^-$ corresponds to the function $t \mapsto \tilde{\mu}(t)$.

Recall that by [Bon21d, Lemma 2.6.4] $u_{\mathfrak{w}^n} \in \text{End}_P C_c(G \parallel K, \mathbb{Z})$ and the coefficients of $H_{G, \mathfrak{c}}$ are in $\mathcal{H}(R) \simeq \text{End}_G C_c(G \parallel K, \mathbb{R})$ [Wed00, 2.8], thus

$$H_{G, \mathfrak{c}}(u_{\mathfrak{w}^n}) \in \text{End}_P C_c(G \parallel K, \mathbb{R}).$$

Using [Bon21a, Theorem 6.5.1], we see that $\hat{\Theta}_{Bern} \circ \tilde{S}_{\sigma T} H_{G, \mathfrak{c}} \in Z(\mathcal{H}_I(R))[X]$. Write $H_{G, \mathfrak{c}} = \sum_{k=1}^\infty h_k X^k$ and $\bar{h}_k = \hat{\Theta}_{Bern} \circ \tilde{S}_{\sigma T} h_k \in Z(\mathcal{H}_I(R))$. So $\bar{h}_k * 1_K = 1_K * \bar{h}_k = h_k$. We then

\textsuperscript{8}The weight spaces in the weyl orbit of the highest weight are one dimensional, but outside this distinguished weyl orbit, there are weight spaces which are not 1 dimensional.
have for any $p \in P$

$$1_{pK} \bullet H_{G,c}(u_{\mathfrak{m}^\nu}) = \sum_{k=1}^{r} (1_{pK} \bullet u_{\mathfrak{m}^\nu}^k) *_K h_k$$

$$= \sum_{k=1}^{r} (1_{pI} *_I i_{\mathfrak{m}^\nu}^k) *_I 1_K *_I h_k$$

$$= \sum_{k=1}^{r} (1_{pI} *_I i_{\mathfrak{m}^\nu}^k) *_I (\frac{1}{|K:I|} 1_K *_I h_k)$$

Lemma 6.0.3 $= 0$.

We have shown $H_{G,c}(u_{\mathfrak{m}^\nu}) = \sum_{k=1}^{r} h_k \circ u_{\mathfrak{m}^\nu}^k = 0 \in \text{End}_F(C_c(G/K,R))$. □

**Remark 6.0.5.** If $\mu_\varepsilon$ is minuscule, then $\Sigma_F(\mu_\varepsilon) = W(G_{\mathfrak{m}}) \mu_\varepsilon$ and accordingly the degree of the Hecke polynomial is $\deg(H_{G,c}) = |W(G_{\mathfrak{m}}, T) \mu_\varepsilon|$. In particular, $\deg(H_{G,c}) \geq \deg(P_\mu) = |W/W_\mu| = |W(G,S) \mu|$, where $P_\mu$ is the minimal polynomial of $u_{\mathfrak{m}^\nu}$ in $Z(HI(R))$ (see proof of [Bou21d, Theorem 2.8.1]). Therefore, if $G$ is a split group, $\mu_\varepsilon$ minuscule and $E = F$, then $H_{G,[\mu]} = P_\mu *_I 1_K$.

7. **Bültel’s annihilation relation**

In this last section we will show how Theorem 6.0.4 lifts (generalizes) a previously known result due to Büttel [Bül97, 1.2.11].

Let $\hat{S}_P: C_c(P/K_P, Q) \to C_c(L/K_L, Q)$ be the canonical homomorphism given by

$$f \mapsto \left( m \mapsto \int_{U_P^\vee} f(nm) d\mu_{U_P^\vee}(n) \right),$$

where $d\mu_{U_P^\vee}$ is the left-invariant Haar measure giving $K_{U_P^\vee}$ volume 1. Both $Q$-modules $C_c(P/K_P, Q)$ and $C_c(L/K_L, Q)$ are actually $Q$-algebras (by [Bou21a, Lemma 3.11.2]) and the transform $\hat{S}_P$ is an
algebra homomorphism. Indeed, let \( f, g \in C_c(P/K_P, \mathbb{Q}) \) then
\[
\hat{S}_P(f \ast_K g)(p) = \int_{U_P^+} \left( \int_P f(a)g(a^{-1}up)d\mu_P(a) \right)d\mu_{U_P^+}(u)
\]
\[
= \int_{U_P^+} \int_P \int_{U_P^+} f(nm)g(m^{-1}n^{-1}up)d\mu_{U_P^+}(n)d\mu_{L}(m)d\mu_{U_P^+}(u)
\]
\[
= \int_{U_P^+} \left( \int_P f(nm)d\mu_{U_P^+}(n) \right) \left( \int_{U_P^+} g(m^{-1}pa)d\mu_{U_P^+}(u) \right)d\mu_{L}(m)
\]
\[
= \hat{S}_P(f) \ast_K \hat{S}_P(g)(p)
\]
where, \( d\mu \) denotes the left invariant Haar measure giving \( K_P \) measure 1.

We also consider the map \( |p| \) sending any function on \( G \) to its restriction to \( P \). Using the Iwasawa decomposition \( G = PK \) ([Bou21a, Proposition 2.2.1]) one shows that this is actually an algebra homomorphism
\[
|p| : \mathcal{H}_K(R) \longrightarrow C_c(P \parallel K_P, R),
\]
and a \( |p|\)-linear module homomorphism
\[
|p| : C_c(G/K, R) \longrightarrow C_c(P/K_P, R).
\]

**Lemma 7.0.1.** Let \( p \in P \) and \( m \in L \), then:
\[
1_{pK} = 1_{pK_P} \text{ and } \hat{S}_P^L(1_{mK_P}) = |mK_{U_P^+}m^{-1}|_{U_P^+}1_{mK_L}.
\]

**Proof.** The first equality is a direct consequence of the Iwasawa decomposition. For the second it is deduced from the fact that \( K_P = K_LK_{U_P^+} \) given in [Bou21a, Proposition 2.2.1]:
\[
\hat{S}_P^L(1_{mK_P})(a) = \int_{U_P^+} 1_{mK_P}(ua)d\mu_{U_P^+}(u).
\]
The integrand is nonzero if and only if \( ua \in mK_P = mK_L \cdot K_{U_P^+} \), but since \( L \cap U_P^+ = \{1\} \), we have
\[
u \in aK_{U_P^+}a^{-1} \text{ and } a \in mK_L,
\]
which is equivalent to \( u \in mK_{U_P^+}m^{-1} \) and \( w \in mK_L \). Therefore,
\[
\hat{S}_P^L(1_{mK_P}) = \left. |mK_{U_P^+}m^{-1}|_{U_P^+}1_{mK_L} \right|_{U_P^+}.
\]

Observe that if \( mK_{U_P^+}m^{-1} \subset K_{U_P^+} \) then
\[
\left. |mK_{U_P^+}m^{-1}|_{U_P^+} = \frac{1}{[K_P : mK_{U_P^+}m^{-1}]} = \frac{1}{[K_P : mK_{U_P^+}m^{-1}]}.
\]

**Lemma 7.0.2.** We have a following commutative diagram of \( R \)-algebras
\[
\mathcal{H}_K(R) \xrightarrow{\hat{S}_P^L} C_c(L \parallel K_L, R)
\]
\[
\hat{S}_P^L \cong \cong \hat{S}_P^L
\]
\[
R[\Lambda_T]^W \xrightarrow{\hat{S}_P^L} R[\Lambda_T]^W
\]
where, \( W_L \) denotes the relative Weyl group of \( L \) (which is equal to the subgroup \( W_\mu \) of elements in \( W \) fixing \( \mu \)). The lowest horizontal arrow is the inclusion of \( W \)-invariants into \( W_L \)-invariants.
Proof. By definition of the parabolic \( P \), multiplication in \( G \) gives a bijection
\[
(U^+ \cap L) \cdot U^+_P \sim U^+.
\]
For any \( m \in L \) and \( h \in \mathcal{H}_K(R) \)
\[
\hat{S}^G_T(h)(m) = \int_{U^+_P} h(um) d\mu_{U^+_P}(u) \quad \text{[Bou21a, Lemma 5.1.2]}
\]
\[
= \int_{U^+ \cap L} h(u_1u_2m) d\mu_{U^+_P}(u_1) d\mu_{U^+ \cap L}(u_2)
\]
\[
= \int_{U^+ \cap L} \hat{S}^G_L(h)(u_1) d\mu_{U^+ \cap L}(u_2)
\]
\[
= \hat{S}^L_L(h)(m).
\]
Therefore, \( \hat{S}^G_T = \hat{S}^L_L \circ \hat{S}^G_L \) which confirms the claimed commutativity of the above diagram. Finally, the vertical maps are isomorphisms by [Bou21a, Theorem 5.2.1]. □

Let us reformulate the above twisted Satake homomorphism \( \hat{S}^G_T \) as a homomorphism of endomorphism rings. We have a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}_K(R) & \xrightarrow{\mid_P} & \mathcal{C}_c(P \parallel K_P, R) \\
\downarrow & & \downarrow \hat{S}_P \\
\text{End}_G \mathcal{C}_c(G/K, R) & \xrightarrow{(1)} & \text{End}_P \mathcal{C}_c(G/K, R) & \xrightarrow{(2)} & \text{End}_L \mathcal{C}_c(L/K_L, R) & \xrightarrow{(3)} & \text{End}_L \mathcal{C}_c(L/K_L, R).
\end{array}
\]

Let us first say few words about the homomorphisms (1) and (2):

- We have used the Iwasawa decomposition \( G = PK \) to identify \( G/K \simeq P/K_P \) for the middle vertical arrow, accordingly the homomorphism \( \mid_P \) induces the canonical injection (1):
  \[
  \text{End}_G \mathcal{C}_c(G/K, R) \longrightarrow \text{End}_P \mathcal{C}_c(G/K, R).
  \]

- We have a homomorphism of rings
  \[
  \text{End}_P \mathcal{C}_c(G/K, R) \longrightarrow \text{End}_P \mathcal{C}_c(U^+_P \setminus G/K, R)
  \]
  \[
  f \longmapsto (U^+_P gK \mapsto \Pi(f(gK)))
  \]
  where \( \Pi \) is the natural obvious map \( R[G/K] \rightarrow R[U^+_P \setminus G/K] \). But since \( P = LU^+_P \), we actually have \( \text{End}_P \mathcal{C}_c(U^+_P \setminus G/K, R) = \text{End}_L \mathcal{C}_c(U^+_P \setminus G/K, R) \).
  Using the Iwasawa decomposition again \( G = U^+_P LK \), we get a bijection
  \[
  U^+_P \setminus G/K \simeq L/K_L.
  \]

Thus, the homomorphism (2) is the composition
\[
\text{End}_P \mathcal{C}_c(G/K, R) \longrightarrow \text{End}_L \mathcal{C}_c(U^+_P \setminus G/K, R) \xrightarrow{\sim} \text{End}_L \mathcal{C}_c(L/K_L, R).
\]

- The homomorphism (3) is the twist by the modulus function \( \delta \).
Lemma 7.0.3. The operator $u_{\varpi^p}$ lives in $\text{End}_P C_c(G/K, R)$ and its image by the composition $(3) \circ (2)$ is precisely $g_{[p]}$.

Proof. Let us first compute the image of the operator $u_{\varpi^p}$ by the map (2). We have for all $a \in L$ (see [Bou21d, Lemma 2.6.4])

$$u_{\varpi^p}(1_{U^+_p a K}) = \sum_{p' \in [U^+_p \cap I^+ / U^+_p \cap \varpi^p I^+ \varpi^{-\mu}]} 1_{U^+_p a p' \varpi^p K}$$

$$= \#(U^+_p \cap I^+ / U^+_p \cap \varpi^p I^+ \varpi^{-\mu}) 1_{U^+_p a \varpi^p K}$$

$$= \#(I^+ / \varpi^p I^+ \varpi^{-\mu}) 1_{U^+_p a \varpi^p K} \quad [\text{Bou21d, Lemma 2.3.2}]$$

Hence, the image of $u_{\varpi^p} \in \text{End}_P C_c(G/K, R)$ by (2) is

$$\#(I^+ / \varpi^p I^+ \varpi^{-\mu}) g_{[p]} = \delta_B(\varpi^{-\mu}) g_{[p]} = q^{2(p-\mu)} g_{[p]}.$$  

Finally, (3) shows that the image of $u_{\varpi^p}$ by the composition $(3) \circ (2)$ is $g_{[p]} \in \text{End}_L C_c(L/K_L, R)$. □

Bültel’s annihilation result we have mentioned earlier is:

Corollary 7.0.4 (Bültel’s annihilation). We have

$$\hat{S}^G_L(H_{G, \varepsilon}(g_{[p]})) = 0 \in C_c(L \parallel K_L, R).$$

Bültel’s result as stated in [Wed00, §2.9] requires the conjugacy class $\varepsilon$ to be minuscule. We will derive this corollary from Theorem 6.0.4, showing that the assumption "minuscule" is superfluous.

Proof. By definition of the ”excursion” pairing [Bou21d, §2.6] and the proof of Lemma 7.0.3, we see that for all $p \in P$:

$$0 \overset{6.0.4}{=} (H_{G, \varepsilon}(u_{\varpi^p}) \bullet 1_{p K_p})|_P$$

$$= 1_{p K_p} * K_p 1_{K_p \varpi^p K_p} * K_p (H_{G, \varepsilon})|_P.$$  

This shows that

$$(H_{G, \varepsilon})|_P (1_{K_p \varpi^p K_p}) = 0,$$

and consequently we conclude

$$\hat{S}^G_L(H_{G, \varepsilon})(g_{[p]}) = \hat{S}_P ((H_{G, \varepsilon})|_P (1_{K_p \varpi^p K_p})) = 0. \quad \square$$

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