EXTENDED KLEIN MODEL AND A BOUND ON CURVES WITH NEGATIVE SELF-INTERSECTION

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ABSTRACT. Let $S$ be an irreducible smooth projective surface and $\mathcal{F}$ a collection of curves with negative self-intersection on $S$ such that no positive combination $aC_1 + bC_2$ is ample. In this paper, we provide an alternate proof that $\mathcal{F}$ is bounded by an exponential function of the Picard number $\rho(S)$ of $S$ using an extended version of the Klein disc model for hyperbolic space.

1. Introduction

Let us call an irreducible curve with negative self-intersection a negative curve. We will prove the following theorem.

**Theorem 1.1.** Let $S$ be an irreducible smooth projective surface and $\mathcal{F}$ a collection of negative curves such that a positive combination of any two is non-ample. Then $|\mathcal{F}|$ is bounded by an exponential function.

In a recent preprint, Chinburg and Stover showed using hyperbolic codes that for sufficiently large $\rho(X)$, $|\mathcal{F}| < 2^{0.902\rho(X)}$ [1]. We will show a similar bound on $|\mathcal{F}|$ using an extension of the Klein disc model for hyperbolic space.

From the Néron-Severi theorem, we know that the Néron-Severi group $\text{NS}(S)$ is a finitely generated abelian group of rank $\rho(S)$ [2]. We may extend the intersection pairing via the map $\text{NS}(S) \rightarrow \text{NS}(S) \otimes \mathbb{Z} \mathbb{R}$ and by the Hodge index theorem, $\text{NS}(S) \otimes \mathbb{Z} \mathbb{R} \cong \mathbb{R}^{1,n}$ as an inner product space where $n \leq \rho(S)$ is the rank of $\text{NS}(S)$ mod torsion and $\mathbb{R}^{1,n}$ is $\mathbb{R}^{n+1}$ endowed a signature $(1,n)$ inner product $H(\cdot,\cdot)$. In other words, tensoring with $\mathbb{R}$ extends the intersection pairing to be a signature $(1,n)$ inner product identified as $H(\cdot,\cdot)$.

2. Extended Klein model

We now introduce a model of $(\mathbb{R}^{1,n} - \{0\})/\mathbb{R}^+$ that easily exhibits orthogonality with respect to $H(\cdot,\cdot)$. We will see later that for Theorem 1.1 we only require the sign of $H(\cdot,\cdot)$. In our notation, let $\mathbb{R}^{n+1}$ be parametrized by coordinates $x_0, \ldots, x_n$. We first remind the reader that the Klein disc model models the points of hyperbolic $n$-space as the disc $K^n = \{1\} \times D^n \subset \mathbb{R}^{n+1}$ where $D^n$ is the open disc of radius 1 centered on the origin [3]. Alternatively, given the hyperboloid model, we may define the Klein disc model as the projection onto the plane $x_0 = 1$ from the origin.
Remark. We will omit discussion of the hyperbolic metric on $\mathcal{K}^n$ since we need only orthogonality to prove 1.1. Unless stated otherwise, all metrics within this paper will default to the Euclidean metric obtained via the isomorphism $\mathbb{R}^{1,n} \cong \mathbb{R}^{n+1}$. Additionally, $\|\cdot\|$ will always denote the Euclidean norm and $\|\cdot\|_H$ the hyperbolic norm induced by $H(\cdot,\cdot)$.

For this construction, define the map $s : (\mathbb{R}^{1,n} - \{0\})/\mathbb{R}^+ \rightarrow \{-1, 0, 1\}$ mapping a point $x$ to the sign of its norm $H(x,x)$. Consider the discs $D^+ = \{1\} \times D^n$, $D^- = \{-1\} \times D^n$. Their union $D = D^+ \cup D^-$ is a section of $s^{-1}(-1)$. Likewise we may consider $C = (-1, 1) \times S^{n-1}$, which is a section of $s^{-1}(1)$. We see that $\partial D = \partial C$ is a section of $s^{-1}(0)$.

Definition 2.1 (Extended Klein model). The extended Klein model is $\mathcal{E} = D \cup C \cup \partial D$. (See figure 1.)

If we take a point $c \in C \subset \mathcal{E}$, we may describe its orthogonal complement as follows. Acting by $O(n) \subset O^+(1,n)$, we may assume that $c = (x_c, 1, 0, \ldots, 0)$. Then the subspace $L \subset \mathbb{R}^{1,n}$ orthogonal to it intersects $D$ in two disjoint $(n-1)$-discs, one within each of $D^\pm$. Without loss of generality, let us work on $D^+$. Call $z$ the projection of $c$ onto $\partial D^+$ and $D = L \cap D^+$. Using a symmetry argument and $H(\cdot,\cdot)$, we see that the central point of $D$ is the point on $D$ closest to $c$. We will call this point $y$. From the coordinate representations $c = (x_c, 1, 0, \ldots, 0), z = (1, 1, 0, \ldots, 0), y = (1, x_y, 0, \ldots, 0)$ and acting by $O(n)$ to generalize, we see that $\|y-z\| = \|c-z\|$. (See figure 2.)

Lemma 2.2. We may represent a point $c \in C$ uniquely by $(z, \theta) \in S^{n-1} \times (0, \pi)$ with the explicit correspondence given in the proof.

Proof. With discussion and notation from the last paragraph, let $S^{n-1}$ be identified with $\partial D^+$ above and $z$ the projection of $c$ onto the aforementioned sphere. Let $\theta$ be the angular size of the cap defined by $L \cap \partial D^+$ containing $z$. Equivalently, $\theta = \arccos(1 - \|y-z\|)$. □
3. Proof of Theorem 1.1

By the Nakai-Moishezon criterion, $\mathcal{F}$ must satisfy that for any $C_i, C_j \in \mathcal{F}$,

(I) $C_i^2 < 0$

(II) $C_i \cdot C_j \geq 0$

(III) $(aC_i + bC_j)^2 \leq 0 \quad \forall a, b \in \mathbb{N}$

[2]. Note that these conditions are invariant under scaling by $\mathbb{R}^+$. Let $\Phi$ be the composition of maps $\text{NS}(S) - \{0\} \rightarrow \text{NS}(S) \otimes \mathbb{Z} \mathbb{R} - \{0\} \rightarrow \mathcal{E}$. We may maps $\mathcal{F}$ through $\Phi$ and denote $c_i = \Phi(C_i)$ and $\mathcal{G} = \Phi(\mathcal{F})$.

**Theorem 3.1.** Let $c_i \in \mathcal{G}$ be represented by $(z_i, \theta_i)$ with correspondence given in 2.2. Let $\delta_{ij}$ be the angular distance between $z_i$ and $z_j$ on $\partial \mathcal{D}^+$. $\mathcal{G}$ must satisfy

(i) $c_i \in \mathcal{C}$

(ii) $\cos \delta_{ij} \leq \cos \theta_i \cos \theta_j$

(iii) $\theta_i + \theta_j \geq \delta_{ij}$

**Proof.** We will show that the conditions denoted by the same roman numerals in Theorem 3.1 and the discussion above it are equivalent. Conditions I and i are equivalent since $\mathcal{C}$ contains precisely the elements of $c \in \mathcal{E}$ such that $\|c\|_H < 0$.

Condition II is equivalent to $H(c_i, c_j) \geq 0$. With notation from page 2, we see that $L_i$ bisects $\mathcal{E}$ into two connected components. By bicontinuity of the inner product, the component containing $z_i$ contains all points $x$ such that $H(c_i, x) < 0$. Acting by $O(n)$, we may without loss of generality assume $c_i = (\cos \theta_i, 1, 0, \ldots, 0)$, $c_j = (\cos \theta_j, \cos \delta_{ij}, \sin \delta_{ij}, 0, \ldots, 0)$. Recall that $L_i \cap \mathcal{D} = \{(\pm 1, \pm \cos \theta_i, x_2, \ldots, x_n)\}$, so $L_i$ intersects $x_0 = \cos \theta_2$ at $\{(\cos \theta_j, \cos \delta_{ij} \sin \delta_{ij}, x_2, \ldots, x_n)\}$. We then argue that $\cos \delta_{ij}$, the $x_1$ coordinate of $c_j$ not greater than $\cos \theta_i \cos \theta_j$, which is condition ii. (See figure 3.)
Now we turn to show conditions III and iii are equivalent. First we note that iii is equivalent to $D_i \cap D_j \neq \emptyset$ with equality case $\dim(D_i \cap D_j) < \dim D_i$, i.e. when the interior $D_i, D_j$ do not intersect. Passing through the map $\Phi$, we see that III is equivalent to $\|ac_i + c_j\|^2_H \leq 0$ for all $a \in \mathbb{R}^+$. Since this norm is invariant under $O^+(1, n)$, we may act by isometry and assume without loss of generality that $c_i = (0, 1, 0, \ldots, 0)$ and $c_j = (\cos \theta_j, \cos \delta_{ij}, \sin \delta_{ij}, 0, \ldots, 0)$. Maximizing with respect to $a$ and differentiating, we find that $a = -\cos \delta_{ij}$ gives the maximum value of $\|ac_i + c_j\|^2_H = \cos^2 \theta_j - \sin^2 \delta_{ij}$. Thus we have condition III equivalent to $|\cos \theta_j| \leq \sin \delta_{ij}$ when $c_i = (0, 1, 0, \ldots, 0)$. From here, it suffices to show equivalence of $|\cos \theta_j| = \sin \delta_{ij}$ and $\dim(D_i \cap D_j) < \dim D_i$ with $n = 2$ since intersection, $\mathcal{D}^+$, and $\|\cdot\|_H$ are all invariant under action by $O^+(1, n)$ and we may vary $\theta_j$ slightly to obtain the equivalence of III and iii.

From here, we parametrize $\mathcal{D}^+ = \{(x_1, x_2) | x_1 + x_2 \leq 1\}$. Recall that $z_i = (1, 0)$. (See figure 4 for the following construction.) Let $O = (0, 0), P = (0, 1), Z = z_j, R = (\cos \delta_{ij}, 0)$.
Let $Q \in \partial D^+$ such that $OZ$ bisects $PQ$ and let $X$ be their intersection. We then have $\triangle ROZ \equiv \triangle XPO$ and a fortiori $|QZ| = |ZO|$. Letting $D_j = PQ$ gives $\cos j = \sin \delta_{ij}$ and $\theta_i + \theta_j = \delta_{ij}$. Small variations on $\theta_j$ show the equivalence of the inequalities III and iii. □

Proof of Theorem 1.1. First, we can reduce condition ii to

(ii*) \[ \theta_i < \delta_{ij}. \]

Next, we require that $c_i \geq 0$ for all $c_i \in G$ by throwing out at most half of $G$. Embedding $\partial D^+ \hookrightarrow \mathbb{R}^n$, we may reduce ii* and iii to bounding a collection $B$ of open balls $B_i = B(z_i, \theta_i)$ of radii $\theta_i$ centered at $z_i$ in Euclidean space such that $z_i \notin B_j$ and $B_i \cap B_j \neq \emptyset$ for any balls $B_i, B_j \in B$. Consistent with prior notation, let $\delta_{ij} = \|z_i - z_j\|$.

A priori, let $|B|$ be finite. Choose $B_0, B_1$ and rescale the ambient space such that $\delta_{01} = 1$. Let $R^- = \{ p \in \mathbb{R}^n \| p - z_0 \| < 2 \}$ and $R^+ = \{ p \in \mathbb{R}^n \| p - z_0 \| \geq 2 \}$ and $B^\pm = \{ B_i \in B \| z_i \in R^\pm \}$. We have a naive bound $|B^-| < 2^{n+1}$. Let $B_i, B_j \in B^+$ and introduce coordinates such that $z_0 = 0$, $z_i = (0, k, \ldots, 0)$ with $k \geq 2$, and $p_j = (x, y, 0, \ldots, 0)$. From conditions ii* and iii, we have $\sqrt{x^2 + y^2} - 1 < \sqrt{x^2 + (y - k)^2}$, which along with $x^2 + y^2 \geq 4$ from $B_j \in B^+$ tells us that $B_i$ restricts $B_j$ outside of a cone of angle at least $2 \arctan \left( \frac{\sqrt{15}}{7} \right)$. Thus, $|B^+|$ is bounded by some exponential function of $n$ and $|F|$ bounded by an exponential function of $\rho(S)$. □

References

[1] T. Chinburg and M. Stover. Negative curves of small genus on surfaces. ArXiv e-prints, May 2011.

[2] Robin Hartshorne. Algebraic Geometry (Graduate Texts in Mathematics). Springer, 1st ed. 1977. corr. 8th printing 1997 edition, 4 1997.

[3] James W. Cannon, William J. Floyd, Richard Kenyon, and Walter R. Parry. Hyperbolic geometry. In Flavors of geometry, volume 31 of Math. Sci. Res. Inst. Publ., pages 59–115. Cambridge Univ. Press, Cambridge, 1997.