LATTICES RELATED TO EXTENSIONS OF PRESENTATIONS OF TRANVERSAL MATROIDS

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Abstract. For a presentation $\mathcal{A}$ of a transversal matroid $M$, we study the set $T_\mathcal{A}$ of single-element transversal extensions of $M$ that have presentations that extend $\mathcal{A}$; we order these extensions by the weak order. We show that $T_\mathcal{A}$ is a distributive lattice, and that each finite distributive lattice is isomorphic to $T_\mathcal{A}$ for some presentation $\mathcal{A}$ of some transversal matroid $M$. We show that $T_\mathcal{A} \cap T_\mathcal{B}$, for any two presentations $\mathcal{A}$ and $\mathcal{B}$ of $M$, is a sublattice of both $T_\mathcal{A}$ and $T_\mathcal{B}$. We prove sharp upper bounds on $|T_\mathcal{A}|$ for presentations $\mathcal{A}$ of rank less than $r(M)$ in the order on presentations; we also give a sharp upper bound on $|T_\mathcal{A} \cap T_\mathcal{B}|$. The main tool we introduce to study $T_\mathcal{A}$ is the lattice $L_\mathcal{A}$ of closed sets of a certain closure operator on the lattice of subsets of $\{1, 2, \ldots, r(M)\}$.

1. Introduction

We continue the investigation, begun in [4], of the extent to which a presentation $\mathcal{A}$ of a transversal matroid $M$ limits the single-element transversal extensions of $M$ that can be obtained by extending $\mathcal{A}$. The following analogy may help orient readers. A matrix $A$, over a field $F$, that represents a matroid $M$ may contain extraneous information; this can limit which $F$-representable single-element extensions of $M$ can be represented by extending (i.e., adjoining another column to) $A$. For instance, for the rank-3 uniform matroid $U_{3,6}$, partition $E(U_{3,6})$ into three 2-point lines, $L_1$, $L_2$, and $L_3$. Let $A$ be a $3 \times 6$ matrix, over $F$, that represents $U_{3,6}$. The line $L_i$ is represented by a pair of columns of $A$, which span a 2-dimensional subspace $V_i$ of $F^3$. While $V_i \cap V_j$, for $\{i, j\} \subset \{1, 2, 3\}$, has dimension 1 (since the corresponding lines of $U_{3,6}$ are coplanar), the intersection $V_1 \cap V_2 \cap V_3$ can, in general, have dimension either 0 or 1: this dimension is extraneous. If $\dim(V_1 \cap V_2 \cap V_3)$ is 1, then no extension of $A$ represents the extension of $M$ that has an element on, say, $L_1$ and $L_2$ but not $L_3$; otherwise, no extension of $A$ represents the extension of $M$ that has a non-loop on all three lines. (The underlying problem is the lack of unique representability, which is a major complicating factor for research on representable matroids. See Oxley [12, Section 14.6] for discussions.)

In this paper, we consider such problems, but for transversal matroids in place of $F$-representable matroids, and presentations in place of matrix representations.

A transversal matroid can be given by a presentation, which is a sequence of sets whose partial transversals are the independent sets. In [4], we introduced the ordered set $T_\mathcal{A}$ of transversal single-element extensions of $M$ that have presentations that extend $\mathcal{A}$ (i.e., the new element is adjoined to some of the sets in $\mathcal{A}$), where we order extensions by the weak order. In Section 3 we introduce a new tool for studying $T_\mathcal{A}$: given a presentation $\mathcal{A}$ of a transversal matroid $M$ with the number, $|\mathcal{A}|$, of terms in the sequence $\mathcal{A}$ being the rank, $r$, of $M$, we define a closure operator on the lattice $2^{|\mathcal{A}|}$ of subsets of the set $[r] = \{1, 2, \ldots, r\}$, and we show that the resulting lattice $L_\mathcal{A}$ of closed sets is a (necessarily

Date: March 5, 2022.

1991 Mathematics Subject Classification. Primary: 05B35.

Key words and phrases. Transversal matroid, presentation, single-element extension, distributive lattice.
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distributive) sublattice of $2^{|I|}$ that is isomorphic to $T_A$. While they are isomorphic, $L_A$ is

give several descriptions of its elements, show that every distributive lattice is isomorphic
to $L_A$, and so to $T_A$, for a suitable choice of $M$ and $A$, and we interpret the join- and
meet-irreducible elements of $L_A$. We show that if $A$ and $B$ are both presentations of $M$,
then $T_A \cap T_B$ is a sublattice of $T_A$ and of $T_B$. In [4], we showed that $|T_A| = 2^r$ if and
only if the presentation $A$ of $M$ is minimal in the natural order on the presentations of $M$;
using $L_A$, in Section [4] we prove upper bounds on $|T_A|$ for the next $r$ lowest ranks in this
order. We also show that $|T_A \cap T_B| \leq \frac{1}{4} \cdot 2^r$ whenever presentations $A$ and $B$ of $M$ differ
by more than just the order of the sets.

The relevant background is recalled in the next section. See Brualdi [5] for more about
transversal matroids, and Oxley [12] for other matroid background.

2. BACKGROUND

A set system $A = (A_i : i \in [r])$ on a set $E$ is a sequence of subsets of $E$. A partial
transversal of $A$ is a subset $X$ of $E$ for which there is an injection $\phi : X \to [r]$ with
$e \in A_{\phi(e)}$ for all $e \in X$; such an injection is an $A$-matching of $X$ into $[r]$. Edmonds
and Fulkerson [9] showed that the partial transversals of $A$ are the independent sets of a
matroid on $E$; we say that $A$ is a presentation of this transversal matroid $M[A]$.

The first lemma is an easy observation.

Lemma 2.1. Let $M$ be $M[A]$ with $A = (A_i : i \in [r])$. For any subset $X$ of $E(M)$, the
restriction $M[X]$ is transversal and $(A_i \cap X : i \in [r])$ is a presentation of $M[X]$.

We focus on presentations $(A_i : i \in [r])$ of $M$ that are of the type guaranteed by the
first part of Lemma 2.2 that is, $r = r(M)$; the second part of the lemma explains why
other presentations are not substantially different.

Lemma 2.2. Each transversal matroid $M$ has a presentation $A$ with $|A| = r(M)$. If $M$
has no coloops, then all presentations of $M$ have exactly $r(M)$ nonempty sets (counting
multiplicity).

Given a presentation $A = (A_i : i \in [r])$ of a transversal matroid $M$ and a subset $X$ of
$E(M)$, the $A$-support, $s_A(X)$, of $X$ is

$$s_A(X) = \{i : X \cap A_i \neq \emptyset\}.$$  

A cyclic set in a matroid $M$ is a (possibly empty) union of circuits; thus, $X \subseteq E(M)$ is
cyclic if and only if $M[X]$ has no coloops. Lemmas 2.1 and 2.2 give the next result.

Corollary 2.3. If $X$ is a cyclic set of $M[A]$, then $|s_A(X)| = r(X)$.

By Hall’s theorem [11, Theorem VIII.8.20], a subset $Y$ of $E(M)$ is independent in $M$ if
and only if $|s_A(Z)| \geq |Z|$ for all subsets $Z$ of $Y$. One can prove the next lemma from this.

Lemma 2.4. Let $A$ be a presentation of $M$.

(1) For any circuit $C$ of $M$ and element $e \in C$, we have

$$|s_A(C)| = |s_A(C \setminus \{e\})| = r(C) = |C| - 1,$$

so $s_A(C) = s_A(C \setminus \{e\})$.

(2) If $X \subseteq E(M)$ with $|s_A(X)| = r(X)$, then its closure, $\text{cl}(X)$, is

$$\text{cl}(X) = \{e : s_A(e) \subseteq s_A(X)\}.$$
Extending a presentation $\mathcal{A} = (A_i : i \in [r])$ of a transversal matroid $M$ consists of adjoining an element $x$ that is not in $E(M)$ to some of the sets in $\mathcal{A}$. More precisely, for an element $x \notin E(M)$ and a subset $I$ of $[r]$, we let $\mathcal{A}^I$ be $(A_i^I : i \in [r])$ where

$$A_i^I = \begin{cases} A_i \cup \{x\}, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

The matroid $M[\mathcal{A}^I]$ on the set $E(M) \cup \{x\}$ is a rank-preserving single-element extension of $M$. (This is the only type of extension we consider, so below we omit the adjectives “rank-preserving” and “single-element”.) Throughout this paper, we reserve $x$ for the element by which we extend a matroid.

We will use principal extensions of matroids, which we now recall. For any matroid $M$ (not necessarily transversal), a subset $Y$ of $E(M)$, and an element $x$ that is not in $E(M)$, the principal extension $M +_Y x$ of $M$ is the matroid on $E(M) \cup \{x\}$ with the rank function $r'$ where, for $Z \subseteq E(M)$, we have $r'(Z) = r_M(Z)$ and

$$r'(Z \cup \{x\}) = \begin{cases} r_M(Z), & \text{if } Y \subseteq \text{cl}_M(Z), \\ r_M(Z) + 1, & \text{otherwise.} \end{cases}$$

Thus, $M +_Y x = M +_Y x$ whenever $\text{cl}_M(Y) = \text{cl}_M(Y')$. Geometrically, $M +_Y x$ is formed by putting $x$ freely in the flat $\text{cl}_M(Y)$. A routine argument using matchings and part (2) of Lemma 2.4 yields the following result.

**Lemma 2.5.** Let $\mathcal{A}$ be a presentation of a transversal matroid $M$. If $Y$ is a subset of $E(M)$ with $|s_\mathcal{A}(Y)| = r(Y)$, then $M[\mathcal{A}^s_\mathcal{A}(Y)]$ is the principal extension $M +_Y x$, and, relative to containment, the least cyclic flat of $M[\mathcal{A}^s_\mathcal{A}(Y)]$ that contains $x$ is $\text{cl}_M(Y) \cup \{x\}$.

A transversal matroid typically has many presentations, and there is a natural order on them. A mild variant of the customary order on presentations best meets our needs. For presentations $\mathcal{A} = (A_i : i \in [r])$ and $\mathcal{B} = (B_i : i \in [r])$ of $M$, we set $\mathcal{A} \preceq \mathcal{B}$ if $A_i \subseteq B_i$ for all $i \in [r]$. We write $\mathcal{A} \prec \mathcal{B}$ if, in addition, at least one of these inclusions is strict. We say that $\mathcal{B}$ covers $\mathcal{A}$, and we write $\mathcal{A} \prec \mathcal{B}$, if $\mathcal{A} \prec \mathcal{B}$ and there is no presentation $\mathcal{C}$ of $M$ with $\mathcal{A} \prec \mathcal{C} \prec \mathcal{B}$. (The customary order identifies $(A_i : i \in [r])$ and $(A_{\tau(i)} : i \in [r])$ for any permutation $\tau$ of $[r]$, and so sets $\mathcal{A} \preceq \mathcal{B}$ if, up to re-indexing, $A_i \subseteq B_i$ for all $i \in [r]$.)

Mason [11] showed that if $(A_i : i \in [r])$ and $(B_i : i \in [r])$ are maximal presentations of the same transversal matroid, then there is a permutation $\tau$ of $[r]$ with $A_{\tau(i)} = B_i$ for all $i \in [r]$. (Minimal presentations, in contrast, are often more varied.) The next lemma, which is due to Bondy and Welsh [2] and plays important roles in this paper, gives a constructive way to find the maximal presentations of a transversal matroid.

**Lemma 2.6.** Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of $M$. Let $i$ be in $[r]$ and $e$ in $E(M) - A_i$. The following statements are equivalent:

1. the set system obtained from $\mathcal{A}$ by replacing $A_i$ by $A_i \cup \{e\}$ is also a presentation of $M$, and
2. $e$ is a coloop of the deletion $M \setminus A_i$.

A routine argument shows that the complement $E(M) - A_i$ of any set $A_i$ in $\mathcal{A}$ is a flat of $M[\mathcal{A}]$. By Lemma 2.6, the complement of each set in a maximal presentation of $M$ is a cyclic flat of $M$. Bondy and Welsh [2] and Las Vergnas [10] proved the next result about the sets in minimal presentations.

**Lemma 2.7.** A presentation $(C_i : i \in [r])$ of $M$ is minimal if and only if each set $C_i$ is a cocircuit of $M$, that is, $E(M) - C_i$ is a hyperplane of $M$. 
Thus, \((C_i : i \in [r])\) is minimal if and only if \(r(M\setminus C_i) = r - 1\) for all \(i \in [r]\). The next result, by Brualdi and Dinolt [6], follows from the last two lemmas.

**Lemma 2.8.** If \(\mathcal{A} = (A_i : i \in [r])\) is a presentation of \(M\) and \(\mathcal{C} = (C_i : i \in [r])\) is a minimal presentation of \(M\) with \(\mathcal{C} \leq \mathcal{A}\), then
\[
|A_i - C_i| = r(M\setminus C_i) - r(M\setminus A_i) = r - 1 - r(M\setminus A_i).
\]

**Corollary 2.9.** The ordered set of presentations of a rank-\(r\) transversal matroid \(M\) is ranked; the rank of a presentation \((A_i : i \in [r])\) is
\[
r(r - 1) - \sum_{i=1}^{r} r(M\setminus A_i).
\]

This corollary applies to both the order we focus on, \(\mathcal{A} \preceq \mathcal{B}\), and the more customary order, \(\mathcal{A} \preceq \mathcal{B}\); the rank of a presentation is the same in both orders.

The weak order \(\preceq_w\) on matroids on the same set \(E\) is defined as follows: \(M \preceq_w N\) if \(r_M(X) \leq r_N(X)\) for all subsets \(X\) of \(E\); equivalently, every independent set of \(M\) is independent in \(N\). This captures the idea that \(N\) is freer than \(M\). The next two lemmas are simple but useful observations.

**Lemma 2.10.** Let \(M = M[[A_i : i \in [r]]]\) and \(N = M[[B_i : i \in [r]]]\), where \(M\) and \(N\) are defined on the same set. If \(A_i \subseteq B_i\) for all \(i \in [r]\), then \(M \preceq_w N\).

**Lemma 2.11.** Assume that \(M \preceq_w N\) and \(M\setminus e = N\setminus e\). If \(e\) is a coloop of \(M\), then \(e\) is a coloop of \(N\), and so \(M = N\).

Lastly, we recall how to think of transversal matroids geometrically and to give affine representations of those of low rank, as in Figures 1 and 2. A set system \(\mathcal{A} = (A_i : i \in [r])\) on \(E\) can be encoded by a 0-1 matrix with \(r\) rows whose columns are indexed by the elements of \(E\) in which the \(i, e\) entry is 1 if and only if \(e \in A_i\). If we replace the 1s in this matrix by distinct variables, say over \(\mathbb{R}\), then it follows from the permutation expansion of determinants that the linearly independent columns are precisely the partial transversals of \(\mathcal{A}\), so this is a matrix representation of \(M[\mathcal{A}]\). One can in turn replace the variables by non-negative real numbers and preserve which square submatrices have nonzero determinants; one can also scale the columns so that the sum of the entries in each nonzero column is 1. In this way, each non-loop of \(M\) is represented by a point in the convex hull of the standard basis vectors. This yields the following geometric picture: label the vertices of a simplex 1, 2, \ldots, \(r\) and think of associating \(A_i\) to the \(i\)-th vertex, then place each point \(e\) of \(E\) freely (relative to the other points) in the face of the simplex spanned by \(s_{\mathcal{A}}(e)\).

3. A CLOSURE OPERATOR AND TWO ISOMORPHIC DISTRIBUTIVE LATTICES

Let \(\mathcal{A}\) be a presentation of \(M\). In [1], we introduced the ordered set \(T_{\mathcal{A}}\) of transversal extensions of \(M\) that have presentations that extend \(\mathcal{A}\), ordering \(T_{\mathcal{A}}\) by the weak order. As the results in this paper demonstrate, the lattice \(L_{\mathcal{A}}\) of subsets of \([r(M)]\) that we define in this section and show to be isomorphic to \(T_{\mathcal{A}}\) is very useful for studying \(T_{\mathcal{A}}\).

Recall that we consider only single-element rank-preserving extensions. Also, \(x\) always denotes the element by which we extend a matroid.

3.1. The lattice \(L_{\mathcal{A}}\). The first lattice we discuss is the lattice of closed sets for a closure operator that we introduce below, so we first recall closure operators (see, e.g., [1, p. 49]). A closure operator on a set \(S\) is a map \(\sigma : 2^S \rightarrow 2^S\) for which

1. \(X \subseteq \sigma(X)\) for all \(X \subseteq S\),
The set of any closure operator. We claim that Theorem 3.1. For any presentation \( A \) of a transversal matroid \( M \), along with the associated lattices \( L_A \).

(2) if \( X \subseteq Y \subseteq S \), then \( \sigma(X) \subseteq \sigma(Y) \), and  
(3) \( \sigma(\sigma(X)) = \sigma(X) \) for all \( X \subseteq S \).

Given a closure operator \( \sigma : 2^S \rightarrow 2^S \), a \( \sigma \)-closed set is a subset \( X \) of \( S \) with \( \sigma(X) = X \). The set of \( \sigma \)-closed sets, ordered by containment, is a lattice; join and meet are given by \( X \lor Y = \sigma(X \cup Y) \) and \( X \land Y = X \cap Y \). By property (1), the set \( S \) is \( \sigma \)-closed.

Let \( A \) be a presentation of a rank-\( r \) transversal matroid \( M \). By Lemma 2.6, for each subset \( I \) of \( [r] \), there is a greatest subset \( K \) of \( [r] \), relative to containment, for which \( \mathcal{M}[A^I] = \mathcal{M}[A^K] \), namely

\[
K = I \cup \{ k \in [r] - I : x \text{ is a coloop of } (\mathcal{M}[A^I]) \setminus A_k \};
\]

define a map \( \sigma_A : 2^{[r]} \rightarrow 2^{[r]} \) by setting \( \sigma_A(I) = K \). We next show that \( \sigma_A \) is a closure operator. We use \( L_A \) to denote the lattice of \( \sigma_A \)-closed sets. See Figure I for examples.

**Theorem 3.1.** For any presentation \( A = (A_i : i \in [r]) \) of a transversal matroid \( M \), the map \( \sigma_A \) defined above is a closure operator on \( [r] \). The join in the lattice \( L_A \) of \( \sigma_A \)-closed sets is given by \( I \lor J = I \cup J \), so \( L_A \) is distributive. Both \( \emptyset \) and \( [r] \) are in \( L_A \).

**Proof.** Properties (1) and (3) of closure operators clearly hold. For property (2), assume \( I \subseteq J \subseteq [r] \) and \( h \in \sigma_A(I) - I \), so \( x \) is a coloop of \( \mathcal{M}[A^I] \setminus A_h \). Lemma 2.10 gives \( \mathcal{M}[A^I] \setminus A_h \leq_M \mathcal{M}[A^J] \setminus A_h \), so \( x \) is a coloop of \( \mathcal{M}[A^J] \setminus A_h \) by Lemma 2.11 so \( h \in \sigma(J) \), as needed.

Let \( I \) and \( J \) be in \( L_A \). Their meet, \( I \land J \), is \( I \cap J \) since, as noted above, this holds for any closure operator. We claim that \( I \lor J = I \cup J \). (The fact that \( L_A \) is distributive then follows since union and intersection distribute over each other.) Since \( I \) and \( J \) are in \( L_A \),

1. if \( h \in [r] - I \), then \( x \) is not a coloop of \( \mathcal{M}[A^I] \setminus A_h \), and
2. if \( h \in [r] - J \), then \( x \) is not a coloop of \( \mathcal{M}[A^J] \setminus A_h \).

\[
A_2 = \{a, b, c, d, e, f\} \\
A_1 = \{a, b, c\} \\
A_3 = \{d, e, f, g, h, i\} \\
A_4 = \{g, h, i\} \\
{1, 2, 3, 4} \quad {1, 2, 3}\{2, 3, 4\} \\
\{1, 2\} \{2, 3\} \{3, 4\} \\
\emptyset
\]

**Figure 1.** Two presentations \( A \) of a transversal matroid \( M \), along with the associated lattices \( L_A \).
Note that the following two statements are equivalent: (i) \( I \lor J = I \cup J \) and (ii) \( I \cup J \) is \( \sigma_A \)-closed. To prove statement (ii), let \( h \) be in \( [r] - (I \cup J) \) and let \( Z \) be a basis of \( M \setminus A_h \). If \( x \) were a coloop of \( M[A_{I \cup J}] \setminus A_h \), then there would be an \( A_{I \cup J} \)-matching \( \phi : Z \cup \{ x \} \to [r] \). Either \( \phi(x) \in I \) or \( \phi(x) \in J \); if \( \phi(x) \in I \), then \( \phi \) shows that \( Z \cup \{ x \} \) is independent in \( M[A_I] \setminus A_h \), contrary to item (1) above; similarly, \( \phi(x) \in J \) contradicts item (2). Thus, as needed, \( x \) is not a coloop of \( M[A_{I \cup J}] \setminus A_h \).

Note that \( \emptyset \) is in \( L_A \) since \( x \) is a loop of \( M[A^I] \) if and only if \( I = \emptyset \).

We now show how the order on presentations relates to the lattices of closed sets.

**Theorem 3.2.** For two presentations \( A = (A_i : i \in [r]) \) and \( B = (B_i : i \in [r]) \) of \( M \), if \( A \leq B \), then \( L_B \) is a sublattice of \( L_A \) and \( M[A^I] = M[B^I] \) for all \( I \subseteq L_B \).

**Proof.** Fix \( I \) in \( L_B \). Set \( M_B = M[B^I] \) and \( M_A = M[A^I] \). For \( i \in [r] - I \), the element \( x \) is not a coloop of \( M_B \setminus B_i \) since \( I \in L_B \). Now \( M_A \setminus B_i \leq_w M_B \setminus B_i \), so \( x \) is not a coloop of \( M_A \setminus B_i \) by Lemma 2.4, so \( x \) is not a coloop of \( M_A \setminus A_i \). Thus, \( I \subseteq L_A \), so \( L_B \) is a sublattice of \( L_A \). Lemma 2.4 and the following two claims give \( M_A = M_B \):

1. For each \( i \in I \), each element of \( (B_i \cup \{ x \}) - (A_i \cup \{ x \}) \) (that is, \( B_i - A_i \)) is a coloop of \( M_A \), \( (A_i \cup \{ x \}) \) (that is, \( M_A \)), and \( A_i \).
2. For each \( i \in [r] - I \), each element of \( B_i - A_i \) is a coloop of \( M_A \).

By the hypothesis and Lemma 2.6 for all \( i \in [r] \), each element of \( B_i - A_i \) is a coloop of \( M_A \), so claim (1) holds. For claim (2), fix \( i \in [r] - I \) and \( y \in B_i - A_i \). As shown above, \( x \) is not a coloop of \( M_A \setminus B_i \); let \( C \) be a circuit of \( M_A \setminus B_i \) with \( x \in C \). Thus, \( y \notin C \). Assume, contrary to claim (2), that some circuit \( C' \) of \( M_A \setminus A_i \) contains \( y \). Now \( x \in C' \) since \( y \) is coloop of \( M_A \). By strong circuit elimination, applied in \( M_A \setminus A_i \), some circuit \( C'' \subseteq (C \cup C') - \{ x \} \) contains \( y \); however, \( C'' \) is a circuit of \( M \setminus A_i \), which contradicts \( y \) being a coloop of \( M \setminus A_i \). Thus, claim (2) holds.

The corollary below is a theorem from \[4\].

**Corollary 3.3.** For each transversal extension \( M' \) of \( M \), there is a minimal presentation of \( M \) that can be extended to a presentation of \( M' \).

### 3.2. The lattice \( T_A \)

The lattice \( T_A \) consists of the set \( \{ M[A^I] : I \in L_A \} \) of transversal extensions of \( M \) that have presentations that extend \( A \), which we order by the weak order. The next result relates \( T_A \) and \( L_A \).

**Theorem 3.4.** Let \( A \) be a presentation of \( M \). For any sets \( I \) and \( J \) in \( L_A \), we have \( M[A^I] \leq_w M[A^J] \) if and only if \( I \subseteq J \). Thus, the bijection \( I \mapsto M[A^I] \) from \( L_A \) onto \( T_A \) is a lattice isomorphism, so \( T_A \) is a distributive lattice.

**Proof.** Assume that \( M[A^I] \leq_w M[A^J] \). Any \( A_{I \cup J} \)-matching \( \phi \) of an independent set \( X \) of \( M[A_{I \cup J}] \) with \( x \in X \) has \( \phi(x) \) in either \( I \) or \( J \), so \( X \) is independent in one of \( M[A^I] \) and \( M[A^J] \), and so, by the assumption, in \( M[A^I] \). Thus, \( M[A^I] \leq_w M[A^J] \). The equality \( M[A^I] = M[A^I_{I \cup J}] \) now follows by Lemma 2.10; thus, \( J = I \cup J \) since \( J \) and \( I \cup J \) are \( \sigma_A \)-closed, so \( I \subseteq J \). The other implication follows from Lemma 2.10.

**Corollary 3.5.** For presentations \( A \) and \( B \) of \( M \), if \( A \leq B \), then \( T_B \) is a sublattice of \( T_A \).

The converse of the corollary fails even under the more common order on presentations as we now show.

**Example 1.** Consider the uniform matroid \( U_{3,4} \) on \( \{a, b, c, d\} \) and its presentations

\[ A = (\{a, b, d\}, \{a, c, d\}, \{b, c, d\}) \quad \text{and} \quad B = (\{a, b, c\}, \{a, b, d\}, \{a, c, d\}). \]
It is easy to check that both $T_A$ and $T_B$ consist of just the extension by a loop, $U_{3,4} \oplus U_{0,0}$, and the free extension, $U_{3,5}$. Thus, $T_A = T_B = T_C$, where $C$ is a maximal presentation of $U_{3,4}$, that is, $C = \{\{a, b, c, d\}, \{a, b, c, d\}, \{a, b, c, d\}\}$.

From the next result, which is a reformulation of [3, Theorem 3.1], we see that we cannot recover the presentation $A$ from $L_A$.

**Theorem 3.6.** A presentation $A = (A_i : i \in [r])$ of a transversal matroid $M$ is minimal if and only if $L_A = 2^{[r]}$, that is, $|L_A| = 2^r$.

**Proof.** If $A$ is not minimal, then $r(M \setminus A_i) < r - 1$ for some $i \in [r]$; thus, $x$ is a coloop of $M[A^{[r]-\{i\}] \setminus A_i$, so $[r] - \{i\} \not\in L_A$. If $A$ is minimal, then $x$ is not a coloop of $M[A^{[r]} \setminus A_j$ for distinct $i, j \in [r]$ since $r(M \setminus A_j) = r - 1$; thus, $\{i\} \in L_A$, so closure under unions gives $L_A = 2^{[r]}$.

As Example 1 shows, we cannot always reconstruct the sets in $A$ from $T_A$; however, in some cases we can. For the matroid in Figure 1, one can check that the sets in each of its 3.3. The sets in $L_A$. The results in this section, other than Corollary 3.8, are used heavily in Section 4. We start with several characterizations of the sets in $L_A$.

**Theorem 3.7.** For a presentation $A$ of a transversal matroid $M$, the sets in $L_A$ are

1. the sets $s_A(X)$, where $X$ is an independent set of $M$ and $|X| = |s_A(X)|$, and
2. all intersections of such sets.

In particular, for $I \in L_A$, if $C_x$ is the set of all circuits of $M[A^I]$ that contain $x$, then

\[
I = \bigcap_{C \in C_x} s_A(C \setminus \{x\}).
\]

Item (1) could be replaced by: (1') the sets $s_A(Y)$ where $r(Y) = |s_A(Y)|$.

**Proof.** Set $r = r(M)$. First assume that $X$ satisfies condition (1). Set $I = s_A(X)$. Thus, $X \cup \{x\}$ is independent in $M[A^I]$ but independent in $M[A^I \cup \{h\}]$ for any $h \in [r] - I$, so $I$ is in $L_A$. Since $L_A$ is closed under intersection, all sets identified above are in $L_A$.

Fix $I$ in $L_A$ and let $C_x$ be as defined above. Let $X$ be $C - \{x\}$ for some $C \in C_x$, so $X$ is independent in $M$. Now $s_A(X) = s_A'(X)$, and Lemma 2.3 gives $|s_A'(X)| = |X|$, so $|X| = |s_A(X)|$. Also, $I = s_A'(x) \subseteq s_A'(C) = s_A(X)$, so to prove equation (3.1) and show that all sets in $L_A$ are given by items (1) and (2), it suffices to show that for each $h$ in $[r] - I$, there is some $C_h \in C_x$ with $h \not\in s_A(C_h \setminus \{x\}).$ Now $M[A^I] \leq_{w} M[A^I \cup \{h\}]$, so some circuit, say $C_h$, of $M[A^I]$ is independent in $M[A^I \cup \{h\}]$. Thus, $C_h \in C_x$ and

\[
|s_{A^I \cup \{h\}}(C_h)| \geq |C_h| > |s_{A^I}(C_h)|,
\]

so $h \not\in s_{A^I}(C_h)$, so $h \not\in s_A(C_h \setminus \{x\})$, as needed.

Item (1') can replace item (1) since, by Lemma 2.3, $r(Y) = |s_A(Y)|$ for a set $Y$ if and only if $|X| = |s_A(X)|$ for some (equivalently, every) basis $X$ of $M[Y]$. \qed
Lemma 2.6. If $H$ has adjoining any elements in particular, if $I$ consists of all such sets (which include 0), along with $[r]$. By Lemma 2.6, since $I \in L_A$, and thus of $L_A$ isomorphism above, $I$ is a coloop of $M$. Thus, Theorem 3.7 gives $|s_A(X) − s_A(F)| = |s_A(X') - s_A(X)|$. Now $s_A(X') = s_A(X \cup F)$ and $J \cup s_A(F) = \bigcap_{X' : X \in \mathcal{J}} s_A(X')$.

Also, $|X'| = |s_A(X')|$. Thus, Theorem 3.7 gives $J \cup s_A(F) \in L_A$.

For the last assertion, take $J = s_A(e) - \{h\}$ and $F = \{e\}$.

The next result gives conditions under which the support of a set is, or is not, closed.

Corollary 3.8. Let $A = (A_i : i \in [r])$ be a presentation of $M$. Fix $F \subseteq E(M)$ and $J \in L_A$, and set $H = s_A(F) - J$. If $|H| \leq |F|$ and $H \subseteq s_A(e)$ for all $e \in F$, then $J \cup s_A(F) \in L_A$. In particular, if $s_A(e) = \{h\}$ is an automorphism $M$ and $h \in s_A(e)$, then $s_A(e) \in L_A$.

Proof. Since $J \in L_A$, there is a set $\mathcal{J}$ of subsets $X$ of $E(M)$, all satisfying condition (1) of Theorem 3.7 with $J = \bigcap_{X \in \mathcal{J}} s_A(X)$. For each set $X \in \mathcal{J}$, form a new set $X'$ by adjoining any $|s_A(F) - s_A(X)|$ elements of $F$ to $X$. Note that $X'$ is independent: match elements in $X' \setminus X$ to $s_A(F) - s_A(X)$.

Corollary 3.9. Let $A$ be a presentation of $M$. Fix $F \subseteq E(M)$ and $J \in L_A$, and set $H = s_A(F) - J$. If $|H| \leq |F|$ and $H \subseteq s_A(e)$ for all $e \in F$, then $J \cup s_A(F) \in L_A$. In particular, if $s_A(e) = \{h\}$ is an automorphism of $M$ and $h \in s_A(e)$, then $s_A(e) \in L_A$.

Proof. Since $J \in L_A$, there is a set $\mathcal{J}$ of subsets $X$ of $E(M)$, all satisfying condition (1) of Theorem 3.7 with $J = \bigcap_{X \in \mathcal{J}} s_A(X)$. For each set $X \in \mathcal{J}$, form a new set $X'$ by adjoining any $|s_A(F) - s_A(X)|$ elements of $F$ to $X$. Note that $X'$ is independent: match elements in $X' \setminus X$ to $s_A(F) - s_A(X)$. Now $s_A(X') = s_A(X \cup F)$ and $J \cup s_A(F) = \bigcap_{X' : X \in \mathcal{J}} s_A(X')$.

Also, $|X'| = |s_A(X')|$. Thus, Theorem 3.7 gives $J \cup s_A(F) \in L_A$.

For the last assertion, take $J = s_A(e) - \{h\}$ and $F = \{e\}$.

The next result gives conditions under which the support of a set is, or is not, closed.

Theorem 3.10. Let $A = (A_i : i \in [r])$ and $B = (B_i : i \in [r])$ be presentations of $M$.

1. If the presentation $A$ is maximal, then $s_A(X) \in L_A$ for all $X \subseteq E(M)$.
2. Assume $A \prec B$. For $X \subseteq E(M)$, if $s_A(X) \neq s_B(X)$, then $s_A(X) \notin L_B$.

Proof. We start with an observation. For an element $e \in E(M)$, set $I = s_A(e)$. Since $e$ and $x$ are in the same sets in $A^l$, the transposition $\phi$ on $E(M) \cup \{x\}$ that switches $e$ and $x$ is an automorphism of $M[A^l]$. Thus, $\phi$ restricted to $E(M)$ is an isomorphism of $M$ onto $M[A^l \setminus e]$.

For part (1), since $L_A$ is closed under unions, it suffices to treat a singleton set $\{e\}$. Since $[r] \in L_A$, we may assume that $s_A(e) \neq [r]$. Set $I = s_A(e)$ and fix $h \in [r] - I$. By Lemma 2.6 since $A$ is maximal, $e$ is not a coloop of $M[A^l \setminus A_h]$, so, by the isomorphism above, $x$ is not a coloop of $M[A^l \setminus (A_h \cup \{e\})]$. Thus, $x$ is not a coloop of $M[A^l \setminus A_h]$, so $I \subseteq L_A$.

For part (2), set $J = s_A(X)$, fix $h \in s_B(X) - J$, and pick $e \in X$ with $h \in s_B(e)$. Set $I = s_A(e)$. Since $A \prec B$, the element $e$ is a coloop of $M[A^l]$. By Lemma 2.6, the isomorphism above, $x$ is a coloop of $M[A^l \setminus (A_h \cup \{e\})]$, and thus of $M[B^l \setminus (A_h \cup \{e\})]$ by Lemma 2.11 and thus of $M[B^l \setminus B_h]$. Thus, $J \notin L_B$. □
Let \( A = \{ A_i : i \in [r] \} \) be a maximal presentation of \( M \). Thus, \( s_A(e) \in L_A \) for all \( e \in E(M) \) by Theorem 3.10. The unions of the sets \( s_A(e) \) include the supports of all cyclic flats, but intersections of supports of cyclic flats, which are in \( L_A \), need not be intersections of the sets \( s_A(e) \), as the example in Figure 2 shows. Each presentation \( A \) of \( M \) is both maximal and minimal, so \( L_A = 2^L \). However, \( \{2, 3\} \) is not an intersection of the \( A \)-supports of singletons. Thus, the sets \( s_A(e) \) generate \( L_A \), but both their unions and the intersections of such unions are needed to obtain all of \( L_A \).

**Corollary 3.11.** Let \( A \) and \( B \) be presentations of \( M \) with \( A \prec B \). The sublattice \( L_B \) of \( L_A \) is a proper sublattice of \( L_A \) if either of the conditions below holds.

1. There is an \( e \in E(M) \) and \( h \in s_A(e) \) with \( s_A(e) - \{h\} \in L_B \) and \( s_A(e) \neq s_B(e) \).
2. For each \( I \in 2^{[r]} - L_B \), there is some \( h \in I \) with \( I - \{h\} \in L_B \).

**Proof.** Condition (1), Corollary 3.9 and Theorem 3.10 give \( s_A(e) \in L_A - L_B \). For the second condition, since \( A \prec B \), there is an \( e \in E(M) \) with \( s_A(e) \neq s_B(e) \), so condition (1) applies. \( \square \)

### 3.4. The intersection of \( T_A \) and \( T_B \).

We show that, for presentations \( A \) and \( B \) of a transversal matroid \( M \), the intersection \( T_A \cap T_B \) is a sublattice of \( T_A \) and of \( T_B \), so for pairs of extensions that are in both of these lattices, their meet in \( T_A \) is their meet in \( T_B \), and likewise for joins. This line of inquiry is motivated in part by the following question [4, Problem 4.1]: is the set of all rank-preserving single-element transversal extensions of a transversal matroid, ordered by the weak order, a lattice? An affirmative answer would provide a transversal counterpart of the well-known result of Crapo [8]: the set of all single-element extensions of a matroid \( M \), ordered by the weak order, is a lattice. (This lattice is called the lattice of extensions of \( M \).) While it is far from addressing the question about the transversal extensions of a transversal matroid \( M \), the next result, from [4], shows that the join in \( T_A \) is the join in the lattice of extensions of \( M \).

**Lemma 3.12.** Let \( A \) be a presentation of \( M \), and \( r = r(M) \). For any subsets \( I \) and \( J \) of \( [r] \), the join of \( M[A^I] \) and \( M[A^J] \) in the lattice of extensions of \( M \) is transversal and is \( M[A^{I\cup J}] \).

**Corollary 3.13.** Let \( A \) and \( B \) be presentations of a transversal matroid \( M \). If \( M_1 \) and \( M_2 \) are in both \( T_A \) and \( T_B \), then their join in \( T_A \) is their join in \( T_B \).

**Proof.** Since \( M_1 \) and \( M_2 \) are in both \( T_A \) and \( T_B \), there are sets \( I_1 \) and \( I_2 \) in \( L_A \), and sets \( J_1 \) and \( J_2 \) in \( L_B \), with \( M[I_1] = M[J_1] = M_1 \) and \( M[I_2] = M[J_2] = M_2 \). By the isomorphism in Theorem 3.4, the join of \( M_1 \) and \( M_2 \) in \( T_A \) is \( M[A^{I_1\cup I_2}] \), and that in \( T_B \) is...
$A_1 = \{a, b, c, d, g\}$

$A_2 = \{c, d, e, f, g\}$

$A_3 = \{a, b, e, f, g\}$

$A_4 = \{g, h\}$

$B_1 = \{a, b, c, d, h\}$

$B_2 = \{c, d, e, f, h\}$

$B_3 = \{a, b, e, f, h\}$

$B_4 = \{g, h\}$

**Figure 3.** The presentations and the meet of the extensions discussed in Example 2. In the first figure, $g$ is in no proper face of the simplex; in the second, $h$ is in no proper face.

The situation for meets is more complex, as the example below illustrates.

**Example 2.** Consider the matroid $M$ shown in the first two diagrams in Figure 3 and the two presentations given there. In the extension $M_1 = M[A^{(1)}_{A}] = M[B^{(1)}_{B}]$, both $\{x, a, b\}$ and $\{x, c, d\}$ are lines. In the extension $M_2 = M[A^{(2)}_{A}] = M[B^{(2)}_{B}]$, both $\{x, c, d\}$ and $\{x, e, f\}$ are lines. In the meet of $M_1$ and $M_2$ in the lattice of extensions of $M$, each of $\{x, a, b\}$, $\{x, c, d\}$ and $\{x, e, f\}$ is dependent; this meet, which is shown in the third diagram in Figure 3, is not transversal. One way to see this is that the three coplanar 3-point lines through $x$ are incompatible with the affine representation described at the end of Section 2. That view also implies that the meet of $M_1$ and $M_2$ in both $T_A$ and $T_B$ is formed by extending $M$ by a loop.

This example illustrates the next result: the meet of $M_1$ and $M_2$ in $T_A$ is their meet in $T_B$ (even though these can differ from their meet in the lattice of all extensions).

**Theorem 3.14.** If $A$ and $B$ are presentations of $M$, then the set

$$L_{A,B} = \{I \in L_A : M[A^I] = M[B^J] \text{ for some } J \in L_B\}$$

is $M[B^{I_1 \cup I_2}]$. As claimed, these matroids are equal since, by Lemma 3.12,

$$M[A^{I_1 \cup I_2}] = M[A^{I_1}] \lor M[A^{I_2}] = M[B^{I_1}] \lor M[B^{I_2}] = M[B^{I_1 \cup I_2}],$$

where $\lor$ denotes the join in the lattice of extensions of $M$. □
is a sublattice of $L_A$. The sublattices $L_{A,B}$, of $L_A$, and $L_{B,A}$, of $L_B$, are isomorphic, and $T_A \cap T_B$ is a sublattice of both $T_A$ and $T_B$.

The proof of this theorem uses the following result from \[4\].

**Lemma 3.15.** Let $M$ be $M[A]$. For subsets $X$ and $Y$ of $E(M)$, if $r(X) = |s_A(X)|$ and $r(Y) = |s_A(Y)|$, then $r(X \cup Y) = |s_A(X \cup Y)|$.

**Proof of Theorem 3.14.** The closure of $L_{A,B}$ under unions follows from the argument that gives equation (3.2). We next show that the closure of $L_{A,B}$ under intersections follows from statement (3.14.1), which we then prove.

(3.14.1) For subsets $X_1, X_2, \ldots, X_t$ of $E(M)$, if $|s_A(X_k)| = r(X_k) = |s_B(X_k)|$ for all $k \in [t]$, then $\bigcap_{k=1}^t s_A(X_k) \in L_{A,B}$.

To see why proving this statement suffices, consider a pair $I_1 \in L_A$ and $J_1 \in L_B$ with $M[A^I] = M[B^J]$; let $M'$ denote this extension of $M$. By equation (3.1),

$$I_1 = \bigcap_{C \in C_x} s_A(C - \{x\}) \quad \text{and} \quad J_1 = \bigcap_{C \in C_x} s_B(C - \{x\}),$$

where $C_x$ is the set of circuits of $M'$ that contain $x$. Now $s_{A^I}(C) = s_A(C - \{x\})$ for all $C \in C_x$, so Lemma 2.4 gives $|s_{A^I}(C - \{x\})| = r(C - \{x\}) = |C - \{x\}|$, and the corresponding statements hold for $s_{B^J}(C - \{x\})$. The corresponding conclusions also hold for any other pair $I_2 \in L_A$ and $J_2 \in L_B$ with $M[A^I] = M[B^J]$, so $I_1 \cap I_2$ has the form $\bigcap_{k=1}^t s_A(X_k)$ that the claim treats.

The case $t = 1$ merits special attention: if $|s_A(X)| = r(X) = |s_B(X)|$ for some $X \subseteq E(M)$, then $s_A(X) \in L_{A,B}$ since $M[A^{s_A(X)}]$ and $M[B^{s_B(X)}]$ are, by Lemma 2.3 both the principal extension $M +_x x$ of $M$.

Let the sets $X_1, X_2, \ldots, X_t$ be as in statement (3.14.1). Set $I = \bigcap_{k=1}^t s_A(X_k)$ and $J = \bigcap_{k=1}^t s_B(X_k)$. To prove the equality $M[A^I] = M[B^J]$, which proves statement (3.14.1), by symmetry it suffices to prove that each circuit $C$ of $M[A^I]$ that contains $x$ is dependent in $M[B^I]$. Fix such a circuit $C$ of $M[A^I]$.

We claim that for each $k \in [t]$, we have

$$|s_A((C - \{x\}) \cup X_k)| = r((C - \{x\}) \cup X_k) = |s_B((C - \{x\}) \cup X_k)|.
$$

To see this, let $\text{cl}$ be the closure operator of $M$, and $\text{cl}_I$ that of $M[A^I]$. For any $y \in C - \{x\}$,

$$\text{cl}_I((C - \{x, y\}) \cup X_k) = \text{cl}_I((C - \{x, y\}) \cup X_k - \{x\}).$$

Lemma 2.4 gives $x \in \text{cl}_I(X_k)$. Thus, $y$ is in $\text{cl}_I((C - \{x, y\}) \cup X_k)$ since $C$ is a circuit of $M[A^I]$. Thus, $y \in \text{cl}(C - \{x, y\} \cup X_k)$. By the formulation of closure in terms of circuits (as in \[12\] Proposition 1.4.11), it follows that each $y \in C - (X_k \cup \{x\})$ is in some circuit, say $C_y$, of $M$ with $C_y \subseteq X_k \cup (C - \{x\})$. Now $|s_A(C_y)| = r(C_y) = |s_B(C_y)|$ by Lemma 2.4. Since this applies for each $y \in C - (X_k \cup \{x\})$, and since we also have $|s_A(X_k)| = r(X_k) = |s_B(X_k)|$, equation (3.3) now follows from Lemma 3.15.

From equation (3.3), another application of Lemma 3.15 gives

$$|s_A((C - \{x\}) \cup \bigcup_{k \in P} X_k)| = r((C - \{x\}) \cup \bigcup_{k \in P} X_k) = |s_B((C - \{x\}) \cup \bigcup_{k \in P} X_k)|$$

for any non-empty subset $P$ of $[t]$. Thus, for any such $P$,

$$\left|\bigcup_{k \in P} s_A((C - \{x\}) \cup X_k)\right| = \left|\bigcup_{k \in P} s_B((C - \{x\}) \cup X_k)\right|. $$
Now
\[ \bigcap_{k=1}^{t} s_{A,I}(C - \{x\} \cup X_k) = \bigcap_{k=1}^{t} (s_{A,I}(C - \{x\}) \cup s_{A,I}(X_k)) \]
\[ = s_{A,I}(C - \{x\}) \cup \left( \bigcap_{k=1}^{t} s_{A,I}(X_k) \right) \]
\[ = s_{A,I}(C - \{x\}) \cup I \]
\[ = s_{A,I}(C). \]

The same argument applies to \( B \) and gives
\[ s_{B,J}(C) = \bigcap_{k=1}^{t} s_{B,J}(C - \{x\} \cup X_k). \]

The deductions in the previous two paragraphs and inclusion-exclusion give
\[ |s_{A,I}(C)| = \left| \bigcap_{k=1}^{t} s_{A,I}(C - \{x\} \cup X_k) \right| \]
\[ = \sum_{P \subseteq [t]: P \neq \emptyset} (-1)^{|P|+1} \left| \bigcup_{k \in P} s_{A,I}(C - \{x\} \cup X_k) \right| \]
\[ = \sum_{P \subseteq [t]: P \neq \emptyset} (-1)^{|P|+1} \left| \bigcup_{k \in P} s_{B,J}(C - \{x\} \cup X_k) \right| \]
\[ = \left| \bigcap_{k=1}^{t} s_{B,J}(C - \{x\} \cup X_k) \right| \]
\[ = |s_{B,J}(C)|. \]

Since \( C \) is a circuit of \( M[A,I] \), we have \(|s_{A,I}(C)| < |C| \). Thus \(|s_{B,J}(C)| < |C| \), so \( C \) is dependent in \( M[B,J] \), as needed.

The assertions about \( L_{B,A} \) and \( T_A \cap T_B \) now follow easily. \( \square \)

The proof of Theorem 3.14 and its reduction to statement 3.14.1 give the following alternative description of \( L_{A,B} \).

**Theorem 3.16.** For presentations \( A \) and \( B \) of \( M \), the sublattice \( L_{A,B} \) of \( L_A \) consists of the sets \( I \in L_A \) that satisfy condition (\( * \)), as well as all intersections of such sets:

(\( * \)) \( I = s_{A,X}(X) \) for some \( X \subseteq E(M) \) with \(|s_{A,X}(X)| = r(X) = |s_{B,X}(X)|\).

The sets \( I \) that satisfy condition (\( * \)) correspond to the principal extensions \( M + x \) of \( M \) that are common to \( T_A \) and \( T_B \).

We conclude this section with two corollaries. Note that we can iterate the operation of extending set systems to get \((A_1)_{i_1} \cup \ldots \cup (A_t)_{i_t}\), where \( x_{i_1} \) is added in \( A_{i_1} \), and \( x_{i_2} \) is added in \((A_1)_{i_2}\). We next show that such extensions, using sets in \( L_{A,B} \), are compatible.

**Corollary 3.17.** If \( M[A_{i_1}] = M[B_{i_1}] \) and \( M[A_{i_2}] = M[B_{i_2}] \), for some sets \( I_1, I_2 \in L_A \) and \( J_1, J_2 \in L_B \), then \( M[(A_{i_1})_{i_2}] = M[(B_{i_1})_{i_2}] \).
Proof. The result follows from two observations: (i) Theorem 3.7 yields \( I_2 \in L_{A_1} \) and \( J_2 \in L_{B_1} \); (ii) if \( I_2 \) and \( X \) satisfy condition (*) above in \( M \), then so do \( I_2 \) and \( X \) in \( M[A^{1}] \), and likewise for intersections of sets that satisfy condition (*). □

Corollary 3.18. For \( I \in L_{A} \) and \( J \in L_{B} \), if \( M[A^{1}] = M[B^{1}] \), then \( |I| = |J| \).

Proof. Apply Corollary 3.17 repeatedly, with each \( I_h = I \) and each \( J_h = J \), until the set of added elements is cyclic in the extension; the rank of this cyclic set must be both \( |I| \) and \( |J| \). □

3.5. How to any finite distributive lattice. We show that each sublattice of \( 2^{[r]} \) that includes both \( \emptyset \) and \( [r] \) is the lattice \( L_{A} \) for some presentation \( A \) of a transversal matroid of rank \( r \); indeed, we prove two refinements of this result. Up to isomorphism, this result covers all finite distributive lattices since each such lattice \( L \) is isomorphic to the lattice of order ideals of some finite ordered set (specifically, the induced order on the set of join-irreducible elements of \( L \); see, e.g., [1, Theorem II.2.5]). Combining the result below with Theorem 3.4 shows any distributive lattice is isomorphic to \( T_{A} \) for some presentation \( A \) of a transversal matroid.

Theorem 3.19. Let \( L \) be a sublattice of \( 2^{[r]} \) that contains both \( \emptyset \) and \( [r] \).

1. There is a rank-\( r \) transversal matroid \( M \) and maximal presentation \( A \) of \( M \) with \( L = L_{A} \).

2. For any \( n \geq r \), there is a presentation \( B \) of the uniform matroid \( U_{r,n} \) with \( L = L_{B} \).

Proof. To prove assertion (1), for each non-empty set \( I \in L \), let \( X_{I} \) be a set of \( |I| + 1 \) elements that is disjoint from all other such sets \( X_{J} \). For \( i \) with \( 1 \leq i \leq r \), let

\[
A_{i} = \bigcup_{I \in L : i \in I} X_{I},
\]

so the elements of \( X_{I} \) are in exactly \( |I| \) of the sets \( A_{i} \) (counting multiplicity; we may have \( A_{i} = A_{j} \) even if \( i \neq j \)). Let \( A = (A_{i} : i \in [r]) \) and let \( M \) be the matroid \( M[A] \) on

\[
E(M) = \bigcup_{I \in L : I \neq \emptyset} X_{I} = \bigcup_{i=1}^{r} A_{i}.
\]

Thus, if \( e \in X_{I} \), then \( s_{A}(e) = I \). The presentation \( A \) of \( M \) is maximal since, with \( |X_{I}| > |I| \) and \( s_{A}(X_{I}) = I \), the set \( X_{I} \) is dependent in \( M \), yet if we adjoin any element of \( X_{I} \) to any set \( A_{j} \) with \( j \notin I \), then the resulting set system \( A' \) has a matching of \( X_{I} \), so \( X_{I} \) is independent in \( M[A'] \). It now follows from Theorem 3.10 that \( L \subseteq L_{A} \). Since \( L \) and \( L_{A} \) are sublattices of \( 2^{[r]} \) and \( s_{A}(e) \in L \) for all \( e \in E(M) \) by construction, we get \( s_{A}(F) \in L \) for each cyclic flat \( F \) of \( M \), so Corollary 3.8 gives \( L_{A} \subseteq L \). Thus, \( L_{A} = L \).

Figure 4 illustrates the proof of assertion (2). Let \( [n] \) be the ground set of \( U_{r,n} \). For \( I \in L \), let \( I_{0} \) be the (possibly empty) set of elements that occur first in \( I \), that is,

\[
I_{0} = I - \bigcup_{J \in L : J \subseteq I} J.
\]

Since \( L \) is closed under intersection, for each \( i \in [r] \), there is exactly one \( I \in L \) with \( i \in I_{0} \); using that \( I \), set

\[
B_{i} = ([n] - [r]) \cup \bigcup_{J \in L : i \in J} J_{0}.
\]

By construction, \( |B| = r \) and \( i \in B_{i} \) for \( [r] \) is a basis of \( M[B] \). Since \( [n] - [r] \subseteq B_{i} \) for all \( i \in [r] \), it follows that \( M[B] \) is the uniform matroid \( U_{r,n} \). For \( i \in I_{0} \) and \( j \in J_{0} \), we
have \( i \in B_j \) if and only if \( J \subseteq I \), so \( s_B(i) = I \). Since \( L \) is closed under unions, we get \( s_B(X) \in L \) for all \( X \subseteq [r] \). Also, each set \( I \in L \) is independent in \( U_{r,n} \) and \( s_B(I) = I \).

From these observations and Theorem 3.7, we get \( L = L_B \). \( \square \)

3.6. Irreducible elements. An element \( a \) in a lattice \( L \) is *join-irreducible* if (i) \( a \) is not the least element of \( L \) and (ii) if \( a = b \lor c \), then \( a \in \{ b, c \} \). Dually, \( a \) is *meet-irreducible* if (i') \( a \) is not the greatest element of \( L \) and (ii') if \( a = b \land c \), then \( a \in \{ b, c \} \). (While not all authors include them, conditions (i) and (i') shorten the wording of results.)

The irreducible elements of a finite distributive lattice \( L \) are of great interest. The order induced on the set of join-irreducibles of \( L \) is isomorphic to that induced on its set of meet-irreducibles, and the lattice of order ideals of each of these induced suborders of \( L \) is isomorphic to \( L \) itself. (See, e.g., [1] Theorem II.2.5 and Corollary II.2.7.) Thus, the rank of \( L \) is the number of join-irreducibles in \( L \), which is also its number of meet-irreducibles.

We now study the irreducible elements of the lattices \( L_A \) introduced above.

The least set \( S_i \) in \( L_A \) that contains a given element \( i \in [r] \) is \( \bigcap_{J \in L_A : i \notin J} J \). The sets \( S_i \) are not limited to the atoms of \( L_A \); see the examples in Figure 1. Clearly \( S_i \) is join-irreducible. Each set \( U \) in \( L_A \) is \( \bigcup_{i \in U} S_i \), so there are no other join-irreducibles of \( L_A \). Thus, the number of join-irreducibles is the number of distinct sets \( S_i \). Note that if \( A_i \) and \( A_j \) in \( A \) are equal, then \( S_i = S_j \) since, for \( X \subseteq E(M) \), we have \( i \in s_A(X) \) if and only if \( j \in s_A(X) \). Thus, the number of join-irreducible sets in \( L_A \) is at most the number of distinct sets in \( A \). As Example 1 shows, this bound can be strict (there, \( A \) has three distinct sets but \( L_A \) has only one join-irreducible; likewise for \( B \)).

The greatest set in \( L_A \) that does not contain a given element \( i \in [r] \) is \( \bigcup_{J \in L_A : i \notin J} J \). An argument like that above, or an application of order-duality, shows that these are the meet-irreducibles of \( L_A \). By the remark after the proof of Theorem 3.7 each meet-irreducible element of \( L_A \) corresponds to a principal extension of \( M \); the converse is false, since for instance, in either example in Figure 1 the set \( \{2, 3\} \) corresponds to a principal extension, but \( \{2, 3\} \) is the meet of the sets \( \{1, 2, 3\} \) and \( \{2, 3, 4\} \) in \( L_A \).

We now identify a join-sublattice \( L_A' \) of \( L_A \) that, by Theorem 3.7, has the same the meet-irreducibles, thereby reducing the problem of finding the meet-irreducibles of \( L_A \) to the same problem on a potentially smaller lattice. Set

\[
L_A' = \{ s_A(X) : X \subseteq E(M), |s_A(X)| = r(X) \}.
\]
(Adding the condition that $X$ is independent would not change $L'_A$.) By Theorem 3.7, $L'_A \subseteq L_A$ and $L'_A$ generates $L_A$ since $L_A$ consists precisely of the intersections of the sets in $L'_A$. Lemma 3.15 shows that $L'_A$ is a join-sublattice of $L_A$.

Each lattice is isomorphic to $L'_A$ for a maximal presentation $A$ of some transversal matroid (see the proof of [3, Theorem 2.1]). By Corollary 3.8 when the presentation $A$ is maximal, the same conclusions hold for the (often smaller) lattice

$$L'_A = \{ s_A(X) : X \text{ is a cyclic flat of } M \} \cup \{ r \}.$$

4. APPLICATIONS

Theorems 4.1 and 4.5 below are applications of the results in Section 3. Both results stem from the observation that proper sublattices of $2^r$ must be substantially smaller than $2^r$. (The special case of maximal proper sublattices of $2^r$ have been studied in other settings, such as finite topologies; see, e.g., Sharp [14] and Stephen [15].)

**Theorem 4.1.** Let $M$ be a transversal matroid of rank $r$, and let $A^i$ be a presentation of $M$ that has rank $i$ in the ordered set of presentations of $M$. If $1 \leq i < r$, then

$$|T_{A^i}| = |L_{A^i}| \leq \left(\frac{1}{2} + \frac{1}{2^{r-i}}\right)2^r;$$

these bounds are sharp. Also, if $i \geq r$, then $|T_{A^i}| = |L_{A^i}| \leq 2^{r-1}$.

We first give examples to show that, for $1 \leq i < r$, the bounds are sharp. (These examples, which play a role in the proof of the bound, have coloops; to get examples without coloops, take free extensions of these.) Let $B = (B_2, B_3, \ldots, B_r)$ be a minimal presentation of a transversal matroid $N$ of rank $r - 1$. Fix an element $e \notin E(M)$ and let $M$ be the direct sum of $N$ and the rank-1 matroid on $\{e\}$. For $0 \leq k < r$, define $A^k = (A^k_i : i \in [r])$ by

$$A^k_i = \begin{cases} \{e\}, & \text{if } i = 1, \\ B_i \cup \{e\}, & \text{if } 2 \leq i \leq k + 1, \\ B_i, & \text{otherwise}. \end{cases}$$

Thus, $s_{A^k}(e) = [k + 1]$. Each $A^k$ is a presentation of $M$, the presentation $A^0$ is minimal, and $A^{k-1} \preceq A^k$ for $k \geq 1$. Thus, $A^k$ has rank $k$ in the ordered set of presentations. Since $B$ is a minimal presentation of $N$, each subset of $\{2, 3, \ldots, r\}$ is in $L_{A^k}$. Thus, since $s_{A^k}(e) = [k + 1]$, Corollary 3.9 implies that all supersets of $[k + 1]$ are in $L_{A^k}$. Since $1 \in s_{A^k}(X)$ if and only if $e \notin X$, by Theorem 3.7 the sets in $L_{A^k}$ that contain 1 must contain all of $[k + 1]$. Thus, $L_{A^k}$ consists of the subsets of $[r]$ that either do not contain 1 or contain all of $[k + 1]$. For reasons that Lemma 4.3 will reveal, it is useful to recast this as follows: $L_{A^k}$ is the complement, in $2^r$, of the union of the intervals

$$\{1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \ldots, \{1, 2, \ldots, k\}, \{k + 1\},$$

where $\overline{X}$ denotes the complement of the set $X$. From the first description of $L_{A^k}$, we get

$$|L_{A^k}| = 2^{r-1} + 2^{r-(k+1)} = \left(\frac{1}{2} + \frac{1}{2^{r-1}}\right)2^r.$$

The proof of the bound in Theorem 4.1 uses Lemma 4.3 which catalogs the sublattices of $2^r$ that have more than $2^{r-1}$ elements. The proof of that lemma uses the following result by Chen, Koh, and Tan [2] (see the proof in Rival [13]).
Lemma 4.2. Let $\mathcal{J}$ be the set of join-irreducibles of a finite distributive lattice $L$, and $\mathcal{M}$ its set of meet-irreducibles. The maximal proper sublattices of $L$ are precisely the differences $L - [a, b]$ where the interval $[a, b]$ in $L$ satisfies $[a, b] \cap \mathcal{J} = \{a\}$ and $[a, b] \cap \mathcal{M} = \{b\}$.

Lemma 4.3. Up to permutations of $[r]$, the sublattices of $2^{[r]}$ that have more than $2^{r-1}$ elements are $L_i = 2^{[r]} - U_i$ and $L'_i = 2^{[r]} - U'_i$, for $1 \leq i < r$, where

$$U_i = \bigcup_{j : 1 \leq j \leq i} \{1, 2, \ldots, j\} \lor \{j + 1\}$$

and

$$L'_i = \bigcup_{j : 1 \leq j \leq i} \{1, 2, \ldots, j\} \land \{j + 1\}.$$

and $L_V = 2^{[r]} - V$ where $V = \{1\} \lor \{2\} \cup \{3\} \lor \{4\}$. Thus, $|L_i| = |L'_i| = \left(\frac{1}{2} + \frac{1}{2^{r+1}}\right)2^r$ and $|L_V| = \frac{9}{16} \cdot 2^r$. Also, $L_V$ is not contained in any sublattice $L$ of $2^{[r]}$ with $|L| = \frac{5}{8} \cdot 2^r$.

Proof. To prove this result, we apply Lemma 4.2 recursively. To simplify the argument, note that $U_i$ is the image of $L_i$ under the complementation map $X \mapsto X$ (which is order-reversing) of $2^{[r]}$; this allows us to pursue only the lattices $L_V$ and $L_1, L_2, \ldots, L_{r-1}$ below.

The join-irreducibles of $2^{[r]}$ are the singleton sets, and the meet-irreducibles are their complements, so by Lemma 4.2 the maximal proper sublattices of $2^{[r]}$ are $L_1$ and its images under permutations of $[r]$ (the lattice $L'_1$ is obtained by such a permutation).

To verify the assertions below about join-irreducibles, note that (i) each join-irreducible of $L_{i-1}$ that is also in $L_i$ is join-irreducible in $L_i$, and (ii) $L_i$ has at most $r$ join-irreducibles. (The second statement holds since the rank of a distributive lattice is its number of join-irreducibles; see [I, Corollary II.2.11].) Similar observations apply to meet-irreducibles.

We now find the maximal proper sublattices of $L_1 = 2^{[r]} - \{1\} \lor \{2\}$. Its join-irreducibles are $\{i\}$, for $2 \leq i \leq r$, along with $\{1, 2\}$; its meet-irreducibles are $\{i\}$, for $i \in [r] - \{2\}$, along with $\{1, 2\}$. Up to the map $X \mapsto X$ (which maps $L_2$ to $L'_2$) and permuting $3, 4, \ldots, r$, there are three maximal proper sublattices, namely

1. $L_2 = L_1 - \{1, 2\} \lor \{3\}$, which has $\frac{3}{8} \cdot 2^r$ elements,
2. $L_V = L_1 - \{3\} \lor \{4\}$, which has $\frac{9}{16} \cdot 2^r$ elements, and
3. $L_1 - \{2\} \lor \{1\}$, which has $2^{r-1}$ elements.

(The join-irreducible $\{1, 2\}$ is in $\{2\} \lor \{3\}$, so this interval is not listed. Likewise for $\{1, 2\}$ and $\{3\} \lor \{1\}$.) Only $L_2$ and $L_V$ are of interest for the lemma.

The join-irreducibles of $L_V$ are $\{i\}$, for $i \in [r] - \{1, 3\}$, along with $\{1, 2\}$ and $\{3, 4\}$; its meet-irreducibles are $\{j\}$, for $j \in [r] - \{2, 4\}$, along with $\{1, 2\}$ and $\{3, 4\}$. Up to switching the pair $\{1, 2\}$ with the pair $\{3, 4\}$, permuting $5, 6, \ldots, r$, and the map $X \mapsto X$, there are three maximal proper sublattices of $L_V$ (omitting the case covered by (3) above):

4. $L_V - \{1, 2\} \lor \{3, 4\}$, which has $2^{r-1}$ elements,
5. $L_V - \{1, 2\} \lor \{5\}$, which has $\frac{15}{64} \cdot 2^r$ elements, and
6. $L_V - \{5\} \lor \{6\}$, which has $\frac{27}{64} \cdot 2^r$ elements.

Thus, no proper sublattices of $L_V$ have more than $2^{r-1}$ elements.

To complete the proof, we induct to show that for $i$ with $3 \leq i < r$, the only maximal proper sublattice $L$ of $L_{i-1}$ with $|L| > 2^{r-1}$ is $L_i$, up to permuting elements. We include the following conditions in the induction argument (see Figure 5):

(i) the join-irreducibles of $L_{i-1}$ are $\{j\}$, for $1 < j \leq r$, along with $\{i\}$, and
(ii) the meet-irreducibles of $L_{i-1}$ are $\{1\}$ and $\{k\}$, for $i < k \leq r$, along with $\{1, t\}$ where $2 \leq t \leq i$.

Conditions (i) and (ii) are easy to see in the base case, $i = 3$. We use the same argument for the base case as for the inductive step. Let $L$ be a maximal proper sublattice of $L_{i-1}$.
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Figure 5. The induced order on the irreducibles of $L_{i-1}$.

If $L = L_{i-1} - [A, B]$ where $|A| = 1$ and $B = \{1, t\}$ with $2 \leq t \leq i$, then $[A, B]$ is disjoint from $U_{i-1}$ and has $2^{r-3}$ elements, so $|L| \leq 2^{r-1}$. If $L = L_{i-1} - \{j\}$, with $j$ and $k$ distinct elements of $\{i + 1, i + 2, \ldots, r\}$, then $|L| \leq \frac{15}{32} \cdot 2^r$ by case (5) (with relabelling). Thus, up to relabelling, only $L_i = L_{i-1} - \{1, 2, \ldots, i\}$ has more than $2^{r-1}$ elements: $|L_i| = \left(\frac{1}{2} + \frac{1}{2^{r-1}}\right) 2^r$. It is easy to check that conditions (i) and (ii) hold for $L_i$, which completes the induction. □

The last background item we need before proving the upper bounds in Theorem 4.1 is the following lemma from [3].

Lemma 4.4. Let $A$ be a presentation of $M$. Fix $Y \subseteq E(M)$. If $r(M \setminus Y) = r(M)$, then $M$ has a minimal presentation $C$ with $C \leq A$ so that $s_C(e) = s_A(e)$ for all $e \in Y$.

Proof of Theorem 4.7. Consider presentations $A^0 \prec A^1 \prec \cdots \prec A^r$ of $M$ where $A^0$ is minimal. Thus, $A^j$ has rank $j$ in the order on presentations, and $L_{A^j}$ is a sublattice of $L_{A^{j-1}}$. By Lemma 4.3, if $|L_{A^j}| > 2^{r-1}$, then $|L_{A^j}| = \left(\frac{1}{2} + \frac{1}{2^{i+1}}\right) 2^r$ for some $i$ with $1 \leq i < r$, so it suffices to prove the following statement:

\[
\text{if } |L_{A^j}| = \left(\frac{1}{2} + \frac{1}{2^{i+1}}\right) 2^r, \text{ then } j \leq i.
\]

For $i = 1$, assume $|L_{A^1}| = \frac{3}{2} \cdot 2^r$. By Lemma 3.3 up to permuting $[r]$, we have $L_{A^1} = 2^r - \{1\} \cup \{2\}$. Condition (2) of Corollary 3.1 holds ($h = 1$), so $L_{A^1}$ is properly contained in $L_{A^{1-1}}$: since $L_{A^1}$ is a proper sublattice only of $2^r$, we have $L_{A^{1-1}} = 2^r$. Thus, $A^{j-1}$ is a minimal presentation by Theorem 3.5 so $j - 1 = 0$, so $j = 1$.

For $i = 2$, if $|L_{A^2}| = \frac{5}{2} \cdot 2^r$, then, by Lemma 3.3 up to permuting $[r]$, the lattice $L_{A^2}$ is either

\[
2^r - (\{1\} \cup \{2\} \cup \{3\}) \quad \text{or} \quad 2^r - (\{2\} \cup \{3\} \cup \{1, 2\}).
\]

Condition (2) of Corollary 3.1 holds ($h = 1$ in the first case and either 2 or 3 in the second), so $L_{A^2}$ is properly contained in $L_{A^{2-1}}$. Thus, $|L_{A^{2-1}}| \geq \frac{5}{4} \cdot 2^r$. The previous case gives $j - 1 \leq 1$, so $j \leq 2$.

The general case with $L_{A^j} = L_i$ or $L_{A^j} = L_i'$ follows inductively in the same manner. We turn to the only case that requires a more involved argument, namely

\[
L_{A^j} = L_V = 2^r - (\{1\} \cup \{2\} \cup \{3\} \cup \{4\}).
\]

Since $A^{j-1} \prec A^j$, we have $s_{A^{j-1}}(e) \subseteq s_{A^{j-1}}(e)$ for some $e \in E(M)$, so $s_{A^{j-1}}(e) \not\subseteq L_V$ by Theorem 3.10. Thus, $s_{A^{j-1}}(e) \not\subseteq \{1\} \cup \{2\} \cup \{3\} \cup \{4\}$. If $s_{A^{j-1}}(e)$ is in only one of $\{1\} \cup \{2\}$ and $\{3\} \cup \{4\}$, then $L_{A^j}$ is a proper sublattice of $L_{A^{j-1}}$ by condition (1) of
Corollary 3.11, thus, \( |L_{A^{j-1}}| \geq \frac{3}{2} \cdot 2^r \), so \( j - 1 \leq 1 \), so \( j < 3 \). We may now assume that \( L_{A^1} = L_{A^{j-1}} \) and that \( s_{A^{j-1}}(e) \in \{ 1 \}, \{ 2 \} \cap \{ 3 \}, \{ 4 \} \).

First assume that for all options for the terms \( A^0, A^1, \ldots, A^{j-1} \), the only element \( d \) with \( s_{A^j}(d) \neq s_{A^k}(d) \) for some \( k < j \) is \( d = e \). Lemma 4.4 then implies that \( e \) is a co-loop of \( M \); also, the presentation of \( M|e \) that is obtained by removing \( e \) from all sets in \( A^0 \) is minimal. This case is covered by the example that we used to show that the bound is sharp, so we may now assume that \( e \) is not a co-loop of \( M \).

In this case, by Lemma 3.4 with \( J = \{ e \} \), we can choose \( A^0, A^1, \ldots, A^{j-2} \) so that \( s_{A^{j-1}}(e) = s_{A^{j-2}}(e) \). Since \( A^{j-2} \prec A^{j-1} \), we have \( s_{A^{j-2}}(e') \subseteq s_{A^{j-1}}(e') \) for some \( e' \in E(M) \). Thus, \( e' \neq e \). Now \( s_{A^{j-2}}(e') \not\in L_v \) by Theorem 3.10 so \( s_{A^{j-2}}(e') \) is in either \( \{ 1 \}, \{ 2 \} \) or \( \{ 3 \}, \{ 4 \} \). If \( s_{A^{j-2}}(e') \) is not in both intervals, then the argument above gives the result, so assume \( s_{A^{j-2}}(e') \in \{ 1 \}, \{ 2 \} \cap \{ 3 \}, \{ 4 \} \). Set \( F = \{ e, e' \} \). Thus,

\[
s_{A^{j-2}}(F) = s_{A^{j-2}}(e) \cup s_{A^{j-2}}(e') \in \{ 1 \}, \{ 2 \} \cap \{ 3 \}, \{ 4 \}.
\]

Corollary 5.9 with \( J = s_{A^{j-2}}(F) \not\subseteq \{ 1 \} \), and so \( H = \{ 1 \} \), gives \( s_{A^{j-2}}(F) \in L_{A^{j-2}} \), so \( L_{A^1} \) is a proper subattice of \( L_{A^{j-2}} \). Lemma 4.3 gives \( |L_{A^{j-2}}| \geq \frac{3}{4} \cdot 2^r \); thus, \( j - 2 \leq 1 \), so \( j \leq 3 \), as needed.

Let \( A \) and \( B \) be presentations of \( M \). In Theorem 3.14 we showed that \( T_A \cap T_B \) is a sublattice of both \( T_A \) and \( T_B \). The smallest that \( |T_A \cap T_B| \) can be is two, with these two common extensions being the free extension and the extension by a loop; for instance, the two minimal presentations

\[
A = (\{ i \} \cup [2r] - \{ r \} : i \in [r]) \quad \text{and} \quad B = ([r] \cup \{ i \} : i \in [2r] - \{ r \})
\]

of \( U_r, 2r \) have this property. We conclude with a sharp upper bound on \( |T_A \cap T_B| \).

**Theorem 4.5.** If the presentations \( A = (A_i : i \in [r]) \) and \( B = (B_i : i \in [r]) \) of \( M \) differ by more than just reindexing the sets, then \( |T_A \cap T_B| \leq \frac{1}{4} \cdot 2^r \). This bound is sharp.

**Proof.** The inequality follows from Theorems 4.1 and 3.14 if either \( A \) or \( B \) is not minimal, so we may assume that both are minimal. As shown in Section 3.2 when \( A \) is minimal, we can reconstruct the sets in \( A \) from \( T_A \); thus, by our assumption, \( T_A \neq T_B \), so \( L_A, L_B \) is a proper sublattice of \( L_A \). Thus, we get the bound by our work above.

To see that this bound is tight, let \( M \) be \( U_{r-2, r-2} \cup U_{2, 3} \), with \( U_{r-2, r-2} \) and \( U_{2, 3} \) on the sets \( \{ e_1, e_2, \ldots, e_{r-2} \} \) and \( \{ e_{r-1}, a, b \} \), respectively. Consider the presentations \( A = (A_i : i \in [r]) \) and \( B = (B_i : i \in [r]) \) where \( A_i = B_i = e_i \) for \( i \in [r-2] \) and

\[
A_{r-1} = \{ e_{r-1}, a \}, \quad B_{r-1} = \{ e_{r-1}, b \}, \quad A_r = B_r = \{ a, b \}.
\]

By Lemma 2.3 if \( I \subseteq [r - 1] \), then both \( M[A^I] \) and \( M[B^I] \) are the principal extension \( M + Y \) where \( Y = \{ e_i : i \in I \} \); also, if \( \{ r - 1 \}, r \} \subseteq I \subseteq [r] \), then \( M[A^I] \) and \( M[B^I] \) are both \( M + Y \) where \( Y = \{ e_i : i \in I - \{ r \} \} \cup \{ a, b \} \). There are \( 2^{r-1} + 2^{r-2} = \frac{3}{4} \cdot 2^r \) such sets \( I \), so the bound is optimal.

**Acknowledgments**

The author thanks Anna de Mier for very useful feedback on the ideas in this paper, for comments that improved the exposition, for catching a flaw in the original proof of Theorem 3.14, and for observations that led to Theorem 3.10.
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