Holographic derivation of a class of short range correlation functions

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Abstract

We construct a class of backgrounds with a warp factor and anti-de Sitter asymptotics, which are dual to boundary systems that have a ground state with a short-range two-point correlation function. The solutions of probe scalar fields on these backgrounds are obtained by means of confluent hypergeometric functions. The explicit analytical expressions of a class of short-range correlation functions on the boundary and the correlation lengths $\xi$ are derived from gravity computation. The two-point function calculated from gravity side is explicitly shown to exponentially decay with respect to separation in the infrared. Such feature inevitably appears in confining gauge theories and certain strongly correlated condensed matter systems.
1 Introduction

The gauge/gravity correspondence [1, 2, 3] has given an extraordinary method to study a quantum system by a higher dimensional gravity, which relates a theory with gravity to a quantum system without gravity in a non-trivial way. This duality indicates the emergence of bulk spacetime geometry from the degrees of freedom living in the boundary [4, 5, 6, 7]. It further provides us a way to compute interesting quantitative features of strongly-coupled quantum systems and non-perturbative effects of quantum field theories, since it allows us to make predictions of observables pertaining to the boundary system, by working in the gravity side.

The gauge/gravity correspondence enables us to compute correlation functions of a boundary conformal field theory by working in the gravity, and the details of this procedure were reviewed in [8]. Most of the gravity computations in the literature are for long-range correlation functions in the conformal field theory. On the other hand, short-range correlation functions are also very interesting and important in both quantum field theories and condensed matter systems. We focus on a class of short-range correlation functions in a D-dimensional system and derive them from gravity computation, using a new class of gravity backgrounds that we construct. The appearance of a holographic direction for a boundary system is also closely related to the renormalization group flow of the boundary system, e.g. [9, 10]. Our gravity ansatz is similar to the ansatz used in the aforementioned holographic renormalization group.

We want to construct a class of new backgrounds that enable us to compute short-range exponential decay two-point functions around the ground state, with a correlation length. The short-range correlation is distinguished from the long-range correlation. The short-range correlation means that the correlation length is much smaller than the overall size of the boundary system. If the overall size of the boundary system is infinite, a finite correlation length will lead to a short-range correlation.

These short-range correlation functions are similar to those that occur in the vacua of massive quantum field theories. On the other hand, vacua of massive quantum field theories have gravity dual descriptions, e.g. [11, 12, 13, 14, 15, 16]. We compute short-range correlated two-point correlation functions from the gravity side with the backgrounds in our paper, and our results are also potentially related to confining gauge theories.

The gauge/gravity correspondence is very useful for studying condensed matter systems, since it can give descriptions in strong coupling regimes, e.g. [17, 18, 4]. The feature that the boundary systems in our case have a short-range correlation function near the ground state, is also similar to that of many strongly correlated condensed matter systems. These aspects are also very interesting for investigations.

The organization of this paper is as follows. In Section 2, we construct a class of backgrounds with a warp factor and anti-de Sitter asymptotics, which we will show that the dual
boundary systems have a short-range two-point correlation function around the ground state. In Section 3, we solve basis solutions of a probe scalar field on these curved backgrounds, by means of confluent hypergeometric functions. Then in Section 4, we compute the bulk-to-bulk propagator and the boundary-to-bulk propagator in these backgrounds. Afterward in Section 5, we derive from the gravity side the short-range correlation function of the boundary system and the corresponding correlation length. In Section 6, we analyze the implication of our gravity computation to the short-range correlation and the correlation length. Finally, we discuss our results and draw some conclusions in Section 7. In Appendix A, we include details of our derivation of the background solutions in matter coupled gravity. In Appendices C and D, we include detailed derivations for the basis solutions and the propagators, respectively.

2 A class of geometries for short range correlation

We desire a short-range correlation function evaluated around the ground state with a correlation length $\xi$ for a $D$-dimensional spacetime. This $D$-dimensional spacetime can be constructed as an asymptotic boundary of a gravity system in higher dimensions. Let us consider that this gravity system is on the spacetime that we denote as $M$. The gravity system contains a holographic direction which we denote by $z$ here. Using a warp factor $a^2(z)$, the metric ansatz of $M$ can have the form

$$ds^2 = a^2(z)(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2), \quad (2.1)$$

where $\mu = 0, \cdots, D - 1$, and $z > 0$ is the holographic radial direction and $x$ is denoted as the spacetime position vector on the $D$-dimensional boundary $\partial M$. This metric ansatz (2.1) is used extensively in the holographic analysis of the renormalization group flow of boundary systems, for a review see e.g. [8].

We want to construct a class of new backgrounds of the form (2.1), that are dual to a quantum theory on the boundary with exponential decay two-point function around the ground state, with leading behavior

$$\langle O(x)O(x') \rangle \sim e^{-|x-x'|/\xi} \quad (2.2)$$

when the separation $|x-x'|$ is much larger than the correlation length $\xi$. The short-range correlation is distinguished from the long-range correlation. Consider that the boundary system has an overall size of the system $l_{\text{sys}}$. The short-range correlation means that $\xi \ll l_{\text{sys}}$. Hence if the overall size of the boundary system is infinite, a finite correlation length $\xi$ will lead to a short-range correlation. In a many-body condensed matter system, $l_{\text{sys}}$ is of order the finite size of the material. In this paper, we focus on a class of short-range correlations and derive them from gravity computation.
Since the energy scale of the boundary system is related to the inverse of the radial direction $z$, there must exists a special radius scale $z = z_0$ in the gravity dual characterizing the energy scale $\xi^{-1}$ in the infrared of the boundary system. On general grounds, we expect that $\xi$ is a function $\xi(z_0)$ of $z_0$, and $\xi$ may also depend on other parameters.

Here we still work on the asymptotically AdS background, where the asymptotic boundary is at $z = 0$. We consider that $a^2(z)$ can be expanded in powers of $z/L$, namely,

$$a^2(z) = \frac{L^2}{z^2} + \frac{\eta(z_0)L}{z} + \gamma(z_0) + O\left(\frac{z}{L}\right), \quad (2.3)$$

where $\eta(z_0)$ and $\gamma(z_0)$ are functions of $z_0$, abbreviated as $\eta$ and $\gamma$ in this paper. In the later sections, we will show that above warp factor is a nice choice we desire.

The above metric with the warp factors (2.3) can be solved in matter coupled gravity. The details of our derivation are in Appendix A. As an example, they can be obtained in scaler coupled gravity with the action

$$S = \frac{1}{2\kappa^2} \int d^{D+1}y \sqrt{-g} \left[ R - \frac{1}{2} \partial_M \varphi \partial^M \varphi - V(\varphi) \right], \quad (2.4)$$

where $\kappa$ is the gravitational coupling constant, and $M = 0, \cdots, D$. The profile of the scalar field $\varphi$ deforms the AdS background.

For

$$a^2(z) = \frac{L^2}{z^2} + \frac{\eta L}{z} + \gamma + O\left(\frac{z}{L}\right), \quad (2.5)$$

to the first three orders in $z$, the scalar and its potential are

$$V = -\frac{D(D-1)}{L^2} + (D-1)(2D-1)\frac{\eta^2}{L^3} + (D-1)(12D\gamma - 12\gamma - 13D\eta^2 + 11\eta^2)\frac{z^2}{4L^2}, \quad (2.6)$$

$$\varphi = \varphi_0 + \frac{1}{6} \sqrt{\frac{(D-1)z}{-2\eta L}} \left[ 24\eta + (12\gamma - 7\eta^2)\frac{z^4}{L^3} \right]. \quad (2.7)$$

Here we require $\eta < 0$. The meaning of $\varphi_0$ is that it is the value of $\varphi$ at $z = 0$. Note that, Eq. (2.6)–(2.7) gives a parametric form of $V(\varphi)$, where $V$ is a function of $\varphi$, written in a parametric representation.

We also find an exact solution, with the warp factor

$$a(z) = L \left( \frac{1}{z} - \frac{1}{z + 2z_0} \right), \quad (2.8)$$

with $z \geq 0$ and $2z_0 > 0$, and the corresponding scalar and its potential are

$$V(\varphi) = -\frac{(D-1)}{8L^2} \left( (2D-1)e^{\varphi_0/2}\varphi_0 + (2D-1)e^{\varphi_0/2} + 2(2D + 1) \right), \quad (2.9)$$
The solution (2.8)–(2.10) is an exact solution. If expanded, it is a special case of the solution (2.5)–(2.7) for \( \eta(z_0) = -Lz_0^{-1} \), \( \gamma(z_0) = \frac{3}{4}L^2z_0^{-2} \).

In the above analysis, the case for maximally symmetric AdS geometry is \( V = -\frac{D(D-1)}{L^2} \) and \( \varphi = \varphi_0 \), corresponding to \( \eta = 0 \) and \( \gamma = 0 \).

Our metrics may be relevant for holographic normalization schemes, e.g. [9, 10] and the relations between our ansatz and that used in the holographic renormalization are described in Appendix B.

### 3 The basis solutions

In this section, we consider a probe scalar field \( \phi \) on these curved backgrounds (2.3), (2.8) whose boundary value is regarded as the source coupling to \( \mathcal{O}(x) \) which has a short-range correlation function as (2.2) in the large separation \( |x - x'| \). We do not consider the back-reaction of the probe scalar field \( \phi \) to the backgrounds. The action of \( \phi \) in the curved background (2.3) reads

\[
S = -\frac{1}{2} \int d^{D+1}y \sqrt{|g|} [g^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2],
\]

where \( y \) parametrizes the coordinates \( (z, x^\mu) \) in \( D + 1 \) dimensions. The equation of motion derived from the action (3.1) is the Klein-Gordon equation, in the form

\[
\frac{1}{\sqrt{|g|}} \partial_M (\sqrt{|g|} g^{MN} \partial_N \phi) - m^2 \phi = 0.
\]

More explicitly, after substituting the metric (2.1), it becomes

\[
a^{-2} \left( -\partial_z^2 - (D-1)(\ln a)'/\partial_z - \partial_\mu \partial^\mu + m^2 a^2 \right) \phi(z, x^\mu) = 0.
\]

One may perform the Fourier transform of \( \phi \) in the \( x^\mu \) coordinates

\[
\phi(z, x^\mu) = \int \frac{d^Dk}{(2\pi)^D} e^{ik \cdot x} \phi(z, k^\mu),
\]

so that the equation could be rewritten in the form of the Helmholtz equation

\[
\left( -\partial_z^2 - (D-1)(\ln a)'/\partial_z + k^2 + m_0^2 a^2 \right) \phi(z, k^\mu) = 0,
\]
where \( k^2 = k_{\mu}k^{\mu} \) and we use the metric \( (2.3) \). It gives the following form

\[
\left( k^2 + m^2\gamma(z_0) \right) + \frac{m^2L^2}{z^2} + m^2\eta(z_0)\frac{L}{z} - (D-1) \left( -\frac{1}{z} + \frac{\eta(z_0)}{2L} + \frac{z}{L^2} \left( \frac{\gamma(z_0) - \eta(z_0)^2}{2} \right) \right) \partial_z - \partial_z^2 \right) \phi(z, k) = 0. \tag{3.6}
\]

This equation has exact solutions if the linear term in the first-order derivative term is neglected. The two linearly independent basis solutions are by means of the confluent hypergeometric functions of the second kind \( U(a, b, x) \) and of the first kind \( _1F_1(a, b, x) \), respectively \[19, 20\]. The detailed derivations are in Appendix C.

Consider this special solution,

\[
\phi(z, k) = z^{D/2+\nu} e^{-\left(\beta + (D-1)\eta\right)z} U \left( \alpha + \nu, 2\nu + 1; \frac{\beta z}{2L} \right). \tag{3.7}
\]

Here \( \nu = \sqrt{\frac{D^2}{4} + m^2L^2} \), and

\[
\alpha = \frac{1}{2} - \frac{\eta}{2\beta} \left[ (D-1)^2 - 4m^2L^2 \right], \tag{3.8}
\]

\[
\beta = 4L(k^2 + \xi^{-2})^{1/2}, \tag{3.9}
\]

where \( \xi \) is a parameter with value

\[
\xi = \left( \frac{(D-1)^2\eta^2}{16L^2} + \gamma m^2 \right)^{-1/2}. \tag{3.10}
\]

We will show that \( \xi \) is the correlation length of the boundary system and that the correlation is short-ranged, in the later sections.

Around the boundary \( z = 0 \), Eq. (3.7) could be expanded as

\[
\phi(z, k) = \phi_0(k) \left( z^{D/2-\nu} (1 + O(z)) + \frac{G(k)}{2\nu} z^{D/2+\nu} (1 + O(z)) \right), \tag{3.11}
\]

with \( \phi_0(k) \) a \( z \)-independent prefactor and

\[
G(k) = 2\nu \frac{4^{-\nu}\Gamma(\alpha + \nu)\Gamma(-2\nu)}{\Gamma(2\nu)\Gamma(\alpha - \nu)} \left( \frac{\beta}{L} \right)^{2\nu}. \tag{3.12}
\]

In the regime \( |\eta| \ll 1 \), we have that \( \frac{1}{2} - \alpha = O(\eta|\xi/8L|) \ll 1 \), i.e. \( \alpha \approx \frac{1}{2} \). Using the Legendre duplication formula \( \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z + \frac{1}{2}) \), we have the identity of Gamma functions

\[
\frac{\Gamma \left( \nu + \frac{1}{2} \right) \Gamma(-2\nu)}{\Gamma(2\nu)\Gamma \left( \frac{1}{2} - \nu \right)} = 2^{4\nu} \frac{\Gamma(-\nu)}{\Gamma(\nu)}. \tag{3.13}
\]

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Hence the response function \( G(k) \) reads

\[
G(k) = -2\nu \frac{2^{-2\nu} \Gamma(1 - \nu) (k^2 + \xi^{-2})^\nu}{\Gamma(1 + \nu)}. \tag{3.14}
\]

In the limit \( \eta \to 0, \gamma \to 0 \) and hence \( \frac{\xi}{2^\nu} \to 2kz \), by the Kummer’s second transformation [19, 20], \( U(a, 2a, x) = e^{x/2} x^{1/4-a} K_{a-1/4} \left( \frac{x}{4} \right) \). In this limit, the above response function \( G(k) \) reduces to \( -2\nu \frac{2^{-2\nu} \Gamma(1 - \nu)}{\Gamma(1 + \nu)} k^{2\nu} \), which is the result in the maximally symmetric AdS geometry, see e.g. [8, 21].

4 Bulk-to-bulk propagator and boundary-to-bulk propagator

Now we compute the bulk-to-bulk propagator and the boundary-to-bulk propagator in the background (2.3) and (2.8). The bulk-to-bulk propagator \( \tilde{G}(k, z; z') \) in momentum space satisfies

\[
\left( \partial_z^2 + (D - 1) \frac{a'(z)}{a(z)} \partial_z - k^2 - a^2(z) m^2 \right) \tilde{G}(k, z; z') = -\frac{1}{a^{D-1}(z)} \delta(z - z'). \tag{4.2}
\]

We find a special solution of \( \tilde{G}(k, z; z') \) which could be checked by substituting into it,

\[
\tilde{G}(k, z; z') = \frac{\theta(z - z') \phi_1(z) \phi_2(z') + \theta(z' - z) \phi_1(z') \phi_2(z)}{a^{D-1}(z') (\phi_1(z') \phi_2(z') - \phi_1'(z') \phi_2(z))}, \tag{4.3}
\]

where \( \phi_1(z) \) and \( \phi_2(z) \) are two linearly independent solutions of the Helmholtz equation (3.6). The boundary-to-bulk propagator \( \tilde{K}(z, k) \) in momentum space could be derived by taking limit \( z' \to 0 \) together with a normalization factor, \( \tilde{K}(z, k) = \lim_{z' \to 0} 2\nu(z')^{D-1} a^{D-1}(z') \tilde{G}(k, z; z') \). Thus the boundary-to-bulk propagator \( K(z, x; x') \) in position space could be gotten by Fourier
transformation,

\[
K(z, x; x') = \int \frac{d^D \tilde{k}}{(2\pi)^D} \tilde{K}(z, k) e^{i\tilde{k}(x-x')}
= \frac{2^{-\nu+1}}{(2\pi)^{D/2}\Gamma(\nu)} \left( \frac{z}{\xi \sqrt{z^2 + |x-x'|^2}} \right)^{\nu+\frac{D}{2}} K_{\nu+\frac{D}{2}} \left( \xi^{-1} \sqrt{z^2 + |x-x'|^2} \right),
\]

(4.4)

where \(K_{\nu+\frac{D}{2}}(\cdot)\) is the modified Bessel function of the second kind. The detailed derivations are in Appendix D.

In the \(\xi^{-1} \to 0\) limit, the above boundary-to-bulk propagator (4.4) reduces to that of AdS case. For small \(\xi^{-1}\),

\[
K_{\nu+\frac{D}{2}} \left( \xi^{-1} \sqrt{z^2 + |x-x'|^2} \right) = \frac{\Gamma(\nu + \frac{D}{2})}{2} \left( \frac{\xi^{-1} \sqrt{z^2 + |x-x'|^2}}{2} \right)^{-\nu - \frac{D}{2}} (1 + O(\xi^{-1})),
\]

(4.5)

consequently,

\[
\lim_{\xi \to +\infty} K(z, x; x') = C_\Delta \left( \frac{z}{\xi \sqrt{z^2 + |x-x'|^2}} \right)^\Delta,
\]

(4.6)

with \(C_\Delta = \frac{\Gamma(\Delta)}{\pi^{D/2} \Gamma(\nu)}\) and \(\Delta = \frac{D}{2} + \nu\).

The boundary-to-boundary propagator \(\beta(x, x')\) could be obtained by taking the limit \(z \to 0\) together with a normalization factor \(z^{-\Delta}\), namely,

\[
\beta(x, x') = \lim_{z \to 0} z^{-\Delta} K(z, x; x') = \frac{2^{-\nu+1} \Gamma(\nu + \frac{D}{2})}{(2\pi)^{D/2}\Gamma(\nu)} \frac{K_{\nu+\frac{D}{2}} (\xi^{-1} |x-x'|)}{(\xi |x-x'|)^{\nu+\frac{D}{2}}}.\]

(4.7)

For large \(|x-x'| \gg \xi\),

\[
\beta(x, x') \approx \frac{2^{-\nu+\frac{D}{2}} \sqrt{\pi}}{(2\pi)^{D/2}\Gamma(\nu)} \frac{1}{\xi^{2(\nu+\frac{D}{2})}} \exp \left[-\frac{|x-x'|}{\xi} + O(\ln(|x-x'|/\xi))\right].
\]

(4.8)

For small \(|x-x'| \ll \xi\),

\[
\beta(x, x') \approx \frac{\Gamma(\nu + \frac{D}{2})}{\pi^{D/2}\Gamma(\nu) |x-x'|^{2(\nu+\frac{D}{2})}}.
\]

(4.9)
The Fourier transform of (4.7) is
\[
\tilde{\beta}(k) = \int d^Dx \beta(x, 0) e^{-ik \cdot x} = (2\pi)^{\frac{D}{2}} k^{1 - \frac{D}{2}} \int_0^\infty dx x^\frac{D}{2} J_{\frac{D}{2}-1}(kx) \beta(x, 0)
\]
\[
= \frac{2^{-\nu+1} k^{1 - \frac{D}{2}}}{\Gamma(\nu) \xi^\Delta} \int_0^\infty dx x^{\frac{D}{2} - \Delta} J_{\frac{D}{2}-1}(kx) K_\Delta(x/\xi).
\]
(4.10)

The integral, however, could be worked out explicitly for \(\Delta - \frac{D}{2} < 0\),
\[
\int_0^\infty dx x^{\frac{D}{2} - \Delta} J_{\frac{D}{2}-1}(kx) K_\Delta(x/\xi) = 2^{-\nu - 1} k^{\frac{D}{2}} \xi^{1 - \nu} \Gamma(-\nu) (k^2 + \xi^{-2})^{\Delta - \frac{D}{2}}
\]
(4.11)
which eventually gives
\[
\tilde{\beta}(k) = \frac{2^{-2\nu} \Gamma(-\nu)}{\Gamma(\nu)} (k^2 + \xi^{-2})^{\Delta - \frac{D}{2}}.
\]
(4.12)

For the case \(\Delta - \frac{D}{2} \geq 0\), we make a regularization with a short-distance regulator, subtracting a regulator dependent piece, and then the resulting integral is precisely (4.12).

In the limit \(\xi^{-1} \to 0\), the results here reduce to the results on the maximally symmetric AdS geometry, see for example [8, 21].

5 Derivation of short range correlation from gravity side

Here we derive the short-range correlation function of the boundary system from the gravity computation. An integration by parts of the scalar field action (3.1) gives
\[
S_{\text{reg}} = -\frac{1}{2} \int_{z \geq \epsilon} d^Dx dz \sqrt{|g|} |\phi(-\nabla^2 + m^2) \phi| + \frac{1}{2} \int d^Dx |\sqrt{|g|} g^{zz} \phi \partial_z \phi|_{z = \epsilon}.
\]
(5.1)
The \(\epsilon\) is a UV regulator of the boundary system. The last term can also be rewritten as
\[
\frac{1}{2} \int d^Dx |\sqrt{|g|} \gamma^{z\mu} \partial_\mu \phi|_{z = \epsilon},
\]
with \(\gamma^{MN} = g^{MN} \partial_\nu \eta^{\mu\\nu} = a^2(z) \eta_{\mu\nu}\) the induced metric on the boundary, and \(n^\mu \partial_\mu = \frac{1}{a(z)} \partial_z\) where \(n^\mu\) is the unit vector normal to the boundary. Near the boundary \(\gamma^{\mu\nu}|_{z = \epsilon} = \frac{\xi^2}{\epsilon^2} \eta_{\mu\nu}\).

The bulk field \(\phi\) is the convolution of the boundary field \(\phi_0\) and the boundary-to-bulk propagator,
\[
\phi(z, x) = \int d^Dx' K(z, x; x') \phi_0(x'),
\]
(5.2)
and \( \phi(z,x)|_{z=\epsilon} = z^{D-\nu} \phi_0(x)|_{z=\epsilon} \).

The first term in the regularized action (5.1) is vanishing for on-shell configurations. Hence the regularized on-shell action is

\[
S_{\text{reg}} = \frac{1}{2} \int d^Dx [a^{D-1}(z) \phi \partial_z \phi] |_{z=\epsilon} = \frac{L^{D-1}}{2} \int d^Dx_1 d^Dx_2 [\phi_0(x_1)A(x_1,x_2)\phi_0(x_2)].
\] (5.3)

Here

\[
A(x_1,x_2) = z^{-D+1} \int d^Dx K(z,x;x_1)\partial_z K(z,x;x_2)|_{z=\epsilon}
= (D - \Delta) \epsilon^{-\nu} \delta^D(x_1 - x_2) + D \frac{2^{-\nu+1}}{(2\pi)^{D/2} \Gamma(\nu)} \frac{K_{\nu+\frac{D}{2}}(\xi^{-1}|x_1 - x_2|)}{(\xi|x_1 - x_2|)^{\nu+\frac{D}{2}}},
\] (5.4)

where we used the property (4.7) of the boundary-to-bulk propagator. In order to precisely cancel the first divergent term when taking the limit \( \epsilon \to 0 \), it is inevitable to add the counter-term

\[
S_{\text{ct}} = - \frac{L^{D-1}}{2} (D - \Delta) \int d^Dx [\sqrt{\gamma} |\phi^2|] |_{z=\epsilon}
= \frac{L^{D-1}}{2} \int d^Dx_1 d^Dx_2 [\phi_0(x_1)A_{\text{ct}}(x_1,x_2)\phi_0(x_2)],
\] (5.5)

with

\[
A_{\text{ct}}(x_1,x_2) = -(D - \Delta) \epsilon^{-\nu} \delta^D(x_1 - x_2) - 2(D - \Delta) \frac{2^{-\nu+1}}{(2\pi)^{D/2} \Gamma(\nu)} \frac{K_{\nu+\frac{D}{2}}(\xi^{-1}|x_1 - x_2|)}{(\xi|x_1 - x_2|)^{\nu+\frac{D}{2}}},
\] (5.6)

The renormalized action \( S_{\text{ren}} = S_{\text{reg}} + S_{\text{ct}} \) reads

\[
S_{\text{ren}} = \frac{L^{D-1}}{2} (2\Delta - D) \int d^Dx_1 d^Dx_2 [\phi_0(x_1)\beta(x_1,x_2)\phi_0(x_2)],
\] (5.7)

where

\[
\beta(x_1,x_2) = \frac{2^{-\nu+1}}{(2\pi)^{D/2} \Gamma(\nu)} \frac{K_{\nu+\frac{D}{2}}(\xi^{-1}|x_1 - x_2|)}{(\xi|x_1 - x_2|)^{\nu+\frac{D}{2}}},
\] (5.8)
In the following calculation, we set the unit of energy such that $L = 1$. The dual operator $O(x)$ in the boundary system is sourced by $\phi_0(x)$, and the generating functional for the correlation function of $O(x)$ is

$$W[\phi_0] = \langle \exp \int d^D x [\phi_0(x) O(x)] \rangle = e^{S_{\text{ren}}[\phi_0]}.$$  

(5.9)

The vacua expectation value is then

$$\langle O(x) \rangle = \frac{\delta}{\delta \phi_0(x)} S_{\text{ren}}[\phi_0] = (2\Delta - D) \int d^D x' [\phi_0(x') \beta(x, x')] = (2\Delta - D) \phi_1(x).$$  

(5.10)

The two-point correlation function is then

$$\langle O(x) O(x') \rangle = \frac{\delta}{\delta \phi_0(x)} \frac{\delta}{\delta \phi_0(x')} S_{\text{ren}}[\phi_0] = (2\Delta - D) \beta(x, x').$$  

(5.11)

Here $\beta(x, x')$ is (5.8), which is short-range correlated, and in the large separation $|x - x'|$ it takes the form of (4.8).

Via the boundary-to-bulk propagator,

$$\phi(z, x) = \int d^D x' K(z, x; x') \phi_0(x').$$  

(5.12)

Performing Fourier transform,

$$\phi(z, k) = \tilde{K}(z, k) \phi_0(k).$$  

(5.13)

Near the boundary, $\phi(z, x)|_{z=\epsilon} = z^{\frac{D - \nu}{2} - \nu} \phi_0(x)|_{z=\epsilon}$, and hence their Fourier transforms satisfy $\phi(z, k)|_{z=\epsilon} = z^{\frac{D - \nu}{2} - \nu} \phi_0(k)|_{z=\epsilon}$. The field near the boundary is $\phi(z, k)|_{z=\epsilon} = [\phi_0(k) u_0(z) + \phi_1(k) u_1(z)]|_{z=\epsilon}$ where the two terms are source-mode and vev-mode respectively. Around $z = 0$, performing the expansion of the propagator,

$$\tilde{K}(z, k)|_{z=\epsilon} = \left[ z^{\frac{D - \nu}{2} - \nu} - \frac{2 - 2\nu}{\Gamma(1 + \nu)} (k^2 + \xi^{2\nu})^{\Delta - \frac{D}{2} + \nu} \right]_{z=\epsilon} = \left[ z^{\frac{D - \nu}{2} - \nu} + \frac{G(k)}{2\nu} z^{\frac{D}{2} + \nu} \right]_{z=\epsilon}.$$  

(5.14)

Hence we see that $\phi_1(k) = \frac{G(k)}{2\nu} \phi_0(k)$. We are still in an asymptotically AdS space, although we are not in a maximally symmetric AdS geometry. Hence $G(k)$ is the response of the vev $\phi_1(k)$ to the source $\phi_0(k)$. Performing the Fourier transform of (5.10),

$$\phi_1(k) = \tilde{\beta}(k) \phi_0(k),$$  

(5.15)
so we see that $G(k) = 2\nu\tilde{\beta}(k)$.

Performing Fourier transform of the fields, we hence have in the momentum space,

$$\langle O(k)O(-k)\rangle = (2\Delta - D)\tilde{\beta}(k) = -2\nu \frac{2^{-2\nu} (1 - \nu)}{\Gamma(1 + \nu)} (k^2 + \xi^{-2})^{\Delta - \frac{D}{2}}. \quad (5.16)$$

This is the Fourier transform of (5.11).

In the limit $\xi^{-1} \to 0$, the formula between the vev and source reduces to the canonical result in the maximally symmetric AdS geometry, e.g. [8, 21, 22, 23].

6 Relation to correlation length

Here we analyze the implication of our gravity computation to the short-range correlation and the correlation length $\xi$. We look at features of the two-point function in position space. For small $|x - x'| \ll \xi$,

$$\langle O(x)O(x')\rangle \approx 2\nu C \frac{1}{|x - x'|^{2\Delta}}. \quad (6.1)$$

The operator $O(x)$ has a UV scaling dimension of $\Delta$.

For large $|x - x'| \gg \xi$,

$$\langle O(x)O(x')\rangle \approx C \frac{1}{\xi^{2\Delta}} \exp \left[ -|x - x'|/\xi + O(\ln(|x - x'|/\xi)) \right], \quad (6.2)$$

where $C = 2\nu \frac{2^{-\nu} + \frac{1}{\sqrt{\pi}}}{(2\pi)^{1/2} \Gamma(\nu)}$. We may rescale the fields such that $\tilde{O} = \xi^\Delta O$, hence in large separation,

$$\langle \tilde{O}(x)\tilde{O}(x')\rangle \approx C \exp \left[ -|x - x'|/\xi + O(\ln(|x - x'|/\xi)) \right]. \quad (6.3)$$

Eq. (6.3) shows that the system has a short-range correlation function around the ground state. The leading behavior of this short-range correlation function is an exponential decay $e^{-|x-x'|/\xi}$ with respect to the separation $|x - x'|$ with a correlation length $\xi$.

The inverse correlation length $\xi^{-1}$ is related to the inverse radius size $z_0^{-1}$ in the holographic dimension. We have that $\beta = 4L(k^2 + \xi^{-2})^{1/2}$. The $\xi^{-1}$ is the inverse correlation length, which is an energy scale in the infrared regime of the boundary system. In the dual geometric side, $z_0^{-1}$ is an infrared energy scale for the boundary system. In the model here, as derived from the response function, we have

$$\xi^{-1} = \sqrt{\frac{(D - 1)^2}{16L^2} \eta(z_0)^2 + \gamma(z_0) m^2}. \quad (6.4)$$

If $\eta(z_0) = \tilde{\eta} \left( \frac{L}{z_0} \right)^a$, $\gamma(z_0) = \tilde{\gamma} \left( \frac{L^2}{z_0} \right)^a$ where $\tilde{\eta}, \tilde{\gamma}$ are dimensionless numbers and $a$ is any non-negative number, then

$$\xi^{-1} = c(\nu)L^{a-1}z_0^{-a}, \quad (6.5)$$

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where \( c(\nu) = \sqrt{\frac{(D-1)^2}{16} \bar{\eta}^2 + \bar{\gamma} m^2 L^2} \) is a model-dependent factor.

Hence the correlation length \( \xi \) of the boundary system is mapped to a geometric radius size \( z_0 \) in the dual gravity description. For example, we expect \( \xi^{-1} \propto z_0^{-a} \), such as in Eq. (6.5). The dual boundary system has a short-range correlation function which is an exponential decay in large separation, evaluated near its ground state. This is a nice feature from our gravity description. On the other hand, short-range correlation functions are very interesting and can be exhibited in many systems, such as in quantum magnetic systems, in quantum vortex models, and in confining gauge theories.

7 Discussion

We constructed a class of backgrounds with a warp factor \( a^2(z) \) and anti-de Sitter asymptotics, which are dual to boundary systems that have a ground state with a short-range two-point correlation function. We produced these metrics in matter coupled gravity using scalar coupled to gravity as an example, and our scalar profile deforms the background. This is a two-parameter class of geometries for short-range correlations on the boundary. Our solutions hold for a general \( D \).

We computed the bulk-to-boundary Green’s function for probe scalar fields in these backgrounds, obtained from our solutions of the fields on these backgrounds by means of confluent hypergeometric functions. Using them, the two-point correlation function for operators in the boundary theory which couple to the boundary values of the probe scalar field, were calculated. This leads to a short-range two-point function with a correlation length \( \xi \). Explicit analytical expressions of the correlation functions with the short-range correlation and a correlation length were obtained. Our analytical expressions may be useful for further aspects of the correspondence in this set-up. We also obtained the relation between source-mode and vev-mode in these backgrounds, which, in the limit when \( \xi^{-1} \) goes to zero, reduce to the canonical results in the maximally symmetric AdS geometry.

The operator aforementioned has a naive scaling dimension in the UV, since our set-up is still in asymptotically AdS geometry. In the IR, its two-point correlation function will decay exponentially with the separation, with a characteristic length scale the correlation length \( \xi \). We computed this correlation length from the gravity side, using gauge/gravity correspondence. Hence our gravity results predict that the boundary system has a short-range correlation with a correlation length derived from gravity.

The inverse correlation length in this paper is not due to a thermal nature. The topology of the metrics does not have a temporal circle whose perimeter would represent the inverse finite temperature. The correlation function here is evaluated near the ground state. Our correlation functions are similar to the short-range two-point correlation functions in a massive quantum field theory at zero temperature or a quantum many-body system near a ground state with a
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\section*{A Matter coupled gravity}

The gravity dual is on the manifold $M^{D+1}$, with the metric of the form

$$ds^2 = a^2(z)(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu),$$

(A.1)

where $\mu = 0, \cdots, D - 1$. We consider the background with the warp factor $a^2(z)$ in our case, to be dual to a boundary system that has a ground state with a short-range two-point correlation function.

By choosing appropriate matter and its potential, we can obtain the geometry with the warp factor $a^2(z)$. These warp factors can be obtained in matter coupled gravity, with appropriate matter sources. The matter sector does not have to be unique. One can choose matter sectors to be scalar field, axion-scalar field, vector field, or fermionic field. These matter fields back-react to the gravity sector, and thus deform the background geometry.

Consider that the back-reacting matter is modeled by a scalar field $\varphi$ with a potential
$V(\varphi)$. The action of the matter with gravity is

$$S = \frac{1}{2\kappa^2} \int d^{D+1}y \sqrt{-g} \left[ R - \frac{1}{2} \partial^M \varphi \partial_M \varphi - V(\varphi) \right],$$

(A.2)

where $\kappa$ is the gravitational coupling constant, and $M = 0, \cdots, D$.

The geometry is a small deformation of AdS$_{D+1}$ spacetime. We consider the warp factor to be

$$a^2(z) = \frac{L^2}{z^2} + \frac{\eta L}{z} + \gamma + O(\frac{z}{L}),$$

(A.3)

where $\eta = \eta(z_0), \gamma = \gamma(z_0)$ are functions of $z_0$.

The equations of motion obtained from (A.2) are

$$R_{MN} - \frac{1}{2} g_{MN} \left( R - \frac{1}{2} \partial^I \varphi \partial_I \varphi - V(\varphi) \right) - \frac{1}{2} \partial_M \varphi \partial_N \varphi = 0.$$  (A.4)

$$\nabla^M \nabla_M \varphi - \frac{dV(\varphi)}{d\varphi} = 0.$$  (A.5)

The equations of the motion simplified from the above (A.4), when we consider the solutions invariant under the isometry of the boundary are

$$(D - 1) \left( \frac{a''(z)}{a(z)} - 2 \left( \frac{a'(z)}{a(z)} \right)^2 \right) + \frac{1}{2} (\varphi'(z))^2 = 0.$$  (A.6)

$$(D - 1) \left( \frac{a''(z)}{a(z)} \right) + (D - 1)(D - 2) \left( \frac{a'(z)}{a(z)} \right)^2 + a^2(z)V(\varphi) = 0.$$  (A.7)

There is a third equation simplified from (A.5),

$$a^{-2}(z) \left( \varphi''(z) + (D - 1) \frac{a'(z)}{a(z)} \varphi'(z) \right) - \frac{dV(\varphi)}{d\varphi} = 0.$$  (A.8)

With appropriate potential, the metric can be generated. The solutions depend on not only the potential $V(\varphi)$ but also the profile $\varphi(z)$. From (A.6), the derivative of the $\varphi(z)$ is

$$\varphi'(z) = \pm \sqrt{\frac{(D - 1) L (4\eta L^2 + L z (12\gamma + \eta^2) + 4\gamma \eta z^2)}{2z (L^2 + \eta L z + \gamma z^2)^2}}.$$  (A.9)

To make sure $\varphi'(z)$ is real,

$$4\eta L^2 + L z (12\gamma + \eta^2) + 4\gamma \eta z^2 < 0, \quad \forall z \geq 0.$$  (A.10)
Take limit $z \to 0^+$, we could get $\eta \leq 0$. So $\eta$ must be non-positive and $\gamma$ is non-negative.

We make a Taylor expansion with respect to $z$ up to the first three orders. Hence, in a parametric representation, for

$$a^2(z) = \frac{L^2}{z^2} + \frac{\eta L}{z} + \gamma + O\left(\frac{z}{L}\right), \quad (A.11)$$

to the first three orders in $z$, the scalar and its potential are

$$V = -\frac{D(D-1)}{L^2} + (D-1)(2D-1)\frac{\eta z}{L^3} + (D-1)(12D\gamma - 12\gamma - 13D\eta^2 + 11\eta^2)\frac{z^2}{4L^2}, \quad (A.12)$$

$$\varphi = \varphi_0 + \frac{1}{6}\sqrt{(D-1)z} \left[ 24\eta + (12\gamma - 7\eta^2)\frac{z}{L} \right]. \quad (A.13)$$

Note that (A.12)–(A.13) is a parametric form of the potential $V(\varphi)$, through the variable $z$. The ratio $\gamma/\eta^2$ can take general values. The meaning of $\varphi_0$ is that it is the value of $\varphi$ at $z = 0$ with $V(\varphi_0) = -\frac{D(D-1)}{L^2}$. The AdS case is $V = -\frac{D(D-1)}{L^2}$ and $\varphi = \varphi_0$, with $\eta = 0$ and $\gamma = 0$.

In particular, we also have an interesting special solution

$$a(z) = L\left(\frac{1}{z} - \frac{1}{z + 2z_0}\right), \quad (A.14)$$

for $z \geq 0$ and $2z_0 > 0$, and we have that

$$V(\varphi) = -\frac{(D-1)}{8L^2}\left((2D-1)e^{\frac{\varphi - \varphi_0}{\sqrt{D-1}}} + (2D-1)e^{\frac{\varphi_0 - \varphi}{\sqrt{D-1}}} + 2(2D+1)\right), \quad (A.15)$$

$$\varphi = \varphi_0 \pm 4\sqrt{D-1}\log\left(\sqrt{\frac{z}{2z_0}} + 1 + \sqrt{\frac{z}{2z_0}}\right). \quad (A.16)$$

And the above solution (A.14)–(A.16) is an exact solution. The above solution can reduce to AdS case in the $z_0^{-1} \to 0$ limit.

The case $a(z) = L/z - L/(z + 2z_0)$ for $z \geq 0$ and $2z_0 > 0$ corresponds to $\eta = -Lz_0^{-1}$, $\gamma = \frac{3}{4}L^2z_0^{-2}$. The inverse function of (A.16) is

$$z = \frac{z_0}{2}e^{-\frac{2\varphi_0}{2\sqrt{D-1}}} \frac{\sqrt{D-1}}{2\sqrt{D-1}} \left(e^{\frac{\varphi_0}{\sqrt{D-1}}} - e^{\frac{\varphi}{\sqrt{D-1}}}\right)^2. \quad (A.17)$$

In this variable,

$$V = -\frac{(D-1)(2D-1)z^2 + 2(2D-1)zz_0 + 2Dz_0^2}{2L^2z_0^2} \quad (A.18)$$

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exactly. The profile (A.16) can be expanded as

\[ \varphi(z) = \varphi_0 \mp \frac{(z - 12z_0)\sqrt{(D - 1)z_0}}{3\sqrt{2}z_0^2}. \]  

(A.19)

It is exactly Eq.(A.13) with \( \eta = -L/z_0 \) and \( \gamma = 3L^2/(4z_0^2) \). Inputting these values in Eq. (A.12) gives (A.18) exactly. This is an example when \( \gamma/\eta^2 = 3/4 \) which is of order 1. On the other hand, we can also have other examples when \( \gamma/\eta^2 \) take more general values including when \( \gamma/\eta^2 \) is large, from the more general solutions in (A.11)–(A.13). Our solutions hold for a general \( D \).

B Relation to holographic renormalization

We are in matter coupled gravity. The \( z \) is a holographic radial direction. Our metrics are relevant for holographic normalization schemes, e.g. [9, 10]. Here we describe relations between our ansatz and that used in the holographic renormalization.

The metric ansatz is

\[ ds^2 = a^2(z) \left( dz^2 + (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \right), \]  

(B.1)

where \( \eta_{\mu\nu} \) is the background metric of the boundary and \( h_{\mu\nu} \) is its fluctuation. We can write this metric in the form used in holographic renormalization schemes [9, 10]

\[ ds^2 = L^2 \left( \frac{dr^2}{r^2} + \frac{1}{r^2}g_{\mu\nu}dx^\mu dx^\nu \right). \]  

(B.2)

In the above, \( r \) is a holographic radial direction, which is related to \( z \) by a change of coordinates. In particular, if the background is AdS, then \( r = z \).

The change of coordinates between (B.1) and (B.2) is

\[ \log r = C + \frac{1}{L} \int a(z)dz, \quad g_{\mu\nu} = r^2a^2(z)\left( \eta_{\mu\nu} + h_{\mu\nu} \right). \]  

(B.3)

For \( a(z) = L/z - L/(z + 2z_0) \), the integral gives

\[ r = \frac{2z_0z}{z + 2z_0} = z - \frac{z^2}{2z_0} + \frac{z^3}{4z_0^2} + O(z^4), \]  

(B.4)

\[ z = \frac{2rz_0}{2z_0 - r} = r + \frac{r^2}{2z_0} + \frac{r^3}{4z_0^2} + O(r^4), \]  

(B.5)
\[
g_{\mu\nu} = \frac{(r - 2z_0)^4}{16z_0^4}(\eta_{\mu\nu} + h_{\mu\nu}) = \left(1 - \frac{2r}{z_0} + \frac{3r^2}{2z_0^2} + O\left(\frac{r^3}{L^3}\right)\right)(\eta_{\mu\nu} + h_{\mu\nu}). \quad (B.6)
\]

We are in matter coupled gravity, hence we keep both even and odd powers of \(r\), in the expansion of \(g_{\mu\nu}\).

For \(a^2(z) = L^2/z^2 + \eta L/z + \gamma + O(z/L)\),
\[
a(z) = \frac{L}{z} + \frac{\eta^2 - 4\gamma}{8L}z + O\left(\frac{z^2}{L^2}\right), \quad (B.7)
\]
\[
r = z + \frac{\eta}{2L}z^2 + \frac{\eta^2 + 4\gamma}{16L^2}z^3 + O(z^4), \quad (B.8)
\]
\[
z = r - \frac{\eta}{2L}z^2 + \frac{7\eta^2 - 4\gamma}{16L^2}z^3 + O(r^4), \quad (B.9)
\]
\[
g_{\mu\nu} = \left(1 + \frac{2\eta}{L}r + \frac{3(\eta^2 + 4\gamma)}{8L^2}r^2 + O\left(\frac{r^3}{L^3}\right)\right)(\eta_{\mu\nu} + h_{\mu\nu}). \quad (B.10)
\]

The expansion of the first case \((B.6)\) is a special case of the above expansion \((B.10)\) by substituting \(\eta = -L/z_0\) and \(\gamma = 3L^2/(4z_0^2)\).

### C Detailed derivation of the basis solutions

The Helmholtz equation \((3.5)\) gives the following form
\[
\left((k^2 + m^2\gamma(z_0)) + \frac{m^2L^2}{z^2} + m^2\eta(z_0)\frac{L}{z}\right)
-(D - 1)\left(-\frac{1}{z} + \frac{\eta(z_0)}{2L} + \frac{z}{L^2}\left(\gamma(z_0) - \frac{\eta(z_0)^2}{2}\right)\right)\partial_z - \partial_{z^2}\right)\phi(z, k) = 0. \quad (C.1)
\]

This equation has exact solutions if the linear term in the first-order derivative term is neglected. The two linearly independent basis solutions are by means of the confluent hypergeometric functions of the second kind \(U(a, b, x)\) and of the first kind \(1F_1(a, b, x)\), respectively, namely
\[
\phi(z, k) = z^{\frac{D}{2} + \nu}e^{-\frac{(\alpha + (\beta - 1)\nu)}{4L}z}\left[c_{(1)}U\left(\alpha + \nu, 2\nu + 1, \frac{\beta z}{2L}\right) + c_{(2)}1F_1\left(\alpha + \nu, 2\nu + 1, \frac{\beta z}{2L}\right)\right] = c_{(1)}h_1(z, k) + c_{(2)}h_2(z, k). \quad (C.2)
\]
The second equal sign gives the definition of $h_1(z, k)$ and $h_2(z, k)$, and $c_{(1)}, c_{(2)}$ are two coefficients. Here $\nu = \sqrt{D^2 + m^2 L^2}$, and we have $\alpha = \frac{1}{2} - \frac{n}{2\eta} [\xi - 1 - 4m^2 L^2]$, $\beta = 4L(k^2 + \xi^{-2})^{1/2}$, where $\xi$ is a parameter,

$$\xi = \left(\frac{(D - 1)^2 \eta^2}{16L^2} + \gamma m^2 \right)^{-1/2}. \quad (C.3)$$

The definition of the confluent hypergeometric functions are \[19, 20\]

$$1 F_1(a, b, x) = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)/\Gamma(a) x^n}{\Gamma(b + n)/\Gamma(b) n!}, \quad (C.4)$$

$$U(a, b, x) = \frac{\Gamma(1 - b)}{\Gamma(a + 1 - b)} 1 F_1(a, b, x) + \frac{\Gamma(b - 1)}{\Gamma(a)} x^{1-b} 1 F_1(a + 1 - b, 2 - b, x). \quad (C.5)$$

The definition in the second line only works for non-integer $b$, but it can be extended to any integers $b$ by continuity. The Kummer's transformations

$$U(a, b, x) = x^{1-b}U(a - b + 1, 2 - b, x) \quad (C.6)$$

can be used to verify the $\nu \leftrightarrow -\nu$ symmetry of the Helmholtz equation.

Around $z = 0$, $h_2(z, k) = z^{\frac{D}{2} + \nu} (1 + O(z))$, which does not contain a $z^{\frac{D}{2} - \nu}$ mode. Whereas, $h_1(z, k)$ is vanishing when $z$ goes to infinity, and its expansion around $z = 0$ contains both the $z^{\frac{D}{2} + \nu}$ mode and the $z^{\frac{D}{2} - \nu}$ mode.

Consider this special solution,

$$\phi(z, k) = z^{\frac{D}{2} + \nu} e^{-\frac{(\beta + (D - 1)\eta)z}{4\nu}} U \left(\alpha + \nu, 2\nu + 1, \beta z \frac{\nu}{2L} \right). \quad (C.7)$$

Around the boundary $z = 0$, Eq. (C.7) could be expanded as

$$\phi(z, k) = z^{\frac{D}{2} - \nu} 2^{2\nu} \frac{\Gamma(2\nu)}{\Gamma(\alpha + \nu)} \left(\frac{\beta}{L} \right)^{-2\nu} (1 + O(z)) + z^{\frac{D}{2} + \nu} \frac{\Gamma(-2\nu)}{\Gamma(\alpha - \nu)} (1 + O(z)). \quad (C.8)$$

Hence it can be rewritten in the form

$$\phi(z, k) = \phi_0(k) \left(z^{\frac{D}{2} - \nu}(1 + O(z)) + \frac{G(k)}{2\nu} z^{\frac{D}{2} + \nu}(1 + O(z)) \right), \quad (C.9)$$

with $\phi_0(k)$ a $z$-independent prefactor and

$$G(k) = 2^{\nu} \frac{4^{-\nu} \Gamma(\alpha + \nu) \Gamma(-2\nu)}{\Gamma(2\nu) \Gamma(\alpha - \nu)} \left(\frac{\beta}{L} \right)^{2\nu}. \quad (C.10)$$
The pieces $O(z)$ in the source mode and in the vev mode in (C.9) have the following property near the boundary, $\lim_{\epsilon \to 0} z^{D-\nu} (1 + O(z)) |_{z=\epsilon} = e^{\frac{\beta z}{2L}}$.

In the regime $|\eta| \ll 1$, we have that $\frac{1}{2} - \alpha = O(|\eta|\frac{\xi}{8L}) \ll 1$, i.e. $\alpha \approx \frac{1}{2}$. Using the Legendre duplication formula $\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2})$, the response function $G(k)$ reads

$$G(k) = -2\nu \frac{2^{-2\nu} \Gamma(1-\nu) (k^2 + \xi^{-2})^\nu}{\Gamma(1+\nu)}. \quad (C.11)$$

Consider the limit $\eta \to 0$, $\gamma \to 0$ and hence $\frac{\beta z}{2L} \to 2kz$. By the Kummer’s second transformation [19, 20],

$$U(a, 2a, x) = \frac{e^{x/2}}{\sqrt{\pi}} x^{\frac{3}{2} - a} K_{a-\frac{1}{2}} \left( \frac{x}{2} \right), \quad (C.12)$$

$$1F_1(a, 2a, x) = 4^{a-\frac{1}{2}} e^{x/2} x^{\frac{1}{2} - a} \Gamma(a + \frac{1}{2}) I_{a-\frac{1}{2}} \left( \frac{x}{2} \right), \quad (C.13)$$

the above solutions reduce to

$$h_1(z, k) = e^{-kz} z^{\nu} U \left( \frac{1}{2} + \nu, 2\nu + 1, 2kz \right) = \frac{(2k)^{-\nu}}{\sqrt{\pi}} z^{D/2} K_{\nu}(kz), \quad (C.14)$$

$$h_2(z, k) = e^{-kz} z^{\nu} 1F_1 \left( \frac{1}{2} + \nu, 2\nu + 1, 2kz \right) = \left( \frac{2}{k} \right)^{\nu} \Gamma(\nu + 1) z^{D/2} I_{\nu}(kz), \quad (C.15)$$

and $G(k)$ reduces to $-2\nu \frac{2^{-2\nu} \Gamma(1-\nu)}{\Gamma(1+\nu)} k^{2\nu}$, which are the results in the maximally symmetric AdS geometry, see e.g. [8, 21].

D Detailed derivation of the bulk-to-bulk and boundary-to-bulk propagators

Here we present details of the derivation in Section 4. The bulk-to-bulk propagator $\tilde{G}(k; z; z')$ could be written as

$$\tilde{G}(k; z; z') = \frac{\theta(z - z') \phi_1(z) \phi_2(z') + \theta(z' - z) \phi_1(z') \phi_2(z)}{a^{D-1}(z') (\phi_1(z') \phi_2'(z') - \phi_1'(z') \phi_2(z'))}, \quad (D.1)$$

where $\phi_1(z)$ and $\phi_2(z)$ are two linearly independent solutions of the Helmholtz equation (3.6).

As derived in Section 3, we let

$$\phi_1(z) = z^{\nu} e^{-\frac{\beta z (D-1)n}{4L}} U \left( \alpha + \nu, 2\nu + 1, \frac{\beta z}{2L} \right), \quad (D.2)$$
\[ \phi_2(z) = z^{\nu} e^{-\frac{z(\alpha + 1 - \nu)}{4L}} \binom{\nu}{1} \] (D.3)

Consequently the bulk-to-bulk propagator can be written as
\[ \tilde{G}(k; z; z') = (2L/\beta)B(z')^{-1}a^{1-D}(z')z^{\nu} \frac{\nu}{2} e^{-\frac{(D-1)\eta[z-z']}{4L}} \]
\[ \left[ \theta(z - z') \binom{\nu}{1} F \left( \alpha + \nu, 2\nu + 1; \frac{z'\beta}{2L} \right) U \left( \alpha + \nu, 2\nu + 1, \frac{z\beta}{2L} \right) \right. \]
\[ + \theta(z' - z) \binom{\nu}{1} F \left( \alpha + \nu, 2\nu + 1; \frac{z\beta}{2L} \right) U \left( \alpha + \nu, 2\nu + 1, \frac{z'\beta}{2L} \right) \] (D.4)

where
\[ B(z') = \frac{(\alpha + \nu)}{(2\nu + 1)} \left( \binom{\nu}{1} F \left( \alpha + \nu + 1, 2\nu + 2; \frac{z'\beta}{2L} \right) U \left( \alpha + \nu, 2\nu + 1, \frac{z\beta}{2L} \right) \right. \]
\[ + (2\nu + 1) \binom{\nu}{1} F \left( \alpha + \nu, 2\nu + 1; \frac{z\beta}{2L} \right) U \left( \alpha + \nu + 1, 2\nu + 2, \frac{z'\beta}{2L} \right) \] (D.5)

which comes from the denominator in (D.1).

As long as \( |\eta| \ll 1 \), the two basis solutions (D.2), (D.3) have approximate forms
\[ \phi_1(z) = z^{\nu} \exp \left(-\frac{(D-1)\eta}{4L}z \right) K_\nu \left( k^2 + \xi^{-2} \frac{1}{4} z \right) , \] (D.6)
\[ \phi_2(z) = z^{\nu} \exp \left(-\frac{(D-1)\eta}{4L}z \right) I_\nu \left( k^2 + \xi^{-2} \frac{1}{4} z \right) , \] (D.7)

where \( K_\nu (\cdot) \) and \( I_\nu (\cdot) \) are the modified Bessel functions of the second and the first kind. Therefore, the propagator is
\[ \tilde{G}(k; z; z') = z^{D/2}(z')^{1-\nu} e^{-\frac{(D-1)\eta[z-z']}{4L}} \left[ \theta(z - z') K_\nu \left( z \sqrt{k^2 + \frac{1}{\xi^2}} \right) I_\nu \left( z' \sqrt{k^2 + \frac{1}{\xi^2}} \right) \right. \]
\[ + \theta(z' - z) K_\nu \left( z' \sqrt{k^2 + \frac{1}{\xi^2}} \right) I_\nu \left( z \sqrt{k^2 + \frac{1}{\xi^2}} \right) \] a^{1-D}(z'). (D.8)

The boundary-to-bulk propagator \( \tilde{K}(z, k) \) in momentum space could be derived by taking limit \( z' \to 0 \) in (D.8) together with a normalization factor,
\[ \tilde{K}(z, k) = \lim_{z' \to 0} 2\nu(z')^{\nu-1} a^{D-1}(z') \tilde{G}(k; z; z') \]
\[ = \frac{2^{-\nu-1}}{\Gamma(\nu)} z^{D/2} e^{-\frac{(D-1)\eta}{4L}} K_\nu \left( z \sqrt{k^2 + \frac{1}{\xi^2}} \right) \left( \sqrt{k^2 + \frac{1}{\xi^2}} \right)^\nu . \] (D.9)
Thus the boundary-to-bulk propagator $K(z, x; x')$ in position space could be gotten by Fourier transformation,

$$K(z, x; x') = \int \frac{d^D \vec{k}}{(2\pi)^D} \tilde{K}(z, k) e^{i\vec{k}(\vec{x} - \vec{x}')}$$

$$= \frac{|x - x'|^{D - \frac{D}{2}}}{(2\pi)^{D/2}} \int_0^\infty dk k^{D/2} J_{\frac{D}{2} - 1}(k|x - x'|) \tilde{K}(z, k)$$

$$= \frac{2^{-\nu + 1}}{(2\pi)^{D/2} \Gamma(\nu)} \left( \frac{z}{\xi \sqrt{z^2 + |x - x'|^2}} \right)^{\nu + \frac{D}{2}} e^{-\frac{(D - 1)\eta z^2}{4\xi}} K_{\nu + \frac{D}{2}} \left( \xi^{-1} \sqrt{z^2 + |x - x'|^2} \right),$$

(D.10)

which we present in Eq. (4.4).

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