Spectra of \((H_1, H_2)-\text{combined subdivision graph of a graph} \)

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Abstract

In this paper, we define a ternary graph operation which generalizes the construction of subdivision graph, \(R\)–graph, central graph. Also, it generalizes the construction of overlay graph (Marius Somodi et al., 2017), and consequently, \(Q\)–graph, total graph, and quasitotal graph. We denote this new graph by \([S(G)]_{H_1}^{H_2}\), where \(G\) is a graph, and \(H_1\) and \(H_2\) are suitable graphs corresponding to \(G\). Further, we define several new unary graph operations which becomes particular cases of this construction. We determine the Adjacency and Laplacian spectrum of \([S(G)]_{H_1}^{H_2}\) for some classes of graphs \(G, H_1\) and \(H_2\). As applications, these results enable us to compute the number of spanning trees and Kirchhoff index of these graphs.

Keywords: Adjacency spectrum; Laplacian spectrum; subdivision graph; spanning trees; Kirchhoff index.

2010 Mathematics Subject Classification: 05C50, 05C76

1 Introduction

All the graphs considered in this paper are undirected and simple. \(K_n, C_n\) and \(P_n\) denotes the complete graph, the cycle graph and the path graph on \(n\) vertices, respectively. The complete bipartite graph whose partite sets having sizes \(p\) and \(q\) is denoted by \(K_{p,q}\). \(J_{n \times m}\) denotes the matrix of size \(n \times m\) in which all the entries are 1. We will denote \(J_{n \times n}\) simply by \(J_n\).

In spectral graph theory, there are several matrices associated to graphs. The properties of graphs are studied by the properties of the associated matrices and its eigenvalues and eigenvectors. For a graph \(G = (V,E)\) with \(V(G) = \{v_1, v_2, \ldots, v_n\}\) and \(E(G) = \{e_1, e_2, \ldots, e_m\}\), the adjacency matrix of \(G\) is the \(n \times n\) matrix \(A(G) = [a_{ij}]\), where \(a_{ij} = 1\), if \(i \neq j\) and \(v_i\) and \(v_j\) are adjacent in \(G\); \(0\), otherwise. The vertex-edge incidence matrix of \(G\) is the \(n \times m\) matrix \(B(G) = [b_{ij}]\), where \(b_{ij} = 1\), if the vertex \(v_i\) is incident with the edge \(e_j\); \(0\), otherwise. The degree matrix \(D(G)\) of \(G\) is the diagonal matrix \(diag(d_1, d_2, \ldots, d_n)\), where \(d_i\) denotes the degree of the vertex \(i\). The Laplacian

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matrix $L(G)$ of $G$ is the matrix $D(G) - A(G)$ and the signless Laplacian matrix $Q(G)$ of $G$ is the matrix $D(G) + A(G)$. Note that $Q(G) = B(G)B(G)^T$.

The characteristic polynomial of $A(G)$, $L(G)$ and $Q(G)$ are denoted by $P_G(x)$, $L_G(x)$ and $Q_G(x)$, respectively. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, are said to be the $A$--spectrum, $L$--spectrum and $Q$--spectrum of $G$, respectively. Two graphs are said to be $A$--cospectral (resp. $L$--cospectral, $Q$--cospectral) if they have same the $A$--spectrum (resp. $L$--spectrum, $Q$--spectrum). The $A$--spectrum, $L$--spectrum and $Q$--spectrum of a graph $G$ with $n$ vertices are denoted by $\lambda_1(G), \mu_i(G)$ and $\nu_i(G)$, $i = 1, 2, \ldots, n$, respectively.

If $\lambda_1, \lambda_2, \ldots, \lambda_t$ are the distinct eigenvalues of a a matrix $M$ with multiplicity $m_1, m_2, \ldots, m_t$, respectively, then the eigenvalues of $M$ are denoted by $\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_t^{m_t}$. If $m_i = 1$, for some $i$, then $\lambda_i^{m_i}$ is simply denoted by $\lambda_i$.

Two graphs are said to be commute if their adjacency matrices commute. Some properties, examples of commuting graphs are studied in [1, 2]. For example, the complete graph $K_n$ commutes with any regular graph of order $n$ and the complete bipartite graph $K_{p,q}$ commutes with any of its regular spanning subgraph (See, [17, Proposition 2.3.6]).

A--spectrum, $L$--spectrum of a graph are powerful tools for analyzing the properties of the corresponding graph. Apart from graph theory, the determination of the various spectra of graphs has found applications in many other fields such as physics, chemistry, computer science etc.; see, for instance [3, 7, 13].

In literature, several graph operations were defined to construct graphs from the given graphs such as the Cartesian product, the Kronecker product, the NEPS, the corona, the edge corona, the join etc. For the results on the spectra of these graphs, we refer the reader to [4, 5, 9, 10, 11, 16, 18, 15, 20, 22, 26] and the references therein. On the other hand, there are several interesting graphs constructed from a single graph (that is by unary graph operations). Some of them are given below for the easy reference of the reader: The line graph $L(G)$ of $G$ is the graph having $E(G)$ as its vertex set and two vertices are adjacent if and only if the corresponding edges are adjacent in $G$. The subdivision graph $S(G)$ of $G$ is the graph obtained by inserting a new vertex into every edge of $G$. The middle graph or $Q$--graph $Q(G)$ of $G$ is the graph obtained from $G$ by inserting a new vertex into each edge of $G$, and joining by edges those pairs of new vertices which lie on adjacent edges of $G$. The total graph $T(G)$ of $G$ is the graph obtained by taking one copy of $R(G)$ and joining the new vertices which lie on the adjacent edges of $G$. The quasitotal graph $QT(G)$ of $G$ is the graph obtained by taking one copy of $Q(G)$ and joining the vertices which are not adjacent in $G$. The determination of $A$--spectra of these graphs have been made in [11, 14, 27, 29]. The central graph $C(G)$ of $G$ is the graph obtained by taking one copy of $S(G)$ and joining the vertices which are not adjacent in $G$.

In [25], Marius Somodi et al. defined the following graph operation which generalizes the constructions of the middle, total, and quasitotal graphs: Let $G$ and $G'$ be two graphs having $n$ vertices with same vertex labeling $\{v_1, v_2, \ldots, v_n\}$. Then the overlay of $G$ and $G'$, denoted by $G \times G'$ is the graph obtained by taking $Q(G)$ and joining the vertices $v_i$ and $v_j$ of $G$ if and only if $v_i$ and $v_j$ are adjacent in $G'$. Therein, they obtained the characteristic polynomial of adjacency and
Laplacian matrices of overlay of two commuting graphs. Among the other results, they determined the number of spanning trees and Kirchhoff index of overlay of two graphs. Also they derived the $A$–spectrum and $L$–spectrum of $Q$–graph, total graph and quasitotal graph of a graph.

Motivated by these, in this paper we define a new ternary graph operation namely, $(H_1, H_2)$–combined subdivision graph of $G$ which is obtained from the subdivision graph of $G$ by combining the suitable graphs $H_1$ and $H_2$. Consequently, this construction generalizes some graph operations defined in the literature, and enables us to define some new unary operations in Section 2. In Section 3, we obtain the $A$–spectrum and $L$–spectrum of the $(H_1, H_2)$–combined subdivision graph for some classes of graphs $G$, $H_1$ and $H_2$. In addition, we deduce the $L$–spectra of the overlay graph for some class of constituting graphs. In Section 4, we obtain the number of spanning trees and the Kirchhoff index of the $(H_1, H_2)$–combined subdivision graph for some classes of graphs $G$, $H_1$ and $H_2$.

2. $(H_1, H_2)$–combined subdivision graph of $G$

First we define the following ternary graph operation:

**Definition 2.1.** Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let $H_1$ and $H_2$ be two graphs with $V(H_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(H_2) = \{w_1, w_2, \ldots, w_m\}$. The $(H_1, H_2)$–combined subdivision graph of $G$, denoted by $[S(G)]_{H_1}^{H_2}$, is the graph obtained by taking $S(G)$ and joining the vertices $v_i$ and $v_j$ if and only if the vertices $u_i$ and $u_j$ are adjacent in $H_1$ and joining the new vertices lie on the edges $e_t$ and $e_s$ if and only if $w_t$ and $w_s$ are adjacent in $H_2$.

Clearly, if $G$ has $n$ vertices and $m$ edges, and $H_1$ and $H_2$ have $m_1$ and $m_2$ edges, respectively, then $[S(G)]_{H_1}^{H_2}$ has $n + m$ vertices and $2m + m_1 + m_2$ edges.

We denote the graphs $[S(G)]_{H}^{\bar{k}_n}$ and $[S(G)]_{H}^{\bar{k}_m}$ simply by $[S(G)]_{H}$ and $[S(G)]_{H}$, respectively.

The construction used in Definition 2.1 generalizes many graph constructions: $S(G) \cong [S(G)]_{\bar{k}_n}^{\bar{k}_n}$, $R(G) \cong [S(G)]^{G}$ and $C(G) \cong [S(G)]^{G}$. Also note that the graph $[S(G)]_{L(G)}^{H(G)}$ is the graph overlay of $G$ and $H$. Consequently, $Q(G) \cong [S(G)]_{L(G)}^{H(G)}$, $T(G) \cong [S(G)]_{L(G)}^{H(G)}$, $QT(G) \cong [S(G)]_{L(G)}^{H(G)}$.

**Example 2.1.** Consider the graphs $G$, $H_1$ and $H_2$ as shown in Figure 1(a), 1(b) and 1(c), respectively. Then the graphs $[S(G)]_{H_1}^{H_2}$, $[S(G)]_{H_1}^{H_2}$ and $[S(G)]_{H_2}^{H_2}$ are shown in Figures 1(d), 1(e) and 1(f), respectively.

Some of the special cases of Definition 2.1 enables us to define some interesting unary graph operations:

**Definition 2.2.** Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$.

1. The *point complete subdivision graph* of $G$ is the graph obtained by taking one copy of $S(G)$ and joining all the vertices $v_i, v_j \in V(G)$.

2. The *$Q$–complemented graph* of $G$ is the graph obtained by taking one copy of $S(G)$ and joining the new vertices lie on the non-adjacent edges of $G$. 

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(3) The total complemented graph of $G$ is the graph obtained by taking one copy of $R(G)$ and joining the new vertices lie on the non-adjacent edges of $G$.

(4) The quasitotal complemented graph of $G$ is the graph obtained by taking one copy of $Q$–complemented graph of $G$ and joining all the vertices $v_i, v_j \in V(G)$ which are not adjacent in $G$.

(5) The complete $Q$–complemented graph of $G$ is the graph obtained by taking one copy of $Q$–complemented graph of $G$ and joining all the vertices of $v_i, v_j \in V(G)$.

(6) The complete subdivision graph of $G$ is the graph obtained by taking one copy of $S(G)$ and joining the all the new vertices lie on the edges of $G$.

(7) The complete $R$–graph of $G$ is the graph obtained by taking one copy of $R(G)$ and joining all the new vertices lie on the edges of $G$.

(8) The complete central graph of $G$ is the graph obtained by taking one copy of central graph of $G$ and joining all the new vertices lie on the edges of $G$.

(9) The fully complete subdivision graph of $G$ is the graph obtained by taking one copy of $S(G)$ and joining all the vertices of $G$ and joining all the new vertices lie on the edges of $G$.

Notice that the graphs mentioned in Definitions 2.2(1)-(9) are isomorphic to $[S(G)]^{K_n}$, $[S(G)]^{G}_{E(G)}$, $[S(G)]^{K_n}_{E(G)}$, $[S(G)]^{G}_{K_n}$, $[S(G)]^{K_m}_{E(G)}$, $[S(G)]^{G}_{K_m}$, $[S(G)]^{K_n}_{K_m}$, respectively.
structure of these graphs for $G = C_4$ are shown in Figures 2(a)-(i), respectively.

![Graphs](image)

**Figure 2:** (a) The point complete subdivision graph of $C_4$, (b) The $Q$–complemented graph of $C_4$, (c) The total complemented graph of $C_4$, (d) The quasitotal complemented graph of $C_4$, (e) The complete $Q$–complemented graph of $C_4$, (f) The complete subdivision graph of $C_4$, (g) The complete $R$–graph of $C_4$, (h) The complete central graph of $C_4$, (i) The fully complete subdivision graph of $C_4$.

3. **$A$–spectra and $L$–spectra of $[S(G)]^{H_1}_{H_2}$**

In this section, we compute the $L$–spectrum of $[S(G)]^{H_1}_{H_2}$ for some classes of graphs $G$, $H_1$ and $H_2$.

The Laplacian matrix of $[S(G)]^{H_1}_{H_2}$ is

$$
\begin{bmatrix}
L(H_1) + D(G) & -B(G) \\
-B(G)^T & L(H_2) + 2I_m
\end{bmatrix}
$$

(3.1)

We first state the following results which will be used later.

**Theorem 3.1.** (3) Let $A$ be an $n \times n$ matrix partitioned as

$$
A = \begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix},
$$

where $A_1, A_4$ are square matrices. If $A_1, A_4$ are invertible, then

$$
|A| = |A_4| \left| A_1 - A_2 A_4^{-1} A_3 \right| = |A_1| \left| A_4 - A_3 A_1^{-1} A_2 \right|.
$$

To obtain the $L$–spectrum of $[S(G)]^H$, $[S(G)]^{K_m}[^H_G]$ and $[S(G)]^{[^H_{L(G)}]}$, where $G$ is a regular, first we obtain the characteristic polynomial of a partitioned matrix:
Proposition 3.1. Let \( A \in M_n(\mathbb{R}) \) and \( B \in M_{n \times m}(\mathbb{R}) \) such that the sum of all entries in each row is equal to \( r \). Then the characteristic polynomial of the matrix

\[
M = \begin{bmatrix}
A & B \\
B^T & t_1I_m + t_2J_m + t_3B^T B \\
\end{bmatrix},
\]

is

\[
(x - t_1)^{m-n} \times \left( (x - t_1)I_n - t_3BB^T - \frac{t_2}{2}rJ_n \right) (xI_n - A) - BB^T.
\]

Proof.

\[
\begin{bmatrix}
xI_n - A & -B \\
-B^T & (x - t_1)I_m - t_2J_m - t_3B^T B \\
\end{bmatrix} = \begin{bmatrix}
xI_n - A & -B \\
-B^T - t_3B^T (xI_n - A) (x - t_1)I_m - t_2J_m \\
\end{bmatrix}
\]

\((R_2 \rightarrow R_2 - t_2B^T R_1)\)

\[
= \begin{bmatrix}
xI_n - A & -B \\
-B^T - \left\{ t_3B^T + \frac{t_2}{2}J_{m \times n} \right\} (xI_n - A) (x - t_1)I_m \\
\end{bmatrix}
\]

\((R_2 \rightarrow R_2 - \frac{t_2}{2}J_{n \times m} R_1)\)

So, the result follows from Theorem 3.1.

The following result gives a characterization of commuting matrices in terms of their eigenvectors.

Proposition 3.2. ([17 Proposition 2.3.2]) Let \( A_1, A_2, \ldots, A_m \) be symmetric matrices of order \( n \). Then the following are equivalent.

1. \( A_iA_j = A_jA_i, \forall i, j \in \{1, 2, \ldots, m\} \).
2. There exists an orthonormal basis \( \{x_1, x_2, \ldots, x_n\} \) of \( \mathbb{R}^n \) such that \( x_1, x_2, \ldots, x_n \) are eigenvectors of \( A_i, \forall i = 1, 2, \ldots, m \).

If \( G \) and \( H \) are two commuting graphs, then by Proposition 3.2, there exists an orthonormal basis \( \{x_1, x_2, \ldots, x_n\} \) of \( \mathbb{R}^n \) such that \( x_i \)'s are eigenvectors of both \( A(G) \) and \( A(H) \). For such graphs, throughout this paper, the \( A \)–spectra of \( G \) and \( H \) are denoted by \( \lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G) \) and \( \lambda_1(H), \lambda_2(H), \ldots, \lambda_n(H) \), respectively, where \( \lambda_i(G) \) and \( \lambda_i(H) \) are the eigenvalues of \( A(G) \) and \( A(H) \) corresponding to the same eigenvector \( x_i, i = 1, 2, \ldots, n \). Further, if \( G \) and \( H \) are commuting graphs which are \( r_1, r_2 \) regular, respectively, then their \( A \)–spectra are denoted by \( \lambda_1(G)(= r_1), \lambda_2(G), \ldots, \lambda_n(G) \) and \( \lambda_1(H)(= r_2), \lambda_2(H), \ldots, \lambda_n(H) \), respectively; their \( L \)–spectra are denoted by \( \mu_1(G)(= 0), \mu_2(G), \ldots, \mu_n(G) \) and \( \mu_1(H)(= 0), \mu_2(H), \ldots, \mu_n(H) \), respectively.

Theorem 3.2. Let \( G \) be an \( r \)--regular graph \((r \geq 2)\) with \( n \) vertices and \( m = \frac{1}{2} nr \) edges. Let \( H \) be a regular graph with \( n \) which commutes with \( G \). Let \( M \) be a matrix of the form

\[
M = \begin{bmatrix}
L(H) + rI_n & B(G) \\
B(G)^T & t_1I_m + t_2J_m + t_3B(G)^T B(G) \\
\end{bmatrix}.
\]
Then the eigenvalues of \( M \) are

\[
t_1^{m-n}, \frac{1}{2} \left( r + t_1 + 2rt_3 + mt_2 \pm \sqrt{(r - t_1 - 2rt_3 - mt_2)^2 + 8r} \right),
\]

\[
\frac{1}{2} \left( r + \mu_1(H) + t_1 + 2rt_3 - 3\mu_1(G) \pm \sqrt{(r + \mu_1(H) - t_1 - 2rt_3 + 3\mu_1(G))^2 + 4(2r - \mu_1(G))} \right).
\]

**Proof.** By using Proposition 3.1, \( |xI_{n+m} - M| \) equals,

\[
(x - t_1)^{m-n} \left\{ (x - t_1)I_n - t_3B(G)B(G)^T - \frac{t_2}{2}J_n \right\} \left\{ (x - r)I_n - L(H) \right\} - B(G)B(G)^T
\]

\[
= (x - t_1)^{m-n} \left\{ (x - t_1)I_n - t_3(2rI_n - L(G)) - \frac{t_2}{2}J_n \right\} \left\{ (x - r)I_n - L(H) \right\} - 2rI_n + L(G)
\]

(3.2)

Let \( D = \left\{ (x - t_1)I_n - t_3(2rI_n - L(G)) - \frac{t_2}{2}J_n \right\} \left\{ (x - r)I_n - L(H) \right\} - 2rI_n + L(G) \).

For any graph \( G' \), the sum of entries in each row (column) of \( L(G') \) is 0. So, we have \( J_nL(G') = 0 = L(G')J_n \). That is, for any graph \( G' \), \( J_n \) commutes with \( L(G') \) and so \( J_n \) commutes with both \( L(G) \) and \( L(H) \). Since \( G \) and \( H \) are regular commuting graphs, \( L(G) \) and \( L(H) \) also commute. So \( J_n \), \( L(G) \) and \( L(H) \) mutually commutes with each other. Then by Proposition 3.2, there exists orthonormal vectors \( x_1, x_2, \ldots, x_n \) which are eigenvectors of \( J_n, L(G) \) and \( L(H) \). Since 0 is an eigenvalue of both \( L(G) \) and \( L(H) \), respectively with eigenvector \( \frac{1}{\sqrt{n}}J_{n\times 1} \), we assume that \( \mu_1(1) = 0 \) and \( \mu_1(H) = 0 \). Since \( \frac{1}{\sqrt{n}}J_{n\times 1} \) is an eigenvector of \( J_n \) corresponding to the eigenvalue \( n \), we take \( \lambda_1(J_n) = n \) and all other eigenvalues of \( J_n \) are 0. Let \( P \) be the matrix with columns \( \frac{1}{\sqrt{n}}J_{n\times 1}, x_2, \ldots, x_n \). Then \( P \) is orthonormal. Consequently, \( P^T L(G)P = diag(0, \mu_2(G), \ldots, \mu_n(G)) \), \( P^T L(H)P = diag(n, \mu_2(H), \ldots, \mu_n(H)) \) and \( P^T J_nP = diag(n, 0, 0, \ldots, 0) \). So, we have

\[
D = |P^T| \left\{ (x - t_1)I_n - t_3(2rI_n - L(G)) - \frac{t_2}{2}J_n \right\} \left\{ (x - r)I_n - L(H) \right\} |P|
\]

\[
= \left\{ (x - t_1)I_n - t_3(2rI_n - P^T L(G)P) - \frac{t_2}{2}P^T J_nP \right\} \left\{ (x - r)I_n - P^T L(H)P \right\} - 2rI_n + L(G)
\]

\[
= \left\{ x^2 - (r + t_1 + 2rt_3 + mt_2) x + r(t_1 + 2rt_3 + mt_2) - 2r \right\}
\]

\[
\times \prod_{i=2}^{\infty} \left\{ x^2 - [r + \mu_i(H) + t_1 + t_3(2r - \mu_i(G))]x + t_3(2r - \mu_i(G))(r - \mu_i(H)) - 2r + \mu_i(G) \right\}.
\]

Substituting this in \( (3.2) \) we get the result.

In the following result we deduce the \( L \)-spectra of some special cases of the graph \( [S(G)]_{H_1}^{H_2} \).

**Corollary 3.1.** Let \( G \) be an \( r \)-regular graph \((r \geq 2)\) with \( n \) vertices and \( m = \frac{1}{2}nr \) edges. Let \( H \) be a regular graph with \( n \) which commutes with \( G \). Then we have the following.

1. The \( L \)-spectrum of \([S(G)]_{H}^{H}\) is

\[
0, r + 2, 2^{m-n}, \frac{1}{2} \left( r + \mu_i(H) + 2 \pm \sqrt{(r + \mu_i(H) + 2)^2 - 8\mu_i(H) - 4\mu_i(G)} \right)
\]

for \( i = 2, 3, \ldots, n \).
(2) The $L$–spectrum of $[S(G)]^{H}_{K_m}$ is

$$0, r + 2, (m + 2)^{m-n}, \frac{1}{2} \left( m + r + \mu_i(H) + 2 \pm \sqrt{(m - r - \mu_i(H) + 2)^2 + 4[2r - \mu_i(G)]} \right)$$

for $i = 2, 3, \ldots, n.$

(3) (22) The $L$–spectrum of $[S(G)]^{H}_{L(G)}$ is

$$0, r + 2, (2r + 2)^{m-n}, \frac{1}{2} \left( r + \mu_i(H) + \mu_i(\overline{\mu_i(\overline{G} - \mu_i(H) + 2)^2 + 4[2r - \mu_i(G)]} \right),$$

where $t_i = m - \mu_i(G) + 2$ and $i = 2, 3, \ldots, n.$

(4) The $L$–spectrum of $[S(G)]^{H}_{L(G)}$ is

$$0, r + 2, (m - 2r + 2)^{m-n}, \frac{1}{2} \left( t_i + r + \mu_i(H) \pm \sqrt{(t_i - r - \mu_i(H))^2 + 4[2r - \mu_i(G)]} \right),$$

where $t_i = m - \mu_i(G) + 2$ and $i = 2, 3, \ldots, n.$

Proof. The Laplacian matrix of $[S(G)]^{H}_{K_m}, [S(G)]^{H}_{K_m}$, $[S(G)]^{H}_{L(G)}$ and $[S(G)]^{H}_{L(G)}$ can be obtained by substituting $L(K_m) = 0$, $L(K_m) = mI_m - J_m$, $L_{L(G)} = 2rI_m - B(G)^TB(G)$ and $L_{L(G)} = (m - 2r)I_m - J_m + B(G)^TB(G)$, respectively in (3.1). The $L$–spectrum of these matrices can be obtained, respectively by taking $t_1, t_2$ and $t_3$ in Theorem 3.2 in the following order: (1) $t_1 = 2$, $t_2 = t_3 = 0$; (2) $t_1 = m + 2$, $t_2 = -1$ and $t_3 = 0$; (3) $t_1 = 2r + 2$, $t_2 = 0$ and $t_3 = -1$; (4) $t_1 = m - 2r + 2$, $t_2 = -1$, and $t_3 = 1.$

Note 3.1. For a given regular graph $G$, by suitably substituting the graph $H_1$ and $H_2$ in Corollary 3.1, we can obtain the $L$–spectra of its $R$–graph, central graph and each of the graphs defined in Definitions 2.2(1)-(9). Also, it can be seen that the $L$–spectrum of these graphs constructed using $G$ are uniquely determined by the $L$–spectrum of $G$. Consequently, if $G$ and $G'$ are two regular $L$–cospectral graphs, then the graphs constructed using them as in Definitions 2.2(1)-(9) are $L$–cospectral.

$(H_1, H_2)$–combined subdivision graph of $K_{p,p}$

In the next result, we deduce the $L$–spectrum of $(H_1, H_2)$–combined subdivision graph of $K_{p,p}$, for $H_2 = K_m, K_m, \overline{L(K_{p,p})}.$

To obtain this, we use the following result:

Proposition 3.3. (23 Proof of Lemma 3.13)] Let $G$ be a bipartite graph with bipartite sets $X$ and $Y$ having $p$ and $q$ vertices, respectively. If $\lambda(G)$ is an eigenvalue of $G$ with eigenvector $[x_1 \ x_2]^T$, then $-\lambda(G)$ is also an eigenvalue of $G$ with eigenvector $[x_1 - x_2]^T$, where $x_1 \in \mathbb{R}^p$ and $x_2 \in \mathbb{R}^q$.

Corollary 3.2. Let $H$ be a spanning $r$–regular subgraph of $K_{p,p}$. Then we have the following.
(1) The \( L \)-spectrum of \([S(K_{p,p})]^H\) is
\[
0, p + 2, p + 2r, 2^{p^2 - 2p^2 + 1}, \frac{1}{2} \left( \mu_i + 1 \right) + 2 \pm \sqrt{(p + \mu_i^2) + 2} - 8 \mu_i - 4p \\
\text{for } i = 3, 4, \ldots, 2p.
\]

(2) The \( L \)-spectrum of \([S(K_{p,p})]^H\) \( K_{p,2} \) is
\[
0, p + 2, (p^2 + 2)^{p^2 - 2p^2 + 1}, p + 2r, \frac{1}{2} \left( p^2 + \mu_i^2 + 2 \pm \sqrt{(p^2 - p - \mu_i^2) + 2} + 8p \\
\text{for } i = 3, 4, \ldots, 2p.
\]

(3) The \( L \)-spectrum of \([S(K_{p,p})]^H\) \( L_{(K_{p,p})} \) is
\[
0, p + 2, (p^2 - 2p + 2)^{p^2 - 2p^2 + 1}, p + 2r, \frac{1}{2} \left( p^2 + \mu_i^2 + 2 \pm \sqrt{(p^2 - 2p - \mu_i^2) + 2} + 4p \\
\text{for } i = 3, 4, \ldots, 2p.
\]

(4) The \( L \)-spectrum of \([S(H)]^{K_{p,p}}\) is
\[
0, r + 2, r + 2p, 2^{m-2p^2 + 1}, \frac{1}{2} \left( r + 2 + p \right) - \sqrt{(r + 2 + p) + 8p - 4\mu_i} \\
\text{for } i = 3, 4, \ldots, 2p.
\]

(5) The \( L \)-spectrum of \([S(H)]^{K_{p,p}}\) \( L_{(K_{p,p})} \) is
\[
0, r + 2, r + 2p, (m + 2)^{m-2p^2 + 1}, \frac{1}{2} \left( m + r + p \right) \pm \sqrt{(m - r + 2) + 4[2r - \mu_i]} \\
\text{for } i = 3, 4, \ldots, 2p.
\]

(6) The \( L \)-spectrum of \([S(H)]^{K_{p,p}}\) \( L_{(H)} \) is
\[
0, r + 2, r + 2p, (m - 2r + 2)^{m-2p^2 + 1}, \frac{1}{2} \left( s_i + k \right) \pm \sqrt{(s_i - k) + 4[2r - \mu_i]} \\
\text{where } k = r + p, \text{ } s_i = m - \mu_i + 2 \text{ and } i = 3, 4, \ldots, 2p,
\]

Proof. Note that the spectrum of \( K_{p,p} \) is \( p, -p, 0^{2p-2} \). Also \( J_{2p \times 1} \) and \( [J_{1 \times p} - J_{1 \times p}^T] \) are the eigenvectors corresponding to the eigenvalues \( p \) and \( -p \), respectively. Since \( H \) is an \( r \)-regular spanning subgraph of \( K_{p,p} \), it is a \( r \)-regular bipartite graph. Since \( H \) is an \( r \)-regular, \( r \) is an eigenvalue of \( A(H) \) corresponding to the eigenvector \( J_{2p \times 1} \). By Proposition \( 3.3 \), \( -r \) is also an eigenvalue of \( H \) corresponding to the eigenvector \( [J_{1 \times p} - J_{1 \times p}^T]^T \), since \( H \) is bipartite. Consequently, 0 is an eigenvalue of \( L(K_{p,p}) \) (resp. \( L(H) \)) corresponding to the eigenvector \( J_{2p \times 1} \) and 2p (resp. 2r) is an eigenvalue of \( L(K_{p,p}) \) (resp. \( L(H) \)) corresponding to the eigenvector \( [J_{1 \times p} - J_{1 \times p}^T]^T \). So, we arrange the \( L \)-spectrum of \( K_{p,p} \) and \( H \) such that \( \mu_1(K_{p,p}) = 0, \mu_2(K_{p,p}) = 2p, \mu_3(K_{p,p}) = p, \ldots, \mu_2p(K_{p,p}) = p \) and \( \mu_1(H) = 0, \mu_2(H) = 2r, \mu_3(H), \ldots, \mu_2p(H) \). Then by using Corollary \( 3.1 \), we get the result.
Note 3.2. The argument used in the proof of Corollary 3.1 can be used to derive the adjacency, signless Laplacian spectrum and the normalized Laplacian spectrum of those graphs.

\((H_1, H_2)\)−combined subdivision graph of \(K_{1,m}\)

Theorem 3.3. If \(H\) is a graph with \(m\) vertices, then we have the following.

1. If \(H\) is \(r\)−regular, then the \(A\)−spectrum of \([S(K_{1,m})]_H\) is

\[0, \frac{1}{2} \left( r \pm \sqrt{r^2 + 4m + 4} \right), \frac{1}{2} \left( \lambda_i(H) \pm \sqrt{\lambda_i(H)^2 + 4} \right) \text{ for } i = 2, 3, \ldots, m.\]

2. The \(L\)−spectrum of \([S(K_{1,m})]_H\) is

\[0, \frac{1}{2} \left( m + 3 \pm \sqrt{(m-1)^2 + 4} \right), \frac{1}{2} \left( \mu_i(H) + 3 \pm \sqrt{\mu_i(H)^2 + 4} \right) \text{ for } i = 2, 3, \ldots, m,\]

Proof. (1) It is easy to see that

\[A([S(G)]_H) = \begin{bmatrix} 0 & B(K_{1,m}) \\ B(K_{1,m})^T & A(H) \end{bmatrix}.\]

By using Theorem 3.1 and the fact \(L(K_{1,m}) = K_m\), we have

\[P_{[S(K_{1,m})]_H}(x) = x^{n-m} |x^2I_m - xA(H) - B(K_{1,m})^TB(K_{1,m})| = x^{n-m} |(x^2 - 2)I_m - xA(H) - A(L(K_{1,m}))| = x^{n-m} |(x^2 - 2)I_m - xA(H) - A(K_m)| = x^{n-m} |(x^2 - 1)I_m - xA(H) - J_m| = x(x^2 - rx - m - 1) \times \left\{ \prod_{i=2}^{m} (x^2 - \lambda_i(H)x - 1) \right\}.\]

So the proof follows.

(2) It is easy to see that

\[L([S(K_{1,m})]_H) = \begin{bmatrix} D(K_{1,m}) & -B(K_{1,m}) \\ -B(K_{1,m})^T & L(H) + 2I_m \end{bmatrix}.\]

By using Theorem 3.1 we have

\[L_{[S(K_{1,m})]_H}(x) = |xI_m - D(K_{1,m})||xI_m - L(H) - 2I_m - B(K_{1,m})[xI_m + D(K_{1,m})]^{-1}B(K_{1,m})^T|\]

Since \(B(K_{1,m})^T = [J_{m \times 1} \quad I_m]\) and \(D(K_{1,m}) = \begin{bmatrix} m & 0 \\ 0 & I_m \end{bmatrix}\), we have

\[|xI_m + D(K_{1,m})| = (x - m)(x - 1)^m\]
and
\[
B(K_{1,m})^T(xI - D(K_{1,m}))^{-1}B(K_{1,m}) = \frac{1}{x-m}J_m + \frac{1}{x-1}I_m.
\]

Applying these in (3.3), we get
\[
L[S(K_{1,m})]_H(x) = (x - m)(x - 1)^m \left| (x - 2)I_m - L(H) - \frac{1}{x-m}J_m - \frac{1}{x-1}I_m \right|
\]
\[
= (x - m)^{1-m} \left| (x - m)(x - 1)[(x - 2)I_m - L(H)] - (x - 1)J_m - (x - m)I_m \right|
\]

Using the eigenvalues of \( J_m \),
\[
L[S(K_{1,m})]_H(x) = \left\{ (x - m)(x - 1)(x - 2) - m(x - 1) - (x - m) \right\}
\]
\[
\times \left\{ \prod_{i=2}^n ((x - 1)[x - \mu_i(H) - 2] - 1) \right\}
\]
\[
= x(x^2 - (m + 3)x + 2m + 1) \times \left\{ \prod_{i=2}^m \left( x^2 - (\mu_i(H) + 3)x + \mu_i(H) + 1 \right) \right\}.
\]

So the proof follows.

\((H_1, H_2)\)-combined subdivision graph of \( P_n \)

The following result is proved in [6].

**Theorem 3.4.** ([6, Theorem 3.2]) Suppose that \( p(x) \) is a polynomial of degree less than \( n \). Then
\( p(A(P_n)) \) is the adjacency matrix of a graph if and only if \( p(x) = P_{P_{2i+1}}(x) \), for some \( i \), \( 0 \leq i \leq \lfloor n/2 \rfloor - 1 \).

Using this result, we prove the following result.

**Corollary 3.3.** Let \( n \geq 3 \) be an integer. If \( H \) is a graph with \( A(H) = P_{P_{2i+1}}(A(P_{n-1})) \), for some \( i \), with \( 0 \leq i \leq \lfloor n/2 \rfloor - 1 \), then the \( A \)-spectrum of \( [S(P_n)]_H \) is
\[
c_j \pm \sqrt{c_j^2 + 8 \left( \cos \frac{\pi j}{n} + 1 \right)},
\]
where \( c_j = \sum_{k=0}^{i} (-1)^k \binom{2i+1-k}{k} \left( 2 \cos \frac{\pi j}{n} \right)^{2(i-k)+1} \) and \( j = 1, 2, \ldots, n-1 \).

**Proof.** It is easy to see that
\[
A([S(P_n)]_H) = \begin{bmatrix} 0 & B(P_n) \\ B(P_n)^T & A(H) \end{bmatrix}
\]

So by Theorem 3.1 we have
\[
P_{[S(P_n)]_H}(x) = x \times \left| x^2I_n - xA(H) - B(P_n)B(P_n)^T \right|
\]
\[
= x \times \left| (x^2 - 2)I_n - xA(H) - A(L(P_n)) \right|.
\]
Using the facts \( \mathcal{L}(P_n) = P_{n-1} \) and \( A(H) = P_{P_2 + 1}(A(P_{n-1})) \), we have

\[
P_{[S(P_n)]H}(x) = x \times \left\{ \prod_{j=1}^{n-1} \left( x^2 - P_{P_2 + 1}(\lambda_j(P_{n-1}))x - \lambda_j(P_{n-1}) - 2 \right) \right\}.
\]

So the proof follows from the facts that \( P_{P_2 + 1}(x) = \sum_{k=0}^{i} (-1)^k \binom{2i + 1 - k}{k} x^{2(i-k)+1} \) and \( \lambda_j(P_{n-1}) = 2 \cos \frac{\pi j}{n} \), where \( j = 1, 2, \ldots, n - 1 \).

\[\Box\]

**Q-complemented graph of a graph**

For a given \( n \times n \) matrix \( M \), its coronal \( \chi_M(x) \) is defined as \( \chi_M(x) = J_{n \times n}(xI_n - M)^{-1}J_{n \times 1}^T \).

For a graph \( G \), the coronal of \( G \) is defined as the coronal of \( A(G) \) and is simply denoted by \( \chi_G(x) \).

**Proposition 3.4.** ([23] Proposition 6]) If \( G \) is an \( r \)-regular graph with \( n \) vertices, then

\[
\chi_G(x) = \frac{n}{x - r}.
\]

**Theorem 3.5.** ([23] Proposition 6]) Let \( M \) be a square matrix of order \( n \) and \( \alpha \) be a scalar. Then

\[
|xI_n - M - \alpha J_n| = (1 - \alpha \chi_M(x)) |xI_n - M|.
\]

**Theorem 3.6.** ([22] Theorem 2.4.1, Equation (2.28)]) Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then the characteristic polynomial of \( A(\mathcal{L}(G)) \) is

\[
(x + 2)^{m-n}Q_G(x + 2).
\]

**Theorem 3.7.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then the characteristic polynomial of \( Q \)-complemented graph of \( G \) is

\[
(-1)^n(x - 1)^m \left( 1 - \frac{x}{1-x} \chi_{\mathcal{L}(G)} \left( \frac{x^2 + x - 2}{1-x} \right) \right) Q_G(-x).
\]

**Proof.** Note that

\[
A \left( \left[ S(G) \right] \overline{\mathcal{L}(G)} \right) = \begin{bmatrix} 0 & B(G) \\ B(G)^T & A(\overline{\mathcal{L}(G)}) \end{bmatrix}
\]

So by Theorem 3.1 and using the fact \( A(\overline{\mathcal{L}(G)}) = J_m - I_m - A(\mathcal{L}(G)) \)

\[
P_{[S(G)]\overline{\mathcal{L}(G)}}(x) = x^{n-m} \times \left| x^2 I_m - - A(\overline{\mathcal{L}(G)})B(G)^T B(G) \right|
\]

\[
= x^{n-m} \times \left| (x^2 + x - 2)I_m - xJ_m - (1-x)A(\mathcal{L}(G)) \right|
\]

By using Theorem 3.5 we have,

\[
P_{[S(G)]\overline{\mathcal{L}(G)}}(x) = x^{n-m} \left( 1 - \frac{x}{1-x} \chi_{\mathcal{L}(G)} \left( \frac{x^2 + x - 2}{1-x} \right) \right) \left| (x^2 + x - 2)I_m - (1-x)A(\mathcal{L}(G)) \right|
\]

\[
= x^{n-m}(1-x)^m \left( 1 - \frac{x}{1-x} \chi_{\mathcal{L}(G)} \left( \frac{x^2 + x - 2}{1-x} \right) \right) P_{\mathcal{L}(G)} \left( \frac{x^2 + x - 2}{1-x} \right).
\]

So the proof follows by Theorem 3.6.

\[\Box\]
In the following result, we show that for a graph $G$ whose line graph is regular, $Q$–complemented graph of $G$ can be completely determined by the $Q$–spectrum of $G$.

**Corollary 3.4.** Let $G$ be a graph with $n$ vertices and $m$ edges whose line graph is $r$–regular ($r \geq 1$). Then the $A$–spectrum of $Q$–complemented graph of $G$ is

$$1^{m-1}, -\nu_i(G), \frac{m-r-1 \pm \sqrt{(m-r-1)^2 + 4r + 8}}{2},$$

where $i = 2, 3, \ldots, n$.

**Proof.** Since $\mathcal{L}(G)$ is $r$–regular, by Proposition 3.4 $\chi_{\mathcal{L}(G)}(x) = \frac{m}{x-r}$. So

$$\chi_{\mathcal{L}(G)} \left( \frac{x^2 + x - 2}{1-x} \right) = \frac{m(1-x)}{(x+r+2)(x-1)}.$$

By Theorem 3.7, the characteristic polynomial of $Q$–complemented graph of $G$ is

$$P_{\{S(G)\}_{\mathcal{L}(G)}}(x) = (-1)^n(x-1)^m \left( 1 - \frac{mx}{(x+r+2)(x-1)} \right) Q_G(-x)$$

$$= (-1)^n(x-1)^m \left( \frac{x^2 - (m-r-1)x - r - 2}{(x+r+2)(x-1)} \right) Q_G(-x) \quad (3.3)$$

Also, since $A(\mathcal{L}(G)) = B(G)^T B(G) - 2I_m$, the sum of the entries in each row of the matrix $B(G)^T B(G)$ is $r+2$ and so $\nu_1(G) = r+2$. From this fact, we get

$$Q_G(-x) = (-1)^n(x+r+2) \times \left\{ \prod_{i=2}^n (x + \nu_i(G)) \right\}.$$

The proof follows by substituting this in (3.3). \qed

**Corollary 3.5.** The $A$–spectrum of $Q$–complemented graph of $K_{p,q}$ is

$$0, 1^{pq-1}, (-p)^{q-1}, (-q)^{p-1}, \frac{1}{2} \left( pq - p - q + 1 \pm \sqrt{(pq - p - q + 1)^2 + 4(p+q)} \right).$$

**Proof.** Since $\mathcal{L}(K_{p,q})$ is $(p+q-2)$–regular, by using Corollary 3.4 and the $Q$–spectrum of $K_{p,q}$ is $p+q, 0, q^{p-1}, p^{q-1}$, we get the $A$–spectrum of $Q$–complemented graph of $K_{p,q}$. \qed

**Complete subdivision graph of a graph**

**Theorem 3.8.** Let $G$ be a graph with $n$ vertices and $m$ edges. Then the characteristic polynomial of the complete subdivision graph of $G$ is

$$(x + 1)^{m-n} (1 - x \chi_{\mathcal{L}(G)}(x^2 + x - 2)) Q_G(x^2 + x);$$

**Proof.** Note that

$$A([S(G)]_{K_m}) = \begin{bmatrix} 0 & B(G) \\ B(G)^T & A(K_m) \end{bmatrix}$$

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So by Theorem 3.1 and using the fact \( A(K_m) = J_m - I_m \), we have

\[
P_{[S(G)]K_m}(x) = x^{n-m} \left| x^2 I_m - x(J_m - I_m) - B(G)^T B(G) \right|
\]

\[
= x^{n-m} \left| (x^2 + x - 2)I_m - xJ_m - A(L(G)) \right|
\]

\[
= x^{n-m} \left( 1 - x \chi(L(G))(x^2 + x - 2) \right) \left| (x^2 + x - 2)I_m - A(L(G)) \right|
\]

\[
= x^{n-m} \left( 1 - x \chi(L(G))(x^2 + x - 2) \right) P_{L(G)}(x^2 + x - 2).
\]

Proof follows by Theorem 3.6.

In the next result, we show that for a graph \( G \) whose line graph is regular, the \( A \)--spectrum of complete subdivision graph of \( G \) can be completely determined by the \( Q \)--spectrum of \( G \).

**Corollary 3.6.** (1) The \( A \)--spectrum of complete subdivision graph of \( tK_{1,2} \) \((t \geq 1)\) is

\[
0^t, \left( \frac{-1 \pm \sqrt{5}}{2} \right)^{t-1}, \frac{1}{2} \left( 2t - 1 \pm \sqrt{(2t-1)^2 + 12} \right)
\]

(2) Let \( G \) be a graph with \( n \) vertices and \( m \) edges whose line graph is \( r \)--regular \((r \geq 2)\). Then the \( A \)--spectrum of complete subdivision graph of \( G \) is

\[
(-1)^{m-n}, \frac{1}{2} \left( m - 1 \pm \sqrt{(m-1)^2 + 4r} \right), \frac{1}{2} \left( -1 \pm \sqrt{4r(G)} \right) \] for \( i = 2, 3, \ldots, n \).

**Proof.** (1) Since \( L(G) \) is \( r \)--regular, by Proposition 3.4, \( \chi(L(G))(x) = \frac{2t}{x - 1} \). Using this fact and the \( Q \)--spectrum of \( tK_{1,2} \) is \( 3^t, 1^t, 0^t \) in the parts (2) and (3) of Theorem 3.8, we get the result.

(2) Since \( L(G) \) is \( r \)--regular, by Proposition 3.4, \( \chi(L(G))(x) = \frac{m}{x - r} \). So by Theorem 3.8, the characteristic polynomial of \( A([S(G)]K_m) \) is

\[
(x + 1)^{m-n} \left( \frac{x^2 - (m-1)x - r - 2}{x^2 + x - r - 2} \right) Q_G(x^2 + x).
\]

Also, since \( A(L(G)) = B(G)^T B(G) - 2I_m \), the sum of the entries in each row of the matrix \( B(G)^T B(G) \) is \( r + 2 \) and so \( \nu_1(G) = r + 2 \). Using this fact, the proof follows from (3.4).

**Corollary 3.7.** Let \((p, q) \neq (1, 2), (2, 1)\). Then the \( A \)--spectrum of complete subdivision graph of \( K_{p,q} \) is

\[
0, (-1)^\alpha, \left( \frac{-1 \pm \sqrt{4pq + 1}}{2} \right)^{q-1}, \left( \frac{-1 \pm \sqrt{4q + 1}}{2} \right)^{p-1}, \frac{pq - 1 \pm \sqrt{(pq - 1)^2 + 4(p + q)}}{2},
\]

where \( \alpha = pq - p - q + 1 \).

**Proof.** Note that \( L(K_{p,q}) \) is \((p + q - 2)\)--regular. So by using Corollary 3.6 (2) and the fact that the \( Q \)--spectrum of \( K_{p,q} \) is \( p + q, 0, p^{q-1}, q^{p-1} \), we get the \( A \)--spectrum of complete subdivision graph of \( K_{p,q} \).
4 Applications

In this section, we determine the number of spanning trees and the Kirchhoff index of $[S(G)]_{H_1}$ for some families of graphs $G$, $H_1$ and $H_2$.

The number of spanning trees of a graph $G$ is denoted by $\tau(G)$. First, we state a well known result to count the number of spanning trees of a graph using Laplacian eigenvalues.

**Theorem 4.1.** ([3, Theorem 4.11]) Let $G$ be a graph with $n$ vertices. Then

$$\tau(G) = \mu_2(G)\mu_3(G)\ldots\mu_n(G)/n.$$  

By using Corollary 3.1 and Theorem 3.3 in Theorem 4.1, we have the following result.

**Corollary 4.1.** Let $G$ be an $r$-regular graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $H$ be a graph with $V(H) = \{u_1, u_2, \ldots, u_n\}$ which commutes with $G$. Then we have the following.

1. $\tau([S(G)]_{H}) = 2^{m-n+1} \times \frac{1}{n} \left\{ \prod_{i=1}^{n} [2\mu_i(H) + \mu_i(G)] \right\}$.
2. $\tau([S(G)]_{K_m}) = (m + 2)^{m-n} \times \frac{2}{n} \left\{ \prod_{i=1}^{n} [(m + 2)[r + \mu_i(H)] + \mu_i(G) - 2r] \right\}$.
3. $\tau([S(G)]_{L(G)}) = (m - 2r + 2)^{m-n} \times \frac{2}{n} \left\{ \prod_{i=1}^{n} [(r + \mu_i(H)][m - \mu_i(G) + 2] + \mu_i(G) - 2r] \right\}$.
4. If $H$ is a graph with $m$ vertices, then

$$\tau([S(K_{1,m})]_{H}) = \left\{ \prod_{i=2}^{m} (\mu_i(H) + 1) \right\}.$$  

Let $G$ be a connected graph with $V(G) = \{1, 2, \ldots, n\}$. Then the resistance distance $r_{ij}$ between vertices $i$ and $j$ of $G$ is defined to be the effective resistance between nodes $i$ and $j$ as computed with Ohm’s law when all the edges of $G$ are considered to be unit resistors. The Kirchhoff index $Kf(G)$ of $G$ is defined as $Kf(G) = \sum_{i<j} r_{ij}$ [19].

The resistance distance and the Kirchhoff index attracted extensive attention due to their wide applications in electric network theory, physics, chemistry, etc. and the Kirchoff index of graphs constructed by graph operations were also obtained; see [8, 16, 21, 28, 30, 31, 32].

The Kirchhoff index of a connected graph can be calculated by using the following result.

**Theorem 4.2.** ([16, Lemma 3.4]) For a connected graph $G$ with $n \geq 2$ vertices,

$$Kf(G) = n \sum_{i=2}^{n} \frac{1}{\mu_i(G)}.$$  

**Corollary 4.2.** Let $G$ be an $r$-regular graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $H$ be a graph with $V(H) = \{u_1, u_2, \ldots, u_n\}$ which commutes with $G$. Then we have the following.
(1) \( K_f([S(G)]^H) = \frac{n}{2} + \frac{m^2 - n^2}{2} + k_1 \sum_{i=2}^{n} \left( \frac{k_2 + \mu_i(H)}{2\mu_i(H) + \mu_i(G)} \right), \)

(2) \( K_f([S(G)]^H_{K_m}) = \frac{n}{2} + \frac{m^2 - n^2}{m + 2} + k_1 \sum_{i=2}^{n} \left( \frac{m + k_2 + \mu_i(H)}{m(r + \mu_i(H)) + 2\mu_i(H) + \mu_i(G)} \right), \)

(3) \( K_f([S(G)]^H_{\bar{E}(G)}) = \frac{n}{2} + \frac{m^2 - n^2}{k_3} + k_1 \sum_{i=2}^{n} \left( \frac{m + k_2 + \mu_i(H) - \mu_i(G)}{[m - \mu_i(G)](r + \mu_i(H)) + 2\mu_i(H) + \mu_i(G)} \right), \)

(4) If \( H \) is a graph with \( m \) vertices, then

\[ K_f([S(K_{1,m})]^H) = m + 3 + (2m + 1) \sum_{j=2}^{m} \left( \frac{\mu_j(H) + 3}{\mu_j(H) + 1} \right), \]

where \( k_1 = m + n \), \( k_2 = r + 2 \) and \( k_3 = m - 2r + 2 \).

Proof. (1) Let \( a_i = r + \mu_i(H) + 2, b_i = 2\mu_i(H) + \mu_i(G) \). Then by applying Corollary 3.1(1) in Theorem 4.2, we have,

\[ K_f([S(G)]^H) = \frac{m + n}{r + 2} + \frac{m^2 - n^2}{2} + (m + n) \sum_{i=2}^{n} \left( \frac{2}{a_i + \sqrt{a_i^2 - 4b_i}} + \frac{2}{a_i - \sqrt{a_i^2 - 4b_i}} \right) \]

\[ = \frac{nr + 2n}{2(r + 2)} + \frac{m^2 - n^2}{2} + (m + n) \sum_{i=2}^{n} \left( \frac{a_i}{b_i} \right). \]

(2)–(4): Proof is analogous to the proof of (1) and by using Corollary 3.1(2), 3.1(4) and 3.3 respectively in Theorem 4.2.

\[ \square \]

Acknowledgment

The second author is supported by INSPIRE Fellowship, Ministry of Science and Technology, Government of India under the grant no. DST/INSPIRE Fellowship/[IF150651] 2015.

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