Boundedness criterion for integral operators on the fractional Fock–Sobolev spaces

Guangfu Cao¹ · Li He¹ · Ji Li² · Minxing Shen³

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Abstract
We provide a boundedness criterion for the integral operator \( S_\varphi \) on the fractional Fock–Sobolev space \( F^{s,2}(\mathbb{C}^n) \), \( s \geq 0 \), where \( S_\varphi \) (introduced by Zhu [18]) is given by

\[
S_\varphi F(z) := \int_{\mathbb{C}^n} F(w)e^{\bar{z} \cdot \bar{w}} \varphi(z - \bar{w}) d\lambda(w)
\]

with \( \varphi \) in the Fock space \( F^2(\mathbb{C}^n) \) and \( d\lambda(w) := \pi^{-n} e^{-|w|^2} dw \) the Gaussian measure on the complex space \( \mathbb{C}^n \). This extends the recent result in Cao et al. (Adv Math 363: 107001, 33 pp, 2020). The main approach is to develop multipliers on the fractional Hermite–Sobolev space \( W^{s,2}_H(\mathbb{R}^n) \).

Keywords Fock–Sobolev space · Hermite–Sobolev space · Integral operator · Hermite operator · Bargmann transform

Mathematics Subject Classification 30H20 · 42A38 · 44A15

1 School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
2 Department of Mathematics, Macquarie University, Sydney, NSW 2109, Australia
3 Department of Mathematics, Sun Yat-sen (Zhongshan) University, Guangzhou 510275, China
1 Introduction

Let $\mathbb{C}^n$ be the complex $n$-dimensional space with the inner product

$$z \cdot \bar{w} = \sum_{j=1}^{n} z_j w_j, \quad z = (z_1, \ldots, z_n), \quad w = (w_1, \ldots, w_n) \in \mathbb{C}^n$$

and modulus $|z| = (z \cdot \bar{z})^{\frac{1}{2}}$. The Fock space $F^2(\mathbb{C}^n)$ is the set of all entire functions $F$ on $\mathbb{C}^n$ such that the norm

$$\|F\|_{F^2(\mathbb{C}^n)} := \left( \int_{\mathbb{C}^n} |F(z)|^2 d\lambda(z) \right)^{\frac{1}{2}} < \infty,$$

where $d\lambda(z) := \pi^{-n} e^{-|z|^2} dz$ is the Gaussian measure on $\mathbb{C}^n$ ([1, 2]).

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $\alpha \in \mathbb{N}_0^n$ we define

$$e_\alpha := \frac{z^\alpha}{\|z^\alpha\|_{F^2}} = \frac{z^\alpha}{\sqrt{\alpha!}}.$$

Then $\{e_\alpha | \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for $F^2(\mathbb{C}^n)$. The Fock space $F^2(\mathbb{C}^n)$ is a Hilbert space with the inner product inherited from $L^2(\mathbb{C}^n, d\lambda)$. The fractional Fock-Sobolev space of order $s \in \mathbb{R}$ is defined by

$$F^{s,2}(\mathbb{C}^n) := \left\{ f = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha : \|f\|_{F^{s,2}(\mathbb{C}^n)} < \infty \right\}$$

with the norm given by

$$\|f\|_{F^{s,2}(\mathbb{C}^n)} := \left[ \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s |c_\alpha|^2 \right]^{\frac{1}{2}}.$$

The Fock-Sobolev space is a convenient tool for many problems in functional analysis, mathematical physics, and engineering. We refer to [1–3, 11, 13, 18, 19] for an introduction.

In [18], Zhu introduced the following integral operator

$$S_\varphi F(z) := \int_{\mathbb{C}^n} F(w) e^{z \cdot \bar{w}} \varphi(z - \bar{w}) d\lambda(w), \quad (1)$$

which recovers (or is linked to) many fundamental examples of integral operators in harmonic analysis and complex analysis with different choices of $\varphi \in F^2(\mathbb{C}^n)$, including the Riesz transform on $\mathbb{R}^n$ and the Ahlfors–Beurling operator on $\mathbb{C}$. Thus, characterizing the boundedness of $S_\varphi$ is interesting and non-trivial. In [7], it was shown by Wick, Yan and the first, third and fourth authors that the integral operator $S_\varphi$ in (1) is bounded on $F^2(\mathbb{C}^n)$ if and only if there exists an $m \in L^\infty(\mathbb{R}^n)$ such that

$$\varphi(z) = \left( \frac{2}{\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x) e^{-2(x - \frac{1}{2} z)^2} dx, \quad z \in \mathbb{C}^n. \quad (2)$$

Moreover, we have that

$$\|S_\varphi\|_{F^2(\mathbb{C}^n) \to F^2(\mathbb{C}^n)} = \|m\|_{L^\infty(\mathbb{R}^n)}.$$
The purpose of the paper is to continue the line in [7, 18] to establish a boundedness criterion for the integral operator $S_{\varphi}$ on the fractional Fock–Sobolev space $F^{s,2}(\mathbb{C}^n)$ by developing the multipliers on the fractional Hermite–Sobolev space $W^{s,2}_H(\mathbb{R}^n)$ (see Sect. 3 below about the multipliers on the fractional Hermite-Sobolev space $W^{s,2}_H(\mathbb{R}^n)$), where $\varphi \in F^2(\mathbb{C}^n)$. Our main result is the following.

**Theorem 1** Let $s \geq 0$, $\varphi \in F^2(\mathbb{C}^n)$. Then the integral operator $S_{\varphi}$ is bounded on $F^{s,2}(\mathbb{C}^n)$ if and only if there exists a multiplier $m$ on the space $W^{s,2}_H(\mathbb{R}^n)$ such that

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x)e^{-2(x-z)^2} dx, \quad z \in \mathbb{C}^n.$$ 

In particular, $\varphi \in F^{s,2}(\mathbb{C}^n)$ if $S_{\varphi}$ is bounded on $F^{s,2}(\mathbb{C}^n)$.

We would like to mention that in a recent paper [16], Wick and Wu obtained an isometry between the Fock–Sobolev space and the Gauss–Sobolev space. As an application, they used multipliers on the Gauss–Sobolev space to characterize the boundedness of the integral operator $S_{\varphi}$ in (1) on the Fock-Sobolev spaces $F^{s,2}(\mathbb{C}^n)$ when $s$ is a positive integer. Note that when $s$ is a positive integer, our result in Theorem 1 coincides with their result in [16, Theorem 4.4].

The layout of the article is as follows. In Sect. 2 we prove some properties of the fractional Hermite–Sobolev spaces and the fractional Fock–Sobolev spaces for proving our main result. In Sect. 3 we develop the multipliers on the fractional Hermite–Sobolev spaces. The proof of our Theorem 1 will be given in Sect. 4 by adapting an argument in [7] to the fractional Fock–Sobolev space $F^{s,2}(\mathbb{C}^n)$ for $s \geq 0$ by using multipliers on the fractional Hermite–Sobolev space $W^{s,2}_H(\mathbb{R}^n)$ in Sect. 3.

Throughout, the letter “c” and “C” will denote (possibly different) constants that are independent of the essential variables.

## 2 Preliminaries

Let $H$ be the Hermite operator (also called the harmonic oscillator) in the $n$-dimensional real space $\mathbb{R}^n$, which is defined by

$$H := -\Delta + |x|^2 := -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + |x|^2, \quad x = (x_1, \ldots, x_n). \quad (3)$$

The Hermite operator arises naturally in mathematical physics (see [10]). For each non-negative integer $k$, the Hermite polynomials $H_k$ on $\mathbb{R}$ are defined by $H_k = e^{x^2} \frac{d^k}{dx^k}(e^{-x^2})$ and by normalization in $L^2(\mathbb{R})$, the Hermite functions

$$h_k(x) = (\sqrt{\pi} 2^k k!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} (-1)^k H_k(x), \quad x \in \mathbb{R}.$$ 

It is not difficult to check that

$$\left(-\frac{d^2}{dx^2} + x^2\right) h_k(x) = (2k + 1) h_k(x). \quad (4)$$
In the higher dimensions, for each multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \), the Hermite function \( h_\alpha \) on \( \mathbb{R}^n \) is defined by

\[
h_\alpha(x) = \prod_{j=1}^{n} h_{\alpha_j}(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]

By (4), we see that

\[
H h_\alpha = (2|\alpha| + n)h_\alpha.
\] (5)

That is, these \( \{h_\alpha\} \) are eigenfunctions of the Hermite operator \( H \). Moreover, \( \{h_\alpha\}_{\alpha \in \mathbb{N}_0^n} \) is an orthonormal basis of \( L^2(\mathbb{R}^n) \). Note there is a constant \( C > 0 \) such that \( \|h_\alpha\|_{L^\infty(\mathbb{R}^n)} \leq C \) for all \( \alpha \in \mathbb{N}_0^n \), and for each \( m \in \mathbb{N} \), we have

\[
|\langle f, h_\alpha \rangle_{L^2(\mathbb{R}^n)}| \leq \|H^m f\|_{L^2(\mathbb{R}^n)}(2|\alpha| + n)^{-m}.
\]

Hence, if \( f \) is a rapidly decreasing function, then the Hermite series expansion

\[
f = \sum_{\alpha \in \mathbb{N}_0^n} \langle f, h_\alpha \rangle_{L^2(\mathbb{R}^n)} h_\alpha
\]

converges to \( f \) uniformly in \( \mathbb{R}^n \), and certainly, also in \( L^2(\mathbb{R}^n) \).

Let \( s \in \mathbb{R} \) and \( f \in S(\mathbb{R}^n) \). One defines the fractional Hermite operator \( H^s \) by

\[
H^s f := \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s \langle f, h_\alpha \rangle_{L^2(\mathbb{R}^n)} h_\alpha.
\]

The fractional Hermite-Sobolev space of order \( s \in \mathbb{R} \) is defined by

\[
W^{s,2}_H(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) : H^{\frac{s}{2}} f \in L^2(\mathbb{R}^n) \right\}
\]

with the norm given by \( \|f\|_{W^{s,2}_H(\mathbb{R}^n)} := \|H^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)} \) (see [5, 8]).

For \( 1 \leq j \leq n \), let

\[
H_j := \frac{\partial}{\partial x_j} + x_j \quad \text{and} \quad H_{-j} := H_j^* = -\frac{\partial}{\partial x_j} + x_j.
\]

Then it is easy to check that

\[
H = \frac{1}{2} \sum_{j=1}^{n} [H_j H_{-j} + H_{-j} H_j].
\]

For any positive integer \( k \) we define \( \tilde{W}^{k,2}_H(\mathbb{R}^n) \) as the space of functions \( f \in L^2(\mathbb{R}^n) \) such that for every \( 1 \leq |j_1|, \ldots, |j_m| \leq n \) and \( 1 \leq m \leq k \),

\[
H_{j_1} \cdots H_{j_m} f \in L^2(\mathbb{R}^n).
\]

The norm on \( \tilde{W}^{k,2}_H(\mathbb{R}^n) \) is given by

\[
\|f\|_{\tilde{W}^{k,2}_H(\mathbb{R}^n)} := \sum_{1 \leq |j_1|, \ldots, |j_m| \leq n, 1 \leq m \leq k} \|H_{j_1} \cdots H_{j_m} f\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)}.
\]

**Lemma 2** ([5] Theorem 4) For \( k \in \mathbb{N} \) we have that \( \tilde{W}^{k,2}_H(\mathbb{R}^n) = W^{k,2}_H(\mathbb{R}^n) \). Moreover, the norms \( \|\cdot\|_{\tilde{W}^{k,2}_H(\mathbb{R}^n)} \) and \( \|\cdot\|_{W^{k,2}_H(\mathbb{R}^n)} \) are equivalent.
For general integer \( s \geq 1 \), we can also characterize \( W^{s,2}_H(\mathbb{R}^n) \) with the help of operators \( H_j \).

**Lemma 3** For \( s \geq 1 \), there holds

\[
\| f \|_{W^{s,2}_H(\mathbb{R}^n)} \approx \sum_{1 \leq |j| \leq n} \| H_j f \|_{W^{s-1,2}_H(\mathbb{R}^n)} + \| f \|_{L^2(\mathbb{R}^n)}.
\]  

**Proof** It is obvious that (6) holds for \( s \in \mathbb{N} \) by Lemma 2. Assume \( k < s < k+1 \), where \( k \geq 1 \) is an integer. Applying (6) to the norm \( \| \cdot \|_{W^{k,2}_H(\mathbb{R}^n)} \) indicates

\[
\| f \|_{W^{k,2}_H(\mathbb{R}^n)} = \| H^{\frac{k-1}{2}} f \|_{W^{k,2}_H(\mathbb{R}^n)} \approx \sum_j \| H_j H^{\frac{k-1}{2}} f \|_{W^{k-1,2}_H(\mathbb{R}^n)} + \| f \|_{L^2(\mathbb{R}^n)}.
\]

It follows from Lemma 4 in [5] that

\[
H_j H^{\frac{k-1}{2}} f = (H + 2)^{\frac{k-1}{2}} H_j f, \quad 1 \leq j \leq n,
\]

and

\[
H_j H^{\frac{k-1}{2}} f = (H - 2)^{\frac{k-1}{2}} H_j f, \quad -n \leq j \leq -1.
\]

By (7), (8) and the definition of the \( W^{k,2}_H(\mathbb{R}^n) \)-norm, we have

\[
\| H_j H^{\frac{k-1}{2}} f \|_{W^{k-1,2}_H(\mathbb{R}^n)}^2 = \| (H \pm 2)^{\frac{k-1}{2}} H_j f \|_{W^{k-1,2}_H(\mathbb{R}^n)}^2
\]

\[
= \| H^{\frac{k-1}{2}} (H \pm 2)^{\frac{k-1}{2}} H_j f \|_{L^2(\mathbb{R}^n)}^2
\]

\[
= \sum_{\alpha} (2|\alpha| + n)^{k-1} (2|\alpha| + n \pm 2)^{s-k} |\langle H_j f, h_\alpha \rangle|^2
\]

\[
\approx \sum_{\alpha} (2|\alpha| + n)^{s-1} |\langle H_j f, h_\alpha \rangle|^2
\]

\[
= \| H_j f \|_{W^{s-1,2}_H(\mathbb{R}^n)}^2.
\]

Thus,

\[
\| f \|_{W^{s,2}_H(\mathbb{R}^n)} \approx \sum_j \| H_j f \|_{W^{s-1,2}_H(\mathbb{R}^n)} + \| f \|_{L^2(\mathbb{R}^n)}.
\]

The proof of Lemma 3 is complete. \( \square \)

For any \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) we use \( \tau_a \) to denote the operator of translation by \( a \), namely, \( \tau_a f(x) = f(x - a) \).

**Lemma 4** Let \( m \in \mathbb{N} \), \( a \in \mathbb{R} \), and \( f \in L^2(\mathbb{R}^n) \). Then

\[
H_{j_1} \cdots H_{j_m} \tau_a f(x) = \sum_{l \leq m} p_l(a) \tau_a H_{j_1} \cdots H_{j_l} f(x),
\]

where

\[
1 \leq |j_1|, \ldots, |j_m| \leq n, 1 \leq |j_1'|, \ldots, |j_l'| \leq n,
\]

and \( p_l(\cdot) \) is a polynomial of order \( l \), \( 1 \leq l \leq m \).
Proof For $1 \leq |j| \leq n$,

$$H_j \tau_a f(x) = \pm \frac{\partial}{\partial x_{|j|}} f(x - a) + x_{|j|} f(x - a)$$

$$= \pm \frac{\partial}{\partial (x - a)_{|j|}} f(x - a) + x_{|j|} f(x - a)$$

$$= H_j f(x - a) + a_{|j|} f(x - a)$$

$$= \tau_a [H_j + a_{|j|}] f(x).$$

By mathematical induction, we obtain the desired result. \qed

The Bargmann transform is the classical tool in mathematics analysis and mathematical physics (see [1, 2, 11, 13, 19] and references therein). Consider $f \in L^2(\mathbb{R}^n)$, and define

$$Bf(z) := \left( \frac{2}{\pi} \right)^{\frac{n}{4}} \int_{\mathbb{R}^n} f(x) e^{2x \cdot z - \frac{x^2}{2}} dx$$

$$= \left( \frac{2}{\pi} \right)^{\frac{n}{4}} e^{\frac{z^2}{2}} \int_{\mathbb{R}^n} f(x) e^{-(x - z)^2} dx, \quad z \in \mathbb{C}^n. \quad (9)$$

For $s \geq 0$ it is clear that $W_{H}^{s,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, so $Bf$ is well-defined on $W_{H}^{s,2}(\mathbb{R}^n)$. Also, it is well known that, for

$$f = \sum_{\alpha \in \mathbb{N}_0^n} c_{\alpha} h_{\alpha},$$

we have

$$Bf = \sum_{\alpha \in \mathbb{N}_0^n} c_{\alpha} Bh_{\alpha} = \sum_{\alpha \in \mathbb{N}_0^n} c_{\alpha} e_{\alpha}.$$

Consequently, the Bargmann transform is a unitary operator from $L^2(\mathbb{R}^n)$ to $F^2(\mathbb{C}^n)$, and it is also a unitary operator from the fractional Hermite–Sobolev space $W_{H}^{s,2}(\mathbb{R}^n)$ to the fractional Fock-Sobolev space $F^s(\mathbb{C}^n)$.

There is an equivalent definition for the Fock–Sobolev spaces, that is, so called the weighted Fock spaces. Given a real number $s$ we define $F^2_s(\mathbb{C}^n)$ as the space of entire functions $f$ on $\mathbb{C}^n$ with

$$\|f\|_{F^2_s(\mathbb{C}^n)}^2 = \omega_{n,s} \int_{\mathbb{C}^n} \left(1 + |z|^2\right)^{2s} |f(z)|^2 e^{-|z|^2} d\bar{z} < \infty,$$

where $\omega_{n,s}$ is a normalizing constant such that the constant function 1 has norm 1. It follows from Lemma 5 below that the fractional Fock-Sobolev spaces are the same as these weighted Fock spaces whose definition does not involve derivatives. Sometimes, it is more convenient to study function theoretic and operator theoretic properties on the weighted Fock spaces instead of the Fock-Sobolev spaces.

**Lemma 5** ([9] Theorem 1.2) For $s \in \mathbb{R}$ we have $F^s(\mathbb{C}^n) = F^2_s(\mathbb{C}^n)$ with equivalent norms.

Recall that the Weyl operators $W_a$, $a \in \mathbb{C}^n$, are defined by

$$W_a f(z) := f(z - a) e^{-\frac{|z|^2}{2} + \bar{a} z}. $$

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Lemma 6 Suppose \( s \in \mathbb{R} \) and \( a \in \mathbb{C}^n \). Then \( W_a \) is bounded on \( F^{s, 2}(\mathbb{C}^n) \). Moreover, for all \( f \in F^{s, 2}(\mathbb{C}^n) \)

\[
\| W_a f \|_{F^{s, 2}(\mathbb{C}^n)} \leq C (1 + |a|^{s|}) \| f \|_{F^{s, 2}(\mathbb{C}^n)}.
\]

**Proof** By Lemma 5, we have that for any \( f \in F^{s, 2}(\mathbb{C}^n) \),

\[
\| W_a f \|^2_{F^{s, 2}(\mathbb{C}^n)} \leq C \| W_a f \|^2_{F^{s, 2}(\mathbb{C}^n)}
\]

\[
= C \int_{\mathbb{C}^n} (1 + |z|)^{2s} |f(z - a)|^2 e^{-|z-a|^2} dz
\]

\[
= C \int_{\mathbb{C}^n} (1 + |z + a|)^{2s} |f(z)|^2 e^{-|z|^2} dz.
\]

Note that \( 1 + |z + a| \leq 1 + |z| + |a| \leq (1 + |z|)(1 + |a|) \). Also \( 1 + |z| = 1 + |z + a| - a \leq (1 + |z + a|)(1 + |a|) \) and so \( (1 + |z + a|)^{-1} \leq (1 + |a|)(1 + |z|)^{-1} \). Thus for \( s \in \mathbb{R} \),

\[
\| W_a f \|^2_{F^{s, 2}(\mathbb{C}^n)} \leq C (1 + |a|)^{2s} \int_{\mathbb{C}^n} (1 + |z|)^{2s} |f(z)|^2 e^{-|z|^2} dz
\]

\[
\leq C (1 + |a|)^{2s} \| f \|^2_{F^{s, 2}(\mathbb{C}^n)}.
\]

The proof of Lemma 6 is complete.

Since \( \tau_a = B^{-1} W_a B \) and the Bargmann transform is a unitary operator from \( W^{s, 2}_{\alpha} (\mathbb{R}^n) \) to \( F^{s, 2}(\mathbb{C}^n) \), we see that \( \tau_a \) is bounded on \( W^{s, 2}_{\alpha} (\mathbb{R}^n) \) for each \( a \in \mathbb{R}^n \) and

\[
\| \tau_a f \|_{W^{s, 2}_{\alpha} (\mathbb{R}^n)} \leq C (1 + |a|^{s|}) \| f \|_{W^{s, 2}_{\alpha} (\mathbb{R}^n)}
\]

(10) for all \( f \in W^{s, 2}_{\alpha} (\mathbb{R}^n) \). A direct computation shows that \( S_{\alpha} \) commutes with \( W_a \) on \( F^{s, 2}(\mathbb{C}^n) \), that is, \( S_{\alpha} W_a = W_a S_{\alpha} \); see [7]. Since \( BW^{s, 2}_{\alpha} (\mathbb{R}^n) = F^{s, 2}(\mathbb{C}^n) \) and \( B h_a = e_a \), we see that

\[
T = B^{-1} S_{\alpha} B
\]

commutes with \( \tau_a \) for \( a \in \mathbb{R}^n \). In fact, for any \( f \in W^{s, 2}_{\alpha} (\mathbb{R}^n) \),

\[
\tau_a T f (z) = \tau_a B^{-1} S_{\alpha} B f
\]

\[
= (B^{-1} W_a B)(B^{-1} S_{\alpha} B) f
\]

\[
= (B^{-1} W_a S_{\alpha} B) f
\]

\[
= (B^{-1} S_{\alpha} W_a B) f
\]

\[
= (B^{-1} S_{\alpha} B)(B^{-1} W_a B) f
\]

\[
= T \tau_a f (z).
\]

In the following, the Fourier transform of a function \( f \) is given by

\[
\mathcal{F} f (x) := \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-2ix \cdot y} f(y) dy, \quad x \in \mathbb{R}^n,
\]

The inverse of the Fourier transform \( \mathcal{F} \) will be denoted by \( \mathcal{F}^{-1} \), i.e., \( \mathcal{F} \mathcal{F}^{-1} = \mathcal{F}^{-1} \mathcal{F} = I d \), the identity operator on \( L^2(\mathbb{R}^n) \). Then we have

**Lemma 7** For any \( s \in \mathbb{R} \), the Fourier transformation \( \mathcal{F} \) is a unitary operator on \( W^{s, 2}_{\alpha} (\mathbb{R}^n) \).

**Proof** Following the proof of Lemma 2.3 in [7], we get

\[
B \mathcal{F} B^{-1} f (z) = f (-iz), \quad \text{and} \quad B \mathcal{F}^{-1} B^{-1} f (z) = f (iz).
\]
It is obvious that the operators $f(z) \mapsto f(i z)$ and $f(z) \mapsto f(-i z)$ are unitary on $F^{s,2}(\mathbb{C}^n)$. Since the Bargmann transform is a unitary from $W^{s,2}_H(\mathbb{R}^n)$ to $F^{s,2}(\mathbb{C}^n)$, we conclude that the Fourier transform $\mathcal{F}$ is also unitary on $W^{s,2}_H(\mathbb{R}^n)$. □

**Lemma 8** ([5] Lemma 3) Let $p \in [1, \infty)$ and $s > 0$. Then the operator $|x|^{2s} H^{-s}$ is bounded on $L^p(\mathbb{R}^n)$, that is, there is a positive constant $M$ such that

$$\int_{\mathbb{R}^n} |x|^{2s} H^{-s} f(x)|^p dx \leq M \int_{\mathbb{R}^n} |f(x)|^p dx$$

for all $f \in L^p(\mathbb{R}^n)$.

### 3 Multipliers on the fractional Hermite–Sobolev spaces

In the following we denote $\mathcal{M}(W^{s,2}_H(\mathbb{R}^n))$ the space of multipliers on $W^{s,2}_H(\mathbb{R}^n)$; i.e., the set of all functions $g$ such that

$$\|g f\|_{W^{s,2}_H(\mathbb{R}^n)} \leq M \|f\|_{W^{s,2}_H(\mathbb{R}^n)}$$

for all $f \in W^{s,2}_H(\mathbb{R}^n)$, equipped with the norm defined as the infimum of all such $M$ that the above inequality holds.

Before we discuss the multipliers, we need some characterizations for Hermite-Sobolev functions at first.

Fix a $K \in \mathbb{N}$, we define

$$G_{s,K}(f)(x) = \left( \int_0^\infty |(I - e^{-t^2 H})^K f(x)|^2 \frac{dt}{t^{1+2i}} \right)^{1/2}.$$  \hspace{1cm} (11)

For simplicity, we will often write $G_s$ in place of $G_{s,H}$. Then we have

**Lemma 9** If $0 < s < 2K$, then for $f \in W^{s,2}_H(\mathbb{R}^n)$

$$\|f\|_{W^{s,2}_H(\mathbb{R}^n)} \sim \|G_{s,K}(f)\|_{L^2(\mathbb{R}^n)}$$

with the implicit equivalent positive constants independent of $f$.

**Proof** It suffices to show that for $0 < s < 2K$,

$$C^{-1} \|H^{s/2} f\|_{L^2(\mathbb{R}^n)} \leq \|G_{s,K}(f)\|_{L^2(\mathbb{R}^n)} \leq C \|H^{s/2} f\|_{L^2(\mathbb{R}^n)},$$

Denote by $\psi(z) = z^{-s}(1 - e^{-z^2})^K$. It follows from the spectral theory ([17]) that for any $f \in L^2(\mathbb{R}^n)$,

$$\|G_{s,K}(H^{-s/2} f)\|_{L^2(\mathbb{R}^n)} = \left\{ \int_0^\infty \|\psi(t \sqrt{H}) f\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \right\}^{1/2}$$

$$= \left\{ \int_0^\infty \left\langle \tilde{\psi}(t \sqrt{H}) \psi(t \sqrt{H}) f, f \right\rangle \frac{dt}{t} \right\}^{1/2}$$

$$\leq \kappa \|f\|_{L^2(\mathbb{R}^n)},$$

\hspace{1cm} (12)
where \( \kappa := \{ f_0^\infty |\psi(t)|^2 dt/t \}^{1/2} \). This shows that \( \| G_{s, K, H} f \|_{L^2(\mathbb{R}^n)} \leq C \| H^{s/2} f \|_{L^2(\mathbb{R}^n)} \).

On the other hand, from the spectral theory ([17]) we see that for any \( f \in L^2(\mathbb{R}^n) \),
\[
f = c \int_{-\infty}^\infty (t^2 H)^{-s} (1 - e^{-t^2 H})^{2K} (f) \frac{dt}{t}
\]
for some constant \( c > 0 \), where the integral converges in \( L^2(\mathbb{R}^n) \). Hence for every \( g \in L^2(\mathbb{R}^n) \) with \( \| g \|_{L^2(\mathbb{R}^n)} \leq 1 \), we apply (12) to get
\[
|\langle f, g \rangle| = c \left| \int_{-\infty}^\infty \langle (t^2 H)^{-s} (1 - e^{-t^2 H})^{2K} f, g \rangle \frac{dt}{t} \right|
\leq c \| G_{s, K, H} (H^{-s/2} f) \|_{L^2(\mathbb{R}^n)} \| G_{s, K, H} (H^{-s/2} g) \|_{L^2(\mathbb{R}^n)}
\leq C \| G_{s, K, H} (H^{-s/2} f) \|_{L^2(\mathbb{R}^n)},
\]
which shows that \( \| H^{s/2} f \|_{L^2(\mathbb{R}^n)} \leq C \| G_{s, K, H} f \|_{L^2(\mathbb{R}^n)} \). This proves the lemma. \( \square \)

In the following, we let \( \eta(x) \in C_0^\infty (\mathbb{R}^n) \) satisfy

(i) \( 0 \leq \eta(x) \leq 1 \),
(ii) \( \eta(x) = 1 \) on the cube \( \{ |x| \leq 1 \} \),
(iii) \( \eta(x) = 0 \) outside the cube \( \{ |x| \leq 2 \} \).
(iv) \( \sum_{m \in \mathbb{Z}_n} \eta_m(x) = c_0 \) for some constant \( c_0 \) and all \( x \in \mathbb{R}^n \), where \( \eta_m(x) = \eta(x + m) \) and \( \mathbb{Z}_n \) denotes the lattice points in \( \mathbb{R}^n \).

**Proposition 10** Let \( s \geq 0 \). The function \( f \) belongs to \( W^{s, 2}_H (\mathbb{R}^n) \) if and only if \( f \eta_m \in W^{s, 2}_H (\mathbb{R}^n) \) for every \( m \in \mathbb{Z}_n \) and
\[
\left( \sum_{m \in \mathbb{Z}_n} \| f \eta_m \|_{W^{s, 2}_H (\mathbb{R}^n)} \right)^{1/2} < \infty, \tag{13}
\]
in which case this expression is equivalent to \( \| f \|_{W^{s, 2}_H (\mathbb{R}^n)} \).

Moreover, the necessity holds for any function \( \eta \) in \( C_0^\infty (\mathbb{R}^n) \).

**Proof** The case \( s = 0 \) is trivial. Let us consider the case \( 0 < s < 1 \). For \( f \in L^2(\mathbb{R}^n) \), define
\[
G_{s, H}^{(1)} f(x) := \left( \int_0^1 |(I - e^{-t^2 H}) f(x)|^2 \frac{dt}{t^{1+2s}} \right)^{1/2},
\]
\[
G_{s, H}^{(2)} f(x) := \left( \int_1^\infty |(I - e^{-t^2 H}) f(x)|^2 \frac{dt}{t^{1+2s}} \right)^{1/2},
\]
and so \( G_{s, H} f(x) \leq G_{s, H}^{(1)} f(x) + G_{s, H}^{(2)} f(x) \). Since the kernel \( K_{e^{-t^2 H}}(x, y) \) of \( e^{-t^2 H} \) satisfies
\[
|K_{e^{-t^2 H}}(x, y)| \leq C t^{-s} e^{-\frac{|x-y|^2}{2t^2}}, \tag{14}
\]
we see that
\[
\| G_{s, H}^{(2)} f \|_{L^2(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}. \tag{15}
\]
Assume that $f \eta_m \in W^{s,2}_H(\mathbb{R}^n)$ for every $m \in \mathbb{Z}_n$ and (13) holds. Let us prove that $f \in W^{s,2}_H(\mathbb{R}^n)$ and

$$\|f\|_{W^{s,2}_H(\mathbb{R}^n)} \leq C \left( \sum_{m \in \mathbb{Z}_n} \|f \eta_m\|_{W^{s,2}_H(\mathbb{R}^n)}^2 \right)^{1/2}. \quad (16)$$

We now prove (16). From (15), we use the properties i), ii) and iii) of $\eta$ to obtain

$$\|G^{(2)}_{s,H} f\|^2_{L^2(\mathbb{R}^n)} \leq C \|f\|^2_{L^2(\mathbb{R}^n)} \leq C \sum_{m \in \mathbb{Z}_n} \|f \eta_m\|^2_{L^2(\mathbb{R}^n)}.$$ 

Now, let $B_m := \{x : |x - m| \leq 4\}$. Since $\sum_{m \in \mathbb{Z}_n} \eta_m(x) = c_0$ for all $x \in \mathbb{R}^n$, we see that $\|G^{(1)}_{s,H} f\|^2_{L^2(\mathbb{R}^n)} \leq (E + F)/c_0^2$, where

$$E := \int_0^1 \int_{\mathbb{R}^n} \left| \sum_{m \in \mathbb{Z}_n} (I - e^{-t^2H})(f \eta_m)(x) \chi_{B_m}(x) \right|^2 \frac{dxdt}{t^{1+2s}}$$

and

$$F := \int_0^1 \int_{\mathbb{R}^n} \left| \sum_{m \in \mathbb{Z}_n} e^{-t^2H}(f \eta_m)(x) \chi_{B_m^c}(x) \right|^2 \frac{dxdt}{t^{1+2s}}$$

For the term $E$, we have

$$E \leq \sum_{m \in \mathbb{Z}_n} \int_0^1 \int_{\mathbb{R}^n} \left| (I - e^{-t^2H})(f \eta_m)(x) \right|^2 \frac{dxdt}{t^{1+2s}} \leq C \sum_{m \in \mathbb{Z}_n} \|f \eta_m\|^2_{W^{s,2}_H(\mathbb{R}^n)}.$$

To estimate the term $F$, we note that

$$F \leq \int_0^1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| K_t(x, y) \right| \left| \sum_m f \eta_m(y) \right| \left| dy \right|^2 \frac{dx dt}{t^{1+2s}},$$

where

$$|K_t(x, y)| \leq Ct^{-n} e^{-c \frac{|x - y|^2}{t^2}} \chi_{|x - y| \geq 2}.$$ 

It follows that $F \leq C \sum_{m \in \mathbb{Z}_n} \|f \eta_m\|^2_{L^2(\mathbb{R}^n)}$. Hence, estimates $E$ and $F$ yield that $\|G^{(1)}_{s,H} f\|^2_{L^2(\mathbb{R}^n)} \leq C \sum_{m \in \mathbb{Z}_n} \|f \eta_m\|^2_{W^{s,2}_H(\mathbb{R}^n)}$. This completes the proof of the sufficiency part in the case $0 < s < 1$.

Now we prove the necessity part of the proposition under a weaker condition on $\eta \in C_0^\infty(\mathbb{R}^n)$. Indeed, for every $f \in W^{s,2}_H(\mathbb{R}^n)$ we will show that if $\eta \in C_0^\infty(\mathbb{R}^n)$ and $\sum_{m \in \mathbb{Z}_n} |\eta_m(x)| \leq C$, for all $x \in \mathbb{R}^n$,
then \( f \eta_m \in W^{s, 2}_H(\mathbb{R}^n) \) for every \( m \in \mathbb{Z}_n \) and
\[
\left( \sum_{m \in \mathbb{Z}_n} \| f \eta_m \|_{W^{s, 2}_H(\mathbb{R}^n)}^2 \right)^{1/2} \leq C \| f \|_{W^{s, 2}_H(\mathbb{R}^n)}.
\] (17)

To prove (17), we write
\[
\left( \int_0^\infty |(I - e^{-t^2 H})(f \eta_m)(x)|^2 \frac{dt}{t^{1+2s}} \right)^{1/2} \leq I_m(x) + J_m^{(1)}(x) + J_m^{(2)}(x),
\]
where
\[
I_m(x) := \left( \int_0^\infty |(I - e^{-t^2 H}) f(x) \eta_m(x)|^2 \frac{dt}{t^{1+2s}} \right)^{1/2},
\]
\[
J_m^{(1)}(x) := \left( \int_0^1 |e^{-t^2 H} (f \eta_m)(x) - e^{-t^2 H} (f)(x) \eta_m(x)|^2 \frac{dt}{t^{1+2s}} \right)^{1/2},
\]
\[
J_m^{(2)}(x) := \left( \int_1^\infty |e^{-t^2 H} (f \eta_m)(x) - e^{-t^2 H} (f)(x) \eta_m(x)|^2 \frac{dt}{t^{1+2s}} \right)^{1/2}.
\]

Now
\[
\sum_{m \in \mathbb{Z}_n} \int_{\mathbb{R}^n} |I_m(x)|^2 dx 
\leq \int_{\mathbb{R}^n} \int_0^\infty |(I - e^{-t^2 H}) f(x)|^2 \left( \sum_{m \in \mathbb{Z}_n} |\eta_m(x)|^2 \right) \frac{dt}{t^{1+2s}} dx
\leq C \| f \|_{W^{s, 2}_H(\mathbb{R}^n)}.
\]

For the term \( J_m^{(1)} \), we apply (14) to get
\[
\sum_{m \in \mathbb{Z}_n} \int_{\mathbb{R}^n} |J_m^{(1)}(x)|^2 dx 
\leq \sum_{m \in \mathbb{Z}_n} \int_{\mathbb{R}^n} \int_0^1 \left( \int_{\mathbb{R}^n} t^{-n} e^{-\frac{|y|^2}{t^2}} |f(x+y)\eta_m(x+y) - \eta_m(x)|^2 dy \right) \frac{dt}{t^{1+2s}} dx.
\]

For each \( x \) the inner is non-zero for at most \( 7^n \) distant \( m \)'s; namely when \( |x_i + m_i| \leq 3 \). Fix one such \( m_i \). By the mean value theorem \( |\eta_m(x+y) - \eta_m(x)| = |y| |\nabla \eta(x_0)| \leq M |y| \) where \( M = \| \nabla \eta \|_{L^\infty(\mathbb{R}^n)} \). Hence,
\[
\sum_{m \in \mathbb{Z}_n} \int_{\mathbb{R}^n} |J_m^{(1)}(x)|^2 dx 
\leq C \int_{\mathbb{R}^n} \int_0^1 \left( \int_{\mathbb{R}^n} t^{-n} e^{-\frac{|y|^2}{t^2}} |f(x+y)| \left( \frac{|y|}{t} \right)^2 dy \right) \frac{dt}{t^{2s-1}} dx
\leq C \| f \|_{L^2(\mathbb{R}^n)}^2
\]
when \( 0 \leq s < 1 \). Further, we use the property (14) of the kernel \( K_{e^{-t^2 H}}(x, y) \) to see that
\[
\sum_{m \in \mathbb{Z}_n} \int_{\mathbb{R}^n} |J_m^{(2)}(x)|^2 dx \leq \| f \|_{L^2(\mathbb{R}^n)}^2.
\]
Estimates of \( I_m, J_m^{(1)} \) and \( J_m^{(2)} \) together, give the proof of the necessity part in the case \( 0 < s < 1 \).
Thus the proposition is proved for $0 \leq s < 1$. For $s \geq 1$, we will prove it by induction. Suppose the proposition is true for $k \leq s < k+1$, where $k$ is an integer. Let $k+1 \leq s < k+2$. By Lemma 3 and the assumption

\[
\|f\|_{W^{s,2}_H(\mathbb{R}^n)}^2 \approx \sum_j \|H_j f\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2 + \|f\|_{L^2(\mathbb{R}^n)}^2
\]

\[
\approx \sum_j \sum_{m \in \mathbb{Z}^n} \|(H_j f) \eta_m\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2 + \|f\|_{L^2(\mathbb{R}^n)}^2.
\]

A direct calculation shows

\[
H_j(f \eta_m) = (H_j f) \eta_m + f \partial_j \eta_m.
\]

Hence, (18) is controlled by

\[
C\left(\sum_j \sum_{m \in \mathbb{Z}^n} \|(H_j f) \eta_m\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2 + \sum_j \sum_{m \in \mathbb{Z}^n} \|f \partial_j \eta_m\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2 + \|f\|_{L^2(\mathbb{R}^n)}^2\right)
\]

\[
\leq C\left(\sum_{m \in \mathbb{Z}^n} \|f \eta_m\|_{W^{s,2}_H(\mathbb{R}^n)}^2 + \|f\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2\right)
\]

\[
\leq C\left(\sum_{m \in \mathbb{Z}^n} \|f \eta_m\|_{W^{s,2}_H(\mathbb{R}^n)}^2 + \sum_{m \in \mathbb{Z}^n} \|f \eta_m\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2\right)
\]

\[
\leq C \sum_{m \in \mathbb{Z}^n} \|f \eta_m\|_{W^{s,2}_H(\mathbb{R}^n)}^2,
\]

where we use Lemma 3 and the assumption again. Thus the sufficiency is proved for $k+1 \leq s < k+2$.

On the other hand, for every $\eta \in C_0^\infty(\mathbb{R}^n)$, we obtain that

\[
\sum_{m \in \mathbb{Z}^n} \|f \eta_m\|_{W^{s,2}_H(\mathbb{R}^n)}^2 \approx \sum_j \sum_{m \in \mathbb{Z}^n} \|(H_j f) \eta_m\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2
\]

\[
\leq C\left(\sum_j \sum_{m \in \mathbb{Z}^n} \|(H_j f) \eta_m\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2 + \sum_j \sum_{m \in \mathbb{Z}^n} \|f \partial_j \eta_m\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2\right)
\]

\[
\leq C\left(\sum_j \|H_j f\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2 + \|f\|_{W^{s-1,2}_H(\mathbb{R}^n)}^2\right)
\]

\[
\leq C \|f\|_{W^{s,2}_H(\mathbb{R}^n)}^2.
\]

This shows Proposition 10 also holds for $k+1 \leq s < k+2$. 

**Lemma 11** \(\mathcal{M}(W^{s,2}_H(\mathbb{R}^n)) \subset \mathcal{M}(W^{t,2}_H(\mathbb{R}^n))\) if $s \geq t \geq 0$. In particular, \(\mathcal{M}(W^{s,2}_H(\mathbb{R}^n)) \subset L^\infty(\mathbb{R}^n)\) if $s \geq 0$. 

\(\square\)

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Proof Let $C_0^\infty(\mathbb{R}^n)$ be the space of smooth functions with compact support on $\mathbb{R}^n$. Suppose $m \in \mathcal{M}(W^{s,2}_H(\mathbb{R}^n))$, then for any positive integer $N$ and any $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\|m^N f\|_{L^2(\mathbb{R}^n)} \leq \|m\|_{\mathcal{M}(W^{s,2}_H(\mathbb{R}^n))} \|f\|_{W^{s,2}_H(\mathbb{R}^n)}.$$  

In addition, for $\Omega := \{x \in \mathbb{R}^n : |m(x)| \geq \|m\|_{\mathcal{M}(W^{s,2}_H(\mathbb{R}^n))} + 1\}$ we have

$$\|m^N f\|_{L^2(\mathbb{R}^n)} \leq \left(\|m\|_{\mathcal{M}(W^{s,2}_H(\mathbb{R}^n))} + 1\right) \left(\int_\Omega |f(x)|^2 \, dx\right)^{\frac{1}{2N}}.$$  

We first claim $m \in L^\infty(\mathbb{R}^n)$. If it’s not true, then $\Omega$ is of positive measure and hence one can find a proper function $f \in C_0^\infty(\mathbb{R}^n)$ such that $\int_\Omega |f(x)|^2 \, dx > 0$. Let $N \to \infty$, then there is a obvious contradiction.

The left of the proof is to use interpolation to the operator of multiplication by $m$ between $W^{s,2}_H(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. □

For $s \geq 0$, let $W^{s,2}(\mathbb{R}^n)$ be the classical Sobolev space, i.e.,

$$W^{s,2}(\mathbb{R}^n) = \left\{ f : \|f\|_{W^{s,2}(\mathbb{R}^n)} = \|(1 + | \cdot |^2)^{\frac{1}{2}} \mathcal{F} f\|_{L^2(\mathbb{R}^n)} < \infty \right\}.$$  

Lemma 12 For $s \geq 0$, we have that $W^{s,2}_H(\mathbb{R}^n) \subset W^{s,2}(\mathbb{R}^n)$. In particular, if $s > n/2$, then $W^{s,2}_H(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$.

Proof Let $f \in W^{s,2}_H(\mathbb{R}^n)$. It suffices to show that $(1 + |\xi|^2)^{\frac{1}{2}} \mathcal{F} f(\xi)$ is an $L^2(\mathbb{R}^n)$ function. Actually we know $f$ belongs to $W^{s,2}_H(\mathbb{R}^n)$ for all $0 \leq s' \leq s$, then so does $\mathcal{F} f$ since the Fourier transform is an isometry on $W^{s',2}_H(\mathbb{R}^n)$. Hence it follows from Lemma 8 that

$$|\xi|^{s'} \mathcal{F} f(\xi) \in L^2(\mathbb{R}^n) \text{ for all } 0 \leq s' \leq s,$$

which implies $(1 + |\xi|^2)^{\frac{1}{2}} \mathcal{F} f(\xi) \in L^2(\mathbb{R}^n)$. This shows $f \in W^{s,2}(\mathbb{R}^n)$. Thus $W^{s,2}_H(\mathbb{R}^n) \subset W^{s,2}(\mathbb{R}^n)$.

As for the $s > n/2$ case, by Sobolev embedding theorem we know $W^{s,2}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$.

Lemma 13 (Leibniz’s Rule) Let $s \in \mathbb{N}$. Then for any $f, g \in \tilde{W}^{s,2}_H(\mathbb{R}^n)$ and any integer $k \leq s$, we have the generalized Leibniz’s rule

$$H_{j_1} \cdots H_{j_k}(fg) = \sum_{h+l+m=k} \sum_{1 \leq |j_1'| \leq n, \ldots, 1 \leq |j_k'| \leq n} \sum_{1 \leq |j_1''| \leq n, \ldots, 1 \leq |j_m''| \leq n} p_h(x)(H_{j_1'} \cdots H_{j_k'} f)(H_{j_1''} \cdots H_{j_m''} g)$$  

$$= \sum_{|\alpha|+m=k} \sum_{1 \leq |j_1'| \leq n, \ldots, 1 \leq |j_m'| \leq n} \left(\frac{\partial^\alpha}{\partial x^\alpha} f\right)(H_{j_1'} \cdots H_{j_m'} g)$$  

$$= \sum_{h+|\alpha|+|\beta|=k} p_h \left(\frac{\partial^\alpha}{\partial x^\alpha} f\right) \left(\frac{\partial^\beta}{\partial x^\beta} g\right),$$

where $p_h$ is a polynomial in $x$ of order $h$.  

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Proof We only give the proof for the first equality since the other two equalities can be obtained by minor modifications with it.

For any $1 \leq |j| \leq n$,

$$H_j(fg) = \pm \frac{\partial}{\partial x_{|j|}} (fg) + x_{|j|} fg$$

$$= \pm \left[ \frac{\partial}{\partial x_{|j|}} f \right] g + f \left[ \frac{\partial}{\partial x_{|j|}} g \right] + x_{|j|} fg$$

$$= \left[ \pm \frac{\partial}{\partial x_{|j|}} f \right] g + f \left[ \pm \frac{\partial}{\partial x_{|j|}} g \right] + x_{|j|} fg$$

$$= (H_j f) g + f (H_j g) - x_{|j|} fg.$$

Assume the first equality holds for any $k'<k$, and for convenience, we omit the subscripts of the sum. Then for any $1 \leq |j| \leq n$,

$$H_j H_{j_1} \cdots H_{j_{k'}} (fg)$$

$$= \sum \left[ \pm \frac{\partial}{\partial x_{|j|}} p_h(x) \cdot (H_{j_1} \cdots H_{j_{k'}} f) (H_{j_1} \cdots H_{j_{m'}} g) \right]$$

$$= \sum \left[ \pm \frac{\partial}{\partial x_{|j|}} p_h(x) \cdot (H_{j_1} \cdots H_{j_{k'}} f) (H_{j_1} \cdots H_{j_{m'}} g) \right]$$

$$= \sum \left[ \pm \frac{\partial}{\partial x_{|j|}} p_h(x) \cdot (H_{j_1} \cdots H_{j_{k'}} f) (H_{j_1} \cdots H_{j_{m'}} g) \right]$$

$$= \sum \left[ \pm \frac{\partial}{\partial x_{|j|}} x_{|j|} p_h(x) \cdot (H_{j_1} \cdots H_{j_{k'}} f) (H_{j_1} \cdots H_{j_{m'}} g) \right]$$

$$= \sum \left[ \pm \frac{\partial}{\partial x_{|j|}} x_{|j|} p_h(x) \cdot (H_{j_1} \cdots H_{j_{k'}} f) (H_{j_1} \cdots H_{j_{m'}} g) \right]$$

where $h + l + m + 1 = k' + 1$. By mathematical induction, the proof of the first equality is complete. The proof of Lemma 13 is complete. □

Proposition 14 Let $k > n/2$ be an integer. Then we have

(i) $W^{k,2}_H(\mathbb{R}^n)$ is an algebra and $W^{k,2}_H(\mathbb{R}^n) \subset \mathcal{M}(W^{k,2}_H(\mathbb{R}^n))$.

(ii) $W^{k,2}_H(\mathbb{R}^n) \subset \mathcal{M}(W^{k,2}_H(\mathbb{R}^n))$.

(iii) $\mathcal{M}(W^{k,2}_H(\mathbb{R}^n)) \subset \mathcal{M}(W^{k,2}_H(\mathbb{R}^n))$. 

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Proof Since $W^{k,2}_H(\mathbb{R}^n) = \tilde{W}^{k,2}_H(\mathbb{R}^n)$ by Lemma 2, we work with functions $f, g \in \tilde{W}^{k,2}_H(\mathbb{R}^n)$. By the Leibniz’s rule, for all $0 \leq k' \leq k$, we have

$$H_{j_1} \cdots H_{j_{k'}}(fg) = \sum_{h+l+m=k'} \sum_{1 \leq |j_i'| \leq n, \cdots, 1 \leq |j_{k'}'| \leq n} \sum_{1 \leq |j_{k'}'| \leq n, \cdots, 1 \leq |j_{k'}'| \leq n} p_h(x) \cdot (H_{j_1} \cdots H_{j_{k'}})(H_{j_{k'}} \cdots H_{j_{m+k'}} g).$$

It follows from Lemma 8 and 12 that

$$p_h H_{j_1} \cdots H_{j_{k'}} f \in \tilde{W}^{k-(h+l),2}_H(\mathbb{R}^n) \subset W^{k-(h+l),2}(\mathbb{R}^n)$$

and

$$H_{j_{k'}} \cdots H_{j_{m+k'}} g \in \tilde{W}^{k-m,2}_H(\mathbb{R}^n) \subset W^{k-m,2}(\mathbb{R}^n).$$

Let $f_0 = p_h H_{j_1} \cdots H_{j_{k'}} f$ and $g_0 = H_{j_{k'}} \cdots H_{j_{m+k'}} g$. If $k - (h + l) > \frac{n}{2}$ or $k - m > \frac{n}{2}$, then $f_0 g_0 \in L^2$ since one of $f_0$ and $g_0$ is bounded. If $k - (h + l), k - m \leq \frac{n}{2}$, then by Sobolev embedding theorem we know $f_0 \in L^q$ for $\frac{1}{2} - \frac{k - (h + l)}{n} \leq \frac{1}{q} < \frac{1}{2}$ and $g_0 \in L^r$ for $\frac{1}{2} - \frac{k - m}{n} \leq \frac{1}{r} < \frac{1}{2}$. Since $h + l + m = k' \leq k$, we can choose $q$ and $r$ such that $\frac{1}{2} = \frac{1}{q} + \frac{1}{r}$, hence by Hölder’s inequality,

$$\|f_0 g_0\|_{L^2(\mathbb{R}^n)} \leq \|f_0\|_{L^q} \|g_0\|_{L^r(\mathbb{R}^n)} \leq C \|\tilde{f}_0\|_{\tilde{W}^{k-(h+l),2}_H(\mathbb{R}^n)} \|g_0\|_{W^{k-m,2}(\mathbb{R}^n)} \leq C \|f\|_{\tilde{W}^{k,2}_H(\mathbb{R}^n)} \|g\|_{\tilde{W}^{k,2}_H(\mathbb{R}^n)}.$$  

This shows $fg \in \tilde{W}^{k,2}_H(\mathbb{R}^n)$ and $f, g \in \mathcal{M}(\tilde{W}^{k,2}_H(\mathbb{R}^n))$, which implies that $W^{k,2}_H(\mathbb{R}^n) = \tilde{W}^{k,2}_H(\mathbb{R}^n)$ is an algebra. This proves (i).

We now prove (ii). Let $f \in W^{k,2}(\mathbb{R}^n)$. For any $g \in \tilde{W}^{k,2}_H(\mathbb{R}^n)$ we have

$$H_{j_1} \cdots H_{j_k}(fg) = \sum_{|\alpha| + m = k} \sum_{1 \leq |j_i'| \leq n, \cdots, 1 \leq |j_{m+k'}| \leq n} \left(\frac{\partial^\alpha}{\partial x^\alpha} f\right)(H_{j_1} \cdots H_{j_{m+k'}} g)$$

by Lemma 13. Similar to the proof above, we see that

$$(H_{j_1} \cdots H_{j_{m+k'}} g) \in \tilde{W}^{k-m,2}_H(\mathbb{R}^n) \subset W^{k-m,2}(\mathbb{R}^n),$$

and

$$\frac{\partial^\alpha}{\partial x^\alpha} f \in W^{k-|\alpha|,2}(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n).$$

Thus $H_{j_1} \cdots H_{j_k}(fg) \in L^2(\mathbb{R}^n)$. Similarly, for any $k' \leq k$, we have $H_{j_1} \cdots H_{j_{k'}}(fg) \in L^2(\mathbb{R}^n)$. Hence $fg \in \tilde{W}^{k,2}_H(\mathbb{R}^n) = W^{k,2}_H(\mathbb{R}^n)$.

Now we turn to prove (iii). Let $\tilde{\eta} \in C_0^\infty$ such that $\tilde{\eta}(x) = 1$ if $|x| \leq 2$; $0$ if $|x| \geq 3$. Denote by $\tilde{\eta}_m(x) = \tilde{\eta}(x + m)$. Let $f \in \mathcal{M}(W^{k,2}_H(\mathbb{R}^n))$ and $g \in \tilde{W}^{k,2}_H(\mathbb{R}^n)$. Since $\tilde{\eta}_m \in W^{k,2}(\mathbb{R}^n)$, we know from (ii) that $f \tilde{\eta}_m \in W^{k,2}(\mathbb{R}^n) \subset \mathcal{M}(W^{k,2}_H(\mathbb{R}^n))$. Then it follows from Proposition 10 that

$$\tilde{\eta}_m f \tilde{\eta}_m \in W^{k,2}(\mathbb{R}^n) \subset \mathcal{M}(W^{k,2}_H(\mathbb{R}^n)).$$
\[
\| fg \|_{W^{k,2}_H(\mathbb{R}^n)}^2 \leq C \sum_{m \in \mathbb{Z}^n} \| fg \eta_m \|_{W^{k,2}_H(\mathbb{R}^n)}^2 \\
\leq C \sum_{m \in \mathbb{Z}^n} \| f \eta_m \|_{W^{k,2}_H(\mathbb{R}^n)}^2 \| g \eta_m \|_{W^{k,2}_H(\mathbb{R}^n)}^2 \\
\leq C \| f \|_{\mathcal{M}(W^{k,2}(\mathbb{R}^n))}^2 \sum_{m \in \mathbb{Z}^n} \| g \eta_m \|_{W^{k,2}_H(\mathbb{R}^n)}^2 \\
\leq C \| f \|_{\mathcal{M}(W^{k,2}(\mathbb{R}^n))}^2 \| g \|_{W^{k,2}_H(\mathbb{R}^n)}^2
\]

This shows \( f \in \mathcal{M}(W^{k,2}_H(\mathbb{R}^n)) \).

The proof of Proposition 14 is complete. \( \square \)

In the end of this section we study the multipliers on the Hermite-Sobolev spaces by localizations. Let \( \eta \) be a function as in Proposition 10. Recall that for the classical Sobolev spaces \( W^{s,2}(\mathbb{R}^n) \),

\[
\| \eta(x + m) \|_{W^{s,2}(\mathbb{R}^n)} = \| \eta(x) \|_{W^{s,2}(\mathbb{R}^n)}
\]

for every \( m \in \mathbb{Z}^n \). In [15, Corollary 3.3], Strichartz proved the following well-known result that \( f \in \mathcal{M}(W^{s,2}(\mathbb{R}^n)) \) if and only if \( f(x) \eta(x + m) \in \mathcal{M}(W^{s,2}(\mathbb{R}^n)) \) for all \( m \in \mathbb{Z}^n \) and

\[
\sup_{m \in \mathbb{Z}^n} \| f \eta_m \|_{\mathcal{M}(W^{s,2}(\mathbb{R}^n))} < \infty.
\]

The supremum is equivalent to \( \| f \|_{\mathcal{M}(W^{s,2}(\mathbb{R}^n))} \). From Theorem 2.1 and Corollary 2.2 in [15], we know that for \( s > n/2 \), \( W^{s,2}(\mathbb{R}^n) \subseteq \mathcal{M}(W^{s,2}(\mathbb{R}^n)) \), and so \( f \in \mathcal{M}(W^{s,2}(\mathbb{R}^n)) \) if and only if \( f \) is a uniformly local function in the sense of norms in \( W^{s,2} \), i.e., \( \| f \eta_m \|_{W^{s,2}(\mathbb{R}^n)} \leq C \) for all \( m \in \mathbb{Z}^n \).

Turning to the Hermite–Sobolev spaces, we have

\[
\| \eta(x + m) \|_{\tilde{W}^{k,2}_H(\mathbb{R}^n)}
= \sum_{1 \leq |j| \leq n} \| H_j \eta(x + m) \|_{L^2(\mathbb{R}^n)} + \| \eta(x + m) \|_{L^2(\mathbb{R}^n)}
\geq \sum_{1 \leq |j| \leq n} \left( \| x_{|j|} \eta(x + m) \|_{L^2(\mathbb{R}^n)} - \| \frac{\partial}{\partial x_{|j|}} \eta \|_{L^2(\mathbb{R}^n)} \right) - \| \eta \|_{L^2(\mathbb{R}^n)}
= \sum_{1 \leq |j| \leq n} \left( \| (x_{|j|} - m_{|j|}) \eta \|_{L^2(\mathbb{R}^n)} - \| \frac{\partial}{\partial x_{|j|}} \eta \|_{L^2(\mathbb{R}^n)} \right) - \| \eta \|_{L^2(\mathbb{R}^n)}
\geq \sum_{1 \leq |j| \leq n} |m_{|j|}| \| \eta \|_{L^2(\mathbb{R}^n)} - \sum_{1 \leq |j| \leq n} \| x_{|j|} \eta \|_{L^2(\mathbb{R}^n)} - \| \eta \|_{L^2(\mathbb{R}^n)}
\rightarrow \infty \quad \text{as} \quad |m| \rightarrow \infty.
\]

This shows although \( 1 \in \mathcal{M}(W^{s,2}_H(\mathbb{R}^n)) \), \( \| 1 \cdot \eta_m \|_{W^{s,2}_H(\mathbb{R}^n)} \) cannot be controlled by a constant, which is different from the case of Sobolev multipliers. We can also find a function which is not a multiplier of \( W^{s,2}(\mathbb{R}^n) \) and not uniformly local in the sense of norms in \( W^{s,2}_H(\mathbb{R}^n) \), but
it is a multiplier of $W^{s,2}_H(\mathbb{R}^n)$. In fact, if $h(x) = e^{ix^2}$, then $h$ is not a uniformly local function but $h$ is a multiplier of $W^{1,2}_H(\mathbb{R})$. To see this, note that for any $f \in W^{1,2}_H(\mathbb{R})$ we have

$$H(hf) = (hf)' + x(hf)$$

$$= \frac{4}{3}x^3 e^{ix^2} f + e^{ix^2} f' + xe^{ix^2} f$$

$$= \left(\frac{4}{3}x^3 + x\right) e^{ix^2} f + e^{ix^2} f'.$$

Since $f \in W^{1,2}_H(\mathbb{R})$, we see that $xf \in L^2(\mathbb{R})$. Furthermore,

$$\left(\frac{4}{3}x^3 + x\right) e^{ix^2} f \in L^2(\mathbb{R}).$$

Thus $H(hf) \in L^2(\mathbb{R})$. This means that $h \in \mathcal{M}(W^{1,2}_H(\mathbb{R}))$. However, it is not difficult to see that $\|h\eta_m\|_{W^{1,2}_H(\mathbb{R})} \to \infty$ as $m \to \infty$. We see easily that $h$ is not a multiplier of $W^{1,2}$ since $\|h\eta_m\|_{W^{1,2}_H(\mathbb{R})} \to \infty$ as $m \to \infty$.

To obtain multipliers on the Hermite-Sobolev spaces, we have the following proposition.

**Proposition 15** Let $s \geq 0$. Then $f \in \mathcal{M}\left(W^{s,2}_H(\mathbb{R}^n)\right)$ if and only if $f \eta_m \in \mathcal{M}\left(W^{s,2}_H(\mathbb{R}^n)\right)$ uniformly in $m \in \mathbb{Z}_n$, i.e.,

$$\sup_{m \in \mathbb{Z}_n} \|f \eta_m\|_{\mathcal{M}\left(W^{s,2}_H(\mathbb{R}^n)\right)} < \infty.$$

The supremum is equipped to $\|f\|_{\mathcal{M}\left(W^{s,2}_H(\mathbb{R}^n)\right)}$.

**Proof** Let $f \in \mathcal{M}\left(W^{s,2}_H(\mathbb{R}^n)\right)$. Then for every $g \in W^{s,2}_H(\mathbb{R}^n)$,

$$\|f \eta_m g\|_{W^{s,2}_H(\mathbb{R}^n)} \leq C \|\eta_m g\|_{W^{s,2}_H(\mathbb{R}^n)} \leq C \|g\|_{W^{s,2}_H(\mathbb{R}^n)}$$

with a constant $C$ independent of $m$, where in the second inequality we used Proposition 10. Conversely, assume that

$$\sup_{m \in \mathbb{Z}_n} \|f \eta_m\|_{\mathcal{M}\left(W^{s,2}_H(\mathbb{R}^n)\right)} =: M < \infty.$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\tilde{\eta}(x) = 1$ on the cube $\{|x| \leq 2\}$ and $\tilde{\eta}_m(x) = \tilde{\eta}(x + m)$. We follow an argument as in Proposition 10 to show that for every $g \in W^{s,2}_H(\mathbb{R}^n)$,

$$\|fg\|_{W^{s,2}_H(\mathbb{R}^n)} \leq C \sum_{m \in \mathbb{Z}_n} \|fg \eta_m \tilde{\eta}_m\|_{W^{s,2}_H(\mathbb{R}^n)}$$

$$\leq CM^2 \sum_{m \in \mathbb{Z}_n} \|g \tilde{\eta}_m\|_{W^{s,2}_H(\mathbb{R}^n)}$$

$$\leq CM^2 \|g\|_{W^{s,2}_H(\mathbb{R}^n)}.$$
Corollary 16 Let $k > n/2$ be an integer. If
\[
\sup_{m \in \mathbb{Z}_n} \| f \eta_m \|_{W^k_H(\mathbb{R}^n)} < \infty,
\]
then $f \in \mathcal{M}(W^k_H(\mathbb{R}^n))$.

**Proof** It follows by Proposition 14 that for an integer $k > n/2$, we have that $W^k_H(\mathbb{R}^n) \subset \mathcal{M}(W^k_H(\mathbb{R}^n))$, and thus there exists some $M > 0$ such that for every $m \in \mathbb{Z}_n$,
\[
\| f \eta_m \|_{\mathcal{M}(W^k_H(\mathbb{R}^n))} \leq \| f \eta_m \|_{W^k_H(\mathbb{R}^n)} \leq M.
\]
Then we apply Proposition 15 to obtain that $f \in \mathcal{M}(W^k_H(\mathbb{R}^n))$. \(\square\)

**Remark 1** It would be interesting to establish a necessary and sufficient condition for $f$ to be in $\mathcal{M}(W^s_H(\mathbb{R}^n))$ for $s > 0$. To the best of our knowledge, it is not clear for us yet.

4 Proof of Theorem 1

In the following, for any $N > 0$, we use $Q_N$ to denote the cube centered at $0 \in \mathbb{R}^n$ with side length $2N$. Let $\Delta = \bigcup \Delta_l$ be an arbitrary partition of $Q_N$ and choose $x_l \in \Delta_l$ for each $l$. Suppose that $f$ is a measurable function on $\mathbb{R}^n$. We define the Riemann sum of $f$ as
\[
S^N_\Delta(f) = \sum_l f(x_l)|\Delta_l|,
\]
where $|\Delta_l|$ denotes the volume of $\Delta_l$. Let diam $(\Delta_l)$ denote the diameter of $\Delta_l$ and $\lambda := \max \text{diam} (\Delta_l)$. If $\lim_{N \to \infty} \lim_{\lambda \to 0} S^N_\Delta(f)$ exists, we say that $f$ is integrable on $\mathbb{R}^n$ and we write
\[
\int_{\mathbb{R}^n} f \, dx = \lim_{N \to \infty} \int_{Q_N} f \, dx = \lim_{N \to \infty} \lim_{\lambda \to 0} S^N_\Delta(f).
\]

**Lemma 17** Let $s \geq 0$ and $T$ be a bounded operator on $W^{s,2}_H(\mathbb{R}^n)$, which commutes with translations $\tau_a$ for all $a \in \mathbb{R}^n$. Then for $f, g \in C^\infty_0(\mathbb{R}^n)$, we have
\[
T(f \ast g) = Tf \ast g = f \ast Tg.
\]

**Proof** Let $f, g \in C^\infty_0(\mathbb{R}^n)$. It follows from the proof of Theorem 2.3.20 in [12] that $S^N_\Delta(f, g) \to f \ast g$ in the Schwarz space $\mathcal{S}$. This implies
\[
H_{j_1} \cdots H_{j_{k'}} S^N_\Delta(f, g) \to H_{j_1} \cdots H_{j_{k'}} (f \ast g) \quad \text{in} \quad L^\infty \quad \text{for} \quad 0 \leq k' \leq k \in \mathbb{N}.
\]
Since $f$ and $g$ have compact supports, we know
\[
H_{j_1} \cdots H_{j_k} S^N_\Delta(f, g) \to H_{j_1} \cdots H_{j_{k'}} (f \ast g) \quad \text{in} \quad L^2(\mathbb{R}^n),
\]
which means that $S^N_\Delta(f, g) \to f \ast g$ in $W^{k}_{H}(\mathbb{R}^n)$. Thus $S^N_\Delta(f, g) \to f \ast g$ in $W^{k,2}_H(\mathbb{R}^n) \subset W^{s,2}_H(\mathbb{R}^n)$ if one let $k = [s] + 1$. \(\square\)

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Since $T$ is bounded on $W^{s,2}_H(\mathbb{R}^n)$ and commutes with translations, we have

$$T(f \ast g)(x) = T(\lim_{N \to \infty} \lim_{\lambda \to 0} S^N_\Delta(f, g))(x)$$

$$= \lim_{N \to \infty} \lim_{\lambda \to 0} T(S^N_\Delta(f, g))(x)$$

$$= \lim_{N \to \infty} \lim_{\lambda \to 0} \sum y Tg(x - y)|\Delta_x|.$$ 

Note $f \in C^\infty_0(\mathbb{R}^n)$ and $Tg \in L^2(\mathbb{R}^n)$, which shows $f \ast g \in L^2(\mathbb{R}^n)$, i.e., the integral defining the convolution of $f$ and $g$ converges. So

$$\lim_{N \to \infty} \lim_{\lambda \to 0} \sum y Tg(x - y)|\Delta_x| = f \ast Tg(x)$$

pointwisely in $x$. This shows $T(f \ast g) = f \ast Tg$. \qed

If $\mathcal{F}(f)$ denotes the Fourier transformation of $f$, then for $f, g \in C^\infty_c(\mathbb{R}^n)$,

$$\mathcal{F}(f) \mathcal{F}(Tg) = \mathcal{F}(Tf) \mathcal{F}(g). \tag{20}$$

**Proposition 18** Let $s \geq 0$. Suppose $T$ is a bounded operator on $W^{s,2}_H(\mathbb{R}^n)$. If $T$ commutes with all translations $\tau_a$, $a \in \mathbb{R}^n$, on $W^{s,2}_H(\mathbb{R}^n)$, then there is an $m \in \mathcal{M}(W^{s,2}_H(\mathbb{R}^n))$ such that

$$\mathcal{F}(Tf) = m \mathcal{F}(f), \quad f \in W^{s,2}_H(\mathbb{R}^n). \tag{21}$$

Conversely, for any $m \in \mathcal{M}(W^{s,2}_H(\mathbb{R}^n))$, $T = \mathcal{F}^{-1} M_m \mathcal{F}$ is bounded on $W^{s,2}_H(\mathbb{R}^n)$ and commutes with translations on $\mathcal{F}(\mathbb{R}^n)$, where $M_m f = mf$ for any $f \in W^{s,2}_H(\mathbb{R}^n)$.

**Proof** For any $f \in W^{s,2}_H(\mathbb{R}^n)$ there is a sequence $\{f_j\} \subset C^\infty_0(\mathbb{R}^n)$, such that

$$\|f_j - f\|_{W^{s,2}_H(\mathbb{R}^n)} \to 0, \quad j \to \infty.$$ 

Since $T$ is bounded on $W^{s,2}_H(\mathbb{R}^n)$, we see that

$$\|Tf_j - Tf\|_{W^{s,2}_H(\mathbb{R}^n)} \to 0, \quad j \to \infty.$$ 

Consequently,

$$\|\mathcal{F}(f_j) - \mathcal{F}(f)\|_{L^2(\mathbb{R}^n)}, \quad \|\mathcal{F}(Tf_j) - \mathcal{F}(Tf)\|_{L^2(\mathbb{R}^n)} \to 0, \quad j \to \infty.$$ 

Then we can find subsequences of $\{\mathcal{F}(f_j)\}$ and $\{\mathcal{F}(Tf_j)\}$, which are still denoted by $\{\mathcal{F}(f_j)\}$ and $\{\mathcal{F}(Tf_j)\}$ respectively, such that $\mathcal{F}(f_j) \to \mathcal{F}(f)$ a.e. and $\mathcal{F}(Tf_j) \to \mathcal{F}(Tf)$ a.e.. By (20), we see that for any $g \in C^\infty_0(\mathbb{R}^n)$,

$$\mathcal{F}(f_j) \mathcal{F}(Tg) = \mathcal{F}(Tf_j) \mathcal{F}(g).$$

Let $j \to \infty$, we have

$$\mathcal{F}(f) \mathcal{F}(Tg) = \mathcal{F}(Tf) \mathcal{F}(g) \quad a.e.$$ 

By the same token, there still holds

$$\mathcal{F}(f) \mathcal{F}(Tg) = \mathcal{F}(Tf) \mathcal{F}(g) \quad a.e.$$
for all \( f, g \in W^{s,2}_H(\mathbb{R}^n) \). We may choose some \( g \in W^{s,2}_H(\mathbb{R}^n) \) such that \( \mathcal{F}(g) \) has no zeros on \( \mathbb{R}^n \) and let \( m = \mathcal{F}(Tg)/\mathcal{F}(g) \). Then
\[
\mathcal{F}(Tf) = m\mathcal{F}(f), \quad f \in W^{s,2}_H(\mathbb{R}^n).
\]

Note that
\[
\|\mathcal{F}(Tf)\|_{W^{s,2}_H(\mathbb{R}^n)} = \|Tf\|_{W^{s,2}_H(\mathbb{R}^n)} \leq \|T\|\|f\|_{W^{s,2}_H(\mathbb{R}^n)} = \|T\|\|\mathcal{F}(f)\|_{W^{s,2}_H(\mathbb{R}^n)}.
\]

Thus
\[
\|m\mathcal{F}(f)\|_{W^{s,2}_H(\mathbb{R}^n)} \leq \|T\|\|\mathcal{F}(f)\|_{W^{s,2}_H(\mathbb{R}^n)}.
\]

This shows that \( m \in \mathcal{M}(W^{s,2}_H(\mathbb{R}^n)) \).

Conversely, if \( m \in \mathcal{M}(W^{s,2}_H(\mathbb{R}^n)) \), then \( T \) defined by (21) is bounded.

The proof of Proposition 18 is complete.

**Proposition 19** Let \( s \geq 0 \). Suppose that \( T \) is a bounded operator on \( W^{s,2}_H(\mathbb{R}^n) \) and it commutes with all translations \( \tau_a, \ a \in \mathbb{R}^n \), on \( W^{s,2}_H(\mathbb{R}^n) \). If
\[
\mathcal{F}(Tf)(x) = m(x)\mathcal{F}(f)(x), \quad f \in W^{s,2}_H(\mathbb{R}^n),
\]
with an \( m \in \mathcal{M}(W^{s,2}_H(\mathbb{R}^n)) \), then for every \( g \in F^{s,2}(\mathbb{C}^n) \),
\[
BTB^{-1}g(z) = \int_{\mathbb{C}^n} g(w)e^{z\bar{w}}\varphi(z - \bar{w})d\lambda(w), \quad z \in \mathbb{C}^n,
\]
where
\[
\varphi(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}}\int_{\mathbb{R}} m(x)e^{-2(x - \frac{i}{2}z)^2}dx \in F^{s,2}(\mathbb{C}^n).
\]

**Proof** Following an argument of Lemma 3.4 in [7], we obtain
\[
\varphi(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}}\int_{\mathbb{R}} m(x)e^{-2(x - \frac{i}{2}z)^2}dx
\]
in terms of \( m \in \mathcal{M}(W^{s,2}_H(\mathbb{R}^n)) \). By Proposition 18, it suffices to show that \( \varphi(z) \in F^{s,2}(\mathbb{C}^n) \) for \( m \in \mathcal{M}(W^{s,2}_H(\mathbb{R}^n)) \). To show it, for \( z \in \mathbb{C}^n \) we write \( z = u + iv \), and the key observation is the following:
\[
\varphi(z) = CF^{-1}[m(x - \frac{1}{2}v)e^{-2x^2}]u e^{-\frac{|u|^2}{2}}.
\]

Notice that
\[
\int_{\mathbb{R}^n} (1 + |\xi|)^2 |\mathcal{F}^{-1}f(\xi)|^2d\xi \leq C\|f\|_{W^{s,2}_H}^2 \leq C\|f\|_{W^{s,2}_H(\mathbb{R}^n)}^2,
\]
and from (10),
\[
\|m(x - \frac{1}{2}v)e^{-2x^2}\|_{W^{s,2}_H(\mathbb{R}^n)}^2 \leq (1 + \frac{|v|}{2})^{2s}\|m(x)e^{-2(2x + \frac{1}{2}v)^2}\|_{W^{s,2}_H(\mathbb{R}^n)}^2.
\]
\[
\|\varphi\|^2_{L^2(\mathbb{C}^n)} \leq C \int_{\mathbb{C}^n} (1 + |z|)^{2s} |\varphi(z)|^2 e^{-|z|^2} \, dz
\]
\[
\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |u|)^{2s} (1 + |v|)^{2s} |\varphi(u + iv)|^2 e^{-((u|^2 + |v|^2)^2) \, du \, dv}
\]
\[
\leq C \int_{\mathbb{R}^n} (1 + |v|)^{2s} e^{-|v|^2} \int_{\mathbb{R}^n} (1 + |u|)^{2s} |F^{-1}[m(x - \frac{1}{2}v)e^{-2x^2}](u)|^2 \, du \, dv
\]
\[
\leq C \int_{\mathbb{R}^n} (1 + |v|)^{2s} e^{-|v|^2} \|m(x - \frac{1}{2}v)e^{-2x^2}\|^2_{L^2(\mathbb{R}^n)} \, dv
\]
\[
\leq C \|m\|^2_{\mathcal{M}(L^2(\mathbb{R}^n))} \int_{\mathbb{R}^n} (1 + |v|)^{2s} \left(1 + \frac{|v|}{2}\right)^{4s} e^{-|v|^2} \, dv
\]
\[
\leq C \|m\|^2_{\mathcal{M}(L^2(\mathbb{R}^n))}.
\]

This proves \(\varphi \in F^{s,2}(\mathbb{C}^n)\). The proof of Proposition 19 is complete. \(\square\)

Finally, we are ready to prove our Theorem 1.

**Proof of Theorem 1** First, we assume that
\[
\varphi(z) = \left(\frac{2}{\pi}\right)^{n} \int_{\mathbb{R}^n} m(x)e^{-2(x - \frac{1}{2}z^2)} \, dx, \quad z \in \mathbb{C}^n
\]
where \(m\) is a multiplier on the space \(W^{s,2}_{H}(\mathbb{R}^n), s \geq 0\). Let \(S_\varphi\) be an integral operator as in (1). To prove that \(S_\varphi\) is bounded on the space \(F^{s,2}(\mathbb{C}^n)\), we notice that from Proposition 19, \(\varphi \in F^{s,2}(\mathbb{C}^n)\) and
\[
S_\varphi = \mathcal{B}\mathcal{T}\mathcal{B}^{-1},
\]
where \(\mathcal{T}\) is given by
\[
\mathcal{F}(T f)(x) = m(x)\mathcal{F}(f)(x) \quad \text{for all} \quad f \in W^{s,2}_{H}(\mathbb{R}^n).
\]
By Lemma 7, the operator \(\mathcal{T}\) is bounded on the space \(W^{s,2}_{H}(\mathbb{R}^n)\). From the properties of the operators \(\mathcal{B}\) and \(\mathcal{B}^{-1}\), we see that \(S_\varphi\) is bounded on the space \(F^{s,2}(\mathbb{C}^n)\).

Conversely, let \(S_\varphi\) be a bounded operator on \(F^{s,2}(\mathbb{C}^n)\) as in (1). Then from the properties of the operators \(\mathcal{B}\) and \(\mathcal{B}^{-1}\), we have that
\[
T = \mathcal{B}^{-1}S_\varphi\mathcal{B}
\]
is bounded on \(W^{s,2}_{H}(\mathbb{R}^n)\). Note that \(S_\varphi W_a f = W_a S_\varphi f\) for any \(a \in \mathbb{R}^n\) and \(f \in F^{s,2}(\mathbb{C}^n)\). It follows that \(T\tau_a f = \tau_a T f\) for any \(a \in \mathbb{R}^n\) and \(f \in W^{s,2}_{H}(\mathbb{R}^n)\). Thus by Proposition 18, there is an \(m \in \mathcal{M}(W^{s,2}_{H}(\mathbb{R}^n))\) such that \(\mathcal{F}(T f) = m\mathcal{F} f\). This implies
\[
S_\varphi = \mathcal{B}\mathcal{F}^{-1}M_m\mathcal{F}\mathcal{B}^{-1}.
\]
By Proposition 19, \( B \mathcal{F}^{-1} M_m \mathcal{F} B^{-1} \) is an integral operator \( S_{\varphi_0} \), where
\[
\varphi_0(z) = \left( \frac{2}{\pi} \right)^\frac{n}{2} \int_{\mathbb{R}} m(x) e^{-2(x - \frac{1}{2}z)^2} \, dx \in F^{s,2}(\mathbb{C}^n).
\]
This implies for all \( g \in F^{s,2}(\mathbb{C}^n) \), there holds
\[
\int_{\mathbb{C}^n} g(w) e^{z \cdot \bar{w}} (\varphi(z - \bar{w}) - \varphi_0(z - \bar{w})) \, d\lambda(w) = 0, \quad z \in \mathbb{C}^n.
\]
To finish the proof, it suffices to show that \( \varphi = \varphi_0 \). Taking \( z = 0 \) in the above equality, we see that for all \( g \in F^{s,2}(\mathbb{C}^n) \),
\[
\int_{\mathbb{C}^n} g(w)(\varphi(-\bar{w}) - \varphi_0(-\bar{w})) \, d\lambda(w) = 0.
\]
Write \( \psi(w) = \varphi(-w) - \varphi_0(-w) \in F^2(\mathbb{C}^n) \). Then \( \psi \) has the series expansion
\[
\psi(w) = \sum_{\alpha} c_{\alpha} e_{\alpha}(w) = \sum_{\alpha} c_{\alpha} \left( \frac{1}{\alpha!} \right)^\frac{1}{2} w^\alpha
\]
with \( \sum_{\alpha} |c_{\alpha}|^2 = \| \psi \|^2_{F^2(\mathbb{C}^n)} \). Letting \( g = e_{\alpha} \) for all \( \alpha \in \mathbb{N}_0^n \), we obtain
\[
c_{\alpha} = \int_{\mathbb{C}^n} e_{\alpha}(w) \psi(\bar{w}) \, d\lambda(w)
= \int_{\mathbb{C}^n} e_{\alpha}(w)(\varphi(-\bar{w}) - \varphi_0(-\bar{w})) \, d\lambda(w) = 0.
\]
This shows \( \varphi = \varphi_0 \). Hence, the proof of Theorem 1 is complete. \( \square \)

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References

1. Bargmann, V.: On a Hilbert space of analytic functions and an associated integral transform. Part I. Comm. Pure Appl. Math. 14, 187–214 (1961)
2. Bargmann, V.: On a Hilbert space of analytic functions and an associated integral transform. Part II. Comm. Pure Appl. Math. 20, 1–101 (1967)
3. Berger, C., Coburn, L.: Toeplitz operators and quantum mechanics. J. Funct. Anal. 68(3), 273–299 (1986)
4. Bongioanni, B., Torrea, J.L.: Regularity theory for the fractional harmonic oscillator. J. Funct. Anal. 260(10), 3097–3131 (2011)
5. Bongioanni, B., Torrea, J.L.: Sobolev spaces associated to the harmonic oscillator. Proc. Indian Acad. Sci. (Math. Sci.) 116(3), 337–360 (2003)
6. Bony, J.-M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. Sci. Ec. Norm. Supér. 14, 209–246 (1982)
7. Cao, G.F., Li, J., Shen, M.X., Wick, B.D., Yan, L.X.: A boundedness criterion for singular integral operators of convolution type on the Fock space. Adv. Math. 363, 107001 (2020). (33 pp)
8. Cho, H. R., Choi, H., Lee, H.W.: Boundedness of the Segal-Bargmann Transform on Fractional Hermite-Sobolev Spaces. J. Function Spaces, Article ID 9176914, 6, 1–6 (2017)
9. Cho, H.R., Park, S.: Fractional Fock-Sobolev spaces. Nagoya Math. J. 237, 79–97 (2020)
10. Feynman, R.P., Hibbs, A.R.: Quantum Mechanics and Path Integrals. Emended edition. Emended and with a preface by Daniel F. Styer. Dover Publications, Inc., Mineola, NY, (2010)
11. Folland, G.: Harmonic Analysis in Phase Space. Princeton University Press (1989)
12. Grafakos, L.: Classical Fourier Analysis. GTM 249. Springer (2000)
13. Gröchenig, K.: Foundations of Time-Frequency Analysis. Applied and Numerical Harmonic Analysis. Birkhauser Boston Inc, Boston, MA (2001)
14. Stinga, P.R., Torrea, J.L.: Extension problem and Harnack’s inequality for some fractional operators. Comm. Partial Diff. Equ. 35(11), 2092–2122 (2010)
15. Strichartz, R.S.: Multipliers on fractional Sobolev spaces. J. Math. Mech. 16(9), 1031–1060 (1967)
16. Wick, B.D., Wu, S.K.: Integral operators on Fock-Sobolev spaces via multipliers on Gauss-Sobolev spaces. Preprint, arXiv: 2004.05231v1 (2020)
17. Yosida, K.: Functional Analysis, 5th edn. Spring-Verlag, Berlin (1978)
18. Zhu, K.H.: Singular integral operators on the Fock space. Integral Equ. Oper. Theory 81, 451–454 (2015)
19. Zhu, K.H.: Analysis on Fock Spaces. Springer, New York (2012)

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