A STUDY OF SATURATED TENSOR CONE FOR
SYMMETRIZABLE KAC-MOODY ALGEBRAS

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1. Introduction

Let \( g \) be a symmetrizable Kac-Moody Lie algebra with the standard Cartan subalgebra \( h \) and the Weyl group \( W \). Let \( P_+ \) be the set of dominant integral weights. For \( \lambda \in P_+ \), let \( L(\lambda) \) be the irreducible, integrable, highest weight representation of \( g \) with highest weight \( \lambda \). For a positive integer \( s \), define the saturated tensor semigroup as

\[
\Gamma_s := \{ (\lambda_1, \ldots, \lambda_s, \mu) \in P_+^{s+1} : \exists N > 1 \text{ with } L(N\mu) \subset L(N\lambda_1) \otimes \cdots \otimes L(N\lambda_s) \}\.
\]

The aim of this paper is to begin a systematic study of \( \Gamma_s \) in the infinite dimensional symmetrizable Kac-Moody case. In this paper, we produce a set of necessary inequalities satisfied by \( \Gamma_s \), which we describe now. Let \( X = G^{\text{min}}/B \) be the standard full KM-flag variety associated to \( g \), where \( G^{\text{min}} \) is the ‘minimal’ Kac-Moody group with Lie algebra \( g \) and \( B \) is the standard Borel subgroup of \( G^{\text{min}} \). For \( w \in W \), let \( X_w = BwB/B \subset X \) be the corresponding Schubert variety. Let \( \{ \varepsilon^w \}_{w \in W} \subset H^*(X, \mathbb{Z}) \) be the (Schubert) basis dual (with respect to the standard pairing) to the basis of the singular homology of \( X \) given by the fundamental classes of \( X_w \). The following result is our first main theorem valid for any symmetrizable \( g \) (cf. Theorem 3.3).

**Theorem 1.1.** Let \( (\lambda_1, \ldots, \lambda_s, \mu) \in \Gamma_s \). Then, for any \( u_1, \ldots, u_s, v \in W \) such that \( n_{u_1,\ldots,u_s}^{u,v} \neq 0 \), where

\[
\varepsilon^{u_1} \cdots \varepsilon^{u_s} = \sum_w n_{u_1,\ldots,u_s}^w \varepsilon^w,
\]

we have

\[
\left( \sum_{j=1}^s \lambda_j(u_j x_i) \right) - \mu(\nu x_i) \geq 0, \text{ for any } x_i,
\]

where \( x_i \in h \) is dual to the simple roots of \( g \).

The proof of the theorem relies on the Kac-Moody analogue of the Borel-Weil theorem and the Geometric Invariant Theory (specifically the Hilbert-Mumford index). We conjecture that the above inequalities are sufficient as well to describe \( \Gamma_s \). In fact, we conjecture a much sharper result, where much fewer inequalities suffice to describe the semigroup \( \Gamma_s \). To explain our conjecture, we need some more notation.
Let $P \supset B$ be a (standard) parabolic subgroup and let $X_P := G^{\text{min}}/P$ be the corresponding partial flag variety. Let $W_P$ be the Weyl group of $P$ (which is, by definition, the Weyl group of the Levi $L$ of $P$) and let $W^P$ be the set of minimal length coset representatives of cosets in $W/W_P$. The projection map $X \to X_P$ induces an injective homomorphism $H^*(X, \mathbb{Z}) \to H^*(X_P, \mathbb{Z})$ and $H^*(X_P, \mathbb{Z})$ has the Schubert basis $\{\varepsilon_{\bar{w}}^p\}_{w \in W^P}$ such that $\varepsilon_{\bar{w}}^p$ goes to $\varepsilon_{w}^P$ for any $w \in W^P$. As defined by Belkale-Kumar [BK, §6] in the finite dimensional case (and extended here in Section 7 for any symmetrizable Kac-Moody case), there is a new deformed product $\circ$ in $H^*(X_P, \mathbb{Z})$, which is commutative and associative. Now, we are ready to state our conjecture (see Conjecture 7.3).

1.2. Conjecture. Let $\mathfrak{g}$ be any indecomposable symmetrizable Kac-Moody Lie algebra and let $(\lambda_1, \ldots, \lambda_s, \mu) \in P^{s+1}_+$. Assume further that none of $\lambda_j$ is $W$-invariant and $\mu - \sum_{j=1}^s \lambda_j \in Q$, where $Q$ is the root lattice of $G$. Then, the following are equivalent:

(a) $(\lambda_1, \ldots, \lambda_s, \mu) \in \Gamma_s$.

(b) For every standard maximal parabolic subgroup $P$ in $G^{\text{min}}$ and every choice of $s + 1$-tuples $(w_1, \ldots, w_s, v) \in (W^P)^{s+1}$ such that $\varepsilon_{\bar{w}}^P$ occurs with coefficient 1 in the deformed product

$$\varepsilon_{\bar{w}}^{w_1}_P \circ_0 \cdots \circ_0 \varepsilon_{\bar{w}}^{w_s}_P \in \left( H^*(X_P, \mathbb{Z}), \circ_0 \right),$$

the following inequality holds:

$$\left( \sum_{j=1}^s \alpha_{w_j}(w_jx_P) \right) - \mu(vx_P) \geq 0, \quad (I_P^{w_1, \ldots, w_s, v})$$

where $\alpha_{w_j}$ is the (unique) simple root not in the Levi of $P$ and $x_P := x_{i_P}$.

This conjecture is motivated from its validity in the finite case due to Belkale-Kumar [BK, Theorem 22]. (For a survey of these results in the finite case, see [K5].) So far, the only evidence of its validity in the infinite dimensional case is shown for $s = 2$ and $\mathfrak{g}$ of types $A_1^{(1)}$ and $A_2^{(2)}$ (cf. Theorems 7.5 and 8.0). In these cases, we explicitly determine $\Gamma_2$ and thereby show the validity of the conjecture.

A positive integer $d_o$ is called a saturation factor for $\mathfrak{g}$ if for any $\Lambda, \Lambda', \Lambda'' \in P_+$ such that $\Lambda - \Lambda' - \Lambda'' \in Q$ and $L(\Lambda \Lambda')$ is a submodule of $L(\Lambda \Lambda') \otimes L(\Lambda \Lambda'')$, for some $N \in \mathbb{Z}_{>0}$, then $L(d_o \Lambda)$ is a submodule of $L(d_o \Lambda') \otimes L(d_o \Lambda'')$.

We prove the following result on saturation factors (cf. Corollaries 6.4 and 8.7).

**Theorem 1.3.** For $A_1^{(1)}$, any integer $d_o > 1$ is a saturation factor. For $A_2^{(2)}$, 4 is a saturation factor.

The proof in these affine rank-2 cases makes use of basic representation theory of the Virasoro algebra (in particular, Lemma 4.1). Let $\delta$ be the smallest positive imaginary root of $\mathfrak{g}$. To determine the saturated tensor
semigroup, we show that it is enough to know the components of $L(\lambda_1) \otimes L(\lambda_2)$ which are $\delta$-maximal, i.e., the components $L(\mu) \subset L(\lambda_1) \otimes L(\lambda_2)$ such that $L(\mu + n\delta) \nsubseteq L(\lambda_1) \otimes L(\lambda_2)$ for any $n > 0$. Let $m^{\mu}_{\lambda_1, \lambda_2}$ be the multiplicity of $L(\mu)$ in $L(\lambda_1) \otimes L(\lambda_2)$. If $L(\mu)$ is a $\delta$-maximal component of $L(\lambda_1) \otimes L(\lambda_2)$, then $\sum_{n \in \mathbb{Z}, n \leq 0} L(\mu + n\delta)^{\otimes m^{\mu}_{\lambda_1, \lambda_2}}$ is a unitarizable coset module for the Virasoro algebra arising from the Sugawara construction for the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$. Proposition 5.5 for $A(1)$ (and the analogous Proposition 8.2 for $A(2)$) determining the maximal $\delta$-components plays a crucial role in the proofs.

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2. **Notation**

We take the base field to be the field of complex numbers $\mathbb{C}$. By a variety, we mean an algebraic variety over $\mathbb{C}$, which is reduced but not necessarily irreducible.

Let $G$ be any symmetrizable Kac-Moody group over $\mathbb{C}$ completed along the negative roots (as opposed to completed along the positive roots as in [K3, Chapter 6]) and $G^{\text{min}} \subset G$ be the ‘minimal’ Kac-Moody group as in [K3, §7.4]. Let $B$ be the standard (positive) Borel subgroup, $B^-$ the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus and $W$ the Weyl group (cf. [K3, Chapter 6]). Let $U$ (resp. $U^-$) be the unipotent radical $[B, B]$ (resp. $[B^-, B^-]$) of $B$ (resp. $B^-$). Let

$$\bar{X} = G/B$$

be the ‘thick’ flag variety which contains the standard KM-flag variety

$$X = G^{\text{min}}/B.$$

If $G$ is not of finite type, $\bar{X}$ is an infinite dimensional non quasi-compact scheme (cf. [Ka, §4]) and $X$ is an ind-projective variety (cf. [K3, §7.1]). The group $G^{\text{min}}$ acts on $\bar{X}$ and $X$.

More generally, for any standard parabolic subgroup $P \supset B$, define the partial flag variety

$$X_P = G^{\text{min}}/P,$$

and

$$\bar{X}_P = G/P.$$

Recall that if $W_P$ is the Weyl group of $P$ (which is, by definition, the Weyl Group $W_L$ of its Levi subgroup $L$), then in each coset of $W/W_P$ we have a unique member $w$ of minimal length. Let $W^P$ be the set of the minimal length representatives in the cosets of $W/W_P$.

For any $w \in W^P$, define the Schubert cell:

$$C^P_w := BwP/P \subset G/P$$
endowed with the reduced subscheme structure. Then, it is a locally closed subvariety of the ind-variety $G/P$ isomorphic with the affine space $k^{\ell(w)}$, $\ell(w)$ being the length of $w$ (cf. [K3, §7.1]). Its closure is denoted by $X^P_w$, which is an irreducible (projective) subvariety of $G/P$ of dimension $\ell(w)$. We denote the point $wP \in C^P_w$ by $\dot{w}$. We abbreviate $C^P_w, X^P_w$ by $C_w, X_w$ respectively.

Similarly, define the opposite Schubert cell

$$C^w_B := B^-wP/P \subset \breve{X}_P,$$

and the opposite Schubert variety

$$X^w_P := \overline{C^w_P} \subset \breve{X}_P,$$

both endowed with the reduced subscheme structures. Then, $X^w_P$ is a finite codimensional irreducible subscheme of $\breve{X}_P$ (cf. [K3, Section 7.1] and [Ka, §4]). As above, we abbreviate $C^w_B, X^w_B$ by $C^w, X^w$ respectively.

For any integral weight $\lambda$ (i.e., any character $e^\lambda$ of $H$), we have a $G^\min$-equivariant line bundle $L_B(\lambda)$ on $X$ associated to the character $e^{-\lambda}$ of $H$. Similarly, we have a $G$-equivariant line bundle $L_{B^-}(\lambda)$ on $X^- := G/B^-$ associated to the character $e^\lambda$ of $H$.

By the Bruhat decomposition

$$X_P = \bigsqcup_{wP \in W^P} C^P_w,$$

the singular homology $H_*(X_P, \mathbb{Z})$ of $X_P$ with integral coefficients has a basis $\{\mu(X^P_w)\}_{wP \in W^P}$, where $\mu(X^P_w) \in H_{2\ell(w)}(X_P, \mathbb{Z})$ denotes the fundamental class of $X^P_w$. Let $\{\epsilon^w_P\}_{wP \in W^P}$ be the dual basis of the singular cohomology $H^*(X_P, \mathbb{Z})$ under the standard pairing of cohomology with homology, i.e.,

$$\epsilon^w_P(\mu(X^P_w)) = \delta_{u,v}, \text{ for any } u, v \in W^P.$$

Thus, $\epsilon^w_P \in H^{2\ell(w)}(X_P, \mathbb{Z})$. If $P = B$, we abbreviate $\epsilon^w_B$ by $e^w$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*$ be the set of simple roots, $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \subset \mathfrak{h}$ the set of simple coroots and $\{s_1, \ldots, s_r\} \subset W$ the corresponding simple reflections, where $\mathfrak{h} := \text{Lie } H$. Let $\rho \in X(H)$ be any weight satisfying

$$\rho(\alpha_i^\vee) = 1, \text{ for all } 1 \leq i \leq r,$$

where $X(H)$ is the character group of $H$ (identified as a subgroup of $\mathfrak{h}^*$ via the derivative). When $G$ is a finite dimensional semisimple group, $\rho$ is unique, but for a general Kac-Moody group $G$, it may not be unique.

Choose elements $x_i \in \mathfrak{h}$ such that

$$\alpha_j(x_i) = \delta_{i,j}, \text{ for any } 1 \leq i, j \leq r. \quad (1)$$

Observe that $x_i$ may not be unique.

Define the set of \textit{dominant integral weights}

$$P_+ := \{\lambda \in X(H) : \lambda(\alpha_i^\vee) \in \mathbb{Z}_+ \forall 1 \leq i \leq r\},$$

and the set of \textit{dominant integral regular weights}

$$P_{++} := \{\lambda \in X(H) : \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 1} \forall 1 \leq i \leq r\},$$
where $\mathbb{Z}_+$ is the set of non-negative integers. The integrable highest weight (irreducible) modules of $G_{\min}$ are parameterized by $P_+$. For $\lambda \in P_+$, let $L(\lambda)$ be the corresponding integrable highest weight (irreducible) $G$-module with highest weight $\lambda$.

3. Necessary Inequalities for the Saturated Tensor Semigroup

Fix a positive integer $s$ and define the saturated tensor semigroup $\Gamma_s = \Gamma_s(G)$:

$$\Gamma_s := \{ (\lambda_1, \ldots, \lambda_s, \mu) \in P_+^{s+1} : \exists N > 1 \text{ with } L(N\mu) \subset L(N\lambda_1) \otimes \cdots \otimes L(N\lambda_s) \}. \quad (2)$$

It is indeed a semigroup by the analogue of the Borel-Weil theorem for the Kac-Moody case (see the identity (3) in the proof of Theorem 3.3). We give a certain set of inequalities satisfied by $\Gamma_s$. But, we first recall some basic results about the Hilbert-Mumford index.

3.1. Definition. Let $S$ be any (not necessarily reductive) algebraic group acting on a (not necessarily projective) variety $X$ and let $\mathbb{L}$ be an $S$-equivariant line bundle on $X$. Let $O(S)$ be the set of all one parameter subgroups (for short OPS) in $S$. Take any $x \in X$ and $\delta \in O(S)$ such that the limit $\lim_{t \to 0} \delta(t)x$ exists in $X$ (i.e., the morphism $\delta_x : \mathbb{G}_m \to X$ given by $t \mapsto \delta(t)x$ extends to a morphism $\tilde{\delta}_x : \mathbb{A}^1 \to X$). Then, following Mumford, define a number $\mu^L(x, \delta)$ as follows: Let $x_o \in X$ be the point $\tilde{\delta}_x(0)$. Since $x_o$ is $\mathbb{G}_m$-invariant via $\delta$, the fiber of $\mathbb{L}$ over $x_o$ is a $\mathbb{G}_m$-module; in particular, it is given by a character of $\mathbb{G}_m$. This integer is defined as $\mu^L(x, \delta)$.

We record the following standard properties of $\mu^L(x, \delta)$ (cf. [MFK, Chap. 2, §1]):

3.2. Proposition. For any $x \in X$ and $\delta \in O(S)$ such that $\lim_{t \to 0} \delta(t)x$ exists in $X$, we have the following (for any $S$-equivariant line bundles $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$):

(a) $\mu^{L_1 \otimes L_2}(x, \delta) = \mu^{L_1}(x, \delta) + \mu^{L_2}(x, \delta)$.

(b) If there exists $\sigma \in H^0(X, \mathbb{L})^S$ such that $\sigma(x) \neq 0$, then $\mu^L(x, \delta) \geq 0$.

(c) If $\mu^L(x, \delta) = 0$, then any element of $H^0(X, \mathbb{L})^S$ which does not vanish at $x$ does not vanish at $\lim_{t \to 0} \delta(t)x$ as well.

(d) For any $S$-variety $X'$ together with an $S$-equivariant morphism $f : X' \to X$ and any $x' \in X'$ such that $\lim_{t \to 0} \delta(t)x'$ exists in $X'$, we have $\mu^{f^*L}(x', \delta) = \mu^L(f(x'), \delta)$.

(e) (Hilbert-Mumford criterion) Assume that $X$ is projective, $S$ is connected and reductive and $\mathbb{L}$ is ample. Then, $x \in X$ is semistable (with respect to $\mathbb{L}$) if and only if $\mu^L(x, \delta) \geq 0$, for all $\delta \in O(S)$.

In particular, if $x \in X$ is semistable and $\delta$-fixed, then $\mu^L(x, \delta) = 0$.

The following theorem is one of our main results giving a collection of necessary inequalities defining the semigroup $\Gamma_s$. 

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3.3. **Theorem.** Let $G$ be any symmetrizable Kac-Moody group and let $(\lambda_1, \cdots, \lambda_s, \mu) \in \Gamma_s$. Then, for any $u_1, \ldots, u_s, v \in W$ such that $n_{u_1, \ldots, u_s}^v \neq 0$, where

$$e^{u_1} \cdots e^{u_s} = \sum w n_{u_1, \ldots, u_s}^w e^w \in H^*(X, \mathbb{Z}),$$

we have

$$\left( \sum_{j=1}^s \lambda_j(u_j x_i) \right) - \mu(v x_i) \geq 0, \quad \text{for any } x_i,$$

where $x_i$ is defined by the equation (1).

**Proof.** Let

$$Z := \{ (g_1, \ldots, g_s) \in (X^-)^s : g_1 X_{u_1} \cap \cdots \cap g_s X_{u_s} \cap X_v \neq \emptyset \},$$

where $X^- := G/B^-$ and $g_j = g_j B^-$. Then, $Z$ contains a nonempty open set by Proposition 3.7 (In fact, by Proposition 8.7, $Z = (X^-)^s$; but we do not need this stronger result.)

Take a nonzero $\sigma \in H^0((X^-)^s \times X, \mathcal{L}^N)^{G_{\min}}$, where

$$\mathcal{L} := L_B(\lambda_1) \boxtimes \cdots \boxtimes L_B(\lambda_s) \boxtimes L_B(\mu).$$

Such a nonzero $\sigma$ exists, for some $N > 0$, since by [K$_3$, Corollary 8.3.12(a) and Lemma 8.3.9],

$$H^0((X^-)^s \times X, \mathcal{L}^N)^{G_{\min}} \cong \text{Hom}_{G_{\min}}(L(N\lambda_1)^{\vee} \otimes \cdots \otimes L(N\mu)^{\vee} \otimes L(N\sigma), \mathbb{C})$$

$$\cong \text{Hom}_{G_{\min}}(L(N\mu), [L(N\lambda_1)^{\vee} \otimes \cdots \otimes L(N\mu)^{\vee}]^{\ast})$$

$$\cong \text{Hom}_{G_{\min}}(L(N\mu), [L(N\lambda_1)^{\vee} \otimes \cdots \otimes L(N\lambda_s)^{\vee}],$$

$$\neq 0,$$

(3)

since $(\lambda_1, \cdots, \lambda_s, \mu) \in \Gamma$, where, for a $G_{\min}$-module $M$, $M^{\vee}$ denotes the direct sum of the $H$-weight spaces of the full dual module $M^*$. Pick $(g_1, \ldots, g_s) \in Z$ such that $\sigma(g_1, \ldots, g_s, \bar{1}) \neq 0$, where $\bar{1} = 1 \cdot B$. Since $(\bar{g}_1, \ldots, \bar{g}_s) \in Z$, there exists $u'_1 \geq u_1, \ldots, u'_s \geq u_s$ and $v' \leq v$ such that $g_1 C_{u'_1} \cap \cdots \cap g_s C_{u'_s} \cap C_{v'}$ is nonempty. Now, pick $g \in G_{\min}$ such that

$$gB = g_1 C_{u'_1} \cap \cdots \cap g_s C_{u'_s} \cap C_{v'}.$$  (4)

By Proposition 3.2, for any $\delta \in O(G_{\min})$, $\mu^\mathcal{L}(\bar{x}, \delta(t)) \geq 0$, where $\bar{x} = (\bar{g}_1, \ldots, \bar{g}_s, \bar{1})$ (since $\sigma(\bar{x}) \neq 0$). By the following Lemma 3.4 applied to the OPS $\delta(t) = g t^x g^{-1}$, we get

$$\left( \sum_{j=1}^s \lambda_j(u'_j x_i) \right) - \mu(v' x_i) \geq 0.$$  (5)

But, by [K$_3$, Lemma 8.3.3],

$$(u'_j)^{-1} \lambda_j \leq u_j^{-1}(\lambda_j).$$
Thus,
\[ \lambda_j(u'_j x_i) \leq \lambda_j(u_j x_i) . \]
Similarly,
\[ \mu(v'_i x_i) \geq \mu(v x_i) . \]
Thus, from (5), we get
\[ \sum_{j=1}^{s} \lambda_j(u_j x_i) - \mu(v x_i) \geq 0 . \]
This proves the theorem. \( \Box \)

3.4. Lemma. Let \( g \in G^{\min} \) be as in the equation \( \Delta \). Consider the one parameter subgroup \( \delta(t) = g t^{x_i} g^{-1} \in O(G^{\min}) \). Then, 
(a) \( \mu^\mathcal{L}_{B^-}(\lambda_j)(g_j B^-, \delta(t)) = \lambda_j(u'_j x_i) \).
(b) \( \mu^\mathcal{L}_{B^-}(\lambda_j)(1 \cdot B, \delta(t)) = -\mu(v'_i x_i) \).

Proof. (a) \( \mu^\mathcal{L}_{B^-}(\lambda_j)(g_j B^-, \delta(t)) = \mu^\mathcal{L}_{B^-}(\lambda_j)(g^{-1} g_j B^-, t^{x_i}) \).

By assumption, \( g_j^{-1} g \in U^- u_j'B \). Write
\[ g_j^{-1} g = b_j^{-1} u_j'p_j, \text{ for some } b_j^{-1} \in U^-, p_j \in B. \]
Thus,
\[ 1 = g^{-1} g_j b_j^{-1} u_j'p_j. \]
Let
\[ b_j(t) = b_j^{-1} u_j' t^{-x_i}(u'_j)^{-1}(b_j^{-1})^{-1} \in B^- . \]
Then,
\[ t^{x_i} g_j^{-1} b_j(t) = t^{x_i} p_j^{-1} t^{-x_i}(u'_j)^{-1}(b_j^{-1})^{-1} . \] \hspace{1cm} (6)

Consider the \( G_m \)-invariant section (via \( t^{x_i} \)) of \( \mathcal{L}_{B^-}(\lambda_j) \):
\[ \bar{\sigma}(t) = (t^{x_i} g_j^{-1} g_j, 1) \mod B^- \]
\[ = (t^{x_i} g_j^{-1} g_j b_j(t), \lambda_j(b_j(t)^{-1})) \mod B^- . \]
Clearly, \( \lim_{t \to 0} t^{x_i} g_j^{-1} g_j b_j(t) \) exists in \( G \) by \( \Delta \).

Now,
\[ \lambda_j(b_j(t)^{-1}) = \lambda_j(b_j^{-1} u_j' t^{x_i}(u'_j)^{-1}(b_j^{-1})^{-1}) \]
\[ = \lambda_j(t^{x_i} u_j') . \]
This gives
\[ \mu^\mathcal{L}_{B^-}(\lambda_j)(g_j B^-, \delta(t)) = \lambda_j(u'_j x_i) . \]
This proves the (a) part of the lemma.
(b) \( \mu^\mathcal{L}_{B^-}(\lambda_j)(1 \cdot B, \delta(t)) = \mu^\mathcal{L}_{B^-}(\lambda_j)(g^{-1} B, t^{x_i}) \).

By assumption,
\[ g \in Bu' \cdot B. \]
Write
\[ g = bv'p, \text{ for } b \in U, p \in B. \]
Thus, \[ 1 = g^{-1}bv'p. \]

Let \[ b(t) = bv't^{-x_i}(v')^{-1}b^{-1} \in B. \]

Now, \[ t^{x_i}g^{-1}b(t) = t^{x_i}p^{-1}t^{-x_i}(v')^{-1}b^{-1}. \]

Thus, \( \lim_{t \to 0} t^{x_i}g^{-1}b(t) \) exists in \( G^\text{min} \).

Consider the \( G_m \)-invariant section (via \( t^{x_i} \))
\[ \hat{\sigma}(t) = (t^{x_i}g^{-1}, 1) \mod B \]
\[ = (t^{x_i}g^{-1}b(t), \mu(b(t))) \mod B. \]

Now,
\[ \mu(b(t)) = \mu(bv't^{-x_i}(v')^{-1}b^{-1}) \]
\[ = \mu(t^{-x_i}). \]

This gives
\[ \mu^C_B(1 \cdot B, \delta(t)) = -\mu(v'(x_i)). \]

This proves the (b)-part and hence the lemma is proved. \( \square \)

3.5. Definition. For a quasi-compact scheme \( Y \), an \( \mathcal{O}_Y \)-module \( S \) is called coherent if it is finitely presented as an \( \mathcal{O}_Y \)-module and any \( \mathcal{O}_Y \)-submodule of finite type admits a finite presentation.

An \( \mathcal{O}_{\bar{X}} \)-module \( S \) is called coherent if \( S|_{V^S} \) is a coherent \( \mathcal{O}_{V^S} \)-module for any finite ideal \( S \subset W \) (where a subset \( S \subset W \) is called an ideal if for \( x \in S \) and \( y \leq x \Rightarrow y \in S \)), where \( V^S \) is the quasi-compact open subset of \( \bar{X} \) defined by
\[ V^S = \bigcup_{w \in S} wU/B. \]

Let \( K^0(\bar{X}) \) denote the Grothendieck group of coherent \( \mathcal{O}_{\bar{X}} \)-modules \( S \).

Similarly, define \( K_0(X) := \lim_{n \to \infty} K_0(X_n), \) where \( \{X_n\}_{n \geq 1} \) is the filtration of \( X \) giving the ind-projective variety structure (i.e., \( X_n = \bigcup_{\ell(w) \leq n} C_w \)) and \( K_0(X_n) \) is the Grothendieck group of coherent sheaves on the projective variety \( X_n \).

We also define
\[ K^{\text{top}}(X) := \text{Invlt}_{n \to \infty} K^{\text{top}}(X_n), \]
where \( K^{\text{top}}(X_n) \) is the topological \( K \)-group of the projective variety \( X_n \).

Let \( * : K^{\text{top}}(X_n) \to K^{\text{top}}(X_n) \) be the involution induced from the operation which takes a vector bundle to its dual. This, of course, induces the involution \( * \) on \( K^{\text{top}}(X) \).

For any \( w \in W, \)
\[ [\mathcal{O}_{X_w}] \in K_0(X). \]

3.6. Lemma. \( \{[\mathcal{O}_{X_w}]\}_{w \in W} \) forms a basis of \( K_0(X) \) as a \( \mathbb{Z} \)-module.
**Proof.** By [CG, §5.2.14 and Theorem 5.4.17], the result follows. □

For $u \in W$, by [KS, §2], $\mathcal{O}_{X^u}$ is a coherent $\mathcal{O}_X$-module. In particular, $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module.

Define a pairing

$$\langle , \rangle : K^0(\bar{X}) \otimes K_0(X) \to \mathbb{Z}, \quad \langle [\mathcal{S}], [\mathcal{F}] \rangle = \sum_i (-1)^i \chi(X_n, Tor_i^{O_X}(\mathcal{S}, \mathcal{F})),$$

if $\mathcal{S}$ is a coherent sheaf on $\bar{X}$ and $\mathcal{F}$ is a coherent sheaf on $X$ supported in $X_n$ (for some $n$), where $\chi$ denotes the Euler-Poincaré characteristic. Then, as in [K4, Lemma 3.4], the above pairing is well defined.

By [KS, Proof of Proposition 3.4], for any $u \in W$,

$$\mathcal{E}xt_k^{O_X}(\mathcal{O}_{X^u}, \mathcal{O}_X) = 0 \quad \forall k \neq \ell(u). \quad (7)$$

Define the sheaf

$$\omega_{X^u} := \mathcal{E}xt^{\ell(u)}_{O_X}(\mathcal{O}_{X^u}, \mathcal{O}_X) \otimes \mathcal{L}(-2\rho),$$

which, by the analogy with the Cohen-Macaulay (for short CM) schemes of finite type, will be called the dualizing sheaf of $X^u$.

Now, set the sheaf on $\bar{X}$

$$\xi^u := \mathcal{L}(\rho)\omega_{X^u}$$

$$= \mathcal{L}(-\rho)\mathcal{E}xt^{\ell(u)}_{O_X}(\mathcal{O}_{X^u}, \mathcal{O}_X).$$

Then, as proved in [K4, Proposition 3.5], for any $u, w \in W$,

$$\langle [\xi^u], [\mathcal{O}_{X^w}] \rangle = \delta_{u,w}. \quad (8)$$

With these preliminaries, we are ready to prove the following result.

**3.7. Proposition.** With the notation as in the proof of Theorem 3.3, $Z = (X^-)^s$, if $e^v$ occurs in $e^{u_1} \cdots e^{u_s}$ with nonzero coefficient.

**Proof.** We give the proof in the case $s = 2$. The proof for general $s$ is similar.

For $u, v \in W$, express

$$e^ue^v = \sum_{\ell(w) = \ell(u) + \ell(v)} n_{u,v}^{w}e^w.$$

Express the product in topological $K$-theory $K^{top}(X)$ of $X = G_{min}/B$:

$$\psi_{\alpha}^{\mu} \psi_{\alpha}^{\nu} = \sum_{\ell(u) + \ell(v)} m_{u,v}^{\alpha} \psi_{\alpha}^{w},$$

where $\psi_{\alpha}^{\alpha} := *\tau_{\alpha}^{-1}$ ($\tau_{\alpha}$ being the Kostant-Kumar ‘basis’ of $K^{top}(X)$ as in [KK, Remark 3.14]) and $\{\psi_{\alpha}^{\alpha}\}_{\alpha \in W}$ is the corresponding ‘basis’ of $K^{top}(X) \cong \mathbb{Z} \otimes_{\mathbb{R}(H)} K^{top}(X)$, cf. [KK, Proposition 3.25]).

Then, by [KK, Proposition 2.30],

$$n_{u,v}^{w} = m_{u,v}^{w}, \quad \text{if } \ell(w) = \ell(u) + \ell(v). \quad (9)$$
Let $\Delta : X \to X \times X$ be the diagonal map. Then, by [K4, Proposition 4.1] and the identity $\boxtimes$, for any $u, v, w \in W$, $g_1, g_2 \in G^{\text{min}},$
\[m_{u,v}^w = \langle [\xi^u \otimes \xi^v], [\Delta_* O_{X_w}] \rangle = \langle [\xi^u \otimes \xi^v], [(g_1^{-1}g_2^{-1}) \cdot (\Delta_* O_{X_w})] \rangle,
\]
since $[(g_1^{-1}g_2^{-1}) \cdot \Delta_* O_{X_w}] = [\Delta_* O_{X_w}]$ as elements of $K_0(X \times X)$. Thus,
\[m_{u,v}^w = \langle [\xi^u \otimes \xi^v], [(g_1^{-1}g_2^{-1}) \cdot (\Delta_* O_{X_w})] \rangle (10) := \sum_i (-1)^i \chi(X \times X, \text{Tor}_i^O X \times X \otimes (\xi^u \otimes \xi^v, (g_1^{-1}g_2^{-1}) \cdot (\Delta_* O_{X_w}))).\]

Now, by definition, the support of $\xi^u$ is contained in $X^u$ and hence the support of the sheaf
\[S_i := \text{Tor}_i^O X \times X \otimes (\xi^u \otimes \xi^v, (g_1^{-1}g_2^{-1}) \cdot \Delta_* O_{X_w})\]
is contained in $X^u \times X^v \cap ((g_1^{-1}g_2^{-1}) \cdot \Delta(X_w))$, (11)
which is empty if
\[(g_1X^u) \cap (g_2X^v) \cap X_w = \emptyset. (12)\]
Thus, if the equation (12) is true, then the Tor sheaf $S_i = 0 \forall i \geq 0$. Thus, if the equation (12) is satisfied,
\[m_{u,v}^w = 0.\]
Now, assume that $\ell(w) = \ell(u) + \ell(v)$. Then, by the equation $[\boxtimes]$, $n_{u,v}^w = 0$, if the equation (12) is satisfied. But, since by assumption, $n_{u,v}^w \neq 0$, we see that
\[(g_1X^u) \cap (g_2X^v) \cap X_w \neq \emptyset, \text{ for any } g_1, g_2 \in G^{\text{min}}.\]
But since $G^{\text{min}} / (G^{\text{min}} \cap B^-) \xrightarrow{\sim} X^-$, we get the proposition. \qed

4. Tensor Product Decomposition for Affine Kac–Moody Lie Algebras

4.1. The Virasoro Algebra. We recall the definition of the Virasoro algebra and its basic representation theory, which we need. The Virasoro algebra $\text{Vir}$ has a basis $\{C, L_n : n \in \mathbb{Z}\}$ over $\mathbb{C}$ and the Lie bracket is given by
\[[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\delta_{m,-n}C \text{ and } [\text{Vir}, C] = 0.\]

Let $\text{Vir}_0 := \mathbb{C}L_0 \oplus \mathbb{C}C$. Then, a $\text{Vir}$ module $V$ is said to be a highest weight representation if there exists a $\text{Vir}_0$-eigenvector $v_0 \in V$ such that $L_nv_0 = 0$ for $n \in \mathbb{Z}_{>0}$ and $U(\bigoplus_{n < 0} \mathbb{C}L_n)v_0 = V$. Such a $V$ is said to have highest weight $\lambda \in \text{Vir}_0^*$ if $Xv_0 = \lambda(X)v_0$, for all $X \in \text{Vir}_0$. (It is easy to see that such a $v_0$ is unique up to a scalar multiple and hence $\lambda$ is unique.) The irreducible highest weight representations of $\text{Vir}$ are in 1-1 correspondence with elements of $\text{Vir}_0^*$ given by the highest weight. Denote the basis of $\text{Vir}_0^*$
If \( \lambda \in \Lambda \), then \( V_{\lambda} := \{ v \in V : X \cdot v = \mu(X)v \ \forall X \in \text{Vir}_0 \} \).

Define a Vir module \( V \) to be unitarizable if there exists a positive definite Hermitian form \((\cdot, \cdot)\) on \( V \) so that \((L_n v, w) = (v, L_{-n} w)\) for all \( n \in \mathbb{Z} \) and \((C v, w) = (v, C w)\). It is easy to see that if \( M \) is a Vir-submodule of \( V \), then \( M^\perp \) is also a submodule. Hence, any unitarizable representation of Vir is completely reducible. Note that for a unitarizable highest weight Vir-representation \( V \) with highest weight \( \lambda \), if \( v_o \) is a highest weight vector, then

\[
0 \leq (L_{-n} v_o, L_{-n} v_o) = (L_n L_{-n} v_o, v_o) = (2n\lambda(L_0) + \frac{1}{12}(n^3 - n)\lambda(C))(v_o, v_o) \tag{13}
\]

for all \( n > 0 \). Therefore, both \( \lambda(L_0) \) and \( \lambda(C) \) must be nonnegative real numbers.

**Lemma 4.1.** Let \( V \) be a unitarizable, highest weight (irreducible) representation of Vir with highest weight \( \lambda \).

(a) If \( \lambda(L_0) \neq 0 \), then \( V_{\lambda + nh} \neq 0 \), for any \( n \in \mathbb{Z}_+ \).

(b) If \( \lambda(L_0) = 0 \) and \( \lambda(C) \neq 0 \), then \( V_{\lambda + nh} \neq 0 \), for any \( n \in \mathbb{Z}_{> 1} \) and \( V_{\lambda + h} = 0 \).

(c) If \( \lambda(L_0) = \lambda(C) = 0 \), then \( V \) is one dimensional.

**Proof.** If \( \lambda(L_0) \neq 0 \), then by the equation \((13)\) (since both of \( \lambda(L_0) \) and \( \lambda(C) \in \mathbb{R}_+ \)), \( L_{-n} v_o \neq 0 \), for any \( n \in \mathbb{Z}_+ \).

If \( \lambda(L_0) = 0 \) and \( \lambda(C) \neq 0 \), then again by the equation \((13)\), \( L_{-n} v_o \neq 0 \), for any \( n \in \mathbb{Z}_{> 1} \). Also, \( L_{-1} v_o = 0 \).

If \( \lambda(L_0) = \lambda(C) = 0 \), then (by the equation \((13)\) again), \( L_{-n} v_o = 0 \), for any \( n \in \mathbb{Z}_{\geq 1} \). This shows that \( V \) is one dimensional. \( \square \)

### 4.2. Tensor product decomposition: A general method

Let \( g \) be the untwisted affine Kac-Moody Lie algebra associated to a finite dimensional simple Lie algebra \( \hat{\mathfrak{g}} \), i.e.,

\[
\mathfrak{g} = (\hat{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus Cd.
\]

Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \hat{\mathfrak{g}} \). Then,

\[
\mathfrak{h} := \hat{\mathfrak{h}} \otimes 1 \oplus \mathbb{C}c \oplus Cd
\]

is the standard Cartan subalgebra of \( g \). Let \( \delta \in \mathfrak{h}^* \) be the smallest positive imaginary root of \( \hat{g} \) (so that the positive imaginary roots of \( g \) are precisely \( \{n\delta, n \in \mathbb{Z}_{> 1}\} \)). Then, \( \delta \) is given by \( \delta|_{\hat{\mathfrak{h}} \oplus \mathbb{C}c} = 0 \) and \( \delta(d) = 1 \). For any \( \Lambda \in P_+ \), let \( P(\Lambda) \) be the set of weights of \( L(\Lambda) \) and let \( P^\alpha(\Lambda) \) be the set of \( \delta \)-maximal weights of \( L(\Lambda) \), i.e.,

\[
P^\alpha(\Lambda) = \{ \lambda \in \mathfrak{h}^* : \lambda \in P(\Lambda) \text{ but } \lambda + n\delta \not\in P(\Lambda) \text{ for any } n > 0 \}.
\]
For any $\lambda \in X(H)$, define the $\delta$-character of $L(\Lambda)$ through $\lambda$ by

$$c_{\Lambda, \lambda} = \sum_{n \in \mathbb{Z}} \dim L(\Lambda)_{\lambda + n\delta} e^{n\delta}.$$  

Since $\delta$ is $W$-invariant,

$$c_{\Lambda, \lambda} = c_{\Lambda, w\lambda}, \text{ for any } w \in W. \tag{14}$$

Moreover, $P^o(\Lambda)$ is $W$-stable. It is obvious that

$$\text{ch} L(\Lambda) = \sum_{\lambda \in P^o(\Lambda)} c_{\Lambda, \lambda} e^\lambda. \tag{15}$$

By [K3, Exercise 13.1.E.8], for any $\lambda \in P(\Lambda')$ and $\Lambda'' \in P_+$, $\Lambda'' + \lambda + \rho$ belongs to the Tits cone. Hence, there exists $v \in W$ such that $v^{-1}(\Lambda'' + \lambda + \rho) \in P_+$. Moreover, if $\Lambda'' + \lambda + \rho$ has nontrivial $W$-isotropy, then its isotropy group must contain a reflection (cf. [K3, Proposition 1.4.2(a)]). Thus, for such a $\lambda \in P(\Lambda')$, i.e., if $\Lambda'' + \lambda + \rho$ has nontrivial $W$-isotropy,

$$\sum_{w \in W} \varepsilon(w) e^{w(\Lambda'' + \lambda + \rho)} = 0. \tag{16}$$

Define

$$\bar{P}_+ := \{ \Lambda \in P_+ : \Lambda(d) = 0 \}.$$  

For any $m \in \mathbb{Z}_+$, let

$$P_+^{(m)} := \{ \Lambda \in P_+ : \Lambda(c) = m \},$$

and let

$$\bar{P}_+^{(m)} := P_+ \cap P_+^{(m)}.$$  

Then, $\bar{P}_+^{(m)}$ provides a set of representatives in $P_+^{(m)} \mod (P_+ \cap \mathbb{C}\delta)$.

For any $\Lambda, \Lambda', \Lambda'' \in P_+$, define

$$T_{\Lambda, \Lambda', \Lambda''}^* = \{ \lambda \in P^o(\Lambda') : \exists v_{\Lambda, \Lambda', \Lambda''} \in W \text{ and } S_{\Lambda, \Lambda', \Lambda''} \in \mathbb{Z} \text{ with }\lambda + \Lambda'' + \rho = v_{\Lambda, \Lambda', \Lambda''}(\Lambda + \rho) + S_{\Lambda, \Lambda', \Lambda''} \delta \}.$$  

Observe that since $\Lambda + \rho + n\delta \in P_{++}$ for any $n \in \mathbb{Z}$, such a $v_{\Lambda, \Lambda', \Lambda''}$ and $S_{\Lambda, \Lambda', \Lambda''}$ are unique by [K3, Proposition 1.4.2 (a), (b)] (if they exist). Also, observe that

$$T_{\Lambda, \Lambda', \Lambda''}^* = \emptyset, \text{ unless } \Lambda(c) = \Lambda'(c) + \Lambda''(c) \text{ and } \Lambda' + \Lambda'' - \Lambda \in Q, \tag{17}$$

where $Q$ is the root lattice of $g$.

**Proposition 4.2.** For any $\Lambda'$ and $\Lambda'' \in P_+$,

$$\text{ch} \left( L(\Lambda') \otimes L(\Lambda'') \right) = \sum_{\Lambda \in P_+^{(m)}} \text{ch} \left( L(\Lambda) \left( \sum_{\lambda \in T_{\Lambda, \Lambda', \Lambda''}^*} \varepsilon(v_{\Lambda, \Lambda', \Lambda''}) e^{S_{\Lambda, \Lambda', \Lambda''}} \right) \right),$$

where $m := \Lambda'(c) + \Lambda''(c)$.  

Moreover, $\sum_{\Lambda \in T^{N',\Lambda''}_A} \varepsilon(v_{\Lambda,\Lambda''}) c_{\Lambda',\Lambda} e^{S_{\Lambda,\Lambda''},\lambda \delta}$ is the character of a unitary representation (though, in general, not irreducible) of the Virasoro algebra Vir with central charge

$$\dim \hat{g} \cdot \left( \frac{m'}{m' + g} + \frac{m''}{m'' + g} - \frac{m}{m + g} \right),$$

where $m' := \Lambda'(e), m'' := \Lambda''(e)$ and $g$ is the dual Coxeter number of $\hat{g}$. 

**Proof.** By the Weyl-Kac character formula (cf. [K3, Theorem 2.2.1]) and the identity (15), for any $\Lambda', \Lambda'' \in P_+$,

$$\left( \sum_{w \in W} \varepsilon(w) e^{w\rho} \right) \cdot \chi \left( L(\Lambda') \right) \cdot \chi \left( L(\Lambda'') \right)$$

$$= \left( \sum_{\lambda \in P^+(\Lambda')} c_{\Lambda',\lambda} e^\lambda \right) \cdot \left( \sum_{w \in W} \varepsilon(w) e^{w(\Lambda'' + \lambda + \rho)} \right)$$

$$= \sum_{\lambda \in P^+(\Lambda')} c_{\Lambda',\lambda} \sum_{w \in W} \varepsilon(w) e^{w(\Lambda'' + \lambda + \rho)}, \text{ by (14)}$$

$$= \sum_{\Lambda \in P^+(\Lambda')} \sum_{\lambda \in T^{N',\Lambda''}_A} c_{\Lambda',\lambda} \sum_{w \in W} \varepsilon(w) e^{w(v_{\Lambda,\Lambda''},\lambda + \Lambda + \rho + S_{\Lambda,\Lambda''},\lambda \delta)}, \text{ by (13)}$$

$$= \sum_{\Lambda \in P^+(\Lambda')} \sum_{\lambda \in T^{N',\Lambda''}_A} \varepsilon(v_{\Lambda,\Lambda''},\lambda) e^{w(\Lambda + \rho)} e^{S_{\Lambda,\Lambda''},\lambda \delta}$$

$$= \sum_{\Lambda \in P^+(\Lambda')} \sum_{w \in W} \varepsilon(w) e^{w(\Lambda + \rho)} \sum_{\lambda \in T^{N',\Lambda''}_A} \varepsilon(v_{\Lambda,\Lambda''},\lambda) c_{\Lambda',\lambda} e^{S_{\Lambda,\Lambda''},\lambda \delta}.$$

Thus,

$$\chi \left( L(\Lambda') \otimes L(\Lambda'') \right) = \sum_{\Lambda \in P^+(\Lambda')} \chi \left( L(\Lambda') \right) \left( \sum_{\lambda \in T^{N',\Lambda''}_A} \varepsilon(v_{\Lambda,\Lambda''},\lambda) c_{\Lambda',\lambda} e^{S_{\Lambda,\Lambda''},\lambda \delta} \right).$$

To prove the second part of the proposition, use [KR, Proposition 10.3]. This proves the proposition. \( \square \)

4.3. **Remark.** For an affine Kac-Moody Lie algebra $\mathfrak{g}$, if we consider the tensor product decomposition of $L(\Lambda') \otimes L(\Lambda'')$ with respect to the derived subalgebra $\mathfrak{g}'$ (i.e., without the $d$-action), then the components $L(\Lambda)$ are precisely of the form $\Lambda \in \Lambda' + \Lambda'' + \hat{Q}$, where $\hat{Q}$ is the root lattice of $\hat{g}$ (cf. [KW]). Thus, the determination of the eigen semigroup and the saturated eigen semigroup is fairly easy for $\mathfrak{g}'$.

Let $\theta = \sum_{i=1}^{\ell} h_i \alpha_i$ be the highest root of $\hat{g}$ (with respect to a choice of the positive roots), written as a linear combination of the simple roots
\{\alpha_1, \ldots, \alpha_\ell\} of \mathfrak{g}. Let

\[ S := \left\{ \sum_{i=0}^{\ell} n_i \alpha_i : n_i \geq 0 \text{ for any } i \text{ and } 0 \leq n_i < h_i \text{ for some } 0 \leq i \leq \ell \right\}, \]

where \( h_0 := 1 \).

**Proposition 4.4.** Let \( \mathfrak{g} \) be an untwisted affine Kac-Moody Lie algebra as above. Then, for any \( \Lambda \in P_+ \) with \( \Lambda(\ell) > 0 \),

\[ P^0(\Lambda)_+ = S(\Lambda) \cap P_+, \]

where \( P^0(\Lambda)_+ := P^0(\Lambda) \cap P_+ \) and \( S(\Lambda) = \{ \Lambda - \beta : \beta \in S \} \).

**Proof.** Take \( \lambda \in S(\Lambda) \). Then, for any \( n \geq 1 \),

\[ \Lambda - (\lambda + n\delta) = \left( \sum_{i=0}^{\ell} n_i \alpha_i \right) - n\delta = (n_0 - n)\alpha_0 + \sum_{i=1}^{\ell} (n_i - nh_i)\alpha_i, \]

since \( \alpha_0 := \delta - \theta \). Now, the coefficient of some \( \alpha_i \) in the above sum is negative, for any positive \( n \), since \( \lambda \in S(\Lambda) \). Thus, \( \lambda + n\delta \) could not be a weight of \( L(\Lambda) \) for any positive \( n \). Therefore, if \( \lambda \in P(\Lambda) \cap S(\Lambda) \), then it is \( \delta \)-maximal.

By [Kac, Proposition 12.5(a)], if \( \Lambda(\ell) \neq 0 \), then \( S(\Lambda) \cap P_+ \subset P(\Lambda) \). Therefore, \( S(\Lambda) \cap P_+ \subset P^0(\Lambda)_+ \).

Conversely, take \( \lambda \in P^0(\Lambda)_+ \). Then, \( \lambda \in P(\Lambda) \cap P_+ \) and \( \lambda + \delta \notin P(\Lambda) \). Express \( \lambda = \Lambda - n_0\alpha_0 - \sum_{i=1}^{\ell} n_i \alpha_i \), for some \( n_i \in \mathbb{Z}_+ \). Then,

\[ \lambda + \delta = \Lambda - (n_0 - 1)\alpha_0 - \sum_{i=1}^{\ell} (n_i - h_i)\alpha_i. \]

Again applying [Kac, Proposition 12.5(a)], \( \lambda + \delta \notin P(\Lambda) \) if and only if \( \lambda + \delta \notin \Lambda \), i.e., for some \( 0 \leq i \leq \ell \), \( n_i < h_i \). Thus, \( \lambda \in S(\Lambda) \). This proves the proposition. \( \square \)

5. \( A_1^{(1)} \) Case

In this section, we consider \( \mathfrak{g} = \widehat{\mathfrak{sl}_2} = (\bigoplus_{n \in \mathbb{Z}} \mathbb{C}l^n \otimes \mathfrak{sl}_2) \oplus \mathbb{C}c \oplus \mathbb{C}d \). In this case \( \mathfrak{h}^* = \mathbb{C}\alpha \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0 \), where \( \alpha \) is the simple root of \( \mathfrak{sl}_2 \) and \( \Lambda_0 \mid_{\mathfrak{h}^* \oplus \mathbb{C}d} = 0 \) and \( \Lambda_0(c) = 1 \). Then, \( \Lambda_0 \) is a zeroeth fundamental weight. The simple roots of \( \widehat{\mathfrak{sl}_2} \) are \( \alpha_0 := \delta - \alpha \) and \( \alpha_1 := \alpha \). The simple coroots are \( \alpha_0^\vee := c - \alpha_1^\vee \) and \( \alpha_1^\vee := \alpha_1^\vee \). It is easy to see that an element of \( \mathfrak{h}^* \) of the form \( m\Lambda_0 + \frac{j}{2}\alpha \) belongs to \( P_+ \) if and only if \( m, j \in \mathbb{Z}_+ \) and \( m \geq j \).

Specializing Proposition 4.4 to the case of \( \mathfrak{g} = \widehat{\mathfrak{sl}_2} \), we get the following.

5.1. **Corollary.** For \( \mathfrak{g} = \widehat{\mathfrak{sl}_2} \) and \( \Lambda = m\Lambda_0 + \frac{j}{2}\alpha \in P_+ \),

\[ P^0(\Lambda)_+ = \left\{ \Lambda - k\alpha, \Lambda - l(\delta - \alpha) : k, l \in \mathbb{Z}_+ \text{ and } k \leq \frac{j}{2}, l \leq \frac{m - j}{2} \right\}. \]
Proof. The corollary follows from Proposition 4.4 since \( m_1\Lambda_0 + \frac{m_2}{2}\alpha + m_3\delta \) belongs to \( P_+ \) if and only if \( m_1, m_2 \in \mathbb{Z}_+ \) and \( m_1 \geq m_2 \).

Let \( \pi \) be the projection \( \mathfrak{h}^* = \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha \oplus \mathbb{C}\delta \to \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha \).

5.2. Lemma. Let \( \mathfrak{g} = \hat{\mathfrak{sl}_2} \). For \( \Lambda = m\Lambda_0 + \frac{j}{2}\alpha \in P_+ \) (i.e., \( m, j \in \mathbb{Z}_+ \) and \( m \geq j \)) such that \( m > 0 \),

\[
\pi(P^\alpha(\Lambda)) = \{\Lambda + k\alpha : k \in \mathbb{Z}\}. \tag{19}
\]

Moreover, for any \( k \in \mathbb{Z} \), let \( n_k \) be the unique integer such that \( \Lambda + k\alpha + n_k\delta \in P^\alpha(\Lambda) \). Then, writing \( k = qm + r, 0 \leq r < m \), we have:

\[
n_k = n_r - q(k + r + j). \tag{20}
\]

Proof. The assertion [19] follows from the identity (18) together with the action of the affine Weyl group \( W \cong \hat{W} \times (\mathbb{Z}_0^\vee) \) on \( \mathfrak{h}^* \), where \( \hat{W} \) is the Weyl group of \( \mathfrak{sl}_2 \) and \( \mathbb{Z}_0^\vee \) acts on \( \mathfrak{h}^* \) via:

\[
T_{n\alpha}(\mu) = \mu + n\mu(c)\alpha - [n\mu(\alpha^\vee) + n^2\mu(c)]\delta, \text{ for } n \in \mathbb{Z}, \mu \in \mathfrak{h}^*. \tag{21}
\]

Since \( P^\alpha(\Lambda) \) is \( W \)-stable, the identity (20) can be established from the action of the affine Weyl group element \( T_{-q\alpha^\vee} \) on \( \Lambda + k\alpha + n_k\delta \). \( \square \)

The value of \( n_r \) for \( 0 \leq r < m \) can be determined from the identity (18) by applying \( T_{\alpha^\vee}, T_{\alpha^\vee} \cdot T_1 \) to \( \Lambda - k\alpha \) and applying \( 1, T_{\alpha^\vee} \cdot T_1 \) to \( \Lambda - l(\delta - \alpha) \), where \( s_1 \) is the nontrivial element of \( \hat{W} \). We record the result in the following lemma.

5.3. Lemma. With the notation as in the above lemma, the value of \( n_r \) for any integer \( 0 \leq r < m \) is given by

\[
n_r = \begin{cases} 
-r, & \text{for } 0 \leq r \leq m - j \\
-m - j - 2r & \text{for } m - j \leq r < m.
\end{cases}
\]

5.4. Lemma. Take the following elements in \( P_+ \):

\[
\Lambda = m\Lambda_0 + \frac{j}{2}\alpha, \quad \Lambda' = m'\Lambda_0 + \frac{j'}{2}\alpha, \quad \Lambda'' = m''\Lambda_0 + \frac{j''}{2}\alpha,
\]

where \( m := m' + m'' \) and we assume that \( m' > 0 \). Then,

\[
\pi \left( T_\Lambda^{\Lambda', \Lambda''} \right) = \{\Lambda + k\alpha : k \in \mathbb{Z}, k \equiv \frac{1}{2}(j - j' - j'') \mod 2M, \text{ or } k \equiv -\frac{1}{2}(j + j' + j'') - 1 \mod M\},
\]

where \( M := m + 2 \). In particular, by the equation (17), \( T_\Lambda^{\Lambda', \Lambda''} \) is nonempty if and only if \( \frac{j - j' - j''}{2} \in \mathbb{Z} \).

Moreover, for \( \lambda = \Lambda' + k\alpha + n_k\delta \in T_\Lambda^{\Lambda', \Lambda''} \),

\[
v_{\Lambda, \Lambda', \Lambda''} = \begin{cases} 
T_{\frac{j'}{M}}(\Lambda'_{\alpha^\vee}) & \text{if } k \equiv \frac{1}{2}(j - j' - j'') \mod M \\
T_{-\frac{j'}{M}}(\Lambda''_{\alpha^\vee}) & \text{if } k \equiv -\frac{1}{2}(j + j' + j'') - 1 \mod M
\end{cases},
\]

where \( M = 2M + 1 \) and \( \Lambda' \) and \( \Lambda'' \) are in \( P_+ \).
where \( T_{\alpha} \) is defined by the equation (21). Further,
\[
S_{\Lambda, \Lambda', \lambda} = n_k + \frac{(k - \frac{1}{2} (j - j') (j'' + 1))}{M}.
\]

Proof. Follows from the fact that \( W = W \times \mathbb{Z} \alpha \) and that \( \rho = 2\Lambda_0 + \frac{1}{2}\alpha \).

We have the following very crucial result.

**Proposition 5.5.** Fix \( \Lambda, \Lambda' \) and \( \Lambda'' \) as in Lemma 5.4 and assume that \( \frac{j - j'}{2} \in \mathbb{Z} \) and both of \( m', m'' > 0 \). Then, the maximum of \( \left\{ S_{\Lambda, \Lambda', \lambda} : \lambda \in T_{\Lambda}^{N', \Lambda''} \text{ and } \varepsilon(v_{\Lambda, \Lambda', \lambda}) = 1 \right\} \) is achieved precisely when \( \pi(\alpha) = \Lambda' + \frac{1}{2} (j - j') (\alpha) \).

Proof. By Lemma 5.4 and the explicit description of the length function of \( T_{\alpha} \) (cf. [K3, Exercise 13.1.E.3]),
\[
\pi\{ \lambda \in T_{\Lambda}^{N', \Lambda''} : \varepsilon(v_{\Lambda, \Lambda', \lambda}) = 1 \} = \{ \Lambda' + k_1 \alpha : l \in \mathbb{Z} \},
\]
where \( M := m + 2 \) and \( k_1 := \frac{j - j'}{2} + lM \). Take \( \lambda = \Lambda' + k_1 \alpha \in \pi(T_{\Lambda}^{N', \Lambda''}) \) for \( l \in \mathbb{Z} \). Write \( k_l = q_lm' + r_l \) for \( q_l \in \mathbb{Z} \) and \( 0 \leq r_l < m' \). Then, by Lemmas 5.2, 5.3 and 5.4 for \( \lambda = \Lambda' + k_1 \alpha \) (setting \( J := \frac{j - j'}{2} \)),
\[
S_{\Lambda, \Lambda', \lambda} = n_{r_l} - \frac{(J + j' + lM + r_l)(J + lM - r_l)}{m'} + l(lM + 1) j
\]
\[
= l^2 M(1 - \frac{M}{m'}) + l(1 + l - \frac{M(j - j' - j'' - 2)}{m'}) - \frac{(j - j')^2 - j'^2}{4m'} + \frac{r_l^2}{m'} + \frac{r_l j'}{m'} + n_{r_l}
\]
\[
= l^2 M(1 - \frac{M}{m'}) + l(1 + l - \frac{M(j - j' - j'')}{m'}) - \frac{(j - j')^2 - j'^2}{4m'} + p(k_l),
\]
where
\[
p(k_l) := \frac{r_l^2}{m'} + \frac{r_l j'}{m'} + n_{k_l}.
\]

Let \( P = P_{m', j'} : \mathbb{R} \to \mathbb{R} \) be the following function:
\[
P(s) := \begin{cases} 
\frac{(s - \frac{m'}{2})^2}{4m'} - \frac{(j')^2}{4m'}, & \text{if } |s - \frac{m'}{2}k| \leq \frac{j'}{2} \text{ for some } k \in \mathbb{Z} \\
\frac{(s - \frac{m'}{2})^2}{4m'} - \frac{(m' - j')^2}{4m'}, & \text{if } |s - \frac{m'}{2}k| \leq \frac{m' - j'}{2} \text{ for some } k \in \mathbb{Z} + 1.
\end{cases}
\]

Let \( k_s \in \mathbb{Z} \) be such a \( k \). (Of course, \( k_s \) depends upon \( m' \) and \( j' \).)

**Claim 5.6.** \( P(s) = p(s - j') \) for \( s \in \frac{j'}{2} + \mathbb{Z} \).

Proof. Clearly, both of \( P \) and \( p \) are periodic with period \( m' \). So, it is enough to show that \( P(s) = p(s - j') \) for \( s - j' \) equal to any of the integral points of the interval \([ -j', m' - j'] \). By Lemma 5.3 and the identity (20), for any integer \(-j' \leq r \leq 0\),
\[
p(r) = \frac{1}{m'} r (r + j'),
\]
and for any integer \(0 \leq r \leq m' - j'\),
\[
p(r) = \frac{r(r + j')}{m'} - r.
\]
From this, the claim follows immediately. \(\square\)

Fix \(m' > 0\). Let
\[
I := \{(t, j', m'', j'', j) \in \mathbb{R}^5 : 0 \leq j' \leq m', \quad 0 \leq j'' \leq m'', \quad 0 \leq j \leq m' + m''\}.
\]
Define \(F : I \rightarrow \mathbb{R}\) by
\[
F : (t, j', m'', j'', j) \mapsto t^2 M(1 - \frac{M}{m'}) + t(j - \frac{M}{m'}j'') + \frac{(j')^2 - (j - j'')^2}{4m'} + P\left(\frac{1}{2} (j - j'') + tM\right).
\]
Thus, \(F\) is a continuous, piecewise smooth function with failure of differentiability along the set
\[
\{(t, j', m'', j'', j) \in I : \frac{1}{2}(j + j' - j'') + tM \in m'\mathbb{Z}\}.
\]

Claim 5.7. Let \(\Delta(t) = \Delta(t, j', m'', j'', j) \coloneqq F(t+1, j', m'', j'', j) - F(t, j', m'', j'', j)\). Then, on \(I\),
(1) \(\Delta\) is a nonincreasing function of \(t\)
(2) \(\Delta\) is increasing with respect to \(j''\)
(3) \(\Delta\) is nonincreasing in \(j\)
(4) \(\Delta(0)\) is decreasing in \(m''\)
(5) \(\Delta(-1)\) is nondecreasing in \(m''\).

Proof. We compute and give bounds for the partial derivatives of \(\Delta\), where they exist.
\[
\Delta(t) = 2tM(1 - \frac{M}{m'}) + ((j + M)(1 - \frac{M}{m'}) + 1 + \frac{M}{m'}j'')
+ P(tM + M + \frac{1}{2}(j - j'')) - P(tM + \frac{1}{2}(j - j'')).
\]
Hence,
\[
\partial_t \Delta(t) = 2M(1 - \frac{M}{m'}) + M(P'(tM + M + \frac{1}{2}(j - j'')) - P'(tM + \frac{1}{2}(j - j''))) \]
\[
= 2M(1 - \frac{M}{m'}) + 2\frac{M}{m'}(M - \frac{m'}{k_1} + \frac{m'}{2} k_0)
\]
\[
= 2M(1 - \frac{k_1 - k_0}{2}),
\]
where \(k_1 := k_{(t+1)M+\frac{1}{2}(j-j'')}\) and \(k_0 := k_{tM+\frac{1}{2}(j-j'')}\). Since \(2 \leq k_1 - k_0\), we see that \(\partial_t \Delta \leq 0\), wherever \(\partial_t \Delta\) exists. Since \(\Delta\) is continuous everywhere
and differentiable on all but a discrete set, $\Delta$ is nonincreasing in $t$.

$$\partial_j \Delta(t) = \frac{M}{m'} - \frac{1}{2} \left( P'(tM + M + \frac{1}{2}(j - j'')) - P'(tM + \frac{1}{2}(j - j'')) \right).$$

Now, $|P'| \leq 1$, so $\frac{M}{m'} + 1 \geq \partial_j \Delta \geq \frac{M}{m'} - 1 = \frac{m'' + 2}{m'} > 0$.

For (3):

$$\partial_j \Delta(t) = 1 - \frac{M}{m'} + \frac{1}{2} \left( M - \frac{m'}{2}k_1 + \frac{m'}{2}k_0 \right) = 1 - \frac{k_1 - k_0}{2} \leq 0.$$ (4) and (5) follow from the following calculation:

$$\partial_{m''} \Delta = 2t(1 - 2\frac{M}{m'}) + (1 - 2\frac{M}{m'} + \frac{1}{m''}(j'' - j)) + (t + 1)P'(tM + M + \frac{1}{2}(j - j'')) - tP'(tM + \frac{1}{2}(j - j'')).$$

Hence,

$$\partial_{m''} \Delta(0) = 1 - 2\frac{M}{m'} + \frac{1}{m''}(j'' - j) + P'(M + \frac{1}{2}(j - j''))$$

$$\leq 1 - 2\frac{M}{m'} + \frac{m''}{m'} + 1$$

$$= -\frac{m'' - 4}{m'} < 0,$$

and

$$\partial_{m''} \Delta(-1) = -2(1 - 2\frac{M}{m'}) + (1 - 2\frac{M}{m'} + \frac{1}{m''}(j'' - j)) + P'(-M + \frac{1}{2}(j - j''))$$

$$= -1 + 2\frac{M}{m'} + \frac{1}{m''}(j'' - j) + P'(-M + \frac{1}{2}(j - j''))$$

$$= -1 + 2\frac{M}{m'} + \frac{1}{m''}(j'' - j) - 2\frac{M}{m'} + \frac{1}{m''}(j - j'') - k_0$$

$$= -1 - k_0.$$ Note that $k_0 \leq -1$ since $-\frac{(j - j'')}{2} - M < -\frac{m'}{2}$. Thus, $\partial_{m''} \Delta(-1) \geq 0$.

**Claim 5.8.** The maximum of $F = F(-, j', m'', j'', j) : \mathbb{Z} \to \mathbb{R}$ occurs at 0.

**Proof.** We show that $\Delta(-1) > 0 > \Delta(0)$. Since $\Delta$ is nonincreasing in $t$, it would follow that $F(0) > F(t)$ for all $t \in \mathbb{Z}_{\neq 0}$.

Let us begin with $\Delta(-1)$. By the previous claim, $\Delta(-1)$ is as small as possible when $m'' = 1$, $j'' = 0$, and $j = m' + 1$. So, let us compute with these values:
\[ \Delta(-1) \geq \frac{6}{m'} + 1 + P\left(\frac{1}{2}m' + \frac{1}{2}\right) - P\left(-2 - \frac{1}{2}m' - \frac{1}{2}\right) \\
= \frac{6}{m'} + 1 + \left(\frac{3m' + \frac{1}{2} - \frac{1}{2}m'k_1}{m'}\right) - \left(\frac{2 + \frac{1}{2}m' + \frac{1}{2} + \frac{1}{2}m'k_0}{m'}\right) \\
+ \begin{cases} \frac{m'}{4} - \frac{j'}{2} & \text{if } k_0 \text{ odd, } k_1 \text{ even} \\
0 & \text{if } k_1 - k_0 \text{ even} \\
\frac{j'}{2} - \frac{m'}{4} & \text{if } k_1 \text{ odd, } k_0 \text{ even.} \end{cases} \]

Note that for \( m' \geq 5 \), the possible values of \((k_1, k_0)\) are \((1, -1)\); \((1, -2)\); or \((2, -2)\). So, the result, that \( \Delta(-1) > 0 \), is established by considering such pairs directly and by cases for smaller \( m' \).

For \( \Delta(0) \), we take \( m'' = 1, j'' = 1, \) and \( j = 0 \).

\[ \Delta(0) = \left(\frac{-3(3 + m')}{m'}\right) + 1 + \left(\frac{3 + m'}{m'}\right) + P\left(\frac{1}{2} + 2 + m'\right) - P\left(-\frac{1}{2}\right) \\
= 1 - \frac{2(3 + m')}{m'} + P\left(\frac{1}{2} + 2 + m'\right) - P\left(-\frac{1}{2}\right) \\
= 1 - \frac{2(3 + m')}{m'} + \left(\frac{\frac{1}{2} + 2 + m' - \frac{1}{2}m'k_1}{m'}\right)^2 - \left(\frac{\frac{1}{2} + \frac{1}{2}m'k_0}{m'}\right)^2 \\
+ \begin{cases} \frac{m'}{4} - \frac{j'}{2} & \text{if } k_0 \text{ odd, } k_1 \text{ even} \\
0 & \text{if } k_1 - k_0 \text{ even} \\
\frac{j'}{2} - \frac{m'}{4} & \text{if } k_1 \text{ odd, } k_0 \text{ even.} \end{cases} \]

For \( m' \geq 5 \), the possible values of \((k_1, k_0)\) are \((3, -1)\); \((3, 0)\); or \((2, 0)\). So, again the result, that \( \Delta(0) < 0 \), is established by considering such pairs directly and by cases for smaller \( m' \).

This completes the proof of the proposition.

\[ \square \]

**Remark 5.9.** We have shown that \( F(l, j', m'', j'', j) = S_{\lambda, \Lambda', \lambda} \) for integral values of \( l \). If \( l \) is not an integer, then \( \lambda_l := \Lambda' + (lM + J)\alpha \) may not be in \( \pi(T^{\Lambda', \Lambda''}_\lambda) \), in which case \( S_{\lambda, \Lambda', \lambda} \) is not defined. On the other hand, if \( \lambda_l \in \pi(T^{\Lambda', \Lambda''}_\lambda) \), we note that the equality \( F(l, j', m'', j'', j) = S_{\lambda, \Lambda', \lambda} \) holds, as can be seen by letting \( k_l = lM - \frac{1}{2}(j + j' + j'') - 1 \) in the above proof.

Now, let us apply the same analysis to the case that \( \epsilon(v_{\lambda, \Lambda', \lambda}) = -1 \). By Lemma 5.4, this corresponds to \( k_l = -\frac{1}{2}(j + j' + j'') - 1 + lM \). For \( \lambda = \Lambda' + k_l \alpha \), let us denote the function \( S_{\lambda, \Lambda', \lambda} \) by \( G_{\mathbb{Z}}(l) = G_{\mathbb{Z}}(l, j', m'', j'', j) \). Thus, \( G_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z} \).

5.10. **Lemma.** Define the function \( G = G(-, j', m'', j'', j) : \mathbb{R} \to \mathbb{R} \) by

\[ G(t, j', m'', j'', j) = F(t - \frac{j + 1}{M}, j', m'', j'', j). \]

Then, \( G_{|\mathbb{Z}} = G_{\mathbb{Z}} \).

Hence, \( S_{\lambda, \Lambda', \lambda} \) has a maximum when \( l = 0 \) or \( l = 1 \).
Proof. By the proof of Proposition 5.5 and Remark 5.9, \( S_{\Lambda,\Lambda''+\lambda} = F(l) \), for \( \lambda = \Lambda + k\alpha. \) Since \( \lambda = \Lambda' + \left( -\frac{1}{2} (j + j') + 1 \right) lM \alpha, \) by Proposition 5.3, \( S_{\Lambda,\Lambda''+\lambda} = F(l - \frac{j + j'}{M}). \) This proves the lemma. \( \square \)

From Lemma 5.10 and the definition of \( F \), it is easy to see that
\[
G(1-t,m',m'',m''-j',m'+m''-j) + \frac{1}{2} (j' + j' - j) = G(t,m',m'',j),
\]
for any \( t \in \mathbb{R} \). Hence, if the maximum of \( G \) occurs at 1, it is equal to
\[
G(0,m'-j',m'',m''-j',m'+m''-j) + \frac{1}{2} (j' + j'' - j).
\]
(22)

We also record the following identity, which is easy to prove from the definition of \( F \).
\[
F(0,m'-j',m'',m''-j',m'+m''-j) + \frac{1}{2} (j' + j'' - j) = F(0,j',m'',j'',j).
\]
(24)

As a corollary of Proposition 5.5 and Lemma 5.10, we get the following ‘Non-Cancellation Lemma’.

5.11. Corollary. Let \( \Lambda, \Lambda', \Lambda'' \) be as in Proposition 5.5 and let
\[
\mu_A^{\Lambda',\Lambda''} := \max \left\{ S_{\Lambda,\Lambda',\lambda} : \lambda \in \mathcal{T}_{\Lambda,\Lambda'}^{\Lambda''} \text{ and } \varepsilon(v_{\Lambda,\Lambda',\lambda}) = 1 \right\},
\]
\[
\bar{\mu}_A^{\Lambda',\Lambda''} := \max \left\{ S_{\Lambda,\Lambda',\lambda} : \lambda \in \mathcal{T}_{\Lambda,\Lambda'}^{\Lambda''} \text{ and } \varepsilon(v_{\Lambda,\Lambda',\lambda}) = -1 \right\}.
\]
Assume that \( \mu_A^{\Lambda',\Lambda''} = \bar{\mu}_A^{\Lambda',\Lambda''} \). Then,
\[
\mu_A^{\Lambda',\Lambda''} \neq \bar{\mu}_A^{\Lambda',\Lambda''}.
\]

Proof. We proceed in two cases:

Case I. Suppose the maximum \( \bar{\mu}_A^{\Lambda',\Lambda''} \) occurs when \( \pi(\lambda) = \Lambda' - \left( \frac{1}{2} (j + j') + 1 \right) lM \alpha \) (cf. Lemma 5.10). This means that the \( \delta \)-maximal weights of \( L(\Lambda') \) through \( \Lambda' - \left( \frac{1}{2} (j + j') + 1 \right) lM \alpha \) and through \( \Lambda' + \frac{1}{2} (j + j'' - j') \alpha \) have the same \( \delta \) coordinate (cf. Proposition 5.5). By (next) Lemma 5.12 we know that this occurs if and only if one of the following two conditions are satisfied:

1. \( \left| \frac{1}{2} (j - j'') \right| \leq \frac{j'}{2} \) and \( \frac{1}{2} (j + j'') + 1 \leq \frac{j'}{2} \), or
2. \( \frac{1}{2} (j + j'') + 1 = \frac{1}{2} (j - j'') \).

The latter is clearly impossible, while the former condition is fulfilled precisely when \( \frac{1}{2} (j + j'') + 1 \leq \frac{j'}{2} \).

So, for the equality \( \mu_A^{\Lambda',\Lambda''} = \bar{\mu}_A^{\Lambda',\Lambda''} \) in this case, the necessary and sufficient condition is:
\[
\frac{1}{2} (j + j'') + 1 \leq \frac{j'}{2}.
\]
(25)
Lemma 5.12. Suppose the maximum $\mu_{\Lambda,\Lambda''}^{\Lambda'}$ occurs when $\pi(\lambda) = \Lambda' - \left(\frac{1}{2}(j + j^* + j'') + 1 - M\right)\alpha$. Then, by the identities (23) and (24), we get
\[
G(0, m' - j', m'', m'' - j', m' + m'' - j) = F(0, m' - j', m'', m'' - j', m' + m'' - j).
\]
(26)

So, from the case I, we get in this case II, $\mu_{\Lambda,\Lambda''}^{\Lambda'} = \mu_{\Lambda}^{\Lambda',\Lambda''}$ if and only if
\[
\frac{1}{2} \left( (m' + m'' - j) + (m'' - j') \right) + 1 \leq \frac{1}{2} (m' - j').
\]
(27)

So, if either of the inequalities (25) or (27) is satisfied, then none of them can be satisfied for the triple $(\Lambda, \Lambda', \Lambda'')$ replaced by $(\Lambda, \Lambda'', \Lambda')$. This proves the corollary. □

Lemma 5.12. Suppose $\Lambda' - \left(\frac{1}{2}(j + j^* + j'') + 1\right)\alpha + n_1\delta$ and $\Lambda' + \frac{1}{2}(j - j' - j'')\alpha + n_2\delta$ are $\delta$-maximal weights of $L(\Lambda')$. Then $n_1 = n_2$ if and only if
\[
\left| \frac{1}{2} (j - j'') \right| \leq \frac{j'}{2} \quad \text{and} \quad \frac{1}{2} (j + j'') + 1 \leq \frac{j'}{2},
\]
or $\frac{1}{2} (j + j'') + 1 = \frac{1}{2} (j - j')$.

Proof. Fix an integer $n$ and consider the set $P_n = \{ \nu \in P(\Lambda') : \Lambda' - \nu = k\alpha + n\delta, k \in \mathbb{Z} \}$. We give a description of $P_n \cap P^0(\Lambda')$. Clearly, $P_n = \{ \lambda, \lambda - \alpha, \ldots, \lambda - (\lambda, \alpha')\alpha \}$ for some $\lambda = \lambda_n$ and that this $\lambda$ is uniquely determined by $n$ (cf. [K3, Exercise 2.3.E.2]). Suppose that some $\mu \in P_n$ is not $\delta$-maximal, then none of $\{\mu, \mu - (\mu, \alpha')\alpha\}$ are $\delta$-maximal, since if $\mu + k\delta \in P(\Lambda')$, then the whole string $\{\mu + k\delta, \ldots, \mu + k\delta - (\mu, \alpha')\alpha\} \subset P(\Lambda')$. In particular, if $\lambda - \alpha$ is $\delta$-maximal, then so is $\lambda$. Hence, $g_{\delta - \alpha}L(\Lambda') = 0$, and $g_{\alpha}L(\Lambda') = 0$. Therefore, $\lambda$ is the highest weight $\Lambda'$. Thus, $P_n \cap P^0(\Lambda')$ is either empty, or $\lambda = \Lambda'$ (in the case that $n = 0$), or the set $\{\lambda, s_1\lambda\}$. From this and Corollary 5.1, the lemma follows easily. □

6. Saturation factor for the $A^{(1)}_1$ case

We assume that $\mathfrak{g} = \widehat{sl}_2$ in this section.

Definition 6.1. Let $\Lambda' \in P(\Lambda', \Lambda'' \in P(\Lambda'' \in P(L(\Lambda', \Lambda'')$ and $\Lambda \in P(L(\Lambda')$ and $\Lambda$. Then, we call $L(\Lambda + n\delta)$ the $\delta$-maximal component of $L(\Lambda') \otimes L(\Lambda'')$ through $\Lambda$ if $L(\Lambda + n\delta)$ is a submodule of $L(\Lambda') \otimes L(\Lambda'')$ but $L(\Lambda + m\delta)$ is not a component for any $m > n$.

Theorem 6.2. Let $\Lambda', \Lambda''$, $\Lambda$ be as in Proposition 5.3. Then, $L(\Lambda + n\delta)$ is a $\delta$-maximal component of $L(\Lambda') \otimes L(\Lambda'')$ if $n = \min(n_1, n_2)$, where $n_1$ is such that $\Lambda - \Lambda'' + n_1\delta \in P^0(\Lambda')$ and $n_2$ is such that $\Lambda - \Lambda' + n_2\delta \in P^0(\Lambda'')$.

Proof. This follows immediately by combining Propositions 5.2, 5.3, and Lemma 5.3. □

Lemma 6.3. Fix a positive integer $N$. Let $\Lambda \in \widehat{P}_+$ and let $\lambda \in \Lambda + Q$, where $Q$ is the root lattice $\mathbb{Z}\alpha \otimes \mathbb{Z}\delta$ of $\widehat{sl}_2$. Then, $N\lambda \in P^0(N\Lambda)$ if and only if $\lambda \in P^0(\Lambda)$. □
Proof. The validity of the lemma is clear for \( \lambda \in P^o(\Lambda)_+ \) from Corollary 5.1. But since \( P^o(\Lambda) = W \cdot (P^o(\Lambda)_+) \), and the action of \( W \) on \( \mathfrak{h}^* \) is linear, the lemma follows for any \( \lambda \in P^o(\Lambda) \). \( \Box \)

**Corollary 6.4.** Let \( d_o \in \mathbb{Z}_{>1} \). Let \( \Lambda, \Lambda', \Lambda'' \in P_+ \) be such that \( \Lambda - \Lambda' - \Lambda'' \in \mathfrak{q} \) and \( L(N\Lambda) \) is a submodule of \( L(N\Lambda') \otimes L(N\Lambda'') \), for some \( N \in \mathbb{Z}_{>0} \). Then, \( L(d_o\Lambda) \) is a submodule of \( L(d_o\Lambda') \otimes L(d_o\Lambda'') \).

Such a \( d_o \) is called a saturation factor.

**Proof.** If \( \Lambda'(c) = 0 \) or \( \Lambda''(c) = 0 \), then

\[
L(N\Lambda') \otimes L(N\Lambda'') \simeq L(N(\Lambda' + \Lambda'')),
\]

for any \( N \geq 1 \). Thus, the corollary is clearly true in this case. So, let us assume that both of \( \Lambda'(c) > 0 \) and \( \Lambda''(c) > 0 \). Let \( L(N\Lambda + n\delta) \) be the \( \delta \)-maximal component of \( L(N\Lambda') \otimes L(N\Lambda'') \) through \( L(N\Lambda) \), for some \( n \geq 0 \). For any \( \Psi \in P_+ \), let \( \bar{\Psi} \in \bar{P}_+ \) be the projection \( \pi(\Psi) \) defined just before Lemma 5.2. Applying Theorem 6.2 to \( \Lambda', \Lambda'', \bar{\lambda} \), and observing that

\[
L(\bar{\Psi} + k\delta) \simeq L(\bar{\Psi}) \otimes L(k\delta)
\]

and \( L(k\delta) \) is one dimensional, we get that there is a \( \delta \)-maximal component \( L(\Lambda + \bar{n}\delta) \) of \( L(\Lambda') \otimes L(\Lambda'') \) through \( L(\Lambda) \), for some (unique) \( \bar{n} \in \mathbb{Z} \).

Again applying Theorem 6.2 to \( N\Lambda', N\Lambda'', N\bar{\lambda} \), and observing (using Corollary 5.1) that

\[
P^o(N\bar{\Psi}) \supset NP^o(\bar{\Psi}),
\]

we get that \( L(N\Lambda + N\bar{n}\delta) \) is the \( \delta \)-maximal component of \( L(N\Lambda') \otimes L(N\Lambda'') \) through \( L(N\Lambda) \). Thus, \( n = N\bar{n} \). In particular,

\[
\bar{n} \geq 0.
\]

Let

\[
\sum_{\lambda \in \mathcal{T}^n_{\Lambda, \Lambda'}} \varepsilon(v_{\Lambda', \Lambda''})c_{\Lambda', \Lambda''}e^{S_{\Lambda', \Lambda''} + \bar{n}\delta} = \sum_{k \in \mathbb{Z}_+} c_k e^{(\Lambda(d) + \bar{n} - k)\delta},
\]

for some \( c_k \in \mathbb{Z}_{>0} \) with \( c_0 \) nonzero. By Proposition 4.2, this is the character of a unitarizable Virasoro representation with each irreducible component having the same nonzero central charge. Thus, by Lemma 4.1 for any \( k > 1 \), we get \( c_k \neq 0 \).

By the above argument, \( L(d_o\Lambda + d_o\bar{n}\delta) \) is the \( \delta \)-maximal component of \( L(d_o\Lambda') \otimes L(d_o\Lambda'') \) through \( L(d_o\Lambda) \). If \( \bar{n} = 0 \), we get that

\[
L(d_o\Lambda) \subset L(d_o\Lambda') \otimes L(d_o\Lambda'').
\]

If \( \bar{n} > 0 \), then \( d_o\bar{n} \) being \( > 1 \), by the analogue of (31) for \( d_o\Lambda', d_o\Lambda'' \) and \( d_o\Lambda, L(d_o\Lambda) \subset L(d_o\Lambda') \otimes L(d_o\Lambda'') \). (Here we have used that \( L_0 = -d + p \) on any \( g \)-isotypical component of \( L(\Lambda') \otimes L(\Lambda'') \) with highest weight in \( \Lambda + Z\delta \), for a number \( p \) depending only upon \( \Lambda, \Lambda' \) and \( \Lambda'' \), cf. [KR, Identity 10.25 on page 116].) This proves the corollary. \( \Box \)
Remark 6.5. We note that \(L(2\Lambda_0 - \delta)\) is not a component of \(L(\Lambda_0) \otimes L(\Lambda_0)\) (cf. [Kac, Exercise 12.16]). But, of course, \(L(2\Lambda_0)\) is a \(\delta\)-maximal component. By the identity (31), we know that \(L(2d_o \Lambda_0 - d_o \delta)\) must be a component of \(L(d_o \Lambda_0) \otimes L(d_o \Lambda_0)\), for any \(d_o > 1\). So \(d_o\) can not be taken to be 1 in Corollary 6.3.

7. A Conjecture

In this section, \(G\) is any symmetrizable Kac-Moody group. We recall the following definition of the deformed product due to Belkale-Kumar [BK]. (Even though they gave the definition in the finite case, the same definition works in the symmetrizable Kac-Moody case, though with only one parameter.)

7.1. Definition. Let \(P\) be any standard parabolic subgroup of \(G\). Recall from Section 2 that \(\{\epsilon_w^P\}_{w \in W_P}\) is a basis of the singular cohomology \(H^*(X_P, \mathbb{Z})\). Write the standard cup product in \(H^*(X_P, \mathbb{Z})\) in this basis as follows:

\[
\epsilon_w^P \cdot \epsilon_v^P = \sum_{w \in W_P} n_{u,v}^w \epsilon_w^P, \quad \text{for some (unique) } n_{u,v}^w \in \mathbb{Z}.
\]

(32)

Introduce the indeterminate \(\tau\) and define a deformed cup product \(\odot\) as follows:

\[
\epsilon_w^P \odot \epsilon_v^P = \sum_{w \in W_P} \tau^{(w^{-1} - 1 + v^{-1})(w^{-1} - 1)(w^{-1} - 1)(w^{-1} - 1)(x_P)} n_{u,v}^w \epsilon_w^P,
\]

(33)

where \(x_P := \sum_{i \in \Delta \setminus \Delta(P)} x_i, \Delta(P)\) is the set of simple roots of the Levi group \(L\) of \(P\) and, as in Section 2, \(\Delta\) is the set of simple roots of \(G\).

The following lemma is a generalization of the corresponding result in the finite case (cf. [BK, Proposition 17]).

7.2. Proposition. (a) The product \(\odot\) is associative and clearly commutative.

(b) Whenever \(n_{u,v}^w\) is nonzero, the exponent of \(\tau\) in the above is a nonnegative integer.

Proof. The proof of the associativity of \(\odot\) is identical to the proof given in [BK, Proof of Proposition 17 (b)].

(b) The proof of this part follows the proof of [BK, Theorem 43]. Consider the decreasing filtration \(A = \{A_m\}_{m \geq 0}\) of \(H^*(X_P, \mathbb{C})\) defined as follows:

\[A_m := \bigoplus_{w \in W_P, (w^{-1} - 1)(w^{-1} - 1)(w^{-1} - 1)(w^{-1} - 1)(x_P) \geq m} \mathbb{C} \epsilon_w^P.\]

A priori \(\{A_m\}_{m \geq 0}\) may not be a multiplicative filtration.

We next introduce another filtration \(\{F_m\}_{m \geq 0}\) of \(H^*(X_P, \mathbb{C})\) in terms of the Lie algebra cohomology. Recall that \(H^*(X_P, \mathbb{C})\) can be identified canonically with the Lie algebra cohomology \(H^*(\mathfrak{g}, I)\), where \(I\) is the Lie algebra of the Levi subgroup \(L\) of \(P\) (cf. [K2, Theorem 1.6]). The underlying cochain complex \(C^* := C^*(\mathfrak{g}, I)\) for \(H^*(\mathfrak{g}, I)\) can be written as

\[C^* := [\wedge^*(\mathfrak{g}/I)^*]^1 = \text{Hom}_I(\wedge^*(u_P) \otimes \wedge^*(u_P), \mathbb{C}),\]
where \(u_P\) (resp. \(u_P^\perp\)) is the nil-radical of the Lie algebra of \(P\) (resp. the opposite parabolic subgroup \(P^\perp\)). Define a decreasing multiplicative filtration \(\mathcal{F} = \{\mathcal{F}_m\}_{m \geq 0}\) of the cochain complex \(C^\bullet\) by subcomplexes:

\[
\mathcal{F}_m := \text{Hom}_t \left( \bigoplus_{s+t \leq m-1} \Lambda^s(u_P) \otimes \Lambda^t(u_P^\perp), \mathbb{C} \right),
\]

where \(\Lambda^s(u_P)\) (resp. \(\Lambda^s(u_P^\perp)\)) denotes the subspace of \(\Lambda^s(u_P)\) (resp. \(\Lambda^s(u_P^\perp)\)) spanned by the \(h\)-weight vectors of weight \(\beta\) with \(P\)-relative height \(\text{ht}_P(\beta) := |\beta(x_P)| = s\).

Now, define the filtration \(\bar{\mathcal{F}} = \{\bar{\mathcal{F}}_m\}_{m \geq 0}\) of \(H^*(g, l) \simeq H^*(X_P, \mathbb{C})\) by

\[
\bar{\mathcal{F}}_m := \text{Image of } H^*(\mathcal{F}_m) \to H^*(C^\bullet).
\]

The filtration \(\mathcal{F}\) of \(C^\bullet\) gives rise to the cohomology spectral sequence \(\{E_r\}_{r \geq 1}\) converging to \(H^*(C^\bullet) = H^*(X_P, \mathbb{C})\). By [K3, Proof of Proposition 3.2.11], for any \(m \geq 0\),

\[
E_1^m = \bigoplus_{s+t=m} [H^s(u_P) \otimes H^t(u_P^\perp)],
\]

where \(H^s(u_P)\) denotes the cohomology of the subcomplex \((\Lambda^s(u_P))^*\) of the standard cochain complex \(\Lambda^s(u_P)^*\) associated to the Lie algebra \(u_P\) and similarly for \(H^t(u_P^\perp)\). Moreover, by loc. cit., the spectral sequence degenerates at the \(E_1\) term, i.e.,

\[
E_1^m = E_\infty^m.
\]

Further, by the definition of \(P\)-relative height,

\[
[H^s(u_P) \otimes H^t(u_P^\perp)] \neq 0 \implies s = t.
\]

Thus,

\[
E_1^m = 0, \quad \text{unless } m \text{ is even and }
E_1^{2m} = [H^m(u_P) \otimes H^m(u_P^\perp)].
\]

In particular, from [34] and the general properties of spectral sequences (cf. [K3, Theorem E.9]), we have a canonical algebra isomorphism:

\[
\text{gr}(\bar{\mathcal{F}}) \simeq \bigoplus_{m \geq 0} [H^m(u_P) \otimes H^m(u_P^\perp)],
\]

where \([H^m(u_P) \otimes H^m(u_P^\perp)]\) sits inside \(\text{gr}(\bar{\mathcal{F}})\) precisely as the homogeneous part of degree \(2m\); homogeneous parts of \(\text{gr}(\bar{\mathcal{F}})\) of odd degree being zero.

Finally, we claim that, for any \(m \geq 0\),

\[
\mathcal{A}_m = \bar{\mathcal{F}}_{2m}.
\]

Following Kumar [K1], take the \(\partial - \partial\) harmonic representative \(\hat{s}^w\) in \(C^\bullet\) for the cohomology class \(c_P^w\). An explicit expression is given in [K1, Proposition 3.17]. From this explicit expression, we easily see that

\[
\mathcal{A}_m \subset \bar{\mathcal{F}}_{2m}, \quad \text{for all } m \geq 0.
\]
Moreover, from the definition of $\mathcal{A}$, for any $m \geq 0$,

$$ \dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \# \{ w \in W^P : (\rho - w^{-1}\rho)(x_P) = m \}. $$

Also, by the isomorphism (35) and [K3, Theorem 3.2.7],

$$ \dim \frac{\mathcal{F}_{2m}}{\mathcal{F}_{2m+1}} = \# \{ w \in W^P : (\rho - w^{-1}\rho)(x_P) = m \}. $$

Thus,

$$ \dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \dim \frac{\mathcal{F}_{2m}}{\mathcal{F}_{2m+1}}. \quad (38) $$

Of course,

$$ \mathcal{A}_0 = \mathcal{F}_0. \quad (39) $$

Thus, combining the equations (37), (38) and (39), we get (38). It is easy to see that the filtration $\{\mathcal{F}_{2m}\}_{m \geq 0}$ is multiplicative and hence so is $\{\mathcal{A}_m\}_{m \geq 0}$. This proves the (b) part of the proposition. \hfill \square

The cohomology of $X_P$ obtained by setting $\tau = 0$ in $(H^*(X_P, \mathbb{Z}) \otimes \mathbb{Z}[\tau], \sigma)$ is denoted by $(H^*(X_P, \mathbb{Z}), \sigma_0)$. Thus, as a $\mathbb{Z}$-module, it is the same as the singular cohomology $H^*(X_P, \mathbb{Z})$ and under the product $\sigma_0$ it is associative (and commutative).

The following conjecture is a generalization of the corresponding result in the finite case due to Belkale-Kumar [BK, Theorem 22].

7.3. **Conjecture.** Let $G$ be any indecomposable symmetrizable Kac-Moody group (i.e., its generalized Cartan matrix is indecomposable, cf. [K3, § 1.1]) and let $(\lambda_1, \ldots, \lambda_s, \mu) \in P_+^{s+1}$. Assume further that none of $\lambda_j$ is $W$-invariant and $-\sum_{j=1}^{s} \lambda_j \in \mathcal{Q}$, where $\mathcal{Q}$ is the root lattice of $G$. Then, the following are equivalent:

(a) $(\lambda_1, \ldots, \lambda_s, \mu) \in \Gamma_s$.

(b) For every standard maximal parabolic subgroup $P$ in $G$ and every choice of $s+1$-tuples $(w_1, \ldots, w_s, v) \in (W^P)^{s+1}$ such that $\epsilon_P^{w_i} \circ \cdots \circ \epsilon_P^{w_s} \in (H^*(X_P, \mathbb{Z}), \sigma_0)$, the following inequality holds:

$$ \left( \sum_{j=1}^{s} \lambda_j(w_jx_P) \right) - \mu(vx_P) \geq 0, \quad (I_{(w_1, \ldots, w_s, v)}^P) $$

where $\alpha_{i_P}$ is the (unique) simple root in $\Delta \setminus \Delta(P)$ and $x_P := x_{i_P}$.

7.4. **Remark.** (a) By Theorem 3.3, the above inequalities $I_{(w_1, \ldots, w_s, v)}^P$ are indeed satisfied for any $(\lambda_1, \ldots, \lambda_s, \mu) \in \Gamma_s$.

(b) If $G$ is an affine Kac-Moody group, then the condition that $\lambda \in P_+$ is $W$-invariant is equivalent to the condition that $\lambda(c) = 0$. 
7.5. Theorem. Let $\mathfrak{g} = \mathfrak{sl}_2$. Let $\lambda, \mu, \nu \in P_+$ be such that $\lambda + \mu - \nu \in Q$ and both of $\lambda(c)$ and $\mu(c)$ are nonzero. Then, the following are equivalent:

(a) $(\lambda, \mu, \nu) \in \Gamma_2$.

(b) The following set of inequalities is satisfied for all $w \in W$ and $i = 0, 1$.

$$\lambda(x_i) + \mu(wx_i) - \nu(wx_i) \geq 0, \quad \text{and} \quad \lambda(wx_i) + \mu(x_i) - \nu(wx_i) \geq 0.$$  

In particular, Conjecture 7.3 is true for $\mathfrak{g} = \mathfrak{sl}_2$ and $s = 2$.

Proof. By Lemma 5.2 there exist (unique) $n_1, n_2 \in \mathbb{Z}$ such that

$$\nu - \mu + n_1 \delta \in P^0(\lambda), \quad \text{and} \quad \nu - \lambda + n_2 \delta \in P^0(\mu).$$

Let $n := \min(n_1, n_2)$. By our description of the $\delta$-maximal components as in Theorem 6.2 applied to $\lambda, \mu, \nu$ and using the identity (28), we see that $L(\nu + n\delta)$ is a $\delta$-maximal component of $L(\lambda) \otimes L(\mu)$. Thus, by the equation (29), for any $N \geq 1$, $L(N\nu + Nn\delta)$ is a $\delta$-maximal component of $L(N\lambda) \otimes L(N\mu)$. In particular, by Proposition 4.2 and Lemma 4.1

$$L(N\nu) \subset L(N\lambda) \otimes L(N\mu) \quad \text{for some } N > 1 \quad \text{if and only if } n \geq 0. \quad (40)$$

By [Kac, Proposition 12.5 (a)], if a weight $\gamma + k\delta \in P(\lambda)$ (for some $k \in \mathbb{Z}_+$), then $\gamma \in P(\lambda)$. Thus,

$$n \geq 0 \quad \text{if and only if } \nu \in (P(\lambda) + \mu) \cap (P(\mu) + \lambda). \quad (41)$$

We next show that

$$P(\lambda) = (\lambda + Q) \cap C_\lambda, \quad (42)$$

where $C_\lambda := \{ \gamma \in \mathfrak{h}^* : \lambda(x_i) - \gamma(wx_i) \geq 0 \text{ for all } w \in W \text{ and all } x_i \}$. Clearly,

$$P(\lambda) \subset (\lambda + Q) \cap C_\lambda.$$  

Since $\lambda + Q$ and $C_\lambda$ are $W$-stable, and $\lambda + Q$ is contained in the Tits cone (by [Kac, Exercise 13.1.E.8(a)]), $(\lambda + Q) \cap C_\lambda = W \cdot ((\lambda + Q) \cap C_\lambda \cap P_+)$. 

Conversely, take $\gamma \in (\lambda + Q) \cap C_\lambda \cap P_+$. Then, $(\lambda - \gamma)(x_i) \geq 0$ and $(\lambda - \gamma)(c) = 0$ and hence $\lambda - \gamma \in \oplus_i \mathbb{Z}_+ \alpha_i$, i.e., $\lambda \geq \gamma$. Thus, by [Kac, Proposition 12.5(a)], $\gamma \in P(\lambda)$. This proves (42). Now, combining (40), (41) and (42), we get $L(N\nu) \subset L(N\lambda) \otimes L(N\mu)$ for some $N > 1$ if and only if for all $w \in W$ and $i = 0, 1$,

$$\lambda(x_i) - (\nu - \mu)(wx_i) \geq 0, \quad \text{and} \quad \mu(x_i) - (\nu - \lambda)(wx_i) \geq 0.$$  

This proves the equivalence of (a) and (b) in the theorem.

To prove the ‘In particular’ statement of the theorem, let $P_0$ (resp. $P_1$) be the maximal parabolic subgroup of $G = \text{SL}_2$ with $\Delta(P_0) = \{ \alpha_1 \}$ (resp. $\Delta(P_1) = \{ \alpha_0 \}$). For any $n \geq 0$, let

$$w_n := \ldots s_0 s_1 s_0 \quad (\text{n-factors}) \quad \text{and} \quad v_n := \ldots s_1 s_0 s_1 \quad (\text{n-factors}).$$
Then, by [K₃, Exercise 11.3.E.4], $H^*(G/P₀)$ has a $\mathbb{Z}$-basis $\{εₙ^{P₀}\}_{n \geq 0}$, where $εₙ^{P₀} := ε^n_{P₀}$. Moreover,

$$εₙ^{P₀} \cdot εₘ^{P₀} = \binom{n + m}{n} ε^{n + m}_{P₀}.$$  

In particular, $ε^{n+m}_{P₀}$ appears with coefficient one in $εₙ^{P₀} \cdot εₘ^{P₀}$ if and only if at least one of $n$ or $m$ is 0.

By using the diagram automorphism of $\text{SL}_2$, one similarly gets that $H^*(G/P₁)$ has a $\mathbb{Z}$-basis $\{εₙ^{P₁}\}_{n \geq 0}$, where $εₙ^{P₁} := ε^n_{P₁}$, with the product given by

$$εₙ^{P₁} \cdot εₘ^{P₁} = \binom{n + m}{n} ε^{n + m}_{P₁}.$$  

Moreover, from the definition of the deformed product $\circ₀$, clearly

$$ε₀^{P₀} \circ₀ εₘ^{P₀} = ε₀^{P₀} \cdot εₘ^{P₀},$$

and similarly for $P₁$. From this the 'In particular' statement of the theorem follows.

7.6. **Remark.** (1) It is easy to see that if $λ = mδ$ for some $m \in \mathbb{Z}$, then the equivalence of (a) and (b) in the above theorem breaks down.

(2) Though we have proved Conjecture 7.3 for $\text{SL}_2$ only for $s = 2$, it is quite likely that a similar proof will prove it for any $s$ (for $\text{SL}_2$).

8. **The $A^2_2$ Case**

By a method similar to that of $A^1_1$, we handle the $A^2_2$ case, with minor modifications where necessary. Write $h = \mathbb{C}c \oplus \mathbb{C}α \oplus \mathbb{C}d$ and $h^* = \mathbb{C}ω₀ \oplus \mathbb{C}α \oplus \mathbb{C}δ$, where $α(α^\vee) = 2$, $δ(d) = 1$, $ω₀(c) = 1$, and all other values are 0. Then $\{α₀ := δ - 2α, α₁ := α\}, \{α₀^\vee := c - \frac{1}{2}α^\vee, α₁^\vee := α^\vee\}$ is a realization of the GCM

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

of $A^2_2$. The fundamental weights are $ω₀$ and $ω₁ = \frac{1}{2}ω₀ + \frac{1}{2}α$. This easily allows one to compute the dominant $δ$-maximal weights. Analogous to Corollary 5.1 we have the following:

8.1. **Lemma.** Let $λ$ be a dominant integral weight. Then, the dominant $δ$-maximal weights of $L(λ)$ are the dominant weights of the form

$$P₊ \cap \{λ - jα, λ + k(2α - δ), λ + α - δ + l(2α - δ) : j, k, l \in \mathbb{Z}_{≥0}\}.$$  

Moreover, $P^α(λ)$ is the $W$-orbit of the above.

Again, to determine the saturated tensor cone, it is enough to describe the $δ$-maximal components. Thus, to determine the $δ$-maximal components, by virtue of proposition 1.2 we must find the highest $δ$-degree term in $\sum_{λ ∈ T^{λ′,λ′′}_δ} ε(v_{λ,λ′,λ′′}) c_{λ′,λ} e^{α}_{λ,λ′′,λ′}$. This computation is done in a somewhat
similar manner as in the $A_1^{(1)}$ case, but there are some important modifications. First, we need to use two different piecewise smooth functions to describe the $\delta$-maximal weights of $L(\lambda)$. An upper function $A^+$ interpolates the $\delta$-maximal weights which are in the $W$-orbit of the dominant weights of the form

$$\{\lambda - j\alpha, \lambda + k(2\alpha - \delta) : j, k \in \mathbb{Z}_{\geq 0}\},$$

while another function $A^-$ interpolates the $\delta$-maximal weights in the $W$-orbit of the dominant weights of the form

$$\{\lambda - j\alpha, \lambda + \alpha - \delta + l(2\alpha - \delta) : j, l \in \mathbb{Z}_{\geq 0}\}.$$

Now, all of the arguments made in the $\widehat{sl}_2$ case must be made for two extensions of $S_{\Lambda,\Lambda',\lambda}$ to non-integral values, using $A^+$ and $A^-$ respectively. Let $\Lambda := m_0\omega_0 + m_1\omega_1$, $\Lambda' := m_0'\omega_0 + m_1'\omega_1$, and $\Lambda'' := m_0''\omega_0 + m_1''\omega_1$. The following result is an analogue of Proposition 5.5 and Lemma 5.10 for the $A_2^{(2)}$ case.

**Proposition 8.2.** Let $\Lambda, \Lambda', \Lambda''$ be as above. Assume that both of $\Lambda'(c)$ and $\Lambda''(c) > 0$ and $\Lambda - \Lambda' - \Lambda'' \in Q$, where $Q = \mathbb{Z}\alpha + \mathbb{Z}\delta$ is the root lattice of $A_2^{(2)}$. Then, the maximum $\mu_{\Lambda,\Lambda',\Lambda''}$ of the set

$$\left\{ S_{\Lambda,\Lambda',\lambda} : \lambda \in T_{\Lambda}^{\Lambda',\Lambda''}, \varepsilon(v_{\Lambda,\Lambda',\lambda}) = 1 \right\}$$

occurs when $\lambda \equiv \Lambda' + \frac{1}{2}(m_1 - m_1' - m_1'')\alpha \mod \delta$. The maximum $\mu_{\Lambda,\Lambda',\Lambda''}$ of the set

$$\left\{ S_{\Lambda,\Lambda',\lambda} : \lambda \in T_{\Lambda}^{\Lambda',\Lambda''}, \varepsilon(v_{\Lambda,\Lambda',\lambda}) = -1 \right\}$$

occurs when $\lambda \equiv \Lambda' - \left(\frac{1}{2}(m_1 + m_1'' + m_1) + 1\right)\alpha \mod \delta$ or when $\lambda \equiv \Lambda' - \left(\frac{1}{2}(m_1 + m_1'' + m_1) - 2(\Lambda'(c) + \Lambda''(c) + 1)\right)\alpha \mod \delta$.

**8.3. Corollary.** Let $\Lambda, \Lambda', \Lambda''$ be as in Proposition 8.2. Assume further that $\Lambda'(c) \geq 2$, $\Lambda''(c) \geq 2$, $m_1, m_1'' \neq 1$. Then, if $\mu_{\Lambda,\Lambda',\Lambda''} = \mu_{\Lambda',\Lambda''}$, we have

$$\mu_{\Lambda,\Lambda',\Lambda''} \neq -\Lambda''(\Lambda').$$

The proof of Corollary 8.3 requires a description of the situations in which $\mu_{\Lambda,\Lambda',\Lambda''} = \mu_{\Lambda',\Lambda'',\Lambda}$. We reduce these situations to certain cases, and show that in most of these cases, if the roles of $\Lambda'$ and $\Lambda''$ are interchanged, then (as in the $\widehat{sl}_2$ case) the equality does not occur. In the remaining cases, we show that $\Lambda'(c) < 2$, $\Lambda''(c) < 2$, $m_1' = 1$, or $m_1'' = 1$.

**Theorem 8.4.** Let $\Lambda, \Lambda', \Lambda''$ be as in Proposition 8.2. Then, $L(\Lambda + n\delta)$ is a $\delta$-maximal component of $L(\Lambda') \otimes L(\Lambda'')$ if $n = \min(n_1, n_2)$, where $n_1$ is such that $\Lambda - \Lambda' + n_1\delta \in P^o(\Lambda')$ and $n_2$ is such that $\Lambda - \Lambda' + n_2\delta \in P^o(\Lambda'')$.

**Lemma 8.5.** Fix a positive integer $N$. Let $\Lambda \in \bar{Z}_+$ and let $\lambda \in \Lambda + Q$. Then, $N\lambda \in P^o(N\Lambda)$ if and only if $\lambda \in P^o(\Lambda)$. 
Combining the above results, we get a description of $\Gamma_2$, which is identical to that of $\mathfrak{sl}_2$ (cf. Theorem 7.5).

8.6. **Theorem.** Let $\mathfrak{g} = A^{(2)}_2$. Let $\lambda, \mu, \nu \in P_+$ be such that $\lambda + \mu - \nu \in Q$ and both of $\lambda(c)$ and $\mu(c)$ are nonzero. Then, the following are equivalent:

(a) $(\lambda, \mu, \nu) \in \Gamma_2$.

(b) The following set of inequalities is satisfied for all $w \in W$ and $i = 0, 1$.

$$
\lambda(x_i) + \mu(wx_i) - \nu(wx_i) \geq 0, \quad \text{and} \quad \\
\lambda(wx_i) + \mu(x_i) - \nu(wx_i) \geq 0.
$$

In particular, Conjecture 7.3 is true for this case as well for $s = 2$.

The ‘In particular’ statement of the above theorem follows by using the description of the cup product in the cohomology of the full flag variety of $A^{(2)}_2$ given by Kitchloo [Ki].

It is clear that if the level of $L(\Lambda')$ or $L(\Lambda'')$ is zero, then the tensor product has a single component. Thus, it is already saturated. Assume now that the levels of both of $L(\Lambda')$ and $L(\Lambda'')$ are $> 0$. Then, since there are representations of level $\frac{1}{2}$, the conditions of Corollary 8.3 are satisfied for any $N\Lambda, N\Lambda', N\Lambda''$ with $\Lambda - \Lambda' - \Lambda'' \in Q$, provided $N \geq 4$. Hence:

**Corollary 8.7.** For $A^{(2)}_2$, 4 is a saturation factor.

8.8. **Remark.** When the Kac-Moody Lie algebra $\mathfrak{g}$ is infinite dimensional, then the saturated tensor semigroup $\Gamma_s$ is not finitely generated, for any $s \geq 2$. Thus, it is not clear a priori that there exists a saturation factor for such a $\mathfrak{g}$.

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