A DECOMPOSITION THEOREM FOR SINGULAR KÄHLER SPACES WITH TRIVIAL FIRST CHERN CLASS OF DIMENSION AT MOST FOUR

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Abstract. Let $X$ be a compact Kähler fourfold with klt singularities and vanishing first Chern class, smooth in codimension two. We show that $X$ admits a Beauville–Bogomolov decomposition: a finite quasi-étale cover of $X$ splits as a product of a complex torus and singular Calabi–Yau and irreducible holomorphic symplectic varieties. We also prove that $X$ has small projective deformations and the fundamental group of $X$ is projective. To obtain these results, we propose and study a new version of the Lipman–Zariski conjecture.

1. Introduction

Let $X$ be a compact Kähler manifold such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. The structure of such $X$ is in many aspects well-understood [Bea83, Bog78, Tia87, Tod89]:

- Geometry: a finite étale cover of $X$ splits as a product of a complex torus, simply connected Calabi–Yau manifolds and irreducible holomorphic symplectic manifolds (Beauville–Bogomolov decomposition).
- Deformation theory: The local deformation space of $X$ is smooth (Bogomolov–Tian–Todorov theorem), and $X$ admits small projective deformations (Kodaira problem).
- Topology: the fundamental group $\pi_1(X)$ is projective, virtually abelian, and finite if the augmented irregularity $\tilde{q}(X)$ of $X$ vanishes.

The BB (= Beauville–Bogomolov) decomposition is a cornerstone in the classification of compact Kähler manifolds up to biholomorphic maps. The subject of birational geometry, however, is rather a structure theory up to birational morphic maps. As is well-known, in this context it is necessary to consider singular analogues of the above manifolds. By this, we mean compact Kähler spaces with klt singularities and $c_1(X) = 0$.

In the projective case, the BB decomposition has been established by [Dru18, GGK19, HP19]. This result is commonly referred to as the BBDGGHKP decomposition. But most of the other properties listed in the beginning remain elusive even for projective varieties.

Very recently, the decomposition theorem has been extended to the Kähler case [GS20, CGGN20, BGL20]. The final statement may therefore be called the BBBCDGGGHKLNPS decomposition.
The goal of this paper is to give an independent proof of the decomposition in dimension four. Our principal result, however, is a generalization of the BTT (= Bogomolov–Tian–Todorov) theorem.

**Theorem 1.1** (Singular BTT theorem in dimension four). Let $X$ be a normal compact Kähler space of dimension $\leq 4$, with klt singularities and such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. Assume that $\dim X_{\text{reg}} \leq 1$. Then the semiuniversal locally trivial deformation space $\text{Def}^{lt}(X)$ is smooth, unless possibly in the case $\tilde{q}(X) = \dim X - 3$.

In all cases where Theorem 1.1 does not apply, $X$ is already projective. This allows us to draw the following consequences.

**Corollary 1.2** (Kodaira problem in dimension four). Let $X$ be as in Theorem 1.1. Then the semiuniversal family $X \to \text{Def}^{lt}(X)$ is a strong locally trivial algebraic approximation. In particular, $X$ is locally algebraic.

This puts [CGGN20, Thm. H] in its final form: we remove both the local algebraicity assumption and the necessity to take a quasi-étale cover before obtaining an algebraic approximation of $X$. Note also that Corollary 1.2 confirms [CGGN20, Conj. K] in this particular case.

We now turn to fundamental groups. In [CGGN20, Thm. G], we showed that in dimension four $\pi_1(X)$ is virtually abelian. In particular, this group is “virtually projective”, i.e. it contains a normal subgroup of finite index which is isomorphic to $\pi_1(Y)$ for some projective manifold $Y$. Corollary 1.3 below shows that passing to a subgroup is in fact not necessary. It also implies that $\pi_1$ of any Kähler fourfold of Kodaira dimension zero admitting a good minimal model is projective. This gives some new evidence towards the conjecture that every Kähler group is projective [ABC+96, (1.26)].

Campana’s Abelianity Conjecture [Cam04, Conj. 7.3] also makes similar predictions for the fundamental group of the smooth locus $\pi_1(X_{\text{reg}})$, but this is currently out of reach even for $X$ projective. Using Corollary 1.2, we can at least confirm the conjecture for the profinite completion $\hat{\pi}_1(X_{\text{reg}})$.

**Corollary 1.3** (Fundamental groups). Let $X$ be as in Theorem 1.1. Then:

1.3.1 The fundamental group $\pi_1(X)$ is projective.

1.3.2 The algebraic fundamental group of the smooth locus $\pi_1(X_{\text{reg}})$ is virtually abelian, and finite if $\tilde{q}(X) = 0$.

Finally, we have the decomposition theorem mentioned in the title. For the definition of singular CY and IHS varieties, we refer to [CGGN20, Def. 6.11].

**Corollary 1.4** (BB decomposition). Let $X$ be as in Theorem 1.1. Then some quasi-étale cover of $X$ splits as a product of a complex torus, Calabi–Yau and irreducible holomorphic symplectic varieties.

The flat factor of $\mathcal{H}_X$. Let us briefly comment on the proof of Theorem 1.1. It relies on an analysis of the flat factor $\mathcal{F}$ in the holonomy decomposition of $\mathcal{H}_X$. To explain this, and also to fix notation, recall that in [CGGN20, Thm. C] we proved in particular the following.

**Setup and Notation 1.5** (Standard setting). Let $X$ be a normal compact Kähler space with klt singularities such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. Then after replacing $X$ by a finite quasi-étale cover, the so-called holonomy cover, the tangent sheaf of $X$ decomposes as

$$\mathcal{H}_X = \mathcal{F} \oplus \bigoplus_{k \in K} \mathcal{E}_k,$$

where the sheaves $\mathcal{F}$ and $\mathcal{E}_k$ satisfy the following:
The restriction \( F \mid_{X_{\text{reg}}} \) is flat, i.e. given by a representation \( \pi_1(X_{\text{reg}}) \to SU(r) \), where \( r = \text{rk}(F) \).

Each summand \( E_k \mid_{X_{\text{reg}}} \) has full holonomy group either \( SU(n_k) \) or \( \text{Sp}(n_k/2) \), with respect to a suitable singular Ricci-flat metric. Here \( n_k = \text{rk}(E_k) \geq 2 \).

The natural conjecture concerning the flat factor \( F \) is that it corresponds to the torus factor in the (conjectural) Beauville–Bogomolov decomposition of \( X \). It is convenient to rephrase this in different but equivalent ways:

- The rank of \( F \) should equal the augmented irregularity of \( X \), i.e. \( r = \tilde{q}(X) \).
- If \( \tilde{q}(X) \) vanishes, then \( F \) should be the zero sheaf.

The conjecture follows from [BGL20], but for the purpose of giving a logically independent proof of Theorem 1.1, the following partial result is key. It bounds the rank of \( F \) from above in terms of the dimension of the singular locus of \( X \) and in fact holds in arbitrary dimension.

**Theorem 1.6** (Bounding the flat factor). In the standard setting 1.5, assume that the augmented irregularity of \( X \) vanishes, \( \tilde{q}(X) = 0 \). Then:

1.6.1 The flat factor \( F \) satisfies \( \text{rk}(F) \leq \dim X_{\text{sg}} \).
1.6.2 If \( \dim X_{\text{sg}} \leq 1 \), then \( F = 0 \).

**Remark.** In (1.6.1), we need to adopt the convention that the empty set has dimension zero (as opposed to \(-1\) or \(-\infty\)) in order for the conclusion to hold also if \( X \) is smooth.

**The Lipman–Zariski Conjecture for direct summands.** To prove Theorem 1.6, we propose and study a new variant of the Lipman–Zariski Conjecture, which we explain now. The classical Lipman–Zariski Conjecture (which is still open) states that if the tangent sheaf \( T_X \) of a complex algebraic variety or complex space is locally free, then \( X \) is smooth. Here we ask what happens if \( T_X \) is not necessarily locally free, but contains a locally free direct summand.

**Question 1.7** (Lipman–Zariski Conjecture for direct summands). Let \( X \) be a complex space. Assume that the tangent sheaf of \( X \) admits a direct sum decomposition

\[ T_X = \mathcal{E} \oplus \mathcal{F}, \]

where \( \mathcal{F} \) is locally free. Under what assumptions on the rank of \( \mathcal{F} \) and on the singularities of \( X \) can we conclude that \( X \) is smooth?

**Question 1.7** is formulated in a deliberately vague way. In this work, we will concentrate on the case of klt singularities, but other classes of singularities would be equally interesting.

**Theorem 1.8** (Question 1.7 for klt singularities). Let \( X \) be a normal complex space with klt singularities such that the tangent sheaf admits a direct sum decomposition

\[ T_X = \mathcal{E} \oplus \mathcal{F}, \]

where \( \mathcal{F} \) is locally free of rank \( r \). If \( \dim X_{\text{sg}} \leq r - 1 \), then \( X \) is smooth.

Easy examples show that the bound on \( r \) in Theorem 1.8 is sharp (Remark 3.2).

2. **Notation and basic facts**

Unless otherwise stated, complex spaces are assumed to be countable at infinity, separated, reduced and connected. Algebraic varieties and schemes are always assumed to be defined over the complex numbers.
\textbf{Definition 2.1 (Torsion-free differentials).} Let $X$ be a reduced complex space and $p \in \mathbb{N}$ a non-negative integer. The sheaf of \textit{torsion-free differential $p$-forms} on $X$ is defined to be

$$\Omega_X^p := \Omega_X^p / \text{tor} \Omega_X^p,$$

where $\Omega_X^p := \bigwedge^p \Omega_X^1$ is the sheaf of Kähler differentials and $\text{tor} \Omega_X^p$ is the subsheaf of $\Omega_X^p$ consisting of the sections vanishing on some dense open subset $U \subset X$. Equivalently, $\text{tor} \Omega_X^p$ consists of those sections whose support is contained in the singular locus $X_{\text{reg}}$.

\textbf{Definition 2.2 (Quasi-étale covers).} A \textit{cover} is a finite, surjective morphism $\gamma: Y \to X$ of normal, connected complex spaces. A cover $\gamma$ is called \textit{quasi-étale} if there exists a closed subset $Z \subset Y$ with $\text{codim}_Y(Z) \geq 2$ such that $\gamma|_{Y \setminus Z}: Y \setminus Z \to X$ is étale.

\textbf{Definition 2.3.} Let $X$ be a normal complex space. A \textit{maximally quasi-étale cover} of $X$ is a quasi-étale Galois cover $\gamma: Y \to X$ satisfying the following equivalent conditions:

\begin{enumerate}
  \item Any étale cover of $Y_{\text{reg}}$ extends to an étale cover of $Y$.
  \item Any quasi-étale cover of $Y$ is étale.
\end{enumerate}

\textbf{Definition 2.4 (Irregularity).} The \textit{irregularity} of a compact complex space $X$ is $q(X) := h^1(Y, \mathcal{O}_Y)$, where $Y \to X$ is any resolution of singularities. The \textit{augmented irregularity} of $X$ is

$$\tilde{q}(X) := \max \left\{ q(\tilde{X}) \mid \tilde{X} \to X \text{ quasi-étale} \right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

\textbf{Vector fields.} Let $X$ be a reduced complex space. A \textit{vector field} on $X$ is a (local) section of the tangent sheaf $\mathcal{T}_X := \mathcal{Hom}(\Omega_X^1, \mathcal{O}_X)$, where $\Omega_X^1$ is the sheaf of Kähler differentials.

Let $v$ be a vector field. For any point $x \in X$, the germ of $v$ at $x$ is a $\mathbb{C}$-linear derivation $v_x: \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ and hence it can be restricted to an element of the Zariski tangent space of $X$ at $x$:

$$v(x) \in \left( \frac{\mathcal{m}_x}{\mathcal{m}_x^2} \right)^\vee := T_xX.$$

Note, however, that in general not every Zariski tangent vector at $x$ is of the form $v(x)$ for some local vector field $v$.

\textbf{Lemma 2.5.} Let $Z \subset X$ be an analytic subset that is fixed by every local automorphism of $X$. Then $v(z) \in T_zZ \subset T_zX$ for every vector field $v$ on $X$ defined near $z \in Z$.

\textbf{Proof.} We will use the correspondence between derivations, vector fields and local $\mathbb{C}$-actions as described in [Akhi95, §1.4, §1.5]. The vector field $v$ induces a local $\mathbb{C}$-action $\Phi: \mathbb{C} \times X \to X$. By the definition of local group action, $\Phi(t, -)$ is an automorphism of germs $(X, z) \cong (X, \Phi(t, z))$ for every sufficiently small $t \in \mathbb{C}$. It then follows from the assumption that $\Phi(t, z) \in Z$ for every $t \in \mathbb{C}$. Now, we can recover the derivation $\delta$ corresponding to $v$ from $\Phi$ by the formula

$$\delta(f)(x) = \left. \frac{d}{dt} \right|_{t=0} f(\Phi(t, x))$$

for every $f \in \mathcal{O}_{X,z}$. Plugging the above statement into (2.5.1), we see that $\delta$ stabilizes the ideal of $Z$, i.e. $\delta(I_{Z,z}) \subset I_{Z,z}$. Hence $\delta$ induces a derivation of $\mathcal{O}_{Z,z}$ and then also an element of $T_zZ$.

\textbf{Example 2.6.} Let $X = \mathbb{C}^2$ and let $0 \in C \subset X$ be a curve such that $0$ is a singular point of $C$. If $v$ is a vector field on $X$ that is tangent to $C \setminus \{0\}$, then its local flows restrict to automorphisms of $C$. These automorphisms necessarily fix the singular point $0 \in C_{\text{reg}}$ and hence $v$ vanishes at the origin, $v(0) = 0$. 

Deformation theory. This is just a very quick reminder. For more details, see for example [GS20].

Definition 2.7 (Deformations of complex spaces). A deformation of a (reduced) compact complex space $X$ is a proper flat morphism $\pi: X \to (S, 0)$ from a (not necessarily reduced) complex space $X$ to a complex space germ $(S, 0)$, equipped with a fixed isomorphism $X_0 := \pi^{-1}(0) \cong X$.

Definition 2.8 (Algebraic approximations). Let $X$ be a compact complex space and $\pi: X \to S$ a deformation of $X$. Consider the set of projective fibres $S^{\text{alg}} := \{ s \in S \mid X_s \text{ is projective} \} \subset S$ and its closure $\overline{S^{\text{alg}}} \subset S$. We say that $X \to S$ is an algebraic approximation of $X$ if $0 \in \overline{S^{\text{alg}}}$. We say that $X \to S$ is a strong algebraic approximation of $X$ if $S^{\text{alg}} = S$ as germs, i.e. $S^{\text{alg}}$ is dense near $0 \in S$.

Definition 2.9 (Locally trivial deformations). A deformation $\pi: X \to S$ is called locally trivial if for every $x \in X_0$ there exist open subsets $0 \in S^o \subset S$ and $x \in U \subset \pi^{-1}(S^o)$ and an isomorphism $U \cong (X_0 \cap U) \times S^o$.

3. The Lipman–Zariski Conjecture for direct summands

In this section, we prove Theorem 1.8 from the introduction and the following corollary. We then deduce Theorem 1.6.

Corollary 3.1 (Spaces with large flat summands). Let $X$ be a normal complex space with klt singularities such that

$$\mathcal{F}_X = \mathcal{E} \oplus \mathcal{F},$$

where $\mathcal{F}|_{X_{\text{reg}}}$ is flat of rank $r$, i.e. given by a representation $\pi_1(X_{\text{reg}}) \to \text{GL}(r, \mathbb{C})$. If $\dim X_{\text{sg}} \leq r - 1$, then $X$ has only quotient singularities.

Remark 3.2 (Sharpness of Theorem 1.8). The bound on $\dim X_{\text{sg}}$ in Theorem 1.8 is sharp, as shown by the (easy) example $X = Y \times \mathbb{C}$, where $Y$ is a (non-smooth) isolated klt singularity. In this case $\mathcal{F}_X$ has a rank one free summand and $\dim X_{\text{sg}} = 1$, but $X$ is not smooth.

Remark 3.3 (Reformulation of Theorem 1.8). The conclusion of Theorem 1.8 could also be formulated in a somewhat oblique manner as follows: every irreducible component of $X_{\text{sg}}$ has dimension at least $r$.

Remark 3.4 (Comparison to previous results). The “usual” Lipman–Zariski Conjecture is well-known for spaces with klt, or even log canonical, singularities [GK14, Cor. 1.3]. If $r = n := \dim X$ in Theorem 1.8, the statement reduces to this result.

In general, $\dim X_{\text{sg}} = n - 2$ and then the only case left where Theorem 1.8 applies is $r = n - 1$. In this case, we may consider a (local) index one cover $X_1 \to X$. There, also the rank one sheaf $\mathcal{E}$ will become locally free, hence $X_1$ is smooth. Summing up, we see that [GK14] only yields the weaker statement that $X$ has quotient singularities (instead of being smooth).
3.A. Proof of Theorem 1.8. Assuming that $Z := X_{\text{sg}}$ is non-empty, we will derive a contradiction. Let $f : Y \to X$ be the functorial resolution. Pick a sufficiently general point $z \in Z$. Then $Z$ is smooth at $z$, i.e. $z \in Z_{\text{reg}}$, and by Generic Smoothness the fibre $F := f^{-1}(z) \subset Y$ will be a simple normal crossings variety (albeit not necessarily a divisor in $Y$). Shrinking $X$ around $z$, we may without loss of generality make the following

Additional Assumption 3.5. The singular locus $Z$ of $X$ is smooth. The sheaf $\mathcal{F}$ is free, isomorphic to $\mathcal{O}_X^{\vee}$.

Let $\{v_1, \ldots, v_r\}$ be a basis of $\mathcal{F}$ and let $\{\alpha_1, \ldots, \alpha_r\}$ be the dual basis of $\mathcal{F}^\vee$, defined by $\alpha_i(v_j) = \delta_{ij}$. Since $\mathcal{F} \subset \mathcal{F}_X$ is a direct summand, so is $\mathcal{F}^\vee \subset \Omega_X^{[1]}$. This enables us to consider the sections $\alpha_i$ as reflexive 1-forms on $X$. Evaluating the vector fields $v_i$ at the point $z$ and taking into account that $Z$ is stabilized by their flows, we obtain $v_i(z) \in T_zZ$. Since $\dim T_zZ = \dim Z \leq r - 1$, there is a non-trivial relation

\[(3.6.1) \quad \sum_{i=1}^r \lambda_i v_i(z) = 0 \in T_zZ \subset T_zX, \quad \lambda_i \in \mathbb{C}.
\]

Some coefficient in (3.6.1), say $\lambda_1$, will be non-zero. Replacing $v_1$ by $\sum_{i=1}^r \lambda_i v_i$, we arrive at the

Additional Assumption 3.6. The free sheaf $\mathcal{F}$ has a basis $\{v_1, \ldots, v_r\}$ with the property that $v_1(z) = 0$.

This means that $z$ is stabilized by the flow of $v_1$. Let $\tilde{v}_1$ be the lift of $v_1$ to $Y$. Then the flow of $\tilde{v}_1$ stabilizes the fibre $F = f^{-1}(z)$, i.e. $\tilde{v}_1$ restricts to a vector field on $F$. The same is then true of any irreducible component of $F$. Fix one such component $P \subset F$, and note that $P$ is smooth because $F$ is an snc variety.

On the other hand, let $\tilde{\alpha}_1$ be the lift of $\alpha_1$ to $Y$, which is a holomorphic 1-form by [KS19]. Since the function $\tilde{\alpha}_1(\tilde{v}_1)$ is identically one, the restricted form $\tilde{\alpha}_1|_P$ cannot be zero. Indeed, if $p \in P$ is arbitrary then $\tilde{\alpha}_1$ evaluated on $\tilde{v}_1(p) \in T_pP$ is non-zero. On the other hand, the restriction map factors as

\[(3.6.1) \quad H^0(Y, \Omega_Y^1) \rightarrow H^0(F, \Omega_F^1) \rightarrow H^0(P, \Omega_P^1)
\]

because $P$ is smooth and not contained in the singular locus of $F$. The middle term vanishes due to [HM07, Cor. 1.5] and [Keb13, Thm. 4.1]. Hence (3.6.1) implies $\tilde{\alpha}_1|_P = 0$, contradicting our previous observation that this form is non-zero and thus ending the proof. Easy.

3.B. Proof of Corollary 3.1. Since the problem is local, we may assume $X$ to be a germ. We argue by induction on $\dim X_{\text{sg}}$. If $X_{\text{sg}} = \emptyset$, there is nothing to show. Otherwise, by [CGGN20, Prop. 5.8] there exists an open subset $X_{\text{reg}} \subset X^\circ \subset X$ admitting a maximally quasi-étaile cover $\gamma^\circ : Y^\circ \to X^\circ$ and satisfying $\dim (X \setminus X^\circ) \leq \dim X_{\text{sg}} - 1$. We may extend $\gamma^\circ$ to a quasi-étaile cover $\gamma : Y \to X$, by [DG94, Thm. 3.4].

Note that $Y$ reproduces all assumptions of Corollary 3.1. In particular, taking the reflexive pullback of the decomposition $\mathcal{F}_X = \mathcal{E} \oplus \mathcal{F}$, we get

$$\mathcal{F}_Y = \mathcal{E}_Y \oplus \mathcal{F}_Y.$$
where $\mathcal{F}_Y|_{Y_{\text{reg}}}$ is flat. Restricting $\mathcal{F}_Y$ to $Y_{\text{reg}}$ and using property (2.3.3) combined with [Pla68], it follows that $\mathcal{F}_Y|_{Y_{\text{reg}}}$ is flat, in particular locally free. Thus $Y^\circ$ is smooth thanks to Theorem 1.8. This implies
\[
\dim Y_{\text{sg}} \leq \dim (Y \setminus Y^\circ) = \dim (X \setminus X^\circ) \leq \dim X_{\text{sg}} - 1.
\]
By the induction hypothesis, $Y$ has only quotient singularities and hence so does $X$.

Easy.

3.C. Proof of Theorem 1.6. Assume that $\text{rk}(\mathcal{F}) \geq \dim X_{\text{sg}} + 1$. By Corollary 3.1, the space $X$ has only quotient singularities. In particular, it is locally algebraic and hence $\text{rk}(\mathcal{F}) = \tilde{q}(X) = 0$ by [CGGN20, Thm. C]. This contradiction proves (1.6.1).

For (1.6.2), we only need to exclude the case that $\text{rk}(\mathcal{F}) = 1$. In this case, $\mathcal{F}|_{X_{\text{reg}}}$ is given by a representation $\pi_1(X_{\text{reg}}) \to \text{SU}(1) = \{1\}$, so it is the trivial line bundle. By reflexivity, this implies $\mathcal{F} \cong \mathcal{O}_X$. In particular, $H^0(X, \mathcal{F}) \neq 0$. Since $\mathcal{F}' \subset \Omega^1_X$, this contradicts the assumption that $\tilde{q}(X) = 0$.

Easy.

4. PROOF OF MAIN RESULTS

4.A. Torus covers revisited. In the standard setting 1.5, recall the notion of torus cover from [CGGN20, Thm. B]: this is a quasi-étale cover $\gamma: T \times Z \to X$, where $T$ is a complex torus of dimension $\tilde{q}(X)$, while $Z$ satisfies $\omega_Z \cong \mathcal{O}_Z$ as well as $\tilde{q}(Z) = 0$.

We do not know if the map $\gamma$ can always be chosen to be quasi-étale. Indeed, this is not obvious from the construction and taking the Galois closure of a given $\gamma$ might destroy the splitting property. The following weaker statement is however sufficient for our purposes:

**Proposition 4.1** (Torus covers revisited). In the standard setting 1.5, the torus cover can be chosen in such a way that it is a composition of quasi-étale Galois morphisms:

\[
\begin{array}{ccc}
T \times Z & \xrightarrow{\text{Galois}} & X' \\
\downarrow & & \downarrow \\
A_2 & \xrightarrow{\text{Galois}} & A
\end{array}
\]

**Proof.** Consider the index one cover $X_1 \to X$ and choose a quasi-étale cover $X_2 \to X_1$ with $q(X_2) = \tilde{q}(X)$. Replacing $X_2 \to X$ by its Galois closure $X' \to X$ yields the first map in the statement to be proven.

For the second map, we know from [CGGN20, Thm. 4.1] that the Albanese map $X' \to A := \text{Alb}(X')$ becomes trivial after a finite étale base change $A_1 \to A$. We may consider the Galois closure $A_2 \to A$ of the latter map. The pullback of $X'$ to $A_2$ still splits, by the transitivity of fibre products.

\[
\begin{array}{ccc}
F \times A_2 & \xrightarrow{\text{Galois}} & F \times A_1 \\
\downarrow & & \downarrow \\
A_2 & \xrightarrow{\text{Galois}} & A
\end{array}
\]

Furthermore, $F \times A_2 \to X'$ is Galois, being the pullback of the Galois morphism $A_2 \to A$ along $X' \to A$. We now set $T := A_2$ and $Z := F$. The proof that this is indeed a torus cover of $X$ is the same as in [CGGN20, proof of Cor. 4.2].

Easy.

**Lemma 4.2.** Notation as above.

(4.2.1) The tangent space $T_0 \text{Def}^t(T \times Z)$ can be calculated as follows:
\[H^1(T \times Z, \mathcal{F}_T) = H^1(T, \mathcal{F}_T) \oplus H^1(Z, \mathcal{F}_Z).\]

(4.2.2) If $\text{Def}^t(Z)$ is smooth, then so is $\text{Def}^t(T \times Z)$. More precisely, in this case $\text{Def}^t(T \times Z) = \text{Def}^t(T) \times \text{Def}^t(Z)$.
Proof. Let $p: T \times Z \to T$ and $r: T \times Z \to Z$ be the projections. The proof is in a series of claims.

Claim 4.3. The tangent sheaf of $T \times Z$ decomposes as

$$\mathcal{F}_{T \times Z} = p^* \mathcal{F}_T \oplus r^* \mathcal{F}_Z.$$

Proof. Clearly the decomposition exists on the smooth locus of $T \times Z$. By reflexivity, it extends to a decomposition $\mathcal{F}_{T \times Z} = p^![1] \mathcal{F}_T \oplus r^![1] \mathcal{F}_Z$, where $p^! \mathcal{F}_T := (p^* \mathcal{F}_T)^{\vee}$ and $r^! \mathcal{F}_Z$ denote the reflexive pullback. Hence it suffices to show that $p^* \mathcal{F}_T$ and $r^* \mathcal{F}_Z$ are already reflexive. For $p^* \mathcal{F}_T$, this is obvious because $\mathcal{F}_T$ is even (locally) free.

For $r^* \mathcal{F}_Z$, we use the characterization of reflexive sheaves as locally 2nd syzygy sheaves, [OSS80, Ch. I, Lemma 1.1.16 and proof of Lemma 1.1.10]\(^1\). That is, on sufficiently small open sets $U \subset Z$ there exists an exact sequence

$$0 \to \mathcal{F}_Z|_U \to \mathcal{O}_U^m \to \mathcal{O}_U^n.$$

The sequence stays exact when pulled back along the flat morphism $r$, showing that also $r^* \mathcal{F}_Z|_{T \times U}$ is reflexive. Of course, this argument shows quite generally that the pullback of a reflexive sheaf via a flat map remains reflexive. \(\text{Easy.}\)

Claim 4.4. For any $i \geq 0$, we have

$$R^i p_* p^* \mathcal{F}_T = \mathcal{F}_T \otimes R^i p_* \mathcal{O}_{T \times Z}$$

and

$$R^i r_* r^* \mathcal{F}_Z = \mathcal{F}_Z \otimes R^i r_* \mathcal{O}_{T \times Z}.$$  

Proof. For $\mathcal{F}_T$, this is simply the projection formula. Regarding $\mathcal{F}_Z$, some care is required because that sheaf is not locally free. So let $U \subset Z$ be a Stein open subset. Then on the level of presheaves, we need to check that

$$H^i(r^{-1}(U), r^* \mathcal{F}_Z) = H^0(U, \mathcal{F}_Z) \otimes H^i(r^{-1}(U), \mathcal{O}_{T \times Z}),$$

where the tensor product is taken over the ring $H^0(U, \mathcal{O}_T)$. But $r^{-1}(U) = T \times U$, so the Künneth formula tells us that the left-hand side equals

$$\bigoplus_{k+l=i} H^k(U, \mathcal{F}_Z) \otimes H^l(T, \mathcal{O}_T) = H^0(U, \mathcal{F}_Z) \otimes H^i(T, \mathcal{O}_T)$$

since the higher cohomology groups on $U$ vanish. For the same reason, the second factor on the right-hand side is

$$\bigoplus_{k+l=i} H^k(U, \mathcal{F}_Z) \otimes H^l(T, \mathcal{O}_T) = H^0(U, \mathcal{F}_Z) \otimes H^i(T, \mathcal{O}_T) = H^i(T, \mathcal{O}_T).$$

(4.4.1) now follows by comparing (4.4.2) and (4.4.3). \(\text{Easy.}\)

By Claim 4.3, statement (4.2.1) is reduced to the following claim.

Claim 4.5. $H^1(T \times Z, p^* \mathcal{F}_T) = H^1(T, \mathcal{F}_T)$ and $H^1(T \times Z, r^* \mathcal{F}_Z) = H^1(Z, \mathcal{F}_Z)$.

Proof. Concerning the first factor, the Leray spectral sequence combined with Claim 4.4 gives

$$0 \to H^1(T, \mathcal{F}_T) \to H^1(T \times Z, p^* \mathcal{F}_T) \to H^0(T, \mathcal{F}_T \otimes R^1 p_* \mathcal{O}_{T \times Z})$$

where the last term vanishes because $q(Z) = 0$. We obtain that $H^1(T \times Z, p^* \mathcal{F}_T) = H^1(T, \mathcal{F}_T)$.

For the second factor, again by Claim 4.4 we have a similar exact sequence

$$0 \to H^1(Z, \mathcal{F}_Z) \to H^1(T \times Z, r^* \mathcal{F}_Z) \to H^0(Z, \mathcal{F}_Z \otimes R^1 r_* \mathcal{O}_{T \times Z}).$$

\(^1\)The cited reference assumes the underlying space to be smooth, but the arguments work verbatim for sheaves on normal complex spaces.
but here $R^1r_*\mathcal{O}_{T\times Z}$ is a trivial vector bundle (of rank equal to $\dim T$). To conclude as before, we therefore need to know that $H^0(Z, \mathcal{F}) = 0$. This can be seen as follows, where $m = \dim Z$ and $\tilde{Z} \to Z$ is a resolution:

$$h^0(Z, \mathcal{F}) = h^0(Z, \Omega_Z^{m-1})$$

contraction and $\omega_Z \cong \mathcal{O}_Z$

$$= h^0(\tilde{Z}, \Omega_{\tilde{Z}}^{m-1})$$

[KS19, Cor. 1.8]

$$= h^{m-1}(\tilde{Z}, \mathcal{O}_{\tilde{Z}})$$

Hodge theory on $\tilde{Z}$

$$= h^{m-1}(Z, \mathcal{O}_Z)$$

$Z$ has rational singularities

$$= h^1(Z, \mathcal{O}_Z)$$

Serre duality [BS76, Ch. VII, Thm. 3.10]

$$= 0$$

because $q(Z) = \tilde{q}(Z) = 0$.

This ends the proof of Claim 4.5. Easy.

For (4.2.2), assume that $\text{Def}^h(Z)$ is smooth. Recall that $\text{Def}^h(T)$ is smooth in any case because $T$ is a complex torus. Hence also $B := \text{Def}^h(T) \times \text{Def}^h(Z)$ is smooth. Consider the product deformation of $T \times Z$ over $B$, i.e. the fibre over a point $(t, s) \in B$ is $T_t \times Z_s$. The Kodaira–Spencer map of this deformation,

$$\kappa: T_0B \longrightarrow H^1(T \times Z, \mathcal{F}_{T \times Z}),$$

is an isomorphism by (4.2.1). In particular, it is surjective and this implies that $\text{Def}^h(T \times Z)$ is smooth. Furthermore, our deformation over $B$ is pulled back from the semiuniversal deformation via a map

$$B \longrightarrow \text{Def}^h(T \times Z),$$

which on tangent spaces induces the isomorphism $\kappa$. Hence the map itself is likewise an isomorphism. This ends the proof. Easy.

4.B. Proof of Theorem 1.1. Keeping notation, note that $\dim Z = \dim X - \tilde{q}(X)$, so that we may assume $\dim Z \neq 3$ for the purpose of this proof. Note also that smoothness

of $\text{Def}^h(Z)$ implies smoothness of $\text{Def}^h(X)$ by Lemma 4.2, Proposition 4.1 and [GS20, Prop. 5.3]. We may therefore replace $X$ by $Z$ and then we only need to show the following: in the standard setting 1.5, if $\tilde{q}(X) = 0$ and $\dim X \in \{1, 2, 4\}$ then $\text{Def}^h(X)$ is smooth.

If $\dim X \leq 2$, then $X$ has only quotient singularities and hence the claim follows directly from [GS20, Cor. 1.7]. If $\dim X = 4$, then we apply Theorem 1.6 to conclude that the flat factor of $\mathcal{F}_X$ vanishes, $\mathcal{F} = 0$. (Note that here we have implicitly replaced $X$ by a cover, but this is harmless by [GS20, Prop. 5.3] again because that cover can be taken to be Galois.) Consequently, there are only two possibilities for the holonomy decomposition of $\mathcal{F}_X$. In slightly abused notation, they are:

- $\text{Sp}(2)$, the strongly stable case, and
- $\text{SU}(2) \oplus \text{SU}(2)$, the non-stable case.

Since $\text{SU}(2) = \text{Sp}(1)$, in both cases $X$ carries a holomorphic symplectic form. This implies smoothness of $\text{Def}^h(X)$ by [BL20, Thm. 4.7], cf. also the footnote in the proof of [CGGN20, Thm. 8.4]. Easy.

4.C. Proof of Corollary 1.2. Again let $T \times Z \to X$ be a torus cover. If $\dim Z = 3$, then $\text{Def}^h(X)$ is smooth by Theorem 1.1. In this case, Corollary 1.2 follows directly from [GS20, Thm. 1.2]. We may therefore assume that $\dim Z = 3$. In this case, $h^2(Z, \mathcal{O}_Z) = h^1(Z, \mathcal{O}_Z) = q(Z) = 0$ by Serre duality. Since also $h^2(T, \mathcal{O}_T) = 0$ for dimension reasons, we conclude from the Künneth formula that $H^2(T \times Z, \mathcal{O}_{T \times Z}) = 0$. This in turn implies $H^2(X, \mathcal{O}_X) = 0$. This Hodge number is constant in locally trivial
families, as one can see e.g. by performing a simultaneous resolution and using that \(X\) has rational singularities. That is, every locally trivial deformation \(X_t\) of \(X\) still satisfies \(H^2(X_t, \mathcal{O}_{X_t}) = 0\). By the Kodaira embedding theorem, \(X_t\) is projective for any \(t\) (including \(X_t = X\)). In particular, any locally trivial deformation of \(X\) is a strong algebraic approximation. Easy.

4.D. Proof of Corollaries 1.3 and 1.4. The projectivity of \(\pi_1(X)\) follows from Thom’s First Isotopy Lemma, which in our situation says that \(X\) and \(X_t\) are homeomorphic. For more details, cf. the proof of [Gra18, Cor. 1.8]. Thus (1.3.1) is proved.

Regarding (1.3.2), we know from Corollary 1.2 that \(X\) is locally algebraic and hence it admits a maximally quasi-étale cover \(\gamma : Y \to X\) by [CGGN20, Prop. 5.9]. By (2.3.3), the map \(\tilde{\pi}_1(Y_{\text{reg}}) \to \tilde{\pi}_1(Y)\) is an isomorphism. Also, \(\tilde{\pi}_1(Y_{\text{reg}}) = \tilde{\pi}_1(\gamma^{-1}(X_{\text{reg}})) \to \tilde{\pi}_1(X_{\text{reg}})\) is injective with image of finite index. It is therefore sufficient to prove the claim for \(\tilde{\pi}_1(Y_{\text{reg}})\). But this reduces to [CGGN20, Thm. G] applied to \(Y\).

We already remarked above that we now know that \(X\) is locally algebraic. Therefore Corollary 1.4 follows immediately from [CGGN20, Thm. H]. Easy.

References

[Akh95] D. N. Akhiezer: Lie group actions in complex analysis, Aspects of Mathematics, E27, Friedr. Vieweg & Sohn, Braunschweig, 1995. ↑ 4
[ABC+96] J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo: Fundamental Groups of Compact Kähler Manifolds, Mathematical Surveys and Monographs, vol. 44, American Mathematical Society, 1996. ↑ 2
[BGL20] B. Bakker, H. Guenancia, and C. Lehmann: Algebraic approximation and the decomposition theorem for Kähler Calabi–Yau varieties, arXiv:2012.00441 [math.AG], December 2020. ↑ 1, 3
[BL20] B. Bakker and C. Lehmann: The global moduli theory of symplectic varieties, arXiv:1812.09748 [math.AG], version 3, June 2020. ↑ 9
[BS76] C. Bănică and O. Stănășilă: Algebraic Methods in the Global Theory of Complex Spaces, John Wiley & Sons, 1976. ↑ 9
[Bea83] A. Beauville: Variétés Kähleriennes dont la première classe de Chern est nulle, J. Differential Geom. 18 (1983), no. 4, 755–782. ↑ 1
[Bog78] F. A. Bogožolov: Hamiltonian Kählerian manifolds, Dokl. Akad. Nauk SSSR 243 (1978), no. 5, 1101–1104. ↑ 1
[Cam04] F. Campana: Orbifolds, special varieties and classification theory, Ann. Inst. Fourier Grenoble 54 (2004), 499–630. ↑ 2
[CGGN20] B. Claudon, P. Graf, H. Guenancia, and P. Naumann: Kähler spaces with zero first Chern class: Bochner principle, fundamental groups, and the Kodaira problem, arXiv:2008.13008 [math.AG], August 2020. ↑ 1, 2, 6, 7, 9, 10
[DG94] G. Dethloff and H. Grauert: Seminormal complex spaces, Several complex variables, VII, Encyclopaedia Math. Sci., vol. 74, Springer, Berlin, 1994, pp. 183–220. ↑ 6
[Dru18] S. Drue: A decomposition theorem for singular spaces with trivial canonical class of dimension at most five, Invent. Math. 211 (2018), no. 1, 245–296. ↑ 1
[Gra18] P. Graf: Algebraic approximation of Kähler threefolds of Kodaira dimension zero, Math. Annalen 371 (2018), 487–516. ↑ 10
[GGK19] D. Greb, H. Guenancia, and S. Kebekus: Klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups, Geom. Topol. 23 (2019), no. 4, 2051–2124. ↑ 11
[HM07] C. D. Hacon and J. McKernan: On Shokurov’s rational connectedness conjecture, Duke Math. J. 138 (2007), no. 1, 119–136. ↑ 6
[HP19] A. Höring and T. Peternell: Algebraic integrability of foliations with numerically trivial canonical bundle, Invent. Math. 216 (2019), no. 2, 395–419. ↑ 1
[Keb13] S. Kebekus: Pull-back morphisms for reflexive differential forms, Adv. Math. 245 (2013), 78–112. ↑ 6
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S. KEBEKUS AND C. SCHNELL: Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities, arXiv:1811.03644 [math.AG], version 3, February 2019.

C. OKONEK, M. SCHNEIDER, AND H. SPINDLER: Vector Bundles on Complex Projective Spaces, Progress in Mathematics, vol. 3, Birkhäuser, 1980.

V. P. PLATONOV: A certain problem for finitely generated groups, Dokl. Akad. Nauk BSSR 12 (1968), 492–494.

G. TIAN: Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, pp. 629–646.

A. N. TODOROV: The Weil-Petersson geometry of the moduli space of SU(n) ≥ 3 (Calabi-Yau) manifolds. I, Comm. Math. Phys. 126 (1989), no. 2, 325–346.

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