Rigid String From QCD Lagrangian

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Abstract

Starting from the 4-dimensional gluodynamics, we discuss the statistical ensemble of torons (the toron is a self-dual solution with fractional topological number; it can be understood as a point defect) which interact strongly. It is shown that the effective Lagrangian describing this statistical ensemble possesses the fourth derivative kinetic term and leads to the area law for the Wilson loop.

Besides that we derive the effective string action containing the rigid (extrinsic curvature) term as a consequence of the fourth derivative action.
1. Introduction.

It is generally believed that the large-distance behavior of the confining phase of QCD is given by an effective string theory, see recent reviews [1], [2], [3]. The main problem of this idea can be formulated as follows. How to find the correct collective variables in terms of the underlying field theory (QCD) which effectively describe the string theory.

In some simple models (it is a very narrow family of all field theories), one has a map from original field theory to stringlike variables, and one can derive properties of the resulting "string theory" from the field theory. One of the well studied system of such kind is 2 + 1 dimensional Ising-like model described by the action

$$S = \frac{1}{2} \int d^3 x [\partial_\mu \phi \partial_\mu \phi + \lambda (\phi^2 - \frac{m^2}{\lambda})^2]$$

As is known this model possesses the Bloch wall solution which is independent of two variables (t and y) and well localized in the transverse direction z. Such solution spontaneously break translation invariance in the $D-2$ transverse directions. As a result, there are Nambu-Goldstone massless excitations about such background, even if the field theory has only massive excitations about trivial background. So the leading term in the effective field theory describing low energy phenomena must take the form

$$S_{\text{string}}(f^i) = \text{const} \int dt dy [ (\partial_t f^i)^2 + (\partial_y f^i)^2 ] + ..., \quad i = 1, 2, D-2$$

where $S_{\text{string}}(f^i)$ is the action governing low energy phenomena, and we have chosen the $t, y$ plane as the plane of the worldsheet.

The derivation of the $S_{\text{string}}(f^i)$ is standard [4], [5], [6], [7], [8]. The steps involved in going from a theory with soliton-like solution of eq.(1) to the effective string action (2) are: a) to rewrite a functional integral $\int D\phi e^{iS(\phi)}$ about a classical solution $f$ to the effective action (2); b) to integrate out these massive modes. Thus one has:

$$\int D\phi e^{iS(\phi)}|_{\text{about a classical solution}} = \int Df^i Dh^i e^{iS_{\text{string}}(f^i, h)} = \int Df e^{iS_{\text{string}}(f^i)}$$

It was shown for particular model (1) by [5] that the long wavelength expansion (2) for this model up to $O(\partial^6)$ reproduces the Nambu-Goto action for the string

$$S_{NG} \sim \sqrt{1 + (\partial f)^2} = \sqrt{\det h_{\mu\nu}}$$

with induced metric

$$h_{\mu\nu} = \delta_{\mu\nu} + \partial_\mu f \partial_\nu f.$$  

Besides that there are additional terms as well, which do not have a geometrical interpretation. Few comments are in order. First of all, the effective action (2) is not renormalizable. This is not a catastrophe, however, because the high frequency oscillations of $f$ are to be

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2 Here and what follows we are discussing the Euclidean version of the models
cut off anyhow: when internal degrees of freedom of the tube are excited, the description in terms of \( f \) breaks down.

As a second remark, let us note that the quantization of the underlying field theory induces a quantization of the effective string theory. As a consequence, Lorentz invariance holds in the effective string theory for any \( D \) (not only for \( D = 26 \)) [8].

As a last remark we note that the absence of the so-called rigid term [4], [10], describing the extrinsic curvature of the world sheet, is the direct consequence of the canonical expression for the kinetic term (without higher derivatives) of the underlying field theory (3). We’ll come back to this point later.

Thus the problem of derivation of the long distances effective Lagrangian from field theory, which possesses the soliton like solution reduces to the calculation of the functional integral (3). The physical sense of the string variable \( f \) in this case is clear and can be understood in terms of the original field variables \( \phi \). Indeed, roughly speaking, the string variable \( f \) describes the fluctuations about the classical solution \( \phi_{cl} \), see [4]–[8] for more details.

The gluodynamics does not belong to this class, and so, the corresponding methods can not be applied to Yang-Mills theory directly. However, it is generally believed that QCD might be represented as a string theory. Many of the hadron properties will be understandable in this case. The standard approach to this problem is a making of a guess what the effective long distances lagrangian is going to be. But in such approach the relation with the original QCD completely lost and the connection of the string variables \( f \) with gauge fields \( A_{\mu}^a \) looks absolutely unclear.

In this letter I shall follow in the opposite direction, from the gauge fields to string theory. In this case, the each step of the reformulation QCD in terms of string variables is going to be under control (at least in principle). Besides that, the string variable \( f \) can be expressed in terms of the gauge theory.

Schematically, the steps involved in going from QCD to the effective string theory look as follow.

\[
S = -\frac{1}{4} \int d^4x G_{\mu\nu} G^{\mu\nu} \quad \Rightarrow \quad Z = \sum_{k=0}^{\infty} \frac{\Lambda^{4(k_1+k_2)}}{(k_1)! (k_2)!} \sum_{I_1, I_2} \prod_{i=1}^{k_1+k_2} d^4x_i \exp(-\epsilon_{\text{int.}})
\]

\[
\langle W \rangle = \int Df \exp(-\int d^2\sigma L_{\text{string}}(f))
\]

Here the first step (1) is related to the consideration of the statistical ensemble of pseudoparticles (point defects) with fractional topological charge, so-called torons, which interact strongly. In the next Section we briefly formulate the basic assumptions of this toron approach. The second step (2) is more or less standard one which allows us to reformulate the statistical mechanics problem in terms of the functional integral over some auxiliary field \( \phi \) with some effective action \( S_{eff} \). It turns out that this \( S_{eff} \) can be considered in the same way as \( S \) from the formula (3) in a sense that in both cases we have some solitonic shape solution. So, the method briefly described above for transition...
from $S(1)$ to $S_{\text{string}}(2)$ by calculating of the functional integral (3) can be applied in the case of the gluodynamics as well. This is just step (3).

2. Review of the toron approach

Before we proceed to the detail consideration of the string representation of QCD, let me briefly formulate the basic assumptions of the toron approach, step (1).

i) I extend the class of admissible gauge transformation in gluodynamics. Thus, I allow the configurations with fractional topological charge (one half for $SU(2)$ group) in the definition of the functional integral. It means that a multivalued functions will appear in the functional integral. However, the main physical requirement is - all gauge invariant values must be singlevalued. Thus, the different cuts accompany the multivalued functions should be unobservable, i.e. the gauge invariant values coincide on the upper and on the lower edges of the cut.

At large distances the toron looks like a singular gauge transformation. At small distances this configuration should be somehow regularized. It can be explicitly done for the separate toron, but a general solution of this problem is still lacking. Fortunately, the long distances pseudoparticle interaction (the expression we are interested in) does not depend on regularization procedure.

The direct consequence of the such definition of the functional integral is the appearing of the new quantum number classifying the vacuum states. Indeed, as soon as we allowed one half topological charge, the number of the classical vacuum states is increased by the same factor two in comparison with a standard classification, counting only integer winding numbers $|n|$. Of course, vacuum transitions eliminate this degeneracy. However the trace of enlargement number of the classical vacuum states does not disappear. Vacuum states now classified by two numbers: $0 \leq \theta < 2\pi$ and $k = 0, 1$. These is in agreement with large $N$ results where the nontrivial $\theta$ dependence in pure YM theory comes through $\theta/N$ at large $N$ (in particular, $\langle \tilde{G}G \rangle \sim \sin(\frac{\theta}{N})$). Such a function can be periodic in $\theta$ with period $2\pi$ only if there are many vacuum states for given values of $\theta$. These vacua should not be degenerated due to the vacua transitions, however the trace of the enlargement number of the vacuum states have to be seen.

ii) The next main point of the toron approach may be formulated as follows. We hope that in the functional integral of the gluodynamics, when the bare charge tends to zero and when we are calculating some long range correlation function, only certain field configurations (the toron of all types) are important. In this case the hopeless problem of integration over all possible fields is reduced to the problem of summation over classical toron configurations. I have no proof that the system of configurations which have been

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3 See [11],[12] for much of the material in this section.
4 We keep the term "toron", introduced in ref.[13]. By this means we emphasize the fact that the considering solution minimizes the action and carries the topological charge $Q = 1/2$, i.e. it possesses all the characteristics ascribed to the standard toron [13]. However I should note from the very beginning of this paper, that our solution has nothing to do with the standard toron and it is formulated in principle in another way than in ref.[13]. The keeping of this term has a historical origin.
taken into account is a complete system. But I would like to stress that a lot of problems (like the \( \theta \) dependence, the \( U(1) \) problem, the counting of the discrete number of vacuum states, the confinement, the nonzero value of the vacuum energy and so on...) can be described in a very simple manner from this uniform point of view.

Both these points are quite nontrivial ones. However, I would like to convince the reader in the consistency of these assumptions by considering a few simple models, where, from the one hand, the results are well known beforehand and, from the other hand they can be reproduced by toron calculations [11].

Let me start by giving a few formulae from ref.[12]. The grand partition function which presumably describes 4 dimensional gluodynamics is given by

\[
Z = \sum_{k=0}^{\infty} \frac{\Lambda^{4(k_1+k_2)}}{(k_1)!(k_2)!} \sum_{l_a,l_a} d^4x_i \exp(-\epsilon_{\text{int.}}),
\]

where two different kinds of torons classified by the weight \( I_i \) of fundamental representation of the \( SU(2) \) group and \( q_i \) is the sign of the topological charge. Besides that, in formula (6) the value \( g^2(M_0) \) is the bare coupling constant and \( M_0 \) is ultraviolet regularization, so that eq.(6) depends on the renormalization invariant combination \( \Lambda \). In obtaining (6) we took into account that the classical contribution to \( Z \) from \( k \) torons is equal to

\[
Z \sim \exp(-\frac{4\pi^2}{g^2(k)}).
\]

Besides that the factor \( d^4x_i \) in eq.(6) is due to the 4 translation coordinates accompany an each toron and combinatorial factor \( k_1!k_2! \) is necessary for avoiding double counting for \( k_1 \) torons and \( k_2 \) antitorons; lastly, the average overall configurations \( q, I \) is an average over all isotopical directions and topological charge signs of torons.

To compute some vacuum expectation values it is convenient to use the correspondence between the grand partition function for the gas (5) and field theory with Sine-Gordon interaction, as it was done by Polyakov in ref.[16] for 3d QED. Let us rewrite (5) in the form:

\[
L_{\text{eff}} = \frac{1}{2} \sum_{l_a} \Lambda^4 \exp(i8\pi/\sqrt{3}I_\alpha \tilde{\phi} + i\theta/2) - \sum_{l_a} \Lambda^4 \exp(-i8\pi/\sqrt{3}I_\alpha \tilde{\phi} - i\theta/2).
\]

In this derivation it was used the fact that the logarithm function which appears in the formula for the interaction (5) is the Green function for the operator \( \Box \). After that we can use the method [14] to express the generating functional in terms of effective field theory [8].

\[5\]

It is quite obviously, that this Lagrangian does not correspond to any fundamental theory. In particular, the kinetic term has a forth derivative form, so this theory does not describe any asymptotic states in Minkowski space (there is no continuation from Euclidean space here). Thus, Lagrangian (8) is understood as effective one, describing the statistical ensemble of pseudoparticles.
In this effective field theory the sum over $\vec{I}_\alpha$ runs over the 2 weights of the fundamental representation of $SU(2)$ group. Note, that the first interaction term is related to torons and the second one to antitorons. Besides that, since we wish to discuss the $\theta$ dependence, we also include a term proportional to the topological charge density $\frac{\theta}{32\pi^2} G_{\mu\nu} \tilde{G}_{\mu\nu}$ to the starting Lagrangian and corresponding track from this to the effective Lagrangian (8).

From this expression it is clear that one of the feature of the effective Lagrangian (8) is the nontrivial dependence on $\theta$ of the topological density and susceptibility, relevant quantities for the solution of the $U(1)$ problem:

$$\delta Z_{\theta}/\delta \theta \sim \langle G_{\mu\nu} \tilde{G}_{\mu\nu} \rangle = i2\Lambda^4 \sin(\theta/2), \quad -\pi \leq \theta \leq \pi.$$  \hspace{1cm} (9)

As discussed above, such a dependence on $\theta$ is in agreement with large $N$ results. We mentioned here these few consequences of the effective Lagrangian (8) just to demonstrate that the system (3) reproduces these very nontrivial, but well established at large $N$ results (the correct $\theta/N$ dependence, the number of vacuum states equals $N$, the nonzero value for topological density $\langle G_{\mu\nu} \tilde{G}_{\mu\nu} \rangle \sim \sin(\theta/N)$ and so on) correctly. From the other hand, we expect (see [16] and references therein) that all problems under consideration (as well as confinement and string reformulation problems) are tightly connected. Thus, any self-consistent dynamical solution of one of them should be necessarily accompanied by the resolution of the rest problems within same approach.

Therefore we expect that the information about confinement and string representation of QCD somehow is coded in the effective Lagrangian (8).

3. String description of the gluodynamics.

Now we want to discuss the relation between string variable $f^i$ and auxiliary field $\phi$ from the effective Lagrangian (8). But before to do so, we would like to understand the physical sense of the $\phi$ field in terms of the underlying gauge theory. To this aim we define the $M$ as an operator that acts on original $A^a_{\mu}$ fields by gauge transforming them by $U^{x_0}(x)$; this gauge transformation is singular at $x_0$ and has the following property: for any plane crossing $x_0$ and for any $x$ at the plane, as soon as $x$ encircles $x_0$, $U$ does not return to its original value (as it happens in the instanton case), but acquires a $Z_N$ phase ($N = 2$ for $SU(2)$ group):

$$U^{x_0}(\alpha = 2\pi) = \exp(-i2\pi/N)U^{x_0}(\alpha = 0), \quad U = \exp(iI\alpha)$$  \hspace{1cm} (10)

where $\alpha$ is an angle variable in the chosen $x - y$ plane and the point $x_0$ lies at the same plane. From its definition it must be clear that $M(x)$ absorbs one half topological unit, so we say that $M(x)$ is the annihilation operator for one point toron at $x$ with weight $I$ and $M^+(x)$ is the creation operator for one toron. It should be clear, that $U$ depends on all $x_\mu$ variables, so that $M$ is the annihilation operator for the point defect. However, at the chosen $x - y$ plane, $U$ depends only on angle variable $\alpha$. The singularity of $A^a_{\mu} = iU^+ \partial_\mu U$ must be smeared over an infinitesimal region around $x$ as it was done for the separate toron solution, but the regularization problem does not influent on the following consideration.
We would like to express the $M$ in terms of the effective field theory (8). To this aim, let us consider the $\epsilon_{int}$ in the formula (3) after the action of the gauge transformation (10) at point $x_0$. Because this gauge transformation creates an additional toron at point $x_0$ with isospin $I_0$ in the system of the other torons placed at $x_i$ with isospins $I_i$ we will obtain an additional contribution to the $\epsilon_{int}$. Namely, after action of the operator $M$ we have an additional interaction term between created toron $I_0$ and torons $I_i$ from the system

$$\Delta\epsilon_{int} \sim \sum_i I_0 I_i \ln(x_0 - x_i)^2.$$  

(11)

It is easy to understand that this interaction after simply repeating the derivation of eq.(8), reduces to the following expression in the effective field theory:

$$\langle M(x_0) \rangle = \int D\phi e^{-\int d^4x L_{eff.}} \exp(i8\pi/\sqrt{3}I_0\phi(x_0) + i\theta/2)$$

(12)

Thus, the operator $M$ under consideration in the effective theory looks like this

$$M_\alpha(x) = e^{i\chi_\alpha(x) + i\theta/2}, \quad \chi_\alpha \equiv \frac{8\pi}{\sqrt{3}} I_\alpha\phi.$$  

(13)

Thus, the operator of large gauge transformation, $M$, which should be highly non-local and nontrivial in terms of the original fields ($A_{\mu}^a$ - gluons) has a very simple form in terms of the auxiliary variables $\chi_\alpha$. It gives the link $A_\mu \Longrightarrow \chi_\alpha$. From the other hand, we shall see in a few moments that the $\chi_\alpha$ field related in a very simple manner to string variable $f_i$. It gives the second wanted link $A_{\mu}^a \Longrightarrow \chi_\alpha \Longrightarrow f_i$.

It is interesting to note that the disorder operator $M$ (13) in gluodynamics has the same exponential form like in 2 + 1 dimensional QED [18].

Our next step is to consider the vacuum expectation value of the Wilson loop. As was explained above, the torons at large distances look like singular pure gauge field with definite isotopical direction and so, the $A_\mu^a$ field is abelian at large distances (in a more detail see [12]). Thus, the standard quasiclassical approximation, when we substitute for $A_\mu^a$ the corresponding classical solution, leads (after simply repeating the derivation of (8)) to the following expression for $\langle W \rangle$ at $\theta = 0$

$$\langle W \rangle = \langle Tr \exp(\oint_i i q A_\mu dx_\mu) \rangle =$$

$$\int D\chi_\alpha e^{-\int d^4x L_{eff.}}, \quad L_{eff} = 1/2(\frac{\sqrt{3}}{4\pi})^2(\square \chi_\alpha)^2 - \sum_i 2\Lambda^4\cos(\chi_\alpha + I_\alpha\Phi).$$

(14)

where the term proportional to $\Phi$ is related to Wilson loop insertion and has the following property : $\Phi(x)$ is equal to the external charge $2\pi q$ if $x \in S$, Wilson plane, and $\Phi = 0$ otherwise. In this derivation we took into account that if the toron is in the $S$ plane, then the integral over $G_{\mu\nu} d\sigma_{\mu\nu}$ is non-zero, and it is equal to zero otherwise (see for a more detail about toron properties the ref . [12]).

At this moment I would like to come back to discussion in the Introduction concerning of the semiclassical calculation of the functional integral in the solitonic background, see
formula (3). For this purpose we will be considering the effective Lagrangian from the formula (14) on the same foot as fundamental Lagrangian from the formula (1). To this aim, we have to look for the solution of the corresponding classical equation:

$$\Box \Box \chi'_d + 4\Lambda^2 \left( \frac{4\pi}{\sqrt{3}} \right)^2 \sin(\chi'_d) = 2\pi \theta_S(z, t) \delta'(x) \delta'(y),$$

(15)

where $$\chi' \equiv \chi + \Phi \vec{I}, \ \delta'(x) \equiv \frac{d\delta(x)}{dx}, \ \theta_S(z, t) = 1$$ if $$z, t \in S$$ and $$\theta_S(z, t) = 0$$ otherwise. The right-hand side of this equation is related to the Wilson loop insertion, i.e. with function $$\Phi(x)$$. 

I do not know an exact solution for this equation, but the physics suggests that a linearization is legitimate, so for qualitative estimation we can substitute $$\chi$$ instead of $$\sin(\chi)$$ and find $$\chi_{cl}$$ by means of Fourier transformation (here and what follows we drop the prime in the notation for $$\chi'$$ field)

$$\chi_{cl} \sim \theta_S(z, t) \int d^2 k e^{ikx} k_x k_y \frac{1}{k^4 + m^4}, \quad m^4 \equiv 4\Lambda^2 \left( \frac{4\pi}{\sqrt{3}} \right)^2$$

(16)

This solution correctly reproduces the discontinuity related to the right hand side of eq. (15). Besides that, the integral (14) can be reduced to the modified Bessel function $$K_0(z)$$ and so we have the exponentially localized in the transverse directions ($$x, y$$) solution, more exactly

$$\chi_{cl}(\vert x \vert \to \infty) \sim i \partial_x \partial_y (\exp(-e^{\frac{i\pi}{4}} m \vert x \vert) - \exp(-e^{-\frac{i\pi}{4}} m \vert x \vert)), \quad \vert x \vert \equiv \sqrt{x^2 + y^2}$$

(17)

Thus, this situation remind us the analysis of the 2 + 1 dimensional model (1) with the well-localized Bloch wall solution. Now we can expand the effective action (14) in the background of the classical solution (16) as it was done for 2 + 1 dimensional model (3), integrate over perpendicular $$x, y$$ directions and end up with some effective 'string' theory. But before to do so, I would like to make a few comments. First of all, there is a big difference between the induced string action derived from the underlying field theory (1) and from the effective Lagrangian (14). In the former case we have the classical solitonic solution corresponding to domain wall with $$\phi(\pm \infty) \neq \phi(0)$$. There are no sources (quarks) at all. So, we have infinitely long string. In the later case we are calculating the vacuum expectation value of the Wilson loop, so we have inserted the heavy quarks into the system. It means that we are describing an open string with the fixed ends.

As a consequence of the Wilson loop insertion, the right hand side of the eq. (15) has a very important significance: the classical solution (16) is appearing together with the

6 More exactly, the right hand side is proportional to $$2\delta'(x)\delta'(y) + \delta''(x)\text{sign}(y)\delta_{x,0} + \delta''(y)\text{sign}(x)\delta_{y,0}$$ where $$\delta_{x,0}$$ is Kroneker symbol. However the last two terms do not play any role in the following calculation and we will skip them for the simplification of formulae. The technical reason for that is the vanishing of the integrals like $$\int dy\delta_{y,0}G(y) = 0$$ for any smooth function $$G(y)$$.

7 It is clear, that we have essentially a nontrivial dependence only on two variables, $$x, y$$. The shape (17) guarantees the convergence of the corresponding integrals $$\int dxdy$$, over the direction perpendicular to the Wilson loop. So, starting from 4 dimensional action we end up with 2 dimensional 'string' theory.
sources. It means that the string will emerge into the theory simultaneously with source insertion.

The same situation takes place in 3-dimensional QED [16], but in this case the equation analogous [13] is essentially one dimensional. Right hand side of the corresponding equation is proportional to $2\pi\theta_S(z,t)\delta'(x)$ as a consequence of the Wilson loop insertion. Such nonlinearity ensures the exponentially localized in $x$ direction solution and reproduces the correct discontinuity of $\chi_{cl}$ solution. As a consequence of it we have an area law in the model.

Now we are in position to describe the low energy effective action. Because our solution [16] spontaneously breaks translation invariance in the $D - 2 = 2$ transverse space dimensions, there are Nambu-Goldstone massless excitations about such background. The derivation of the corresponding $S_{string}$ is standard and shortly discussed in the Introduction. However, for our purpose it is enough to keep only a few leading terms in the low energy expansion, so we can write the fluctuations of the $\chi$ field in the following form

$$\chi(t, z, x_i) = \chi_{cl}(x'_i = x_i + f_i(t, z)) + 0(f^2), \quad i = 1, 2$$  \hspace{1cm} (18)

where vector field $f^i$ can be treated as a string variable and represents the deflection of the thin flux tube from its rest position \footnote{Actually the thickness of the string is order $\Lambda$, the only dimensional parameter we have in gluodynamics. For justification of procedure, using in the text, see discussion at the end of this Section}. Using the decomposition [13] the $(\Box \chi)^2$ can be represented in the following form

$$\Box \chi = (\frac{\partial^2 \chi_{cl}}{\partial x_k^2})^2 + 2(\frac{\partial^2 \chi_{cl}}{\partial x_k^2})(\frac{\partial^2 \chi_{cl}}{\partial x_i \partial x_j \partial x_k}) + \frac{\partial \chi_{cl}}{\partial x_i} \frac{\partial f_i}{\partial x_k} + \frac{\partial \chi_{cl}}{\partial x_i} \frac{\partial f_i}{\partial x_k} + 0(f^2) \hspace{1cm} (19)$$

Here $x_i, \quad i = 1, 2$ are variables, perpendicular to Wilson loop. They describe the classical solution [13]. At the same time $x_\mu, \quad \mu = 0, 3$ variables describe the string plane. The position of the string is specified by a two-component vector field $f_i$, depending on $x_\mu$.

If we now insert this configuration [18] into the functional integral [13] and perform the $x_i$ integrations we shall obtain some effective string Lagrangian which, from the very general arguments, must be local and invariant under the Poincare transformations in $x_\mu \equiv (z, t)$ plane. Besides that this Lagrangian must be invariant under $0(2)$ rotations and translations of the vector field $f_i$. Of course, all these properties will be fulfilled automatically in our scheme. Explicit integration over $d^2 x_i$ gives the following formula for effective string action describing the long wavelength fluctuations of the string (we are keeping only leading terms of the expansion, proportional to $(f^2)$):

$$S_{string} = \Lambda^2 \int d^2 x_\mu \{c_1 + c_2(\frac{\partial f_i}{\partial x_\mu} \frac{\partial f_i}{\partial x_\mu}) + c_3 \Lambda^{-2}(\frac{\partial^2 f_i}{\partial x_\mu \partial x_\nu \partial x_\nu} - \frac{\partial^2 f_i}{\partial x_i \partial x_\mu \partial x_\mu}) + 0(f^4) + \ldots\}. \hspace{1cm} (20)$$

Here, $c_1, c_2, c_3$ are dimensionless constants determined by classical solution $\chi_{cl}(x_i)$. In particular, the constant $c_1$ is two-dimensional classical action:

$$c_1 = \Lambda^{-2} \int d^2 x_i L_{eff}(\chi_{cl}) \hspace{1cm} L_{eff} = (\frac{\sqrt{3}}{4\pi})(\chi_{cl} - \eta)\Box \chi - \Lambda^4[\cos(\chi_{cl}) - 1] \hspace{1cm} (21)$$
and $c_2$ and $c_3$ are defined by the following integrals

$$
c_2 = \Lambda^{-2} \left( \frac{\sqrt{3}}{4\pi} \right)^2 \int d^2 x_i (\chi_{cl} - \eta) \Box \Box (\chi_{cl} - \eta) \tag{22}
$$

$$
c_3 = \left( \frac{\sqrt{3}}{4\pi} \right)^2 \int d^2 x_i (\chi_{cl} - \eta) \Box (\chi_{cl} - \eta). \tag{23}
$$

Here $\eta$ is related to Wilson loop insertion and has the property that $\Box \Box \eta \sim \delta'(x)\delta'(y)$ and $\eta \sim \text{sign}(y)\delta_{y,0}\text{sign}(x)\delta_{x,0}$, see footnote after formula (13). Now we can estimate these coefficients using the approximate expression for the $\chi_{cl}$ from (16). We expect that the accuracy for such procedure is not very high and the numerical coefficient given below should be considered as an estimation of the order of value. With these remarks in mind we obtain:

$$
c_1 \simeq 2c_2 \simeq \frac{\sqrt{3\pi}}{32}, \quad c_3 \simeq \frac{3}{256\pi}. \tag{23}
$$

We emphasize that the convergent result for these coefficients is the direct consequence of the fourth derivative term in the action as well as correct magnitude for the discontinuity related to Wilson insertion. Technically it can be seen from the following expression for the $\Box \Box (\chi_{cl} - \eta)$:

$$
\Box \Box (\chi_{cl} - \eta)(x) \sim \int d^2 k e^{ikx} k_y \left( \frac{k^4}{k^4 + m^4} - 1 \right) \sim \int d^2 k e^{ikx} k_y \left( \frac{m^4}{k^4 + m^4} \right). \tag{24}
$$

On substitution of (24) into the formulae (21,22) we obtain the convergent integral at large $k$ ($x, y \to 0$) as was announced.

Now we want to rewrite the effective string action (20) in geometrical terms. With this aim we note that the first two terms in the derivative expansion can be represented in the Nambu-Goto form:

$$
S_{NG} = \frac{1}{2\pi \alpha'} \int d^2 x_\mu \sqrt{1 + \left( \frac{\partial f_i}{\partial x_\mu} \frac{\partial f_i}{\partial x_\mu} \right)} = \frac{1}{2\pi \alpha'} \int d^2 x_\mu \sqrt{\text{det} h} \tag{25}
$$

with induced metric

$$
h_{\mu\nu} = \delta_{\mu\nu} + \frac{\partial f_i}{\partial x_\mu} \frac{\partial f_i}{\partial x_\mu}, \quad h \equiv \text{det} h_{\mu\nu} \tag{26}
$$

and string tension $(2\pi \alpha')^{-1} \equiv c_1 \Lambda^2$. The third term of the expansion (20) can be rewritten (up to higher corrections) as the extrinsic curvature

$$
S_K = \frac{1}{\alpha_r} \int d^2 x_\mu \sqrt{h} K_{\mu\nu}^i K^{i\mu\nu} = \frac{1}{\alpha_r} \int d^2 x_\mu \sqrt{h} (\Delta(h) f^i)^2, \quad \Delta(h) f^i = \frac{1}{\sqrt{h}} \partial_\mu (\sqrt{h} h^{i\mu} \partial_\nu f^i), \tag{27}
$$

where $K_{\mu\nu}^i$ is known as the second fundamental form and $\alpha_r^{-1} \equiv c_3$ is the rigidity parameter.

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9The calculation of $c_3$ can be easily done by means of Fourier transformation, using the identity $\Box = \Box^{-1} \Box \Box$ and acting by operator $\Box \Box$ on $(\chi_{cl} - \eta)(x)$ as shown in (24) and substituting instead of $\Box^{-1}$ its Fourier transformed expression $k^{-2}$. 

10
Thus, our effective string action in the leading order in $f$ can be written as follows

$$S_{\text{string}} = S_{NG} + S_K + ...$$

(28)

This new, rigid term was introduced to string theory in refs.[9],[10]. It is easy to see that $S_K$ is the invariant under the scale transformation and $\alpha_r$ is dimensionless constant. The motivation [9] for the inclusion of the extrinsic curvature term to the string action is some desire to get a “smooth” string. Indeed with only intrinsic terms, surfaces can crumple up over arbitrary short distances, as long as their total area is preserved. The extrinsic curvature acts to give the surface rigidity, smoothing it out over short distances. This property is quite desirable for QCD.

The formula (28) with calculable (in principle) coefficients $\alpha', \alpha_r$ is the main result of this letter. Now several comments are in order.

i) The quantization of the underlying field theory (4-dimensional YM) induces a quantization of the induced string theory (28). In particularly, it is clear, that the light cone quantization of the fundamental string is not relevant to quantization of the effective string (28). For instance, we would expect that the Lorentz invariance of the effective string is a good symmetry not only for dimension $D = 26$, but for $D = 4$ as well. It has been checked explicitly for a more simple $2 + 1$ dimensional model [8] and I believe that the same is true in our case as well.

ii) It is well-known that the higher derivative Lagrangian, defined in Minkowski space-time, violates the unitarity because of exponentially growing modes in time (see, e.g. [2],[19]). However, a priori, there is nothing wrong with theories of surfaces embedded in Euclidean space-time and described by the extrinsic curvature term. Anyhow, we are considering the Lagrangian (14) with higher derivative terms, as an effective one, describing our statistical ensemble of pseudoparticles (6). The appearance of the fourth derivative term in this Lagrangian (as well as extrinsic curvature term $S_K$ in the string effective action (27)) is the direct consequence of the strong $\sim \ln(x_i - x_j)^2$ pseudoparticle interaction at large distances.

iii) It is useful to treat our effective string Lagrangian like the chiral Lagrangian (describing $\pi$-meson physics) in a sense that only lightest degrees of freedom are relevant to the problem. In such treatment the small-distance physics (regularization, renormalization, loop calculation and so on...) is coded in the magnitude of constants $c_i$.

iv) As a next remark, I would like to comment the result [20], concerning the rigid string. It was shown that starting from renormalizable unitary field theory one can get the higher derivative terms in the effective action by integrating out heavy fields from underlying field theory. However, in such procedure one gets the wrong sign for the fourth-order derivative terms in effective action and the corresponding strings are not smooth.

We would like to note, that the effective action (14), we are dealing with, has different origin. It describes the statistical ensemble (6) of pseudoparticle. The corresponding Partition Function is well-defined; the field $\chi$ which appears in the effective action is auxiliary one and was introduced just to describe this statistical ensemble. It is clear that $\chi$ field does not describe the asymptotic states and thus, the argumentation of [20] can not be applied to this case. So, we should not be surprised
that the string action we derived (28) has the correct sign for rigid term. The technical reason for that can be easily seen from the expression for \((\Box \chi)^2\), (12). The relative positive sign in this decomposition leads to the positive sign between Nambu-Goto \(S_{NG}\) and rigid \(S_K\) terms in the string action (28).

v) We keep only a few leading terms in the low energy expansion. To include higher order effects in \(f\), one needs to integrate over all massive excitations in the classical background, as it was done for simple models in [1]-[8]. It is clear that in such procedure we will get an infinite number of terms. Some of these terms have a geometrical interpretation, some of them— not. But we expect that in the long wavelength limit only a few leading terms (which have a very clear geometrical and physical meaning) of this expansion are important and they are given by formula (28). Here some arguments in favor of this hope.

But before to give these arguments, let us try to answer on the following question. Whether the effective string description, obtained from underlying field theory describes the long wavelength limit correctly? In other words, is it possible to choose some parameters of the original theory so that the energy of fluctuations to be too small to excite the internal structure of the string. In this case the string can be considered as the structureless with zero width. Nielson and Olesen [21] addressed this question in the Abelian Higgs model and they have shown that the string is effectively of zero width when the length scale of the energy levels for excitations of the string (defined by string tension \(\alpha'\)) is much greater than the length scales characterizing the width of the string (the penetration depth and the correlation length in the Abelian Higgs model). In the model this requirement corresponds to the electric charge much bigger than one, \(e \gg 1\), thus the thin string condition is the strong coupling limit, \(\bar{\hbar} \rightarrow \infty\), which makes semiclassical approximation very doubtful.

The same situation takes place for the model (1), see [8]. Indeed in this case the classical string solution in eq. (1) has a width of order \(m^{-1}\); the energy scale for excitations of the string is given by

\[
\frac{1}{2\pi\alpha'} \sim \int dz L_{cl}(z) \sim \frac{m^3}{\lambda}.
\]

The string can be considered as thin one when the internal modes will not be excited, i.e.

\[
m^2 \gg \frac{1}{2\pi\alpha'}, \quad \Rightarrow \lambda \gg m
\]

As before, this condition means the strong coupling limit and it is not clear whether the standard semiclassical approximation can be applied to this system.

In contrast with these explicit strings, I would expect that QCD (implicit) string, we are interested in, has different features. Let me demonstrate this point by considering the relation analogous to (30). But first of all I should note, that in contract with 2 + 1 dimensional models discussed above, in gluodynamics we have the only parameter in the theory, \(\Lambda\), and thus, all relations like (30) have numerical and not parametrical meaning. In our approximation the string tension \((2\pi\alpha')^{-1}\) is equal to \(c_1\Lambda^2\). At the same time, the characteristic scale of the internal excitations of the string is determined by its width end is equal to \(m\) ([16][17]). The criterion for string to be thin when fluctuations of the string
will not excite the internal modes, looks as follows

\[ m^2 \gg \frac{1}{2\pi \alpha'}, \quad \Rightarrow 2\Lambda^2 \left(\frac{4\pi}{\sqrt{3}}\right) \gg c_1 \Lambda^2 \quad (31) \]

With our very rough estimation for coefficient \( c_1 \) (23), the criterion (31) is satisfied. Probably this numerical smallness for \( \frac{1}{2\pi \alpha'} \) is related, somehow, to \( 1/N \) expansion. We are considering this numerical game as a hint suggesting how the structureless string could appear. In some sense, the inequality (31) is justification for our expansion (18), see footnote after this formula.

4. Final remarks.

The main point of this letter can be formulated as follows. We believe that most of the fundamental problems in QCD, such as the \( \theta/N \) dependence, \( N \) vacuum states, confinement, string representation of QCD, and so on, should be solved at the same time within the same dynamical approach. Some of these problems can be understood within so-called "toron approach". In particular, from the corresponding effective Lagrangian (8) it is possible to reproduce the correct behavior for the vacuum expectation value of the topological density \( <\tilde{G}G>_k \sim \sin\left(\frac{\theta + 2\pi k}{N}\right) \) and number of vacuum states \( N \). Therefore, we would expect the information about confinement (and string representation as a consequence of it) is coded in the same Lagrangian.

We have demonstrated the possible way of extracting this information from Lagrangian. The result is the formula (28). Let us note that the string fields \( f_i \) in the formula are not the space-time coordinates, but some variables related to color space. This is hardly surprising because the 't Hooft's analysis (22) of the large \( N \) behavior (this is the main motivation of our belief in string picture of QCD) ensures planarity in index space and not in real space-time. The main fundamental (not technical) assumptions I made in this derivation are following:

i) The multivalued functions are admissible in the definition of the functional integral. This assumption is related to a new classification of the vacuum states.

ii) The only certain field configurations (torons) are important and the problem of integration over all possible fields is reduced to the problem of summation over classical toron configurations.

It is quite possible that the technical realization of it can be given in a different, more appropriate way than we discussed. But I believe that the main point of this Letter, the new classification of gauge fields, will emerge in the formulation of the Theory.

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