Simulating Parity Reasoning (extended version)*

Tero Laitinen, Tommi Junttila, and Ilkka Niemelä

Aalto University
Department of Information and Computer Science
PO Box 15400, FI-00076 Aalto, Finland
Tero.Laitinen,Tommi.Junttila,Ilkka.Niemela@aalto.fi

Abstract. Propositional satisfiability (SAT) solvers, which typically operate using conjunctive normal form (CNF), have been successfully applied in many domains. However, in some application areas such as circuit verification, bounded model checking, and logical cryptanalysis, instances can have many parity (xor) constraints which may not be handled efficiently if translated to CNF. Thus, extensions to the CNF-driven search with various parity reasoning engines ranging from equivalence reasoning to incremental Gaussian elimination have been proposed. This paper studies how stronger parity reasoning techniques in the DPLL(XOR) framework can be simulated by simpler systems: resolution, unit propagation, and parity explanations. Such simulations are interesting, for example, for developing the next generation SAT solvers capable of handling parity constraints efficiently.

1 Introduction

Propositional satisfiability (SAT) solver technology has developed rapidly providing a powerful solution technique in many industrial application domains (see e.g. [1]). The efficiency of SAT solvers is partly due to efficient data structures and algorithms that allow very efficient Boolean constraint propagation and conflict-driven clause learning in conjunctive normal form (CNF). Straightforward Tseitin-translation [2] of a problem instance to CNF may result in poor performance, especially in the case of parity (xor) constraints, that can be abundant in applications such as circuit verification, bounded model checking, and logical cryptanalysis. Although pure parity constraints (linear arithmetic modulo two) can be efficiently solved with Gaussian elimination, they can be very difficult for resolution [3] and thus for state-of-the-art conflict-driven clause learning (CDCL) satisfiability solvers as their underlying proof system is equivalent to resolution [4]. Due to this inherent hardness of parity constraints, several approaches to combining CNF-level and xor-constraint reasoning have been proposed [5][6][7][8][9][10][11][12][13][14][15][16][17][18] (see [19] for an alternative state-based approach). In these approaches, CNF-driven search has been extended with various parity reasoning techniques, ranging from plain unit propagation via equivalence reasoning

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to Gaussian elimination. Stronger parity reasoning may prune the search space effectively but often at the expense of high computational overhead, so resorting to simpler but more efficiently implementable systems, e.g. unit propagation, may lead to better performance.

In this paper, we study to what extent such simpler systems can simulate stronger parity reasoning engines in the DPLL(XOR) framework [13]. The DPLL(XOR), similar to the DPLL(\(T\)) approach [20] to Satisfiability Modulo Theories, is a framework to integrate a parity reasoning engine to a CDCL SAT solver. The aim is to offer generalizable results that provide a foundation for developing techniques to handle xor-constraints in next generation SAT solvers. Instead of developing yet another propagation engine and assessing it through an experimental comparison we believe that useful insights can be acquired by considering unanswered questions on how some existing propagation engines and proof systems relate to each other on a more fundamental level. Several experimental studies have already shown that SAT solvers extended with different parity reasoning engines can outperform unmodified solvers on some instance families, so we focus on more general results on the relationships between resolution, unit propagation, equivalence reasoning, parity explanations, and Gauss-Jordan elimination, which is a complete parity reasoning technique.

We show that resolution can simulate equivalence reasoning efficiently, which raises a question whether significant reductions in solving time can be gained by integrating specialized equivalence reasoning in a SAT solver since in theory it does not strengthen the underlying proof system of the SAT solver. In practice, though, the performance of the SAT solver is largely governed by variable selection and other heuristics that are likely to be non-optimal, which may justify the pragmatic use of equivalence reasoning.

Although equivalence reasoning alone is not enough to cross the “exponential gap” between resolution and Gauss-Jordan elimination, another light-weight parity reasoning technique comes intriguingly close at simulating complete parity reasoning. We show that parity explanations, an efficiently implementable conflict explanation technique, on nondeterministic unit propagation derivations can simulate Gauss-Jordan elimination on a restricted yet practically relevant class of xor-constraint conjunctions. Choosing assumptions and unit propagation steps nondeterministically may not be possible in an actual implementation with greedy propagation strategies. However, we present further experimental results indicating that the simulation may still work in an actual implementation to some degree provided that parity explanations are stored as learned xor-constraints as described in [16].

Additional xor-constraints can also be added to the formula in a preprocessing step in order to enable unit propagation to deduce more implied literals, which has the benefit of not requiring modifications to the SAT solver. We present a translation that enables unit propagation to simulate parity reasoning systems stronger than equivalence reasoning through the use of additional xor-constraints on auxiliary variables. The translation takes into account the structure of the original conjunction of xor-constraints and can produce compact formulas for sparsely connected instances. Using the translation to simulate full Gauss-Jordan elimination with plain unit propagation requires an exponential number of additional xor-constraints in the worst case, but we show that the translation is polynomial for instance families of bounded treewidth. Recently, it has
been shown in [21] that a conjunction of xor-constraints does not have a polynomial-size “arc consistent” CNF-representation, which implies it is not feasible to simulate Gauss-Jordan elimination by unit propagation in the general case. On many instances, though, better solver performance can be obtained by simulating a weaker parity reasoning system as it reduces the size of the translation substantially. By applying our previous results on detecting whether unit propagation or equivalence reasoning is enough to deduce all implied literals, the size of the translation can be optimized further. The experimental evaluation on a challenging benchmark set suggests that the translation can lead to significant reduction in the solving time for some instances.

The proofs of lemmas and theorems are in the appendix.

2 Preliminaries

Let $\mathbb{B} = \{\bot, \top\}$ be the set of truth values “false” and “true”. A literal is a Boolean variable $x$ or its negation $\neg x$ (as usual, $\neg x$ will mean $x$), and a clause is a disjunction of literals. If $\phi$ is any kind of formula or equation, (i) $\text{vars}(\phi)$ is the set of variables occurring in it, (ii) $\text{lits}(\phi) = \{x, \neg x \mid x \in \text{vars}(\phi)\}$ is the set of literals over $\text{vars}(\phi)$, and (iii) a truth assignment for $\phi$ is a, possibly partial, function $\tau : \text{vars}(\phi) \to \mathbb{B}$. A truth assignment satisfies (i) a variable $x$ if $\tau(x) = \top$, (ii) a literal $\neg x$ if $\tau(x) = \bot$, and (iii) a clause $(l_1 \lor \ldots \lor l_k)$ if it satisfies at least one literal $l_i$ in the clause.

Resolution. Given two clauses, $x \lor C$ and $\neg x \lor D$ for arbitrary disjunctions of literals $C$ and $D$, their resolvent is $C \lor D$. Given a CNF formula $\phi$, a resolution derivation on $\phi$ is a finite sequence $\pi = C_1 C_2 \ldots C_m$ of clauses such that for all $1 \leq i \leq m$ it holds that either (i) $C_i$ is a clause in $\phi$, or (ii) $C_i$ is the resolvent of two clauses, $C_j$ and $C_k$, in $\pi$ with $1 \leq j,k < i$. A clause $C$ is resolution derivable from $\phi$ if there is resolution derivation on $\phi$ including $C$. The formula $\phi$ is unsatisfiable if and only if the empty clause is resolution derivable from $\phi$.

Xor-constraints. An xor-constraint is an equation of the form $x_1 \oplus \ldots \oplus x_k \equiv p$, where the $x_i$s are Boolean variables and $p \in \mathbb{B}$ is the parity[1]. We implicitly assume that duplicate variables are always removed from the equations, e.g. $x_1 \oplus x_2 \oplus x_1 \oplus x_3 \equiv \top$ is always simplified into $x_3 \equiv \top$. If the left hand side does not have variables, then it equals to $\bot$; the equation $\bot \equiv \top$ is a contradiction and $\bot \equiv \bot$ a tautology. We identify the xor-constraint $x \equiv \top$ with the literal $x$, $x \equiv \bot$ with $\neg x$, $\bot \equiv \bot$ with $\bot$, and $\top \equiv \bot$ with $\bot$. A truth assignment $\tau$ satisfies an xor-constraint $x_1 \oplus \ldots \oplus x_k \equiv p$ if $\tau(x_1) \oplus \ldots \oplus \tau(x_k) = p$. We use $D[x/Y]$ to denote the xor-constraint obtained from $D$ by substituting the variable $x$ in it with $Y$. For instance, $(x_1 \oplus x_2 \oplus x_3 \equiv \top)[x_1/x_2 \equiv \top] = x_2 \oplus \top \oplus \top \equiv \top$. The straightforward CNF translation of an xor-constraint $D$ is denoted by $\text{cnf}(D)$; for instance, $\text{cnf}(x_1 \oplus x_2 \oplus x_3 \equiv \bot) = (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land \neg x_1 \lor x_2 \lor x_3 \land x_1 \lor \neg x_2 \land \neg x_3$. We define the linear combination of two xor-constraints, $D = (x_1 \oplus \ldots \oplus x_k \equiv p)$ and $D' = (x_1 \oplus \ldots \oplus x_k \equiv q)$, as $D \land D' = (x_1 \oplus \ldots \oplus x_k \equiv p \land q)$.

[1] The correspondence of xor-constraints to the “xor-clause” representation used e.g. in [13,15,16] is straightforward: $x_1 \oplus \ldots \oplus x_k \equiv \top$ corresponds to the xor-clause $(x_1 \oplus \ldots \oplus x_k)$ and $x_1 \oplus \ldots \oplus x_k \equiv \bot$ to $(x_1 \oplus \ldots \oplus x_k \oplus \top)$. 
\( E = (y_1 \oplus \ldots \oplus y_l \equiv q) \), by \( D + E = (x_1 \oplus \ldots \oplus x_k \oplus y_1 \oplus \ldots \oplus y_l \equiv p \oplus q) \). An xor-constraint \( E = (x_1 \oplus \ldots \oplus x_k \equiv p) \) with \( k \geq 1 \) is a prime implicate of a satisfiable xor-constraint conjunction \( \phi_{xor} \) if (i) \( \phi_{xor} \models E \) but (ii) \( \phi_{xor} \not\models E' \) for all xor-constraints \( E' \) for which \( \operatorname{vars}(E') \) is a proper subset of \( \operatorname{vars}(E) \).

A cnf-xor formula is a conjunction \( \phi_{or} \land \phi_{xor} \), where \( \phi_{or} \) is a conjunction of clauses and \( \phi_{xor} \) is a conjunction of xor-constraints. A truth assignment satisfies \( \phi_{or} \land \phi_{xor} \) if it satisfies every clause and xor-constraint in it.

### 2.1 DPLL(XOR) and Xor-Reasoning Modules

We are interested in solving the satisfiability of cnf-xor formulas of the form \( \phi_{or} \land \phi_{xor} \) defined above. Similarly to the DPLL(T) approach for Satisfiability Modulo Theories, see e.g. [20][22], the DPLL(XOR) approach [13] for solving cnf-xor formulas consists of (i) a conflict-driven clause learning (CDCL) SAT solver that takes care of solving the CNF-part \( \phi_{or} \), and (ii) an xor-reasoning module that handles the xor-part \( \phi_{xor} \). The CDCL solver is the master process, responsible of guessing values for the variables according to some heuristics (“branching”), performing propagation in the CNF-part, conflict analysis, restarts etc. The xor-reasoning module receives variable values, called xor-assumptions, from the CDCL solver and checks (i) whether the xor-part can still be satisfied under the xor-assumptions, and (ii) whether some variable values, called xor-implied literals, are implied by the xor-part and the xor-assumptions. These checks can be incomplete, like in [13][15][12] for the implication checks, as long as the satisfiability check is complete when all the variables have values. The very basic interface for an xor-reasoning module can consist of the following methods:

- \( \text{init}(\phi_{xor}) \) initializes the module with \( \phi_{xor} \). It may return “unsat” if it finds \( \phi_{xor} \) unsatisfiable, or a set of xor-implied literals, i.e. literals \( \hat{l} \) such that \( \phi_{xor} \models \hat{l} \) holds.
- \( \text{assume}(l) \) is used to communicate a new variable value \( l \) deduced in the CNF solver part to the xor-reasoning module. This value, called xor-assumption literal \( l \), is added to the list of current xor-assumptions. If \([l_1, \ldots, l_k]\) are the current xor-assumptions, the module then tries to (i) deduce whether \( \phi_{xor} \land l_1 \land \ldots \land l_k \) became unsatisfiable, i.e. whether an xor-conflict was encountered, and if this was not the case, (ii) find xor-implied literals, i.e. literals \( \hat{l} \) for which \( \phi_{xor} \land l_1 \land \ldots \land l_k \models \hat{l} \) holds. The xor-conflict or the xor-implied literals are then returned to the CNF solver part so that it can start conflict analysis (in the case of xor-conflict) or extend its current partial truth assignment with the xor-implied literals.

In order to facilitate conflict-driven backjumping and clause learning in the CNF solver part, the xor-reasoning module has to provide a clausal explanation for each xor-conflict and xor-implied literal it reports. That is,

- if \( \phi_{xor} \land l_1 \land \ldots \land l_k \) is deduced to be unsatisfiable, then the module must report a (possibly empty) clause \( (\neg l'_1 \lor \ldots \lor \neg l'_m) \) such that (i) each \( l'_i \) is an xor-assumption or an xor-implied literal, and (ii) \( \phi_{xor} \land l'_1 \land \ldots \land l'_m \) is unsatisfiable (i.e. \( \phi_{xor} \not\models (\neg l'_1 \lor \ldots \lor \neg l'_m) \));
- if it was deduced that \( \phi_{xor} \land l_1 \land \ldots \land l_k \models \hat{l} \) for some \( \hat{l} \), then the module must report a clause \( (\neg l'_1 \lor \ldots \lor \neg l'_m \lor \hat{l}) \) such that (i) each \( l'_i \) is an xor-assumption
\[
\begin{array}{cccc}
D[x/\top] & D[x/\bot] & D[x/y] & D[x/y/\top] \\
\oplus\text{-Unit}^+ & \oplus\text{-Unit}^- & \oplus\text{-Eqv}^+ & \oplus\text{-Eqv}^-
\end{array}
\]

Fig. 1. Inference rules of Subst; \(x\) and \(y\) are variables, \(D\) is an xor-constraint, and \(x\) occurs in \(D\).

or an xor-implied literal reported earlier, and (ii) \(\phi_{\text{xor}} \land l'_1 \land \ldots \land l'_m \models \hat{l}\), i.e. \(\phi_{\text{xor}} \models (\neg l'_1 \lor \ldots \lor \neg l'_m \lor \hat{l})\).

- backtrack() retracts the latest xor-assumption and all the xor-implied literals deduced after it.

Naturally, variants of this interface are easily conceivable. For instance, a larger set of xor-assumptions can be given with the assume method at once instead of only one.

For xor-reasoning modules based on equivalence reasoning, see [13,15]. The Gaussian and Gauss-Jordan elimination processes in [12,14,23,18] can also be easily seen as xor-reasoning modules.

3 Equivalence Reasoning and Resolution

We know that there exist infinite families of xor-constraint conjunctions \(\phi_{\text{xor}}\) whose CNF translations \(\bigwedge_{D \in \phi_{\text{xor}}} \text{cnf}(D)\) have no polynomial size resolution proofs [3]. On the other hand, Gaussian elimination [14] can solve the satisfiability of xor-constraint conjunctions in polynomial time (and Gauss-Jordan [23,18] can detect all xor-implied literals as well). As these elimination procedures can be computationally heavy, more light-weight “equivalence reasoning” systems have been proposed [6,10,13,15].

Here we study how the equivalence reasoning systems Subst [13] and EC [15] relate to resolution. These systems are equally powerful in detecting unsatisfiability and xor-implied literals (we’ll use Subst due to its notational simplicity); they are more powerful than unit propagation but weaker than Gaussian/Gauss-Jordan elimination.

The Subst deduction system consists of the inference rules in Fig. 1. Given a conjunction \(\psi\) of xor-constraints, a Subst-derivation on it is a vertex-labeled directed acyclic graph \(G = (V, E, L)\) such that for each vertex \(v \in V\) it holds that (i) if \(v\) has no incoming edges, then \(L(v)\) is an xor-constraint in \(\psi\), and (ii) otherwise \(v\) has two incoming edges, say from \(v'\) and \(v''\), and \(L(v)\) is obtained from \(L(v')\) and \(L(v'')\) by applying one of the inference rules. As an example, Fig. 2(a) shows a Subst-derivation on \((x \oplus y \oplus z \equiv \top) \land (x \oplus z \oplus w \equiv \bot) \land (y \oplus w \oplus t \equiv \top) \land (x)\), please ignore the “Cut W” line for now.

If we can derive an xor-constraint \(D\) with Subst, we can derive (in the CNF translated instance) a CNF translation of \(D\) with resolution relatively compactly:

**Theorem 1.** Assume a Subst-derivation \(G = (V, E, L)\) on a conjunction \(\psi\) of xor-constraints. There is a resolution derivation \(\pi\) on \(\bigwedge_{D \in \psi} \text{cnf}(D)\) such that (i) if \(v \in V\) and \(L(v) \neq \top\), then the clauses \(\text{cnf}(L(v))\) occur in \(\pi\), and (ii) \(\pi\) has at most \(|V|^2 m - 1\) clauses, where \(m\) is the number of variables in the largest xor-constraint in \(\psi\).
A similar result is already observed in [6] when restricted on binary and ternary xor-constraints. Recalling that for each xor-constraint $D$ the CNF translation $cnf(D)$ is exponentially large in the number of variables in $D$, we can say that resolution simulates Subst-derivations “pseudo-linearly”. Furthermore, the natural encodings in many application domains (e.g. logical cryptanalysis) seem to employ xor-constraints with only few (typically 3) variables only.

3.1 Implicative Explanations

In the DPLL(XOR) framework, the clausal explanations for the xor-implied literals and xor-conflicts are vital for the CDCL solver when it performs its conflict analysis and clause learning. We next show that the implicative explanation procedure described in [13] can also be simulated with resolution, and discuss the consequence of this result.

Like the conflict resolution methods in modern CNF-level CDCL solvers, the explanation method is based on taking certain cuts in derivations. Assume a Subst-derivation $G = (V, E, L)$ on $\phi_{xor} \land l_1 \land \ldots \land l_k$, where $\phi_{xor}$ is a conjunction of xor-constraints and $l_1, \ldots, l_k$ are some xor-assumption literals. For a non-input vertex $v \in V$, a cut for $v$ is a partitioning $(V_a, V_b)$ of $V$ such that (i) $v \in V_b$, and (ii) if $v' \in V$ is an input vertex and there is a path from $v'$ to $v$, then $v' \in V_a$. As an example, the line “cut $W$” shows a cut for the vertex $v_8$ in Fig. 2(a). The implicative explanation of the vertex $v$ under the cut $W$ is the conjunction $Expl(v, W) = f_W(v)$, there $f_W$ is recursively defined as:

E1 If $u$ is an input vertex with $L(u) \in \phi_{xor}$, then $f_W(u) = T$.
E2 If $u$ is an input vertex with $L(u) \in \{l_1, \ldots, l_k\}$, then $f_W(u) = L(u)$.
E3 If $u$ is a non-input vertex in $V_a$, then $f_W(u) = L(u)$.
E4 If $u$ is a non-input vertex in $V_b$, then $f_W(u) = f_W(u_1) \land f_W(u_2)$, where $u_1$ and $u_2$ are the source vertices of the two edges incoming to $u$.

If the cut is $cnf$-compatible, meaning that all the vertices in $V_b$ having an edge to a vertex in $V_a$ are either (i) xor-constraints in $\phi_{xor}$ or (ii) unary xor-constraints, then the explanation $Expl(v, W)$ is a conjunction of literals and the clausal explanation of the xor-implied literal $L(v)$ returned to the CDCL part is $Expl(v, W) \Rightarrow L(v)$. As an example,
for the vertex $v_8$ and cnf-compatible cut $W$ in Fig. 2(a), we have $\text{Expl}(v_8, W) = (x)$ and the clausal explanation is thus $x \Rightarrow \neg t$, i.e., $\neg x \vee \neg t$.

We now prove that all such clausal explanations can in fact be derived with resolution from the CNF translation of the original xor-constraints $\phi_{\text{xor}}$ only, without the use of xor-assumptions. To illustrate some parts of the construction, Fig. 2(b) shows how the clausal explanation $x \Rightarrow \neg t$ above can be derived.

**Theorem 2.** Assume a Subst-derivation $G = (V, E, L)$ on $\phi_{\text{xor}} \land l_1 \land \cdots \land l_k$ and a cnf-compatible cut $W = (V_a, V_b)$. There is a resolution derivation $\pi$ on $\bigwedge_{D \in \phi_{\text{xor}}} \text{cnf}(D)$ such that (i) for each vertex $v \in V_b$ with $L(v) \neq \top$, $\pi$ includes all the clauses in \{ $\text{Expl}(v, W) \Rightarrow C \mid C \in \text{cnf}(L(v))$ \}, and (ii) $\pi$ has at most $|V|2^{m-1}$ clauses, where $m$ is the number of variables in the largest xor-constraint in $\phi_{\text{xor}}$.

As modern CDCL solvers can be seen as resolution proof producing engines [24,25], a DPLL(XOR) solver with Subst or EC as the xor-reasoning module can thus also be seen as such engine: the clausal explanations used by the CDCL part can be first obtained with resolution and then treated as normal clauses when producing the resolution proof corresponding to the execution of the CDCL part. And, recalling that modern CDCL solvers can polynomially simulate resolution [4], we have the following:

**Corollary 1.** For cnf-xor instances with fixed width xor-constraints, the underlying proof system of a DPLL(XOR) solver using Subst or EC as the xor-reasoning module is polynomially equivalent to resolution.

### 4 Parity Explanations and Gauss-Jordan Elimination

A key observation made in [16] was that the inference rules in Fig. 1 (and some others, as explained in [16]) could not only be read as “the premises imply the consequence” but also as “the linear combination of premises equals the consequence”. This led to the introduction of an improved explanation method, called parity explanations, which can produce (i) smaller clausal explanations, and (ii) new xor-constraints that are logical consequences of the original ones. As shown in [16], even when applied on a very weak deduction system UP, which only uses the unit propagation rules $\oplus$-Unit$^+$ and $\oplus$-Unit$^-$ in Fig. 1, the parity explanation method can quickly detect the unsatisfiability of some instances whose CNF translations have no polynomial size resolution refutations [3].

We now strengthen this result and prove that parity explanations on UP-derivations can in fact produce xor-constraints corresponding to the explanations produced by Gauss-Jordan elimination, provided that one can make the xor-assumptions suitably and each variable in the xor-constraint conjunction occurs at most three times (Thm. 3 below).

Formally, assume a UP-derivation $G = (V, E, L)$ for $\phi_{\text{xor}} \land l_1 \land \cdots \land l_k$. For each non-input vertex $v$ of $G$, and each cut $W = (V_a, V_b)$ of $G$ for $v$, the parity explanation of $v$ under $W$ is $\text{Expl}_\oplus(v, W) = f_W(v)$, where $f_W$ is recursively defined as earlier for $\text{Expl}(v, W)$ except that the case “E4” is replaced by

E4 If $u$ is a non-input node in $V_b$, then $f_W(u) = f_W(u_1) + f_W(u_2)$, where $u_1$ and $u_2$ are the source nodes of the two edges incoming to $u$. 
As shown in [16], \( \phi_{\text{xor}} \models \Expl(v, W) + L(v) \) and the clausal explanation for \( L(v) \) can be obtained from \( \text{cnf}(\Expl(v, W) + L(v)) \). As an example, the parity explanation \( \Expl(v_8, W) \) of the vertex \( v_8 \) in Fig. 2(a) is \( (\perp \equiv \perp) \), i.e. \( \top \), and indeed \( (x \oplus y \oplus z \equiv \top) \land (x \oplus z \oplus w \equiv \bot) \land (y \oplus w \oplus t \equiv \top) \models (\perp 
hdots \perp) + L(v_8) = (t \equiv \bot) \). Note that \( x \) does not occur in the parity explanation or in the clausal explanation \( (\neg t) \) returned.

For instances in which each variable occurs at most three times we can prove that, by selecting the xor-assumptions appropriately, parity explanations can in fact produce all prime implicate xor-constraints:

**Theorem 3.** Let \( \phi_{\text{xor}} \) be a conjunction of xor-constraints such that each variable occurs in at most three xor-constraints.

If \( \phi_{\text{xor}} \) is unsatisfiable, then there is a UP-derivation on \( \phi_{\text{xor}} \land y_1 \land \ldots \land y_m \) with some \( y_1, \ldots, y_m \in \text{vars}(\phi_{\text{xor}}) \), a vertex \( v \) with \( L(v) = (\perp \equiv \perp) \) in it, and a cut \( W \) for \( v \) such that \( \Expl(v, W) = (\perp \equiv \perp) \) and thus \( \Expl(v, W) + L(v) = (\perp \equiv \perp) \).

If \( \phi_{\text{xor}} \) is satisfiable and \( \phi_{\text{xor}} \models (x_1 \oplus \ldots \oplus x_k \equiv p) \), then there is a UP-derivation on \( \phi_{\text{xor}} \land (x_1 \equiv p_1) \land \ldots \land (x_k \equiv p_k) \land y_1 \land \ldots \land y_m \) with some \( y_1, \ldots, y_m \in \text{vars}(\phi_{\text{xor}}) \) \( \setminus \{x_1, \ldots, x_k\} \), a vertex \( v \) with \( L(v) = (\perp \equiv \top) \) in it, and a cut \( W \) for \( v \) such that \( \Expl(v, W) + L(v) = (x_1' \oplus \ldots \oplus x_l' \equiv p') \) for some \( \{x_1', \ldots, x_l'\} \subseteq \{x_1, \ldots, x_k\} \) and \( p' \in \{\perp, \top\} \) such that \( \phi_{\text{xor}} \models (x_1' \oplus \ldots \oplus x_l' \equiv p') \).

Now observe that the clausal explanations provided by the complete Gauss-Jordan elimination propagation engine of [13] are based on prime implicate xor-constraints (this follows from the fact that reduced row-echelon form matrices are used and the explanations are derived from the rows of such matrices). As a consequence, for instances in which each variable occurs at most three times, parity explanations on UP-derivations can in theory simulate the complete Gauss-Jordan elimination propagation engine [13] in the DPLL(XOR) framework if we allow unlimited restarts in the CDCL part and xor-constraint learning [16]: we can first learn all the linear combinations that the Gauss-Jordan engine would use to detect xor-implied literals and conflicts.

### 4.1 Experimental Evaluation

To evaluate the practical applicability of parity explanations further and to compare it to the xor-reasoning module using incremental Gauss-Jordan elimination presented in [13], we used our prototype solver based on minisat [26] (version 2.0 core) extended with four different xor-reasoning modules: (i) UP deduction system with implicative explanations, (ii) UP with parity explanations (UP+PEXP), (iii) UP with parity explanations and xor-constraint learning (UP+PEXP+learn) as described in [16], and (iv) incremental Gauss-Jordan elimination with biconnected component decomposition (UP+Gauss-Jordan) as described in [13]. We ran the solver configurations on two benchmark sets. The first benchmark set consists of instances in “crafted” and “industrial/application” categories of the SAT Competitions 2005, 2007, and 2009 as well as all the instances in the SAT Competition 2011 (see http://www.satcompetition.org/). We applied the xor-constraint extraction algorithm described in [13] to these CNF instances and found a large number of instances with xor-constraints. To get rid of some “trivial” xor-constraints, we eliminated unary clauses and binary xor-constraints from each instance by unit propagation and substitution, respectively. After this easy...
preprocessing, 474 instances (with some duplicates due to overlap in the competitions) having xor-constraints remained. In the second benchmark set we focus on the domain of logical cryptanalysis by modeling a “known cipher stream” attack on stream cipher Hitag2. The task is to recover the full key when a small number of cipher stream bits (33-38 bits, 51 instances / stream length) are given. In the attack, the IV and a number of cipher stream bits are given. There are only a few more generated cipher stream bits than key bits, so a number of keys probably produce the same prefix of the cipher stream.

The results for the SAT Competition benchmarks are shown in Fig. 3 and the results for Hitag2 in Fig. 4. The number of solved instances is shown in Fig. 5. For both benchmark sets, parity explanations without learning do not seem to reduce the num-

|                      | 2005 | 2007 | 2009 | 2011 | all | Grain | AS/1 | Trivium |
|----------------------|------|------|------|------|-----|-------|------|---------|
| instances            | 123  | 100  | 140  | 111  | 474 | 301   | 357  | 640     | 1020 |
| UP                   | 79   | 66   | 82   | 41   | 268 | 264   | 305  | 605     | 879  |
| UP+PEXP              | 78   | 70   | 85   | 48   | 281 | 257   | 301  | 610     | 867  |
| UP+PEXP+learn        | 96   | 69   | 88   | 48   | 301 | 274   | 257  | 635     | 909  |
| UP+Gauss-Jordan      | 97   | 61   | 82   | 39   | 279 | 115   | 84   | 640     | 880  |

Fig. 5. Number of instances solved within the time limit (3600s)
ber of decisions nor the solving time. However, storing parity explanations as learned xor-constraints results in a significant reduction in the number of decisions and this is also reflected in the solving time. Most variables have at most three occurrences (98% of variables in Hitag2, and 97% in SAT instances), so in most cases a parity explanation that is equivalent to the “Gauss-Jordan explanation” could be found using nondeterministic unit propagation. The SAT competition benchmarks has 64 instances consisting entirely of parity constraints which were of course solved without branching by Gauss-Jordan elimination. The results of the other instances that require searching on the CNF part illustrate that when parity explanations are learned, many instances can be solved much faster than with Gauss-Jordan elimination. It remains open whether the theoretical power of parity explanations could be exploited to an even higher degree by employing different propagation heuristics.

We also evaluated the performance of the four xor-reasoning modules on three other ciphers, Grain, A5/1, and Trivium, by encoding a similar “known cipher stream” attack as with Hitag2 above. For Grain, the simplest method, plain unit propagation, works the best. Gauss-Jordan elimination does not reduce the number of decisions enough to compensate for the computational overhead of complete parity reasoning. Parity explanations reduce the number of decisions slightly, but the small computational overhead is still too much. For A5/1, the solver using Gauss-Jordan elimination works the best. The solvers using parity explanations perform better than plain unit propagation, too, but not as well as the solver with Gauss-Jordan elimination. For Trivium, the solver using parity explanations with learning solves the most instances.

5 Simulating Stronger Parity Reasoning with Unit Propagation

An efficient translation for simulating equivalence reasoning with unit propagation has been presented in our earlier work [17]. We now present a translation that adds redundant xor-constraints and auxiliary variables in the problem guaranteeing that unit propagation is enough to always deduce all xor-implied literals in the resulting xor-constraint conjunction. The translation thus effectively simulates a complete parity reasoning engine based on incremental Gauss-Jordan elimination presented in [18,23]. The translation can be seen as an arc-consistent encoding of the xor-reasoning theory (also compare to the eager approach to SMT [22]). The translation is based on ensuring that each relevant linear combination of original variables has a corresponding “alias” variable, and adding xor-constraints that enable unit propagation to infer values of “alias” variables when corresponding linear combinations are implied. The translation, which is exponential in the worst-case, can be made polynomial by bounding the length of linear combinations to consider. While unit propagation may not be able then to deduce all xor-implied literals, the overall performance can be improved greatly.

The redundant xor-constraint conjunction, called a GE-simulation formula $\psi$, added to $\phi_{xor}$ by the translation should satisfy the following: (i) the satisfying truth assignments of $\phi_{xor}$ are exactly the ones of $\phi_{xor} \land \psi$ when projected to $\text{vars}(\phi_{xor})$, and (ii) if $\phi_{xor}$ is satisfiable and $\phi_{xor} \land \Lambda_{1} \land \cdots \land \Lambda_{k} \models \hat{l}$, then $\hat{l}$ is UP-derivable from $(\phi_{xor} \land \psi) \land \Lambda_{1} \land \cdots \land \Lambda_{k}$, and (iii) if $\phi_{xor}$ is unsatisfiable, then $(\phi_{xor} \land \psi) \vdash_{\text{UP}} (\bot \equiv \top)$.
ptable\( (Y, \phi_{\text{ex}} , k) \): start with \( \phi'_{\text{ex}} = \phi_{\text{ex}} \)
1. for each \( Y'' \subseteq Y \) such that \( |Y''| \leq k \) and \( Y'' \neq \emptyset \)
2. if there is no \( a \in \text{vars}(\phi'_{\text{ex}}) \) such that \( (a \oplus Y'') \equiv \perp \) is in \( \phi'_{\text{ex}} \)
3. \( \phi'_{\text{ex}} \leftarrow \phi'_{\text{ex}} \land (a \oplus Y'') \equiv \perp \) where \( a \) is a new “alias” variable for \( Y'' \)
4. if \( (Y'' \equiv \perp) \) is in \( \phi'_{\text{ex}} \) and \( (a \equiv \perp) \) is not in \( \phi_{\text{ex}} \) where \( p \in \{\perp, \top\} \)
5. \( \phi'_{\text{ex}} \leftarrow \phi'_{\text{ex}} \land (a \equiv \perp) \)
6. for each pair of subsets \( Y_1, Y_2 \subseteq Y \) such that \( |Y_1| \leq k, |Y_2| \leq k, \) and \( Y_1 \neq Y_2 \)
7. if there is an “alias” variable \( a_3 \in \text{vars}(\phi'_{\text{ex}}) \) such that \( (a_3 \oplus (Y_1 \oplus Y_2)) \equiv \perp \) is in \( \phi'_{\text{ex}} \)
8. \( a_1 \leftarrow \) the “alias” variable \( v \) such that \( (v \oplus Y_1) \equiv \perp \) is in \( \phi_{\text{ex}} \)
9. \( a_2 \leftarrow \) the “alias” variable \( v \) such that \( (v \oplus Y_2) \equiv \perp \) is in \( \phi_{\text{ex}} \)
10. if \( (a_1 \oplus a_2 \oplus a_3) \equiv \perp \) is not in \( \phi_{\text{ex}} \)
11. \( \phi'_{\text{ex}} \leftarrow \phi'_{\text{ex}} \land (a_1 \oplus a_2 \oplus a_3) \equiv \perp \)
12. return \( \phi'_{\text{ex}} \setminus \phi_{\text{ex}} \)

Fig. 6. The ptable translation

\( k\text{-Ge}(\phi_{\text{ex}}) \): start with \( \phi'_{\text{ex}} = \phi_{\text{ex}} \) and \( V = \text{vars}(\phi_{\text{ex}}) \)
1. while \( (V \neq \emptyset) \):
2. Let \( \text{clauses}(x, \phi'_{\text{ex}}) = \{D \mid D \in \phi'_{\text{ex}} \text{ and } x \in \text{vars}(D)\} \)
3. Let \( x \) be a variable in \( V \) minimizing \( |\text{vars}(\text{clauses}(x, \phi'_{\text{ex}})) \cap V| \)
4. \( \phi'_{\text{ex}} \leftarrow \phi'_{\text{ex}} \land \text{ptable}(\text{clauses}(x, \phi'_{\text{ex}})) \cap V, \phi_{\text{ex}}, k) \)
5. Remove \( x \) from \( V \)
6. return \( \phi'_{\text{ex}} \setminus \phi_{\text{ex}} \)

Fig. 7. The \( k\text{-Ge} \) translation

The translation \( k\text{-Ge} \), presented in Fig. 7, where \( k \) stands for the maximum length of linear combinations to consider, “eliminates” each variable of the xor-constraint conjunction \( \phi_{\text{ex}} \) at a time and adds xor-constraints produced by the subroutine translation \( \text{ptable} \), presented in Fig. 6. Although the choice of variable to eliminate does not affect the correctness of the translation, we employ a greedy heuristic to pick a variable that shares xor-constraints with the fewest variables because the number of xor-constraints produced in the subroutine \( \text{ptable} \) is then the smallest. The translation \( \text{ptable}(Y, \psi, k) \) adds “alias” variables and at most \( O(2^{2k}) + |\phi_{\text{ex}}| \) xor-constraints to \( \psi \) with the aim to simulate Gauss-Jordan row operations involving at most \( k \) variables in the xor-constraints of the eliminated variable (the set \( Y \)) and no other variables. Provided that the maximum length of linear combinations to consider, the parameter \( k \), is high enough, the resulting xor-constraint conjunction \( \psi \land \text{ptable}(Y, \psi, k) \) has a UP-propagation table for the set of variables \( Y \subseteq \text{vars}(\phi_{\text{ex}}) \), meaning that the following conditions hold for all \( Y', Y_1, Y_2 \subseteq Y \):

**PT1:** There is an “alias” variable for every non-empty subset of \( Y \): if \( Y' \) is a non-empty subset of \( Y \), then there is a variable \( a \in \text{vars}(\psi) \) such that \( (a \oplus Y') \equiv \perp \) is in \( \psi \), where \( (a \oplus Y') \equiv \perp \) for \( Y' = \{y'_1, \ldots , y'_n\} \) means \( a \oplus y'_1 \oplus \cdots \oplus y'_n \equiv \perp \).

**PT2:** There is an xor-constraint for propagating the symmetric difference of any two subsets of \( Y \): if \( Y_1 \subseteq Y \) and \( Y_2 \subseteq Y \), then there are variables \( a_1, a_2, a_3 \in \text{vars}(\psi) \).
PT3: Alias variables of original xor-constraints having only variables of \( Y \) are assigned:

if \( (Y' \equiv p) \) is an xor-constraint in \( \psi \) such that \( Y' \subseteq Y \), then there is a variable \( a \in \text{vars}(\psi) \) such that \((a + Y' \equiv \perp)\) it holds that \((a \equiv p)\) is in \( \psi \).

A UP-propagation table for a set of variables \( Y \) in \( \psi \) guarantees that if some alias variables \( a_1, \ldots, a_n \in \text{vars}(\psi) \) binding the variable sets \( Y_1, \ldots, Y_n \subseteq Y \) are assigned, the alias variable \( a \in \text{vars}(\psi) \) bound to the linear combination \((Y_1 \oplus \cdots \oplus Y_n)\) is UP-deducible: \( \psi \land (a_1 \equiv p_1) \land \cdots \land (a_n \equiv p_n) \vdash_{\text{UP}} (a \equiv p_1 \oplus \cdots \oplus p_n) \). Provided that sufficiently long linear combinations are considered (the parameter \( k \)), UP-propagation tables added by the \( k \)-Ge enable unit propagation to always deduce all xor-implied literals, and thus simulate a complete Gauss-Jordan propagation engine:

Theorem 4. If \( \phi_{\text{xor}} \) is an xor-constraint conjunction, then \( k \)-Ge(\( \phi_{\text{xor}} \)) is a GE-simulation formula for \( \phi_{\text{xor}} \) provided that \( k = |\text{vars}(\phi_{\text{xor}})| \).

Example 1. Consider the xor-constraint conjunction \( \phi_{\text{xor}}^{(0)} = (x_1 \oplus x_6 \oplus x_7 \equiv T) \land (x_2 \oplus x_3 \oplus x_7 \equiv \perp) \land (x_2 \oplus x_5 \oplus x_8 \equiv \perp) \land (x_3 \oplus x_4 \oplus x_5 \equiv T) \land (x_4 \oplus x_6 \oplus x_8 \equiv \perp) \) illustrated in Fig. 8. It is clear that \( \phi_{\text{xor}} \models \phi_{\text{xor}}^{(0)} \).

With the elimination order \((x_1, x_7, x_4, x_5, x_2, x_3, x_6, x_8)\) and \( k = 4 \), the translation \( k \)-Ge first extends \( \phi_{\text{xor}} \) to \( \phi_{\text{xor}}^{(1)} \) with the xor-constraints in ptable\( \{x_1, x_6, x_7\} \), \( \phi_{\text{xor}}, k \). These include (i) the “alias binding constraints” \( a_1 \oplus x_1 \equiv \perp, a_{6,7} \oplus x_6 \oplus x_7 \equiv \perp, a_{1,6,7} \oplus x_1 \oplus x_6 \oplus x_7 \equiv \perp \), (ii) the “linear combination constraint” \( a_1 \oplus a_{6,7} \oplus a_{1,6,7} \equiv \perp \), and (iii) the “original constraint binder” \( a_{1,6,7} \equiv T \), where \( a_{i,...} \) is the alias for the subset \( \{x_i,...\} \) of the original variables. After unit propagation, these constraints imply the binary constraint \( a_1 \oplus a_{6,7} \equiv T \) allowing us to deduce \( x_1 \) from the parity \( a_{6,7} \) of \( x_6 \) and \( x_7 \).

The translation next “eliminates” \( x_7 \) and adds ptable\( \{x_2, x_3, x_6, x_7\} \), \( \phi_{\text{xor}}^{(1)} \), \( k \) including the linear combination constraint \( a_{6,7} \oplus a_{2,3,6} \equiv \perp \) and the original constraint binder \( a_{2,3,7} \equiv T \), propagating the binary constraint \( a_{6,7} \oplus a_{2,3,6} \equiv T \) allowing us to deduce the parity of \( \{x_6, x_7\} \) from the parity of \( \{x_2, x_3, x_6\} \).

Eliminating \( x_4 \) adds ptable\( \{x_3, x_4, x_5, x_6, x_8\} \), \( \phi_{\text{xor}}^{(2)} \), \( k \) including the constraints \( a_{3,4,5} \oplus a_{4,6,8} \oplus a_{5,6,8} \equiv \perp, a_{3,4,5} \equiv T \), and \( a_{4,6,8} \equiv T \), propagating \( a_{3,5,6,8} \equiv T \).

Eliminating \( x_5 \) adds ptable\( \{x_2, x_3, x_5, x_6, x_8\} \), \( \phi_{\text{xor}}^{(3)} \) (observe that \( x_6 \) is in the set as it occurs in the constraint \( a_{3,5,6,8} \oplus x_3 \oplus x_5 \oplus x_6 \oplus x_8 \equiv \perp \) added in the previous step), including \( a_{2,5,8} \oplus a_{2,3,6} \oplus a_{3,5,6,8} \equiv \perp \) and \( a_{2,5,8} \equiv \perp \).
At this point we could already unit propagate \( x_1 \equiv \top \) (from \( a_{3,5,6,8} \equiv \top, a_{2,5,8} \equiv \bot \), and \( a_{2,5,8} \oplus a_{2,3,6} \oplus a_{3,5,6,8} \equiv \bot \) we get \( a_{2,3,6} \equiv \top \) and from this then \( a_{6,7} \equiv \bot \) and finally \( a_1 \equiv \top \), i.e. \( x_1 \equiv \top \).

Note that the translation \( 3\text{-Ge}(\phi_{\text{xor}}) \) is not a GE-simulation formula for \( \phi_{\text{xor}} \) because `ptable` does not add “alias” variables for any 4-subset of original variables and the linear combination of any two original xor-constraints has at least four variables.

The translation `ptable` as presented in [6] for illustration purposes adds new “alias” variables for all relevant linear combinations involving at most \( k \) original variables. However, in an actual implementation, the original variables of the xor-constraint conjunction can be used as “alias” variables. For example, the variable \( x_1 \) in the xor-constraint \( (x_1 \oplus x_2 \oplus x_3 \equiv \top) \) can be used as an “alias” variable for \( (x_2 \oplus x_3 \equiv \bot) \).

The translation \( k\text{-Ge} \) is a generalization of the translation \( Eq^* \), which simulates equivalence reasoning with unit propagation, presented in [17]. Provided that original variables are treated as “alias” variables as above and all xor-constraints have at most three variables, the translation \( 2\text{-Ge} \), that considers only (in)equivalences between pairs of variables, enables unit propagation to simulate equivalence reasoning.

The size of the GE-simulation formula for \( \phi_{\text{xor}} \) may be reduced considerably if \( \phi_{\text{xor}} \) is partitioned into disjoint xor-constraint conjunctions \( \phi_{\text{xor}}^1 \wedge \cdots \wedge \phi_{\text{xor}}^n \) according to the connected components of the xor-constraint graph, and then combining the component-wise GE-simulation formulas \( k_1\text{-Ge}(\phi_{\text{xor}}^1) \wedge \cdots \wedge k_n\text{-Ge}(\phi_{\text{xor}}^n) \). Efficient structural tests for deciding whether unit propagation or equivalence reasoning is enough to achieve full propagation in an xor-constraint conjunction, presented in [17], can indicate appropriate values for some of the parameters \( k_1, \ldots, k_n \).

### 5.1 Propagation-preserving xor-simplification

Some of the xor-constraints added by \( k\text{-Ge} \) can be redundant regarding unit propagation. We now present a simplification method that preserves literals that can be implied by unit propagation. There are two simplification rules, given a pair of xor-constraint conjunctions \( \langle \phi_a, \phi_b \rangle \) (initially \( \langle \phi_{\text{xor}}, \emptyset \rangle \)): [S1] an xor-constraint \( D \) in \( \phi_a \) can be moved to \( \phi_b \), resulting in \( \langle \phi_a \setminus \{D\}, \phi_b \cup \{D\} \rangle \), and [S2] an xor-constraint \( D \) in \( \phi_a \) can be simplified with an xor-constraint \( D' \) in \( \phi_b \) to \( \langle D + D' \rangle \) provided that \( |\text{vars}(D') \cap \text{vars}(D)| \leq |\text{vars}(D')| - 1 \), resulting in \( \langle \phi_a \setminus \{D\} \cup \{D + D'\}, \phi_b \rangle \).

**Theorem 5.** If \( \langle \phi'_a, \phi'_b \rangle \) is the result of applying one of the simplification rules to \( \langle \phi_a, \phi_b \rangle \) and \( \phi_a \wedge \phi_b \wedge l_1 \wedge \cdots \wedge l_k \vDash_{\text{UP}} \tilde{I} \), then \( \phi'_a \wedge \phi'_b \wedge l_1 \wedge \cdots \wedge l_k \vDash_{\text{UP}} \tilde{I} \).

**Example 2.** The conjunction \( 3\text{-Ge}((x_1 \oplus x_2 \oplus x_3 \oplus x_4 \equiv \bot)) \) contains the alias binding constraints \( D_1 := (a_{1,2,3,4} \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \equiv \bot), D_2 := (a_{1,2} \oplus x_1 \oplus x_2 \equiv \bot) \), \( D_3 := (a_{3,4} \oplus x_3 \oplus x_4 \equiv \bot) \), as well as the linear combination constraint \( D_4 := (a_{1,2} \oplus a_{3,4} \oplus a_{1,2,3,4} \equiv \bot) \). The alias binding constraint \( D_1 \) can in fact be eliminated by first applying the rule S1 to the xor-constraints \( D_2, D_3, \) and \( D_4 \). Then, by using the rule S2, the xor-constraint \( D_1 \) is simplified first with \( D_2 \) to \( (a_{1,2,3,4} \oplus a_{1,2} \oplus x_3 \oplus x_4 \equiv \bot) \) and then with \( D_3 \) to \( (a_{1,2,3,4} \oplus a_{1,2} \oplus a_{3,4} \equiv \bot) \), and finally with \( D_4 \) to \( (\bot \equiv \bot) \).
5.2 Experimental evaluation

To evaluate the translation $k$-Ge, we studied the benchmark instances in "crafted" and "industrial/application" categories of the SAT Competitions 2005, 2007, 2009, and 2011. We ran cryptominisat 2.9.6, glucose 2.3, and zenn 0.1.0 on the same 474 SAT Competition cnf-xor instances as in Section 4.1 with the translations $k$-Ge and $Eq^\ast$. It is intractable to simulate full Gauss-Jordan elimination for these instances, so we adjusted the $k$-value of each call to the subroutine $ptable(Y, \psi, k)$ to limit the number of additional xor-constraints. The translation was computed for each connected component separately. We found good performance by (i) stopping when $|Y| > 66$, (ii) setting $k = 1$ when it was detected that unit propagation deduces all xor-implied literals, (iii) setting $k = 2$ when $|Y| \in [10, 66]$ or when $|Y| < 10$ and it was detected that equivalence reasoning deduces all xor-implied literals, (iv) setting $k = 3$ when $|Y| \in [6, 9]$, and (v) setting $k = |Y|$ when $|Y| \leq 5$. With these parameters, the worst-case number of xor-constraints added by the subroutine $ptable$ is 2145. Figure 9 shows the increase in formula size by the translation $k$-Ge. Propagation-preserving xor-simplification was used to simplify the instances reducing the formula size in 404 instances with the median reduction being 16%. The translation $Eq^\ast$ was computed in a similar way. The results are shown in Fig. 11 including the time spent in computing the translations. Using xor-simplification increases the number of solved instances for both translations. The detailed solving time comparison in Fig. 10 shows that that the translation $k$-Ge can incur some overhead, but also allows great speedups, enabling the three solvers to solve the highest number of instances for the whole benchmark set.

6 Connection to Treewidth

The number of xor-constraints produced by the translation $k$-Ge depends strongly on the instance, as shown in Fig. 9. Now we connect the worst-case size of a $ptable$-based GE-simulation formula to treewidth, a well-known structural property of (constraint) graphs used often to characterize the hardness of solving a problem, e.g. an instance of CSP with bounded treewidth can be solved in polynomial time [27]. We develop a new decomposition method that we can apply to a tree decomposition to produce a
polynomial-size GE-simulation formula for instances of bounded treewidth. We also present some found upper bounds for treewidth in SAT Competition instances that illustrate to what extent parity reasoning can be simulated through unit propagation.

The new decomposition technique is a generalization of the method in [18], which states that, in order to guarantee full propagation, it is enough to (i) propagate only values through “cut variables”, and (ii) have full propagation for the “biconnected components”. Now we extend the technique to larger cuts. Given an xor-constraint conjunction $\phi_{\text{xor}}$, a cut variable set is a set of variables $X \subseteq \text{vars}(\phi_{\text{xor}})$ for which there is a partition $(V_a, V_b)$ of xor-constraints in $\phi_{\text{xor}}$ with $\text{vars}(V_a) \cap \text{vars}(V_b) = X$; such a partition $(V_a, V_b)$ is called an X-cut partition of $\phi_{\text{xor}}$. If full propagation can be guaranteed for both sides of an X-cut partition, then communicating the implied linear combinations involving cut variables is enough to guarantee full propagation for the whole instance:
Theorem 6. Let \((V_a, V_b)\) be an X-cut partition of \(\phi_{\text{xor}}\). Let \(\phi_{\text{xor}}^a = \bigwedge_{D \in V_a} D\), \(\phi_{\text{xor}}^b = \bigwedge_{D \in V_b} D\), and \(l_1, \ldots, l_k \in \text{lits}(\phi_{\text{xor}})\). Then it holds that:

- If \(\phi_{\text{xor}} \land l_1 \land \cdots \land l_k\) is unsatisfiable, then
  1. \(\phi_{\text{xor}}^a \land l_1 \land \cdots \land l_k\) or \(\phi_{\text{xor}}^b \land l_1 \land \cdots \land l_k\) is unsatisfiable; or
  2. \(\phi_{\text{xor}}^a \land l_1 \land \cdots \land l_k \models (X' \equiv p')\) and \(\phi_{\text{xor}}^b \land l_1 \land \cdots \land l_k \models (X' \equiv p' \oplus \top)\) for some \(X' \subseteq X\) and \(p', p' \in \{\top, \bot\}\).

- If \(\phi_{\text{xor}} \land l_1 \land \cdots \land l_k\) is satisfiable and \(\phi_{\text{xor}} \land l_1 \land \cdots \land l_k \models (Y \equiv p)\) for some \(Y \subseteq \text{vars}(\phi_{\text{xor}}^a)\), \(Y \cap (\text{vars}(\phi_{\text{xor}}^b) \setminus \text{vars}(\phi_{\text{xor}}^a)) = \emptyset\), and \(p \in \{\top, \bot\}\) where \(\alpha \in \{a, b\}\) and \(\beta \in \{a, b\} \setminus \{\alpha\}\), then
  1. \(\phi_{\text{xor}}^a \land l_1 \land \cdots \land l_k \models (Y \equiv p)\) or \(\phi_{\text{xor}}^b \land l_1 \land \cdots \land l_k \models (Y \equiv p)\); or
  2. \(\phi_{\text{xor}}^a \land l_1 \land \cdots \land l_k \models (X' \equiv p')\) and \(\phi_{\text{xor}}^b \land l_1 \land \cdots \land l_k \models (X' \equiv p' \oplus \top)\) for some \(X' \subseteq X\), \(p', p' \in \{\top, \bot\}\), \(\alpha \in \{a, b\}\), and \(\beta \in \{a, b\} \setminus \{\alpha\}\).

Example 3. Consider the constraint graph in Fig. 12. The cut variable set \(\{x_2, x_3, x_6\}\) partitions the xor-constraints into two conjunctions \(\phi_{\text{xor}}^a = (x_1 \oplus x_6 \oplus x_7 \equiv \top) \land (x_2 \oplus x_3 \oplus x_7 \equiv \top)\) and \(\phi_{\text{xor}}^b = (x_2 \oplus x_5 \plusr x_8 \equiv \top) \land (x_4 \oplus x_5 \oplus x_9 \equiv \top) \land (x_4 \oplus x_6 \oplus x_8 \equiv \top)\).

It holds that \(\phi_{\text{xor}}^a \models (x_2 \oplus x_3 \oplus x_6 \equiv \top)\) and \(\phi_{\text{xor}}^b \land (x_2 \oplus x_3 \oplus x_6 \equiv \top) \models (x_1 \equiv \top)\).

![Fig. 12. Primal graph](image-url)
variables have an occurrence in the same xor-constraint. The primal graph of the xor-constraint conjunction shown in Fig. 8 and a tree decomposition for it are shown in Fig. 12 and in Fig. 13. If an xor-constraint conjunction has a bounded treewidth, the tree decomposition can be used to construct a polynomial-size GE-simulation formula:

**Theorem 7.** If \( \{X_1, \ldots, X_n\} \) is the family of variable sets in the tree decomposition of the primal graph of an xor-constraint conjunction \( \phi_{\text{xor}} \) and \( \phi_0, \ldots, \phi_n \) is a sequence of xor-constraint conjunctions such that \( \phi_0 = \phi_{\text{xor}} \) and \( \phi_i = \phi_{i-1} \land \text{ptable}(X_i, \phi_{i-1}, |X_i|) \) for \( i \in \{1, \ldots, n\} \), then \( \phi_n \setminus \phi_{\text{xor}} \) is a GE-simulation formula for \( \phi_{\text{xor}} \) with \( O(n2^{\max(|X_i|, \ldots, |X_n|)}) \) xor-constraints, where \( k = \max(|X_1|, \ldots, |X_n|) \).

To find out to what extent unit propagation can simulate stronger parity reasoning, we studied the 474 benchmark instances in “crafted” and “industrial/application” categories of the SAT Competitions 2005, 2007, 2009, and 2011. Computing the exact value of treewidth is an NP-complete problem [28], so we applied the junction tree algorithm described in [29] to get an upper bound for treewidth. The found treewidths are shown in Fig. 14. There are some instances that have compact GE-simulation formulas, but for the majority of the instances, full GE-simulation formula is likely to be intractably large. For these instances a powerful solution technique can be to choose a suitable propagation method for each biconnected component separately, either through a translation or an xor-reasoning module.

## 7 Conclusions

We have studied how stronger parity reasoning techniques in the DPLL(XOR) framework can be simulated by simpler systems. We have shown that resolution simulates
equivalence reasoning efficiently. We have proven that parity explanations on non-deterministic unit propagation derivations can simulate Gauss-Jordan elimination on a restricted yet practically relevant class of instances. We have shown that Gauss-Jordan elimination can be simulated by unit propagation by adding additional xor-constraints, and for instance families of bounded treewidth, a polynomial number of additional xor-constraints suffices.

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8 Proofs

8.1 Fundamental Properties of Linear Combinations

Some fundamental, easy to verify properties are $D + D + E = E$, $D \land E \models D + E$, $D \land E \models D \land (D + E)$, and $D \land (D + E) \models D \land E$.

The logical consequence xor-constraints of an xor-constraint conjunction $\psi$ are exactly those that are linear combinations of the xor-constraints in $\psi$.

Lemma 1 (from [30]). Let $\psi$ be a conjunction of xor-constraints. Now $\psi$ is unsatisfiable if and only if there is a subset $S$ of xor-constraints in $\psi$ such that $\sum_{D \in S} D = (\bot \equiv \top)$. If $\psi$ is satisfiable and $E$ is an xor-constraint, then $\psi \models E$ if and only if there is a subset $S$ of xor-constraints in $\psi$ such that $\sum_{D \in S} D = E$.

8.2 Proof of Theorem 1

Theorem 1. Assume a Subst-derivation $G = \langle V, E, L \rangle$ on a conjunction $\psi$ of xor-constraints. There is a resolution derivation $\pi$ on $\bigwedge_{D \in \psi} \text{cnf}(D)$ such that (i) if $v \in V$ and $L(v) \neq \top$, then the clauses $\text{cnf}(L(v))$ occur in $\pi$, and (ii) $\pi$ has at most $|V|2^{m-1}$ clauses, where $m$ is the number of variables in the largest xor-constraint in $\psi$.

Proof. We construct a resolution derivation $\pi$ with the desired properties by first inductively associating each vertex $v$ in $G$ with a set $L_\pi(v)$ of clauses such that

1. if $L(v) \neq \top$, then all the clauses in $\text{cnf}(L(v))$ occur in $L_\pi(v)$,
2. all the clauses in $L_\pi(v)$ either occur in the CNF translation $\bigwedge_{D \in \psi} \text{cnf}(D)$ or can be obtained with one resolution step (i) from the ones in $L_\pi(v')$ and $L_\pi(v'')$, where $v'$ and $v''$ are the source vertices of the two edges incoming to $v$, or (ii) from the ones produced as in (i) above, and
3. $|L_\pi(v)| \leq 2^{m-1}$.

A resolution derivation can be obtained directly from this construction by just listing the clauses in the $L_\pi$-sets in appropriate order.

For each input vertex $v$ we have that $L(v)$ occurs in $\psi$ and thus we simply set $L_\pi(v) = \{C \mid C \text{ occurs in } \text{cnf}(L(v))\}$.

For non-input vertices, we apply the following construction.

1. $L(v)$ is obtained from $L(v')$ and $L(v'')$ by using $\oplus$-Unit$^+$.
   Suppose that $L(v') = (x \equiv \top)$ for some variable $x$. Thus $L_\pi(v') \supseteq \{(x)\}$.
   - If $L(v'') = (x \equiv \top)$, then $L(v) = \top$ and we set $L_\pi(v) = \emptyset$.
   - If $L(v'') = (x \equiv \bot)$, then $L_\pi(v') \supseteq \{(-x)\}$, $L(v) = \bot$ and we set $L_\pi(v)$ to be the resolvent of $(x) \in L_\pi(v')$ and $(-x) \in L_\pi(v'')$. Thus $L_\pi(v) = (x)$ = $\text{cnf}(\bot \equiv \top)$.
   - If $L(v'') = (x \oplus y \oplus \ldots \equiv p)$, then $L_\pi(v') \supseteq \{C \mid C \text{ occurs in } \text{cnf}(L(v''))\}$, $L(v) = L(v'')[x/\top]$ and we set $L_\pi(v)$ to be the set of all clauses obtained by resolving $(x) \in L_\pi(v')$ with the clauses of form $(-x \lor \ldots)$ occurring in $\text{cnf}(L(v''))$. One can verify that indeed $L_\pi(v) = \{C \mid C \text{ occurs in } \text{cnf}(L(v))\}$.
2. \(L(v)\) is obtained from \(L(v')\) and \(L(v'')\) by using \(\oplus\)-Unit\(^-\).
   This case is similar to the previous one.
3. \(L(v)\) is obtained from \(L(v')\) and \(L(v'')\) by using \(\oplus\)-Eqy\(^+\).
   Suppose that \(L(v') = (x \oplus \cdots \equiv p)\) such that \(y\) does not occur in it, then we set \(L(v)\) to consist of all the clauses obtained by (i) resolving \((-x \lor y)\) with each clause of form \((x \lor \cdots)\) occurring in \(\text{cnf}(L(v'))\) and thus also in \(L(v'')\), and (ii) resolving \((x \lor -y)\) with each clause of form \((-x \lor \cdots)\) occurring in \(\text{cnf}(L(v''))\) [and thus also in \(L(v'')\)]. It is straightforward to verify that \(L(v) = \{C \mid C \text{ occurs in } \text{cnf}(L(v))\}\).
   - If \(L(v') = (x \lor y \equiv \bot)\), then \(L(v) = \top\) and we set \(L(v) = \emptyset\).
   - If \(L(v') = (x \lor y \equiv \top)\), then \(L(v') \supseteq \{(x \lor y), (-x \lor -y)\}\), \(L(v) = \bot\) and we set \(L(v) = \{(y), (-y)\}\) [all these clauses can be obtained with resolution from the ones in \(L(v')\) and \(L(v'')\)].
   - If \(L(v'') = (x \oplus y \oplus z_1 \oplus \cdots \oplus z_k \equiv p)\), then we first resolve (i) \((-x \lor y)\) with each of the \(2^{k-1}\) clauses in \(\{(x \lor y \lor C) \mid C \text{ occurs in } \text{cnf}(z_1 \oplus \cdots \oplus z_k \equiv p)\}\) and (ii) \((x \lor -y)\) with each of the \(2^{k-1}\) clauses in \(\{(-x \lor -y \lor C) \mid C \text{ occurs in } \text{cnf}(z_1 \oplus \cdots \oplus z_k \equiv p)\}\).
   We then resolve, for each \(C\) occurring in \(\text{cnf}(z_1 \oplus \cdots \oplus z_k \equiv p)\), the clauses \((y \lor C)\) and \((-y \lor C)\) obtained above; the result is the \(2^{k-1}\) clauses in \(\text{cnf}(z_1 \oplus \cdots \oplus z_k \equiv p)\), as required to represent \(L(v) = (z_1 \oplus \cdots \oplus z_k \equiv p)\).

4. \(L(v)\) is obtained from \(L(v')\) and \(L(v'')\) by using \(\oplus\)-Eqy\(^-\).
   This case is similar to the previous one.

\(\square\)

### 8.3 Proof of Theorem\(^2\)

We start by giving some auxiliary results and lemmas.

First, observe that \((x \equiv \top) + D = D [x/\top], (x \equiv \bot) + D = D [x/\bot], (x \oplus y \equiv \bot) + D = D [x/y], and (x \oplus y \equiv \top) + D = D [x/y \oplus \top]\) when \(x\) occurs in \(D\) and thus the Subst-rules in Fig.\([1]\) are special cases of a more general linear combination rule.

The next lemmas show that these special cases, when conditioned with some conjunctions of literals, can be derived with resolution with a linear number of steps.

**Lemma 2.** Let \(\phi\) and \(\phi'\) be conjunctions of literals and take some xor-constraints \(D = (x \equiv p)\) and \(D' = (x \oplus z_1 \oplus \cdots \oplus z_k \equiv p')\). Given the sets \(S = \{\phi \Rightarrow C \mid C \in \text{cnf}(D)\}\) and \(S' = \{\phi' \Rightarrow C \mid C \in \text{cnf}(D')\}\) of clauses, the set \(\{(\phi \land \phi') \Rightarrow C \mid C \in \text{cnf}(D + D')\}\) has \(2^{k-1}\) clauses and we can derive them from those in \(S\) and \(S'\) with \(2^{k-1}\) resolution steps.

**Proof.** Take the only clause \(l_1 \land \cdots \land l_m \Rightarrow (x \equiv p)\) in \(S\) and resolve it with each clause \(l'_1 \land \cdots \land l'_n \Rightarrow C\) with \(C = ((x \equiv -p) \lor \cdots) \in \text{cnf}(D')\) in \(S'\) (there are \(2^{k-1}\) of them). Each resulting clause forces that either (i) one of the literals \(l_1, \ldots, l_m, l'_1, \ldots, l'_n\) is false or (ii) that the parity of \(z_1, \ldots, z_k\) is not one of the \(2^{k-1}\) ones not allowed by \((p \oplus z_1 \oplus \cdots \oplus z_k \equiv p') = D + D'\).

\(\square\)
Lemma 3. Let $\phi$ and $\phi'$ be conjunctions of literals and take some xor-constraints $D = (x \oplus y \equiv p)$ and $D' = (x \oplus z_1 \oplus ... \oplus z_k \equiv p')$. Given the sets $S = \{\phi \Rightarrow C \mid C \in \text{cnf}(D)\}$ and $S' = \{\phi' \Rightarrow C \mid C \in \text{cnf}(D')\}$ of clauses, the clause set $\{(\phi \land \phi') \Rightarrow C \mid C \in \text{cnf}(D + D')\}$ has $2^k$ clauses and we can derive them from those in $S$ and $S'$ with $2^k$ resolution steps.

Proof. Take the clause $l_1 \land ... \land l_m \Rightarrow ((x \equiv \top) \lor (y \equiv p))$ in $S$ and resolve it with each clause $l'_1 \land ... \land l'_n \Rightarrow C$ with $C = ((x \equiv \bot) \lor ...)$ in $\text{cnf}(D)$ in $S'$ (there are at most $2^k-1$ of them). Each resulting clause $l_1 \land ... \land l_m \land l'_1 \land ... \land l'_n \Rightarrow ((y \equiv p) \lor ...)$ forces that either (i) one of the literals $l_1, ..., l_m, l'_1, ..., l'_n$ is false, or (ii) $y \equiv \neg p$ implies that the parity of $z_1, ..., z_k$ is not one of the $2^k-1$ ones not allowed by $((T \lor z_1 \lor ... \lor z_k) \equiv p')$.

Similarly, take the clause $l_1 \land ... \land l_m \Rightarrow ((x \equiv \bot) \lor (y \equiv p))$ in $S$ and resolve it with each clause $l'_1 \land ... \land l'_n \Rightarrow C$ with $C = ((x \equiv \top) \lor ...)$ in $\text{cnf}(D')$ in $S'$ (there are at most $2^k-1$ of them). Each resulting clause $l_1 \land ... \land l_m \land l'_1 \land ... \land l'_n \Rightarrow ((y \equiv p) \lor ...)$ forces that either (i) one of the literals $l_1, ..., l_m, l'_1, ..., l'_n$ is false, or (ii) $y \equiv p$ implies that the parity of $z_1, ..., z_k$ is not one of the $2^k-1$ ones not allowed by $((\bot \lor z_1 \lor ... \lor z_k) \equiv p' \lor p)$.

As $D + D' = (y \oplus z_1 \oplus ... \oplus z_k \equiv p \lor p')$, the $2^k$ clauses above are the ones in $\{(\phi \land \phi') \Rightarrow C \mid C \in \text{cnf}(D + D')\}$. ☐

Lemma 4. Let $\phi$ and $\phi'$ be conjunctions of literals and take some xor-constraints $D = (x \oplus y \equiv p)$ and $D' = (x \oplus y \oplus z_1 \oplus ... \oplus z_k \equiv p')$. Given the sets $S = \{\phi \Rightarrow C \mid C \in \text{cnf}(D)\}$ and $S' = \{\phi' \Rightarrow C \mid C \in \text{cnf}(D')\}$ of clauses, the clause set $\{(\phi \land \phi') \Rightarrow C \mid C \in \text{cnf}(D + D')\}$ has $2^k$-1 clauses and we can derive them from the ones in $S$ and $S'$ with $2^k$ resolution steps.

Proof. Take the clause $l_1 \land ... \land l_m \Rightarrow ((x \equiv \top) \lor (y \equiv p))$ in $S$ and resolve it with each clause $l'_1 \land ... \land l'_n \Rightarrow C$ with $C = ((x \equiv \bot) \lor ...)$ in $\text{cnf}(D)$ in $S'$ (there are at most $2^k-1$ of them). Each resulting clause $l_1 \land ... \land l_m \land l'_1 \land ... \land l'_n \Rightarrow ((y \equiv p) \lor ...)$ forces that either (i) one of the literals $l_1, ..., l_m, l'_1, ..., l'_n$ is false, or (ii) $y \equiv \neg p$ implies that the parity of $z_1, ..., z_k$ is not one of the $2^k-1$ ones not allowed by $((T \lor z_1 \lor ... \lor z_k) \equiv p' \lor p)$, i.e., $(z_1 \oplus ... \oplus z_k \equiv p' \lor p)$.

Similarly, take the clause $l_1 \land ... \land l_m \Rightarrow ((x \equiv \bot) \lor (y \equiv p))$ in $S$ and resolve it with each clause $l'_1 \land ... \land l'_n \Rightarrow C$ with $C = ((x \equiv \top) \lor (y \equiv \neg p) \lor ...)$ in $\text{cnf}(D')$ in $S'$ (there are at most $2^k-1$ of them). Each resulting clause $l_1 \land ... \land l_m \land l'_1 \land ... \land l'_n \Rightarrow ((y \equiv \neg p) \lor ...)$ forces that either (i) one of the literals $l_1, ..., l_m, l'_1, ..., l'_n$ is false, or (ii) $y \equiv p$ implies that the parity of $z_1, ..., z_k$ is not one of the $2^k-1$ ones not allowed by $((\bot \lor z_1 \lor ... \lor z_k) \equiv p' \lor p)$, i.e., $(z_1 \oplus ... \oplus z_k \equiv p' \lor p)$.

Finally, resolve each obtained clause $l_1 \land ... \land l_m \land l'_1 \land ... \land l'_n \Rightarrow ((y \equiv p) \lor C)$, $C$ being disjunction of literals, with the corresponding clause $l_1 \land ... \land l_m \land l'_1 \land ... \land l'_n \Rightarrow ((y \equiv \neg p) \lor \neg C)$. The resulting $2^k$-1 clauses together force that either (i) one of the literals $l_1, ..., l_m, l'_1, ..., l'_n$ is false, or (ii) $(z_1 \oplus ... \oplus z_k \equiv p' \lor p)$ holds.

As $D + D' = (z_1 \oplus ... \oplus z_k \equiv p \lor p')$, the $2^k$-1 clauses above are the ones in $\{(\phi \land \phi') \Rightarrow C \mid C \in \text{cnf}(D + D')\}$. ☐

Theorem 2. Assume a Subst-derivation $G = (V, E, L)$ on $\phi_{out} \land l_1 \land ... \land l_k$ and a cnf-compatible cut $W = (V_2, V_3)$. There is a resolution derivation $\pi$ on $\bigwedge_{D \in \phi_{out}} \text{cnf}(D)$.
such that (i) for each vertex \( v \in V_b \) with \( L(v) \neq \top \), \( \pi \) includes all the clauses in \( \{ \text{Exp}(v, W) \Rightarrow C \mid C \in \text{cnf}(L(v)) \} \), and (ii) \( \pi \) has at most \( |V|2^{m-1} \) clauses, where \( m \) is the number of variables in the largest xor-constraint in \( \phi_{\text{xor}} \).

**Proof.** Iteratively on the structure of \( G \), we show how to derive the clauses \( \text{der}(v) = \{ \text{Exp}(v, W) \Rightarrow C \mid C \in \text{cnf}(L(v)) \} \) for each \( v \in V_b \) with \( L(v) \neq \top \). First, we case split by the rule type and have the following two cases.

**Case I:** \( L(v) \) is obtained from \( L(v') \) and \( L(v'') \) by using \( + \)-Unit\(^+ \) or \( + \)-Unit\(^- \). Thus \( L(v') = (x \equiv p) \) for some variable \( x \) and parity \( p \). We have the following cases depending on the role of \( v' \).

1. \( v' \) is an input vertex with \( L(v') \in \phi_{\text{xor}} \).
   - Now \( f_W(v') = \top \) and the clauses in \( S' = \{ f_W(v') \Rightarrow C \mid C \in \text{cnf}(L(v')) \} = \{(x \equiv p)\} \) occur in \( \bigwedge_{D \in \phi_{\text{xor}}} \text{cnf}(D) \).
   - We then case split by the role of \( v'' \).
     - \( v'' \) is an input vertex with \( L(v'') \in \phi_{\text{xor}} \).
       - Now \( f_W(v'') = \top \) and \( S'' = \{ f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v'')) \} \), equalling to \( \{ C \mid C \in \text{cnf}(L(v'')) \} \), consists of clauses already in \( \bigwedge_{D \in \phi_{\text{xor}}} \text{cnf}(D) \).
       - By Lemma 2, the clauses \( \text{der}(v) = \{ f_W(v') \land f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v)) \} \) can thus be derived from the ones in \( S' \) and \( S'' \).
     - \( v'' \) is an input vertex with \( L(v'') \in \{ l_1, ..., l_k \} \).
       - If \( L(v'') = (x \equiv p) \), then \( L(v) = \top \) and there is nothing to prove.
       - If \( L(v'') = (x \equiv \neg p) \), then \( f_W(v'') = (x \equiv \neg p) \), \( L(v) = \bot \), and \( \text{der}(v) = \{ \top \land (x \equiv \neg p) \Rightarrow \bot \} = \{(x \equiv p)\} \) occurring in \( S' \).
     - \( v'' \) is a non-input vertex in \( V_b \).
       - As the cut is cnf-compatible, \( L(v'') \) is either \( (x \equiv p) \) or \( (x \equiv \neg p) \).
       - If \( L(v'') = (x \equiv p) \), then \( L(v) = \top \) and there is nothing to prove.
       - If \( L(v'') = (x \equiv \neg p) \), then \( f_W(v'') = (x \equiv \neg p) \), \( L(v) = \bot \), and \( \text{der}(v) = \{ \top \land (x \equiv \neg p) \Rightarrow \bot \} = \{(x \equiv p)\} \) occurring in \( S' \).
     - \( v'' \) is a non-input vertex in \( V_b \).
       - Now \( f_W(v'') \) is a conjunction of literals as the cut is cnf-compatible. We already have derived the clauses in \( S'' = \{ f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v'')) \} \). By Lemma 2, the clauses \( \text{der}(v) = \{ f_W(v') \land f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v)) \} \) can thus be derived from the ones in \( S' \) and \( S'' \).

2. \( v' \) is an input vertex with \( L(v') \in \{ l_1, ..., l_k \} \).
   - Now \( f_W(v') = L(v') = (x \equiv p) \).
   - We next case split by the role of \( v'' \).
     - \( v'' \) is an input vertex with \( L(v'') \in \phi_{\text{xor}} \).
       - Now \( f_W(v'') = \top \) and \( \text{der}(v) = \{ f_W(v') \land f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v)) \} = \{(x \equiv p) \Rightarrow C \mid C \in \text{cnf}(L(v')) \} \). The clauses in \( \text{der}(v) \) are thus a subset of those occurring in \( \text{cnf}(L(v'')) \) and thus also in \( \bigwedge_{D \in \phi_{\text{xor}}} \text{cnf}(D) \).
     - \( v'' \) is an input vertex with \( L(v'') \in \{ l_1, ..., l_k \} \).
       - Now \( L(v'') \) is either \( (x \equiv p) \) or \( (x \equiv \neg p) \).
       - If \( L(v'') = (x \equiv p) \), then \( L(v) = \top \) and there is nothing to prove.
       - If \( L(v'') = (x \equiv \neg p) \), then \( f_W(v'') = (x \equiv \neg p) \), \( L(v) = \bot \), and \( \text{der}(v) = \{(x \equiv p) \land (x \equiv \neg p) \Rightarrow \bot \} = \emptyset \). This is fine because the clausal explanation is also the tautology stating that \( x \) cannot be true and false at the same time.
Case II:

(c) \( v'' \) is a non-input vertex in \( V'_a \).
Now \( L(v') \) is either \((x \equiv p)\) or \((x \equiv \neg p)\) because the cut is cnf-compatible.
The rest is thus similar to the previous sub-case.

(d) \( v'' \) is a non-input vertex in \( V'_b \).
Now \( f_W(v'') \) is a conjunction of literals as the cut is cnf-compatible and we have already derived the clauses in \( S'' = \{ f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v'')) \} \).
The clause set \( \text{der}(v) = \{ f_W(v') \land f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v)) \} \) equals to \( \{ f_W(v') \Rightarrow (x \equiv \neg p) \lor C \mid C \in \text{cnf}(L(v'[x/p])) \} \). The clauses in \( \text{der}(v) \) are thus a subset of those occurring in \( S'' \).

3. \( v' \) is a non-input vertex in \( V'_a \).
Now \( f_W(v') = L(v') = (x \equiv p) \) as the cut is cnf-compatible.
The rest of this sub-case is similar to the previous sub-case.

4. \( v' \) is a non-input vertex in \( V'_b \).
Now \( f_W(v') \) is a conjunction of literals as the cut is cnf-compatible and we have already derived \( S' = \{ f_W(v') \Rightarrow C \mid C \in \text{cnf}(L(v')) \} = \{ (f_W(v') \Rightarrow (x \equiv p)) \} \).
We then case split by the role of \( v'' \).

(a) \( v'' \) is an input vertex with \( L(v'') \in \phi_{\text{xor}} \).
Now \( f_W(v'') = \top \) and \( S'' = \{ f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v'')) \} \), equaling to \( \{ C \mid C \in \text{cnf}(L(v')) \} \), consists of clauses already in \( \bigwedge_{D \in \phi_{\text{xor}}} \text{cnf}(D) \). By Lemma \ref{lem: xor-clauses}, the clauses \( \text{der}(v) = \{ f_W(v') \land f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v')) \} \) can thus be derived from the clauses in \( S' \) and \( S'' \).

(b) \( v'' \) is an input vertex with \( L(v'') \in \{ l_1, ..., l_k \} \).
If \( L(v') = (x \equiv p) \), then \( L(v) = \top \) and there is nothing to prove.
If \( L(v') = (x \equiv \neg p) \), then \( f_W(v') = (x \equiv \neg p) \), \( L(v) = \bot \), and \( \text{der}(v) = \{ (f_W(v') \land (x \equiv \neg p) \Rightarrow \bot) \} \) equals to \( \{ (f_W(v') \Rightarrow (x \equiv p)) \} \) occurring in \( S' \).

(c) \( v'' \) is a non-input vertex in \( V'_a \).
As the cut is cnf-compatible, \( L(v'') \) is either \((x \equiv p)\) or \((x \equiv \neg p)\).
The rest is thus similar to the previous sub-case.

(d) \( v'' \) is a non-input vertex in \( V'_b \).
Now \( f_W(v'') \) is a conjunction of literals as the cut is cnf-compatible. We already have derived the clauses in \( S'' = \{ f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v'')) \} \). By Lemma \ref{lem: xor-clauses}, the clauses \( \text{der}(v) = \{ f_W(v') \land f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v')) \} \) can thus be derived from the clauses in \( S' \) and \( S'' \).

Case II: \( L(v) \) is obtained from \( L(v') \) and \( L(v'') \) by using \( \oplus\text{-Eqv}^+ \) or \( \oplus\text{-Eqv}^- \). Thus \( L(v') = (x \oplus y \equiv p) \) for some variables \( x, y \) and parity \( p \). We have the following cases depending on the role of \( v' \).

1. \( v' \) is an input vertex with \( L(v') \in \phi_{\text{xor}} \).
Now \( f_W(v') = \top \) and the clauses in \( S' = \{ f_W(v') \Rightarrow C \mid C \in \text{cnf}(L(v')) \} \) occur in \( \bigwedge_{D \in \phi_{\text{xor}}} \text{cnf}(D) \).
We then case split by the role of \( v'' \).

(a) \( v'' \) is an input vertex with \( L(v'') \in \phi_{\text{xor}} \).
Now \( f_W(v'') = \top \) and \( S'' = \{ f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v'')) \} \) consists of clauses already occurring in \( \bigwedge_{D \in \phi_{\text{xor}}} \text{cnf}(D) \). By Lemmas \ref{lem: xor-clauses} and \ref{lem: xor-clauses-2}, we can thus derive the clauses in \( \text{der}(v) = \{ f_W(v') \land f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v)) \} \).
\( v'' \) is an input vertex with \( L(v'') \in \{l_1, ..., l_k\} \).

Now \( f_W(v'') = L(v''). \) As \( x \) must occur in \( L(v'') \), \( L(v'') = (x \equiv p'' \text{ for a } p'' \in \{\bot, \top\}) \).

Thus \( \text{der}(v) = \{ f_W(v') \land (x \equiv p'') \Rightarrow C \mid C \in \text{cnf}(L(v') \cup \{x \equiv p''\})\} \).
This equals to \( \{ f_W(v') \Rightarrow (x \equiv \neg p'') \lor C \mid C \in \text{cnf}(L(v') \cup \{x \equiv p''\})\} \) and thus all the clauses in \( v' \) are already in \( S' = \{ f_W(v') \Rightarrow C \mid C \in \text{cnf}(L(v'))\} \).

\( v'' \) is a non-input vertex in \( V_a \).

Now \( f_W(v'') = L(v''). \) As \( x \) must occur in \( L(v'') \) and the cut is cnf-compatible, \( L(v'') = (x \equiv p'') \) for a \( p'' \in \{\bot, \top\} \).

The rest of the case is similar to the previous one.

\( v'' \) is a non-input vertex in \( V_b \).

Now \( f_W(v'') \) is a conjunction of literals as the cut is cnf-compatible and we have already derived the clauses in \( S'' = \{ f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v''))\} \).

The clauses in \( \text{der}(v) = \{ f_W(v') \land f_W(v'') \Rightarrow C \mid C \in \text{cnf}(L(v))\} \) can thus be derived by Lemmas 3 and 4.

2. \( v' \) is an input vertex with \( L(v') \in \{l_1, ..., l_k\} \).
This case is not possible because \( L(v) \) should be of form \( (x \oplus y \equiv p) \).

3. \( v' \) is a non-input vertex in \( V_a \).
As the cut is cnf-compatible, \( L(v') \) must be of form \( (x \equiv p) \), not of \( (x \oplus y \equiv p) \) as required. Therefore, this case is impossible.

4. \( v' \) is a non-input vertex in \( V_b \).
Now \( f_W(v') \) is a conjunction of literals as the cut is cnf-compatible and we have already derived the clauses in \( S' = \{ f_W(v') \Rightarrow C \mid C \in \text{cnf}(L(v'))\} \).
The rest of the sub-case is similar to the sub-case “\( v' \) is an input vertex with \( L(v') \in \phi_{xor} \)” proven above.

\( \square \)

8.4 Proof of Theorem 3

The constructs in the proof are illustrated in Figures 15 and 16.

**Theorem 3** Let \( \phi_{xor} \) be a conjunction of xor-constraints such that each variable occurs in at most three xor-constraints.

If \( \phi_{xor} \) is unsatisfiable, then there is a UP-derivation on \( \phi_{xor} \land y_1 \land ... \land y_m \) with some \( y_1, ..., y_m \in \text{vars}(\phi_{xor}) \), a vertex \( v \) with \( L(v) = (\bot \equiv \top) \) in it, and a cut \( W \) for \( v \) such that \( \text{Expl}(v, W) = (\bot \equiv \top) \) and thus \( \text{Expl}(v, W) + L(v) = (\bot \equiv \top) \).

If \( \phi_{xor} \) is satisfiable and \( \phi_{xor} \models (x_1 \oplus ... \oplus x_k \equiv p) \), then there is a UP-derivation on \( \phi_{xor} \land (x_1 \equiv p_1) \land ... \land (x_k \equiv p_k) \land y_1 \land ... \land y_m \) with some \( y_1, ..., y_m \in \text{vars}(\phi_{xor}) \) \{\( x_1, ..., x_k \), a vertex \( v \) with \( L(v) = (\bot \equiv \top) \) in it, and a cut \( W \) for \( v \) such that \( \text{Expl}(v, W) + L(v) = (x_1' \oplus ... \oplus x_k' \equiv p') \) for some \( \{x_1', ..., x_k'\} \subseteq \{x_1, ..., x_k\} \) and \( p' \in \{\bot, \top\} \) such that \( \phi_{xor} \models (x_1' \oplus ... \oplus x_k' \equiv p') \).

**Proof.** Let \( \phi_{xor} = D_1 \land ... \land D_n \) be a conjunction of xor-constraints such that each variable occurs in at most three xor-constraints.

We construct the required UP-derivations by starting from the one consisting of \( n \) input vertices (one for each xor-constraint in \( \phi_{xor} \)) and then transforming each “current vertex for the xor-constraint \( D_i \)” into a new one by applying unit propagation to it.
Case I: $\phi_{\text{xor}}$ is unsatisfiable. First, as long as the current xor-constraint vertices contain unary xor-constraints whose variable is occurring in other current xor-constraint vertices, apply the unit propagation rule to eliminate the other occurrences. If the false vertex $\bot \equiv \top$ is derived, then the parity explanation for it will be $\bot \equiv \bot$ under the furthest cut (i.e., the cut $\langle V_a, V_b \rangle$ with the smallest “reason side” $V_a$), as required.

Otherwise, the set $S'$ of current xor-constraint vertices with binary or longer xor-constraint labels induces an unsatisfiable conjunction of xor-constraints. By Lemma 1 there is a subset $S''$ of $S'$ such that $\sum_{v \in S''} L(v) = (\bot \equiv \top)$. We can, and will, assume that $S''$ is minimal, i.e. that there is no subset of $S''$ whose labels’ linear combination is $(\bot \equiv \top)$. Each variable occurring in the labels of $S''$ occurs there exactly two times: it occurs an even number of times because the linear combination of the labels is empty and it cannot occur more than three times due to the assumption we have made in the theorem.

We next consider the “dual graph” for $S''$, meaning the edge-labeled multi-graph $\langle \{ L(v) \mid v \in S'' \} , \{ \langle D, D' \rangle , x \mid x \in \text{vars}(D) \cap \text{vars}(D') \} \rangle$ and take any spanning tree of it. As each variable occurs at exactly two times in the labels of $S''$, it occurs in exactly one edge in the dual graph.
To complete the UP-derivation, we proceed in two phases. In phase one, we make an xor-assumption \((x \equiv \top)\) for each variable occurring in an edge of the dual graph not belonging to the spanning tree. We apply unit propagation so that the variable is removed from the two xor-constraint labels it occurs in the current version of xor-constraints of \(S''\). Thus the out-degree of the xor-assumption vertex is thus two. In phase two, we unit propagate the remaining variables in \(S''\), starting from the leafs of the spanning tree, and obtain conflict on some variable occurring in an edge of the spanning tree. Take the furthest cut of the constructed UP-derivation for the conflict vertex. As all the xor-constraints in \(S''\) were required to obtain the conflict, all the occurrences of the variables in the edges not in the spanning tree (i.e. xor-assumptions made) were required, too. The out-degrees of the other vertices in the last two phases are one. Thus each xor-assumption occurs twice in when computing the parity explanation and these occurrences cancel each other out, resulting in the empty parity explanation as required.

Case II: \(\phi_{xor}\) is satisfiable and \(\phi_{xor} \models (x_1 \oplus ... \oplus x_k \equiv p)\). First, choose some values \(p_1, ..., p_k\) for the variables \(x_1, ..., x_k\) so that \(p_1 \oplus ... \oplus p_k \neq p\). Now clearly \(\phi_{xor} \land (x_1 \equiv p_1) \land ... \land (x_k \equiv p_k)\) is unsatisfiable.

Next, make the xor-assumption \((x_i \equiv p_i)\) for each \(x_i\) and apply unit propagation as long as possible. If the falsity vertex \(\bot \equiv \top\) is derived, then the parity explanation of \(\bot \equiv \top\) under the furthest cut will be \((x'_1 \oplus ... \oplus x'_l \equiv p'')\) for some \(\{x'_1, ..., x'_l\} \subseteq \{x_1, ..., x_k\}\) and \(p'' \in \{\bot, \top\}\). As \(\phi_{xor} \models (x'_1 \oplus ... \oplus x'_l \equiv p'') + (\bot \equiv \top)\), we have the desired result.

Otherwise, the set \(S'\) of current xor-constraint vertices with binary or longer xor-constraint labels induces an unsatisfiable conjunction of xor-constraints. We can thus proceed as in Case I after the initial unit propagation. The conflict vertex obtained eventually may depend on the xor-assumptions \((x_i \equiv p_i)\) we made above and these may also
occur in the parity explanation under the furthest cut. Thus the parity explanation will be \((x'_1 \oplus \ldots \oplus x'_l \equiv p')\) for some \(\{x'_1, \ldots, x'_l\} \subseteq \{x_1, \ldots, x_k\}\) and \(p' \in \{\bot, \top\}\). As \(\phi_{\text{xor}} \models (x'_1 \oplus \ldots \oplus x'_l \equiv p') + (\bot \equiv \top)\), we have the desired result.

\[ \square \]

8.5 Proof of Theorem 4

If an xor-constraint conjunction \(\psi\) has a \(\text{UP}-\text{propagation table}\) for the set of variables \(Y \subseteq \text{vars}(\psi)\), we denote this by \(Y \subseteq_{\text{UP}} \psi\).

**Lemma 5.** Let \(\phi\) be a satisfiable conjunction of xor-constraints such that \(Y \subseteq_{\text{UP}} \phi\) for some \(Y \subseteq \text{vars}(\phi)\), and \(a, a_1, \ldots, a_n \in \text{vars}(\phi)\) “alias” variables for the subsets \(Y', Y_1, \ldots, Y_n \subseteq Y\), respectively, and \(Y' = Y_1 \oplus \cdots \oplus Y_n\). It holds that \(\phi \land (a_1 \equiv p_1) \land \cdots \land (a_n \equiv p_n) \vdash_{\text{UP}} (a \equiv p_1 \oplus \cdots \oplus p_n)\).

**Proof.** We prove the lemma by induction on the sequence \(a_1, \ldots, a_n\). The induction hypothesis is that Lemma 5 holds for the case \(a_1, \ldots, a_{n-1}\).

Base case: \(n = 1\). The claim holds trivially, because \(a = a_1\).

Induction step for \(n > 1\). By the property PT1, the “alias” variable \(a'\) for the set of variables \((Y_1 \oplus \cdots \oplus Y_{n-1})\) is present in \(\text{vars}(\phi)\) and the xor-constraint \((a' \oplus Y_1 \oplus \cdots \oplus Y_{n-1}) \equiv \bot\) is in \(\phi\). By the induction hypothesis, it holds that \(\phi \land (a_1 \equiv p_1) \land \cdots \land (a_{n-1} \equiv p_{n-1}) \vdash_{\text{UP}} (a' \equiv p_1 \oplus \cdots \oplus p_{n-1})\). By the property PT2, it holds that the xor-constraint \((a \oplus a_{n-1} + a' \equiv \bot)\) is in \(\phi\). It follows that \(\phi \land (a_1 \equiv p_1) \land \cdots \land (a_n \equiv p_n) \vdash_{\text{UP}} (a \equiv p_1 \oplus \cdots \oplus p_n)\).

**Lemma 6.** Let \(\phi\) be a conjunction of xor-constraints such that \(Y \subseteq_{\text{UP}} \phi\) for some \(Y \subseteq \text{vars}(\phi)\) of variables in \(\phi\), and \(\phi'\) be a satisfiable conjunction of xor-constraints in \(\phi\) such that \(\text{vars}(\phi') \subseteq Y\). If \(\phi' \models (Y' \equiv p)\) for some \(Y' \subseteq Y\), then it holds for the “alias” variable \(a \in \text{vars}(\phi')\) for the subset \(Y'\) that \(\phi \vdash_{\text{UP}} (a \equiv p)\).

**Proof.** By Lemma 1 there is a subset \(S = (Y_1 \equiv p_1) \land \cdots \land (Y_n \equiv p_n)\) of xor-constraints in \(\phi'\) such that \(\sum_{D \in S} D = (Y' \equiv p)\). By the property PT1, it holds that the “alias” variable \(a\) for the set of variables \(Y'\) is present in \(\text{vars}(\phi)\) and the xor-constraint \((a \oplus Y' \equiv \bot)\) is in \(\phi\). Also by the property PT1, it holds for each xor-constraint \((Y_i \equiv p_i)\) in \(S\) that the “alias” variable \(a_i\) for the set of variables \(Y_i\) is present in \(\text{vars}(\phi)\) and the xor-constraint \((a_i \oplus Y_i \equiv \bot)\) is in \(\phi\). By the property PT3 the xor-constraint \((a \equiv p_i)\) is in \(\phi\). It holds by Lemma 5 that \(\phi \land (a_1 \land p_1) \land \cdots \land (a_n \equiv p_n) \vdash_{\text{UP}} (a \equiv p)\).

**Lemma 7.** If \(\phi_{\text{xor}}\) is an xor-constraint conjunction and \(Y \subseteq \text{vars}(\phi_{\text{xor}})\), then \(Y \subseteq_{\text{UP}} \phi_{\text{xor}} \land \text{ptable}(Y, \phi_{\text{xor}}, \{Y\})\).

**Proof.** Consider the pseudo code for the algorithm ptable in Fig. 6. The variable \(Y'\) takes the value of each subset of \(Y\) in the loop in lines 1-5, and as the result \(\phi_{\text{xor}}'\) has a variable \(a\) for each non-empty subset \(Y'\) of \(Y\) such that \((a \oplus Y' \equiv p)\) is in \(\phi_{\text{xor}}'\). The property PT1 is satisfied by the lines 2-3 and the property PT3 by the lines 4-5.

In the loop in lines 6-11 is iterated for every pair of subsets \(Y_1, Y_2 \subset Y\) such that \(Y_1 \neq Y_2\). It holds for the smallest-indexed variables \(a_1, a_2, a_3 \in \text{vars}(\phi_{\text{xor}})\) such that
the xor-constraints \((a_1 \oplus Y_1 \equiv \perp), (a_2 \oplus Y_2 \equiv \perp),\) and \((a_3 \oplus (Y_1 \oplus Y_2) \equiv \perp)\) that the xor-constraint \((a_1 \oplus a_2 \oplus a_3 \equiv \perp)\) is in \(\phi'_{\text{xor}}\). This satisfies PT2.

**Lemma 8.** Given an xor-constraint conjunction \(\phi_0\) and an elimination order \(\langle x_1, \ldots, x_n \rangle\) for the variables of \(\phi_0\) for the algorithm \(k\)-Ge where \(k = |\text{vars}(\phi_0)|\), it holds that there is a sequence of xor-constraint conjunctions \(\langle \phi_1, \ldots, \phi_n \rangle\) in \(\psi = \phi_0 \land k\text{-Ge}(\phi_0)\) and a sequence of sets of variables \(\langle Y_1, \ldots, Y_n \rangle\) such that it holds for each triple \(\langle x_i, Y_i, \phi_i \rangle\):

- \(Y_i = \text{vars}(\text{clauses}(x_i, \phi_{i-1})) \cap \{x_i, \ldots, x_n\}\)
- \(Y_i \subseteq \text{UP} \psi\)
- \(\phi_i = \phi_{i-1} \land \text{ptable}(Y_i, \phi_{i-1}, k)\)
- \(\phi_n = \phi_0 \land k\text{-Ge}(\phi_0)\)

**Proof.** Assume an xor-constraint conjunction \(\phi_0\) and an elimination order \(\langle x_1, \ldots, x_n \rangle\) for the variables \(\phi_0\) for the algorithm \(k\)-Ge where \(k = |\text{vars}(\phi_0)|\). The translation \(k\text{-Ge}(\phi_0)\) in Figure 7 is initialized with \(\phi'_{\text{xor}} = \phi_0\) and \(V = \text{vars}(\phi_0)\). The loop in lines 1-5 is run \(n\) times and \(V\) takes the values \(V_1, \ldots, V_n\). In the first iteration of the loop, all variables of \(\phi_0\) are in the set \(V_1 = V\). Then for each successive iteration \(i\) it holds that \(V_i = V_{i-1} \setminus \{x_i\}\) because \(x_i\) is removed from the set \(V\) in the line 4. We now argue that the xor-constraints in the conjunction \(\phi_1 \land \cdots \land \phi_n\) are in \(\psi = \phi_0 \land k\text{-Ge}(\phi_0)\) after choosing to “eliminate” the variable \(x_i\) in the line 3, the xor-constraint conjunction \(\phi'_{\text{xor}}\) is augmented with \(\text{ptable}(\text{vars}(\text{clauses}(x_i, \phi'_{\text{xor}})) \cap V_i, \phi'_{\text{xor}}, k)\), so \(\phi_i = \phi_{i-1} \land \text{ptable}(Y_i, \phi_{i-1}, k)\). It is clear that \(V_i = \{x_1, \ldots, x_n\}\), so \(\phi_i\) is identical to the xor-constraint conjunction \(\phi'_{\text{xor}}\) after the \(i\)th iteration of the loop. Upon \(i\)th iteration of the loop in the lines 1-5, the translation \(\text{ptable}\) in Figure 6 is initialized with \(Y = Y_i\) and \(\phi'_{\text{xor}} = \phi_{i-1}\). After all the \(n\) iterations are done it is clear that \(\phi_n = \phi_0 \land k\text{-Ge}(\phi_0)\). By Lemma 7 it holds that \(Y_i \subseteq \text{UP} \phi'_{\text{xor}}\) and also \(Y_i \subseteq \text{UP} \psi\), because adding xor-constraints cannot break any conditions of the UP-propagation table.

**Lemma 9.** Given a satisfiable xor-constraint conjunction \(\phi'_0\) in an xor-constraint conjunction \(\phi_0\) and an elimination order \(\langle x_1, \ldots, x_n \rangle\) for the variables of \(\phi_0\) for the algorithm \(k\)-Ge where \(k = |\text{vars}(\phi_0)|\), it holds that there is a sequence of xor-constraint conjunctions \(\langle \phi'_1, \ldots, \phi'_n \rangle\) in \(\psi = \phi_0 \land k\text{-Ge}(\phi_0)\) such that for each \(\phi'_i\) in \(\langle \phi'_0, \ldots, \phi'_n \rangle\) it holds that

- given literals \(l_1, \ldots, l_k, \hat{l}\) such that \((\sum_{D \in \phi'_i} D) \land l_1 \land \cdots \land l_k \models \hat{l}\), it holds that \(\psi \land l_1 \land \cdots \land l_k \models \text{UP} \hat{l}\).

**Proof.** Assume a satisfiable xor-constraint clause conjunction \(\phi'_0\) in an xor-constraint conjunction \(\phi_0\) and an elimination order \(\langle x_1, \ldots, x_n \rangle\) for the variables of \(\phi_0\) for the algorithm \(k\)-Ge.

By Lemma 5 it holds that there is a sequence of xor-constraint conjunctions \(\langle \phi_1, \ldots, \phi_n \rangle\) in \(\psi = \phi_0 \land k\text{-Ge}(\phi_0)\) and a sequence of sets of variables \(\langle Y_1, \ldots, Y_n \rangle\) such that it holds for each triple \(\langle x_i, Y_i, \phi_i \rangle\):

- \(Y_i = \text{vars}(\text{clauses}(x_i, \phi_{i-1})) \cap \{x_i, \ldots, x_n\}\)
- \(Y_i \subseteq \text{UP} \psi\)
- \(\phi_i = \phi_{i-1} \land \text{ptable}(Y_i, \phi_{i-1}, k)\)
Let \( \text{vars}^\times(\phi_{i-1}') = \text{vars}(\phi_{i-1}') \setminus \text{vars}(\sum_{D \in \phi_{i-1}'} D) \) be the set of variables the “disappear” in the normal form of the linear combination of the xor-constraints in \( \phi_{i-1}' \).

We define a corresponding sequence of \( n \) tuples \( \langle Y_i', X_i, V_i, a_i, \phi_i' \rangle \) as follows:

- \( Y_i' = \text{vars}(\text{clauses}(x_i, \phi_{i-1}')) \cap \{ x_i, \ldots, x_n \} \), and
- \( X_i = \text{vars}(\sum_{D \in \phi_{i-1}'} D) \cap \text{vars}(\text{clauses}(x_i, \phi_i')) \) be the set of variables having occurrences in the xor-constraints of the variable \( x_i \) and also remain in the normal form of the linear combination of \( \phi_{i-1}' \), and
- \( V_i = \text{vars}(\sum_{D \in \text{clauses}(x_i, \phi_{i-1}')} D) \cap \text{vars}^\times(\phi_{i-1}') \) be the set of variables remain in the normal form of the linear combination of the xor-constraints of the variable \( x_i \) that also disappear in the normal form of the linear combination of \( \phi_{i-1}' \), and
- \( a_i \) is a variable such that the xor-constraint \( (a_i \oplus V_i \equiv p_i) \) is in \( \phi_i \cap (\bot \equiv \bot) \) (it exists because \( V_i \subseteq Y_i \) and \( Y_i \subseteq \text{UP} \phi_i \)), and
- if \( x_i \not\in \text{vars}(\phi_i') \) or \( V_i = \emptyset \), then \( \phi_i' = \phi_{i-1}' \), otherwise
  - if \( (a_i \oplus V_i \equiv p_i \oplus p_i') \) is in \( \phi_{i-1}' \), then \( \phi_i' = \phi_{i-1}' \setminus \text{clauses}(x_i, \phi_{i-1}') \), otherwise
    - \( \phi_i' = \phi_{i-1}' \setminus \text{clauses}(x_i, \phi_{i-1}') \land (a_i \oplus V_i \equiv p_i \oplus p_i') \).

We prove the lemma by induction on the structure of the xor-constraint conjunction sequence \( \langle \phi_0', \ldots, \phi_n' \rangle \).

The induction hypothesis is that the lemma holds for the xor-constraint conjunction sequence \( \langle \phi_1', \ldots, \phi_n' \rangle \).

Base case: \( i = n \). Assume any literals \( l_1, \ldots, l_k \) such that \( \sum_{D \in \phi_i'} D \land l_1 \land \cdots \land l_k \models \hat{l} \). It holds that \( \text{vars}(\phi_i') = \emptyset \), so \( \text{vars}(\hat{l}) \in \text{vars}(l_1, \ldots, l_k) \). It clearly holds that \( \phi_0 \land l_1 \land \cdots \land l_k \vdash_{\text{UP}} \hat{l} \).

Induction step: \( 0 \leq i-1 < n \). Assume any literals \( l_1, \ldots, l_k \) such that \( \text{vars}(l_1, \ldots, l_k, \hat{l}) \subseteq \text{vars}(\phi_0) \) and \( \sum_{D \in \phi_{i-1}'} D \land l_1 \land \cdots \land l_k \models \hat{l} \). If \( \phi_{i-1}' = \phi_i' \), then it holds by the induction hypothesis that \( \phi_0 \land l_1 \land \cdots \land l_k \vdash_{\text{UP}} \hat{l} \).

We have two cases to consider:

- Case I: \( \text{vars}(\hat{l}) \in X_i \). It holds that \( \text{vars}(\sum_{D \in \phi_i'} D) \subseteq \text{vars}(l_1, \ldots, l_k) \cup \{ a_i \} \) and \( \sum_{D \in \phi_i'} D \land l_1 \land \cdots \land l_k \models (V_i \equiv p_i) \), so by induction hypothesis it holds that \( \phi_0 \land l_1 \land \cdots \land l_k \vdash_{\text{UP}} (a_i \equiv p_i') \). It holds that \( \sum_{D \in \text{clauses}(x_i, \phi_{i-1}')} D \land l_1 \land \cdots \land l_k \land (a_i \equiv p_i') \models \hat{l} \). By Lemma 6, it holds that \( \phi_i \land l_1 \land \cdots \land l_k \land (a_i \equiv p_i') \vdash_{\text{UP}} \hat{l} \).
- Case II: \( \text{vars}(\hat{l}) \not\in X_i \). It holds that \( \sum_{D \in \text{clauses}(x_i, \phi_{i-1}')} D \land l_1 \land \cdots \land l_k \models (V_i \equiv p_i) \). By Lemma 6, it holds that \( \phi_i \land l_1 \land \cdots \land l_k \vdash_{\text{UP}} (a_i \equiv p_i') \). It holds that \( \text{vars}(\sum_{D \in \phi_i'} D) \subseteq \text{vars}(l_1, \ldots, l_k, \hat{l}) \cup \{ a_i \} \), so \( \sum_{D \in \phi_i'} D \land l_1 \land \cdots \land l_k \land (a_i \equiv p_i') \models \hat{l} \). It holds by induction hypothesis that \( \phi_0 \land l_1 \land \cdots \land l_k \land (a_i \equiv p_i') \vdash_{\text{UP}} \hat{l} \).

The following lemma states that \( k \)-Ge translation refutes any unsatisfiable xor-constraint conjunctions.

**Lemma 10.** If \( \phi_{xor} \) is an unsatisfiable xor-constraint conjunction, then \( \phi_{xor} \land k \text{-Ge}(\phi_{xor}) \vdash_{\text{UP}} (\bot \equiv \top) \) where \( k = | \text{vars}(\phi_{xor}) | \).
Proof. Assume an unsatisfiable xor-constraint conjunction $ϕ_{xor}$. By Lemma \[\text{1}\] there is a subset $S = C_1 \land \cdots \land C_m$ of xor-constraints in $ϕ_{xor}$ such that $\sum_{D \in S} D = (⊥ \equiv ⊤)$. Let $(X_m \equiv p_m) = C_m$. It holds that $\sum_{D \in C_1 \land \cdots \land C_{m-1}} D = (X_m \equiv p_m \lor ⊤)$. Thus, $ψ = C_1 \land \cdots \land C_{m-1}$ is satisfiable. Let $ψ = φ_{xor} \land k-Ge(ϕ_{xor})$. By Lemma \[\text{8}\] it holds that:

- there is a set of variables $Y \subseteq \text{vars}(ψ)$ such that $\text{vars}(C_m) \subseteq Y$ and $Y \subseteq UP ψ$.
- there is a variable $y \in \text{vars}(ψ)$ such that the xor-constraint $(y \lor X_m \equiv p_m \lor p'_m)$ is in $ψ$.
- the xor-constraint $(y \equiv p'_m)$ is in $ψ$.

Because $ψ \models (y \equiv p'_m \lor ⊤)$, it holds by Lemma \[\text{9}\] that $ψ \models_{UP} (y \equiv p'_m \lor ⊤)$. Since $ψ \models_{UP} (y \equiv p_m)$ and $ψ \land k-Ge(ϕ_{xor}) \models_{UP} (y \equiv p_m \lor ⊤)$, it follows that $ψ \models_{UP} (⊥ \equiv ⊤)$.

**Lemma 11.** The satisfying truth assignments of $ϕ_{xor}$ are exactly the ones of $ϕ_{xor} \land \text{ptable}(Y,ϕ_{xor},k)$ when projected to $\text{vars}(ϕ_{xor})$ where $Y \subseteq \text{vars}(ϕ_{xor})$.

**Proof.** It holds by definition that $ϕ_{xor} \land \text{ptable}(Y,ϕ_{xor},k) \models ϕ_{xor}$, so it suffices to show that if $τ$ is a satisfying truth assignment for $ϕ_{xor}$, it can be extended to a satisfying truth assignment $τ'$ for $\text{ptable}(Y,ϕ_{xor},k)$. Assume that $τ$ is a truth assignment such that $τ \models ϕ_{xor}$. Let $τ'$ be a truth assignment identical to $τ$ except for the following additions.

The translation $\text{ptable}(Y,ϕ,k)$ in Figure \[\text{6}\] adds four kinds of xor-constraints.

1. $(y \lor Y' \equiv ⊥)$ where $Y'$ is a non-empty subset of $Y$ and $y$ is a new variable. If $τ \models (Y' \equiv ⊤)$, add $y$ to $τ'$, otherwise add $¬y$ to $τ'$. It is clear that $τ' \models (y \lor Y' \equiv ⊥)$.
2. $(a_1 \lor a_2 \lor a_3 \equiv p_1 \lor p_2 \lor p_3)$ if the xor-constraints $(a_1 \lor Y_1 \equiv p_1)$, $(a_2 \lor Y_2 \equiv p_2)$, and $(a_3 \lor (Y_1 \lor Y_2) \equiv p_3)$ are in $ϕ$ augmented with xor-constraints from the previous step. From the previous step it is clear that $τ' \models (a_1 \lor Y_1 \equiv p_1)$, $τ' \models (a_2 \lor Y_2 \equiv p_2)$, and $τ' \models (a_3 \lor Y_3 \equiv p_3)$. It follows that $τ' \models (a_1 \lor a_2 \lor a_3 \equiv p_1 \lor p_2 \lor p_3)$.
3. $(y \equiv p')$ where $y \in \text{vars}(ϕ)$ such that the xor-constraints $(y \lor Y' \equiv p \lor p')$ and $(Y' \equiv p)$ are in $ϕ_{xor}$ augmented with xor-constraints from the previous step. Since $(Y' \equiv p)$ is an original xor-constraint in $ϕ$, it holds that $τ' \models (Y' \equiv p)$. It follows that $τ' \models (y \equiv p')$.
4. $(y \lor y' \equiv p \lor p')$ where $y,y' \in \text{vars}(ϕ)$ and $p,p' \in \{⊤,⊥\}$ such that the xor-constraints $(y \lor Y' \equiv p)$ and $(y' \lor Y' \equiv p')$ where $Y'$ is a non-empty subset of $Y$ are in $ϕ_{xor}$ augmented with xor-constraints from the previous step. If $τ' \models (Y' \equiv p \lor ⊤)$, then add $y$ to $τ'$, otherwise add $¬y$ to $τ'$. If $τ' \models (Y' \equiv p \lor ⊤)$, then add $y'$ to $τ'$, otherwise add $¬y'$ to $τ'$. It follows that $τ' \models (y \lor y' \equiv p \lor p')$.

**Theorem 4** If $ϕ_{xor}$ is an xor-constraint conjunction, then $k-Ge(ϕ_{xor})$ is a GE-simulation formula for $ϕ_{xor}$ provided that $k = |\text{vars}(ϕ_{xor})|$.

**Proof.** We first prove that the satisfying truth assignments of $ϕ_{xor}$ are exactly the ones of $ψ = φ_{xor} \land k-Ge(φ_{xor})$ when projected to $\text{vars}(ϕ_{xor})$. The translation $k-Ge$ in Figure \[\text{7}\] only adds xor-constraint conjunctions of the type $\text{ptable}(Y,ϕ,k)$ for some set of
variables $Y \subseteq \text{vars}(\phi_{\text{xor}})$ and some xor-constraint conjunction $\phi$ and by Lemma 11 the satisfying assignments of $\phi$ are exactly the ones of $\phi \land \text{ptable}(Y, \phi, k)$ when projected to $\text{vars}(\phi)$. It follows by induction that the satisfying truth assignment for $\phi_{\text{xor}}$ are exactly to the ones of $\phi_{\text{xor}} \land k$-Ge$(\phi_{\text{xor}})$ when projected to $\text{vars}(\phi_{\text{xor}})$.

Next we show that if $\phi_{\text{xor}}$ is satisfiable and $\phi_{\text{xor}} \land l_1 \land \cdots \land l_k \models \hat{l}$, then $\hat{l}$ is UP-derivable from $\phi_{\text{xor}} \land k$-Ge$(\phi_{\text{xor}}) \land l_1 \land \cdots \land l_k$. By Lemma 1 there is a subset $S$ of xor-constraints in $\phi \land l_1 \land \cdots \land l_k$ such that $\sum_{D \in S} D = \hat{l}$. By Lemma 9, it holds that $\sum_{D \in S} D \models_{\text{UP}} \hat{l}$.

It remains to show that if $\phi_{\text{xor}}$ is unsatisfiable, then $\psi \models_{\text{UP}} (\bot \equiv \top)$. Assume that $\phi_{\text{xor}}$ is unsatisfiable. By Lemma 10, it holds that $\psi \models_{\text{UP}} (\bot \equiv \top)$. All the requirements for GE-simulation formula are satisfied, so $k$-Ge$(\phi_{\text{xor}})$ is a GE-simulation formula for $\phi_{\text{xor}}$.

8.6 Proof of Theorem 5

Theorem 5 If $\langle \phi_a', \phi_b' \rangle$ is the result of applying one of the simplification rules to $\langle \phi_a, \phi_b \rangle$ and $\phi_a \land \phi_b \land l_1 \land \cdots \land l_k \models_{\text{UP}} \hat{l}$, then $\phi_a' \land \phi_b' \land l_1 \land \cdots \land l_k \models_{\text{UP}} \hat{l}$.

Proof. Let $\langle \phi_a', \phi_b' \rangle$ be the result of applying one of the simplification rules to $\langle \phi_a, \phi_b \rangle$, $\phi_a \land \phi_b \land l_1 \land \cdots \land l_k \models_{\text{UP}} \hat{l}$, and $\hat{l} = (x \equiv p)$. If S1 was the simplification rule used, then it clearly holds that $\phi_a' \land \phi_b' \land l_1 \land \cdots \land l_k \models_{\text{UP}} \hat{l}$.

Otherwise, S2 was used to simplify an xor-constraint $D$ in $\phi_a$ with an xor-constraint $D'$ in $\phi_b$ such that $|\text{vars}(D) \cap \text{vars}(D')| \geq |\text{vars}(D')| - 1$. It holds that $\phi_a' = \phi_a \setminus \{D\} \cup \{D + D'\}$ and $\phi_b' = \phi_b$. It must hold that there is an xor-clause $C = (x + y_1 + \cdots + y_n \equiv p + p_1 + \cdots + p_n)$ in $\psi$ such that for each $y_i \in \{y_1, \ldots, y_n\}$ it holds that $\psi \models_{\text{UP}} (y_i \equiv p_i)$. We prove by induction $\hat{l}$ is UP-derivable from $\psi' = \phi_a' \land \phi_b' \land l_1 \land \cdots \land l_k$. The induction hypothesis is that for each $y_i \in \{y_1, \ldots, y_n\}$ it holds that $\psi \models_{\text{UP}} (y_i \equiv p_i)$.

Base case: $C = \hat{l} = (x \equiv p)$. If $C \neq D$, then $(x \equiv p)$ is in $\psi'$ and $\psi' \models_{\text{UP}} \hat{l}$. Otherwise, $C = D$. Since $|\text{vars}(D') \cap \text{vars}(D)| \geq |\text{vars}(D')| - 1$, it holds that $\text{vars}(D + D') = \{x'\}$ for some $x' \in \text{vars}(\psi) \text{vars}(D') = \{x, x'\}$. The xor-constraint $D + D'$ is in $\phi_a'$, and the xor-constraint $D' \phi_b'$. It clearly holds that $\psi' \models_{\text{UP}} \hat{l}$.

Induction step: $C \neq \hat{l}$. If $C \neq D$, then $C$ is in $\psi'$ and $\psi' \models_{\text{UP}} \hat{l}$. Otherwise $C = D$.

We have two cases to consider:

- Case 1: $x \in \text{vars}(D')$. By induction hypothesis it holds for each $y_i \in \{y_1, \ldots, y_n\}$ that $\psi' \models_{\text{UP}} (y_i \equiv p_i)$. If there is a variable $z \in \text{vars}(D')$ such that $z \notin \text{vars}(D)$, then $(D + D') \land (y_1 \equiv p_1) \land \cdots \land (y_n \equiv p_n) \models_{\text{UP}} (z \equiv p')$ and then $D \land (y_1 \equiv p_1) \land \cdots \land (y_n \equiv p_n) \land (z \equiv p') \models_{\text{UP}} (x \equiv p)$, so $\psi' \models_{\text{UP}} \hat{l}$. Otherwise, it holds $\text{vars}(D') \subseteq \text{vars}(D)$, and it clearly holds that $\psi' \models_{\text{UP}} \hat{l}$.

- Case 2: $x \notin \text{vars}(D')$. By induction hypothesis it holds for each $y_i \in \{y_1, \ldots, y_n\}$ that $\psi' \models_{\text{UP}} (y_i \equiv p_i)$. If there is a variable $z \in \text{vars}(D')$ such that $z \notin \text{vars}(D)$, then $D' \land (y_1 \equiv p_1) \land \cdots \land (y_n \equiv p_n) \models_{\text{UP}} (z \equiv p')$ and then $(D + D') \land (y_1 \equiv p_1) \land \cdots \land (y_n \equiv p_n) \land (z \equiv p') \models_{\text{UP}} (x \equiv p)$, so $\psi' \models_{\text{UP}} \hat{l}$. Otherwise, it holds $\text{vars}(D') \subseteq \text{vars}(D)$, and it clearly holds that $\psi' \models_{\text{UP}} \hat{l}$. 
8.7 Proof of Theorem 6

**Theorem** Let \((V_a, V_b)\) be an X-cut partition of \(\phi_{\text{xor}}\). Let \(\phi_{\text{xor}}^a = \bigwedge_{D \in V_a} D\), \(\phi_{\text{xor}}^b = \bigwedge_{D \in V_b} D\), and \(l_1, \ldots, l_k \in \text{lits}(\phi_{\text{xor}})\). Then it holds that:

- If \(\phi_{\text{xor}} \land l_1 \land \cdots \land l_k\) is unsatisfiable, then
  1. \(\phi_{\text{xor}}^a \land l_1 \land \cdots \land l_k \lor \phi_{\text{xor}}^b \land l_1 \land \cdots \land l_k\) is unsatisfiable; or
  2. \(\phi_{\text{xor}}^a \land l_1 \land \cdots \land l_k \models (X' \equiv p')\) and \(\phi_{\text{xor}}^b \land l_1 \land \cdots \land l_k \models (X' \equiv p' \oplus \top)\) for some \(X' \subseteq X\) and \(p', p' \in \{\top, \bot\}\).

- If \(\phi_{\text{xor}} \land l_1 \land \cdots \land l_k\) is satisfiable and \(\phi_{\text{xor}} \land l_1 \land \cdots \land l_k \models (Y \equiv p)\) for some \(Y \subseteq \text{vars}(\phi_{\text{xor}})\), \(Y \subseteq \text{vars}(\phi_{\text{xor}}^a) \cap \text{vars}(\phi_{\text{xor}}^b)\), \(\alpha \in \{a, b\}\) and \(\beta \in \{a, b\} \setminus \{\alpha\}\), then
  1. \(\phi_{\text{xor}}^a \land l_1 \land \cdots \land l_k \models (Y \equiv p)\) or \(\phi_{\text{xor}}^b \land l_1 \land \cdots \land l_k \models (Y \equiv p)\); or
  2. \(\phi_{\text{xor}}^a \land l_1 \land \cdots \land l_k \models (X' \equiv p')\) and \(\phi_{\text{xor}}^b \land l_1 \land \cdots \land l_k \models (X' \equiv p') \lor (Y \equiv p)\) for some \(X' \subseteq X\) and \(p', p' \in \{\top, \bot\}\).

\[\text{Proof.}\] Let \((V'_a, V'_b)\) be an X-cut partition of \(\phi_{\text{xor}} \land (l_1) \land \ldots \land (l_k)\) with \(\text{vars}(V'_a) = \text{vars}(V_a)\), \(\text{vars}(V'_b) = \text{vars}(V_b)\), \(V'_a \subseteq V'_b\), and \(V'_b \subseteq V_b\). Such partition exists because the xor-assumption literals \(l_i\) are unit xor-constraints.

- Case I: \(\phi_{\text{xor}} \land l_1 \land \ldots \land l_k\) is unsatisfiable. By Lemma 1 there is a subset \(S\) of xor-constraints in \(\phi_{\text{xor}} \land (l_1) \land \ldots \land (l_k)\) such that \(\sum_{D \in S} D = (\bot \equiv \top)\). Observe that \(\sum_{D \in S} D = \sum_{D \in V'_a \cap S} D + \sum_{D \in V'_b \cap S} D\). If \(\sum_{D \in V'_b \cap S} D = (\bot \equiv \top)\), then \(\phi_{\text{xor}}^a \land l_1 \land \ldots \land l_k\) is unsatisfiable. Otherwise, it must be that \(\sum_{D \in V'_a \cap S} D = (X' \equiv p')\) and \(\sum_{D \in V'_b \cap S} D = (X' \equiv p' \oplus \top)\) with \(p' \in \{\bot, \top\}\). This gives two cases:

  1. evaluating \((\sum_{D \in V'_a \cap S} D)\) gives an empty expression and the simplified equation is then \((Y \equiv p) = (\sum_{D \in V'_a \cap S} D)\), so it follows that \(\phi_{\text{xor}} \land l_1 \land \ldots \land l_k \models (Y \equiv p)\).

  2. evaluating \((\sum_{D \in V'_b \cap S} D)\) gives an xor-constraint \((X' \equiv p')\) for some \(X' \subseteq X\) and \(p' \in \{\top, \bot\}\) because \(V'_a \cap V'_b = \emptyset\), \(\text{vars}(V'_a) \cap \text{vars}(V'_b) = X'\) and \((\sum_{D \in V'_a \cap S} D) + (\sum_{D \in V'_b \cap S} D) = (Y \equiv p)\). The simplified equation is then \((Y \equiv p) = (X' \equiv p') + (\sum_{D \in V'_a \cap S} D)\), so it follows that \(\phi_{\text{xor}} \land l_1 \land \ldots \land l_k \land (X' \equiv p') \models (Y \equiv p)\).
8.8 Proof of Theorem

Lemma 12. If \( \phi \) is a satisfiable conjunction in \( \phi_{\text{xor}} \land \psi \) such that \( \text{vars}(\phi) \subseteq Y, Y \subseteq \text{vars}(\phi_{\text{xor}}) \), \( Y \subseteq_{UP} \phi_{\text{xor}} \land \psi, \) and \( \phi \land (Y_1 \equiv p_1) \land \cdots \land (Y_n \equiv p_n) \models (Y' \equiv p') \) where \( Y_1, \ldots, Y_n, Y' \subseteq Y \) and \( p_1, \ldots, p_n, p' \in \{ \top, \bot \} \), then \( \phi_{\text{xor}} \land \psi \land a_1 \equiv p_1 \land \cdots \land a_n \equiv p_n \vdash_{UP} a' \equiv p' \) where \( a_1, \ldots, a_n, a' \) are the “alias” variables for the sets \( Y_1, \ldots, Y_n, Y' \), respectively.

Proof. By Lemma 1, there is a subset \( \phi' \) of xor-constraints in \( \phi \land (Y_1 \equiv p_1) \land \cdots \land (Y_n \equiv p_n) \) such that \( \sum_{D \subseteq \phi'} D = (Y' \equiv p') \). By the property PT1, it holds for each xor-constraint \( (Y'' \equiv p'') \) in \( \phi' \) that the corresponding “alias” variable \( a'' \) for the set of variables \( Y'' \) is present in \( \text{vars}(\phi) \) and by the property PT3 the xor-constraint \( (a'' \equiv p'') \) is in \( \phi_{\text{xor}} \land \psi \). It holds by Lemma 5 that \( \phi_{\text{xor}} \land \psi \land (a_1 \equiv p_1) \land \cdots \land (a_n \equiv p_n) \vdash_{UP} (a' \equiv p') \).

Theorem 7. If \( \{X_1, \ldots, X_n\} \) is the family of variable sets in the tree decomposition of the primal graph of an xor-constraint conjunction \( \phi_{\text{xor}} \) and \( \phi_0, \ldots, \phi_n \) is a sequence of xor-constraint conjunctions such that \( \phi_0 = \phi_{\text{xor}} \land \psi \) and \( \phi_i = \phi_{i-1} \land \text{ptable}(X_i, \phi_{i-1}, |X_i|) \) for \( i \in \{1, \ldots, n\} \), then \( \phi_n \land \phi_{\text{xor}} \) is a GE-simulation formula for \( \phi_{\text{xor}} \) with \( O(n2^k) \) + \( |\phi_{\text{xor}}| \) xor-constraints, where \( k = \max(|X_1|, \ldots, |X_n|) \).

Proof. The construction is illustrated in Figures 17, 18, 19.

Let \( \psi = \phi_n \land \phi_{\text{xor}} \). We first prove that the satisfying truth assignments of \( \phi_{\text{xor}} \) are exactly the ones of \( \phi_{\text{xor}} \land \psi \) when projected to \( \text{vars}(\phi_{\text{xor}}) \). By Lemma 1, the satisfying truth assignments of \( \phi \) are exactly the ones of \( \phi \land \text{ptable}(Y, \phi, k) \) when projected to \( \text{vars}(\phi) \), so by induction the satisfying truth assignments of \( \phi_{\text{xor}} \) are exactly the ones of \( \phi_{\text{xor}} \land \psi \) when projected to \( \text{vars}(\phi_{\text{xor}}) \). The number of xor-constraints in \( \text{ptable}(Y, \phi, k) \) is \( O(2^k) + |\phi| \), so the number of xor-constraints in \( \psi \) is \( O(n2^k) + |\phi_{\text{xor}}| \).

It holds for each \( X_i \in \{X_1, \ldots, X_n\} \) by Lemma 7 that \( X_i \subseteq_{UP} \phi_{\text{xor}} \land \psi \). Next we show that if \( \phi_{\text{xor}} \) is satisfiable and \( \phi_{\text{xor}} \land l_1 \land \cdots \land l_k \models \hat{l} \) then \( \hat{l} \) is UP-derivable from \( \phi_{\text{xor}} \land \psi \land l_1 \land \cdots \land l_k \). Assume that \( \phi_{\text{xor}} \) is satisfiable and \( \phi_{\text{xor}} \land l_1 \land \cdots \land l_k \models \hat{l} \). We prove by induction on the structure of the tree decomposition that the following property holds for each subtree \( T' \) of the tree decomposition having the set of variables \( X_{T'} \) and the root node of \( T' \) with the set of variables \( X_r \):

- If \( \phi \) is a satisfiable conjunction in \( \phi_{\text{xor}} \land \psi \land l_1 \land \cdots \land l_k \) such that \( \text{vars}(\phi) \subseteq X_{T'} \), and \( \phi \land (Y_1 \equiv p_1) \land \cdots \land (Y_m \equiv p_m) \models (Y' \equiv p') \) where \( Y' \subseteq X \), and for each \( Y_j \in \{Y_1, \ldots, Y_m\} \) there is a \( k \in \{1, \ldots, n\} \) for which it holds that \( Y_j \subseteq X_k \) and \( p_1, \ldots, p_m, p' \in \{ \top, \bot \} \), then \( \phi_{\text{xor}} \land \psi \land (a_1 \equiv p_1) \land \cdots \land (a_n \equiv p_n) \vdash_{UP} (a' \equiv p') \) where \( a_1, \ldots, a_n, a' \) are the “alias” variables for the variable sets \( Y_1, \ldots, Y_n, Y' \), respectively.

The induction hypothesis is that the property holds for each proper subtree of \( T' \).

Base case: \( T' \) has only one node. The property holds by Lemma 12.

Induction step: \( T' \) has more than one node. Let \( \phi' = \phi \land (Y_1 \equiv p_1) \land \cdots \land (Y_m \equiv p_m) \). The idea is to remove xor-constraints involving variables other than in \( X_r \) from \( \phi' \) and add additional xor-constraints of the type involving variables only in \( X_r \). This
is done by considering each direct child node of the root node of \( T' \) at a time possibly rewriting \( \phi' \) by substituting a sub-conjunction of \( \phi' \) with at most one xor-constraint having only variables in \( X_r \). Let \( T'' \) be the subtree induced by one direct child node of the root node having the set of variables \( X_{T''} \). The per-child substitution operation of \( \phi' \) is defined as follows. Let \( \phi_a \) be the maximal conjunction of xor-constraints in \( \phi' \) such that \( \text{vars}(\phi_a) \subseteq X_{T''} \), and \( \phi_b \) be the conjunction of xor-constraints in \( \phi' \) but not in \( \phi_a \). If \( \phi_a \) is empty, then nothing needs to be removed from \( \phi' \). Otherwise, \( \phi_a \) is non-empty and there is an \( X \)-cut partition \((V_A, V_B)\) of \( \phi' \) such that \( \phi_a \equiv \bigwedge_{D \in V_A} D \), \( \phi_b = \bigwedge_{D \in V_B} D \) and \( \text{vars}(\phi_a) \cap \text{vars}(\phi_b) = X \subseteq X_r \cap X_c \). By Theorem 6, it holds that

1. \( \phi_a \models (Y' \equiv p') \) or \( \phi_b \models (Y' \equiv p') \); or
2. \( \phi_a \models (X'' \equiv p'') \) and \( \phi_b \models (X'' \equiv p'') \) \( \models (Y' \equiv p') \) for some \( X'' \subseteq X \), \( p' \in \{\top, \bot\} \), \( \alpha \in \{a, b\} \), and \( \beta \in \{a, b\} \setminus \{\alpha\} \).

We analyze the cases:

Case 1: \( \phi_a \models (Y' \equiv p') \) or \( \phi_b \models (Y' \equiv p') \). Since \( Y' \subseteq X_r \), it must be that \( \phi_b \models (Y' \equiv p') \). In this case, set \( \phi' \leftarrow \phi_b \).

Case 2: \( \phi_a \models (X'' \equiv p'') \) and \( \phi_b \models (X'' \equiv p'') \) \( \models (Y' \equiv p') \) for some \( X'' \subseteq X \), \( p' \in \{\top, \bot\} \), \( \alpha \in \{a, b\} \), and \( \beta \in \{a, b\} \setminus \{\alpha\} \). Again since \( Y' \subseteq X_r \), it must be that \( \alpha = a \). In this case, set \( \phi' \leftarrow \phi_b \wedge (X'' \equiv p'') \).
After each child node has been processed in this way, it holds that $\phi' \models (Y \equiv p)$ and $\text{vars}(\phi') \subseteq X_r$. It also holds by the induction hypothesis for each xor-constraint $(X_i' \equiv p_i')$ in the sequence $(X_1' \equiv p_1'), \ldots, (X_q' \equiv p_q')$ of added xor-constraints that the corresponding “alias” variables $a_1', a_2', \ldots, a_l'$ for $X_1', \ldots, X_q'$, respectively, that, since $\phi \land (Y_1 \equiv p_1) \land \cdots \land (Y_m \equiv p_m) \land (X_1' \equiv p_1') \land \cdots \land (X_{m-1}' \equiv p_{m-1}') \models (X_1' \equiv p_1')$, then it holds that $\phi_{\text{xor}} \land \psi \land l_1 \land \cdots \land l_k \land (a_1 \equiv p_1) \land \cdots \land (a_m \equiv p_m) \land (a_1' \equiv p_1') \land \cdots \land (a_{m-1}' \equiv p_{m-1}') \models \phi_r' \models \phi_r' \models (X_1' \equiv p_1')$. Now $\phi' \models (Y \equiv p)$, $\text{vars}(\phi') \subseteq X_r$, and each xor-constraint $(X'' \equiv p'')$ in $\phi'$ has its corresponding “alias” variable $a''$ implied by unit propagation, that is, $\phi_{\text{xor}} \land \psi \land l_1 \land \cdots \land l_k \models (a' \equiv p')$. By induction it follows that $\phi_{\text{xor}} \land \psi \land l_1 \land \cdots \land l_k \models a' \equiv p'$.

It remains to show that if $\phi_{\text{xor}}$ is unsatisfiable, then $\psi \models \bot$. Assume that $\phi_{\text{xor}}$ is unsatisfiable. By Lemma 1, there is a minimal subset $S$ of xor-constraints in $\phi_{\text{xor}} \land l_1 \land \cdots \land l_k$ such that $\sum_{D \in S} D = (\bot \equiv \top)$. Now, let $S' = S \setminus D$ be a subset of $S$ identical to $S$ except that one xor-constraint $D = (X' \equiv p) \subseteq S$ is removed. It clearly holds that $S'$ is satisfiable and $\sum_{D \in S'} D = (X' \equiv p' \equiv \top)$. There is a node in $T'$ that has the variables $X_i$ such that $X_i' \subseteq X_i$. It holds that $X_i \subseteq \uparrow_{UP} \phi_{\text{xor}} \land \psi$, so by Lemma 6 it holds for the “alias” variable $a'$ for $X'$ that $\phi_{\text{xor}} \land \psi \land l_1 \land \cdots \land l_k \models (a' \equiv p \equiv \bot)$. Repeat the proof as above for the satisfiable case and for the subset $S'$ showing that $\phi_{\text{xor}} \land \psi \land l_1 \land \cdots \land l_k \models (a' \equiv p)$. It follows that $\phi_{\text{xor}} \land \psi \land l_1 \land \cdots \land l_k \models (\bot \equiv \top)$. All the requirements for GE-simulation formula are satisfied, so $\psi$ is a GE-simulation formula for $\phi_{\text{xor}}$. 

![Fig. 18. Primal graph for the instance whose constraint graph is shown in Fig. 17(a)](image-url)
Assume that

\[ X_1 = \{x_1, x_2, x_3, x_4, x_6\}, X_2 = \{x_3, x_4, x_5, x_6\}, X_3 = \{x_2, x_4, x_5, x_6\}, X_4 = \{x_2, x_3, x_5, x_7\}, X_5 = \{x_2, x_3, x_4, x_5, x_6\}, X_6 = \{x_4, x_5, x_6, x_10, x_{11}\} \]

\[ \varphi_0 = \varphi_{\text{xor}}, \varphi_1 = \varphi_0 \land \text{ptable}(X_1, \varphi_0, |X_1|), \varphi_2 = \varphi_1 \land \text{ptable}(X_2, \varphi_1, |X_2|), \varphi_3 = \varphi_2 \land \text{ptable}(X_3, \varphi_2, |X_3|), \varphi_4 = \varphi_3 \land \text{ptable}(X_4, \varphi_3, |X_4|), \varphi_5 = \varphi_4 \land \text{ptable}(X_5, \varphi_4, |X_5|), \text{and } \varphi_6 = \varphi_5 \land \text{ptable}(X_6, \varphi_5, |X_6|), \text{and } \psi = \varphi_6 \setminus \varphi_{\text{xor}}. \]

It holds that

\[ X_1 \subseteq_{\text{UP}} \varphi_{\text{xor}} \land \psi, \ldots, X_6 \subseteq_{\text{UP}} \varphi_{\text{xor}} \land \psi. \]

The UP system can deduce \((x_1 \equiv \top)\), i.e.

\[ \varphi_{\text{xor}} \land \psi \vdash_{\text{UP}} (x_1 \equiv \top) \]

by “propagating” intermediate linear combinations starting from the leaves of the tree decomposition towards the root node (the node with the set of variables \(X_1\)). Since \(x_6 \equiv \top\) is in \(\varphi_{\text{xor}}\) it holds that

\[ \varphi_{\text{xor}} \land \psi \vdash_{\text{UP}} a_{2,3,4} \equiv \bot. \]

And in a similar way because \(x_7 \equiv \top, x_8 \equiv \top,\) and \(x_9 \equiv \top\) are in \(\varphi_{\text{xor}},\) then

\[ \varphi_{\text{xor}} \land \psi \vdash_{\text{UP}} a_{2,3,5} \equiv \bot, \varphi_{\text{xor}} \land \psi \vdash_{\text{UP}} a_{2,4,5} \equiv \bot, \text{and } \varphi_{\text{xor}} \land \psi \vdash_{\text{UP}} a_{3,4,5} \equiv \bot. \]

By combining these intermediate results, it holds that

\[ \varphi_{\text{xor}} \land \psi \vdash_{\text{UP}} a_{2,3,4,5} \equiv \bot \text{ and finally } \varphi_{\text{xor}} \land \psi \vdash_{\text{UP}} x_1 \equiv \top. \]