PARTIAL CLASSIFICATION RESULTS FOR POSITIVE QUATERNION KÄHLER MANIFOLDS

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ABSTRACT. Positive Quaternion Kähler Manifolds are Riemannian manifolds with holonomy contained in $\text{Sp}(n)\text{Sp}(1)$ and with positive scalar curvature. Conjecturally, they are symmetric spaces. We prove this conjecture in dimension 20 under additional assumptions and we provide recognition theorems for quaternionic projective spaces (in low dimensions) as well as the real Grassmanian $\text{Gr}_4(\mathbb{R}^{n+4})$.

INTRODUCTION

Quaternion Kähler Manifolds settle in the highly remarkable class of special geometries. Hereby one refers to Riemannian manifolds with special holonomy among which Kähler manifolds, Calabi–Yau manifolds or Joyce manifolds are to be mentioned as the most prominent examples. Quaternion Kähler Manifolds have holonomy contained in $\text{Sp}(n)\text{Sp}(1)$; they are called positive, if their scalar curvature is positive.

The only known examples of Positive Quaternion Kähler Manifolds are given by the so-called Wolf-spaces, which are all symmetric and the only homogeneous examples due to Alekseevski. Indeed, they are given by the infinite series $\mathbb{H}P^n$, $\text{Gr}_2(\mathbb{C}^{n+2})$ and $\text{Gr}_4(\mathbb{R}^{n+4})$ (the Grassmanian of oriented real 4-planes) and the exceptional spaces $G_2/\text{SO}(4)$, $F_4/\text{Sp}(3)\text{Sp}(1)$, $E_6/\text{SU}(6)\text{Sp}(1)$, $E_7/\text{Spin}(12)\text{Sp}(1)$, $E_8/E_7\text{Sp}(1)$. Besides, it is known that in each dimension there are only finitely many Positive Quaternion Kähler Manifolds. This endorses the fundamental conjecture

Conjecture (LeBrun, Salamon). Every Positive Quaternion Kähler Manifold is a Wolf space.

A confirmation of the conjecture has been achieved in dimensions four (Hitchin) and eight (Poon–Salamon, LeBrun–Salamon).

Recently this field of study has received a lot of attention with several contributions via completely different approaches and methods ranging from Ricci-flow to complex geometry. However, the LeBrun–Salamon conjecture still seems to be open. This article is devoted to an investigation of low-dimensional Positive Quaternion Kähler Manifolds—dimensions 16 to 24, with a clear emphasis on dimension 20—as well as high-dimensional ones with symmetries.

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In this article we provide several classification results under different assumptions coming from Index Theory, algebraic topology or the study of symmetries.

There used to be a classification of Positive Quaternion Kähler Manifolds in dimension 12 by Haydeé and Rafael Herrera (cf. [15]) confirming the main conjecture. In the second part of that article they showed that any 12-dimensional Positive Quaternion Kähler Manifold $M$ is symmetric if the $\hat{A}$-genus of $M$ vanishes. If $M$ is a spin manifold, this condition is always fulfilled by a classical result of Lichnerowicz, since a Positive Quaternion Kähler Manifold has positive scalar curvature. Positive Quaternion Kähler Manifolds $M^n \not\cong \mathbb{H}P^n$ are known to admit a spin structure if and only if $n$ is even (cf. proposition [22], 2.3, p. 148). One also knows that $\hat{A}(M)[M]$ vanishes on the symmetric examples with finite second homotopy group (cf. theorem [6], 23.3). Atiyah and Hirzebruch (cf. [3]) showed that the $\hat{A}$-genus vanishes on spin manifolds with smooth effective $S^1$-action.

In the first part of their article [15] Haydeé and Rafael Herrera claim a similar result for simply-connected manifolds with finite second homotopy group instead of a spin structure. Unfortunately, we found the proof of this assertion to be erroneous and—as we show together with Anand Dessai in [2]—it is not true in general that the $\hat{A}$-genus of such a simply-connected $\pi_2$-finite manifold with smooth effective $S^1$-action vanishes. Thus the classification in dimension twelve can no longer be sustained—see also [14]. However, it still remains an open question whether the $\hat{A}$-genus vanishes on $\pi_2$-finite Positive Quaternion Kähler Manifolds.

These observations actually where a byproduct of our work in dimension 20—with rather detrimental effects on our own arguments and theorems. This is the reason why the case of dimension 12 is neglected in this article and why the assumption $\hat{A}(M)[M] = 0$ appears in

**Theorem A.** A 20-dimensional Positive Quaternion Kähler Manifold $M$ with $\hat{A}(M)[M] = 0$ satisfying

$$\dim \text{Isom}(M) \notin \{15, 22, 29\}$$

(where possible groups $\text{Isom}_0(M)$ in these dimensions can be read off from table 8) is a Wolf space.

The proof of this theorem essentially splits into two parts: On the one hand we combine relations from Index Theory with further properties of Positive Quaternion Kähler Manifolds to restrict possible identity components of the isometry group to a small list of relatively large groups. In the second part we use Lie theoretic arguments to provide the classification result. Both approaches path the way towards further results:

In the vein of the index computations we obtain a classification result concerning the quaternionic projective spaces.

**Theorem B.** Suppose $2 \neq n \leq 6$. Let $M^{4n}$ be a Positive Quaternion Kähler Manifold with $b_4(M) = 1$. Then $M$ is homothetic to $\mathbb{H}P^n$.

This result has been proven in dimensions $\dim M \leq 16$ in [12]. The methods applied there permit a generalisation to dimensions 20 and 24.
Note that the exceptional Wolf space $F_4/Sp(3)Sp(1)$ has dimensions 28 (and $\dim G_2/SO(4) = 8$)—both of them satisfy $b_4 = 1$.

The arguments arising from the theory of transformation groups can be generalised to yield a recognition theorem for the real Grassmannian by means of the dimension of its isometry group. The complex Grassmannian and the quaternionic projective space are topologically well-identifiable by means of the second homotopy group, i.e. $\pi_2(M^{4n}) = 0$ implies $M^{4n} \cong \mathbb{HP}^n$ and $\pi_2(M) = \mathbb{Z}$ leads to $M^{4n} \cong \mathbb{Gr}_2(C^{n+2})$. These identities make it possible to recognize these two spaces by means of the rank of the isometry group (cf. [10]). No similar characterisations seem to be known for the real Grassmannian which makes our recognition theorem a first one of its kind.

**Theorem C.** Let $M^{4n}$ be a Positive Quaternion Kähler Manifold. Suppose that the dimension of the isometry group $\dim \text{Isom}(M^{4n})$ satisfies the respective condition depicted in Table 1. Then $M$ is symmetric and it holds:

$\begin{align*}
M &\cong \tilde{\text{Gr}}_4(\mathbb{R}^{n+1}) \iff \dim \text{Isom}(M) = \frac{n^2 + 7n + 12}{2} \\
M &\cong \text{Gr}_2(C^{n+2}) \iff \dim \text{Isom}(M) = n^2 + 4n + 3 \\
M &\cong \mathbb{HP}^n \iff \dim \text{Isom}(M) = 2n^2 + 5n + 3
\end{align*}$

In particular, if the isometry group satisfies that

$$\dim \text{Isom}(M) > \frac{n^2 + 5n + 12}{2}$$

for $n \geq 22$ and $n \notin \{27, 28\}$, then $M$ is symmetric and we recognise the real Grassmannian by the dimension of its isometry group.

The symbols $\mathbb{H}, \mathbb{C}, \mathbb{R}$ in the column “recognising” in Table 1 refer to whether we may identify the quaternionic projective space, the complex Grassmannian or the real Grassmannian in this dimension by the theorem.

We remark that in general the dimension of the isometry group $\dim \text{Isom}(M^{4n})$ takes values in

$$[0, \dim Sp(n + 1)] = [0, 2n^2 + 5n + 3]$$

(cf. 1.4). Hence—apart from recognising the real Grassmannian—this theorems rules out approximately three quarters of all possible values.

As the dimension of the isometry group of a Positive Quaternion Kähler Manifold may be interpreted as the index of a certain twisted Dirac operator (cf. theorem 1.3) we can make the following observation.

The question whether a Positive Quaternion Kähler Manifold $M^{4n}$ is symmetric or not can (almost always) be decided from the index

$$\text{ind}(D(S^{n+2}H)) = \langle \hat{A}(M) \cdot \text{ch}(S^{n+2}H), [M] \rangle$$

(For the bundle $H$ we refer to the next section.)

**Structure of the article.** In section 1 we shall give a very brief introduction to Positive Quaternion Kähler Geometry focussing on properties obtained via Index Theory or transformation groups. Section 2 is devoted to computations of several twisted $\hat{A}$-genera via characteristic classes. In section 3 we shall prove theorem B. Moreover, we shall identify further
Table 1. A recognition theorem

| n = | dim Isom($M^{2n}$) | recognising |
|-----|-------------------|-------------|
| 3   | 28                | H           |
| 4   | 52                | H           |
| 5   | 55                | H           |
| 6   | 55                | H, C        |
| 7   | 78                | H, C        |
| 8   | 78                | H, C        |
| 9   | 133               | H           |
| 10  | 133               | H, C        |
| 11  | 248               | H           |
| 12  | 248               | H           |
| 13  | 251               | H           |
| 14  | 251               | H, C        |
| 15  | 262               | H, C        |
| 16  | 262               | H, C        |
| 17  | 269               | H, C        |
| 18  | 269               | H, C        |
| 19  | 300               | H, C        |
| 20  | 300               | H, C        |
| 21  | 303               | H, C        |
| 22  | 303               | H, C, R     |
| 23  | 328               | H, C, R     |
| 24  | 354               | H, C, R     |
| 25  | 381               | H, C, R     |
| 26  | 409               | H, C, R     |
| 27  | 496               | H, C        |
| 28  | 496               | H, C        |

\[ n \geq 29, \quad \frac{n^2 + 3n + 12}{2} \]

properties related to the methods of the proof. Section 4 will be used to present further results that come out of our index computations. On the one hand we present theorems (e.g. on the existence of isometric $S^1$-actions) that are of interest for their own sake. On the other hand this section will establish the first part of the proof of theorem A; namely it will yield the existence of large isometry groups under mild assumptions. Section 5 is devoted to classification of 20-dimensional Positive Quaternion Kähler Manifolds with large isometry groups. The results here combine with the ones from chapter 4 to complete the proof of theorem A. The proof of theorem C will be given in 6.

We remark that a more elaborate introduction to the subject as well as detailed proofs can be found in [1].

Several arguments involve heavy computations. All of these were done with the help of Mathematica 6.01 or Maple 9 or later programme versions respectively.
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1. Positive Quaternion Kähler Manifolds

Due to Berger’s celebrated theorem the holonomy group $\text{Hol}(M,g)$ of a simply-connected, irreducible and non-symmetric Riemannian manifold $(M,g)$ is one of $\text{SO}(n)$, $\text{U}(n)$, $\text{SU}(n)$, $\text{Sp}(n)$, $\text{Sp}(n)\text{Sp}(1)$, $G_2$ and $\text{Spin}(7)$. A connected oriented Riemannian manifold $(M, g)$ is called a Quaternion Kähler Manifold if

$$\text{Hol}(M,g) \subseteq \text{Sp}(n)\text{Sp}(1) = \text{Sp}(n) \times \text{Sp}(1)/\langle -\text{id}, -1 \rangle$$

(In the case $n = 1$ one additionally requires $M$ to be Einstein and self-dual.) Quaternion Kähler Manifolds are Einstein (cf. [4].14.39, p. 403). In particular, their scalar curvature is constant.

Definition 1.1. A Positive Quaternion Kähler Manifold is a Quaternion Kähler Manifold with complete metric and with positive scalar curvature.

For an elaborate depiction of the subject we recommend the survey articles [22] and [24]. We shall content ourselves with mentioning a few properties that will be of importance throughout the article:

Foremost, we note that Positive Quaternion Kähler Manifolds $M$ clearly are not necessarily Kählerian, as the name might suggest. Moreover, the manifold $M$ is compact and simply-connected (cf. [22], p. 158 and [22].6.6, p. 163).

Locally the structure bundle with fibre $\text{Sp}(n)\text{Sp}(1)$ may be lifted to its double covering with fibre $\text{Sp}(n) \times \text{Sp}(1)$. The bundles associated to the standard complex representations of $\text{Sp}(n)$ on $\mathbb{C}^{2n}$ and of $\text{Sp}(1)$ on $\mathbb{C}^2$ will be called $E$ respectively $H$. Recall that $\mathbb{P}_\mathbb{C}(H)$ is called the twistor space of $M$. This space is a Fano contact Kähler Einstein manifold (cf. theorem [20].1.2, p. 113).

We obtain the following formula for the complexified tangent bundle $T_C M$ of the Positive Quaternion Kähler Manifold $M$ (cf. [24], p. 93):

$$T_C M = E \otimes H$$

The bundles $E$ and $H$ arise from self-dual representations and so their odd-degree Chern classes vanish. The Chern classes of $E$ will be denoted by

$$c_{2i} := c_{2i}(E) \in H^{2i}(M)$$

and

$$u := -c_2(H) \in H^4(M)$$

The quaternionic volume

$$v = (4u)^n \in H^{4n}(M^{4n})$$

is integral and satisfies $1 \leq v \leq 4^n$—cf. [24], p. 114 and corollary [25].3.5, p. 7.

Recall the definiteness of the intersection form on Positive Quaternion Kähler Manifolds (cf. [11]), [21]) (which is a consequence of the Hodge–Riemann bilinear relations on the twistor space). The quoted articles differ
in the formulation of positive/negative definiteness. We use the form \( u = -c_2(H) \) to state the theorem. The orientation on \( M \) is naturally given by \( u^n \).

**Theorem 1.2.** The generalised intersection form

\[
Q(x, y) = (-1)^{r/2} \int_M x \wedge y \wedge u^{n-r/2}
\]

for \([x], [y] \in H^r(M^{4n}, \mathbb{R})\) with even \( r \geq 0 \) is positive definite. In particular, the signature of the manifold satisfies

\[
\text{sign}(M) = (-1)^n b_{2n}(M)
\]

\[\square\]

An oriented compact manifold \( M^{4n} \) is called *spin* if its \( \text{SO}(4n) \)-structure bundle lifts to a \( \text{Spin}(4n) \)-bundle, or equivalently, its second Stiefel-Whitney class \( w_2(M) = 0 \) vanishes. A vector bundle \( E \to M \) is called *spin* if \( w_1(E) = w_2(E) = 0 \). Positive Quaternion Kähler Manifolds \( M^{4n} \neq \mathbb{H}P^n \) are spin if and only if \( n \) is even (cf. proposition [22], p. 148).

On a Positive Quaternion Kähler Manifold \( M^{4n} \) we have the locally associated bundles \( E \) and \( H \) from above. Now form the (virtual) bundles

\[
\bigwedge^k_0 E := \bigwedge^k E - \bigwedge^{k-2} E
\]

of exterior powers and the bundles

\[
S^l H := \text{Sym}^l H
\]

of symmetric powers. In general, the bundles \( \bigwedge^k_0 E \otimes S^l H \) exist globally if and only if \( n + k + l \) is even. In this case, using the index theorem one obtains the following relations (cf. [24], p. 117) where \( i^{k,l} := \text{ind}(\bigwedge^k_0 E \otimes S^l H) \) is the index of a twisted Dirac operator.

**Theorem 1.3.** It holds:

\[
i^{k,l} = \begin{cases} 0 & \text{if } k + l < n \\ (-1)^k(b_{2k}(M) + b_{2k-2}(M)) & \text{if } k + l = n \\ d & \text{if } k = 0, \ l = n + 2 \end{cases}
\]

where \( d = \dim \text{Isom}(M) \) is the dimension of the isometry group of \( M \) and the \( b_i(M) \) are the Betti numbers of \( M \) as usual.

\[\square\]

As a consequence of the Atiyah–Singer Index Theorem we then may express these indices topologically via genera:

\[
(1) \quad \text{ind}(\bigwedge^k_0 E \otimes S^l H) = \langle \hat{A}(M) \cdot \text{ch} \left( \bigwedge^k_0 E \right) \cdot \text{ch} \left( S^l H \right) \rangle, [M]
\]

Let us now collect some information on isometry groups:

**Theorem 1.4.** Let \( M^{4n} \) be a Positive Quaternion Kähler Manifold with isometry group \( \text{Isom}(M) \). We obtain:
The rank \( \text{rk} \text{Isom}(M) \) may not exceed \( n + 1 \). If \( \text{rk} \text{Isom}(M) = n + 1 \), then \( M \in \{ \mathbb{H} \mathbb{P}^n, \text{Gr}_2(\mathbb{C}^{n+2}) \} \).

If \( \text{rk} \text{Isom}(M) \geq \frac{n}{2} + 3 \), then \( M \) is isometric to \( \mathbb{H} \mathbb{P}^n \) or to \( \text{Gr}_2(\mathbb{C}^{n+2}) \).

It holds that \( \dim \text{Isom}(M^{4n}) \leq \dim \text{Sp}(n+1) = (n+1)(2n+3) \). Equality holds if and only if \( M \cong \mathbb{H} \mathbb{P}^n \).

If \( n = 3 \), then \( \dim \text{Isom}(M) \geq 5 \); if \( n = 4 \), then \( \dim \text{Isom}(M) \geq 8 \).

**Proof.** The first assertion is due to theorem [24],2.1, p. 89. The second item is theorem [10],1.1, p. 642. The inequality in the third assertion follows from corollary [25],3.3, p. 6. In case \( \dim \text{Isom}(\mathbb{R}) = (n+1)(2n+3) \) it was already observed on [22], p. 161 that \( M \) is homothetic to the quaternionic projective space. The fourth point is due to theorem [22],7.5, p. 169. \( \square \)

The isometry group \( \text{Isom}(M) \) of \( M \) is a compact Lie group. Due to theorems [7],V.8.1, p. 233, and [7],V.7.13, p. 229, we may assume up to finite coverings that \( \text{Isom}_0(M) \) — the component of the identity — is the product of a simply-connected semi-simple Lie group and a torus.

For the convenience of the reader we give a table of simple Lie groups by table 2, which will support our future arguments involving dimensions and ranks of Lie groups. From theorem [18],2.2, p. 13, we cite tables 3,

| type | corresponding Lie group | dimension |
|------|-------------------------|-----------|
| \( \mathbb{A}_n, n = 1, 2, \ldots \) | \( \text{SU}(n+1) \) | \( n(n+2) \) |
| \( \mathbb{B}_n, n = 1, 2, \ldots \) | \( \text{SO}(2n+1) \) | \( n(2n+1) \) |
| \( \mathbb{C}_n, n = 1, 2, \ldots \) | \( \text{Sp}(n) \) | \( n(2n+1) \) |
| \( \mathbb{D}_n, n = 3, 4, \ldots \) | \( \text{SO}(2n) \) | \( n(2n-1) \) |
| \( \mathbb{G}_2 \) | \( \text{Aut}(\mathbb{O}) \) | 14 |
| \( \mathbb{F}_4 \) | \( \text{Isom}(\mathbb{O} \mathbb{P}^2) \) | 52 |
| \( \mathbb{E}_6 \) | \( \text{Isom}((\mathbb{C} \otimes \mathbb{O}) \mathbb{P}^2) \) | 78 |
| \( \mathbb{E}_7 \) | \( \text{Isom}((\mathbb{H} \otimes \mathbb{O}) \mathbb{P}^2) \) | 133 |
| \( \mathbb{E}_8 \) | \( \text{Isom}((\mathbb{O} \otimes \mathbb{O}) \mathbb{P}^2) \) | 248 |

The index \( n \) denotes the rank

4, 5 of maximal connected subgroups (up to conjugation) of the classical Lie groups. (By \( \text{Irr}_\mathbb{R}, \text{Irr}_\mathbb{C}, \text{Irr}_\mathbb{H} \) real, complex and quaternionic irreducible representations are denoted. The tensor product “\( \otimes \)” of matrix Lie groups is induced by the Kronecker product of matrices.)

| subgroup | for |
|----------|-----|
| \( \text{SO}(k) \times \text{SO}(n-k) \) | \( 1 \leq k \leq n-1 \) |
| \( \text{SO}(p) \otimes \text{SO}(q) \) | \( pq = n, 3 \leq p \leq q \) |
| \( \mathbb{U}(k) \) | \( 2k = n \) |
| \( \text{Sp}(p) \otimes \text{Sp}(q) \) | \( 4pq = n \) |
| \( \varrho(H) \) | \( H \) simple, \( \varrho \in \text{Irr}_\mathbb{R}(H) \), \( \deg \varrho = n \) |
Table 4. Maximal connected subgroups of $\text{SU}(n)$

| subgroup          | for |
|-------------------|-----|
| $\text{SO}(n)$   |     |
| $\text{Sp}(m)$   | $2m = n$ |
| $\text{SU}(k) \times \text{U}(n-k)$ | $1 \leq k \leq n - 1$ |
| $\text{SU}(p) \otimes \text{SU}(q)$ | $pq = n, p \geq 3, q \geq 2$ |
| $\varrho(H)$     | $H$ simple, $\varrho \in \text{Irr}(H)$, $\deg \varrho = n$ |

Table 5. Maximal connected subgroups of $\text{Sp}(n)$

| subgroup          | for |
|-------------------|-----|
| $\text{Sp}(k) \times \text{Sp}(n-k)$ | $1 \leq k \leq n - 1$ |
| $\text{SO}(p) \otimes \text{Sp}(q)$ | $pq = n, p \geq 3, q \geq 1$ |
| $\text{U}(n)$    | $H$ simple, $\varrho \in \text{Irr}(H)$, $\deg \varrho = 2n$ |

From [5], p. 219, we cite table 6 of maximal rank maximal connected subgroups. From table [18].2.1 we cite subgroups of maximal dimension in

Table 6. Maximal rank maximal connected subgroups

| ambient group | subgroup |
|---------------|----------|
| $\text{SU}(n)$ | $\text{SU}(i) \times \text{U}(n-i-1)$ for $i \geq 1$ |
| $\text{SO}(2n+1)$ | $\text{SO}(2n), \text{SO}(2(n+1)) \times \text{SO}(2(n-i))$ for $1 \leq i \leq n-1$ |
| $\text{Sp}(n)$ | $\text{Sp}(i) \times \text{Sp}(n-i)$ for $i \geq 1$, $\text{U}(n)$ |
| $\text{SO}(2n)$ | $\text{SO}(2(n-i))$ for $i \geq 1$, $\text{U}(n)$ |
| $G_2$          | $\text{SO}(4), \text{SU}(3)$ |

table 7.

Table 7. Subgroups of maximal dimension

| ambient group | subgroup |
|---------------|----------|
| $\text{SU}(n), n \neq 4$ | $\text{SU}(1) \times \text{U}(n+1)$ |
| $\text{SU}(4)$ | $\text{Sp}(2)$ |
| $\text{SO}(n)$ | $\text{SO}(n-1)$ |
| $\text{Sp}(n), n \geq 2$ | $\text{Sp}(n-1) \times \text{Sp}(1)$ |
| $G_2$          | $\text{SU}(3)$ |

Let $H \subseteq \text{Isom}(M)$ be either $\mathbb{S}^1$ or $\mathbb{Z}_2$. Consider the isotropy representation at an $H$-fixed-point $x \in M$ composed with the canonical projection $\text{Sp}(1) \to \text{SO}(3)$:

$$\varphi : H \hookrightarrow \text{Sp}(n) \text{Sp}(1) \to \text{SO}(3)$$

Theorems [8].4.4, p. 602, and [8].5.1, p. 606, (together with [8], p. 600) show that the type of the fixed-point component $F$ of $H$ around $x$ depends on the image of $\varphi$: If $\varphi(H) = 1$, the component $F$ is quaternionic for
If $\varphi(H) \neq 1$, the component $F$ is locally Kählerian for $H = \mathbb{Z}_2$ and Kählerian for $H = S^1$. A result by Gray (cf. [13]) shows that a quaternionic submanifold is totally geodesic. Formula [4].14.42b, p. 406, then yields that the quaternionic components are again Positive Quaternion Kähler Manifolds.

It is easy to see that the dimension of $F$ is exactly $2n$ if $H = \mathbb{Z}_2$ and given that $F$ is locally Kählerian. The dimension of $F$ is smaller than or equal to $2n$ if $H = S^1$ and provided that $F$ is Kählerian.

Let us finally state some cohomological properties of Positive Quaternion Kähler Manifolds.

**Theorem 1.5 (Cohomological properties).** A Positive Quaternion Kähler Manifold $M$ satisfies:

- Odd-degree Betti numbers vanish, i.e. $b_{2i+1} = 0$ for $i \geq 0$.
- The identity
  \[
  \sum_{p=0}^{n-1} (6p(n - 1 - p) - (n - 1)(n - 3))b_{2p} = \frac{1}{2}n(n - 1)b_{2n}
  \]
  holds and specialises to
  \[
  (2) \quad -1 + 3b_2 + 3b_4 - b_6 = 2b_8
  \]
  \[
  (3) \quad -4 + 5b_2 + 8b_4 + 5b_6 - 4b_8 = 5b_{10}
  \]
in dimensions $16$ and $20$ respectively.
- A Positive Quaternion Kähler Manifold $M^{4n} \not\sim{\text{Gr}}_2(\mathbb{C}^{n+2})$ is rationally $3$-connected.
- The real cohomology algebra possesses an analogue of the Hard-Lefschetz property, i.e. with the four-form $u \in H^4(M, \mathbb{R})$ from above the morphism
  \[
  L^k : H^{n-k}(M, \mathbb{R}) \to H^{n+k}(M, \mathbb{R}) \quad L^k(\alpha) = u^k \wedge \alpha
  \]
is an isomorphism. In particular, we obtain
  \[
  b_{i-4} \leq b_i
  \]
for (even) $i \leq 2n$. A generator in top cohomology $H^{4n}(M)$ is given by $u^n$. This defines a canonical orientation.

**Proof.** The first point is proven in theorem [22].6.6, p. 163, where it is shown that the Hodge decomposition of the twistor space is concentrated in terms $H^{p,p}(Z)$. The second item is due to [23].5.4, p. 403.

The next item basically follows from theorem [24].5.5, p. 103 where it is proven that $b_2 = 0$ for $M^{4n} \not\sim{\text{Gr}}_2(\mathbb{C}^{n+2})$.

The Hard-Lefschetz property of $M$ follows from the Hard-Lefschetz property of the twistor space. Indeed, a Positive Quaternion Kähler Manifold has this property with respect to $u$. \qed

So for a Positive Quaternion Kähler Manifold $M$ it is equivalent to demand that $M$ be rationally $3$-connected—i.e. to have that $\pi_1(M) \otimes \mathbb{Q} = \pi_2(M) \otimes \mathbb{Q} = \pi_3(M) \otimes \mathbb{Q} = 0$—and to require that $M$ be $\pi_2$-finite—i.e. to suppose that $\pi_2(M) < \infty$. 

2. Preparations

This section is devoted to a computation of several indices $i^{p,q}$ in terms of the characteristic numbers of the complexified tangent bundle $T_{\mathbb{C}}M$ for a Positive Quaternion Kähler Manifold $M$ of dimension 20. That is, we compute

$$i^{p,q} = \langle \hat{A}(M) \cdot \text{ch}(R^{p,q}), M \rangle$$

with $R^{p,q} = \bigwedge^p E \otimes S^qH$ as usual. The formulas relating these indices to other invariants are given in theorem 1.3. Combining these equations with our computations yields the fundamental system of equations we shall mainly be concerned with in the following. It is linear in the characteristic numbers of $M$.

We shall compute these indices in terms of characteristic classes $u = -c_2(H)$ respectively $c_2, c_4, \ldots, c_{10}$ of the bundles $H$ and $E$. Using the formula $\text{ch}(E) = \sum_{i=1}^{10} e^{x_i}$ (for the formal roots $x_i$), the analogue for the bundle $H$ and the fact that Chern classes may be described as the elementary symmetric polynomials in the formal roots one obtains easily:

$$\text{ch}(H) = 2 + u + \frac{u^2}{12} + \frac{u^3}{360} + \frac{u^4}{20160} + \frac{u^5}{1814400}$$

$$\text{ch}(E) = 10 - c_2 + \frac{1}{12} (c_2^2 - 2c_4) + \frac{1}{360} (-c_3^2 + 3c_2c_4 - 3c_6)$$

$$+ \frac{1}{20160} (c_4^2 - 4c_2^2c_4 + 2c_4^2 + 4c_2c_6 - 4c_8)$$

$$+ \frac{1}{1814400} (-5c_{10} - c_5^2 + 5c_3^2c_4 - 5c_2c_4^2 - 5c_2c_6 + 5c_4c_6 + 5c_2c_8)$$

Now use the formula $T_{\mathbb{C}}M = E \otimes H$ to successively compute the Chern classes of the complexified tangent bundle and the Pontryagin classes $p_i$ of $M$. Filling in these Pontryagin classes into the characteristic series of the $\hat{A}$-genus yields

$$\hat{A}(M) = 1 + \frac{1}{12} (c_2 - 5u) + \frac{1}{720} (3c_2^2 - c_4 - 28c_2u + 65u^2)$$

$$+ \frac{1}{60480} (10c_2^3 - 9c_2c_4 + 2c_6 - 136c_2^2u + 55c_4u + 570c_2u^2 - 820u^3)$$

$$+ \frac{1}{3628800} (21c_4^2 - 34c_2c_4 + 5c_4^2 + 13c_2c_6 - 3c_8 - 384c_3u)$$

$$+ 409c_2c_4u - 113c_6u + 2274c_2^2u^2 - 1060c_4u^2 - 5736c_2u^3 + 5760u^4)$$

$$+ \frac{1}{479001600} (90c_5^2 - 219c_2^3c_4 + 87c_2c_4^2 + 109c_2c_6 - 32c_4c_6 - 43c_2c_8$$

$$+ 10c_{10} - 2136c_2^4u + 3990c_2^2c_4u - 675c_2^3u - 1834c_2c_6u + 525c_8u$$

$$+ 16524c_2^2u^2 - 19740c_2c_4u^2 + 6155c_6u^2 - 57576c_2u^3 + 29935c_4u^3$$

$$+ 98815c_2u^4 - 73985u^5)$$

We now compute the Chern characters of the exterior powers of $E$. For this we use that the roots of $\bigwedge^k E$ are given by $y_{i_1, \ldots, i_k} = x_{i_1} + \cdots + x_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq 10$. It remains to compute the Chern characters of the
symmetric bundles, which can be done in a similar fashion e.g. using the
roots given by the $y_{i_1,\ldots,i_k} = x_{i_1} + \cdots + x_{i_k}$ for $1 \leq i_1 \leq \cdots \leq i_k \leq 10$.
This enables us to compute the following indices in terms of characteristic
numbers via (1). (The “indices” $i^{p,q}$ with $p + q + 5$ odd are to be regarded as
formal expressions, as they do not necessarily correspond to twisted Dirac
operators.)

\[
\begin{align*}
i^{0,0} &= \frac{1}{479001600} (10c_{10} + 90c_5^2 - 32c_4c_6 - 2136c_2^4u - 675c_2^2u + 525c_8u \\
&\quad + 6155c_6u^2 + 29935c_4u^3 - 73985u^4 - 3c_2(73c_4 - 5508u^2) + c_2^2(109c_6 \\
&\quad + 3990u^2 - 5775u^3) + c_2(87c_4^2 - 43c_8 - 1834c_6u - 19740c_4u^2 \\
&\quad + 98815u^4) \}
\end{align*}
\]

\[
\begin{align*}
i^{0,1} &= \frac{1}{239500800} (10c_{10} + 90c_5^2 - 32c_4c_6 - 750c_2^4u - 345c_2^2u + 327c_8u \\
&\quad - 643c_6u^2 - 22799c_4u^3 + 90817u^5 - 3c_2(73c_4 + 1840u^2) + c_2^2(109c_6 \\
&\quad + 1746c_4u + 50400u^3) + c_2(87c_4^2 - 43c_8 - 976c_6u + 4284c_4u^2 \\
&\quad - 116543u^4) \}
\end{align*}
\]

\[
\begin{align*}
i^{0,2} &= \frac{1}{119750400} (10c_{10} + 90c_5^2 - 32c_4c_6 + 4794c_2^4u + 975c_2^2u - 465c_8u \\
&\quad - 4075c_6u^2 + 87025c_4u^3 + 310465u^5 - 3c_2(73c_4 - 8368u^2) \\
&\quad + c_2^2(109c_6 - 7230c_4u - 135456u^3) \\
&\quad + c_2(87c_4^2 - 43c_8 + 2456c_6u - 6540c_4u^2 - 277055u^4) \}
\end{align*}
\]

\[
\begin{align*}
i^{0,3} &= \frac{1}{79833600} (10c_{10} + 90c_5^2 - 32c_4c_6 + 14034c_2^4u + 3175c_2^2u - 1785c_8u \\
&\quad + 74685c_6u^2 - 154625c_4u^3 + 44944065u^5 + c_2^2(-219c_4 + 498544u^2) \\
&\quad + c_2^2(109c_6 - 22190c_4u + 6228704u^3) \\
&\quad + c_2(87c_4^2 - 43c_8 + 8176c_6u - 404740c_4u^2 + 30037185u^4) \}
\end{align*}
\]

\[
\begin{align*}
i^{0,4} &= \frac{1}{59875200} (10c_{10} + 90c_5^2 - 32c_4c_6 + 26970c_2^4u + 6255c_2^2u - 3633c_8u \\
&\quad + 362357c_6u^2 - 18291599c_4u^3 + 3830160577u^5 - 3c_2^2(73c_4 \\
&\quad - 682800u^2) + c_2^2(109c_6 - 43134c_4u + 61087200u^3) \\
&\quad + c_2(87c_4^2 - 43c_8 + 16184c_6u - 1760556c_4u^2 + 796656577u^4) \}
\end{align*}
\]

\[
\begin{align*}
i^{0,5} &= \frac{1}{239500800} (-610c_{10} + 450c_2^2 + 1952c_4c_6 - 29294c_2^4u - 94575c_2^2u \\
&\quad + 22425c_8u - 149405c_6u^2 + 690875c_4u^3 - 369925u^5 + c_2^2(-2481c_4 \\
&\quad + 60576u^2) + c_2^2(-4669c_6 + 43050c_4u - 179904u^3) \\
&\quad + c_2(4593c_4^2 - 3317c_8 + 56104c_6u - 224760c_4u^2 + 113915u^4) \}
\end{align*}
\]
\[ \begin{align*}
\dot{i}^{1,2} &= \frac{1}{79833600} \left( -610c_{10} + 450c_5^2 + 1952c_4c_6 + 9186c_4^2u + 60425c_3^2u \\
&\quad - 43575c_8u + 8115c_6u^2 + 750715c_4u^3 - 693765u^5 - c_2(2481c_4 \\
&\quad + 48544u^2) - c_2^2(4669c_6 + 39670c_4u + 144704u^3) \\
&\quad + c_2(4593c_4^2 - 3317c_8 - 101416c_6u + 203800c_4u^2 + 43515u^4) \right) \\
\dot{i}^{1,4} &= \frac{1}{47900160} \left( -610c_{10} + 450c_5^2 + 1952c_4c_6 + 46146c_4^2u + 280425c_3^2u \\
&\quad - 175575c_8u - 5810093c_6u^2 - 31189765c_4u^3 - 6917125u^5 \\
&\quad - 3c_2^2(827c_4 - 333472u^2) \\
&\quad - c_2(-4593c_4^2 + 3317c_8 + 416456c_6u + 3272904c_4u^2 + 8410117u^4) \\
&\quad - c_2^2(4669c_6 + 4770(43c_4u - 1504u^3)) \right) \\
\dot{i}^{2,1} &= \frac{1}{5443200} \left( 15070c_{10} + 90c_5^2 - 2864c_4c_6 - 246c_4^2u + 855c_2^2u + 21567c_8u \\
&\quad - 153103c_6u^2 - 79439c_4u^3 + 90817u^5 - 3c_2^2(241c_4 + 2112u^2) \\
&\quad + c_2^2(13c_6 + 1146c_4u + 2198u^3) \\
&\quad + c_2(1599c_4^2 + 8369c_8 - 7840c_6u + 32784c_4u^2 + 56257u^4) \right) \\
\dot{i}^{2,3} &= \frac{1}{2721600} \left( 15070c_{10} + 90c_5^2 - 2864c_4c_6 + 5298c_4^2u + 59775c_2^2u \\
&\quad + 530535c_8u + 1506665c_6u^2 + 60625c_4u^3 + 310465u^5 \\
&\quad + c_2^2(-723c_4 + 53088u^2) \\
&\quad + c_2^2(13c_6 - 36630c_4u + 109728u^3) \\
&\quad + c_2(1599c_4^2 + 8369c_8 - 5848c_6u - 307800c_4u^2 + 414145u^4) \right) \\
\dot{i}^{3,0} &= \frac{1}{21772800} \left( -876370c_{10} + 450c_5^2 - 38176c_4c_6 - 7278c_4^2u - 73575c_3^2u \\
&\quad + 1714425c_8u + 571315c_6u^2 - 293125c_4u^3 - 369925u^5 \\
&\quad + c_2^2(-4497c_4 + 28512u^2) \\
&\quad + c_2^2(9347c_6 + 55050c_4u - 22848u^3) \\
&\quad + c_2(10641c_4^2 + 15931c_8 - 121112c_6u - 159720c_4u^2 - 439045u^4) \right) \\
\dot{i}^{3,2} &= \frac{1}{7257600} \left( -876370c_{10} + 450c_5^2 - 38176c_4c_6 + 1120c_3^2u + 190025c_2^2u \\
&\quad - 3765975c_8u - 158205c_6u^2 - 636485c_4u^3 - 693765u^5 \\
&\quad - c_2^2(4497c_4 + 3808u^2) \\
&\quad + c_2^2(9347c_6 - 104470c_4u - 225728u^3) \\
&\quad + c_2(10641c_4^2 + 15931c_8 + 201368c_6u - 126680c_4u^2 - 1062405u^4) \right) 
\end{align*}\]
Using this information one may form the described linear system of equations.

Now compute the Hilbert Polynomial $f$ of $M$ in the parameters $d$, $v$ and $i_{0,0} \in \mathbb{Q}$, i.e. in the dimension of the isometry group, the quaternionic volume and the $\hat{A}$-genus. The Hilbert Polynomial $f$ on $M$ is given by

$$f(q) = \text{ind} D(S^q H) = \langle \hat{A} \cdot \text{ch}(S^q H), [M] \rangle = i_{0,q}$$

and has degree 11. We use the formula

$$(H - 2)^\otimes m = \sum_{j=0}^{m} (-1)^j \left( \binom{2m}{j} - \binom{2m}{j-2} \right) S^{m-j} H$$

resulting from the Glebsch-Gordan formula. The leading term of the power series of the $\hat{A}$-genus is 1 and the first non-zero coefficient of the power series $\text{ch}(H - 2)^m$ lies in degree $m$. Thus we obtain that $\langle \hat{A}(M) \cdot \text{ch}(H - 2)^{\otimes 5}, [M] \rangle = u^5$ and all the higher terms vanish. Combining this with theorem 1.3, i.e. with $f(0) = i_{0,0}^0$, $f(1) = f(3) = 0$, $f(5) = 1$ and $f(7) = d$ permits
us to compute the following identities.

\[
\begin{align*}
f(0) &= i^{0,0} \\
f(1) &= 0 \\
f(2) &= \frac{-2816 + 128d - 360448i^{0,0} - 7v}{229376} \\
f(3) &= 0 \\
f(4) &= \frac{269568 - 7040d + 4685824i^{0,0} + 273v}{1146880} \\
f(5) &= 1 \\
f(6) &= \frac{228096 + 18304d - 2342912i^{0,0} - 273v}{114688} \\
f(7) &= d \\
f(8) &= \frac{13(-143616 + 35200d + 3063808i^{0,0} + 595v)}{114688} \\
f(9) &= \frac{1}{140}(-10692 + 1760d + 262144i^{0,0} + 63v) \\
f(10) &= \frac{13(-4333824 + 598400d + 116424704i^{0,0} + 33915v)}{229376} \\
f(11) &= \frac{1}{14}(-9152 + 1144d + 262144i^{0,0} + 91v)
\end{align*}
\]

The Hilbert polynomial has degree 11 and thus can be computed from these values.

From theorem [25].1.1, p. 2, we are given the formula

\[
0 \leq f_M(5 + 2q) \leq f_{\mathbb{H}^5}(5 + 2q) = \binom{11 + 2q}{11}
\]

for \( q \in \mathbb{N}_0 \). So we may compute for each \( q \) a lower and an upper bound for \( i^{0,0} \)—depending on \( d \) and \( v \). Unfortunately, with \( q \) growing, these bounds seem to become worse so that we use low values of \( q \)—i.e. \( q = 3 \) respectively \( q = 2 \)—to obtain:

\[
\begin{align*}
(4) & \quad \frac{1}{140}(-9152 + 262144i^{0,0} + 1144d + 91v) \leq 12376 \\
(5) & \quad \frac{1}{140}(-10692 + 262144i^{0,0} + 1760d + 63v) \geq 0
\end{align*}
\]

Let us now compute further relations involving the \( \hat{A} \)-genus of a Positive Quaternion Kähler Manifold \( M \). We adapt lemma [22].7.6, p. 169, to dimension 20 and plug in the expression \(- (2c_2 - 10u)\) for the first Pontryagin class \( p_1 \):

\[
8u^5 - p_1u^4 \geq 0 \iff 8u^5 + (2c_2 - 10u)u^4 \geq 0 \iff c_2u^4 - u^5 \geq 0
\]

From the solution of the fundamental system of equations we cite

\[
c_2u^4 = \frac{81}{70} + \frac{3d}{28} + \frac{1536\hat{A}(M)[M]}{35} - \frac{31u^5}{5}
\]
Combining this with formula \((6)\) yields
\[
-\frac{81}{70} + \frac{3d}{28} - \frac{36u^5}{5} + \frac{1536\hat{A}(M)[M]}{35} \geq 0
\]
(7)

Recall that \(1 \leq v \leq 1024\). So for \(d = 0\) the equation becomes
\[
-\frac{81}{70} - \frac{36}{5 \cdot 1024} + \frac{1536\hat{A}(M)[M]}{35} \geq 0
\]
(8)

3. Special cases and the proof of theorem B

This section will combine further observations with the proof of theorem B. Indeed, we shall deal with each dimension—i.e. \(\dim M \in \{20, 24\}\)—in theorem B separately thereby proving slightly more general assertions. The theorem itself is then a combination of corollary 3.6 and theorem 3.7. As we already remarked, in dimension 28 there is an exceptional Wolf space, which makes further generalisation more difficult. Nonetheless, as we were told by Gregor Weingart, a similar recognition theorem—which also identifies the exceptional Wolf space \(F_4/Sp(3)Sp(1)\)—seems to be possible.

Foremost we recall

Theorem 3.1. Let \(M\) be a Positive Quaternion Kähler Manifold. If \(8 \neq \dim M \leq 16\) and \(b_4(M) = 1\), then \(M\) is homothetic to the quaternionic projective space.

Proof. See theorem [24].2.1.ii, p. 89.

Before generalising this theorem, we shall reconsider the problem in dimension 16 and we shall illustrate the used methods by pointing out certain additional results.

3.1. Dimension 16. In dimension 16 the relation on Betti numbers given in (2) of theorem 1.5 together with the Hard-Lefschetz property (cf. theorem 1.5) have the following consequence: If \(b_4 = 1\) and if we assume \(M\) to be rationally 3-connected (cf. 1.5), we obtain \(b_0 = b_4 = b_8 = b_{12} = b_{16} = 1\) with all the other Betti numbers vanishing. So necessarily every Pontryagin class is a multiple of the corresponding power of the form \(u\). This motivates the following slight improvement.

Proposition 3.2. If each of the Chern classes \(c_i\) of the bundle \(E\) over a 16-dimensional Positive Quaternion Kähler Manifold is a (scalar) multiple of the corresponding power of \(u\), then the manifold already is homothetic to \(\mathbb{H}P^4\).

Proof. We form a linear system of equations as we did in section 2. By assumption we may now replace every Chern class \(c_i\) by some \(x_i u^{n_i}\) for \(x_i \in \mathbb{R}\).

If one focuses on the case \(b_2 = 0\) (cf. 1.5), the system of equations can be solved and it yields \(d = 55\), \(b_4 = b_8 = 1\), \(b_6 = 0\), \(u^4 = 1\) (with all the factors \(x_i\) equal to one). We then observe that in dimension 55 only semi-simple Lie groups of rank at least 5 appear. Theorem 1.4 then yields the assertion; i.e. the isometry group becomes very large and permits to identify \(M\) as the quaternionic projective space.
If one does not assume $b_2 = 0$, the list of possible configurations for $(d, b_2, b_4, b_6, b_8, u^4)$ becomes a little larger. However, the configuration from above remains the only one with integral $d \in \mathbb{Z}$. □

Assume $b_2 = 0$. Then the same proof works if one only requires $c_2$ and $c_4$ to be scalar multiples of $u$ respectively $u^2$. In this case a numerical solving procedure leads to six different solutions of which the only one with an integral value for $d$ is the requested one—as in proposition 3.2.

Focussing on the case that only $c_2$ is a multiple $x \in \mathbb{R}$ of $u$ leads to the two equations

\begin{align}
(9) & \quad d = 7 + \frac{v}{6} + \frac{vx}{48} \\
(10) & \quad b_4 = \frac{783}{2} - \frac{7}{8}v - \frac{9}{16}vx - \frac{11}{128}vx^2 - \frac{1}{512}vx^3
\end{align}

where $v = (4u)^4$ is the quaternionic volume. The element $x$ is integral by the same reasoning as in the original proof, i.e. the proof of theorem [12], p. 62. In [19] it is proven that $i^{1,n+1} \leq 0$. In the survey article [24], p. 117, it is suggested that this index vanishes unless $M$ is the quaternionic projective space. In the following we assume the vanishing of $i^{1,5}$ in the case $M \neq H\mathbb{P}^4$, which produces

\begin{align}
(11) & \quad d = \frac{7(304 + 56x + 3x^2)}{16 + 20x + 3x^2} \\
(12) & \quad b_4 = -\frac{27(1280 - 304x - 40x^2 + 7x^3)}{8(16 + 20x + 3x^2)} \\
(13) & \quad b_6 = \frac{1}{36} \left(3289 - 294x + 63x^2 - 6(c_4u^2)x^2 - \frac{53200}{16 + 20x + 3x^2}
\right.
\left. - \frac{93544x}{16 + 20x + 3x^2}\right) \\
(14) & \quad b_8 = \frac{1}{144} \left(14410 - 1113x - (126x^2 - 12(c_4u^2))x^2 - \frac{1163680}{16 + 20x + 3x^2}
\right.
\left. + \frac{14728x}{16 + 20x + 3x^2}\right) \\
(15) & \quad c_4^2 = \frac{1}{16} \left(-3546 + 567x - 378x^2 + 44(c_4u^2)x^2 - \frac{821520}{16 + 20x + 3x^2}
\right.
\left. + \frac{445032x}{16 + 20x + 3x^2}\right)
\end{align}

The only integral solution for $8 \leq d < 55 = \dim \text{Sp}(5)$ (cf. theorem 1.4) is given by $x = 4$ and $d = 28$. Then we directly obtain $b_4 = 3$ by (12) and also $v = 84$ by (9). Indeed, by the relations on Betti numbers in 1.5 only two possibilities for $(b_4, b_6, b_8)$ remain, namely $(3, 0, 4)$ or $(3, 2, 3)$. Equations (13) and (14) yield $c_4u^2 = \frac{47}{32}$ in the first case. By theorem 1.2 we may use the positive definiteness of the generalised intersection form $Q$ to see $0 \leq Q(c_4, c_4) = c_4^2$. Yet, in the case $(b_4, b_6, b_8) = (3, 2, 3)$ we obtain the contradiction $c_4^2 = -\frac{75}{16}$ by (15). As a consequence, we have the following theorem:
Theorem 3.3. If $M$ is a rationally 3-connected 16-dimensional Positive Quaternion Kähler Manifold with $i^{1,5} = 0$ and if the class $c_2$ is a scalar multiple of $u$, then either $M \cong \mathbb{H}P^4$ or the datum $(d, v, b_4, b_6, b_8) = (28, 84, 3, 0, 4)$ is exactly the one of $\tilde{\text{Gr}}_4(\mathbb{R}^8)$.

We remark that the property that $c_2$ is a multiple of $u$ seems to be a special feature of $\tilde{\text{Gr}}_4(\mathbb{R}^{n+4})$ for $n = 4$ among the infinite series of Wolf spaces other than the quaternionic projective space.

3.2. Dimension 20. The following consequence is as simple as it is astonishing.

Lemma 3.4. Let $M$ be rationally 3-connected of dimension 20 with $b_4 \leq 5$. Then it holds:

\begin{equation}
(16) \quad b_6 = b_{10} \lor (b_4, b_8) \in \{(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)\}
\end{equation}

Proof. By assumption $b_2 = 0$. Equation (3) becomes

\[4(2b_4 - b_8 - 1) = 5(b_{10} - b_6)\]

where the right hand side is non-negative due to Hard-Lefschetz (cf. 1.5). Hence the term $2b_4 - b_8 - 1$ must either be a positive multiple of 5 or zero. Since also $b_8 \geq b_4$, the first case may not occur for $b_4 \leq 5$. Thus it holds that $b_6 = b_{10}$. The existence of the form $0 \neq u \in H^4(M)$ shows that $b_4 \geq 1$. □

Let us now prove the analogue of proposition 3.2 in dimension 20. Again, without restriction, we focus on rationally 3-connected Positive Quaternion Kähler Manifolds $M^{20}$.

Theorem 3.5. If each of the Chern classes $c_{2i}$ of $E$ over $M^{20}$ is a (scalar) multiple of the corresponding power of $u$, then $M^{20} \cong \mathbb{H}P^5$.

Proof. We proceed as before in dimension 16; i.e. we solve the system of equations (cf. section 2) setting all characteristic classes to rational multiples of a suitable power of the form $u$. As a result we obtain $b_4 = b_8 = 1$, $b_6 = b_{10} = 0$, $u^5 = 1$, all the scalars are 1 and $d = 78$. However, such a large isometry group can only occur for the quaternionic projective space—cf. theorem 1.4. □

We remark that for this proof to work we do not need that $c_6$ is a scalar multiple of $u^3$. We shall now use the observation we stated in lemma 3.4 to finish the reasoning.

Corollary 3.6. If $b_4(M^{20}) = 1$, then $M^{20} \cong \mathbb{H}P^5$.

Proof. Lemma 3.4 tells us that $b_4 = 1$ implies $b_8 = 1$. By Poincaré Duality we may conclude that all the $c_{2i} \in H^{4i}(M)$ satisfy the condition from theorem 3.5. □

Observe that clearly by this corollary we have ruled out an infinite number of possible configurations of Betti numbers, since it automatically follows that $b_4 = 1$ not only implies $b_8 = 1$ but also $b_6 = b_{10} = 0$.

Observe that one may prove this corollary in a slightly different way: Assume only that $c_2$ and $u$ are scalar multiples and do the same for monomials...
containing \(c_2\) and \(u\). Use the equations with the additional information \(b_4 = 1\) (and \(b_2 = 0\)) and the result follows directly.

3.3. Dimension 24. We apply similar techniques as before to prove

**Theorem 3.7.** A 24-dimensional Positive Quaternion Kähler Manifold \(M^{24}\) with \(b_4(M) = 1\) is homothetic to \(\mathbb{H}P^6\).

**Proof.** Again we replace \(c_2\) by a scalar multiple of \(u\) and do the same for all the monomials containing \(c_2\) as a factor. This and the additional information \(b_4 = 1\) (respectively \(b_2 = 0\)) simplifies the system of equations (cf. 1.3) that we build up as we did for dimensions 16 and 20. Solving it yields a list of possibilities, which we run through in order to isolate the one leading to \(\mathbb{H}P^6\):

We may rule out the first solution, since it yields \(d = \frac{244724}{2891} \notin \mathbb{Z}\) for the dimension of the isometry group. The second solution gives \(d = 105 = \dim \text{Sp}(7)\). Thus we directly know that \(M \cong \mathbb{H}P^6\) in this case—cf. theorem 1.4. So what is left is to rule out the following configuration of solutions, which is marked by

\[
\begin{align*}
    d &= \frac{7937019926774969}{402874803650560} x_1 - \frac{19592196959405797}{2417248821903360} x_2^2 + \frac{457452279096536909}{2417248821903360} x_4^4 \\
    &\quad - \frac{263256496233}{805749607301120} x_3^6 + \frac{402874803650560}{161149921460224} x_5^4 \\
    &\quad - \frac{604312205475840}{282904843313851} x_6^3 \
\end{align*}
\]

where each \(x_i\) is a root of

\[
\begin{align*}
    &29223x^7 - 358275x^6 - 6960405x^5 + 67759961x^4 + 579930789x^3 \\
    &\quad - 4142432537x^2 - 9711667063x + 33284884867
\end{align*}
\]

Numerically, the roots of this polynomial are given by

\[
\begin{align*}
    &2.156753156, 7.720829360, 11.12408307, 15.23992325, \\
    &- 10.15093795 + 2.570319306i, -3.679678028, \\
    &- 10.15093795 - 2.570319306i
\end{align*}
\]

A computer-based check on all the possible combinations now shows that there are no integral solutions for \(d\) in all these cases. So we are done. □

4. Properties of interest

In this section we shall show that under slight assumptions some surprising results on the degree of symmetry of 20-dimensional Positive Quaternion Kähler Manifolds \(M\) are obtained. This will lead us to the existence of large isometry groups under mild assumptions thereby making a first step towards the proof of theorem A.

**Proposition 4.1.** Unless \(M\) admits an isometric \(S^1\)-action, the \(\hat{A}\)-genus of \(M\) is restricted by

\[
0.0321350097 < \hat{A}(M)[M] < 0.6955146790
\]
Proof. The upper bound clearly results from equation (4) when substituting in the extremal value $v = 1$. For the lower bound we form the linear combination

$$\frac{1}{448}(-1053 + 136d + 32768r_0^0) \geq 0$$

out of equations (5) and (7). The result follows from setting $d = 0$. □

Let us now use the fact that the terms $f(5 + 2q) = r_0^{5+2q}$ (for $q \geq 0$) are indices of the twisted Dirac operator $D(S^{5+2q}H)$; i.e. in particular they are integral. This leads to congruence relations for the dimension of the isometry group and the quaternionic volume.

Theorem 4.2. A 20-dimensional rationally 3-connected Positive Quaternion Kähler Manifold with $\hat{A}(M)[M] = 0$ satisfies

$$d \equiv 1 \mod 7 \quad \text{and} \quad v \equiv 4 \mod 20$$

Proof. We use the Hilbert Polynomial $f$ of $M$. Thus, under the assumption that $\hat{A}(M)[M] = 0$ we obtain

$$Z \ni r^{0.9} = f(9) = \frac{1}{140}(-10692 + 1760d + 63v).$$

This implies that

$$-10692 + 1760d + 63v \equiv 0 \mod 140 \quad \iff \quad 88 + 80d + 63v \equiv 0 \mod 140 \quad \iff \quad (d \equiv 1 \mod 7) \lor (v \equiv 4 \mod 20)$$

Remark 4.3. • Any computation of further indices seems to result in the fact that only denominators appear that divide $2^2 \cdot 5 \cdot 7$. Since $v_{\mathbb{H}P^5} = 1024$ and since $v_{\mathbb{F}\mathbb{G}_2(R)} = 264$, we see that the relations found in the theorem are the only ones that may hold on the quaternionic volume when focussing on congruence modulo $m$ for $m|140$.

• The dimension $d_{\mathbb{H}P^n}$ of $\mathbb{H}P^n$ is given by $(n + 1)(2n + 3)$ with

$$d_{\mathbb{H}P^n} \equiv (n + 1)((n + 1) + (n + 2)) \equiv 1 \mod n + 2.$$

The dimension $d_{\mathbb{F}\mathbb{G}_2(R^{n+4})}$ of $\mathbb{F}\mathbb{G}_2(R^{n+4})$ is given by

$$d_{\mathbb{F}\mathbb{G}_2(R^{n+4})} = \begin{cases} \frac{1}{2}(n + 3)(n + 4) & \text{for } n \text{ odd} \\ \frac{1}{2}(n + 3)(n + 4) & \text{for } n \text{ even} \end{cases}$$

So in any case we have $d_{\mathbb{F}\mathbb{G}_2(R^{n+4})} \equiv 1 \mod n + 2$. Thus in dimensions without exceptional Wolf spaces—for these it is not true—by the main conjecture it should hold on a rationally 3-connected Positive Quaternion Kähler Manifold that the dimension of the isometry group is congruent 1 modulo $n + 2$.

Corollary 4.4. A 20-dimensional Positive Quaternion Kähler Manifold satisfying $\hat{A}(M)[M] = 0$ and possessing an isometry group of dimension greater than 36 is isometric to the complex Grassmannian or the quaternionic projective space.
PROOF. Again we assume the manifold to be rationally 3-connected. There are no compact Lie groups in dimensions 43, 50, 57, 64 and 71 with rank smaller than or equal to 5. Thus theorems 1.4 and 4.2 yield the result. □

We shall now prove the existence of $S^1$-actions on 20-dimensional Positive Quaternion Kähler Manifolds which satisfy some slight assumptions. For this we use the definiteness of the intersection form to establish

**Lemma 4.5.** We have:
\[
u^5 \geq 0 \quad c_2^3 u \geq 0 \quad c_2^2 u^3 \geq 0 \quad c_2^4 u \geq 0
\]
More generally, the same holds for
\[
(kc_2^2 + lc_2 u + mc_2^2 + nc_4)^2 u \geq 0
\]
with $k, l, m, n \in \mathbb{R}$.

**Proof.** Recall the generalised intersection form $Q$ from theorem 1.2. All the classes $y$ from the assertion may be written as $y = Q(x, x)$ for some $x \in H^r(M)$ with $r \in \{4, 8\}$. However, the intersection form $Q$ is positive definite in degrees divisible by 4 and it results that $Q(x, x) \geq 0$. □

Lemma 4.5 yields that $c_2^3 u^3 \geq 0$, which translates to
\[
495392 - 14240d - 35651584i_{0,0} - 1120i_{1,6} + 707v \geq 0
\]
after solving the linear system of equations on indices (cf. section 2) and substituting in the special solution for $i_{1,6}$. So we obtain:
\[
i_{1,6} \leq \frac{495392 - 14240d - 35651584i_{0,0} + 707v}{1120}
\]
which already shows that for extremal values of $v$ and $d$, i.e. $v = 1024$ and $d = 0$, the index $i_{1,6}$ becomes very small.

Now consider the term $(c_2 u + mu^2)^2 u$ together with the solution from the system of equations (cf. section 2) and the solution for $i_{1,6}$. By lemma 4.5 we have $(c_2 u + mu^2)^2 u \geq 0$. Suppose $d = 0$. This has the consequence that
\[
i_{1,6} \leq \frac{1}{1120} (495392 - 35651584i_{0,0} - 82944m + 3145728i_{0,0}m + 707v - 434mv + 35m^2v)
\]
for all $m \in \mathbb{R}$. The right hand side is a parabola in $m$. Determine the apex of this parabola as $m_0 = \frac{41472 - 1572864i_{0,0} + 217v}{39v}$, put it into the inequality and obtain:
\[
i_{1,6} \leq -\frac{1}{4900v} (214990848 + 309237645312(i_{0,0})^2 + 82516v + 2793v^2 + 131072v_{0,0}(-124416 + 539v))
\]
The right hand side is a function in $i_{0,0}$ and $v$ which has no critical point in the interior of the square $[0.0321350097, 0.695514790] \times [1, 1024]$—cf. proposition 4.1. Thus its maximum lies in the boundary of the square. A direct check reveals that on the border of the square the function is decreasing monotonously in $i_{0,0}$ for $v \in \{1, 1024\}$. Analogously, we see that for $i_{0,0} = 0.0321350097$ the function has the only maximum $-549.348$ for $v = 61$.
and for \( i^{0,0} = 0.695514790 \) it is increasing in \([1, 1024]\). So it takes its maximum \(-549.348\) on the square in \( v = 61 \) and \( i^{0,0} = 0.695514790 \). So, in particular, we obtain the following theorem:

**Theorem 4.6.** A 20-dimensional Positive Quaternion Kähler Manifold with

\[ i^{1,6} \geq -549 \]

admits an effective isometric \( S^1 \)-action.

\[ \square \]

As we remarked already for \( M \not\simeq \mathbb{H}P^n \), the index \( i^{1,6} \) is smaller or equal to zero and it is conjectured to equal zero. On \( \mathbb{H}P^n \) it equals \( i^{1,n+1} = n(2n+3) \).

Next we shall link the existence of an isometric \( S^1 \)-action to the Euler characteristic.

**Theorem 4.7.** A 20-dimensional Positive Quaternion Kähler Manifold \( M \) with Euler characteristic restricted by

\[ \chi(M) < 16236 \]

admits an effective isometric \( S^1 \)-action. The same holds if the Betti numbers of \( M \) satisfy either

\[ b_4 - \frac{b_6}{4} < 842.5 \]

or

\[ \frac{59b_4}{3} - \frac{25b_6}{4} < 3027.93 \]

or—as a combination of both inequalities—if

\[ b_4 \leq 3381 \]

**Proof.** The theorem is trivial for \( \text{Gr}_2(\mathbb{C}^7) \). Thus we assume \( M \) to be rationally 3-connected. We give a proof by contradiction and assume \( d = \dim \text{Isom}(M) = 0 \). We shall choose special values for \( k, l, m, n \) from lemma 4.5. These coefficients determine an element \( y \). We shall obtain the contradiction \( Q(y, y) < 0 \) under the assumptions from the assertion.

Use the linear combination in corollary 4.5 with coefficients \( n = -0.168, m = 4.99, k = -n, l = -2\sqrt{-mn} - 18n^2 \) under the assumption of \( d = 0, b_2 = 0 \) together with our solution to the system of indices (cf. section 2) and the relation on Betti numbers (3). This results in the formula

\[ 19.9668 + 0.254016b_4 - 0.063504b_6 - 9835.62e^{0,0} + 0.0801763v \geq 0 \]

(where coefficients are rounded off.) Substituting in the lower bound \( e^{0,0} \geq 0.0321350097 \) from proposition 4.1 and the upper bound \( v = 1024 \) yields \( b_4 - \frac{b_6}{4} \geq 842.468 \). This contradicts our assumption \( b_4 - \frac{b_6}{4} < 842.5 \). Thus we obtain \( d \neq 0 \).
The second formula involving Betti numbers results from similar arguments with coefficients \( l = 22k \), \( n = -\frac{39n}{22} \), \( m = -\frac{239n}{3} \), \( n = 1 \). This yields
\[
-467.202 + \frac{59b_4}{3} - \frac{25b_6}{4} - 104.025i^{0,0} + 4.72222i^{1,6} - 0.439931v \geq 0
\]
Once more we assume there is no \( S^1 \)-action on \( M \). Thus theorem 4.6 gives us \( i^{1,6} \leq -551 \). So in this case we additionally substitute the other known bounds \( i^{0,0} \geq 0.0321350097 \) and \( v \geq 1 \). This eventually yields that \( \frac{59b_4}{3} - \frac{25b_6}{4} \geq 3027.93 \) contradicting our assumption.

Assume \( d = 0 \). Thus from the previous two relations on Betti numbers we compute
\[
25 \cdot b_4 - \frac{59b_4}{3} \geq 25 \cdot 842.5 - 3027.93 \Leftrightarrow b_4 \geq 3381.48
\]
Hence, whenever \( b_4 \leq 3381 \), we obtain a contradiction and \( d \neq 0 \).

The result on the Euler characteristic results from the formula \( b_4 \leq 3381 \) and the Hard-Lefschetz property by a computer-based check on all possible configurations of \( (b_4, b_6, b_8, b_{10}) \)—in a suitable range—that satisfy relation (3). That is, we start with \( b_4 = 3382 \) and figure out the configuration of \( (b_4, b_6, b_8, b_{10}) \)—satisfying all the properties from theorem 1.5—with smallest Euler characteristic. This configuration is given by
\[
(b_4, b_6, b_8, b_{10}) = (3382, 0, 3383, 2704)
\]
and Euler characteristic \( \chi(M) = 16236 \). So whenever \( \chi(M) < 16236 \) we necessarily have \( b_4 \leq 3381 \). The result follows by our previous reasoning. □

**Theorem 4.8.** Let \( M^{20} \notin \{\mathbb{H}P^5, Gr_2(C^7)\} \) be a (rationally 3-connected) Positive Quaternion Kähler Manifold with \( \hat{A}(M)[M] = 0 \). Then it holds:

- The dimension \( d \) of the isometry group of \( M \) satisfies
  \[
  d \in \{15, 22, 29, 36\}
  \]

- The pair \((d, v)\) of the dimension of the isometry group and the quaternionic volume is one of
  \[
  (15, 4), (15, 24), (15, 44), (15, 64), (22, 24), \ldots, (22, 164), (29, 24), \ldots, (29, 264), (36, 24), \ldots, (36, 384)
  \]
  where \( v \) increases by steps of 20.

- The connected component \( Isom_0(M) \) of the isometry group of \( M \) is as given in table 8 up to finite coverings.

**Proof.** Recall equation (7)
\[
-\frac{81}{70} + \frac{3d}{28} - \frac{36u^5}{5} + \frac{1536\hat{A}(M)[M]}{35} \geq 0
\]
and set the \( \hat{A} \)-genus to zero. This results in
\[
\frac{162}{35} + \frac{3d}{7} - \frac{144v}{1024} \geq 0
\]
Table 8. Possible isometry groups

| dim Isom(M) | type of Isom(M) up to finite coverings |
|-------------|----------------------------------------|
| 15          | $SO(6), G_2 \times S^1, SO(4) \times SO(4) \times SO(3), Sp(2) \times Sp(1) \times S^1 \times S^1, SU(3) \times SO(4) \times S^1$ |
| 22          | $Sp(3) \times S^1, SO(7) \times S^1, G_2 \times SU(3)$ |
| 29          | $SO(8) \times S^1, SO(6) \times G_2, G_2 \times G_2 \times S^1, SO(7) \times SU(3), Sp(3) \times SU(3)$ |
| 36          | $SO(9), Sp(4)$ |

From computations with the Hilbert Polynomial we know that $d \equiv 1 \mod 7$ and $v \equiv 4 \mod 20$ for the dimension of the isometry group and the quaternionic volume—cf. theorem 4.2. Since $\frac{3d}{7} - \frac{162}{35} < 0$ for all values of $d$ with $d \equiv 1 \mod 7$ and $d < 15$, we may thus rule them all out.

Now use the classification of compact Lie groups. The connected component of the identity of Isom(M) permits a finite covering by a product of a semi-simple Lie group and a torus (cf. section 1). Now we figure out all those products $G$ of simple Lie groups and tori that satisfy

- $\dim G \equiv 1 \mod 7$,
- $15 \leq \dim G \leq 36$ (cf. corollary 4.4),
- $rkG \leq 5$. (By theorem 1.4 we know that $rkG \geq 6$ already implies $M \cong \mathbb{HP}^5$, as $M$ is rationally 3-connected.)

Congruence classes modulo 7 of the dimensions of the relevant different types of Lie groups are as described in table 9. The list of Lie groups $G$ then is

Table 9. Dimensions modulo 7

| type | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ |
|------|---------|---------|---------|---------|---------|
| $A_n$ | 3       | 1       | 1       | 3       | 0       |
| $B_n$ | 3       | 3       | 0       | 1       | 6       |
| $C_n$ | 3       | 3       | 0       | 1       | 6       |
| $D_n$ | 1       | 6       | 1       | 0       | 3       |

$\dim G_2 \equiv 0 \mod 7$

$\dim F_4 \equiv 3 \mod 7$

given as in the assertion.

Let us finally establish the list of pairs $(d, v)$. For this we apply equation (17) once more. Additionally, note that the total deficiency

$$\Delta = 2n + 1 + 2v - d \geq 0$$

is non-negative (cf. [16], p. 208), which translates to

$$11 + 2v - d \geq 0$$
for dimension 20. These conditions already reduce the list of pairs \((d,v)\) due to the following restrictions: It holds \(4 \leq v \leq 64\), \(24 \leq v \leq 164\), \(24 \leq v \leq 264\) and \(24 \leq v \leq 384\) when \(d = 15\), \(d = 22\), \(d = 29\) and \(d = 36\) respectively.

We remark that all the Lie groups in the theorem have rank at least 3. Note that if one focuses on simple groups \(G\), then only dimensions 15 and 36 occur. Moreover, we easily derive the following recognition theorem for the quaternionic volume:

**Corollary 4.9.** A 20-dimensional Positive Quaternion Kähler Manifold with \(\hat{A}(M)[M] = 0\) and quaternionic volume \(v > 384\) is symmetric.

The \(\hat{A}\)-genus of \(M\) fits into the elliptic genus \(\Phi(M)\) as a first coefficient. The latter vanishes on the rationally 3-connected Wolf spaces of dimension \(4n\) with odd \(n\). This implies \(\hat{A}(M)[M] = \hat{A}(M,T\mathbb{C}M)[M] = \text{sign}(M) = 0\) on these spaces. So it is reasonable to assume this last chain of equations. This assumption can be combined with further slight conditions to yield a confirmation of the main conjecture in dimension 20. Note that the original techniques in [15]—before we spotted the error in there—would have permitted to derive this vanishing of indices and to actually confirm the conjecture (in dimension 20) under a very mild upper bound on the Euler characteristic.

5. Isometry Groups

In this section we finish the proof of theorem A. By theorem 4.8 and table 8 it remains to show that a Positive Quaternion Kähler Manifold of dimension 20 with isometry group one of \(\text{Sp}(4)\) or \(\text{SO}(9)\) up to finite coverings is homothetic to the real Grassmannian \(\tilde{\text{Gr}}_4(\mathbb{R}^9)\). The arguments given in this section will be Lie theoretic mainly.

In this section we obey the following convention: Every statement on equalities, inclusions or decompositions of groups is a statement up to finite coverings.

First of all note the relation

\[
\text{SO}(n) \otimes \text{Sp}(1) \cong (\text{SO}(n) \times \text{Sp}(1))/\langle((-1)^{n+1},(-\text{id})^{n+1})\rangle
\]

This can be seen by computing the Lie algebras (cf. [18], p. 25) and determining the fibre of the covering \(\text{SO}(n) \times \text{Sp}(1) \to \text{SO}(n) \otimes \text{Sp}(1)\) over the identity.

From the tables in appendix B in [18], p. 63–68, we cite that all irreducible representations \(\varrho(\text{G}_2)\) of \(\text{G}_2\) in degrees smaller than or equal to

\[
\begin{align*}
\text{deg} \varrho &\leq 30 \text{ if } \varrho \in \text{Irr}_\mathbb{R}(\text{G}_2) \\
\text{deg} \varrho &\leq 15 \text{ if } \varrho \in \text{Irr}_\mathbb{C}(\text{G}_2) \\
\text{deg} \varrho &\leq 12 \text{ if } \varrho \in \text{Irr}_\mathbb{H}(\text{G}_2)
\end{align*}
\]

are real and have degree

\[
\text{deg} \varrho \in \{7, 14, 27\}
\]
This is considered an exemplary citation of the most important case. We shall use this information in order to shed more light on inclusions of Lie groups.

**Lemma 5.1.** Let $G = G_1 \times G_2$ be a decomposition of Lie groups. Let further $H \neq 1$ be a simple Lie subgroup of $G$. Then (up to finite coverings) $H$ is also a subgroup of one of $G_1$ and $G_2$ (by the canonical projection).

**Proof.** Compose the inclusion $H \hookrightarrow G_1 \times G_2$ with the canonical projection $G \rightarrow G_i$ (with $i \in \{1, 2\}$) to obtain a morphism $f_i : H \rightarrow G_i$. The kernel of $f_i$ is a normal subgroup of $H$, i.e. we have ker $f_i \in \{1, H\}$ (up to finite coverings). If ker $f_i = 1$, the morphism $f_i$ is an injection and we are done. Otherwise, if ker $f_i = H$, the morphism $f_i$ is constant. So if both $f_1$ and $f_2$ are constant, the original inclusion $H \hookrightarrow G$ is a constant map, too. This contradicts $H \neq 1$. □

**Lemma 5.2.** There is no inclusion of Lie groups $\text{SO}(7) \hookrightarrow \text{Sp}(5)$ (not even up to finite coverings).

**Proof.** By table 5 the group $\text{SO}(7)$ has to be contained in one of

$$\text{U}(5), \text{Sp}(4) \times \text{Sp}(1), \text{Sp}(3) \times \text{Sp}(2), \text{SO}(5) \otimes \text{Sp}(1)$$

as there is no quaternionic representation of a simple Lie group $H$ in degree 10 (other than the standard representation of $\text{Sp}(5)$) by the tables in appendix B in [18], p. 63–68. (The tables neglect the cases of Lie groups with dimensions smaller than 11. Clearly, so can we.)

Thus, by lemma 5.1 and for dimension reasons, we see that $\text{SO}(7)$ has to be a subgroup of one of

$$\text{SU}(5), \text{Sp}(4)$$

Suppose first that $\text{SO}(7)$ is a subgroup of $\text{SU}(5)$. By table 4, lemma 5.1 and for dimension reasons this is not possible. (Again, there are no further irreducible complex representations of degree 5 of interest.)

Assume $\text{SO}(7)$ is a subgroup of $\text{Sp}(4)$. We argue in the analogous way to get a contradiction. Alternatively, one may quote table 7 for subgroups of maximal dimension. □

The following lemmas can fairly easily be proved using similar arguments. We leave this to the reader.

**Lemma 5.3.** There is no inclusion of Lie groups $\text{SU}(5) \hookrightarrow \text{SO}(9)$ and equally, $\text{SU}(5)$ is not a subgroup of $\text{Sp}(4)$ either—not even up to finite coverings.

**Lemma 5.4.** There is no inclusion of Lie groups $\text{Sp}(2) \times \text{SU}(3) \hookrightarrow \text{SO}(9)$ and equally, the group $\text{Sp}(2) \times \text{SU}(3)$ also is not a subgroup of $\text{Sp}(4)$—not even up to finite coverings. □
Lemma 5.5. Suppose that \( k \in \{3, 4\} \). The only inclusion of Lie groups \( \text{Sp}(k) \hookrightarrow \text{Sp}(5) \) is given by the canonical blockwise inclusion up to conjugation.

\[ \square \]

Lemma 5.6. There is no inclusion of Lie groups \( \text{Sp}(2) \times \text{Sp}(2) \hookrightarrow \text{Sp}(5) \) unless one of the \( \text{Sp}(2) \)-factors includes blockwise (up to conjugation).

Proof. By table 5 and by dimension \( \text{Sp}(2) \times \text{Sp}(2) \) includes into one of

\[ U(5), \ \text{Sp}(4) \times \text{Sp}(1), \ \text{Sp}(3) \times \text{Sp}(2) \]

Thus by lemma 5.1 we see that \( \text{Sp}(2) \times \text{Sp}(2) \) is a subgroup of one of

\[ \text{SU}(5), \ \text{Sp}(3) \times \text{Sp}(2) \]

It cannot be a subgroup of \( \text{SU}(5) \), as a subgroup of \( \text{SU}(5) \) of maximal dimension is of dimension \( 16 < 20 = \dim \text{Sp}(2) \times \text{Sp}(2) \). Thus \( \text{Sp}(2) \times \text{Sp}(2) \subseteq \text{Sp}(3) \times \text{Sp}(2) \) and we need to determine all the possible inclusions \( i \).

The only inclusion of \( \text{Sp}(2) \) into \( \text{Sp}(3) \) possible is given by the canonical blockwise inclusion (up to conjugation). Suppose now that both \( \text{Sp}(2) \)-factors do not include blockwise. We shall lead this to a contradiction.

We obtain that the inclusion composed with the canonical projection

\[ i_1 : \text{Sp}(2) \times \{1\} \hookrightarrow \text{Sp}(3) \times \text{Sp}(2) \rightarrow \text{Sp}(2) \]

again is an inclusion. This is due to the fact that the kernel of this map has to be trivial, as \( \text{Sp}(2) \) is simple. Thus by our assumption of non-blockwise inclusion the kernel has to be the trivial group and \( i_1 \) is an inclusion. The same holds for

\[ i_2 : \{1\} \times \text{Sp}(2) \hookrightarrow \text{Sp}(3) \times \text{Sp}(2) \rightarrow \text{Sp}(2) \]

Without restriction, we may suppose that \( i_1 = i_2 = \text{id} \). Thus we obtain that \( i \) is an inclusion if and only if

\[ \text{Sp}(2) \times \text{Sp}(2) \hookrightarrow \text{Sp}(3) \times \text{Sp}(2) \rightarrow \text{Sp}(3) \]

is an inclusion. By consideration of rank this is impossible. This yields a contradiction and at least one \( \text{Sp}(2) \)-factor is canonically included. \( \square \)

Lemma 5.7. The only inclusion of Lie groups \( \text{SU}(4) \hookrightarrow \text{Sp}(5) \) respectively \( \text{SU}(4) \times \text{Sp}(1) \hookrightarrow \text{Sp}(5) \) is given by the canonical blockwise inclusion up to conjugation.

Proof. By table 5, by the tables in appendix B in [18], p. 63–68, and by dimension we see that \( \text{SU}(4) \) respectively \( \text{SU}(4) \times \text{Sp}(1) \) lies in one of

\[ U(5), \ \text{Sp}(4) \times \text{Sp}(1), \ \text{Sp}(3) \times \text{Sp}(2) \]

The group \( \text{SU}(4) \) is not a subgroup of \( \text{Sp}(3) \) by table 5 and by dimension. The only inclusion of \( \text{SU}(4) \) into \( U(5) \) is given by the canonical blockwise one due to the usual arguments. The group \( \text{SU}(4) \times \text{Sp}(1) \) is not a subgroup of \( U(5) \); indeed, this is impossible by table 4, the tables in appendix B in [18], p. 63–68, and by dimension.
Case 1. Thus in the case of the inclusion $\text{SU}(4) \hookrightarrow \text{Sp}(5)$ we observe that either the inclusion factors over $\text{SU}(4) \hookrightarrow \text{U}(4) \hookrightarrow \text{Sp}(5)$ and is given blockwise or the inclusion is given via $\text{SU}(4) \hookrightarrow \text{Sp}(4) \hookrightarrow \text{Sp}(5)$. Again, the inclusion necessarily is given blockwise. For this we realize that $\text{SU}(4)$ cannot be included into any maximal subgroup of $\text{Sp}(4)$ other than $\text{U}(4)$.

Case 2. As for the inclusion $\text{SU}(4) \times \text{Sp}(1) \hookrightarrow \text{Sp}(5)$ we note that $\text{SU}(4) \times \text{Sp}(1)$ maps into $\text{Sp}(4) \times \text{Sp}(1)$ by dimension. By lemma 5.1 and by dimension we see that $\text{SU}(4)$ again lies in $\text{Sp}(4)$. As we have seen this inclusion necessarily is blockwise. As $\text{Sp}(4)$ maps into $\text{Sp}(5)$ by blockwise inclusion, the inclusion $\text{SU}(4) \hookrightarrow \text{Sp}(5)$ is the canonical one.

It then remains to see that $\text{Sp}(1)$ includes into the $\text{Sp}(1)$-factor of $\text{Sp}(4) \times \text{Sp}(1)$ (and not into the $\text{Sp}(4)$-factor). Assume this is not the case. This means that the group $\text{Sp}(1)$ necessarily does include into the $\text{Sp}(4)$-factor. Thus there is a homomorphism of groups

$$i : \text{SU}(4) \times \text{Sp}(1) \rightarrow \text{Sp}(4)$$

with the property that $i|_{\text{SU}(4)}$ as well as $i|_{\text{Sp}(1)}$ are injective. (Clearly, the morphism $i$ itself cannot be injective.) However, as we shall show, this contradicts the fact that $i$ is a homomorphism: For $(x_1, x_2), (y_1, y_2) \in \text{SU}(4) \times \text{Sp}(1)$ we compute

$$i((x_1, x_2) \cdot (y_1, y_2)) = i(x_1 y_1, x_2 y_2) = i(x_1, 1)i(y_1, 1)i(1, x_2)i(1, y_2)$$
$$i((x_1, x_2) \cdot (y_1, y_2)) = i(x_1, x_2)i(y_1, y_2) = i(x_1, 1)i(1, x_2)i(y_1, 1)i(1, y_2)$$

Thus we necessarily have that $i(1, y_1)i(1, x_2) = i(1, x_2)i(y_1, 1)$. As $i|_{\text{Sp}(1)}$ is injective, we realize that—whatever the inclusion $i|_{\text{Sp}(1)}$ will be—the group $i(\text{Sp}(1))$ always contains elements that do not commute with every element of $\text{SU}(4) \subseteq \text{Sp}(4)$. (Clearly, $\text{Sp}(1)$ cannot be included into the centre $T^4$ of $\text{SU}(4)$.)

Consequently, we obtain that the inclusion of $\text{Sp}(1)$ into $\text{Sp}(4) \times \text{Sp}(1)$ maps $\text{Sp}(1)$ to the $\text{Sp}(1)$-factor and the projection $\text{Sp}(1) \rightarrow \text{Sp}(4) \times \text{Sp}(1) \rightarrow \text{Sp}(4)$ is the trivial map. This proves the assertion. \hfill \Box

Let us now prove the classification result.

**Theorem 5.8.** A 20-dimensional Positive Quaternion Kähler Manifold $M$ which has an isometry group $\text{Isom}(M)$ that satisfies

$$\text{Isom}_0(M) \in \{\text{SO}(9), \text{Sp}(4)\}$$

up to finite coverings is homothetic to the real Grassmannian $M \cong \overline{\text{Gr}}_4(\mathbb{R}^9)$

**Proof.** We proceed in three steps. First we shall establish a list of stabilizer groups in a $T^4$-fixed-point—where $T^4$ is the maximal torus of $\text{Isom}_0(M)$—that might occur unless $M$ is a Wolf space. As a second step we reduce the list by inclusions into the isometry group and the holonomy group. Finally, in the third step we show by more distinguished arguments that also
the remaining stabilisers from the list cannot occur, whence \( M \) has to be symmetric.

**Step 1.** Both groups \( \text{SO}(9) \) as well as \( \text{Sp}(4) \) have rank 4, i.e. they contain a 4-torus \( T^4 = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \). The Positive Quaternion Kähler Manifold \( M \) has positive Euler characteristic by theorem 1.5. Thus by the Lefschetz fixed-point theorem we derive that there exists a \( T^4 \)-fixed-point \( x \in M \). Let \( H_x \) denote the (identity-component of the) isotropy group of the \( G \)-action in \( x \) for \( G \in \{ \text{SO}(9), \text{Sp}(4) \} \). Since \( G \) is of dimension 36 and since \( \dim M = 20 \), we obtain that \( \dim H_x \geq 17 \) unless the action of \( G \) on \( M \) is transitive. If this is the case, a result by Alekseevskii (cf. \([4].14.56, p. 409\)) yields the symmetry of \( M \). If \( M \) is symmetric, it is a Wolf space and the dimension of the isometry group then yields that \( M \cong \tilde{\text{Gr}}_4(\mathbb{R}^9) \). We may even assume \( \dim H_x \geq 18 \) by the classification of cohomogeneity one Positive Quaternion Kähler Manifolds in theorem \([9].7.4, p. 24\).

Since \( \text{rk} \, G = 4 \) and since \( x \) is a \( T^4 \)-fixed-point, we also obtain \( \text{rk} \, H_x = 4 \). Moreover, \( H_x \) is a closed subgroup. Thus we may give a list of all products of semi-simple Lie groups and tori of dimension \( 36 \geq \dim H_x \geq 18 \) and with \( \text{rk} \, H_x = 4 \) up to finite coverings:

\[
\begin{align*}
&\text{SU}(5), \; \text{SO}(9), \; \text{SO}(8), \; \text{Sp}(4), \; \text{Sp}(3) \times \text{Sp}(1), \; \text{Sp}(3) \times \mathbb{S}^1, \\
&\text{SO}(7) \times \text{Sp}(1), \; \text{SO}(7) \times \mathbb{S}^1, \; \text{SO}(6) \times \text{Sp}(1), \; \text{Sp}(2) \times \text{Sp}(2), \\
&\text{Sp}(2) \times \text{SU}(3), \; \text{Sp}(2) \times \text{G}_2, \; \text{G}_2 \times \text{G}_2, \; \text{G}_2 \times \text{SU}(3), \\
&\text{G}_2 \times \text{Sp}(1) \times \text{Sp}(1), \; \text{G}_2 \times \text{Sp}(1) \times \mathbb{S}^1
\end{align*}
\]

**Step 2.** We now apply two criteria by which we may reduce the list:

- On the one hand we have that \( H_x \) is a Lie subgroup of \( G \).
- On the other hand by the isotropy representation \( H_x \) is a Lie subgroup of \( \text{Sp}(5) \text{Sp}(1) \)—cf. theorem \([17].VI.4.6, p. 248\).

We use lemma 5.1 to see that every group \( H_x \) in the list contains a factor that has to include into \( \text{Sp}(5) \) up to finite coverings.

An iterative application of table 6 yields that every maximal rank subgroup of the classical group \( G \in \{ \text{SO}(9), \text{Sp}(4) \} \) again is a product of classical groups. Thus \( H_x \) may not be one of the groups

\[
\begin{align*}
&\text{Sp}(2) \times \text{G}_2, \; \text{G}_2 \times \text{G}_2, \; \text{G}_2 \times \text{SU}(3), \\
&\text{G}_2 \times \text{Sp}(1) \times \text{Sp}(1), \; \text{G}_2 \times \text{Sp}(1) \times \mathbb{S}^1
\end{align*}
\]

(not even up to finite coverings).

Now apply lemma 5.2 in the respective cases to reduce the list of potential stabilisers to

\[
\begin{align*}
&\text{SU}(5), \; \text{Sp}(4), \; \text{Sp}(3) \times \text{Sp}(1), \; \text{Sp}(3) \times \mathbb{S}^1, \\
&\text{SO}(6) \times \text{Sp}(1), \; \text{Sp}(2) \times \text{Sp}(2), \; \text{Sp}(2) \times \text{SU}(3)
\end{align*}
\]

Indeed, this lemma rules out all the groups \( H_x \) that contain a factor of the form \( \text{SO}(7) \); and as we see that \( \text{SO}(7) \) is not a subgroup of \( \text{Sp}(5) \), also \( \text{SO}(8) \) and \( \text{SO}(9) \) cannot be subgroups of \( \text{Sp}(5) \).

Now apply lemmas 5.3 and 5.4 by which potential inclusions into the isometry group \( G \) are made clearer. That is, they rule out the groups \( \text{SU}(5) \)
and $\text{Sp}(2) \times \text{SU}(3)$. Thus the list of possible isotropy groups reduces further to
\[
\text{Sp}(4), \text{Sp}(3) \times \text{Sp}(1), \text{Sp}(3) \times S^1, \text{SO}(6) \times \text{Sp}(1), \text{Sp}(2) \times \text{Sp}(2)
\]

**Step 3.** Let us consider the inclusion of the groups $H_x$ into the holonomy group $\text{Sp}(5) \text{Sp}(1)$. By 5.1 the largest direct factor of the candidates in our list has to be a subgroup of $\text{Sp}(5)$ (up to finite coverings), as it cannot be included into $\text{Sp}(1)$. By lemma 5.5 we see that the inclusion of $\text{Sp}(4)$ into $\text{Sp}(5)$ and the one of the $\text{Sp}(3)$-factor of $\text{Sp}(3) \times \text{Sp}(1)$ respectively of $\text{Sp}(3) \times S^1$ has to be blockwise. Due to lemma 5.6 we observe that there is also an $\text{Sp}(2)$-factor of $\text{Sp}(2) \times \text{Sp}(2)$ that includes blockwise into $\text{Sp}(5)$. Thus we obtain that every group from our list which contains a factor of the form $\text{Sp}(k)$ for $k \geq 2$ has a circle subgroup $S^1 \subseteq \text{Sp}(k)$ that includes into $\text{Sp}(5)$ by
\[
\text{diag}(S^1, 1, 1, 1, 1) \subseteq \text{diag}(\text{Sp}(k), 1, \ldots, 1) \subseteq \text{Sp}(5)
\]
Thus this circle group fixes a codimension 4 Positive Quaternion Kähler component. Due to theorem [10].1.2, p. 2, we obtain that $M \cong \mathbb{HP}^5$ or $M \cong \text{Gr}_2(\mathbb{C}^7)$; a contradiction by our assumption on the isometry group.

This leaves us with $H_x = \text{SO}(6) \times \text{SO}(3)$, which is $\text{SU}(4) \times \text{Sp}(1)$ up to $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$-covering. Equivalently, we consider an orbit of the form
\[
X := \frac{\text{SO}(9)}{\text{SO}(6) \times \text{SO}(3)} = \frac{(\text{Spin}(9))/\langle(-\text{id}, -\text{id})\rangle}{\text{Spin}(9)}/\langle(-\text{id}, -1)\rangle
\]
Now consider the isotropy representation of the stabiliser group. There are basically two possibilities: Either the whole stabiliser includes into the $\text{Sp}(5)$-factor or the $\text{Sp}(1)$-factor includes into the $\text{Sp}(1)$-factor of $\text{Sp}(5)\text{Sp}(1)$ (and not into the $\text{Sp}(5)$-factor).

In the first case we apply lemma 5.7 to see that the inclusion of $\text{SU}(4) \times \text{Sp}(1)$ into $\text{Sp}(5)$ is blockwise. So is the inclusion of the $\text{Sp}(1)$-factor in particular. Thus again we obtain a sphere which is represented by $\text{diag}(S^1, 1, 1, 1, 1)$ and which fixes a codimension four quaternionic component. We proceed as above.

Let us now deal with the second case. Again we cite lemma 5.7 to see that the $\text{SU}(4)$-factor includes into $\text{Sp}(5)$ in a blockwise way. Observe now that the tangent bundle $TM$ of $M$ splits as
\[
TM = TX \oplus NX
\]
over $X$, where $NX$ denotes the normal bundle. Since
\[
\dim X = \dim \text{SO}(9) - \dim \text{SO}(6) \times \text{SO}(3) = 36 - 18 = 18
\]
we obtain that the normal bundle is two-dimensional. Thus the slice representation of the isotropy group $(\text{SU}(4) \times \text{Sp}(1))/\langle(-\text{id}, -1)\rangle$ at a fixed-point, i.e. the representation on $NX$, is necessarily trivial. That is, the action of
the isotropy group \((\text{SU}(4) \times \text{Sp}(1))/(-\text{id}, -1)\) at a fixed-point has to leave the normal bundle pointwise fixed.

The \(\text{Sp}(1)\)-factor of \(\text{SU}(4) \times \text{Sp}(1)\) maps isomorphically (up to finite coverings) into the \(\text{Sp}(1)\)-factor of the holonomy group; the \(\text{SU}(4)\)-factor maps into \(\text{Sp}(5)\). Thus the action of this \(\text{SU}(4) \times \text{Sp}(1)\) on the tangent space \(T_x M \cong \mathbb{H}^5\) at \(x \in M\) is given by \((A, h)(v) = Avh^{-1}\). This action, however, has no 18-dimensional (respectively 2-dimensional) invariant subspace as the \(\text{Sp}(1)\)-factor acts transitively on each \(\mathbb{H}\)-component. Hence the normal bundle does not remain fixed under the action of the stabiliser. Thus \(\text{SU}(4) \times \text{Sp}(1)\) cannot occur as an isotropy group.

Hence we have excluded all the cases that arose from the assumption \(\dim H_x \geq 18\). This was equivalent to the action of the isometry group neither being transitive nor of cohomogeneity one. In the latter two cases—as already observed—the manifold \(M\) has to be symmetric. More precisely, since there is no 20-dimensional Wolf space with \(\text{Isom}_0(M) = \text{Sp}(4)\) (up to finite coverings), we obtain \(\text{Isom}_0 M = \text{SO}(9)\) and \(M \cong \text{Gr}_4(\mathbb{R}^9)\). \(\square\)

Theorem A now follows from theorems 4.8 and 5.8.

Clearly, as for \(\dim \text{Isom}_0(M) \not\in \{15, 22, 29\}\) one hopes an approach by similar techniques as in the proof of 5.8 to be likewise successful. Yet, we remark that for example in dimension 29 one will have to cope with five different isometry groups due to table 8. All these groups are of rank 5. So one lists all the possible stabilisers at a \(T^5\)-fixed-point on \(M\) that do not necessarily make the action of \(\text{Isom}(M)\) transitive or of cohomogeneity one. That is, one computes all the products \(H\) of semi-simple Lie groups and tori that satisfy \(\text{rk} H = 5\) and \(11 \leq \dim H \leq 29\). This results in a list of 45 possible groups \(H\) (up to finite coverings). Following our previous line of argument we then try to rule out stabilisers by showing that they either may not include into a respective isometry group or that they may not be a subgroup of \(\text{Sp}(5)\text{Sp}(1)\). If both is not the case, as a next step we try to show that the way \(H\) includes into the holonomy group already implies the existence of an \(S^1\)-fixed-point component of codimension 4. This would imply the symmetry of the ambient manifold \(M^{20}\). We observe that by far the biggest part of this procedure is covered by the arguments we applied before and we encourage the reader to provide the concrete reasoning. Nonetheless, we encounter new difficulties: For example, the group \(\text{SU}(4) \times S^1 \times S^1\) includes into the isometry group \(\text{SU}(4) \times G_2 \cong \text{SO}(6) \times G_2\). If its inclusion into \(\text{Sp}(5)\text{Sp}(1)\) is induced by the blockwise inclusion of \(\text{SU}(4)\) into \(\text{Sp}(5)\), the canonical inclusion of \(S^1\) into \(\text{Sp}(1)\) and the diagonal inclusion of \(S^1\) into \(\text{Sp}(5)\), we realise that there is no codimension four \(S^1\)-fixed-point component. Then methods more particular in nature will have to be provided—as we did in step 3 of the proof of theorem 5.8. We leave this to the reader.

6. Proof of theorem C

Since one may relate the index \(i^{0,n+2}\) (cf. 1.3) directly to the dimension of \(\text{Isom}(M^{4n})\), it seems to be pretty natural to try to provide a recognition
theorems on this information. This will result in theorem C, the first one to identify the real Grassmanian.

For the proof of theorem C we imitate and generalise the techniques and results from section 5. Again, we shall neglect finite coverings.

Lemma 6.1. For \( n \geq 6 \) there is no inclusion of Lie groups \( \text{SO}(n+1) \hookrightarrow \text{Sp}(n) \), not even up to finite coverings.

Proof. Due to table 5 the group \( \text{SO}(n+1) \) either has to be contained in \( U(n) \), \( \text{Sp}(k) \times \text{Sp}(n-k) \) with \( 1 \leq k \leq n-1 \), some \( \text{SO}(p) \otimes \text{Sp}(q) \) with \( pq = n \), \( p \geq 3 \), \( q \geq 1 \) or in \( \varrho(H) \) for a simple Lie group \( H \) and an irreducible quaternionic representation \( \varrho \in \text{Irr}(H) \) of dimension \( \deg \varrho = 2n \). The cases with direct product or tensor product yield an inclusion of \( \text{SO}(n+1) \) in either some \( \text{SO}(k) \) with \( k \geq n \)—which is impossible by dimension—or into some smaller symplectic group by lemma 5.1.

Assume there is an inclusion into \( U(n) = (SU(n) \times U(1))/\mathbb{Z}_n \). Then again lemma 5.1 yields an inclusion into \( SU(n) \). By table 4 the maximal subgroups of \( SU(n) \) are given by \( \text{SO}(n) \), \( \text{Sp}(m) \) with \( 2m = n \), \( S(U(k) \times U(n-k)) \) for \( 1 \leq k \leq n-1 \), \( SU(p) \otimes SU(q) \) with \( pq = n \), \( p \geq 3 \), \( q \geq 1 \) and by \( \varrho(H) \) for a simple Lie group \( H \) and an irreducible quaternionic representation \( \varrho \in \text{Irr}(H) \) of dimension \( \deg \varrho = n \). An inclusion in the first case is impossible due to dimension. Cases two to four lead to inclusions into smaller symplectic or special unitary groups by lemma 5.1.

Hence we need to have a closer look at irreducible quaternionic and complex representations of simple Lie groups \( H \). The tables in [18], appendix B, p. 63–68, give all the representations of simple Lie groups satisfying a certain dimension bound, which is given by

\[
2 \dim H \geq \deg \varrho - 2
\]

\[
\dim H \geq \deg \varrho - 1
\]

\[
\dim H \geq \frac{3}{2} \deg \varrho - 4
\]

for real, complex and quaternionic representations respectively.

First of all for \( n \geq 3 \) the tables together with our previous reasoning yield that \( k = n \) is the maximal number for which \( SU(k) \) is a maximal subgroup of \( \text{Sp}(n) \). Equally, for \( n \geq 7 \) we obtain that \( k = n \) is the maximal number for which \( \text{SO}(k) \) is a maximal subgroup of \( \text{Sp}(n) \) or of \( SU(n) \). This means in particular that \( \text{SO}(n+1) \) cannot be included into \( \text{Sp}(n) \) by a chain

\[
\text{SO}(n+1) \subseteq G_1 \subseteq \ldots \subseteq G_l \subseteq \text{Sp}(n)
\]

of (irreducible representations of) classical groups \( G_1, \ldots, G_l \) for \( n \geq 6 \).

It remains to prove that there is no such chain involving (representations of) exceptional Lie groups \( G_i \). For this it suffices to realise that there are no exceptional Lie groups \( H \) satisfying

\[
\dim \text{SO}(n+1) \leq \dim H \leq \dim \text{Sp}(n)
\]

with \( H \) admitting a quaternionic or complex representation of degree smaller than or equal to \( 2n \) or \( n \) respectively. (We clearly may neglect the real representations \( \varrho \) of degree \( k \) with \( k \leq n \), as there evidently cannot be inclusions \( \text{SO}(n+1) \subseteq \varrho(H) \subseteq \text{SO}(k) \) by dimension.)
In table 10 for each exceptional Lie group $H$ we give the values of $n$ for which the inequalities (20) are satisfied. Additionally, we note the corresponding maximal degree $\deg \varrho = 2n$ ($\deg \varrho = n$) of an irreducible quaternionic (complex) representation $\varrho$ by which $H$ might become the subgroup $\varrho(H) \subseteq \text{Sp}(k)$ ($\varrho(H) \subseteq \text{SU}(k)$) with $k \leq n$ for the given values of $n$. That is, for example in the case of $G_2$ we see that if there is a quaternionic representation (a complex representation) of degree smaller than or equal to 22 (to 11), then there is an inclusion of $G_2$ into $\text{Sp}(k)$ (into $\text{SU}(k)$) for $k \leq 11$ and now also conversely: If there is no such representation, then $G_2$ cannot be a subgroup satisfying $\text{SO}(n+1) \subseteq G_2 \subseteq \text{Sp}(n)$ for any $n \in \mathbb{N}$.

Now the tables in [18] yield that there are no quaternionic respectively complex representations of $H$ in the degrees depicted in table 10. This amounts to the fact that for the relevant values of $n$ from table 10 there is no inclusion $\text{SO}(n+1) \subseteq H \subseteq \text{Sp}(n)$. Thus by (20) there are no inclusions of exceptional Lie groups $H$ with $\text{SO}(n+1) \subseteq H \subseteq \text{Sp}(n)$ for any $n \in \mathbb{N}$.

Thus we have proved that there cannot be a chain of the form (19) with an exceptional Lie group $G_i$. Combining this with our previous arguments proves the assertion. \hfill \qed

| Lie group | $n \in$ | $\deg \varrho \leq$ |
|-----------|--------|-----------------|
| $G_2$     | $\{3, 4\}$ | 8, 4           |
| $F_4$     | $\{5, 6, 7, 8, 9\}$ | 18, 9        |
| $E_6$     | $\{7, 8, 9, 10, 11\}$ | 22, 11       |
| $E_7$     | $\{8, 9, 10, 11, 12, 13, 14, 15\}$ | 30, 15       |
| $E_8$     | $\{11, 12, \ldots, 20, 21\}$ | 42, 21       |

Note that the bound $n \geq 7$ in the lemma is necessary since the universal two-sheeted covering of $\text{SO}(6)$ is $\text{SU}(4)$. In higher dimensions no such exceptional identities occur as can be seen from the corresponding Dynkin diagrams.

**Lemma 6.2.** For $n \geq 3$ the only inclusion of Lie groups $\text{Sp}\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \hookrightarrow \text{Sp}(n)$ is given by the canonical blockwise one up to conjugation.

**Proof.** We proceed as in lemma 6.1. Indeed, by the same arguments as above we see that every chain of classical groups

\begin{equation}
\text{Sp}\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \subseteq G_1 \subseteq \ldots \subseteq G_l \subseteq \text{Sp}(n)
\end{equation}

involves symplectic or special unitary groups $G_i$ of rank smaller than or equal to $n$ only. For this we use that there is no inclusion $\text{Sp}(\left\lfloor \frac{n}{2} \right\rfloor + 1) \subseteq \text{SO}(n)$ by dimension; indeed

$$\dim \text{SO}(n) = \frac{n(n-1)}{2} < \begin{cases} \frac{(n+1)(n+3)}{2} & \text{for } n \text{ even} \\ \frac{(n+1)(n+2)}{2} & \text{for } n \text{ odd} \end{cases}$$

$$= \dim \text{Sp}\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)$$
More precisely, in such a chain of classical groups the inclusion of $\text{Sp}\left(\left\lfloor \frac{n}{2} \right\rfloor +1\right)$ is necessarily blockwise since $n \geq 3$. This is due to the fact that actually only symplectic groups $G_i$ which are included in a blockwise way may appear by table 4; i.e. the subgroups of $\text{SU}(n)$ are to small to permit the inclusion of $\text{Sp}\left(\left\lfloor \frac{n}{2} \right\rfloor +1\right)$.

We now have to realise that there is no chain as in (21) with an exceptional Lie group $G_i$. As in the proof of lemma 6.1 we depict the values of $n$ for which an inclusion $\text{Sp}\left(\left\lfloor \frac{n}{2} \right\rfloor +1\right) \subseteq H \subseteq \text{Sp}(n)$ of an exceptional Lie group $H$ might be possible—when merely considering dimensions—in table 11. The table again also yields degree bounds for the degrees of quaternionic and complex representations. Then the tables in appendix [18].B, p. 63–68, yield

| Lie group | $n \in$ | $\deg \varrho \leq$ |
|-----------|--------|------------------|
| $G_2$     | $\{3\}$ | 6, 3             |
| $F_4$     | $\{5, 6, 7\}$ | 14, 7           |
| $E_6$     | $\{7, 8, 9\}$ | 18, 9           |
| $E_7$     | $\{8, 9, 10, 11, 12, 13\}$ | 26, 13 |
| $E_8$     | $\{11, 12, 13, 14, 15, 16, 17, 18, 19\}$ | 38, 19 |

that under these respective restrictions no representations of exceptional Lie groups can be found. This implies that each $G_i$ in the chain is classical. Thus the inclusion of $\text{Sp}\left(\left\lfloor \frac{n}{2} \right\rfloor +1\right)$ into $\text{Sp}(n)$ is necessarily given blockwise. □

Note that $\text{Sp}(1) \cong \text{SU}(2)$, whence the inclusion $\text{Sp}(1) \hookrightarrow \text{Sp}(2)$ is not necessarily blockwise.

**Lemma 6.3.** Let $n \geq 7$ be odd. Every inclusion $\text{Sp}\left(\frac{n+1}{2} - 1\right) \times \text{Sp}(2) \hookrightarrow \text{Sp}(n)$ restricts to the canonical blockwise one (up to conjugation and finite coverings) on the first factor.

**Proof.** By table 5 and by dimension we see that $\text{Sp}\left(\frac{n+1}{2} - 1\right) \times \text{Sp}(2)$ lies in one of $\text{U}(n)$, $\text{Sp}(k) \times \text{Sp}(n-k)$ for $1 \leq k \leq n-1$, $\varrho(H)$ for a simple Lie group $H$ and an irreducible quaternionic representation $\varrho$ of degree $\deg \varrho = 2n$.

If $\text{Sp}\left(\frac{n+1}{2} - 1\right) \times \text{Sp}(2)$ should happen to appear as a subgroup of $\text{SU}(n)$, then table 4 would show that

$$\text{Sp}\left(\frac{n+1}{2} - 1\right) \hookrightarrow \text{SU}(n-1) \hookrightarrow \text{SU}(n)$$

necessarily is included in the standard “diagonal” way induced by the standard inclusion $\mathbb{H} \hookrightarrow \mathbb{C}^{2\times2}$. For this we observe the following facts that result when additionally taking into account the tables in appendix [18].B, p. 63–68. The largest special orthogonal subgroup (up to finite coverings) of $\text{SU}(n)$ is $\text{SO}(n)$ for $n \geq 6$. The group $\text{SO}(n)$ does not permit $\text{Sp}\left(\frac{n+1}{2} - 1\right)$ as a subgroup for $n \geq 4$. Moreover, there are no irreducible complex representations by which a simple Lie group $H$ might include into some $\text{SU}(k)$ (for $k \leq n$) satisfying $\text{Sp}\left(\frac{n+1}{2} - 1\right) \subseteq H$. 
Now we see that whenever $\text{Sp} \left( \frac{n+1}{2} - 1 \right)$ is included diagonally into $\text{SU}(n)$ as depicted, there is no inclusion of $\text{Sp} \left( \frac{n+1}{2} - 1 \right) \times \text{Sp}(2)$ possible. That is, for the inclusion of this direct product to be a homomorphism we need the group $\text{Sp}(2)$ to map into the centraliser $C_{\text{SU}(n)}(\text{Sp}(\frac{n+1}{2} - 1))$ of $\text{Sp}(\frac{n+1}{2} - 1)$ in $\text{SU}(n)$. Yet, we obtain

$$C_{\text{SU}(n)} \left( \text{Sp} \left( \frac{n+1}{2} - 1 \right) \right) \cong S^1 \times S^1$$

and thus no inclusion of $\text{Sp}(2)$ is possible. Hence $\text{Sp}(\frac{n+1}{2} - 1) \times \text{Sp}(2)$ cannot be a subgroup of $\text{U}(n)$.

The tables in [18] again yield that whenever a simple Lie group $H$ is included into $\text{Sp}(k)$ (for $k \leq n$) via an irreducible quaternionic representation $\varrho$, the inclusion is one of

$\text{SU}(6) \hookrightarrow \text{Sp}(10), \text{SO}(11) \hookrightarrow \text{Sp}(16), \text{SO}(12) \hookrightarrow \text{Sp}(16), \text{E}_7 \hookrightarrow \text{Sp}(28)$

or an inclusion of a symplectic group of rank smaller than $k$ unless the degree of the representation is far too large to be of interest for our purposes. Indeed, already the depicted inclusions are not relevant, since $\text{Sp}(\frac{n+1}{2} - 1)$ cannot be included into $\text{SU}(6), \text{SO}(11), \text{SO}(12), \text{E}_7$ respectively when $n \geq 10, 16, 16, 28$.

Thus we see that every inclusion of $\text{Sp}(\frac{n+1}{2} - 1) \times \text{Sp}(2)$ has to factor through one of $\text{Sp}(n-k) \times \text{Sp}(k)$ for $1 \leq k \leq n-1$. Hence for $1 \leq k < \frac{n+1}{2} - 1$ we obtain that the only inclusion of $\text{Sp}(\frac{n+1}{2} - 1)$ into $\text{Sp}(n)$ factoring through $\text{Sp}(n-k) \times \text{Sp}(k)$ is given by the standard blockwise inclusion

$$\text{Sp} \left( \frac{n+1}{2} - 1 \right) \hookrightarrow \text{Sp}(n-k) \hookrightarrow \text{Sp}(k)$$

Thus we may assume without restriction that $k = \frac{n+1}{2} - 1$ (and $n-k = \frac{n+1}{2}$) and that the inclusion of $\text{Sp}(\frac{n+1}{2} - 1) \hookrightarrow \text{Sp}(n)$ is not the standard blockwise one. Thus we see that the inclusion necessarily factors over

$$\text{Sp} \left( \frac{n+1}{2} - 1 \right) \hookrightarrow \text{Sp} \left( \frac{n+1}{2} - 1 \right) \times \text{Sp} \left( \frac{n+1}{2} \right) \hookrightarrow \text{Sp}(n)$$

where the first inclusion splits as a product of the standard blockwise inclusions

$$\text{Sp} \left( \frac{n+1}{2} - 1 \right) \hookrightarrow \text{Sp} \left( \frac{n+1}{2} - 1 \right) \hookrightarrow \text{Sp} \left( \frac{n+1}{2} \right)$$

So regard $\text{Sp}(\frac{n+1}{2} - 1)$ as the subgroup of $\text{Sp}(n)$ given by this inclusion. Again we make use of the fact that the $\text{Sp}(2)$-factor has to include into the centraliser

$$C_{\text{Sp}(n)} \left( \text{Sp} \left( \frac{n+1}{2} - 1 \right) \right) \cong S^1 \times S^1 \times \text{Sp}(1)$$
Such an inclusion clearly is impossible and we obtain a contradiction. Thus the inclusion of $\text{Sp}(\frac{n+1}{2} - 1) \times \text{Sp}(2)$ is the standard blockwise one when restricted to the $\text{Sp}(\frac{n+1}{2} - 1)$-factor. □

**Lemma 6.4.** In table 12 the semi-simple Lie groups of maximal dimension with respect to a fixed rank (from rank 1 to rank 12) are given up to isomorphisms and finite coverings. From rank 13 on the groups that are maximal in this sense are given by the two infinite series $\text{Sp}(n)$ and $\text{SO}(2n+1)$ only.

**Table 12.** Largest Lie groups with respect to fixed rank

| rk $G$ | extremal Lie groups $G$ | dim $G$ |
|--------|-------------------------|---------|
| 1      | $\text{Sp}(1)$          | 3       |
| 2      | $G_2$                   | 14      |
| 3      | $\text{Sp}(3), \text{SO}(7)$ | 21     |
| 4      | $F_4$                   | 52      |
| 5      | $\text{Sp}(5), F_4 \times \text{Sp}(1), \text{SO}(11)$ | 55     |
| 6      | $E_6, \text{Sp}(6)$     | 78      |
| 7      | $E_7$                   | 133     |
| 8      | $E_8$                   | 248     |
| 9      | $E_8 \times \text{Sp}(1)$ | 251   |
| 10     | $E_8 \times G_2$        | 262     |
| 11     | $E_8 \times \text{Sp}(3), E_8 \times \text{SO}(7)$ | 269   |
| 12     | $\text{Sp}(12), \text{SO}(25), E_8 \times F_4$ | 300   |

**Proof.** Table 12 results from a case by case check using table 2. From dimension 13 on semi-simple Lie groups involving factors that are exceptional Lie groups are smaller than the largest classical groups. Among products of classical groups the ratio between dimension and rank is maximal for the types $B_n$ and $C_n$. □

Now we provided all the tools that will permit to prove theorem C.

**Proof of Theorem C. Step 1.** By theorem 1.4 we may suppose that $\text{rk Isom}(M^{4n}) \leq \lceil \frac{n}{2} \rceil + 2$, since otherwise $M \in \{\mathbb{P}^n, \text{Gr}_2(\mathbb{C}^{n+2})\}$. The dimension bounds in table 1 for $4 \leq n \leq 20$ result from table 12: That is, for each such $n$ the bound is the dimension of the largest group that satisfies this rank condition. Thus every group with larger dimension has rank large enough to identify $M^{4n}$ as one of $\mathbb{P}^n$ and $\text{Gr}_2(\mathbb{C}^{n+2})$.

In degree $n = 3$ we use that there is no semi-simple Lie group of rank smaller than or equal to 4 in dimensions 29 to 36 = dim $\text{Sp}(4)$. By theorem 1.4 we have dim $\text{Isom}(M^{12}) \leq 36$.

**Step 2.** Now we determine all (the one-components of) the isometry groups $G = \text{Isom}_0(M^{4n})$ (up to finite coverings) with $\text{rk } G \leq \lceil \frac{n}{2} \rceil + 2$ satisfying the dimension bound for $n \geq 22$ and $n \notin \{27, 28\}$ as

$$G \in \left\{ \text{SO}(n+4), \text{SO}(n+5), \text{Sp}\left(\frac{n}{2} + 2\right) \right\}$$
for $n$ even and as
\[
G \in \left\{ \text{SO}(n + 4), \text{SO}(n + 4) \times \text{SO}(2), \text{SO}(n + 4) \times \text{SO}(3), \right.
\]
\[
\text{SO}(n + 5), \ \text{SO}(n + 6), \ \text{Sp}\left(\frac{n + 1}{2} + 2\right), \ \text{Sp}\left(\frac{n + 1}{2} + 1\right),
\]
\[
\text{Sp}\left(\frac{n + 1}{2} + 1\right) \times \text{SO}(2), \ \text{Sp}\left(\frac{n + 1}{2} + 1\right) \times \text{Sp}(1) \right\}
\]
for $n$ odd. This can be achieved as follows: We see that whenever we have a product of classical groups we may replace it by a simple classical group of the same rank and of larger dimension. The classical groups for which the ratio between dimension and rank is maximal are given by the groups of type $B$ and $C$. Moreover, the series $\dim B_n = \dim C_n$ is strictly increasing in $n$. We compute
\[
\dim \text{SO}(n + 3) \times \text{SO}(3) = \frac{n^2 + 5n + 12}{2}
\]
whilst
\[
\text{rk} \text{SO}(n + 3) \times \text{SO}(3) = \left\lfloor \frac{n + 3}{2} \right\rfloor + 1 = \left\lceil \frac{n}{2} \right\rceil + 2
\]
Consequently, by our assumption on the dimension of $G$ we need to find all the groups that are larger in dimension but not larger in rank than $\text{SO}(n + 3) \times \text{SO}(3)$. This process results in the list we gave.

We still need to see when there are groups $G$ that are larger in dimension than $\text{SO}(n + 3) \times \text{SO}(3)$ but not larger in rank and that have exceptional Lie groups as direct factors (up to finite coverings). Clearly, this can only happen in low dimensions. So we use lemma 6.4 and table 12 to see that unless $n \in \{27, 28\}$ there do no appear exceptional Lie groups as factors. As for degrees $n \in \{27, 28\}$ the group $E_8 \times E_8$ has dimension
\[
\dim(E_8 \times E_8) = 496 > \begin{cases} 
438 = \dim \text{SO}(30) \times \text{SO}(3) \\
468 = \dim \text{SO}(31) \times \text{SO}(3)
\end{cases}
\]
Therefore in these degrees we want to assume that $\dim \text{Isom}(M^{4n}) > 496$. This will make it impossible to identify the real Grassmannian, since $\dim \text{SO}(31) = 465$ and since $\dim \text{SO}(32) = 496$. Nonetheless the following arguments hold as well.

**Step 3.** We now prove that whenever $G$ is taken out of the list we gave, then actually $G = \text{SO}(n + 4)$ and $M \cong \text{Gr}_4(\mathbb{R}^{n+4})$. In order to establish this we shall have a closer look at orbits around a fixed-point of the maximal torus of $G$ for each respective possibility of $G$. (Such a point exists due to the Lefschetz fixed-point theorem and the fact that $\chi(M) > 0$—cf. 1.5.) This will lead to the observation that a potential action of $G$ has to be transitive, whence $M$ is homogeneous. Due to Alekseevski homogeneous Positive Quaternion Kähler Manifolds are Wolf spaces—cf. [4],14.56, p. 409.

Since $\dim M = 4n$, we necessarily obtain that the orbit $G/H$ of $G$ has dimension at most $\dim G/H \leq 4n$. Assume first that $G$ is a direct product.
from the list with $\text{SO}(n+4)$ as a factor (up to finite coverings). Thus all the maximal rank subgroups $H$ of $G$ satisfying $\dim G/H \leq 4n$ for $n \geq 22$, necessarily contain one of the groups

$$\text{SO}(n) \times \text{SO}(4), \text{SO}(n+3), \text{SO}(n+2) \times \text{SO}(2)$$

as a factor—which includes into $\text{SO}(n+4)$—due to table 6. (Note that whether $\text{SO}(n+3)$ is a maximal rank subgroup of $\text{SO}(n+4)$ or not depends on the parity of $n$ being odd or even.) By the same arguments we see that for $G = \text{SO}(n+5)$ only the following subgroups $H$ may appear:

$$\text{SO}(n+4), \text{SO}(n+3) \times \text{SO}(2), \text{SO}(n+2) \times \text{SO}(2), \text{SO}(n+2) \times \text{SO}(3)$$

For $G = \text{SO}(n+6)$ the group $H$ is out of the following list:

$$\text{SO}(n+5), \text{SO}(n+4) \times \text{SO}(2), \text{SO}(n+3) \times \text{SO}(3), \text{SO}(n+3) \times \text{SO}(2)$$

In any of the cases there has to be an inclusion $\text{SO}(n+k)$ for $k \geq 0$ into $\text{Sp}(n)$ by the isotropy representation—cf. [17], theorem VI.4.6.(2). By lemma 6.1, however, this is impossible unless $k = 0$. Thus we see that

$$G/H = \frac{\text{SO}(n+4)}{\text{SO}(n) \times \text{SO}(4)} = M$$

since the action of $G$ thus necessarily is transitive.

Now suppose that $G = \text{Sp}(\frac{n}{2} + 2)$ for $n$ even. Again we use table 6 to list maximal rank subgroups $H$ with $\dim G/H \leq 4n$:

$$\text{Sp}\left(\frac{n}{2} + 1\right) \times \text{Sp}(1), \text{Sp}\left(\frac{n}{2} + 1\right) \times \text{U}(1), \text{Sp}\left(\frac{n}{2}\right) \times \text{Sp}(2)$$

If $H = \text{Sp}\left(\frac{n}{2}\right) \times \text{Sp}(2)$, we see that $\dim G/H = 4n$ and that the action of $G$ is transitive. Thus $M$ is homogeneous and symmetric. Yet, by the classification of Wolf spaces we derive that

$$M = \frac{\text{Sp}\left(\frac{n}{2} + 2\right)}{\text{Sp}\left(\frac{n}{4}\right) \times \text{Sp}(2)}$$

cannot be the case.

Now apply lemma 6.2 in the other cases and derive that the isotropy representation of $H$ is marked by a blockwise included $\text{Sp}(\frac{n}{2} + 1)$ $\hookrightarrow \text{Sp}(n)$ for $k > 0$. This implies that the sphere

$$S^1 \times \{1\} \times (n/2) \times \{1\} \hookrightarrow T^{n/2+1} \hookrightarrow \text{Sp}\left(\frac{n}{2} + 1\right)$$

is represented by $S^1 \times \{1\} \times (n-1) \times \{1\}$ in the $\text{Sp}(n)$-factor of the holonomy group $\text{Sp}(n)\text{Sp}(1)$. Thus it fixes a quaternionic fixed-point component of codimension 4. Thus by theorem [10].1.2, p. 2, we obtain that $M^{4n} \in \{\mathbb{H}P^{4n}, \text{Gr}_2(\mathbb{C}^{n+2})\}$.

If $G = \text{Sp}\left(\frac{n+1}{2} + 2\right)$ and $n$ is odd, virtually the same arguments apply. That is, the list of isotropy subgroups $H$ is given by

$$\text{Sp}\left(\frac{n+1}{2} + 1\right) \times \text{Sp}(1), \text{Sp}\left(\frac{n+1}{2} + 1\right) \times \text{U}(1)$$

As above this leads to a codimension four quaternionic $S^1$-fixed-point component.
Finally, suppose $n$ to be odd and the group $G$ to contain a factor of the form $\text{Sp}(\frac{n+1}{2}+1)$. The list of possible stabilisers is given as

$$
\text{Sp}(\frac{n+1}{2}) \times \text{Sp}(1), \quad \text{Sp}(\frac{n+1}{2}) \times U(1), \quad \text{Sp}(\frac{n+1}{2} - 1) \times \text{Sp}(2),
$$

If $H = \text{Sp}(\frac{n+1}{2} - 1) \times \text{Sp}(1) \times \text{Sp}(1)$, the action of $G$ again is transitive which is impossible by the classification of Wolf spaces. If

$$
H \in \left\{ \text{Sp}(\frac{n+1}{2}) \times \text{Sp}(1), \quad \text{Sp}(\frac{n+1}{2}) \times U(1) \right\}
$$

we note that its $\text{Sp}(\frac{n+1}{2})$-factor again maps into $\text{Sp}(n)$ in a blockwise way—cf. lemma 6.2—by the isotropy representation. This leads to a codimension four quaternionic $S^1$-fixed point component once more. Now suppose

$$
H = \text{Sp}(\frac{n+1}{2} - 1) \times \text{Sp}(2)
$$

Then the holonomy representation necessarily makes $H$ a subgroup of $\text{Sp}(n)$. By lemma 6.3 this can only occur in the standard blockwise way. Again this yields a quaternionic codimension four $S^1$-fixed-point component which leads to $M \in \{ \mathbb{H}P^n, \text{Gr}_2(\mathbb{C}^{n+2}) \}$.

In degree $n = 21$ we see that similar arguments apply as for $n \geq 22$. However, we have that $\dim(\text{SO}(25)) = 300 < 303 = \dim E_6 \times F_4 \times \text{Sp}(1)$. Thus due to the assumption that $\dim \text{Isom}(M^{21}) > 303$ we may not identify the real Grassmannian $\tilde{\text{Gr}}_4(\mathbb{R}^{25})$ but only the quaternionic projective space and the complex Grassmannian. □

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