Classification of infinite-dimensional irreducible Hermitian-symmetric affine coadjoint orbits

Alice Barbara TUMPACH

Abstract

In the finite-dimensional setting, every Hermitian-symmetric space of compact type is a coadjoint orbit of a finite-dimensional Lie group. It is natural to ask whether every infinite-dimensional Hermitian-symmetric space of compact type, which is a particular example of an Hilbert manifold, is transitivity acted upon by a Hilbert Lie group of isometries. In this paper we give the classification of infinite-dimensional irreducible Hermitian-symmetric affine coadjoint orbits of \( L^* \)-groups of compact type using the notion of simple roots of non-compact type. The key step is, given an infinite-dimensional symmetric pair \((\mathfrak{g}, \mathfrak{t})\), where \( \mathfrak{g} \) is a simple \( L^* \)-algebra and \( \mathfrak{t} \) a subalgebra of \( \mathfrak{g} \), to construct an increasing sequence of finite-dimensional subalgebras \( \mathfrak{g}_n \) of \( \mathfrak{g} \) together with an increasing sequence of finite-dimensional subalgebras \( \mathfrak{t}_n \) of \( \mathfrak{t} \) such that \( \mathfrak{g} = \bigcup \mathfrak{g}_n \), \( \mathfrak{t} = \bigcup \mathfrak{t}_n \), and such that the pairs \((\mathfrak{g}_n, \mathfrak{t}_n)\) are symmetric. Comparing with the classification of Hermitian-symmetric spaces given by W. Kaup, it follows that any Hermitian-symmetric space of compact type is an affine-coadjoint orbit of an Hilbert Lie group.

1 Introduction

Let us introduce some notation. For any complex Hilbert space \( \mathcal{F} \) endowed with a distinguished basis \( \{ f_j, j \in J \} \), \( \mathcal{F}_\mathbb{R} \) will denote the real Hilbert space with basis \( \{ f_j, i f_j, j \in J \} \). The Hilbert space of Hilbert-Schmidt operators on \( \mathcal{F} \) will be denoted by \( L^2(\mathcal{F}) \), the Banach space of trace class operators on \( \mathcal{F} \) by \( L^1(\mathcal{F}) \). The group of invertible operators on \( \mathcal{F} \) will be denoted by \( GL(\mathcal{F}) \), and the group of unitary operators on \( \mathcal{F} \) by \( U(\mathcal{F}) \). In the sequel, \( \mathcal{H} \) will denote a separable complex Hilbert space endowed with an orthonormal basis \( \{ e_n, n \in \mathbb{Z} \setminus \{ 0 \} \} \).

The Hermitian scalar product on \( \mathcal{H} \) will be denoted by \( \langle \cdot, \cdot \rangle_\mathcal{H} \) and will be \( \mathbb{C} \)-linear with respect to the first variable, and \( \mathbb{C} \)-linear with respect to the second variable. For a bounded operator \( x \) on \( \mathcal{H} \), denote by \( x^T \) the transpose of \( x \) defined by \( \langle x^T e_i, e_j \rangle_\mathcal{H} = \langle x e_j, e_i \rangle_\mathcal{H} \), and by \( x^* \) the adjoint of \( x \) defined by \( \langle x^* e_i, e_j \rangle_\mathcal{H} = \langle e_i, x e_j \rangle_\mathcal{H} \). The closed infinite-dimensional subspace of \( \mathcal{H} \) generated by the \( e_n \)'s for \( n > 0 \) will be called \( \mathcal{H}_+ \), and its orthogonal \( \mathcal{H}_- \). For \( 0 < p < +\infty \), the \( p \)-dimensional subspace of \( \mathcal{H} \) generated by the \( e_n \)'s for \( 0 < n \leq p \) will be denoted \( \mathcal{H}_p \). Let \( J_0 \) be the bounded operator on \( \mathcal{H} \) defined by \( J_0 e_i = - e_{-i} \) if \( i < 0 \) and \( J_0 e_i = e_{-i} \) if \( i > 0 \). For \( \mathcal{F} = \mathcal{H}, \mathcal{H}_+, \mathcal{H}_p \), or \( \mathcal{H}_+^\perp \) define the following Hilbert Lie groups and the associated Lie algebras

\[
\begin{align*}
GL_2(\mathcal{F}) & := \{ g \in GL(\mathcal{F}) \mid g - i d \in L^2(\mathcal{F}) \}, & gl_2(\mathcal{F}) & := L^2(\mathcal{F}), \\
U_2(\mathcal{F}) & := \{ g \in U(\mathcal{F}) \mid g - i d \in L^2(\mathcal{F}) \}, & u_2(\mathcal{F}) & := \{ a \in L^2(\mathcal{F}) \mid a^* + a = 0 \}, \\
O_2(\mathcal{F}) & := \{ g \in U_2(\mathcal{F}) \mid g^T g = i d \}, & o_2(\mathcal{F}) & := \{ a \in L^2(\mathcal{F}) \mid a^T + a = 0 \}.
\end{align*}
\]

At last define

\[
Sp_2(\mathcal{H}) := \{ g \in U_2(\mathcal{H}) \mid g^T J_0 g = J_0 \}, & sp_2(\mathcal{H}) := \{ a \in L^2(\mathcal{H}) \mid a^T J_0 + J_0 a = 0 \}.
\]

On the Lie algebras \( \mathfrak{g} \) listed above, the bracket is the commutator of operators and the Hermitian product \( \langle \cdot, \cdot \rangle_\mathcal{H} \) is defined using the trace by

\[
\langle A, B \rangle := \text{Tr} A^* B.
\]
These Lie algebras are $L^*$-algebras in the sense that the following property is satisfied:

$$\langle [x, y], z \rangle = \langle y, [x^*, z] \rangle$$

for every $x, y$ and $z$. In fact, $u_2(\mathcal{H}), \phi_2(\mathcal{H})$ and $sp_2(\mathcal{H})$ are the unique infinite-dimensional simple $L^*$-algebras of compact type modulo isomorphisms (see below for the corresponding definition and [11], [6], or [17] for the proof of this statement). The group associated to an $L^*$-algebra is called an $L^*$-group. The $L^*$-groups $GL_2(\mathcal{H}), U_2(\mathcal{H})$ and $Sp_2(\mathcal{H})$ are connected, but $O_2(\mathbb{H})$ admits two connected components (see Proposition 12.4.2 on page 245 in [12]). The connected component of $O_2(\mathbb{H})$ containing the special orthogonal group

$$SO_1(\mathbb{H}) := \{ g \in O_2(\mathbb{H}) \mid g = \text{id} \in L^1(\mathcal{H}), \det(g) = 1 \}$$

will be denoted by $O^+_{2}(\mathcal{H})$. The aim of this paper is to prove the following statement.

**Theorem 1.1** Every infinite-dimensional affine Hermitian-symmetric irreducible coadjoint orbit of a simple $L^*$-group of compact type is isomorphic to one of the following homogeneous space:

1. the Grassmannian $Gr^{(p)} = U_2(\mathcal{H})/(U_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp))$ of $p$-dimensional subspaces of $\mathcal{H}$ with $\dim(\mathcal{H}_p) = p < +\infty$

2. the connected component of the restricted Grassmannian $Gr^{0}_{\text{res}} = U_2(\mathcal{H})/(U_2(\mathcal{H}_+) \times U_2(\mathcal{H}_-))$ of the polarized Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\dim \mathcal{H}_+ = \dim \mathcal{H}_- = +\infty$

3. the Grassmannian $Gr^{(2)} = O^+_{2}(\mathcal{H})/((SO((\mathcal{H}_2)_0) \times O^+_{2}(\mathcal{H}_2)_0)))$ of oriented 2-planes in $\mathcal{H}_2$.

4. the Grassmannian $Z(\mathcal{H}) = O^+_{2}(\mathcal{H})/U_2(\mathcal{H})$ of orientation-preserving orthogonal complex structures close to the distinguished complex structure on $\mathcal{H}$.

5. the Grassmannian $L(\mathcal{H}) = Sp_2(\mathcal{H})/U_2(\mathcal{H}_+)$ of Lagrangian subspaces close to $\mathcal{H}_+$.

In the finite-dimensional case, every Hermitian-symmetric space of compact type is a coadjoint orbit of its connected group of isometries (see Proposition 8.89 in [3]). In the infinite-dimensional setting, the biggest group of isometries of a given Hermitian-symmetric space is not a Hilbert Lie group in general. For example the restricted unitary group $U_{\text{res}}(\mathcal{H})$ (see [12] for its definition) is a Banach Lie group acting by isometries on the restricted Grassmannian. It is a non trivial fact that the unitary Hilbert Lie group $U_2(\mathcal{H})$, strictly contained in $U_{\text{res}}(\mathcal{H})$, acts transitively on each connected component of the restricted Grassmannian (see Proposition 5.2 in [24]). Theorem 1.1 above compared to the work of W. Kaup ([8], [9]), leads to the following generalization:

**Corollary 1.2** Every Hermitian-symmetric space of compact type is an homogeneous space of an Hilbert Lie group. More precisely, every Hermitian-symmetric space of compact type is an affine-coadjoint orbit of an $L^*$-group.

2 Root Theory of complex $L^*$-algebra

The root theory of complex $L^*$-algebras has been developed by J. R. Schue in [13] and [14]. Let us first recall that an $L^*$-algebra $\mathfrak{g}$ over $\mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \}$ is a Lie algebra over $\mathbb{K}$, which is also a Hilbert space over $\mathbb{K}$ such that for every element $x \in \mathfrak{g}$, there exists $x^* \in \mathfrak{g}$ with the following property

$$\langle [x, y], z \rangle = \langle y, [x^*, z] \rangle,$$

for every $y, z$ in $\mathfrak{g}$. In the case when $\mathbb{K} = \mathbb{C}$, our convention for the Hermitian product $\langle \cdot, \cdot \rangle$ is that it is $\mathbb{C}$-skew-linear with respect to the first variable, and $\mathbb{C}$-linear with respect to the second variable. The first example of $L^*$-algebra is the semi-simple finite-dimensional complex Lie algebra $\mathfrak{g}_0$ endowed with an involution $\sigma$, which defines a compact real form of $\mathfrak{g}_0$. In this example, the involutions $*$ and $\sigma$ are related by $x^* = -\sigma(x)$ and the Hermitian scalar product is given by $\langle x, y \rangle = B(x^*, y)$, where $B$ denotes the Killing form of $\mathfrak{g}_0$. An $L^*$-algebra is called of compact type if $x^* = -x$ for every $x$ in $\mathfrak{g}$, and for a given $L^*$-algebra $\mathfrak{g}$ the subspace

$$\mathfrak{f} := \{ x \in \mathfrak{g} \mid x^* = -x \}$$

is a real $L^*$-algebra of compact type. Thus an $L^*$-algebra can be thought as an Hilbert Lie algebra together with a distinguished compact real form.
For every subsets $A$ and $B$ of an $L^*$-algebra $\mathfrak{g}$, $[A, B]$ will denote the \textit{closure} of the vector space spanned by $\{[a, b], a \in A, b \in B\}$. With this notation, an $L^*$-algebra is called semi-simple if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and simple if $\mathfrak{g}$ is non-commutative and if every closed ideal of $\mathfrak{g}$ is trivial. Every $L^*$-algebra can be decomposed into an orthogonal sum of its center and a semi-simple closed ideal (see [13], 2.2.13). A Cartan subalgebra of a complex semi-simple $L^*$-algebra $\mathfrak{g}^R$ is defined as a maximal Abelian $*$-stable subalgebra of $\mathfrak{g}^R$. Note that the condition of being $*$-stable is added in comparison to the finite-dimensional setting, hence a Cartan subalgebra may not be maximal in the set of Abelian subalgebras. It is noteworthy that a Cartan subalgebra of an $L^*$-algebra is in fact maximal Abelian (see [14], 1.1).

Remark that a finite-dimensional Cartan subalgebra of a complex semi-simple Lie algebra $\mathfrak{g}$ (for the usual definition) is contained in a compact real form of $\mathfrak{g}$, thus is also a Cartan subalgebra of the corresponding finite-dimensional $L^*$-algebra. The existence of Cartan subalgebras of $L^*$-algebras is guarantied by Zorn’s Lemma. Every semi-simple $L^*$-algebra is an Hilbert sum of closed $*$-stable simple ideals (see Theorem 1 in [13] for the initial proof. Some missing arguments can be found in [18]).

In the sequel, $\mathfrak{g}^C$ will denote a semi-simple complex $L^*$-algebra and $\mathfrak{h}^C$ a Cartan subalgebra of $\mathfrak{g}^C$. A root of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$ is defined, as in the finite dimensional case, as an element $\alpha$ in the dual of $\mathfrak{h}^C$ such that the corresponding “eigenspace”

$$V_{\alpha} := \{v \in \mathfrak{g}^C \mid \forall h \in \mathfrak{h}^C, [h, v] = \alpha(h)v\}.$$ 

is non-empty. In the following the set of non-zero roots with respect to a given Cartan subalgebra will be denoted by $\mathcal{R}$. Let us remark that a root has operator norm less than 1 and that for a non-zero root $\alpha$, the vector space $V_{\alpha}$ is one-dimensional (see [13]). The Jacobi identity implies that

$$[V_{\alpha}, V_{\beta}] \subset V_{\alpha+\beta}. \tag{2}$$

By relation (1), $V^*_\alpha = V_{-\alpha}$. The main achievement in [13] is to prove that a semi-simple complex $L^*$-algebra $\mathfrak{g}^C$ admits a Cartan Decomposition with respect to a given Cartan subalgebra $\mathfrak{h}^C$ in the sense that $\mathfrak{g}^C$ is the Hilbert sum

$$\mathfrak{g}^C = \mathfrak{h}^C \oplus \sum_{\alpha \in \mathcal{R}} V_{\alpha}. \tag{3}$$

Let us remark that in a separable $L^*$-algebra, the set of root is countable or finite.

By Zorn’s Lemma, one can decompose the set $\mathcal{R}$ of non-zero roots into two disjoint subsets $\mathcal{R}_+$ and $\mathcal{R}_-$ such that $\alpha \in \mathcal{R}_+ \Leftrightarrow -\alpha \in \mathcal{R}_-$. Such a decomposition defines a strict partial ordering on $\mathcal{R}$ by

$$\alpha > \beta \Leftrightarrow \alpha - \beta > 0.$$ 

In the sequel, a decomposition $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ as before and the induced ordering on the set of non-zero roots will be identified. The elements in $\mathcal{R}_+$ will be called \textit{positive} roots.

For every positive root $\alpha$, one can choose $e_\alpha \in V_{\alpha}$ such that $\|e_\alpha\| = 1$. Then $e_{-\alpha}^* \in V_{-\alpha}$ and $\|e_{-\alpha}^*\| = 1$. This choice made, we define $e_\alpha := e_{-\alpha}^*$ for $\alpha \in \mathcal{R}_-$, in order to have, for every $\alpha \in \mathcal{R}$, the following relation $e_\alpha^* = e_{-\alpha}$. By (3), the set $\{e_\alpha, \alpha \in \mathcal{R}\}$ is an Hilbert basis of $(\mathfrak{h}^C)\perp$, and by (2), $[e_\alpha, e_\alpha^*]$ belongs to $\mathfrak{h}^C$. We define the following elements in the Cartan subalgebra $\mathfrak{h}^C$:

$$h_\alpha := [e_\alpha, e_\alpha^*]. \tag{4}$$

A positive root is called \textit{simple} if it can not be written as the sum of two positive roots. The set of simple roots will be denoted by $\mathcal{S}$. A subset $\mathcal{N}$ of the set of non-zero roots $\mathcal{R}$ is called a \textit{root system}, if it satisfies the following conditions:

1. $\alpha \in \mathcal{N} \Rightarrow -\alpha \in \mathcal{N}$,
2. $(\alpha, \beta \in \mathcal{N} \text{ and } \alpha + \beta \in \mathcal{R}) \Rightarrow \alpha + \beta \in \mathcal{N}$.

A subset $\mathcal{N} \subset \mathcal{R}$ is called \textit{indecomposable} if it can not be written as the union of two orthogonal non-empty subsets. As in the classical theory, one has the following facts. The set $\mathcal{R}$ of non-zero roots of a simple $L^*$-algebra is indecomposable. If $F$ is an indecomposable subset of the set of non-zero roots $\mathcal{R}$, then it generates a root system $\mathcal{N}_F$, which is again indecomposable. The simple $L^*$-algebra generated by $\{e_\alpha, \alpha \in \mathcal{N}_F\}$ will be denoted by $\mathfrak{g}(\mathcal{N}_F)$.

For the classification of Hermitian-symmetric affine coadjoint orbits given in next section, we will need the following results. They were proved by J. R. Schue in [13] in order to classify the complex simple infinite-dimensional $L^*$-algebras.
Proposition 2.1 ([13]) For every finite subset $F$ of the set of non-zero roots $\mathcal{R}$ of a simple $L^*$-algebra, there exists a finite indecomposable system of non-zero roots containing $F$.

Theorem 2.2 ([13], 3.2) Let $\mathfrak{g}_C$ be a simple complex separable $L^*$-algebra and $\mathcal{R} = \{\alpha_i, i \in \mathbb{N} \setminus \{0\}\}$ the set of non-zero roots with respect to a given Cartan subalgebra of $\mathfrak{g}_C$. For every $n \in \mathbb{N} \setminus \{0\}$, set $F_n := \{\alpha_1, \ldots, \alpha_n\}$. Then there exists a sequence $\{N_n\}_{n \in \mathbb{N} \setminus \{0\}}$ of finite subsets of $\mathcal{R}$ such that

1. $F_n \subset N_n \subset N_{n+1}$;
2. $N_n$ is an indecomposable root system;
3. $\mathcal{R} = \cup_{n \in \mathbb{N} \setminus \{0\}} N_n$
4. the simple subalgebras $\mathfrak{g}(N_n)$ generated by $N_n$ form a strictly increasing sequence with

$$\mathfrak{g}^C_C = \cup_{n \in \mathbb{N} \setminus \{0\}} \mathfrak{g}(N_n);$$
5. The simple complex finite-dimensional algebras $\mathfrak{g}(N_n)$ are of the same Cartan type $A$, $B$, $C$, or $D$.

Proposition 2.3 ([13], 3.2) Given a sequence $\{N_n\}_{n \in \mathbb{N} \setminus \{0\}}$ as in the previous Theorem, it is possible to define a total ordering on the vector space generated by the set of roots such that:

1. $\alpha > 0 \Rightarrow -\alpha < 0$;
2. $\alpha > 0, \beta > 0 \Rightarrow \alpha + \beta > 0$;
3. If $\alpha > 0$ and $\alpha \notin N_n$ then $\alpha > \beta$ for all $\beta \in N_n$;
4. The induced ordering on $N_n$ is a lexicographical ordering with respect to a basis of roots.

Proposition 2.4 ([13], 3.3) Let $S$ be the set of simple roots of $\mathfrak{g}_C$ with respect to the ordering defined in the previous Proposition. The following assertions hold:

1. If $S \cap N_n$ is a complete system of simple roots of the finite-dimensional algebra $\mathfrak{g}(N_n)$, i.e., every positive root $\alpha$ of $N_n$ can be written as a linear combination of elements in $S \cap N_n$ with non-negative integral coefficients;
2. If $\alpha$ and $\beta$ belong to $S$, $\alpha - \beta$ is a root if and only if $\alpha = \beta$;
3. The elements in $S$ are linearly independent on the reals and every positive root $\alpha \in \mathcal{R}_+$ is a linear combination of elements in $S$ with non-negative integral coefficients which are almost all zero.

3 Classification of irreducible Hermitian-symmetric affine coadjoint orbits

Affine coadjoint orbits have been introduced in particular by K.-H. Neeb in [10]. The classification of finite-dimensional Hermitian-symmetric coadjoint orbits using the notion of roots of non-compact type has been carried out by J. A. Wolf in [19]. In the sequel, $\mathfrak{g}$ will denote an infinite-dimensional separable simple $L^*$-algebra of compact type. According to [1], [3] or [17], it can be realized as a subalgebra of the $L^*$-algebra $GL_2(\mathcal{H})$ consisting of Hilbert-Schmidt operators on a separable complex Hilbert space $\mathcal{H}$. Let $G$ be the connected $L^*$-group with Lie algebra $\mathfrak{g}$, and $G^C := \mathfrak{g} \oplus ig$. By the duality $\mathfrak{g}_C = \mathfrak{g}$ given by the trace, we can identify affine adjoint and affine coadjoint orbits of $G$. Let $\mathcal{D}$ be a derivation of $\mathfrak{g}$ such that the affine (co-)adjoint orbit $\mathcal{O}$ of $0$ in $\mathfrak{g}$ associated to the affine adjoint action of $G$ defined by $\mathcal{D}$ is strongly Kähler (see [10]). By Theorem 4.4 in [10], there exists $D \in B(\mathcal{H})$ satisfying $D^* = -D$ such that for all $x$ in $\mathfrak{g}$, $\mathcal{D}x = [D, x]$, as well as a Cartan subalgebra $\mathcal{H}$ of $\mathfrak{g}^C$ which is contained in $\ker \mathcal{D}$. To emphasize the relation between the orbit and the bounded operator $D$, we will often write $\mathcal{O} = \mathcal{O}_D$. Abusing slightly the notation, we will sometimes denote $\mathcal{D}$ by $\text{ad}(D)$. An alternative definition of $\mathcal{O}_D$ is

$$\mathcal{O}_D = \{gDg^{-1} - D, g \in G\},$$
and the affine adjoint action of $G$ on $\mathfrak{g}$ is given by

$$g \cdot a = \text{Ad}(g)(a) + gDg^{-1} - D$$
where \( g \in G \) and \( a \in \mathfrak{g} \). The subalgebra \( \mathfrak{k} \) of \( \mathfrak{g} \) which fixes 0 is
\[
\mathfrak{k} := \{ x \in \mathfrak{g}, [D, x] = 0 \}.
\]
It is an \( L^* \)-subalgebra of \( \mathfrak{g} \). Let \( K \) be the associated connected \( L^* \)-group.

The affine (co-)adjoint orbit \( O_D \) is called (locally-)symmetric if the orthogonal \( \mathfrak{m} \) of \( \mathfrak{k} \) in \( \mathfrak{g} \) satisfies
\[
(\mathfrak{m}, \mathfrak{m}) \subset \mathfrak{k}.
\]
The strongly Kähler orbit \( O_D \) is called Hermitian-symmetric if, for every \( x \in O_D \), there exists a globally defined isometry \( s_x \) (the symmetry with respect to \( x \)) preserving the complex structure, such that \( x \) is a fixed point of \( s_x \), and such that the differential of \( s_x \) at \( x \) is minus the identity of \( T_x O_D \). An Hermitian-symmetric orbit is (locally-)symmetric. In the following, we will be interested in Hermitian-symmetric orbits \( O_D \). Let \( \mathfrak{f}^c \) and \( \mathfrak{m}^c \) denote the complexifications of \( \mathfrak{f} \) and \( \mathfrak{m} \) respectively. Note that \( \mathfrak{g}^c \) is the orthogonal sum of \( \mathfrak{f}^c \) and \( \mathfrak{m}^c \) with respect to the Hermitian product of the \( L^* \)-algebra \( \mathfrak{g}^c \).

**Proposition 3.1** Let \( \mathfrak{h}^c \) be a Cartan subalgebra of \( \mathfrak{g}^c \) that is contained in \( \ker \text{ad} D \), and let
\[
\mathfrak{g}^c = \mathfrak{h}^c \oplus \sum_{\alpha \in \mathcal{R}} V_\alpha
\]
be the associated Cartan decomposition of \( \mathfrak{g}^c \), where \( \mathcal{R} \) denotes the set of non-zero roots with respect to \( \mathfrak{h}^c \). Suppose that \( O_D \) is Hermitian-symmetric. Then there exists two subsets \( A \) and \( B \) of \( \mathcal{R} \) such that
\[
A \cup B = \mathcal{R} \quad \text{and} \quad \mathfrak{f}^c = \mathfrak{h}^c \oplus \sum_{\alpha \in A} V_\alpha, \quad \mathfrak{m}^c = \sum_{\alpha \in B} V_\alpha.
\]

\[\square\text{ Proof of Proposition 3.1} :\]
Since \( O_D \) is (locally-)symmetric, one has \( \mathfrak{g}^c = \mathfrak{f}^c \oplus \mathfrak{m}^c \) with
\[
[\mathfrak{f}^c, \mathfrak{f}^c] \subset \mathfrak{f}^c ; \quad [\mathfrak{f}^c, \mathfrak{m}^c] \subset \mathfrak{m}^c ; \quad [\mathfrak{m}^c, \mathfrak{m}^c] \subset \mathfrak{f}^c.
\]
Let \( v \) be a non-zero vector in \( V_\alpha \), and \( v = v_0 + v_1 \) his decomposition with respect to the direct sum \( \mathfrak{g}^c = \mathfrak{f}^c \oplus \mathfrak{m}^c \). For every \( h \in \mathfrak{h}^c \), one has
\[
[h, v] = [h, v_0 + v_1] = \alpha(h)(v_0 + v_1) = \alpha(h)v_0 + \alpha(h)v_1 = [h, v_0] + [h, v_1].
\]
Since \( [\mathfrak{h}^c, \mathfrak{f}^c] \subset \mathfrak{f}^c \) and \( [\mathfrak{h}^c, \mathfrak{m}^c] \subset \mathfrak{m}^c \), it follows that
\[
[h, v_0] = \alpha(h)v_0 \quad \text{et} \quad [h, v_1] = \alpha(h)v_1.
\]
But \( V_\alpha \) is one-dimensional, hence either \( v_0 = 0 \), or \( v_1 = 0 \). Consequently \( V_\alpha \) is contained either in \( \mathfrak{f}^c \) or in \( \mathfrak{m}^c \). \[\square\]

**Proposition 3.2** For every \( \alpha \in \mathcal{R} \), there exists a constant \( c_\alpha \in \mathbb{R} \) such that \( [D, e_\alpha] = ic_\alpha e_\alpha \). Moreover \( c_{-\alpha} = -c_\alpha \).

\[\square\text{ Proof of Proposition 3.2} :\]
For every \( \alpha \in \mathcal{R} \) and every \( h \in \mathfrak{h}^c \), one has
\[
[h, [D, e_\alpha]] = [[h, D], e_\alpha] + [D, [h, e_\alpha]] = \alpha(h) [D, e_\alpha].
\]
The space \( V_\alpha \) being one-dimensional, it follows that \( [D, e_\alpha] \) is proportional to \( e_\alpha \). Since \( D \) satisfies \( D^* = -D \), one has, for every \( \alpha \in \mathcal{R} \), the following relation
\[
\langle [D, e_\alpha], e_\alpha \rangle = -\langle e_\alpha, [D, e_\alpha] \rangle = -\langle [D, e_\alpha], e_\alpha \rangle.
\]
Thus there exists a real constant \( c_\alpha \) such that
\[
[D, e_\alpha] = ic_\alpha e_\alpha
\]
On the other hand,
\[
[D, e_\alpha]^* = [e_\alpha^*, D^*] = -[e_\alpha^*, D] = [D, e_\alpha^*].
\]
Whence
\[
\langle [D, e_\alpha^*], e_\alpha \rangle = \langle e_\alpha, [D, e_\alpha^*] \rangle = \langle e_\alpha, [D, e_\alpha] \rangle = ic_\alpha.
\]
Consequently \( [D, e_\alpha^*] = -ic_\alpha e_\alpha \). \[\square\]
Remark 3.3 Let us denote by \( m_+ \) (resp. \( m_- \)) the closed subspace of \( \mathfrak{g}^C \) generated by the \( e_\alpha \)'s, where \( \alpha \) runs over the set of roots for which \( e_\alpha > 0 \) (resp. \( e_\alpha < 0 \)). Let \( \mathcal{B}_+ \) (resp. \( \mathcal{B}_- \)) be the set of roots \( \beta \) in \( \mathcal{B} \) such that \( V_\beta \in m_+ \) (resp. \( V_\beta \in m_- \)).

Definition 3.4 The affine adjoint orbit \( \mathcal{O}_D \) is called (isotropy-)irreducible if \( m \in \) a non-zero irreducible \( \text{Ad}(K) \)-module.

Proposition 3.5 If the affine adjoint orbit \( \mathcal{O}_D \) is irreducible, then \( m_+ \) and \( m_- \) are irreducible \( \text{Ad}(K) \)-modules, and there exists a constant \( c > 0 \) such that \( \text{ad}(D)|_{m_+} = ic \text{id}|_{m_+} \) and \( \text{ad}(D)|_{m_-} = -ic \text{id}|_{m_-} \). In particular, the spectrum of \( \text{ad}(D) \) is \( \{0, ic, -ic\} \), hence \( D \) admits exactly two distinct eigenvalues.

\[ \square \text{ Proof of Proposition 3.5.} \]

For every \( k \in \mathfrak{k} \) and every \( e_\alpha \in m_\pm \), one has

\[ [D, [k, e_\alpha]] = [[D, k], e_\alpha] + [k, [D, e_\alpha]] = ic_\alpha[k, e_\alpha]. \]

It follows that \([\mathfrak{t}, m_\pm] \subset m_\pm \) and that \( m_\pm \) is stable under the adjoint action of \( K \). Let us suppose that \( m_+ \) decomposes into a sum of two non-zero \( \text{Ad}(K) \)-modules \( m_1 \) and \( m_2 \).

\[ m_- = m_1^\ast \oplus m_2^\ast, \]

and it follows that \( m \) decomposes also into the sum of two non-zero \( \text{Ad}(K) \)-modules, namely \( \mathfrak{g} \cap (m_1 \oplus m_1^\ast) \) and \( \mathfrak{g} \cap (m_2 \oplus m_2^\ast) \). The orbit \( \mathcal{O}_D \) being irreducible, \( m \) is an irreducible \( \text{Ad}(K) \)-module and this leads to a contradiction. So the irreducibility of \( m_\pm \) is proved. Let \( e_\alpha \) be an element in \( m_+ \) and set \( c = e_\alpha : [D, e_\alpha] = ic e_\alpha. \)

The kernel \( \ker(D - ic) \) being an \( \text{Ad}(K) \)-module of \( m_+ \), one has \( \text{ad}(D)|_{m_+} = ic \text{id}|_{m_+} \). The relation \( e_{-\alpha} = -c e_\alpha \) implies that \( \text{ad}(D)|_{m_-} = -ic \text{id}|_{m_-} \). \( \square \)

Definition 3.6 Given an ordering on the set of non-zero roots \( \mathcal{R} \) of \( \mathfrak{g}^C \), a simple roots \( \phi \) is called of non-compact type (see [16]) if every root \( \alpha \in \mathcal{R} \) is of the form

\[ \alpha = \pm \sum_{\psi \in \mathcal{S} - \{\phi\}} a_\psi \psi, a_\psi \geq 0, \]

or of the form

\[ \alpha = \pm \left( \phi + \sum_{\psi \in \mathcal{S} - \{\phi\}} a_\psi \psi \right), a_\psi \geq 0. \]

Lemma 3.7 Let \( \mathcal{O}_D \) be a Hermitian-symmetric affine adjoint irreducible orbit of a simple \( L^* \)-algebra \( \mathfrak{g}, \mathfrak{h}^C \) be a Cartan subalgebra of \( \mathfrak{g}^C \) contained in \( \ker \text{ad}D \), and

\[ \mathfrak{g}^C = \mathfrak{h}^C \oplus \sum_{\alpha \in \mathcal{A}} V_\alpha \oplus \sum_{\beta \in \mathcal{B}} V_\beta \]

be the associated Cartan decomposition of \( \mathfrak{g}^C \) with

\[ \mathfrak{t}^C = \mathfrak{h}^C \oplus \sum_{\alpha \in \mathcal{A}} V_\alpha, \text{ and } \mathfrak{m}^C = \sum_{\beta \in \mathcal{B}} V_\beta. \]

For every ordering \( \mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_- \) on the set of roots, there exists a unique simple root \( \phi \) belonging to \( \mathcal{B} \).

\[ \triangle \text{ Proof of Lemma 3.7.} \]

Let \( \{\phi_i, \psi_j\}_{i \in I, j \in J} \) be the set of simple roots with \( \phi_i \in \mathcal{B} \) and \( \psi_j \in \mathcal{A} \). Let us suppose that \( I \) is empty. The relation \([\mathfrak{t}^C, \mathfrak{t}^C] \subset \mathfrak{t}^C \) implies that every positive root belongs to \( \mathcal{A} \) and consequently \( m = \{0\} \), which contradicts the hypothesis that \( m \) is a non-zero irreducible \( \text{Ad}(K) \)-module. Let \( \phi \) be a simple root in \( \mathcal{B} \). The closed vector space spanned by the adjoint action of \( \mathfrak{k} \) on \( e_\phi \) is a non-zero irreducible \( \text{Ad}(K) \)-module of \( \mathfrak{m}^C \). It follows that \( \phi \) is necessarily unique. \( \triangle \)
Lemma 3.8 Under the hypothesis of Lemma 3.7, there exists an increasing sequence of finite indecomposable root systems \( \mathcal{N}_n \) such that

1. \( \mathcal{R} = \bigcup_{n \in \mathbb{N}} \mathcal{N}_n \);
2. all the finite-dimensional subalgebras \( \mathfrak{g}(\mathcal{N}_n) \) generated by \( \mathcal{N}_n \) belong to the same type \( A, B, C, \) or \( D \) and \( \mathfrak{g}^0 \) is the closure of the union of the subalgebras \( \mathfrak{g}(\mathcal{N}_n) \);
3. \( \phi \) is a simple root of non-compact type for each subalgebra \( \mathfrak{g}(\mathcal{N}_n) \) with respect to the ordering on the roots of \( \mathfrak{g}(\mathcal{N}_n) \) induced by the ordering on \( \mathcal{R} \) defined in Proposition 3.9.

**Proof of Lemma 3.8**

Let \( \{\alpha_1, \ldots, \alpha_n, \ldots\} \) be a numbering of the roots in \( \mathcal{A} \). Set \( F_n = \{\alpha_1, \ldots, \alpha_n\} \). Let us construct by induction an increasing sequence of finite indecomposable root systems \( \mathcal{N}_n \) as follows. By Proposition 2.1, there exists a finite indecomposable root system \( \mathcal{N}_1 \) containing \( \{\phi\} \cup F_1 \). Suppose that \( \mathcal{N}_{n-1} \) is constructed, then there exists a finite indecomposable root system \( \mathcal{N}_n \) containing \( F_n \cup \mathcal{N}_{n-1} \). Since every root in \( \mathcal{B} \) is the sum of \( \phi \) and roots in \( \mathcal{A} \), \( \mathcal{R} = \bigcup_{n \in \mathbb{N}} \mathcal{N}_n \). The sequence of finite-dimensional simple subalgebras \( \mathfrak{g}(\mathcal{N}_n) \) generated by the root systems \( \mathcal{N}_n \) is increasing and such that \( \mathfrak{g}^0 = \bigcup_{n \in \mathbb{N}} \mathfrak{g}(\mathcal{N}_n) \).

Since there exists only 9 types of finite-dimensional simple algebras, at least one type occurs an infinite number of times. Since \( \mathfrak{g}^0 \) is infinite-dimensional and since only the types \( A, B, C, \) or \( D \) corresponds to algebras of arbitrary dimension, at least one of the types \( A, B, C, \) or \( D \) occurs an infinite number of times. It follows that there exists a subsequence \( \mathcal{N}_{n_k} \) of \( \mathcal{N}_n \) such that all the subalgebras \( \mathfrak{g}(\mathcal{N}_{n_k}) \) are of the same type \( A, B, C, \) or \( D \). Let \( S_{n_k} \) be the set of simple roots of \( \mathfrak{g}(\mathcal{N}_{n_k}) \) with respect to the ordering induced by the ordering on \( \mathcal{R} \) defined in Proposition 3.9. By Proposition 3.8, \( S_{n_k} = S \cap \mathfrak{g}(\mathcal{N}_{n_k}) \), where \( S \) is the set of simple roots of \( \mathfrak{g}^0 \). For every positive root \( \gamma \) in \( \mathcal{N}_{n_k} \), there exists a finite sequence \( \{\gamma_i, i = 1, \ldots, k\} \) of roots in \( S_{n_k} \) such that

\[
\gamma = \gamma_1 + \cdots + \gamma_k,
\]
and such that the partial sums \( \gamma_1 + \cdots + \gamma_j, 1 \leq j \leq k \) are roots (see [4]). Hence the vector space \( V_\gamma \) is generated by

\[
v = [e_{\gamma_1}, e_{\gamma_1 - \gamma_2}, \ldots, e_{\gamma_1 - \cdots - \gamma_k}].
\]

The orbit \( \mathcal{O}_\gamma \) being irreducible, \( [D, e_\phi] = \epsilon_\phi ie_\phi \) with \( \epsilon_\phi = +1 \) (resp. \( -1 \)) if \( V_\phi \subset \mathfrak{m}_+ \) (resp. \( \mathfrak{m}_- \)). Whence

\[
[D, v] = \operatorname{card} \{i, \gamma_i = \phi\} \epsilon_\phi iv.
\]
Since \( \operatorname{ad}(D) \) preserves \( \mathfrak{g}^0, \mathfrak{m}_+ \) and \( \mathfrak{m}_- \), it follows that for \( \gamma \) in \( \mathcal{A} \cap \mathcal{R}_+ \), \( \operatorname{card} \{i, \gamma_i = \phi\} = 0 \) and for \( \gamma \) in \( \mathcal{B} \cap \mathcal{R}_+ \), \( \operatorname{card} \{i, \gamma_i = \phi\} = 1 \). Consequently \( \phi \) is of non-compact type.

**Proposition 3.9** Let \( \mathcal{O} = G/K \) be a Hermitian-symmetric irreducible affine coadjoint orbit of an \( L^* \)-group \( G \) of compact type, and \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) the associated decomposition of the Lie algebra \( \mathfrak{g} \) of \( G \), where \( \mathfrak{k} \) is the Lie algebra of the isometry group \( K \). Then there exists an increasing sequence of finite-dimensional subalgebras \( \mathfrak{g}_n \) of \( \mathfrak{g} \) of the same type \( A, B, C \) or \( D \), and an increasing sequence of subalgebras \( \mathfrak{t}_n \) of \( \mathfrak{k} \) such that

1. \( \mathfrak{g} = \bigcup_{n \in \mathbb{N}} \mathfrak{g}_n \)
2. \( \mathfrak{k} = \bigcup_{n \in \mathbb{N}} \mathfrak{k}_n \)
3. for every \( n \in \mathbb{N} \setminus \{0\} \), the orthogonal \( \mathfrak{m}_n \) of \( \mathfrak{t}_n \) in \( \mathfrak{g}_n \) satisfies

\[
[\mathfrak{t}_n, \mathfrak{m}_n] \subset \mathfrak{m}_n \quad \text{and} \quad [\mathfrak{m}_n, \mathfrak{m}_n] \subset \mathfrak{t}_n,
\]

hence \( (\mathfrak{g}_n, \mathfrak{t}_n) \) is a symmetric pair.

**Proof of Proposition 3.9**

This is a direct consequence of Lemma 3.8 with \( \mathfrak{g}_n = \mathfrak{g} \cap \mathfrak{g}(\mathcal{N}_n) \) and \( \mathfrak{t}_n = \mathfrak{k} \cap \mathfrak{g}(\mathcal{N}_n) \).

From the discussion above it follows that the classification of Hermitian-symmetric irreducible affine coadjoint orbits of \( L^* \)-groups of compact type can be deduced from the knowledge of the simple roots of non-compact type of finite-dimensional simple algebras (see the proof of Theorem 1.1 below). A simple root of a simple finite-dimensional algebra is of non-compact type if and only if it appears with the coefficient +1 in the expression of the greatest root. We recall the list of simple roots of non-compact type in the finite-dimensional Lie algebras of type \( A, B, C \), or \( D \) in tabular [6] (see [7] or [10]).
and D.

Simple roots of non-compact type in the simple finite-dimensional Lie algebras of type A, B, C and D. identified as complex manifold with the conic

onto

H

D

bounded operators

orthogonal projection onto

By Lemma 3.7, there exists a unique simple root \( \phi \)

one-parameter family of Kähler structure (encoded by \((l, g)\)) for the derivations defined by the bounded operators \( D^{(p)}_{k,l} = ikp_{H_p} - ilp_{H_{-l}} \), where \( k, l \in \mathbb{R}, k \neq -l \), and \( p_{H_p} \) (resp. \( p_{H_{-l}} \)) is the orthogonal projection onto \( H_p \) (resp. \( H_{-l} \)). The homogeneous space \( G \) is therefore endowed with a one-parameter family of Kähler structure (encoded by \((k + l)\)). The derivation \( D^{(p)}_{k,l} \) is inner if and only if \( l = 0 \). For \( p = 1 \), \( G^{(p)} \) is the projective space of \( H \).

The restricted Grassmannian \( G_{\text{res}} \) has been studied in [12] and [22]. The connected component \( G_{\text{res}} \) containing \( H_+ \) is the affine adjoint orbit of \( U_2(\mathbb{H}) \) for the derivations defined by the bounded operators \( D^{(p)}_{k,l} = ikp_{H_p} - ilp_{H_{-l}} \), where \( k, l \in \mathbb{R}, k \neq -l \), and \( p_{H_p} \) (resp. \( p_{H_{-l}} \)) is the orthogonal projection onto \( H_+ \). None of these derivations is inner.

The Grassmannian \( G_{\text{res}}^{(2)} = O^2_2(\mathbb{H}) / (SO(2) \times O^2_2((\mathbb{H})_2)) \) of oriented 2-planes in \( \mathbb{H} \) is the \( O^2_2(\mathbb{H}) \)-adjoint orbit of \( kJ \) where \( k \neq 0 \) and \( J \) is the natural complex structure on \( (\mathbb{H})_2 \). It can be identified as complex manifold with the conic \( C \) in the complex projective space \( P(\mathcal{H}) \) defined by

\[
C := \{ (z_1: \ldots : z_n: \ldots) \in P(\mathcal{H}), \sum_{i \in \mathbb{N}} z_i^2 = 0 \}.
\]

Denote by \( \langle \cdot, \cdot \rangle \) the real part of the Hermitian scalar product on \( \mathcal{H} \). The Grassmannian \( Z(\mathcal{H}) =

Table 1: Simple roots of non-compact type in the simple finite-dimensional Lie algebras of type A, B, C and D.

Proof of Theorem 1.1:

By Lemma 3.7 there exists a unique simple root \( \phi \) in \( B \) regardless to the ordering chosen on the set of non-zero roots \( R \). By Lemma 3.8 part 3., \( \phi \) is a simple root of non-compact type for each finite-dimensional subalgebras \( g(\mathcal{N}) \) constructed in Lemma 3.8 part 2. when \( R \) is endowed with the particular ordering constructed in Proposition 2.3. For this ordering, simple roots of \( g(\mathcal{N}) \) are simple roots of \( g \). It follows that the set of possible roots \( \phi \) can be deduced from the Dynkin diagram of \( g \). Each simple root \( \phi \) defines a symmetric pair \((g, \mathfrak{k})\) where \( \mathfrak{k} \) is the \( \mathfrak{L} \) -algebra of compact type whose Dynkin diagram is obtained by removing \( \phi \) from the Dynkin diagram of \( g \) (\( \mathfrak{k} \) is the orthogonal of the vector space generated by the \( e_{\phi + \alpha} \) 's).

Proof of Corollary 1.2:

In [11], W. Kaup gives the classification of Hermitian-symmetric spaces of arbitrary dimension using the theory of Hermitian-Jordan triple system developed in the Banach context in [8]. It is straightforward to verify that both lists coincide.

Remark 3.10 The Grassmannian \( G^{(p)} = U_2(\mathcal{H}) / (U_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p)) \) of \( p \)-dimensional subspaces of \( \mathcal{H} \) with \( \text{dim}(\mathcal{H}_p) = p < +\infty \), is the affine adjoint orbit of \( U_2(\mathcal{H}) \) for the derivations defined by the bounded operators \( D^{(p)}_{k,l} = ikp_{\mathcal{H}_p} - ilp_{\mathcal{H}_{-l}} \), where \( k, l \in \mathbb{R}, k \neq -l \), and \( p_{\mathcal{H}_p} \) (resp. \( p_{\mathcal{H}_{-l}} \)) is the orthogonal projection onto \( \mathcal{H}_p \) (resp. \( \mathcal{H}_{-l} \)). The homogeneous space \( G^{(p)} \) is therefore endowed with a one-parameter family of Kähler structure (encoded by \((k + l)\)). The derivation \( D^{(p)}_{k,l} \) is inner if and only if \( l = 0 \). For \( p = 1 \), \( G^{(p)} \) is the projective space of \( \mathcal{H} \).

The restricted Grassmannian \( G_{\text{res}}^{(p)} \) has been studied in [12] and [22]. The connected component \( G_{\text{res}}^{(p)} \) containing \( \mathcal{H}_+ \) is the affine adjoint orbit of \( U_2(\mathcal{H}) \) for the derivations defined by the bounded operators \( D^{(p)}_{k,l} = ikp_{\mathcal{H}_p} - ilp_{\mathcal{H}_{-l}} \), where \( k, l \in \mathbb{R}, k \neq -l \), and \( p_{\mathcal{H}_p} \) (resp. \( p_{\mathcal{H}_{-l}} \)) is the orthogonal projection onto \( \mathcal{H}_+ \). None of these derivations is inner.

The Grassmannian \( G_{\text{res}}^{(2)} = O^2_2(\mathcal{H}) / (SO(2) \times O^2_2((\mathcal{H})_2)) \) of oriented 2-planes in \( \mathcal{H} \) is the \( O^2_2(\mathcal{H}) \)-adjoint orbit of \( kJ \) where \( k \neq 0 \) and \( J \) is the natural complex structure on \( (\mathcal{H})_2 \). It can be identified as complex manifold with the conic \( C \) in the complex projective space \( P(\mathcal{H}) \) defined by

\[
C := \{ (z_1: \ldots : z_n: \ldots) \in P(\mathcal{H}), \sum_{i \in \mathbb{N}} z_i^2 = 0 \}.
\]

Denote by \( \langle \cdot, \cdot \rangle \) the real part of the Hermitian scalar product on \( \mathcal{H} \). The Grassmannian \( Z(\mathcal{H}) =

8
$O_2^+(\mathcal{H}^\mathbb{R})/U_2(\mathcal{H})$ is the space of complex structures $I$ on $\mathcal{H}^\mathbb{R}$ such that

$$(IX, IY) = (X, Y),$$

defining the same orientation as the distinguished complex structure $I_0$ on $\mathcal{H}$ and being closed to it. For every $k \neq 0$, the space $\mathcal{Z}(\mathcal{H})$ can be identified with the $O_2^+(\mathcal{H}^\mathbb{R})$-affine adjoint orbit of 0 for the the bounded operator $D_k^{(0)} = kI_0$. Denote by $\mathcal{H}^C$ the $\mathbb{C}$-extension of $\mathcal{H}^\mathbb{R}$ and by $Z_+$ (resp. $Z_-$) the eigenspace of the $\mathbb{C}$-linear extension of $I_0$ with eigenvalue $+i$ (resp. $-i$). One has $\mathcal{H}^C = Z_+ \oplus Z_-$. The homogeneous space $\mathcal{Z}(\mathcal{H})$ injects into the restricted Grassmannian of the polarized Hilbert space $\mathcal{H}^C = Z_+ \oplus Z_-$ as a totally geodesic submanifold via the application which maps a complex structure $I$ to the subspace consisting of $(1,0)$-type vectors $X$ with respect to $I$, i.e. satisfying $IX = iX$. The Grassmannian $\mathcal{L}(\mathcal{H}) = \text{Sp}_2(\mathcal{H})/U_2(\mathcal{H}_+)$ of Lagrangian subspaces close to $\mathcal{H}_+$ is the $\text{Sp}_2(\mathcal{H})$-affine adjoint orbit of 0 for the derivations given by the bounded operators $D_l^{(\infty)} = ilp_+ - ilp_-$. It is a totally geodesic submanifold of the restricted Grassmannian $\text{Gr}_{\text{res}}$. 

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