Bounded State Solutions of Kirchhoff Type Problems with a Critical Exponent in High Dimension

Qilin Xie and Jianshe Yu

School of Mathematics and Information Science
Guangzhou University, Guangzhou 510006, Guangdong, China

Abstract. In the present paper, we consider the following Kirchhoff type problem

\[
\begin{cases}
-\left( a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + V(x)u = |u|^{2^* - 2}u & \text{in } \mathbb{R}^N, \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases}
\]

where \( a \) is a positive constant, \( \lambda \) is a positive parameter, \( V \in L^{\frac{N}{2}}(\mathbb{R}^N) \) is a given nonnegative function and \( 2^* \) is the critical exponent. The existence of bounded state solutions for Kirchhoff type problem with critical exponents in the whole \( \mathbb{R}^N (N \geq 5) \) has never been considered so far. We obtain sufficient conditions on the existence of bounded state solutions in high dimension \( N \geq 4 \), and especially it is the first time to consider the case when \( N \geq 5 \) in the literature.

1. Introduction and main results. In the present paper, we consider the existence of bounded state solutions for the Kirchhoff type problem

\[
\begin{cases}
-\left( a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + V(x)u = |u|^{2^* - 2}u & \text{in } \mathbb{R}^N, \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases}
\]

(\( SK^* \))

where \( a \) is a positive constant, \( \lambda \) is a positive parameter, \( V \in L^{\frac{N}{2}}(\mathbb{R}^N) \) is a given nonnegative function and \( 2^* = \frac{2N}{N - 2} \), \( (N \geq 4) \), is the critical exponent.

When \( \lambda = 0 \), this problem reduces to the scalar field equation. A large number of papers have been published about the scalar field equation both in \( H^1(\mathbb{R}^N) \) and in \( D^{1,2}(\mathbb{R}^N) \). Felmer et al. [6] obtained some monotonicity properties for ground states of the scalar field equation with subcritical exponent. Benci and Cerami [2] studied the existence of bounded state solutions for the scalar field equation with critical exponents.

2000 Mathematics Subject Classification. Primary: 35J60; Secondary: 47J30, 35J20.

Key words and phrases. Kirchhoff type problems, critical exponent, bounded state solutions, high dimension.

The first author is supported by National Natural Science Foundation of China grant 11701113 and China Postdoctoral Science Foundation funded project grant 2016M600647. The second author is supported by National Natural Science Foundation of China grant 11471085, Program for Changjiang Scholars and Innovative Research Team in University grant IRT1226 and Guangdong Innovative Research Team Program grant 2011S009.

* Corresponding author.
Recently, the following Kirchhoff type problem with critical exponent in $\mathbb{R}^3$
\[
\begin{aligned}
-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\Delta u + V(x)u &= f(x,u) + u^5 \quad \text{in } \mathbb{R}^3, \\
u u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] (1.1)
has been widely studied by many authors under different conditions on $V$ and $f$. We refer the readers to [1, 4, 7, 8, 11, 12, 13, 17, 21, 22, 23, 24] for related results.

However, quite a few papers have been published about this problem in 4 dimension. In Naimen [18], the author considered the following Kirchhoff type problem
\[
\begin{aligned}
-\left(a+b\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u &= \nu u^q + \mu u^3, \quad u > 0 \quad \text{in } \Omega, \\
u u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where $\Omega \subset \mathbb{R}^4$ is a bounded domain with smooth boundary $\partial \Omega$. The author obtained some existence results with some suitable conditions on $a, \nu, \mu > 0, b \geq 0$ and $1 \leq q < 3$. Naimen [18] is the first author investigating the Kirchhoff type problem with critical exponent in 4 dimension. We refer the readers to [9, 10, 14, 15, 16] for related results.

As we all know, the existence of bounded state solutions for Kirchhoff type problems with critical exponents in the whole $\mathbb{R}^N (N \geq 4)$ has never been considered so far. To state our result, we denote the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ by $S$. We also denote the norm in $L^p(\mathbb{R}^N)$ by $|\cdot|_p$.

When $N = 4$, we have the following result.

**Theorem 1.1.** Let $a > 0$, $N = 4$ and $V \in L^2(\mathbb{R}^4)$ be a nonnegative function. If the following inequalities hold,
\[
0 < \lambda < S^{-2} \quad \text{and} \quad 0 < |V|_2 < \left(\sqrt{2} - 1\right) aS,
\] (1.2)
then problem $(SK^*)$ has at least one bounded state solution.

**Remark 1.2.** It should be mentioned that the basic idea in the proof follows from those in Benci and Cerami [2] and Xie et al. [25]. For problem $(SK^*)$ with $a = 1, \lambda = 0$ in 4 dimension, Benci and Cerami [2] obtained a positive solution under the following assumptions: $V(x) \geq 0$ for any $x \in \mathbb{R}^4$, there exist two positive constants $p_1 < 2 < p_2$ such that
\[
V \in L^p(\mathbb{R}^4) \quad \text{for all } p \in [p_1, p_2]
\] (1.3)
and $0 < |V|_2 < (\sqrt{2} - 1)S$. Clearly, our theorem improves the main result in Benci and Cerami [2] when $N = 4$.

For $N \geq 5$, if we set
\[
\Lambda_0 := \frac{2}{N-2} \left(\frac{N-4}{N-2}\right)^{\frac{N-4}{2}} S^{-\frac{N}{2}} a^{-\frac{N-4}{4}},
\] (1.4)
then we have the following result.

**Theorem 1.3.** Let $a > 0$, $N \geq 5$ and $V \in L^{\frac{N}{2}}(\mathbb{R}^N)$ be a nonnegative function. There exist two positive constants $\mu := \mu(a, N) < 1/2 \Lambda_0$ and $\nu := \nu(a, N, \lambda)$ such that problem $(SK^*)$ has at least two bounded state solutions for $\Lambda_0 - \mu < \lambda < \Lambda_0$ and $0 < |V|_2^{\frac{N}{2}} < \nu$. 
Remark 1.4. To the best of our knowledge, our theorem is the first result concerning about the bounded state solutions of Kirchhoff type problems with critical exponents in \( \mathbb{R}^N \), \( N \geq 5 \). Compared with Benci and Cerami [2], we obtain two distinct bounded state solutions which are quite different from those of \( N = 3 \) and \( 4 \). Actually, in view of the corresponding energy functional, the interaction between the nonlocal term \( \|u\|_{D^{1,2}}^4 \) and the critical non-linearity \( |u|^{2^*}_2 \) is crucial. If \( N \geq 5 \) holds, then the critical exponent \( 2^* \) is strictly less than 4. The limit equation \((SK^*)\) with \( V \equiv 0 \) has two positive solutions (see Figure 3 in Remark 2.5), which are two minimum points of two parts in the Nehari manifold and one of the solutions is the least energy solution for some suitable \( \lambda \) (see Propositions 2.8-2.10). As Naimen pointed out in Remark 4.4 of [18], the least energy level can be negative (see the end of Remark 2.4). Moreover, the Nehari manifold will be a bounded set in \( D^{1,2}(\mathbb{R}^N) \). All these phenomena will appear when \( N \geq 5 \). We believe that it is an interesting and important feature of Kirchhoff type problems in high dimension.

If the parameter \( \lambda \) is suitably large, then we have the following nonexistence result.

**Theorem 1.5.** Let \( a > 0 \) and \( V \in L^{\frac{N}{2}}(\mathbb{R}^N) \) be a nonnegative function. If one of the following cases holds,

(i) \( N = 4 \) and \( \lambda > S^{-2} \);
(ii) \( N \geq 5 \) and \( \lambda > \Lambda_0 \);

then problem \((SK^*)\) has no nontrivial solution.

**Remark 1.6.** Now, we have partially solved the existence of nontrivial solutions for the Kirchhoff type problem \((SK^*)\). When \( N = 4 \), it follows from Theorem 1.1 and Theorem 1.5 that problem has at least one nontrivial solution for \( 0 < \lambda < S^{-2} \); and problem has no nontrivial solution for \( \lambda > S^{-2} \). We do not know what will happen for \( \lambda = S^{-2} \). So does the case when \( N \geq 5 \) and \( \lambda = \Lambda_0 \). Because of the limitation of our methods and technique, Theorem 1.3 only obtains nontrivial solutions for problem \((SK^*)\) with \( \Lambda_0 - \mu < \lambda < \Lambda_0 \). A natural and interesting question is whether we can establish multiplicity theorems for problem \((SK^*)\) with small positive parameter \( \lambda \). We guess that the answer to the above problem is positive. The main difficulties when we investigate problem \((SK^*)\) are that for Kirchhoff type problems with critical exponents, it is not easy to verify a range where Palais-Smale condition holds, especially when the energy level may be negative.

This paper is organized as follows. In Section 2, we give the variational setting for our problem and investigate the solutions for the limit equation of \((SK^*)\). After that, we prove that problem \((SK^*)\) can not be solved by minimization on Nehari manifold. At the end of Section 2, we prove our Theorem 1.5 indirectly. In Section 3, by a description of Palais-Smale sequences, we obtain a local compactness result Proposition 3.2 with \( N = 4 \) and a global compactness result Proposition 3.3 with \( N \geq 5 \). By a standard argument, we obtain the existence of the bounded state solutions in Section 4. In Section 5, we prove some necessary lemmas, which have been used in our main context.

2. Preliminaries. In this section, we give the variational setting for problem \((SK^*)\) and some basic information on the limit equation of \((SK^*)\). The main works in this paper are considered in the Hilbert space \( D^{1,2}(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \)
\(|\nabla u| \in L^2(\mathbb{R}^N)\) with the inner product and norm, respectively,

\[(u, v)_{D^{1,2}} = \int_{\mathbb{R}^N} \nabla u \nabla v dx, \quad \|u\|_{D^{1,2}} = (u, u)_{D^{1,2}}^{\frac{1}{2}}.\]

As we know, the following classical Schrödinger equation

\[\begin{cases}
-\Delta u = |u|^{2^*-2} u & \text{in } \mathbb{R}^N, \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases}\]  

\((S^*)\)

has been well studied. On the one hand, positive solutions must be the form

\[U_{\delta,y}(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(\delta + |x-y|^2)^{\frac{N-2}{4}}} , \quad \delta > 0, \quad y \in \mathbb{R}^N.\]  

\((2.1)\)

This class of functions achieves the best Sobolev constant \(S\),

\[S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} |u|^2 dx)^{\frac{1}{2}}};\]  

\((2.2)\)

and \(S = |U_{\delta,y}|^{\frac{2}{2^*}} = \|U_{\delta,y}\|_{D^{1,2}}^{\frac{2}{2^*}}.\) On the other hand, infinitely many sign-changing solutions are obtained by Ding [5] when \(N \geq 3\).

Through out this paper, we define a positive constant \(c_{a,\lambda,4}^*\) by

\[c_{a,\lambda,4}^* := \frac{(aS)^2}{4(1 - \lambda S^2)}.\]

The limit equation of \((SK^*)\) is

\[\begin{cases}
-\left( a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u = |u|^{2^*-2} u & \text{in } \mathbb{R}^N, \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases}\]  

\((SK^*_\infty)\)

Now the functionals \(I\) and \(I_\infty\) related to \((SK^*)\) and \((SK^*_\infty)\) are introduced, respectively,

\[I(u) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 dx,\]

\[I_\infty(u) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 - \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 dx.\]

Firstly, we will consider the existence of positive and sign-changing solutions for this limit equation \((SK^*_\infty)\) with \(N \geq 4\), which is heavily dependent on the dimension \(N\) and the parameter \(\lambda\). What should be mentioned is that the case when \(N = 3\) has been considered in Xie et al. [25]. Before that, we need the following lemma.

**Lemma 2.1.** For \(N \geq 4, a, \lambda > 0\). Set

\[g(t) = \lambda S^\frac{N}{2} t^2 - t^{2^*-2} + a, \quad t \geq 0.\]

Then \(g(t) = 0\) has a unique solution \(K_-\) if and only if either \(N = 4\) and \(\lambda < S^{-2}\) or \(N \geq 5\) and \(\lambda = \Lambda_0\). Moreover, \(g(t) = 0\) has two different solutions \(K_-\) and \(K_+\), \((0 < K_- < K_+)\) if and only if \(N \geq 5\) and \(\lambda < \Lambda_0\). These results are also valid for

\[f(t) = \lambda S^\frac{N}{2} t^2 - t^{2^*-2} + aS^\frac{N}{2}, \quad t \geq 0.\]
Proof. We only prove our results for $g$. When $N = 4$, one obtains
\[ g(t) = -(1 - \lambda S^2) t^2 + a. \]
It is easy to prove that $g(t) = 0$ has a unique positive solution
\[ K_- = \sqrt{\frac{a}{1 - \lambda S^2}} \] (2.3)
if and only if $\lambda < S^{-2}$. When $N \geq 5$, we know $0 < \frac{4}{N-2} \leq \frac{4}{3}$. Moreover, $g(0) = a$ and $g(t) \to \infty$ as $t \to \infty$. It is easy to prove that
\[ K_{\text{min}} = \left( \frac{2}{N-2} \right)^{\frac{N-2}{N-1}} (\lambda S^\frac{2}{N})^{-\frac{N-2}{(N-1)N}} \] (2.4)
is the unique global minima point of $g$. Thus $\min_{t \geq 0} g(t) = g(K_{\text{min}}) = 0$ if and only if $\lambda = \Lambda_0$. Moreover,
there exist $0 < K_- < K_{\text{min}} < K_+$ such that $g(K_-) = g(K_+) = 0$ \iff $\lambda < \Lambda_0$.
This completes the proof. \qed

**Proposition 2.2.** Assume that $a > 0$ and $\lambda > 0$ hold. Then we have the following results:

(a) Problem $(SK^*_\infty)$ has at least a positive solution if and only if either $N = 4$ and $\lambda < S^{-2}$ or $N \geq 5$ and $\lambda \leq \Lambda_0$.

(b) If $N = 4$ and $\lambda \geq \frac{1}{2}S^{-2}$ or $N \geq 5$ and $\lambda > \frac{1}{2}\Lambda_0$ hold, then problem $(SK^*_\infty)$ admits no sign-changing solutions.

Proof. (a) On the one hand, if $w$ is a positive solution of problem $(S^*)$, then there exists at least one positive constant $t$ such that $u = tw$ solves problem $(SK^*_\infty)$. Actually, if $u = tw$ solves problem $(SK^*_\infty)$, then one obtains
\[ -(a + \lambda t^2 \|w\|^2_{D^1,2}) t \Delta w = t^{2^* - 1} w^{2^* - 1}. \]
Combining the uniqueness of positive solutions for problem $(S^*)$ and $\|w\|^2_{D^1,2} = S^{\frac{2}{N}}$, then
\[ \lambda S\frac{2}{N} t^2 - t^{\frac{1}{N}} + a = 0; \]
namely $g(t) = 0$. If $N = 4$ and $\lambda < S^{-2}$ or $N \geq 5$ and $\lambda \leq \Lambda_0$ hold, then we get the existence of the constant $t$ by Lemma 2.1.

On the other hand, if $u$ is a positive solution of problem $(SK^*_\infty)$, then
\[ w = (a + \lambda \|u\|^2_{D^1,2})^{-\frac{N-2}{N}} u \] (2.5)
is also a positive solution of problem $(S^*)$. It follows from the uniqueness of positive solutions for problem $(S^*)$ and $\|w\|^2_{D^1,2} = S^{\frac{2}{N}}$ that there exists at least a $\xi = \|u\|_{D^1,2} > 0$ solving the following problem
\[ t^2 = (a + \lambda t^2)^{-\frac{N-2}{N}} S^{\frac{2}{N}}; \]
namely
\[ \lambda S\frac{2}{N} t^2 - t^{\frac{1}{N}} + aS^{\frac{2}{N}} = 0. \] (2.6)
In other words, $f(t) = 0$ has at least a positive solution $\xi$. By Lemma 2.1, we have $N = 4$ and $\lambda < S^{-2}$ or $N \geq 5$ and $\lambda \leq \Lambda_0$. 

**Theorem 2.3.** Assume that $a > 0$ and $\lambda > 0$ hold. Then we have the following results:

(a) Problem $(SK^*_\infty)$ has at least a positive solution if and only if either $N = 4$ and $\lambda < S^{-2}$ or $N \geq 5$ and $\lambda \leq \Lambda_0$.

(b) If $N = 4$ and $\lambda \geq \frac{1}{2}S^{-2}$ or $N \geq 5$ and $\lambda > \frac{1}{2}\Lambda_0$ hold, then problem $(SK^*_\infty)$ admits no sign-changing solutions.
(b) Arguing indirectly, if \( u \) is a sign-changing solution of problem \((SK_\infty^+)\), then \( w = (a + \lambda \|u\|_{D_1,2}^2)^{-\frac{N-2}{2}} u \) is also a sign-changing solution of problem \((S^+)\). With the help of \( \|w\|_{D_1,2}^2 \geq 2S^\frac{N}{2} \), we know
\[
\|u\|_{D_1,2}^2 \geq 2S^\frac{N}{2} \left(a + \lambda \|u\|_{D_1,2}^2\right)^{\frac{N-2}{2}},
\]
which implies that
\[
\lambda S^{\frac{N}{2}} \|u\|_{D_1,2}^2 - 2^{-\frac{N-2}{2}} \|u\|_{D_1,2}^\frac{N}{2} + aS^{\frac{N}{2}} \leq 0. \tag{2.7}
\]
If \( N = 4 \) holds, then
\[
\left(\lambda S^2 - \frac{1}{2}\right) \|u\|_{D_1,2}^2 + aS^2 \leq 0,
\]
which is impossible because of \( \lambda \geq \frac{1}{2}S^{-2} \). If \( N \geq 5 \) holds, then
\[
f_1(t) := \lambda S^{\frac{N}{2}} t^2 - 2^{-\frac{N-2}{2}} t^{N-2} + aS^{\frac{N}{2}}.
\]
Clearly, \( \xi_{\text{min}} = \left(\frac{N-2}{2} 2\frac{2}{N} \lambda S^{\frac{N}{2}}\right)^{-\frac{N-2}{2}} \) is the unique global minima point of \( f_1 \). It follows from \( \lambda > \frac{1}{2} \Lambda_0 \) that
\[
\min_{t \geq 0} f_1(t) = f_1(\xi_{\text{min}}) > 0,
\]
which contradicts with (2.7). \( \Box \)

**Remark 2.3.** If \( N = 4 \) and \( \lambda < S^{-2} \) hold, then it follows from the proof of (a) in Proposition 2.2 that problem \((SK_\infty^+)\) has a positive least energy solution (unique up to translation and scaling)
\[
\varphi_{\delta,y} := K_-U_{\delta,y} = \sqrt{\frac{a}{1 - \lambda S^2}} U_{\delta,y}, \tag{2.8}
\]
where \( K_- \) is the positive solution of \( g(t) = 0 \) in Lemma 2.1. In fact, the energy of this positive solution is
\[
\begin{align*}
I_\infty(\varphi_{\delta,y}) &= I_\infty(\varphi_{\delta,y}) - \frac{1}{4}(I_\infty'(\varphi_{\delta,y}), \varphi_{\delta,y}) = \frac{a}{4} K^2 \|U_{\delta,y}\|_{D_1,2}^2 = \frac{(aS)^2}{4(1 - \lambda S^2)} = c_{\alpha,\lambda}^\delta. 
\end{align*}
\tag{2.9}
\]
Moreover, let \( u_s \) be a sign-changing solution of problem \((SK_\infty^+)\). Then it follows from (2.7) that
\[
\|u_s\|_{D_1,2}^2 \geq \frac{2aS^2}{1 - 2\lambda S^2} > \frac{2aS^2}{1 - \lambda S^2},
\]
which implies
\[
I_\infty(u_s) = I_\infty(u_s) - \frac{1}{4}(I'_\infty(u_s), u_s) = \frac{a}{4} \|u_s\|_{D_1,2}^2 \geq \frac{(aS)^2}{2(1 - \lambda S^2)} = 2c_{\alpha,\lambda}^\delta. \tag{2.10}
\]
This, combined with (2.9), gives that the unique positive solution \( \varphi_{\delta,y} \) is the least energy solution. Actually, there is no essential difference between \( N = 4 \) in the above situation and \( N = 3 \) in Proposition 2.1 of Xie et al. [25]. The constant \( c_{\alpha,\lambda}^\delta \) has also been given in Naimen [18].
Remark 2.4. Problem \((SK_\infty^*)\) with \(N \geq 5\) will be quite different from those of \(N = 3\) and \(4\). If \(N \geq 5\) and \(\lambda = \Lambda_0\) hold, then problem \((SK_\infty^*)\) has only one nontrivial solution
\[
\varphi_{\delta,y} := K \min U_{\delta,y} = \left( \frac{2}{N-2} \right)^{\frac{N-2}{N-4}} (\Lambda_0 S_N^N)^{-\frac{N-2}{N-4}} U_{\delta,y}.
\] (2.11)

If \(N \geq 5\) and \(\lambda < \Lambda_0\) hold, this problem \((SK_\infty^*)\) will be more complicated. Problem \((SK_\infty^*)\) has two positive solutions and one of them must be the least energy solution. In fact, if \(N \geq 5\) and \(\lambda < \Lambda_0\) hold, in this case, we denote the positive solutions of the following problem
\[
f(t) = \lambda S \frac{N}{\lambda} t^2 - t^{2^* - 2} + a S \frac{N}{\lambda} = 0
\] (2.12)
by \(\xi_1\) and \(\xi_2\) \((0 < \xi_1 < \xi_2)\). Thus there exist two positive solutions \((\varphi^\pm)\) of problem \((SK_\infty^*)\) satisfying \(\|\varphi^-\|_{D^{1,2}} = \xi_1\) and \(\|\varphi^+\|_{D^{1,2}} = \xi_2\). More precisely,
\[
\varphi^-_{\delta,y} := K_- U_{\delta,y} \quad \text{and} \quad \varphi^+_{\delta,y} := K_+ U_{\delta,y},
\] (2.13)
where \(K_\pm\) are the positive solutions of \(g(t) = 0\) in Lemma 2.1. If \(\lambda < \frac{1}{2} \Lambda_0\) and \(u_s\) is a sign-changing solution of problem \((SK_\infty^*)\), then we denote the positive solutions of the following problem
\[
f_1(t) = \lambda S \frac{N}{\lambda} t^2 - 2 \frac{N}{\lambda} t^{2^* - 2} + a S \frac{N}{\lambda} = 0
\] (2.14)
by \(\varsigma_1\) and \(\varsigma_2\) \((0 < \varsigma_1 < \varsigma_2)\). Obviously, \(\xi_1 < \varsigma_1 < \varsigma_2 < \xi_2\). Noting (2.7), one obtains \(\varsigma_1 \leq \|u_s\|_{D^{1,2}} \leq \varsigma_2\). Set
\[
I_\infty(u) = I_\infty(u) - \frac{1}{2} \langle I_\infty'(u), u \rangle = \frac{1}{N} a \|u\|_{D^{1,2}}^2 - \frac{N-4}{4N} \lambda \|u\|_{D^{1,2}}^4 := I_\infty(\|u\|_{D^{1,2}}),
\] (2.15)
where
\[
I_\infty(t) = \frac{1}{N} a t^2 - \frac{N-4}{4N} \lambda t^4 \quad \text{for} \quad t \geq 0.
\]
It is easy to prove that \(I_\infty\) achieves its maximum at \(t_{\text{max}}^2 = \frac{2a}{(N-4)\lambda}\). If \(N \geq 6\) holds, then it follows from \(I_\infty(t) > 0\) for \(t \in (0, t_{\text{max}})\) and \(\xi_1^2 < \frac{2a}{(N-4)\lambda} < \xi_2^2\) that
\[
I_\infty(\xi_1) > 0, \quad \text{i.e.,} \quad I_\infty(\varphi^-) > 0.
\]
Moreover, by
\[
\xi_1^2 + \xi_2^2 - \frac{4a}{(N-4)\lambda} > 0 \quad \text{(seeLemma 5.1)}
\]
we obtain
\[
I_\infty(\xi_2) < I_\infty(\xi_1) \quad \text{and} \quad I_\infty(\xi_2) < I_\infty(\|u_s\|_{D^{1,2}});
\]
namely
\[
I_\infty(\varphi^+) < I_\infty(\varphi^-) \quad \text{and} \quad I_\infty(\varphi^+) < I_\infty(u_s).
\]
If \(N = 5\) holds, it follows from \(\xi_1^2 < \frac{2a}{(N-4)\lambda} < \xi_2^2\) and \(I_\infty(t) > 0\) for \(t \in (0, t_{\text{max}})\) that
\[
I_\infty(\varphi^-) > 0 \quad \text{and} \quad \min \{I_\infty(\varphi^+), I_\infty(\varphi^-)\} < I_\infty(u_s).
\]
Either \(\varphi^-\) or \(\varphi^+\) must be the least energy solution. Through out this paper, we only consider the case where \(\varphi^+\) is the least energy solution. Since the other case can be dealt with similarly. It should be mentioned that both of them can be the least energy solution for some \(\lambda_0\) at the same time and we only avoid those \(\lambda_0\).

Thus, for \(N \geq 5\), we know that \(\varphi^+\) is the least energy solution. We define
\[
I_\infty(\varphi^-_{\delta,y}) := c_{-}^{a,\lambda,N} > 0 \quad \text{and} \quad I_\infty(\varphi^+_{\delta,y}) := c_{+}^{a,\lambda,N}.
\] (2.16)
As Naimen pointed out in Remark 4.4 of [18], the least energy level \( c_{+}^{a,\lambda,N} \) can be negative. For example, when \( N = 6 \), by direct calculations, we can obtain the following results

\[
\varphi_{\delta,y}^{-} = K_{-}U_{\delta,y} = \frac{1 - \sqrt{1 - 4a\lambda S^3}}{2\lambda S^3}U_{\delta,y}; \quad \varphi_{\delta,y}^{+} = K_{+}U_{\delta,y} = \frac{1 + \sqrt{1 - 4a\lambda S^3}}{2\lambda S^3}U_{\delta,y}
\]

and

\[
I_{\infty}(\varphi_{\delta,y}^{-}) = \frac{6a\lambda S^3 - 1 + \sqrt{1 - 4a\lambda S^3}}{24\lambda S^3}\left(\frac{1 - \sqrt{1 - 4a\lambda S^3}}{2\lambda S^3}\right)^2 = c_{-}^{a,\lambda,6} > 0;
\]

\[
I_{\infty}(\varphi_{\delta,y}^{+}) = \frac{6a\lambda S^3 - 1 - \sqrt{1 - 4a\lambda S^3}}{24\lambda S^3}\left(\frac{1 + \sqrt{1 - 4a\lambda S^3}}{2\lambda S^3}\right)^2 = c_{+}^{a,\lambda,6}.
\]

Obviously, for some suitable \( \lambda \), then one can easily see \( c_{+}^{a,\lambda,6} < 0 \).

**Remark 2.5.** As to problem \((SK^*_{\infty})\) with \( N = 3 \), from Xie et al. [25], we obtain the following branching diagram, in which the blue one is the positive solution and the red one is a sign-changing solution of \((SK^*_{\infty})\).

![Figure 1. N=3.](image1)

When \( N = 4 \), we will have a similar bifurcation result from Proposition 2.2 and Remark 2.3. Note that \( u_s \) is a sign-changing solution of \((S^*\)).

![Figure 2. N=4.](image2)
However, by Proposition 2.2 and Remark 2.4, problem $(SK^*_\infty)$ in high dimension will be quite different from those of $N = 3$ and 4.

We set the Nehari manifolds as follows,

\[
N^N_N := \{ u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} : (I_\infty(u), u) = 0 \};
\]

\[
N_{N,\pm}^\infty := \left\{ u \in N^N_N : \pm \left( a \|u\|_{D^{1,2}}^2 + 3\lambda \|u\|_{D^{1,2}}^4 - (2^* - 1)\|u\|_{2^*}^2 \right) > 0 \right\};
\]

\[
N^N_N := \{ u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} : (I'(u), u) = 0 \};
\]

and

\[
N_{N,\pm} := \left\{ u \in N^N_N : \pm \left( a \|u\|_{D^{1,2}}^2 + \int V(x)u^2 + 3\lambda \|u\|_{D^{1,2}}^4 - (2^* - 1)\|u\|_{2^*}^2 \right) > 0 \right\}.
\]

It is easy to see that

\[
N^N_N = N^{N,-}_N, \quad N^{N,+}_N = \emptyset \quad \text{and} \quad \varphi_{\delta,y}^+ \in N^{N,-}_N,
\]

where $\varphi^-$ is defined in (2.8). Moreover, if $N \geq 5$ and $0 < \lambda < \Lambda_0$ hold, then

\[
N^{N,-}_N \cup N^{N,+}_N \subset N^{N}_N, \quad \varphi_{\delta,y}^- \in N^{N,-}_N \quad \text{and} \quad \varphi_{\delta,y}^+ \in N^{N,+}_N,
\]

where $\varphi^\pm$ are defined in (2.13).

**Lemma 2.6.** Assume that $V \in L^2(\mathbb{R}^4)$. Then, for any $u \in N^{N,-}_N$, there exists a unique constant $t_u^- \geq 1$ such that $t_u^- u \in N^{N,-}_N$. Moreover, for any $u \in N^{N,-}_N$, there exists a unique constant $0 < t_u^N \leq 1$ such that $t_u^N u \in N^{N,-}_N$.

**Proof.** It follows from $u \in N^{N,-}_N$ that

\[
\int_{\mathbb{R}^4} |u|^4dx - \lambda \left( \int_{\mathbb{R}^4} |\nabla u|^2dx \right)^2 = a \int_{\mathbb{R}^4} |\nabla u|^2dx > 0,
\]

which implies that

\[
\frac{dI(tu)}{dt} = at \int_{\mathbb{R}^4} |\nabla u|^2dx + t \int_{\mathbb{R}^4} V(x)u^2dx + \lambda t^3 \left( \int_{\mathbb{R}^4} |\nabla u|^2dx \right)^2 - t^3 \int_{\mathbb{R}^4} |u|^4dx = 0
\]
has a unique positive solution
\[ t_u^- = \sqrt{1 + \frac{\int_{R^l} V(x) u^2 dx}{a \int_{R^l} |\nabla u|^2 dx}}. \] (2.18)

Equivalently, \( t_u^\pm u \in \mathcal{N}^{4,-} \). Similarly, taking
\[ t_\infty = \sqrt{\frac{a \int_{R^l} |\nabla u|^2 dx + \int_{R^l} V(x) u^2 dx}{a \int_{R^l} |\nabla u|^2 dx}}. \] (2.19)
we have \( t_\infty^- u \in \mathcal{N}^{4,-}_\infty \). This completes the proof. \( \square \)

**Lemma 2.7.** Assume that \( N \geq 5 \) and \( V \in L^\frac{N}{N-2}(\mathbb{R}^N) \). Then, for any \( u \in \mathcal{N}^{N,\pm} \), there exist two constants \( t_\infty^- \leq t_\infty^+ \) such that \( t_\infty^\pm u \in \mathcal{N}^{N,\pm}_\infty \). Moreover, there exists \( \nu_1 := \nu_1(a, N, \lambda) > 0 \) such that \( t_\phi^\pm \varphi_{\delta,y}^\pm \in \mathcal{N}^{N,\pm} \) for any \( V \in L^\frac{N}{N-2}(\mathbb{R}^N) \) with \( |V|_\frac{N}{N-2} < \nu_1 \), where \( \varphi^\pm \) are defined in (2.13) and \( t_\phi^\pm \geq 1 \) are two constants.

**Proof.** It follows from \( u \in \mathcal{N}^{N,\pm}_\infty \) that
\[
\begin{align*}
&\begin{cases}
\lambda u\|u\|^4_{D^1,1} t^2 - |u|_{2^*}^2 t^{2^*-2} + a\|u\|_{D^1,2}^2 = 0; \\
3\lambda \|u\|^4_{D^1,1} - (2^*-1)|u|_{2^*}^2 + a\|u\|_{D^1,2}^2 + \int_{R^N} V(x) u^2 dx < 0.
\end{cases}
\end{align*}
\]
These imply that
\[ \lambda\|u\|^4_{D^1,1} t^2 - |u|_{2^*}^2 t^{2^*-2} + a\|u\|_{D^1,2}^2 = 0 \]
has two different positive solutions \( t_1 \) and \( t_2 \), \((1 \leq t_1 \leq t_2)\). Set \( t_\infty^- = t_1 \), then \( t_\infty^- u \in \mathcal{N}^{N,\pm}_\infty \). Similarly, it follows from \( u \in \mathcal{N}^{N,\pm}_\infty \) that
\[
\begin{align*}
&\begin{cases}
\lambda u\|u\|^4_{D^1,1} t^2 - |u|_{2^*}^2 t^{2^*-2} + a\|u\|_{D^1,2}^2 = 0; \\
3\lambda \|u\|^4_{D^1,1} - (2^*-1)|u|_{2^*}^2 + a\|u\|_{D^1,2}^2 + \int_{R^N} V(x) u^2 dx > 0.
\end{cases}
\end{align*}
\]
These imply that
\[ \lambda\|u\|^4_{D^1,1} t^2 - |u|_{2^*}^2 t^{2^*-2} + a\|u\|_{D^1,2}^2 = 0 \]
has two different positive solutions \( t_1' \) and \( t_2' \), \((1' \leq t_1' \leq t_2')\). Set \( t_\infty^+ = t_2' \), then \( t_\infty^+ u \in \mathcal{N}^{N,\pm}_\infty \). Obviously, if \( \int_{R^N} V(x) u^2 dx \to 0 \), then \( t_\infty^- \to 1 \).

Noting \( \varphi_{\delta,y}^- \) is defined in (2.13), then \( \varphi_{\delta,y}^- \in \mathcal{N}^{N,\pm}_\infty \) and
\[\begin{align*}
&\begin{cases}
\lambda \|\varphi_{\delta,y}^-\|^4_{D^1,1} - |\varphi_{\delta,y}^-|_{2^*}^2 + a\|\varphi_{\delta,y}^-\|_{D^1,2}^2 = 0; \\
3\lambda \|\varphi_{\delta,y}^-\|^4_{D^1,1} - (2^*-1)|\varphi_{\delta,y}^-|_{2^*}^2 + a\|\varphi_{\delta,y}^-\|_{D^1,2}^2 < 0.
\end{cases}
\end{align*}\] (2.20)
According to
\[ \int_{R^N} V(x) |\varphi_{\delta,y}^-|^2 dx \leq S^{-1} |V|_\frac{N}{N-2} \|\varphi_{\delta,y}^-\|^2_{D^1,2} \]
and (2.20), if \( |V|_\frac{N}{N-2} \) is suitably small, then
\[ \lambda \|\varphi_{\delta,y}^-\|^4_{D^1,2} t^2 - |\varphi_{\delta,y}^-|_{2^*}^2 t^{2^*-2} + a\|\varphi_{\delta,y}^-\|_{D^1,2}^2 + \int_{R^N} V(x) |\varphi_{\delta,y}^-|^2 dx = 0 \]
has two different positive solutions \( t_3 \) and \( t_4 \), \((1 \leq t_3 < t_4)\). Set \( t_\infty^- = t_3 \), then \( t_\infty^- \varphi_{\delta,y}^- \in \mathcal{N}^{N,\pm}_\infty \). Similarly, for \( \varphi_{\delta,y}^+ \) defined in (2.13), we know \( \varphi_{\delta,y}^+ \in \mathcal{N}^{N,\pm}_\infty \) and
\[\begin{align*}
&\begin{cases}
\lambda \|\varphi_{\delta,y}^+\|^4_{D^1,1} - |\varphi_{\delta,y}^+|_{2^*}^2 + a\|\varphi_{\delta,y}^+\|_{D^1,2}^2 = 0; \\
3\lambda \|\varphi_{\delta,y}^+\|^4_{D^1,1} - (2^*-1)|\varphi_{\delta,y}^+|_{2^*}^2 + a\|\varphi_{\delta,y}^+\|_{D^1,2}^2 > 0.
\end{cases}
\end{align*}\]
If $|V|_\frac{2}{N}$ is suitably small, then

$$
\lambda \|\varphi_{\delta,y}\|_{D^{1,2}}^2 t^2 - |\varphi_{\delta,y}^+|^2_2 t^{2^*-2} + a \|\varphi_{\delta,y}^+\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} V(x) |\varphi_{\delta,y}^+|^2 dx = 0
$$

has two different positive solutions $t'_3$ and $t'_4$, $(1 \leq t'_3 < t'_4)$. Set $t'_\varphi = t'_4$, then $t'_\varphi \varphi_{\delta,y}^+ \in \mathcal{N}^{N,+}$. Obviously, if $\int_{\mathbb{R}^N} V(x) |\varphi_{\delta,y}^+|^2 dx \to 0$, then $t'_\varphi \to 1$. This completes the proof.

From the above two lemmas, we can define

$$
m_{\infty,-}^N := \inf_{u \in \mathcal{N}^{N,-}_\infty} I_\infty(u), \quad m_{N,-} := \inf_{u \in \mathcal{N}^{N,-}} I(u) \quad \text{for} \quad N \geq 4 \quad (2.21)
$$

and

$$
m_{\infty,+}^N := \inf_{u \in \mathcal{N}^{N,+}_\infty} I_\infty(u), \quad m_{N,+} := \inf_{u \in \mathcal{N}^{N,+}} I(u) \quad \text{for} \quad N \geq 5. \quad (2.22)
$$

When $N = 4$, we can easily obtain the following result.

**Proposition 2.8.** Let $a > 0$, $0 < \lambda < S^{-2}$ and $V \in L^2(\mathbb{R}^4)$ be a nonnegative function. Then the relation $m_{\infty}^{N,-} = m_{\infty}^{N,-} = c_{a,\lambda,N}^{-}$ holds and $m_{\infty}^{N,-}$ is not attained.

This proof is similar to that of the next Proposition 2.9 or Proposition 2.5 in Xie et al. [25]. So we omit it here. When $N \geq 5$, the following two results can be proved.

**Proposition 2.9.** Let $a > 0$, $N \geq 5$ and $V \in L^\frac{N}{N-4}(\mathbb{R}^N)$ be a nonnegative function. Then there exist two positive constants $\mu_1 := \mu_1(a,N)$ and $\nu_1 := \nu_1(a,N,\lambda)$ such that the relation $m_{\infty}^{N,-} = m_{\infty}^{N,-} = c_{a,\lambda,N}^{-}$ holds for $\lambda_0 - \mu_1 < \lambda < \lambda_0$ and $0 < |V|_\frac{2}{N} < \nu_1$. Moreover, $m_{\infty}^{N,-}$ is not attained.

**Proof.** Firstly, we show $m_{\infty}^{N,-} = c_{a,\lambda,N}^{-}$. Because of $\varphi_{\delta,y}^- \in \mathcal{N}^{N,-}_{\infty}$, then

$$
m_{\infty}^{N,-} = \inf_{u \in \mathcal{N}^{N,-}_{\infty}} I_\infty(u) \leq I_\infty(\varphi_{\delta,y}^-) = c_{a,\lambda,N}^{-}.
$$

Hence, $m_{\infty}^{N,-} \leq c_{a,\lambda,N}^{-}$. It follows from $u \in \mathcal{N}^{N,-}_{\infty}$ that

$$
\begin{align*}
& a \|u\|_{D^{1,2}}^2 + \lambda \|u\|_{D^{1,2}}^4 = \|u\|_{2^*}^2 \leq S^{-\frac{2}{2^*}} \|u\|_{D^{1,2}}^2; \\
& a \|u\|_{D^{1,2}}^2 + 3\lambda \|u\|_{D^{1,2}}^4 < (2^* - 1)\|u\|_{2^*}^2 \leq (2^* - 1)S^{-\frac{2}{2^*}} \|u\|_{D^{1,2}}^2.
\end{align*}
$$

According to (2.23) and Lemma 5.2, one obtains

$$
\begin{align*}
& f(\|u\|_{D^{1,2}}) = \lambda S^2 \|u\|_{D^{1,2}}^2 - \|u\|_{D^{1,2}}^{2^* - 2} + a S^2 \leq 0; \\
& f_2(\|u\|_{D^{1,2}}) = 3\lambda S^2 \|u\|_{D^{1,2}}^2 - (2^* - 1)\|u\|_{D^{1,2}}^{2^* - 2} + a S^2 < 0,
\end{align*}
$$

which imply that $\xi_1 \leq \|u\|_{D^{1,2}} < \xi_4$, where $\xi_i$ are defined in Lemma 5.2. It follows from (5.9) that

$$
I_\infty(u) = \frac{1}{N} a \|u\|_{D^{1,2}}^2 - \frac{N - 4}{4N} \lambda \|u\|_{D^{1,2}}^4 = I(\|u\|_{D^{1,2}}) \geq I(\xi_1) = c_{a,\lambda,N}^{-}.
$$

Thus $m_{\infty}^{N,-} = c_{a,\lambda,N}^{-}$.

Secondly, we show $m_{\infty}^{N,-} = m_{N,-}$. On the one hand, by Lemma 2.7, for any $u \in \mathcal{N}^{N,-}_\infty$, there exists $t^\infty \leq 1$ such that $t^\infty \in \mathcal{N}^{N,-}_\infty$ and then

$$
m_{\infty}^{N,-} \leq I_\infty(t^\infty u) \leq I(t^\infty u) \leq \max_{0 \leq t \leq 1} I(tu) = I(u).
$$
Then there exist two positive constants \(\mu\). Proposition 2.10. Let \(m_{N^+} = m_{N^-} \leq m_{N^-}\). On the other hand, \(m_{N^+} \leq m_{N^-}\) needs to be proved. By Lemma 2.7, take the sequence \(u_n = t_n \phi_{\delta,z_n} \in \mathcal{A}_{N^-}\) with \(\phi_{\delta,z_n}\) defined by (2.13), \(\{z_n\} \subset \mathbb{R}^N\) such that \(|z_n| \to \infty\) as \(n \to \infty\) and \(t_n = t_{\phi_{\delta,z_n}}\). We need to show

\[
\lim_{n \to \infty} I(u_n) = m_{N^-}. \tag{2.24}
\]

Obviously, \(\|\phi_{\delta,z_n}\|^2_{D^{1,2}} = K^2 S^2\). It follows from Lemma 2.13 in \([20]\) and \(\phi_{\delta,z_n} \to 0\) as \(n \to \infty\) weakly in \(D^{1,2} (\mathbb{R}^N)\) that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x)|\phi_{\delta,z_n}|^2 dx = 0. \tag{2.25}
\]

Therefore, to prove (2.24), we only need to show \(t_n \to 1\) as \(n \to \infty\), which is obvious because of Lemma 2.7 and (2.25). Then \(m_{N^+} = m_{N^-}\) has been proved.

Lastly, we show that \(m_{N^-}\) is not attained. Arguing indirectly, assuming \(v_0 \in \mathcal{A}_{N^-}\) such that \(I(v_0) = m_{N^-} = m_{N^-}\), by Lemma 2.7, there exists \(t_\infty \leq 1\) such that \(t_\infty v_0 \in \mathcal{A}_{N^+}\). Then

\[
m_{N^-} \leq I(t_\infty v_0) = \frac{a}{2} \|t_\infty v_0\|^2_{D^{1,2}} + \frac{\lambda}{4} \|t_\infty v_0\|^4_{D^{1,2}} - \frac{1}{2} \|t_\infty v_0\|_{2^*}^2,
\]

\[
\leq \frac{a}{2} \|t_\infty v_0\|^2_{D^{1,2}} + \frac{1}{2} \int_{\mathbb{R}^N} V(x)(t_\infty v_0)^2 dx + \frac{\lambda}{4} \|t_\infty v_0\|^4_{D^{1,2}} - \frac{1}{2} \|t_\infty v_0\|_{2^*}^2,
\]

\[
= I(t_\infty v_0) \leq \max_{0 \leq t \leq 1} I(tv_0) = I(v_0) = m_{N^-} = m_{N^-},
\]

which implies that \(t_\infty = 1\) and

\[
\int_{\mathbb{R}^N} V(x)|v_0|^2 dx = 0. \tag{2.26}
\]

Thus \(v_0 \in \mathcal{A}_{N^-}\) and \(I_\infty (v_0) = m_{N^-}\). By Remark 2.4 and the uniqueness of the family realizing \(c_{a,\Lambda,\lambda}^{-}\), \(v_0(\cdot) = \phi_{\delta,x_0}(\cdot) > 0\), where \(\phi_{\delta,x_0}\) is defined in (2.13) for some \(\delta > 0\) and \(x_0 \in \mathbb{R}^N\), we have

\[
\int_{\mathbb{R}^N} V(x)|v_0|^2 dx = \int_{\mathbb{R}^N} V(x)|\phi_{\delta,x_0}|^2 dx > 0,
\]

contradicting with (2.26). This completes the proof. \(\square\)

**Proposition 2.10.** Let \(a > 0\), \(N \geq 5\) and \(V \in L^\infty (\mathbb{R}^N)\) be a nonnegative function. Then there exist two positive constants \(\mu_1 := \mu_1 (a,N)\) and \(\nu_1 := \nu_1 (a,N,\lambda)\) such that the relation \(m_{N^+} = m_{N^-} = c_{a,\Lambda,\lambda}^{-}\) holds for \(\Lambda_0 - \mu_1 < \lambda < \Lambda_0\) and \(0 < |V|_{\frac{2}{2}} < \nu_1\). Moreover, \(m_{N^+}^{-}\) is not attained.

**Proof.** This proof is mainly similar to that of Proposition 2.9. Firstly, we show \(m_{N^-}^{+} = c_{a,\Lambda,\lambda}^{+}\). It follows from \(\phi_{\delta,y} \in \mathcal{A}_{N^+}\) that

\[
m_{N^-}^{+} = \inf_{u \in \mathcal{A}_{N^+}} I_\infty (u) \leq I_\infty (\phi_{\delta,y}^{-}) = c_{a,\Lambda,\lambda}^{+}.
\]

Hence, \(m_{N^-}^{+} \leq c_{a,\Lambda,\lambda}^{-}\). For any \(u \in \mathcal{A}_{N^-}\), one has

\[
\begin{aligned}
&\left\{\begin{array}{l}
a\|u\|^2_{D^{1,2}} + \lambda \|u\|^4_{D^{1,2}} = |u|^2_{2^*} \leq S^{-\frac{2}{2^*}} \|u\|^2_{2^*}, \\
a\|u\|^2_{D^{1,2}} + 3\lambda \|u\|^4_{D^{1,2}} > (2^* - 1)|u|^2_{2^*}.
\end{array}\right.
\end{aligned}
\]
Lemma 5.2. This, combined with Lemma 5.2, gives

\[ I_\infty(u) = \frac{1}{N} \epsilon_1 \|u\|_{D_1^2}^N - \frac{4 - 4}{4N} \lambda \|u\|_{D_1^2}^4 = I(\|u\|_{D_1^2}) \geq \mathcal{I}(\xi_2) = c^*_{+N}. \]

Thus \( m^{N,+}_\infty = c^*_{+N}. \)

Secondly, we show \( m^{N,+}_\infty = m^{N,+}. \) On the one hand, by Lemma 2.7, for any \( u \in \mathcal{N}^{N,+} \), there exists \( t^*_+ \geq 1 \) such that \( I_{t^*_+}(u) \leq I_\infty(tu) \leq I_{\infty}(u) \leq I(u) \).

The arbitrariness of \( u \) from \( \mathcal{N}^{N,+} \) deduces \( m^{N,+}_\infty \leq m^{N,+}. \) On the other hand, \( m^{N,+} \leq m^{N,+}_\infty \) needs to be proved. By Lemma 2.7, consider the sequence \( u_n = t_n \varphi_{\delta_n}^+ \in \mathcal{N}^{N,+} \), with \( \varphi_{\delta_n}^+ \) defined by (2.13), \( \{z_n\} \subset \mathbb{R}^N \) such that \( |z_n| \to \infty \) as \( n \to \infty \) and \( t_n = t^*_{\varphi_{\delta_n}^+}. \) Applying a similar argument as that of (2.24), we can obtain

\[ \lim_{n \to \infty} I(u_n) = m^{N,+}_\infty. \]

Then \( m^{N,+} = m^{N,+}_\infty \) has been proved.

Lastly, we show that \( m^{N,+} \) is not attained. Arguing indirectly, assuming \( v_0 \in \mathcal{N}^{N,+} \) such that \( I(v_0) = m^{N,+} = m^{N,+}_\infty \), by Lemma 2.7, there exists \( t^*_+ \geq 1 \) such that \( I_{t^*_+}(v_0) \in \mathcal{N}^{N,+}. \) Then

\[
m^{N,+}_\infty \leq I_{t^*_+}(v_0) = \frac{a}{2} \|t^*_+ v_0\|_{D_1^2}^2 + \frac{\lambda}{4} \|t^*_+ v_0\|_{D_1^2}^4 - \frac{1}{2} \|t^*_+ v_0\|_{H^2}^2 \leq I_{t^*_+}(u_0) = m^{N,+}_\infty,
\]

which implies that \( t^*_+ = 1 \) and \( \int_{\mathbb{R}^N} V(x) |v_0|^2 dx = 0. \) Similarly to the proof of Proposition 2.9, we can get a contradiction. This completes the proof.

Proof of Theorem 1.5. If either \( N = 4 \) and \( \lambda > S^{-2} \) or \( N \geq 5 \) and \( \lambda > \Lambda_0 \) hold, then with the help of Lemma 2.1, one obtains

\[ f(t) = \lambda S \frac{N}{N-2} t^2 - t^{2^*-2} + aS \frac{N}{N-2} > 0 \]

for all \( t \geq 0. \) On the other hand, if \( u_0 \) is a nontrivial solution of problem \( (SK^*) \), then

\[ (a + \lambda \|u_0\|_{H^1}^2) \|u_0\|_{D_1^2}^2 + \int_{\mathbb{R}^N} V(x) |u_0|^2 dx = |u_0|_{H_2^2}^2. \]

It follows from \( V(x) \geq 0 \) that

\[ (a + \lambda \|u_0\|_{H^1}^2) \|u_0\|_{D_1^2}^2 \leq S^{-2^*} \|u_0\|_{H_2^2}^2. \]

Setting \( t_0 = \|u_0\|_{D_1^2} > 0 \), we have

\[ f(t_0) = \lambda S \frac{N}{N-2} t_0^2 - t_0^{2^*-2} + aS \frac{N}{N-2} \leq 0, \]

which is impossible. The proof is completed. \( \square \)
3. Compactness results. In this section, we investigate the behavior of Palais-\-Smale sequences of $I$. The following proposition is a description of Palais-\-Smale sequences of $I$, which comes from Proposition 5.1 in Xie et al. [25].

**Proposition 3.1.** Under the assumption $V \in L^{\frac{4}{3}}(\mathbb{R}^N)$, let $\{u_n\}$ be a Palais-\-Smale sequence of $I$. Then $\{u_n\}$ has a subsequence which strongly converges in $D^{1,2}(\mathbb{R}^N)$, or, otherwise, replacing $\{u_n\}$ if necessary by a subsequence, there exist a function $u_0 \in D^{1,2}(\mathbb{R}^N)$, a number $A \in \mathbb{R}$, a number $l \in \mathbb{N}$, $l$ sequences of number $\{\sigma_n^i\} \subset \mathbb{R}^+$, points $\{y_n^i\} \subset \mathbb{R}^N$ and functions $w^i \in D^{1,2}(\mathbb{R}^N)$, $i \in \{1, 2, \cdots, l\}$, satisfying

$$-(a + \lambda A^2)\Delta u_0 + V(x)u_0 = |u_0|^{2^* - 2}u_0 \quad \text{in} \ \mathbb{R}^N$$

and

$$-(a + \lambda A^2)\Delta u^i = |u^i|^{2^* - 2}u^i \quad \text{in} \ \mathbb{R}^N,$$

such that, up to subsequences, there hold

$$\left\| u_n - u_0 - \sum_{i=1}^{l} (\sigma_n^i)^{-\frac{N+2}{2}} u^i (\frac{\cdot - y_n^i}{\sigma_n^i}) \right\|_{\mathcal{D}^{1,2}} \to 0,$$

$$\|u_n\|^2_{\mathcal{D}^{1,2}} \to A^2 = \|u_0\|^2_{\mathcal{D}^{1,2}} + \sum_{i=1}^{l} \|u^i\|^2_{\mathcal{D}^{1,2}},$$

and

$$I(u_n) \to J(u_0) + \sum_{i=1}^{l} J_{\infty}(u^i),$$

as $n \to \infty$ and we define that

$$J(u_0) = \left(\frac{a}{2} + \frac{\lambda A^2}{4}\right) \|u_0\|^2_{\mathcal{D}^{1,2}} + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u_0|^2 \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u_0|^{2^*} \, dx;$$

$$J_{\infty}(u_i) = \left(\frac{a}{2} + \frac{\lambda A^2}{4}\right) \|u^i\|^2_{\mathcal{D}^{1,2}} - \frac{1}{2^*} \int_{\mathbb{R}^N} |u^i|^{2^*} \, dx.$$

As a consequence, we have the following compactness result in $N = 4$.

**Proposition 3.2.** Let $a > 0$, $0 < \lambda < S^{-2}$ and $V \in L^{2}(\mathbb{R}^4)$ be a nonnegative function. Then $I$ satisfies a local $(PS)_c$ condition with $c \in (c^-_{\lambda, A^4}, 2c^-_{\lambda, A^4})$.

**Proof.** Arguing indirectly, for any Palais-\-Smale sequence $\{u_n\}$ of $I$ at level $c$ with $c \in (c^-_{\lambda, A^4}, 2c^-_{\lambda, A^4})$, $\{u_n\}$ contains no subsequence strongly convergent in $D^{1,2}(\mathbb{R}^4)$. Without loss of generality, according to Proposition 3.1, assume that $u^i$ are constant-\-sign solutions for $i \in \{1, 2, \cdots, k\}$ ($k \leq \min\{\lfloor \frac{1}{\lambda S^2} \rfloor, l\}$) and $u^j$ are sign-\-changing solutions for $j \in \{k + 1, k + 2, \cdots, l\}$. Define the integer part of a constant $Q$ by $[Q]$. We claim that

$$\|u^i\|^2_{\mathcal{D}^{1,2}} = \|u^i\|^2_{\mathcal{D}^{1,2}} \quad \text{and} \quad J_{\infty}(u^i) = J_{\infty}(u^i)$$

for any $i, i' \in \{1, 2, \cdots, k\}$ with $i \neq i'$. Actually, from an argument similar to that in the proof of Proposition 2.2, we have that, for any positive solution $u^i$ of problem (3.2), the function

$$w^i(x) := (a + \lambda A^2)^{-\frac{1}{2}} u^i \in D^{1,2}(\mathbb{R}^4)$$

is a solution of problem $(S^*)$. The uniqueness of the positive solution of problem $(S^*)$ implies that $\|w^i\|^2_{\mathcal{D}^{1,2}} = S^2$ and

$$\|u^i\|^2_{\mathcal{D}^{1,2}} = (a + \lambda A^2)^2 S^2 \quad \text{for any} \ i \in \{1, 2, \cdots, k\},$$

and
which implies that
\[ \|u^i\|_{D^{1,2}}^2 = \left( a + \lambda \|u_0\|_{D^{1,2}}^2 + k \lambda \|u^i\|_{D^{1,2}}^2 + \lambda \sum_{j=k+1}^l \|u^j\|_{D^{1,2}}^2 \right) S^2. \]

Equivalently,
\[ \|u^i\|_{D^{1,2}}^2 = \frac{S^2}{1 - k \lambda S^2} \left( a + \lambda \|u_0\|_{D^{1,2}}^2 + \lambda \sum_{j=k+1}^l \|u^j\|_{D^{1,2}}^2 \right) \]
for any \( i \in \{1, 2, \ldots, k\} \). This, combined with (3.2) and (3.5), gives us
\[ J_\infty(u^i) = J_\infty(u^i) - \frac{1}{4} (a\|u^i\|_{D^{1,2}}^2 + b \lambda^2 \|u^i\|_{D^{1,2}}^2 - |u^i|_4^4) \]
\[ = \frac{a S^2}{4 (1 - k \lambda S^2)} \left( a + \lambda \|u_0\|_{D^{1,2}}^2 + \lambda \sum_{j=k+1}^l \|u^j\|_{D^{1,2}}^2 \right) \]
for any \( i \in \{1, 2, \ldots, k\} \).
Assume that there exist \( l \geq 1 \) and \( k \leq \min\{\frac{1}{\lambda S^2}, \ l\} \) such that
\[ c = J(u_0) + k c^*(a, \lambda, k, l, \|u_0\|_{D^{1,2}}) + \sum_{j=k+1}^l J_\infty(u^j). \]

In what follows, we will estimate every term in the right hand side of the above equality.

Firstly, estimate the value of \( J(u_0) \). It is obvious that \( J(0) = 0 \). If \( u_0 \neq 0 \), it follows from \( A^2 \geq \|u_0\|_{D^{1,2}}^2 > 0 \), \( V(x) \geq 0 \) and (3.1) that
\[ (a + \lambda \|u_0\|_{D^{1,2}}^2) \|u_0\|_{D^{1,2}}^2 \leq \int_{\mathbb{R}^4} |u_0|^4 dx \leq S^{-2} \|u_0\|_{D^{1,2}}^4, \]
which implies that
\[ \|u_0\|_{D^{1,2}}^2 \geq \frac{a S^2}{1 - \lambda S^2}. \]
(3.9)
By (3.1), (3.5) and (3.9), one obtains
\[ J(u_0) = J(u_0) - \frac{1}{4} \left( (a + \lambda A^2) \|u_0\|_{D^{1,2}}^2 + \int_{\mathbb{R}^4} V(x)|u_0|^2 dx - |u_0|_4^4 \right) \]
\[ = \frac{a S^2}{4 (1 - b S^2)} = c^{a, \lambda, 4}. \]
(3.10)
Secondly, note that
\[ c^*(a, \lambda, k, l, \|u_0\|_{D^{1,2}}) \geq c^*(a, \lambda, 1, 1, 0) = \frac{(a S^2)^2}{4 (1 - \lambda S^2)} = c^{a, \lambda, 4}. \]
(3.11)
Lastly, estimate the level of \( J_\infty(u^j) \) with \( k + 1 \leq j \leq l \). By Remark 2.3, \( w^j = (a + \lambda A^2)^{-\frac{1}{2}} u^j \) is a sign-changing solution of problem (\( S^* \)). Thus \( \|w^j\|_{D^{1,2}}^2 \geq 2 S^2 \) and
\[ \|w^j\|_{D^{1,2}}^2 \geq 2 (a + \lambda A^2) S^2 \geq 2 (a + \lambda \|u^j\|_{D^{1,2}}^2) S^2, \]
which implies that \(1 - 2\lambda S^2 > 0\) and
\[
\|u^j\|^2_{D^{1,2}} \geq \frac{2\alpha S^2}{1 - 2\lambda S^2} > \frac{2aS^2}{1 - \lambda S^2}.
\]
After a simple calculation, we have
\[
J_\infty (u^j) = J_\infty (u^j) - \frac{1}{4} (a\|u^j\|^2_{D^{1,2}} + \lambda A^2\|u^j\|^2_{D^{1,2}} - |u^j|^4)
\]
\[
= \frac{a}{4}\|u^j\|^2_{D^{1,2}} \geq \frac{2(\alpha S^2)}{4(1 - \lambda S^2)} = 2c_{-}^{a,\lambda,4}
\]
for \(k + 1 \leq j \leq l\).

Now, if the equality (3.8) holds, then one of the following four cases will occur:

(i) \(u_0 = 0\) and \(l = k = 1\). Then \(c = c^*(a, \lambda, 1, 1, 0) = c_{-}^{a,\lambda,4}\);

(ii) \(u_0 = 0\) and \(l \geq k \geq 2\). Then \(c = kc^*(a, \lambda, k, k, 0) \geq 2c_{-}^{a,\lambda,4}\);

(iii) \(u_0 = 0\) and \(l > k\). Then
\[
c = kc^*(a, \lambda, k, l, 0) + \sum_{j=k+1}^{l} J_\infty (u^j) \geq J_\infty (u^l) \geq 2c_{-}^{a,\lambda,4};
\]

(iv) \(u_0 \neq 0\) and \(l \geq 1\). Then
\[
c = J(u_0) + kc^*(a, \lambda, k, l, \|u_0\|_{D^{1,2}}) + \sum_{j=k+1}^{l} J_\infty (u^j) \geq 2c_{-}^{a,\lambda,4}.
\]

Because of \(c \in (c_{-}^{a,\lambda,4}, 2c_{-}^{a,\lambda,4})\), none of the possibilities (i)-(iv) can be true. The proof is completed.

When \(N \geq 5\), we will have a quite different result.

**Proposition 3.3.** Let \(a > 0\), \(N \geq 5\) and \(V \in L^\infty_\pi (\mathbb{R}^N)\) be a nonnegative function. Then there exist two positive constants \(\mu_2 := \mu_2(a, N) < \frac{1}{2}\Lambda_0\) and \(\nu_1 := \nu_1(a, N, \lambda)\). If \(c\) is neither \(c_{-}^{a,\lambda,N}\) nor \(c_{-}^{a,\lambda,N}\), then \(I\) satisfies the (PS)_c condition for \(\Lambda_0 - \mu < \lambda < \Lambda_0\) and \(0 < |V|_{\pi} < \nu_1\).

**Proof.** Arguing indirectly, for any Palais-Smale sequence \(\{u_n\}\) of \(I\) at level \(c\), in which \(c\) is neither \(c_{-}^{a,\lambda,N}\) nor \(c_{-}^{a,\lambda,N}\), \(\{u_n\}\) contains no subsequence strongly convergent in \(D^{1,2}(\mathbb{R}^N)\). According to Proposition 3.1, without loss of generality, we can assume that \(u^j\) are constant-sign solutions for \(j \in \{1, 2, \cdots, k\}\) and \(u^j\) are sign-changing solutions for \(j \in \{k + 1, k + 2, \cdots, l\}\), \(l \geq 1\). We claim that

1. \(k = l = 1\);  
2. \(u_0 = 0\) and \(u^1 \neq 0\).

To prove (1): Firstly, prove \(k = l\). Namely, there is no sign-changing solution \(u^j\).

In fact, if \(u^j\) is a sign-changing solution of (3.2), then \(w^j = (a + \lambda A^2)^{\frac{1}{2}} u^j\) is a sign-changing solution of problem (\(S^\ast\)). Thus \(\|w^j\|^2_{D^{1,2}} \geq 2S^\ast\) and
\[
\|w^j\|^2_{D^{1,2}} \geq 2 (a + \lambda A^2)^{\frac{N}{2}} S^\ast \geq 2 (a + \lambda\|u^j\|^2_{D^{1,2}})^{\frac{N}{2}} S^\ast.
\]
Evidently,
\[
\lambda S^\ast \frac{N}{2} \|w^j\|^2_{D^{1,2}} - 2^\ast \frac{N}{2} \|u^j\|^2_{D^{1,2}} + aS^\ast \frac{N}{2} \leq 0,
\]
which is impossible for \(\lambda > \frac{1}{2}\Lambda_0\). Secondly, prove \(k = 1\). For any positive solution \(u^j\) of problem (3.2), the function \(w^i(x) := (a + \lambda A^2)^{-\frac{N}{2}} u^i\) is a solution of
problem \((S^\ast)\). The uniqueness of the positive solution of problem \((S^\ast)\) implies that 
\[ ||u^\ast||^2_{D^2} = S^\ast \].
If \(k \geq 2\), then
\[ ||u||^2_{D^1} = (a + \lambda^2)^{\frac{\alpha-2}{\alpha}} S\lambda \geq (a + k\lambda^2 ||u||^2_{D^1})^{\frac{\alpha-2}{\alpha}} S\lambda \]
\[ \geq (a + 2\lambda^2 ||u||^2_{D^1})^{\frac{\alpha-2}{\alpha}} S\lambda \].

Evidently,
\[ 2\lambda S^{\frac{\alpha}{2}} ||u||^2_{D^1} - ||u^\ast||^2_{D^1} + aS^{\frac{\alpha}{2}} \leq 0, \tag{3.15} \]
which is impossible for \(\lambda > \frac{1}{2}A_0\). Thus \(k = l = 1\) has been proved.

To prove (2), by the fact that \(k = l = 1\), we only have to prove \(u_0 = 0\). Due to Lemma 2.1, we have
\[ \min_{t \geq 0} f(t) \to 0 \] as \(\lambda \to A_0\).

Thus there exists \(0 < \mu_2 < \frac{1}{2}A_0\) such that, for \(\lambda_0 - \mu_2 < \lambda < \lambda_0\),
\[ f(||u||^2_{D^1}) = \lambda S^{\frac{\alpha}{2}} ||u||^2_{D^1} - ||u^\ast||^2_{D^1} + aS^{\frac{\alpha}{2}} \leq -\lambda a^{\frac{\alpha}{2}} S\lambda. \tag{3.16} \]
If \(u_0 \neq 0\), then it follows from (3.1), \(A^2 \geq ||u_0||^2_{D^1} > 0\) and \(V(x) \geq 0\) that
\[ a||u_0||^2_{D^1} \leq (a + \lambda||u_0||^2_{D^1})||u_0||^2_{D^1} \leq ||u_0||^2_{D^1} \leq \lambda a^{\frac{\alpha}{2}} ||u_0||^2_{D^1}, \]
which implies that
\[ ||u_0||^2_{D^1} \geq \left(a S^{\frac{\alpha}{2}}\right)^\frac{\alpha}{\alpha-2}. \tag{3.17} \]

Similarly to (3.14) with \(k = 1\), one obtains
\[ ||u^1||^2_{D^1} = (a + \lambda||u_0||^2_{D^1} + \alpha||u^1||^2_{D^1})^{\frac{\alpha-2}{\alpha}} S\lambda \]
\[ \geq \left(a + \lambda \left(a S^{\frac{\alpha}{2}}\right)^\frac{\alpha}{\alpha-2} + \alpha||u^1||^2_{D^1}\right)^{\frac{\alpha-2}{\alpha}} S\lambda. \tag{3.18} \]

Evidently,
\[ \lambda S^{\frac{\alpha}{2}} ||u^1||^2_{D^1} - ||u^\ast||^2_{D^1} + aS^{\frac{\alpha}{2}} \leq -\lambda a^{\frac{\alpha}{2}} S\lambda, \tag{3.19} \]
which contradicts with (3.16). Hence, \(u_0 = 0\) has been proved.

By our claims (1) and (2), and (3.2), we have \(A^2 = ||u^1||^2_{D^1}\) and \(u^1\) is a constant-sign solution of
\[ -(a + \lambda||u^1||^2_{D^1}) \Delta u^1 = |u^1|^{\alpha-2} u^1 \] in \(\mathbb{R}^N\).

It follows from Remark 2.4 and (2.13) that \(\varphi_{y,\gamma}^\pm\) are the two unique positive solutions of (3.20). Thus \(u^1 = \pm \varphi_{\delta,y}^-\) or \(u^1 = \pm \varphi_{\delta,y}^+\) and accordingly
\[ c = c_{-\lambda,N}^- \text{ or } c = c_{\lambda,N}^- \],
which is impossible. The proof is completed.

**Corollary 3.4.** If \(\{u_n\}\) is a minimizing sequence for \(I\) on \(\mathcal{N}^{N,\pm}\) (or \(\mathcal{N}^{4,-}\)) for \(N \geq 5\), then there exist a sequence \(\{y_n\} \subset \mathbb{R}^N\), a sequence of positive numbers \(\{\delta_n\}\) and a sequence \(\{w_n\} \subset D^{1,2}(\mathbb{R}^N)\) such that
\[ u_n(x) = w_n(x) + \varphi_{\delta_n,y_n}^\pm(x), \tag{3.21} \]
where \(\varphi_{\delta_n,y_n}^\pm(x)\) are functions defined in (2.13) (or (2.8)) and \(w_n \to 0\) as \(n \to \infty\) strongly in \(D^{1,2}(\mathbb{R}^N)\).
Proof. We only prove the result on $N^N_{N-}$ for $N \geq 5$. The following methods are similar to the proof of Theorem 2.1 in [3] or the proof of Theorem 1 in [19]. By Ekeland’s Variational Principle, there exists a minimising subsequence $\{u_n\} \subset N^N_{N-}$ of the minimisation problem such that
\[
I(u_n) < c_{-}^{a,N} + \frac{1}{n}
\]
and
\[
I(v) \geq I(u_n) - \frac{1}{n} \|v - u_n\|_{D^{1,2}} \quad \text{for} \quad v \in N^N_{N-}. \tag{3.22}
\]
Since $\{u_n\} \subset N^N_{N-}$, similar to (2.23), we get
\[
\xi_1 \leq \|u_n\|_{D^{1,2}} < \xi_4, \tag{3.23}
\]
where $\xi_i$, $i = 1, 4$, come from Lemma 5.2. It follows from $I(u_n) \to c_{-}^{a,N}$ that
\[
I(u_n) = \frac{1}{N} u\|u_n\|_{D^{1,2}}^2 - \frac{N - 4}{4N} \lambda \|u_n\|_{D^{1,2}}^4 \to c_{-}^{a,N} \quad \text{as} \quad n \to \infty. \tag{3.24}
\]
With the help of (5.9), (3.23) and (3.24), one obtains
\[
\|u_n\|_{D^{1,2}} \to \xi_1. \tag{3.25}
\]
Now we will show
\[
\|I'(u_n)\| \to 0. \tag{3.26}
\]
Since $\{u_n\} \subset N^N_{N-}$, by Lemma 5.3, we can find $\varepsilon_n > 0$ and a differentiable functional $t_n(w) > 0$, $w \in D^{1,2}(\mathbb{R}^N)$ satisfying that
\[
t_n(w)(u_n - w) \in N^N_{N-} \quad \text{for} \quad \|w\|_{D^{1,2}} < \varepsilon_n.
\]
By the continuity of $t_n(w)$ and $t_n(0) = 1$, without loss of generality, we can assume $\varepsilon_n$ is small enough such that $\frac{1}{2} \leq t_n(w) \leq \frac{3}{2}$ for $\|w\|_{D^{1,2}} < \varepsilon_n$. By (3.22), we obtain
\[
I(t_n(w)(u_n - w)) - I(u_n) \geq \frac{1}{n} \|t_n(w)(u_n - w) - u_n\|_{D^{1,2}},
\]
that is,
\[
\langle I'(u_n), t_n(w)(u_n - w) - u_n \rangle + o(\|t_n(w)(u_n - w) - u_n\|_{D^{1,2}})
\geq \frac{1}{n} \|t_n(w)(u_n - w) - u_n\|_{D^{1,2}}.
\]
Then it follows that
\[
t_n(w) \langle I'(u_n), w \rangle + (1 - t_n(w)) \langle I'(u_n), u_n \rangle \leq \frac{1}{n} \|t_n(w) - 1\| u_n - t_n(w)w\|_{D^{1,2}}
+ o(\|t_n(w)(u_n - w) - u_n\|_{D^{1,2}}).
\]
By the choice of $\varepsilon_n$, we obtain
\[
\langle I'(u_n), w \rangle \leq \frac{C}{n} \|t_n'(0), w\| + o(\|w\|_{D^{1,2}})
+ \frac{1}{n} o(\|w\|_{D^{1,2}}) + o(\|t_n'(0), w\|) (\|u_n\|_{D^{1,2}} + \|w\|_{D^{1,2}}). \tag{3.27}
\]
If we prove that
\[
\|t_n'(0), w\| \leq C \|w\|_{D^{1,2}}, \tag{3.28}
\]
then by (3.27), we get
\[
\langle I'(u_n), w \rangle \leq \frac{C}{n} \|w\|_{D^{1,2}} + o(\|w\|_{D^{1,2}}) \quad \text{for} \quad \|w\|_{D^{1,2}} < \varepsilon_n.
\]
Hence, for any \(0 < \varepsilon < \varepsilon_n\), we have

\[
\|I'(u_n)\| \leq \frac{1}{\varepsilon} \sup_{w \in \mathcal{D}^{1,2}} \langle I'(u_n), w \rangle \leq \frac{C}{n} + \frac{1}{\varepsilon} o(\varepsilon),
\]

for some \(C > 0\) independent of \(\varepsilon\) and \(n\). Taking \(\varepsilon \to 0\), we obtain (3.26). We now turn to proving (3.28). Indeed, by (5.12), we have

\[
\langle t'_n(0), w \rangle = \frac{(4\lambda\|u_n\|_{\mathcal{D}^{1,2}}^2 - 2a) \int_{\mathbb{R}^N} \nabla u_n \nabla w dx + 2 \int_{\mathbb{R}^N} V(x) u_n w dx}{2\lambda\|u_n\|_{\mathcal{D}^{1,2}}^2 - (2^* - 2)\|u_n\|_{2^*}^2} + 2\|\int_{\mathbb{R}^N} |u_n|^{2^* - 2} u_n w dx\|\leq \left( \frac{(4\lambda\|u_n\|_{\mathcal{D}^{1,2}}^2 - 2a)\|u_n\|_{\mathcal{D}^{1,2}}^2 + \frac{2}{S} V(\frac{1}{2})\|u_n\|_{\mathcal{D}^{1,2}}^2 + 2S^{-\frac{2}{2^*}}\|u_n\|_{\mathcal{D}^{1,2}}^{2^* - 1} \right)\|w\|_{\mathcal{D}^{1,2}}
\]

\[
\leq \frac{C\|w\|_{\mathcal{D}^{1,2}}}{|2\lambda\|u_n\|_{\mathcal{D}^{1,2}}^2 - (2^* - 2)\|u_n\|_{2^*}^2|}.
\]

To prove (3.28), we only need to show that

\[
\left|2\lambda\|u_n\|_{\mathcal{D}^{1,2}}^2 - (2^* - 2)\|u_n\|_{2^*}^2\right| > \delta,
\]

for some positive constant \(\delta\) and \(n\) large. By the definition of \(f\), we get

\[
f \left( \left( \frac{2}{N - 4 - \lambda} \right)^\frac{2}{2} \right) = \lambda S^\frac{a}{2^*} \left( \frac{2}{N - 4 - \lambda} \right) ^\frac{2^* - 2}{2} + aS^\frac{a}{2^*} - \left( \frac{a}{N - 4 - \lambda} \right) ^\frac{2^* - 2}{2} = 0.
\]

By the continuity of \(f(t)\), there exists \(M > 0\) such that \(A_n := \left( \frac{2}{N - 4 - \lambda} \right)^\frac{2}{2}\) satisfying \(f(A_n) < 0\) for \(n > M\). Combining with \(f(\lambda) = 0\), we get that \(\lambda < A_n\).

Arguing indirectly, assume that, for a subsequence (still denoted by \(\{u_n\}\)), we have, for \(n > M\),

\[
\left|2\lambda\|u_n\|_{\mathcal{D}^{1,2}}^2 - (2^* - 2)\|u_n\|_{2^*}^2\right| < \frac{(2^* - 2)\|u_n\|_{\mathcal{D}^{1,2}}^2}{n},
\]

which implies that

\[
\|u_n\|_{2^*}^2 \leq \frac{2\lambda}{2^* - 2}\|u_n\|_{\mathcal{D}^{1,2}}^2 + \frac{\|u_n\|_{\mathcal{D}^{1,2}}^2}{n}.
\]

It follows from \(\{u_n\} \subset \mathcal{N}^{N, -}\) that

\[
a\|u_n\|_{\mathcal{D}^{1,2}}^2 + \lambda\|u_n\|_{\mathcal{D}^{1,2}}^2 \leq a\|u_n\|_{\mathcal{D}^{1,2}}^2 + \int_{\mathbb{R}^N} V(x) u_n^2 dx + \lambda\|u_n\|_{\mathcal{D}^{1,2}}^2 = \|u_n\|_{2^*}^2
\]

\[
\leq \frac{2\lambda}{2^* - 2}\|u_n\|_{\mathcal{D}^{1,2}}^2 + \frac{\|u_n\|_{\mathcal{D}^{1,2}}^2}{n}
\]

which implies \(\|u_n\|_{\mathcal{D}^{1,2}}^2 \geq A_n \geq \xi_1\). This is impossible because of (3.25). Therefore, we conclude that \(I'(u_n) \to 0\) as \(n \to \infty\) in the dual space of \(\mathcal{D}^{1,2}(\mathbb{R}^N)\). What has been proved is that the minimizing sequence \(\{u_n\}\) for \(I\) on \(\mathcal{N}^{N, -}\) is a Palais-Smale sequence of \(I\) at level \(c^{-\lambda, N}_{-}\). This result follows from Proposition 2.9 and Proposition 3.3 immediately.
4. **Existence of bounded state solutions.** In this section, we build a suitable min-max scheme for our problem. Firstly, it should be recalled the definition of the barycenter of a function. Setting

\[
\sigma(x) = \begin{cases} 
0 & \text{if } |x| < 1; \\
1 & \text{if } |x| \geq 1,
\end{cases}
\]

define \(\alpha : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}^N \times \mathbb{R}^+\) by

\[
\alpha(u) = \frac{1}{S^2 K^2} \int_{\mathbb{R}^N} \frac{x}{|x|} \sigma(x) |\nabla u|^2 dx := (\beta(u), \gamma(u)),
\]

(4.1)

where

\[
\beta(u) = \frac{1}{S^2 K^2} \int_{\mathbb{R}^N} \frac{x}{|x|} |\nabla u|^2 dx, \quad \gamma(u) = \frac{1}{S^2 K^2} \int_{\mathbb{R}^N} \sigma(x) |\nabla u|^2 dx
\]

and \(K = K_-\) or \(K = K_+\) is a positive solution of \(g(t) = 0\) in Lemma 2.1.

Recall that \(N^4^+ = \emptyset\). In order to facilitate presentation, in the following context, we define

\[N^4^\pm := N^4^- \quad \text{and} \quad c^a,\lambda,4^\pm := c^a,\lambda,4^-.
\]

With the help of Lemmas 2.6 and 2.7, we can define the map \(\theta^\pm : (y, \delta) \in \mathbb{R}^N \times \mathbb{R}^+ \to D^{1,2}(\mathbb{R}^N)\) by

\[
\theta^\pm(y, \delta) = t^\pm \Phi^\pm y \in N^{N,\pm} \quad \text{for } N \geq 4,
\]

where \(\Phi^\pm y\) is defined in (2.8) or (2.13).

**Lemma 4.1.** Under the assumptions of Theorem 1.1, we have

\[
\sup_{(y, \delta) \in \mathbb{R}^N \times (0, \infty)} I(\theta^- (y, \delta)) < 2c^a,\lambda,4^-.
\]

(4.2)

**Proof.** It follows from \(t^- \varphi^- y \in N^{4^-}\) that

\[
I(\theta^- (y, \delta)) = I(\theta^- (y, \delta)) - \frac{1}{2} \langle I'(\theta^- (y, \delta)), \theta^- (y, \delta) \rangle
\]

\[
= \frac{1}{4} (t^-)^4 \left( |\varphi^- y|^4 - \lambda \|\varphi^- y\|_{D^{1,2}}^4 \right)
\]

\[
= \frac{1}{4} (t^-)^4 K^4 \left( |U_{\delta,y}|^4 - \lambda \|U_{\delta,y}\|_{D^{1,2}}^4 \right)
\]

(4.3)

and

\[
(t^-)^2 \left( |\varphi^- y|^4 - \lambda \|\varphi^- y\|_{D^{1,2}}^4 \right) = a \|\varphi^- y\|_{D^{1,2}}^2 + \int_{\mathbb{R}^4} V(x)|\varphi^- y|^2 dx.
\]

A direct computation gives

\[
(t^-)^2 \leq \frac{a \|\varphi^- y\|_{D^{1,2}}^2 + \int_{\mathbb{R}^4} V(x)|\varphi^- y|^2 dx}{|\varphi^- y|^4 - \lambda \|\varphi^- y\|_{D^{1,2}}^4} \leq 1 + \frac{|V|_2}{aS}.
\]
By (2.3), (4.3) and the assumption that \(|V|_2 < (\sqrt{2} - 1) S\), one obtains
\[
\sup_{(y, \delta) \in \mathbb{R}^N \times (0, \infty)} I(\theta^-(y, \delta)) \leq \frac{1}{4} (t_\varphi^-)^4 K_\varphi^- \left( |U_{\delta, y}|_4^4 - \lambda \|U_{\delta, y}\|_{D^1,2}^4 \right) \\
\leq \frac{1}{4} \left( 1 + \frac{|V|_2}{aS} \right)^2 aK^2 \|U_{\delta, y}\|_{D^1,2}^2 \\
\leq \left( 1 + \frac{|V|_2}{aS} \right)^2 c^{\alpha, \lambda, 4} < 2c^{\alpha, \lambda, 4}.
\]

This completes the proof. \(\square\)

**Lemma 4.2.** Let \(a > 0\), \(N \geq 5\) and \(V \in L^\infty(\mathbb{R}^N)\) be a nonnegative function. Then there exists a positive constant \(\nu_2 := \nu_2(a, N, \lambda)\) such that, for \(0 < \lambda < \Lambda_0\) and \(0 < |V|_\infty < \nu_2\),
\[
\sup_{(y, \delta) \in \mathbb{R}^N \times (0, \infty)} I(\theta^+(y, \delta)) < c^{\alpha, \lambda, N}.
\]

**Proof.** Actually,
\[
I(\theta^+(y, \delta)) = I(\theta^+(y, \delta)) - \frac{1}{2} \langle I'(\theta^+(y, \delta)), \theta^+(y, \delta) \rangle \\
= \frac{a}{N} (t_\varphi^+)^2 \|\varphi_{\delta, y}^+\|_{D^1,2}^2 - \frac{N - 4}{4N} \lambda (t_\varphi^+)^4 \|\varphi_{\delta, y}^+\|_{D^1,2}^4 \\
+ \frac{1}{N} (t_\varphi^+)^2 \int_{\mathbb{R}^N} V(x) |\varphi_{\delta, y}^+|^2 dx \\
\leq (t_\varphi^+)^4 \left[ \frac{a}{N} \|\varphi_{\delta, y}^+\|_{D^1,2}^2 - \frac{N - 4}{4N} \lambda \|\varphi_{\delta, y}^+\|_{D^1,2}^4 \right] \\
+ \frac{1}{N} (t_\varphi^+)^2 \int_{\mathbb{R}^N} V(x) |\varphi_{\delta, y}^+|^2 dx \\
\leq (t_\varphi^+)^4 \left( c^{\alpha, \lambda, N} + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |\varphi_{\delta, y}^+|^2 dx \right).
\]

By Lemma 2.7, we have \(t_\varphi^+ = 1 + o(1)\) as \(|V|_\infty \to 0\). Thus, there exists \(\nu_2 = \nu_2(a, N, \lambda)\) such that, for \(|V|_\infty < \nu_2\),
\[
(t_\varphi^+)^4 \left( c^{\alpha, \lambda, N} + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |\varphi_{\delta, y}^+|^2 dx \right) < c^{\alpha, \lambda, N}.
\]

Hence, the result (4.4) holds. \(\square\)

Similarly to the proof of Lemma 4.2 in Xie et al. [25], we can easily prove the following results.

**Lemma 4.3.** If \(|y| \geq \frac{1}{2}\), then
\[
\beta(\varphi_{\delta, y}^+) = \frac{y}{|y|} + o(1) \quad \text{as} \quad \delta \to 0,
\]
where \(\varphi_{\delta, y}^+\) is defined in (2.8) or (2.13).

Define a subset of the Nehari manifold by
\[
\mathcal{M}^N,\pm = \left\{ u \in \mathcal{N}^N,\pm : (\beta(u), \gamma(u)) = (0, \frac{1}{2}) \right\} \quad \text{for} \quad N \geq 4.
\]
Obviously, $\mathcal{M}^{N,\pm}$ are not empty, because there exists $\delta^*$ such that $\theta^\pm(0, \delta^*) \in \mathcal{M}^{N,\pm}$. Then we set

$$c_{\mathcal{M}^{N,\pm}} = \inf_{u \in \mathcal{M}^{N,\pm}} I(u) \quad \text{for} \quad N \geq 4.$$  

(4.5)

Lemma 4.4. Under the assumptions of Theorem 1.1 or Theorem 1.3, the following inequalities hold,

$$c_{\mathcal{M}^{N,\pm}} > c_{\pm}^{a,\lambda, N} \quad \text{for} \quad N \geq 4.$$  

(4.6)

Proof. It is clear that $c_{\mathcal{M}^{N,\pm}} \geq c_{\pm}^{a,\lambda, N}$ and we need to show that the equality cannot hold. Arguing indirectly, assume a sequence $\{u_n\} \subset \mathcal{M}^{N,\pm}$ such that

$$\lim_{n \to \infty} I(u_n) = c_{\pm}^{a,\lambda, N};$$  

(4.7)

$$\beta(u_n) = 0 \quad \text{and} \quad \gamma(u_n) = \frac{1}{2} \quad \text{for any} \quad n \in \mathbb{N}.$$  

(4.8)

According to Corollary 3.4, there exist a sequence $\{y_n\} \subset \mathbb{R}^N$, a sequence of positive numbers $\{\delta_n\}$ and $w_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$ such that

$$u_n(x) = w_n(x) + \varphi_{\delta_n,y_n}^\pm(x).$$

By the fact that $w_n \to 0$ strongly in $D^{1,2}(\mathbb{R}^N)$, we have

$$\alpha(u_n) = \alpha(\varphi_{\delta_n,y_n}^\pm) + o(1) \quad \text{as} \quad n \to \infty.$$  

It follows from (4.8) that

$$\beta(\varphi_{\delta_n,y_n}^\pm) \to 0 \quad \text{and} \quad \gamma(\varphi_{\delta_n,y_n}^\pm) \to \frac{1}{2} \quad \text{as} \quad n \to \infty.$$  

(4.9)

Then there exists a subsequence $(\delta_{n_k}, y_{n_k})$ such that one of the following four cases occurs:

1. $\delta_{n_k} \to \infty$ as $n \to \infty$;
2. $\delta_{n_k} \to \bar{\delta} \neq 0$ as $n \to \infty$;
3. $\delta_{n_k} \to 0$ and $y_{n_k} \to \bar{y}$ as $n \to \infty$, where $|\bar{y}| < \frac{1}{2}$;
4. $\delta_{n_k} \to 0$ as $n \to \infty$ and $|y_{n_k}| \geq \frac{1}{2}$ for $n$ large.

It suffices to show that none of the possibilities (1)-(4) can be true. If (1) holds, then

$$\gamma(\varphi_{\delta_{n_k},y_{n_k}}^\pm) = \frac{1}{S^2 K^2_\pm} \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla \varphi_{\delta_{n_k},y_{n_k}}^\pm|^2 dx = \frac{1}{S^2 K^2_\pm} \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla U_{\delta_{n_k},y_{n_k}}|^2 dx$$

$$= 1 - \frac{1}{S^2 K^2_\pm} \int_{B_1(0)} |\nabla U_{\delta_{n_k},y_{n_k}}|^2 dx = 1 - o(1) \quad \text{as} \quad n \to \infty,$$

which contradicts with (4.9). If (2) holds, then $|y_{n_k}| \to \infty$. Otherwise, $\varphi_{\delta_{n_k},y_{n_k}}^\pm$ would converge strongly in $D^{1,2}(\mathbb{R}^N)$, which contradicts with Proposition 2.9 or Proposition 2.10. By $\delta_{n_k} \to \bar{\delta} \neq 0$ and $|y_{n_k}| \to \infty$, we have

$$\gamma(\varphi_{\delta_{n_k},y_{n_k}}^\pm) = \gamma(\varphi_{\delta_{n_k},y_{n_k}}^\pm) + o(1) = \frac{1}{S^2 K^2_\pm} \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla \varphi_{\delta_{n_k},y_{n_k}}^\pm|^2 dx + o(1)$$

$$= \frac{1}{S^2 K^2_\pm} \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla U_{\bar{\delta},y_{n_k}}|^2 dx + o(1) = \frac{1}{S^2 K^2_\pm} \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla U_{\bar{\delta},0}|^2 dx + o(1)$$

$$= 1 - \frac{1}{S^2 K^2_\pm} \int_{B_1(0)} |\nabla U_{\bar{\delta},0}|^2 dx + o(1) = 1 + o(1) \quad \text{as} \quad n \to \infty.$$
Thus, $\gamma(\varphi_{\delta_n,y_n}^\pm) \to 1$, which contradicts with (4.9). If (3) holds, then
\[
\gamma(\varphi_{\delta_n,y_n}^\pm) = \frac{1}{S^{\frac{N}{2}}} \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla \varphi_{\delta_n,y_n}^\pm|^2 dx = \frac{1}{S^{\frac{N}{2}}} \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla U_{\delta_n,y_n}|^2 dx
\]
\[
= \frac{1}{S^{\frac{N}{2}}} \int_{\mathbb{R}^N \setminus B_1(y_n)} |\nabla U_{\delta_n}|^2 dx = o(1) \text{ as } n \to \infty,
\]
which contradicts with (4.9). Finally, if (4) holds, then it follows from Lemma 4.3 that
\[
\beta(\varphi_{\delta_n,y_n}^\pm) = \frac{\gamma_n}{|y_n|} + o(1) \text{ as } n \to \infty,
\]
which again contradicts with (4.9). This completes the proof. ☐

**Lemma 4.5.** Under the assumptions of Theorem 1.1 or Theorem 1.3, for any fixed $y \in \mathbb{R}^N$, the following equality holds
\[
\lim_{\delta \to 0^+} I(\theta^\pm(y, \delta)) = \lim_{\delta \to \infty} I(\theta^\pm(y, \delta)) = c^\pm_{\delta,y,N} \text{ for } N \geq 4.
\] (4.10)

**Proof.** It follows from the proof of Lemma 4.4 in Xie et al. [25] that, for any fixed $y \in \mathbb{R}^N$,
\[
\lim_{\delta \to 0^+} \int_{\mathbb{R}^N} V(x)U_{\delta,y}^2 dx = \lim_{\delta \to \infty} \int_{\mathbb{R}^N} V(x)U_{\delta,y}^2 dx = 0.
\] (4.11)
By Lemma 2.6, Lemma 2.7 and (4.11), for any fixed $y \in \mathbb{R}^N$, we have $t^\pm_{\varphi} \varphi_{\delta,y}^\pm \in \mathcal{N}^N,y$ and
\[
\lim_{\delta \to 0^+} t^\pm_{\varphi} = \lim_{\delta \to \infty} t^\pm_{\varphi} = 1.
\] (4.12)
It follows from (4.11), (4.12), and $t^\pm_{\varphi} \geq 1$ that
\[
\begin{align*}
c^\pm_{\delta,y,N} & \leq I(\theta^\pm(y, \delta)) = I(\theta^\pm(y, \delta)) - \frac{1}{2} \langle I'(\theta^\pm(y, \delta)), \theta^\pm(y, \delta) \rangle \\
& = \frac{a}{N}(t^\pm_{\varphi})^2 \|\varphi_{\delta,y}^\pm\|_{D^{1,2}}^2 - \frac{N - 4}{4N} \lambda(t^\pm_{\varphi})^4 \|\varphi_{\delta,y}^\pm\|_{D^{1,2}}^4 + \frac{1}{N}(t^\pm_{\varphi})^2 \int_{\mathbb{R}^N} V(x)|\varphi_{\delta,y}^\pm|^2 dx \\
& \leq (t^\pm_{\varphi})^4 \left[ \frac{a}{N} \|\varphi_{\delta,y}^\pm\|_{D^{1,2}}^2 - \frac{N - 4}{4N} \lambda \|\varphi_{\delta,y}^\pm\|_{D^{1,2}}^4 \right] + \frac{1}{N}(t^\pm_{\varphi})^2 \int_{\mathbb{R}^N} V(x)|\varphi_{\delta,y}^\pm|^2 dx \\
& = (t^\pm_{\varphi})^4 c^\pm_{\delta,y,N} + \frac{1}{N}(t^\pm_{\varphi})^2 K^2 \int_{\mathbb{R}^N} V(x)U_{\delta,y}^2 dx \\
& = c^\pm_{\delta,y,N} + o(1)
\end{align*}
\] (4.13)
as $\delta \to 0$ or $\delta \to \infty$. This completes the proof. ☐

**Lemma 4.6.** Under the assumptions of Theorem 1.1 or Theorem 1.3, for any fixed $\delta \in \mathbb{R}^+$, the following equality holds,
\[
\lim_{|y| \to \infty} I(\theta^\pm(y, \delta)) = c^\pm_{\delta,y,N} \text{ for } N \geq 4.
\] (4.14)

**Proof.** It follows from the proof of Lemma 4.5 in Xie et al. [25] that, for any fixed $\delta > 0$,
\[
\lim_{|y| \to \infty} \int_{\mathbb{R}^N} V(x)U_{\delta,y}^2 dx = 0.
\] (4.15)
By Lemma 2.6, Lemma 2.7, and (4.15), for any fixed $\delta \in \mathbb{R}^+$, we have $t^\pm_{\varphi} \varphi_{\delta,y}^\pm \in \mathcal{N}^N,y$ and
\[
\lim_{|y| \to \infty} t^\pm_{\varphi} = 1.
\] (4.16)
Similarly to (4.13), our result (4.14) follows from (4.15) and (4.16). This completes the proof.

By Lemma 4.4, there exists a constant \( \kappa > 0 \) such that
\[
c^{\alpha,\lambda,N}_{\pm} + \kappa < c_{M,N,\pm} \quad \text{for } N \geq 4.
\] (4.17)

**Lemma 4.7.** There exists a constant \( \delta_1 \in (0, \frac{1}{2}) \) such that, for \( \delta \in (0, \delta_1) \),
\[\begin{align*}
(a) \quad & I(\theta^+(y, \delta)) < c^{\alpha,\lambda,N}_{\pm} + \kappa \quad \text{for any } y \in \mathbb{R}^N; \\
(b) \quad & \gamma(\theta^+(y, \delta)) < \frac{1}{2} \quad \text{for any } y \in \mathbb{R}^N \text{ with } |y| < \frac{1}{2}; \\
(c) \quad & |\beta(\theta^+(y, \delta)) - \frac{y}{|y|}| < \frac{1}{4} \quad \text{for any } y \in \mathbb{R}^N \text{ with } |y| \geq \frac{1}{2}.
\end{align*}\]

**Proof.** Lemma 4.5 implies (a). It follows from \( |y| < \frac{1}{2} \) and \( t^\pm \to 1 \) as \( \delta \to 0 \) that
\[
\gamma(\theta^+(y, \delta)) = \frac{1}{S^N} (t^\pm)^2 \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla \varphi^\pm_{\delta,y}|^2 dx = \frac{1}{S^N} (t^\pm)^2 \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla U_{\delta,y}|^2 dx
\]
\[
= \frac{1}{S^N} (t^\pm)^2 \int_{B_1(y)} |\nabla U_{\delta,0}|^2 dx = o(1) \quad \text{as } \delta \to 0,
\]
which implies (b). With the help of Lemma 4.3 and (4.12), one obtains (c). \( \square \)

**Lemma 4.8.** There exists a constant \( \delta_2 \geq \frac{1}{2} \) such that, for \( \delta \geq \delta_2 \),
\[\begin{align*}
(a) \quad & I(\theta^+(y, \delta)) < c^{\alpha,\lambda,N}_{\pm} + \kappa \quad \text{for any } y \in \mathbb{R}^N; \\
(b) \quad & \gamma(\theta^+(y, \delta)) > \frac{1}{2} \quad \text{for any } y \in \mathbb{R}^N.
\end{align*}\]

**Proof.** Lemma 4.5 implies (a). It follows from (4.12) that
\[
\gamma(\theta^+(y, \delta)) = (t^\pm)^2 \left(1 - \frac{1}{S^N} \int_{B_1(0)} |\nabla U_{\delta,y}|^2 dx \right) = 1 - o(1) \quad \text{as } \delta \to \infty,
\]
which implies (b). \( \square \)

**Lemma 4.9.** There exists a positive constant \( R > \frac{1}{2} \) such that
\[\begin{align*}
(a) \quad & I(\theta^+(y, \delta)) < c^{\alpha,\lambda,N}_{\pm} + \kappa \quad \text{for any } y \in \mathbb{R}^N \text{ with } |y| \geq R \text{ and } \delta \in [\delta_1, \delta_2]; \\
(b) \quad & (\beta(\theta^+(y, \delta)) \cdot y) > 0 \quad \text{for any } y \in \mathbb{R}^N \text{ with } |y| \geq R \text{ and } \delta \in [\delta_1, \delta_2].
\end{align*}\]

**Proof.** By the compactness of the interval \([\delta_1, \delta_2]\) and Lemma 4.6, there exists a positive constant \( R \) such that
\[
I(\theta^+(y, \delta)) < c^{\alpha,\lambda,N}_{\pm} + \kappa \quad \text{for any } \delta \in [\delta_1, \delta_2] \text{ and } y \in \mathbb{R}^N \text{ with } |y| \geq R. \tag{4.18}
\]
Similarly, (b) follows from (4.16), Lemma 4.3, and the compactness of the interval \([\delta_1, \delta_2]\). \( \square \)

Noting \( \delta_1, \delta_2 \) and \( R \) from Lemma 4.7, Lemma 4.8, and Lemma 4.9, respectively, we define a bounded domain \( D \subset \mathbb{R}^N \) by
\[
D := \{(y, \delta) \in \mathbb{R}^N \times [0, \infty) : |y| \leq R, \quad \delta_1 \leq \delta \leq \delta_2\}. \tag{4.19}
\]

**Lemma 4.10.** Define the map \( \theta : D \to \mathbb{R}^N \times \mathbb{R}^+ \) by
\[
H(y, \delta) := (\beta \circ \theta^+(y, \delta), \gamma \circ \theta^+(y, \delta)).
\]
Then the following equality holds
\[
\deg(H, \ D^0, \ (0, \ \frac{1}{2})) = 1, \tag{4.20}
\]
where \( D^0 \) is the interior of the set \( D \).
Proof. Consider the homotopy
\[ \zeta((y, \delta), s) = (1 - s)id(y, \delta) + sH(y, \delta). \]
According to the homotopy invariance of the topological degree and the fact that
\[ \deg \left( id, \ D^\circ, \ (0, \frac{1}{2}) \right) = 1, \]
it suffices to prove
\[ \zeta((y, \delta), s) \neq (0, \frac{1}{2}) \text{ for any } (y, \delta) \in \partial D \text{ and } s \in [0, 1]. \] (4.21)
In fact, if \(|y| < \frac{1}{2}\) and \(\delta = \delta_1\) hold, then by Lemma 4.7 (b), we have
\[ (1 - s)\delta_1 + s\gamma \circ \theta^\pm(y, \delta_1) < \frac{1}{2} \text{ for any } s \in [0, 1]. \]
If \(\frac{1}{2} \leq |y| \leq R\) and \(\delta = \delta_1\) hold, it follows from Lemma 4.7 (c) that
\[ \left| ((1 - s)y + s\beta(\theta^\pm(y, \delta_1)) \right| \geq \left| (1 - s)y + s |y| - s\beta(\theta^\pm(y, \delta_1)) \right| \geq s + (1 - s)|y| - \frac{1}{4} \geq \frac{1}{4} \neq 0. \]
If \(|y| \leq R\) and \(\delta = \delta_2\) hold, then by Lemma 4.8 (b), we have
\[ (1 - s)\delta_2 + s\gamma \circ \theta^\pm(y, \delta_2) > \frac{1}{2} \text{ for any } s \in [0, 1]. \]
If \(|y| = R\) and \(\delta_1 \leq \delta \leq \delta_2\) hold, it follows from Lemma 4.9 (b) that
\[ \left( ((1 - s)y + s\beta(\theta^\pm(y, \delta)) \cdot y \right) > 0 \text{ for any } s \in [0, 1]. \]
So (4.21) has been proved and the proof is completed. \(\square\)

Set \(I^e = \{ u \in \mathcal{N}^{4, -} : I(u) \leq c \} \) and \(I_c = \{ u \in \mathcal{N}^{4, -} : I(u) \geq c \} \).

Proof of Theorems 1.1 and 1.3. To obtain the bounded state solutions, we set
\[ Q^\pm = \theta^\pm(D), \]
where \(D\) is defined in (4.19) and
\[ \Gamma^\pm = \{ h \in C(\mathcal{N}^{N, \pm}, \mathcal{N}^{N, \pm}) : h(u) = u \text{ for } I(u) < c_{\pm}^{a, \lambda, N} + \kappa \} \]
Moreover,
\[ \Sigma^\pm = \{ A^\pm \subset \mathcal{N}^{N, \pm} : A^\pm = h(Q^\pm), h \in \Gamma^\pm \}. \]
Firstly, we claim that \(A^\pm \cap \mathcal{M}^{N, \pm} \neq \emptyset \) for any \(A^\pm \in \Sigma^\pm \). In fact, let us consider \(h \in \Gamma^\pm\) and define \(\eta : D \to \mathbb{R}^N \times R^+\) by
\[ \eta(y, \delta) = \left( \beta \circ h \circ \theta^\pm(y, \delta), \gamma \circ h \circ \theta^\pm(y, \delta) \right). \]
Then \(\eta\) is a continuous function. If \((y, \delta) \in \partial D\), then it follows from Lemma 4.7(a), Lemma 4.8(a), Lemma 4.9(a) that
\[ I(\theta^\pm(y, \delta)) < c_{\pm}^{a, \lambda, N} + \kappa. \]
Thus we obtain
\[ h \circ \theta^\pm(y, \delta) = \theta^\pm(y, \delta), \]
which implies that
\[ \eta(y, \delta) = \left( \beta \circ \theta^\pm(y, \delta), \gamma \circ \theta^\pm(y, \delta) \right) = H(y, \delta) \text{ for } (y, \delta) \in \partial D. \]
By the property of topological degree and (4.20), we have
\[ \deg(\eta, D^0, (0, \frac{1}{2})) = \deg(H, D^0, (0, \frac{1}{2})) = 1, \]
which implies that there exists \((y', \delta') \in D^0\) such that \(h \circ \theta^\pm(y', \delta') \in A \cap \mathcal{M}^{N, \pm}\).

Now, we suppose \(N = 4\). Let us define
\[ d := \inf_{A \in \Sigma^-} \sup_{u \in A} I(u). \]
By \(A^- \cap \mathcal{M}^{4, -} \neq \emptyset\) for any \(A \in \Sigma^-\), we have
\[ d \geq \inf_{u \in \mathcal{M}^{4, -}} I(u) = c_{A^+, -} > c_{A, -}^{a, \lambda, A}. \]
It follows from Lemma 4.1 that
\[ d \leq \sup_{u \in Q^-} I(u) \leq \sup_{(y, \delta) \in \mathbb{R}^4 \times (0, \infty)} I(\theta^- (y, \delta)) < 2c_{A, -}^{a, \lambda, A}. \]
Hence, \(c_{A, -}^{a, \lambda, A} < d < 2c_{A, -}^{a, \lambda, A}\). Suppose \(K_d := \{u \in \mathcal{N}^{4, -} : I(u) = d, \ I|_{\mathcal{N}^{4, -}}(u) = 0\} = \emptyset\). Since a local \((PS)_c\) condition in \((c_{A, -}^{a, \lambda, A}, 2c_{A, -}^{a, \lambda, A})\) holds, using a variant of a well-known deformation Lemma (see [20]), one can find a continuous map
\[ g : [0, 1] \times \mathcal{N}^{4, -} \rightarrow \mathcal{N}^{4, -} \]
and a positive number \(\varepsilon_0\) such that
\[
\begin{align*}
(i) & \ I^{d + \varepsilon_0} \bigcap I_{d - \varepsilon_0} \subseteq I^{2c_{A, -}^{a, \lambda, A}} \bigcap I_{c_{A, -}^{a, \lambda, A} + \varepsilon}_-; \\
(ii) & \ g(0, u) = u; \\
(iii) & \ g(t, u) = u \text{ for any } u \in I^{d - \varepsilon_0} \bigcup (\mathcal{N}^{4, -} \setminus I^{d + \varepsilon_0}) \text{ and } t \in (0, 1); \\
(iv) & \ g(1, I^{d + \varepsilon_0}) \subseteq I^{d - \varepsilon_0}.
\end{align*}
\]
Now let \(\tilde{A}^- \in \Sigma^-\) be such that
\[ d \leq \sup_{u \in \tilde{A}^-} I(u) < d + \frac{\varepsilon_0}{2}. \]
Then \(g(1, \tilde{A}^-) \in \Gamma^-\) and
\[ \sup_{u \in g(1, \tilde{A}^-)} I(u) \leq d - \frac{\varepsilon_0}{2}, \]
which is contradicting the definition of \(d\) on \(\mathcal{N}^{4, -}\). Thus, by the similar and simpler argument of Corollary 3.4, \(d\) is a critical value of \(I\).

Now suppose \(N \geq 5\). Set \(\mu := \min\{\mu_1, \mu_2\}, \nu = \min\{\nu_1, \nu_2\}, \) where \(\mu_1, \mu_2, \nu_1\) and \(\nu_2\) are taken from Proposition 2.9, Proposition 3.3, Lemma 2.7 and Lemma 4.2, respectively. Let \(\Lambda_0 - \mu < \lambda < \Lambda_0\) and \(0 < |V| \frac{\mu}{4} < \nu\). We define
\[ d^\pm := \inf_{A \in \Sigma^+} \sup_{u \in A^\pm} I(u). \]
One obtains \(d^\pm \geq c_{A, N, \pm} > c_{A, -}^{a, \lambda, N}\). It follows from (4.4) that
\[ c_{A, -}^{a, \lambda, N} < c_{A, N, \pm} \leq d^+ \leq \sup_{u \in A^+} I(u) \leq \sup_{(y, \delta) \in \mathbb{R}^N \times (0, \infty)} I(\theta^+(y, \delta)) < c_{A, -}^{a, \lambda, N}. \]
By \(c_{A, -}^{a, \lambda, N} > c_{A, -}^{a, \lambda, N}\) and Proposition 3.3, we know that the \((PS)_c\) condition holds for \(c > c_{A, -}^{a, \lambda, N}\) and \(c \in (c_{A, -}^{a, \lambda, N}, c_{A, -}^{a, \lambda, N})\). Thus, \(d^- > c_{A, -}^{a, \lambda, N}\) and \(d^+ \in (c_{A, -}^{a, \lambda, N}, c_{A, -}^{a, \lambda, N})\) are two different critical values of \(I\) by the same method. \(\square\)
5. Some necessary lemmas.

Lemma 5.1. Let \( N \geq 6 \) and 
\[
  f(t) = \lambda S^{\frac{2^*}{2^* - 2}} t^2 - t^{2^* - 2} + a S^{\frac{2^*}{2^* - 2}}, \quad t \geq 0.
\]
When \( 0 < \lambda < \Lambda_0 \), we denote the solutions of \( f(t) = 0 \) by \( \xi_1 \) and \( \xi_2 \) \( (0 < \xi_1 < \xi_2) \).
Then the following inequality holds,
\[
  \xi_1^2 + \xi_2^2 - \frac{4a}{(N - 4)\lambda} > 0 \quad \text{for} \quad 0 < \lambda < \Lambda_0.  \quad (5.1)
\]

Proof. When \( 0 < \lambda < \Lambda_0 \), it is easy to prove that 
\[
  \min_{t \geq 0} f(t) = f(\eta_0), \quad \text{where} \quad \eta_0 = \left( \frac{2}{2^* - 2} \lambda S^{\frac{2^*}{2^* - 2}} \right)^{\frac{1}{2^* - 2}}
\]
and \( \xi_1 < \eta_0 < \xi_2 \). Moreover, \( \frac{2a}{(N - 4)\lambda} < \eta_0^2 \). The proof is divided into the following three steps.

Step 1. We have 
\[
  -f'(\eta_0 - s) \geq f'(\eta_0 + s) \quad \text{for any} \quad 0 < s < \eta_0. \quad (5.2)
\]
In fact, it can be easily obtained that 
\[
  f'(t) = 2\lambda S^{\frac{2^*}{2^* - 2}} t - (2^* - 2)t^{2^* - 3}; \\
  f''(t) = 2\lambda S^{\frac{2^*}{2^* - 2}} + (2^* - 2)(3 - 2^*)t^{2^* - 4}; \\
  f'''(t) = (2^* - 2)(3 - 2^*)(2^* - 4)t^{2^* - 5}.
\]
As \( N \geq 6 \), \( f'''(t) \leq 0 \) for any \( t > 0 \) implies 
\[
  f''(\eta_0 - t) \geq f''(\eta_0 + t) \quad \text{for any} \quad 0 \leq t < \eta_0.
\]
Take \( 0 < s < \eta_0 \). Integrating the above inequality from 0 to \( s \) gives us
\[
  \int_{\eta_0 - s}^{\eta_0} f''(t)dt \geq \int_{\eta_0}^{\eta_0 + s} f''(t)dt. \quad (5.3)
\]
By the fact that \( f'(\eta_0) = 0 \),
\[
  \begin{cases}
    f'(\eta_0) = f'(\eta_0 - s) + \int_{\eta_0 - s}^{\eta_0} f''(t)dt; \\
    f'(\eta_0 + s) = f'(\eta_0) + \int_{\eta_0}^{\eta_0 + s} f''(t)dt,
  \end{cases}
\]
and the inequality (5.3), then (5.2) holds.

Step 2. We claim that 
\[
  \xi_2 - \eta_0 \geq \eta_0 - \xi_1. \quad (5.4)
\]
Arguing indirectly, if \( \xi_2 - \eta_0 < \eta_0 - \xi_1 \) holds, then set \( \delta := \xi_2 - \eta_0 \). Obviously, \( \delta < \eta_0 \). Integrate the inequality (5.2) from 0 to \( \delta \) to obtain
\[
  -\int_{\eta_0 - \delta}^{\eta_0} f'(t)dt \geq \int_{\eta_0}^{\eta_0 + \delta} f'(t)dt. \quad (5.5)
\]
The inequality (5.5) implies that
\[
  \int_{\eta_0 - \delta}^{\eta_0 + \delta} f'(t)dt = \int_{\eta_0 - \delta}^{\eta_0} f'(t)dt + \int_{\eta_0}^{\eta_0 + \delta} f'(t)dt \leq 0. \quad (5.6)
\]
However, it follows from $0 < \eta_0 - \delta - \xi_1 < \eta_0 - \xi_1$ and (5.6) that
\[ 0 > \int_{\xi_1}^{\xi_1 + (\eta_0 - \delta - \xi_1)} f'(t)dt = \int_{\eta_0}^{\eta_0 - \delta} f'(t)dt = - \int_{\eta_0 - \delta}^{\eta_0} f'(t)dt \geq 0, \tag{5.7} \]
which is a contradiction. Thus, the claim (5.4) holds.

**Step 3.** It follows from $\xi_1 < \eta_0 < \xi_2$ and (5.4) that
\[ \xi_2^2 - \eta_0^2 = (\xi_2 - \eta_0)(\xi_2 + \eta_0) \geq (\eta_0 - \xi_1)(\eta_0 + \xi_1) = \eta_0^2 - \xi_1^2. \tag{5.8} \]
Combining this with $\frac{2a}{N-4} < \eta_0^2$, we have
\[ \xi_1^2 + \xi_2^2 - \frac{4a}{N-4} > \xi_1^2 + \xi_2^2 - 2\eta_0^2 \geq 0. \]
This completes the proof. \[ \square \]

**Lemma 5.2.** Let $N \geq 5$ and
\[ f(t) = \lambda S^{\frac{2}{2*}} t^2 - t^{2* - 2} + aS^{\frac{2}{2*}}, \ t \geq 0; \]
\[ f_2(t) = 3\lambda S^{\frac{2}{2*}} t^2 - (2^* - 1)t^{2* - 2} + aS^{\frac{2}{2*}}, \ t \geq 0. \]
For $\lambda \in (0, \Lambda_0)$, we denote the solutions of $f(t) = 0$ by $\xi_1$ and $\xi_2$ ($0 < \xi_1 < \xi_2$), the solutions of $f_2(t) = 0$ by $\xi_3$ and $\xi_4$ ($0 < \xi_3 < \xi_4$). Then there exists $\mu_1 := \mu_1(a,N) > 0$ such that
\[ \frac{4a}{(N-4)\lambda} - \xi_1^2 - \xi_2^2 > 0 \quad \text{for} \quad \Lambda_0 - \mu_1 < \lambda < \Lambda_0. \tag{5.9} \]

**Proof.** For $\lambda \in (0, \Lambda_0)$, it is easy to prove the existence of $\xi_i$, $i = 1, 2, 3, 4$ and $\xi_3 < \xi_1 < \xi_4 < \xi_2$. Moreover,
\[ \min_{t \geq 0} f(t) = f(\eta_0), \quad \text{where} \quad \eta_0 = \left( \frac{2}{2^* - 2} \lambda S^{\frac{2}{2*}} \right)^{\frac{1}{2^* - 2}} \]
and $\xi_1 < \eta_0 < \xi_2$. Furthermore, $|\xi_i - \eta_0| \to 0$ as $\lambda \to \Lambda_0$, $i = 1, 2, 4$. Since $\xi_i$ depend on $\lambda$, we rewrite $\xi_i$ by $\xi_i(\lambda)$ in the following context. It is not difficult to obtain $\xi_i(\lambda) \in C^1((0, \Lambda_0))$. Let
\[ P(\lambda) := \frac{4a}{N-4}\lambda - \xi_1^2(\lambda) - \xi_2^2(\lambda), \quad 0 < \lambda \leq \Lambda_0. \]
Then $P(\lambda) \in C^1((0, \Lambda_0))$ and
\[ \lim_{\lambda \to \Lambda_0} P(\lambda) = 0 = P(\Lambda_0). \]
To obtain (5.9), it suffices to prove that there exists $\mu_1 > 0$ such that
\[ P'(\lambda) = -2 \left( \frac{2a}{(N-4)\lambda^2} + \xi_1(\lambda)\xi_1'(\lambda) + \xi_4(\lambda)\xi_4'(\lambda) \right) < 0 \quad \text{for} \quad \Lambda_0 - \mu_1 < \lambda < \Lambda_0. \tag{5.10} \]
In fact, it follows from $f(\xi_1(\lambda)) = f_2(\xi_4(\lambda)) = 0$ and $\xi_i(\lambda) > 0$ that
\[ S^{2*}\xi_1(\lambda) + 2\lambda S^{\frac{2}{2*}}\xi_1'(\lambda) - (2^* - 2)\xi_1^{2* - 4}(\lambda)\xi_1'(\lambda) = 0; \]
\[ 3S^{2*}\xi_4(\lambda) + 6\lambda S^{\frac{2}{2*}}\xi_4'(\lambda) - (2^* - 1)(2^* - 2)\xi_4^{2* - 4}(\lambda)\xi_4'(\lambda) = 0. \]
As a consequence,
\[
\xi_1(\lambda)\xi'_1(\lambda) = \frac{S^{2\gamma} \xi_1^2(\lambda)}{(2^* - 2)\xi_1^2(\lambda) - 2\lambda S^{2\gamma}}; \\
\xi_4(\lambda)\xi'_4(\lambda) = \frac{3S^{2\gamma} \xi_1^2(\lambda)}{(2^* - 1)(2^* - 2)\xi_1^2(\lambda) - 6\lambda S^{2\gamma}}.
\]
By the fact that \(\xi_1(\lambda) < \eta_0\) and \(|\xi_i(\lambda) - \eta_0| \rightarrow 0\) as \(\lambda \rightarrow \Lambda_0\), \(i = 1, 4\), one obtains
\[0 < (2^* - 2)\xi_1^{2\gamma - 4}(\lambda) - 2\lambda S^{2\gamma} \rightarrow 0;\]
\[(2^* - 1)(2^* - 2)\xi_1^{2\gamma - 4}(\lambda) - 6\lambda S^{2\gamma} \rightarrow \frac{2}{3}(2^* - 4)\Lambda_0 S^{2\gamma}\]
as \(\lambda \rightarrow \Lambda_0\), which imply that
\[\xi_1(\lambda)\xi'_1(\lambda) \rightarrow \infty \text{ and } |\xi_4(\lambda)\xi'_4(\lambda)| < C \text{ as } \lambda \rightarrow \Lambda_0. \tag{5.11}\]
According to (5.11), (5.10) has been proved. This completes the proof.

**Lemma 5.3.** Given \(u \in \mathcal{N}_{N-1}\) for \(N \geq 5\), there exist \(\varepsilon > 0\) and a differentiable functional \(t(w) > 0\), \(w \in D^{1,2}(\mathbb{R}^N)\), \(\|w\|_{D^{1,2}} < \varepsilon\) satisfying that
\[t(0) = 1, \ t(w)(u - w) \in \mathcal{N}_{N-1}, \text{ for } \|w\|_{D^{1,2}} < \varepsilon\]
and
\[
(t'(0), w) = \frac{(4\lambda\|u\|_{D^{1,2}}^2 - 2a)\int_{\mathbb{R}^N} \nabla u \nabla w dx + 2\int_{\mathbb{R}^N} V(x)u^2wdx + 2\int_{\mathbb{R}^N} |u|^{2\gamma - 2}uwdx - 2\lambda\|u\|_{D^{1,2}}^2 - (2^* - 2)|u|^2\gamma}{2\lambda\|u\|_{D^{1,2}}^2 - (2^* - 2)|u|^2\gamma}. \tag{5.12}
\]

**Proof.** This methods are taken from [3, 19]. By the similar argument of Lemma 2.7, it is not difficult to prove the existence of \(t(w)\). Then we define \(F : \mathbb{R} \times D^{1,2} \rightarrow \mathbb{R}\) by
\[
F(t, w) = a\|u - w\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} V(x)(u - w)^2dx + t^2\lambda\|u - w\|_{D^{1,2}}^4 - t^{2\gamma - 2}\|u - w\|_{D^{1,2}}^{2\gamma}.
\]
Since \(F(1, 0) = 0\) and
\[
F_t(1, 0) = 2\lambda\|u\|_{D^{1,2}}^2 - (2^* - 2)|u|^2\gamma = a\|u\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} V(x)(u - w)^2dx + 3\lambda\|u\|_{D^{1,2}}^4 - (2^* - 1)|u|^2\gamma < 0,
\]
we can apply the Implicit Function Theorem at the point \((1, 0)\) and get the result.

**Acknowledgments.** We would like to thank the referees and the editor for their valuable comments which have led to an improvement of the presentation of this paper.

**REFERENCES**

[1] C. O. Alves, F. J. S. A. Corrêa and G. M. Figueiredo, On a class of nonlocal elliptic problems with critical growth, *Differ. Equ. Appl.*, 2 (2010), 409–417.
[2] V. Benci and G. Cerami, Existence of positive solutions of the equation \(-\Delta u + a(x)u = u^{(N+2)/(N-2)}\) in \(\mathbb{R}^N\), *J. Funct. Anal.*, 88 (1990), 90–117.
[3] D. M. Cao and H. S. Zhou, Multiple positive solutions of nonhomogeneous semilinear elliptic equations in \(\mathbb{R}^N\), *Proc. Roy. Soc. Edinburgh Sect. A*, 126 (1996), 443–463.
[4] P. Chen and X. Liu, Multiplicity of solutions to Kirchhoff type equations with critical Sobolev exponent, *Commun. Pure Appl. Anal.*, 17 (2018), 113–125.

[5] W. Y. Ding, On a Conformally Invariant Elliptic Equation on \( \mathbb{R}^N \), *Commun. Math. Phys.*, 107 (1986), 331–335.

[6] P. L. Felmer, A. Quaas, M. X. Tang and J. S. Yu, Monotonicity properties for ground states of the scalar field equation, *Ann. I. H. Poincaré-AN.*, 25 (2008), 105–119.

[7] G. M. Figueiredo, R. C. Morales, J. J. R. Santos and A. Suárez, Study of a nonlinear Kirchhoff equation with non-homogeneous material, *J. Math. Anal. Appl.*, 416 (2014), 597–608.

[8] X. M. He and W. M. Zou, Ground states for nonlinear Kirchhoff equations with critical growth, *Ann. Mat. Pura Appl.*, 193 (2014), 473–500.

[9] Y. S. Huang, Z. Liu and Y. Z. Wu, On Kirchhoff type equations with critical Sobolev exponent and Naimen’s open problems, *Mathematics*, 7 (2015), 97–114.

[10] Y. S. Huang, Z. Liu and Y. Z. Wu, On finding solutions of a Kirchhoff type problem, *Proc. Amer. Math. Soc.*, 144 (2016), 3019–3033.

[11] G. B. Li and H. Y. Ye, Existence of positive solutions for nonlinear Kirchhoff type problems in \( \mathbb{R}^N \) with critical Sobolev exponent, *Math. Methods Appl. Sci.*, 37 (2014), 2570–2584.

[12] Z. S. Liu and S. J. Guo, On ground states for the Kirchhoff-type problem with a general critical nonlinearity, *J. Math. Anal. Appl.*, 426 (2015), 267–287.

[13] Z. S. Liu and S. J. Guo, Existence and concentration of positive ground state for a Kirchhoff equation involving critical Sobolev exponent, *Z. Angew. Math. Phys.*, 66 (2015), 747–769.

[14] J. Liu, J. F. Liao and C. L. Tang, Positive solutions for Kirchhoff-type equations with critical exponent in \( \mathbb{R}^N \), *J. Math. Anal. Appl.*, 429 (2015), 1153–1172.

[15] Z. Liu, S. Guo and Y. Fang, Positive solutions of Kirchhoff type elliptic equations in \( \mathbb{R}^4 \) with critical growth, *Mathematische Nachrichten*, 290 (2017), 367–381.

[16] R. Q. Liu, C. L. Tang, J. F. Liao and X. P. Wu, Positive solutions of Kirchhoff type problem with singular and critical nonlinearities in dimension four, *Commun. Pure Appl. Anal.*, 15 (2016), 1841–1856.

[17] D. Naimen, Positive solutions of Kirchhoff type elliptic equations involving a critical Sobolev exponent, *Nonlinear Differ. Equ. Appl.*, 21 (2014), 885–914.

[18] D. Naimen, The critical problem of Kirchhoff type elliptic equations in dimension four, *J. Differential Equations*, 257 (2014), 1168–1193.

[19] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, *Ann. I. H. Poincaré-AN.*, 9 (1992), 281–304.

[20] M. Willem, *Minimax Theorems*, Birkhäuser, 1996.

[21] J. Wang, L. X. Tian, J. X. Xu and F. B. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, *J. Differential Equations*, 253 (2012), 2314–2351.

[22] J. Wang and L. Xiao, Existence and concentration of solutions for a Kirchhoff type problem with potentials, *Discrete Contin. Dyn. Syst. A.*, 12 (2016), 7137–7168.

[23] Y. J. Sun and X. Liu, Existence of positive solutions for Kirchhoff type problems with critical exponent, *J. Partial Differ. Equ.*, 25 (2012), 187–198.

[24] Q. L. Xie, X. P. Wu and C. L. Tang, Existence and multiplicity of solutions for Kirchhoff type problem with critical exponent, *Commun. Pure Appl. Anal.*, 12 (2013), 2773–2786.

[25] Q. L. Xie, S. W. Ma and X. Zhang, Bound state solutions of Kirchhoff type problems with critical exponent, *J. Differential Equations*, 261 (2016), 890–924.

Received September 2017; revised October 2017.

*E-mail address: xieqilinsxdt@163.com*

*E-mail address: jsyu@gzhu.edu.cn*