Two-Grid $hp$-DGFEMs on Agglomerated Coarse Meshes

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We generalise the a priori error analysis of two-grid $hp$-version discontinuous Galerkin finite element methods for strongly monotone second-order quasilinear elliptic partial differential equations to the case when coarse meshes consisting of general agglomerated polytopic elements are employed.

1 Introduction

We study the $hp$-version of the two-grid incomplete interior penalty (IIP) discontinuous Galerkin finite element method (DGFEM) using an agglomerated coarse mesh, for the numerical approximation of the following problem: find $u$ such that

$$\nabla \cdot (\mu(x)|\nabla u|\nabla u) = f(x) \quad \text{in} \Omega, \quad u = 0 \quad \text{on} \Gamma,$$

where $\Omega$ is a bounded polygonal/polyhedral Lipschitz domain in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma := \partial \Omega$ and $f \in L^2(\Omega)$. We assume that $\mu \in C^0([\Omega \times [0, \infty)($, and there exists positive constants $m_\mu$ and $M_\mu$ such that $m_\mu(t - s) \leq \mu(x, t) t - \mu(x, s) s \leq M_\mu(t - s)$, $t \geq s \geq 0$, $x \in \Omega$. For ease of notation write $\mu(t)$ instead of $\mu(x, t)$.

The two-grid method was originally introduced by Xu [1, 2]. The key idea of this approach, in the context of numerically approximating nonlinear partial differential equations (PDEs), is to first compute a numerical approximation of the nonlinear PDE on a coarse mesh/approximation space, and subsequently employ this solution to linearize the underlying problem on the fine mesh/approximation space; in this way only a linear solve is required on the fine mesh/approximation space. In the context of $hp$-version DGFEMs, in [3] and [4] we have considered the application of the two-grid approach to both scalar strongly monotone second-order quasilinear PDEs of the form (1) and non-Newtonian fluids, respectively; in both cases the coarse and fine spaces employ standard meshes employing simplices/tensor-product elements. In this article, we generalize this to the case when general polytopic coarse elements, generated by agglomerating fine mesh elements, are employed.

2 Two-grid $hp$-version IIP DGFEM

We write $T_h = \{\kappa\}$ to denote the fine mesh consisting of simplices/tensor-product elements of local mesh size $h_\kappa = \text{diam}(\kappa)$, $\kappa \in T_h$. Similarly, $T_H = \{K\}$ denotes the coarse mesh consisting of polytopic elements $K$ constructed by agglomerating elements $\kappa \in T_h$: $H_K = \text{diam}(K)$, $K \in T_H$. We assume that $T_{h}$ is of bounded local variation. Writing $p = \{p_\kappa : \kappa \in T_h\}$ and $P = \{P_K : K \in T_H\}$ to denote the polynomial orders defined over $T_h$ and $T_H$, respectively, (p is assumed to be of bounded local variation) we write $V_{h_p} = \{v \in L^2(\Omega) \mid v |_{P_\kappa} = \text{p}_\kappa \kappa \in T_h\}$ and $V_{H_p} = \{v \in L^2(\Omega) \mid v |_{P_K} = \text{p}_K K \in T_H\}$, where $P_\kappa$ denotes the space of all polynomials of total degree $p$ on $\kappa$.

We write $F_h$ and $F_H$ to denote the set of all faces in the meshes $T_h$ and $T_H$, respectively. Furthermore, we write $[\ ]$ and $\{\}$ to denote suitable average and jump operators, respectively, which are defined on either $F_h$ or $F_H$; see [3] for details. With this notation, we first introduce the following standard IIP DGFEM on the fine mesh $T_h$, for the numerical approximation of the problem (1): find $u_{h_p} \in V_{h_p}$ such that $A_{h_p}(u_{h_p}; u_{h_p}, v_{h_p}) = \sum_{\kappa \in T_h} \int_{\kappa} f v_{h_p} d\kappa$ for all $v_{h_p} \in V_{h_p}$, where

$$A_{h_p}(\phi; u, v) = \sum_{\kappa \in T_h} \int_{\kappa} \mu(\phi) \nabla u \cdot \nabla v d\kappa - \sum_{F \in F_h} \int_{F} \mu(\phi) \nabla u \cdot [v] d\kappa + \sum_{F \in F_h} \int_{F} \sigma_{h_p} [u] \cdot [v] d\kappa$$

and $\nabla h$ is used to denote the broken gradient operator, defined elementwise. Given a face polynomial degree function $p_F$ and a face mesh size function $h_F$, $F \in F_h$ the interior penalty parameter $\sigma_{h_p}$ is given by $\sigma_{h_p}|_F = \gamma_{h_p} p_F^{-1} h_F^{-1}$, $F \in F_h$, where $\gamma_{h_p}$ is a sufficiently large constant, cf. [3]. The two-grid IIP DGFEM is given by:

1. Compute $u_{H_p} \in V_{H_p}$ such that $A_{H_p}(u_{H_p}; u_{H_p}, v_{H_p}) = \sum_{K \in T_H} \int_{K} f v_{H_p} d\kappa$ for all $v_{H_p} \in V_{H_p}$.
2. Find $u_{2G} \in V_{h_p}$ such that $A_{h_p}(u_{2G}; u_{2G}, v_{h_p}) = \sum_{\kappa \in T_h} \int_{\kappa} f v_{h_p} d\kappa$ for all $v_{h_p} \in V_{h_p}$.

Here, $A_{H_p}(u; u)$ is defined analogously to $A_{h_p}(u; u)$, but with a modified interior penalty parameter $\sigma_{H_p}$, cf. [5].

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3 Error analysis

For the proceeding error analysis, we require the following definitions and assumptions, cf. [5].

**Definition 3.1** For $K \in \mathcal{T}_h$ we write $\mathcal{F}_F^K$ to be the set of all possible $d$-simplices contained in $K$ and having at least one face in common with $K$; we write $\mathcal{K}_0^F$ to denote a simplex belonging to $\mathcal{F}_F^K$ which shares with $K \in \mathcal{T}_h$ the face $F \subset \partial K$.

**Assumption 3.2** For any $K \in \mathcal{T}_h$, there exists a set of non-overlapping $d$-dimensional simplices $\{K_k^F\} \subset \mathcal{F}_F^K$ contained within $K$, such that for all $F \subset \partial K$, the condition $H_K \leq C_s|K_F^F||F|^{-1}$ holds, where $C_s$ is a positive constant, which is independent of the discretization parameters, the number of faces that the element possesses, and the measure of $F$.

**Definition 3.3** The covering $\mathcal{T}_h^d = \{K\}$ related to $\mathcal{T}_h$ is a set of open shape-regular $d$-simplices $K$, such that, for each $K \in \mathcal{T}_h$, there exists a $K \in \mathcal{T}_h^d$, such that $K \subseteq K$. Given $\mathcal{T}_h^d$ we denote by $\partial K$ the covering domain given by $\partial K = \bigcup_{K \in \mathcal{T}_h^d} \partial K$.

**Assumption 3.4** We assume a covering $\mathcal{T}_h^d$ of $\mathcal{T}_h$ and positive constant $O(\Omega)$ exists, independent of the mesh, such that $\max_{K \in \mathcal{T}_h} \text{card}(\{K \in \mathcal{T}_h : K' \cap \partial K \neq \emptyset, K \in \mathcal{T}_h^d \} K \subseteq \mathcal{T}_h^d)$, for each pair $K \in \mathcal{T}_h^d, K \in \mathcal{T}_h^d$ with $K \subset K$, for a constant $C_D > 0$, uniformly with respect to the mesh size.

We now state the main result of this article; see [6] for details.

**Theorem 3.5** Let $\mathcal{T}_h^d$ be a coarse agglomerated mesh satisfying Assumptions 3.2 and 3.4, with $\mathcal{T}_h^d = \{K\}$ an associated covering of $\mathcal{T}_h$ consisting of $d$-simplices; cf. Definition 3.3. If the analytical solution $u \in H^1(\Omega)$ to (1) satisfies $u|_K \in H^{l_K}(\Omega)$, $l_K \geq 2$, and $u|_K \in H^2(K), L_K \geq 1/2$, for $K \in \mathcal{T}_h$, such that $\mathcal{E}u|_K \in H^{l_K}(\Omega)$, where $K \in \mathcal{T}_h^d$ with $K \subset K$; then, writing $\|v\|_{\mathcal{V}_h^p} = \|\nabla_h v\|^2_{L^2(\Omega)} + \sum_{F \in \mathcal{F}_h} \int_F \sigma_{hp}[v]^2 \, ds$, the solution $u |_{\mathcal{V}_h^p}$ satisfies the error bound

$$\|u - u_{2G}\|_{L^2(\Omega)} \leq C \left( \sum_{K \in \mathcal{T}_h} h_K^{s_K-2} \right) \|u\|_{H^{l_K}(\Omega)} + C \left( \sum_{K \in \mathcal{T}_h^d} H_K^{s_K-2} \right) \|u\|_{H^2(K)}.$$

where $G_K(H_K, P_K) := (P_K + P_K^T + 1) \max_{F \subset \partial K} \nabla H_K^F \sigma_{H_K} |F| + H_K P_K^{-1} \max_{F \subset \partial K} \sigma_{H_K} |F|$, $S_K = \min(P_K + 1, L_K)$, for $K \in \mathcal{T}_h$, $s_K = \min(p_k + 1, l_K)$, for $K \in \mathcal{T}_h^d$, and $C$ is a positive constant independent of $u$, $h$, $P$, and $P$, but depends on the constants $\mu_0, \mu_1$ from the monotonicity properties of $\mu(\cdot)$. Finally, $\mathcal{E}$ denotes the extension operator defined in [7].

To confirm Theorem 3.5, we set $\Omega = (0, 1)^2 \subset \mathbb{R}^2$, $\mu(\nabla u) = 2 + (1 + |\nabla u|^2)^{-1}$, and select $f$ so that $u(x, y) = x(1 - x)y(1 - y)(1 - 2y)e^{-20(2x-1)^2}$. Firstly, we consider a sequence of uniform fine meshes satisfying $n \times n$, $n = 4, 8, 16, 32, 64, 128, 256$, square elements with a coarse mesh containing elements constructed by agglomerating 4 fine elements, i.e., $H \approx O(h)$ for all meshes; see Figure 1(a) which confirms the optimal convergence rate of $O(h^2)$, for $p$ fixed. By selecting $H \approx O(h^{1/2})$, cf. Figure 1(b), we observe a deterioration in the order of convergence to $O(h^{3/2})$, as expected.

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