Hölder conditions and τ-spikes for analytic Lipschitz functions

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Abstract

Let $U$ be an open subset of $\mathbb{C}$ with boundary point $x_0$ and let $A_\alpha(U)$ be the space of functions analytic on $U$ that belong to $\text{lip}_\alpha(U)$, the “little Lipschitz class”. We consider the condition $S = \sum_{n=1}^{\infty} 2^{(t+\lambda+1)n} M_{s+\alpha}^1(A_n \setminus U) < \infty$, where $t$ is a non-negative integer, $0 < \lambda < 1$, $M_{s+\alpha}^1$ is the lower $1+\alpha$ dimensional Hausdorff content, and $A_n = \{ z : 2^{-n-1} < |z-x_0| < 2^{-n} \}$. This is similar to a necessary and sufficient condition for bounded point derivations on $A_\alpha(U)$ at $x_0$. We show that $S = \infty$ implies that $x_0$ is a $(t+\lambda)$-spike for $A_\alpha(U)$ and that if $S < \infty$ and $U$ satisfies a cone condition, then the $t$-th derivatives of functions in $A_\alpha(U)$ satisfy a Hölder condition at $x_0$ for a non-tangential approach.

1 Introduction

This paper concerns necessary and sufficient conditions for bounded point derivations on various function spaces. Given a compact subset $X$ in $\mathbb{C}$ and a Banach Space $B$ on $X$, a point $x_0 \in X$ is said to admit a bounded point derivation for $B$ if there exists a constant $C > 0$ such that $|f'(x_0)| \leq C||f||$, for all $f \in B$, where $|| \cdot ||$ is the norm of the Banach space. Bounded point derivations were originally studied in the case of the space $R(X)$, the uniform closure of rational functions with poles off $X$. An important problem in the theory of rational approximation was to determine conditions for which $R(X) = C(X)$, the space of continuous functions on $X$, because if $R(X) = C(X)$, then every continuous function on $X$ can be uniformly approximated.

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by rational functions with poles off $X$. To solve this problem, the concepts of peak points and non-peak points were developed in papers such as [1,3,4]. A point $x_0 \in X$ is said to be a peak point for a uniform algebra $A$ on $X$ if there exists $f \in A$ such that $f(x_0) = 1$ and $|f(x)| < 1$ for all other $x \in A$; otherwise it is a non-peak point. Bounded point derivations on $R(X)$ are generalizations of non-peak points, and thus provide information about approximation of derivatives of rational functions. Some of the earlier results in connection with this problem can be found in [2] and [11]. Hence it is of great value to determine necessary and sufficient conditions for bounded point derivations. Moreover, the conditions mentioned in this paper depend only on the point $x_0$ under consideration and the geometry of the set $X$.

Hallstrom was the first to determine necessary and sufficient conditions for bounded point derivations, which he determined for the space $R(X)$, the uniform closure of rational functions with poles outside $X$ [5]. These conditions are given in terms of a quantity known as analytic capacity, which is defined as follows. Let $X$ be a compact subset of $\mathbb{C}$. A function $f$ is said to be admissible on $X$ if

1. $|f(z)| \leq 1$ for $z \in \mathbb{C} \setminus X$
2. $f(\infty) = 0$
3. $f$ is analytic outside $X$

and the analytic capacity of $X$ is denoted by $\gamma(X)$ and defined by

$$\gamma(X) = \sup f'(\infty) = \sup \lim_{z \to \infty} zf(z)$$

where the supremum is taken over all admissible functions. Hallstrom’s conditions for bounded point derivations are summarized in Theorem 1. In it, and throughout the rest of the paper $A_n$ denotes the annulus $\{z : 2^{-n-1} < |z - x_0| < 2^{-n}\}$.

**Theorem 1.** Let $X$ be a compact subset of the complex plane, $x_0 \in X$, and let $t$ be a non-negative integer. Then there exists a bounded point derivation on $R(X)$ at $x_0$ of order $t$ if and only if

$$\sum_{n=1}^{\infty} 2^{(t+1)n} \gamma(A_n \setminus X) < \infty.$$
In [7], O’Farrell considered the problem of what happens if the integer \( t \) in Hallstrom’s theorem is replaced with a non-integer. He was able to show two results in opposite directions. The first result concerns sets that satisfy a cone condition. A set \( X \) is said to satisfy a cone condition at a point \( x_0 \) if there exists a cone \( \mathcal{C} \) with vertex at \( x_0 \) and midline \( J \) that satisfies the following property: there exists a constant \( C > 0 \) such that if \( x \in J \) and \( z \) is outside \( \mathcal{C} \) then \( |x - x_0| \leq C|z - x| \). The midline \( J \) is also known as a non-tangential ray to \( x_0 \) and the limit as \( x \to x_0, x \in J \) is called a non-tangential limit to \( x_0 \). O’Farrell’s first result shows that replacing the \( t \) in Hallstrom’s theorem with \( t + \lambda \), where \( 0 < \lambda < 1 \), implies a Hölder condition for the \( t \)-th derivative of \( f \), as long as the set \( X \) satisfies a cone condition.

**Theorem 2.** Suppose \( X \) is a compact subset of \( \mathbb{C} \) which satisfies a cone condition at \( x_0 \) and \( J \) is a non-tangential ray to \( x_0 \). Let \( t \) be a non-negative integer and let \( 0 < \lambda < 1 \). If

\[
\sum_{n=1}^{\infty} 2^{(t+\lambda+1)n} \gamma(A_n \setminus X) < \infty
\]

then there is a constant \( C > 0 \) such that

\[
\frac{|f^{(t)}(x) - f^{(t)}(x_0)|}{|x - x_0|^\lambda} \leq C ||f||_\infty
\]

for all \( x \) in \( J \) and \( f \in R(X) \).

O’Farrell’s second result involves representing measures and \( \tau \)-spikes. Let \( \mu \) be a measure and let \( \mu^\tau = \int \frac{d|\mu(\zeta)|}{|\zeta - z|^\tau} \). Then \( \mu \) is a representing measure on for a point \( x_0 \in X \) on a Banach space \( B \) if \( \int f d\mu = f(x_0) \) for all \( f \) in \( B \) and a point \( x_0 \in X \) is a \( \tau \)-spike for \( B \) if \( \mu^\tau(x_0) = \infty \) whenever \( \mu \) is a representing measure for \( x_0 \) on \( B \). On \( R(X) \) a peak point is a \( \tau \)-spike for all \( \tau > 0 \). O’Farrell’s second result shows that if \( x_0 \) is a \((t + \lambda)\)-spike for \( R(X) \) only if (1) holds.

**Theorem 3.** Suppose \( X \) is a compact subset of \( \mathbb{C} \), \( x_0 \in X \), \( 0 < \lambda < 1 \) and let \( t \) be a non-negative integer. If \( \sum_{n=1}^{\infty} 2^{(t+\lambda+1)n} \gamma(A_n \setminus X) = \infty \) then \( \mu^{t+\lambda} = \infty \) whenever \( \mu \) is a representing measure for \( x_0 \) on \( R(X) \).

For the remainder of the paper, we consider bounded point derivations on \( A_\alpha(U) \), the space of functions that are analytic on an open set \( U \) and belong to the “little Lipschitz class”. Let
$U$ be an open subset in the complex plane and let $0 < \alpha < 1$. A function $f : U \rightarrow \mathbb{C}$ satisfies a Lipschitz condition with exponent $\alpha$ on $U$ if there exists $k > 0$ such that for all $z, w \in U$

$$|f(z) - f(w)| \leq k|z - w|^\alpha. \quad (2)$$

Let $\text{Lip}_\alpha(U)$ denote the space of functions that satisfy a Lipschitz condition with exponent $\alpha$ on $U$. $\text{Lip}_\alpha(U)$ is a Banach space with norm given by $||f||_{\text{Lip}_\alpha(U)} = \sup_U |f| + k(f)$, where $k(f)$ is the smallest constant that satisfies (2). If we let $||f||'_{\text{Lip}_\alpha(U)} = k(f)$ then $||f||'_{\text{Lip}_\alpha(U)}$ is a seminorm on $\text{Lip}_\alpha(U)$.

The little Lipschitz class, $\text{lipo}_\alpha(U)$, is the subspace of $\text{Lip}_\alpha(U)$ which consists of those functions in $\text{Lip}_\alpha(U)$ that also satisfy the additional property that for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $z, w$ in $U$, $|f(z) - f(w)| \leq \epsilon|z - w|^\alpha$ whenever $|z - w| < \delta$. Lipschitz functions form an important class of functions and much work has been done on approximations of Lipschitz functions by rational functions in papers such as [8–10].

Let $A_\alpha(U)$ denote the space of functions that are analytic on $U$ and belong to $\text{lipo}_\alpha(U)$. A point $x_0 \in \overline{U}$ is said to admit a bounded point derivation on $A_\alpha(U)$ if there exists a constant $C > 0$ such that $|f'(x_0)| \leq C||f||_{\text{Lip}_\alpha(\mathbb{C})}$ for all $f \in A_\alpha(U)$. In [6], Lord and O’Farrell determined the necessary and sufficient conditions for bounded point derivations on $A_\alpha(U)$, which is given in terms of an appropriate Hausdorff content.

The Hausdorff content of a set is defined using measure functions. A measure function is a monotone nondecreasing function $h : [0, \infty) \rightarrow [0, \infty)$. For example, $r^\beta$ is a measure function for $0 \leq \beta < \infty$. If $h$ is a measure function then the Hausdorff content $M_h$ associated to $h$ is defined by

$$M_h(E) = \inf \sum h(\text{diam } B),$$

where the infimum is taken over all countable coverings of $E$ by balls and the sum is taken over all the balls in the covering. If $h(r) = r^\beta$ then we denote $M_h$ by $M^\beta$. The lower $(1 + \alpha)$-dimensional Hausdorff content is denoted by $M_\ast^{1+\alpha}(E)$ and defined by

$$M_\ast^{1+\alpha}(E) = \sup M_h(E),$$
where the supremum is taken over all measurable functions \( h \) such that \( h(r) \leq r^{1+\alpha} \) and \( r^{-1-\alpha}h(r) \) converges to 0 as \( r \) tends to 0. The lower \((1 + \alpha)\)-dimensional Hausdorff content is a monotone set function; i.e. if \( E \subseteq F \) then \( M_{*}^{1+\alpha}(E) \leq M_{*}^{1+\alpha}(F) \). The result of Lord and O’Farrell characterizing bounded point derivations on \( A_{\alpha}(U) \) in terms of Hausdorff content is summarized in the following theorem.

**Theorem 4.** Let \( U \) be an open subset of \( \mathbb{C} \), \( x_0 \in \partial U \) and let \( t \) be a non-negative integer. Then there exists a bounded point derivation of order \( t \) on \( A_{\alpha}(U) \) if and only if

\[
\sum_{n=1}^{\infty} 2^{(t+1)n} M_{*}^{1+\alpha}(A_n \setminus U) < \infty. \tag{3}
\]

A natural question to ask is what is the significance of a non-integer \( t \) in (3). In analogy with \( R(X) \), results similar to Theorem 2 and Theorem 3 should hold for \( A_{\alpha}(U) \). In this paper we prove the following results.

**Theorem 5.** Suppose \( U \) is an open subset of \( \mathbb{C} \) which satisfies a cone condition at \( x_0 \) and \( J \) is a non-tangential ray to \( x_0 \). Let \( t \) be a non-negative integer and let \( 0 < \lambda < 1 \). If

\[
\sum_{n=1}^{\infty} 2^{(t+1+n)} M_{*}^{1+\alpha}(A_n \setminus U) < \infty
\]

then there is a constant \( C > 0 \) such that

\[
\frac{|f^{(t)}(x) - f^{(t)}(x_0)|}{|x - x_0|^\lambda} \leq C||f||_{\text{Lip}_\alpha(\mathbb{C})}
\]

for all \( x \in J \) and \( f \in A_{\alpha}(U) \).

**Theorem 6.** Suppose \( U \) is an open subset of \( \mathbb{C} \), \( x_0 \in \overline{U} \), \( 0 < \lambda < 1 \), and let \( t \) be a non-negative integer. Also let \( \mu_{\beta}(z) = \int \frac{d|\mu(\zeta)|}{|\zeta - z|^\beta} \). If \( \sum_{n=1}^{\infty} 2^{(t+\lambda+1)n} M_{*}^{1+\alpha}(A_n \setminus U) = \infty \) then \( \mu^{t+\lambda} = \infty \) whenever \( \mu \) is a representing measure for \( x_0 \) on \( A_{\alpha}(U) \).

The next section contains some preliminary lemmas that are used in the proofs of Theorems 5 and 6. In Section 3 we prove Theorem 5 and in Section 4 we prove Theorem 6.
2 Preliminary lemmas

Throughout the remainder of the paper, we make use of the following factorization lemma.

**Lemma 7.** For complex numbers $a$ and $b$ and positive integer $n$,

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

We will also make use of the following closely related lemma which is less well known.

**Lemma 8.** Let $a$ and $b$ be non-zero complex numbers and let $n$ be a negative integer. Then

$$a^n - b^n = \frac{b - a}{ab} \cdot (a^{n+1} + a^{n+2}b^{-1} + \ldots + b^{n+1}).$$

**Proof.** Let $z = a^{-1}$ and $w = b^{-1}$. Then it follows from Lemma 7 that

$$a^n - b^n = z^n - w^n$$

$$= (z - w)(z^{n-1} + z^{n-2}w + \ldots + w^{n-1})$$

$$= \frac{b - a}{ab} \cdot (a^{n+1} + a^{n+2}b^{-1} + \ldots + b^{n+1}).$$

\[\square\]

Another key lemma is the following Cauchy type theorem for Lipschitz functions which appears in the paper of Lord and O’Farrell [6, pg.110].

**Lemma 9.** Let $\Gamma$ be a piecewise analytic curve bounding a region $\Omega \in \mathbb{C}$, and suppose that $\Gamma$ is free of outward pointing cusps. Let $0 < \alpha < 1$ and suppose that $f \in \text{lip}_\alpha(\mathbb{C})$ is analytic outside a closed region $S$. Then there exists a constant $\kappa > 0$ such that

$$\left| \int f(z)dz \right| \leq \kappa \cdot M_\ast^{1+\alpha}(\Omega \cap S) \cdot ||f||_{\text{Lip}_\alpha(\Omega)}.$$

The constant $\kappa$ only depends on $\alpha$ and the equivalence class of $\Gamma$ under the action of the conformal group of $\mathbb{C}$. In particular this means that $\kappa$ is the same for any curve obtained from $\Gamma$ by rotation or scaling.
The next lemma is an immediate corollary of the cone condition that is applicable in a wide variety of situations.

**Lemma 10.** Suppose that $J$ is a non-tangential ray to $x_0$ in a cone $\mathcal{C}$, $x \in J$ and $z$ is outside $\mathcal{C}$, then for all positive integers $t$, there exists a constant $C > 0$ depending only on $t$ such that

$$\frac{1}{|z-x|^t} \leq \frac{C}{|z-x_0|^t}.$$  

**Proof.** Since $x$ lies on $J$, which is a non-tangential ray to $x_0$, there exists a constant $K > 0$ such that for $z \notin \mathcal{C}$, $\frac{|x-x_0|}{|z-x|} \leq K$. Thus for $z \notin U$, $\frac{|z-x_0|}{|z-x|} \leq 1 + \frac{|x-x_0|}{|z-x|} \leq 1 + K$. Hence

$$\frac{1}{|z-x|^t} \leq \frac{(1+K)^t}{|z-x_0|^t}.$$  

Finally, we will need the following decay lemma which was first proved by Lord and O’Farrell [6, pg.109].

**Lemma 11.** Let $\alpha$ be such that $0 < \alpha < 1$, let $K$ be a compact subset of $\mathbb{C}$ and let $f \in \text{Lip}_\alpha(\mathbb{C})$ be analytic outside $K$ and vanish at $\infty$. Then there is a constant $C$ depending on $\alpha$ but not on $K$ or $f$ such that the following estimates hold.

1. $||f||_\infty \leq C||f||_{\text{Lip}_\alpha(\mathbb{C})} \cdot M_1^{1+\alpha}(K)^{\frac{\alpha}{1+\alpha}}$

2. For $z \notin K$, $|f(z)| \leq \frac{C||f||'_{\text{Lip}_\alpha(\mathbb{C})} \cdot M_1^{1+\alpha}(K)}{\text{dist}(z,K)}$

### 3 A Hölder condition for derivatives

In this section we present the proof of Theorem 5.4

**Proof.** To prove Theorem 5.4 we first note that by translation invariance we may suppose that $x_0 = 0$. Moreover by replacing $f$ by $f - f(0)$ if needed, we may suppose that $f(0) = 0$. In addition, we may suppose that $U$ is contained in the ball $\{z : |z| < \frac{3}{2}\}$.

It is a result of Lord and O’Farrell [6, Lemma 1.1] that $A_\alpha(U \cup \{x_0\})$ is dense in $A_\alpha(U)$. (Note that in this paper what we refer to as $A_\alpha(U)$ is denoted by $a(U)$.) Hence we may suppose...
that \( f \) is analytic at 0 and thus there is a neighborhood \( \Omega \) of 0 such that \( f \) is analytic on \( \Omega \). We can further suppose that \( U \subseteq \Omega \). Let \( B_n \) denote the ball centered at 0 with radius \( 2^{-n} \). Then there exists an integer \( N > 0 \) such that \( \Omega \) contains \( B_N \) and hence \( f \) is analytic inside the ball \( B_N \). Since \( J \) is a non-tangential ray to 0, it follows that there is a sector in \( U \) with vertex at 0 that contains \( J \). Let \( C \) denote this sector. It follows from the Cauchy integral formula that

\[
f^{(t)}(x) - f^{(t)}(0) = \frac{t!}{2\pi i} \int_{\partial(C \cup B_N)} \frac{f(z)}{(z-x)^{t+1}} - \frac{f(z)}{z^{t+1}} \, dz
\]

where the boundary is oriented so that the interior of \( C \cup B_N \) lies always to the left of the path of integration. (See Figure 1.) We can factor an \( x \) out of the integrand.

\[
f^{(t)}(x) - f^{(t)}(0) = \frac{t!}{2\pi i} \int_{\partial(C \cup B_N)} \frac{f(z)}{(z-x)^{t+1}} - \frac{f(z)}{z^{t+1}} \, dz
\]

\[
= \frac{t!}{2\pi i} \int_{\partial(C \cup B_N)} \frac{f(z) \cdot (z^{t+1} - (z-x)^{t+1})}{z^{t+1}(z-x)^{t+1}} \, dz
\]

\[
= \frac{t!}{2\pi i} \int_{\partial(C \cup B_N)} \frac{f(z) \cdot (z - (z-x)) \cdot (z^t + z^{t-1}(z-x) + \ldots + (z-x)^t)}{z^{t+1}(z-x)^{t+1}} \, dz
\]

\[
= \frac{t!}{2\pi i} \int_{\partial(C \cup B_N)} f(z) \cdot x \sum_{k=0}^{t} z^{k-t-1}(z-x)^{-k-1} \, dz.
\]

Thus

\[
\frac{f^{(t)}(x) - f^{(t)}(0)}{x^\lambda} = \frac{t!x^{1-\lambda}}{2\pi i} \int_{\partial(C \cup B_N)} f(z) \sum_{k=0}^{t} z^{k-t-1}(z-x)^{-k-1} \, dz.
\]
Let $D_n = A_n \setminus C$. Then

$$\frac{f^{(t)}(x) - f^{(t)}(0)}{x^\lambda} = \frac{t! x^{1-\lambda}}{2\pi i} \sum_{n=1}^{N} \int_{\partial D_n} f(z) \sum_{k=0}^{t} z^{k-t-1} (z - x)^{-k-1} dz + \frac{t! x^{1-\lambda}}{2\pi i} \int_{|z| = \frac{1}{2}} f(z) \sum_{k=0}^{t} z^{k-t-1} (z - x)^{-k-1} dz.$$ 

We first bound the second integral. Since $x$ lies on $J$, which is a non-tangential ray to $x_0$, it follows from Lemma 10 that $|z - x|^{-k-1} \leq C |z|^{-k-1}$. Thus by the triangle inequality,

$$\left| \frac{t! x^{1-\lambda}}{2\pi i} \int_{|z| = \frac{1}{2}} f(z) \sum_{k=0}^{t} z^{k-t-1} (z - x)^{-k-1} dz \right| \leq \frac{t! |x|^{1-\lambda}}{2\pi} \int_{|z| = \frac{1}{2}} \sum_{k=0}^{t} |z|^{-t-2} dz \leq C \frac{t! |x|^{1-\lambda}}{2\pi} (t + 1) 2^{t+2} \sup_U f \leq C \sup_U f. \tag{4}$$

To bound the sum, we note that since $x^{1-\lambda} f(z) \sum_{k=0}^{t} z^{k-t-1} (z - x)^{-k-1}$ is analytic on $D_n \setminus U$ for $M \leq n \leq N$, an application of Lemma 9 shows that

$$\left| x^{1-\lambda} \int_{\partial D_n} f(z) \sum_{k=0}^{t} z^{k-t-1} (z - x)^{-k-1} dz \right| \leq \kappa M^{1+\alpha} (D_n \setminus U) \cdot \left\| x^{1-\lambda} f(z) \sum_{k=0}^{t} z^{k-t-1} (z - x)^{-k-1} \right\|_{Lip^\alpha(D_n)} \tag{5}.$$ 

Recall that the constant $\kappa$ is the same for curves in the same equivalence class. Since the regions $D_n$ differ from each other by a scaling it follows that $\kappa$ doesn’t depend on $n$ in (5).

The remainder of the proof is to show that $\left\| x^{1-\lambda} f(z) \sum_{k=0}^{t} z^{k-t-1} (z - x)^{-k-1} \right\|_{Lip^\alpha(D_n)}$ can be bounded by a constant independent of $f$ and $x$. It follows from the definition of the Lipschitz seminorm that

$$\left\| x^{1-\lambda} f(z) \sum_{k=0}^{t} z^{k-t-1} (z - x)^{-k-1} \right\|_{Lip^\alpha(D_n)} = \sup_{z,w \in D_n, z \neq w} \frac{|x^{1-\lambda} f(z) \sum_{k=0}^{t} z^{k-t-1} (z - x)^{-k-1} - x^{1-\lambda} f(w) \sum_{k=0}^{t} w^{k-t-1} (w - x)^{-k-1}|}{|z - w|^\alpha} \tag{6}.$$
and it follows from an application of the triangle inequality that (6) is bounded by

\[ \sup_{z,w \in D_n; z \neq w} \frac{|x^{1-\lambda}(f(z) - f(w)) \sum_{k=0}^{t} z^{k-t-1}(z - x)^{-k-1}|}{|z - w|^\alpha} \]

(7)

\[ \sup_{z,w \in D_n; z \neq w} \frac{|x^{1-\lambda} \cdot |f(w)| \cdot \sum_{k=0}^{t} z^{k-t-1}(z - x)^{-k-1} - w^{k-t-1}(w - x)^{-k-1}|}{|z - w|^\alpha} \]

(8)

We can determine upper bounds for both (7) and (8). To bound (7) we recall that

\[ \|f\|_{Lip_n(D_n)} = \sup_{z,w \in D_n; z \neq w} \frac{|f(z) - f(w)|}{|z - w|^\alpha} \]

It follows from Lemma 10 that since \( x \) is on a non-tangential ray and \( z \in D_n \), and \( |z - x|^{-k-\lambda} \leq C|z|^{-k-\lambda} \), and it follows from the cone condition that \( |x^{1-\lambda}| \leq C|z - x|^{1-\lambda} \). Hence

\[ \frac{|x^{1-\lambda}(f(z) - f(w)) \sum_{k=0}^{t} z^{k-t-1}(z - x)^{-k-1}|}{|z - w|^\alpha} \leq C\|f]\|_{Lip_n(D_n)}|z|^{-t-1-\lambda} \leq C2^{n(t+1+\lambda)}\|f\|_{Lip_n(C)}' \]

(9)

Now we bound (8). By the triangle inequality, this is bounded by

\[ \sup_{z,w \in D_n; z \neq w} \frac{|x^{1-\lambda}f(w) \cdot \sum_{k=0}^{t} z^{k-t-1} - w^{k-t-1}(z - x)^{-k-1}|}{|z - w|^\alpha} \]

(10)

\[ + \frac{|x^{1-\lambda}f(w) \cdot \sum_{k=0}^{t} w^{k-t-1}((z - x)^{-k-1} - (w - x)^{-k-1})|}{|z - w|^\alpha} \]

(11)

We first obtain a bound for (10). Since \( z, w \in D_n, |z| \leq 2|w| \) and \( |w| \leq 2|z| \). Also it follows from Lemma 10 that for all integers \( m < 0, |z - x|^m \leq C|z|^m \) and it follows from the cone condition that \( |x^{1-\lambda}| \leq C|z - x|^{1-\lambda} \). Hence it follows from Lemma 10 that
and it follows from the cone condition that

\[ |z - w|^\alpha \leq \sup_{z,w \in D_n; z \neq w} \frac{|z - w|^\alpha}{|z - w|} \]

Since \( f(0) = 0 \), it follows that for \( w \in \mathbb{C} \), \( \frac{|f(w)|}{|w|^{\alpha}} \leq ||f||'_{\text{Lip}_\alpha(\mathbb{C})} \). Hence

\[ \sup_{z,w \in D_n; z \neq w} \frac{|f(w)|}{|z - w|^{1-\alpha}} \leq \sup_{z,w \in D_n; z \neq w} \frac{||f||'_{\text{Lip}_\alpha(\mathbb{C})}}{||z - w|^{1-\alpha} - 2 - \lambda} \]

Since \( z \) and \( w \) both belong to \( D_n \), \( |z - w| \leq C|w| \) and hence

\[ \sup_{z,w \in D_n; z \neq w} ||f||'_{\text{Lip}_\alpha(\mathbb{C})} \cdot |z - w|^{1-\alpha} \leq C2^{n(t+1+\lambda)}||f||'_{\text{Lip}_\alpha(\mathbb{C})}. \]

Thus

\[ \sup_{z,w \in D_n; z \neq w} |x^{1-\lambda} f(w)| \leq C2^{n(t+1+\lambda)}||f||'_{\text{Lip}_\alpha(\mathbb{C})}. \hspace{1cm} (12) \]

We next obtain a bound for \((11)\). Since \( z, w \in D_n \), \( |z| \leq 2|w| \), and \( |w| \leq 2|z| \). Also it follows from Lemma\(\ref{lemma10}\) that for all integers \( m < 0 \), \( |z - x|^m \leq C|z|^m \) and \( |w - x|^m \leq C|w|^m \) and it follows from the cone condition that \( |x^{1-\lambda}| \leq C|z - x|^{1-\lambda} \). Hence it follows from Lemma\(\ref{lemma8}\) that
Since \( f(0) = 0 \) it follows that for \( w \in \mathbb{C}, \frac{|f(w)|}{|w|^\alpha} \leq ||f||_{\text{Lip}(\mathbb{C})} \). Hence

\[
\sup_{z,w \in D_n; z \neq w} |f(w)| \cdot |z - w|^{-\alpha} \leq \sup_{z,w \in D_n; z \neq w} ||f||'_{\text{Lip}(\mathbb{C})} \cdot |z - w|^{-t-2-\lambda + \alpha}.
\]

Since \( z \) and \( w \) both belong to \( D_n \), \( |z - w| \leq C|w| \) and hence

\[
\sup_{z,w \in D_n; z \neq w} ||f||'_{\text{Lip}(\mathbb{C})} \cdot |z - w|^{-t-2-\lambda + \alpha} \leq C 2^{n(t+1+\lambda)} ||f||'_{\text{Lip}(\mathbb{C})}.
\]

Thus

\[
\sup_{z,w \in D_n; z \neq w} \frac{|x^1 \cdot f(w)| \cdot \sum_{k=0}^{t} w^{k-1}((z - x)^{-k-1} - (w - x)^{-k-1})}{|z - w|^\alpha} \leq 2^{n(t+1+\lambda)} ||f||'_{\text{Lip}(\mathbb{C})}.
\]

(13)

By applying (12) and (13) it follows that (8) is bounded by \( C 2^{n(t+1+\lambda)} ||f||'_{\text{Lip}(\mathbb{C})} \) and it follows from this and (9) that

\[
\left| x^{1-\lambda} \int_{\partial D_n} f(z) \sum_{k=0}^{t} z^{k-t-1}(z - x)^{-k-1} dz \right| \leq C 2^{n(t+1+\lambda)} M_{1+\alpha}^1(D_n \setminus U) ||f||'_{\text{Lip}(\mathbb{C})}.
\]

Since Hausdorff content is monotone, \( M_{1+\alpha}^1(D_n \setminus U) \leq M_{1+\alpha}^1(A_n \setminus U) \) and hence by the hypothesis of the theorem,
\[ x^{1-\lambda} \sum_{n=M}^{N} \int_{\partial D_n} f(z) \sum_{k=0}^{t} z^{k-t-1}(z-x)^{-k-1}dz \leq C \sum_{n=1}^{\infty} 2^{\eta(t+1+\lambda)} M_{*}^{1+\alpha} (A_n \setminus U) ||f||'_{Lip(\mathbb{C})} \]
\[ \leq C ||f||'_{Lip(\mathbb{C})}, \]

and thus it follows from this and (4) that

\[ \frac{|f^{(t)}(x) - f^{(t)}(0)|}{|x|^{\lambda}} \leq C ||f||_{Lip(\mathbb{C})} \]

for all \( f \in A_{\alpha}(U) \) and \( x \in J \).

4 Representing measures and bounded point derivations

We now prove Theorem 6.

Proof. As before we will assume that \( x_0 = 0 \) and \( X \) is contained in the ball \( \{ z : |z| < \frac{1}{2} \} \). Choose a sequence \( \epsilon_n \to 0 \) such that \( \sum_{n=1}^{\infty} 2^{(t+\lambda+1)n} \epsilon_n M_{*}^{1+\alpha} (A_n \setminus U) = \infty \) and \( 2^{(t+\lambda+1)n} \epsilon_n M_{*}^{1+\alpha} (A_n \setminus U) \leq 1 \) for all \( n \). Then for each integer \( N \), there exists an integer \( M > N \) such that

\[ 1 \leq \sum_{n=N}^{M} 2^{(t+\lambda+1)n} \epsilon_n M_{*}^{1+\alpha} (A_n \setminus U) \leq 2. \]

By Frostman’s Lemma [6 2.2], for each integer \( n \) with \( N \leq n \leq M \) there exists a positive measure \( \nu_n \) with support on \( A_n \setminus U \) such that \( \int \nu_n = C \epsilon_n M_{*}^{1+\alpha} (A_n \setminus U) \), where the constant \( C > 0 \) and does not depend on \( n \) or \( U \). Now define a function \( f_n \) by

\[ f_n(z) = \int \frac{d\nu_n(\zeta)}{z-\zeta}. \]

\( f_n \) belongs to \( A_{\alpha}(U) \) and is analytic off \( A_n \). In addition, it follows from Lemma [11] that \( |f_n(z)| \leq C 2^{-\lambda n} \) and if \( z \notin A_{n-1} \cup A_n \cup A_{n+1} \), then \( |f_n(z)| \leq \frac{CM_{*}^{1+\alpha} (A_n \setminus U)}{dist(z, A_n)}. \)
Now define $g_N(z)$ by

$$g_N(z) = |z|^\lambda z^{t+1} \sum_{n=N}^{M} 2^{(t+\lambda+1)n} f_n(z)$$

and consider the sequence $\{g_N\}_1^\infty$. We will show that this sequence is uniformly bounded on the unit disk.

Suppose that $z \in A_j$ for some positive integer $j$. By Lemma [11] if $n \neq j-1, j, j+1$ then

$$|f_n(z)| \leq C2^j M^{1+\alpha}(A_n \setminus U)$$

and if $n = j-1, j, j+1$ then

$$|f_n(z)| \leq C2^{-\lambda n} \leq C$$

since $n$ is a positive integer. From these estimates it follows that

$$|g_N(z)| \leq |z|^{\lambda+t+1} \left( 3C \cdot 2^{(t+\lambda+1)j} + \sum_{n=N}^{M} C2^{(t+\lambda+1)n} 2^j M^{1+\alpha}(A_n \setminus U) \right).$$

Since $z \in A_j$, $|z|^{\lambda+t+1} \leq C2^{-(t+\lambda+1)j}$ and

$$|g_N(z)| \leq 3C + |z|^{\lambda+t} 2C.$$

Thus $\{g_N(z)\}$ is uniformly bounded on the unit disk. Next define $h_N(z)$ by $h_N(z) = |z|^{-\lambda} z g_N(z)$, and consider the sequence $\{h_N(z)\}$. Because $g_N(z)$ is uniformly bounded on the unit disk, so also is $h_N(z)$ and since $h_N(z)$ is analytic outside the ball $\{|z| \leq 2^{-N}\}$, a subsequence (also denoted $\{h_N\}$) converges pointwise on $\mathbb{C} \setminus \{0\}$ to a function $h$ which is also analytic on $\mathbb{C} \setminus \{0\}$. Moreover, $g_N$ is uniformly bounded near 0 for each $N$ and hence $h(0) = 0$. Thus $h$ is entire.

Let $k_N(z)$ be defined by $k_N(z) = z^{t-2} h_N(z)$. Then
\[ k'_N(\infty) = \lim_{z \to \infty} zk_N(z) \]
\[ = \lim_{z \to \infty} z^{-t-1} h_N(z) \]
\[ = \lim_{z \to \infty} |z|^{-\lambda} z^{-t} g_N(z) \]
\[ = \lim_{z \to \infty} z \sum_{n=N}^{M} 2^{(t+\lambda+1)n} f_n(z). \]

Since \( \lim_{z \to \infty} zf_n(z) = \int \nu_n = C e_n M_1^{1+\alpha}(A_n \setminus U) \), it follows that \( 0 < C \leq |k'_N(\infty)| \leq 2C \). Thus passing to a second subsequence, still denoted by \( \{k'_N\} \), we find that \( \{k'_N(\infty)\} \) converges to \( \beta \) for some \( C \leq \beta \leq 2C \). Thus \( \lim_{z \to \infty} zk_N(z) = \beta \) and hence \( g_N(z) \) converges pointwise to \( \beta |z|^\lambda z^t \).

Assume that there exists a measure \( \mu \) which represents 0 on \( A_\alpha(U) \) such that \( \mu^{t+\lambda} < \infty \). Then \( |z|^{-\lambda} z^{-t} \mu \) is a finite measure. Let \( L_N(z) = |z|^{-\lambda} g_N(z) \). Then \( L_N(z) \) is analytic in a neighborhood of 0 and belongs to \( A_\alpha(U) \). Hence

\[ 0 = L_N^{(t)}(0) = t! \int \frac{L_N(z)}{z^t} d\mu(z) \]
\[ = t! \int \frac{g_N(z)}{|z|^\lambda z^t} d\mu(z) \]
\[ \rightarrow t! \int \beta d\mu(z) = t! \cdot \beta, \]

which is a contradiction. Hence \( \mu^{t+\lambda} = \infty \).

\[ \square \]

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