RESUMMATION OF PERTURBATIVELY ENHANCED
GLUON RADIATIVE CORRECTIONS

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Abstract

By remaining strictly within the confines of QCD, i.e., \textit{without} invoking the parton model or any other exogenous element, we identify and resum perturbatively to leading order, along with a correction term, enhanced radiative gluon contributions to the Drell-Yan type ($q\bar{q}$ pair annihilation) and deep-inelastic-scattering type ($eq \rightarrow eq + X$) cross-sections. The key feature of the adopted approach is the recasting of QCD in terms of a space-time mode of description, which employs a path-integral formulation of field theories, as originally implied in works of Fock, Feynman and Schwinger.

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I. INTRODUCTION

Studies surrounding the infrared (IR) behavior of transition amplitudes in gauge field theories present special interest not only for the fundamental – of interpretational nature – questions they pose but also for the calculable physical implications they entail. To become more specific and, at the same time, identify the object of interest in the present investigation, we are here alluding to resummation procedures of perturbatively enhanced pure QCD contributions to physical cross-sections. These matters have come under systematic scrutiny by Sterman and collaborators [1–3], following earlier realizations [4] on the physical relevance of soft gluon radiation.

The wide acceptance of QCD as the microscopic theory of the strong interactions, despite the fact that its fundamental field quanta do not register as observed asymptotic states, rests largely on one’s ability to isolate and identify its contribution to a given hadronic process by means of factorization theorems. With respect to the impact of IR properties of QCD on measurable quantities, the studies conducted in [1–3], which commence at the hadronic level and methodically proceed to the microscopic one [5], encourage the idea to pursue IR issues through a direct reference to QCD, i.e., through considerations which base themselves on the Green’s functions of the theory. Recognizing, at the same time, the key role played by the parton model in connection with applications of QCD, we shall employ for its description a Fock-Feynman-Schwinger [6–8], or equivalently worldline casting [9–11] (for recent reviews, we refer to [12,13]), wherein a space-time interpretation – as opposed to a Feynman diagrammatic one – of fundamental matter field quanta propagation is attained [14]. In this way, one aspires to achieve consistency with the parton model mode of description without, however, invoking its probabilistic content. Our efforts, in this paper, will be directed towards determining pure QCD contributions related to processes induced by electroweak interactions – Drell-Yan (DY) and deep inelastic scattering (DIS), in particular.

For the IR domain of QCD, a central issue of concern is where the perturbative picture...
ends and to what extent non-perturbative effects start to become dominant (for a review and references, we refer to [13]). As a matter of fact, the very concept of soft gluon radiation looses its meaning below some momentum scale of the order of $\Lambda_{\text{QCD}}$ (recent discussions of these issues are given in [16,17]). One consistent approach for attacking this problem strictly within the framework of the microscopic theory itself has been articulated by Ciafaloni [18].

As already mentioned, our own considerations will rely upon the worldline casting of QCD, keeping a more pragmatic course and focusing our attention on cross-section expressions.

One might reasonably wonder as what constitutes the innovative tool, provided by the worldline casting of QCD, that facilitates calculations which “flirt” with the edge of the perturbative domain of the theory. A direct answer can be given at this point though its full content will become more transparent through the main exposition in the sections to follow. In short, it is the ability to isolate a special set of space-time paths having a very simple geometrical profile shared in a restricted (but directly relevant to the physics of the process) neighborhood by each and every contour entering the path integral. The single (multiplicative) renormalization constant carried by this special family of paths automatically factorizes their contribution to amplitudes/cross-sections given that it also accompanies the rest of the paths. The more complex geometrical structure of the latter simply implicates additional ultraviolet (UV) singularities which can be absorbed into conventional wave-function and coupling-constant renormalizations. This clean, geometrically based, argument, which singularly underlines the worldline description, not only constitutes a notable simplification over the type of reasoning that has been promoted by Collins et al. [5] within the Feynman diagram perturbative approach to realize factorization in QCD, but, potentially, it paves the way to further applications. We especially have in mind issues pertaining to projections of our approach into the non-perturbative domain of the theory.

Letting these comments suffice for an introductory exposition, we now proceed to display the organization of this paper, which is as follows. In the next section, we exhibit the worldline expression for the full fermionic Green’s function and subsequently employ it to construct corresponding expressions both for DY and DIS type QCD amplitudes/cross-sections. Sec-
tion 3 furnishes, with the aid of an Appendix, our basic calculations associated with one virtual gluon exchanges for a special set of trajectories. The resulting expression explicitly reveals the threshold enhancement factor while the task of virtual gluon re-summation is performed, via the aid of the renormalization group, in Section 4. Section 5 deals with the re-summation of contributions from real gluons, whereas our conclusions are presented in Section 6.

II. BASIC WORLDLINE EXPRESSIONS FOR AMPLITUDES AND CROSS-SECTIONS

Consider the full two-point (fermionic) Green’s function in the presence of an external gluonic field. The expression, in Euclidean space-time,

\[ iG_{ij}(x, y|A) = \int_0^\infty dt e^{-\frac{tm^2}{2}} \int_{x(T) = y}^{x(0) = x} D\chi(t) \left[ m - \frac{1}{2} \gamma \cdot \dot{x}(T) \right] P \exp \left( \frac{i}{4} \int_0^T dt \sigma_{\mu\nu} \omega_{\mu\nu} \right) \times \exp \left[ -\frac{1}{4} \int_0^T dt \dot{x}^2(t) \right] P \exp \left[ ig \int_0^T dt \dot{x} \cdot A(x(t)) \right] \]

displays the basic worldline features pertaining, more generally, to \( n \)-point Green’s functions and, by extension, to amplitudes. Here, and below, \( P \) denotes the usual path ordering of the integrals. The first thing to point out is that a given path of the matter field quantum, starting at \( x \) and ending at \( y \) between respective “proper-time” values 0 and \( T \), also enters a Wilson line factor. The latter, being the sole carrier of the dynamics, separates itself from the rest of the factors in the path integral which are associated with geometrical properties of paths traversed by spin-1/2 particle entities. The most notable such quantity is the so-called spin factor \([19]\), \( P \exp \left[ (i/4) \int_0^T dt \sigma \cdot \omega \right] \), where \( \omega_{\mu\nu} = (T/2)(\dot{x}_\mu \dot{x}_\nu - \dot{x}_\mu x_\nu) \), accounting, in a geometrical way, for the spin-1/2 nature of the propagating particle. Accordingly, our perturbative expansions should be perceived of in terms of (Euclidean) space-time paths.

\[ ^1 \text{In this paper we shall be exclusively concerned with electroweak vertex functions of quarks which include gluon radiative effects.} \]
involving a “proper time” parameter and not in terms of Feynman diagrams. As it turns out [20], in the perturbative context, the structure of matter particle contours, entering the path integral, is determined by the points, where a momentum change takes place, i.e., points where a gauge field line (real or virtual) attaches itself on the (fermionic) matter field path. The almost everywhere non-differentiability of these contours, is residing precisely at these points. A major effort, in this paper, will be devoted to the extension of the worldline formalism to expressions for cross-sections corresponding to the particular processes of \( q \bar{q} \) annihilation and \( e + q \rightarrow e + q + X \).

From the worldline point of view, the processes we intend to study involve fermionic matter particle (quarks) paths that commence at \( x \) and end at \( y \), being forced to pass through an intermediate point \( z \), where a momentum transfer \( Q \) takes place. This means that the Green’s (vertex-type) function we shall be dealing with has the following form (\( \Gamma_\mu \) denotes some Clifford-Dirac algebra element)

\[
V_{\mu,ij}(y, z, x|A) = G_{ik}(y, z|A) \Gamma_\mu G_{kj}(z, x|A) \\
= \int_0^\infty dTe^{-Tm^2} \int_0^T ds \int_{x(T)=y} \mathcal{D}x(t) \delta(x(s) - z) \Gamma_\mu (\dot{x}, s) \exp \left[ -\frac{1}{4} \int_0^T dt \dot{x}^2(t) \right] \\
\times e^{ig \int_0^T dt \dot{x}(t) \cdot A(x(t))} \\
\times \exp \left[ i \frac{4}{4} \int_0^s dt \dot{x}(t) \cdot \omega \right],
\]

(2)

where

\[
\Gamma_\mu (\dot{x}, s) \equiv \left[ m - \frac{1}{2} \gamma \cdot \dot{x}(T) \right] P \exp \left( \frac{i}{4} \int_s^T dt \sigma \cdot \omega \right) P \exp \left( \frac{i}{4} \int_0^s dt \sigma \cdot \omega \right) \\
\times \Gamma_\mu \left[ m - \frac{1}{2} \gamma \cdot \dot{x}(s) \right].
\]

(3)

It is especially important to realize that in our approach off-shellness is naturally parameterized in terms of the finite size of the matter particle contours and realistically accounts for the fact that quarks reside inside a hadron (\( m \) can be viewed as an effective quark mass).

Going over to momentum space, we write

\[
\tilde{V}_{\mu,ij}(p, p'|z|A) = \int_0^\infty dTe^{-Tm^2} \int_0^T ds \int \mathcal{D}x(t) \delta(x(s) - z) \Gamma_\mu (\dot{x}, s)
\]

5
\[ \times \exp \left[ -\frac{1}{4} \int_0^T dt \dot{x}^2(t) + ip \cdot x(0) + ip' \cdot x(T) \right] \]
\[ \times P \exp \left[ ig \int_0^T dt \dot{x}(t) \cdot A(x(t)) \right]_{ij} \]
\[ \equiv \sum_{C_z} \bar{\Gamma}[C^z] P \exp \left[ ig \int_{C_z} dx \cdot A(x) \right]_{ij}, \quad (4) \]

where \( C^z \) denotes a generic path forced to pass through point \( z \), at which the momentum \( Q \) is imparted.

For a process of the type (DY) \( q + \bar{q} \rightarrow \text{lepton pair} + X \) the “amplitude” expression reads
\[ \Delta^{(\text{DY})}_{\mu,ij} = \bar{v}(p', s')(i\gamma \cdot p' + m) \bar{V}_{\mu,ij}(i\gamma \cdot p + m) u(p, s) \]
\[ \equiv \sum_{C_z} \bar{I}^{(\text{DY})}_{\mu,p'} [C^z] P \exp \left[ ig \int_{C_z} dx \cdot A(x) \right]_{ij}, \quad (5) \]

with the second, comprehensive, expression to be understood having recourse to Eq. (4).

Similarly, for the DIS-type process \( e + q \rightarrow e + q + X \), one writes
\[ \Delta^{(\text{DIS})}_{\mu,ij} = \bar{u}(p', s')(i\gamma \cdot p' + m) \bar{V}_{\mu,ij}(i\gamma \cdot p + m) u(p, s) \]
\[ \equiv \sum_{C_z} \bar{I}^{(\text{DIS})}_{\mu,p'} [C^z] P \exp \left[ ig \int_{C_z} dx \cdot A(x) \right]_{ij}, \quad (6) \]

with the replacement \( p' \rightarrow -p' \) to be also made in Eq. (4).

For the cross-section, we need to employ the following quantity, which we implicitly display in Minkowski space-time after straightforward adjustments,
\[ \Delta^{(\alpha)}_\mu \Delta^{(\alpha)}_\nu = \sum_{C_z} \sum_{C_{z'}} \bar{I}^{(\alpha)\dagger}_{\mu,p'} [\bar{C}^{z'}] \bar{I}^{(\alpha)}_{\nu,p'} [C^z] \]
\[ \times Tr \left\{ \bar{P} \exp \left[ ig \int_{C_{z'}} dx' \mu A_{\mu}(x') \right] P \exp \left[-ig \int_{C_z} dx'' A_{\mu'}(x) \right] \right\}, \quad (7) \]

where \( \bar{P} \) denotes anti-path ordering and the index \( \alpha \) stands for either DY or DIS. Even though not explicitly displayed, the cross-section acquires a path-integral form, which has the following characteristics: 1) Paths \( C^z \) and \( \bar{C}^{z'} \) are forced to pass through points \( z \) and \( z' \), respectively, where the momentum transfer occurs (see Fig. [1]). The distance \( b \equiv |z - z'| \) serves as a measure of how far apart the two conjugate contours can venture away from each
to each other at points $z$ and $z'$, where the momentum transfer for the physical process takes place. The distance $|z - z'|$ is referred to as the impact parameter.

other and will be referred to as the impact parameter. 2) The traversal of $\bar{C}z'$ is made in the opposite sense relative to $Cz$, pretending the two paths join at one end (using translational invariance), while allowing the other two ends of the contour to close at infinity, giving rise to the formation of a Wilson loop. 3) Under the circumstance just described, the Wilson loop formation guarantees the gauge invariance of the expression for the cross section. On the other hand, by keeping the contour lengths finite, but very large, thereby placing the quarks off-mass-shell, gauge invariance will still continue to hold to the order of approximation we employ in our computations, given that the off-mass-shellness also serves as an IR cutoff.

Up to this point our considerations have been centered around the geometrical profile of the worldline casting of QCD. In terms of our comprehensive formulas (3) and (4) for the amplitude and cross-section, respectively, the primary emphasis has been placed on the first factors of the respective sums, the main conclusion being that the relevant contours entering the path integral are open for the amplitude and closed (or almost so) for the cross-section. Armed with this information, we now turn our attention to the Wilson factor which
contains all the dynamics of the given process. The obvious task in front of us is to assess its implications once the gauge fields are quantized, i.e., once the Wilson factor is inserted into a \textit{functional} integral weighted by the exponential of the Yang-Mills action. We display the quantity of interest as follows

\begin{equation}
W = \left\langle \text{Tr} \left\{ \bar{P} \exp \left[ ig \int_{C^z} dx^\mu A_\mu (x) \right] \right\} \right\rangle_\Lambda \left\{ \text{P exp} \left[ -ig \int_{C^z} dx^\mu A_\mu (x) \right] \right\} \right\rangle_\Lambda \\
\equiv \left\langle \text{Tr} (U (\bar{C}^z) U (C^z)) \right\rangle.
\end{equation}

In the above expression, \{ \cdots \}_\Lambda signifies the expectation value with respect to the gauge field functional integral which, in this work, will be considered in the context of perturbation theory. Note in the same context that a virtual gluon attaching itself with both ends to the fermionic worldline, entering the amplitude, corresponds to a correlator between a pair of gauge fields originating from the expansion of the Wilson factor. On the other hand, for an emitted “real” gluon from the fermionic line, the correlator is between an “external” and a Wilson-line gauge field.\footnote{One will, of course, also encounter correlators that involve gauge fields from the non-linear terms of the Yang-Mills action. These, however, do not enter the leading logarithmic considerations.} The overall situation is depicted in Fig. 2. At the cross-section level, now, “real” gluons are integrated with respect to “propagators” linking together the two conjugate contours, while their polarization vectors are summed over (cut propagators). This is precisely what \langle \cdots \rangle signifies in the last equation, as it covers both Wilson line factors.

In the light of the above remarks, let us proceed to display the first-order (in perturbation theory) expression for \( W \), which receives contributions both from virtual gluons, viz.; ones attached at both ends either to worldline contour \( C^z \) or to contour \( \bar{C}^z \), as well as from “real” gluons linking these contours to each other (cf. Fig. 2). This expression reads

\begin{equation}
W^{(2)} = \text{Tr} \left[ I - g^2 C_F \int_0^T dt \int_0^T dt' \theta (t_2 - t_1) \dot{x}^\mu (t_2) \dot{x}'^\nu (t_1) D_{\mu \nu} (x(t_2) - x(t_1)) \right. \\
- g^2 C_F \int_0^{T'} dt' \int_0^{T'} dt'' \theta (t'_2 - t'_1) \dot{x}'^\mu (t'_1) \dot{x}''^\nu (t'_1) D_{\mu \nu} (x(t'_2) - x(t'_1)) \\
- g^2 C_F \int_0^T dt \int_0^{T'} dt' \dot{x}(t) \cdot \dot{x}' (t') D_{\text{cut}} (x(t) - x(t')) + O \left( g^4 \right).
\end{equation}

FIG. 2. Virtual gluon radiative corrections of various sorts and “real” gluon lines with their ends attached on each of the two depicted contours at the cross-section level.

It becomes obvious from their structure that the first two non-trivial terms correspond to virtual gluon contributions – one per conjugate branch –, while the third one is associated with “real” gluon emission. Finally, concerning the gluon propagators entering the above equation, we shall be employing their Feynman-gauge form without loss of generality due to gauge invariance. In particular we have, in $D$-dimensions,

$$D_{\mu \nu}(x) = -ig_{\mu \nu} \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{e^{-ik \cdot x}}{k^2 + i0^+} = g_{\mu \nu} \frac{1}{4\pi^2} \left(-\pi \mu^2\right)^{(4-D)/2} \frac{\Gamma(D/2 - 1)}{(x^2 - i0^+)^{(D/2)-1}}, \quad (10)$$

whereas

$$D_{\text{cut}}(x) = \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} 2\pi \delta(q^2) \theta(q^0) e^{-i q \cdot x} = \frac{1}{4\pi^2} \left(-\pi \mu^2\right)^{(4-D)/2} \frac{\Gamma(D/2 - 1)}{[(x_0^2 - i0^+)^2 - x^2]^{(D/2)-1}}. \quad (11)$$

As already established by other methods, the perturbative expansion (3) is plagued by large threshold logarithms leading to the need for factorization and resummation. This is precisely the task we are about to undertake within the framework we have developed and which does not take us outside QCD.
III. FIRST-ORDER VIRTUAL GLUON CORRECTIONS UNIVERSAL TO THE VICINITIES OF POINTS $z$ AND $z'$

The space-time mode of description of our worldline casting puts us into the position to promote the following argument: The point $z$ (or $z'$), where the momentum transfer $Q$ is imparted, marks the presence of a neighborhood around it, no matter how infinitesimal in size this might be, whose geometrical structure is shared by all fermionic paths entering the path-integral. Specifically, there will be a derailment (cusp formation), whose opening angle will be fixed unambiguously, since it is determined by the momentum transfer. It follows that the contributions to the amplitude and cross-section from the immediate vicinities of these points is a common feature of all contours and eventually factorizes. In this section we shall determine the first-order perturbative term corresponding exactly to this factor.

Consider now the neighborhood of point $z$ on the contour $C^z$. Expanding around this point, we write

$$x^\mu(t) = x^\mu(s) + t \dot{x}^\mu(s \pm 0) + \ldots$$ \hspace{1cm} (12)

with $v^\mu = \dot{x}^\mu(s - 0)$ and $v'^\mu = \dot{x}^\mu(s + 0)$ being entrance and exit four-velocities, respectively, with respect to $z$.

Adjusting our notation by re-parameterizing the contour so that the zero value is assigned to point $z$, the relevant quantity to compute, to first perturbative order, becomes

$$U^{(2)}_{C,S} = 1 - g^2 C_F \left[ \int_{-\sigma}^{0} dt_1 \int_{-\sigma}^{0} dt_2 \theta(t_2 - t_1) v^\mu v'^\nu D_{\mu\nu}(vt_2 - vt_1) + \int_{0}^{\sigma} dt_1 \int_{0}^{\sigma} dt_2 \theta(t_2 - t_1) \
+ v'^\mu v'^\nu D_{\mu\nu}(v't_2 - v't_1) + \int_{-\sigma}^{0} dt_1 \int_{-\sigma}^{0} dt_2 \theta(t_2 - t_1) v'^\mu v'^\nu D_{\mu\nu}(v't_2 - vt_1) \right].$$ \hspace{1cm} (13)

It is clear that the above expression corresponds to the first term in Eq. (9), which monitors virtual gluon exchanges occurring on contour $C$.

From the above considerations it follows that the main contribution to each double integral comes from the common limit $t_1, t_2 \to 0$. Suppose now, the other limit is to be determined by demanding that its contribution to the integrals is of vanishing importance.
Then, such a requirement automatically isolates those contours, whose only significant geometrical characteristic is that the four-velocities to approach and depart from point \( z \) are fixed, denoted by \( v^\mu \) and \( v'^\mu \) respectively; the same, of course, happens for point \( z' \), but in reverse order. This justifies the subscript \( S \) in \( U_{C,S} \), which stands for “smooth”. More complex geometrical properties such as non-differentiability almost everywhere (which most certainly characterizes the vast majority of paths) do not enter Eq. (13). The point is, on the other hand, that every path will share the geometrical structure we are focusing on in some neighborhood of the point \( z \), or \( z' \), no matter how close to these points one has to come. At the same time, the UV singularities of this restricted set of paths will entail expressions that solely depend on the two four-velocities and the opening angle. Paths of more complex geometrical structure, on the other hand, will certainly exhibit these UV singularities plus additional ones.\(^3\) It naturally follows that our restricted set of trajectories, by exclusively determining the corresponding (multiplicative) renormalization constant will factorize from the rest of the expression for the amplitude and/or the cross-section. From dimensional considerations it follows that the omitted terms in Eq. (12) will contain negative powers of \( \sigma \), whose dimension in the denominator is \((\text{mass})^2\). Neglecting their presence means that \( \sigma \) should be very large in magnitude and hence should be related to the IR cutoff, i.e., \( \sigma \simeq \lambda^{-2}, \lambda > \Lambda_{\text{QCD}} \).

Going over to Minkowski space-time, there are two distinct possibilities for defining an infinitesimally small neighborhood around \( z \). The first one, to be labelled \((a)\), reads

\[
(x - x')^2 = \mathcal{O}(\epsilon^2), \quad \text{with} \quad v_\mu \simeq v'_\mu, \quad \text{for all} \quad \mu,
\]

where \( \epsilon (\leq Q^{-1}) \) is a small length scale. The second alternative, to be labelled \((b)\), can be typically represented by

\[
(14)
\]

\(^3\) Actually, the standard UV singularities of perturbative field theories associated with \( \beta \)-functions, coupling constant and wave-function renormalization pertain to almost everywhere non-differentiable paths.
\[(x - x')^2 = \mathcal{O}(\varepsilon^2)\] with \(|v - v'|^2 = \mathcal{O}(\lambda^2)\) but \((v_+ - v'_+) \simeq \mathcal{O}(Q)\) and \((v_- - v'_-) \simeq \mathcal{O}\left(\frac{\lambda^2}{Q}\right)\)

\[
\Rightarrow (v_+ - v'_+)(v_- - v'_-) = \mathcal{O}(\lambda^2)
\]

that is equivalently effected via the condition \(v_+ \gg v'_+, v_- \simeq v'_-\). All in all, there are four different configurations: \(+ \leftrightarrow -\) and prime \(\leftrightarrow\) no-prime entering this case.

We characterize case (a) as “uniformly soft”, given that the considered gluon exchanges take place in a neighborhood whose smallness pertains to all directions. Case (b), on the other hand, will be referred to as “jet” since gluon emission occurs under circumstances, where entrance and exit four-velocities differ between them significantly along one or the other of the light-cone directions. We point out that what here is termed as “jet” does not coincide with the notion of Collins et al. [5]. The actual connection will become evident later on.

Let us commence our calculations by taking up the first \(\mathcal{O}(g^2)\) term entering the right hand side of Eq. (13). Since this only involves the branch of the contour \(C^z\) entering point \(z\), we obtain the same expression regardless of whether or not a uniformly soft or a jet configuration is being considered. It reads

\[
I_1 = \int_{-\sigma}^{0} dt_1 \int_{-\sigma}^{0} dt_2 \theta(t_2 - t_1) v^\mu v'^\nu D_{\mu\nu} (vt_2 - vt_1)
\]

\[
= -\frac{1}{8\pi^2} \left(-\pi \mu^2 L_1^2\right)^{(4-D)/2} \Gamma \left(\frac{D}{2} - 1\right) \frac{1}{D - 3} \frac{1}{2 - D/2}, \tag{16}
\]

where \(L_1 = \sigma|v|\). The second term has the same structure (it involves the exiting branch of \(C^z\)) and therefore produces a similar result:

\[
I_2 = -\frac{1}{8\pi^2} \left(-\pi \mu^2 L_2^2\right)^{(4-D)/2} \Gamma \left(\frac{D}{2} - 1\right) \frac{1}{D - 3} \frac{1}{2 - D/2}, \tag{17}
\]

with \(L_2 = \sigma|v'|\).

A couple of remarks are in order at this point. First, even though the length scales \(L_1\) and \(L_2\) are both large, being proportional to \(\sigma\), they will be of the same order of magnitude.

\(^4\)Note that \(v\) has dimensions of mass as our “time” parameter \(\sigma\) has dimensions of \((\text{mass})^{-2}\).
for case (a), whereas for case (b), one scale will be negligible in comparison with the other. Accordingly, the total expression for the uniformly soft amplitude will be twice as large as that of the jet-like one. This being said, we shall denote the dominant length scale by $L \approx L_1$ and/or $L_2$, when it enters our final expressions, and set it equal to $\frac{1}{\lambda}$, recognizing that it is of the same order as the IR cutoff. Second, in order to avoid the double counting resulting from the fact that each branch has been “cut-off” at corresponding distances away from $z$ (implying that gluon emission will occur at the endpoints to be offset by a similar one but opposite in sign from that portion of the contour that continues to stretch out to infinity), the final expressions for the amplitudes/cross-sections should be multiplied by a factor of $1/2$. Equivalently, one might think of this compensation as actually identifying the missing energy of the gluon emission at the extremities with the off-mass-shellness. In fact, this is what we have been implying all along when claiming that finite contours signify off-mass-shellness.

Turning our attention to the contribution resulting from a virtual gluon exchange from the entrance to the exit branch, with respect to $z$, we consider the quantity

$$I_3 = \int_{-\sigma}^{\sigma} dt_1 \int_{t_1}^{\sigma} dt_2 \, v^\nu v'^\nu D_{\nu\nu} (v't_2 - vt_1) = \frac{1}{4\pi^2} (-\pi \mu^2)^{(4-D)/2} \Gamma \left( \frac{D}{2} - 1 \right)$$

$$\times v \cdot v' \int_{-\sigma}^{\sigma} dt_1 \int_{t_1}^{\sigma} dt_2 \left( t_1^2 v^2 + t_2^2 v'^2 + 2t_1t_2 v \cdot v' - i0_+ \right)^{1-(D/2)}. \quad (18)$$

For case (a) it assumes the form (recall that $v \cdot v'$ is negative for the DY and positive for the DIS type of process)

$$I_{3}^{(a)} = \frac{1}{4\pi^2} (-\pi \mu^2)^{(4-D)/2} \Gamma \left( \frac{D}{2} - 1 \right) \frac{v \cdot v'}{|v||v'|} \int_{0}^{1} dt_1$$

$$\times \int_{0}^{1} dt_2 \left( t_1^2 + t_2^2 + 2t_1t_2 \frac{v \cdot v'}{|v||v'|} - i0_+ \right)^{1-(D/2)}. \quad (19)$$

As shown in the Appendix, for DY, one determines ($\gamma_E$ is Euler’s constant)

$$I_{3,\text{DY}}^{(a)} = \frac{1}{8\pi^2} \gamma_{\text{DY}} \coth \gamma_{\text{DY}} \frac{1}{2 - \frac{D}{2}} + \frac{1}{8\pi^2} \gamma_{\text{DY}} \coth \gamma_{\text{DY}} \ln \left( \frac{\mu^2}{\lambda^2 \pi e^{2\gamma_E}} \right), \quad (20)$$

where $\cosh \gamma_{\text{DY}} = w_{\text{DY}} = \frac{-v \cdot v'}{|v||v'|} \geq 1$, while for DIS ($w_{\text{DIS}} = \frac{v \cdot v'}{|v||v'|} \geq 1$)
\[ I^{(a)}_{3,\text{DIS}} = \frac{1}{8\pi^2} \gamma_{\text{DIS}} \coth \gamma_{\text{DIS}} \frac{1}{2 - D/2} + \frac{1}{8\pi^2} \gamma_{\text{DIS}} \coth \gamma_{\text{DIS}} \ln \left( \frac{\mu^2}{\lambda^2} \pi e^{2+\gamma_E} \right). \]  \hspace{1cm} (21)

In all of the above expressions, as well as in those that will follow, we have ignored: (i) all imaginary terms that will drop out when contributions (for virtual gluons) from the conjugate contour are taken into account and (ii) finite, \( \mu \)-independent terms that will cancel out when real-gluon contributions to the cross-section are included.

Collecting all terms, we deduce, for the “uniformly smooth” part,

\[ I^{(a)}_1 + I^{(a)}_2 + I^{(a)}_3 = \frac{1}{8\pi^2} (\gamma \coth \gamma - 1) \frac{1}{2 - D/2} + \frac{1}{8\pi^2} (\gamma \coth \gamma - 1) \ln \left( \frac{\mu^2}{\lambda^2} \pi e^{2+\gamma_E} \right), \]  \hspace{1cm} (22)

where the angle \( \gamma \) is to be adjusted either to the DY or the DIS situation.

Concerning the “jet” part of the computation, we only need to consider \( I^{(b)}_3 \) because \( I^{(b)}_1 + I^{(b)}_2 \) is simply one half of that of \( I^{(a)}_1 + I^{(a)}_2 \). A typical term entering \( I^{(b)}_3 \) \( (v_- \gg v'_+) \) is

\[ I^{(b)}_3 = \frac{1}{4\pi} (-\pi \mu^2)^{(4-D)/2} \Gamma \left( \frac{D}{2} - 1 \right) v \cdot v' \int_0^\sigma dt_1 \int_0^\sigma dt_2 \left( t_1^2 v^2 + 2t_1t_2 v \cdot v' - i0_+ \right)^{1-(D/2)}, \]  \hspace{1cm} (23)

whose computation suffices to furnish each of the other three terms as well.

It is shown in the latter part of the Appendix that for either the DY or the DIS type process one obtains

\[ I^{(b)}_3 = \frac{1}{16\pi^2} \frac{1}{(2 - D/2)^2} + \frac{1}{16\pi^2} \frac{1}{2 - D/2} \ln \left( \frac{\mu^2}{\lambda^2} \pi e^{\gamma_E} \right) + \frac{1}{32\pi^2} \ln^2 \left( \frac{\mu^2}{\lambda^2} \pi e^{\gamma_E} \right) + \text{const.} \]  \hspace{1cm} (24)

Subtracting the pole terms in the \( \overline{\text{MS}} \) scheme, we arrive at the finite part of the overall result. For the uniformly soft contribution, in particular, we get

\[ (I^{(a)}_1 + I^{(a)}_2 + I^{(a)}_3)_{\text{fin}} = \frac{1}{8\pi^2} (\gamma \coth \gamma - 1) \ln \left( \frac{\mu^2}{\lambda^2} \right), \]  \hspace{1cm} (25)

while the jet contribution reads

\[ \text{Recall the remark following Eq. (17).} \]
\[
(I_1^{(b)} + I_2^{(b)} + I_3^{(b)})_{\text{fin}} = \frac{1}{16\pi^2} \ln^2 \left( \frac{\mu^2}{\lambda^2} \right),
\]
(26)

where we have set \( \bar{\lambda}^2 \equiv 4\lambda^2 e^{-2\eta_c} \). The above relation takes into account all four different configurations contributing to \( I_3^{(b)} \).

Putting everything together we arrive at the following overall result for the second order contribution stemming from contour \( C^z \)

\[
U_{C,S}^{(2)} = 1 - \frac{\alpha_s}{2\pi} C_F \left[ \gamma \coth \gamma \ln \left( \frac{\mu^2}{\lambda^2} \right) - \frac{3}{2} \ln \left( \frac{\mu^2}{\lambda^2} \right) + \ln^2 \left( \frac{\mu^2}{\lambda^2} \right) \right].
\]
(27)

A similar result is obtained also for contour \( \bar{C}^{z'} \).

Noting that \( \gamma \coth \gamma = \ln (Q^2/m^2) \) (for \( Q^2 \gg m^2 \)), with \( Q^2 = (p + p')^2 \) for the DY and \( Q^2 = -(p' - p)^2 \) for the DIS type of process, respectively, we recognize that the well known perturbative enhancements occurring as \( Q^2 \to \infty \) are associated with the eikonal-type trajectories upon which our present calculations have been based. One, now, realizes that these trajectories define threshold conditions, with respect to the given momentum exchange \( Q \), for the processes under consideration since they leave no room for space-time contour fluctuations. In the following section we shall deal with the summation of these enhanced contributions to leading logarithmic order. We shall, furthermore, identify a correction factor associated with those terms in Eq. (27) not involving the enhancement factor \( \ln(Q^2/m^2) \).

**IV. SUMMATION OF ENHANCED CONTRIBUTIONS FROM VIRTUAL GLUONS**

The family of worldline paths to which the considerations in the previous section have referred to recognize all (virtual) single-gluon exchanges, consistent with the simple geometrical configuration of two constant four-velocities making a fixed angle \( \gamma \) between them (in Euclidean formulation). Among these gluons there will be “hard” ones (upper limit \( Q \)) and “soft” ones (lower limit set by \( \bar{\lambda} \)). What is debited to the former and what to the latter group of gluons is, of course, relative. It is precisely the role of the renormalization scale \( \mu \), entering through the need to face UV divergences arising even for the restricted family
of paths, to provide the dividing line. The corresponding renormalization-group equation reflects the fact that the scale $\mu$ is arbitrary and that physical results do not depend on it. A straightforward application of this fact will enable us to sum the enhanced, virtual gluon contribution to the amplitude in leading logarithmic order as well as to obtain a bona-fide correction term.

To bring the above discussion into a concrete form, consider a factorization, good to order $1/Q^2$, between the “soft” and the “hard”, relative to the scale $\mu$, gluon contributions to the amplitude $U_C$ entering Eq. (8) for the restricted set of contours. We write

$$U_C = U_{C,S} \left( \frac{Q^2}{m^2}, \frac{\mu^2}{\lambda^2} \right) U_{C,H} \left( \frac{Q^2}{\mu^2} \right) + \mathcal{O}\left( \frac{1}{Q^2} \right)$$

noting that the soft part $U_{C,S}$ coincides, to second order, with $U_{C,S}^{(2)}$.

It is further convenient to separate the part of the “soft” term involving the enhancement factor from the one which does not. To this end we write, to second order in perturbation theory,

$$U_{C,S}^{(2)} = U_{C,cusp}^{(2)} U_{C,coll}^{(2)}$$

with

$$U_{C,cusp}^{(2)} = 1 - \frac{\alpha_s}{2\pi} C_F \ln \left( \frac{Q^2}{m^2} \right) \ln \left( \frac{\mu^2}{\lambda^2} \right)$$

and

$$U_{C,coll}^{(2)} = 1 - \frac{\alpha_s}{2\pi} C_F \left[ \ln^2 \left( \frac{\mu^2}{\lambda^2} \right) - \frac{3}{2} \ln \left( \frac{\mu^2}{\lambda^2} \right) \right].$$

In the above relations, the designation “cusp” refers to that part of the soft factor which recognizes the angle $\gamma$, whereas the characterization “collinear” signifies independence from $\gamma$. From a physical standpoint, it represents the contribution from those gluons which

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6 For the purposes of the following discussion, it suffices to use the generic notation $C$ for either contour.
are emitted collinearly with incoming or outgoing four-velocities due to the sudden acceleration/deceleration occurring at the cusp point. It should be further pointed out that collinear emission occurring in the above context pertains to the soft sector and does not coincide in the full with what is designated as “collinear” by Collins et al. in [5]. Full agreement on “collinearity” will arise a posteriori, i.e., once resummation is performed.

From Eqs. (28)-(31) we obtain a relation which does not involve $U_{\text{coll}}$ by simply taking the logarithmic derivative with respect to $Q^2$:

$$\frac{d}{d \ln Q^2} \ln U = \frac{d}{d \ln Q^2} \ln U_{\text{cusp}} + \frac{d}{d \ln Q^2} \ln U_{\text{H}},$$  \hspace{1cm} (32)

where we have dropped the subscript $C$ as being superfluous in the considerations to follow.

The $\mu$-independence of physical results leads to the renormalization group equation

$$\frac{d}{d \ln \mu} \frac{d}{d \ln Q^2} \ln U_{\text{H}} = - \frac{d}{d \ln \mu} \frac{d}{d \ln Q^2} \ln U_{\text{cusp}} = - \frac{1}{2} \Gamma_{\text{cusp}}(\alpha_s)$$  \hspace{1cm} (33)

with $\Gamma_{\text{cusp}}$ to be read off from Eqs. (22)-(24) and (13); specifically,

$$\Gamma_{\text{cusp}}(\alpha_s) = \frac{\alpha_s}{\pi} C_F + \mathcal{O}(\alpha_s^2).$$  \hspace{1cm} (34)

From the second leg of Eq. (33), one obtains

$$\frac{d}{d \ln Q^2} \ln U_{\text{cusp}} = - \int_{\frac{Q^2}{\mu^2}}^{\mu^2} \frac{dt}{2t} \Gamma_{\text{cusp}}(\alpha_s(t))$$ \hspace{1cm} (35)

which, in turn, gives

$$\frac{d}{d \ln Q^2} \ln U_{\text{H}} = - \int_{\frac{Q^2}{\mu^2}}^{\mu^2} \frac{dt}{2t} \Gamma_{\text{cusp}}(\alpha_s(t)) + \Gamma \left[ \alpha_s(Q^2) \right],$$ \hspace{1cm} (36)

where we have defined

$$\Gamma \left[ \alpha_s(Q^2) \right] \equiv \frac{d}{d \ln Q^2} \ln U_{\text{H}} \left( \frac{Q^2}{\mu^2} \right)_{\mu^2=Q^2}. \hspace{1cm} (37)$$

Combining the last three equations we have

$$\frac{d}{d \ln Q^2} \ln U = - \int_{\frac{Q^2}{\mu^2}}^{\mu^2} \frac{dt}{2t} \Gamma_{\text{cusp}}(\alpha_s(t)) + \Gamma \left[ \alpha_s(Q^2) \right],$$ \hspace{1cm} (38)
where, in terms of \( \ln U \), we write
\[
\Gamma[\alpha_s(Q^2)] \equiv \frac{d}{d \ln Q^2} \ln U|_{\lambda^2=Q^2}.
\] (39)

Setting \( Q^2 = \mu^2 \), we are led to the isolation of the “cusp” and “collinear” terms according to
\[
\ln U = \ln U_{\text{cusp}} \left( \frac{Q^2}{m^2}, \frac{Q^2}{\lambda^2} \right) + \ln U_{\text{coll}} \left( \frac{Q^2}{\lambda^2} \right) + \ln U_H \left( \frac{Q^2}{\mu^2} \right)_{\mu^2=Q^2}
\] (40)
from which, once appealing to Eq. (35), we obtain
\[
\frac{d}{d \ln Q^2} \ln U|_{\lambda^2=Q^2} = \frac{d}{d \ln Q^2} \ln U_{\text{coll}}|_{\lambda^2=Q^2}
\] (41)
which leads to the identification
\[
\Gamma[\alpha_s(Q^2)] = \frac{d}{d \ln Q^2} \ln U_{\text{coll}}|_{\lambda^2=Q^2} = \frac{3}{4} \alpha_s(Q^2) + \mathcal{O}(\alpha_s^2).
\] (42)

Collecting together all our findings, we obtain our final, re-summed result corresponding to the contour \( C \). It reads
\[
U = \exp \left\{ -\int_{\lambda^2}^{Q^2} \frac{dt}{2t} \left[ \ln \frac{Q^2}{t} \Gamma_{\text{cusp}}(\alpha_s(t)) - \Gamma(\alpha_s(t)) \right] \right\} U_0(\alpha_s(Q^2)).
\] (43)

One notes that the correction factor is associated with collinear emission (cf. Eq. (42)). Secondly, the factor \( U_0(\alpha_s(Q^2)) \) represents initial conditions input at the QCD level. One cannot help but bring to mind evolution equations at the partonic level where the probabilistic interpretation prevails. The connection between evolution and renormalization group equations is widely taken for granted. The worldline approach might provide the space-time description framework for understanding the precise connection between the two types of equations. Finally, let us remark that the conjugate-contour term \( U^\dagger(\bar{C}z') \) can be treated in a completely analogous fashion.
V. SUMMATION OF ENHANCED CONTRIBUTIONS FROM REAL GLUON EMISSION

We shall now turn our attention to real gluons and attempt to factorize cross-section contributions from neighborhoods around points $z$ and $z'$. Note that we now have to deal with gluons which connect two “opposite” neighborhoods while crossing the unitarity line. (this situation is depicted in Fig. 3) The relevant scale promptly entering our considerations is the impact parameter $b = z - z'$, which must be eventually integrated over in order to get the cross-section. Naturally, the short-distance cutoff in this integration will be provided by the (length) scale $1/|Q|$.

For the eikonal-type family of paths and in first order perturbation theory, the relevant quantity on which our quantitative considerations are to be based, i.e., the counterpart of Eq. (13), is given by

$$U^{(2)}_{CC,S} = 1 + g^2 C_F \left[ \int_{-\sigma}^{0} dt_1 \int_{-\sigma}^{0} dt_2 v \cdot \bar{v} D (t_1 v - t_2 \bar{v} + b) + \int_{0}^{\sigma} dt_1 \int_{0}^{\sigma} dt_2 v' \cdot \bar{v}' D (t_1 v' - t_2 \bar{v}' + b) \right]$$

FIG. 3. Neighborhoods of respective points on two conjugate contours, where the momentum transfer takes place, and associated four-velocities.
\begin{align} 
&+ \int_{-\sigma}^{0} dt_1 \int_{-\sigma}^{0} dt_2 v \cdot \bar{v}' D (t_1 v - t_2 \bar{v} + b) \\
&+ \int_{0}^{\sigma} dt_1 \int_{-\sigma}^{0} dt_2 v' \cdot \bar{v} D (t_1 v' - t_2 \bar{v} + b), \\
\end{align}

where the bar denotes four-velocities for the conjugate contour and the subscript cut is henceforth omitted.

To identify the leading behavior of \( U_{CC,S}^{(2)} \), with respect to \( b \), we shall consider first the situation corresponding to \( b = 0 \). The subsequent emergence of UV divergences, once handled through dimensional regularization, will introduce a mass scale \( \bar{\mu} \) that will be bounded from below by an IR cutoff \( \lambda \) and from above by the (mass) scale \( 1/b \). The resulting renormalization group equation will facilitate the resummation of the leading terms, just as in the virtual-gluon case.

Let us commence our quantitative considerations by looking at the term

\begin{align}
J_1(b) &\equiv v \cdot \bar{v} \int_{-\sigma}^{0} dt_1 \int_{-\sigma}^{0} dt_2 D (t_1 v - t_2 \bar{v} + b) \\
&\equiv J_1(0) = -\frac{1}{4\pi^2} \left( -\pi \bar{\mu}^2 L_1^2 \right)^{(2-D/2)} \Gamma \left( \frac{D}{2} - 1 \right) \frac{1}{D-3} \frac{1}{4-D} \left[ 1 - (2^{4-D} - 1) \right] \\
\end{align}

which actually coincides with what one would obtain if the regular propagator was substituted. The significance of this occurrence is that it leads to the same anomalous dimensions for the running of the real gluon contribution to the cross-section as for the virtual part. This fact can be immediately verified via a direct comparison with Eq. (16).

Isolating the finite part of the above expression, we write

\begin{align}
J_1^{(a),\text{fin}} &= -\frac{1}{8\pi^2} \ln \left( \frac{\bar{\mu}^2}{\lambda^2} \right). \\
\end{align}

It is trivial to see that the same result holds also for \( J_2^{(a),\text{fin}} \).

We next turn our attention to the term
\[ J_3(b) \equiv v \cdot \bar{v}' \int_{-\sigma}^{0} dt_1 \int_{0}^{\sigma} dt_2 \left( t_1 v - t_2 \bar{v}' + b \right). \]  

Its computation will concurrently allow us to determine \( J_4(b) \) which corresponds to the exchange prime\( \leftrightarrow \)no-prime in the expression above.

Dimensionally regularizing the cut propagator, we obtain

\[
J_3(0) = \frac{1}{4\pi^2} (-\pi \bar{\mu}^2)^{(4-D)/2} \Gamma \left( \frac{D}{2} - 1 \right) v \cdot \bar{v}' \int_{0}^{\sigma} dt_1 \int_{0}^{\sigma} dt_2 \left( t_1^2 + t_2^2 + 2v \cdot \bar{v}' - i0_+ \right)^{1-D/2}. 
\]

Once again we record, by referring to Eq. (23), coincidence of the singularities and, by extension, of associated anomalous dimensions between virtual and real gluon expressions that contribute to the cross section.

For the “uniformly soft” configuration the corresponding result is

\[
J_{3,DY}^{(a)}(0) = \frac{1}{4\pi^2} (-\pi \bar{\mu}^2)^{(4-D)/2} \Gamma \left( \frac{D}{2} - 1 \right) v \cdot \bar{v}' \int_{0}^{1} dt_1 \int_{0}^{1} dt_2 \left( t_1^2 + t_2^2 + 2t_1 t_2 \frac{v \cdot \bar{v}'}{|v||\bar{v}'|} - i0_+ \right)^{1-D/2}. 
\]

Taking into consideration that for the DY case \( \frac{v \cdot \bar{v}'}{|v||\bar{v}'|} = \frac{v' \cdot \bar{v}}{|v'||\bar{v}|} = \cosh \gamma_{DY} > 0 \) we obtain

\[
J_{3,DY}^{(a)}(0) = J_{4,DY}^{(a)}(0) = \frac{1}{4\pi^2} (-\pi \bar{\mu}^2)^{(4-D)/2} \Gamma \left( \frac{D}{2} - 1 \right) \cosh \gamma_{DY} \\
\times \int_{0}^{1} dt_1 \int_{0}^{1} dt_2 \left( t_1^2 + t_2^2 + 2t_1 t_2 \cosh \gamma_{DY} - i0_+ \right)^{1-D/2}, 
\]

whose finite part reads

\[
J_{3,DY}^{(a),\text{fin}}(0) = J_{4,DY}^{(a),\text{fin}}(0) = \frac{1}{8\pi^2} \gamma_{DY} \coth \gamma_{DY} \ln \left( \frac{\bar{\mu}^2}{\lambda^2} \right). 
\]

For the DIS case, where \( \frac{v \cdot \bar{v}'}{|v||\bar{v}'|} = \cosh(\gamma_{DIS} - i\pi) < 0 \), and \( \frac{v' \cdot \bar{v}}{|v'||\bar{v}|} = \cosh(\gamma_{DIS} + i\pi) < 0 \), we find

\[
J_{3,DIS}^{(a),\text{fin}}(0) = \frac{1}{8\pi^2} (\gamma_{DIS} - i\pi) \coth \gamma_{DIS} \ln \left( \frac{\bar{\mu}^2}{\lambda^2} \right), 
\]

whereas
\begin{align}
J_{4,\text{DIS}}^{(a),\text{fin}}(0) &= \frac{1}{8\pi^2} (\gamma_{\text{DIS}} + i\pi) \coth \gamma_{\text{DIS}} \ln \left( \frac{\bar{\mu}^2}{\lambda^2} \right), \quad (54)
\end{align}

Given that the imaginary parts in the last two equations will eventually drop out, from now on we shall write, generically,

\begin{align}
J_{3}^{(a),\text{fin}}(0) &= J_{4}^{(a),\text{fin}}(0) = \frac{1}{4\pi^2} \gamma \coth \gamma \ln \left( \frac{\bar{\mu}^2}{\lambda^2} \right), \quad (55)
\end{align}

where \(\gamma\) can be adjusted to the appropriate case.

Turning our attention to the “jet” configuration, we can go directly to \(J_{3}^{(b)}(0)\), since \(J_{1}^{(b)}(0) + J_{2}^{(b)}(0)\) furnishes half the contribution of its uniformly soft counterpart, the reason being the same as the one given in the virtual gluon case. We thus have

\begin{align}
J_{3}^{(b)}(0) &= -4 \frac{1}{4\pi^2} \left(-\pi \bar{\mu}^2\right)^{(4-D)/2} \Gamma \left( \frac{D}{2} - 1 \right) \frac{v \cdot \bar{v}'}{|v|} \int_0^1 dt_1 \int_0^1 dt_2 \left( t_1^2 + 2t_1t_2 \frac{v \cdot \bar{v}'}{|v|} - i0^+ \right)^{1-D/2} \quad (56)
\end{align}

with a corresponding expression holding also for \(J_{4}^{(b)}(0)\).

For the finite parts one obtains

\begin{align}
J_{3}^{(b),\text{fin}}(0) + J_{4}^{(b),\text{fin}}(0) &= \frac{1}{4\pi^2} \ln^2 \left( \frac{\bar{\mu}^2}{\lambda^2} \right), \quad (57)
\end{align}

which holds true for both the DY and the DIS case.

Collecting our findings from the real-gluon analysis to the second-order level, we write for the finite contribution to the cross-section

\begin{align}
U_{C\bar{C},S}^{(2)} &= 1 + \frac{\alpha_s}{\pi} C_F \left[ \gamma \coth \gamma \ln \left( \frac{\bar{\mu}^2}{\lambda^2} \right) - \frac{3}{2} \ln \left( \frac{\bar{\mu}^2}{\lambda^2} \right) + \ln^2 \left( \frac{\bar{\mu}^2}{\lambda^2} \right) \right] \\
&= U_{C\bar{C},\text{cusp}}^{(2)} U_{C\bar{C},\text{coll}}^{(2)} , \quad (58)
\end{align}

where (to the order we have been calculating) the results for the “cusp” and the “collinear” terms read

\begin{align}
U_{C\bar{C},\text{cusp}} &= 1 + \frac{\alpha_s}{\pi} C_F \gamma \coth \gamma \ln \left( \frac{\bar{\mu}^2}{\lambda^2} \right) \quad (59)
\end{align}

and
\[ U_{C\bar{C},\text{coll}}^{(2)} = 1 + \frac{\alpha_s}{\pi} C_F \left[ -\frac{3}{2} \ln \left( \frac{\bar{\mu}^2}{\lambda^2} \right) + \ln^2 \left( \frac{\bar{\mu}^2}{\lambda^2} \right) \right]. \] (60)

At the same time, the singularity structure of the full expression for the cross-section entails a multiplicative renormalization factor, which is common to all “Wilson loop” configurations entering its description, but which is the only one that pertains to the family of eikonal-type paths under consideration. The reasoning is, of course, identical to the one promoted for the virtual gluon case. Therefore, the corresponding contribution to the cross-section factorizes and the same resummation procedure can be employed as for the virtual-gluon case. As already observed, the anomalous dimension is in both cases the same. There are, however, the following notable differences. First, the upper limit for the momentum of real-gluon emission is \( 1/b^2 \) instead of \( Q^2 \). Second, there is a difference of sign, which becomes evident by comparing Eqs. (30) and (31) with Eqs. (59) and (60) and, finally, no hard real-gluon emission enters our considerations - by definition. In this light, it is practically self-evident that the resummed expression for real-gluon emission becomes

\[ U_{CC} = \exp \left\{ \int_{1/b^2}^{Q^2} \frac{dt}{t} \left[ \ln \frac{Q^2}{t} \Gamma_{\text{cusp}}(\alpha_s(t)) - \Gamma(\alpha_s(t)) \right] \right\} U_{C\bar{C},0}. \] (61)

We can now bring together real and virtual gluon results by referring to our generic expression for the cross-section as given by Eq. (8). We write

\[ \mathcal{W} = \left\langle Tr \left( U^\dagger(\bar{C}z')U(Cz) \right) \right\rangle = U_C U_{\bar{C}} U_C \bar{C}, \] (62)

where we have first extracted the resumed expression corresponding to the curly bracket expectation values from virtual gluon exchanges in the amplitude and its conjugate. Clearly, the last factor corresponds to the resummed expression resulting from the real-gluon “averaging”.

Now, at the cross-section level, our threshold resummation of the virtual gluons reads

\[ U_C U_{\bar{C}} = \exp \left\{ -\int_{\lambda^2}^{Q^2} \frac{dt}{t} \left[ \ln \frac{Q^2}{t} \Gamma_{\text{cusp}}(\alpha_s(t)) - \Gamma(\alpha_s(t)) \right] \right\} U_{C,0} U_{\bar{C},0}. \] (63)

Finally, by making the re-adjustment \( \ln (\bar{\mu}^2/\lambda^2) \rightarrow \ln (\mu^2/\bar{\lambda}^2) \) we obtain our final result, which reads
\[ \mathcal{W}_{CC} = \exp \left\{ - \int_{c/b}^{Q^2} \frac{dt}{t} \left[ \ln \frac{Q^2}{t} \Gamma_{\text{cusp}}(\alpha_s(t)) - \Gamma(\alpha_s(t)) \right] \right\} \mathcal{W}_0, \tag{64} \]

with \( \Gamma_{\text{cusp}} \) and \( \Gamma \) given by Eqs. (34) and (42), respectively and where \( c = 4e^{-2\gamma_E} \).

An expression whose previous derivation, see, e.g., Ref. [21] and references therein, has employed Wilson lines, as an exogenous element attached to quark operators.

VI. CONCLUDING REMARKS

In this paper we have studied the threshold resummation behavior of DY and DIS type of processes, widely pursued by Sterman and others (see, for example, in [1,3]), by staying strictly within the framework of QCD. From a physical viewpoint, we based our considerations on soft-gluon radiation, restricting ourselves to an energy regime whose lower cutoff is high enough to justify an analysis in which reference to “gluons”, as dynamical degrees of freedom, continues to make sense. In this context we have implicitly assumed the pre-confinement property, originally articulated in the first work of reference [4] (see also [22]), according to which the non-perturbative dynamics responsible for confinement screens color up to the infrared scale \( \lambda \) which sets the lower limit for the perturbative regime.

Our analysis has been conducted via the adoption of the worldline casting of QCD whose emphasis on a space-time propagation content facilitates the isolation and eventually factorization of IR contributions to cross sections through eikonal-type paths. In this connection, let us remark that we have already extensively applied this approach to the calculation of various fermionic Green’s functions [23] (see, also [24]), the Sudakov form factor [25], as well as to derive gluon Reggeization in connection with the four-point fermionic Green’s function - for the latter case in the forward regime [26]. The present paper presents the first application of the worldline methodology to “cross-sections” in which real-gluon radiation effects must also be taken into account - and so far have received much less systematic treatment.

The common renormalization-group running of virtual and real - uniformly soft and jet-type - gluon contributions to the cross-section has led to a final result in which the IR-cutoff
momentum is provided by the inverse impact parameter which can be vested with concrete physical meaning related to the process itself. A number of possible physical applications, ranging from extensions to exclusive processes to intrusions into the non-perturbative regime of QCD, as per, for example, along the lines of Ref. [21] come to mind which will occupy our attention in the immediate future.
APPENDIX A:

Our task is to establish Eqs. (20) and (21) in the text. Performing the integration entering the right hand side of Eq. (19), one obtains

\[ I^{(a)}_{3,DY} = \frac{1}{4\pi^2} (-\pi \mu^2)^{(4-D)/2} \Gamma \left( \frac{D}{2} - 1 \right) \frac{1}{4-D} \frac{1}{D-3} 2w_{DY} \]
\[ \times \left\{ w_{DY} F \left( 1, \frac{D}{2} - 1; \frac{D-1}{2}; 1 - w_{DY}^2 \right) \right. \]
\[ + \frac{1}{2} [2(1 - w_{DY})]^{2-D/2} F \left( 1, \frac{D}{2} - 1; \frac{D-1}{2}; \frac{1 + w_{DY}}{2} \right) \} \] \hspace{1cm} (A1)

for DY kinematics and

\[ I^{(a)}_{3,DIS} = \frac{1}{4\pi^2} (-\pi \mu^2)^{(4-D)/2} \Gamma \left( \frac{D}{2} - 1 \right) \frac{1}{4-D} \frac{1}{D-3} 2w_{DIS} \]
\[ \times \left\{ w_{DIS} F \left( 1, \frac{D}{2} - 1; \frac{D-1}{2}; 1 - w_{DIS}^2 \right) \right. \]
\[ - \frac{1}{2} [2(1 + w_{DIS})]^{2-D/2} F \left( 1, \frac{D}{2} - 1; \frac{D-1}{2}; \frac{1 - w_{DIS}}{2} \right) \} \] \hspace{1cm} (A2)

for the DIS situation.

Setting \( D = 4 \), we obtain

\[ F \left( 1, 1; 3/2; 1 - w_{DY}^2 \right) = \frac{\gamma_{DY}}{\sinh \gamma_{DY} \cosh \gamma_{DY}} \] \hspace{1cm} (A3)

and

\[ F \left( 1, 1; 3/2; \frac{1 + w_{DY}}{2} \right) = \frac{\gamma_{DY}}{\sinh \gamma_{DY}} - i \frac{\pi}{\sinh \gamma_{DY}}. \] \hspace{1cm} (A4)

As the imaginary part in the above expression will cancel against its counterpart in the conjugate expression, it can be dropped as far as the cross-section is concerned.

Denoting the expression inside the curly brackets on the rhs of Eq. (A1) by \( f_{D}^{DY}(w) \) and setting

\[ f_{D}^{DY}(w) = f_{4}^{DY}(w) + (4 - D) \frac{f_{D}^{DY}(w) - f_{4}^{DY}(w)}{4 - D}, \] \hspace{1cm} (A5)

we realize that the second term on the rhs will lead to finite terms that depend solely on \( w \) and which will cancel from similar contributions of the same sort coming from the other
terms entering Eq. (13). Putting everything together, one finally arrives at Eq. (20). Clearly, the DIS situation is similarly confronted.

To establish the result given by Eq. (24), we first note that Eq. (23) gives

$$I_3^{(b)} = \frac{1}{4\pi^2} \left(-\pi \mu^2 \right)^{(4-D)/2} \Gamma \left(\frac{D}{2} - 1\right) \frac{1}{(4-D)^2} \left(\frac{2v \cdot v'}{|v|^2}\right)^{(4-D)/2}$$

$$\times \left[ F \left(\frac{D}{2} - 1, 2 - \frac{D}{2} ; 3 - \frac{D}{2} ; -\frac{2v \cdot v'}{|v|^2}\right) + \left(1 + \frac{2v \cdot v'}{|v|^2}\right)^{2-D/2} ight] \left(\frac{2v \cdot v'}{|v|^2}\right)^{2-D/2} \right]. \tag{A6}$$

Now, the distinction between the DY and DIS like processes resides in the sign of $v \cdot v'$ (positive and negative, respectively). On the other hand, what is relevant for the “jet” situation is that the absolute value of $2v \cdot v' \frac{1}{|v|^2}$ is of the order of unity. One, then, easily determines that in the limit $D \to 4$ Eq. (24) is retrieved.
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