Boundary term contribution to the volume of a small causal diamond

Surbhi Khetrapal\textsuperscript{1} and Sumati Surya\textsuperscript{2}

\textsuperscript{1} BITS Pilani, K K Birla Goa Campus, Goa, India
\textsuperscript{2} Raman Research Institute, Bangalore, Karnataka, India

E-mail: ssurya@rri.res.in

Received 4 December 2012, in final form 23 January 2013
Published 21 February 2013
Online at stacks.iop.org/CQG/30/065005

Abstract
In his calculation of the spacetime volume of a small Alexandrov interval in four dimensions, Myrheim introduced a term which he referred to as a surface integral (Myrheim 1978 CERN preprint TH-2538). The evaluation of this term has remained opaque and led subsequent authors to obtain a formula for the volume using other techniques (Gibbons and Solodukhin 2007 Phys. Lett. B \textbf{649} 317). It is the purpose of this work to explicitly calculate this integral and in the process complete the proof for the volume formula in arbitrary dimensions. We point out that it arises from the difference in the flat spacetime volumes of the curved and flat spacetime intervals. We use first-order degenerate perturbation theory to evaluate this difference and find that it adds a dimension-independent factor to the volume of the flat spacetime interval as the lowest order correction. Our analysis admits a simple extension to a more general class of integrals over the same domain. Using a combination of techniques we also find that the next-order correction to the volume vanishes.

PACS number: 04.20.Gz

1. Introduction

Spacetime volume plays a crucial role in the discrete-continuum correspondence of causal set quantum gravity [3]. In this approach, the fundamental entity underlying the continuum is a locally finite partially ordered set or causal set. The order relation of the causal set corresponds to the causal structure of the spacetime, while the condition of local finiteness means that underlying every $N$ Planck volumes of a spacetime region there are, on average, $N$ elements of the causal set. Calculations of the volumes of generic but causally well-defined spacetime regions are therefore of particular interest to causal set theory.

In an insightful CERN preprint on discrete statistical geometry, Myrheim [1] introduced several key concepts that were subsequently adopted by the causal set approach [3]. An important calculation in this work is of the volume $V$ of a small Alexandrov interval $I[p, q]$ between two chronologically related events $p$ and $q$ (see figure 1). The metric is expanded about the mid-point $r$ of the flat spacetime geodesic $\gamma_0$ from $p$ to $q$ using Riemann normal
coordinates (RNCs), with the smallness parameter being the proper time $T$ from $p$ to $q$. In four spacetime dimensions Myrheim obtained an expression for $V$ to order $T^2$ which depends on the scalar curvature as well as the curvature components along $y_0$. While the calculation is straightforward to set up, the evaluation of one of the terms, referred to as a ‘surface integral’ in [1], has remained opaque. This led Gibbons and Solodukhin to an evaluation of $V$ which side steps this integral by instead calculating the universal coefficients that appear in an order $T^2$ expansion of the volume for the Einstein static universe and de Sitter spacetime [2].

It is the main purpose of this work to evaluate Myrheim’s ‘surface’ integral explicitly and thus complete the proof for the volume formula to order $T^2$. Rather than a surface term, this integral is the difference in the flat spacetime volumes of the regions defined by the curved spacetime interval $I[p, q]$ and the flat spacetime interval $I_0[p, q]$. As we will show in section 2, the boundary of $I[p, q]$ can be determined to order $T^2$ from the modified light cones in the tangent spaces $T_pM$ and $T_qM$ of $p$ and $q$, respectively; the effect of the coordinate acceleration of the null geodesics emanating from $p$ and $q$ can be shown to be sub-leading. The boundary of the modified light cones in $T_pM$ are calculated using first-order degenerate perturbation theory. In the RNC, this curvature-dependent perturbation is restricted to the spatial directions and can transform the spherical spatial cross sections of the light cone to ellipsoidal ones. Making a coordinate transformation to the principal directions of the resulting ellipsoid we find the characteristic equation for the perturbations in the eigenvalues. In particular, we find that the perturbations in the eigenvalues sum to

$$-\frac{1}{12} R_{00},$$

(1)

independent of dimension. In section 3, we show that this sum appears crucially in the Jacobian of transformation relating the flat spacetime integral to Myrheim’s term. This implies that Myrheim’s integral evaluates to

$$V_0 \left(\frac{T^2}{24} R_{00}(0)\right).$$

(2)

in all dimensions. This allows us to complete the RNC calculation of the volume $V$ of $I[p, q]$ up to order $T^2$ for arbitrary $n$. Our expression agrees with that obtained by Myrheim [1] for $n = 4$ as well as that obtained by Gibbons and Solodukhin [2] for general $n$ using a different technique. An important by-product of our analysis is that it allows us to calculate more generic
integrals which crop up in causal set theory. We give an example of one such integral which has recently been used to obtain a new expression for the scalar curvature of a small causal set [4] using a curved spacetime generalization of the work of [5].

What about higher order corrections to the volume? In [6], this question was partially addressed in the discussion on the volumes of large causal diamonds and their asymptotic behaviour. As evident from the RNC expansion, higher order corrections to the volume will include higher derivatives of the curvature. In section 4, we will examine these terms using the next-order perturbation analysis, the acceleration of the null geodesics is non-negligible. Hence it does not suffice to look at $T_pM$ and $T_qM$, although it does help determine the types of higher derivative terms appearing to this order. Using a combination of the analysis of light cones in $T_pM$, $T_qM$ and the approach of Gibbons and Solodukhin [2], we evaluate the volume to $O(T^3)$ in FRW spacetimes. We find that the $O(T^3)$ correction to the volume vanishes altogether.

Because of the appearance of higher derivatives of the curvature, it is tempting to ask if the volume expansion can have any significance in determining effective actions in quantum gravity. This idea is not so far-fetched in causal set theory where spacetime volume plays a fundamental role, making it possible to speculate that higher derivative corrections to the action must be determined by these corrections to the volume. This could distinguish the causal set approach from other approaches to quantum gravity. While our primary focus in this note is to give a proof for the volume formula and to extend the analysis to the calculation of a second derivatives of the curvature, modulo boundary terms. However, a systematic approach to prove such a conjecture is currently beyond our scope.

2. The boundary of $I[p, q]$

The RNC about a point $r \in M$ is defined within a convex normal neighbourhood $Q$ of $r$ in the spacetime $(M, g)$, i.e. a region $Q \subset M$ in which the exponential map $\exp : T_pM \to Q$ is a diffeomorphism for any $p \in Q$. In the RNC, the geodesics emanating from $r$ are used to coordinatize $Q$, and the spacetime metric at $r$ is given by $g_{ab}(r) = \eta_{ab}$. In the RNC, the Christoffel connection $\Gamma^c_{ab}(r) = 0$, so that the metric at any $x \in Q$ can be expanded as

$$g_{ab}(x) = \eta_{ab}(0) - \frac{1}{2} x^c R_{a\beta c\delta}(0) + O(x^3),$$

where we have used the RNC identity $\delta_{\beta \gamma} T^d_{a\gamma j}(0) = 0$. Here, and in the future, we will often use the short form $0$ to denote the origin $r = (0, 0, \ldots, 0)$ of the RNC.

The volume of a small Alexandrov interval $I[p, q]$ in $n$ spacetime dimensions between the points $p = (-T/2, 0, 0, \ldots, 0)$ and $q = (T/2, 0, 0, \ldots, 0)$ in the RNC is therefore given by the integral

$$V = \int_{I[p, q]} \sqrt{-g} d^n x = \int_{I[p, q]} \left( 1 - \frac{1}{6} x^c R_{a\beta c\delta}(0) + O(x^3) \right) d^n x,$$

which was first calculated for $n = 4$ by Myrheim [1]. Importantly, the integration is over a region $I[p, q]$ which itself is determined by $g_{ab}(x)$ and hence contains corrections to the flat spacetime interval $I_0[p, q]$. Up to $O(x^3)$, this integral can be split into two parts $V = I_M + I_2$, where

$$I_M = \int_{I[p, q]} d^n x, \quad I_2 = \int_{I[p, q]} \left( -\frac{1}{6} x^c R_{a\beta c\delta}(0) \right) d^n x.$$
As we will show in the following section, the second term is straightforward to evaluate in arbitrary dimensions, much of the simplification arising from the fact that odd terms do not contribute to \( I_0[p, q] \) because of its symmetries.

The integral \( I_M \) itself comprises two pieces: the volume \( V_0 \) of \( I_0[p, q] \) plus a contribution \( I_{\Delta I[p, q]} \) which Myrheim referred to as a ‘boundary’ term. This term was evaluated for \( n = 4 \) in [1] without any explanation, although how to evaluate it has been far from obvious to subsequent researchers [2]. Indeed, it is this precise integral that we wish to decode in this work. We first note that this term is not a boundary integral but simply the difference in the flat spacetime volumes of the interval \( I[p, q] \) with respect to \( \eta_{ab} \) and \( g_{ab} \) as realized also in [2].

Thus, in order to determine \( I_M \), we need to first obtain the boundary of \( I[p, q] \) since the difference in the volumes \( I_{\Delta I[p, q]} \) roughly arises from the difference in the range of integration. In flat spacetime the boundary of \( I_0[p, q] \) is that of a pair of uniform (light)cones with base radius \( T/2 \), one emanating to the past from \( q \) and the other to the future from \( p \) so that we can evaluate

\[
V_0 = \int_{I_0[p, q]} d^nx = 2 \int_0^{T/2} dr \int_0^{T/2-t} r^{n-2} dr \int_{S^{n-2}} d\Omega_{n-2} = \frac{2A_{n-2}}{n(n-1)} \left( \frac{T}{2} \right)^n,
\]

where \( A_{n-2} \) is the volume of a uniform \( n-2 \) sphere \( S^{n-2} \).

The presence of curvature obviously modifies the boundaries of the two light cones, and it is this that we now attempt to quantify. The boundary of \( I[p, q] \) is ruled by future-directed and past-directed null geodesics emanating from \( p \) and \( q \), respectively. The tangents to these null geodesics in turn lie along the future and past light cones in \( T_pM \) and \( T_qM \), respectively. The effect of curvature in general would be to accelerate these null geodesics, so that they no longer lie along \( T_pM \) once they leave \( p \), and similarly for \( q \). Thus, one would no longer expect as simple an integration as in equation (6).

Nevertheless, the first step to take is to determine the future and past light cones in \( T_pM \) and \( T_qM \), respectively. Because of the symmetry of \( I_0[p, q] \) it suffices to restrict our attention to \( T_qM \). Using the RNC expansion of the metric at \( q \),

\[
g_{ab}(q) = \eta_{ab}(0) - \frac{1}{12} T^2 R_{0a0b}(0),
\]

the tangents \( \xi^a = \frac{dx^a}{dr} = (\xi^0, \xi^1, \ldots, \xi^{n-1}) \) to the past and future directed null geodesics at \( q \) satisfy

\[
g_{ab}(q)\xi^a \xi^b = 0 \quad \Rightarrow - (\xi^0)^2 + \sum_{i=1}^{n-1} (\xi^i)^2 = \frac{1}{12} T^2 R_{0i0j}(0) \xi^i \xi^j,
\]

where we have used the symmetries of the Riemann tensor to simplify the expression. Thus the light cone in the tangent space \( T_qM \) is given by the matrix equation

\[
\tilde{\xi}^T M \tilde{\xi} = (\xi^0)^2,
\]

where \( \tilde{\xi} \) is the spatial part of \( \xi^a \) and

\[
M_{ij} = \delta_{ij} - T^2 \frac{1}{12} R_{0i0j}.
\]

When there is no curvature, equation (9) reduces to \( (\tilde{\xi})^2 = (\xi^0)^2 \) which is the equation for a uniform (light)cone, i.e. a cone with \( (n-2) \)-dimensional spherical cross-sections with radii \( \xi^0 \).

Since \( M \) is a real symmetric matrix, it can be diagonalized by an orthogonal matrix \( Q \), so that inserting \( Q^T Q = 1 \) in equation (9) yields

\[
\xi^T \Lambda \xi = (\xi^0)^2 \quad \Rightarrow \sum_j \lambda_j (\xi^j)^2 = (\xi^0)^2,
\]

where \( \lambda_j \) are the eigenvalues of \( M \). This is the result of Myrheim and collaborators [2] that we wish to verify in this work.
and the properties of the curvature tensor, we see that to this order. In other words, the spatial components of any vector have an order cone in the RNC. In other words, to lowest order, the (coordinate) acceleration of geodesics is zero, as expected. Expanding along the affine parameter \( s \),

\[
\xi^a_s = \xi^a_{0} - \frac{T}{2} \partial_0 \Gamma^a_{bc}(0) \xi^b \xi^c + O(s^2),
\]

we see that for small \( s \), the second term is sub-leading to order \( O(T^2) \) but not to order \( O(T^3) \). Thus, if we find that the first correction to \( \xi^a \) is of order \( O(T^2) \), then the acceleration is sub-leading, so that the boundary of \( \Gamma[p, q] \) is determined by the light cones in \( T_{p,q}M \) alone, to order \( T^2 \).

We find the leading-order correction to \( \xi^a \) by parallel transporting it from \( r = (0, 0, \ldots, 0) \) to \( q \) along a geodesic. Indeed, as is easily shown, up to order \( T^2 \) the curve \( \gamma = (t, 0, \ldots, 0) \) from \( r \) to \( q \) is a geodesic. The tangent to \( \gamma \) is \( U^a = (1, 0, \ldots, 0) \), so that \( \partial_0 U^b = 0 \) all along \( \gamma \). For \( \gamma \) to be a geodesic therefore

\[
U^a \nabla_a U^b(t) = \partial_t U^b(t) + \Gamma^b_{00}(t) = i \partial_0 \Gamma^b_{00}(0) + O(t^2)
\]

must vanish to the required order.

Using the RNC identity

\[
\partial_0 \Gamma^a_{bc}(0) = -\frac{1}{2} (R_{abc} - R_{am} + R_{mb})
\]

and the properties of the Riemann tensor, we see that \( U^a \nabla_a U^b(t) = O(t^2) \), so that \( \gamma \) is a geodesic up to \( O(T^2) \).

Now consider the parallel transport of a vector \( \omega^a \) from \( r \) to \( q \) along \( \gamma \), \( U^b \nabla_b \omega^a = 0 \):

\[
\partial_t \omega^a + i \partial_0 \Gamma^a_{00}(0) \omega^b = 0 \Rightarrow \partial_t \omega^a = -\frac{t}{3} R_{000} \omega^a = 0,
\]

where we have used identity (15). The symmetries of the Riemann tensor imply that the time-component of any vector does not change along \( \gamma \), i.e. \( \partial_0 \xi^0 = 0 \) up to this order. Since \( R_{000} \) is symmetric and real (and hence Hermitian), let us use its eigenfunctions \( \{ e^{(k)} \} \) of \( R_{000} \) to give us a (spatial) orthonormal basis vectors. Thus the parallel transport equation reduces to

\[
\partial_t e^{(k)} = -\frac{t}{3} R_{000} e^{(k)} = 0
\]

\[
\Rightarrow e^{(k)}(t) = e^{(k)}(0) \exp \frac{t}{3} R_{000}
\]

\[
\Rightarrow e^{(k)}(T/2) \approx \left( \delta^k_j + \frac{T^2}{24} R_{000}(0) \right) e^{(k)}(0)
\]

to this order. In other words, the spatial components of any vector have an order \( O(T^2) \) correction from flat spacetime. Thus, we have shown that the acceleration of the null geodesics at \( q \) is negligible to this order and can therefore be ignored.

We thus find that to \( O(T^2) \) the modified past light cone from \( q \) is identical to that in \( T_{p}M \) and hence can have ellipsoidal instead of spherical sections at constant time. In order to evaluate the boundary of \( \Gamma[p, q] \), it therefore suffices to find the relevant properties of this light cone in \( T_{p}M \), in particular to find the eigenvalues \( \lambda_i \) of \( M \).
The $\lambda_i$ can be determined to order $T^2$ using first-order perturbation theory in $T^2$, or equivalently, second-order perturbation in $T$ where the first-order perturbation is zero. We prefer to use the latter terminology since it will make it easier to adapt to the $O(T^3)$ perturbations. Rewriting

$$M = M^{(0)} + T^2 M^{(2)},$$

where $M^{(0)} = I$ and $M^{(2)}_{ij} = -\frac{1}{12} R_{00ij}$, we note that the zeroth-order eigenfunctions are simply the unit vectors $\psi^{(0)}_i$ in the principal directions, with eigenvalues $\lambda^{(0)}_{ij} = 1$. The first correction $\lambda^{(2)}_{ij}$ to the $\lambda^{(0)}_{ij}$ are then the eigenvalues of the operator $\psi^{(0)}_i T^2 M^{(2)} \psi^{(0)}_j$, which in this case is simply $M^{(2)}$ itself.

The characteristic equation $||M^{(2)} - \lambda I|| = 0$ gives rise to an $(n-1)$th-order polynomial

$$\lambda^{n-1} - \lambda^{n-2} \sum M^{(2)}_{ii} + O(\lambda^{n-3}) = 0.$$  

As we will show in the following section, the volume calculation does not require us to explicitly solve this eigenvalue problem in order to obtain an expression for the volume up to order $T^2$. Indeed, all that is required is the sum $\sum \lambda^{(2)}_{ij}$ of its roots $\lambda^{(2)}_{ij}$. Rewriting equation (19),

$$\prod_{i} (\lambda - \lambda^{(2)}_{ij}) = 0,$$

we see that the coefficient of the $(n-2)$th-order term is

$$- \sum \lambda^{(2)}_{ij} = - \sum M^{(2)}_{ii} = \sum \frac{1}{12} R_{00ii} = \frac{1}{12} R_{00}.$$  

It is nevertheless instructive to see the effect of curvature on the light cones in a simple example. For $n = 3$,

$$\lambda^{(2)}_{\pm} = \frac{1}{2} (M^{(2)}_{11} + M^{(2)}_{22}) \pm \sqrt{(M^{(2)}_{11} - M^{(2)}_{22})^2 + 4(M^{(2)}_{12})^2}.$$  

Thus, the light cone in $T_{\theta\varphi}M$ is transformed from a symmetric cone to one with elliptical sections at constant $\theta^0$ with the lengths of the semi-major and semi-minor axes being $\theta^0/\sqrt{\lambda_+}$ and $\theta^0/\sqrt{\lambda_-}$, respectively. The principal directions are degenerate, i.e. $\lambda^{(2)}_{\pm} = \lambda^{(2)}_{\mp}$ iff $M^{(2)}_{11} = M^{(2)}_{22}$ and $M^{(2)}_{12} = 0$, i.e. the curvature is isotropic. In this case $\lambda_{\pm} = 1 - \frac{1}{4} T^2 R_{00}$, which means that the light cone, though still symmetric, is narrower than in flat spacetime if $R_{00} > 0$ and wider otherwise. We can expect a straightforward generalization of this behaviour in higher dimensions—the light cone could be wider in some spatial directions and narrower in others depending on the curvature. These various possibilities are already evident in $n = 3$. For example, if $\sum M^{(2)}_{ii} = -\frac{1}{12} R_{00}$ is equal to $\Delta = (M^{(2)}_{11} - M^{(2)}_{22})^2 + 4(M^{(2)}_{12})^2)^{1/2}$ (i.e. $M_{12} = \sqrt{M^{(2)}_{11} M^{(2)}_{22}}$), $\lambda_+ < 1$ for $R_{00} > 0$ and $\lambda_+ > 1$ for $R_{00} < 0$, while $\lambda_- = 1$ in both cases. If $R_{00} > 0$ and $\Delta > \frac{1}{4} T^2 R_{00}$, then $\lambda_+ > 1$ and $\lambda_- < 1$, etc. Figure 2 shows one of these possibilities.

3. The volume calculation

We are now in a position to calculate the volume of $I[p,q]$ in arbitrary dimensions. But first let us compare this approach with the one in [2]. The starting point in [2] was to assume a form for the volume

$$V = V_0 (1 + \alpha(n) R(0) T^2 + \beta(n) R_{00}(0) T^2 + O(T^3)),$$

based on Myrheim’s calculation, thus bypassing the explicit evaluation of equation (4). The universal constant $\alpha(n)$ was evaluated by calculating the volume of a causal diamond explicitly
for the Einstein static universe for which $R_{00}$ is identically zero, and expanding in powers of $T$. Using this, the universal constant $\beta(n)$ was calculated from the volume of a causal diamond de Sitter spacetime ($R, R_{00} \neq 0$), again by expanding in powers of $T$. Thus, the authors of [2] obtained a general formula for the volume of a small causal diamond in $n$ spacetime dimensions to order $T^2$, and this will serve as a check for our calculations. Moreover, this technique of [2] will prove to be useful in our exploration of the next higher order corrections to the volume in section 4.

As shown in the previous section, the contribution from $I_M$ to the volume to $O(T^2)$ comes from a region bounded by light cones whose tangent vectors at $q$ satisfy equation (11). The boundary of the backward light cone from $q$ is therefore given by

$$\left(\frac{T}{2} - t\right)^2 = \sum \lambda_i y_i^2,$$

where we have chosen the spatial coordinates $y_i$ along the principal axes of the spatial ellipsoids. We may therefore construct the nested integral

$$I_M = 2 \int_0^{T/2} dz_0 \int_{-z_0}^{z_0} dy_1 \int_{-w_1}^{w_1} dy_2 \ldots \int_{-w_{n-2}}^{w_{n-2}} dy_{n-1},$$

where $z_0 = T/2 - t, \sqrt{\lambda_{k+1} w_k} = \sqrt{\lambda_{k+1} w_k} = \sqrt{\lambda_{k+1} w_k}$. The coordinate transformation $z_i = \sqrt{\lambda_i} y_i$ simplifies the integral to that of a regular(uniform) cone along with a factor corresponding to the Jacobian of transformation

$$J = \frac{1}{\sqrt{\lambda_1 \lambda_2 \ldots \lambda_{(n-1)}}} = 1 - \frac{T^2}{2} \sum_{i=0}^{n-1} \lambda_{(i)}^{(2)} + O(T^3),$$

since $\lambda_i = 1 + T^2 \lambda_{(i)}^{(2)} + O(T^3)$. Thus,

$$I_M = V_0 \left(1 - \frac{T^2}{2} \sum_{i=0}^{n-1} \lambda_{(i)}^{(2)}\right) = V_0 \left(1 + \frac{T^2}{24} R_{00}(0)\right)$$

(27)
where we have used equation (21). Thus, we have shown that in any dimension the boundary term of Myrheim [1] takes the form

\[ I_{\Delta[p,q]} = V_0 \left( \frac{T^2}{24} R_{00}(0) \right). \]  

(28)

We complete this section by including the computation of the \( I_2 \) integral in arbitrary dimensions. Let us rewrite

\[ I_2 = \int_{B[p,q]} \left( -\frac{1}{6} x' x' R_{\alpha\beta}(0) \right) d^p x = -\frac{1}{6} (J_1 + J_2 + J_3), \]  

(29)

where

\[ J_1 = \int_{B[p,q]} d^p x' t^2 R_{00}(0), \quad J_2 = 2 \sum_{i=1}^{n-1} \int_{B[p,q]} d^p x' t x^i R_{0i}(0), \]

\[ J_3 = \sum_{i,j=1}^{n-1} \int_{B[p,q]} d^p x' x^i R_{ij}(0). \]  

(30)

Evaluating

\[ J_1 = 2R_{00}(0) \int_0^\pi \int_0^2 \int_0^{\pi-t} R_{00}(0) \int_0^\pi \int_0^{\pi-t} \int_0^{\pi-t} d\Omega_{n-2} \]

\[ = \frac{2A_{n-2}}{n-1} R_{00}(0) \int_0^\pi \int_0^{\pi-t} R_{00}(0) \left( \frac{T}{2} - t \right)^{n-1} \]

\[ = \frac{4A_{n-2}}{n(n-1)(n+1)(n+2)} R_{00}(0) \left( \frac{T}{2} \right)^{n+2}. \]  

(31)

To evaluate \( J_2, J_3 \), it is convenient to transform from the Cartesian coordinates \( x^i \) to spherical polar coordinates \( x' = r f'(\Omega) \), where

\[ f'(\theta_1, \theta_2, \ldots, \theta_{n-2}) = \begin{cases} \prod_{k=1}^{n-i-1} \sin \theta_k \cos \theta_{n-i} & (i > 1) \\ \prod_{k=1}^{n-2} \sin \theta_k & (i = 1). \end{cases} \]

Since the angular contribution to \( J_2 \) is odd, it must therefore vanish. Similarly, the only contribution to \( J_3 \) comes from the even terms. The contribution from each \( x^{n-1} = r \cos \theta_1 \)

\[ J_3 = \sum_{i=1}^{n-1} \int_{B[p,q]} d^p x' (x')^2 R_{ii}(0) \]

\[ = 2 \left( \sum_{i=1}^{n-1} R_{ii}(0) \right) \int_0^\pi \int_0^{\pi-t} \int_0^\pi \int_0^{\pi-t} d\Omega_{n-2} \cos^2 \theta_1, \]  

(32)

where

\[ \int_{\Omega_{n-2}} d\Omega_{n-2} \cos^2 \theta_1 = \int d\theta_1 \cdots d\theta_{n-2} \left( \prod_{i=1}^{n-3} \sin^{n-3} \theta_i \right) \cos^2 \theta_1 \]

\[ = \int_0^\pi d\theta_1 \sin^{n-3} \theta_1 \cos^2 \theta_1 \int_0^\pi d\theta_2 \sin^{n-4} \theta_2 \cdots \int_0^\pi d\theta_{n-2} \]

\[ = \frac{A_{n-2}}{n-1}. \]  

(33)
so that
\[
J_3 = \frac{2A_{n-2}}{(n-1)(n+1)(n+2)} \left( \sum_{j=1}^{n-1} R_{ij} \right) \left( \frac{T}{2} \right)^{n+2}.
\] (34)

Thus
\[
I_2 = -\frac{1}{3} \frac{A_{n-2}}{(n-1)(n+1)(n+2)} \left( \frac{T}{2} \right)^{n+2} \left( \left( \frac{2}{n} + 1 \right) R_{00}(0) + R(0) \right)
\]
\[
= -\frac{1}{24(n+1)(n+2)} T^2 \left( \frac{2}{n} + 1 \right) R_{00}(0) + R(0) \right). (35)
\]

Adding this to \( I_{MMII} \) we obtain
\[
V = V_0 \left( 1 + \frac{n}{24(n+1)} T^2 R_{00}(0) - \frac{n}{24(n+1)(n+2)} T^2 R(0) \right)
\] (36)

which matches Myrheim’s expression for \( n = 4 \) [1] as well as the expression in [2] for arbitrary \( n \).

As discussed in the introduction, an important reason to be able to calculate the integral \( I_M \) explicitly is so that we can extend the analysis to other integrals that make their appearance in causal set theory. In [4], for example, the following integral must be evaluated for integer \( m \geq 0 \):
\[
K = \int_{I_{[p,q]}} d^n y \ \tau^m,
\] (37)

where \( \tau \) is the proper time from the event that one is integrating over and the future most point \( q \) of \( I_{[p,q]} \). Since the boundary of the light cones are given by equation (11), we see that \( \tau^2 = (T/2 - t)^2 - \sum_i \lambda_i \gamma_i^2 \). As in the evaluation of \( I_{MM} \) the limits of the integrals are again given as in equation (25). Again, performing a change of coordinates \( z_i = \sqrt{x_i \gamma_i} \) simplifies the integral to
\[
K = \frac{1}{\sqrt{(1 - \lambda_i \gamma_i)}} \int_{I_{[p,q]}} d^n z \ \tau^m = \left( 1 + \frac{T^2}{24} R_{00}(0) \right) \int_{I_{[p,q]}} d^n z \ \tau^m. (38)
\]

The universality of this term proves to be crucial in determining a recurrence formula for the average numbers of ‘\( k \)-chains’ in a causal set \( C \) which is approximated by \( I_{[p,q]} \). This in turn allows us to find an expression for the scalar curvature as well as the dimension in purely order theoretic terms [4].

4. On the order \( T^3 \) corrections to the volume

In this section, we show that the next higher order correction \( O(T^3) \) to the volume is in fact zero.

To order \( T^3 \), the volume is given by
\[
V = \int_{I_{[p,q]}} \sqrt{-g} \ d^n x = \int_{I_{[p,q]}} \left( 1 - \frac{1}{6} \gamma^i \gamma^j R_{i,j}(0) - \frac{1}{12} \gamma^i \gamma^j \gamma^k \gamma^l \partial_{i,j;k,l}(0) + O(\gamma^4) \right) d^n x,
\] (39)

where we have used the \( O(\gamma^4) \) expansion in the RNC
\[
g_{ab}(x) = g_{ab}(0) - \frac{1}{2} \gamma^i \gamma^j R_{i,j}(0) - \frac{1}{6} \gamma^i \gamma^j \gamma^k \gamma^l \partial_{i,j;k,l}(0) + O(\gamma^4)
\] (40)

which arises from the identity \( \partial_{[a} \partial_{b]} \Gamma^{e}_{cd} (0) = 0 \) or equivalently
\[
\partial_{a} \partial_{b} \Gamma^{e}_{cd} (0) = -\frac{1}{3} \left( 2 \partial_{a} R^{e}_{bd} (0) + \partial_{b} R^{e}_{cd} (0) + \partial_{d} R^{e}_{ab} (0) \right). (41)
\]
Again, we may split up the above integral into a piece that comes entirely from the flat spacetime interval $I_0[p, q]$ and a ‘remainder’ coming from the difference in the two integrals

$$V = I_1 + I_2 + I_3$$

$$I_1 = \int_{I[p, q]} \text{d}^3x$$

$$I_2 = -\int_{I[p, q]} \frac{1}{6} \zeta^2 x^d R_{cd}(0) \text{d}^3x,$$

$$I_3 = \int_{I[p, q]} \left( -\frac{1}{12} \zeta x^d \partial_c R_{cd} + O(x^4) \right) \text{d}^3x. \quad (42)$$

The symmetry of $I_0[p, q]$ means that the leading-order contribution to $I_1$, being odd, vanishes. In addition, since the leading-order correction to $I[p, q]$ is of order $T^3$ and the integrand of $I_2$ is $O(x^2)$, this term must be $O(T^4)$. Hence the only $O(x^2)$ contribution can come from $I_1$ which is analogous to the Myrheim integral.

The $O(T^3)$ corrections to $V$ should be of the form $V''W^3Z^i \partial_i R_{0\alpha0\beta}$ where the $V''$, $W^a$, $Z^i$ can be time-like or space-like vectors. Since to $O(T^2)$, one only has terms of $R$ and $R_{0\alpha0\beta}$, a first guess is to include terms like $\partial_3 R$ and $\partial_3 R_{0\alpha0\beta}$. We show that these are precisely the sorts of terms that appear when examining the light cones in $T_0^M$ and $T_q^M$ to this order.

To order $T^3$, the metric at $q$ is

$$g_{ab}(q) = \eta_{ab}(0) - \frac{1}{12} T^2 R_{0000}(0) - \frac{1}{38} T^3 \partial_3 R_{0000}(0), \quad (43)$$

so that for $\zeta^a \in T_q^M$ satisfying the null condition $g_{ab}(q) \zeta^a \zeta^b = 0$,

$$- (\zeta^0)^2 + \sum_{i=1}^{n-1} (\zeta^i)^2 = \left( \frac{1}{12} T^2 R_{0000}(0) + \frac{1}{48} T^3 \partial_3 R_{0000}(0) \right) \zeta^0 \zeta^0, \quad (44)$$

or $\tilde{\zeta}^T M \tilde{\zeta} = (\zeta^0)^2$ which is the next-order modification of equation (9), where now

$$M_{ij} = \delta_{ij} - \frac{1}{27} T^2 R_{0000} - \frac{1}{38} T^3 \partial_3 R_{0000} = M_{ij}^{(0)} + T^2 M_{ij}^{(2)} + T^3 M_{ij}^{(3)}. \quad (45)$$

This suggests that only the time derivatives of the Riemann tensor are relevant to the volume calculation. Moreover, using equation (41) we see that the curve $y$ from $r$ to $q$ along the $t$-axis is also a geodesic to $O(T^3)$. This follows from the fact that

$$U^a \nabla_a U^b = \frac{t^2}{2} \partial_0^3 \Gamma_{00}^0(0) + O(t^3) = O(t^3), \quad (46)$$

since $\partial_0^2 \Gamma_{00}^0(0) = 0$ from (41). Moreover, again to this order the time-component of any vector remains unchanged under parallel transport, while the spatial components satisfy

$$\partial_t e^{(k)i} - \frac{t}{3} R_{0\alpha0}^j(0) e^{(k)i} - \frac{t^2}{6} \partial_3 R_{0\alpha0}^j(0) e^{(k)i} = 0. \quad (47)$$

Thus, at least in $T_q^M$, even to order $T^3$, the only derivative of the curvature that appears is the time derivative. If we were to ignore the acceleration of the null geodesics, and use third-order perturbation theory to find the eigenvalues of $M$, then again, $I_1$ differs from the flat space integral purely by the Jacobian $J$ (equation (26)), which to this order is

$$J = 1 - \frac{T^2}{2} \sum_{i=1}^{n-1} \lambda_{(i)2}^{(2)} - T^3 \sum_{i=1}^{n-1} \lambda_{(i)3}. \quad (48)$$
Given the form of \( M^{(3)} \), this suggests that only the trace \( \sum_{i=1}^{n-1} \partial_0 R_{i00} = \lambda \partial_0 R_{00} \) will contribute to \( \mathcal{I} \). We show that this is indeed the case, using a simple extension of standard third-order perturbation theory. Expanding \( M \), its eigenvalues and its eigenfunctions in powers of \( T \),

\[
(M^{(0)} + TM^{(1)} + T^2M^{(2)} + T^3M^{(3)} + \ldots)(\psi_0 + T\psi_1 + T^2\psi_2 + T^3\psi_3)
\]

\[
= (\lambda_0 + T\lambda_1 + T^2\lambda_2 + T^3\lambda_3 + O(T^4))(\psi_0 + T\psi_1 + T^2\psi_2 + T^3\psi_3),
\]

gives the following set of equations:

\[
(M^{(0)} - \lambda_0)\psi_0 = 0
\]
\[
(M^{(0)} - \lambda_0)\psi_1 = (\lambda_1 - M^{(1)})\psi_0
\]
\[
(M^{(0)} - \lambda_0)\psi_2 = (\lambda_2 - M^{(2)})\psi_1 + (\lambda_1 - M^{(1)})\psi_0
\]
\[
(M^{(0)} - \lambda_0)\psi_3 = (\lambda_3 - M^{(3)})\psi_2 + (\lambda_2 - M^{(2)})\psi_1 + (\lambda_1 - M^{(1)})\psi_0.
\]

Because \( M^{(1)} = 0 \), the second equation tells us that \( \lambda_1 = 0 \). This in turn means that \( \psi_1 \) is an eigenfunction of \( M^{(0)} \). Using the freedom to add any multiple of \( \psi_0 \) to \( \psi_s \), \( s > 0 \), we can arrange \( \langle \psi_0, \psi_s \rangle = 0 \) for all \( s > 0 \) [7]. In particular, we may choose \( \psi_1 = 0 \). Hence, contracting the last equation with \( \psi_0 \) gives us

\[
\lambda_3(\psi_0|\psi_0) = (\psi_0|M^{(3)}|\psi_0).
\]

Going back to our notation

\[
\sum_{i=1}^{n-1} \lambda_i^{(3)} = \sum_{i=0}^{n-1} M^{(3)}_{ii} = -\frac{1}{48} \partial_0 R_{00}.
\]

However, as discussed earlier, the acceleration of a null geodesics at \( q \), though sub-leading in \( O(T^2) \) is not sub-leading to \( O(T^3) \) and hence cannot be ignored. It is at present not clear to us how to evaluate this contribution to the volume due to the acceleration. Instead, we resort to the Gibbons–Solodukhin approach by calculating the volumes of small causal diamonds in FRW spacetimes for which \( \partial_0 R \) and \( \partial_0 R_{00} \) do not vanish.

Based on the considerations above, we take the \( O(T^3) \) correction to the volume to be of the form

\[
V_0(\chi(n)\partial_0 R(0) + \kappa(n)\partial_0 R_{00}(0)),
\]

where again \( \chi(n) \) and \( \kappa(n) \) are universal constants to be evaluated. A more covariant version of the order \( T^3 \) terms is \( T^{\alpha\beta}T^T\partial_\alpha R(0) + T^{\sigma\beta}T^\tau\partial_\sigma R_{\tau\rho}(0) \) which reduces to the above when the proper time is aligned along the time axis.

We now calculate \( \chi(n) \) and \( \kappa(n) \) by finding the volume of a small diamond in \( n \)-dimensional FRW spacetimes. It suffices to consider a spatially flat class of FRW spacetimes

\[
dx^2 = -dt^2 + t^{2\sigma} \sum_{i=1}^{n-1} (dx^i)^2,
\]

where we have taken the scale factor to be of the form \( a(t) = t^\sigma \). The Alexandrov interval we will evaluate is centred at some \( t = t_0 \), with \( p \) and \( q \) chosen appropriately. We will examine two special cases \( \sigma = 1 \) and \( \sigma = 1/2 \).

**Case I: \( \sigma = 1 \).**

We will work in conformal time, \( \eta = \ln(1/t) \) or \( t = t_0 \exp^\eta \), where we have chosen the integration constant so that \( \eta = 0 \) at \( t = t_0 \), and \( t_0 > 0 \):

\[
dx^2 = \hat{a}(\eta)^2 \left( -d\eta^2 + \sum_{i=1}^{n-1} (dx^i)^2 \right).
\]

11
where \( \tilde{a}(\eta) = \tilde{t}_0 e^{\eta} \). For this spacetime
\[
R(t) = \frac{(n-1)(n-2)}{t^2}, \quad R_0 = 0
\]
\[
\Rightarrow \partial_t R(t) = -\frac{2(n-1)(n-2)}{t^3}, \quad \partial_0 R_{00} = 0, \tag{56}
\]
which allows us to calculate \( \chi (n) \).

We now calculate the volume of an interval \( I[p, q] \) where in conformal coordinates \( p = (-\frac{N}{2}, 0, \ldots , 0) \) and \( q = (\frac{N}{2}, 0, \ldots , 0) \), so that the proper time \( T \) between \( p \) and \( q \) is
\[
T = t_q - t_p = \tilde{t}_0 (e^{\frac{N}{2}} - e^{-\frac{N}{2}}) = 2\tilde{t}_0 \sinh \left(\frac{N}{2}\right), \tag{57}
\]
or
\[
N = 2 \sinh^{-1} \left(\frac{T}{2\tilde{t}_0}\right) = \frac{T}{\tilde{t}_0} \left(1 - \frac{T^2}{24\tilde{t}_0^2} + O(T^4)\right). \tag{58}
\]
Since \( g_{ab} \) is conformally flat the boundary of \( I[p, q] \) is determined by the equation \( r = \frac{N}{2} - |\eta|\):
\[
V(\tau) = V_+ + V_-
\]
\[
= \int_0^{\frac{N}{2}} d\eta \tilde{a}(\eta)^n \int_0^{\frac{N}{2}-\eta} d\tau \tau^{n-2} \int_{N/2-2\tau}^{N/2} d\Omega_n-2 + \int_0^{\frac{N}{2}} d\eta \tilde{a}(\eta)^n \int_0^{\frac{N}{2}-\eta} d\tau \tau^{n-2} \int_{N/2-2\tau}^{N/2} d\Omega_n-2
\]
\[
= \frac{2A_{n-2}}{n-1} \tilde{t}_0^n \int_0^{\frac{N}{2}} \cosh(n\eta) \left(\frac{N}{2} - \eta\right)^{n-1}. \tag{59}
\]
The integral
\[
I = \int_0^{\frac{N}{2}} \cosh(n\eta) \left(\frac{N}{2} - \eta\right)^{n-1}
\]
\[
= \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \frac{n^k}{(2k)!} \left(\frac{N}{2}\right)^{n-k-l} (-1)^l \binom{n-1}{l} \int_0^{\frac{N}{2}} d\eta \eta^{2k+1}
\]
\[
= \frac{n}{N} \frac{1}{n-1} \left(1 + \frac{n^2N^2}{4(n+1)(n+2)} + O(N^4)\right), \tag{60}
\]
where we have used the relation
\[
\sum_{l=0}^{n} \frac{(-1)^l}{l+a} \binom{n-1}{l} = \frac{1}{a} \left(n - 1 + a\right)^{-1}. \tag{61}
\]
Using the expansion
\[
\left(\frac{\tilde{t}_0 N}{2}\right)^n = \left(\frac{T}{2}\right)^n \left(1 - \frac{n}{24} \left(\frac{T}{\tilde{t}_0}\right)^2 + O(T^4)\right), \tag{62}
\]
we find the expression for the volume
\[
V = V_0 \left(1 - T^2 \frac{n(n-1)(n-2)}{24(n+1)(n+2)\tilde{t}_0^2} + O(T^4)\right)
\]
\[
= V_0 \left(1 - T^2 \frac{n}{24(n+1)(n+2)} R(\tilde{t}_0) + O(T^4)\right). \tag{63}
\]
Clearly, even though \( \partial_0 R(\tilde{t}_0) = -2(n-1)(n-2)\tilde{t}_0^{-3} \neq 0 \), the \( O(T^3) \) terms vanish, hence implying that \( \chi (n) = 0 \).

**Case II:** \( \sigma = 1/2 \).

Conformal time in this case is given by \( \eta = 2(\sqrt{t} - \sqrt{\tilde{t}_0}) \), where again \( \eta = 0 \) at \( t = \tilde{t}_0, \tilde{t}_0 > 0 \), so that \( t = \tilde{t}_0 (1 + \sqrt{\tilde{t}_0})^2 \). The conformal timescale factor is therefore \( \tilde{a}(\eta) = \sqrt{\tilde{t}_0} + \frac{\eta}{2} \). For
The dimension independence of the factor multiplying \( V \) Note that the spacetime, this stems from the fact that to this order it is only the sum of the eigenvalues of first-order perturbation theory that contribute. This sum is itself dimension independent since it involves the trace \( \sum_{i=1}^{n-1} R_{00i}(0) = \sum_{i=1}^{n} R_{00i}(0) = R_{00}(0) \). Whether there is a more profound reason for this universality is however unclear.

### 5. Conclusions

In this note we have shown explicitly how to calculate the volume up to \( O(T^2) \) for a small causal diamond \( \Pi[p, q] \) in arbitrary dimensions using Myrheim’s approach. Our main result is that in any dimension the boundary term of Myrheim [1] takes the form

\[
I_{\Delta \Pi[p, q]} = V_0 \left( \frac{T^2}{24} R_{00}(0) \right). \tag{68}
\]

The dimension independence of the factor multiplying \( V_0 \) is intriguing. At its most mundane, this stems from the fact that to this order it is only the \textit{sum} of the eigenvalues of first-order perturbation theory that contribute. This sum is itself dimension independent since it involves the trace \( \sum_{i=1}^{n-1} R_{00i}(0) = \sum_{i=1}^{n} R_{00i}(0) = R_{00}(0) \). Whether there is a more profound reason for this universality is however unclear.

\footnote{Note that the \( O(T^2) \) terms in both cases are compatible with our expression for the volume (equation (36)) to that order.}
Importantly, our analysis allows us to generalize this result to other integrals over the region $I[p,q]$, so that for any integrable function $\Phi(x^\mu)$ in $I[p,q]$, 

$$
\int_{I[p,q]} d^n x \, \Phi(x^\mu) = \left(1 + \frac{T^2}{24}R_{00}(0)\right) \int_{I[p,q]} d^n z \, \tilde{\Phi}(z^\mu) + O(T^{3+m}), \quad (69)
$$

where $m$ is the order of the flat spacetime integral. This has useful implications for calculations in causal set theory. Recently, it has been used to find an expression for the discrete Ricci scalar in a causal set in terms of the abundance of ‘k-chains’, a construction which differs substantially from that previously obtained by Benincasa and Dowker [8].

We have also extended our analysis to examine the order $O(T^3)$ contribution. Using a combination of our analysis and the approach of Gibbons and Solodukhin [2], we find from calculations of the spacetime volume in two types of FRW spacetimes that the $O(T^3)$ contribution to the volume in fact vanishes. Since the spacetime volume plays such a key role in the continuum approximation of causal set theory, we conjecture that the first non-trivial higher derivative correction to the Einstein–Hilbert action from causal set theory must contain at least second derivatives of the curvature (modulo boundary terms), thus distinguishing the causal set effective action from other higher derivative theories. Our conjecture has obvious limitations since it rests solely on the assumption that these higher order corrections obtain only from contributions to the volume. We have neglected entirely the effect of other geometric and topological contributions that might arise in a more subtle fashion in causal set theory, although these are at present hard to construe. To substantiate the conjecture requires a far better understanding than we have at present of the discrete-continuum correspondence, and in particular the precise manner in which locality is recovered from causal set quantum gravity.

References

[1] Myrheim J 1978 Statistical geometry CERN report number TH-2538
[2] Gibbons G W and Solodukhin S N 2007 Phys. Lett. B {649} 317
[3] Bombelli L, Lee J, Meyer D and Sorkin R D 1987 Phys. Rev. Lett. {59} 521–4
[4] Roy M, Sinha D and Surya S 2012 arXiv:1212.0631 [gr-qc]
[5] Meyer D A 1988 The dimension of causal sets PhD Thesis MIT
[6] Gibbons G W and Solodukhin S N 2007 Phys. Lett. B {652} 103
[7] Schiff Leonard I 1968 Quantum Mechanics (New York: McGraw-Hill)
[8] Benincasa D M T and Dowker F 2010 Phys. Rev. Lett. {104} 181301