Strictly elliptic operators with Dirichlet boundary conditions on spaces of continuous functions on manifolds

TíM BINZ

Abstract. We study strictly elliptic differential operators with Dirichlet boundary conditions on the space $C(M)$ of continuous functions on a compact Riemannian manifold $M$ with boundary and prove sectoriality with optimal angle $\frac{\pi}{2}$.

1. Introduction

Our starting point is a smooth compact Riemannian manifold $M$ of dimension $n$ with smooth boundary $\partial M$ and Riemannian metric $g$ and the initial value-boundary problem

\[
\begin{align*}
\frac{d}{dt} u(t) &= \sqrt{\vert a \vert} \text{div}_g \left( \frac{1}{\sqrt{\vert a \vert}} a \nabla^g_M u(t) \right) + \langle b, \nabla^g_M u(t) \rangle + cu(t) \quad \text{for } t > 0, \\
\left. u(t) \right|_{\partial M} &= 0 \quad \text{for } t > 0, \\
u(0) &= u_0.
\end{align*}
\]

(IBP)

Here, $a$ is a smooth $(1, 1)$-tensorfield, $b \in C(M, \mathbb{R}^n)$ and $c \in C(M, \mathbb{R})$. We are interested in existence, uniqueness and qualitative behaviour of the solution of this initial value-boundary problem. To study these properties systematically, the theory of operator semigroups (cf. [4,11,13,18]) can be used. We choose the Banach space $C(M)$ and define the differential operator with Dirichlet boundary condition

\[
A_0 f := \sqrt{\vert a \vert} \text{div}_g \left( \frac{1}{\sqrt{\vert a \vert}} a \nabla^g_M f \right) + \langle b, \nabla^g_M u(t) \rangle + cf
\]

with domain

\[
D(A_0) := \left\{ f \in \bigcap_{p \geq 1} W^{2,p}(M) \cap C_0(M) : A_0 f \in C(M) \right\}.
\]

Mathematics Subject Classification: 47D06, 34G10, 47E05, 47F05

Keywords: Dirichlet boundary conditions, Analytic semigroup, Riemannian manifolds.
Then, the initial value-boundary problem (IBP) is equivalent to the abstract Cauchy problem

\[
\begin{cases}
\frac{d}{dt} u(t) = A_0 u(t) & \text{for } t > 0, \\
 u(0) = u_0
\end{cases}
\]  

(ACP)

in \( C(\overline{M}) \). In this paper, we show that the solution \( u \) of the above problems can be extended analytically in the time variable \( t \) to the open complex right half-plane. In operator theoretic terms this corresponds to the fact that \( A_0 \) is sectorial of angle \( \frac{\pi}{2} \).

Here is our main theorem.

**Theorem 1.1.** The operator \( A_0 \) is sectorial of angle \( \frac{\pi}{2} \) and has compact resolvent on \( C(\overline{M}) \).

For domains \( \Omega \subset \mathbb{R}^n \), the generation of analytic semigroups by elliptic operators with Dirichlet boundary conditions on different spaces is well known. It was first shown by Browder in [8] for \( L^2(\Omega) \), by Agmon in [3] for \( L^p(\Omega) \) (see also [18, Chap. 3.1.1]) and by Stewart in [22] for \( C(\overline{\Omega}) \) (see also [18, Chap. 3.1.5]). By Stewart’s method, one even gets the angle of analyticity. Later Arendt proved in [5] (see also [1, Chap. III. 6]), using the Poisson operator, that the angle of the analytic semigroup generated by the Laplacian on the space \( C(\overline{\Omega}) \) is \( \frac{\pi}{2} \). However, this method does not work on manifolds with boundary.

The angle \( \frac{\pi}{2} \) of analyticity of \( A_0 \) plays an important role in the generation of analytic semigroups by elliptic differential operators with Wentzell boundary conditions on spaces of continuous functions. Many authors are interested in this topic, and we refer, e.g. to [9,10,12,14,15]. In this context, one starts from the “maximal” operator \( A_m : D(A_m) \subseteq C(\overline{M}) \rightarrow C(\overline{M}) \) in divergence form, given by

\[
A_m f := \sqrt{|a|} \text{div}_g \left( \frac{1}{\sqrt{|a|}} a^{\gamma} \nabla_M^g f \right) + \langle b, \nabla_M^g f \rangle + cf
\]

with domain

\[
D(A_m) := \left\{ f \in \bigcap_{p \geq 1} W^{2,p}(M) : A_m f \in C(\overline{M}) \right\}.
\]

Moreover, using the outer co-normal derivative \( \frac{\partial a}{\partial n} : D(\frac{\partial a}{\partial n}) \subset C(\overline{M}) \rightarrow C(\partial M) \), a constant \( \beta < 0 \) and \( \gamma \in C(\partial M) \), one defines the differential operator \( A \) with generalized Wentzell boundary conditions by requiring

\[
f \in D(A) \iff f \in D(A_m) \text{ and } A_m f \big|_{\partial M} = \beta \cdot \frac{\partial a}{\partial n} f + \gamma \cdot f \big|_{\partial M} .
\]

The main theorem in [6] shows that this operator \( A \) can be split into the operator \( A_0 \) with Dirichlet boundary conditions on \( C(\overline{M}) \) and the Dirichlet-to-Neumann operator \( N := \beta \cdot \frac{\partial a}{\partial n} L_0 \) on \( C(\partial M) \), where \( L_0 \varphi = f \) denotes the unique solution of

\[
\begin{cases}
A_m f = 0, \\
 f \big|_{\partial \Omega} = \varphi.
\end{cases}
\]
Using Theorem 1.1 and [6, Thm. 3.1 & Cor. 3.2], one obtains the following result.

**Corollary 1.2.** The operator $A$ with Wentzell boundary conditions generates a compact and analytic semigroup of angle $\theta > 0$ on $C(\overline{M})$ if and only if the Dirichlet-to-Neumann operator $N$ does so on $C(\partial M)$.

In an upcoming paper [7], we prove the latter statement with the optimal angle $\pi/2$ and conclude that elliptic differential operators with Wentzell boundary conditions generate compact and analytic semigroups of angle $\pi/2$ on $C(\overline{M})$.

This paper is organized as follows.

In Sect. 2, we study the special case where $A_0$ is the Laplace–Beltrami operator with Dirichlet boundary conditions. We approximate its resolvents by modifying the Green’s functions of the Laplace operator on $\mathbb{R}^n$, study the scaling of the error of the Laplace–Beltrami operator and prove estimates for the associated Green’s functions. Finally, one obtains the sectoriality of angle $\pi/2$ for the Laplace–Beltrami operator with Dirichlet boundary conditions on $C(\overline{M})$.

In Sect. 3, the main result from Sect. 2 is extended to arbitrary strictly elliptic operators. Introducing a new Riemannian metric, induced by the coefficients of the second-order part of the elliptic operator, the operator takes a simpler form: Up to a relatively bounded perturbation of bound 0, it is a Laplace–Beltrami operator for the new metric. Regularity and perturbation theory yield the main theorem in its full generality.

In this paper, the following notation is used. For a closed operator $T : D(T) \subset X \to X$ on a Banach space $X$, we denote by $[D(T)]$ the Banach space $D(T)$ equipped with the graph norm $\| \cdot \|_T := \| \cdot \|_X + \| T(\cdot) \|_X$ and indicate by $\hookrightarrow$ a continuous and by $\hookrightarrow$ a compact embedding. Moreover, we use Einstein’s notation of sums, i.e.

$$x_k y^k := \sum_{k=1}^n x_k y^k$$

for $x := (x_1, \ldots, x_n)$, $y := (y_1, \ldots, y_n)$. Furthermore, we denote by $\mathbb{R}_+ := \{ r \in \mathbb{R} : r > 0 \}$ the positive real numbers and by $\mathbb{R}_- := \mathbb{R} \setminus \mathbb{R}_+$ the non-positive real numbers. Besides one defines the sector by $\Sigma_\theta := \{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta \}$. Using the distance function $d$ on $\overline{M}$, we denote by $B_R(x) := \{ y \in \overline{M} : d(x, y) < R \}$.

### 2. Laplace–Beltrami operators with Dirichlet boundary conditions

In this section, we consider the special case where $A_0$ is the Laplace–Beltrami operator with Dirichlet boundary conditions, i.e.

$$\Delta^g_0 f := \Delta g f = \text{div}_g(\nabla g f) = g^{ij} \partial^2_{ij} f - g^{ij} \Gamma^k_{ij} \partial_k f,$$

$$D(\Delta^g_0) := \left\{ f \in \bigcap_{p \geq 1} W^{2,p}(M) \cap C_0(\overline{M}) : \Delta^g f \in C(\overline{M}) \right\}$$

(2.1)
on the space $C(M)$ of continuous functions on $\overline{M}$. Here,
\[ g^k_i j := \frac{1}{2} g^{kl} \left( \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \]
denote the Christoffel symbols of the Riemannian metric $g$.

**Proposition 2.1.** For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the operator $\lambda - \Delta^g_0$ is injective.

**Proof.** Considering the equation
\[ \begin{cases} \lambda f = \Delta^g f, \\ f|_{\partial M} = 0 \end{cases} \tag{2.2} \]
for $f \in C(\overline{M})$, one obtains that $\lambda - \Delta^g_0$ is injective if the only solution of (2.2) is zero.

Since $\overline{M}$ is compact, the domain $D(\Delta^g_0)$ is contained in $L^2(M)$ and $\Delta^g f \in L^2(M)$. Hence, Green’s formula implies
\[ \lambda \| f \|^2_{L^2(M)} = \lambda \int_M f \overline{f} \, \text{dvol}^g_M = \int_M \Delta^g f \overline{f} \, \text{dvol}^g_M = -\int_M g(\nabla^g f, \nabla^g \overline{f}) \, \text{dvol}^g_M \in \mathbb{R}. \]
Since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the term $\lambda \| f \|^2_{L^2(M)}$ can be in $\mathbb{R}$ only if $f = 0$. □

In the next step, we construct Green’s functions such that the associated integral operators approximate the resolvent of $A_0$.

To this end, it is necessary to smooth the distance function $d$ on $\overline{M}$. We consider a sufficiently small $\varepsilon > 0$ and define
\[ \rho(x, y) := d(x, y) \chi \left( \frac{d(x, y)}{\varepsilon} \right) + 2\varepsilon \left( 1 - \chi \left( \frac{d(x, y)}{\varepsilon} \right) \right), \]
where $\chi$ is a smooth cut-off function with $\chi(s) = 1$ if $s < 1$ and $\chi(s) = 0$ if $s > 2$. Then, $\rho \equiv d$ for $d(x, y) < \varepsilon$ and $\rho \in C^\infty((\overline{M} \times \overline{M}) \setminus \{(x, x) : x \in \overline{M}\}, \mathbb{R})$.

Next, we extend the smoothed distance function $\rho$ on $\overline{M}$ beyond the boundary $\partial M$. To this end, the set $S_{2\varepsilon} := \{ x \in \overline{M} : d(x, \partial M) < 2\varepsilon \}$ is identified via the normal exponential map with $\partial M \times [0, 2\varepsilon)$. Considering $\overline{M} \cup (\partial M \times (-2\varepsilon, 0])$ and identifying $\partial M$ with $\partial M \times \{0\}$ via $x \sim (x, 0)$, one obtains a smooth manifold $\tilde{M}$. By Whitney’s extension theorem (see [21]), the metric $g$ can be extended to a smooth metric $\tilde{g}$ on $\tilde{M}$ and hence the smoothed distance function $\rho$ can be extended to a smooth function $\tilde{\rho}$ on $\tilde{M} \times \tilde{M} \setminus \{(x, x) : x \in \tilde{M}\}$.

For $x \in S_{2\varepsilon}$, we consider the reflected point $x^* \in \tilde{M} \setminus M$ with
\[ \tilde{\rho}(x, \partial M) = \tilde{\rho}(x^*, \partial M) \]
such that the nearest neighbour of $x$ on $\partial M$ and the nearest neighbour of $x^*$ on $\partial M$ coincide.
Here and in the following, we denote by $n := \dim(\mathcal{M})$ the dimension of the manifold. The kernels are defined by

$$K_\lambda(x, y) = \begin{cases} \sqrt{\pi}^{-1} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda} \rho(x, y))}{\rho(x, y)^{\frac{n}{2}-1}}, & \text{if } d(x, \partial \mathcal{M}) < \varepsilon, \\ \sqrt{\pi}^{-1} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda} \rho(x, y))}{\rho(x, y)^{\frac{n}{2}-1}}, & \text{if } d(x, \partial \mathcal{M}) \in [\varepsilon, 2\varepsilon], \\ \sqrt{\pi}^{-1} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda} \rho(x, y))}{\rho(x, y)^{\frac{n}{2}-1}}, & \text{if } d(x, \partial \mathcal{M}) > 2\varepsilon, \end{cases}$$

for $x \in \mathcal{M}$, $y \in \mathcal{M}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$, where $K_{\frac{n}{2}-1}$ is the modified Bessel function of the second kind (cf. Proposition A.1) of order $\frac{n}{2} - 1$. Moreover, the associated integral operators are given by

$$(G_\lambda f)(x) := \int_{\mathcal{M}} K_\lambda(x, y) f(y) \, dy.$$ 

We now prove that the integral operators $G_\lambda$ satisfy similar estimates as the resolvents of a sectorial operator.

**Proposition 2.2.** Let $\eta > 0$. For $\lambda \in \Sigma_{\pi - \eta}$ with $|\lambda| \geq 1$, the integral operators $G_\lambda$ fulfil

$$\|G_\lambda f\|_{L^\infty(\mathcal{M})} \leq \frac{C(\eta)}{|\lambda|} \|f\|_{L^\infty(\mathcal{M})}$$

for all $f \in C(\mathcal{M})$ and $C(\eta) > 0$.

**Proof.** By Lemmas A.2 and A.3, we obtain

$$\|G_\lambda f\|_{L^\infty(\mathcal{M} \setminus S_{2\varepsilon})} \leq C \sqrt{|\lambda|}^{\frac{n}{2}-1} \sup_{x \in \mathcal{M} \setminus S_{2\varepsilon}} \int_{\mathcal{M}} \frac{K_{\frac{n}{2}-1}(C(\eta) \sqrt{|\lambda|} \rho(x, y))}{\rho(x, y)^{\frac{n}{2}-1}} \, dy \cdot \|f\|_{L^\infty(\mathcal{M})} \leq \frac{C'(\eta)}{|\lambda|} \|f\|_{L^\infty(\mathcal{M})} \quad (2.3)$$

for $f \in C(\mathcal{M})$. Moreover, Lemmas A.2, A.3 and Corollary A.4 imply

$$\|G_\lambda f\|_{L^\infty(S_{\varepsilon})} \leq C \sqrt{|\lambda|}^{\frac{n}{2}-1} \left( \sup_{x \in S_{\varepsilon}} \int_{\mathcal{M}} \frac{K_{\frac{n}{2}-1}(C(\eta) \sqrt{|\lambda|} \bar{\rho}(x, y))}{\bar{\rho}(x, y)^{\frac{n}{2}-1}} \, dy \right) \|f\|_{L^\infty(\mathcal{M})} \leq \frac{C'(\eta)}{|\lambda|} \|f\|_{L^\infty(\mathcal{M})} \quad (2.4)$$
for \( f \in C(\overline{M}) \). Furthermore, Lemmas A.2, A.3 and Corollary A.4 yield
\[
\| G_\lambda f \|_{L^\infty(S_{2\epsilon}\setminus S_\epsilon)} \leq C \sqrt{|\lambda|^{\frac{n}{2}-1}} \left( \sup_{x \in S_{2\epsilon}\setminus S_\epsilon} \int_M K^{n-1}_\varphi \left( \frac{\rho(x, y)}{\rho(x, y)^{\frac{n}{2}-1}} \right) dy \right)
\]
\[
+ \sup_{x \in S_{2\epsilon}\setminus S_\epsilon} \chi \left( \frac{\rho(x, \partial M)}{\epsilon} \right)
\]
\[
\sup_{x \in S_{2\epsilon}\setminus S_\epsilon} \int_M K^{n-1}_\varphi \left( \frac{\rho(x^*, y)}{\rho(x^*, y)^{\frac{n}{2}-1}} \right) dy \| f \|_{L^\infty(M)}
\]
\[
\leq C' \left( \frac{\eta}{|\lambda|} \right) \| f \|_{L^\infty(M)} \tag{2.5}
\]
for \( f \in C(\overline{M}) \). Summing up it follows that
\[
\| G_\lambda f \|_{L^\infty(M)} = \| G_\lambda f \|_{L^\infty(M \setminus S_{2\epsilon})} + \| G_\lambda f \|_{L^\infty(S_\epsilon)} + \| G_\lambda f \|_{L^\infty(S_{2\epsilon}\setminus S_\epsilon)}
\]
\[
\leq C(\eta) \frac{|\lambda|}{|\lambda|} \| f \|_{L^\infty(M)}
\]
for \( f \in C(\overline{M}) \) as claimed. \( \square \)

To show that the kernel \( K_\lambda \) is approximately a Green’s function for \( \lambda - \Delta^g_0 \), we need the following lemmata.

**Lemma 2.3.** Let \( \eta > 0 \). For \( \lambda \in \Sigma_{\pi-\eta} \) with \( |\lambda| \geq 1 \), we have
\[
(\lambda - \Delta^g_0) \left( \frac{\sqrt{\lambda^{\frac{n}{2}-1}} K^{n-1}_\varphi \left( \frac{\rho(x, y)}{\rho(x, y)^{\frac{n}{2}-1}} \right)}{\sqrt{2\pi}^{\frac{n}{2}}} \right)
\]
\[
= \delta(x) + O\left( \frac{\sqrt{|\lambda|^{\frac{n}{2}-1}} K^{n-1}_\varphi \left( \frac{\rho(x, y)}{\rho(x, y)^{\frac{n}{2}-1}} \right)}{\rho(x, y)^{\frac{n}{2}-2}} \right)
\]
\[
+ C(\eta) \sqrt{|\lambda|^{\frac{n}{2}-1}} + e^{-C(\eta)\sqrt{|\lambda|^{\frac{n}{2}}}}
\]
for \( x, y \in \overline{M} \).

**Proof.** Considering
\[
K(r) := \frac{\sqrt{\lambda^{\frac{n}{2}-1}} K^{n-1}_\varphi \left( \frac{\sqrt{\lambda} r}{r^{\frac{n}{2}-1}} \right)}{\sqrt{2\pi}^{\frac{n}{2}}} \tag{2.6}
\]
one obtains

\[ K'(r) = \frac{\sqrt{\lambda}^{\frac{n}{2} - 1}}{2\pi^{\frac{n}{2}}} \left( \frac{\sqrt{\lambda} \, K'_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{r^\frac{n}{2} - 1} - \frac{(\frac{n}{2} - 1) \, K_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{r^\frac{n}{2}} \right) \]

\[ = -\frac{\sqrt{\lambda}^{\frac{n}{2} - 1}}{2\pi^{\frac{n}{2}}} \left( \frac{\sqrt{\lambda} \, K_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{2r^\frac{n}{2} - 1} + \frac{\sqrt{\lambda} \, K_{\frac{n}{2} - 2}(\sqrt{\lambda}r)}{2r^\frac{n}{2} - 1} \right) \]

\[ + \frac{(\frac{n}{2} - 1) \, K_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{r^\frac{n}{2} - 1} \]

(2.7)

and hence

\[ K''(r) = \frac{\sqrt{\lambda}^{\frac{n}{2} - 1}}{2\pi^{\frac{n}{2}}} \left( \frac{\lambda \, K''_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{r^\frac{n}{2} - 1} - (n - 2) \frac{\sqrt{\lambda} \, K'_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{r^\frac{n}{2} - 1} \right) \]

\[ + \left( \frac{n^2}{4} - \frac{n}{2} \right) \frac{K_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{r^\frac{n}{2} + 1} \]

(2.8)

These imply

\[ K''(r) + \frac{n - 1}{r} K'(r) - \lambda K(r) \]

\[ = -\frac{\sqrt{\lambda}^{\frac{n}{2} - 1}}{2\pi^{\frac{n}{2}}} \left( \frac{\lambda \, K''_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{r^\frac{n}{2} - 1} + \frac{\sqrt{\lambda} \, K'_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{r^\frac{n}{2} - 1} + \left( \frac{n^2}{4} - \frac{n}{2} \right) \frac{K_{\frac{n}{2} - 1}(\sqrt{\lambda}r)}{r^\frac{n}{2} + 1} \right) \]

Remark that the kernel is rotation symmetric, and hence, the Laplacian is given by

\[ \Delta_x K(|x|) = \partial_x^2 K(|x|) + \frac{n-1}{r} \partial_r K(|x|). \]

Using (3.4), we conclude

\[ \Delta_x K(|x|) - \lambda K(|x|) = -C \cdot \delta_0(x). \]

(2.9)

Next, we determine the constant \( C \). For sufficient small \( R > 0 \) one has by Gauss Divergence Theorem

\[ C = \int_{B_R(0)} C \cdot \delta_0(x) \, d\text{vol}_{B_R(0)} = -\int_{B_R(0)} \Delta_x K(|x|) - \lambda K(|x|) \, d\text{vol}_{B_R(0)} \]

\[ = -\int_{S_{R}^{n-1}} \frac{\partial}{\partial n} K(|x|) \, d\text{vol}_{S_R^{n-1}} + \lambda \int_{B_R(0)} K(|x|) \, d\text{vol}_{B_R(0)} \]

Vol. 20 (2020) Elliptic operators with Dirichlet boundary conditions 1011
\[ \begin{align*}
&= - \int_{S_{R}^{n-1}} K'(R) \, d\text{vol}_{S_{R}^{n-1}} + \lambda \int_{B_{R}(0)} K(|x|) \, d\text{vol}_{B_{R}(0)} \\
&= -\text{vol}(S^{n-1}) K'(R) R^{n-1} + \lambda \int_{B_{R}(0)} K(|x|) \, d\text{vol}_{B_{R}(0)}. 
\end{align*} \]

Using (2.7), we obtain

\[ C = \frac{\sqrt{\lambda}^{\frac{n}{2}}}{{\sqrt{2\pi}}^{\frac{n}{2}}} \cdot \text{vol}(S^{n-1}) \left( \frac{\sqrt{\lambda} K_{\frac{n}{2}}(\sqrt{\lambda} R)}{2 R^{\frac{n}{2}-1}} + \frac{\sqrt{\lambda} K_{\frac{n-2}{2}}(\sqrt{\lambda} R)}{2 R^{\frac{n-2}{2}-1}} \right) \\
\left. + \frac{\left( \frac{n}{2} - 1 \right) K_{\frac{n-1}{2}}(\sqrt{\lambda} R)}{R^{\frac{n}{2}}} \right) R^{n-1} + \lambda \int_{B_{R}(0)} K(|x|) \, d\text{vol}_{B_{R}(0)}. \]

Since \( K_{\alpha}(r) = O(r^{-\alpha}) \) for small \( r \in \mathbb{R}_{+} \), the second and the fourth term vanishes by taking the limit \( R \to 0 \). Since

\[ \lim_{r \to 0} r^{\alpha} K_{\alpha}(r) = 2^{\alpha-1} \Gamma(\alpha) \]

and

\[ \text{vol}(S^{n-1}) = n \cdot \frac{\sqrt{\pi}^{n}}{\Gamma(n/2 + 1)} \]

the limit of the first term is given by

\[ \left( \frac{n}{2} - 1 \right) \cdot \frac{\sqrt{\lambda}^{\frac{n-1}{2}}}{{\sqrt{2\pi}}^{\frac{n}{2}}} \cdot \text{vol}(S^{n-1}) \cdot \lim_{R \to 0} K_{\frac{n-1}{2}}(\sqrt{\lambda} R) \cdot R^{\frac{n-1}{2}} \]

\[ = \left( \frac{n}{2} - 1 \right) \cdot \frac{\sqrt{\lambda}^{\frac{n-1}{2}}}{{\sqrt{2\pi}}^{\frac{n}{2}}} \cdot \frac{n}{\Gamma(n/2 + 1)} \cdot \lim_{R \to 0} K_{\frac{n-1}{2}}(\sqrt{\lambda} R) \cdot R^{\frac{n-1}{2}} \]

\[ = \left( \frac{n}{2} - 1 \right) \cdot \frac{\sqrt{\lambda}^{\frac{n-1}{2}}}{{\sqrt{2\pi}}^{\frac{n}{2}}} \cdot \frac{n}{\Gamma(n/2 + 1)} \cdot 2^{\frac{n-1}{2}} \Gamma\left( \frac{n}{2} \right) \]

\[ = \left( \frac{n}{2} - 1 \right) \cdot \frac{\sqrt{\lambda}^{\frac{n-1}{2}}}{{\sqrt{2\pi}}^{\frac{n}{2}}} \cdot \frac{n}{\Gamma(n/2 + 1)} \cdot 2^{\frac{n-1}{2}} \Gamma\left( \frac{n}{2} \right) = \frac{1}{2}, \]

where we used in the last line, that the Gamma function satisfies \( \Gamma(x + 1) = x \Gamma(x) \).

Similar the limit of the third term is

\[ \left( \frac{n}{2} - 1 \right) \cdot \frac{\sqrt{\lambda}^{\frac{n-1}{2}}}{{\sqrt{2\pi}}^{\frac{n}{2}}} \cdot \text{vol}(S^{n-1}) \cdot \lim_{R \to 0} K_{\frac{n-1}{2}}(\sqrt{\lambda} R) \cdot R^{\frac{n-1}{2}} \]

\[ = \left( \frac{n}{2} - 1 \right) \cdot \frac{\sqrt{\lambda}^{\frac{n-1}{2}}}{{\sqrt{2\pi}}^{\frac{n}{2}}} \cdot \frac{n}{\Gamma(n/2 + 1)} \cdot \lim_{R \to 0} K_{\frac{n-1}{2}}(\sqrt{\lambda} R) \cdot R^{\frac{n-1}{2}} \]
\[
\begin{align*}
&= \left( \frac{n}{2} - 1 \right) \cdot \frac{1}{\sqrt{2^n}} \cdot \frac{n}{\Gamma \left( \frac{n}{2} + 1 \right)} \cdot \lim_{R' \to 0} K_{\frac{n}{2} - 1}^n (R') \cdot (R')_{\frac{n}{2} - 1}^2 \\
&= \left( \frac{n}{2} - 1 \right) \cdot \frac{1}{\sqrt{2^n}} \cdot \frac{n}{\Gamma \left( \frac{n}{2} + 1 \right)} \cdot 2^{\frac{n}{2} - 2} \Gamma \left( \frac{n}{2} - 1 \right) \\
&= \left( \frac{n}{2} - 1 \right) \cdot \frac{1}{\sqrt{2^n}} \cdot \frac{n}{\frac{n}{2} - 1} \cdot \Gamma \left( \frac{n}{2} - 1 \right) \cdot 2^{\frac{n}{2} - 2} \Gamma \left( \frac{n}{2} - 1 \right) = \frac{1}{2}.
\end{align*}
\]

Hence \( C = \frac{1}{2} + \frac{1}{2} = 1 \). Moreover, we have
\[
(K \circ \rho)(x, y) = \frac{\sqrt{\lambda}^{\frac{n}{2} - 1} K_{\frac{n}{2} - 1} (\sqrt{\lambda} \rho(x, y))}{\sqrt{2^n} \rho(x, y)^{\frac{n}{2} - 1}}
\]
for \( x, y \in \overline{M} \). Using geodetic normal coordinates, the metric is given by
\[
g_{ij}(x) = \delta_{ij} + \mathcal{O}(\rho(x, y)^2).
\]

From \( \Delta_g^\delta |x - y|^2 = 2n \), it follows
\[
\Delta_g^\delta (\rho(x, y)^2) = 2n + \mathcal{O}(\rho(x, y)^2).
\]

Using
\[
\Delta_g^\delta (\rho(x, y)^2) = 2|\nabla_g^\delta \rho(x, y)|_g^2 + 2\rho(x, y) \Delta_g^\delta \rho(x, y) \tag{2.10}
\]
First, we consider \( y \in B_x(x) \). Since \( |\nabla_g^\delta \rho(x, y)|_g = 1 \), one obtains
\[
\Delta_g^\delta (\rho(x, y)) = \frac{n - 1}{\rho(x, y)} + \mathcal{O}(\rho(x, y)).
\]

Therefore, we obtain
\[
\begin{align*}
\Delta_g^\delta (K \circ \rho)(x, y) \\
&= K''(\rho(x, y)) |\nabla_g^\delta \rho(x, y)|_g^2 + K'(\rho(x, y)) \Delta_g^\delta \rho(x, y) \\
&= K''(\rho(x, y)) + K'(\rho(x, y)) \Delta_g^\delta \rho(x, y) \\
&= K''(\rho(x, y)) + \frac{n - 1}{\rho(x, y)} K'(\rho(x, y)) + \mathcal{O}(\rho(x, y)|K'(\rho(x, y))|).
\end{align*}
\]

Using (2.9) and Lemma A.2, it follows that
\[
(\lambda - \Delta_g^\delta)(K \circ \rho)(x, y) = \delta_x(y) + \mathcal{O}(\rho(x, y)|K'(\rho(x, y))|)
\]
\[
\begin{align*}
&= \delta_x(y) + \mathcal{O} \left( \sqrt{|\lambda|} \cdot \frac{K_{\frac{n}{2} - 1}(C(\eta) \sqrt{|\lambda|} \rho(x, y))}{\rho(x, y)^{\frac{n}{2} - 2}} + \frac{\left( \frac{n}{2} - 1 \right) \cdot K_{\frac{n}{2} - 1}(C(\eta) \sqrt{|\lambda|} \rho(x, y))}{\rho(x, y)^{\frac{n}{2} - 1}} \right).
\end{align*}
\]
Now let \( y \in B_{2\varepsilon}(x) \setminus B_{\varepsilon}(x) \). Since \( \rho \) is smooth on \( M \setminus B_{\varepsilon}(x) \), we have \( |\nabla_x^g \rho(x, y)|^2 \leq C \) and therefore by (2.10)

\[
|\Delta^g_x(K \circ \rho)(x, y)| \leq C|K''(\rho(x, y))| + C(n) \frac{|K'(\rho(x, y))|}{\rho(x, y)} + O(\rho(x, y)|K'(\rho(x, y))|).
\]

Moreover, one obtains by Lemma A.2

\[
|K_{n-1}^g(\sqrt{\lambda} \rho(x, y))| \leq K_{n-1}^g(C(\eta)\sqrt{|\lambda|}\varepsilon) = O(e^{-C(n)\sqrt{|\lambda|}\varepsilon})
\]

\[
|K_{n-1}^g(\sqrt{\lambda} \rho(x, y))| \leq |K_{n}^g(\sqrt{\lambda} \rho(x, y))| + |K_{n}^g(\sqrt{\lambda} \rho(x, y))| \\
\leq |K_{n}^g(C(\eta)\sqrt{|\lambda|}\varepsilon)| + |K_{n}^g(C(\eta)\sqrt{|\lambda|}\varepsilon)| \\
= O(e^{-C(n)\sqrt{|\lambda|}\varepsilon})
\]

\[
|K_{n-1}^g(\sqrt{\lambda} \rho(x, y))| \leq |K_{n+1}^g(\sqrt{\lambda} \rho(x, y))| + |K_{n-1}^g(\sqrt{\lambda} \rho(x, y))| \\
+ |K_{n-3}^g(\sqrt{\lambda} \rho(x, y))| \\
= |K_{n+1}^g(C(\eta)\sqrt{|\lambda|}\varepsilon)| + |K_{n-1}^g(C(\eta)\sqrt{|\lambda|}\varepsilon)| \\
+ |K_{n-3}^g(C(\eta)\sqrt{|\lambda|}\varepsilon)| \\
= O(e^{-C(n)\sqrt{|\lambda|}\varepsilon})
\]

for \( |\lambda| \) and \( \lambda \in \Sigma_{n-1}^g \). Since \( \rho(x, y) \geq \varepsilon \), it follows

\[
(\lambda - \Delta^g_x)(K \circ \rho)(x, y) = O(e^{-C(n)\sqrt{|\lambda|}\varepsilon})
\]

for \( |\lambda| \geq 1 \).

Finally, we consider \( y \in M \setminus B_{2\varepsilon}(x) \). Since \( \rho \) is constant on \( M \setminus B_{2\varepsilon}(x) \), it follows that \( \Delta^g_x(K \circ \rho) = 0 \) and therefore as before

\[
(\lambda - \Delta^g_x)(K \circ \rho)(x, y) = O(e^{-C(n)\sqrt{|\lambda|}\varepsilon})
\]

for \( |\lambda| \geq 1 \). \( \square \)

**Lemma 2.4.** Let \( \eta > 0 \). For \( \lambda \in \Sigma_{n-1}^g \) with \( |\lambda| \geq 1 \), we have

\[
(\lambda - \Delta^g_x) \left( \frac{\sqrt{\lambda}^{n-1}}{\sqrt{2\pi}^n} K_{n-1}^{n-1}(\sqrt{\lambda} \rho(x^*, y)) \right) \\
= \delta_{x^*}(y) \\
+ O\left(\sqrt{|\lambda|}\frac{K_{n-1}^{n-1}(\sqrt{\lambda} \rho(x^*, y))}{\rho(x^*, y)^{n-1}}\right) \\
+ O\left(\sqrt{|\lambda|}^{n-1} \frac{K_{n-1}^{n-1}(\sqrt{\lambda} \rho(x^*, y))}{\rho(x^*, y)^{n-1}}\right)
\]
\[\begin{align*}
&+ d(x, \partial M) \left( \sqrt{|\lambda|}^{-1} \left( \frac{K_{2-1}(C(\eta) \sqrt{|\lambda|} \rho(x^*, y))}{\rho(x^*, y)^{2+1}} \right) \\
&+ \sqrt{|\lambda|}^{\frac{n}{2}} \frac{K_2(C(\eta) \sqrt{|\lambda|} \rho(x^*, y))}{\rho(x^*, y)^2} \\
&+ \sqrt{|\lambda|}^{\frac{n}{2}+1} \left( \frac{K_{2+1}(C(\eta) \sqrt{|\lambda|} \rho(x^*, y))}{\rho(x^*, y)^{2-1}} \right) \\
&+ e^{-C\sqrt{|\lambda|} \varepsilon} \right) 
\end{align*}\]

for \(x \in S_{2\varepsilon}\) and \(y \in \overline{M}\).

**Proof.** Considering the reflection \(\sigma : S_{2\varepsilon} \to \tilde{M} : x \mapsto x^*\) and taking a point on the boundary \(p \in \partial M\) every normal vector is an eigenvector for the eigenvalue \(-1\) for the differential \(D\sigma_p : T_p M \to T_p M\) and all tangential vectors on \(\partial M\) eigenvectors with eigenvalue \(1\). In particular, \(D\sigma_p\) is a linear local isometry, i.e. \(\sigma^* g = g\) for \(p \in \partial M\). Since \(\sigma^* g - g\) is smooth, we conclude that

\[\sigma^* g = g + O(d(x, \partial M)).\]

Hence, one obtains \(\nabla^\delta_x \sigma^* g = \nabla^\delta_x g + O(1)\) and

\[\Delta^\delta \tilde{h} = \Delta^\delta (h \circ \sigma) = (\Delta^{\sigma^* g} h) \circ \sigma\]

for \(\tilde{h}(x) := h(x^*)\). Therefore,

\[(\lambda - \Delta^\delta) \tilde{h} = ((\lambda - \Delta^{\sigma^* g}) h) \circ \sigma + (\Delta^{\sigma^* g} h - \Delta^\delta h) \circ \sigma. \hspace{1cm} (2.11)\]

Using

\[\Delta^\delta f = g^{ij} (\partial_i^2 f - \Gamma^k_{ij} \partial_k f)\]

we obtain

\[|\Delta^\delta f - \Delta^{\sigma^* g} f|(x) \leq C \cdot |g - \sigma^* g|_g(x) \cdot \sum_{i,j=1}^n |\partial_i^2 f|(x) + C \cdot |\nabla g - \nabla (\sigma^* g)|_g(x) \cdot |\nabla f|_g(x).\]

Since \(|g - \sigma^* g|_g(x) = O(d(x, \partial M))\) and \(|\nabla g - \nabla (\sigma^* g)|_g(x) = O(1)\), we consider the derivatives of the kernel. Define \(K\) as in (2.6), we consider

\[\partial_i (K \circ \rho)(x^*, y) = K'(\rho(x^*, y)) \cdot \partial_i \rho(x^*, y)\]

\[\partial_{ij} (K \circ \rho)(x^*, y) = K''(\rho(x^*, y)) \cdot \partial_i \rho(x^*, y) \cdot \partial_j \rho(x^*, y) + K'(\rho(x^*, y)) \cdot \partial_{ij} \rho(x^*, y).\]
Since $\partial_t \overline{\rho}(x^*, y) = O(1)$ and (2.7), we obtain

\[
\nabla_x (K \circ \overline{\rho})(x^*, y) = + O \left( \sqrt{|\lambda|^2} - 1 \left( \frac{K_{\frac{n-1}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \right) \right.
\]

\[
+ \left. \sqrt{|\lambda|^2} \left( \frac{K_{\frac{n}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n-1}{2}}} \right) \right),
\]

where we used that by Lemma A.2 and the monotonicity of Bessel functions

\[
|K_{\frac{n-2}{2}}(\sqrt{\lambda} \overline{\rho}(x^*, y))| \leq K_{\frac{n-2}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y)) \leq K_{\frac{n}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))
\]

holds. Similar we obtain from (2.8)

\[
K''(\overline{\rho}(x^*, y)) = O \left( \sqrt{|\lambda|^2} - 1 \left( \frac{K_{\frac{n-1}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}+1}} \right) \right.
\]

\[
+ \left. \sqrt{|\lambda|^2} \left( \frac{K_{\frac{n}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \right) \right.
\]

\[
+ \left. \sqrt{|\lambda|^2} \left( \frac{K_{\frac{n+1}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n-1}{2}}} \right) \right).
\]

Using

\[
\partial_{ij}^2 \overline{\rho}(x^*, y)^2 = 2 \partial_i \overline{\rho}(x^*, y) \partial_j \overline{\rho}(x^*, y) + 2 \overline{\rho}(x^*, y) \partial_{ij}^2 \overline{\rho}(x^*, y)
\]

and $\partial_i \overline{\rho}(x^*, y) = O(1)$ and $\partial_{ij}^2 \overline{\rho}(x^*, y)^2 = O(1)$ one has

\[
\partial_{ij}^2 \overline{\rho}(x^*, y) = O \left( \frac{1}{\overline{\rho}(x^*, y)} \right).
\]

Hence,

\[
\partial_{ij}^2 (K \circ \overline{\rho})(x^*, y) = O \left( \sqrt{|\lambda|^2} - 1 \left( \frac{K_{\frac{n-1}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}+1}} \right) \right.
\]

\[
+ \left. \sqrt{|\lambda|^2} \left( \frac{K_{\frac{n}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \right) \right.
\]

\[
+ \left. \sqrt{|\lambda|^2} \left( \frac{K_{\frac{n+1}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n-1}{2}}} \right) \right).
\]

Finally, we conclude

\[
\left( (\Delta^{\pi^\nu^g} \Delta^{\pi^\nu^g}) \left( \overline{\rho}(x^*, y) \frac{\sqrt{\lambda}^{\frac{n-1}{2}} K_{\frac{n-1}{2}}(\sqrt{\lambda} \overline{\rho}(x^*, y))}{\nabla_\lambda \overline{\rho}(x^*, y)^{\frac{n-1}{2}}} \right) \right)(x^*)
\]

\[
= O \left( \sqrt{|\lambda|^2} - 1 \left( \frac{K_{\frac{n-1}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \right) \right).
\]
\[
+ \sqrt{|\lambda|^\frac{n}{2}} \frac{K_\frac{n}{2} (C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n-1}{2}}} \\
+ d(x, \partial M) \left( \sqrt{|\lambda|^\frac{n}{2}-1} K_{\frac{n}{2}-1} (C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y)) \right) \frac{\overline{\rho}(x^*, y)^{\frac{n}{2}+1}}{\bar{g}(x^*, y) \cdot g(x^*, y)^{\frac{n}{2}+1}} \\
+ \sqrt{|\lambda|^2} \frac{K_\frac{n}{2} (C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \\
+ \sqrt{|\lambda|^\frac{n}{2}+1} \frac{K_{\frac{n}{2}+1} (C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n-1}{2}}} \right). \tag{2.12}
\]

Now, the claim follows by Lemma 2.3 (for \(\sigma^* g\) instead of \(g\)), using (2.11) and (2.12).

\[\bbox[white,2pt,10pt]{\text{Lemma 2.5. Let } \eta > 0. \text{ We obtain}}\]

\[
(\lambda - \Delta_{x}^{\frac{n}{2}}) \left( \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{2\pi^{n}} \chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) = O \left( e^{-(C(\eta) \sqrt{|\lambda|})} \right)
\]

for \(y \in \overline{M}, x \in S_{2\varepsilon} \setminus S_\varepsilon\) and for \(\lambda \in \Sigma_{\pi - \eta}\) with \(|\lambda| \geq 1\).

**Proof.** By the product rule an easy calculation yields

\[
(\lambda - \Delta_{x}^{\frac{n}{2}}) \left( \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{2\pi^{n}} \chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) = \chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) (\lambda - \Delta_{x}^{\frac{n}{2}}) \left( \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{2\pi^{n}} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) \\
- \Delta_{x}^{\frac{n}{2}} \left( \chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) \right) \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{2\pi^{n}} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \\
- 2 \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{2\pi^{n}} \left( \nabla_{x}^{\frac{n}{2}} \chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right), \nabla_{x}^{\frac{n}{2}} \left( \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) \right).
\]

Using Lemmas 2.3 and 2.4, one obtains for the first term

\[
\chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) (\lambda - \Delta_{x}^{\frac{n}{2}}) \left( \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{2\pi^{n}} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) = O \left( \frac{\sqrt{|\lambda|}^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \right) \\
+ \frac{\sqrt{|\lambda|^{2}} K_{\frac{n}{2}}(C(\eta) \sqrt{|\lambda|} \overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}}.
\]

\[ d(x, \partial M) \left( \sqrt{|\lambda|^{2}} -1 \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|p(x^*, y)})}{p(x^*, y)^{\frac{n}{2}+1}} + \sqrt{|\lambda|^{2}} \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|p(x^*, y)})}{p(x^*, y)^{\frac{n}{2}}} + \sqrt{|\lambda|^{2}+1} \frac{K_{\frac{n}{2}+1}(C(\eta)\sqrt{|\lambda|p(x^*, y)})}{p(x^*, y)^{\frac{n}{2}-1}} \right) + e^{-C\sqrt{|\lambda|\varepsilon}} \]

Since \( d(x, \partial M) \in [\varepsilon, 2\varepsilon] \) is bounded away from 0, Lemma A.2 yields
\[ \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|p(x^*, y)})}{p(x^*, y)^{\frac{n}{2}+1}} \leq \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|\varepsilon})}{\varepsilon^{\frac{n}{2}+1}}. \]

Since \( d(x, \partial M) < 2\varepsilon \) and
\[ K_\alpha(\sqrt{|\lambda|\varepsilon}) = \mathcal{O}(e^{-\sqrt{|\lambda|\varepsilon}}), \]

one concludes that
\[ \chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) (\lambda - \Delta^g_x) \left( \frac{\sqrt{|\lambda|^{2}} -1 \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}p(x^*, y))}{\sqrt{2\pi^n} p(x^*, y)^{\frac{n}{2}}} + \sqrt{|\lambda|^{2}} \frac{K_{\frac{n}{2}}(\sqrt{\lambda}p(x^*, y))}{\sqrt{2\pi^n} p(x^*, y)^{\frac{n}{2}}} + \sqrt{|\lambda|^{2}+1} \frac{K_{\frac{n}{2}+1}(\sqrt{\lambda}p(x^*, y))}{\sqrt{2\pi^n} p(x^*, y)^{\frac{n}{2}-1}} \right) = \mathcal{O}(e^{-C(\eta)\sqrt{|\lambda|\varepsilon}}) \]

for \( \lambda \in \Sigma_{\pi-\eta} \) with \( |\lambda| \geq 1 \). Since \(|\nabla^g_x \rho|_g \) is bounded on \( S_{2\varepsilon} \setminus S_\varepsilon \) and \(|\Delta^g_x \rho|_g \leq \frac{C}{\rho} \) on \( S_{2\varepsilon} \setminus S_\varepsilon \), it follows that
\[ \nabla^g_x \left( \chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) \right) = \chi' \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) \frac{\nabla^g_x \rho(x, \partial M)}{\varepsilon} = \mathcal{O}(1) \]

and
\[ \Delta^g_x \left( \chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) \right) = \chi'' \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) \frac{|\nabla^g_x \rho(x, \partial M)|^2}{\varepsilon^2} + \chi' \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) \Delta^g_x \frac{\rho(x, \partial M)}{\varepsilon} = \mathcal{O}(1). \]

Hence, the second term satisfies
\[ \Delta^g_x \left( \chi \left( \frac{\rho(x, \partial M)}{\varepsilon} \right) \right) \frac{\sqrt{|\lambda|^{2}} -1 \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}p(x^*, y))}{\sqrt{2\pi^n} p(x^*, y)^{\frac{n}{2}}} + \sqrt{|\lambda|^{2}} \frac{K_{\frac{n}{2}}(\sqrt{\lambda}p(x^*, y))}{\sqrt{2\pi^n} p(x^*, y)^{\frac{n}{2}}} + \sqrt{|\lambda|^{2}+1} \frac{K_{\frac{n}{2}+1}(\sqrt{\lambda}p(x^*, y))}{\sqrt{2\pi^n} p(x^*, y)^{\frac{n}{2}-1}} \right) = \mathcal{O}(e^{-C(\eta)\sqrt{|\lambda|\varepsilon}}) \]

for \( \lambda \in \Sigma_{\pi-\eta} \) with \( |\lambda| \geq 1 \). Since
\[ \nabla^g_x \left( \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}p(x^*, y))}{\sqrt{2\pi^n} p(x^*, y)^{\frac{n}{2}}} \right) = \sqrt{\lambda} K'_{\frac{n}{2}-1}(\sqrt{\lambda}p(x^*, y)) \nabla^g_x \frac{p(x^*, y)}{p(x^*, y)^{\frac{n}{2}}} - \left( \frac{n}{2} - 1 \right) \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}p(x^*, y)) \nabla^g_x p(x^*, y)}{p(x^*, y)^{\frac{n}{2}}}, \]

...
\[
\sqrt{\lambda} K_\frac{n}{2}^2 \left( \sqrt{\lambda} \rho(x, y) \right) \nabla_x^\frac{n}{2} \rho(x, y) \\
+ \sqrt{\lambda} K_{\frac{n}{2} - 1} (\sqrt{\lambda} \rho(x, y)) \nabla_x^\frac{n}{2} \rho(x, y) \\
- \left( \frac{n}{2} - 1 \right) K_{\frac{n}{2} - 1} (\sqrt{\lambda} \rho(x, y)) \frac{\nabla_x^\frac{n}{2} \rho(x, y)}{\rho(x, y)^{\frac{n}{2} - 1}} \\
= O \left( e^{-C(\eta) \sqrt{|\lambda|} \varepsilon} \right),
\]
we conclude
\[
\sqrt{\lambda} \frac{n}{2} - 1 \sqrt{2} \pi \langle \nabla g_x \chi(\rho(x, \partial M) \varepsilon), \nabla g_x (K \frac{n}{2}^2 - 1 (\sqrt{\lambda} \rho(x, y)) \rho(x, y)^{\frac{n}{2} - 1}) \rangle = O \left( e^{-C(\eta) \sqrt{|\lambda|} \varepsilon} \right)
\]
for \( \lambda \in \Sigma_{\pi - \eta} \) for \( |\lambda| \geq 1 \). Summing up the claim follows. \( \square \)

Now, we are prepared to show that \( K_{\lambda} \) is approximately a Green’s function for \( \lambda - \Delta^{\frac{n}{2}}_x \).

**Theorem 2.6.** The integral operators \( G_{\lambda} \) satisfy
\[
\| (\lambda - \Delta^{\frac{n}{2}}_x) G_{\lambda} f - f \|_{L^\infty(M)} \leq \frac{C(\eta)}{\sqrt{|\lambda|}} \| f \|_{L^\infty(M)}
\]
for \( \lambda \in \Sigma_{\pi - \eta} \) with \( |\lambda| \geq 1 \), \( \eta > 0 \), and \( f \in C(M) \).

**Proof.** For \( x \in \overline{M \setminus S_{2\varepsilon}} \) Lemma 2.3 yields
\[
\| (\lambda - \Delta^{\frac{n}{2}}_x) G_{\lambda} f - f \|_{L^\infty(M \setminus S_{2\varepsilon})} \\
\leq \sup_{x \in M \setminus S_{2\varepsilon}} \left| \int_M \delta_x(y) f(y) \, dy - f(x) \right| \\
+ O \left( \sup_{x \in M \setminus S_{2\varepsilon}} \sqrt{|\lambda|} \frac{2}{\rho(x, y)^{\frac{n}{2} - 2}} \right) \int_M \frac{K_{\frac{n}{2}} (C(\eta) \sqrt{|\lambda|} \rho(x, y))}{\rho(x, y)^{\frac{n}{2} - 1}} \, dy \\
+ \sup_{x \in M \setminus S_{2\varepsilon}} \sqrt{|\lambda|} \frac{n}{2} - 1 \int_M \frac{K_{\frac{n}{2} - 1} (C(\eta) \sqrt{|\lambda|} \rho(x, y))}{\rho(x, y)^{\frac{n}{2} - 1}} \, dy \\
+ \int_M e^{-C(\eta) \sqrt{|\lambda|} \varepsilon} \, dy \| f \|_{L^\infty(M)} \right) \\
\]
for \( \lambda \in \Sigma_{\pi - \eta} \) with \( |\lambda| \geq 1 \) and \( f \in C(M) \). Therefore, by Lemma A.3 it follows that
\[
\| (\lambda - \Delta^{\frac{n}{2}}_x) G_{\lambda} f - f \|_{L^\infty(M \setminus S_{2\varepsilon})} \leq \frac{C(\eta)}{\sqrt{|\lambda|}} \| f \|_{L^\infty(M)}
\]
for \( \lambda \in \Sigma_{\pi - \eta} \) with \( |\lambda| \geq 1 \) and \( f \in C(\overline{M}) \). For \( x \in S_e \) we obtain by Lemmas 2.3 and 2.4

\[
\| (\lambda - \Delta^g_\lambda) G_{\lambda} f - f \|_{L^\infty(S_e)} \leq \sup_{x \in S_e} \left| \int_M \delta_x(y) f(y) \, dy - f(x) \right| \\
+ \mathcal{O} \left( \sup \sqrt{|\lambda|} \int_M K^\frac{n}{2}_g(C(\eta)\sqrt{|\lambda|} \rho(x, y)) \frac{\rho(x, y)^{n-2}}{\|C\|} \, dy \right) \\
+ \sup \sqrt{|\lambda|} \int_M K^\frac{n}{2}_g(C(\eta)\sqrt{|\lambda|} \rho(x, y)) \frac{\rho(x, y)^{n-1}}{\|C\|} \, dy \\
+ \sup \sqrt{|\lambda|} \int_M K^\frac{n}{2}_g(C(\eta)\sqrt{|\lambda|} \rho(x, y)) \frac{\rho(x, y)^{n-2}}{\|C\|} \, dy \\
+ \sup \sqrt{|\lambda|} \int_M K^\frac{n}{2}_g(C(\eta)\sqrt{|\lambda|} \rho(x, y)) \frac{\rho(x, y)^{n-1}}{\|C\|} \, dy \\
+ \sup \sqrt{|\lambda|} \int_M K^\frac{n}{2}_g(C(\eta)\sqrt{|\lambda|} \rho(x, y)) \frac{\rho(x, y)^{n-2}}{\|C\|} \, dy \\
+ \int_M e^{-C(\eta)\sqrt{|\lambda|} \rho(x, y)} \, dy \|f\|_{L^\infty(M)} \right)
\]

for \( f \in C(\overline{M}) \). Since \( \overline{\rho}(x, y) \) only vanish if \( x, y \in \partial M \) and \( d(x, \partial M) = d(x, \partial M) \leq \overline{\rho}(x, y) \) for \( x, y \) near \( \partial M \), Lemma A.3 and Corollary A.4 imply

\[
\| (\lambda - \Delta^g_\lambda) G_{\lambda} f - f \|_{L^\infty(S_{2e})} \leq \frac{C(\eta)}{\sqrt{|\lambda|}} \|f\|_{L^\infty(M)}
\]

for \( |\lambda| \) and \( f \in C(\overline{M}) \). Moreover, we have for \( x \in S_{2e} \setminus S_e \) by Lemmas 2.3 and 2.5

\[
\| (\lambda - \Delta^g_\lambda) G_{\lambda} f - f \|_{L^\infty(S_{2e} \setminus S_e)} \leq \sup_{x \in S_{2e} \setminus S_e} \left| \int_M \delta_x(y) f(y) \, dy - f(x) \right| \\
+ \mathcal{O} \left( \sup \sqrt{|\lambda|} \int_M K^\frac{n}{2}_g(C(\eta)\sqrt{|\lambda|} \rho(x, y)) \frac{\rho(x, y)^{n-2}}{\|C\|} \, dy \right) \\
+ \sup \sqrt{|\lambda|} \int_M K^\frac{n}{2}_g(C(\eta)\sqrt{|\lambda|} \rho(x, y)) \frac{\rho(x, y)^{n-1}}{\|C\|} \, dy \\
+ \sup \sqrt{|\lambda|} \int_M K^\frac{n}{2}_g(C(\eta)\sqrt{|\lambda|} \rho(x, y)) \frac{\rho(x, y)^{n-2}}{\|C\|} \, dy \\
+ \sup \sqrt{|\lambda|} \int_M K^\frac{n}{2}_g(C(\eta)\sqrt{|\lambda|} \rho(x, y)) \frac{\rho(x, y)^{n-1}}{\|C\|} \, dy \\
+ \int_M e^{-C(\eta)\sqrt{|\lambda|} \rho(x, y)} \, dy \|f\|_{L^\infty(M)} \right)
\]
\[
+ \int_M e^{-C\sqrt{|\lambda|}\varepsilon} \, dy \| f \|_{L^\infty(M)}
\]
for \( f \in C(\overline{M}) \).

Since \( \overline{M} \) is compact, it follows that
\[
\int_M e^{-C(\eta)\sqrt{|\lambda|}\varepsilon} \sqrt{|\lambda|} \, dy \leq \tilde{C}(\eta) \frac{\sqrt{|\lambda|}}{|\lambda|},
\]
for \(|\lambda| \geq 1\). Hence, as a consequence of Lemma A.3 one obtains
\[
\| G_\lambda f - f \|_{L^\infty(S_\varepsilon \setminus S_{\varepsilon/2})} \leq C(\eta) \sqrt{|\lambda|} \| f \|_{L^\infty(M)},
\]
for \(|\lambda| \geq 1\) and \( f \in C(\overline{M}) \). Summing up we conclude that
\[
\| (\lambda - \Delta^g_\lambda) f - f \|_{L^\infty(M)} \leq C(\eta) \sqrt{|\lambda|} \| f \|_{L^\infty(M)}
\]
for \(|\lambda| \geq 1\) and \( f \in C(\overline{M}) \). \( \square \)

Finally, we obtain the main theorem by combining the estimates from Proposition 2.2 and Theorem 2.6.

**Theorem 2.7.** The operator \( \Delta^g_0 \) is sectorial of angle \( \pi/2 \) on \( C(\overline{M}) \).

**Proof.** For \( \lambda \in \Sigma_{\pi - \eta} \) with sufficient large absolute value \(|\lambda| \) Lemma 2.6 implies that
\[
\| (\lambda - \Delta^g) G_\lambda - \text{Id} \| \leq \frac{\tilde{C}(\eta)}{\sqrt{|\lambda|}} < 1,
\]
hence \( (\lambda - \Delta^g) G_\lambda \) is invertible. Therefore,
\[
\text{Id} = (\lambda - \Delta^g) G_\lambda (\lambda - \Delta^g) G_\lambda)^{-1}
\]
and \( \lambda - \Delta^g \) is right-invertible with right-inverse
\[
(\lambda - \Delta^g)^{-1} = G_\lambda (\lambda - \Delta^g) G_\lambda)^{-1}.
\]
Hence, by Proposition 2.1 the operator \( \lambda - \Delta^g \) is invertible and
\[
(\lambda - \Delta^g)^{-1} = G_\lambda (\lambda - \Delta^g) G_\lambda)^{-1}.
\]
In particular, we obtain
\[
\Delta^g G_\lambda (\lambda - \Delta^g) G_\lambda)^{-1} f = \lambda G_\lambda (\lambda - \Delta^g) G_\lambda)^{-1} f - f \in C(\overline{M})
\]
for all \( f \in C(\overline{M}) \). Moreover \( G_\lambda (\lambda - \Delta^g) G_\lambda)^{-1} f \) is a solution of
\[
\begin{cases}
\Delta^g_\lambda u = \lambda u - f, \\
u|_{\partial M} = 0
\end{cases}
\]
for $\lambda \in \Sigma_{\pi-\eta}$ with sufficient large absolute value $|\lambda|$. Since $f \in C(M) \subset L^p(M)$ for every $p \geq 1$, elliptic regularity (cf. [16, Thm. 8.12]) implies $G_\lambda((\lambda - \Delta^g)G_\lambda)^{-1} f \in \bigcap_{p \geq 1} W^{2,p}(M)$. Therefore $G_\lambda((\lambda - \Delta^g)G_\lambda)^{-1} f \in D(A_0)$ and one concludes $R(\lambda, \Delta^g_0) = G_\lambda((\lambda - \Delta^g)G_\lambda)^{-1} f$ for $\lambda \in \Sigma_{\pi-\eta}$ with sufficient large absolute value $|\lambda|$. Thus by Proposition 2.2 it follows that

$$\| R(\lambda, \Delta^g_0) \| \leq \| G_\lambda \| \cdot \| ((\lambda - \Delta^g)G_\lambda)^{-1} \| \leq C(\eta) \frac{|\lambda|}{|\lambda|}$$

for $\lambda \in \Sigma_{\pi-\eta}$ with sufficient large absolute value $|\lambda|$. By [1, Thm. 3.7.11] and [1, Cor. 3.7.17], $\Delta^g_0$ is sectorial of angle $\frac{\pi}{2}$.

3. Strictly elliptic operators with Dirichlet boundary conditions

In this section, we consider strictly elliptic second-order differential operators with Dirichlet boundary conditions on the space $C(M)$ of the continuous functions for a smooth, compact, Riemannian manifold $(M, g)$ with smooth boundary $\partial M$. To this end, take real-valued functions

$$a_j^k = a_k^j \in C^\infty(M), \quad b_j, c \in C(M), \quad 1 \leq j, k \leq n.$$ 

satisfying the strict ellipticity condition

$$a_j^k(q)g^{jl}(q)X_k(q)X_l(q) > 0 \quad \text{for all } q \in M$$

for all co-vector fields $X_k, X_l$ on $M$ with $(X_1(q), \ldots, X_n(q)) \neq (0, \ldots, 0)$ and define on $C(M)$ the differential operator in divergence form with Dirichlet boundary conditions as

$$A_0 f := \sqrt{|a|} \text{div}_g \left( \frac{1}{\sqrt{|a|}} a \nabla^g_M f \right) + \langle b, \nabla^g_M f \rangle + cf$$

with domain

$$D(A_0) := \left\{ f \in \bigcap_{p \geq 1} W^{2,p}(M) \cap C_0(M) : A_0 f \in C(M) \right\}, \quad (3.1)$$

where $a = a_j^k, |a| = \det(a_j^k)$ and $b = (b_1, \ldots, b_n)$.

The key idea is to reduce the strictly elliptic operator on $M$, equipped by $g$, to the Laplace–Beltrami operator on $M$, corresponding to a new metric $\tilde{g}$.

For this purpose, we consider a $(2, 0)$-tensorfield on $M$ given by

$$\tilde{g}^{kl} = a_i^k g^{jl}.$$ 

Its inverse $\tilde{g}$ is a $(0, 2)$-tensorfield on $M$, which is a Riemannian metric since $a_j^k g^{jl}$ is strictly elliptic on $M$. We denote $M$ with the old metric by $M^g$ and with the new metric
by $\tilde{M}^\tilde{g}$ and remark that $\tilde{M}^\tilde{g}$ is a smooth, compact, orientable Riemannian manifold with smooth boundary $\partial M$. Since the differentiable structures of $\tilde{M}^\tilde{g}$ and $M^\tilde{g}$ coincide, the identity

$$\text{Id}: \tilde{M}^\tilde{g} \longrightarrow M^\tilde{g}$$

is a $C^\infty$-diffeomorphism. Hence, the spaces

$$C(\tilde{M}) := C(\tilde{M}^\tilde{g}) = C(M^\tilde{g})$$

coincide. Moreover, [17, Prop. 2.2] implies that the spaces

$$L^p(M) := L^p(M^\tilde{g}) = L^p(M^\tilde{g}),$$
$$W^{k,p}(M) := W^{k,p}(M^\tilde{g}) = W^{k,p}(M^\tilde{g}),$$

for all $p \geq 1$ and $k \in \mathbb{N}$ coincide. We now denote by $\Delta_0^\tilde{g}$ the operator defined as in (2.1) respecting $\tilde{g}$. Moreover, we denote by $\tilde{A}_0$ the operator given in (3.1) for $b_k = c = 0$.

**Lemma 3.1.** The operator $A_0$ and $\tilde{A}_0$ differ only by a relatively bounded perturbation of bound 0.

**Proof.** Consider

$$P f := g^{k\ell} b_k \partial_{\ell} f + cf$$

for $f \in D(A_0) \cap D(\tilde{A}_0)$. Since $D(\tilde{A}_0)$ is contained in $\bigcap_{p>1} W^{2,p}(M)$, Morrey’s embedding (cf. [2, Chap. V. and Rem. 5.5.2]) and the closed graph theorem imply

$$[D(\tilde{A}_0)]^C \ominus C^1(\tilde{M}) \ominus C(\tilde{M}),$$

in particular $D(\tilde{A}_0)$ and $D(A_0)$ coincide. Since $P \in \mathcal{L}(C^1(\tilde{M}), C(\tilde{M}))$ and it follows by (3.3) and Ehrling’s Lemma (see [20, Thm. 6.99]) that $P$ is relatively $\tilde{A}_0$-bounded with bound 0.

**Lemma 3.2.** The operator $\tilde{A}_0$ equals the Laplace–Beltrami operator $\Delta_0^\tilde{g}$ with respect to $\tilde{g}$.

**Proof.** Using (3.2), we calculate in local coordinates

$$\tilde{A}_0 f = \frac{1}{\sqrt{|g|}} a_j \partial_j \left( \frac{\sqrt{|g|}}{|a|} a_j g^{k\ell} \partial_k f \right)$$
$$= \frac{1}{\sqrt{|\tilde{g}|}} \partial_j \left( \frac{\sqrt{|\tilde{g}|}}{|a|} \tilde{g}^{k\ell} \partial_k f \right)$$

for $f \in D(\tilde{A}_0)$, since $|g| = |a| \cdot |\tilde{g}|$.

**Theorem 3.3.** The operator $A_0$ is sectorial of angle $\frac{\pi}{2}$ on $C(\tilde{M})$. 
Proof. By Theorem 2.7 and Lemma 3.2 it follows that $\tilde{A}_0$ generates an analytic semigroup of angle $\frac{\pi}{2}$ on $C(\bar{M})$. Finally, Lemma 3.1 and [11, Thm. III. 2.10] implies the claim.

Remark 3.4. This generalizes [18, Cor. 3.1.21.(ii)] to manifolds with boundary.

By Theorem 3.3, the abstract Cauchy problem (ACP) is well posed. This implies the existence and uniqueness of a continuous solution $u$ of the initial value-boundary problem (IBP), having an analytic extension in a right half space in the time variable. Moreover, $u(t), A_0u(t) \in C^{\infty}(M) \cap C(\bar{M})$ for all $t > 0$.

Corollary 3.5. The resolvents $R(\lambda, A_0)$ are compact operators for all $\lambda \in \rho(A_0)$.

Proof. This follows immediately by (3.3) and [11, Prop. II. 4.25].

We finish this section with the special case of closed manifolds, i.e. $\partial M = \emptyset$. Then, the Dirichlet boundary conditions gets an empty condition. Hence, the operator $A_0$ becomes

$$Af := \sqrt{|a|} \mathrm{div}_g \left( \frac{1}{\sqrt{|a|}} a \nabla^g_M f \right) + \langle b, \nabla^g_M f \rangle + cf,$$

with domain

$$D(A) := \left\{ f \in \bigcap_{p \geq 1} W^{2,p}(M) : A_0 f \in C(M) \right\}.$$

Remark that then $d(x, \partial M) = d(x, \emptyset) = \infty$ and the kernel $K_\lambda$ becomes much easier.

Corollary 3.6. If the manifold $M$ is closed, the operator $A$ generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $C(M)$.

Proof. Since $C^2(M) \subset D(A)$ and $C^2(M) \subset C(M)$ dense, it follows that $A$ is densely defined. Now Theorem 3.3 and [11, Thm. III.4.6] imply that $A$ generates an analytic semigroup of angle $\frac{\pi}{2}$ on $C(M)$. Finally, the compactness of the semigroup follows by Corollary 3.5 and [11, Thm. II.4.29].

Acknowledgements

The author wishes to thank Professor Simon Brendle and Professor Klaus Engel for important suggestions and fruitful discussions. Moreover the author thanks the referee for his many helpful comments.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
Appendix A. Bessel functions

The solutions of the ordinary differential equation
\[ z^2 \frac{d^2}{dz^2} f(z) + z \frac{d}{dz} f(z) = (z^2 + \alpha^2) f(z) \]  \hspace{1cm} (3.4)
for \( z \in \mathbb{C} \) are called modified Bessel functions of order \( \alpha \in \mathbb{R} \). In particular, we have the following.

**Proposition A.1.** The modified Bessel functions of first kind of order \( \alpha \in \mathbb{R} \) are given by
\[ I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k+\alpha}}{\Gamma(k+\alpha+1)k!} \]
for \( z \in \mathbb{C} \setminus \mathbb{R} \), where \( \Gamma \) denotes the Gamma function. Moreover, we obtain the modified Bessel function of second kind of order \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \) by
\[ K_\alpha(z) = \frac{\pi}{2} \cdot \frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin(\pi \alpha)} \]
for \( z \in \mathbb{C} \setminus \mathbb{R} \). If \( \alpha \in \mathbb{Z} \), there exists a sequence \( (\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus \mathbb{Z} \) such that \( \alpha_n \to \alpha \) and \( K_\alpha \) is the limit
\[ K_\alpha(z) := \lim_{n \to \infty} K_{\alpha_n}(z) \]
for \( z \in \mathbb{C} \setminus \mathbb{R} \).

First, we prove an estimate for the modified Bessel function of second kind.

**Lemma A.2.** Let \( \alpha \in \mathbb{R} \) and \( \eta > 0 \). Then, there exists a constant \( C(\eta) > 0 \) such that
\[ |K_\alpha(z)| \leq K_\alpha(C(\eta)|z|) \]
for all \( z \in \Sigma_{\frac{\pi}{2}-\eta} \).

**Proof.** Since \( \text{Re}(z) > 0 \) for all \( z \in \Sigma_{\frac{\pi}{2}-\eta} \) and \( \alpha \in \mathbb{R} \) it follows by [23, p. 181] that
\[ |K_\alpha(z)| = \left| \int_0^{\infty} e^{-z \cosh(t)} \cosh(\alpha t) \, dt \right| \leq \int_0^{\infty} e^{-\text{Re}(z) \cosh(t)} \cosh(\alpha t) \, dt. \]
Note that \( z = |z|e^{i\varphi} \) with \( |\varphi| \in [0, \pi/2 - \eta) \). The monotony of the cosinus implies
\[ \frac{\text{Re}(z)}{|z|} = \cos(\varphi) \geq \cos(\pi/2 - \eta) = \sin(\eta) =: C(\eta) > 0. \]
Using the monotony of the exponential function and the positivity of \( \cosh \), we conclude
\[ \int_0^{\infty} e^{-\text{Re}(z) \cosh(t)} \cosh(\alpha t) \, dt \leq \int_0^{\infty} e^{-C(\psi)|z| \cosh(t)} \cosh(\alpha t) \, dt = K_\alpha(C(\eta)|z|) \]
for all \( z \in \Sigma_{\frac{\pi}{2}-\eta} \).

\[ \square \]
Therefore, we obtain an estimate for the kernel.

**Lemma A.3.** Let $\alpha \in \mathbb{R}$, $k \in [0, \infty)$ and $\lambda \in \Sigma_{\pi - \eta}$ for $\eta > 0$. If $k + \alpha < n$, we obtain

$$\sup_{x \in M} \int_{M} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x, y))}{\rho(x, y)^k} dy \leq C(\eta)\sqrt{|\lambda|}^k |^{k-n}$$

for $|\lambda| \geq 1$.

**Proof.** Remark that

$$\int_{M} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x, y))}{\rho(x, y)^k} dy = \int_{B_R(x)} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x, y))}{\rho(x, y)^k} dy + \int_{M \setminus B_R(x)} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x, y))}{\rho(x, y)^k} dy.$$

For the first term, one obtains

$$\int_{B_R(x)} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x, y))}{\rho(x, y)^k} dy \leq \check{C} \int_{\mathbb{R}^n} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}|y|)}{|y|^k} dy = \check{C}(\eta)\sqrt{|\lambda|} \int_{\mathbb{R}^n} \frac{1}{|y|^k} K_\alpha(|z|) |z|^k d|z| = \check{C}(\eta)\sqrt{|\lambda|}^{k-n} \int_{\mathbb{S}_{n-1}} \frac{K_\alpha(r)}{r^k} \, d\nu_{\mathbb{S}_{n-1}} \, dr = \check{C}(\eta)\sqrt{|\lambda|}^{k-n} \int_{0}^{\infty} K_\alpha(r) r^{n-1-k} dr.$$

Since

$$K_\alpha(r) = \mathcal{O}(r^{-\alpha})$$

for small $r \in \mathbb{R}_+$ and

$$K_\alpha(r) = \mathcal{O}\left(\frac{e^{-r}}{\sqrt{r}}\right)$$

for large $r \in \mathbb{R}_+$, we have

$$r^{n-1-k} K_\alpha(r) = \mathcal{O}(r^{n-1-k-\alpha})$$

for small $r \in \mathbb{R}_+$ and

$$r^{n-1-k} K_\alpha(r) = \mathcal{O}(r^{n-3-k} e^{-r})$$

for large $r \in \mathbb{R}_+$.

Hence, there exists a constant $\check{C} < \infty$ such that

$$\int_{0}^{\infty} K_\alpha(r) r^{n-1-k} dr < \check{C}.$$
and we conclude that
\[
\int_{B_R(x)} K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x, y)) \frac{1}{\rho(x, y)^k} \, dy \leq C(\eta) \sqrt{|\lambda|}^{k-n}.
\]

If \( y \in \overline{M} \setminus B_R(x) \), we have \( \rho(x, y) \geq R \) and therefore
\[
\int_{\overline{M} \setminus B_R(x)} K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x, y)) \frac{1}{\rho(x, y)^k} \, dy \leq \frac{K_\alpha(C(\eta) R \sqrt{|\lambda|})}{R^k} \text{vol}_g(M \setminus B_R(x))
\]
\[
\leq \hat{C}(\eta)e^{-\tilde{C}(\eta)\sqrt{|\lambda|}}
\]
\[
\leq \bar{C}(\eta)\sqrt{|\lambda|}^{k-n}
\]
for \( |\lambda| \) since
\[
K_\alpha(r) = \mathcal{O}\left(\frac{e^{-r}}{\sqrt{r}}\right)
\]
for large \( r \in \mathbb{R}_+ \).

Replacing \( x \) by \( x^* \) this yields an estimate for the reflected kernel.

**Corollary A.4.** Let \( \alpha \in \mathbb{R} \), \( k \in [0, \infty) \) and \( \lambda \in \Sigma_{\pi-\eta} \) for \( \eta > 0 \). Moreover, let \( x \in S_{2\varepsilon} \). If \( k + \alpha < n \), we obtain
\[
\sup_{x \in S_{2\varepsilon}} \int_M K_\alpha(C(\eta)\sqrt{\lambda\bar{\rho}(x^*, y))} \frac{1}{\bar{\rho}(x^*, y)^k} \, dy \leq C\sqrt{|\lambda|}^{k-n}
\]
for \( |\lambda| \geq 1 \).

**REFERENCES**

[1] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-Valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, vol. 96. Birkhäuser (2001).

[2] R. A. Adams. *Sobolev Spaces*, Academic Press, New York-London (1975).

[3] S. Agmon. *On th eigenfunctions and the eigenvalues of general boundary value problems*. Comm. Pure Appl. Math. 25 (1962).

[4] H. Amann. *Linear and Quasilinear Parabolic Problems*, vol. 1. Birkhäuser (2001).

[5] W. Arendt. *Resolvent positive operators and inhomogeneous boundary value problems*. Ann. Scuola Norm. Sup. Pisa 24.70 (2000), 639–670.

[6] T. Binz and K. Engel *Operators with Wentzell boundary conditions and the Dirichlet-to-Neumann Operator*. Math. Nachr. (to appear 2018).

[7] T. Binz *Strictly elliptic operators with Wentzell boundary conditions on spaces of continuous functions on manifolds*. (preprint 2018).

[8] F. Browder. *On the spectral theory of elliptic differential operators I*. Math. Ann. 142.1 (1961), 22–130.

[9] M. Campiti and G. Metafune. *Ventcel’s boundary conditions and analytic semigroups*. Arch. Math. 70 (1998), 377–390.

[10] K.-J. Engel and G. Fragnelli. *Analyticity of semigroups generated by operators with generalized Wentzell boundary conditions*. Adv. Differential Equations 10 (2005), 1301–1320.

[11] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Math., vol. 194. Springer (2000).
[12] K.-J. Engel. *The Laplacian on $C(\overline{\Omega})$ with generalized Wentzell boundary conditions*. Arch. Math. 81 (2003), 548–558.

[13] L. C. Evans. *Partial Differential Equations*, Graduate Studies in Mathematics., vol. 19. Amer. Math. Soc. (1998).

[14] A. Favini, G. R. Goldstein, J. A. Goldstein, E. Obrecht, and S. Romanelli. *Elliptic operators with general Wentzell boundary conditions, analytic semigroups and the angle concavity theorem*. Math. Nachr. 283 (2010), 504–521.

[15] A. Favini, G. R. Goldstein, J. A. Goldstein, and S. Romanelli. *The heat equation with generalized Wentzell boundary condition*. J. Evol. Equ. 2 (2002), 1–19.

[16] Gilbarg, D. and Trudinger, N. S. *Elliptic partial differential equations of second order*, Classics in Mathematics. Springer (2001).

[17] E. Hebey. *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Lecture Notes. Amer. Math. Soc. (2000).

[18] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser (1995).

[19] M. Rudin. *Real and Complex Analysis*, Higher Mathematics Series, vol 3. McGraw-Hill (1986).

[20] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations*, Texts in Appl. Math., vol 13. Springer (1993).

[21] R. T. Seeley. *Extension of $C^\infty$ functions defined in a half space*. Proc. Amer. Math. Soc. 15 (1964), 625–626.

[22] B. Stewart. *Generation of analytic semigroups by strongly elliptic operators*. Trans. Amer. Math. Soc. 199 (1974), 141–161.

[23] G. N. Watson *A Treatise on the Theory of Bessel Functions*, Cambridge University Press (1995).

Tim Binz  
Department of Mathematics  
University of Tübingen  
Auf der Morgenstelle 10  
72076 Tübingen  
Germany  
E-mail: tibi@fa.uni-tuebingen.de