ON SINGULAR POISSON STERNBERG SPACES

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Abstract. We obtain a theory of stratified Sternberg spaces thereby extending the theory of cotangent bundle reduction for free actions to the singular case where the action on the base manifold consists of only one orbit type. We find that the symplectic reduced spaces are stratified topological fiber bundles over the cotangent bundle of the orbit space. We also obtain a Poisson stratification of the Sternberg space. To construct the singular Poisson Sternberg space we develop an appropriate theory of singular connections for proper group actions on a single orbit type manifold including a theory of holonomy extending the usual Ambrose-Singer theorem for principal bundles.

1. Introduction

In this paper we consider the problem of cotangent bundle reduction for a proper action of a Lie group on a manifold with the simplifying assumption that the base manifold on which the group acts consists of just one orbit type. This is a major simplifying assumption to the more general problem where there are multiple orbit types on the base manifold. However, the resulting theory is already interesting and leads to a generalization of the theory of connections on a principal bundle. We will motivate the theory with a class of examples generated by homogeneous spaces where a group $G$ acts on $G=H$ and then on its cotangent bundle by the lifted action. Since this is a transitive action it is clear that there is just one orbit type. This example will appear later as a fundamental ingredient of the theory.

After reviewing some preliminary results on cotangent bundle reduction in Section 2, we begin, in Section 3, with the transitive case of a base manifold that is a homogeneous space. We consider the quotient of $G=H$ by the action of the group $G$. Using commuting reduction by stages, we show that this is a Poisson stratified space with strata determined by the coadjoint induced action of $H$ on $g$. If we denote an element of the isotropy lattice for this action by $(K)$, then the Poisson strata are given by $h_{\mathfrak{k}}=H$. The Poisson bracket on each stratum is induced by the Lie-Poisson structure on $g$. We also show that the symplectic leaves of each stratum are given by $(\mathfrak{o} \setminus h_{\mathfrak{k}})=H$. Notice that this result is the singular generalization of the Lie-Poisson structures and coadjoint orbits, in the sense that if the action is free, then $H=\mathfrak{e}$ and Poisson and symplectic reduction produce the Lie-Poisson structure and the Kostant-Kirillov-Souriau (KKS) structure on the coadjoint orbits respectively.

Significantly, this particular case of singular cotangent bundle reduction does not require a connection precisely because the action on the base manifold is transitive so every tangent vector on the G-principal bundle $G=H$ if $g$ is vertical and

1991 Mathematics Subject Classification. Primary 53E20; Secondary 53D17, 57N80.

Keywords and phrases. Symplectic geometry, stratified spaces, reduction, momentum maps, cotangent bundles.
therefore every nonzero covector has nonzero m on entum, i.e. the splitting of the cotangent bundle into zero m on entum and nonzero m on entum covectors is trivial.

Next we consider the non-transitive case. We need to first split the tangent bundle of the base manifold into vertical and horizontal distributions. This can be done with a G-invariant metric guaranteed by the properness of the action. The next step, undertaken in Section 4, is to associate a connection to this splitting.

At first glance this can't be done in the usual way since a surjective mapping from the tangent space at any point to the Lie algebra will have kernel with dimension equal to the codimension of the algebra. On the other hand, the horizontal spaces have codimension equal to dim g - dim g_2 where g_2 is the nontrivial stabilizer algebra at the point m. The resolution is to form a vector bundle \( M \), thanks to the properness of the action, whose fibers are isomorphic to g_2 and denote a connection, which we call a singular connection, as a surjective bundle map covering the identity from TM to \( M \). This leads to an invariant splitting of the tangent bundle.

With this in place we study the orbit type stratification of TM and prove, in Theorem 3, that the isotropy lattice is determined by the action of \( h \) on g-h where h is the stabilizer algebra at some point \( m \in M \). Relative to the splitting induced from the connection, we explicitly determine the strata of TM and its quotient \( (TM)/G \) and we also write down a stratified version of the Atiyah sequence for a principal bundle.

In 4.3 we introduce the curvature of the singular connection. Since the singular connection is an Ehresmann connection, we can use the curvature theory for an Ehresmann connection rather than attempt to define an exterior derivative of a bundle map. Using this as a starting point, we are able to prove, in Proposition 5, that the curvature is a G-equivariant bundle map \( \nabla TM \) that takes values in the stratum of \( TM \) that contains the zero section.

In addition, in Theorem 6 we prove an Ambrose-Singer theorem for the singular connection which demonstrates that the holonomy group at a point \( m \in M \) is contained as a subgroup in N (\( h \)) = H where H is the stabilizer of m. Along the way to doing this, we obtain a one-to-one correspondence between singular connections on \( M \) and principal connections on the bundle \( M \times G \). This confirms our study of the geometry of the singular connection.

In Section 5, we apply the singular connection to the construction of a connection dependent realization of the symplectic structure on the reduced spaces \( J^1(O)=G \), where \( O \in G \) is a chosen coadjoint orbit. In the image of the m on entum map \( J : TM \) associated to the cotangent lifted action of G (given by \( \mathbb{H}_i(m) \); \( i = h \in H \), \( \mathbb{H}(m) \)). The construction follows the one for the Steinberg space when the action is free \([10,11]\). This is the content of Theorem 7, which shows that the singular reduced Steinberg space is a bundle over T (\( M \times G \)) whose symplectic basis are \( (O \setminus h_{g \in G}) = H \) which are shown, in Section 3, to be the symplectic leaves of the Poisson strata of h = H.

Finally, in Section 6 we compute the full Poisson stratification of (\( TM \)) = G in the Steinberg representation. The strata of \( (TM) = G \) are determined by the H-isotropy lattice of h. We show in Theorem 8 that, using the singular connection, we can realize each stratum as a bundle over T (\( M \times G \)) with fibers isomorphic to the Poisson strata in the homogeneous Lie-Poisson problem, that is \( h_{g \in G} = H \). The Poisson bracket obtained on this space and it generalizes the gauge Poisson bracket.
in the free theory \([6,7]\). The bracket consists of a canonical term associated to the canonical symplectic structure on \(T(M=\mathbb{G})\), a coupling term that involves the reduced curvature of the singular connection and a term involving the homogeneous Lie-Poisson structure on the bases \(h_{(\xi)}=H\).

The theory developed for the problem with a single orbit type will play an important role in the solution to the general problem of singular cotangent bundle reduction for base manifolds admitting multiple orbit type which is the subject of a forthcoming paper [3].

Finally we remark that a related approach to the problem studied in this paper carried out in [2], following the alternative realization of \((T(M=\mathbb{G}) due to W. Einstein [11] in the free case.

Acknowledgments. M. Perlmutter wishes to acknowledge the generous support of the Bernoulli Center, where part of the research for this paper was completed.

2. Background and preliminaries

2.1. Proper actions with single orbit type. Let \(M\) be a single orbit type manifold with respect to the proper action of the Lie group \(G\). Thus \(M = M_{\{G\}}\) (where \(M_{\{G\}} = \{m \in M : \mathbb{G} \cdot m = g \mathbb{G} \cdot m \text{ for some } g \in \mathbb{G}\}) for some compact subgroup \(\mathbb{G}\). It is well known (see [3]) that the orbit space \(M_{\{G\}} = G\) is then a smooth manifold. One way to see this is to consider the smooth submanifold \(M_{G}\) of \(M\) consisting of the points in \(M\) with stabilizer precisely equal to \(H\). It is easy to see that the subgroup \(N(H)\), the normalizer group of \(H\) in \(G\), acts on \(M_{G}\) and that every orbit in \(M\) intersects \(M_{G}\) on an \(N(H)\) orbit. Furthermore, the quotient group \(N(H) = H\) acts freely on \(M_{G}\) and generates the same orbit space. Therefore we have

\[ M_{\{G\}} = G \quad M_{G} = (N(H) = H) \]

and the right hand side is the base space of the principal \(N(H)=H\)-bundle \(M_{G} \quad M_{G} = (N(H) = H)\). We will see that the subgroup \(N(H)\) plays a crucial role in the geometry of connections that we are going to define.

2.2. The Sternberg space for a free action. Here we recall an important realization of the Poisson reduced space \((TQ=\mathbb{G})\) obtainable once a principal connection \(A\) on the principal bundle : \(A = \mathbb{G}\) is fixed. (To distinguish the free case from the singular one, we denote the manifold on which \(G\) acts freely by \(Q\) instead of \(M\)). The connection allows us to realize the reduced space as a fiber bundle over the reduced cotangent bundle \(T(Q=\mathbb{G})\). Detailed proofs of the results in this section are found in [5].

The construction of the Sternberg space proceeds in two steps. First one pulls back the configuration space bundle : \(Q = \mathbb{G}\) by the cotangent bundle projection \(\pi : T(Q=\mathbb{G})\) \(Q = \mathbb{G}\) to obtain the \(G\)-principal bundle

\[ Q = f([\xi];g) T(Q=\mathbb{G}) = \{(\xi);\pi \quad \pi \mathbb{G} \}

over \(T(Q=\mathbb{G})\) with fiber over \([\xi]\) isomorphic to \((\xi)(H)\). Recall that the \(G\)-action on \(Q\) is given by \(g((\xi);g) = (\xi)(g^{-1})\) for any \(g \in \mathbb{G}\) and \((\xi);g\) \(Q\).

The diagram defining this pullback bundle and the associated maps is given by
A bus is not at 1 on we will note the ber projection

\[ \mathcal{Q} \xrightarrow{\sim} Q \]

where \( \sim \) are the projections onto the rst and second factors respectively. The following fact about \( \mathcal{Q} \) will be often used in the sequel.

\( \mathbf{P} \) roposition 1. \( \mathcal{Q} \) is also a vector bundle over \( Q \) isomorphic to the annihilator \( V(Q) \) of the vertical bundle \( V(Q) \) of \( TQ \). The bers of these vector subbundles are given by \( V(Q)_q = \ker T_q \) and \( V(Q)_q = f_{q} 2 \) \( T_qQ \) for each \( q \in Q \). Consequently, \( \mathcal{Q} \) is bundle isomorphic to \( J^1(0) \).

The second step is to form the coadjoint bundle of \( \mathcal{Q} \), that is, the associated vector bundle to the \( G \)-principal bundle \( \mathcal{Q} \) given by the coadjoint representation of \( G \) on \( q \). The Sternberg space, denoted by \( S \), is thus defined by

\[ S = G \circ q : \]

Abusing notation we will denote the ber projection \( \mathcal{Q} \) with the same symbol as the quotient map \( \mathcal{Q} \). Using the connection \( A \) we then construct the bundle isomorphism to the Poisson reduced space \( (TQ)_G \) as follows.

\( \mathbf{P} \) roposition 2. The map \( 'A : \mathcal{Q} \) given by

\[ 'A \quad (g q) = T_q (Tq) + A(q) \]

is a \( G \)-equivariant vector bundle isomorphism over \( Q \). It descends to a vector bundle isomorphism over \( Q = G \)

\[ 'A : S \to (TQ)_G = G \]

The gauge Poisson bracket on \( S \) is the pullback by \( 'A \) of the reduced Poisson structure on \( (TQ)_G \). In order to study \( 'A \) we rst introduce the necessary notions of horizontal lifts and covariant derivatives in the context needed for our purposes. First one constructs the horizontal lift on the \( G \)-bundle \( \mathcal{Q} \) endowed with the connection from \( \mathcal{A} \) that is \( A \). Given a curve \( (g) \) in \( Q = G \), one has \( q_t, q_t \), the horizontal lift of \( q \) of the curve \( q_t = G (g_t) \) relative to \( A \). Then the curve \( (g_t) \) lies in \( \mathcal{Q} \), is horizontal (relative to \( A \)) and covers \( q_t \). Denoting horizontal lift operators by \( \text{hor} \), it follows that

\[ \text{hor}(g_t) = T(\text{hor}) = T(G) \]

Now, \( S \) is an associated bundle to \( \mathcal{Q} \), therefore, for \( s = [l_g] \in g \),

\[ \text{hor}(v) = T(TQ)_G = T(2TQ)_G \]

\[ \text{hor}(v) = T(TQ)_G = T(2TQ)_G \]
where \( Q \rightarrow g \) is the orbit projection. Finally, for \( f, g : C^1(S) \) and \( s, t \in S \), define \( d^f_{\alpha} f(s) T^1_{\alpha}(s) T^0_{\alpha}(Q = G) \) by

\[
d^f_{\alpha} f(s) v_{\alpha} = df(s) h_{\alpha} v_{\alpha},
\]

Denote the curvature of the connection \( A \) by \( \text{Curv}_A \). The reduced curvature form is a bundle map from \( \wedge^2(T(Q = G)) \) to the adjoint bundle, \( g = Q \rightarrow g \). Recall that the adjoint bundle \( g \) is defined as the quotient \( g = (Q \rightarrow g)/G \) relative to the diagonal left \( G \)-action \( (g;T) := (g,g \cdot d) \) on \( Q \rightarrow g \), where \( g \in G, Q \neq Q \), \( g \neq g \), and \( A_d \) is the adjoint representation of \( G \) on \( g \). The adjoint bundle is a Lie algebra bundle with base \( Q = G \), that is, each fiber has a Lie algebra bracket depending smoothly on the base. The reduced curvature is then defined by

\[
B[\{g\};(u_1,v_1)] \in \{g\} \text{Curv}_A(u_1,v_1);
\]

where \( u_1,v_1 \in T_Q Q \) are arbitrary vectors satisfying \( T_Q u_1 = u_1, T_Q v_1 = v_1 \) respectively and \( \{g\} \) is the \( G \)-class through \( \{g\} \).

The reduced gauge Poisson bracket is then given by the following result.

**Theorem 1.** Let \( s = [(g);Q] \in 2 \mathcal{S} \) and \( v = [(g);Q] \in 2 \mathcal{G} \). The Poisson bracket of \( f;g : C^1(S) \) is given by

\[
f f; g g(s) = g_{=0} (f; g) = \frac{d^f_{\alpha} f(s)}{d^g_{\alpha} g(s)} E + v; B[\{g\};(u_1,v_1)] = \frac{d^f_{\alpha} f(s)}{d^g_{\alpha} g(s)} E + \frac{v}{s} \frac{g}{s};
\]

where \( g_{=0} \) is the canonical symplectic form on \( T(Q = G), B(1) : \wedge^2(T(Q = G);g) \) is the \( g \)-valued two-form on \( T(Q = G) \) given by

\[B = g_{=0} B;\]

with \( B(1) : \wedge^2(Q = G;g) \) defined in \[2.2, \] \( : T(T(Q = G)) \), \( : T(Q = G) \) is the vector bundle isomorphism induced by \( g_{=0} \), and \( f(s) \) is the usual derivative of \( f \) at the point \( s \in 2 \mathcal{S} \), that is,

\[s^0; f = \frac{df}{ds}_{s=0} \]

for any \( s^0 = [(g);Q] \in 2 \mathcal{S} \).

2.3. MMM in all coupling form. S. The symplectic leaves of the Sternberg space are given by the submanifolds in \( S \) of the form \( Q \rightarrow G \), where \( G \rightarrow G \) is the cotangent orbit through \( G \). To describe the symplectic forms on these spaces we need to recall the minimal coupling form due to Sternberg [10], which is a functorial construction of a presymplectic manifold associated to a principal bundle with a connection and a Hamiltonian \( G \)-space.

Let \( Z \rightarrow B \) be a left principal \( G \)-bundle over the symplectic manifold \( (B;\mathcal{G}) \), \( L \rightarrow L(1) \) a connection one-form on \( Z \), \( (F;!) \) a Hamiltonian \( G \)-space with equivariant momentum map \( : F \rightarrow G \), and denote by \( Z : F \rightarrow Z \) and \( F : F \rightarrow F \) the two projections.

It can be shown that the closed two-form \( \{h,F \}_{F}^{g} \) descends to a closed two-form \( \{L,F \}_{F}^{g} \), that is, \( \{L,F \}_{F}^{g} \) is characterized by the relation

\[\{L,F \}_{F}^{g} = \{h,F \}_{F}^{g} - \{Z,F \}_{F}^{g} ;
\]

where \( Z : F \rightarrow Z \rightarrow g F \) is the projection to the orbit space.
Now denote by \( F : Z \to F \) the associated bundle projection given by \( F (z; f) = (z) \). Then \( F \) is also a closed two-form on \( Z \to F \) and one gets the minimal coupling presymplectic form \( !^1 + F \). In general, this presymplectic form is degenerate, but in the crucial case below it is in fact a reduced symplectic form.

2.4. Symplectic leaves of the Sternberg Space. Let us apply the previous construction to the situation \( Z = \mathcal{Q}; B = T \mathcal{Q}, \quad q = q \) (the canonical symplectic form on the cotangent bundle \( T (\mathcal{Q} = \mathbb{G}) ) \), \( \mathcal{Q} = \mathcal{G} \) \( !^1 \in \mathcal{Q} \). Then \( \mathcal{Q} = \mathcal{G} \) \( !^1 \) in this situation by \( !^1 \) and hence it is uniquely characterized by the relation

\[
\begin{align*}
D &= E \\
\mathcal{Q} = \mathcal{G} \end{align*}
\]

The minimal coupling form in this situation is

\[
\begin{align*}
!^1_{\text{min}} &= !^1_{\text{min}} + (\mathcal{Q} = \mathcal{G}) \\
&= !^1_{\text{min}} + (\mathcal{Q} = \mathcal{G})
\end{align*}
\]

We then have the following theorem that says that the minimal coupling form coincides with the reduced symplectic form on the leaves of the Sternberg spaces.

Theorem 2. The symplectic leaves of the Sternberg space \( (\mathcal{Q} ; f; \mathcal{G}) \) are given by \( (\mathcal{Q} ; \mathcal{Q} = \mathcal{G}; !^1_{\text{min}}) \) where \( \mathcal{Q} = \mathcal{G} \) is a coadjoint orbit of \( \mathcal{G} \). The minimal coupling two-form \( !^1_{\text{min}} \) is the reduced symplectic form on the leaf obtained by orbit reduction, i.e., \( !^1_{\text{min}} \) given in equation (2.4) is the unique two-form on \( \mathcal{Q} \to \mathcal{Q} \) that satisfies

\[
!^1_{\text{min}} = !^1_{\text{min}} + (\mathcal{Q} = \mathcal{G})
\]

where \( !^1_{\text{min}} \) and \( \mathcal{Q} = \mathcal{G} \) is the inclusion of \( \mathcal{Q} = \mathcal{G} \) into \( \mathcal{Q} = \mathcal{G} \).
Proof. Let \( f \in \mathfrak{c}^H (g) \). Then its Hamiltonian vector \( \mathfrak{h} \) evaluated at \( 2h \) is \( \text{ad}_\mathfrak{h} f \). Let \( 2h \). Then

\[
\mathfrak{h} f; g (2h) = \mathfrak{h} f (2h) = \mathfrak{h} f (\mathfrak{h} g) = \mathfrak{h} f (gg) = \mathfrak{h} (gg) = \mathfrak{h} g g
\]

since \( f \) is \( H \)-invariant. Thus \( \text{ad}_\mathfrak{h} 2h = T_{\mathfrak{h}} \), so \( h \) is left invariant by the Hamiltonian vector \( f \). Hence \( C^H (g) \) is a reduced Poisson manifold and \( C^H (h) \) is a reduced Poisson subalgebra. Since the algebra of smooth functions on \( H \) is a singular reduced space is defined precisely as \( C^H (h) \), we have that there is a reduced Poisson bracket on \( C^H (h) \) given by

\[
\{g; h\} = \mathfrak{F} \mathfrak{g} \mathfrak{F} \mathfrak{g} \mathfrak{F}
\]

where \( [ \] \) \( 2h = H, f; g 2C^H (h) = g (g) \) and \( f; g \) are smooth \( H \)-invariant extensions to \( C^H (g) \) off and \( g \) respectively. Then, in view of Lemma 2, and since for any orbit type \( h_{K} = H \) smooth, this reduced Poisson algebra restricts to a reduced smooth Poisson structure on the smooth stratum \( h_{K} = H \) of \( H \), making it a Poisson manifold.

In the case of a \( G \)-homogeneous space \( M \) we can identify \( M = G \) where \( H \) is a compact isotropy group and we take the quotient to be for the right action of \( G \) on \( G \). Then \( G \) acts on the left on \( G = H \) according to \( g \), \( g^{-1} = [gg] \). It is clear that the stabilizer group of the point \( g \) is then \( gH g^{-1} \) and therefore \( M = M_{g} \).

We next consider the problem of singular symplectic reduction for the cotangent lifted action \( G \times T (G = H) \) \( T (G = H) \). The bulk of this paper is devoted to obtaining gauge realizations of the singular reduced Poisson and symplectic spaces for more general singular orbit type base manifolds. We will see that this particular example of a homogeneous space will appear and play a role analogous to that of a coadjoint orbit in the free case. In this sense the symplectic reduction of the homogeneous space is precisely the correct generalization in the singular setting of a coadjoint orbit in the free case, which, recall, is obtained by the regular symplectic reduction for the action \( G \times T G \times T G \), which is the cotangent lift of the left translation of \( G \) on itself.

We can carry out the reduction using the technique of commuting reduction by stages (see example 3). We consider the left action of \( G \) on \( T G \), which is the cotangent lift of the action on \( G \) given by

\[
(g; h) \mathfrak{F} g = gg \mathfrak{F} h^{-1}
\]

It is then clear that the restricted actions of the two subgroups of \( G \), \( H \), \( G \), and \( H \) commute. While the total action is not free, the restricted actions are free actions. Consider reduction at the moment value \( (0) 2g h \). We first reduce by the \( H \)-action at zero moment to obtain \( T (G = H) \), equipped with the canonical symplectic form, by the regular cotangent bundle reduction theorem at zero moment (see [1]). The remaining action is then given by
We know by the Sjamaar-Lehm an theory \cite{9} of symplectic stratifications that the symplectic quotient $J_H^{-1}(0) = H$ is a stratified topological space with symplectic strata given by $(J_H^{-1}(0))_\kappa = H$ with $\kappa$ where the $K$ are the stabilizer subgroups determined from the action $H \times h \rightarrow h$ (which is just the coadjoint action restricted to the subgroup $H$) and $\kappa$. In other words the image of this momentum map takes values that satisfy $0 \nabla h = 0$.

Remark 1. The symplectic structure on the strata, $(0 \nabla h)_\kappa = H$ is given by the quotient of the restriction of the symplectic form of the coadjoint orbit. Therefore, denoting this structure by $!_{(\kappa)}$ one has the following formula

\begin{equation}
(0 \nabla h)_\kappa = !_{(\kappa)}
\end{equation}

where $(0 \nabla h)_\kappa : (0 \nabla h)_\kappa = H$, $!_{(\kappa)} : (0 \nabla h)_\kappa \rightarrow 0$ and $!_{(\kappa)}$ is the ( ) KKS symplectic structure on $O$. From our construction it follows that these symplectic strata are also symplectic leaves of $h_{(\kappa)} = H$ for the smooth Poisson structure induced by the Poisson structure \cite{5.1} in $h = H$. We will refer to the structures defined by \cite{3.1} and \cite{3.2} by homogeneous Lie-Poisson bracket and homogeneous KKS form, respectively.

4. Singular Connections

We wish to extend the concept of a principal connection to the setting of a single orbit type manifold. In the rest of the paper we will have $M = M_{(\kappa)}$. Let $G \rightarrow M$ be a proper action. Then each $\kappa$ has isotropy group $G_{\kappa}$ conjugate to $H$ in $G$ with Lie algebra $g_{\kappa}$. We next need to generalize the target space for a connection in this singular setting as we no longer have a fixed Lie algebra, but rather a family of spaces $g_{\kappa}$ for each $\kappa$. We must project $g_{\kappa}$ onto which the connection must project.

We denote by $G = G_{m \times 2}$ and for $G$ we can prove that this space is a vector bundle as follows.
Proposition 3. The set $m$ is a smooth vector bundle over $M$. We call it the stabilizer bundle over $M$.

Proof. This is a simple application of the tube theorem. We must show that $m$ is locally trivializable. Fix $m \in M$. Without loss of generality we can assume that $G_m = H$. Let $S_m = T_m G \cap m$ with respect to some $G$-invariant metric on $M$ (available by the properness of the action). Then $S_m$ is then a linear slice for the $G$-action at $m$ and $\exp_m$ is an $H$-equivariant diffeomorphism from an open ball $B$ containing the origin in $S_m$ to an $H$-invariant submanifold transverse to $G_m$ at $m$. We have that $G \times B = U$ given by

$$\{ (g;v) = g \exp v :$$

is a $G$-equivariant local diffeomorphism onto a $G$-invariant neighborhood of the orbit $G \cdot m$. Note that $\{ e;0 \} = x$. Choose now a $H$-invariant Riemannian metric on $G$ and split $g = T_g G = g h k$. Then $k = h^\gamma$ is a slice at the identity for the free $H$-action on $G$, and therefore we can use again the tube theorem to construct a local $H$-diffeomorphism $H \cdot 0 \to U$ onto a neighborhood of $H$ in $G$. Here $0$ is a sufficiently small open ball around $0$ in $k$. This diffeomorphism is explicitly given by $(h;k) \mapsto h \exp k$, where $\exp_k$ is the associated Riemannian exponential on $G$, for which $0$ lies inside its domain of injectivity. This diffeomorphism drops to a diffeomorphism $H \cdot 0 = H \cdot 0 \to G \cdot 0 \to G = H$ where $O^0$ is a neighborhood in $G = H$ containing $[e]$. We can therefore identify each element $[g]$ of $O^0$ with $\exp_k$ for some $k \in O^0$. Shrink $O^0$ if necessary so that it becomes a trivializing neighborhood for the associated bundle $G \times H \times B$ over $O^0$ and call the induced trivializing bundle $C$. Thus $U = M$ is a (not invariant) neighborhood of $x$. Finally, we can construct a trivialization of $m$ over $U$, as follows: $f : U \to M$ is given by

$$f(\exp_k \exp s) = (\exp_k \exp s; A \exp_k x).$$

Next, we form the bundle that will play the role of the Lie algebra in the standard theory of connections on principal bundles. The fibers of this bundle, over each point in $M$, should be isomorphic to the tangent space of the group orbit at that point. The natural candidate for this fiber over $2M$ is simply $g \cdot g$. Consider the trivial bundle over $M$, $M \times g$. We then have the following injective inclusion of vector bundles over $M$ covering the identity on the base $m / M \times g$.

Definition 1. Let $M$ be the quotient bundle defined by,

$$m \to M \times g.$$

Note that the fiber over $m \times 2M$ is simply $m = g \cdot g$. We then have the following properties of the vector bundle $m$.

Proposition 4. The vector bundle satisfies:

1. $\hat{g} = M \times g = h$, i.e. the restriction to the constant stabilizer manifold $M_H$ is a trivial bundle.
There is a smooth action of the group $G$ on $M$ which is linear on the fibers and covers the $G$-action on $M$. This action is defined by

\[(1)\quad g \cdot [m] = Ad_g \cdot \frac{1}{h} \cdot m:\]

(3) With respect to this action, $M$ is a saturated vector bundle, i.e.

\[G = \text{ad}_H: \]

Proof. For (1), because the isotropy algebra for any point $m \in M_H$ is just $h$, the stabilizer bundle $m$ restricted to $M_H$ is the trivial bundle, $m \frac{1}{h} = M_H \cdot h$; and therefore

\[m \frac{1}{h} = \frac{M_H}{M_H \cdot h} g = M_H \cdot g = h: \]

For (2), it is clear that if equation (1) is well defined, then it is an action. To see that it is well defined, choose another representative $0$ of the class $\frac{1}{h}$, so that $0 = g \cdot 2g\cdot 0 = Ad_g \cdot 0 = Ad_g \cdot 0 = Ad_g \cdot 0$. But, $Ad_g \cdot 0 = g$ and therefore $\frac{1}{h} \cdot m = \frac{1}{h} \cdot m$ as required. To prove (3), it is clear that since $M = G \cdot M_{H_H}$, any point $m \in M$ can be written as $m \cdot g = g \cdot m$ for some $g \in M_{H_H}$ and therefore since $g \cdot m = A \cdot d_g \cdot g = g \cdot h$.

Remark 2. When the action is free, $h = 0$ and therefore the stabilizer bundle $m$ is 0 so that $m = M \cdot g$ and therefore its quotient $\cdot G$ is the adjoint bundle $g = \frac{M_H}{M_H \cdot h}$.

We next define the main object of this section, a singular connection for single orbit type manifolds.

Definition 2. A singular connection $A$ for the single orbit type manifold $M$ is a smooth surjective vector bundle map $A : TM \rightarrow M$, covering the identity, with the properties:

(i) $A$ is $G$-equivariant: $A(g \cdot v) = g \cdot A(v)$ for any $v \in TM$.

(ii) For all $g$, $A (m (tm)) = \frac{1}{h}$.

Consequently for each $m \in M$, ker $A (m)$ is a complement to $g \cdot m \in TM$ and together form a $G$-invariant subbundle $H M$ of $TM$. Such connections always exist with our assumption of a proper action, since it is well known (see [3]) that, assuming the action is proper, there exists a $G$-invariant Riemannian metric on $M$. Using this metric, we can simply declare the horizontal space at a point $m \in M$ to be $H_m := \{ g \cdot m \}$. It is clear that these spaces form a subbundle of $TM$ invariant under the $G$-action, and satisfy $TM = H (M) \vee (TM) \cdot H (M)$. We obtain the following result.

4.1. Stratification of $TM$. In this section we use the singular connection to determine the orbit type stratification for the tangent lifted action $G \cdot TM \rightarrow TM$. We will obtain an analogous result when we dualize the action for the cotangent bundle. In studying the strata of the tangent lifted action we can use the connection to reduce the problem to studying the strata for the $G$-action on $M$. We obtain the following result.

Theorem 3. The isotropy lattice for the action of $G$ on $TM$ is one-to-one correspondence with the lattice determined by the $\text{Ad}$-induced action

$H \cdot g=h \iff g=h$. 


Let A be a singular connection on M. Then we have a connection dependent G-equivariant di eomorphism \( A : TM \to H(M) \) such that, for each \((k)(h)\) in the previous isotropy lattice it restricts to an equivariant di eomorphism

\[
A |_{TM_{(k)}} : (TM)_{(k)} \to H(M)_{(k)};
\]

where \(H(M)\) is the horizontal subbundle in TM determined by \(K\) and \((k)\) refers to the conjugacy class of \(K\) in \(G\).

Since \(A\) is \(G\)-equivariant, then \(A : TM \to H(M)\) is a stratified morphism respecting the orbit type strata of TM and \(H\). Also, \((k) = G\) is a smooth fiber bundle over \(M = G\) with typical fiber isomorphic to \((g = h)_{k} = H\). Finally, \((TM) = G\) has the structure of a stratified vector bundle over \(M = G\) with smooth strata \((TM)_{(k)} = G\) isomorphic to

\[
T M = G \quad TM_{(k)} = G \quad H(M)_{(k)} = M;
\]

since \(G\) acts on \(TM\) by tangent lifts and the base action satisfies \(G = M = M\). Because of these saturations, it is enough to study the strata of the fiber over some \(m \in 2M_{H}\) since the strata over any other fiber will be \((g)\)-translations. So, we restrict our attention to the diagonal \(H\)-action on \(H(M)_{m}\) and notice that \(H(M)_{m}\) is isomorphic to a linear slice for the \(G\)-action on \(M\) at the point \(m\). Since \(M\) is a single orbit type manifold, the \(H\)-action on the linear slice can only have one orbit type, and therefore the entire space must be运送 by \(H\) since \(H\) acts on the base point. Therefore, since the \(H\)-action is diagonal, we have reduced the study of the strata of TM to the study of the strata of the \(H\)-action on \(m\). Recall that \(g = h\) and the action is given by \(g \cdot h = (g \cdot h)_{m}\) since \(H\) acts on the base point. It follows that if we denote by \((k), (h) = (H)\), the elements of the isotropy lattice for the \(H\)-action, \((g = h)_{m}\) is isomorphic to \((g = h)_{m}\) and then this lattice is in one-to-one correspondence with the isotropy lattice for the \(G\)-action on \(H(M)\).

Furthermore, since \(A : TM \to H(M)\) is a \(G\)-equivariant bundle isomorphism, \(A\) restricts to a smooth iso eomorphism

\[
(TM)_{(k)} : H(M)_{(k)}.
\]

From this isomorphism, it is now clear that the map \(A : TM \to H(M)\) is a stratified morphism mapping each orbit type stratum \((TM)_{(k)}\) onto the orbit type stratum \((H)_{(k)}\) and covering the identity map on the base.

Denote the quotient map for the \(G\)-action on \(M\) by \(\pi : M \to M/G\). Of course \(M = G\) is a smooth manifold since \(M\) is a single orbit type manifold, and is in fact di eomorphic to the orbit space for the free and proper action of \(H = H_{m}\) on \(M_{m}\). Consider the restriction of the \(G\)-action to the subbundle \(H(M)\). The quotient by this action is a manifold since \(H(M)\) has just one orbit type. Furthermore, the quotient is a bundle over \(M = G\) since the action covers the action of \(G\) on \(M\). The fiber of this bundle over a point \(m \in 2M = G\) with \(m \in 2M_{H}\) is \(H_{m}(M) = H_{m}TM_{H}(M)\) since the \(H\)-action fixes every point in \(H_{m}(M)\) (by the argument given earlier).
Therefore, the isomorphism $T_m : H_m (\mathbb{M}) / \mathbb{M} \to (\mathbb{M} = \mathbb{G})$ induces a bundle isomorphism $H (\mathbb{M}) = \mathbb{G} \to T (\mathbb{M}) = \mathbb{G}$. Similarly the bundle $(\mathbb{M})$ with a smooth structure is given by a smooth manifold which is also a bundle over $\mathbb{M} = \mathbb{G}$. The fiber of this bundle over a point $[n] \mathbb{M}$ is the quotient by the $H$-action on $(\mathbb{M}_m)$ which is $(g = h) (\mathbb{M}_m)^H$. Next, since the $G$-action on the product bundle $H (\mathbb{M})$, this bundle has a fiber bundle over $\mathbb{M} = \mathbb{G}$ and its fiber over a point $[n] \mathbb{M}$ is given by the quotient of the $H$-action on the fiber $H (\mathbb{M})$. Since $H$ acts diagonally, the fiber is just $H_m (\mathbb{M}) = \mathbb{G} \cdot (g = h) (\mathbb{M}) = \mathbb{G} \cdot [m] \mathbb{M}$. It follows that the quotient, $\mathbb{G} (\mathbb{M}) = \mathbb{G}$ is isomorphic to the direct product bundle over $\mathbb{M} = \mathbb{G}$ given by $T (\mathbb{M} = \mathbb{G})$. 

4.2. Singular Atiyah Sequence. It is useful to describe a singular version of the standard Atiyah sequence for a principal bundle. In this singular case, i.e., of a single orbit type manifold, the analogous sequence is no longer a sequence of vector bundles over the quotient space, but rather a sequence of strata in vector bundles and each arrow corresponds to a strata morphism. The singular connection establishes a splitting of the sequence of vector bundles over $\mathbb{M}$,

$$0 \to T (\mathbb{M} = \mathbb{G}) \to H (\mathbb{M}) \to T (\mathbb{M} = \mathbb{G}) \to 0$$

where $T (\mathbb{M} = \mathbb{G})$ is the pullback bundle of $\mathbb{M}$ over $\mathbb{M} = \mathbb{G}$ with respect to the tangent projection $\mathbb{M} = \mathbb{G} : T (\mathbb{M} = \mathbb{G}) \to \mathbb{M} = \mathbb{G}$. That is,

$$T (\mathbb{M} = \mathbb{G}) = \{(m ; v_m) : (m) = \mathbb{M} \to (v_m)\} ;$$

and $pr (v_m) = \{(m ; T_m (v_m))\}$. Notice that $T (\mathbb{M} = \mathbb{G})$ is a fiber bundle over $\mathbb{M} = \mathbb{G}$ with fiber over $[n] \mathbb{M}$, $v_m (\mathbb{M}) = \mathbb{M} = \mathbb{G}$, and a fiber bundle over $T (\mathbb{M} = \mathbb{G})$ with fiber over $v_m$, the orbifolds which where $\{(m) = \mathbb{M} \to (v_m)\}$. Further, $T (\mathbb{M} = \mathbb{G})$ carries a proper $G$-action given by $g (m ; v_m) = (g \cdot m ; v_m)$. It is easy to see that this space has only a single orbit type, $(\mathbb{M})$, identical to the one for the action of $G$ on $\mathbb{M}$. Now, as in the regular case, the singular connection $A$ determines an injective horizontal lift map for each $m \mathbb{M}$, $hor_m : T (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G} \to T (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G}$. Consequently there is an injective bundle map $T (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G} \to T (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G}$. Notice that the splitting of the sequence is identical to the strata and this fact depends crucially on the property that the strata action of $T (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G}$ is along the vertical part of the Whitney sum in $T (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G}$, the bundle. Each strata ed sequence, with smooth morphisms is just

$$0 \to (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G} \to T (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G} \to 0;$$

Following the construction in the regular case, we consider the quotient of the strata ed sequence $0 \to (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G} \to T (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G} \to 0$ with smooth strata and morphisms given by

$$0 \to (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G} \to T (\mathbb{M} = \mathbb{G}) / \mathbb{M} = \mathbb{G} \to 0.$$
4.3. Curvature. Recall that the usual theory of curvature begins with the definition of the covariant differential of a connection

\[ DA(u;v) = dA(Hu;Hv) \]

often denoted \( B = DA \) and subsequent proof that this is a tensor, and that it verifies the identity

\[ B(X;Y) = A(HoX;YoY) \]

for vector fields \( X \) and \( Y \). This last identity gives the curvature the interpretation as the measure of non-integrability of the horizontal distribution of the connection \( A \). In our case the singular connection is a bundle map from \( TM \) to \( \). Rather than define a covariant differential for this object, we make the following definition of its curvature form. In Remark \( 3 \) we take an alternative approach of defining the covariant differential by realizing the singular connection as an Ehresmann connection.

**Definition 3.** The curvature of \( A : TM \) is defined to be

\[ (4.3) \quad B(u_m;v_n) = A([HoX;YoY]) \] \( (m) \)

where \( X \) is a vector field extending \( u_m \) and \( Y \) is a vector field extending \( v_n \). Recall that \( Ho: TM \to \) \( H(M) \) is the projection relative to the singular connection.

**Proposition 5.** The curvature \( B \) given in the previous definition is well-defined. Also, \( B \) is a \( G \)-equivariant bundle map \( B: \hat{^*}TM \to \) and it takes values in the stratum of containing the zero section, \( \) \( \). Further, \( B \) uniquely determines a reduced curvature form \( B: \hat{^*}T \to \) \( \) \( M \) \( =G \) which is a bundle map covering the identity in \( M \) \( =G \).

**Proof.** We demonstrate that equation (4.3) uniquely determines a well-defined valued 2-form on \( M \) by showing that it is tensorial, i.e., that \( B(FX;FY) = fB(X;Y) \). Recall that \( [FX;FY] = f[X;Y] \) \( Y \) \( fX \) so that

\[ B(FX;FY) = A([HoX;YoY]) = A(f[HoX;YoY])HoY(f)HoX = fA([HoX;YoY]) = fB(X;Y); \]

as required.

Denote by \( g \) the di eomorphism on \( M \) corresponding to the group element \( 2G \). To check \( G \)-equivariance of \( B \), given \( X \) and \( Y \) vector fields extending \( u_m \) \( \nu_n \) \( 2 \) \( TM \), note that \( (\); \( X \)(\( g \) \( m \)) = \( T^g[X](\( m \)) = g \) \( Y \) and similarly for \( (\); \( Y \). Therefore,

\[ (\); (\( u_m \); \( \nu_n \)) = B(g \( Y \); \( g \) \( Y \)) = A([HoX;HoY])g \( m \) = A((\( g \) \( X \))(\( g \) \( m \))) = A(g) ([HoX;HoY] \( m \)) \]

as required. This equivariance has the following consequence for the values of the curvature form. Let \( m \) \( M \) and \( u_m \) \( \nu_n \) \( 2 \) \( TM \). Recall that since the manifold consists of a single orbit type, the \( H \)-action on \( M \). Further, since

\[ u_m = Ho(u_m) + (A(u_m))_H \] \( \) \( m \) \( =Ho(u_m) + Ver(u_m); \]
we have \( h \cdot y = \text{Hor}(u_m) + h \cdot \text{Ver}(y) \). Now let \( X \) extend \( u_m \). Then the vector \( \text{Hor}X + \tilde{X} \) extends \( y \) since
\[
X(h \cdot m) = \text{Hor}(h \cdot m) + h \cdot \text{Ver}(h \cdot m) = h \cdot \text{Hor}(m) + h \cdot \text{Ver}(m)
\]
which is
\[
h \cdot (\text{Hor}(m) + \text{Ver}(m)) = \tilde{X}.
\]

Following a similar construction for \( v_m \) extended by \( Y \), and a comparable definition of \( Y \), we have
\[
B(h \cdot y;h \cdot y) = A(\text{Hor}X;\text{Hor}Y)(h \cdot m)) = B(u_m;v_m);
\]
so that we are forced to conclude that for every \( h \in H \), \( B(u_m;v_m) = h \cdot B(u_m;v_m) \); and therefore the curvature takes values in the \( H \)-fixed set of \( m = g = h \), which is the base of the stratum containing \( \mathcal{Q} \), that is, \( \mathcal{Q} \).

For the reduced curvature form, let \( u_m;v_m \in \mathcal{T}_m \), \( M = G \). We define
\[
B(u_m;v_m) = [B(u_m;v_m)];
\]
where \( \mathcal{T}_m \) denotes the element of \( \mathcal{Q} \) determined by \( B(u_m;v_m) \). An easy calculation using \( G \)-equivalence of \( B \) shows that this is well defined.

Remark 3. Alternatively, we can approach the covariant derivative of the connection by realizing that a singular connection is equivalent to an Ehresmann connection as follows. An Ehresmann connection is simply a choice of horizontal distribution compatible to the vertical distribution that is \( G \)-invariant. Given a singular connection \( A \), one defines an Ehresmann connection \( 2 ~ \mathcal{T}_m = V(G \otimes M) \) (a \( V(G) \)-valued one form on \( M \), where \( V(G) \) is the vertical distribution) by \( B(u_m;v_m) = \tilde{g} \cdot A(v_m) = [\tilde{g}]^{-1} \cdot (v_m) \). Now, recall that for \( 2 ~ \mathcal{T}_m = V(G \otimes M) \), a \( V(G) \)-valued \( k \)-form on \( M \), the definition of the covariant derivative of \( D \) is
\[
D(X_0;\ldots;X_k) := \sum_{j=0}^{k} (1)^j \tilde{g} \cdot \text{Hor}, X_0;\ldots;X_{i-1};X_i;\ldots;X_{j-1};X_j;\ldots;X_k \text{Hor})
\]
where \( X_0;\ldots;X_k \) are vector fields on \( M \), and \( \text{Hor} \) and \( \text{Ver} \) is the horizontal and vertical projection of \( X \).

We can then alternatively denote
\[
DA = [\tilde{g}] D
\]
curv $X \cdot Y = D (X \cdot Y) = (X^{\text{hor}}, Y^{\text{hor}})$ which satisfies $B = [ ]$ curv where $B$ is the curvature of the singular connection $A$ as defined in equation (4.3). Finally we remark that using definition (4.5) the Bianchi identity for $B$ follows immediately since $D \text{curv} = 0$ implies $\text{DB} = 0$.

4.4 A holonomy theorem. Our first result concerns the lowest dimensional stratum of the $\text{Ad}$-induced $H$-action on $g=H$. As the next lemma shows, this stratum turns out to be the Lie algebra of the group $N (H) = H$ which is precisely the group that acts freely on the submanifold $M_H$. We will then establish a bundle reduction theorem that will enable us to prove the Ambrose-Singer theorem for singular connections; that the Lie algebra of the holonomy group for a singular connection is given by the image of the curvature of the connection. By Lemma 2 this holonomy group is then contained in the group $N (H) = H$.

Lemma 2. The stratum containing $0 \in g$ corresponding to the stabilizer group $H$, i.e., the lowest dimensional stratum for the $H$-induced stratification, is the Lie algebra of the group $N (H) = H$.

Proof. By definition $(g = h|g) = (g = h)H$ the fixed set by the linear $H$-action. Let us denote this action by $h [ ]$. We then have $h [ ] = \text{Ad}$ and therefore $(g = h)H = f[2] g = h : \text{Ad} = [ ]$ for all $h \in \text{H}$ $g$

$$= f[2] g = h : \text{Ad} = 2 \text{h for all } h \in \text{H} : g$$

Next, we prove that the set $f[2] g : \text{Ad} = 2 \text{h for all } h \in \text{H} : g$ is in fact $\text{Lie}(N (H))$. First suppose $\Rightarrow \text{Lie}(N (H))$. Then we have $\exp(t) \exp(h \exp(t)) 2 \text{H}$ for all $t$ and therefore $\exp(t) h \exp(t) 2 \text{H}$ for all $t$ which is a curve passing through $e$ at $t = 0$. Therefore we have

$$\frac{d}{dt} \bigg|_{t=0} h \exp(t) h \exp(t) = \text{Ad} : 2 \text{h}$$

as required. Conversely, suppose $2 \text{g}$ satisfies

$$\text{Ad} = 2 \text{h for all } h : g$$

We need to show that $\exp(t) \exp(h \exp(t)) 2 \text{H}$ for all $t$. Notice that, differentiating equation (4.3) at $e$ in the direction $2 \text{h}$, we have $[ ] 2 \text{h}$. Now, because $\exp$ intertwines the $\text{Ad}$ action with the $\text{AD}$ action (the action by inner automorphisms of $G$ on itself) we have

$$h \exp(t) h \exp(t) = h \exp(t) \exp(t) = \exp(t \text{Ad} : 2 \text{h})$$

We compute this last expression using the Baker-Campbell-Hausdorff formula as follows: $\exp(t \text{Ad} : 2 \text{h}) \exp(t) \exp(t) = \exp(t \text{Ad} : 2 \text{h} + O(t^2))$, where $O(t^2)$ is a convergent series in $g$ each term of which is a composition of brackets containing a bracket of $\text{Ad} = 2 \text{h}$, or with $[ ] 2 \text{h}$ so if we take a bracket of $\text{Ad} = 2 \text{h}$, we again get an element in $h$. For the other type of term, involving a bracket of $\text{Ad} = 2 \text{h}$, we have just seen that $\text{Ad} = 2 \text{h}$, so if $t^2 2 \text{h}$,

$$\text{Ad} = \text{Ad} + [ ; ] 2 \text{h}$$
since $\beta = 0$ by (4.3). It follows that $0 (t^2) 2 \ h$ and therefore $h \exp \ (t) \ h^\top \exp \ (t) \ 2 \ H$. Multiplying on the left by $h^\top$ we conclude that $h \exp \ (t) \ h \exp \ (t) \ 2 \ H$ as required.

The next theorem addresses bundle reduction to a principal bundle for single orbit type manifolds and establishes a one-to-one correspondence with singular connections and principal bundle connections of the reduced bundle.

**Theorem 4.** Consider the following commutative diagram.

$$
\begin{array}{c}
M_H \\
\downarrow g \downarrow \\
M_H = N (H) \\
\downarrow g \downarrow \\
M = G
\end{array}
$$

Then $N (H) : M_H \leq N (H)$ is a bundle reduction of $M \leq G$ and $\bar{\partial} : M_H \leq N (H) = N (H)$. There is a one-to-one correspondence between principal bundle connections on $M_H \leq N (H) = N (H)$ and singular connections on $M$. Furthermore, the curvature form for the singular connection restricted to $M_H$ is equal to the curvature form of the principal connection.

**Proof.** As we have already remarked, every $G$-orbit in $M$ intersects $M_H$ in a unique $N (H)$-orbit. Therefore for $m \in M_H$, $N (H) (m) = (m)$, and then $M = G \leq M_H = N (H)$. Furthermore, since $H$ acts every point of $M_H$, the free action of $N (H) = H$ on $M_H$ has the same orbits as the action of $N (H) = H$ on $M_H$ and therefore the bundle $M_H \leq N (H) = N (H)$ is a principal bundle reduction of $M_H \leq N (H)$. To set up the one-to-one correspondence of connections, we start with a connection on the principal bundle $\partial : M_H \leq N (H) = N (H)$. Denote this connection by a horizontal data consisting of horizontal spaces $Q_m$ for each $m \in M_H$ and its connection form by $\partial : TM \leq n \ h$. We show how to induce from this data a connection on $M \leq G$ where the connection data will consist of a horizontal distribution invariant with respect to the group action. Given $m_0$ in $M_H$, we have $m_0 = g \cdot m$ for some $m \in M_H$, and $g \in G$. Denoting as before the action of $G$ on $M$ by $\cdot$, and by slight abuse of notation, denoting the restricted action of $N (H)$ on $M_H$ also by $\cdot$, one takes

$$
H = \frac{T_m \cdot g}{\ T_n (Q_m)}.
$$

To check it is well defined, take another realization, $m_0 = g_1 \cdot m_1$ where $m_1 \in M_H$. Then, $g \cdot m = g \cdot m_1$ so that $g_1 \cdot g \cdot m = m_1$ and therefore $h = g_1 \cdot g \cdot N (H)$. Now, by the $N (H)$-invariance of the connection on $M_H$ we know that $Q_m = T_m \cdot h \cdot Q_m$ and of course for $h \cdot N (H)$,

$$
T_m \cdot h \cdot Q_m = T_m \cdot g \cdot T_n (Q_m) = T_m \cdot g \cdot T_n (Q_m) = T_m \cdot g \cdot T_n (Q_m);
$$

proving that equation (4.7) is well defined. We next establish that these horizontal spaces are complementary to the vertical spaces in $M \leq G$. For any $m_0 \in M_H$,
represented by $m^0 = g_m$ for $M$, we have the following commutative diagram,

![Diagram](attachment:image.png)

Since $T_m^0 : Q_m \to T_m \phi = \mathcal{N} = H = H'$ is an isomorphism and the diagram commutes, we must have that $T_m^0 : H_m \to T_m \phi = \mathcal{N} = H$ is also an isomorphism, proving that $H_m$ is a complement to $\ker T_m^0$, so that the all the $H_m$ spaces define a smooth distribution $H(M)$ transversal to the $G$-action on $M$. By (4.7), this distribution is $G$-invariant. Therefore it defines a singular connection on $M$ with corresponding connection form $A : TM \to M$, given by

$$A(v) = P(v)$$

where $P : TM \to M$ is the projection induced from the splitting.

Conversely, starting with a singular connection $A : TM \to M^0$, we induce a principal connection on $M^0$ $M = G/H = H$, as before, let $M = \ker A(m)$. For $M$, in fact $H_m = TM/M$, we see that $M = G/M$. So that, for $M$, in fact $H_m = TM/M + g_m$. We see this by taking an arbitrary complement, $\mathcal{N}$, of $\phi = \mathcal{N} = H$ in $g$ so that $g = n$. Now, since $m \in M$, we have $T_m M = T_m M/H$ if and only if $T_m H = T_m M/H$. On the other hand, since $T_m M = T_m M/H + nH$, it follows that $T_m M/H = T_m M/H$. In fact we get the inner splitting $T_m M = T_m M/H + nH$. Finally notice that since the distribution defined by the spaces $H_m$ is $G$-invariant, then it is also $G$-invariant so that we can define the horizontal spaces on $TM$ by $Q_m = T_m M/H$. For the correspondence, let be the $n$-valued curvature form of $M$. Let $B$ be the curvature of $A$ as defined in Definition 3. For $M$, given $u_m, v_m : T_m M$, let $u_m = \text{Hor } u_m$, and $v_m = \text{Hor } v_m$. Using the definition of $B$, let $X$ and $Y$ be arbitrary extensions of $u_m, v_m$, chosen to be tangent to $M$. In fact we can construct these vector fields by first projecting $u_m, v_m$ to the quotient $M = G$ and then taking arbitrary extensions in $M = G$. Next, lift the vector fields horizontally with respect to the principal bundle $0 : M \to M = G/H = H$, and then extend them to the entire manifold by the $G$-action, which we can do since $M$ is saturated by $M$. Since the $G$-action preserves the horizontal distribution, this will define globally two horizontal vector fields on $M$, denoted by $X$ and $Y$, extending $u_m$ and $v_m$. Since they are
horizontal, and the horizontal distribution, restricted to $M_H$, is contained in $TM_H$, their restrictions to $M_H$ are smooth vector fields on $M_H$. We then have

$$B(\xi_m;\nu_m) = A(\{H_{mX};H_{mY}\})\xi_m$$

where for the third equality, we use the fact that the Jacobibracket of $H_{mX}$ and $H_{mY}$ in $M$ evaluated at $m \ 2 M_H$ coincides with their bracket as vector fields in $M_H$ (evaluated at $m$) so that the third equality holds. The inequality is just a consequence of the usual curvature identity for principal bundles.

**Theorem 5 (Ambrose-Singer).** The horizontal lift of a curve in the base space $M = G$ through a point $m \ 2 M_H$ lies entirely in $M_H$ and the holonomy group through $m$, $H(\cdot m)$, of the singular connection is contained in $N(H) = H$. The Lie algebra of the holonomy group through $m$ is given by the image of the curvature of the singular connection at $m$.

**Proof.** Given $[n] = [m] \ 2 M = G$ and a loop $l(t)$ in $M = G$ through $[m]$, consider $m \ 2 M_H$, such that $[m] = [n]$. From Proposition 4, we can view the loop $l$ as a loop in the base manifold $M_H = [N(H)]$. Consider its horizontal lift, $L(t)$, in the usual sense of principal bundles, through the point $m \ 2 M_H$. Since, by the proof of Theorem 6, the horizontal spaces of the bundle $M_H$ are mapped into horizontal spaces of $M$, $M = G$, it follows that $L(t)$ is a horizontal curve in $M$ for the singular connection through the point $m$ and projects to $l(t)$ by construction. However, the horizontal lift of a curve in $M = G$ to $M$, through a specified point is unique. To prove this, one uses approximately the same argument as in the free case. Suppose $L_2(t)$ is another curve through $m$ which projects to $l(t)$. It follows that $L_2(t) = g(t)L(t)$ for some curve $g(t) \ 2 G$. This curve is not unique since the action is not free. However, $\frac{d}{dt}_{t=0} L_2(t) = g(t)L(t) + g(t)L(t)$ by the Leibniz identity, and therefore, applying the connection to this expression one finds

$$0 = A \frac{d}{dt}_{t=0} L_2(t) = A(g(t)L(t))$$

since $L(t)$ is horizontal. The only way $g(t)L(t)$ can be horizontal is if it is zero and therefore $g(t)$ must be a curve that lies for all time in the stabilizer of $L(t)$ and therefore $L_2(t) = L(t)$ as required.

Next, we show that the Lie algebra of the holonomy group $H(\cdot m)$ is the image of the curvature of the singular connection at $m$. Since the horizontal lift of a loop in $M = G$ through $m$ lies entirely in $M_H$, then $H(\cdot m) \ 2 N(H) = H$. Now we can apply the standard holonomy theorem to the principal bundle $M_H$ with the unique principal connection induced from the singular connection. The conclusion is that the Lie algebra of the holonomy group is given by the curvature of this induced principal connection at $m$. However, the curvature of $B$, evaluated at $m$ coincides with the curvature of the induced principal connection. Since this can be done for any $m$, the statement follows.

5. Singular Sternberg

The singular connection from $A$ allows us to write down a $G$-equivariant diffeomorphism,

$$A : M \rightarrow TM$$

which will play the fundamental role in establishing the connection dependent realization of the Poisson strati ed space $TM = G$. The mapping $\bar{A}$ is a
product factorization of phase space into zero momentum and non-zero momentums respectively. We call the quotient of the domain in $\mathfrak{g}$, $M^*_{\mathfrak{g}}$ the singular Sternberg space, as it generalizes the original representation due to Sternberg of $G(M) = G$ in the free category. Using $A$, we will also determine the minimal coupling form on the strata of the symplectic quotients.

The construction of the Sternberg space reviewed in Section 2 generalizes to single orbit type manifolds as follows. We start by constructing the zero momentum space $M^*$ by taking the pullback of $M! = G$ by the cotangent projection $M = G : T \ni (M = G) ! M = G$ so that we have

$$
\begin{array}{c}
M^*
\xrightarrow{\sim}
T \ni (M = G) \xrightarrow{\pi^{-1}} M = G
\end{array}
$$

We then have

**Proposition 6.** $M^*$ is a her bundle over $T \ni (M = G)$ whose bases are the $G$-orbits of $M$. Furthermore, $M^*$ is a $G$-space with single orbit type and as a bundle over $M$ is bundle isomorphic to the zero momentum space, $J^{-1}(0) \cap V(M)$ where $V(M)$ is the annihilator of the vertical bases in $M! = G$. Then, $M^*$ naturally inherits a singular connection form given by $A = A \cdot T M^*$. 

**Proof.** By definition $M^* = f(\{m, \pi\} : (m) = G(m, \pi)) = [m, g]$, and therefore, for $\pi \in 2 T M(M = G)$, $(\pi')^{-1}(\pi) = f(\{m, \pi\} : m \in 2 G G m)^{-1}$ but $m$ is any representative of $[m, g]$. Similarly, the bases of $\sim -1(\pi)$, $T \pi(M = G)$. Notice that $M^*$ inherits a $G$-action defined by $g(\{m, \pi\}) = (\{m, g \cdot \pi\})$, clearly well defined, smooth, and proper since the action of $G$ on $M$ is so. Furthermore, since $G(\{m, \pi\} = G m$. It follows that $M^*$ is a single orbit type manifold with orbit type $G$, the same as for $M$, and therefore the quotient of $M^*$ by this action is a smooth manifold. By inspection this quotient is simply $T \ni (M = G)$. Consider the map $\pi : M! = G$ given by $(\{m, \pi\} : (T_M m) \ni \pi)$. This map takes the base of $M^*$ over $m$ into $T M_M$ and takes values in $V(M)$ since $(T_M m) \ni \pi = \pi \in T M_T \ni (\pi = \pi_M) = 0$. Since the map restricted to $\sim -1(\pi)$ is injective (being the dual to the surjective map $T_M : T_M M \ni (M = G)$), and since $\dim V(M) = \dim M + \dim G + \dim H = \dim (M = G)$ it follows that $j : (m) : T \ni (M = G) ! V(M) \ni m$ is an isomorphism and therefore $M^! = V(M) \ni m$ is a bundle isomorphism covering the identity on $M$. Finally, consider the map $A : T M^*$. First notice that is the correct target bundle for a singular connection on $M^*$ since it is a single orbit type manifold with orbit type $G$. Also, we have for each $g, j : (m) : T \ni G(M) = G$, $j : (m) : (A \cdot \pi) = (A \cdot \pi_M) = \pi_M$.

Equivariance follows from equivariance of the projection $\sim$ and equivariance of $A$.

Denote by $M^*$ the corresponding product bundle over $M$. We will use the following notation for bundle projections, $\pi : M! = G$ and $\pi : M^* 
\pi : M^*$. Note then, that $\pi : T \ni (M = G)$ which we abbreviate, below, as simply $\pi$ to economize notation.
Remark 4. Recall that, by definition, the monentum map for the G-action on M is a map, J : T M ! g. However, since the image of \( \mathcal{J}_\lambda \) is \( \mathfrak{g}_m \) \( \sim \), we can also regard J as a bundle map T M ! over M. In the following we will sometimss implicitly use this point of view.

Proposition 7. Associated to the singular connection A there is an equivariant bundle isomorphism, \( \lambda : M^* \rightarrow T M \) covering the identity on M given by

\[
\mathcal{D} M = (\mathcal{J}_\lambda) + A \ (m) \ m \ ;
\]

Furthermore, the pullback of the canonical one-form on T M by \( \lambda \) is given by,

\[
\mathcal{D} E M = (\mathcal{J}_\lambda) + A \ (m) \ m \ ;
\]

where \( h ; i \) indicates the natural pairing on the bers with the corresponding bers on M.

J \( \lambda : M^* \rightarrow T M \) is just the bundle projection so that \( J \ M \lambda \) \( \sim \) is the one-form on M given by

\[
\mathcal{J} M = (\mathcal{J}_\lambda) + A \ (m) \ m \ ;
\]

with \( \mathcal{J} = ((\mathcal{J}_\lambda) + A \ (m) \ m \ ; \ m \ ; \ m \).

Proof. Equivariance of \( \lambda \) follows immediately from the equivariance of the singular connection A and the definition of the G-action on M.

Next, observe that for any \( g \) we have

\[
J \lambda ((\mathcal{J}_\lambda) + A \ (m) \ m ; \ m ; \ m) = (\mathcal{J}_\lambda) + A \ (m) \ m ; \ m ; \ m = h \ ; i = h ; i ;
\]

where \( \mathcal{J} \ (m) \) and therefore \( J \lambda \), restricted to the ber over m, takes values in the ber of over m, and is surjective on this ber since \( J \lambda \) is the dual of the injective map \( \mathcal{J} \). Let \( \nu \ 2 \ T M \) with \( \mathcal{J} = ((\mathcal{J}_\lambda) + A \ (m) \ m ; \ m \). We have

\[
(\mathcal{J}_\lambda) \ (\mathcal{J}_\lambda + A \ (m) \ m ; \ m) = (\mathcal{J}_\lambda \ (\mathcal{J}_\lambda + A \ (m) \ m ; \ m) = h \ ; i = h ; i ;
\]

using the facts that \( M \lambda = M = M \lambda - M \) in the third equality.

In the next two theorems, we prove that the minimal coupling form due to Stemberg generalizes to the singular setting. Care must be taken to prove the generalization since we need to deal with a ber product bundle, M \( \tilde{\Omega} \) and not just the product of manifolds as in the free case. The proof of the extension to the singular setting will make repeated use of the fact that the bundles over M, M \( \tilde{\Omega} \) and J \( \lambda \) \( \sim \) (\( \tilde{\Omega} \)), are each G-saturated bers bundles over M.
Theorem 6. Let $O$ be a coadjoint orbit through a point in the image of $J : T \mathcal{M} \rightarrow \mathbb{R}$. The map $\Lambda$ restricts to an equivariant stratified bundle isomorphism, $J^{-1}(O) \rightarrow \mathcal{M} \setminus \mathcal{O}$, where $\mathcal{M} \setminus \mathcal{O}$ is the under product bundle over $\mathcal{M}$, and $\mathcal{O}$ is a sub-bundle with fiber over $m \in M$ given by $O \setminus h$. The orbit type strata of $\mathcal{M} \setminus \mathcal{O}$ are $\mathcal{M} \setminus \mathcal{O}_{(K)}$, where $K$ are the subgroups of $H$ determined by the isotropy lattice for the action $H \acts \mathcal{M}$ on $\mathcal{M}$ and $(K)$ denote their conjugacy classes in $G$. Denote by $(k) : M' \setminus \mathcal{O}_{(K)} \rightarrow M'$ the inclusion. The restriction of the form $\Lambda_{(k)}^\dagger$ to the pullback of the canonical symplectic form on $T \mathcal{M}$, to each stratum $M' \setminus \mathcal{O}_{(K)}$, is given by

$$J_{\mathcal{M} \setminus \mathcal{O}_{(K)}}^*(\Lambda^\dagger) \rightarrow \mathbb{R}$$

where $\Lambda_{(k)}^\dagger$ is the canonical symplectic form on $T \mathcal{M} \setminus \mathcal{O}_{(K)}$ and $J_{\mathcal{M} \setminus \mathcal{O}_{(K)}}$ is the restriction of $J \Lambda$ to $M' \setminus \mathcal{O}_{(K)}$. Denote the two-form on $M' \setminus \mathcal{O}_{(K)}$ by

$$J_{\mathcal{M} \setminus \mathcal{O}_{(K)}}^*(\Lambda^\dagger) = \sum_{i=1}^n J_{\mathcal{M} \setminus \mathcal{O}_{(K)}}^*(\Lambda^\dagger)_i$$

where $\Lambda_{(k)}^\dagger$ is the canonical symplectic form on $T \mathcal{M} \setminus \mathcal{O}_{(K)}$, and $!_{\mathcal{O}}^\dagger$ is the $(+)$ orbit symplectic form on $O$. The two-form $!_{\mathcal{O}}^\dagger$ satisfies the following:

(i) It is basic, i.e. $G$-invariant and annihilates $G$-vertical vectors.

(ii) It drops to a unique two-form $!_{\mathcal{M}}^\dagger$ on $M' \setminus \mathcal{O}_{(K)}$.

(iii) Denoting by $(k) : M' \setminus \mathcal{O}_{(K)} \rightarrow M' \setminus \mathcal{O}_{(K)}$ the orbit map, the reduced form $!_{\mathcal{M}}^\dagger$ on $M' \setminus \mathcal{O}_{(K)}$, (denoted by $(k)^\dagger = !_{\mathcal{O}}^\dagger$) satisfies

$$J_{\mathcal{M} \setminus \mathcal{O}_{(K)}}^*(\Lambda^\dagger) \rightarrow \mathbb{R}$$

Proof. First recall that $\mathcal{M}$ is a single orbit type manifold with orbit type $(H)$ identical to that of $M$. By definition, $\mathcal{O}$ is the bundle over $M$ whose fiber over $m$ is $G \setminus \mathcal{O}$. It is clear that $(J \Lambda)^\dagger = \Lambda^\dagger$ since the restriction of $J \Lambda$ to each fiber is given simply by $\mathcal{O}$. It follows that $J_{\mathcal{M}}(\Lambda^\dagger_{(k)} ; m ; m) = m \setminus \mathcal{O} \setminus \mathcal{O}$.

On the other hand, the bundle $\mathcal{O}$ over $M$ has the structure of a stratified fiber bundle which also trivializes over $M$. Restricted to $M$, we have

$$\mathcal{O} \setminus \mathcal{M} = M \setminus (O \setminus h)$$

and further over $O = G \setminus \mathcal{O}$ since the fiber over $g \in M$ is just $O \setminus (\mathcal{O} \setminus h) = O \setminus g \setminus (O \setminus h)$ according to the definition of the $G$-action on $O$.

Equation (5.3) follows by taking the restriction of equation (5.2) of the previous Proposition to the stratum $M' \setminus \mathcal{O}_{(K)}$ and then taking the exterior derivative. For (i), note that $\mathcal{O} \setminus \mathcal{M}$ is $G$-equivariant, and $!_{\mathcal{O}}^\dagger$ is $G$-invariant and therefore $!_{\mathcal{M}}^\dagger = (\Lambda^\dagger)^\dagger$ on $\mathcal{O} \setminus \mathcal{M} \setminus \mathcal{O}_{(K)}$ is $G$-invariant. It follows that the pulled back form $\Lambda_{(k)}^\dagger$ to the $G$-manifold $M' \setminus \mathcal{O}_{(K)}$, is also $G$-invariant. In fact, each term in $\Lambda_{(k)}^\dagger$ is independently invariant. To check this, let $!_{\mathcal{O}}^\dagger = ((\Lambda^\dagger}_{(k)} ; m) \rightarrow \mathbb{R}$.
from which it follows that, by infinitesimal invariance,

$$\rho = 0 = \frac{d}{dt} \left( J_{M,(x)} J_{\sigma_{(x)}} \right);$$

where $L_X$ denotes the Lie derivative, and equation (5.6) follows from Cartan's magic formula. Now, consider the form $\Omega^{\dagger} J_{\sigma_{(x)}}$. The sum of the first two terms is simply $\Lambda$! and each is $G$-invariant, and the first term, $(\Lambda)$! is basic. The third term of $\Omega^{\dagger} J_{\sigma_{(x)}}$ is easily checked to also be $G$-invariant using equivariance of $J_{\sigma_{(x)}}$ and invariance of the orbit symplectic form $\Omega^{\dagger}$! Therefore, $\Omega^{\dagger} J_{\sigma_{(x)}}$ is $G$-invariant.

To see that $\Omega^{\dagger} J_{\sigma_{(x)}}$ annihilates vertical vectors we must show that

$$\frac{d}{dt} \left( \left[ J_{M,(x)} J_{\sigma_{(x)}} \right] ; \right) = 0;$$

From equation (5.5) we have

$$\frac{d}{dt} \left( J_{M,(x)} J_{\sigma_{(x)}} \right) = \left[ J_{M,(x)} J_{\sigma_{(x)}} \right] ;$$

where $\left[ \right]$ is the section of given by $\left[ \right]_{\sigma_{(x)}} = \left[ \rho \right]_{\sigma_{(x)}}$. We have,

$$\left[ J_{M,(x)} J_{\sigma_{(x)}} \right] (t) = \frac{d}{dt} \left( \left( \rho \right)_{\sigma_{(x)}} \right)_{\exp \left( t \right)} m; \Ad \left( t \right) m$$

and therefore,

$$\frac{d}{dt} \left( J_{M,(x)} J_{\sigma_{(x)}} \right) (t) = \frac{d}{dt} \left( \left( \rho \right)_{\sigma_{(x)}} \right)_{\exp \left( t \right)} m = \Ad \left( t \right) m.$$
On the other hand,

\[ \begin{align*}
D & \quad E \\
\text{d} J_{\mathcal{O}(K)} [v] & = T J_{\mathcal{O}(K)} [v] [;] \\
& = \text{ad}_m [;] + h_m [;] [;]
\end{align*} \]

so that equation (5.7) is satis ed. Consequently, \( \mathcal{O}(K) \) drops to a unique form \( !_{\text{m in}}^{O(K)} \) on \( M^* \circ \mathcal{O}(K) \), and this form satis es equation (5.5) since it is the unique orbit reduced form by construction.

Remark 5. The above theorem generalizes the minimal coupling constrictions for the regular case given in Section 2 as follows. The form \( (1) \) \( !_{\text{m in}}^{O(K)} \) is the singular generalization of \( t_0 \) given in equation (2.2). The form \( !_{\text{m in}}^{O(K)} \) is the singular generalization of the minimal coupling form \( !_{\text{m in}}^{O(K)} \) in equation (2.4).

We then have the following result on the reduced symplectic form \( !_{\text{m in}}^{O(K)} \) determined on each \( M^* \circ \mathcal{O}(K) \) coupling the canonical symplectic structure on the homogeneous Kostant-Kirillov form on the orbit bers via the reduced curvature \( B \) of the singular connection.

The involved mappings are summarized in the following diagram

\[ \begin{array}{ccc}
M^* & \circ \mathcal{O}(K) & \xrightarrow{\pi} M^* \\
\downarrow & & \downarrow \text{\ } \pi \\
M^* \circ \mathcal{O}(K) & \xrightarrow{\text{ker}} M = \mathbb{G} & \xrightarrow{\text{G}} M = \mathbb{G} \\
\end{array} \]

We will proceed as before to write \( 1 = 1 \) \( M^* \) in order to economize notation.

Theorem 7. The reduced minimal coupling form \( !_{\text{m in}}^{O(K)} \) on \( M^* \circ \mathcal{O}(K) \) can be expressed in terms of the reduced curvature, \( B \), of \( \mathbb{G} \) as follows.

\[ \begin{align*}
D & \quad E \\
\text{d} J_{\mathcal{O}(K)} [v] & = T J_{\mathcal{O}(K)} [v] [;] \\
& = \text{ad}_m [;] + h_m [;] [;]
\end{align*} \]

where \( \pi : M^* \circ \mathcal{O}(K) \xrightarrow{p} T (M = \mathbb{G}) \) is the submersion given by \( p((\text{in} [;m]; [m]) = \text{in} [;] \) and \( \sim ((\text{in} [;m]; [m]) = (\text{in} [;] [m]) \) in \( 2 \mathcal{O}(K) \).

The two-form \( !_{\text{m in}}^{O(K)} \) is equivalent at each point to the homogeneous reduced symplectic form needed in (3.2).

Proof. First, we observe that, as in the free case, using the connection \( A \) we can induce from the two-form \( J_{\mathcal{O}(K)} [v] \) on \( M^* \circ \mathcal{O}(K) \), a form \( t_{\mathcal{O}(K)} \) on the quotient space \( M^* \circ \mathcal{O}(K) \). To construct this form, we need to rst obtain a splitting of the tangent bundle of \( M^* \circ \mathcal{O}(K) \). Note that the vertical bers of \( p \) are given by

\[ p^{-1}(\text{in})' \circ (\mathbb{G} \setminus h_{\mathcal{O}(K)}) = \mathbb{H} \]

where \( \mathbb{G} = H \). These bers carry a symplectic structure as we have determined in (3.2). The tangent spaces to these bers determine the vertical subbundle \( V \) of \( T M^* \circ \mathcal{O}(K) \); for each \( [1] = ([\text{in} [;m]; [m]) \) \( 2 M^* \circ \mathcal{O}(K), V_{[1]} = \ker T_{[1]} p \).

The connection, \( A \), determines a complement as follows. Given a vector \( v_{[1]} \) 2
T_{(\pi, m)}(M = G) \text{ tangent to some curve } \{p(t) \text{, denote the projected curve to } M = G \text{ by } \hat{p}(t) = \pi_* \hat{p}(t). \text{ Then, letting } m(t) \text{ denote the horizontal lift of } \hat{p}(t) \text{ to } M \text{ through the point } m. \text{ Notice that, by construction, the curve } (\hat{p}(t), m(t)) \text{ is contained in the } \pi \text{-orbit through } \hat{p}(t) \text{, and, furthermore, the curve is not contained in the } \pi \text{-orbit through } (\hat{p}(t), m(t)). \text{ Finally, the tangent vector to the curve } ((\pi_* \hat{p}(t), m(t)); m) \in \mathbb{R}^\infty \text{ through } \hat{T}_{(\pi, m)}(p) \text{ and therefore we have constructed an injective map } \frac{1}{\pi}: \pi_{\{p\}}(M = G) \rightarrow T_{\{p\}}(M^\infty \subset \mathcal{O}). \text{ The image of } \frac{1}{\pi} \text{ then has dimension complementary to } \ker T_{\{p\}}(p). \text{ Now ranging over } \{\pi\}, \text{ this defines the horizontal distribution } H \text{ in } T_{\pi, m} \mathbb{R}^\infty \subset \mathcal{O}. \text{ We then have a projection } P_{\{\pi\}} \text{ onto the vertical space } V_{\{\pi\}} \text{ corresponding to this splitting. The form } b_{\{\pi\}} \text{ is then defined on } \mathbb{R}^\infty \subset \mathcal{O}_{\{\pi\}} \text{ by }

\begin{equation}
\begin{aligned}
b_{\{\pi\}}(v_1; w_1) = & \langle J_{\mathcal{O}_{\{\pi\}}}; v \rangle(0; v_1); (0; w_1) \rangle
\end{aligned}
\end{equation}

where } v_2 T_{\pi, \mathcal{O}_{\{\pi\}}} \text{ satisfies } T_{\{\pi\}}(0; v) = P_{\{\pi\}}(v_1) \text{ and analogously for } w. \text{ This is well defined due to the } G\text{-invariance of the form } J_{\mathcal{O}_{\{\pi\}}}. \text{ Next, consider the basic form (from equation } [5.7]) \text{ on } \mathbb{R}^\infty \subset \mathcal{O}_{\{\pi\}}, \text{ so that } J_{\mathcal{O}_{\{\pi\}}}; A \text{, and } J_{\mathcal{O}_{\{\pi\}}}; A. \text{ We need to show that }

\begin{equation}
\begin{aligned}
D_{\pi} \sim (\pi \subset \mathcal{O}) \quad b_{\{\pi\}} = & \langle J_{\mathcal{O}_{\{\pi\}}}; v \rangle(0; v_1); (0; w_1)\rangle
\end{aligned}
\end{equation}

Fix a point } \pi = ((\pi, m); m) \in \mathbb{R}^\infty \subset \mathcal{O}_{\{\pi\}}. \text{ We consider three types of tangent vectors at this point: } h_1((\pi, m); 0), \text{, and } h_2((\pi, m); 0) \text{ where } h_1((\pi, m); 0) \text{ is a horizontal vector at } T_{(\pi, m)} \mathbb{R}^\infty \subset \mathcal{O}_{\{\pi\}} \text{ relative to } A, \text{ and } u_2 \text{ is a tangent vector at } T_{(\pi, m)} \mathbb{R}^\infty \subset \mathcal{O}_{\{\pi\}}. \text{ We prove equation } [5.8] \text{ by checking equality on all six pairs of these types of tangent vectors. Note that on vectors of the form } h_1((\pi, m); 0) \text{ both sides of equation } [5.7] \text{ are zero: trivially for the left hand side, and for the right hand side, by equation } [5.7]. \text{ Therefore it will suffice to verify equation } [5.8] \text{ on the three types of pairs generated by types } h_1((\pi, m); 0) \text{ and } (0; u), \text{ which we call horizontal and momentum vectors respectively.}}
have,
\[
d \left( J_{\sigma, \alpha} \right) \mathbb{R}^{\mathbb{R}} \left( \{u_1; 0\}; \{u_2; 0\} \right) = \begin{array}{ll}
D & E \\
\{U_1; 0\} & J_{\sigma, \alpha} \mathbb{R}^{\mathbb{R}} \left( \{U_1; 0\}; \{U_2; 0\} \right) \\
\{U_2; 0\} & J_{\sigma, \alpha} \mathbb{R} \left( \{U_1; 0\} \right)
\end{array}
\]

where we have used the fact that \(T \sim U_1 = U_2 \sim\), so that \(T \sim U_1; U_2 = \{U_1; U_2\} \equiv\), and also the definition of the curvature of the singular connection in the n-equality.

We now compute the left hand side of equation \(5.2\) on the horizontal vectors,
\[
d \left( \Delta \right) \left( (u_1; 0); (u_2; 0) \right) = \begin{array}{ll}
D & E \\
\{U_1; 0\} & \{u_1; 0\}; (u_2; 0) \circ \left( (u_1; 0); (u_2; 0) \right)
\end{array}
\]

agreeing with the right hand side.

The next equality holds since \(T \circ \left( (0; v); (0; w) \right) \equiv\) is in the kernel of \(T\). The second equality holds by definition of the form \(\Delta\) and last follows by extending \((0; v); (0; w)\) to vector \(\exp((0; V); (0; W))\) where \(V\) and \(W\) are vector \(\exp\) on \(\mathbb{R}^{\mathbb{R}}\) and then using the fact that the bracket of these \(\exp\) is just \(0; [V; W]\).

H horizontal, M om entum . On a pair of m ixed vectors, both sides of equation \(5.2\) vanish since horizontal vector \(\exp\) commutes with momentum vector \(\exp\).

We have now proven equation \(5.2\), and therefore by equation \(5.4\), the theorem follows.
6. The Poisson stratification

In this section we compute the Poisson stratification of the reduced Poisson Sternberg space \( S = M^* G \), and we write down the reduced gauge bracket on each of the Poisson strata.

Following the setup of Section 5, the \( G \)-equivariant symplectic morphism \( \lambda \) descends to a stratified isomorphism \( \lambda : S ! (T M)^* G \). Since \( S \) is a quotient of a smooth manifold by a proper group action, it has a natural stratification with strata \( S^{(k)} = \{ F^k \}_{k \in G} \). Since \( M^* \) is a single orbit type manifold, the orbit types of the pre-quotiented space correspond to the orbit types of the total space of the bundle, which were shown, in the proof of Theorem \( \text{[3]} \), to be in one-to-one correspondence with the isotropy lattice for the action \( H \triangleright h \). Therefore the strata of \( S \) are given by

\[
S^{(k)} = M^* \oplus (G_k) \text{.}
\]

Moreover, since \( M^* \) is a symplectic manifold with symplectic structure \( \lambda \), it follows from the general theory that the orbit type strata of \( S \) are actually Poisson, the Poisson structure coming from singular reduction. The symplectic lifts of these Poisson structures are exactly the manifolds \( M^* \oplus (G_k) \) equipped with the minimal coupling form from \( (G/m, \rho) \) of Theorem \( \text{[4]} \). Instead of using the theory of singular reduction to compute the reduced Poisson strata on the strata \( S^{(k)} \), in Theorem \( \text{[5]} \) we will postulate these brackets and then verify that they actually produce the minimal symplectic foliation of Theorem \( \text{[7]} \). By the uniqueness of the symplectic foliation of a Poisson manifold it follows that the postulated brackets are actually the reduced gauge Poisson brackets.

**Theorem 8.** Let \( s = [(\xi); x] \in S^{(k)} \), where without loss of generality we have chosen \( G_x = H \) and \( H_x = K \). Let \( f; g \in C^1(S^{(k)}) \). Then the reduced Poisson structure on \( S^{(k)} \) is given by

\[
[\xi]_s = \sum_{\gamma \in G} \sum_{\alpha \in \Lambda} \frac{\partial}{\partial \alpha^\gamma} f(s) \frac{\partial}{\partial \alpha^\gamma} g(s) + \int_x \{ \gamma \} \left( \frac{\partial}{\partial \alpha^\gamma} f(s) \frac{\partial}{\partial \alpha^\gamma} g(s) \right)
\]

where the sharp operator in the first term is with respect to \( !_{\xi} = \alpha \) and the covariant derivative \( \frac{\partial}{\partial \alpha^\gamma} \) is computed using the singular connection as in equation \( \text{(2.1)} \) for the regular case. In the second term, \( \{ \gamma \} = \frac{\partial}{\partial \alpha^\gamma} \) where \( \frac{\partial}{\partial \alpha^\gamma} \) is the reduced curvature of the singular connection, \( A \). For the third term, \( \{ \gamma \} \) is the restriction of \( f \) to the fiber of \( \text{ker} \), which is isomorphic to \( \text{ker} \), and \( \{ \gamma \} \) is a \( H \)-invariant extension of the lift of \( \{ \gamma \} \) to \( h \). Note that the last term is simply a ( ) homogeneous Lie-Poisson bracket

\[
\{ \gamma \} \{ \gamma \} = \{ \gamma \}
\]

on the fiber of \( \text{ker} \), as introduced in Section 3, in equation \( \text{(3.1)} \).

**Proof.** Since \( S^{(k)} \) is Poisson, the theorem is proved if for any pair \( f; g \in C^1(S^{(k)}) \) and \( s = \xi \in S^{(k)} \), \( [\xi]_s = \{ \xi \} \), where \( X_f, X_g \) are the \( H \)-invariant vector fields associated to the restrictions of \( f \) and \( g \) to \( M^* \oplus (G_k) \). Let \( U = 0 \times R^0 \) be a trivializing neighborhood of \( T(M = G) \) and shrink \( 0
and $R^n$ if necessary so that $S^{(k)} \cong U$ like $S^{(m)}_U = U$ $h_{(k)} = H$.

If $x^i; p_i; i = 1; \ldots; n$ are bundle coordinates on $U$ we will consider the family of functions $x^i; p_i$ and $f 2 C^1 (\mathbb{R}^m - H)$, whose differentials span the cotangent bundle of $S^{(k)}$ at any point of $S^{(m)}_U$.

Let $(x^1; \ldots; x^n; g^1; \ldots; g^k)$ be local coordinates on $M$ over $O$ and $A_{ij} = 1; \ldots; n + k; 1 = 1; \ldots; k$ the components of the connection $A$. The local coordinates $f^j g$ on $G$ are chosen in a way that $A (\xi g) = g_i$, where $f_1; \ldots; k g$ is a basis for $g$ for which, for $r < k$, $f_r; \ldots; k g$ is a basis for $g_r$. Then the horizontal lift of a local vector eld $\xi_i$ on $O$ is $\xi_i A_{ij} g^i$.

Therefore, we have $h_{ij} (\xi_i) = \xi_i$ and $h_{ij} (\xi_i) = \xi_i$. Consequently we obtain

$$
\begin{align*}
\frac{d}{\xi_i} x^i (s) &= dx^i \\
\frac{d}{\xi_i} p_i (s) &= dp_i \\
\frac{d}{\xi_i} f (s) &= 0
\end{align*}
$$

Since in this trivialization $\xi_i = g^i$ is given by $dx^i \wedge dp_i$ it follows that

$$
\begin{align*}
\frac{d}{\xi_i} x^i (s) &= 0 \\
\frac{d}{\xi_i} p_i (s) &= 0 \\
\frac{d}{\xi_i} f (s) &= 0
\end{align*}
$$

Next, linear coordinates for $g$ with respect to the dual basis $f_1; \ldots; k g$ are given by $f_1; \ldots; k g$. Let $B_{ij} = 1; \ldots; n; = r + 1; \ldots; k$

be the local expression for the components of $B$, the reduced curvature of $A$.

Then the local expressions for the bracket in the statement of the theorem are given by

$$
\begin{align*}
fx^i ; p_i g (s) &= \frac{i}{j} \\
fp_i ; p_i g (s) &= B_{ij} = r + 1; \ldots; k \\
fx^i ; x^j g (s) &= 0 \\
fx^i ; fg (s) &= 0 \\
fp_i ; f g (s) &= 0 \\
ff g (s) &= 0
\end{align*}
$$

(6.1)

This is easily checked to be a Poisson tensor. The only point that requires a straightforward calculation is to check the Jacobi identity on a bracket of type $fp_i; f p_i; g g$. But, since the singular curvature satisfies the Bianchi identity (see Remark [3], it follows that the two-form $h [ ] \xi_i B_i$ is closed which implies Jacobi.

Antisymmetry is implied by the antisymmetry of the reduced curvature form.

Note that the last expression is nothing but the homogenous Lie-Poisson bracket of Section 3, and that it restricts on each moment weber of $M'^{}; G^{(k)}$ over $T' = G$ to the homogeneous reduced symplectic form in $(3.2)$.

We now compute the Hamiltonian vector eld on a given typical symplectic leaf $M' \subset G^{(k)}$ with respect to $h_m^{(k)}$, as well as the Poisson structure on these leaves induced by $h_m^{(k)}$.

It easily follows from Theorem [7] that in our local coordinates,

$$
\begin{align*}
h_m^{(k)} &= dx^i \wedge dp_i \\
B_{ij} dx^i \wedge dx^j + h_{(k)}; i j = 1; \ldots; n; = r + 1; \ldots; k
\end{align*}
$$
From here, we immediately obtain the associated Hamiltonian vector fields
\[ X_{x_i} = \theta_{p_i}, \]
\[ X_{p_i} = \theta_{x_i} - B_{ij} \theta_{p_j}, \]
for \( i, j = 1; \ldots; n; \) \( = r + 1; \ldots; k. \) If \( f \) is \( C^1 \) and \( \mathcal{O}_{\mathcal{U}^\prime} \), we denote also by \( f \) its restriction to \( 0 \in \mathcal{U} \subseteq \mathcal{U}^\prime \), the typical fibre of the projection \( \mathcal{U}^\prime \) to \( T \) \( (\mathcal{M} = \mathcal{G}). \)

Then \( X_f \) is defined by
\[ X_f (X_{x_i}) = 0; \]
\[ X_f (X_{p_i}) = B_{ij} \]
\[ X_f (X_{x_j}) = 0; \]
\[ X_f (X_{p_j}) = 0; \]
\[ X_f (X_{x_k}) = X_f (X_{x_k}) = f; \]

which agree with (6.1) by (3.1) and the discussion of Section 3.

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