Majorana meets Coxeter:
Non-Abelian Majorana Fermions and Non-Abelian Statistics

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We discuss statistics of vortices having zero-energy non-Abelian Majorana fermions inside them. Considering the system of multiple non-Abelian vortices, we derive a non-Abelian statistics that differs from the previously derived non-Abelian statistics. The new non-Abelian statistics presented here is given by a tensor product of two different groups, namely the non-Abelian statistics obeyed by the Abelian Majorana fermions and the Coxeter group. The Coxeter group is a symmetric group related to the symmetry of polytopes in a high-dimensional space. As the simplest example, we consider the case in which a vortex contains three Majorana fermions that are mixed with each other under other other under the SO(3) transformations. We concretely present the representation of the Coxeter group in our case and its geometrical expressions in the high-dimensional Hilbert space constructed from non-Abelian Majorana fermions.

I. INTRODUCTION

Majorana fermions proposed by Ettore Majorana in the 20th century have a peculiar property in that an antiparticle is equivalent to a particle. His original conjecture that neutrinos might be such particles has been rejected; however, Majorana fermions are now attracting much attention in condensed matter physics. It has been recently recognized that Majorana fermions with exact zero energy are trapped inside the core of half-quantized vortices in chiral $p$-wave superconductors or superfluids and that they appear on the edge of topological superconductors and insulators. A system of multiple vortices provides an exchange statistics that differs from that of bosons, fermions and anyons. Such statistics, called “non-Abelian statistics,” is mathematically described by the braid group. More precisely, one Dirac fermion is defined by a set of two Majorana fermions trapped in two different vortices, and the Hilbert space is constructed from these Dirac fermions. The dimension ($2^m$) of the Hilbert space increases exponentially with an increase in the even number ($2m$) of vortices. Consequently, non-Abelian statistics provides one possible candidate for quantum computation, and hence, $p$-wave superconductors or superfluids may be used as a device for quantum computers. This is why Majorana fermions and non-Abelian statistics have attracted so much attention in recent years. Recently, it has been proposed that even in three dimensions, non-Abelian statistics is realized by Majorana fermions trapped on monopole-like objects.

Though the exchange statistics discussed above is non-Abelian, Majorana fermions on vortices or other defects studied thus far are all “Abelian” in the sense that only a single Majorana fermion is trapped in each defect. On the other hand, if multiple Majorana fermions are trapped inside a defect and can continuously mix with each other, one may be able to interpret them as a single “non-Abelian Majorana fermion,” namely a Majorana fermion having a non-Abelian internal symmetry. Then, these fermions must belong to representations of the underlying Lie group.

In this article, we show that non-Abelian Majorana fermions obey a non-Abelian exchange statistics that has not been identified before. Since Majorana fermions are real (i.e. not complex) fields, their group representation must also be real. The simplest of such representation is the vector representation, triplet, of the SO(3) Lie group. We explicitly construct the non-Abelian exchange statistics of non-Abelian Majorana fermions belonging to the vector representation of SO(3). We show not only that the Hilbert space for $2m$ vortices has a dimension $2^{3m}$ that is much larger than the $2^m$ of Abelian fermions but also that it contains a new component. In addition to the non-Abelian statistics already derived by Ivanov, we find another structure, i.e. the Coxeter group: the entire non-Abelian statistics is a tensor product of these two. The Coxeter group is a symmetry group of higher-dimensional generalization of polytopes such as a triangle or a tetrahedron, which was found by Harold Scott MacDonald (“Donald”) Coxeter, one of the great mathematicians of the 20th century. The large Hilbert space spanned by non-Abelian Majorana fermions contains high-dimensional internal spaces of various representations of SO(3), not only singlets and triplets but also quintets and higher representations in general, where the Coxeter group acts on them to exchange multiple states in the same representations.

One question arising immediately may be whether there exist physical systems realizing such non-Abelian Majorana fermions in reality. The answer is yes, such systems probably exist in the universe, i.e. in quark matter...
at extremely high density in neutron stars or quark stars, which are expected to exhibit the so-called superconductivity\textsuperscript{11,12}. These stars rotate rapidly, and consequently, stable vortices are created; these vortices are non-Abelian vortices with color magnetic fluxes confined inside them\textsuperscript{13}. We have shown in our previous paper\textsuperscript{14} that non-Abelian Majorana fermions of an $SO(3)$ triplet indeed exist in the core of a non-Abelian vortex. The origin of this $SO(3)$ group is the color-flavor locked symmetry $SU(3)$ in the ground state of a color superconductor\textsuperscript{11,12}, which is spontaneously broken down to its subgroup $SU(2) \times U(1)$ in the core of a vortex\textsuperscript{13}. Since there remains an unbroken symmetry $SU(2) \sim SO(3)$ inside the core, the zero modes trapped in it must belong to the representations of $SO(3)$. We have found triplet and singlet Majorana fermions\textsuperscript{14}; however, only the triplet is a new object to be considered in this article. Theoretically, fermionic modes in vortices are treated by the Bogoliubov-de Gennes equation, which is an equation for fermions coupled to the vortex profile. For example, the particle ($\varphi$) and hole ($\eta$) components of the triplet zero modes (chirally right-handed) are approximately given, in two-component Weyl spinor representation, as

\begin{align*}
\varphi(r, \theta) &= e^{-i\Delta_{CFL}/r} \begin{pmatrix} J_0(\mu r) e^{i\varphi} \\ iJ_1(\mu r) e^{i\varphi} \end{pmatrix}, \\
\eta(r, \theta) &= e^{-i\Delta_{CFL}/r} \begin{pmatrix} -J_1(\mu r) e^{-i\varphi} \\ iJ_0(\mu r) \end{pmatrix},
\end{align*}

where $\Delta_{CFL}$ is the gap value of the bulk color superconductor (in the color-flavor locked phase), $\mu$ is the chemical potential, and $r, \theta$ are the polar coordinates perpendicular to the $z$ direction. The triplet zero modes $\psi^a (a = 1, 2, 3)$ are then compactly expressed as $\psi^a \propto (\varphi, (-1)^a + \eta)^t$, and they satisfy the Majorana condition $\psi^a = (\psi^a)^C$, where $(\psi^a)^C \propto (\eta^t, (-1)^a - \varphi)^t$ with $C$ being the charge conjugation. It should be noted that these zero-mode solutions (non-Abelian Majorana fermions) are well localized around the center of the vortex and do not depend on the $z$ coordinate, indicating that they are essentially local objects on a two-dimensional plane. Hence, the non-Abelian vortices appearing in high-density matter provide an example of realization of the non-Abelian Majorana fermions.

However, it should be emphasized that our conclusion in this article does not rely on any specific model; all that we need is non-Abelian Majorana fermions of an $SO(3)$ triplet. Thus, our analysis raises a possibility to realize such a statistics in table top samples, for instance, in cold atomic gases that can be well controlled through experiments. It should also be noted that the $SO(3)$ group and its vector representation are chosen only for illustration as the simplest example and that our method works for arbitrary Lie groups and arbitrary (real) representations, opening up a new possibility of Majorana fermions, non-Abelian statistics, and quantum computations.

This paper is organized as follows. After recalling non-Abelian statistics of Abelian vortices in Sec. \textsuperscript{II} we construct the matrix representation of exchange operators of two and four non-Abelian vortices in Sec. \textsuperscript{III}. We then identify the Coxeter group from the decomposition of those matrices in Sec. \textsuperscript{IV}. Section \textsuperscript{V} is devoted to summary and discussion. In Appendix \textsuperscript{A} we present explicit forms of the basis of the Hilbert spaces for four non-Abelian vortices. In Appendix \textsuperscript{B} we give a decomposition of exchange matrices at the operator level.

\section{Statistics on Two-Dimensional Plane}

The exchange of particles on a two-dimensional plane is described by the braid group. Let us suppose $n$ particles (braids) and label them as shown in Fig. 1. The braid group is defined as a set of operations $T_k$ for $k = 1, \cdots, n - 1$ that involve the exchange of the positions of the neighboring $k$th and $(k + 1)$th braids in a way such that the $k$th braid always goes around the $(k + 1)$th braid in an anti-clockwise direction. The operations $T_k$ satisfy the following braid relations: (i) $T_k T_l = T_l T_k$ for $|k - l| > 1$ and (ii) $T_k T_{k+1} T_k = T_{k+1} T_k T_k$ for $|k - l| = 1$, which are schematically shown in Fig. 2. It should be noted that $T_k^{-1} \neq T_k$ because the operation is directed. This simple definition of the braid group allows for various representations with rich non-trivial structures, as we will see below.

The representation of the braid group is expressed by a linear group \{\tau_k | k = 1, \cdots, n - 1\} acting on a vector space. Here, $\tau_k$’s also satisfy the following braid relations: (i) $\tau_k \tau_l = \tau_l \tau_k$ for $|k - l| > 1$ and (ii) $\tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_k$ for $|k - l| = 1$. The representations contain information about the statistics on the exchange of particles. For example, the one-dimensional representation allows for the anyon statistics, which gives a wave func-

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Schematic of $n$ particles on a two-dimensional plane and exchange of the $k$th and $(k + 1)$th particles denoted by $T_k$ ($k = 1, \cdots, n - 1$).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Schematic of the braid relations (i) $T_k T_l = T_l T_k$ with $|k - l| > 1$ and (ii) $T_k T_{k+1} T_k = T_{k+1} T_k T_k$.}
\end{figure}
tion a complex factor under the exchange of particles; \( \tau_1 = \tau_2 = \cdots = \tau_{n-1} = e^{i\theta} \) with \( 0 \leq \theta < 2\pi \) being a real number. Although the anyon statistics is a characteristic statistics in two dimensions, it is still Abelian. More generally, the braid group allows for non-Abelian statistics in which neighboring \( \tau_k \)'s are noncommutative: \( [\tau_k, \tau_l] \neq 0 \) for \( |k - l| = 1 \).

III. NON-ABELIAN STATISTICS OF MULTIPLE NON-ABELIAN VORTICES

Let us consider \( n \) non-Abelian vortices with the \( SO(3) \) symmetry. There are correspondingly \( n \) non-Abelian Majorana fermions belonging to the triplet of \( SO(3) \) at each vortex site. Let us introduce operators \( \gamma^a_k \) \( (a = 1, 2, 3) \) for creation operators of the triplet non-Abelian Majorana fermions in the \( k \)th non-Abelian vortex. These operators satisfy the anticommutation relation \( \{ \gamma^a_k, \gamma^b_k \} = 2\delta^a_b \delta^{ab} \) and the self-conjugation relation \( \gamma^a_k = \gamma^a_k \). This formulation is an extension of the Abelian Majorana fermions discussed in Ref. [6] to the non-Abelian case. An exchange of neighboring \( k \)th and \( (k+1) \)th non-Abelian vortices, denoted by \( T_k \) as shown in Fig. 3 induces an exchange of the non-Abelian Majorana fermions. Because the Majorana fermion turning around a vortex changes the sign of the wave function owing to the odd winding number and the Majorana condition, a cut should be introduced to keep the phase single valued [7]. Consequently, the operation \( T_k \) induces the following transformation (see Fig. 3):

\[
T_k : \begin{cases}
\gamma^a_k &\rightarrow \gamma^a_{k+1} \\
\gamma^a_{k+1} &\rightarrow -\gamma^a_k
\end{cases}
\]

with the rest \( \gamma^a_{l} \) \( (l \neq k, k+1) \) unchanged. It will turn out that the presence of this minus sign is crucial to induce the non-Abelian statistics.

Now, we discuss the representation \( \tau_k \) for the exchange of non-Abelian Majorana fermions. First, it should be noted that transformation (3) can be realized by \( SO(3) \) invariant unitary operators \( \tilde{\tau}_k = \frac{1}{2} \gamma^a_k \gamma^{a+1}_k \), where \( \gamma^a_k \) is made of non-Abelian Majorana operators \( \gamma^a_k \) and \( \gamma^a_{k+1} \) as

\[
\tilde{\tau}_k^a = \frac{1 + \gamma^a_{k+1} \gamma^a_k}{\sqrt{2}} \quad (\text{not summed over} \, a).
\]

It is easily verified that \( \tilde{\tau}_k \gamma^a_k \tilde{\tau}_k^{-1} \) indeed generates the above transformation. Once the Hilbert space is defined, one obtains the representations of \( \tilde{\tau}_k \) as matrices. In order to define the bases of the Hilbert space, we first introduce an operator of non-Abelian Dirac fermions \( \hat{\Psi}^a_k \) \( (k = 1, \cdots, n/2) \) using an even number \( (n) \) of non-Abelian vortices. The non-Abelian Dirac fermions satisfy the anti-commutation relations \( \{ \hat{\Psi}^a_k, \hat{\Psi}^b_{k'} \} = \delta^a_b \delta^{ab} \), \( \{ \hat{\Psi}^a_k, \hat{\Psi}^b_{k'} \} = \{ \hat{\Psi}^a_k, \hat{\Psi}^b_{k'} \} = 0 \), and the operators \( \hat{\Psi}^a_k \) correspond to annihilation and creation operators, respectively. Then, we can construct the Hilbert space by acting successively creation operators \( \hat{\Psi}^a_k \)'s on the “vacuum state” \( |0\rangle \) defined by \( \hat{\Psi}^a_k |0\rangle = 0 \). In what follows, we concretely show representations \( \tau_k \) in two cases with two \( (n = 2) \) and four \( (n = 4) \) non-Abelian vortices.

First, we discuss the case with two non-Abelian vortices \( (n = 2) \), where we can define only one operation \( T_1 \) and only one non-Abelian Dirac fermion \( \hat{\Psi}^1_1 \). The Hilbert space is spanned by the following four basis states: singlet-even \( |1_1\rangle \) \( (0\rangle \) (vacuum), singlet-odd \( |1_3\rangle \) \( = \frac{1}{\sqrt{2}} e^{i\pi/4} \hat{\Psi}^1_1 \hat{\Psi}^2_{1\bar{1}} |0\rangle \) (filled by three Dirac fermions), triplet-even \( |3_1\rangle \) \( = \frac{1}{\sqrt{2}} e^{i\pi/4} \hat{\Psi}^1_1 \hat{\Psi}^2_{1\bar{1}} |0\rangle \) (occupied by two Dirac fermions), and triplet-odd \( |3_3\rangle \) \( = \hat{\Psi}^1_1 |0\rangle \) (occupied by one Dirac fermion), where the subscript denotes the number of non-Abelian Dirac fermions. With these bases \( \{1_1, 1_3, 3_1, 3_3\} \), we find the representation \( \tau_1 \) to be an \( 8 \times 8 \) matrix: \( \tau_1 = \text{diag}(\omega, \omega^3, 1, 1, 1, 1, 1, 1) \), where \( \omega \) is a 3rd root of unity.

\[
\tau_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \omega^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \omega^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega
\end{pmatrix}
\]

The 3 \times 3 unit matrix \( I_{3 \times 3} \) means the three components in each triplet state. The obtained matrix \( \tau_1 \) is diagonal and, thus, the system of two non-Abelian Majorana fermions follows the (anyonlike) Abelian statistics.

Second, we discuss the representation \( \tau_k \) \( (k = 1, 2, 3) \) for four non-Abelian vortices \( (n = 4) \). We can construct the Hilbert space in a manner similar to that described above, and this is expressed by singlet (1), triplet (3), and quintet (5) states, which are further specified by even (E) and odd (O) numbers of Dirac fermions (see Appendix A). Then, we obtain the representations \( \tau_k \) \( (k = 1, 2, 3) \) as 64 \times 64 matrices; \( \tau_k = \text{diag}(\tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1, \tau_k^1) \). The submatrices \( \tau_k^{M,P} \) \( (M = 1, 3, 5, P = E, O) \) are found to have the following structure:

\[
\tau_k^{M,P} = \sigma_k^M \otimes h_k^P.
\]

Here, the first matrices \( \sigma_k^M \) have different expressions depending on the multiplets:

\[
\sigma_1 = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{pmatrix}
1 & \sqrt{3} \\
\sqrt{3} & -1
\end{pmatrix}, \quad \sigma_3 = \sigma_1.
\]
for the singlet states,

\[
\sigma_1^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2^3 = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},
\]

\[
\sigma_3^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

(6)

for the triplet states, and

\[
\sigma_1^5 = \sigma_2^5 = \sigma_3^5 = 1,
\]

(7)

for the quintet states, and

\[
\tau
\]

The Coxeter matrices for the other representations are given by submatrices in \(M_3\). The Coxeter matrix for the singlet is the top-left (bottom-right) 2 \times 2 submatrix

\[
\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}
\]

in \(M_3\), because there are only two distinct operators \(\sigma_1^5\) and \(\sigma_2^5\) (\(\sigma_3^5\) and \(\sigma_4^5\)) in Eq. (7). The Coxeter matrix for the quintet is the top-left (middle or bottom-right) 1 \times 1 submatrix (1) in \(M_3\), because there is only one distinct operator \(\sigma_1^5\) (\(\sigma_2^5\) or \(\sigma_3^5\)) in Eq. (7).

The Coxeter group is closely related to the geometry in a high-dimensional space, because conditions (a) and (b) can be interpreted as geometrical operations in the Hilbert space: Condition (a) corresponds to a reflection and condition (b) corresponds to a rotation by an angle \(2\pi/m_{ij}\). Therefore, the Coxeter group leads to the existence of several polytopes that are invariant under reflections and rotations, such as a 2-simplex (triangle) under the reflections \(\sigma_1^5\) and \(\sigma_2^5\) for the singlet and a 3-simplex (tetrahedron) under the reflections \(\sigma_1^3\), \(\sigma_2^3\), and \(\sigma_3^3\) for the triplet, as shown in Fig. 4. The fact that the Coxeter group appears in the exchange statistics of the system of the non-Abelian vortices provides a new insight that the non-Abelian statistics of non-Abelian Majorana fermions can be intuitively understood with the help of the geometry of polytopes.

We may extend our discussion to the system of any even number \((n = 2m)\) of non-Abelian vortices. Because an interchange of vortices does not mix the \(i\)th and \(j\)th vortices for \(|i - j| > 1\), the essential structure is already determined from the analysis for four vortices. Therefore, we find \(m_{i,j} = 3\) when \(|i - j| = 1\) and \(m_{i,j} = 2\) when \(|i - j| > 1\) for the representation matrices \(\sigma_i (i = 1, \ldots, 2m - 1)\), which are obtained from the decomposition of \(\tau_k\). Hence, the Coxeter matrix will be
given by the \((2m - 1) \times (2m - 1)\) matrix as follows:

\[
\mathbf{M}_{2m-1} = \begin{pmatrix}
1 & 3 & 2 & 2 & \cdots \\
3 & 1 & 3 & 2 & \cdots \\
2 & 3 & 1 & 3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\] (10)

In other words, a product \(\sigma_i \sigma_j\) of reflections \(\sigma_i\) and \(\sigma_j\) makes a rotation by an angle \(2\pi/3\) for \(|i - j| = 1\) and \(2\pi/2 = \pi\) for \(|i - j| > 1\). Therefore, the system of an even number \((2m)\) of non-Abelian vortices leads to the existence of a \((2m - 1)\)-simplex as the highest-dimensional object (see also Appendix B). The Coxeter group summarized by the matrix \((10)\) is called \(A_{2m-1}\), which is in fact isomorphic to the permutation group of \(2m\) elements.

V. SUMMARY AND DISCUSSION

In conclusion, we have explicitly constructed the non-Abelian statistics, described by the Coxeter group, that appears in a system of non-Abelian vortices having zero-energy non-Abelian Majorana fermions. Although color superconductors are only one example realizing such non-Abelian vortices thus far, our result is robust: Any system admitting vortices with three zero-energy Majorana fermions will have this non-Abelian statistics. Cold atomic gasses such as Bose-Fermi mixture may offer such examples. We hope that not only the large dimension of the Hilbert space but also the Coxeter group should play important roles in quantum computations.

It is known that the Coxeter group is classified into several types \([10]\). As denoted above, the Coxeter group summarized by the matrix \((10)\) is called \(A_{2m-1}\). Then, it would be interesting to ask whether the system of non-Abelian Majorana fermions can lead to other types of Coxeter groups, such as \(D_l (l \geq 2)\), \(E_l (l \geq 4)\), \(F_l (l = 6, 7, 8)\), \(G_2\), \(H_l (l = 3, 4)\), and \(I_2(l) (l = 5, l \geq 7)\), or which new symmetry of the non-Abelian Majorana fermions would be connected to them. These questions would open an extensive view about the relationship between the non-Abelian Majorana fermions and the Coxeter groups.

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Appendix A: Bases of the Hilbert space for the system of four non-Abelian vortices

We explicitly present the bases of the Hilbert space of four non-Abelian vortices. There are four bases,

\[
\begin{align*}
|1_{00}\rangle &= |0\rangle, \\
|1_{33}\rangle &= i\frac{1}{3!}e^{abc}1\frac{1}{3!}e^{def}\hat{\psi}_1^a\hat{\psi}_1^b\hat{\psi}_2^c\hat{\psi}_2^d|\Psi_1\rangle|\Psi_2\rangle|\Psi_3\rangle|\Psi_4\rangle, \\
|1_{11}\rangle &= i\frac{1}{\sqrt{3}}\hat{\psi}_1^a|\Psi_1^a\rangle|0\rangle, \\
|1_{22}\rangle &= i\frac{1}{\sqrt{3}}|\Psi_1^a\rangle|\Psi_1^b\rangle|\Psi_1^c\rangle|\Psi_1^d\rangle,
\end{align*}
\] (A1)

for the singlet-even \((1, \mathcal{E})\) states and

\[
\begin{align*}
|1_{03}\rangle &= \frac{1}{3!}e^{abc}\hat{\psi}_2^a\hat{\psi}_1^b\hat{\psi}_2^c|0\rangle, \\
|1_{30}\rangle &= -i\frac{1}{3!}e^{abc}\hat{\psi}_1^a\hat{\psi}_1^b\hat{\psi}_1^c|0\rangle, \\
|1_{21}\rangle &= -i\frac{1}{3!}e^{abc}\hat{\psi}_1^a\hat{\psi}_1^b\hat{\psi}_2^c|0\rangle, \\
|1_{12}\rangle &= i\frac{1}{3!}e^{abc}\hat{\psi}_1^a\hat{\psi}_2^b\hat{\psi}_2^c|0\rangle,
\end{align*}
\] (A2)

for the singlet-odd \((1, \mathcal{O})\) states. There are six bases,

\[
\begin{align*}
|3_{02}\rangle &= \frac{1}{2!}e^{abc}\hat{\psi}_2^a\hat{\psi}_2^b|0\rangle, \\
|3_{31}\rangle &= -i\frac{1}{3!}e^{bcd}\hat{\psi}_1^b\hat{\psi}_1^c\hat{\psi}_1^d|0\rangle, \\
|3_{22}\rangle &= \frac{1}{2!}e^{abc}\frac{1}{2!}e^{bde}e^{cfd}\hat{\psi}_1^c\hat{\psi}_2^d|0\rangle, \\
|3_{11}\rangle &= i\frac{1}{2!}e^{abc}\hat{\psi}_1^a\hat{\psi}_2^b\hat{\psi}_2^c|0\rangle, \\
|3_{20}\rangle &= -\frac{1}{2!}e^{abc}\hat{\psi}_2^a\hat{\psi}_1^b\hat{\psi}_1^c|0\rangle, \\
|3_{33}\rangle &= i\frac{1}{3!}e^{bcd}\hat{\psi}_1^a\hat{\psi}_2^b\hat{\psi}_2^c\hat{\psi}_2^d|0\rangle,
\end{align*}
\] (A3)

for the triplet-even \((3, \mathcal{E})\) states and

\[
\begin{align*}
|3_{01}\rangle &= \hat{\psi}_2^a|0\rangle, \\
|3_{32}\rangle &= i\frac{1}{3!}e^{bcd}\hat{\psi}_1^b\hat{\psi}_1^c\hat{\psi}_1^d\frac{1}{2!}e^{ace}\hat{\psi}_2^e|0\rangle, \\
|3_{21}\rangle &= \frac{1}{2!}e^{abc}\frac{1}{2!}e^{bde}\hat{\psi}_1^c\hat{\psi}_2^d|0\rangle, \\
|3_{12}\rangle &= -i\frac{1}{\sqrt{2}}\frac{1}{2!}e^{bcd}\hat{\psi}_2^b\hat{\psi}_2^c\hat{\psi}_2^d|0\rangle, \\
|3_{33}\rangle &= \frac{1}{\sqrt{2}}e^{abc}\hat{\psi}_1^a\hat{\psi}_2^b\hat{\psi}_2^c\hat{\psi}_2^d|0\rangle, \\
|3_{10}\rangle &= i\hat{\psi}_1^a|0\rangle.
\end{align*}
\] (A4)
for the quintet-odd \((3, \mathcal{O})\) states. There are two bases,

\[
|5_{21}\rangle = i\mathcal{N}\left\{ \frac{1}{2}\left( \frac{1}{2!}e^{acd}\hat{\Psi}^a_1\hat{\Psi}^b_2 + \frac{1}{2!}e^{bde}\hat{\Psi}^a_1\hat{\Psi}^b_2 + \frac{1}{2!}e^{cde}\hat{\Psi}^a_1\hat{\Psi}^b_2 \right) - \frac{\delta_{ab}}{3}\hat{\Psi}^a_1\hat{\Psi}^a_2 \right\}|0\rangle,
\]

(A21)

\[
|5_{12}\rangle = -\mathcal{N}\left\{ \frac{1}{2}\left( \frac{1}{2!}e^{acd}\hat{\Psi}^a_1\hat{\Psi}^b_2 + \frac{1}{2!}e^{bde}\hat{\Psi}^a_1\hat{\Psi}^b_2 + \frac{1}{2!}e^{cde}\hat{\Psi}^a_1\hat{\Psi}^b_2 \right) - \frac{\delta_{ab}}{3}\hat{\Psi}^a_1\hat{\Psi}^a_2 \right\}|0\rangle,
\]

(A22)

for the quintet-even \((5, \mathcal{E})\) states and

\[
|5_{21}\rangle = -i\mathcal{N}\left\{ \frac{1}{2}\left( \frac{1}{2!}e^{acd}\hat{\Psi}^a_1\hat{\Psi}^b_2 + \frac{1}{2!}e^{bde}\hat{\Psi}^a_1\hat{\Psi}^b_2 + \frac{1}{2!}e^{cde}\hat{\Psi}^a_1\hat{\Psi}^b_2 \right) - \frac{\delta_{ab}}{3}\hat{\Psi}^a_1\hat{\Psi}^a_2 \right\}|0\rangle,
\]

(A23)

\[
|5_{12}\rangle = -\mathcal{N}\left\{ \frac{1}{2}\left( \frac{1}{2!}e^{acd}\hat{\Psi}^a_1\hat{\Psi}^b_2 + \frac{1}{2!}e^{bde}\hat{\Psi}^a_1\hat{\Psi}^b_2 + \frac{1}{2!}e^{cde}\hat{\Psi}^a_1\hat{\Psi}^b_2 \right) - \frac{\delta_{ab}}{3}\hat{\Psi}^a_1\hat{\Psi}^a_2 \right\}|0\rangle,
\]

(A24)

for the quintet-odd \((5, \mathcal{O})\) states with \(\mathcal{N} = \sqrt{3/2}\) for \(a = b\) and \(\mathcal{N} = \sqrt{2}\) for \(a \neq b\). With these bases we obtain the matrices \(\tau_k (k = 1, 2, 3)\) as presented in Eqs. (4)–(8).

**Appendix B: Coxeter matrix for arbitrary number of non-Abelian vortices**

In the text, we show the tensor-product structure of the matrices \(\tau_k = \sigma_k^M \otimes \hat{h}_k^P \) \((M = 1, 3, 5, \text{ and } P = \mathcal{E}, \mathcal{O})\) obtained in the Hilbert space with bases presented in Appendix A. In fact, such a product structure holds even at the operator level. The operator \(\hat{\tau}_k = \hat{\tau}_k^{1/2}\hat{\tau}_k^{3/2}\) with \(\hat{\tau}_k = (1 + \gamma_k^a + \gamma_k^a)/\sqrt{2}\) is expressed as a product of two \(SO(3)\) invariant unitary operators, i.e.

\[
\hat{\tau}_k = \hat{\sigma}_k \hat{h}_k,
\]

(B1)

where

\[
\hat{\sigma}_k = \frac{1}{2}\left( 1 - \gamma_k^a + \gamma_k^b + \gamma_k^c - \gamma_k^a \gamma_k^b \gamma_k^c - \gamma_k^a \gamma_k^b \gamma_k^c \gamma_k^d \right),
\]

(B2)

and

\[
\hat{h}_k = \frac{1}{\sqrt{2}}\left( 1 - \gamma_k^a \gamma_k^b \gamma_k^c \gamma_k^d \right).
\]

(B3)

It is easily verified that operators \(\hat{\sigma}_k\) satisfy relations (a) and (b) of the Coxeter group, such that

\[
\sigma_k^2 = 1, \quad (\hat{\sigma}_k \hat{\sigma}_l)^3 = 1 \quad \text{for} \quad |k - l| = 1, \quad (\hat{\sigma}_k \hat{\sigma}_l)^2 = 1 \quad \text{for} \quad |k - l| > 1.
\]

(B4)

Therefore, we confirm the Coxeter matrix [10] for an arbitrary number of non-Abelian vortices.

[1] E. Majorana, Nuovo Cim. 14, 171 (1937).
[2] F. Wilczek, Nature Phys. 5, 614 (2009).
[3] G. E. Volovik, JETP Lett. 70, 609–614 (1999).
[4] L. Fu and C. L. Kane, Phys. Rev. Lett. 100, 096407 (2008).
[5] N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
[6] D. A. Ivanov, Phys. Rev. Lett. 86, 268 (2001).
[7] For a review, see C. Nayak, S. H. Simon, A. Stern, M. Freedman and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[8] J. C. Y. Teo and C. L. Kane, Phys. Rev. Lett. 104, 046401 (2010).
[9] H. S. M. Coxeter, Ann. Of Math. 35, 588–621 (1934); J. London Math. Soc. 10, 21–25 (1935).
[10] J. E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics, Vol.29 (Cambridge University Press, Cambridge, UK, 1990).
[11] M. G. Alford, K. Rajagopal and F. Wilczek, Phys. Rev. Lett. 80, 242 (1998); Nucl. Phys. B 537, 443 (1999).
[12] For a review, see M. G. Alford, A. Schmitt, K. Rajagopal and T. Schafer, Rev. Mod. Phys. 80, 1455 (2008).
[13] A. P. Balachandran, S. Digal and T. Matsuura, Phys. Rev. D 73, 074009 (2006); M. Eto and M. Nitta, Phys. Rev. D 80, 125007 (2009).
[14] S. Yasui, K. Itakura and M. Nitta, Phys. Rev. D 81, 105003 (2010).
[15] E. Nakano, M. Nitta and T. Matsuura, Phys. Rev. D 78, 045002 (2008); M. Eto, E. Nakano and M. Nitta, Phys. Rev. D 80, 125011 (2009); M. Eto, M. Nitta and N. Yamamoto, Phys. Rev. Lett. 104, 161601 (2010); Y. Hirota, T. Kanazawa and M. Nitta, Phys. Rev. D 83, 085018 (2011); M. Eto, M. Nitta and N. Yamamoto, Phys. Rev. D 83, 085005 (2011).