ON THE DISCRETENESS OF SOME CLASS OF MAPPINGS ON THE BOUNDARY

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Abstract

We consider a class of open discrete and closed mappings between domains of the Euclidean space that satisfy the inverse Poletsky-type inequality. We have proved that under certain conditions such mappings have a continuous extension to the boundary of the domain, and this extension is a discrete mapping of the boundary. We separately considered the cases of domains with bad and good boundaries.

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1  Introduction

In [Vu], some issues related to the discreteness of a closed quasiregular map \( f : \mathbb{B}^n \to \mathbb{R}^n \) in \( \mathbb{B}^n \) are considered, see [Vu, Lemma 4.4, Corollary 4.5 and Theorem 4.7]. In particular, the following result holds (see [Vu, Theorem 4.7]).

Theorem. Let \( f : \mathbb{B}^n \to G' \) be a closed non-constant quasiregular mapping and let \( G' \) be locally connected on the boundary. Then \( f \) can be extended to a continuous mapping \( f : \mathbb{B}^n \to \mathbb{R}^n \) such that \( N(f) = N(\overline{f}) \) and hence \( \overline{f} \) is discrete.

This article is devoted to a deeper study of this fact, more precisely, we extend the mentioned result not only to a wider class of mappings, but also to a wider class of domains. In this case, two fundamentally different situations will be considered: when mappings have the usual continuous extension, and when this extension should be understood in terms of prime ends.

Let us turn to the definitions. In what follows, \( M_p(\Gamma) \) denotes the \( p \)-modulus of a family \( \Gamma \) (see [Val, Section 6]). We write \( M(\Gamma) \) instead \( M_n(\Gamma) \). Let \( y_0 \in \mathbb{R}^n \), \( 0 < r_1 < r_2 < \infty \) and

\[
A = A(y_0, r_1, r_2) = \{ y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2 \} .
\] (1.1)
Given \( x_0 \in \mathbb{R}^n \), we put
\[
B(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| < r \}, \quad B^n = B(0, 1),
\]
\[
S(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| = r \}.
\]
Given sets \( E, F \subset \mathbb{R}^n \) and a domain \( D \subset \mathbb{R}^n \) we denote by \( \Gamma(E, F, D) \) a family of all paths \( \gamma : [a, b] \to \mathbb{R}^n \) such that \( \gamma(a) \in E, \gamma(b) \in F \) and \( \gamma(t) \in D \) for \( t \in [a, b] \). Given a mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \), a point \( y_0 \in f(D) \setminus \{ \infty \} \), and \( 0 < r_1 < r_2 < r_0 = \sup_{y \in f(D)} |y - y_0| \), we denote by \( \Gamma_f(y_0, r_1, r_2) \) a family of all paths \( \gamma \) in \( D \) such that \( f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2)) \). Let \( Q : \mathbb{R}^n \to [0, \infty) \) be a Lebesgue measurable function. We say that \( f \) satisfies the inverse Poletsky inequality at a point \( y_0 \in \overline{f(D)} \setminus \{ \infty \} \) if the relation
\[
M(\Gamma_f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^n(|y - y_0|) \, dm(y) \tag{1.2}
\]
holds for any Lebesgue measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that
\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \tag{1.3}
\]
Using the inversion \( \psi(y) = \frac{y}{|y|^2} \), we may also define the relation \((1.2)\) at the point \( y_0 = \infty \).

A mapping \( f : D \to \mathbb{R}^n \) is called discrete if the pre-image \( \{ f^{-1}(y) \} \) of any point \( y \in \mathbb{R}^n \) consists of isolated points, and open if the image of any open set \( U \subset D \) is an open set in \( \mathbb{R}^n \). A mapping \( f \) of \( D \) onto \( D' \) is called closed if \( f(E) \) is closed in \( D' \) for any closed set \( E \subset D \) (see, e.g., [Vul, Chapter 3]). Let \( h \) be a chordal metric in \( \mathbb{R}^n \),
\[
h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}} ,
\]
\[
h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}} \quad x \neq \infty \neq y . \tag{1.4}
\]
and let \( h(E) := \sup_{x,y \in E} h(x, y) \) be a chordal diameter of a set \( E \subset \mathbb{R}^n \) (see, e.g., [Val, Definition 12.1]). Everywhere further the boundary \( \partial A \) of the set \( A \) and the closure \( \overline{A} \) should be understood in the sense extended Euclidean space \( \mathbb{R}^n \). A continuous extension of the mapping \( f : D \to \mathbb{R}^n \) also should be understood in terms of mapping with values in \( \mathbb{R}^n \) and relative to the metric \( h \) in \((1.4)\) (if a misunderstanding is impossible). Recall that a domain \( D \subset \mathbb{R}^n \) is called locally connected at the point \( x_0 \in \partial D \), if for any neighborhood \( U \) of a point \( x_0 \) there is a neighborhood \( V \subset U \) of \( x_0 \) such that \( V \cap D \) is connected. A domain \( D \) is locally connected at \( \partial D \), if \( D \) is locally connected at any point \( x_0 \in \partial D \). The boundary of the domain \( D \) is called weakly flat at the point \( x_0 \in \partial D \), if for any \( P > 0 \) and for any neighborhood \( U \) of a point \( x_0 \) there is a neighborhood \( V \subset U \) of the same point such that
$M(\Gamma(E, F, D)) > P$ for any continua $E, F \subset D$, which intersect $\partial U$ and $\partial V$. The boundary of the domain $D$ is called weakly flat if the corresponding property is fulfilled at any point of the boundary $D$. Given a mapping $f : D \to \mathbb{R}^n$, a set $E \subset D$ and $y \in \mathbb{R}^n$, we define the multiplicity function $N(y, f, E)$ as a number of preimages of the point $y$ in a set $E$, i.e.

$$N(y, f, E) = \text{card } \{x \in E : f(x) = y\},$$

$$N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E).$$

(1.5)

Note that, the concept of a multiplicity function may also be extended to sets belonging to the closure of a given domain. We put

$$q_{y_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(y_0, r)} Q(y) d\mathcal{H}^{n-1}(y),$$

(1.6)

and $\omega_{n-1}$ denotes the area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. We say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at a point $x_0 \in D$, write $\varphi \in FMO(x_0)$, if

$$\limsup_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int_{B(x_0, \varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \, dm(x) < \infty,$$

where $\overline{\varphi}_\varepsilon = \frac{1}{\Omega_n \varepsilon^n} \int_{B(x_0, \varepsilon)} \varphi(x) \, dm(x)$. We also say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at $A \subset \overline{D}$, write $\varphi \in FMO(A)$, if $\varphi$ has a finite mean oscillation at any point $x_0 \in A$. The following result is true.

**Theorem 1.1.** Let $n \geq 2$, let $D$ be a domain with a weakly flat boundary and let $D'$ be a domain which is locally connected on its boundary. Let $f$ be open discrete and closed mapping of $D$ onto $D'$ for which there is a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$, equal to zero outside $D'$, such that the relations (1.2)–(1.3) hold at any point $y_0 \in \partial D'$. Assume that, one of the following conditions hold:

1) $Q \in FMO(\partial D')$;

2) for any $y_0 \in \partial D'$ there is $\delta(y_0) > 0$ such that

$$\int_{\varepsilon}^{\delta(y_0)} \frac{dt}{t q_{y_0}^{\frac{n-1}{n}}(t)} < \infty, \quad \int_{0}^{\delta(y_0)} \frac{dt}{t q_{y_0}^{\frac{n-1}{n}}(t)} = \infty$$

(1.7)

for sufficiently small $\varepsilon > 0$.

Then $f$ has a continuous extension $\overline{f} : \overline{D} \to \overline{\mathbb{R}^n}$ such that $N(f, D) = N(f, \overline{D}) < \infty$. In particular, $\overline{f}$ is discrete in $\overline{D}$.

Let us also state the same result for the case of "bad boundaries". Recall some definitions (see, for example, [KR1] and [KR2]). Let $\omega$ be an open set in $\mathbb{R}^k$, $k = 1, \ldots, n - 1$. A
continuous mapping \( \sigma : \omega \to \mathbb{R}^n \) is called a \( k \)-dimensional surface in \( \mathbb{R}^n \). A surface is an arbitrary \((n-1)\)-dimensional surface \( \sigma \) in \( \mathbb{R}^n \). A surface \( \sigma \) is called a Jordan surface, if \( \sigma(x) \neq \sigma(y) \) for \( x \neq y \). In the following, we will use \( \sigma \) instead of \( \sigma(\omega) \subset \mathbb{R}^n \), \( \overline{\sigma} \) instead of \( \overline{\sigma(\omega)} \) and \( \partial \sigma \) instead of \( \partial \sigma(\omega) \setminus \sigma(\omega) \). A Jordan surface \( \sigma : \omega \to D \) is called a cut of \( D \), if \( \sigma \) separates \( D \), that is \( D \setminus \sigma \) has more than one component, \( \partial \sigma \cap \partial D = \emptyset \) and \( \partial \sigma \cap \partial D \neq \emptyset \).

A sequence of cuts \( \sigma_1, \sigma_2, \ldots, \sigma_m, \ldots \) in \( D \) is called a chain, if:

(i) the set \( \sigma_{m+1} \) is contained in exactly one component \( d_m \) of the set \( D \setminus \sigma_m \), wherein \( \sigma_{m-1} \subset D \setminus (\sigma_m \cup d_m) \); (ii) \( \bigcap_{m=1}^{\infty} d_m = \emptyset \).

Two chains of cuts \( \{\sigma_m\} \) and \( \{\sigma'_k\} \) are called equivalent, if for each \( m = 1, 2, \ldots \) the domain \( d_m \) contains all the domains \( d'_k \), except for a finite number, and for each \( k = 1, 2, \ldots \) the domain \( d'_k \) also contains all domains \( d_m \), except for a finite number.

The end of the domain \( D \) is the class of equivalent chains of cuts in \( D \). Let \( K \) be the end of \( D \) in \( \mathbb{R}^n \), then the set \( I(K) = \bigcap_{m=1}^{\infty} d_m \) is called the impression of the end \( K \). Throughout the paper, \( \Gamma(E, F, D) \) denotes the family of all paths \( \gamma : [a, b] \to \mathbb{R}^n \) such that \( \gamma(a) \in E \), \( \gamma(b) \in F \) and \( \gamma(t) \in D \) for every \( t \in [a, b] \). In what follows, \( M \) denotes the modulus of a family of paths, and the element \( dm(x) \) corresponds to the Lebesgue measure in \( \mathbb{R}^n \), \( n \geq 2 \), see [Na1]. Following [Na2], we say that the end \( K \) is a prime end, if \( K \) contains a chain of cuts \( \{\sigma_m\} \) such that \( \lim_{m \to \infty} M(\Gamma(C, \sigma_m, D)) = 0 \) for some continuum \( C \) in \( D \). In the following, the following notation is used: the set of prime ends corresponding to the domain \( D \), is denoted by \( E_D \), and the completion of the domain \( D \) by its prime ends is denoted \( \overline{D}_p \).

Consider the following definition, which goes back to Näkki [Na1], see also [KR1]–[KR3]. We say that the boundary of the domain \( D \) in \( \mathbb{R}^n \) is locally quasiconformal, if each point \( x_0 \in \partial D \) has a neighborhood \( U \) in \( \mathbb{R}^n \), which can be mapped by a quasiconformal mapping \( \varphi \) onto the unit ball \( \mathbb{B}^n \subset \mathbb{R}^n \) so that \( \varphi(\partial D \cup U) \) is the intersection of \( \mathbb{B}^n \) with the coordinate hyperplane.

For a given set \( E \subset \mathbb{R}^n \), we set \( d(E) := \sup_{x, y \in E} |x - y| \). The sequence of cuts \( \sigma_m, m = 1, 2, \ldots \), is called regular, if \( \overline{\sigma_m} \cap \overline{\sigma_{m+1}} = \emptyset \) for \( m \in \mathbb{N} \) and, in addition, \( d(\sigma_m) \to 0 \) as \( m \to \infty \). If the end \( K \) contains at least one regular chain, then \( K \) will be called regular. We say that a bounded domain \( D \) in \( \mathbb{R}^n \) is regular, if \( D \) can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in \( \mathbb{R}^n \), and, besides that, every prime end in \( D \) is regular. Note that space \( \overline{D}_p = D \cup E_D \) is metric, which can be demonstrated as follows. If \( g : D_0 \to D \) is a quasiconformal mapping of a domain \( D_0 \) with a locally quasiconformal boundary onto some domain \( D \), then for \( x, y \in \overline{D}_p \) we put:

\[
\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|, \tag{1.8}
\]

where the element \( g^{-1}(x), x \in E_D \), is to be understood as some (single) boundary point of the domain \( D_0 \). The specified boundary point is unique and well-defined by [IS2], Theorem 2.1, Remark 2.1], cf. [Na2, Theorem 4.1]. It is easy to verify that \( \rho \) in (1.8) is a metric on \( \overline{D}_p \),
and that the topology on $\overline{D}_P$, defined by such a method, does not depend on the choice of the map $g$ with the indicated property.

We say that a sequence $x_m \in D$, $m = 1, 2, \ldots$, converges to a prime end of $P \in E_D$ as $m \to \infty$, if for any $k \in \mathbb{N}$ all elements $x_m$ belong to $d_k$ except for a finite number. Here $d_k$ denotes a sequence of nested domains corresponding to the definition of the prime end $P$. Note that for a homeomorphism of a domain $D$ onto $D'$, one of the most important statements of the manuscript is the following.

**Theorem 1.2.** Let $n \geq 2$, let $D$ be a domain with a weakly flat boundary and let $D'$ be a regular domain. Let $f$ be open discrete and closed mapping of $D$ onto $D'$ for which there is a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$, equal to zero outside $D'$, such that the relations (1.2)–(1.3) hold at any point $y_0 \in \partial D'$. Assume that, one of the following conditions hold:

1) $Q \in FMO(\partial D')$;

2) for any $y_0 \in \partial D'$ there is $\delta(y_0) > 0$ such that

$$\int_{\varepsilon}^{\delta(y_0)} \frac{dt}{t^{q_n-1}(t)} < \infty, \quad \int_{0}^{\delta(y_0)} \frac{dt}{t^{q_n-1}(t)} = \infty \quad (1.9)$$

for sufficiently small $\varepsilon > 0$.

Then $f$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}_P$ such that $N(f, D) = N(f, \overline{D}) < \infty$. In particular, $\overline{f}$ is discrete in $\overline{D}$, that is, $\overline{f}^{-1}(P_0)$ consists only from isolated points for any $P_0 \in E_{D'}$.

### 2 A continuous extension of mappings in the case of good boundaries

In [SSD], we studied in sufficient detail the problem of the local and global behavior of mappings satisfying the so-called inverse Poletsky inequality, in which the corresponding majorant is integrable. In particular, the possibility of continuous extension of these mappings to the boundary of the domain was shown. In this article, we will show a little more, namely that this result holds not only for integrable $Q$, but also for those that have finite integrals on spheres centered at a fixed point on a set of radii some "not very small" measure. Let us point to examples of non-integrable functions that have these finite integrals by spheres and mappings that correspond to them (see, for example, [SevSkv3, Examples 1,2]).

Let $D \subset \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ be a discrete open mapping, $\beta : [a, b) \to \mathbb{R}^n$ be a path, and $x \in f^{-1}(\beta(a))$. A path $\alpha : [a, c) \to D$ is called a maximal $f$-lifting of $\beta$ starting at $x$, if

1. $\alpha(a) = x$;
2. $f \circ \alpha = \beta|_{[a,c)}$;
3. for $c < c' \leq b$, there is no a path $\alpha' : [a, c') \to D$ such that $\alpha = \alpha'|_{[a,c)}$ and $f \circ \alpha' = \beta|_{[a,c')}$. The following statement holds (see [R1, Corollary II.3.3]).
Proposition 2.1. Let \( f : D \rightarrow \mathbb{R}^n \) be a discrete open mapping, \( \beta : [a, b] \rightarrow \mathbb{R}^n \) be a path, and \( x \in f^{-1}(\beta(a)) \). Then \( \beta \) has a maximal \( f \)-lifting starting at \( x \). If \( \beta : (a, b] \rightarrow f(D) \) be a path, and \( x \in f^{-1}(\beta(b)) \), then \( \beta \) has a maximal \( f \)-lifting ending at \( x \).

A path \( \alpha : [a, b] \rightarrow D \) is called a total \( f \)-lifting of \( \beta \) starting at \( x \), if (1) \( \alpha(a) = x \); (2) \( (f \circ \alpha)(t) = \beta(t) \) for any \( t \in [a, b] \). In the case when the mapping \( f \) is also closed, we have a strengthened version of Proposition 2.1 (see, for example, [Vu, Lemma 3.7]).

Proposition 2.2. Let \( f : D \rightarrow \mathbb{R}^n \) be a discrete open and closed mapping, \( \beta : [a, b] \rightarrow f(D) \) be a path, and \( x \in f^{-1}(\beta(a)) \). Then \( \beta \) has a total \( f \)-lifting starting at \( x \).

The following statement holds.

Theorem 2.1. Let \( D \subset \mathbb{R}^n \), \( n \geq 2 \), be a domain with a weakly flat boundary, and let \( D' \subset \mathbb{R}^n \) be locally connected at its boundary. Suppose that \( f \) is open discrete and closed mapping of \( D \) onto \( D' \) satisfying the relation (1.2) at any point \( y_0 \in D' \). Suppose that, for each point \( y_0 \in D' \) and \( 0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0| \) there is a set \( E \subset [r_1, r_2] \) of a positive linear Lebesgue measure such that the function \( Q \) is integrable on \( S(y_0, r) \) for any \( r \in E \). Then \( f \) has a continuous extension \( \overline{f} : \overline{D} \rightarrow \overline{D'} \), while \( \overline{f}(D) = D' \).

Proof. Put \( x_0 \in \partial D \). It is necessary to show the possibility of continuous extension of the mapping \( f \) to the point \( x_0 \). Using the Möbius transform \( \varphi : \infty \mapsto 0 \) and the invariance of the modulus \( M \) under a conformal mapping on the left side of the relation (1.2) (see [Vu, Theorem 8.1]), we may assume that \( x_0 \neq \infty \).

Assume that the conclusion about the continuous extension of the mapping \( f \) to the point \( x_0 \) is not correct. Then there are sequences \( x_i, y_i \in D \), \( i = 1, 2, \ldots \), such that \( x_i, y_i \to x_0 \) as \( i \to \infty \), and

\[
h(f(x_i), f(y_i)) \geq a > 0
\]

for some \( a > 0 \) and any \( i \in \mathbb{N} \), where \( h \) is a chordal metric. Since \( \mathbb{R}^n \) is a compact space, we may assume that the sequences \( f(x_i) \) and \( f(y_i) \) converge to \( z_1 \) and \( z_2 \) as \( i \to \infty \), respectively. We may assume also that \( z_1 \neq \infty \). Since \( f \) is closed, it is boundary preserving (see [Vu, Theorem 3.3]). Thus, \( z_1, z_2 \in \partial D' \). Since \( D' \) is locally connected on the boundary, there are neighborhoods \( U_1 \) and \( U_2 \) of points \( z_1 \) and \( z_2 \) such that \( W_1 = D' \cap U_1 \) and \( W_2 = D' \cap U_2 \) are connected. We may assume that \( W_1 \) and \( W_2 \) are locally connected, because \( U_1 \) and \( U_2 \) may be chosen open (see, e.g., [MRSY, Proposition 13.2]; see Figure 1). We may assume that

\[
U_1 \subset B(z_*, R_0), \quad \overline{B(z_*, 2R_0)} \cap \overline{U_2} = \emptyset, \quad R_0 > 0,
\]

where \( z_* \in D' \) is some point sufficiently close to \( z_1 \). We also may assume that \( f(x_i) \in W_1 \) and \( f(y_i) \in W_2 \) for any \( i = 1, 2, \ldots \). Join the points \( f(x_i) \) and \( f(x_i) \) by a path \( \alpha_i : [0, 1] \rightarrow D' \), and points \( f(y_i) \) and \( f(y_i) \) by a path \( \beta_i : [0, 1] \rightarrow D' \) such that \( |\alpha_i| \subset W_1 \) and \( |\beta_i| \subset W_2 \) as \( i = 1, 2, \ldots \). Let \( \bar{\alpha}_i : [0, 1] \rightarrow D' \) and \( \bar{\beta}_i : [0, 1] \rightarrow D' \) be total liftings of paths \( \alpha_i \) and \( \beta_i \).
ON THE DISCRETENESS OF SOME CLASS OF MAPPINGS...

starting at points \( x_i \) and \( y_i \), respectively (these liftings exist due to [Vu, Lemma 3.7]). Note that the points \( f(x_1) \) and \( f(y_1) \) may have no more than a finite number of pre-images under the mapping \( f \) in the domain \( D \), see [Vu, Lemma 3.2]. Then there exists \( r_0 > 0 \) such that \( \tilde{\alpha}_i(1), \tilde{\beta}_i(1) \in D \setminus B(x_0, r_0) \) for any \( i = 1, 2, \ldots \). Since the boundary of \( D \) is weakly flat, for any \( P > 0 \) there is \( i = i_P \geq 1 \) such that

\[
M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) > P \quad \forall i \geq i_P . \tag{2.3}
\]

Let us to show that, the condition (2.3) contradicts the definition of \( f \) in (1.2). Indeed, due to the relation (2.2) and by [Ku, Theorem 1.1.5.46] we obtain that

\[
f(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) > \Gamma(S(z_*, R_0), S(z_*, 2R_0), A(z_*, R_0, 2R_0)). \tag{2.4}
\]

It follows from (2.4) that

\[
\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D) > \Gamma_f(z_*, R_0, 2R_0) . \tag{2.5}
\]

In addition, by (2.5) we obtain that

\[
M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) \leq M(\Gamma_f(z_*, R_0, 2R_0)) \leq \int_A Q(y) \cdot \eta^n(|y - z_*|) \, dm(y) , \tag{2.6}
\]

where \( A = A(z_*, R_0, 2R_0) \) and \( \eta \) is any Lebesgue measurable function satisfying the relation (1.3) for \( r_1 := R_0 \) and \( r_2 := 2R_0 \). Below we use the following conventions: \( a/\infty = 0 \) for \( a \neq \infty \), \( a/0 = \infty \) for \( a > 0 \) and \( 0 \cdot \infty = 0 \) (see, e.g., [Sal 3.1]). Put

\[
I = \int_{R_0}^{2R_0} \frac{dt}{t^{q_2^{1/(n-1)}}(t)} , \tag{2.7}
\]
where \( q_z(t) \) is defined in (1.6) for \( y_0 := z_\ast \). By the assumption, there is a set \( E \subset [R_0, 2R_0] \) of a positive measure such that \( q_z(t) \) is finite for all \( t \in E \). Thus \( I \neq 0 \) in (2.7.1). In this case, a function \( \eta_0(t) = \frac{1}{t q_z(t)} \) satisfy the relation (1.3) for \( r_1 := R_1 \) and \( r_2 := 2R_0 \). Subsisting this function in the right-hand part of (2.6) and using the Fubini theorem, we obtain that

\[
M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) \leq \frac{\omega_{n-1}}{n-1} < \infty.
\]

The relation (2.8) contradicts with (2.3). The contradiction obtained above disproves the assumption made in (2.1).

The proof of the equality \( \overline{f(D)} = D' \) is similar to the second part of the proof of Theorem 3.1 in [SSD]. \( \square \)

**Remark 2.1.** A slightly different formulation of Theorem 2.1 is also true.

Let \( D \subset \mathbb{R}^n, n \geq 2, \) be a domain which has a weakly flat boundary at a point \( x_0 \in \partial D \), and let \( D' \subset \mathbb{R}^n \) be a locally connected at any point \( z \in C(f, x_0) \). Assume that, \( f \) is an open discrete and closed mapping of \( D \) onto \( D' \) satisfying the relation (1.2) at any point \( y_0 \). Assume also that there exists at least one finite point \( z_1 \in C(f, x_0) \) for which there is \( 0 < r_1 = r_1(z_1) \) such that a function \( Q \) is integrable on \( S(z_1, r) \) for any \( r \in (0, r_1) \). Then \( f \) has a continuous extension \( \overline{f} : D \cup \{x_0\} \rightarrow D' \).

The proof of this statement with minor modifications repeats the proof of the theorem 2.1. We present this proof below.

Assume that the conclusion about the continuous extension of the mapping \( f \) to the point \( x_0 \) is not correct. Then there are at least two sequences \( x_i, y_i \in D, i = 1, 2, \ldots, \) such that \( x_i, y_i \rightarrow x_0 \) as \( i \rightarrow \infty \), while the relation (2.1) holds for some \( a > 0 \) and all \( i \in \mathbb{N} \). Since \( z_1 \in C(f, x_0) \), we may choose the sequences \( f(x_i) \) and \( f(y_i) \) converge to \( z_1 \) and \( z_2 \) as \( i \rightarrow \infty \) respectively, where \( z_1 \in \partial D' \subset \mathbb{R}^n \). Since \( D' \) is locally connected at its boundary, there are disjoint neighborhoods of \( U_1 \) and \( U_2 \) of points \( z_1 \) and \( z_2 \) such that \( W_1 = D' \cap U_1 \) and \( W_2 = D' \cap U_2 \) are connected. We may assume that \( W_1 \) and \( W_2 \) are path connected. We may assume that

\[
U_1 \subset B(z_1, R_\ast), \quad \overline{B(z_1, 2R_0)} \cap \overline{U_2} = \emptyset, \quad 0 < R_\ast < 2R_0 < r_1. \tag{2.9}
\]

Join the points \( f(x_i) \) and \( f(x_1) \) by a path \( \alpha_i : [0, 1] \rightarrow D' \), and points \( f(y_i) \) and \( f(y_1) \) by a path \( \beta_i : [0, 1] \rightarrow D' \) such that \( |\alpha_i| \subset W_1 \) and \( |\beta_i| \subset W_2 \) for \( i = 1, 2, \ldots \). Let \( \tilde{\alpha}_i : [0, 1] \rightarrow D' \) and \( \tilde{\beta}_i : [0, 1] \rightarrow D' \) be whole liftings of \( \alpha_i \) and \( \beta_i \) starting at points \( x_i \) and \( y_i \), respectively (these liftings exist due to [Vil Lemma 3.7]). Observe that, the points \( f(x_1) \) and \( f(y_1) \) have at least a finite number of pre-images in \( D \) under \( f \), see [Vil Lemma 3.2]. Then there is \( r_\ast > 0 \) such that \( \tilde{\alpha}_i(1), \tilde{\beta}_i(1) \in D \setminus B(x_0, r_\ast) \) for any \( i = 1, 2, \ldots \). Since the boundary of \( D \) is weakly flat, for any \( P > 0 \) there is \( i = i_P \geq 1 \) such that the relation (2.3) holds. On the other hand, due to [Ku Theorem 1.1.5.46]

\[
f(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) > \Gamma(S(z_1, R_\ast), S(z_1, 2R_0), A(z_1, R_\ast, 2R_0)). \tag{2.10}
\]
It follows from \( (2.10) \) that
\[
\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D) > \Gamma_f(z_0, R_*, 2R_0). \tag{2.11}
\]

In turn, by \( (2.11) \) we have the following:
\[
M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) \leq M(\Gamma_f(z_1, R_*, 2R_0)) \leq \int_A Q(y) \cdot \eta^n(|y - z_1|) \, dm(y), \tag{2.12}
\]
where \( A = A(z_1, R_*, 2R_0) \) and \( \eta \) is nonnegative Lebesgue measurable function satisfying the relation \( (1.3) \) for \( r_1 := R_* \) and \( r_2 := 2R_0 \).

Put
\[
I = \int_{R_*}^{2R_0} \frac{dt}{t q_{z_1}^{(n-1)}(t)}, \tag{2.13}
\]
where \( q_{z_1} \) is defined by the relation \( (1.6) \). By the assumption, \( q_{z_1}(t) \) is finite for any \( t \in [R_*, 2R_0] \subset [0, r_1] \). Thus, \( I \neq 0 \) in \( (2.13) \). In this case, the function \( \eta_0(t) = \frac{1}{t q_{z_1}^{(n-1)}(t)} \) satisfies the relation \( (1.3) \). Substituting this function in \( (2.12) \) and applying the Fubini theorem, we obtain that
\[
M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) \leq \omega_{n-1} I_n < \infty. \tag{2.14}
\]
The relation \( (2.14) \) contradicts with \( (2.3) \). The contradiction obtained above disproves the assumption made in \( (2.1) \). \( \square \)

**Remark 2.2.** The statement given in Remark \( (2.1) \) remains true, if in its formulation instead of the specified conditions on function \( Q \) to require that there is \( \delta(z_1) > 0 \) such that
\[
\delta(z_1) \int_{\varepsilon}^{\varepsilon_0} \frac{dt}{t q_{z_1}^{(n-1)}(t)} < \infty, \quad \delta(z_1) \int_{0}^{\varepsilon_0} \frac{dt}{t q_{z_1}^{(n-1)}(t)} = \infty \tag{2.15}
\]
for \( \varepsilon > 0 \) sufficiently small.

Indeed, literally repeating the proof of the statement given in Remark \( (2.1) \) we choose \( R_* \) so small that \( I \) in \( (2.13) \) is strictly positive (this is possible due to the conditions in \( (2.13) \)). The rest of the reasoning will not change. \( \square \)

**Remark 2.3.** The statement given in Remark \( (2.1) \) remains true, if in its formulation instead of the specified conditions on function \( Q \) to require that there is \( \varepsilon_0 = \varepsilon_0(z_1) > 0 \) and a Lebesgue measurable function \( \psi : (0, \varepsilon_0] \to [0, \infty] \) such that
\[
I(\varepsilon, \varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t) \, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0), \quad I(\varepsilon, \varepsilon_0) > 0 \text{ as } \varepsilon \to 0, \tag{2.16}
\]
and, in addition,
\[
\int_{A(z_1, \varepsilon, \varepsilon_0)} Q(x) \cdot \psi^n(|x - x_0|) \, dm(x) \leq C_0 I^n(\varepsilon, \varepsilon_0), \tag{2.17}
\]
as \( \varepsilon \to 0 \), where \( C_0 \) is some constant, and \( A(x_0, \varepsilon, \varepsilon_0) \) is defined in (1.1).

Indeed, literally repeating the proof of the statement given in Remark 2.1 to the ratio (2.12) inclusive, we put

\[
\eta(t) = \begin{cases} 
\psi(t)/I(R_*, 2R_0), & t \in (R_*, 2R_0), \\
0, & t \not\in (R_*, 2R_0), 
\end{cases}
\]

where \( I(1/l, \varepsilon_0) = \int_1^{\varepsilon_0} \psi(t) \, dt \). Observe that \( \int_1^{\varepsilon_0} \eta(t) \, dt = 1 \). Now, by the definition of \( f \) in (1.2) and due to the relation (2.12) we obtain that

\[
M(\Gamma(|\bar{\alpha}_i|, |\bar{\beta}_i|, D)) \leq C_0 < \infty. \tag{2.18}
\]

The relation (2.18) contradicts with (2.3). The resulting contradiction indicates the falsity of the assumption made in (2.1).

\[\blacksquare\]

3 Examples

Example 1. First of all, let us use the construction given in Example 1 in [SevSkv3]. Consider the function \( \varphi : [0, 1] \to \mathbb{R} \), defined by equality

\[
\varphi(t) = \begin{cases} 
1, & t \in \left(\frac{1}{2k+1}, \frac{1}{2k}\right), \quad k = 1, 2, \ldots, \\
\frac{1}{2k}, & t \in \left[\frac{1}{2k}, \frac{2k-1}{2k-1}\right], \quad k = 1, 2, \ldots,
\end{cases}
\]

\[
Q(x) = \varphi(|x|), \quad Q : \mathbb{B}^n \setminus \{0\} \to [0, \infty). \tag{3.1}
\]

By the Fubini theorem and by the countable additivity of the Lebesgue integral, we obtain that

\[
\int_{\mathbb{B}^n} Q(x) \, dm(x) = \int_0^1 \int_{S(0,r)} Q(x) \, d\mathcal{H}^{n-1}dr = \\
\omega_n-1 \int_0^1 r^{n-1} \varphi(r) \, dr \geq \omega_n-1 \sum_{k=1}^{\infty} \int_{1/(2k)}^{1/(2k-1)} \frac{dr}{r} = \omega_n-1 \sum_{k=1}^{\infty} \ln \left( \frac{2k}{2k-1} \right). \tag{3.2}
\]

Note that the series on the right side of the relation (3.2) diverges. Indeed, according to Lagrange’s theorem on the mean we obtain that \( \ln \left( \frac{2k}{2k-1} \right) = \ln(2k) - \ln(2k-1) = \frac{1}{\theta(k)} \geq \frac{1}{2k} \), where \( \theta(k) \in [2k - 1, 2k] \). Since \( \sum_{k=1}^{\infty} \frac{1}{2k} = \infty \), we obtain that \( \sum_{k=1}^{\infty} \ln \left( \frac{2k}{2k-1} \right) = \infty \) and thus

\[
\int_{\mathbb{B}^n} Q(x) \, dm(x) = \infty.
\]
On the other hand,
\[
\int_0^1 \frac{dt}{t q_0^{1/(n-1)}(t)} \geq \sum_{k=1}^{\infty} \int_{1/(2k+1)}^{1/(2k)} \frac{dr}{r} = \sum_{k=1}^{\infty} \ln \frac{2k+1}{2k} = \infty. \tag{3.3}
\]
Set
\[
g(x) = \frac{x}{|x|} \rho(|x|), \quad g(0) := 0,
\]
where
\[
\rho(r) = \exp \left\{ - \int_r^1 \frac{dt}{t q_0^{1/(n-1)}(t)} \right\}. \tag{3.4}
\]
Observe that, \(g\) is a homeomorphism of the unit ball \(B^n\) onto itself. Let us establish that \(g\) satisfies the relation
\[
M(g(\Gamma(S(x_0, r_1), S(x_0, r_2), D))) \leq \int_D Q(x) \cdot \eta^n(|x - x_0|) \, dm(x) \tag{3.5}
\]
for each nonnegative Lebesgue measurable function \(\eta\), which satisfies the relation \((1.3)\).

Indeed, \(g\) belongs to the class \(ACL\), and its Jacobian and the operator norm of the derivative are calculated by the formulae
\[
\|g'(x)\| = \frac{\exp \left\{ - \int_0^1 \frac{dt}{t q_0^{1/(n-1)}(t)} \right\}}{|x|}, \quad |J(x, g)| = \frac{\exp \left\{ -n \int_0^1 \frac{dt}{t q_0^{1/(n-1)}(t)} \right\}}{|x|^n q_0^{1/(n-1)}(|x|)},
\]
see e.g. [IS1, Proof of Theorem 5.2]. Thus, \(g \in W^{1,n}_{loc}(B^n \setminus 0)\). Moreover, the so-called inner dilatation \(K_I(x, g)\) of the mapping \(g\) at \(x\) is calculated as follows: \(K_I(x, g) = q_0(|x|)\). In this case, \(g\) satisfies the relation \((3.5)\) for \(Q = K_I(x, g) = q_0(|x|)\) (see, e.g., [MRSY, Corollary 8.5 and Theorem 8.6].

Therefore, the mapping \(f = g^{-1}\) satisfies the relation \((1.2)\) in \(B^n\). Observe that, the function \(Q\), is extended by zero outside the unit ball, is integrable over almost all spheres with center at any point \(x_0 \in B^n\), because this function is locally bounded in \(B^n \setminus \{0\}\). Observe also that, the corresponding integrable functions \(Q\) in \(B^n\) do not exist. Indeed, otherwise we would have that \(K_I(x, g) \leq c_n Q(x)\) (see, e.g., [SalSev, Theorem 3.1]), but then the inner dilatation \(K_I(x, g)\) would also be integrable in \(B^n\), which, due to the above, is not true.

Since \(g\) is a homeomorphism, \(g\) is open, discrete and closed. Obviously, the unit ball \(B^n\) is locally connected on the boundary. In addition, \(B^n\) has a weakly flat boundary (see, e.g., [Val, Theorem 17.12]). Thus, all of the conditions of Theorem 2.1 are fulfilled.

**Example 2.** We may also specify an example of a mapping with the corresponding function \(Q\) in \((1.2)\), which has a singularity at the boundary of the unit sphere. For simplification, we limit ourselves to the case \(n = 2\). We arbitrarily choose the point \(z_0 \in \partial B^2\) and
For this purpose, in the notations of Example 1, we put: 

\[ f(\rho(|z - z_0|)) \], \quad g_1(z) := 0, \]

where the function \( \rho \) is still defined in (3.1). For \( z \in \mathbb{B}^2 \setminus B(z_0, 1) \) we set \( f(z) = z \). Note that \( g_1 \) satisfies the relation (3.2), where \( Q_1(z) = q_0(|z - z_0|) \) and \( q_0(z) \) is defined in (3.1). For the same reasons, \( Q_1(z) \) has finite integrals for almost all spheres with centers in \( \mathbb{B}^2 \), where, as usual, the function \( Q_1 \) vanishes outside \( \mathbb{B}^2 \). We show that the function \( Q_1 \) has infinite integrals over sufficient small balls \( B(z_0, \varepsilon_0) \). For this purpose, we introduce the polar coordinates \( z = (r, \varphi) \) centered at the point \( z_0 \), where \( r \) denotes the Euclidean distance from \( z_0 \) to \( z \), and \( \varphi \) is the angle between the radius vector \( z_0 - z \) and tangent to the disk \( \mathbb{B}^2 \), passing through the point \( z_0 \). Let

\[ \theta_1 = \inf_{z \in \mathbb{B}^2 \cap S(z_0, \varepsilon)} \varphi, \quad \theta_2 = \sup_{z \in \mathbb{B}^2 \cap S(z_0, \varepsilon)} \varphi. \]

Using elementary methods of geometry, we will have that \( \sin \theta_1 = \varepsilon/2, \sin(\pi - \theta_2) = \varepsilon/2 \). Then, for \( \varepsilon \to 0 \), we obtain that \( \theta_1 \to 0, \theta_2 \to \pi \). From here it follows that the interval of change of angles \( \varphi \) is close to \( \pi \) for \( z \in \mathbb{B}^2 \cap B(z_0, \varepsilon_0) \). In particular, for some (rather small) \( \varepsilon_0 > 0 \) we have that \( \theta_2 - \theta_1 \geq 5\pi/6 \). Then, by the relation (3.2), we obtain that

\[
\int_{\mathbb{B}^2 \cap B(z_0, \varepsilon_0)} Q_1(z) \, dm(z) = \int_0^{\varepsilon_0} \int_{S(z_0, r) \cap B(z_0, \varepsilon_0)} Q_1(z) \, |dz| \, dr = \\
\int_0^{\varepsilon_0} \int_{\theta_1}^{\theta_2} r \varphi(r) \, dr \geq \frac{5\pi}{6} \int_0^{\varepsilon_0} r \varphi(r) \, dr = \infty.
\]

Since \( g_1(\mathbb{B}^2) \) is simply connected, according to Riemann’s theorem on conformal mapping we may find a mapping \( \varphi \) such that \( (\varphi \circ g_1)(\mathbb{B}^2) = \mathbb{B}^2 \). Put \( f_1 := g_1^{-1} \circ \varphi^{-1} \). Then the mapping \( f_1 \) satisfies all the conditions of Theorem 2.1, in particular, the inequality (1.2) for \( Q = Q_1(z) \).

**Example 3.** Finally, let us construct relevant examples of mappings with a branching. For this purpose, in the notations of Example 1 we put: \( f_2(z) = (\varphi_1 \circ f)(z) \), where \( \varphi_1(z) = z^2 \). Observe that, a mapping \( f_2 \) is open, discrete and closed, in addition, it satisfies the relation (1.2) for \( Q := 2K_1(x, g) = 2q_0(x) \) (see, e.g., [MRV] Theorem 3.2).

### 4 On the discreteness of mappings with the inverse Pol-letsky inequality at the boundary of a domain

In this section we talk about the discreteness of mappings that satisfy the condition (1.2). Note that here we consider mappings \( f \) with, generally speaking, an unbounded function \( Q \).
in (1.2), in addition, we consider the case of an arbitrary domain \( D \) with some additional conditions on its boundary.

Here are some definitions. Let \( p \geq 1 \). Consider a more general definition compared to (1.2). We will say that \( f \) satisfies the inverse Poletsky inequality at a point \( y_0 \in \overline{f(D)} \setminus \{\infty\} \) relative to \( p \)-modulus, if the relation

\[
M_p(\Gamma f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2)} Q(y) \cdot \eta^p(|y - y_0|) \, dm(y)
\]

(4.1)

holds for any Lebesgue measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that

\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.
\]

(4.2)

Using the inversion \( \psi(y) = \frac{y}{|y|^{2}} \), we also may defined the relation (4.1) at the point \( y_0 = \infty \).

Following \cite{NP} Section 2.4, we say that a domain \( D \subset \mathbb{R}^n, n \geq 2 \), is uniform with respect to \( p \)-modulus, if for any \( r > 0 \) there is \( \delta > 0 \) such that the inequality

\[
M_p(\Gamma(F^*, F, D)) \geq \delta
\]

(4.3)

holds for any continua \( F, F^* \subset D \) with \( h(F) \geq r \) and \( h(F^*) \geq r \). When \( p = n \), the prefix "relative to \( p \)-modulus" is omitted. Note that this is the definition slightly different from the "classical" given in \cite{NP} Chapter 2.4, where the sets \( F \) and \( F^* \subset D \) are assumed to be arbitrary connected. We prove the following statement (see its analogue for quasiregular mappings of the unit ball in \cite{Vil} Lemma 4.4).

**Lemma 4.1.** Let \( n \geq 2, n - 1 < p \leq n \), and let \( D \) be a domain which is uniform with respect to \( p \)-modulus. Let \( f : D \to \mathbb{R}^n \) be an open discrete and closed mapping in \( D \), for which there is a Lebesgue measurable function \( Q : \mathbb{R}^n \to [0, \infty] \), equals to zero outside of \( f(D) \), such that the relations (4.1)–(4.2) hold for some \( y_0 \in \overline{f(D)} \). Assume that, there is \( \varepsilon_0 = \varepsilon_0(y_0) > 0 \) and a Lebesgue measurable function \( \psi : (0, \varepsilon_0) \to [0, \infty] \) such that

\[
I(\varepsilon, \varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t) \, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0), \quad I(\varepsilon, \varepsilon_0) \to \infty \text{ as } \varepsilon \to 0,
\]

(4.4)

and, in addition,

\[
\int_{A(y_0, \varepsilon, \varepsilon_0)} Q(y) \cdot \psi^p(|y - y_0|) \, dm(y) = o(I^p(\varepsilon, \varepsilon_0)),
\]

(4.5)

as \( \varepsilon \to 0 \), where \( A(y_0, \varepsilon, \varepsilon_0) \) is defined in (1.1). Let \( C_j, j = 1, 2, \ldots, \) be a sequence of continua such that \( h(C_j) \geq \delta > 0 \) for some \( \delta > 0 \) and any \( j \in \mathbb{N} \) and, in addition, \( h(f(C_j)) \to 0 \) as \( j \to \infty \). Then \( h(f(C_j), y_0) \geq \delta_1 > 0 \) for any \( j \in \mathbb{N} \) and some \( \delta_1 > 0 \).
Proof. We may assume that \( y_0 \neq \infty \). Suppose the opposite, namely, let \( h(f(C_{j_k}), y_0) \rightarrow 0 \) as \( k \rightarrow \infty \) for some increasing sequence of numbers \( j_k, k = 1, 2, \ldots \). Let \( F \subset D \) be any continuum in \( D \) such that \( y_0 \not\in f(F) \). Let \( \Gamma_k := \Gamma(F, C_{j_k}, D) \). Then, due to the definition of the uniformity of the domain with respect to \( p \)-modulus, we obtain that

\[
M_p(\Gamma_k) \geq \delta_2 > 0
\]

for any \( k \in \mathbb{N} \) and some \( \delta_2 > 0 \). On the other hand, let us to consider the family of paths \( f(\Gamma_k) \).

Let us to prove that, for any \( l \in \mathbb{N} \) there is a number \( k = k_l \) such that

\[
f(C_{j_k}) \subset B(y_0, 1/l), \quad k \geq k_l.
\]  

(4.7)

Suppose the opposite. Then there is \( l_0 \in \mathbb{N} \) such that

\[
f(C_{j_{m_l}}) \cap (\mathbb{R}^n \setminus B(y_0, 1/l_0)) \neq \emptyset
\]

(4.8)

for some increasing sequence of numbers \( m_l, l = 1, 2, \ldots \). In this case, there is a sequence \( x_{m_l} \in f(C_{j_{m_l}} \cap (\mathbb{R}^n \setminus B(y_0, 1/l_0))), l \in \mathbb{N} \). Since by the assumption \( h(f(C_{j_k}), y_0) \rightarrow 0 \) for some sequence of numbers \( j_k, k = 1, 2, \ldots \), we obtain that

\[
h(f(C_{j_{m_l}}), y_0) \rightarrow 0 \quad \text{as} \quad l \rightarrow \infty.
\]

(4.9)

Since \( h(f(C_{j_{m_l}}), y_0) = \inf_{y \in f(C_{j_{m_l}})} h(y, y_0) \) and \( f(C_{j_{m_l}}) \) is a compact as a continuous image of the compact set \( C_{j_{m_l}} \) under the mapping \( f \), it follows that \( h(f(C_{j_{m_l}}), y_0) = h(y_l, y_0) \), where \( y_l \in f(C_{j_{m_l}}) \). Due to the relation (4.9) we obtain that \( y_l \rightarrow y_0 \) as \( l \rightarrow \infty \). Since by the assumption \( h(f(C_j)) = \sup_{y,z \in f(C_j)} h(y, z) \rightarrow 0 \) as \( j \rightarrow \infty \), we have that \( h(y_l, x_{m_l}) \leq h(f(C_{j_{m_l}})) \rightarrow 0 \) as \( l \rightarrow \infty \). Now, by the triangle inequality, we obtain that

\[
h(x_{m_l}, y_0) \leq h(x_{m_l}, y_l) + h(y_l, y_0) \rightarrow 0 \quad \text{as} \quad l \rightarrow \infty.
\]

The latter contradicts with (4.8). The contradiction obtained above proves (4.7).

The following considerations are similar to the second part of the proof of Lemma 2.1 in [Sev1]. Without loss of generality we may consider that the number \( l_0 \in \mathbb{N} \) is such that \( 1/l < \varepsilon_0 \) for any \( l \geq l_0 \), and

\[
f(F) \subset \mathbb{R}^n \setminus B(y_0, 1/l_0).
\]

(4.10)

In this case, we observe that

\[
f(\Gamma_{k_l}) > \Gamma(S(y_0, 1/l), S(y_0, \varepsilon_0), A(y_0, 1/l, \varepsilon_0)).
\]

(4.11)

Indeed, let \( \tilde{\gamma} \in f(\Gamma_{k_l}) \). Then \( \tilde{\gamma}(t) = f(\gamma(t)) \), where \( \gamma \in \Gamma_{k_l}, \gamma : [0, 1] \rightarrow D, \gamma(0) \in F, \gamma(1) \in C_{j_{k_l}} \). Due to the relation (4.10), we obtain that \( f(\gamma(0)) \in f(F) \subset \mathbb{R}^n \setminus B(y_0, 1/l_0) \). In
addition, by (4.7) we have that $\gamma(1) \in C_{k_1} \subset B(y_0, 1/l_0)$. Thus, $|f(\gamma(t))| \cap B(y_0, 1/l_0) \neq \emptyset \neq |f(\gamma(t))| \cap (\mathbb{R}^n \setminus B(y_0, 1/l_0))$. Now, by [Ku] Theorem 1.1.5.46 we obtain that, there is $0 < t_1 < 1$ such that $f(\gamma(t_1)) \in S(y_0, 1/l_0)$. Set $\gamma_1 := \gamma|_{[t_1, 1]}$. We may consider that $f(\gamma(t)) \in B(y_0, \varepsilon_0)$ for any $t \geq t_1$. Arguing similarly, we obtain $t_2 \in [t_1, 1]$ such that $f(\gamma(t_2)) \in S(y_0, 1/l)$. Put $\gamma_2 := \gamma|_{[t_1, t_2]}$. We may consider that $f(\gamma(t)) \in B(y_0, 1/l)$ for any $t \in [t_1, t_2]$. Now, a path $f(\gamma_2)$ is a subpath of $f(\gamma) = \tilde{\gamma}$, which belongs to $\Gamma(S(y_0, 1/l), S(y_0, \varepsilon_0), A(y_0, 1/l, \varepsilon_0))$. The relation (4.11) is established.

It follows from (4.11) that
\[
\Gamma_{k_1} > \Gamma_f(S(y_0, 1/l), S(y_0, \varepsilon_0), A(y_0, 1/l, \varepsilon_0)).
\] (4.12)

Set
\[
\eta_\varepsilon(t) = \begin{cases} 
\psi(t)/I(1/l, \varepsilon_0), & t \in (1/l, \varepsilon_0), \\
0, & t \not\in (1/l, \varepsilon_0),
\end{cases}
\]
where $I(1/l, \varepsilon_0) = \int_{1/l}^{\varepsilon_0} \psi(t) \, dt$. Observe that $\int_0^{\varepsilon_0} \eta_\varepsilon(t) \, dt = 1$. Now, by the relations (4.5) and (4.12), and due to the definition of $f$ in (4.1), we obtain that
\[
M_p(\Gamma_{k_1}) \leq M_p(\Gamma_f(S(y_0, 1/l), S(y_0, \varepsilon_0), A(y_0, 1/l, \varepsilon_0))) \leq \frac{1}{I_p(1/l, \varepsilon_0)} \int_{A(y_0, 1/l, \varepsilon_0)} Q(y) \cdot \psi^p(|y - y_0|) \, dm(y) \to 0 \quad \text{as} \quad l \to \infty.
\] (4.13)
The latter contradicts with (4.6). The contradiction obtained above proves the lemma. \(\square\)

Let $X$ and $Y$ be metric spaces. Recall that, a mapping $f : X \to Y$ is called light, if for any point $y \in Y$, the set $f^{-1}(y)$ does not contain any nondegenerate continuum $K \subset X$. The following lemma generalizes Corollary 4.5 in [Vu] for the case of mappings with unbounded characteristic.

**Lemma 4.2.** Let $n \geq 2$, $n - 1 < p \leq n$ and let $D$ be a domain which is uniform with respect to $\nu$-modulus. Let $f : D \to \mathbb{R}^n$ be an open discrete and closed mapping of $D$ for which there is a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$ equals to zero outside $f(D)$ such that the conditions (4.1)–(4.2) hold for any point $y_0 \in \partial f(D)$. Assume that, there is $\varepsilon_0 = \varepsilon_0(y_0) > 0$ and a Lebesgue measurable function $\psi : (0, \varepsilon_0) \to [0, \infty]$ such that the relations (4.3)–(4.5) hold, where $A(y_0, \varepsilon, \varepsilon_0)$ is defined in (4.1). Assume also that, a domain $D$ is locally connected on its boundary, and that $f$ has a continuous extension $\overline{f} : \overline{D} \to \mathbb{R}^n$. Then $\overline{f}$ is light.

**Proof.** Assume the contrary, namely, let $y_0 \in \partial f(D)$ be some point such that $f^{-1}(y_0) \supset K_0$, where $K_0 \subset \partial D$ is some nondegenerate continuum. Then, in particular, $f(K_0) = y_0$. Since $\overline{D}$ is a compactum in $\mathbb{R}^n$ and, in addition, $\overline{f}$ is continuous in $\overline{D}$, the mapping $\overline{f}$ is uniformly continuous in $\overline{D}$. In this case, for any $j \in \mathbb{N}$ there is $\delta_j < 1/j$ such that
\[
h(\overline{f}(x), \overline{f}(x_0)) = h(\overline{f}(x), y_0) < 1/j \quad \forall \, x, x_0 \in \overline{D}, \quad h(x, x_0) < \delta_j, \quad \delta_j < 1/j.
\] (4.14)
Denote by \( B_h(x_0, r) = \{x \in \mathbb{R}^n : h(x, x_0) < r\} \). Then, given \( j \in \mathbb{N} \), we set
\[
B_j := \bigcup_{x_0 \in K_0} B_h(x_0, \delta_j), \quad j \in \mathbb{N}.
\]
Since the set \( B_j \) is a neighborhood of \( K_0 \), by [HK] Lemma 2.2 there is a neighborhood \( U_j \) of the set \( K_0 \) such that \( U_j \subset B_j \) and the set \( U_j \cap D \) is connected. Without loss of generality, we may assume that \( U_j \) is open. Then the set \( U_j \cap D \) is path connected, as well (see [MRSY] Proposition 13.1]). Since \( K_0 \) is a compact set, there are \( z_0, w_0 \in K_0 \) such that
\[
h(K_0) = h(z_0, w_0).
\]
It follows from this, that there are \( z_j \in U_j \cap D \) and \( w_j \in U_j \cap D \) such that \( z_j \to z_0 \) and \( w_j \to w_0 \) as \( j \to \infty \). We may assume that
\[
h(z_j, w_j) > h(K_0)/2 \quad \forall \; j \in \mathbb{N}.
\]
Since the set \( U_j \cap D \) is path connected, we may join points \( z_j \) and \( w_j \) by some path \( \gamma_j \in U_j \cap D \). Set \( C_j := |\gamma_j| \).

Observe that, \( h(f(C_j)) \to 0 \) as \( j \to \infty \). Indeed, since \( f(C_j) \) is a continuum in \( \mathbb{R}^n \), there are points \( y_j, y'_j \in f(C_j) \) such that \( h(f(C_j)) = h(y_j, y'_j) \). Then there are \( x_j, x'_j \in C_j \) such that \( y_j = f(x_j) \) and \( y'_j = f(x'_j) \). Then points \( x_j \) and \( x'_j \) belong to \( U_j \subset B_j \). Therefore, there are \( x^j_1 \) and \( x^j_2 \in K_0 \) such that \( x_j \in B(x^j_1, \delta_j) \) and \( x'_j \in B(x^j_2, \delta_j) \). In this case, by the relation (4.14) and due to the triangle inequality we obtain that
\[
h(f(C_j)) = h(y_j, y'_j) = h(f(x_j), f(x'_j)) \leq
\]
\[
\leq h(f(x_j), f(x^j_1)) + h(f(x^j_1), f(x^j_2)) + h(f(x^j_2), f(x'_j)) < 2/j \to 0 \quad \text{as} \quad j \to \infty.
\]
It follows from (4.15) and (4.16) that, the continua \( C_j, j = 1, 2, \ldots \), satisfy the conditions of Lemma 4.1. By this lemma we may obtain that \( h(f(C_j), y_0) \geq \delta_1 > 0 \) for any \( j \in \mathbb{N} \). On the other hand, by the proving above \( x_j \in B(x^j_1, \delta_j) \). Now, by the relation (4.14) we obtain that \( h(f(x_j), y_0) < 1/j, \; j = 1, 2, \ldots \). The resulting contradiction indicates the incorrectness of the assumption that \( f \) is not light in \( \partial D \). Lemma is proved. \( \square \)

**Corollary 4.1.** The statements of Lemmas 4.1 and 4.2 are fulfilled if we put \( D = \mathbb{B}^n \).

**Proof.** Obviously, the domain \( D = \mathbb{B}^n \) is locally connected at its boundary. We prove that this domain is uniform with respect to the \( p \)-modulus for \( p \in (n-1, n) \). Indeed, since \( \mathbb{B}^n \) is a Loewner space (see [He] Example 8.24(a)], the set \( \mathbb{B}^n \) is Ahlfors regular with respect to the Euclidean metric \( d \) and Lebesgue measure in \( \mathbb{R}^n \) (see [He] Proposition 8.19]). In addition, in \( \mathbb{B}^n \), \((1; p)\)-Poincaré inequality holds for any \( p \geq 1 \) (see e.g. [HaK] Theorem 10.5]). Now, by [AS] Proposition 4.7 we obtain that the relation
\[
M_p(\Gamma(E, F, \mathbb{B}^n)) \geq \frac{1}{C} \min\{\text{diam} \; E, \text{diam} \; F\},
\]
holds for any \( n-1 < p \leq n \) and for any continua \( E, F \subset \mathbb{B}^n \), where \( C > 0 \) is some constant, and \( \text{diam} \) denotes the Euclidean diameter. Since the Euclidean distance is equivalent to the
chordal distance on bounded sets, the uniformity of the domain \( D = \mathbb{R}^n \) with respect to the p-modulus follows directly from (4.17). □

We need the following statement (see [Na1] Theorem 4.2]).

**Proposition 4.1.** Let \( \mathcal{F} \) be a family of connected sets in \( D \) such that \( \inf_{F \subseteq \mathcal{F}} h(F) > 0 \), and let \( \inf_{F,F^* \subseteq \mathcal{F}} M(\Gamma(F,A,D)) > 0 \) for some continuum \( A \subset D \). Then

\[
\inf_{F,F^* \subseteq \mathcal{F}} M(\Gamma(F,F^*,D)) > 0.
\]

Let \( p \geq 1 \). Due to [MRSY, Section 3] we say that a boundary \( D \) is called strongly accessible with respect to p-modulus at \( x_0 \in \partial D \), if for any neighborhood \( U \) of the point \( x_0 \in \partial D \) there is a neighborhood \( V \subset U \) of this point, a compactum \( F \subset D \) and a number \( \delta > 0 \) such that \( M_p(\Gamma(E,F,D)) \geq \delta \) for any continua \( E \subset D \) such that \( E \cap \partial U \neq \emptyset \neq E \cap \partial V \). The boundary of a domain \( D \) is called strongly accessible with respect to p-modulus, if this is true for any \( x_0 \in \partial D \). When \( p = n \), prefix "relative to p-modulus" is omitted. The following lemma is valid (see the statement similar in content to [Na1] Theorem 6.2]).

**Lemma 4.3.** A domain \( D \subset \mathbb{R}^n \) has a strongly accessible boundary if and only if \( D \) is uniform.

**Proof.** The fact that uniform domains have strongly accessible boundaries has been proved in [SevSkv1] Remark 1]. It remains to prove that domains with strongly accessible boundaries are uniform.

We will prove this statement from the opposite. Let \( D \) be a domain which has a strongly accessible boundary, but it is not uniform. Then there is \( r > 0 \) such that, for any \( k \in \mathbb{N} \) there are continua \( F_k \) and \( F_k^* \subset D \) such that \( h(F_k) \geq r \), \( h(F_k^*) \geq r \), however,

\[
M(\Gamma(F_k,F_k^*,D)) < 1/k.
\] (4.18)

Let \( x_k \in F_k \). Since \( \overline{D} \) is compact in \( \mathbb{R}^n \), we may assume that \( x_k \to x_0 \in \overline{D} \). Note that the strongly accessibility of the domain \( D \) at the boundary points is assumed to be, and at the inner points it is even weakly flat, which is the result of Väisälä’s lemma (see e.g. [Va Sect. 10.12], cf. [SevSkv2 Lemma 2.2]). Let \( U \) be a neighborhood of the point \( x_0 \) such that \( h(x_0,\partial U) \leq r/2 \). Then there is a neighborhood \( V \subset U \), a compactum \( F \subset D \) and a number \( \delta > 0 \) such that the relation \( M(\Gamma(E,F,D)) \geq \delta \) holds for any continuum \( E \subset D \) such that \( E \cap \partial U \neq \emptyset \neq E \cap \partial V \). By the choice of the neighborhood \( U \), we obtain that \( F_k \cap U \neq \emptyset \neq F_k \cap (D \setminus U) \) for sufficiently large \( k \in \mathbb{N} \). Observe that, for the same \( k \in \mathbb{N} \), the condition \( F_k \cap V \neq \emptyset \neq F_k \cap (D \setminus V) \) holds. Then, by [Ku Theorem 1.1.5.46] we obtain that \( F_k \cap \partial U \neq \emptyset \neq F_k \cap \partial V \). Observe that, a compactum \( F \) can be imbedded in some continuum \( A \subset D \) (see [Sm Lemma 1]). Then the inequality \( M(\Gamma(E,A,D)) \geq \delta \) will only increase. Given the above, we obtain that

\[
M(\Gamma(F_k,A,D)) \geq \delta \quad \forall k \geq k_0
\] (4.19)
for some. Taking \( \inf \) over all \( k \geq k_0 \) in (4.19), we obtain that
\[
\inf_{k \geq k_0} M(\Gamma(F_k, A, D)) \geq \delta.
\] (4.20)

Set \( \mathfrak{F} := \{F_k\}_{k = k_0}^{\infty} \). Now, by the condition (4.20) and by Proposition 4.1, we obtain that
\[
\inf_{k \geq k_0} M(\Gamma(F_k, F_k^*, D)) > 0,
\]
that contradicts the assumption made in (4.18). The resulting contradiction completes the proof of the lemma. \( \square \)

Obviously, weakly flat boundaries are strongly accessible. Now, by Lemma 4.3 we obtain the following.

**Corollary 4.2.** If \( D \subset \mathbb{R}^n \) has a weakly flat boundary, then \( D \) is uniform.

The following lemma holds.

**Lemma 4.4.** Suppose that, under the conditions of Lemma 4.2 \( p = n \), the domain \( D \) is weakly flat and the domain \( f(D) \) is locally connected at its boundary. Then the mapping \( f \) has a continuous extension \( \overline{f} : \overline{D} \to \mathbb{R}^n \) such that \( N(f, D) = N(f, \overline{D}) < \infty \). In particular, \( \overline{f} \) is discrete in \( \overline{D} \).

**Proof.** First of all, the possibility of continuous extension of \( f \) to a mapping \( \overline{f} : \overline{D} \to \mathbb{R}^n \) follows by Remark 2.3. Note also that \( N(f, D) < \infty \), see [MS, Theorem 2.8]. Let us prove that \( N(f, D) = N(f, \overline{D}) \). Next we will reason using the scheme proof of Theorem 4.7 in [Vms]. Assume the contrary. Then there are points \( y_0 \in \partial f(D) \) and \( x_1, x_2, \ldots, x_k, x_{k+1} \in \partial D \) such that \( f(x_i) = y_0, i = 1, 2, \ldots, k+1 \) and \( k := N(f, D) \). We may assume that \( y_0 \neq \infty \). Since by the assumption \( f(D) \) is locally connected at any point of its boundary, for any \( p \in \mathbb{N} \) there is a neighborhood \( U'_p \subset B(y_0, 1/p) \) such that the set \( U'_p \cap f(D) = U'_p \) is connected.

Let us prove that, for any \( i = 1, 2, \ldots, k+1 \) there is a component \( V^i_p \) of the set \( f^{-1}(U'_p) \) such that \( x_i \in V^i_p \). Fix \( i = 1, 2, \ldots, k+1 \). By the continuity of \( \overline{f} \) in \( \overline{D} \), there is \( r_i = r_i(x_i) > 0 \) such that \( f(B(x_i, r_i) \cap D) \subseteq U'_p \). By [MRSY, Lemma 3.15], a domain with a weakly flat boundary is locally connected on its boundary. Thus, we may find a neighborhood \( W_i \subset B(x_i, r_i) \) of the point \( x_i \) such that \( W_i \cap D \) is connected. Then \( W_i \cap D \) belongs to one and only one component \( V^p_i \) of the set \( f^{-1}(U'_p) \), while \( x_i \in W_i \cap D \subset V^p_i \), as required.

Next we show that the sets \( V^i_p \) are disjoint for any \( i = 1, 2, \ldots, k+1 \) and large enough \( p \in \mathbb{N} \). In turn, we prove for this that \( h(V^i_p) \to 0 \) as \( p \to \infty \) for each fixed \( i = 1, 2, \ldots, k+1 \). Let us prove the opposite. Then there is \( 1 \leq \bar{i}_0 \leq k+1 \), a number \( r_0 > 0 \), \( r_0 < \frac{1}{2} \min_{1 \leq i,j \leq k+1, i \neq j} h(x_i, x_j) \) and an increasing sequence of numbers \( p_m, m = 1, 2, \ldots \), such that \( S_h(x_{\bar{i}_0}, r_0) \cap V^\bar{i}_0 p_m \neq \emptyset \), where \( S_h(x_0, r) = \{x \in \mathbb{R}^n : h(x, x_0) = r\} \). In this case, there are \( a_m, b_m \in V^\bar{i}_0 p_m \) such that \( a_m \to x_{\bar{i}_0} \) as \( m \to \infty \) and \( h(a_m, b_m) \geq r_0/2 \). Join the points \( a_m \) and \( b_m \) by a path \( C_m \), which entirely belongs to \( V^\bar{i}_0 p_m \). Then \( h(|C_m|) \geq r_0/2 \) for \( m = 1, 2, \ldots \). On the other hand, since \( |C_m| \subset f(V^\bar{i}_0 p_m) \subset B(y_0, 1/p_m) \), then simultaneously \( h(f(|C_m|)) \to 0 \) as \( m \to \infty \) and
h(|C_m|, y_0) \to 0 \text{ as } m \to \infty, \text{ that contradicts with Lemma 4.1. The resulting contradiction indicates the incorrectness of the above assumption.}

By [Vu, Lemma 3.6] f is a mapping of \( V^i_p \) onto \( U'_p \) for any \( i = 1, 2, \ldots, k, k + 1 \). Thus, \( N(f, D) \geq k+1 \), which contradicts the definition of the number \( k \). The obtained contradiction refutes the assumption that \( N(f, \overline{D}) > N(f, D) \). The lemma is proved. \[\square\]

**Proof of Theorem 1.1.** In the case 1), we choose \( \psi(t) = \frac{1}{t \log t} \), and in the case 2), we set

\[
\psi(t) = \begin{cases} 
1/[t^{\frac{n-1}{n} q_0^{-1}}(t)], & t \in (\varepsilon, \varepsilon_0), \\
0, & t \notin (\varepsilon, \varepsilon_0),
\end{cases}
\]

Observe that, the relations (1.4)–(1.5) hold for these functions \( \psi \), where \( p = n \) (the proof of this facts may be found in [Sev1, Proof of Theorem 1.1]). The desired conclusion follows from Lemma 4.4. \[\square\]

5 A continuous extension in terms of prime ends

The following result holds.

**Theorem 5.1.** Let \( D \subset \mathbb{R}^n, n \geq 2 \), be a domain with a weakly flat boundary, and let \( D' \subset \mathbb{R}^n \) be a regular domain. Suppose that \( f \) is open discrete and closed mapping of \( D \) onto \( D' \) satisfying the relation (1.2) at any point \( y_0 \in \partial D' \). Suppose that, for each point \( y_0 \in \partial D' \) there is \( 0 < r_* = r_*(y_0) < \sup_{y \in D'} |y - y_0| \) such that, for any \( 0 < r_1 < r_2 < r_* \), there is a set \( E \subset [r_1, r_2] \) of a positive linear Lebesgue measure such that the function \( Q \) is integrable on \( S(y_0, r) \) for any \( r \in E \). Then \( f \) has a continuous extension \( \overline{f} : \overline{D} \to \overline{D'}_P \), while \( \overline{f}(\overline{D}) = \overline{D'}_P \).

**Proof.** We carry out the proof according to a scheme similar to the proof of Theorem 1 in [Sev2]. Fix \( x_0 \in \partial D \). It is necessary to show the possibility of continuous extension of the mapping \( f \) to the point \( x_0 \). Using, if necessary, the transformation \( \varphi : \infty \mapsto 0 \) and taking into account the invariance of the modulus \( M \) in the left part of the relation \( \varphi : \infty \mapsto 0 \) (see [Va, Theorem 8.1]), we may assume that \( x_0 \neq \infty \).

Assume that the conclusion about the continuous extension of the mapping \( f \) to the point \( x_0 \) is not correct. Then any prime end \( P_0 \in E_{D'} \) is not a limit of \( f \) at \( x_0 \), in other words, there is a sequence \( x_k, k = 1, 2, \ldots, x_k \to x_0 \) as \( k \to \infty \) and a number \( \varepsilon_0 > 0 \) such that \( \rho(f(x_k), P_0) \geq \varepsilon_0 \) for any \( k \in \mathbb{N} \), where \( \rho \) is one of the metrics in (1.8). Since \( D' \) is a regular domain by the assumption, it may be mapped on some bounded domain \( D_* \) with a locally quasiconformal boundary using some a mapping \( h : D' \to D_* \). Note that, there is a one-to-one correspondence between boundary points and prime ends of domains with locally quasiconformal boundaries (see, e.g., [IS2, Theorem 2.1]; cf. [Na2, Theorem 4.1]). Since \( \overline{D}_* \) is a compactum in \( \mathbb{R}^n \), we conclude from the above that a metric space \( (\overline{D'}_P, \rho) \) is compact.
Thus, we may assume that \( f(x_k) \) converges to some element \( P_1 \neq P_0, P_1 \in D'_P \) as \( k \to \infty \). Since, by the assumption, \( f \) has no a limit at \( x_0 \), there is at least one a sequence \( y_k \to x_0 \) as \( k \to \infty \) such that \( \rho(f(y_k), P_1) \geq \varepsilon_1 \) for any \( k \in \mathbb{N} \) and some \( \varepsilon_1 > 0 \). Again, since the metric space \( (D'_P, \rho) \) is compact, we may assume that \( f(y_k) \to P_2 \) as \( k \to \infty \), \( P_1 \neq P_2, P_2 \in D'_P \). Since \( f \) is closed, it preserves the boundary of a domain, see [Vu, Theorem 3.3]. Thus, \( P_1, P_2 \in E_{D'} \).

Let \( \sigma_m \) and let \( \sigma'_m, m = 0, 1, 2, \ldots \), be a sequence of cuts corresponding to prime ends \( P_1 \) and \( P_2 \), respectively. Let also cuts \( \sigma_m, m = 0, 1, 2, \ldots \), lie on spheres \( S(z_0, r_m) \) centered at a point \( z_0 \in \partial D' \), where \( r_m \to 0 \) as \( m \to \infty \) (such a sequence \( \sigma_m \) exists by [IS2, Lemma 3.1], cf. [KR2 Lemma 1]). We may assume that \( r_0 < r_* = r_*(z_0) \), where \( r_* \) is the number from conditions of the theorem. Let \( d_m \) and \( g_m, m = 0, 1, 2, \ldots \), be sequences of domains in \( D' \) corresponding to cuts \( \sigma_m \) and \( \sigma'_m \), respectively. Since \( (D'_P, \rho) \) is a metric space, we may consider that \( d_m \) and \( g_m \) disjoint for any \( m = 0, 1, 2, \ldots \), in particular,

\[
d_0 \cap g_0 = \emptyset. \tag{5.1}
\]

Since \( f(x_k) \) converges to \( P_1 \) as \( k \to \infty \), for any \( m \in \mathbb{N} \) there is \( k = k(m) \) such that \( f(x_k) \in d_m \) for \( k \geq k = k(m) \). By renumbering the sequence \( x_k \) if necessary, we may assume that \( f(x_k) \in d_k \) for any natural \( k \). Similarly, we may assume that \( f(y_k) \in g_k \) for any \( k \in \mathbb{N} \). Fix \( f(x_1) \) and \( f(y_1) \). Since, by the definition of a prime end, \( \bigcap_{k=1}^{\infty} d_k = \bigcap_{l=1}^{\infty} g_l = \emptyset \), there are numbers \( k_1 \) and \( k_2 \in \mathbb{N} \) such that \( f(x_1) \notin d_{k_1} \) and \( f(y_1) \notin g_{k_2} \). Since, by the definition, \( d_k \subset d_{k_0} \) for any \( k \geq k_1 \) and \( g_k \subset g_{k_2} \) for \( k \geq k_2 \), we obtain that

\[
f(x_1) \notin d_k, \quad f(y_1) \notin g_k, \quad k \geq \max\{k_1, k_2\}. \tag{5.2}
\]

Let \( \gamma_k \) be a path joining \( f(x_1) \) and \( f(x_k) \) in \( d_1 \), and let \( \gamma'_k \) be a path joining \( f(y_1) \) and \( f(y_k) \) in \( g_1 \). Let also \( \alpha_k \) and \( \beta_k \) be total \( f \)-liftings of \( \gamma_k \) and \( \gamma'_k \) in \( D \) starting at \( x_k \) and \( y_k \), respectively (such liftings exist by Proposition 2.2, see Figure 2). Note that the points \( f(x_1) \) and \( f(y_1) \) may have no more than a finite number of pre-images under the mapping \( f \) in the domain \( D \), see [Vu, Lemma 3.2]. Then there exists \( R_0 > 0 \) such that \( \alpha_k(1), \beta_k(1) \in D \setminus B(x_0, R_0) \) for any \( k = 1, 2, \ldots \). Since the boundary of \( D \) is weakly flat, for any \( P > 0 \) there is \( i = i_P \geq 1 \) such that

\[
M(\Gamma(|\alpha_k|, |\beta_k|, D)) > P \quad \forall \ k \geq k_P. \tag{5.3}
\]

Let us to show that, the condition (5.3) contradicts the definition of \( f \) in (1.2). Indeed, let \( \gamma \in \Gamma(|\alpha_k|, |\beta_k|, D) \). Then \( \gamma : [0, 1] \to D, \gamma(0) \in |\alpha_k| \) and \( \gamma(1) \in |\beta_k| \). In particular, \( f(\gamma(0)) \in |\gamma'| \) and \( f(\gamma(1)) \in |\gamma'| \). In this case, it follows from the relations (5.1) and (5.3) that \( |f(\gamma)| \cap d_1 \\
\neq \emptyset \neq |f(\gamma)| \cap (D' \setminus d_1) \) for \( k \geq \max\{k_1, k_2\} \). By [Ku Theorem 1.I.5.46] \(|f(\gamma)| \cap \partial d_1 \neq \emptyset \), in other words, \(|f(\gamma)| \cap S(z_0, r_1) \neq \emptyset \), because \( \partial d_1 \cap D' \subset \sigma_1 \subset S(z_0, r_1) \) by the definition of a cut \( \sigma_1 \). Let \( t_1 \in (0, 1) \) be such that \( f(\gamma(t_1)) \in S(z_0, r_1) \) and \( f(\gamma)|_{[1]} := f(\gamma)|_{[t_1, 1]} \). Without loss of generality, we may assume that \( f(\gamma)|_{[1]} \subset \mathbb{R}^n \setminus B(z_0, r_1) \). Arguing similarly for a path \( f(\gamma)|_{[1]} \),
we may find a point \( t_2 \in (t_1, 1) \) such that \( f(\gamma(t_2)) \in S(z_0, r_0) \). Put \( f(\gamma)|_{t_2} := f(\gamma)|_{[t_1, t_2]} \). Then \( f(\gamma)|_{t_2} \) is a subpath of \( f(\gamma) \) and, in addition, \( f(\gamma)|_{t_2} \in \Gamma(S(z_0, r_1), S(z_0, r_0), D') \). Without loss of generality, we may assume that \( f(\gamma)|_{t_2} \subset B(z_0, r_0) \). Therefore, \( \Gamma(|\alpha_k|, |\beta_k|, D) > \Gamma_f(z_0, r_1, r_0) \). From the latter relation, due to the minority of the modulus of families of paths (see e.g. [Fu, Theorem 1(c)]) we obtain that

\[
M(\Gamma(|\alpha_k|, |\beta_k|, D)) \leq M(\Gamma_f(z_0, r_1, r_0)).
\]

Combining (5.4) with (1.2), we obtain that

\[
M(\Gamma(|\alpha_k|, |\beta_k|, D)) \leq \int_{A(y_0, r_1, r_0) \cap f(D)} Q(y) \cdot \eta^n(|y - y_0|) \, dm(y),
\]

where \( \eta : (r_1, r_2) \to [0, \infty] \) is any Lebesgue measurable function with \( \int_{r_1}^{r_0} \eta(r) \, dr \geq 1 \).

Below we use the following conventions: \( a/\infty = 0 \) for \( a \neq \infty \), \( a/0 = \infty \) for \( a > 0 \) and \( 0 \cdot \infty = 0 \) (see, e.g., [Sa, 3.I]). Put

\[
I = \int_{r_1}^{r_0} \frac{dt}{t^{1/(n-1)}(t)}. \tag{5.6}
\]

By the assumption, there is a set \( E \subset [r_1, r_0] \) of a positive measure such that \( q_{z_0}(t) \) is finite for all \( t \in E \). In this case, a function \( \eta_0(t) = \frac{1}{t^{1/(n-1)}(t)} \) satisfies the relation (1.3). Substituting this function in the right-hand part of (5.3) and using the Fubini theorem, we obtain that

\[
M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) \leq \frac{\omega_{n-1}}{\int_{r_1}^{r_0} t^{1/(n-1)}(t)} < \infty. \tag{5.7}
\]

The relation (5.7) contradicts with (5.3). The contradiction obtained above disproves the assumption on the absence of a continuous extension of the mapping \( f \) to the boundary of
the domain \( D \). The proof of the equality \( \overline{f(D)} = D' \) is similar to the second part of the proof of Theorem 3.1 in [SSD]. \( \square \)

**Remark 5.1.** The statement of Theorem 5.1 remains true, if in its formulation instead of the specified conditions on function \( Q \) to require that \( Q \in L^1_{\text{loc}}(\mathbb{R}^n) \), \( Q(y) \equiv 0 \) for \( y \in \mathbb{R}^n \setminus f(D) \). The justification of this fact is carried out in exactly the same way as the justification of Remark 2.1.

**Remark 5.2.** The statement of Theorem 5.1 remains true, if in its formulation instead of the specified conditions on function \( Q \) to require that for any \( y_0 \in \partial D' \) there is \( \delta(y_0) > 0 \) such that

\[
\int_0^{\delta(y_0)} \frac{dt}{t^{\frac{n-p}{n}}} = \infty,
\]

for sufficiently small \( \varepsilon > 0 \). This statement may be proved by the choosing of the admissible function \( \eta \) in (5.5) and by the using the fact that the second condition in (5.8) is possible only if the inequality \( q_{y_0}(t) < \infty \) holds for some set \( E \subset [\varepsilon, \delta(y_0)] \) of a positive linear measure.

**Remark 5.3.** The statement of Theorem 5.1 remains true, if in its formulation instead of the specified conditions on function \( Q \) to require that, for any \( y_0 \in \partial D' \) there is \( \varepsilon_0 = \varepsilon_0(y_0) > 0 \) and a Lebesgue measurable function \( \psi : (0, \varepsilon_0) \rightarrow [0, \infty] \) such that the relations (2.16)–(2.17) hold, where \( C_0 \) is some constant, and \( A(y_0, \varepsilon, \varepsilon_0) \) is defined in (1.1). This assertion can be obtained in exactly the same way as the proof of Remark 5.3.

### 6 On the discreteness of mappings with the inverse Poletsky inequality at the boundary of a domain

We prove the following statement (see its analogue for quasiregular mappings of the unit ball in [Vu, Lemma 4.4]).

**Lemma 6.1.** Let \( n \geq 2, n-1 < p \leq n \), let \( D \) be a domain which is uniform with respect to \( p \)-modulus, and let \( D' \) be a regular domain. Let \( f : D \rightarrow \mathbb{R}^n \) be an open discrete and closed mapping in \( D \), for which there is a Lebesgue measurable function \( Q : \mathbb{R}^n \rightarrow [0, \infty] \) equals to zero outside of \( D' \), such that the relations (1.1)–(1.2) hold for any \( y_0 \in \partial D' \). Assume that, for any \( y_0 \in \partial D' \) there is \( \varepsilon_0 = \varepsilon_0(y_0) > 0 \) and a Lebesgue measurable function \( \psi : (0, \varepsilon_0) \rightarrow [0, \infty] \) such that the relations (4.4)–(4.5) hold, where \( A(y_0, \varepsilon_0) \) is defined in (1.1). Let \( C_j, j = 1, 2, \ldots, \) be a sequence of continua such that \( h(C_j) \geq \delta > 0 \) for some \( \delta > 0 \) and any \( j \in \mathbb{N} \) and, in addition, \( \rho(f(C_j)) \rightarrow 0 \) as \( j \rightarrow \infty \). Then there is \( \delta_1 > 0 \) such that \( \rho(f(C_j), P_0) \geq \delta_1 > 0 \) for any \( j \in \mathbb{N} \) and for any \( P_0 \in E_D \), where the metrics \( \rho \) is defined in (1.8).
Here, as usually,

\[ \rho(A) = \sup_{x,y \in A} \rho(x,y), \]
\[ \rho(A, B) = \inf_{x \in A, y \in B} \rho(x,y). \]

**Proof.** Suppose the opposite, namely, let \( \rho(f(C_{j_k}), P_0) \to 0 \) as \( k \to \infty \) for some \( P_0 \in E_{D'} \), and for some increasing sequence of numbers \( j_k, k = 1, 2, \ldots \). Let \( F \subset D \) be any continuum in \( D \), and let \( \Gamma_k := \Gamma(F, C_{j_k}, D) \). Due to the definition of the uniformity of the domain with respect to \( p \)-modulus, we obtain that

\[ M_p(\Gamma_k) \geq \delta_2 > 0 \quad (6.1) \]

for any \( k \in \mathbb{N} \) and some \( \delta_2 > 0 \). On the other hand, let us to consider the family of paths \( f(\Gamma_k) \). Let \( d_l, l = 1, 2, \ldots \), be a sequence of domains which corresponds to the prime end \( P_0 \), and let \( \sigma_l \) be a cut corresponding to \( d_l \). We may assume that \( \sigma_l, l = 1, 2, \ldots \), lie on spheres \( S(y_0, r_l) \) centered at some point \( y_0 \in \partial D' \), where \( r_l \to 0 \) as \( l \to \infty \) (see [IS2], Lemma 3.1), cf. [KR2] Lemma 1).

Let us to prove that, for any \( l \in \mathbb{N} \) there is a number \( k = k_l \) such that

\[ f(C_{j_{k_l}}) \subset d_l, \quad k \geq k_l. \quad (6.2) \]

Suppose the opposite. Then there is \( l_0 \in \mathbb{N} \) such that

\[ f(C_{j_{m_l}}) \cap (\mathbb{R}^n \setminus d_{l_0}) \neq \emptyset \quad (6.3) \]

for some increasing sequence of numbers \( m_l, l = 1, 2, \ldots \). In this case, there is a sequence \( x_{m_l} \in f(C_{j_{m_l}} \cap (\mathbb{R}^n \setminus d_{l_0}), l \in \mathbb{N} \). Since by the assumption \( \rho(f(C_{j_k}), P_0) \to 0 \) for some sequence of numbers \( j_k, k = 1, 2, \ldots \), we obtain that

\[ \rho(f(C_{j_{m_l}}), P_0) \to 0 \quad \text{as} \quad l \to \infty. \quad (6.4) \]

Since \( \rho(f(C_{j_{m_l}}), P_0) = \inf_{y \in f(C_{j_{m_l}})} h(y, P_0) \) and \( f(C_{j_{m_l}}) \) is a compact set in \( D_{D'} \) as a continuous image of the compactum \( C_{j_{m_l}} \) under the mapping \( f \), it follows that \( \rho(f(C_{j_{m_l}}), P_0) = \rho(y_l, P_0) \), where \( y_l \in f(C_{j_{m_l}}) \). Due to the relation \( (6.4) \) we obtain that \( y_l \to y_0 \) as \( l \to \infty \) in the metric \( \rho \). Since by the assumption \( \rho(f(C_j)) = \sup_{y,z \in f(C_j)} \rho(y, z) \to 0 \) as \( j \to \infty \), we have that

\[ \rho(y_l, x_{m_l}) \leq \rho(f(C_{j_{m_l}})) \to 0 \quad \text{as} \quad l \to \infty. \]

Now, by the triangle inequality, we obtain that

\[ \rho(x_{m_l}, P_0) \leq \rho(x_{m_l}, y_l) + \rho(y_l, P_0) \to 0 \quad \text{as} \quad l \to \infty. \]

The latter contradicts with \( (6.3) \). The contradiction obtained above proves \( (4.7) \).

The following considerations are similar to the second part of the proof of Lemma 2.1 in [Sev1]. Without loss of generality we may consider that the number \( l_0 \in \mathbb{N} \) is such that \( r_l < \varepsilon_0 \) for any \( l \geq l_0 \), and

\[ f(F) \subset \mathbb{R}^n \setminus d_1. \quad (6.5) \]
In this case, we observe that, for \( l \geq 2 \)
\[
f(\Gamma_{k_l}) > \Gamma(S(y_0, r_1), S(y_0, r_1), A(y_0, r_1, r_1)). \tag{6.6}
\]
Indeed, let \( \tilde{\gamma} \in f(\Gamma_{k_l}) \). Then \( \tilde{\gamma}(t) = f(\gamma(t)) \), where \( \gamma \in \Gamma_{k_l}, \gamma : [0, 1] \to D, \gamma(0) \in F, \gamma(1) \in C_{\delta_{k_l}} \). Due to the relation (6.5), we obtain that \( f(\gamma(0)) \in f(F) \subset \mathbb{R}^n \setminus B(y_0, \epsilon_0) \). On the other hand, by (6.2), \( \gamma(1) \in C_{\delta_{k_l}} \subset d_1 \subset d_1 \). Thus, \( |f(\gamma(t))| \cap d_1 \neq \emptyset \neq |f(\gamma(t))| \cap (\mathbb{R}^n \setminus d_1) \).

Now, by [Ku] Theorem 1.1.5.46, we obtain that there is \( 0 < t_1 < 1 \) such that \( f(\gamma(t_1)) \in \partial d_1 \cap D \subset S(y_0, r_1) \). Set \( \gamma_1 := \gamma|_{[t_1, 1]} \). We may consider that \( f(\gamma(t)) \in d_1 \) for any \( t \geq t_1 \).

Arguing similarly, we obtain \( t_2 \in [t_1, 1] \) such that \( f(\gamma(t_2)) \in S(y_0, r_1) \). Put \( \gamma_2 := \gamma|_{[t_1, t_2]} \). We may consider that \( f(\gamma(t)) \in d_1 \) for any \( t \in [t_1, t_2] \). Now, a path \( f(\gamma_2) \) is a subpath of \( f(\gamma) = \tilde{\gamma} \), which belongs to \( \Gamma(S(y_0, r_1), S(y_0, r_1), A(y_0, r_1, r_1)) \). The relation (6.6) is established.

It follows from (6.6) that
\[
\Gamma_{k_l} > \Gamma_f(S(y_0, r_1), S(y_0, r_1), A(y_0, r_1, r_1)). \tag{6.7}
\]
Set
\[
\eta_l(t) = \begin{cases} 
\psi(t)/I(r_1, r_1), & t \in (r_1, r_1), \\
0, & t \notin (r_1, r_1), 
\end{cases}
\]
where \( I(r_1, r_1) = \int_{r_1}^{r_1} \psi(t) \, dt \). Observe that \( \int \eta_l(t) \, dt = 1 \). Now, by the relations (4.5) and (6.7), and due to the definition of \( f \) in (4.1), we obtain that
\[
M_p(\Gamma_{k_l}) \leq M_p(\Gamma_f(S(y_0, r_1), S(y_0, r_1), A(y_0, r_1, r_1))) \leq \frac{1}{I_p(r_1, r_1)} \int_{A(y_0, r_1, r_1)} Q(y) \cdot \psi^p(|y - y_0|) \, dm(y) \to 0 \text{ as } l \to \infty. \tag{6.8}
\]
The relation (6.8) contradicts with (6.1). The contradiction obtained above proves the lemma. \( \Box \)

Let \( X \) and \( Y \) be metric spaces. Recall that, a mapping \( f : X \to Y \) is called light, if for any point \( y \in Y \), the set \( f^{-1}(y) \) does not contain any nondegenerate continuum \( K \subset X \). The following lemma generalizes Corollary 4.5 in [Vu] for the case of mappings with unbounded characteristic.

**Lemma 6.2.** Let \( n \geq 2, n - 1 < p \leq n \) and let \( D \) be a domain which is uniform with a respect to \( p \)-modulus. Let \( f : D \to \mathbb{R}^n \) be an open discrete and closed mapping of \( D \) onto \( D' \) for which there is a Lebesgue measurable function \( Q : \mathbb{R}^n \to [0, \infty] \) equals to zero outside \( D' \) such that the conditions (4.1)–(4.2) hold for any point \( y_0 \in \partial D' \). Assume that, there is \( \delta_0 = \delta_0(y_0) > 0 \) and a Lebesgue measurable function \( \psi : (0, \delta_0) \to [0, \infty] \) such that the relations (4.3)–(4.5) hold, where \( A(y_0, \epsilon, \epsilon_0) \) is defined in (4.7). Assume also that, a domain \( D \) is locally connected on its boundary, and \( D' \) is a regular domain. Then \( f \) has a continuous extension \( \tilde{f} : \overline{D} \to \overline{D'}^p \), and \( \tilde{f} \) is light.
Proof. By Theorem 5.1 and Remark 5.3, \( f \) has a continuous extension \( \overline{f} : \overline{D} \to D' \). It remains to prove that \( \overline{f} \) is light, in other words, we need to prove that the set \( \overline{D}^{-1}(P_0) \) does not contain a continuum for any \( P_0 \subset D'_0 \).

Assume the contrary, namely, let \( P_0 \in E_D \), be some point such that \( \overline{D}^{-1}(P_0) \supseteq K_0 \), where \( K_0 \subset \partial D \) is some nondegenerate continuum. Then, in particular, \( \overline{f}(K_0) = P_0 \). Since \( \overline{D} \) is a compactum in \( \mathbb{R}^n \) and, in addition, \( \overline{f} \) is continuous in \( \overline{D} \), this mapping is uniformly continuous in \( \overline{D} \). In this case, for any \( j \in \mathbb{N} \) there is \( \delta_j < 1/j \) such that

\[
\rho(\overline{f}(x), \overline{f}(x_0)) = \rho(\overline{f}(x), y_0) < 1/j \quad \forall \, x, x_0 \in \overline{D}, \quad h(x, x_0) < \delta_j, \quad \delta_j < 1/j.
\] (6.9)

Denote by \( B_h(x_0, r) = \{ x \in \mathbb{R}^n : h(x, x_0) < r \} \). Then, given \( j \in \mathbb{N} \), we set

\[
B_j := \bigcup_{x_0 \in K_0} B_h(x_0, \delta_j), \quad j \in \mathbb{N}.
\]

Since the set \( B_j \) is a neighborhood of \( K_0 \), by [HK, Lemma 2.2] there is a neighborhood \( U_j \) of the set \( K_0 \) such that \( U_j \subset B_j \) and the set \( U_j \cap D \) is connected. Without loss of generality, we may assume that \( U_j \) is open. Then the set \( U_j \cap D \) is path connected, as well (see [MRSY, Proposition 13.1]). Since \( K_0 \) is a compact set, there are \( z_0, w_0 \in K_0 \) such that \( h(K_0) = h(z_0, w_0) \). It follows from this, that there are \( z_j \in U_j \cap D \) and \( w_j \in U_j \cap D \) such that \( z_j \to z_0 \) and \( w_j \to w_0 \) as \( j \to \infty \). We may assume that

\[
h(z_j, w_j) > h(K_0)/2 \quad \forall \, j \in \mathbb{N}.
\] (6.10)

Since the set \( U_j \cap D \) is path connected, we may join points \( z_j \) and \( w_j \) by some path \( \gamma_j \in U_j \cap D \). Set \( C_j := |\gamma_j| \).

Observe that, \( \rho(f(C_j)) \to 0 \) as \( j \to \infty \). Indeed, \( f(C_j) \) is a continuum in \( (D', \rho) \) because \( D' \) may be mapped onto a domain \( D_0 \) with a locally quasiconformal boundary, and \( (D', \rho) \) is homeomorphic to \( (D_0, |\cdot|) \). Therefore, there are points \( y_j, y'_j \in f(C_j) \) such that \( \rho(f(C_j)) = \rho(y_j, y'_j) \). Then there are \( x_j, x'_j \in C_j \) such that \( y_j = f(x_j) \) and \( y'_j = f(x'_j) \). Then points \( x_j \) and \( x'_j \) belong to \( U_j \subset B_j \). Therefore, there are \( x_j^1 \) and \( x_j^2 \in K_0 \) such that \( x_j \in B(x_j^1, \delta_j) \) and \( x'_j \in B(x_j^2, \delta_j) \). In this case, by the relation (6.9) and due to the triangle inequality we obtain that

\[
\rho(f(C_j)) = \rho(y_j, y'_j) = \rho(f(x_j), f(x'_j)) \leq \rho(f(x_j), f(x_j^1)) + \rho(f(x_j^1), f(x_j^2)) + \rho(f(x_j^2), f(x'_j)) < 2/j \to 0 \quad \text{as} \quad j \to \infty.
\] (6.11)

It follows from (6.10) and (6.11) that, the continua \( C_j, \ j = 1, 2, \ldots \), satisfy the conditions of Lemma 6.1. By this lemma we may obtain that \( \rho(f(C_j), P_0) > \delta_1 > 0 \) for any \( j \in \mathbb{N} \). On the other hand, by the proving above \( x_j \in B(x_j^1, \delta_j) \). Now, by the relation (6.9) we obtain that \( \rho(f(x_j), P_0) < 1/j, \ j = 1, 2, \ldots \). The resulting contradiction indicates the incorrectness of the assumption that \( \overline{f} \) is not light in \( \partial D \). Lemma is proved. \( \square \)

Corollary 6.1. The statements of Lemmas 6.1 and 6.2 are fulfilled if we put \( D = \mathbb{R}^n \).
Proof. Obviously, the domain \( D = \mathbb{B}^n \) is locally connected at its boundary. We prove that this domain is uniform with respect to the \( p \)-modulus for \( p \in (n - 1, n) \). Indeed, since \( \mathbb{B}^n \) is a Loewner space (see [He, Example 8.24(a)]), the set \( \mathbb{B}^n \) is Ahlfors regular with respect to the Euclidean metric \( d \) and Lebesgue measure in \( \mathbb{R}^n \) (see [He, Proposition 8.19]). In addition, in \( \mathbb{B}^n \), \((1; p)\)-Poincaré inequality holds for any \( p > 1 \) (see e.g. [HaK] Theorem 10.5). Now, by [AS] Proposition 4.7 we obtain that the relation
\[
M_p(\Gamma(E, F, \mathbb{B}^n)) \geq \frac{1}{C} \min \{\text{diam } E, \text{diam } F\},
\]
holds for any \( n - 1 < p \leq n \) and for any continua \( E, F \subset \mathbb{B}^n \), where \( C > 0 \) is some constant, and \( \text{diam} \) denotes the Euclidean diameter. Since the Euclidean distance is equivalent to the chordal distance on bounded sets, the uniformity of the domain \( D = \mathbb{B}^n \) with respect to the \( p \)-modulus follows directly from (6.12). \( \square \)

Finally, we formulate and prove a key statement about the discreteness of mapping (see [Vu] Theorem 4.7).

Lemma 6.3. Suppose that, under the conditions of Lemma 6.2, \( p = n \), the domain \( D \) is weakly flat and the domain \( D' \) is regular. Then the mapping \( f \) has a continuous extension \( \overline{f} : \overline{D} \rightarrow \overline{D'}^p \) such that \( N(f, D) = N(f, \overline{D}) < \infty \). In particular, \( \overline{f} \) is discrete in \( \overline{D} \), that is, \( \overline{f}^{-1}(P_0) \) consists only from isolated points for any \( P_0 \subset E_{D'} \).

Proof. First of all, the possibility of continuous extension of \( f \) to a mapping \( \overline{f} : \overline{D} \rightarrow \overline{D'}^p \) follows by Remark 5.3. Note also that \( N(f, D) < \infty \), see [MS] Theorem 2.8]. Let us to prove that \( N(f, D) = N(f, \overline{D}) \). Next we will reason using the scheme proof of Theorem 4.7 in [Vu]. Assume the contrary. Then there are points \( P_0 \in E_{D'} \) and \( x_1, x_2, \dots, x_k, x_{k+1} \in \partial D \) such that \( f(x_i) = P_0, i = 1, 2, \dots, k+1 \) and \( k := N(f, D) \). Since by the assumption \( D' \) is regular, there is a mapping \( g \) of some domain with a locally quasiconformal boundary \( D_0 \) onto \( D' \). Let us consider the mapping \( F := f \circ g^{-1} \). Note that, by the definition of a domain with a locally quasiconformal boundary, \( D_0 \) is locally connected on \( \partial D_0 \). Note that, the mapping \( F \) has a continuous extension \( \overline{F} : \overline{D} \rightarrow \overline{D_0} \) to \( \overline{D} \), and \( \overline{F}(\overline{D}) = \overline{D_0} \) (this follows from the fact that each of the mappings \( f \) and \( g^{-1} \) has a continuous extension to \( \overline{D} \) and \( \overline{D'}^p \), respectively). Set \( y_0 := \overline{F}(P_0) \in D_0 \). Now, for any \( p \in \mathbb{N} \) there is a neighborhood \( \overline{U}_p' \subset B(y_0, 1/p) \) of \( y_0 \) such that the set \( \overline{U}_p' \cap D_0 = U_p' \) is connected.

Let us to prove that, for any \( i = 1, 2, \dots, k+1 \) there is a component \( V^i_p \) of the set \( F^{-1}(U_p') \) such that \( x_i \in \overline{V^i_p} \). Fix \( i = 1, 2, \dots, k+1 \). By the continuity of \( F \) in \( \overline{D} \), there is \( r_i = r_i(x_i) > 0 \) such that \( f(B(x_i, r_i) \cap D) \subset U_p' \). By [MRSY] Lemma 3.15], a domain with a weakly flat boundary is locally connected on its boundary. Thus, we may find a neighborhood \( W_i \subset B(x_i, r_i) \) of the point \( x_i \) such that \( W_i \cap D \) is connected. Then \( W_i \cap D \) belongs to one and only one component \( V^i_p \) of the set \( F^{-1}(U_p') \), while \( x_i \in W_i \cap D \subset \overline{V^i_p} \), as required.
Next we show that the sets $\overline{V^i_p}$ are disjoint for any $i = 1, 2, \ldots, k+1$ and large enough $p \in \mathbb{N}$. In turn, we prove for this that $h(\overline{V^i_p}) \to 0$ as $p \to \infty$ for each fixed $i = 1, 2, \ldots, k+1$. Let us prove the opposite. Then there is $1 \leq i_0 \leq k+1$, a number $r_0 > 0$, $r_0 < \frac{1}{2} \min_{1 \leq i,j \leq k+1, i \neq j} h(x_i, x_j)$ and an increasing sequence of numbers $p_m$, $m = 1, 2, \ldots$, such that $S_h(x_{i_0}, r_0) \cap \overline{V_{p_m}^{i_0}} \neq \emptyset$, where $S_h(x_0, r) = \{ x \in \mathbb{R}^m : h(x, x_0) = r \}$, and $h$ denotes the chordal metric in $\mathbb{R}^m$. In this case, there are $a_m, b_m \in \overline{V_{p_m}^{i_0}}$ such that $a_m \to x_{i_0}$ as $m \to \infty$ and $h(a_m, b_m) \geq r_0/2$. Join the points $a_m$ and $b_m$ by a path $C_m$, which entirely belongs to $\overline{V_{p_m}^{i_0}}$. Then $h(|C_m|) \geq r_0/2$ for $m = 1, 2, \ldots$. On the other hand, since $|C_m| \subset f(\overline{V_{p_m}^{i_0}}) \subset B(y_0, 1/p_m)$, then simultaneously $h(F(|C_m|)) \to 0$ as $m \to \infty$ and $h(F(|C_m|), y_0) \to 0$ as $m \to \infty$. Now, by the definition of the metric $\rho$ in (1.8) and of the mapping $g$, we obtain that $\rho(f(|C_m|)) \to 0$ as $m \to \infty$ and $\rho(f(|C_m|), y_0) \to 0$ as $m \to \infty$, that contradicts with Lemma [6.1]. The resulting contradiction indicates the incorrectness of the above assumption.

By [V1] Lemma 3.6] $F$ is a mapping of $\overline{V^i_p}$ onto $U^i_p$ for any $i = 1, 2, \ldots, k, k+1$. Thus, $N(f, D) = N(F, D) \geq k+1$, which contradicts the definition of the number $k$. The obtained contradiction refutes the assumption that $N(f, D) > N(f, D)$. The lemma is proved. □

Let us now turn to the main results of this section.

**Proof of Theorem 1.2** In the case 1), we choose $\psi(t) = \frac{1}{t \log \frac{1}{t}}$, and in the case 2), we set

$$
\psi(t) = \begin{cases} 
1/[t^{\alpha-1} q_0^{-1}(t)] , & t \in (\varepsilon, \varepsilon_0) , \\
0 , & t \notin (\varepsilon, \varepsilon_0) ,
\end{cases}
$$

Observe that, the relations (4.4)–(4.5) hold for these functions $\psi$, where $p = n$ (the proof of this facts may be found in [Sev]). The desired conclusion follows from Lemma [6.3]. □

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